A note on real forms of the complex N=4 supersymmetric Toda chain hierarchy in real N=2 and N=4 superspaces

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Abstract

Three inequivalent real forms of the complex N=4 supersymmetric Toda chain hierarchy (Nucl. Phys. B558 (1999) 545, solv-int/9907004) in the real N = 2 superspace with one even and two odd real coordinates are presented. It is demonstrated that the first of them possesses a global N = 4 supersymmetry, while the other two admit a twisted N = 4 supersymmetry. A new superfield basis in which supersymmetry transformations are local is discussed and a manifest N = 4 supersymmetric representation of the N = 4 Toda chain in terms of a chiral and an anti-chiral N = 4 superfield is constructed. Its relation to the complex N = 4 supersymmetric KdV hierarchy is established. Darboux-Backlund symmetries and a new real form of this last hierarchy possessing a twisted N = 4 supersymmetry are derived.

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1. Introduction. Recently the Lax pair representation of the even and odd flows of the complex $N = 4$ supersymmetric Toda chain hierarchy in $N = 2$ superspace were constructed in [1]. The corresponding local and nonlocal Hamiltonians, the finite and infinite discrete symmetries, the first two Hamiltonian structures and the recursion operator connecting all evolution equations and the Hamiltonian structures were also studied. The goal of the present letter is first to analyse the possible real forms of the $N = 4$ Toda chain hierarchy in $N = 2$ superspace, second to derive a manifest $N = 4$ supersymmetric representation of its first nontrivial even flows in the real $N = 4$ superspace, and third to clarify its relation (if any) with the $N = 4$ supersymmetric KdV hierarchy.

Let us start with a short summary of the results that we shall need concerning the complex $N = 4$ supersymmetric Toda chain hierarchy (see [1, 2, 3, 4] for more details).

The complex $N = 4$ supersymmetric Toda chain hierarchy in the complex $N = 2$ superspace comprises an infinite set of even and odd flows for two complex even $u(z, \theta^+, \theta^-)$ and $v(z, \theta^+, \theta^-)$, where $z$ and $\theta^\pm$ are complex even and odd coordinates, respectively. The flows are generated by complex even and odd evolution derivatives \( \{ \frac{\partial}{\partial z_k}, U^\pm_k, U_k, U_k^- \} \) and \( \{ D^\pm_k, Q_k^\pm \} \) \((k \in \mathbb{N})\), respectively, with the following length dimensions:

\[
[\frac{\partial}{\partial z_k}] = [U^\pm_k] = [U_k^-] = -k, \quad [D^\pm_k] = [Q_k^\pm] = -k + \frac{1}{2}. \tag{1}
\]

The first few of these flows are:

\[
\frac{\partial}{\partial t_3} \left( \begin{array}{c} v \\ u \end{array} \right) = \partial \left( \begin{array}{c} v \\ u \end{array} \right), \tag{2}
\]

\[
\frac{\partial}{\partial t_2} v = +v'' + 2uv(D_+D_-v) - (D_+D_-v^2u) - v^2(D_+D_-u) + 2v(\theta^+), \quad \frac{\partial}{\partial t_2} u = -u'' + 2uv(D_+D_-u) - (D_+D_-u^2v) - u^2(D_+D_-v) - 2u(\theta^-), \tag{3}
\]

\[
\frac{\partial}{\partial t_3} v = v''' - 3(D_+v)'(D_-uv) + 3(D_-v)'(D_+uv) - 3v'(D_+uv)(D_-u) + 3v'(D_-uv)(D_+u) - 6uv'(D_+D_-u) + 6(\theta^+), \quad \frac{\partial}{\partial t_3} u = u''' - 3(D_-u)'(D_+uv) + 3(D_+u)'(D_-uv) - 3u'(D_-uv)(D_+u) + 3u'(D_+uv)(D_-u) - 6uu'(D_-D_+u) + 6(\theta^-), \tag{4}
\]

\[
D^+_1 v = -D_\pm v \pm 2vD^{-1}_\mp(uv), \quad D^+_1 u = -D_\pm u \mp 2uD^{-1}_\mp(uv), \tag{5}
\]

\[
Q^\pm_1 \left( \begin{array}{c} v \\ u \end{array} \right) = Q^\pm \left( \begin{array}{c} v \\ u \end{array} \right), \tag{6}
\]

\[
U^\pm_0 v = \frac{1}{2} v - \theta^\pm(D_\pm v \mp 2vD^{-1}_\mp(uv)), \quad U^\pm_0 u = \frac{1}{2} u - \theta^\pm(D_\pm u \mp 2uD^{-1}_\mp(uv)), \tag{7}
\]

\[
U_0 v = -\theta^+(D_\pm v + 2vD^{-1}_\mp(uv)) - \theta^-(D_\mp v - 2vD^{-1}_\mp(uv)), \quad U_0 u = -\theta^+(D_\mp u - 2uD^{-1}_\pm(uv)) - \theta^-(D_\pm u + 2uD^{-1}_\pm(uv)), \tag{8}
\]

