Time variation of Equation of State for Dark Energy

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Abstract

The time variation of the equation of state \( w_Q \) for the dark energy are analyzed by the present values of parameters \( \Omega_Q, w_Q \) and its time derivatives. In the future the detailed feature of the dark energy could be observed, so we have considered the second derivatives of \( w_Q \) for two typical potentials \( V = M^{4+\alpha}/Q^\alpha \) and \( V = M^4 \exp (\beta M/Q) \). The first derivative \( dw_Q/da \) and the second derivative \( d^2w_Q/da^2 \) for both potentials are derived. The first derivative is estimated by the observed two parameters \( \Delta = w_Q + 1 \) and \( \Omega_Q \), with the assuming for \( Q_0 \). In the limit \( \Delta \to 0 \), the first derivative is null and, under the tracker approximation, the second derivative becomes also null. For the inverse power potential \( V = M^{4+\alpha}/Q^\alpha \), the observed first and second derivatives are used to determine the potential parameter \( M \) and \( \alpha \). For the potential of \( V = M^4 \exp (\beta M/Q) \), the second derivative is estimated by the observed parameters \( \Delta, \Omega_Q \) and \( dw_Q/da \).
I. INTRODUCTION

Even though it has passed almost one and half decade after the detection of the acceleration of the universe, it is not well understood the dark energy \cite{1}. We do not yet know whether it is the cosmological constant or not \cite{2,3}. Then it is searched to observe the variation of the equation of state ($w = w_Q$) for the dark energy. Many works have been done on the study of dark energy in the form of a slowly rolling scalar field and time variation of the equation of state for the dark energy. Usually it is taken the parameter to denote the variation of $w$ as \cite{1,4-9}

$$w(a) = w_0 + w_a (1 - a),$$

(1)

where $a$, $w_0$ and $w_a$ are the scale factor ($a = 1$ at present), the present value of $w(a)$ and the first derivative of $w(a)$ by $w_a = -dw/da$, respectively.

Even now it is not so much well known about the value of $w_0$ and $w_a$, we extend the parameter space as

$$w(a) = w_0 + w_a (1 - a) + \frac{1}{2} w_{a2}(1 - a)^2,$$

(2)

including the second derivative of $w(a)$ as $w_{a2} = d^2 w/da^2$. Although it may be hard to observe the parameter $w_{a2}$, it must be a good clue to understand the feature of the dark energy in the future.

We follow the single scalar field formalism of Steinhardt et al. \cite{10} and take the potential of two type as $V = M^{4+\alpha}/Q^\alpha$ and $V = M^4 \exp(\beta M/Q)$. These potentials are supported by Wang et al. \cite{11}.

In §2, the equation of state for scalar field and parameters to describe potentials are presented. The First derivatives of $w_Q$ for two type potentials are calculated in §3. In §4, the second derivatives are presented, where the detailed calculations are shown in the Appendix A. The results and discussion are presented in §5.
II. EQUATION OF STATE

A. Scalar field

For the dark energy, we consider the scalar field $Q(x, t)$, where the action for this field in the gravitational field is described by

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{16\pi G} R + \frac{1}{2} g^{\mu\nu} \partial_\mu Q \partial_\nu Q - V(Q) \right] + S_M,$$

(3)

where $S_M$ is the action of matter field and $G$ is the gravitational constant, occasionally putting $G = 1$. Neglecting the coordinate dependence, the equation for $Q(t)$ becomes as

$$\ddot{Q} + 3H\dot{Q} + V' = 0,$$

(4)

where $H$ is the Hubble parameter and $V'$ is the derivative of $V$ by $Q$. Being $\kappa = 8\pi/3$, $H$ satisfies the following equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \kappa(\rho_B + \rho_Q) = \kappa \rho_c,$$

(5)

where $\rho_B$, $\rho_Q$ and $\rho_c$ are the energy density of the background, scalar field and the critical density of the universe. The energy density and pressure for the scalar field are written as

$$\rho_Q = \frac{1}{2} \dot{Q}^2 + V,$$

(6)

and

$$p_Q = \frac{1}{2} \dot{Q}^2 - V,$$

(7)

respectively. Then the parameter $w_Q$ for the equation of state is described by

$$w_Q \equiv \frac{p_Q}{\rho_Q} = \frac{\frac{1}{2} \dot{Q}^2 - V}{\frac{1}{2} \dot{Q}^2 + V}.$$

(8)

B. Time variation of $w_Q$

It is assumed that the present value of $w_Q$ is slightly different from a negative unity by $\Delta(>0)$ as

$$w_Q = -1 + \Delta.$$

(9)
By using Eq. (8), \( \dot{Q}^2 \) is written by

\[
\dot{Q}^2 = \frac{2\Delta V}{2 - \Delta},
\]

(10)

which is also written by using the density parameter \( \Omega_Q = \rho_Q / \rho_c \) as

\[
\dot{Q}^2 = 2(\rho_c \Omega_Q - V).
\]

(11)

Combining Eq. (10) and Eq. (11), \( V \) is given by

\[
V = \rho_c \Omega_Q (1 - \frac{\Delta}{2}).
\]

(12)

From Eq. (11) and Eq. (12), \( \dot{Q} \) is given as

\[
\dot{Q} = \sqrt{\Delta (\rho_c \Omega_Q)}.
\]

(13)

If we determine the potential, the parameters to describe the evolution of the scalar field is the value of \( Q \) and \( \dot{Q} \) at some fixed time, because Eq. (4) is the second derivative equation. Then the evolution or backward variation could be estimated from this fixed point. In the following we take this fixed time is at present and estimate backward the accelerating behavior in the near past.

