A modular equality for Cameron-Liebler line classes
in projective and affine spaces of odd dimension

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January 26, 2022

Abstract

In this article we study Cameron-Liebler line classes in PG(n, q) and AG(n, q), objects also known as boolean degree one functions. A Cameron-Liebler line class \( L \) is known to have a parameter \( x \) that depends on the size of \( L \). One of the main questions on Cameron-Liebler line classes is the (non)-existence of these sets for certain parameters \( x \). In particularly it is proven in [12] for \( n = 3 \), that the parameter \( x \) should satisfy a modular equality. This equality excludes about half of the possible parameters. We generalize this result to a modular equality for Cameron-Liebler line classes in PG(\( n, q \)), and AG(\( n, q \)) respectively. Since it is known that a Cameron-Liebler line class in AG(\( n, q \)) is also a Cameron-Liebler line class in its projective closure, we end this paper with proving that the modular equality in AG(\( n, q \)) is a stronger condition than the condition for the projective case.

1 Introduction

The study of Cameron-Liebler sets of \( k \)-spaces, (or Cameron-Liebler \( k \)-sets for short), in finite affine or projective spaces of general dimension, originated from the study Cameron-Liebler line classes in PG(3, q). Cameron and Liebler in [3] studied irreducible collineation groups of PG(\( d, q \)), having equally many point orbits as line orbits. This work resulted in several equivalent definitions of line sets (that were later called Cameron-Liebler line classes) having particular combinatorial properties, or, equivalently, having very distinctive algebraic properties. One of the questions posed in [3] is whether non-trivial examples of Cameron-Liebler line classes exist. The conjecture answering this question negatively, was later shown to be not true by a constructive result. However, non-trivial Cameron-Liebler line classes are very rare. Hence not surprisingly, there are a lot of non-existence results for particular values of the parameter of the Cameron-Liebler line class, the number which determines its main combinatorial characteristic. More precisely, a Cameron-Liebler line class with parameter \( x \) is a set of lines of PG(3, q) meeting every spread in exactly \( x \) lines. Trivial examples with parameter 1 are the set of lines through a fixed point and dually the set...
of lines in a fixed plane. One of the strongest non-existence results on Cameron-Liebler line classes is the following result.

**Theorem 1.1.** [12, Theorem 1.1] Suppose that $L$ is a Cameron-Liebler line class with parameter $x$ of $\text{PG}(3, q)$. Then for every plane and every point of $\text{PG}(3, q)$,

$$\binom{x}{2} + m(m - x) \equiv 0 \mod (q + 1),$$  \hspace{1cm} (1)

where $m$ is the number of lines of $L$ in the plane, respectively through the point.

Cameron-Liebler line classes in $\text{PG}(3, q)$ have been generalized to Cameron-Liebler line classes in $\text{PG}(n, q)$, see [8], and very recently to Cameron-Liebler $k$-sets in $\text{PG}(n, q)$, $n \geq 2k + 1$, see [2]. In both cases, classification of such objects based on their parameter remains of great interest. Cameron-Liebler line classes and Cameron-Liebler $k$-sets have been defined and studied in affine spaces as well, see [6, 7]. Once again, classification of these objects has been motivating to study restrictions on the parameter. From [6] we recall the following result, which is in principle an easy corollary of Theorem 1.1.

**Theorem 1.2 (6, Corollary 4.3).** Suppose that $L$ is a Cameron-Liebler line class in $\text{AG}(3, q)$ with parameter $x$. Then

$$x(x - 1) \equiv 0 \mod 2(q + 1).$$  \hspace{1cm} (2)

Such modular conditions on the parameter imply non-existence for particular values of $x$. This observation has been the motivation to investigate similar results for Cameron-Liebler line classes in $\text{PG}(n, q)$ and $\text{AG}(n, q)$. The main results of this paper are the following theorems.

**Theorem 1.3.** Suppose that $L$ is a Cameron-Liebler line class with parameter $x$ in $\text{PG}(n, q)$, $n \geq 7$ odd. Then for any point $p$,

$$x(x - 1) + 2\overline{m}(\overline{m} - x) \equiv 0 \mod (q + 1),$$

where $\overline{m}$ is the number of lines of $L$ through $p$.

**Theorem 1.4.** Suppose that $L$ is a Cameron-Liebler line class in $\text{AG}(n, q)$, $n \geq 3$, with parameter $x$, then

$$x(x - 1)q^{n-2} - 1 \equiv 0 \mod (q + 1).$$

2 Preliminaries

Recall that a line spread of $\text{PG}(3, q)$, respectively $\text{AG}(3, q)$ is a set of lines of $\text{PG}(3, q)$, respectively $\text{AG}(3, q)$ partitioning the point set of $\text{PG}(3, q)$, respectively $\text{AG}(3, q)$. We start with formally defining Cameron-Liebler line classes are their generalizations.

