CLUSTER EXPANSION FOR THE ISING MODEL IN THE CANONICAL ENSEMBLE

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ABSTRACT. We show the validity of the cluster expansion in the canonical ensemble for the Ising model. We compare its radius of convergence with the one computed by the virial expansion working in the grand-canonical ensemble. Using the cluster expansion we give direct proofs with quantification of the higher order error terms for the decay of correlations, central limit theorem and large deviations.

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1. INTRODUCTION

The analysis of the relation between thermodynamic quantities and their use for quantitative prediction of macroscopic properties of matter through its microscopic structure, is the fundamental goal of statistical mechanics. A key tool in this direction is the cluster expansion initially developed by Mayer [9] for non-ideal gases viewed as a perturbation around the ideal gas. This method allows to express the thermodynamic quantities as absolutely convergent power series. Over the last years many methods and generalizations have been developed mostly adapted to the grand-canonical ensemble as the canonical constraint seemed restrictive to
the product structure of the underlying system. As it was proved in [13] one can remove this constraint by viewing it as a hard-core system of clusters and then it fits beautifully in the existing theory of the abstract polymer model [5], [7]. For the case of the Ising model, the constraint seems even more restrictive as particles are indexed by their lattice position, with the constraint of fixed magnetization destroying the product structure. Hence, in the present paper

1. We view the Ising model as a lattice gas [3] and indexing the spins rather than their position (as in the continuous case) we can treat the canonical constraint in a similar manner as in [13].

2. We compare the convergence condition of the virial expansion working in canonical vs the grand-canonical ensemble. Moreover we also compare the convergence condition for the cluster expansion of the grand-canonical partition function for the Ising model with the contour representation (see for example [4]) and with the alternative one [3].

3. We get a decay of correlations estimate working directly in the canonical ensemble.

4. We prove moderate and large deviations with quantification of the higher order error terms and compare with the results developed previously in [1] and [2] for the case of the Ising model in the grand-canonical ensemble.

5. It is worth noticing that despite the fact that we focus in the Ising model, we expect that our approach is applicable to more general lattice systems with more complicated interactions and the key idea of indexing the spins rather than their position remains valid.

The structure of the paper is as follows: in Section 2 we present model and results. The main theorem about the validity of the cluster expansion for the canonical partition function under an appropriate condition on the density (Theorem 2.1) is given in Subsection 2.1. The proof is given in Section 3. In Subsection 2.2 we compare graphically the canonical and the grand-canonical convergence conditions of the virial expansion as well as the convergence conditions of the grand-canonical partition function for the Ising model with the two different representations: the contour and the lattice gas. The related calculations are given in Subsection 3.1. The decay of the 2-point correlation for the canonical ensemble is presented in Theorem 2.2 (Subsection 2.1) and it is proved in Section 4. We conclude with Section 5 where we compare our approach for the study of Precise Large Deviations and Local Moderate Deviations, as presented in Theorems 2.3, 2.4 and Corollary 2.5, with the one in the grand-canonical ensemble from [1] and [2].

### 2. Notation, model and results

#### 2.1. Cluster expansion and 2-point correlation function in canonical ensemble.**

We consider the ferromagnetic Ising Model on a finite volume $\Lambda \subset \mathbb{Z}^d$ at small inverse temperature $\beta$. We denote with $\sigma = (\sigma(x_1), ..., \sigma(x_{|\Lambda|})) \in \{-1, 1\}^\Lambda$ a spins vector on $\Lambda$ and with $\sigma^c \in \{-1, 1\}^{\Lambda^c}$ a spins vector on $\Lambda^c := \mathbb{Z}^d \setminus \Lambda$ with fixed value $\sigma^c$, i.e., such that $\sigma(x) = \sigma^c$ for all $x \in \Lambda^c$. Hence, defining $\mathcal{E}_\Lambda := \{\{x, x'\} \subset \mathbb{Z}^d | \{x, x'\} \cap \Lambda \neq \emptyset, |x - x'| = 1\}$, where $|\cdot|$ is the Euclidean distance, the Hamiltonian is given by

$$
\mathcal{H}_\Lambda^\sigma(\sigma) := -J \sum_{\{x, x'\} \in \mathcal{E}_\Lambda} \sigma(x)\sigma(x')
$$

(2.1)
with $J \in \mathbb{R}^+$. The canonical partition function at fixed magnetization $m \in (-1, -1 + \epsilon)$, $0 < \epsilon < 1$, is defined as
\[
\bar{Z}_{\Lambda, \beta}^m(m) := \sum_{\sigma \in \{-1, 1\}^\Lambda : \sum_{x \in \Lambda} \sigma(x) = m} e^{-\beta \mathcal{H}_\Lambda^c(\sigma)},
\] (2.2)

We reduce the canonical partition function for the Ising model given by (2.2) to the one for a classic lattice gas system using the identity
\[
\sigma(x) = 2\eta(x) - 1,
\] (2.3)
with $\eta : \mathbb{Z}^d \mapsto \{0, 1\}$. Then (2.1) and (2.2) become
\[
\mathcal{H}_\Lambda^\sigma(\sigma) \equiv \mathcal{H}_\Lambda^\eta(\eta) := 4Jm' |\mathcal{E}_\Lambda| - J|\mathcal{E}_\Lambda| - 4J \sum_{\{x, x'\} \in \mathcal{E}_\Lambda} \eta(x)\eta(x')
\]
and
\[
\bar{Z}_{\Lambda, \beta}^\sigma(m') \equiv \bar{Z}_{\Lambda, \beta}^\eta(m') := \sum_{\eta \in \{0, 1\}^\Lambda : \sum_{x \in \Lambda} \eta(x) = m'|\Lambda|} e^{-\beta \mathcal{H}_\Lambda^\eta(\eta)},
\] (2.4)

where $m' := (m + 1)/2$.

We denote with $N \equiv N(m') := m'|\Lambda|$ the number of (indistinguishable) particles of the system, with $\mathbf{x} = (x_1, ..., x_N) \in \Lambda^N$ a configuration vector and we introduce the “hard-core” potential [3]
\[
V(x, x') := \begin{cases} 
\infty & \text{if } x = x', \\
-4J & \text{if } |x - x'| = 1, \\
0 & \text{otherwise},
\end{cases}
\] (2.5)
for all $x, x' \in \mathbb{Z}^d$. In this way, from (2.4) we can write
\[
\bar{Z}_{\Lambda, \beta}^\sigma(m') = \exp \left\{ -\beta |\Lambda| \left[ 4Jm' \left| \mathcal{E}_\Lambda \right| - J \left| \mathcal{E}_\Lambda \right| \right] \right\} Z_{\Lambda, \beta}^\gamma(N),
\] (2.6)
where
\[
Z_{\Lambda, \beta}^\gamma(N) := \frac{1}{N!} \sum_{\mathbf{x} \in \Lambda^N} e^{-\beta H_\Lambda^\gamma(\mathbf{x})},
\] (2.7)
with $H_\Lambda^\gamma(\mathbf{x})$ Hamiltonian given by
\[
H_\Lambda^\gamma(\mathbf{x}) : (\mathbb{Z}^d)^N \rightarrow \mathbb{R}
\]
\[
\mathbf{x} \mapsto \sum_{1 \leq i < j \leq N} V(x_i, x_j) + \sum_{1 \leq i \leq N, j \geq 1} V(x_i, \gamma_j),
\] (2.8)
and $\gamma = (\gamma_1, ..., \gamma_i, ...) \in \Lambda^N$ an appropriate fixed configuration outside $\Lambda$. In order to simplify the calculation we consider zero boundary conditions such that our Hamiltonian is given by $H_\Lambda^\gamma(\mathbf{x}) = \sum_{1 \leq i < j \leq N} V(x_i, x_j)$. Notice that, as it is explained in Remark 3.1, the following results hold also for $\gamma \neq \mathbf{0}$ boundary conditions.

The potential defined in (2.5) satisfies the usual regularity and stability conditions needed for the cluster expansion. Indeed, for all fixed $x^* \in \mathbb{Z}^d$ we get
\[
\sum_{1 \leq j \leq N} V(x^*, x_j) \geq -4J \sum_{1 \leq j \leq N} 1_{\{|x^*-x_j|=1\}}(x_j) \geq -8Jd =: -B,
\] (2.9)
and
\[ \sum_{x' \in \mathbb{Z}^d} \left| e^{-\beta V(x',x)} - 1 \right| = \sum_{x' \in \mathbb{Z}^d : |x' - x| \leq 1} \left| e^{-\beta V(x',x)} - 1 \right| =: C_{J,d}(\beta), \tag{2.10} \]
where
\[ C_{J,d}(\beta) = 2d(e^{4\beta J} - 1) + 1 < \infty \tag{2.11} \]
for all finite \( \beta \geq 0. \)

Remark 2.1. The following study can be also done if the potential (2.5) acts when \( |x - x'| \leq R \) with \( R^d < |\Lambda| \) (for example in the case of Kac potential) and if we consider, with the necessary technical reformulations, magnetization \( m \in (1 - \epsilon, 1). \)

Defining the finite volume free energy as
\[ f_{\beta,\Lambda,0}(N) := -\frac{1}{|\Lambda|} \log Z_{\Lambda,\beta}^0(N), \tag{2.12} \]
the thermodynamic free energy is given by
\[ f_{\beta}(\rho) := \lim_{N/|\Lambda| \to \rho \in (0,1)} f_{\beta,\Lambda,0}(N). \tag{2.13} \]

The main result of this paper is the cluster expansion of (2.7) presented in Theorem 2.1 below. Thanks to it we also derive an expression for the thermodynamic free energy as an absolutely convergent power series with respect to the density with coefficients given by the discrete form of the (2-connected) Mayer’s coefficients [9]. These are defined as
\[ \beta_n := \frac{1}{n!} \sum_{g \in B_{n+1}} \sum_{x \in (\mathbb{Z}^d)^n} \prod_{(i,j) \in E(g)} (e^{-\beta V(x_i,x_j)} - 1), \tag{2.14} \]
where the set \( B_{n+1} \) is the set of the graphs with \( n + 1 \) vertices which remain connected when a vertex is removed (called also 2-connected), and \( E(g) \) and \( V(g) \) are respectively the set of edges and vertices of a graph \( g. \)

