Transverse momentum resummation in soft-collinear effective theory

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Abstract

We present a universal formalism for transverse momentum resummation in the view of soft-collinear effective theory (SCET), and establish the relation between our SCET formula and the well known Collins-Soper-Sterman’s pQCD formula at the next-to-leading logarithmic order (NLLO). We also briefly discuss the reformulation of joint resummation in SCET.

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I. INTRODUCTION

Recently, the soft-collinear effective theory (SCET) has made great simplifications on the proof of factorization in B meson decays \cite{1} and high energy hard scattering processes \cite{2,3}, including resummation of large logarithms in certain regions of phase space, for example, $e^+e^-$ annihilation into two jets of thrust $T \rightarrow 1$ \cite{3}, the deep inelastic scattering (DIS) in the threshold region $x \rightarrow 1$ \cite{4} and Drell-Yan (DY) process in the case of $z \rightarrow 1$ \cite{5}. The reason for these facts is that SCET can be viewed as an operator realization of the pQCD analysis when the modes participating the interactions of interest are soft and collinear, just like chiral dynamics vs QCD at low energy region. This effective field theory (EFT) provides a simple and systematic method for factorization of hard, collinear and usoft or soft degrees of freedom at operator level, especially usoft modes can be decoupled from collinear modes in the Lagrangian at leading order by making a field redefinition, and the large double logarithms such as $(\alpha_s \log^2 \frac{Q^2}{\Lambda^2})^n$, where $Q, \Lambda$ are two typical scales that characterize a process, can be resummed naturally through the running of renormalization group equation (RGE).

However, all the above works have not discussed the transverse momentum ($Q_T$) distributions of high energy hard scattering processes. In this paper, we will investigate the resummed $Q_T$ distributions \cite{6}, taking the Higgs-boson production via gluon fusion in small $Q_T$ region \cite{7,8} as an example, within the framework of SCET. It can be seen that in SCET the $Q_T$ resummation formula automatically separates the process-dependent Wilson coefficient and universal anomalous dimension of the effective operator in a process, which once has been studied by the authors of \cite{9} within the Collins-Soper-Sterman (CSS) frame.

The paper is organized as follows. In section II we start by reviewing the basic steps for factorization and resummation in SCET. In section III we apply it to derive the $Q_T$ distribution at small $Q_T$ region directly, which confirms the CSS formula. In section IV, we also discuss a similar formula for joint resummation. Section V contains our concluding remarks. The details of calculation are given in Appendix.

II. PRELIMINARIES

SCET is appropriate for the kinematic regions of collinear and usoft (soft) modes with momenta scaling: $p_c = (p^+, p^-, p_\perp) = (n \cdot p, \bar{n} \cdot p, p_\perp) \sim Q(\lambda^2, 1, \lambda)$ and $p_{us} \sim Q(\lambda^2, \lambda^2, \lambda^2)$
or \( p_s \sim Q(\lambda, \lambda, \lambda) \), where \( \lambda \ll 1 \) is the scaling parameter, and the light-like vectors \( n = (1, 0, 0, 1) \), \( \bar{n} = (1, 0, 0, -1) \) satisfy \( n \cdot \bar{n} = 2 \), and the perpendicular components of any four vector \( V \) are defined by \( V^\mu_\perp = V^\mu - (n \cdot V)\bar{n}^\mu/2 - (\bar{n} \cdot V)n^\mu/2 \).

In constructing SCET, one should first identify the scaling of all possible modes of initial and final states with soft and collinear degrees of freedom, then integrate other degrees of freedom, and the remaining modes must reproduce all the infrared physics of the full theory in the region where SCET is valid, which is ensured by the method of regions for Feynman integrals with massless quarks and gluons\(^1\) [10]. The EFT describing the usoft(soft) and collinear modes is known as SCET\(_I\)(II), and to distinguish the two theories, the scaling parameter corresponding to SCET\(_II\) is denoted by \( \eta \sim \lambda^2 \), i.e., \( p_c \sim Q(\eta^2, 1, \eta) \) and \( p_s \sim Q(\eta, \eta, \eta) \).

The elements of SCET\(_I\) consist of usoft sectors \( \{q_{us}, A_{us}\} \) and collinear sectors \( \{\xi_n, A_n\} \) moving in the \( n\)-direction, which are expanded as

\[
\phi_n(x) = \sum_p e^{-i\bar{p} \cdot x} \phi_{n,p}(x), \quad p = \bar{p} + k, \tag{1}
\]

where \( k \sim Q\lambda^2 \) resides in the space-time dependence of \( \phi_{n,p}(x) \), i.e., \( \partial \phi_{n,p}(x) \sim (Q\lambda^2)\phi_{n,p}(x) \), and \( \bar{p} \sim Q(0, 1, \lambda) \) is called label momentum, and the label operators \( \bar{P}, P_\perp \) are defined by picking out \( \bar{p}^-, \bar{p}_\perp \) momenta for collinear fields \( \phi_n(x)^2 \), respectively. The Wilson line for \( n\)-collinear fields has the form of

\[
W_n(x) = \left[ \sum_{\text{perms}} \exp \left( -\frac{g}{\bar{P}} \bar{n} \cdot A_n(x) \right) \right],
\]

which is required to ensure collinear gauge invariance. The Lagrangian of collinear sectors, which is invariant under the usoft and collinear gauge transformation, at leading order\(^3\) (LO) in \( \lambda \) is \( [1] \),

\[
\mathcal{L}_c = \mathcal{L}_{cg} + \mathcal{L}_{cq},
\]

\[
\mathcal{L}_{cg} = \frac{1}{2g^2} \Tr \left[ \left[ iD^\mu + gA^\mu_n, iD^\nu + gA^\nu_n \right] \right]^2
+ 2\Tr \left[ \bar{c}_n \left[ iD^\mu, A^\mu_n, c_n \right] \right] + \frac{1}{\alpha} \Tr \left[ \left[ iD^\mu, A^\mu_n \right] \right]^2,
\]

\[
\mathcal{L}_{cq} = \bar{\xi}_n \left[ i\bar{n} \cdot D + i \bar{D}_c \frac{1}{\bar{m} \cdot D_c} i \bar{D}_c \right] \bar{\xi}_n. \tag{3}
\]

\(^1\) In the presence of masses, the regions analysis is very complicated, and we only discuss the massless case.

\(^2\) The convention \( \phi_n(x) = \phi_{n,p}(x) \) for collinear fields will be used for convenience.

\(^3\) We’ll restrict our discussion only at this order through the paper.
Here the third line are the gauge fixing terms with parameter $\alpha$ and $c_n$ denotes collinear ghost field, and

$$
 iD^\mu = \bar{P}n^\mu/2 + P_\perp^\mu + (in \cdot \partial + gn \cdot A_{us})\bar{n}^\mu/2,
$$

$$
 in \cdot D = in \cdot D_{us} + gn \cdot A_n, \quad iD_{us} = i\partial + gA_{us},
$$

$$
 i\bar{n} \cdot D^c = \bar{P} + g\bar{n} \cdot A_n, \quad iD^c_{\perp} = P_\perp + gA^c_\perp.
$$

The Lagrangian of soft sectors in SCET is identical to that of QCD.

As for SCET$_{II}$, it was emphasized that it can also be viewed as the EFT of SCET$_I$, and is the final theory [11]. This suggests a short path to go into SCET$_{II}$ from SCET$_I$, if the following matching and running steps are taken [11]:

1. Matching QCD onto SCET$_I$ at a scale $\mu^2 \sim Q^2$ with $p_c^2 \sim Q^2 \lambda^2$;
2. Decoupling the usoft-collinear interactions with the field redefinitions, $\xi_n = Y_n^\dagger \xi_n^{(0)}$ and $A_n = Y_n^\dagger A_n^{(0)}Y_n$. Here $Y_n(x) = P \exp(ig \int ds n \cdot A_{us}(ns + x))$ is the usoft Wilson line of usoft gluons in $n$ direction from $s = 0$ to $s = \infty$ for final state particles, and $P$ means path-ordered product, while for initial state particles, $Y_n$ is from $s = -\infty$ to $s = 0$ and the daggers are reversed. This step leads to $\mathcal{L}_c(\xi_n, A_n, n \cdot A_{us}) = \mathcal{L}_c(\xi_n^{(0)}, A_n^{(0)}, 0)$;
3. Matching SCET$_I$ onto SCET$_{II}$ at a scale $\mu^2 \sim Q^2 \lambda^2$ with $p_c^2 \sim Q^2 \eta^2$. Thus, the soft and collinear modes are decoupled in the Lagrangian of SCET$_{II}$.

