On BC-Subtrees in Multi-Fan and Multi-Wheel Graphs

Yu Yang 1, Long Li 1,2, Wenhu Wang 1,* and Hua Wang 3

1 School of Computer Science, Henan Province Key Laboratory of Germplasm Innovation and Utilization of Eco-economic Woody Plant, Pingdingshan University, Pingdingshan 467000, China; yangyu@sjtu.edu.cn (Y.Y.); lilong@home.hpu.edu.cn (L.L.)
2 College of Computer Science and Technology, Henan Polytechnic University, Jiaozuo 454003, China
3 Department of Mathematical Sciences, Georgia Southern University, Statesboro, GA 30460, USA; hwang@georgiasouthern.edu
* Correspondence: whwang@pdsu.edu.cn

Abstract: The BC-subtree (a subtree in which any two leaves are at even distance apart) number index is the total number of non-empty BC-subtrees of a graph, and is defined as a counting-based topological index that incorporates the leaf distance constraint. In this paper, we provide recursive formulas for computing the BC-subtree generating functions of multi-fan and multi-wheel graphs. As an application, we obtain the BC-subtree numbers of multi-fan graphs, multi-wheel (wheel) graphs, and discuss the change of the BC-subtree numbers between different multi-fan or multi-wheel graphs. We also consider the behavior of the BC-subtree number in these structures through the study of extremal problems and BC-subtree density. Our study offers a new perspective on understanding new structural properties of cyclic graphs.

Keywords: BC-subtree number index; generating function; multi-fan graph; multi-wheel graph; BC-subtree density

1. Introduction

A topological index is a numerical graph invariant that can quantitatively characterize the properties of the corresponding structure. Topological indices in general have applications in numerous areas such as network design, compounds synthesis, pharmacology, and biology. Consequently a large number (approximately four hundred) of topological indices have been introduced during previous decades.

Among various topological indices, distance-based [1–4], degree-based [5–8], counting-based [9–13] have received much attention [14–17], and many indices have been employed as tools for characterization of molecular topology [18–21].

Compared to the subtree number index [22–26], the BC-subtree number index is a counting-based topological index that incorporates leaf distance constraint. First of all, a BC-tree is a tree in which any two leaves are at even distance apart. This important structure was introduced by F. Harary et al. [27,28]. The BC-subtree number index or simply the BC-subtree index was first introduced in [13], as the number of all non-empty BC-subtrees (subtrees that are also BC-trees) of a graph.

Much work has been done on the BC-trees and BC-subtree number index. Some examples include generic chemical structures storage problem [29,30], fault-tolerant computing and parallel scheduling [31,32], recognizing and identifying special substructures [33–35], the flow network simplification [36], Markovian queueing systems decomposition [37], matching problem [38,39], extremal and density problems [13,40], and enumerating algorithm design [13,41].

We are particularly interested in the enumeration of BC-subtrees. Using a generating functional approach proposed by Yan and Yeh in [12] (for general subtrees), Yang et al. [13] presented algorithms of enumerating BC-subtrees of trees. Later Yang et al. [41] solved...
the BC-subtree enumeration problem on uni-cyclic and edge-disjoint bicyclic graphs and further generalized this study to spiro and polyphenyl hexagonal chains [40].

Previous studies show that the multi-fan and multi-wheel graphs (Definitions 1 and 2) have many interesting properties and applications in circuit layout and interconnection network design. The $W_{n+1}$ is also useful in designing efficient wireless ad hoc networks [42]. The wheel-transposition graphs (graphs generated by wheel graphs) could be used in parallel and distributed system [43]. Both the multi-fan graphs and the wheel graphs $W_{n+1}(n \neq 6)$ are among the graphs that can be determined by its Laplacian spectra [44,45]. Moreover, the Hilbert series of fan graphs, multi-fan graphs and multi-wheel graphs were derived in [46]. Subtrees of Fan Graphs, Wheel Graphs, and “Partitions” of Wheel Graphs under Dynamic Evolution was studied in [47]. In this paper we will consider the BC-subtree number index for these two structures.

We first introduce related preliminary information in Section 2. The BC-subtree generating functions of multi-fan graphs and multi-wheel graphs are provided in Sections 3 and 4. Some special cases are also discussed as immediate consequences. In Section 5 we make use of our findings to consider the behavior of the BC-subtree number in the above mentioned structures. Extremal problems are studied. We summarize our work in Section 6.

2. Preliminaries

Before introducing our main results there are quite some preparations to do. Let $G = (V(G), E(G); f, g)$ be a weighted graph with vertex set $V(G)$, edge set $E(G)$, vertex-weight function $f = (f_o(u), f_e(u))$ ($f_o(u)$ and $f_e(u)$ are the odd, even weight of $u \in V(G)$, respectively) and edge-weight function $g = g(e)$ for $e \in E(G)$. First we list the necessary notations and terminologies.

2.1. Basic Notations

- $d_G(u, v)$: the distance between $u \in V(G)$ and $v \in V(G)$.
- $G \setminus Y$: the graph after removing $Y$ from $G$.
- $L(T)$: the leaf set of $T$.
- $f_o(u), f_e(u)$: the odd, even weight of $u \in V(G)$, respectively.
- $S(G; X)$: the subtree set of $G$ containing $X$, where $X$ can be a vertex set or an edge set (or both), or a subtree of $G$.
- $S(G; v; odd)$: the subtree set of $G$ containing $v$ such that all leaves (excluding $v$) have odd distance from $v$; the subtrees in $S(G; v; odd)$ are called the $v_{odd}$-subtrees of $G$.
- $S(G; v; even)$: the subtree set of $G$ containing $v$ such that all leaves (excluding $v$) have even distance from $v$; Note that the single vertex tree $v$ itself is included in this set and we call subtrees in $S(G; v; even)$ the $v_{even}$-subtrees of $G$.
- $\omega^o_e(T_1), \omega^o_o(T_1)$: the $\omega^o_e, \omega^o_o$ weight of subtree $T_1 \in S(G; v)$, respectively.
- $S_{BC}(G)$: the BC-subtree set of $G$.
- $S_{BC}(G; X)$: the set of BC-subtrees of $G$ containing $X$, where $X$ can be a vertex set or an edge set (or both), or a subtree of $G$.
- $\omega_{bc}(T_2)$: the BC-weight of $T_2 \in S_{BC}(G)$.
- $F_{BC}(\cdot)$: the sum of BC-weight of BC-subtrees in $S_{BC}(\cdot)$.
- $\eta_{BC}(\cdot)$: the number of BC-subtrees in set $S_{BC}(\cdot)$.

Given $T_1 \in S(G; v)$ and $v$ a fixed vertex of $V(G)$, let

$$S_o(T_1) = \{ u | u \in V(T_1) \land d_{T_1}(v, u) \equiv 1(\text{mod } 2) \}$$

and

$$S_e(T_1) = \{ u | u \in V(T_1) \land d_{T_1}(v, u) \equiv 0(\text{mod } 2) \}.$$

Then:

- the $\omega^o_e$ weight of $T_1$, denoted by $\omega^o_e(T_1)$, is:
2.2 Facts

Let \( T = (V(T), E(T); f, g) \) be a weighted tree of order \( n > 1 \) with vertex weight function \( f(u) = (f_0(u), f_1(u)) \) for \( u \in V(T) \) and edge weight function \( g = g(e) \) for \( e \in E(T) \), assume \( \bar{v} \neq v \) be a pendant vertex and \( \bar{e} = (\bar{v}, \bar{u}) \) the pendant edge of \( T \), let

- If \( T_1 \) is a weighted single vertex \( v \), then \( \omega^0_v(T_1) = f_0(v) \);
- otherwise,
  \[
  \omega^0_v(T_1) = (1 + f_0(v)) \prod_{u \in S_v(T_1)} f_e(u) \prod_{u \in L(T_1)} (1 + f_0(u)) \prod_{u \in S_v(T_1) \setminus v} f_e(u) \prod_{e \in E(T_1)} g(e).
  \]

- the \( \omega^0_v \) weight of \( T_1 \), denoted by \( \omega^0_v(T_1) \), is defined as:
  - If \( T_1 \) is a weighted single vertex \( v \), then \( \omega^0_v(T_1) = f_e(v) \);
  - otherwise, \( \omega^0_v(T_1) = \prod_{u \in S_v(T_1)} f_e(u) \prod_{u \in L(T_1)} (1 + f_0(u)) \prod_{u \in S_v(T_1) \setminus v} f_e(u) \prod_{e \in E(T_1)} g(e) \).

The odd, even generating function of \( S(G; v) \) are respectively defined as

\[
F(G; f, g; v, \text{odd}) = \sum_{T_1 \in S(G; v)} \omega^0_v(T_1),
\]

and

\[
F(G; f, g; v, \text{even}) = \sum_{T_1 \in S(G; v)} \omega^0_v(T_1).
\]

Similarly, for a given BC-subtree \( T_2 \) of a weighted graph \( G \), we define

\[
BS_\text{odd}(T_2) = \{u | u \in V(T_2) \land d_{T_2}(u, v_1) \equiv 0 \text{ mod } 2\}
\]

and

\[
BS_\text{even}(T_2) = \{u | u \in V(T_2) \land d_{T_2}(u, v_1) \equiv 1 \text{ mod } 2\}
\]

where \( v_1 \in L(T_2) \).

The BC-weight of \( T_2 \in S_{BC}(G) \) is

\[
\omega_{bc}(T_2) = \prod_{u \in BS_\text{odd}(T_2)} f_e(u) \prod_{e \in E(T_2)} g(e)
\]

and the BC-subtree generating function of \( G \) is

\[
F_{BC}(G; f, g) = \sum_{T \in S_{BC}(G)} \omega_{bc}(T).
\]

Similarly,

\[
F_{BC}(G; f, g; X) = \sum_{T \in S_{BC}(G; X)} \omega_{bc}(T).
\]

Let \( \eta(G; v, \text{odd}) \) (resp. \( \eta(G; v, \text{even}) \)) denote the number of \( v_{\text{odd}} \)-subtrees (resp. \( v_{\text{even}} \)-subtrees) in \( S(G; v, \text{odd}) \) (resp. \( S(G; v, \text{even}) \)). Then

\[
\eta(G; v, \text{odd}) = F(G; (0, 1), 1; v, \text{odd}),
\]

\[
\eta(G; v, \text{even}) = F(G; (0, 1), 1; v, \text{even})
\]

and

\[
\eta_{BC}(G) = F_{BC}(G; (0, 1), 1), \quad \eta_{BC}(G; X) = F_{BC}(G; (0, 1), 1; X)
\]

2.2. Facts

With the above notations we introduce some previously established results that will be used in our arguments.

Let \( T = (V(T), E(T); f, g) \) be a weighted tree of order \( n > 1 \) with vertex weight function \( f(u) = (f_0(u), f_1(u)) \) for \( u \in V(T) \) and edge weight function \( g = g(e) \) for \( e \in E(T) \), assume \( \bar{v} \neq v \) be a pendant vertex and \( \bar{e} = (\bar{v}, \bar{u}) \) the pendant edge of \( T \), let
Let $T' = (V(T'), E(T'); f', g')$ of order $n - 1$ be the weighted tree constructed from $T$ through “contracting” $\emptyset$ as follows: $V(T') = V(T) \setminus \emptyset$, $E(T') = E(T) \setminus \emptyset$, and

$$f'(v_s) = \begin{cases} f_o(u)(1 + g'(e)f_o(\emptyset)) + g(\emptyset)f_o(\emptyset) & \text{if } v_s = u, \\ f_o(v_s) & \text{otherwise.} \end{cases}$$

(1)

$$f'(v_s) = \begin{cases} f_o(u)(1 + g'(e)f_o(\emptyset)) & \text{if } v_s = u, \\ f_o(v_s) & \text{otherwise.} \end{cases}$$

(2)

for any $v_s \in V(T')$, and $g'(e) = g(e)$ for any $e \in E(T')$.

**Lemma 1** ([13]). Following the above notations, we have $F(T; f, g; v, \text{odd}) = F(T'; f', g'; v, \text{odd})$ and $F(T; f, g; v, \text{even}) = F(T'; f', g'; v, \text{even})$.

**Lemma 2** ([13]). Let $P_n$ be a path on $n$ vertices with vertex weight function $f(v) = (0, y)$ for all $v \in V(P_n)$ and edge weight function $g(e) = z$ for all $e \in E(P_n)$. Then,

$$F_{BC}(P_n; f, g) = \sum_{i=1}^{\lceil n/2 \rceil - 1} (n - 2i) y^2 z^2 (yz^2)^{i-1}$$

**Lemma 3** ([13]). Let $K_{1,n}$ be a star on $n + 1$ vertices with vertex weight function $f(v) = (0, y)$ for all $v \in V(K_{1,n})$ and edge weight function $g(e) = z$ for all $e \in E(K_{1,n})$. Then,

$$F_{BC}(K_{1,n}; f, g) = \sum_{i=2}^{n} i y^i z^i$$

A unicyclic graph is a connected graph whose number of edges is equal to the number of vertices.

