THE $L^p$-NORMS OF THE BEURLING-AHLFORS TRANSFORM ON RADIAL FUNCTIONS

MICHAL STRZELECKI

Abstract. We calculate the norms of the operators connected to the action of the Beurling-Ahlfors transform on radial function subspaces introduced by Bañuelos and Janakiraman. In particular, we find the norm of the Beurling-Ahlfors transform acting on radial functions for $p > 2$, extending the results obtained by Bañuelos and Janakiraman, Bañuelos and Osekiowski, and Volberg for $1 < p \leq 2$.

1. Introduction and main results

The Beurling-Ahlfors transform is a singular operator defined by

$$BF(z) = -\frac{1}{\pi} \text{p.v.} \int_{\mathbb{C}} \frac{f(w)}{(z-w)^2} dw,$$

where the integration is with respect to the Lebesgue measure on the complex plane $\mathbb{C}$. It plays an important role in the study of quasiconformal mappings and partial differential equations (see e.g. [1, 13]).

A longstanding conjecture of Iwaniec [13] states that for $1 < p < \infty$,

$$\|B\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})} = p^* - 1,$$

where $p^* = \max \{p, \frac{p}{p'} \}$. While the lower bound $\|B\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})} \geq p^* - 1$ was already known to Lehto [14], the question about the opposite estimate remains open. Most results rely on the ideas of Burkholder and the Bellman function technique [7, 18, 2, 11, 3, 8], with the current best being $\|B\|_{L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})} \leq 1.575(p^* - 1)$ due to Bañuelos and Janakiraman [3] (see also [8] for an asymptotically better estimate as $p \to \infty$).

However, some sharp results are known for the Beurling-Ahlfors transform restricted to the class of radial functions [4, 12, 5, 16, 19, 6]. In this case we have the representation (see [4])

$$BF(z) = \frac{z}{z} (H_0 f(|z|^2) - f(|z|^2)),$$

where $f : [0, \infty) \to \mathbb{C}$ is an integrable function, $F(z) = f(|z|^2)$ is the associated radial function, and $H_0$ is the Hardy operator defined by the formula

$$H_0 f(t) = \frac{1}{t} \int_0^t f(s) ds.$$

Bañuelos and Janakiraman [4, Theorem 4.1] (and later, using other techniques, Bañuelos and Osekiowski [5, Theorem 5.1], Volberg [19]) proved that for $1 < p \leq 2$ and any radial function $F \in L^p(\mathbb{C})$, we have $\|BF\|_p \leq \frac{1}{p-1} \|F\|_p$. The constant $1/(p-1)$ is the best possible and coincides with the constant from Iwaniec’s conjecture. As for $p > 2$, Bañuelos and Osekiowski [5] observed that $\|BF\|_p \leq \frac{2p}{p-1} \|F\|_p$.
This bound is asymptotically sharp (and does not agree with the behavior conjectured in the case of all, not only radial, functions).

In their paper, Bañuelos and Janakiraman [4] went a step further and considered for $m \in \mathbb{N}$ the operators

$$
(I - (1 + m)H_m)f(t) = f(t) - \frac{1 + m}{t^{1+m/2}} \int_0^t f(s)s^{m/2}ds, \quad f \in L_{loc}([0, \infty)),
$$

which correspond to the action of the Beurling-Ahlfors transform on the radial function subspaces

$$
\{ F \in L^p(\mathbb{C}) : F(re^{i\theta}) = f(r)e^{-im\theta} \}.
$$

They proved [4, Section 5] that

$$
\|H_m\|_{L^p([0, \infty)) \rightarrow L^p([0, \infty))} = \frac{1}{m/2 + (p-1)/p}, \quad 1 < p < \infty,
$$

(with the extremal family $f_\varepsilon(t) = t^{-1/p-1}(1_{t \in (0,1)})$), and conjectured [4, Conjecture 1] that the $L^p$-norm of the operator $I - (1 + m)H_m$ is equal to

$$
(1 + m)\|H_m\|_{L^p([0, \infty)) \rightarrow L^p([0, \infty))} - 1 = \frac{m/2 + 1/p}{m/2 + (p-1)/p}
$$

for $1 < p < 2$. For $p > 2$, this number is smaller than one, and cannot be a candidate for the norm of $I - (1 + m)H_m$, since the operator $(1 + m)H_m : L^p([0, \infty)) \rightarrow L^p([0, \infty))$ is not invertible (see Remark 3.4).

In fact, the formula

$$
H_m f(t) = \frac{1}{t^{1+m/2}} \int_0^t f(s)s^{m/2}ds
$$

defines a bounded operator on the space $L^p([0, \infty))$ ($1 < p < \infty$) not only for natural $m$, but for all $m > -2(p-1)/p$ (see Proposition 3.1). The main goal of this article is to find the $L^p$-norm of the operator $I - \lambda H_m$ for $1 < p < \infty$, $m > -2(p-1)/p$, and $\lambda \in \mathbb{R}$. The case $\lambda = 1 + m$, $m \in \mathbb{N}$, corresponds to the action of the Beurling-Ahlfors transform on radial function subspaces considered by Bañuelos and Janakiraman, but it turns out that [4, Conjecture 1] does not hold.

For the formulation of the main result we denote $g_{p,m} = m/2 + (p-1)/p$ for $m > -2(p-1)/p$ and $1 < p < \infty$.

**Theorem 1.1.** If $1 < p < \infty$, $m > -2(p-1)/p$, and $\lambda \in \mathbb{R}$, then

$$
(1.1) \quad \|f - \lambda H_m f\|_p \leq C_{p,m,\lambda}\|f\|_p, \quad f \in L^p([0, \infty)),
$$

where

$$
C_{p,m,\lambda} = \sup \left\{ \frac{(\beta - g_{p,m})|\alpha - \lambda|^p + (g_{p,m} - \alpha)|\beta - \lambda|^p}{(\beta - g_{p,m})|\alpha|^p + (g_{p,m} - \alpha)|\beta|^p} : \alpha < g_{p,m} < \beta \right\}.
$$

The inequality is sharp. Moreover, the constant $C_{p,m=0,\lambda=1}^p$ is equal to

$$
(1.2) \quad C_{p}^p := \sup_{\alpha \leq (p-1)/p} \frac{|\alpha - 1|^p}{p(1-\alpha) - 1 + |\alpha|^p} = \begin{cases} 
\frac{1-p}{p} & \text{if } 1 < p \leq 2, \\
\frac{1}{p}\left(\frac{1}{1+\alpha_p}\right)^{p-2} & \text{if } p > 2,
\end{cases}
$$

where, for $2 < p < \infty$, $\alpha_p \in \mathbb{R}$ is the unique negative solution to the equation

$$(p-1)\alpha_p + 2 - p = |\alpha|^p/\alpha_p^p.
$$

**Remark 1.2.** Even for $1 < p < 2$ (and $\lambda = 1 + m$), the norm of the operator $I - (1 + m)H_m$ is sometimes greater than the conjectured value $(1 + m)g_{p,m}^{-1} - 1$. E.g. for $p = 3/2$, $m = 1$, $\lambda = 2$, we have $C_{p,m,\lambda}^p \approx 1.81$ (attained in the neighbourhood of $g_{p,m} = (0.4, 5.7)$), whereas $((1 + m)g_{p,m}^{-1} - 1)^p = (7/5)^{3/2} \approx 1.66$. 

Remark 1.3. Apart from the case \( m = 0, \lambda = 1 \), there are simple formulas for \( C_{p,m,\lambda} \) if \( \lambda \leq 0 \) or \( p = 2 \) (see Section 3.3). Moreover, a sufficient and necessary condition for \( C_{p,m,\lambda} = |\lambda g_{p,m}^{-1} - 1| \) to hold can be formulated (see the proof of Proposition 3.7 and Section 3.3). Note also that \( C_{p,m,\lambda} \geq \max\{|\lambda g_{p,m}^{-1} - 1|, 1\} \) (see Lemma 3.3).

Remark 1.4. Throughout the paper we work with real-valued functions, but Theorem 1.1 also holds (with the same constant) for complex-valued functions (see Lemma 3.8).

The results of Theorem 1.1 are new already for \( p > 2, m = 0, \) and \( \lambda = 1 \), and give immediately the following extension of results obtained by other authors [4, 5, 19].

Corollary 1.5. For \( 1 < p < \infty \) and any complex-valued radial function \( F \in L^p(\mathbb{C}) \), we have the sharp inequality \( \|BF\|_p \leq C_p\|F\|_p \).

The article is organized as follows. A complete and purely analytical proof of inequality (1.1) is contained in Section 3. Section 2 is designed to show a bigger picture. We prove a maximal martingale inequality connected to the special case \( m = 0 \) and \( \lambda = 1 \). We also identify the constant \( C_{p,0,1} \) and try to explain the main ideas behind the construction of the special functions used in the proofs.

2. Backstage: the martingale inequality

2.1. Motivation and results. For a martingale \( f = (f_n)_{n=0}^\infty \) denote its one-sided maximal function by \( f^*_n = \sup_{0 \leq j \leq n} f_j \). We also use the notation \( f^*_n = \sup_{0 \leq n} f_n \) and \( f_\infty = \lim_{n \to \infty} f_n \) (if the limit exist a.s.).

Recall that the \( L^p \)-norm, \( 1 < p < \infty \), of the Hardy operator \( H_0 \) is equal to \( p/(p-1) \). This number is also the best constant in Doob’s inequality: for a martingale \( (f_k)_{k=0}^\infty \) we have \( \|f^*_n\|_p \leq \frac{p}{p-1}\|f_n\|_p \). It turns out that the martingale inequality can be used to derive the estimate \( \|H_0f\|_p \leq \frac{p}{p-1}\|f\|_p \) for nonnegative and nonincreasing functions [10]; a simple rearrangement argument gives then \( \|H_0f\|_p \leq \frac{p}{p-1}\|f\|_p \) for all real-valued \( f \in L^p([0,\infty)) \).

