Vortex Velocities in the $O(n)$ Symmetric TDGL Model

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Abstract

An explicit expression for the vortex velocity field as a function of the order parameter field is derived for the case of point defects in the $O(n)$ symmetric time-dependent Ginzburg-Landau model. This expression is used to find the vortex velocity probability distribution in the gaussian closure approximation in the case of phase ordering kinetics for a nonconserved order parameter. The velocity scales as $L^{-1}$ in scaling regime where $L \approx t^{1/2}$ and $t$ is the time after the quench.

The importance of the role of defects in understanding a variety of problems in physics is clear. In certain cosmological [1] and phase ordering [2] problems key questions involve an understanding of the evolution and correlation among defects like vortices, monopoles, disclinations, etc. In studying such objects in a field theory questions arises as to how one can define quantities like the density of vortices and an associated vortex velocity field. The purpose of this paper is to identify the appropriate vortex-velocity field in the context of an $O(n)$ symmetric time-dependent Ginzburg-Landau (TDGL) model for the case of point defects where $n = d$ and $d$ is the spatial dimensionality. Using this rather general definition for the velocity field the distribution of velocities is determined in the case of the late state phase ordering using the gaussian closure approximation for a nonconserved order parameter. The physical results are that the velocity scales as $L(t)^{-1}$ where $L(t) \approx t^{1/2}$ is
the characteristic scaling length for the order parameter correlation function which grows with time \( t \) after the quench. The vortex velocity probability distribution function is given in this approximation by

\[
P(\vec{v}_0) = \frac{\Gamma\left(\frac{n}{2} + 1\right)}{(\pi \bar{v}^2)^{n/2}} \frac{1}{\left(1 + (\vec{v}_0)^2 / \bar{v}^2\right)^{(n+2)/2}}
\]

where the parameter \( \bar{v} \) is defined below and varies as \( L^{-1} \) for long times.

The focus here is on the defect dynamics generated by the TDGL model satisfied by a nonconserved \( n \)-component vector order parameter \( \vec{\psi}(\vec{r}, t) \):

\[
\frac{\partial \vec{\psi}}{\partial t} = \vec{K} \equiv -\Gamma \frac{\delta F}{\delta \vec{\psi}} + \vec{\eta}
\]

where \( \Gamma \) is a kinetic coefficient, \( F \) is a Ginzburg-Landau effective free energy assumed to be of the form

\[
F = \int d^d r \left( \frac{c}{2} (\nabla \vec{\psi})^2 + V(|\vec{\psi}|) \right)
\]

where \( c > 0 \) and the potential is assumed to be of the degenerate double-well form. \( \vec{\eta} \) is a thermal noise which is related to \( \Gamma \) by a fluctuation-dissipation theorem.

Consider a system with \( n = d \) where there are topologically stable point defects \( \vec{\psi} \) formed in a phase ordering system (quenched for example from a high temperature disordered state to a temperature below the order temperature). As pointed out by Halperin \( \cite{4} \) and exploited by Liu and Mazenko \( \cite{5} \), the vortex density for such a system can be written as

\[
\rho = \delta(\vec{\psi}) \mathcal{D}
\]

where \( \mathcal{D} \) is the Jacobian (determinant) for the change of variables from the set of vortex positions \( r_i(t) \) (where \( \vec{\psi} \) vanishes) to the field \( \vec{\psi} \):

\[
\mathcal{D} = \frac{1}{n!} \epsilon_{\mu_1,\mu_2,...,\mu_n} \epsilon_{\nu_1,\nu_2,...,\nu_n} \nabla_{\mu_1} \psi_{\nu_1} \nabla_{\mu_2} \psi_{\nu_2} \cdots \nabla_{\mu_n} \psi_{\nu_n}
\]

where \( \epsilon_{\mu_1,\mu_2,...,\mu_n} \) is the \( n \)-dimensional fully anti-symmetric tensor and summation over repeated indices here and below is implied.
The first goal here is to derive the equation of motion satisfied by $\rho$. Toward this end one needs two identities whose proof is relatively straightforward. The first identity is given by:

$$\frac{\partial D}{\partial t} = \nabla \alpha J^{(K)}_\alpha \quad \text{Identity I}$$

where, for a general vector $\vec{A}$, the current $J^{(A)}_\alpha$ is defined as

$$J^{(A)}_\alpha = \frac{1}{(n-1)!} \epsilon_{\alpha,\mu_2,...,\mu_n,\nu_1,\nu_2,...,\nu_n} A_{\nu_1} \nabla_{\mu_2} \psi_{\nu_2} \cdots \nabla_{\mu_n} \psi_{\nu_n} \quad \text{(7)}$$

