On the Existence of Weak Solutions for the 2D Incompressible Euler Equations with In–Out Flow and Source and Sink Points

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Communicated by A. L. Mazzucato

Abstract. Well-posedness for the two dimensional Euler system with given initial vorticity is known since the works of Judovič. In this paper we show existence of solutions in the case where we allow the fluid to enter in and exit from the boundaries and from some points of the fluid domain. In particular we derive the equations of the model as the limit when we replace the points by some small holes. To do that we extend the existence results for the two dimensional Euler system with in–out flow to time-dependent domains and we derive the system that models a fluid which is allowed to enter in and exit from the boundary and some points. The solutions are characterized by the presence of source, sink and vortex points.

Keywords. Fluid dynamics, Euler equation, In–out flow, Source and sink.

1. Introduction

In this paper we study the existence of solutions for a system that describes the flow of a two dimensional incompressible inviscid fluid which is allowed to enter in and exit from the boundary and from some points called source and sink respectively.

The fluid is described at a mathematical level by the incompressible Euler equations that read

\[
\begin{align*}
\partial_t v + v \cdot \nabla v + \nabla p &= 0 \quad \text{for } x \in \mathcal{F}, \\
\text{div}(v) &= 0 \quad \text{for } x \in \mathcal{F},
\end{align*}
\]

where \( \mathcal{F} \subset \mathbb{R}^2 \) denotes the fluid domains, \( v = (v_1, v_2) : \mathbb{R}^+ \times \mathcal{F} \rightarrow \mathbb{R}^2 \) is the velocity of the fluid and describes the speed of a particle of the fluid at time \( t \in \mathbb{R}^+ \) and position \( x \in \mathcal{F} \). Finally \( p : \mathbb{R}^+ \times \mathcal{F} \rightarrow \mathbb{R} \) is the pressure which is a scalar quantity.

If the fluid domain \( \mathcal{F} \subset \mathbb{R}^2 \) is bounded and with smooth enough boundary, some conditions have to be prescribed on \( \partial \mathcal{F} \) to show a well-posedness result. Classically the no-permeable boundary condition is imposed and it writes

\[
v \cdot n = 0 \quad \text{for } x \in \partial \mathcal{F},
\]

where \( n \) is the normal vector to the boundary that exits from the fluid domain. This condition describes the fact that the fluid is not allowed to enter in or exit from the boundary of the domain.

Let us recall that a classical well-posedness result for system (1) and (2) is [30], where Judovič showed existence and uniqueness of solutions for initial velocity fields \( v^{in} \) such that \( \text{curl}(v^{in}) = \partial_1 v^{in}_2 - \partial_2 v^{in}_1 \in L^\infty(\mathcal{F}) \). The existence result was then extended to the case where \( \text{curl}(v^{in}) \in L^p(\mathcal{F}) \) for \( p \in (1, \infty) \) in [15] but uniqueness is still an open problem. Actually one can expect non-uniqueness, see [1]. In this work the authors show non-uniqueness for the forced Euler system, more precisely for system (1) with on the right hand side an ad hoc forcing term. Finally for \( \text{curl}(v^{in}) \in L^1(\mathcal{F}) \cap H^{-1} \) existence of Delort type solutions is proved in [29]. This concept of solutions was introduced in [13] to consider initial data in the class of non-negative Radon measures.
In this paper we will focus in the case where the fluid is allowed to enter and exit from the boundary of the domain, in particular we will prescribe the normal component of the velocity on the boundaries. This condition reads

$$v \cdot n = g \quad \text{for } x \in \partial \mathcal{F},$$

where $g$ is a given scalar function. Let us notice that due to the divergence free condition of $v$, the integral of $g$ on the boundary has to be zero. The sign of $g$ decomposes $\partial \mathcal{F}$ into three zones

$$\partial \mathcal{F}^+ = \{ x \in \partial \mathcal{F} \text{ such that } g < 0 \}, \quad \partial \mathcal{F}^- = \{ x \in \partial \mathcal{F} \text{ such that } g > 0 \}$$

and

$$\partial \mathcal{F}^0 = \{ x \in \partial \mathcal{F} \text{ such that } g = 0 \}.$$  

Notice that $\partial \mathcal{F}^+$ is the part of the boundary where the fluid is entering in $\mathcal{F}$. On the contrary in $\partial \mathcal{F}^-$ the fluid is exiting. Finally, $\partial \mathcal{F}^0$ is the part of the boundary which is impermeable.

As noticed in [31], system (1)–(3) is under-determined, so we do not expect that a well-posedness result holds. To see this, let us introduce the vorticity $\omega = \text{curl}(v) = \partial_1 v_2 - \partial_2 v_1$. If we apply the curl to the first equation of (1), we deduce

$$\partial_t \omega + v \cdot \nabla \omega = 0 \quad \text{for } x \in \mathcal{F},$$

where we used in essential manner the fact that we restrict to dimension two. Equation (5) tells that the vorticity is transported by the velocity field $v$. If we assume that $v$ is regular enough, the solution to (5) is given by the method of characteristics. Notice that on $\partial \mathcal{F}^+$ the characteristics are entering in the fluid domain and the value of the vorticity cannot be deduced by the initial datum. On the contrary on $\partial \mathcal{F}^-$ the characteristics are exiting and the vorticity is already prescribed. In [31], Judović proposed to prescribe the value of the entering vorticity on $\partial \mathcal{F}^+$. The system then reads

$$\partial_t v + v \cdot \nabla v + \nabla p = 0 \quad \text{for } x \in \mathcal{F},$$
$$\text{div}(v) = 0 \quad \text{for } x \in \mathcal{F},$$
$$v \cdot n = g \quad \text{for } x \in \partial \mathcal{F},$$
$$\text{curl}(v) = \omega^+ \quad \text{for } x \in \partial \mathcal{F}^+.$$  

For the above equations existence and uniqueness of classical solutions were showed in Theorem 1 of [31]. Later in [2], Alekseev extended the existence result to the case of initial and entering vorticity in $L^\infty$ and in the same class, uniqueness was proved recently in [25]. For initial and entering vorticity in $L^p$ for $p \in [1, \infty)$ existence of solutions in an appropriate sense was shown in [9].

To be more specific in the work [31], Judović considers also the case where the fluid domain $\mathcal{F} = \mathcal{F}(t) \subset \mathbb{R}^2$ is time dependent. In particular system (6) holds in

$$\mathcal{F} = \bigcup_{t \in \mathbb{R}^+} t \times \mathcal{F}(t).$$

To show well-posedness, the author assumed enough regularity of $\partial \mathcal{F}$ as a two dimensional surface. As mentioned in [31], in some interesting situations this regularity condition is not satisfied by $\partial \mathcal{F}$. An important example is when the fluid domain $\mathcal{F}(t)$ has some holes that become points at a certain times or conversely from points they become holes of non-empty interior.

The goal of this paper is to study system (6) when we allow some connected component of $\partial \mathcal{F}(t)$ to be points. In this part of the boundary the last two equations of (6) have no sense. So, the first step is to derive a system satisfied by the fluid and then to show existence of solutions. As we will see in the next sections, when we allowed to consider points as connected component of the boundary, some singularities appear in the velocity field so we will work with the concept of weak solutions. The results are stated in rigorous form in Sect. 3.4.

Let us conclude this introduction by recalling that for fluid domains with regular boundary, the choice to impose the entering vorticity is not the only one to obtain a well-posedness result for system (1)–(3). For a survey on other possible conditions we refer to [24]. In dimension three some well-posedness results in Hölder space are available in [3,18,19,27] where different boundary conditions are imposed.
2. Formal Derivation of the Model

As we present in the introduction we will study system (6) when we allowed some connected components of the boundary to be points. In this case the last equation of (6) have not sense and we have to look for an appropriate model that describes the fluid. To do that let us consider a prototype of the fluid domain that contains all the difficulties of the problem but allow us to simplify the notations. We consider a fluid domain of the form

\[ \mathcal{F}(t) = \Omega \setminus \text{int}(S^+(t) \cup S^-(t)). \]  

Here \( \Omega \subset \mathbb{R}^2 \) is open, bounded, connected, simply-connected and with smooth enough boundary and \( S^+(t), S^-(t) \subset \Omega \) are two closed, connected, simply-connected, disjoint subset of \( \Omega \). For \( A \subset \mathbb{R}^2 \), we denote \( \text{int}(A) \) the interior of \( A \) and with \( \overline{A} \) the closure.

Moreover we assume \( \partial \Omega \) to be impermeable, we allow the fluid to enter from \( \partial S^+(t) \) and to exit from \( \partial S^-(t) \). This explains the notation in (7) where we use sign + and − as exponent of \( S \) to recall the fact that the fluid is entering in and exiting from the fluid domain respectively. In the same spirit we often call \( S^+(t) \) the source and \( S^-(t) \) the sink.

2.1. The Euler System with In–Out Flow in a Regular Domain

As we have already mentioned, in the case where \( S^+(t) \) and \( S^-(t) \) have no empty interiors and \( \partial \mathcal{F} = \partial (\bigcup t \times \partial \mathcal{F}(t)) \) is regular enough, the motion of an inviscid incompressible fluid which is entering from \( \partial S^+(t) \) and exiting from \( \partial S^-(t) \) is described by (6). Let us notice that the assumption that the fluid is entering from \( \partial S^+(t) \) and exiting from \( \partial S^-(t) \) is encoded in the hypothesis that

\[ g = 0 \quad \text{on} \quad \partial \Omega, \quad g - q < 0 \quad \text{on} \quad \partial S^+ \quad \text{and} \quad g - q > 0 \quad \text{on} \quad \partial S^- . \]  

Here \( q \) is the normal velocity of the boundary, which in local coordinates writes

\[ q(\gamma(t, \sigma)) = \partial_t \gamma \cdot \left( e_1 \wedge \partial_\sigma \gamma \right) \]

where \( \wedge \) is the vectorial product and \( \gamma : [0, T] \times \partial B_1(0) \longrightarrow \bigcup_{t \in [0, T]} \{ t \} \times \partial S^i(t) \) is a \( C^2 \) diffeomorphism such that \( \gamma(t, \cdot) : \partial B_1(0) \longrightarrow \{ t \} \times \partial S^i(t) \) is also a \( C^2 \) diffeomorphism for any \( t \). Let us remark that in (4) \( q \) does not appear because we were assuming \( \mathcal{F} \) to be time independent which implied \( q = 0 \). In the following, for a fluid domain \( \mathcal{F}(t) \) of the form of (7) and a normal component of the velocity \( g \) defined on \( \partial \mathcal{F} \), we will say that \( g \) is source sink compatible if it satisfies (8).

To study system (6), it is useful to rewrite it in the vorticity form. In this formulation the unknown is \( \omega = \text{curl}(v) \). The velocity field \( v \) is recovered by \( \omega \) and the circulations \( C_i \) around \( S^i(t) \) as the solution of the div–curl system

\[ \begin{align*}
\text{div} \ v &= 0 & \text{for} \quad x \in \mathcal{F}(t), \\
\text{curl} \ v &= \omega & \text{for} \quad x \in \mathcal{F}(t), \\
v \cdot n &= g & \text{for} \quad x \in \partial \mathcal{F}(t), \\
\oint_{\partial S^+(t)} v \cdot \tau &= C_i(t). 
\end{align*} \]  

Here \( \tau \) is the counterclockwise tangent vector to the boundary. Let us recall that in the case \( v \cdot n - q = 0 \), Kelvin’s circulation Theorem applies and \( C_i(t) \) are time independent. In our setting it holds

\[ \frac{d}{dt} \oint_{\partial S^+(t)} v(t, \cdot) \cdot \tau = -\oint_{\partial S^+(t)} (v \cdot n - q) \text{curl} \ v. \]  

\( \Box \)
In fact for smooth enough solutions of (6), we rewrite the first equation of (6) in the form
\[ \partial_t v + \text{curl}(v)v^\perp + \nabla \left( p + \frac{1}{2} |v|^2 \right) = 0. \]

(11)

We deduce from Reynolds transport theorem that
\[ \frac{d}{dt} \oint_{\partial S^i(t)} v(t, \cdot) \cdot \tau = \oint_{\partial S^i(t)} \partial_t v(t, \cdot) \cdot \tau + \oint_{\partial S^i(t)} \text{curl}(v(t, \cdot))q \]
\[ = - \oint_{\partial S^i(t)} \text{curl}(v(t, \cdot))v(t, \cdot) \cdot \nu + \oint_{\partial S^i(t)} \text{curl}(v(t, \cdot))q \]
\[ = - \oint_{\partial S^i(t)} (v \cdot \nu - q) \text{curl} v \]

where we used (11) and the fact that the integral on a closed curve of the tangential derivative of a \( C^1 \) function is zero, to deduce the second equality. Let us notice that if we apply formula (10) for the circulation around \( \partial \Omega \), we deduce
\[ \frac{d}{dt} C_{\partial \Omega}(t) = \frac{d}{dt} \oint_{\partial \Omega} v(t, \cdot) \cdot \tau = 0. \]

By an integration by parts, we notice that
\[ \int_{\mathcal{F}(t)} \omega(t, \cdot) = \int_{\mathcal{F}(t)} \text{curl}(v(t, \cdot)) = \int_{\partial \mathcal{F}(t)} v(t, \cdot) \cdot \tau = C_{\partial \Omega} + C_+(t) + C_-(t). \]

This allows us to write the circulation around \( \partial S^-(t) \) in dependence of \( C_{\partial \Omega}, C_+(t) \) and the integral of \( \omega \) in \( \mathcal{F}(t) \). We have
\[ C_-(t) = \int_{\mathcal{F}(t)} \omega(t, \cdot) - C_{\partial \Omega} - C_+(t) \]
\[ = C_-^{in} + \int_{\mathcal{F}(t)} \omega(t, \cdot) - C_{\partial \Omega} - C_-^{in} - C_+^{in} + \int_0^t \oint_{\partial S^+(t)} \omega^+(g - q) \]
\[ = C_-^{in} + \int_{\mathcal{F}(t)} \omega(t, \cdot) - \int_{\mathcal{F}(t)} \omega^{in} + \int_0^t \oint_{\partial S^+(t)} \omega^+(g - q) \]
(12)

where in second equality we used the integrated version of (10) and the boundary conditions for (6). Moreover \( C_-^{in} \) are the initial circulations.

Finally let us introduce two linear operators \( \mathcal{K}_{\mathcal{F}(t)}[\omega, \mathcal{C}_i] \) and \( \mathcal{J}_{\mathcal{F}(t)}[f, g] \) which are right inverses of respectively the curl and the div operator. They are defined as the unique solutions of
\[ \text{div} \left( \mathcal{K}_{\mathcal{F}(t)}[\omega, \mathcal{C}_i] \right) = 0 \quad \text{for} \ x \in \mathcal{F}(t), \]
\[ \text{curl} \left( \mathcal{K}_{\mathcal{F}(t)}[\omega, \mathcal{C}_i] \right) = \omega \quad \text{for} \ x \in \mathcal{F}(t), \]
\[ \mathcal{K}_{\mathcal{F}(t)}[\omega, \mathcal{C}_i] \cdot \nu = 0 \quad \text{for} \ x \in \partial \mathcal{F}(t), \]
\[ \oint_{\partial S^+(t)} \left( \mathcal{K}_{\mathcal{F}(t)}[\omega, \mathcal{C}_i] \right) \cdot \tau = \mathcal{C}_i, \]

and
\[ \text{div} \left( \mathcal{J}_{\mathcal{F}(t)}[f, g] \right) = f \quad \text{for} \ x \in \mathcal{F}(t), \]
\[ \text{curl} \left( \mathcal{J}_{\mathcal{F}(t)}[0, g] \right) = 0 \quad \text{for} \ x \in \mathcal{F}(t), \]
\[ \left( \mathcal{J}_{\mathcal{F}(t)}[0, g] \right) \cdot \nu = g \quad \text{for} \ x \in \partial \mathcal{F}(t), \]
\[ \oint_{\partial S^+(t)} \left( \mathcal{J}_{\mathcal{F}(t)}[0, g] \right) \cdot \tau = 0. \]
Notice that $K_{F(t)}[\omega, C]$ is called Biot–Savart operator in the literature. We are now able to write (6) in the vorticity form. If we apply the curl operator to the first equation of (6) and we use (9), (10) and (12), the system reads
\[
\begin{align*}
\partial_t \omega + v \cdot \nabla \omega &= 0 \\
\omega &= \omega^+ \\
v &= K_{F(t)}[\omega, C] + F_{\mathcal{F}(t)}[0, g] \\
C_+(t) &= C^{in}_+ - \int_0^t \int_{\partial S^+(t)} \omega^+(g - q), \\
C_-(t) &= C^{in}_- + \int_{\mathcal{F}(t)} \omega - \int_{\mathcal{F}(t)} C^o_+ \omega^+ + \int_0^t \int_{\partial S^+(t)} \omega^+(g - q).
\end{align*}
\] (13)

Notice that in the works [9,25,31] the authors took advantage of the above formulation to prove their results. To show existence and uniqueness of classical solution of (6), Judović proved in [31] existence of solutions for (13) with $\omega \in C^0_{loc}(\mathbb{R}^+; W^{1,r}(\mathcal{F}(t)))$ and $\partial_t \omega \in C^0_{loc}(\mathbb{R}^+; L^r(\mathcal{F}(t)))$ for any $r \in (1, \infty)$ via a Pichard iterative method. More precisely given $\omega_0$ the $n$–th iteration, he defined $v_n+1$ to be the solution of the last three equations of (13). In particular $v_n+1$ is Lipschitz in the space variables. Then $\omega_{n+1}$ was the solution of the first two equation of (13) with $v$ replaced by $v_{n+1}$. Uniqueness of solutions was proved separately.

In [9,25], the authors restrict their attention in case where the fluid domain is time independent, i.e. $\mathcal{F}(t) = \mathcal{F}(0)$. In particular in [9] they showed existence of solutions with $\omega \in C^0_{loc}(\mathbb{R}^+; L^r(\mathcal{F}(t)))$ in an appropriate sense via a vanishing viscosity method. In [25] they proved uniqueness for solutions in $C^0_{loc}(\mathbb{R}^+; L^\infty – w(\mathcal{F}(t)))$ via a non-trivial adaptation of the corresponding uniqueness result for the no-permeable case.

In the literature has already been consider a special case where it is allowed to some part of the boundary to be points, for example in [8]. In the next section we will recall this result.

