Correction to the paper
“Some remarks on Davie’s uniqueness theorem”

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Abstract

The property 4 in Proposition 2.3 from the paper “Some remarks on Davie’s uniqueness theorem” is replaced with a weaker assertion which is sufficient for the proof of the main results. Technical details and improvements are given.

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1. Introduction

We consider the stochastic differential equation

\[ X_t = x + W_t + \int_0^t b(s, X_s) \, ds. \]

In the paper \cite{1} the following theorem was proved:

**Theorem 1.1.** Let \( b : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d \) be a Borel measurable bounded mapping. Then for almost all Brownian paths the equation \( \mathbb{I} \) has exactly one solution.

In the work \cite{11} an alternative approach was proposed. However as it was pointed out in \cite{10} (see Remark 5.3, p. 24) the uniform Hölder continuity (the property 4 from Proposition 2.3 in \cite{11}) doesn’t immediately follow from Kolmogorov continuity theorem and the moments estimates established in \cite{11}. Below we present a simple modification of Kolmogorov continuity theorem and adjust the proofs of the main results from \cite{11} accordingly. Some other observations regarding the regularity of the flow, in particular, a simple treatment of the case of a bounded drift, are not included into this short note and will be discussed in a separate paper.

2. Auxiliary results

**Proposition 2.3.** Let

\[ b \in L^q([0, T], L^p(\mathbb{R}^d)), \quad \frac{d}{p} + \frac{2}{q} < 1. \]

Then, there exists a Hölder flow of solutions to the equation \( \mathbb{I} \). More precisely, for any filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)\) and a Brownian motion \( W \), there exists a mapping \((s, t, x, \omega) \mapsto \varphi_{s,t}(x)(\omega)\) with values in \( \mathbb{R}^d \), defined for \( 0 \leq s \leq t \leq T, \ x \in \mathbb{R}^d, \ \omega \in \Omega \), such that for each \( s \in [0, T] \) the following conditions hold:

1. for any \( x \in \mathbb{R}^d \) the process \( X_{s,t}^x = \varphi_{s,t}(x) \) is a continuous \( \mathcal{F}_{s,t} \) adapted solution to the equation \( \mathbb{I} \)

2. \( P \)-almost surely the mapping \( x \mapsto \varphi_{s,t}(x) \) is a homeomorphism,

3. \( P \)-almost surely for all \( x \in \mathbb{R}^d \) and \( 0 \leq s \leq u \leq t \leq 1 \)

\[ \varphi_{s,t}(x) = \varphi_{u,t}(\varphi_{s,u}(x)), \]

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4. For any \( \alpha \in (0, 1) \), \( \eta > 0 \), \( N > 0 \) and a given increasing sequence \( S \) of finite sets \( \{S_n\}_{n=0}^{\infty} \) with \( |S_n| \leq 2^{\eta n} \) there exists a set \( \Omega' \) of probability 1 such that for any \( s \in S_n \), \( x, y \in \mathbb{R}^d \) with \( |x|, |y| < N, |x - y| \leq 2^{-n} \) and each \( t \in [s, T] \)

\[
|\varphi_{s,t}(x) - \varphi_{s,t}(y)| \leq C(\alpha, T, N, S, \omega)|x - y|^{\alpha}.
\]

Following the proof given in [11] we consider the process

\[
Y_t := \psi_t(t, X_t) = X_t + U(t, X_t)
\]

which is the unique solution of the transformed equation

\[
dY_t = \tilde{b}(t, Y_t) \, dt + \tilde{\sigma}(t, Y_t) \, dW_t,
\]

for details see [11]. In the work [11] the following bound was established:

\[
\mathbb{E} \sup_{t \in [0, T]} |Y_t^x - Y_t^y|^{\alpha} \leq C(a, T)(|x - y|^{\alpha} + |x - y|^{\alpha-1}), \tag{2}
\]

It is easy to see that the same arguments provide the estimate

\[
\sup_{s \in [0, T]} \mathbb{E} \sup_{t \in [s, T]} |Y_{s,t}^x - Y_{s,t}^y|^{\alpha} \leq C(a, T)(|x - y|^{\alpha} + |x - y|^{\alpha-1})
\]