Next, we extend SCET to include the possibility of collinear fields moving in different light-cone directions $n_1, n_2, n_3, \ldots$. These directions defined by $n_i$ and $n_j$ satisfy $n_i \cdot n_j \gg \lambda^2$ for $i \neq j$. For simplicity we will only consider the case of head-on jets corresponding to collinear particles moving in the $n$ and $\bar{n}$ directions. Since the effective theory only takes account the interactions of the modes in the local way, the Lagrangian of the effective theory contains no direct coupling of collinear particles moving in the two separate directions, however the usoft gluons can mediate between them in SCET$_I$. Hence the Lagrangian in this case can be written by

$$
 \mathcal{L}^c_{\{n, \bar{n}\}} = \mathcal{L}^c_n + \mathcal{L}^c_{\bar{n}},
$$

and the soft parts are unchanged, so the decoupling transformations are also valid here.

To illustrate the application of SCET and warm up, we consider the Sudakov effect of quark electromagnetic form factor in QCD [12], i.e., the double logarithmic asymptotic of conservative current $j^\mu = \bar{\psi}\gamma^\mu \psi$ in the following kinematics:
(a) nearly on-shell case

\[ Q^2 = -(p_1 - p_2)^2 \gg -p_1^2 = -p_2^2 \sim \Lambda_{QCD}^2, \]
\[ p_1 \sim Q(\eta^2, 1, \eta), \quad p_2 \sim Q(1, \eta^2), \quad \eta \sim \Lambda_{QCD}/Q; \]

(b) off-shell case

\[ Q^2 = -(p_1 - p_2)^2 \gg -p_1^2 = -p_2^2 \sim Q\Lambda_{QCD}, \]
\[ p_1 \sim Q(\lambda^2, 1, \lambda), \quad p_2 \sim Q(1, \lambda^2), \quad \lambda \sim \sqrt{\Lambda_{QCD}/Q}. \]

Here \( p_1, p_2 \) are the momenta of the initial and final quarks. In the above two cases we have omitted quark mass effects.

Following the treatment of heavy-to-collinear current discussed in \[11, 13\] for case (a), we first match the full current onto the corresponding operator in SCET \[I\]. At LO in \( \lambda \), it gives

\[ j^\mu = [\bar{\xi}_n W_n] \gamma^\mu C_q(P^\dagger, \bar{P}, \mu^2)[W_n^\dagger \xi_n]. \] (5)

By the requirement of collinear and usoft gauge invariance, the LO effective operator is determined uniquely, and the re-parameterization invariance (RPI) \[11\] implies that the Wilson coefficient satisfies \( C_q(P^\dagger, \bar{P}, \mu^2) = C_q(P^\dagger \cdot \bar{P}, \mu^2). \)

Obviously, the tree level matching condition for \( j^\mu \) leads to \( C_q(Q^2, \mu^2) = 1 + O(\alpha_s) \). We certainly can determine \( O(\alpha_s) \) correction by adopting dimensional regularization\[4\] (DR) to regulate UV and IR divergences to compute on-shell matrix elements on both sides of Eq.\[13\], for which is valid at the operator level and the matching calculation is independent of regularization method. With this choice, the fact \( IR_{QCD} = IR_{SCET} \) provides us a direct matching calculation to read off the Wilson coefficients and anomalous dimensions of the operators in SCET. As all the on-shell loop integrals in SCET are scaleless and vanish, \( IR_{SCET} = -UV_{SCET} \), and then \( IR_{QCD} = -UV_{SCET} \). Furthermore, the self energy diagrams in full QCD with massless quarks are also vanish, and all the wave function renormalization constants and the residues of the related propagators are equal to unity, and we also note that the conserved current in full QCD needs not to be renormalized. Thus, what we need

\[ \gamma_E \]

\[ \frac{\mu^2 e^{-\gamma_E}/4\pi}{\mu^2} \]

\[ \text{is used through this paper, where } \gamma_E \text{ is Euler’s constant.} \]
to calculate is an one particle irreducible diagram, Fig.1 in the full theory, which is given by
\[
\langle p_2|j^\mu|p_1 \rangle = \langle p_2|j^\mu|p_1 \rangle^{\text{tree}} + \langle p_2|j^\mu|p_1 \rangle^{\text{one-loop}} = \bar{u}(p_2)\gamma^\mu(1 + V_q)u(p_1),
\]
(6)

\[
V_q = \frac{\alpha_s C_F}{4\pi} \left( \frac{\mu^2 e^{\gamma_E}}{Q^2} \right)^\epsilon \frac{\Gamma(1-\epsilon)}{\Gamma(2-\epsilon)} \left( -\frac{2}{\epsilon^2} - \frac{3}{\epsilon} - 8 - \frac{\pi^2}{3} \right)
= \frac{\alpha_s C_F}{4\pi} \left[ -\frac{2}{\epsilon^2} - \frac{1}{\epsilon} \left( 2 \log \frac{\mu^2}{Q^2} + 3 \right) - \log^2 \frac{\mu^2}{Q^2} - 3 \log \frac{\mu^2}{Q^2} - 8 + \frac{\pi^2}{6} \right].
\]
(7)

Here \( C_F = (N_c^2 - 1)/2N_c \) for \( SU(N_c) \), and \( N_c = 3 \) for QCD. The \( \epsilon \)-poles in Eq.(7) are of IR character, whose opposition are just the UV poles in SCET. Thus the matching calculation at one-loop level gives
\[
C_q(Q^2, \mu^2) = 1 + \frac{\alpha_s C_F}{4\pi} \left( -8 + \frac{\pi^2}{6} \right),
\]
(8)

\[
Z_V \equiv \sum_n \frac{Z_V^{(n)}}{e^n} = 1 + \frac{\alpha_s C_F}{4\pi} \left[ \frac{2}{\epsilon^2} + \frac{1}{\epsilon} \left( 2 \log \frac{\mu^2}{Q^2} + 3 \right) \right].
\]
(9)

Here \( Z_V \) is defined as \( \overline{\text{MS}} \) renormalization constant of the effective operator, and \( \mu \) has been set to \( Q \) to minimize the logarithms in the Wilson coefficient. It was pointed out [1] that the anomalous dimension of the effective operator is independent of its spin structure, for which can be factorized out from loop integrals. This means the evolution equation in SCET is universal, and only the Wilson coefficient is process-dependent.

From Eq.(9), we obtain the RGE of \( C_q(Q^2, \mu^2) \),
\[
\frac{d\log C_q(Q^2, \mu^2)}{d\log(\mu)} = \gamma_1(\mu) = -g \frac{\partial Z_V^{(1)}}{\partial g},
\]
(10)
FIG. 2: Factorization of on-shell form factor (a) and off-shell form factor (b) in SCET. In (b), the soft Wilson lines are terminated and the external quarks are amputated, therefore, (b) can be taken as a sub-diagram of (a). Both diagrams are depicted under the gauge $\vec{n} \cdot A_n = n \cdot A_{\bar{n}} = 0$.

\[ \gamma_1(\mu) \equiv A_q(\alpha_s) \log \frac{Q^2}{\mu^2} + B_q(\alpha_s) \]
\[ = -\frac{\alpha_s C_F}{4\pi}(4 \log \frac{\mu^2}{Q^2} + 6). \]  

(11)

Here $A_q^{(1)} = C_F$ and $B_q^{(1)} = -\frac{3}{2} C_F$. With Eq.(10), we can resum the terms such as double logarithms from the scale $\sim Q^2$ down to the scale $\sim Q^2 \lambda^2$, we abbreviate this matching step as a chain $QCD|_{Q^2} \rightarrow SCET_I|_{Q^2 \lambda^2}$.

