On stability of generalised systems of difference equation with non-consistent initial conditions

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Abstract: For given non-consistent initial conditions, we study the stability of a class of generalised linear systems of difference equations with constant coefficients and taking into account that the leading coefficient can be a singular matrix. We focus on the optimal solutions of the system and derive easily testable conditions for stability.

Keywords: singular system, stability, difference equations, optimal, non-consistent.

1 Introduction

Singular systems of difference/differential equation have been studied by many authors in the past years. See \cite{1-28}, and \cite{29-36} for recent applications of such systems. For an extended version of this type of systems using fractional operators, see \cite{37-45}. In this article, we consider the following initial value problem:

\begin{equation}
FY_{k+1} = GY_k, \quad k = 1, 2, ..., \quad Y_0.
\end{equation}

Where $F, G \in \mathbb{R}^{m \times m}$ and $Y_k \in \mathbb{R}^m$. The matrix $F$ is singular ($\det F = 0$). The initial conditions $Y_0$ are considered to be non-consistent. Note that the initial conditions are called consistent if there exists a solution for the system which satisfies the given conditions. We will also assume that the pencil of the system is regular, i.e. for an arbitrary $s \in \mathbb{C}$ we have $\det(sF - G) \neq 0$, see \cite{46-53}.

There are stability results in the literature dealing with regular systems and for generalised systems with consistent conditions, see \cite{11, 12, 47, 48}. As already mentioned, we consider initial conditions that are non-consistent. This means that the system has infinite many solutions and an optimal solution is required for this case, see \cite{54, 55}. The aim of this paper is to study the stability of the optimal solution of (1).

2 Preliminaries

Some tools from matrix pencil theory will be used throughout the paper. Since in this article we consider the system (1) with a regular pencil, the class of $sF - G$ is characterized by a uniquely defined element, known as the Weierstrass canonical form, see \cite{46-53},
specified by the complete set of invariants of $sF - G$. This is the set of elementary divisors of type $(s - a_j)^{p_j}$, called finite elementary divisors, where $a_j$ is a finite eigenvalue of algebraic multiplicity $p_j$ ($1 \leq j \leq \nu$), and the set of elementary divisors of type $s^q = \frac{1}{p}$, called infinite elementary divisors, where $q$ is the algebraic multiplicity of the infinite eigenvalue. \[ \sum_{j=1}^{\nu} p_j = p \text{ and } p + q = m. \]

From the regularity of $sF - G$, there exist non-singular matrices $P, Q \in \mathbb{R}^{m \times m}$ such that

\[
PFQ = \begin{bmatrix} I_p & 0_{p,q} \\ 0_{q,p} & H_q \end{bmatrix},
\]

\[
PGQ = \begin{bmatrix} J_p & 0_{p,q} \\ 0_{q,p} & I_q \end{bmatrix}.
\]

$J_p, H_q$ are appropriate matrices with $H_q$ a nilpotent matrix with index $q*$, $J_p$ a Jordan matrix and $p + q = m$. With $0_{q,p}$ we denote the zero matrix of $q \times p$. The matrix $Q$ can be written as

\[
Q = \begin{bmatrix} Q_p & Q_q \end{bmatrix}.
\]

$Q_p \in \mathbb{R}^{m \times p}$ and $Q_q \in \mathbb{R}^{m \times q}$. The following results have been proved.

**Theorem 2.1.** (See [1-28]) We consider the systems (1) with a regular pencil. Then, its solution exists and for $k \geq 0$, is given by the formula

\[
Y_k = Q_p J_p^k C.
\]

Where $C \in \mathbb{R}^p$ is a constant vector. The matrices $Q_p, Q_q, J_p, H_q$ are defined by (2), (3).

**Non-consistent initial conditions**

The following proposition identifies if the initial conditions are non-consistent:

**Proposition 2.1.** The initial conditions of system (1) are non-consistent if and only if

\[
Y_0 \notin \text{colspan}Q_p.
\]

We can state the following Theorem, see [54, 55].

**Theorem 2.2.** We consider the system (1) with known non-consistent initial conditions. For the case that the pencil $sF - G$ is regular, after a perturbation to the non-consistent initial conditions accordingly

\[
\min \| Y_0 - \tilde{Y}_0 \|_2,
\]

or, equivalently,

\[
\| Y_0 - Q_p (Q_p^*Q_p)^{-1} Y_0 \|_2.
\]
an optimal solution of the initial value problem (1) is given by

\[ \hat{Y}_k = Q_p J_p^k (Q_p^* Q_p)^{-1} Q_p^* Y_0. \]  

(5)

The matrices \( Q_p, J_p \) are given by by (2), (3).