### 2.2. The Euler System with Source and Sink Points in a Regular Domain

In this section we recall the result presented in Chapter 5 of [8]. In this work the author studied system (6) in the case $S^+(t) = h^+ \neq h^- = S^-(t)$ and the fluid was allowed to enter from $h^+$ and exit from $h^-$. Notice that $\mathcal{F}(t) = \Omega$ by definition (7). In this case the fluid is modeled by the system
\[
\begin{align*}
\partial_t \omega + \text{div}(v \omega) &= j \delta_{h^+} + \left( \frac{d}{dt} \int_{\Omega} \omega - j \right) \delta_{h^-} \\
v &= J_{\Omega} [\mu \delta_{h^+} - \mu \delta_{h^-}] + K_{\Omega} [\omega + C_+ \delta_{h^+} + C_- \delta_{h^-}] \\
C_+(t) &= C^{in}_+ - \int_0^t j, \\
C_-(t) &= C^{in}_- + \int_{\Omega} \omega - \int_{\Omega} C^o_+ \omega + \int_0^t j,
\end{align*}
\] (14)

where $\mu$ and $j$ are quantities associated with respectively the entering flow and the entering vorticity. First of all, let us notice that $\text{div}(v) \neq 0$ in fact the operator $J_{\Omega}$, which is a right inverse of the divergence, is applied to the non-zero measure $\mu \delta_{h^+} - \mu \delta_{h^-}$, in particular $\text{div}(v) \neq v \cdot \nabla \omega$. Secondly the boundary condition $v \cdot n = g$ on $\partial S^+(t)$ is replaced by $\mu \delta_{h^+}$ which appears in (14) in the equation for the velocity and describes the quantity of fluid that is entering ($\mu \geq 0$). The connection between $g$ and $\mu$ is given in (15) and it will be explained later. The singular term $\mu \delta_{h^+}$ inside $J_{\Omega}$ corresponds to a point source in the velocity which means that $J_{\Omega}[\mu \delta_{h^+} - \mu \delta_{h^-}]$ behaves like
\[
\frac{H\mu x - h^+}{2\pi|x - h^+|^2}
\]
closed to $h^+$. Similarly the condition $v \cdot n = g$ on $\partial S^-(t)$ is replaced by $-\mu \delta_{h^-}$ and corresponds to a point sink for $v$ because $-\mu \leq 0$. The fact that we have $\mu$ in front of $\delta_{h^+}$ and its opposite $-\mu$ in front of $\delta_{h^-}$ is a compatibility condition and corresponds to the balance between the quantity of fluid which is entering and the one which is exiting. The condition on the entering vorticity $\omega = \omega^+$ on $\partial S^+(t)$ is replaced by $j \delta_{h^+}$ which appears in (14) as a source term for the equation for the vorticity. The connection between $\omega^+$ and $j$ is given in (15). The term

$$\left( \frac{d}{dt} \int_\Omega \omega - j \right) \delta_{h^-},$$

on the right hand side of (14) is a compatibility condition and tell us that the exiting vorticity is an unknown of the problem and cannot be fixed a priori. Finally the circulations $C_i$ are not extra constrains to recover uniquely the velocity field but rather they are singular terms for the vorticity. This terms corresponds to point vortices for $v$, in fact $K_\Omega [C_+ \delta_+ + C_- \delta_-]$ behaves like

$$C_+ \frac{x - h^+}{2\pi |x - h^+|^2} \quad \text{and} \quad C_- \frac{x - h^-}{2\pi |x - h^-|^2}$$
closed to $h^+$ and $h^-$ respectively.

System (14) was not deduced by physical observations but rather mathematically from (6). More precisely, in [8], the author considers approximate fluid domains of the type $F_\varepsilon = \Omega \setminus B_\varepsilon(h^+) \cup B_\varepsilon(h^-)$ and boundary conditions of the type $v_\varepsilon \cdot n = g_\varepsilon$ on $\partial F_\varepsilon$ and $\omega_\varepsilon = \omega^+_\varepsilon$ on $\partial B_\varepsilon(h^+)$. Then he study the limit as $\varepsilon$ goes to zero for solutions of system (6) associated with $F_\varepsilon$, $g_\varepsilon$ and $\omega_\varepsilon$ under some mild assumptions on $g_\varepsilon$ and $\omega_\varepsilon$. He deduced that solutions $(\omega_\varepsilon, v_\varepsilon)$ of (6) converges to a solution $(\omega, v)$ of system (14) where $j$ and $\mu$ are defined as

$$\int_{\partial B_\varepsilon(h^+)} g_\varepsilon \rightarrow \mu \quad \text{and} \quad \int_{\partial B_\varepsilon(h^+)} g_\varepsilon \omega^+_\varepsilon \rightarrow j$$

where the convergence holds in some appropriate sense.

We are now ready to present the model when we allow $S^i(t)$ to have either non-empty interior or to be points. The system that we deduce is a mixture of the Eqs. (13) and (14) to describe the behaviour of the vorticity in dependence of the geometric properties of $S^i(t)$. More precisely close to an hole with non-empty interior the system is described by (13), in the case it is a point by (16) which is characterized by the presence of a point source/sink and a point vortex.

### 2.3. The Euler System with In–Out Flow and Source and Sink Points

We move to the study of a system that describes an incompressible inviscid fluid which is allowed to enter from $\partial S^+(t)$ and exit from $\partial S^-(t)$ where we allow $S^i(t)$ to have either non-empty interior or to be points. The equations that describe this model have not been considered in the literature yet. The goal of this paper is then to derive mathematically this system and to show existence of solutions in an appropriate sense. The idea is to consider approximate fluid domains for which the fluid is described by (6) and to pass to the limit. As in Sect. 2.2, let us start by writing down the equations.

Let us introduce

$$T^+ = \{ t \in \mathbb{R}^+ \text{ such that } S^+(t) = h^+(t) \text{ a point in } \Omega \},$$

$$T^- = \{ t \in \mathbb{R}^+ \text{ such that } S^-(t) = h^-(t) \text{ a point in } \Omega \},$$

and their complements $T^+_N = \mathbb{R}^+ \setminus T^+$ and $T^-_N = \mathbb{R}^+ \setminus T^-$. The equations in the vorticity form read as

$$\partial_t \omega + \text{div}(v \omega) = j \mathbb{1}_{T^+} \delta_{h^+(t)}$$

$$+ \left( \frac{d}{dt} \int \omega - \int_{\partial S^+(t)} \omega^+(g - q) \mathbb{1}_{T^+_N} - j \mathbb{1}_{T^+} \right) \mathbb{1}_{T^-} \delta_{h^-(t)} \quad \text{for } x \in F(t),$$

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of that the right hand side of the first equation of (16) is zero and that

Let us start by presenting the geometry that we allow for the domain Ω and the holes

3. Definition of Solutions and Main Result

We will now define the concept of solutions for (16) and the main theorems of the paper. Before doing

when

where for $A \subset \mathbb{R}^+$, we define $1_A$ as the characteristic function of $A$, in other words $1_A = 1$ for $x \in A$ and 0 elsewhere.

Let us start to notice that for $t \in T^+_N \cap T^-_N$, system (16) is equal to (13). One can easily verify

that the right hand side of the first equation of (16) is zero and that $v = \mathcal{J}_{\mathcal{T}(i)}[0, g] + \mathcal{K}_{\mathcal{F}(i)}[\omega, C_i]$. For

t $\in T^+ \cap T^-$ then (16) is equivalent to (14). For this reason we can say that the system that is a mixture of

the Eqs. (13) and (14) to describe the behaviour of the vorticity in dependence of the geometric properties

of $\mathcal{S}^i(t)$.

By looking to system (16) we can answer the question posed by Judović in [31] by noticing that the

vector field $v$ becomes singular when the holes $\mathcal{S}^i(t)$ become points. In particular we cannot expect a well-

posedness result for classical solutions to hold. Moreover we notice that the right quantity to prescribe are

$j$ and $\mu$ when $\mathcal{S}^+ (t)$ is a point. Nevertheless we are able to find a class of functions for which existence of

solutions for (16) holds. Finally we can notice that in presence of source and sink points it is natural the

appearance of points vortexes associated with the entering vorticity. This extend the model considered in

[10].

We will now define the concept of solutions for (16) and the main theorems of the paper. Before doing

that we will give a precise definition of what we mean by regular enough boundary for $\mathcal{F}(t)$.

3. Definition of Solutions and Main Result

Let us start by presenting the geometry that we allow for the domain $\Omega$ and the holes $\mathcal{S}^i$. The properties

that we require are basically two. The first one is that the set $\Omega, \mathcal{S}^+$ and $\mathcal{S}^-$ have regular enough

boundaries in space and time variables to prove elliptic estimates uniformly in time and to have the

velocity of the boundary regular enough. The second one is that $\Omega, \mathcal{S}^+$ and $\mathcal{S}^-$ are compatible in the

sense that the mutual distance is greater than a positive constant in any compact time interval $[0, T]$.

To give an idea, we are considering $\Omega, \mathcal{S}^+$ and $\mathcal{S}^-$ such that $\bar{\mathcal{S}^+}, \bar{\mathcal{S}^-} \subset \text{int}(\Omega), \bar{\mathcal{S}^+} \cap \bar{\mathcal{S}^-} = \emptyset$ and the

boundaries are smooth except the transition time where an hole of non-empty interior becomes a point

or the other way around.

In the next subsection we formalize this idea by introducing the definition of regular compatible

gometry. In a first reading this subsection can be skipped.

3.1. Assumptions on the Fluid Domain

The holes $\mathcal{S}^i \subset \Omega$ can have non-empty interior or can be points. In the transition times where an hole

passes from a subset of $\mathbb{R}^2$ with non-empty interior to a point it is not clear how to study the regularity

of the boundary. To avoid this issue we restrict only to holes $\mathcal{S}^i$ that are given as follow. There exist a

shape map $\mathcal{S}^i : \mathbb{R}^+ \times \partial B_1(0) \to \mathbb{R}^2$, a radius map $r_i : \mathbb{R}^+ \to \mathbb{R}^+$ and a position map $h^i : \mathbb{R}^+ \to \Omega$ such that

the boundary of the i-th hole is

$$\partial \mathcal{S}^i(t) = \{ r^i(t) \mathcal{S}^i(t, x) + h^i(t) \text{ such that } x \in \partial B_1(0) \}.$$ (17)
The set $S^i(t)$ is the closure of the bounded component of $\mathbb{R}^2$ separated by $\partial S^i(t,.)$ if $r^i(t) > 0$. It is the point $h^i(t)$ if $r^i(t) = 0$. With this notation $T^i = \{t \in \mathbb{R}^+ \text{ such that } r^i(t) = 0\}$ and the normal velocity of the boundary $q(t,x) = \partial_n (r^i(t)S^i(t,x) + h^i(t)) \cdot n$ for $(t,x) \in T^i \times \partial S^i(t)$.

**Definition 1** (Regular septuple). Let $\Omega \subset \mathbb{R}^2$ a connected simply-connected bounded domain. For $i \in +, -$ let $S^i : \mathbb{R}^+ \times \partial B_1(0) \rightarrow \mathbb{R}^2$ the shape maps, $r_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ the radius map and $h^i : \mathbb{R}^+ \rightarrow \Omega$ the position map. Then we say that the septuple $(\Omega, S^+, S^-, r^+, r^-, h^+, h^-)$ is regular if for some $\alpha > 0$

- $\Omega$ has $C^{2,\alpha}$ boundary,
- $S^i$ is a $C^{1,\alpha}_{loc}(\mathbb{R}^+; C^{1,\alpha}(\partial B_1(0); \mathbb{R}^2)) \cap C^{0,\alpha}_{loc}(\mathbb{R}^3; C^{0,\alpha}(\partial B_1(0); \mathbb{R}^2))$ embedding,
- $r_1 \in C^{1,\alpha}_{loc}(\mathbb{R}^+; \mathbb{R}^+)$ and $h_{loc}^1 \in C^{1,\alpha}(\mathbb{R}^+; \mathbb{R}^2),$
- the origin is contained in the bounded component of $\mathbb{R}^2$ separated by the image of $S^i(t,.)$.

We define now the concept of compatibility.

**Definition 2** (Compatible geometry). We say that a triple of collection of subsets of $\mathbb{R}^2$ indexed by $t \in \mathbb{R}^+$ $(\Omega, S^+, S^-)$ or a septuple $(\Omega, S^+, r^+, h^+, S^-, r^-, h^-)$ is a compatible geometry if for any $t$

- $\overline{S^i(t)} \subset \text{int}(\Omega)$ for $i \in \{+, -\}$
- $S^+(t) \cap S^-(t) = \emptyset$,

where we recall that given a septuple $(\Omega, S^+, r^+, h^+, S^-, r^-, h^-)$, $S^+$ and $S^-$ are the bounded domain with boundary defined in (17).

**Definition 3** (Regular compatible geometry). Let $(\Omega, S^+, S^-)$ a triple of collection of subsets of $\mathbb{R}^2$ indexed by $t \in \mathbb{R}^+$. Then we say that it is a regular compatible geometry if there exists a regular and compatible septuple such that $S^i$ is defined via (17).

**Remark 1.** In the following we always consider regular compatible geometry. In particular the mutual distances between the source, the sink and the boundary of $\Omega$ is always lower bounded by a positive constant in any compact time interval $[0,T]$. 

### 3.2. Existence Results in the Case of Regular Boundary

In the section we restrict our attention to regular compatible geometry $(\Omega, S^+, S^-)$ such that $r^+, r^- \geq c > 0$, in particular $S^+$ and $S^-$ have no empty interior and the fluid is modeled by (13) which is the vorticity formulation of (6).

In this setting we will show existence of solutions with initial and entering vorticity in $L^p$ for $p \in [1, \infty)$. There results are an extension of [9] to time dependent domains of [20] to permeable boundary conditions and they answer a question left open in [17].

Let us recall that distributional solutions for (13) with data in $L^p$ are defined only for $p \geq 4/3$. In fact the first equation of (13) is satisfied in the sense

$$\int_{\mathbb{R}} \omega \partial_t \varphi + \omega v \cdot \nabla \varphi = 0,$$

for any $\varphi \in C^\infty_c(\mathbb{R})$. For $p < 4/3$ the term $\omega t$ is not in general $L^1$ so it is not clear how to give sense to the above equality. For this reason we introduce the definition of renormalized solutions for (13) which is a natural extension of the concept of renormalized solution for the transport equation introduced by DiPerna and Lions [14]. Let us recall that in [14] the authors studied the transport equation associated with a velocity field which is tangent to the boundary, i.e. it satisfies (2). This result was extended in [6] where the author allowed the normal component of the velocity field to be non-zero on the boundary. In Appendix A we extend section 3 of [6] to the case of time dependent domain.

Before introducing the notation of renormalized solution for (13), let us use introduce the following notation. For a collection of sets $X(t) \subset \mathbb{R}^2$, we denote

$$\mathbb{R}^+ \cap X(t) = \bigcup_{t \in \mathbb{R}^+} t \times X(t)$$

and

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\[ L^p_{\text{loc}}(\mathbb{R}^+; W^{s,p}(X(t))) = \{ u : \mathbb{R}^+ \odot X(t) \to \mathbb{R} \text{ s.t. } \| u(t,.) \|_{W^{s,p}(X(t))} \in L^p_{\text{loc}} \}, \]

for \( p, q \in [1, \infty] \). Moreover for any measurable subset \( I \subset \mathbb{R}^+ \) and any function \( f \) defined on \( I \odot \partial S^i(t) \), we also denote by \( f \) the extension by zero on \( \mathbb{R}^+ \odot \partial S^i(t) \). Let us now introduce the definition of compatible in–out velocity.

**Definition 4 (Compatible in–out velocity).** Let \((\Omega, S^+, S^-)\) a regular compatible geometry such that \( r^+, r^- \geq c > 0 \). Then a measurable function \( g : \mathbb{R}^+ \odot \partial F(t) \to \mathbb{R} \) is called compatible in–out velocity if

\[
\int_{\partial F(t)} g = 0, \quad g = 0 \text{ in } \partial \Omega, \quad g - q < 0 \text{ in } \partial S^+(t) \quad \text{and} \quad g - q > 0 \text{ in } \partial S^-(t),
\]

for almost any time \( t \in \mathbb{R}^+ \).

Recall that for the system (13), the data that we have to prescribe are the initial vorticity \( \omega^i \), the initial circulations \( C^i \) around \( \partial S^i \), the compatible in–out velocity \( g \) and the entering vorticity \( \omega^- \). Moreover we say that they are \( L^p \) given data if they satisfy

\[
\omega^i \in L^p(\mathcal{F}(0)), \quad C^i \in \mathbb{R}, \quad g \in L^p_{\text{loc}}(\mathbb{R}^+; W^{1-1/p,p}(\partial \mathcal{F}(t)))
\]

and \( \omega^- \in L^p_{\text{loc}}(\mathbb{R}^+ \odot \partial S^i(t), |g - q| \, dt \, ds) \),

for \( p > 1 \).

After all this preliminary we are able to present the definition of renormalized solution for (13).

**Definition 5.** Let \( p > 1 \) and \( r \geq 1 \). Let \( \omega^i, C^i, g \) and \( \omega^- \) given data that satisfy (18). Then a triple \((\omega, \omega^-, v) \in \mathcal{C}_c^\infty(\mathbb{R}^+, L^p(\mathcal{F}(t))) \times L^p_{\text{loc}}(\mathbb{R}^+ \odot \partial S^-; W^{1-1/p,p}(\partial \mathcal{F}(t))) \times L^p_{\text{loc}}(\mathbb{R}^+; W^{1-1/p,p}(\partial \mathcal{F}(t))) \) is a renormalized solution of system (13) if for any \( \varphi \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^2) \) and \( \beta \in C^1(\mathbb{R}) \), it holds

\[
\int_{\mathcal{F}(0)} \beta(\omega^i) \varphi(0,.) + \int_{\mathbb{R}^+} \int_{\mathcal{F}(t)} \beta(\omega) \partial_t \varphi + \int_{\mathbb{R}^+} \int_{\mathcal{F}(t)} \beta(\omega) v \cdot \nabla \varphi = \sum_{i \in \{+, -\}} \int_{\mathbb{R}^+} \int_{\partial S^i} \beta(\omega^i)(v \cdot n - q) \varphi, \tag{19}
\]

the velocity field \( v \) satisfies in a strong sense the Div-Curl system

\[
\begin{align*}
\text{div } v &= 0, \quad \text{curl } v = \omega \quad \text{in } \mathcal{F}(t), \\
v \cdot n &= g \quad \text{on } \partial \mathcal{F}(t) \quad \text{and} \quad \int_{\partial \mathcal{F}^i(t)} v(t,.) \cdot \tau = \mathcal{C}_i(t),
\end{align*}
\tag{20}
\]

where \( \mathcal{C}_+ \) and \( \mathcal{C}_- \) satisfy the last two equations of (13).

Let us notice the right hand side of (19) comes from the integration by parts of term associated with \( v \cdot \nabla \omega \). Moreover on \( \mathbb{R}^+ \odot S^+(t) \), the entering vorticity \( \omega^- \) is a prescribed quantity while on \( \mathbb{R}^+ \odot S^+(t) \) the exiting vorticity \( \omega^- \) is an unknown of the problem.

The existence result reads as follows.

**Theorem 1.** There exists a solution \((\omega, \omega^-, v)\) to system (13) in the sense of Definition 5.

Let us conclude the section with the limit case \( p = 1 \). For this case we prove existence of Delort type solutions introduced in [13], see also [28]. The idea is to rewrite the non linear term \( \omega v \) as an integral of a kernel multiplied by the vorticity. To do that we decompose the velocity

\[
v = v_g + K_{\mathcal{F}(t)}^0[\omega] + \sum_i \left( \int_{\mathcal{F}} \Psi_i \omega + \mathcal{C}_i(t) \right) X_i \tag{21}
\]

where \( v_g = J_{\mathcal{F}(t)}[0, g] = \nabla \varphi, K_{\mathcal{F}}^0[\omega] = \nabla^\perp \phi \) and \( \Psi_i \) are respectively the solutions of

\[
\begin{align*}
-\Delta \varphi &= 0 \quad \text{in } \mathcal{F}(t), \\
\varphi &= 0 \quad \text{in } \partial \mathcal{F}(t),
\end{align*}
\]

\[
\begin{align*}
\nabla \varphi \cdot n &= g \quad \text{in } \partial \mathcal{F}(t), \\
\phi &= 0 \quad \text{in } \partial \mathcal{F}(t),
\end{align*}
\]
Finally, $X_i$ is the unique harmonic vector field with circulation around $\partial S_i$ equal to the Kronecker delta $\delta_{ij}$.