Next, we decouple the usoft and collinear modes by the field redefinitions, which results in

\[ \langle p_2| [\bar{\xi}_n W_{\bar{n}}]|\gamma^\mu[W_{\bar{n}}^\dagger \xi_n]|p_1 \rangle \rightarrow \langle p_2| [\bar{\xi}_n^{(0)} W_{\bar{n}}^{(0)}]|\Omega \rangle \gamma^\mu \langle \Omega| T[Y_n Y_{\bar{n}}]|\Omega \rangle \langle \Omega| [W_{\bar{n}}^{(0)}\dagger \xi_n^{(0)}]|p_1 \rangle, \]  

(12)

where $T$ means time ordering operator.

For the final step we integrate out all the off-shell modes of order $\sqrt{Q \Lambda_{QCD}}$ and go into SCET_{II}. We can rename the usoft fields as soft fields for the usoft degrees of freedom scaling as soft ones, and then lower the the off-shellness of the collinear fields that would be matched onto SCET_{II}. Since the leading collinear Lagrangians in SCET_{I} and SCET_{II} are the same and (u)soft and collinear fields are decoupled at LO in $(\lambda)\eta$, all possible time-ordered products involve collinear fields agree exactly and we can simply replace $\bar{\xi}_n^{(0)} W_{\bar{n}}^{(0)} \rightarrow \bar{\xi}_n^{II} W_{\bar{n}}^{II}$

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5 The notion $A \equiv \sum_n (\alpha_s/\pi)^n A^{(n)}$, we adopt $\{A, B\}$ to distinguish the well known coefficients $\{A, B\}$ in pQCD.
and $W^{(0)\dagger}_n \xi^{(0)}_n \rightarrow W^{II\dagger}_n \xi^{\Pi}_n$, where the superscript II denotes SCET\textsubscript{II} will be dropped from now on.

Because $p_1, p_2$ of (a) are described by the collinear modes in SCET\textsubscript{II}, the general matching structure of SCET\textsubscript{II} diagram is shown in Fig.2.(a). The Wilson coefficient at this step is unity and anomalous dimension is the same as the first step, except that it runs from the scale $\sim Q^2 \lambda^2$ to the scale $\sim Q^2 \eta^2$. We abbreviate this step as $SCET_I|_{Q^2 \lambda^2} \Rightarrow SCET_{II}|_{Q^2 \eta^2}$. Collecting all the results above, we obtain the known Sudakov form factor $S_q^{(a)}(\Lambda_{\text{QCD}}, Q)$, leaving other coefficients omitted,

$$S_q^{(a)}(\Lambda_{\text{QCD}}, Q) = \exp \left( - \int_{\Lambda_{\text{QCD}}}^{Q} \gamma_1(\mu) d \log \mu \right). \quad (13)$$

For case (b), it can be taken as a sub-diagram of the on-shell case, from kinematical considerations, of which the external legs are amputated. Thus, step (1) is unchanged, and in step (2), $\langle 0 | T[\bar{Y}_n Y_n] | 0 \rangle$ changes into

$$- \int_0^{\infty} ds dt e^{iQ\lambda^2(s+t)}/\int_0^{\infty} ds dt e^{iQ\lambda^2(s+t)} \langle \Omega | T[\bar{Y}_n(0, \bar{n}s) Y_n^\dagger(0, -nt)] | \Omega \rangle, \quad (14)$$

where $1/(Q\lambda^2)$ is the effective contour length \cite{12} and

$$Y_n(0, \bar{n}s) \equiv P \exp \left( ig \int_0^s d\beta \bar{n} \cdot A_{us}(\bar{n}\beta) \right),$$

$$Y_n(0, -nt) \equiv P \exp \left( - ig \int_0^t d\beta n \cdot A_{us}(-n\beta) \right). \quad (16)$$

Because of the jets $J_1, J_2$ with fluctuations $-p_1^2 = -p_2^2 = Q\Lambda_{\text{QCD}} \gg Q^2 \eta^2$ in Fig.2.(b), they must be integrated out in SCET\textsubscript{II}, and only (14) is left over after step (3) associated with renaming the usoft modes in SCET\textsubscript{I} as the soft modes in SCET\textsubscript{II}, of which the running behavior is the same as $F_{IR}$ of \cite{12}. Finally, the Sudakov factor in the off-shell case is

$$S_q^{(b)}(Q\eta, Q) = \exp \left( - \int_{Q\lambda}^{Q} \gamma_1(\mu) d \log \mu - \int_{Q\eta}^{Q\lambda} \gamma_2(\mu) d \log \mu \right), \quad (17)$$

$$\gamma_2(\mu) = \frac{\alpha_s}{\pi} C_F \log \frac{\mu^2}{Q^2 \eta^2} + \mathcal{O}(\alpha_s^2), \quad (18)$$

where $\gamma_2(\mu)$ is the anomalous dimension of (14). We conclude this section with a chain for the off-shell case, $QCD|_{Q^2} \rightarrow SCET_I|_{Q^2 \lambda^2} \rightarrow SCET_{II}|_{Q^2 \eta^2}$. Now we are ready to turn into the $Q_T$ resummation in the following.
III. METHOD OF $Q_T$ RESUMMATION IN SCET

Since SCET is powerful to disentangle the soft and collinear interaction, and IR power counting [14] tells that the singular terms of $Q_T$ distribution for DY-like processes in the limit of $Q_T \to 0$ originate from soft and collinear modes, which are emitted by partons from hadrons $p_1, p_2$, it is not unexpected that SCET can be applied to treat them and to derive the resummed part of full transverse momentum distribution for these semi-inclusive processes, while the remaining regular terms $Y_{[6,7,8]}$ and the prescription of incorporating non-perturbative region ($Q_T \sim \Lambda_{QCD}$) are neglected in this paper.

For the sake of simplicity, the process of Higgs-boson production is taken as a demonstration, but the method we used is not confined to this example. The dominant process for Higgs-boson production at the Large Hadron Collider (LHC) in the Standard Model are gluon fusion through a heavy quark loop, mainly the top quark, $p_1(p_1) + p_2(p_2) \to gg \to \phi(Q) + X$ with $P_1 = (0, 2p, 0)$, $P_2 = (2p, 0, 0)$ and $S = P_1^- P_2^+$. It is convenient to start from the effective Lagrangian for one Higgs-boson and gluons coupling [15],

$$\mathcal{L}_{\phi gg} = \tau(\alpha_s) \phi G_{\mu\nu}^a G^{\mu\nu}_a, \quad (19)$$

where $\tau(\alpha_s) = \frac{\alpha_s(Q)}{12\pi}(\sqrt{2}G_F)^{1/2} + \mathcal{O}(\alpha_s^2)$ and $Q = m_\phi$. Therefore, the operator for Higgs-boson production is $\mathcal{H} = G_{\mu\nu}^a G^{\mu\nu}_a$. Here the coupling $\alpha_s$ suffers the QCD correction, which is unlike the case of electro-charge coupling. Furthermore, because the renormalization constant of $\alpha_s G_{\mu\nu}^a G^{\mu\nu}_a$ is unity up to $\mathcal{O}(\alpha_s)$, the renormalization constant of $\mathcal{H}$ is just $Z_g^{-2}$, where $Z_g$ is the renormalization constant of gauge coupling-$g$.

If we set $\lambda^2 \sim Q_T/Q$ with $Q \gg Q_T \gg \Lambda_{QCD}$, the situation is much like that of quark form factor-(a) discussed in last section, and the matching and running procedure can be followed. The operator $\mathcal{H}$ can match at LO in $\lambda$ onto

$$\mathcal{H} = \frac{1}{2} \mathcal{B}_n^\mu C_g(P^\dagger \cdot \vec{P}, \mu^2) \mathcal{B}_n^\mu, \quad (20)$$

where

$$\mathcal{B}_n^\mu = \bar{n}_\nu G_\nu^{\mu},$$

$$G_\nu^{\mu} = W_n^\dagger [i\not{D}_n^\mu + g A_n^\mu, i\not{D}_n^\nu + g A_n^\nu] W_n, \quad (21)$$
FIG. 3: Graphical representation for gluon current matching.

and $n \leftrightarrow \bar{n}$ for $G_{\mu\nu}^{\mu\nu}$.

