A parabolic Monge-Ampère type equation of Gauduchon metrics

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Abstract. We prove the long time existence and uniqueness of solution to a parabolic Monge-Ampère type equation on compact Hermitian manifolds. We also show that the normalization of the solution converges to a smooth function in the smooth topology as $t$ approaches infinity which, up to scaling, is the solution to a Monge-Ampère type equation. This gives a parabolic proof of the Gauduchon conjecture based on the solution of Székelyhidi, Tosatti and Weinkove to this conjecture.

1. Introduction

Let $(M, \alpha)$ be a compact complex Hermitian manifold with dim$_{\mathbb{C}} M = n \geq 2$. Then the real $(1,1)$ form associated to the Hermitian metric $\alpha$ (denoted by itself) is defined by

$$\alpha = \sqrt{-1} \alpha_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$  

The Hermitian metric $\alpha$ is called Kähler if $d\alpha = 0$, Astheno-Kähler (see [24]) if $\partial \bar{\partial} \alpha^{n-2} = 0$, balanced (see [29]) if $d\alpha^{n-1} = 0$, Gauduchon (see [13]) if $\partial \bar{\partial} \alpha^{n-1} = 0$, and strongly Gauduchon (see [30]) if $\bar{\partial} \alpha^{n-1}$ is $\partial$-exact.

If $(M, \alpha)$ is a Kähler manifold, then Yau’s solution [49] to the Calabi conjecture (see [2] for a parabolic proof using the estimates in [49]) says that for any given smooth positive volume form $\sigma$ on $M$ satisfying $\int_M \sigma = \int_M \alpha^n$, there exists a unique Kähler metric $\omega$ with $[\omega] = [\alpha] \in H^2(M, \mathbb{R})$ such that

$$\omega^n = \sigma.$$  

Moreover, Yau’s theorem is equivalent to the following statement. Given any $\Psi \in c_1(M)$, the first Chern class, we can find a unique Kähler metric $\omega$ with $[\omega] = [\alpha] \in H^2(M, \mathbb{R})$ such that

$$\text{Ric}(\omega) = \Psi,$$  

where $\text{Ric}(\omega)$ is the Ricci form of the Kähler metric $\omega$ and can be defined as

$$\text{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log \det \omega.$$  

It is natural to ask whether there hold similar results when $M$ does not admit a Kähler metric, but only a Hermitian metric $\alpha$. If there is no any restriction on the class of Hermitian metrics, then we can solve (1.1) trivially by a conformal change of metric.

Tosatti and Weinkove [39] proved that for any Hermitian metric $\alpha$ on $M$, there exists a Hermitian metric $\omega$ of the form $\omega = \alpha + \sqrt{-1} \partial \bar{\partial} u$ with $u \in C^\infty(M, \mathbb{R})$ such that (1.2) holds. (cf. [3, 17, 38]  



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and see [17] for a parabolic proof based on [39]). Chu, Tosatti and Weinkove [6] recently proved similar results on almost Hermitian manifolds, based on which Chu [5] gave a parabolic proof.

Gauduchon [13] showed that there exists a unique Gauduchon metric up to scaling (when $n \geq 2$) in the conformal class of any Hermitian metric $\alpha$.

Motivated by Yau’s solution to the Calabi conjecture, Gauduchon [14, Chapter IV.5] proposed the following conjecture.

**Conjecture 1.1** (Gauduchon [14]; 1984). Let $M$ be a compact Hermitian manifold and $\Psi$ be a closed real $(1,1)$ form on $M$ with $[\Psi] = c_{1BC}(M) \in H_{BC}^{1,1}(M, \mathbb{R})$. Then there exists a Gauduchon metric satisfying (1.2), where $\text{Ric}(\omega)$ is the Chern-Ricci form of $\omega$ and can be expressed as

$$\text{Ric}(\omega) = -\sqrt{-1} \partial \bar{\partial} \log \det \omega.$$ 

Here $H_{BC}^{1,1}(M, \mathbb{R})$ is the Bott-Chern cohomology group and its definition can be found in Section 2.

This conjecture can also be restated as follows.

**Conjecture 1.2.** Let $M$ be a compact complex Hermitian manifold and $\sigma$ be a smooth positive volume form. Then there exists a Gauduchon metric $\omega$ on $M$ satisfying (1.1).

Very recently, Székelyhidi, Tosatti and Weinkove [36] solved this conjecture based on their previous works [35, 42, 43]. More precisely, they proved

**Theorem 1.3** (Székelyhidi, Tosatti and Weinkove [36]; 2015). Let $M$ be a compact complex Hermitian manifold with a Gauduchon metric $\alpha_0$, and $\Psi$ be a closed real $(1,1)$ form on $M$ with $[\Psi] = c_{1BC}(M, \mathbb{R}) \in H_{BC}^{1,1}(M, \mathbb{R})$. Then there exists a Gauduchon metric $\omega$ with $[\omega] = [\alpha_0^{n-1}]_A \in H_{A}^{n-1,n-1}(M, \mathbb{R})$ solves (1.2).

Here $H_{A}^{n-1,n-1}(M, \mathbb{R})$ is the Aeppli cohomology group and its definition can be found in Section 2.

Tosatti and Weinkove [43] deduced that to obtain this theorem it is sufficient to solve the following partial differential equation, which was also independently introduced by Popovici [32]. They sought a Hermitian metric $\omega$ on $M$ with the property that

$$\omega^{n-1} = \alpha_0^{n-1} + \partial \left( \frac{\sqrt{-1}}{2} \partial \bar{\partial} u \wedge \alpha^{n-2} \right) + \bar{\partial} \left( \frac{\sqrt{-1}}{2} \bar{\partial} \bar{\partial} u \wedge \alpha^{n-2} \right)$$

$$= \alpha_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u \wedge \alpha^{n-2} + \text{Re} \left( \sqrt{-1} \partial \bar{\partial} u \wedge \bar{\partial} (\alpha^{n-2}) \right),$$

where $u \in C^\infty(M, \mathbb{R})$ and $\alpha$ is a background Gauduchon metric. If $\alpha_0$ is Gauduchon, then the metric $\omega$ is also Gauduchon. Note that there exists an $F \in C^\infty(M, \mathbb{R})$ such that $\text{Ric}(\alpha) = \Psi + \sqrt{-1} \partial \bar{\partial} F$. Now we can deduce that (1.3) is equivalent to

$$\omega^n = e^F + b \alpha^n$$

with some constant $b \in \mathbb{R}$. Note that Székelyhidi, Tosatti and Weinkove [36] solved the following equivalent equation (their paper solved a family of Monge-Ampère type equations including this one)

$$\log \left( \frac{\det \omega^{n-1}}{\det \alpha} \right) = \log \left( \frac{\det \omega}{\det \alpha} \right)^{n-1} = (n-1)(F+b),$$
It is easy to deduce that (1.7) is equivalent to the following flow with
\begin{equation}
\psi = \frac{1}{(n-1)!} * \alpha_0^{n-1}, \Delta u = \zeta^i \partial_i \partial_j u, \tag{1.5}
\end{equation}
where \( \zeta \) is the normalization of \( (n-1) \) plurisubharmonic (Psh) functions (see [21, 22]).

More remarks about Conjecture 1.1 (also Conjecture 1.2) and applications of the methods in the proof of Theorem 1.3 can be found in [36] and references therein.

In this paper, we consider a parabolic version of (1.4), analogs to [2, 17, 5], as follows.

\begin{equation}
\frac{\partial}{\partial t} u = \log \left( \frac{\varpi + \frac{1}{n-1} \left( (\Delta u) \alpha - \sqrt{-1} \partial \bar{\partial} u \right) + Z(u) }{\alpha^n} \right) - \psi \tag{1.7}
\end{equation}
with \( u(0) = u_0 \in C^\infty(M, \mathbb{R}) \), and (1.5) and (1.6) with \( u \) evolved by the time \( t \geq 0 \).

It is easy to deduce that (1.7) is equivalent to the following flow

\begin{align*}
\frac{\partial}{\partial t} \alpha_t^{n-1} &= -(n-1) \left( \text{Ric}(\alpha_t) - \text{Ric}(\alpha) + \frac{1}{n-1} \sqrt{-1} \partial \bar{\partial} \psi \right) \wedge \alpha^{n-2} \\
&\quad + (n-1) \text{Re} \left( \sqrt{-1} \partial \right) \left( \log \frac{\alpha_t^n}{e^{\psi / (n-1) \alpha^n}} \right) \wedge \bar{\partial} (\alpha^{n-2}) \biggr),
\end{align*}
with initial metric \( \alpha(0) = \alpha_0^{n-1} + \sqrt{-1} \partial \bar{\partial} u_0 \wedge \alpha^{n-2} + \text{Re} \left( \sqrt{-1} \partial u_0 \wedge \bar{\partial} (\alpha^{n-2}) \right) > 0 \).

This flow preserves the Gauduchon condition if the initial metric \( \alpha_0 \) is Gauduchon. Indeed, taking \( \partial \bar{\partial} \) on both sides of (1.8), we can obtain

\begin{equation}
\frac{\partial}{\partial t} \partial \bar{\partial} \alpha_t^{n-1} = (n-1) \partial \bar{\partial} (\partial \gamma + \bar{\partial} \gamma) = 0,
\end{equation}

where

\[ \gamma = \frac{\sqrt{-1}}{2} \bar{\partial} \left( \log \frac{\alpha_t^n}{e^{\psi / (n-1) \alpha^n}} \right) \wedge \alpha^{n-2}, \]

as required. When \( n = 2 \) this flow can be seen as the “twisted” Chern-Ricci flow (cf. [9, 17, 18, 19, 40, 41, 45, 48, 50]). We show

**Theorem 1.4.** Let \((M, \alpha_0)\) be a compact Hermitian manifold with \( \dim \mathbb{C} M = n \geq 3 \) and \( \alpha \) be a Gauduchon metric on \( M \). Then there exists a unique solution \( u \) to (1.7) on \( M \times [0, \infty) \) and if we define the normalization of \( u \) by

\[ \tilde{u}(x, t) := u(x, t) - \frac{1}{\text{Vol}_\alpha(M)} \int_M u(y, t) \alpha^n(y), \]

then \( \tilde{u} \) converge smoothly to a function \( \tilde{u}_\infty \) as \( t \to \infty \), and \( \tilde{u}_\infty \) is the unique solution to (1.4) by taking \( \psi = (n-1)F \), up to adding a constant \( b \in \mathbb{R} \) defined as in (8.3).
This gives a parabolic proof of the Gauduchon conjecture based on the solution of Szekelyhidi, Tosatti and Weinkove to this conjecture [36]. It is analogous to H.-D. Cao’s parabolic proof of the Calabi conjecture [2] based on Yau’s work [49], and to Gill’s result in the Hermitian case [17] based on Tosatti and Weinkove [39].

**Remark 1.5.** We can also consider another kind of parabolic flow of Gauduchon metrics, a revised version of Gill [20], given by

\[ \frac{\partial}{\partial t} \alpha_t^{n-1} = -(n-1)\text{Ric}(\alpha_t) \wedge \alpha^{n-2} + (n-1)\text{Re} \left( \sqrt{-1} \partial \left( \frac{\alpha_t^n}{\alpha^n} \right) \wedge \overline{\partial}(\alpha^{n-2}) \right) \]  

(1.8)

with \( \alpha(0) = \alpha_0 \). Note that in the case \( n = 2 \) this flow is exactly the Chern-Ricci flow.

If the initial metric \( \alpha_0 \) is Gauduchon and \( \alpha \) is Astheno-Kähler, then it is easy to deduce that the flow (1.8) preserves the Gauduchon condition. Taking Aeppli cohomology, we can deduce

\[ \frac{d}{dt} [\alpha_t^{n-1}]_A = -(n-1)\text{Ric}(\alpha) \wedge \alpha^{n-2} ]_A, \]

the right side of which is independent of time \( t \). Note that Gill [20] also introduced a parabolic flow suggested by Tosatti and Weinkove [42, 43]

\[ \frac{\partial}{\partial t} \alpha_t^{n-1} = -(n-1)\text{Ric}(\alpha_t) \wedge \alpha^{n-2} \]  

(1.9)

which also preserves the Gauduchon condition under the same assumptions as above. However, if we take Aeppli Cohomology on the both sides of (1.9), then we get

\[ \frac{d}{dt} [\alpha_t^{n-1}]_A = -(n-1)\text{Ric}(\alpha_t) \wedge \alpha^{n-2} ]_A, \]

the right side of which is dependent on the time \( t \).

If \( M \) is non-Kähler Calabi-Yau manifold, i.e., \( c^{BC}_1(M) = 0 \), then we can take \( \alpha \) to be the Chern-Ricci flat Hermitian metric by Tosatti and Weinkove [39] and then it is easy to see that the flow (1.8) also preserves the Gauduchon condition.

Using the method in this paper originated from [42, 43, 36], it follows that there exists a unique solution to (1.8) on \( M \times [0, T) \), where

\[ T := \sup \left\{ t \geq 0 : \exists \psi \in C^\infty(M, \mathbb{R}) \text{ such that} \right\} \]

\[ \Phi_t + \sqrt{-1} \partial \overline{\partial} \psi \wedge \alpha^{n-2} + \text{Re} \left( \sqrt{-1} \partial \psi \wedge \overline{\partial}(\alpha^{n-2}) \right) > 0, \]

with

\[ \Phi_t := \alpha_t^{n-1} - t(n-1)\text{Ric}(\alpha) \wedge \alpha^{n-2}. \]

This solves a conjecture revised from [20, Conjecture 1.2].

