AG-GROUPS AND OTHER CLASSES OF RIGHT BOL QUASIGROUPS

M. SHAH, S. SHPECTOROV, AND A. ALI

Abstract. By a result of Sharma, right Bol quasigroups are obtainable from right Bol loops via an involutive automorphism. We prove that the class of AG-groups, introduced by Kamran, is obtained via the same construction from abelian groups. We further introduce a new class of Bol∗ quasigroups, which turns out to correspond, as above, to the class of groups.

Sharma’s correspondence allows an efficient implementation and we present some enumeration results for the above three classes.

1. Introduction

By definition, an AG-group (also called an LA-group) G is a set with a binary operation satisfying the left invertive law : (xy)z = (zy)x for all x, y, z ∈ G and also having left identity and left inverses. From these axioms it follows that the left inverse also is the right inverse and thus it becomes a two sided inverse. In particular, AG-groups belong to the class of quasigroups [10]. AG-groups were introduced in the PhD thesis of Kamran [2] and they first appeared in print in [3]. Some of the basic properties of AG-groups were discussed in [8]. For example, associativity, commutativity, and the existence of identity are equivalent properties for AG-groups. In particular, among the groups only the abelian groups are AG-groups. For a geometric interpretation of AG-groups see [12]. AG-groupoids (also called LA-semigroups), which generalize AG-groups, have applications in flock theory, see [6]. For additional sources on AG-groupoids, we suggest [4], and also [13].

It was noticed in [10] that AG-groups belong to the class of right Bol quasigroups. It is well known that right Bol quasigroups and right Bol loops have applications in differential geometry [7]. In [8] enumeration of AG-groups was proposed as an interesting problem. In [9] the enumeration was carried out computationally up to order 12. In this paper we completely classify AG-groups by showing that every AG-group arises from an abelian group via an involutive automorphism.

Theorem 1. Suppose G is an abelian group and α ∈ Aut(G) satisfying α² = 1. Define a new binary operation on G by a · b := α(a) + b. Then Gα = (G, ·) is an AG-group. Furthermore, every AG-group is obtainable in this way. Finally, the AG-groups Gα and Hβ are isomorphic if and only if the abelian groups G and H are isomorphic and automorphisms α and β are conjugate.

This description of the class of AG-groups allows us to classify various subclasses of them. For example, it easily follows from Theorem 1 that the AG-group Gα is a group if and only if α is the identity automorphism of the abelian group G. In

1991 Mathematics Subject Classification. 20N99.
Key words and phrases. AG-group, AG-groupoid, LA-group, counting.
the similar spirit let an AG-group be called involutory if its every element is an involution, that is, it satisfies $a^2 = e$, where $e$ is the (left) identity element. The following is a corollary of Theorem 1.

Theorem 2. An AG-group $G_\alpha$ is involutory if and only if $\alpha$ is the minus identity automorphism that is $\alpha(g) = -g$ for all $g \in G$. In particular, there is a natural bijection between abelian groups and involutory AG-groups.

The groups of order one and two are the only cyclic groups for which the identity automorphism is the same as minus identity. In particular, for all orders $n > 2$ there exists a non-associative AG-group.

There have been a lot of publications (see for example, [1]) about the multiplication groups of loops and quasigroups. By definition, the multiplication group $M(Q)$ of a quasigroup $Q$ is the subgroup of $\text{Sym}(Q)$ generated by all left and right translations. The multiplication group of an AG-group was studied in [11] where it was established that for a nonassociative AG-group of order $n$ its multiplication group is nonabelian of order $2^n$ and, correspondingly, the so called inner mapping group has order two. Based on Theorem 1, we can give a more precise description of the multiplication group.

Theorem 3. Suppose $G_\alpha$ is a non-associative AG-group, that is, $\alpha$ is non-identity. Then $M(G_\alpha)$ is isomorphic to the semidirect product $G : \langle \alpha \rangle$. Note also that the order two group $\langle \alpha \rangle$ is the inner mapping group $I(G_\alpha)$, that is, the stabilizer in $M(G_\alpha)$ of the identity element.

The construction of the AG-groups from the abelian groups, as described in Theorem 1 can easily be implemented in a computer algebra system. In fact, we implemented it in GAP [5] and were able to enumerate all AG-groups up to the order 2009. We stopped at the number because we used the small group library of GAP as our source of abelian groups. The method can easily be extended to much greater orders, as long as the abelian groups of that order are available. As a sample of the computation, we provide here (see Table 1) the information about the number of AG-groups up to order 20.

| Order | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-------|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|
| Group | 1 | 2 | 1 | 1 | 3 | 2 | 1 | 1 | 1 | 1 | 5 | 1 | 2 | 1 | 2 | 2 | 2 | 2 |
| Other | 1 | 1 | 1 | 1 | 3 | 1 | 1 | 1 | 1 | 4 | 1 | 3 | 1 | 2 | 2 | 2 | 2 | 2 |
| Total | 2 | 3 | 2 | 2 | 4 | 3 | 2 | 2 | 2 | 2 | 7 | 3 | 2 | 2 | 2 | 2 | 2 | 2 |