**Lemma 4** ([41]). Let $U_n = (V(U_n), E(U_n); f, g)$ be a weighted unicyclic graph of order $n$ with no pendant vertices, whose vertex weight function $f(v) = (0, y)$ for all $v \in V(U_n)$ and edge weight function $g(e) = z$ for all $e \in E(U_n)$. Then,

$$F_{BC}(U_n; f, g) = \sum_{i=1}^{\lceil n/2 \rceil - 1} ny^2 z^2 (yz^2)^{i-1}$$

2.3. Observations

We now move on to establish some new observations to facilitate our discussion of the main results later.

Let $T = (V(T), E(T); f, g)$ be a weighted tree on $n \geq 2$ vertices, with the vertex weight function $f(u) = (f_o(u), f_e(u))$ for $u \in V(T)$ and the edge weight function $g = g(e)$ for $e \in E(T)$. Let $T_o$ be a subtree of $T$ and define $T_o^* = (V(T_o^*), E(T_o^*); f^*, g^*)$ the weighted subtree constructed from $T$ through contracting leaves of $T$ recursively with $V(T_o^*) = V(T_o)$ and $E(T_o^*) = E(T_o)$ as follows:

- Choose a pendant vertex $\emptyset \in L(T) \setminus V(T_o)$ and let $e = (\emptyset, a)$ denote the pendant edge;
- Update the odd, even weight of $a$, and edge weight with rule as described in Lemma 1;
- Remove the vertex $\emptyset$, edge $e$ and set $T' = T \setminus \{\emptyset, e\}$;
- Repeat the contracting process until the remaining tree is the weighted tree $T_o^* = (V(T_o^*), E(T_o^*); f^*, g^*)$ with $V(T_o^*) = V(T_o)$, and $E(T_o^*) = E(T_o)$.

From Lemma 1 we have the following two observations as immediate consequences. We skip the repetitive details.
Theorem 1. Given $T_v$ a subtree of $T = (V(T), E(T); f, g)$ containing vertex $v$, and let $T_v^o = (V(T_v^o), E(T_v^o); f^*, g^*)$ be the weighted subtree defined above, then, the odd and even generating functions of $S(T; T_v^o)$, denoted by $F(T; f, g; T_v^o, odd)$ and $F(T; f, g; T_v^o, even)$, respectively, are

$$F(T; f, g; T_v^o, odd) = \left( \prod_{u \in V(T_v^o)} f^*(u) \prod_{u \in V(T_v^o)} (1 + f^*_v(u)) \prod_{u \in L(T_v^o)} f^*_v(u) \right) \prod_{e \in E(T_v^o)} g^*(e),$$

$$F(T; f, g; T_v^o, even) = \prod_{e \in E(T_v^o)} g^*(e) \left( \prod_{u \in V(T_v^o)} f^*_v(u) \prod_{u \in V(T_v^o)} (1 + f^*_v(u)) \prod_{u \in L(T_v^o)} f^*_v(u) \right),$$

where $V_v(T_v^o) = \{ u | u \in V(T_v^o) \land d_{T_v^o}(u, v) \equiv 1 \text{ (mod 2)} \}$, and $V_v(T_v^o) = \{ u | u \in V(T_v^o) \land d_{T_v^o}(u, v) \equiv 0 \text{ (mod 2)} \}$.

Theorem 2. Given $T_v$ a subtree of $T = (V(T), E(T); f, g)$, and let $T_v^o = (V(T_v^o), E(T_v^o); f^*, g^*)$ be the weighted subtree obtained from $T$ defined above, then, the BC-subtree generating function of $T$ containing the subtree $T_v^o$ is

$$F_{BC}(T; f, g; T_v) = \left( f^*_v(v_1) \prod_{u \in V(T_v^o)} f^*_v(u) \prod_{u \in L(T_v^o)} (1 + f^*_v(u)) \prod_{u \in L(T_v^o)} f^*_v(u) \right) \prod_{e \in E(T_v^o)} g^*(e),$$

where $v_1 \in L(T_v^o)$ is a leaf, $V_v(T_v^o) = \{ u | u \in V(T_v^o) \land d_{T_v^o}(u, v_1) \equiv 1 \text{ (mod 2)} \}$, and $V_v(T_v^o) = \{ u | u \in V(T_v^o) \land d_{T_v^o}(u, v_1) \equiv 0 \text{ (mod 2)} \}$.

We now establish the following for general graphs.

Theorem 3. Let graph $G = (V(G), E(G); f, g)$ be obtained from $G_1$ and $G_2$ by identifying the unique common vertex $c$ (see Figure 1), with the vertex weight function $f(v) = (f_1(v), f_2(v))$ for $v \in V(G)$ and the edge weight function $g(e) = z$ for $e \in E(G)$. Then

$$F_{BC}(G; f, g; c) = F_{BC}(G_1; f, g; c) \frac{1}{1 + f_0(c)} \prod_{i=1}^2 F(G_i; f, g; c, odd)$$

$$+ \frac{1}{f_0(c)} \sum_{i=1}^2 (F(G_i; f, g; c, even) - f_0(c)) + F_{BC}(G_2; f, g; c).$$

If the vertex weight function $f(v) = (0, y)$ for $v \in V(G)$, then we have

$$F_{BC}(G; f, g; c) = F_{BC}(G_1; f, g; c) + \sum_{i=1}^2 F(G_i; f, g; c, odd)$$

$$+ \frac{2}{y} \sum_{i=1}^2 (F(G_i; f, g; c, even) - y) + F_{BC}(G_2; f, g; c).$$

**Figure 1.** Weighted graph $G$. 
Theorem 4. Let graph \( G = (V(G), E(G); f, g) \) be obtained from \( G_1 \) and \( G_2 \) with the unique common vertex \( c \) (see Figure 1), with the vertex weight function \( f(v) = (f_o(v), f_e(v)) \) for \( v \in V(G) \) and the edge weight function \( g(e) = z \) for \( e \in E(G) \). Then

\[
F(G; f, g; c, odd) = F(G_1; f, g; c, odd) + F(G_2; f, g; c, odd) + \frac{1}{(1 + f_o(c))} \prod_{i=1}^{2} F(G_i; f, g; c, odd)
\]

and

\[
F(G; f, g; c, even) = \frac{1}{f_e(c)} \prod_{i=1}^{2} F(G_i; f, g; c, even).
\]

If the vertex weight function \( f(v) = (0, y) \) for \( v \in V(G) \), then we have

\[
F(G; f, g; c, odd) = F(G_1; f, g; c, odd) + F(G_2; f, g; c, odd) + \prod_{i=1}^{2} F(G_i; f, g; c, odd)
\]
and

\[ F(G; f, g; c, \text{even}) = \frac{1}{y} \prod_{i=1}^{2} F(G_i; f, g; c, \text{even}). \]  

(11)

2.4. Further Definitions

We now introduce the graph structures under consideration in this paper. Recall that \( K_{1,n} \) is a star on \( n + 1 \) vertices, \( P_n \) is a path on \( n \) vertices, and \( C_n \) is a cycle on \( n \) vertices. A fan graph, denoted by \( F_{n+1} \), is a graph formed by adding an additional vertex adjacent to every vertex of \( P_n \). A wheel graph \( W_{n+1} \) is a graph formed from a cycle \( C_n \) by adding a vertex adjacent to every vertex of \( C_n \).

Assume \( G_1 \) and \( G_2 \) are two disjoint graphs, a graph \( G = G_1 + G_2 \) is called the disjoint union of \( G_1 \) and \( G_2 \), if \( V(G) = V(G_1) \cup V(G_2) \) and \( E(G) = E(G_1) \cup E(G_2) \). Moreover, let the product \( G_1 \times G_2 \) denote the graph obtained from \( G_1 + G_2 \) by adding edges \((a, b)\) with \( a \in V(G_1) \) and \( b \in V(G_2) \). In the special case when \( G_2 \) is a single vertex \( c \), we write \( G_1 \times G_2 \) as \( G_1 \times c \).

**Definition 1.** Let \( P_{l_i} \) (\( l_i \geq 1 \), and \( l_i \neq l_j \), if \( 1 \leq i \neq j \leq k \)) be \( k \) distinct paths on \( l_i \) vertices, and each \( P_{l_i} \) has \( n_i \) copies with \( \sum_{i=1}^{k} n_i l_i = n \), then, the graph \( (n_1 P_{l_1} + n_2 P_{l_2} + \cdots + n_k P_{l_k}) \times c_0 \)

is called a multi-fan graph, where \( n_1 P_{l_1} + n_2 P_{l_2} + \cdots + n_k P_{l_k} \) is the disjoint union of \( \sum_{i=1}^{k} n_i \) paths (\( n_i \) is the number of paths of length \( l_i - 1 \)) and \( c_0 \) is the center vertex (see Figure 2).

Clearly, in the case of \( k = 1 \) and \( n_1 = 1 \), the multi-fan graph is just the fan graph \( F_{n+1} \). For convenience we call the subgraph \( P_{l_i} \times c_0 \) (\( i = 1, 2, \ldots, k \)) of multi-fan graph \( (n_1 P_{l_1} + n_2 P_{l_2} + \cdots + n_k P_{l_k}) \times c_0 \) the sub-fan graph \( F_{l_i+1} \). It is easy to see that the fan graph \( F_1 \) is the single vertex \( c_0 \) for the case \( l_1 = 0 \), and \( F_2 \) is an edge for the case \( l_1 = 1 \).

![Figure 2. The multi-fan graph \( (iP_1 + P_{n_i+1} + \cdots + P_{n_k}) \times c_0 \).](image-url)
Definition 2. Let $C_l$ ($l_i \geq 3$, and $l_i \neq l_j$, if $1 \leq i \neq j \leq k$) be $k$ distinct cycles on $l_i$ vertices. Suppose each $C_l$ has $n_i$ copies with $\sum_{i=1}^{k} n_i l_i = n$. Then, the graph

$$(n_1 C_{l_1} + n_2 C_{l_2} + \cdots + n_k C_{l_k}) \times c_0$$

is called a multi-wheel graph, where $n_1 C_{l_1} + n_2 C_{l_2} + \cdots + n_k C_{l_k}$ is the disjoint union of $\sum_{i=1}^{k} n_i$ cycles ($n_i$ is the number of cycles of length $l_i$) and $c_0$ is the center vertex.

Clearly, in the case of $k = 1$ and $n_1 = 1$, the multi-wheel graph is just the wheel graph $W_{n+1}$. Similarly, we call the subgraph $C_l \times c_0$ ($l = 1, 2, \ldots, k$) of multi-wheel graph $(n_1 C_{l_1} + n_2 C_{l_2} + \cdots + n_k C_{l_k}) \times c_0$ the sub-wheel graph $W_{l+1} + 1$.

Definition 3. Let $G = (n_1 P_{l_1} + n_2 P_{l_2} + \cdots + n_k P_{l_k}) \times c_0$ be the multi-fan graph with $\sum_{i=1}^{k} n_i l_i = n$. Then we also call $G$ the $r$ ($1 \leq r < n, r$ is an integer) multi-fan graph, and is denoted by $F_r^{\ast}$.

If $(n \mod r) \neq 0$, then let $k = 2, l_1 = r, n_1 = \lfloor \frac{n}{r} \rfloor, l_2 = 1, n_2 = (n \mod r)$, and we call $F_{n+1}^r$ the $r$ quasi-regular multi-fan graph.

Otherwise, let $k = 1, l_1 = r, n_1 = \lfloor \frac{n}{r} \rfloor$, and we call $F_{n+1}^r$ the $r$ regular multi-fan graph.

3. BC-Subtree Generating Functions of Multi-Fan Graphs

We now move on to study the BC-subtree generating functions. First we consider the multi-fan graphs in this section.