**Remark 1.6.** Let \( M \) be a compact complex manifold with two Hermitian metrics \( \omega_0 \) and \( \omega \), and \( \dim_{\mathbb{C}} M = n \). Then for any \( F \in C^\infty(M, \mathbb{R}) \), Tosatti and Weinkove [42, 43] proved that there exists a unique pair \((u, b)\) with \( u \in C^\infty(M, \mathbb{R}) \) and \( b \in \mathbb{R} \) such that

\[ \text{det} \left( \omega_0^{n-1} + \sqrt{-1} \partial \overline{\partial} u \wedge \omega^{n-2} \right) = e^{F+b} \text{det} \left( \omega^{n-1} \right), \]

(1.10)

with

\[ \omega_0^{n-1} + \sqrt{-1} \partial \overline{\partial} u \wedge \omega^{n-2} > 0, \sup_M u = 0. \]
If $\omega$ is Kähler, then this is a conjecture of Fu and Xiao [12] (see also [10, 11, 31]). We can also consider the parabolic version of (1.10)

$$(1.11) \quad \frac{\partial}{\partial t} u = \log \left( \omega_h + \frac{1}{n-1} ((\Delta u) \omega - \sqrt{-1} \partial \bar{\partial} u) \right)^n \Omega,$$

for a fixed volume form $\Omega$ and $\omega_h = \frac{1}{(n-1)!} \omega_0^{n-1}$, where $\ast$ is respect to $\omega$. Tosatti and Weinkove [42, Remark 1.5] conjectured that the solutions to (1.11) exist for all time and converge (after normalization) to give solutions to (1.10) up to a constant. Using the method in this paper, we can confirm this conjecture.

The paper is organized as follows. In section 2, we collect some basic concepts about Hermitian manifolds. In section 3, we give the uniform bounds of the normalization of the solution $\tilde{u}$ to (1.7). In section 4 and Section 5, we give the second and first order priori estimates of the solution $u$ to (1.7) respectively. In Section 6, we prove the long time existence and uniqueness of the equation (1.7) claimed as in the first part of Theorem 1.4. In Section 7, we give the Harnack inequality for the equation (3.1) which will be used to prove the convergence of the normalization of the solution to (1.7) claimed as in the second part of Theorem 1.4 in Section 8.

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2. Preliminaries

In this section, to avoid confusions, we collect some preliminaries about Hermitian geometry which will be used in this paper.

Let $(M, J, g)$ be a Hermitian manifold with $\dim_{\mathbb{C}} M = n$, $J$ be the canonical complex structure and $g$ be the Riemannian metric with $g(JX, JY) = g(X, Y)$ for any vector fields $X, Y \in \mathfrak{X}(M)$. Then in the real local coordinates $(x^1, \cdots, x^{2n})$ with

$$J \left( \frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^{n+i}}, \quad J \left( \frac{\partial}{\partial x^{n+i}} \right) = -\frac{\partial}{\partial x^i}, \quad i = 1, \cdots, n,$$

we have

$$g_{ij} = g_{n+i,n+j}, \quad g_{i,n+j} = -g_{n+i,j}, \quad g_{\alpha\beta} = g_{\beta\alpha}, \quad i, j = 1, \cdots, n, \quad \alpha, \beta = 1, \cdots, 2n,$$

where $g_{\alpha\beta} = g(\partial/\partial x^\alpha, \partial/\partial x^\beta)$. We can define a real 2-form $\omega$ by

$$\omega(X, Y) := g(JX, Y), \quad \forall X, Y \in \mathfrak{X}(M).$$

This form is determined uniquely by $g$ and vise versa. The volume element is usually defined by

$$dV = \sqrt{\det(g_{\alpha\beta})} dx^1 \wedge \cdots \wedge dx^{2n}.$$

For any $p$ form $\varphi$

$$\varphi = \frac{1}{p!} \varphi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p},$$

the star $\ast$ operator is defined by

$$(2.1) \quad \psi \wedge \ast \varphi = \frac{1}{p!} \psi_{j_1 \cdots j_p} g^{j_1 \ell_1} \cdots g^{j_p \ell_p} \varphi_{\ell_1 \cdots \ell_p} \ast 1 = \frac{1}{p!} \psi_{j_1 \cdots j_p} g^{j_1 \ell_1} \cdots g^{j_p \ell_p} \varphi_{\ell_1 \cdots \ell_p} dV.$$
where \( \psi = \frac{1}{p!} \psi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \) is another \( p \) form. We define inner product by

\[
\langle \psi, \varphi \rangle := \frac{1}{p!} \psi_{j_1 \cdots j_p} g^{j_1 \ell_1} \cdots g^{j_p \ell_p} \varphi_{\ell_1 \cdots \ell_p}.
\]

It is easy to deduce that

\[
* \varphi = \frac{1}{p!(2n-p)!} \sqrt{\det(g_{\alpha \beta})} \delta_{j_1 \cdots j_p k_1 \cdots k_{2n-p}} g^{j_1 \ell_1} \cdots g^{j_p \ell_p} \varphi_{\ell_1 \cdots \ell_p} \, dx^{k_1} \wedge \cdots \wedge dx^{k_{2n-p}},
\]

where \( \delta_{j_1 \cdots j_p k_1 \cdots k_{2n-p}} \) is the general Kronecker symbol. It is easy to get

\[
* * \varphi = (-1)^{2np+p} \varphi,
\]

where we do not omit \( 2np \) since the dimension can actually be any positive integer \( m \).

In the complex local coordinates

\[
z = (z^1, \ldots, z^n) = (x^1 + \sqrt{-1} x^{n+1}, \ldots, x^n + \sqrt{-1} x^{2n}),
\]

we denote \( \partial_i = \partial / \partial z^i, \partial_j = \partial / \partial \bar{z}^j, \) \( i, j = 1, \ldots, n \). Then we have

\[
g = \sum_{i,j=1}^n g_{ij} (dz^i \otimes d\bar{z}^j + d\bar{z}^j \otimes dz^i),
\]

\[
\omega = \sqrt{-1} \sum_{i,j=1}^n g_{ij} dz^i \wedge d\bar{z}^j,
\]

where \( g_{ij} = \frac{1}{2} (g_{i,j} + \sqrt{-1} g_{i,n+j}) \). Then we can choose \( \frac{\omega^n}{n!} \) as the volume element. Therefore, for any \((p,q)\) form

\[
\phi = \frac{1}{p!q!} \phi_{i_1 \cdots i_p j_1 \cdots j_q} \, dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q},
\]

using (2.1) and (2.2), we can deduce (see for example [28])

\[
* \phi = \frac{(-1)^n (n-1)^{p+q(n-1)/2}}{(n-p)! (n-q)! p! q!} \det g_{i_1 \cdots i_p j_1 \cdots j_q} \phi_{i_1 \cdots i_p j_1 \cdots j_q} \, g^{i_1 k_1} \cdots g^{i_p k_p} g^{j_1 k_1} \cdots g^{j_q k_q}
\]

\[
\delta_{\ell_1 \cdots \ell_p k_1 \cdots k_{2n-p}} \delta_{\ell_1 \cdots \ell_p b_1 \cdots b_{2n-p}} \, dz^{a_1} \wedge \cdots \wedge dz^{a_q} \wedge d\bar{z}^{b_1} \wedge \cdots \wedge d\bar{z}^{b_{2n-p}}
\]

and

\[
\varpi \wedge * \phi = \frac{1}{p! q!} \varpi_{i_1 \cdots i_p j_1 \cdots j_q} \phi_{i_1 \cdots i_p j_1 \cdots j_q} \, g^{i_1 k_1} \cdots g^{i_p k_p} g^{j_1 k_1} \cdots g^{j_q k_q} \frac{\omega^n}{n!},
\]

where \( \varpi = \frac{1}{p! q!} \varpi_{i_1 \cdots i_p j_1 \cdots j_q} \, dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_q} \) is another \((p,q)\) form and \( \det g = \det(g_{ij}) \).

We also define inner product by

\[
\langle \varpi, \phi \rangle := \frac{1}{p! q!} \varpi_{i_1 \cdots i_p k_1 \cdots k_q} \phi_{i_1 \cdots i_p j_1 \cdots j_q} \, g^{i_1 k_1} \cdots g^{i_p k_p} g^{j_1 k_1} \cdots g^{j_q k_q},
\]

Note that

\[
* 1 = \frac{\omega^n}{n!}, \quad * \phi = * \phi,
\]

where the second equality shows that \( * \) is a real operator. From (2.3), we can deduce

\[
* * \phi = (-1)^{p+q} \phi.
\]

The following basic concepts of positivity can be found in for example [7, Chapter III].

A \((p,p)\) form \( \varphi \) is said to be positive if for any \((1,0)\) forms \( \gamma_j, 1 \leq j \leq n-p \), then

\[
\varphi \wedge \sqrt{-1} \gamma_1 \wedge \gamma_1 \wedge \cdots \wedge \sqrt{-1} \gamma_{n-p} \wedge \gamma_{n-p}
\]
is a positive \((n,n)\) form. Any positive \((p,p)\) form \(\varphi\) is real, i.e., \(\overline{\varphi} = \varphi\). In particular, in the local coordinates, a real \((1,1)\) form

\[(2.6) \quad \phi = \sqrt{-1} \phi_{ij} dz^i \wedge d\overline{z}^j\]

is positive if and only if \((\phi_{ij})\) is a semi-positive Hermitian matrix and we denote \(\det \phi := \det (\phi_{ij})\).

Similarly, a real \((n-1,n-1)\) form

\[(2.7) \quad \psi = \sqrt{-1} (n-1) \sum_{i,j=1}^{n} (-1)^{i+j+1} \psi_{ij} dz^i \wedge \cdots \wedge d\overline{z}^n d\overline{z}^1 \wedge \cdots \wedge d\overline{z}^{n-1}\]

is positive if and only if \((\psi_{ij})\) is a semi-positive Hermitian matrix and we denote \(\det \psi := \det (\psi_{ij})\).

We remark that for \((1,1)\) and \((n-1,n-1)\) forms one also has the stronger notion of positive definiteness, which is to require that the Hermitian matrix \((\phi_{ij})\) (resp. \((\psi_{ij})\)) is positive definite. In this paper, we need this stronger notion.

For a positive \((1,1)\) form \(\phi\) defined as in \((2.6)\), we can deduce a positive \((n-1,n-1)\) form

\[(2.8) \quad \frac{\phi^{n-1}}{(n-1)!} = \sqrt{-1} (n-1) \sum_{k,l=1}^{n} (-1)^{\frac{n(n+1)}{2}+k+l+1} \det (\phi_{ij}) \phi^{kl}_{ij} \]

\[dz^1 \wedge \cdots \wedge d\overline{z}^k \wedge \cdots \wedge d\overline{z}^n d\overline{z}^1 \wedge \cdots \wedge d\overline{z}^{n-1},\]

where \((\phi^{kl})\) is the inverse matrix of \((\phi_{ij})\), i.e., \(\sum_{l=1}^{n} \phi^{kl} \phi_{kj} = \delta^k_l\). Hence we have

\[(2.9) \quad \det \left( \frac{\phi^{n-1}}{(n-1)!} \right) = \det \phi^{n-1}.\]

Furthermore, we have

**Lemma 2.1.** Let \((M,g)\) be a complex \(n\)-dimensional Hermitian manifold. Then there exists a bijection from the space of positive definite \((1,1)\) forms to positive definite \((n-1,n-1)\) forms, given by

\[\phi \mapsto \frac{\phi^{n-1}}{(n-1)!}.\]

The above bijection can be found in [29] and proved by orthonormal basis. We can also use \((2.7)\) and \((2.8)\) gives the explicit formulae involved (cf. [42]).

