Optimality of function spaces for kernel integral operators

Jakub Takáč

Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Czech Republic

Correspondence
Jakub Takáč, Department of Mathematics, University of Warwick, Zeeman Building, Coventry, CV4 7HP, UK.
Email: jakub.takac404@gmail.com

Present Address
Jakub Takáč, Department of Mathematics, University of Warwick, Zeeman Building, Coventry, CV4 7HP, UK.

Abstract
We explore boundedness properties of kernel integral operators acting on rearrangement-invariant (r.i.) spaces. In particular, for a given r.i. space $X$ we characterize its optimal range partner, that is, the smallest r.i. space $Y$ such that the operator is bounded from $X$ to $Y$. We apply the general results to Lorentz spaces to illustrate their strength.

KEYWORDS
kernel integral operator, Lorentz space, Marcinkiewicz space, optimal range, Peetre $K$-functional, rearrangement-invariant space

MSC (2020)
Primary: 47B34, 46E30

1 | INTRODUCTION

The Laplace transform is defined by

$$\mathcal{L}f(t) = \int_0^\infty f(s)e^{-ts} \, ds$$

for every real-valued function $f$ on $(0, \infty)$ for which the integral makes sense, and for every $t \in (0, \infty)$. It is well known that $\mathcal{L}$ is an important integral operator with plenty of applications for example in the theory of differential equations, probability theory, investigation of spectral properties of pseudo-differential operators, or the study of Fredholm integral equations (see, for instance, [1, 5, 22, 26, 29]). The Laplace transform can be viewed as a particular instance of a fairly more general class of kernel integral operators

$$Kf(t) = \int_0^\infty f(s)k(s, t) \, ds,$$

where $k$ is an appropriate measurable function of two variables (to obtain the Laplace transform we set $k(s, t) = e^{-st}$ for $s, t \in (0, \infty)$).

In this text, we focus on problems concerning sharp action of kernel operators of a particular type, namely operators defined by

$$S_{\alpha}f(t) = \int_0^\infty f(s)a(st) \, ds,$$

for all real-valued functions $f$ on $(0, \infty)$ for which the integral makes sense, where $a : (0, \infty) \to [0, \infty)$ is an appropriate function of one variable. Various particular types of such operators and their numerous modifications have been studied by many authors. To name just a few classical ones, see [2, 8, 21], and the references therein.
Our goal here is to investigate fine properties of operators of type $S_a$ acting on the so-called rearrangement-invariant (r.i. for short) spaces. In these spaces, sometimes in literature called also symmetric spaces, or Köthe spaces, the decisive parameter is always the size of a function (rather than other properties such as continuity, smoothness, or oscillation). They are built upon an idea of axiomatization of good properties enjoyed by Lebesgue spaces, which constitute a central subclass of r.i. spaces, but in themselves are not rich enough in order to satisfactorily describe action of important operators and embeddings, especially in their various critical or limiting states. With a certain licence it can be said that the norms in r.i. spaces take into account only the measure of level sets of a given function.

To be more precise, we will concentrate on the questions of existence and eventual characterization of the optimal partner space for an operator of type $S_a$. By the optimal range partner space, supposing a domain space $X$ is given, we mean a space $Y$ such that $S_a$ takes boundedly $X$ to $Y$, and $Y$ is the smallest possible such space. Analogously we define the optimal domain partner when the range space is fixed. To make such questions sensible one has to, of course, state the pool of competing spaces—here this will always be that of r.i. spaces.

Let us note that while action of kernel operators on function spaces has been studied, little attention has been so far paid to optimality of such results. On the other hand, an extensive research of optimality of function spaces in different situations (e.g., in Sobolev embeddings, trace embeddings, Gaussian measure space, probability spaces etc., see, e.g., [6, 9–13, 15, 16, 19, 24, 25, 27]) could be seen mainly during last two decades. There are few exceptions, namely papers in which optimality of spaces for integral operators is studied, for example, [7, 17] or [27, 30].

To fulfill our goal, we will combine known techniques with certain new ones which we have to develop. In particular, we will calculate the Peetre $K$-functional for certain specific pairs of spaces (this has been known only partially) and we will, in an efficient way, combine the notion of the Marcinkiewicz-type endpoint space with the norm of the dilation operator. Such methods have not been known before and we shall show that they lead to quite a fruitful theory.

The paper is structured as follows. In Section 2, we fix notation and collect all the preliminary stuff including all the definitions and basic knowledge about the function spaces, operators, and related topics. We give a detailed definition of r.i. spaces and recall everything we shall need to know about them. We also define particular function spaces which will be used in examples to illustrate the abstract results. In Section 3, we present several background results that will be needed in the proofs of the main results. In particular, we introduce here two types of Marcinkiewicz endpoint spaces and study their elementary but useful properties. We also establish certain important relations concerning the Peetre $K$-functional in the spirit of [23].

Main results are stated and proved in Section 4. We tackle the problem of the very existence of the partner space, and in the affirmative cases we characterize or construct the optimal one. Furthermore, we present a series of results based on spaces given in terms of the norm of the dilation operator on the domain space. To this end, we establish several results of independent interest, providing Calderón-type estimates for the operators in question. The final section contains a thorough and comprehensive analysis of the action of operators $S_a$ on Lorentz spaces.

2 | PRELIMINARIES

In this section, we recall some definitions and basic properties of rearrangement-invariant spaces. The standard reference is [4].