Note that $v_g$ and $[\int_{\mathcal{F}} \Psi\omega + C_i(t)]$ $X_i$ are $L^\infty$ vector fields if we assume $g$ smooth enough. We are left with the term $\omega K^0_{\mathcal{F}_1(t)}[\omega]$ which is not in general $L^1$, in particular the following expression, that appears in the weak formulation,

$$\int_0^t \int_{\mathcal{F}} \omega K^0_{\mathcal{F}_1(t)}[\omega] \cdot \nabla \varphi$$

does not make sense. To avoid this issue we rewrite it in the same spirit of [13]. Let us introduce the trilinear map

$$\langle w, \omega, \varphi \rangle = \int_{\mathcal{F}} \int_{\mathcal{F}} H_\varphi(x,y) w(t,x) \omega(t,y) \, dx \, dy,$$  

where

$$H_\varphi(t,x,y) = -\nabla^\perp G(t,x,y) \cdot \frac{\nabla \varphi(t,x) - \nabla \varphi(t,y)}{2}.$$  

It holds

$$\langle \omega, \omega, \varphi \rangle = \int_0^t \int_{\mathcal{F}} \omega K^0_{\mathcal{F}_1(t)}[\omega] \cdot \nabla \varphi$$

for smooth enough $\omega$. Let us also notice that $H_\varphi$ is in $L^\infty(\mathcal{F}(t) \times \mathcal{F}(t))$ if $\varphi \in \mathcal{C} = \{ \varphi \in W^{2,+\infty}(\mathcal{F}(t)) \}$ such that $\varphi$ is constant in any connected component of $\partial \mathcal{F}(t)$, see Lemma 2. This implies that (22) makes sense for any vorticity $w = \omega \in L^1(\mathcal{F}(t))$.

**Lemma 2.** Let $\varphi \in W^{2,+\infty}(\mathcal{F}(t))$ then

$$\| H_\varphi \|_{L^\infty(\mathcal{F}(t) \times \mathcal{F}(t))} \leq M \| \varphi \|_{W^{2,+\infty}(\mathcal{F}(t))},$$

where $M$ does not depend on time $t$.

Let us postpone the proof to the Appendix B. We are now able to define weak solutions for system (13) with vorticity initially in $L^1$. As for the case $p > 1$, we prescribe the initial vorticity $\omega^{in}$, the initial circulations $C_i^{in}$ around $\partial S^i$, the compatible in–out velocity $g$ and the entering vorticity $\omega^+$ but they satisfy the regularity assumptions

$$\omega^{in} \in L^1(\mathcal{F}(0)), \quad C_i^{in} \in \mathbb{R}, \quad g \in L^1_{loc}(\mathbb{R}^+; W^{1/2,2+\varepsilon}((\partial \mathcal{F}(t)))$$

and

$$\omega^+ \in L^1_{loc}(\mathbb{R}^+ \cap \partial S^i(t), |g - q| \, dt \, ds),$$

for some $\varepsilon > 0$.

**Definition 6.** Let $r \geq 1$. Let $\omega^{in}$, $C_i^{in}$, $g$ and $\omega^+$ given data that satisfy the regularity condition (24). Then a couple $(\omega, \omega^-) \in L^\infty_{loc}(\mathbb{R}^+; L^1(\mathcal{F}(t))) \times L^1_{loc}(\mathbb{R}^+ \cap \partial S^-(t), |g - q| \, dt \, ds)$ is a weak solution of (13) if for any $\varphi \in C^\infty_c(\mathbb{R}^+ \times \mathcal{F}) \cap \mathcal{C}$,

$$\int_{\mathcal{F}(t)} \omega^{in} \varphi(0,.) \, dx + \int_{\mathbb{R}^+} \int_{\mathcal{F}(t)} (\omega \partial_t \varphi + \omega v_g \cdot \nabla \varphi) \, dx \, dt + \int_{\mathbb{R}^+} \langle \omega, \omega, \varphi \rangle \, dt$$

$$+ \int_{\mathbb{R}^+} \sum_i \left[ \int_{\mathcal{F}(t)} \Psi_i \omega + C_i(t) \right] \int_{\mathcal{F}(t)} X_i \cdot \nabla \varphi \, dx \, dt$$

$$= \int_{\mathbb{R}^+} \int_{\partial S^+(t)} (g - q) \omega^+ \varphi \, ds \, dt + \int_{\mathbb{R}^+} \int_{\partial S^-(t)} (g - q) \omega^- \varphi \, ds \, dt,$$  

(25)
and the following estimate holds for almost any \( t \in \mathbb{R}^+ \).

\[
\int_{\mathcal{F}} |\omega(t,.)| + \int_0^t \int_{\partial \mathcal{F}} (g - q)|\omega^-| \leq \int_{\mathcal{F}} |\omega^{in}| - \int_0^t \int_{\partial \mathcal{F}} (g - q)|\omega^+|.
\]

**Remark 2.** In contrast to the case \( p > 1 \), we need to assume a better integrability on the normal component of the velocity \( g \) on \( \partial \mathcal{F}(t) \) due to the fact that we need \( v_g \in L^\infty \) to make sense of the term \( \int \omega v_g \cdot \nabla \varphi \).

**Remark 3.** In the case the fluid domain is \( \mathbb{R}^2 \) and \( p = 1 \) existence of renormalized solutions for the transport equation of the vorticity was shown in [12]. It is not clear how to extend this result in our setting.

The existence result reads as follow.

**Theorem 3.** There exists a solution \((\omega,\omega^-)\) to system (13) in the sense of Definition 6.

Note that the above theorem answers to the question left open at the end of [17] where the authors consider the special case with \( g = q \).

Let us now go back to the case where we allow \( S^i(.) \) to be also points.

### 3.3. Boundary Data and Definition of Weak Solutions for System (16)

Let us start by discussing the boundary data that we require in the definition and in the existence result of weak solutions for system (16). On \( \partial S^+(t) \), we impose the normal component of the velocity \( g \) for \( t \in T^+_{NP} \). Moreover because \( g - q < 0 \), we are allowed to prescribe also the entering vorticity \( \omega^+ \). For \( t \in T^+ \), \( \partial S^+(t) = h^+(t) \) and we impose \( \mu \) and \( j \), the strengths respectively of the point source and of the point vortex. On \( \partial S^-(t) \), we prescribe the normal component of the velocity \( g \) for \( t \in T^-_{NP} \). In this case \( g - q > 0 \), which implies that the exiting vorticity is an unknown of the problem. For \( t \in T^- \), no data are required, in fact both the strength of the point sink and point vortex can be recovered due to respectively the mass conservation and the conservation of the vorticity, where here by conservation we mean the sum of what is entering minus what is exiting plus what is inside is constant. For example for \( t \in T^-_{NP} \), the compatibility condition

\[
\left( \oint_{\partial S^+(t)} (g - q) \right) \mathbf{1}_{T^+_{NP}} + \mu \mathbf{1}_{T^+} + \left( \oint_{\partial S^-(t)} (g - q) \right) = 0,
\]

have to be satisfied. Let us extend the concept of compatible in–out velocity in this setting.

**Definition 7** (Compatible in–out velocities). Let \((\Omega,S^+(.),S^-(.))\) a regular compatible geometry. A couple \((g,\mu)\) with

\[
g : (\mathbb{R}^+ \times \partial \Omega) \cup \bigcup_{i \in \{+,-\}} T^i_{NP} \odot \partial S^i(t) \longrightarrow \mathbb{R} \quad \text{and} \quad \mu : T^+ \longrightarrow \mathbb{R}^+
\]

is a compatible in–out velocities if

- it holds \( g - q < 0 \) on \( T^+_{NP} \odot \partial S^+ \), \( g - q > 0 \) on \( T^-_{NP} \odot \partial S^- \) and \( g = 0 \) on \( \mathbb{R}^+ \times \partial \Omega \).
- for any \( t \in T^-_{NP} \) equality (26) holds.

Let us remark that we do not define the strength \( \mu^- \) of the sink point because for \( t \in T^- \) it holds

\[
\mu^- = -\left( \oint_{\partial S^+(t)} (g - q) \right) \mathbf{1}_{T^+_{NP}} - \mu \mathbf{1}_{T^+},
\]

due to the incompressibility of the velocity field.
The first part of this section can be summarized as follows. For a compatible geometry \((\Omega, S^+, S^-)\) the initial and entering data for the system (16) are the initial vorticity \(\omega^{in}\), the initial circulations \(C_i^{in}\) around \(\partial S^i\), the compatible in–out velocity \((g, \mu)\) and the entering vorticities \(\omega^+\) and \(j\). Moreover we assume that they satisfy the regularity assumptions

\[
\begin{align*}
\omega^{in} &\in L^p(\mathcal{F}(0)), \quad C_i^{in} \in \mathbb{R}, \quad g \in L^r_{loc}(\mathbb{R}^+; W^{1-1/p, p}(\partial \mathcal{F}(t))), \\
\mu &\in L^p_{loc}(T^+), \quad \omega^+ \in L^p_{loc}(T^+_N \cap \partial S^i(t), |g - q| dt ds) \\
\text{and} \quad j &\in L^p_{loc}(\mathbb{R}^+),
\end{align*}
\]

such that \(\text{supp}(j) \subset \text{supp}(\mu)\) and \(j/\mu \in L^p_{loc}(\mathbb{R}^+)\) where \(j/\mu\) is defined 0 when \(\mu = 0\).

We present the definition of weak solutions for system (16).

**Definition 8.** Let \((\Omega, S^+, S^-)\) a regular compatible geometry. Let \(p \in (2, +\infty), q \in [p/(p-1), 2)\) and \(r > 1\). Let \(\omega^{in}, C_i^{in}, (g, \mu), \omega^+\) and \(j\) given data for the system (16) satisfying the regularity hypothesis (27). Then we say that a triple \((\omega, \omega^-, v)\) is a weak solution of system (16) if \(\omega \in L^p_{loc}(\mathbb{R}^+, L^p(\mathcal{F}(t)))\), \(\omega^- \in L^p_{loc}(T^-_N \cap \partial S^-(t), |g - q| dt ds)\), \(v \in L^p_{loc}(\mathbb{R}^+; L^p(\mathcal{F}(t)))\), \(\text{in} \mathcal{F}(t) \omega \in W^{1,1}_{loc}(\mathbb{R}^+)\) and it holds

\[
\begin{align*}
\int_{\mathcal{F}(0)} \omega^\varphi(0,.) + \int_{\mathcal{R}^+} \int_{\mathcal{F}(t)} \omega \partial_t \varphi + \int_{\mathcal{R}^+} \int_{\mathcal{F}(t)} \omega v \cdot \nabla \varphi \\
= \sum_{i \in \{+,-\}} \int_{T^+_{NP}} \int_{\partial S^i(t)} \omega^i(g - q) \varphi + \int_{T^+_{NP}} j \varphi(t, h^+(t)) \\
- \int_{T^-} \left( \frac{d}{dt} \int_{\mathcal{F}(t)} \omega + \int_{\partial S^+(t)} \omega^+(g - q) 1_{T^+_{NP}} + j 1_{T^+} \right) \varphi(t, h^-(t)),
\end{align*}
\]

for any \(\varphi \in C^\infty_c(\mathbb{R}^+ \times \Omega)\),

\[
\begin{align*}
\int_{\mathcal{F}(t)} v(t,.) \cdot \nabla \xi = \sum_{i \in \{+,-\}} \left( \int_{\partial S^i(t)} g(t,.) \xi \right) 1_{T^+_{NP}}(t) + \mu(t) 1_{T^+} \xi(h^+(t)) \\
- \left( \int_{\partial S^i(t)} g(t,.) 1_{T^+_{NP}} + \mu(t) 1_{T^+}(t) \right) 1_{T^-}(t) \xi(h^-(t)),
\end{align*}
\]

for any \(\xi \in C^\infty(\Omega)\),

\[
\begin{align*}
\int_{\mathcal{F}(t)} v(t,.) \cdot \nabla \zeta = &\ - \int_{\mathcal{F}(t)} \omega(t,.) \zeta \\
&\ + \left[ C^+_i - \int_{0}^{t} \left( \int_{\partial S^i(t)} \omega^+(g - q) 1_{T^+_{NP}} + j 1_{T^+} \right) \right] \zeta \partial S^+(t) \\
&\ + \left[ C^-_i + \int_{\mathcal{F}(t)} \omega - \int_{\mathcal{F}(0)} w^i + \int_{0}^{t} \left( \int_{\partial S^i(t)} \omega^+(g - q) 1_{T^+_{NP}} + j 1_{T^+} \right) \right] \zeta \partial S^-(t),
\end{align*}
\]

for \(\zeta \in C^\infty(\mathcal{F}(t))\) such that \(\zeta = 0\) on \(\partial \Omega\) and constant on any \(\partial S^i(t)\). Such constants are denoted by \(\zeta \partial S^i(t)\), in particular for \(t \in T^i\), it holds \(\zeta \partial S^i(t) = \zeta(h^i(t))\).

### 3.4. Existence Result and Justification of System (16)

In this section we present the main results of the paper. The first one is the existence of weak solutions of (16) in the sense of Definition 8. The second one is the justification of the model (16). To show the second result we consider a sequence of approximated fluid domain \(\mathcal{F}_x\) for which the fluid is modelled by (13). Then we study the limit as \(\varepsilon\) goes to zero for the solutions \((\omega_\varepsilon, \omega^-_\varepsilon, v_\varepsilon)\) of (13) under some mild assumptions on the data \(\omega^{in}_\varepsilon, \omega^+_\varepsilon\) and \(g_\varepsilon\). We deduce that \((\omega_\varepsilon, \omega^-_\varepsilon, v_\varepsilon)\) converges to \((\omega, \omega^-, v)\) a weak
solution of (16). Notice that the existence result for (16) becomes a corollary of the justification of the model.

**Theorem 4.** There exists a weak solution \((\omega, \omega^-, v)\) of system (16) in the sense of Definition 8.

Let us now present the justification result. Let us start by considering a sequence of approximate geometries such that \(S^+_\varepsilon(t)\) and \(S^-_\varepsilon(t)\) have no empty interior for any time.

**Definition 9 (Sequence of approximate geometries).** Let \((\Omega, S^+, S^-)\) a regular compatible geometry associated with the septuple \((\Omega, S^+, h^+, r^+, S^-, h^-, r^-)\). For \(\varepsilon > 0\), we say that \((\Omega, S^+_\varepsilon, S^-_\varepsilon)\) is a sequence of approximate geometries if there exist \(\Xi_\varepsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) such that \(\Xi_\varepsilon\) is smooth, increasing, \(\Xi_\varepsilon(x) = \varepsilon\) for \(x \in [0, \varepsilon]\), \(\Xi_\varepsilon(x) = x\) for \(x \in [2\varepsilon, \infty)\) and \((\Omega, S^+_\varepsilon, S^-_\varepsilon)\) is associated with the septuple \((\Omega, S^+, h^+, \Xi_\varepsilon(r^+), S^-, h^-, \Xi_\varepsilon(r^-))\).

**Remark 4.** Note that for any \(T > 0\), there exist \(\varepsilon_T\) such that for \(0 < \varepsilon < \varepsilon_T\) the triple \((\Omega, S^+_\varepsilon, S^-_\varepsilon)\) is a regular compatible geometry for \(t \in [0, T]\). In the following we will assume without loss of generality that there exists \(\varepsilon_\infty\) such that \((\Omega, S^+_\varepsilon, S^-_\varepsilon)\) is a regular compatible geometry for all times \(t \in \mathbb{R}^+\) and for \(0 < \varepsilon < \varepsilon_\infty\) because the results that we will prove hold only locally in time.

Theorem 1 ensures that there exists a solution \((\omega^\varepsilon, \omega^-\varepsilon, v_\varepsilon)\) to system (13) associated with \(F_\varepsilon, \omega^\varepsilon, C_\varepsilon, g_\varepsilon\) and \(\omega^\varepsilon\). We look for some mild conditions on the data that allow us to show that solutions \((\omega^\varepsilon, \omega^-\varepsilon, v_\varepsilon)\) of (13) converge to a weak solution of (16) in an appropriate sense.

Let \(p \in (2, +\infty)\), let \(q \in [p/(p-1), 2)\) and for \(\delta > 0\) denote by \(T^i_{NP, \delta} = \{t \in \mathbb{R}^+ : r^i > \delta\}\). Regarding the initial datum and the initial circulation, we assume that
\[
\omega^\varepsilon \in \omega^i \text{ in } L^p(F_\varepsilon) \quad \text{and} \quad C^\varepsilon \in C^i. \tag{28}
\]

Regarding the boundary conditions, we assume that for any \(\delta\)
\[
ge_\varepsilon \rightarrow g \text{ in } L^p_\loc(T^i_{NP, \delta}; L^p(\partial S^i_\varepsilon(t))) \quad \text{and} \quad (g_\varepsilon - q_\varepsilon)^{1/p}_\varepsilon \rightarrow (g - q)^{1/p}_\varepsilon \omega^+ \text{ in } L^p_\loc(T^i_{NP, \delta}; L^p(\partial S^+_\varepsilon(t))). \tag{29}
\]

And for \(t \in T^+\), we assume
\[
\int_{\partial S^i_\varepsilon(t)} g_\varepsilon \rightarrow \mu \text{ in } L^p_\loc(T^+) \quad \text{and} \quad \int_{\partial S^+_\varepsilon(t)} (g_\varepsilon - q_\varepsilon) \omega^+_\varepsilon \rightarrow j \text{ in } L^p_\loc(T^+). \tag{30}
\]
The above conditions are natural and they ensure the convergence of the initial and boundary data. We have also to assume that the boundary data do not concentrate in space and time variables. These hypothesis read
\[
\varepsilon^{1/q} \|g_\varepsilon\|_{L^q(\partial S^i_\varepsilon(t))} \rightarrow 0 \text{ in } L^p_\loc(T^i) \quad \text{and} \quad \left\|(g_\varepsilon - q_\varepsilon)^{1/p}_\varepsilon \omega^+_\varepsilon\right\|_{L^p(0, T) \cap \partial S^+_\varepsilon(t)} \leq C_T. \tag{31}
\]
Finally we assume the following equi-integrability property. For any compact \(K \subset \mathbb{R}^+\) and any small parameter \(\varepsilon > 0\), there exist \(\delta\) such that for any \(A \subset K\) with Lebesgue measure \(L(A) < \delta\), it holds
\[
\sup_\varepsilon \left\|(r^+\varepsilon)^{1-1/q}(t) g_\varepsilon(t)\right\|_{L^q(\partial S^i_\varepsilon(t))} \left\|_{L^r(A)} < \varepsilon. \tag{32}
\]

Let us now state the main result of the paper.

**Theorem 5.** Let \((\Omega, S^+, S^-)\) a regular compatible geometry and let \((\Omega, S^+_\varepsilon, S^-_\varepsilon)\) a sequence of approximate geometry for \(\varepsilon > 0\). Let \(p \in (2, +\infty)\), \(q \in [p/(p-1), 2)\) and \(r > 1\). Let \(\omega^\varepsilon, C^\varepsilon_\varepsilon, g_\varepsilon\) and \(\omega^\varepsilon\) a sequence of given data for the system (13) in the fluid domain \(F_\varepsilon\) satisfying the regularity hypothesis (18). Suppose there exists \(\omega^i, C^i, (g, \mu), \omega^+\) and \(j\) given data for the system (16) in the fluid domain \(F\) satisfying the regularity assumptions (27) such that (28)-(29)-(30)-(31)-(32) hold. Then up to a subsequence the
Theorem 5 implies the existence of a weak solution of (16) in the sense of Definition 8 with initial and entering data \( \omega^{in}, C^i_{i,\varepsilon}, g^{i}_{\varepsilon}, \omega^{+}_{\varepsilon} \) converges to \( (\omega, \omega^{-}, v) \) a solution of (16) in the sense of Definition 8 with initial and entering data \( \omega^{in}, C^i_{i,\varepsilon}, (g, \mu) \), \( \omega^{+} \) and \( j \). Moreover, the convergence holds in the following sense.

- \( \omega_{\varepsilon} \xrightarrow{wk} \omega \) in \( L^p_{\infty}(\mathbb{R}^+; L^p(\mathcal{F}(t))) \),
- \( (g_{\varepsilon} - q_{\varepsilon})^{1/p} \omega_{\varepsilon} \xrightarrow{wk} (g - q)^{1/p} \omega^{-} \) in \( L^p_{loc}(T_{NP,\delta}^{-} \cap \partial S^{-}(t)) \) for any \( \delta > 0 \).

Moreover, the convergence holds in the following sense.

