On the cohomology ring of chains in $\mathbb{R}^d$

(Walker’s conjecture for chains in $\mathbb{R}^d$)

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Abstract

A chain is a configuration in $\mathbb{R}^d$ of segments of length $\ell_1, \ldots, \ell_{n-1}$ joining two points at a fixed distance $\ell_n$. When $d \geq 3$, we prove that the spaces of such chains are determined up to diffeomorphism by their (mod 2)-cohomology rings.

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Introduction

For $\ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}_{>0}^n$ and $d$ a positive integer, define the subspace $C_d^n(\ell)$ of $(S^{d-1})^{n-1}$ by

$$C_d^n(\ell) = \{ z = (z_1, \ldots, z_{n-1}) \in (S^{d-1})^{n-1} \mid \sum_{i=1}^{n-1} \ell_i z_i = \ell_n e_1 \},$$

where $e_1 = (1, 0, \ldots, 0)$ is the first vector of the standard basis $e_1, \ldots, e_d$ of $\mathbb{R}^d$. An element of $C_d^n(\ell)$, called a chain, can be visualised as a configuration of $(n-1)$-segments in $\mathbb{R}^d$, of length $\ell_1, \ldots, \ell_{n-1}$, joining the origin to $\ell_n e_1$. The vector $\ell$ is called the length-vector.
The group $O(d - 1)$, viewed as the subgroup of $O(d)$ stabilising the first axis, acts naturally (on the left) upon $C^n_d(\ell)$. The quotient $SO(d - 1) \backslash C^n_d(\ell)$ is the polygon space $N^n_d(\ell)$, usually defined as

$$N^n_d(\ell) = SO(d) \backslash \left\{ z \in (S^{d-1})^n \bigg| \sum_{i=1}^n \ell_i z_i = 0 \right\}.$$ 

When $d = 2$ the space of chains $C^n_2(\ell)$ coincides with the polygon space $N^n_2(\ell)$. Descriptions of several chain and polygon spaces are provided in [8] (see also [7] for a classification of $C^n_4(\ell)$).

A length-vector $\ell \in \mathbb{R}^n_{>0}$ is generic if $C^n_1(\ell) = \emptyset$, that is to say there is no collinear chain. It is proven in e.g. [7] that, for $\ell$ generic, $C^n_d(\ell)$ is a smooth closed manifold of dimension

$$\dim C^n_d(\ell) = (n - 2)(d - 1) - 1.$$ 

Another known fact is that if $\ell, \ell' \in \mathbb{R}^n_{>0}$ satisfy

$$(\ell'_1, \ldots, \ell'_{n-1}, \ell'_n) = (\ell_{\sigma(1)}, \ldots, \ell_{\sigma(n-1)}, \ell_n)$$

for some permutation $\sigma$ of $\{1, \ldots, n-1\}$, then $C^n_d(\ell')$ and $C^n_d(\ell)$ are $O(d - 1)$-equivariantly diffeomorphic, see [8, 1.5].

A length-vector $\ell \in \mathbb{R}^n_{>0}$ is dominated if $\ell_n \geq \ell_i$ for all $i < n$. The goal of this paper is to show that the spaces $C^n_d(\ell)$ are, for $\ell$ generic and dominated and when $d \geq 3$, determined up to $O(d - 1)$-equivariant diffeomorphism by their mod2-cohomology rings:

**Theorem A.** Let $d \in \mathbb{N}$, $d \geq 3$. Then, the following properties of generic and dominated length-vectors $\ell, \ell' \in \mathbb{R}^n_{>0}$ are equivalent:

- (a) $C^n_d(\ell)$ and $C^n_d(\ell')$ are $O(d - 1)$-equivariantly diffeomorphic.
- (b) $H^*(C^n_d(\ell); \mathbb{Z})$ and $H^*(C^n_d(\ell'); \mathbb{Z})$ are isomorphic as graded rings.
- (c) $H^*(C^n_d(\ell); \mathbb{Z}_2)$ and $H^*(C^n_d(\ell'); \mathbb{Z}_2)$ are isomorphic as graded rings.

In the case $d = 2$ we do not know if (c) $\Rightarrow$ (a) although the equivalence (a) $\sim$ (b) is true. This is related to a conjecture of K. Walker [13] who suggested that planar polygon spaces are determined by their integral cohomology rings. The conjecture was proven for a large class of length-vectors in [4] and the
(difficult) remaining cases were settled in [11]. The spatial polygon spaces $\mathcal{N}_3^n$ are also determined up to diffeomorphism by their mod2-cohomology ring if $n > 4$, see [3, Theorem 3]. No such result is known for $\mathcal{N}_d^n$ when $d > 3$.

We now give the scheme of the proof of Theorem A. We first recall that the $O(d - 1)$-diffeomorphism type of $\mathcal{C}_d^n(\ell)$ is determined by $d$ and the sets of $\ell$-short (or long) subsets, which play an important role all along this paper. A subset $J$ of $\{1, \ldots, n\}$ is $\ell$-short, or just short, if

$$\sum_{i \in J} \ell_i < \sum_{i \notin J} \ell_i.$$ 

The reverse inequality defines long (or $\ell$-long) subsets. Observe that $\ell$ is generic if and only if any subset of $\{1, \ldots, n\}$ is either short or long.

The family of subsets of $\{1, \ldots, n\}$ which are long is denoted by $L = L(\ell)$. Short subsets form a poset under inclusion, which we denote by $S = S(\ell)$. We are interested in the subposet

$$\breve{S} = \breve{S}(\ell) = \{J \subset \{1, \ldots, n - 1\} \mid J \cup \{n\} \in S\}. \quad (1)$$

The following lemma is proven in [3, Lemma 1.2] (this reference uses the poset $S_n(\ell) = \{J \in S \mid n \in J\}$ which is determined by $\breve{S}(\ell)$).

**Lemma 0.1.** Let $\ell, \ell' \in \mathbb{R}_{>0}^n$ be generic length-vectors. Suppose that $\breve{S}(\ell)$ and $\breve{S}(\ell')$ are isomorphic as simplicial complexes. Then $\mathcal{C}_d^m(\ell)$ and $\mathcal{C}_d^m(\ell')$ are $O(d - 1)$-equivariantly diffeomorphic. \qed

Note that $H^*(\mathcal{C}_d^n; \mathbb{Z}_2) = 0$ if and only if $\mathcal{C}_d^n = \emptyset$, which happens if and only if $\{n\}$ is long. We can thus suppose that $\{n\}$ is short and hence $\breve{S}(\ell)$ is determined by its subposet

$$\breve{S} = \breve{S}(\ell) = \breve{S}(\ell) - \{\emptyset\}. \quad (2)$$

The poset $\breve{S}$ is an abstract simplicial complex (as a subset of a short subset is short) with vertex set contained in $\{1, \ldots, n - 1\}$. To prove Theorem A, it then suffices to show that the graded ring $H^*(\mathcal{C}_d^n(\ell); \mathbb{Z}_2)$ determines $\breve{S}(\ell)$ when $\ell$ is dominated.