We shall exclusively work with the Lebesgue measure on $(0, \infty)$ and we define

\[ \mathcal{M} = \{ f : (0, \infty) \to [-\infty, \infty] : f \text{ is Lebesgue-measurable in } (0, \infty) \}, \]

and

\[ \mathcal{M}_+ = \{ f \in \mathcal{M} : f \geq 0 \text{ a.e.} \}. \]

The distribution function $f_* : (0, \infty) \to [0, \infty]$ of a function $f \in \mathcal{M}$ is defined as

\[ f_*(\lambda) = |\{ x \in (0, \infty) : |f(x)| > \lambda \} |, \lambda \in (0, \infty), \]
where \(| \cdot |\) denotes the Lebesgue measure, and the non-increasing rearrangement \(f^* : (0, \infty) \rightarrow [0, \infty)\) of a function \(f \in \mathcal{M}\) is defined as

\[
f^*(t) = \inf\{\lambda \in (0, \infty) : f_*(\lambda) \leq t\}, \quad t \in (0, \infty).
\]

Here, we define \(\inf \emptyset = \infty\) and we remark that both \(f_\ast\) and \(f^\ast\) are allowed to attain \(\infty\). The operation \(f \mapsto f^\ast\) is monotone in the sense that \(|f| \leq |g|\) a.e. in \((0, \infty)\) implies \(f^\ast \leq g^\ast\). We define the elementary maximal function \(f^{**} : (0, \infty) \rightarrow [0, \infty]\) of a function \(f \in \mathcal{M}\) as

\[
f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) \, ds.
\]

Once again, \(f^{**}\) is well defined for any \(f \in \mathcal{M}\), but it may be identically equal to \(\infty\). While the operation \(f \mapsto f^{**}\) is subadditive, that is, for any \(f, g \in \mathcal{M}_+\) and \(t \in (0, \infty)\) one has

\[
(f + g)^{**}(t) \leq f^{**}(t) + g^{**}(t),
\]

for \(f \mapsto f^\ast\) one only has the following property. Let \(s, t \in (0, \infty)\) and \(f, g \in \mathcal{M}_+\), then

\[
(f + g)^\ast(s + t) \leq f^\ast(t) + g^\ast(s).
\]

We recall that for every \(f \in \mathcal{M}_+\) and every \(t \in (0, \infty)\), one has

\[
|[s \in (0, \infty) : f(s) > f^\ast(t)]| \leq t.
\]

The Hardy–Littlewood inequality asserts that if \(f, g \in \mathcal{M}\) attain finite values a.e., then

\[
\int_0^\infty |f(t)g(t)| \, dt \leq \int_0^\infty f^*(t)g^*(t) \, dt.
\]

Let us fix \(f_1, f_2 \in \mathcal{M}_+\). If for any \(t \in (0, \infty)\) it holds that

\[
\int_0^t f_1(s) \, ds \leq \int_0^t f_2(s) \, ds,
\]

then Hardy’s lemma asserts that for any \(g \in \mathcal{M}_+\) non-increasing, the following inequality holds:

\[
\int_0^\infty f_1(s)g(s) \, ds \leq \int_0^\infty f_2(s)g(s) \, ds.
\]

Hardy’s inequality asserts that if \(p \in (1, \infty)\) and \(\alpha > -(1 - \frac{1}{p})\), then

\[
\int_0^\infty \left( t^{-\alpha - 1} \int_0^t f(s) \, ds \right)^p \, dt \leq \left( \frac{1}{1 + \alpha - \frac{1}{p}} \right)^p \int_0^\infty (f(t)t^{-\alpha})^p \, dt \quad \text{for } f \in \mathcal{M}_+,
\]

and if \(\alpha > 1 - \frac{1}{p}\), then

\[
\int_0^\infty \left( t^{-\alpha + 1} \int_0^t f(s) \, ds \right)^p \, dt \leq \left( \frac{1}{-\alpha + \alpha + \frac{1}{p}} \right)^p \int_0^\infty (f(t)t^\alpha)^p \, dt \quad \text{for } f \in \mathcal{M}_+.
\]
This inequality can be found for example in [4, Chapter 3, Lemma 3.9], where a slightly different, but equivalent, formulation is used.

Following [4], we say that a functional \( \rho : \mathcal{M}_+ \to [0, \infty] \) is a **Banach function norm**, if for all \( f, g \) and \( \{f_j\}_{j \in \mathbb{N}} \) in \( \mathcal{M}_+ \), and every \( \lambda \geq 0 \), the following properties hold:

\begin{enumerate}[\text{(P1)}]
\item \( \rho(f) = 0 \) if and only if \( f = 0 \); \( \rho(\lambda f) = \lambda \rho(f) \);
\item \( f \leq g \) a.e. implies \( \rho(f) \leq \rho(g) \);
\item \( f_j \not\to f \) a.e. implies \( \rho(f_j) \not\to \rho(f) \);
\item \( \rho(\chi_G) < \infty \) for every \( G \subset (0, \infty) \) of finite measure;
\item \( \rho \) for every \( G \subset (0, \infty) \) of finite measure there is a constant \( C_G \) such that \( \int_G f(t)dt \leq C_G \rho(f) \).
\end{enumerate}

If also the property

\( \text{(P6)} \) \( \rho(f) = \rho(g) \) whenever \( f^* = g^* \),

holds, we say that \( \rho \) is a **rearrangement-invariant Banach function norm**, or just a **rearrangement-invariant norm**.

It is useful to remark here that conventions regarding rearrangement-invariant norms are not unified throughout the literature. These spaces are sometimes also called symmetric, or Köthe spaces and sometimes not all of the properties (P1)–(P5) are assumed. Mainly the property (P3), called also the **Fatou property** is often not assumed to hold for these spaces. Usually, it is replaced by the assumption that the resulting space is a Banach space, which is weaker than assuming (P3). We, however, restrict our attention only to spaces satisfying all of the properties (P1)-(P6).

