On the ergoregion in the Kerr spacetime: properties of the equatorial circular motion

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We investigate in detail the circular motion of test particles on the equatorial plane of the ergoregion in the Kerr spacetime. We find all the regions inside the ergoregion where circular motion is allowed, and analyze their stability properties and the energy and angular momentum of the test particles. We show that the structure of the stability regions has definite features that make it possible to distinguish between black holes and naked singularities. The naked singularity case presents a very structured non-connected set of regions of orbital stability. The properties of the circular orbits turn out to be so distinctive that they allow the introduction of a complete classification of Kerr spacetimes, each class of which is characterized by different physical effects that could be of particular relevance in observational astrophysics. The presence of counterrotating particles and zero angular momentum particles inside the ergoregion of a specific class of naked singularities is interpreted as due to the presence of a repulsive field generated by the central source of gravity.

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I. INTRODUCTION

Black holes are very probably the central engines of quasars, active galactic nuclei, and gamma ray bursts. Consequently, the mechanism by which energy is extracted from them is of great astrophysical interest. While the exact form of this mechanism is not known, it seems that the effects occurring inside the ergoregion of black holes are essential for understanding the central engine mechanism [1, 2]. It is therefore very important to study all the physical properties of the ergoregion.

In astrophysics, it is particularly interesting to study the general features of the motion of test particles moving along circular orbits around the central source. In fact, one can imagine a thin disk made only of test particles as a hypothetical accretion disk of matter surrounding the central source. Although this is a very idealized model for an accretion disk, one can nevertheless extract some valuable information about the dynamics of particles in the corresponding gravitational field and the amount of energy that can be released by matter when falling into the central mass distribution [3] and [6], with the equatorial circular geodesics being then relevant for the Keplerian accretion disks. In addition, recently we formulated the question about the geometric structure of such an idealized accretion disk and how it can depend on the values of the physical parameters that determine the gravitational field. This is also an important issue since it could lead to physical effects that depend on the structure of the accretion disk, with the corresponding possible observational consequences. Indeed, in a series of previous studies [5–12], it was established that the motion of test particles on the equatorial plane of black hole spacetimes can be used to derive information about the structure of the central source of gravitation (see also discussions in [13–15] and [16]). In [9] and [12], we discussed some of the characteristics of the circular motion, both in the Kerr and Kerr-Newman spacetimes for black holes and naked singularities. Typical effects of repulsive gravity emerged in the naked singularity ergoregion. In this work, we clarify and deepen those results, providing a classification of attractor sources on the basis of the properties of this type of dynamics (see also [14–17] and [18]). It is the main aim of the present work to analyze the circular motion of test particles in the equatorial plane of the Kerr spacetime. In particular, we will investigate in detail the differences that appear when the central source is a black hole or a naked singularity.

One of the main features of the ergoregion is that test particles must rotate together with the central source, an effect known as frame dragging. However, there is an increasing interest in the properties of the matter dynamics in

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the ergoregion, especially as a possible source of data discriminating between black holes and super-spinning objects (see [9] and [14, 17, 18], also [19, 20]).

This paper is organized as follows. In Sec. II we present the Kerr line element and briefly discuss the main physical properties of the corresponding spacetime. We present the effective potential that governs the dynamics of the test particle motion on the equatorial plane, and derive the conditions for the existence and stability of circular orbits. Our approach consists in analyzing the behavior of the energy and angular momentum of the test particles as functions of the radial distance and of the intrinsic angular momentum of the central source, which for the sake of brevity will be often referred as the “spin” of the source. The results are then compared with the behavior of the effective potential. This procedure allows us to carry out a methodical and detailed physical analysis of all the regions inside the ergoregion in which circular motion is allowed. In Secs. III and IV, we apply our method to explore all the relevant orbital regions in the case of black holes and naked singularities, respectively. It turns out that it is necessary to introduce a classification of these spacetimes in terms of the values of the rotational parameter. In the case of black holes, we find three different classes which are denoted as BH-I, BH-II and BH-III, whereas for naked singularities four classes are introduced and denoted by NS-I, etc. It turns out that circular orbits in extreme black holes and circular orbits on the ergosphere surface have a particular rich structure which is investigated in Secs. V and VI respectively. In Sec. VII we perform a comparison of our main results in the case of black holes and naked singularities, emphasizing the differences in the structure of the accretion disks. Finally, in Sec. VIII we discuss the results and perspectives of our work.

II. GENERAL PROPERTIES

The Kerr spacetime is an exact solution of Einstein’s equations in vacuum that describes an axisymmetric, stationary (nonstatic), asymptotically flat gravitational field. In Boyer–Lindquist (BL) coordinates with signature (− + + +), the Kerr line element has the form

\[ ds^2 = -dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2) \sin^2 \theta d\phi^2 + \frac{2M}{\rho^2} r (dt - a \sin^2 \theta d\phi)^2 , \]

where

\[ \Delta \equiv r^2 - 2Mr + a^2, \quad \rho^2 \equiv r^2 + a^2 \cos^2 \theta . \]

The redshift infinity surface and event horizons are determined respectively by the equations \( g_{tt} = 0, \) and \( g^{rr} = 0 . \) The corresponding solutions are interpreted as the outer and inner ergosurfaces with

\[ r_{\pm}^e = M \pm \sqrt{M^2 - a^2 \cos^2 \theta} \]

and the outer and inner horizons with

\[ r_{\pm} = M \pm \sqrt{M^2 - a^2} . \]

Here \( M \) is a mass parameter and the specific angular momentum is given as \( a = J/M, \) where \( J \) is the total angular momentum of the gravitational source. In this work, we will consider the Kerr black hole (BH) case defined by \( a \in [0, M[ , \) the extreme black hole source \( a = M, \) and the naked singularity (NS) case for which \( a > M. \) The limiting case \( a = 0 \) is the Schwarzschild solution.

A. The black hole case \((0 < a < M)\)

In this spacetime the \( r \)-coordinate is spacelike in the intervals \( r \in [0, r_-] \cup r > r_+ , \) and timelike in the region \( r \in [r_-, r_+]. \) This means that the surfaces \([2] \) of constant \( r, \) say \( \Sigma_r, \) are timelike for \( \Delta > 0, \) spacelike for \( \Delta < 0, \) and null for \( \Delta = 0. \) Moreover, we have that

\[ g_{tt} < 0, \quad \text{for} \quad r \in [0, r_{-}^-] \cup [r_{+}^+, \infty] . \]

Inside the interval \( r_{-}^- < r < r_{+}^+ \) the metric component \( g_{tt} \) changes its sign. Moreover, \( g_{tt} \) vanishes for \( r = r_{-}^\pm \) and \( 0 < \cos^2 \theta \leq 1, \) and also at \( r = 2M \) for \( \theta = \pi/2. \) The location of these hypersurfaces is such that:

\[ r_{-}^- < r_- < r_+ < r_{+}^+. \]

The region \( r_+ < r < r_{+}^+, \) where \( g_{tt} > 0, \) is called ergoregion. In this region the Killing vector \( \xi_t^a = (1, 0, 0, 0) \) becomes spacelike, i.e., \( g_{ab} \xi_a^a \xi_t^b = g_{tt} > 0. \) This fact implies in particular that a static observer, i.e., an observer with four velocity proportional to \( \xi_t^a \) so that \( \dot{\theta} = \dot{\phi} = 0, \) (where the dot denotes the derivative with respect to the proper time \( \tau \) along the curve), cannot exist inside the ergoregion: an observer inside this region is forced to move.
B. The extreme black hole case \((a = M)\)

In this spacetime, the horizons coincide, \(r_- = r_+ = M\), and if \(0 \leq \cos^2 \theta < 1\), then
\[
g_{tt} < 0 \quad \text{for} \quad 0 < r < r_+^\epsilon \quad \text{and} \quad r > r_+^\epsilon,  
\] (7)
while if \(\cos^2 \theta = 1\), i.e., on the axis,
\[
g_{tt} < 0 \quad \text{for} \quad 0 < r < M \quad \text{and} \quad r > M.  
\] (8)

Furthermore, the geometry of this case is such that
\[
r_+^\epsilon < r_- = r_+ < r_+^\epsilon, \quad \text{for} \quad 0 \leq \cos^2 \theta < 1,  
\] (9)
\[
r_+^\epsilon = r_- = r_+ = r_+^\epsilon \quad \text{for} \quad \cos^2 \theta = 1.  
\] (10)

C. The naked singularity case \((a > M)\)

In naked singularities, there are no solutions for the equation \(g^{rr} = 0\) and the radii \(r_{\pm}\) are not real. The singularity at \(\rho = 0\) is not covered by a horizon. On the equatorial plane, we have that \(\rho = r\) and the spacetime singularity is located at \(r = 0\). However, the ergospheres \(r_{\pm}^\epsilon\) are well defined and
\[
g_{tt} < 0  
\]
\[
\text{in} \quad 0 \leq \cos \theta^2 \leq \frac{M^2}{a^2} \quad \text{for} \quad 0 < r < r_-^\epsilon \cup r > r_+^\epsilon  
\] (11)
\[
\text{in} \quad \frac{M^2}{a^2} < \cos \theta^2 \leq 1 \quad \text{for} \quad r > 0.  
\] (12)

In particular, on the equatorial plane, where \(r_+^\epsilon = 2M\) and \(r_-^\epsilon = 0\), \(g_{tt} < 0\) for \(r > 2M\).