1
\[ \mathcal{U}_0 \begin{pmatrix} v \\ u \end{pmatrix} = \frac{1}{2} \left( \theta^-(D_+ + Q_+) - \theta^+(D_- + Q_-) \right) \begin{pmatrix} v \\ u \end{pmatrix}. \]

(9)

Throughout this letter, we shall use the notations \( u' = \partial u = \frac{\partial}{\partial z} u \). \( D_\pm \) and \( Q_\pm \) are odd covariant derivatives and supersymmetry generators,

\[ D_\pm \equiv \frac{\partial}{\partial \theta^\pm} + \theta^\pm \partial, \quad Q_\pm \equiv \frac{\partial}{\partial \theta^\pm} - \theta^\pm \partial. \]

(10)

They form the algebra\(^1\)

\[ \{D_\pm, D_\pm\} = +2 \partial, \quad \{Q_\pm, Q_\pm\} = -2 \partial. \]

(11)

Using the explicit expressions of the flows (2–9), one can calculate their algebra which has the following nonzero brackets:

\[ \left[ U_0, U_0^\pm \right] = \pm \mathcal{U}_0, \quad \left[ \mathcal{U}_0, U_0^\pm \right] = \pm U_0, \quad \left[ U_0, \mathcal{U}_0 \right] = 2(U_0^+ - U_0^-), \]

(13)

\[ \left[ U_0^\pm, D_1^\pm \right] = -Q_1^\pm, \quad \left[ U_0^\pm, Q_1^\pm \right] = -D_1^\pm, \]
\[ \left[ U_0^\pm, D_1^\mp \right] = +Q_1^\mp, \quad \left[ U_0^\pm, Q_1^\mp \right] = +D_1^\mp, \]
\[ \left[ \mathcal{U}_0^\pm, D_1^\pm \right] = \pm D_1^\mp, \quad \left[ \mathcal{U}_0^\pm, Q_1^\mp \right] = \pm Q_1^\mp. \]

(14)

This algebra reproduces the algebra of the global complex \( N = 4 \) supersymmetry, together with its \( gl(2, \mathbb{C}) \) automorphisms. It is the algebra of symmetries of the nonlinear even flows (3–4). It may be realized in the superspace \( \{ t_k, \theta_k^\pm, \rho_k^\pm, h_k^\pm, \mathcal{h}_k \} \), where \( t_k, h_k^\pm, h_k, \mathcal{h}_k \) (\( \theta_k^\pm, \rho_k^\pm \)) are complex even (odd) abelian evolution times with the length dimensions

\[ [t_k] = [h_k^0] = [h_k] = [\mathcal{h}_k] = k, \quad [\theta_k^\pm] = [\rho_k^\pm] = k - \frac{1}{2} \]

(15)

which are in one-to-one correspondence with the length dimensions \( [\mathbb{I}] \) of the corresponding evolution derivatives.

2. Real forms of the \( N=4 \) Toda chain hierarchy. It is well known that different real forms derived from the same complex integrable hierarchy are inequivalent in general. Keeping this in mind it seems important to find as many different real forms of the \( N = 4 \) Toda chain hierarchy as possible.

With this aim let us discuss various inequivalent complex conjugations of the superfields \( u(z, \theta^+, \theta^-) \) and \( v(z, \theta^+, \theta^-) \), of the superspace coordinates \( \{ z, \theta^\pm \} \), and of the evolution derivatives \( \{ \frac{\partial}{\partial t_k}, U_k^\pm, U_k, \mathcal{U}_k^\pm, D_k^\pm, Q_k^\pm \} \) which should be consistent with the flows (3–4). We restrict our considerations to the case when \( iz \) and \( \theta^\pm \) are coordinates of the real \( N = 2 \) superspace which satisfy the following standard complex conjugation properties:

\[ (iz, \theta^\pm)^* = (iz, \theta^\mp), \]

(16)

\(^1\)Hereafter, we explicitly present only non-zero brackets.
where \( i \) is the imaginary unity. We will also use the standard convention regarding complex conjugation of products involving odd operators and functions (see, e.g., the books [3]). In particular, if \( \mathcal{O} \) is some even differential operator acting on a superfield \( F \), we define the complex conjugate of \( \mathcal{O} \) by \( (\mathcal{O}F)^* = \mathcal{O}^*F^* \). Then, in the case under consideration one can derive, for example, the following relations

\[
\partial^* = -\partial, \quad (\epsilon^*)^* = \epsilon^\mp, \quad (\epsilon^\mp \epsilon^-)^* = -\epsilon^\pm \epsilon^-, \quad (\epsilon^\mp D_\pm)^* = \epsilon^\pm D_\pm, \quad (D_+D_-)^* = -D_+D_-
\]

which we use in what follows. Here, \( \epsilon^\pm \) are constant odd real parameters.