- \( v_{\varepsilon} \rightarrow v \) in \( L^1_{loc}(\mathbb{R}^+; L^q(\mathcal{F}(t))) \).

First of all, we show Theorem 4 as a corollary of the above Theorem 5.

**Proof of Theorem 4.** Let \( \Xi_{\varepsilon} : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) some smooth increasing functions such that \( \Xi_{\varepsilon}(x) = \varepsilon \) for \( x \in [0, \varepsilon] \), \( \Xi_{\varepsilon}(x) = x \) for \( x \in (2\varepsilon, \infty) \) and let \( (\Omega, S^+_\varepsilon, S^-_{\varepsilon}) \) the sequence of approximate geometries associated with \( \Xi_{\varepsilon} \).

For the approximate fluid domain \( \mathcal{F}_{\varepsilon}(t) = \Omega \setminus (S^+_\varepsilon(t) \cup S^-_{\varepsilon}(t)) \), we show that there exists a sequence of data \( \omega^{in}_{\varepsilon}, C^i_{i,\varepsilon}, g^{i}_{\varepsilon}, \omega^{+}_{\varepsilon} \) that satisfy the hypothesis of Theorem 5 and in particular (28)-(29)-(30)-(31)-(32). Theorem 5 implies the existence of a weak solution of (16).

We consider \( \omega^{in}_{\varepsilon} = \omega^{in}|_{\mathcal{F}_{\varepsilon}(0)} \) and \( C^i_{i,\varepsilon} = C^i_{i,\varepsilon} \), for which (28) is clear. We use the notation \( x = h^i(t) + \frac{r^i(t)}{r_\varepsilon(t)}(x - h^i(t)) \), we define

\[
\begin{align*}
    g_{\varepsilon}(t, x) &= \begin{cases} 
    g(t, x) & \text{in } T_{NP,2\varepsilon}^+ \cap \partial S^i_{\varepsilon}(t), \\
    \left( \frac{r^i(t)}{r_\varepsilon(t)} \right)^{1/q} (g(t, x) - q(t, x)) + q_\varepsilon & \text{in } (T_{NP} \setminus T_{NP,2\varepsilon}) \cap \partial S^i_{\varepsilon}(t), \\
    \frac{\mu}{L_{N_{\varepsilon}(t)}} + q_\varepsilon & \text{in } \{ t | \mu(t) \neq 0 \} \cap \partial S^i_{\varepsilon}(t),
    \end{cases}
\end{align*}
\]

and

\[
\omega^+_{\varepsilon} = \begin{cases} 
    \omega^+_{\varepsilon}(t, x) & \text{for } (t, x) \in T_{NP,2\varepsilon}^+ \cap \partial S^i_{\varepsilon}(t), \\
    \frac{\mu}{L_{N_{\varepsilon}(t)}} & \text{for } (t, x) \in T^+ \cap \partial S^i_{\varepsilon}(t), \\
    0 & \text{else}.
\end{cases}
\]

Notice that \( \mathcal{F}_{\varepsilon}, g_{\varepsilon} \) and \( \omega^+_{\varepsilon} \) satisfy (29)-(30)-(31)-(32). We apply Theorem 5 to deduce the existence of a solution for the system (16). This shows Theorem 4.

In the remaining part of the paper we show Theorems 1, 3 and 5. In particular in Sect. 4, we recall some well-posedness results for the Div–Curl system in time dependent domains and some a priori estimates in the case where the holes \( S^i \) are assumed small. In Sect. 5, we show existence of solutions for the system (13) with given data in \( L^p \) and \( L^1 \) which correspond to Theorem 1 and 3 respectively. Finally in Sect. 6, we prove Theorem 5 which is the main result of this work.

### 4. The Div–Curl System in Time Dependent Domain

Before moving to the proof of Theorem 1, 3 and 5 let us study well-posedness and regularity for the Div–Curl system in the time dependent domain \( \mathcal{F}(t) \) associated with a regular compatible geometry \( (\Omega, S^+, S^-) \). In the first subsection we deal with the case where the boundary of \( \mathcal{F}(t) \) is regular. In the second one we show uniform estimates independent of the radii \( r^i \) of \( S^i \) in the case they are sufficiently small.
4.1. Well-Posedness for the Div–Curl System in Time-Depending domain

In this subsection we study the Div–Curl system in the case the holes are not allowed to become points. More precisely the fluid domain $\mathcal{F}(t)$ associated with a regular compatible geometry $(\Omega, S^+, S^-)$ and we suppose $r_i(t) \geq c > 0$ for any $t$. In particular we are looking for estimates independent of the shape of $\mathcal{F}(t)$. All these results are taken from [31].

The Div–Curl system we are interested in is

$$
\begin{align*}
\text{div} \ u &= 0 \\
\text{curl} \ u &= \omega \\
\dot{t} u &= g \\
\int_{\partial S_i(t)} u \cdot n &= 0 \\
\int_{\partial S_i(t)} u \cdot \tau &= C_i(t) \\
\end{align*}
$$

(33)

The following lemmas holds.

Lemma 6. Let $l \geq 1$, let $p \in (1, +\infty)$ and let $\partial \mathcal{F}$ of class $C^{l+1}$. If $\omega(t, \cdot) \in W^{l-1, p}(\mathcal{F}(t))$, $g \in W^{l-1/p, p}(\partial \mathcal{F}(t))$ such that $\int g = 0$ and $C_i(t) \in \mathbb{R}$. Then there exists a unique solution $u(t, \cdot) \in W^{l, p}(\mathcal{F}(t))$ of (33) such that

$$
\|u(t, \cdot)\|_{W^{l, p}(\mathcal{F}(t))} \leq C \frac{p^2}{p-1} \left( \|\omega(t, \cdot)\|_{W^{l-1, p}(\mathcal{F}(t))} + \|g(t, \cdot)\|_{W^{l-1/p, p}(\partial \mathcal{F}(t))} + \sum_{i=+, -} |C_i(t)| \right),
$$

where $C$ does not depend on time.

Let us denote by $\bar{D}_t$ the convective derivative following the boundary of the domain $\partial \mathcal{F}(t)$, i.e. $\bar{D}_t g = \frac{\partial}{\partial t} g + \dot{x} \cdot \nabla_x g$ where $\tau$ is the tangent vector to $\partial \mathcal{F}(0)$ and $x(t, y) = h^i(t) + r^i(t) S^i(t, y)$ on $\partial S^i(0)$ and $x(t, y) = y$ on $\partial \Omega$. The following lemmas holds.

Lemma 7. Let $p \in (1, +\infty)$. Let $\omega(t, \cdot) \in W^{1, p}(\mathcal{F}(t))$ such that $\frac{\partial}{\partial t} \omega(t, \cdot) \in L^p(\mathcal{F}(t))$, let $g \in W^{2-1/p, p}(\partial \mathcal{F}(t))$ with $\bar{D}_t g \in W^{1-1/p, p}(\mathcal{F}(t))$ and let $C_i(t) \in \mathbb{R}$ with $\dot{C}_i(t) \in \mathbb{R}$. Then there exists a unique solution $u$ of (33) such that $u(t, \cdot) \in W^{2, p}(\mathcal{F}(t))$ and $\frac{\partial}{\partial t} u(t, \cdot) \in W^{1, p}(\mathcal{F}(t))$ such that

$$
\|u(t, \cdot)\|_{W^{2, p}(\mathcal{F}(t))} + \|\frac{\partial}{\partial t} u(t, \cdot)\|_{W^{1, p}(\mathcal{F}(t))} \leq C \frac{p^2}{p-1} \left( \|\omega(t, \cdot)\|_{W^{1, p}(\mathcal{F}(t))} + \|\frac{\partial}{\partial t} \omega(t, \cdot)\|_{L^p(\mathcal{F}(t))} + \|g(t, \cdot)\|_{W^{2-1/p, p}(\partial \mathcal{F}(t))} \\
+ \|\bar{D}_t g(t, \cdot)\|_{W^{1-1/p, p}(\partial \mathcal{F}(t))} + \sum_{i=+, -} \left( |\dot{C}_i(t)| + |\dot{C}_i(t)| \right) \right),
$$

where $C$ does not depend on time.

For the proof we refer to [31].

4.2. Reflection Method for the Div–Curl System

Let us recall that a regular compatible geometry $(\Omega, S^+, S^-)$ is defined by a septuple $(\Omega, S^+, r^+, h^+, S^-, r^-, h^-)$. In this section we study how solutions to the Div-Curl system (33) depends on $r^+$ and $r^-$ when they are assumed to be small. For this reason we introduce the notation $\mathcal{F}_{r^+, r^-}(t) = \mathcal{F}(t)$ and $S^i_{r^+}(t) = S^i(t)$ to emphasize the dependence on $r^+$ and $r^-$ in the definition of $\mathcal{F}(t)$ and $S^i(t)$. Finally for $i \in \{+, -\}$, we denote by $\bot i$ the only element of $\{+, -\}\setminus i$. 

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Lemma 8. Let \( q \in (1, +\infty), T > 0 \) and let \((\Omega, S^+, S^-)\) regular compatible geometries. Then there exists \( \kappa_T > 0 \) such that for any \( 0 < r^+, r^- < \kappa_T \), any \( g^i \in L^2(\partial S^i_T(0,T)) \), where \( \sim \) means that \( \int_{\partial S^i_T(t)} g^i = 0 \) and any \( t \in [0, T] \), the following estimate holds

\[
\|w\|_{L^q(F_{r^+, r^-}(t))} \leq C_T \sum_{i \in \{±, -\}} (r^i)^{1/q} \|g^i\|_{L^q(\partial S^i_T(t))}
\]

where \( w \) is the solution of \( \text{div} \, w = 0 \) and \( \text{curl} \, w = 0 \) on \( F_{r^+, r^-}(t) \), \( w \cdot n = g^i \) on \( \partial S^i_T \), \( w \cdot n = 0 \) on \( \partial \Omega \) and \( \int_{\partial S^i_T} w \cdot \tau = 0 \) for \( i \in \{±, -\} \).

Moreover for \( r^{op} \in C^{1,\alpha}(0, T) \) with \( r^{op} \geq \bar{c} > 0 \), there exist \( \kappa_T^{op} \) such that for \( r^i < \kappa_T^{op} \) and for \( g^i \in L^q(\partial S^i_T(t)) \), the following estimate holds

\[
\|w\|_{L^q(F_{r^+, r^-}(t))} \leq C_T (r^i)^{1/q} \|g^i\|_{L^q(\partial S^i_T(t))}
\]

where \( w \) is the solution of \( \text{div} \, w = 0 \) and \( \text{curl} \, w = 0 \) on \( F_{r^+, r^-}(t) \), \( w \cdot n = g^i \) on \( \partial S^i_T \), \( w \cdot n = 0 \) on \( \partial \Omega \) and \( \int_{\partial S^i_T} w \cdot \tau = 0 \) for \( j \in \{±, -\} \).

The proof of the above lemma is a consequence of the reflection method. The only delicate part is the fact that the domains are not only shrinking but also changing shape in time.

For time independent domain Lemma 8 was shown in Corollary 5.3.1 of [8] and the proof relies on the Lemma 5.3.1 and 5.3.2. Due to the fact that the proof of Lemma 8 is exactly the same of Corollary 5.3.1, we only present here the proof of the analogous version of Lemma 5.3.1 and 5.3.2 for time independent domain.

Lemma 9 (Analogous of Lemma 5.3.1). For \( t \in [0, T] \) let \( F_{r^+, r^-}(t) \) such that

1. either \( r^+ = r^- = 0 \), i.e. \( F_{r^+, r^-}(t) = \Omega \),
2. or \( 0 < c \leq r^i \) and \( r^i \in C^{1,\alpha}([0, T]) \) and \( r^{op}_i = 0 \).

Let \( g \in L^q(\partial F_{r^+, r^-}(t)) \) such that \( \int_{\partial F_{r^+, r^-}(t)} g = 0 \). Then there exists a unique solution \( h[g] \in L^q(F_{r^+, r^-}(t)) \) such that \( -\Delta h[g] = 0 \) in \( F_{r^+, r^-}(t) \) and \( \nabla h[g] \cdot n = g \) on \( \partial F_{r^+, r^-}(t) \). Moreover for any compact subset \( K \) of \( [0, T] \cap F_{r^+, r^-}(t) \), we denote by \( K_t = \{x \in \Omega \} \) such that \( (t, x) \in K \cap t \times \Omega \) and we have that \( h[g] \) is of class \( C^k(K_t) \) and

\[
\|h[g]\|_{C^k(K_t)} \leq C \|g\|_{L^q(\partial F_{r^+, r^-}(t))},
\]

with \( C \) independent of time.

Proof. The proof is base on improved interior estimates that are well-know from Lemma 6. \( \square \)

Let us introduce the space

\[
\mathcal{W}^{1,\beta}(\mathbb{R}^2 \setminus S^i_T) = \{ u \in D'(\mathbb{R}^2 \setminus S^i_T) \text{ such that } \|1 + |.|^2|^{-1/2}|u_\cdot|\|_{L^\beta(\mathbb{R}^2 \setminus S^i_T)} + \|\nabla u\|_{L^\beta(\mathbb{R}^2 \setminus S^i_T)} < \infty \}
\]

with norm given by

\[
\|u\|_{\mathcal{W}^{1,\beta}(\mathbb{R}^2 \setminus S^i_T)} = \|1 + |.|^2|^{-1/2}|u_\cdot|\|_{L^\beta(\mathbb{R}^2 \setminus S^i_T)} + \|\nabla u\|_{L^\beta(\mathbb{R}^2 \setminus S^i_T)}
\]
Lemma 10. (Analogous of Lemma 5.3.2)
Let $E(S_r^i) = \mathbb{R}^2 \setminus S_r^i$, let $g \in L^q(\partial S_r^i)$ such that $\int_{\partial S_r^i} g = 0$, let $\beta > 2$, let $\hat{f}[g]$ to be the unique solution in $W^{1,\beta}(E(S_r^i))$ of $-\Delta \hat{f}[g] = 0$ in $E(S_r^i)$, $\nabla \hat{f}[g] \cdot n = g$ on $\partial S_r^i$ and $|\hat{f}[g]| \to 0$ as $|x| \to +\infty$. Then it holds
\[
\|\nabla \hat{f}[g]\|_{L^q(E(S_r^i))} \leq C (r^i)^{1/q} \|g\|_{L^q(\partial S_r^i)} \quad \text{and} \quad \|\nabla \hat{f}[g]\|_{L^q(\partial S_r^i)} \leq C \frac{(r^i)^{2-1/p}}{|x-h(t)|^2} \|g\|_{L^q(\partial S_r^i)} \quad \text{for } x \text{ s. t. } |x - h(t)| \geq Cr^i,
\]
where $C$ and $\hat{C}$ do not depend on $t$ and $r^i$.

Proof. To show the result we use scaling estimates, so it is enough to show that in the case $r^i = 1$ there exist a unique solution in $W^{1,\beta}(E(S^i))$ of $-\Delta \hat{f}[g] = 0$ in $E(S^i)$, $\nabla \hat{f}[g] \cdot n = 0$ on $\partial S^i$ and $|\hat{f}[g]| \to 0$ as $|x| \to +\infty$ such that
\[
\|\nabla \hat{f}[g]\|_{L^q(E(S^i))} \leq C \|g\|_{L^q(\partial S^i)} \quad \text{and} \quad \|\nabla \hat{f}[g]\|_{L^q(\partial S^i)} \leq C \frac{1}{|x-h(t)|^2} \|g\|_{L^q(\partial S^i)} \quad \text{for } x \text{ s. t. } |x - h(t)| \geq C,
\]
where $C$ and $\hat{C}$ do not depend on $t$.

In the case $q \geq 2$, it is enough to consider $\beta = p$ and existence and uniqueness in $W^{1,q}$ follows from Theorem 3.1 and Proposition 3.3 of [4]. Moreover the estimates (34) holds with a constant independent of time because the proof is based on a Poincaré inequality that is proved via a partition of unity which can be chosen smooth in the time variables.

Regarding the decay at infinity, let $B_R(h(t))$ a ball such that $S^i \subset B_{R/2}(h(t))$ for $t \in [0,T]$. Let $w$ the solution of
\[
-\Delta w = 0 \quad \text{in } \mathbb{R}^2 \setminus B_R(h(t)), \quad \nabla w \cdot n = \nabla \hat{f}[g] \cdot n \text{ on } \partial B_R(h(t)) \quad \text{and} \quad |w| \to 0 \text{ for } |x| \to +\infty.
\]
First of all notice that the only solution in $W^{1,q}$ is given by $\hat{f}[g]$. By interior estimates we have $\|\nabla \hat{f}[g] \cdot n\|_{L^q(\partial B_R(h(t)))} \leq C \|g\|_{L^q(\partial S^i)}$ and $w(t, x) = \int_{\partial B_R(h(t))} G(x - h(t), y - h(t))g$, where $G$ are the Green function associated with the Newman Laplacian on the exterior of a ball. It is possible to compute explicitly $G$ and deduce that (35) holds. Let us conclude with the interesting case $q \in (1,2)$. First of all let us notice that $L^q(\partial S^i) \subset W^{-1/\beta,\beta}$ for any $\beta > 2$. Theorem 3.1 and Proposition 3.3 of [4] imply existence and uniqueness of solutions in $W^{1,\beta}$. As in the case $q \geq 2$, we deduce that (35) holds, which implies that $\|\nabla \hat{f}[g]\|_{L^q(\mathbb{R}^2 \setminus B_{R/2}(h(t)))} \leq C \|g\|_{L^q(\partial S^i)}$. Let us now notice that $B_R(h(t)) \setminus S^i$ is bounded so the solution of a Laplace problem with Newman boundary condition satisfies
\[
\|\nabla \hat{f}[g]\|_{L^q(B_R(h(t)) \setminus S^i)} \leq C \|g\|_{L^q(\partial S^i)} + \|\nabla \hat{f}[g]\|_{L^q(\partial B_R(h(t)))} \leq (C + \hat{C}) \|g\|_{L^q(\partial S^i)}.
\]

After this preliminary result on the Div–Curl system we move to the proof of Theorem 1 and 3.

## 5. Proof of Theorem 1 and 3

In this section we prove Theorem 1 and 3. The idea is to consider a viscous approximation of the Euler system, which is a Navier-Stokes system with non-physical boundary conditions, and to pass to the limit as the viscosity parameter $\nu$ tends to zero. To do that we will take advantage of a duality formula. Notice that these proofs follow the same strategy of the time independent case that was considered in [9]. We decide to briefly present this argument for completeness and to recall a compactness argument for the velocity that will be used in the proof of Theorem 5.
5.1. Well-Posedness for System (36)

We show existence and uniqueness for the viscous approximating system (36). Let

\[
\begin{align*}
\partial_t \omega_\nu + v_\nu \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu &= 0 & \text{for } x \in \mathcal{F}(t), \\
\nu \partial_n \omega_\nu - (\omega_\nu - \omega_\nu^+) (g_\nu - q) 1_{\partial S^+} &= 0 & \text{for } x \in \partial \mathcal{F}(t), \\
\text{div } v_\nu &= 0 & \text{for } x \in \mathcal{F}(t), \\
\text{curl } v_\nu &= \omega_\nu & \text{for } x \in \mathcal{F}(t), \\
v_\nu \cdot n &= g_\nu & \text{for } x \in \partial \mathcal{F}(t),
\end{align*}
\]

\(\int_{\partial S_i} v_\nu \cdot \tau = C_i^{\text{int}} - \int_0^t \int_{\partial S_i} (\omega_\nu^+ 1_{\partial S^+} + \omega_\nu 1_{\partial S^-}) (g_\nu - q).\) (36)

For smooth enough data system (36) admits regular solutions in the sense Lemma 11.

We show existence and uniqueness for the viscous approximating system (36).