D. The equatorial plane

Since in this work we are interested in the dynamics inside the ergoregion on the plane \(\theta = \pi/2\), for the sake of completeness we mention here the main properties of the resulting spacetime. The line element is given by
\[
ds^2 = -dt^2 + \frac{\Delta}{r^2} dr^2 + (r^2 + a^2)d\phi^2 + \frac{2M}{r}(dt - ad\phi)^2,  
\] (14)
and the non-vanishing components of the Christoffel symbols of the first kind are
\[
\Gamma_{trr} = -\Gamma_{ttr} = -\frac{M^2}{r^2}, \quad \Gamma_{tr\phi} = -\Gamma_{rt\phi} = \frac{Ma}{r^2}, \quad \Gamma_{r\phi r} = \frac{M}{r^2}(Mr - a^2), \quad \Gamma_{r\phi \phi} = -\Gamma_{\phi r\phi} = \frac{r^3 - Ma^2}{Mr^2},  
\] (15)
which are then used to compute the geodesics equations. Notice that in the static limit, \(r \to r_+^\epsilon\), all the components of the Christoffel symbols are well behaved in this coordinate system and, consequently, the geodesics equations are also well defined.

E. Symmetries and Killing vectors

In the investigation we will perform below, several symmetries and properties of the Kerr spacetime will be used in order to simplify the analysis. We review here the most important aspects.

- The Kerr metric \([1]\) is invariant under the application of any two different transformations: \(P(Q) : Q \to -Q\), where \(Q\) as one of the coordinates \((t, \phi)\) or the metric parameter \(a\). On the other hand, a single transformation leads to a spacetime with an opposite rotation respect to the unchanged metric.

- The Kerr metric is symmetric under reflection through the equatorial hyperplane \(\theta = \pi/2\). We limit ourselves to the case of equatorial trajectories because they are confined in the equatorial geodesic plane (geodesics starting in the equatorial plane are planar) as a consequence of the reflection symmetry.
- Killing vectors and constant of motion: Let \( u^a = dx^a/d\tau = \dot{x}^a \) be the tangent vector to a curve \( x^a(\tau) \). The momentum \( p^a = \mu \dot{x}^a \) of a particle with mass \( \mu \) is normalized so that \( g_{ab} \dot{x}^a \dot{x}^b = -k \), where \( k = 0, -1, 1 \) for null, spacelike and timelike curves, respectively.

Since the metric is independent of \( \phi \) and \( t \), the covariant components \( p_\phi \) and \( p_t \) of the particle’s four-momentum are conserved along its geodesic. Thus, we use the fact that the quantity

\[
E \equiv -g_{ab} \xi^a p^b = -(gt^t + g_{\phi \phi} p^\phi),
\]

\[
L \equiv g_{ab} \xi^a p^b = g_{\phi \phi} p^\phi + g_{t \phi} p^t.
\]

is a constant of motion, where \( \xi_t = \partial_t \) is the Killing field representing stationarity. In general, we may interpret \( E \) for timelike geodesics, as representing the total energy of the test particle for a particle coming from radial infinity, as measured by a static observer at infinity. On the other hand, the rotational Killing field \( \xi_\phi = \partial_\phi \) yields the constant of motion \( L \) interpreted as the angular momentum of the particle as measured by an observer at infinity.

F. The energetic inside the Kerr ergoregion

As we have already noted, a fundamental property of the ergoregion from the point of view of the matter and field dynamics configuration is that any matter can be at rest (as seen by a faraway observer or, in other words, from infinity in a BL coordinate frame) \([21]\). Then, the metric is no longer stationary. In this paper, we will use mainly the BL coordinates for which the following statements are valid:

- The surfaces of constant \((r, \theta, \phi)\), with line element \( ds |_{\Sigma_r, \theta, \phi} \), are spacelike inside the ergoregion, that is, the “time” interval becomes spacelike, and in terms of BL coordinates this means that \( t \) is spacelike. This is why we are forced to change the coordinate system inside the ergoregion and any motion projected into \( \Sigma_{r, \theta, \phi} \) is forbidden.

- We focus our attention on the plane \( \theta = \pi/2 \) where \( r_+^+ | r/2 = r_+ |_{a=0} = 2M \). We compare the dynamics in the regions \( [r_+, r_+^+] \) and \( [0, r_+ |_{a=0}] \) for the BH and NS cases, respectively. Thus, in the static (BH) spacetime \((a = 0)\) the region within the interval \([0, r_+ |_{a=0}] \) coincides with the zone inside the horizon; then, no particle can stay at rest (with respect to an observer located at infinity) neither at \( r = \text{constant} \), i.e., any particle is forced to fall down into the singularity. On the other hand, for the stationary spacetimes \((a \neq 0)\) in the region \( [r_+, r_+^+] \) the motion with \( \phi = \text{constant} \) is not possible and all particles are forced to rotate with the source \( \dot{\phi}a > 0 \). Nevertheless, trajectories with \( r = \text{constant} \) and \( \dot{r} > 0 \) are possible. Another important point is that for an observer at infinity, the particle will reach and penetrate the surface \( r = r_+^+ \), in general, in a finite time \( t \). For this reason, the ergosphere is not a surface of infinite redshift, except for the axis of rotation where the ergosphere coincides with the event horizon.

- Concerning the frequency of a signal emitted by a source in motion along the boundary of the ergoregion \( r_+^+ \), it is clear that the proper time of the source particle is not null. This means that the observer at infinity will see a non-zero emission frequency. In the spherical symmetric case, however, as \( g_{t \phi} = 0 \) the proper time interval \( d\tau = \sqrt{|g_{tt}|} dt \), goes to zero as one approaches \( r = r_+ \). Thus, on the equatorial plane as \( a \to 0 \) and the geometry “smoothly” resembles the spherical symmetric case the frequency of the emitted signals, as seen by the observer at infinity, goes to 0.

G. The particle’s energy and effective potential

In general, the particle’s energy can be defined in two different ways: \( E_\tau \) in terms of \( \partial S/\partial \tau \) and \( E = E_t \) in terms of \( \partial S/\partial t \), where \( S \) is the particle action. The energy \( E_\tau \) is defined with respect to the proper time of the particle synchronized along the trajectory. This quantity is always positive but not conserved, in general. \( E_t \), on the other hand, contains the derivative with respect to the universal time and takes account of the symmetries of the stationary spacetime; it is constant along the orbit of the time-Killing vector. This quantity, as defined in Eq. \([16]\), is conserved, but can be negative inside the ergoregion, where \( t \) is no more a timelike coordinate (see, for example, \([21]\)). From the normalization condition of the particle four-velocity, and using \( \dot{\theta} = 0 \), we get

\[
g_{tt} \dot{t}^2 + g_{\phi \phi} \dot{\phi}^2 + 2g_{t \phi} \dot{t} \dot{\phi} + g_{rr} r^2 = -k.
\]
Furthermore, using Eqs. (16) for a particle in circular motion, i.e. $\dot{r} = 0$, we obtain from Eq. (18)

$$V^\pm = -g_{\phi\phi}L \pm \sqrt{\left(g_{\phi\phi}^2 - g_{tt}g_{\phi\phi}\right)(L^2 + g_{\phi\phi}k\mu^2)} / g_{\phi\phi},$$  \hspace{1cm} (19)$$

where $V^\pm/\mu$ is the effective potential which represents that value of $E/\mu$ that makes $r$ into a “turning point” ($V = E/\mu$); in other words, it is the value of $E/\mu$ at which the (radial) kinetic energy of the particle vanishes. The (positive) effective potential can be written explicitly as

$$V \equiv V^+ = -\frac{\beta}{2\alpha} + \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha},$$ \hspace{1cm} (20)$$

where $\alpha \equiv (r^2 + a^2)^2 - a^2\Delta$, $\beta \equiv -2aL(r^2 + a^2 - \Delta)$, $\gamma \equiv a^2L^2 - (M^2r^2 + L^2)\Delta$. \hspace{1cm} (21)$$

The behavior of the effective potential $V^-$ can be studied by using the following symmetry $V^+(L) = -V^-(L)$. Moreover, we note that the potential function (20) is invariant under the mutual transformation of the parameters $(a, L) \rightarrow (-a, -L)$. Therefore, we will limit our analysis to the case of positive values of $a$ for corotating ($L > 0$) and counterrotating ($L < 0$) orbits.

The investigation of the motion of test particles in the spacetime [1] was thus reduced to the study of motion in the effective potential $V$. We will focus on (timelike) circular orbits for which (see also [9])

$$\dot{r} = 0, \quad V = \frac{E}{\mu}, \quad \frac{dV}{dr} = V' = 0.$$ \hspace{1cm} (22)$$

We solve the condition $V' = 0$ with respect to the angular momentum and find

$$L^\pm \equiv \frac{\left|\frac{r^2}{M^2} \pm 2\frac{r}{M} \sqrt{\frac{r}{M}} + \frac{r^2}{M^2}\right|}{\sqrt{\frac{r^2}{M^2} \left(\frac{r}{M} - 3\right) + 2\frac{r}{M} \sqrt{\frac{r^3}{M^2}}} \mu M}.$$ \hspace{1cm} (23)$$

This means that any particle moving along a circular orbit in these spacetime must have as an angular momentum of magnitude either $L^+$ or $L^-$. If we now insert the above expression into the effective potential, we obtain the different types of energy the test particle can have, namely,

$$E^{(\pm)}_\pm \equiv E(L\pm), \quad E^{(\pm)}_{\mp} \equiv E(-L\pm)$$ \hspace{1cm} (24)$$

for corotating and counterrotating orbits, respectively. The angular momenta $\pm L^\pm$ and energies $E^{(\pm)}$, together with the corresponding radii, determine entirely the properties of the test particles moving along circular orbits. These are the main quantities that will be used below to explore the physical properties of spacetimes describe by the Kerr metric.