For a real \((1,1)\) form \(\phi\) defined as in \((2.6)\) (no need to be positive), \((2.5)\) implies

\[(2.10) \quad \ast \phi = \sqrt{-1} (n-1) \sum_{k,l=1}^{n} (-1)^{\frac{n(n-1)}{2}+n+k+l+1} \det (\omega) \phi^{kl}_{ij} \]

\[dz^1 \wedge \cdots \wedge d\overline{z}^{k} \wedge \cdots \wedge d\overline{z}^{n} d\overline{z}^1 \wedge \cdots \wedge d\overline{z}^{n-1},\]

where \(\phi^{kl} = g^{kl} \phi_{ij} \phi^{ji}\). Hence, if \(\xi\) is another real \((1,1)\) form with \(\det \xi \neq 0\), then we can deduce

\[(2.11) \quad \frac{\det \ast \phi}{\det \ast \xi} = \frac{\det \phi}{\det \xi}.\]

We need the following useful formulae and the proofs are direct and complicated computation.
The Christoffel symbols, torsion and curvature of Chern connection (see for example [40]) are
\[ \Gamma^k_{ij} = g^{kl} \partial_i g_{lj}, \quad \mathbf{T}^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji}, \]
\[ R^q_{ij} = - \partial_j \mathbf{T}^q_i - R^q_{ji} = R^q_{ij} g^{qp}. \]

Denote
\[ T^k_{ij} := T^k_{ij} g_{k\ell} = g^{\ell q} (\partial_i g_{j\ell} - \partial_j g_{i\ell}) g_{k\ell} = \partial_i g_{j\ell} - \partial_j g_{i\ell}. \]

For a (1, 0) form \( \alpha = a_\ell dx^\ell \), define covariant derivative \( \nabla_i a_\ell \) by
\[ \nabla_i a_\ell := \partial_i a_\ell - \Gamma^p_{\ell i} a_p. \]

Then we can deduce
\[ \nabla_i \nabla_j^\ell a_\ell = - R^q_{ij} \nabla^p a_p, \quad \nabla_i \nabla_j a_{\ell m} = R^q_{ij} \nabla^p g_{q\ell m}, \]
where \( R^q_{ij} = R^q_{ij} g^{qp} g_{q\ell m}. \) For any \( u \in C^\infty(M, \mathbb{R}) \), we have
\[ \nabla_i u = \partial_i u =: u_i, \quad \nabla_\ell u = \partial_\ell u =: u_\ell, \quad \nabla_\ell \nabla_i u = \partial_\ell \partial_i u =: u_{\ell i}, \quad \nabla_i \nabla_j u = - T^k_{ij} u_\ell. \]

Using (2.13), we can get the following commutation formulae:
\[ \nabla_\ell \nabla_i u_\ell = \nabla_\ell \nabla_\ell u_i - R^q_{ij} u_\ell, \quad \nabla_\ell \nabla_{\ell m} u_{\ell n} = \nabla_\ell u_{\ell m} - T^p_{m \ell} u_{\ell p}, \]
\[ \nabla_\ell \nabla_{\ell m} u_{\ell n} = \nabla_\ell \nabla_{\ell n} u_{\ell m} + R^q_{ij} u_{\ell p} - R^q_{ij} u_{\ell p} - T^p_{m \ell} \nabla_\ell u_{\ell p} - T^p_{m \ell} \nabla_\ell u_{\ell p} - T^p_{m \ell} \nabla_\ell u_{\ell p}. \]

Lemma 2.3. For any \( u \in C^\infty(M, \mathbb{R}) \), we have
\[ *(\partial u \wedge \mathbf{\Omega}(\omega^{n-2})) = (n-2)! \left( u_\ell T^\ell_{1q} - T^\ell_{1q} g^{ij} u_i g_{jp} - g_{jq} g^{ij} u_k T^\ell_{1q} \right) dz^p \wedge d\mathbf{\Omega}. \]

Note that \( * \) is a real operator, from Lemma 2.3, we can deduce the formula in [36] as follows.

Corollary 2.4. For any \( u \in C^\infty(M, \mathbb{R}) \), we have
\[ Z(u) := \frac{1}{(n-1)!} \left\{ \Re \left( \sqrt{-1} \partial u \wedge \mathbf{\Omega}(\omega^{n-2}) \right) \right\} = \sqrt{-1} Z_{\mathbf{\Omega}} dz^p \wedge d\mathbf{\Omega}, \]
where
\[ Z_{\mathbf{\Omega}} = \frac{1}{2(n-1)} \left[ u_\ell T^\ell_{1q} - T^\ell_{1q} g^{ij} u_i g_{jp} - g_{jq} g^{ij} u_k T^\ell_{1q} \right]. \]

We can also write
\[ Z_{\mathbf{\Omega}} = Z^1_{\mathbf{\Omega}} u_i + Z^2_{\mathbf{\Omega}} u_\ell, \]
where
\[ Z^1_{\mathbf{\Omega}} = \frac{1}{2(n-1)} \left( \delta_\ell^i T^\ell_{1q} - T^\ell_{1q} g^{ij} u_k - g_{jq} g^{ij} T^\ell_{1q} \right). \]

Note that \( Z(u) \) is linear in \( \nabla u \). The following useful lemma is simple and we will use it without pointing it out again and again (cf.[20]).

Lemma 2.5. For any \( f \in C^\infty(M, \mathbb{R}) \), at the point where \( \sqrt{-1} \partial f \leq (\geq)0 \), we have
\[ (\Delta_g f) \omega - \sqrt{-1} \partial \overline{\partial} f \leq (\geq)0, \]
where \( \Delta_g f = g^{ij} \partial_i \partial_j f \). At the point where \( \nabla f = 0 \), we have \( Z(f) = 0. \)
To end this section, we introduce some terminology concerning cohomology classes of \((n-1, n-1)\) forms. Define the Aeppli cohomology group (see [43])

\[ H^{n-1,n-1}_A(M, \mathbb{R}) := \{ \partial\overline{\partial}-\text{closed real } (n-1, n-1) \text{ forms} \} \]

This space is naturally in duality with the finite dimensional Bott-Chern cohomology group with the nondegenerated pairing

\[ H^{n-1,n-1}_A(M, \mathbb{R}) \otimes H^{1,1}_{BC}(M, \mathbb{R}) \rightarrow \mathbb{R} \]

given by wedge product and integration over on \(M\) (see [1]), where

\[ H^{1,1}_{BC}(M, \mathbb{R}) = \{ \text{d-closed real (1, 1) forms} \} \]

\[ \{ \sqrt{-1} \partial \overline{\partial} \psi : \psi \in C^\infty(M, \mathbb{R}) \} \].

For any \(u \in C^\infty(M, \mathbb{R})\), define

\[ \gamma := \frac{\sqrt{-1}}{2} \partial u \wedge \chi^{n-2}, \]

where \(\chi\) is a real (1, 1) form. Then we have

\[ \beta_u := \partial \gamma + \overline{\partial} \gamma = \sqrt{-1} \partial \overline{\partial} u \wedge \chi^{n-2} + \text{Re} \left( \sqrt{-1} \partial u \wedge \overline{\partial} (\chi^{n-2}) \right). \]

\(\beta_u\) is \(\partial\overline{\partial}\)-closed. Indeed, it is the \((n-1, n-1)\) part of the \(d\)-exact \((2n-2)\) form \(d \left( d^c u \wedge \chi^{n-2} \right)\), where

\[ d^c = \frac{\sqrt{-1}}{2} (\overline{\partial} - \partial) \]

with \(dd^c = \sqrt{-1} \partial \overline{\partial}\). Let \(\alpha\) and \(\alpha'\) be Hermitian metrics on \(M\). Then

\[ \sqrt{-1} \partial \overline{\partial} \left( \log \frac{\alpha^n}{\alpha'^n} \right) \wedge \chi^{n-2} + \text{Re} \left[ \sqrt{-1} \partial \left( \log \frac{\alpha^n}{\alpha'^n} \right) \wedge \overline{\partial} (\chi^{n-2}) \right], \]

is well-defined \(\partial\overline{\partial}\)-closed since

\[ \log \frac{\alpha^n}{\alpha'^n} \in C^\infty(M, \mathbb{R}). \]

3. Preliminary estimates

Define a linear operator

\[ L(\varphi) := \Theta \partial_i \partial_j \varphi + \tilde{g}^0 Z(\varphi)_{ij} = \Theta \partial_i \partial_j \varphi + \text{tr}_{\omega} Z(\varphi) \]

with

\[ \Theta \tilde{\nabla}^i = \frac{1}{n-1} \left( (\text{tr}_{\omega} \alpha) \alpha \tilde{\nabla}^i - \tilde{g}^i \right) > 0. \]

Obviously, \(L\) is a second order elliptic operator. Noting that \(L\) is the linearized operator of (1.7), standard parabolic theory implies that there exists a smooth solution \(u\) to (1.7) on \([0, T)\), where \([0, T)\) is the maximal time interval with \(T \in (0, \infty]\). We will prove \(T = \infty\). First, we give a preliminary estimate as follows.

**Lemma 3.1.** Let \(u\) be the solution to (1.7) on \(M \times [0, T)\). Then there exists a uniform constant \(C\), i.e., depending only on the initial data on \(M\), such that

\[ \sup_{M \times [0, T)} \left| \frac{\partial u}{\partial t} (x, t) \right| \leq C. \]
Proof. From (1.7), we get the evolution equation for \( \dot{u} := \frac{\partial}{\partial t} u \)

\[
\frac{\partial}{\partial t} \dot{u} = L(\dot{u}).
\]  

(3.3)

By the maximum principle, we get

\[
\sup_{M \times [0, T)} \left| \frac{\partial u}{\partial t} (x, t) \right| \leq \sup_{M} \left| \frac{\partial u}{\partial t} (x, 0) \right| + \left\| \psi \right\|_{L^\infty(M)},
\]

as required.

Next, using Lemma 3.1, we can get the estimate of \( \tilde{u} \).

**Proposition 3.2.** Let \( u \) be the solution to (1.7) on \( M \times [0, T) \). Then there exists a uniform constant \( C \) such that

\[
\sup_{M \times [0, T)} |\tilde{u}(x, t)| \leq C.
\]  

(3.4)

Proof. We can rewrite (1.7) as

\[
\log \left( \frac{\varpi + \frac{1}{n-1} \left[ (\Delta u_0) \alpha - \sqrt{-1} \partial \bar{\partial} u_0 \right] + Z(u_0)}{\alpha^n} \right)^n = \psi + \dot{u}.
\]  

(3.5)

Then by [43, Theorem 1.6] (see also [35, Remark 12]), there exists a constant \( C' \) depending only on the initial data on \( M \) and \( \sup_M |\psi + \dot{u}| \), such that

\[
\sup_M |u(x, t) - u(y, t)| \leq C'
\]  

(3.6)

By Lemma 3.1, if follows that \( \sup_M |\psi + \dot{u}| \) is uniformly bounded, and hence \( C' \) in (3.6) is a uniform constant. Since we have \( \int_M \tilde{u} \alpha^n = 0 \) by definition, there exists a point \((y, t)\) such that \( \tilde{u}(y, t) = 0 \) and hence for any \((x, t) \in M \times [0, T)\), we get

\[
|\tilde{u}(x, t)| = |\tilde{u}(x, t) - \tilde{u}(y, t)| = |u(x, t) - u(y, t)| \leq C,
\]

(3.7)

as required.

\[\square\]

4. Second order estimate

We can use the ideas from [36] in the elliptic setting to the prove the second estimate.

**Theorem 4.1.** Let \( u \) be the solution to (1.7) on \( M \times [0, T) \). Then there exists a uniform \( C > 0 \) such that

\[
\sup_{M \times [0, T)} |\sqrt{-1} \partial \bar{\partial} u|_{\alpha} \leq CK,
\]  

(4.1)

where

\[
K = 1 + \sup_{M \times [0, T)} |\nabla u|^2_{\alpha}.
\]  

\[10\]
We need some preliminaries. For any real (1, 1) form $\xi$, we define
\[
P_\alpha(\xi) := \frac{1}{n-1} \left( \operatorname{tr}_\alpha \xi \alpha - \xi \right) = \frac{1}{(n-1)!} \times (\xi \wedge \alpha^{n-2}).
\]
Note that $\operatorname{tr}_\alpha \xi = \operatorname{tr}_\alpha (P_\alpha(\xi))$ and
\[
\xi = (\operatorname{tr}_\alpha (P_\alpha(\xi))) \alpha - (n-1) P_\alpha(\xi).
\]
Denote
\[
x_{\bar{\alpha}} = (\operatorname{tr}_\alpha \alpha) \alpha - (n-1) \alpha,
\]
with $P_\alpha(x) = \varpi$, and
\[
W(u)_{\bar{\alpha}} = (\operatorname{tr}_\alpha Z(u)) \alpha - (n-1) Z(u)_{\bar{\alpha}} = W_{\bar{\alpha}}^\varpi u_p + W_{\bar{\alpha}}^\varpi u_p.
\]
Then we define
\[
g_{\bar{\alpha}} = x_{\bar{\alpha}} + u_{\bar{\alpha}} + W(u)_{\bar{\alpha}}
\]
with $P_\alpha(g_{\bar{\alpha}}) = \tilde{g}_{\bar{\alpha}}$. 

In orthonormal coordinates for $\alpha$ at any given point, it follows that the component $Z_{\bar{\alpha}}$ is independent of $u_\bar{\alpha}$ and $u_j$, and that $\nabla_i Z_{\bar{\alpha}}$ is independent of $u_i, u_\bar{\alpha}, u_\bar{\alpha}, \nabla_i u_i$. Indeed, in such local coordinates, from (2.18), we have
\[
Z_{\bar{\alpha}} = \frac{1}{2(n-1)} \left( \sum_{p \neq i} \sum_{k \neq i} u_p \overline{T}_{pk}^k + \sum_{p \neq i} \sum_{k \neq i} u_p \overline{T}_{pk}^k \right),
\]

\[
\nabla_i Z_{\bar{\alpha}} = \frac{1}{2(n-1)} \left( \sum_{p \neq i} \sum_{k \neq i} \left( \nabla_i u_p \overline{T}_{pk}^k + u_p \overline{T}_{pk}^k \right) + \sum_{p \neq i} \sum_{k \neq i} \left( u_p \overline{T}_{pk}^k + u_p \overline{T}_{pk}^k \right) \right),
\]

\[
\nabla_i Z_{\bar{\alpha}} = \frac{1}{2(n-1)} \left( - \sum_{k \neq i} \left( u_i \overline{T}_{jk}^k + u_k \overline{T}_{jk}^k \right) - \sum_{k \neq i} \left( u_\bar{\alpha} \overline{T}_{jk}^k + u_{\bar{\alpha}} \overline{T}_{jk}^k \right) \right),
\]
where $i \neq j$ are fixed indices and in the last equality, we use the skew-symmetry of the torsion, as desired.

Furthermore, $\nabla_i \nabla_{\bar{\alpha}} Z_{\bar{\alpha}}$ is independent of $\nabla_i u_i, \nabla_{\bar{\alpha}} u_\bar{\alpha}, \nabla_{\bar{\alpha}} u_i$ and $\nabla_i u_\bar{\alpha}$. For any index $p$, $\nabla_i Z_{\bar{\alpha}}$ is independent of $\nabla_i u_i$. In this following part of this section, we will use such properties directly and do not prove them again.

