Abstract

In this paper we begin the study of some important Banach spaces of slice hyperholomorphic functions, namely the Bloch, Besov and weighted Bergman spaces, and we also consider the Dirichlet space, which is a Hilbert space. The importance of these spaces is well known, and thus their study in the framework of slice hyperholomorphic functions is relevant, especially in view of the fact that this class of functions has recently found several applications in operator theory and in Schur analysis. We also discuss the property of invariance of this function spaces with respect to Möbius maps by using a suitable notion of composition.

AMS Classification: 32K05, 30G35.

Key words: Bloch spaces, Besov spaces, Dirichlet spaces, Slice hyperholomorphic functions.

1 Introduction and preliminary results

Since their introduction in 2006, see [21], slice hyperholomorphic function have found several applications, e.g. in operator theory, see [16], and in Schur analysis. Such functions are also called slice regular in the case they are defined on quaternions and are quaternionic-valued. In the case of functions $f : \Omega \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$, where $\mathbb{R}^{n+1}$ is the Euclidean space and $\mathbb{R}_n$ is the real Clifford algebra of real dimension $2^n$, these functions are also called slice monogenic. The main advantage of this class of functions, with respect to the more classical function theories like the one of Fueter regular functions or of Dirac regular functions, see [3] [19] [20] [24], is that it contains power series expansions of the quaternionic variable or of the paravector variable in the case of Clifford algebras-valued functions. In this paper we will consider the quaternionic setting.

In our recent works in Schur analysis (see [2] for an overview) we have studied the Hardy space $H_2(\Omega)$ where $\Omega$ is the quaternionic unit ball $\mathbb{B}$ or the half space $\mathbb{H}^+$ of quaternions with
The plan of the paper is as follows. In Section 2 we treat the Bloch space \( B \) on the unit ball \( \mathbb{B} \). We recall the notion of \( \circ \)-composition and we show the invariance of \( B \) under Möbius transformations with respect to this composition. We also prove some conditions on the coefficients of a converging power series belonging to the Bloch space which generalize to this setting the analogous inequalities in the case of a function belonging to the complex Bloch space. Finally, we introduce the little Bloch space which turns out to be separable. In Section 3 we deepen the study of Bergman spaces by introducing their weighted versions.

Then, in Section 4 we move to the Besov spaces \( \mathfrak{B}_p \). We show their invariance under Möbius transformations, using the \( \circ \)-composition, then we introduce suitable seminorms and we study the property of being a Banach space. We also show the relation between the Besov spaces and the weighted Bergman spaces and \( L^p \) spaces (via a Bergman type projection) and we conclude with some duality results. Finally, in Section 5 we introduce the Dirichlet space and we show that it is a Hilbert space.

We now recall the main results of slice regular functions that we will need in the sequel. For more details see the book [10]. We consider the space \( \mathbb{R}^3 \) embedded in \( \mathbb{H} \) as follows

\[(a_1, a_2, a_3) \mapsto a_1e_1 + a_2e_2 + a_3e_3,\]

where \( \{e_0 = 1, e_1, e_2, e_3\} \) is the usual basis of the quaternions. Let \( S^2 \) be the sphere of purely imaginary unit quaternions and let \( i \in S^2 \). The space generated by \( \{1, i\} \), denoted by \( \mathbb{C}(i) \), is isomorphic, not only as a linear space but even as a field, to the field of the complex numbers. Given a domain \( \Omega \subset \mathbb{H} \), let \( \Omega_i = \Omega \cap \mathbb{C}(i) \) and \( Hol(\Omega_i) \) represents the complex linear space of holomorphic functions from \( \Omega_i \) to \( \mathbb{C}(i) \).

Any nonreal quaternion \( q = x_0 + e_1x_1 + e_2x_2 + e_3x_3 := x_0 + \bar{q} \) can be uniquely written in the form \( q = x + I_qy \) where \( x = x_0, I_q = \frac{q}{\|q\|} \in S^2 \), and \( y = \|q\| \) thus it belongs to the complex plane \( \mathbb{C}(I_q) \).
Definition 1.1 (Slice regular or slice hyperholomorphic functions). A real differentiable quaternionic-valued function \( f \) defined on an open set \( \Omega \subset \mathbb{H} \) is called (left) slice regular on \( \Omega \) if, for any \( i \in S^2 \), the function \( f|_{\Omega_i} \) is such that
\[
\left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) f|_{\Omega_i} (x + yi) = 0, \quad \text{on } \Omega_i.
\]
We denote by \( SR(\Omega) \) the set of slice regular functions on \( \Omega \).

Lemma 1.2 (Splitting Lemma). If \( f \) is a slice regular function on an open set \( \Omega \), then for every \( i \in S^2 \), and every \( j \in S^2 \), orthogonal to \( i \), there are two holomorphic functions \( F, G : \Omega \cap \mathbb{C}(i) \to \mathbb{C}(i) \) such that for any \( z = x + yi \), it is
\[
f|_{\Omega_i} (z) = F(z) + G(z)j.
\]

The open sets on which slice regular functions are naturally defined are described below.

Definition 1.3. Let \( \Omega \subseteq \mathbb{H} \). We say that \( \Omega \) is axially symmetric if whenever \( q = x + I_qy \) belongs to \( \Omega \) all the elements \( x + iy \) belong to \( \Omega \) for all \( i \in S^2 \). We say that \( \Omega \) is a slice domain, or s-domain for short, if it is a domain intersecting the real axis and such that \( \Omega \cap \mathbb{C}(i) \) is connected for all \( i \in S^2 \).

A main property of slice hyperholomorphic functions is the Representation Formula, see [16].

Theorem 1.4 (Representation Formula). Let \( f \) be a slice regular function on an axially symmetric s-domain \( \Omega \subseteq \mathbb{H} \). Choose any \( j \in S^2 \). Then the following equality holds for all \( q = x + iy \in \Omega \):
\[
f(x + yi) = \frac{1}{2} \left[ (1 + ij) f(x - yj) + (1 - ij) f(x + yj) \right].
\]

Let \( i, j \in S^2 \) be mutually orthogonal vectors and \( \Omega \subset \mathbb{H} \) an axially symmetric s-domain, then the Splitting Lemma and the Representation Formula, imply the good definition of the following operators, which relate the slice regular space with the space of pairs of holomorphic functions on \( \Omega_i \), denoted by \( Hol(\Omega_i) \). We define:
\[
Q_i : \ SR(\Omega) \longrightarrow Hol(\Omega_i) + Hol(\Omega_i)j
\]
\[
Q_i : \ f \longmapsto f|_{\Omega_i},
\]
and
\[
P_i : Hol(\Omega_i) + Hol(\Omega_i)j \longrightarrow SR(\Omega),
\]
\[
P_i[f](q) = P_i[f](x + yI_q) = \frac{1}{2} \left[ (1 + I_qi) f(x - yi) + (1 - I_qi) f(x + yi) \right],
\]
where \( f \in Hol(\Omega_i) + Hol(\Omega_i)j \). Moreover, we have that
\[
P_i \circ Q_i = I_{SR(\Omega)} \quad \text{and} \quad Q_i \circ P_i = I_{Hol(\Omega_i) + Hol(\Omega_i)j}
\]
where \( I \) denotes the identity operator.
2 Bloch space

Let $B = B(0, 1) \subset \mathbb{H}$ denote the unit ball centered at zero. Then, for each $i \in S^2$, we denote by $B_i = B \cap C(i)$ the unit disk in the plane $C(i)$.

In usual complex analysis, the Bloch space $B_C$ is defined to be the space of analytic functions on the unit disc $D$ in the complex plane such that

$$\|f\|_{B_C} = |f(0)| + \sup \{(1 - |z|^2)|f'(z)| : z \in D\} < \infty.$$ 

Analogously, we give the following definitions.

Definition 2.1. The quaternionic slice regular Bloch space $B$ associated with $B$ is the quaternionic right linear space of slice regular functions $f$ on $B$ such that

$$\sup \{(1 - |q|^2)|\frac{\partial f}{\partial x_0}(q)| : q \in B\} < \infty.$$ 

We define a norm on this space by

$$\|f\|_B = |f(0)| + \sup \{(1 - |q|^2)|\frac{\partial f}{\partial x_0}(q)| : q \in B\} < \infty.$$ 

In the sequel, we will be also in need of the following definition:

Definition 2.2. By $B_i$ we denote the quaternionic right linear space of slice regular functions $f$ on $B$ such that

$$\sup \{(1 - |z|^2)|Q_i[f](z)| : z \in B_i\} < \infty.$$ 

We define a norm on this space by

$$\|f\|_{B_i} = |f(0)| + \sup \{(1 - |z|^2)|Q_i[f](z)| : z \in B_i\}.$$ 

Remark 2.3. The fact that $\|f\|_B$ and $\|f\|_{B_i}$ are norms can be verified by a direct computation.

Remark 2.4. Note that for any $i \in S^2$ the function $Q_i[f]$ is a holomorphic map of a complex variable $z = x_0 + iy$ and that for its derivative we have $Q_i[f]'(z) = \frac{\partial Q_i[f]}{\partial x_0}(z)$.

Remark 2.5. Let $i \in S^2$, and let $f \in B_i$. Then, for any $j \in S^2$ with $j \perp i$, there exist holomorphic functions $f_1, f_2 : B_i \to C(i)$ such that $Q_i[f] = f_1 + f_2 j$. Moreover, as

$$|Q_i[f]'(z)|^2 = |f_1'(z)|^2 + |f_2'(z)|^2,$$

the condition $f \in B_i$ is equivalent to $f_1, f_2$ belonging to the one dimensional complex Bloch space. As this is a Banach space one can see directly that $(B_i, \| \cdot \|_{B_i})$ is also a Banach space.

Proposition 2.6. Let $i \in S^2$. Then $f \in B_i$ if and only if $f \in B$. The space $(B, \| \cdot \|_{B})$ is a Banach space and $(B, \| \cdot \|_{B})$ and $(B_i, \| \cdot \|_{B_i})$ have equivalent norms. Precisely, one has

$$\|f\|_{B_i} \leq \|f\|_{B} \leq 2\|f\|_{B_i}.$$
Proof. As \(B_i \subset B\), for any \(f \in B\), one has \(\|f\|_{B_i} \leq \|f\|_B\) by definition. Therefore \(B \subset B_i\).

On the other hand, let \(f \in B_i\). Then \(\frac{\partial f}{\partial x_0}\) is slice regular. Thus, we can apply the Representation Formula for slice regular functions and we obtain

\[
\frac{\partial f}{\partial x_0}(q) = \frac{1}{2}[(1 - I_i)\frac{\partial f}{\partial x_0}(z) + (1 + I_i)\frac{\partial f}{\partial x_0}(\bar{z})],
\]

where \(q = x + Iy\) and \(z = x + iy\). Applying the triangle inequality, as \(|1 - I_i| \leq 2\), \(|1 + I_i| \leq 2\), we obtain

\[
|\frac{\partial f}{\partial x_0}(q)| \leq |\frac{\partial f}{\partial x_0}(z)| + |\frac{\partial f}{\partial x_0}(\bar{z})|,
\]

and as \(|q| = |z| = |\bar{z}|\) we get

\[
(1 - |q|^2)|\frac{\partial f}{\partial x_0}(q)| \leq (1 - |z|^2)|\frac{\partial f}{\partial x_0}(z)| + (1 - |\bar{z}|^2)|\frac{\partial f}{\partial x_0}(\bar{z})| \leq 2\|f\|_{B_i}.
\]

Taking the supremum over all \(q \in B\), one concludes that

\[
\|f\|_B \leq 2\|f\|_{B_i}.
\]

Thus, \(f \in B\). Furthermore, the norms \(\| \cdot \|_{B_i}\) and \(\| \cdot \|_B\) are equivalent and \(B\) turns out to be a Banach space.

Remark 2.7. For \(z = x + iy \in B_i\) and \(w = x + jy \in B_j\), from the proof of the previous result, in particular, we have that

\[
(1 - |z|^2)|Q_i[f]'(z)| \leq (1 - |w|^2)|Q_j[f]'(w)| + (1 - |\bar{w}|^2)|Q_j[f]'(\bar{w})| \leq 2\|f\|_{B_j}.
\]

Taking the supremum over all \(z \in B_i\) we get

\[
\|f\|_{B_i} \leq 2\|f\|_{B_j}.
\]

Thus, we have that \(f \in B_i\) if and only if \(f \in B_j\) and that the norms \(\| \cdot \|_{B_i}\) and \(\| \cdot \|_{B_j}\) are equivalent.

Remark 2.8. In usual complex analysis it’s well known that the \(B_i\) is not separable, therefore \(B\) is also not separable. In fact, following the complex case, see [18], one can consider the family of slice regular functions on \(B\)

\[
f_n(q) = P_i\left[\frac{e^{-in}}{2} \log \left(\frac{1 + e^{-in}z}{1 - e^{-in}z} \right)\right](q), \quad n \in \mathbb{N}
\]

where \(q = x + Iy\) and \(z = x + iy\), which has the following property:

\[
\|f_n - f_m\|_B \geq 1,
\]

so \(B\) is not separable.

Similar to the usual complex case, see [29], we formulate the following definition, which can be found also in [25].

Definition 2.9. We define the space \(H^\infty(B)\) as the quaternionic right linear space of bounded slice regular functions \(f\) defined on \(B\), i.e., of all slice regular functions such that

\[
\|f\|_\infty = \sup\{|f(q)| : q \in B\} < \infty.
\]
We will be also in need of the following:

**Definition 2.10.** We denote by $\mathcal{H}_i^\infty(\mathbb{B})$ the quaternionic right linear space of slice regular functions $f$ on $\mathbb{B}$ such that 
\[ \|f\|_{\infty,i} = \sup \{|Q_i[f](z)| : z \in \mathbb{B}_i\} < \infty. \]

**Remark 2.11.** Let $i \in \mathbb{S}^2$ and let $f \in \mathcal{H}_i^\infty(\mathbb{B}_i)$. Then for any $j \in \mathbb{S}^2$ with $j \perp i$ there exist holomorphic functions $f_1, f_2: \mathbb{B}_i \to \mathbb{C}(i)$ such that $Q_i[f] = f_1 + f_2 j$. It is easy to show that $f \in \mathcal{H}_i^\infty(\mathbb{B}_i)$ if and only if $f_1$ and $f_2$ belong to the usual complex space $\mathcal{H}_i^\infty(\mathbb{C})$ of bounded holomorphic functions defined in $\mathbb{B}_i$. Moreover, as $\mathcal{H}_i^\infty(\mathbb{B}_i)$ is a Banach space, one can see that $\mathcal{H}_i^\infty(\mathbb{B}_i)$ is also a Banach space.