H. Stability of circular orbits and notable radii

For both NS- and BH-sources, the last circular orbits are located at

$$r_\gamma^\pm \equiv 2M \left(1 + \cos\left[\frac{2\arccos\left(\mp \frac{a}{M}\right)}{3}\right]\right),$$ \hspace{1cm} (25)$$

for orbits with angular momentum $L = \pm L^\pm$, respectively. No circular orbits can be located at $r > r_\gamma$.

For all the circular orbits we will investigate the stability properties which are determined by the value of the second derivative of the effective potential. In this connection, the inflection points determined by the condition

$$\frac{d^2V}{dr^2} = V'' = 0,$$ \hspace{1cm} (26)$$
are of special interest since they determine the radii of the last stable circular orbits \( \text{lsco} \). In the case of naked singularities, the last stable circular orbits correspond to the following radii:

\[
\begin{align*}
\text{lsco}^{\text{NS}}^+ & = M \left( 3 - Z_2 + \sqrt{(3 - Z_1)(3 + Z_1 - 2Z_2)} \right) \quad \text{for } a/M > 1.28112 \quad (\text{NS} : \text{L}_-), \\
\text{lsco}^{\text{NS}}^- & = M \left( 3 - Z_2 - \sqrt{(3 - Z_1)(3 + Z_1 - 2Z_2)} \right) \quad \text{for } a/M \in [1, 1.28112] \quad (\text{NS} : -\text{L}_-)
\end{align*}
\]

where \( Z_1 \equiv 1 + \left[ 1 - (a/M)^2 \right]^{1/3} \left[ (1 + a/M)^{1/3} + (1 - a/M)^{1/3} \right] \) and \( Z_2 \equiv \sqrt{3(a/M)^2 + Z_1^2} \) (see Fig. 1). \( \text{lsco}^{\text{NS}}_r \) is the last stable circular orbit for the orbits with \( L = -L_- \) and \( \text{lsco}^{\text{NS}}_L \) with \( L = L_- \). In the case of black holes, we obtain

\[
\begin{align*}
\text{lsco}^{\text{BH}}_r & = M \left( 3 + Z_2 + \sqrt{(3 - Z_1)(3 + Z_1 + 2Z_2)} \right), \\
\text{lsco}_r^- & = (\text{BH} : \text{L}_-), \\
\text{lsco}_r^+ & = (\text{BH/NS} : -\text{L}_+)
\end{align*}
\]

for corotating \( L_- \) and counterrotating orbits \((-L_+)\), respectively. For the investigation of the different regions inside the ergoregion where circular orbits are allowed, it is convenient to introduce various radii as follows

\[
\hat{r}_\pm \equiv \frac{1}{\sqrt{6}} \left[ \mathcal{S} \pm \sqrt{\frac{6\sqrt{6}a^2M}{\mathcal{S}}} - \mathcal{S}^2 - 6a^2 \right], \quad \mathcal{S} = \sqrt{\frac{4a^4}{3} + s^{1/3} - 2a^2},
\]

where \( s = \left[ 27M^2a^4 - 8a^6 + 3M\sqrt{81M^2a^8 - 48a^{10}} \right] \).

The radii \( \hat{r}_\pm \) are solutions of the equations \( V' = 0 \quad L = 0 \) (see also [9]). The \textit{marginally bounded orbits} are located at

\[
\begin{align*}
\text{lsco}_b^{\text{NS}} & \equiv 2M + a - 2\sqrt{M\sqrt{M} + a} \quad (\text{NS} : \text{L}_-) \\
\text{lsco}_b^{\text{NS}} & \equiv 2M + a \pm 2\sqrt{M\sqrt{M} \pm a}, \quad \text{lsco}_b^- \equiv (\text{BH} : \text{L}_-), \quad \text{lsco}_b^+ \equiv (\text{BH/NS} : -\text{L}_+)
\end{align*}
\]

for black hole sources with angular momentum \( L = L_- \) and naked singularity at \( L = -L_+ \). The meaning and origin of these radii will be explained in the sections below. In Fig. 2 we show the behavior of the radii in terms of the angular momentum \( a \). The different values of the radii and the points of intersection determine three intervals for black holes \((a \leq M)\) and four intervals for naked singularities \((a > M)\) in which the properties of the circular orbits must be investigated. The intervals are characterized by different values of the parameter \( a \) as illustrated in Fig. 2. The analysis of circular orbits in the region \( r > r_b^+ \) can be found in [9]. In the following subsections, we will investigate the properties of circular orbits inside the ergosurface radius \( r < r_b^+ \). Figures 4 and 5 summarize the main results of this analysis. For a better presentation of all the details of our results, we will now discuss each case separately.
FIG. 2: Arrangement of the radii determining the properties of circular orbits around a rotating central mass. The right plot shows the regions of orbital radii $r_\pm$ and $r_{lsco}$ for arbitrary $r$ and $a$ and find:

$$\frac{L}{\mu} = -\sqrt{-\frac{r\Delta}{r - 2M}}, \quad r < 2M.$$ (32)

Moreover, if we solve the equation $V(r; L, a) = 0$ for arbitrary $r$ and $a$ and find:

$$L^+_{\gamma} = \frac{2}{3} \left[ 1 + \frac{\sqrt{4M^4 - 3a^2M^2 - 3L^2}}{M^2} \right] \cos \left( \frac{1}{3} \arccos \left( \frac{M^2(8M^4 - 9a^2M^2 + 18L^2)}{(4M^4 - 3M^2a^2 - 3L^2)^{3/2}} \right) \right), \quad a < a^\mu, \quad r > 2M.$$ (33)

$$L^-_{\gamma} = \frac{2}{3} \left[ 1 - \frac{\sqrt{4M^4 - 3a^2M^2 - 3L^2}}{M^2} \right] \sin \left( \frac{1}{3} \arcsin \left( \frac{M^2(8M^4 - 9a^2M^2 + 18L^2)}{(4M^4 - 3M^2a^2 - 3L^2)^{3/2}} \right) \right), \quad a > M, \quad r > 2M.$$ (34)

$$L_{lsco}^+ = \frac{2}{3} \left[ 1 - \frac{\sqrt{4M^4 - 3a^2M^2 - 3L^2}}{M^2} \right] \sin \left( \frac{1}{6} \pi + 2 \arccos \left( \frac{M^2(8M^4 - 9a^2M^2 + 18L^2)}{(4M^4 - 3M^2a^2 - 3L^2)^{3/2}} \right) \right), \quad a \in [M, a^\mu], \quad r > 2M.$$ (35)

where $a^\mu = 4\sqrt{2/3}/3M \approx 1.08866M$. We note that the radii $r_{\gamma}$ are even in $a$ and $L$; this means that the solutions are independent of the corotating or counterrotating nature of the orbit. The boundary of the regions of orbital momentum where $r_{\gamma}$ exists, are given by $L = \pm L^\pm_{\gamma}$, which can be written as

$$L^+_{\gamma} = \frac{\sqrt{-8M^2 - 3a^2 + 4M\sqrt{4M^4 + 27a^2} \cos \left( \frac{1}{3} \arccos \left( \frac{128M^4 - 27a^2(8M^2 + 27a^2)}{16M(4M^2 + 27a^2)^{3/2}} \right) \right)}}{\sqrt{3M}}, \quad r > 2M.$$ (36)

$$L^-_{\gamma} = \frac{\sqrt{-8M^2 - 3a^2 - 4M\sqrt{4M^4 + 27a^2} \sin \left( \frac{1}{3} \arcsin \left( \frac{128M^4 - 27a^2(8M^2 + 27a^2)}{16M(4M^2 + 27a^2)^{3/2}} \right) \right)}}{\sqrt{3M}}, \quad r > 2M.$$ (37)

These values of angular momenta are valid only for $a < a^\mu$. At the limiting value $a = a^\mu$, they reduce to $L^+_{\gamma} = L^-_{\gamma} = 2/3\sqrt{3M}\mu$: Moreover, $L^-_{\gamma} = 0$ at $a = M$ (see Fig. 3). We will see below that indeed there are particles with
zero energy in the case of black holes (not circular orbits but spiraling motion as discussed in Sec. III). In naked singularities, instead, there are two possible types of zero-energy particles moving along circular orbits.

III. BLACK HOLES

We will consider in this case three different intervals of values of the source angular momentum which define three different sets of black holes denoted by BH – I: $a \in [0, \tilde{a}_1]$, BH – II: $a \in [\tilde{a}_1, \tilde{a}_2]$, and BH – III: $a \in [\tilde{a}_2, M]$. For each of these sets we will now investigate the properties of the circular orbits, the effective potential, and the energy of the orbiting test particle.

A. The set BH – I: $a \in [0, \tilde{a}_1]$

Orbital regions:
In general, corotating orbits with $L = L_-$ exist for $r > r_\gamma^-$, and counterrotating orbits with $L = -L_+$ for $r > r_\gamma^+$ (cf. Fig. 2, see also [9]). In the region $(r_\gamma^-, r_\gamma^+)$, there are timelike orbits. For the limiting orbits with $r = r_\gamma^\pm$, the circular corotating $(E_-, L_-)$ and counterrotating $(E_+, -L_+)$ orbital energies and angular momenta, respectively, are not well defined. The radius $r_\gamma^-$ crosses the static limit, $r_\gamma^+ = r_\gamma^-$, on the equatorial plane only if the rotation of the source is $\tilde{a}_1 \equiv 1/\sqrt{2} M \approx 0.707107 M$, i.e., no circular motion can occur for $r < r_\gamma^+$ in the spacetime with spin $\tilde{a}_1$. The orbital amplitude of this region reaches its maximum $\Delta_{I}^\text{max} \equiv (r_\gamma^+ - r_\gamma^-)|_{\tilde{a}_1} \approx 0.458804 M$ in its upper extreme (cf. Fig. 3, upper-left and Figs. 2).