The one loop calculation, Fig. 3, is similar to quark current, except for dividing the final result by $Z_{H}^{-2}$. Finally,

$$\langle g_{1}|\mathcal{H}|g_{2}\rangle = \langle g_{1}|\mathcal{H}|g_{2}\rangle^\text{tree} + \langle g_{1}|\mathcal{H}|g_{2}\rangle^\text{one-loop} + \text{c.t.}$$

$$V_{g} = \frac{\alpha_{s}}{4\pi} \left[ \frac{\Gamma(1 - \epsilon)}{\Gamma(1 - 2\epsilon)} \left( \frac{\mu^{2} e^{\gamma_{E}}}{Q^{2}} \right)^{\epsilon} \left( -\frac{2C_{A}}{\epsilon^{2}} - \frac{2\beta_{0}}{\epsilon} + A_{g}^{H} \right) \right]$$

$$= \frac{\alpha_{s}}{4\pi} \left[ -\frac{2C_{A}}{\epsilon^{2}} - \frac{1}{\epsilon} \left( 2C_{A} \log \frac{\mu^{2}}{Q^{2}} + 2\beta_{0} \right) - C_{A} \log^{2} \frac{\mu^{2}}{Q^{2}} - 2\beta_{0} \log \frac{\mu^{2}}{Q^{2}} + A_{g}^{H} + \frac{C_{A} \pi^{2}}{2} \right]$$

is obtained\(^{6}\), from which we read

$$C_{g}(Q^{2}, Q^{2}) = 1 + \frac{\alpha_{s}}{4\pi} (A_{g}^{H} + \frac{C_{A} \pi^{2}}{2})$$

$$Z_{H} \equiv \sum_{n} \frac{Z_{H}^{(n)}(\epsilon^{n})}{\epsilon^{n}} = 1 + \frac{\alpha_{s}}{4\pi} \left[ \frac{2C_{A}}{\epsilon^{2}} + \frac{1}{\epsilon} \left( 2C_{A} \log \frac{\mu^{2}}{Q^{2}} + 2\beta_{0} \right) \right].$$

Here $C_{A} = N_{c}/6$ and $\beta_{0} = -\frac{2}{3}n_{f}T_{R}$. $A_{g}^{H} = 11 + 2\pi^{2}$, $T_{R} = 1/2$ and $n_{f} = 5$ is the number of active quark flavors. Then the RGE of $C_{g}(Q^{2}, \mu^{2})$ is

$$\frac{d \log C_{g}(Q^{2}, \mu^{2})}{d \log(\mu)} = \gamma_{1}(\mu) = -g \frac{\partial Z_{H}^{(1)}}{\partial g},$$

$$\gamma_{1}(\mu) \equiv A_{g}(\alpha_{s}) \log \frac{Q^{2}}{\mu^{2}} + B_{g}(\alpha_{s})$$

$$= -\frac{(\alpha_{s}/\pi)(C_{A} \log \frac{\mu^{2}}{Q^{2}} + \beta_{0})}{2},$$

with $A_{g}^{(1)} = C_{A}$ and $B_{g}^{(1)} = -\beta_{0}$. Thus the evolution from the scale $\sim Q^{2}$ to the scale $\sim Q^{2}\lambda^{2}$ gives

$$C_{g}(Q^{2}, Q^{2}\lambda^{2}) = C_{g}(Q^{2}, Q^{2}) \exp \left( -\int_{Q^{2}\lambda^{2}}^{Q^{2}} \frac{d\mu^{2}}{2\mu^{2}} \gamma_{1}(\mu) \right).$$

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\(^{6}\) Here we have absorbed the scale dependence of $\alpha_{s}$ into that of $Z_{g}$. 

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As shown above, the extraction of $A, B$ in SCET is different from that of $A, B$ in pQCD, i.e., there is no need to calculate real correction which is more difficult to handle. Using the virtual part of higher order calculation, such as the two loop on-shell quark and gluon form factor \cite{16}, we can find the $O(\alpha_s^2)$ universal anomalous dimension. For example,

$$A_a^{(2)} = \frac{1}{2} C_a K, \quad K = C_A \left( \frac{67}{18} - \frac{\pi^2}{6} \right) - \frac{10}{9} T_R n_f,$$

where $C_q = C_F, C_g = C_A$.

After performing field redefinitions, Eq.\,(26) can be directly used to running $\mathcal{H}$ from the scale $\sim Q^2$ to the scale $\sim Q^2 \eta^2$ without loss of degrees of freedom. So the relevant operator for Higgs-boson production at the scale $\sim Q^2 \eta^2 (Q_T^2)$ is

$$\mathcal{H} = C_g (Q^2, Q^2 \eta^2) \text{Tr}\{ T[Y_n^{\dagger} Y_n B_n^{\mu} Y_n^{\dagger} B_n^{\mu}] \}$$

$$= C_g (Q^2, Q^2 \eta^2) \frac{1}{2} \text{Tr}\{ Y_n^{ab} Y_n^{ac} B_n^{\mu} B_n^{\mu} \}$$

$$\equiv C_g (Q^2, Q^2 \eta^2) \hat{\mathcal{H}}.$$  \hspace{1cm} (30)

Here $Y_n(\bar{n})$ is the adjoint soft Wilson line from $-\infty$ to $0$ in $n(\bar{n})$ direction for incoming fields.

Now, we have completed the procedures corresponding to step (1), (2) and (3).

To obtain the differential cross section, we relate it to the composite operator $\mathcal{H}$ at the renormalization scale $\mu^2 \sim Q^2 \eta^2$ in SCET$_H$, where the cross section can be written as

$$\frac{1}{\sigma_{gg}^{(0)}} \frac{d\sigma_{\text{resum}}}{dQ^2 dy dQ_T^2} = H_g^\phi (Q) e^{-S_g(\mu, Q)} \sigma_{\text{SCET}}(Q_T, Q, \mu),$$  \hspace{1cm} (31)

where $H_g^\phi (Q) = |C_g (Q^2, Q^2)|^2$ is a function of $\alpha_s (Q)$, and

$$\sigma_{gg}^{(0)} = (\sqrt{2} G_F)^2 \frac{\alpha_s^2 (Q) m_H^2}{576 \pi^3} \delta (Q^2 - m_H^2),$$  \hspace{1cm} (32)

$$S_g (\mu, Q) = \int_{\mu^2}^{Q^2} \frac{d\mu^2}{\mu^2} \left[ A_g (\alpha_s) \log \frac{Q^2}{\mu^2} + B_g (\alpha_s) \right],$$  \hspace{1cm} (33)

and

$$\sigma_{\text{SCET}}(Q_T, Q, \mu) = \frac{1}{\sigma_{gg}^{(0)}} \frac{d\sigma_{\text{SCET}}(\mu)}{dQ^2 dy dQ_T^2}$$  \hspace{1cm} (34)

represents the normalized differential cross section calculated in SCET$_H$ with the composite operator $\hat{\mathcal{H}}$. The general structure of relevant diagram is shown in Fig.\,

\begin{figure}
\centering
\includegraphics[width=\textwidth]{figure.png}
\caption{Diagram for relevant process.}
\end{figure}

where the soft and collinear modes are decoupled and the spin and color are summed over in the matrix.
element of hadron, from which the SCET cross section can be written in the form of multiple convolution,

\[
\sigma_{\text{SCET}}(Q_T, Q, \mu) = \int d^2 \vec{k}_{T1} d^2 \vec{k}_{T2} d^2 \vec{k}_{TS} \delta^2(\vec{k}_{T1} + \vec{k}_{T2} + \vec{k}_{TS} - \vec{Q}_T) \\
\times J_{p_1}(x_1, k_{T1}, \mu) J_{p_2}(x_2, k_{T2}, \mu) S(k_{TS}, \mu),
\]