**Proposition 2.12.** Let $i \in \mathbb{S}^2$. Then $f \in \mathcal{H}_i^\infty(\mathbb{B})$ if and only if $f \in \mathcal{H}_i^\infty(\mathbb{B})$. The space $(\mathcal{H}_i^\infty(\mathbb{B}), \| \cdot \|_\infty)$ is a Banach space. Moreover, the norms $\| \cdot \|_\infty$ and $\| \cdot \|_{\infty,i}$ are equivalent. Precisely, one has 
\[ \|f\|_{\infty,i} \leq \|f\|_\infty \leq 2\|f\|_{\infty,i}. \]

**Proof.** By definition, we have $\|f\|_{\infty,i} \leq \|f\|_\infty$. Furthermore, from the Representation Formula it follows that for any $f \in \mathcal{S}\mathcal{R}(\Omega)$ one has 
\[ \|f\|_\infty \leq 2\|f\|_{\infty,i}. \]

Therefore, $\|f\|_\infty < \infty$ if and only $\|f\|_{\infty,i} < \infty$, the norms are equivalent and, as $\mathcal{H}_i^\infty(\mathbb{B})$ is a Banach space, $\mathcal{H}_i^\infty(\mathbb{B})$ is a Banach space too. \hfill \Box

**Proposition 2.13.** Let $f \in \mathcal{H}_i^\infty(\mathbb{B})$. Then $f \in \mathcal{B}$ and $\|f\|_{\mathcal{B}} \leq 4\|f\|_\infty$.

**Proof.** Let $f \in \mathcal{H}_i^\infty(\mathbb{B})$ and $i, j \in \mathbb{S}^2$ such that $i \perp j$. Then there exist holomorphic functions $f_1, f_2: \mathbb{B}_i \to \mathbb{C}(i)$ such that $Q_i[f] = f_1 + f_2 j$. From usual complex analysis, it is well known that, for any holomorphic function $g$ defined on the complex unit disc, the estimates $\|g\|_{\mathcal{B}_C} \leq \|g\|_\infty$ holds (see e.g. [29]). Thus one has 
\[ \|f\|_{\mathcal{B}_i} \leq |f_1(0)| + \sup_{z \in \mathbb{B}_i} (1 - |z|^2)|f_1'(z)| + |f_2(0)| + \sup_{z \in \mathbb{B}_i} (1 - |z|^2)|f_2'(z)| \]
\[ = \|f_1\|_{\mathcal{B}_C} + \|f_2\|_{\mathcal{B}_C} \leq \|f_1\|_\infty + \|f_2\|_\infty \leq 2\|f\|_{\infty,i}. \]

Therefore, from the previous propositions, one obtains that 
\[ \|f\|_{\mathcal{B}} \leq 2\|f\|_{\mathcal{B}_i} \leq 4\|f\|_{\infty,i} \leq 4\|f\|_\infty. \]

In particular, $\|f\|_{\mathcal{B}} < \infty$ if $\|f\|_\infty < \infty$. \hfill \Box

In general, as in the complex case (see [29]), $f \in \mathcal{B}$ does not imply $f \in \mathcal{H}_i^\infty(\mathbb{B})$. For example the function 
\[ f(q) = P_i[\log(1 - z)](q), \]
belongs to $\mathcal{B}$, but $Q_i[f]$ does not belong to $\mathcal{H}_i^\infty(\mathbb{B}_i)$ and so $f \notin \mathcal{H}_i^\infty(\mathbb{B})$.

The complex Bloch space is important because of its invariance with respect to M"obius transformation. As it is well known, when dealing with hyperholomorphic functions (not only in the slice regular setting but also in more classical settings, like in the Cauchy-Fueter regular setting) the composition of hyperholomorphic functions does not give, in general, a function of
the same type. The Möbius transformation we will consider are those which are slice regular, see [26], and we will define a suitable notion of composition which will allow us to prove invariance under Möbius transformation.

For the $*$-product of slice hyperholomorphic functions used below, we refer the reader to [16].

**Definition 2.14.** For any $a \in \mathbb{B} \setminus \mathbb{R}$ we define the slice regular Möbius transformation as

$$T_a(q) = (1 - qa)^{*} (a - q), \quad q \in \mathbb{B},$$

where $*$ denotes the slice regular product.

**Proposition 2.15.** Let $a \in \mathbb{B} \setminus \mathbb{R}$ and $I = \frac{a}{\|a\|}$, or $a \in \mathbb{R}$ and $I$ is any element in $\mathbb{S}^2$. Then the slice regular Möbius transformation $T_a$ has the following properties:

(i) $T_a$ is a bijective mapping of $\mathbb{B}$ onto itself.

(ii) $T_a(z) = \frac{a - z}{1 - \overline{a}z}$ for all $z \in \mathbb{B}_I$, i.e. on $\mathbb{B}_I$ the slice regular Möbius transformation coincides with the usual one dimensional complex Möbius transformation.

(iii) $T_a(0) = a$, $T_a(a) = 0$ and $T_a \circ T_a(q) = q$ for all $q \in \mathbb{C}(I)$.

We note that the function $T_a$ is such that $T_a : \mathbb{B} \cap \mathbb{C}_I \to \mathbb{C}_I$. Thus, using Proposition 2.9 in [13], we can give the following definition.

**Definition 2.16.** Let $a \in \mathbb{B} \setminus \mathbb{R}$ and $I = \frac{a}{\|a\|}$, or $a \in \mathbb{R}$ and $I$ is any element in $\mathbb{S}^2$. Let $T_a$ be the associated Möbius transformation. For any $f \in \mathcal{SR}(\mathbb{B})$ we define the function $f \circ I T_a \in \mathcal{SR}(\mathbb{B})$ as

$$f \circ I T_a := P_I[ ( f_1 \circ Q_I[T_a] ) ] + P_I[ ( f_2 \circ Q_I[T_a] ) ] j, $$

where $j \in \mathbb{S}^2$ with $j \perp I$ and $f_1, f_2 : \mathbb{B}_I \to \mathbb{C}(I)$ are holomorphic functions such that $Q_I[f] = f_1 + f_2 j$.

Using this notion of composition, we can prove a result on invariance under Möbius transformation.

**Proposition 2.17.** Let $a \in \mathbb{B} \setminus \mathbb{R}$ and $I = \frac{a}{\|a\|}$, or $a \in \mathbb{R}$ and $I$ is any element in $\mathbb{S}^2$. Then for any $f \in \mathcal{B}$, one has $f \circ I T_a \in \mathcal{B}$.

**Proof.** Let $f \in \mathcal{B}$ and let $I = \frac{a}{\|a\|}$, if $a \in \mathbb{R}$ then in the computations below we can use any $I \in \mathbb{S}$. For $q = x_0 + iy \in \mathbb{B}$ let $z = x_0 + Iy$. As $g := \frac{\partial}{\partial x_0} (f \circ I T_a)$ is slice regular on $\mathbb{B}$, we can apply the Representation Formula. Thus, as $|q| = |z| = |\bar{z}|$, we have

$$(1 - |q|^2)^2 |g(q)|^2 = (1 - |q|^2)^2 \frac{1}{2} (1 - iI)g(z) + \frac{1}{2} (1 + iI)g(\bar{z})^2$$

$$= (1 - |q|^2)^2 \frac{1}{4} \left( |1 - iI|^2 |g(z)|^2 + 2 \text{Re} \left( (1 - iI)g(z) \overline{g(\bar{z})} (1 + iI) \right) + |1 + iI|^2 |g(\bar{z})|^2 \right)$$

$$= \frac{1}{4} \left( |1 - iI|^2 (1 - |z|^2)^2 |g(z)|^2 + 2 \text{Re} \left( (1 - iI)(1 - |z|^2)g(z) \overline{g(\bar{z})}(1 + iI) \right) \right.$$

$$\left. + |1 + iI|^2 (1 - |\bar{z}|^2)^2 |g(\bar{z})|^2 \right).$$
Now let us denote 
\[ f'|_{C_I} = \frac{\partial}{\partial x_0} Q_I[f] \quad \text{and} \quad T'_a|_{C_I} = \frac{\partial}{\partial x_0} Q_I[T_a]. \]

Observe that \( f'|_{C_I} \) and \( T'_a|_{C_I} \), defined above, depend on \( I \in \mathbb{S}^2 \), but we omit the subscript in the rest of the proof. Then we have \( g(z) = f'(T_a(z))T'_a(z) \) as \( z \in \mathbb{C}(I) \). If we set \( w = T_a(z) \), we know that
\[
|1 - |w|^2| = (1 - |z|^2)|T'_a(z)|
\]
as the function \( Q_I[T_a] \) is nothing but the complex Möbius transformation on \( \mathbb{B}_I \). Thus, if we put \( w^* = T_a(z) \), we have
\[
(1 - |z|^2)|g(z)| = (1 - |z|^2)|f(T_a(z))||T'_a(z)| = (1 - |w|^2)|f'(w)|
\]
\[
(1 - |\bar{z}|^2)|g(\bar{z})| = (1 - |\bar{z}|^2)|f(T_a(\bar{z}))||T'_a(\bar{z})| = (1 - |w^*|^2)|f'(w^*)|.
\]

Plugging this into the above equation, we obtain
\[
(1 - |q|^2)^2|g(q)|^2 = \frac{1}{4}
\left[
(1 - |w|^2)^2|f'(w)|^2 + 2\text{Re} \left( (1 - |w|^2)(1 - |w^*|^2)g(w^*)^2 (1 + I i) \right)
\right.
\]
\[
+ |1 + I i|^2(1 - |w^*|^2)^2|f'(w^*)|^2
\]
\[
= \frac{1}{4} \left( (1 - |w|^2)|f'(w)| + (1 - |w^*|^2)|f'(w^*)| \right)^2
\]
\[
\leq ((1 - |w|^2)|f'(w)| + (1 - |w^*|^2)|f'(w^*)|)^2.
\]

Recalling the definitions of \( g \) and \( f' \) we get
\[
(1 - |q|^2)^2 \left| \frac{\partial}{\partial x_0} (f \circ I T_a)(q) \right| \leq (1 - |w|^2)|f'(w)| + (1 - |w^*|^2)|f'(w^*)|
\]
\[
\leq 2 \sup \left\{ (1 - |x|^2) \left| \frac{\partial f}{\partial x_0}(x) \right| : x \in \mathbb{B}_I \right\} < \infty.
\]

Therefore, \( \|f \circ I T_a\|_{B_I} \) is bounded, that is \( f \circ I T_a \in B_I \) and so \( f \circ I T_a \in B \).

The condition on a function in order to belong to the Bloch space involves a first derivative. Here we show that it is equivalent to a condition on the \( n \)-th derivative.

**Proposition 2.18.** For any \( f \in B \) and any \( n \in \mathbb{N} \) with \( n \geq 2 \) the following inequality holds:
\[
\sup \{ (1 - |q|^2)^n |\partial_{x_0}^n f(q)| : q \in \mathbb{B} \} \leq 2^{n+2}(n-1)!\|f\|_B.
\]

Conversely, if \( f \in SR(\mathbb{B}) \) for any \( n \in \mathbb{N} \) with \( n \geq 2 \) one has
\[
\sup \{ (1 - |q|^2)^n |\partial_{x_0}^n f(q)| : q \in \mathbb{B} \} < \infty
\]
then \( f \in B \).
Proof. Let $f \in B$ and $i, j \in S^2$ with $j \perp i$. Then there exist holomorphic functions $f_1, f_2: B_i \to \mathbb{C}(i)$ such that $Q_i[f] = f_1 + f_2j$. From Remark 2.15 it follows that $f_1$ and $f_2$ lie in the complex Bloch space. Due to Theorem 1 in [18], we have

$$\sup\{(1 - |q|^2)^n|\partial^n_{x_0} f(q)|: q \in B_i\} \leq \sup\{(1 - |z|^2)^n|\partial^n_{x_0} f_1(z)|: z \in B_i\}$$

\[+ \sup\{(1 - |z|^2)^n|\partial^n_{x_0} f_2(z)|: z \in B_i\}\]

\[\leq (n - 1)!2^n\|f_1\|_{B_C} + (n - 1)!2^n\|f_2\|_{B_C}\]

\[\leq (n - 1)!2^{n+1}\|f\|_B\]

Now let $z = x + Iy$ where $I \in S^2$ and $q = x + iy$. As $|1 + Ii| \leq 2$, $|1 - Ii| \leq 2$, by applying the Representation Formula, one obtains

$$(1 - |z|^2)^n|\partial^n_{x_0} f(z)| = (1 - |q|^2)^n\left((1 - Ii)\partial^n_{x_0} f(q) + (1 + Ii)\partial^n_{x_0} f(q)\right)$$

$$\leq (1 - |q|^2)^n|\partial^n_{x_0} f(q)| + (1 - |q|^2)^n|\partial^n_{x_0} f(q)|$$

\[\leq 2\sup\{(1 - |q|^2)^n|\partial^n_{x_0} f(q)|: q \in B_i\}.

Thus, one has

$$\sup\{(1 - |z|^2)^n|\partial^n_{x_0} f(z)|: z \in B\} \leq 2\sup\{(1 - |q|^2)^n|\partial^n_{x_0} f(q)|: q \in B_i\}$$

\[\leq 2(n - 1)!2^{n+1}\|f\|_B.

Conversely, if

$$\sup\{(1 - |q|^2)^n|\partial^n_{x_0} f(q)|: q \in B\} < \infty,$$

one has

$$\sup\{(1 - |q|^2)^n|\partial^n_{x_0} f_k(q)|: q \in B\} < \infty, \quad k = 1, 2$$

as $|\partial^n_{x_0} f_k(z)| \leq |\partial^n_{x_0} f(z)|, z \in B$. But then, from Theorem 1 in [18], it follows that $f_1, f_2$ belong to the complex Bloch space $B_C$ and so $f \in B$.

The condition on the derivatives studied in the above proposition translates into a condition on the coefficients of the series expansion of a function in $B$.

**Proposition 2.19.** Let $f \in B$ and let $(a_n)_{n \in \mathbb{N}} \subset \mathbb{H}$ such that

$$f(q) = \sum_{n=0}^{\infty} q^n a_n.$$  

Then

$$|a_n| \leq \frac{e}{\sqrt{2}}\|f\|_B,$$

for any $n \in \mathbb{N} \cup \{0\}$.

**Proof.** Let $i, j \in S^2$ such that $i \perp j$, let $z \in \mathbb{C}(i)$ and let $f_1, f_2: B_i \to \mathbb{C}(i)$ such that $f = f_1 + f_2j$. Furthermore, for any $n \in \mathbb{N}$, let $\alpha_n, \beta_n \in \mathbb{C}(i)$ such that $a_n = \alpha_n + \beta_n j$. Then we have

$$f(z) = \sum_{n=0}^{\infty} z^n \alpha_n + \sum_{n=0}^{\infty} z^n \beta_n j = f_1(z) + f_2(z)j.$$
From Remark 2.5 it follows that \( f_1, f_2 \) lie in the complex Bloch space on \( \mathbb{B}_i \). Therefore, from Theorem 2, [18], it follows that for any \( n \in \mathbb{N} \) one has

\[ |\alpha_n| \leq \frac{e}{2} \| f_1 \|_{\mathcal{B}_C}, \quad |\beta_n| \leq \frac{e}{2} \| f_2 \|_{\mathcal{B}_C} \]

and so, as \( \| f_k \|_{\mathcal{B}_C} \leq \| f \|_{\mathcal{B}_i}, k = 1, 2 \), one obtains

\[ |a_n|^2 = |\alpha_n|^2 + |\beta_n|^2 \leq \frac{e^2}{4} (\| f_1 \|^2_{\mathcal{B}_C} + \| f_2 \|^2_{\mathcal{B}_C}) \leq \frac{e^2}{2} \| f \|^2_{\mathcal{B}_i}. \]

Thus, by Proposition 2.6 one concludes that

\[ \| a_n \| \leq \frac{e}{\sqrt{2}} \| f \|_{\mathcal{B}_i} \leq \frac{e}{\sqrt{2}} \| f \|_{\mathcal{B}}. \]

\[ \square \]

Another property of the slice regular Bloch space is the following:

**Proposition 2.20.** Let \( f \in \mathcal{SR}(\mathbb{B}) \), let \( (n_k)_{k \in \mathbb{N}} \subset \mathbb{N} \cup \{0\} \) and \( (a_{nk})_{k \in \mathbb{N}} \subset \mathbb{H} \) be a sequence of quaternions such that

\[ f(q) = \sum_{k=0}^{\infty} q^{n_k} a_{nk}. \]

If there exist constants \( \alpha > 1 \) and \( M > 0 \) such that

\[ \frac{n_{k+1}}{n_k} \geq \alpha, \quad |a_{nk}| \leq M, \quad \forall k \in \mathbb{N}, \tag{2} \]

then \( f \in \mathcal{B} \).

