Abstract. New solutions of DHOST theories can be generated by applying a disformal transformation to a known seed solution. We examine the nature of spherically symmetric solutions of DHOST gravity obtained by disforming static spherical scalar field solutions, or stealth solutions, of general relativity. It is shown that, in these cases, black hole horizons are never created by disforming a black hole seed. New DHOST solutions are then created by disforming two lesser known scalar field solutions of general relativity: Wyman’s “other” solution and the Husain-Martinez-Nuñez one. These new solutions demonstrate that one can obtain black hole horizons, wormhole throats, or horizonless geometries by disforming non-stealth, time-dependent, seeds.

Keywords: dark energy theory, modified gravity
1 Introduction

The theoretical need to explain the present acceleration of the cosmic expansion without invoking an ad hoc dark energy \cite{1} keeps stimulating the study of modifications of gravity with respect to Einstein’s general relativity (GR). It is quite possible that the observed cosmic acceleration is the manifestation of deviations from GR at large scales. One of the most popular candidates for modified gravity is the $f(R)$ family of theories (where $R$ is the Ricci scalar), a subclass of scalar-tensor gravity \cite{2–6} (see \cite{7–9} for reviews). However, new scalar-tensor theories of gravity (in addition to the original Brans-Dicke theory \cite{2} and its “first generation” generalizations \cite{3–6}) have emerged and have been the subject of intense study for almost a decade, beginning with the rediscovery and reformulation \cite{10–12} of Horndeski gravity \cite{13}. The field equations of Horndeski gravity are of second order, thus avoiding the notorious Ostrogradski instability affecting higher order theories. Surprisingly, certain theories with higher order equations of motion reduce, under the imposition of certain degeneracy conditions, to healthy theories in this regard \cite{14, 15}. These Degenerate Higher Order Scalar-Tensor (or “DHOST”) theories were developed in \cite{16, 18, 18–22} and have generated a rich literature (see Refs. \cite{23, 24} for reviews). The recent multi-messenger observation of a simultaneous gravitational wave and gamma-ray burst GW170817/GRB170817 \cite{25, 26} sets severe constraints on the space of DHOST theories by restricting the difference between the propagation speeds of gravitational and electromagnetic waves \cite{27}. Theoretical constraints avoiding graviton decay into scalar field perturbations further restrict the space of allowable theories \cite{28}.

We follow the notation of Ref. \cite{29}: the metric signature is $-++++$ and units are used in which the speed of light and Newton’s constant are unity. In this notation, the general
DHOST action is
\[
S_{\text{(DHOST)}}[g_{ab}, \phi] = \int d^4x \sqrt{-g} \left\{ f_0(\phi, \bar{X}) + f_1(\phi, \bar{X}) \Box \phi + f_2(\phi, \bar{X}) R \\
+ A^{abcd}_{(2)} \nabla_a \nabla_b \phi \nabla_c \nabla_d \phi + f_3(\phi, \bar{X}) G_{ab} \nabla^a \nabla^b \phi \\
+ A^{abcdef}_{(3)} \nabla_a \nabla_b \phi \nabla_c \nabla_d \phi \nabla_e \nabla_f \phi \right\}, \tag{1.1}
\]
where \( \phi \) is the scalar field degree of freedom, \( X \equiv \nabla^c \phi \nabla_c \phi \), \( \nabla_a \) is the covariant derivative of the metric \( g_{ab} \) (which has determinant \( g \)), and \( \Box \equiv g^{ab} \nabla_a \nabla_b \) is d’Alembert’s operator, while \( G_{ab} \) denotes the Einstein tensor.

The quadratic terms in the action (1.1) are usually written as
\[
A^{abcd}_{(2)} \nabla_a \nabla_b \phi \nabla_c \nabla_d \phi = \sum_{i=1}^{5} \alpha_i(\phi, \bar{X}) \mathcal{L}^i_{(2)}, \tag{1.2}
\]
and
\[
\mathcal{L}^1_{(2)} = \nabla_a \nabla_b \phi \nabla^a \nabla^b \phi, \tag{1.3}
\]
\[
\mathcal{L}^2_{(2)} = (\Box \phi)^2, \tag{1.4}
\]
\[
\mathcal{L}^3_{(2)} = (\Box \phi) \nabla^a \phi \nabla^b \phi \nabla_a \nabla_b \phi, \tag{1.5}
\]
\[
\mathcal{L}^4_{(2)} = \nabla^a \phi \nabla^b \phi \nabla_a \nabla_c \phi \nabla^c \nabla^b \phi, \tag{1.6}
\]
\[
\mathcal{L}^5_{(2)} = \left( \nabla^a \phi \nabla^b \phi \nabla_a \nabla_b \phi \right)^2, \tag{1.7}
\]
while the cubic terms are written as
\[
A^{abcdef}_{(3)} \nabla_a \nabla_b \phi \nabla_c \nabla_d \phi \nabla_e \nabla_f \phi = \sum_{i=1}^{10} \beta_i(\phi, \bar{X}) \mathcal{L}^i_{(3)}, \tag{1.8}
\]
where

\[ L_1^{(3)} = (\Box \phi)^3, \]  
\[ L_2^{(3)} = (\Box \phi) \nabla_a \nabla_b \phi \nabla^a \nabla_b \phi, \]  
\[ L_3^{(3)} = \nabla_a \nabla_b \phi \nabla^b \phi \nabla^a \nabla \phi, \]  
\[ L_4^{(3)} = (\Box \phi)^2 \nabla_a \phi \nabla_b \phi \nabla^a \nabla_b \phi, \]  
\[ L_5^{(3)} = (\Box \phi) \nabla_a \phi \nabla^a \nabla^b \phi \nabla_b \nabla \phi, \]  
\[ L_6^{(3)} = \left( \nabla_a \nabla_b \phi \nabla^a \nabla^b \phi \right) \nabla_c \phi \nabla_d \phi \nabla^c \nabla^d \phi, \]  
\[ L_7^{(3)} = \nabla_a \phi \nabla_b \phi \nabla^a \nabla^b \phi \nabla_c \phi \nabla_d \phi \nabla^c \nabla^d \phi, \]  
\[ L_8^{(3)} = \nabla_a \phi \nabla_c \phi \nabla^a \nabla^b \phi \nabla_b \nabla \phi \nabla^c \nabla \phi, \]  
\[ L_9^{(3)} = (\Box \phi) \left( \nabla_a \phi \nabla_b \phi \nabla^a \nabla^b \phi \right)^2, \]  
\[ L_{10}^{(3)} = \left( \nabla_a \phi \nabla_b \phi \nabla^a \nabla^b \phi \right)^3. \]  