As \( \rho_c \) is given from the observation through Hubble parameter \( H \), \( \dot{Q} \) is determined by \( \Omega_Q \) and \( \Delta \) which also determine the value of \( V \). If we adopt the form and parameter of the potential, the value of \( V \) could be used to estimate the value of \( Q \). In reality, the evolution of \( H \) in Eq. (4) depends on the background densities which include radiation density. The effect of radiation density could be ignored in the near past \( (z \leq 10^3) \) and so is not considered in this work.

In the following, we investigate the power inverse potential \( V = M^4(M/Q)^\alpha; (\alpha > 0) \) and the exponential potential \( V = M^4 \exp(\beta M/Q); (\beta > 0) \), respectively.

1. \( V = M^4(M/Q)^\alpha \)

The parameters of the potential are \( M \) and \( \alpha \). If we take \( M = M_* \), \( V \) becomes as

\[
V = M_*^4 \left( \frac{M_*}{Q} \right)^\alpha.
\]

(14)
From Eq. (12) as \( M^4(M_*/Q)^\alpha = \rho_c \Omega_Q(1 - \Delta/2) \), \( Q \) is given as

\[
Q = \left( \frac{M^{4+\alpha}}{\rho_c \Omega_Q(1 - \Delta/2)} \right)^{1/\alpha}.
\]

(15)

If we take \( Q = Q_0 M_{pl} \) at present, \( M_{pl} \) being the planck mass, \( M_* \) becomes

\[
M_* = M_{pl} \left( Q_0^\alpha \rho_c \Omega_Q(1 - \Delta/2) \right)^{1/(4+\alpha)}.
\]

(16)

Then \( Q_0, \Omega_Q, \Delta \) and \( \alpha \) determine the parameter \( M_* \), which means that parameters to determine the accelerating behavior are \( Q_0, \Omega_Q, \Delta \) and \( \alpha \).

The difference of the observed value \( \rho_c \) and the value \( M_{pl}^4 \) is described by the observed value \( N \) as

\[
\rho_c = M_{pl}^4 \times 10^{-N},
\]

(17)

being \( N \approx 122 \), \( M_* \) becomes as

\[
M_* = M_{pl} \left( Q_0^\alpha \Omega_Q(1 - \Delta/2) \right)^{1/(4+\alpha)} \times 10^{-N/(4+\alpha)}.
\]

(18)

The problem is how to estimate \( Q_0 \) and \( \alpha \).

2. \( V = M^4 \exp(\beta M/Q) \)

If we put \( M_{pl} = \beta M \), \( V \) is written as

\[
V = \left( \frac{M_{pl}}{\beta} \right)^4 \exp \left( \frac{M_{pl}}{Q} \right).
\]

(19)

In essence, as \( \beta \) is combined with \( M \), the parameter of this potential is \( \beta \). From Eq.(12), \( \exp(M_{pl}/Q) \) is given by

\[
\exp \left( \frac{M_{pl}}{Q} \right) = \left( \frac{\beta}{M_{pl}} \right)^4 \rho_c \Omega_Q(1 - \Delta/2),
\]

(20)

Then \( Q \) is estimated as

\[
Q = \frac{M_{pl}}{\ln(\beta^4 \rho_c \Omega_Q(1 - \Delta/2)/M_{pl}^4)}.
\]

(21)
If we take $Q = Q_0 M_{\text{pl}}$ at present, $Q_0$ determines the parameter $\beta$ as,

$$\beta = \left( \frac{M_{\text{pl}}^4}{\rho_c \Omega_Q (1 - \Delta/2)} \right)^{1/4} \exp \left( Q_0^{-1/4} \right). \quad (22)$$

In this potential, the parameters to determine the accelerating behavior are $Q_0$, $\Omega_Q$ and $\Delta$. The problem how to estimate $Q_0$ is also left.