We consider the following maximal inequality.

Theorem 2.1. For any martingale \( (f_n)_{n=0}^\infty \), we have

\[
\|f_n - f^*_n\|_p \leq C_p\|f_n\|_p, \quad 1 < p < \infty, n \geq 0,
\]

where \( C_p \) is defined in (1.2). The inequality is sharp.

The quantity \( \|f_n - f^*_n\|_p \) seems natural to study, but the main motivation is the aforementioned link to the Hardy operator (see Section 2.6 for the proof). Note that this approach is different from that of Bañuelos and Osękowski [5], who used estimates for pure-jump martingales, and the analytical approaches of Bañuelos and Janakiraman [4], and Volberg [19].

Corollary 2.2. Let \( 1 < p < \infty \). If \( f \in L^p([0,\infty)) \) is real-valued and nonincreasing, then

\[
\|f - H_0f\|_p \leq C_p\|f\|_p.
\]

Quite unexpectedly, some difficulties arise at the stage of rearrangements. In our setting it is possible that

\[
\|g - H_0g\|_p < \|f - H_0f\|_p,
\]

where \( g \) denotes the nonincreasing rearrangement of a real-valued function \( f \in L^p([0,\infty)) \) (examples can be found with \( f \) being a (positive) step function, in which case \( H_0g, \|g - H_0g\|_p, \|f - H_0f\|_p \) can be explicitly calculated). Hence, it seems that Corollary 2.2 does not directly imply Theorem 1.1 (for \( m = 0, \lambda = 1 \)). Fortunately,
it is possible to use the tools from the proof of the martingale inequality (and adapt them to work not only for \( m = 0 \), but for \( m > -2(p-1)/p \) and all \( \lambda \in \mathbb{R} \)) to obtain our main result (see Sections 2.7 and 3).

2.2. Method of the proof of Theorem 2.1 and a lower bound for the best constant. We follow Burkholder’s approach to the Doob inequality [9, p. 578]: in order to prove inequality (2.1), it suffices to find an appropriate special function (for further reading about maximal martingale inequalities see also [15, Chapter 7]).

**Proposition 2.3.** Let \( V(x, y) = |x - y|^p - C\sigma|x|^p \) and suppose that \( U : \mathbb{R}^2 \to \mathbb{R} \) satisfies the following conditions.

1. (Majorization) If \( x \leq y \), then \( V(x, y) \leq U(x, y) \).
2. (Initial condition) For all \( x \in \mathbb{R} \), we have \( U(x, x) \leq 0 \).
3. (Maximal condition) If \( x \leq y, h \in \mathbb{R} \), then
   \[
   U(x + h, (x + h) \vee y) \leq U(x + h, y).
   \]
4. (Concavity) For all \( y \in \mathbb{R} \), the function \( U(\cdot, y) : \mathbb{R} \to \mathbb{R} \) is concave.

Then \( \|f_n - f_n^*\|_p \leq C\|f_n\|_p \) for any martingale \( (f_n)_{n=0}^\infty \) and any \( n \geq 0 \).

**Proof.** It suffices to consider the inequality for simple martingales (in which case all expressions below are integrable). Conditions 3 and 4 imply that

\[
\mathbb{E}U(f_n, f_n^*) = \mathbb{E}U(f_n - 1 + (f_n - f_n - 1), (f_n - 1 + (f_n - f_n - 1)) \vee f_n^*) \\
\leq \mathbb{E}U(f_n - 1 + (f_n - f_n - 1), f_n^*) \\
\leq \mathbb{E}U(f_n - 1, f_n^*) + \mathbb{E}(f_n - f_n - 1)U_x^+(f_n - 1, f_n^*),
\]

where \( U_x^+ \) denotes the right derivative. Moreover, \( \mathbb{E}(f_n - f_n - 1)U_x^+(f_n - 1, f_n^*) = 0 \) because \( f \) is a martingale. Hence, \( \mathbb{E}U(f_n, f_n^*) \leq \mathbb{E}U(f_n - 1, f_n^*) \). Thus, using Conditions 1 and 2, we arrive at

\[
\|f_n - f_n^*\|_p \leq \mathbb{E}V(f_n, f_n^*) \\
\leq \mathbb{E}U(f_n, f_n^*) \leq \ldots \leq \mathbb{E}U(f_0, f_0^*) = \mathbb{E}(f_0, f_0) \leq 0.
\]

This ends the proof. \( \square \)

**Remark 2.4.** In the above proof it is enough to have \( \mathbb{E}(f_n - f_n - 1)U_x^+(f_n - 1, f_n^*) \leq 0 \). This inequality holds if \( f \) is a nonnegative submartingale and \( U_x^+(x, y) \leq 0 \) for \( y \geq 0 \). This additional assumption is satisfied by the function \( U \) which we construct in Section 2.5. In particular, for any martingale \( (f_n)_{n=0}^\infty \) also

\[
\|f_n - f_n^*\|_p \leq \mathbb{E}V(f_n, f_n^*) \leq \mathbb{E}U(f_n, f_n^*) \leq \ldots \leq \mathbb{E}U(f_0, f_0^*) = \mathbb{E}(f_0, f_0) \leq 0.
\]

This holds, since \( \|f_n\|_p \) is a nonnegative submartingale whenever \( (f_n)_{n=0}^\infty \) is a martingale. This bound is sharp in the case \( 1 < p \leq 2 \) (see the example in Section 2.6), but the constant \( C_p \) does not seem to be the best possible for \( p > 2 \).

There is an abstract way for finding a candidate for the function from Proposition 2.3. Namely, let \( V(x, y) = |x - y|^p - C\sigma|x|^p \) and define

\[
U^0(x, y) = \sup\{\mathbb{E}V(f_n, f_n^* \vee y) : f_0 = x\},
\]

where the supremum is taken over the class \( \mathcal{M} \) consisting of all simple martingales \( f = (f_n)_{n=0}^\infty \) on the probability space [0, 1] equipped with the Borel \( \sigma \)-algebra and the Lebesgue measure (the filtration may vary). This approach has one main drawback: the expression defining \( U^0 \) is hard to work with. Nonetheless, we can use the function \( U^0 \) to extract important information: a lower bound for the constant \( C = C(p) \), with which the martingale inequality holds (and on which the function \( V \) depends). For explicit examples of extremal martingales see Section 2.6.
Sharpness of (2.1). Let $1 < p < \infty$ be fixed. First note that by the triangle and Doob’s inequality the estimate (2.1) holds with some finite constant. Let us denote it by $C$ (of course it may depend on $p$) and let $V, U^{0}$ be the functions defined in the preceding paragraph. Note that, as for now, we do not claim that $U^{0} < +\infty$.

Clearly, $U^{0}(x, y) \geq V(x, x \vee y)$ (since a constant martingale, $f_{n} \equiv x_{n}$, belongs to $\mathcal{M}$), $U^{0}(x, y) = U^{0}(x, x \vee y)$ (since $f_{0} \leq f_{\infty}^{*}$), and $U^{0}(ax, ay) = |a|^{p}U^{0}(x, y)$. A “splicing” argument (cf. [15]) gives us concavity of $U^{0}(\cdot, y)$: if $\lambda \in (0, 1)$, $f, g \in \mathcal{M}$, $f_{0} = x_{1}$, and $g_{0} = x_{2}$, then the process defined by $h_{0} = \lambda x_{1} + (1 - \lambda)x_{2}$ and $h_{n}(\omega) = \lambda f_{n-1}(\omega/\lambda)1_{\omega \in [0, \lambda)} + (1 - \lambda)g_{n-1}((\omega - \lambda)/(1 - \lambda))1_{\omega \in [\lambda, 1)}$, $n \geq 1$, is a simple martingale starting from $x = \lambda x_{1} + (1 - \lambda)x_{2}$. Hence

$$U^{0}(x, y) \geq \mathbb{E}V(h_{\infty}, h_{\infty}^{*} \vee y) = \lambda \mathbb{E}V(f_{\infty}, f_{\infty}^{*} \vee y) + (1 - \lambda)\mathbb{E}V(g_{\infty}, g_{\infty}^{*} \vee y),$$

which after taking the suprema over $f$ and $g$ yields the claim.

Moreover, if $f \in \mathcal{M}$ satisfies $f_{0} = y$, then $\mathbb{E}V(f_{\infty}, f_{\infty}^{*} \vee y) = \mathbb{E}V(f_{\infty}, f_{\infty}^{*}) \leq 0$, where the inequality follows from the assumption that the martingale inequality is satisfied with constant $C$. Therefore $U^{0}(y, y) \leq 0$ for all $y \in \mathbb{R}$, and hence $U^{0}(x, y) < +\infty$ for any $x, y \in \mathbb{R}$. Indeed, $U^{0}$ is concave with respect to the first variable, and a concave function on the real line, which takes values in the set $(-\infty, +\infty)$, is equal to $+\infty$ at some point, is identically equal to $+\infty$.