**Lemma 11.** Let \(\omega_\nu^{\text{int}} \in H^1(\mathcal{F}(0)), \omega_\nu^+ \in C^1([0,T] \cap \partial S^+(t))\) and \(g_\nu \in C^1([0,T] \cap \partial \mathcal{F}(t))\), the system admits a unique solution of (36) with \(\omega_\nu \in L^p_{\text{loc}}(\mathbb{R}^+; H^2(\mathcal{F}(t))) \cap C^0_{\text{loc}}(\mathbb{R}^+; H^1(\mathcal{F}(t)))\) with \(\partial \omega_\nu \in L^2_{\text{loc}}(\mathbb{R}^+; L^2(\mathcal{F}(t)))\). Moreover the following estimate holds. For any positive convex function \(G\),

\[
\int_{\mathcal{F}(t)} G(|\omega_\nu|) + \int_0^t \int_{\partial S^-(t)} (g_\nu - q) G(|\omega_\nu|) \leq \int_{\mathcal{F}(t)} G(|\omega_\nu^{\text{int}}|) - \int_0^t \int_{\partial S^+(t)} (g_\nu - q) G(|\omega_\nu^+|).
\]

In particular for \(p > 1\) and \(G(x) = |x|^p\) the above inequality reads

\[
\begin{align*}
\frac{d}{dt} \int_{\mathcal{F}(t)} |\omega_\nu|^p + \int_{\partial S^-(t)} (g_\nu - q)|\omega_\nu|^p + \nu \frac{4(p-1)}{p} \int_{\mathcal{F}(t)} \left(\nabla |\omega_\nu|^2\right)^2 & \\
\leq -\int_{\partial S^+(t)} (g_\nu - q)|\omega_\nu^+|^p.
\end{align*}
\]

The existence of regular solutions for system (36) was done in [9, Lemma 4], in the case the domain \(\mathcal{F}(t) = \mathcal{F}(0)\) does not depend on time. Let us recall this argument for completeness and let us extend it to the case where \(\mathcal{F}(t)\) is time dependent.

**Proof.** The proof is based on a Schauder fixed point argument. Regarding the estimate (37), it follows formally by multiplying the first equation of (36) by \(G(|\omega_\nu|)|\omega_\nu|/|\omega_\nu|\), integrate in \((0,T) \cap \mathcal{F}(t)\) and some integrations by part. For a rigorous proof see Proposition 2 of [9].

Let us recall that the Schauder fixed point Theorem asserts that if \(Z\) is a non-empty convex closed subset of a normed space \(X\) and \(F : Z \rightarrow Z\) is a continuous mapping such that \(F(Z)\) is contained in a compact subset of \(B\), then \(F\) has a fixed point.

In our setting the space \(Z\) is

\[
Z = \left\{ \omega \in \bigcup_{i=0}^1 H^i(0,T; H^{2-2i}(\mathcal{F}(t))) \right. \\
\text{such that } \|\omega\|_Z := \|\omega\| \leq R \right\},
\]

for an \(R > 0\) big enough. The map \(F : Z \rightarrow \bigcup_{i=0}^1 H^i(0,T; H^{2-2i}(\mathcal{F}(t)))\) is defined by \(F(\omega) = \bar{\omega}\) which is the solution of

\[
\begin{align*}
\partial_t \bar{\omega} + u_\omega \cdot \nabla \bar{\omega} - \nu \Delta \bar{\omega} &= 0 & \text{for } x \in \mathcal{F}(t), \\
\partial_n \bar{\omega} &= (\bar{\omega} - \omega_\nu^+) g_\nu & \text{for } x \in \partial \mathcal{F}(t)
\end{align*}
\]
Lemma 12. Let us consider a sequence of regular initial data with the following properties. Let \( (\omega_j) \) a bounded sequence in \( \mathcal{Z} \). Then, up to a subsequence, \( \omega_j \rightarrow \omega \) in \( \mathcal{Z} \). By Aubin–Lions’s theorem the convergence is strong in \( L^2(0,T;H^1(\mathcal{F}(t))) \). Moreover for any \( j \), the function \( w_j = \bar{\omega} - \tilde{\omega}_j = F(\omega) - F(\omega_j) \) satisfies the system

\[
\partial_t w_j + v \cdot \nabla w_j - \Delta w_j = -(v - v_j) \cdot \nabla \bar{\omega}_j \quad \text{for } x \in \mathcal{F}(t),
\]

\[
\partial_n w_j = w_j g \quad \text{for } x \in \partial \mathcal{F}(t),
\]

with zero initial data. We observe that

\[
\| (v - v_j) \cdot \nabla \bar{\omega}_j \|_{L^2(0,T;L^2(\mathcal{F}(t)))} \leq \| (v - v_j) \|_{L^2(0,T;L^\infty(\mathcal{F}))} \| \nabla \bar{\omega}_j \|_{L^\infty(0,T;L^2(\mathcal{F}))}
\]

which converges to zero. Then by using the a priori estimates for the transport diffusion equation, we deduce that \( w_j \) converges to 0 in \( \mathcal{Z} \). Thus \( F \) is relatively compact. The continuity of \( F \) can be proved along the same lines. Thus Schauder’s fixed point theorem can be applied. It implies that \( F \) has a fixed point in \( \mathcal{Z} \). This proves the local in time existence of strong solutions. Moreover the existence to all \([0,T]\) can be deduced from the a priori estimates. In fact if we suppose by contradiction that there exists a maximal time of existence \( t_* < T \), the a priori estimates ensure that \( \omega(t_*) \) is enough regular to apply again the local existence result and we obtain a contradiction. Uniqueness follows from the energy estimate and Grönwall’s lemma.

Before going in the proof of Theorem 1 and 3, let us explain how to regularize the initial data to be able to apply Lemma 11.

5.2. Regularization of the Initial Data

Under the hypothesis of Theorem 1 and 3 the initial data are not regular enough to apply Lemma 11. To tackle this issue we consider a sequence of regular initial data with the following properties.

Lemma 12. Let \( p \in (1, +\infty) \). Let \( g \in L^1_{loc}(\mathbb{R}^d;W^{1-1/p,p}(\mathcal{F}(0))) \) be a compatible in–out velocity. Let \( \omega^{in} \) and \( \omega^+ \) respectively in \( L^p(\mathcal{F}(0)) \) and \( L^p_{loc}(\mathbb{R}^d \cap \partial S^+(t);|g - q|\,dt\,ds) \). Then there exist \( g_\nu, \omega^{in}_\nu \) and \( \omega^+_\nu \) such that \( g_\nu \in C^{1,\alpha}_{loc}(\mathbb{R}^d;C^{2,\alpha}(\partial \mathcal{F}(t))) \). Such that \( \omega^{in}_\nu \in H^1(\mathcal{F}(0)) \) and such that \( \omega^+_\nu \in C^{1}_{loc}(\mathbb{R}^d \cap \partial S^+(t)) \). Moreover

\[
g_\nu \rightarrow g \quad \text{in } L^1(0,T;W^{1-1/p,p}(\partial \mathcal{F}(t))),
\]

\[
\omega^{in}_\nu \rightarrow \omega^{in} \quad \text{in } L^p(\mathcal{F}(0)),
\]

\[
\omega^+_\nu (g_\nu - q)^{1/p} \rightarrow \omega^+(g - q)^{1/p} \quad \text{in } L^p_{loc}(\mathbb{R}^d \cap \partial S^+(t)).
\]

And

\[
\int_{\mathcal{F}(0)} \| \omega^{in}_\nu \|^2 \, dx - \int_{\mathbb{R}^d} \int_{\partial S^+(t)} \| \omega^+_\nu \|^2 (v_\nu \cdot n - q) \, ds \, dt
\]
Recall that $\omega$ by dominate convergence, we fix $M$ 105. In the previous section we show existence and a priori bounds for the viscous approximation. In this section we show the convergence associated with $\omega^+$, in fact we do not know if $\omega^+$ is in $L^1$ because we do not have a lower bound for $|g - q|$. To avoid this issue, for $M > 0$, we denote by $f \wedge M(x) = f(x)$ if $|f(x)| \leq M$ and $f \wedge M(x) = \text{sign}(f(x))M$ if $|f(x)| > M$. Let us consider the following inequality

$$
\|\omega^+(g \cdot n - q)|^{1/p} - \omega^+(v_i \cdot n - q)|^{1/p}\|_{L^p((0,T) \cap \partial S^+)}
\leq \|\omega^+(v \cdot n - q)|^{1/p} - (\omega^+ \wedge M)(v \cdot n - q)|^{1/p}\|_{L^p((0,T) \cap \partial S^+)}
+ \|\rho^+ \wedge M)(v \cdot n - q)|^{1/p} - (\rho^+ \wedge M)(v_i \cdot n - q)|^{1/p}\|_{L^p((0,T) \cap \partial S^+)}
+ \|\rho^+ \wedge M)(v_i \cdot n - q)|^{1/p} - \rho^+_v(v_i \cdot n - q)|^{1/p}\|_{L^p((0,T) \cap \partial S^+)}
$$

By dominate convergence, we fix $M$ such that the first term on the right hand side is less or equal to $\nu/3$. Then we denote by $v^\nu_i = v_i$, such that the second term is less the $\nu/3$. Finally $\rho^+ \wedge M$ is $L^p$ so by density of smooth functions we choose one such that the last term of the right hand side is smaller then $\nu/3$. With these choices the second convergence of (39) holds true and the approximation is smooth. Regarding $\omega^\nu$ we use the density of smooth functions in $L^p$. Finally (40) holds true after relabelling the sequences.

We are now able to prove Theorem 1 and 3. We start from the first one.

5.3. End of the Proof of Theorem 1 and 3

In the previous section we show existence and a priori bounds for the viscous approximation. In this section we study the limit as $\nu$ converges to zero. This is the last step of the proof of Theorem 1 and 3.

Proof of Theorem 1. Let $\omega^\nu$, $\zeta_i^\nu$, $g$ and $\omega^+$ the given data. Then Lemma 12 ensures the existence of regular approximate data $\omega^\nu \in H^1(\mathcal{F}(0))$, $g \in C_{1}^{\alpha}([R^+; C^{2,\alpha} \partial \mathcal{F}(t))]$ and $\omega^+ \in C_{1}^{\alpha}([R^+ \cap \partial S^+])$ satisfying (39) and (40).

For the data $\omega^\nu$, $\zeta_i^\nu$, $g$ and $\omega^+$ Lemma 11 applies. We deduce the existence of solutions $(\omega^\nu, v^\nu_i)$ such that (38) holds true. Moreover $\omega^\nu_i$ is uniformly bounded in $\mathcal{L}^\infty_{loc}(R^+; L^p(\mathcal{F}(t)))$ and in $L^p_{loc}(R^+ \cap \partial S^-(t); |g^\nu - q| ds))$ and up to subsequence

$$
\omega^\nu \overset{w^\nu}{\longrightarrow} \omega \text{ in } \mathcal{L}^\infty_{loc}(R^+; L^p(\mathcal{F}(t))) \text{ and } \omega^\nu(g^\nu - q)^{1/p} \overset{w^\nu}{\longrightarrow} \omega^-(g - q)^{1/p} \text{ in } L^p_{loc}(R^+ \cap \partial S^-(t)).
$$

The next step is to show strong convergence for the velocity. To do that notice that $\partial_i \omega^\nu$ is uniformly bounded in $\mathcal{L}^\nu_{loc}(R^+; H_0^{\nu,s}(\mathcal{F}(t)))$ for some $s$ big enough, where $H^{-s}$ is the dual of $H_0^s$. To see this let us decompose the velocity field as in (21), i.e.

$$
v^\nu_i = v^\nu_{gi} + \mathcal{K}_i^\nu(\cdot)[\omega^\nu] + \sum_i \left[ \int_{\mathcal{F}} \Psi_i \omega^\nu + C_i(t) \right] X_i
$$

Recall that $\omega^\nu$ is a solution of (36), in particular it holds

$$
\int_0^T \langle \partial_i \omega^\nu, \varphi \rangle dt = \int_0^T \int_{\mathcal{F}(t)} \omega^\nu_i \cdot \nabla \varphi dx dt - \nu \int_0^T \int_{\mathcal{F}(t)} |\nabla \omega^\nu_i|^2 dx dt
= \int_0^T \int_{\mathcal{F}(t)} \omega^\nu_i \left( v^\nu_{gi} + \mathcal{K}_i^\nu(\cdot)[\omega^\nu] \right) \sum_i \left[ \int_{\mathcal{F}} \Psi_i \omega^\nu + C_i(t) \right] h^i \cdot \nabla \varphi dx dt
$$
Let us estimate the four terms of the right-hand side separately. Regarding the first one we denote by \( \eta_\nu \) the solution of \( \Delta \eta_\nu = \omega_\nu \) in \( \mathcal{F}(t) \) and \( \eta_\nu = 0 \) on \( \partial \mathcal{F}(t) \). We have

\[
\int_0^T \int_{\mathcal{F}(t)} \omega_\nu v_{g_\nu} \cdot \nabla \varphi \, dx dt = \int_0^T \int_{\mathcal{F}(t)} \Delta \eta_\nu v_{g_\nu} \cdot \nabla \varphi \, dx dt
\]

\[
= \int_0^T \int_{\mathcal{F}(t)} \eta_\nu \Delta (v_{g_\nu} \cdot \nabla \varphi) \, dx dt
\]

\[
= 2 \int_0^T \int_{\mathcal{F}(t)} \eta_\nu \nabla v_{g_\nu} : \nabla^2 \varphi \, dx dt
\]

\[
+ \int_0^T \int_{\mathcal{F}(t)} \eta_\nu v_{g_\nu} \cdot \nabla \Delta \varphi \, dx dt,
\]

where we use that \( v_{g_\nu} = \nabla \psi \) with \( -\Delta \psi = 0 \) in \( \mathcal{F}(t) \) and \( \nabla \psi \cdot n = g_\nu \) on \( \partial \mathcal{F}(t) \). It is now easy to see that

\[
\left| \int_0^T \int_{\mathcal{F}(t)} \omega_\nu v_{g_\nu} \cdot \nabla \varphi \, dx dt \right|
\]

\[
\leq \| \eta_\nu \|_{L^\infty(0,T;L^p(\mathcal{F}(t)))} \| v_{g_\nu} \|_{L^{r}(0,T;W^{1,p}(\mathcal{F}(t)))} \| \varphi \|_{L^{r'}(0,T;W^{3,\infty}(\mathcal{F}(t)))}
\]

\[
\leq C \| \eta_\nu \|_{L^\infty(0,T;L^p(\mathcal{F}(t)))} \| v_{g_\nu} \|_{L^{r}(0,T;W^{1,p}(\mathcal{F}(t)))} \| \varphi \|_{L^{r'}(0,T;W^{3,\infty}(\mathcal{F}(t)))},
\]

\[
\leq C \| \omega_\nu \|_{L^\infty(0,T;L^p(\mathcal{F}(t)))} \| v_{g_\nu} \|_{L^{r}(0,T;W^{1,p}(\mathcal{F}(t)))} \| \varphi \|_{L^{r'}(0,T;W^{3,\infty}(\mathcal{F}(t)))},
\]

where we use \( p > 1 \). Regarding the second term we recall that it can be rewritten as

\[
\int_0^T \int_{\mathcal{F}(t)} \omega_\nu \mathcal{K}_0^\mathcal{F}(t)[\omega_\nu] \cdot \nabla \varphi \, dx dt = \int_0^T \int_{\mathcal{F}(t)} \int_{\mathcal{F}(t)} H_\varphi(t,x,y)\omega_\nu(t,x)\omega_\nu(t,y) \, dx dy dt,
\]

where \( H_\varphi \) is defined in (23) and satisfy \( \| H_\varphi \|_{L^\infty(\mathcal{F}(t))} \leq \| \varphi \|_{W^{2,\infty}(\mathcal{F}(t))} \). We deduce that

\[
\left| \int_0^T \int_{\mathcal{F}(t)} \omega_\nu \mathcal{K}_0^\mathcal{F}(t)[\omega_\nu] \cdot \nabla \varphi \, dx dt \right|
\]

\[
\leq C \| \varphi \|_{L^{1}(0,T;W^{2,\infty}(\mathcal{F}(t)))} \| \omega_\nu \|_{L^{\infty}(0,T;L^p(\mathcal{F}(t)))}^2.
\]

In the third term the velocity is the linear combination of finitely many vector fields which are bounded. This implies that this term can be tackle easily. Finally the last term is bounded by energy estimates.

We showed that \( \partial_t \omega_\nu \) is uniformly bounded in \( \mathcal{L}^1_{loc}(\mathbb{R}^+;H^{r-\delta}(\mathcal{F}(t))) \), which implies together with (41) that

\[
\omega_\nu \longrightarrow \omega \quad \text{in} \quad \mathcal{L}^1_{loc}(\mathbb{R}^+;L^p - w(\mathcal{F}(t))),
\]

where \( L^p - w(\mathcal{F}(t)) \) denotes the \( L^p(\mathcal{F}(t)) \) space endowed with the weak topology. This implies that

\[
\mathcal{K}_0^\mathcal{F}(t)[\omega_\nu] \longrightarrow \mathcal{K}_0^\mathcal{F}(t)[\omega] \quad \text{in} \quad \mathcal{L}^r_{loc}(\mathbb{R}^+;L^p(\mathcal{F}(t))),
\]

as a consequence of Lemma 6.4 of [26]. Due to the \( \mathcal{L}^1_{loc}(0,T;W^{1-\delta/p,p}(\mathcal{F}(t))) \) convergence of \( g_\nu \) towards \( g \), we have that \( v_{g_\nu} \) converges to \( v_g \) in \( \mathcal{L}^r_{loc}(\mathbb{R}^+;L^p(\mathcal{F}(t))) \). Finally the convergence

\[
\sum_i \left[ \int_\mathcal{F} \Psi_i \omega_\nu + C_i(t) \right] X_i \longrightarrow \sum_i \left[ \int_\mathcal{F} \Psi_i \omega + C_i(t) \right] X_i \quad \text{in} \quad \mathcal{L}^r_{loc}(\mathbb{R}^+;L^p(\mathcal{F}(t)))
\]

is straight-forward. We deduce that

\[
v_\nu \longrightarrow v \quad \text{in} \quad \mathcal{L}^1_{loc}(\mathbb{R}^+;L^p(\mathcal{F}(t)))
\]

and by linearity of Div–Curl system it holds (20). Instead to pass to the limit in the weak formulation associated with system (36), we use a duality argument for the transport equation. The idea is that Theorem 18 ensures the existence of a solution \( (\bar{\omega}, \bar{\omega}^-) \) associated with the velocity field \( v \) and the data.
It remains to show that \((\omega, \omega^-) = (\bar{\omega}, \bar{\omega}^-)\). To do that we use a backward flow \((\phi_v, \phi_v^-)\), see for instance [11] or [9], solution of

\[
-\partial_t \phi_v - v_v \cdot \nabla \phi_v - \nu \Delta \phi_v = \chi \\
\nabla \phi_v \cdot n = - (\phi_v - \Psi)(g_v - q) \mathbb{1}_{\partial \mathcal{F}_v(t)} \\
\phi_v(T, x) = 0,
\]

where \(\chi\) and \(\Psi\) are smooth functions. And

\[
-\partial_t \phi - v \cdot \nabla \phi = \chi \\
\phi = \Psi \\
\phi(T, x) = 0,
\]

Then there exists a solution \(\phi_v\) of (42) in the sense of Lemma 11 and a distributional solution of (43).