Theorem 5. Let $G = (n_1 P_{l_1} + n_2 P_{l_2} + \cdots + n_k P_{l_k}) \times c_0$ be a multi-fan graph with center vertex $c_0$, the vertex weight function $f(v) = (0, y)$ for $v \in V(G)$ and the edge weight function $g(e) = z$ for $e \in E(G)$. For convenience we denote $F(F_{i_1+1}; f, g; c_0, \text{even})$ by $F_{i_1+1}^{\text{even}}(c_0)$ and $F(F_{i_1+1}; f, g; c_0, \text{odd})$ by $F_{i_1+1}^{\text{odd}}(c_0)$. Then

$$F_{BC}(G; f, g) = \sum_{i=1}^{n-1} \left(1 + F_{i+1}^{\text{odd}}(c_0)^{n_i} - 1\right) \sum_{s_1 \leq t_1 \leq \cdots \leq s_{k-1} \leq t_{k-1} \leq s_k} \prod_{i=1}^{k} \left(1 + F_{i+1}^{\text{odd}}(c_0)^{n_i} - 1\right)

+ \sum_{i=1}^{n-1} \left(F_{i+1}^{\text{odd}}(c_0) - y \right) \left[ \left(y - F_{i+1}^{\text{odd}}(c_0)\right)^{n_i} - 1\right]

+ \sum_{i=1}^{n-1} \left(F_{i+1}^{\text{odd}}(c_0) - y \right) \left[ \left(y - F_{i+1}^{\text{odd}}(c_0)\right)^{n_i} - 1\right]

+ F_{i+1}^{\text{odd}}(c_0) \sum_{j=1}^{n_i-1} \left(1 + F_{i+1}^{\text{odd}}(c_0)^{j-1} + n_i F_{BC}(F_{i_1+1}; f, g; c_0)\right)

+ \sum_{i=1}^{n-1} \left(1 + F_{i+1}^{\text{odd}}(c_0)^{j-1} - 1\right)\right)
and
\[
F_{BC}(F_{i+1}; f, g; c_0) = F_{BC}(F_{i}; f, g; c_0) + \sum_{j=0}^{\left\lfloor \frac{i-2}{2} \right\rfloor} y^{j+1} z^{2j+1} F_{BC}^{\text{odd}}(c_0) \\
+ \sum_{p=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor} \sum_{q=1}^{\left\lfloor \frac{i-2}{2} \right\rfloor} y^{2p+2q} z F_{BC}^{\text{odd}}(c_0) \\
+ \sum_{p=1}^{\left\lfloor \frac{i}{2} \right\rfloor} \sum_{q=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor} y^{2p+2q} z F_{BC}^{\text{even}}(c_0) \\
+ 2 \sum_{p=1}^{\left\lfloor \frac{i}{2} \right\rfloor} y^{2p} z F_{BC}^{\text{even}}(c_0)
\]

with
\[
F_{BC}^{\text{odd}}(c_0) = y z + (yz + 1) F_{BC}^{\text{odd}}(c_0) + \sum_{k=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor} y^{k+1} z^{2k+1} (1 + F_{BC}^{\text{odd}}(c_0)) \\
+ \sum_{p=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor} \sum_{q=1}^{\left\lfloor \frac{i-2}{2} \right\rfloor} y^{2p+2q} z (1 + F_{BC}^{\text{odd}}(c_0)),
\]
\[
F_{BC}^{\text{even}}(c_0) = \sum_{p=1}^{\left\lfloor \frac{i}{2} \right\rfloor} y^{2p} z F_{BC}^{\text{even}}(c_0) + \sum_{q=1}^{\left\lfloor \frac{i-1}{2} \right\rfloor} y^{2q-1} z F_{BC}^{\text{even}}(c_0) \\
+ \sum_{k=1}^{\left\lfloor \frac{i}{2} \right\rfloor} y^{k+1} z^{2k} F_{BC}^{\text{even}}(c_0) + F_{BC}^{\text{even}}(c_0),
\]

and \(F_{BC}(F_2; f, g; c_0) = 0, F_{BC}^{\text{odd}}(c_0) = 0, F_{BC}^{\text{even}}(c_0) = y\).

**Proof.** First we consider two cases for the BC-subtrees of multi-fan graph \(G = (n_1 P_{l_1} + n_2 P_{l_2} + \cdots + n_k P_{l_k}) \times c_0\):

(i) ones not containing the center \(c_0\);

(ii) ones containing the center \(c_0\).

With Lemma 2, we have the BC-subtree generating function of case (i) as
\[
\sum_{i=1}^{k} (n_i \sum_{j=1}^{\left\lfloor \frac{i-2}{2} \right\rfloor} y^{j+1} z^{2j+1})^{-1}. 
\]

With slightly more complicated structure analysis, Theorems 3 and 4, we have the BC-subtree generating function of case (ii) as
\[
\sum_{i=1}^{k} \left( F((n_i P_{l_i}) \times c_0; f, g; c_0, \text{odd}) \right) \\
+ \sum_{i=1}^{k} \left( F((n_i P_{l_i}) \times c_0; f, g; c_0, \text{even}) - y \right) \\
+ \sum_{i=1}^{k} F_{BC}((n_i P_{l_i}) \times c_0; f, g; c_0).
\]
Similarly, the odd and even generating functions, the BC-subtree generating functions of multi-fan graph \((n_i P_k) \times c_0 (i = 1, 2, \ldots, k)\) containing \(c_0\) are, respectively,
\[
F((n_i P_k) \times c_0; f, g; c_0, \text{odd}) = (1 + F(n_{i+1}; f, g; c_0, \text{odd}))^n - 1,
\]
(18)
\[
F((n_i P_k) \times c_0; f, g; c_0, \text{even}) = y^{1-n} F(n_{i+1}; f, g; c_0, \text{even})^{n},
\]
(19)
\[
F_{BC}((n_i P_k) \times c_0; f, g; c_0) = F(n_{i+1}; f, g; c_0, \text{odd}) \sum_{j=1}^{n-1} [(1 + F(n_{i+1}; f, g; c_0, \text{odd}))^j - 1] + (F(n_{i+1}; f, g; c_0, \text{even}) - y) \sum_{j=1}^{n-1} [y^{-j} F(n_{i+1}; f, g; c_0, \text{even})^j - 1] + n F_{BC}(n_{i+1}; f, g; c_0).
\]
(20)
We now label the non-center vertices of the sub-fan graph \(F_{i+1} = P_l \times c_0 (i = 1, 2, \ldots, k)\) as \(c_1, c_2, \ldots, c_l\) in counterclockwise order. In what follows, we further focus on computing the odd and even generating functions, and the BC-subtree generating functions of \(F_{i+1} (i = 1, 2, \ldots, k)\) containing \(c_0\).

Let \(c_i = (c_0, c_i)\), then we have
\[
S(F_{i+1}; c_0) = S_1 \cup S_2 \cup S_3 \cup S_4
\]
and
\[
S_{BC}(F_{i+1}; c_0) = S_{1*} \cup S_{2*} \cup S_{3*} \cup S_{4*}
\]
where

- \(S_1\) (resp. \(S_{1*}\)) is the set of subtrees (resp. BC-subtrees) that contain \(c_0\), but not \((c_0, c_l)\) or \((c_{l-1}, c_l)\);
- \(S_2\) (resp. \(S_{2*}\)) is the set of subtrees (resp. BC-subtrees) that contain \(c_0\), \((c_0, c_l)\), but not \((c_{l-1}, c_l)\);
- \(S_3\) (resp. \(S_{3*}\)) is the set of subtrees (resp. BC-subtrees) that contain \(c_0\), \((c_{l-1}, c_l)\), but not \((c_0, c_l)\);
- \(S_4\) (resp. \(S_{4*}\)) is the set of subtrees (resp. BC-subtrees) that contain \(c_0\), \((c_0, c_l)\) and \((c_{l-1}, c_l)\).

We now study each case:

(a) \(S_1 = S(F_l; c_0)\);
(b) \(S_2 = \{ T_1 + e_i | T_1 \in S_1 \}\), where \(T_1 + e_i\) is the tree obtained from \(T_1\), by attaching an edge \(e_i\) at vertex \(c_0\);
(c) The set \(S_3\) can be written as
\[
S_3 = \{ T + (c_0, c_{l-k}) \cup \bigcup_{h=2-k}^{r} (c_{l-k-h+1}, c_{l-k-h+2}) | T \in S(F_{i-k-r+1}; c_0) \},
\]
where \(k = 1, 2, \ldots, l_i - 1, r = 1, 2, \ldots, l_i - k\);
(d) Evidently, each \(T_4 \in S_4\) must not contain the edge \((c_0, c_{l-1})\). We further consider these subtrees by cases of containing edges \((c_0, c_l) \cup \bigcup_{r=1}^{k} (c_{l-r}, c_{l-r+1})\) but not \((c_{l-r-1}, c_{l-r})\), for \(k = 1, 2, \ldots, l_i - 1\), which can be rewritten as:
\[
S_4 = \{ T + (c_0, c_l) \cup \bigcup_{h=1}^{k} (c_{l-h}, c_{l-h+1}) | T \in S(F_{k}; c_0) \}
\]
where \(k = 1, 2, \ldots, l_i - 1\). Note that \(S(F_1; c_0)\) is the single vertex \(c_0\).
By the definitions of $\omega_0^*$ weight, $\omega_0^e$ weight, odd, even generating function of subtrees containing a fixed vertex, (a)–(d), and Theorem 1, we have the followings.

$$\sum_{T_1 \in S_1} \omega_0^*(T_1) = F(F_i; f, g; c_0, \text{odd}),$$

$$\sum_{T_1 \in S_1} \omega_0^e(T_1) = F(F_i; f, g; c_0, \text{even}),$$

$$\sum_{T_2 \in S_2} \omega_0^*(T_2) = (1 + \sum_{T_1 \in S_1} \omega_0^*(T_1)) f_e(c_i) g(e_i)$$

$$= y^2 (1 + F(F_i; f, g; c_0, \text{odd})),$$

$$\sum_{T_2 \in S_2} \omega_0^e(T_2) = F(F_i; f, g; c_0, \text{even} f_0(c_i) g(e_i) = 0,$$

$$\sum_{T_3 \in S_3} \omega_0^*(T_3) = \sum_{k=1}^{l-1} \left[ f_e(c_{l-k}) f_e(c_k) \frac{1-1+2}{1} \prod_{s=1}^{l-1} \left( 1 + f_e(c_{l-k-2s}) f_e(c_{l-k-2s}) \right) \right]$$

$$\prod_{s=0}^{l-1} g((c_{l-k+s}, c_{l-k+s+1})) \left[ \prod_{s=1}^{l-1} \left( 1 + f_e(c_{l-k-2s}) f_e(c_{l-k-2s}) \right) f_e(c_{l-k-r}) \frac{1-1+2}{1} \right]$$

$$g((c_0, c_{l-k})) (1 + F(F_i; f, g; c_0, \text{odd}))$$

$$= \sum_{p=1}^{l-1} \left( \sum_{q=1}^{l-1} z^{2p+2q-1} y^q (1 + F(F_i; 2p-2q+2; f, g; c_0, \text{odd})) \right),$$

$$\sum_{T_3 \in S_3} \omega_0^e(T_3) = \sum_{k=1}^{l-1} g((c_0, c_{l-k})) \left[ f_e(c_k) \frac{1-1+2}{1} \prod_{s=0}^{l-1} \left( 1 + f_e(c_{l-k-2s}) f_e(c_{l-k-2s}) \right) \right]$$

$$\prod_{s=0}^{l-1} g((c_{l-k+s}, c_{l-k+s+1})) \left[ F(F_i; f, g; c_0, \text{even}) + f_e(c_{l-k-1}) \sum_{r=1}^{l-1} F(F_i; f, g; c_0, \text{even}) f_e(c_{l-k-r}) \frac{1-1+2}{1} \right]$$

$$\left( \prod_{s=0}^{l-1} g((c_{l-k+s}, c_{l-k+s+1})) \prod_{s=1}^{l-1} \left( 1 + f_e(c_{l-k-s}) f_e(c_{l-k-s}) \right) \right)$$

$$= \sum_{p=1}^{l-1} z^{2p} y^p \left( F(F_i; 2p+1; f, g; c_0, \text{even}) + \sum_{q=1}^{l-1} z^{2q+2} y^q \right) F(F_i; 2p-2q+2; f, g; c_0, \text{even}),$$

$$\sum_{T_4 \in S_4} \omega_0^*(T_4) = f_e(c_0) g(c_i) \sum_{k=1}^{l-1} \left[ f_e(c_{l-2k+1}) \prod_{r=1}^{l-1} \left( 1 + f_e(c_{l-2k+2}) \right) \right]$$

$$\prod_{s=0}^{l-1} g((c_{l-s}, c_{l-s+1})) \left( 1 + F(F_i; 2k+1; f, g; c_0, \text{odd}) \right)$$

$$+ \sum_{k=1}^{l-1} \left[ \prod_{s=0}^{l-1} \left( 1 + f_e(c_{l-2s}) f_e(c_{l-2s}) \right) \prod_{s=1}^{l-1} g((c_{l-s}, c_{l-s+1})) \right]$$

$$f_e(c_i) g(c_i) \left( 1 + F(F_i; 2k; f, g; c_0, \text{odd}) \right)$$

$$= \sum_{k=1}^{l-1} z^{2k+1} y^{k+1} (1 + F(F_i; 2k; f, g; c_0, \text{odd})),.$$
with \( F(T_1; f, g; c_0, \text{odd}) = 0 \), \( F(T_1; f, g; c_0, \text{even}) = y \).

