SMOOTH MONOMIAL TOGLIATTI SYSTEMS OF CUBICS

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Abstract. The goal of this paper is to solve the conjecture stated in [6, Remark 6.2] and classify all smooth minimal monomial Togliatti systems of cubics.

More precisely, we classify all minimal monomial artinian ideals $I \subset k[x_0, \ldots, x_n]$ generated by cubics, failing the weak Lefschetz property and whose apolar cubic system $I^{-1}$ defines a smooth toric variety or, equivalently, we classify all minimal monomial artinian ideals $I \subset k[x_0, \ldots, x_n]$ generated by cubics whose apolar cubic system $I^{-1}$ defines a smooth toric variety satisfying at least a Laplace equation of order 2.

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1. Introduction

An $n$-dimensional variety $X$ whose $d$-osculating space at a general point $x$ has dimension $\binom{n+d}{d} - 1 - \delta$ is said to satisfy $\delta$ independent Laplace equations of order $d$. The classification of $n$-dimensional varieties satisfying at least one Laplace equation of order $d$ is an open problem that has attracted the attention of a huge number of geometers. The roots of this problem go back to work of Togliatti who in [11] and [12] classified all projections of Veronese surfaces in $\mathbb{P}^9$ which satisfy at least one Laplace equation of order 2. He proved: The osculating space $T^2_x X$ at a general point $x$ of the rational surface $X = \text{Im}(\psi)$ where $\psi$ is the rational map

$$\psi : \mathbb{P}^2 \longrightarrow \mathbb{P}^5$$

$$(x_0, x_1, x_2) \mapsto (x_0^2 x_1, x_0 x_1 x_2, x_0 x_2^2, x_0 x_2, x_1 x_2, x_1 x_2^2)$$

has dimension 4 instead of the expected dimension 5.

This example appears also as one of few smooth toric Legendrian varieties in [2, Example 3]. Inspired by the famous example by Togliatti, in [6] Mezzetti, Miró-Roig and Ottaviani have recently established a link between projections $X$ of the Veronese variety $V(n, d)$ satisfying a Laplace equation of order $d - 1$ and artinian ideals $I \subset k[x_0, x_1, \ldots, x_n]$ generated by forms of degree $d$ which fail the weak Lefschetz property (see Definition 2.1). It turns out that

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an artinian ideal $I = (F_1, \cdots, F_r) \subset k[x_0, x_1, \cdots, x_n]$ generated by $r \leq \binom{n+d-1}{n-1}$ forms of degree $d$ fails the weak Lefschetz property in degree $d-1$ if and only if the projection $X^I_{n,d}$ of the Veronese variety $V(n,d)$ from the linear space $\langle F_1, \cdots, F_r \rangle \subset |\mathcal{O}_{\mathbb{P}^n}(d)| = k[x_0, x_1, \cdots, x_n]_d$ satisfies a Laplace equation of order $d-1$. In this case we say that $I$ is a Togliatti system of forms of degree $d$. We also say that $I$ is a smooth (resp. monomial) Togliatti system if, in addition, $X^I_{n,d}$ is a smooth variety (resp. $I$ can be generated by monomials).

The classification of smooth Togliatti systems is a challenging problem far of being solved and so far only partial results have been achieved. In [11] and [12] (see also [1]) Togliatti gave a complete classification of smooth minimal Togliatti systems of cubics in the case $n = 2$. In [6, Theorem 4.11], Mezzetti, Miró-Roig and Ottaviani classified smooth minimal Togliatti systems of cubics in the case $n = 3$ and stated a conjecture for arbitrary $n$ [6, Remark 6.2]. The goal of our work is to prove this conjecture.

Let us briefly outline how this paper is organized. In Section 2, we fix notation/definitions and we collect the background and basic results needed in the sequel. In particular, we recall a close relationship between a priori two unrelated problems: (1) the existence of artinian ideals which fail the weak Lefschetz property, and (2) the existence of smooth varieties satisfying at least a Laplace equation. Section 3 is the heart of the paper. Let $P$ be a monomial system of cubics in $k[x_0, x_1, \cdots, x_n]$ defining a smooth toric variety and assume that the apolar cubic system fails the weak Lefschetz property and it is minimal (in the sense of Definition 2.6). Consider $P$ as a subset of $3\Delta$ where $\Delta$ is the standard $n$-dimensional simplex in the lattice $\mathbb{Z}^{n+1}$. We associate to $P$ two graphs $G$ and $G_P$. We start section 3 with a series of technical Lemmas/Propositions gathering the properties of these graphs. These properties will allow us to conclude that all vertices of $3\Delta$ belong to the lattice spanned by $P$ (Proposition 3.5) and $P$ contains $x_i^2x_j$ if and only if also contains $x_ix_j^2$ (Proposition 3.17). Finally, we derive the full classification of minimal smooth monomial Togliatti systems of cubics (see Theorem 3.4).

**Notation.** $V(n,d)$ will denote the image of the projective space $\mathbb{P}^n$ in the $d$-tuple Veronese embedding $\mathbb{P}^n \rightarrow \mathbb{P}^{n+d-1}$. $(F_1, \ldots, F_r)$ denotes the ideal generated by $F_1, \ldots, F_r$, while $\langle F_1, \ldots, F_r \rangle$ denotes the vector space they generate.

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## 2. Definitions and preliminary results

In this section we recall some standard terminology and notation from commutative algebra and algebraic geometry, as well as some results needed later on.
A. The Weak Lefschetz Property. Let $R := k[x_0, x_1, \ldots, x_n] = \oplus_t R_t$ be the graded polynomial ring in $n + 1$ variables over an algebraically closed field $k$ of characteristic zero. We consider a homogeneous ideal $I$ of $R$. The Hilbert function $h_{R/I}$ of $R/I$ is defined by $h_{R/I}(t) := \dim_k(R/I)_t$. Note that the Hilbert function of an artinian $k$-algebra $R/I$ has finite support and is captured in its $h$-vector $\underline{h} = (h_0, h_1, \ldots, h_e)$ where $h_0 = 1$, $h_i = h_{R/I}(i) > 0$ and $e$ is the last index with this property.

Definition 2.1. Let $I \subset R$ be a homogeneous artinian ideal. We will say that the standard graded artinian algebra $R/I$ has the weak Lefschetz property (WLP) if there is a linear form $L \in (R/I)_1$ such that, for all integers $j$, the multiplication map $\times L : (R/I)_j \to (R/I)_{j+1}$ has maximal rank, i.e. it is injective or surjective. (We will often abuse notation and say that the ideal $I$ has the WLP.) In this case, the linear form $L$ is called a Lefschetz element of $R/I$. If for the general form $L \in (R/I)_1$ and for an integer number $j$ the map $\times L$ has not maximal rank we will say that the ideal $I$ fails the WLP in degree $j$.

The Lefschetz elements of $R/I$ form a Zariski open, possibly empty, subset of $(R/I)_1$. Part of the great interest in the WLP stems from the fact that its presence puts severe constraints on the possible Hilbert functions, which can appear in various disguises (see, e.g., [9]). Though many algebras are expected to have the WLP, establishing this property is often rather difficult. For example, it was shown by R. Stanley [10] and J. Watanabe [13] that a monomial artinian complete intersection ideal $I \subset R$ has the WLP. By semicontinuity, it follows that a general artinian complete intersection ideal $I \subset R$ has the WLP but it is open whether every artinian complete intersection of height $\geq 4$ over a field of characteristic zero has the WLP. It is worthwhile to point out that in positive characteristic, there are examples of artinian complete intersection ideals $I \subset k[x, y, z]$ failing the WLP (see, e.g., Remark 7.10 in [7]).