Identity I is just a statement that the determinant $D$ is a conserved invariant. Notice that the superscript $K$ on $J$ in this identity is defined by the right hand side of Eq.(2). The second identity takes the form for general vector $\vec{A}$:

$$J^{(A)}_\alpha \nabla_\alpha \psi_\beta = A_\beta D \quad \text{Identity II}$$

Identity II, after using the chain-rule for differentiation, leads directly to the result

$$D \frac{\partial}{\partial t} \delta(\vec{\psi}) = J^{(K)}_\beta \nabla_\beta \delta(\vec{\psi}) \quad \text{(9)}$$

When this result is combined with Identity I, one easily obtains the equation of motion for the vortex density

$$\frac{\partial \rho}{\partial t} = \nabla_\beta \left[ \delta(\vec{\psi}) J^{(K)}_\beta \right] \quad \text{(10)}$$

This continuity equation reflects the fact that the vortex charge is conserved. A key point here is that $J^{(K)}_\beta$ is multiplied by the vortex locating $\delta$-function. This means that one can replace $\vec{K}$ in $\vec{J}^{(K)}$ by the part of $\vec{K}$ which does not vanish as $\vec{\psi} \to 0$. Thus in the case of a nonconserved order parameter one can replace $J^{(K)}_\beta$ in the continuity equation by

$$J^{(2)}_\beta = \frac{1}{(n-1)!} \epsilon_{\beta,\mu_2,...,\mu_n,\nu_1,\nu_2,...,\nu_n} \left[ \Gamma e \nabla^2 \psi_{\nu_1} + \eta_{\nu_1} \right] \nabla_{\mu_2} \psi_{\nu_2} \cdots \nabla_{\mu_n} \psi_{\nu_n} \quad \text{(11)}$$

In the case of a conserved order parameter the current for $\rho$ is more complicated because of the overall gradients acting in $\vec{K}$. Because of the standard form of the continuity equation Eq.(10), it is clear that one can identify the vortex velocity field as
where it is assumed that the velocity field is used inside expressions multiplied by the vortex locating \( \delta \)-function. This is the primary result of this paper. It gives one an explicit expression for the vortex velocity field in terms of the original order parameter field. In the particular case of \( n = d = 2 \) one has the more explicit result

\[
v_\alpha = \Gamma_c \epsilon_{\alpha\mu} \left( \nabla_\mu \psi_x \nabla^2 \psi_y - \nabla_\mu \psi_y \nabla^2 \psi_x \right) \epsilon_{\nu\sigma} \nabla_\nu \psi_x \nabla_\sigma \psi_y .
\]

Notice that the result given by Eq.(10) does not depend on the details of the TDGL model, only that the equation of motion is first order in time.

As an important application of the result Eq.(12) for \( \vec{v} \) consider the velocity probability distribution function defined by

\[
n_0 P(\vec{v}_0) \equiv \langle n \delta(\vec{v}_0 - \vec{v}) \rangle
\]

where \( \vec{v}_0 \) is a reference velocity, \( n = \delta(\vec{\psi}) |D| \) is the unsigned defect density, and \( n_0 = \langle n \rangle \). We determine \( P \) using the gaussian closure method [6–9] which has been successful in determining the scaling function for the order parameter correlation function. The first step is to express the order parameter in terms of an auxiliary field \( \vec{m} \) which is assumed, to a first approximation, to have a gaussian distribution. In the theory developed in Ref. [4], the relationship between the order parameter and the auxiliary field is given as a solution to the classical interface equation

\[
\nabla^2 \vec{\psi}(\vec{m}) = V'(|\psi|) \hat{\psi}
\]

where the auxilliary field serves as the coordinate labelling the distance to the defect nearest to space point \( \vec{r} \) at time \( t \). The solution of this equation for a charge one vortex is of the form

\[
\vec{\psi}(\vec{m}) = A(|\vec{m}|) \hat{m}
\]
where $A(|\vec{m}|)$ vanishes linearly with $m$ for small $m$ with the next term of $O(m^3)$. It is then easy to show that one can replace $\vec{\psi}$ by $\vec{m}$ in the expression for $\vec{v}$. One can determine $P(\vec{v}_0)$ by first evaluating the more general probability distribution

$$G(\xi, \vec{b}) = \langle \delta(\vec{m})\delta(\xi_\mu - \nabla_\mu m_\nu)\delta(\vec{b} - \nabla^2 \vec{m}) \rangle$$