where \( x_1 = Q e^y / \sqrt{S} \), \( x_2 = Q e^{-y} / \sqrt{S} \) for \( Q_T^2 \ll Q^2 \), and

\[
J_{p_1}(x_1, k_{T1}, \mu) = \frac{2}{x_1 P^{-}} \frac{1}{(2\pi)^3} \int dy^+ d^2 \vec{y}_\perp e^{-i(x_1 P^- y^- - \vec{k}_{T1} \cdot \vec{y}_\perp)} \\
\times \langle p_1 | \text{Tr} [B^\alpha_n(y^+, 0, \vec{y}_\perp) B^\alpha_n(0)] | p_1 \rangle,
\]

\[
J_{p_2}(x_2, k_{T2}, \mu) = \frac{2}{x_2 P^+} \frac{1}{(2\pi)^3} \int dy^- d^2 \vec{y}_\perp e^{-i(x_2 P^+ y^- - \vec{k}_{T2} \cdot \vec{y}_\perp)} \\
\times \langle p_2 | \text{Tr} [B^\alpha_{\bar{n}}(0, y^-, \vec{y}_\perp) B^\alpha_{\bar{n}}(0)] | p_2 \rangle,
\]

\[
S(k_{TS}, \mu) = \frac{1}{(2\pi)^2} \int d^2 \vec{y}_\perp e^{i\vec{k}_{TS} \cdot \vec{y}_\perp} \langle \Omega | \tilde{T} [\mathcal{Y}^{a\alpha}_n \mathcal{Y}^{\alpha b}_n](0, 0, \vec{y}_\perp) \\
\times T[\mathcal{Y}^{ab}_n \mathcal{Y}^{ac}_n](0) | \Omega \rangle,
\]

with \( \tilde{T} \) denoting the anti-time ordering operator. Obviously, in Eq.(38) the matrix element has been factorized, and the delta function is imposed by momentum conservation.

Next, to factorize the phase space, the trick of Fourier transforming to impact parameter space is significant \[6\],

\[
\int d^2 \vec{Q}_T e^{i\vec{b} \cdot \vec{Q}_T} \delta^2(\sum_i \vec{k}_{T_i} - \vec{Q}_T) = \prod_i e^{i\vec{b} \cdot \vec{k}_{T_i}}.
\]

Then, for each transverse momentum \( \vec{k}_{T_i} \), one obtains

\[
\int d^2 \vec{k}_{T_i} e^{i\vec{b} \cdot \vec{k}_{T_i}} f(\vec{k}_{T_i}) = \tilde{f}(b).
\]
This produces the simple product

$$\tilde{\sigma}_{\text{SCET}}(b, Q, \mu) = \tilde{J}_{p_1}(x_1, b, \mu) \tilde{J}_{p_2}(x_2, b, \mu) \tilde{S}(b, \mu). \quad (41)$$

Because of KLN theorem, the contributions from the soft modes are free of IR divergences. So only the collinear divergences are survived, therefore after matching the SCET cross section onto a product of two parton distribution functions (PDFs) given by [2], which are equivalent to the conventional PDFs $f_{a/p_i}(x_i, \mu)$ at LO in $\lambda$, the remaining IR divergences can be absorbed into these nonperturbative inputs, of which the evolutions are controlled by the DGLAP equations. This leads to

$$\tilde{J}_{p_i}(x_i, b, \mu) = \sum_a (f_{a/p_i} \otimes c_{ga})(x_i, b, \mu) = \sum_a \int_{x_i}^1 d\xi f_{a/p_i}(\xi, \mu)c_{ga}(\frac{x_i}{\xi}, b, \mu), \quad (42)$$

$$c_{ga} \equiv \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n c_{ga}^{(n)}.$$

If we define

$$C_{ga}(z, \frac{b_0}{b}, \mu) = c_{ga}(z, b, \mu)[\tilde{S}(b, \mu)]^{1/2}, \quad b_0 = 2e^{-\gamma_E}, \quad (43)$$

then

$$\tilde{\sigma}_{\text{SCET}}(b, Q, \mu) = (f_{a/p_1} \otimes C_{ga})(x_1, \frac{b_0}{b}, \mu)(f_{b/p_2} \otimes C_{gb})(x_2, \frac{b_0}{b}, \mu). \quad (44)$$

Obviously, $C_{ga}^{(0)}(z) = \delta_{ga}\delta(1 - z)$, and we have derived in Appendix that

$$C_{ga}^{(1)}(z) = -2P_{ga}^e(z) + C_A \frac{\pi^2}{6}\delta_{ga}\delta(1 - z), \quad (45)$$

where $\mu$ has been set to $b_0/b$ to eliminate large constant factors in $C_{ga}^{(1)}(z, \mu)$, and $P_{ga}^e(z)$ represent the $O(\epsilon)$ terms of the DGLAP splitting kernels.

Combining Eq. (31)-(44) and Fourier transforming back to $Q_T$ space, we obtain the resummed formula of transverse momentum distribution for Higgs-boson production in SCET,

$$\frac{1}{\sigma_{gg}^{(0)}} \frac{d\sigma_{\text{resum}}}{dQ^2dydQ_T^2} = H_g^{\phi}(Q) \int_0^\infty \frac{db}{2\pi} b J_0(bQ_T) \sum_{ab} e^{-S_{ab}(b, Q)}$$

$$\times (f_{a/p_1} \otimes C_{ga})(x_1, \frac{b_0}{b})(f_{b/p_2} \otimes C_{gb})(x_2, \frac{b_0}{b}). \quad (46)$$
The similar reasoning leads to the general form for $Q_T$ resummation,
\[
\frac{1}{\sigma^{(0)}} \frac{d\sigma^\text{resum}}{dQ^2dydQ_T^2} = H_c^F (Q) \int_0^\infty \frac{db}{2\pi} b J_0 (bQ_T) \sum_{ab} e^{-S_c(\frac{b}{b_0}, Q)} \\
\times (f_{a/p_1} \otimes C_{ca})(x_1, \frac{b_0}{b})(f_{b/p_2} \otimes C_{cb})(x_2, \frac{b_0}{b}), \quad (47)
\]
where $F$ and $c$ stand for the type of process and of parton participating the elementary sub-process, for example, $F = DY$ and $c = q$ for Drell-Yan process. The formula (47) is a little different from the known CSS formula [6],
\[
\frac{1}{\sigma^{(0)}} \frac{d\sigma^\text{resum}}{dQ^2dydQ_T^2} = \int_0^\infty \frac{db}{2\pi} b J_0 (bQ_T) \sum_{ab} e^{-S_c(\frac{b}{b_0}, Q)} \\
\times (f_{a/p_1} \otimes C_{ca})(x_1, \frac{b_0}{b})(f_{b/p_2} \otimes C_{cb})(x_2, \frac{b_0}{b}). \quad (48)
\]
Transforming Eq.(47) into the form as Eq.(48) by the identity
\[
H_c^F (Q) = \exp \left[ \int_{b_0^2/b^2}^{Q^2} \frac{d\mu^2}{\mu^2} \beta(\alpha_s) \frac{d\log H_c^F}{d\log \alpha_s} \right] H_c^F (b_0/b) \quad (49)
\]
with $\beta(\alpha_s)$ denoting the QCD $\beta$-function, one finds
\[
\frac{1}{\sigma^{(0)}} \frac{d\sigma^\text{resum}}{dQ^2dydQ_T^2} = \int_0^\infty \frac{db}{2\pi} b J_0 (bQ_T) \sum_{ab} e^{-S_c(\frac{b}{b_0}, Q)} \\
\times (f_{a/p_1} \otimes \bar{C}_{ca})(x_1, \frac{b_0}{b})(f_{b/p_2} \otimes \bar{C}_{cb})(x_2, \frac{b_0}{b}). \quad (50)
\]
The corresponding coefficients $\{\bar{A}, \bar{B}, \bar{C}\}$ are related to $\{A, B, C\}$ through equation
\[
\bar{A}(\alpha_s) = A(\alpha_s), \\
\bar{B}(\alpha_s) = B(\alpha_s) - \beta(\alpha_s) \frac{d\log H_c^F}{d\log \alpha_s}, \quad (51)
\]
\[
\bar{C}_{ab}(z) = C_{ab}(z)[H_a^F(b_0/b)]^{\frac{3}{2}},
\]
Up to next-to-leading-logarithmic-order (NLLO), $\{A^{(2)}, B^{(1)}, C^{(1)}\}$ and $\{A^{(2)}, B^{(1)}, C^{(1)}\}$ is compatible with each other, and further calculation and confirmation are required at higher order.