3 Main Results

We will focus on the stability of equilibrium state(s) of homogeneous singular discrete time systems:

Definition 3.1. For any system of the form (1), \( Y_* \) is an equilibrium state if it does not change under the initial condition, i.e.: \( Y_* \) is an equilibrium state if and only if \( Y_0 = Y_* \) implies that \( Y_k = Y_* \) for all \( k \geq 1 \).

The set of equilibrium states for a given singular linear system in the form of (1) are given by the following Proposition, see [11, 12]:

Proposition 3.1. Consider the system (1). Then if 1 is not an eigenvalue of the pencil \( sF - G \) then

\[ Y_* = 0_{m,1} \]

is the unique equilibrium state of the system (1). If 1 is a finite eigenvalue of \( sF - G \), then the set \( E \) of the equilibrium points of the system (1) is the vector space defined by

\[ E = N_r(F - G) \cap \text{colspan} Q_p. \]

Where \( N_r \) is the right null space of the matrix \( F - G \), \( Q_p \) is a matrix with columns the \( p \) linear independent (generalized) eigenvectors of the \( p \) finite eigenvalues of the pencil.

Proof. If \( Y_* \) is an equilibrium state of system (1), then this implies that for

\[ Y_0 = Y_* \]

we have

\[ Y_* = Y_k = Y_{k+1}. \]

If 1 is not an eigenvalue of the pencil then \( \text{det}(F - G) \neq 0 \) and

\[ FY_* = GY*, \]

or, equivalently,

\[ (F - G)Y_* = 0_{m,1}. \]

Then the above algebraic system has the unique solution

\[ Y_* = 0_{m,1}. \]
which is the unique equilibrium state of the system. If 1 is a finite eigenvalue of the pencil then \( \det(F - G) = 0 \). If \( Y_* \) is an equilibrium state of the system, then this implies that for

\[
Y_0 = Y_*
\]

we have

\[
Y_* = Y_k = Y_{k+1}.
\]

This requires that \( Y_* \) must be a consistent initial condition which from Proposition 2.1 is equal to

\[
Y_* \in \text{colspan}Q_p.
\]

Moreover we have

\[
FY_* = GY_*,
\]

or, equivalently,

\[
(F - G)Y_* = 0_{m,1},
\]

or, equivalently,

\[
Y_* \in N_r(F - G).
\]

Hence,

\[
Y_* \subseteq N_r(F - G) \cap \text{colspan}Q_p,
\]

or, equivalently,

\[
E \subseteq N_r(F - G) \cap \text{colspan}Q_p.
\]

Let now \( Y_* \in N_r(F - G) \cap \text{colspan}Q_p \) then we can consider

\[
Y_0 = Y_*
\]

as a consistent initial condition and

\[
(F - G)Y_* = 0_{m,1}
\]

or, equivalently,

\[
FY_* = GY_*,
\]

where \( Y_* \) is solution of the system and combined with \( Y_0 = Y_* \) we have \( Y_* \in E \), or, equivalently,

\[
N_r(F - G) \cap \text{colspan}Q_p \subseteq E.
\]

The proof is completed.

**Theorem 3.1.** We consider the system \([1]\) with non-consistent initial conditions. An optimal solution is then given by \([3] \). Then an equilibrium state \( Y_* \in E \) is stable in the sense of Lyapounov, if and only if, there exist a constant \( c \in (0, +\infty) \), such that

\[
\|J^k\| \leq c < +\infty, \text{ for all } k \geq 0.
\]

**Proof.** An optimal solution is then given by \([3] \):

\[
\hat{Y}_k = Q_p J^k_p (Q^*_p Q_p)^{-1} Q^*_p Y_0.
\]
Let \( J \) be a finite eigenvalue of the pencil with algebraic multiplicity \( j \). Then the Jordan block
diag
\[
\begin{bmatrix}
J_{p_1}(a_1) & J_{p_2}(a_2) & \ldots & J_{p_\nu}(a_\nu)
\end{bmatrix}
\]
can be written as
\[
\begin{bmatrix}
J_{p}^k(a)
\end{bmatrix}
\]
and easy we obtain
\[
\hat{Y}_k - Y_* = Q_p J_p^k (Q_p^* Q_p)^{-1} Q_p^* (\hat{Y}_k - Y_*)
\]
or, equivalently,

\[
\hat{Y}_k - Y_\nu = \begin{bmatrix} Q_p & 0_{m,q} \end{bmatrix} \begin{bmatrix} J_{p}^k & 0_{p,q} \end{bmatrix} \begin{bmatrix} (Q_p^* Q_p)^{-1} Q_p^* \\ 0_{q,m} \end{bmatrix} (\hat{Y}_k - Y_*).
\]