(17)

since

$$n_0 P(\vec{v}_0) = \int d^n b \prod_{\mu, \nu} d\xi_\nu |\mathcal{D}(\xi)| \delta(\vec{v}_0 - \vec{v}(\vec{b}, \xi)) G(\xi, \vec{b})$$

(18)

where

$$\vec{v}(\vec{b}, \xi) = -\frac{\vec{J}^{(2)}(\vec{b}, \xi)}{\mathcal{D}(\xi)}$$

(19)

with

$$\mathcal{D}(\xi) = \frac{1}{n!} \epsilon_{\mu_1, \mu_2, \ldots, \mu_n} \epsilon_{\nu_1, \nu_2, \ldots, \nu_n} \epsilon^{\nu_1}_{\mu_1} \epsilon^{\nu_2}_{\mu_2} \ldots \epsilon^{\nu_n}_{\mu_n}$$

(20)

and

$$J^{(2)}_\alpha(\vec{b}, \xi) = \frac{1}{(n-1)!} \epsilon^{\alpha, \mu_2, \ldots, \mu_n} \epsilon^{\nu_1, \nu_2, \ldots, \nu_n} \Gamma c b_{\nu_1} \epsilon^{\nu_2}_{\mu_2} \ldots \epsilon^{\nu_n}_{\mu_n}. \tag{21}$$

In this last expression it has been assumed that the quench is to zero temperature [10] so that the noise can be set to zero. The gaussian average determining $G(\xi, \vec{b})$ is relatively straightforward to evaluate with the result:

$$G(\xi, \vec{b}) = \frac{1}{(2\pi S_0)^{n/2}} \frac{1}{(2\pi S^{(2)})^{n/2}} \exp \left[ -\frac{1}{2 S^{(2)}} \sum_{\mu, \nu} (\xi_\mu)^2 \right]$$

(22)

where, $S_0 = \frac{1}{n} \langle \vec{m}^2 \rangle$ is proportional to $L^2$, 

$$S^{(2)} = \frac{1}{n^2} \langle (\nabla \vec{m})^2 \rangle \tag{23}$$

and

$$\bar{S}_4 = \frac{1}{n} \langle (\nabla^2 \vec{m})^2 \rangle - \left( \frac{n S^{(2)}}{S_0} \right)^2. \tag{24}$$
The quantities $S_0, S^{(2)}, \bar{S}_4$ are determined from the theory for the order parameter correlation function. Using this result for $G(\xi, \vec{b})$ in the expression for the probability distribution one can use the usual integral representation for the $\delta$-function to perform the integration over the $\vec{b}$ field to obtain

$$n_0 P(\vec{v}_0) = \int \prod_{\mu,\nu} d\xi_{\mu}^{\nu} \frac{|D(\xi)|}{(2\pi S^{(2)})^{n/2}} \exp \left[ -\frac{1}{2S^{(2)}} \sum_{\mu,\nu} (\xi_{\mu}^{\nu})^2 \right] \frac{1}{(4\pi^2 S_0 \gamma)^{n/2}} \frac{1}{\sqrt{\det M}} \exp \left[ -\frac{1}{2\gamma} \sum_{\mu,\nu} v_0^\mu [M^{-1}]_{\mu,\nu} v_0^\nu \right]$$

(25)

where $\gamma = \bar{S}_4 (\Gamma c)^2$, and the matrix $M$ is given by

$$M_{\alpha,\beta} = \frac{1}{D^2 [(n-1)!]^2} \epsilon_{\alpha,\mu_2,\ldots,\mu_n} \epsilon_{\nu,\mu_2,\ldots,\nu_n} \epsilon_{\mu_2} \epsilon_{\beta,\mu_2,\ldots,\mu_n} \epsilon_{\nu,\mu_2,\ldots,\nu_n} \epsilon_{\mu_2} \ldots \epsilon_{\mu_n} \epsilon_{\nu} \epsilon_{\nu} \ldots \epsilon_{\nu_n} .$$

(26)