For a finite simplicial complex $\Delta$ whose vertex set $V(\Delta)$ is contained in $\{1, \ldots, n\}$, consider the graded ring

$$\Lambda(\Delta) = \mathbb{Z}_2[X_1, \ldots, X_n] / \mathcal{I}(\Delta),$$

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where \( I(\Delta) \) is the ideal of the polynomial ring \( \mathbb{Z}_2[X_1, \ldots, X_n] \) generated by \( X_i^2 \) and the monomials \( X_{j_1} \cdots X_{j_k} \) when \( \{j_1, \ldots, j_k\} \) is not a simplex of \( \Delta \). Let \( \Delta \) and \( \Delta' \) be two finite simplicial complexes with vertex sets contained in \( \{1, \ldots, n\} \). By a result of J. Gubeladze, any graded ring isomorphism \( \Lambda(\Delta) \approx \Lambda(\Delta') \) is induced by a simplicial isomorphism \( \Delta \approx \Delta' \) (see [6, Example 3.6]; for more details, see [4, Theorem 8]). Hence, the implication (c) \( \Rightarrow \) (a) of Theorem A will be established if we prove the following result:

**Theorem B.** Let \( \ell \in \mathbb{R}_{>0}^n \) be a generic length-vector which is dominated. When \( d \geq 3 \), the subring \( H^{(d-1)}(C^n_d(\ell); \mathbb{Z}_2) \) of \( H^*(C^n_d(\ell); \mathbb{Z}_2) \) is isomorphic to \( \Lambda(\tilde{S}(\ell)) \).

The implication (b) \( \Rightarrow \) (c) follows since, under the condition that \( \ell \) is dominated, \( H^*(C^n_d(\ell); \mathbb{Z}) \) is torsion free (see Theorem 2.1). Note also Remark 2.2 which shows that the condition that \( \ell \) is dominated is necessary.

The proof of Theorem B is given in Section 4. The preceding sections are preliminaries for this goal. For instance, the computation of \( H^*(C^n_d(\ell); \mathbb{Z}) \) as a graded abelian group, is given in Theorem 2.1.

### 1 Robot arms in \( \mathbb{R}^d \)

Let

\[
S = S^{n}_d = \{ \rho: \{1, \ldots, n\} \to S^{d-1} \} \approx (S^{d-1})^n .
\]

By post-composition, the orthogonal group \( O(d) \) acts smoothly on the left upon \( S \). In [5, §4–5], the quotient \( W = SO(2) \backslash S^n_2 \approx (S^1)^{n-1} \) is used to get cohomological informations about \( C^n_d \). In this section, we extend these results for \( d > 2 \). The quotient \( SO(d) \backslash S^n_d \) is no longer a convenient object to work with, so we replace it by the fundamental domain for the \( O(d) \)-action given by the submanifold

\[
Z = Z^{n}_d = \{ \rho \in S \mid \rho(n) = -e_1 \} \approx (S^{d-1})^{n-1} .
\]

Observe that \( Z \) inherits an action of \( O(d - 1) \).

For a length-vector \( \ell = (\ell_1, \ldots, \ell_n) \in \mathbb{R}_{>0}^n \) the \( \ell \)-robot arm is the smooth map \( \tilde{F}_\ell: S \to \mathbb{R}^d \) defined by

\[
\tilde{F}_\ell(\rho) = \sum_{i=1}^n \ell_i \rho(i) .
\]
Observe that the point $\rho \in C_d^4$ in the above figure lies in $Z$. We also define an $O(d)$-invariant smooth map $\tilde{f}_\ell: \mathbb{S} \to \mathbb{R}$ by

$$\tilde{f}_\ell(\rho) = -|F_\ell(\rho)|^2.$$ 

The restrictions of $\tilde{F}$ and $\tilde{f}$ to $Z$ are denoted by $F$ and $f$ respectively. Observe that

$$C = \mathcal{C}_d^n(\ell) = f^{-1}(0) \subset Z.$$ 

Define

$$S' = S - C \quad \text{and} \quad Z' = Z - C.$$ 

The restriction of $\tilde{f}$ and $f$ to $S'$ and $Z'$ are denoted by $\tilde{f}'$ and $f'$ respectively.

Denote by $\text{Crit}(g)$ be the subspace of critical points of a real value map $g$. One has $\text{Crit}(\tilde{f}) = \mathcal{C} \cup \text{Crit}(\tilde{f}')$ and $\text{Crit}(f) = \mathcal{C} \cup \text{Crit}(f')$, where $\cup$ denotes the disjoint union. It is easy and well known that $\rho \in \text{Crit}(\tilde{f}')$ if and only if $\rho$ is a collinear configuration, i.e. $\rho(i) = \pm \rho(j)$ for all $i, j \in \{1, \ldots, n\}$.

We will index the critical points of $\tilde{f}'$ and $f'$ by the long subsets. For each $J \in \mathcal{L}$, let $\text{Crit}_J(\tilde{f}') \subset \text{Crit}(\tilde{f}')$ be defined by

$$\text{Crit}_J(\tilde{f}') = \{ \rho \in \mathbb{S} | \kappa_J(j)\rho(j) = \kappa_J(i)\rho(i) \quad \text{for all} \quad i, j \in \{1, \ldots, n\} \}.$$ 

where $\kappa_J: \{1, \ldots, n\} \to \{-1, 1\}$ the multiplicative characteristic function of $J$, defined by:

$$\kappa_J(i) = \begin{cases} 
-1 & \text{if} \ i \in J \\
1 & \text{if} \ i \notin J.
\end{cases}$$