If we set \( \|Q_p\| = \|Q_p^* Q_p\| \) and \( \|Q_p^* Q_p\|^{-1} \) then

\[
\|Y_0 - Y_*\| \leq \|Q_p\| \|J_p^k\| \|Q_p^* Q_p\|^{-1} \leq \epsilon,
\]

or, equivalently,

\[
\|\hat{Y}_k - Y_*\| \leq \|Q_p\| \epsilon \|Q_p^* Q_p\|^{-1} c \leq \epsilon.
\]

The proof is completed.

**Theorem 3.2.** The system \( \text{(I)} \) with non-consistent initial conditions has an optimal solution is then given by \( \text{(M)} \). Then it is asymptotic stable at large, if and only if, all the finite eigenvalues of \( s F - G \) lie within the open disc,

\[
|s| < 1.
\]

**Proof.** The system \( \text{(I)} \) with non-consistent initial conditions has an optimal solution is then given by \( \text{(M)} \):

\[
\hat{Y}_k = Q_p J_p^k (Q_p^* Q_p)^{-1} Q_p^* Y_0.
\]

Let \( a_j \) be a finite eigenvalue of the pencil with algebraic multiplicity \( p_j \). Then the Jordan matrix \( J_p^k \) can be written as

\[
J_p^k = \text{blockdiag} \begin{bmatrix} J_{p_1}^k(a_1) & J_{p_2}^k(a_2) & \ldots & J_{p_\nu}^k(a_\nu) \end{bmatrix},
\]
with $J^k_p \in \mathcal{M}_{p,j}$ be a Jordan block. Every element of this matrix has the specific form

$$k^p_j a_j^k.$$

The sequence

$$k^p_j |a_j^k|,$$

can be written as

$$k^p_j e^{k \ln |a_j|}.$$

The system is asymptotic stable at large, when

$$\lim_{k \to \infty} \hat{Y}_k = Y_*.$$

Thus this holds if and only if

$$\ln |a_j| < 0,$$

or, equivalently,

$$|a_j| < 1.$$

Then for $k \to +\infty$:

$$k^p_j e^{k |\ln a_j|} \to 0,$$

or, equivalently,

$$k^p_j |a_j|^k \to 0,$$

or, equivalently, for every $k \geq 0$

$$J^k_p \to 0_{p,p}.$$

Then for every initial condition $Y_0$

$$\lim_{k \to \infty} \hat{Y}_k = 0_{m,1}.$$

The proof is completed.

**Conclusions**

In this article we focused and provided properties for the stability of the optimal solutions of a linear generalized discrete time system in the form of (1) for given non-consistent initial conditions.

**References**

[1] T. M. Apostol; *Explicit formulas for solutions of the second order matrix differential equation $Y'' = AY$*, Amer. Math. Monthly 82 (1975), pp. 159-162.

[2] Apostolopoulos, N., Ortega, F. and Kalogeropoulos, G., 2015. *Causality of singular linear discrete time systems*. arXiv preprint [arXiv:1512.04740](http://arxiv.org/).
[3] R. Ben Taher and M. Rachidi; Linear matrix differential equations of higher-order and applications, E. J. of Differential Eq., Vol. 2008 (2008), No. 95, pp. 1-12.

[4] C. Kontzalis, G. Kalogeropoulos. A note on the relation between a singular linear discrete time system and a singular linear system of fractional nabla difference equations. arXiv preprint arXiv:1412.2380 (2014).

[5] C. Kontzalis, G. Kalogeropoulos. Homogeneous linear matrix difference equations of higher order: Singular case. arXiv preprint arXiv:1510.04071 (2015).

[6] C. Kontzalis, G. Kalogeropoulos. Solutions of Generalized Linear Matrix Differential Equations with Boundary conditions. (2015).