### III. FIRST DERIVATIVE OF $w_Q$

To investigate the variation of $w_Q$, we calculate $dw_Q/da$, using Eqs. (4), (6) and (7),

$$\frac{dw_Q}{da} = \frac{1}{\dot{a}} \frac{d}{dt} \left( \frac{p_Q}{\rho_Q} \right) = \frac{1}{\dot{a}} \frac{\dot{p}_Q \rho_Q - p_Q \dot{\rho}_Q}{\rho_Q^2}$$

$$= \frac{1}{a H \rho_Q^2} (\dot{Q}(\ddot{Q} - V') \rho_Q - p_Q \dot{Q}(\ddot{Q} + V'))$$

$$= \frac{\dot{Q}}{a H \rho_Q^2} ((-3 \dot{Q} - 2 V') \rho_Q - p_Q (-3 \dot{Q}))$$

$$= \frac{\dot{Q}}{a H \rho_Q^2} (3 \dot{Q}(p_Q - \rho_Q) - 2 V' \rho_Q)$$

$$= \frac{2V' \dot{Q}}{a H \rho_Q^2} \left( -3 \dot{Q} - \frac{V'}{V} \rho_Q \right). \quad (23)$$

In the limit $\Delta \to 0$ where $\dot{Q} = \sqrt{\Delta \rho_c \Omega_Q} \to 0$, $dw_Q/da$ becomes null. To investigate further, we must consider the potential form.

#### A. $V = M^{4+\alpha}/Q^\alpha$

Being $V'/V = -\alpha/Q$, Eq.(23) becomes

$$\frac{dw_Q}{da} = \frac{2V' \dot{Q}}{a H \rho_Q^2} \left( \frac{\alpha \rho_Q}{Q} - 3 \dot{Q} \right)$$

$$= \frac{2V' \dot{Q}}{a H Q \rho_Q} \left( \alpha - \frac{3 \dot{Q} Q}{\rho_Q} \right). \quad (24)$$

To investigate the signature of $dw_Q/da$, we must estimate the following term,

$$\frac{3 \dot{Q} Q}{\rho_Q} = \frac{3H \dot{Q} Q}{\rho_c \Omega_Q} \sqrt{\Delta (\rho_c \Omega_Q) Q_0 M_{\text{pl}}} = \sqrt{\frac{24\pi \Delta}{\Omega_Q}} \times Q_0. \quad (25)$$
To estimate $Q_0$, we consider the tracker approximation that $w_Q$ is almost constant as

$$-\sqrt{\frac{\Omega_Q}{24\pi(1+w_Q)}} \frac{V'}{V} = 1,$$

which is given by Eq. (9) of Steinhardt et al. [10]. If we adopt this approximation, $Q/M_{pl}$ becomes as

$$Q_\alpha = \sqrt{\frac{\Omega_Q}{24\pi \Delta}} \times \alpha.$$  \hspace{1cm} (27)

We approximate the present value $Q_0$ as $Q_0 = (1+\varepsilon) \times Q_\alpha$. Then from Eq. (25), $\sqrt{\frac{24\pi \Delta}{\Omega_Q}} \times Q_0$ becomes $(1+\varepsilon) \alpha$. If $\varepsilon > 0$, $dw_Q/da < 0$, vice versa.

From Eq. (25), $Q$ is derived as

$$Q = \frac{2\alpha \rho Q \dot{Q}}{H(a \rho Q^2 \frac{dw_Q}{da} + 6V \dot{Q}^2)},$$

then $Q_0$ is given by

$$Q_0 = \alpha \sqrt{\frac{3\Delta \Omega_Q}{2\pi}} \frac{(1-\Delta/2)}{(\frac{dw_Q}{da} + 6\Delta(1-\Delta/2))}. \hspace{1cm} (29)$$

If $dw_Q/da$ is observed, $Q_0/\alpha$ is determined by the observed values $\Omega_Q, \Delta, \text{ and } dw_Q/da$.

For the positive value of $Q_0$, $\frac{dw_Q}{da}$ must be greater than $-6\Delta(1-\Delta/2)$. Assuming $dw_Q/da \lesssim 1$ and $\Delta \lesssim 0.1$, it is estimated $0 \lesssim Q_0 \lesssim \infty$.

**B.** $V = (\frac{M_{pl}}{\rho})^4 \exp(\frac{M_{pl}}{Q})$

As $V'/V = -M_{pl}/Q^2$, Eq. (23) becomes

$$\frac{dw_Q}{da} = \frac{2V}{a H \rho Q} \left[ -3H \dot{Q}^2 - \frac{V'}{V} \dot{Q} \right] = \frac{2V}{a \rho Q} \left[ -3\dot{Q}^2 - \frac{V' \dot{Q}}{VH} \right] = \frac{2V}{a \rho Q} \left[ -3\rho Q \Delta \Omega_Q - \frac{M_{pl} \sqrt{\Delta \Omega_Q M_{pl}^4}}{Q^2 \sqrt{8\pi/3}} \right] \right.$$  

$$= \frac{2V}{a \rho Q} \left[ -3\Delta \Omega_Q + \frac{\sqrt{\Delta \Omega_Q}}{(Q/M_{pl})^2 \sqrt{8\pi/3}} \right] = \frac{2V}{a \rho Q (Q/M_{pl})^2} \sqrt{\frac{3\Delta \Omega_Q}{8\pi}} \left[ 1 - \sqrt{\Delta \Omega_Q (Q/M_{pl})^2 \sqrt{24\pi}} \right]. \hspace{1cm} (30)$$
If we adopt the tracker approximation in Eq. (26), \( Q \) becomes as

\[
Q_\beta = \left( \frac{\Omega Q}{24\pi \Delta} \right)^{1/4}.
\]

(31)

We approximate the present value \( Q_0 \) as \( Q_0 = (1 + \varepsilon) \times Q_\beta \). Then from Eq. (30), the above value within \([\ ]\) becomes \(-\varepsilon\). If \( \varepsilon > 0 \), \( dw_q/da < 0 \), vice versa.

