Mass and CKM Matrices of Quarks and Leptons, the Leptonic CP-phase in Neutrino Oscillations

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Abstract

A general approach for construction of quark and lepton mass matrices is formulated. The hierarchy of quarks and charged leptons ("electrons") is large, it leads using the experimental values of mixing angles to the hierarchical mass matrix slightly deviating from one's suggested earlier by Stech and including naturally the CP-phase.

The same method based on the rotation of generation numbers in the diagonal mass matrix is used in the electron-neutrino sector of theory, where neutrino mass matrix is determined by the Majorano see-saw approach. The hierarchy of neutrino masses, much smaller than for quarks, was used including all existing (even preliminary) experimental data on neutrinos mixing.

The leptonic mass matrix found in this way includes not known value of the leptonic CP-phase. It leads to a large $\nu_\mu\nu_\tau$ oscillations and suppresses the $\nu_e\nu_\tau$ and also $\nu_e\nu_\mu$ oscillations. The explicit expressions for the probabilities of neutrino oscillation were obtained in order to specify the role of leptonic CP-phase. The value of time reversal effect (proportional to $\sin \delta'$) was found to be small $\sim 1\%$. However, a dependence of the values of $\nu_e\nu_\mu, \nu_e\nu_\tau$ transition probabilities, averaged over oscillations, on the leptonic CP-phase has found to be not small - of order of tens percent.

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1. Introduction

Serious efforts have been invested recently in the natural understanding of experimental results on neutrino oscillations. They have shown that neutrinos of three generations have, perhaps, non vanishing small masses. The heaviest of them, the neutrino of the third generation, seems to have a mass of the order of $\sim (1/20)\text{eV}$ and, as the Super Kamiokande data on atmospheric neutrinos show, has the maximal possible mixing with the neutrino of the second generation, and may be, also not too small mixing with that of the first generation. This was not expected a priori, since all similar mixing angles of quarks are small.

It is the challenge of modern particle physics to include naturally these results into the framework of Grand Unification Theory together with the data on quark masses and mixing angles. For the quarks these angles are small and are known already [1].

We begin this paper by reminding the well-known picture of masses, CP-phase and mixings for the quark sector of a theory. A general method will be developed which allows one to construct consistently the $3 \times 3$ mass-matrix and the CKM mixing matrix for quarks. The same general approach will be used later for the electron-neutrino sector of a theory.

Let us consider, as a useful introduction to a consistent theory of quark and lepton masses and mixings, a simple phenomenological approach suggested by B. Stech [2]. He has noticed the following quark and charged lepton (“electrons”) mass hierarchies:

$$m_t : m_e : m_u \simeq 1 : \sigma^2 : \sigma^4, \quad m_b : m_s : m_d \simeq 1 : \frac{1}{2} \sigma : 8 \sigma^3, \quad m_\tau : m_\mu : m_e \simeq 1 : \sigma : \frac{3}{2} \sigma^3,$$  \hspace{1cm} (1)

with a very small $\sigma^2 \simeq 1/300$, $\sigma \simeq 0.058$. He has also introduced the following mass matrices which reproduced approximately the masses of all the quarks as well as theirs mixing angles:

$$\hat{M}_u = \begin{pmatrix}
0 & \frac{1}{\sqrt{2}} \sigma^3 \eta & \sigma^2 \eta \\
\frac{1}{\sqrt{2}} \sigma^3 \eta^* & -\frac{1}{2} \sigma^2 & \frac{1}{\sqrt{2}} \sigma \\
\sigma^2 \eta^* & \frac{1}{\sqrt{2}} \sigma & 1
\end{pmatrix} m_t, \quad \hat{M}_d = \begin{pmatrix}
0 & a_d \sigma^3 & 0 \\
0 & -\frac{\sigma^2}{2} & 0 \\
0 & 0 & \sigma
\end{pmatrix} \frac{m_b}{\sigma},$$  \hspace{1cm} (2)

$$\hat{M}_e = \begin{pmatrix}
0 & a_e \sigma^3 & 0 \\
a_e \sigma^3 & -\sigma^2 & 0 \\
0 & 0 & \sigma
\end{pmatrix} \frac{m_\tau}{\sigma},$$  \hspace{1cm} (3)

Here $\eta = e^{i\delta}$ represents the CP violating phase $\delta$ in the quark sector, while the values of the constants $a_d \simeq 2$, $a_e \simeq \sqrt{\frac{3}{2}}$ correspond to the best fit of all masses of quarks and electrons (their central values, see below).

The diagonalization of the matrices (2) and (3) by means of an unitary matrix $\hat{U}_a$:

$$\hat{M}_a^{\text{diag}} = \hat{U}_a \hat{M}_a \hat{U}_a^+ \quad a = u, d, e,$$  \hspace{1cm} (4)
(where $\hat{U}_a\hat{U}_a^+ = \hat{U}_a^+\hat{U}_a = 1$), reproduces simultaneously both the experimental (central) values of running masses of quarks and electrons and all quark mixing angles (their sines):

$$s_{12} = \sin \vartheta_{12}^q, \quad s_{23} = \sin \vartheta_{23}^q, \quad s_{13} = \sin \vartheta_{13}^q$$

in the CKM matrix:

$$\hat{V}_{CKM} = \hat{U}_u^+\hat{U}_d = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta'} \\
    -s_{12}c_{13} - c_{12}s_{23}s_{13}e^{i\delta'} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta'} & s_{23}c_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta'} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta'} & c_{23}c_{13}
\end{pmatrix}. \quad (5)$$

Here, according to experimental data \cite{1,6}, one has:

$$s_{12} \approx \sin \vartheta_{12}^q \approx 0.221 \pm 0.004, \quad s_{23} = \sin \vartheta_{23}^q = 0.039 \pm 0.003,$$

$$s_{13} = \sin \vartheta_{13}^q \approx s_{12}s_{23}|R| = 0.0032 \pm 0.0014, \quad |R| = s_{13}/s_{12}s_{23} \approx 0.38 \pm 0.19 \quad (6)$$

and $\delta = \pi/2 \pm \pi/4$.

The factors $1/\sqrt{2}, 1/2, 2, \sqrt{3}/2$, etc. in Eq.(2),(3) were adjusted so as to reproduce all these mixing angles and all the masses of quarks and leptons. As has been mentioned, for quarks the mixing angles are small, i.e. $s_{13} \ll s_{23} \ll s_{12} \ll 1$ and therefore the cosines $c_{13}, c_{23}, c_{12}$ of them can be put approximately equal to unity.

The major ideas of Stech’s approach were: 1) to consider the running masses of all quarks at the same scale (instead of the pole ones) and 2) to obtain these masses and mixing angles in terms of few parameters. However, the matrices (2) and (3) have been written phenomenologically, just by hand.

Below we develop an approach to quark and lepton masses and mixings through the following steps:

a) A general method is suggested which allows one to construct the quark mass matrices consistently (without using Eq.(2)) in terms of their masses and mixing angles (6). The resulting mass matrices coincide with those given by Eq.(2). They are diagonalized in an analytical form of decomposition in powers of small parameter $\sigma$ and the quark CKM matrix is reproduced analytically.

b) The same approach is applied to the reconstruction the neutrino mass matrices $\hat{M}_\nu$. This is performed using the Eq.(4) and the unitary matrix $\hat{U}_\nu$. Preliminary results on masses of neutrinos and their mixing angles, extracted from neutrino oscillation experiments have been used.

c) The leptonic CKM matrix $\hat{V}_{CKM}^l = \hat{U}_\nu^+\hat{U}_e$ containing the leptonic CP-phase $\delta'$ is introduced. It leads to $\nu_\mu\nu_\tau$ oscillation of the type observed at Super-Kamiokande and also to some suppression of $\nu_e\nu_\mu$ oscillation.

\footnote{The upper index $q$ is used to emphasize that the angles $\vartheta_{ij} = \vartheta_{ij}^q$, or $s_{ij} = \sin \vartheta_{ij}^q$ determine just by the quarks and not by the leptonic mixings. The corresponding leptonic angles and their sines are denoted below by $\vartheta_{ij}^l$ and by $s_{ij} = \sin \vartheta_{ij}^l$ without any upper index (see sect. 3 and 4). The leptonic CP-phase is denoted (in sect.3, 4) by $\delta'$.}
d) The expressions for the three type neutrino oscillations are specified and some realistic numerical examples of oscillation dependence on the leptonic CP-phase are presented.

Finishing this section let us note that the electron and quark masses emerge from the usual Yukawa interaction:

\[
L_{\text{Yuk}}' = \varphi_2 \bar{u}_i \hat{h}_{ij} q_j + \varphi_1 \bar{d}_i \hat{h}_{ij} q_j + \varphi_1 \bar{e}_i \hat{h}_{ij} l_j,
\]

where our mass matrices (2) and (3) are:

\[
(\hat{M}_u) = \langle \varphi_2 \rangle \hat{h}_{u}, \quad (\hat{M}_d) = \langle \varphi_1 \rangle \hat{h}_{d}, \quad (\hat{M}_e) = \langle \varphi_1 \rangle \hat{h}_{e},
\]

and \(\langle \varphi_1 \rangle = v \cos \beta, \langle \varphi_2 \rangle = v \sin \beta\) are V.E.V of SUSY two neutral CP-even Higgs fields \((v = 174\text{GeV}, \text{and } \tan \beta)\) are two well known SUSY parameters) and \(\hat{h}_{ij} \sim \hat{h}_{ij} \sim h_{ij} \sim 1\) are the Yukawa coupling constants.