[7] L. Dai, Impulsive modes and causality in singular systems, International Journal of Control, Vol 50, number 4 (1989).

[8] I. Dassios, On a boundary value problem of a class of generalized linear discrete time systems, Advances in Difference Equations, Springer, 2011:51 (2011).

[9] I.K. Dassios, On non-homogeneous linear generalized linear discrete time systems, Circuits systems and signal processing, Volume 31, Number 5, 1699-1712 (2012).

[10] I. Dassios, On solutions and algebraic duality of generalized linear discrete time systems, Discrete Mathematics and Applications, Volume 22, No. 5-6, 665–682 (2012).

[11] I. Dassios, On stability and state feedback stabilization of singular linear matrix difference equations, Advances in difference equations, 2012:75 (2012).

[12] I. Dassios, On robust stability of autonomous singular linear matrix difference equations, Applied Mathematics and Computation, Volume 218, Issue 12, 6912–6920 (2012).

[13] I.K. Dassios, G. Kalogeropoulos, On a non-homogeneous singular linear discrete time system with a singular matrix pencil, Circuits systems and signal processing, Volume 32, Issue 4, 1615–1635 (2013).

[14] I. Dassios, G. Kalogeropoulos, On the relation between consistent and non consistent initial conditions of singular discrete time systems, Dynamics of continuous, discrete and impulsive systems Series A: Mathematical Analysis, Volume 20, Number 4, pp. 447–458 (2013).

[15] Dassios I., On a Boundary Value Problem of a Singular Discrete Time System with a Singular Pencil, Dynamics of continuous. Discrete and Impulsive Systems Series A: Mathematical Analysis, 22(3): 211-231 (2015).

[16] I. K. Dassios, K. Szajowski, Bayesian optimal control for a non-autonomous stochastic discrete time system, Applied Mathematics and Computation, Volume 274, 556–564 (2016).

[17] E. Grispos, S. Giotopoulos, G. Kalogeropoulos; On generalised linear discrete-time regular delay systems., J. Inst. Math. Comput. Sci., Math. Ser. 13, No.2, 179-187, (2000).
[18] E. Grispos, *Singular generalised autonomous linear differential systems*, Bull. Greek Math. Soc. 34, 25-43 (1992).

[19] Kaczorek, T.; *General response formula for two-dimensional linear systems with variable coefficients*. IEEE Trans. Aurom. Control Ac-31, 278-283, (1986).

[20] Kaczorek, T.; *Equivalence of singular 2-D linear models*. Bull. Polish Academy Sci., Electr. Electrotechnics, 37, (1989).

[21] Kalogeropoulos, Grigoris, and Charalambos Kontzalis. *Solutions of Higher Order Homogeneous Linear Matrix Differential Equations: Singular Case*. arXiv preprint arXiv:1501.05667 (2015).

[22] J. Klamka, J. Wyrwa, *Controllability of second-order infinite-dimensional systems*. Syst. Control Lett. 57, No. 5, 386–391 (2008).

[23] J. Klamka, *Controllability of dynamical systems*, Matematyka Stosowana, 50, no.9, pp.57-75, (2008).

[24] C. Kontzalis, G. Kalogeropoulos. *Controllability and reachability of singular linear discrete time systems*. arXiv preprint arXiv:1406.1489 (2014).

[25] F. L. Lewis; *A survey of linear singular systems*, Circuits Syst. Signal Process. 5, 3-36, (1986).

[26] F.L. Lewis; *Recent work in singular systems*, Proc. Int. Symp. Singular systems, pp. 20-24, Atlanta, GA, (1987).

[27] F. Milano; I. Dassios, *Small-Signal Stability Analysis for Non-Index 1 Hessenberg Form Systems of Delay Differential-Algebraic Equations*, Circuits and Systems I: Regular Papers, IEEE Transactions on 63(9):1521-1530 (2016).

[28] L. Verde-Star; *Operator identities and the solution of linear matrix difference and differential equations*, Studies in Applied Mathematics 91 (1994), pp. 153-177.

[29] I. Dassios, A. Zimbidis, *The classical Samuelson’s model in a multi-country context under a delayed framework with interaction*, Dynamics of continuous, discrete and impulsive systems Series B: Applications & Algorithms, Volume 21, Number 4-5b pp. 261–274 (2014).

[30] I. Dassios, A. Zimbidis, C. Kontzalis. *The Delay Effect in a Stochastic Multiplier-Accelerator Model*. Journal of Economic Structures 2014, 3:7.

