Abstract—We propose the application of multiple-bases belief-propagation, an optimized iterative decoding method, to a set of rate-1/2 LDPC codes from the IEEE 802.16e WiMAX standard. The presented approach allows for improved decoding performance when signaling over the AWGN channel. As all required operations for this method can be run in parallel, the decoding delay of this method and standard belief-propagation decoding are equal. The obtained results are compared to the performance of LDPC codes optimized with the progressive edge-growth algorithm and to bounds from information theory. It will be shown that the discussed method mitigates the gap to the well-known random coding bound by about 20 percent.

I. INTRODUCTION

The use of belief-propagation (BP) decoding [1] with redundant parity-check matrix representations has drawn a lot of attention. Several authors [2], [3], [4], [5], [6] presented pioneering work on the binary erasure channel (BEC) and provided results on the number of redundant parity-check equations required to prevent certain decoder failures. The concepts used on the BEC cannot be transferred to the additive white Gaussian noise (AWGN) channel in a straightforward manner. For this reason, several authors designed algorithms to use redundant code descriptions for BP decoding of data signaled over the AWGN channel. A proof of concept using the extended Golay code of length 24 was already given in [7]. In [8] and [9], adaptive BP algorithms were proposed. These algorithms adjust the parity-check information for each iteration, taking into account the current decoder state. They require additional operations which cannot be parallelized and hence increase the delay of the data stream. The random redundant decoding (RRD) algorithm [10] uses multiple parity-check matrix representations in a serial fashion to decode block codes with dense parity-check matrices [17]. This algorithm is motivated by the fact that different parity-check matrices allow for decoding of different error patterns is a modified BP decoding algorithm. It was shown that the combination of MBBP and the Leaking algorithm is a valuable tool to improve the decoding performance if a low number of redundant parity checks is available. In this paper, we extend the field of application of MBBP to iteratively decoded channel codes from the Worldwide Interoperability for Microwave Access (WiMAX) standard [15] and demonstrate the effectiveness of the algorithm for this class of codes. Also, we compare the performance of these codes to the performance of optimized PEG codes of comparable length, both for BP and MBBP decoding.

The paper is structured as follows. In Section II we describe the transmission setup and review the basic principles of MBBP decoding. Section III states how a set of different parity-check matrix representations is generated, and Section IV presents a selection of results including the comparison to PEG codes with equal rate and similar length.

II. TRANSMISSION SETUP AND CHANNEL CODING

In this section we introduce a consistent notation and give a proper definition of the channel setup. A source emits non-redundant binary information symbols \( u \), which are encoded and mapped to binary antipodal symbols \( x \). As systematic encoding leads to several advantages [16], this type of encoding is used throughout this work. Due to the fact that exclusively \( [n, k, d] \) block codes are used in this investigation, the encoded symbols are denoted as vectors \( x \) of length \( n \). These vectors are transmitted over the AWGN channel. In this context, \( y \) denotes the noisy received vector corresponding to \( x \). At the receiver, an iterative decoding scheme is used to estimate \( x \) and the corresponding source symbols. This scheme is either a standard BP decoder or the MBBP decoding setup. Let us briefly review the basic properties of MBBP, which allows for the performance improvements discussed in this paper. MBBP is an iterative decoding scheme, originally designed to decode block codes with dense parity-check matrices [17]. To this end, it runs multiple instances of the BP decoding algorithm in parallel. Each of these decoders is provided with the received signal \( y \) and a different parity-check matrix for the code. In this context, we denote the \( l \) parity-check matrix representations by \( H_1 \) to \( H_l \). The corresponding codeword estimates are \( \hat{x}_1 \) to \( \hat{x}_l \) and the candidate forwarded to the information sink is \( \hat{x} \).

This algorithm is motivated by the fact that different parity-check matrices allow for decoding of different error patterns.
A. Set of matrix representations for MBBP decoding

The most crucial parameter for the success of MBBP decoding is the set of parity-check matrices used in the decoding instances. Especially simulations with PEG codes [13] have shown that the applied matrices need to fulfill two criteria. First, the Tanner graphs [20] of the matrices need to differ sufficiently in their structure such that the decoders obtain a decoding diversity and a performance improvement. Second, the decoders running on the parity-check matrices need to obtain comparable performance results. Adding a representation to an existing MBBP setup can only increase the overall performance if its standard BP performance is comparable to the performance of the current MBBP setup.

In [13] a general method to construct a set of redundant parity-check matrices for a given code was presented. This method was originally intended to provide good redundant parity-check matrices for PEG-constructed codes of short length and makes use of the fact that there exist cycles of length 4 and 6 in the Tanner graph. Especially for code lengths $n \leq 1000$, many additional parity-check equations can be found with this method. The aim of this approach, to be described in detail shortly, is to approximate the property “low-density” for the additional checks. Let $c$ be the length of the considered cycle and let $G_c$ be one set of indices of parity checks closing a cycle of length $c$. A linear combination of the parity checks indexed by the set $G_c$ leads to a novel parity-check equation with a (Hamming) weight of at most

$$w_i = \sum_{i \in G_c} w_i - c,$$

where $w_i$ denotes the weight of parity check $i$. This is a general approach and can be used for any parity-check matrix with a local cycle length of $c$. It was shown in [13] and [14] that this approach leads to desirable performance results when using PEG codes. However, the parity-check matrix of the WiMAX codes show more structure, what allows for a better construction algorithm.

