Multipartite entanglement purification with quantum nondemolition detectors

Yu-Bo Sheng\textsuperscript{1,2,3}, Fu-Guo Deng\textsuperscript{4,1}, Bao-Kui Zhao\textsuperscript{1,2,3}, Tie-Jun Wang\textsuperscript{1,2,3}, and Hong-Yu Zhou\textsuperscript{1,2,3}

\textsuperscript{1} Key Laboratory of Beam Technology and Material Modification of Ministry of Education, Beijing Normal University, Beijing 100875, China
\textsuperscript{2} College of Nuclear Science and Technology, Beijing Normal University, Beijing 100875, China
\textsuperscript{3} Beijing Radiation Center, Beijing 100875, China
\textsuperscript{4} Department of Physics, Applied Optics Beijing Area Major Laboratory, Beijing Normal University, Beijing 100875, China

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We present a scheme for multipartite entanglement purification of quantum systems in a Greenberger-Horne-Zeilinger state with quantum nondemolition detectors (QNDs). This scheme does not require the controlled-not gates which cannot be implemented perfectly with linear optical elements at present, but QNDs based on cross-Kerr nonlinearities. It works with two steps, i.e., the bit-flipping error correction and the phase-flipping error correction. These two steps can be iterated perfectly with parity checks and simple single-photon measurements. This scheme does not require the parties to possess sophisticated single photon detectors. These features maybe make this scheme more efficient and feasible than others in practical applications.

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I. INTRODUCTION

Entanglement plays an important role in quantum information processing \cite{1}. For example, bipartite entangled states provide some novel ways for quantum cryptography \cite{2, 3, 4, 5, 6}, quantum teleportation \cite{7} and quantum dense coding \cite{8, 9, 10}. Multipartite entangled states have many important applications in quantum computation and quantum communication. It provides the superpower of quantum computer \cite{11} and the resource for quantum error correction codes \cite{12}. Some important branches of quantum communication require multipartite entangled states to set up the quantum channel, such as controlled teleportation \cite{13, 14, 15, 16}, quantum secret sharing \cite{17, 18, 19, 20, 21} and quantum state sharing \cite{22, 23, 24, 25}.

In experiment, the implement of quantum communication depends on the transmission of quantum systems. However, the noise in quantum channel will degrade the entanglement of the quantum system transmitted, even make it in a mixed state, which will decrease the fidelity of quantum teleportation or make quantum communication insecure. In this time, the parties of quantum communication usually exploit entanglement purification \cite{22, 23, 24, 25, 26, 27, 28, 29} or entanglement concentration \cite{30, 31, 32, 33, 34, 35} to obtain some maximally entangled states from a less-entanglement ensemble. Entanglement purification is used to increase the entanglement of quantum systems in a mixed state, while entanglement concentration is only used to obtain some maximally entangled states from a set of pure entangled quantum systems. The former is more general than the latter in a practical quantum communication. By far, entanglement purification has been studied not only for bipartite entangled quantum systems \cite{22, 23, 24, 25, 26} but also for multipartite entangled quantum systems \cite{27, 28, 29}. As multipartite entanglement purification is far more difficult than that for two-particle Bell states, there are only several multipartite entanglement purification schemes \cite{27, 28, 29}, including that for high-dimensional quantum systems.

In 1998, Murao et al. \cite{27} presented a multipartite entanglement purification protocol for quantum systems in a Greenberger-Horne-Zeilinger (GHZ) state with controlled-\textsc{not} (CNOT) gates and local Hadamard transformations. Their protocol has been generalized to high-dimensional multipartite quantum systems by Cheong et al. \cite{28} in 2007. In the latter \cite{28}, they use some generalized \textsc{xor} gates in high-dimensional systems, instead of the common CNOT gates in two-dimensional systems, and the Hadamard transformation is substituted by the quantum Fourier transformation. It has been shown that with only single photon sources and linear optical elements, the maximal probability for achieving the CNOT gate is $3/4$ \cite{30}. So the CNOT gate based on linear optics is beyond the reach of current technology. These obstacles make the multipartite entanglement purification protocols \cite{27, 28, 29} hard to realize at present.