It can be seen that SCET provides a natural framework of $Q_T$ resummation by conventional RGE in EFT. We have noted that the matched effective operator is determined by collinear and soft gauge invariance and is unique. In addition, the corresponding anomalous dimension is independent of its spin structure and is universal, while the process-dependent
quantity resides in the Wilson coefficient. Even more, the coefficients $C_{ab}$ defined in SCET are process-independent too. So the matching and running procedure in EFT naturally separated the process-dependent and universal contributions to a process, i.e., \{A, B, C\} are universal and only $H^F_{ab}$ is process-dependent. In pQCD, the formula and relation like Eq.(47) and Eq.(51) have been proposed by the authors of [9].

Compared with SCET, pQCD analysis invoke gauge invariance and a new evolution equation [6] which comes from differentiating the jet-function from the factorized cross section with respect to the axial parameter in axial gauge to separate soft and collinear contributions, which is crucial to resum the double logarithms, since RGE in full theory only resums single logarithms between two scales.

We conclude this section with a chain for the $Q_T$ resummation in this section,

$$QCD|_{Q^2} \rightarrow SCET_I|_{Q^2 \lambda^2} \rightarrow SCET_{II}|_{Q^2 \eta^2} \leftarrow DGLAP|_{\mu_0^2},$$

where the last arrow indicates that the PDFs used at the scale $Q^2\eta^2$ can be obtained from those at some fixed scale $\mu_0^2$ by the evolutions of the DGLAP equations.

IV. DISCUSSION

(I) Applying the formula (47) to the production of lepton pair via virtual photon, one can find,

$$\frac{1}{\sigma_{qq}^{(0)}} \frac{d\sigma_{\text{resum}}}{dQ^2dydQ_T} = H_q^{DY}(Q) \int_0^\infty \frac{db}{2\pi} bJ_0(bQ_T) \sum_{ab} e^{-S_q(b, Q)} \times (f_{a/p_1} \otimes C_{qa})(x_1, \frac{b_0}{b}) (f_{b/p_2} \otimes C_{qb})(x_2, \frac{b_0}{b}),$$

(52)

where

$$\sigma_{qq}^{(0)} = e_a^2 \frac{4\pi^2 \alpha_s^2}{9 SQ^2}, \quad H_q^{DY}(Q) = 1 + \frac{\alpha_s C_F}{2\pi} (-8 + \frac{7}{6} \pi^2),$$

$$S_q(b, Q) = \int_{\frac{b_0}{b}}^{Q^2} \frac{d\mu^2}{\mu^2} \left[ A_q(\alpha_s) \log \frac{Q^2}{\mu^2} + B_q(\alpha_s) \right],$$

$$C_{qa}^{(0)} = \delta_{qa} \delta(1 - z), \quad C_{qa}^{(1)} = -2P_{qa}(z) - C_F \frac{\pi^2}{6} \delta_{qa} \delta(1 - z).$$

(II) The reformulation of joint resummation can also be made straightforwardly in SCET. In fact, threshold resummation [5] under $z = Q^2/S \rightarrow 1$ for $d\sigma_{\text{resum}}/dQ^2$ is performed in
moment-$N$ space, and the relevant $\lambda^2 = 1/\bar{N} \sim 1 - z$ with $\bar{N} = e^{\gamma_E} N$. The conclusion of \[5\] can be represented by a chain,

$$QCD|Q^2 \rightarrow SCET_I|Q^2\chi \Rightarrow SCET_{II}|Q^2\chi \leftarrow DGLAP|\mu_0^2.$$  

We observe that the two chains for $Q_T$ and threshold resummation in SCET have identical structure. This suggests that we can do threshold and $Q_T$ resummation for $d\sigma^\text{resum}/dQ^2dQ_T^2$ simultaneously. The relevant $\lambda^2 \sim 1/\chi(\bar{N}, \bar{b})$ with $\bar{b} \equiv bQ/b_0$ is an interpolation of $\lambda^2 \sim 1/\bar{N}$ and $\lambda^2 \sim 1/\bar{b}$, let us say \[17\]

$$\chi(\bar{N}, \bar{b}) = \bar{b} + \frac{\bar{N}}{1 + \rho \bar{b}/\bar{N}}, \quad \rho = \frac{1}{4}, \quad (54)$$

which approaches to $\bar{N}$ for $\bar{b} \ll \bar{N}$ and to $\bar{b}$ for $\bar{b} \gg \bar{N}$, respectively. The matching steps for joint resummation then can be written as

$$QCD|Q^2 \rightarrow SCET_I|Q^2/\chi \Rightarrow SCET_{II}|Q^2/\chi \leftarrow DGLAP|\mu_0^2,$$

which leads to similar result as Eq.\[17\] corresponding to that of \[17\], and the Mellin transformed and jointly resummed cross section follows,

$$\frac{1}{\sigma^{(0)}} \frac{d\sigma^\text{resum}(N)}{dQ^2dQ_T^2} = H_c^F(Q) \int_0^\infty \frac{db}{2\pi} b J_0(bQ_T) \sum_{ab} e^{-S_c(Q/\chi, Q)} \times \left( C_{ca}(N)f_{a/p_1}(N, Q/\chi) \right) \left( C_{cb}(N)f_{b/p_2}(N, Q/\chi) \right), \quad (55)$$

where $\phi(N) \equiv \int_0^1 d\xi \xi^N \phi(\xi)$ for any function $\phi(\xi)$ with $0 \leq \xi \leq 1$ is used.

V. CONCLUSION

We have presented the method of $Q_T$ resummation in the framework of SCET and given a simple correspondence between $\{A, B, C\}$ in SCET and the well known coefficients $\{A, B, C\}$ in pQCD, with which the available information is compatible. The equivalence of the two framework can be confirmed by higher order computation. We have also shown that the reformulation of joint resummation can be performed in SCET directly. So any process, which is confined to the soft and collinear regions by dynamics or kinematics, can be treated in SCET following the steps outlined above.
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Note added. -After an expanded version of our manuscript was completed, a new paper [18] by A. Idilbi, X.D. Ji, and F. Yuan appeared, in which they also discussed the transverse momentum distribution in $b$ space and derived a similar result to ours.

Appendix A

In this appendix, the details of the calculation to extract $C^{(1)}$ in SCET are given explicitly.