**Definition 2.1.** A Cameron-Liebler line class with parameter $x$ in $\text{PG}(3, q)$, respectively $\text{AG}(3, q)$, is a set $L$ of lines of $\text{PG}(3, q)$, respectively $\text{AG}(3, q)$, such that $|L \cap S| = x$ for every spread $S$ of $\text{PG}(3, q)$, respectively $\text{AG}(3, q)$.
Classical examples of Cameron-Liebler line classes in $\text{PG}(3, q)$ are (1) the empty set, with parameter $x = 0$, (2) all lines through a fixed point, with parameter $x = 1$, (3) all lines contained in a fixed plane, with parameter $x = 1$ and (3) the disjoint union of (2) and (3), with parameter $x = 2$. These examples and their complements are also known as trivial examples and are the only possibilities for their corresponding parameters. In the affine case, only examples (1) and (2) and the corresponding complements remain.

Non-trivial examples of Cameron-Liebler line classes are rare. The first example of an infinite family was given in [1]. More recently, some infinite families have been described in [4, 9, 10]. It is noteworthy that the last three examples are affine, i.e. they are constructed in $\text{PG}(3, q)$, but it turns out that there exists always a plane not containing any line of the Cameron-Liebler line class. Hence, see [6] Theorem 3.8, these examples are also examples of non-trivial Cameron-Liebler line classes in $\text{AG}(3, q)$. As mentioned in the introduction, non-existence results are of great interest as well, and one of the most consequential non-existence conditions is Theorem [11] cited in Section 1.

We now state the definition of Cameron-Liebler $k$-sets in general dimension. A point-pencil of $k$-spaces is the set of all $k$-spaces through a given point.

**Definition 2.2.** Let $n \geq 4$, and let $1 \leq k \leq n - 1$. A Cameron-Liebler $k$-set in $\text{PG}(n, q)$, is a set $\mathcal{L}$ of $k$-spaces such that its characteristic vector $\chi$ can be written as a linear combination of the characteristic vectors of point-pencils. Its parameter $x$ is defined as $|\mathcal{L}|/\begin{bmatrix} n \end{bmatrix}_q$.

**Remark 2.3.** These objects are also known as boolean degree 1 functions, see [11, 14].

It can be shown that when there exists $k$-spreads in $\text{PG}(n, q)$, a Cameron-Liebler $k$-set is characterized by its constant intersection property with $k$-spreads. The intersection number is then exactly its parameter, which is henceforth a natural number. Also note that the trivial examples of Cameron-Liebler line classes can be generalized easily to Cameron-Liebler $k$-sets.

When no $k$-spreads exist, the parameter of a Cameron-Liebler $k$-set is a rational number. The following theorem provides more information on the parameter in this case, and it will turn out to be very useful.

**Theorem 2.4.** [5] Theorem 5.1 for $k = 1$ and $t = 3$] Suppose that $\mathcal{L}$ is a non-empty Cameron-Liebler line class in $\text{PG}(n, q)$, $n \geq 4$ even, with parameter $x$. Then

$$x = 1 + \frac{C}{\begin{bmatrix} n-2 \end{bmatrix}_q},$$

for some $C \in \mathbb{N}$.

The following three lemmas will be of use in Section 4. The set of $k$-spaces in the subspace $\pi$ (of dimension at least $k$), will be denoted as $[\pi]_k$.

**Lemma 2.5.** (Folklore, [5] Theorem 3.1) Suppose that $\mathcal{L}$ is a Cameron-Liebler $k$-set in $\text{PG}(n, q)$, $n > k \geq 1$. Then for every $i$-dimensional subspace $\pi$, with $i > k$ the set $\mathcal{L} \cap [\pi]_k$ is a Cameron-Liebler $k$-set of a certain parameter $x_\pi$ in $\pi$. 


Lemma 2.6. [2 Theorem 2.9] Suppose that $\mathcal{L}$ is a Cameron-Liebler line class with parameter $x$ in $\text{PG}(n, q)$, with $n \geq 3$. If $\ell$ is an arbitrary line in $\text{PG}(n, q)$ then there are in total $q^2 2^{n-2} - 1 (x - \chi(\ell))$ lines of $\mathcal{L}$ skew to $\ell$. Here $\chi(\ell)$ equals one if $\ell \in \mathcal{L}$ or zero otherwise.