The \( \alpha_i(\phi, \bar{X}) \) and \( \beta_i(\phi, \bar{X}) \) are arbitrary (regular) functions of their arguments, and so are the \( \mathcal{A}_{(2)}^{abcd} \) and \( \mathcal{A}_{(3)}^{abcdef} \). The requirement that the Ostrogradski instability is avoided imposes relations between these functions [30].

Naturally, the search for exact solutions describing hairy black holes, stars, and other condensations has been a part of developing DHOST theories. Beginning with spherically symmetric solutions, and continuing with axially symmetric ones, the catalogue of analytical solutions is expanding, although slowly (see [31] for a recent review of spherically symmetric solutions of GR and of general scalar-tensor gravity), including three-dimensional spacetimes [32]. Most of the currently known solutions are stealth ones with a static geometry and a scalar field that depends linearly on time [33–39]. In Refs. [40–42], general conditions for stealth solutions beyond shift symmetry have been obtained. However, the search for solutions beyond stealth ones has been started [35, 37, 43], the recent Refs. [44–46] leading the way in this effort. The novel approach used in [44–46] is based on the use of disformal transformations [47–50] to generate new solutions from known “seed” ones; this technique is linked to the fact that the degeneracy classes of quadratic DHOST theories are stable under disformal transformations. Moreover, solutions of the coupled Einstein-Klein-Gordon equations are mapped into DHOST solutions and can be used as seeds. Ref. [51] discusses the Petrov classification of spacetimes in relation with the construction of disformal solutions of DHOST theories and how the Petrov classes are mapped by disformal transformations. We refer to [44, 51] and the references therein for the transformation properties of the many terms in the DHOST action under disformal transformations and their inverses.
In particular, Ref. [44] establishes the impossibility of “disforming” the Fisher-Janis-Newman-Winicour-Wyman solution of GR coupled with a free scalar field (which contains a naked singularity) [52–56] into a black hole.¹

Starting with a DHOST theory containing a metric tensor \( g_{ab} \) and a scalar field \( \phi \), a generic disformal transformation of the metric has the form [47–50]

\[
g_{ab} \to \tilde{g}_{ab} = \Omega^2 (\phi, X) g_{ab} + W (\phi, X) \nabla_a \phi \nabla_b \phi,
\]

where \( \Omega > 0 \) and \( W \) are regular functions of the scalar field and of its gradient squared. The functions \( \Omega(\phi, X) \) and \( W(\phi, X) \) must satisfy the inequalities

\[
\Omega \neq 0,
\]

\[
\Omega^2 - X (\Omega^2)_X - X^2 W_X \neq 0,
\]

to ensure that the map \( g_{ab} \to \tilde{g}_{ab} \) is invertible [44].

When \( W = 0 \) the disformal transformation reduces to the usual conformal transformation (usually attributed to Bekenstein [57], but somehow anticipated in [58, 59]) of “old” scalar-tensor gravity. This conformal transformation has been widely used as a solution-generating technique in this context (e.g., [60–67] and references therein).

Here we want to establish, as much as is practically feasible, the nature of the image under disformal transformations of spherically symmetric seeds describing, respectively, black holes, wormhole throats, or naked singularities (or, more in general, spherical solutions without horizons). In other words, we would like to know whether a black hole (or a wormhole, or a horizonless solution) is mapped into another black hole, or a wormhole, or a horizonless solution. The corresponding results for pure conformal transformations (i.e., \( W = 0 \)) are reported in [73] (see also [74]); the discussion becomes significantly more complicated for disformal transformations (1.19). As exposed in the next section, it is possible to give a complete classification when starting with static seed solutions, but the analysis is not as conclusive (although some result can still be obtained) for stealth solutions in which \( g_{ab} \) is static and the scalar \( \phi \) has a linear dependence on time.

The authors of [44] assess the presence or absence of black hole horizons by studying the norm squared of the Kodama vector, which is always defined in spherical symmetry, and changes sign on an (apparent) horizon. Our procedure of this section is equivalent but, in practice, streamlined and has the benefit of a more compact discussion. We use the fact that, in spherical symmetry, when (apparent) horizons exist they are located by the real and positive roots of the equation

\[
g^{ab} \nabla_a R \nabla_b R = 0,
\]

where \( R \) is the areal radius of the spherical geometry [75, 77, 80]. A black hole horizon corresponds to a single real positive root while a double root identifies a wormhole throat, and no roots correspond to no horizons. We refer to apparent horizons because in time-dependent situations, which are of potential interest for DHOST solutions, horizons are not the null event horizons familiar from stationary black holes in GR, but they can be timelike or spacelike, or change their causal nature (see [71, 72, 75] for reviews). Apparent horizons have the drawback that they are foliation-dependent [69, 70]. However, in spherical symmetry,

¹This solution is the general static, spherical, and asymptotically flat solution of the Einstein equations sourced by a free scalar field.
they become gauge-independent if one restricts oneself to spherically symmetric foliations [68], which will be assumed in the following.

As shown in the next sections, results can be established for static and stealth seed solutions, but the significant freedom of choosing the two functions $\Omega(\phi, X)$ and $W(\phi, X)$ allows for practically any outcome one desires when starting with general seeds and assigning these two functions. In Sec. 3 and 4 we provide two examples that are not static or stealth and show how to obtain new solutions of varied nature.