Denote $\left( \tilde{B}_i^j \right) = \left( \tilde{g}_{\bar{\alpha}} \alpha^{\bar{\alpha}} \right)$ which can be seen as the endomorphism of $T^{1,0} M$. This endomorphism is Hermitian with respect to the Hermitian metric $\alpha$, i.e., for any tangent vectors $X = X^i \partial_i$ and $Y = Y^j \partial_j$, we have
\[
\langle \tilde{B} X, Y \rangle_{\alpha} = \tilde{B}_i^k X^i \alpha^{k \bar{\alpha}} Y^\bar{\alpha} = \tilde{g}_{\bar{\alpha}} \alpha^{\bar{\alpha}} X^i \alpha^{k \bar{\alpha}} Y^\bar{\alpha} = \tilde{g}_{\bar{\alpha}} X^\bar{\alpha} Y^\bar{\alpha} = X^i \alpha_{\bar{\alpha}} \tilde{B}_i^{\bar{\alpha}} = \langle X, \tilde{B} Y \rangle_{\alpha}.
\]
We define
\[
\tilde{F}(\tilde{B}) = \log \det \tilde{B} = \log(\mu_1 \cdots \mu_n) =: \tilde{f}(\mu_1, \cdots, \mu_n),
\]
where $\mu_1, \cdots, \mu_n$ are the eigenvalues of $\left( \tilde{B}_i^j \right)$. Then (1.7) can be rewritten as
\[
(4.3) \quad \tilde{F}(\tilde{B}) = \dot{u} + \psi =: h.
\]
For $\dot{f}$ and $h$, we have
(i) \( \tilde{f} \) is defined on 
\[ \tilde{\Gamma} := \Gamma_n = \left\{ (x_1, \cdots, x_n) \in \mathbb{R}^n : x_i > 0, \ i = 1, \cdots, n \right\}. \]

(ii) \( \tilde{f} \) is symmetric, smooth, concave and increasing, i.e., \( \tilde{f}_i > 0 \) for all \( i \).

(iii) \( \sup_{\partial \tilde{\Gamma}} \tilde{f} \leq \inf_{M \times [0, T)} h. \)

(iv) For any \( \mu \in \tilde{\Gamma} \), we get \( \lim_{t \to \infty} \tilde{f}(t\mu) = \sup_{\tilde{\Gamma}} \tilde{f} = \infty. \)

(v) \( h \) is bounded on \( M \times [0, T) \) thanks to the estimate of \( |\dot{u}| \) in Lemma 3.1.

We also define 
\[ F(A) := \tilde{F}(\tilde{B}) =: f(\lambda_1, \cdots, \lambda_n), \]
where \((A_{ij}) = \left(g_{\pi \sigma}^{\tilde{B}_{ij}}\right)\), which is also an endomorphism of \( T^{1,0}M \) with respect to the Hermitian metric \( \alpha \), and \( \lambda_1, \cdots, \lambda_n \) are its eigenvalues. There exists a map 
\[ P : \mathbb{R}^n \longrightarrow \mathbb{R}^n, \quad \mu_k = \frac{1}{n-1} \sum_{i \neq k} \lambda_i, \]
induced by \( P_\alpha \) above. Then we have 
\[ f(\lambda_1, \cdots, \lambda_n) = \tilde{f} \circ P(\lambda_1, \cdots, \lambda_n) \]
defined on \( \Gamma := P^{-1}(\tilde{\Gamma}) \). Clearly, \( f \) satisfies the same conditions as \( \tilde{f} \). Then (4.3) can also be rewritten as
\[ (4.4) \quad F(A) = h. \]
We make some simple calculation about \( \tilde{f} \) and \( f \). Since
\[ (4.5) \quad \tilde{f}_i = \frac{1}{\mu_i}, \]
we can get
\[ (4.6) \quad f_i = \frac{1}{n-1} \sum_{k \neq i} \frac{1}{\mu_k}. \]
Suppose that \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \in \Gamma \). From the definition of \( P \), (4.5) and (4.6), we have
\[ 0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n, \]
\[ \tilde{f}_1 \geq \tilde{f}_2 \geq \cdots \geq \tilde{f}_n, \]
\[ 0 < f_1 \leq f_2 \leq \cdots \leq f_n, \]
\[ (4.7) \quad \sum_{k=1}^{n} \lambda_k f_k = \sum_{k=1}^{n} \mu_k \tilde{f}_k = n. \]
where we also use the fact that \( \left(\tilde{B}_{i\tilde{j}}\right) \) is positive definite. For \( k \geq 2 \), we have
\[ (4.8) \quad 0 < \frac{\tilde{f}_1}{n-1} \leq f_k \leq \tilde{f}_1. \]
and
\[ (4.9) \quad \tilde{f}_k \leq (n-1)f_1. \]
Proposition 4.2. For any \( x \in M \), choose orthonormal coordinates for \( \alpha \) at \( x \), with \( g \) defined as in (4.2) is diagonal with eigenvalues \((\lambda_1, \cdots, \lambda_n)\). Then we have
\[
|\lambda| \geq R,
\]
and there also holds two possibilities as follows.

(a) We have
\[
\sum_k f_k(\lambda) (\chi_k \bar{\kappa} - \lambda_k) - |\dot{u}| > \kappa \sum_k f_k(\lambda).
\]

(b) Or we have
\[
f_k(\lambda) > \kappa \sum_{i=1}^n f_i(\lambda)
\]
for all \( k = 1, 2, \cdots, n \).

In addition, we have \( \sum_{k=1}^n f_k(\lambda) > \kappa \). Here \( 0 < \kappa < 1 \) is a uniform constant.

Proof. By the Cauchy-Schwarz inequality and the inequality of arithmetic and geometric means, we get
\[
|\lambda|^2 \geq \frac{(\lambda_1 + \cdots + \lambda_n)^2}{n} \geq \frac{(\mu_1 + \cdots + \mu_n)^2}{n} \geq n(\mu_1 \cdots \mu_n)^\frac{2}{n} = ne^{2h/n} \geq R^2,
\]
where for the last inequality we use the fact that \( \tilde{F}(\tilde{B}) = \dot{u} + \psi \) is uniformly bounded by Lemma 3.1.

Since \( \chi_{k\bar{k}} = \left( \frac{\sum_{i=1}^n \omega_i^2}{n} \right) - (n-1)\omega_{k\bar{k}} \), (4.6) implies
\[
\sum_k f_k(\lambda) \chi_{k\bar{k}} \geq \sum_k \tilde{f}_k(\lambda) \omega_{k\bar{k}} = \sum_k \frac{1}{\mu_k} \omega_{k\bar{k}} > \tau \sum_k \frac{1}{\mu_k} = \tau \sum_k f_k(\lambda),
\]
where we use the fact that \( \omega \) has a uniform lower bound. If \( \lambda_1 \) is relatively large, i.e., \( \mu_1 \) is relatively small such that \( \frac{\tau}{\mu_1} > n + C \geq \sum_{k=1}^n \lambda_k f_k(\lambda) + |\dot{u}| \), where \( C \) is the uniform bound of \( \dot{u} \) in Lemma 3.1, then we have
\[
\sum_k f_k(\lambda) \chi_{k\bar{k}} \geq \frac{\tau}{2} \sum_k \frac{1}{\mu_k} + \frac{\tau}{2} \sum_k \frac{1}{\mu_k}
\geq \frac{\tau}{2} \sum_k \frac{1}{\mu_k} + \frac{\tau}{2\mu_1}
\geq \frac{\tau}{2} \sum_k \frac{1}{\mu_k} + n + C
\geq \frac{\tau}{2} \sum_k f_k(\lambda) + \sum_{k=1}^n \lambda_k f_k(\lambda) + |\dot{u}|,
\]
as required. Otherwise, there exists a large constant \( A > 1 \) such that \( \mu_2 \leq \cdots \leq \mu_n \leq A\mu_1 \) since \( \dot{u} + \psi = \log(\mu_1 \cdots \mu_n) \) is bounded and we just need to prove
\[
f_n \geq \cdots \geq f_1 = \sum_{i=2}^n \frac{1}{\mu_i} \geq \kappa \sum_{i=1}^n \frac{1}{\mu_i}.
\]
To get this, we just need choose $0 < \kappa < 1$ such that

\[
(1 - \kappa) \sum_{i=2}^{n} \frac{1}{\mu_i} \geq \frac{(1 - \kappa)(n - 1)}{A\mu_1} \geq \frac{\kappa}{\mu_1},
\]

as required.

In addition, using the inequality of arithmetic and geometric means, we can get

\[
\sum_{k=1}^{n} f_k(\lambda) = \sum_{k=1}^{n} \frac{1}{\mu_k} \geq n(\mu_1 \cdot \cdots \cdot \mu_n)^{-1/n} = ne^{-h/n} \geq \kappa,
\]

where we use the fact that $h$ is uniformly bounded by Lemma 3.1. □

Now we need some basic formulae for the derivatives of eigenvalues (see for example [34]).

**Lemma 4.3** (Spruck [34]). The derivative of the eigenvalue $\lambda_i$ at a diagonal matrix $(A_{ij})$ with distinct eigenvalue are

\[
\lambda_i^{pq} = \delta_{pi}\delta_{qi}, \quad \lambda_i^{pq,rs} = (1 - \delta_{ip})\frac{\delta_{iq}\delta_{ir}\delta_{ps}}{\lambda_i - \lambda_p} + (1 - \delta_{ir})\frac{\delta_{is}\delta_{ip}\delta_{rq}}{\lambda_i - \lambda_r},
\]

where

\[
\lambda_i^p = \frac{\partial \lambda_i}{\partial A_{ij}}, \quad \lambda_i^{pq,rs} = \frac{\partial^2 \lambda_i}{\partial A_{ij} \partial A_{rs}}.
\]

**Lemma 4.4** (Gerhardt [16]). If $F(A) = f(\lambda_1, \cdots, \lambda_n)$ in terms of a symmetric function of the eigenvalues, then at a diagonal matrix $(A_{ij})$ with distinct eigenvalues we have

\[
F_{ij} = \delta_{ij} f_i, \quad F_{ij,rs} = f_{ir}\delta_{ij}\delta_{rs} + \frac{f_i - f_j}{\lambda_i - \lambda_j}(1 - \delta_{ij})\delta_{is}\delta_{jr},
\]

where

\[
F_{ij} = \frac{\partial F}{\partial A_{ij}}, \quad F_{pq,rs} = \frac{\partial^2 F}{\partial A_{ij} \partial A_{rs}}.
\]

These formulae make sense even if the eigenvalues are not distinct, since if $f$ is symmetric then $F$ is a smooth function on the space of matrices. In particular, we have $f_i \to f_j$ as $\lambda_i \to \lambda_j$. If $f$ is concave and symmetric, then we have (see [34]) that $f_i - f_j \leq 0$. In particular, if $\lambda_i \leq \lambda_j$, then we have $f_i \geq f_j$. Also it follows that

\[
F_{ij,rs} \left( \nabla_k u_{ij} \right) \left( \nabla_k u_{rs} \right) = f_{ij} \left( \nabla_k u_{ij} \right) \left( \nabla_k u_{rs} \right) + \sum_{p \neq q} \frac{f_p - f_q}{\lambda_p - \lambda_q} |\nabla_k u_{pq}|^2
\]

\[
\leq f_{ij} \left( \nabla_k u_{ij} \right) \left( \nabla_k u_{ij} \right) + \sum_{p > 1} \frac{f_1 - f_p}{\lambda_1 - \lambda_p} |\nabla_k u_{pq}|^2.
\]