Lemma 2.7. [15 Section 170] The number of $j$-spaces disjoint to a fixed $m$-space in $\text{PG}(n, q)$ is equal to $q^{(m+1)(j+1)} \left[ \frac{n - m}{j + 1} \right]_q$.

In the affine case, we can easily define Cameron-Liebler $k$-sets using spreads.

Definition 2.8. A Cameron-Liebler $k$-set with parameter $x$ in $\text{AG}(n, q)$, $n \geq 4$ is a set $L$ of lines of $\text{AG}(n, q)$ such that $|L \cap S| = x$ for every $k$-spread $S$ of $\text{AG}(n, q)$.

Example 2.9. A straightforward example of a $k$-spread in $\text{AG}(n, q)$ consists of all skew $k$-spaces in $\text{AG}(n, q)$ that have a $(k-1)$-space in the projective closure in common. In the case of lines this is often called a parallel class of lines.

Lemma 2.10. (Folklore, [5 Theorem 3.6]) Suppose that $\mathcal{L}$ is a Cameron-Liebler $k$-set in $\text{AG}(n, q)$, $n > k \geq 1$. Then for every $i$-dimensional subspace $\pi$, with $i > k$ the set $\mathcal{L} \cap [\pi]_k$ is a Cameron-Liebler $k$-set of a certain parameter $x_\pi$ in $\pi$.

Finally, the following facts will be used to obtain the modular equations.

Lemma 2.11. Let $a, x \in \mathbb{N}$. Let $q$ be a prime power. Then

- $2q^a - 1 \equiv 0 \mod 2(q + 1)$ if $a \equiv 0 \mod 2$.
- $aq^2 \equiv a \mod 2(q + 1)$ if $a \equiv 0 \mod 2$.
- $x(x - 1) \equiv 0 \mod 2$.

3 The affine case

In this short section, we will generalize Theorem 1.2 to $\text{AG}(n, q)$.

Theorem 3.1. Suppose that $\mathcal{L}$ is a Cameron-Liebler line class in $\text{AG}(n, q)$ with parameter $x$, then

$$x(x - 1)\frac{q^{n-2} - 1}{q - 1} \equiv 0 \mod 2(q + 1).$$

Proof. Denote by $\pi_\infty$ the hyperplane at infinity of the projective closure of $\text{AG}(n, q)$. Fix a point $p \in \pi_\infty$. Now count triples $(\ell_1, \ell_2, \pi)$, for which

- $\ell_1 \cap \ell_2 = p$
- $\ell_1, \ell_2 \in \mathcal{L}$
- $\langle \ell_1, \ell_2 \rangle \subset \pi$ and $\dim \pi = 3$. 


First, for any affine 3-space \( \pi \), the set of lines \( \mathcal{L} \cap [\pi]_1 \) is a Cameron-Liebler line class with parameter \( x_\pi \) by Lemma\textsuperscript{2.10}. Since the lines of the parallel class in \( \pi \) through \( p \) are a line spread of \( \pi \) (Example\textsuperscript{2.9}), exactly \( x_\pi \) of these lines are contained in \( \mathcal{L} \cap [\pi]_1 \). Hence there are \( x_\pi(x_\pi - 1) \) choices for the pair \((\ell_1, \ell_2)\).

Secondly, for a fixed pair \((\ell_1, \ell_2)\), the number of affine 3-spaces containing \((\ell_1, \ell_2)\) equals the number of 3-spaces through a plane in \( \text{PG}(n, q) \), which is \( \frac{q^{n-2} - 1}{q - 1} \).

Since the lines of the parallel class in \( \pi \) through \( p \) are a line spread of \( \text{AG}(n, q) \), there are \( x(x - 1) \) choices for a pair \((\ell_1, \ell_2)\) of lines of \( \mathcal{L} \) through \( p \). Hence, if we denote \( \Phi_3 \) as the set of all affine 3-spaces, we obtain that

\[
\sum_{\pi \in \Phi_3} x_\pi(x_\pi - 1) = \sum_{(\ell_1, \ell_2)} \frac{q^{n-2} - 1}{q - 1} = \frac{q^{n-2} - 1}{q - 1} x(x - 1).
\]

Using Corollary\textsuperscript{1.2} on the affine space \( \pi \), the left hand side reduces to \( 0 \mod 2(q + 1) \) and we indeed obtain the assertion.