In particular, $\kappa_J = -\kappa_J$ if $\bar{J}$ is the complement of $J$ in $\{1, \ldots, n\}$. In words, $\text{Crit}_J(\tilde{f}')$ is the space of collinear configurations $\rho$ which take constant values on $J$ and $\bar{J}$ and such that $\rho(J) = -\rho(\bar{J})$. The space $\text{Crit}_J(\tilde{f}')$ is a submanifold of $\mathbb{S}$ diffeomorphic, via $\tilde{F}$, to the sphere in $\mathbb{R}^d$ of radius $\sum_{j \in J} \ell_j - \sum_{j \notin J} \ell_j$ (this number is positive since $J$ is long). One has

$$\text{Crit}(\tilde{f}') = \bigcup_{J \in \mathcal{L}} \text{Crit}_J(\tilde{f}').$$
The $O(d)$-invariance of $\tilde{f}'$ has two consequences: each sphere $\text{Crit}_J(\tilde{f}')$ intersects $Z$ transversally in the single point $\rho_J$ and $\text{Crit}(f') = \text{Crit}(\tilde{f}') \cap Z$. Hence

$$\text{Crit}(f') = \{\rho_J \mid J \in \mathcal{L}\} \quad (3)$$

(note that $\rho_J \notin \mathcal{C}$ as $\ell$ is generic). As $\rho(n) = -e_1$ if $\rho \in Z$, the critical points $\rho_J$ are of two types, depending on $n \in J$ or not:

$$\rho_J(i) = \begin{cases} \kappa_J(i) e_1 & \text{if } n \in J \\ -\kappa_J(i) e_1 & \text{if } n \notin J \end{cases} \quad (4)$$

**Lemma 1.1.** The map $f': Z' \to (-\infty, 0)$ is a proper Morse function with set of critical points $\{\rho_J \mid J \in \mathcal{L}\}$. The index of $\rho_J$ is $(d-1)(n-|J|)$.

**Proof.** Because $f'$ extends to $f: (Z, \mathcal{C}) \to ((-\infty, 0], 0)$, the map $f'$ is proper. We already described $\text{Crit}(f')$ in (3). The non-degeneracy of $\rho_J$ and the computation of its index are classical in topological robotics using arguments as in [7, proof of Theorem 3.2].

Consider the axial involution $\hat{\tau}$ on $\mathbb{R}^d = \mathbb{R} \times \mathbb{R}^{d-1}$ defined by $\hat{\tau}(x, y) = (x, -y)$. It induces an involution $\tau$ on $S$ and on $Z$. The maps $\tilde{f}$ and $f$ are $\tau$-invariant. Moreover, the critical set of $f': Z' \to (-\infty, 0)$ coincides with the fixed point set $Z^\tau$. By Lemma 1.1 and [5, Theorem 4], this proves the following

**Lemma 1.2.** The Morse function $f': Z' \to (-\infty, 0)$ is $\mathbb{Z}$-perfect, in the sense that $H_i(Z')$ is free abelian of rank equal to the number of critical points of index $i$. Moreover, the induced map $\tau_*: H_i(Z') \to H_i(Z')$ is multiplication by $(-1)^i$.

(Theorem 4 of [5] is stated for a Morse function $f: M \to \mathbb{R}$ where $M$ is a compact manifold with boundary. To use it in the proof of Lemma 1.2, just replace $Z'$ by $Z - N$ where $N$ is a small open tubular neighbourhood of $\mathcal{C}$.)

We now represent a homology basis for $Z$ and $Z'$ by convenient closed manifolds. For $J \subset \{1, \ldots, n\}$, define

$$\mathbb{S}_J = \{\rho \in S \mid |\rho(J)| = 1\}$$

(the condition $|\rho(J)| = 1$ is another way to say that $\rho$ is constant on $J$). The space $\mathbb{S}_J$ is a closed submanifold of $S$ diffeomorphic to $(S^{d-1})^{n-|J|+1}$. As it is $O(d)$-invariant, it intersects $Z$ transversally. Let

$$W_J = \mathbb{S}_J \cap Z \approx (S^{d-1})^{n-|J|}.$$
The manifold $W_J$ is $O(d - 1)$-invariant and then is $\tau$-invariant. As in Formula (4), the dichotomy “$n \in J$ or not” occurs:

$$W_J = \begin{cases} 
\{ \rho \in Z \mid |\rho(J)| = -e_1 \} & \text{if } n \in J \\
\{ \rho \in Z \mid |\rho(J)| = 1 \} & \text{if } n \notin J.
\end{cases} \tag{5}$$

We denote by $[W_J] \in H_{(d - 1)(n - |J|)}(Z'; Z)$ the class represented by $W_J$ (for some chosen orientation of $W_J$). If $J$ is long, then $W_J \subset Z'$ and we also denote by $[W_J]$ the class represented by $W_J$ in $H_{(d - 1)(n - |J|)}(Z'; Z)$.

**Lemma 1.3.** (a) $H_*(Z'; Z)$ is freely generated by the classes $[W_J]$ for $J \in \mathcal{L}$.

(b) $H_*(Z'; Z)$ is freely generated by the classes $[W_J]$ for all $J \in \{1, \ldots, n\}$ with $n \in J$.

**Proof.** For (a), we invoke [5] Theorem 5. Indeed, the collection of $\tau$-invariant manifolds $\{W_J \mid J \in \mathcal{L}\}$ satisfies all the hypotheses of this theorem (see also [5] Lemma 8).

Let $K = \{1, \ldots, n - 1\}$. The restriction of $\rho \in Z$ to $K$ gives a diffeomorphism from $h: Z \xrightarrow{\approx} S_K \approx (S^{d - 1})^{n - 1}$. By the K"unneth formula, $H_*(S_K'; Z)$ is freely generated by the classes $[W_I]$ for all $I \subset K$. If $n \in J$, $h(W_J) = W_{J - \{n\}}$, which proves (b).

Let $J, J' \subset \{1, \ldots, n\}$. When $|J| + |J'| = n + 1$, one has $\dim W_J + \dim W_{J'} = \dim Z = \dim Z'$ and the intersection number $[W_J] \cdot [W_{J'}] \in \mathbb{Z}$ is defined (intersection in $Z$). We shall use the following formulae.

**Lemma 1.4.** $J, J' \subset \{1, \ldots, n\}$ with $|J| + |J'| = n + 1$. Then

$$[W_J] \cdot [W_{J'}] = \begin{cases} 
\pm 1 & \text{if } |J \cap J'| = 1 \\
0 & \text{if } |J \cap J'| > 1 \text{ and } n \in J \cup J'.
\end{cases}$$

**Proof.** Suppose that $J \cap J' = \{q\}$. Then $|J \cup J'| = |J| + |J'| - |J \cap J'| = n$. Then, $n \in J \cup J'$ and $W_J \cap W_{J'}$ consists of the single point $\rho_{J,J'}$ (satisfying $\rho_{J,J'}(i) = -e_1$ for all $i \in \{1, \ldots, n\}$). It is not hard to check that the intersection is transversal (see [5] proof of (34)), so $[W_J] \cdot [W_{J'}] = \pm 1$.