Example 2.2. (Brenner-Kaid’s example [11]) The ideal $I = (x_0^3, x_1^3, x_2^3, x_0x_1x_2) \subset k[x_0, x_1, x_2]$ fails to have the WLP, because for any linear form $L = a x_0 + bx_1 + cx_2$ the multiplication map $\times L : (k[x_0, x_1, x_2]/I)_2 \to (k[x_0, x_1, x_2]/I)_3$ is neither injective nor surjective. Indeed, since it is a map between two $k$-vector spaces of dimension 6, to show the latter assertion it is enough to exhibit a nontrivial element in its kernel. If $f = a^2 x_0^2 + b^2 x_1^2 + c^2 x_2^2 - ab x_0 x_1 - ac x_0 x_2 - bc x_1 x_2$, then $f \notin I$ and we easily check that $L \cdot f \in I$. 


In [6], Mezzetti, Miró-Roig, and Ottaviani showed that the failure of the WLP can be used to produce varieties satisfying at least a Laplace equation. Let us review the needed concepts from differential geometry in order to state their result.

B. Laplace Equations. Let \( X \subset \mathbb{P}^N \) be a projective variety of dimension \( n \) and let \( x \in X \) be a smooth point. Choose affine coordinates and a local parametrization \( \phi \) around \( x \) where \( x = \phi(0,...,0) \) and the \( N \) components of \( \phi \) are formal power series. The \( s \)-th osculating space \( T_{x}^{(s)}X \) to \( X \) at \( x \) is the projectivised span of all partial derivatives of \( \phi \) of order \( \leq s \). The expected dimension of \( T_{x}^{(s)}X \) is \( \binom{n+s}{s} - 1 \), but in general \( \dim T_{x}^{(s)}X \leq \binom{n+s}{s} - 1 \); if strict inequality holds for all smooth points of \( X \), and \( \dim T_{x}^{(s)}X = \binom{n+s}{s} - 1 - \delta \) for a general point \( x \), then \( X \) is said to satisfy \( \delta \) Laplace equations of order \( s \).

**Remark 2.3.** It is clear that if \( N < \binom{n+s}{s} - 1 \) then \( X \) satisfies at least one Laplace equation of order \( s \), but this case is not interesting and will not be considered in the following.

Very recently Mezzetti, Miró-Roig and Ottaviani [6] have established a link between projections of \( n \)-dimensional Veronese varieties \( V(n,d) \) satisfying a Laplace equation and artinian graded rings \( R/I \) failing the WLP. Let us finish this preliminary section highlighting the existence of this relationship between a pure algebraic problem: the existence of artinian ideals \( I \subset R \) generated by homogeneous forms of degree \( d \) and failing the WLP; and a pure geometric problem: the existence of projections of the Veronese variety \( V(n,d) \subset \mathbb{P}^{\binom{n+d}{d}-1} \) in \( X \subset \mathbb{P}^N \) satisfying at least one Laplace equation of order \( d - 1 \).

Let \( I \) be an artinian ideal generated by \( r \) forms \( F_1, \ldots, F_r \in R \) of degree \( d \). Associated to \( I_d \) there is a morphism

\[
\varphi_{I_d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{r-1}.
\]

Its image \( X_{n,I_d} := \text{Im}(\varphi_{I_d}) \subset \mathbb{P}^{r-1} \) is the projection of the \( n \)-dimensional Veronese variety \( V(n,d) \) from the linear system \( \langle (I^{-1})_d \rangle \subset |O_{\mathbb{P}^n}(d)| = R_d \) where \( I^{-1} \) is the ideal generated by the Macaulay inverse system of \( I \). Analogously, associated to \( (I^{-1})_d \) there is a rational map

\[
\varphi_{(I^{-1})_d} : \mathbb{P}^n \longrightarrow \mathbb{P}^{\binom{n+d}{d}-r-1}.
\]

The closure of its image \( X_{n,(I^{-1})_d} := \text{Im}(\varphi_{(I^{-1})_d}) \subset \mathbb{P}^{\binom{n+d}{d}-r-1} \) is the projection of the \( n \)-dimensional Veronese variety \( V(n,d) \) from the linear system \( \langle F_1, \ldots, F_r \rangle \subset |O_{\mathbb{P}^n}(d)| = R_d \). The varieties \( X_{n,I_d} \) and \( X_{n,(I^{-1})_d} \) are usually called apolar.

Note that when \( I \subset R \) is an artinian monomial ideal the inverse system \( I_d^{-1} \) is spanned by the monomials in \( R \) not in \( I \).

We are now ready to state the main result of this section. We have:
Theorem 2.4. Let $I \subset R$ be an artinian ideal generated by $r$ homogeneous polynomials $F_1, \ldots, F_r$ of degree $d$. If $r \leq \binom{n+d-1}{n-1}$, then the following conditions are equivalent:

1. The ideal $I$ fails the WLP in degree $d - 1$,
2. The homogeneous forms $F_1, \ldots, F_r$ become $k$-linearly dependent on a general hyperplane $H$ of $\mathbb{P}^n$,
3. The $n$-dimensional variety $X_{n,(I^{-1})d}$ satisfies at least one Laplace equation of order $d - 1$.

Proof. See [6, Theorem 3.2]. □

Remark 2.5. Note that, in view of Remark 2.3, the assumption $r \leq \binom{n+d-1}{n-1}$ ensures that the Laplace equations obtained in (3) are not obvious in the sense of Remark 2.3. In the particular case $n = 2$, this assumption gives $r \leq d + 1$.

Definition 2.6. With notation as above, we will say that $I^{-1}$ (or $I$) defines a Togliatti system if it satisfies the three equivalent conditions in Theorem 2.4.

We will say that $I^{-1}$ (or $I$) is a monomial Togliatti system if, in addition, $I^{-1}$ (and hence $I$) can be generated by monomials.

We will say that $I^{-1}$ (or $I$) is a smooth Togliatti system if, in addition, $n$-dimensional variety $X_{n,(I^{-1})d}$ is smooth.

A monomial Togliatti system $I^{-1}$ (or $I$) is said to be minimal if $I$ is generated by monomials $m_1, \ldots, m_r$ and there is no proper subset $m_{i_1}, \ldots, m_{i_{r-1}}$ defining a monomial Togliatti system.

Remark 2.7. Note that we do not require the larger monomial Togliatti system to be smooth. Indeed, consider the system:

$$P := \{x_0^2 x_1, x_0 x_2^2, x_0 x_1 x_2, x_0^2 x_3, x_0 x_2 x_3, x_2^2 x_3, x_1 x_2 x_3, x_1^2 x_3, x_0 x_1 x_3, x_0^2 x_4, x_0 x_1 x_4, x_1^2 x_4, x_0 x_2 x_4, x_2^2 x_4, x_1 x_2 x_4\}.$$

It is a smooth monomial Togliatti system of cubics not contained in any of the Togliatti systems described in Example 3.3. However, it is not minimal in the sense of Definition 2.6 as it can be extended by all cubic monomials not divisible by $x_1^2$ different from $x_0^3, x_1^3, x_2^3, x_3^3$. Such an extended system is a Togliatti system, but not a smooth one.