Table 1. Number of AG-groups of order $n$, $3 \leq n \leq 20$

The correspondence between the classes of abelian groups and AG-groups is very simple, so naturally, we were wondering whether a similar construction had been known. And indeed, we found a paper by Sharma [14] establishing a correspondence between the classes of left Bol loops and left Bol quasigroups. By duality, there is a similar correspondence between right Bol loops and right Bol quasigroups. This dual correspondence is essentially the same as our correspondence. Clearly, the class of abelian groups is a subclass of the class of right Bol loops. It is not so immediately clear, but still can be shown that the class of AG-groups is a subclass of the class of right Bol quasigroups. Hence our correspondence is simply a special
case of Sharma’s correspondence adjusted for the case of right Bol loops. In this sense, our Theorem 1 shows that the class of AG-groups is the counterpart of the class of abelian groups under Sharma’s correspondence. We consider it an interesting problem to determine which classes of quasigroups are the counterparts of other subclasses of right Bol loops, such as say, the class of groups or the class of Moufang loops. In this paper we give an answer to the first of these questions, namely, we provide the axioms for the class of quasigroups corresponding to the class of groups.

**Definition 1.** A right Bol* quasigroup is a quasigroup satisfying
\[ a(bc \cdot d) = (ab \cdot c)d \]
for all elements \( a, b, c, d \).

Note that the substitution \( d = b \) turns the above equality into the right Bol law, which shows that the class of the right Bol* quasigroups is a subclass of right Bol quasigroups. In the future we will just speak of Bol quasigroups and Bol* quasigroups, skipping ‘right’.

**Theorem 4.** Suppose \( G \) is a group and \( \alpha \in \text{Aut}(G) \) satisfying \( \alpha^2 = 1 \). Define a new binary operation on \( G \) by \( a \ast b := \alpha(a)b \). Then \( G\alpha = (G, \ast) \) is a Bol* quasigroup. Furthermore, every Bol* quasigroup is obtainable in this way. Finally, the Bol* quasigroups \( G\alpha \) and \( H\beta \) are isomorphic if and only if the groups \( G \) and \( H \) are isomorphic and automorphisms \( \alpha \) and \( \beta \) are conjugate.

2. Preliminaries

The following property of AG-groups was established in [10].

**Lemma 1.** Every AG-group satisfies the identity \((ab \cdot c)d = a(bc \cdot d)\). In other words, every AG-group is a Bol* quasigroup.

We now embark on proving Theorem 1. We start with the first claim in that theorem.

**Proposition 1.** Let \( G \) be an abelian group under addition and let \( \alpha \in \text{Aut}(G) \) be such that \( \alpha^2 = 1 \). Define \( x \cdot y = \alpha(x) + y \) for all \( x, y \in G \). Then \( G\alpha = (G, \cdot) \) is an AG-group with left identity \( e = 0 \).

**Proof.** We start by checking the left invertive law in \( G\alpha \). Let \( x, y, z \in G \). Then \( xy \cdot z = \alpha(\alpha(x) + y) + z = \alpha^2(x) + \alpha(y) + z = x + \alpha(y) + z \), since \( \alpha^2 = 1 \). Similarly, \( zy \cdot x = z + \alpha(y) + x \), and so \( zy \cdot x = z + \alpha(y) + x = x + \alpha(y) + z = xy \cdot z \).

It is easy to see that \( 0 \) is the left identity in \( G\alpha \). Indeed, \( 0x = \alpha(0) + x = 0 + x \), for all \( x \in G \). Finally, we claim that \( \alpha(-x) \) is the left inverse of \( x \). Indeed, \( \alpha(-x)x = \alpha(\alpha(-x)) + x = -x + x = 0 \).

This shows that \( G\alpha \) is an AG-group. \( \square \)

We next need to show that every AG-group can be obtained as above. Let \( G \) be an AG-group with a left identity \( e \). We first show how to build an abelian group from \( G \).

**Proposition 2.** Consider the set \( G \) together with the new operation + defined as follows:
\[ x + y := xe \cdot y, \]
for all \(x, y \in G\). Then \((G, +)\) is an abelian group. The zero element of this group is \(e\) and, for every \(x \in G\), the inverse \(-x\) is equal to \(x^{-1}e\).

**Proof.** We start by checking associativity of addition. Let \(x, y, z \in G\). Then \((x + y) + z = (xe \cdot y)e \cdot z\). Using Lemma 1 with \(a = xe\), \(b = ye\), \(c = e\), and \(d = z\), we get that \((xe \cdot y)e \cdot z = xe \cdot (ye \cdot z) = x + (y + z)\). Therefore, \((x + y) + z = x + (y + z)\), proving associativity.

Commutativity of addition follows essentially by the definition. Indeed, \(x + y = xe \cdot ye = ye \cdot x = y + x\) by the left invertive law. Similarly, \(e + x = ee \cdot x = ex = x\). Now by commutativity \(e\) is the zero element of \((G, +)\). Finally, \(x^{-1}e + x = (x^{-1}e)e \cdot x = ((ee)x^{-1})x = ex^{-1} \cdot x = x^{-1}x = e\). Again, commutativity shows that \(x^{-1}e\) is the inverse \(-x\).

