INTEGRAL AND BOUNDARY ESTIMATES
FOR CRITICAL METRICS OF THE VOLUME FUNCTIONAL

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ABSTRACT. In this article, we investigate the geometry of critical metrics of the volume functional on compact manifolds with boundary. We use the generalized Reilly’s formula to derive new sharp integral estimates for critical metrics of the volume functional on \( n \)-dimensional compact manifolds with boundary. As application, we establish new boundary estimates for such manifolds.

1. Introduction

A classical method to find canonical metrics on a given manifold is to investigate critical metrics which arise as solutions of the Euler-Lagrange equations for curvature functionals. For example, Einstein and Hilbert showed that the critical metrics of the scalar curvature functional on compact manifolds with unitary volume are Einstein (see [8, Theorem 4.21]). In a similar context and also motivated by a volume comparison result obtained by Fan, Shi and Tam [15] on asymptotically flat 3-dimensional manifolds, Miao and Tam [19, 20] and Corvino, Eichmair and Miao [12] investigated the modified problem of finding stationary points for the volume functional constrained to the space of metrics of constant scalar curvature on a given compact manifold with boundary.

In order to proceed, it is appropriate to fix some terminology (cf. [5], [7]).

Definition 1. Let \( (M^n, g) \) be a connected compact Riemannian manifold with boundary \( \partial M \) and dimension \( n \geq 3 \). We say that \( g \) is, for brevity, a Miao-Tam critical metric (or simply, critical metric), if there is a nonnegative smooth function \( f \) on \( M^n \) such that \( f^{-1}(0) = \partial M \) satisfying the equation

\[
(1.1) \quad \mathfrak{L}_g^*(f) = -(\Delta f) g + \nabla^2 f - f \text{Ric} = g.
\]

Here, \( \mathfrak{L}_g^* \) is the formal \( L^2 \)-adjoint of the linearization of the scalar curvature operator \( \mathfrak{L}_g \). Moreover, \( \text{Ric}, \Delta \) and \( \nabla^2 \) stand for the Ricci tensor, the Laplacian operator and the Hessian on \( (M^n, g) \), respectively.

Miao and Tam [20] showed that such critical metrics defined in (1.1) arise as critical points of the volume functional on \( M^n \) when restricted to the class of metrics \( g \) with prescribed constant scalar curvature such that \( g|_{T\partial M} = \gamma \) for a prescribed Riemannian metric \( \gamma \) on the boundary; see also [12]. Interestingly, it follows from [20, Theorem 7] that connected Riemannian manifolds satisfying the critical metric equation (1.1) have necessarily constant scalar curvature \( R \). In particular, for any given metric \( g \) in the space of the metrics with constant scalar curvature \( R \), the map \( \mathfrak{L}_g^* \) defined from \( C^\infty \) to \( \mathcal{M} \) is an over determined-elliptic operator. Such metrics are relevant in understanding the influence of the scalar curvature

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in controlling the volume of a given manifold. In this scenario, Corvino, Eichmair and Miao \cite{12} used the study of critical metrics to establish a deformation result which suggests that the information of the scalar curvature is not sufficient in giving volume comparison. In particular, it is known by the works \cite{12,20} that the Schoen’s conjecture can not be generalized directly to manifolds with boundary if only the Dirichlet boundary condition is imposed (see also \cite{33}). Recall that the Schoen’s conjecture \cite{28} asserts: Let \((M^n, g)\) be a closed hyperbolic manifold and let \(g\) be another metric on \(M^n\) with scalar curvature \(R(g) \geq R(\overline{g})\), then \(Vol(g) \geq Vol(\overline{g})\). The 3-dimensional case of the conjecture follows as a consequence of the works of Hamilton on nonsingular Ricci flow and Perelman on geometrization of 3-manifolds.

Some explicit examples of critical metrics can be found in \cite{19,20}. They include the spatial Schwarzschild metrics and AdS-Schwarzschild metrics restricted to certain domains containing their horizon and bounded by two spherically symmetric spheres. Besides, standard metrics on geodesic balls in space forms \(\mathbb{R}^n\), \(\mathbb{H}^n\) or \(\mathbb{S}^n\) are also critical metrics. These last ones are simply connected and have attracted a lot of attention in the last few years. As observed by Miao and Tam in \cite{19}, it is interesting to know whether the standard metrics on geodesic balls in space forms are the only critical metrics on simply connected compact manifolds with connected boundary. There have been a lot of advances concerning the rigidity of critical metrics of the volume functional; see, e.g., \cite{3,4,5,6,7,12,14}.

Boundary estimates are classical objects of study in geometry and physics. Besides being interesting on their own, such estimates play a fundamental role in proving classification results and discarding some possible new examples of special metrics on a given manifold. In recent years, it was established some useful boundary and volume estimates for critical metrics of the volume functional, as for example, isoparametric and Shen-Boucher-Gibbons-Horowitz type inequalities (see, e.g., \cite{1,3,6,7,12,14}). At the same time, it is natural to explore integral estimates in order to obtain new obstruction results. In this context, the Reilly’s formula \cite{24} has been shown to be a promising tool. Such a formula was used in solving interesting problems in differential geometry and has numerous applications. For instance, Ros \cite{26} used the Reilly’s formula to prove an integral inequality which was applied to show the Alexandrov’s rigidity theorem for high order mean curvatures. Miao, Tam and Xie \cite{21} employed the Reilly’s formula to obtain an integral inequality that relates the mean curvature \(H\) and the second fundamental form \(II\) on the boundary \(\partial \Omega\) of bounded domains \(\Omega\) in the Euclidean space \(\mathbb{R}^n\). To be precise, by assuming that the mean curvature of the boundary \(H > 0\), they proved that

\[
\int_{\partial \Omega} \left[ \left( \nabla_{\partial \Omega} \eta \right)^2 \frac{1}{H} - II(\nabla_{\partial \Omega} \eta, \nabla_{\partial \Omega} \eta) \right] dS \geq 0,
\]

for any smooth function \(\eta\) on \(\partial \Omega\), where \(\nabla_{\partial \Omega}\), \(\Delta_{\partial \Omega}\) and \(dS\) denote the gradient, Laplacian and volume form on \(\partial \Omega\), respectively. Moreover, equality holds in (1.2) if and only if \(\eta = a_0 + \sum_{i=1}^{n} a_i x_i\) for some constants \(a_0, a_1, ..., a_n\). Here, \(\{x_1, x_2, ..., x_n\}\) are the standard coordinate functions on \(\mathbb{R}^n\). In other words, (1.2) can be seen as a stability inequality for Wang-Yau energy on \(\partial \Omega\), which is quite important in general relativity. For more details on this subject, see, e.g., \cite{11,30,31}. Subsequently, Kwong and Miao \cite{17} obtained a generalization of the estimate (1.2) to the boundary of compact Riemannian manifolds whose metric is static. In \cite{23}, Qiu and Xia established a generalized Reilly’s formula that was used to give an alternative proof of the Alexandrov’s theorem and prove a new Heintze-Karcher inequality for Riemannian manifolds with boundary and sectional curvature bounded from below. We refer the reader to \cite{17,18,22,23,52} for further interesting applications.