Theorem 2.1. There exists a constant \( R^C \equiv R^C(d,J,\beta) \) independent of \( N \) and \( \Lambda \) (see Lemma 3.1 for the explicit value), such that if \( N/|\Lambda| < R^C \) then
\[ \frac{1}{|\Lambda|} \log Z_{\Lambda,\beta}^0(N) = \frac{1}{|\Lambda|} \log |\Lambda|^N + \frac{N}{|\Lambda|} \sum_{n \geq 1} F_{\beta,N,\Lambda}(n) \tag{2.15} \]
and in the thermodynamic limit
\[ \lim_{\Lambda \to \mathbb{Z}^d} \frac{N}{|\Lambda|} F_{\beta,N,\Lambda}(n) = \frac{1}{n + 1} \rho^{n+1} \beta_n, \tag{2.16} \]
for all \( n \geq 1, \beta_n \) given by (2.14). Furthermore, there exist constants \( C, c > 0 \) such that for every \( N \) and \( \Lambda \) and for all \( n \geq 1: \)
\[ |F_{\beta,N,\Lambda}(n)| \leq Ce^{-cn}. \tag{2.17} \]

Remark 2.2. We make an expansion around \( m = -1 \), which is the equivalent density expansion presented in [13].
As we will see in Section 3 the term \( F_{\beta,N,\Lambda}(n) \) is given by
\[
F_{\beta,N,\Lambda}(n) = \frac{1}{n+1} P_{N,|\Lambda|}(n) B_{\Lambda,\beta}(n) \tag{2.18}
\]
where
\[
P_{N,|\Lambda|} := \begin{cases} \frac{(N-1)\cdots(N-n)}{|\Lambda|^n} & \text{if } n < N, \\ 0 & \text{otherwise} \end{cases} \tag{2.19}
\]
and \( B_{\Lambda,\beta}(n) \to \beta_n \text{ as } \Lambda \to \mathbb{Z}^d \). Let \( P_{n+1}(\rho) \) be a polynomial of degree \( n+1 \) in \( \rho \) such that
\[
P_{n+1}(\rho) := \frac{N}{|\Lambda|} P_{N,|\Lambda|}(n) \tag{2.20}
\]
for \( \rho_\Lambda = N/|\Lambda| \). Then, defining
\[
\mathcal{F}_{\Lambda,\beta,0}(\rho) := \frac{1}{\beta} \left\{ \rho (\log \rho - 1) - \sum_{n \geq 1} \frac{1}{n+1} P_{n+1}(\rho) B_{\Lambda,\beta}(n) \right\} \tag{2.21}
\]
with \( \rho \in [0,1] \), using Stirling’s approximation we get
\[
f_{\Lambda,\beta,0}(\rho) = \mathcal{F}_{\Lambda,\beta,0}(\rho_\Lambda) + S_{|\Lambda|}(\rho_\Lambda), \tag{2.22}
\]
where \( S_{|\Lambda|}(\rho_\Lambda) \) is an error term of order \( \log \sqrt{|\Lambda|/|\Lambda|} \) (see Appendix B of [15]). Hence
\[
\beta f_\beta(\rho) = \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{\beta |\Lambda|} \log \tilde{Z}_{\Lambda,\beta}(m)
\]
where \( \tilde{Z}_{\Lambda,\beta}(m) \) is given by (2.2) with constant \(-1\) boundary conditions. From (2.4), (2.6) and Theorem 2.1, we obtain that
\[
\beta f_\beta(\rho) = \beta \phi_\beta(m) - 4d \beta J \left( \frac{m+1}{2} \right) + d \beta J, \tag{2.24}
\]
since \( |\mathcal{E}_\Lambda|/|\Lambda| \to d \) as \( |\Lambda| \to \infty \) and where \( \rho = \rho_\Lambda = (m_1 + 1)/2 \).

Furthermore, thanks to the validity of the cluster expansion we study the behavior of the truncated 2-point (“canonical”) correlation function. Given \( q_1, q_2 \in \Lambda \) we define:
\[
u_{\Lambda,\beta}^{(2)}(q_1, q_2) := \rho_{\Lambda,\beta}^{(2)}(q_1, q_2) - \rho_{\Lambda,\beta}^{(1)}(q_1) \rho_{\Lambda,\beta}^{(1)}(q_2), \tag{2.25}
\]
where
\[
\rho_{\Lambda,\beta}^{(1)}(q) := \frac{1}{(N-1)!} \sum_{\mathbf{x} \in \Lambda_{N-1}} \frac{1}{Z_{\Lambda,\beta}(N)} e^{-\beta H_{\Lambda,\beta}(q, \mathbf{x})} \tag{2.26}
\]
and
\[
\rho_{\Lambda,\beta}^{(2)}(q_1, q_2) := \frac{1}{(N-2)!} \sum_{\mathbf{x} \in \Lambda_{N-2}} \frac{1}{Z_{\Lambda,\beta}(N)} e^{-\beta H_{\Lambda,\beta}(q_1, q_2, \mathbf{x})} \tag{2.27}
\]
where \( Z_{\Lambda,\beta}(N) \) is given by (2.7) with the difference that, for simplicity, we consider here periodic boundary conditions. We have:
Theorem 2.2. Let $q_1, q_2$ be two fixed points in the domain $\Lambda$, then there exist positive constants $C$ and $C_1$, independent of $N$ and $\Lambda$, such that, when \( N/|\Lambda| \) is small enough, we have

\[
|u^{(2)}(q_1, q_2)| \leq \left( \frac{N}{|\Lambda|} \right)^2 \left[ (e^{4\beta J} - 1)1_{|q_1 - q_2| = 1} + 1_{q_1 = q_2} \right] + \left( e^{4\beta J} - 1 \right) \frac{1_{|q_1 - q_2| = 1} + 1_{q_1 = q_2}}{|\Lambda|} + Ce^{-|q_1 - q_2|} + C_1 \frac{1}{|\Lambda|}. \tag{2.28}
\]

The proof of the theorem will be given in Section 4.

2.2. Grand-canonical description and related discussion. Considering now the grand-canonical ensemble we define the thermodynamic pressure as

\[
p_\beta(\mu) := \lim_{\Lambda \to \mathbb{R}^d} \frac{1}{\beta |\Lambda|} \log \Xi^{0}_{\Lambda, \beta}(\mu) \tag{2.29}
\]

with $\mu \in \mathbb{R}$ chemical potential, and where $\Xi^{0}_{\Lambda, \beta}(\mu)$ is the grand-canonical partition function with zero boundary conditions, given by:

\[
\Xi^{0}_{\Lambda, \beta}(\mu) := \sum_{N \geq 0} e^{\beta \mu N} Z^{0}_{\Lambda, \beta}(N). \tag{2.30}
\]

As it will be discussed in detail in Subsection 3.1, one can also define the density as a function of the activity (equation (3.14)) and by inverting this formula, obtain an expression for the pressure with respect to the density, which is the so called virial expansion. In the figure below we compare the radius of convergence this expansion, denoted with $R^{LG}_{V} \equiv R^{LG}_{V}(\beta, d, J)$, with the one of the canonical expansion presented in Theorem 2.1. We can see that $R^{LG}_{V}$ gives us a bigger convergence density region than $R^{C}$. In Figure 1, we represent $R^{LG}_{V}$ and $R^{C}$ with $J = 1$ and $\beta \in [0, 1]$, in dimension 1, 2 and 3. We can also observe the same behavior if we fix the dimension and we consider different values of $J$ as it is shown in Figure 3 in Subsection 3.1.
Next we compare two equivalent descriptions in grand-canonical ensemble: the lattice gas representation given by equation (2.30) and the Ising model with external magnetic field $h \in \mathbb{R}$. Since we will work close to the $-1$ phase we will be considering $h, \mu \leq 0$.

Hence, we define the grand-canonical partition function for the Ising model with -1 boundary conditions as

$$
\hat{\Xi}_{\Lambda, \beta}^-(h) := \sum_{\sigma \in \{-1, 1\}^\Lambda} e^{\beta h \sum_{x \in \Lambda} \sigma(x) - \beta h_{\Lambda}^{-1}(\sigma)} = \sum_{m : m | \sigma \in \Lambda \sigma(x)} e^{\beta h_{\Lambda}|\Lambda|} \hat{Z}_{\Lambda, \beta}^-(m). \tag{2.31}
$$

Note that, starting from (2.30) and using (2.3) - (2.6), we have

$$
\Xi_{\Lambda, \beta}^0(\mu) = \exp \left\{ \beta |\Lambda| \left[ \frac{\mu}{2} + J \frac{|E_{\Lambda}|}{|\Lambda|} \right] \right\} \hat{\Xi}_{\Lambda, \beta}^-(h_{\Lambda}), \tag{2.32}
$$

where

$$
h_{\Lambda} \equiv h_{\Lambda}(\mu) := \frac{\mu}{2} + \frac{|E_{\Lambda}|}{|\Lambda|}. \tag{2.33}
$$
On the other hand, in a similar way, if we start from the Ising model, i.e., given \( h \in \mathbb{R} \) we get

\[
\tilde{\Xi}_{\Lambda, \beta}(h) = \exp \left\{ -\beta |\Lambda| \left[ h - J \frac{|E_{\Lambda}|}{|\Lambda|} \right] \right\} \Xi_{\Lambda, \beta}^{0}(\mu_{\Lambda}),
\]

where

\[
\mu_{\Lambda} \equiv \mu_{\Lambda}(h) := 2h - 4J \frac{|E_{\Lambda}|}{|\Lambda|}.
\]

We call \( \mathcal{M}_{LG} \equiv \mathcal{M}_{LG}(\beta, d, J) \) the radius of convergence of the cluster expansion of (2.30) obtained applying the result presented in [12]. For the other representation, being close to the negative phase, one can represent the partition function (2.31) using the contour ensemble, as, for example, in Chapter 5 of [4] and we denote with \( \mathcal{M}_{IS} \equiv \mathcal{M}_{IS}(\beta, d, J) \) its radius of convergence.

In Figure 2, we compare the two radius of convergence for \( J = 1, d = 1, 3 \) and \( \beta \in [0, 1] \). We observe that there exists \( \bar{\beta} \equiv \bar{\beta}(d, J) \) which decreases as \( d \) increases, such that \( \mathcal{M}_{IS} \leq \mathcal{M}_{LG} \) for all \( \beta \leq \bar{\beta} \) and \( \mathcal{M}_{LG} < \mathcal{M}_{IS} \) when \( \beta > \bar{\beta} \). The same behavior can also be observed if we fix the dimension and we consider different values of \( J \), as it is shown in Figure 4 in Subsection 3.1.