2 Disformal mappings of static and of stealth solutions

Consider a spherically symmetric seed solution which, without loss of generality, can be written in the form

\begin{equation}
\begin{aligned}
ds^2 &= g_{ab} dx^a dx^b = -A(t, r) dt^2 + \frac{dr^2}{B(t, r)} + r^2 d\Omega^2, \\
\phi &= \phi(t, r),
\end{aligned}
\end{equation}

where $d\Omega^2 \equiv d\theta^2 + \sin^2 \theta \, d\varphi^2$ is the line element on the unit 2-sphere, $r$ is the areal radius, and $\phi$ is the scalar field acting as the only matter source. If this geometry possesses apparent horizons, they are located by the roots of the equation

\begin{equation}
\nabla_c \nabla_c \tilde{R} = 0 \quad \text{(2.7)}
\end{equation}

One can, of course, transform to new coordinates using the areal radius $\tilde{R}$ as the radial coordinate, but this is unnecessary since eq. (2.7) locating the apparent horizons (when they exist) is a scalar equation independent of the coordinate system.

2.1 Static seed solutions

Often a static and spherically symmetric solution, i.e., one with $A = A(r), B = B(r), \phi = \phi(r)$ (for example, the Schwarzschild black hole [44]) is used as a seed. In this case $X(r) = B\phi^2$ and the new line element, which is diagonal, is simply

\begin{equation}
\begin{aligned}
ds^2 &= -\left(\Omega^2 A - W\phi^2\right) dt^2 + 2W\phi r dr + \left(\frac{\Omega^2}{B} + W\phi^2\right) dr^2 + \tilde{R}^2 d\Omega^2, \\
\tilde{R}(t, r) &= \Omega(\phi(t, r), X(t, r))^r
\end{aligned}
\end{equation}

is the areal radius of the new metric $\tilde{g}_{ab}$.
(since these horizons are static, they are null event horizons). Since
\[ \nabla_\mu \tilde{R} = \left\{ \Omega \phi' + \Omega_X (B\phi'^2)' \right\} r + \Omega \delta_{\mu 1}, \] (2.8)
eq (2.7) becomes
\[ \frac{B}{\Omega^2 + BW \phi'^2} \left[ \Omega \phi' + \Omega_X (B\phi'^2)' + \frac{\Omega}{r} \right]^2 = 0. \] (2.9)
The possible roots are all those of the seed geometry, which are identified by \( B = 0 \), plus the roots of the new equation
\[ \Omega \phi' + \Omega_X (B\phi'^2)' + \frac{\Omega}{r} = 0. \] (2.10)
As is clear from eq. (2.9), if they exist, these new roots introduced by the disformal transformation are always double roots associated with wormhole throats. Therefore, the disformal transformation cannot generate new black hole horizons. In particular, this implies that:

- It is not possible to map a geometry without horizons into one with black hole horizons. The newly generated geometry can only have a wormhole throat or no horizons at all.
- The disformal transformation cannot map a geometry characterized by a wormhole throat into a black hole.
- It is possible that a black hole seed geometry is mapped into a wormhole throat. This can in principle happen when a new wormhole throat (a double root \( \tilde{R}_{\text{WHT}} \) of eq. (2.10)) is introduced by the disformal transformation and the radius of the would-be black hole horizon \( r_H \), which corresponds to the areal radius \( \tilde{R}_H = \Omega(\phi(r_H), \phi'(r_H))r_H \), lies below the wormhole throat, i.e., \( \Omega(\phi(r_H))r_H < \tilde{R}_{\text{WHT}} \).

Equation (2.10) can be written as
\[ \frac{d\Omega}{dr} + \frac{\Omega}{r} = 0; \] (2.11)
it is easy to see that this equation cannot be satisfied identically. In fact, its integration would give \( \Omega(r) = C/r \), where \( C \) is an integration constant. This conformal factor is singular at \( r = 0 \) and tends to zero as \( r \to +\infty \), while \( \Omega \) is instead required to be regular everywhere and go to unity as \( r \to +\infty \) for asymptotically flat solutions.

We can discuss three cases separately.

1. If the seed solution has no horizons, there are no roots of \( B(r) = 0 \), then there can only be a wormhole throat if real positive roots of eq. (2.10) exist. If no such throat exists, there are no horizons (in particular, if the seed solutions contains a naked singularity, it is mapped into another naked singularity spacetime).

2. If the seed solution has a black hole (event) horizon at \( r = r_{\text{BH}} \), then
\[ B(r_{\text{BH}}) = 0, \quad B'(r_{\text{BH}}) > 0. \] (2.12)
The equation locating the horizons in the new geometry is
\[ B \left[ \Omega \phi' + \Omega_X (B\phi'^2)' + \frac{\Omega}{r} \right]^2 = 0, \] (2.13)
and all the roots $r_{BH}$ are still roots but, as explained above, a new wormhole throat could be created by the disformal transformation, that relegates a black hole horizon to a region that is no longer part of the physical spacetime in the new geometry, if $\Omega(\phi(r_H))r_H < \tilde{R}_{WHT}$.

3. Finally, suppose that the seed geometry has a wormhole throat at $r_W$, where $B(r_W) = B'(r_W) = 0$. This remains a double root of

$$B \left[ \Omega_\phi \phi' + \Omega_X (B \phi'^2)' + \frac{\Omega r}{r} \right] = 0, \quad (2.14)$$

corresponding to the physical (areal) radius $\tilde{R}_W = \Omega(\phi(r_W), X(r_W))r_W$. If extra (real and positive) roots $\tilde{R}_{WW}$ exist, they are always double roots and they could obliterate the previous ones if $\tilde{R}_W < \tilde{R}_{WW}$, in the manner described above for black hole horizons.