Effective potential:
There is no solution for Eqs. (22), because the effective potential has no extreme point and it is always increasing ($V' > 0$) in this region. This implies that black holes with spin value in the interval $a \in [0, \tilde{a}_1]$ does not allow circular motion inside the ergoregion. For the sake of completeness, however, we describe here the behavior of the effective potential for this class of black holes. As has been described in Sec. II the potential is always positive definite for $L > 0$. For $L = 0$, the potential is positive and vanishes as the horizon is approached. For negative values of $L$, the potential can be negative or zero at $L = L_0^\pm$ (see Fig. 2 and Eqs. (33)). Thus, for particles with angular momentum $L < L_0^\pm$, we have that $V \leq 0$ in the interval $r \in [r_\gamma^-, r_0^\pm]$, whereas for $L_0^+ < L < 0$ the potential $V \leq 0$ in the interval $r \in [r_\gamma^+, r_0^\pm]$. Notice, however, that in this case $L$ is just a parameter that controls the behavior of the potential function, as there are no orbits for these sources in the ergoregion.

B. The set BH – II: $a \in [\tilde{a}_1, \tilde{a}_2]$

Orbital regions:
FIG. 4: Arrangement of the radii determining the properties of circular orbits around a rotating central mass. Different regions covering the black hole BH and naked singularity NS sources are enlightened. The region outside the static limit \( r > r_p^+ \) is in gray, and the one with \( r < r_+ \) is in black. The arrow covers the region \( a \in [\tilde{a}_2, \tilde{a}_1] \), where stability for orbits inside the ergoregion can occur.

FIG. 5: Arrangement of the radii determining the properties of circular orbits around a rotating central mass. Different regions covering the black hole BH and naked singularity NS sources are enlightened. The region outside the static limit \( r > r_p^+ \) is in gray, and the one with \( r < r_+ \) is in black. The orbital stability in each region is analyzed and explicitly shown: The angular momentum is quoted with its stability property as \( |l_\alpha| \) for stable and \( |l_u| \) for unstable orbits. The last stable circular orbit \( r_{\text{lsco}} \) is plotted as a function of the spin \( a/M \) source (see text). The right plot shows a subset of sources of the classes BH-III and NS-I. The gray region is a neighborhood of the extremal case \( a = M \). Orbits with radii \( \tilde{r}_\pm \) have \( L = 0 \), and with radii \( r_p^\pm \) are counterrotating orbits \( (L = -L_\pm) \) with \( E = 0 \). The region \( [r_u^-, r_u^+] \) contains only counterrotating orbits with \( E < 0 \). At the radii \( r_p^\pm \) (gray dotted lines) defined in Eqs. \[30-31\] there are corotating orbits \( (L = L_+ \) with \( E = \mu \).
The existence of a region of instability in these spacetimes is important because it should give rise to the decay phenomena in which a particle may either escape, spiraling, into the outer region and therefore could become observable, or be captured by the source changing its spin-to-mass ratio (see, for example, [22]). We conclude that no critical points for the energy function and, therefore, the class BH-III we also plot $E_\gamma \equiv V(L_-)$ as a function of $a/M$ and $r/M$. The horizon surface $r_+$ (black) and $r_{lsco}$ (gray) are also plotted. The gap in the energy profile in the region $r < r_\gamma$ is evident.

In these spacetimes, there are only unstable circular orbits in the ergoregion with $L = L_-$ and $r > r_\gamma^-$ (see Fig. 4, upper-left). The width of the unstable circular orbit region increases with the spin, and the upper limit of this region is for sources with $a_2/M = 2\sqrt{2}/3 \approx 0.942809$ which corresponds to the value where $r_{lsco}(L_-) = r_\gamma^-$. The maximum extension of the instability region is $\Delta_{\gamma}^{\text{inst}} \equiv (r_\gamma^- - r_\gamma^+) |_{a_2} \approx 0.585384 M$. The extension of the complementary orbital region, where circular orbits are not possible, is $\Delta_\gamma \equiv [r_+, r_\gamma^-]$. This length has an upper and lower extreme which decreases with $a/M$; its extension therefore varies in the range $|\Delta_\gamma|/M \in [0.585384, 0.17547]$ in units of mass. The analysis of the orbit stability outside the ergoregion for this set of spacetimes has been studied in detail in [9]. In general, the existence of a region of instability in these spacetimes is important because it should give rise to the decay phenomena in which a particle may either escape, spiraling, into the outer region and therefore could become observable, or be captured by the source changing its spin-to-mass ratio (see, for example, [22]). We conclude that no disk of test particles can be formed inside the ergoregion due to the instability.

Effective potential and particle energy:

We first consider the sign of the potential function. The results are similar to those obtained for sources within the class BH-I: For $L \geq 0$, we have $V > 0$, and $V \to 0$ as the horizon is approached. For the solution with negative energy we obtain similar results. The novelty is the presence of critical points for the energy function and, therefore, the presence of unstable orbits, as can be seen in Fig. 6. Clearly, for this case $E_\gamma > 0$, and, as the orbital angular momentum $(L_-)$ increases, it approaches $r_\gamma^-$ where it is not well defined. On the other hand, it decreases with the source spin, indicating that at a fixed orbit the energy decreases as the spin of the source increases. The instability region also covers a larger orbital region. In general, in the instability region the orbiting particles can possibly fall into the singularity, giving its energy and angular momentum to the black hole. Since the particle energy and the angular momentum in such orbits cannot be negative, they cannot contribute negatively to the black hole energy. However, the particle might run away to infinity, because the energy to decay into a lower circular orbit is higher than the energy required for spiraling outward to an exterior orbit. This could give rise to the ejection of positive energy outside the static limit. The presence of these orbits might be used to identify the black hole spin. However, considering the energy of the unstable orbits, it is evident from Figs. 8 and 9 that the class BH-II is crossed by the radius $r_\gamma^-$ in the interval $[r_\gamma^-, r_+^+]$, where $V(r_\gamma^-, L_-) \equiv E_\gamma(r_\gamma^-) = \mu$ (cf. Eq. 31)).

Orbital regions:

For the third set of sources the orbital stability in the ergoregion needs to be discussed in more detail. In fact, since the radius $r_{lsco}$ lies entirely in this region, it becomes split into two parts, one with only stable circular orbits and the other one with only unstable circular orbits, i.e., in the interval $[r_+^+, r_{lsco}]$ there are only stable orbits with $L = L_-$. The extension of this region increases with the intrinsic spin of the source. It should be noted that this
set of sources is characterized by large values of the spin-to-mass ratio, i.e. $a/M \in [0.942809,1]$, and it includes the extreme Kerr-BH. The maximum orbital extension of this region occurs then in the extreme-BH case where $r_{\text{isco}}(a = M) = M$. It interesting to note that close to the boundary spacetimes (at $a = \tilde{a}_2$ and $a = M$), the properties around the limiting spin value are rather subject to a sort of fine-tuning in the sense that, at a fixed radial distance from the source, the dynamical properties of the test particles are completely different for a source situated to the left or to the right within the interval of spins $a = \tilde{a}_2$ and $a = M$. For this reason, the case of extreme black holes deserve a detailed analysis which will be carried out in Sec. [11]. For the BH-III set of black holes we have that $\Delta_{\text{Ext}} = M$. On the other hand, however, the internal region determined by the interval $|r_{\gamma}, r_{\text{isco}}|$ is characterized by unstable circular orbits with $L = L_-$, followed by the region in the interval $|r_+, r_{\gamma}|$ which is forbidden for this type of orbits. Nevertheless, in contrast to BH-II and BH-I spacetimes, the extension of the orbital region and the region of stability for BH-III sources decreases with the spin of the source, having a maximum extension at the spin $\tilde{a}_2$. However, the remarkable feature of this region is the presence of a stability zone where the following conditions are valid $V(L_-) = E_-, V'(L_-) = 0$, $V''(L_-) > 0$ (see Fig. 4-upper-left).

Effective potential and particle energy:
As for the BH-I and BH-II sources the potential $V$ is always positive if the angular momentum is positive and goes to zero as $L$ approaches zero on the horizon. On the other hand, for $L \leq 0$ the potential increases with the distance from the source; thus, there are no (stable or unstable) orbits with $L = 0$ (zero angular momentum observers) or counterrotating orbits. Consequently, there are no orbits with negative or zero energy. The angular momentum and energy of the orbits are always decreasing with the spin and $L_- > E_-$ (see Fig. 4).

D. Some final notes on the BH-case

In the ergoregion, BH-sources are characterized by a unique family of corotating orbits with $L = L_-$. Spacetimes of the type BH-I are characterized by lower spins up to a maximum value of $\tilde{a}_1 \approx 0.707M$. No circular orbits are allowed in this case. In the second set of sources BH-II, with spins contained between $\tilde{a}_1$ and $\tilde{a}_2 \approx 0.943M$, there exists a region where circular orbits are allowed, but all of them are unstable. Finally, only in the third set of sources BH-III, the ergoregion can be filled with stable circular orbits. The stability region increases as the intrinsic spin increases. It is clear that only black holes with larger spin can allow the existence of stable circular orbits. This fact is very important in view of the following analysis which investigates black holes with superextreme spin. Finally, we notice that the ergoregion in back holes is characterized by the presence of a photon-like orbit at $r = r_{\gamma}$. The energy and angular momentum of the particle diverge as the photon-like orbit is approached. The properties of circular orbits outside the ergoregion of black holes are sketched in Fig. 4 (left and lower plots), and have been analyzed in detail in [9].

