Lorentz invariant “potential magnetic field” and magnetic flux conservation in an ideal relativistic plasma

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Abstract

Lorentz invariant scalar functions of the magnetic field are defined in an ideal relativistic plasma. These invariants are advected by the plasma fluid motion and play the role of the potential magnetic field introduced by R. Hide in Ann. Geophys., 1, 59 (1983) on the line of Ertel’s theorem. From these invariants we recover the Cauchy conditions for the magnetic field components in the Eulerian-Lagrangian variable mapping. In addition the adopted procedure allows us to formulate Alfvén theorem for the conservation of the magnetic flux through a surface comoving with the plasma in a Lorentz invariant form.

1 Introduction

The nonlinear dynamics of relativistic plasmas is presently under extensive theoretical and experimental investigation, both in the context of laboratory plasmas such as laser-produced plasmas and in the context of high-energy astrophysics. While the description of the relativistic plasma dynamics would require a fully kinetic treatment involving the relativistic Vlasov equation coupled to Maxwell’s equation, on the large spatial and temporal scales that are of interest for astrophysical plasmas fluid type approximations can be usefully adopted. This is in particular the case in physically complex settings such as high energy plasmas in curved space time, see e.g. in the recent article [1] for the case of magnetized neutron stars and pulsar winds. Relativistic fluid descriptions have also been used in a simplified modelling of magnetic reconnection in high energy plasmas, see e.g., [2].

A common feature of these descriptions, in the limit where dissipation and microscopic effects are disregarded, is the occurrence of topological invariants, such as the conservation of the magnetic flux through a surface comoving with the plasma (Alfvén theorem) in ideal Magnetohydrodynamics, that restrict the plasma dynamics. In fact the process of magnetic reconnection mentioned above arises from the local violation of these constraints due to the local violation of the ideal Ohm’s law, that is of the condition \( E + (v/c) \times B = 0 \), where \( E \) and \( B \) are the electric and magnetic field and \( v \) is the fluid plasma velocity.

In general, even for relativistic plasmas, these topological constraints are formulated in a form that is not explicitly invariant under Lorentz transformations even if the ideal plasma condition

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\[ \mathbf{E} + (\mathbf{v}/c) \times \mathbf{B} = 0 \] is fully relativistic and covariant, as can be seen explicitly by rewriting it in the 4 dimensional form \( F_{\mu\nu} \mathbf{u}^\nu = 0 \) where \( F_{\mu\nu} \) is the electromagnetic field tensor and \( \mathbf{u}^\nu \) is the plasma fluid 4-velocity.

In order to bypass this limitation in two previous papers [3, 4] the concept of covariant magnetic connections between fluid elements has been introduced. A basic ingredient in the definition of invariant connections is a procedure of time resetting along the fluid trajectories that is compatible with the ideal Ohm’s law and that is required in order to restore the simultaneity between two fluid elements at different spatial locations that is not preserved by a Lorentz transformation.

Following a different angle of approach, but under the same ideal plasma assumption, R. Hide defined [5, 6], by analogy to the potential vorticity introduced by Ertel [7] for fluids, a potential magnetic field that is advected by the plasma velocity field.

In the present article we provide a relativistic definition of the potential magnetic field that is explicitly Lorentz invariant. This generalization turns out to be rather convenient as it allows us on the one hand to recover the well known Cauchy conditions for the magnetic field components in the Eulerian-Lagrangian variable mapping (see [8]) and on the other to prove a Lorentz invariant form of the Alfvén theorem that, as in the case of the covariant connections in [3, 4], requires a time resetting procedure in order to restore simultaneity among the different points of the comoving surface through which the magnetic flux is computed.