2.2 Stealth seed geometry

Let us turn now to spherical stealth solutions [33–39] with

$$A = A(r), \quad B = B(r), \quad \phi(t, r) = qt + \chi(r), \quad (2.15)$$

where $q \neq 0$ is a constant, giving $X(r) = -\frac{q^2}{A} + B \phi'^2$, $\phi' = \chi'$, and the line element

$$d\tilde{s}^2 = - (\Omega^2 A - q^2 W) dt^2 + 2qW \phi'dt dr + \left( \frac{\Omega^2}{B} + W \phi'^2 \right) dr^2 + \tilde{R}^2 d\Omega^2(\tilde{t}), \quad (2.16)$$

with $\tilde{R}(r) = \Omega(\phi(r), X(r))r$. We now have

$$\nabla_\mu \tilde{R} = qr \Omega_{\phi} \delta_{\mu 0} + r \left[ \frac{\Omega}{r} + \Omega_\phi \phi' + \Omega_X \left( -\frac{q^2}{A} + B \phi'^2 \right) \right] \delta_{\mu 1}, \quad (2.17)$$

and the equation locating the apparent horizons is now

$$\tilde{g}^{ab} \nabla_a \tilde{R} \nabla_b \tilde{R} = -q^2 \Omega_{\phi}^2 \left( \Omega^2 + BW \phi'^2 \right) + 2q^2 \Omega_{\phi} BW \phi' \left[ \frac{\Omega}{r} + \Omega_\phi \phi' + \Omega_X \left( -\frac{q^2}{A} + B \phi'^2 \right) \right]$$

$$+ (\Omega^2 A - q^2 W) B \left[ \frac{\Omega}{r} + \Omega_\phi \phi' + \Omega_X \left( -\frac{q^2}{A} + B \phi'^2 \right) \right]^2 = 0, \quad (2.18)$$

where we used the inverse metric

$$\left( \tilde{g}^{\mu\nu} \right) = \begin{pmatrix}
\frac{\Omega^2}{g^{(\phi \phi)}}, & -\frac{qW \phi'}{g^{(\phi \phi)}}, & 0 & 0 \\
-\frac{qW \phi'}{g^{(\phi \phi)}}, & \frac{q^2 W - \Omega A}{g^{(\phi \phi)}}, & 0 & 0 \\
0 & 0 & \frac{1}{\tilde{R}^2} & 0 \\
0 & 0 & 0 & \frac{1}{\tilde{R}^2 \sin^2 \varphi}
\end{pmatrix}, \quad (2.19)$$
and where
\[ g^{(2)} = \Omega^2 \left( \frac{q^2 W}{B} - W A \phi'^2 - \frac{\Omega^2 A}{B} \right) < 0, \] (2.20)
is the determinant of the restriction of the seed metric \( g_{ab} \) to the 2-dimensional subspace \((t, r)\). In general, if they exist and they are real and positive, the roots of eq. (2.18) are time-dependent due to the time-dependence of \( \Omega, \Omega_\phi, \Omega_X, \) and \( W \). As a consequence, the corresponding horizons (when they exist) are dynamical apparent horizons (see, e.g., [75]). The analysis of the roots of eq. (2.18) (and, therefore, of the nature of the associated horizons) is more difficult and not much can be said in general without specifying the form of the functions \( \Omega(\phi, X) \) and \( W(\phi, X) \). We limit ourselves to the following considerations.

Suppose that the seed geometry has a black hole (event) horizon at \( r_H \), with \( B(r_H) = 0 \) and \( B'(r_H) > 0 \): then eq. (2.18) evaluated at \( r_H \) yields
\[ (q \Omega \Omega_\phi)^2 \bigg|_{r_H} = 0. \] (2.21)
Since \( q \neq 0 \) and \( \Omega > 0 \), for the black hole horizon at \( r_H \) to remain a (apparent) horizon (at \( \hat{R}_H = \Omega(r_H) r_H \)) it is necessary that
\[ \Omega_\phi(\phi(t, r_H), X(r_H)) = 0, \] (2.22)
and, when this happens, we necessarily have a double root of eq. (2.21). Therefore, the (single root) black hole horizon of the “old” spacetime is either eliminated by the disformal transformation or it is converted into a (double root) wormhole throat in the disformed world.

Due to the large amount of literature on shift-symmetric theories, it is relevant to restrict oneself to invertible disformal transformations and shift-symmetric potentials, as done in [44], in which case
\[ \Omega = \Omega(X), \quad W = W(X). \] (2.23)
Under this assumption, eq. (2.22) is identically satisfied and a black hole horizon of the seed geometry is transformed into a wormhole throat. Moreover, eq. (2.18) simplifies to
\[ (\Omega^2 A - q^2 W) B \left[ \frac{\Omega}{r} + \Omega_X \left( -\frac{q^2}{A} + B \phi'^2 \right) \right]^2 = 0. \] (2.24)
In addition to the horizons of the seed metric (roots of \( B = 0 \)) already discussed, the additional potential roots given by the vanishing of the expression in square brackets \( \Omega + \Omega_X \left( -\frac{q^2}{A} + B \phi'^2 \right) \) (if they exist and are real and positive), can only be double roots and, therefore, wormhole throats. The only possibility left to obtain a new root \( \bar{R}_H = r_H \Omega(\phi(r_H)) \) corresponds to the vanishing of the first bracket \( \Omega^2 A - q^2 W \) in eq. (2.24), which happens when the functions appearing in the disformal transformation (1.19) satisfy the very specific relation
\[ W(X) - \frac{A(r)}{q^2} \Omega^2(X) = 0 \quad \text{at} \quad r_H = 0. \] (2.25)
When it exists, this root is a black hole horizon (a single root) if the derivative with respect to \( r \) of the left hand side of eq. (2.25) is non-vanishing, or
\[ X' \left( W_X - \frac{2 A \Omega^2 \Omega_X}{q^2} \right) - \frac{A' \Omega^2}{q^2} \neq 0 \quad \text{at} \quad r_H. \] (2.26)
This possibility can be usefully exploited to generate new DHOST solutions by choosing the functions \( \Omega(X) \) and \( W(X) \) appropriately.