**Remark 3.2.** The same result can be obtained by double counting the pairs \((\ell_1, \ell_1, \pi)\), with \( p \in \ell_1, \ell_1 \cap \ell_2 = \emptyset, \ell_1, \ell_2 \in \mathcal{L} \) and \((\ell_1, \ell_2) = \pi \) a 3-dimensional space. This proof is left to the reader.

Theorem\textsuperscript{3.1} can be reformulated for Cameron-Liebler \( k \)-sets in \( \text{AG}(n, q) \), based on the following result.

**Theorem 3.3.** [7 Theorem 6.15] Let \( \mathcal{L} \) be a Cameron-Liebler \( k \)-set in \( \text{AG}(n, q) \), with \( n \geq k + 2 \). Suppose now that \( \mathcal{L} \) has parameter \( x \), then \( x \) satisfies every condition which holds for Cameron-Liebler line classes in \( \text{AG}(n - k + 1, q) \).

**Theorem 3.4.** Let \( \mathcal{L} \) be a Cameron-Liebler \( k \)-set in \( \text{AG}(n, q) \), with \( n \geq k + 5, k > 1 \), with parameter \( x \). Then

\[
x(x - 1) \frac{q^{n-k-1} - 1}{q - 1} \equiv 0 \mod 2(q + 1).
\]

Obviously, Theorem\textsuperscript{3.4} becomes trivial for \( n - k = 3 \). Typical non-existence results give a lower or upper bound on the parameter \( x \). Theorem\textsuperscript{3.4} can be combined with the following theorem.

**Theorem 3.5.** [5 Theorem 1.2] Suppose that \( n \geq 2k + 2 \) and \( k \geq 1 \). Let \( \mathcal{L} \) be a Cameron-Liebler \( k \)-set with parameter \( x \) in \( \text{AG}(n, q) \) such that \( \mathcal{L} \) is not a point-pencil, nor the empty set. Then

\[
x \geq 2 \left( \frac{q^{n-k} - 1}{q^{k+1} - 1} \right) + 1.
\]

To show the significance of this theorem, we give the following small example.

**Example 3.6.** Let \( n = 5, k = 1 \) and \( q = 7 \). We can restrict to \( x \leq 1201 \sim \frac{1}{5} q^{n-1} \) since the complement of a Cameron-Liebler line class is also a Cameron-Liebler line class. Theorem\textsuperscript{3.5} implies \( x \in \{0, 1\} \) of \( x \geq 101 \). Theorem\textsuperscript{3.4} gives \( 9x(x - 1) \equiv 0 \mod 16 \), which reduces the number of possibilities for \( x \) in a significant way. Finally note that for \( q < 7 \) the classification is complete in [11].
4 The projective case

In this section, we will prove a generalization of Theorem 1.1. For a given Cameron-Liebler line class \( L \) with parameter \( x \) in \( \text{PG}(n, q) \) and a fixed point \( p \), the number of lines of \( L \) through \( p \) will be denoted by \( m \).

**Theorem 4.1.** Suppose that \( L \) is a Cameron-Liebler line class with parameter \( x \) in \( \text{PG}(n, q) \), \( n \geq 7 \) odd, then
\[
x(x - 1) + 2m(m - x) \equiv 0 \pmod{q + 1}.
\]

Consider a Cameron-Liebler line class \( L \), a fixed point \( p \), and a 3-dimensional subspace \( \pi \) through \( p \). By Lemma 2.5, \( L_\pi := L \cap [\pi]_1 \) is a Cameron-Liebler line class in \( \pi \). Its parameter will be denoted by \( x_\pi \), and by \( m_\pi \) we denote the number of lines \( L_\pi \) through \( p \).

**Lemma 4.2.** Suppose that \( L \) is a Cameron-Liebler line class in \( \text{PG}(n, q) \), \( n \geq 4 \). Then
\[
\sum_{\pi \ni p} m_\pi = \frac{q^{n-1} - 1}{q - 1}.
\]

**Remark 4.3.** The sum of the equation above runs over all 3-spaces \( \pi \) through the point \( p \). We will make the convention that the summation is always over the first object from the left.

**Proof of Lemma 4.2.** This follows by counting pairs \((\ell, \pi)\), where \( \ell \in L, p \in \ell \) and \( \pi \) a 3-dimensional space through \( p \).