We remark that in place of the identity \(e\) we could use any fixed element \(c \in G\). Namely, if we define addition via: \(x \oplus y := xc \cdot y\) then we again get an abelian group, whose zero element is \(c\) and where the inverses are computed as follows: \(\ominus x := x^{-1}c\). The proof is essentially the same. Furthermore, the groups obtained for different elements \(c\) are all isomorphic. Namely, the isomorphism between \((G, +)\) and \((G, \oplus)\) is given by \(x \mapsto x * c\).

Our next step is to prove that the mapping \(\alpha : G \rightarrow G\) defined by \(g \mapsto ge\) is an involutive automorphism of the abelian group \((G, +)\).

**Proposition 3.** For all \(x, y \in G\), we have \(\alpha(x + y) = \alpha(x) + \alpha(y)\) and, furthermore, \(\alpha^2 = 1\). Therefore, \(\alpha\) is an involutive automorphism of \((G, +)\).

**Proof.** We first note that by the left invertive law \(\alpha^2(x) = xe \cdot e = ee \cdot x = ex = x\) for all \(x \in G\). Therefore, \(\alpha^2 = 1\), the identity mapping on \(G\). By Lemma 1 \(\alpha(x + y) = \alpha(xe \cdot ye) = (xe \cdot ye) = x(ye) = x(e ye)\). On the other hand, \(\alpha(x) + \alpha(y) = (xe) \cdot ye\). We saw above that \(xe \cdot e = x\), hence \(\alpha(x) + \alpha(y) = x(ye)\), which we have shown to be equal to \(\alpha(x + y)\). Therefore, \(\alpha\) is an automorphism. \(\square\)

The last two results show that every AG-group \(G\) canonically defines an abelian group \((G, +)\) and its involutive automorphism \(\alpha\). It remains to see that the AG-group \(G\) can be recovered from \((G, +)\) and \(\alpha\) as in Proposition 1.

**Proposition 4.** Suppose \(G\) is an AG-group and let \((G, +)\) and \(\alpha\) be the corresponding abelian group and its involutive automorphism. Then for all \(x, y \in G\) we have \(xy = \alpha(x) + y\). That is, \(G = G_\alpha\).

**Proof.** This is clear: indeed, \(\alpha(x) + y = (xe) \cdot ye = xy\). We used the identity \(xe \cdot e = x\), which we showed before. \(\square\)

We now turn to homomorphisms between AG-groups.

**Proposition 5.** Suppose \(G\) and \(H\) are abelian groups and let \(\alpha \in \text{Aut}(G)\) with \(\alpha^2 = 1\) and \(\beta \in \text{Aut}(H)\) with \(\beta^2 = 1\). Then the set of homomorphisms between AG-groups \(G_\alpha\) and \(H_\beta\) coincides with the set of group homomorphisms \(\pi : G \rightarrow H\) satisfying \(\pi \alpha = \beta \pi\).

**Proof.** Suppose \(\pi : G_\alpha \rightarrow H_\beta\) is a homomorphism of AG-groups, that is, it is a mapping \(G \rightarrow H\) such that \(\pi(gh) = \pi(g)\pi(h)\). By cancellativity, \(\pi\) sends the left identity of \(G_\alpha\) to the left identity of \(H_\beta\). Therefore, for \(x, y \in G\), we have \(\pi(x + y) = \pi(xe \cdot y) = \pi(x)\pi(y) = \pi(x)\pi(e \cdot y) = \pi(x)e \cdot \pi(y) = \pi(x) + \pi(y)\). This shows that \(\pi\) is a homomorphism of abelian groups. Next, let \(x \in G\). Then
\[ \pi \alpha (x) = \pi (xe) = \pi (x)e = \beta \pi (x). \] Since \( x \in G \) is arbitrary, we conclude that \( \pi \alpha = \beta \pi \).

For the converse, suppose that \( \pi : G \to H \) is a homomorphism of abelian groups and that \( \pi \) satisfies \( \pi \alpha = \beta \pi \). Then, for \( x, y \in G \), we have \( \pi (xy) = \pi (\alpha (x) + y) = \pi \alpha (x) + \pi (y) = \beta \pi (x) + \pi (y) = \pi (x) \pi (y) \). Hence \( \pi \) is a homomorphism of AG-groups. \( \square \)

This allows to complete the proof of Theorem 1.

**Corollary 1.** Two AG-groups \( G_\alpha \) and \( H_\beta \) are isomorphic if and only if there is an isomorphism \( \pi \) between \( G \) and \( H \), satisfying \( \pi \alpha \pi ^{-1} = \beta \).

**Proof.** Immediately follows from Proposition 5. Indeed, if \( \pi \) is bijective then the condition \( \pi \alpha = \beta \pi \) is equivalent to \( \pi \alpha \pi ^{-1} = \beta \). \( \square \)

We record here a further corollary of Proposition 5, which describes the full automorphism group of the AG-group \( G_\alpha \).