We now exploit the function $U^{0}$ to get an estimate of the constant $C$. Fix $\alpha \leq (p - 1)/p$ and $\delta, t \in (0, 1)$. The properties of $U^{0}$ imply

$$U^{0}(1, 1) \geq \frac{\delta}{1 - \alpha + \delta}U^{0}(\alpha, 1) + \frac{1 - \alpha}{1 - \alpha + \delta}U^{0}(1 + \delta, 1) \geq \frac{\delta}{1 - \alpha + \delta}V(\alpha, 1) + \frac{1 - \alpha}{1 - \alpha + \delta}U^{0}(1 + \delta, 1 + \delta) \geq \frac{\delta}{1 - \alpha + \delta}V(\alpha, 1) + \frac{1 - \alpha}{1 - \alpha + \delta}U^{0}(1, 1),$$

which can be rewritten in the form

$$U^{0}(1, 1) \frac{1 - \alpha + \delta - (1 - t)(1 - \alpha)(1 + \delta)^{p}}{\delta} \geq V(\alpha, 1) + \frac{1 - \alpha}{1 - \alpha + \delta}(1 + \delta)^{p}V(1, 1).$$

Now, for $\delta < 1/p$, we put $t = \delta(p - 1/(1 - \alpha))$ (note that $t \in [0, 1]$, since $\alpha \leq (p - 1)/p$), take $\delta \to 0^{+}$, and arrive at

$$0 \geq V(\alpha, 1) + (p(1 - \alpha) - 1)V(1, 1).$$

Using the definition of the function $V$ we can solve this inequality with respect to $C$. Taking the supremum over $\alpha \leq (p - 1)/p$ yields then

$$C^{p} \geq \sup_{\alpha \leq (p - 1)/p} \frac{|\alpha - 1|^{p}}{p(1 - \alpha) - 1 + |\alpha|^{p}}$$

(note that $p(1 - \alpha) - 1 + |\alpha|^{p}$ is strictly positive for $\alpha \neq 1$, since the function $\alpha \mapsto p\alpha - p + 1$ is tangent to the convex function $\alpha \mapsto |\alpha|^{p}$ at $\alpha = 1$). Hence the best constant with which the martingale inequality is satisfied is not smaller than the right-hand side of the above inequality.

2.3. Finding the concave majorant. In this subsection we give some informal reasoning, which is helpful in guessing an explicit formula for the function $U$ satisfying the assumptions of Proposition 2.3. As before, we denote $V(x, y) = |x - y|^{p} - C^{p}|x|^{p}$. We look for a function $U$ such that $U(\cdot, y)$ is not only concave, but even affine: let

$$U(x, y) = p(|\alpha x - y|^{p-2}(\alpha y - y) - C^{p}|\alpha y|^{p-2}\alpha y)(x - \alpha y) + |\alpha y - y|^{p} - C^{p}|\alpha y|^{p}.$$
be the tangent to \( V(\cdot, y) \) at the point \( x = ay \) (for some \( \alpha \), yet to be determined). Note that if \( V \) was concave with respect to the first variable, then such a choice of \( U \) would automatically guarantee the majorization property (i.e. \( V(x, y) \leq U(x, y) \)). Unfortunately, this is not the case in our setting (cf. Lemma 3.6).

The maximal condition states that \( U(x + h, x + h) \leq U(x, y) \) for \( x + h > y \), and implies \( U_y(x, x) \leq 0 \). Let us assume that \( U_y(x, x) = 0 \). Some calculations reveal that this condition is equivalent to

\[
C^p = |\alpha^{-1} - 1|^{p-2}(\alpha^{-1} - 1)(((p-1)(1-\alpha))^{-1} - 1)
\]

(provided that \( \alpha \notin (0, 1) \)). Taking such \( C \) we arrive at

\[
U(x, y) = -\frac{|1-\alpha|^{p-2}}{p-1}y^{p-2}(px-y(p-1)).
\]

Note that for this choice the initial condition (i.e. \( U(x, x) \leq 0 \)) is also satisfied.

Moreover, the preceding subsection suggests that for the right choice of \( \alpha \) we should have

\[
|\alpha^{-1} - 1|^{p-2}(\alpha^{-1} - 1)((p-1)(1-\alpha))^{-1} - 1) = \sup_{\alpha' \leq (p-1)/p} \frac{|\alpha'-1|^p}{p(1-\alpha') - 1 + |\alpha'|^p}.
\]

We identify the correct values of \( \alpha = \alpha(p) \) and \( C = C(p) \) in some technical lemmas in the next section. This is relatively easy in the case \( 1 < p \leq 2 \), where one can simply take \( \alpha(p) = (p - 1)/p \).

In Section 2.5, we check that for these choices the function \( U \) is indeed the majorant of \( V \). We prove the martingale inequality (2.1) in Section 2.6.

2.4. Technical lemmas. The first three results are needed to identify the value of the optimal constant in the martingale inequality (2.1).

**Lemma 2.5.** For each \( p \in (2, \infty) \), there exists exactly one number \( \alpha_p \leq (p - 1)/p \) such that

\[
\frac{|\alpha_p - 1|}{p(1-\alpha_p) - 1 + |\alpha_p|^p} = \sup_{\alpha \leq (p-1)/p} \frac{|\alpha - 1|}{p(1-\alpha) - 1 + |\alpha|^p} = \sup_{\alpha \neq 1} \frac{|\alpha - 1|}{p(1-\alpha) - 1 + |\alpha|^p}.
\]

Moreover, \( (p-1)\alpha_p + 2 - p = |\alpha_p|^{p-2}\alpha_p \) and \( \alpha_p < -(p-1)^{1/(p-2)} < 0 \).

**Proof.** Recall that \( p(1-\alpha) - 1 + |\alpha|^p \) is strictly positive for \( \alpha \neq 1 \) (since the function \( \alpha \mapsto px - p + 1 \) is tangent to the convex function \( \alpha \mapsto |\alpha|^p \) at \( \alpha = 1 \)). Moreover, \( \lim_{\alpha \to 1} |\alpha - 1|^p / (p(1-\alpha) - 1 + |\alpha|^p) = 0 \). Hence the function

\[
h(\alpha) = \frac{|\alpha - 1|}{p(1-\alpha) - 1 + |\alpha|^p} 1\{\alpha \neq 1\}
\]

is continuous. Its derivative (for \( \alpha \neq 1 \)) is equal to

\[
h'(\alpha) = \frac{p|\alpha - 1|^{p-2}(\alpha - 1)(p(1-\alpha) - 1 + |\alpha|^p) - |\alpha - 1|^{p-1}(-p + p|\alpha|^{-2}\alpha)}{(p(1-\alpha) - 1 + |\alpha|^p)^2},
\]

which is nonpositive if and only if

\[
(\alpha - 1)((p(1-\alpha) - 1 + |\alpha|^p) - (\alpha - 1)(-1 + |\alpha|^{-2}\alpha)) \leq 0,
\]

which we can simplify to

\[
(\alpha - 1)((|\alpha|^{-2}\alpha - (p-1)\alpha - 2 + p) \leq 0
\]

The function \( \alpha \mapsto (p - 1)\alpha + 2 - p \) is linear and tangent (at \( \alpha = 1 \)) to the function \( \alpha \mapsto |\alpha|^{-2}\alpha \), which is strictly concave on \((-\infty, 0]\) and strictly convex on \([0, \infty)\). Therefore, the equation \((p-1)\alpha + 2 - p = |\alpha|^{-2}\alpha \) has exactly one negative solution, which we denote by \( \alpha_p \). Moreover, the inequality (2.2) holds if and only if \( \alpha \in [\alpha_p, 1] \). Hence, the function \( h \) is increasing on \((-\infty, \alpha_p]\), decreasing on \([\alpha_p, 1]\),
and increasing on $[1, \infty)$. The observation that $\lim_{\alpha \to \pm \infty} h(\alpha) = 1$ ends the proof of the first part of the lemma.

Moreover, the inequality (2.2) does hold for $\alpha = -(p-1)^{1/(p-2)}$ and hence $\alpha_p < -(p-1)^{1/(p-2)}$.

**Lemma 2.6.** Let $\alpha_p$ be the number defined for $p > 2$ in Lemma 2.5. Then
\[
|\alpha_p^{-1} - 1|^{p-2}(\alpha_p^{-1} - 1)((p-1)(1-\alpha_p))^{-1} - 1) = \frac{|\alpha_p - 1|^p}{p(1-\alpha_p) - 1 + |\alpha_p|^p} = \frac{(1 + |\alpha_p|)^{p-2}}{p-1} > 1.
\]

**Proof.** We have
\[
((p-1)(1-\alpha_p))^{-1} - 1 = \frac{(p-1)\alpha_p + 2 - p}{(p-1)(1-\alpha_p)} = \frac{|\alpha_p|^{p-2}\alpha_p}{(p-1)(1-\alpha_p)}
\]
and hence the first and third expressions are equal. Also
\[
p(1-\alpha_p) - 1 + |\alpha_p|^p = p(1-\alpha_p) - 1 + \alpha_p((p-1)\alpha_p + 2-p) = (p-1)(\alpha_p - 1)^2
\]
and hence the second and third expressions are equal. Finally, the inequality follows directly from the estimate for $\alpha_p$ from the preceding lemma.

**Lemma 2.7.** Let $p \in (1, 2)$. If we denote $\alpha_p = (p-1)/p$, then
\[
\frac{1}{(p-1)^p} \leq \frac{|\alpha_p - 1|^p}{p(1-\alpha_p) - 1 + |\alpha_p|^p} = \frac{\sup_{\alpha \in (p-1)/p} |\alpha - 1|^p}{p(1-\alpha) - 1 + |\alpha|^p}
\]
\[
= |\alpha_p^{-1} - 1|^{p-2}(\alpha_p^{-1} - 1)((p-1)(1-\alpha_p))^{-1} - 1).
\]

The proof is less involved than in the case $p > 2$. Therefore we leave the details of checking that the function
\[
\alpha \mapsto \frac{|\alpha - 1|^p}{p(1-\alpha) - 1 + |\alpha|^p}, \quad \alpha \in (-\infty, (p-1)/p],
\]
attains its maximum at $\alpha = (p-1)/p$ to the Readers. Let us only remark that in contrast to the case $p > 2$, for $1 < p < 2$ we have
\[
\sup_{\alpha \neq 1} |\alpha - 1|^p \frac{p(1-\alpha) - 1 + |\alpha|^p}{p(1-\alpha) - 1 + |\alpha|^p} = \infty.
\]

We also need the following technical lemma.

**Lemma 2.8.** If $p \in (1, 2)$, then
\[
p^{p-2} \geq (p-1)^{p-1}, \quad (p+1)^{p-1} \geq (2p-1)(p-1)^{p-1}.
\]

**Proof.** Both inequalities are satisfied in the limit for $p \to 1^+$ and $p \to 2^-$. To prove the first, we notice that the difference of the logarithms of both sides is a concave function since
\[
((p-2)\ln(p) - (p-1)\ln(p-1))^\prime = -\frac{2}{(p-1)^2} \leq 0,
\]
for $p \in (1, 2)$.