From Eq. (30), \( Q \) is determined as

\[
Q = \left( \frac{2V \rho_Q \dot{Q}}{H(a\rho_Q^2 \frac{dw_Q}{da} + 6V\dot{Q}^2)} \right)^{1/2}.
\]

(32)

Then \( Q_0 \) is estimated by observable parameters \( \Omega Q, \Delta \), and \( dw_q/da \) as

\[
Q_0 = \left( \sqrt{\frac{3\Delta \Omega Q}{8\pi}} \frac{2(1 - \Delta/2)}{a\frac{dw_Q}{da} + 6\Delta(1 - \Delta/2)} \right)^{1/2},
\]

(33)

\( Q_0 \) does not depend on the potential parameter \( \beta \), which is determined by Eq. (22).

For the real value of \( Q_0 \), \( \frac{dw_Q}{da} \) must be greater than \(-6\Delta(1 - \Delta/2)\). Assuming \( \frac{dw_Q}{da} \lesssim 1 \) and \( \Delta \lesssim 0.1 \), it is estimated \( 0 \lesssim Q_0 \lesssim \infty \).

IV. THE SECOND DERIVATIVE OF \( w_Q \)

From Eq. (23), the second derivative of \( w_Q \) is given by

\[
\frac{d^2 w_Q}{da^2} = \frac{1}{\dot{a}^2 \rho_Q^4} [(\ddot{p}_Q \rho_Q - p_Q \dot{p}_Q) \dot{a} \rho_Q^2 + \dot{a} \rho_Q^2 (\ddot{p}_Q \rho_Q - p_Q \dot{p}_Q)] \nonumber
\]

\[
- (\ddot{p}_Q \rho_Q - p_Q \dot{p}_Q)(\dot{a} \rho_Q^2 + 2\dot{a} \rho_Q \dot{p}_Q)].
\]

(34)

The time derivatives of \( p_Q \) and \( \rho_Q \) are written as

\[
\dot{p}_Q = -3H \dot{Q}^2 - 2V' \dot{Q},
\]

\[
\ddot{p}_Q = \left(-3\frac{\ddot{a}}{\dot{a}} + 21H^2 - 2V'' \dot{Q}^2 + 12H \dot{Q} V' + 2(V')^2, \right.
\]

\[
\dot{\rho}_Q = -3H \dot{Q}^2, \quad \ddot{\rho}_Q = \left(-3\frac{\ddot{a}}{\dot{a}} + 21H^2 \dot{Q}^2 + 6H \dot{Q} V', \right.
\]

(35)

where \( \ddot{a}/\dot{a} = -4\pi G(\rho_c + p_Q). \) By using these equations, the detailed calculation of Eq. (34) is described in the Appendix A.
From Eq. (A21), $d^{2}w/da^{2}$ becomes
\[ \frac{d^{2}w_{Q}}{da^{2}} = \frac{3}{4\pi} \frac{\Omega_{Q}}{a^{2}} \left( 1 - \frac{\Delta}{2} \right) \times \left[ -\Delta M_{pl}^{2} \frac{V''}{V} + \sqrt{\frac{6\pi \Delta}{\Omega_{Q}}} \left( 1 - \Delta \right) \left( 6 + \Omega_{Q} \right) - \frac{1}{3} \right] \]
\times M_{pl} \left( \frac{V''}{V} \right) + \left( 1 - \frac{\Delta}{2} \right) M_{pl}^{2} \left( \frac{V''}{V} \right)^{2} + \frac{8\pi \Delta}{\Omega_{Q}} (7 - 6\Delta) \]
\[ (37) \]

From this equation, the value of $d^{2}w_{Q}/da^{2}$ could be estimated. In the limit $\Delta \to 0$, the signature of $d^{2}w_{Q}/da^{2}$ is positive under the condition $V'/V \neq 0$. In principle, if the first derivative is observed, $Q_{0}$ is estimated through Eq. (29) or (33). Then the second derivative is estimated by the observed parameters $\Delta$, $\Omega_{Q}$ and $Q_{0}$.

From this equation, we estimate $d^{2}w_{Q}/da^{2}$ for each potential in the following.

