ATLAS OF TWO-DIMENSIONAL
IRREVERSIBLE CONSERVATIVE
LAGRANGIAN MECHANICAL SYSTEMS
WITH A SECOND QUADRATIC INTEGRAL

H. M. YEHIA

Department of Mathematics, Faculty of Science,
Mansoura University, Mansoura 35516, Egypt
E-mail: hyehia@mans.edu.eg

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Abstract

This paper aims at the most comprehensive and systematic construction and tabulation of mechanical systems that admit a second invariant, quadratic in velocities, other than the Hamiltonian. The configuration space is in general a 2D Riemannian or pseudo-Riemannian manifold and the determination of its geometry is a part of the process of solution. Forces acting on the system include a part derived from a scalar potential and a part derived from a vector potential, associated with terms linear in velocities in the Lagrangian function of the system. The last cause time-irreversibility of the system.

We construct 41 multi-parameter integrable systems of the type de-
scribed in the title mostly on Riemannian manifolds. They are mostly new and cover all previously known systems as special cases, corresponding to special values of the parameters. Those include all known cases of motion of a particle in the plane and all known cases in the dynamics of rigid body. In the last field we introduce a new integrable case related to Steklov’s case of motion of a body in a liquid. Several new cases of motion in the plane, on the sphere and on the pseudo-sphere or in the hyperbolic plane are found as special cases. Prospective applications in mathematics and physics are also pointed out.
1 Introduction

1.1 Historical

The direct method for constructing integrable time-reversible mechanical systems was initiated by Bertrand in his papers [1] and [2]. In the first one he posed and partially solved the problem of finding forces that should act on a particle in the plane from the knowledge of an additional integral of motion of a prescribed form. In the second paper [2] Bertrand formulated the problem of finding forces acting on a particle in the plane so that the Lagrangian

\[ L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) - V(x, y) \]  

admits a first integral of a prescribed form. He obtained the equations satisfied by the coefficients of an integral, polynomial or rational (linear to linear) in the velocities and the components of force acting on the system and succeeded to solve them in certain partial cases. For the case of quadratic integral, Bertrand obtained the necessary and sufficient condition is that \( V \) satisfies the partial differential equation

\[ (2axy + by + b'x - c_1)(V_{xx} - V_{yy}) + 2[a(y^2 - x^2) + by - b'x + c)V_{xy} + 3(2ay + b)V_x - 3(2ax + b')V_y = 0 \]  

where \( a, b, ... \) are constants. Darboux [3] gave the generic solution of this equation for the potential in the form

\[ V = \frac{f(\alpha) - \varphi(\beta)}{\alpha^2 - \beta^2} \]  

where \( f, \varphi \) are arbitrary functions and \( \alpha, \beta \) are the parameters of confocal ellipses and hyperbolas in the plane. The cases of degeneracy of elliptic coordinates to
parabolic, polar and Cartesian coordinates were considered in complete form only recently [4]. The last cases were also rediscovered more than once in later times [5], [6] [7]. The same result was drawn from different points of view in [8], where some complex cases were pointed out (see also [9] for complete description of possible real and complex cases) and [23], which applied differential geometric concepts to the iso-energetic Jacobi metric. It turns out that in those four cases the Hamilton-Jacobi equation of the system [1] is separable [11], [12] (see also [13]).

In the present paper we deal with a significant generalization of the system of (1), described by the Lagrangian

$$L = \frac{1}{2} \sum_{i,j=1}^{2} a_{ij} \dot{q}_i \dot{q}_j + \sum_{i=1}^{2} a_i \dot{q}_i - V$$  \hspace{1cm} (4)

in which $a_{ij}, a_i, V$ depend only on the coordinates $q_1, q_2$. This system differs from (1) in two important aspects:

1. It accommodates Riemannian two-dimensional manifolds as possible configuration spaces. This allows application to various problems in dynamics involving motion of a rigid body about a fixed point and motion of a particle on a fixed smooth curved surface.

2. The terms linear in velocities account for gyroscopic forces. Those are forces that do not do work during motion of the system. They appear naturally in gyroscopic systems as a result of ignoring cyclic coordinates by the Routh procedure. Linear terms may arise also when the system has some charged components moving in a stationary magnetic field.

\[1\]In classical mechanics we are usually interested in systems whose kinetic energy (equivalently, metric) is positive definite. Certain results below involve indefinite (pseudo-Riemannian) metrics. These are included for the sake of completeness of enumeration of all possible cases.
this context the vector \((a_1, a_2)\) may be called "vector potential" while \(V\) is the scalar potential. The system \((4)\) is usually called a system with velocity-dependent potential. For other circumstances see [16].

Birkhoff [29] raised and completely solved the problem of constructing systems of the type \((4)\), which admit an integral linear in velocities in addition to the Jacobi integral (the Hamiltonian of the system). The first cases of irreversible systems in the Euclidean plane, having an integral quadratic in velocities were obtained by Vandervoort [14] and by Dorizzi et al [15].

The first systematic study of the full irreversible system of type \((4)\) for existence of a quadratic integral was done in our work [16], where possible integrable systems were classified according to whether the normal form of the metric has the Euclidean form, the form of a metric on a surface of revolution or a generic Liouville form. Several new many-parameter families of systems were found. Most of the known systems of the same type were either recovered or shown to be special cases of the new systems, corresponding to certain particular values of the parameters. Some main systems were written down explicitly, but many others were only briefly pointed out. Some cases of integrable motions in the plane generalizing the special plane versions of [16] were presented in [17].

Cases of irreversible motion on a sphere corresponding to integrable problems due to Clebsch and Lyapunov in rigid body dynamics were obtained in [18]. A case of irreversible motion of a particle on a smooth ellipsoid corresponding to Clebsch’s case of asymmetric body in a liquid was also obtained in [19]. In the last two works the Lagrangian for each motion was expressed in redundant Cartesian coordinates on the surface. They can be simply reduced to the form \((4)\) by using spher-conic coordinates on the sphere and elliptic coordinates on the ellipsoid.

After the publication of [16] few works appeared dealing with the same prob-
lem, but mainly investigating the case of Euclidean plane. A nearly complete list is composed of the following papers: [20], [21] (devoted to motion of a particle in a rotating plane) and [23], [25], [26] dealing with particles moving in the plane under the action of potential and "forces with vector potential" or "velocity-dependent" forces. Only two works considered irreversible motions on curved manifolds: [21] from the point of view of separation of variables and [22] in which a classification of possible cases on Riemannian manifolds is tried, but most of the results are very special cases of those presented in [16].

The aim of the present paper is to make the most exhaustive enumeration of irreversible mechanical systems, which have an invariant quadratic in velocities and functionally independent of the Hamiltonian of the system. We have constructed a total of 41 different several-parameter families of those systems. Some of those families or their special cases or degenerations were listed in our previous works but the majority are listed for the first time. We give briefly the very essential moments in the different methods used in the derivation of the families and then provide for every family the minimal information enough for its characterization. Usually, we give the Lagrangian and the second invariant and occasionally the Gaussian curvature of the configuration space, which we use for interpretation of suitable results as cases of motion on a plane or a space of constant curvature. The last section is devoted to application in the two model problems of motion of a rigid body under various circumstances and of motion of a particle under potential forces on a smooth fixed ellipsoid.

1.2 Formulation of the problem

It is well known that the system (4) can always be referred to some isometric coordinates \(x, y\) (say), in which the Lagrangian takes the form

\[
L = \frac{1}{2} \Lambda_0 (\dot{x}^2 + \dot{y}^2) + l_1 \dot{x} + l_2 \dot{y} - V
\]

(5)
containing one coefficient in the kinetic energy term, instead of three in (4). The pair of coefficients $l_1, l_2$ are not unique for a given system, since one can add the time derivative of an arbitrary function of the position to the Lagrangian. The Lagrangian -more precisely, the Lagrangian equations of motion- is thus characterized by the triplet of functions $(\Lambda_0, \Omega_0, V)$, where $\Omega_0 = \frac{\partial l_1}{\partial y} - \frac{\partial l_2}{\partial x}$.

The system (5) transforms under conformal mapping

$$x + iy = z(\zeta = \xi + i\eta)$$

(6)

to

$$L = \frac{1}{2} \Lambda(\dot{\xi}^2 + \dot{\eta}^2) + l_1' \dot{\xi} + l_2' \dot{\eta} - V$$

(7)

where $\Lambda = |\frac{d\zeta}{d\xi}|^2 \Lambda_0, \Omega = |\frac{d\zeta}{d\xi}|^2 \Omega_0$. The last system admits the Jacobi integral of motion (the Hamiltonian expressed in the state variables)

$$H = \frac{1}{2} \Lambda(\dot{\xi}^2 + \dot{\eta}^2) + V = K \text{ (arbitrary constant.)}$$

(8)

Suppose that the system (5) admits an integral of general quadratic form

$$I = \frac{1}{2}(b_{11}\dot{x}^2 + 2b_{12}\dot{x}\dot{y} + b_{22}\dot{y}^2) + b_1\dot{x} + b_2\dot{y} + b_0$$

(9)

In our previous work [16] we have shown that for a system of the type under consideration one can always find a conformal mapping of the plane and a time transformation of the type

$$dt = \Lambda d\tau$$

(10)

such that the Lagrangian, Jacobi’s integral and the complementary quadratic integral take, with the proper use of (10), the form [16]:

$$L = \frac{1}{2}(\xi'^2 + \eta'^2) + P\xi' - Q\eta' + U$$

(11)

$$H = \frac{1}{2}(\xi'^2 + \eta'^2) - U$$

(12)

$$I = \frac{1}{2}\xi'^2 + P\xi' + Q\eta' + R$$

(13)
where primes denote differentiation with respect to $\tau$,

\[
P = \phi_{\eta}, Q = \phi_{\xi},
\]

(14a)

\[
U = \Lambda(K - V),
\]

(14b)

\[
R = \int_{\eta_0}^{\eta} \Omega P d\eta - (\Omega Q + U\dot{\xi})_{\eta = \eta_0} d\xi,
\]

(14c)

\[
\Omega = \phi_{\xi\xi} + \phi_{\eta\eta}
\]

(14d)

and $\Lambda, \phi, V$ satisfy the over-determined system of four equations

\[
\Lambda_{\xi\eta} = 0, \quad (15a)
\]

\[
\phi_{\eta} \Lambda_{\xi} + \phi_{\xi} \Lambda_{\eta} + 2\Lambda \phi_{\xi\eta} = 0, \quad (15b)
\]

\[
\phi_{\eta} V_{\xi} + \phi_{\xi} V_{\eta} = 0, \quad (15c)
\]

\[
\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right](\phi_{\xi} \phi_{\eta}) - (\Lambda V)_{\xi\eta} = 0 \quad (15d)
\]

The equations of motion corresponding to the Lagrangian [11] acquire the simplest form

\[
\xi'' + \Omega_{\eta} = \frac{\partial U}{\partial \xi}, \quad \eta'' - \Omega_{\xi} = \frac{\partial U}{\partial \eta},
\]

(16)

In the original (natural) time variable the Lagrangian and integrals take the form

\[
L = \frac{1}{2} \Lambda(\dot{\xi}^2 + \dot{\eta}^2) + P\dot{\xi} - Q\dot{\eta} + K - V \quad (17a)
\]

\[
H = \frac{1}{2} \Lambda(\dot{\xi}^2 + \dot{\eta}^2) + V - K = 0 \quad (17b)
\]

\[
I = \frac{1}{2} \Lambda^2 \dot{\xi}^2 + \Lambda(P\dot{\xi} + Q\dot{\eta}) + R \quad (17c)
\]

In the last form of the integral [11c] the function $R$ may involve the energy-parameter (more precisely, Jacobi’s parameter) $K$ as a linear multiplier in certain terms. This is obvious from the way of construction. This parameter is interpreted as its numerical value on a given motion, or may be substituted by
its expression as a function of the state variables from (17b), resulting in an unconditional integral in the state space.

The function $\Lambda$, the solution of equation (15a) completely characterizes the metric on the configuration space of the system (17a). This gives a natural basis for a classification of all systems of the type under consideration, depending on whether $\Lambda$ has one of three possible forms:

1) The factor $\Lambda$ is a constant and, without loss of generality, one can set

$$\Lambda = 1$$  \hspace{1cm} (18)

In that case the configuration space has Euclidean metric. It may be interpreted as a plane or a developable surface.

2) The factor $\Lambda$ depends only on one variable

$$\Lambda = \mu(\eta)$$  \hspace{1cm} (19)

In that case the metric on the configuration space has the structure of a metric on a surface of revolution. This does not mean that one can always realize this metric on a surface of revolution, but this may be done in certain special circumstances.

3) $\Lambda$ has the generic form of a Liouville metric

$$\Lambda = \lambda(\xi) - \mu(\eta)$$  \hspace{1cm} (20)

The solution of equations (15b-15d) will depend mainly on the form of chosen solution for $\Lambda$. The three cases will be dealt with separately in the following sections.