In the case $|J \cap J'| > 1$, there exists $q \in J \cap J'$ with $q \neq n$. Let $\alpha$ be a rotation of $\mathbb{R}^d$ such that $\alpha(e_1) \neq e_1$. Let $h: Z \to Z$ be the diffeomorphism such that $h(\rho)(k) = \rho(k)$ if $k \neq q$ and $h(\rho)(q) = \alpha \rho(q)$. We now use that $n \in J \cup J'$, say $n \in J'$. Then, $\rho(q) = -e_1$ for $\rho \in W_{J'}$. Hence, $h(W_J) \cap W_{J'} = \emptyset$. As $h$ is isotopic to the identity of $Z$, this implies that $[W_J] \cdot [W_{J'}] = 0$. 

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Remark 1.5. In Lemma 1.4, the hypothesis $n \in J \cup J'$ is not necessary if $d$ is even, by the above proof, since there exists a diffeomorphism of $S^{d-1}$ isotopic to the identity and without fixed point. But, for example, if $n = d = 3$, one checks that $[W_J] \cdot [W_{J'}] = \pm 2$ for $J = J' = \{1, 2\}$.

In the case $n \in J \cap J'$ and $|J| + |J'| = n+1$, Lemma 1.4 takes the following form:

$$[W_J] \cdot [W_{J'}] = \begin{cases} \pm 1 & \text{if } J \cap J' = \{n\} \\ 0 & \text{otherwise} \end{cases}$$

(6)

Therefore, the basis $\{[W_J] | |J| = n - k, n \in J\}$ of $H_k(d-1)(Z; \mathbb{Z})$ has a dual basis (up to sign) $\{[W_J]^{\#} \in H_k(n-k)(d-1)(Z; \mathbb{Z}) | |J| = n - k, n \in J\}$ for the intersection form, defined by $[W_J]^{\#} = [W_K]$, where $K = J \cup \{n\}$.

We are now in position to express the homomorphism $\phi_* : H_*(Z'; \mathbb{Z}) \to H_*(Z; \mathbb{Z})$ induced by the inclusion $Z' \subset Z$. By Lemma 1.3 one has a direct sum decomposition

$$H_*(Z'; \mathbb{Z}) = A_* \oplus B_*,$$

where

- $A_*$ is the free abelian group generated by $[W_J]$ with $J \subset \{1, \ldots, n\}$ long and $n \in J$.
- $B_*$ is the free abelian group generated by $[W_J]$ with $J \subset \{1, \ldots, n\}$ long and $n \notin J$.

Lemma 1.3 also gives a direct sum decomposition

$$H_*(Z; \mathbb{Z}) = A_* \oplus C_*,$$

where

- $A_*$ is the free abelian group generated by $[W_J]$ with $J \subset \{1, \ldots, n\}$ with $n \in J$ and $J$ long.
- $C_*$ is the free abelian group generated by $[W_J]$ with $J \subset \{1, \ldots, n\}$ with $n \in J$ and $J$ short.

Lemma 1.6. (a) $\phi_*$ restricted to $A_*$ coincides with the identity of $A_*$. 

(b) Suppose that $\ell$ is dominated. Then $\phi_*(B_*) \subset A_*$. 

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Proof. Point (a) is obvious. For (b), let \([W_J] \in B_{(n-|J|)(d-1)}\). By what has been seen, it suffices to show that \([W_J] \cdot [W_K]^2 = 0\) for all \([W_K] \in C_s\) with \(|K| = |J|\). Suppose that there exists \([W_K] \in C_s\) with \(|K| = |J|\) such that \([W_J] \cdot [W_K]^2 = \pm 1\). One has \([W_K]^2 = [W_L]\) where \(L = K \cup \{n\}\) by Lemma 1.4, this means that \(J \cap (K \cup \{n\}) = J - K = \{i\}\), with \(i < n\). As \(|K| = |J|\), this is equivalent to \(K = (J - \{i\}) \cup \{n\}\). As \(\ell_n \geq \ell_j\), this contradicts the fact that \(J\) is long and \(K\) is short. \(\square\)

2 \hspace{1em} The Betti numbers of the chain space

Let \(\ell = (\ell_1, \ldots, \ell_n)\) be a dominated length-vector. Let \(a_k = a_k(\ell)\) be the number of short subsets \(J\) containing \(n\) with \(|J| = k + 1\). Alternatively, \(a_k\) is the number of sets \(J \in \mathcal{S}\) with \(|J| = k\).

Theorem 2.1. Let \(\ell = (\ell_1, \ldots, \ell_n)\) be a dominated length-vector. Then, if \(d \geq 3\), \(H^*(\mathcal{C}^n_d(\ell); \mathbb{Z})\) is free abelian with rank

\[
\text{rank } H^k(\mathcal{C}^n_d(\ell); \mathbb{Z}) = \begin{cases} 
  a_s & \text{if } k = s(d-1), \quad s = 0, 1, \ldots, n-3, \\
  a_{n-s-2} & \text{if } k = s(d-1) - 1, \quad s = 0, \ldots, n-2, \\
  0 & \text{otherwise.}
\end{cases}
\]

Proof. Let \(N\) be a closed tubular neighbourhood of \(\mathcal{C} = \mathcal{C}^n_d(\ell)\) in \(Z = Z^n_d\). Let \(Z' = Z - \mathcal{C}\). By Poincaré-Lefschetz duality and excision, one has the isomorphisms on integral homology

\[
H^k(\mathcal{C}) \approx H^k(N) \approx H_{(n-1)(d-1)-k}(N, \partial N) \approx H_{(n-1)(d-1)-k}(Z, Z')
\]

and

\[
H^k(Z, \mathcal{C}) \approx H^k(Z, N) \approx H^k(Z - \text{int } N, \partial N) \\
\approx H_{(n-1)(d-1)-k}(Z - \text{int } N) \approx H_{(n-1)(d-1)-k}(Z').
\]

The homology of \(Z\) and \(Z'\) are concentrated in degrees which are multiples of \((d - 1)\). Hence, \(H^k(\mathcal{C}) = 0\) if \(k \neq 0, -1 \mod (d - 1)\). The possibly non-vanishing \(H^k(\mathcal{C})\) sit in a diagram

\[
\begin{array}{cccccccc}
0 & \rightarrow & H^{s(d-1)-1}(\mathcal{C}) & \rightarrow & H^{s(d-1)}(Z, \mathcal{C}) & \rightarrow & H^{s(d-1)}(Z) & \rightarrow & H^{s(d-1)}(\mathcal{C}) & \rightarrow & 0 \\
\downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & \\
0 & \rightarrow & H_{r(d-1)+1}(Z, Z') & \rightarrow & H_{r(d-1)}(Z') & \rightarrow & H_{r(d-1)}(Z) & \rightarrow & H_{r(d-1)}(Z, Z') & \rightarrow & 0
\end{array}
\]
with \( r = n - 1 - s \). The horizontal sequences are exact. The (co)homology is with integral coefficients and the diagram commutes up to sign \([\text{Theorem I.2.2}]\).