The scale of neutrino masses \((10^{11} - 10^{12} \text{ times smaller than the electron and quark masses})\) can be a result of the see–saw approach [3–5]. It leads after the integrating out of the corresponding heavy states (e.g. the right handed Majorano neutrinos with heavy masses of an order of \(M \sim 10^{14} - 10^{15}\text{GeV}\)) to the appearance of higher order effective Majorano mass operator (see [3],[5]):

\[
(\nu_i \hat{M}_\nu)^{ij} \nu_j = (\varphi_2^2/M)(\nu_i h^{ij} \nu_j),
\]

where \(h^{ij} \sim 1\) are the neutrino coupling constants and \(\hat{M}_\nu = (\varphi_2^2/M)\hat{h}\) is the neutrino mass matrix.

2. Quarks mass matrices and the analytical form of the CKM matrix.

The running of u- and d-quark masses calculated [7] in the first (dashed curves) and in the fourth (solid curves) order of QCD perturbation theory are shown [8] in Fig. 1. The quark running masses \(m_i(\mu)\) are related in \(\overline{\text{MS}}\) renormalization scheme at the scale \(\mu = M_t\) to their pole masses \(M_i\) by the well known relation:

\[
m_i(M_t) = M_i/[1 + \frac{4\alpha_s(M_t)}{3\pi} + K(\frac{\alpha_s(M_t)}{\pi})^2],
\]

where \(K=11,2\). The scale \(\mu = M_t = 174.4\text{ GeV}\) is the most natural for the SUSY Standard Model. The running quark masses at this scale, (see in Fig. 1) have the following values (in GeV’s):

\[
\begin{align*}
m_u(M_t) & = (0.21 \pm 0.1) \cdot 10^{-2}, \\
m_d(M_t) & = (0.42 \pm 0.21) \cdot 10^{-2}, \\
m_e & = 0.511 \cdot 10^{-3}, \\
m_c(M_t) & = 0.59 \pm 0.07, \\
m_s(M_t) & = 0.082 \pm 0.041, \\
m_\mu & = 105.66 \cdot 10^{-3}, \\
m_t(M_t) & = 163 \pm 4, \\
m_b(M_t) & = 2.80 \pm 0.40, \\
m_\tau & = 1.777 \pm 0.0003.
\end{align*}
\]

These values differ strongly from the ones used in Ref.[2] which are determined at a very small scale \(\mu = 1\text{ GeV}\). However, the Stech’s relation between them still holds.
Note that the matrices $\hat{M}_d$ and $\hat{M}_e$ in Eqs.(2),(3) have the block structure and their diagonalization is trivial. In fact, for any $2 \times 2$ matrix $\hat{m} = \begin{pmatrix} a & \rho \\ \rho & b \end{pmatrix}$, which $\hat{M}_d$ and $\hat{M}_e$ contain in their left-upper part, one has:
\[
\hat{m}_{\text{diag}} = \hat{u} \hat{m} \hat{u}^+ = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}, \quad \hat{u} = \begin{pmatrix} c & s \\ -s & c \end{pmatrix}, \quad \mu_{1,2} = \frac{1}{2}(a + b \pm \sqrt{(a - b)^2 + 4\rho^2}). \tag{8}
\]

Here $t_{2\vartheta} = \tan 2\vartheta = 2\rho/(a - b)$ and $s = \sin \vartheta = \frac{1}{\sqrt{2}}(1 - (1 + t_{2\vartheta}^2)^{-\frac{1}{2}})$, $c = \cos \vartheta = \frac{1}{\sqrt{2}}(1 + (1 + t_{2\vartheta}^2)^{-\frac{1}{2}})$, and $\vartheta$ is the mixing angle. In Eqs.(2),(3) for the matrices $\hat{M}_d$ and $\hat{M}_e$ one has $a = 0$ and the mixing angles $\vartheta_d, \vartheta_e$ are small since $t_{2\vartheta_d} = \frac{2\sigma^3}{\sigma^2/2} = 8\sigma < 1$ and $t_{2\vartheta_e} = 2\sqrt{\frac{3\sigma^3}{2\sigma^2}} = \sqrt{6}\sigma$ is even smaller. Due to the block structure of $\hat{M}_d$ and $\hat{M}_e$ the unitary matrices $\hat{U}_d$ and $\hat{U}_e$ also have the following block structure:
\[
\hat{U}_d = \begin{pmatrix} c_d & s_d & 0 \\ -s_d & c_d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{U}_e = \begin{pmatrix} c_e & s_e & 0 \\ -s_e & c_e & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{9}
\]

where $s_d = \frac{1}{\sqrt{2}}(1 - (1 + (8\sigma^2)^{-\frac{1}{2}})^{-\frac{1}{2}}) \approx 4\sigma(1 - 24\sigma^2) \approx 0.214$, $s_e = \frac{1}{\sqrt{2}}(1 - (1 + 6\sigma^2)^{-\frac{1}{2}})^{-\frac{1}{2}} \approx \sqrt{3/2}\sigma(1 - (3/2\sigma^2)^2) \approx 0.0705$ up to the terms of the order of $\sigma^4$.

Let us note that Stech’s matrices $\hat{M}_d, \hat{M}_e$ in Eqs.(2),(3) can be reconstructed using their diagonal form (i.e. the physical masses of d-quarks and electrons):
\[
\hat{M}_d^{\text{diag}} = \begin{pmatrix} m_d & -m_s \\ -m_s & m_b \end{pmatrix}, \quad \hat{M}_e^{\text{diag}} = \begin{pmatrix} m_e & -m_\mu \\ -m_\mu & m_\tau \end{pmatrix}, \tag{10}
\]

by means of Eq.(4), which states:
\[
\hat{M}_d = \hat{U}_d^+ \hat{M}_d^{\text{diag}} \hat{U}_d, \quad \hat{M}_e = \hat{U}_e^+ \hat{M}_e^{\text{diag}} \hat{U}_e. \tag{11}
\]

The matrices $\hat{U}_d = \hat{O}_{12}^d$ (or $\hat{U}_e = \hat{O}_{12}^e$) in Eq.(9) can be considered as rotating the 12 generations of d-quarks (or electrons).

A bit more complicated (but much more instructive) is the diagonalization of the matrix $\hat{M}_u$. Similarly to $\hat{M}_d$ and $\hat{M}_e$ this matrix can be represented as:
\[
\hat{M}_u = \hat{U}_u^+ \hat{M}_u^{\text{diag}} \hat{U}_u, \quad \text{where } \hat{U}_u^+ = \hat{O}_{23}^+ \hat{O}_{13}^+ (\delta) (\hat{O}_{12}^u)^+ \tag{12}
\]

Here the matrices:
\[
(\hat{O}_{12}^u)^+ = \begin{pmatrix} c_u & -s_u & 0 \\ s_u & c_u & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \hat{O}_{13}^+ (\delta) = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}, \quad \hat{O}_{23}^+ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \tag{13}
\]
rotates 12, 13 and 23 generations, respectively. The 13 rotation includes naturally the complex phase $\delta$ which violates the CP-parity conservation of a theory. It can not be removed by a trivial phase transformation of the $u$-quark or $t$-quark fields. However, the value of $s_u$ turns out to be very small ($s_u \sim \sigma^5 \ll 1$) and one can really put $\hat{O}^t_{12} \simeq \hat{1}$. The quark CKM matrix is:

$$\hat{V}^q_{CKM} = \hat{U}^-_u \hat{U}^-_d = \hat{O}^+_{23} \hat{O}^-_{13}(\delta) \hat{O}^+_{12}, \text{ where } \hat{O}^+_{12} = (\hat{O}^-_{12})^* \hat{O}^d_{12}. \quad (14)$$

Here in general $(\hat{O}^v_{12})^+ = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$, with $s_{12} = s^d_{12}c_u - c^d_{12}s_u = \sin(\vartheta_d - \vartheta_{12}^u) \simeq \sin \vartheta_d$ since $s_u \sim \sigma^5$ (see below) is negligibly small.