This article is organized as follows. In Sec.2 following closely [4], we recall for the sake of self-containedness the basic features of the Lichnerowicz-Anile (LA) representation [11, 12] of the electromagnetic field tensor \( F_{\mu\nu} \), the definition of the magnetic 4-vector \( \mathbf{b}^\mu \) and of the electric 4-vector \( \mathbf{e}^\mu \), the ideal plasma limit where the electric 4-vector \( \mathbf{e}^\mu \) vanishes and the expression of the divergentless dual tensor \( G_{\mu\nu} \) in this ideal limit in terms of the 4-vectors \( \mathbf{b}^\mu \) and \( \mathbf{e}^\mu \). Then we recall the gauge freedom (see also [13]) in the definition of the magnetic 4-vector. This gauge freedom plays a very important role in the derivation of the potential magnetic field and of the Alfvén theorem as it allows us to move easily from a Lorentz invariant 4-dimensional formulation to a 3+1-dimensional formulation in a chosen frame and back, whichever formulation is more convenient through the different steps of the proofs. In Sec.3 we derive the explicit relativistic invariant expression of the potential magnetic field. In Sec.2 using the gauge freedom mentioned above, we show that the relativistic expression of the potential magnetic field reduces to Hide’s definition when expressed in a 3+1-dimensional form. In Sec.5 by introducing appropriately conserved “charges” that are closely related to the potential magnetic field defined in Sec.3 and by using the gauge freedom explicitly, we recover in a Lorentz invariant form the Alfvén theorem for the conservation of the magnetic flux through a surface comoving with the plasma. Finally in Sec.6 the main results of this article are summarised and possible extensions are indicated.

## 2 Electric and magnetic 4-vectors

Following [4, 10] we adopt the so called Lichnerowicz-Anile (LA) representation [11, 12] (see also [13]) of the relativistic e.m. field tensor \( F_{\mu\nu} \)

\[ F_{\mu\nu} = \varepsilon_{\mu\nu\lambda\sigma} \mathbf{b}^\lambda \mathbf{u}^\sigma + [\mathbf{u}_\mu \mathbf{e}_\nu - \mathbf{u}_\nu \mathbf{e}_\mu], \]  

(1)

where \( \mathbf{b}^\mu \) is the 4-vector magnetic field and \( \mathbf{e}^\mu \) is the 4-vector electric field, with \( \mathbf{u}^\mu \mathbf{e}_\mu = 0 \) and \( \mathbf{u}_\mu \mathbf{b}^\mu = 0 \). The 4-vectors \( \mathbf{e}^\mu \) and \( \mathbf{b}^\mu \) are related to the standard electric and magnetic fields \( \mathbf{E} \) and \( \mathbf{B} \) respectively.

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1This name is a generalization to a magnetic field of a term originally used for the vorticity in an incompressible fluid [9].
B in 3-D space by
\[ b^\mu = \gamma (B + E \times v , B \cdot v ) , \] (2)
and
\[ e_\mu = \gamma (E + v \times B , -E \cdot v ) , \] (3)
with \( e_\mu b^\mu = E \cdot B \). We have adopted the Minkowski metric tensor \( \eta_{\mu\nu} \) defined by \((+, +, +, -)\) and normalized 3-D velocities \( v \) to the speed of light: \( \gamma \) is the relativistic Lorentz factor and we have used \( u^\mu = \gamma (v, 1) \) and \( u_\mu u^\mu = -1 \). The orthogonality conditions \( u^\mu e_\mu = u_\mu b^\mu = 0 \) make the LA representation unique.

The LA representation allows us to separate covariantly the magnetic and the electric parts of the e.m. field tensor relative to a given plasma element moving with 4-velocity \( u^\mu \). In the local rest frame of this plasma element the time components of \( e^\mu \) and of \( b^\mu \) vanish, while their space components reduce to the standard 3-D electric and magnetic fields. A corresponding representation holds for the dual tensor \( G^\mu\nu \equiv \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} / 2 \) with \( e_\mu \) and \( b^\mu \) interchanged. Thus:
\[ G^\mu\nu = \varepsilon^{\mu\nu\lambda\sigma} u_\lambda e_\sigma + [u^\mu b^\nu - u^\nu b^\mu] , \quad \text{with} \quad e_\mu = F_{\mu\nu} u^\nu \quad \text{and} \quad b^\mu = G^{\mu\nu} u_\nu . \] (4)

If the ideal Ohm’s law \( F_{\mu\nu} u^\nu = 0 \) holds, the electric 4-vector \( e_\mu \) vanishes, the tensors \( F_{\mu\nu} \) and \( G^{\mu\nu} \) have rank two and can be written as
\[ F_{\mu\nu} = \varepsilon_{\mu\nu\lambda\sigma} b^\lambda u^\sigma , \quad G^{\mu\nu} = [u^\mu b^\nu - u^\nu b^\mu] , \] (5)
with
\[ F_{\mu\nu} b^\nu = F_{\mu\nu} u^\nu = 0 , \] (6)
\[ F_{\mu\nu} G^{\nu\mu} = 0 \rightarrow E \cdot B = 0 , \quad \text{and} \quad b_\mu b^\mu = G_{\mu\nu} G^{\nu\mu} / 2 = F_{\mu\nu} F^{\nu\mu} / 2 . \] (7)