**Proof of Theorem 4.1.** We use the ideas from [36] in the elliptic setting. Since here we consider the parabolic equation, there are some new terms involved time $t$ which we need to estimate and we need Proposition 4.2 which is slightly different from [36, Proposition 2.3]. Also note that here
we use covariant derivatives instead of partial derivatives used in [36]. Hence we can be brief and point out the main differences in the following part. It is sufficient to prove
\[ \lambda_1 \leq CK. \]
Indeed, since \( \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{k} \mu_i > 0 \), if \( \lambda_1 \leq CK \) then so \( |\lambda_k|, k = 2, \cdots, n \), which implies (4.1).
To obtain this, we consider the function
\[ H = \log \lambda_1 + \phi \left( |\nabla u|_g^2 \right) + \varphi(u), \]
where
\[ \phi(s) = -\frac{1}{2} \log \left( 1 - \frac{s}{2K} \right), \quad \varphi(s) = D_1 e^{-D_2 s}, \]
with sufficiently large uniform constants \( D_1, D_2 > 0 \) to be determined later. Note that \( \phi \left( |\nabla u|_g^2 \right) \in [0, 2 \log 2] \)
and
\[ \frac{1}{4K} < \phi' < \frac{1}{2K}, \quad \phi'' = 2(\phi')^2. \]
We work at a point \((x_0, t_0)\) where \( H \) achieves its maximum. Without loss of generality, we assume \( 0 < t_0 < T \). Choose orthonormal complex coordinates such that \( x_0 \) is the origin, \( \alpha \) is a unit matrix and \( g \) is diagonal with \( \lambda_1 = g_{11} \). To make sure that \( H \) is smooth at this point, we fix a diagonal matrix \( B \) with \( B_{11} = 0 \), \( B_{22} < \cdots < B_{nn} \), and define \( \tilde{A} = A - B \) with eigenvalues denoted by \( \tilde{\lambda}_1, \cdots, \tilde{\lambda}_n \). Clearly, at this point \((x_0, t_0)\), we have
\[ \tilde{\lambda}_1 = \lambda_1, \quad \tilde{\lambda}_i = \lambda_i - B_{ii}, \quad i = 2, \cdots, n \]
and \( \tilde{\lambda}_1 > \cdots > \tilde{\lambda}_n \). Noting that \( \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \mu_i > 0 \), we can assume that \( B \) is small enough such that
\[ \sum_{i=1}^{n} \tilde{\lambda}_i > -1 \]
and
\[ (4.18) \]
\[ \sum_{p>1} \frac{1}{\lambda_1 - \lambda_p} \leq C \]
where \( C \) is a fixed constant depending only on \( n \). We give some remarks about \( B \). It can also be considered as an endomorphism of \( T^{1,0}M \), and is represented by a constant diagonal matrix \((B_{ij})\) in these local coordinates. Hence, we can deduce
\[ \nabla_\bar{z} B_r^s = \partial_{\bar{z}} B_r^s = 0, \quad \nabla_i \nabla_\bar{z} B_r^s = 0, \]
\[ \nabla_i B_r^s = \partial_i B_r^s - \Gamma^p_{ir} B_p^s + \Gamma^s_{ip} B_r^p = \Gamma^s_{ir} (B_r^r - B_s^s), \]
\[ \nabla_\bar{z} \nabla_i B_r^s = \partial_{\bar{z}} \nabla_i B_r^s = R_{\bar{z}r}^s (B_s^s - B_r^r). \]
Now consider the quantity
\[ \tilde{H} = \log \tilde{\lambda}_1 + \phi \left( |\nabla u|_g^2 \right) + \varphi(u), \]

\[ ^{1}\text{The author thanks Professor Valentino Tosatti suggesting him this point.} \]
which is smooth in this chart and attains its maximum at the point \((x_0, t_0)\). The following calculation is at this point. We may assume \(\lambda_1 \gg K > 1\). We use subscripts \(k\) and \(\bar{\tau}\) to denote the partial derivatives \(\partial/\partial z^k\) and \(\partial/\partial \bar{\tau}\). As in [36], we have

\[
(4.19) \quad \tilde{H}_q = \lambda_{1,q} \frac{\lambda_1}{\lambda_1} + \varphi' V_q + \varphi' u_q = 0, \quad \text{for } V_q := u_r u_{\tau_0} + u_{\tau} \nabla q u_r.
\]

\[
(4.20) \quad \tilde{H}_{q,\bar{\tau}} = \lambda_{1,q,\bar{\tau}} - \frac{\lambda_{1,q}^2}{\lambda_1^2} + \varphi' \left( u_r \nabla q u_{\tau_0} + u_{\tau} \nabla \tau \nabla q u_r + |\nabla q u_r|^2 + |u_{\tau q}|^2 \right)
+ \varphi'' |V_q|^2 + \varphi'' |u_q|^2 + \varphi' u_{\tau q}.
\]

From (4.13), we get

\[
(4.21) \quad \tilde{\lambda}_{1,p} = \tilde{\lambda}_1^{rs} \left( \alpha \tilde{n}^s \nabla_p g r_{\tau j} - \nabla_p B r^s \right) = \nabla_p g r_{1\bar{\tau}} - \nabla_p B_{1} = \nabla_p g r_{1\bar{\tau}},
\]

where we use the fact that \(\nabla_p B_{1} = 0\). Then using this formula and (4.13) and (4.14), we can deduce

\[
(4.22) \quad \tilde{\lambda}_{1,q} = \nabla q g r_{1\bar{\tau}} + \sum_{r > 1} \frac{|\nabla_q g r_{1\bar{\tau}}|^2 + |\nabla_q g r_{1\bar{\tau}}|^2}{\lambda_1 - \lambda_r} - \sum_{r > 1} \left( \frac{\nabla q B_{1}}{\lambda_1 - \lambda_r} \frac{1}{\nabla q g r_{1\bar{\tau}}} \right) \left( \nabla q g r_{1\bar{\tau}} \right) \left( \nabla q g r_{1\bar{\tau}} \right) \frac{1}{\lambda_1 - \lambda_r}.
\]

From (4.4) and (4.15), we have

\[
(4.23) \quad \dot{u}_k = F^{rs} \nabla_k \left( g r_{1\bar{\tau}} \alpha \tilde{n}^s \right) - \psi_k = F^{rs} \alpha \tilde{n}^s \nabla_k g r_{1\bar{\tau}} - \psi_k = F^{rs} \nabla_k g r_{1\bar{\tau}} - \psi_k.
\]

Also from (4.15), we know that \(F^{kk} = f_k\), \(\tilde{F}^{kk} = \tilde{f}_k\). For later use, we write \(F = \sum_{k=1}^{n} F^{kk} \geq \kappa\).

The formula (4.23) together with (4.15) and (4.16) implies

\[
(4.24) \quad \dot{u}_{k\bar{\tau}} = F^{rs,ab} \nabla_k \left( g r_{1\bar{\tau}} \alpha \tilde{n}^s \right) \nabla_{\bar{\tau}} \left( g a \alpha \tilde{n}^a \right) + F^{rs} \nabla_r \nabla_k \left( g r_{1\bar{\tau}} \alpha \tilde{n}^s \right) - \psi_{k\bar{\tau}}
= F^{rs,ab} \left( \alpha \tilde{n}^s \nabla_k g r_{1\bar{\tau}} \right) \left( \alpha \tilde{n}^a \nabla_{\bar{\tau}} g a \right) + F^{rs} \alpha \tilde{n}^s \nabla_{\bar{\tau}} \nabla_k g r_{1\bar{\tau}} - \psi_{k\bar{\tau}}
= F^{rs,ab} \left( \nabla_k g r_{1\bar{\tau}} \right) \left( \nabla_{\bar{\tau}} g a \right) + F^{rs} \nabla_{\bar{\tau}} \nabla_k g r_{1\bar{\tau}} - \psi_{k\bar{\tau}}.
\]

Combining (4.13), (4.23) and (4.24) gives

\[
(4.25) \quad \partial_t \hat{\lambda}_1 = \hat{\lambda}_1^{pq} \partial_t \left( g r_{1\bar{\tau}} \alpha \tilde{n}^q \right)
= \hat{\lambda}_1^{pq} \left( \dot{u}_{pq} + W^p_{rq} u_r + W^q_{pr} u_{\tau r} \right) - \psi_{1\bar{\tau}}
= W^r_{1\bar{\tau}} u_r + W^q_{1\bar{\tau}} u_{\tau r} + \dot{u}_{1\bar{\tau}}
= W^r_{1\bar{\tau}} \left( F^{pq} \nabla r g q - \psi_r \right) + W^q_{1\bar{\tau}} \left( F^{pq} \nabla \tau g q - \psi_{\tau} \right)
+ F^{rs,ab} \left( \nabla l g r_{1\bar{\tau}} \right) \left( \nabla_{\bar{\tau}} g a \right) + F^{pq} \nabla_{\bar{\tau}} \nabla_1 g q - \psi_{1\bar{\tau}}
\]
Thanks to (4.23) and (4.25), it follows

\begin{equation}
\partial_t \tilde{H} = \frac{\partial \lambda_1}{\lambda_1} + \phi'(u_\tau \tilde{u}_\tau + \dot{u}_r u_r) + \phi' \dot{u},
\end{equation}

where

- \( \frac{1}{\lambda_1} \left[ W_{1T}^{(r)} (F^{qq} \nabla_q g_{1q} - \psi_r) + \nabla_{g_{1q}} (F^{qq} \nabla_q g_{1q} - \psi_r) + F^{qq} \nabla_{1T} g_{1q} - \psi_{1T} \right] \)

- \( \frac{1}{\lambda_1} F^{rs,ab} (\nabla_1 g_{ab}) (\nabla_{1T} g_{ab}) \)

- \( \phi' u_r (F^{qq} \nabla_q g_{1q} - \psi_r) + \phi' \dot{u}_r (F^{qq} \nabla_q g_{1q} - \psi_r) + \phi' \dot{u} \).

Combining (4.20), (4.22) and (4.26), we get

\begin{equation}
0 \geq F^{qq} \tilde{H}_{q_\tau} - \partial_t \tilde{H}
\end{equation}

Using (2.15) implies

\begin{equation}
\nabla_q \nabla_q W_{1T} = u_r \nabla_q \nabla_q W_{1T}^{r} + u_r \nabla_q \nabla_q W_{1T}^{r} + (\nabla_q W_{1T}^{r}) (\nabla_q u_r) + W_{1T}^{r} \nabla_q \nabla_q u_r
\end{equation}

Since \( W_{1T} = W_{1T}^{r} u_r + \nabla_{q_{1T}} u_r \), using Ricci identity (2.15), we can get

\begin{align*}
\nabla_q W_{1T}^{r} &= u_r \nabla_q \nabla_q W_{1T}^{r} + u_r \nabla_q \nabla_q W_{1T}^{r} + (\nabla_q W_{1T}^{r} (\nabla_q u_r) + W_{1T}^{r} \nabla_q \nabla_q u_r
\end{align*}
which implies
\begin{equation}
F_{qq}^q \nabla_q \nabla_q W_{41} - F_{qq}^q \nabla_{q} W_{41} g_{q4} - F_{qq}^q \nabla_{q} g_{q4} \geq - C \left( \sum_r F_{qq}^q |\nabla_q u_r| + \lambda_1 F \right),
\end{equation}
where we use the fact that \( \lambda_1 \gg K > 1 \) and hence that \( |u_{4q}| \) can be controlled by \( \lambda_1 \).

Since
\[ W(u)_{4q} = (\text{tr}_\alpha Z(u)) \alpha_{4q} - (n - 1) Z(u)_{4q}, \]
(4.6) implies (as in [36])
\begin{equation}
F_{qq}^q \nabla_q \nabla_q W(u)_{4q} \leq C \left( \sum_r F_{qq}^q |\nabla_q u_r| + |\nabla_q g_{41}| + \lambda_1 F \right),
\end{equation}
From Young’s inequality, it follows that
\[
\sum_{r>1} \frac{|\nabla_q g_{r1}|^2 + |\nabla_q g_{r1}|^2}{\lambda_1 - \tilde{\lambda}_r} \leq \sum_{r>1} \frac{(\nabla_p B_r^1) (\nabla_{q} g_{r1}) + (\nabla_p B_1^r) (\nabla_{q} g_{r1})}{\lambda_1 - \tilde{\lambda}_r} \geq \frac{1}{2} \sum_{r>1} \frac{|\nabla_q g_{r1}|^2 + |\nabla_q g_{r1}|^2}{\lambda_1 - \tilde{\lambda}_r} - C,
\]
where we also use (4.18). Since
\[(n-1)\lambda_1 + \tilde{\lambda}_r \geq \sum_{r=1}^n \tilde{\lambda}_r > -1, \]
we get \((\lambda_1 - \tilde{\lambda}_r)^{-1} \geq (n\lambda_1 + 1)^{-1}\) for \( r > 1 \). Then Young’s inequality gives
\begin{equation}
\sum_{r>1} \frac{|\nabla_q g_{r1}|^2 + |\nabla_q g_{r1}|^2}{\lambda_1 - \tilde{\lambda}_r} \geq \frac{1}{2(n\lambda_1 + 1)} \sum_{r>1} \left( |\nabla_q g_{r1}|^2 + |\nabla_q g_{r1}|^2 \right) - C
\]
\[
\geq \frac{1}{4n\lambda_1} \sum_{r>1} \left( |\nabla_q g_{r1}|^2 + |\nabla_q g_{r1}|^2 \right) - C,
\]
where for the second inequality we use the fact that \( \lambda_1 > 1 \).

Using Ricci identity, we can obtain
\[
\nabla_{q1} V_{u1} = \nabla_{q1} V_{q1} u_{1\tau} - \nabla_{q1} T_{q1}^{p1} u_{p\tau}
\]
\[
= \nabla_{q1} V_{q1} u_{1\tau} - \nabla_{q1} V_{q1} u_{1\tau} - \nabla_{q1} V_{q1} W(u)_{1\tau} - \nabla_{q1} V_{q1} W(u)_{1\tau},
\]
which implies
\begin{equation}
- \frac{2}{\lambda_1} \text{Re} \left( F_{qq}^q \nabla_{q1} V_{u1\tau} \right) \geq - \frac{2 F_{qq}^q}{\lambda_1} \text{Re} \left( \nabla_{q1} V_{q1} u_{1\tau} \right) - C \left( \frac{1}{\lambda_1} \sum_r F_{qq}^q |\nabla_q u_r| + F \right),
\end{equation}
where we use the fact that \( |u_{ij}| \) can be controlled by \( \lambda_1 \). Then we have
\begin{equation}
\frac{\lambda_1}{(4.31)} + (4.32) \geq - \frac{2 F_{qq}^q}{\lambda_1} \text{Re} \left( \nabla_{q1} V_{q1} g_{14} \right) - C \left( \frac{1}{\lambda_1} \sum_r F_{qq}^q |\nabla_q u_r| + F \right),
\end{equation}
where we use that \( F \geq \kappa \) and hence absorb the constant \( C \) into \( C F \).
Substituting (4.28), (4.30), (4.29) and (4.33) into (4.27), it follows that

\[ 0 \geq - \frac{F^{qq}\lambda_1q}{\lambda_1^2} - \frac{1}{\lambda_1} F^{rs,ab} (\nabla_1 g_{r\tau}) (\nabla_\tau g_{ab}) \]

\[ - CF - \frac{C}{\lambda_1} \left( \sum_r F^{qq} |\nabla g_{ur}| + F^{qq} |\nabla g_{1T}| \right) \]

\[ + \phi' F^{qq} \left( u_r \nabla_\tau u_q + u_\tau \nabla_\tau \nabla_\tau u_r + |\nabla g_{ur}|^2 + |u_\tau|^2 \right) \]

\[ + \phi'' F^{qq}|V_q|^2 + \varphi'' F^{qq}|u_q|^2 + \varphi' F^{qq} u_q \]

\[ - \phi' u_r \left( F^{qq} \nabla_\tau g_{qr} - \psi_r \right) - \phi' u_r \left( F^{qq} \nabla_\tau g_{qr} - \psi_r \right) - \varphi' \dot{u}. \]

Using Ricci identity, we can get

\[ \phi' F^{qq} u_r \nabla_\tau u_q = \phi' F^{qq} u_r \left( \nabla_\tau u_q - T_q^{qr} u_p \right) \]

\[ = \phi' F^{qq} u_r \left( \nabla_\tau u_q - \nabla_\tau u_q - \nabla_\tau W(u)_{qr} - T_q^{qr} u_p \right), \]

which implies

\[ \phi' F^{qq} u_r (\nabla_\tau u_q - \nabla_\tau g_{qr}) \geq - \frac{C}{K^{1/2}} \left( \sum_r |u_q| F^{qq} + \sum_r |\nabla g_{ur}| F^{qq} + \mathcal{F} \right) \]

where we use the fact that \( \phi' \leq 1/(2K) \). Here we also use that \( |u_r| \) can be controlled by \( \lambda_1 \) and hence can also be controlled by \( |u_q| \).