Thus for the given upper quark masses $\hat{M}^{\text{diag}}_u = \begin{pmatrix} m_u & m_c & m_t \end{pmatrix}$ and the given quark mixing angles (from the quark CKM mixing matrix $\hat{V}^q_{CKM}$, see below) the form of $\hat{U}^-_u$ is fixed:

$$\hat{U}^-_u = \hat{O}^-_{23} \hat{O}^-_{13}(\delta) = \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ s_{13}s_{23}e^{i\delta} & c_{23} & c_{13}s_{23} \\ -s_{13}c_{23}e^{i\delta} & -s_{23} & c_{13}c_{23} \end{pmatrix} \quad (15)$$

The same matrix $\hat{U}^-_u$ can be obtained in the form of decomposition in powers of $\sigma$ (up to $\sigma^4$ terms) by a direct calculation:

$$\hat{U}^-_u = \begin{pmatrix} 1 - \frac{\sigma^4}{2} & 0 & \sigma^2(1 - \frac{\sigma^2}{4})e^{-i\delta} \\ -\frac{\sigma^3}{\sqrt{2}}e^{i\delta} & 1 - \frac{\sigma^2}{4} & \frac{\sigma}{\sqrt{2}}(1 - \frac{5}{4}\sigma^2) \\ -\sigma(1 - \frac{\sigma^2}{2})e^{i\delta} & -\frac{\sigma}{\sqrt{2}}(1 - \frac{5}{4}\sigma^2) & 1 - \frac{\sigma^2}{4} \end{pmatrix} \quad (16)$$

Multiplying $\hat{U}^-_u$ in this form by $\hat{U}^-_d$ one obtains with the same accuracy the following CKM quark matrix $\hat{V}^q_{CKM} = \hat{U}^-_u \hat{U}^-_d$:

$$\hat{V}^q_{CKM} = \begin{pmatrix} (1 - \frac{\sigma^4}{2})c_{12} & (1 - \frac{\sigma^4}{2})s_{12} & \sigma^2(1 - \frac{\sigma^2}{4})e^{-i\delta} \\ -(1 - \frac{\sigma^2}{4})s_{12} - \frac{\sigma^3}{\sqrt{2}}c_{12}e^{i\delta} & (1 - \frac{\sigma^2}{4})c_{12} - \frac{\sigma^3}{\sqrt{2}}s_{12}e^{i\delta} & \frac{\sigma}{\sqrt{2}}(1 - \frac{5}{4}\sigma^2) \\ \frac{\sigma}{\sqrt{2}}(1 - \frac{5}{4}\sigma^2)s_{12} - \sigma^2c_{12}e^{i\delta} & -\frac{\sigma}{\sqrt{2}}(1 - \frac{5}{4}\sigma^2)c_{12} - \sigma^2s_{12}e^{i\delta} & 1 - \frac{\sigma^2}{4} \end{pmatrix} \quad (17)$$

Comparing the CKM matrix (5) with Eq.(17) one finds:

$$s_{12} = s_d \simeq 4\sigma(1 - 24\sigma^2) \approx 0.215, \quad s_{23} = \frac{\sigma}{\sqrt{2}}(1 - \frac{5}{4}\sigma^2) \approx 0.0408,$$

$$s_{13} = s_{12}s_{23}|R| = \sigma^2(1 - \sigma^2/4) \approx 3.36 \cdot 10^{-3} \quad (18)$$

implying that $|R| = \frac{s_{13}}{s_{12}s_{23}} = \frac{1}{\sqrt{8(1 - 25\sigma^2)}} \approx 0.38$
Calculating $\hat{M}_u$ by means of Eq.(12) one first obtains $\hat{M}_u' = \hat{O}_{13}^+ M_u^{diag} \hat{O}_{13}$ and then finds $\hat{M}_u = \hat{O}_{23}^+ \hat{M}_u' \hat{O}_{23}$ in the form:

$$
\hat{M}_u = \begin{pmatrix}
0 & s_{13}\sigma_{23}e^{-i\delta} & s_{13}\sigma_{23}e^{-i\delta} \\
0 & s_{23}^{2} + \frac{m_2}{m_3} & s_{23} \\
s_{13}e^{i\delta} & s_{13}e^{i\delta} & 1
\end{pmatrix} m_3 \tag{19}
$$

up to the terms of an order of $\sigma^4$.

Eq.(19) really reproduces the Stech’s matrix $\hat{M}_u$ in Eq.(2) only for negative sign of $m_2 = -|m_2|$; for positive $m_2 = |m_2|$ the ratio $\mu_{22} = (\hat{M}_u)_{22}/m_t$ is $\frac{3}{2}\sigma^2$ instead of $-\frac{\sigma^2}{2}$ in Eq.(2). Considering higher order $\sigma^2$ corrections to the matrix (19) one finds that $\mu_{11} = (\hat{M}_u)_{11}/m_t$ turns out to be $2\sigma^4$ for positive $m_u = |m_u|$ and is much smaller, $-\frac{\sigma^6}{2}$, for negative lightest eigenvalue when $m_u = -|m_u|$.

Therefore, literally the Stech’s matrix $\hat{M}_u$ is reproduced in the form (2) only for negative $m_1$ and $m_2$, e.g. at: $\hat{M}_u^{diag} = \begin{pmatrix} -|m_u| & |m_c| & m_t \end{pmatrix}$

Thus, the u-quark mass matrix $\hat{M}_u$ can be reproduced by the former of Eqs.(12) using the $\hat{U}_u$ and $\hat{U}_u^+$ matrices in the form of the latter of Eqs.(12) (or by Eq.(15) if $(\hat{O}_u^+)^* \approx 1$). Actually this method will be very useful later for the restoration of the neutrino mass matrix.

All these results are in a good agreement with the experimental data presented above. Also the diagonalisation (4) of $\hat{M}_u$ and $\hat{M}_d$ with the help of $\hat{U}_u$ and $\hat{U}_d$ matrices leads to the following reasonable quark masses (in GeV’s, as in Eq.(1)):

$$
\begin{align*}
m_u &= \sigma^4(1 - \sigma^4)m_t(M_t) \approx 2.00 \cdot 10^{-3} \\
m_c &= \sigma(1 - \sigma^2/2)m_t(M_t) \approx 0.56 \\
m_t &= (1 - \sigma^2/2)m_t(M_t) \approx m_t(M_t) = 163
\end{align*}
$$

$$
\begin{align*}
m_d &= \frac{\sigma}{4}(\sqrt{1 + (8\sigma)^2} - 1)m_b = 0.416 \cdot 10^{-3} \\
m_s &= \frac{\sigma}{4}(\sqrt{1 + (8\sigma)^2} + 1)m_b = 0.085 \\
m_b &= m_b(M_b) = 2.83
\end{align*}
$$

These results correspond fairly well, as Stech has remarked [2], to the experimental data (7).

The minus sign of some eigenvalues in Eq.(10) and in $\hat{M}_u^{diag}$ can be easily removed by redefinition of the corresponding quark field: $q_k \rightarrow \gamma_5 q_k$.

3. Neutrino mass matrix, the leptonic CKM matrix and CP-phase

To proceed further let us introduce the neutrino mass matrix $\hat{M}_\nu$ with a structure similar to that of $\hat{M}_u$. The Super Kamiokande data [9] suggest a large $\nu_\mu - \nu_\tau$ neutrino mixing i.e. large $t_{23} = \text{tg} 2\theta_{23} \gg 1$, or $\sin^2 2\theta_{23} = (1 + t_{23}^{-2})^{-1} \simeq 1$ (i.e. $s_{23} \simeq c_{23} \simeq \frac{1}{\sqrt{2}} \simeq 0.71$) and not too small value of $\Delta m^2_{32} = m^2_3 - m^2_2 \simeq (0.59 \pm 0.20)^2 10^{-2}$eV. As $m_3 \gg m_2$ (see below) one has:

$$
m_3 \simeq \sqrt{\Delta m^2_{32}} \simeq (0.059 \pm 0.020) \text{eV} \tag{21}
$$
Simultaneously the atmosphere and solar neutrino data [10 - 12] (see the discussion in Refs.[13,14]) show a large suppression of $\nu_e\nu_\mu$ and also $\nu_e\nu_\tau$ oscillations which have not yet been observed. This can be a result of small mixing angles $s_{12} \lesssim s_{13} \ll s_{23} \sim 1/\sqrt{2}$ (see Ref. [10] and also [12 - 14]):

$$s_{12} \simeq 0.035 \pm 0.020, \quad s_{13} \simeq 0.25 \pm 0.10, \quad s_{23} = 0.7 \simeq 1/\sqrt{2}$$ (22)

and some (obviously not too large) hierarchy of three neutrino masses $m_1 \ll m_2 \ll m_3$ of a type considered above for the quarks and elections:

$$m_3 : m_2 : m_1 = 1 : \sigma^2_\nu : \sigma^4_\nu.$$ (23)

Here $\sigma^2_\nu$ is an unknown parameter which we can choose to be equal to $\sigma = 0.058$ to avoid the introduction of additional new parameter $\sigma_\nu = \sqrt{0.058} \simeq 0.24$. This gives $m_2 \simeq 0.34 \cdot 10^{-2}$eV and together with Eq.(22) seems to be in approximate agreement with the atmospheric neutrino data [12].

Essentially the other possibility has been suggested in a recent paper [15], where the neutrinos $\nu_1, \nu_2$ and $\nu_3$ have been considered with almost equal masses $m_3 \simeq m_2 \simeq m_1 \simeq 3$eV but a large hierarchy has been introduced in their mixing angles

$$s_{13} \simeq 0 \ll s_{12} \simeq s_{23} \simeq 1/\sqrt{2}$$ (24)

This situation is not considered below as it seems more natural that neutrino have small mass hierarchy of the type (1) like the u-quarks, but with much smaller power of hierarchy: $\sigma_\nu \simeq 0.24 \gg \sigma$. It is very suitable to describe the situation in neutrino physics since the condition (23) $m_2 < m_3$ (and $m_1 < m_2$) together with Eqs.(22) suppress strongly the non observed (at least for a moment) $\nu_e\nu_\mu$ oscillations from electron neutrino sources on the Earth.

Therefore, let us take approximately the central values of the data given in Eqs.(21)–(23) as a basis of our approach (see e.g. Ref.[5]) putting also $\sigma_\nu \simeq \sqrt{0.058}$ in Eq.(23) and taking the following values for the neutrino masses:

$$M^{\text{diag}}_\nu = \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} 0.019 \\ 0.34 \\ 5.90 \end{pmatrix} 10^{-2} \text{eV}$$ (25)

where $m_2 \simeq \sigma^2_\nu m_3$, $m_1 \simeq \sigma^2_\nu m_2$ and the central value of $m_3$ has been taken from Eq.(21).