**Proof.** Let \( i, j \in \mathbb{S}^2 \) such that \( i \perp j \) and let \( a_{nk} = \alpha_{nk} + \beta_{nk}j \), for \( k \in \mathbb{N} \). Then

\[ Q_i[f](z) = \sum_{k=0}^{\infty} z^{n_k} \alpha_{nk} + \sum_{k=0}^{\infty} z^{n_k} \beta_{nk}j. \]

Note that the coefficients \( \alpha_{nk} \) and \( \beta_{nk} \) satisfy (2). Thus by Theorem 4 in [18] it follows, that the functions

\[ f_1(z) = \sum_{k=0}^{\infty} z^{n_k} \alpha_{nk}, \quad f_2(z) = \sum_{k=0}^{\infty} z^{n_k} \beta_{nk}, \]

belong to the complex Bloch space \( \mathcal{B}_C \). Thus, \( f \in \mathcal{B}_i \) and \( f \in \mathcal{B} \) follows from Remark 2.5 and Proposition 2.6.

\[ \square \]

As we pointed out before, the Bloch space \( \mathcal{B} \) is not separable. In usual complex analysis, the little Bloch space \( \mathcal{B}_C^0 \) of all functions \( f \in \mathcal{B}_C \) such that \( \lim_{|z| \rightarrow 1} (1 - |z|^2)f'(z) = 0 \) is a separable subspace of \( \mathcal{B}_C \) which is of interest on its own. Thus we give the following definition.

**Definition 2.21.** The slice regular little Bloch space \( \mathcal{B}^0 \) is the space of all functions \( f \in \mathcal{B} \) such that

\[ \lim_{|q| \rightarrow 1} (1 - |q|^2)|\partial_q f(q)| = 0. \]
**Remark 2.22.** Let $f ∈ \mathcal{SR}(B)$ and $i, j ∈ S^2$ with $i \perp j$ and let $f_1, f_2 : B_i \to \mathbb{C}(i)$ be holomorphic functions such that $Q_i[f] = f_1 + f_2j$. Then one has

$$|\partial_{x_0} f(z)|^2 = |\partial_{x_0} f_1(z)|^2 + |\partial_{x_0} f_2(z)|^2, \quad \forall z \in B_i.$$ 

Due to the Representation Formula, this implies that $f ∈ B^0$ if and only if $f_1, f_2$ belong to the complex little Bloch space $B^0_C$.

**Proposition 2.23.** The little Bloch space $B^0$ is the closure with respect to $\| \cdot \|_B$ of the set of quaternionic polynomials of the form

$$P(q) = \sum_{k=0}^n q^k a_k, \quad a_k ∈ \mathbb{H}.$$ 

In particular $B^0$ is separable.

**Proof.** Let $f ∈ B^0$ and $i, j ∈ S$ such that $i \perp j$. Furthermore, let $f_1, f_2 : B_i \to \mathbb{C}(i)$ be holomorphic functions such that $Q_i[f] = f_1 + f_2j$. As the set of complex polynomials is dense in the complex little Bloch space (see Corollary 5.10 in [29]), by the above remark, there exist complex polynomials $p_{1,n}(z) = \sum_{k=0}^n z^k α_{n,k}$ and $p_{2,n}(z) = \sum_{k=0}^n z^k β_{n,k}$ such that $\|p_k - p_{k,n}\|_{B_C} → 0$ as $n → ∞$ for $k = 1, 2$. Let $a_{n,k} = α_{n,k} + β_{n,k}j$ and $p_n(q) = \sum_{k=0}^n q^k a_{n,k}$. Then one has

$$\|p_n - f\|_B ≤ 2\|p_n - f\|_{B_i} ≤ 2\|p_{1,n} - f\|_{B_C} + \|p_{2,n} - f\|_{B_C} \xrightarrow{n → ∞} 0.$$ 

□

**Proposition 2.24.** Let $f \in B$, then $f ∈ B^0$ if and only if one has that

$$\lim_{r \to 1} \|f_r - f\|_B = 0,$$

where $f_r(q) = f(rq)$ for all $q ∈ B$ and $r ∈ (0, 1)$.

**Proof.** Let $f ∈ B$ and $i, j ∈ S^2$ with $i \perp j$ and let $f_1, f_2 : B_i \to \mathbb{C}(i)$ holomorphic functions such that $Q_i[f] = f_1 + f_2j$. Then, for $k = 1, 2$, one has

$$\|f_{k,r} - f_k\|_{B_C} ≤ \|f_r - f\|_{B_i} ≤ \|f_r - f\|_B ≤ 2\|f_r - f\|_{B_i} ≤ 2\|f_{1,r} - f_1\|_{B_C} + 2\|f_{2,r} - f_2\|_{B_C}.$$ 

Thus, $[3]$ is satisfied if and only if

$$\lim_{r \to 1} \|f_{k,r} - f_k\|_{B_C} = 0, \quad k = 1, 2,$$

where $f_{k,r}(z) = f_k(rz), k = 1, 2$ for all $z ∈ B_i$ and $r ∈ (0, 1)$. But this is equivalent to $f_1, f_2 ∈ B^0_C$, see Theorem 5.9 in [29]. Thus, from Remark 2.22 it follows that $f ∈ B^0$ if and only if $[3]$ holds. □

We conclude this section with a result that holds on a slice.

**Proposition 2.25.** Let $f ∈ B$ and $i ∈ S^2$ be fixed. Then for all $q, u ∈ B_i$, one has

$$|f(q) - f(u)| ≤ \|f\|_B d(q, u),$$

where

$$d(q, u) = \frac{1}{2} \log \left( \frac{1 + \frac{|q - u|}{|1 - qu|}}{1 - \frac{|q - u|}{|1 - qu|}} \right).$$
Proof. If \( q \) and \( u \) lie on the same complex plane, i.e. \( q, u \in \mathbb{B}_i \) for some \( i \in S^2 \), then again for \( j \in S \) with \( j \perp i \) there exist holomorphic functions \( f_1, f_2 : \mathbb{B}_i \to \mathbb{C}(i) \) such that \( Q_i[f] = f_1 + f_2j \). Thus, as \( \|f_k\|_{B_i} \leq \|f\|_{B_i}, k = 1, 2 \) we have
\[
|f(q) - f(u)|^2 = |f_1(q) - f_1(u)|^2 + |f_2(q) - f_2(u)|^2 \\
\leq \|f_1\|_{B_i}^2 d(q, u)^2 + \|f_2\|_{B_i}^2 d(q, u)^2 \\
\leq 2\|f\|_{B_i}^2 d(q, u)^2
\]
in which we have used the Theorem 3 in [18].

3 Weighted Bergman spaces

Bergman spaces in the slice hyperholomorphic setting have been studied in [13, 14, 15]. Here we deepen this study and we introduce weighted slice regular Bergman spaces.

Let \( dA \) be the area measure on the unit ball of the complex plane \( \mathbb{D} \), normalized so that the area 0f \( \mathbb{D} \) is 1, i.e. \( dA = \frac{1}{\pi} dxdy \). For \( \alpha > -1 \) let \( dA_\alpha = (\alpha + 1)(1 - |z|^2)^\alpha dA(z) \).

Then the complex Bergman space \( A_p^{\alpha, C} \) is defined as the space of all holomorphic functions \( f \) on \( \mathbb{D} \) such that \( f \in L^p(\mathbb{D}, dA_\alpha) \). With \( \| \cdot \|_{p, \alpha} \) we will note the \( L^p \)-norm on \( \mathbb{D} \) with respect to \( dA_\alpha \). We begin with the following definitions.

Definition 3.1. For \( i \in S^2 \) let \( dA_i \) be the normalized differential of area in the plane \( \mathbb{C}(i) \) such that the area of \( \mathbb{B}_i \) is equal to 1. Moreover, for \( \alpha > -1 \) let
\[
dA_{\alpha,i}(z) = (\alpha + 1)(1 - |z|^2)^\alpha dA_i(z).
\]

Then, for \( p > 0 \), we define the weighted slice regular Bergman space \( A_p^{\alpha, i} \) as the quaternionic right linear space of all slice regular functions on \( \mathbb{B} \) such that
\[
\sup_{i \in S^2} \int_{\mathbb{B}_i} |f(z)|^p dA_{\alpha,i}(z) < \infty.
\]
Furthermore, for each function \( f \in A_p^{\alpha, i} \) we define
\[
\|f\|_{p, \alpha,i} = \left( \int_{\mathbb{B}_i} |f(z)|^p dA_{\alpha,i}(z) \right)^{\frac{1}{p}}.
\]

Definition 3.2. For \( i \in S^2, p > 0 \), we define \( A_p^{\alpha, i} \) as the quaternionic right linear space of all slice regular functions on \( \mathbb{B} \) such that
\[
\int_{\mathbb{B}_i} |f(z)|^p dA_{\alpha,i}(z) < \infty,
\]
that is \( A_p^{\alpha, i} = \mathcal{SR}(\mathbb{B}) \cap L^p(\mathbb{B}, dA_{\alpha,i}) \). Furthermore, for each function \( f \in A_p^{\alpha, i} \) we define
\[
\|f\|_{p, \alpha,i} = \left( \int_{\mathbb{B}_i} |f(z)|^p dA_{\alpha,i}(z) \right)^{\frac{1}{p}}.
\]

As we shall see in the next section, the weighted slice regular Bergman spaces are related with the Besov spaces.
Remark 3.3. Let \( j \in \mathbb{S}^2 \) be such that \( j \perp i \). Then there exist holomorphic functions \( f_1, f_2 : \mathbb{B}_i \to \mathbb{C}(i) \) such that \( Q_i[f] = f_1 + f_2j \). Furthermore, for all \( z \in \mathbb{B}_i \), we have
\[
|f_k(z)|^p \leq |f(z)|^p \leq 2^{\max\{0,p-1\}} (|f_1(z)|^p + |f_2(z)|^p) \quad k = 1, 2. 
\]
Here, the first inequality is trivial. For \( p > 1 \), the second one follows from the convexity of the function \( x \mapsto x^p \) on \( \mathbb{R}^+ \), in fact
\[
|f(z)|^p \leq 2^p \left( \frac{1}{2} |f_1(z)| + \frac{1}{2} |f_2(z)| \right)^p 
\leq 2^p \left( \frac{1}{2} |f_1(z)|^p + \frac{1}{2} |f_2(z)|^p \right) 
= 2^{p-1} (|f_1(z)|^p + |f_2(z)|^p). 
\]
On the other hand, for \( 0 < p \leq 1 \) and \( a, b \geq 0 \), the inequality \( (a + b)^p \leq a^p + b^p \) holds. Therefore, in this case, we have
\[
|f(z)|^p \leq (|f_1(z)| + |f_2(z)|)^p \leq |f_1(z)|^p + |f_2(z)|^p. 
\]
Note that the inequality (4) implies that \( f \) is in \( \mathcal{A}^p_{\alpha,i} \) if and only if \( f_1 \) and \( f_2 \) are in \( \mathcal{A}^p_{\alpha,i} \).

Moreover we have:

Proposition 3.4. Let \( i, j \in \mathbb{S}^2 \), let \( p > 0 \) and \( \alpha > -1 \) and \( f \in \mathcal{SR} \mathbb{B} \). Then \( f \in \mathcal{A}^p_{\alpha,i} \) if and only if \( \mathcal{A}^p_{\alpha,j} \).

Proof. Let \( f \in \mathcal{SR} \mathbb{B} \) and let \( w = x + yj \in \mathbb{B}_j \), \( z = x + yi \in \mathbb{B}_i \). Note that \( |z| = |w| \). Then, the Representation Formula implies
\[
|f(w)| = \frac{1}{2} |(1 + ji)f(z) + (1 - ji)f(\bar{z})| \leq |f(z)| + |f(\bar{z})|. 
\]
This yields
\[
\int_{\mathbb{B}_j} |f(w)|^p (\alpha + 1)(1 - |w|^2)^\alpha \ dA_j(w) 
\leq 2^{\max\{p-1,0\}} (\alpha + 1) \left( \int_{\mathbb{B}_i} |(1 - |z|^2)^\alpha f(z)|^p \ dA_i(z) + \int_{\mathbb{B}_i} |(1 - |z|^2)^\alpha f(\bar{z})|^p \ dA_i(z) \right). 
\]
Moreover, the change of coordinates \( \bar{z} \to z \) gives
\[
\int_{\mathbb{B}_j} |(1 - |z|^2)^\alpha f(\bar{z})|^p \ dA_i(z) = \int_{\mathbb{B}_j} |(1 - |z|^2)^\alpha f(z)|^p \ dA_i(z), 
\]
and so
\[
\int_{\mathbb{B}_j} |(1 - |w|^2)^\alpha f(w)|^p (\alpha + 1) \ dA_j(w) \leq 2^{\max\{p,1\}} \left( \int_{\mathbb{B}_i} |(1 - |z|^2)^\alpha f(z)|^p (\alpha + 1) \ dA_i(z) \right). 
\]
Thus, for any \( f \in \mathcal{A}^p_{\alpha,i} \) we have that \( f \in \mathcal{A}^p_{\alpha,i} \). By exchanging the roles of \( i \) and \( j \), we obtain the other inclusion. \( \square \)
Next results describe the growth of slice regular functions in the ball and, in particular, of functions belonging to the weighted Bergman spaces.

**Proposition 3.5.** Let $p > 0$, let $i \in S^2$ and let $f \in \mathcal{SR}(\mathbb{B})$.

(i) For any $0 < r < 1$ we have
\[
|f(0)|^p \leq \frac{2 \max\{p,1\}}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.
\]

(ii) For $\alpha > -1$ we have
\[
|f(0)|^p \leq 2 \max\{p,1\} \int_{\mathcal{B}_i} |f(z)|^p dA_{i,\alpha}(z).
\]

**Proof.** Let $j \in S^2$ such that $i \perp j$ and let $f_1, f_2: \mathcal{B}_i \to \mathbb{C}(i)$ be holomorphic functions such that $Q_i[f] = f_1 + f_2 i$. Then, as a direct consequence of the Lemma 4.11 in [29] and of the inequality (4) in the previous remark, one obtains
\[
|f(0)|^p \leq 2 \max\{p-1,0\} \left( |f_1(0)|^p + |f_2(0)|^p \right)
\]
\[
\leq 2 \max\{p-1,0\} \left( \frac{1}{2\pi} \int_0^{2\pi} |f_1(re^{i\theta})|^p d\theta + \frac{1}{2\pi} \int_0^{2\pi} |f_2(re^{i\theta})|^p d\theta \right)
\]
\[
\leq \frac{2 \max\{p,1\}}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.
\]

The second inequality follows in the same way from the analogous result for holomorphic functions, Lemma 4.12 in [29].