When the seed solution is not static or stealth, the situation is more complicated. Due to the freedom of choosing the two functions \( \Omega(\phi, X) \) and \( W(\phi, X) \), the disformed universe can be endowed with black hole horizons even if the seed solution has none, as shown in the next section.

### 3 Disforming Wyman’s “other” solution

Here we use as seed a little-known and simple solution of Einstein gravity with a free minimally coupled scalar field found long ago by Wyman [56].\(^3\) The geometry and scalar field are

\[
ds^2 = -\kappa r^2 dt^2 + 2dr^2 + r^2 d\Omega^2_2,
\]

\[
\phi(t) = \phi_0 t,
\]

where \( \kappa = 8\pi G \), \( \phi_0 \) is a dimensionless constant, and \( r \) is the areal radius. We refer to eqs. (3.1) and (3.2) as Wyman’s “other” solution to distinguish it from the better known Fisher-Janis-Newman-Winicour-Wyman solution [52–56] which is the general solution of the Einstein equations sourced by a free scalar field with the properties of being spherical, static, and asymptotically flat.

The geometry is static (but the scalar field is not) and it is not asymptotically flat: the spatial sections \( t = \text{const.} \) are curved. This is not a stealth solution. The Wyman geometry (3.1) can also be recovered as a special case of a solution of scalar-tensor gravity found by Carloni and Dunsby [79] which, however, has a scalar field that depends only on \( r \), instead of \( t \), and is subject to a power-law potential.

The Ricci scalar of the Wyman solution (3.1), (3.2)

\[
\mathcal{R} = \kappa g^{ab} \nabla_a \phi \nabla_b \phi = -\frac{\phi_0^2}{r^2},
\]

diverges as \( r \to 0^+ \). Since the equation \( g^{ab} \nabla_a r \nabla_b r = 0 \) has no roots because \( g^{ab} \nabla_a r \nabla_b r = g^{rr} = 1/2 \), there are no horizons and the Wyman spacetime hosts a central naked singularity. The geometry is not asymptotically flat since the Ricci tensor

\[
\mathcal{R}_{ab} = \kappa \nabla_a \phi \nabla_b \phi = \kappa \phi_0^2 \delta_{ab},
\]

(which is independent of \( r \)) does not vanish as \( r \to +\infty \). The Misner-Sharp-Hernandez mass of a sphere of radius \( r \), defined by \( 1 - 2M_{\text{MSH}}/r \equiv \nabla^a r \nabla_a r \) [80, 81], is \( M_{\text{MSH}}(r) = r/4 \).

Under the disformal transformation (1.19), the Wyman line element (3.1) is mapped to

\[
ds^2 = -(\kappa r^2 \Omega^2 - W \phi_0^2) dt^2 + 2\Omega^2 dr^2 + \tilde{R}^2 d\Omega^2_2
\]

(which is still diagonal), where \( \tilde{R}(t, r) = \Omega r \). Since

\[
\nabla_\mu \tilde{R} = \delta_{\mu0} \phi_0 \Omega \phi r + \delta_{\mu1} \left( \Omega + \frac{2\phi_0^2}{\kappa r^2} \Omega X \right),
\]

\(^3\)Wyman’s solution has been generalized to include a positive cosmological constant [78] and has been used to generate a new family of dynamical solutions of Brans-Dicke gravity [67].
the equation $\tilde{g}^{ab}\nabla_a\tilde{R}\nabla_b\tilde{R} = 0$ locating the apparent horizons reads

$$-2\phi_0^2\Omega^2_0\Omega^2_0r^2 + \left(\Omega + \frac{2\phi_0^2\Omega X}{\kappa r^2}\right)^2 \left(\kappa r^2\Omega^2 - W\phi_0^2\right) = 0.$$  

(3.7)

We can now exploit the freedom given by $\Omega(\phi, X)$ and $W(\phi, X)$ to map the naked singularity into a black hole or a wormhole throat with corresponding static or dynamical horizons. All the choices of $\Omega(\phi, X)$ and $W(\phi, X)$ in this section satisfy the inequalities (1.20) and (1.21).

3.1 $\Omega(X) = -X$

First, we consider the conformal factor

$$\Omega(X) = -X = \frac{\phi_0^2}{\kappa r^2};$$  

(3.8)

eq. (3.7) locating the apparent horizons becomes

$$\frac{\phi_0^6}{\kappa^2 r^4} \left(\frac{\phi_0^2}{\kappa X} - W\right) = 0.$$  

(3.9)

The new spacetime will still be horizonless if, for example

$$W(X) = -X - \left(\frac{\phi_0^2}{\kappa X}\right)^2 = \frac{\phi_0^2}{\kappa r^2} - r^4,$$  

(3.10)

because then eq. (3.9) has no roots.

Alternatively, the naked singularity is mapped to a black hole with static apparent horizon if

$$W(X) = \frac{r_0\sqrt{\kappa}}{\phi_0} \sqrt{|X|} = \frac{r_0}{r},$$  

(3.11)

(with $r_0$ a length scale), which yields the single root

$$r_H = \frac{\phi_0}{\kappa r_0},$$  

(3.12)

corresponding to areal radius

$$\tilde{R}_H = \Omega_H r_H = r_0.$$  

(3.13)

Finally, the singular seed ends up as a wormhole throat with static apparent horizon if $W = 1$, which generates the double root $r_H = \phi_0/\sqrt{\kappa}$ and

$$\tilde{R}_H = \frac{\phi_0}{\sqrt{\kappa}}.$$  

(3.14)

Therefore, the central naked singularity of Wyman’s “other” spacetime can be mapped to another horizonless geometry, or to a black hole with static apparent horizon, or to a static wormhole throat apparent horizon according to specific choices of the conformal and disformal factors $\Omega$ and $W$. 