The main point of this paper is that approach of the type of Eqs.(12)–(16) together with the electron matrix $\hat{M}_e$ (determined by Eq.(3)) allows one to construct in general analytical form the neutrino mass matrix $\hat{M}_\nu$ and also the leptonic CKM matrix. The matrix $\hat{M}_\nu$ so obtained will have all desired properties reproducing naturally Eqs.(21) - (23) and will naturally include the CP-phase.
To this end let us remind that \( \hat{M}_e \) in Eq.(3) has the form:

\[
\hat{M}_e = \hat{U}_e \hat{M}^{diag}_e \hat{U}_e^\dagger,
\]  

(26)

where \( \hat{U}_e = \hat{O}^e_{12} \) has been determined in Eq.(9) with \( s_\nu \simeq 0.0705 \). Here \( \hat{M}^{diag}_e = \begin{pmatrix} m_e & m_\mu & m_\tau \end{pmatrix} \) has the form:

\[
\hat{M}^{diag}_e = \begin{pmatrix} \hat{M}_e & m_\mu & m_\tau \end{pmatrix} = \begin{pmatrix} 0.5175 & 103.6 & 1777 \end{pmatrix} \text{MeV}
\]

represents the rotation of 13 and 23 generations of \( \hat{\nu} \) and \( \nu \) with very small \( \beta \). where \( \hat{M}^{diag}_e \) is defined similarly to \( \hat{O}^e_{12} \) with the substitution \( s_\nu = \sin \vartheta^e_{12} \) for \( s_\nu \) in \( \hat{O}^e_{12} \) defined above (the value of \( \vartheta^e_{12} \) will be determined later on). The matrices:

\[
\hat{O}^{l+}_{13}(\delta') = \begin{pmatrix} c_{13} & 0 & s_{13} e^{-i\delta'} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta'} & 0 & c_{13} \end{pmatrix}, \quad \hat{O}^{l+}_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}
\]

(28)

represent the rotation of 13 and 23 generations of \( \hat{M}^{diag}_\nu \), respectively, and \( \delta' \) is the leptonic CP-phase which has appeared here naturally (clearly \( s_{ij}, c_{ij} \) from here and later on means the sines and cosines of the leptonic and not quark mixing angles).

To calculate \( \hat{M}_\nu \) in Eq.(27) in explicit form we note that according to Eq.(8) one has

\[
\hat{O}^{\nu^+}_{12} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \hat{O}^{\nu^+}_{12} = \begin{pmatrix} a & \rho & 0 \\ \rho & b & 0 \\ 0 & 0 & m_3 \end{pmatrix}
\]

(29)

with \( a = s_\nu^2 m_2 + c_\nu^2 m_1 \simeq m_1, \quad b = c_\nu^2 m_2 + s_\nu^2 m_1 \simeq m_2, \quad \rho = -c_\nu s_\nu (m_2 - m_1) \) and \( |\rho| \ll m_2 \) is small since \( s_\nu \) in Eq.(22) is very small (actually \( s_\nu \simeq s_{12} - s_e \simeq 0.036 \) see below). Then it is

\[\begin{pmatrix} 0 \sqrt{\frac{3}{2} \sigma - \beta \sigma^2} & \sqrt{\frac{3}{2} \sigma^3} \\ 0 & 0 & \sigma \end{pmatrix} \frac{m_e}{\beta}, \]

where \( \beta = 1 + \lambda_1 \) with very small \( \lambda_1 = 0.0202308 \) and \( \lambda = 0.0106891 \) adjusted to reproduce exactly the well known masses of all three electrons: \( \hat{M}^{diag}_e = \hat{U}_e \hat{M}_e \hat{U}_e^\dagger = \begin{pmatrix} 0.510999 \\ 105.6584 \\ 1777 \end{pmatrix} \text{MeV}. \) This modification will change negligibly the leptonic CKM matrix, leading to \( s_e \simeq 0.0689 \) and is not important at all as the neutrino basic parameters are determined very roughly in Eqs.(21)–(23) (see also the Conclusion).
easy to calculate the matrix:

\[
\hat{M}' = \hat{O}_{13}^+ (\delta') \begin{pmatrix} a & \rho & 0 \\ \rho & b & 0 \\ 0 & 0 & m_3 \end{pmatrix} \hat{O}_{13} (\delta')
\]

and further to find the following result for the neutrino mass matrix \(\hat{M}_\nu = \hat{O}_{23}^+ \hat{M}' \hat{O}_{23}^\dagger\):

\[
\hat{M}_\nu = \begin{pmatrix}
ac_{13}^2 + m_3 s_{13}^2 & c_{13}s_{13}s_{23}(m_3 - a)e^{-i\delta'} & c_{23}c_{13}s_{13}(m_3 - a)e^{-i\delta'} \\
+\rho c_{13}c_{23} & s_{23}^2(m_3c_{13}^2 + a^2s_{13}^2) & c_{23}s_{23}(m_3c_{13}^2 - b + as_{13}^2) \\
-\rho s_{33}c_{13} & c_{23}s_{23}(m_3c_{13}^2 - b + as_{13}^2) & +\Delta_{33}
\end{pmatrix}
\]

(30)

The values of \(\Delta_{22} = -\rho s_{13} \sin 2\varphi \sin \delta'\), \(\Delta_{23} = -\rho s_{13}(e^{-i\delta'} - 2s_{23}^2 \cos \delta') = \Delta_{32}^+\) and of \(\Delta_{33} = -\Delta_{22}\) are small and can be neglected. Also since \(a \ll b \ll m_3\), \(s_\nu \simeq s_{12}\) (see below) and \(s_{12}^2 \ll s_{13}^2 \ll 1\) are very small, one can disregard in the matrix \(\hat{M}_\nu\) all the terms containing \(a, b\) and put \(c_{12} \simeq c_{13} \simeq 1\). The value of \(|\rho| \simeq s_{12}m_2\) is also very small \(|\rho| \ll s_{13}m_3\), but the terms containing it in the matrix (30) can not be omitted as that would violate the normal complex structure of the matrix \(\hat{M}_\nu\) and of CKM matrix considered below. Therefore omitting small terms one obtains the matrix \(\hat{M}_\nu\) in the following simple form:

\[
\hat{M}_\nu = \begin{pmatrix}
s_{13}^2m_3 + ac_{13} & c_{13}(s_{13}s_{23}m_3e^{-i\delta'} + \rho c_{23}) & c_{13}(s_{13}c_{23}m_3e^{-i\delta'} - \rho s_{23}) \\
c_{13}(s_{13}s_{23}m_3e^{i\delta'} + \rho c_{23}) & s_{23}^2m_3 + c_{23}^2m_2 & s_{23}c_{23}(m_3c_{13}^2 - m_2) \\
c_{13}(s_{13}c_{23}m_3e^{i\delta'} - \rho s_{23}) & s_{23}c_{23}(m_3c_{13}^2 - m_2) & +\Delta_{33}
\end{pmatrix}
\]

(31)

Here, \(s_{23} \simeq c_{23} \simeq 1/\sqrt{2}\) and not too small values of \(s_{13}\) are determined in Eqs.(22).

In conclusion of this section let us construct the leptonic CKM matrix:

\[
\hat{V}_{CKM}^l = \hat{U}_\nu^+ \hat{U}_e = \hat{O}_{13}^+ \hat{O}_{13}^\dagger (\delta') \hat{O}_{12}^1,
\]

(32)

where

\[
\hat{O}_{12}^1 = \hat{O}_{12}^\nu + \hat{O}_{12}^e = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

and \(s_{12} = \sin(\varphi_1^l - \varphi_2^l) = \sin \varphi_{12}^l \simeq 0.035 \pm 0.020\) as it is determined by Eq.(22). This gives for the neutrino \(\nu_1\nu_2\) mixing angle: \(\varphi_{12}^\nu = \varphi_1^l - \varphi_2^l \simeq 0.036 \pm 0.020\) (in radians, or \((2.1 \pm 1.1)\)).

Multiplying the matrices \(\hat{O}_{ij}^l\) in Eq (32) one obtains \(\hat{V}_{CKM}^l\) just in the form of Eq.(5) with \(s_{23} \simeq c_{23} \simeq 1/\sqrt{2}\) and \(s_{12}, \ s_{13}\) given by Eq.(22).
Let us emphasize that the approach of this section, in particular the basic parameters in (21),(22), and also the relation (23), are approximate and have a model nature: also the hierarchy parameter $\sigma_\nu$ is the main one and its value $\sigma_\nu = 0.24$ is determined only approximately by modern experimental data.

Numerically the neutrino mass matrix (31) is represented in our approach by:

$$
\hat{M}_\nu = \begin{pmatrix}
0.130 & 0.335e^{-i\delta'} + 0.27 \cdot 10^{-2} & 0.341e^{-i\delta'} + 0.26 \cdot 10^{-2} \\
0.335e^{i\delta'} + 0.27 \cdot 10^{-2} & 0.965 & 0.868 \\
0.341e^{i\delta'} - 0.26 \cdot 10^{-2} & 0.868 & 1 \\
\end{pmatrix}
m_3/1.975
$$

(33)

and the CKM leptonic matrix (32) is

$$
\hat{V}_{CKM}^l = \begin{pmatrix}
0.968 & 0.0339 & 0.25e^{-i\delta'} \\
-0.175e^{i\delta'} - 0.025 & 0.714 - 0.061e^{i\delta'} & 0.678 \\
-0.178e^{i\delta'} + 0.025 & -0.699 - 0.0863e^{i\delta'} & 0.692 \\
\end{pmatrix}
$$

(34)

Unfortunately instead of predicting the neutrino mixing angles $s_{12}$, $s_{13}$, $s_{23}$ we have used their values (22) which are badly defined by experiment. The hierarchy parameter $\sigma_\nu$ remains, as has been mentioned above, practically free and we have put it to $\sqrt{\sigma} \approx 0.24$ just by hand.