It is straightforward to obtain the rather clean results

$$\det(M) = \frac{1}{(D)^2}$$

(27)

and

$$M^{-1}_{\alpha,\beta} = \sum_{\nu} \xi_{\alpha}^{\nu} \xi_{\beta}^{\nu}$$

(28)

so that

$$n_0 P(\vec{v}_0) = \int \prod_{\mu,\nu} d\xi_{\mu}^{\nu} \frac{1}{(2\pi S^{(2)})^{n/2}} \exp \left[ -\frac{1}{2S^{(2)}} \sum_{\mu,\nu} (\xi_{\mu}^{\nu})^2 \right] \frac{D^2}{(4\pi^2 S_0 \gamma)^{n/2}} \exp \left[ -\frac{1}{2\gamma} \sum_{\alpha,\beta,\nu} v_0^\alpha \xi_{\alpha}^{\nu} \xi_{\beta}^{\nu} v_0^\beta \right] .$$

(29)

The remaining integrals look formidable but can be carried out if one makes a transformation from $\xi_{\alpha}^{\nu}$ to $\chi_{\alpha}^{\nu}$ via

$$\xi_{\alpha}^{\nu} = N_{\alpha,\beta} \chi_{\beta}^{\nu}$$

(30)

such that

$$\frac{1}{S^{(2)}} \sum_{\alpha,\nu} (\xi_{\alpha}^{\nu})^2 + \frac{1}{\gamma} \sum_{\alpha,\beta,\nu} v_0^\alpha \xi_{\alpha}^{\nu} \xi_{\beta}^{\nu} v_0^\beta = \sum_{\alpha,\nu} (\chi_{\alpha}^{\nu})^2 .$$

(31)

One easily finds that
\[ N_{\alpha \beta} = \sqrt{S^{(2)}} \left[ \delta_{\alpha \beta} + \frac{1}{\sqrt{1 + (\bar{v}_0)^2/v^2}} - 1 \right] \bar{v}_0^\alpha \bar{v}_0^\beta \]  \hspace{1cm} (32)

where

\[ \bar{v}^2 = \gamma / S^{(2)} = (\Gamma c)^2 \frac{\bar{S}_4}{S^{(2)}} \]  \hspace{1cm} (33)

After changing variables from \( \xi \) to \( \chi \) one obtains

\[ n_0 P(\bar{v}_0) = \left( \frac{S^{(2)}}{2\pi S_0} \right)^{n/2} \left( \frac{1}{2\pi \bar{v}^2} \right)^{n/2} \frac{\bar{J}}{[1 + (\bar{v}_0)^2/v^2]^{(n+2)/2}} \]  \hspace{1cm} (34)

where \( \bar{J} \) is the remaining dimensionless integral over the \( \chi \)'s which can be be evaluated directly as a separable product of gaussian integrals with the simple result \( \bar{J} = n! \). Since \( P(\nu_0) \) is normalized to one, we find on integration over \( \bar{v}_0 \) the result

\[ n_0 = \left( \frac{S^{(2)}}{2\pi S_0} \right)^{n/2} \frac{n!}{2^{n/2} \Gamma\left( \frac{n}{2} + 1 \right)} \]  \hspace{1cm} (35)

which agrees with the result found by Liu and Mazenko \[5\] using a more indirect method. After using this result for \( n_0 \) one finally obtains the result given by Eq.(1). This result basically says that the probability of finding a large velocity decreases with time. However, since this distribution falls off only as \( \nu_0^{-(n+2)} \) for large \( \nu_0 \) only the first moment beyond the normalization integral exists. This seems to imply the existence of a source of large velocities.

It seems likely \[11\] that this is associated with vortex-antivortex final annihilation.

The determination of \( S_0, S^{(2)}, \bar{S}_4 \) and \( \bar{v} \) requires a theory for the auxiliary field correlation function

\[ C_0(12) = \frac{1}{n} \langle \vec{m}(1) \cdot \vec{m}(2) \rangle \]  \hspace{1cm} (36)

There are two theories available and both are of the gaussian closure type assumed above. One, due to Ohta, Jasnow, and Kawasaki(OJK) \[12\], essentially postulates that \( C_0(12) \) is a gaussian

\[ C_0(12) = S_0 e^{-\bar{r}^2/(2L^2)} \]  \hspace{1cm} (37)

where \( \bar{r} = \bar{r}_1 - \bar{r}_2 \). In this case one easily finds that
where the coefficient of $L^2/t$ is undetermined in the theory of OJK. Using the theory developed in Ref. [13] for $n = 2$ one finds self-consistently [14] that

$$
\tilde{v}^2 = \frac{2d}{L^2}(\Gamma c)^2
$$

(38)

where $\mu = 0.53721..$ is the eigenvalue determined within the theory [13].

One can go forward and extend these ideas to treat two-point velocity correlation functions and string-like defects ($n = d - 1$) as will be discussed elsewhere.

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