In our first result, motivated by the above discussion, we will make use of the generalized Reilly’s formula by Qiu and Xia \cite{23} and a suitable boundary value problem in order to derive the following integral inequality for critical metrics of the volume functional, which
is similar to Miao-Tam-Xie and Kwong-Miao estimates (cf. [21 Corollary 3.1] and [17 Theorem 3]).

**Theorem 1.** Let \((M^n, g, f)\) be an \(n\)-dimensional connected compact oriented critical metric with connected boundary \(\partial M\). Suppose that \(\text{Ric} \geq (n-1)\kappa g\), where \(\kappa\) is a constant. Then we have:

\[
\frac{1}{H} \int_{\partial M} \left[ |\nabla_{\partial M} \eta|^2 - (n-1)\kappa \eta^2 + H^2 \eta \langle \nabla u, \nabla f \rangle \right] dS \geq -\kappa (R - n(n-1)\kappa) \int_M f u^2 dV_g,
\]

for any function \(\eta\) on \(\partial M\), where \(H\) stands for the mean curvature of \(\partial M\) and \(u\) is a solution of

\[
\begin{cases}
\Delta u + n\kappa u = 0 & \text{in } M, \\
u = \eta & \text{on } \partial M.
\end{cases}
\]

In particular, for \(\kappa \leq 0\), one has

\[
\int_{\partial M} (|\nabla_{\partial M} \eta|^2 - (n-1)\kappa \eta^2 + H^2 \eta \langle \nabla u, \nabla f \rangle) \, dS \geq 0.
\]

Moreover, equality holds in (1.5) if and only if \((M^n, g)\) is isometric to a geodesic ball in the simply connected space form \(\mathbb{H}^n\) or \(\mathbb{R}^n\).

**Remark 1.** We point out that, for \(\kappa \leq 0\), given any nontrivial \(\eta\) on \(\partial M\), there exists a unique solution \(u\) to the problem (1.4). Moreover, according to [25 Theorem 4], in the case \(\kappa > 0\), one can still solve (1.4), provided that \((M^n, g)\) is not isometric to the standard hemisphere of radius \(\kappa\). In particular, given a critical metric \((M^n, g, f)\), by assuming that \(\eta = -\frac{n}{n-1}\), one easily verifies that \(u = -\frac{R+2n}{n(n-1)}\) is a solution of (1.4) with \(\kappa = \frac{R}{n(n-1)}\).

In light of the above, it is natural to ask whether the assumption on the Ricci curvature \(\text{Ric} \geq (n-1)\kappa g\) in Theorem 1 can be replaced by a lower bound condition on the scalar curvature, as for example, \(R \geq n(n-1)\kappa\). In such a situation, we have the following result.

**Theorem 2.** Let \((M^n, g, f)\) be an \(n\)-dimensional connected compact oriented critical metric with connected boundary \(\partial M\). Suppose that \(R \geq n(n-1)\kappa\), where \(\kappa\) is a non-positive constant. Then we have:

\[
H \int_{\partial M} \langle \nabla f, \nabla u \rangle^2 dS - \frac{(n-1)\kappa}{H} \int_{\partial M} \eta^2 dS \geq \frac{1}{n-1} \int_M |\nabla u|^2 dV_g + n\kappa \int_M u^2 (2n\kappa f - 1) dV_g - (3n - 2)\kappa \int_M f |\nabla u|^2 dV_g,
\]

for any function \(\eta\) on \(\partial M\) and \(u\) a solution of

\[
\begin{cases}
\Delta u + n\kappa u = 0 & \text{in } M, \\
u = \eta & \text{on } \partial M.
\end{cases}
\]

Moreover, equality holds in (1.6) if and only if \(R = n(n-1)\kappa\) and \(\nabla^2 u + \kappa g u = 0\).

Before discussing our next result, we recall that the Yamabe constant for a Riemannian manifold \(M^n\) with boundary \(\partial M\) is given by

\[
\mathcal{Y}(M, \partial M, [g]) = \inf_{0 \neq \phi \in C^\infty(M)} \frac{\int_M \left( \frac{4(n-1)}{n-2} |\nabla \phi|^2 + R \phi^2 \right) dV_g + 2 \int_{\partial M} H \phi^2 dS}{\left( \int_M |\phi|^{\frac{2n}{n-2}} dV_g \right)^{\frac{n-2}{n}}},
\]
where \( H \) is the mean curvature of \( \partial M \) and \( \phi \) is a positive smooth function on \( M^n \). We highlight that \( \mathcal{Y}(M, \partial M, [g]) \) is invariant under a conformal change of the metric \( g \). We refer to [9, 13] for a general discussion on this subject.

It is well-known that the Yamabe invariant plays a crucial role in the study of prescribed metrics. In [6], Barros and Silva showed that a critical metric of the volume functional \((M^n, g, f)\) with connected Einstein boundary \( \partial M \) of positive scalar curvature must satisfy

\[
|\partial M|^{\frac{n-2}{n-1}} \leq \frac{\mathcal{Y}(S^{n-1}, [g_{can}])}{C(R)},
\]

where \( C(R) = \frac{2n-2}{n} R + \frac{n-2}{n-1} H^2 \) is a positive constant and \( \mathcal{Y}(S^{n-1}, [g_{can}]) \) denotes the Yamabe constant of a standard sphere \( S^{n-1} \). In the specific dimension \( n = 4 \), the Yamabe invariant alone is too weak to classify a given manifold. Accordingly, it is natural to impose an additional condition in order to obtain a classification theorem. A recent result due to Baltazar, Diógenes and Ribeiro [2], inspired by the work in [10], establishes a sharp integral curvature estimate involving the Yamabe constant for \( 4 \)-dimensional critical metrics of the volume functional with positive scalar curvature. In the same spirit, as a consequence of Theorem 2, we shall derive the following boundary estimate involving the Yamabe constant for critical metrics of arbitrary dimension \( n \geq 3 \), which can be also seen as an obstruction result for the existence of new examples of critical metrics.

**Theorem 3.** Let \((M^n, g, f)\) be an \( n \)-dimensional connected compact oriented critical metric with connected boundary \( \partial M \) and non-negative scalar curvature. Then we have:

\[
(1.9) \quad \mathcal{Y}(M, \partial M, [g]) \Psi \leq \frac{4(n-1)^2 R^2 + 4n(n^2 - 1)RH^2 + 2n^2(n-2)H^4}{(n-1)^2(n-2)H^3} |\partial M|,
\]

where \( \Psi = \left( \int_M (\Delta_f)^{\frac{n-2}{n-1}} dV_g \right)^{\frac{n-2}{n}} \). Furthermore, if the equality holds in (1.9), then \((M^n, g)\) has zero scalar curvature.

In the work [12, Proposition 2.5], Corvino, Eichmair and Miao showed that the area of the boundary \( \partial M \) of an \( n \)-dimensional scalar flat critical metric must have an upper bound depending on the volume of \( M^n \). In particular, it follows from (2.2) that scalar flat critical metrics must satisfy

\[
\Delta f = -\frac{n}{n-1} f.
\]

Consequently, by the Stokes’ formula, one sees that

\[
Vol(M) = \frac{n-1}{nH} |\partial M|.
\]

This raised the question about the cases of negative and positive scalar curvature. In [3], Baltazar, Diógenes and Ribeiro established a sharp isoperimetric inequality for critical metrics with non-negative scalar curvature. In spite of that, the isoperimetric constant obtained by them depends on the potential function \( f \). It would be interesting to see if such a constant can be improved to depend only on the dimension and mean curvature of the boundary. Other boundary estimates for critical metrics were obtained in, e.g., [6, 7, 12, 14].