– 10 –
3.2 \( \Omega(X, \phi) = -\alpha X \phi \)

Another possible choice for the conformal factor is

\[
\Omega(X, \phi) = -\alpha X \phi = \frac{\alpha \phi_0^4 t}{\kappa r_0^2},
\]

(3.15)

where \( \alpha > 0 \) has the dimensions of an inverse length. Eq. (3.7) becomes

\[
\frac{\alpha^2 \phi_0^4}{\kappa} \left( t^2 - \frac{2}{\kappa} \right) - Wr^2 = 0.
\]

(3.16)

If we further choose \( W = 1 \), we find a double root corresponding to areal radius

\[
\tilde{R}_H(t) = \frac{\phi_0 t}{\sqrt{\kappa t^2 - 2}}.
\]

(3.17)

for \( t > \sqrt{2/\kappa} \); it decreases with time and stabilizes as

\[
\tilde{R}_H(t) \longrightarrow \frac{\phi_0}{\sqrt{\kappa}}.
\]

(3.18)

If we choose instead

\[
W(X) = \frac{r_0 \sqrt{\kappa}}{\phi_0} \sqrt{-X} = \frac{r_0}{\kappa},
\]

(3.19)

we find a single root corresponding to a dynamical black hole horizon with coordinate radius

\[
r_H(t) = \frac{\alpha^2 \phi_0^4}{\kappa r_0^2} \left( t^2 - \frac{2}{\kappa} \right)
\]

(3.20)

for \( t > \sqrt{2/\kappa} \). This dynamical black hole apparent horizon in the new spacetime has physical radius

\[
\tilde{R}_H(t) = \frac{r_0 t}{\alpha \phi_0 (t^2 - 2/\kappa)},
\]

(3.21)

which shrinks to zero as \( t \to +\infty \).

3.3 \( \Omega(\phi) = \alpha \phi \)

We finally consider a conformal factor of the form

\[
\Omega(\phi) = \alpha \phi = \alpha \phi_0 t,
\]

(3.22)

which determines the equation for the horizon radii

\[
-2\alpha^2 r^2 + \kappa r \alpha^2 t^2 - W = 0.
\]

(3.23)

If

\[
W(\phi, X) = -\frac{\alpha^2 \phi_0^2}{X} - \frac{\phi_0}{\kappa r_0} |X|^{-1/2} = \kappa \alpha^2 r^2 t^2 - \frac{r}{r_0},
\]

(3.24)

one obtains the single root apparent horizon radius \( r_H = 1/(2\alpha^2 r_0) \) and the associated areal radius

\[
\tilde{R}_H(t) = \frac{\phi_0 t}{2\alpha r_0}
\]

(3.25)
describing a linearly expanding black hole apparent horizon.

By choosing instead the constant disformal factor $W = \phi_0^2$, one finds a wormhole throat at radius

$$r_H(t) = \frac{\phi_0}{\alpha \sqrt{\kappa t^2 - 2}},$$

which is a double root at times $t > \sqrt{2/\kappa}$. Therefore, the areal radius of this horizon

$$\tilde{R}_H(t) = \alpha^2 t \sqrt{\kappa t^2 - 2}$$

(3.27)

expands. Coincidentally, the different choice

$$W(\phi) = \frac{\kappa \phi^2 - 2\phi_0^2}{(2\alpha \phi_0 r_0)^2} = \frac{\kappa t^2 - 2}{4\alpha^2 r_0^2}$$

(3.28)

leads to the same radius (3.25) and describes again a (double root) wormhole throat.

4 Disforming the Husain-Martinez-Nuñez solution

We can use another analytical solution of the Einstein equations found by Husain, Martinez and Nuñez [82] as a seed to disform. This seed solution describes a spherically symmetric geometry sourced by a free scalar field that is time-dependent. The geometry is also time-dependent, asymptotically Friedmann-Lemaître-Robertson-Walker (FLRW), and conformal to the Fisher-Janis-Newman-Winicour-Wyman solution. The Husain-Martinez-Nuñez line element in comoving time and the associated scalar field are [75]

$$ds^2 = -\left(1 - \frac{2C}{r}\right)^\alpha dt^2 + \frac{a^2(t)}{(1 - \frac{2C}{r})^\alpha} dr^2 + a^2(t)r^2 \left(1 - \frac{2C}{r}\right)^{1-\alpha} d\Omega^2_2,$$

(4.1)

$$\phi(t, r) = \pm \frac{1}{4\sqrt{3\pi}} \ln \left[ D \left(1 - \frac{2C}{r}\right)^{\frac{\alpha}{2}} a^2 \sqrt{3} (t) \right],$$

(4.2)

where $\alpha = \pm \sqrt{3}/2$, $a(t) = a_0 t^{1/3}$, and $a_0, C$ and $D$ are non-negative constants. The areal radius is

$$R(t) = a(t)r \left(1 - \frac{2C}{r}\right)^{\frac{1-\alpha}{2}}.$$

(4.3)

Applying the disformal transformation (1.19) to this geometry and using

$$\nabla_\mu \phi = \pm \frac{1}{2\sqrt{3\pi}} \left[ \frac{\delta_0}{t} + \frac{\alpha C}{r^2 (1 - 2C/r)} \right]$$

(4.4)

gives

$$ds^2 = -\left[ \Omega^2 \left(1 - \frac{2C}{r}\right)^\alpha - \frac{W}{r^2} \right] dt^2 + \frac{2\alpha WC}{tr^2(1 - 2C/r)} dt dr + \left[ \frac{\Omega^2 a_0^2 t^{2/3}}{(1 - 2C/r)^\alpha} + \frac{\alpha^2 C^2 W}{r^4(1 - 2C/r)^2} \right] dr^2$$

$$+ \frac{\Omega^2 a_0^2 t^{2/3} r^2}{(1 - \frac{2C}{r})^{1-\alpha}} d\Omega^2_2,$$