Direct verification shows that the flows (12–14) admit the three inequivalent complex conjugations:

\[
(v, u)^* = (-v, u), \quad (z, \theta^\pm)^* = (-z, \theta^\pm),
\]

\[
(t_p, U^\pm_p, U_p, i\epsilon^\pm_p D^\pm_p, \epsilon^\pm_p Q^\pm_p)^* = (-1)^p(t_p, U^\pm_p, U_p, -\epsilon^\pm_p D^\pm_p, -\epsilon^\pm_p Q^\pm_p), \quad \text{(18)}
\]

\[
(v, u)^* = (u, v), \quad (z, \theta^\pm)^* = (-z, \theta^\pm),
\]

\[
(t_p, U^\pm_p, U_p, i\epsilon^\pm_p D^\pm_p, \epsilon^\pm_p Q^\pm_p)^* = (-t_p, U^\pm_p, U_p, \epsilon^\pm_p D^\pm_p, \epsilon^\pm_p Q^\pm_p), \quad \text{(19)}
\]

\[
(v, u)^* = (-u(D_+D_\pm \ln u + uv), \frac{1}{u}), \quad (z, \theta^\pm)^* = (-z, \theta^\pm),
\]

\[
(t_p, U^\pm_p, U_p, i\epsilon^\pm_p D^\pm_p, \epsilon^\pm_p Q^\pm_p)^* = (-t_p, -U^\pm_p, -U_p, -\epsilon^\pm_p D^\pm_p, \epsilon^\pm_p Q^\pm_p), \quad \text{(20)}
\]

where \( \epsilon^\pm_p \) and \( \epsilon^\pm_p \) are constant odd real parameters. We would like to underline that the complex conjugations of the evolution derivatives (the second lines of eqs. (18–20)) are defined and fixed completely by the explicit expressions (12–14) for the flows. These complex conjugations extract different real forms of the algebra (12–14). The real forms of the algebra (12–14) with the involutions (18–19) correspond to a twisted real \( N = 4 \) supersymmetry, while the real form corresponding to the involution (20) reproduces the algebra of the real \( N = 4 \) supersymmetry. This last fact becomes more obvious if one uses the \( N = 2 \) basis of the algebra with the generators

\[
\Sigma_1 \equiv U_0, \quad \Sigma_2 \equiv -i\overline{U}_0, \quad \Sigma_3 \equiv U_0^\mp - U_0^\pm, \quad \Sigma \equiv U_0^- + U_0^+,
\]

\[
\mathcal{D}_1 \equiv Q_1^+ + D_1^+, \quad \mathcal{D}_2 \equiv Q_1^- + D_1^-, \quad \overline{\mathcal{D}}^1 \equiv Q_1^+ - D_1^+, \quad \overline{\mathcal{D}}^2 \equiv Q_1^- - D_1^-.
\]

(21)

Then, the nonzero algebra brackets (12–14) and the complex conjugation rule (20) are the standard ones for the real \( N = 4 \) supersymmetry algebra together with its \( u(2) \) automorphisms

\[
\{\mathcal{D}_\alpha, \overline{\mathcal{D}}^\beta\} = 4\delta_\alpha^\beta \frac{\partial}{\partial t_1}, \quad [\Sigma_\alpha, \Sigma_\beta] = 2i\epsilon_{abc}\Sigma_c,
\]

\[
[\Sigma_\alpha, \mathcal{D}_\beta] = (\sigma_\alpha)_\beta^\gamma \mathcal{D}_\gamma, \quad [\Sigma_\alpha, \overline{\mathcal{D}}^\beta] = -\overline{\mathcal{D}}^\gamma (\sigma_\alpha)_\beta^\gamma,
\]

\[
[\Sigma, \mathcal{D}_\alpha] = -\mathcal{D}_\alpha, \quad \Sigma, \overline{\mathcal{D}}^\alpha] = \overline{\mathcal{D}}^\alpha,
\]

(22)

\[
\left(\frac{\partial}{\partial t_1}, \Sigma_\alpha, \Sigma, \epsilon^\alpha \mathcal{D}_\alpha, \epsilon^\alpha \overline{\mathcal{D}}^\alpha\right)^* = -\left(\frac{\partial}{\partial t_1}, \Sigma_\alpha, \Sigma, -\epsilon^\alpha \overline{\mathcal{D}}^\alpha, -\epsilon^\alpha \mathcal{D}_\alpha\right), \quad (\epsilon^\alpha, \epsilon^\alpha)^* = (\epsilon^\alpha, \epsilon^\alpha).
\]

(23)

Here, \( \epsilon^\alpha, \epsilon^\alpha \) are constant odd parameters, a summation over repeated indices \( \alpha, \beta = 1, 2 \) and \( a, b, c = 1, 2, 3 \) is understood in eqs. (22), and \( \sigma_\alpha \) are the Pauli matrices

\[
\sigma_1 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_a \sigma_b = \delta_{ab} + i\epsilon_{abc}\sigma_c,
\]

(24)
and $\epsilon_{abc}$ is a totally antisymmetric tensor ($\epsilon_{123} = 1$). Therefore, we conclude that the complex $N = 4$ supersymmetric Toda chain hierarchy with the complex conjugation \((20)\) possesses a real $N = 4$ supersymmetry, and due to this remarkable fact it can be called the $N = 4$ supersymmetric Toda chain hierarchy (for some examples of $N = 4$ supersymmetric integrable systems, see \[4\]–\[12\] and references therein).