Cross-kerr nonlinearity provides a good tool to complete a parity-check measurement \cite{37, 38}. With quantum language, the cross-Kerr nonlinearities can be described with the Hamiltonian as follows \cite{37, 38}:

\begin{equation}
H_{ck} = \hbar \chi a_s^\dagger a_s a_p^\dagger a_p,
\end{equation}

where $a_s^\dagger$ and $a_p^\dagger$ denote the creation operations, and $a_s$ and $a_p$ are the annihilation operations. $\hbar \chi$ is the coupling strength of the nonlinearity, which is decided by the property of material. For a quantum signal in a Fock
when the parity of the two photons is even, the coherent state of the two photons injected into the two spatial modes \( |\alpha\rangle_p \) after the interaction with the cross-Kerr nonlinear medium the whole system evolves as

\[
U_{ck}|\Psi\rangle_s|\alpha\rangle_p = e^{iH_{ck}t/\hbar}[c_0|0\rangle_s + c_1|1\rangle_s]|\alpha\rangle_p = c_0|0\rangle_s|\alpha\rangle_p + c_1|1\rangle_s|e^{i\theta}\rangle_p,
\]

where \( \theta = \chi t \) and \( t \) is the interaction time. It is shown that the coherent beam picks up a phase shift proportional to the number of the photons in the Fock state.

In this paper, we present a feasible scheme for multipartite entanglement purification of quantum systems in a GHZ state by constructing nondestructive quantum nondemolition detectors (QND) with cross-Kerr nonlinearities. The task of multipartite entanglement purification can be completed with two steps which can be iterated perfectly. The first one is to purify the bit-flipping errors in multipartite quantum systems, and the second one is to purify their phase-flipping errors. This protocol does not require the CNOT gate based on linear optics and sophisticated single-photon detectors, which makes it more feasible in practical applications. Moreover, it has the same yield as that with CNOT gates but reduces a large number of quantum resources in principle.

II. MULTIPARTITE ENTANGLEMENT PURIFICATION WITH QUANTUM NONDEMOLITION DETECTORS

A. quantum nondemolition detector and description of errors

The principle of our nondestructive quantum nondemolition detector (QND) is shown in Fig.1. It is composed of two cross-Kerr nonlinearities (\( c_{k1} \) and \( c_{k2} \)), four polarization beam splitters (PBSs), a coherent beam \( |\alpha\rangle_p \), and an X homodyne measurement. \( b_1 \) and \( b_2 \) represent the up spatial mode and the down spatial mode, respectively. Each polarization beam splitter (PBS) is used to pass through the horizontal polarization photons \( |H\rangle \) and reflect the vertical polarization photons \( |V\rangle \). The cross-Kerr nonlinearity will make the coherent beam \( |\alpha\rangle_p \) pick up a phase shift \( \theta \) if there is a photon in the mode. The probe beam \( |\alpha\rangle_p \) will pick up a phase shift \( \theta \) if the state of the two photons injected into the two spatial modes \( b_1 \) and \( b_2 \) is \( |HH\rangle \) or \( |VV\rangle \); otherwise it picks up a phase shift 0 (for \( |VH\rangle \) or \( |HV\rangle \)). That is, when the parity of the two photons is even, the coherent beam \( |\alpha\rangle_p \) will pick up a phase shift \( \theta \); otherwise it will pick up 0 or \( 2\theta \). Each party of quantum communication can determine the parity of his two photons with an X homodyne measurement. With this QND, we can distinguish superpositions and mixtures of \( |HH\rangle \) and \( |VV\rangle \) from \( |HV\rangle \) and \( |VH\rangle \).

A multipartite GHZ state for spin 1/2 systems can be written as

\[
|\phi^+\rangle_s = \frac{1}{\sqrt{2}}(|00\cdots 0\rangle + |11\cdots 1\rangle).
\]

Here \( |0\rangle \equiv |H\rangle \) and \( |1\rangle \equiv |V\rangle \) represent the horizontal polarization state and the vertical polarization one, respectively. They are the two eigenvectors of the basis \( Z \). In the following, we first use three-particle GHZ-state systems as an example for demonstrating the principle of our multipartite entanglement purification scheme and then discuss the case for \( N \)-particle systems in a GHZ state. This scheme includes two steps: one for the bit-flipping error correction and the other for phase-flipping error correction. As this scheme works with quantum nondemolition detectors, instead of CNOT gates, we denote it QND scheme.