2 Time-reversible plane systems: The Bertrand-
Darboux project

When the function $\Omega \equiv 0$ equations (16) become time-reversible and one can take $\phi \equiv 0$, so that in virtue of (15a, 15d) the Lagrangian can be written as

$$L = \frac{1}{2} \left( \lambda (\xi) - \mu (\eta) \right) [\dot{\xi}^2 + \dot{\eta}^2] - \frac{v_1(\xi) - v_2(\eta)}{(\lambda (\xi) - \mu (\eta))} + h \quad (21)$$

This is the general two-dimensional system of Liouville type, and it can be put in the common form.

$$L = \frac{1}{2} (\lambda - \mu) \left[ \frac{\dot{\lambda}^2}{F(\lambda)} + \frac{\dot{\mu}^2}{G(\mu)} \right] - \frac{v_1(\lambda) - v_2(\mu)}{\lambda - \mu} + h \quad (22)$$

It involves four arbitrary functions of each of one variable each. Separation of variables in this case is obvious.

Many cases of physical interest can be identified in the form of the Liouville system (21).

If the original configuration space of (5) is the ordinary Euclidean plane, then $\Lambda_0 = 1$ and we obtain a different and very compact formulation of the Bertrand-Darboux problem presented in the introduction. Solutions of this problem are systems of the form (21) with the function $z(\zeta)$ in the transformation (9) satisfying the equation

$$\Lambda \equiv \left| \frac{dz}{d\zeta} \right|^2 = \lambda (\xi) - \mu (\eta)$$

It was shown in [16] that this equation has the general solution

$$\zeta = \int \frac{dz}{\sqrt{\alpha z^2 + \beta z + \gamma}} \quad (23)$$

in which $\alpha$ is a non-negative constant and the arbitrary constants $\beta, \gamma$ can be made real by a suitable choice of axes in the $z-$plane. The generic case of two distinct roots defines the elliptic coordinates in the $z-$plane, reproduces Darboux’s result. The degenerate cases of two equal roots or with one or both roots
coalescing with the point at infinity define the polar, parabolic and Cartesian
coordinates in the plane. This gives a much simpler and transparent way to the
completion of the Bertrand-Darboux project than that relying on the cases of
solution of Bertrand’s equation [2].

3 The first type of irreversible systems - The
metric is an Euclidean plane one (System 1)

We begin the classification by considering the first and simplest case, when
\( \Lambda = \text{const.} = 1 \). It is quite straightforward to solve the system [15] as in [16] and
construct the Lagrangian in the form

\[
L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \sqrt{G(\mu)}\dot{x} - \sqrt{F(\lambda)}\dot{y} \\
+ \frac{1}{2}a(\lambda - \mu)^3 + \frac{1}{2}(b_1 + b_2)(\lambda - \mu)^2 + K_1(\lambda - \mu) + h
\]  

(24)

where

\[
x = \int^\lambda \frac{d\lambda}{\sqrt{F(\lambda)}}, \quad y = \int^\mu \frac{d\mu}{\sqrt{G(\mu)}} \\
F(\lambda) = a\lambda^3 + b_1\lambda^2 + c_1\lambda + d_1, \\
G(\mu) = -a\mu^3 + b_2\mu^2 + c_2\mu + d_2
\]  

(25)

and \( a, b_1, b_2, c_1, c_2, d_1, d_2, K_1, h \) are arbitrary constants. This is equivalent to the
less explicit result provided in [16]. The equations of motion have the form

\[
\ddot{x} + \Omega \dot{y} = -\frac{\partial V}{\partial x}, \quad \ddot{y} - \Omega \dot{x} = -\frac{\partial V}{\partial y}
\]  

(26)

where

\[
\Omega = \frac{1}{2}3a(\lambda^2 - \mu^2) + 2(b_1\lambda + b_2\mu) + c_1 + c_2
\]  

(27)
They admit the complementary quadratic integral
\[
I = \frac{1}{2} \dot{x}^2 + \sqrt{G(\mu)} \dot{x} + \sqrt{F(\lambda)} \dot{y} \\
- \frac{1}{2}[a (\lambda - \mu)^2 (2\lambda + \mu) + (c_1 + c_2)(\lambda - \mu) + b_2(\lambda^2 - \mu^2)] - b_1 \lambda (\lambda - \mu) + 128
\]

The system with the Lagrangian (24) describes the motion of a particle in the plane, with arbitrary Jacobi’s constant \( h \). An equivalent of this system was pointed out in [15] and discussed also in [16]. The Lagrangian can be explicitly expressed in terms of the Cartesian coordinates \( x, y \) using Weierstrass’ elliptic functions of two independent sets of invariants. Different possible real interpretations are discussed in [16] in terms of Jacobi’s elliptic functions. Except some degenerations, the potential \( V \) and the gyroscopic functions \( \Omega \) are periodic in both \( x, y \) directions. The generic system (24) may be used as a model for the study of motion of an electrically charged particle in the plane under the action of an electric field in the plane and a magnetic field orthogonal to it. Both fields may be regarded as resulting from a spatially periodic distribution of sources.

The simplest periodic case, when elliptic functions degenerate into trigonometric ones is characterized by
\[
V = \frac{1}{2}(\alpha^2 + \beta^2)[A \cos(\alpha x) + B \cos(\beta y)]^2 - k[A \cos(\alpha x) + B \cos(\beta y)]
\]
\[
\Omega = \alpha^2 A \cos(\alpha x) + \beta^2 B \cos(\beta y)
\]
\[
I = \dot{x}^2 - \dot{y}^2 + 4\beta B \sin(\beta y) \dot{x} + 4\alpha A \sin(\alpha x) \dot{y} - 2k[A \cos(\alpha x) + B \cos(\beta y)]
\]
\[
+ [A \cos(\alpha x) - B \cos(\beta y)](3\alpha^2 + \beta^2) \cos(\alpha x) + B(\alpha^2 + 3\beta^2) \cos(\beta y)
\]

Even this case is a quite complicated integrable one. The following two graphs 1, 2 show the functions \( V \) and \( \Omega \) on an area \( 4\pi \times 4\pi \), containing four primitive
periodic cells of the plane in the special case $\alpha = \beta = A = B = k = 1$.

Figure 1: $V(x, y)$ the electric potential in the plane of motion.

Figure 2: $\Omega(x, y)$ the magnetic field orthogonal to the plane.
4 Second type

The problem of constructing systems of this type was considered in [16], where only few cases were given in detail. Here we give a more systematic treatment and the final results cover a much wider hierarchy of possible different cases, some of which were overlooked in [16].

In fact, putting $\Lambda = \mu(\eta)$ in (15) we obtain the general solution of (15b) as $\varphi = \frac{M(\xi)}{\sqrt{\mu}} + N(\eta)$, where $M, N$ are arbitrary functions. Also, from (15c) we deduce that $V$ must have the structure

$$V = V(\psi), \psi = \sqrt{\mu}M(\xi) + v(\eta)$$

where $v$ is related to $N$ by the condition $v'(\eta) = -\mu N'(\eta)$. In order to make the remaining equation (15d) more tractable we have used $\lambda = M(\xi)$ and $\mu$ as variables instead of $\xi, \eta$, and introduced the notation

$$\left(\frac{d\lambda}{d\xi}\right)^2 = F(\lambda), \left(\frac{d\mu}{d\eta}\right)^2 = G(\mu)$$

where $F$ and $G$ are functions to be determined. Now we can write

$$\psi = \sqrt{\frac{\mu}{\lambda}} + v(\mu), \quad \varphi = \frac{1}{\sqrt{\lambda \mu}} - \int \frac{v'(\mu)}{\mu} d\mu$$

and equation (15d) becomes a consistency condition which has to be satisfied by four functions in one variable each $V(\psi), F(\lambda), G(\mu)$ and $v(\lambda)$. The last equation can be shown to have the structure

$$\mu \psi V''(\psi) + \frac{3}{2} V'(\psi) = S$$

To solve this condition, i.e. to find compatible choices of the four functions we have applied two ways that led to different findings.

4.1 The first way: (33) is a differential equation
For (33) to be a differential equation in $V(\psi)$, the coefficient $\mu \psi_\mu$ and the function $S$ must be functions of the single variable $\psi$. The first implies that $v(\mu) = k \sqrt{\mu}$. Turning to the function $S$ we have found that $k = 0$ and also

$$F(\lambda) = \lambda^2(a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0)$$

$$G(\mu) = \mu^2(b_3 \mu^3 + b_2 \mu^2 + b_1 \mu - a_0) \quad (34)$$

Returning to equation (33) we find

$$V(\psi) = -\frac{1}{8}(b_3 \psi^2 + \frac{a_3}{\psi^4}) + \frac{A}{\psi^2}, \quad \psi = \sqrt{\frac{\mu}{\lambda}} \quad (35)$$

By this information we are now able to construct the system we are looking for. We call it System 2. It has the Lagrangian

$$L = \frac{1}{2} \mu \left[ \frac{\lambda^2}{F(\lambda)} + \frac{\mu^2}{G(\mu)} \right] + \frac{1}{2} J \sqrt{\frac{F(\lambda) G(\mu)}{\lambda^3 \mu^3}} \left[ \lambda \frac{\dot{\lambda}}{F(\lambda)} - \mu \frac{\dot{\mu}}{G(\mu)} \right] - \frac{A \lambda}{\mu} + J^2 \frac{1}{8} (a_3 \frac{\lambda^2}{\mu^2} + b_3 \frac{\mu}{\lambda}) \quad (36)$$

4.2 The second way: (33) is a consistency condition:

We deal with (33) assuming for $V(\psi)$ a preassigned form like a polynomial and try to find compatible combinations of coefficients that help separate equations for $F, G$ and $v$. The result found in this way can be formulated as in the following

Theorem 1
The system with the Lagrangian

$$L = \frac{1}{2} \left[ \frac{\mu \dot{\lambda}^2}{a_0 + a_1 \lambda - c_0 \lambda^2} + \frac{\dot{\mu}^2}{4 \mu (c_3 \mu^3 + c_2 \mu^2 + c_1 \mu + c_0)} \right]$$

$$- \left[ \frac{J \lambda}{\sqrt{\mu}} + 2v'(\mu) \right] \frac{\sqrt{c_3 \mu^3 + c_2 \mu^2 + c_1 \mu + c_0 \dot{\lambda}}}{\sqrt{a_0 + a_1 \lambda - c_0 \lambda^2}}$$

$$- \frac{J \sqrt{a_0 + a_1 \lambda - c_0 \lambda^2} \dot{\mu}}{2 \mu^{3/2} \sqrt{c_3 \mu^3 + c_2 \mu^2 + c_1 \mu + c_0}}$$

$$- b \left[ J \sqrt{\mu \lambda} + v(\mu) \right] + \frac{1}{2} c_3 [J \sqrt{\mu \lambda} + v(\mu)]^2$$

(37)

where $c_j, a_i, b, J$ are arbitrary constants and $v(\mu)$ is the solution of the linear differential equation

$$8(c_3 \mu^3 + c_2 \mu^2 + c_1 \mu + c_0) v'''(\mu) + 12(3c_3 \mu^2 + 2c_2 \mu + c_1) v''(\mu)$$

$$+ 6(3c_3 \mu + c_2) v'(\mu) - 3c_3 v(\mu) = -3b - \frac{3Ja_1}{2 \mu^{5/2}}$$

(38)

admits the second quadratic integral

$$I = \frac{1}{2} \frac{\mu^2 \dot{\lambda}^2}{a_0 + a_1 \lambda - c_0 \lambda^2}$$

$$- \mu \left[ \frac{J \lambda}{\sqrt{\mu}} + 2v'(\mu) \right] \frac{\sqrt{c_3 \mu^3 + c_2 \mu^2 + c_1 \mu + c_0 \dot{\lambda}}}{\sqrt{a_0 + a_1 \lambda - c_0 \lambda^2}}$$

$$+ \frac{J \mu \sqrt{a_0 + a_1 \lambda - c_0 \lambda^2} \dot{\mu}}{2 \mu^{3/2} \sqrt{c_3 \mu^3 + c_2 \mu^2 + c_1 \mu + c_0}}$$

$$+ \frac{1}{2} \lambda^2 \left[ \lambda \mu (c_3 \mu + c_2) - \frac{a_1}{\mu} \right] - \frac{1}{2} Ja_1 \int \frac{v'(\mu)}{\mu^{3/2}} d\mu$$

$$+ J \lambda \sqrt{\mu} (4(c_3 \mu^3 + c_2 \mu^2 + c_1 \mu + c_0) v''(\mu))$$

$$+ 2(3c_3 \mu^2 + 2c_2 \mu + c_1) v'(\mu) - 3c_3 v(\mu) + b \mu$$

$$+ 2v'(\mu)^2 (c_3 \mu^3 + c_2 \mu^2 + c_1 \mu + c_0)$$

(39)

When $c_0 \neq 0$ the above theorem gives the same result as in [16], as it is possible in this case to eliminate $a_1$ by a suitable shift in the variable $\lambda$. But when $c_0 = 0$ the presence of $a_1$ is significant and leads to seven new cases in dependence on the combination of roots of $G(\mu)$. The complete hierarchy is given in the next subsection.
4.2.1 Classification