**Corollary 2.** The automorphism group of the AG-group \( G_\alpha \) coincides with \( C_{\text{Aut}(G)}(\alpha) \), the centralizer in \( \text{Aut}(G) \) of the involution \( \alpha \).

**Proof.** If \( G_\alpha = H_\beta \) (and so \( G = H \) and \( \alpha = \beta \)) then the condition \( \pi \alpha = \beta \pi = \alpha \pi \) means simply that \( \pi \in \text{Aut}(G) \) must commute with \( \alpha \). \( \square \)

It is interesting that the involutory twist construction can be used repeatedly.

**Proposition 6.** Let \((G, \circ)\) be an AG-group with a left identity \( e \). Let \( \alpha \in \text{Aut}(G) \) such that \( \alpha^2 = 1 \). Define \( x \circ y = \alpha (x) \circ y \) for all \( x, y \in G \). Then \((G, \cdot)\) is again an AG-group.

**Proof.** Initially this had an independent proof. However, with all the theory that we have developed, this result follows easily. Indeed, by Theorem 1 the AG-group \((G, \circ)\) must be equal to \( G_\beta \), for an abelian group \( G \) and its involutory automorphism \( \beta \).

Note that this means that \( x \circ y = \beta (x) + y \), where, as usual, plus indicates addition in the abelian group \( G \). According to Corollary 2 \( \alpha \) is an automorphism of the group \( G \) commuting with \( \beta \). In particular, \( \gamma = \beta \alpha \) is again an involutory automorphism of \( G \).

We now notice that \( x \cdot y = \alpha (x) \circ y = \beta (\alpha (x)) + y = \gamma (x) + y \). This means that \((G, \cdot)\) is simply the AG-group \( G_\gamma \). \( \square \)

### 3. Particular classes of AG-groups

It is natural to ask when the AG-group \( G_\alpha \) is associative, that is, a group. It was shown in [8] that for AG-groups associativity is equivalent to commutativity and also to the property that the left identity \( e \) is a two-sided identity. We can show that, in fact, \( G_\alpha \) is never a group, when \( \alpha \neq 1 \).

**Proposition 7.** Suppose \( G \) is an abelian group and \( \alpha \in \text{Aut}(G) \) with \( \alpha^2 = 1 \). Then \( G_\alpha \) is a group if and only if \( \alpha = 1 \).

**Proof.** If \( \alpha = 1 \) then \( \alpha (x) + y = x + y \), hence \( G_\alpha \) is simply the group \( G \). Conversely, assume that \( G_\alpha \) is a group. Note that \( e = 0 \) is the left identity of \( G_\alpha \), since \( 0 \cdot x = \alpha (0) + x = 0 + x = x \). However, in a group the left identity is the same as the right identity. Therefore, for all \( x \in G \), we must have \( x \cdot 0 = x \). However, \( x \cdot 0 = \alpha (x) + 0 = \alpha (x) \). Hence, \( \alpha (x) = x \) for all \( x \in G \), which means that \( \alpha = 1 \). \( \square \)
This proof already verifies that $G_\alpha$ is a group whenever it has a two-sided identity. Quite similarly, if $G_\alpha$ is commutative then for every $x \in G$ we have $x \cdot 0 = 0 \cdot x = x$. On the other hand, $x \cdot 0 = \alpha(x) + 0 = \alpha(x)$. Hence we must have that $\alpha(x) = x$ for all $x \in G$, and so $G_\alpha = G$ is a group. This shows that indeed commutativity is also equivalent to associativity.

The second interesting class of AG-groups is the class of involutory AG-groups. Recall from the introduction that an AG-group $G$ is called involutory if its every nontrivial element is an involution, i.e., $x^2 = e$ for all $x \in G$ where $e$ is the left identity of $G$.

**Proposition 8.** Suppose $G$ is an abelian group and $\alpha \in \text{Aut}(G)$ with $\alpha^2 = 1$. Then $G_\alpha$ is an involutory if and only if $\alpha = -1$. (This means that $\alpha(x) = -x$ for all $x \in G$.)

**Proof.** Recall that $x \cdot x = \alpha(x) + x$, so $x \cdot x = e = 0$ if and only if $\alpha(x) + x = 0$, that is, $\alpha(x) = -x$, so $G_\alpha$ is involutory if and only if $\alpha(x) = -x$ for all $x$. □

As a consequence, we get the following.

**Corollary 3.** For every order $n \geq 3$ there exists a non-associative AG-group of order $n$.

**Proof.** Indeed, we can take $G = C_n$, the cyclic group of order $n$, and $\alpha = -1$. When $n \geq 3$, we have $\alpha \neq 1$, which means that $G_\alpha$ is non-associative by Proposition 7. □

Since for every prime order $n = p > 2$ there exists exactly one abelian group, the cyclic group $C_p$, and since $\text{Aut}(C_p) \cong C_p - 1$, which has a unique element of order two, we have the following result.