Hence, by Equations (21)–(28) we have

\[
\begin{align*}
F(T_{i+1}; f, g; c_0, \text{even}) &= \sum_{T_1 \in \mathcal{S}_1} \omega^c_0(T_1) + \sum_{T_2 \in \mathcal{S}_2} \omega^c_0(T_2) + \sum_{T_3 \in \mathcal{S}_3} \omega^c_0(T_3) + \sum_{T_4 \in \mathcal{S}_4} \omega^c_0(T_4) \\
&= \sum_{p=1}^{\lfloor \frac{i}{2} \rfloor} z^{2p} y^p \left( F(T_{i-2p+1}; f, g; c_0, \text{even}) \right) + \\
&\quad \sum_{q=1}^{\lfloor \frac{i}{2} \rfloor} z^{2q-1} y^q \left( F(T_{i-2q+2}; f, g; c_0, \text{even}) \right) + \\
&\quad \sum_{k=1}^{\lfloor \frac{i}{2} \rfloor} y^k z^{2k} F(T_{i-2k+1}; f, g; c_0, \text{even}) \\
&\quad + F(T_1; f, g; c_0, \text{even})
\end{align*}
\]

with \( F(T_1; f, g; c_0, \text{odd}) = 0 \), \( F(T_1; f, g; c_0, \text{even}) = y \).

Similarly for the BC-subtrees, through case analysis of BC-subtree generating function, BC-weight \( \omega_{bc}(T) \) of \( T \in \mathcal{S}_{BC}(G) \), and by Theorem 2, we have

\[
\begin{align*}
\sum_{T_1 \in \mathcal{S}_1} \omega_{bc}(T_1) &= F_{BC}(T_1; f, g; c_0), \\
\sum_{T_2 \in \mathcal{S}_2} \omega_{bc}(T_2) &= yz F(T_1; f, g; c_0, \text{odd}),
\end{align*}
\]
\[
\sum_{T^*_3 \in S_3^*} \omega_{bc}(T^*_3) = \sum_{p=1}^{\frac{\lfloor \frac{n}{2} \rfloor}{2}} \sum_{q=1}^{\lfloor \frac{n}{2} \rfloor} z^{2p+2q-1}y^{q+p}F(F_{l-2p-2q+2}; f, g; c_0, \text{odd}) + \sum_{p=1}^{\frac{\lfloor \frac{n}{2} \rfloor}{2}} \sum_{q=1}^{\lfloor \frac{n}{2} \rfloor} z^{2p+2q-1}y^{q+p}F(F_{l-2p-2q+2}; f, g; c_0, \text{even}) \\
+ \sum_{p=1}^{\frac{\lfloor \frac{n}{2} \rfloor}{2}} z^{2p}y^pF(F_{l-2p+1}; f, g; c_0, \text{even}),
\]

and
\[
\sum_{T^*_4 \in S_4^*} \omega_{bc}(T^*_4) = \sum_{s=1}^{\lfloor \frac{n-1}{2} \rfloor} y^s z^{2s+1}F(F_{l-2s}; f, g; c_0, \text{odd}) + \sum_{s=1}^{\lfloor \frac{n-1}{2} \rfloor} y^s z^{2s+1}F(F_{l-2s+1}; f, g; c_0, \text{even}),
\]

Combining the Equations (31)–(34), we have
\[
F_{BC}(F_{l+1}; f, g; c_0) = \sum_{T^*_1 \in S_1^*} \omega_{bc}(T^*_1) + \sum_{T^*_2 \in S_2^*} \omega_{bc}(T^*_2) + \sum_{T^*_3 \in S_3^*} \omega_{bc}(T^*_3) + \sum_{T^*_4 \in S_4^*} \omega_{bc}(T^*_4)
\]
\[
= F_{BC}(F_l; f, g; c_0) + \sum_{s=0}^{\lfloor \frac{n-1}{2} \rfloor} y^s z^{2s+1}F(F_{l-2s}; f, g; c_0, \text{odd}) + \sum_{s=1}^{\lfloor \frac{n-1}{2} \rfloor} y^s z^{2s+1}F(F_{l-2s+1}; f, g; c_0, \text{even}) + 2 \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} z^{2p}y^pF(F_{l-2p+1}; f, g; c_0, \text{even})
\]

with \( F_{BC}(F_2; f, g; c_0) = 0, F(F_{l+1}; f, g; c_0, \text{odd}), F(F_{l+1}; f, g; c_0, \text{even}) \) given in Equations (29) and (30), respectively, and \( F(F_l; f, g; c_0, \text{odd}) = 0, F(F_l; f, g; c_0, \text{even}) = y \).

Combining Equations (16)–(20), (29), (30), and (35), we have the BC-subtree generating function of multi-fan graph \( G = (n_1P_{l_1} + n_2P_{l_2} + \cdots + n_kP_{l_k}) \times c_0 \) as claimed in (12).

Next we consider the BC-subtree generating function of some special cases of multi-fan graphs. First of all the BC-subtree generating function of the \( r \) multi-fan graph \( F_{n+1}^r \) (1 \( \leq r \leq n \), \( r \) is an integer) follows from Theorem 5.

**Theorem 6.** Let \( F_{n+1}^r \) (1 \( \leq r \leq n \) is a positive integer) be the \( r \) multi-fan graph defined in Definition 3 with vertex weight function \( f(v) = (0, y) \) for \( v \in V(F_{n+1}^r) \) and the edge weight function \( g(e) = z \) for \( e \in E(F_{n+1}^r) \), then
\[
F_{BC}(F_{n+1}^r; f, g) = \left( \frac{n}{r} \right) (F_{BC}(F_{r+1}; f, g; c_0) - r^{\text{odd}}(c_0) - r^{\text{even}}(c_0) + \sum_{j=1}^{\lfloor \frac{r}{2} \rfloor} (r-2j)y^{j+1}z^{2j})
\]
\[
+ (1 + yz)^{n-r} + (1 - F^{\text{odd}}(c_0)) + y^{\lfloor \frac{r}{2} \rfloor} - (n - r \frac{n}{r})y + (\frac{n}{r} - 1)y - 1
\]
with

\[ F_{BC}(F_{r+1}; f, g; c_0) = F_{BC}(F_r; f, g; c_0) + \sum_{s=0}^{\frac{\lfloor \frac{r}{2} \rfloor - 1}{2}} y^{s+1} z^{2s+1} F_{r+2s}^{\text{odd}}(c_0) \]

\[ + \sum_{p=1}^{\frac{\lfloor \frac{r}{2} \rfloor - 1}{2}} \sum_{q=1}^{\lfloor \frac{r}{2} \rfloor - p} z^{2p+2q-1} y^{p} F_{r-2p-2q+2}^{\text{odd}}(c_0) \]

\[ + \sum_{p=1}^{\frac{\lfloor \frac{r}{2} \rfloor - 1}{2}} \sum_{q=1}^{\lfloor \frac{r}{2} \rfloor - p} z^{2p+2q-1} y^{p} F_{r-2p-2q+2}^{\text{even}}(c_0) \]

\[ + 2 \sum_{p=1}^{\frac{\lfloor \frac{r}{2} \rfloor - 1}{2}} z^{2p} y^{p} F_{r-2p+1}^{\text{even}}(c_0), \] (37)

\[ F_{r+1}^{\text{odd}}(c_0) = yz + (yz + 1) F_{r}^{\text{odd}}(c_0) + \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor - 1} y^{k+1} z^{2k+1} (1 + F_{r-2k}^{\text{odd}}(c_0)) \]

\[ + \sum_{p=1}^{\frac{\lfloor \frac{r}{2} \rfloor - 1}{2}} \sum_{q=1}^{\lfloor \frac{r}{2} \rfloor - p} z^{2p+2q-1} y^{p} (1 + F_{r-2p-2q+2}^{\text{odd}}(c_0)), \] (38)

\[ F_{r+1}^{\text{even}}(c_0) = \sum_{p=1}^{\frac{\lfloor \frac{r}{2} \rfloor - 1}{2}} z^{2p} y^{p} \left( F_{r-2p+1}^{\text{even}}(c_0) + \sum_{q=1}^{\lfloor \frac{r}{2} \rfloor - p} z^{2q-1} y^{q} F_{r-2p-2q+2}^{\text{even}}(c_0) \right) \]

\[ + \sum_{k=1}^{\lfloor \frac{r}{2} \rfloor - 1} y^{k} z^{2k} F_{r-2k+1}^{\text{even}}(c_0) + F_{r}^{\text{even}}(c_0), \] (39)

and \( F_{BC}(F_2; f, g; c_0) = 0, F_{F_1}^{\text{odd}}(c_0) = 0, F_{F_1}^{\text{even}}(c_0) = y. \)

Letting \( y = z = 1 \) in the above statements we immediately have the following two consequences.

**Corollary 1.** Let \( G = (n_1P_1 + n_2P_2 + \cdots + n_kP_k) \times c_0 \) be a multi-fan graph, then the BC-subtree number of \( G \) is

\[ \eta_{BC}(G) = \sum_{i=1}^{k_1} \left( (1 + a(i_{i+1}))^{n_{i+1}} - 1 \right) \sum_{j=1}^{k_1} \left[ \prod_{i=1}^{i_{i+1}} (1 + a(i_{i+1})^{n_{i+1}} - 1) \right] \]

\[ + \sum_{i=1}^{k_1} \left( b(i_{i+1}) - 1 \right) \left( \prod_{i=1}^{k_1} b(i_{i+1})^{n_{i+1}} - 1 \right) + \sum_{i=1}^{k_1} \left( b(i_{i+1}) - 1 \right) \sum_{j=1}^{n_{i+1}} [b(i_{i+1})^{n_{i+1}} - 1] \]

\[ + a(i_{i+1}) \sum_{j=1}^{n_{i+1}} [(1 + a(i_{i+1}))^{n_{i+1}} - 1] + n_{i+1} \eta_{BC}(F_{i_{i+1}}; c_0) \]

\[ + \sum_{i=1}^{k_1} \sum_{j=1}^{n_{i+1}} n_{i+1}(l_i - 2j) \] (40)

with

\[ \eta_{BC}(F_{r+1}; c_0) = \eta_{BC}(F_r; c_0) + \sum_{p=1}^{\lfloor \frac{r-1}{2} \rfloor} \sum_{q=1}^{\lfloor \frac{r-2p}{2} \rfloor} a(i_{r-2p-2q+2}) + \sum_{s=0}^{\lfloor \frac{r-1}{2} \rfloor} a(i_{r-2s}) \]

\[ + \sum_{p=1}^{\lfloor \frac{r}{2} \rfloor} \sum_{q=1}^{\lfloor \frac{r-2p}{2} \rfloor} b(i_{r-2p-2q+2}) + 2 \sum_{p=1}^{\lfloor \frac{r}{2} \rfloor} b(i_{r-2p+1}), \] (41)
Let $G = \left(n_1 C_{i_1} + n_2 C_{i_2} + \cdots + n_k C_{i_k}\right) \times c_0$ be a multi-wheel graph with center vertex $c_0$, the vertex weight function $f(v) = (0, y)$ for $v \in V(G)$ and the edge weight function $g(e) =
abla
z for \( e \in E(G) \). Let \( F(W_{l+1}; f, g; c_0, \text{odd}) \) be denoted by \( F_{W_{l+1}}^{\text{odd}}(c_0) \), \( F(W_{l+1}; f, g; c_0, \text{even}) \) by \( F_{W_{l+1}}^{\text{even}}(c_0) \), \( F(F_{l+1}; f, g; c_0, \text{odd}) \) by \( F_{F_{l+1}}^{\text{odd}}(c_0) \) and \( F(F_{l+1}; f, g; c_0, \text{even}) \) by \( F_{F_{l+1}}^{\text{even}}(c_0) \). Then

\[
F_{BC}(G; f, g) = \sum_{i=1}^{k-1} \left( (1 + F_{W_{l+1}}^{\text{odd}}(c_0)) \right)^{n_i} - 1 \right] \\
+ \sum_{i=1}^{k-1} \left( (1 + F_{W_{l+1}}^{\text{even}}(c_0)) \right)^{n_i} - 1 \right] \\
+ \sum_{i=1}^{k-1} \left( y^{-i} \left( F_{W_{l+1}}^{\text{even}}(c_0) \right)^{n_i} - 1 \right]
\]

with

\[
F_{W_{l+1}}^{\text{odd}}(c_0) = 2 \sum_{k=0}^{\lfloor \frac{k-1}{2} \rfloor} \sum_{p=0}^{k} \left( \sum_{q=0}^{\lfloor \frac{k-2p}{2} \rfloor} z^{p+2q-1}y^{q+r} (1 + F_{F_{l+2p-2q+2}}^{\text{odd}}(c_0)) \right)
\]

and

\[
F_{W_{l+1}}^{\text{even}}(c_0) = 2 \sum_{k=0}^{\lfloor \frac{k}{2} \rfloor} \sum_{p=0}^{k} \left( \sum_{q=0}^{\lfloor \frac{k-2p}{2} \rfloor} z^{p+2q-1}y^{q+r} (1 + F_{F_{l+2p-2q+2}}^{\text{odd}}(c_0)) \right) + \sum_{p=0}^{\lfloor \frac{k}{2} \rfloor} \left( y^{k+1}z^{2k-1} (1 + F_{F_{l+2p}}^{\text{odd}}(c_0)) + (yz + 1)F_{F_{l+2p}}^{\text{odd}}(c_0) + yz, \right)
\]

with \( F_{BC}(F_{l+1}; f, g; c_0), F_{F_{l+1}}^{\text{odd}}(c_0), F_{F_{l+1}}^{\text{even}}(c_0) \) as in Equations (13)–(15), respectively, and \( F_{BC}(F_{2}; f, g; c_0) = 0, F_{F_{1}}^{\text{odd}}(c_0) = 0, F_{F_{1}}^{\text{even}}(c_0) = y. \)

**Proof.** Firstly, consider the BC-subtrees of multi-wheel graph \( G = (n_1C_1 + n_2C_2 + \cdots + n_kC_k) \times c_0 \) in two cases:
(i) ones not containing the center \( c_0 \);
(ii) ones containing the center \( c_0 \).