There are eight three-particle GHZ states, i.e.,

\[
\begin{align*}
|\Phi^\pm\rangle_{ABC} &= \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)_{ABC}, \\
|\Phi^+\rangle_{ABC} &= \frac{1}{\sqrt{2}}(|100\rangle \pm |011\rangle)_{ABC}, \\
|\Phi^\mp\rangle_{ABC} &= \frac{1}{\sqrt{2}}(|010\rangle \pm |101\rangle)_{ABC}, \\
|\Phi^\mp\rangle_{ABC} &= \frac{1}{\sqrt{2}}(|001\rangle \pm |110\rangle)_{ABC}.
\end{align*}
\]

Here the subscripts \( A, B, \) and \( C \) represent the particles belonging to the three parties, say Alice, Bob, and Charlie, respectively. Suppose that the original GHZ state transmitted among the three parties is \( |\Phi^+\rangle_{ABC} \). If a bit-flipping error takes place on the particle in this GHZ state after it is transmitted in a noisy channel, the three-particle system is in the state \( |\Phi^+\rangle_{ABC} \). We label that a bit-flipping error occurs on the first particle. If \( |\Phi^+\rangle \) becomes \( |\Phi^-\rangle \), there is a phase-flipping error. Sometimes, both a bit-flipping error and a phase-flipping error will take place on a three-particle quantum system transmis-
To purify three-particle entangled systems, the task requires correcting both bit-flipping errors and phase-flipping errors on the quantum systems.

B. bit-flipping error correction

Suppose that Alice, Bob, and Charlie share an ensemble \( \rho \) after the transmission of particles, i.e.,
\[
\rho = F|\Phi^+\rangle\langle \Phi^+| + (1 - F)|\Phi^{-}\rangle\langle \Phi^{-}|.
\]
It means that there is a bit-flipping error on the quantum system with a probability of \( 1 - F \). Here \( F(\geq \frac{1}{2}) \) is the fidelity of the quantum systems transmitted. For correcting this error, the three parties divide their quantum systems in the ensemble \( \rho \) into many groups and each group is composed of a pair of three-photon quantum systems, same as the first multipartite entanglement purification protocol by Murao et al. \([27]\) in 1998. We label each group with \( A_1 B_1 C_1 A_2 B_2 C_2 \) (the two three-photon quantum systems \( A_1 B_1 C_1 \) and \( A_2 B_2 C_2 \)).

![Diagram of QND process](image)

**FIG. 2:** The principle of the bit-flipping error correction with QND. The 45° wave plate \( R_{45} \) is used to transform the state \( |0\rangle \) to \( \sqrt{2}|0\rangle + |1\rangle \) and \( |1\rangle \) to \( \sqrt{2}|0\rangle - |1\rangle \). \( M \) represents a single-photon measurement with the basis \( Z \). We denote this process \( P_1 \).

The principle of our scheme for correcting a bit-flipping error is shown in Fig. 2. The state of the two quantum systems \( A_1 B_1 C_1 A_2 B_2 C_2 \) can be viewed as the mixture of four pure states, i.e., \( |\Phi^+\rangle \otimes |\Phi^+\rangle \) with a probability of \( F^2 \), both \( |\Phi^+\rangle \otimes |\Phi^{-}\rangle \) and \( |\Phi^{-}\rangle \otimes |\Phi^+\rangle \) with an equal probability of \( F(1 - F) \), and \( |\Phi^+\rangle \otimes |\Phi^{-}\rangle \) with a probability of \( (1 - F)^2 \). For each group, Alice takes her two photons \( A_1 A_2 \) to pass through the setup shown in Fig. 2. The photon \( A_1 \) enters the upper spatial mode and the photon \( A_2 \) enters the down spatial mode. So do the other parties Bob and Charlie. After the QNDs, the three parties compare the parity of their photons. They only keep the groups for which all the three parties get an even parity. After these operations, the quantum systems are in a new mixed state which is composed of the two states
\[
|\phi\rangle = \frac{1}{\sqrt{2}}((000000) + |111111\rangle)_{A_1 B_1 C_1 A_2 B_2 C_2}
\]
with a probability of \( \frac{1}{2}F^2 \) and
\[
|\phi_1\rangle = \frac{1}{\sqrt{2}}((100100) + |011011\rangle)_{A_1 B_1 C_1 A_2 B_2 C_2}
\]
with a probability of \( \frac{1}{4}(1 - F)^2 \) as the two cross-combinations \( |\Phi^+\rangle \otimes |\Phi^+\rangle \) and \( |\Phi^{-}\rangle \otimes |\Phi^+\rangle \) never lead all the three parties to have the same parity.