**Proposition 3.6.** Let $p > 0$ and $\alpha > -1$ and let $f \in \mathcal{A}_\alpha^p$, with $i \in S^2$. Then for any $z \in \mathbb{B}_i$ we have
\[
|f(z)| \leq \frac{2 ||f||_{p,\alpha,i}}{(1 - |z|^2)^{\frac{2+\alpha}{p}}}.
\]

**Proof.** Let $z \in \mathbb{B}$ and $j \in S^2$ such that $i \perp j$ and let $f_1, f_2: \mathbb{B}_i \to \mathbb{C}(i)$ be holomorphic functions such that $Q_i[f] = f_1 + f_2 i$. Then, by Theorem 4.14 in [29], we have
\[
|f(z)| \leq |f_1(z)| + |f_2(z)|
\]
\[
\leq \frac{||f_1||_{p,\alpha}}{(1 - |z|^2)^{\frac{2+\alpha}{p}}} + \frac{||f_2||_{p,\alpha}}{(1 - |z|^2)^{\frac{2+\alpha}{p}}}
\]
\[
\leq \frac{2 ||f||_{p,\alpha,i}}{(1 - |z|^2)^{\frac{2+\alpha}{p}}}.
\]

**Corollary 3.7.** Let $p > 0$ and $\alpha > -1$ and let $f \in \mathcal{A}_\alpha^p$. Then
\[
|f(q)| \leq \frac{4 ||f||_{p,\alpha}}{(1 - |q|^2)^{\frac{2+\alpha}{p}}}
\]
for all $q \in \mathbb{B}$. 

14
Proof. The result follows by the Representation Formula and taking the supremum for \( i \in S^2 \) in (5).

**Proposition 3.8.** Let \( p > 0 \) and \( \alpha > -1 \). Then

\[
\sup \{ |f(q)| : \|f\|_{p,\alpha} \leq 1, q \in S \} < \infty
\]

for every \( f \in A_{p,\alpha}^p \) and for any axially symmetric compact set \( S \subseteq \mathbb{B} \).

Proof. Let \( i, j \in S^2 \) be such that \( i \perp j \) and let \( S_i = S \cap \mathbb{C}(i) \). Since \( \|f\|_{p,\alpha} \leq 1 \) then, by definition, \( \|f\|_{p,\alpha,i} \leq 1 \) and \( f \in A_{p,\alpha,i}^p \). Let \( f_1, f_2 : \mathbb{B}_i \to \mathbb{C}(i) \) be holomorphic functions such that \( Q_i[f] = f_1 + f_2j \). Then, from the corresponding result in the complex case, Corollary 4.15 in [29], we obtain

\[
\sup \{ |f(z)| : \|f\|_{p,\alpha,i} \leq 1, z \in S_i \} \leq \sup \{ |f_1(z)| : \|f_1\|_{p,\alpha} \leq 1, z \in S_i \} + \sup \{ |f_2(z)| : \|f_2\|_{p,\alpha} \leq 1, z \in S_i \} < \infty.
\]

The result now follows from the Representation Formula, in fact, for any \( q = x + iy \in S \), \( z = x + iy, \bar{z} = x - iy \in S_i \) we have

\[
|f(q)| \leq |f(z)| + |f(\bar{z})| < \infty,
\]

thus

\[
\sup \{ |f(q)| : \|f\|_{p,\alpha} \leq 1, q \in S \} \leq \sup \{ |f(z)| : \|f\|_{p,\alpha,i} \leq 1, z \in S_i \} + \sup \{ |f(\bar{z})| : \|f\|_{p,\alpha,i} \leq 1, z \in S_i \} < \infty,
\]

and this concludes the proof.

**Proposition 3.9.** The space \( A_{p,\alpha}^p \), \( p > 0 \) and \( \alpha > -1 \), is complete.

Proof. Let \( f_n \) be a Cauchy sequence in \( A_{p,\alpha}^p \). Then \( f_n \) is a Cauchy sequence in \( A_{p,\alpha,i}^p \) for some \( i \in S^2 \). Let \( j \in S^2 \) be such that \( i \perp j \) and let \( f_{n,1}, f_{n,2} : \mathbb{B}_i \to \mathbb{C}(i) \) be holomorphic functions such that \( Q_i[f_n] = f_{n,1} + f_{n,2}j \).

As the complex Bergman space \( A_{p,\alpha,C}^p \) is complete, see Corollary 4.16 in [29], there exist functions \( f_1, f_2 \in A_{p,\alpha,C}^p \) such that \( f_{n,1} \to f_1 \) and \( f_{n,2} \to f_2 \) in \( A_{p,\alpha,C}^p \) as \( n \to \infty \). Let \( f = P_i(f_1 + f_2j) \), then we have \( f \in A_{p,\alpha,i}^p \). Furthermore, as

\[
\|f_n - f\|_{p,\alpha,i} \leq \|f_{n,1} - f_1\|_{p,\alpha} + \|f_{n,2} - f_2\|_{p,\alpha} \xrightarrow{n \to \infty} 0,
\]

for \( p \geq 1 \) and

\[
\|f_n - f\|_{p,\alpha,i} \leq \|f_{n,1} - f_1\|_{p,\alpha} + \|f_{n,2} - f_2\|_{p,\alpha} \xrightarrow{n \to \infty} 0,
\]

for \( 0 < p \leq 1 \), we have \( f_n \to f \) in \( A_{p,\alpha,i}^p \). Thus, \( f \in A_{p,\alpha,i}^p \) but also \( f \in A_{p,\alpha}^p \) which is then complete.

In complex analysis, the distance function in the Bergman metric on the unit disc \( \mathbb{D} \) is

\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}, \text{ for all } z, w \in \mathbb{D},
\]

where \( \rho(z, w) = \frac{|z - \bar{w}|}{1 - \bar{z}w} \). This motivates the following definition.
Definition 3.10. For $i \in S^2$ we define

$$\beta_i(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)}, \quad z, w \in \mathbb{B}_i,$$

where

$$\rho(z, w) = \frac{|z - w|}{1 - z\overline{w}}.$$

Note that, as in the complex case, $\beta_i$ is invariant under any slice regular Möbius transformation $T_a$ with $a \in \mathbb{B}_i$, that is $\beta_i(T_a(z), T_a(w)) = \beta_i(z, w)$ for all $z, w \in \mathbb{B}_i$.

Proposition 3.11. Let $p > 0$, $\alpha > -1$ and $r > 0$. For $z \in \mathbb{B}_i$, let $D_i(z, r) = \{w \in \mathbb{B}_i: \beta_i(z, w) < r\}$. Then there exists a positive constant $C$ such that, for all $f \in SR(\mathbb{B})$ and all $z \in \mathbb{B}_i$, we have

$$|f(z)|^p \leq \frac{2^{\max\{p,1\}}C}{(1 - |z|^2)^{2+\alpha}} \int_{D_i(z,r)} |f(w)|^p dA_{\alpha,i}(w).$$

Proof. Owing to Proposition 4.13 in [29], there exists a constant $C$ such that

$$|g(z)|^p \leq \frac{C}{(1 - |z|^2)^{2+\alpha}} \int_{D_i(z,r)} |g(w)|^p dA_{\alpha,i}(w)$$

for all $z \in \mathbb{B}_i$ and all holomorphic functions $g: \mathbb{B}_i \to \mathbb{C}(i)$. Now let $j \in S^2$ be such that $i \perp j$ and for any $f \in SR(\mathbb{B})$ let $f_1, f_2: \mathbb{B}_i \to \mathbb{C}(i)$ be holomorphic functions such that $Q_i[f] = f_1 + f_2$. Then, because of the inequality (4) in Remark 8.3, we have

$$|f(z)|^p \leq 2^{\max\{p-1,0\}}(|f_1(z)|^p + |f_2(z)|^p)$$

$$\leq 2^{\max\{p-1,0\}} \left( \frac{C}{(1 - |z|^2)^{2+\alpha}} \int_{D_i(z,r)} |f_1(w)|^p dA_{\alpha,i}(w) \right)$$

$$+ \frac{C}{(1 - |z|^2)^{2+\alpha}} \int_{D_i(z,r)} |f_2(w)|^p dA_{\alpha,i}(w) \right)$$

$$\leq \frac{2^{\max\{p,1\}}C}{(1 - |z|^2)^{2+\alpha}} \int_{D_i(z,r)} |f(w)|^p dA_{\alpha,i}(w).$$

\[ \square \]

4 Besov spaces

In complex analysis, the Besov spaces are analogue to Bergman spaces and they are studied with similar techniques. The Besov space $\mathfrak{B}_{p,\alpha}$, for $p > 0$, is defined as the set of all analytic functions on the complex unit disc $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |(1 - |z|^2)^n f^{(n)}(z)|^p d\lambda(z) < \infty,$$

for some $n$ with $np > 1$ where $d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2}$.

To move to the setting of quaternionic Besov spaces, we define a suitable differential. For $i \in S^2$ let

$$d\lambda_i(z) = \frac{dA_i(z)}{(1 - |z|^2)^2},$$

where $dA_i(z)$ is again the normalized differential of area in the plane $\mathbb{C}(i)$. Note that, as in the complex case, $d\lambda_i$ is invariant under slice regular Möbius transformations $T_a$ with $a \in \mathbb{B}_i$. 

16
Proposition 4.1. Let \( i \in \mathbb{S}^2 \), let \( p > 0 \) and let \( m, n \in \mathbb{N} \) such that \( np > 1 \) and \( mp > 1 \). Then for any \( f \in \mathcal{SR}(\mathbb{B}) \) the inequality
\[
\int_{\mathbb{B}_i} |(1 - |z|^2)^n \partial_{z_0}^n f(z)|^p d\lambda_i(z) < \infty
\]
holds if and only if
\[
\int_{\mathbb{B}_i} |(1 - |z|^2)^m \partial_{z_0}^m f(z)|^p d\lambda_i(z) < \infty.
\]

Proof. Let \( f \in \mathcal{SR}(\mathbb{B}) \), let \( j \in \mathbb{S}^2 \) such that \( i \perp j \) and let \( f_1, f_2 \colon \mathbb{B}_i \to \mathbb{C}(i) \) be holomorphic functions such that \( Q_i[f] = f_1 + f_2 j \). Note that \( Q_i[f] \) is a function of the complex variable \( z = x_0 + y_i \). So for all \( k \in \mathbb{N} \) we have \( \partial_{x_0}^k f(z) = Q_i[f]^{(k)}(z) = f_1^{(k)}(z) + f_2^{(k)}(z) j \) and, from the inequality (1) in Remark 3.3 we obtain
\[
|(1 - |z|^2)^k f_1^{(k)}(z)|^p \leq |(1 - |z|^2)^k \partial_{x_0}^k f(z)|^p \leq 2^{\max(p-1,0)} |(1 - |z|^2)^k f_1^{(k)}(z)|^p + 2^{\max(p-1,0)} |(1 - |z|^2)^k f_2^{(k)}(z)|^p
\]
for \( l = 1, 2 \) and \( z \in \mathbb{B}_i \). Therefore, the inequality
\[
\int_{\mathbb{B}_i} |(1 - |z|^2)^k f_1^{(k)}(z)|^p d\lambda_i(z) < \infty, \quad l = 1, 2,
\]
holds if and only if \( \int_{\mathbb{B}_i} |(1 - |z|^2)^k \partial_{x_0}^k f(z)|^p d\lambda_i(z) < \infty \). But from Lemma 5.16 in [29], we have that (6) holds for \( k = n \) if and only if it holds for \( k = m \), thus the statement is true.

The following definition will be useful in the sequel.

Definition 4.2. By \( \mathcal{B}_{p,i} \) we denote the quaternionic right linear space of slice regular functions on \( \mathbb{B} \) such that
\[
\int_{\mathbb{B}_i} |(1 - |z|^2)^n \partial_{x_0}^n f(z)|^p d\lambda_i(z) < \infty,
\]
for some \( n \in \mathbb{N} \) such that \( np > 1 \). Moreover we set \( \mathcal{B}_{\infty,i} = \mathcal{B}_i \), where \( \mathcal{B}_i \) is as in Definition 2.2.

Note that by Proposition 4.1, this definition is independent of the choice of \( n \).

Remark 4.3. Let \( j \in \mathbb{S}^2 \) be such that \( j \perp i \). Then there exist holomorphic functions \( f_1, f_2 \colon \mathbb{B}_i \to \mathbb{C}(i) \) such that \( Q_i[f] = f_1 + f_2 j \) and so \( \partial_{x_0}^n f(z) = f_1^{(n)}(z) + f_2^{(n)}(z) j \) for \( z \in \mathbb{B}_i \). Thus, for \( z \in \mathbb{B}_i \), we have
\[
|f_k^{(n)}(z)|^p \leq |\partial_{x_0}^n f(z)|^p \leq 2^{\max(0,p-1)} \left( |f_1^{(n)}(z)|^p + |f_2^{(n)}(z)|^p \right), \quad k = 1, 2.
\]
Thus, the function \( f \) is in \( \mathcal{B}_{p,i} \) if and only if \( f_1 \) and \( f_2 \) are in the complex Besov space \( \mathcal{B}_{p,C} \).

Proposition 4.4. Let \( p > 0 \) and \( i, j \in \mathbb{S}^2 \). Then \( \mathcal{B}_{p,i} \) and \( \mathcal{B}_{p,j} \) are equal as sets.

Proof. Let \( f \in \mathcal{SR}(\mathbb{B}) \) and let \( n \in \mathbb{N} \) such that \( pn > 1 \). For \( w = x_0 + y_j \in \mathbb{B}_j \), let \( z = x_0 + y_i \in \mathbb{B}_i \). Then, from the Representation Formula, one obtains
\[
|\partial_{x_0}^n f(w)| = \frac{1}{2} \left| (1 - ji) \partial_{x_0}^n f(z) + (1 + ji) \partial_{x_0}^n f(\bar{z}) \right| \leq |\partial_{x_0}^n f(z)| + |\partial_{x_0}^n f(\bar{z})|.
\]
This implies
\[ \int_{B_j} |(1 - |w|^2)^n \partial_{x_0} f(w)|^p \, d\lambda_j(w) \leq 2^{\max\{p-1,0\}} \left( \int_{B_j} |(1 - |z|^2)^n \partial_{x_0} f(z)|^p \, d\lambda_i(z) + \int_{B_j} |(1 - |z|^2)^n \partial_{x_0} f(\bar{z})|^p \, d\lambda_i(z) \right). \]

By changing coordinates $\bar{z} \to z$, we obtain
\[ \int_{B_j} |(1 - |z|^2)^n \partial_{x_0} f(z)|^p \, d\lambda_i(z) = \int_{B_j} |(1 - |z|^2)^n \partial_{x_0} f(\bar{z})|^p \, d\lambda_i(z), \]
and so
\[ \int_{B_j} |(1 - |w|^2)^n \partial_{x_0} f(w)|^p \, d\lambda_j(w) \leq 2^{\max\{p,1\}} \int_{B_j} |(1 - |z|^2)^n \partial_{x_0} f(z)|^p \, d\lambda_i(z). \]

Thus, for any $f \in \mathcal{B}_{p,i}$ we have that $f \in \mathcal{B}_{p,j}$. By exchanging the roles of $i$ and $j$, we obtain the other inclusion. \(\square\)

By virtue of the previous result we can now give the definition of Besov space in this setting:

**Definition 4.5.** Let $p > 0$, let $i \in S^2$ and let $n \in \mathbb{N}$ with $pn > 1$. The slice regular Besov space $\mathcal{B}_p$ is the quaternionic right linear space of slice regular functions on $\mathbb{B}$ such that
\[ \sup_{z \in S^2} \int_{B_i} |(1 - |z|^2)^n \partial_{x_0} f(z)|^p \, d\lambda_i(z) < \infty. \]

We also define the Besov space $\mathcal{B}_\infty = \mathcal{B}$, where $\mathcal{B}$ is the Bloch space.