We deduce that \( H_{r(d-1)}(Z, Z') \approx \ker \phi_{r(d-1)} \) which is isomorphic to \( C_{r(d-1)} \) by Lemma \([1.6]\) Therefore, \( H^{s(d-1)}(C^d_d(\ell)) \) is free abelian with rank

\[
\text{rank } H^{s(d-1)}(C^d_d(\ell)) = \text{rank } C_{(n-1-s)(d-1)} = a_s.
\]

On the other hand, \( H_{r(d-1)+1}(Z, Z') \approx \ker \phi_{r(d-1)} \) which, by Lemma \([1.6]\) is isomorphic (though not equal, in general) to \( B_{r(d-1)} \). Therefore, \( H^{s(d-1)-1}(C^d_d(\ell)) \) is free abelian with rank

\[
\text{rank } H^{s(d-1)-1}(C^d_d(\ell)) = \text{rank } B_{(n-1-s)(d-1)} = a_{n-s-2}.
\]

\( \square \)

**Remark 2.2.** Theorem \([2.1]\) is wrong if \( \ell \) is not dominated. For example, let \( \ell = (1, 1, 1, \varepsilon) \) with \( \varepsilon < 1 \). Then, \( C^d_d(\ell) \) is diffeomorphic to the unit tangent bundle \( T^1S^{d-1} \) of \( S^{d-1} \): a map \( g: C^d_d(\ell) \to T^1S^{d-1} \) is given by \( g(\rho) = (\rho(1), \hat{\rho}(2)) \), where the latter is obtained from \( (\rho(1), \rho(2)) \) by the Gram-Schmidt orthonormalization process. The map \( g \) is clearly a diffeomorphism for \( \varepsilon = 0 \) and the robot arm \( F_{(1,1,1)}: S^3_0 \to \mathbb{R}^d \) of Section \([4]\) has no critical value in the disk \( \{|x| < 1\} \subset \mathbb{R}^d \). In particular, \( C_3^d(\ell) \) is diffeomorphic to \( SO(3) \), and thus \( H^2(C_3^d(\ell); \mathbb{Z}) = \mathbb{Z}_2 \). What goes wrong is Point (b) of Lemma \([1.6]\) for instance \( A_2 = 0 \), \( B_2 = H_2(Z, Z') \approx H^2(Z) = C_2 \approx \mathbb{Z}_3 \) and, by the proof of Theorem \([2.1]\) \( \phi: H^2(Z') \to H^2(Z) \) must be injective with cokernel \( \mathbb{Z}_2 \). To obtain this fine result with our technique would require to control the signs in Lemma \([1.4]\).

## 3 The manifold \( V_d(\ell) \)

Let \( \ell \in \mathbb{R}^n_0 \) be a length-vector. In \([4, 8]\), a manifold \( V_d(\ell) \) is introduced, whose boundary is \( \partial C = C^d_d(\ell) \), and Morse Theory on \( V_d(\ell) \) provides some information on \( C \). In this section, we further study the manifold \( V_d(\ell) \) in order to compute the ring \( H^{d-1}(C) \) when \( d \geq 3 \).

Presented as a submanifold of \( Z = Z^d_d \), the manifold \( V_d(\ell) \) is

\[
V_d(\ell) = \{ \rho \in Z \mid \sum_{i=1}^{n-1} \ell_i \rho(i) = t e_1 \text{ with } t \geq \ell_n \}.
\]
Observe that \( V_d(\ell) \) is \( O(d-1) \)-invariant. Let \( g : V_d(\ell) \to \mathbb{R} \) defined by \( g(z) = -|\sum_{i=1}^{n-1} \ell_i z_i| \). The following proposition is proven in \([7, \text{Th. 3.2}]\).

**Proposition 3.1.** Suppose that the length-vector \( \ell \in \mathbb{R}_n^+ \) is generic. Then

(i) \( V_d(\ell) \) is a smooth \( O(d-1) \)-submanifold of \( Z \), of dimension \( (n-2)(d-1) \), with boundary \( C \).

(ii) \( g : V_d(\ell) \to \mathbb{R} \) is an \( O(d-1) \)-equivariant Morse function, with critical points \( \{ \rho_J \mid J \text{ short and } n \in J \} \) (see \([4] \) for the definition of \( \rho_J \)). The index of \( \rho_J \) is equal to \( (d-1)(|J| - 1) \). □

**Corollary 3.2.** The cohomology group \( H^*(V_d(\ell);\mathbb{Z}) \) is free abelian and

\[
\text{rank } H^k(V_d(\ell);\mathbb{Z}) = \begin{cases} a_s & \text{if } k = s(d-1) \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** The number of critical point of \( g \) is equal to \( a_s \). Corollary 3.2 is then obvious if \( d \geq 3 \). When \( d = 2 \), one uses \([5, \text{Theorem 4}]\), the Morse function \( g \) being \( \tau \)-invariant and its critical set being the the fixed point set \( V_d(\ell)^\tau \). □

For each \( J \subset \{1, \ldots, n-1\} \), define the submanifold \( \mathcal{R}_d(J) \) of \( Z_d^n = Z \) by

\[
\mathcal{R}_d(J) = \{ \rho \in Z \mid \rho(i) = e_1 \text{ if } i \notin J \} \approx (S^{d-1})^J.
\]

Consider the space

\[
\mathcal{R}_d(\ell) = \bigcup_{J \in \hat{S}} \mathcal{R}_d(J) \subset Z.
\]

As \( \hat{S} \) is a simplicial complex, the family \( \{[\mathcal{R}_d(J)] \mid J \in \hat{S}\} \) is a free basis for \( H_*(\mathcal{R}_d(\ell);\mathbb{Z}) \) (homology classes of \( \mathcal{R}_d(J) \) in lower degrees coincide in \( H_*(\mathcal{R}_d(\ell)) \) with \([\mathcal{R}_d(J')]) \) for \( J' \subset J \). Thus, \( H_*(\mathcal{R}_d(\ell)) \) is free abelian and

\[
\text{rank } H_k(\mathcal{R}_d(\ell);\mathbb{Z}) = \begin{cases} a_s & \text{if } k = s(d-1) \\ 0 & \text{otherwise.} \end{cases}
\]

**Lemma 3.3.** For \( d \geq 2 \), there exists a map \( \mu : \mathcal{R}_d(\ell) \to V_d(\ell) \) such that \( H^*\mu : H^*(V_d(\ell);\mathbb{Z}) \to H^*(\mathcal{R}_d(\ell);\mathbb{Z}) \) is a ring isomorphism.