After the rotation \( R_{45} \) on each photon in the down spatial mode, the three parties measure their photons out of the down spatial modes with the basis \( Z \). The wave plate \( R_{45} \) is used to rotate the horizontal and vertical polarizations by 45° (this task can be completed by a half-wave plate whose orientation is 22.5°), i.e., it acts as a Hadamard (\( H \)) gate,
\[
|0\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle),
\]
\[
|1\rangle \rightarrow \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).
\]

After the rotations, \( |\phi\rangle \) becomes
\[
|\phi\rangle' = \frac{1}{\sqrt{2}}((000)(1\sqrt{2})^{\otimes 3}|0\rangle + |1\rangle)^{\otimes 3}
\]
\[
+|111(1\sqrt{2})^{\otimes 3}|0\rangle - |1\rangle)^{\otimes 3}
\]
and \( |\phi_1\rangle \) becomes
\[
|\phi_1\rangle' = \frac{1}{\sqrt{2}}((100)(1\sqrt{2})^{\otimes 3}|0\rangle - |1\rangle)(0\rangle + |1\rangle)^{\otimes 2}
\]
\[
+|011(1\sqrt{2})^{\otimes 3}|0\rangle + |1\rangle)(0\rangle - |1\rangle)^{\otimes 2}.
\]

After the measurements on the photons \( A_2, B_2, \) and \( C_2 \), the three parties will obtain the state \( |\Phi^+\rangle_{A_1 B_1 C_1} \) with a fidelity of \( F' = \frac{F^2 + F^2 (1 - F)}{F^2 + F^2 (1 - F)} > F \) if their outcome is \( |000\rangle_{A_2 B_2 C_2}, |011\rangle_{A_2 B_2 C_2}, |101\rangle_{A_2 B_2 C_2} \) or \( |110\rangle_{A_2 B_2 C_2} \). If their outcome is \( |001\rangle, |010\rangle, |100\rangle \) or \( |111\rangle \), they need only to flip the phase of the quantum systems kept and get the same result above. This task can be accomplished with a 90° rotation on a photon in each three-photon quantum system. In detail, one of the three parties let his photon pass through a half-wave plate whose orientation is 90°.

In the process above, each party chooses the phase shift \( \theta \) for purification and gets the even parity \( |HH\rangle \) or \( |VV\rangle \). Another outcome for each one of \( |\Phi^+\rangle \otimes |\Phi^+\rangle \) is an odd parity, i.e., \( |000111\rangle_{A_1 B_1 C_1 A_2 B_2 C_2} \) or \( |110000\rangle_{A_1 B_1 C_1 A_2 B_2 C_2} \). The two photons of each party are in the state \( |HV\rangle \) or \( |VH\rangle \), which leads to a phase shift 20 or 0, respectively. These photons are discarded in the process above. In this way, the yield of this scheme
is half of the original protocol proposed by Murao et al. [27] with CNOT gates. However, if we choose a proper material for the QND and make $\theta = \pi$, we cannot distinguish the phase shift $2\theta$ and 0. In this time, the three parties can also keep their photons when they all get an odd parity. The state of the six photons becomes

$$|\phi\rangle^o = \frac{1}{\sqrt{2}}(|000111\rangle + |111000\rangle)_{A_1B_1C_1A_2B_2C_2}$$

(12)

with a probability of $\frac{1}{2}F^2$ and

$$|\phi_1\rangle^o = \frac{1}{\sqrt{2}}(|001110\rangle + |110001\rangle)_{A_1B_1C_1A_2B_2C_2}$$

(13)

with a probability of $\frac{1}{2}(1 - F)^2$. With a bit-flipping operation on each photon in the quantum system $A_2B_2C_2$, we get the same result as that with an even parity, which will double the yield.