**Corollary 4.** For every prime order $n = p \geq 3$, there is only one non-associative AG-group of order $n$.

### 4. Some examples

For illustration we provide some examples. The case of the prime order has been dealt with in the preceding section.

**Example 1.** For order 6, we have only one abelian group, namely $C_6$. Since $\text{Aut}(C_6)$ has only one nontrivial involution, there are exactly two AG-groups of order 6, one associative, $C_6$, and one non-associative, namely, $(C_6)_\alpha$, where $\alpha = -1$.

The same is true for all orders $2p$, where $p$ is an odd prime. So this case is similar to the case of the odd prime order.

**Example 2.** For order 12, there are exactly two abelian groups, namely $C_{12}$ and $C_6 \times C_2$. In the first case, $\text{Aut}(C_{12})$ is an elementary abelian group of order four. Hence its every element can be used to construct a new AG-group. This gives us four AG-groups (one associative, one non-associative involutory, and two further non-associative non-involutory). In the second case, $\text{Aut}(C_6 \times C_2)$ is isomorphic to $C_2 \times \text{Sym}(3)$, and so is nonabelian of order 12. In addition to the identity element, this group has three conjugacy classes of involutions. Hence, in this case, too, we get four different AG-groups.

In total, we obtain eight AG-groups of order 12, out of which six are non-associative.
Example 3. Let us consider the order $2009 = 7^2 \cdot 41$. Again, there are two abelian groups of this order, $C_{2009}$ and $C_{287} \times C_7$. In the first case the automorphism group is abelian, containing three involutions. Hence this group leads to four AG-groups. The automorphism group of $C_{287} \times C_7$ is isomorphic to $C_{40} \times GL(2, 7)$. This group has five conjugacy classes on involutions in addition to the identity element, hence in this case we obtain six different AG-groups.

In total, there are 10 different AG-groups of order 2009, out of which eight are non-associative.

5. A GAP package for computing with AG-groups

V. Sorge and the first author developed a GAP package AGGROUPOIDS which, in particular, contains functions dealing with AG-groups. They are based on the theory developed in this paper.

There are four main functions:

- The function $\text{NrAllSmallNonassociativeAGGroups}(n)$ returns the total number of nonassociative AG-groups of order $n$ provided that the SmallGroups library contains the list of groups of order $n$. This restriction will be lifted in the future, since all abelian groups of a given order are easy to construct.
- The function $\text{AllSmallNonassociativeAGGroups}(n)$ returns the list of all non-associative AG-groups of the given order. Each AG-group is represented as a GAP quasigroup.
- The function $\text{NrAllSmallNonassociativeAGGroupsFromAnAbelianGroup}(G)$ returns the total number of non-associative AG-groups that can be obtained from the abelian group $G$. This is equal to the number of conjugacy classes of involutions in $\text{Aut}(G)$.
- The function $\text{AllSmallNonassociativeAGGroupsFromAnAbelianGroup}(G)$ returns the list of non-associative AG-groups obtainable from $G$, again as GAP quasigroups.

The entire package (not limited to these four functions) will shortly be available from the GAP repository.

6. Multiplication group of an AG-group

The concept of the multiplication group of a loop and, more generally, a quasigroup is well known. In a quasigroup $Q$, multiplication on the left (or right) by an element $x \in Q$ is a permutation $L_x$ (respectively, $R_x$) of $Q$ called the left (respectively, right) translation by $x$. The set of all left translations is called the left section, and similarly, the set of right translations is called the right section of $Q$. We will write $LSec$ and $RSec$ for the left and right sections, respectively. Therefore, $LSec = \{L_x \mid x \in Q\}$ and $RSec = \{R_x \mid x \in Q\}$.

The multiplication group $M(Q)$ is the subgroup of the symmetric group $\text{Sym}(Q)$ generated by $LSec \cup RSec$. If $Q$ is a loop, the stabilizer in $M(Q)$ of the identity is called the inner mapping group and denoted $\text{Inn}(Q)$.

Since every AG-group $G_\alpha$ is a quasigroup we can consider its multiplication group $M(G_\alpha)$. Since $G_\alpha$ has a left identity 0, we can generalize the concept of the inner mapping group to the class of AG-groups by setting $\text{Inn}(G_\alpha)$ to be the stabilizer of 0 in $M(G_\alpha)$. 
Proposition 9. Let $G$ be an abelian group and $\alpha \in \text{Aut}(G)$ with $\alpha^2 = 1$. Then the following hold:

1. $M(G_\alpha) = \text{LSec} \cup \text{RSec}$;
2. $\text{Inn}(G_\alpha) = \langle \alpha \rangle$;
3. $\text{LSec}$ is a normal subgroup of $M(G_\alpha)$ and it is naturally isomorphic to $G$;
4. $\text{RSec} = \alpha \text{LSec}$; and
5. $M(G_\alpha)$ is isomorphic to the semidirect product of $G$ with the cyclic group $\langle \alpha \rangle$.