In order to prove the second inequality, we substitute $s = p - 1$, divide both sides by $(2s+1)s^s$, take the logarithm of both sides, and arrive at the following equivalent formulation of the assertion:
\[
s \ln(1 + 2/s) - \ln(2s + 1) \geq 0, \quad s \in (0, 1).
\]
The left-hand side is a concave function, since for \( s \in (0, 1) \),
\[
(s \ln(1 + 2/s) - \ln(2s + 1))^\prime = \frac{4(s^3 - 1)}{s(s + 2)^2(2s + 1)^2} \leq 0.
\]
Hence the assertion of the lemma holds. \( \square \)

2.5. **The special function.** Define the constant \( C_p \) by the formula
\[
C_p^p = \sup_{\alpha \leq (p-1)/p} \frac{|\alpha - 1|^p}{p(1 - \alpha) - 1 + |\alpha|^p} = \begin{cases} \frac{1}{|\alpha|^p} & \text{if } 1 < p \leq 2, \\ \frac{1 + |\alpha|^p}{p} & \text{if } p > 2. \end{cases}
\]
Here \( \alpha_p \in \mathbb{R} \), \( 2 < p < \infty \) is the unique negative solution to the equation \((p-1)\alpha_p + 2 - p = |\alpha_p|^{p-2}\alpha_p \) (see Lemma 2.5). We also denote \( \alpha_p = (p-1)/p \) for \( 1 < p \leq 2 \).

We introduce the special functions
\[
V(x, y) = |x - y|^p - C_p^p |x|^p,
\]
\[
U(x, y) = -\frac{|\alpha|^p}{p-1} |y|^{p-2} y(px - (p-1)y)
\]
\[
= \begin{cases} -\frac{1}{|\alpha|^p} |y|^{p-2} y(px - (p-1)y) & \text{if } 1 < p \leq 2, \\ -C_p^p |y|^{p-2} y(px - (p-1)y) & \text{if } p > 2. \end{cases}
\]
The following proposition is the core of the proof of the martingale inequality and the main result (in the case \( m = 0, \lambda = 1 \)). Note that the assertion is stronger than the majorization condition from Proposition 2.3.

**Proposition 2.9.** For \( 1 < p < \infty \) and any \( x, y \in \mathbb{R} \), we have \( V(x, y) \leq U(x, y) \).

**Proof for** \( 2 < p < \infty \). The inequality is satisfied for \( y = 0 \) and \( x = y \), so by homogeneity it is enough to consider it for \( y = 1 \) and \( x \neq 1 \). We can rewrite it as
\[
C_p^p (|x|^p - px + p - 1) \geq |x - 1|^p.
\]
For \( x \neq 1 \) the left-hand side is positive (the function \( x \mapsto px - p + 1 \) is tangent to the convex function \( x \mapsto |x|^p \) at \( x = 1 \)), and therefore we conclude that the assertion is equivalent to
\[
C_p^p \geq \sup_{x \neq 1} \frac{|x - 1|^p}{p(1 - x) - 1 + |x|^p},
\]
which is true by Lemma 2.5. \( \square \)

**Remark 2.10.** The above proof stresses the fact that \( C_p \) is chosen exactly so, that the statement is true, but we can also use a slightly different approach. Again, it is enough to consider \( y = 1 \). The function \( V(., 1) \) is continuously differentiable, its second derivative exists in all but two points, and moreover \( V_{xx}(x, 1) = 0 \) if and only if \( |x - 1|^{p-2} = C_p^p |x|^{p-2} \) or equivalently \( x = 1/(1 + C_p^p/(p-2)) \). Hence the function \( V(., 1) \) is concave on the interval \( (-\infty, a] \), convex on the interval \( [a, b] \), and again concave on the interval \([b, \infty)\), where \( a = 1/(1 - C_p^p/(p-2)) \), \( b = 1/(1 + C_p^p/(p-2)) \). Moreover \( U(., 1) \) is the tangent to \( V(., 1) \) at the points \( \alpha_p \) and 1. This implies the inequality \( V(x, 1) \leq U(x, 1) \) for \( x \in \mathbb{R} \) (see Lemma 3.6 below).

We turn to the case \( 1 < p < 2 \) (for \( p = 2 \) the assertion is trivial). The argument is similar to that above, but slightly more complicated.

**Proof for** \( 1 < p < 2 \). The inequality is satisfied for \( y = 0 \), so by homogeneity it is enough to consider it for \( y = 1 \). We can rewrite it as (recall that \( \alpha_p = (p-1)/p \))
\[
C_p^p (|x|^p - p \frac{\alpha_p}{1 - \alpha_p} (x - \alpha_p)) \geq |x - 1|^p.
\]
It is easy to see that left-hand side is strictly positive (the global minimum is attained for \( x = \alpha_p (1 - \alpha_p)^{-1/(p-1)} \), for which the expression in the brackets on the left-hand side is equal to
\[
|\alpha_p|^p (1 - \alpha_p)^{-p/(p-1)} (1 - p + p (1 - \alpha_p)^{-1/(p-1)}),
\]
which is positive by the first inequality from Lemma 2.8). Therefore we conclude that the assertion is equivalent to the inequality
\[
(2.3) \quad C_p^p \geq \frac{|x - 1|^p}{|x|^p - p \alpha_p \alpha_p^{-1} (x - \alpha_p)}
\]
holding for every \( x \in \mathbb{R} \). We denote the right-hand side of the above inequality by \( R(x) \). A calculation shows that \( R'(x) \) is positive if and only if
\[
p|x - 1|^{p-2} (x - 1) \left( |x|^p - \frac{\alpha_p^{-1}}{1 - \alpha_p} (x - \alpha_p) \right) - |x - 1|^p \left( p|x|^{p-2} x - p \frac{\alpha_p^{-1}}{1 - \alpha_p} \alpha_p \right) = p|x - 1|^{p-2} (x - 1) \left( T(x) - S(x) \right) \geq 0,
\]
where we have denoted
\[
S(x) = \frac{\alpha_p^{-1}}{1 - \alpha_p} ((p-1)x - p\alpha_p + 1) = p\alpha_p^{-1} ((p-1)x - p + 2),
\]
\[
T(x) = |x|^{p-2} x.
\]
The function \( T \) is convex on \((-\infty, 0)\) and concave on \((0, \infty)\), since \( 1 < p < 2 \). Moreover, \( S(\alpha_p) = T(\alpha_p) \) and
\[
S'(\alpha_p) = p(p-1)\alpha_p^{-1} < (p-1)\alpha_p^{-2} = T'(\alpha_p).
\]
We conclude that the equation \( S(x) = T(x) \) has three solutions: \( x_1 < 0, x_2 = \alpha_p, \) and \( x_3 > \alpha_p \). Moreover, \( x_3 \geq (p+1)/p > 1 \) since \( S((p+1)/p) \leq T((p+1)/p) \) by the second inequality from Lemma 2.8.

Therefore, the function \( R \) is decreasing on each of the intervals \((-\infty, x_1), (\alpha_p, 1), (x_3, \infty), \) and increasing on \((x_1, \alpha_p)\) and \((1, x_3)\). Since \( R(\alpha_p) = (1/\alpha_p - 1)^p = C_p^p \) and \( \lim_{x \to \infty} R(x) = 1 \leq C_p^p \), in order to prove (2.3) it is enough to check that \( C_p^p \geq R(x_3) \). But \( S(x_3) = T(x_3) = x_3^{p-1} \), so
\[
|x_3|^p - p \frac{\alpha_p^{-1}}{1 - \alpha_p} (x - \alpha_p) = x_3 S(x_3) - p \frac{\alpha_p^{-1}}{1 - \alpha_p} (x - \alpha_p) = \frac{\alpha_p^{-1}}{1 - \alpha_p} (p-1)(x_3 - 1)^2
\]
and consequently
\[
R(x_3) = \frac{1 - \alpha_p}{\alpha_p^{-1}} \frac{(x_3 - 1)^{p-2}}{(p-1)} = \frac{p^{p-2}}{(p-1)^p} (x_3 - 1)^{p-2}.
\]
Hence, \( C_p^p \geq R(x_3) \) is equivalent to \( 1 \leq p(x_3 - 1) \) (recall that \( p - 2 < 0 \)). Since we already know that \( x_3 \geq (p+1)/p \), the proof is finished. \( \square \)

2.6. **Proof of the martingale inequality.** In order to prove the martingale inequality, we just gather the results of the preceding sections.

*Proof of inequality (2.1).* For \( p \in (1, \infty) \), the functions \( V \) and \( U \) defined in Section 2.5 satisfy all assumptions of Proposition 2.3. Indeed, the majorization property follows from Proposition 2.9. The initial condition is satisfied, since \( U(x, x) = -|x|^p |1 - \alpha_p|^{p-2}/(p-1) \leq 0 \). If \( y \leq 0 \leq x + h \), then \( U(x + h, x + h) \leq 0 \leq U(x + h, y) \).

If on the other hand \( y < x + h < 0 \) or \( 0 < y < x + h \), then there exists \( \xi \in (y, x + h) \) such that
\[
U(x + h, x + h) - U(x + h, y) = U_y(x + h, \xi)(x + h - y)
\]
= p|1 - α|p^{-2}(ξ - x)(x + h - y) \leq 0.

This implies the maximal condition. Finally, \( U \) is clearly concave with respect to the first variable. Hence the martingale inequality (2.1) holds by Proposition 2.3. □

Moreover, from the abstract argument in Subsection 2.2 we already know that the constant \( C_p \) is optimal. Let us however give explicit extremal examples here.