Next result gives a nice characterization of the functions in $\mathcal{B}_p$.

**Proposition 4.6.** Let $i \in S^2$. For any $p > 0$ there exists a sequence $(a_k)_{k \geq 1}$ in $\mathbb{B}_i$ with the following property: if $b > \max\{0,(p-1)/p\}$, then $f \in \mathcal{B}_p$ if and only if there exists a sequence $(d_k)_{k \geq 1} \in l^p(\mathbb{H})$ such that
\[ f = \sum_{k=1}^{\infty} P_i \left( \frac{1 - |a_k|^2}{1 - z \overline{a_k}} \right)^b d_k. \]  

(7)

**Proof.** Let $i \in S^2$ and let us identify $B_i$ with the unit disc $\mathbb{D}$. Theorem 5.17 in [29] yields the existence of a sequence $(a_k)_{k \geq 1} \subset \mathbb{B}_i$ such that the complex Besov space $\mathcal{B}_{p,\mathbb{C}}$ on $\mathbb{B}_i$ consists exactly of the functions of the form
\[ g(z) = \sum_{k=1}^{\infty} \left( \frac{1 - |a_k|^2}{1 - z \overline{a_k}} \right)^b c_k \]  

(8)
with $(c_k)_{k \geq 1} \in l^p(\mathbb{C}(i))$. Now let $j \in S^2$ such that $j \perp i$. For any $f \in \mathcal{B}_p$ there exist two holomorphic functions $f_1, f_2: \mathbb{B}_i \to \mathbb{C}(i)$ such that $Q_i[f] = f_1 + f_2j$. Since $f \in \mathcal{B}_{p,i}$, for $i \in S^2$, from Remark 4.3 it follows that $f_1$ and $f_2$ belong to the complex Besov space $\mathcal{B}_{p,\mathbb{C}}$ and so there exist sequences $(c_{1,k})_{k \geq 1}, (c_{2,k})_{k \geq 1} \in l^p(\mathbb{C}(i))$ that give the representation [30] of $f_1$ and $f_2$, respectively. Therefore, we have
\[ Q_i[f](z) = f_1(z) + f_2(z)j = \sum_{k=1}^{\infty} \left( \frac{1 - |a_k|^2}{1 - z \overline{a_k}} \right)^b c_{1,k} + \sum_{k=1}^{\infty} \left( \frac{1 - |a_k|^2}{1 - z \overline{a_k}} \right)^b c_{2,k,j}. \]
As \( P_i \circ Q_i = I_{S^R(\mathcal{B})} \), we obtain the desired representation

\[
f = P_i \circ Q_i[f] = P_i \left[ \sum_{k=1}^{\infty} \left( \frac{1 - |a_k|^2}{1 - z\overline{a_k}} \right)^b (c_{1,k} + c_{2,k,j}) \right] = \sum_{k=1}^{\infty} P_i \left[ \left( \frac{1 - |a_k|^2}{1 - z\overline{a_k}} \right)^b \right] (c_{1,k} + c_{2,k,j}),
\]

where the sequence of the coefficients \( d_k = c_{1,k} + c_{2,k,j} \) lie in \( \ell^p(\mathbb{H}) \) because \( (c_{1,k})_{k \geq 1} \) and \( (c_{2,k})_{k \geq 1} \) belong to \( \ell^p(\mathbb{C}(i)) \).

If, on the other hand, \( f \) has the form (7), then there exist sequences \( (c_{1,k})_{k \geq 1} \) and \( (c_{2,k})_{k \geq 1} \) such that \( d_k = c_{1,k} + c_{2,k,j} \). Thus, we have

\[
Q_i[f] = f_1 + f_2 = \sum_{k=1}^{\infty} \left( \frac{1 - |a_k|^2}{1 - z\overline{a_k}} \right)^b c_{1,k} + \sum_{k=1}^{\infty} \left( \frac{1 - |a_k|^2}{1 - z\overline{a_k}} \right)^b c_{2,k,j}
\]

and as \( (d_k)_{k \geq 1} \) lies in \( \ell^p(\mathbb{H}) \), it follows that \( (c_{1,k})_{k \geq 1} \) and \( (c_{2,k})_{k \geq 1} \) are in \( \ell^p(\mathbb{C}(i)) \). Therefore, \( f_1 \) and \( f_2 \) are of the form (8) and so they lie in the complex Besov space \( \mathfrak{B}_{p,\mathbb{C}} \), which implies \( f \in \mathfrak{B}_{p,i} \) and since \( i \in \mathbb{S}^2 \) is arbitrary, we conclude that \( f \in \mathfrak{B}_p \).

The following result shows that the space \( \mathfrak{B}_{p,i} \) is invariant under Möbius transformation if one takes the \( \circ_i \) composition.

**Proposition 4.7.** Let \( i \in \mathbb{S}^2 \), let \( a \in \mathfrak{B}_i \) and let \( T_a \) be a slice regular Möbius transformation. Then for \( f \in \mathfrak{B}_{p,i} \) we have \( f \circ_i T_a \in \mathfrak{B}_{p,i} \).

**Proof.** Let \( j \in \mathbb{S}^2 \) with \( j \perp i \) and let \( f_1, f_2 : \mathfrak{B}_i \rightarrow \mathbb{C}(i) \) be holomorphic functions such that \( Q_i[f] = f_1 + f_2 \). Moreover, the functions \( f_1 \) and \( f_2 \) lie in the complex Besov space \( \mathfrak{B}_{p,\mathbb{C}} \) because of Remark 4.3. By the definition of the \( \circ_i \)-composition, we have

\[
Q_i[f \circ_i T_a] = f_1 \circ Q_i[T_a] + f_2 \circ Q_i[T_a].
\]

But the function \( Q_i[T_a] \) is nothing but the complex Möbius transformation associated with \( a \in \mathfrak{B}_i \). Moreover, because of Theorem 5.18 in [29], the complex Besov space \( \mathfrak{B}_{p,\mathbb{C}} \) is Möbius invariant. Thus, \( f_1 \circ Q_i[T_a] \) and \( f_2 \circ Q_i[T_a] \) lie in the complex Besov space, which is equivalent to \( f \circ_i T_a \in \mathfrak{B}_{p,j} \).

**Definition 4.8.** Let \( i \in \mathbb{S}^2 \). For \( p > 1 \), we define

\[
\rho_{p,i}(f) = \left[ \int_{\mathfrak{B}_i} (1 - |z|^2)^p |\partial_z f(z)|^p d\lambda_i(z) \right]^\frac{1}{p}
\]

for all \( f \in \mathfrak{B}_{p,i} \). For \( 0 < p \leq 1 \) and \( n \in \mathbb{N} \) with \( np > 1 \), we define

\[
\rho_{p,i,n}(f) = \sup_{z \in \mathfrak{B}_i} \left| f(z) \right| + \sup_{a \in \mathfrak{B}_i} \left[ \int_{\mathfrak{B}_i} (1 - |z|^2)^{np} |\partial_z^n (f \circ_i T_a)(z)|^p d\lambda_i(z) \right]^\frac{1}{p},
\]

for all \( f \in \mathfrak{B}_{p,i} \).

In the following result we show that, for \( p > 1 \), \( \mathfrak{B}_{p,i} \) is a Banach space invariant under the \( \circ_i \)-composition with a Möbius map.
Lemma 4.9. Let \( i \in \mathbb{S}^2 \) and \( p > 1 \). Then \( \rho_{p,i} \) is a complete seminorm on \( \mathfrak{B}_{p,i} \) such that \( \mathfrak{B}_{p,i} \) modulo the constant functions endowed with the norm induced by \( \rho_{p,i} \) is a Banach space, and it satisfies

\[
\rho_{p,i}(f \circ_i T_a) = \rho_{p,i}(f)
\]

for all \( f \in \mathfrak{B}_{p,i} \) and all \( a \in \mathfrak{B}_i \).

Proof. It is clear, that \( \rho_{p,i} \) actually is a seminorm. Let \( j \in \mathbb{S}^2 \) with \( i \perp j \) and for \( f \in \mathfrak{B}_{p,i} \) let \( f_1, f_2 \in \mathfrak{B}_p \) such that \( Q_i[f] = f_1 + f_2 j \). Then, the inequality

\[
|f_k'(z)|^p \leq |\partial_{x_0} f(z)|^p \leq 2^{p-1} \left(|f_1'(z)|^p + |f_2'(z)|^p\right) \quad \text{for } k = 1, 2,
\]

implies

\[
\rho_p(f_k)^p \leq \rho_{p,i}(f)^p \leq 2^{p-1} \left(\rho_p(f_1)^p + \rho_p(f_2)^p\right) \quad \text{for } k = 1, 2,
\]

where \( \rho_p \) is the corresponding seminorm on the complex Besov space \( \mathfrak{B}_{p,\mathbb{C}} \) on \( \mathfrak{B}_i \). Thus, as \( \rho_p \) is a complete seminorm on the complex Besov space (see Theorem 5.18 in [29]), the seminorm \( \rho_{p,i} \) on is complete too.

Now we want to prove the Möbius invariance of \( \rho_{p,i} \), so let \( a \in \mathfrak{B}_i \). If we denote

\[
(f \circ T_a)' = \partial_0 Q_i[f \circ T_a], \quad f' = \partial_{x_0} Q_i[f], \quad \text{and} \quad T'_a = \partial_{x_0} Q_i[T_a],
\]

we have

\[
|(f \circ T_a)'(z)| = |(f_1 \circ T_a)'(z) + (f_2 \circ T_a)'(z) j| = |T'_a(z) : f'(T_a(z)) j| = |T'_a(z) | f'(T_a(z)) |.
\]

Now recall that \( (1 - |z|^2)|T'_a(z)| = 1 - |T_a(z)|^2 \) for \( z \in \mathfrak{B}_i \) as \( Q_i[T_a] \) is the usual complex Möbius transformation on \( \mathfrak{B}_i \) associated with \( a \). So, as \( \lambda_i \) is invariant under \( T_a \), we obtain

\[
\rho_{p,i}(f \circ_i T_a)^p = \int_{\mathfrak{B}_i} (1 - |z|^2)^p |(f \circ T_a)'(z)|^p d\lambda_i(z)
\]

\[
= \int_{\mathfrak{B}_i} (1 - |z|^2)^p |T'_a(z) | f'(T_a(z))|^p d\lambda_i(z)
\]

\[
= \int_{\mathfrak{B}_i} (1 - |T_a(z)|^2)^p |f'(T_a(z))|^p d\lambda_i(T_a(z)) = \rho_{p,i}(f)^p.
\]

\( \square \)

Corollary 4.10. Let \( p > 1 \) and let \( \rho_p(f) = \sup_{i \in \mathbb{S}^2} \rho_{p,i}(f) \). Then \( \rho_p \) is a complete seminorm on \( \mathfrak{B}_p \) such that \( \mathfrak{B}_p \) modulo the constant functions endowed with the norm induced by \( \rho_p \) is a Banach space.

Proof. The result follows from the previous Lemma. \( \square \)

Remark 4.11. We observe that, in general, \( \rho_p(f \circ_i T_a)^p \neq \rho_p(f)^p \). In fact for \( w = x_0 + i x_1 \) let \( z_i = x_0 + i x_1 \) and assume that \( f' \) denotes the derivative the restriction of \( f \) to the plane \( \mathbb{C}(i) \), then

\[
\rho_p(f \circ_i T_a)^p = \sup_{j \in \mathbb{S}} \rho_{p,j}(f \circ_i T_a)^p = \sup_{j \in \mathbb{S}} \int_{\mathfrak{B}_j} (1 - |w|^2)^p |\partial_0 f \circ_i T_a|^p d\lambda_j(w)
\]

\[
= \sup_{j \in \mathbb{S}} \int_{\mathfrak{B}_j} \frac{1}{2^p} (1 - |z|^2)^p |(1 - ji)(f \circ_i T_a)'(z) + (1 + ji)(f \circ_i T_a)'(z)j|^p d\lambda_i(z)
\]

\[
= \sup_{j \in \mathbb{S}} \int_{\mathfrak{B}_j} \frac{1}{2^p} (1 - |z|^2)^p |(1 - ji)f'(T_a(z))T'_a(z) + (1 + ji)f'(T_a(z))T'_a(z)j|^p d\lambda_i(z).
\]

20
But, in general, it is $|T'_a(z)| \neq |T'_a(\bar{z})|$ thus we obtain

$$
\rho_p(f \circ T_a)^p \neq \sup_{j \in \mathbb{S}} \int_{\mathbb{B}_i} \frac{1}{2p} (1 - |z|^2)^p |T'_a(z)|^p \left|(1 - ji)f'(T_a(z)) + (1 + ji)f'(T_a(\bar{z}))\right|^p d\lambda_i(z)
$$

$$
= \sup_{j \in \mathbb{S}} \int_{\mathbb{B}_i} \frac{1}{2p} (1 - |T_a(z)|^2)^p \left|(1 - ji)f'(T_a(z)) + (1 + ji)f'(T_a(\bar{z}))\right|^p d\lambda_i(T_a(z))
$$

$$
= \sup_{j \in \mathbb{S}} \rho_{p,i}(f)^p = \rho_p(f)^p,
$$

where we used the fact that $(1 - |T_a(z)|^2) = (1 - |z|^2)|T'_a(z)|$ and that $d\lambda_i$ is invariant under $T_a$.

**Corollary 4.12.** For $p > 1$, the Besov space $\mathcal{B}_p$ is a Banach space with the norm

$$
\|f\|_{\mathcal{B}_p} = |f(0)| + \sup_{i \in \mathbb{S}^2} \left[ (1 - |z|^2)^p |\partial_{x_0} f(z)|^p d\lambda_i(z) \right]^{\frac{1}{p}}.
$$

**Proof.** It is an immediate consequence of the above lemma and of the fact that $f \in \mathcal{B}_p$ if and only if $f \in \mathcal{B}_{p,i}$.