Both the matrices $\hat{O}_{12}^\nu$ and $\hat{O}_{12}^e$ represent a simple Abelian rotation: $\hat{O}_{12}^\nu$ by the angle $\vartheta_{12}^\nu$, $\hat{O}_{12}^e$ by the angle $\vartheta_{12}^e$. Therefore the product $\hat{O}_{12}^{\nu+}\hat{O}_{12}^e = \hat{O}_{12}^e\hat{O}_{12}^{\nu+}$ leads to a rotation by the angle $\vartheta_{12}^l = \vartheta_{12}^e - \vartheta_{12}^\nu$, where $\vartheta_{12}^e \approx 2\vartheta_{12}^l$ (the value of $s_e$ has been given above just after Eq.(9) and $s_{12}$ in Eq.(18)) and therefore $\vartheta_{12}^\nu \approx \vartheta_{12}^l$, or $s_{12} \approx s_{\nu}$.

Similarly to the case of the quark CKM matrix, the most natural value of the leptonic CP-phase leading to the largest possible CP violation can be $\delta' = \pi/2$ or $\eta' = e^{i\delta'} = i$. This CP-phase $\delta'$ can manifest itself in the Pontecorvo neutrino oscillation experiments. It is very difficult to observe it now. Below we discuss shortly the possibility of these observations. The exact expressions for probabilities of neutrino oscillations are given in Appendix, since they are very cumbersome. Some of them have been obtained earlier in a number of papers [16 - 20].

4. The leptonic CP-phase in neutrino oscillations experiments

Many papers were devoted to the studies, pioneered by Bruno Pontecorvo [16], of two and of three [17 - 20],[5] neutrino oscillations. We consider them below shortly in order mainly to specify the role of the leptonic CP-phase [17] in these oscillations.

Let us express $\nu_e, \nu_\mu$ and $\nu_\tau$ fields entering the weak interaction Lagrangian in terms of neutrino states $\nu_1, \nu_2, \nu_3$ with definite masses $m_1, m_2, m_3$ using the leptonic CKM matrix (5)
(or (A1) from Appendix) as follows:

\[
\begin{align*}
\nu_e(ct) &= c_{13} \nu_1(0)e^{-i\epsilon_1 t} + s_{12} c_{13} \nu_2(0)e^{-i\epsilon_2 t} + s_{13} \nu_3(0)e^{-i\epsilon_3 t - i\delta'} \\
\nu_\mu(ct) &= -(s_{12} c_{23} + s_{13} s_{23} e^{i\delta'})\nu_1(0)e^{-i\epsilon_1 t} + c_{23} \nu_2(0)e^{-i\epsilon_2 t} + c_{13} s_{23} \nu_3(0)e^{-i\epsilon_3 t} \\
\nu_\tau(ct) &= (s_{12} s_{23} - s_{13} c_{23} e^{i\delta'})\nu_1(0)e^{-i\epsilon_1 t} - s_{23} \nu_2(0)e^{-i\epsilon_2 t} + c_{13} c_{23} \nu_3(0)e^{-i\epsilon_3 t}
\end{align*}
\]

(35)

up to the terms of the second order in small quantities \(s_{12}, s_{13} \ll 1\). (see Appendix for the exact \(V_{CKM}^l\) matrix) Here \(t = L/c\) is the time when neutrinos are observed at a distance \(L = ct\) from their source; the probabilities of neutrino observation at this distance from the source is \(P_{ab}(t) = |(\nu_a(t)\bar{\nu}_b(0))|^2\). Multiplying \(\nu_a(t)\) from Eq.(35) by \(\bar{\nu}_b(0)\) and taking into account the orthogonality of \(\nu_a(0)\) states \(\nu_a(0)\bar{\nu}_b(0) = \delta_{ab}\), it is easy to find:

\[
\begin{align*}
P(\nu_e\nu_e) &= |c_{13}^2 + s_{12}^2 c_{13}^2 e^{i\varphi_{21}} + s_{13}^2 e^{i\varphi_{31}}|^2 \\
P(\nu_\mu\nu_e) &= |s_{12} c_{23} + s_{13} s_{23} e^{i\delta'}|^2 + c_{23}^2 + s_{13}^2 s_{23}^2 e^{i\varphi_{31}}|^2 \\
P(\nu_\tau\nu_e) &= |s_{12} s_{23} - s_{13} c_{23} e^{i\delta'}|^2 + s_{23}^2 + c_{13}^2 s_{23}^2 e^{i\varphi_{31}}|^2
\end{align*}
\]

(36)

\[
\begin{align*}
P(\nu_e\nu_\mu) &= |c_{13}(c_{23}s_{12} + s_{13}s_{23} e^{i\delta'}) - s_{12} c_{13} c_{23} e^{i\varphi_{21}} - s_{13} c_{13} s_{23} e^{i(\delta' + \varphi_{31})}|^2 \\
P(\nu_e\nu_\tau) &= |c_{13}(s_{23}s_{12} - c_{23}s_{13} e^{i\delta'}) - s_{12} c_{13} c_{23} e^{i\varphi_{21}} + s_{13} c_{13} s_{23} e^{i(\delta' + \varphi_{31})}|^2 \\
P(\nu_\mu\nu_\tau) &= |(s_{12} s_{23} - s_{13} c_{13} c_{23} e^{i\delta'}) (s_{12} c_{23} + s_{13} c_{12} s_{23} e^{i\delta'}) + c_{23} s_{23} c_{13} e^{i(\delta' + \varphi_{31})}|^2
\end{align*}
\]

(37)

where

\[
\varphi_{ij} = (\varepsilon_i - \varepsilon_j)L/c = \frac{\Delta m^2}{2p_\nu}L = 2.54 \frac{L(m)}{E_\nu(MeV)} \Delta m^2(eV^2)
\]

(38)

and

\[
\Delta m^2_{ij} = m_i^2 - m_j^2, \quad i, j = 1, 2, 3 \quad \text{(as } \varepsilon_i \simeq p_\nu + \frac{m_i^2}{2p_\nu} \text{ at } cp \gg m_i).\]

Eqs.(36)–(38) determine the neutrino oscillation probabilities in vacuum ignoring the very important in some cases MSW effect of the medium influence (see in ref.[10]). This effect has been well studied and can be included separately (e.g. for \(\nu_e\nu_e\), \(\nu_e\nu_\mu\), \(\nu_e\nu_\tau\) in the solar neutrino case). The expressions for \(P(\nu_\mu\nu_e), P(\nu_\tau\nu_e), P(\nu_\mu\nu_\mu)\) coincide with Eqs.(37) with the substitution \(\delta' \rightarrow -\delta'\).

The simple algebra allows one to reduce Eqs.(36),(37) to the partially known [17 - 20] expressions containing the leptonic CP-phase \(\delta'\) and given below in Appendix.

Let us rewrite the Eqs.(36),(37) in much more simple forms calculating numerically the coefficient in front of \(\cos \varphi_{ij} = 1 - 2 \sin^2(\varphi_{ij}/2)\) using for example the central values of leptonic mixing angles \(\theta'_{12}, \theta'_{13}, \theta'_{23}\) given in Eqs.(22), i.e. their sines \(s_{12}, s_{13}, s_{23}\), respectively. These equations in their numerical form show clearly the influence of the leptonic CP-phase \(\delta'\) on
the neutrino oscillations patterns:

\[
1 - P(\nu_e \nu_e) \simeq c_{12}^2 \sin^2(2\delta_{13}) \sin^2(\varphi_{31}/2) + c_{13}^2 \sin^2(2\delta_{12}) \sin^2(\varphi_{21}/2) \\
+ s_{12}^2 \sin^2(2\delta_{13}) \sin^2(\varphi_{32}/2) \\
0.23 \sin^2(\varphi_{31}/2) + 4.3 \cdot 10^{-3} \sin^2(\varphi_{21}/2) + 0.29 \cdot 10^{-3} \sin^2(\varphi_{32}/2)
\]

\[
1 - P(\nu_\mu \nu_\mu) \simeq (0.94 + 0.017 \cos \delta') \sin^2(\varphi_{32}/2) + (0.062 - 0.017 \cos \delta') \sin^2(\varphi_{31}/2) \\
(0.063 + 0.016 \cos \delta' - 0.31 \cdot 10^{-3} \cos^2 \delta') \sin^2(\varphi_{21}/2)
\]

\[
1 - P(\nu_\tau \nu_\tau) \simeq (0.94 + 0.017 \cos \delta') \sin^2(\varphi_{32}/2) + (0.062 - 0.017 \cos \delta') \sin^2(\varphi_{31}/2) \\
(0.064 - 0.016 \cos \delta' - 0.31 \cdot 10^{-3} \cos^2 \delta') \sin^2(\varphi_{21}/2)
\]

Eqs.(37) determine the probabilities for neutrino of a given sort \(\nu_i\) to change its type (i.e. to transform into the other sort \(\nu_j\)) at a distance \(L\) from the source of neutrinos. For \(\nu_e\) this probability does not depend on the CP-phase \(\delta'\) at all as can be seen already from Eqs.(36),(37), however, for transformations of \(\nu_\mu\) and \(\nu_\tau\) this dependence is not too small.