[31] I. Dassios, G. Kalogeropoulos, *On the stability of equilibrium for a reformulated foreign trade model of three countries*. Journal of Industrial Engineering International, Springer, Volume 10, Issue 3, pp. 1-9 (2014). 10:71 DOI 10.1007/s40092-014-0071-9.

[32] I. Dassios, M. Devine. *A macroeconomic mathematical model for the national income of a union of countries with interaction and trade*. Journal of Economic Structures 2016, 5:18.

[33] Ogata, K: *Discrete Time Control Systems*. Prentice Hall, (1987)

[34] W.J. Rugh; *Linear system theory*, Prentice Hall International (Uk), London (1996).
[35] J.T. Sandefur; *Discrete Dynamical Systems*, Academic Press, (1990).

[36] A. P. Schinnar, *The Leontief dynamic generalized inverse*. The Quarterly Journal of Economics 92.4 pp. 641-652 (1978).

[37] D. Baleanu, K. Diethelm, E. Scalas, *Fractional Calculus: Models and Numerical Methods*, World Scientific (2012).

[38] I.K. Dassios, *Optimal solutions for non-consistent singular linear systems of fractional nabla difference equations*, Circuits, Systems and Signal Processing, Springer, Volume 34, Issue 6, pp. 1769-1797 (2015). DOI 10.1007/s00034-014-9930-2

[39] I.K. Dassios, D. Baleanu, *On a singular system of fractional nabla difference equations with boundary conditions*, Boundary Value Problems, 2013:148 (2013).

[40] I.K. Dassios, D.I. Baleanu. *Duality of singular linear systems of fractional nabla difference equations*. Applied Mathematical Modeling, Elsevier, Volume 39, Issue 14, pp. 4180-4195 (2015). DOI 10.1016/j.apm.2014.12.039

[41] I. Dassios, D. Baleanu, G. Kalogeropoulos, *On non-homogeneous singular systems of fractional nabla difference equations*, Applied Mathematics and Computation, Volume 227, 112–131 (2014).

[42] I. Dassios, *Geometric relation between two different types of initial conditions of singular systems of fractional nabla difference equations*, Math. Meth. Appl. Sci., 2015, doi: 10.1002/mma.3771.

[43] I. Dassios, *Stability and robustness of singular systems of fractional nabla difference equations*. Circuits, Systems and Signal Processing (2016). doi:10.1007/s00034-016-0291-x

[44] T. Kaczorek, *Application of the Drazin inverse to the analysis of descriptor fractional discrete-time linear systems with regular pencils*. Int. J. Appl. Math. Comput. Sci 23.1, 2013: 29–33 (2014).

[45] I. Podlubny, *Fractional Differential Equations, Mathematics in Science and Engineering*.p. xxiv+340. Academic Press, San Diego, Calif, USA (1999).

[46] H.-W. Cheng and S. S.-T. Yau; *More explicit formulas for the matrix exponential*, Linear Algebra Appl. 262 (1997), pp. 131-163.

[47] B.N. Datta; *Numerical Linear Algebra and Applications*, Cole Publishing Company, 1995.

[48] L. Dai, *Singular Control Systems*, Lecture Notes in Control and information Sciences Edited by M.Thoma and A.Wyner (1988).

[49] R. F. Gantmacher; *The theory of matrices I, II*, Chelsea, New York, (1959).

[50] G. I. Kalogeropoulos; *Matrix pencils and linear systems*, Ph.D Thesis, City University, London, (1985).
[51] Kontzalis, Charalambos P., and Panayiotis Vlamos. Solutions of Generalized Linear Matrix Differential Equations which Satisfy Boundary Conditions at Two Points. Applied Mathematical Sciences 9.10 (2015): 493-505.

[52] I. E. Leonard; The matrix exponential, SIAM Review Vol. 38, No. 3 (1996), pp. 507-512.

[53] G.W. Steward and J.G. Sun; Matrix Perturbation Theory, Oxford University Press, (1990).

[54] Apostolopoulos, N., Ortega, F. and Kalogeropoulos, G., 2016. The case of a generalised linear discrete time system with infinite many solutions. [arXiv:1610.00927].

[55] Apostolopoulos, N., Ortega, F. and Kalogeropoulos, G., 2016. A boundary value problem of a generalised linear discrete time system with no solutions and infinitely many solutions. [arXiv:1610.08277].