**Definition 4.13.** A seminorm $\rho$ is said to be $i$-Möbius invariant on $\mathcal{B}_{p,i}$ if it satisfies the condition in Lemma 4.9

According to Definition 4.8, we now study the case $0 < p \leq 1$. We have the following result:

**Proposition 4.14.** Let $i \in \mathbb{S}^2$ and $0 < p \leq 1$ and let $n \in \mathbb{N}$ with $np > 1$. Then $\rho_{p,n,i}(f)$ is finite for any $f \in \mathcal{B}_{p,i}$. Furthermore, we have

$$
\rho_{p,n,i}(f \circ T_a) = \rho_{p,n,i}(f), \quad \text{for all } a \in \mathbb{B}_i.
$$

**Proof.** As for $a \in \mathbb{B}_i$ the slice regular Möbius transformation maps $\mathbb{B}_i$ bijectively onto itself, by the definition of $\rho_{p,n,i}$, it is obvious that (9) holds, provided that $\rho_{p,n,i}$ is finite.

Now let $f \in \mathcal{B}_{p,i}$. To show that $\rho_{p,n,i}(f)$ is finite, we chose again $j \in \mathbb{S}^2$ with $i \perp j$ and apply the Splitting Lemma to obtain functions $f_1$ and $f_2$ in the complex Besov space $\mathcal{B}_{p,\mathbb{C}}$ on $\mathbb{B}_i$ such that $Q_i[f] = f_1 + f_2 j$. From Theorem 5.18 in [29], we know that, for any function $g$ in $\mathcal{B}_p$, we have

$$
\rho_{p,n}(g) = \sup_{z \in \mathbb{B}_i} |g(z)| + \sup_{a \in \mathbb{B}_i} \left[ \int_{\mathbb{B}_i} (1 - |z|^2)^{np} |(g \circ \varphi_a)^{(n)}(z)|^p d\lambda_i(z) \right]^{\frac{1}{p}} < \infty,
$$

where $\varphi_a$ denotes the complex Möbius transformation associated with $a$. Thus, on one hand we have

$$
\sup_{z \in \mathbb{B}_i} |f(z)| \leq \sup_{z \in \mathbb{B}_i} |f_1(z)| + \sup_{z \in \mathbb{B}_i} |f_2(z)| \leq \rho_{p,n}(f_1) + \rho_{p,n}(f_2) < \infty.
$$

On the other hand, as $0 < p \leq 1$, we have $(\alpha + \beta)^p < \alpha^p + \beta^p$ for $\alpha, \beta > 0$. Moreover, for any $a$ in $\mathbb{B}_i$, we have $\varphi_a = Q_i[T_a]$. As $x \mapsto x^{1/p}$ is increasing on $\mathbb{R}^+$, we can exchange it with the supremum and so we obtain

$$
\sup_{a \in \mathbb{B}_i} \int_{\mathbb{B}_i} (1 - |z|^2)^{np} |\partial_{x_0} (f \circ T_a)(z)|^p d\lambda_i(z) \leq \sup_{a \in \mathbb{B}_i} \int_{\mathbb{B}_i} (1 - |z|^2)^{np} |(f_1 \circ \varphi_a)^{(n)}(z)|^p d\lambda_i(z)
$$

$$
+ \sup_{a \in \mathbb{B}_i} \int_{\mathbb{B}_i} (1 - |z|^2)^{np} |(f_2 \circ \varphi_a)^{(n)}(z)|^p d\lambda_i(z) \leq \rho_{p,n}(f_1)^p + \rho_{p,n}(f_2)^p < \infty.
$$

Putting all this together, we obtain that $\rho_{p,n,i}(f)$ is finite. 

---

21
The space $\mathcal{B}_1$ requires some special attention.

**Proposition 4.15.** Let $f \in S\mathcal{R}(\mathbb{B})$ and let $i \in S^2$. Then the following facts are equivalent

(i) $f$ belongs to $\mathcal{B}_1$.

(ii) There exists a sequence $(\gamma_k)_{k \geq 0} \in \ell^1(\mathbb{H})$ and a sequence $(a_k)_{k \geq 1} \in \mathbb{B}_i$ such that

$$f(q) = \gamma_0 + \sum_{k=1}^{\infty} T_{a_k}(q)\gamma_k.$$  \hspace{1cm} (10)

(iii) There exists a finite $\mathbb{H}$-valued Borel measure $\mu$ on $\mathbb{B}_i$ such that

$$f(q) = \int_{\mathbb{B}_i} T_z(q) d\mu(z).$$

**Proof.** We first show that (i) implies (ii), so let $f \in \mathcal{B}_1$. For $a \in \mathbb{B}_i$, let $T_a(z) = \frac{a - z}{1 - za}$ be the complex Möbius transformation on $\mathbb{B}_i$ associated with $a$. Then we have

$$\frac{1 - |a|^2}{1 - za} = 1 - T_a(z)a.$$  

Thus, we can apply Proposition 4.16 with $b = 1$ and obtain sequences $(d_k)_{k \geq 1} \in \ell^1(\mathbb{H})$ and $(a_k)_{k \geq 1} \in \mathbb{B}_i$ such that

$$f = \sum_{k=1}^{\infty} P_i \left[ \frac{1 - |a_k|^2}{1 - za_k} \right] d_k = \sum_{k=1}^{\infty} P_i [1 - T_{a_k}a_k] d_k = \sum_{k=1}^{\infty} d_k - \sum_{k=1}^{\infty} P_i[T_{a_k}]a_k d_k.$$  

So, if we define $\gamma_0 = \sum_{k=1}^{\infty} d_k$ and $\gamma_k = -\overline{a_k} d_k$ for $k \geq 1$, as $P_i[T_{a_k}] = T_{a_k}$, we obtain (10). Moreover, we have $(\gamma_k)_{k \geq 0} \in \ell^1(\mathbb{H})$ because $|a_k| < 1$ and so

$$\sum_{k=0}^{\infty} |\gamma_k| = |\gamma_0| + \sum_{k=1}^{\infty} |a_k||d_k| \leq 2 \sum_{k=1}^{\infty} |d_k| < \infty.$$  

Now we will show that (ii) implies (iii), so let us suppose that (ii) holds. Then there exist sequences $(\gamma_{1,k})_{k \in \mathbb{N}_0}, (\gamma_{2,k})_{k \in \mathbb{N}_0} \subset \mathbb{C}(i)$ such that $\gamma_k = \gamma_{1,k} + \gamma_{2,k}j$ and, as $|\gamma_{1,k}| \leq |\gamma_k|$ and $|\gamma_{2,k}| \leq |\gamma_k|$ for $k \geq 0$, they are in $\ell^1(\mathbb{C}(i))$. Moreover, let $j \in S^2$ with $i \perp j$ and let $f_1, f_2 : \mathbb{B}_i \rightarrow \mathbb{C}(i)$ be holomorphic functions such that $Q_i[f] = f_1 + f_2j$. Then, as $Q_i[T_a] = T_a$ for $a \in \mathbb{B}_i$, we have,

$$f_1(z) + f_2(z)j = Q_i[f](z) = \gamma_{1,0} + \sum_{k=1}^{\infty} T_{a_k}(z)\gamma_{1,k} + \left( \gamma_{2,0} + \sum_{k=1}^{\infty} T_{a_k}(z)\gamma_{2,k} \right)j.$$  

So $f_i(z) = \gamma_{1,0} + \sum_{k=1}^{\infty} T_{a_k}(z)\gamma_{1,k}$ for $l = 1, 2$. By Theorem 5.19 in [29], there exist two finite Borel measures $\mu_1$ and $\mu_2$ on $\mathbb{B}_i$ with values in $\mathbb{C}(i)$ such that

$$f_1(z) = \int_{\mathbb{B}_i} T_w(z) d\mu_1(w) \quad \text{and} \quad f_2(z) = \int_{\mathbb{B}_i} T_w(z) d\mu_2(w).$$

Thus, if we set $\mu = \mu_1 + \mu_2j$, we have

$$f(q) = P_i[f_1 + f_2j](q) = \int_{\mathbb{B}_i} T_w(q) d\mu_1(w) + \int_{\mathbb{B}_i} T_w(q) d\mu_2(w)j = \int_{\mathbb{B}_i} T_w(q) d\mu(w).$$
Finally, we will show that (iii) implies (i), so let us assume that (iii) holds. Then there exist two finite \( \mathbb{C} \)-valued Borel measures \( \mu_1, \mu_2 \) such that \( \mu = \mu_1 + \mu_2 \). For all \( z \in \mathbb{B}_i \), we have
\[
f_1(z) + f_2(z) = Q_i[f](z) = \int_{\mathbb{B}_i} T_w(q) d\mu_1(w) + \int_{\mathbb{B}_i} T_w(q) d\mu_2(w),
\]
so \( f_l(z) = \int_{\mathbb{B}_i} T_w(z) d\mu_l(w), l = 1, 2 \). Because of Theorem 5.19 in [29] this implies that \( f_1 \) and \( f_2 \) lie in the complex Besov space \( \mathfrak{B}_{1,i} \) which is equivalent to \( f \in \mathfrak{B}_{1,i} \) and so \( f \in \mathfrak{B}_1 \).

As in the complex case, the representation (10) allows to define an \( i \)-Möbius invariant Banach space structure on \( \mathfrak{B}_{1,i} \).

**Remark 4.16.** In the next result we introduce a norm on \( \mathfrak{B}_{1,i} \) which is different from the one in Definition 1.8.

**Proposition 4.17.** Let \( i \in \mathbb{S}^2 \). Then \( \mathfrak{B}_{1,i} \) is a Banach space with the norm
\[
\|f\|_{\mathfrak{B}_{1,i}} = \inf \left\{ \sum_{k=0}^{\infty} |\gamma_k| : \exists (a_k)_{k \geq 1} \subset \mathbb{B}_i : f = \gamma_0 + \sum_{k=1}^{\infty} T_{a_k} \gamma_k \right\}.
\]
Furthermore, for all \( a \in \mathbb{B}_i \), the following equation holds
\[
\|f \circ T_a\|_{\mathfrak{B}_{1,i}} = \|f\|_{\mathfrak{B}_{1,i}}.
\]

**Proof.** It is clear that \( \|f\lambda\|_{\mathfrak{B}_{1,i}} = \|f\|_{\mathfrak{B}_{1,i}} |\lambda| \) for any \( \lambda \in \mathbb{H} \) and that \( \|f\|_{\mathfrak{B}_{1,i}} \geq 0 \) with equality if and only if \( f = 0 \). Now let \( f, g \in \mathfrak{B}_{1,i} \) and let \( f + g = \gamma_0 + \sum_{k=1}^{\infty} T_{a_k} \gamma_k \). Then \( f = \omega_0 + \sum_{k=1}^{\infty} T_{a_k} \omega_k \) if and only if \( g = \omega_0 \) and \( \sum_{k=1}^{\infty} T_{a_k} \omega_k \). Thus, in (13), we can vary \( \omega_k \) and \( v_k \) independently, which allows us to split the infimum. So we obtain
\[
\|f + g\|_{\mathfrak{B}_{1,i}} \leq \inf \left\{ \sum_{k=0}^{\infty} |\omega_k| : \exists (b_k)_{k \geq 1} \subset \mathbb{B}_i : f = \omega_0 + \sum_{k=1}^{\infty} T_{b_k} \omega_k \right\} + \inf \left\{ \sum_{k=0}^{\infty} |v_k| : \exists (c_k)_{k \geq 1} \subset \mathbb{B}_i : g = \omega_0 + \sum_{k=1}^{\infty} T_{c_k} v_k \right\} = \|f\|_{\mathfrak{B}_{1,i}} + \|g\|_{\mathfrak{B}_{1,i}}.
\]
Hence \( \| \cdot \|_{\mathfrak{B}_{1,i}} \) is actually a norm.
Furthermore, for $a, b \in \mathbb{B}_i$ the $i$-composition $T_b \circ_i T_a$ is again a slice regular Möbius transformation $T_c$ with $c \in \mathbb{B}_i$. Thus, $f = \gamma_0 + \sum_{k=1}^{\infty} T_{c_k} \gamma_k$ if and only if $f \circ_i T_a = \gamma_0 + \sum_{k=1}^{\infty} T_{c_k} \gamma_k$, where $T_{c_k} = T_{b_k} \circ T_a$. From the definition of $\| \cdot \|_{\mathcal{B}_{1,i}}$, it is therefore clear that this norm is invariant under Möbius transformations $T_a$ with $a \in \mathbb{B}_i$.

Finally, to show that the space is complete, we choose again $j \in S^2$ such that $i \perp j$. Then, for any $\gamma_k \in \mathbb{H}$ there exist $\gamma_{k,1}, \gamma_{k,2} \in \mathbb{C}(i)$ such that $\gamma_k = \alpha_k + \beta_k j$ and for any $f \in \mathcal{B}_{1,i}$ there exist $f_1, f_2$ in the space $\mathcal{B}_{1,i}$ on $\mathbb{B}_i$ such that $f = f_1 + f_2 j$. Furthermore, if $f = \lambda_0 + \sum_{k=1}^{\infty} T_{\theta_k} \lambda_k$, then we have $f_1 = \lambda_0 + \sum_{k=1}^{\infty} \varphi_{a_k} \lambda_k$ and $f_2 = \beta_0 + \sum_{k=1}^{\infty} \varphi_{a_k} \beta_k$, where $\varphi_{a_k}$ denotes the complex Möbius transformation on $\mathbb{B}_i$ associated with $a_k$. Therefore, we get

$$\|f\|_{\mathcal{B}_{1,i}} = \inf \left\{ \sum_{k=0}^{\infty} |a_k + \beta_k j| \mid \exists (a_k)_{k \geq 1} \subset \mathbb{B}_i : f = \alpha_0 + \sum_{k=1}^{\infty} T_{\theta_k} (a_k + \beta_k j) \right\}$$

$$\leq \inf \left\{ \sum_{k=0}^{\infty} |a_k| + \sum_{k=0}^{\infty} |\beta_k| \mid \exists (a_k)_{k \geq 1} \subset \mathbb{B}_i : f_1 = \alpha_0 + \sum_{k=1}^{\infty} \varphi_{a_k} \lambda_k, f_2 = \beta_0 + \sum_{k=1}^{\infty} \varphi_{a_k} \beta_k \right\}. \quad (14)$$

On the other hand, an argument like the one used for the triangle inequality shows that we can vary $\alpha_k$ and $\beta_k$ independently, which allows us to split the infimum. So we get

$$\|f\|_{\mathcal{B}_{1,i}} \leq \inf \left\{ \sum_{k=0}^{\infty} |a_k| \mid \exists (b_k)_{k \geq 1} \subset \mathbb{B}_i : f_1 = \alpha_0 + \sum_{k=1}^{\infty} \varphi_{b_k} \lambda_k \right\}$$

$$+ \inf \left\{ \sum_{k=0}^{\infty} |\beta_k| \mid \exists (c_k)_{k \geq 1} \subset \mathbb{B}_i : f_2 = \beta_0 + \sum_{k=1}^{\infty} \varphi_{c_k} \beta_k \right\} \leq \|f_1\|_{\mathcal{B}_{1,\mathcal{C}}} + \|f_2\|_{\mathcal{B}_{1,\mathcal{C}}},$$

where $\| \cdot \|_{\mathcal{B}_{1,\mathcal{C}}}$ is norm on the complex Besov space $\mathcal{B}_{1,\mathcal{C}}$ on $\mathbb{B}_i$, which is defined analogously to $\mathcal{B}_{1,i}$.