Since \( \psi_t, \psi_t^{-1} \) are Lipschitz continuous uniformly in time an analogous bound holds for \( X_{s,t}^x \)

\[
\sup_{s \in [0, T]} \mathbb{E} \sup_{t \in [s, T]} |X_{s,t}^x - X_{s,t}^y|^{\alpha} \leq C(a, T)(|x - y|^{\alpha} + |x - y|^{\alpha-1})
\]

We can assume (see [2]) that for each \( s \) the mapping \( X_{s,t}^x \) is jointly continuous with respect to \( t, x \). To complete the proof we will need the following lemma:

**Lemma 2.1.** Let \( X(s, x) \) be a continuous with respect to \( x \) process with values in a complete metric space \( (M, \varrho_{M}) \) on \( S \times [0, 1]^d \). Assume that for some \( a, b > 0 \)

\[
\sup_{s \in S} \mathbb{E} \varrho_{M}(X_s(u), X_s(v))^{\alpha} \leq |u - v|^{d+b}, \quad u, v \in [0, 1]^d
\]

For any \( \alpha \in (0, b/a), \eta \in (0, b - \alpha a) \) and any increasing sequence \( S \) of finite subsets \( \{S_n\}_{n=0}^{\infty} \) with \( |S_n| \leq 2^{\eta n} \) there exists a set \( \Omega' \) of probability 1 such that

\[
\varrho_{M}(X_s(u), X_s(v)) \leq C(\alpha, \eta, S, \omega)|u - v|^{\alpha} \quad s \in S_n, u, v \in [0, 1]^d, |u - v| \leq 2^{-n}, \omega \in \Omega',
\]

The proof is a minor modification of the standard proof of Kolmogorov continuity theorem, for details see [9].

**Proof.** Let \( \alpha \in (0, b/a) \). Define \( D_n \) as

\[
D_n := \{(k_1, \ldots, k_d)2^{-n}; k_1, \ldots, k_d \in \{1, \ldots, 2^n\}\}
\]

Let

\[
Y(s, n) := \max \left\{ \varrho_{M}(X_s(u), X_s(v)); u, v \in D_n, |u - v| = 2^{-n} \right\}
\]

Then

\[
\mathbb{E}(2^{an}Y(s, n))^{\alpha} \leq C2^{an}2^{dn}(2^{-n})^{d+b} \leq C2^{(a\alpha-b)n}
\]

Now one readily sees that

\[
\mathbb{E} \sum_{n=1}^{\infty} \sum_{s \in S_n} (2^{an}Y(s, n))^{\alpha} < \infty
\]
Consequently, there exists a set \( \Omega' \) of full measure such that
\[
\sum_{n=1}^{\infty} \sum_{s \in S_n} (2^{an} Y(s, n))^a < C(\omega) < \infty, \omega \in \Omega'.
\]
in particular
\[
Y(s, n)(\omega) \leq C'(\omega) 2^{-an}, s \in S_n, \omega \in \Omega'.
\]
Using the fact that the sequence \( S \) is increasing we obtain the bound
\[
Y(s, m)(\omega) \leq C'(\omega) 2^{-am}, s \in S_n, m \geq n, \omega \in \Omega'.
\]
Now let \( s \) be a fixed point in \( S_n \). Applying the standard arguments one can see that for each \( m \geq n \) and any \( u, v \in D_m \) such that \( |u - v| \leq 2^{-n} \) the following inequality holds:
\[
\varrho(M(X_s(u), X_s(v)) \leq C'(\omega) 2^{-a}, s \in S_n, m \geq n, \omega \in \Omega'.
\]
Now it is easy to complete the proof. \( \square \)

Now let us come back to the proof of the property 4. Define a random mapping \( J \) from \([0,T] \times [-N,N]^d\) to the Banach space \( \mathcal{C}([0,T], \mathbb{R}^d) \) equipped with the standard sup-norm as follows:
\[
J(\omega, s, x)(t) := X_{s,t}(\omega).
\]
The joint continuity of \( X_{s,t} \) with respect to \( t, x \) immediately implies the mapping \( J \) is continuous. Next, the estimate
\[
\sup_{s \in [0,T]} \mathbb{E} \sup_{t \in [s,T]} |X_s^x - X_s^y|^a \leq C(a, T)(|x - y|^a + |x - y|^{a-1})
\]
can be written as
\[
\sup_{s \in [0,T]} \mathbb{E}||J(s, x) - J(s, y)||^a \leq C(a, T)(|x - y|^a + |x - y|^{a-1})
\]
For any \( \alpha \in (0, 1) \) and \( \eta > 0 \) one can find sufficiently large \( a > 0 \) such that
\[
\alpha < a - 1 - d, \quad \eta < a - 1 - d - \alpha a
\]
so now it is easy to complete the proof applying Lemma \( \ref{lemma2.1} \).