If \( \rho \) is a rearrangement-invariant norm, then the collection

\[ X = X(\rho) = \{ f \in \mathcal{M} : \rho(|f|) < \infty \} \]

is called a **rearrangement-invariant Banach function space**, or just a **rearrangement-invariant space**. The norm on the space \( X \) is given by \( \|f\|_X = \rho(|f|) \). Note that \( \rho(|f|) \) is defined for every \( f \in \mathcal{M} \), and

\[ f \in X \iff \rho(|f|) < \infty. \]

We recall that it follows from the axioms that such \( X \) is always complete (even if (P6) does not hold), hence the name Banach function space. For a rearrangement-invariant norm \( \rho \), we define its **associate norm** by

\[ \rho'(g) = \sup \left\{ \int_0^\infty f(t)g(t)dt : f \in \mathcal{M}_+, \rho(f) \leq 1 \right\} \quad \text{for } g \in \mathcal{M}_+. \]

This \( \rho' \) is also a rearrangement-invariant norm. Furthermore, it also holds that \( \rho'' = \rho \). If \( X = X(\rho) \) is a rearrangement-invariant space and \( \rho' \) is the norm associated with \( \rho \), then \( X(\rho') \) is the associate space of \( X \) and is denoted by \( X' \).

If \( X, Y \) are rearrangement-invariant spaces, we denote by \( X \hookrightarrow Y \) the continuous embedding of \( X \) into \( Y \) and by \( T : X \to Y \) the boundedness of an operator \( T \) from \( X \) to \( Y \). We have

\[ X \hookrightarrow Y \iff Y' \hookrightarrow X'. \quad (2.6) \]

Given a linear operator \( T \) defined on some subspace of \( \mathcal{M} \), we call the operator \( T' \) adjoint operator to the operator \( T \) if

\[ \int_0^\infty T(f)g = \int_0^\infty fT'(g) \]

for all \( f, g \in \mathcal{M} \) for which the left-hand side makes sense. Recall that \( T : X \to Y \) holds if and only if \( T' : Y' \to X' \). We say that the rearrangement-invariant space \( Y \) is the **optimal range partner** for the linear operator \( T \) and a given domain rearrangement-invariant space \( X \) if \( T : X \to Y \) and for every rearrangement-invariant space \( Z \) such that \( T : X \to Z \) it holds that \( Y \hookrightarrow Z \). An operator which will be used extensively throughout this work is the dilation operator \( E_t \) defined for
any \( t \in (0, \infty) \) by the formula

\[ E_t f(s) = f\left( \frac{s}{t} \right). \]

We recall that \( E_t \) is bounded on every rearrangement-invariant space for any \( t \in (0, \infty) \).

We define the fundamental function, \( \varphi_X \), of a given rearrangement-invariant space \( X \) by \( \varphi_X(t) = \| \chi_{(0,t)} \|_X, t \in (0, \infty) \). We say that a function \( \varphi : (0, \infty) \to (0, \infty) \) is quasiconcave if it is non-decreasing and \( \frac{t}{\varphi(t)} \) is non-decreasing. We say that the function \( \varphi : (0, \infty) \to (0, \infty) \) satisfies the \( \Delta_2 \) condition, if it is non-decreasing and there exists a constant \( C > 0 \) such that \( \varphi(2t) \leq C \varphi(t) \) for all \( t > 0 \). The fundamental function of any rearrangement-invariant space is quasiconcave. Given a quasiconcave function \( \varphi \), we define the rearrangement-invariant spaces \( M_\varphi, \Lambda_\varphi \) with the rearrangement-invariant norms given by

\[
\| f \|_{M_\varphi} = \sup_{t \in (0, \infty)} \varphi(t) f^{**}(t), f \in M, \quad (2.7)
\]

and

\[
\| f \|_{\Lambda_\varphi} = \int_0^\infty f^*(t) d\varphi(t), f \in M. \quad (2.8)
\]

The first expression indeed defines a rearrangement-invariant norm, while the second expression is equivalent to one up to a multiplicative constant. The former of the spaces is called the Marcinkiewicz endpoint space and the latter is called the Lorentz endpoint space. It is also known that both \( M_\varphi \) and \( \Lambda_\varphi \) have a common fundamental function which is equal to \( \varphi \). For a rearrangement-invariant space \( X \), we denote

\[ M(X) = M_{\varphi_X}, \Lambda(X) = \Lambda_{\varphi_X}, \]

where \( \varphi_X \) is the fundamental function of \( X \). We recall that for a rearrangement-invariant space \( X \) we have

\[ \Lambda(X) \hookrightarrow X \hookrightarrow M(X), \]

the latter of the embeddings holding with norm 1. In other words, the spaces \( M_\varphi, \Lambda_\varphi \) are respectively the largest and the smallest rearrangement-invariant space with the fixed fundamental function equal to \( \varphi \).

One of the basic examples of an important class of rearrangement-invariant spaces can be obtained by considering general Lorentz \( L^{p,q} \) spaces with \( p, q \in (0, \infty) \), governed for \( f \in M \) by the functional

\[
P_{p,q}(f) = \begin{cases} \left( \int_0^{\infty} \left( \int_0^t f^*(s) \right)^q \frac{dt}{t} \right)^{\frac{1}{q}} & \text{if } 0 < q < \infty, \\ \sup_{t \in (0, \infty)} t^\frac{1}{p} f^*(t) & \text{if } q = \infty. \end{cases}
\]