**Proof.** Let \( J \in \hat{S} \). Elementary Euclidean geometry shows that, for \( \rho \in \mathcal{R}_d(J) \), there is a unique \( \hat{\rho} \in V_d(\ell) \) satisfying the three conditions

\[
\text{(1)} \quad \sum_{i \in J} |\ell_i z_i| = \sum_{i \notin J} |\ell_i z_i|
\]
\[
\text{(2)} \quad |\hat{\rho}(i)| = e_1 \text{ if } i \notin J
\]
\[
\text{(3)} \quad \sum_{i \in J} |\ell_i z_i| = a_s
\]
(a) $\hat{\rho}(i) = \rho(i)$ if $i \in J$ and

(b) $|\hat{\rho}(\bar{J})| = 1$, where $\bar{J} = \{1, \ldots, n - 1\} - J$.

(c) $\langle \hat{\rho}(i), e_1 \rangle > 0$ if $i \in \bar{J}$.

This defines an embedding $\mu_J: \mathcal{R}_d(J) \to V_d(\ell)$ by $\mu_J(\rho) = \hat{\rho}$. An example is drawn below with $n = 6$ and $J = \{1, 2, 3\}$ (the last segments $\ell_n(\rho) = -\ell_ne_1$ of the configurations are not drawn).

![Diagram](image)

We shall construct the map $\mu: \mathcal{R}_d(\ell) \to V_d(\ell)$ so that its restriction to $\mathcal{R}_d(J)$ is homotopic to $\mu_J$ for each $J \in \mathring{S}$. Unfortunately, when $J \subset J'$, the restriction of $\mu_J'$ to $\mathcal{R}_d(J)$ is not equal to $\mu_J$ so the construction of $\mu$ requires some work.

For $J \in \mathring{S}$, consider the space of embeddings

$\mathfrak{N}_J = \{\alpha: \mathcal{R}_d(J) \to V_d(\ell) \mid \alpha(\rho) \text{ satisfies (a) and (c)}\}$

We claim that $\mathfrak{N}_J$ retracts by deformation onto its one-point subspace $\{\mu_J\}$. Indeed, let $\alpha \in \mathfrak{N}_J$ and let $\rho \in \mathcal{R}_d(\ell)$. For $J \in \mathring{S}$, consider the space

$A_\rho = \{\zeta: \bar{J} \to S^{d-1} \mid \langle \zeta(i), e_1 \rangle > 0 \text{ and } \sum_{i \in J} \rho(i) + \sum_{i \in \bar{J}} \zeta(i) = \lambda e_1 \text{ with } \lambda > 0\}$

Obviously, $\alpha(\rho), \zeta_j \in A_\rho$. The space $A_\rho$ is a submanifold of $(S^{d-1})^{|J|}$ which can be endowed with the induced Riemannian metric. The parameter $\lambda$ provides a function $\lambda: A_\rho \to \mathbb{R}$. As usual, this is a Morse function with critical points the lined configurations $\zeta(i) = \pm \zeta(j)$. But, as $\langle \zeta(i), e_1 \rangle > 0$, the only critical point is a maximum, the restriction of $\mu_J(\rho)$ to $\bar{J}$. Following the gradient line at constant speed thus produces a deformation retraction of $A_\rho$ onto $\mu_J(\rho)|_{\bar{J}}$. The manifold $A_\rho$ and its gradient vector field depending smoothly on $\rho$, this provides the required deformation retraction of $\mathfrak{N}_J$ onto $\{\mu_J\}$. 
Let $\mathcal{BS}_n$ be the first barycentric subdivision of $\mathcal{S}$. Recall that the vertices of $\mathcal{BS}_n$ are the barycenters $\mathcal{J} \in |\mathcal{S}|$ of the simplexes $J$ of $\mathcal{S}$, where $\sigma$ denotes the geometric realization. A family $\{\mathcal{J}_0, \ldots, \mathcal{J}_k\}$ of distinct barycenters forms a $k$-simplex $\sigma \in \mathcal{BS}_n$ if $J_0 \subset J_1 \subset \cdots \subset J_k$. Set $\min(\sigma) = J_0$. For $J \in \mathcal{S}$, we also see $\mathcal{J}$ as a point of $|\mathcal{BS}_n| = |\mathcal{S}|$.

Let us consider the quotient space:

$$
\hat{\mathcal{R}}_d(\ell) = \coprod_{\sigma \in \mathcal{BS}_n} |\sigma| \times \mathcal{R}_d(\min(\sigma))/\sim,
$$

where $(t, \rho) \sim (t', \rho')$ if $\sigma < \sigma'$, $t = t' \in |\sigma| \subset |\sigma'|$ and $\rho \sim \rho'$ under the inclusion $\mathcal{R}_d(\min(\sigma)) \hookrightarrow \mathcal{R}_d(\min(\sigma'))$. The projections onto the first factors in [8] provide a map $q: \hat{\mathcal{R}}_d \to |\mathcal{BS}_n|$ such that $q^{-1}(J) = \{J\} \times \mathcal{R}_d(J)$ $\hat{\mathcal{R}}_d(J) \approx \mathcal{R}_d(J)$. Over a 1-simplex $e = \{J, \{J, J'\}\}$ of $\mathcal{BS}_n$, one has $q^{-1}(\{J\}) \approx \mathcal{R}_d(J)$, $q^{-1}(\{J'\}) \approx \mathcal{R}_d(J')$ and $q^{-1}(\{e\})$ is the mapping cylinder of the inclusion $\mathcal{R}_d(J) \hookrightarrow \mathcal{R}_d(J')$.

We now define a map $\hat{\mu}: \hat{\mathcal{R}}_d \to \mathcal{V}_d(\ell)$ by giving its restriction

$$
\hat{\mu}^k: q^{-1}(|\mathcal{BS}_d(\ell)|^k) \to \mathcal{V}_d(\ell)
$$

over the $k$-skeleton of $\mathcal{BS}_n$. We proceed by induction on $k$. The restriction of $\hat{\mu}$ to $q^{-1}(J) = \mathcal{R}_d(J)$ is equal to $\mu_J \in \mathcal{R}_J$. This defines $\hat{\mu}^0$. For an edge $e = \{J, \{J, J'\}\}$, we use that $\mathcal{R}_J$ is contractible, as seen above. The restriction of $\mu_{J'}$ to $\mathcal{R}_d(J)$ is thus homotopic to $\mu_J$ and we can use a homotopy to extend $\hat{\mu}^0$ over $|e|$. Thus $\hat{\mu}^1$ is defined. Suppose that $\hat{\mu}^{k-1}$ is defined. Let $\sigma = \{J_0, \ldots, J_k\}$ be a $k$-simplex of $\mathcal{BS}_d(\ell)$ with $\min(\sigma) = J_0$ and with boundary $\partial \sigma$. As $\mathcal{R}_{J_0}$ is contractible, the restriction of $\hat{\mu}^{k-1}$ to $q^{-1}(|\partial \sigma|)$ extends to $q^{-1}(|\sigma|)$. This process permits us to define $\hat{\mu}^k$.