The names are in honor to Togliatti who proved that for $n = 2$ the only smooth monomial Togliatti system of cubics is $I = (x_0^3, x_1^3, x_2^3, x_0 x_1 x_2) \subset k[x_0, x_1, x_2]$ (see [11, 12]). The main goal of our paper is to classify all smooth minimal monomial Togliatti systems of cubics and it will be achieved in the next section.
3. The Main Theorem

In this section, we will restrict our attention to the monomial case and we will classify all minimal monomial systems of cubics \( I \subset R \) failing WLP in degree two, such that the apolar system defines an \( n \)-dimensional smooth rational variety satisfying at least a Laplace equation of order 2. In other words, we will classify all smooth minimal monomial Togliatti systems of cubics.

The following assumptions and notation are valid from now on. Let \( P = \{m_1, \ldots, m_s\} \) be a monomial system of cubics in \( \mathbb{k}[x_0, \ldots, x_n] \), defining a smooth toric variety. Let \( S = \{m'_1, \ldots, m'_r\} \) be the apolar cubic system. Therefore, we have \( r + s = \binom{n+3}{3} \). We always suppose that \( S \) defines an artinian ring, so it contains \( x_i^3, 0 \leq i \leq n \). Since we are interested in smooth toric varieties satisfying at least one non-trivial (in the sense of Remark 2.3) Laplace equation of order 2, we will also assume \( r \leq \binom{n+2}{3} \) (see Remark 2.5). We also assume that \( S \) is minimal, i.e. \( S \) fails the weak Lefschetz property in degree 2 and no proper subset of \( S \) generates an artinian ideal failing weak Lefschetz property in degree 2. By [8, Proposition 1.1], this is equivalent to assume that there is a hyperquadric \( Q \) containing all integral points of \( P \) and no integral point of \( 3\Delta \setminus P \) and the same is true for any other hyperquadric \( Q' \) containing all integral points of \( P \). First of all, we want to point out that for monomial ideals (i.e. the ideals invariants for the natural toric action of \((\mathbb{k}^*)^n\) on \( \mathbb{k}[x_0, \ldots, x_n] \)) to test the WLP there is no need to consider a general linear form. In fact, we have

**Proposition 3.1.** Let \( I \subset R := \mathbb{k}[x_0, x_1, \ldots, x_n] \) be an artinian monomial ideal. Then \( R/I \) has the WLP if and only if \( x_0 + x_1 + \cdots + x_n \) is a Lefschetz element for \( R/I \).

**Proof.** See [7, Proposition 2.2]. \( \square \)

**The example of the truncated simplex:** Consider the linear system of cubics

\[
P = \{x_i^2 x_j\}_{0 \leq i \neq j \leq n}.
\]

Note that \( \dim P = n(n+1) \). Let \( \varphi_P : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n(n+1)-1} \) be the rational map associated to \( P \). The closure of its image \( X := \overline{\text{Im}(\varphi_P)} \subset \mathbb{P}^{n(n+1)-1} \) is (projectively equivalent to) the projection of the Veronese variety \( V(n, 3) \) from the linear subspace

\[
S := \langle x_0^3, x_1^3, \ldots, x_n^3, \{x_i x_j x_k\}_{0 \leq i < j < k \leq n} \rangle
\]

of \( \mathbb{P}^{n(n+3)/3-1} \). \( X \) is smooth and it satisfies a Laplace equation of order 2.

In [5], p. 12, G. Ilardi conjectured that the above example was the only smooth monomial Togliatti system of cubics of dimension \( n(n+1) - 1 \). This conjecture has been recently disapproved by Mezzetti, Miró-Roig and Ottaviani in [6] who gave the following example.
Example 3.2. Consider the linear system of cubics

\[ P = \{ x_i^2 x_j \}_{0 \leq i \neq j \leq n, (i, j) \neq (0, 1)} \cup \{ x_0 x_1 x_i \}_{2 \leq i \leq n} \]

Note that \( \dim < P > = n^2 + 2n - 3 \). Let \( \varphi_P : \mathbb{P}^n \dasharrow \mathbb{P}^{n^2 + 2n - 4} \) be the rational map associated to \( P \). The closure of its image \( X := \overline{\text{Im}(\varphi_P)} \subset \mathbb{P}^{n^2 + 2n - 4} \) is (projectively equivalent to) the projection of the Veronese variety \( V(n, 3) \) from the linear subspace

\[ S := \langle x_0^3, x_1^3, \ldots, x_n^3, x_0^2 x_1, x_0 x_1^2, \{ x_i x_j x_k \}_{0 \leq i < j < k \leq n, (i, j) \neq (0, 1)} \rangle \]

of \( \mathbb{P}(\binom{n+3}{3})^{-1} \). Again \( X \) is smooth and it satisfies a Laplace equation of order 2.

Notice that \( n^2 + 2n - 4 = n^2 + n - 1 \) if and only if \( n = 3 \). Hence, for \( n = 3 \), Example 3.2 provides a counterexample to Ilardi’s conjecture. Nevertheless \( X \) cannot be further projected without acquiring singularities; therefore for \( n > 3 \), this example does not give a counterexample to Ilardi’s conjecture.

The following series of examples were also presented in [6] and provide counterexamples to Ilardi’s conjecture for any \( n \geq 3 \).

Example 3.3. Let us consider a partition of \( n + 1 \): \( n + 1 = a_1 + a_2 + \cdots + a_s \) with \( n - 1 \geq a_1 \geq a_2 \geq \cdots \geq a_s \geq 1 \) and the monomial ideal

\[ S = (x_0, \ldots, x_{a_1-1})^3 + \cdots + (x_{n+1-a_s}, \ldots, x_n)^3 + J \]

where

\[ J := (x_i x_j x_k \mid i < j < k \text{ and } \forall 1 \leq \lambda \leq s \#(\{i, j, k\} \cap \{ \sum_{a \leq \lambda-1} a_\alpha, \cdots, \sum_{a \leq \lambda} a_\alpha - 1 \}) \leq 1) \]

First of all we observe that \( S \) is a monomial artinian ideal generated by

\[ \mu_{a_1, \ldots, a_s} := \binom{a_1 + 2}{3} + \cdots + \binom{a_s + 2}{3} + \sum_{1 \leq i < j < k \leq s} a_i a_j a_k \]

cubics. The ideal \( S \) fails the WLP in degree 2 since all the vertex points in \( \mathbb{Z}^{n+1} \) corresponding to monomials in the apolar system \( P \) are contained in the quadric \( Q \) of equation

\[ Q = 2 \sum_{i=0}^{n} x_i^2 - 5 \sum_{0 \leq i < j \leq n} x_i x_j + 9 \sum_{0 \leq i < j \leq a_1-1} x_i x_j + 9 \sum_{a_1 \leq i < j \leq a_1+a_2-1} x_i x_j + \cdots + 9 \sum_{n+1-a_s \leq i < j \leq a_s} x_i x_j \]

Alternative, the restriction of all cubics in \( S \) to the hyperplane \( x_0 + \cdots + x_n \) become \( k \)-linearly dependent (Proposition 3.1). Moreover,

\[ \beta_{a_1, \ldots, a_s} := \dim P = \binom{n + 3}{3} - \mu_{a_1, \ldots, a_s} \]
and the closure of the image of the rational map $\varphi_P : \mathbb{P}^n \rightarrow \mathbb{P}^{\beta_{a_1, \ldots, a_s} - 1}$ is a smooth variety $X$ of dimension $n$ which can be seen as the projection of $V(n, 3)$ from the linear space generated by all cubic monomials in $S$.