We recall that these are equivalent to rearrangement-invariant norms in cases when either \( p \in (1, \infty) \) and \( q \in [1, \infty) \) or \( p = q = 1 \) or \( p = q = \infty \). In case \( p = \infty \) and \( q < \infty \), the resulting space is a trivial set containing only the zero function. Furthermore, note that \( L^{1,1} = L^1 \), where \( L^1 \) is the classical Lebesgue space. For \( p \in [1, \infty] \), we define the associated exponent \( p' \) by \( \frac{1}{p} + \frac{1}{p'} = 1 \). The following equalities hold up to an equivalence of norms and further in this paper will be treated as equalities, disregarding the equivalence constants.

\[
L^{p,1} = \Lambda(L^p) \quad \text{for } p \in [1, \infty),
\]

\[
L^{p,\infty} = M(L^p) \quad \text{for } p \in (1, \infty],
\]

\[
(L^{p,q})' = L^{p',q'} \quad \text{for } p \in (1, \infty), q \in [1, \infty] \text{ or } p = q = \infty \text{ or } p = q = 1.
\]
We will also be using the following embeddings:

(a) if \( q_1, q_2 \in [1, \infty] \) and \( q_1 \leq q_2 \), then

\[
L^{p; q_1} \hookrightarrow L^{p; q_2} \quad \text{for all } p \in (1, \infty),
\]

(b) if \( q_1, q_2 \in [1, \infty] \), \( p_1, p_2 \in (1, \infty) \) and \( p_1 < p_2 \), then

\[
L^1 \cap L^{p_1; q_1} \hookrightarrow L^1 \cap L^{p_2; q_2} \quad \text{and} \quad L^\infty \cap L^{p_1; q_1} \hookrightarrow L^\infty \cap L^{p_2; q_2},
\]

(c) if \( q_1, q_2 \in [1, \infty] \), \( p_1, p_2 \in (1, \infty) \) and \( p_1 < p_2 \), then

\[
L^1 + L^{p_1; q_1} \hookrightarrow L^1 + L^{p_2; q_2} \quad \text{and} \quad L^\infty + L^{p_1; q_1} \hookrightarrow L^\infty + L^{p_2; q_2},
\]

where, as per usual, for \( X_0 \) and \( X_1 \) rearrangement-invariant spaces, the norm in \( X_0 \cap X_1 \) is given by \( \max\{\| \cdot \|_{X_0}, \| \cdot \|_{X_1}\} \).

Let \( X_0 \) and \( X_1 \) be quasi-normed spaces, which are compatible in the sense that they are embedded in some common Hausdorff topological vector space (in our case we are working with the space \( \{ f \in \mathcal{M}, |f| < \infty \text{ a.e.}\} \)). By \( X_0 + X_1 \) we denote the set of all functions \( f \in \mathcal{M} \) for which there exists a decomposition \( f = g + h \) such that \( g \in X_0 \) and \( h \in X_1 \). We equip the space \( X_0 + X_1 \) with the quasinorm

\[
\| f \|_{X_0+X_1} = \inf_{f=g+h} (\| g \|_{X_0} + \| h \|_{X_1}),
\]

where the infimum is taken over all such decompositions. Recall that if \( X_0 \) and \( X_1 \) are normed, then \( X_0 + X_1 \) is also normed. For \( f \in X_0 + X_1 \), the Peetre \( K \)-functional is defined by

\[
K(t, f; X_0, X_1) := \inf_{f=g+h} (\| g \|_{X_0} + t\| h \|_{X_1}) \quad \text{for } t > 0.
\]

The function \( K \) as a function of variable \( t \) is increasing and concave on \( (0, \infty) \). Furthermore, the function \( t^{-1}K(t, f; X_0, X_1) \) is non-increasing on \( (0, \infty) \). Observe that

\[
\frac{1}{t}K(f, t; X_0, X_1) = K \left( f, \frac{1}{t}X_1, X_0 \right). \tag{2.9}
\]

Recall that in the case when \( X_0 = L^1 \) and \( X_1 = L^\infty \), an exact formula for the \( K \)-functional is known (see, e.g., [4, Chapter 5, Theorem 1.6]), namely,

\[
K(f, t; L^1, L^\infty) = \int_0^t f^*(s) \, ds \quad \text{for } t \in (0, \infty) \text{ and } f \in (L^1 + L^\infty). \tag{2.10}
\]

## 3 BACKGROUND RESULTS

In this section, we shall establish some results on rearrangement-invariant spaces and \( K \)-functionals which will be useful later. Let us begin with the definition of a Marcinkiewicz-type space \( m_\varphi \) similar to \( M_\varphi \).

**Definition 3.1.** Let \( \varphi : (0, \infty) \to (0, \infty) \) be a function satisfying the \( \Delta_2 \) condition. We define the functional \( \| \cdot \|_{m_\varphi} \) for \( f \in \mathcal{M} \) by the formula

\[
\| f \|_{m_\varphi} = \sup_{t \in (0, \infty)} \varphi(t)f^*(t).
\]

We define the space \( m_\varphi \) as the set of all functions \( f \) for which the functional \( \| f \|_{m_\varphi} \) is finite.
The $\Delta_2$ condition imposed on the function $\varphi$ guarantees that $\| \cdot \|_{m_\varphi}$ is a quasinorm (see also [14]), therefore we will consider $m_\varphi$ to be a quasinormed space. We say that a linear set $M \subset \mathcal{M}$ together with a functional $F : M \to [0, \infty)$ can be equivalently renormed with a rearrangement-invariant norm, if there exists a rearrangement-invariant space $X$ and constants $C_1$ and $C_2$ such that $X = M$ and

$$C_1\|f\|_X \leq F(f) \leq C_2\|f\|_X \quad \text{for } f \in X.$$  

If that is the case, we will identify $M = X$ not only set-wise but also as spaces, in other words, we consider $M$ to be a rearrangement-invariant space. Let $\varphi$ be a quasi-concave function. Then, $m_\varphi$ can be equivalently renormed with a rearrangement-invariant norm $\| \cdot \|_{M_\varphi}$ if and only if there exists a constant $C > 0$ such that

$$\int_0^t \frac{1}{\varphi(s)} \, ds \leq C \frac{t}{\varphi(t)} \quad \text{for every } t \in (0, \infty). \quad (3.1)$$

One of the implications can be found, for example, in [28, Proposition 7.10.5] and the other is an easy exercise. A more general result can be found in [10].