IV. NAKED SINGULARITIES

The different intervals for the intrinsic angular momentum define different sets of naked singularities which we will denote as $\text{NS} - \text{I}: a \in [1, \tilde{a}_3]$, $\text{NS} - \text{II}: a \in [\tilde{a}_3, \tilde{a}_4]$, $\text{NS} - \text{III}: a \in [\tilde{a}_4, \tilde{a}_5]$, and $\text{NS} - \text{IV}: a \in [\tilde{a}_5, \infty]$. For each of these sets we will now analyze the main physical properties of the test particles moving along circular orbits.

A. The set $\text{NS} - \text{I}: a \in [1, \tilde{a}_3]$

Orbital regions:
The spin $\tilde{a}_3 \equiv 3\sqrt{3}/4M \approx 1.29904M$ arises as the solution of the equation $\hat{r}_+ = \hat{r}_- = 3/4M$ (see Fig. 4-right). The radii $\hat{r}_\pm$ are solutions of the equations $V' = 0$ and $L = 0$. These radii determine a close region whose maximum extension is $\Delta_{\text{Ext}} = 0.704402M$ when $a = M$, and null at $a = \tilde{a}_3$ (see Figs. [9]). Naked singularities contained in this set present rather complex orbital stability properties and, as they limit with the extreme-BH case at $a = M$, they could be involved in the hypothetical transitions between the two main BH and NS classes, implying a dynamical processes with a shift in the central object spin.

Effective potential and particle energy:
For orbits at the radius $r = \hat{r}_-$ the energy is positive and increases with the source spin. For the higher orbit with $r = \hat{r}_+$ the energy also increases with the source rotation, however, $\hat{r}_+$ decreases as $a/M$ increases. One can also show that $E(\hat{r}_-) > E(\hat{r}_+)$. A characterization of the orbital energy outside the ergosphere was presented with some details in [9]. The results of the present analysis are based upon the interpretation of Figs. 5, 8 and 9, and can be summarized in the following points:
In the interval \( |0, \hat{r}_-| \) there are unstable circular orbits with angular momentum \( L = L_- \). Notice, however, that the curvature singularity is located within this interval. This region reaches its maximum orbital extension, \( \Delta_{NS-I} = 3/4M \), for the maximum spin \( \hat{a}_3 \). This is a major difference with the BH-case where regions very close to the horizon \( r_+ \) are always prohibited for any circular orbit.

2. In contrast with the BH-case, there is a region of counterrotating orbits with angular momentum \( L = -L_- < 0 \). This property characterizes only this set of naked singularities, and it can be interpreted as the result of the existence of a repulsive gravitational field, an effect that has been detected also in other spacetimes with naked singularities represented by axisymmetric solutions of the Einstein equations [8, 10, 12]. This orbital region is split by the radius \( r_{NS-\text{ISCO}} \), which is a solution of the equation \( V'' = 0 \). The orbits with \( r > r_{NS-\text{ISCO}} \) and \( L = -L_- \) are stable, as for them it holds that \( V'' > 0 \); lower orbits, included in \( ]r_-, \hat{r}_+[ \), are unstable. The energy of these orbits can be positive, negative or even null.

3. Objects within the class NS-I are characterized by the presence of a subset with \( a \in [M, a_\mu] \) which allows counterrotating orbits with negative energy up to the limit \( E = 0 \) (see Fig. 5). The radii of the counterrotating orbits are located at

\[
r^+_\nu \equiv \frac{4}{3} M \left( 1 + \sin \left[ \frac{1}{3} \arcsin \left( 1 - \frac{27a^2}{16M^2} \right) \right] \right), \quad r^-_\nu \equiv \frac{8}{3} M \sin \left[ \frac{1}{6} \arccos \left( 1 - \frac{27a^2}{16M^2} \right) \right]^{\frac{3}{2}},
\]

and are included in the orbital region \( ]\hat{r}_-, \hat{r}_+[ \). Moreover, \( r^+_\nu (a_\mu) = (2/3)M \) and \( r^+_\nu |_{a=M} = M \), \( r^-_\nu |_{a=M} = 0.381966M \). This region is always separated from the source by a region of corotating unstable orbits. The angular momenta \( L^\nu_L \equiv -L_- (r^\nu_+) \) of these orbits are plotted as functions of the spin parameter in Fig. 7.

4. Finally, the orbits in \( r \in ]\hat{r}_+, r^+_\nu[ \) are once again corotating with \( L = L_- \) and unstable. The largest orbital extension of this region is \( \Delta_{NS-I,L} = 1.25M \) and it is reached at its upper limit \( \hat{a}_3 \).

The region of stable orbits is disconnected in the sense discussed in [8, 12]. This means that if we imagine a hypothetical accretion disk made of test particles only, the disconnected stability regions form a ring-like configuration around the central object. This is an intrinsic characteristic of NS-sources that has been also highlighted for other axisymmetric exact solutions of the Einstein equations, in particular, for the electrovacuum spacetime described by the Reissner-Nordström (RN) solution. The circular orbit configuration on the equatorial plane of this spacetime has been addressed in detail in [8, 10, 11]. It is indeed particularly interesting to compare the results obtained for the circular motion of an electrically charged test particle moving in the charged non-rotating RN spacetime, with the

![FIG. 7: Angular momenta \( L^\nu_L \equiv -L_- (r^\nu_+) \) as functions of the source spin \( a \in [M, a_\mu] \). The region \( ]r^-_\nu, r^+_\nu[ \) contains only counterrotating orbits with \( E < 0 \), and the radii \( r^\pm_\nu \) sets counterrotating orbits with \( E = 0 \). The behavior of the radii \( r^\pm_\nu \) is depicted in Figs. 5. The values of the \( L^\nu_L \equiv -L_- (r^\nu_+) \) are bounded by a minimum value at \( L^\nu_L (a_\mu)/M_\mu = -2/\sqrt{3} \) and a maximum value at \( L^\nu_L (a = M)/M_\mu = 0 \).](image-url)
present analysis on the plane \( \theta = \pi/2 \). In general, these two axisymmetric solutions have some remarkable similar geometrical features which can be seen, for instance, in the expressions for the outer and inner horizons \( r_{\pm} \), where one sees the spin parameter \( a/M \) of the Kerr spacetime corresponding to an electric charge parameter \( Q/M \) of the RN solution. However, even if the global structure of the Kerr spacetime is \( \theta \)-dependent, on the equatorial plane the structure of different aspects of its geometry is similar to the RN spherical solution, where the spin-orbit coupling terms in the Kerr source correspond to the electrodynamic interaction between the test charge \( q \) and the intrinsic charge of the source. Thus, on the equatorial plane the conformal diagram for the maximally extended Kerr (BH and NS) spacetimes is identical to that of the RN solution [23]. The analogy with the case of charged test particles in the RN spacetime extends also to the definition of an effective ergoregion \( r_{\text{eff}}^{+}, r_{\text{eff}}^{-} \), where \( r_{\text{eff}}^{+} \equiv M + \sqrt{M^{2} - Q^{2}} \) is the outer horizon in the RN geometry and \( r_{\text{eff}}^{-} \equiv M - \sqrt{M^{2} - Q^{2}}(1 - q^{2}/\mu^{2}) \) is the effective ergosurface in the RN solution [24]. In particular, the definition of an effective ergoregion is introduced in the description of the energy extraction phenomena that equally characterizes the particle dynamics in the RN spacetime and Kerr solution. This region is thus not induced (also in the case of BH spacetimes) by a rotation or a deformation of the source, but it is due to the attractive interaction \( (Qq < 0) \) between the two charges that results in negative energy states for test particles [10, 11]. As discussed in detail in these works, this region is well defined even for the naked singularity case as long as certain constraints on the values of the charges are fulfilled. However, although this situation appears very complex, it essentially evidences the existence of a limited region of values of \( Q \) and \( q \) for which (stable or unstable) orbits with negative energy are possible. Again, these phenomena only involve a limited region of sources with parameter values very close to the black hole case; this situation was also found in the charged and rotating axisymmetric solution [12]. One can extrapolate the analogy further to the correspondence between the case of corotating or counterrotating orbits in the effective ergoregion and the motion of test particles in charged spacetimes with \( Qq < 0 \) or \( Qq > 0 \), respectively. Notice that the effective ergosurface is located at a radius with \( E = 0 \); thus, we can equally introduce the concept of an outer effective ergosurface and an inner effective ergosurface defined by the radii \( r^{-}_{\text{eff}} < r^{+}_{\text{eff}} \) which satisfy Eq. (38). However, these radii are independent of the orbital angular momentum and for each value of the spacetime rotation in the interval \( [M, a_{\mu}] \), they include a region of counterrotating orbits with negative energy inside the ergoregion. Furthermore, this effective ergoregion is actually a surface on the equatorial plane \( (\theta = \pi/2) \) only. Finally, we note that \( r_{\text{eff}}^{\pm}(a) \), where \( E = 0 \), are particular sections of the curves \( r_{\text{eff}}^{\pm}(a, L) \) defined by the condition \( V = 0 \). For the counterrotating orbit \( L = -L_{-} \), we have considered the analysis reported in Fig. [3]. For the spin interval \( a/M \in [a_{\mu}, \tilde{a}_{3}] \), we have \( a_{\mu}/M \equiv 4\sqrt{2}/3 \) it holds that \( V < 0 \) in the interval \( r < r_{\text{eff}}^{-} \) or in the case of circular orbits \( r \in ]r^{-}_{\text{eff}}, r^{+}_{\text{eff}}[ \). This leads to the possibility of realizing regions of orbits with particles having negative conserved energy (as defined by means of observers at infinity). In the outer region, \( r > r_{\text{eff}}^{+} \), where in fact there are two distinct angular momenta for the zeros of the potential function, the situation is more varied than in the BH case. The potential is always negative in the region \( r < r_{\text{eff}}^{-} \), for different values of the angular momentum \( L < L_{-}^{+} \) and \( L_{-}^{0} < L < 0 \). The situation is more complex for \( L^{-}_{0} < L < L_{+}^{+} \). The existence of counterrotating orbits can be seen as a “repulsive gravity” effect.