![Figure 2. \( \mathcal{M}_{IS} \) (continuous line) and \( \mathcal{M}_{LG} \) (dashed line) with \( J = 1 \) and \( \beta \in [0, 1] \), in dimension 1 (red lines) and 3 (blue lines).](image)

The precise definitions and analysis of the quantities involved in the above figures will be given in Lemma 3.1 in Section 3 and in Subsection 3.1.
2.3. Application of Theorem 2.1 to Precise Large and Local Moderate Deviations Theorems. In the last part of the paper, using the validity of the cluster expansion in the canonical ensemble (Theorem 2.1), we prove theorems on Local Moderate and Precise Large Deviations. Then we compare with the analogous results presented in [1] and [2] but using a different expansion. For that, we follow the strategy presented in [15] for the case of the lattice-free gas, to which we refer for proofs and details.

Fixing a chemical potential $\mu_0$ the grand-canonical probability measure at finite volume with zero boundary condition is defined as

$$P_{\Lambda,\mu_0}^0(x) := \bigotimes_{N \geq 0} \frac{1}{\Xi_0^\Lambda,\beta(\mu_0)} \frac{e^{\beta \mu_0 N} e^{-\beta H_0^\Lambda(x)}}{N!}. \quad (2.36)$$

Note that, thanks to (2.30), (2.32), (2.33) and (2.34), the previous probability can be expressed via the grand-canonical probability measure for the Ising model with $-\mathbf{1}$ boundary conditions.

Hence, using the results presented in Theorem 2.1 we can study directly the probability of the set

$$A_N := \{x \equiv \{x_i\}_{i \geq 1}, \ x_i \in \mathbb{Z}^d \mid |x \cap \Lambda| = N\}, \quad (2.37)$$

for $N$ taking the value $\tilde{N}$ being a general deviation from the mean value $\bar{N}_\Lambda$ of order $\alpha \in [1/2,1]$, i.e,

$$\tilde{N} := \bar{N}_\Lambda + u|\Lambda|^\alpha, \quad \alpha \in [1/2,1], \ u \in \mathbb{R}^+ \quad (2.38)$$

where

$$\bar{N} := \bar{N}_\Lambda := |\bar{\rho}_\Lambda|\Lambda| \quad (2.39)$$

We have that

$$P_{\Lambda,\mu_0}^0(A_{\tilde{N}}) = \frac{e^{\beta \mu_0 \tilde{N}} Z_{\Lambda,\beta}^0(\tilde{N})}{e^{\beta \mu_0 N} Z_{\Lambda,\beta}^0(N)} \quad (2.40)$$

and the key point is that we can now compute it using Theorem 2.1. For that purpose, we also define $N^*$ as the number of particles such that

$$\sup_N \left\{e^{\beta \mu_0 N} Z_{\Lambda,\beta}^0(N)\right\} = e^{\beta \mu_0 N^*} Z_{\Lambda,\beta}^0(N^*). \quad (2.41)$$

The previous quantity is the one which allows to express the chemical potential $\mu_0$ using quantities on the canonical ensemble. Indeed $N^*$ has the following properties [15]:

$$|\bar{N}_\Lambda - N^*| \leq C, \quad (2.42)$$

with $C$ positive constant independent of $\Lambda$ and

$$\mu_0 = \mathcal{F}_{\Lambda,\beta,0}(\rho_\Lambda^*) + S'_0|\Lambda|(|\Lambda|), \quad (2.43)$$

where $S'_0|\Lambda|(|\Lambda|)$ has order $|\Lambda|^{-1}$ for all $\rho_\Lambda = N/|\Lambda|$. Note that from (2.42) we can rewrite $\bar{N}$ given by (2.38) as

$$\tilde{N} = N^* + u'|\Lambda|^\alpha \quad (2.44)$$

with $u' \sim u$ and where $\sim$ means “asymptotically” as $|\Lambda| \to \infty$.

We have the following results
Theorem 2.3 (Precise Large Deviations). Let \( \mu_0 \in \mathbb{R} \) be a chemical potential and let us fix zero boundary conditions. Let \( \hat{N} \) be a fluctuation given by (2.38) with \( \alpha = 1 \) such that Theorem 2.1 holds. Moreover let \( \tilde{\mu}_\Lambda \in \mathbb{R} \) be the chemical potential such that \( \tilde{\rho}_\Lambda = \hat{N}/|\Lambda| =: E^0_{\Lambda,\tilde{\rho}_\Lambda}[N/|\Lambda|]. \) We have:

\[
\begin{align*}
\left| \mathcal{P}^0_{\Lambda,\mu_0}(A_{\hat{N}}) - e^{-|\Lambda| f^\text{GC}_{\Lambda,0}(\tilde{\rho}_\Lambda)} \right| & \leq \frac{C e^{-|\Lambda| f^\text{GC}_{\Lambda,0}(\tilde{\rho}_\Lambda)}}{|\Lambda|}, \\
\end{align*}
\]

where

\[
\begin{align*}
f^\text{GC}_{\Lambda,0}(\tilde{\rho}_\Lambda) & := \beta \left[ f^\text{GC}_{\Lambda,0}(\tilde{\rho}_\Lambda) - f^\text{GC}_{\Lambda,0}(\rho_0) - \mu_0 (\tilde{\rho}_\Lambda - \rho_0) \right] \\
\end{align*}
\]

and

\[
\begin{align*}
D_{\Lambda,0}(\tilde{\rho}_\Lambda) & := \left[ \beta \mathcal{F}'_{\Lambda,0}(\tilde{\rho}_\Lambda) \right]^{-1} \\
\end{align*}
\]

Here \( \tilde{\rho}_\Lambda = \hat{N}/|\Lambda| \) where \( \hat{N}^* \) satisfies (2.41) with \( \tilde{\mu}_\Lambda \) instead of \( \mu_0 \) and \( f^\text{GC}_{\Lambda,0}(\cdot) \) is the grand-canonical free energy given by

\[
\beta f^\text{GC}_{\Lambda,0}(\rho_\Lambda) := \sup_{\mu} \{ \beta \rho_\Lambda \mu - \beta p_{\Lambda,\beta,0}(\mu) \}.
\]

Remark 2.3. Note that: \( \beta f^\text{GC}_{\Lambda,0}(\tilde{\rho}_\Lambda) = \beta \tilde{\rho}_\Lambda \mu_0 - \beta p_{\Lambda,\beta,0}(\mu_0) \), \( (f^\text{GC}_{\Lambda,0}(\cdot))' \) is the \( \alpha \)th derivative of \( f_{\Lambda,0}\), \( \rho^*_{\Lambda} = \hat{N}^*/|\Lambda| \) and \( E_{\Lambda}(\alpha, u', \rho^*_\Lambda) \)

Theorem 2.4 (Local Moderate Deviations). Let \( \mu_0 \in \mathbb{R} \) be a chemical potential and let us fix zero boundary conditions. Let \( \hat{N} \) be as in (2.41) such that Theorem 2.1 holds. For \( \hat{N} \) and the set \( A_{\hat{N}} \) respectively given by (2.44) and (2.37) with \( \alpha \in [1/2, 1] \), we have:

\[
\begin{align*}
\left| \mathcal{P}^0_{\Lambda,\mu_0}(A_{\hat{N}}) - e^{-|\Lambda| f^\text{GC}_{\Lambda,0}(\tilde{\rho}_\Lambda)} \right| & \leq \frac{2e^{-|\Lambda| f^\text{GC}_{\Lambda,0}(\tilde{\rho}_\Lambda)}}{|\Lambda|^{1/2} |\Lambda|}, \\
\end{align*}
\]

where

\[
\begin{align*}
D_{\Lambda,0}(\rho^*_\Lambda) & := \left[ \beta \mathcal{F}'_{\Lambda,0}(\rho^*_\Lambda) \right]^{-1} \\
\end{align*}
\]

and

\[
\begin{align*}
D_{\Lambda,0}(\rho^*_\Lambda) & := \sum_{m=3}^{\infty} \frac{2(u')^{m-2} \mathcal{F}_{\Lambda,0}(\rho^*_\Lambda)}{m! |\Lambda|^{[(m-2)(1-\alpha)]}} \\
\end{align*}
\]

Here, \( m(\alpha) \) is given by \( m(\alpha) := \min \{ m \in \mathbb{N} \mid m(1-\alpha) - 1 > 0 \} \) and \( E_{\Lambda}(\alpha, u', \rho^*_\Lambda) \) is an error term of order \( |\Lambda|^{-[(m(\alpha)-1)-1]} \) defined via cluster expansion as

\[
\begin{align*}
E_{\Lambda}(\alpha, u', \rho^*_\Lambda) & := \beta \frac{(u')^{m(\alpha)} \mathcal{F}_{\Lambda,0}(\rho^*_\Lambda)}{m(\alpha)!} + \frac{(u' \mu_0 - \mathcal{F}_{\Lambda,0}(\rho^*_\Lambda))}{|\Lambda|^{1-m(\alpha)(1-\alpha)-1}} \\
\end{align*}
\]

where \( \mathcal{F}_{\Lambda,0}(\cdot) \) is the \( m \)-th derivative of \( \mathcal{F}_{\Lambda,0}(\cdot) \).
Corollary 2.5 (Local Central Limit Theorem.). Under the same assumptions as in Theorem 2.4 for $\alpha = 1/2$ we have that
\[
\left| \mathbb{E}^0_{\Lambda, \mu_\alpha}(A_N) - \frac{\exp \left\{ - \frac{(w_i')^2}{2D_{\Lambda,0}(\rho_{\Lambda})} \right\}}{\sqrt{2\pi D_{\Lambda,0}(\rho_{\Lambda})|\Lambda|}} \right| \leq \frac{2e^{-\frac{(w_i')^2}{2\pi D_{\Lambda,0}(\rho_{\Lambda})}} E_{|\Lambda|}(1/2, u', \rho_{\Lambda})}{\sqrt{2\pi D_{\Lambda,0}(\rho_{\Lambda})|\Lambda|}},
\]
(2.53)
where, using (2.47),
\[
D_{\Lambda,0}(\rho_{\Lambda}) = [\beta F_{\Lambda,0}(\rho_{\Lambda})]^{-1}
\]
(2.54)
and $E_{|\Lambda|}(1/2, u', \rho_{\Lambda})$ is an error term of order $|\Lambda|^{-1/2}$ defined via cluster expansion and given by (2.52).

For the proofs and the discussion related to the previous results we refer to Section 5.