In our next result, motivated by these aforementioned facts and the approach used in the proof of Theorem 1 we have established the following estimate.

**Theorem 4.** Let \((M^n, g, f)\) be an \( n \)-dimensional connected compact oriented critical metric with connected boundary \( \partial M \) and positive scalar curvature. Then we have:

\[
(1.10) \quad Vol(M) \geq \frac{n-1}{n} \sqrt{\frac{n(n+2)}{n(n+2)H^2 + 2(n-1)R}} |\partial M|.
\]
Moreover, if the equality holds in \((1.10)\), then \((M^n, g)\) is isometric to a geodesic ball in a simply connected space form \(S^n\).

**Remark 2.** We point out that in the case of negative scalar curvature, as an application of Theorem 2, one can further derive the estimate

\[
Vol(M) \geq \left(\frac{n-1}{nH^2 - R}\right) |\partial M|.
\]

For convenience, the proof of (1.11) will be presented in Proposition 3 of Section 3. It turns out that this estimate is not sharp in the sense that it is not achieved by the geodesic ball in the hyperbolic space \(\mathbb{H}^n\).

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## 2. Background

In this section, we review some basic facts and present some key results that will play a crucial role in the proof of the main theorems.

We start by recalling that a Riemannian manifold \((M^n, g)\) is a critical metric of the volume functional (or simply, critical metric), if there exists a smooth function \(f\) such that

\[
-(\Delta f)g + \nabla^2 f - fRic = g
\]

for \(f \geq 0\) and \(f^{-1}(0) = \partial M\) is the boundary of \(M\). In particular, tracing (2.1) we deduce that the potential function \(f\) also satisfies the equation

\[
\Delta f = -\frac{Rf + n}{n-1}.
\]

From this, we have

\[
\nabla^2 f - fRic = -\frac{Rf + 1}{n-1}g
\]

and

\[
fRic = \hat{\nabla}^2 f,
\]

where \(\hat{T} = T - \frac{1}{n}g\) stands for the traceless of tensor \(T\).

It should be also mentioned that, choosing appropriate coordinates, \(f\) and \(g\) are analytic. Thus, the set of regular points of \(f\) is dense in \(M^n\) (see [12, Proposition 2.1]). Besides, at regular points of \(f\), the vector field \(\nu = -\frac{\nabla f}{|\nabla f|}\) is normal to \(\partial M\) and it is known from [20, Theorem 7] that \(|\nabla f|\) is constant (non null) on each connected component of \(\partial M\). Furthermore, the second fundamental form of \(\partial M\) is given by

\[
\Pi(e_i, e_j) = \langle \nabla_{e_i}\nu, e_j \rangle,
\]

where \(\{e_1, \ldots, e_{n-1}\}\) is an orthonormal frame on \(\partial M\). Thus, one obtains from (2.3) that

\[
\Pi(e_i, e_j) = -\left\langle \nabla_{e_i} \left|\nabla f\right|, e_j \right\rangle = \frac{1}{(n-1)|\nabla f|^2}g_{ij}.
\]

Consequently, the mean curvature is constant \(H = \frac{1}{|\nabla f|}\) and therefore, \(\partial M\) is totally umbilical. For more details, see [7, Sec. 3].

The following generalized Reilly’s formula, obtained previously by Qiu and Xia [23], will be very useful.
Proposition 1 (23). Let \((M^n, g)\) be an \(n\)-dimensional, compact Riemannian manifold with boundary \(\partial M\). Given two functions \(f\) and \(u\) on \(M^n\) and a constant \(\kappa\), we have

\[
\int_M f (\Delta u + \kappa nu)^2 - |\nabla^2 u + \kappa u|^2 \, dV_g = (n - 1) \kappa \int_M (\Delta f + n \kappa f) u^2 \, dV_g
\]

\[
+ \int_M (\nabla^2 f - (\Delta f) g - 2(n - 1) \kappa f g + f \text{Ric}) (\nabla u, \nabla u) \, dV_g
\]

\[
+ \int_{\partial M} f \left[ 2 \left( \frac{\partial u}{\partial \nu} \right) \Delta_{\partial M} u + H \left( \frac{\partial u}{\partial \nu} \right)^2 + \Pi(\nabla_{\partial M} u, \nabla_{\partial M} u) + 2(n - 1) \kappa \left( \frac{\partial u}{\partial \nu} \right) u \right] \, dS
\]

\[
+ \int_{\partial M} \frac{\partial f}{\partial \nu} \left( |\nabla_{\partial M} u|^2 - (n - 1) \kappa u^2 \right) \, dS,
\]

where \(H\) and \(\Pi\) stand for the mean curvature and second fundamental form of \(\partial M\), respectively.

The classical Reilly’s formula is obtained by considering \(f = 1\) and \(\kappa = 0\) in the above expression. For the reader’s convenience, we shall provide a proof of Proposition [17, Proposition 1].

Proof. To begin with, upon integrating by parts over \(M\), one sees that

\[
\int_M \langle \nabla f, \nabla |\nabla u|^2 \rangle \, dV_g = \frac{3}{2} \int_M \langle \nabla f, \nabla |\nabla u|^2 \rangle \, dV_g - \frac{1}{2} \int_M \langle \nabla f, \nabla |\nabla u|^2 \rangle \, dV_g
\]

\[
= \frac{3}{2} \int_M \langle \nabla f, \nabla |\nabla u|^2 \rangle \, dV_g - \int_M \nabla^2 u (\nabla u, \nabla f) \, dV_g
\]

\[
= -\frac{3}{2} \int_M (\Delta f) |\nabla u|^2 \, dV_g + \frac{3}{2} \int_{\partial M} \frac{\partial f}{\partial \nu} |\nabla u|^2 \, dS
\]

\[
- \int_M \nabla^2 u (\nabla u, \nabla f) \, dV_g,
\]

and taking into account that

\[
\nabla^2 f (\nabla u, \nabla u) = \text{div} \left( (\nabla f, \nabla u) \nabla u \right) - \Delta u (\nabla f, \nabla u) - \nabla^2 u (\nabla u, \nabla f),
\]

one obtains that

\[
\int_M \langle \nabla f, \nabla |\nabla u|^2 \rangle \, dV_g = -\frac{3}{2} \int_M (\Delta f) |\nabla u|^2 \, dV_g + \frac{3}{2} \int_{\partial M} \frac{\partial f}{\partial \nu} |\nabla u|^2 \, dS
\]

\[
- \int_{\partial M} \langle \nabla f, \nabla u \rangle \frac{\partial u}{\partial \nu} \, dS + \int_M \nabla^2 f (\nabla u, \nabla u) \, dV_g
\]

\[
+ \int_M \langle \nabla f, \nabla u \rangle \Delta u \, dV_g.
\]

(2.6)