3. Main results

In this section we adjust the proofs of the main results stated in the paper \( \cite{11} \) using the corrected version of the property 4 from Proposition \( \ref{prop2.3} \).

**Theorem 3.1.** Assume that the coefficient \( b \) satisfies the following conditions:
1. there exists \( M_1 \in L^{q_1}([0,T], \mathbb{R}) \) such that
   \[
   |b(t, x)| \leq M_1(t), \quad t \in [0,T], \quad x \in \mathbb{R}^d
   \]
2. there exists \( M_2 \in L^{q_2}([0,T], \mathbb{R}) \) and \( \beta > 0 \) such that
   \[
   |b(t, x) - b(t, y)| \leq M_2(t)|x - y|^\beta, \quad t \in [0,T], \quad x, y \in \mathbb{R}^d
   \]
3. one has
   \[
   q_1 \geq q_2 > 2, \quad \beta > 0, \quad \frac{\beta}{p_1} + \frac{1}{p_2} > 1, \quad \text{where} \quad \frac{1}{p_1} + \frac{1}{q_1} = 1, \quad \frac{1}{p_2} + \frac{1}{q_2} = 1.
   \]
Then there exist a set \( \Omega' \) with \( P(\Omega') = 1 \) such that for each \( \omega \in \Omega' \) the equation \( \ref{equation1} \) has exactly one solution.
Proof. Let $Y_t$ be a solution to the equation for a fixed Brownian trajectory $W$. Then the following estimate holds:

$$
\max_{t \in [0,T]} |Y_t| \leq |x| + \max_{t \in [0,T]} |W_t| + T^{1/p_1} \| M_1 \|_{L^{\infty}[0,T]} =: M(x,W),
$$

so without loss of generality we can assume that $b(t,x) = b(t,x)I_{|x| < N}$ for some $N > 0$. Then Proposition 2.3 (it is clear that one can take $q_1$ for $q$ and any sufficiently large positive number for $p$) yields that $P$-almost surely the equation has a Hölder-continuous flow of solutions which will be denoted by $X(s,t,x,W), s \leq t, x \in \mathbb{R}^d$.

$$
1 + \gamma := \frac{\beta}{p_1} + \frac{1}{p_2}, \gamma > 0.
$$

Let us pick $\alpha \in (0,1)$ such that

$$
\frac{\alpha \beta}{p_1} + \frac{\alpha}{p_2} = 1 + \delta, \delta > 0.
$$

Let us estimate $|Y_r - X(u,r,Y_u,W)|$. It is clear that we have the following trivial bound:

$$
|Y_r - X(u,r,Y_u,W)| \leq \int_u^r |b(s,Y_s) - b(s,X(u,s,Y_u,W))| \, ds \leq 2 \int_u^r M_1(s) \, ds \leq 2 \| M_1 \|_{L^{\infty}[0,T]} |r - u|^{\frac{1}{p_1}}
$$

The previous estimate can be improved if we take into account the Hölder-continuity of the coefficient $b$:

$$
|Y_r - X(u,r,Y_u,W)| \leq \int_u^r |b(s,Y_s) - b(s,X(u,s,Y_u,W))| \, ds \leq \int_u^r M_2(s) |Y_s - X(u,s,Y_u,W)|^\beta \, ds \leq K' \int_u^r M_2(s) |r - u|^{\frac{2}{p_1}} \, ds \leq K' \| M_2 \|_{L^{\infty}[0,T]} |r - u|^{\frac{2}{p_1} + \frac{1}{p_2}} = K' \| M_2 \|_{L^{\infty}[0,T]} |r - u|^{1+\gamma}.
$$