A. $V = M^{4+\alpha}/Q^{\alpha}$

As the following relations are derived
\[ \frac{V''}{V} = \frac{\alpha (\alpha + 1)}{Q^{2}}, \quad \frac{V'}{V} = -\frac{\alpha}{Q}, \quad \left( \frac{V'}{V} \right)^{2} = \frac{\alpha^{2}}{Q^{2}}, \]
we put them in Eq. (37) and get
\[ d^{2}w_{Q}/da^{2} = \frac{3}{4\pi} \frac{\Omega_{Q}}{a^{2}} \left( 1 - \frac{\Delta}{2} \right) \times \left[ -\Delta \frac{\alpha (\alpha + 1)}{\left( Q/M_{pl} \right)^{2}} - \sqrt{\frac{6\pi \Delta}{\Omega_{Q}}} \left( 1 - \Delta \right) \left( 6 + \Omega_{Q} \right) - \frac{1}{3} \right] \left( \frac{\alpha}{Q/M_{pl}} \right) \]
\[ + \left( 1 - \frac{\Delta}{2} \right) \left( \frac{\alpha}{Q/M_{pl}} \right)^{2} + \frac{8\pi \Delta}{\Omega_{Q}} (7 - 6\Delta) \]
\[ (38) \]

If $dw_{Q}/da$ is observed, $Q/\alpha$ is determined by Eq. (29). If $d^{2}w_{Q}/da^{2}$ is observed, one could estimate the value of $\alpha$ from the above equation.

If we take the tracker approximation as $Q_{0} \simeq (1 + \varepsilon) \times Q_{\alpha} \simeq (1 + \varepsilon) \times \sqrt{\Omega_{Q}/(24\pi \Delta)} \times \alpha$ of Eq. (27), the part within [ ] of the above equation becomes
\[ \frac{24\pi \Delta}{\Omega_{Q}} \left\{ -\frac{\Delta (1 + 1/\alpha)}{(1 + \varepsilon)^{2}} - \frac{1}{2(1 + \varepsilon)} \left( (1 - \Delta) (6 + \Omega_{Q}) - \frac{1}{3} \right) + \frac{1}{(1 + \varepsilon)^{2}} (1 - \frac{\Delta}{2}) + \frac{1}{3} (7 - 6\Delta) \right\} \]
\[ (39) \]

In the limit $\Delta \to 0$ where $Q_{\alpha} \to \infty$, $dw_{Q}/da^{2}$ becomes null in this approximation.
B. \( V = \left(\frac{M_{pl}}{\beta}\right)^4 \exp\left(\frac{M_{pl}}{Q}\right) \)

As the following relations are derived

\[
\frac{V''}{V} = \frac{2M_{pl}}{Q^3} + \frac{M_{pl}^2}{Q^4}, \quad \frac{V'}{V} = -\frac{M_{pl}}{Q^2}, \quad \left(\frac{V'}{V}\right)^2 = \frac{M_{pl}^2}{Q^4}.
\]

we put them in Eq. (37) and get

\[
\frac{d^2w_Q}{da^2} = \frac{3}{4\pi} \frac{\Omega_Q}{a^2} \left(1 - \frac{\Delta}{2}\right) \times \left[-\Delta M_{pl}^2 \left(2 \left(\frac{M_{pl}}{Q}\right)^3 + \left(\frac{M_{pl}}{Q}\right)^4\right) \right] \\
- \sqrt{\frac{6\pi\Delta}{\Omega_Q}} \left(1 - \Delta\right) \left(6 + \Omega_Q\right) - \frac{1}{3} \left(\frac{M_{pl}}{Q}\right)^2 \\
+ (1 - \Delta) \left(\frac{M_{pl}}{Q}\right)^4 + \frac{8\pi\Delta}{\Omega_Q} \left(7 - 6\Delta\right).
\]

If we take the tracker approximation as \( Q_0 \simeq (1 + \varepsilon) \times Q_\beta = (1 + \varepsilon) \times (\Omega_Q/(24\pi\Delta))^{1/4} \) in Eq. (31), the part within [ ] of the above equation becomes

\[
\left[ \frac{24\pi\Delta}{\Omega_Q} \left\{ -\Delta \left(\frac{2}{(1 + \varepsilon)^3} \left(\frac{24\pi\Delta}{\Omega_Q}\right)^{-1/4} + \frac{1}{(1 + \varepsilon)^4}\right) + \frac{1}{2(1 + \varepsilon)^2} \left(1 - \Delta\right) \left(6 + \Omega_Q\right) - \frac{1}{3} \right\} \\
+ \frac{1}{(1 + \varepsilon)^3} (1 - \Delta) \left(\frac{M_{pl}}{Q}\right)^4 + \frac{1}{3} \left(7 - 6\Delta\right) \right].
\]

In the limit \( \Delta \to 0 \) where \( Q_\beta \to \infty \), \( dw_Q^2/da^2 \) becomes null.