It remains to prove that $S$ is minimal. First we easily see that the above quadric $Q$ that contains all integral points in $P$ does not contain any integral point of $3\Delta \setminus P$. Let us now check that $Q$ is unique. Assume that there is another one

$$Q' = \sum_{i=0}^{n} \mu_i x_i^2 + \sum_{0 \leq i < j \leq n} \mu_{i,j} x_i x_j.$$  

We first consider two indexes $i < j$ such that $\#(\{i, j\} \cap \{\sum_{a \leq \lambda - 1} a_{\alpha}, \ldots, \sum_{a \leq \lambda} a_{\alpha} - 1\}) \leq 1$, $\forall 1 \leq \lambda \leq s$. Then $x_i^2 x_j$ and $x_j^2 x_i$ belongs to $P$ and we get

$$\begin{align*}
4\mu_i + \mu_j + 2\mu_{i,j} &= 0 \\
\mu_i + 4\mu_j + 2\mu_{i,j} &= 0
\end{align*}$$

which gives us

$$\mu_i = \mu_j = \frac{-2\mu_{i,j}}{5}. \tag{3}$$

Now, we consider two indexes $i < j$ such that $\{i, j\} \subset \{\sum_{a \leq \lambda - 1} a_{\alpha}, \ldots, \sum_{a \leq \lambda} a_{\alpha} - 1\}$ for certain $1 \leq \lambda \leq s$. Then $x_i x_j x_k \in P$ for all $k \notin \{\sum_{a \leq \lambda - 1} a_{\alpha}, \ldots, \sum_{a \leq \lambda} a_{\alpha} - 1\}$. Therefore, we have

$$\mu_i + \mu_j + \mu_k + \mu_{i,j} + \mu_{i,k} + \mu_{j,k} = 0. \tag{4}$$

By (3), we have $\mu_i = \mu_k = \mu_j = \frac{-2\mu_{i,k}}{5} = \frac{-2\mu_{j,k}}{5}$. Substituting in (4) we obtain $\mu_{i,j} = \frac{-4\mu_{i,k}}{5}$ which finish the proof.

So, we have a series of examples of smooth monomial Togliatti system of cubics (see Definition 2.6) and, if $a_1 = n - 1$ and $a_2 = a_3 = 1$ or $a_1 = a_2 = \cdots = a_{n+1} = 1$, they have dimension $n(n + 1) - 1$.

In [6, Remark 6.2], it was conjectured that all smooth monomial Togliatti systems of cubics are obtained by the above procedure. The main goal of our work is to prove this conjecture. In fact, we have got:

**Theorem 3.4.** Let $P$ (or its inverse system $S$) be a minimal smooth monomial Togliatti system of cubics. Then, up to a permutation of the coordinates, the pair $(P, S)$ is one of the examples presented in 3.3. Moreover, $|S| \leq \binom{n+1}{3} + n + 1$ and if $|S| = \binom{n+1}{3} + n + 1$ then it corresponds to one of the following partitions:

1. $n + 1 = (n - 1) + 1 + 1$,
2. $n + 1 = 1 + 1 + \cdots + 1$, 

\[4 = 2 + 2.\]

The proof of our main result will follow from a series of technical lemmas/propositions. Before we start the proof let us present motivation. There is a combinatorial criterion \[4, \text{ Corollary 3.2}\] to check if a subset \(P\) of points in a lattice \(L\) defines a smooth toric variety. However, the set of points \(P\) should always be regarded in the lattice that it spans. This assumption is not automatically satisfied in our case, as \(P\) may span a proper sublattice. It is hard to directly prove that this is not the case. Thus, we start with a weaker statement, Proposition \[3.5\]. The first part of the proof is entirely devoted to proving it. The second step is to prove that if \(P\) contains a monomial \(xy^2\) then it also contains \(x^2y\) - Proposition \[3.17\]. Having this two results, the main theorem follows easily.

Let \(\Delta\) be the standard \(n\)-dimensional simplex in the lattice \(\mathbb{Z}^{n+1}\). We may consider \(P\) as a subset of \(3\Delta\). By choosing a point \(Z \in 3\Delta\) we may consider a lattice spanned by \(3\Delta\) with \(Z\) as the origin. Let \(M\) be the sublattice spanned by the points in \(P\).

**Proposition 3.5.** Let \(P\) be a set satisfying the hypothesis of Theorem \[3.4\]. Then, all vertices of \(3\Delta\) belong to \(M\).

**Proof.** Suppose \(x^3_0 \notin M\). Then on every edge of \(3\Delta\) adjacent to \(x^3_0\) there may be at most 1 point belonging to \(M\). We say that an edge between \(x^3_0\) and \(x^3_i\) is of:

- type a) if \(x^2_0x_i \in M\),
- type b) if \(x_0x^2_i \in M\),
- type c) if it is neither of type a) or b).

Let us define a graph \(G\). It has \(n\) vertices corresponding to edges of \(3\Delta\) adjacent to \(x^3_0\). Thus the vertices of \(G\) are also of type a), b) and c). There is an edge joining vertices \(v_i\) and \(v_j\) if and only if \(x_0x_ix_j \in M\).

The proof of Proposition \[3.5\] will be a consequence of properties of the graph \(G\) listed in the following lemmas.

**Lemma 3.6.** Consider the graph \(G\). It holds:

1. There are no edges between two vertices of type a).
2. There are no edges between a vertex of type a) and a vertex of type b).
3. No vertex of type c) is included in any triangle.
4. There is no edge between a vertex of type b) and a vertex of type c).

**Proof.** In any of the mentioned cases \(x^3_0\) would belong to the lattice spanned by \(P\). \(\square\)

**Lemma 3.7.** Consider two vertices \(v, w\) of type a) and a vertex \(u\) of type c). If there is an edge between \(u\) and \(v\) in \(G\), then so is between \(u\) and \(w\).
Proof. Follows by inspection. □

Claim 1: All points in \( M \) corresponding to monomials divisible by \( x_0 \) belong to a hyperplane. In particular, all points of \( P \) corresponding to monomials divisible by \( x_0 \) belong to a hyperplane.

The points of \( M \) corresponding to monomials divisible by \( x_0 \) correspond either to vertices of type a) and b) in \( G \) or to edges in \( G \). We will construct a hyperplane that contains all these points.

Let \( G_c \) be the restriction of \( G \) to vertices of type c).

Lemma 3.8. Consider four vertices \( v_1, v_2, v_3, v_4 \) of any type in \( G \). If there is path of length 3 joining them, then this path is a part of a 4 cycle.

Proof. The points of \( M \) corresponding to four edges of a cycle form a 2-dimensional rhombus and any 3 vertices of a rhombus generate the fourth one. □

The following is an easy graph theoretic consequence of previous lemmas.