3. CLUSTER EXPANSION AND ITS CONVERGENCE, PROOF OF THEOREM 2.1

The proof follows closely the strategy in [13]. For completeness of the presentation we repeat the main steps keeping track of the main modifications due to the lattice. The key idea is to view the canonical partition function (2.7) as a perturbation around the ideal case. Renormalizing with $|\Lambda|^N$ and defining
\[
Z^{\text{ideal}}_{\Lambda,N} := \frac{|\Lambda|^N}{N!} \quad \text{and} \quad Z^{\text{int}}_{\Lambda,\beta,0}(N) := \frac{1}{|\Lambda|^N} \sum_{\mathbf{x} \in \Lambda^N} e^{-\beta H^0_{\Lambda}(\mathbf{x})},
\]
(3.1)
we rewrite (2.7) as
\[
Z^0_{\Lambda,\beta}(N) = Z^{\text{ideal}}_{\Lambda,N} Z^{\text{int}}_{\Lambda,\beta,0}(N).
\]
Calling now $\mathcal{E}(N) := \{\{i, j\} \mid i, j \in \{1, ..., N\}\}$ and $f_{i,j} := e^{-\beta V(x_i, x_j)} - 1$ we have that the factor $e^{-\beta H^0_{\Lambda}(\mathbf{x})}$ can be expressed as
\[
e^{-\beta H^0_{\Lambda}(\mathbf{x})} = \prod_{1 \leq i < j \leq N} (f_{i,j} + 1) = \sum_{E \subseteq \mathcal{E}(N)} \prod_{(i,j) \in E} f_{i,j}
\]
where the term $+1$ is given by $E = \emptyset \subseteq \mathcal{E}(N)$. Note that we can associate to any set $E \subseteq \mathcal{E}(N)$ a graph $g \equiv (V(g), E)$ where $V(g) := \{i \in \{1, ..., N\} \mid \exists e \in E \text{ s.t. } i \in e\}$ is the set of its vertices and $E$ is the set of its edges. Moreover, a graph created from $E$ does not contain isolated vertices and can be viewed as the pairwise compatible (non-ordered) collection of its connected components, where two graphs $g$, $g'$ are called compatible ($g \sim g'$) if and only if $V(g) \cap V(g') = \emptyset$. In other terms given $E$ we can find a graph $g$ such that $g \equiv \{g_1, ..., g_k\}_\sim$ with $k \geq 1$, where, denoting by $\mathcal{C}_m$ the set of connected graphs with $m$ vertices, $g_l \in \mathcal{C}_m$ for all $l = 1, ..., k$ and $2 \leq m \leq N$. In this way we have that
\[
e^{-\beta H^0_{\Lambda}(\mathbf{x})} = \sum_{g \in \mathcal{C}_m} \prod_{g_l \text{ connected}} \prod_{(i,j) \in E(g_l)} f_{i,j},
\]
(3.2)
where the collection $\{g_1, ..., g_k\}_\sim = \emptyset$ gives the term 1 in the sum.

Hence, defining
\[
\zeta_{\Lambda}(V) := \sum_{\mathbf{g} \in \mathcal{C}_m} \frac{1}{|\Lambda||V|} \sum_{\mathbf{x} \in \Lambda^{|V|}} \prod_{(i,j) \in E(g)} f_{i,j},
\]
(3.3)
we get
\[
Z_{A,\beta,0}^{\text{int}}(N) = \sum_{\{V_1, \ldots, V_k\}_{i=1}^k} \prod_{|V_i| \geq 2} \zeta_{A}(V_i) = \exp \left\{ \sum_{I \in \mathcal{I}} c_I \zeta_{A}^{I} \right\}
\] (3.4)
where \(V_i \equiv V(g_i), \ l = 1, \ldots, k\) and the second equality of (3.4) holds under the validity of Lemma 3.1 and where we used
\[
c_I := \frac{1}{I!} \sum_{G \in \mathcal{G}_I} (-1)^{|E(G)|}
\] (3.5)
which comes from the polymer model representation described below (see also Section 3 of [13]).

An abstract polymer model \((\Delta, \mathcal{G}_{\Delta}, \omega)\) consists of a set of polymers \(\Delta := \{\delta_1, \ldots, \delta_{|\Delta|}\}\), a compatibility graph \(\mathcal{G}_{\Delta}\) with set of vertices \(\Delta\) and set of edges \(E_{\Delta}\) such that \(\{i, j\} \in E_{\Delta}\) if and only if \(\delta_i \neq \delta_j\) (i.e. \(\delta_i \cap \delta_j \neq \emptyset\)) and a weight function \(\omega : \Delta \rightarrow \mathbb{R}\).

In our case the set of polymers is given by \(\mathcal{V} := \{\{V_1, V_2, \ldots, V_k\} | V_i \subset \{1, \ldots, N\} \text{ and } |V_i| \geq 2 \ \forall \ i = 1, \ldots, k\}\) and the weight function is \(\zeta_{A}(V)\) defined in (3.3). In the second equality in (3.4) the sum in the exponent is over the set \(\mathcal{I}\) of all multi-indices \(I : \mathcal{V} \rightarrow \{0, 1, \ldots\}\), with \(\zeta_{A}^{I} = \prod_{V \in \mathcal{V}} \zeta(V)^{I(V)}\). Denoting also with \(\supp I := \{V \in \mathcal{V} | I(V) > 0\}\), \(\mathcal{G}_I\) is the graph with \(\sum_{V \in \supp I} I(V)\) vertices induced from \(\mathcal{G}_{\supp I} \subset \mathcal{G}_{\mathcal{V}}\) by replacing each vertex \(V\) by the complete graph on \(I(V)\) vertices.

We recall that (as it is observed in [13]), the sum in (3.5) is over all connected subgraphs \(G\) of \(\mathcal{G}_I\) spanning the whole set of vertices of \(\mathcal{G}_I\) and \(I! = \prod_{V \in \supp I} I(V)!\), indeed if \(I\) is not a cluster (i.e. \(\mathcal{G}_{\supp I}\) is not connected), then \(c_I = 0\).

Then from Sections 5 and 6 of [13] and using the representation above we have that \(F_{A,\beta,0}(n)\) is given by (2.18) where now we can define rigorously \(B_{\beta,\Lambda}(n)\) as
\[
B_{\beta,\Lambda}(n) := \frac{|\Lambda|^n}{n!} \sum_{A(I) = [n+1]} c_I \zeta_{A}^{I}\n\] (3.6)
where \(A(I) := \bigcup_{V \in \supp I} V \subset \{1, \ldots, N\}\) and \([n+1] := \{1, \ldots, n+1\}\).

The convergence of the cluster expansion is guaranteed from the following Lemma, in which we follow Theorem 1 - (ii) in [10].

**Lemma 3.1.** There exist constants \(C^r\) and \(a > 0\), such that when \(N/|\Lambda| < C^r\), the following holds:
\[
\sup_{i \in \{1, \ldots, N\}, V \in \mathcal{V}(N)} \sum_{i \in V} |\zeta_{A}(V)|e^{|V|} \leq e^a - 1.
\] (3.7)

**Proof.** Denoting with \(\mathcal{T}_n\) the set of trees with \(n = |V|\) vertices such that for a rooted tree \(T \in \mathcal{T}_n\) the set of the edges is given by \(E(T) = \{(i_1, j_1), \ldots, (i_{n-1}, j_{n-1})\}\) we have:
\[
\sum_{x \in \Lambda^n} \frac{1}{|\Lambda|^n} \prod_{(i,j) \in E(T)} |f_{i,j}| = \frac{1}{|\Lambda|^n} \sum_{x \in \Lambda^n} \prod_{k=1}^{n-1} |f_{i_k,j_k}|
\]
\[
\leq \frac{1}{|\Lambda|^n} \sum_{i_1} \sum_{y \in \Lambda^{n-1}} \prod_{k=2}^{n} |e^{-\beta V(y_k)} - 1| \leq \frac{1}{|\Lambda|^{n-1}} |C_{J,d}(\beta)|^{n-1}
\] (3.8)
where $i_1$ is considered as a root and $y$ is a vector in $\Lambda^{n-1}$ with components $y_k = x_{i_k} - x_{j_k}$, $\forall k = 2, ..., n$. In the previous estimate the boundary term gives a similar contribution which, however, is multiplied for $|\partial\Lambda|/|\Lambda|$ as in equations (4.18)-(4.21) in [14]. Hence, using the tree graphs inequality [12], we have

$$|\zeta_\Lambda(V)| \leq \frac{n^{n-2}}{|\Lambda|^{n-1}} e^{2\beta B(n-2)} [C_{J,d}(\beta)]^{n-1},$$

where $B$ is the stability constant defined in (2.9). Fixing now $i \in \{1, ..., N\}$, and using the fact that $\zeta_\Lambda(V)$ depends only on $|V|$, from (3.9), for the left hand side of (3.7) we get

$$\sup_{i \in \{1, ..., N\}} \sum_{V \in \mathcal{V}(N) : i \in V} |\zeta_\Lambda(V)| e^{a|V|} \leq e^{a-2\beta B} \sum_{n=2}^{N} \binom{N-1}{n-1} \frac{n^{n-2}}{|\Lambda|^{n-1}} \left( e^{(2\beta B+a)C_{J,d}(\beta)} \right)^{n-1}.$$ 

Then using the result from [10] (equations (3.12)-(3.15)) we have that the cluster expansion is absolutely convergent (uniformly in $N$ and $\Lambda$) when

$$\frac{N}{|\Lambda|} \leq R^C,$$

where

$$R^C := [e^{2\beta B C_{J,d}(\beta)}]^{-1} \left\{ \max_{a > 0} \frac{\ln[1 + u(1 - e^{-a})]}{a[1 + u(1 - e^{-a})]} \right\}.$$ (3.10)

For the conclusion of the proof of Theorem 2.1 we refer the reader to [13], Sections 5 and 6.