For correcting the bit-flipping errors in multipartite entangled quantum systems, we can follow the same step of that for the three-particle GHZ-state systems. We should only increase the number of the QND equipments. For instance, for $N$-particle quantum systems whose original states are

$$|\Phi^+\rangle_N = \frac{1}{\sqrt{2}}(|00\cdots0\rangle + |11\cdots1\rangle),$$

(14)

if a bit-flipping error occurs on the first particle, the $N$ parties choose the same phase shift $\theta$ after their QNDs and the state of a group of the quantum systems kept (two GHZ-state quantum systems) becomes

$$|\phi\rangle_{2N} = \frac{1}{\sqrt{2}}(|00\cdots0\cdots00\rangle + |11\cdots11\cdots111\rangle)$$

(15)

with a probability of $\frac{1}{2}(1 - F)^2$ and

$$|\phi_1\rangle_{2N} = \frac{1}{\sqrt{2}}(|10\cdots010\cdots0\rangle + |01\cdots010\cdots1\rangle)$$

(16)

with a probability of $\frac{1}{2}(1 - F)^2$. After a 45° rotation on each photon in the second $N$-particle quantum system, Eq. (15) becomes

$$|\phi\rangle'_{2N} = \frac{1}{\sqrt{2}}(|0\cdots0\rangle (\frac{1}{\sqrt{2}})^\otimes N |0\rangle + |1\rangle)\otimes N$$

$$+ |1\cdots1\rangle (\frac{1}{\sqrt{2}})^\otimes N |0\rangle - |1\rangle)\otimes N)$$

(17)

and Eq. (16) becomes

$$|\phi_1\rangle'_{2N} = \frac{1}{\sqrt{2}}(|10\cdots0\rangle (\frac{1}{\sqrt{2}})^\otimes N |0\rangle - |1\rangle)\otimes (N-1)$$

$$+ |01\cdots1\rangle (\frac{1}{\sqrt{2}})^\otimes N |0\rangle + |1\rangle)\otimes (N-1)).$$

(18)

After the measurements on the photons in the second quantum system with the basis $Z$, we will get the state $|\Phi^+\rangle_N$ with a fidelity of $\frac{\sqrt{2}}{1 - F}$ if the number of $|1\rangle$ is even; otherwise, we get the state $|\Phi^-\rangle_N$ with the same fidelity if the number of the outcomes $|1\rangle$ is odd. Here

$$|\Phi^-\rangle_N = \frac{1}{\sqrt{2}}(|00\cdots0\rangle - |11\cdots1\rangle).$$

(19)

With a phase-flipping operation, the state $|\Phi^-\rangle_N$ is transformed into the state $|\Phi^\perp\rangle_N$.

In essence, the process above is used to purify the bit-flipping error occurring on the first particle. Those on the other particles can also be corrected in the same way and one will get the same result above easily. That is, they can correct the bit-flipping errors in the state $\rho_N$, here

$$\rho_N = F_0|\Phi^+\rangle\langle\Phi^+| + F_1|\Phi^+_1\rangle\langle\Phi^+_1| + F_2|\Phi^+_2\rangle\langle\Phi^+_2|$$

$$+ \cdots + F_N|\Phi^+_N\rangle\langle\Phi^+_N|.\quad (20)$$

C. phase-flipping error correction

A phase-flipping error cannot be corrected directly, different from a bit-flipping error, but it can be transformed into a bit-flipping error with H operations. With a H operation on each photon, the states shown in Eq. (1) are transformed into the following ones

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle + |01\rangle + |11\rangle),$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |01\rangle + |10\rangle + |11\rangle),$$

$$|\Psi^+_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |01\rangle - |01\rangle - |10\rangle),$$

$$|\Psi^-_1\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |01\rangle - |10\rangle - |11\rangle),$$

$$|\Psi^+_2\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle + |10\rangle - |11\rangle),$$

$$|\Psi^-_2\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |01\rangle + |10\rangle - |11\rangle),$$

$$|\Psi^+_3\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |01\rangle - |10\rangle + |11\rangle),$$

$$|\Psi^-_3\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |01\rangle - |10\rangle + |11\rangle).\quad (21)$$