Let us remark that a combination of the two involutions \((20)\) and \((19)\) generates the infinite-dimensional group of discrete Darboux transformations \[2, 3, 1\]

\[
(v, u)^* = (v(D_+ D_- \ln v - uv), \frac{1}{v}), \quad (z, \theta^\pm)^* = (z, \theta^\pm),
\]

\[
(t_p, U^\pm_p, U_p, D^\pm_p, Q^\pm_p)^* = (t_p, -U^\pm_p, -U_p, D^\pm_p, Q^\pm_p).
\]

This way of deriving discrete symmetries was proposed in \[13\] and applied to the construction of discrete symmetry transformations of the $N = 2$ supersymmetric GNLS hierarchies.

3. A KdV-like basis with locally realized supersymmetries. The third complex conjugation \((20)\) looks rather complicated when compared to the first two ones \((18–19)\). However, in another superfield basis defined as

\[
J \equiv uv + D_- D_+ \ln u, \quad \overline{J} \equiv -uv,
\]

where $J$ and $\overline{J}$ ($[J] = [\overline{J}] = -1$) are unconstrained even $N = 2$ superfields, it drastically simplifies. In this basis the complex conjugations \((18–20)\) are given by

\[
(J, \overline{J})^* = -(J, \overline{J}),
\]

\[
(J, \overline{J})^* = (J - D_- D_+ \ln \overline{J}, \overline{J}),
\]

\[
(J, \overline{J})^* = (\overline{J}, J),
\]

and the equations \((3–9)\) become simpler as well,

\[
\frac{\partial}{\partial t_2} J = -J'' - 2(J D_-^{-1} D_+ \overline{J})' + D_+ D_- (D_-^{-1} D_+ J)^2,
\]

\[
\frac{\partial}{\partial t_2} \overline{J} = +\overline{J}'' - 2(\overline{J} D_+^{-1} D_- J)' + D_+ D_- (D_+^{-1} D_- \overline{J})^2,
\]

\[
\frac{\partial}{\partial t_3} J = J''' + 3\left[ J' D_-^{-1} D_-(J + \overline{J}) + J D_-^{-1} D_-(J + \overline{J})^2 + (D_- J) D_+ J - \overline{J} J^2 - \frac{1}{3} J^3 \right]'
\]

\[
- 3D_+ D_- \left[ J^2 D_-^{-1} D_-(J + \overline{J}) \right],
\]

\[
\frac{\partial}{\partial t_3} \overline{J} = \overline{J}''' - 3\left[ \overline{J}' D_+^{-1} D_- (J + \overline{J}) - \overline{J} (D_+^{-1} D_- (J + \overline{J})^2 + (D_- \overline{J}) D_+ \overline{J} + \overline{J} \overline{J}^2 + \frac{1}{3} \overline{J}^3 \right]'
\]

\[
- 3D_+ D_- \left[ \overline{J}^2 D_-^{-1} D_-(J + \overline{J}) \right],
\]

\[
D_1^\pm \left( \frac{J}{\overline{J}} \right) = D_\pm \left( \frac{+J}{-\overline{J}} \right), \quad Q_1^\pm \left( \frac{J}{\overline{J}} \right) = Q_\pm \left( \frac{J}{\overline{J}} \right),
\]

\[
(25)
\]
\[U_0^\pm \left( \frac{J}{\bar{J}} \right) = -D_\pm \theta^\pm \left( \frac{+J}{-\bar{J}} \right), \quad U_0 \left( \frac{J}{\bar{J}} \right) = (\theta^+ D_- + \theta^- D_+) \left( \frac{+J}{-\bar{J}} \right), \]

\[\bar{U}_0 \left( \frac{J}{\bar{J}} \right) = \frac{1}{2} (\theta^- (D_+ + Q_+) - \theta^+ (D_- + Q_-)) \left( \frac{J}{\bar{J}} \right). \]

Notice that the supersymmetry and \(u(2)\) transformations \([32,34]\) of the superfields \(J, \bar{J}\) are local functions of the superfields, while the evolution equations \([31,33]\) become nonlocal.