Remark 3.1. Lemma 3.1 and then Theorem 2.1 also hold if we consider $\gamma \neq 0$ fixed boundary conditions. Indeed, defining $\nu_\Lambda(x_i|\gamma) := e^{-\beta \sum_{j \geq 1} V(x_i, \gamma_j)} > 0$, which is 1 if $d_1(x, \Lambda^c) := \inf_{x' \in \Lambda^c} \{ |x - x'| | x \in \Lambda \} > 1$, we can write (2.7) as

$$Z_\Lambda^\gamma(N) = \frac{1}{N!} \sum_{\mathbf{x} \in \Lambda^N} e^{-\beta H_\Lambda^\gamma(x)} \prod_{i=1}^{N} \nu_\Lambda(x_i|\gamma),$$

where we used

$$H_\Lambda^\gamma(x) = H_\Lambda^0(x) + \sum_{1 \leq i \leq N, x_i \in \Lambda \atop j \geq 1, \gamma_j \in \Lambda^c} V(x_i, \gamma_j).$$

Then noting that

$$e^{\beta B} \leq \nu_\Lambda(x_i|\gamma) \leq e^{\beta dB}$$

estimate (3.8) is here given by

$$\sum_{\mathbf{x} \in \Lambda^N} \prod_{i=1}^{N} \frac{\nu_\Lambda(x_i|\gamma)}{|\Lambda|^n} \prod_{(i,j) \in E(T)} |f_{i,j}| \leq \begin{cases} \frac{C_{J,d}(\beta)^{n-1}}{|\Lambda|^{n-1}}, \text{if } d_1(x_i, \Lambda^c) > 1 \forall i = 1, ..., n, \\
\frac{e^{\beta dB}}{|\Lambda|^{n-1}} \left\{ \frac{|\partial\Lambda|}{|\Lambda|} \left[ e^{\beta dB C_{J,d}(\beta)} \right]^{n-1} \right\}, \text{otherwise.} \end{cases}$$

The latter implies that near the boundary the convergence condition is better (for $\Lambda$ large enough) and also that the sum on the clusters close to $\Lambda^c$ gives a contribution of order $|\partial\Lambda|/|\Lambda|$ which vanishes as $|\Lambda| \to \infty$. 
Density convergence of free energy and pressure. We derive now the density expansion working in the grand-canonical ensemble and starting from the Ising model. Then let us consider the presence of a negative external magnetic field \( h \in \mathbb{R}^- \) such that there is a very strong incentive for the spins to take value \(-1\). Below we recall the relation (2.34) between the grand-canonical partition function of the Ising model and the one defined in (2.30)

\[
\tilde{\Xi}_{\Lambda,\beta}^-(h) = \exp \left\{ -\beta |\Lambda| \left[ h - J \frac{|E_{\Lambda}|}{|\Lambda|} \right] \right\} \Xi_{\Lambda,\beta}^0(\mu_{\Lambda}), \quad \text{with} \quad \mu_{\Lambda} = 2h - 4J \frac{|E_{\Lambda}|}{|\Lambda|}.
\]

For the cluster expansion of \( \Xi_{\Lambda,\beta}^0(\mu_{\Lambda}) \) we can follow [12]. Hence, when

\[
e^{\beta \mu_{\Lambda} + \beta B \tilde{C}_{J,d}(\beta)} < e^{-1} \iff \mu_{\Lambda} \leq -\frac{1}{\beta} \log \left( e^{\beta B + 1} \tilde{C}_{J,d}(\beta) \right) =: \mathcal{M}_{LG},
\]

where we defined

\[
\tilde{C}_{J,d}(\beta) := \sum_{x \in \mathbb{Z}^d} \left( 1 - e^{-\beta |V(x)|} \right) = 1 + 2d(1 - e^{-4\beta J})
\]

with \( B \) given by (2.9), (2.30) can be written as

\[
\Xi_{\Lambda,\beta}^0(\mu_{\Lambda}) = \exp \left\{ \sum_{N \geq 1} \frac{e^{\beta \mu_{\Lambda} N}}{N!} \sum_{g \in C_N} \left( \sum_{x \in \Lambda^N} \prod_{\{i,j\} \in E(g)} f_{i,j} \right) \right\},
\]

where the series in the exponent is absolutely convergent. Then from (2.29), and (3.13) we find

\[
\beta p_\beta(\mu) = \sum_{n \geq 1} e^{\beta \mu n} b_n,
\]

with \( \mu := \lim_{\Lambda \to \mathbb{Z}^d} \mu_{\Lambda} = 2h - 4Jd \) and where \( b_n \)'s are the discrete version of the connected Mayer’s coefficient [9] given by

\[
b_n := \frac{1}{n!} \sum_{g \in C_n} \sum_{x \in \mathbb{Z}^d} \prod_{\{i,j\} \in E(g)} f_{i,j}.
\]

Defining now the density as

\[
\rho \equiv \rho(\mu) := \beta \frac{\partial p_\beta(\mu)}{\partial \log(e^{\beta \rho})} = \frac{\partial p_\beta(\mu)}{\partial \mu} = \sum_{n \geq 1} n e^{\beta \mu n} b_n,
\]

(3.14)

we find

\[
B^* := 4J \quad \text{and applying Theorem 4.1 in [6] we get}
\]

\[
\beta \mu \equiv \beta \rho(\rho) = \log \rho - \sum_{n \geq 1} \beta_n \rho^n \quad \text{and} \quad \beta p_\beta(\rho) = \rho + \sum_{n \geq 1} \frac{n \beta_n}{n+1} \rho^{n+1},
\]

(3.15)

when

\[
\rho \leq \mathcal{R}_{\psi}^{LG} := \left( 2e^{1+\beta[J(2d+1)]} \tilde{C}_{J,d}(\beta) \right)^{-1},
\]

(3.16)

where \( \beta_n \)'s are given by (2.14) and \( \tilde{C}_{J,d}(\beta) \) is defined in (3.12). Moreover from (2.23) and (3.15) we can recover (explicitly) the Legendre transform relations between the thermodynamic free energy and the thermodynamic pressure

\[
\beta f_\beta(\rho) = \sup_{\mu} \left\{ \rho \mu - \beta p_\beta(\mu) \right\},
\]

(3.17)

\[
\beta p_\beta(\mu) = \sup_{\rho} \left\{ \rho \mu - \beta f_\beta(\rho) \right\}.
\]

(3.18)

Let us also observe that, as for the free energy, defining the thermodynamic pressure for the Ising model as

\[
\psi_\beta(h) := \lim_{\Lambda \to \mathbb{Z}^d} \frac{1}{|\Lambda|} \log \tilde{\Xi}_{\Lambda,\beta}^-(h)
\]
we have
\[ \beta p_\beta (\mu) = \beta \psi_\beta (h) - \beta Jd + \beta h. \quad \text{(3.19)} \]

Then (2.24), (3.17) and (3.19) give us the following relation between the thermodynamic free energy and the thermodynamic pressure for the Ising model
\[ \beta \phi_\beta (m) = \sup_h \left\{ 2h \left( \frac{m+1}{2} \right) - \beta h - \beta \psi_\beta (h) \right\}. \]

In the next Figure we compare \( R^C \) and \( R^{\text{LG}} \) in dimension 1, with \( J = 1, 2, 5 \) and \( \beta \in [0, 1] \). We have that, as in Figure 1, the grand-canonical radius of converge is bigger than the one obtained in canonical ensemble.

![Figure 3. \( R^C \) (continuous line) and \( R^{\text{LG}} \) (dashed line) in dimension 1 with \( J = 1 \) (green lines), \( J = 2 \) (blue lines), \( J = 5 \) (red lines) and \( \beta \in [0, 1] \).](image)

**Remark 3.2.** Below we recall the classical contour cluster expansion for the grand-canonical partition function for the Ising model in order to do the comparisons expressed in Figures 2 and 4. We rewrite (2.31) as
\[ \tilde{\Xi}_n^\beta (h) = \exp \left\{ \beta |\Lambda| \left[ J \frac{|E_\Lambda|}{|\Lambda|} - h \right] \right\} \Xi_n^{\text{nt}} (z_h), \]
where

\[ \Xi_{\Lambda, \beta}^{inh}(z_h) := 1 + \sum_{n \geq 1} \frac{1}{n!} \sum S_i \cdots \sum S_n \prod_{1 \leq i < j \leq n} (\hat{f}_{i,j} + 1) \prod_{i=1}^{n} w(S_i) z_h^{[S_i]} . \]

In the last definition we used the notation: \( S_i := \{ x \in \Lambda \mid \sigma(x) = +1 \text{ and } |x - x'| = 1, \forall x, x' \in S_i \} , \)

\[ \hat{f}_{i,j} \equiv \hat{f}(S_i, S_j) := \begin{cases} -1 & \text{if } \inf \{|x - x'| x \in S_i, x' \in S_j\} \leq 1, \\ 0 & \text{otherwise,} \end{cases} \]

\[ z_h = \exp\{2\beta h\} \text{ and } w(S) := \exp\{-2\beta J[\partial_c S]\} , \text{ where } \partial_c S := \{ \{x, x'\} \mid |x - x'| = 1, x \in S, x' \not\in S \}. \]

In this case, denoting with \( [S]_1 := \{ x \in \mathbb{Z}^d \mid d_1(x, S) \leq 1 \} , \) we have

\[ \log \Xi_{\Lambda, \beta}^{inh}(h) = \sum_{n \geq 1} \sum_{S \subset \Lambda} \cdots \sum_{S \subset \Lambda} \frac{1}{n!} \sum_{g \in \mathcal{C}_n} \prod_{(i,j) \in E(g)} \hat{f}_{i,j} \prod_{i=1}^{n} w(S_i) z_h^{[S_i]} \]

and

\[ 1 + \sum_{n \geq 2} \frac{1}{(n-1)!} \sum_{S \subset \Lambda} \cdots \sum_{S \subset \Lambda} \sum_{g \in \mathcal{C}_n} \prod_{(i,j) \in E(g)} \hat{f}_{i,j} \prod_{i=2}^{n} z_h(S_i) \leq e^{[S^*_1]}, \]

under the condition (Section 5.7.1 in [4])

\[ \sum_{S \subset \Lambda} |w_h(S) z_h^{[S]}||\hat{f}(S, S^*)|e^{[S^*_1]} \leq [[S^*_1]], \forall S^* \subset \Lambda. \quad (3.20) \]

Hence, having that \( [[S^*_1]] \leq (2d + 1)||S|| \) and

\[ \sum_{S \subset \Lambda} |w_h(S) z_h^{[S]}||\hat{f}(S, S^*)|e^{[S^*_1]} \leq [[S^*_1]] \sum_{S \geq 0} |w_h(S) z_h^{[S]}|e^{[S^*_1]} \]

\[ \leq [[S^*_1]] \sum_{n \geq 1} e^{n[2\beta h + 2d + 1 + 2 \log(2d)]}, \]

(3.20) is valid when

\[ K(h, d) := [e^{-(2\beta h + 2d + 1 + 2 \log(2d))} - 1]^{-1} = \sum_{n \geq 1} e^{n[2\beta h + 2d + 1 + 2 \log(2d)]} \leq 1, \]

i.e., for all \( h \) such that \( h \leq h_{IS} := -\frac{1}{2\beta} (2d + 1 + 2 \log(2d) + \log 2) . \)

From (2.33), \( h_{IS} \) gives us

\[ M_{IS} := 2h_{IS} - 4dJ. \quad (3.21) \]

Below we compare \( M_{LG} \) and \( M_{IS} \) in dimension 1, with \( J = 1, 3 \) and \( \beta \in [0, 1] \) finding a similar result to the one represented in Figure 2.
4. Decay of correlations in the canonical ensemble, proof of Theorem 2.2

For the proof of Theorem 2.2 we follow the strategy of [8]. Let \( n \in \mathbb{N}_0 \) and \( k \in \mathbb{N} \). We denote with \( C_{n, n+k} \) the set of connected graphs with \( n + k \) vertices, where we singled out \( n \) vertices which will be called “white” and the remaining \( k \) vertices will be called “black”. Moreover, we call articulation vertex a vertex such that removing it the graph is decomposed in two or more separate part, where at least one of them does not contain white vertices. Hence, we denote with \( \mathcal{B}^{AP}_{n, n+k} \) the set of graphs with \( n \) white and \( k \) black vertices and without articulation vertices.