Now, observe that

\[
\frac{1}{2} \Delta (f |\nabla u|^2) = \frac{1}{2} (\Delta f) |\nabla u|^2 + \frac{1}{2} f \Delta |\nabla u|^2 + \langle \nabla f, \nabla |\nabla u|^2 \rangle.
\]

This jointly with (2.6) and the Bochner’s formula:

\[
\frac{1}{2} \Delta |\nabla u|^2 = \text{Ric}(\nabla u, \nabla u) + |\nabla^2 u|^2 + \langle \nabla u, \nabla \Delta u \rangle,
\]
gives
\[
\frac{1}{2} \int_{\partial M} \frac{\partial}{\partial \nu} (f |\nabla u|^2) dS - \int_M f \left( |\nabla^2 u|^2 + \langle \text{Ric} (\nabla u), \nabla u \rangle + \langle \nabla \Delta u, \nabla u \rangle \right) dV_g
\]
\[
= - \int_M (\Delta f) |\nabla u|^2 dV_g - \frac{3}{2} \int_{\partial M} \frac{\partial f}{\partial \nu} |\nabla u|^2 dS - \int_{\partial M} \langle \nabla f, \nabla u \rangle \frac{\partial u}{\partial \nu} dS
\]
\[
(2.7) \quad + \int_M \nabla^2 f (\nabla u, \nabla u) dV_g + \frac{3}{2} \int_{\partial M} \frac{\partial f}{\partial \nu} |\nabla u|^2 dS,
\]

In another direction, upon integrating by parts, we achieve
\[
\int_M f \langle \Delta u, \nabla u \rangle dV_g = - \int_M f (\Delta u)^2 dV_g - \int_M \langle \nabla f, \nabla u \rangle \Delta u dV_g + \int_{\partial M} f (\Delta u) \frac{\partial u}{\partial \nu} dS,
\]
moreover, by using Fermi coordinates, with \( \nabla_{\nu, \nu} = 0 \), it is not difficult to check that
\[
\frac{1}{2} \frac{\partial}{\partial \nu} |\nabla u|^2 = \left\langle \nabla_{\nu, \nabla} \left( \frac{\partial u}{\partial \nu} \right) \right\rangle - \left\langle \nabla_{\nu, \nabla} u, \nabla_{\nu, \nabla} u \right\rangle + \frac{\partial u}{\partial \nu} \left( \Delta u - \Delta_{\nu, \nabla} u - H \frac{\partial u}{\partial \nu} \right).
\]

Plugging these two above expressions into (2.7) yields
\[
\int_{\partial M} f \left[ - \left\langle \nabla_{\nu, \nabla} u, \nabla_{\nu, \nabla} u \right\rangle + \frac{\partial u}{\partial \nu} \left( -2 \Delta_{\nu, \nabla} u - H \frac{\partial u}{\partial \nu} \right) \right] dS - \int_{\partial M} \frac{\partial f}{\partial \nu} |\nabla_{\nu, \nabla} u|^2 dS
\]
\[
= \int_M f \left( |\nabla^2 u|^2 - (\Delta u)^2 \right) dV_g + \int_M - (\Delta f) g + \nabla^2 f + f \text{Ric} (\nabla u, \nabla u) dV_g.
\]

To conclude, it suffices to use the fact
\[
\int_M f \left( |\nabla^2 u|^2 - (\Delta u)^2 \right) dV_g = \int_M f \left[ |\nabla^2 u + \kappa g|^2 - (\Delta u + n \kappa u)^2 \right] dV_g
\]
\[
+ (n - 1) \kappa \left[ \int_{\partial M} \left( 2 f u \frac{\partial u}{\partial \nu} - u^2 \frac{\partial f}{\partial \nu} \right) dS + \int_M (\Delta f) u^2 - 2 f |\nabla u|^2 \right] dV_g
\]
\[
+ n(n - 1) \kappa^2 \int_M f u^2 dV_g.
\]
So, the proof is finished. \( \square \)

Next, we shall establish a key lemma that will be used in the proofs of Theorems 1 and 4.

**Lemma 1.** Let \((M^n, g, f)\) be an \(n\)-dimensional connected compact oriented critical metric with connected boundary \(\partial M\). Then we have:
\[
\frac{1}{H} \int_{\partial M} (|\nabla_{\nu, \nabla} \eta|^2 - (n - 1) \kappa \eta^2) dS = \int_M f |\nabla^2 u + \kappa g|^2 dV_g + \int_M |\nabla u|^2 dV_g
\]
\[
+ 2 \int_M f \left[ \text{Ric} - (n - 1) \kappa g \right] (\nabla u, \nabla u) dV_g
\]
\[
- \kappa (R - n(n - 1) \kappa) \int_M f u^2 dV_g
\]
\[
- n \kappa \int_M u^2 dV_g,
\]
where \(\eta\) is any function on \(\partial M\), \(H\) stands for the mean curvature of \(\partial M\) and \(u\) is a solution of
\[
\begin{cases}
\Delta u + n \kappa u = 0 & \text{in } M, \\
u = \eta & \text{on } \partial M.
\end{cases}
\]
Proof. Initially, we use Proposition 1 and the fact that \( f = 0 \) on \( \partial M \) to infer
\[
\int_M f \left[ (\Delta u + \kappa nu)^2 - |\nabla^2 u + \kappa u g|^2 \right] dV_g = (n - 1) \kappa \int_M (\Delta f + n \kappa f) u^2 dV_g \\
+ \int_M \left[ \nabla^2 f - (\Delta f) g - f \text{Ric} \right] (\nabla u, \nabla u) dV_g \\
+ 2 \int_M f [\text{Ric} - (n - 1) \kappa g] (\nabla u, \nabla u) dV_g \\
+ \int_{\partial M} \frac{\partial f}{\partial \nu} \left[ |\nabla_{\partial M} u|^2 - (n - 1) \kappa u^2 \right] dS.
\]
(2.10)
Substituting (2.1) and (2.2) into (2.10) yields
\[
\int_M f \left[ (\Delta u + \kappa nu)^2 - |\nabla^2 u + \kappa u g|^2 \right] dV_g = 2 \int_M f [\text{Ric} - (n - 1) \kappa g] (\nabla u, \nabla u) dV_g \\
+ \int_M |\nabla u|^2 dV_g - \kappa |R - n(n - 1)\kappa| \int_M u^2 dV_g \\
+ \int_{\partial M} \frac{\partial f}{\partial \nu} \left[ |\nabla_{\partial M} u|^2 - (n - 1) \kappa u^2 \right] dS.
\]
(2.11)
Taking into account that \( \nu = -\frac{\nabla f}{|\nabla f|} \) is the outward unit normal to \( \partial M \) and \( H = \frac{1}{|\nabla f|} \) is the mean curvature of \( \partial M \) with respect to \( \nu \), one sees that \( \frac{\partial f}{\partial \nu} = -|\nabla f| = -\frac{1}{H} \). Hence, applying (2.9) and (2.11), one obtains (2.8), which gives the desired result.
\[\square\]

Now we are ready to present the proofs of the main results.

3. Proof of the Main Results

In this section, we will present the proofs of Theorems 1, 2, 3 and 4.