Our next aim is to characterize the $K$-functional for a pair of Marcinkiewicz spaces. Results in this direction can be found in the literature (see, e.g., [3, 18, 23, 31]). Here, we need to obtain $K(m_\varphi, m_\psi)$ with rather mild conditions on the functions $\varphi, \psi$. In particular, we require little more than that they satisfy the $\Delta_2$ condition. Our proof of the Proposition 3.2 is in the spirit of that of [23, Theorem 4.1].

**Proposition 3.2.** Let $\varphi, \psi \in \mathcal{M}_+$ satisfy the $\Delta_2$ condition. Denote by $s$ the function $s(t) = \frac{\varphi(t)}{\psi(t)}$ for $t \in (0, \infty)$. If $s$ is monotone, then there exist constants $C_1, C_2 > 0$ such that for every $t > 0$ and $f \in \mathcal{M}$

$$C_1(\| (E_1^2 f)^* \chi_A \|_{m_\varphi} + t \| (E_1^2 f)^* \chi_B \|_{m_\psi}) \leq K(f, t, m_\varphi, m_\psi) \leq C_2(\| f^* \chi_A \|_{m_\varphi} + t \| f^* \chi_B \|_{m_\psi}), \quad (3.2)$$

where $A = \{ u : s(u) < t \}$ and $B = (0, \infty) \setminus A$.

**Proof.** First, let $f = f_0 + f_1, f_0 \in m_\varphi, f_1 \in m_\psi$ and $t > 0$. Then, we have

$$\|f_1\|_{m_\varphi} + t\|f_1\|_{m_\psi} \geq \frac{1}{2}(I_1 + I_2), \quad (3.3)$$

where

$$I_1 = \sup_{u > 0} f_0^*(u)\chi_A(u)\varphi(u) + t \sup_{u > 0} f_1^*(u)\chi_A(u)\psi(u) \quad (3.4)$$

and

$$I_2 = \sup_{u > 0} f_0^*(u)\chi_B(u)\varphi(u) + t \sup_{u > 0} f_1^*(u)\chi_B(u)\psi(u). \quad (3.5)$$

From the definition of the set $A$, we have

$$t \sup_{u > 0} f_1^*(u)\chi_A(u)\psi(u) \geq \sup_{u > 0} f_1^*(u)\chi_A(u)\varphi(u), \quad (3.6)$$

therefore, combining Equation (3.4) with (3.6), we obtain

$$I_1 \geq \sup_{u > 0} (f_0^* + f_1^*)^*(u)\chi_A(u)\varphi(u) \geq \sup_{u > 0} (f_0 + f_1)^*(2u)\chi_A(u)\varphi(u) = \|(E_1^2 f)^* \chi_A\|_{m_\varphi}. \quad (3.7)$$

for some positive constant $C$. 
Now from the definition of the set \( B \), it is easy to see that

\[
\sup_{u>0} f_0^*(u) \chi_B(u) \varphi(u) \geq t \sup_{u>0} f_0^*(u) \chi_B(u) \psi(u),
\]

which in combination with Equation (3.5) immediately gives

\[
I_2 \geq t \sup_{u>0} f_0^*(u) \chi_B(u) \psi(u) + t \sup_{u>0} f_1^*(u) \chi_B(u) \psi(u) \\
\geq t \sup_{u>0} (f_0^* + f_1^*)(u) \chi_B(u) \psi(u) \\
\geq t \sup_{u>0} (f_0^* + f_1^*)(2u) \chi_B(u) \psi(u) \\
= t \| (E; f^*) \chi_B(u) \|_{m^\psi}.
\]

Since the decomposition \( f = f_0 + f_1 \) was arbitrary, combining Equations (3.7) and (3.8) together with Equation (3.3) gives the first inequality.

For the second inequality, we will assume \( s \) to be non-decreasing and define \( s^\dagger(t) = \inf s^{-1}(t) \). If \( s \) is non-increasing the proof is analogous with \( s^\dagger \) defined using supremum. Decompose \( f = f_0 + f_1 \), where

\[
f_0 = \begin{cases} 
  f - f^*(s^\dagger(t)) & \text{if } f > f^*(s^\dagger(t)) \\
  0 & \text{otherwise}.
\end{cases}
\]

Then, we have

\[
t \| f_0^* \|_{m^\psi} \leq t f^*(s^\dagger(t)) \psi(s^\dagger(t)) + t \| f^* \chi_B \|_{m^\psi} \\
\leq \sup_{u>0} f^*(u) \chi_A(u) \varphi(u) + t \| f^* \chi_B \|_{m^\psi},
\]

where the last inequality is a consequence of continuity of \( \psi \). We also have

\[
\| f_0^* \|_{m^\psi} \leq \sup_{u>0} f^*(u) \chi_A(u) \varphi(u)
\]

Combining Equations (3.9) and (3.10) gives the second inequality.

Let us point out that for the first inequality in Equation (3.2) the assumption of monotonicity of \( s \) is not needed.

In the proofs of the main results, we will also require a characterization of the \( K \)-functional for a Marcinkiewicz space and \( L^\infty \). Again, various versions of this result are known and scattered in the literature but not in the precise form which we need. Therefore, for a reader’s convenience we insert the proof.