B. Parity-check matrices for codes specified in the IEEE 802.16e standard

The LDPC codes of rate 1/2 standardized in [15] are all deduced from one base matrix $H_b'$. The realizations of different lengths are created from this matrix by lifting [20]. Prior to this step, a renormalization is done, i.e. the lifting procedure is applied to the elements

$$H_b(i,j) = \begin{cases} \frac{H_b'(i,j) \cdot z}{96} & \text{if } H_b'(i,j) > 0 \\ H_b'(i,j) & \text{if } H_b'(i,j) \leq 0 \end{cases}$$

(2)

of the matrix $H_b$.

In this context, $z$ is the expansion factor and depends on the code realization. The lifting procedure, from which the parity-check matrix $H$ results, is described as follows. Each negative entry in the base matrix $H_b'$ is replaced by a $z \times z$ zero matrix and each non-negative element $H_b'(i,j)$ is substituted by an identity matrix which is cyclically shifted to the right.
by \( H_b(i, j) \) positions. Equation (3) specifies the base matrix \( H'_b \) for the rate-1/2 LDPC code [15, p. 628].

Performing the lifting approach leads to the binary matrix \( H \). Considering that any entry in \( H_b \) is replaced by a permutation matrix with constant row weight one, it is easy to see from Equation (3) that the weight of any parity check of \( H \) is 6 or 7, regardless of the actual length \( n \) of the code. The girth of the code was found to be 6 for all lengths considered. It is now our task to determine parity checks which are linear combinations of the given parity checks and have as low as possible weight.

\[ H'_b = \begin{pmatrix}
-1 & 94 & 73 & -1 & -1 & -1 & -1 & 55 & -1 & -1 & -1 & 7 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 27 & -1 & -1 & 22 & 79 & 9 & -1 & -1 & 12 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 1 & 24 & 22 & 81 & -1 & 33 & -1 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
61 & -1 & 47 & -1 & -1 & -1 & 1 & 65 & 25 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 39 & -1 & -1 & -1 & 84 & -1 & 41 & 72 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 46 & 40 & -1 & 82 & -1 & -1 & 79 & 0 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & 95 & 53 & -1 & -1 & -1 & 14 & 18 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
-1 & 11 & 73 & -1 & -1 & -1 & 2 & 1 & -1 & 47 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 & -1 \\
12 & -1 & -1 & -1 & 83 & 24 & -1 & 43 & -1 & -1 & 51 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 & -1 \\
-1 & -1 & -1 & -1 & 1 & 94 & -1 & 59 & -1 & -1 & 70 & 72 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & -1 & -1 & -1 & -1 \\
-1 & -1 & 7 & 65 & -1 & -1 & -1 & 39 & 49 & -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
43 & -1 & -1 & -1 & 66 & -1 & -1 & 41 & -1 & -1 & 26 & 7 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 
\end{pmatrix} \]

(3)

C. Redundant parity checks from \( H_b \)

The novel approach for creating redundant parity-check equations uses the base matrix \( H_b \) instead of the binary matrix \( H \) to find valid linear combinations. Subsequently it performs the lifting operation on the redundant checks. Let us elaborate on the generation of these checks. In a binary matrix, a redundant check can be found as a linear combination of two or more existing checks. This proceeding is in general not possible when the base matrix \( H_b \) is considered, as the addition of two entries is not defined. However, the addition of a negative and a non-negative element, as well as the addition of two zero elements is a straightforward task. The result of the addition is the non-negative element and the element \(-1\), respectively. Using this approach, redundant checks can be created by the linear combination of two existing checks, which do not share a positive element in any column. Lifting a redundant check leads to a set of \( z \) checks for the binary matrix \( H \), which are subsequently used to create sets of non-equal, binary parity check matrices. As an example, we state that the linear combination of rows 11 and 12 in \( H_b \), leads to \( z \) binary redundant checks of weight 10, since the non-negative entries in rows 11 and 12 have disjoint column positions, except for the last column, which contains zero entries.

Depending on the length of the code, we replace 10 to 16 parity checks in the existing parity-check matrix to generate a new representation. At this step, we ensure that the resulting parity-check matrix has full rank.

Let us now compare this result to the approach from Section III-A by means of the WiMAX code of length \( n = 576 \). The local girth of its parity-check matrix varies between \( c = 6 \) and \( c = 8 \). Using Equation (1) and \( c = 6 \), it can be deduced that additional parity-check representations have a weight of at most 12 to 15, depending on the actual parity checks used to create the linear combinations. It was verified by computer simulations that this bound is met by the realizations.