3.1. Proof of Theorem 1

Proof. First of all, a direct computation shows
\[
\text{div}(u \nabla u) = u \Delta u + |\nabla u|^2 = -n \kappa u^2 + |\nabla u|^2,
\]
where we have used (1.4). Hence, on integrating over \( M^n \), one has
\[
\int_M (|\nabla u|^2 - n \kappa u^2) dV_g = \int_M \text{div}(u \nabla u) dV_g = \int_{\partial M} u(\nabla u, \nu) dS = -H \int_{\partial M} \eta(\nabla u, \nabla f) dS.
\]
This, together with Lemma 1, gives
\[
\frac{1}{H} \int_{\partial M} \left[ |\nabla u|^2 - (n - 1) \kappa \eta^2 + H^2 \eta(\nabla u, \nabla f) \right] dS = \int_M f |\nabla^2 u + \kappa u g|^2 dV_g + 2 \int_M f [\text{Ric} - (n - 1) \kappa g] (\nabla u, \nabla u) dV_g \\
- \kappa (R - n(n - 1)\kappa) \int_M u^2 dV_g.
\]
(3.1)
Thereby, since \( \text{Ric} \geq (n - 1) \kappa g \), one obtains that
\[
\frac{1}{H} \int_{\partial M} \left[ |\nabla \partial_M \eta|^2 - (n-1)\kappa \eta^2 + H^2 \eta (\nabla u, \nabla f) \right] dS \geq -\kappa (R - n(n-1)\kappa) \int_M f u^2 dV_g.
\]

Hence, (1.3) is proved.

We now deal with the equality case. Observe that our assumption also implies that \(R \geq n(n-1)\kappa\) and then, for \(\kappa \leq 0\), one concludes from (3.1) that

\[
(3.2) \quad \int_{\partial M} \left( |\nabla \partial_M \eta|^2 - (n-1)\kappa \eta^2 + H^2 \eta (\nabla u, \nabla f) \right) dS \geq 0.
\]

Whence, equality holds in (3.2) if and only if

\[
\text{Ric}(\nabla u, \nabla u) = (n-1)\kappa |\nabla u|^2,
\]

(3.3)

\[
R = n(n-1)\kappa
\]

and

\[
\nabla^2 u + u\kappa g = 0.
\]

Besides, since \(\text{Ric} \geq (n-1)\kappa g\), one obtains from (3.3) that

\[
\text{Ric} = (n-1)\kappa g = \frac{R}{n} g,
\]

that is, \(g\) is an Einstein metric. To finish the proof, we apply Theorem 1.1 of [19] to conclude that \((M^n, g)\) is isometric to a geodesic ball in a simply connected space form \(\mathbb{H}^n\) or \(\mathbb{R}^n\). \(\Box\)

3.2. Proof of Theorem 2.

Proof. To begin with, one inserts the critical equation (2.1) into the identity from Lemma 1 in order to obtain

\[
\frac{1}{H} \int_{\partial M} \left( |\nabla \partial_M \eta|^2 - (n-1)\kappa \eta^2 \right) dS = \int_M f |\nabla^2 u + \kappa u g|^2 dV_g - \int_M |\nabla u|^2 dV_g
\]

\[
+ 2 \int_M \left[ (-\Delta f) g + \nabla^2 f \right] (\nabla u, \nabla u) dV_g
\]

\[
- 2(n-1)\kappa \int_M f |\nabla u|^2 dV_g
\]

\[
- \kappa (R - n(n-1)\kappa) \int_M f u^2 dV_g - n\kappa \int_M u^2 dV_g.
\]

(3.4)

On the other hand, with aid of (2.5), one sees that

\[
\int_M \nabla^2 f (\nabla u, \nabla u) dV_g = \int_{\partial M} \langle \nabla f, \nabla u \rangle \langle \nabla u, \nu \rangle dS - \int_M \Delta u (\nabla f, \nabla u) dV_g
\]

\[
- \frac{1}{2} \int_M \langle |\nabla u|^2, \nabla f \rangle dV_g
\]

\[
= \int_{\partial M} \langle \nabla f, \nabla u \rangle \langle \nabla u, \nu \rangle dS - \int_M \Delta u (\nabla f, \nabla u) dV_g
\]

\[
- \frac{1}{2} \int_{\partial M} |\nabla u|^2 \langle \nabla f, \nu \rangle dS + \frac{1}{2} \int_M |\nabla u|^2 \Delta f dV_g.
\]

Rearranging terms we get
\[
\int_M \left[ (-\Delta f)g + \nabla^2 f \right] (\nabla u, \nabla u) dV_g = \int_{\partial M} \langle \nabla f, \nabla u \rangle \langle \nabla u, \nu \rangle dS - \int_M \langle \nabla f, \nabla u \rangle \Delta u dV_g \\
- \frac{1}{2} \int_{\partial M} |\nabla u|^2 \langle \nabla f, \nu \rangle dS - \frac{1}{2} \int_M |\nabla u|^2 \Delta f dV_g.
\]

This substituted into (3.4) gives

\[
\frac{1}{H} \int_{\partial M} \left( |\nabla_{\partial M} \eta|^2 - (n - 1)\kappa \eta^2 \right) dS = \int_M f |\nabla^2 u + \kappa ug|^2 dV_g - \int_M |\nabla u|^2 dV_g \\
+ 2 \int_{\partial M} \langle \nabla f, \nabla u \rangle \langle \nabla u, \nu \rangle dS - 2 \int_M \langle \nabla f, \nabla u \rangle \Delta u dV_g \\
- \int_{\partial M} |\nabla u|^2 \langle \nabla f, \nu \rangle dS - \int_M |\nabla u|^2 \Delta f dV_g \\
- 2(n - 1)\kappa \int_M f |\nabla u|^2 dV_g \\
- \kappa (R - n(n - 1)) \int_M f u^2 dV_g - n\kappa \int_M u^2 dV_g.
\]

(3.5)

By using that \( \nu = -\frac{\nabla f}{|\nabla f|} \), \( \Delta u = -n\kappa u \), \( H = \frac{1}{|\nabla f|} \) and (2.2), one obtains that

\[
\frac{1}{H} \int_{\partial M} \left( |\nabla_{\partial M} \eta|^2 - (n - 1)\kappa \eta^2 \right) dS = \int_M f |\nabla^2 u + \kappa ug|^2 dV_g - \int_M |\nabla u|^2 dV_g \\
+ 2 H \int_{\partial M} \langle \nabla f, \nabla u \rangle^2 dS + 2n\kappa \int_M u \langle \nabla f, \nabla u \rangle dV_g \\
+ \frac{1}{H} \int_{\partial M} |\nabla u|^2 dS + R \frac{2(n - 1)^2\kappa}{n-1} \int_M f |\nabla u|^2 dV_g \\
- \kappa (R - n(n - 1)) \int_M f u^2 dV_g - n\kappa \int_M u^2 dV_g.
\]

(3.6)

Proceeding, it is easy to check from (1.7) that

\[
\text{div} (fu \nabla u) = -n\kappa fu^2 + f |\nabla u|^2 + u \langle \nabla f, \nabla u \rangle.
\]

This leads to

\[
\int_M u \langle \nabla f, \nabla u \rangle dV_g = n\kappa \int_M f u^2 dV_g - \int_M f |\nabla u|^2 dV_g,
\]