Proof. First note that the mapping $\psi : x \mapsto L_x$ is a homomorphism from $G$ to $\text{Sym}(G)$. Indeed, $L_{x+y}(z) = (x+y) \cdot z = \alpha(x+y) + z = \alpha(x) + \alpha(y) + z = x \cdot (\alpha(y) + z) = x \cdot (y \cdot z) = L_x(L_y(z))$ for all $z \in G$. This means that $L_{x+y}$ is indeed the product of $L_x$ and $L_y$. Since $\psi$ is a homomorphism, its image $\text{LSec}$ is a subgroup of $\text{Sym}(G)$. Furthermore, if $L_x(z) = z$ for some $z \in G$ then $\alpha(x) + z = z$, which implies that $x = 0$. Therefore, $\psi$ is injective and so it is an isomorphism from $G$ onto $\text{LSec}$.

Next, note that $\alpha(z) = \alpha(z) + 0 = z \cdot 0 = R_0(z)$. This means that $\alpha = R_0$ is an element of $\text{RSec}$. Furthermore, $R_x(z) = \alpha(z) + x = \alpha(z) + \alpha^2(x) = \alpha(\alpha(x)) + \alpha(z) = \alpha(\alpha(x) + z) = (\alpha L_x)(z)$. This means that $R_x = \alpha L_x$ for all $x \in G$, that is, $\text{RSec}$ is the coset of $\text{LSec}$ containing $\alpha$.

We now turn to part (1). We claim that $\alpha$ normalizes $\text{LSec}$. Indeed, $(\alpha L_x \alpha)(z) = \alpha L_x(\alpha(z)) = \alpha(\alpha(x) + \alpha(z)) = x + z = \alpha(\alpha(x)) + z = L_\alpha(x)(z)$. Thus, $\alpha L_x \alpha = L_\alpha(x)$, proving that $\alpha$ normalizes the subgroup $\text{LSec}$. Since $\text{RSec} = \alpha \text{LSec}$, we conclude that every element of $\text{RSec}$ normalizes $\text{LSec}$, which means that $\text{LSec}$ is normal in $M(G_\alpha)$. Also, it means that $M(G_\alpha) = \langle \text{LSec}, \alpha \rangle$, which implies that $\text{LSec}$ has index at most two in $M(G_\alpha)$. (This proves (1).) To be more precise, the index is two if and only if $\alpha \not\in \text{LSec}$. Clearly, $\alpha$ fixes 0 and, as we have already seen, the only element of $\text{LSec}$ fixing 0 is $L_0$, the identity element of $\text{LSec}$. Hence $\text{LSec}$ has index two in $M(G_\alpha)$ if and only if $\alpha \neq 1$.

From the above, we also have that $|\text{Inn}(G_\alpha)| = |\alpha|$, since $\text{LSec}$ is regular on $G$ and so $|\text{Inn}(G_\alpha)|$ is equal to the index of $\text{LSec}$ in $M(G_\alpha)$. Since $\alpha$ fixes 0, we have $\alpha \in \text{Inn}(G_\alpha)$, which implies (2). Parts (3) and (4) have already been proven. Finally, since $\alpha \not\in \text{LSec}$ and $M(G_\alpha) = \langle \text{LSec}, \alpha \rangle$, (5) follows as well. $\square$

As an example of how the multiplication group can be used to identify the AG-group, we present the following result.

Theorem 5. Suppose $M = M(G_\alpha)$ for a non-associative AG-group $G_\alpha$ and $M \cong D_{2n}$. Then either $G$ is the Klein four-group (and so $n = 4$) or $G \cong C_n$ is cyclic. In the latter case $\alpha = -1$, and hence $G_\alpha$ is involutory.

Proof. First of all, since $G_\alpha$ is non-associative, $\alpha$ is a nontrivial automorphism of $G$ and so $n = |G| \geq 3$. By Theorem 9, the abelian group $G$ is isomorphic to an index two subgroup of $M$. From this, it immediately follows that either $n = 4$ and $G$ is the Klein four-group, or $n \geq 3$ is arbitrary and $G$ is cyclic. Finally, in the cyclic case, since $M(G_\alpha)$ is isomorphic to the semidirect product of $G$ and $\langle \alpha \rangle$, we conclude that $\alpha$ inverts every element of $G$ and so $\alpha = -1$. $\square$

We also give a general characterization of all groups that arise as multiplication group of an AG-group.
Theorem 6. A nonabelian group $M$ is isomorphic to a multiplication group of some non-associative AG-group if and only if $M \cong T \times R$ where $T$ is abelian and $|R| = 2$.

Proof. If $M = M(G_{\alpha})$ then $M = G \times \langle \alpha \rangle$ and so all the claimed properties hold. Conversely, suppose $M = T \times R$ where $T$ is abelian and $|R| = 2$. Let $\alpha \in \text{Aut}(T)$ be the automorphism induced by the generator of $R$ on $T$. Then $M \cong M(T_{\alpha})$ by Proposition 3.5.