From Eq. (21), one can see that the transformation between phase-flipping errors and bit-flipping errors in three-particle GHZ states is more complex than that in Bell states [22, 23, 24]. We can use the equipment shown in Fig. 2 to purify the states in Eq. (21) directly. That is, we cannot exploit simply H operations to complete the transformation between phase-flipping errors and bit-flipping errors in three-particle GHZ states perfectly, different from Bell states. Fortunately, the eight states can be divided into two groups. In the GHZ states with the superscript +, the number of $|1\rangle$ is even such as $|\Psi^+, \Psi^+_1, \Psi^+_2, \Psi^+_3\rangle$. In the other group, the number of $|1\rangle$ is odd such as $|\Psi^-, \Psi^-_1, \Psi^-_2, \Psi^-_3\rangle$, and
can be viewed as the mixture of four pure states, i.e., $|\Psi^+\rangle \otimes |\Psi^+\rangle$, $|\Psi^+\rangle \otimes |\Psi^-\rangle$, $|\Psi^-\rangle \otimes |\Psi^+\rangle$, and $|\Psi^-\rangle \otimes |\Psi^-\rangle$, similar to the case for the bit-flipping error correction. After passing through the QND of $P_2$ shown in Fig.3, the three parties measure the phase shifts of their coherent beams with X homodyne measurements and keep the six photons if their phase shifts all are $\theta$; otherwise, they discard their six photons. By choosing the samples with even parities, the cross-combinations $|\Psi^+\rangle \otimes |\Psi^-\rangle$ and $|\Psi^-\rangle \otimes |\Psi^+\rangle$ will never appear. The remaining items are

$$|\varphi\rangle = \frac{1}{2}(000000 + |011011\rangle + |101101\rangle + |110110\rangle)$$

(23)

with a probability of $\frac{1}{4}F^2$ and

$$|\varphi'\rangle = \frac{1}{2}(001001 + |010010\rangle + |100100\rangle + |111111\rangle)$$

(24)

with a probability of $\frac{1}{4}(1 - F)^2$. Eq. (23) and Eq. (24) are both six-photon entangled states. In order to get the three-photon GHZ state $|\Psi^+\rangle$, each party needs to measure this photon out of the low spatial mode of the setup $P_2$ with the basis $X = \{\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\}$. That is, the three parties rotate the three photons $A_2$, $B_2$ and $C_2$ by 45°, which will complete the transformations

$$|\varphi\rangle \rightarrow \frac{1}{4\sqrt{2}}(000)_{A_2}B_{2}C_{2}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle)_{A_1}B_{1}C_{1}$$

$$|\varphi\rangle \rightarrow \frac{1}{4\sqrt{2}}|001\rangle_{A_2}B_{2}C_{2}(|000\rangle - |001\rangle - |010\rangle + |011\rangle + |100\rangle - |101\rangle - |110\rangle + |111\rangle)_{A_1}B_{1}C_{1}$$

(25)

and then they measure their photons $A_2$, $B_2$ and $C_2$ with the basis $Z$.

From Eqs. (25) and (26), one can see that the three parties will get the state $|\Psi^+\rangle_{A_1}B_{1}C_{1}$ with some uni-
inary operations if the outcome of the measurements on the three particles \( A_2, B_2, \) and \( C_2 \) is \( |000\rangle A_2 B_2 C_2, |011\rangle A_2 B_2 C_2, |101\rangle A_2 B_2 C_2, \) or \( |110\rangle A_2 B_2 C_2 \), which takes place with a probability of \( \frac{1}{2} F^2 \); otherwise, they get the state \( |\Psi^-\rangle_{A_2 B_2 C_2} \) with a probability of \( \frac{1}{2} (1 - F)^2 \). The three parties can transform the states \( |\Psi^+\rangle \) and \( |\Psi^-\rangle \) into \( |\Phi^+\rangle \) and \( |\Phi^-\rangle \), respectively, by adding a Hadamard transformation (45° rotations) on each photon in the first three-photon quantum system \( A_1 B_1 C_1 \). In this way, they will get a new mixed entangled state with the fidelity of \( \frac{F^2}{2} \), same as that for bit-flipping error correction.