We note that the last stable orbit \( r_{\text{ISCO}}^{\text{NS}} \), for particles with \( L = -L_{-} \) has a minimum for the source \( a_{\mu} \) where \( r_{\text{ISCO}}^{\text{NS}} / M = 2/3 \). This naked singularity has peculiar properties: the maximum extension of the stability region is \( \Delta_{\text{NS}}^{\text{stab Max}} = 0.324601M \) and then decreases to zero at the boundary. The stability properties of these regions can be derived from Fig. [4] and Fig. [5] it follows that there are stable orbits, even with negative or zero energies, only for sources with sufficiently small spin, i.e., \( a \in [M, a_{\mu}] \). It is important to note that counterrotating orbits are confined in a bounded orbital region. The stability of these orbits would suggest the presence of a belt of material covering the singularity (see Figs. [8] and Figs. [9]). The existence of stable and unstable circular orbits with \( L < 0 \) and \( E < 0 \), although located on an orbital region far from the source, can be important for the phenomena of accretion from the equatorial plane, because it would imply dropping “test” material into the singularity with a negative contribution to the total energy and momentum.

### B. The set NS – II: \( a \in [\tilde{a}_{3}, \tilde{a}_{4}] \)

**Orbital regions:**

The upper boundary \( \tilde{a}_{4} = 2\sqrt{2}M \approx 2.82843M \) corresponds to the intersection of the radius \( r_{\text{ISCO}} \) with the static limit. There is a last stable orbit with \( L = L_{-} \). The stability region for these orbits decreases with the spin until it disappears at the boundary spin for this set (see Fig. [4] right). The behavior of these orbits outside the ergoregion is analyzed in [9] and summarized schematically in the lower plot of Fig. [4]. The analysis inside the ergosphere leads to the following results. There are only corotating orbits with \( L = L_{-} \). In the region \( r < r_{\text{ISCO}} \) which extends, in principle, to the singularity, the orbits are unstable. In \( r \in [r_{\text{ISCO}}, r_{+}^{+}] \), circular orbits are always allowed and all of them are stable. The case \( r = r_{+}^{+} \) will be analyzed separately. We note that for this kind of sources there are
no counterrotating orbits, circular orbits are always allowed inside the stability region, and the unstable orbits are expected to end in the naked singularity. There are no orbits with $L = 0$ or orbits with negative energy.

**Effective potential and particle energy**

For $L \leq 0$ the effective potential is positive. For $L < 0$, no circular orbits are possible. The effective potential is zero at $r_{lsco}^{NS}$, and $V < 0$ for $r \in [0, r_{lsco}^{NS}]$. At $r_0^+$, the potential is positive. The orbital region decreases as the magnitude of the orbital angular momentum $|L|$ decreases and as the source spin $a/M$ increases (see Figs. 8 and 9). For $L \leq 0$ the effective potential is an increasing function of the orbital radius.

### C. The set NS $-$ III: $a \in ]a_4, a_5]$

**Orbital regions:**

These spacetimes are characterized by unstable corotating orbits with $L = L_-$ which theoretically can reach the singularity. The maximum spin for this set is $a_5 = 9M$ which is the spin value where $r_0 = r_{lsco}^{NS}$ (see Fig. 4-right).

**Effective potential and particle energy:**

As in the case of NS-II sources, for $L \leq 0$ the effective potential is always positive and increases with the radius. For $L < 0$ there is a radius $r_{lsco}^{NS}$ (not a circular orbit) where the energy function is zero, and for $r < r_{lsco}^{NS}$ it becomes
negative. The energy and the angular momentum decrease with the radius and increase with the spacetime spin, and \( L_- > E_- \). Both the energy and the angular momentum diverge as the source is approached.

D. The set \( \text{NS} - \text{IV} : a \in [\tilde{a}_5, +\infty] \)

Orbital regions:
Finally, the set \( \text{NS-IV} \) includes spacetimes with a spin greater than \( \tilde{a}_5 = 9M \). For these spacetimes, \( r_{\text{ISCO}}^{\text{NS}} > r_+^* > r_-^* \), meaning that in the region \( r < r_+^* < r_+^* \) there are only unstable orbits with \( L = L_- \). This situation is quite similar to the \( \text{BH-II} \) and \( \text{NS-III} \) cases. We can conclude that naked singularities with sufficiently high spins do not allow any orbiting test particles in the ergoregion. This is in contrast with the \( \text{BH} \) case where only sources with spin sufficiently close to the extreme case \( \text{BH-III} \) allow stable orbits below the static limit (see Fig. 4).

Effective potential:
The potential is positive definite for \( L \leq 0 \), it vanishes at \( r = r_{\text{ISCO}}^{\text{NS}} \), and is negative in the interval \( r < r_{\text{ISCO}}^{\text{NS}} \) with \( L < 0 \). The derivative \( V' \) is always positive for \( L \leq 0 \). The functions \( E_- \) and \( L_- \) decrease with \( r/M \) and increase with the spin \( a/M \). It then holds that \( E_- < L_- \).

V. THE EXTREME BLACK HOLE

The extreme Kerr black hole, \( a = M \), is the limiting case that separates the class of \( \text{BH} \) and \( \text{NS} \) classes. We classified it as the upper boundary of \( \text{BH-III} \) sources. It possesses distinctive features from the point of view of
stability of the orbiting particles. These characteristics may give rise to entirely different effects even for a slight change in the spin. We therefore now investigate the properties of circular orbits located within a neighborhood of the spin value $a = M$.

**Orbital regions:**

For $a = M$ the effective potential is an increasing function of the radius orbits with $L \leq 0$.

Corotating orbits are allowed only for the angular momentum interval $[2/\sqrt{3}, 1.68707\mu M]$. For particles with angular momentum $L/M \mu \in [0, 2/\sqrt{3})$, the effective potential increases with the radius $r/M$, whereas for $L > 1.68707\mu M$ the potential decreases with the radius $r/M$.

The orbital arrangement in this spacetime follows the main features of the BH-I sources: There are circular orbits with $L = L_\pm \in [2/\sqrt{3}, 1.68707\mu M]$. For these angular momenta the effective potential increases with $r$ within the interval $r \in [r_{+}^a, r_{+}^a]$, and decreases with $r$ for $r \in [L_{+}^a, r_{-}^a]$. Therefore, the radius $r_{+}^a(L)$ corresponds to a stable orbit with angular momentum $L = L_{+}(r_{+}^a)$. The radius is implicitly defined as a solution of the equation $L_{+}(r) = L$ for values of the spin-parameter $a < M$, i.e., $r_{+}^a(L) : L_{+}(r_{+}^a) = L$.

There are no last stable circular orbits, i.e., no solutions to the equations $V' = 0$, $V'' = 0$. This means that circular orbits are stable even close to the horizon. The energy of the rotating particles is positive so that one can imagine a rotating ring inside the ergosphere which extends up to the horizon. However, it must be emphasized that this configuration is highly unstable with respect to infinitesimal changes of the black hole spin. It is important to note that the radius $r_{+}$ for naked singularities can be defined at any value $a = M + \epsilon$ with $\epsilon > 0$: the black hole counterpart $\tilde{r}_{+} = M + f(\epsilon)$, where $f(\epsilon) < 0$, can be easily evaluated and is of the order of $(a - M)^2$; the orbital structures in the two cases $a \lesssim M$ and $a \gtrsim M$ are completely different. Moreover, the radius $r_{+}^a$ has a maximum at $a = M$, and the surface is orthogonal to the horizon $a = M$.

**Effective potential and particle energy:**

For $L \geq 0$, the potential is positive, $V(a = M) > 0$ (see also Fig. 10). We investigate the energy of circular orbits in the BH-case with $a/M = 1 - \epsilon$ and in the NS-case with $a/M = 1 + \epsilon$ and $\epsilon = 10^{-5}$, considering all the angular momenta in the interval $L \in (L_+, -L_-)$. The plot shows the analogies and differences between the orbits in the two cases. We note certain asymmetry for the energies and angular momenta around the orbit $r = M$. Nevertheless, close to the left-hand side of $r/M = 1$ all the orbits have negative energy and angular momentum, whereas on the opposite side those quantities are positive. In Fig. 10 we present several details of the behavior of the energies and angular momenta in terms of the radius of the circular orbits.