For the complete determination of the Lagrangian (37) and the integral (39) we need only the solutions of (38) for \( v(\mu) \). This turned out to be depending on the distribution of roots of the polynomial \( G(\mu)/\mu^2 = c_3\mu^3 + c_2\mu^2 + c_1\mu + c_0 \), which we denote by \( \mu_1, \mu_2, \mu_3 \). We now proceed to enumerate the fourteen possible cases. For each case we give its conditions and the corresponding solution \( v(\mu) \).
| Conditions on coefficients | Roots | $v(\mu)$ |
|---------------------------|-------|----------|
| $c_3 \neq 0$ | $c_0 \neq 0$ | $\mu_3 \neq \mu_2 \neq \mu_1 \neq 0$ | $\frac{b}{c_3} + K_1 \sqrt{\mu - \mu_1}$ $+ K_2 \sqrt{\mu - \mu_2} + K_3 \sqrt{\mu - \mu_3}$ |
| $c_3 \neq 0$ | $c_0 = 0, c_1 \neq 0$ | $\mu_3 = \mu_2 \neq \mu_1 \neq 0$ | $\frac{b}{c_3} + K_1 \sqrt{\mu - \mu_1} + K_2 / \sqrt{\mu - \mu_1}$ $+ K_3 / (\mu - \mu_1)^{3/2}$ |
| $c_3 = 0, c_2 \neq 0$ | $c_0 \neq 0$ | $\mu_2 \neq \mu_1 \neq 0$ | $-\frac{b\mu}{c_2} + K_1 \sqrt{\mu - \mu_1} + K_2 \sqrt{\mu - \mu_2}$ |
| $c_3 = c_2 = 0$ | $c_0 \neq 0$ | $\mu_2 = \mu_1 \neq 0$ | $-\frac{b\mu}{c_2} + K_1 \sqrt{\mu - \mu_1} + K_2 / \sqrt{\mu - \mu_1}$ |
| $c_3 = c_2$ | $c_0 \neq 0$ | $\mu_2 \neq \mu_1 \neq 0$ | $-\frac{b\mu^2}{8c_1} + K_1 \mu + K_2 \sqrt{c_1 \mu + c_0}$ |
| $c_3 = c_2$ | $c_0 = 0$ | $\mu_2 = \mu_1 \neq 0$ | $-\frac{b\mu^2}{8c_1} + K_1 \mu + \frac{Ja_1}{4c_1 \sqrt{\mu}}$ |
| $c_3 = c_2$ | $c_0 \neq 0$ | $c_0 \neq 0$ | $-\frac{b\mu^3}{16c_0} + K_2 \mu^2 + K_1 \mu$ |

Table I

The Gaussian curvature of the configuration space of (37) is

$$\kappa = -(3c_3\mu^2 + 2c_2\mu + c_1)$$

$$= -F'(\mu)$$

(40)
The variable \( \mu \) should be restricted to an interval \([a, b]\) (say), where \( a, b \in \{-\infty, 0, \infty, \mu_1, \mu_2, \mu_3\} \) on which \( F(\mu) \geq 0 \). The Gaussian curvature \( \kappa \) on the configuration manifold is of variable sign in general. But it is possible to have manifolds of curvature of definite sign. An example is when one of the two bounds is taken to be the point at infinity. We note also that in order that the Lagrangian was real, the coefficients \( K_i \) should be taken real, imaginary or including a complex conjugate pair, depending on the nature and relative order of the roots of \( F \). From (40) we also conclude that only case 16 lives on a flat manifold and only cases 14, 15 has a manifold of constant non-zero curvature.

In the last two cases the system describes either:

1. a motion on a sphere (when \( c_1 < 0 \)), which can be recognized as the axisymmetric version of Clebsch’s integrable case of motion in a liquid of a body with a spherical inertia tensor. The full Clebsch case is obtained as a special version of another new integrable system in [6.2.1]

2. or a motion on a pseudo-sphere or in the hyperbolic plane \( H^2 \) when \( c_1 > 0 \).

4.2.2 Example: A case of motion on a Riemannian manifold of positive Gaussian curvature

In order that the Lagrangian (37) represents a real system, the kinetic energy should be positive definite. In case 6 this implies that the variable \( \mu \) should be confined to an interval of the non-negative half of the \( \mu \)-axis on which \( c_2 \mu^2 + c_1 \mu + c_0 \geq 0 \). Various possible choices of the interval lead to different configuration manifolds. We give an example of motion on a convex manifold:

In case 10 of table I, let \( c_0 (= c_2 \mu_1 \mu_2) = 1, a_1 = 0, a_0 = \alpha^2, 0 \leq \mu \leq \mu_1 \leq \mu_2 \).

The substitution \( \lambda = \alpha \cos \phi, \mu = \mu_1 \text{sn}^2 u, K_1 = ik_1, K_2 = ik_2 (\sqrt{\mu - \mu_1} = i\sqrt{\mu_1} \text{cn} u, \sqrt{\mu - \mu_2} = i\sqrt{\mu_2} \text{dn} u) \), where the modulus of the elliptic functions
\[ k = \sqrt{\frac{\mu_1}{\mu_2}} \] transforms the Lagrangian to the form

\[
L = \frac{1}{2} \mu_1 (\dot{u}^2 + \text{sn}^2 u \dot{\varphi}^2) + \frac{\alpha \sin \varphi}{\sqrt{c_2 \mu_1 \text{sn}^2 u}} \dot{u} \\
+ \sqrt{c_2}[k_1 \sqrt{\mu_2} \text{dn} u + k_2 \sqrt{\mu_1} \text{cn} u + \frac{b}{c_2} \frac{\alpha \cos \varphi}{\sqrt{\mu_1 \text{sn} u}} \text{cn} u \text{dn} u] \dot{\varphi} \\
- b[\alpha \sqrt{\mu_1} \text{sn} u \cos \varphi - \frac{b \mu_1}{2c_2} \text{sn}^2 u - k_1 \sqrt{\mu_1} \text{cn} u - k_2 \sqrt{\mu_2} \text{dn} u] \\
(41)
\]

The Gaussian curvature of the metric of this system is strictly positive

\[ \kappa = c_2 (\mu_1 \text{cn}^2 u + \mu_2 \text{dn}^2 u) \geq c_2 (\mu_2 - \mu_1) > 0 \] \hspace{1cm} (42)

It is possible to realize the Riemannian metric of the last system on a closed surface of revolution. In fact, if \((\rho(u), \varphi, z(u))\) is a current point of the surface in cylindrical coordinates, then we have

\[
d\rho^2 + dz^2 + \rho^2 d\varphi^2 = \mu_1 (du^2 + \text{sn}^2 u \, d\varphi^2) \] \hspace{1cm} (43)

Comparing both sides we find that

\[ \rho = \sqrt{\mu_1} \text{sn} u \] \hspace{1cm} (44)

and hence we get an equation for \(z\) as

\[ \mu_1 \text{cn}^2 u \, d\varphi^2 + \left(\frac{dz}{du}\right)^2 = \mu_1 \]

which, on separation and integration, gives

\[ z = \sqrt{\mu_1} \int_0^u \sqrt{1 + k^2 - k^2 \text{sn}^2 u \, \text{sn} u \, du} \] \hspace{1cm} (45)

Noting that this is a periodic function with period \(4K(k)\), we deduce that the surface thus constructed is a closed one. This family of surfaces of revolution depends on two parameters \(\mu_1\) and \(k\). The first can be absorbed by a suitable scaling, but \(k\) changes the form of the surface. Figure 3 shows this surface for
\( k = \frac{1}{\sqrt{2}}. \)

Figure 3: The surface of revolution described by (44) and (45) for \( k = \frac{1}{\sqrt{2}} \)

5  **The third type - Systems with a Liouville-type metric**

5.1 General considerations

Now we consider the solution of the system of equations (15) in the generic case when \( \lambda'(\xi)\mu'(\eta) \neq 0 \). It will be easier in this case to use \( \lambda, \mu \) as variables. Let

\[
\lambda'^2(\xi) = F(\lambda), \mu'^2(\eta) = G(\mu) \quad (46)
\]

so that

\[
\xi = \int \frac{d\lambda}{\sqrt{F(\lambda)}}, \eta = \int \frac{d\mu}{\sqrt{G(\mu)}} \quad (47)
\]
Equations (15b-15d) can be written as

\[ \phi_\lambda - \phi_\mu - 2(\lambda - \mu)\phi_{\lambda\mu} = 0 \]  
\[ \phi_\mu V_\lambda + \phi_\lambda V_\mu = 0 \]  
\[ [(\sqrt{F(\lambda)}\frac{\partial}{\partial \lambda})^2 + (\sqrt{G(\mu)}\frac{\partial}{\partial \mu})^2][\sqrt{F(\lambda)}G(\mu)\phi_\lambda \phi_\mu] \]

\[ -\sqrt{F(\lambda)G(\mu)}[(\lambda - \mu)V]_{\lambda\mu} = 0 \]  

(48c)

One of those equations can be readily integrated. In fact, the equation of the characteristics of (48b) for \( V \) is

\[ \phi_\lambda d\lambda - \phi_\mu d\mu = 0 \]  

(49)

Multiplying this equation by \( (\lambda - \mu) \) one can verify that the resulting expression is a total differential, in virtue of (48a), so that one can write

\[ d\psi = (\lambda - \mu)(\phi_\lambda d\lambda - \phi_\mu d\mu) \]  

(50)

Thus, the potential \( V \) can be expressed as \( V = V(\psi) \), where \( \psi \) is related to \( \phi \) by the relations

\[ \psi_\lambda = (\lambda - \mu)\phi_\lambda, \quad \psi_\mu = -(\lambda - \mu)\phi_\mu \]  

(51)

Since only the derivatives of \( \phi \) enter in (48a) and (48c), one can completely eliminate \( \phi \) from those equations. This gives for \( \psi \) the equation

\[ \psi_\lambda - \psi_\mu + 2(\lambda - \mu)\psi_{\lambda\mu} = 0 \]  

(52)

\[ [(\sqrt{F(\lambda)}\frac{\partial}{\partial \lambda})^2 + (\sqrt{G(\mu)}\frac{\partial}{\partial \mu})^2][\sqrt{F(\lambda)}G(\mu)\psi_\lambda \psi_\mu] \]

\[ -(\lambda - \mu)\sqrt{F(\lambda)G(\mu)}[V^{''}(\psi)\psi_\lambda \psi_\mu + 3V^{'}(\psi)\psi_{\lambda\mu}] \]

\[ = 0 \]  

(53)
The Lagrangian describing the system (in the fictitious time) may now be written as

\[
L = \frac{1}{2} \left[ \frac{\psi'}{F(\lambda)} \frac{\psi}{G(\mu)} \lambda' \left( \frac{\psi}{F(\lambda)} \right)^2 + \frac{\psi'}{G(\mu)} \mu' \left( \frac{\psi}{G(\mu)} \right)^2 \right] \\
- \frac{1}{(\lambda - \mu)} \sqrt{F(\lambda)G(\mu)} \left[ \frac{\psi}{F(\lambda)} \lambda' + \frac{\psi}{G(\mu)} \mu' \right] \\
+ (\lambda - \mu) \left[ K - V(\psi) \right]
\]  

(54)

and in the natural time it takes the form

\[
L = \frac{1}{2} (\lambda - \mu) \left[ \frac{\lambda^2}{F(\lambda)} + \frac{\mu^2}{G(\mu)} \right] \\
- \frac{1}{(\lambda - \mu)} \sqrt{F(\lambda)G(\mu)} \left[ \frac{\psi}{F(\lambda)} \lambda' + \frac{\psi}{G(\mu)} \mu' \right] \\
+ (\lambda - \mu) \left[ K - V(\psi) \right]
\]  

(55)

In certain circumstances we need also the expression

\[
\Omega = F(\lambda) \frac{\partial}{\partial \lambda} \left( \frac{\psi}{\lambda - \mu} \right) - G(\mu) \frac{\partial}{\partial \mu} \left( \frac{\psi}{\lambda - \mu} \right) + \frac{F'(\lambda) \psi - G'(\mu) \psi}{2(\lambda - \mu)}
\]

(56)

The process of constructing time-irreversible mechanical systems with a quadratic integral is now reduced to the simultaneous solution of the pair of equations (52) and (53), which involve the function \( \psi(\lambda, \mu) \) together with three functions of one variable each: \( F(\lambda), G(\mu) \) and \( V(\psi) \). It is to be noted also that the three one-variable functions occur only linearly in the last equation. This becomes obvious if we write down this equation in the expanded form

\[
\left[ F''(\lambda) + G''(\mu) \right] \frac{\psi \psi_{\mu}}{\lambda - \mu} \\
+ 3 \left\{ \left[ \frac{\psi \psi_{\mu}}{(\lambda - \mu)^2} \right] \lambda F'(\lambda) + \left[ \frac{\psi \psi_{\mu}}{(\lambda - \mu)^2} \right] \mu G'(\mu) \right\} \\
+ 2 \left\{ \left[ \frac{\psi \psi_{\mu}}{(\lambda - \mu)^2} \right] \lambda \lambda F(\lambda) + \left[ \frac{\psi \psi_{\mu}}{(\lambda - \mu)^2} \right] \mu \mu G(\mu) \right\} \\
+ 2(\lambda - \mu) \left[ V'(\psi) \psi_{\lambda} \psi_{\mu} + 3V'(\psi) \psi_{\lambda} \psi_{\mu} \right] \\
= 0
\]

(57)
5.2 Solving the equation for $\psi$

Equation (52) and (48a) were given an interesting interpretation in [16]. We provide it here and use it in a more systematic way to construct simple solutions of that equation, future candidates of satisfying (53). In fact, if we introduce new variables $\rho = \lambda - \mu$ and $z = i(\lambda + \mu)$ then (48a), (52) and (51) reduce to

$$\phi_{\rho\rho} + \frac{1}{\rho} \phi_{\rho} + \phi_{zz} = 0 \quad (58a)$$

$$\psi_{\rho\rho} - \frac{1}{\rho} \psi_{\rho} + \psi_{zz} = 0 \quad (58b)$$

$$(-i\psi)_z = -\rho\phi_{\rho}, \quad (-i\psi)_\rho = \rho\phi_z \quad (58c)$$

Imagine $z$ as the axis of cylindrical coordinates in a three-dimensional complex space and $\rho$ is the radial distance of the current point from that axis. Equation (58a) is just Laplace’s equation satisfied by $\phi$ — the velocity potential of a virtual flow of an ideal incompressible fluid, symmetric around the $z$ axis, (58b) is the equation for $\psi$ — Stokes stream function of that flow and (58c) are the well-known relations between the two functions (see e.g. [41]). In [16] we have utilized some known axisymmetric hydrodynamic flows to construct certain solutions of (52).