**Lemma 4.4.** Suppose that \( L \) is a Cameron-Liebler line class in \( \text{PG}(n, q) \), \( n \geq 4 \). Let \( p \) be a fixed point in \( \text{PG}(n, q) \). Then
\[
\sum_{\pi \ni p} m_\pi^2 = \frac{q^{n-2} - 1}{q - 1} + \frac{q^{n-1} - 1}{q - 1} m
\]

and
\[
\sum_{\pi \ni p} m_\pi x_\pi = m(x - 1) \frac{q^{n-2} - 1}{q - 1} + \frac{q^{n-1} - 1}{q - 1} m
\]

**Proof.** First we count the triples \((\ell_1, \ell_2, \pi)\), \( \ell_1, \ell_2 \in L \), such that \( \ell_1 \cap \ell_2 = p \), and \( \ell_1, \ell_2 \subseteq \pi \) for a 3-space \( \pi \), both elements of \( L \). For a fixed 3-dimensional subspace \( \pi \), the number of possible pairs \((\ell_1, \ell_2)\) satisfying the conditions, equals \( m_\pi (m_\pi - 1) \).

For a given pair of lines \((\ell_1, \ell_2)\), there are exactly \( \frac{q^{n-2} - 1}{q - 1} \) possible choices for \( \pi \). The total number of pairs \((\ell_1, \ell_2)\) equals \( \frac{m(m-1)}{2} \). Hence
\[
\sum_{\pi \ni p} m_\pi (m_\pi - 1) = \sum_{(\ell_1, \ell_2)} \frac{q^{n-2} - 1}{q - 1} = \frac{m(m-1)}{2} \frac{q^{n-2} - 1}{q - 1}.
\]

Using Lemma 4.2, we find the first equation of the lemma,
\[ \sum_{\pi \ni p} m_{\pi}^2 = \frac{m(m - 1)}{q} \left( \frac{q^{n-2} - 1}{q - 1} + \frac{q^{n-1} - 1}{q - 1} \right) \]  \hfill (3)

Secondly, we count the triples \((\ell_1, \ell_2, \pi), \ell_1, \ell_2 \in \mathcal{L}, \text{ such that } \ell_1 \cap \ell_2 = \emptyset, \ p \in \ell_1 \text{ and } \ell_1, \ell_2 \subseteq \pi, \text{ both elements of } \mathcal{L} \). For a fixed 3-dimensional subspace \(\pi\), we have \(m_{\pi}^2\) possibilities for \(\ell_1\) and, due to Lemma 2.6, we have \(q^2(x_{\pi} - 1)\) possibilities for \(\ell_2\). For a fixed pair \((\ell_1, \ell_2)\) we have only one possible \(3\)-space \(\pi\). Hence,

\[ \sum_{\pi \ni p} m_{\pi}(x_{\pi} - 1)q^2 = \sum_{(\ell_1, \ell_2)} 1 \]

We have in total \(m\) possibilities for \(\ell_1\) and, by Lemma 2.6, the number of lines of \(\mathcal{L}\) skew to \(\ell_1\) equals \((x - 1)q^2\frac{q^{n-2} - 1}{q - 1}\). Now using Lemma 4.2 we can conclude that

\[ \sum_{\pi \ni p} m_{\pi}x_{\pi} = m(x - 1)\frac{q^{n-2} - 1}{q - 1} + \frac{q^{n-1} - 1}{q - 1} \]  \hfill (4)

For a Cameron-Liebler line class \(\mathcal{L}\) in \(\text{PG}(n, q)\), Lemma 2.6 gives precise information on the number of lines of \(\mathcal{L}\) skew to a given line of \(\mathcal{L}\). In the of proof of Lemma 4.6 we will need to control the number of lines of \(\mathcal{L}\) skew to a plane \((\ell_1, p)\), \(p \notin \ell_1 \in \mathcal{L}\). It seems hard to get precise information on this number. However, the next lemma provides us with a modular equation on this number, which will be essential for Lemma 4.6

**Lemma 4.5.** Suppose that \(\mathcal{L}\) is a Cameron-Liebler line class in \(\text{PG}(n, q)\), \(n \geq 7\) odd. Let \(\pi\) be a fixed plane in \(\text{PG}(n, q)\) and let \(\mathcal{L} \cap [\pi]_1 \neq \emptyset\). Denote by \(T\) the number of lines of \(\mathcal{L}\) skew to \(\pi\). Then \(T \equiv 0 \mod (q + 1)\).