7. Sharma’s Correspondence

In his paper [8] from 1976 Sharma proved the following theorem. We recall that the identity $(ab \cdot c)b = a(bc \cdot b)$ is known as the right Bol identity. The loops (respectively, quasigroups) satisfying this identity are called the right Bol loops (respectively, right Bol quasigroups).

Theorem 7. Suppose $G$ is a right Bol loop and $\alpha \in \text{Aut}(G)$ satisfying $\alpha^2 = 1$. Define a new binary operation on $G$ by $a * b := \alpha(a)b$. Then $G_{\alpha} = (G, *)$ is a right Bol quasigroup. Furthermore, every right Bol quasigroup is obtainable in this way. Finally, the right Bol quasigroups $G_{\alpha}$ and $H_{\beta}$ are isomorphic if and only if the right Bol loops $G$ and $H$ are isomorphic and automorphisms $\alpha$ and $\beta$ are conjugate.

In reality Sharma proved the “left” version of this theorem, but we switched to the above, “right” version because it matches better our own results.

In particular, Sharma’s theorem implies that every right Bol quasigroup automatically has a left identity element.

We note that Sharma’s construction is essentially the same as ours, except it is done for a different, larger class of objects, the Bol loops instead of abelian groups. In other words, what we proved in Theorem 1 means simply that the class of AG-groups is the counterpart of the subclass of abelian groups under Sharma’s correspondence. It would be interesting to ask what are the counterparts of other subclasses of Bol loops, such as, say, groups or Moufang loops. We leave the Moufang loops case as an open question, but we have an answer for the class of groups.

Recall from the introduction that by a Bol* quasigroup we mean a quasigroup satisfying

$$a(bc \cdot d) = (ab \cdot c)d$$

for all $a, b, c, d$. Note that this is clearly a subclass of Bol quasigroups. In particular, every Bol* quasigroup automatically has a left identity element.

Theorem 8. Suppose $G$ is a group and $\alpha \in \text{Aut}(G)$ satisfying $\alpha^2 = 1$. Define a new binary operation on $G$ by $a * b := \alpha(a)b$. Then $G_{\alpha} = (G, *)$ is a Bol* quasigroup. Furthermore, every Bol* quasigroup is obtainable in this way. Finally, the Bol* quasigroups $G_{\alpha}$ and $H_{\beta}$ are isomorphic if and only if the groups $G$ and $H$ are isomorphic and automorphisms $\alpha$ and $\beta$ are conjugate.

Proof. Let us first see that $G_{\alpha}$ as above satisfies the identity

$$a * ((b * c) * d) = ((a * b) * c) * d.$$ 

Indeed, $a * ((b * c) * d) = \alpha(a)\alpha(\alpha(b)c)d = \alpha(a)\alpha^2(b)\alpha(c)d = \alpha(\alpha(b)a)c)d$, since $\alpha^2 = 1$. Similarly, $((a * b) * c) * d = \alpha(\alpha(a)b)c)\alpha(c)d = \alpha^3(a)\alpha^2(b)\alpha(c)d = \alpha(\alpha(b)a)c)d$. So the identity holds, proving that $G_{\alpha}$ is a Bol* quasigroup.
Conversely, assume that \((G, \ast)\) is a Bol\(^*\) quasigroup with left identity \(e\). For \(x \in G\), define \(\alpha(x) = x \ast e\) and also, for \(x, y \in G\), define \(xy = \alpha(x)y\). We need to see that (1) \(G\) with this new product is a group; (2) \(\alpha\) is an automorphism of this group of order two; and (3) \((G, \ast) = G_\alpha\).

First of all, for \(x, y, z \in G\), \(x(yz) = \alpha(x) \ast (\alpha(y) \ast z) = (x \ast e) \ast ((y \ast e) \ast z)\). By the identity, the latter is equal to \(((x \ast e) \ast y) \ast e) \ast z\) and this is equal to \((x \ast ((e \ast y) \ast e) \ast z)\).

On the other hand, \((xy)z = \alpha(\alpha(x) \ast y) \ast z = (((x \ast e) \ast y) \ast e) \ast z\), so we have \(x(yz) = (xy)z\) for all \(x, y, z \in G\), proving that the new operation is associative. Cancellativity is clear, so we have an associative quasigroup, hence a group. Note that \(e\) is the identity element of the group, since \(ex = \alpha(e) \ast x = (e \ast e) \ast x = e \ast x = x\).

For (2), we first need to show that \(\alpha\) is a permutation of order two: \(\alpha^2(x) \ast z = ((x \ast e) \ast e) \ast z = x \ast ((e \ast e) \ast z) = x \ast z\), and so by cancellativity, \(\alpha^2(x) = x\). Thus, \(\alpha^2 = 1\). To show that \(\alpha\) is an automorphism, we compute: \(\alpha(xy) = (xy) \ast e = (\alpha(x) \ast y) \ast e = ((x \ast e) \ast y) \ast e = x \ast ((e \ast y) \ast e) = x \ast (y \ast e)\) and \(\alpha(xy) = \alpha(\alpha(x)) \ast \alpha(y) = x \ast (y \ast e)\). Thus, \(\alpha(xy) = \alpha(x) \ast \alpha(y)\). Finally, (3) is clear since \(x \ast y = \alpha^2(x) \ast y = \alpha(x)y\). Hence \(x \ast y = \alpha(x)y\), which means that \((G, \ast) = G_\alpha\).