On the other hand it is clear that $\|f_k\|_{\mathcal{B}_{1,\mathcal{C}}} \leq \|f\|_{\mathcal{B}_{1,i}}$ for $k = 1, 2$. Thus, putting all these inequalities together, we obtain

$$\|f_k\|_{\mathcal{B}_{1,\mathcal{C}}} \leq \|f\|_{\mathcal{B}_{1,i}} \leq \|f_1\|_{\mathcal{B}_{1,\mathcal{C}}} + \|f_2\|_{\mathcal{B}_{1,\mathcal{C}}}.$$}

Therefore, for any Cauchy sequence $(f_n)_{n \geq 0}$ in $\mathcal{B}_{1,i}$, the sequences of the component functions $(f_{n,1})_{n \geq 1}$ and $(f_{n,2})_{n \geq 1}$ are Cauchy sequences in $\mathcal{B}_{1,\mathcal{C}}$. But as the complex Besov space $\mathcal{B}_{1,\mathcal{C}}$ is complete with the norm $\| \cdot \|_{\mathcal{B}_{1,\mathcal{C}}}$, see [8], there exist limit functions $f_1$ and $f_2$ of $(f_{n,1})_{n \geq 1}$ and $(f_{n,2})_{n \geq 1}$ in $\mathcal{B}_{1,\mathcal{C}}$ and the above inequality implies that the function $f = f_1 + f_2 j$ is the limit of $(f_n)_{n \geq 1}$ in $\mathcal{B}_{1,i}$. Thus, $\mathcal{B}_{1,i}$ is complete.

\[\square\]

Remark 4.18. If we work modulo constants and modulo linear terms, then we obtain that, for $i, j \in S^2$, the norms on $\mathcal{B}_{1,i}$ and $\mathcal{B}_{1,j}$ are equivalent.

Lemma 4.19. Let $i \in S^2$. Then, for $f \in \mathcal{B}_{1,i}$, we have

$$\frac{1}{16\pi} \int_0^1 \int_0^{2\pi} |\partial^2_{x_0} f(re^{i\theta})| \, d\theta \, dr \leq \|f - f(0) - z \partial_{x_0} f(0)\|_{\mathcal{B}_{1,i}} \leq \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |\partial^2_{x_0} f(re^{i\theta})| \, d\theta \, dr.$$

Proof. Let $j \in S^2$ with $i \perp j$. Then there exist $f_1$ and $f_2$ in the complex Besov space $\mathcal{B}_{1,\mathcal{C}}$ on $\mathbb{B}_i$ such that $f = f_1 + f_2 j$. Moreover, if $T_a$ denotes again the complex Möbius transformation on
Moreover, we can define a norm on $\mathcal{B}_i$ associated with $a$, then $f = \gamma_0 + \sum_{k=1}^{\infty} T_{a_k} \gamma_k$ if and only if $f_k = \gamma_{0,l} + \sum_{k=1}^{\infty} T_{a_k} \gamma_k$, $l=1,2$, with $\gamma_k = \gamma_{k,1} + \gamma_{k,2}j$. Thus, we have

$$\|f_i\|_{\mathcal{B}_i} \leq \|f\|_{\mathcal{B}_i} \leq \|f_i\|_{\mathcal{B}_i} + \|f_2\|_{\mathcal{B}_i}. \quad \text{for } l = 1, 2.$$  

Because of Theorem 8 in [8], we have

$$\frac{1}{8\pi} \int_0^1 \int_0^{2\pi} |g''(re^{i\theta})| \, d\theta \, dr \leq \|g - g'(0) - g''(0)z\|_{\mathcal{B}_i} \leq \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |g''(re^{i\theta})| \, d\theta \, dr$$

for any function $g \in \mathcal{B}_i$. Thus, on one hand, we have

$$\frac{1}{16\pi} \int_0^1 \int_0^{2\pi} |\partial_{x_0} f(re^{i\theta})| \, d\theta \, dr \leq \frac{1}{16\pi} \int_0^1 \int_0^{2\pi} |f_1''(re^{i\theta})| \, d\theta \, dr + \frac{1}{16\pi} \int_0^1 \int_0^{2\pi} |f_2''(re^{i\theta})| \, d\theta \, dr$$

$$\leq \frac{1}{2} \|f_1 - f_1(0) - f_1'(0)z\|_{\mathcal{B}_i,1} + \frac{1}{2} \|f_2 - f_2(0) - f_2'(0)z\|_{\mathcal{B}_i} \leq \|f - f(0) - z\partial_{x_0} f(0)\|_{\mathcal{B}_i,1}.$$  

On the other hand, we have

$$\|f - f(0) - z\partial_{x_0} f(0)\|_{\mathcal{B}_i,1} \leq \|f_1 - f_1(0) - f_1'(0)z\|_{\mathcal{B}_i,1} + \|f_2 - f_2(0) - f_2'(0)z\|_{\mathcal{B}_i}$$

$$\leq \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |f_1'(re^{i\theta})| \, d\theta \, dr + \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |f_2'(re^{i\theta})| \, d\theta \, dr \leq \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |\partial_{x_0} f(re^{i\theta})| \, d\theta \, dr.$$  

\[ \Box \]

**Corollary 4.20.** Let $i, j \in S^2$ and let $f \in \mathcal{B}_i$. Then

$$\|f - f(0) - z\partial_{x_0} f(0)\|_{\mathcal{B}_i,1} \leq 32 \|f - f(0) - z\partial_{x_0} f(0)\|_{\mathcal{B}_i,1}.$$  

Moreover, we can define a norm on $\mathcal{B}_i$ by

$$\|f\|_{\mathcal{B}_i} = \sup_{i \in S^2} \|f\|_{\mathcal{B}_i,1}.$$  

**Proof.** By the representation formula and the previous lemma, we obtain

$$\|f - f(0) - z\partial_{x_0} f(0)\|_{\mathcal{B}_i,1} \leq \frac{1}{\pi} \int_0^1 \int_0^{2\pi} |\partial_{x_0} f(re^{i\theta})| \, d\theta \, dr$$

$$\leq \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |(1 - ji)\partial_{x_0} f(re^{i\theta})| \, d\theta \, dr + \frac{1}{2\pi} \int_0^1 \int_0^{2\pi} |(1 + ji)\partial_{x_0} f(re^{i\theta})| \, d\theta \, dr$$

$$= \frac{2}{\pi} \int_0^1 \int_0^{2\pi} |\partial_{x_0} f(re^{i\theta})| \, d\theta \, dr \leq 32 \|f - f(0) - z\partial_{x_0} f(0)\|_{\mathcal{B}_i,1}.$$  

Moreover, because of this inequality, the function $\|f\|_{\mathcal{B}_i} = \sup_{i \in S^2} \|f\|_{\mathcal{B}_i,1}$ is well defined. It is trivial to check that it actually is a norm.  

\[ \Box \]

**Remark 4.21.** Note that the local norm on a slice $\|f\|_{\mathcal{B}_i,1}$ is invariant under Möbius transformations $T_a$ with $a \in \mathcal{B}_i$, but that $\|f\|_{\mathcal{B}_i}$ is not necessarily invariant under Möbius transformations.
In usual complex analysis, the reproducing kernel of the holomorphic weighted Bergman space for $p = 2$ with weight $(\alpha + 1)(1 - |z|^2)^\alpha$ is

$$K^C_\alpha(z, w) = \frac{1}{(1 - z\bar{w})^{2+\alpha}},$$

see for instance Corollary 4.20 in [29]. This motivates the following definition.

**Definition 4.22.** Let $\alpha > -1$. We denote by $K_\alpha(\cdot, \cdot)$ the $\alpha$-weighted slice regular Bergman kernel defined on $\mathbb{B} \times \mathbb{B}$. This kernel is defined by

$$K_\alpha(\cdot, w) = P_{i_z} \left[ \frac{1}{(1 - z\bar{w})^{2+\alpha}} \right],$$

where $i_z = \frac{z}{\|z\|}$.

**Remark 4.23.** Observe that the function $K_\alpha(\cdot, w)$ is the left slice regular extension of $K^C_\alpha$ in the $z$ variable. In alternative, it can be computed as the right slice regular extension in the variable $\bar{w}$.

As customary, by $L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H})$, we denote the set of functions $h: \mathbb{B}_i \to \mathbb{H}$ such that

$$\int_{\mathbb{B}_i} |h(w)|^p d\lambda_i(w) < \infty.$$
where we use the reproducing property of \( K^C_\alpha(\cdot, \cdot) \) on the complex Bergman space \( \mathcal{A}_{\alpha,\mathbb{C}}^p \), see Proposition 4.23 in [29]. Thus

\[
f(z) = \int_{\mathbb{B}_i} K^C_\alpha(z, w) f(w) \, dA_{\alpha,i}(w)
\]

holds for any \( z \in \mathbb{B}_i \) and so by taking the extension with respect to the variable \( z \) we have

\[
f(q) = P_i \left[ \int_{\mathbb{B}_i} K^C_\alpha(z, w) f(w) \, dA_{\alpha,i}(w) \right]
\]

\[
= \int_{\mathbb{B}_i} P_i[K^C_\alpha(z, w)] f(w) \, dA_{\alpha,i}(w) = \int_{\mathbb{B}_i} K_\alpha(q, w) f(w) \, dA_{\alpha,i}(w).
\]

\[ \square \]

**Proposition 4.26.** Let \( \alpha > -1 \) and \( p \geq 1 \) and let \( f \in S\mathcal{R}(\mathbb{B}) \). Then \( f \in \mathfrak{B}_p \) if and only if \( f \in \mathfrak{K}_{\alpha,i} L^p(\mathbb{B}_i, d\lambda, \mathbb{H}) \) for \( i \in \mathbb{S}^2 \).

**Proof.** Since \( f \in \mathfrak{B}_p \) then \( f \in \mathfrak{B}_{p,i} \) for \( i \in \mathbb{S}^2 \). Let \( j \in \mathbb{S}^2 \) be such that \( j \perp i \). Then for \( f \in S\mathcal{R}(\mathbb{B}) \) there exist holomorphic functions \( f_1, f_2 : \mathbb{B}_i \to \mathbb{C}(i) \) such that \( Q_i[f] = f_1 + f_2 j \). By Remark 4.3 we have \( f \in \mathfrak{B}_{p,i} \) if and only if \( f_1 \) and \( f_2 \) belong to the complex Besov space \( \mathfrak{B}_{p,\mathbb{C}}(\mathbb{B}_i) \), where \( B_i \) is identified with \( \mathbb{D} \subset \mathbb{C}_i \).

From Theorem 5.20 in [29], it follows that \( f_1, f_2 \in \mathfrak{B}_{p,\mathbb{C}}(\mathbb{B}_i) \) if and only if there exist functions \( g_1, g_2 \in L^p(\mathbb{B}_i, d\lambda, \mathbb{C}(i)) \) such that \( f_k = \mathfrak{K}_\alpha[g_k] \) for \( k = 1, 2 \) where \( \mathfrak{K}_\alpha \) is the corresponding complex operator, that is

\[
\mathfrak{K}_\alpha[g_i](z) = \int_{\mathbb{B}_i} K^C_\alpha(z, w) g_i(w) dA_{\alpha,i}(w) \quad \text{for } z \in \mathbb{B}_i.
\]

Note that \( \mathfrak{K}_{\alpha,i}[h] = P_i \circ \mathfrak{K}_\alpha[h] \) if \( h \) has only values in \( \mathbb{C}(i) \). So we have

\[
f = P_i[f_1 + f_2 j] = P_i[\mathfrak{K}_\alpha[g_1]] + P_i[\mathfrak{K}_\alpha[g_2]] j = \mathfrak{K}_{\alpha,i}[g_1] + \mathfrak{K}_{\alpha,i}[g_2] j = \mathfrak{K}_{\alpha,i}[g_1 + g_2 j].
\]

Thus, as \( g = g_1 + g_2 j \) is in \( L^p(\mathbb{B}_i, d\lambda, \mathbb{H}) \) if and only if the components \( g_1, g_2 \) are in \( L^p(\mathbb{B}_i, d\lambda, \mathbb{C}(i)) \), the statement is true.

\[ \square \]

Therefore we have \( \mathfrak{B}_p \cong \mathfrak{K}_{\alpha,i} L^p(\mathbb{B}_i, d\lambda, \mathbb{H}) \) as topological vector spaces, if \( \mathfrak{K}_{\alpha,i} L^p(\mathbb{B}_i, d\lambda, \mathbb{H}) \) is given with the quotient norm.

**Proposition 4.27.** Let \( p > 1 \) and \( \alpha > -1 \) and let \( f \in S\mathcal{R}(\mathbb{B}) \). Then \( f \in \mathfrak{B}_p \) if and only if

\[
\int_{\mathbb{B}_i} \int_{\mathbb{B}_i} \frac{|f(z) - f(w)|^p}{|1 - z w|^{2(2+\alpha)}} dA_{\alpha,i}(z) dA_{\alpha,i}(w) < +\infty,
\]

for \( i \in \mathbb{S}^2 \).

**Proof.** Let \( i, j \in \mathbb{S}^2 \) such that \( i \perp j \) and let \( f_1, f_2 : \mathbb{B}_i \to \mathbb{C}(i) \) be holomorphic functions such that \( Q_i[f] = f_1 + f_2 j \). Then, for \( z, w \in \mathbb{B}_i \), we have

\[
|f_i(z) - f_i(w)|^p \leq |f(z) - f(w)|^p \leq 2^{p-1} \left( |f_1(z) - f_1(w)|^p + |f_2(z) - f_2(w)|^p \right),
\]

\[ \square \]
for $l = 1, 2$. Thus, the condition (16) is satisfied if and only if
\[
\int_{B_i} \int_{B_i} \frac{|f_l(z) - f_l(w)|^p}{|1 - z\bar{w}|^{2(2+\alpha)}} dA_{\alpha,i}(z)dA_{\alpha,i}(w) < +\infty
\]
for $l = 1, 2$. But because of Theorem 5.21 in [29] this holds if and only if $f_1$ and $f_2$ belong to the complex Besov space, which is equivalent to $f \in \mathcal{B}_{p,i}$ and so also equivalent to $f \in \mathcal{B}_p$.

Proposition 4.28. Let $i \in S^2$ and let $p \geq 1$, $pt > 1$, and $\alpha > -1$. Then the integral operator $T = T_{\alpha,t,i}$

\[
Tf(z) = (1 - |z|^2)^t \int_{B_i} \frac{(1 - |w|^2)^\alpha}{(1 - z\bar{w})^{2+t+\alpha}} f(w) dA_i(w)
\]

is an embedding of $\mathcal{B}_{p,i}$ into $L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H})$.

Proof. Let $\mathcal{B}_{p,C}$ be the complex Besov space on $\mathbb{B}_i$ identified with $\mathbb{D}$. By Theorem 5.22 in [29], the operator $T$ is an embedding of $\mathcal{B}_{p,C}$ into $L^p(\mathbb{B}_i, d\lambda_i, \mathbb{C}(i))$, thus

\[T \otimes T : \mathcal{B}_{p,C}(\mathbb{B}_i)^2 \rightarrow L^p(\mathbb{B}_i, d\lambda_i, \mathbb{C}(i))^2\]

is also an embedding.