Now the projections to the second factors in [8] give rise to a surjective map $p: \hat{\mathcal{R}}_d(\ell) \to \mathcal{R}_d(\ell)$. Let $x \in \mathcal{R}_d(\ell)$. Let $J \in \mathcal{S}$ minimal such that $x \in \mathcal{R}_d(J)$. Then

$$
p^{-1}(\{x\}) = |\text{Star}(\mathcal{J}, \mathcal{BS}_n)| \times \{x\}
$$

is a contractible polyhedron. The preimages of points of $p$ are then all contractible and locally contractible, which implies that $p$ is a homotopy equivalence [12]. Using a homotopy inverse for $p$ and the map $\hat{\mu}$, we get a map $\mu: \mathcal{R}_d(\ell) \to \mathcal{V}_d(\ell)$.

For $J \in \mathcal{S}$, let us compose $\mu_J$ with the inclusion $\beta: \mathcal{V}_d(\ell) \hookrightarrow Z$. When $\rho \in \mathcal{R}_d(J)$, the common value $\hat{\rho}(i)$ for $i \notin J$ is not equal to $-(e_1, e_1, \ldots, e_1)$. However,
Using arcs of geodesics in $S^{d-1}$ enables us to construct a homotopy from $\beta \circ \mu_{J}$ to the inclusion of $\mathcal{R}_{d}(J)$ into $Z$. This implies that $H_{s} \mu: H_{s}(\mathcal{R}_{d}(\ell); \mathbb{Z}) \to H_{s}(V_{d}(\ell); \mathbb{Z})$ is injective. By Corollary 3.2 and (7), $H_{s} \mu$ is an isomorphism. Hence, $H^{s} \mu: H^{s}(V_{d}(\ell); \mathbb{Z}) \to H^{s}(\mathcal{R}_{d}(\ell); \mathbb{Z})$ is a ring isomorphism. □

Remark 3.4. When $d \geq 3$, Lemma 3.3 implies that $\mu: \mathcal{R}_{d}(\ell) \to V_{d}(\ell)$ is a homotopy equivalence, since the spaces under consideration are simply connected. We do not know if $\mu$ is also a homotopy equivalence when $d = 2$.

4 Proof of Theorm B

Theorem B is a direct consequence of Propositions 4.1 and 4.3 below.

Proposition 4.1. Let $\ell \in \mathbb{R}^{n}_{>0}$ be a generic length-vector which is dominated. Then the inclusion $\mathcal{C}_{d}^{\ell}(\ell) \subset V_{d}(\ell)$ induces an injective ring homomorphism

$$H^{s}(\mathcal{R}_{d}(\ell); \mathbb{Z}) \approx H^{s}(V_{d}(\ell); \mathbb{Z}) \hookrightarrow H^{s}(\mathcal{C}_{d}^{\ell}(\ell); \mathbb{Z}). \tag{9}$$

When $d \geq 3$ its image is equal to the subring $H^{s(d-1)}(\mathcal{C}_{d}^{\ell}(\ell); \mathbb{Z})$.

Proof. By Theorem 2.1 and its proof, the homomorphism $H^{s(d-1)}(\mathbb{Z}_{d}; \mathbb{Z}) \to H^{s(d-1)}(\mathcal{C}_{d}^{\ell}(\ell); \mathbb{Z})$ induced by the inclusion is surjective and rank $H^{s(d-1)}(\mathcal{C}_{d}^{\ell}(\ell); \mathbb{Z}) = a_{s}$ (recall that $\mathbb{Z}_{d} = \mathbb{Z}$). As the inclusion $\mathcal{C}_{d}^{\ell}(\ell) \subset \mathbb{Z}_{d}$ factors through $V_{d}(\ell)$, the homomorphism $H^{s(d-1)}(V_{d}(\ell); \mathbb{Z}) \to H^{s(d-1)}(\mathcal{C}_{d}^{\ell}(\ell); \mathbb{Z})$ induced by the inclusion is also surjective. As $\text{rank } H^{s(d-1)}(V_{d}(\ell); \mathbb{Z}) = a_{s}$ by Corollary 3.2, this proves the proposition. □

Remark 4.2. Proposition 4.1 is wrong if $\ell$ is not dominated. For example, let $\ell = (1, 1, 1, \varepsilon)$ with $\varepsilon < 1$. Then $a_{1} = 3$, so $H^{d-1}(V_{d}(\ell); \mathbb{Z}) \approx \mathbb{Z}^{3}$. But, for $d = 3$, we saw in Remark 2.2 that $H^{2}(\mathcal{C}_{d}^{\ell}(\ell); \mathbb{Z}) = \mathbb{Z}_{2}$.

As in the introduction, consider the polynomial ring $\mathbb{Z}_{2}[X_{1}, \ldots, X_{n-1}]$ with formal variables $X_{1}, \ldots, X_{n-1}$. If $J \subset \{1, \ldots, n - 1\}$, we denote by $X_{J} \in \mathbb{Z}_{2}[X_{1}, \ldots, X_{n-1}]$ the monomial $\prod_{j \in J} X_{j}$. Let $\mathcal{I}(\mathcal{S}(\ell))$ be the ideal of $\mathbb{Z}_{2}[X_{1}, \ldots, X_{n-1}]$ generated by the squares $X_{i}^{2}$ of the variables and the monomials $X_{J}$ for $J \notin \mathcal{S}(\ell)$ (non-simplex monomials).