**4. A manifest \(N = 4\) supersymmetric representation.** The equations \([31,33]\) admit a manifestly \(N = 4\) supersymmetric representation

\[
\frac{\partial}{\partial z} \mathcal{J} = -\mathcal{J}'' - D_+ D_- \left[ 2(\mathcal{J} \partial^{-1} \mathcal{J})' - (\overline{D}^+ \overline{D}^- \partial^{-1} \mathcal{J})^2 \right],
\]

\[
\frac{\partial}{\partial \bar{z}} \mathcal{J} = +\mathcal{J}'' - \overline{D}^+ \overline{D}^- \left[ 2(\overline{\mathcal{J}} \partial^{-1} \overline{\mathcal{J}})' - (D_+ D_- \partial^{-1} \mathcal{J})^2 \right],
\]

\[
\frac{\partial}{\partial \theta^\pm} \mathcal{J} = \mathcal{J}''' + \partial \mathcal{J} + 3 \partial \mathcal{J} \partial^{-1} \mathcal{J} D_+ D_- \partial^{-1} \mathcal{J} - \frac{1}{2} (\partial^+ \partial^- \partial^{-1} \mathcal{J})^2,
\]

\[
\frac{\partial}{\partial \bar{\theta}^\pm} \mathcal{J} = \overline{\mathcal{J}}''' + \partial \overline{\mathcal{J}} + 3 \partial \overline{\mathcal{J}} \partial^{-1} \overline{\mathcal{J}} D_+ D_- \partial^{-1} \mathcal{J} + \frac{1}{2} (\partial^+ \partial^- \partial^{-1} \mathcal{J})^2,
\]

in terms of one chiral \(\mathcal{J}(z, \theta^+, \theta^-, \eta^+, \eta^-)\) and one antichiral \(\overline{\mathcal{J}}(z, \bar{\theta}^+, \bar{\theta}^-, \bar{\eta}^+, \bar{\eta}^-)\) even \(N = 4\) superfield,

\[D_\pm \mathcal{J} = 0, \quad \overline{D}^\pm \mathcal{J} = 0. \]

Here \(D_\pm, \overline{D}^\pm\) are \(N = 4\) odd covariant derivatives,

\[D_\pm \equiv \frac{1}{2} \left( \frac{\partial}{\partial \theta^\pm} + i \frac{\partial}{\partial \eta^\pm} \right), \quad \overline{D}^\pm \equiv \frac{1}{2} \left( \frac{\partial}{\partial \bar{\theta}^\pm} - i \frac{\partial}{\partial \bar{\eta}^\pm} \right), \]

\[\{D_k, \overline{D}^m\} = \delta^m_k \partial, \quad \{D_k, D_m\} = \{\overline{D}^k, \overline{D}^m\} = 0, \quad k, m = \pm, \]

and \(\eta^\pm\) are two additional real odd coordinates. The relations between the independent components of the \(N = 2\) superfields \(\{\mathcal{J}(z, \theta^+, \theta^-), \overline{\mathcal{J}}(z, \bar{\theta}^+, \bar{\theta}^-)\}\) and those of the \(N = 4\) superfields \(\{\mathcal{J}(z, \theta^+, \theta^-, \eta^+, \eta^-), \overline{\mathcal{J}}(z, \bar{\theta}^+, \bar{\theta}^-, \bar{\eta}^+, \bar{\eta}^-)\}\) are the following:

\[\mathcal{J}|_{\eta^\pm = 0} = J, \quad \overline{D}^+ \mathcal{J}|_{\eta^\pm = 0} = D_+ J, \quad \overline{D}^+ \overline{\mathcal{J}}|_{\eta^\pm = 0} = D_+ D_- J, \]

\[\overline{\mathcal{J}}|_{\eta^\pm = 0} = J, \quad D_\pm \mathcal{J}|_{\eta^\pm = 0} = D_\pm J, \quad D_+ \overline{D}^- \mathcal{J}|_{\eta^\pm = 0} = D_+ D_- J. \]

Let us also present a manifestly \(N = 4\) supersymmetric form of the discrete Darboux transformations

\[\mathcal{J}^{\bullet \bullet} = \mathcal{J} - D_+ D_+ \ln \mathcal{J}, \quad \overline{\mathcal{J}}^{\bullet \bullet} = \overline{\mathcal{J}} - \overline{D}^+ \overline{D}^+ \ln \mathcal{J}^{\bullet \bullet} \]

which can easily be derived using eqs. \([28,29]\) and \([33]\). They are discrete symmetries of the even and odd flows of the \(N = 4\) supersymmetric Toda chain hierarchy. In other words, if the
set \{\mathcal{J}, \overline{\mathcal{J}}\} is a solution of the \(N = 4\) Toda chain hierarchy, then the set \{\mathcal{J}^{\bullet\bullet}, \overline{\mathcal{J}}^{\bullet\bullet}\}, related to the former by eqs. (40), is a solution of the hierarchy as well. The equations (40) reproduce the one-dimensional reduction of the two-dimensional \(N = (2|2)\) superconformal Toda lattice [14, 15].