**Proof.** We will count the pairs \((\ell_2, \tau), \tau\) a subspace of dimension \(n - 3\), \(\ell_2 \in [\tau]_1 \cap \mathcal{L}\), and \(\tau \cap \pi = \emptyset\). Consider a fixed line \(\ell_2\). The number of \((n - 3)\)-spaces through \(\ell_2\) skew to \(\pi\) equals the number of \((n - 5)\)-spaces in \(\text{PG}(n - 2, q)\) skew to a plane. By Lemma 2.7 this equals \(q^{3(n-4)}\). Since the lines of \(\mathcal{L} \cap [\tau]_1\) induce a Cameron-Liebler line class with parameter \(x_{\tau}\) in \(\tau\), and using Theorem 2.4 we find

\[ q^{3(n-4)}T = \sum_{\tau} x_{\tau} q^{n-3} - 1 \quad \frac{q - 1}{q - 1} = \sum_{\tau} \left(1 + \frac{C_{\tau}}{q^{n-2} - 1} \right) \frac{q^{n-3} - 1}{q - 1} \],  \hfill (5)

where \(C_{\tau} \in \mathbb{N}\) for each \(\tau\). Hence

\[ \sum_{\tau} \frac{C_{\tau} q^{n-3} - 1}{q - 1} = q^{3(n-4)}T - \sum_{\tau} \frac{q^{n-3} - 1}{q - 1} \in \mathbb{N} \],  \hfill (6)

and it follows hat for some integer \(K \in \mathbb{N}\),

\[ \sum_{\tau} C_{\tau} \frac{q^{n-3} - 1}{q - 1} = \frac{q^{n-2} - 1}{q - 1} K \].
Since gcd\((q^{n-2} - 1, q^{n-3} - 1) = q - 1\), clearly \(\frac{q^{n-2} - 1}{q-1}\) \(\sum\limits_{\tau} C_{\tau}\), and now by Lemma 2.11(1),

\[
\sum_{\tau} C_{\tau} \frac{q^{n-3} - 1}{q-1} = \sum_{\tau} C_{\tau} \frac{q^{n-3} - 1}{q-1} \equiv 0 \mod (q + 1).
\]

Thus, by Equation (5), we indeed have that \(T \equiv 0 \mod (q + 1)\). \(\Box\)

**Lemma 4.6.** Suppose that \(\mathcal{L}\) is a Cameron-Liebler line class in \(\text{PG}(n, q)\), \(n \geq 7\) odd. Let \(p\) be a fixed point in \(\text{PG}(n, q)\). Then

\[
2 \sum_{\pi \ni p} (x_\pi - 1)x_\pi \equiv 2 \frac{q^n - 1}{q - 1} \frac{q^{n-2} - 1}{q - 1} x(x - 1) \mod 2(q + 1).
\]

**Proof.** We will count the triples \((\ell_1, \ell_2, \pi), \ell_1, \ell_2 \in \mathcal{L}\), such that for the lines \(\ell_1, \ell_2, \ell_1 \cap \ell_2 = \emptyset\), \(p \notin \ell_1, \ell_2 \cap (\ell_1, p) \neq \emptyset\) and \(\pi = (\ell_1, \ell_2)\).

Fix a 3-dimensional subspace \(\pi\). Then there are \(|\mathcal{L}_x| - m_\pi\) choices for \(\ell_1\). For a chosen \(\ell_1\), by Lemma 2.6, there are \((x_\pi - 1)q^2\) suitable lines skew to \(\ell_1\). For a fixed pair of lines \((\ell_1, \ell_2)\), the 3-dimensional space \(\pi\) is uniquely determined. Hence, with \(N\) the number of pairs \((\ell_1, \ell_2)\) satisfying the conditions,

\[
N = \sum_{\pi \ni p} (x_\pi - 1)q^2(x_\pi(q^2 + q + 1) - m_\pi). \quad (7)
\]

By Lemma 2.6 there are \((x_\pi - 1)q^2\frac{q^{n-2} - 1}{q - 1}\) lines of \(\mathcal{L}\) skew to a given line \(\ell_1\). Denote by \(T_{\ell_1}\) the number of lines of \(\mathcal{L}\) skew to \(\langle \ell_1, p \rangle\). Then

\[
N = \sum_{\ell_1 \in \mathcal{L}, p \notin \ell_1} \left( (x_\pi - 1)q^2\frac{q^{n-2} - 1}{q - 1} - T_{\ell_1} \right) \quad (8)
\]

Now we can reduce the expressions (7) and (8) modulo \(2(q + 1)\) and get

\[
\sum_{\pi \ni p} (x_\pi - 1)x_\pi q^2(q^2 + q + 1) - \sum_{\pi \ni p} q^2(x_\pi - 1)m_\pi = \sum_{\ell_1} (x_\pi - 1)q^2\frac{q^{n-2} - 1}{q - 1} - \sum_{\ell_1} T_{\ell_1} \mod 2(q + 1).
\]