(3.7)

where we have used that \( f = 0 \) on \( \partial M \). Furthermore, on \( \partial M \), we have

\[
|\nabla u|^2 = \langle \nabla u, \nu \rangle^2 + |\nabla_{\partial M} u|^2 = H^2 \langle \nabla f, \nabla u \rangle^2 + |\nabla_{\partial M} \eta|^2,
\]

so that

\[
\int_{\partial M} |\nabla_{\partial M} \eta|^2 dS - \int_{\partial M} |\nabla u|^2 dS = -H^2 \int_{\partial M} \langle \nabla f, \nabla u \rangle^2 dS.
\]

(3.8)
Hence, substituting (3.7) and (3.8) into (3.6) yields

\[
H \int_{\partial M} \langle \nabla f, \nabla u \rangle^2 dS - \frac{(n-1)\kappa}{H} \int_{\partial M} \eta^2 dS = \int_M |\nabla^2 u + \kappa u|^2 dV_g + \frac{1}{n-1} \int_M |\nabla u|^2 dV_g \\
\quad + 2n^2\kappa^2 \int_M f u^2 dV_g - n\kappa \int_M u^2 dV_g \\
\quad + \frac{R - 2(n-1)(2n-1)\kappa}{n-1} \int_M f |\nabla u|^2 dV_g \\
\quad - \kappa (R - n(n-1)\kappa) \int_M f u^2 dV_g.
\]

(3.9)

Since \( R \geq n(n-1)\kappa \) and \( f > 0 \) in \( M \), it holds that

\[
H \int_{\partial M} \langle \nabla f, \nabla u \rangle^2 dS - \frac{(n-1)\kappa}{H} \int_{\partial M} \eta^2 dS \geq \frac{1}{n-1} \int_M |\nabla u|^2 dV_g - n\kappa \int_M u^2 dV_g \\
\quad + 2n^2\kappa^2 \int_M f u^2 dV_g \\
\quad - (3n-2)\kappa \int_M f |\nabla u|^2 dV_g.
\]

In particular, observe that the equality holds in the above inequality if and only if \( R = n(n-1)\kappa \) and \( \nabla^2 u + \kappa u = 0 \). So, the proof is finished. \( \square \)

3.3. Proof of Theorem [3]

Proof. First of all, one easily verifies that under the hypotheses of Theorem 3, the derivation of (3.9) holds. Moreover, observe that, for

\[
\kappa = \frac{R}{n(n-1)} \quad \text{and} \quad u = \Delta f = - \frac{Rf + n}{n-1},
\]

we have

\[
\begin{cases}
\Delta u + n\kappa u = 0 & \text{in } M, \\
\eta = - \frac{n}{n-1} & \text{on } \partial M,
\end{cases}
\]

that is, \( u \) satisfies the problem (1.7). This fact jointly with (3.9) allows us to infer

\[
\frac{R^2 - nH^2 R}{(n-1)^2 H^3 |\partial M|} = \int_M |\nabla^2 \Delta f|^2 dV_g + \frac{R^2}{(n-1)^3} \int_M |\nabla f|^2 dV_g + \frac{2R^2}{n(n-1)^3} \int_M f |\nabla f|^2 dV_g \\
\quad - \frac{R}{n-1} \int_M (\Delta f)^2 dV_g - \frac{(3n-2)R^3}{n(n-1)^3} \int_M f |\nabla f|^2 dV_g \\
\quad = \int_M |\nabla^2 \Delta f|^2 dV_g + \frac{R^2}{(n-1)^3} \int_M |\nabla f|^2 dV_g + \frac{2R^2}{n(n-1)^3} \int_M f |\nabla f|^2 dV_g \\
\quad - \frac{R}{n-1} \int_M (\Delta f)^2 dV_g - \frac{(3n-2)R^3}{2n(n-1)^3} \int_M f^2 \Delta f dV_g,
\]

(3.12)

where we have used integration by parts.

From the assumption \( R \geq 0 \) and choosing \( \phi = -\Delta f > 0 \) in (1.8), we then deduce the following inequality

\[
\frac{R^2}{(n-1)^2} \int_M |\nabla f|^2 dV_g \geq \frac{n-2}{4(n-1)} \mathcal{V}(M, \partial M, [g]) \Psi - \frac{(n-2)R}{4(n-1)} \int_M (\Delta f)^2 dV_g \\
\quad - \frac{n^2(n-2)H}{2(n-1)^3} |\partial M|,
\]

(3.13)
where \( \Psi = \left( \int_M (\Delta f)^{\frac{n-2}{2}} dV_g \right)^{\frac{n-2}{n-4}} \).

Plugging (3.13) into (3.12) yields

\[
\frac{n-2}{4(n-1)^2} \Upsilon(M, \partial M, [g]) \Psi \leq \frac{R^2 - nH^2 R}{(n-1)^2 H^3} |\partial M| - \int_M f |\nabla^2 \Delta f|^2 dV_g
\]

\[
+ \frac{(n-2)R}{4(n-1)^2} \int_M (\Delta f)^2 dV_g + \frac{n^2(n-2)H}{2(n-1)^4} |\partial M|
\]

\[
- \frac{2R^2}{(n-1)^2} \int_M f (\Delta f)^2 dV_g + \frac{R}{n-1} \int_M (\Delta f)^2 dV_g
\]

\[
- \frac{3n-2}{2(n-1)^3} \int_M f^2 \Delta f dV_g.
\]

By using (2.2) and rearranging terms, one sees that

\[
\frac{n-2}{4} \Upsilon(M, \partial M, [g]) \Psi \leq \frac{2(n-1)^2 R(R - nH^2) + n^2(n-2)H^4}{2(n-1)^2 H^3} |\partial M|
\]

\[
- \int_M f |\nabla^2 \Delta f|^2 dV_g + \frac{(5n-6)R}{4(n-1)^2} \int_M (\Delta f)^2 dV_g
\]

\[
- \frac{2R^2}{(n-1)^2} \int_M f (\Delta f)^2 dV_g - \frac{(3n-2)R^3}{2n(n-1)^3} \int_M f^2 \Delta f dV_g.
\]

By using (2.2) guarantees that

\[
Rf \Delta f = -(n-1)(\Delta f)^2 - n \Delta f,
\]

which substituted into (3.14) gives

\[
\frac{n-2}{4} \Upsilon(M, \partial M, [g]) \Psi \leq \frac{2(n-1)^2 R(R - nH^2) + n^2(n-2)H^4}{2(n-1)^2 H^3} |\partial M|
\]

\[
- \int_M f |\nabla^2 \Delta f|^2 dV_g + \frac{(5n-6)R}{4(n-1)^2} \int_M (\Delta f)^2 dV_g
\]

\[
+ \frac{(n+2)R^3}{2n(n-1)^2} \int_M f^2 \Delta f dV_g - 2n \int_M (\Delta f)^2 dV_g
\]

\[
- \frac{2n^2 R}{n-1} \int_M \Delta f dV_g.
\]

Hence, we arrive at
\[ \frac{n-2}{4} \mathcal{Y}(M, \partial M, [g]) \Psi \leq \frac{2(n-1)^2 R^2 + 2n(n^2 - 1)RH^2 + n^2(n-2)H^4}{2(n-1)^2 H^3} |\partial M| \]

\[-(n-1)^2 \int_M f |\nabla^2 \Delta f|^2 dV_g \leq \frac{3(n+2)R}{4} \int_M (\Delta f)^2 dV_g \]

\[+ \frac{(n+2)R^3}{2n(n-1)} \int_M f^2 \Delta f dV_g. \]

Again, since \( R \geq 0 \), we have from (2.2) that \( \Delta f \leq 0 \) and therefore, we obtain

\[(3.15) \quad \mathcal{Y}(M, \partial M, [g]) \Psi \leq \frac{4(n-1)^2 R^2 + 4n(n^2 - 1)RH^2 + 2n^2(n-2)H^4}{(n-1)^2(n-2)H^4} |\partial M|, \]

which proves the asserted inequality.