The functions \(\phi_v\) satisfy estimates of the type (38), which imply a uniform bound in \(L^\infty(0, T; L^p(\mathcal{F}))\), moreover due to (42) \(\phi_v\) is uniformly continuous in some \(W^{-1, r}(\mathcal{F})\) for some \(r\). It follows that for a subsequence

\[
\phi_v \rightarrow \bar{\phi} = \phi \quad \text{in} \quad C^0([0, T]; L^q - w(\mathcal{F})) \quad \text{and} \quad (g_v - q)^{1/q} \overset{w}{\rightarrow} \bar{\psi} = |g - q|^{1/q} \text{ in } L^q((0, T) \cap \partial S^+(t))
\]

and the identification \((\bar{\phi}, |g - q|^{-1/q} \bar{\psi}) = (\phi, \phi^+)\) comes from the fact that we can pass to the limit in the weak formulation satisfied by \(\phi_v\) to show that \((\bar{\phi}, |g - q|^{-1/q} \bar{\psi})\) is a weak solution of (43) and conclude by uniqueness. Note that here we use the fact that \(\phi v \in L^1(\mathcal{F})\).

Using the duality formula from Theorem 20 with \(u = \bar{\omega}\) and \(r(t, \cdot) = \phi(T - t)\), we have

\[
\int_0^T \int_\mathcal{F} (\bar{\omega} - \omega) \chi + \int_0^T \int_{\partial \mathcal{F}^+} (g - q) \omega^- \Psi = \int_\mathcal{F} \omega^{in} \phi(0, \cdot) - \int_0^T \int_{\partial \mathcal{F}^+} (g - q) \omega^+ \phi^+
\]

Consider now the equation satisfied by \(\omega_v\), tested with \(\phi_v\). Due to the convergences previously showed we deduce

\[
\int_0^T \int_\mathcal{F} \omega^{in} \phi(0, \cdot) - \int_0^T \int_{\partial \mathcal{F}^+} (g - q) \omega^+ \phi^+.
\]

The right hand side of the two above equalities are the same. We deduce that

\[
\int_0^T \int_\mathcal{F} (\bar{\omega} - \omega) \chi + \int_0^T \int_{\partial \mathcal{F}^+} (g - q) (\omega^- - \bar{\omega}^-) \Psi = 0
\]

for any smooth function \(\chi\) and \(\Psi\). This shows \((\omega, \omega^-) = (\bar{\omega}, \bar{\omega}^-)\).

We move to the case of \(L^1\) vorticity.

**Proof of Theorem 3.** The proof of this result is similar to Theorem 4 of [9]. From (37) we deduce that \(G(\omega_v)\) and \(G(\omega^-_v)(g_v - q)\) are uniformly bounded respectively in \(L^\infty(\mathbb{R}^+; L^1(\mathcal{F}(t)))\) and \(L^1_{loc}(\mathbb{R}^+ \cap \partial S^-(t))\) for some superlinear convex functions \(G\). This together with De la Vallée Poussin’s Lemma, see Lemma 2 of [9], implies that \(\omega_v\) and \(\omega_v(g_v - q)\) converge respectively to \(\omega\) weakly-star in \(L^\infty(\mathbb{R}^+; L^1(\mathcal{F}(t)))\) and to \(\omega^- (g - q)\) weakly in \(L^1_{loc}(\mathbb{R}^+ \cap \partial S^-(t))\). Moreover using equation (36) to show some a priori bounds on the time derivative of \(\omega_v\), we deduce that, after extending by zero \(\omega_v\) in \(\mathbb{R}^2\), the product \(\omega_v(t, x) \omega_v(t, y) \overset{w}{\rightarrow} \omega(t, x) \omega(t, y)\) in \(L^\infty(0, T, \mathcal{M}(\mathbb{R}^2))\), where \(\mathcal{M}(\mathbb{R}^2)\) is the space of Random measures.

We can now pass to the limit in the weak formulation satisfied by (36) to conclude the proof. The only convergence which is not straight-forward is

\[
\int_{\mathbb{R}^+} \int_{\mathcal{F}(t)} \int_{\mathcal{F}(t)} H_p(x, y) \omega_v(t, x) \omega_v(t, y) \, dx \, dy \, dt
\]

\[
\rightarrow \int_{\mathbb{R}^+} \int_{\mathcal{F}(t)} \int_{\mathcal{F}(t)} H_p(x, y) \omega(t, x) \omega(t, y) \, dx \, dy \, dt,
\]

\(\square\)
in fact $H_\varepsilon$ is bounded from Lemma 2 but not continuous. Notice $H_\varepsilon$ is discontinuous only on the point of the diagonal. To pass to the limit is than enough to show that $\omega_\varepsilon(t,x)\omega_\varepsilon(t,y)$ does not concentrate along the diagonal, but this follow from (37). See also [9, 28] for more details.

6. Proof of Theorem 5

In this section we prove Theorem 5. The proof is dived in four steps. In the first one we show a priori bounds for $\omega_\varepsilon$, $\omega_\varepsilon^-$ and $v_\varepsilon$ from which we deduce weak convergence of the vorticities $\omega_\varepsilon$ and $\omega_\varepsilon^-$ in step two. In the third one we show strong convergence of the velocity $v_\varepsilon$. In the last one we explain how to pass to the limit in the weak formulation of (13) satisfied by $(\omega_\varepsilon, \omega_\varepsilon^-, v_\varepsilon)$ and derive system (16).

6.1. Uniform Bounds

In this subsection we prove uniform bounds for $\omega_\varepsilon$, $\omega_\varepsilon^-$ and $v_\varepsilon$ in appropriate spaces.

Lemma 13. Let $(\omega_\varepsilon, \omega_\varepsilon^-, v_\varepsilon)$ a weak solution of (16) satisfying the hypothesis of Theorem 5 and let $T > 0$. Then $\|\omega_\varepsilon\|_{L^\infty(0,T;L^p(F_\varepsilon(t)))}$ and $\|g_\varepsilon - q_\varepsilon\|^{1/p}_{L^p(0,T;L^p(\partial S^+_\varepsilon(t)))}$ are uniformly bounded respect to the parameter $\varepsilon$.

Proof. The proof is a direct consequence of the fact that $(\omega_\varepsilon, \omega_\varepsilon^-, v_\varepsilon)$ are renormalized solution to the transport equation of the vorticity.

We now prove a bound for the velocity field. To do that we decompose the time $\mathbb{R}^+$ and the velocity field in dependence of the size of the holes $S^i$.

6.1.1. Decomposition of the Time and of the Velocity Field.

In this section we prove a bound independent of $\varepsilon$ for the velocity field $v_\varepsilon$. Let us recall that $v_\varepsilon$ satisfies the Div–Curl system (33). Lemma 6 ensures that $v_\varepsilon$ is bounded in $W^{1, p}(F_\varepsilon(t))$ but this estimate depends on $\varepsilon$ and the constant explodes for times $t \in T^+ \cup T^-$, i.e. when $S^+(t)$ or $S^-(t)$ are points. The idea is to decompose $\mathbb{R}^+$ in three main regions. In the first one $r_\varepsilon^+, r_\varepsilon^- \geq c > 0$ and Lemma 6 implies an a priori bound for $v_\varepsilon$ independent of $\varepsilon$. The second one is when $r_\varepsilon^+ \sim r_\varepsilon^- \sim 0$ and we use the first part of Lemma 8. The last one when $r_\varepsilon^+ \sim 0$ and $r_\varepsilon^- \geq c > 0$, in this case we use the second part of Lemma 8. To do that we introduce some notation.

Given $(\Omega, S^+, S^-)$ we decompose the time $\mathbb{R}^+$ in dependence of the size of the holes $S^i$. We have already introduced $T^i = \{t \in \mathbb{R}^+ \text{ s.t. } r^i(t) = 0\}$, $T^i_{NP} = \{t \in \mathbb{R}^+ \text{ s.t. } r^i(t) > 0\}$, $T^i_{NP, \delta} = \{t \in \mathbb{R}^+ \text{ s.t. } r^i(t) \geq \delta\}$. We now define the “transition times” $T^i_{TR, \delta} = \{t \in \mathbb{R}^+ \text{ s.t. } 0 < r^i(t) < \delta\}$ and $T^i_{TR} = T^i_{TR, \delta} \cup T^i_{TR, \delta}$. Finally the “no-transition times” are $T^i_{NTR, \delta} = \mathbb{R}^+ \setminus T^i_{TR, \delta}$. All this informations are resumed in Figure 1, where $A = T^+ \cup T^-$, $B^i = T^i_{TR, \delta} \cap T^i_{NP, \delta}$, $C^i = T^i_{NP, \delta} \cap T^i_{NP}$, $D = T^i_{TR} \cap T^i_{TR, \delta}$, $E^i = T^i_{TR, \delta} \cap T^i_{NT, \delta}$, $F = T^i_{NP, \delta} \cap T^i_{NP, \delta}$ and for $i \in \{+,-\}$, we denote $g^i$ the unique element of $\{+, -\} \setminus \{i\}$.

In the non transition times $T^i_{NTR, \delta} = A \cup C^+ \cup C^- \cup F$ three situations can occur:

A. both the holes are points, in particular $t \in T^+ \cap T^- = A$.

\[ \begin{array}{c|c|c|c} \hline T^+ \iff r^+ = 0 & T^+_{TR, \delta} \iff 0 < r^+ < \delta & T^+_{NP, \delta} \iff r^+ \geq \delta \\ \hline T^- \iff r^- = 0 & A & B^+ \\ \hline T^+_{TR, \delta} \iff 0 < r^- < \delta & B^- & D \\ \hline T^+_{NP, \delta} \iff r^- \geq \delta & C^- & E^- \\ \hline \end{array} \]

FIG. 1. Table of times
C. one hole is a point the other has size greater or equal to \( \delta \). In this case for \( i \in \{+,-\} \), we denote 
\[
op_i := \text{the unique element of } \{+,-\}\setminus \{i\} \]
and \( t \in \bigcup_{i \in \{+,-\}} T^i \cap \mathcal{T}^i_{NP,\delta} = C^+ \cup C^- = T_{NP,\delta} \).
F. both the holes has size greater of equal to \( \delta \), in particular \( t \in T^+_{NP,\delta} \cap \mathcal{T}^+_{NP,\delta} = F \).

We are now able to rewrite the velocity field \( v_\varepsilon \) in dependence of the time interval. Recall that \( v_\varepsilon \) depends on \( \mathcal{F}_\varepsilon, \omega_\varepsilon, \mathcal{C}_{i,\varepsilon} \) and \( g_\varepsilon \) through the formula
\[
v_\varepsilon = K_{\mathcal{F}_\varepsilon}(t)[\omega_\varepsilon, \mathcal{C}_{i,\varepsilon}] + \mathcal{J}_{\mathcal{F}_\varepsilon}(t)[0, g_\varepsilon].
\]
To prove Theorem 5 we will show weak convergence for \( \omega_\varepsilon \) while for \( \mathcal{C}_{i,\varepsilon} \) and \( g_\varepsilon \) the convergence is a strong. We decompose
\[
v_\varepsilon = K_{\mathcal{F}_\varepsilon}(t)[\omega_\varepsilon, 0] + (K_{\mathcal{F}_\varepsilon}(t)[0, \mathcal{C}_{i,\varepsilon}] + \mathcal{J}_{\mathcal{F}_\varepsilon}(t)[0, g_\varepsilon]). \tag{44}
\]
to treat separately the more delicate term involving \( \omega_\varepsilon \). In the following we rewrite the right hand side of (44) in such a way that we can show some a priori estimates independent of \( \varepsilon \).

**Decomposition of** \( v_\varepsilon \) **in** \( A, B^+, B^- \) **and** \( D \).

For \( t \in A \cup B^+ \cup B^- \cup D \) the radii \( r^i_\varepsilon(t) \) are small, in particular \( \mathcal{F}_\varepsilon(t) \sim \Omega \). We expect that
\[
K_{\mathcal{F}_\varepsilon}(t)[\omega_\varepsilon, 0] \sim K_\Omega[\omega_\varepsilon], \quad K_{\mathcal{F}_\varepsilon}(t)[0, \mathcal{C}_{i,\varepsilon}] \sim K_\Omega[\mathcal{C}_{i,\varepsilon} \delta_{h^+} + \mathcal{C}_{i,\varepsilon} \delta_{h^-} - \mathcal{C}_{1,\varepsilon} \delta_{h^+}]
\]
and \( \mathcal{J}_{\mathcal{F}_\varepsilon}(t)[0, g_\varepsilon] \sim \mathcal{J}_\Omega \left[ \left( \int_{i,\varepsilon} \mathcal{C}_{i,\varepsilon} \delta_{h^+} - \int_{i,\varepsilon} \mathcal{C}_{i,\varepsilon} \delta_{h^-} \right) \right] \)
in fact any similar couple of velocity fields has the same divergence and vorticity in \( \mathcal{F}_\varepsilon(t) \), the same circulations around \( \partial S^i_\varepsilon(t) \) and the same quantity of entering and exiting fluid from \( \partial S^i_\varepsilon(t) \). To short the notation and make more understandable the proof we write explicitly the terms
\[
K_\Omega[\mathcal{C}_{i,\varepsilon} \delta_{h^+} + \mathcal{C}_{i,\varepsilon} \delta_{h^-}] + \mathcal{J}_\Omega \left[ \left( \int_{i,\varepsilon} \mathcal{C}_{i,\varepsilon} \delta_{h^+} - \int_{i,\varepsilon} \mathcal{C}_{i,\varepsilon} \delta_{h^-} \right) \right] = L_\varepsilon + B_{\Omega\varepsilon}[\varepsilon \cdot n]
\]
with
\[
L_\varepsilon(t, x) = \sum_{i \in \{+,-\}} \left( \int_{\partial S^i_\varepsilon(t)} g_\varepsilon \frac{x - h^i(t)}{2\pi|x - h^i(t)|^2} \right) + \sum_{i \in \{+,-\}} \left( C_{i,\varepsilon}^n - \int_0^t \int_{\partial S^i_\varepsilon(t)} \omega^i_\varepsilon(g_\varepsilon - q_\varepsilon) \right) \frac{(x - h^i(t))^i}{2\pi|x - h^i(t)|^2} \tag{45}
\]
And with \( B_{\Omega\varepsilon}[\varepsilon \cdot n] \) the solution of
\[
div (B_{\Omega\varepsilon}[\varepsilon \cdot n]) = 0 \quad \text{and} \quad \text{curl} (B_{\Omega\varepsilon}[\varepsilon \cdot n]) = 0 \quad \text{in} \quad \Omega
\]
and \( B_{\Omega\varepsilon}[\varepsilon \cdot n] \cdot n = -L_\varepsilon \cdot n \) on \( \partial \Omega \).

After this considerations, the equality (44) rewrites
\[
v_\varepsilon = K_\Omega[\omega_\varepsilon] + L_\varepsilon + B_{\Omega\varepsilon}[\varepsilon \cdot n] + w^1_\varepsilon, \tag{46}
\]
where \( w_\varepsilon \) satisfies
\[
div w^1_\varepsilon = 0 \quad \text{and} \quad \text{curl} w^1_\varepsilon = 0 \quad \text{in} \quad \mathcal{F}_\varepsilon(t),
\]
\[
w^1_\varepsilon \cdot n = g_\varepsilon - (K_\Omega[\omega_\varepsilon] + L_\varepsilon + B_{\Omega\varepsilon}[\varepsilon \cdot n]) \cdot n \quad \text{on} \quad \partial \mathcal{F}_\varepsilon(t)
\]
and
\[
\int_{\partial S^i_\varepsilon(t)} w^1_\varepsilon \cdot \tau = 0.
\]

The advantage of decomposition (46) is that \( K_\Omega[\omega_\varepsilon], L_\varepsilon \) and \( B_{\Omega\varepsilon}[\varepsilon \cdot n] \) do not depend on \( \mathcal{F}_\varepsilon(t) \).

**Decomposition of** \( v_\varepsilon \) **in** \( C^+, C^-, E^+ \) **and** \( E^- \).
For $i \in \{+,-\}$ and $t \in C^i \cup E^i$, we set $\tilde{F}(t) = \Omega \setminus S^i(t)$. Notice that for $t \in C^i$ we have $\tilde{F}(t) = F(t)$. In the same spirit as before, we decompose the velocity field

$$v_\varepsilon = \mathcal{K}_{\tilde{F}(t)}[\omega_\varepsilon, 0] + L^{op}_\varepsilon + \mathcal{B}_{\tilde{F}(t)}[g_\varepsilon - L^{op}_\varepsilon \cdot n] + w^2_\varepsilon$$

where

$$L^{op}_\varepsilon(t, x) = \left( \int_{\partial S^{op}_\varepsilon(t)} g_\varepsilon \right) \frac{x - h^{op}_\varepsilon(t)}{2\pi |x - h^{op}_\varepsilon(t)|^2} + \left[ \mathcal{C}^{in}_{op, i} - \int_0^t \int_{\partial S^{op}_\varepsilon(t)} \omega^{op}_\varepsilon (g_\varepsilon - q_\varepsilon) \right] \frac{(x - h^{op}_\varepsilon(t))^\perp}{2\pi |x - h^{op}_\varepsilon(t)|^2}.$$ 

The velocity field $\mathcal{B}_{\tilde{F}(t)}[g_\varepsilon - L^{op}_\varepsilon \cdot n]$ satisfies

$$\text{div} \mathcal{B}_{\tilde{F}(t)}[g_\varepsilon - L^{op}_\varepsilon \cdot n] = 0, \quad \text{curl} \mathcal{B}_{\tilde{F}(t)}[g_\varepsilon - L^{op}_\varepsilon \cdot n] = 0 \quad \text{in} \quad \tilde{F}(t),$$

$$\mathcal{B}_{\tilde{F}(t)}[g_\varepsilon - L^{op}_\varepsilon \cdot n] \cdot n = g_\varepsilon - L_\varepsilon \cdot n \quad \text{on} \quad \partial \tilde{F}(t)$$

and

$$\int_{\partial S^{op}_\varepsilon(t)} \mathcal{B}_{\tilde{F}}[g_\varepsilon - L^{op}_\varepsilon \cdot n] \cdot \tau = C_{i, \varepsilon}(t).$$

and finally

$$\text{div} w^2_\varepsilon = 0 \quad \text{and} \quad \text{curl} w^2_\varepsilon = 0 \quad \text{in} \quad F_\varepsilon(t), \quad w^2_\varepsilon \cdot n = 0 \quad \text{on} \quad \partial S^i(t)$$

$$w^2_\varepsilon \cdot n = g_\varepsilon - (\mathcal{K}^0_{\tilde{F}(t)}[\omega_\varepsilon] + L^{op}_\varepsilon + \mathcal{B}_{\tilde{F}(t)}[g_\varepsilon - L^{op}_\varepsilon \cdot n]) \cdot n \quad \text{on} \quad \partial S^{op}_\varepsilon(t)$$

and

$$\int_{\partial S^{op}_\varepsilon(t)} w^2_\varepsilon \cdot \tau = 0,$$

for $j \in \{+,-\}$.

**Decomposition of $v_\varepsilon$ in $F$.**

For $t \in T_{NP, \delta}^- \cap T_{NP, \delta}^- = F$ and small enough $\varepsilon$ we have $F_\varepsilon(t) = F(t)$. If we write

$$\mathcal{B}_{\tilde{F}(t)}[g_\varepsilon] = \mathcal{K}_{\tilde{F}(t)}[0, C_{i, \varepsilon}] + J_{\tilde{F}(t)}[0, g_\varepsilon],$$

then (44) rewrites

$$v_\varepsilon = \mathcal{K}_{\tilde{F}(t)}[\omega_\varepsilon, 0] + \mathcal{B}_{\tilde{F}(t)}[g_\varepsilon].$$

In Fig. 2, we resume the decompositions of the velocity field $v_\varepsilon$ in dependence of time.

Now we will show that the velocity field $v_\varepsilon$ is uniformly bounded in $L^r_{loc}(\mathbb{R}^+; L^q(F(t)))$ and if we restrict to the “transition times” $T_{TR, \delta}$ then $L^r_{loc}(T_{TR, \delta}; L^q(F(t)))$ norm of $v_\varepsilon$ converges to zero uniformly in $\varepsilon$ as $\delta$ goes to zero.