Similarly, we can also deduce

\[ \phi' F^{qq} u_r \nabla_\tau u_q = \phi' F^{qq} u_r \nabla_\tau \left( \nabla_\tau u_q - T_q^{qr} u_p \right) \]

\[ = \phi' F^{qq} u_r \left( \nabla_\tau u_q - R_{\tau q}^{qr} u_p - u_p \nabla_\tau T_q^{qr} - T_q^{qr} u_{r\tau} \right) \]

\[ = \phi' F^{qq} u_r \left( \nabla_\tau g_{qr} - \nabla_\tau \tau_{qr} - \nabla_\tau W(u)_{qr} - R_{\tau q}^{qr} u_p - u_p \nabla_\tau T_q^{qr} - T_q^{qr} u_{r\tau} \right), \]

which implies

\[ \phi' F^{qq} u_r (\nabla_\tau u_q - \nabla_\tau g_{qr}) \geq - \frac{C}{K^{1/2}} \left( \sum_r |u_q| F^{qq} + \sum_r |\nabla g_{ur}| F^{qq} + \mathcal{F} \right). \]

From (4.34), (4.35), (4.36) and the fact that \( \phi' \geq 1/(4K) \), we can get

\[ 0 \geq - \frac{F^{qq}\lambda_1q}{\lambda_1^2} - \frac{1}{\lambda_1} F^{rs,ab} (\nabla_1 g_{r\tau}) (\nabla_\tau g_{ab}) \]

\[ - C \left( \mathcal{F} + \lambda_1^{-1} F^{qq} |\nabla g_{1T}| \right) + \sum_r \frac{F^{qq}}{6K} \left( |\nabla g_{ur}|^2 + |u_\tau|^2 \right) \]

\[ + \phi'' F^{qq}|V_q|^2 + \varphi'' F^{qq}|u_q|^2 + \varphi' F^{qq} u_q \]

\[ - \phi' u_r \left( F^{qq} \nabla_\tau g_{qr} - \psi_r \right) - \phi' u_r \left( F^{qq} \nabla_\tau g_{qr} - \psi_r \right) - \varphi' \dot{u}. \]

where we use the fact that \( K > 1 \) and hence \( K^{-1/2} < 1 \) and absorb the constant into \( C \mathcal{F} \).

Next, we can use the ideas of [36] in the elliptic setting to deal with two cases separately (cf. [23, 35]).

**Case 1.** Suppose that \( \delta \lambda_1 \geq -\lambda_n \), where the constant \( \delta \) will be determined later. First we define the set

\[ I := \left\{ \bar{i} : F^{ii} > \delta^{-1} F^{11} \right\}. \]
First we remark that for $i \in I$, we have $F^{kk} > F^{11}$ and hence it follows that $i > 1$ and $\lambda_1 > \lambda_i$. Using the same discussion as in [36], from (4.37), we can deduce

\begin{equation}
0 \geq - (1 - 2\delta) \sum_{q \in I} \frac{F^{qq} (|\frac{1}{\lambda_1} \frac{\partial}{\partial x} g_q|)^2 - |\nabla_1 g_q|^2}{\lambda_1^2}
- C \left( F + \lambda_1^{-1} F^{kk} |\nabla_k g_q| / \right) + \frac{F^{qq}}{6K} \sum_r \left( |\nabla_q u_r|^2 + |u_{\theta q}|^2 \right)
- 2\varphi'^2 \delta^{-1} F^{11} K + \frac{1}{2} \varphi'' F^{qq} |u_q|^2 + \varphi' F^{qq} u_{\theta q} - \varphi' \dot{u}.
\end{equation}

Here we choose the positive constant $\delta$ small enough so that

\begin{equation}
4\delta \varphi'^2 \leq \frac{1}{2} \varphi'',
\end{equation}

which can be easy obtained from the definition of $\varphi$.

Then we need the first claim whose proof is the same as the one in [36].

**Claim.** If $\lambda_1 / K$ is sufficiently large compared to $\varphi'$ (i.e., the choices of $D_1$ and $D_2$ which will be determined later), then for any $\varepsilon > 0$, there exists a constant $C_\varepsilon$ with

\begin{equation}
\sum_{q \in I} \frac{F^{qq} |\nabla_1 g_q|^2}{\lambda_1^2} \geq \frac{F^{qq} |\frac{1}{\lambda_1} \frac{\partial}{\partial x} g_q|^2 - |\nabla_q u_r|^2}{\lambda_1^2} - \sum_r \frac{F^{qq}}{12K} \left( |u_{\theta q}|^2 + |\nabla_q u_r|^2 \right)
+ C_\varepsilon \varphi' F^{qq} |u_q|^2 + \varepsilon C_\varepsilon' \varphi - C_\varepsilon' F.
\end{equation}

Substituting (4.40) into (4.38) gives

\begin{equation}
0 \geq \sum_r \frac{F^{qq}}{12K} \left( |\nabla_q u_r|^2 + |u_{\theta q}|^2 \right) + C_\varepsilon \varphi' F^{qq} |u_q|^2 + \varepsilon C_\varepsilon' \varphi - C_\varepsilon' F
- C \left( F + \lambda_1^{-1} F^{qq} |\nabla q g_1| \right) - 2\varphi'^2 \delta^{-1} F^{11} K
+ \frac{1}{2} \varphi'' F^{qq} |u_q|^2 + \varphi' F^{qq} u_{\theta q} - \varphi' \dot{u}.
\end{equation}

Using (4.19) again implies

\begin{equation}
\frac{|\lambda_1 q|}{\lambda_1} = |\varphi'(u_{\tau q} + u_{\theta q} | u_{\tau q}|) + \varphi' u_q|
\leq \frac{1}{2K^{1/2}} U + \frac{1}{2K^{1/2}} \sum_r |u_{\tau q}| - C_\varepsilon \varphi' |u_q|^2 - \varepsilon \varphi',
\end{equation}

where we use Young’s inequality and the facts that $\varphi' \leq (2K)^{-1}$ and that $\varphi' < 0$. This inequality and (4.21) imply

\begin{equation}
F^{qq} \lambda_1^{-1} |\nabla q g_1| \leq \frac{1}{2K^{1/2}} \sum_r (|u_{\tau q}| + |\nabla q u_r|) - C_\varepsilon \varphi' F^{qq} |u_q|^2 - \varepsilon \varphi' F.
\end{equation}

Substituting this into (4.41) implies

\begin{equation}
0 \geq \sum_r \frac{F^{qq}}{20K} \left( |\nabla_q u_r|^2 + |u_{\theta q}|^2 \right) + C_\varepsilon \varphi' F^{qq} |u_q|^2 + \varepsilon C_\varepsilon' \varphi - C_\varepsilon' F
- C \varphi' K^{-1} + \frac{1}{2} \varphi'' F^{qq} |u_q|^2 + \varphi' F^{qq} u_{\theta q} - \varphi' \dot{u}.
\end{equation}
Since \( W(u)_\bar{\gamma} = (\text{tr}_\alpha Z(u)) \alpha_\bar{\gamma} - (n - 1)Z(u)_\bar{\gamma} \), (2.19) and (4.6) gives
\[
\sum_q F^{qq} W(u)_{\bar{\gamma}q} = \sum_q \tilde{F}^{qq} Z(u)_{\bar{\gamma}q}
\]
\[
= \sum_{q=1}^n \tilde{F}^{qq} \left( \sum_{r \neq q} Z_{\bar{\gamma}qq} u_r \right) + \sum_{q=1}^n \tilde{F}^{qq} \left( \sum_{r \neq q} Z_{\bar{\gamma}qr} u_r \right)
\]
\[
= \sum_{r=1}^n \left( \sum_{q \neq r} \tilde{F}^{qq} Z_{\bar{\gamma}qq} \right) u_r + \sum_{r=1}^n \left( \sum_{q \neq r} \tilde{F}^{qq} Z_{\bar{\gamma}rr} \right) u_r.
\]
This together with (4.8) and (4.9) implies
\[
\left| \sum_q F^{qq} W(u)_{\bar{\gamma}q} \right| \leq C \sum_q F^{qq} |u_q|^2 + \varepsilon F,
\]
where in the last step we use Young’s inequality. Then it follows that
\[
\varphi' F^{qq} u_{\bar{\gamma}q} = \varphi' F^{qq} (g_{\bar{\gamma}q} - \chi_{\bar{\gamma}q} - W(u)_{\bar{\gamma}q}) \\
\geq \varphi' F^{qq} (g_{\bar{\gamma}q} - \chi_{\bar{\gamma}q}) + C \varphi' F^{qq} |u_q|^2 + \varepsilon \varphi',
\]
where we use the fact that \( \varphi' < 0 \). Substituting (4.44) into (4.43) gives
\[
0 \geq F^{11} \left( \frac{\lambda_1^2}{40 K} - 2 \varphi'^2 \delta^{-1} K \right) + \left( C \varphi' + \frac{1}{2} \varphi'' \right) F^{qq} |u_q|^2 + \varepsilon C \varphi' F - C F - \varphi' \kappa F,
\]
where we use the fact that \( |u_1| \geq \frac{1}{2} \lambda_1^2 - C K \).

Now we need to use Proposition 4.2.

(a) Suppose that we have \( F^{qq} (\chi_{\bar{\gamma}q} - g_{\bar{\gamma}q}) - |\dot{u}| > \kappa F \). Then (4.45) becomes
\[
0 \geq F^{11} \left( \frac{\lambda_1^2}{40 K} - 2 \varphi'^2 \delta^{-1} K \right) + \left( C \varphi' + \frac{1}{2} \varphi'' \right) F^{qq} |u_q|^2 + \varepsilon C \varphi' F - C F - \varphi' \kappa F.
\]
Choose \( \varepsilon > 0 \) sufficiently small such that \( \varepsilon C < \frac{\kappa}{2} \). Next we choose \( D_2 \) in the definition \( \varphi(t) = D_1 e^{-D_2 t} \) to be large enough with
\[
\frac{1}{2} \varphi'' > C \varphi'.
\]
Then we arrive at
\[
0 \geq F^{11} \left( \frac{\lambda_1^2}{40 K} - 2 \varphi'^2 \delta^{-1} K \right) - C F - \frac{1}{2} \varphi' \kappa F.
\]
Now we choose \( D_1 \) large enough such that \( -\frac{1}{2} \kappa \varphi' > C \), then we can get
\[
0 \geq \frac{\lambda_1^2}{40 K} - 2 \varphi'^2 \delta^{-1} K,
\]
which implies the bound of \( \lambda_1 / K \), as required.

(b) Suppose that we have \( F^{11} > \kappa F \). With the constants \( D_1 \) and \( D_2 \) determined above, \( |\dot{u}| \)
and \( \varphi' \) is uniformly bounded and hence can be absorbed in \( C F \). Then using the same arguments in [36], we can get the uniform upper bound of \( \lambda_1 / K \) and omit the details here.
Case 2. Suppose that $\delta \lambda_1 \leq -\lambda_n$, and all the constants $D_1, D_2$ and $\delta$ are fixed as in the previous case. Since in this case we can absorb the uniformly bounded term $-\varphi' \dot{u}$ into $CF$, using the same discussion as in [36], we can get the uniform upper bound of $\lambda_1/K$. Here we omit the details.

5. First order estimate

Given the form of the second estimate, we need the first order estimate for $u$.

**Theorem 5.1.** Let $u$ be the solution to (1.7) on $M \times [0, T)$. Then there exists a uniform $C > 0$ such that

$$
\sup_{M \times [0, T)} |\nabla u|^2_\alpha \leq C.
$$

First we recall some notations from [42]. Let $\beta$ be the Euclidean Kähler metric on $\mathbb{C}^n$, $n \geq 2$ and $\Delta$ be the Laplace operator with respect to $\beta$. Let $\Omega \subset \mathbb{C}^n$ be a domain. Suppose that $u \rightarrow \mathbb{R} \cup \{-\infty\}$ is an upper semicontinuous function which is in $L^1_{\text{loc}}(\Omega)$. If

$$
P(u) := \frac{1}{n-1} \left( (\Delta u) \beta - \sqrt{-1} \partial \bar{\partial} u \right) \geq 0
$$

as a real $(1,1)$ current, then we will say that $u$ is $(n-1)$-Psh.