We define the \( n \)-point correlation function with \( n \leq N \) as:

\[
\rho^{(n)}_{\Lambda, N}(q_1, ..., q_n) := \frac{1}{(N-n)!} \sum_{\mathbf{x} \in \Lambda^{N-n}} \frac{1}{Z^{\text{per}}_{\Lambda, \beta}(N)} e^{-\beta H_{\Lambda, \beta}^{\text{per}}(q_1, ..., q_n, \mathbf{x})},
\]

where with \( \{q_i\}_{i=1}^n \subset \Lambda \) we denote the fixed particles and \( Z^{\text{per}}_{\Lambda, \beta}(N) \) is given by (2.7) with periodic boundary conditions. When we will do the cluster expansion, the fixed particles \( \{q_i\}_{i=1}^n \) will correspond to the white vertices in the connected graphs (clusters).
Denoting with $\mu_{\Lambda,\beta,N}(\cdot)$ the canonical Gibbs measure in the volume $\Lambda$, i.e.,
$$\mu_{\Lambda,\beta,N}(C) := \frac{1}{Z_{\beta,\Lambda}(N)} \frac{1}{N!} \sum_{x \in \Lambda^N \cap C} e^{-\beta H_{\per}(x)},$$
where $C \subset (\mathbb{Z}^d)^N$, we define for a test function $\varphi$, the Bogoliubov functional $L_B(\varphi)$ as
$$L_B(\varphi) := \sum_{x \in \Lambda^N} \prod_{k=1}^N (1 + \varphi(x_k)) \mu_{\Lambda,\beta,N}(\{x\}).$$

We can define implicitly the truncated $n$-point correlation function $u_{\Lambda,\beta,N}^{(n)}(\cdot)$ by its generating function which is the logarithm of the Bogoliubov functional, i.e.,
$$\log L_B(\varphi) := \sum_{n \geq 1} \frac{1}{n!} \sum_{x \in \Lambda^n} \varphi(x_1) \cdots \varphi(x_n) u_{\Lambda,\beta,N}^{(n)}(x_1,\ldots,x_n), \quad (4.1)$$
where, for example, when $n = 2$ and fixing $q_1, q_2 \in \Lambda$, $u_{\Lambda,\beta,N}^{(2)}(q_1, q_2)$ is given by (2.25).

The extended (canonical) partition function is defined as
$$Z_{\Lambda,\beta,N}^{\per}(\alpha \varphi) := \frac{1}{N!} \sum_{x \in \Lambda^N} \prod_{i=1}^N (1 + \alpha \varphi(x_i)) e^{-\beta H_{\per}(x)}$$
with $\alpha \in \mathbb{R}$, such that
$$L_B(\alpha \varphi) = \frac{Z_{\Lambda,\beta,N}^{\per}(\alpha \varphi)}{Z_{\Lambda,\beta,N}^{\per}(0)}, \quad \text{where } Z_{\Lambda,\beta,N}^{\per}(0) = \sum_{n \geq 1} \frac{1}{n!} \sum_{x \in \Lambda^n} \varphi(x_1) \cdots \varphi(x_n) u_{\Lambda,\beta,N}^{(n)}(x_1,\ldots,x_n).$$

and then, thanks to (4.1) for all $n \geq 1$, we have
$$\sum_{x \in \Lambda^n} \varphi(x_1) \cdots \varphi(x_n) u_{\Lambda,\beta,N}^{(n)}(x_1,\ldots,x_n) = \frac{\partial^n}{\partial \alpha^n} \log Z_{\Lambda,\beta,N}^{\per}(\alpha \varphi) \bigg|_{\alpha=0} \cdot \quad (4.2)$$

Using the polymer model representation recalled in Section 3, with set of polymers $V_N := \{(V_1, A_1), \ldots, (V_k, A_k)\} \mid V_i \in \{1, \ldots, N\}, |V_i| \geq 2$, and $A_i \subset V_i \forall i = 1, \ldots, k$ where the compatibility relation is here given by $(V_i, A_i) \sim (V_j, A_j) \Leftrightarrow V_i \cap V_j = \emptyset$ and with weights
$$\tilde{\zeta}_\Lambda((V,A)) := \alpha^{|A|} \sum_{g \in C_V} \frac{1}{|A||V(g)|} \sum_{x \in \Lambda^{|V(g)|}} \prod_{\{i,j\} \in E(g)} f_{i,j} \prod_{i \in A} \varphi(x_i),$$
for $N/|\Lambda|$ small enough (see Theorem 2.1 in [8]), we have
$$\log Z_{\Lambda,\beta,N}^{\per}(\alpha \varphi) = \log Z_{\Lambda,\beta,N}^{\per}(N) + \sum_{n=1}^N \sum_{m=1}^{N-m} \sum_{k=0}^m \binom{N}{m+k} \binom{m+k}{m} \alpha^n \sum_{I : \bigcup_{(V,A) \in \supp I} A = [m]} \sum_{\bigcup_{(V,A) \in \supp I} V = [m+k]} \sum_{\sum_{(V,A) \in \supp I} |A| I((V,A)) = n} c_I \tilde{\zeta}_\Lambda$$
$$= \log Z_{\Lambda,\beta,N}^{\per}(N) + \sum_{n=1}^N \sum_{m=1}^{N-m} \sum_{k=0}^m \alpha^n \hat{P}_{N,|\Lambda|}(m+k) \hat{B}_{\Lambda,\beta}(n, m, k), \quad (4.3)$$
where
$$\hat{P}_{N,\Lambda}(n) := \begin{cases} N(N-1)(N-2) \cdots (N-n+1), & \text{for } n \leq N, \\ 0, & \text{otherwise}, \end{cases}$$
and

\[ \hat{B}_{\Lambda,\beta}(n, m, k) := \frac{|\Lambda|^{(m+k)}}{m!k!} \sum_{I: \cup_{(V,A) \in \text{supp} I} V = [m] \cup_{(V,A) \in \text{supp} I} \Lambda = [n] \cup_{(V,A) \in \text{supp} I} |A| f((V,A)) = n} c_I \tilde{\zeta}_I. \]

The term \( \hat{B}_{\Lambda,\beta}(n, m, k) \) can be written as

\[ \hat{B}_{\Lambda,\beta}(n, m, k) = \hat{B}_{\Lambda,\beta}(n, k) \delta_{n,m} + R_{\Lambda,\beta}(n, m, k) \quad (4.4) \]

with

\[ \hat{B}_{\Lambda,\beta}(n, k) := \frac{|\Lambda|^{(n+k)}}{n!k!} \sum_{I: A(I) = [n+k]} c_I \tilde{\zeta}_I = \frac{1}{n!k!} \sum_{g \in B_{\Lambda,\beta}^{n+k}} \sum_{x \in \Lambda^{n+k}} \prod_{i=1}^n f_{i,j} \prod_{i=1}^n \varphi(x_i). \quad (4.5) \]

In the previous definition with * we mean that the sum runs over all multi-indices which satisfy \( n + k = |V_n| + \sum_{(V,A) \in \text{supp} I, V \neq V_0} (|V| - 1) \) and \( I((V,A)) = 1 \) for all \( (V,A) \in \text{supp} I \) and where \( V_0 \) contains the indices 1, 2, ..., \( n \). The second form of \( \hat{B}_{\Lambda,\beta}(n, k) \) expressed in (4.5) is due to the fact that we consider here periodic boundary conditions (Lemma 4.1 in [8]).

Hence, from (4.2), (4.3) and (4.4) we get:

\[ \frac{1}{2} \sum_{(x_1,x_2) \in \Lambda^2} \varphi(x_1)\varphi(x_2)u^{(2)}_{\Lambda,N}(x_1,x_2) = \sum_{k=0}^{N-1} \hat{P}_{N,|\Lambda|}(1 + k)R_{\Lambda,\beta}(2,1,k) \]

\[ + \sum_{k=0}^{N/2} \hat{P}_{N,|\Lambda|}(2 + k)\hat{B}_{\Lambda,\beta}(2,k). \quad (4.6) \]

The first sum gives a contribution of order \( |\Lambda|^{-1} \). This estimate comes from the fact that the term \( R_{\Lambda,\beta}(n, m, k) \) consists of lower order terms and in particular from [14] (as it is also recalled [8]), we have

\[ |R_{\Lambda,\beta}(n, m, k)| \leq C \frac{1}{|\Lambda|}, \quad (4.7) \]

for all \( n, k \) and uniformly on \( \varphi \).

For the first term \( (k = 0) \) in the second sum \( (n = m = 2) \) we have

\[ \frac{1}{2} \left| \sum_{(x_1,x_2) \in \Lambda^2} \frac{N(N-1)}{|\Lambda|^2} f_{1,2} \varphi(x_1)\varphi(x_2) \right| \]

\[ \leq \frac{1}{2} \left[ \left( \frac{N}{|\Lambda|} \right)^2 + \frac{N}{|\Lambda|^2} \right] \sum_{(x_1,x_2) \in \Lambda^2} |\varphi(x_1)\varphi(x_2)| \left[ (e^{4|\Lambda|} - 1)1_{|x_1-x_2| = 1} + 1_{x_1 = x_2} \right]. \quad (4.8) \]

For \( k \geq 1 \) we will use the analogous of Lemma 4.2 (which is recalled below) in [8] in order to exchange the sum over \( k \) and the one over \( x \).