**Proposition 14.** Let $\delta$ small enough. Then $v_\varepsilon$ is uniformly bounded in $L^r_{loc}(\mathbb{R}^+; L^q(F(t)))$. Moreover $w^1_\varepsilon \rightarrow 0$ in $L^r_{loc}(T^+ \cap T^-; L^q(F(t)))$, $w^2_\varepsilon \rightarrow 0$ in $L^r_{loc}(T_{PN, \delta}; L^q(F(t)))$ as $\varepsilon$ converges to zero and

$$\|v_\varepsilon\|_{L^r_{loc}(T_{TR, \delta}, L^q(F(t)))} \rightarrow 0 \quad \text{as} \quad \delta \rightarrow 0$$

uniformly in $\varepsilon$. 

![Fig. 2. Decomposition of the velocity field](image-url)
Proof. Fix a time $T > 0$, in the following we prove the estimates in the compact time interval $[0, T]$. To show the uniform bound we show the bound for the right hand sides of (48)-(46)-(47). Fix $2 \delta \leq \kappa_T$ with $\kappa_T$ from Lemma 8.

Case $t \in F$. 
For $t \in T_{NP, \delta}^+ \cap T_{NP, \delta}^- \cap [0, T]$ the decomposition (48) holds, in particular
\[
\|v_\varepsilon\|_{L^q(F(t))} \leq \|\mathcal{K}_\Omega[\omega_\varepsilon] + B_{\mathcal{F}(t)}[g_\varepsilon]\|_{L^q(F(t))} \\
\leq C \left( \|\omega_\varepsilon\|_{L^q(F(t))} + \|g_\varepsilon\|_{L^q(\partial F(t))} + |C_{+\varepsilon}| + |C_{-\varepsilon}| \right)
\]
and after taking the $L^r$-norm in time and using Lemma 13 and the hypothesis on $g_\varepsilon$, we deduce the desired estimates in this zone.

Case $t \in A$. 
For $t \in T^+ \cap T^-$ and for $\varepsilon < \delta$, we use the decomposition (46) in the following way. For $\mathcal{K}_\Omega[\omega_\varepsilon]$ the estimates is clear, the uniform bound of $L_\varepsilon$ comes from the fact that it is the sum of the products of an $L^r$ in time functions for example $\int_{\partial S_t^i} g_\varepsilon$ and vector fields of the type $(x - h^i(t))/|x - h^i(t)|^2$ which are bounded in $L^q(\Omega)$ ($q < 2$) uniformly in $h^i(t) \in \Omega$. The vector field $B_{\mathcal{F}(t)}[-L_\varepsilon \cdot n] = \mathcal{B}_{\mathcal{F}(t)}[-L_\varepsilon \cdot n]$ is associated with $L_\varepsilon$ throw its values on $\partial \Omega$, in particular they are bounded in $L^r_{loc}(L^q(\partial \Omega))$ because $h^i(t) \in K$ a set compactly contained in $\Omega$. We are left with the uniform bound for $w_\varepsilon$. In this case we actually prove that $\omega_\varepsilon^1 \rightarrow 0$ in $C^r_{loc}(T^+ \cap T^-, L^q(\mathcal{F}(t)))$. To prove this convergence we notice that for $\varepsilon < 2 \delta$ we are in the setting to apply Lemma 8, in particular it holds
\[
\|w_\varepsilon^1\|_{L^q(F_\varepsilon(t))} \leq C \varepsilon^{1/q} \|g_\varepsilon - (\mathcal{K}_\Omega[\omega_\varepsilon] + L_\varepsilon + B_{\mathcal{F}(t)}[-L_\varepsilon \cdot n])\|_{L^q(\partial F_\varepsilon(t))} \\
\leq C \varepsilon^{1/q} \left( \|g_\varepsilon\|_{L^q(\partial F_\varepsilon(t))} + \|\mathcal{K}_\Omega[\omega_\varepsilon]\|_{L^q(\partial F_\varepsilon(t))} \right) \\
+ \|L_\varepsilon \cdot n\|_{L^q(\partial F_\varepsilon(t))} + \|B_{\mathcal{F}(t)}[-L_\varepsilon \cdot n]\|_{L^q(\partial F_\varepsilon(t))}
\]
and the right hand side tends to zero in $L^r$, in fact $\varepsilon^{1/q} \|g_\varepsilon\|_{L^q(\partial F_\varepsilon(t))} \rightarrow 0$ by hypothesis (31), $\mathcal{K}_\Omega[\omega_\varepsilon]$ and $B_{\mathcal{F}(t)}[-L_\varepsilon \cdot n]$ are uniformly bounded in $L^\infty$ in any compact subset of $\Omega$ which implies that $\varepsilon^{1/q} \|\mathcal{K}_\Omega[\omega_\varepsilon] \cdot n\|_{L^q(\partial F_\varepsilon(t))} \rightarrow 0$ and the estimates for $L_\varepsilon$ follows from
\[
\varepsilon^{1/q} \left\| \frac{x - h^i(t)}{2\pi |x - h^i(t)|^2} \right\|_{L^q(\partial S_t^i)} = \varepsilon^{1/q} \varepsilon^{1/q - 1} \left\| \frac{x - h^i(t)}{2\pi |x - h^i(t)|^2} \right\|_{L^q(\partial S_t^i)} \rightarrow 0,
\]
because $q < 2$.

Case $t \in C^+ \cup C^-$. 
For $t \in T_{NP, \delta} = C^+ \cup C^-$ we assume, without loss of generality, that the i-th hole is a point and we assume $\varepsilon$ sufficiently small to apply the second part of Lemma 8. As usual we rewrite $v_\varepsilon$ with the help of (47). The estimate of the right hand side of (47) can be performed in the similar way to the case $t \in T^+ \cap T^- = A$, in particular we estimate
\[
\|B_{\mathcal{F}(t)}[g_\varepsilon - L_\varepsilon \cdot n]\|_{L^q(\mathcal{F}(t))} \leq C \|g_\varepsilon - L_\varepsilon \cdot n\|_{L^q(\partial \mathcal{F}(t))}
\]
and for $w_\varepsilon^2$ we apply the second part of Lemma 8.

Case $t \in B^+ \cup B^- \cup C \cup E^+ \cup E^-$. 
For $t \in B^+ \cup B^- \cup C$, we use the decomposition (46) and note that we are in the setting to use Lemma 8. We estimate the right hand side of (46) as follow,
\[
\|\mathcal{K}_\Omega[\omega_\varepsilon]\|_{L^r(T_{R, A}; L^\infty(\Omega))} \leq C \|\omega_\varepsilon\|_{L^r(T_{R, A}; L^\infty(\Omega))} \\
\leq C \left( \int_{\Omega} \left| B_{\mathcal{F}(t)}[-L_\varepsilon \cdot n] \right| \right)^{1/r} \|\omega_\varepsilon\|_{L^\infty(T_{R, A}; L^\infty(\Omega))}.
\]
Note that the last term in the series of inequalities is bounded uniformly in $\varepsilon$ due to Lemma 13 and $\operatorname{Leb}^1(T_{R, A})$, which denotes the Lebesgue measure of $T_{R, A}$, converges to zero. To estimate $L_\varepsilon$ and $B_{\mathcal{F}(t)}[-L_\varepsilon \cdot n]$ we notice that
\[
\|L_\varepsilon + B_{\mathcal{F}(t)}[-L_\varepsilon \cdot n]\|_{L^r(T_{R, A}; L^\infty(\Omega))}
\]
\[
\leq C \sum_{i \in \{+,-\}} \left( \left\| \oint_{\partial S_i^1(t)} g_{\varepsilon} \right\|_{L^r(T_{TR,d})} + \left\| C_{i,e}^{in} - \int_0^T \oint_{\partial S_i^1(t)} \omega_{\varepsilon}^i (g_{\varepsilon} - q_{\varepsilon}) \right\|_{L^r(T_{TR,d})} \right)
\]

\[
\leq C \sum_{i \in \{+,-\}} \left( \left( r_{\varepsilon}^i (t) \right)^{1-1/q} \left\| g_{\varepsilon} \right\|_{L^p(\partial S_i^1(t))} \left\| \omega_{\varepsilon}^i \right\|_{L^r(T_{TR,d})} \right)
\]

\[
+ C \left( \text{Leb}^1(T_{TR,d}) \right)^{1/r} \left[ \left\| \omega_{\varepsilon}^{in} \right\| \right]
\]

\[
+ C \left( \text{Leb}^1(T_{TR,d}) \right)^{1/r} \left[ \left| \int_0^T \oint_{\partial S_i^1(t)} |\omega_{\varepsilon}^i|^p g_{\varepsilon} - q_{\varepsilon} |^{1/p} + \left| \int_0^T \oint_{\partial S_i^1(t)} |g_{\varepsilon} - q_{\varepsilon}|^{1/q} \right| \right] ,
\]

which converges uniformly in \( \varepsilon \) to zero due to hypothesis (32), Lemma 13 and the fact that the measure of \( T_{TR,d} \) converges to zero.

We are left with the reminder \( w_\varepsilon^2 \) for which the uniform convergence to zero follows from Lemma 8, together with Lemma 13 and hypothesis (32) applied in the same style as above to estimate

\[
\sum_{i \in \{+,-\}} r_{\varepsilon}^i (t)^{1/q} \left\| g_{\varepsilon} - (K_{\Omega} \omega_{\varepsilon} + L_{\varepsilon} + \mathcal{B}_F [-L_{\varepsilon} \cdot n] \cdot n) \right\|_{L^p(\partial S_i^1(t))} .
\]

For \( t \in E^+ \cup E^- \) the estimates are similar. \( \square \)

We now present an a priori bound for the time derivative of the vorticity.

**Lemma 15.** Let \((\omega_\varepsilon, \omega_{\varepsilon}^-, v_\varepsilon)\) a weak solution of (13) satisfying the hypothesis of Theorem 5 and let \( T > 0 \). Then \( \| \partial_t \omega_\varepsilon \|_{L^1(0,T;W^{-2, r}(\mathbb{R}^2))} \) is uniformly bounded in \( \varepsilon \) for \( l, s \) big enough.

**Proof.** This is a consequence of equation (13), Lemma 13 and Proposition 14. The triple \((\omega_\varepsilon, \omega_{\varepsilon}^-, v_\varepsilon)\) is a weak solution of system (13), we deduce that

\[
\left| \langle \partial_t \omega_\varepsilon, \varphi \rangle \right| = \left| \int_0^T \int_{\mathcal{F}_e} \omega_\varepsilon \partial_t \varphi dx dt - \int_{\mathcal{F}_e(T, \cdot)} \omega_\varepsilon (T, \cdot) \varphi (T, \cdot) dx + \int_{\mathcal{F}_e(t)} \omega_{\varepsilon}^{in} \varphi (0, \cdot) dx \right|
\]

\[
\leq \left| \int_0^T \int_{\mathcal{F}_e(t)} v_\varepsilon \cdot \nabla \varphi \omega_\varepsilon dx dt \right| + \sum_{i \in \{+,-\}} \left| \int_0^T \oint_{\partial S_i^1(t)} (g_{\varepsilon} - q_{\varepsilon}) \omega_{\varepsilon}^i \varphi ds dt \right| .
\]

The terms in the right hand side above can be estimated as follows.

\[
\left| \int_0^T \int_{\mathcal{F}_e(t)} v_\varepsilon \cdot \nabla \varphi \omega_\varepsilon \right| \leq C \left\| \omega_\varepsilon \right\|_{L^\infty(0,T;L^p(\mathcal{F}_e(t)))} \left\| v_\varepsilon \right\|_{L^r(0,T;L^s(\mathcal{F}_e(t)))} \left\| \nabla \varphi \right\|_{L^{r'}(0,T;L^\infty(\mathcal{F}_e(t)))} ,
\]

and

\[
\left| \int_0^T \oint_{\partial S_i^1(t)} (g_{\varepsilon} - q_{\varepsilon}) \omega_{\varepsilon}^i \varphi \right| \leq C \left\| g_{\varepsilon} - q_{\varepsilon} \right\|_{L^p(0,T;L^1(\partial S_i^1(t)))} \left\| \varphi \right\|_{L^{r'}(0,T;L^\infty(\partial S_i^1(t)))} .
\]

The uniform boundedness of the above quantities follows from Lemma 13, Proposition 14 and hypothesis (29)-(31)-(32). \( \square \)

**Remark 5.** Let us notice that in contrast to the proof of Theorem 1, the uniform bound of the time derivative of the vorticity is much easier because the term \( \omega_\varepsilon v_\varepsilon \) is uniformly bounded in \( L^1 \) which was not the case in the proof of Theorem 1.
6.2. Weak and Strong Convergence of the Vorticities

From the uniform bounds of Lemma 13 we derive weak convergences of the vorticity in the following sense.

**Proposition 16.** Let \((\omega_\varepsilon, \omega^-_\varepsilon, v_\varepsilon)\) a weak solution of (13) satisfying the hypothesis of Theorem 5. Then up to subsequence \(\omega_\varepsilon\) converges weakly-star to \(\omega\) in \(L^p_{loc}(\mathbb{R}^+; L^p(F(t)))\) and strongly in \(C^0_{loc}(\mathbb{R}^+; L^p - w(F(t)))\) and for any small enough \(\delta\) the function \((g_\varepsilon - q_\varepsilon)^{1/p} \omega^-_\varepsilon\) converges weakly to \((g - q)^{1/p} \omega^-\) in \(L^p_{loc}(T_{NP,\delta}; L^p(\partial S^-)(t))\).

**Proof.** The weak-star convergence of \(\omega_\varepsilon\) is a direct consequence of Lemma 13, after noticing that

\[
L^\infty_{loc}(\mathbb{R}^+; L^p(F(t))) \cong \left( L^1_{loc}(\mathbb{R}^+; L^{p/(p-1)}(F(t))) \right)^*.
\]

The strong convergence in \(C^0_{loc}(\mathbb{R}^+; L^p - w(F(t)))\) is a consequence of Lemma 13 and 15 together with Lemma C.1 of [23].

6.3. Strong Convergence for the Velocity

This subsection is dedicated to the proof of strong convergence for the the velocity field \(v_\varepsilon\).

**Proposition 17.** Let \((\omega_\varepsilon, \omega^-_\varepsilon, v_\varepsilon)\) a weak solution of (13) satisfying the hypothesis of Theorem 5. Then up to subsequence \(v_\varepsilon\) converges strongly to \(v\) in \(L^r_{loc}(\mathbb{R}^+; L^q(F(t)))\). Moreover

\[
v = \mathcal{J}_{F(t)} \left[ \mu \mathbf{1}_{T^+} \delta_{h^+(t)} - \left( \int_{\partial S^+(t)} g \mathbf{1}_{T^+} + \mu \mathbf{1}_{T^-} \right) \mathbf{1}_{T^-} \delta_{h^-(t)} \right] + \mathcal{K}_{F(t)} \left[ \omega + \mathcal{C}_+ \mathbf{1}_{T^+} \delta_{h^+(t)} + \mathcal{C}_- \mathbf{1}_{T^-} \delta_{h^-(t)} \right].
\]

**Proof.** Proposition 14 shows that

\[
\|v_\varepsilon\|_{L^r_{loc}(T_{TR,\delta}; L^q(F(t)))} \to 0 \quad \text{as} \quad \delta \to 0
\]

uniformly in \(\varepsilon\). This implies that it is enough to show the convergence of \(v_\varepsilon\) in the non-transition zone \(T_{NP,\delta}\) for any small enough fixed \(\delta\). We divide the non-transition zone in three parts as in the beginning of Sect. 6.1.

**Case** \(t \in F\).

In \(T_{NP,\delta}^+ \cap T_{NP,\delta}^- = F\), the velocity field \(v_\varepsilon = \mathcal{K}_{F(t)}[\omega_\varepsilon, 0] + \mathcal{B}_{F(t)}[g_\varepsilon]\), moreover

\[
\|B_{F(t)}[g_\varepsilon] - B_{F(t)}[g]\|_{L^q(F(t))} = \|B_{F(t)}[g_\varepsilon] - g\|_{L^q(F(t))} \leq C\|g_\varepsilon - g\|_{L^1(\partial F(t))} \to 0,
\]

in \(L^1_{loc}(T_{NP,\delta}^+ \cap T_{NP,\delta}^-)\) by hypothesis (29).

Regarding \(\mathcal{K}_{F(t)}[\omega_\varepsilon, 0]\), the strong convergence in \(L^r_{loc}(\mathbb{R}^+; L^q(F(t)))\) follows from the \(C^0_{loc}(\mathbb{R}^+; L^p - w(F(t)))\) convergence of \(\omega_\varepsilon\) together with Lemma 6.4 of [26].

**Case** \(t \in A\).

In \(T^+ \cap T^-\), the velocity field \(v_\varepsilon = \mathcal{K}_\Omega[\omega_\varepsilon] + L_\varepsilon + \mathcal{B}_\Omega[-L_\varepsilon \cdot n] + w_\varepsilon^1\). We have already shown that \(\mathcal{K}_\Omega[\omega_\varepsilon] \to \mathcal{K}_\Omega[\omega]\) and \(w_\varepsilon^1 \to 0\) in \(L^1_{loc}(T^+ \cap T^-, L^q(\Omega))\). It remains to show the strong convergence of \(L_\varepsilon\) and \(\mathcal{B}_\Omega[-L_\varepsilon \cdot n]\), in particular the strong convergence of \(L_\varepsilon\) implies also the one of \(\mathcal{B}_\Omega[-L_\varepsilon \cdot n]\). The vector field \(L_\varepsilon\) is defined in (45). From hypothesis (30), we have

\[
\sum_{i \in \{+,-\}} \int_{\partial S_i(t)} g_\varepsilon \frac{x - h^i(t)}{2\pi|x - h^i(t)|^2} \to \sum_{i \in \{+,-\}} i \mu \frac{x - h^i(t)}{2\pi|x - h^i(t)|^2}
\]

\[\varepsilon\] Birkhäuser
in $L^r_{loc}(T^+ \cap T^-, L^q(\Omega))$, it remains to show that
\[
\mathcal{C}^{in}_{+, \varepsilon} - \int_0^t \int_{\partial S^+(t)} \omega^+_{\varepsilon}(g_\varepsilon - q_\varepsilon) \, d\mathcal{H}^1 \exp(f_{\partial S^+(t)}) \mathbf{1}_{T^+} + j \mathbf{1}_{T^+})
\]
and
\[
\mathcal{C}^{in}_{-, \varepsilon} - \int_0^t \int_{\partial S^-_+(t)} \omega^-_{\varepsilon}(g_\varepsilon - q_\varepsilon) \, d\mathcal{H}^1 \exp(f_{\partial S^-_+(t)}) \mathbf{1}_{T^+} + j \mathbf{1}_{T^+})
\]
in $L^r_{loc}(T^+ \cap T^-)$. Note that the second convergence follows from the first one plus the weak formulation of the first equation of (13) tested with the constant function 1. For $\delta > 0$, we have
\[
\int_0^t \int_{\partial S^+_+(t)} \omega^+_{\varepsilon}(g_\varepsilon - q_\varepsilon) \mathbf{1}_{T^+} \exp(f_{\partial S^+_+(t)}) \mathbf{1}_{T^+} + j \mathbf{1}_{T^+}) + \int_{\partial S^-_+(t)} \omega^-_{\varepsilon}(g_\varepsilon - q_\varepsilon) \mathbf{1}_{T^+} + j \mathbf{1}_{T^+})\]
\[
\mathbf{1}_{T^+} \exp(f_{\partial S^+_+(t)}) \mathbf{1}_{T^+} + j \mathbf{1}_{T^+})
\]
\[
\int_0^t \int_{\partial S^-_+(t)} \omega^-_{\varepsilon}(g_\varepsilon - q_\varepsilon) \mathbf{1}_{T^+} + j \mathbf{1}_{T^+})
\]
Finally we use (31)-(32) to show that the remaining term converge to zero in $L^\infty$ and in particular in $L^r_{loc}(\mathbb{R}^+)$. It holds
\[
\left| \int_{T_{R,2\delta}} \int_{\partial S^+_+(t)} (g_\varepsilon - q_\varepsilon) \omega^+_{\varepsilon} \right| \leq C \left| \int_{T_{R,2\delta}} \int_{\partial S^+_+(t)} |g_\varepsilon - q_\varepsilon| |\omega^+_{\varepsilon}|^p |\mathbf{1}_{T^+} + j \mathbf{1}_{T^+}) + \int_{\partial S^-_+(t)} |g_\varepsilon - q_\varepsilon||1/q \leq C \left( \text{Leb}^1(T_{R,2\delta}) \right)^{1-1/r} \int_{T_{R,2\delta}} \left( \frac{r_\varepsilon}{\varepsilon} \right)^{r-1/r} \|g_\varepsilon - q_\varepsilon\|_{L^q(\partial S^+_+(t))} \right)^{r/q}
\]
we deduce that the right hand side converge to zero as $\delta$ converges to zero uniformly in $\varepsilon$. This allows us to conclude that the convergence holds.