From Lemma 4, we have the BC-subtree generating function for case (i) as

\[
\sum_{i=1}^{k} \left( \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor - 1} l_i y^j z^2 (yz^2)^{j-1} \right) .
\]  

(53)

Similar to the previous section, we have the BC-subtree generating function of case (ii) as

\[
\sum_{i=1}^{k} F((n_i C_1) \times c_0; f, g; c_0, \text{odd}) \sum_{i=1}^{k} \left[ \sum_{j=1}^{\lfloor \frac{k}{2} \rfloor - 1} I_{i} F((n_s C_1) \times c_0; f, g; c_0, \text{odd}) \right] \\
+ \sum_{i=1}^{k} \left( F((n_i C_1) \times c_0; f, g; c_0, \text{even}) - y \right) \left[ y^{j-k} \prod_{s=1}^{i} F((n_s C_1) \times c_0; f, g; c_0, \text{even}) - 1 \right] \\
+ \sum_{i=1}^{k} F_{BC}(n_i C_1) \times c_0; f, g; c_0)
\]

where

\[
F((n_i C_1) \times c_0; f, g; c_0, \text{odd}) = (1 + F(W_{i+1}; f, g; c_0, \text{odd}))^{n_i} - 1,
\]

(55)

\[
F((n_i C_1) \times c_0; f, g; c_0, \text{even}) = y^{1-n_i} F(W_{i+1}; f, g; c_0, \text{even})^{n_i},
\]

(56)

\[
F_{BC}(n_i C_1) \times c_0; f, g; c_0) = F(W_{i+1}; f, g; c_0, \text{odd}) \sum_{i=1}^{n_i-1} [(1 + F(W_{i+1}; f, g; c_0, \text{odd}))^{j} - 1] \\
+ (F(W_{i+1}; f, g; c_0, \text{even}) - y) \sum_{j=1}^{n_i-1} [y^{j-k}F(W_{i+1}; f, g; c_0, \text{even})^{j} - 1] \\
+ n_i F_{BC}(W_{i+1}; f, g; c_0).
\]

(57)

Now label the \( l_i \) non-center vertices of wheel graph \( W_{i+1} = C_i \times c_0 \) \((i = 1, 2, \ldots, k)\) with \( c_1, c_2, \ldots, c_{l_i} \). For convenience we let \( e_t = (c_0, c_t) \) \((t = 1, 2, \ldots, l_i)\), \( e_t^{*} = (c_1, c_t) \) and \( e_{l_i-r}^{*} = (c_{l_i-r}, c_{l_i-r+1}) \) \((r = 1, 2, \ldots, l_i - 1)\). We partition the set of \( S(W_{i+1}; c_0) \) and \( S_{BC}(W_{i+1}; c_0) \) into five cases:

\[
S(W_{i+1}; c_0) = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5
\]

\[
S_{BC}(W_{i+1}; c_0) = S_1^{*} \cup S_2^{*} \cup S_3^{*} \cup S_4^{*} \cup S_5^{*}
\]

where

- \( S_1 \) (resp. \( S_1^{*} \)) is the set of subtrees (resp. BC-subtrees) that contain \( c_0 \), but not \((c_1, c_l)\);
- \( S_2 \) (resp. \( S_2^{*} \)) is the set of subtrees (resp. BC-subtrees) that contains \( c_0 \) and \((c_1, c_l)\), but neither \((c_0, c_l)\) nor \((c_{l-1}, c_l)\);
- \( S_3 \) (resp. \( S_3^{*} \)) is the set of subtrees (resp. BC-subtrees) that contain \( c_0 \), \((c_0, c_l)\) and \((c_1, c_l)\), but not \((c_{l-1}, c_l)\);
- \( S_4 \) (resp. \( S_4^{*} \)) is the set of subtrees (resp. BC-subtrees) that contain \( c_0 \), \((c_0, c_l)\), \((c_{l-1}, c_l)\) and \((c_1, c_l)\);
- \( S_5 \) (resp. \( S_5^{*} \)) is the set of subtrees (resp. BC-subtrees) that contain \( c_0 \), \((c_{l-1}, c_l)\) and \((c_1, c_l)\), but not \((c_0, c_l)\).

Studying each cases, we have:

(a) \( S_1 = S(F_{i+1}; c_0) \);
(b) \( \mathcal{S}_2 = S(W_{l+1} \setminus \{e_{l-1}^* \cup e_{l}; e_l^* \} \). Note that \( W_{l+1} \setminus e_{l-1}^* \setminus F_{l+1} \), thus we have \( \mathcal{S}_2 \cong \mathcal{S}_3 \), where \( S_3(S(F_{l+1}; c_0)) \);

(c) \( \mathcal{S}_3 = S(W_{l+1} \setminus e_{l-1}^* \cup e_l^* \} \). Similarly, \( \mathcal{S}_3 \cong \mathcal{S}_4 \), where \( S_4(S(F_{l+1}; c_0)) \);

(d) We can further divide the subtree set \( \mathcal{S}_4 \) into cases of subtrees containing edge set \( \{e_i, k, \overline{e_{i-r}} \} \), but not \( \{e_{i-k-1}, k, e_i \} \) (recursively for \( k = 1, 2, \ldots, l - 2 \)). That is,

\[
\mathcal{S}_4 = \{ T | T \in S(W_{l+1} \setminus (e_{l-1}^* \cup e_l^*); e_i \cup e_i^*) \};
\]

where \( k = 1, 2, \ldots, l - 2 \);

(e) \( \mathcal{S}_5 \) is the subtree set of \( S(W_{l+1} \setminus e_{l}; c_0 \cup \bigcup_{0} e_{l-1}^*) \). Denote by \( \tilde{W}_{l+1}^{l-k} (k = 0, 1, \ldots, l - 2) \) the graph after removing edge set \( \bigcup_{r=1} e_i \setminus e_i^* \) from \( W_{l+1} \). Thus \( \tilde{W}_{l+1}^{l-k} = W_{l+1} \setminus \bigcup_{r=1} e_i \setminus e_i^* \).

2) the graph after removing edge set \( \bigcup_{r=1} e_i \setminus e_i^* \) from \( W_{l+1} \). Thus \( \tilde{W}_{l+1}^{l-k} = W_{l+1} \setminus \bigcup_{r=1} e_i \setminus e_i^* \) and evidently \( \tilde{W}_{l+1}^{l-k} = W_{l+1} \).

Similar to the previous case, we partition the set of \( S(W_{l+1} \setminus e_{l}; c_0 \cup \bigcup_{0} e_{l-1}^*) \) and \( S_{BC} \)

\( \tilde{W}_{l+1}^{l-k}; c_0 \cup \bigcup_{0} e_{l-1}^* \) into four cases:

\[
S(W_{l+1} \setminus e_{l}; c_0 \cup \bigcup_{0} e_{l-1}^*) = S_{l-k,1,1} \cup S_{l-k,2,2} \cup S_{l-k,3,3} \cup S_{l-k,4,4}
\]

where

- \( S_{l-k,1} \) (resp. \( S_{l-k,1}^* \)) is the set of subtrees (resp. BC-subtrees) that contain \( c_0 \cup k \bigcup_{r=0} e_{l-1}^* \), but not \( e_{l-k}^* \) or \( e_{l-k-1,1}^* \);

- \( S_{l-k,2} \) (resp. \( S_{l-k,2}^* \)) is the set of subtrees (resp. BC-subtrees) that contain \( c_0 \cup k \bigcup_{r=0} e_{l-1}^* \) and \( e_{l-k}^* \), but not \( e_{l-k-1,1}^* \);

- \( S_{l-k,3} \) (resp. \( S_{l-k,3}^* \)) is the set of subtrees (resp. BC-subtrees) that contain \( c_0 \cup k \bigcup_{r=0} e_{l-1}^* \) \( e_{l-k}^* \) and \( e_{l-k-1,1}^* \);

- \( S_{l-k,4} \) (resp. \( S_{l-k,4}^* \)) is the set of subtrees (resp. BC-subtrees) that contain \( c_0 \cup k \bigcup_{r=0} e_{l-1}^* \) and \( e_{l-k-1,1}^* \), but not \( e_{l-k}^* \).

Again, we have that \( \tilde{W}_{l+1}^{l-k} \setminus (e_{l-k} \cup e_{l-k-1}^*) \) \( (k = 0, 1, \ldots, l - 2) \) is the graph obtained from \( W_{l+1} \) after removing edge set \( \bigcup_{r=1} e_{l-k-1} \). Clearly, \( \tilde{W}_{l+1}^{l-k} \setminus (e_{l-k} \cup e_{l-k-1}^*) \cong F_{l+1} \setminus \bigcup_{r=0} (c_0; e_i \setminus e_i^* \). \)

We also have

\[
\sum_{T_2 \in \mathcal{S}_{l-k,2,2}} \omega_{e_i}^c(T_2) + \sum_{T_3 \in \mathcal{S}_{l-k,3,3}} \omega_{e_i}^c(T_3) = \sum_{T_1 \in \mathcal{S}_{l-k,1,1}} \omega_{e_i}^c(T_1), \tag{58}
\]

\[
\sum_{T_2 \in \mathcal{S}_{l-k,2,2}} \omega_{e_i}^c(T_2) + \sum_{T_3 \in \mathcal{S}_{l-k,3,3}} \omega_{e_i}^c(T_3) = \sum_{T_1 \in \mathcal{S}_{l-k,1,1}} \omega_{e_i}^c(T_1), \tag{59}
\]
and
\begin{align}
\sum_{T \in S(\tilde{S}_{l_1-1}^{l-1} \cup \bigcup_{r=1}^{l_1} S_{r-1}^{l_1-1})} \omega_0^c(T) &= 2 \sum_{T_1 \in S_{l_1-1}} \omega_0^c(T_1) + \sum_{T \in S(\tilde{S}_{l_1-1}^{l-1} \cup \bigcup_{r=1}^{l_1} S_{r-1}^{l_1-1})} \omega_0^c(T), \\
\sum_{T \in S(\tilde{S}_{l_1-1}^{l-1} \cup \bigcup_{r=1}^{l_1} S_{r-1}^{l_1-1})} \omega_1^c(T) &= 2 \sum_{T_1 \in S_{l_1-1}} \omega_1^c(T_1) + \sum_{T \in S(\tilde{S}_{l_1-1}^{l-1} \cup \bigcup_{r=1}^{l_1} S_{r-1}^{l_1-1})} \omega_1^c(T),
\end{align}
\hspace{1cm} (60)
\hspace{1cm} (61)