**Lemma 4.1.** For any \( n \geq 2 \) and \( k \geq 1 \) we have that

\[ \hat{P}_{N,|\Lambda|}(n + k)\hat{B}_{\Lambda,\beta}(n;k) \leq C \left( \frac{N}{|\Lambda|} \right)^2 e^{-ck}, \]
where
\[ \hat{B}_{\Lambda,\beta}(n; k) := \frac{1}{n!k!} \sum_{x \in \Lambda^k} \sum_{g \in B_{n,n+k}^{\Lambda}} \prod_{i,j} f_{i,j}, \]
for some \( c > 1 \) and \( C > 0 \) independent on \( k, N \) and \( \Lambda \).

**Proof.** The proof follows immediately from [8]. Indeed the calculation is similar to the one presented by the authors for the proof of Lemma 4.2 and the fact that we can choose \( c > 1 \) is possible thanks to their Theorem 3.1.

Moreover we multiply and divide for \( e^{x_1-x_2} \). Hence from the fact that \( |x_1-x_2| \leq |V_0| - 1 \leq k \) and using the second equality of (4.5), we find
\[
\begin{align*}
\left| \sum_{k=1}^{N-2} \tilde{P}_{N,|\Lambda|}(2+k) \hat{B}_{\Lambda,\beta}(2,k) \right| &\leq \frac{1}{2} \sum_{(x_1,x_2) \in \Lambda^2} |\varphi(x_1)\varphi(x_2)| e^{-|x_1-x_2|} |V_0| \sum_{k=1}^{N-2} \tilde{P}_{N,|\Lambda|}(2+k) \hat{B}_{\Lambda,\beta}(2,k) \\
&\leq C \left( \frac{N}{|\Lambda|} \right)^2 \sum_{(x_1,x_2) \in \Lambda^2} |\varphi(x_1)\varphi(x_2)| e^{-|x_1-x_2|} \sum_{k=1}^{N-2} e^{-(c-1)k} \\
&\leq C_1 \left( \frac{N}{|\Lambda|} \right)^2 \left[ \sum_{(x_1,x_2) \in \Lambda^2} |\varphi(x_1)\varphi(x_2)| e^{-|x_1-x_2|} \right], \quad (4.9)
\end{align*}
\]
where \( c \) and \( C \) are the constants of Lemma 4.1 and \( C_1 \) is a positive constant bigger than \( C \) and independents on \( N, \Lambda \).

Then from (4.6), (4.7), (4.8) and (4.9) we have
\[
\begin{align*}
\sum_{(x_1,x_2) \in \Lambda^2} |\varphi(x_1)\varphi(x_2)||u_{L,N}^{(2)}(x_1,x_2)| &\leq \sum_{(x_1,x_2) \in \Lambda^2} |\varphi(x_1)\varphi(x_2)| \left\{ \left( \frac{N}{|\Lambda|} \right)^2 \left[ (e^{4\beta J} - 1) \mathbf{1}_{|x_1-x_2|=1} + \mathbf{1}_{x_1=x_2} \right] \\
&\quad + \frac{(e^{4\beta J} - 1) \mathbf{1}_{|x_1-x_2|=1} + \mathbf{1}_{x_1=x_2}}{|\Lambda|} + Ce^{-|x_1-x_2|} \right\} + C_1 \frac{1}{|\Lambda|} \right\}
\end{align*}
\]
with \( C, C_1 \in \mathbb{R}^+ \). Then the conclusion follows choosing as test functions the Kronecker deltas in \( q_1 \) and \( q_2 \).

5. **Precise large and local moderate deviations, proofs of Theorems 2.3, 2.4 and Corollary 2.5**

In this section we compare our approach for the study of precise large and local moderate deviations (Theorems 2.3, 2.4 and Corollary 2.5) with the ones presented in [1] and, in particular, in [2]. For the proofs of the Theorems we refer to [15], since once one can write \( \log Z_{\Lambda,\beta}^{n} \) as a power series of the density (Theorem 2.1) then the proof is the same. Furthermore, and differently from [15], thanks to the lattice structure, we can have general boundary conditions (Remark 3.1) while for the continuous case an extra assumption (superstable potential) would be necessary for a not zero boundary conditions.
**Proof of Theorem 2.3.** Follows from Theorem 2.1 and the proof of Theorem 2.1 in [15]. □

**Proof of Theorem 2.4.** Follows from Theorem 2.1 and the proof of Theorem 2.2 in [15]. □

**Proof of Corollary 2.5.** Follows from Theorem 2.1 and the proof of Corollary 2.2.1 in [15]. □

In order to do the comparison, we briefly recall the approach followed in [1] and [2]. For a fixed chemical potential $\mu_0$, we define the logarithmic generating function for the moments at finite volume associated to the probability given by (2.36) as

$$L^0_{\Lambda,\beta,\mu_0}(\mu) := \log \left[ \sum_{N \geq 0} \mathbb{P}^0_{\Lambda,\mu_0}(A_N)e^{\beta \mu N} \right],$$  \hspace{1cm} (5.1)

with $A_N$ given by (2.37). From (2.39) and (5.1) we have

$$\bar{\rho}_\Lambda := \mathbb{E}^0_{\Lambda,\mu_0} \left[ \frac{N}{|\Lambda|} \right] = \frac{1}{|\Lambda|} \frac{1}{\beta} \frac{d}{d\mu} L^0_{\Lambda,\beta,\mu_0}(\mu) \bigg|_{\mu=0},$$  \hspace{1cm} (5.2)

and

$$\sigma^2_{\Lambda,0}(\mu_0) := \mathbb{E}^0_{\Lambda,\mu_0} \left[ \frac{(N - \bar{\rho}_\Lambda |\Lambda|)^2}{|\Lambda|} \right] = \frac{1}{|\Lambda|} \frac{1}{\beta^2} \frac{d^2}{d\mu^2} L^0_{\Lambda,\beta,\mu_0}(\mu) \bigg|_{\mu=0}. \hspace{1cm} (5.3)$$

In general, denoting by $G^m_{\Lambda,0}$ the $m$-th moment per unit of volume, we have:

$$G^m_{\Lambda,0} = \frac{1}{\beta^m} \frac{d^m}{d\mu^m} L^0_{\Lambda,\beta,\mu_0}(\mu) \bigg|_{\mu=0}. \hspace{1cm} (5.4)$$

Let us define the characteristic function as

$$\psi_{\Lambda,\mu'}(t) := \sum_{N \geq 0} \mathbb{P}^0_{\Lambda,\mu}(A_N)e^{itN}, \hspace{1cm} (5.5)$$

where for $\mu' = \mu + \mu_0$, the “excess (by $\mu$) probability measure” is given by

$$\mathbb{P}^0_{\Lambda,\mu+\mu_0}(A_N) := \exp \left\{ -L^0_{\Lambda,\beta,\mu_0}(\mu) + \beta \mu N \right\} \mathbb{P}^0_{\Lambda,\mu_0}(A_N).$$

First, for the large deviations, i.e., considering a deviation $\tilde{N}$ given by (2.38) with $\alpha = 1$, the probability of $A_{\tilde{N}}$ can be expressed using the excess measure optimizing over $\mu$ such that $\mathbb{P}^0_{\Lambda,\mu+\mu_0}(A_{\tilde{N}}) \sim 1$, i.e., by making $\tilde{N}$ “central” with respect to the new measure. In this way we obtain

$$\lim_{|\Lambda| \to \infty} (\beta |\Lambda|)^{-1}\log \mathbb{P}^0_{\Lambda,\mu_0}(A_{\tilde{N}}) = \lim_{|\Lambda| \to \infty} \frac{-T^0_{\Lambda,\beta,\mu_0}(\tilde{N})}{\beta |\Lambda|} = -\frac{I_\beta(\bar{\rho}; \rho_0)}{\beta},$$

where

$$T^0_{\Lambda,\beta,\mu_0}(\tilde{N}) := \sup_{\mu} \left\{ \beta \mu \tilde{N} - L^0_{\Lambda,\beta,\mu_0}(\mu) \right\} \hspace{1cm} (5.6)$$

and

$$I_\beta(\bar{\rho}; \rho_0) := \beta f_\beta(\bar{\rho}) - \beta f_\beta(\rho_0) - \beta f'_\beta(\rho_0)(\bar{\rho} - \rho_0).$$

In the previous formulas we have that the quantity $\rho_0$, which is the limit of $\bar{\rho}_\Lambda$ as $\Lambda \to \infty$, is also such that

$$f'_\beta(\rho_0) = \mu_0 \Leftrightarrow p'_\beta(\mu_0) = \rho_0.$$
where the last relations follow from (3.17), (3.18) and the fact that we are far from the phase transition (see also (2.39)). Moreover, let us note that from (2.42) \( \rho_0 \) is also the thermodynamic limit of \( \rho_* \). For later use, from (2.48), we have that (2.46) is the “volume normalized version” of (5.6), i.e
\[
\mathcal{I}_{\beta,\Lambda}^{GC}(\rho_0; \tilde{\rho}_\Lambda) = \mathcal{I}_{\beta,\Lambda}^0(N)|\Lambda|^{-1}.
\]  
(5.7)

A more precise formula at finite volume as well as the higher order corrections come from the inversion of (5.5):
\[
\mathcal{P}_{\Lambda,\beta,0}(A_N) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i t N} \varphi_{\Lambda,\beta,0}(t) dt
\]  
(5.8)

where by \( \tilde{\rho}_\Lambda \) we denote the optimal chemical potential found in (5.6). This is also the approach of [1] for \( \alpha = 1/2 \) and \( \tilde{\rho}_\Lambda = \mu_0 \). In [1] and [2] the authors, starting from (5.8), recover the inversion of the characteristic function of the Gaussian distribution which gives them, calculating the integral, a finite volume formula with an approximation for the high order correction term. This can be done by the Taylor expansion at the second order of the characteristic function around \( t = 0 \) and applying, for instance, the Gaeßken’s method to estimate the integral. In particular, we refer to equations (4.1)-(4.10) of Section 4 in [1] and equations (2.1.30)-(2.1.34) Subsection 2.1 in [2]. On the other hand, our results come from a direct approach without passing form the calculation of the integral in (5.8), which also gives us an explicit formulation of the error terms. In fact, considering Theorem 2.3, the numerator in the fraction in the left hand side of (2.45) comes immediately from definition (2.48) and the Radon-Nikodym derivative of our probability measure with respect to the one with \( \tilde{\rho}_\Lambda \) (instead of \( \mu_0 \)). This can be clearly observed in equation (2.53) in [15]. Moreover, thanks to the explicit formula that we have for the finite volume free energy (Theorem 2.1), together with \( Z_{\Lambda,\beta}^0(N) = \exp \left\{ -|\Lambda| f_{\Lambda,\beta,0}(N) \right\} \) and (2.40), we can also obtain in an explicit and direct way both the normalization as well as the error terms as it is shown in Lemmas 3.3 in [15].