Define sets $\{S_n\}$ as

$$
S_n := \left\{ k/2^n; k \in \{0,1,\ldots,2^n - 1\} \right\}, \ |S_n| = 2^n
$$

Using the property 4 from Proposition 2.3 with $\eta = 1$ and $S = \{S_n\}_{n=1}^{\infty}$ we obtain $\Omega'$ with $P(\Omega') = 1$ such that the following estimate holds:

$$
|X(s,t,x,W) - X(s,t,y,W)| \leq C(\alpha, T, N, \omega) |x - y|^\alpha, \ |x - y| \leq \frac{1}{2^n}, s \in S_n.
$$

Now let us prove that for each trajectory $W \in \Omega'$ the equation has exactly one solution. Let us choose a sufficiently large number $K$. Let $t \in S_{k'}$, where $k' \geq K$. Define an auxiliary function $f$ by the formula

$$
f(s) = X(s,t,Y_s,W) - X(0,t,x,W), \ s \in [0,t].
$$

Let $k \geq k'$ and $u, r$ be of the form $\frac{k}{2^n}$, $\frac{k'}{2^n}$ respectively, in particular $u, r \in S_k$. Recall that

$$
|Y_r - X(u,r,Y_u,W)| \leq C|r - u|^{1+\gamma} \leq C2^{-k\gamma}2^{-k}
$$
Since $K$ is supposed to be sufficiently large we may assume that $C2^{-K\gamma} \leq 1$. Consequently, 

$$|Y_r - X(u, r, Y_u, W)| \leq 2^{-k}$$

Then

$$|f(r) - f(u)| = |X(r, t, Y_r, W) - X(u, t, Y_u, W)| =$$

$$= |X(r, t, Y_r, W) - X(r, t, X(u, r, Y_u, W), W)| \leq$$

$$\leq C(\alpha, S, T, M(x, W), \omega)|Y_r - X(u, r, Y_u, W)|^\alpha.$$ 

Finally,

$$|f(r) - f(u)| \leq C(\alpha, S, T, M(x, W), \omega)|r - u|^{1+\delta}.$$ 

Due to the arbitrariness of $k$ we conclude

$$f(t) = X(x, 0, t, W) - Y_t = 0.$$ 

Since $t$ was an arbitrary dyadic number in $[0, 1]$ with a sufficiently large denominator, the continuity of $Y_t$ and $X(x, 0, t, W)$ implies the equality $Y_t = X(x, 0, t, W)$ for each $t \in [0, 1]$. The proof is complete. 

Now we show how to prove uniqueness in the case of a Borel measurable drift following [11]. Similarly to the proof of Theorem 3.1, it is readily seen that without loss of generality we can assume that $b(t, x) = b(t, x)I_{\{|x|<N\}}$ and $\|b\|_{\infty} \leq 1$.

Below we will need the following set of functions:

$$Lip_N([r, u], \mathbb{R}^d) := \{h \in C([r, u], \mathbb{R}^d) \mid |h(t) - h(s)| \leq |t - s|, s, t \in [r, u], \max_{s \in [r, u]} |h(s)| \leq N\}$$

with the uniform metric $\rho(h_1, h_2) = \|h_1 - h_2\|_{\infty}$.

The following result was proved in [11] and the corresponding arguments remain unchanged.

**Lemma 3.6.** There exist constants $C, \zeta > 0$, independent of $l = u - r$, and a set $\Omega'$ such that

$$P(\Omega \setminus \Omega') \leq C \exp(-l^{-\zeta})$$

and for any $h_1, h_2 \in Lip_N([r, u], \mathbb{R}^d)$ with $\|h_1 - h_2\|_{\infty} \leq 4l$, $W \in \Omega'$ the following inequality holds:

$$|\varphi(h_1, W) - \varphi(h_2, W)| \leq C l^{\frac{1}{\delta}}.$$ 

We can now proceed to the proof of Theorem 1.1.