The authors are aware of the fact that this novel method is still a suboptimal approach and therefore assess it in a more general manner. The methods provided in [21] allow for an efficient search of low-weight codewords. Using these methods to the dual of the IEEE 802.16e rate-1/2-code of length 576 did not return any codewords which are not already present in the rows of the original parity-check matrix of weight below 10. This allows us to conclude that the proposed method is well suited for the considered class of codes.

IV. RESULTS AND COMPARISON

We present simulation results for codes from the WiMAX standard of different length. In this context, we apply standard BP decoding as well as the (L)-MBBP approach to show the performance improvements obtained with this method. We allow all BP decoding units to perform at most 200 iterations. This is a proper choice for which a further increase does not improve the standard BP decoding performance significantly. We also limit the number of different parity-check matrices in an MBBP setup to 15 and allow leaking with an initial setting of \( p_L = 0.9 \), what results to a maximum number of 30 decoders in parallel. The current development of multiprocessor techniques [22] allows us to state that this setting can easily be parallelized with upcoming microcontroller techniques. Furthermore we set the parameter \( I_{\text{max}} = 300 \), as this setting leads to desirable results in our computer simulations.

Figure 2 shows performance results for the WiMAX codes of length \( n = 576 \) and \( n = 960 \). In order to emphasize that the bigger part of the decoding gain is already obtained by a low number of decoder representations, we show different MBBP settings. To be precise, we allow \( l = 7, l = 15, \) and \( l = 30 \) representations to run in parallel. In Figure 2 we observe that the most prominent part of the decoding gain is already achieved with 7 decoders in parallel and another small gain is achieved for \( l = 15 \). The setup using L-MBBP and utilizing 30 decoders in total compares favorably but the difference is small in relation to the number of decoders additionally required. Using all decoding units, the proposed multi-decoding approach improves the performance of WiMAX codes for about 0.15
we estimate the minimum distance and compare the codes defined in the WiMAX standard to the Gallager bound [16]. Errors happen in an erroneously decoded frame. Details on this approach can be found in [16].

Within the error region of interest, the created ensembles show strictly concentrated behavior, what allows us to study the subsequent results independent of the random seed used for the construction algorithm.

Figure 3 shows the signal-to-noise ratio $10 \cdot \log_{10}(E_b/N_0)$, which is required to obtain the reliability criterion $\text{BER} = 10^{-3}$ and $\text{FER} = 10^{-3}$, respectively. Plotted are results for WiMAX codes and PEG-optimized codes for both BP and L-MBBP decoding as well as the Gallager bound. In order to keep an appropriate presentation for the numerical results, but still provide the reader with an idea of the position of the sphere packing bound (SPB), we choose to plot the left-most part of it and state that its shape is shown to be similar to the shape of the Gallager bound in [16].

Let us first consider the results for the WiMAX codes. In our simulations, the codes showed slightly different error-floor behavior. This holds in particular for the code of length $n = 864$. It is to observe that a gain of about 0.15 dB is achieved for all code lengths considered. From the plot for $\text{FER} = 10^{-3}$ and the code of length $n = 960$ we observe that the gap to the Gallager bound reads about 0.7 dB, which can be lowered by 0.14 dB (or 20 %) with the L-MBBP approach.

Similar results are presented for the PEG-optimized LDPC codes, where we also restrict the maximum number of decoders in parallel to 30. The actual number is however lower due to lack of well-performing presentations. The PEG codes show the desired performance results at about 0.15 dB lower signal-to-noise ratios. Again, the L-MBBP approach mitigates the gap to the random coding bound by about 20 %.

It is worth mentioning that the codes defined in the WiMAX standard have a significantly lower density compared to the considered PEG codes. This allows for faster decoding with the BP algorithm. If one considers not only the length but also the decoding speed as a system parameter, the standardized codes are comparable to the PEG-optimized codes discussed in this work. Detailed results on this comparison can be found in [14].

V. Conclusions
The contribution of this paper is two-fold. First, we adapted the scheme of MBBP decoding to codes from the WiMAX standard what allowed us to improve the decoding performance for about 0.15 dB. As a second contribution, we compared the performance of the WiMAX codes with the performance of PEG codes. Both for BP decoding and MBBP decoding, the PEG codes obtained a measurable performance improvement compared to the codes in the WiMAX standard.

REFERENCES
[1] J. Pearl, Probabilistic reasoning in intelligent systems: networks of plausible inference. Morgan Kaufmann Publishers, September 1988.

$$L(x) = 0.5043865558 \cdot x^2 + 0.2955760529 \cdot x^3 +$$

$$0.0572634080 \cdot x^5 + 0.0362602194 \cdot x^6 +$$

$$0.0049622081 \cdot x^7 + 0.0292344776 \cdot x^9 +$$

$$0.0650312477 \cdot x^{11} + 0.0072858305 \cdot x^{12}$$
Fig. 3. Required SNR to meet given quality constraints BER = 10^{-5} and FER = 10^{-3}, respectively, for WiMAX codes (576 ≤ n ≤ 960) and PEG-optimized codes (500 ≤ n ≤ 1000). For comparison, the Gallager bound and the SPB are shown.