In this case we can use \( e_\mu = 0 \) in order to express \( b^\mu \) in terms of \( B \) and \( v \) only as
\[ b^\mu = \gamma (B / \gamma^2 + v (v \cdot B) , v \cdot B) . \] (8)

Note that in general \( \partial_\mu b^\mu \neq 0 \) while from Maxwell’s equations we have
\[ \partial_\mu G^{\mu\nu} = 0 . \] (9)

### 2.1 Gauge freedom

As shown in detail in \([4, 13]\) a gauge freedom is allowed in the definition of the magnetic 4-vector field \( b^\mu \) in the LA representation provided we relax the orthogonality condition \( b^\mu u_\mu = 0 \):
\[ b^\mu \rightarrow h^\mu \equiv b^\mu + g u^\mu , \] (10)
where \( g \) is a free scalar field and the velocity 4-vector \( u^\mu \) satisfies the continuity equation
\[ \partial_\mu (N u^\mu) = 0 , \] (11)
with \( N \) is the proper density of the plasma element and \( N u^\mu \) of the density 4-vector. Different choices of the gauge field \( g \) allow us to impose specific conditions on \( h^\mu \). If we take in a given frame\(^2\) the magnetic gauge
\[ g = -v \cdot B , \] (12)
\(^2\)Actually this gauge is Lorentz invariant since the quantity \( -v \cdot B \) can be written as a Lorentz scalar. Its expression in a frame moving with respect to the chosen frame with velocity 4-vector \( V_\mu \) is given by \( -(V_\mu b^\mu) / (V_\mu u^\nu) \).
we can make the time component of \( h^\mu \) vanish and \( h \| B \) in that frame i.e.,

\[
h^\mu = (B / \gamma, 0).
\]  

(13)

Note that the expression for \( G^{\mu \nu} \) in Eq.(5) is unchanged if we insert \( h^\mu \) for \( b^\mu \) in Eq.(5).

3 Advected relativistic “potential magnetic field”

Let \( S \) be a scalar function in Minkowski space-time, then

\[
(\partial_{\nu} S) \partial_{\mu} G^{\mu \nu} = \partial_{\mu}[(\partial_{\nu} S) G^{\mu \nu}] - (\partial_{\mu} \partial_{\nu} S) G^{\mu \nu} = \partial_{\mu}[(\partial_{\nu} S) G^{\mu \nu}] = 0.
\]  

(14)

This is a generalization of the solenoidal property of the magnetic field (for \( S = t \), Eq.(14) reduces to \( \nabla \cdot B = 0 \)) and includes the induction equation whose components are recovered for \( S = x, y, z \), respectively.

In the ideal MHD case from Eq.(5) we obtain

\[
\partial_{\mu}[(\partial_{\nu} S) G^{\mu \nu}] = -\partial_{\mu}[b^\mu \partial_{\nu} S - u^\mu b^\nu \partial_{\nu} S] = 0.
\]  

(15)

We find it convenient to choose \( S \) such that it is advected by the 4-velocity field \( u^\mu \) (e.g. any function of the initial spatial positions \( a \) at \( t = 0 \)). Then

\[
\partial_{\tau} S = u^\mu \partial_{\mu} S = 0,
\]  

(16)

and

\[
\partial_{\mu}[u^\mu b^\nu \partial_{\nu} S] = 0.
\]  

(17)

Using Eq.(11), from Eq.(17) we obtain

\[
\partial_{\mu}[(N/N)u^\mu b^\nu \partial_{\nu} S] = N \partial_{\tau}[(b^\nu \partial_{\nu} S)/N] = 0.
\]  

(18)

Equation (18) is gauge invariant under \( b^\mu \rightarrow b^\mu + g u^\mu \) for any scalar function \( g \) since \( \partial_{\tau} S = 0 \).

The scalar quantity \( (b^\nu \partial_{\nu} S)/N \) is a relativistic Lorentz invariant generalization of the potential magnetic field introduced by R. Hide in [5, 6] along the lines of the potential vorticity and of Ertel theorem [7] in fluid dynamics.