**Proof.** Let us fix a positive number $N$. Let $C, \zeta$ be constants found in Lemma 3.6. For each $k$ we split the interval $[0, 1]$ into $M = 2^k$ closed subintervals

$$[0, \frac{1}{M}], \ldots, \left[\frac{M - 1}{M}, M\right].$$

Applying Lemma 3.6 to each interval $[\frac{i}{M}, \frac{i+1}{M}]$ we can find the corresponding sets $\Omega_{k,i}$. Let

$$\Omega_k := \bigcap_{i=0}^{M-1} \Omega_{k,i}.$$
With the help of the Borel–Cantelli lemma it is easy to show that the set
\[
\Omega' := \liminf_{k \to \infty} \Omega_k = \bigcup_{K=1}^{\infty} \bigcap_{k=K}^{\infty} \Omega_k
\]
has probability 1.
Define \( S_n \) as
\[
S_n := \left\{ k/2^n; k \in \{0, 1, \ldots, 2^n - 1\} \right\}, \mid S_n \mid = 2^n
\]
Using the property 4 from Proposition 2.3 with \( \eta = 1 \) and \( S = \{S_n\}_{n=1}^{\infty} \) we may assume (removing, if necessary, a set of zero probability) that on the set \( \Omega' \) the following estimate holds:
\[
|X(s, t, x, W) - X(s, t, y, W)| \leq C(\alpha, T, N, \omega)|x - y|^\alpha, \quad |x - y| \leq \frac{1}{2^n}, s \in S_n
\]
Let us show that for each \( W \in \Omega' \) such that
\[
|x| + \max_{t \in [0,1]} |W_t| + 1 \leq N,
\]
the equation 1 has a unique solution. Indeed, let \( Y_t \) be a solution to the equation 1. It is not difficult to see that \( |Y_t| \leq N \) for each \( t \in [0,1] \). Due to our choice of \( \Omega' \) there exists \( K = K(\omega) \) such that for all \( k \geq K \) the Brownian trajectory \( W \) belongs to \( \Omega_k \). Let
\[
M' = 2^{k'}, \quad r = \frac{i}{M'}, \quad \text{where} \quad k' \geq K.
\]
Let us define an auxiliary function \( f \) on the interval \( [0, r] \) by the following formula:
\[
f(t) := X(x, 0, r, W) - X(Y_t, t, r, W).
\]
We observe that for any \( s \leq t, \) by the definition of a flow we have
\[
f(t) - f(s) = -X(Y_t, t, r, W) + X(Y_s, s, r, W) =
= -X(Y_t, t, r, W) + X(X(Y_s, s, t, W), r, W).
\]
The difference \( Y_t - X(Y_s, s, t, W) \) can be represented as follows:
\[
Y_t - X(Y_s, s, t, W) =
= \int_s^t b\left(u, Y_s + W_u - W_s + \int_s^u b(r, Y_r) \, dr\right) \, du -
\int_s^t b\left(u, Y_s + W_u - W_s + \int_s^u b(r, X_r) \, dr\right) \, du =
= \int_s^t b\left(u, W_u + h_1(u)\right) \, du - \int_s^t b\left(u, W_u + h_2(u)\right) \, du,
\]
where
\[
h_1(u) = Y_s - W_s + \int_s^u b(r, Y_r) \, dr, \quad h_2(u) = Y_s - W_s + \int_s^u b(r, X_r) \, dr.
\]
Let \( k \geq k' \quad M = 2^k \). If we take \( s, t \) of the form \( \frac{i}{M} \) and \( \frac{i+1}{M} \), respectively, then we obtain the following estimate:
\[
|Y_t - X(Y_s, s, t, W)| \leq \frac{C}{M^4}
\]
Since we may assume that $M$ is sufficiently large this inequality implies the bound
$$|Y_t - X(Y_s, s, t, W)| \leq \frac{1}{M}.$$ Hence there exists a positive constant $C = C(N, S, W)$ such that
$$|f(t) - f(s)| \leq C|Y_t - X(Y_s, s, t, W)|^{\frac{4}{5}}.$$ and consequently
$$|f(r)| \leq \frac{C}{M^{\frac{4}{5}}}.$$ Due to the arbitrariness of $k$ we conclude
$$f(r) = X(x, 0, r, W) - Y_r = 0.$$ Since $r$ was an arbitrary dyadic number in $[0, 1]$ with a sufficiently large denominator, the continuity of $Y_t$ and $X(x, 0, t, W)$ implies the equality $Y_t = X(x, 0, t, W)$ for each $t \in [0, 1]$. The proof is complete. □

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