**Sharpness of (2.1).** Fix \( 1 < p < \infty \), \( α \in (-∞, (p - 1)/p) \setminus \{0\} \) and \( s \in (0, 1) \), and let \( β = β(s) \) be given by the relation \( sα + (1 - s)β = 1 \). Observe that if \( s \) is sufficiently small (depending on \( p, α \)), then \((1 - s)β > 1\) (indeed, the inequality \((1 - sα)p > (1 - s)p - 1\) can be verified by comparison of derivatives at \( s = 0 \)). We consider a martingale \( (f_n)_{n=0}^∞ \) such that \( f_0 = 1 \), and such that conditioned on the event \( \{(f_n, f^n_*) = (x, x)\} \), one of the following events occurs:

1. With probability \( s \) we have \( f_{n+1} = αx \) and the martingale stops, i.e. \( f_{n+1} = f_{n+2} = \ldots \); note that in this case \( f^*_n = f_n = x \).
2. With probability \( 1 - s \), we have \( f_{n+1} = βx \) and the evolution continues accordingly to our rules. In this case \( f^*_n = f_{n+1} \).

Note that \( f_n \) takes values in the set \( \{α, αβ, αβ^2, \ldots, αβ^{n-1}, β^n\} \). Moreover, \( E(f_n = αβ^k) = s(1 - s)^k \) for \( k \in \{0, \ldots, n - 1\} \), \( E(f_n = β^n) = (1 - s)^n \), and

\[
E|f_n|^p_{1(f_n\neq β^n)} = \sum_{k=0}^{n-1} s(1 - s)^k(αβ^k)^p = s|α|^p \frac{1 - (1 - s)^nβ^p}{1 - (1 - s)β^p}.
\]

Also, \( f_n - f^*_n = (1 - 1/α)f_n \) if \( f_n \neq β^n \), and \( f_n - f^*_n = 0 \) if \( f_n = β^n \). Hence, the \( p \)-th power of the constant with which the martingale inequality (2.1) holds has to be equal at least

\[
\lim_{s \to 0^+, \ n \to \infty} \lim_{n \to \infty} \frac{\|f_n - f^*_n\|_p}{\|f_n\|_p} = \lim_{s \to 0^+, \ n \to \infty} \frac{|1 - 1/α|^p s|α|^p \frac{1 - (1 - s)^nβ^p}{1 - (1 - s)β^p} + (1 - s)^nβ^p}{s|α|^p + (1 - s)β^p - 1} = \frac{|α - 1|^p}{|α|^p - pα + p - 1},
\]

where we used \((1 - s)β > 1\) in the second last equality, and \( β = (1 - sα)/(1 - s) \) in the last equality. To obtain the sharpness of (2.1) we take \( α \to (p - 1)/p^+ \) in the case \( 1 < p \leq 2 \) or take \( α = α_p \) (see Lemma 2.5) in the case \( p > 2 \). □

**Remark 2.11.** In the above example \( f_n \) converges a.s. to a random variable \( f_∞ \), but \( E|f_∞|^p = +\infty \). In the case \( 1 < p \leq 2 \) one can consider \( α > (p - 1)/p \) instead of \( α < (p - 1)/p \) to obtain an example in which \( E|f_∞|^p < \infty \).

**2.7. Relation to Theorem 1.1 for m = 0.** As announced before, the martingale inequality implies the main result for \( m = 0 \) and \( λ = 1 \) in the special case of nonincreasing functions.

**Proof of Corollary 2.2.** First note that a standard approximation arguments yields a continuous time version of (2.1): for any martingale \( (X_t)_{t \geq 0} \) with right-continuous trajectories, we have

\[
\|X_t - \sup_{0 \leq s \leq t} X_s\|_p < C_p\|X_t\|_p.
\]

Let \( f \in L^p([0, 1]) \) be a nonincreasing function. On the probability space \([0, 1]\), equipped with the Lebesgue measure, consider the filtration

\[
F_t = \sigma([0, 1 - t], B([1 - t, 1])), \quad t \in [0, 1],
\]

where \( B([a, b]) \) denotes the Borel σ-algebra on \([a, b] \).
Clearly, $g$ is a positive number. Moreover, we have the following slight extension of [4, Proposition 5.1].
Proposition 3.1. For $1 < p < \infty$ and $m > -2(p-1)/p$, the formula
\[ H_m f(t) = \frac{1}{t^{1+m/2}} \int_0^t f(s)s^{m/2}ds \]
defines a bounded operator on the space $L^p([0, \infty))$. Moreover,
\[ \|H_m\|_{L^p([0, \infty))\to L^p([0, \infty))} = g_{p,m}^{-1}. \]

Proof. If $f \in L^p([0, \infty))$ is bounded, then the function $H_m f$ is well defined (since $m/2 > -1$), and the Minkowski integral inequality yields
\[ \|H_m f\|_p = \left( \int_0^\infty \left| \frac{1}{t^{1+m/2}} \int_0^t f(s)s^{m/2}ds \right|^p dt \right)^{1/p} \]
\[ = \left( \int_0^1 \left( \int_0^1 |f(ut)|u^{m/2}du \right|^p dt \right)^{1/p} \]
\[ \leq \int_0^1 \left( \int_0^\infty (|f(ut)|)^p dt \right)^{1/p} u^{m/2} du = \int_0^1 u^{m/2-1/p} \|f\|_p du = g_{p,m}^{-1} \|f\|_p. \]

A standard density argument implies the claim. The family $t \mapsto t^{-1/p+\varepsilon} f_1(t \in [0,1])$ extremizes the norm as $\varepsilon \to 0^+$. \qed

We define the set $\Omega_{p,m}$, the function $c_{p,m,\lambda} : \Omega_{p,m} \to \mathbb{R}$, and the constant $C_{p,m,\lambda}$ by the formulas:
\[ \Omega_{p,m} = \{ (\alpha, \beta) \in \mathbb{R}^2 : \alpha < g_{p,m} < \beta \}, \]
\[ c_{p,m,\lambda}(\alpha, \beta) = \left( \frac{(\beta - g_{p,m})|\alpha - \lambda|^p + (g_{p,m} - \alpha)|\beta - \lambda|^p}{(\beta - g_{p,m})|\alpha|^p + (g_{p,m} - \alpha)|\beta|^p} \right)^{1/p}, \quad (\alpha, \beta) \in \Omega_{p,m}, \]
\[ C_{p,m,\lambda} = \sup \{ c_{p,m,\lambda}(\alpha, \beta) : (\alpha, \beta) \in \Omega_{p,m} \}. \]

Recall that our aim is to show that $C_{p,m,\lambda} = \|I - \lambda H_m\|_{L^p([0, \infty))\to L^p([0, \infty))}$. The first lemma shows that $C_{p,m,\lambda}$ is a lower bound for the norm of the operator $I - \lambda H_m$. It also serves as a proof that $C_{p,m,\lambda}$ is finite.

Lemma 3.2. If $1 < p < \infty$, $m > -2(p-1)/p$, $\lambda \in \mathbb{R}$, then
\[ \|I - \lambda H_m\|_{L^p([0, \infty))\to L^p([0, \infty))} \geq C_{p,m,\lambda}. \]

Proof. Fix $1 < p < \infty$ and $m > -2(p-1)/p$. For $\alpha < g_{p,m} < \beta$, consider the function
\[ f_{\alpha,\beta}(t) = \beta t^{\beta-g_{p,m}-1/p} \mathbf{1}_{\{t \in [0,1]\}} + \alpha t^{\alpha-g_{p,m}-1/p} \mathbf{1}_{\{t \in [1,\infty]\}}, \]
which clearly belongs to the space $L^p([0, \infty))$. We have
\[ H_m f_{\alpha,\beta}(t) = \beta t^{\beta-g_{p,m}-1/p} \mathbf{1}_{\{t \in [0,1]\}} + \alpha t^{\alpha-g_{p,m}-1/p} \mathbf{1}_{\{t \in [1,\infty]\}} \]
and
\[ \|f_{\alpha,\beta} - \lambda H_m f_{\alpha,\beta}\|_p \leq \left( \frac{\beta - \lambda}{p(\beta - g_{p,m})} + \frac{\alpha - \lambda}{p(\alpha - g_{p,m})} \right) \]
\[ \|f_{\alpha,\beta}\|_p = \left( \frac{\beta}{p(\beta - g_{p,m})} + \frac{\alpha}{p(\alpha - g_{p,m})} \right). \]

Thus we see that $\|I - \lambda H_m\|_{L^p([0, \infty))\to L^p([0, \infty))} \geq C_{p,m,\lambda}$. \qed

The next lemma summarizes further observations about the constant $C_{p,m,\lambda}$.

Lemma 3.3. If $1 < p < \infty$, $m > -2(p-1)/p$, and $\lambda \in \mathbb{R}$, then
\[ C_{p,m,\lambda} \geq \max \{ 1 - \lambda g_{p,m}^{-1}, 1 \}. \]
Also, if the above inequality is strict, then the supremum in the definition of $C_{p,m,\lambda}$ is attained at some point of the set $\Omega_{p,m}$. Moreover, $C_{p,m,\lambda} > 1$ unless $\lambda = 0$ or $p = 2$.

Proof. Throughout the proof we consider only $(\alpha, \beta) \in \Omega_{p,m}$. For $\alpha \neq 0$, we can write the function $c_{p,m,\lambda}$ as a convex combination:

$$c_{p,m,\lambda}(\alpha, \beta) = w_1(\alpha, \beta) \cdot |1 - \lambda/\alpha|^p + w_2(\alpha, \beta) \cdot |1 - \lambda/\beta|^p,$$

where

$$w_1(\alpha, \beta) = \frac{(\beta - g_{p,m})|\alpha|^p}{(\beta - g_{p,m})|\alpha|^p + (g_{p,m} - \alpha)|\beta|^p},$$

$$w_2(\alpha, \beta) = \frac{(g_{p,m} - \alpha)|\beta|^p}{(\beta - g_{p,m})|\alpha|^p + (g_{p,m} - \alpha)|\beta|^p}.$$

Using (3.2) we see that

$$\limsup_{\alpha, \beta} c_{p,m,\lambda}(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha \to -\infty \text{ and } \beta \to \infty, \\ |1 - \lambda g_{p,m}^{-1}|^p & \text{if } \alpha \to g_{p,m}^- \text{ and } \beta \to g_{p,m}^+ \end{cases}$$

which implies the first part of the assertion.