Here we provide a systematic way to construct solutions of (52) and (48a).

It is well known that (58a) admits the spherically symmetric solution $\phi = \frac{1}{r} = \frac{1}{\sqrt{z^2 + \rho^2}}$ representing a source at the origin. For a source on the $z$-axis at a distance $z_0$ from the origin the velocity potential becomes

$$\phi = \frac{1}{\sqrt{z_0^2 - 2z_0z + z^2 + \rho^2}} \quad (59)$$

$$= \frac{1}{\sqrt{z_0^2 - 2iz_0(\lambda + \mu) - 4\lambda\mu}} \quad (60)$$

This function has two types of expansions:

$$\phi = \sum_{n=0}^{\infty} \frac{k_n}{z_0^{n+1}} \phi_n \quad (61)$$
for large \( z_0 \) and
\[
\phi = \sum_{n=0}^{\infty} k'_n z_0^{n} \frac{\phi_n}{(\lambda \mu)^{n+1/2}} \tag{62}
\]
for small \( z_0 \), where \( \phi_n \) is a homogeneous polynomial of degree \( n \) in \( \lambda, \mu \) and \( \{k_n\}, \{k'_n\} \) are numerical (real or imaginary) coefficients, which can be used to give the polynomials \( \phi_n \) the most convenient form. Since the expansion is valid for arbitrary (large or small) values of \( z_0 \), we conclude that the two infinite sequences of functions \( \{\phi_n\}, \{\phi_n / (\lambda \mu)^{n+1/2}\} \) are solutions of equation (48a). It is noteworthy to mention that those functions correspond to the familiar sequences of axisymmetric solutions \( r^n P_n(\cos \theta), r^{-(n+1)} P_n(\cos \theta) \) of Laplace’s equation in spherical coordinates.

We now follow a parallel line for solving equation (52). The spherically symmetric solution of Stokes equation (58b) is easily seen to be \( \psi = r = \sqrt{z^2 + \rho^2} \). Displacing the source as indicated above we get Stokes function
\[
\psi = \sqrt{z_0^2 - 2t z_0 (\lambda + \mu) - 4 \lambda \mu} \tag{63}
\]
For this function we obtain the two expansions
\[
\psi = \sum_{n=0}^{\infty} K_n z_0^n \psi_n \tag{64}
\]
and
\[
\psi = \sqrt{\lambda \mu} \sum_{n=0}^{\infty} K'_n z_0^n \frac{\psi_n}{(\lambda \mu)^n} \tag{65}
\]
thus giving the two infinite sequences of solutions of Stokes equation (58b) \( \{\psi_n, n = 0, \ldots, \infty\}, \{\tilde{\psi}_n = \sqrt{\lambda \mu} \frac{\psi_n}{(\lambda \mu)^n}, n = 1, \ldots, \infty\} \). The first few non-constant
The first few of the second sequence \( \{ \tilde{\psi}_n \} \) are

\[
\begin{align*}
\psi_0 &= 1 \\
\psi_1 &= \lambda + \mu \\
\psi_2 &= (\lambda - \mu)^2 \\
\psi_3 &= (\lambda - \mu)^2(\lambda + \mu) \\
\psi_4 &= (\lambda - \mu)^2(5\lambda^2 + 5\mu^2 + 6\lambda\mu) \\
\psi_5 &= (\lambda - \mu)^2(\lambda + \mu)(7\lambda^2 + 7\mu^2 + 2\lambda\mu)
\end{align*}
\]

(66)

Finally, we note that equation (52) is invariant with respect to shifting \((\lambda, \mu)\) to \((\lambda - \nu, \mu - \nu)\) for all constant \(\nu\), so that the sequences of solutions (66, 67) remain valid after this shift. We shall use this freedom in certain circumstances.

5.3 Adapting a solution

A possible way of solving (52-57) is choosing a solution \(\psi(\lambda, \mu)\) for the linear equation (52) and inserting this solution in (57), which will serve as a consistency condition. A working choice \(\psi(\lambda, \mu)\) should allow the existence of \(F, G, V\), so that this equation is satisfied. It is natural that some solutions of (52) will be accepted in (57), while others will be excluded. Till now, we have not been able to separate any part of the last equation to operate independently on one or two functions. It remains a matter of guess and trial to find possible cases. It is not possible in this way to claim that this way will lead to construction of all possible cases of the type under consideration. However, it has lead to several rich families of systems in one of the ways described below.
5.3.1 The first procedure:

In general, equation (57), which may be written in the form

\[ V''(\psi) + \frac{\psi_{\lambda\mu} \psi_{\lambda} \psi_{\mu}}{\psi_{\lambda} \psi_{\mu}} \psi' = N(\lambda, \mu) \]  

(68)

where

\[
N = \left\{ \frac{[F''(\lambda) + G''(\mu)] \psi_{\lambda\mu}}{(\lambda - \mu)^2} \right. \\
\left. + 3\left[ \frac{\psi_{\lambda\mu} \psi_{\lambda}}{(\lambda - \mu)^2} \lambda F'(\lambda) + \frac{\psi_{\lambda\mu} \psi_{\mu}}{(\lambda - \mu)^2} \mu G'(\mu) \right] \right. \\
\left. + 2\left[ \frac{\psi_{\lambda\mu} \psi_{\lambda}}{(\lambda - \mu)^2} \lambda \lambda F(\lambda) - \frac{\psi_{\lambda\mu} \psi_{\mu}}{2(\lambda - \mu)^2} \psi_{\lambda} \psi_{\mu} \right] \right\} / 2(\lambda - \mu) \psi_{\lambda} \psi_{\mu} 
\]

(69)

is a consistency condition which should be satisfied by the four functions \( F(\lambda), G(\mu), V(\psi), \) and \( \psi(\lambda, \mu), \) of which only \( \psi \) is subject to another equation, namely, (52). An important situation is met when (68) becomes a differential equation in \( \psi' \).

For this to happen, the coefficient of \( \psi' \) as well as the function \( N \) should be functions of \( \psi \). That is

\[
(\frac{\psi_{\lambda\mu}}{\psi_{\lambda} \psi_{\mu}}) \lambda \psi_{\mu} - (\frac{\psi_{\lambda\mu}}{\psi_{\lambda} \psi_{\mu}}) \mu \psi_{\lambda} = 0 \quad (70a) \\
N_{\lambda} \psi_{\mu} - N_{\mu} \psi_{\lambda} = 0 \quad (70b)
\]

Thus, for the present procedure to work, \( \psi \) must satisfy simultaneously the linear second-order equation (52) and the nonlinear third-order equation (70a).

We now begin with trying a solution of (52) in the form of one of the functions \( \psi_n \) and \( \tilde{\psi}_n \) from (66,67). It turned out that only four functions \( \psi_1, \psi_2, \tilde{\psi}_1, \tilde{\psi}_2 \) satisfy (70a). Moreover, we have tried solutions in the form of linear combinations

\[
\psi = \sum_{i=1}^{5} (J_i \psi_i + K_i \tilde{\psi}_i) 
\]

(71)

This yielded the above four functions and two more functions \( \psi_1 \pm 2 \tilde{\psi}_1 = (\sqrt{\lambda} \pm \sqrt{\mu})^2 \), so that the present procedure can be applied for the six cases:
Those cases will be discussed in the next section. For each case:

a) The function $\psi$ is substituted in (70b) and separation of variables is achieved after affecting a differential operator.

b) $F(\lambda), G(\mu)$ are determined. It turned out that $F$ is rational in case 17 and polynomial in all other cases and also $G(\mu) = -F(\mu)$ in all cases except 19 and 20.

c) $F(\lambda), G(\mu)$ are substituted in (69) and the expression $N$ is expressed as a function of $\psi$ only.

d) $N(\psi)$ is inserted in (68), and the last is solved for $V(\psi)$.

e) The final form of $V$ is obtained by expressing $\psi$ again in terms of $\lambda, \mu$.

Some of the indicated steps are quite cumbersome, even with the use of computer algebra packages. For space saving most of those details are not reflected here. We shall give only the data necessary to unambiguously reproduce every case.

### Table II

| Case | $\psi$ |
|------|--------|
| 17   | $2J\sqrt{\lambda\mu}$ |
| 18   | $J(\lambda + \mu)$ |
| 19   | $J(\lambda - \mu)^2$ |
| 20   | $J\frac{\lambda + \mu}{\sqrt{\lambda\mu}}$ |
| 21   | $J(\sqrt{\lambda} + \sqrt{\mu})^2$ |
| 22   | $J(\sqrt{\lambda} - \sqrt{\mu})^2$ |
5.3.2 The second procedure:

The last procedure determined cases in which the potential $V$ is a solution of a second order ODE depending linearly on one significant arbitrary integration constant. The other being an immaterial additive constant. This does not exclude the existence of other potentials not containing this arbitrary constant. It may of course depend on the parameters appearing in other functions $\psi, F, G$.

For such cases $\psi$ is not required to satisfy \(70a \, 70b\). One may try expressing $V(\psi)$ in the form of a polynomial (say)

$$V = \sum_{i=1}^{N} v_i \psi^i,$$

choose a solution $\psi$ of \(52\) and insert this expression in \(57\) and try to find combinations of parameters that lead to separation of the two unknown functions $F(\lambda), G(\mu)$. Obviously, the outcome of this method will depend on the ability to perform necessary operations on functions $V, \psi$ involving as much coefficients as possible.

We have made three trials:

A) For $\psi = \sum_{i=1}^{N} J_i \psi_i$, the separation resulted in 12th-degree polynomials for $F$ and $G$, but the final analysis led to 6 cases

(a) Two special versions of cases 18 and 19 of table II.

(b) $\psi$ is a combination of $\psi_1, \psi_2, v_2 = 0, F, G$ are cubic and $G(\mu) = -F(\mu)$.

(c) $\psi$ is a combination of $\psi_1, \ldots, \psi_3, v_2 = 0, F, G$ are quadratic and $G(\mu) = -F(\mu)$.

(d) $\psi$ is a combination of $\psi_1, \ldots, \psi_4, v_2 = 0, F, G$ are linear and $G(\mu) = -F(\mu)$.  

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\( \psi \) is a combination of \( \psi_1, ..., \psi_5, v_2 = 0, F(\mu) = 1 = -G(\mu). \)

It was noted that the presence of any of the few next \( \psi_i \) with \( ii > 5 \)
led to the inconsistent result \( G(\mu) = F(\mu) = 0. \)

B) For \( \psi = \sum_{i=1}^{5} J_i \tilde{\psi}_i \) we have been lead to 5 cases:

(a) A special version of case 17 of table II.