The probabilities for different neutrino transitions \(P(\nu_i \nu_j) \neq P(\nu_j \nu_i)\) depend also on the value of \(\delta'\):

\[
P(\nu_e \nu_\mu) \simeq 0.059\{1 - 0.14 \cos \delta' - 0.98[\cos(\varphi_{31}) + 0.019 \cos(\varphi_{21}) + 0.0012 \cos(\varphi_{32})] + 0.140[\cos(\varphi_{32} + \delta') - \cos(\varphi_{31} + \delta') - \cos(\varphi_{21} - \delta')]\}
\]

\[
P(\nu_e \nu_\tau) \simeq 0.061\{1 - 0.13 \cos \delta' - 0.98[\cos(\varphi_{31}) + 0.018 \cos(\varphi_{21}) + 0.0012 \cos(\varphi_{32})] - 0.135[\cos(\varphi_{31} + \delta') + \cos(\varphi_{32} + \delta') - \cos(\varphi_{21} - \delta')]\}
\]

\[
P(\nu_\mu \nu_\tau) \simeq 0.47\{1 - 0.99[\cos(\varphi_{32}) - 0.066 \cos(\varphi_{21}) + 0.061 \cos(\varphi_{31})] + 0.035 \sin \delta' \cos(\varphi_{21}) + 0.035 \sin \delta' \cos(\varphi_{31} + \varphi_{32}) \sin(\varphi_{21}/2) - 0.017 \sin \delta' \sin(\varphi_{21})\}
\]

It is interesting to consider the time reversal effect which reveals in the difference between the probabilities \(P(\nu_\mu \nu_\mu)\) and \(P(\nu_\mu \nu_\tau)\) or \(P(\nu_\mu \nu_\tau)\) and \(P(\nu_\tau \nu_\tau)\) etc. [17]. Taking the difference between \(P(\nu_e \nu_\mu) - P(\nu_\mu \nu_e)\) and the same for \(\nu_e \nu_\tau, \nu_\mu \nu_\tau\) one obtains:

\[
P(\nu_e \nu_\mu) - P(\nu_\mu \nu_e) = 0.0164(\sin \varphi_{21} + \sin \varphi_{32} - \sin \varphi_{31}) \sin \delta' \\
P(\nu_e \nu_\tau) - P(\nu_\tau \nu_e) = -0.0164(\sin \varphi_{21} + \sin \varphi_{32} - \sin \varphi_{31}) \sin \delta' \\
P(\nu_\mu \nu_\mu) - P(\nu_\mu \nu_\tau) = 0.0328(\cos \varphi_{21}/2 - \cos (\varphi_{31} + \varphi_{32})/2) \sin \delta'
\]

where 0.0328 = \(c_{13} \sin 2\delta_{12} \sin 2\delta_{13} \sin 2\delta_{23}\) for the numerical coefficients given above. So, the leptonic CP-phase can manifest itself in time reversal neutrino transitions \(\nu_\mu \nu_\mu\) and \(\nu_\tau \nu_\tau\) experiments. However, these experiments need the neutrino beams with a fixed value of \(L/2E_\nu\) what is very difficult to organize because usually one deals with continuum spectra of produced neutrino.

We emphasize once more that the numerical coefficients in Eqs.(39),(41) can be determined by future experimental data only, but the general form of them (obtained algebraically from exact Eqs.(A2)–(A6) of Appendix) remains always valid.
Let us average Eqs.(39),(40) over $L/2E_{\nu}$ considering the case of large $L/2E_{\nu} > (\Delta m_{21}^2)^{-1}$, i.e. of large angles $\varphi_{ij} = \Delta m_{ij}^2 L/2E_{\nu}$. This nicely corresponds to the real experimental situation where one works with a continuum spectra of neutrino. This averaging gives:

$$\langle 1 - P(\nu_e\nu_e) \rangle \simeq 0.1193,$$

$$\langle 1 - P(\nu_\mu\nu_\mu) \rangle \simeq 0.529(1 + 0.016 \cos \delta' - 0.289 \cdot 10^{-3} \cos^2 \delta'),$$

$$\langle 1 - P(\nu_\tau\nu_\tau) \rangle \simeq 0.531(1 - 0.015 \cos \delta' - 0.288 \cdot 10^{-3} \cos^2 \delta')$$

and

$$\langle P(\nu_e\nu_\mu) \rangle \simeq 0.0585(1 + 0.140 \cos \delta'), \quad \langle P(\nu_e\nu_\tau) \rangle \simeq 0.0608(1 - 0.134 \cos \delta'),$$

$$\langle P(\nu_\mu\nu_\tau) \rangle \simeq 0.470(1 + 0.394 \cdot 10^{-3} \cos \delta' - 0.163 \cdot 10^{-3} \cos 2\delta')$$

where brackets mean the averaging over all $\varphi_{ij}$.

Comparing Eqs.(39) and (40) (and also (A2)–(A6) in the Appendix) one finds: $P(\nu_i\nu_i) + P(\nu_i\nu_j) + P(\nu_i\nu_k) = 1$ for different $\nu_i, \nu_j$ and $\nu_k$ (e.g. $1 - P(\nu_e\nu_\mu) = P(\nu_e\nu_\mu) + P(\nu_e\nu_\tau)$).

Eqs.(42)–(43) show also that the leptonic CP-phase $\delta'$ can be observed experimentally, in principle, by measuring the average $\nu_e\nu_\mu$, or $\nu_e\nu_\tau$ transition rates with 14% accuracy which is much larger than the effect (41) of time reversal.

This is illustrated in Figs. 2(a,b) where the averaged probabilities of $\nu_i\nu_j$ transitions $\langle P(\nu_i\nu_j) \rangle$ are calculated for the values of $s_{12}, s_{13}, s_{23}$ determined in Eqs.(22). These figures show a large (about 30%) difference between $\langle P(\nu_\mu\nu_e) \rangle$ and $\langle P(\nu_\tau\nu_e) \rangle$ probabilities dependencies on the CP-phase $\delta'$. For example the value of $\langle P(\nu_\mu\nu_e) \rangle \simeq 0.067$ at $\delta' = 0$ turns out to be larger than $\langle P(\nu_\tau\nu_e) \rangle \simeq 0.057$ by 17%, i.e. $\langle P(\nu_\mu\nu_e) \rangle / \langle P(\nu_\tau\nu_e) \rangle \simeq 1.17$, while at $\delta' = \pi$ vice versa $\langle P(\nu_\mu\nu_e) \rangle \simeq 0.050$ becomes smaller than $\langle P(\nu_\tau\nu_e) \rangle \simeq 0.059$ by 15%: $\langle P(\nu_\mu\nu_e) \rangle / \langle P(\nu_\tau\nu_e) \rangle \simeq 0.85$.

Unfortunately, the absolute values of these probabilities are small of about 1/20, but nevertheless they be really observed experimentally. Even weaker is the $\delta'$ dependence of the average value of the probability of $\nu_\mu\nu_\tau$ transition rate (averaged over the oscillations connected with different $L/E_{\nu}$ values, or over $\varphi_{ij}$ – as in Figs. 2(a,b)), shown in Fig. 2c. This figure shows that $\langle P(\nu_\mu\nu_\tau) \rangle$ changes by 0.05% only when $\delta'$ changes from 0 to $\pi$.

5. Conclusion

The following three simple problems were discussed and solved in this paper:

a) The u- and d-quarks and electrons mass matrices were obtained in a simple hierarchical form, including quark’s CP-phase $\delta$ and correcting the matrices suggested by B.Stech,

b) The recent data on neutrino masses and mixing angles were discussed shortly and used for construction of the neutrino and leptonic mass matrices and CKM matrix, both including the
leptonic CP-phase $\delta'$,  
c) The CKM matrix obtained was used to investigate the three neutrino oscillations in the vacuum. The method of determination of the leptonic CP-phase from their observation (averaged over the oscillations) was presented.

Note, that there is a vast ambiguity in the determination of quark’s (or leptonic) mass matrices. E.g. the pair of matrices $\hat{M}'_u = \hat{U}_o \hat{M}_d \hat{U}_o^+$, with any unitary matrix $\hat{U}_o$, leads exactly to the same mass eigenvalues and to the same mixing angles as $\hat{M}_u$, $\hat{M}_d$.