From the cases (a)–(e), Theorem 1, and by the definitions of \( \omega_0^c \) weight, \( \omega_0^e \) weight, odd, even generating function of subtrees containing a fixed vertex, we have
\begin{align}
\sum_{T_1 \in S_1} \omega_0^c(T_1) &= F(F_{l_1+1}; f, g; c_0, \text{odd}), \hspace{2cm} (62) \\
\sum_{T_1 \in S_1} \omega_0^e(T_1) &= F(F_{l_1+1}; f, g; c_0, \text{even}), \hspace{2cm} (63) \\
\sum_{T_3 \in S_3} \omega_0^c(T_3) + \sum_{T_4 \in S_4} \omega_0^c(T_4) &= \sum_{T_2 \in S_2} \omega_0^c(T_2) = \sum_{T \in S_{l_1-1}} \omega_0^c(T), \hspace{2cm} (64)
\end{align}
\hspace{1cm} (65)

where \( k = 0, 1, \ldots, l_1 - 2; \)
\begin{align}
\sum_{T_3 \in S_3} \omega_0^c(T_3) + \sum_{T_4 \in S_4} \omega_0^c(T_4) &= \sum_{T_2 \in S_2} \omega_0^c(T_2) = \sum_{T \in S_{l_1-1}} \omega_0^c(T), \hspace{2cm} (66)
\end{align}
\hspace{1cm} (67)

where \( k = 0, 1, \ldots, l_1 - 2. \)
Thus, by Equations (29) and (30) and (62)–(67) we have
\[
F(W_{l+1}; f, g; c_0, \text{odd}) = \sum_{T_1 \in S_1} \omega_{bc}(T_1) + \sum_{T_2 \in S_2} \omega_{bc}(T_2) + \sum_{T_3 \in S_3} \omega_{bc}(T_3) + \sum_{T_4 \in S_4} \omega_{bc}(T_4)
\]
\[
= 2 \sum_{k=0}^{l-2} \sum_{p=\left\lfloor \frac{k}{2} \right\rfloor}^{\left\lceil \frac{k+1}{2} \right\rceil} \left( \sum_{q=1}^{\left\lfloor \frac{k}{2} \right\rfloor} 2^{p+2q-1} g^{q+p} \left( 1 + F(F_{l-2p-2q+2}; f, g, c_0, \text{odd}) \right) \right)
\]
\[
+ \sum_{p=1}^{\left\lceil \frac{k}{2} \right\rceil} \left( \sum_{q=1}^{\left\lfloor \frac{k}{2} \right\rfloor} 2^{p+2q-1} g^{q+p} \left( 1 + F(F_{l-2p-2q+2}; f, g, c_0, \text{odd}) \right) \right)
\]
\[
+ \sum_{k=1}^{\left\lceil \frac{1}{2} \right\rceil} y^{k+1} 2^{k+1} \left( 1 + F(F_{l-2k}; f, g, c_0, \text{odd}) \right)
\]
\[
+ (yz + 1) F(F_l; f, g, c_0, \text{odd}) + yz.
\]
and
\[
F(W_{l+1}; f, g; c_0, \text{even}) = \sum_{T_1 \in S_1} \omega_{bc}(T_1) + \sum_{T_2 \in S_2} \omega_{bc}(T_2) + \sum_{T_3 \in S_3} \omega_{bc}(T_3) + \sum_{T_4 \in S_4} \omega_{bc}(T_4)
\]
\[
= 2 \sum_{k=0}^{l-2} \sum_{p=\left\lfloor \frac{k}{2} \right\rfloor}^{\left\lceil \frac{k+1}{2} \right\rceil} 2^{p+q} g^p \left( F(F_{l-2p+1}; f, g, c_0, \text{even}) + \right.
\]
\[
\left. + \sum_{q=1}^{\left\lfloor \frac{k}{2} \right\rfloor} 2^{q-1} g^q \left( F(F_{l-2p-2q+1}; f, g, c_0, \text{even}) \right) \right)
\]
\[
+ \sum_{p=1}^{\left\lceil \frac{k}{2} \right\rceil} 2^{p+q} g^p \left( F(F_{l-2p+1}; f, g, c_0, \text{even}) + \right.
\]
\[
\left. + \sum_{q=1}^{\left\lfloor \frac{k}{2} \right\rfloor} 2^{q-1} g^q \left( F(F_{l-2p-2q+1}; f, g, c_0, \text{even}) \right) \right)
\]
\[
+ \sum_{k=1}^{\left\lceil \frac{1}{2} \right\rceil} y^k 2^{k} F(F_{l-2k+1}; f, g, c_0, \text{even})
\]
\[
+ F(F_l; f, g, c_0, \text{even}).
\]

with \( F(F_{l+1}; f, g; c_0, \text{odd}), F(F_{l+1}; f, g; c_0, \text{even}) \) as in Equations (29) and (30), respectively, and \( F(F_l; f, g; c_0, \text{odd}) = 0, F(F_l; f, g; c_0, \text{even}) = y \).

Through similar reasoning, we have
\[
\sum_{T_1 \in S_1} \omega_{bc}(T_1) = F_{BC}(W_{l+1}; f, g; c_0),
\]
\[
\sum_{T_3 \in S_3} \omega_{bc}(T_3) + \sum_{T_4 \in S_4} \omega_{bc}(T_4) = \sum_{T_2 \in S_2} \omega_{bc}(T_2) = \sum_{T_1 \in S_1} \omega_{bc}(T_1),
\]
\[
\sum_{T_1 \in S_1} \omega_{bc}(T_1) = \sum_{T_2 \in S_2} \omega_{bc}(T_2) = \sum_{T_3 \in S_3} \omega_{bc}(T_3) = \sum_{T_4 \in S_4} \omega_{bc}(T_4),
\]
\[
\sum_{T_1 \in S_1} \omega_{bc}(T_1) = \sum_{T_2 \in S_2} \omega_{bc}(T_2) = \sum_{T_3 \in S_3} \omega_{bc}(T_3) = \sum_{T_4 \in S_4} \omega_{bc}(T_4),
\]
\[
\sum_{T_1 \in S_1} \omega_{bc}(T_1) = \sum_{T_2 \in S_2} \omega_{bc}(T_2) = \sum_{T_3 \in S_3} \omega_{bc}(T_3) = \sum_{T_4 \in S_4} \omega_{bc}(T_4),
\]
\[
\sum_{T_1 \in S_1} \omega_{bc}(T_1) = \sum_{T_2 \in S_2} \omega_{bc}(T_2) = \sum_{T_3 \in S_3} \omega_{bc}(T_3) = \sum_{T_4 \in S_4} \omega_{bc}(T_4),
\]
\[
\sum_{T_1 \in S_1} \omega_{bc}(T_1) = \sum_{T_2 \in S_2} \omega_{bc}(T_2) = \sum_{T_3 \in S_3} \omega_{bc}(T_3) = \sum_{T_4 \in S_4} \omega_{bc}(T_4),
\]
\[
\sum_{T_1 \in S_1} \omega_{bc}(T_1) = \sum_{T_2 \in S_2} \omega_{bc}(T_2) = \sum_{T_3 \in S_3} \omega_{bc}(T_3) = \sum_{T_4 \in S_4} \omega_{bc}(T_4),
\]
\[
\sum_{T_1 \in S_1} \omega_{bc}(T_1) = \sum_{T_2 \in S_2} \omega_{bc}(T_2) = \sum_{T_3 \in S_3} \omega_{bc}(T_3) = \sum_{T_4 \in S_4} \omega_{bc}(T_4),
\]
\[
\sum_{T_1 \in S_1} \omega_{bc}(T_1) = \sum_{T_2 \in S_2} \omega_{bc}(T_2) = \sum_{T_3 \in S_3} \omega_{bc}(T_3) = \sum_{T_4 \in S_4} \omega_{bc}(T_4),
\]
\[
\sum_{T_1 \in S_1} \omega_{bc}(T_1) = \sum_{T_2 \in S_2} \omega_{bc}(T_2) = \sum_{T_3 \in S_3} \omega_{bc}(T_3) = \sum_{T_4 \in S_4} \omega_{bc}(T_4),
\]

where \( k = 0, 1, \ldots, l - 2 \).
Combining Equations (35), (70)–(72), we have

$$F_{BC}(W_{n+1}; f, g; c_0) = 5 \sum_{j=1}^{5} \sum_{k \in S_j^5} a_k(T_j^5)$$

$$= 2 \left( \begin{array}{c} n-2 \\ 0 \end{array} \right) \sum_{p=\lfloor \frac{n}{2} \rfloor}^{\lceil \frac{n}{2} \rceil} \sum_{q=1}^{\lceil \frac{n}{2} \rceil} \left( \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \frac{1}{|k|} \right) \frac{1}{|k|} \frac{1}{|k-2q|} \frac{1}{|k-2p|} y^{k+p} F(F_{n-2p-2q+2}; f, g; c_0, \text{odd})$$

$$+ \frac{1}{2} \sum_{p=\lfloor \frac{n}{2} \rfloor}^{\lceil \frac{n}{2} \rceil} \sum_{q=1}^{\lceil \frac{n}{2} \rceil} z^{q+p} F(F_{n-2p-2q+2}; f, g; c_0, \text{even})$$

$$+ \frac{1}{2} \sum_{p=\lfloor \frac{n}{2} \rfloor}^{\lceil \frac{n}{2} \rceil} z^{q+p} F(F_{n-2p+1}; f, g; c_0, \text{even})$$

with $F_{BC}(F_{n+1}; f, g; c_0)$, $F(F_{n+1}; f, g; c_0, \text{odd})$, $F(F_{n+1}; f, g; c_0, \text{even})$ as in Equations (29), (30), (35), respectively, and $F_{BC}(F_2; f, g; c_0) = 0$, $F(F_1; f, g; c_0, \text{odd}) = 0$, $F(F_1; f, g; c_0, \text{even}) = y$.

Our conclusion now follows from Equations (53)–(57), (68), (69), and (73).

The BC-subtree generating function of the wheel graph $W_{n+1}$ ($n \geq 3$) (where $n$ is a positive integer) follows from Theorem 7 and Lemma 4.

**Theorem 8.** Let $W_{n+1}$ ($n \geq 3$) be a wheel graph on $n + 1$ vertices with vertex weight function $f(v) = (0, y)$ for $v \in V(W_{n+1})$ and the edge weight function $g(e) = z$ for $e \in E(W_{n+1})$, then

$$F_{BC}(W_{n+1}; f, g) = 2 \sum_{k=0}^{n-2} \sum_{p=\lfloor \frac{n}{2} \rfloor}^{\lceil \frac{n}{2} \rceil} \sum_{q=1}^{\lceil \frac{n}{2} \rceil} \left( \sum_{k=0}^{\lceil \frac{n}{2} \rceil} \frac{1}{|k|} \right) \frac{1}{|k|} \frac{1}{|k-2q|} \frac{1}{|k-2p|} z^{k+p} F_{n-2p-2q+2}^{\text{odd}}(c_0)$$

$$+ \frac{1}{2} \sum_{p=\lfloor \frac{n}{2} \rfloor}^{\lceil \frac{n}{2} \rceil} \sum_{q=1}^{\lceil \frac{n}{2} \rceil} z^{q+p} F_{n-2p-2q+2}^{\text{even}}(c_0)$$

$$+ \frac{1}{2} \sum_{p=\lfloor \frac{n}{2} \rfloor}^{\lceil \frac{n}{2} \rceil} z^{q+p} F_{n-2p+1}^{\text{even}}(c_0) + F_{BC}(F_{n+1}; f, g, c_0)$$

with $F_{BC}(F_{n+1}; f, g; c_0)$, $F_{n+1}^{\text{odd}}(c_0)$, $F_{n+1}^{\text{even}}(c_0)$ given in Equations (13)–(15), respectively, and $F_{BC}(F_2; f, g; c_0) = 0$, $F_1^{\text{odd}}(c_0) = 0$, $F_1^{\text{even}}(c_0) = y$.

Letting $y = z = 1$ in Theorems 7 and 8, respectively, we immediately have the following.

**Corollary 3.** Let $G = (n_1C_1 + n_2C_2 + \cdots + n_kC_k) \cdot c_0$ be the multi-wheel graph. Then the BC-subtree number of $G$ is
\[ \eta_{BC}(G) = \sum_{i=1}^{k-1} \left( (1 + \pi_{i+1}) \right)^m - 1 \right] + \sum_{i=1}^{k-1} \left( \prod_{j=k+1}^{t} \left( 1 + \pi_{j+1} \right) \right)^{m_j} - 1 \right] + \sum_{i=1}^{k-1} \pi_{i+1} \left[ (1 + \pi_{i+1})^i - 1 \right] + n \eta_{BC}(W_{l+1}; c_0) \right] \\
+ \sum_{i=1}^{k} \sum_{j=1}^{n} n_j l_i, \tag{75} \]

with

\[ \eta_{BC}(W_{l+1}; c_0) = \eta_{BC}(F_{l+1}; c_0) + 2 \sum_{k=0}^{l-2} \left[ \sum_{p=\left[ \frac{l-2}{2} \right]}^{\left[ \frac{l-2}{2} \right]} \sum_{q=1}^{\left[ \frac{l-2}{2} \right]} a(l, -2p - 2q + 2) \right] \]

\[ + \sum_{p=\left[ \frac{l-2}{2} \right]}^{\left[ \frac{l-2}{2} \right]} b(l, -2p - 2q + 2) + \sum_{p=\left[ \frac{l-2}{2} \right]}^{\left[ \frac{l-2}{2} \right]} b(l, -2p + 1) \right] \]

\[ \pi_{l+1} = 2 \sum_{k=0}^{l-2} \sum_{p=\left[ \frac{l-2}{2} \right]}^{\left[ \frac{l-2}{2} \right]} \left( \prod_{q=1}^{\left[ \frac{l-2}{2} \right]} \left( 1 + a(l, -2p - 2q + 2) \right) \right) + \sum_{p=\left[ \frac{l-2}{2} \right]}^{\left[ \frac{l-2}{2} \right]} \left( \sum_{q=1}^{\left[ \frac{l-2}{2} \right]} (1 + a(l, -2p - 2q + 2)) \right) \]

\[ + \sum_{k=1}^{\left[ \frac{l-2}{2} \right]} (1 + a(l, -2k)) + 2a_{l_1} + 1, \tag{77} \]

and

\[ F_{l+1} = 2 \sum_{k=0}^{l-2} \sum_{p=\left[ \frac{l-2}{2} \right]}^{\left[ \frac{l-2}{2} \right]} \left( b(l, -2p + 1) + \sum_{q=1}^{\left[ \frac{l-2}{2} \right]} b(l, -2p - 2q + 2) \right) \]

\[ + \sum_{p=\left[ \frac{l-2}{2} \right]}^{\left[ \frac{l-2}{2} \right]} b(l, -2p + 1) + \sum_{q=1}^{\left[ \frac{l-2}{2} \right]} b(l, -2p - 2q + 2) \right] + \sum_{k=1}^{\left[ \frac{l-2}{2} \right]} b(l, -2k + 1) + b_{l_1} \right] \]

with \( \eta_{BC}(F_{l+1}; c_0) \) as in Equation (41), \( \eta_{BC}(F_{2l}; c_0) = 0, a_{l_1}, b_{l_1} \) as in Equations (42) and (43), respectively, and \( a_1 = 0, b_1 = 1. \)