**Proposition 3.3.** Let \( \varphi : (0, \infty) \to (0, \infty) \) be an invertible, continuous, increasing function satisfying the \( \Delta_2 \) condition. Then

\[
\| \chi_{(0, \varphi^{-1}(t))} f^* \|_{m^\psi} \leq K(f, t; m^\psi, L^\infty) \leq 2 \| \chi_{(0, \varphi^{-1}(t))} f^* \|_{m^\psi}
\]

for every \( f \in \mathcal{M} \) and every \( t \in (0, \infty) \).

**Proof.** Let \( f \in (m^\psi + L^\infty) \) and \( t > 0 \). Both \( L^\infty \) and \( m^\psi \) norms are defined in terms of \( f^* \) so it will suffice to prove the assertion assuming that \( f \geq 0 \). First, decompose \( f = f_0 + f_1 \), where

\[
f_0 = \begin{cases} 
  f - f^*(\varphi^{-1}(t)) & \text{if } f > f^*(\varphi^{-1}(t)) \\
  0 & \text{otherwise}.
\end{cases}
\]
Then since $\varphi$ is continuous, we have

$$
\sup_{0 < s < \varphi^{-1}(t)} f^*(s)\varphi(s) \geq \lim_{s \to \varphi^{-1}(t)_-} f^*(s)\varphi(s) = \lim_{s \to \varphi^{-1}(t)_-} f^*(s) \lim_{s \to \varphi^{-1}(t)_-} \varphi(s) \geq f^*(\varphi^{-1}(t))\varphi(\varphi^{-1}(t)) = f^*(\varphi^{-1}(t))t.
$$

And so from the definition of $f_0$ and the above calculation

$$
t\|f_1\|_\infty \leq f^*(\varphi^{-1}(t))t \leq \sup_{0 < s < \varphi^{-1}(t)} f^*(s)\varphi(s) = \|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi}.
$$

(3.12)

We continue by estimating $\|f_0\|_{m_\varphi}$. By definition of $f_0$,

$$
\|f_0\|_{m_\varphi} = \sup_{t \in (0,\infty)} \varphi(t)f_0^*(t) \leq \sup_{0 < s < \varphi^{-1}(t)} f^*(s)\varphi(s) = \|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi}.
$$

(3.13)

Combining Equations (3.12) and (3.13), we obtain

$$
K(f, t; m_\varphi, L_\infty) \leq 2\|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi},
$$

(3.14)

establishing the second inequality in Equation (3.11). For the first one, once again, fix $f \in (m_\varphi + L_\infty)$ non-negative and let $f = g + h$, where $g \in m_\varphi$ and $h \in L_\infty$. First, we shall assert that

$$
f^*(t) \leq g^*(t) + \|h\|_\infty \text{ for every } t \in (0, \infty).
$$

(3.15)

For $t \in (0, \infty)$, set $\lambda = g^*(t) + \|h\|_\infty$ and $y = \{|s \in (0, \infty), f(s) > \lambda\}$. Then

\[
\begin{align*}
y &= \{|s \in (0, \infty), g(s) + h(s) > \lambda\} \\
    &= \{|s \in (0, \infty), g(s) > g^*(t) + \|h\|_\infty\} \\
    &\leq \{|s \in (0, \infty), g(s) > g^*(t)\} + \{|s \in (0, \infty), h(s) > \|h\|_\infty\} \\
    &= \{|s \in (0, \infty), g(s) > g^*(t)\},
\end{align*}
\]

since the set $\{|s \in (0, \infty), h(s) > \|h\|_\infty\}$ obviously has zero measure. By Equation (2.3), we obtain $y \leq t$. By definition of the decreasing rearrangement we get Equation (3.15). Consequently, from subadditivity of supremum and because $\varphi$ is increasing, we obtain

$$
\sup_{0 < s < \varphi^{-1}(t)} f^*(s)\varphi(s) \leq \sup_{0 < s < \varphi^{-1}(t)} g^*(s)\varphi(s) + \sup_{0 < s < \varphi^{-1}(t)} \|h\|_\infty\varphi(s) \\
\leq \sup_{0 < s < \infty} g^*(s)\varphi(s) + \|h\|_\infty\varphi(\varphi^{-1}(t)) = \|g\|_{m_\varphi} + t\|h\|_\infty.
$$

(3.16)

Taking infimum over all such representations $f = g + h$, we arrive at

$$
\|\chi_{(0,\varphi^{-1}(t))} f^*\|_{m_\varphi} \leq K(f, t; m_\varphi, L_\infty),
$$

as desired. The assertion now follows from the combination of Equations (3.14) and (3.16).
4 | THE KERNEL OPERATOR ON REARRANGEMENT-INARIANT SPACES

Definition 4.1. For $a \in \mathcal{M}_+$, we define the operator $S_a$ by the formula

$$S_a f(t) = \int_0^\infty a(st)f(s)\,ds$$

for those $f \in \mathcal{M}$ for which the integral on the right is defined.