Lemma 3.9. Every connected component of \( G_c \) is a complete bipartite graph.

Proof. Since a graph is bipartite if and only if there is no odd cycle, if the component is not bipartite we could choose the smallest odd cycle in it. By Lemma 3.8 it has to be a triangle, which contradicts Lemma 3.6 (3). Hence, the connected component is a bipartite graph. Choose two vertices in different parts. Consider the shortest path joining them. By Lemma 3.8 it has to be of length one. Hence, the bipartite graph is complete. □

Lemma 3.10. Consider a complete bipartite graph \( C = (A, B) \) that is a connected component of \( G_c \). If part \( A \) is connected to some vertex of type \( A \), then part \( B \) is not. Moreover, in such a case all vertices in part \( A \) are connected to all vertices of type \( a \).

Proof. If a vertex \( v \) of type a) is connected to some vertex \( w \in C \), then all vertices of type a) are connected to \( w \) by Lemma 3.7. Hence, if both parts would be connected to some vertex of type a), they would be connected to the same vertex. This contradicts Lemma 3.6 (3).

By choosing any \( u \in B \) by Lemma 3.8 the vertex \( v \) is connected to all vertices in \( A \). By Lemma 3.7 all vertices of type a) are connected to all vertices in \( A \). □

Proof of the Claim 1. Let \( a, b, c \) be respectively the number of points of type a), b) and c). Let \( c_1 \) (resp. \( c_2 \)) be the number of points of type c) that (resp. do not) belong to a connected component connected to a vertex of type a). Hence, \( c_1 + c_2 = c \). Let \( D \) be the set of vertices either of type b) or of type c), but not in a connected component connected to a vertex of type a). We have \( |D| = b + c_2 \). One can consider a \((b + c_2 - 1)\)-dimensional hyperplane \( H' \)
containing all points corresponding to vertices in \( D \) and edges between them, as all these monomials are divisible by \( x_0 \), but not by \( x_0^2 \). We will now construct a hyperplane that contains all edges and vertices in the complement of \( D \). There is an \((a - 1)\)-dimensional hyperplane \( H_1 \) containing all vertices of type a). We now inductively extend \( H_1 \) adding vertices of type c) and edges between them in such a way that the dimension of \( H_1 \) is always equal to \( a - 1 \) plus the number of considered vertices of type c).

Fix a connected component \( C \) of \( G_c \) that is connected to a vertex of type a). By Lemma 3.9 the graph \( C \) is a complete bipartite graph \((A, B)\). By Lemma 3.10 all vertices of type a) are connected to all vertices in \( A \). Choose \( v \in A \). We may extend \( H_1 \) by a point corresponding to any edge joining \( v \) with a vertex of type a). This increases the dimension of \( H_1 \) by one. Note that all points corresponding to edges joining \( v \) with any other vertex of type a) are automatically in this extension, as \( H_1 \) contains all vertices of type a). In this way we may extend \( H_1 \) using all vertices in part \( A \). Now consider \( w \in B \). Extend \( H_1 \) by the point corresponding to edge \((v, w)\). We claim that all other points corresponding to edges between \( w \) and any vertex in \( A \) belong to the extension. Indeed, let \( v' \in A \) and let \( u \) be any vertex of type a). The extension contains the points corresponding to the three edges: \((w, v), (v, u), (u, v')\). Thus, as in the proof of Lemma 3.8 it must also contain the point corresponding to the fourth edge \((w, v')\). Hence we may extend \( H_1 \) by all vertices in \( B \). Proceeding component by component we obtain a hyperplane \( H'' \) of dimension \( a + c_1 \). Notice that there are no edges between vertices in \( D \) and its complement. Hence, the span of \( H' \) and \( H'' \) satisfies the claim.

We can now finish the proof of Proposition 3.5. By the minimality of \( S \), by Claim 1), \( P \) would have to contain all points in two hyperplanes, apart from vertices of \( 3\Delta \). One of this hyperplanes contains all monomials not divisible by \( x_0 \). If \( G \) would contain any edge or a vertex of type b), then \( x_0^3 \in M \). If there is a vertex of type c), then \( S \) would not be minimal. Hence, \( P \) contains all monomials not divisible by \( x_0 \) (apart from pure cubes) and all monomials divisible by \( x_0^2 \). The points divisible by \( x_0^2 \) form an \((n - 1)\)-dimensional simplex. Hence, if \( P \) defines a smooth variety, each of \( n \) of these vertices must have exactly one additional edge. The points not divisible by \( x_0 \) form also a smooth \((n - 1)\)-dimensional polytope, but with \( n(n - 1) \) vertices. Hence, if \( P \) defines a smooth toric variety we must have \( n(n - 1) = n \), hence \( n = 2 \). For \( n = 2 \) we indeed obtain a smooth polytope, contained in two hyperplanes, and spanning a proper sublattice. However, in this case \( S \) is not minimal.

Let us present the definition/construction of a directed graph \( G_P \) associated to the polytope \( P \).
Definition 3.11. Given a polytope $P$ satisfying the hypothesis of Theorem 3.4, we define the graph $G_P$ which has $n+1$ vertices $v_0, \ldots, v_n$, they correspond to the cubics $x_i^3$. Moreover, there is an edge from $v_i$ to $v_j$ if and only if $x_i^2x_j \in P$.

Example 3.12. (1) Consider $P = \{x_i^2x_j\}_{0 \leq i \neq j \leq 2}$. Then, the directed graph $G_P$ associated to $P$ is:

```
\begin{tabular}{c}
\text{v0} \rightarrow \text{v1} \\
\text{v2} \\
\end{tabular}
```

(2) Consider $P = \{x_i^2x_j\}_{0 \leq i \neq j \leq 3, \{i,j\} \neq \{0,1\}} \cup \{x_0x_1x_i\}_{2 \leq i \leq n}$. Then, the directed graph $G_P$ associated to $P$ is:

```
\begin{tabular}{c}
\text{v1} \rightarrow \rightarrow \downarrow \downarrow \\
\text{v2} \rightarrow \rightarrow \uparrow \uparrow \\
\text{v3} \rightarrow \rightarrow \downarrow \downarrow \\
\text{v0} \\
\end{tabular}
```

Often we will be using the following, very easy lemma.

Lemma 3.13. Consider two polytopes $P_1 \subset P_2$. Let $F$ be a face of $P_2$. If $H$ is an $i$-dimensional face of $F \cap P_1$ then $H$ is an $i$-dimensional face of $P_1$.

Proof. Choose a hyperplane $L$ that is supporting for $F$. As $P_1 \subset P_2$ we have $F \cap P_1 = L \cap P_1$. Hence, $F \cap P_1$ is a face of $P_1$ and $H$ is a face of a face, thus a face. □

Lemma 3.14. Let $G_P$ be the graph associated to a polytope $P$ satisfying the hypothesis of Theorem 3.4. Suppose that the edges $(v_i, v_j), (v_j, v_k), (v_k, v_j)$ belong to $G_P$. Then so does $(v_j, v_i)$.