Finally, we would like to remark that the equations (35–36) can be rewritten in a local form, if one introduces a new superfield basis defined by the following invertible transformations:

\[
\begin{align*}
\mathcal{J} &\equiv \mathcal{D}_+ \overline{\Psi}, \\
\overline{\mathcal{J}} &\equiv \mathcal{D}_- \partial^{-1} \mathcal{J}, \\
\overline{\Psi} &\equiv \mathcal{D}_+^{-1} \partial \mathcal{J}, \\
\Psi &\equiv \mathcal{D}_-^{-1} \partial \mathcal{J},
\end{align*}
\]

(41)

where \(\Psi, \overline{\Psi} ([\Psi] = [\overline{\Psi}] = -1/2)\) are new constrained odd \(N = 2\) superfields

\[
\mathcal{D}_- \Psi = \mathcal{D}_+ \overline{\Psi} = 0, \quad \mathcal{D}_- \overline{\Psi} = \mathcal{D}_+ \Psi = 0
\]

(42)

with the reality conditions which can be derived from eqs. (27–29) and (41). Then, these equations become

\[
\begin{align*}
\frac{i}{\partial t} \Psi &= +\Psi'' + 2 \mathcal{D}_- \mathcal{D}_+ (\overline{\Psi} \mathcal{D}_- \Psi) - \mathcal{D}_+ (\mathcal{D}_+ \Psi)^2, \\
\frac{i}{\partial t} \overline{\Psi} &= -\overline{\Psi}'' - 2 \mathcal{D}_+ \mathcal{D}_- (\mathcal{D}_+ \Psi + \mathcal{D}_- \Psi) + \mathcal{D}_- (\mathcal{D}_- \Psi)^2,
\end{align*}
\]

(43)

\[
\begin{align*}
\frac{i}{\partial t_3} \Psi &= \Psi''' + 3\mathcal{D}_- [(\mathcal{D}_+ \Psi)' \mathcal{D}_+ \Psi + (\mathcal{D}_- \Psi)(\mathcal{D}_+ \Psi)^2 + \frac{1}{2} \mathcal{D}_+ \mathcal{D}_- (\mathcal{D}_+ \Psi)^2] \\
&+ \mathcal{D}_+ [(\mathcal{D}_+ \Psi)^3 - 3(\mathcal{D}_+ \Psi)^2 \mathcal{D}_- \Psi - 6(\mathcal{D}_+ \Psi)(\mathcal{D}_- \Psi)\mathcal{D}_+ \Psi], \\
\frac{i}{\partial t_3} \overline{\Psi} &= \overline{\Psi}''' + 3\mathcal{D}_+ [(\mathcal{D}_+ \Psi)' \mathcal{D}_+ \Psi + (\mathcal{D}_- \Psi)(\mathcal{D}_+ \Psi)^2 - \frac{1}{2} \mathcal{D}_+ \mathcal{D}_- (\mathcal{D}_- \Psi)^2] \\
&+ \mathcal{D}_- [(\mathcal{D}_+ \Psi)^3 - 3(\mathcal{D}_+ \Psi)^2 \mathcal{D}_+ \Psi - 6(\mathcal{D}_+ \Psi)(\mathcal{D}_- \Psi)\mathcal{D}_+ \Psi].
\end{align*}
\]

(44)

5. Relation with the \(N = 4\) supersymmetric KdV hierarchy. It is well known that there are often hidden relationships between a priori unrelated hierarchies. Some examples are the \(N = 2\) NLS and \(N = 2 \alpha = 4\) KdV [16], the "quasi" \(N = 4\) KdV and \(N = 2 \alpha = -2\) Boussinesq [9], the \(N = 2\) (1,1)-GNLS and \(N = 4\) KdV [13, 11]. These relationships may lead to a deeper understanding of the hierarchies. They may help to obtain a more complete description and to derive solutions.

The existence of a manifestly \(N = 4\) supersymmetric, local form (43–44) of equations belonging to the \(N = 4\) supersymmetric Toda chain hierarchy, gives an additional evidence in favour of the existence of a hidden relationship with the \(N = 4\) supersymmetric KdV hierarchy [6, 7].

It turns out that such a relationship indeed exists. Let us present it at the level of the second flow equations (30) which in a new superfield basis \{\tilde{\mathcal{J}}, \Phi, \overline{\Phi}\} take the following local form:

\[
\begin{align*}
-i \frac{\partial}{\partial t_2} \tilde{\mathcal{J}} &= -\frac{1}{2}(\Phi + \overline{\Phi})'' - 2(\tilde{\mathcal{J}}(\Phi - \overline{\Phi}))' + [\mathcal{D}, \mathcal{D}](\tilde{\mathcal{J}}(\Phi + \overline{\Phi})), \\
-i \frac{\partial}{\partial t_2} \Phi &= 2 \mathcal{D} \mathcal{D} (\tilde{\mathcal{J}}' - \tilde{\mathcal{J}}^2 - \frac{3}{4} \Phi^2 + \frac{1}{2} \Phi \overline{\Phi}), \\
-i \frac{\partial}{\partial t_2} \overline{\Phi} &= 2 \mathcal{D} \mathcal{D} (\tilde{\mathcal{J}}' + \tilde{\mathcal{J}}^2 + \frac{3}{4} \Phi^2 - \frac{1}{2} \Phi \overline{\Phi}),
\end{align*}
\]