**VI. THE STATIC LIMIT**

The effective potential $V(r; L, a)$ is well defined at the radius of the ergosphere $r_{+}^a$, and reduces to:

$$V(2M; L, a) = \frac{aL + \sqrt{a^2 (4M^2 + 2M^2 a^2 + L^2)}}{2(2M^2 + a^2)} > 0, \quad \forall a > 0. \quad (39)$$

It is null only in the limit $a = 0$ (see Fig. 11).

**Circular orbits:**

We study the effective potential $V_{+} = V(r_{+}^a; L, a)$ around the static limit for both NS- and BH-sources. Particles that cross the ergosphere on the equatorial plane should be able to detect some specific aspects of the source nature, because their behavior should depend on the intrinsic spin source. It turns out that the upper boundary $\tilde{a}_1$ of the BH-I-sources (see Fig. 2), can be used to classify different orbits. In fact, the derivative of the effective potential in the static limit is zero at $L = L_{+}$ for $a > \tilde{a}_1$, i.e., $V'(r; L, a)|_{r_{+}^a} = 0$ at $L_{+}^\prime = L_{-}(r_{+}^a; a)$. If the momentum parameter is $L < L_{+}^\prime$, then the effective potential increases with the source spin; on the other hand, if $L > L_{+}^\prime$, then for $a > \tilde{a}_1$ the effective potential decreases with $a/M$. In the limiting case $a = \tilde{a}_1$, the energy $E_{\pm}$ diverges as shown in Fig. 11.

A detailed analysis of the involved quantities leads to the following results:

1. The potential has always an extreme on the static limit for sources not belonging to BH-I and, therefore, on $r = r_{+}^a$ there exists a circular orbit with angular momentum $L_{-}(r_{+}^a)$. The potential decreases with the radius, i.e., $V'_{+} < 0$ only for particles with angular momentum higher than $L > L_{+}^\prime$ in the BH-I-class.

2. For particles in the BH-I-class, the effective potential is increasing for any value of the angular momentum, while for any sources not belonging to the BH-I-class, $V_{+}^\prime$ grows if the angular momentum is sufficiently small within the interval $L \in [0, L_{+}^\prime]$. The nature of these orbits is investigated by considering the second derivative of the potential at that point. It is then necessary to consider the two extreme spins $\tilde{a}_2$ and $\tilde{a}_4$. The orbits located on the static limit correspond to a minimum point of the effective potential (stable orbit) with angular momentum $L = L_{-}$, only for sources with spins in the interval $a \in [\tilde{a}_2, \tilde{a}_4]$, which include the sources of the sets BH-III, NS-I and NS-II. These sources are precisely in the region of admissible orbital stability within the ergoregion (see Fig. 4). Then, the static
limit allows an orbit with an inflection point in the potential, last stable circular orbit, for the extreme spins $\tilde{a}_2$ and $\tilde{a}_4$. These particular spins in fact are defined as solutions of the equation $r_{lsc0} = r^+ = 2M$. Unstable orbits on the ergosphere are allowed in spacetimes of the classes BH-II, NS-III and NS-IV.

Angular momentum and energy of the particle

The energy function is always increasing with the orbital angular momentum on the static limit. We note that the radius of the static limit on the equatorial plane is independent of the spacetime spin and, therefore, it can be considered as invariant with respect to a slow change of the source spin. It is therefore interesting to study the variation of the orbital energy and angular momentum with respect to the spin. To this end, let us introduce the particular rotation parameter $a_5 \equiv \sqrt{2}M \approx 1.41421M$ and the angular momentum

$$\frac{L_5}{\mu M} \equiv 2\sqrt{2} \sqrt{\frac{M^2}{a^2 - 2M^2}},$$

which diverges at $a = a_5$.

The angular momentum $L_5$ increases as the spacetime rotation decreases and reaches its asymptotic value at $a = a_5$, as shown in Fig. 11. The values of the spin parameters imply $a_3 < \tilde{a}_3 < a_5 < \tilde{a}_4$ so that a source with $a = a_3$ is a

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**FIG. 10:** Extreme black hole ($a = M$). Bottom: Radii $r^+_{l\nu}$ (black curve) and $r^-_{l\nu}$ (gray curve) as functions of the orbital angular momentum $L/M$. The effective potential is zero on $r^+_{l\nu}$, negative in $r < r^+_{l\nu}$ (gray region), and positive for $r > r^+_{l\nu}$ (white region) with $r^+_{l\nu} = r^{BH}_{l\nu, a = M}$. Top: Orbital energies and angular momenta of the orbits in the BH-case $a/M = 1 - \epsilon$ (solid curve), and the NS-cases $a/M = 1 + \epsilon$ with $\epsilon = 10^{-5}$ for $E_{l\nu}^{NS} \equiv E(-L_\nu)$ (dotted-dashed curve) and $E_{l\nu}^{NS} \equiv E(L_\nu)$ (gray curve) as functions of $r/M$. The orbit $r_-$ where $L = 0$ is shown (dashed line), the maximum and minimum of $E_{l\nu}^{NS}$ and $E_{l\nu}^{NS}$, respectively, are represented by a dashed line. It is $r^{NS}_{l\nu, a = M} = 1.03523M$, $r^{NS}_{l\nu, a = M} = 0.966815M$, and $r_+ = M$, $r_- = 0.295605M$, inside plots are zooms for the region close to the BH horizon. Stable counterrotating orbits for NS-sources are located in the interval $r^{NS}_{l\nu, a = M} < r < r_+$. For BH-sources the stable circular orbits are located at $r > r^{BH}_{l\nu, a = M}$. Notice the existence of orbits with negative or null energy and orbits with negative and zero angular momentum. The values ($E_{l\nu}^{NS}, E_{l\nu}^{BH}$) and ($L_{l\nu}^{NS}, L_{l\nu}^{BH}$) in the region $M < r < 2M$ converge, even if in NS-spacetimes orbits are stable and in the BH-spacetimes orbits are unstable.
NS-II-spacetime. The energy function in the static limit decreases with the source spin for \( a > a_0 \) and for a particle angular momentum \( L > L_0 \); for \( L = L_0 \), the potential \( V_+^\ast \) is constant, independently of the source spin. For lower spins \( (a < a_0) \), regardless of the orbital angular momentum, the energy function increases with the spin of the BH- or NS-sources.

We now consider the conditions under which the static limit corresponds to an admissible stable or unstable orbit. To this end, we analyze the behavior of the energy \( E_\ast \) of the orbiting particle and its angular momentum \( L_\ast \). These quantities will be only functions of the spacetime spin. It appears that for a source with \( a = a_0 \) the particle energy on the static limit reaches a minimum value, and therefore the particle energy increases as the spacetime rotation increases, until it reaches its asymptotic value for \( a \to a_1 \) (upper extreme of BH-I sources). This behavior is illustrated in Fig. 11. A different situation appears if we consider the variation of the potential with respect to the orbital angular momentum: The source which corresponds to the minimum of the particle orbital angular momentum has in fact the spin \( a_2 \equiv 2a_2 \approx 1.88562M \) and belongs to the NS-II class, as \( a_3 < a_0 < a_2 < a_4 \).

We summarize our results as follows. We have shown that in any sources, except for BH-I-sources, at \( r = r_+^\ast \) there can be in fact a circular orbit in the sense of Eq. (22) with angular momentum \( L = L_\ast \) and energy \( E_\ast > 0 \) with the following property: The orbit is unstable in BH-II, NS-III and NS-IV sources, and stable in BH-III, NS-II and NS-III sources. The circular orbits on \( r_+^\ast \) are of the type \( L = L_\ast \). A distant observer at infinity will verify that a particle moving along an unstable orbit will eventually cross the static limit and fall into the source, which can be either a black hole or a naked singularity. However the function

\[
\frac{L_\ast(r;a)}{\mu M} = \frac{L_\ast(2M;a)}{\mu M} = \sqrt{\frac{4M^2 - 2\sqrt{2}M + a^2}{4M^3(\sqrt{2}a - M)}}
\]

is regular, and has a minimum value at \( a = a_\ast \). The energy function \( E_\ast \) has instead a minimum at \( a = a_0 \). Thus, the values of the energy and angular momentum of the unstable orbits do not have a monotonically increasing behavior on the static limit with respect to a variation of the source spin, but they reach a minimum in the NS-region. Thus, formally \( r = 2M \) is a possible unstable circular orbit on the equatorial plane for specific sets of black holes and naked singularities. The details are characterized by the spin parameter \( \tilde{a}_1 \) which is determined from the equation \( r_\gamma^- = r_\gamma^+ \) (see Fig. 11). On the other hand, \( V(r_\gamma^+; L, a) \) increases with the orbital angular momentum \( L \), but not with the intrinsic spin. Thus, the angular momentum \( L_\ast(r_\gamma^+) \) can be used to distinguish between black holes and naked singularities. Finally, we have identified in the naked singularity regions the two limiting spins \( (a_0 \) and \( a_\ast \)) on the basis of the orbital nature of the static limit.

VII. SUMMARY OF BLACK HOLES AND NAKED SINGULARITIES

In this section, we review the results of the analysis performed above for the dynamics of test particles moving along circular orbits, emphasizing the similitude and difference between the case of NS-sources and BH-sources. We also formulate some ideas regarding possible scenarios for gravitational collapse and transitions between black holes and naked singularities.

- In NS spacetimes, there is no last circular orbit \( (r_{\text{acc}} = r_{\text{\gamma}}^-) \) for corotating particles with \( L = L_\gamma^- \), indicating that the particles can theoretically reach the singularity. In other words, there is at least one circular orbit as long as the particle angular moment assume certain values. On the other hand, counterrotating orbits \( (L = -L_\gamma^-) \), which are a characteristic of specific NS-I sources, can exist only in the bounded orbital region: \( r \in [\hat{r}_-, \hat{r}_+] \).