Note that \(q^2 + q + 1 = q(q + 1) + 1\). Since \((x_\pi - 1)x_\pi\) is even, \(2(q + 1)\mid (x_\pi - 1)x_\pi q(q + 1)\), and by Lemma 2.11 it follows that \((x_\pi - 1)x_\pi q^2 \equiv (x_\pi - 1)x_\pi \mod 2(q + 1)\). Hence we find

\[
\sum_{\pi \ni p} (x_\pi - 1)x_\pi - \sum_{\pi \ni p} q^2(x_\pi - 1)m_\pi \equiv \sum_{\ell_1} (x_\pi - 1)q^2\frac{q^{n-2} - 1}{q - 1} - \sum_{\ell_1} T_{\ell_1} \mod 2(q + 1) \quad (9)
\]

Note that there are \(|\mathcal{L}| - m = \frac{x^2}{q - 1} - m\) candidates for \(\ell_1\). Furthermore, by Lemma 4.4

\[
\sum_{\pi \ni p} q^2(x_\pi - 1)m_\pi = m(x - 1)q^2\frac{q^{n-2} - 1}{q - 1}.
\]
Hence Equation 9 reduces to
\[ \sum_{\pi \ni p} (x_{\pi} - 1)x_{\pi} \equiv q^2 q^n - 1 q^{n-2} - 1 \frac{x(x - 1)}{q - 1} - \sum_{\ell_1} T_{\ell_1} \mod 2(q + 1) \quad (10) \]

From Lemma 4.5 it follows that \( 2T_{\ell_1} \equiv 0 \mod 2(q + 1) \). So multiplying Equation (10) with 2, and combined with Lemma 2.11 we obtain
\[ 2 \sum_{\pi \ni p} (x_{\pi} - 1)x_{\pi} \equiv 2q^n - 1 q^{n-2} - 1 \frac{x(x - 1)}{q - 1} \mod 2(q + 1). \]

Now we are ready to prove the main theorem of this section.

**Theorem 4.7.** Suppose that \( L \) is a Cameron-Liebler line class with parameter \( x \) in \( PG(n, q) \), with \( n \geq 7 \) odd, then
\[ x(x - 1) + 2m(x - x) \equiv 0 \mod (q + 1). \]

**Proof.** Assume that \( \pi \) is a 3-space. Then the line set \( L_\pi = L \cap \pi \) is a Cameron-Liebler line class in \( \pi \) of a certain parameter \( x_\pi \). By Theorem 4.1, \[ x_\pi(x_\pi - 1) + 2m_\pi(m_\pi - x_\pi) \equiv 0 \mod 2(q + 1). \]

So in particular, for a fixed point \( p \),
\[ \sum_{\pi \ni p} (x_{\pi} - 1)x_{\pi} + 2m_\pi(m_\pi - x_\pi) \equiv 0 \mod 2(q + 1) \]
\[ \iff \sum_{\pi \ni p} x_{\pi} - 1 + 2 \sum_{\pi \ni p} m_\pi^2 - 2 \sum_{\pi \ni p} m_\pi x_{\pi} \equiv 0 \mod 2(q + 1) \]

Filling in the equations from Lemma 4.4 we have that
\[ \sum_{\pi \ni p} x_{\pi} - 1 + 2q^{n-2} - 1 \frac{x(x - 1)}{q - 1} \equiv 0 \mod 2(q + 1). \]

Multiplying this equation by 2 and using Lemma 4.6 we find
\[ 2q^n - 1 \frac{q^{n-2} - 1}{q - 1} x(x - 1) + 4q^{n-2} - 1 \frac{x(x - 1)}{q - 1} \equiv 0 \mod 2(q + 1) \]
\[ \Rightarrow q^n - 1 \frac{q^{n-2} - 1}{q - 1} x(x - 1) + 2q^{n-2} - 1 \frac{x(x - 1)}{q - 1} \equiv 0 \mod (q + 1) \]

Now for \( n \geq 7 \) odd, we have
\[ \frac{q^{n-2} - 1}{q - 1} \equiv \frac{q^n - 1}{q - 1} \equiv 1 \mod (q + 1). \]

Hence we obtain
\[ x(x - 1) + 2m(x - x) \equiv 0 \mod (q + 1), \]
which proves the statement of the theorem. \( \square \)
The following corollary is easy to prove using the principle of duality in PG\((n, q)\).