Note that $S_a$ is a generalization of the Laplace transform, the Laplace transform is $S_a$ for the choice $a(t) = e^{-t}$. In this section, we will formulate a characterization of whether a target space for the operator $S_a$ exists with a domain space fixed. We will also show that whenever a target space exists, optimal space also exists and we can find an implicit definition of said space using the kernel $a$. We will then attempt to find equivalent definitions of target and optimal spaces using Calderón-type operators rather than the function $a$. We shall need some preliminary work. First, note that if $X$ and $Y$ are rearrangement-invariant spaces and $a \in \mathcal{M}$, then

$$S_a : X \to Y \iff S_a : Y' \to X'.$$  \hspace{1cm} (4.1)

This follows from the fact that $S_a$ is a self-adjoint operator, that is,

$$\int_0^\infty S_a f g = \int_0^\infty f S_a g \quad \text{for all } f, g \in \mathcal{M}.$$  \hspace{1cm} (4.2)

Lemma 4.2. Let $a$ be a non-increasing, non-negative function on $(0, \infty)$. Then

$$(S_a f)^* \leq S_a (f^*) \quad \text{for all } f \in \mathcal{M}.$$  

Proof. Taking $t > 0$ and $f \in \mathcal{M}$, we obtain, by the Hardy–Littlewood inequality (recall that $a$ is non-increasing), that

$$(S_a f)^*(t) = (|S_a f|)^*(t) \leq (S_a |f|)^*(t) = \int_0^\infty |f(s)|a(st)\,ds$$

$$\leq \int_0^\infty f^*(s)a(st)\,ds = S_a(f^*)(t). \quad \square$$

Theorem 4.3. Let $X$ be a rearrangement-invariant space and let $a \in \mathcal{M}$ be non-increasing. Then $S_a : X \to L^1 + L^\infty$ if and only if $a^{**} \in X'$. If that be the case, then the expression $\|S_a(f^*)\|_{X'}$ defines a rearrangement-invariant norm. Let $Z$ be the rearrangement-invariant space such that $\|f\|_{Z'} = \|S_a(f^*)\|_{X'}$. We have $S_a : X \to Z$.

Proof. We have $a^{**} = S_a(\chi_{(0,1)})$ and by definition

$$\|a^{**}\|_{X'} = \sup_{\|f\|_X \leq 1} \int_0^\infty |fa^{**}|.$$  

Since $a^{**}$ is non-increasing, it follows from the fact that $S_a$ is self-adjoint that

$$\|a^{**}\|_{X'} = \sup_{\|f\|_X \leq 1} \int_0^\infty f^*a^{**} = \sup_{\|f\|_X \leq 1} \int_0^\infty f^*S_a(\chi_{(0,1)})$$

$$= \sup_{\|f\|_X \leq 1} \int_0^\infty S_a(f^*)\chi_{(0,1)} = \sup_{\|f\|_X \leq 1} \|S_a(f^*)\|_{L^1 + L^\infty},$$
where the last equality follows from Equation (2.10) and the fact that $S_a(f^*)$ is non-increasing. Now, we get from Lemma 4.2 that

$$\|a^{**}\|_{X'} = \sup_{\|f\|_{X} \leq 1} \|S_a(f)\|_{L^1 + L^\infty} = \|S_a\|_{X \to L^1 + L^\infty}.$$  

The first equivalence follows. The proof of the fact that the expression $\|S_a(f^*)\|_{X'}$ defines a rearrangement-invariant norm is standard and we omit it.

Since $S_a f^*$ is a non-increasing function we have by Lemma 4.2, the Hardy–Littlewood inequality, Equation (4.2), and the Hölder inequality that

$$\|S_a f\|_Z = \|(S_a f)^*\|_Z \leq \|S_a f^*\|_Z = \sup_{\|g\|_{X'} \leq 1} \int_0^\infty S_a f^* g^*$$

$$= \sup_{\|g\|_{X'} \leq 1} \int_0^\infty f^* S_a g^* \leq \sup_{\|g\|_{X'} \leq 1} \|f\|_X \|S_a g^*\|_{X'} = \|f\|_X.$$  

In other words, $S_a : X \to Z$. □

Norms defined using the kernel of some integral operator, similarly as in the preceding theorem, were studied before. For a differently defined operator of a type similar to $S_a$ this approach was used, for example, in [20, Theorem 2.2] (only in the proof), but optimality of the resulting space (see Corollary 4.5) is not shown. This is the subject of the following remark.

**Remark 4.4.** Let $a \in \mathcal{M}_+$ be non-increasing and non-trivial. Let $X$ be a rearrangement-invariant space with $a^{**} \in X'$ and $R$ be an operator defined at least on $\mathcal{M}_+$. Define

$$\rho(f) = \|R f^*\|_{X'},$$  

and assume that this $\rho$ is a rearrangement-invariant norm. Define $Z$ as the rearrangement-invariant space determined by $\rho'$. Consider the following two properties of $R$.

(i) There exists $C > 0$ such that for all $f \in \mathcal{M}$ it holds that

$$S_a f^*(t) C \leq R f^*(t) \quad \text{for a.e. } t \in (0, \infty)$$

(ii) There exists $C > 0$ such that for all $f \in \mathcal{M}$ it holds that

$$S_a f^*(t) C \geq R f^*(t) \quad \text{for a.e. } t \in (0, \infty)$$

If (i) holds, then $S_a : X \to Z$. If (ii) holds and $Y$ is a rearrangement-invariant space such that $S_a : X \to Y$, then $Z \hookrightarrow Y$.

The proof of the remark is easy and left to the reader. Setting $R = S_a$ and using Theorem 4.3 and Remark 4.4, we arrive at the following corollary.

**Corollary 4.5.** Let $a \in \mathcal{M}_+$ be non-increasing and let $X$ be a rearrangement-invariant space such that $a^{**} \in X'$. The rearrangement-invariant $Z$ space satisfying $\|f\|_{Z'} = \|S_a f^*\|_{X'}$, $f \in \mathcal{M}$, is the smallest rearrangement-invariant space for which $S_a : X \to Z$.