For the final claim in the theorem, we note that the proofs of Proposition 5 and Corollary 2 depend neither on commutativity of the group operation, nor on the left invertive identity, so they fully apply in our present case.

The package AGGROUPOIDS mentioned above also contains functions enumerating Bol\(^*\) quasigroups and Bol quasigroups based on Theorem 8 and Sharma’s Theorem 7. In Tables 2 and 3 we provide the counting for the Bol\(^*\) quasigroups and Bol quasigroups up to order 20 and 30, respectively.

### Table 2. Number of Bol\(^*\) quasigroups of order \(n\), \(3 \leq n \leq 20\)

| Order | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|-------|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|
| Group | 1 | 2 | 1 | 2 | 4 | 2 | 1 | 5 | 1 | 2 | 1 | 1 | 6 | 1 | 1 | 22 | |
| Non-group | 1 | 2 | 1 | 2 | 4 | 2 | 1 | 5 | 1 | 2 | 1 | 1 | 6 | 1 | 1 | 22 | |
| Total | 2 | 4 | 2 | 4 | 8 | 4 | 2 | 10 | 2 | 3 | 2 | 2 | 12 | 2 | 2 | 44 | |

### Table 3. Number of Bol quasigroups of order \(n\), \(3 \leq n \leq 30\)

| Order | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
|-------|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Bol loop | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 5 | |
| Other | 2 | 4 | 2 | 4 | 8 | 4 | 2 | 10 | 2 | 3 | 2 | 2 | 12 | 2 | 2 | 44 | |
| Total | 3 | 6 | 3 | 6 | 16 | 8 | 6 | 20 | 4 | 5 | 4 | 4 | 24 | 4 | 4 | 48 | |

It might be worth mentioning that we can enumerate Bol\(^*\) quasigroups for much larger orders, as long as the list of groups of that order is available. For Bol quasigroups, we can only go up to the order 31, as the list of Bol loops of order 32 is an open problem.
AG-GROUPS AND OTHER CLASSES OF RIGHT BOL QUASIGROUPS

REFERENCES

[1] A. Drápal, T. Kepka, P. Maršílek, Multiplication groups of quasigroups and loops. II. Acta Univ. Carolin. Math. Phys. 35 (1994), no. 1, 9-29.

[2] M.S. Kamran, Conditions for LA-semigroups to resemble associative structures, Ph.D. Thesis, Quaid-i-Azam University, Islamabad, 1993. Available at http://eprints.hec.gov.pk/2370/1/2225.htm.

[3] Q. Mushtaq and M.S. Kamran, On left almost groups, Proc. Pak. Acad. of Sciences, 33 (1996), 1–2.

[4] Q. Mushtaq and S. M. Yusuf, On Locally Associative LA-semigroup, J. Nat. Sci. Math. Vol. XIX, No.1, April 1979, pp. 57–62.

[5] G. P. Nagy and P. Vojtechovsky, LOOPS: Computing with quasigroups and loops in GAP, version 1.0.0, computational package for GAP; http://www.math.du.edu/loop.

[6] M. Naseeruddin, Some studies on almost semigroups and flocks, Ph.D Thesis, The Aligarh Muslim University, India, 1970.

[7] L. Sabinin, Smooth Quasigroups and Loops, Kluwer, 1999.

[8] M. Shah, A. Ali, Some structural properties of AG-group, International Mathematical Forum 6 (2011), no. 34, 1661–1667.

[9] M. Shah, C. Gretton, Enumerating AG-groups with Finder, submitted.

[10] M. Shah, A. Ali, V. Sorge, A study of AG-groups as quasigroups, submitted.

[11] M. Shah, A. Ali, V. Sorge, Multiplication group of an AG-group, submitted.

[12] M. Shah, V. Sorge, A. Ali, A study of AG-groups as parallelogram spaces, submitted.

[13] M. Shah, T. Shah, A. Ali, On the cancellativity of AG-groupoids, International Mathematical Forum 6 (2011), no. 44, 2187–2194.

[14] B. L. Sharma, Left Loops which satisfy the left Bol identity, Proc. AMS 61 (1976), 189–195.

DEPARTMENT OF MATHEMATICS, QUAID-E-AZAM UNIVERSITY, ISLAMABAD, PAKISTAN.

E-mail address: shahmaths_problem@hotmail.com

E-mail address: sergeys@for.mat.bham.ac.uk

SCHOOL OF MATHEMATICS, UNIVERSITY OF BIRMINGHAM, UK.

E-mail address: s.shpectorov@bham.ac.uk

E-mail address: dr.asif.ali@hotmail.com

DEPARTMENT OF MATHEMATICS, QUAID-E-AZAM UNIVERSITY, ISLAMABAD, PAKISTAN.