In essence, in the process of purifying the phase-flipping error in the quantum systems, the parties of quantum communication first transform the phase-flipping errors into the bit-flipping errors and then correct them by comparing their parities. Although the transformation on multi-particle GHZ states makes them more complex than Bell states, the principles of the bit-flipping error correction and the phase-flipping error correction for multi-particle GHZ states with parity checks are similar to those for Bell states [20]. The difference is just that Bell states are more symmetrical than multi-particle GHZ states under a Hadamard transformation on each particle, which makes the entanglement purification of Bell states easier than that of multi-particle GHZ states. Same as the bit-flipping error correction, the parties can also exploit proper QNDs to improve their yield.

### III. Multipartite Entanglement Purification with Polarizing Beam Splitters

So far, there are mainly three types of principles for the entanglement purification of two-photon Bell states. One is based on CNOT gates, which is the pioneer for entanglement purification. The second one is based on PBSs, which is more feasible than the first one at present if it is used to improve partially the entanglement of the entangled quantum systems transmitted although its yield is in principle half of that with CNOT gates. The third type is based on parity checks with cross-Kerr nonlinearities. It is more feasible than the first type and has the same yield as the latter. We will show that these differences exist for the entanglement purification of multi-particle GHZ states yet.

The principle of the entanglement purification of multi-particle entangled quantum systems with PBSs is similar to that with QNDs discussed above. We call it MPBS protocol. It also contains two steps: a bit-flipping error correction \( (P_1) \) and a phase-flipping error correction \( (P_2) \). We describe the first step with an example of purifying an ensemble \( \rho \) shown in Eq. (35) and the second step with \( \rho' \) shown in Eq. (22) below. The multipartite entanglement purification for other cases are the same as that for an ensemble \( \rho \) with or without a little modification.

The principle of the bit-flipping error correction on three-particle GHZ states with PBSs is shown in Fig.4.

![FIG. 4: The principle of the bit-flipping error correction on three-particle GHZ states with PBSs and sophisticated single-photon detectors, similar to the Pan’s protocol [24]. Each party has a setup for the bit-flipping error correction, and the three parties choose the six-mode events to ensure that they all get an even parity by classical communication.](image)

Each party of quantum communication first lets his two photons coming from two quantum systems \( (A_1 B_1 C_1 \) and \( A_2 B_2 C_2 \) ) pass through the setup shown in Fig.4 from the two spatial modes, respectively, and then they pick up the events in which there is one and only one photon in each spatial mode (call it a six-mode event), similar to the four-mode events in Ref. [24]. The quantum systems kept are in the mixture of the state

\[
|\phi\rangle = \frac{1}{\sqrt{2}} (|000000\rangle + |111111\rangle)_{A_1 B_1 C_1 A_2 B_2 C_2} \tag{27}
\]

with a probability of \( \frac{1}{2} F^2 \) and

\[
|\phi_1\rangle = \frac{1}{\sqrt{2}} (|100100\rangle + |011010\rangle)_{A_1 B_1 C_1 A_2 B_2 C_2} \tag{28}
\]

with a probability of \( \frac{1}{2} (1 - F)^2 \), as same as the case in which all the three parties get an even parity in the entanglement purification of the bit-flipping errors with QNDs in Sec. [III]. In this way, the three parties can get a new ensemble of the fidelity of \( F' \) with some single-photon measurements and unitary operations, same as that with QNDs. The phase-flipping error correction can also be
accomplished with the setup shown in Fig. 5 by picking up only the six-mode events. The other processes are same as those with QNDs except for exploiting sophisticated single-photon detectors to distinguish the six-mode events from others.