(4.5)
where \( W \equiv W/(12\pi) \). The new areal radius is

\[
\tilde{R}(t) = \Omega r a_0 t^{1/3} \left( 1 - \frac{2C}{r} \right)^{\frac{1-\alpha}{2}}.
\]  

(4.6)

If we choose

\[
\Omega(t, r) = \frac{(1 - \frac{2C}{r})^{\alpha/2}}{a_0 t^{1/3}}
\]  

(4.7)

(unfortunately, there is no explicit expression of \( \Omega \) as a function of \( \phi \) and \( X \)), the line element becomes

\[
\begin{aligned}
d\tilde{s}^2 &= - \left[ \frac{(1 - 2C/r)^{2\alpha}}{a_0^2 t^{2/3}} - \frac{W}{t^2} \right] dt^2 + \frac{2\alpha C W}{tr^2(1 - 2C/r)} dtdr + \left[ 1 + \frac{\alpha^2 C^2 W}{r^4(1 - 2C/r)^2} \right] dr^2 \\
&+ r^2 \left( 1 - \frac{2C}{r} \right) d\Omega^2(2)
\end{aligned}
\]  

(4.8)

and is no longer conformal to the Fisher solution.

The new areal radius is

\[
\tilde{R}(r) = r \sqrt{1 - \frac{2C}{r}}
\]  

(4.9)

and the equation locating the apparent horizons (when they exist) reads

\[
\tilde{g}^{ab} \nabla_a \tilde{R} \nabla_b \tilde{R} = \left[ \frac{(1 - 2C/r)^{2\alpha}}{a_0^2 t^{2/3}} - \frac{W}{t^2} \right] \times a_0^2 t^2 r^4 \left( 1 - \frac{C}{r} \right)^2 \left( 1 - \frac{2C}{r} \right)^{1-2\alpha} = 0.
\]  

(4.10)

This equation is satisfied if \( r_1 = C \), or \( r_2 = 2C \), or if the term in square brackets vanishes. We discard the first root \( r_1 \) because \( r_1 < 2C \) corresponds to imaginary \( \tilde{R}_1 \), while the second root is not a horizon since \( r_2 = 2C \) corresponds to physical radius \( \tilde{R}_2 = 0 \).

The choice \( W = 12\pi l_0^2 \) with \( l_0 \) a length scale yields

\[
r_H(t) = \frac{2C t^{\frac{3}{2\alpha}}}{t^{\frac{1}{\alpha}} - (l_0 a_0)^{\frac{1}{\alpha}}},
\]  

(4.11)

with corresponding physical radius

\[
\tilde{R}_H(t) = \frac{2C(l_0 a_0)^{\frac{1}{2\alpha}} t^{\frac{3}{2\alpha}}}{t^{\frac{1}{\alpha}} - (l_0 a_0)^{\frac{1}{\alpha}}}.
\]  

(4.12)

For the parameter value \( \alpha = \sqrt{3}/2 \) of the original Husain-Martinez-Núñez spacetime, the disformed metric is asymptotically flat and the horizon radius is

\[
\tilde{R}_H(t) = \frac{2C(l_0 a_0)^{\frac{3}{2\alpha}} t^{\frac{3}{2\alpha}}}{t^{\frac{1}{\alpha}} - (l_0 a_0)^{\frac{1}{\alpha}}}.
\]  

(4.13)

This radius is positive after a critical time \( t_* \equiv (l_0 a_0)^{3/2} \) (note that, since the scale factor \( a(t) = a_0 t^{1/3} \) is dimensionless, the dimensions of \( a_0 \) are \( [a_0] = [L^{-1/3}] \) and \( t_* \) has the dimensions of a time or length).
We can now deduce the history of the dynamical apparent horizon. Based on the existence of a single root one concludes that, when $0 < t < t_\star$, the disformed spacetime has no horizons since $\tilde{R}_H < 0$; a black hole apparent horizon begins to appear at $t = t_\star$ with $\tilde{R}_H(t_\star) = +\infty$. It is always present at physical radius (4.13) for $t > t_\star$ and is approximated by

$$\tilde{R}_H(t) \simeq \frac{2C(l_0a_0)^{1/\sqrt{3}}}{t^{2\sqrt{3}}},$$

as $t \to +\infty$. This apparent horizon shrinks to zero at late times.

The other parameter value $\alpha = -\sqrt{3}/2$ gives

$$\tilde{R}_H(t) = \frac{2C (l_0a_0)^{1/\sqrt{3}}}{(l_0a_0)^{2/\sqrt{3}} - t^{2/\sqrt{3}}}.$$  \hspace{1cm} (4.15)

Since the sign of the denominator is reversed with respect to eq. (4.13), the horizon history is the time-reverse of that occurring for $\alpha = \sqrt{3}/2$.

The asymptotic behavior of the line element as $r \to +\infty$ (corresponding to $\tilde{R} \to +\infty$) is

$$ds^2 \simeq -\left(\frac{1}{a^2} - \frac{l_0^2}{t^2}\right) dt^2 + dr^2 + r^2 d\Omega^2_{(2)}.$$  \hspace{1cm} (4.16)

We can redefine the time coordinate for this asymptotic metric according to

$$dT \equiv \sqrt{\frac{1}{a_0^2 t^{2/3}} - \frac{l_0^2}{t^2}} dt,$$  \hspace{1cm} (4.17)

which turns the asymptotic line element (4.16) into the Minkowski metric in coordinates $(\tau, r, \vartheta, \varphi)$, hence the new solution is asymptotically flat.

One can diagonalize the full line element (4.8), obtaining (see Appendix A)

$$ds^2 = -F^2 \left(\frac{A^{2\alpha}}{a^2} - \frac{l_0^2}{t^2}\right) dT^2 + \frac{\alpha^2 C^2 l_0^2}{A^{\alpha+1}(1-C/r)^2} \left[\frac{A^2}{r^2} + \frac{A^{\alpha+2}}{\alpha^2 C^2 l_0^2} \frac{l_0^2}{A - 2C/r - t^2 A^\alpha/a^2} \right] d\tilde{R}^2$$

$$+ \tilde{R}^2 d\Omega^2_{(2)},$$

where $F$ is an integrating factor and $A \equiv 1 - 2C/r$. As $\tilde{R} \to +\infty$, $F \to 1$ and the line element (4.18) asymptotes to the Minkowski metric (see Appendix A).