To conclude, if the equality holds, then one has

\[ \nabla^2 \Delta f = \frac{\Delta(\Delta f)}{n} g = -\frac{R}{n(n-1)} \Delta f g, \]

and

\[ \frac{(n+2)R^3}{2n(n-1)} \int_M f^2 \Delta f dV_g = 0, \]

(3.16)

Finally, if \( R \) is not zero, we conclude from (3.16) and (2.2) that \( f \) is constant (non-zero), but it must vanish along the boundary, which leads to a contradiction. Hence, \( R = 0 \) and \( \Delta f = -\frac{n}{n-1} \) on \( M^n \). This finishes the proof of the theorem.

\[ \square \]

3.4. Proof of Theorem 4.

Proof. By using the classical Bochner’s formula, we have

\[ 2f Ric(\nabla u, \nabla u) = f \Delta |\nabla u|^2 - 2f |\nabla^2 u|^2 - 2f \langle \nabla \Delta u, \nabla u \rangle \]

\[ = f \Delta |\nabla u|^2 - 2f |\nabla^2 u|^2 + 2nf |\nabla u|^2. \]

From this, it follows that

\[ 2 \int_M f [Ric - (n-1)\kappa g] (\nabla u, \nabla u) dV_g = \int_M f \Delta |\nabla u|^2 dV_g - 2 \int_M f |\nabla^2 u|^2 dV_g \]

\[+ 2\kappa \int_M f |\nabla u|^2 dV_g. \]

(3.17)

Moreover, by Green’s identity and the fact that \( f = 0 \) on \( \partial M \), one sees that

\[ \int_M f \Delta |\nabla u|^2 dV_g = \int_M |\nabla u|^2 \Delta f dV_g \]

\[+ \int_{\partial M} \left( f \left| \nabla |\nabla u|^2 - \frac{\nabla f}{\nabla |\nabla u|^2} \right| - |\nabla u|^2 \frac{\nabla f}{\nabla |\nabla u|^2} \right) dS \]

\[= -\frac{R}{n-1} \int_M f |\nabla u|^2 dV_g - \frac{n}{n-1} \int_M |\nabla u|^2 dV_g + \frac{1}{H} \int_{\partial M} |\nabla u|^2 dS. \]

This, together with (3.17), yields
\[ 2 \int_M f [Ric - (n - 1)\kappa] (\nabla u, \nabla u) dV_g = \frac{R}{n - 1} \int_M f|\nabla u|^2 dV_g - \frac{n}{n - 1} \int_M |\nabla u|^2 dV_g + \frac{1}{H} \int_{\partial M} |\nabla u|^2 dS - 2 \int_M f|\nabla^2 u|^2 dV_g + 2\kappa \int_M f|\nabla u|^2 dV_g, \]

which compared with Lemma 1 gives

\[ -\frac{1}{H} \int_{\partial M} [|\nabla_{\partial M}\eta|^2 - (n - 1)\kappa\eta^2] dS = -\int_M f|\nabla^2 u + \kappa u g|^2 dV_g - \frac{1}{n - 1} \int_M |\nabla u|^2 dV_g - \frac{R - 2(n - 1)\kappa}{n - 1} \int_M f|\nabla u|^2 dV_g + \frac{1}{H} \int_{\partial M} |\nabla u|^2 dS - 2\kappa \int_M f|\nabla u|^2 dV_g \]

Now, choosing \( \kappa \) and \( u \) as in (3.10), we conclude that \( u \) must satisfy

\[ \begin{cases} 
\Delta u + n\kappa u = 0 & \text{in } M, \\
u = -\frac{n}{n - 1} & \text{on } \partial M.
\end{cases} \]

Plugging this fact into (3.19) we infer

\[ -\frac{nR}{(n - 1)^2 H} |\partial M| = -\int_M f|\nabla^2 f|^2 dV_g + \frac{R\Delta f}{n(n - 1)} g|f|^2 dV_g - \frac{R^2}{(n - 1)^3} \int_M |\nabla f|^2 dV_g \]

Since

\[ \int_M |\nabla f|^2 dV_g = -\int_M f\Delta f dV_g \quad \text{and} \quad \int_M f|\nabla f|^2 dV_g = -\frac{1}{2} \int_M f^2 \Delta f dV_g, \]

\[ -\frac{nR}{(n - 1)^2 H} |\partial M| = -\int_M f|\nabla^2 f|^2 dV_g + \frac{R\Delta f}{n(n - 1)} g|f|^2 dV_g + \frac{R^2}{(n - 1)^3} \int_M |\nabla f|^2 dV_g - \frac{R^2}{n(n - 1)^2} \int_M (\Delta f)^2 dV_g, \]

\[ -\frac{2R^2}{n(n - 1)^2} \int_M f|\nabla f|^2 dV_g - \frac{R}{n - 1} \int_M (\Delta f)^2 dV_g. \]
we then have
\[- \frac{(nH^2 + R)R}{(n-1)^2 H^2} |\partial M| = - \int_M f |\nabla^2 \Delta f|^2 dV_g + \frac{R^2}{(n-1)^2} \int_M f \Delta f dV_g
+ \frac{(n-2)R^3}{2n(n-1)^3} \int_M f^2 dV_g - \frac{2R^2}{n(n-1)^2} \int_M f(\Delta f)^2 dV_g
- \frac{R}{n-1} \int_M (\Delta f)^2 dV_g\]
\[- \int_M f |\nabla^2 \Delta f|^2 dV_g + \frac{R^2}{(n-1)^2} \int_M f \Delta f dV_g
+ \frac{(n-2)R^3}{2n(n-1)^3} \int_M f^2 dV_g - \frac{2R^2}{n(n-1)^2} \int_M f(\Delta f)^2 dV_g
- \frac{R}{n-1} \int_M (\Delta f)^2 dV_g\]
\[- \int_M f |\nabla^2 \Delta f|^2 dV_g + \frac{3R^3}{(n-1)^3} \int_M f \Delta f dV_g
+ \frac{(n+2)R^3}{2n(n-1)^3} \int_M f^2 dV_g - \frac{R}{n-1} \int_M (\Delta f)^2 dV_g.\]

Taking into account that
\[f \Delta f = - \frac{n-1}{R} (\Delta f)^2 - \frac{n}{R} \Delta f,\]
once sees that
\[- \frac{(nH^2 + R)R}{(n-1)^2 H^2} |\partial M| = - \int_M f |\nabla^2 \Delta f|^2 dV_g + \frac{(n+2)R}{(n-1)^2} \int_M (\Delta f)^2 dV_g
- \frac{3nR}{(n-1)^3} \int_M \Delta f dV_g + \frac{(n+2)R^3}{2n(n-1)^3} \int_M f^2 \Delta f dV_g.\]
(3.21)