We now claim, that if $(\alpha, \beta) \to \partial \Omega_{p,m}$ or $\alpha^2 + \beta^2 \to \infty$, then

$$\limsup_{\alpha, \beta} c_{p,m,\lambda}(\alpha, \beta) \leq \max\{1, |1 - \lambda g_{p,m}^{-1}|\}.$$  

It follows from (3.2) that (3.3) holds if $\alpha \to -\infty$ and $\beta \to g_{p,m}^+$, or $\alpha \to g_{p,m}^-$ and $\beta \to \infty$. If on the other hand $\alpha \to -\infty$ or $\alpha \to g_{p,m}^-$, and $\beta \to \beta_{\infty} \in (g_{p,m}, \infty)$, then $w_2(\alpha, \beta) \to 0$, and consequently

$$\lim_{\alpha, \beta} c_{p,m,\lambda}(\alpha, \beta) \in \{1, |1 - \lambda g_{p,m}^{-1}|\}.$$  

Similarly, (3.3) also holds if $\alpha \to \alpha_{\infty} \in (-\infty, g_{p,m}) \setminus \{0\}$ and $\beta \to g_{p,m}^+$ or $\beta \to \infty$ (because then $w_1(\alpha, \beta) \to 0$). Finally, if $\alpha \to 0$ and $\beta \to g_{p,m}^+$ or $\beta \to \infty$, then

$$\limsup_{\alpha, \beta} c_{p,m,\lambda}(\alpha, \beta) \leq \lim_{\alpha, \beta} \frac{(\beta - g_{p,m})|\alpha - \lambda|^p + (g_{p,m} - \alpha)|\beta - \lambda|^p}{(g_{p,m} - \alpha)|\beta|^p} = \max\{1, |1 - \lambda g_{p,m}^{-1}|\}.$$  

These observations imply that if the supremum in the definition of $C_{p,m,\lambda}$ is strictly greater than $\max\{1, |1 - \lambda g_{p,m}^{-1}|\}$, then it is attained at some point of the set $\Omega_{p,m}$.

The last part of the assertion clearly holds if $\lambda > 0$. Assume henceforth that $\lambda > 0$ and $p \neq 2$. Choose $A > 0$ and $B > \max\{\lambda - g_{p,m}, 0\}$ so that the inequality

$$A^{p-1}B > (g_{p,m} + A)(B + g_{p,m})^{p-1}$$

is satisfied (i.e. pick $A$ sufficiently large if $p > 2$ or $B$ sufficiently large if $1 < p < 2$). Since

$$(A + \lambda)^p - A^p \geq p\lambda A^{p-1},$$

$$p\lambda(B + g_{p,m})^{p-1} \geq (g_{p,m} + B)^p - (g_{p,m} + B - \lambda)^p,$$

such a choice of $A, B$ implies that

$$(A + \lambda)^p - A^p > (g_{p,m} + A)((g_{p,m} + B)^p - (g_{p,m} + B - \lambda)^p),$$

which is equivalent to $c_{p,m,\lambda}(-A, g_{p,m} + B) > 1$. □
Hence integration by parts yields

\[ H_m f_n = \int_0^1 \left( \frac{1}{t^{1+m/2}} - \frac{1}{(1+m/2)t^{1+m/2}} \right) (t \in [n, n+1)) \left( \frac{n+1}{1+m/2} t^{1+m/2} - \frac{1}{t} \right) dt \]

\[ \leq \int_n^\infty \left( \frac{(n+1)^{1+m/2} - n^{1+m/2}}{(1+m/2)n^{1+m/2}} \right)^p dt \]

\[ = \frac{(n+1)^{1+m/2} - n^{1+m/2}}{(1+m/2)n^{1+m/2}} \xrightarrow{n \to \infty} 0. \]

Hence the operator \( \lambda H_m : L^p([0, \infty)) \to L^p([0, \infty)) \) is not invertible, and consequently we cannot have \( \|I - \lambda H_m\|_{L^p([0,\infty))} \to L^p([0,\infty)) < 1 \).

3.2. Key tools. The first result generalizes Lemma 2.12 presented above without proof.

**Lemma 3.5.** If \( 1 < p < \infty, m > -2(p-1)/p, \) and \( f : [0, 1] \to \mathbb{R} \) is continuous, then

\[ (p(1+m/2) - 1) \int_0^1 |H_m f(t)|^p dt \leq p \int_0^1 |H_m f(t)|^{p-2} H_m f(t) f(t) dt. \]

**Proof.** Define \( F(t) = \int_0^1 f(s) s^{m/2} ds \). Since \( f \) is continuous, we have \( F'(t) = f(t)^{m/2} \) (in particular, \( H_m f(t) = F(t)/t^{1+m/2} \to f(0)/(1+m/2) \) as \( t \to 0^+ \)). Hence integration by parts yields

\[ (p(1+m/2) - 1) \int_0^1 |H_m f(t)|^p dt = (p(1+m/2) - 1) \int_0^1 t^{-p(1+m/2)} |F(t)|^p dt \]

\[ = [ - t^{-p(1+m/2)+1} |F(t)|^p ]_0^1 + p \int_0^1 t^{-p(1+m/2)+1} |F(t)|^{p-2} F(t) f(t) t^{m/2} dt \]

\[ = \lim_{t \to 0^+} |F(t)|^p + p \int_0^1 |H_m f(t)|^{p-2} H_m f(t) f(t) dt \]

\[ \leq |F(1)|^p + p \int_0^1 |H_m f(t)|^{p-2} H_m f(t) f(t) dt. \]

This implies the assertion of the lemma. \( \square \)

Moreover, the following elementary lemma is useful for us.

**Lemma 3.6.** Suppose that \( v : \mathbb{R} \to \mathbb{R} \) is continuously differentiable and strictly concave on \( (-\infty, a) \), strictly convex on \( (a, b) \), and strictly concave on \( (b, \infty) \) for some \( a, b \in \mathbb{R} \). Let \( u : \mathbb{R} \to \mathbb{R} \) be an affine function tangent to \( v \) at two points. Then \( v(x) \leq u(x) \) for \( x \in \mathbb{R} \).

**Proof.** Denote \( c = u'(x), x \in \mathbb{R} \). There exist \( \alpha < \beta \), such that \( v(\alpha) = u(\alpha), v(\beta) = u(\beta), \) and \( v'(\alpha) = v'(\beta) = c \). By Rolle’s theorem applied to the function \( v - u \), there exists \( \gamma \in (\alpha, \beta) \) such that \( v'(\gamma) = c \). Since the function \( v' \) attains every value at most thrice, we conclude that \( \alpha \in (-\infty, a], \gamma \in (a, b), \) and \( \beta \in [b, \infty) \). The assertion follows from the assumption about concavity (respectively convexity) of \( v \) on those intervals. \( \square \)

Finally, we have the following analog and generalization of Proposition 2.9.

**Proposition 3.7.** If \( 1 < p < \infty, p \neq 2, m > -2(p-1)/p, \) and \( \lambda > 0 \), then there exists a positive constant \( D_{p,m,\lambda} \), such that the inequality

\[ |x - Ay|^p - C_{p,m,\lambda} |x|^p \leq -D_{p,m,\lambda} |y|^{p-2} y(x - g_{p,m} y) \]
holds for all \(x, y \in \mathbb{R}\).

**Proof.** We shall consider two cases. As will be clear from the proof (and the following results) they correspond to the situation when the supremum in the definition of \(C_{p,m,\lambda}\) is equal to \(|\lambda g_{p,m}^{-1}| - 1\), and the situation when the supremum in definition of \(C_{p,m,\lambda}\) is attained in the interior of the set \(\Omega_{p,m}\) (cf. Lemma 3.3).

For the first case, we assume that \(\lambda > 2g_{p,m}\) and the inequality

\[
|x - \lambda|^p - (\lambda g_{p,m}^{-1} - 1)^p|x|^p \leq -p(\lambda - g_{p,m})^{p-1}\lambda g_{p,m}^{-1}(x - g_{p,m})
\]

holds for all \(x \in \mathbb{R}\). Define

\[
V(x, y) = |x - \lambda y|^p - (\lambda g_{p,m}^{-1} - 1)^p|x|^p,
\]

\[
U(x, y) = -p(\lambda - g_{p,m})^{p-1}\lambda g_{p,m}^{-1}|y|^{p-2}y(x - g_{p,m}y).
\]

The inequality \(V(x, y) \leq U(x, y)\) holds for \(y = 0\) (since \(\lambda g_{p,m}^{-1} - 1 > 1\)) and for \(y = 1\) (due to (3.4)), and hence by homogeneity for all \(x, y \in \mathbb{R}\). Hence the assertion is satisfied with \(D_{p,m,\lambda} = p(\lambda - g_{p,m})^{p-1}\lambda g_{p,m}^{-1} > 0\) (and with \(\lambda g_{p,m}^{-1} - 1\), which is not greater than \(C_{p,m,\lambda}\), in the place of \(C_{p,m,\lambda}\)). This finishes the proof in the first case.