V. CONCLUSIONS AND DISCUSSION

It is important to know the variation of the equation of state \( w \) of the back ground field for the investigation of the expansion of the universe. It is known that \( \ddot{a} \) is described by the following equation

\[
\ddot{a} = -\frac{4\pi G}{3} (1 + 3w)a\rho.
\]

At present, it has been proceed the look back observation of large scale structure of the universe to estimate \( w_Q \) at the age \( (1 + z) \). For the moment, it has been persued the value of \( w(a = a_0) \) and \( dw/da \)

\[
w(a) = w(a = a_0) + \frac{dw}{da} da.
\]

(42)
It could be expected the observation of the second derivative of \( w \) in the future

\[
w(a) = w(a_0 = 1) + \frac{dw}{da} da + \frac{1}{2} \frac{d^2 w}{da^2} (da)^2 + \cdots ,
\]  

so we estimate the second derivative of \( w \) with \( a \) for typical two potentials in this work.

The first derivative \( dw_Q/da \) and the second derivative \( d^2 w_Q/da^2 \) for the power inverse and exponential potentials are calculated. The first derivative is estimated by the observed two parameters \( \Delta = w_Q + 1 \) and \( \Omega_Q \), with the assuming parameters \( Q_0 \). In the limit \( \Delta \to 0 \), the first derivative is null and, under the tracker approximation, the second derivative also becomes null. For the inverse power potential \( V = M^{4+\alpha}/Q^\alpha \), the observed first and second derivatives are used to determine the potential parameter \( M_* \) and \( \alpha \). For the exponential potential \( V = M^4 \exp(\beta M/Q) \), the second derivative is estimated by the observed parameters \( \Delta, \Omega_Q \) and \( dw_Q/da \). If the parameters of the potentials are determined, the time variation of the dark energy such as \( d^3 w_Q/da^3, d^4 w_Q/da^4 \) and so on could be estimated.

If we assume matter dominant approximation, we could estimate \( \alpha \) and \( M_* \) from the attractor solution \[12\] which is outlined in the Appendix B.

If \( \Delta < 0 \), we must consider utterly different models such as phantoms or k-essence \[13], [14].

**Appendix A: Derivation for Eq. (34)**

By using Eqs. (35) and (36), the terms within Eq. (34) becomes

\[
\dot{p}_Q \rho_Q - p_Q \dot{\rho}_Q = [-2V'' \dot{Q}^2 + 6H \dot{Q} V' + 2(V')^2 \frac{1}{2} \dot{Q}^2 \\
+ [(-6 \ddot{a}/a + 42H^2 - 2V'') \dot{Q}^2 + 18H \dot{Q} V' + 2(V')^2] V] \tag{A1}
\]

\[
\dot{p}_Q \rho_Q - p_Q \dot{\rho}_Q = \dot{Q}[3H \dot{Q}(p_Q - \rho_Q) - 2V' \rho_Q]. \tag{A2}
\]

Next, we try to calculate the second term within \[ \] of Eq. (34). It is given \[12\]

\[
\frac{\dot{a}}{a} = 4\pi G \left( - \frac{\rho_c}{3} - p_Q \right), \tag{A3}
\]

where we neglect the radiation pressure. By using the relation of \( p_Q = \rho_Q - 2V = \rho_c \Omega_Q - 2V \) and Eq. (12), we get

\[
p_Q = \rho_c \Omega_Q(-1 + \Delta). \tag{A4}
\]
Putting this relation into Eq. (A3), it becomes
\[ \frac{\ddot{a}}{a} = 4\pi G \rho_c \left[ \Omega_Q (1 - \Delta) - \frac{1}{3} \Omega_Q \right]. \] (A5)

Being \( \dot{Q}^2 = \rho_Q \Delta \), the first derivative of \( p_Q \) and \( \rho_Q \) become
\[ \dot{p}_Q = -3H \rho_Q \Delta - 2V' \dot{Q}, \quad \dot{\rho}_Q = -3H \rho_Q \Delta. \] (A6)

Using \( H^2 = (8\pi G/3)\rho_c \), the term within the right side \( (\ ) \) of the second term in the \( \left[ \right] \) of Eq. (34) becomes
\[ 
\dot{p}_Q \rho_Q - p_Q \dot{\rho}_Q = -2V \rho_Q \sqrt{\rho_c} \left[ \sqrt{24\pi G \Delta} + \frac{V'}{V} \sqrt{\Omega_Q \Delta} \right]. 
\] (A8)

From Eqs. (A7) and (A8), the second term in the \( \left[ \right] \) of Eq. (34) becomes
\[ 
(\dot{p}_Q \rho_Q - p_Q \dot{\rho}_Q)(\dot{a} \rho_Q^2 + 2\dot{a} \rho_Q \dot{p}_Q) = - \frac{8\pi G \rho_c \rho_Q^{1/2}}{\Omega_Q} \left( \frac{1}{2} \right) \times \left\{ \left[ \sqrt{24\pi G \Delta} + \frac{V'}{V} \sqrt{\Omega_Q \Delta} \right] \left[ \Omega_Q (1 - \Delta) - \frac{1}{3} \Omega_Q - 4\Delta \right] \right\}. 
\] (A9)