Second, starting from (5.6) and considering the approach expressed in [2], one can go a step further and study the local moderate deviations (\( \alpha \in [1/2, 1] \) in (2.38) by taking the Taylor expansion of (5.6) around \( \tilde{\rho}_\Lambda |\Lambda| \) and obtaining:
\[
\mathcal{I}_{\Lambda,\beta,\mu_0}(N) = \beta^2 \frac{(N - \tilde{\rho}_\Lambda |\Lambda|)^2}{2|\Lambda|\sigma_{\Lambda,0}^2(\mu_0)} + \sum_{j \geq 3} \frac{Q_{\Lambda,0}^{(j)}}{j!} \left( \frac{N - \tilde{\rho}_\Lambda |\Lambda|}{|\Lambda|} \right)^j,
\]  
(5.9)

where the coefficients \( Q_{\Lambda,0}^{(j)} \) are polynomials which can be computed via the moments (5.4) as it is explained next. In the previous equation we used (5.7) and the fact that, from (2.48), we have \( (f_{\Lambda,\beta,0}^{GC})'(\tilde{\rho}_\Lambda) = [\sigma_{\Lambda,0}^2(\mu_0)]^{-1} = \beta [\sigma_{\Lambda,0}^2(\mu_0)]^{-1} \). Note also that in (5.9) we do not have the terms \( \mathcal{I}_{\Lambda,\beta,\mu_0}^0(\tilde{\rho}_\Lambda |\Lambda|) \) and \( (\mathcal{I}_{\Lambda,\beta,\mu_0}^0)'(\tilde{\rho}_\Lambda |\Lambda|) \). This happens because, using the fact that \( L_{\Lambda,\beta,\mu_0}^0(\mu) \) is a strictly convex function of \( \mu \), the supremum in (5.6) is obtained at \( \mu = 0 \) when we consider \( \tilde{\rho}_\Lambda |\Lambda| \) instead of \( \bar{N} \) (see also (5.7)).

In [2] (equations (1.2.18)-(1.2.23)), the polynomials \( Q_{\Lambda,0}^{(j)} \) are calculated substituting
\[
\beta(N - \tilde{\rho}_\Lambda |\Lambda|) = (L_{\Lambda,\beta,\mu_0}^0)'(\tilde{\rho}_\Lambda) - (L_{\Lambda,\beta,\mu_0}^0)'(0) = \beta \tilde{\rho}_\Lambda \sigma_{\Lambda,0}^2(\mu_0)|\Lambda| + \sum_{m \geq 3} \frac{(\beta \tilde{\rho}_\Lambda)^{m-1} G_{\Lambda,0}^m}{(m - 1)!}
\]
where the meaning of \((T^{(m)}_{\Lambda,\beta,\mu_0})'(\tilde{N})\) is expressed in (5.11) for \(N = \tilde{N}\), so that one obtains
\[
Q^{(m)}_{\Lambda,0} = \frac{G^3_{\Lambda,0}}{(\sigma^2_{\Lambda,0}(\mu_0)|\Lambda|^3)},
\]
where the expression \(P(x_1,\ldots,x_n)\) means polynomial in \(x_1,\ldots,x_n\). For example
\[
Q^{(3)}_{\Lambda,0} = \frac{-G^3_{\Lambda,0}}{(\sigma^2_{\Lambda,0}(\mu_0)|\Lambda|^3)} + 3 \left(\frac{G^3_{\Lambda,0}}{(\sigma^2_{\Lambda,0}(\mu_0)|\Lambda|^3)}\right)^2 + 5 \left(\frac{G^3_{\Lambda,0}}{(\sigma^2_{\Lambda,0}(\mu_0)|\Lambda|^3)}\right)^3.
\]
We observe that also in this case our results expressed in Theorem 2.4 and Corollary 2.5, follow directly from Theorem 2.1, (2.22), (2.40) and the Taylor expansion of the free energy defined in (2.21) around \(\rho^*_\Lambda\) (instead of the above indirect procedure). For more details about this we refer in particular to Lemmas 3.1, 3.2 in [15].

Remark 5.1. Let us note that another way for determining the terms \(Q^{(j)}_{\Lambda,0}\), can be derived directly from (5.6). Indeed, let us define for all \(x \in \mathbb{R}^+\) the function
\[
x \mapsto J^{(0)}_{\Lambda,\beta,\mu_0}(x) := \sup_{\mu \in \mathbb{R}} \{\beta x \mu - L^0_{\Lambda,\beta,\mu_0}(\mu)\} = \beta x \mu(x) - L^0_{\Lambda,\beta,\mu_0}(\mu(x)),
\]
where \(\mu(x)\) is implicitly defined by \(\beta x = (L^0_{\Lambda,\beta,\mu_0})'(\mu)\). Note that, when \(x = N \in \mathbb{N}\), we get \(J^{(0)}_{\Lambda,\beta,\mu_0}(x) = T^{(0)}_{\Lambda,\beta,\mu_0}(N)\), which happens if and only if \(N = (L^0_{\Lambda,\beta,\mu_0})'(\mu(N))\). Hence we have:
\[
(J^{(0)}_{\Lambda,\beta,\mu_0})'(x) = \beta \mu(x) = \beta \mu(x) + \mu'(x)[\beta x - (L^0_{\Lambda,\beta,\mu_0})'(\mu(x))]
\]
and
\[
(J^{(0)}_{\Lambda,\beta,\mu_0})''(x) = \beta \mu'(x) = 2 \beta \mu'(x) - (\mu'(x))^2(L^0_{\Lambda,\beta,\mu_0})''(\mu(x)) + \mu''(x)[\beta x - (L^0_{\Lambda,\beta,\mu_0})'(\mu(x))],
\]
which gives
\[
(J^{(0)}_{\Lambda,\beta,\mu_0})''(x) = \beta \mu'(x) = (L^0_{\Lambda,\beta,\mu_0})''(\mu(x))^{-1}.
\]
In this way we have:
\[
\frac{\partial^m J^{(0)}_{\Lambda,\beta,\mu_0}(x)}{\partial x^m} = \frac{\partial^{m-2}[\beta^2(L^0_{\Lambda,\beta,\mu_0})''(\mu(x))]^{-1}}{\partial x^{m-2}},
\]
with
\[
\mu'(x) = \beta[(L^0_{\Lambda,\beta,\mu_0})''(\mu(x))]^{-1}.
\]
Then, the coefficient \(Q^{(j)}_{\Lambda,0}\) - which is the derivative of order \(j\) of \(J^{(0)}_{\Lambda,\beta,\mu_0}(x)\) for \(x = \tilde{N}_\Lambda\) - can be obtained from (5.12) and (5.13) taking into account that, when \(x = \tilde{N}_\Lambda\), the quantities in the right hand side of (5.12) and (5.13) are given by (5.3) and (5.4).

Note that the relations expressed in (5.12) and (5.13) are the same which exist between \(f_\beta(\rho)\) and \(p_\beta(\mu)\) as well as their grand-canonical finite volume versions \((f^{GC}_{\Lambda,\beta,0}(\rho_\Lambda))\) and \((p^{GC}_{\Lambda,\beta,0}(\mu_\Lambda))\).
We conclude the discussion by noting that the formulations expressed in [1] and [2] are equivalent to our formulation (Theorems 2.3, 2.4 and Corollary 2.5). This is due to the fact that for all $\hat{\rho}_\Lambda$ and $\hat{\rho}^*_\Lambda$ which satisfy (2.39) and (2.41) (with the appropriate chemical potential $\mu(\hat{\rho}_\Lambda)$), from (2.43) and Remark 2.3 we have ([15]):

$$|f_{\Lambda,\beta,0}(\hat{\rho}_\Lambda) - F_{\Lambda,\beta,0}(\hat{\rho}^*_\Lambda)| \leq C \frac{\log \sqrt{|\Lambda|}}{|\Lambda|}$$

and

$$|(f_{\Lambda,\beta,0}'(\hat{\rho}_\Lambda) - F_{\Lambda,\beta,0}'(\hat{\rho}^*_\Lambda))| \leq C_1 \frac{1}{|\Lambda|}.$$

Then, defining $I^C_{\Lambda,\beta,0}(\rho_\Lambda; \rho^*_\Lambda) := \beta F_{\Lambda,\beta,0}(\rho_\Lambda) - \beta F_{\Lambda,\beta,0}(\rho^*_\Lambda) + \beta F_{\Lambda,\beta,0}'(\rho^*_\Lambda)(\rho_\Lambda - \rho^*_\Lambda)$ and remembering that $\rho_\Lambda = N/|\Lambda|$, from (2.22) and (2.43) we have

$$\left| [I^C_{\Lambda,\beta,0}(\hat{\rho}_\Lambda; \rho^*_\Lambda) - \beta F_{\Lambda,\beta,0}'(\hat{\rho}_\Lambda) - \beta F_{\Lambda,\beta,0}'(\rho^*_\Lambda)] - [\beta f_{\Lambda,\beta,0}(N^*) + \beta f_{\Lambda,\beta,0}(\hat{N}^*)] \right| \leq C \frac{\log \sqrt{|\Lambda|}}{|\Lambda|},$$

as well as

$$\left| I^C_{\Lambda,\beta,0}(\hat{\rho}^*_\Lambda; \rho_\Lambda) - I^C_{\Lambda,\beta,0}(\rho_\Lambda; \hat{\rho}_\Lambda) \right| \leq C_1 \frac{\log \sqrt{|\Lambda|}}{|\Lambda|},$$

with $C, C_1 \in \mathbb{R}^+$. Moreover, this equivalence is also true in the thermodynamic limit, which is proved in Sections 3 and 4 of [2] for the quantities defined in (5.1), (5.4) and (5.9), where in our case it comes from Theorem 2.1 and [15]. Indeed from Appendix B in [15] we have:

$$|f_{\beta}^{(m)}(\rho_0) - F_{\Lambda,\beta,0}^{(m)}(\rho^*_\Lambda)| \leq C \frac{\rho^{\beta}}{|\Lambda|},$$

for all $m \geq 0$.

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