Also the following correction has to be added to the central part of the paper. It was noted there (in Section 2, after Eq. (21)) that Stech’s matrices (2),(3) have negative value of some masses and that positivity can be restored by a simple $\gamma_5$ transformation. However this will violate the symmetry of different generations fields and also will change the form of mass matrices. Better is to avoid this shortcoming and construct the particles mass matrices with only positive eigenvalues $\hat{M}^\text{diag}_a = \left( \begin{array}{ccc} m_{a_1} & m_{a_2} & m_{a_3} \end{array} \right)$ directly from Eqs. (11) [or (29) and (12)], e.g. for d-quark and electrons one finds:

$$
\hat{M}'_d = \begin{pmatrix} a & \rho & 0 \\ \rho & b & 0 \\ 0 & 0 & m_b \end{pmatrix}, \quad \text{where} \quad \begin{cases} a = s_{12}^2 m_2 + c_{12}^2 m_1 \simeq 15 \sigma^3 m_b \\ b = c_{12}^2 m_2 + s_{12}^2 m_1 \simeq c_{12}^2 m_2 \simeq \frac{1}{2} \sigma^2 m_b \\ \rho = -c_{12} s_{12} (m_2 - m_1) \simeq \sqrt{\frac{7}{2}} \sigma^2 m_b, \quad \text{or:} \\
\end{cases}
$$

$$
\hat{M}'_d = \begin{pmatrix} 15 \sigma^4 & -\sqrt{\frac{7}{2}} \sigma^3 & 0 \\ -\sqrt{\frac{7}{2}} \sigma^3 & \frac{1}{2} \sigma^2 & 0 \\ 0 & 0 & \sigma \end{pmatrix} \frac{m_b}{\sigma}, \quad \text{and similarly:} \quad \hat{M}'_e = \begin{pmatrix} 3 \lambda' \sigma^4 & \sqrt{\frac{3}{2}} \sigma^3 & 0 \\ \sqrt{\frac{3}{2}} \sigma^3 & (1 + \beta') \sigma^2 & 0 \\ 0 & 0 & \sigma \end{pmatrix} \frac{m_e}{\sigma},
$$

with small $\lambda' = 0.020$, $\beta' = 0.016$. Both $\hat{M}_d$, $\hat{M}_e$ differs slightly from Stech’s forms given in Section 1. The u-quark mass matrix is determined by Eqs. (12),(19) and also slightly deviates from the Stech’s form given by Eq. (2):

$$
\hat{M}'_u = \hat{U}_u^+ \hat{M}^\text{diag}_u \hat{U}_u \simeq \begin{pmatrix} 2 \sigma^4 & \frac{1}{\sqrt{2}} \sigma^3 \eta^+ & \sigma^2 \eta^+ \\ \frac{1}{\sqrt{2}} \sigma^3 \eta & 3 \sigma^2 & 1 \\ \sigma^2 \eta & \frac{1}{\sqrt{2}} \sigma & 1 \end{pmatrix} m_t, \quad \text{where} \quad \hat{U}_u^+ = \hat{O}_{23}^+ \hat{O}_{13}^+ (\hat{O}_u \hat{O}_u^+).
$$

The diagonalization of all these matrices $\hat{M}^\text{diag}_a = \hat{U}_a \hat{M}'_a \hat{U}_a^+$ can be done by the same unitary matrices $\hat{U}_d$, $\hat{U}_u$, $\hat{U}_e$, $\hat{U}_v$ with the same mixing angles (used at they construction) as were used above for the Stech’s matrix case.
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6. Appendix

We give below the probabilities rates $P(\nu_i\nu_j)$ in general algebraically form using the well known exact leptonic CKM matrix (given above for quarks in Eq.(5)):

$$V_{CKM}^l = \begin{pmatrix}
    c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta'} \\
    -s_{12}c_{13} - c_{12}s_{23}s_{13}e^{i\delta'} & c_{12}c_{23} + s_{12}s_{23}s_{13}e^{i\delta'} & s_{23}s_{13} \\
    s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta'} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta'} & c_{23}s_{13}
\end{pmatrix}. \quad (A1)$$

At $\delta' \neq 0$ it leads to:

$$1 - P(\nu_\mu\nu_\mu) = \{c_{13}^2 \sin^2(2\theta'_{12}) + s_{12}^2 s_{13}^2 \sin^2(2\theta'_{23}) + s_{13}^2 \sin^2(2\theta'_{12}) + c_{12}^2 s_{13}^2 \sin^2(2\theta'_{23})
+ \cos \delta' \sin(4\theta'_{12}) \sin^2(2\theta'_{23})(s_{13}^2 c_{23} - s_{13}^2 s_{23}) - \cos^2 \delta' s_{13}^2 \sin^2(2\theta'_{23}) \sin^2(2\theta'_{12}) \} \sin^2(\varphi_{21}/2)$$

$$+ \{s_{12} c_{13}^2 \sin^2(2\theta'_{23}) + c_{12}^2 s_{23}^2 \sin^2(2\theta'_{12}) + \cos \delta' \sin c_{23}^2 \sin(2\theta'_{12}) \sin(2\theta'_{23}) \sin(2\theta'_{13}) \sin^2(\varphi_{31}/2)$$

$$+ \{c_{12}^2 c_{13}^2 \sin^2(2\theta'_{23}) + s_{12}^2 s_{23}^2 \sin^2(2\theta'_{12}) - \cos \delta' s_{23}^2 c_{13} \sin(2\theta'_{12}) \sin(2\theta'_{23}) \sin(2\theta'_{13}) \} \sin^2(\varphi_{31}/2)$$

$$= \{c_{13}^2 \sin^2(2\theta'_{12}) + s_{12}^2 s_{13}^2 \sin^2(2\theta'_{23}) + c_{12}^2 s_{13}^2 \sin^2(2\theta'_{23})
+ \cos \delta' \sin(4\theta'_{12}) \sin^2(2\theta'_{23})(s_{13}^2 c_{23} - s_{13}^2 s_{23}) - \cos^2 \delta' s_{13}^2 \sin^2(2\theta'_{23}) \sin^2(2\theta'_{12}) \} \sin^2(\varphi_{21}/2)$$

$$+ \{s_{12} c_{13}^2 \sin^2(2\theta'_{23}) + c_{12}^2 s_{23}^2 \sin^2(2\theta'_{12}) - \cos \delta' c_{23}^2 c_{13} \sin(2\theta'_{12}) \sin(2\theta'_{23}) \sin(2\theta'_{13}) \} \sin^2(\varphi_{31}/2)$$

$$+ \{c_{12}^2 c_{13}^2 \sin^2(2\theta'_{23}) + s_{12}^2 s_{23}^2 \sin^2(2\theta'_{12}) + \cos \delta' c_{23}^2 c_{13} \sin(2\theta'_{12}) \sin(2\theta'_{23}) \sin(2\theta'_{13}) \} \sin^2(\varphi_{32}/2)$$

$$= \{c_{13}^2 \sin^2(2\theta'_{12}) + s_{12}^2 s_{13}^2 \sin^2(2\theta'_{23}) + c_{12}^2 s_{13}^2 \sin^2(2\theta'_{23})
+ \cos \delta' \sin(4\theta'_{12}) \sin^2(2\theta'_{23})(s_{13}^2 c_{23} - s_{13}^2 s_{23}) - \cos^2 \delta' s_{13}^2 \sin^2(2\theta'_{23}) \sin^2(2\theta'_{12}) \} \sin^2(\varphi_{21}/2)$$

$$+ \{s_{12} c_{13}^2 \sin^2(2\theta'_{23}) + c_{12}^2 s_{23}^2 \sin^2(2\theta'_{12}) - \cos \delta' c_{23}^2 c_{13} \sin(2\theta'_{12}) \sin(2\theta'_{23}) \sin(2\theta'_{13}) \} \sin^2(\varphi_{31}/2)$$

$$+ \{c_{12}^2 c_{13}^2 \sin^2(2\theta'_{23}) + s_{12}^2 s_{23}^2 \sin^2(2\theta'_{12}) + \cos \delta' c_{23}^2 c_{13} \sin(2\theta'_{12}) \sin(2\theta'_{23}) \sin(2\theta'_{13}) \} \sin^2(\varphi_{32}/2)$$

for $1 - P(\nu_e\nu_e)$ see Eqs.(39). As before here $\varphi_{ij} = \Delta m_{ij}^2 L/2E_\nu$ and:

$$P(\nu_\mu\nu_\mu) = \frac{1}{4} \{\sin^2(2\theta'_{13}) (s_{23}^2 + c_{12}^2 s_{23}^2 + s_{12}^2 s_{23}^2 + \frac{1}{2} c_{13} \sin(2\theta'_{23}) \sin(4\theta'_{12}) \cos \delta')$$

$$- 2 c_{13}^2 \sin^2(2\theta'_{12}) (c_{23}^2 - s_{13}^2 s_{23}^2) \cos(\varphi_{21}) - 2 s_{23}^2 \sin^2(2\theta'_{13}) (c_{12} \cos(\varphi_{31}) + c_{13}^2 \sin(2\theta'_{13}))$$

$$+ c_{13} \sin(2\theta'_{12}) \sin(2\theta'_{13}) \sin(2\theta'_{23}) (s_{12}^2 \cos(\delta' + \varphi_{21}) - c_{12}^2 \cos(\delta' - \varphi_{21}))$$

$$+ c_{13} \sin(2\theta'_{12}) \sin(2\theta'_{13}) \sin(2\theta'_{23}) (\cos(\delta' + \varphi_{31}) - \cos(\delta' - \varphi_{31}))$$

$$+ 2 c_{13}^2 c_{23}^2 \sin^2(2\theta'_{12}) \} \quad (A4)$$

$$P(\nu_e\nu_e) = \frac{1}{4} \{\sin^2(2\theta'_{13}) (c_{23}^2 + c_{12}^2 c_{23}^2 + s_{12}^2 c_{23}^2 - \frac{1}{2} c_{13} \sin(2\theta'_{23}) \sin(4\theta'_{12}) \cos \delta')$$

$$+ 2 c_{13}^2 \sin^2(2\theta'_{12}) (s_{23}^2 - c_{13}^2 s_{23}^2) \cos(\varphi_{21}) - 2 c_{23}^2 \sin^2(2\theta'_{13}) (c_{12} \cos(\varphi_{31}) + s_{12}^2 \cos(\varphi_{32}))$$