(b) \( \psi \) is a combination of \( \tilde{\psi}_1, \tilde{\psi}_2, v_1 = 0, F, G \) are of 5th degree with two
equal roots and \( G(\mu) = -F(\mu). \)

(c) \( \psi \) is a combination of \( \tilde{\psi}_1, ..., \tilde{\psi}_3, v_1 = 0, F, G \) are of 5th degree with
three equal roots and \( G(\mu) = -F(\mu). \)

(d) \( \psi \) is a combination of \( \tilde{\psi}_1, ..., \tilde{\psi}_4, v_1 = 0, F, G \) are of 5th degree with
four equal roots and \( G(\mu) = -F(\mu). \)

(e) \( \psi \) is a combination of \( \tilde{\psi}_1, ..., \tilde{\psi}_5, v_1 = 0, F, G \) are of 5th degree with
all five roots equal and \( G(\mu) = -F(\mu). \)

C) For \( \psi = \sum_{i=1}^{5} J_i \psi_i(x - c_i, y - c_i), v_1 = 0, F, G \) are of 5th degree and \( G(\mu) =
-F(\mu), \) it was found that the only condition is that \( \{c_i\} \) are roots of \( F. \)

6 The basic cases

6.1 Case 17: The generating system with algebraic \( \psi \)

Here we try the first term of the sequence \( (67): \)

\[ \psi = 2J \sqrt{\lambda \mu} \] (73)

one is lead by equations \( (70b) \) and then \( (68) \) to rational expression for \( F \)

\[ F(\lambda) = a_5 \lambda^5 + a_4 \lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 + \frac{b}{\lambda}, \quad G(\mu) = -F(\mu) \] (74)
The Lagrangian is given by

\[ L = \frac{1}{2}(\lambda - \mu)\left[ \frac{\lambda^2}{F(\lambda)} - \frac{\mu^2}{F(\mu)} \right] - J\sqrt{-F(\lambda)F(\mu)}\left[ \frac{\lambda}{F(\lambda)} \dot{\lambda} - \frac{\mu}{F(\mu)} \dot{\mu} \right] + \frac{1}{2}a_5J^2\lambda\mu - \frac{bJ^2}{2\lambda^2\mu^2} + \frac{c}{\lambda\mu} \]  

(75)
i, b, c being integration constants. The second integral is

\[ I = \frac{1}{2}(\lambda - \mu)\left[ \frac{\mu^2}{F(\lambda)} - \frac{\lambda^2}{F(\mu)} \right] + J\sqrt{-F(\lambda)F(\mu)}\left[ \frac{\lambda}{F(\lambda)} \dot{\lambda} + \frac{\mu}{F(\mu)} \dot{\mu} \right] + \frac{1}{2}J^2\left\{ \frac{2b(\lambda + \mu)}{\lambda^2\mu^2} + \frac{a_0}{\lambda\mu} + \lambda\mu[a_5(\lambda + \mu) + a_4]\right\} + \frac{c(\lambda + \mu)}{\lambda\mu} \]  

(76)
The Gaussian curvature of the configuration space carrying (75) is

\[ \kappa = -\frac{1}{4}\left[a_5(3\lambda^2 + 3\mu^2 + 4\lambda\mu) + 2a_4(\lambda + \mu) + a_3 - \frac{b}{\lambda^2\mu^2}\right] \]  

(77)

In order that the system described by the Lagrangian (75) was real, it is necessary that the variable \( \lambda \) varies on an interval on which \( F(\lambda) \geq 0 \) and \( \mu \) on one of the intervals where \( F(\mu) \leq 0 \). For the quadratic part of the kinetic energy to be positive definite the factor \( \lambda - \mu \) must be non-negative. The interval chosen for \( \lambda \) should lie on the right of that for \( \mu \). The number of working combinations of intervals depends on the number of real roots of \( F \). Figure 4 shows possible choices of intervals for \( \lambda \) and \( \mu \) in the case of five real distinct roots for \( a_5 > 0 \). When \( a_5 < 0 \) the intervals are interchanged and some of them are excluded to
sustain the above conditions.

Figure 4: Possible choices of intervals

The possible 9 combinations are listed in the following table

\[
\begin{align*}
\lambda & \quad \mu \\
I & \quad i, ii, iii \\
II & \quad ii, iii \\
III & \quad iii \\
\end{align*}
\]

\[
\begin{align*}
a_5 > 0 & \quad \{ \\
I & \quad i, ii, iii \\
II & \quad ii, iii \\
III & \quad iii \\
\} \\
\end{align*}
\]

\[
\begin{align*}
a_5 < 0 & \quad \{ \\
i & \quad II, III \\
ii & \quad III \\
\} \\
\end{align*}
\]

The general picture changes when \( F \) has equal roots or has pairs of complex conjugate roots. Degenerate cases will be treated in the same way.

In a next section we shall point out an important application of the system I, Clebsch’s case of motion of a solid in a liquid obtained as a special case. Here we examine integrable motions on spaces of constant Gaussian curvature. In
fact, when $b = a_5 = a_4 = 0$ the Gaussian curvature (77) is constant $\kappa = -\frac{a_3}{4}$ and the Lagrangian takes the form

$$L = \frac{1}{2}\left(\dot{\lambda}\frac{\lambda^2}{a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0} - \frac{\mu^2}{a_3\mu^3 + a_2\mu^2 + a_1\mu + a_0}\right)$$

$$- \sqrt{a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0}\sqrt{-\left(a_3\mu^3 + a_2\mu^2 + a_1\mu + a_0\right)}$$

$$\frac{\lambda \dot{\lambda}}{a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0} - \frac{\mu \dot{\mu}}{a_3\mu^3 + a_2\mu^2 + a_1\mu + a_0}$$

$$+ \frac{h^2}{\lambda\mu}(+h)$$

(78)

For this system one may use the substitutions $\lambda = P_1(\xi), \mu = P_1(i\eta)$, where $P_1$ is the Weierstrass elliptic function generated by the inversion of the integral

$$\xi = \int^\lambda \frac{d\lambda}{\sqrt{a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0}}$$

(79)

to transform to isometric coordinates. However, this procedure is not transparent enough to isolate cases of real Lagrangian systems. Suitable substitutions vary according to the number of real roots of the cubic polynomial, their relative order and to the intervals on which $\lambda$ and $\mu$ vary. We consider somewhat further the case of three different real roots $a, b, c$ ($a > b > c$), say. Noting the symmetry of the structure of the Lagrangian with respect to the variables $\lambda, \mu$, and to guarantee that the kinetic energy is positive definite, there is no loss in generality in assuming that $\lambda - \mu \geq 0$. The variables $\lambda, \mu$ must also take their values on intervals, on which the cubic function has different signs.

1. When $a_3 > 0$ we have three working combinations of intervals for which the Lagrangian describes a motion on the pseudo sphere (or on the hyperbolic plane $H^2$):

(a) i. $\lambda \in [a, \infty), \mu \in [b, a]$

ii. $\lambda \in [a, \infty), \mu \in (-\infty, c]$
iii. $\lambda \in [c, b], \mu \in (-\infty, c]$

2. When $a_3 < 0$ we have only one working combination $\lambda \in [b, a], \mu \in [c, b]$.

This leads to a motion on the sphere, which we will discuss in more detail in the next subsection.

6.1.1 An integrable motion on a sphere

In the case 2 of positive constant curvature, without loss of generality one can set $a_3 = -1$. The Lagrangian and the integral can then be put in the form

$$L = \frac{1}{2} \frac{\dot{\lambda}^2}{(a - \lambda)(\lambda - b)(\lambda - c)} + \frac{\dot{\mu}^2}{(a - \mu)(b - \mu)(\mu - c)}$$

$$- \frac{J}{4} \frac{\sqrt{(a - \lambda)(\lambda - b)(\lambda - c)} \sqrt{(a - \mu)(b - \mu)(\mu - c)}}{(\lambda - \mu) \sqrt{\lambda \mu}} \times$$

$$\times \left[ \frac{\lambda \dot{\lambda}}{(a - \lambda)(\lambda - b)(\lambda - c)} - \frac{\mu \dot{\mu}}{(a - \mu)(b - \mu)(\mu - c)} \right]$$

$$+ \frac{J_2}{4 \lambda \mu} \left( + \frac{h}{4} \right)$$

and

$$I = (\lambda - \mu) \frac{\mu \dot{\lambda}^2}{(a - \lambda)(\lambda - b)(\lambda - c)} + \frac{\lambda \dot{\mu}^2}{(a - \mu)(b - \mu)(\mu - c)}$$

$$- \frac{J}{\sqrt{\lambda \mu}} \frac{\sqrt{(a - \lambda)(\lambda - b)(\lambda - c)} \sqrt{(a - \mu)(b - \mu)(\mu - c)}}{(a - \lambda)(\lambda - b)(\lambda - c)} \times$$

$$\times \left[ \frac{\lambda \dot{\lambda}}{(a - \lambda)(\lambda - b)(\lambda - c)} - \frac{\mu \dot{\mu}}{(a - \mu)(b - \mu)(\mu - c)} \right]$$

$$- \frac{J_2 (\lambda + \mu)}{\lambda \mu} + \frac{abcJ^2}{\lambda \mu} \left( + h \right)$$

In the last system it is quite easy to recognize that $\lambda, \mu$ are the conical coordinates on the unit sphere. They are related to 3D Cartesian coordinates through the relations

$$x = \sqrt{\frac{(a - \lambda)(a - \mu)}{(a - b)(a - c)}}, \quad y = \sqrt{\frac{(\lambda - b)(b - \mu)}{(a - b)(b - c)}}, \quad z = \sqrt{\frac{(\lambda - c)(\mu - c)}{(a - c)(b - c)}}$$

(82)
Note also that
\[
\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = \frac{\lambda \mu}{abc} \quad (83)
\]
and
\[
(b + c)x^2 + (c + a)y^2 + (a + b)z^2 = \lambda + \mu \quad (84)
\]
so that \(\lambda, \mu\) are the solutions of the quadratic equation
\[
\zeta^2 - [(b + c)x^2 + (c + a)y^2 + (a + b)z^2]\zeta + abc\left[\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}\right] = 0 \quad (85)
\]
The potential term in the Lagrangian (80)
\[
V_0 = \frac{-h_2}{4\lambda\mu} = \frac{-h_2}{4abc\left(\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}\right)} \quad (86)
\]
is a separable potential. The Lagrangian (80) thus generalizes the separable system by the introduction of the gyroscopic terms. Comparing the Lagrangian (80) to the results of our paper [18] we conclude that the first gives a new integrable case of the motion of a particle on a sphere or, alternatively, a new integrable case in rigid body dynamics under forces with scalar and vector potential.

### 6.1.2 Systems on a flat manifold

The condition of vanishing Gaussian curvature \(\kappa = 0\) leads to the plane case
\[
V = \frac{C\alpha (\alpha - a)}{(a - \alpha)x^2 - ay^2 + a\alpha (\alpha - a)} \quad (a - \alpha)x^2 - ay^2 + a\alpha (\alpha - a) \quad (87)
\]
discussed firstly in section 3.4 of our work [17]. A case presented in [23] can be easily obtained as a limiting case of this system. An equivalent system was discussed in [26] where one more limiting case is pointed out.
6.2 Case 17: $\psi$ is a linear function

The choice $\psi = J(\lambda + \mu)$ leads to the expressions

\[
F(\lambda) = a_6\lambda^6 + a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0, \quad G(\mu) = -F(\mu) \tag{88}
\]

\[
L = \frac{1}{2}(\lambda - \mu)[\frac{\dot{\lambda}^2}{F(\lambda)} - \frac{\mu^2}{F(\mu)}] - \frac{J\sqrt{-F(\lambda)F(\mu)}}{\lambda - \mu}[\frac{\dot{\lambda}}{F(\lambda)} - \frac{\dot{\mu}}{F(\mu)}] + \frac{1}{2}J^2[a_6(\lambda + \mu)^3 + a_5(\lambda + \mu)^2] - K_1(\lambda + \mu) \tag{89}
\]

and the integral

\[
I = \frac{1}{2}(\lambda - \mu)[\frac{\lambda^2}{F(\lambda)} - \frac{\mu^2}{F(\mu)}] + J\sqrt{-F(\lambda)F(\mu)}[\frac{\dot{\lambda}}{F(\lambda)} + \frac{\dot{\mu}}{F(\mu)}] + \frac{1}{2}J^2\{a_6(\lambda + \mu)(\lambda^2 + \mu^2 - \lambda\mu) + a_5(\lambda^2 + \mu^2)\} + a_4(\lambda + \mu)^2 + a_3(\lambda + \mu)\} - 2K_1\lambda\mu \tag{90}
\]

\[
\kappa = -\frac{1}{4} [2a_6(\lambda + \mu)(2\lambda^2 + 2\mu^2 + \lambda\mu) + a_5(3\lambda^2 + 3\mu^2 + 4\lambda\mu) + 2a_4(\lambda + \mu) + a_3] \tag{91}
\]

6.2.1 The second case of Clebsch of motion of a solid in a liquid

Under the restrictions $a_6 = a_5 = a_4 = 0$, the Gaussian curvature becomes $\kappa = -\frac{a_3}{4}$. In order that the configuration manifold was an ordinary sphere one may set $a_3 = -1$ (say). The cubic polynomial must have three different real roots $a, b, c$($a > b > c$) and the variables $\lambda, \mu$ should be confined to the intervals
and \([c, b]\), respectively, so that \(\lambda - \mu \geq 0\). We obtain

\[
L = \frac{1}{2} (\lambda - \mu)^2 \left[ \frac{\dot{\lambda}^2}{(a - \lambda)(b - \lambda)(\lambda - c)} + \frac{\dot{\mu}^2}{(a - \mu)(b - \mu)(\mu - c)} \right] - J \sqrt{(a - \lambda)(b - \lambda)(\lambda - c)} \sqrt{(a - \mu)(b - \mu)(\mu - c)} \times
\]

\[
\times \left[ \frac{\dot{\lambda}}{(a - \lambda)(b - \lambda)(\lambda - c)} + \frac{\dot{\mu}}{(a - \mu)(b - \mu)(\mu - c)} \right] + h_1 (\lambda + \mu) \quad (92)
\]

Comparing this to formulas of subsection (6.1.1) above, we find that the potential is

\[
V = h_1 (\lambda + \mu)
\]

\[
= h_1 [(b + c)x^2 + (c + a)y^2 + (a + b)z^2]
\]

\[
= \text{const} - h_1 (ax^2 + by^2 + cz^2) \quad (93)
\]

Comparing potential and gyroscopic terms to the list of integrable motions on a sphere in [18], one can show that we are dealing with the famous case of motion of a rigid body in a liquid named as the second case of Clebsch, characterized by a spherically symmetric inertia matrix. We shall not reproduce the integral in this case, which is well known.