Proof. Consider the face $F$ of $3\Delta$ spanned by $x_i^3, x_j^3, x_k^3$. By Proposition 3.13 and the assumptions the lattice $M$ contains all points in $F$. If $x_j^2x_i \not\in P$ then $P$ does not define a smooth variety. Indeed, let $\bar{P}$ be the convex hull of $P$. Consider the vertex $v := x_j^2x_k$. On the face $F$ there are two edges adjacent to $v$ going to $x_k^2x_j$ and $x_j^2x_i$. These do not form a basis of the lattice, as the sublattice they generate is of index 2. □

Lemma 3.15. Let $G_P$ be the graph associated to a polytope $P$ satisfying the hypothesis of Theorem 3.4. For every vertex of $G_P$ there is an outgoing edge.

Proof. Choose a vertex corresponding to $x_i^3$. If non of the edges are outgoing from it, the system $S$ contains all cubics divisible by $x_i^3$. But this is already a subsystem failing WLP.
By minimality, $S$ would have to be equal to this system. The compliment of $S$ in such a case is not a smooth polytope, unless $n = 2$. For $n = 2$ the cardinality assumption is not satisfied.

□

**Lemma 3.16.** Let $G_P$ be the graph associated to a polytope $P$ satisfying the hypothesis of Theorem 3.4. Suppose there is an edge $(v_i, v_j)$ in $G_P$ and $(v_j, v_i)$ is not in $G_P$. Then there is a cycle $(v_{a_1}, \ldots, v_{a_l})$ in $G_P$ of length at least 3, such that there are no other edges in $G_P$ between the vertices in the cycle.

Proof. We may start with an edge $(v_i, v_j)$ and follow the path by Lemma 3.15. On such a path there are no returning edges by Lemma 3.14. At some point we must obtain a closed cycle $C$. Suppose there is an edge $(w, w')$ between two nonconsecutive vertices of $C$. By Lemma 3.14 the edge $(w', w)$ does not belong to $G_P$. Hence, we can consider a smaller cycle, containing $(w, w')$. The minimal cycle satisfies the conditions of the lemma. □

**Proposition 3.17.** Let $G_P$ be the graph associated to a polytope $P$ satisfying the hypothesis of Theorem 3.4. If an edge $(v_i, v_j)$ belongs to $G_P$, then so does $(v_j, v_i)$.

Proof. Suppose this is not true. Consider the cycle $C$ from Lemma 3.16. We may suppose $C = (v_1, \ldots, v_l)$. Let $D$ be the face of $3\Delta$ with vertices given by $x_1^3, \ldots, x_l^3$. The cubics corresponding to edges in $C$ are vertices of an $(l-1)$-dimensional simplex $\tilde{Q} \subset D$. However, the cubics $x_i^3$ do not belong to the lattice spanned by $\tilde{Q}$. By Proposition 3.5 the intersection $P \cap D$ spans the same lattice as $D$. Hence, $P \cap D$ must contain other points than those in $Q$. By the assumption on the cycle $C$ these must be cubics of the form $m := x_i x_j x_k$, where $1 \leq i < j < k \leq l$. We will prove that $P$ cannot also contain such cubics.

**Claim 2:** If $m \in P$ then:

1. $m$ is a vertex of $P$,
2. $P \cap D$ contains at least $l$ edges adjacent to $m$.

Note that the Claim contradicts smoothness of $P$ as $P \cap D$ is $l-1$ dimensional.

Proof of the Claim 2. For each edge in the cycle $C$ we will do one of the following:

(i) we will assign 1 to one edge adjacent to $m$ of the convex hull of $P \cap D$,
(ii) we will assign $1/2$ to two edges adjacent to $m$ of the convex hull of $P \cap D$,
(iii) for at most one edge, we will assign $1/2$ to one edge adjacent to $m$ of the convex hull of $P \cap D$.

At the end of the procedure we will show that the sum of the numbers assigned to each edge of the convex hull is at most 1. As we have assigned numbers summing up at least to $l - 1/2$, the claim will follow.
As we restrict to variables with indices less or equal to \( l \), lying on a cycle, it is more convenient to use cyclic notation module \( l \). Thus, although as numbers \( i < j < k \), from now on we have \( k = i - \delta \), where \( \delta \) is the length of the directed path from \( k \) to \( i \).

Let \((v_s, v_{s+1})\) be an edge of the cycle \( C \). Let \( B \) be the face of \( 3\Delta \) spanned by \( x_i^3, x_j^3, x_k^3, x_s^3, x_{s+1}^3 \).

**Lemma 3.18.** Let \( Q \) be the set containing points corresponding to all edges between the vertices \( v_i, v_j, v_k, v_s, v_{s+1} \) (some of these vertices may coincide) and \( m = x_i x_j x_k \). Let \( L \) be the linear span of \( Q \). Suppose that:

1. \( Q \) forms a simplex,
2. \( L \cap 3\Delta \) does not contain squarefree monomials different from \( m \),
3. \( m \) is a vertex of \( \tilde{P} \).

Then either:

1. there is an edge of \( \tilde{P} \) from \( m \) to \( x_s^2 x_{s+1} \),
2. there are two edges of \( \tilde{P} \) from \( m \) to squarefree monomials in variables \( x_i, x_j, x_k, x_s, x_{s+1} \).

**Proof.** As \( Q \) is a simplex, we have \( \dim \tilde{P} \cap B \geq |Q| - 1 \). If \( \dim \tilde{P} \cap B = |Q| - 1 \) then the first conclusion is satisfied, as \( \tilde{P} \cap B \) must be the simplex \( Q \). Otherwise, by the smoothness assumption, there are at least \( |Q| \) edges adjacent to \( m \) in \( \tilde{P} \cap B \). If none of them is adjacent to \( x_s^2 x_{s+1} \) then at least two of them must be adjacent to squarefree monomials. \( \square \)

The proof of the claim consists of three parts. We often used Polymake software \[8\] to find all lattice points in the intersection of a hyperplane with a simplex and to check if particular polytopes are smooth. In the **first part** we assume that non two of vertices \( v_i, v_j, v_k \) are consecutive on the cycle \( C \). Equivalently, we exclude the case of monomials \( x_i x_j x_k \) where \( i, j, k \) do not differ by one (in cyclic notation).

Without loss of generality, we may assume \( i \leq s < j \). We have to consider 3 different cases:

**Case 1:** Suppose \( i + 3 < j \) (i.e. the directed path from \( v_i \) to \( v_j \) is of length at least 4).

By considering the 2-dimensional face of \( 3\Delta \) with vertices \( x_i^3, x_j^3, x_k^3 \) we see that \( m \) is a vertex of \( \tilde{P} \).

a) Suppose \( i + 2 \leq s < j - 2 \) or \( s = i \) or \( s = j - 1 \). The assumptions of Lemma 3.18 hold, as \( Q \) contains only two points: \( m \) and \( x_s^2 x_{s+1} \).

b) \( s = i + 1 \). The assumptions of Lemma 3.18 hold, as only the three lattice points \( m, x_i^2 x_s, x_s^2 x_{s+1} \) belong to their linear span intersected with \( B \).

c) \( s = j - 2 \). The same reasoning applies as in point b), as only the three lattice points \( m, x_s^2 x_{s+1}, x_{s+1}^2 x_j \) belong to their linear span intersected with \( B \).