Consequently,
\[- \frac{[n(n+2)H^2 + (n-1)R]R}{(n-1)^3 H^3} |\partial M| = - \int_M f |\nabla^2 \Delta f|^2 dV_g + \frac{(n+2)R}{(n-1)^2} \int_M (\Delta f)^2 dV_g
+ \frac{(n+2)R^3}{2n(n-1)^3} \int_M f^2 \Delta f dV_g.\]
(3.22)

Next, since \(R > 0\) and \(f \geq 0\), it follows from (2.2) that \(-\Delta f \geq 0\) and hence, by Holder’s inequality, we achieve
\[\left( \int_M (-\Delta f) (\sqrt{-\Delta f}) dV_g \right)^2 \leq \int_M f^2 (-\Delta f) dV_g \cdot \int_M (-\Delta f) dV_g,\]
so that
\[\left( \int_M f \Delta f dV_g \right)^2 \leq - \frac{1}{H} |\partial M| \int_M f^2 \Delta f dV_g.\]
Thereby, one has

\[-\frac{1}{H} |\partial M| \int_M f^2 \Delta f \, dV_g \geq \left( -\frac{n-1}{R} \int_M (\Delta f)^2 \, dV_g - \frac{n}{R} \int_M \Delta f \, dV_g \right)^2 \]

\[= \left( -\frac{n-1}{R} \int_M (\Delta f)^2 \, dV_g + \frac{n}{HR} |\partial M| \right)^2 \]

\[= \frac{(n-1)^2}{R^2} \left( \int_M (\Delta f)^2 \, dV_g \right)^2 - \frac{2(n-1)}{HR^2} |\partial M| \int_M (\Delta f)^2 \, dV_g \]

\[+ \frac{n^2}{HR^2} |\partial M|^2 , \]

which can be rewritten as

\[\int_M f^2 \Delta f \, dV_g \leq -\frac{(n-1)^2 H}{R^2 |\partial M|} \left( \int_M (\Delta f)^2 \, dV_g \right)^2 + \frac{2(n-1)}{R^2} \int_M (\Delta f)^2 \, dV_g \]

(3.23)

Substituting (3.23) into (3.22), one obtains that

\[-\frac{(n+2)H^2 + (n-1)R^2}{(n-1)^3 H^3} |\partial M| \leq -\int_M f |\nabla^2 \Delta f|^2 \, dV_g - \frac{(n+2)R}{(n-1)^2} \int_M \Delta f \, dV_g \]

\[+ \frac{(n+2)R}{(n-1)^2} \int_M (\Delta f)^2 \, dV_g \]

\[-\frac{n(n+2)R}{2(n-1)^3 H} |\partial M| , \]

Rearranging terms we get

\[-\frac{n(n+2)H^2 + 2(n-1)R |\partial M|}{2(n-1)^3 H^3} \leq -\int_M f |\nabla^2 \Delta f|^2 \, dV_g \]

(3.24)

Again, by Holder’s inequality, one sees that

\[\text{Vol}(M) \int_M (\Delta f)^2 \, dV_g \geq \left( \int_M \Delta f \, dV_g \right)^2 = \frac{1}{H^2} |\partial M|^2 . \]

This jointly with (3.24) gives

\[-\frac{(n+2)H^2 + 2(n-1)R}{2(n-1)^3 H^3} |\partial M| \leq -\int_M f |\nabla^2 \Delta f|^2 \, dV_g - \frac{2(n-1) |\partial M|^3}{(n+2)R^2 (\text{Vol}(M))^2} \]

(3.25)

Thus, since \( R > 0 \), one concludes that

\[\text{Vol}(M) \geq \frac{n-1}{n} \sqrt{\frac{n(n+2)}{n(n+2)H^2 + 2(n-1)R} |\partial M|} , \]

which proves (1.10).
Finally, if the equality holds, then
\[ \nabla^2 \Delta f = \frac{\Delta(\Delta f)}{n} g = -\frac{R}{n(n-1)} \Delta f g. \]
Moreover, since \( \Delta f = \frac{-n}{n-1} \) on the boundary \( \partial M \) (i.e., \( \Delta f \) is constant on \( \partial M \)), we may use Theorem B (II) of \[25\] to deduce that \((M^n, g)\) has constant sectional curvature \( \frac{R}{n(n-1)} \) and this forces \((M^n, g)\) to be an Einstein manifold. Finally, it suffices to invoke Theorem 1.1 of \[19\] to conclude that \((M^n, g)\) is isometric to a geodesic ball in a simply connected space form \( S^n \). Hence, Theorem 4 is proved. \( \square \)

To conclude, we will present the proof of the estimate \((1.11)\) stated in Remark 2. More precisely, we have the following proposition.

**Proposition 2.** Let \((M^n, g, f)\) be a connected compact oriented critical metric with connected boundary \(\partial M\) and negative scalar curvature. Then we have:

\[
Vol(M) \geq \frac{(n-1)H}{nH^2 - R} |\partial M|.
\]

**Proof.** Initially, taking \( \kappa = \frac{R}{n(n-1)} < 0 \) in Theorem \[2\] one obtains that

\[
H \int_{\partial M} \langle \nabla f, \nabla u \rangle^2 dS - \frac{R}{nH} \int_{\partial M} \eta^2 dS \geq \frac{1}{n-1} \int_M |\nabla u|^2 dV_g - \frac{R}{n-1} \int_M u^2 dV_g + \frac{2R^2}{(n-1)^2} \int_M f u^2 dV_g - \frac{(3n-2)R}{n(n-1)} \int_M f |\nabla u|^2 dV_g
\]

(3.26)

Next, choosing \( u = \Delta f = \frac{-Rf + n}{n-1} \), we infer that \( u \) must satisfy

\[
\begin{cases} 
\Delta u + n\kappa u = 0 & \text{in } M, \\
u = -\frac{n}{n-1} & \text{on } \partial M.
\end{cases}
\]

Plugging this into (3.26), one sees that

\[
H \frac{R^2}{(n-1)^2} \int_{\partial M} |\nabla f|^4 dS - \frac{nR}{(n-1)^2H} |\partial M| \geq -\frac{R}{n-1} \int_M (\Delta f)^2 dV_g.
\]

Since \( H = \frac{1}{|\nabla f|} \), we deduce

(3.27)

\[
\frac{R(R-nH^2)}{(n-1)^2H^2} |\partial M| \geq -\frac{R}{n-1} \int_M (\Delta f)^2 dV_g.
\]

On the other hand, by Holder’s inequality we get

\[
Vol(M) \int_M (\Delta f)^2 dV_g \geq \left( \int_M \Delta f dV_g \right)^2 = \left( \int_{\partial M} \langle \nabla f, -\nabla f \rangle dS \right)^2 = \frac{1}{H^2} |\partial M|^2.
\]

This jointly with (3.27) yields

\[
Vol(M) \geq \frac{(n-1)H}{nH^2 - R} |\partial M|,
\]

which proves the asserted result. \( \square \)
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