IV. DISCUSSION AND SUMMARY

In our scheme, we detail the multipartite entanglement purification with two steps: one is the bit-flipping error correction and the other is the phase-flipping error correction. When entangled qubits are transmitted in a practical channel, it is possible to take place both a bit-flipping error and a phase-flipping error on qubits. A simple example is called the "Werner-type" state \( |\phi^+\rangle \), i.e.,

\[
\rho_w = x|\phi^+\rangle\langle \phi^+| + \frac{1-x}{2^N} I. \tag{29}
\]

Its fidelity is

\[
f_\rho = \langle \phi^+| \rho_w |\phi^+\rangle = x + (1-x)/2^N. \tag{30}
\]

If we want to purify this "Werner-type" state for obtaining the maximally entangled state \( |\phi^+\rangle \), the bit-flipping error correction \( P_1 \) and the phase-flipping error correction \( P_2 \) are both needed. In the Murao’s protocol \([27]\), they use \( P_1 + P_2 \) to purify this mixed state. That is, the whole process for purification should be \( P_1 P_2 P_1 P_2 P_1 \ldots \). They found that the \( P_1 + P_2 \) protocol is not optimal for two-particle quantum systems, so it may be not optimal for the purification of multipartite entangled systems. In our protocol, we do not use the \( P_1 + P_2 \) process, the order of \( P_1 \) and \( P_2 \) is arbitrary. However, we do not know which order is the optimal one. The purification of a "Werner-type" state is more complicated than that with a single error correction, a bit-flipping error or a phase-flipping one. We cannot get a deterministic expression for describing the iteration of the fidelity of ensembles kept like those in Refs. \([22, 23]\). It should be studied with some numerical methods according to the noise of the channel.

In the MPBS protocol, the parties only keep the events in which each spatial mode has one and only one photon, which requires each party to possess at least a sophisticated single-photon detector. At present, sophisticated single-photon detectors are not feasible. However, cross-Kerr nonlinearities provide a good way for the parity check of the polarization states of two photons. This feature can be used to construct a QND for entanglement purification of multipartite quantum systems. In our QND protocol, the QND acts as not only the role of a CNOT gate but also that of a photon-number detector, which makes the process for entanglement purification can be iterated perfectly. As it does not require a CNOT gate with linear optical elements and sophisticated single-photon detectors, this protocol is more convenient than the MPBS protocol in practical applications.

Certainly, cross-Kerr nonlinear can be used to construct a CNOT gate \([37]\). With CNOT gates, multipartite entanglement purification can be completed with the Murao’s protocol \([27]\). In fact, parity gates is enough for entanglement purification \([24]\) and it requires less quantum resources largely, compared with CNOT gates \([37]\). In our scheme, we use cross-Kerr nonlinear to construct a QND which acts as the role of parity check and photon number detector. One can also exploit other QNDs \([39, 40, 41]\) to accomplish these tasks.

Same as all existing multipartite entanglement purification protocols \([27, 28, 29]\) (as well as the entanglement purification protocols for two-particle systems \([22, 23, 24, 27, 26]\)), our scheme requires the parties possess the capability of storing the entangled photons in principle for improving the fidelity of multipartite GHZ-state quantum systems. At present, storing a quantum state for a long time is not easy. The parties can exploit optical delays to complete the task of storing a quantum state for a short time. On the other hand, our scheme requires some Hadamard gates on photons, same as the first multipartite entanglement purification protocol by Murao et al. \([27]\). Of course, the task is in principle not difficult to be accomplished with a half-wave plate whose orientation is 22.5°.

In summary, we have presented a multipartite entanglement purification protocol for quantum systems in a GHZ state. The task of entanglement purification can be accomplished with two steps. The first step is to decrease the rate of bit-flipping errors and the other is used for phase-flipping errors. In our protocol, we use a QND to check the parity of the polarization states of two photons. Each QND detector acts as the role of both a CNOT gate and a photon-number detector, which makes this protocol feasible for the iteration of purification. With a weak cross-Kerr medium, the parties of quantum communication can keep the events in which they all get an even parity for a pair of multipartite entangled quantum systems. In this time, this protocol has the same yield as that with PBSs and sophisticated single-photon detectors. If the parties can choose a proper cross-Kerr nonlinear medium and a strong coherent beam, they can also exploit the events in which they all get an odd parity. With this modification, this protocol has the same yield as that with CNOT gates. Compared with the Murao’s protocol \([27]\), this protocol provides a practical way to realize entanglement purification of multipartite entangled quantum systems which are very useful in a long-distance quantum communication.

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