**Case $t \in C^+ \cup C^-$.**

In $T_{NP, \delta}$, we recall from (47) that $v_\varepsilon = K_{\mathcal{F}(t)}[\omega_\varepsilon, 0] + L_\varepsilon + B_{\mathcal{F}(t)}[g_\varepsilon - L_\varepsilon \cdot n] + w^2$. We start by noticing that the convergence of $L_\varepsilon$ and $B_{\mathcal{F}(t)}[-L_\varepsilon \cdot n]$ follows analogously to the time $T^+ \cap T^-$, then the convergence of $K_{\mathcal{F}(t)}[\omega_\varepsilon, 0]$ and $B_{\mathcal{F}(t)}[g_\varepsilon]$ is shown as for $t \in T_{NP, \delta}$. The term $w^2$ converges to zero due to Proposition 14.

**6.4. Passing to the Limit in the Weak Formulation**

Consider a sequence of approximate solutions $(\omega_\varepsilon, \omega^-_\varepsilon, v_\varepsilon)$ satisfying the hypothesis of Theorem 5. We use the weak convergence of the vorticities and the strong one of the velocity to pass to the limit in the
weak formation. Recall that a weak solution of (13) satisfies the integral equation
\[ \int_{F_\varepsilon(0)} \omega_\varepsilon^0 \varphi(0, \cdot) + \int_{\mathbb{R}^+} \int_{F_\varepsilon(t)} \omega_\varepsilon \varphi \, dt + \int_{\mathbb{R}^+} \int_{F_\varepsilon(t)} v_\varepsilon \cdot \nabla \varphi_\varepsilon = \int_{\mathbb{R}^+} \int_{\partial S_\varepsilon^+(t)} (g_\varepsilon - q_\varepsilon) \omega_\varepsilon^+ \varphi + \int_{\mathbb{R}^+} \int_{\partial S_\varepsilon^-(t)} (g_\varepsilon - q_\varepsilon) \omega_\varepsilon^- \varphi. \]

From the weak convergence of the vorticity \( \omega_\varepsilon \), together with the strong convergence of the velocity proved in Proposition 17, we can pass to the limit in the left hand side. Regarding the right hand side the first term passes to the limit by hypothesis, in fact \( \omega_\varepsilon^\pm \) is not an unknown of the problem so we need only to show that
\[ \int_{\mathbb{R}^+} \int_{\partial S_\varepsilon^-(t)} (g_\varepsilon - q_\varepsilon) \omega_\varepsilon^- \varphi \rightarrow \int_{T_{NP}^-} \int_{\partial S^-(t)} \omega^- (g - q) \varphi \]
\[ + \int_{T^-} \left( \frac{d}{dt} \int \omega + \int_{\partial S^+(t)} \omega^+ (g - q) \mathbf{1}_{T_{NP}^+} + \frac{\mathbf{1}_{T^+}}{2} \right) \varphi(t, h^-(t)) \]

First of all for transition times we have
\[ \left| \int_{T_{TR}, \delta} \int_{\partial S_\varepsilon^-(t)} (g_\varepsilon - q_\varepsilon) \omega_\varepsilon^- \varphi \right| \]
\[ \leq \left\| (g_\varepsilon - q_\varepsilon)^{1/p} \omega_\varepsilon^- \right\|_{L^p_{loc}(T_{TR}, \delta; L^p(\partial S_\varepsilon(t)))} \left\| (g_\varepsilon - q_\varepsilon)^{1/q} \right\|_{L^q_{loc}(T_{TR}, \delta; L^1(\partial S_\varepsilon(t)))}
\leq \left\| (g_\varepsilon - q_\varepsilon)^{1/p} \omega_\varepsilon^- \right\|_{L^p_{loc}(T_{TR}, \delta; L^p(\partial S_\varepsilon(t)))}
\times \left\| (r_\varepsilon)^{1-1/q} \left\| g_\varepsilon - q_\varepsilon \right\|_{L^q(\partial S_\varepsilon^-(t))} \right\|_{L^q_{loc}(T_{TR}, \delta)} \]
\[ \rightarrow 0, \]
uniformly in \( \delta \) where we use hypothesis (32). For time in \( T_{NP, \delta}^- \) it holds
\[ \int_{T_{NP, \delta}^-} \int_{\partial S_\varepsilon^-(t)} (g_\varepsilon - q_\varepsilon) \omega_\varepsilon^- \varphi \rightarrow \int_{T_{NP, \delta}^-} \int_{\partial S_\varepsilon^-(t)} (g - q) \omega^\varphi \varphi, \]
from weak convergence. We are left with the times in \( T^- \). We have
\[ \int_{T^-} \int_{\partial S_\varepsilon^-(t)} (g_\varepsilon - q_\varepsilon) \omega_\varepsilon^- \varphi = \int_{T^-} \int_{\partial S_\varepsilon^-(t)} (g_\varepsilon - q_\varepsilon) \omega_\varepsilon^- \varphi(\cdot, h^-(\cdot)) \]
\[ + \int_{T^-} \int_{\partial S_\varepsilon^-(t)} (g_\varepsilon - q_\varepsilon) \omega_\varepsilon^- (\varphi - \varphi(\cdot, h^-(\cdot))). \]

The last term converge to zero, in fact
\[ \left| \int_{T^-} \int_{\partial S_\varepsilon^-(t)} (g_\varepsilon - q_\varepsilon) \omega_\varepsilon^- (\varphi - \varphi(\cdot, h^-(\cdot))) \right| \]
\[ \leq \left\| (g_\varepsilon - q_\varepsilon)^{1/p} \omega_\varepsilon^- \right\|_{L^p_{loc}(T^-; L^p(\partial S_\varepsilon(t)))}
\times \left\| (g_\varepsilon - q_\varepsilon)(\varphi - \varphi(\cdot, h^-(\cdot))) \right\|_{L^1(T^-; L^1(\partial S_\varepsilon(t)))}
\]
\[ \leq \left\| (g_\varepsilon - q_\varepsilon)^{1/p} \omega_\varepsilon^- \right\|_{L^p_{loc}(T^-; L^p(\partial S_\varepsilon(t)))}
\times \left\| (g_\varepsilon - q_\varepsilon) \right\|_{L^q_{loc}(T^-; L^1(\partial S_\varepsilon(t)))} \]
\[ \rightarrow 0, \]
where $L$ is the Lipschitz constant of $\varphi$. For the remaining term we have

$$\int_{\partial S^{-}_e(t)} (g_e - q_e) \omega^-_e$$

is uniformly bounded in $L^p_{loc}(T^-)$, so

$$\int_{T^-} \int_{\partial S^-_e(t)} (g_e - q_e) \omega^-_e \varphi(\cdot, h^-(\cdot)) \longrightarrow \int_{T^-} j^- \varphi(\cdot, h^-(\cdot))$$

up to subsequence. We have now to identify the limit. Note that

$$\frac{d}{dt} \int_{\mathcal{F}(t)} \omega \in L^s_{loc}(\mathbb{R}^+),$$

for some $s > 1$. Moreover using the equation for any $\psi \in C^\infty_c(\text{int}(T^-))$, it holds

$$\int_{\mathbb{R}^+} j1_{T^-} \psi = - \int_{\mathbb{R}^+} \left( \frac{d}{dt} \int_{\mathcal{F}(t)} \omega + \int_{\partial S^+ (t)} \omega^+(g - q) 1_{T^+_NP} + j1_{T^+} \right) \psi.$$
A. Renormalized Solution for Transport Equation with In–Out Flow and Time Dependent Domains

In this appendix we notice that the DiPerna-Lions theory [14] on renormalized solutions for the transport equation extends to the case of smooth enough time dependent domains with incoming and outgoing flows. In particular we will extend Section 3 of [6] for time depending domains.

In this section we do not restrict to regular compatible geometry \((\Omega, S^+, S^-)\) but we consider a domain \(\mathcal{F}(t)\) such that there exists \((\text{Id,} b) : \mathbb{R}^+ \times \partial \mathcal{F}(0) \to \mathbb{R}^+ \odot \partial \mathcal{F}(t)\) a \(C_{loc}^{1,\alpha}(\mathbb{R}^+, C^{1,\alpha}(\partial \mathcal{F}(t))) \cap C_{loc}^{0,\alpha}(\mathbb{R}^+, C^{3,\alpha}(\partial \mathcal{F}(t)))\) diffeomorphism with \(b(0, y) = y\) in \(\partial \mathcal{F}(0)\). For the velocity field \(v\) we assume, for \(p \in [1, \infty)\) that

\[
v \in \mathcal{L}_{loc}^1(\mathbb{R}^+, W^{1,p}(\mathcal{F}(t))) \quad \text{and} \quad \text{div} \, v = 0.
\]

Moreover, \(v\) satisfies one of the following hypothesis.

1. (Geometric condition). There exists \(\psi \in C^\infty(\mathbb{F})\) such that \(\psi = 1\) on \(\partial \mathcal{F}^+\) and \(\psi = 0\) on \(\partial \mathcal{F}^-\).
2. (Extra regularity). There exists \(\alpha > 1\) such that

\[
v \cdot n \in \mathcal{L}_{loc}^\alpha(\mathbb{R}^+ \odot \partial \mathcal{F}(t)),
\]

where we denote by \(\partial \mathcal{F}^+ = \{(t, x) \in \mathbb{R}^+ \odot \partial \mathcal{F}(t)\text{ such that } v \cdot n - q < 0\}\) and analogously \(\partial \mathcal{F}^- = \{(t, x) \in \mathbb{R}^+ \odot \partial \mathcal{F}(t)\text{ such that } v \cdot n - q > 0\}\). Note that we use this convention to recall that in \(\partial \mathcal{F}^+\) the flow is entering and in \(\partial \mathcal{F}^-\) is exiting.

Given \(\rho^{in}\) and \(\rho^+\) measurable functions respectively on \(\partial \mathcal{F}(0)\) and \(\partial \mathcal{F}^+\), we look for solutions \((\rho, \rho^-)\) to the transport equation

\[
\partial_t \rho + v \cdot \nabla \rho = f \quad \text{for } x \in \mathcal{F}(t),
\rho = \rho^+ \quad \text{for } x \in \partial \mathcal{F}^+(t),
\rho(0, \cdot) = \rho^{in} \quad \text{for } \mathcal{F}(0).
\]

Let \(\mu > 0\) a measurable function on \(\Omega\) such that \(\int_\Omega \mu < +\infty\). We denote by

\[
L^0(\Omega, \mu dL) = \{ f : \Omega \to \mathbb{R} \text{ L-measurable} \}.
\]

We are now stating a well-posedness result for (50) with initial and boundary data in \(L^0\) spaces.

**Theorem 18.** Let \(v\) a vector field in \(\mathbb{R}^+ \odot \mathcal{F}(t)\) satisfying the hypothesis (49) and either the geometric condition or the extra regularity. Let \(\rho^{in} \in L^0(\mathcal{F}(0))\) an initial datum, \(\rho^+ \in \mathcal{L}_{loc}^1(\mathbb{R}^+ \odot \partial \mathcal{F}^+(t), |v \cdot n - q| \, dt \, ds)\) an entering information and let \(f \in \mathcal{L}_{loc}^1(\mathbb{R}^+ \odot \partial \mathcal{F}(t))\) a source term, then there exists a unique renormalized solution \((\rho, \rho^-) \in \mathcal{L}_{loc}^1(\mathbb{R}^+; L^1(\mathcal{F}(t))) \times \mathcal{L}_{loc}^1(\mathbb{R}^+ \odot \partial \mathcal{F}^-(t), |v \cdot n - q| \, dt \, ds)\). More precisely

for any \(\varphi \in C^\infty_c(\mathbb{R}^+ \times \mathbb{R}^2)\), it holds

\[
\int_{\mathcal{F}(0)} \beta(\rho^{in}) \varphi(0, \cdot) + \int_{\mathbb{R}^+} \int_{\mathcal{F}(t)} \beta(\rho) \partial_t \varphi + \int_{\mathbb{R}^+} \int_{\mathcal{F}(t)} \beta(\rho) v \cdot \nabla \varphi
\]

\[
- \sum_{i \in \{+,-\}} \int_{\mathbb{R}^+} \int_{\partial \mathcal{F}^i} \beta(\rho^i) (v \cdot n - q) \varphi - \int_{\mathbb{R}^+} \int_{\mathcal{F}(t)} f \beta(\rho) \varphi,
\]

for any \(\beta \in C^1_b(\mathbb{R}) = \{ \beta \in C^0(\mathbb{R}; \mathbb{R}) \text{ such that } ||\beta||_{C^1} < +\infty \} \).

Let us write a straightforward corollary.

**Corollary 19.** Under the hypothesis of Theorem 18, if we assume also that \(\rho^{in} \in L^p(\mathcal{F}(0)), \rho^+ \in \mathcal{L}_{loc}^1(\mathbb{R}^+ \odot \partial \mathcal{F}^+(t), |v \cdot n - q| \, dt \, ds)\) and \(f \in \mathcal{L}_{loc}^1(\mathbb{R}^+ ; L^p(\mathcal{F}(t)))\), then the unique renormalized solution \((\rho, \rho^-) \in C_{loc}^0(\mathbb{R}^+; L^p(\mathcal{F}(t))) \times \mathcal{L}_{loc}^1(\mathbb{R}^+ \odot \partial \mathcal{F}^-(t), |v \cdot n - q| \, dt \, ds)\). Moreover for any \(T \in \mathbb{R}^+\), it holds

\[
\int_{\mathcal{F}(T)} |\rho(T, \cdot)|^p + \int_0^T \int_{\partial \mathcal{F}^-} |\rho^-|^p (v \cdot n - q)
\]
Finally the following duality formula holds true.

**Theorem 20.** Let $p, q \in [1, +\infty)$ conjugate. Let $v \in L^1(\mathbb{R}^+; W^{1,1}(\mathcal{F}(t)))$ with $\text{div}(v) = 0$. Let $u \in L^\infty(0, T; L^p(\mathcal{F}(t)))$ a renormalized solution to (51) and let $r \in L^\infty(0, T; L^q(\mathcal{F}(t)))$ a renormalized solution to the transport equation (50) with source term $f \in L^1(0, T; L^q(\mathcal{F}(t)))$. Then we have the following duality formula

$$
\int_{\mathcal{F}(t)} u(t, .)r(t, .) \, dx - \int_{\mathcal{F}(0)} u(0, .)r(0, .) \, dx + \sum_{i=+, -} \int_0^T \int_{\mathcal{F}_i} u^ir^i(v \cdot n - q) \, dsdt = \int_0^T \int_{\mathcal{F}(t)} fu.
$$

The proofs of the above results follow the original work [14] with some tools introduced in [6, 7].

**B. Proof of Lemma 2**

The proof of Lemma 2 is based on the following estimate.

**Lemma 21.** Let $\Omega, S^+, S^-$ a regular and compatible geometry. For $t \in [0, T]$, let $G(t, x, y)$ the Green function in $\mathcal{F}(t)$ with Dirichlet boundary condition, i.e. $G$ is the solution of $-\Delta_x G(t, ., y) = \delta_0(x - y)$ in $\mathcal{F}(t)$ and $G(t, ., y) = 0$ on $\partial \mathcal{F}(t)$. Then

$$
|G(t, x, y)| \leq M \left(1 + |\log|x - y||\right) \quad \text{and} \quad |\nabla_x G(t, x, y)| \leq \frac{M}{|x - y|^\alpha}, \tag{52}
$$

for any $x, y \in \mathcal{F}(t)$ and $M$ independent of time.

Let us briefly show Lemma 2.

**Proof of Lemma 2.** The proof is a consequence of the estimates (52) and the fact that $\varphi$ is constant in any connected component of the boundary in such a way that $\nabla \varphi$ is a multiple of the normal to boundary in any point of $\partial \mathcal{F}(t)$. We refer to [21] for a complete proof of the result.

We conclude the section with the proof of Lemma 21. First of all let us recall that Lemma 21 has been shown in [22] for any fixed time and (52) holds true with constant $M = M(t)$. In the case of a domain with smooth boundary a proof is available in the appendix of [21] and relies on the Riemann mapping Theorem. Let us now explain how to get (52) with a constant independent of time.

**Proof of Lemma 21.** In the study of Green’s functions for the Dirichlet Laplacian, the case of dimension two is special in the sense that the fundamental solution in $\mathbb{R}^2$ is given by $\Phi_2(x) = -\log|x|/2\pi$ while in dimension $n \geq 3$ is $\Phi_n(x) = |x|^{2-n}/w(n)$ where $w(n)$ is $n(n-2)$ times the volume of the unit ball in $\mathbb{R}^n$. For this reason often the authors restrict their proof to the case $n \geq 3$.

In the case $n = 3$, Theorem 1 of [16] states that given $\mathcal{O} \subset \mathbb{R}^3$ a bounded domain with $C^{2, \alpha}$ boundary and given $G^\mathcal{O}$ the Green’s function associated with the Dirichlet Laplacian on $\mathcal{O}$, then

$$
|G^\mathcal{O}(t, \bar{x}, \bar{y})| \leq \frac{M}{|\bar{x} - \bar{y}|} \quad \text{and} \quad |\nabla_x G^\mathcal{O}(t, \bar{x}, \bar{y})| \leq \frac{M}{|\bar{x} - \bar{y}|^2},
$$

where $\bar{x}, \bar{y} \in \mathcal{O} \subset \mathbb{R}^3$. The proof is based on interior and boundary estimates. This implies that the proof can be extended to Green function associated with the three dimensional domain $\mathcal{F}(t) \times S^1$ where $S^1$ is the circle with length 1, in fact the boundary is in $C^{1, \alpha}_\text{loc}(\mathbb{R}^+; C^{2, \alpha})$ so the constant $M$ can be chosen independent of the time.
Let us now recall that from the proof of Theorem 13 part ii. of [5] the Green’s function $G(t, ., .)$
associated with the two dimensional domain $\mathcal{F}(t)$ can be obtain from $G^{\mathcal{F}(t)\times S^1}$
through the formula

$$G(t, x, y) = \int_0^1 G^{\mathcal{F}(t)\times S^1}((x, 0), (y, y_3)) dy_3.$$ 

We have

$$|\nabla_x G(t, x, y)| = \left| \int_0^1 \nabla_x G^{\mathcal{F}(t)\times S^1}((x, 0), (y, y_3)) dy_3 \right|$$

$$\leq \int_0^1 |\nabla_x G^{\mathcal{F}(t)\times S^1}((x, 0), (y, y_3))| dy_3$$

$$\leq \int_0^1 \frac{M}{|x - y|^2 + |y_3|^2} dy_3 \leq \frac{1}{|x - y|} \int_0^1 \frac{M}{1 + |s|^2} ds$$

$$\leq C \frac{M}{|x - y|}.$$ 

For the estimates of $|G|$, we follow the proof of Theorem 13 part ii. of [5].

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