4 3+1 formulation in a chosen reference frame

From \( \partial_{\tau}[(b^\nu \partial_{\nu} S)/N] = 0 \) and \( \partial_{\tau} S = 0 \), we obtain

\[
(b^\nu \partial_{\nu} S)/N = K(S, a),
\]  

(19)

where \( a \) gives in the chosen reference frame the spatial initial conditions (at \( t = t_o \)) of the trajectories of the plasma fluid element, i.e., the 3D Lagrangian coordinates, and \( K(S, a) \) is a function of the initial conditions and of the advected scalar \( S \) that we are free to chose as convenient.

Using the gauge freedom (10) in the form given by Eq.(13) we can rewrite Eq.(19) as

\[
\frac{B^i}{\gamma N} \frac{\partial a^j}{\partial x^i} \frac{\partial S}{\partial a^j} = K(S, a).
\]  

(20)
Choosing $S = a^k$ with $k = 1, 2, 3$, respectively, and recalling that $\gamma N$ is the density $n(x, t) = n_o(a)/J(a, t)$ in 3-D space where $J(a, t) = \det|\partial x/\partial a|$ is the 3-D Jacobian determinant, with self-explaining notation, for each choice $S = a^k$ we obtain

$$\frac{B^i}{\gamma N} \frac{\partial a^k}{\partial x^i} = K^k(a),$$

(21)

i.e.

$$B^i(x, t) = \frac{\partial x^i}{\partial a^k} \frac{n_o(a) K^k(a)}{J(a, t)} = \frac{\partial x^i}{\partial a^k} B^k_o(a),$$

(22)

which corresponds to the “Cauchy condition” given by Eq.(2.22) of [8] (see also Eq.(19) of [14]) that can be derived from the conservation of the flux of the 3D magnetic field $B$ through a surface comoving with an ideal plasma.

Note that, knowing $u^\mu(x, t)$ in terms of $u^\mu(a, t_o)$ and imposing $b^\mu u_\mu = 0$ for all $t$, it is possible to reverse the gauge transformation both on the r.h.s. (using Eq.(1) with $e_\mu = 0$) and on the l.h.s. of Eq.(22) (using Eq.(8)) and obtain the Cauchy condition for the 4-vector $b^\mu$ in agreement with Eq.(19).

5 Invariant Alfvèn flux theorem

5.1 Preliminary proposition

We will make use of the following result (see e.g. [15]): if a 4-vector field $Q^\mu$ satisfies the continuity equation $\partial_\mu Q^\mu = 0$ and if its component vanish outside a finite spatial region, then the “charge” $Q$ defined as the space integral of its time component $Q^0$

$$Q = \int d^3x \, Q^0,$$

(23)

is constant in time and is a Lorentz invariant. We will make use of this result in combination with Eq.(17) for different choices of the advected scalar $S$ in order to rephrase in a Lorentz invariant framework the magnetic flux conservation (Alfvèn theorem) that is usually proven in a fixed frame.

5.2 Flux conservation through a comoving 2D surface

First we take

$$Q^\mu_A = u^\mu b^\nu \partial_\nu S_A,$$

(24)

where the function $S_A$ is defined in a given frame at $t = 0$ in terms of the characteristic function of a finite size, disk-like domain bounded by a 2D spatial surface $S(a)$ of the initial conditions (the base of the disk), by the corresponding surface displaced by the infinitesimal shift $\Delta x^\mu = u^\mu \Delta \tau$ (the top of the disk) and by the “ribbon” connecting the rims of the two surfaces (the side of the disk). The function $S_A$ is advected by the flow 4-velocity $u^\mu$, i.e. for all times $t$ in the chosen frame $S_A = 1$ inside the advected domain and $S_A = 0$ outside.

Since $Q^\mu_A$ is left invariant by the gauge transformation (12) it can be rewritten as