**Corollary 4.** The BC-subtree number of \( W_{n+1} \) \((n \geq 3)\) is

\[ \eta_{BC}(W_{n+1}) = \eta_{BC}(F_{n+1}; c_0) + 2 \sum_{k=0}^{n-2} \left[ \sum_{p=\left[ \frac{n-2}{2} \right]}^{\left[ \frac{n-2}{2} \right]} a(n, -2p - 2q + 2) \right] \]

\[ + \sum_{p=\left[ \frac{n-2}{2} \right]}^{\left[ \frac{n-2}{2} \right]} b(n, -2p - 2q + 2) + \sum_{p=\left[ \frac{n-2}{2} \right]}^{\left[ \frac{n-2}{2} \right]} b(n, -2p + 1) \right] \]

\[ + n(\left[ \frac{n}{2} \right] - 1), \tag{79} \]

where

\[ \eta_{BC}(F_{n+1}; c_0) = \eta_{BC}(F_{n}; c_0) + \sum_{p=1}^{\left[ \frac{n}{2} \right]} \sum_{q=1}^{\left[ \frac{n}{2} \right]} a(n, -2p - 2q + 2) + \sum_{s=0}^{\left[ \frac{n}{2} \right]} a(n, -2s) \]

\[ + \sum_{p=1}^{\left[ \frac{n}{2} \right]} \sum_{q=1}^{\left[ \frac{n}{2} \right]} b(n, -2p - 2q + 2) + 2 \sum_{p=1}^{\left[ \frac{n}{2} \right]} b(n, -2p + 1) \right], \tag{80} \]
and

\[ a_{n+1} = 1 + 2a_n + \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor - 1} (1 + a_{(n-2k)}) + \sum_{p=1}^{n-1} \sum_{q=1}^{\lfloor \frac{n-2p}{2} \rfloor} (1 + a_{(n-2p-2q+2)}), \]

\[ b_{n+1} = \frac{1}{p} \sum_{p=1}^{\lfloor \frac{n}{2} \rfloor} (b_{(n-2p+1)}) + \frac{1}{p} \sum_{q=1}^{\lfloor \frac{n}{2} \rfloor} b_{(n-2p-2q+2)} + \frac{1}{p} \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} b_{(n-2k+1)} + b_n, \]

with \( \eta_{BC}(F_2; c_0) = 0, a_1 = 0, b_1 = 1. \)

Directly from Corollary 4 we may generate the BC-subtree number of \( W_{n+1} \) (see Table 1).

Table 1. The BC-subtree number of wheel graph \( W_{n+1} \) (\( n = 3, 4, \ldots, 50 \)).

| \( n \) | \( \eta_{BC}(W_n) \) | \( n \) | \( \eta_{BC}(W_n) \) | \( n \) | \( \eta_{BC}(W_n) \) |
|---|---|---|---|---|---|
| 3 | 16 | 19 | 32,983,507 | 35 | 44,956,112,462,965 |
| 4 | 51 | 20 | 79,507,829 | 36 | 109,018,361,948,171 |
| 5 | 131 | 21 | 191,713,586 | 37 | 264,446,533,681,852 |
| 6 | 340 | 22 | 462,509,681 | 38 | 641,651,556,370,739 |
| 7 | 841 | 23 | 1,116,289,936 | 39 | 1,557,326,469,215,175 |
| 8 | 2067 | 24 | 2,695,367,516 | 40 | 3,780,724,965,944,533 |
| 9 | 5026 | 25 | 6,510,870,551 | 41 | 9,180,821,777,073,118 |
| 10 | 12,147 | 26 | 15,733,892,896 | 42 | 22,299,504,128,100,416 |
| 11 | 29,305 | 27 | 38,036,865,379 | 43 | 54,176,661,898,045,614 |
| 12 | 70,508 | 28 | 91,989,932,352 | 44 | 131,652,455,330,130,629 |
| 13 | 169,664 | 29 | 222,555,514,089 | 45 | 319,993,939,502,050,010 |
| 14 | 407,988 | 30 | 538,634,836,904 | 46 | 777,941,764,394,823,593 |
| 15 | 981,517 | 31 | 1,304,079,385,141 | 47 | 1,891,653,259,171,010,731 |
| 16 | 2,361,611 | 32 | 3,158,367,596,891 | 48 | 4,600,677,046,844,511,460 |
| 17 | 5,684,920 | 33 | 7,651,840,948,554 | 49 | 11,191,401,704,559,183,703 |
| 18 | 13,690,201 | 34 | 18,544,255,820,839 | 50 | 27,228,679,901,334,157,132 |

5. The Behavior of the BC-Subtrees

With the theoretical foundation that was established in the previous sections, we will study the behavior of the BC-subtree number in the multi-fan and multi-wheel graphs. We first mention an extremal result as a simple consequence. We also briefly discuss the change of the BC-subtree numbers between different multi-fan or multi-wheel graphs. Lastly we consider the BC-subtree density in these structures.

5.1. BC-Subtree Number of \( F_{n+1}^r \)

The extremely problems with respect to a topological index concerns finding the extremal structures, among a given class of graphs, that maximize or minimize the index. These graphs always possess the best or worst of some desired properties [48–50]. We first point out the following simple fact.

**Proposition 1.** The \( F_{n+1}^r \) has \( 2^n - n - 1 \) BC-subtrees, fewer than any other \( F_{n+1}^r (r \neq 1) \); the \( F_{n+1}^n \) has more BC-subtrees than any other \( F_{n+1}^r (r \neq n) \).

It is often interesting to know, when studying extremal problems, which graph structures have the second or third largest or smallest values of a certain index. To shed some light on this we ran some simulation, whose result is shown in Figure 3, with Cartesian and semi-log (Log-Y) coordinate, respectively.
(a) BC-subtree numbers of $F_{n+1}^r$ with $n = 33, 58, 75, 80$, $r$ from 1 to $n$, respectively in Cartesian coordinate.

(b) BC-subtree numbers of $F_{n+1}^r$ with $n = 33, 58, 75, 80$, $r$ from 1 to $n$, respectively in semi-log (Log-Y) coordinate.

**Figure 3.** BC-subtree numbers of $F_{n+1}^r$ with $n = 33, 58, 75, 80$, $r$ from 1 to $n$, respectively.

From Figure 3a, it appears that among all $F_{n+1}^r$ ($2 \leq r \leq n - 1$), $F_{n+1}^{n-1}$ seems to have the second largest BC-subtree number, and the $F_{n+1}^{n-2}$ seems to be the third largest BC-subtree number for odd (or sufficient large) $n$.

It is also interesting to observe the change of the BC-subtree number as $r$ changes, with the “local minimum” spread out for smaller values of $r$.

From Corollary 4 we can obtain data of similar nature for $W_{n+1}$ ($n \geq 3$). Figure 4 confirms the simple fact that the BC-subtree numbers of $W_{n+1}$ increase very fast. The asymptotic expression of the BC-subtree number of $W_{n+1}$ seems to be $\eta_{BC}(W_{n+1}) \approx \exp(0.0909 + 0.893n)$. This is something that can be further verified through analytic combinatorics approaches.
5.2. BC-Subtree Density

The generating function approach provides us with much more information than just the BC-subtree number. In particular, we will examine the BC-subtree density here, for $r$ multi-fan graph $F_r^{n+1}$ ($1 \leq r \leq n$, $r$ is an integer) and $W_{n+1}$, respectively. For some of the work on this topic one may see [40].

**Definition 4 ([40]):** Assume $G$ is a graph with $n$ vertices and $k$ BC-subtrees of orders $n_1, n_2, \ldots, n_k$, then $\mu_{BC}(G) = \frac{1}{k} \sum_{i=1}^{k} n_i$ denotes the average order of BC-subtrees of $G$, and the BC-subtree density of $G$ is defined as $D_{BC}(G) = \frac{\mu_{BC}(G)}{n}$.

First it is easy to see that

$$n(F_r^{n+1}) = n(W_{n+1}) = n + 1$$  \hfill (83)

Letting $y = 1$ in Theorem 6 and Theorem 8, we could obtain the so called edge generating function of BC-subtrees of $F_r^{n+1}$ and $W_{n+1}$, respectively, i.e., $F_{BC}(F_r^{n+1}; (0, 1), z)$ and $F_{BC}(W_{n+1}; (0, 1), z)$.

By the definition of BC-subtree density and Equation (83), the BC-subtree density of $G^*$ ($G^* = F_r^{n+1}$ or $W_{n+1}$) is simply

$$D_{BC}(G^*) = \left. \frac{\partial (F_{BC}(G^*; (0, 1), z) \times z)}{\partial z} \right|_{z=1} n(G^*).$$  \hfill (84)

We may now provide the BC-subtree densities of $F_r^{n+1}$ ($1 \leq r \leq n$) in Figure 5, with related data in Table 2.
Figure 5. BC-subtree densities of $r$ multi-fan $F_{22+1}$ ($1 \leq r \leq 22$).

Table 2. Data related to BC-subtrees of $F_{n+1}$, with $n = 22$ and $r = 1, 2, \ldots, 22$.

| $r$  | $P_z(F_{n+1})$ | $\eta_{BC}(F_{n+1})$ | $D_{BC}(F_{n+1})$ |
|------|----------------|-----------------------|-------------------|
| 1    | 50,331,603     | 4,194,281             | 0.5217415249997297 |
| 2    | 53,106,905     | 4,371,427             | 0.5282018593848684 |
| 3    | 285,872,367    | 20,279,905            | 0.6128841997941354 |
| 4    | 473,222,721    | 32,609,837            | 0.6309415442046974 |
| 5    | 744,621,994    | 49,805,736            | 0.6500229070874051 |
| 6    | 717,819,208    | 48,287,696            | 0.6463247031419493 |
| 7    | 1,460,830,943  | 93,663,206            | 0.6781146146433033 |
| 8    | 703,383,187    | 47,521,825            | 0.6435333174946917 |
| 9    | 1,091,991,586  | 71,647,232            | 0.662661958015046  |
| 10   | 1,742,448,206  | 110,913,152           | 0.6830444927953532 |
| 11   | 3,063,687,819  | 187,270,907           | 0.7112894381260796 |
| 12   | 448,577,818    | 31,383,926            | 0.621444980120178  |
| 13   | 556,641,580    | 38,350,250            | 0.6310730132420768 |
| 14   | 690,781,206    | 46,878,694            | 0.6406741083329001 |
| 15   | 857,549,247    | 57,340,131            | 0.6502383098769903 |
| 16   | 1,065,571,361  | 70,221,005            | 0.6597625540775962 |
| 17   | 1,326,780,941  | 86,192,884            | 0.6692676890344194 |
| 18   | 1,659,024,388  | 106,256,288           | 0.67884485235874   |
| 19   | 2,091,899,843  | 132,053,383           | 0.6887530256377947 |
| 20   | 2,680,301,328  | 166,568,142           | 0.699622626010619  |
| 21   | 3,536,505,225  | 215,724,226           | 0.7127669413407951 |
| 22   | 4,906,885,913  | 292,065,325           | 0.73046283663632   |

$P_z(F_{n+1})$ stands for $\frac{\partial (f_{BC}(F_{n+1};(0,1),z)) \times z}{\partial z}|_{z=1}$.

Similarly, with Theorem 8 we obtain the generating function $F_{BC}(W_{n+1};(0,1),z)$. Together with Equation (84) we can obtain the BC-subtree densities of $W_{n+1}$ (see Figure 6 and Table 3).
Figure 6. BC-subtree densities of wheel graph $W_{n+1}$ ($3 \leq n \leq 24$).