Let us now consider the second case: we have either \(\lambda \in (0, 2g_{p,m}]\), or we have \(\lambda > 2g_{p,m}\), but the inequality (3.4) does not hold for all \(x \in \mathbb{R}\). We claim that the supremum in the definition of the constant \(C_{p,m,\lambda}\) is attained at some point of the set \(\Omega_{p,m}\). Indeed, if \(\lambda \in (0, 2g_{p,m}]\), then \(\max\{1 - |\lambda g_{p,m}^{-1}|, 1\} = 1\) and Lemma 3.3 implies the claim. If on the other hand \(\lambda > 2g_{p,m}\) and the inequality (3.4) does not hold for every \(x \in \mathbb{R}\), then there exists some \(x_0 \in \mathbb{R}\), such that

\[
|x_0 - \lambda|^p - (\lambda g_{p,m}^{-1} - 1)^p|x_0|^p > -p(\lambda - g_{p,m})^{p-1}\lambda g_{p,m}^{-1}(x_0 - g_{p,m}).
\]

Of course we cannot have \(x_0 = g_{p,m}\). Suppose first, that \(x_0 > g_{p,m}\). Since

\[
\lim_{x \to g_{p,m}} \frac{|x - \lambda|^p - (\lambda g_{p,m}^{-1} - 1)^p|x|^p}{x - g_{p,m}} = -p(\lambda - g_{p,m})^{p-1}\lambda g_{p,m}^{-1},
\]

we conclude from (3.5), that for some \((\alpha, \beta) \in \Omega_{p,m}\) we have

\[
|\beta - \lambda|^p - (\lambda g_{p,m}^{-1} - 1)^p|\beta|^p > \frac{|\alpha - \lambda|^p - (\lambda g_{p,m}^{-1} - 1)^p|\alpha|^p}{\alpha - g_{p,m}}(\beta - g_{p,m})
\]

(it suffices to take \(\beta = x_0\) and \(\alpha\) smaller than, but close to \(g_{p,m}\), or equivalently

\[
\frac{(\beta - g_{p,m})|\alpha - \lambda|^p + (g_{p,m} - \alpha)|\beta - \lambda|^p}{(\beta - g_{p,m})|\alpha|^p + (g_{p,m} - \alpha)|\beta|^p} > (\lambda g_{p,m}^{-1} - 1)^p.
\]

We arrive at the same conclusion, if \(x_0 < g_{p,m}\) (it suffices to take \(\alpha = x_0\) and \(\beta\) greater than, but close to \(g_{p,m}\)). This finishes the proof of the claim: in the second case we always have

\[
C_{p,m,\lambda} \leq c_{p,m,\lambda}(\alpha_0, \beta_0),
\]

for some point \((\alpha_0, \beta_0) \in \Omega_{p,m}\) (of course \(\alpha_0, \beta_0\) may depend of \(p, m, \lambda\); uniqueness is not important to us).

After denoting

\[
K(\alpha, \beta) = (\beta - g_{p,m})|\alpha - \lambda|^p + (g_{p,m} - \alpha)|\beta - \lambda|^p,
\]

\[
L(\alpha, \beta) = (\beta - g_{p,m})|\alpha|^p + (g_{p,m} - \alpha)|\beta|^p,
\]

we can rewrite the condition \(\frac{\partial}{\partial \beta} c_{p,m,\lambda}(\alpha_0, \beta_0) = 0\) as

\[
(p(\beta_0 - g_{p,m})|\alpha_0 - \lambda|^{p-2}(\alpha_0 - \lambda) - |\beta_0 - \lambda|^p) \cdot L(\alpha_0, \beta_0)
\]

\[
- K(\alpha_0, \beta_0) \cdot (p(\beta_0 - g_{p,m})|\alpha_0|^{p-2}\alpha_0 - |\beta_0|^p) = 0.
\]

The condition \(\frac{\partial}{\partial \alpha} c_{p,m,\lambda}(\alpha_0, \beta_0) = 0\) implies a similar equation.
Define now
\[ V(x, y) = |x - \lambda y|^p - C^p_{p, \alpha, \lambda}|x|^p, \]
\[ U(x, y) = \frac{V(\beta_0, 1) - V(\alpha_0, 1)}{\beta_0 - \alpha_0} |y|^{p-2} y(x - g_{p, \alpha, \lambda} y). \]
Using the fact that \( C^p_{p, \alpha, \lambda} = K(\alpha_0, \beta_0)/L(\alpha_0, \beta_0), \) we see that
\[ V(\alpha_0, 1) = \frac{(g_{p, \alpha, \lambda} - \alpha_0)(|\alpha_0 - \lambda|^p|\beta_0|^p - |\alpha_0|^p|\beta_0 - \lambda|^p)}{L(\alpha_0, \beta_0)}, \]
\[ V(\beta_0, 1) = \frac{(\beta_0 - g_{p, \alpha, \lambda})(|\alpha_0|^p|\beta_0 - \lambda|^p - |\alpha_0 - \lambda|^p|\beta_0|^p)}{L(\alpha_0, \beta_0)}, \]
and consequently
\[ (3.7) \quad \frac{V(\beta_0, 1) - V(\alpha_0, 1)}{\beta_0 - \alpha_0} = \frac{|\alpha_0|^p|\beta_0 - \lambda|^p - |\alpha_0 - \lambda|^p|\beta_0|^p}{L(\alpha_0, \beta_0)}. \]
Hence \( V(\alpha_0, 1) = U(\alpha_0, 1) \) and \( V(\beta_0, 1) = U(\beta_0, 1). \)

On the other hand (we use (3.6) in the second equality, and the definitions of \( K \) and \( L \) in the third),
\[ V_x(\alpha_0, 1) = \frac{p|\alpha_0 - \lambda|^p|\beta_0 - \lambda|^p - K(\alpha_0, \beta_0)p|\alpha_0|^p - L(\alpha_0, \beta_0)}{L(\alpha_0, \beta_0)} \]
\[ = \frac{L(\alpha_0, \beta_0)|\beta_0 - \lambda|^p - K(\alpha_0, \beta_0)|\beta_0|^p}{L(\alpha_0, \beta_0)} \]
By (3.7), we conclude that \( V_x(\alpha_0, 1) = U_x(\alpha_0, 1). \) Similarly, \( V_x(\beta_0, 1) = U_x(\beta_0, 1) \)
follows from \( \frac{p|\beta_0 - \lambda|^p - K(\alpha_0, \beta_0)|\beta_0|^p}{L(\alpha_0, \beta_0)} = 0. \)

Therefore, \( U(x, 1) : x \mapsto U(x, 1) \) is tangent to \( V(x, 1) : x \mapsto V(x, 1) \) at \( x = \alpha_0 \)
and \( x = \beta_0. \) Hence Lemma 3.6 implies that \( V(x, 1) \leq U(x, 1) \) for any \( x \in \mathbb{R}, \)
and by homogeneity also \( V(x, y) \leq U(x, y) \) for any \( x, y \in \mathbb{R} \) (for \( y = 0 \) the inequality holds, since \( C_{p, \alpha, \lambda} > 1). \)

Finally, let us notice that
\[ V_x(\beta_0, 1) = \frac{p(|\beta_0 - 1|^p - C^p_{p, \alpha, \lambda}|\beta_0 - 1|^p - |\beta_0|^p - 1 - C^p_{p, \alpha, \lambda})}{L(\alpha_0, \beta_0)}, \]
(we used the fact that \( \beta_0 > 0 \) and \( C_{p, \alpha, \lambda} > 1) \) and hence the assertion is satisfied
with
\[ D_{p, \alpha, \lambda} := U_x(\beta_0, 1) = V_x(\beta_0, 1) > 0. \]

3.3. Proof of the main result. We start with the following observation (cf. [17]).

Lemma 3.8. Let \( T : L^p([0, \infty)) \to L^p([0, \infty)) \) be a linear operator which maps
real-valued functions to real-valued functions. If the inequality \( ||Tf||_p \leq C||f||_p \)
holds for any real-valued function \( f \in L^p([0, \infty)), \) then it also holds (with the same constant)
for any complex-valued function \( f \in L^p([0, \infty)). \)

Proof. Suppose that \( f = u + iv \in L^p([0, \infty)), \) where \( u, v \) are real-valued. Let \( G_1, \)
\( G_2 \) be independent Gaussian random variables with mean zero and variance one.
Using the fact that for \( a_1, a_2 \in \mathbb{R} \) the random variable \( a_1 G_1 + a_2 G_2 \) has the same
distribution as \( \sqrt{a_1^2 + a_2^2} G_1, \) we arrive at
\[ ||Tf||_p^p E|G_1|^p = \int_0^\infty |T(u)^2 + T(v)^2|^{p/2} E|G_1|^p dt = \int_0^\infty E|T(u)G_1 + T(v)G_2|^p dt \]
\[ = E \int_0^\infty |T(uG_1 + vG_2)|^p dt \leq C^p E \int_0^\infty |uG_1 + vG_2|^p dt = C^p ||f||_p^p E|G_1|^p \]
we have suppressed the dependence of the functions on the argument \( t \in [0, \infty) \) in
the notation). This finishes the proof of the lemma. \( \square \)
Henceforth we assume without loss of generality that all functions are real-valued. The proof of Theorem 1.1 is divided into three parts.

Proof of inequality (1.1) for \( \lambda > 0 \) and \( p \neq 2 \). Fix \( 1 < p < \infty \), \( p \neq 2 \), \( m > 2(p-1)/p \), \( \lambda \in \mathbb{R} \), and denote

\[
V(x, y) = |x - \lambda y|^p - C_{p,m,\lambda}^p|x|^p,
\]

\[
U(x, y) = -D_{p,m,\lambda}|y|^{p-2}y(x - g_{p,m}y),
\]

where \( D_{p,m,\lambda} \) is the positive number from Proposition 3.7.