Because $C^{(1)}$ is related to the emission of a soft or collinear gluon, and the phase space is already factorized in this case, we will exploit a special form at $O(\alpha_s)$, for which there is no need to cover the non-perturbative region ($Q_T \sim \Lambda_{QCD}$),

$$\begin{align*}
\frac{1}{\sigma^{(0)}} \frac{d\sigma^{\text{resum}}}{dQ^2 dy dQ_T^2} &= H_c^F(Q) \frac{d}{dQ_T^2} \left[ e^{-S_c(Q_T,Q)} \hat{\sigma}^{\text{SCET}}(Q_T,Q,\mu) \right], \\
\hat{\sigma}^{\text{SCET}}(Q_T,Q,\mu) &= \int_0^{Q_T^2} dq_T^2 \frac{1}{\sigma^{(0)}} \frac{d\sigma^{\text{SCET}}(\mu)}{dQ^2 dy dQ_T^2} \\
&= J_{p_1}(x_1,Q_T,\mu) J_{p_2}(x_2,Q_T,\mu) S(Q_T,\mu).
\end{align*}$$

(56)

The same reasoning as in section 3 leads to

$$J_{p_1}(x_1,Q_T,\mu) J_{p_2}(x_2,Q_T,\mu) S(Q_T,\mu) = \sum_{a,b} (f_{a/p_1} \otimes \hat{C}_{ca})(x_1,Q_T,\mu) \times (f_{b/p_2} \otimes \hat{C}_{cb})(x_2,Q_T,\mu).$$

(57)

Then we get the differential form for $Q_T$ resummation,

$$\begin{align*}
\frac{1}{\sigma^{(0)}} \frac{d\sigma^{\text{resum}}}{dQ^2 dy dQ_T^2} &= H_c^F(Q) \frac{d}{dQ_T^2} \sum_{ab} e^{-S_c(Q_T,Q)} \\
&\times (f_{a/p_1} \otimes \hat{C}_{ca})(x_1,Q_T) (f_{b/p_2} \otimes \hat{C}_{cb})(x_2,Q_T).
\end{align*}$$

(59)

Here $\mu = Q_T$ is set to minimize large factor in $\hat{C}^{(1)}$. Taking the $Q_T^2$ integral of Eq.(57) and Eq.(59) from 0 to $Q_T^2$ and then expanding them at $O(\alpha_s)$, we find $C^{(1)}(z, \frac{b}{\nu}) = \hat{C}^{(1)}(z, Q_T)$,
\[ p \rightarrow \frac{j\bar{n} \cdot p}{2 p^2} = i \frac{g \bar{n} \cdot p}{2 p^2} \]

\[ \mu, a \quad k \quad \nu, b \quad = -i g^{\mu \nu} \delta_{ab} \frac{1}{k^2} \]

\[ = i g T^a \frac{\bar{n}_\mu}{2} \left( \bar{n} \cdot \bar{n}_p + \frac{\gamma_\perp \gamma_\mu^\perp}{\bar{n} \cdot p} + \frac{\gamma_\mu \gamma_\perp}{\bar{n} \cdot q} - \frac{\gamma_\perp \gamma_\mu^\perp}{\bar{n} \cdot p \bar{n} \cdot q} \right) \]

\[ \mu, a \quad k \quad \nu, b \quad = -g T^a \frac{\bar{n}_\mu}{\bar{n} \cdot k} \]

\[ \mu, a \quad k \quad \nu, b \quad = -g T^a \frac{\bar{n}_\mu}{\bar{n} \cdot k} \]

FIG. 5: Feynman rules for \( n \)-direction collinear particles in Feynman gauge \( \alpha = 1 \) \[ 1 \]. Here the direction of gluon momentum in the Wilson line is along the collinear particle. The rules for soft fields and collinear gluon are the same as that for QCD, and \( n \leftrightarrow \bar{n} \) for \( \bar{n} \)-direction collinear particles.

Thus we will use Eq. \[ 57 \] to calculate \( C^{(1)} \). However, it should be emphasized that this formula is valid only with \{\( A^{(1)}, B^{(1)}, \bar{C}^{(1)} \}\}, and the two formulas must be equal at this order, which suggests a way to adjust the parameters in \( b^* \) prescription \[ 4 \]. Previously, the authors (DDT) of \[ 19 \] have derived a similar formula, which is corresponding to our result at leading-logarithmic-order (LLO). Later, the extended DDT formula in the CSS frame with \( \{A^{(2)}, B^{(2)}, C^{(1)}\} \) was suggested in \[ 20 \], whose coefficients \( \{\bar{A}, \bar{B}, C\} \) at \( O(\alpha_s) \) are just our \( \{A^{(1)}, B^{(1)}, \bar{C}^{(1)}\} \). The real radiative contribution to the differential cross section at the scale \( Q_T^2 \) is

\[
\frac{d\sigma_{\text{SCET}}^{(\mu)}(\mu)}{dQ^2 dy dq_T^2} = \frac{1}{\Gamma(1-\epsilon)} \left( \frac{4\pi}{\alpha_s} \right)^\epsilon \int \frac{du}{u} \delta \left( \frac{1}{u} (u - u_{\min})(u - u_{\max}) \right) \times \frac{|M_{ab}^{(0)}(p_1, p_2, k, y)|^2}{8s(2\pi)^2}. \tag{60}
\]

Here \( M_{ab}^{(0)}(p_1, p_2, k, y) \) denotes the corresponding matrix element, and the usual invariants
and the two roots of the equation \((p_1 + p_2 - q)^2 = 0\) are

\[
\begin{align*}
  u_{\text{min}} &= Q^2z - 1 - \sqrt{(1 - z)^2 - 4|q|^2/Q^2} \\
  u_{\text{max}} &= Q^2z - 1 + \sqrt{(1 - z)^2 - 4|q|^2/Q^2}
\end{align*}
\]

The expression \(|M_{ab}^{(0)}(p_1, p_2, k, y)|^2\) in SCET can be written down by cut diagrams, and the Feynman rules we need are shown in Fig.5. For example, in Fig.6 the real contribution of the collinear gluon in the \(n\) direction is

\[
|M_{qq}^{(0)}(p_1, p_2, k, y)|^2 = -4\pi\alpha_s \mu^2 \frac{1}{36} C_F \left\{ \text{Tr} \left[ s \gamma_\mu \frac{\hat{h} \cdot \bar{n} \cdot (p_1 - k) \frac{\hat{p}}{2}}{(p_1 - k)^2} \right. \right.
\]

\[
\times \left( \frac{\hat{n} \cdot (p_1 - k)}{\bar{n} \cdot (p_1 - k)} \frac{\hat{f}_\mu}{2} + \frac{\hat{f}_\mu}{n \cdot p_1} \right) + \frac{\hat{f}_\mu}{\bar{n} \cdot p_1} \frac{\hat{p}_1}{2} \left. - \frac{\hat{f}_\mu}{\bar{n} \cdot (p_1 - k) \bar{n} \cdot \bar{n}} \right) \frac{\hat{f}_\mu}{2} \right)
\]

\[
\times \bar{n} \cdot (p_1 - k) \frac{1}{(p_1 - k)^2} \gamma_\mu \frac{\hat{f}_\mu}{2} \right) + 2\text{Tr} \left[ s \gamma_\mu \frac{\hat{h} \cdot \bar{n} \cdot \bar{n} \cdot \hat{p}}{\bar{n} \cdot (p_1 - k)} \frac{\hat{f}_\mu}{2} \right] \frac{\hat{f}_\mu}{2} \right]
\]

\[
= -4\pi\alpha_s \mu^2 \frac{1}{36} \text{Tr} \left[ s \gamma_\mu \frac{\hat{h} \cdot \bar{n} \cdot \bar{n} \cdot \hat{p}}{\bar{n} \cdot (p_1 - k)} \frac{\hat{f}_\mu}{2} \right]2C_F \left\{ (1 - \epsilon) \left[ 2 \frac{(p_1 - k) \cdot \hat{p}_1}{\bar{n} \cdot (p_1 - k) \bar{n} \cdot \bar{n}} + 2 \frac{\bar{n} \cdot (p_1 - k) \bar{n} \cdot (p_1 - k)}{(p_1 - k)^2} \right] \right.
\]

\[
- \frac{(p_1 - k)^2}{(\bar{n} \cdot (p_1 - k))^2} \left. \frac{\hat{f}_\mu}{2} \right] \frac{\hat{f}_\mu}{2} \right]
\]

If we drop the common factor

\[
\mathcal{M} \equiv 4\pi\alpha_s \mu^2 \frac{1}{36} \text{Tr} \left[ Q^2 \gamma_\mu \frac{\hat{h} \cdot \bar{n} \cdot (p_1 - k)}{\bar{n} \cdot (p_1 - k)} \right],
\]