- Photon-like circular orbits are a feature of BH-II and BH-III sources only; there is no equivalent in the case of NS-sources.

- The lower the spin, the more structured are the stability regions. The orbit stability is limited to only one possible source class with spin values within the interval \( a \in [\tilde{a}_2, \tilde{a}_4] \), which includes the sets BH-III, NS-I and NS-II (see Fig. 11). In other words, stable circular orbits are allowed only inside certain range of source spins.

- A final remark concerns the possibility that during its evolution, the source can pass through stages of adjustment of its spin, i.e., spin-down or spin-up processes, associated with different stages of the source life where it can interact differently with the surrounding matter and fields, see also [25, 33]. In any case, these phenomena involve, in the majority of scenarios considered in high energy astrophysics, the interaction of the accretor with matter and fields in the ergoregion (see for example[11]). In order to make an analysis of this kind of dynamical transition between different kind of spacetimes, one should look at the timescale transition and the geometric
properties of the spacetime regions, considering a small variation of the spin. This analysis would imply serious problems in the transition regions where the properties of spacetime are completely different, especially near the spins $\tilde{a}_i$ with $i \in \{1, \ldots, 5\}$ and $a = M$ which define the BH and NS classes. An example of this behavior was found in Sec. V where we analyzed the orbital properties of a source with intrinsic rotation around the extreme spin $a = M$.

- Questioning about the possible disruption or formation of an horizon and the formation and existence of a NS-spacetime, it is obviously important to consider the possibility of a transition through the BH-III and NS-I regions. These are indeed the source classes that are mostly affected during a hypothetical collapse or “spacetimes-transition” between rotating geometries at different spin that could lead to a disruption of the event horizon, but preserving axial symmetry. The analysis of the orbital configuration and its stability shown in Figs. 3 and 5 can give a hint on what might happen in this kind of phenomenon. In particular, it is seen that some source characteristics can change profoundly the behavior of matter in a spacetime with horizon and in a horizon-free spacetime. In particular, we notice i) the presence of counterrotating orbits, stable or not, enclosed in bounded orbital and spacetime ranges; ii) the presence of negative and zero-energy states for a set of counterrotating
orbits and spacetimes; iii) the existence of orbits with $L = 0$, and, finally, iv) a band of orbiting test particles which extends, in principle, up to the singularity (see Fig. 5).

The possibility of having a spin transition in the early stage of the evolution of a Kerr source (during the gravitational collapse or the subsequent stages involving the interaction of the source with the surrounding matter) should be of special interest in particular within the range of values determined by a subset of the gray region in Fig. 5-right, i.e., in the interval of sources close to the extreme-Kerr solution and to $r = M$ ($I \equiv I_{a=M} \times I_{r=M}$). However, regarding the phenomena of extraction of energy and angular momentum, one might look at these regions as mainly involved during the gravitational collapse of compact objects by dragging mechanisms in which the angular momentum of the collapsing object is reduced to $a < M$. The same kind of mechanism should finally fall within those phenomena which hinder the destruction of a horizon by the accretion of matter (for a more thorough investigation of this aspect see, for example, [34, 35]).

Some further considerations were briefly addressed also in the study of some properties in the extremal case $a = M$. For instance, if we focus on the NS neighborhood, $r = M$ is the point of convergence of counter rotating stable orbits with negative energy. Any hypothetical spin-shift $a = M + \epsilon$ with $\epsilon > 0$, could perhaps give rise to retroactive effects and bring hypothetically to spin-up or spin-down process.

VIII. CONCLUSIONS

In this work, we performed a complete analysis of the properties of circular motion inside the ergoregion of a Kerr spacetime. From the physical point of view, the ergosurface is a particularly interesting surface, because it represents the limit at which an observer can stay at rest. In the ergoregion, an observer is forced to move due to the rotation of the gravitational source.

The dynamics inside the ergoregion is relevant in astrophysics for the possible observational effects, as the matter can eventually be captured by the accretion, increasing or removing part of its energy and angular momentum, prompting a shift of its spin, and inducing an unstable phase in which the intrinsic spin changes (spin-down and spin-up processes with the consequent change in the causal structure). For further consideration of a possible destruction of the horizon and naked singularity formation see, for example, [34, 39]. A discussion on the ergoregion stability can be found in [40, 41]. It is therefore possible that, during the evolutionary phases of the rotating object, the interaction with orbiting matter could lead to evolutionary stages of spin adjustment, in particular for example in the proximity of the extreme Kerr solution (with $a \lesssim M$) where the speculated spin-down of the BH can occur preventing the formation of a naked singularity with $a \gtrsim M$ (see also [41, 26, 33]). On the other hand, the accreting matter can even get out, giving rise, for example, to jets of matter or radiation [1].

In our analysis, we limit ourselves to the study of circular orbits located on the equatorial plane of the Kerr spacetime. On the equatorial plane, the static limit does not depend on the source spin, but for any Kerr spacetime it is $r^+ = 2M$ and it coincides with the event horizon of the Schwarzschild spacetime. This is a simple setting that allows an immediate comparison with the limiting case $a = 0$. Furthermore, matter configurations in accretion are typically axially symmetric and many of the geometrical and dynamical characteristics of the disk are determined by the properties of the configuration on the accretion equatorial section.

Our approach consists in rewriting the geodesic equations in such a form that the motion along circular orbits is governed by one single ordinary differential equation whose properties are completely determined by an effective potential. The conditions imposed on the effective potential for the existence of circular motion allow us to derive explicit expressions for the energy and angular momentum of the test particle. The behavior of these physical quantities determine the main properties of the circular orbits in terms of the radial distance which, in this case, coincides with the radius of the orbit, and the intrinsic angular momentum of the gravitational source. We performed a very detailed investigation of all the spatial regions inside the ergoregion where circular motion is allowed. In addition, we investigate the stability properties of all the existing circular orbits.

The distribution of circular orbits inside the ergoregion turns out to depend very strongly on the rotation parameter $a$ of the source, and makes it necessary to split the analysis into two parts: black holes and naked singularities. In addition, the behavior of the effective potential in the ergoregion in terms of the rotational parameter suggests an additional split by means of which black holes become classified in three classes, BH-I, etc., and naked singularities become classified in four classes NS-I, etc. We then investigate in detail for each class the behavior of the energy and angular momentum of the test particle, as well as the properties of the effective potential. In this manner, the analysis of circular motion allows us to derive physical information about entire sets of black holes and naked singularities.

Our results prove that circular motion is possible inside the ergoregion in black holes and naked singularities as well. However, there are fundamental differences if we consider the stability properties. In the case of black holes, only the set BH-III can support a spatial region with particles moving along circular corotating stable orbits. The
BH-III class includes all the black holes whose rotation parameter is contained within the interval $a/M \in [2\sqrt{2}/3, 1]$, i.e., rapidly rotating black holes which include the extreme black hole. The spatial region with stable particles extends from the radius of the ergoregion ($r^+_E = 2M$) to the radius of the last stable circular orbit, so that the maximum radial extension of this region is $M$ for an extreme black hole, and the minimum extension is zero for a black hole with $a = 2\sqrt{2}/3M$. The last case corresponds to particles moving on the last stable circular orbit. If we imagine a hypothetical accretion disk made of test particles only, then we conclude that black holes can support inside the ergoregion only one corotating disk with a maximum extension of $M$.

The case of naked singularities is more complex. In fact, one of the interesting features is that inside the ergoregion there can exist particles moving along counterrotating stable orbits. As a consequence, the location and structure of the regions with stable particles is much more complex than in the case of black holes. In particular, it was shown that there can exist several stability regions, separated by instability regions. This implies a discontinuous structure of the stability regions so that, if certain energetic conditions are satisfied, an accretion disk made of test particles would show a ring-like structure. This makes naked singularities essentially different from black holes. The characteristics of the rings and their extensions depend on the explicit value of the rotation parameter. Finally, we found that there exists a maximum value of the spin for which no more stable configurations can exist, namely, $a = 2\sqrt{2}M \approx 2.828M$ for which the radius of the last stable circular orbit coincides with the radius of the ergoregion. Naked singularities with spins greater than this critical value do not support any disk-like or ring-like configurations inside the ergoregion.

In NS-I spacetimes, there are both stable and unstable counterrotating orbits inside the ergoregion. This fact can be understood as the effect of repulsive gravity, but it is interesting to note that this phenomenon can occur only in sufficiently slow rotating naked singularities with spin values close to the value of the extreme BH-case. In NS-II spacetimes, there can exist stable corotating orbits; this is the major difference with the BH case and it represents also the main difference with the other NS sources. We have introduced the concept of inner and outer effective ergosurface defined by the radii $r^+_E$ and $r^-_E$, where $E = 0$. The effective ergoregion (at $r \in [r^-_E, r^+_E]$) is defined for supercritical configurations with $a \in [M, 1.008866M]$ in NS-I spacetimes, where $E < 0$.

For the sake of completeness, we also investigated all the properties of circular orbits in the limiting case of extreme black holes, and classify all the sources that allow circular orbits on the radius of the static limit. In both cases, we used the available physical parameters to perform a detailed analysis which confirms the rich structure of the gravitational sources described by the Kerr spacetime.

Our results show that the complex stability properties of circular orbits inside the ergoregion of naked singularities is due to the presence of effects that can be interpreted as generated by repulsive gravitational fields. The nature of this type of fields is not known. We expect to investigate this problem in future works.

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