Let \( f : [0, 1] \to \mathbb{R} \) be a continuous function. By Proposition 3.7, for every \( t \in [0, 1] \) we have \( V(f(t), H_m f(t)) \leq U(f(t), H_m f(t)) \). After integrating over the interval \([0, 1]\) and applying Lemma 3.5, we arrive at

\[
\int_0^1 |f(t) - \lambda H_m f(t)|^p dt - C_{p,m,\lambda}^p \int_0^1 |f(t)|^p dt
\]

\[
\leq -D_{p,m,\lambda} \int_0^1 |H_m f(t)|^{p-2} H_m f(t) (f(t) - g_{p,m} H_m f(t)) dt \leq 0.
\]

A standard approximation argument gives \( \|f - \lambda H_m f\|_{L^p([0, 1])} \leq C_{p,m,\lambda} \|f\|_{L^p([0, 1])} \) for \( f \in L^p([0, 1]) \). If \( f \in L^p([0, \infty)) \), then

\[
\int_0^n |f(t) - \lambda H_m f(t)|^p dt = n \int_0^1 |f(nt) - \lambda H_m (f(nt))(t)|^p dt
\]

\[
\leq n C_{p,m,\lambda}^p \int_0^1 |f(nt)|^p dt = C_{p,m,\lambda} \int_0^n |f(t)|^p dt,
\]

and it suffices to take \( n \to \infty \) to arrive at \( \|f - \lambda H_m f\|_{L^p([0, \infty))} \leq C_{p,m,\lambda} \|f\|_{L^p([0, \infty))} \). This ends the proof of inequality (1.1).

Proof of inequality (1.1) for \( \lambda > 0 \) and \( p = 2 \). By an argument similar to that for \( p \neq 2 \), we show that \( \|I - \lambda H_m\|_{L^2([0, \infty))} \to L^2([0, \infty)) \leq \lambda g_{p=2,m}^{-1} - 1 \) for \( \lambda > 2g_{p=2,m} = 1 + m \). Indeed, we only need to use the functions

\[
V(x, y) = (x - \lambda y)^2 - (\lambda g_{p=2,m}^{-1} - 1)^2 x^2,
\]

\[
U(x, y) = -2(\lambda - g_{p=2,m}) \lambda g_{p=2,m} y(x - g_{p=2,m} y),
\]

for which checking the majorization is straightforward \((x \mapsto U(x, 1)\) is the tangent to the concave function \( x \mapsto V(x, 1) \)) at \( x = g_{p=2,m} \). We conclude that \( \|I - \lambda H_m\|_{L^2([0, \infty))} \to L^2([0, \infty)) \leq \lambda g_{p=2,m}^{-1} - 1 = C_{p=2,m,\lambda} \) for \( \lambda > 1 + m \).

Moreover, the \( L^2 \)-norm of the operator \( I - \lambda H_m \) is clearly a convex function of the variable \( \lambda \):

\[
\|I - (s\lambda_1 + (1-s)\lambda_2) H_m\|_{L^2([0, \infty))} \to L^2([0, \infty)) \]

\[
= \|s(I - \lambda_1 H_m) + (1-s)(I - \lambda_2 H_m)\|_{L^2([0, \infty))} \to L^2([0, \infty)) \]

\[
\leq s\|I - \lambda_1 H_m\|_{L^2([0, \infty))} \to L^2([0, \infty)) + (1-s)\|I - \lambda_2 H_m\|_{L^2([0, \infty))} \to L^2([0, \infty)) \] \]

Since this norm is equal to 1 for \( \lambda = 0 \), tends to 1 as \( \lambda \to (1 + m)^+ \), and is always at least 1 (by Lemmas 3.2 and 3.3, or Remark 3.4), we conclude that \( \|I - \lambda H_m\|_{L^2([0, \infty))} \to L^2([0, \infty)) = 1 \) for \( \lambda \in [0, 1 + m] \).

Remark 3.9. The operator \( I - (1 + m)H_m \) is an isometry on \( L^2([0, \infty)) \) (since the Beurling-Ahlfors transform is an \( L^2 \)-isometry; see [4, Theorem 1.1]).

Proof of inequality (1.1) for \( \lambda \leq 0 \). The triangle inequality and Lemma 3.3 yield

\[
\|I - \lambda H_m\|_{L^p([0, \infty))} \to L^p([0, \infty)) \leq |\lambda| g_{p,m}^{-1} + 1 \leq C_{p,m,\lambda}.
\]
Moreover, the opposite inequality is also true by Lemma 3.2, so we in fact have equalities above. □

Sharpness of inequality (1.1) follows from Lemma 3.2. In order to complete the proof of Theorem 1.1, we have to explain why $C_{p,0,1} = 0$ is equal to

$$
C_p = \sup \{ c_{p,m=0,\lambda=1} : \alpha < (p-1)/p \} = \begin{cases} 
\frac{p-1}{1+|\alpha|} & \text{if } 1 < p \leq 2, \\
\frac{p-1}{p} & \text{if } p > 2
\end{cases}
$$

(see Section 2 for the definition of $\alpha_p$ and details).

Rather than to check directly that the supremum in the definition of $C_{p,0,1}$ is attained in the point $(\alpha_p, 1)$ we refer to results obtained in Section 2. For $m = 0$ and $\lambda = 1$, the above proof of inequality (1.1), can be repeated with the functions $V, U$ being defined like in Subsection 2.5 (of course, instead of using Proposition 3.7, we use Proposition 2.9). This gives us $\|f - H_0f\|_{L^p([0,\infty))} \leq C_p \|f\|_{L^p([0,\infty))}$, but since clearly $C_p \leq C_{p,0,1}$, and the constant $C_{p,0,1}$ is best possible in this inequality, we conclude that $C_p = C_{p,0,1}$.

Acknowledgments

I thank Adam Osękowski for suggesting the study of the Hardy operator by the means of maximal martingale inequalities, as well as for his invaluable help, and many fruitful and inspiring discussions.

I also thank Tomasz Tkocz, whose nicely written bachelor thesis helped me to understand the connection between Doob’s inequality and the Hardy operator, when I was first learning about them.

References

1. Kari Astala, Area distortion of quasiconformal mappings, Acta Math. 173 (1994), no. 1, 37–60. MR 1294669 (95m:30028b)
2. R. Bañuelos and P. J. Méndez-Hernández, Space-time Brownian motion and the Beurling-Ahlfors transform, Indiana Univ. Math. J. 52 (2003), no. 4, 981–990. MR 2001941 (2004h:60067)
3. Rodrigo Bañuelos and Prabhakar Janakiraman, $L^p$-bounds for the Beurling-Ahlfors transform, Trans. Amer. Math. Soc. 360 (2008), no. 7, 3603–3612. MR 2386238 (2009d:42032)
4. , On the weak-type constant of the Beurling-Ahlfors transform, Michigan Math. J. 58 (2009), no. 2, 459–477. MR 2595549 (2011a:47096)
5. Rodrigo Bañuelos and Adam Osękowski, Sharp inequalities for the Beurling-Ahlfors transform on radial functions, Duke Math. J. 162 (2013), no. 2, 417–434. MR 3018958
6. , On the operator $\Lambda^*$ and the Beurling-Ahlfors transform on radial functions, Michigan Math. J. 63 (2014), no. 1, 213–221. MR 3189475
7. Rodrigo Bañuelos and Gang Wang, Sharp inequalities for martingales with applications to the Beurling-Ahlfors and Riesz transforms, Duke Math. J. 80 (1995), no. 3, 575–600. MR 1370109 (96k:60108)
8. Alexander Borichev, Prabhu Janakiraman, and Alexander Volberg, Subordination by conformal martingales in $L^p$ and zeros of Laguerre polynomials, Duke Math. J. 162 (2013), no. 5, 889–924. MR 3047469
9. Donald L. Burkholder, Explorations in martingale theory and its applications, École d’Été de Probabilités de Saint-Flour XIX—1989, Lecture Notes in Math., vol. 1464, Springer, Berlin, 1991, pp. i–x. MR 1108183 (92m:60037)
10. S. D. Chatterji, Some comments on the maximal inequality in martingale theory, Measure theory, Oberwolfach 1979 (Proc. Conf., Oberwolfach, 1979), Lecture Notes in Math., vol. 794, Springer, Berlin, 1980, pp. 361–364. MR 577984 (82a:60068)
11. Oliver Dragičević and Alexander Volberg, Bellman function, Littlewood-Paley estimates and asymptotics for the Ahlfors-Beurling operator in $L^p(C)$, Indiana Univ. Math. J. 54 (2005), no. 4, 971–995. MR 2164413 (2006j:30025)
12. James T. Gill, On the Beurling-Ahlfors transform’s weak-type constant, Michigan Math. J. 59 (2010), no. 2, 353–363. MR 2677626 (2012b:42022)
13. T. Iwaniec, *Extremal inequalities in Sobolev spaces and quasiconformal mappings*, Z. Anal. Anwendungen 1 (1982), no. 6, 1–16. MR 719167 (85g:30027)

14. Olli Lehto, *Remarks on the integrability of the derivatives of quasiconformal mappings*, Ann. Acad. Sci. Fenn. Ser. A I No. 371 (1965), 8. MR 0181748 (31 #5975)

15. Adam Osękowski, *Sharp martingale and semimartingale inequalities*, Instytut Matematyczny Polskiej Akademii Nauk. Monografie Matematyczne (New Series) [Mathematics Institute of the Polish Academy of Sciences. Mathematical Monographs (New Series)], vol. 72, Birkhäuser/Springer Basel AG, Basel, 2012. MR 2964297

16. Adam Osękowski, *Sharp logarithmic bounds for Beurling-Ahlfors operator restricted to the class of radial functions*, Mediterr. J. Math. 10 (2013), no. 4, 1883–1894. MR 3119338

17. A. Pelczyński, *Norms of classical operators in function spaces*, Astérisque (1985), no. 131, 137–162, Colloquium in honor of Laurent Schwartz, Vol. 1 (Palaiseau, 1983). MR 816744 (87b:47036)

18. A. Volberg and F. Nazarov, *Heat extension of the Beurling operator and estimates for its norm*, Algebra i Analiz 15 (2003), no. 4, 142–158. MR 2068982 (2005f:30042)

19. Alexander Volberg, *Ahlfors-Beurling operator on radial functions*, Preprint (2012), arXiv:1203.2291.

Institute of Mathematics, University of Warsaw, Banacha 2, 02–097 Warsaw, Poland. E-mail address: m.strzelecki@mimuw.edu.pl