Proposition 4.3. The ring $H^{s}(\mathcal{R}_{d}(\ell); \mathbb{Z}_{2})$ is isomorphic to the quotient ring $\mathbb{Z}_{2}[X_{1}, \ldots, X_{n-1}] / \mathcal{I}(\mathcal{S}(\ell))$ (The degree of $X_{i}$ being $d - 1$).
Proof. The coefficients of the (co)homology groups are \(\mathbb{Z}_2\) and are omitted in the notation. Consider the inclusion \(\beta: V_d(\ell) \hookrightarrow Z = Z_d^n\). The map \(\rho \mapsto (\rho(1), \ldots, \rho(n - 1))\) is a diffeomorphism from \(Z\) to \((S^{d-1})^n\). Using this identification, the homology \(H_* (Z)\) is the \(\mathbb{Z}_2\)-vector space with basis the classes \([R_d(I)]\) for \(I \subset \{1, \ldots, n - 1\}\). (To compare with the basis of Lemma 1.3, the submanifolds \(R_d(J)\) and \(W_j\) are isotopic, where \(J\) is the complement of \(J\) in \(\{1, \ldots, n\}\).) The homology \(H_*(\mathcal{R}_d(\ell))\) has basis \([\mathcal{R}_d(J)]\) for \(J \in \hat{\mathcal{S}}(\ell)\). The homomorphism \(H_*\beta: H_*(\mathcal{R}_d(\ell)) \to H_*(Z)\) is induced by the inclusion of the above bases. Hence, \(H_j\beta: H_j(\mathcal{R}_d(\ell)) \to H_j(Z)\) is injective and \(\text{coker} \ H_j\) is freely generated by the classes \([\mathcal{R}_d(J)]\) for \(|J| = j\) and \(J \notin \hat{\mathcal{S}}(\ell)\).

In particular, the classes \([\mathcal{R}_d(I)]\), for \(i = 1, \ldots, n - 1\), form a basis of \(H_{d-1}(Z)\). Let \(\{\xi_1, \ldots, \xi_{n-1}\} \subset H^{d-1}(Z) = \text{hom}(H_{d-1}(Z), \mathbb{Z}_2)\) be the Kronecker dual basis. By the Künneth formula, the correspondence \(X_i \mapsto \xi_i\) extends to a ring isomorphism \(\mathbb{Z}_2[X_1, \ldots, X_{n-1}] \xrightarrow{\sim} H^*(Z)\). The the family of monomials \(\{X_J | J \subset \{1, \ldots, n - 1\}\}\) is sent to the the Kronecker dual basis to \(\{[\mathcal{R}_d(J)] | J \subset \{1, \ldots, n - 1\}\}\). The properties of \(H_*\beta\) mentioned above then imply that the composed ring homomorphism

\[
\mathbb{Z}_2[X_1, \ldots, X_{n-1}] \xrightarrow{\sim} H^*(Z) \xrightarrow{H^*\beta} H^*(\mathcal{R}_d(\ell))
\]

is surjective with kernel \(\mathcal{I}(\hat{\mathcal{S}}(\ell))\). \(\square\)

The proof of Theorem B is thus complete which, as seen in the introduction, implies Theorem A.

5 Comments

5.1. The authors are trying to unify the notations used for the various posets of short subsets. Our notation \(\hat{\mathcal{S}} \subset \hat{\mathcal{S}} \subset \mathcal{S}\) are identical to that of \([11]\). In \([9]\), \(\hat{\mathcal{S}}\) is denoted by \(\mathcal{S}_n\) but, in the more recent papers \([10] [8]\), \(\mathcal{S}_n = \{J \in \mathcal{S} | n \in J\}\). This is not used here but could have been naturally in e.g. Theorem 2.4.

5.2. When \(d = 2\), Assertion \([9]\) still holds true but not the last assertion of Proposition 4.1. The image \(J_2^n(\ell)\) of the homomorphism \(H^*(V_2(\ell); \mathbb{Z}) \to H^*(\mathcal{C}_2^5(\ell); \mathbb{Z})\) induced by the inclusion is just some subring of \(H^*_1(\mathcal{C}_2^5(\ell); \mathbb{Z})\), where the latter denotes the subring of \(H^*(\mathcal{C}_2^5(\ell); \mathbb{Z})\) generated by the elements of degree 1. For length-vectors such that \(J_2^n(\ell) = H^*_1(\mathcal{C}_2^5(\ell); \mathbb{Z})\), our
proof of Theorem B (and then of Theorem A) holds. Such length-vectors are called normal in [4].

5.3. The ring structure of $H^\ast(C_d^n(\ell);\mathbb{Z}_2)$ is necessary to differentiate the chain spaces up to diffeomorphism: the Betti numbers are not enough. The first example occurs for $n = 6$ with $\ell = (1,1,1,2,3,3)$ and $\ell' = (\varepsilon,1,1,1,2,2)$, where $0 < \varepsilon < 1$. (The chamber of $\ell$ is $\langle 632,64 \rangle$ and that of $\ell'$ is $\langle 641 \rangle$, see [8, Table C].) Then, $\tilde{S}(\ell)$ and $\tilde{S}(\ell')$ are both graphs with 4 vertices and 3 edges. Therefore, $a_s(\ell) = a_s(\ell')$ for all $s$ which, by Theorem 2.1, implies that $C_d^n(\ell)$ and $C_d^n(\ell')$ have the same Betti numbers. However, $\tilde{S}(\ell)$ and $\tilde{S}(\ell')$ are not poset isomorphic: the former is not connected while the latter is.

5.4. It would be interesting to know if, in Theorem A, the ring $\mathbb{Z}_2$ could be replaced by any other coefficient ring. In the corresponding result for spatial polygon spaces $\mathcal{N}_d^n(\ell)$, which are distinguished by their $\mathbb{Z}_2$-cohomology rings if $n > 4$ [4, Theorem 3], the ring $\mathbb{Z}_2$ cannot be replaced by $\mathbb{R}$. Indeed, $\mathcal{N}_d^3(\varepsilon,1,1,1,2) \approx \mathbb{CP}^2 \# \mathbb{CP}^2$ while $\mathcal{N}_d^3(\varepsilon,\varepsilon,1,1,1) \approx S^2 \times S^2$ ($\varepsilon$ small; see [8, Table B]). These two manifolds have non-isomorphic $\mathbb{Z}_2$-cohomology rings but isomorphic real cohomology rings. One can of course replace $\mathbb{Z}_2$ by $\mathbb{Z}$ in Theorem A since, by Theorem 2.1 $H^\ast(C_d^n(\ell);\mathbb{Z})$ determines $H^\ast(C_d^n(\ell);\mathbb{Z}_2)$ when $\ell$ is dominated.

5.5. We do not know if Theorem A is true for generic length vectors which are not dominated. The techniques developped in [3] might useful to study this more general case.

5.6. Let $K$ be a flag simplicial complex (i.e. if $K$ contains a graph $L$ isomorphic to the 1-skeleton of a $r$-simplex, then $L$ is contained in a $r$-simplex of $K$). Then the complex $\mathcal{R}_1(K)$ is the Salvetti complex of the right-angled Coxeter group determined by the 1-skeleton of $K$, see [2].

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