(45)
where $\tilde{J}, \Phi, \bar{\Phi}$ ($|\tilde{J}| = |\Phi| = |\bar{\Phi}| = -1$) are new unconstrained, chiral ($D \Phi = 0$) and antichiral ($\overline{D} \bar{\Phi} = 0$) even $N = 2$ superfields, respectively, related to the superfields $J, \overline{J}$ (26) by the following invertible transformations:

$$J \equiv \frac{1}{2}(\Phi + \bar{\Phi}) - i\tilde{J}, \quad \overline{J} \equiv \frac{1}{2}(\Phi + \bar{\Phi}) + i\tilde{J},$$

$$\tilde{J} \equiv \frac{i}{2}(J - \overline{J}), \quad \Phi \equiv D\overline{D}\partial^{-1}(J + \overline{J}), \quad \bar{\Phi} \equiv \overline{D}D\partial^{-1}(J + \overline{J}),$$

and $D, \overline{D}$ are $N = 2$ odd covariant derivatives,

$$D \equiv \frac{1}{2}(D_+ + iD_-), \quad \overline{D} \equiv \frac{1}{2}(D_+ - iD_-),$$

$$\{D, D\} = \partial, \quad \{D, \overline{D}\} = \{\overline{D}, D\} = 0.$$ (46)

Now, one can easily recognize that eq. (45) is the second flow of the $N = 4$ KdV hierarchy in a particular "$SU(2)$ frame" (compare eqs. (45) with eqs. (4.5) and (4.3c) from ref. [7]). Moreover, in this basis the second Hamiltonian structure of the $N = 4$ Toda chain hierarchy [1] reproduces the $N = 4$ $SU(2)$ superconformal algebra and the flow (45) is generated by the Hamiltonian $H^2_2$

$$H^2_2 = \int dzd\theta^+ d\theta^- uv' \equiv i \int dzd\theta^+ d\theta^- J[D, \overline{D}]\partial^{-1}(J + \overline{J}) \equiv \int dzd\theta^+ d\theta^- \tilde{J}(\Phi - \bar{\Phi}).$$ (48)

The same relationship is certainly valid for any other flow of the $N = 4$ Toda and $N = 4$ KdV hierarchies both in the $N = 2$ and $N = 4$ superspaces.

The relationship just established allows to apply the formalism of ref. [4], developed for the case of the $N = 4$ Toda chain hierarchy, for a more complete description of the $N = 4$ KdV hierarchy. It can be used to construct new bosonic and fermionic flows and Hamiltonians, new finite and infinite discrete symmetries, the tau function, etc.. Let us present as an example the three involutions

$$(\Phi, \bar{\Phi}, \tilde{J})^* = (-\bar{\Phi}, -\Phi, \tilde{J}),$$

$$\Phi^* = \bar{\Phi} - i\overline{D}D\ln(\Phi + \bar{\Phi} + 2i\tilde{J}), \quad \bar{\Phi} = \Phi + iD\overline{D}\ln(\Phi + \bar{\Phi} + 2i\tilde{J}),$$

$$\tilde{J}^* = -\tilde{J} + \frac{1}{2}[D, \overline{D}]\ln(\Phi + \bar{\Phi} + 2i\tilde{J}),$$

$$(\Phi, \bar{\Phi}, \tilde{J})^* = (\bar{\Phi}, \Phi, \tilde{J})$$ (51)

and the Darboux-Backlund symmetries

$$\Phi^{**} = \Phi + iD\overline{D}\ln(\Phi + \bar{\Phi} + 2i\tilde{J}), \quad \bar{\Phi}^{**} = \bar{\Phi} - iD\overline{D}\ln(\Phi + \bar{\Phi} + 2i\tilde{J}),$$

$$\tilde{J}^{**} = -\tilde{J} + \frac{1}{2}[D, \overline{D}]\ln(\Phi + \bar{\Phi} + 2i\tilde{J})$$

(52)

derived using the relationship (46) and eqs. (27–29). The involution (50) looks rather complicated, however let us remember that in the original superfield basis $\{u, v\}$ it has a very simple form (19). The involutions (19) and (51) were discussed in [8], to our knowledge but
the involution (50) and the corresponding real form of the $N = 4$ KdV hierarchy as well as its Darboux-Backlund symmetries (52) are presented here for the first time.

6. Conclusion. In this letter, we have described three distinct real forms of the $N = 4$ Toda chain hierarchy introduced in [1]. It has been shown that the symmetry algebra of one of these real forms contains the usual (untwisted) real $N = 4$ supersymmetry algebra. A set of $N = 2$ superfields with simple conjugation properties in the untwisted case have been introduced. It has then been shown how to extend these superfields to superfields in $N = 4$ superspace, and write all flows and conjugation rules directly in $N = 4$ superspace. Finally, a change of basis in $N = 4$ superspace has allowed us to eliminate all nonlocalities in the flows. As a byproduct, a relationship between the complex $N = 4$ Toda chain and $N = 4$ KdV hierarchies has been established, which allows to derive Darboux-Backlund symmetries and a new real form of the last hierarchy possessing a twisted $N = 4$ supersymmetry.