### 6.3 Case 19: \(\psi = J(\lambda - \mu)^2\)

For this family of systems

\[
F(\lambda) = a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0,
\]

\[
G(\mu) = -a_3 \mu^3 + b_2 \mu^2 + b_1 \mu + b_0 \quad (94)
\]

\[
L = \frac{1}{2} (\lambda - \mu)^2 \left[ \frac{\dot{\lambda}^2}{F(\lambda)} + \frac{\dot{\mu}^2}{G(\mu)} \right] + J \sqrt{\frac{G(\mu)}{F(\lambda)}} \dot{\lambda} - J \sqrt{\frac{F(\lambda)}{G(\mu)}} \dot{\mu} + \frac{1}{2} J^2 [a_3 (\lambda - \mu)^2 + (a_2 + b_2)(\lambda - \mu)] - \frac{K_1}{(\lambda - \mu)} \quad (95)
\]
\[ I = \frac{1}{2} (\lambda - \mu) \left[ \frac{\mu \lambda^2}{F(\lambda)} + \frac{\lambda \mu^2}{G(\mu)} \right] - J(\lambda - \mu) \left( \sqrt{\frac{G(\mu)}{F(\lambda)}} \frac{\lambda}{\lambda} + \sqrt{\frac{F(\lambda)}{G(\mu)}} \frac{\mu}{\mu} \right) \\
+ \frac{1}{2} J^2 (\lambda - \mu) \left[ a_3 (\lambda^2 - \mu^2) + a_2 \lambda + b_2 \mu + a_1 + b_1 \right] + \frac{1}{2} K \frac{\lambda + \mu}{\lambda - \mu} \]

and

\[ \kappa = \frac{1}{4} \left[ -a_3 + \frac{2(a_2 + b_2) \lambda \mu + (a_1 + b_1)(\lambda + \mu) + 2(a_0 + b_0)}{(\lambda - \mu)^3} \right] \quad (96) \]

6.4 Case 20: \( \psi = J \frac{\lambda + \mu}{\sqrt{\lambda \mu}} \)

For this case we get

\[ F(\lambda) = \lambda^2 (a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0), \quad G(\mu) = \mu^2 (b_3 \mu^3 + b_2 \mu^2 + b_1 \mu - a_0) \quad (97) \]

Note that \( F(0) + G(0) = 0 \). After some tedious calculations we obtain as final result of (68)

\[ V(\psi) = \frac{16 A \psi - (a_3 + b_3) \psi^3}{16 \sqrt{\psi^3 - 4 J^2}} - \frac{1}{16} (a_3 - b_3) \psi^2 \quad (98) \]

The Lagrangian of the present system has the form

\[ L = \frac{1}{2} (\lambda - \mu) \left[ \frac{\lambda^2}{F(\lambda)} + \frac{\mu^2}{G(\mu)} \right] \\
+ \frac{1}{2} J \sqrt{\frac{F(\lambda)G(\mu)}{\lambda^3 \mu^3}} \frac{\lambda \lambda}{F(\lambda)} - \frac{\mu \mu}{F(\mu)} \]

\[ -A \frac{\lambda + \mu}{\lambda - \mu} + J^2 (\lambda + \mu)^2 \left( \frac{a_3 \lambda}{\lambda} + \frac{b_3 \mu}{\mu} \right) \quad (+K) \quad (99) \]

This Lagrangian admits the integral

\[ I = \frac{1}{2} (\lambda - \mu) \left[ \frac{\mu \lambda^2}{F(\lambda)} + \frac{\lambda \mu^2}{G(\mu)} \right] \\
- \frac{1}{2} J(\lambda - \mu) \sqrt{\frac{F(\lambda)G(\mu)}{\lambda^3 \mu^3}} \frac{\lambda \lambda}{F(\lambda)} - \frac{\mu \mu}{F(\mu)} \]

\[ + A \frac{2 \lambda \mu - a(\lambda + \mu)}{\lambda - \mu} \]

\[ + \frac{J^2}{8} \left[ a_2 \frac{\lambda}{\mu} - b_2 \frac{\mu}{\lambda} + \frac{\lambda + \mu}{\lambda - \mu} \left( a_3 \frac{\lambda}{\mu} (\lambda - 3 \mu) - b_3 \frac{\mu}{\lambda} (3 \lambda - \mu) \right) \right] \quad (100) \]
The system (99) is completely new. The fifth degree polynomials $F, G$ have an identical double zero root but the other three are independent for each polynomial. The Gaussian curvature corresponding to (99) can be put in the form
\[ \kappa = -\frac{1}{4}\left[a_3(3\lambda^2 + 3\mu^2 + 4\lambda\mu) + 2a_4(\lambda + \mu) + a_3\right] \]
\[ + \frac{(a_3 + b_3)y^3(5\lambda - 3\mu) + (a_2 + b_2)y^3(2\lambda - \mu) + (a_1 + b_1)y^2(3\lambda - \mu)}{4(\lambda - \mu)^3} \]
(101)

6.5 Cases 21 and 22: $\psi = J(\sqrt{\lambda} \pm \sqrt{\mu})^2$

As in the previous section, for the present two cases we obtain in final form
\[ L = \frac{1}{2}(\lambda - \mu)[\frac{\lambda^2}{F(\lambda)} - \frac{\mu^2}{F(\mu)}] \]
\[ - J(\frac{\sqrt{\lambda} \pm \sqrt{\mu}}{\lambda - \mu}) \sqrt{-\frac{F(\lambda)F(\mu)}{\lambda\mu}} (\frac{\sqrt{\lambda}\lambda}{F(\lambda)} + \frac{\sqrt{\mu}\mu}{F(\mu)}) \]
\[ - \frac{A}{\sqrt{\lambda} \pm \sqrt{\mu}} + \frac{1}{2}J^2a_4(\sqrt{\lambda} \pm \sqrt{\mu})^2 \]
(102)
\[ F(\lambda) = \lambda(a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0) \]
(103)

This system will have the integral
\[ I = \frac{1}{2}(\lambda - \mu)[\frac{\mu\lambda^2}{F(\lambda)} - \frac{\lambda\mu^2}{F(\mu)}] \]
\[ - J(\frac{\sqrt{\lambda} \pm \sqrt{\mu}}{\lambda - \mu}) \sqrt{-\frac{F(\lambda)F(\mu)}{\lambda\mu}} (\frac{\sqrt{\lambda}\lambda}{F(\lambda)} - \frac{\sqrt{\mu}\mu}{F(\mu)}) \]
\[ - A\frac{\sqrt{\lambda}\mu}{\sqrt{\lambda} \pm \sqrt{\mu}} + \frac{1}{2}J^2[a_3(\lambda + \mu) + a_2](\sqrt{\lambda} \pm \sqrt{\mu})^2 \]
(104)

The two systems just obtained are different, in the sense that the difference in sign is significant. In fact, one can see that as $\lambda, \mu$ approach one and the same root of $F$ (different from 0) the potential term $\frac{A}{\sqrt{\lambda} \pm \sqrt{\mu}}$ remains finite for the positive sign but becomes infinite with the negative sign. The present two families of systems are new. Both families share the same configuration space. The Lagrangians can be explicitly in terms of global isometric variables $\xi, \eta$
using elliptic functions of complementary moduli. The Gaussian curvature of the configuration space can be calculated as

\[ \kappa = -\frac{1}{4} [2a_3(\lambda + \mu) + a_2] \]  

(105)

When \( a_3 = 0 \) this space becomes of constant curvature.

7 Variations of cases 17 and 18

It was noted in [16] that if we choose \( F(\lambda) \) as the polynomial part of \( (74) \), i.e.

\[ F(\lambda) = a_5\lambda^5 + a_4\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 \]  

(106)

then one can take \( \psi \) as a linear combination of five terms of the type \( (73) \)

\[ \sum_j J_j \sqrt{(\lambda - \nu_j)(\mu - \nu_j)} \]  

(107)

provided \( \{ \nu_j, j = 1, ..., 5 \} \) are the roots of the equation \( F(\nu) = 0 \). Here \( J_j \) are arbitrary constants which may be chosen real or complex in such a way that \( \psi \) is real. The system constructed in this way involves 11 arbitrary parameters \( \{a_0, ..., a_5, J_1, ..., J_5\} \). When some of the roots \( \{\nu_j\} \) are equal, some of the terms in \( (107) \) becomes dependent of the others and the number of significant parameters \( \{J_j\} \) is reduced by the number of extra repetition of roots. It turned out that the coalescence of roots can be made in such a way to preserve the total number of free parameters in the system.

1. The first procedure was indicated in [16]. A root \( \nu \) repeated \( (k + 1) \) times leads to the replacement of \( k \) terms in \( (107) \) by terms of the form

\[ \sum_{i=1}^{k} J_i^{'(d)} \frac{d}{d\nu^{'(d)}} \sqrt{(\lambda - \nu)(\mu - \nu)} \]  

so that we have the sequence of terms that can appear at possible coalescence combinations are obtained from \( (107) \) by a shift \( \nu \) in both variables.
2. When the leading coefficient $a_5$ of the polynomial \((106)\) vanishes one of the roots goes to infinity. The same happens with every vanishing leading coefficient. It turned out that for every infinite root one can associate one of the terms in $\psi$ of the sequence obtained by expanding the function $\sqrt{(\lambda - \nu)(\mu - \nu)/\nu}$ in powers of $\frac{1}{\nu}$. This gives again the sequence of expressions \((106)\). Terms of this sequence appear in $\psi$ according to the number of infinite roots.

In the following subsections we provide information, sufficient for clear and easy characterization of all the possible 19 cases with different combinations of coalescence of roots of the above two types. For every case we give only the forms of $F, \psi$ and the potential function $V(\psi)$. The Lagrangian is given by \((55)\). The integral can be constructed in a way similar to that followed in the above cases.

We shall identify each case by an ordered triplet expressing, respectively, the number of equal finite roots, the number of finite roots which are not equal to any others and the number of infinite roots. The first number may consist of two parts, when there are two sets of equal finite roots. Possible numbers are \((2, 2)\) and \((2, 3)\) only. In this classification the primitive case of five distinct roots will carry the triplet \([0, 5, 0]\). The case of five infinite roots \([0, 0, 5]\), the same as $F(\lambda) = 1$, leads to a case of pseudo-Riemannian metric (on which the kinetic energy is not positive definite) and we shall keep such cases out of the main list and list them separately. The 19 cases follow in the next two subsections.