**Case 2:** Suppose \( j = i + 3 \).
The only edge of new type is \((v_{i+1}, v_{i+2})\). The assumptions of Lemma 3.18 hold, as only the four lattice points \(m, x_i^2x_{i+1}, x_{i+1}^2x_{i+2}, x_{i+2}^2x_{j}\) belong to their linear span intersected with \(B\).

**Case 3:** Suppose \(j = i + 2\).

The simplex \(B\) is 3-dimensional. We will consider both edges \((v_i, v_{i+1})\) and \((v_{i+1}, v_j)\) simultaneously. If \(P \cap B\) is two dimensional, then there are both edges from \(m\) to \(x_i^2x_{i+1}\) and \(x_{i+1}^2x_{j}\), as there are no other points of \(B\) in the linear span of the three points. The only case left is when \(P \cap B\) is three dimensional. If there is exactly one edge from \(m\) to a non-squarefree monomial, then the other two must be squarefree monomials. In this case we can associate 1 to the first edge and 1/2 to the other two. The only case left is when all three squarefree monomials different from \(m\), but in \(B\) belong to \(P\). In this case \(P \cap B\) would not be smooth.

This finishes the first part of the proof. We pass to the **second part**. We assume that exactly two vertices \(v_i, v_j, v_k\) are consecutive on the cycle \(C\). Equivalently, we suppose \(j = i + 1\), but \(k \neq i + 2, i - 1\) (in cyclic notation). By considering a face spanned by \(x_i^3, x_j^3, x_k^3\) we see that \(m\) is a vertex of \(\tilde{P}\). We have to consider 7 different cases:

**Case 1:** Suppose the edge \((v_s, v_{s+1})\) is not adjacent to any edge adjacent to \(v_i, v_{i+1}, v_k\), i.e. either:

1. \(v_s\) is on the path from \(v_{i+1}\) to \(v_k\) and \(i + 2 < s < k - 2\),
2. \(v_s\) is on the path from \(v_k\) to \(v_i\) and \(k + 1 < s < i - 2\).

The only non-squarefree monomials in \(B \cap P\) are \(x_i^2x_{i+1}\) and \(x_{s}^2x_{s+1}\). This two points, together with \(m\) span a two dimensional space that intersects \(B\) in only one more lattice point: \(x_k^2x_{i+1}\), that is not squarefree. Hence, Lemma 3.18 applies. Analogously Lemma 3.18 applies in cases:

1. \(s = i + 2\) and \(k > i + 4\),
2. \(s = k - 2\) and \(k > i + 4\),
3. \(s = i - 2\) and \(i > k + 3\),
4. \(s = i + 1\) and \(k > i + 2\),
5. \(s = k - 1\) and \(k > i + 2\),
6. \(s = i - 1\) and \(i > k + 2\).

**Case 2:** \(s = k + 1\) and \(i > k + 3\).

There are two squarefree polynomials: \(x_{i+1}x_{k+1}x_{k+2}, x_i x_k x_{k+1}\) in the intersection of \(B\) with the affine span of \(m, x_i^2x_{i+1}, x_{k+1}^2x_{k+2}, x_k^2x_{k+1}\). Extending this system by 1 or 2 of these squarefree monomials does not give a smooth 3-dimensional polytope. Hence \(B \cap P\) must be
4-dimensional. Hence, from $m$ either there is an edge to $x_{k+1}^2x_{k+2}$ or two edges to squarefree monomials.

**Case 3:** $s = k$ and $i > k + 2$.

There is one squarefree monomial: $x_ix_kx_{k+1}$ in the intersection of $B$ with the linear span of $m, x_{i+1}^2x_i, x_{k+1}^2x_k$. If $B \cap P$ is 2-dimensional then in both cases there is an edge between $m$ and $x_{k+1}^2x_k$. If it is higher dimensional, we can find two edges adjacent to squarefree monomials or to $x_k^2x_{k+1}$.

**Case 4:** $s = i + 1, i + 2, i + 3$ and $k = i + 4$.

The polytope $B \cap P$ is 4-dimensional. The polytope spanned by

$$A := \{m, x_i^2x_{i+1}, x_{i+1}^2x_{i+2}, x_{i+2}^2x_{i+3}, x_{i+3}^2x_k\}$$

is not smooth. We also know that $x_ix_kx_{i+3}$ is not in $P$, from first part of the proof. The span of $A$ and all squarefree monomials apart from $x_ix_kx_{i+3}$ has an edge between $m$ and $x_{i+1}^2x_{i+2}$ and $m$ and $x_{i+3}^2x_k$. Thus for $s = i + 1$ and $i + 3$ we can associate 1 to these edges. There must be another edge from $m$ to a non-squarefree monomial or to $x_{i+2}^2x_{i+3}$. We can associate one also to this edge.

**Case 5:** $s = k + 1$ and $i = k + 3$.

The system obtained only from edges of $C$ and $m$ without additional squarefree monomials defines a proper sublattice. Hence, we can associate 1/2 to some edge between a squarefree monomial and $m$. Notice, that this is the only case in which we associate 1/2 only to one edge.

**Case 6:** $s = i + 1, i + 2$ and $k = i + 3$.

It is not possible to extend the system to a smooth polytope (the number of vertices has to be even, so there are only 3 possibilities).

**Case 7:** $s = k, k + 1$ and $i = k + 2$.

It also is not possible to strictly extend the system to a smooth polytope. If the system is not extended, we can assign 1 to both edges from $m$ to $x_k^2x_{k+1}, x_{k+1}^2x_i$.

In the last, **third part** of the proof we suppose $j = i + 1$ and $k = i + 2$.

We may exclude this case more directly than the previous cases. Suppose that the cycle is of length at least 5. $B \cap P$ contains $m, x_i^2x_{i+1}, x_{i+1}^2x_{i+2}, x_{i+2}^2x_{i+3}$. This is not a smooth polytope, so it must contain some other monomials. Due to previous cases it cannot contain $x_{i-1}x_ix_{i+2}$ and $x_{i-1}x_{i+1}x_{i+2}$. Hence, it may only contain $x_{i-1}x_ix_{i+1}$. This is also not a smooth polytope.

If $C$ is of length 3 the contradiction is obvious. If $C$ is of length 4 we may assume $m = x_1x_2x_3$. Then $x_1^2x_2$ must have three edges to: $m, x_2^2x_3$ and $x_4^2x_1$. They do not form a lattice basis.
We have associated in all cases numbers summing up to at least \( l \) (or \( l - 1/2 \) in case 5) of the second part. Moreover, each edge to a squarefree monomial was considered at most twice. Thus, the sum of numbers on each edge does not exceed 1. The Claim follows. \( \square \)

By the claim we can see that \( P \cap D \) cannot be smooth, which gives a contradiction. \( \square \)

We have seen that in a graph \( G_P \) associated to a polytope \( P \) satisfying the hypothesis of Theorem 3.4 either there are no edges between two vertices or there are edges in both directions.

**Proposition 3.19.** Let \( P \) be the polytope satisfying the hypothesis of Theorem 3.4. Points of \( P \) span the whole lattice, that is not a strict sublattice.