(62)
and use the following parametrization for momenta $p_1, p_2, k$,

$$
\begin{align*}
p_1 &= (0, P^-, 0), & p_2 &= (P^+, 0, 0), \\
k &= \left(\frac{q_T^2}{(1 - z_1)}P^-, (1 - z_1)P^-, -q_T\right), \\
s &= P^+ P^-, & t &= \frac{-1}{1 - z_1} q_T^2, \\
u &= -(1 - z_1) s, & tu &= s q_T^2
\end{align*}
$$

(63)

the above expression can be simplified to

$$
|\mathcal{M}_{qq}^{c^{(1)}(0)}(p_1, p_2, k, y)|^2 \to \frac{2 C_F [(1 - \epsilon)(1 - z_1)^2 + 2 z_1]}{z q_T^2}.
$$

(65)

The contribution of $\bar{n}$ collinear gluon is given by $n \leftrightarrow \bar{n}$ and $z_1 \leftrightarrow z_2$,

$$
|\mathcal{M}_{qq}^{c^{(2)}(0)}(p_1, p_2, k, y)|^2 \to \frac{2 C_F [(1 - \epsilon)(1 - z_2)^2 + 2 z_2]}{z q_T^2},
$$

(66)

and the soft gluon contribution is

$$
|\mathcal{M}_{qq}^{s^{(0)}(0)}(p_1, p_2, k, y)|^2 \to \frac{4 C_F}{z q_T^2}.
$$

(67)

The condition that the emitted gluon is to be collinear is $(1 - z_i)P^- \gg q_T$ or $z_i \to 1$ for $i = 1, 2$, so only half of the phase space is covered, i.e., $u = u_{\text{min}}$ for $u = -(1 - z_i) s$, under which we could safely make the substitution $z_i \to z$. The soft gluon is guaranteed by $(1 - z_i)P^- \sim q_T$ or $z \to 1$. Note that $|\mathcal{M}_{c^{(0)}}|^2 \to |\mathcal{M}_{s^{(0)}}|^2$ when $z_i \to z \to 1$, which results in

$$
|\mathcal{M}_{qq}^{i^{(0)}(0)}(p_1, p_2, k, y)|^2 = |\mathcal{M}_{qq}^{c^{(0)}(0)}(p_1, p_2, k, y)|^2_{1 - z_i \gg \frac{q_T}{P}} \\
+ |\mathcal{M}_{qq}^{s^{(0)}(0)}(p_1, p_2, k, y)|^2_{1 - z_i \sim \frac{q_T}{P}} \\
\to \frac{2 C_F [(1 - \epsilon)(1 - z)^2 + 2 z]}{z q_T^2} \\
\equiv 2(1 - z) P_{qq}(z, \epsilon)
$$

(68)

similarly,

$$
|\mathcal{M}_{ab}^{i^{(0)}(0)}(p_1, p_2, k, y)|^2 \to \frac{2 (1 - z) P_{ab}(z, \epsilon)}{z q_T^2},
$$

\footnote{We have divided soft contribution into two parts.}
where

\[ P_{qq}(z, \epsilon) = C_F \left[ \frac{1 + z^2}{1 - z} - \frac{\epsilon(1 - z)}{1 - \epsilon} \right], \]
\[ P_{gq}(z, \epsilon) = C_F \left[ \frac{1 + (1 - z)^2}{z} - \epsilon z \right], \]
\[ P_{gg}(z, \epsilon) = 2C_A \left[ \frac{1 - z}{z} + \frac{z}{1 - z} + z(1 - z) \right], \]
\[ P_{qq}(z, \epsilon) = T_R \left[ 1 - \frac{2z(1 - z)}{1 - \epsilon} \right]. \] (69)

These are the corresponding results of [8].

Next we match the \( Q_T \) integral Eq.(72) onto PDFs after making Mellin transformation,

\[
\Sigma_{qq}(n) \equiv \frac{\alpha_s}{2\pi} \Sigma_{qq}^{(1)}(n) = \int_{0}^{1-\frac{2q_T}{Q_T}} dz z^n \frac{1}{\sigma^{(0)}} d\sigma^\text{SCET}(\mu) dQ^2 dy dq_T^2, \\
\Sigma_{qq}^{(1)}(n) = \frac{1}{\Gamma(1-\epsilon)} \frac{1}{q_T^2} (\frac{\mu^2 e^{\gamma_E}}{q_T^2})^\epsilon I_n, \] (70)
\[ I_n = \int_{0}^{1-\frac{2q_T}{Q_T}} dz z^n \frac{2(1 - z) P_{qq}(z, \epsilon)}{\sqrt{(1 - z)^2 - 4z \frac{q_T^2}{Q_T^2}}}, \] (71)

where the upper limit \( z = 1 - 2q_T/Q \) is to make the integrand meaningful, and the integral \( \gamma_{qq}(n) \) can be evaluated easily if we retain only the singular terms as \( q_T^2 \to 0, \)
\[ I_n \to \int_{0}^{1-\frac{2q_T}{Q_T}} dz \frac{2(1 - z) P_{qq}(z, \epsilon)}{\sqrt{(1 - z)^2 - 4z \frac{q_T^2}{Q_T^2}}} + \int_{0}^{1} dz (z^n - 1) 2 P_{qq}(z, \epsilon) \]
\[ \to 2C_F \log \frac{Q_T^2}{q_T^2} - 3C_F + 2\gamma_{qq}(n) + 2\epsilon \gamma_{qq}^\epsilon(n), \] (72)

where \( \gamma_{qq}(n) \) and \( \gamma_{qq}^\epsilon(n) \) are the moment of regularized splitting function and of \( P_{qq}^\epsilon(z). \) So

\[
\int_{0}^{Q_T^2} dq_T^2 \Sigma_{qq}^{(1)}(n) = \frac{1}{\Gamma(1-\epsilon)} \int_{0}^{Q_T^2} \frac{dq_T^2}{q_T^2} (\frac{\mu^2 e^{\gamma_E}}{q_T^2})^\epsilon \left[ 2C_F \log \frac{Q_T^2}{q_T^2} - 3C_F + 2\gamma_{qq}(n) + 2\epsilon \gamma_{qq}^\epsilon(n) \right] \\
= \frac{2C_F}{\epsilon^2} + \frac{1}{\epsilon} (3C_F + 2C_F \log \frac{\mu^2}{Q_T^2} - \frac{\gamma_{qq}(n)}{\epsilon} - 2\gamma_{qq}^\epsilon(n) - \frac{C_F \pi^2}{6}) \] (73)
\[ + C_F \log \frac{\mu^2}{Q_T^2} + 3C_F \log \frac{\mu^2}{Q_T^2} - 2\gamma_{qq}(n) \log \frac{\mu^2}{Q_T^2} - 2C_F \log \frac{Q_T^2}{Q_T^2} \log \frac{\mu^2}{Q_T^2}. \]

The virtual correction comes from the UV renormalization constant,
\[ 2\delta Z_V = \frac{\alpha_s}{2\pi} \left[ \frac{2C_F}{\epsilon^2} + \frac{1}{\epsilon} (3C_F + 2C_F \log \frac{\mu^2}{Q_T^2}) \right], \]
which cancels the first two $\epsilon$-poles. The remaining is cancelled by the renormalization of PDF, i.e.,

\[
f_q(n) = f_q(n) = 1 + \frac{\alpha_s}{4\pi} \left[ 2\gamma_{qq}(n) \left( \frac{1}{\epsilon_{UV}} - \frac{1}{\epsilon_{IR}} \right) \right],
\]

where $\mu = \mu_F$ is implied. Finally, we get

\[
\hat{C}_{qq}^{(1)}(n, Q_T) = -2\gamma_{qq}(n) - \frac{C_F \pi^2}{6},
\]

\[
C_{qq}^{(1)}(z, \frac{b_0}{b}) = -2P_{qq}^e(z) - \frac{C_F \pi^2}{6} \delta(1 - z),
\]

Similarly,

\[
C_{ab}^{(1)}(z, \frac{b_0}{b}) = -2P_{ab}^e(z) - \frac{C_a \pi^2}{6} \delta_{ab} \delta(1 - z).
\]

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