**Definition 5.2.** Let $u$ be a continuous $(n-1)$-Psh function. Then we say $u$ is maximal if for any relatively compact set $\Omega' \subset \subset \Omega$ and any continuous $(n-1)$-Psh function $v$ on a domain $\Omega''$ with $\Omega' \subset \subset \Omega'' \subset \subset \Omega$ and $v \leq u$ on $\partial \Omega'$, then we must have $v \leq u$ on $\Omega'$.

We need the following Liouville type theorem from [42].

**Theorem 5.3** (Tosatti and Weinkove [42]; 2013). Let $u: \mathbb{C}^n \rightarrow \mathbb{R}$ be an $(n-1)$-Psh function which is Lipschitz continuous, maximal, and satisfies

$$
\sup_{\mathbb{C}^n} (|u| + |\nabla u|) < \infty.
$$

Then $u$ is a constant.

Using the idea of Dinew and Kołodziej [8] and the Liouville type Theorem 5.3, the argument is identical to [20] which can be obtained by modifying the argument of [42] in the elliptic setting. Hence, we can be brief and just point out the main differences.

**Proof of Theorem 5.1.** Suppose that (5.1) does not hold. Then there exists a sequence $(x_j, t_j) \in M \times [0, T)$ with $t_j \rightarrow T$ such that

$$
\lim_{j \rightarrow \infty} |\nabla u(x_j, t_j)|_\alpha = +\infty.
$$

Without loss of generality, we assume that

$$
C_j := |\nabla u(x_j, t_j)|_\alpha = \sup_{(x,t) \in M \times [0, t_j]} |\nabla u(x, t)|_\alpha \rightarrow +\infty, \quad \text{as} \quad j \rightarrow \infty
$$

and $\lim_{j \rightarrow \infty} x_j = x$. Fix holomorphic coordinates $z = (z^1, \cdots, z^n)$ centered at $x$ and with $z(x) = 0$ and $\alpha(x) = \beta$ identifying with $B_2(0)$. Also assume that $j$ large enough so that $x_j \in B_1(0)$. Define

$$
u_j : M \rightarrow \mathbb{C}, \quad \text{given by} \quad u_j(y) := u(y, t_j),$$

then...
\( \Phi_j : \mathbb{C}^n \to \mathbb{C}^n \), given by \( \Phi_j(z) := C_j^{-1}z + x_j =: w, \)
\( \hat{u}_j : \mathbb{C}^n \supset B_{C_j}(0) \to \mathbb{C}, \) given by \( \hat{u}_j(z) := u_j \circ \Phi_j(z) = u_j \left( C_j^{-1}z + x_j \right). \)

Note that for convenience, we confuse \( z(x) \) and \( z \) by the local coordinates. The function \( \hat{u}_j \) satisfies
\[
\sup_{B_{C_j}(0)} |\hat{u}_j| \leq C \quad \text{and} \quad \sup_{B_{C_j}(0)} |\nabla \hat{u}_j|_\beta \leq CC_j^{-1}|\nabla u_j|_\alpha (x_j + C_j^{-1}z) \leq C,
\]
where we use the fact that the Euclidean metric \( \beta \) is equivalent to the Hermitian metric \( \alpha \) in \( B_2(0) \). In particular, we have
\[
(5.2) \quad \sup_{B_{C_j}(0)} |\nabla \hat{u}_j|_\beta (0) \geq C^{-1}C_j^{-1}|\nabla u_j|_\alpha (x_j) = C^{-1} > 0.
\]

Thanks to Theorem 4.1, we know that
\[
|\sqrt{-1}\partial \bar{\partial} \hat{u}_j|_\beta \leq CC_j^{-2} |\sqrt{-1}\partial \bar{\partial} u_j|_\alpha \leq C'.
\]
Then the elliptic estimate for \( \Delta \) and Sobolev embedding gives that for each compact \( K \subset \mathbb{C}^n \), each \( \gamma \in (0, 1) \) and \( p > 1 \), there exists a constant \( C \) with
\[
\|\hat{u}_j\|_{C^{1,\gamma}(K)} + \|\hat{u}_j\|_{W^{2,p}(K)} \leq C.
\]
Therefore, there exists a subsequence of \( \hat{u}_j \) converges strongly in \( C^{1,\gamma}(\mathbb{C}^n) \) and weakly in \( W^{2,p}_{loc}(\mathbb{C}^n) \) to a function \( u \in W^{2,p}_{loc}(\mathbb{C}^n) \) with \( \sup_{\mathbb{C}^n}|u| + |\nabla u|_\beta \leq C \) and (5.2) implies that \( |\nabla u|_\beta \neq 0. \) In particular, \( u \) is not a constant.

Note that (see [42])
\[
(5.3) \quad \beta_j := C_j^2 \Phi_j^* \alpha = \sqrt{-1}\alpha_q \partial_{q}(x_j + C_j^{-1}z)dz^p \wedge d\bar{z}^q \to \alpha(x) = \beta,
\]
\[
(5.4) \quad \Phi_j^* \alpha = C_j^{-2} \sqrt{-1}\alpha_q \partial_{q}(x_j + C_j^{-1}z)dz^p \wedge d\bar{z}^q \to 0
\]
\[
(5.5) \quad \Phi_j^* \omega = C_j^{-2} \sqrt{-1}\omega_q \partial_{q}(x_j + C_j^{-1}z,t)dz^p \wedge d\bar{z}^q \to 0,
\]
smoothly on compact sets of \( \mathbb{C}^n \). Also it is easy to get
\[
(5.6) \quad \Phi_j^* Z(u_j) = \sqrt{-1} \left( Z^p_{q} u_{j,r} + \overline{Z^p_{q}} u_j,\tau \right) (x_j + C_j^{-1}z)C_j^{-2}dz^p \wedge d\bar{z}^q
\]
\[
= \sqrt{-1} \left( Z^p_{q} (x_j + C_j^{-1}z) \hat{u}_{j,r}(z) + \overline{Z^p_{q}} (x_j + C_j^{-1}z) \hat{u}_j,\tau (z) \right) C_j^{-1}dz^p \wedge d\bar{z}^q \to 0
\]
uniformly on compact sets of \( \mathbb{C}^n \). Then we can deduce
\[
(5.7) \quad \Phi_j^* ((\Delta_\alpha u_j) \alpha) = \sqrt{-1} (\Delta_\beta \hat{u}_j) \beta_j \to (\Delta u) \beta \quad \text{(see [36])}
\]
\[
(5.8) \quad \Phi_j^* \sqrt{-1}\partial \bar{\partial} u_j = \sqrt{-1} \left( \frac{\partial^2 u_j}{\partial w^k \partial \bar{w}^l} \right) (x_j + C_j^{-1}z)C_j^{-2}dz^p \wedge d\bar{z}^q
\]
\[
= \sqrt{-1} \left( \frac{\partial^2 \hat{u}_j}{\partial z^p \partial \bar{z}^q} \right) (z)dz^p \wedge d\bar{z}^q
\]
\[
= \sqrt{-1}\partial \bar{\partial} \hat{u}_j \to \sqrt{-1}\partial \bar{\partial} u
\]
weakly in \( L^p_{loc}(\mathbb{C}^n) \) of the coefficients. In particular, we have
\[
\Phi_j^* \omega_j := \Phi_j^* \omega(t_j) = \Phi_j^* \left( \omega(t_j) + \frac{1}{n-1} \left( ((\Delta_\alpha u_j) \alpha - \sqrt{-1}\partial \bar{\partial} u_j) + Z(u_j) \right) \right)
\]
\[
\to \frac{1}{n-1} \left( ((\Delta u) \beta - \sqrt{-1}\partial \bar{\partial} u) = P(u) \right)
\]
weakly as currents. Since $\Phi_j^*\omega_j > 0$, it follows that $P(u) \geq 0$ as currents. From Lemma 3.1, it follows that functions $\dot{u}_j \circ \Phi_j$ are uniformly bounded. Hence, we have
\begin{equation}
\Phi_j^*\omega_j^n = e^{\dot{u}_j \circ \Phi_j} \Phi_j^*\Omega
\end{equation}
uniformly on compact sets of $\mathbb{C}^n$. From (5.3) and (5.7), we have, for any compact set $K \subset \mathbb{C}^n$,
\[
\sup_{K} |\Delta_{\beta_j} \dot{u}_j - \Delta u_j| \to 0, \quad \text{as} \quad j \to \infty.
\]
From (5.3), (5.4), (5.6), (5.7) and (5.8), we have
\begin{equation}
\sup_{K} |P(\dot{u}_j) - \Phi_j^*\omega_j|_\beta
= \sup_{K} \left| \frac{1}{n-1} ( (\Delta_{\beta_j} \dot{u}_j) )_\beta - (\Delta_{\beta_j} u_j) )_\beta \right|
\leq \sup_{K} \left( \frac{1}{n-1} | (\Delta_{\beta_j} \dot{u}_j) )_\beta - (\Delta_{\beta_j} u_j) )_\beta| + |\Phi_j^*\omega_j|_\beta + |\Phi_j^*Z(u_j)|_\beta \right)
\end{equation}
converges to zero as $j \to \infty$. Noting that $P(\dot{u}_j)$ and $\Phi_j^*\omega_j$ are locally uniformly bounded, (5.9) and (5.10) implies that $P(\dot{u}_j)_n$ converges to zero uniformly on compact sets of $\mathbb{C}^n$, since we have
\[
P(\dot{u}_j)_n - \Phi_j^*\omega_j^n = P(\dot{u}_j)_n - (\Phi_j^*\omega_j)_n = (P(\dot{u}_j) - \Phi_j^*\omega_j) \sum_{r=0}^{n-1} (P(\dot{u}_j)_r \wedge (\Phi_j^*\omega_j)^{n-1-r}).
\]
Then use the arguments in [42], we know that $u$ is maximal. From Theorem 5.3, we know that $u$ is a constant, a contradiction to Lemma 3.1.

6. Proof of the uniqueness and long time existence of the main theorem

In this section, we will give the proof of the uniqueness and long time existence of Theorem 1.4. To get this, we need the following lemma.

**Lemma 6.1.** Let $u$ be the solution to (1.7) on $M \times [0, T)$. Then for any $\epsilon \in (0, T)$ and every positive integer $k$, there is a constant $C_k$, depending only on $k$, $\epsilon$ and the initial data on $M$ such that
\begin{equation}
\sup_{M \times [\epsilon, T)} |\nabla^k u(x, t)| \leq C_k.
\end{equation}

**Proof.** Thanks to Theorem 4.1 and Theorem 5.1, we can deduce a priori $C^{2,\gamma}$ estimates of $u$ from [4, Theorem 5.3] and the discussion in the proof of [5, Lemma 6.1] directly (cf. [44]). Higher order estimates can be obtained after differentiating the equation and applying the usual bootstrapping method.

Then we can prove the long time existence and uniqueness of the solution in the main theorem.

**Proof of the uniqueness and long time existence of Theorem 1.4.** The uniqueness is the result of standard parabolic equation theory. Suppose that $T < \infty$. From Lemma 3.1, there exists a uniform constant $C$ such that
\begin{equation}
|u(x, t)| \leq |u_0(x)| + T \sup_{M \times [0, T)} |\dot{u}| \leq C(T + 1) < \infty.
\end{equation}
This together with Lemma 6.1 and short time existence theorem implies that we can extend the solution to (1.7) on $[0, T + \varepsilon)$ with \(\varepsilon > 0\), absurd. We can find more details about this standard discussion in the proof of [37, Theorem 3.1] (cf. [33, 47]).

\[ \Box \]

7. The Harnack inequality

In this section we consider the Harnack inequalities of the parabolic equation

\[ \frac{\partial}{\partial t} \varphi = L(\varphi) \]

where \(L\) is defined as in (3.1), which are analogs to Li and Yau [26]. Note that Cao [2] stated this result on Kähler manifolds and Gill [17] also proved it on Hermitian manifolds (cf. [5, 46]). This is another necessary preparation for the proof of our main theorem. For convenience, we give a lemma which can be easily obtained from Lemma 6.1 as follows.

**Lemma 7.1.** Let \(u\) be the solution to (1.7) on \(M \times [0, \infty)\). Then for every positive integer \(k\), there is a constant \(C_k\), depending only on \(k\) and the initial data on \(M\) such that

\[ \sup_{M \times [0, \infty)} |\nabla^k u(x, t)| \leq C_k. \]

Then we give our Harnack inequality as follows.

**Lemma 7.2.** Assume that \(\varphi\) is a positive solution to (7.1) and define \(f = \log \varphi\) and \(F = t (|\partial f|^2 - \alpha f_t)\). There holds

\[ L(F) - F_t \geq \frac{1}{2n} (|\partial f|^2 - f_t)^2 - 2 \text{Re} \langle \partial f, \partial F \rangle - (|\partial f|^2 - \alpha f_t) - Ct|\partial f|^2 - Ct, \]

where the definition of \(|\partial f|^2\) can be found in the proof.