**Proof.** We say that a subset \( A \) of vertices of \( 3\Delta \) has property (\( * \)) if on a face spanned by \( A \) the polytope \( P \) spans the whole lattice. Suppose \( A, B \) are subsets having the property (\( * \)). If \( A \) and \( B \) have non-empty intersection, then \( A \cup B \) has also this property. Thus we may define the most coarse partition of vertices of \( 3\Delta \), such that each part has (\( * \)). Our aim is to prove that the partition is trivial. Suppose the partition is given by \( A_1, \ldots, A_k \). Notice that \( P \) cannot contain any monomials that would be of degree 2 in variables from one part and of degree one in variables from the other part. If \( P \) contains monomials in variables from more than one part, then it has to be a product of three variables belonging to different parts. Let us prove that such monomials do not belong to \( P \). Suppose that a monomial contains \( x_i \in A_i \) for \( i = 1, 2, 3 \). By Lemma 3.15 we may find \( x_i' \in A_i \) such that \( x_i^2 x_i', x_i^2 x_i' \) belong to \( P \). Consider the 5-dimensional simplex \( B \) spanned by \( x_i^3, x_i^2 \) for \( i = 1, 2, 3 \). If \( x_1 x_2 x_3 \) belongs to \( P \), the convex hull of \( P \cap B \) must have the six edges from this monomial. Indeed, for each fixed \( i \) there are at least two edges in the simplex spanned by \( x_i^3, x_i^2, x_i^2, x_i^2 \). This contradicts the smoothness. Thus \( P \) contains only monomials that are in variables belonging to precisely one group. If there is more than one group this contradicts the minimality of the complement of \( P \). \( \square \)

Associated to any polytope \( P \) satisfying the hypothesis of Theorem 3.4 we construct a new graph \( G'_P \) that will be a (undirected) complement of \( G_P \).

**Definition 3.20.** Given a polytope \( P \) satisfying the hypothesis of Theorem 3.4 we define the graph \( G'_P \) as follows: The vertices \( v_i \) of \( G'_P \) correspond to cubes \( x_i^3 \). An undirected edge \( (v_i, v_j) \) belongs to \( G'_P \) if and only if both (or equivalently any) cubes \( x_i^2 x_j, x_j^2 x_i \) do not belong to \( P \).
Example 3.21. (1) Consider $P = \{x_i^2 x_j\}_{0 \leq i < j \leq 2}$. Then, the undirected graph $G_P$ associated to $P$ is:

```
   v0 -- v1
   |   |
   v2
```

(2) Consider $P = \{x_i^2 x_j\}_{0 \leq i < j < 3 \setminus \{i,j\} \neq \{0,1\}} \cup \{x_0 x_1 x_i\}_{2 \leq i \leq n}$. Then, the directed graph $G_P$ associated to $P$ is:

```
   v1
   |   |
   v2 -- v3
   |   |
   v0
```

Lemma 3.22 (transitivity of edges). Let $G'_P$ be the graph associated to a polytope $P$ satisfying the hypothesis of Theorem 3.4. If $(v_i, v_j), (v_j, v_k)$ are edges of $G'_P$ then so is $(v_i, v_k)$.

Proof. We know that $x_j^3, x_i^3, x_k^3, x_j^2 x_i, x_i^2 x_k, x_k^2 x_j$ do not belong to $P$. Suppose that $(v_i, v_k)$ is not an edge of $G'_P$, i.e. $x_i^2 x_k, x_k^2 x_i$ belong to $P$. By Lemma 3.15 we may find $l$ such that $x_l^2 x_i \in P$, hence also $x_l^2 x_j \in P$ (Proposition 3.17). This would not define a smooth polytope by Proposition 3.19. Let $D$ be the 3-dimensional simplex spanned by $x_i^3, x_j^3, x_k^3, x_l^3$. Consider $D \cap P$. We know that $x_i^2 x_k, x_k^2 x_i, x_j^2 x_l, x_l^2 x_j \in D \cap P$. The additional points form a nonempty subset of

$$x_i^2 x_l, x_i^2 x_i, x_k^2 x_l, x_k^2 x_k, x_j x_i x_k, x_j x_k x_l, x_i x_k x_l, x_j x_i x_l.$$ 

As the subset is nonempty and the configuration is symmetric we may assume without loss of generality that $x_i x_j x_k \in D \cap P$. By the smoothness there must be other points in $P \cap D$. If all the nonsquarefree monomials belong to $D \cap P$ then it is not smooth. Hence without loss of generality we may assume that $x_i^2 x_l, x_i^2 x_i \notin P$. The remaining cases are easily verified using Polymake [8] and do not give smooth polytopes.

We may now finish the proof of the main theorem. By Lemma 3.22 the graph $G'_P$ associated to $P$ is made of disjoint complete graphs. These complete graphs form a partition $A_1, \ldots, A_k$ of the vertices of $3\Delta$. First, let us note that $k = 1$ would contradict the minimality of $P$. Suppose $|A_1| = n$ and $|A_2| = 1$. By smoothness, $P$ contains all the monomials divisible by $x_0$, different from $x_0^3$. If $P$ does not contain other monomials, then the assumption on the cardinality of $P$ is not satisfied. If it contains, then this must be squarefree monomials that are vertices of $\tilde{P}$. In such a case some vertex of $P$, corresponding to a monomial divisible by $x_0$ would have an additional edge, contradicting the smoothness assumption.
For all other partitions we may consider the construction presented in Example 3.3 obtaining \( P' \). Let us note that \( P' \subset P \). Indeed, both contain the same non-squarefree monomials. A squarefree monomial belongs to \( P' \) if and only if two of the variables belong to one group \( A_i \) and the third one to a different \( A_j \). By smoothness, such a monomial must also belong to \( P \). As we have shown in Example 3.3, the complement of \( P' \) is minimal among smooth monomial Togliatti systems, thus \( P = P' \).

Finally, a straightforward computation using the fact that for any partition \( n + 1 = a_1 + a_2 + \cdots + a_s \), \( n - 1 \geq a_1 \geq a_2 \geq \cdots \geq a_s \) we have (see (1)):

\[
\dim S = \left( \frac{a_1 + 2}{3} \right) + \cdots + \left( \frac{a_s + 2}{3} \right) + \sum_{1 \leq i < j < h \leq s} a_i a_j a_h;
\]

we conclude that \( \dim S \leq \left( \frac{n+1}{3} \right) + n + 1 \) and equality holds if and only if \( n + 1 = (n-1)+1+1 \) or \( n + 1 = 1 + 1 + \cdots + 1 \) or \( 4 = 2 + 2 \).

**Remark 3.23.** If we remove the hypothesis of being smooth the list of monomial Togliatti systems of cubics can be enlarged. For instance,

\[
P = (acd, bcd, a^2c, a^2d, ac^2, ad^2, b^2c, b^2d, bc^2, bd^2, c^2d, cd^2)
\]

is a quasi-smooth monomial Togliatti systems of cubics. The closure of the image of the rational map \( \varphi_P : \mathbb{P}^3 \rightarrow \mathbb{P}^{11} \) is a 3-fold \( X \subset \mathbb{P}^{11} \) of degree 18 and its normalization is isomorphic to \( \mathbb{P}^3 \) blown up in the line \( \{c = d = 0\} \) and in the two points \((0,0,1,0)\) and \((0,0,0,1)\).

A further interesting project is the classification of all smooth monomial Togliatti linear systems of forms of degree \( d \) on \( \mathbb{P}^n \) accomplished here for \( n \geq 3 \) and \( d = 3 \) (see Theorem 3.4).

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