Next, we try to calculate the first term within \( \left[ \right] \) of Eq. (34). Eq. (A11) could be changed to
\[ 
\dot{p}_Q \rho_Q - p \dot{\rho}_Q = \left[ -2V'' \frac{\dot{Q}^2}{V} + \frac{6H \dot{Q} V'}{V} + 2(\frac{V'}{V})^2 \right] \frac{1}{2} \dot{Q}^2 V^2 + \left[ \left( -\frac{\dot{a}}{a} \frac{1}{V} + 42 \frac{H^2}{V} - 2 \frac{V''}{V} \right) \frac{\dot{Q}^2}{V} + 18 \frac{H \dot{Q} V'}{V} + 2(\frac{V'}{V})^2 \right] V^3. 
\] (A10)

Here we calculate elements separately in the above equation as
\[ \frac{\dot{Q}^2}{V} = \frac{\rho_Q \Delta}{\rho_Q (1 - \Delta/2)} = \frac{\Delta}{1 - \Delta/2}, \] (A11)
\[ \frac{H \dot{Q}}{V} = \frac{\sqrt{(8\pi G/3)\rho_c \sqrt{\rho_c \Omega_Q \Delta}}}{\rho_c \Omega_Q (1 - \Delta/2)} = \frac{\sqrt{(8\pi G/3)\Delta}}{\sqrt{\Omega_Q (1 - \Delta/2)}}, \] (A12)
\[ \frac{1}{2} \dot{Q}^2 V^2 = \frac{1}{2} \rho_Q \Delta \left[ \rho (1 - \Delta) \right]^2 = \frac{1}{2} \rho_Q^3 \Delta (1 - \frac{\Delta}{2})^2, \] (A13)
\[ \frac{\dot{a}}{a} \frac{1}{V} = \frac{4\pi G \rho_c \left[ \Omega_Q (1 - \Delta) - \frac{1}{3} \right]}{\rho_c \Omega_Q (1 - \Delta/2)} = \frac{4\pi G \left[ \Omega_Q (1 - \Delta) - \frac{1}{3} \right]}{\Omega_Q (1 - \Delta/2)}, \] (A14)
\[ \frac{H^2}{V} = \frac{(8\pi G/3)\rho_c}{\rho_c \Omega_Q (1 - \Delta/2)} = \frac{8\pi G}{3} \frac{1}{\Omega_Q (1 - \Delta/2)} \] (A15)
Then Eq. (A10) could be described as

\[ \ddot{p}_Q \rho_Q - p_Q \dot{p}_Q = \frac{1}{2} \rho_Q^3 \Delta^2 \left( 1 - \frac{2}{\Delta} \right) \left[ \left( -2 - \frac{4}{\Delta} \left( 1 - \frac{2}{\Delta} \right) \right) \frac{V''}{V} \right. \\
+ \left( 6 \sqrt{\frac{8\pi G/3}{\rho_c}} + \frac{36 \sqrt{8\pi G/3(1 - \Delta/2)}}{\Delta \sqrt{\Delta \Omega}} \right) V' + \left( \frac{2}{\Delta} \left( 1 - \frac{2}{\Delta} \right) + \frac{4}{\Delta^2} \left( 1 - \frac{2}{\Delta} \right)^2 \right) \left( \frac{V'}{V} \right)^2 \\
+ \left. \frac{2}{\Delta \Omega} \left( -24\pi G \left( \Omega_Q (1 - \Delta) - \frac{1}{3} \right) + 112\pi G \right) \right] \] (A16)

Being \( \dot{a} \rho_Q^2 = aH \rho_Q^2 = a \sqrt{(8\pi/3) \rho_c \rho_Q^2} \), the first term within [ ] of Eq. (34) becomes

\[ \langle \ddot{p}_Q \rho_Q - p_Q \dot{p}_Q \rangle \dot{a} \rho_Q^2 = \frac{1}{2} a \rho_Q^5 \sqrt{(8\pi G/3) \rho_c} \Delta^2 \left( 1 - \frac{2}{\Delta} \right) \left[ \left( -4 \frac{4}{\Delta} \right) \frac{V''}{V} \right. \\
+ \left. \left( 6 \sqrt{\frac{8\pi G/3}{\rho_c}} + \frac{36 \sqrt{8\pi G/3(1 - \Delta/2)}}{\rho_Q^3 \Delta} \right) V' + \left( \frac{2}{\Delta} \left( 1 - \frac{2}{\Delta} \right) + \frac{4}{\Delta^2} \left( 1 - \frac{2}{\Delta} \right)^2 \right) \left( \frac{V'}{V} \right)^2 \\
+ \left. \frac{2}{\Delta \Omega} \left( -24\pi G \left( \Omega_Q (1 - \Delta) - \frac{1}{3} \right) + 112\pi G \right) \right] \] (A17)