It is obvious that there remain a lot of work to do in order to improve our understanding of the hierarchy in $N = 4$ superspace. A first step in this direction would be to derive a Lax formulation of the flows in terms of $N = 4$ operators.

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References

[1] F. Delduc, L. Gallot and A. Sorin, $N = 2$ local and $N = 4$ nonlocal reductions of supersymmetric KP hierarchy in $N = 2$ superspace, Nucl. Phys. B558 (1999) 545, solv-int/9907004.

[2] A.N. Leznov and A.S. Sorin, Two-dimensional superintegrable mappings and integrable hierarchies in the (2|2) superspace, Phys. Lett. B389 (1996) 494, hep-th/9608166; Integrable mappings and hierarchies in the (2|2) superspace, Nucl. Phys. (Proc. Suppl.) B56 (1997) 258.

[3] O. Lechtenfeld and A. Sorin, Fermionic flows and tau function of the $N=(1|1)$ superconformal Toda lattice hierarchy, Nucl. Phys. B557 (1999) 535, solv-int/9810009.

[4] O. Lechtenfeld and A. Sorin, Supersymmetric KP hierarchy in $N = 1$ superspace and its $N = 2$ reductions, ITP-UH-14/99, JINR E2-99-216, solv-int/9907021; Nucl. Phys. B, to appear.

[5] S.J. Gates, Jr., M.T. Grisaru, M. Roček and W. Siegel, Superspace or one thousand and one lessons in supersymmetry, the Benjamin/Cummings Publishing Company, Inc, 1983, pgs. 58-59; P. West, Introduction to supersymmetry and supergravity, extended second edition, World Scientific, 1990, pgs. 393-394.

[6] F. Delduc and E. Ivanov, $N = 4$ super KdV equation, Phys. Lett. B309 (1993) 312, hep-th/9301024.
[7] F. Delduc, E. Ivanov and S. Krivonos, \textit{N=4 super KdV hierarchy in N=4 and N=2 superspaces}, J. Math. Phys. \textbf{37} (1996) 1356; Erratum-ibid. \textbf{38} (1997) 1224, \texttt{hep-th/9510033}.

[8] E. Ivanov and S. Krivonos, \textit{New integrable extensions of N=2 KdV and Boussinesq hierarchies}, Phys. Lett \textbf{A231} (1997) 75, \texttt{hep-th/9609191}.

[9] F. Delduc, L. Gallot and E. Ivanov, \textit{New super KdV system with the N=4 SCA as the hamiltonian structure}, Phys. Lett. \textbf{B396} (1997) 122, \texttt{hep-th/9611033}.

[10] E. Ivanov, S. Krivonos and F. Toppan, \textit{N=4 super-NLS-mKdV hierarchies}, Phys. Lett. \textbf{B405} (1997) 85, \texttt{hep-th/9703224}.

[11] L. Bonora and A. Sorin, \textit{The Hamiltonian structure of the N=2 supersymmetric GNLS hierarchy}, Phys. Lett. \textbf{B407} (1997) 131, \texttt{hep-th/9704130}.

[12] Z. Popowicz, \textit{Odd bihamiltonian structure of new supersymmetric N = 2, 4 Korteweg de Vries equation and odd SYSY Virasoro-like algebra}, Phys. Lett. \textbf{B459} (1999) 150, \texttt{hep-th/9903198}.

[13] A. Sorin, \textit{The discrete symmetry of the N=2 supersymmetric modified NLS hierarchy}, Phys. Lett. \textbf{B395} (1997) 218, \texttt{hep-th/9611148}.
\textit{Discrete symmetries of the N=2 supersymmetric Generalized Nonlinear Schroedinger hierarchies}, Phys. Atom. Nucl. \textbf{61} (1998) 1768, \texttt{solv-int/9701020}.

[14] J. Evans, T. Hollowood, \textit{Supersymmetric Toda field theories}, Nucl. Phys. \textbf{B352} (1991) 723.

[15] V.B. Derjagin, A.N. Leznov and A. Sorin, \textit{The solution of the N = (0|2) superconformal f-Toda lattice}, Nucl. Phys. \textbf{B527} (1998) 643, \texttt{solv-int/9803010}.

[16] S. Krivonos and A. Sorin, \textit{The minimal N = 2 superextension of the NLS equation}, Phys. Lett. \textbf{B357} (1995) 94, \texttt{hep-th/9504084}.
S. Krivonos, A. Sorin and F. Toppan, \textit{On the super-NLS equation and its relation with N = 2 super-KdV within coset approach}, Phys. Lett. \textbf{A206} (1995) 146, \texttt{hep-th/9504135}. 