$$+ c_{13} \sin(2\theta'_{12}) \sin(2\theta'_{13}) \sin(2\theta'_{23}) (c_{12}^2 \cos(\delta' - \varphi_{21}) - s_{12}^2 \cos(\delta' + \varphi_{21}))$$

$$+ c_{13} \sin(2\theta'_{12}) \sin(2\theta'_{13}) \sin(2\theta'_{23}) (\cos(\delta' + \varphi_{31}) - \cos(\delta' + \varphi_{32}))$$

$$+ 2 c_{13}^2 s_{23}^2 \sin^2(2\theta'_{12}) \} \quad (A5)$$
\[ P(\nu_\mu \nu_\tau) = \frac{1}{4} \left[ 2 s_{13}^2 \sin^2(2\theta_{12}^l) \cos^2(2\theta_{23}^l) + \left( c_{13}^4 + c_{12}^4 + s_{12}^4 + (c_{13}^2 + s_{13}^2)s_{12}^2 \right) \sin^2(2\theta_{23}^l) \\
- [2 s_{13}^2 (c_{23}^4 + s_{23}^4) \sin^2(2\theta_{12}^l) + [2 s_{13}^2 (c_{23}^4 + s_{23}^4) - (1 + s_{13}^4) \sin^2(2\theta_{12}^l)] \sin^2(2\theta_{23}^l) \cos(\varphi_{21}) \\
- [2c_{13}^2 (s_{12}^2 + c_{12}^2 s_{13}^2) \sin^2(2\theta_{23}^l) - \frac{1}{2} c_{13} \sin(2\theta_{12}^l) \sin(2\theta_{13}^l) \sin(4\theta_{23}^l)] \cos(\varphi_{31}) \\
+ [2 c_{13}^2 \sin^2(2\theta_{23}^l) / (s_{12}^2 - c_{12}^2)] - \frac{1}{2} c_{13} \sin(2\theta_{12}^l) \sin(2\theta_{13}^l) \sin(4\theta_{23}^l) \cos(\varphi_{32}) \\
+ 2 c_{13} \sin(2\theta_{12}^l) \sin(2\theta_{13}^l) \sin(2\theta_{23}^l) \sin(\delta') \sin(\varphi_{21}/2) \cos(\frac{\varphi_{31} + \varphi_{32}}{2}) \\
+ s_{13} \sin(4\theta_{12}^l) \sin(4\theta_{23}^l) \cos(\delta'[1 + s_{13}^2]) \sin(\varphi_{21}/2) \\
- c_{13} \sin(2\theta_{12}^l) \sin(2\theta_{13}^l) \sin(2\theta_{23}^l) \sin(\varphi_{21}/2) \\
- 2 s_{13} \sin(2\theta_{12}^l) \sin(2\theta_{23}^l) \cos(\delta') \sin(\varphi_{21}/2) \} \right] \tag{A6} \]

For the average values of these probabilities over all \(\varphi_{ij}\) (i.e. over \(E_\nu\), or over \(L\) at large \(\varphi_{ij}\)) one obtains the values the (42), (43) slightly dependent on the leptonic CP-phase \(\delta'\):

\[
\begin{align*}
\langle 1 - P(\nu_e \nu_e) \rangle &= A_{ee} \simeq 0.1193 \\
\langle 1 - P(\nu_\mu \nu_\mu) \rangle &= A_{\mu\mu} + B_{\mu\mu} \cos \delta' + C_{\mu\mu} \cos^2 \delta' \\
\langle 1 - P(\nu_\tau \nu_\tau) \rangle &= A_{\tau\tau} + B_{\tau\tau} \cos \delta' + C_{\tau\tau} \cos^2 \delta' \\
\langle P(\nu_e \nu_\mu) \rangle &= A_{e\mu} + B_{e\mu} \cos \delta' \\
\langle P(\nu_e \nu_\tau) \rangle &= A_{e\tau} + B_{e\tau} \cos \delta' \\
\langle P(\nu_\mu \nu_\tau) \rangle &= A_{\mu\tau} + B_{\mu\tau} \cos \delta' + C_{\mu\tau} \cos(2\delta') \tag{A7}
\end{align*}
\]

where the coefficients are:

\[
\begin{align*}
A_{ee} &= \frac{1}{2} \left[ c_{13}^2 \sin^2(2\theta_{12}^l) + \sin^2(2\theta_{13}^l) \right] \\
A_{\mu\mu} &= \frac{1}{2} \left[ (c_{13}^2 + (c_{12}^4 + s_{12}^4)s_{13}^2) \sin^2(2\theta_{23}^l) \\
&\quad + (s_{13}^2 \sin^2(2\theta_{12}^l) + \sin^2(2\theta_{13}^l)c_{23}^2 \\
&\quad + c_{23}^2 \sin^2(2\theta_{12}^l)) \right] \\
A_{\tau\tau} &= \frac{1}{2} \left[ (c_{13}^2 + (c_{12}^4 + s_{12}^4)s_{13}^2) \sin^2(2\theta_{23}^l) \\
&\quad + (s_{13}^2 \sin^2(2\theta_{12}^l) + \sin^2(2\theta_{13}^l)c_{23}^2 \\
&\quad + s_{23}^2 \sin^2(2\theta_{12}^l)) \right] \\
A_{e\mu} &= \frac{1}{4} \left[ (1 + c_{12}^4 + s_{12}^4) c_{23}^2 \sin^2(2\theta_{13}^l) \\
&\quad + 2 c_{13}^2 c_{23}^2 \sin^2(2\theta_{12}^l)) \right] \\
A_{e\tau} &= \frac{1}{4} \left[ (1 + c_{12}^4 + s_{12}^4) c_{23}^2 \sin^2(2\theta_{13}^l) \\
&\quad + 2 c_{13}^2 c_{23}^2 \sin^2(2\theta_{12}^l)) \right] \\
A_{\mu\tau} &= \frac{1}{4} \left[ 2 s_{13}^2 \sin^2(2\theta_{12}^l) \cos^2(2\theta_{23}^l) \\
&\quad + \sin^2(2\theta_{23}^l) / \{(c_{12}^4 + s_{12}^4)s_{13}^2 \\
&\quad + c_{13}^2 + c_{12}^4 + s_{12}^4) \} \right] \\
B_{\mu\mu} &= -\frac{1}{2} s_{13} \sin(2\theta_{13}^l)(s_{23}^2 s_{13}^2 - c_{23}^2) \sin(4\theta_{12}^l) \\
C_{\mu\mu} &= -\frac{1}{2} s_{13} \sin(2\theta_{13}^l)s_{13} \sin(2\theta_{13}^l) \sin^2(2\theta_{12}^l) \\
B_{\tau\tau} &= -\frac{1}{2} s_{13} \sin(2\theta_{13}^l)(s_{23}^2 - c_{23}^2 s_{13}^2) \sin(4\theta_{12}^l) \\
C_{\tau\tau} &= -\frac{1}{2} s_{13} \sin(2\theta_{13}^l)s_{13} \sin(2\theta_{13}^l) \sin^2(2\theta_{12}^l) \\
B_{e\mu} &= \frac{1}{8} c_{13} \sin(2\theta_{13}^l) \sin(2\theta_{23}^l) \sin(4\theta_{12}^l) \\
B_{e\tau} &= -\frac{1}{8} c_{13} \sin(2\theta_{13}^l) \sin(2\theta_{23}^l) \sin(4\theta_{12}^l) \\
B_{\mu\tau} &= \frac{1}{8} (1 + s_{13}^2) s_{13} \sin(4\theta_{12}^l) \sin(4\theta_{23}^l) \\
C_{\mu\tau} &= -\frac{1}{8} s_{13}^2 \sin^2(2\theta_{12}^l) \sin^2(2\theta_{23}^l) \tag{A8}
\end{align*}
\]

The numerical value of these coefficient are given above in the text in (42), (43) for neutrino mixing angles \(s_{12}, \ s_{13}, \ s_{23}\) values determined in Eqs.(22).
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Figures captions.

Fig.1. a) The run of the upper quark masses $m_t, m_c, m_u$ calculated in Refs.[8] in the first order of QCD perturbation theory (the dashed lines) and in the fourth order of it (solid lines). The vertical line corresponds to the scale $\mu = M_t \simeq 174.4$GeV used in the paper. The run of electron’s masses $m_{\tau}, m_{\mu}, m_e$ is disregarded (while it can be easy taken into account and is not essential).

b) The same for the masses $m_b, m_s, m_d$ of the lower quarks.

Fig.2. a) Dependence of the average (over oscillations) value $\langle P(\nu_e\nu_\mu) \rangle$ of $\nu_e\nu_\mu$ transition probability on the CP-phase $\delta'$; at $\delta' = \pi$ it has about 13% minimum.

b) The same for the average value $\langle P(\nu_e\nu_\tau) \rangle$ of $\nu_e\nu_\tau$ transition probability. As is seen instead of the minimum at $\delta' = \pi$, as was for the $\nu_e\nu_\mu$ transition case, it has here the 14% maximum.

c) The same dependence on the CP-phase as was shown in the cases a), b) but for the average $\nu_\mu\nu_\tau$ probability $\langle P(\nu_\mu\nu_\tau) \rangle$. Its dependence on CP-phase $\delta'$ here is much more flat – about in hundred times smaller then in the $\nu_e\nu_\mu$ and $\nu_e\nu_\tau$ cases.
Fig. 1(a,b)

Fig. 2(a,b,c)