Table 3. Data related to BC-subtrees of wheel graph $W_{n+1}$ ($n = 3, 4, \ldots, 24$).

| $k$ | $P_z(W_{k+1})$ | $D_{BC}(W_{k+1})$ | $k$ | $P_z(W_{k+1})$ | $D_{BC}(W_{k+1})$ |
|-----|----------------|-------------------|-----|----------------|-------------------|
| 3   | 0.8125         | 0.7610            | 14  | 4,657.240      | 0.76103173614     |
| 4   | 0.8274509803921568 | 0.7593403247218337 | 15  | 11,924,887     | 0.75773399908885  |
| 5   | 0.7900763358778626 | 0.749499052182229 | 16  | 30,421,819     | 0.7563098024637501 |
| 6   | 0.7907563025210084 | 0.754949905218229 | 17  | 77,392,093     | 0.7536807635394871 |
| 7   | 0.7804696789536266 | 0.7536807635394871 | 18  | 196,372,903    | 0.7524833703900728 |
| 8   | 0.7772402300704188 | 0.751352570735758  | 19  | 497,180,696    | 0.749262321576641  |
| 9   | 0.7731197771587743 | 0.7482982145521214 | 20  | 1,256,299,889  | 0.747599052182229  |
| 10  | 0.7699394538120149 | 0.7462512145521214 | 21  | 497,180,696    | 0.7452833703900728 |
| 11  | 0.7674002300704188 | 0.7442512145521214 | 22  | 79,217,895     | 0.743262321576641  |
| 12  | 0.764923568376313  | 0.74226321576641   | 23  | 20,073,455,736 | 0.7412798214552124 |
| 13  | 0.7629423869698766 | 0.740289905218229  | 24  | 50,423,103,620 | 0.7392982145521214 |

$P_z(W_{k+1})$ stands for $\frac{\partial F_{BC}(W_{k+1}; (0,1), z)}{\partial z}$ $\big|_{z=1}$.

From Table 2 and Figure 5, we see that $F^1_{22+1}$, $F^2_{22+1}$, and $F^3_{22+1}$ ranks the first to the third with the smallest BC-subtree density, respectively. From Table 3 and Figure 6, we see that the BC-subtree density of $W_{n+1}$ ($3 \leq n \leq 24$) maximized at $n = 4$, and decreases gradually when $n \geq 6$.

6. Concluding Remarks

Motivated from the past studies of BC-trees, BC-subtrees, as well as the applications of structural properties of network graphs [51], we provide recursive formulae for computing the BC-subtree generating functions of multi-fan and multi-wheel graphs, and also derive the BC-subtree numbers of multi-fan graphs, multi-wheel (wheel) graphs. Moreover, the behavior of the BC-subtree numbers between different multi-fan or multi-wheel graphs, and extremal problems and BC-subtree density are also briefly discussed. These findings are likely useful in further understanding the properties and behaviors of these graphs. For future work it would be interesting to consider other well known topological indices on the multi-fan and multi-wheel graphs and compare their behaviors with that of the BC-subtree number.

Author Contributions: H.W. and W.W. contribute for supervision, project administration and formal analysis. L.L. and Y.Y. contribute for methodology and writing original draft preparation. The final
draft we written by H.W. and Y.Y. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work is supported by the National Natural Science Foundation of China (grant nos. 61702291,11801371); the Program for Science & Technology Innovation Talents in Universities of Henan Province (grant no. 19HASTIT029), the Key Research Project in Universities of Henan Province (grant nos. 19B110011, 19B630015),the Scientific Research Starting Foundation for High-Level Talents of Pingdingshan University (grant no.PXY-BSDK2017006).

**Conflicts of Interest:** The authors declare no conflict of interest.

References

1. Wiener, H. Structural determination of paraffin boiling points. J. Am. Chem. Soc. 1947, 1, 17–20. [CrossRef]
2. Xu, K.; Das, K.C. On Harary index of graphs. Discret. Appl. Math. 2011, 159, 1631–1640. [CrossRef]
3. Sun, Q.; Ikica, B.; Škrekovski, R.; Vukašinović, V. Graphs with a given diameter that maximise the Wiener index. Appl. Math. Comput. 2019, 356, 438–448. [CrossRef]
4. Binu, M.; Mathew, S.; Mordeson, J. Wiener index of a fuzzy graph and application to illegal immigration networks. Fuzzy Sets Syst. 2020, 384, 132–147. [CrossRef]
5. Dimitrov, D.; Milosavljević, N. Efficient computation of trees with minimal atom-bond connectivity Index revisited. Match Commun. Math. Comput. Chem. 2018, 79, 431–450.
6. Milovanović, E.; Milovanović, I.; Matejić, M. Remark on spectral study of the geometric-arithmetic index and some generalizations. Appl. Math. Comput. 2018, 334, 206–213. [CrossRef]
7. Sah, A.; Sawhney, M. On the Discrepancy Between Two Zagreb Indices. Discret. Math. 2018, 341, 2575–2589. [CrossRef]
8. Zhang, X.; Yang, Y.; Wang, H.; Zhang, X. Maximum atom-bond connectivity index with given graph parameters. Discret. Appl. Math. 2016, 215, 208–217. [CrossRef]
9. Chen, X.; Zhang, J.; Sun, W. On the Hosoya index of a family of deterministic recursive trees. Phys. Stat. Mech. Its Appl. 2017, 465, 449–453. [CrossRef]
10. De Ita Luna, G.; Raymundo Marcial-Romero, J.; Bello Lopez, P.; Contreras Gonzalez, M. Linear-time algorithms for computing the Merrifield-Simmons index on polygonal trees. Match Commun. Math. Comput. Chem. 2018, 79, 55–78.
11. Huang, Y.; Shi, L.; Xu, X. The Hosoya index and the Merrifield–Simmons index. J. Math. Chem. 2018, 56, 3136–3146. [CrossRef]
12. Yan, W.; Yeh, Y. Enumeration of subtrees of trees. Theor. Comput. Sci. 2006, 369, 256–268. [CrossRef]
13. Yang, Y.; Liu, H.; Wang, H.; Makeig, S. Enumeration of BC-subtrees of trees. Theor. Comput. Sci. 2015, 580, 59–74. [CrossRef]
14. Lin, W.; Chen, J.; Wu, Z.; Dimitrov, D.; Huang, L. Computer search for large trees with minimal ABC index. Appl. Math. Comput. 2018, 338, 221–230. [CrossRef]
15. Wagner, S.G. Correlation of graph-theoretical indices. SIAM J. Discret. Math. 2007, 21, 33–46. [CrossRef]
16. Yang, Y.; Liu, H.; Wang, H.; Deng, A.; Magnant, C. On Algorithms for Enumerating Subtrees of Hexagonal and Phenylene Chains. Comput. J. 2017, 60, 690–710. [CrossRef]
17. Zhou, Q.; Wang, L.; Lu, Y. Wiener index and Harary index on Hamilton-connected graphs with large minimum degree. Discret. Appl. Math. 2018, 247, 180–185. [CrossRef]
18. Joija, D.M.; Jántschi, L. Extending the characteristic polynomial for characterization of C20 fullerene congeners. Mathematics 2017, 5, 64. [CrossRef]
19. García-Pereira, I.; Zanni, R.; Galvez-Llopart, M.; Galvez, J.; García-Domenech, R. DesMol2, an effective tool for the construction of molecular libraries and its application to QSAR using molecular topology. Molecules 2019, 24, 736. [CrossRef]
20. Bonchev, D. The overall Wiener index–a new tool for characterization of molecular topology. J. Chem. Inf. Model. 2001, 41, 582–592.
21. Bonchev, D. Chapter 3—On the Concept for Overall Topological Representation of Molecular Structure. In Advances in Mathematical Chemistry and Applications; Basak, S.C., Restrepo, G., Villaveces, J.L., Eds.; Bentham Science Publishers: Sharjah, UAE, 2015; pp. 42–75.
22. Yang, Y.; Sun, X.J.; Cao, J.Y.; Wang, H.; Zhang, X.D. The expected subtree number index in random polyphenylene and spiro chains. Discret. Appl. Math. 2020, 285, 483–492. [CrossRef]
23. Sills, A.V.; Wang, H. The minimal number of subtrees of a tree. Graphs Comb. 2015, 31, 255–264. [CrossRef]
24. Szekeley, L.; Wang, H. On subtrees of trees. Adv. Appl. Math. 2005, 34, 138–155. [CrossRef]
25. Zhang, X.; Zhang, X. The minimal number of subtrees with a given degree sequence. Graphs Comb. 2015, 31, 309–318. [CrossRef]
26. Zhang, X.; Zhang, X.; Gray, D.; Wang, H. The number of subtrees of trees with given degree sequence. J. Graph Theory 2013, 73, 280–295. [CrossRef]
27. Harary, F.; Plummer, M. On the core of a graph. Proc. Lond. Math. Soc. 1967, 17, 249–257. [CrossRef]
28. Harary, F.; Prins, G. The block-cutpoint-tree of a graph. Publ. Math. Debr. 1966, 13, 103–107.
29. Nakayama, T.; Fujiwara, Y. BCT Representation of Chemical Structures. J. Chem. Inf. Comput. Sci. 1980, 20, 23–28. [CrossRef]
30. Nakayama, T.; Fujiwara, Y. Computer representation of generic chemical structures by an extended block-cutpoint tree. J. Chem. Inf. Comput. Sci. 1983, 23, 80–87. [CrossRef]
31. Frederickson, G.N.; Hambrusch, S.E. Planar linear arrangements of outerplanar graphs. *IEEE Trans. Circuits Syst.* **1988**, *35*, 323–333. [CrossRef]
32. Wada, K.; Luo, Y.; Kawaguchi, K. Optimal fault-tolerant routings for connected graphs. *Inf. Process. Lett.* **1992**, *41*, 169–174. [CrossRef]
33. Gagarin, A.; Labelle, G. Two-connected graphs with prescribed three-connected components. *Adv. Appl. Math.* **2009**, *43*, 46–74. [CrossRef]
34. Heath, L.; Pemmaraju, S. Stack and queue layouts of directed acyclic graphs: Part II. *SIAM J. Comput.* **1999**, *28*, 1588–1626. [CrossRef]
35. Paton, K. An algorithm for the blocks and cutnodes of a graph. *Commun. ACM* **1971**, *14*, 468–475. [CrossRef]
36. Misiolek, E.; Chen, D.Z. Two flow network simplification algorithms. *Inf. Process. Lett.* **2006**, *97*, 197–202. [CrossRef]
37. Fox, D. Block cutpoint decomposition for markovian queueing systems. *Appl. Stoch. Model. Data Anal.* **1988**, *4*, 101–114. [CrossRef]
38. Barefoot, C. Block-cutvertex trees and block-cutvertex partitions. *Discret. Math.* **2002**, *256*, 35–54. [CrossRef]
39. Mkrtchyan, V. On trees with a maximum proper partial 0–1 coloring containing a maximum matching. *Discret. Math.* **2009**, *306*, 456–459. [CrossRef]
40. Yang, Y.; Liu, H.; Wang, H.; Sun, S. On Spiro and polyphenyl hexagonal chains with respect to the number of BC-subtrees. *Int. J. Comput. Math.* **2017**, *94*, 774–799. [CrossRef]
41. Yang, Y.; Liu, H.; Wang, H.; Feng, S. On algorithms for enumerating BC-subtrees of unicyclic and edge-disjoint bicyclic graphs. *Discret. Appl. Math.* **2016**, *203*, 184–203. [CrossRef]
42. Yang, Z.; Liu, Y.; Li, X.Y. Beyond trilateration: On the localizability of wireless ad hoc networks. *IEEE/ACM Trans. Netw. (ToN)* **2010**, *18*, 1806–1814. [CrossRef]
43. Tu, J.; Zhou, Y.; Su, G. A kind of conditional connectivity of Cayley graphs generated by wheel graphs. *Appl. Math. Comput.* **2017**, *301*, 177–186. [CrossRef]
44. Liu, X.; Zhang, Y.; Gui, X. The multi-fan graphs are determined by their Laplacian spectra. *Discret. Math.* **2008**, *308*, 4267–4271. [CrossRef]
45. Zhang, Y.; Liu, X.; Yong, X. Which wheel graphs are determined by their Laplacian spectra? *Comput. Math. Appl.* **2009**, *58*, 1887–1890. [CrossRef]
46. Kumar, A.; Sarkar, R. Hilbert series of binomial edge ideals. *Commun. Algebra* **2019**, *47*, 3830–3841. [CrossRef]
47. Yang, Y.; Wang, A.; Wang, H.; Zhao, W.T.; Sun, D.Q. On Subtrees of Fan Graphs, Wheel Graphs, and “Partitions” of Wheel Graphs Under Dynamic Evolution. *Mathematics* **2019**, *5*, 472. [CrossRef]
48. Cao, S.; Dehmer, M.; Shi, Y. Extremality of degree-based graph entropies. *Inf. Sci.* **2014**, *278*, 22–33. [CrossRef]
49. Székely, L.A.; Wang, H. Extremal values of ratios: distance problems vs. subtree problems in trees. *Electron. J. Comb.* **2013**, *20*, P67. [CrossRef]
50. Wang, H. The extremal values of the Wiener index of a tree with given degree sequence. *Discret. Appl. Math.* **2008**, *156*, 2647–2654. [CrossRef]
51. Allen, B.; Lippner, G.; Chen, Y.T.; Fotouhi, B.; Momeni, N.; Yau, S.T.; Nowak, M.A. Evolutionary dynamics on any population structure. *Nature* **2017**, *544*, 227. [CrossRef]