$$Q^\mu_A = (u^\mu/\gamma) B^i \partial_i S_A,$$

so that

$$Q_A = \int d^3x \, B^i \partial_i S_A = 0.$$  

(25)
The last equality in Eq. (25) follows from the fact that $B$ (which is taken at $t = 0$) is divergence-free and that, because of the choice of the function $S_A$, the space integral reduces to the surface integral over the closed surface $\partial \bar{A}$ delimiting the spatial projection $\bar{A}$ of the domain $A$ (only spatial derivatives are present in Eq. (25) because of the gauge used). For the sake of clarity we will now refer to a 3+1 notation and use the fields $v$, $B$ and $E$ so as to make the correspondence with the standard derivation of the Alfvén theorem explicit. There are three contributions to the vanishing flux in Eq. (25): the flux through the base surface, that through the ribbon and that through the top surface. From the space components of $\partial_i S_A$ at the ribbon and using $E = B \times (v/c)$ we see that the flux of $B$ through the ribbon equals $\Delta t = \gamma \Delta \tau$ times the circulation of $E$ along the rim of the domain $\bar{A}$. By virtue of the induction equation this flux cancels the difference between the flux of $B(t)$ and that of $B(t + \Delta t) \sim B(t) + [\partial_t B(t)] \Delta t$ through the top surface. Thus Eq. (25) implies that the flux of $B(t = 0)$ through the base surface (having inverted the direction of the normal to this surface) and that of $B(t = \Delta t)$ through the same surface shifted by $\Delta x = v \Delta t$ (the top surface) are the same. This equality, being valid for all times $t$, recovers the flux conservation theorem in an ideal plasma and makes it Lorentz invariant by virtue of the proposition in Sec. 5.1.

Finally we note when the domain $A$ is Lorentz boosted to a moving frame the points on the boosted base surface (and similarly for the points on the top surface) are no longer simultaneous because simultaneity between points that are spatially separated is not preserved by a Lorentz transformation. This lack of simultaneity in the boosted frame can be corrected, exploiting the flux conservation result derived above for each infinitesimal portion of the base and top surfaces, by a procedure of “time resetting” along the trajectories of the 4-velocity $u^\mu$ that follows the method adopted in [4, 3], see section below Eq. (9) in [3] or Sec. 5.1 in [4].

### 5.3 Invariant vanishing of the flux through an advected closed 2D surface

Here take

$$Q^\mu_D = u^\mu b^\nu \partial_\nu S_D,$$

where the function $S_D$ is defined in terms of the characteristic function of a finite spatial domain $D(a)$ of the initial conditions.

The derivation now follows with only minor changes the one in Sec. (5.2) and we rewrite $Q^\mu_D$ as

$$Q^\mu_D = (u^\mu / \gamma) B^i \partial_i S_D,$$

so that

$$Q_D = \int d^3 x B^i \partial_i S_D = 0.$$

We take the simple case where $\partial D$ is isomorphic to a sphere (i.e., it has no “holes” and thus can be deformed into a sphere without changing its topological structure) and consider a closed curve $\ell$ on $\partial D$ that splits $\partial D$ into two separate parts. Then Eq. (27) recovers the classical result according to which the flux through a 2D surface bounded by a closed curve $\ell$ is independent of the specific surface that is chosen and shows that this result is Lorentz invariant and that, obviously, it is conserved by the domain advection. Clearly, when the domain $D$ is Lorentz boosted to a moving frame, it does not remain purely spatial (i.e., at constant time). This lack of simultaneity in the busted frame can be corrected by the procedure of “time resetting” mentioned above in Sec 5.2.

### 6 Conclusions

A first aim of this article was the completion of the Lorentz invariant formulation of the topological invariants initiated in [3, 4], where the concepts of covariant magnetic connection and of 2D con-
nection hypersurfaces were introduced, by proving that the time resetting procedure can be applied in order to obtain a Lorentz invariant formulation of the Alfvén theorem for the conservation of the magnetic flux through a surface comoving with an ideal plasma.

The procedure adopted in this derivation moves from the definition of a Lorentz invariant “potential magnetic field” similar to the potential vorticity in the non-relativistic Ertel theorem. As an additional bonus this definition makes it possible to set in Lorentz invariant form the so called Cauchy conditions for the magnetic field components in the mapping between Eulerian and Lagrangian variables.

We remark that a Lorentz invariant definition of the magnetic flux conservation in an ideal plasma is important both from a theoretical and in particular from an experimental point of view. This is the case for example when observing a relativistically expanding plasma, because the distinction between electric and magnetic fields is frame dependent and because simultaneity between events at different spatial locations is not maintained in reference frames moving with different velocities with respect to the observer.

Finally we notice that the results obtained in this article can be extended to non ideal plasmas that obey a generalised ideal Ohm’s law, including e.g., electron inertia, as discussed explicitly in [10]. A possible extension to more general fluid theories (see e.g., [16] in the non relativistic limit) and in particular to those that involve an antisymmetric tensor that unifies the electromagnetic and the fluid fields, [17, 18] should also be investigated. On the contrary a possible extension to an ideal plasma in curved space-time must include the fact that in General Relativity covariant derivatives do not commute (which would result in a modification of Eq. [14]).

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