Then Eq. (34) could be written as

\[ \frac{d^2 w_Q}{da^2} = \frac{1}{a^3 \rho_Q^2} \times \frac{1}{2} \sqrt{\frac{8\pi G}{3}} \rho_c^{1/2} \rho_Q^5 a \Delta^2 \left( 1 - \frac{2}{\Delta} \right) \left[ \left[ \right. \right] \text{in Eq. (A17)} \\
+ \left. \frac{2\sqrt{24\pi G}}{\Omega_Q \Delta^2} \right] \text{[ ] in Eq. (A17).} \] (A18)

The term within [ ] is calculated as

\[ - \frac{4}{\Delta} \frac{V''}{V} + \frac{6}{\Delta^2} \sqrt{\frac{8\pi G}{3\Omega_Q}} \Delta \left( 1 - \Delta \right) \left( 6 + \Omega_Q \right) \frac{1}{3} \Omega_Q \left( \frac{V'}{V} \right) \\
+ \left( \frac{2}{\Delta} \right)^2 \left( 1 - \frac{2}{\Delta} \right) \left( \frac{V'}{V} \right)^2 + \frac{16\pi G}{\Delta \Omega_Q} (14 - 12\Delta). \] (A19)

The outside factor of [ ] is

\[ \frac{3}{16\pi G} \frac{\Omega_Q}{a^2} \left( 1 - \frac{2}{\Delta} \right) \Delta^2. \] (A20)

Using Eqs. (A19) and (A20), \( d^2 w/da^2 \) becomes

\[ \frac{d^2 w_Q}{da^2} = \frac{3\Omega_Q}{16\pi a^2} \left( 1 - \frac{2}{\Delta} \right) \left[ -4\Delta M_{pl} \frac{V''}{V} + 4 \left( \frac{6\pi \Delta}{\Omega_Q} \right)^{0.5} \left( (1 - \Delta) (6 + \Omega_Q) - \frac{1}{3} \right) M_{pl} \left( \frac{V'}{V} \right) \right. \\
+ \left. 4(1 - \frac{2}{\Delta}) M_{pl}^2 \left( \frac{V'}{V} \right)^2 + \frac{32\pi \Delta}{\Omega_Q} (7 - 6\Delta) \right]. \] (A21)
Appendix B: Matter Dominant and Attractor Solution Approximation

For the era $\rho_M \geq \rho_Q$, we could use Eq. (1) as matter dominant approximation where $a \propto t^{2/3}$. For the inverse power potential, it becomes

$$\ddot{Q} + \frac{2}{t} \dot{Q} + M^{4+\alpha} Q^{-\alpha} = 0,$$

(B1)

which has the attractor solution as

$$Q = \left(\frac{\alpha(2 + \alpha)^2 M^{4+\alpha} t^2}{2(4 + \alpha)}\right)^{1/(2+\alpha)}.$$

(B2)

If this solution could be used, parameter $\alpha$ and $M$ would be derived by $\Delta$ and the time $t_c$ at which $\rho_M = \rho_Q$.

For this solution, it is derived as

$$\frac{1}{2} \dot{Q}^2 / V = \frac{\alpha}{4 + \alpha},$$

(B3)

then

$$w_Q = (\frac{\alpha}{4 + \alpha} - 1)/(\frac{\alpha}{4 + \alpha} + 1) = -1 + \frac{\alpha}{2(2 + \alpha)}.$$  

(B4)

As $\Delta = \frac{\alpha}{2(2+\alpha)}$, $\alpha$ could be estimated by $\Delta$ as

$$\alpha = 4 \frac{\Delta}{1 - 2\Delta}.$$  

(B5)

If $a \simeq t^{2/3}$ and $\Delta \leq 0.1$ could be approximated until present $t_0$, it could be approximated as $t_c = t_0/(\Omega_Q/(1 - \Omega_Q))^{1/2}$. The parameter $M$ is derived by the relation $\rho_M = \rho_Q$ at $t_c$ as

$$\frac{1}{2} \cdot \frac{1}{6\pi G t_c^2} = M^{4+\alpha} \left(\frac{\alpha(2 + \alpha)^2 M^{4+\alpha} t_c^2}{2(4 + \alpha)}\right)^{-\alpha/(2+\alpha)}$$

(B6)

$$= \left(\frac{2^{1+\alpha}(2 + \alpha)^{2-\alpha}}{\alpha^\alpha(4 + \alpha)^2}\right)^{1/(2+\alpha)} M^{2(4+\alpha)/(2+\alpha)} t_c^{-2\alpha/(2+\alpha)},$$

(B7)

where $\rho_M = 1/(6\pi G t^2)$ and $\rho_Q = \dot{Q}^2/2 + V$ are used. So $M_s$ is determined by $\alpha$ and $t_c$ as

$$M_s = \left((3 \cdot 2^3 \pi)^{-2+\alpha}) \alpha^{\alpha}(2 + \alpha)^{-2+\alpha}(4 + \alpha)^2\right)^{1/(2(4+\alpha))} \times t_c^{-2/(4+\alpha)}.$$  

(B8)

Then the observed value $t_0, \Omega_Q$ and $\Delta$ could determine the potential parameter $\alpha$ and $M$ under the matter dominant and attractor solution approximation.

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