### 7.1 Classification of the resulting cases

With the appropriate choice of the intervals for $\lambda, \mu$, the following 19 cases admit interpretation as motions on Riemannian or pseudo-Riemannian manifolds:
| Class       | $F(\lambda)$                             | $\psi$                                      | $V$                                      |
|------------|------------------------------------------|---------------------------------------------|------------------------------------------|
| 23 [0, 5, 0] | $\sum_{j=0}^{5} a_i \lambda^i$           | $2 \sum_{j=1}^{5} J_j \sqrt{(\lambda - \nu_j)(\mu - \nu_j)}$ | $-\frac{a_0}{8} \psi^2$                  |
| 24 [2, 3, 0] | $a_5 \prod_{i=1}^{3} (\lambda - \nu_i)(\lambda - \nu_4)^2$ | $2 \sum_{j=1}^{4} J_j \sqrt{(\lambda - \nu_j)(\mu - \nu_j)}$ | $-\frac{a_5}{8} \psi^2$                  |
| 25 [3, 2, 0] | $a_5 (\lambda - \nu_1)(\lambda - \nu_2) \times (\lambda - \nu)^3$ | $2 \sum_{j=1}^{3} J_j \sqrt{(\lambda - \nu_j)(\mu - \nu_j)}$ | $-\frac{a_5}{8} \psi^2$                  |
| 26 [4, 1, 0] | $a_5 (\lambda - \nu_1)(\lambda - \nu)^4$ | $2 J_1 \sqrt{(\lambda - \nu_1)(\mu - \nu_1)}$ | $-\frac{a_5}{8} \psi^2$                  |
| 27 [(2, 2), 1, 0] | $a_5 (\lambda - \nu_1)^2 \times (\lambda - \nu_2)^2 (\lambda - \nu_3)$ | $2 \sum_{j=1}^{2} J_j \sqrt{(\lambda - \nu_j)(\mu - \nu_j)}$ | $-\frac{a_5}{8} \psi^2$                  |
| 28 [(2, 3), 0, 0] | $a_5 (\lambda - \nu_1)^2 (\lambda - \nu_2)^3$ | $2 \sum_{j=1}^{3} J_j \sqrt{(\lambda - \nu_j)(\mu - \nu_j)}$ | $-\frac{a_5}{8} \psi^2$                  |
| 29 [5, 0, 0] | $a_5 (\lambda - \nu_1)^5$ | $J_1 \sqrt{\lambda \mu} + J_2 \frac{\lambda + \mu}{\sqrt{\lambda \mu}} + J_3 \frac{(\lambda - \nu_1)^2 (\lambda - \nu_5)^2}{\lambda \mu} + J_4 \frac{(\lambda - \nu_1)^2 (\lambda + \mu)}{\lambda \mu} + J_5 (\lambda - \nu_1)^2 (5 \lambda^2 + 5 \mu^2 + 6 \lambda \mu)$ | $-\frac{a_5}{8} \psi^2$                  |
| 30 [0, 4, 1] | $a_4 \lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0$ | $2 \sum_{j=1}^{4} J_j \sqrt{(\lambda - \nu_j)(\mu - \nu_j)} + J_5 (\lambda + \mu)$ | $-\frac{a_4}{2} J_5 \psi$ |
| 31 [2, 2, 1] | $a_4 (\lambda - \nu_1)(\lambda - \nu_2) \times (\lambda - \nu_3)^2$ | $2 \sum_{j=1}^{2} J_j \sqrt{(\lambda - \nu_j)(\mu - \nu_j)}$ | $-\frac{a_4}{2} J_5 \psi$ |
| 32 [3, 1, 1] | $a_4 (\lambda - \nu_1)(\lambda - \nu)^3$ | $2 J_1 \sqrt{(\lambda - \nu_1)(\mu - \nu)} + J_2 \frac{\lambda + \mu - 2 \nu}{\sqrt{\lambda - \nu_1}(\mu - \nu)} + J_3 \frac{(\lambda - \mu)^2}{\sqrt{\lambda - \nu_1}(\mu - \nu)} + J_4 \frac{(\lambda - \mu)\sqrt{(\lambda - \nu_1)(\mu - \nu)}}{\lambda \mu} + J_5 (\lambda - \nu_1)(\mu - \nu)$ | $-\frac{a_4}{2} J_5 \psi$ |
| 33 [0, 3, 2] | $a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0$ | $2 \sum_{j=1}^{3} J_j \sqrt{(\lambda - \nu_j)(\mu - \nu_j)} + J_4 (\lambda + \mu)$ | $-2 a_3 J_5 \psi$ |
| $I$   | $J$  | Expression                                                                                                                                                                                                 | Value                      |
|-------|------|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|----------------------------|
| 34    | $[2,1,2]$ | $a_3(\lambda - \nu_1)(\lambda - \nu_2)^2$                                                                                                                                                    | $2J_1 \sqrt{(\lambda - \nu_1)(\mu - \nu_1)} + 2J_2 \sqrt{(\lambda - \nu_2)(\mu - \nu_2)} - J_3 \frac{\lambda + \mu - 2\nu_2}{\sqrt{(\lambda - \nu_2)(\mu - \nu_2)}} + J_4(\lambda + \mu) + J_5(\lambda - \mu)^2$ $-2a_3J_5\psi$ |
| 35    | $[3,0,2]$ | $a_3\lambda^3$                                                                                                                                                                                             | $iJ_1 \sqrt{\lambda \mu} - 4iJ_2 \frac{\lambda + \mu}{\sqrt{\lambda \mu}} - 16iJ_4(\lambda - \mu)^2 + J_4(\lambda + \mu) + J_5(\lambda - \mu)^2$ $-2a_3J_5\psi$ |
| 36    | $[0,2,3]$ | $a_2\lambda^2 + a_1\lambda + a_0$                                                                                                                                                                   | $J_1 \sqrt{(\lambda - \nu_1)(\mu - \nu_1)} + J_2 \sqrt{(\lambda - \nu_2)(\mu - \nu_2)} + J_3(\lambda + \mu) + J_4(\lambda - \mu)^2 + J_5(\lambda + \mu)(\lambda - \mu)^2$ $-4J_2a_2\psi$ |
| 37    | $[0,1,4]$ | $a_1\lambda + a_0$                                                                                                                                                                                     | $J_1 \sqrt{-(\lambda + a_0)(\mu + a_0)} + J_2(\lambda + \mu) + (\lambda - \mu)^2[J_3 + J_4(\lambda + \mu) + J_5(5\lambda^2 + 6\lambda \mu + 5\mu^2)]$ $-32a_1J_5\psi$ |
| 38    | $[(2,2),0,1]$ | $a_4(\lambda - \nu_1)^2 \times (\lambda - \nu_2)^2$                                                                                                                                             | $\sum_{j=1}^{2} \left[J_j \sqrt{(\lambda - \nu_j)(\mu - \nu_j)} + J_{j+2} \frac{\lambda + \mu - 2\nu_j}{\sqrt{(\lambda - \nu_j)(\mu - \nu_j)}} + J_5(\lambda + \mu)\right]$ $-\frac{1}{2}a_4J_5\Psi$ |
| 39    | $[4,0,1]$  | $a_4\lambda^4$                                                                                                                                                                                          | $J_1 \sqrt{\lambda \mu} + J_2 \frac{\lambda + \mu}{\sqrt{\lambda \mu}} + J_3 \frac{(\lambda - \mu)^2}{(\lambda \mu)^{3/2}} + J_4 \frac{(\lambda - \mu)^2(\lambda + \mu)}{(\lambda \mu)^{1/2}} + J_5(\lambda + \mu)$ $-\frac{1}{2}a_4J_5\Psi$ |
| 40    | $[2,0,3]$  | $a_2\lambda^2$                                                                                                                                                                                          | $J_1 \sqrt{\lambda \mu} + J_2 \frac{\lambda + \mu}{\sqrt{\lambda \mu}} + J_3(\lambda + \mu) + J_4(\lambda - \mu)^2 + J_5(\lambda - \mu)^2(\lambda + \mu)$ $-\frac{1}{2}a_2J_5\Psi$ |
| 41    | $[0,0,5]$  | 1                                                                                                                                                                                                      | $J_1(\lambda + \mu) + (\lambda - \mu)^2 \times J_2 + J_3(\lambda + \mu) + J_4(\lambda + \mu) + J_5(5\lambda^2 + 6\lambda \mu + 5\mu^2)$ $+ J_5(\lambda + \mu)(7\lambda^2 + 7\mu^2 + 2\lambda \mu)$ $64J_5\Psi$ |

Table IV
Among those cases one can easily recognize the three cases 33, 34 and 35 as
cases of motion on spaces of constant curvature. The first of those accom-
dates spheres and pseudo spheres. In fact, if the roots \(\nu_i\) are real and ordered
such that \(\nu_1 > \nu_2 > \nu_3\), then one can choose for \(\lambda\) and \(\mu\) the intervals \([\nu_2, \nu_1]\)
and \([\nu_3, \nu_2]\), respectively, and put \(a_3 = -1\) to be on a sphere. Similarly, if
\(a_3 = 1, \lambda \in [\nu_1, \infty)\) and \(\mu \in (-\infty, \nu_3]\) we have a pseudo-sphere. We do not
write Lagrangians in specific coordinates adapted to each of the possible choices
of intervals. In the other two cases only pseudo-sphere is possible. The system
34 produces two possible Lagrangians. In case 35 the only possible Lagrangian
can be written as

\[
L = \frac{1}{2} \frac{u^2 + v^2}{u^2 v^2} (u^2 + v^2) \\
+ 2 \left\{ \frac{8J_5}{v^3} + \frac{J_4 u^2}{v(u^2 + v^2)} \right\} \dot{u} + \left\{ \frac{8J_5}{u^3} - \frac{J_4 v^2}{u(u^2 + v^2)} \right\} \dot{v} \\
+ 128J_5 \frac{(u^2 + v^2)^2}{u^4 v^4} \\
+ J_5 \left[ 32J_4 \frac{u^2 - v^2}{u^2 v^2} + J_3 \frac{(u^2 + v^2)^2}{2uv} - 4J_2 \frac{(u^2 - v^2)}{uv} - 32 \frac{J_1}{uv} \right]
\]

(108)

Here we have omitted terms in the linear part of the Lagrangian, which have
no contribution to the gyroscopic function \(\Omega\).

Cases 36 and 37 are Euclidean plane cases in coordinates of elliptic and
parabolic types. Case 36 is equivalent to the seven -parametric case given in
section 3.1 of [17]. Case 37 can be shown to be representing in the \(xy\)– plane a
system with

\[
V = Ax + By - a(x^2 + y^2)(C + bx + \frac{1}{2} a(5x^2 + y^2)), \quad \Omega = 6ax + b
\]

(109)

Trying to eliminate the parameter \(B\) by a rotation of the axes would result in
new terms in both functions \(\Omega, V\). Thus, this is a significant generalization of a
case found in [16] by introducing the parameter \(B\) and freeing the parameter \(A\)
from a relation to $a$. The integral for this system is

$$I = a \left[ 2(x\dot{y} - y\dot{x}) - (4ax + b) (x^2 + y^2) \right] \left[ \dot{y} - C - bx - a \left( 3x^2 + y^2 \right) \right]$$

$$+ A \left[ \dot{y} - bx - a \left( 3x^2 + y^2 \right) \right] - B \left[ \dot{x} + y (b + 2ax) \right]$$

(110)

In the first fifteen cases of table IV the intervals on which the variables $\lambda, \mu$ take their values could be chosen so that the configuration manifold become real Riemannian. The remaining four cases, in which the metric of the manifold has the single signature (1,-1) are listed as 38 to 41. We have just reduced the Lagrangians for those cases to real form by a substitution $\psi = i\Psi$, where $\Psi$ is real. No effort is done to reduce the cases obtained explicitly to the simplest form.

8 More about interpretations

In the previous sections we have constructed forty one different integrable systems admitting a quadratic integral independent of Jacobi’s integral. The presence of several parameters in the structures of those systems gives a wide possibility to adapt those parameters to fit concrete applications in mechanics and physics. So far, we have identified some possible cases of motion in the plane or developable surface, sphere or pseudo-sphere (or hyperbolic plane) by analyzing the Gaussian curvature of the metric of the system under consideration. The question arises, how to apply the above results to a given problem in mechanics. The first and principal step is to identify the metric of the configuration space of the given system with that of one of the systems constructed in sections 3 - 7. This may require the use of certain transformation of the generalized co-ordinates of the original mechanical system to reach an isometric form with a coefficient $\Lambda$ of one of the types \[15\] - \[20\] and try to adapt the candidate integrable system by imposing some restrictions on its parameters to fit it to the
given one. Once this step has succeeded, we are already dealing with an inte-
grable case of the given system and it only remains to characterize the potential
and electromagnetic forces in in terms of the original coordinates of this sys-
tem. If the original given mechanical system is obtained from one of more than
two degrees of freedom by means of reduction in the sense of Routh, the cyclic
constants of the motion would appear in the linear and potential terms in a
certain specified form and the parameters of the candidate system may allow to
accommodate those constants either completely or partially. Extra-parameters
not coming from cyclic constant may then be considered for further explanation
of their origins. In the rest of this section we indicate certain applications of
the above results. For size considerations only some of them are considered in
some detail and others are pointed out.

8.1 Application to rigid body dynamics

As an example, we try here to find all cases of the above systems, relevant
to the dynamics of a rigid body moving about a fixed point. Reduction of
the most general problem of motion under the action of potential forces and
forces of electromagnetic type was reduced to the form involving a Liouville-
type metric in [16]. We give the final form of the Lagrangian for the case of
tri-axial ellipsoid of inertia. Let
\[ I = \text{diag}(A, B, C) \]
be the inertia matrix of the
body in the principal axes of inertia at the fixed point, \( \omega = (p, q, r) \) its angular
velocity, \( \gamma = (\gamma_1, \gamma_2, \gamma_3) \) a unit vector fixed in space and \( V_0(\gamma), m(\gamma) \) be the
scalar and vector potentials of forces acting on the body. The Lagrangian of the
system has the form
\[ L = \frac{1}{2} \omega I : \omega + m : \omega - V_0 \]
(111)
The Euler-Poisson equations of motion can be written as

\[ \dot{\omega}I + \omega \times (\omega I + M) = \gamma \times \frac{\partial V_0}{\partial \gamma}, \quad \dot{\gamma} + \omega \times \gamma = 0 \]  

(112)

where

\[ M = \frac{\partial}{\partial \gamma}(m \cdot \gamma) - (\frac{\partial}{\partial \gamma} m)\gamma \]  

(113)

This system admits the cyclic integral

\[ (\omega I + m) \cdot \gamma = f \]  

(114)

corresponding to the angle of precession \( \psi \) around the vector \( \gamma \) and allows reduction by the Routhian procedure to a system of two degrees of freedom.