5 Conclusions

Relatively few solutions of DHOST theories are known, even when symmetries such as spherical symmetry are imposed. Disforming a scalar field solution of Einstein theory selected as a seed constitutes a valuable solution-generating technique that has provided new analytical DHOST solutions and new insight into these theories, which are complicated and difficult to grasp. Most of the solutions generated using the disformal transformation used as seeds either static or stealth scalar field solutions of GR. We have discussed the disformal images of such seeds that describe black hole horizons, wormhole throats, or horizonless geometries. To move away from these rather limited situations, we have applied the disformal
transformation to two lesser known dynamical scalar field solutions of Einstein theory, i.e., Wyman’s “other” solution and the Husain-Martinez-Nuñez geometry. The choices of conformal and disformal factors $\Omega$ and $W$ used were arbitrary, and designed to obtain horizons (single roots of eq. (2.7)), wormhole throats (double roots), or horizonless (no real positive roots) geometries. These examples demonstrate that pretty much any desired nature of the disformed solution with respect to horizons can be obtained thanks to the enormous freedom of choosing the functions $\Omega(\phi, X)$ and $W(\phi, X)$ arbitrarily.

Apart from seeds that are static or stealth solutions (considered here and, previously, in Ref. [44]), no prediction can be made on the result of the disformal transformation applied to arbitrary seeds, unless restrictions on $\Omega(\phi, X)$ and $W(\phi, X)$ are imposed. Due to the large number of free functions appearing in the DHOST action (1.1), it is not clear at present how to impose meaningful restrictions on $\Omega$ and $W$. Future research will hopefully provide some guidelines on how to choose these functions meaningfully from the physical point of view to restrict the scope of mathematical possibilities.

A Diagonalization of the disformed Husain-Martinez-Nuñez line element (4.8)

Here we diagonalize the metric (4.8). We begin by using the areal radius

$$\tilde{R}(t, r) = r \sqrt{1 - \frac{2C}{r}},$$

(A.1)

as the radial coordinate. Then, we replace $dr$ in eq. (4.8) with

$$dr = \frac{\sqrt{1 - 2C/r}}{1 - C/r} d\tilde{R},$$

(A.2)

which leads to

$$d\tilde{s}^2 = -\left[\frac{(1 - 2C/r)^{2\alpha}}{a^2} - \frac{l_0^2}{r^2}\right] dt^2 + \frac{2l_0^3 \alpha C}{tr^2 \sqrt{1 - 2C/r} (1 - C/r)} dt d\tilde{R} + \left[1 + \frac{\alpha^2 C^2 l_0^2}{r^4 (1 - 2C/r)^{\alpha}}\right] \frac{(1 - 2C/r)}{(1 - C/r)^2} d\tilde{R}^2 + \tilde{R}^2 d\Omega^2_{(2)}.$$

(A.3)

In order to eliminate the cross-term in $dt d\tilde{R}$, we redefine the time coordinate according to $t \to T$, with

$$dT = \frac{1}{F} \left( dt + \beta d\tilde{R} \right),$$

(A.4)

where $\beta(t, r)$ is a function to be determined and $F$ is an integrating factor guaranteeing that the differential $dT$ is exact and obeying the equation

$$\frac{\partial}{\partial \tilde{R}} \left( \frac{1}{F} \right) = \frac{\partial}{\partial r} \left( \frac{\beta}{F} \right).$$

(A.5)
Using the new time, the line element becomes
\[
\begin{align*}
\tilde{s}^2 &= -F^2 \left( \frac{A^{2\alpha}}{a^2} - \frac{l_0^2}{t_0^2} \right) dT^2 + 2F \left[ \frac{\alpha C l_0^2}{tr^2 \sqrt{A(1-C/r)}} \right] dT d\tilde{R} \\
&\quad + \left[ - \frac{\beta^2}{tr^2} \left( \frac{A^{2\alpha}}{a^2} - \frac{l_0^2}{t_0^2} \right) - 2\beta \frac{\alpha C l_0^2}{tr^2 \sqrt{A(1-C/r)}} + \frac{A}{(1-C/r)^2} \left( 1 + \frac{\alpha^2 C l_0^2}{r^4 A^{\alpha}} \right) \right] d\tilde{R}^2 \\
&\quad + \tilde{R}^2 d\Omega^2_{(2)}.
\end{align*}
\]  
(A.6)

If we now set
\[
\beta(t, r) \equiv \frac{\alpha C l_0^2}{tr^2 \sqrt{A(1-C/r)} \left( \frac{l_0^2}{t_0^2} - \frac{A^{2\alpha}}{a^2} \right)},
\]  
(A.7)

the cross-term in \( dtd\tilde{R} \) vanishes and we recover eq. (4.18).

As \( \tilde{R} \to +\infty \), it is
\[
\frac{\partial}{\partial \tilde{R}} \left( \frac{1}{F} \right) \simeq 0,
\]  
(A.8)

asymptotically; then \( F \) asymptotes to a constant and, by rescaling the time coordinate, it can be set to unity and \( dt \simeq dT \). In this regime one redefines the time coordinate \( t \simeq T \to \tau \) according to
\[
d\tau = \sqrt{\frac{1}{a^2} - \frac{l_0^2}{t_0^2}} dT,
\]  
(A.9)

and the line element reduces to the Minkowski one in coordinates \( (\tau, \tilde{R}, \vartheta, \varphi) \),
\[
\tilde{s}_\infty^2 \simeq -d\tau^2 + d\tilde{R}^2 + \tilde{R}^2 d\Omega^2_{(2)}.
\]  
(A.10)

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