Now, let $j \in S^2$ with $i \perp j$ and let us define the operators

\[
Q_{\mathcal{B}_{p,i}} : \begin{cases} 
\mathcal{B}_{p,i} \rightarrow \mathcal{B}_{p,C}(\mathbb{B}_i)^2 \\
f \mapsto (f_1, f_2) \quad \text{with } Q_i[f] = f_1 + f_2j 
\end{cases}
\]

and

\[
J : \begin{cases} 
L^p(\mathbb{B}_i, d\lambda_i, \mathbb{C}(i))^2 \rightarrow L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H}) \\
(f_1, f_2) \mapsto f_1 + f_2j 
\end{cases}
\]

For any $\mathbb{H}$-valued function $f = f_1 + f_2j$ we have

\[|f(z)|^p \leq 2^{p-1}(|f_1(z)|^p + |f_2(z)|^p) \leq 2^p|f(z)|^p,
\]

for any $z \in \mathbb{B}_i$. Thus, one easily obtains corresponding estimates for the norms for the considered Besov and $L^p$-spaces. Therefore, the operators $Q_{\mathcal{B}_{p,i}}$ and $J$ are embeddings too.

Finally, as $Tf = Tf_1 + Tf_2j$, we have $T = J \circ T \otimes T \circ Q_{\mathcal{B}_{p,i}}$, that is the following diagram commutes

\[
\begin{array}{ccc}
\mathcal{B}_{p,i} & \xrightarrow{T} & L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H}) \\
Q_{\mathcal{B}_{p,i}} \downarrow & & \uparrow J \\
\mathcal{B}_{p,C}(\mathbb{B}_i)^2 & \xrightarrow{T \otimes T} & L^p(\mathbb{B}_i, d\lambda_i, \mathbb{C}(i))^2
\end{array}
\]

Thus, $T$ is an embedding.

We conclude this section observing that operator $T$ is important to study the duality theorems for Besov spaces that will be investigated elsewhere.

28
5 Dirichlet space

In usual complex analysis, the Dirichlet space $\mathcal{D}_\mathbb{C}$ is defined as the set of analytic functions in on the unit disc $\mathbb{D}$ such that

$$\int_{\mathbb{D}} |f'(z)|^2 d\Omega(z) < +\infty, \quad (17)$$

where $d\Omega$ is the differential of area in the complex plane, that is $d\Omega = dx dy$ if $z = x + iy$. This motivates the following definition.

**Definition 5.1.** The slice regular Dirichlet space $\mathcal{D}$ is defined as the quaternionic right linear space of slice regular functions $f$ on $\mathbb{B}$ such that

$$\sup_{i \in \mathbb{S}^2} \int_{\mathbb{B}_i} |\partial_{x_0} f(z)|^2 d\Omega_i(z) < \infty \quad (18)$$

where $d\Omega_i(z)$ is the differential of area in the plane $\mathbb{C}(i)$.

**Remark 5.2.** Let $i, j \in \mathbb{S}^2$. Then from the Representation Formula one obtains

$$|\partial_{x_0} f(x + jy)|^2 \leq 4 \left[ |\partial_{x_0} f(x + iy)|^2 + |\partial_{x_0} f(x - iy)|^2 \right]$$

and by integration one gets

$$\int_{\mathbb{B}_j} |\partial_{x_0} f(w)|^2 d\Omega_j(w) \leq 8 \int_{\mathbb{B}_i} |\partial_{x_0} f(z)|^2 d\Omega_i(z).$$

Thus, if $|18|$ is finite for some $i \in \mathbb{S}^2$ then it is finite for all $j \in \mathbb{S}^2$.

**Remark 5.3.** Let $j \in \mathbb{S}^2$ with $j \perp i$ and let $f_1, f_2 : \mathbb{B}_i \to \mathbb{C}(i)$ be holomorphic functions such that $Q_1[f] = f_1 + f_2j$. The identity

$$|\partial_{x_0} f(z)|^2 = |f_1(z)|^2 + |f_2(z)|^2$$

for $z \in \mathbb{B}_i$ implies that $f \in \mathcal{D}$ if and only if $f_1$ and $f_2$ belong to the usual complex Dirichlet space $\mathcal{D}_\mathbb{C}$.

**Proposition 5.4.** Let $f \in S\mathcal{R}(\mathbb{B})$ and let $a_n \in \mathbb{H}$ for $n \geq 0$ such that $f(q) = \sum_{n=0}^{\infty} q^n a_n$. If $f \in \mathcal{D}$ then

$$\frac{1}{\pi} \int_{\mathbb{B}_i} |\partial_{x_0} f(z)|^2 d\Omega_i(z) = \sum_{n=1}^{\infty} n|a_n|^2$$

for any $i \in \mathbb{S}^2$.

**Proof.** Let $i, j \in \mathbb{S}^2$ with $j \perp i$. So it is

1. $Q_i[f] = f_1 + f_2j$, where $f_1, f_2 \in Hol(\mathbb{B}_i)$ and
2. $a_n = a_{1,n} + a_{2,n}j$, where $a_{1,n}, a_{2,n} \in \mathbb{C}(i)$ for $n \geq 0$.

Then we have $f_k(q) = \sum_{n=0}^{\infty} q^n a_{k,n}$ for $k = 1, 2$ and from Remark 5.3 we have that $f_1$ and $f_2$ belong to the complex Dirichlet space $\mathcal{D}_\mathbb{C}$. Thus, from Section 4 of [18], we have that

$$\frac{1}{\pi} \int_{\mathbb{B}_i} |f_k(z)|^2 d\Omega_i(z) = \sum_{n=1}^{\infty} n|a_{k,n}|^2, \quad \text{for } k = 1, 2.$$
and so
\[
\frac{1}{\pi} \int_{B_i} |\partial_{x_0} f(z)|^2 d\Omega_i(z) = \frac{1}{\pi} \int_{B_i} |f'_1(z)|^2 d\Omega_i(z) + \frac{1}{\pi} \int_{B_i} |f'_2(z)|^2 d\Omega_i(z)
= \sum_{n=1}^{\infty} n|a_{1,n}|^2 + \sum_{n=1}^{\infty} n|a_{2,n}|^2
= \sum_{n=1}^{\infty} n|a_n|^2.
\]

□

**Definition 5.5.** On the slice regular Dirichlet space we define a norm by
\[
\|f\|_D = \left( |f(0)|^2 + \sup_{i \in S^2} \int_{B_i} |\partial_{x_0} f(z)| d\Omega_i(z) \right)^{\frac{1}{2}}.
\]

**Remark 5.6.** Let \(i, j \in S^2\) with \(i \bot j\) and let \(f \in D\). Furthermore, let \(f_1\) and \(f_2\) be functions in the complex Dirichlet space \(D_C\) on \(B_i\) such that \(Q_i[f] = f_1 + f_2 j\). Then the equalities
\[
|f(0)|^2 = |f_1(0)|^2 + |f_2(0)|^2
\]
and
\[
\int_{B_i} |\partial_{x_0} f(z)|^2 d\Omega = \int_{B_i} |f'_1(z)|^2 d\Omega + \int_{B_i} |f'_2(z)|^2 d\Omega
\]
(19)

imply
\[
\|f\|_D^2 = |f_1(0)|^2 + \int_{B_i} |f'_1(z)|^2 d\Omega_i(z) + |f_2(0)|^2 + \int_{B_i} |f'_2(z)|^2 d\Omega_i(z) = \|f_1\|_{D_C}^2 + \|f_2\|_{D_C}^2,
\]
where \(\| \cdot \|_{D_C}^2\) denotes the norm on the complex Dirichlet space \(D_C\) on \(B_i\).

**Proposition 5.7.** The function space \((D, \| \cdot \|_D)\) is a complete normed space.

**Proof.** The relation (19) implies that \(\| \cdot \|_D\) is actually a norm. Now let \(i, j \in S\) with \(i \bot j\). Moreover, let \((f_n)_{n \geq 0}\) be a Cauchy sequence in \(D\) and for \(n \in \mathbb{N}\) let \(f_{n,1}\) and \(f_{n,2}\) be two functions in the complex Dirichlet space \(D_C\) on \(B_i\) such that \(Q_i[f_n] = f_{n,1} + f_{n,2} j\). Because of (19), the sequences \((f_{n,1})_{n \geq 1}\) and \((f_{n,2})_{n \geq 1}\) are Cauchy sequences in \(D_C\). As showed in Section 4 of [18] the complex Dirichlet space is complete and so there exist functions \(f_1\) and \(f_2\) such that \(f_{n,1} \rightarrow f_1\) and \(f_{n,2} \rightarrow f_2\) in \(D_C\) as \(n \rightarrow \infty\). Now set \(f = P_i[f_1 + f_2 j]\). Then \(f \in D\) and
\[
\|f - f_n\|_D^2 = \|f_1 - f_{n,1}\|_{D_C}^2 - \|f_2 - f_{n,2}\|_{D_C}^2 \rightarrow 0.
\]

Thus, the slice regular Dirichlet space is complete.

□

Moreover, the slice regular Dirichlet space has the structure of a quaternionic Hilbert space.

**Definition 5.8.** Let \(i \in S^2\). For \(f, g \in D\) we define their inner product as
\[
\langle f, g \rangle_D := f(0)g(0) + \sup_{i \in S^2} \int_{B_i} \partial_{x_0} f(z) \partial_{x_0} g(z) d\Omega_i(z).
\]

30
Because of the Cauchy Schwarz inequality, this inner product is well defined.

**Proposition 5.9.** The function $\langle \cdot, \cdot \rangle_D$ is a quaternionic right linear inner product on $D$. Precisely, for all $f, g, h \in D$ and all $\lambda \in \mathbb{H}$, we have

(i) right linearity: $\langle f, g\lambda + h \rangle_D = \langle f, g \rangle_D \lambda + \langle f, h \rangle_D$

(ii) quaternionic hermiticity: $\langle g, f \rangle_D = \overline{\langle f, g \rangle_D}$

(iii) positivity: $\langle f, f \rangle_D \geq 0$ and $\langle f, f \rangle_D = 0$ if and only if $f = 0$.

**Proposition 5.10.** The space $(D, \langle \cdot, \cdot \rangle_D)$ is a quaternionic right Hilbert space.

**Proof.** The previous proposition shows that $\langle \cdot, \cdot \rangle_D$ is a quaternionic right linear inner product. Furthermore, the induced norm $\sqrt{\langle f, f \rangle_D}$ coincides with $\|f\|_D$, so the space is also complete. ■

**References**

[1] S. Adler, *Quaternionic Quantum Field Theory*, Oxford University Press, 1995.

[2] D. Alpay, *The Schur algorithm, reproducing kernel spaces and system theory*, American Mathematical Society, Providence, RI, 2001. Translated from the 1998 French original by Stephen S. Wilson, Panoramas et Synthèses.

[3] D. Alpay, F. Colombo, I. Sabadini, *Schur functions and their realizations in the slice hyperholomorphic setting*, Integral Equations Operator Theory, 72 (2012), 253–289.

[4] D. Alpay, F. Colombo, I. Sabadini, *Pontryagin de Branges Rovnyak spaces of slice hyperholomorphic functions*, J. Anal. Math., 121 (2013), 87-125.

[5] D. Alpay, F. Colombo, I. Sabadini, *Perturbation of the generator of a quaternionic evolution operator*, to appear in Analysis and Applications, (2014).

[6] D. Alpay, F. Colombo, I. Sabadini, G. Salomon, *Fock space in the slice hyperholomorphic setting*, to appear in Trends in Mathematics 2014/15.

[7] D. Alpay, F. Colombo, J. Gantner, I. Sabadini, *A new resolvent equation for the S-functional calculus*, arxiv 1310.7626v1, to appear in J. Geom. Anal..

[8] J. Arazy, S. Fisher, J. Peetre, *Möbius invariant function spaces*, J. Reine Angew. Math., 363 (1985), 110–145.

[9] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis*, Pitman Res. Notes in Math., 76, 1982.

[10] F. Colombo, I. Sabadini, *On some properties of the quaternionic functional calculus*, J. Geom. Anal., 19 (2009), 601-627.

[11] F. Colombo, I. Sabadini, *The quaternionic evolution operator*, Adv. Math., 227 (2011), 1772–1805.

[12] F. Colombo, I. Sabadini, *The Cauchy formula with s-monogenic kernel and a functional calculus for noncommuting operators*, J. Math. Anal. Appl., 373 (2011), 655–679.
[13] F. Colombo, J. O. González-Cervantes, M. E. Luna-Elizarraras, I. Sabadini, M. Shapiro, *On two approaches to the Bergman theory for slice regular functions*, Advances in hypercomplex analysis, 3954, Springer INdAM Ser., 1, Springer, Milan, 2013.

[14] F. Colombo, J. O. González-Cervantes, I. Sabadini, *On slice biregular functions and isomorphisms of Bergman spaces*, Complex Var. Elliptic Equ., 57 (2012), 825–839.

[15] F. Colombo, J. O. González-Cervantes, I. Sabadini, *The C-property for slice regular functions and applications to the Bergman space*, Compl. Var. Ell. Equa., 58 (2013), 1355–1372.

[16] F. Colombo, I. Sabadini, D.C. Struppa, *Noncommutative Functional Calculus, Theory and Applications of Slice Hyperholomorphic Functions*, Progress in Mathematics V. 289, Birkhäuser Basel 2011.

[17] F. Colombo, I. Sabadini, D.C. Struppa, *A new functional calculus for noncommuting operators*, J. Funct. Anal., 254 (2008), 2255–2274.

[18] N. Danikas, *Some Banach spaces of analytic functions*, In: Function spaces and complex analysis, Joensuu 1997 (Ilomantsi). (Eds.: R. Aulaskari and I. Laine), Univ. Joensuu Dept. Math. Rep. Ser., 2, Univ. Joensuu, Joensuu, 1999, pp. 935.

[19] R. Fueter, *Analytische Funktionen einer Quaternionenvariable*, Comm. Math. Helv., 4 (1932), 9–20.

[20] R. Fueter, *Die Funktionentheorie der Differentialgleichungen Δu = 0 und ΔΔu = 0 mit vier reellen Variablen*, Comm. Math. Helv., 7 (1934), 307–330.

[21] G. Gentili, D.C. Struppa, *A new approach to Cullen-regular functions of a quaternionic variable*, C.R. Acad. Sci. Paris, 342 (2006), 741–744.

[22] R. Ghiloni, V. Moretti, A. Perotti, *Continuous slice functional calculus in the quaternionic Hilbert spaces*, Rev. Math. Phys., 25 (2013), 1350006, 83 pp.

[23] R. Ghiloni, V. Recupero, *Semigroups over real alternative *-algebras: generation theorems and spherical sectorial operators*, Preprint 2013, to appear in Trans. Amer. Math. Soc..

[24] K. Gürlebeck, K. Habetha, W. Sprößig, *Holomorphic Functions in the Plane and n-dimensional space*, Birkhäuser, Basel, 2008.

[25] G. Sarfatti, *Elements of function theory in the unit ball of quaternions*, PhD thesis, Università di Firenze, 2013.

[26] C. Stoppato, *Regular Möbius transformations of the space of quaternions*, Ann. Global Anal. Geom., 39 (2011), 387–401.

[27] N. Vasilevski, *Commutative algebras of Toeplitz operators on the Bergman space*, Operator Theory: Advances and Applications, 185. Birkhäuser Verlag, Basel, 2008.

[28] I. Porteus, *Geometric Topology*, Cambridge University 1981.

[29] K. Zhu, *Operator theory in function spaces*, Second edition. Mathematical Surveys and Monographs, 138. American Mathematical Society, Providence, RI, 2007.