### 8.1.1 Reduction in the case of a tri-axial body

To describe the position of the current point on the configuration space of the reduced system in this case, which is the Poisson sphere \( |\gamma|^2 = 1 \) we shall use the same variables \( \lambda, \mu \) as in (116). Those variables are equivalent to elliptic coordinates on the ellipsoid of inertia with the standard metric. They are solutions of the quadratic equation

\[ D\lambda^2 - N\lambda + ABC = 0 \]  

(115)

\[ N = A(B + C)\gamma_1^2 + B(C + A)\gamma_2^2 + C(A + B)\gamma_3^2, \]

\[ D = A\gamma_1^2 + B\gamma_2^2 + C\gamma_3^2 \]  

(116)

We get

\[ R = \frac{1}{2} ABC(\lambda - \mu)[\lambda^2(\lambda(\lambda - A)(\lambda - B)(\lambda - C)) + \mu^2(A - \mu)(B - \mu)(C - \mu)] \]

\[ + \sqrt{ABC[P\lambda \lambda^2(\lambda - A)(\lambda - B)(\lambda - C) - Q\mu \mu^2(\lambda - A)(\lambda - B)(\lambda - C)]} \]

\[ - V \]  

(117)
where
\[
V = \frac{1}{4}[V_0 + \frac{(f - m \cdot \gamma)^2}{2D}]
\] (118)

We do not express \( P, Q \) here explicitly as functions of \( \lambda, \mu \) and only compare the function
\[
\Omega = \frac{1}{\sqrt{ABC}}[\mu \sqrt{(A - \mu)(B - \mu)(C - \mu)} \frac{\partial P}{\partial \mu} - \lambda \sqrt{(A - \lambda)(B - \lambda)(C - \lambda)} \frac{\partial Q}{\partial \lambda}]
\]
\[
= \frac{(\lambda - \mu) \sqrt{\lambda \mu}}{4ABC} \{ f [A + B + C - 2(\lambda + \mu)] + S \}
\] (119)
in which \( f \) is the cyclic constant and \( S \) is a function depending on vector-potential terms
\[
S = D \frac{\partial}{\partial \gamma} \cdot \left[ \frac{1}{D} \gamma \times (\gamma I \times m) \right]
\] (120)

8.1.2 Integrable cases of a tri-axial body

We now look for cases of the constructed systems, which can accommodate the metric in (117). Note that here the functions \( F \) and \( G \) are fifth-degree polynomials with a double zero root satisfying \( G(\mu) = -F(\mu) \), a condition common to most cases, but not all, of the previous sections. We find that:

1. The case of 6.1 under the restrictions \( a_1 = a_0 = b = 0, a_2 = 1, a_3 = -(1/A + 1/B + 1/C), a_4 = \frac{A+B+C}{ABC}, a_5 = \frac{1}{ABC} \).

2. Case 24 of table IV under the restriction \( \nu_4 = 0 \).

3. The case of 6.2 under the same restrictions, but with \( a_6 \) instead of \( b \).

4. The case of 6.4 under the restrictions \( a_0 = -b_0 = 1, a_1 = -b_1 = -(1/A + 1/B + 1/C), a_2 = -b_2 = \frac{A+B+C}{ABC}, a_3 = -b_3 = \frac{1}{ABC} \).

It was shown in [16] that case 1 is a case of motion of a multi-connected tri-axial solid in a liquid, known after Clebsch [42] and case 2 is the case due
originally to Steklov [43] in its generalized form due to Rubanovsky [44] and involving circulations. Case 3 was also pointed out as a new case. The reader is referred to [16] for full detail.

The last case turns out to be a new one different from known cases of rigid body motion. For this case one can find

\[ V_0 = \frac{kN}{2ABC\sqrt{N^2 - 4ABCD}} \]
\[ m = \frac{K}{2ABC} (A(B + C)\gamma_1, B(C + A)\gamma_2, C(A + B)\gamma_3) \]
\[ M = -K(\frac{\gamma_1}{A}, \frac{\gamma_2}{B}, \frac{\gamma_3}{C}) \]  (121)

where Equations (112) for this choice admit the conditional integral

\[ I = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) - K(p\gamma_1 + q\gamma_2 + r\gamma_3) + \frac{k}{\sqrt{N^2 - 4ABCD}} \]  (122)

In fact, differentiating \( I \) with respect to \( t \) in virtue of (112) and using the geometric and cyclic integrals

\[ \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1, \]
\[ Ap\gamma_1 + Bq\gamma_2 + Cr\gamma_3 + \frac{K}{2ABC}N = f \]  (123)

we find

\[ \frac{dI}{dt} = 4kfC(A - B)(B - C)(C - A)\gamma_1\gamma_2\gamma_3^2 \]  (124)

and hence \( I \) vanishes identically in two cases:

1. When \( k = 0 \), and this reproduces the original Steklov case which is a general integrable case for all initial conditions.

2. When \( k \neq 0, f = 0 \) and this gives a new conditional case valid only on the zero level of the cyclic integral [123].
8.2 On time-reversible integrable problems in rigid body dynamics

For the system (117) to be time-reversible one must have \( m = 0, f = 0 \). The Lagrangian becomes

\[
R = \frac{1}{2}ABC(\lambda - \mu)[\lambda^2(A - \lambda)(\lambda - B)(\lambda - C) + \mu^2(A - \mu)(B - \mu)(\mu - C)] - \frac{1}{4}V_0
\]

(125)

It is clear from the consideration of section 1 that the existence of a quadratic integral is associated with the separation of variables. Comparing the last system to (22) and using (115) we note that it admits a quadratic integral if and only if

\[
V_0 = Dv_1(N + \sqrt{\beta}) + v_2(N - \sqrt{\beta})
\]

(126)

where \( \beta = N^2 - 4ABC\). and \( v_1, v_2 \) are two arbitrary functions. The explicit form of the integral in the Euler-Poisson variables is

\[
I = \frac{1}{2}(Ap^2 + Bq^2 + Cr^2) + \frac{(N + \sqrt{\beta})v_1(N + \sqrt{\beta}) + (N - \sqrt{\beta})v_2(N - \sqrt{\beta})}{\sqrt{\beta}}
\]

(127)

This can be verified by direct calculation. This result was obtained in [30] and independently in equivalent form in [51]. In [49] an infinite sequence of potentials \( \{U_n, n = 1...\infty\} \) that admit quadratic integral, using symbolic computation was found. It can be easily shown that this sequence corresponds to the choices \( \{v_1(x) = -v_2(x) = x^n\} \). This is exactly a case discussed explicitly in [51]. Moreover, if in (126) we take \( v_1(x) = -v_2(x) \) as a full power series we obtain the potential

\[
V_0 = \sum_{n=1}^{\infty} A_n U_n
\]

(128)

admitting the quadratic integral. This could be expected from the separation property overlooked in [49]. It should be mentioned here that the authors of [49]
have also argued that the sequence they obtained is different from another one, obtained earlier by Wojciechowski in 1985 [50]. However, the latter sequence as provided in [49] is not separable for the tri-axial rigid body and is inconsistent with the existence of a quadratic integral for rigid body dynamics.

### 8.3 The problem of motion of a particle on a smooth ellipsoid

Some mechanical systems do not have much importance in their own as much as in other problems related to them, for which they serve as geometrization. A typical example of a useful model in mechanics is the problem of motion on a smooth fixed surface. The motion on a sphere is used for modeling motion of a rigid body [18], in the study of B-phase of the superfluid $^3$He, in the construction of certain wave solutions of the Landau-Lifshitz non-linear equation (e.g. [46]) and in the treatment of Dyson’s fluid dynamical model of spinning gas clouds maintaining ellipsoidal shape [45]. Recently, the motion of a particle on a smooth tilted cone was found to be a generalization covering the problem of motion of the swinging Atwood machine as a special case [47].

The ellipsoid is one of the favorite models. It is closely related to the dynamics of rigid bodies. Let a particle of unit mass be moving on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

under the action of forces with potential $V$ and vector potential $\mathbf{l}$. The Lagrangian for this problem is

$$L = \frac{1}{2} \dot{\mathbf{r}}^2 + \mathbf{l} \cdot \dot{\mathbf{r}} - V$$
We shall only express $\dot{r}^2$ in terms of elliptic coordinates $u, v$ on the ellipsoid

\[
\frac{1}{4}(u - v) \left[ \frac{u \dot{v}^2}{(a^2 - u)(b^2 - u)(c^2 - u)} - \frac{v \dot{u}^2}{(a^2 - v)(b^2 - v)(c^2 - v)} \right]
\]  

(131)

Here the functions $F$ and $G$ are rational (cubic to linear). One can immediately notice that case I (6.1), in which $F$ is rational (degree 6 to one) gives a case of motion on the ellipsoid if coefficients are suitably adapted. This case was found earlier in our work [19] by a different method.

### 8.4 Geometrical interpretation of systems of the type (17)

Using Hamilton’s principle in the form of Jacobi (e.g. [34], [35]), one can derive the equations of the trajectories of motion of the system described by the Lagrangian (17) from the variational problem

\[
\delta \int \sqrt{2\Lambda(h - V)(d\xi^2 + d\eta^2) + Pd\xi - Qd\eta} = 0
\]  

(132)

This means those trajectories are geodesics of the Randers metric

\[
ds = \sqrt{2\Lambda(h - V)(d\xi^2 + d\eta^2) + Pd\xi - Qd\eta}
\]  

(133)

Randers metrics were first studied by physicist, G.Randers [36] from the standpoint of general relativity. Since then, many geometers have made efforts in investigating the geometric properties of Randers metrics as an important and rich class of Finsler metrics and to explore their prospective applications (see e.g. [37] - [40]).

Every integrable mechanical system of the type (17) corresponds to an integrable geodesic flow of the Randers Metric (133) admitting the integral

\[
I = \Lambda(h - V)(\frac{d\xi}{dl})^2 + \sqrt{2\Lambda(h - V)(P\frac{d\xi}{dl} + Q\frac{d\eta}{dl})} + R
\]  

(134)

in which $dl = \sqrt{d\xi^2 + d\eta^2}$. 

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8.5 The Fokker-Plank systems

Stochastic dynamical models described by Fokker-Planck equations, in the limit of weak noise, can be formally associated with Hamiltonian dynamical systems. The Hamiltonian consists of a kinetic energy quadratic term and terms linear in momenta, with zero scalar potential term (e.g. [9], [52]). The study of such Hamiltonian systems has received some attention from both points of view of integrability [53] - [56] and existence of invariant relations [57].

Gauge transformation of the Lagrangian was also used by Hietarinta [54] to construct integrable Fokker-Planck equations from known integrable Hamiltonian systems. The Hamiltonian function corresponding to the Lagrangian (17a), with a gauge term \( \frac{dZ(\xi, \eta)}{dt} \), has the form

\[
H = \frac{1}{2\Lambda} \left[ \left( p_\xi - P - \frac{dZ}{d\xi} \right)^2 + \left( p_\eta - Q - \frac{dZ}{d\eta} \right)^2 \right] + V = 0 \quad (135)
\]

where \( p_\xi, p_\eta \) are the momenta conjugate to the coordinates \( \xi, \eta \) respectively. An integrable system (17) generates a Fokker-Planck system if there exists a function \( Z(\xi, \eta) \), satisfying

\[
\frac{1}{2\Lambda} \left[ \left( P + \frac{dZ}{d\xi} \right)^2 + \left( Q - \frac{dZ}{d\eta} \right)^2 \right] + V = 0 \quad (136)
\]

The results we obtained for this problem are promising and will be presented elsewhere.

9 Conclusion

So far, we have made the most comprehensive analysis of the problem of constructing 2D conservative Lagrangian systems involving terms linear in the velocities, which admit a complementary quadratic integral. The problem under consideration has proved unexpectedly rich. We have constructed and tabulated
41 major several-parameter families of integrable systems of this type. In most cases the parametrization of the system can be chosen, usually in more than one way, so that the configuration manifold becomes Riemannian or pseudo-Riemannian. The remaining few cases are real only on pseudo-Riemannian manifolds.

Some of those families or their special cases or degenerations were listed in our previous works but the majority are either listed for the first time or involve more parameters in their structure. Usually, we give the Lagrangian and the second invariant and occasionally the Gaussian curvature of the configuration space, which we use for interpretation of simple results as cases of motion on a plane or a space of constant curvature. In application to the problem of motion of a rigid body under various circumstances, all known integrable cases of the type under consideration for this problem are restored and a new one is added. A systematic analysis of the results of this paper is still needed to isolate all possible subcases, which admit a preassigned physical or mechanical interpretation. Many of them are tentatively candidates for the study of motion of a charged particle under the action of spatially periodic electric and magnetic fields on various 2D spaces.

It is worthy to say few words about the methods we used throughout the paper, in order to make clear their limitations. The difficulty in solving the problem stems from the fact that we deal with a linear PDE for $\psi(\lambda, \mu)$ coupled with a differential condition involving nonlinearly the same function $\psi$ and linearly three functions of a single variable each $F(\lambda), G(\mu), V(\psi)$. This system is so involved that we may not be able to tell at present how far we are from its general solution. As we see from the last sections, $F$ and $G$ turned out to be rational in one case and polynomial of degree less than six in all others. We see from tables 4 and 5 how the degree of $F(\lambda)$ is interlinked with the number
of terms taken from the sequence of functions \{\psi_n, \tilde{\psi}_n\}. It seems unlikely that \( F \) can be a polynomial of degree six or higher. However, we are working on few cases in which \( F, G \) are algebraic functions. Using certain uniformizing transformations, one can reduce \( F, G \) to polynomial form, but this changes the form of the metric coefficient \((\lambda - \mu)\). This will be pursued elsewhere.

In the more compact case considered in 5.3.1 we deal with simultaneous solutions of the linear PDE (52) and the nonlinear one (70a) in the same unknown function \( \psi \). It is unlikely that those equations admit other solutions than the list of table II. However, we have no proof yet that no other solutions exist, for example, as full infinite series of \{\psi_n, \tilde{\psi}_n\}.

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