**Proof.** The ideas are originated from Li and Yau [26], hence we can be brief and just point out the main differences. For convenience, we introduce some notations as follows.

\[ \langle \chi, \xi \rangle := \Theta^j_i \chi_i \xi_j, \quad |\partial f|^2 := \langle \partial f, \partial f \rangle, \quad |\partial^2 f|^2 := \Theta^j_i \Theta^k_l f_i f_j f_k f_l, \quad |D^2 f|^2 := \Theta^j_i \Theta^l_k f_i f_j f_k f_l, \]

where \(\chi\) and \(\xi\) are \((1, 0)\) forms. By direct computation, we get

\[ L(f) - f_t = -|\partial f|^2, \]

i.e.,

\[ F = -tL(f) - t(\alpha - 1)f_t. \]

Then we can deduce

\[ (L(f))_t = \frac{1}{t^2} F - \frac{1}{t} F_t - (\alpha - 1)f_{tt}. \]

Again direct computation implies

\[ F_t = |\partial f|^2 - \alpha f_t + 2t \text{Re} \langle \partial f, \partial f_t \rangle + t \left( \frac{\partial}{\partial t} \Theta^j_i \right) f_i f_j - \alpha f_{tt}, \]

and

\[ F_{k\bar{\ell}} = t \left[ \left( \Theta^j_i \right)_{k\bar{\ell}} f_i f_j + \left( \Theta^j_i \right)_k f_{\bar{\ell}} f_j + \left( \Theta^j_i \right)_\bar{\ell} f_i f_j - \alpha f_{k\bar{\ell}} \right], \]

where
Note that
\begin{equation}
(7.6) \quad t\Theta^{7k}\Theta^{ji}f_{ik}\bar{f}_{j} + t\Theta^{7k}\Theta^{ji}f_{jk}\bar{f}_{j}
=2t\text{Re} \left\langle \partial f, \partial \left( \Theta^{7k} f_{ik}\bar{f}_{j} \right) \right\rangle - t\Theta^{ji} \left( \Theta^{7k} \right)_{i} f_{k}\bar{f}_{j} - t\Theta^{ji} \left( \Theta^{7k} \right)_{j} f_{k}\bar{f}_{j},
\end{equation}
\begin{equation}
\geq 2t\text{Re} \left\langle \partial f, \partial \left( \Theta^{7k} f_{ik}\bar{f}_{j} \right) \right\rangle - \frac{C_{1}t}{\varepsilon} |\partial f|^{2} - \varepsilon t|\partial \bar{f}|^{2}
\end{equation}
\begin{equation}
= -2 \text{Re} \left\langle \partial f, \partial F \right\rangle - 2t\text{Re} \left\langle \partial f, \partial (\text{tr}_{\omega} Z(f)) \right\rangle - 2t(\alpha - 1)\text{Re} \left\langle \partial f, \partial f_{1} \right\rangle - \frac{C_{1}t}{\varepsilon} |\partial f|^{2} - \varepsilon t|\partial \bar{f}|^{2}
\end{equation}
\begin{equation}
\geq -2 \text{Re} \left\langle \partial f, \partial F \right\rangle - 2t\text{Re} \left\langle \partial f, \partial (\text{tr}_{\omega} Z(f)) \right\rangle - \frac{C_{1}t}{\varepsilon} |\partial f|^{2} - \varepsilon t|\partial \bar{f}|^{2}
\end{equation}
\begin{equation}
\geq -\left( \alpha - 1 \right) F_{1} + (\alpha - 1) (|\partial f|^{2} - \alpha f_{1}) + (\alpha - 1)t \left( \frac{\partial}{\partial t} \Theta^{ji} \right) f_{i}\bar{f}_{j} - (\alpha - 1)\alpha t f_{i}f_{i},
\end{equation}
where for the first inequality we use Young’s inequality, for the second equality we use (7.3), for the third equality we use (7.4), and for the second inequality we use Lemma 7.1, Young’s inequality and the facts that $Z(f)$ is linear of $\partial f$ and $\partial \bar{f}$.

Also using Young’s inequality, we have
\begin{equation}
(7.7) \quad -\alpha t\Theta^{7k}f_{tk}\bar{f}_{t} = \alpha t \left( \frac{\partial}{\partial t} \Theta^{7k} \right) f_{k}\bar{f}_{t} - \alpha t \frac{\partial}{\partial t} \left( \Theta^{7k} f_{k}\bar{f}_{t} \right)
\end{equation}
\begin{equation}
\geq - \frac{C_{1}t}{\varepsilon} - \varepsilon t|\partial \bar{f}|^{2} - \alpha t \frac{\partial}{\partial t} \left( \text{tr}_{\omega} Z(f) \right) - \frac{\alpha}{t} F + \alpha F_{t} + \alpha(\alpha - 1)t f_{i}f_{i}
\end{equation}
\begin{equation}
\geq - \frac{C_{1}t}{\varepsilon} - \varepsilon t|\partial \bar{f}|^{2} - C_{6}t|\partial f|^{2} + \alpha t \left( \text{tr}_{\omega} Z(f_{i}) \right) - \frac{\alpha}{t} F + \alpha F_{t} + \alpha(\alpha - 1)t f_{i}f_{i},
\end{equation}
where for the second inequality we also use Lemma 7.1.

According to the definition of $F$, Lemma 7.1 and Young’s inequality, we can deduce
\begin{equation}
(7.8) \quad \text{tr}_{\omega} Z(F) = t \left( \text{tr}_{\omega} Z \left( \Theta^{ji} f_{i}\bar{f}_{j} \right) \right) - \alpha t \left( \text{tr}_{\omega} Z(f_{i}) \right)
\end{equation}
\begin{equation}
\geq - \frac{C_{7}t}{\varepsilon} |\partial f|^{2} - C_{8}t|\partial f|^{2} - \varepsilon t|\partial \bar{f}|^{2} - \varepsilon t|D^{2}f|^{2} - \alpha t \left( \text{tr}_{\omega} Z(f_{i}) \right).
\end{equation}

Again using Young’s inequality, we get
\begin{equation}
(7.9) \quad t\Theta^{7k} \left[ \left( \Theta^{ji} \right)_{k} f_{i}\bar{f}_{j} + \left( \Theta^{ji} \right)_{k} f_{i}\bar{f}_{j} + \left( \Theta^{ji} \right)_{k} f_{i}\bar{f}_{j} + \left( \Theta^{ji} \right)_{k} f_{i}\bar{f}_{j} \right]
\end{equation}
\begin{equation}
\geq - C_{9}t|\partial f|^{2} - \frac{C_{10}t}{\varepsilon} - \varepsilon t|\partial \bar{f}|^{2} - \varepsilon t|D^{2}f|^{2}.
\end{equation}

Combining (7.4), (7.5), (7.6), (7.7), (7.8) and (7.9), we can deduce
\begin{equation}
(7.10) \quad L(F) - F_{1} \geq - \frac{C_{5} + C_{10}}{\varepsilon} t - 2\text{Re} \left\langle \partial f, \partial F \right\rangle + (1 - 4\varepsilon)t|\partial \bar{f}|^{2} + (1 - 3\varepsilon)t|D^{2}f|^{2}
\end{equation}
\begin{equation}
+ \left( C_{3} + C_{6} + C_{8} + C_{9} + \frac{C_{2} + C_{7}}{\varepsilon} \right) t|\partial f|^{2} - \left( |\partial f|^{2} - \alpha f_{i} \right).
\end{equation}

From the Cauchy-Schwarz inequality, we have
\begin{equation}
(7.11) \quad |\partial \bar{f}|^{2} \geq \frac{1}{n} \left( \Theta^{ji} f_{i}f_{j} \right)^{2} = \frac{1}{26} \left( |\partial f|^{2} - f_{i} + \text{tr}_{\omega} Z(f) \right)^{2}
\end{equation}
\[ \frac{1}{n}(1-\varepsilon)(|\partial f|^2 - f_t)^2 - C\varepsilon |\partial f|^2, \]

where for the second inequality we use Young’s inequality.

Then from (7.10) and (7.11), we can deduce (7.2). \qed

Using Lemma 7.2 and standard discussion, we can get

**Lemma 7.3.** Using the notations as in Lemma 7.2, for any \( t \in [0, \infty) \), there exist uniform constants \( C_1 \) and \( C_2 \) such that
\[ |\partial f|^2 - \alpha_f t \leq C_1 + C_2 t. \tag{7.12} \]

Then by this lemma we can deduce the following theorem which will be used in the convergence discussion.

**Theorem 7.4.** Using the notations as in Lemma 7.2, for any \( 0 < t_1 < t_2 \), there holds
\[ \sup_{x \in M} \varphi(x, t_1) \leq \inf_{x \in M} \varphi(x, t_2) \left( \frac{t_2}{t_1} \right)^{C_1} \exp \left( \frac{C_2}{t_2 - t_1} + C_3(t_2 - t_1) \right), \]
where \( C_1, C_2 \) and \( C_3 \) are uniform constants.

We remark that the proofs of Lemma 7.3 and Theorem 7.4 are routine discussion. Here considering the length of the paper, we omit them (see for example [5, 17]).

8. Proof of the convergence in the main theorem

In this section, we will give the proof of the second part of Theorem 1.4. Since the discussion is standard, we will be brief and just point out the main differences.

**Proof of the convergence of Theorem 1.4.** For any positive integer \( m \), we define
\[ \xi_m(x, t) = \sup_{y \in M} \hat{u}(y, m - 1) - \hat{u}(x, m - 1) + t \]
\[ \psi_m(x, t) = \hat{u}(x, m - 1) - \inf_{y \in M} \hat{u}(y, m - 1). \]

These functions satisfy the equation (7.1).

Without loss of generality, we assume that \( \hat{u}(x, m - 1) \) is not constant. The maximum principle implies that \( \xi_m \) and \( \psi_m \) are positive solutions to (7.1). Using Theorem 7.4 with \( t_1 = 1/2 \) and \( t_2 = 1 \), we get
\[ \sup_{y \in M} \hat{u}(y, m - 1) - \inf_{y \in M} \hat{u}(x, m - 1/2) \leq C \left( \sup_{y \in M} \hat{u}(y, m - 1) - \sup_{y \in M} \hat{u}(x, m) \right), \]
\[ \sup_{y \in M} \hat{u}(y, m - 1/2) - \inf_{y \in M} \hat{u}(x, m - 1) \leq C \left( \inf_{y \in M} \hat{u}(y, m) - \inf_{y \in M} \hat{u}(x, m - 1) \right), \]
which implies
\[ \theta(m - 1) + \theta(m - 1/2) \leq C (\theta(m - 1) - \theta(m)), \]
where \( \theta(t) = \sup_{y \in M} \hat{u}(y, t) - \inf_{y \in M} \hat{u}(y, t) \). Then by induction, we get
\[ \theta(t) \leq C e^{-\eta t}, \tag{8.1} \]
where \( \eta = \log \frac{C}{C - 1} \).
Since \( \int_M \tilde{\omega}^n = 0 \), we obtain \( \int_M \frac{\partial \tilde{u}}{\partial t} \alpha^n = 0 \), which implies that there exists a point \( y \in M \) with \( \frac{\partial \tilde{u}}{\partial t}(y, t) = 0 \). Then we have

\[
\frac{\partial \tilde{u}}{\partial t}(x, t) = \left| \frac{\partial \tilde{u}}{\partial t}(x, t) - \frac{\partial \tilde{u}}{\partial t}(y, t) \right| = \left| \frac{\partial u}{\partial t}(x, t) - \frac{\partial u}{\partial t}(y, t) \right| \leq Ce^{-\eta t}.
\]

Consider the quantity \( Q = \tilde{u} + \frac{C}{\eta} e^{-\eta t} \). Then we have

\[
\frac{\partial Q}{\partial t} = \frac{\partial \tilde{u}}{\partial t} - Ce^{-\eta t} \leq 0,
\]

which implies that \( Q \) converges uniformly as \( t \to \infty \) since \( Q \) is uniformly bounded and denote the limit by \( \tilde{u}_\infty \). Furthermore, we have

\[
\lim_{t \to \infty} \tilde{u} = \lim_{t \to \infty} Q - \lim_{t \to \infty} \frac{C}{\eta} e^{-\eta t} = \tilde{u}_\infty.
\]

It follows that the convergence of \( \tilde{u} \) to \( \tilde{u}_\infty \) is actually \( C^\infty \) using the argument by contradiction and the Arzela-Ascoli theorem. Note that

\[
\frac{\partial \tilde{u}}{\partial t} = \log \left( \frac{\omega + \frac{1}{n-1} \left[ (\Delta \tilde{u})_\alpha - \sqrt{-1} \partial \bar{\partial} \tilde{u} \right] + Z(\tilde{u})}{\alpha^n} \right) - \psi - \frac{1}{\text{Vol}_\alpha(M)} \int_M \frac{\partial u}{\partial t} \alpha^n.
\]

Letting \( t \to \infty \), (8.2) implies

\[
\log \left( \frac{\omega + \frac{1}{n-1} \left[ (\Delta \tilde{u}_\infty)_\alpha - \sqrt{-1} \partial \bar{\partial} \tilde{u}_\infty \right] + Z(\tilde{u}_\infty)}{\alpha^n} \right) = \psi + \tilde{b},
\]

where

\[
\tilde{b} = \frac{1}{\text{Vol}_\alpha(M)} \int_M \left( \log \left( \frac{\omega + \frac{1}{n-1} \left[ (\Delta \tilde{u}_\infty)_\alpha - \sqrt{-1} \partial \bar{\partial} \tilde{u}_\infty \right] + Z(\tilde{u}_\infty)}{\alpha^n} \right) - \psi \right) \alpha^n,
\]

as required.

\[\square\]

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