**Corollary 4.8.** Suppose that \(\mathcal{L}\) is a Cameron-Liebler \((n-2)\)-set with parameter \(x\) in PG\((n, q)\), \(n \geq 7\) odd, then

\[
x(x - 1) + 2m(m - x) \equiv 0 \mod (q + 1).
\]

Here \(m\) denotes the number of \((n-2)\)-spaces of \(\mathcal{L}\) inside a fixed hyperplane \(\pi\).

## 5 Final remarks

In this final section we compare both modular conditions in AG\((n, q)\). Since a Cameron-Liebler line class in AG\((n, q)\), induces also a Cameron-Liebler line class in PG\((n, q)\) with the same parameter (see [7, Theorem 2]), its parameter must satisfy Theorems 3.1 and 4.1.

Let \(\mathcal{L}\) be a Cameron-Liebler line class with parameter \(x\) in AG\((n, q)\), then by Theorem 3.1

\[
q^{n-2} - 1 \equiv 0 \mod (q + 1).
\]

Now let \(n \geq 7\), odd, then by Theorem 4.1

\[
x(x - 1) + 2m(m - x) \equiv 0 \mod (q + 1),
\]

where \(m\) is the number of lines of \(\mathcal{L}\) through a chosen point \(p\). Choosing \(p\) at infinity, gives \(m = x\), and hence only weaker information compared with the first equation is obtained. To compute \(m\) for affine points, we need to do some more work, using a similar strategy as in [6, Lemma 4.4]. First we need the following lemma.

**Lemma 5.1.** [2, Lemma 2.12] Let \(\mathcal{L}\) be a Cameron-Liebler \(k\)-set in PG\((n, q)\), then for every point \(p\) and every \(i\)-dimensional subspace \(\tau\), with \(p \in \tau\) and \(i \geq k + 1\),

\[
|p|_k \cap \mathcal{L} + \left[\frac{n-1}{k} \frac{q(q^k - 1)}{q^i - 1} |[\tau]|_k \cap \mathcal{L}\right] = \left[\frac{n-1}{k} \frac{q}{q^i - 1} |[p, \tau]|_k \cap \mathcal{L}\right] + \frac{q^k - 1}{q^n - 1} |\mathcal{L}|.
\]

Using this lemma, we find the following result in Lemma 5.2. It shows that Equation 12 considered in AG\((n, q)\) will always reduce to \(x(x - 1) \equiv 0 \mod (q + 1)\), which is a weaker condition than Equation 11 for affine Cameron-Liebler line classes.

**Lemma 5.2.** Suppose that \(\mathcal{L}\) is a Cameron-Liebler line class in AG\((n, q)\), for \(n \geq 3\) odd, then for any hyperplane \(\pi\) and any point \(p\),

\[
|[\pi]|_1 \cap \mathcal{L} \equiv 0 \mod (q + 1) \text{ and } |[p]|_1 \cap \mathcal{L} \equiv x \mod (q + 1).
\]

**Proof.** Choose an hyperplane \(\pi\) and fix a point \(p \in \pi\) at infinity in the closure of AG\((n, q)\). By Lemma 5.1 \(|p|_k \cap \mathcal{L} = x\). Using this in Equation 13 and multiplying with \(\frac{q^{n-2}-1}{q-1}\), we obtain, for \(i = n - 1\) and \(k = 1\)

\[
|[\pi]|_1 \cap \mathcal{L} = \frac{q^{n-1}-1}{q-1} |[p, [\pi]|_1 \cap \mathcal{L}|.
\]
Now by Lemma 2.11 if follows for $n$ odd that

$$|\pi_1 \cap L| = \frac{q^{n-1} - 1}{q - 1} |\lfloor p, \pi \rfloor_1 \cap L| \equiv 0 \mod (q + 1).$$

This proves the first part of the lemma. Secondly, choose an arbitrary affine point $p$, and pick an hyperplane $\pi$ through $p$. Using our previous observations, we obtain that $|\lfloor \pi_1 \cap L| \equiv 0 \mod (q + 1)$. If we take a look at Equation (13), and first multiply this equation with $\frac{q^{n-2} - 1}{q - 1}$, we obtain that:

$$|\lfloor p \rfloor_1 \cap L| \frac{q^{n-2} - 1}{q - 1} = \frac{q^{n-1} - 1}{q - 1} |\lfloor p, \pi \rfloor_1 \cap L| + \frac{q^{n-2} - 1}{q - 1} x.$$

Using Lemma 2.11 and noticing that $n$ is odd, we obtain indeed that

$$|\lfloor p \rfloor_1 \cap L| \equiv x \mod (q + 1).$$

This proves the assertion. 

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