**Definition 4.6.** Assume that $X$ is a rearrangement-invariant space. We denote by $E(X)$ the function given by

$$E(X)(t) = \frac{t}{\|E_t\|_{X \to X}} \quad \text{for } t \in (0, \infty).$$
Theorem 4.7. If $X$ is a rearrangement-invariant space, then $m_{E(X)}$ is well defined. Let $a \in M_+$ be non-increasing. If $a \in X'$ then

$$S_a : X \to m_{E(X)}.$$ 

Proof. The proof of the fact that $E(X)$ is quasi-concave is standard. It follows therefore that $E(X)$ satisfies the $\Delta_2$ condition. Indeed since $t \mapsto \frac{E(X)(2t)}{2t}$ is non-increasing, one has

$$E(X)(2t) = \frac{E(X)(2t)}{2t} 2t \leq \frac{E(X)(t)}{t} 2t,$$

which is the $\Delta_2$ condition. We have shown that $m_{E(X)}$ is well defined. Now, assume that $a \in X'$ is non-increasing. Let $f \in M_+$. We first note that since $a$ is non-increasing, so is $Saf$. Thus, using the change of variables $st = u$ and the Hölder inequality, we obtain

$$\|Saf\|_{m_{E(X)}} = \sup_{t \in (0,1)} \frac{t}{\|E_t\|_{X \to X}} S_a f(t) = \sup_{t \in (0,\infty)} \frac{t}{\|E_t\|_{X \to X}} \int_0^{\infty} a(st)f(s)ds$$

$$\leq \sup_{t \in (0,\infty)} \frac{1}{\|E_t\|_{X \to X}} \|E_t f\|_X \|a\|_{X'}, \leq \|a\|_{X'} \|f\|_X.$$ 

□

Definition 4.8. Let $X$ be a rearrangement-invariant space such that $\varphi_{X'}$, the fundamental function of $X'$, is strictly increasing and unbounded. Let $\varphi : (0, \infty) \to (0, \infty)$ be quasi-concave, unbounded function with $\varphi(0_+) = 0$. Then, we define the function $\psi$ with the formula

$$\psi(t) = \varphi^{-1}_X \left( \frac{1}{\varphi(t)} \right) \text{ for } t \in (0, \infty). \quad (4.3)$$

For $f \in M$ and $t \in (0, \infty)$ we define the functions $\alpha(f), \beta(f)$ with the formulas

$$\alpha(f)(t) = \int_0^{\psi(t)} f^+(s)ds, \quad (4.4)$$

$$\beta(f)(t) = \frac{1}{\varphi(t)} \|f^+\chi_{(\psi(t), \infty)}\|_X, \quad (4.5)$$

and we set $R^X_{\infty}(f) = \alpha(f) + \beta(f)$.

Definition 4.9. Let $X$ be a rearrangement-invariant space such that $\varphi_X$ is strictly increasing and unbounded. Let $\varphi$ be a function such that if we define the function $\varphi_k$ with the formula $\varphi_k(u) = \frac{u}{\varphi(u)}$, then the function $\varphi_k$ is quasi-concave and unbounded with $\varphi_k(0+) = 0$. Then, we define the function $\tilde{\psi}$ with the formula

$$\tilde{\psi}(t) = \varphi^{-1}_X \left( \frac{1}{\varphi(t)} \right).$$

For $f \in M$ and $t \in (0, \infty)$, we define the function $R^X_1 f(t)$ with the formula

$$R^X_1(f)(t) = \frac{1}{\varphi(t)} \|f^+\chi_{(0, \tilde{\psi}(t))}\|_X.$$ 

In the following theorems, we will show that, under some conditions, both of the operators $R^X_1$, $R^X_{\infty}$ define rearrangement-invariant spaces in a way we would expect from Remark 4.4. Note that both $R^X_1$ and $R^X_{\infty}$ are defined only
by means of the space $X$ and the function $\varphi$. If we define $\varphi = E(X)$ then $R^X_1$ and $R^X_\infty$ are defined only through $X$. This means that the obtained range partners depend only on $X$ and the fixed domain space, but not directly on the kernel $a$. These spaces, of course, are candidates for the optimal range partner, but in the general setting of rearrangement-invariant spaces, it seems very difficult to prove that they, under reasonable conditions, indeed are optimal. We can, however, obtain optimality of said spaces in the special case of $a \in L^\infty \cap L^1$, see Theorem 4.18, or when the domain space is a Lorentz space, whose exponents satisfy certain conditions. This is further explored in Section 5.

**Lemma 4.10.** Assume that $X$ and $\varphi$ are as in Definition 4.8. Let $W$ be a rearrangement-invariant space and set

$$\rho(f) = \|R^X_\infty f\|_W, f \in M_+.$$  

If the condition

$$\min\left\{ \frac{1}{\varphi}, 1 \right\} \in W \tag{4.6}$$

holds, then $\rho$ is a rearrangement-invariant Banach function norm.

**Proof.** To prove the triangle inequality, fix $f_1, f_2 \in M_+$ and $t \in (0, \infty)$. From the definition of the associate norm, we have

$$R^X_\infty(f_1 + f_2)(t) = \int_0^{\varphi(t)} (f_1 + f_2)^* + \frac{1}{\varphi(t)} \sup \left\{ \int_0^\infty (f_1 + f_2)^* g^*, \|g\|_{X'} \leq 1 \right\}.$$  

Take arbitrary $g \in X'$ with $\|g\|_{X'} \leq 1$ and set

$$h(s) = \begin{cases} 1, & s \in (0, \psi(t)) \\ \frac{1}{\psi(t)} g^*(s), & s \in [\psi(t), \infty) \end{cases}.$$  

We know that $X' \hookrightarrow M(X')$ with the norm of the embedding equal to 1. Therefore, we have

$$\sup_{s \in (0, \infty)} g^*(s) \varphi_{X'}(s) \leq \sup_{s \in (0, \infty)} g^{**}(s) \varphi_{X'}(s) \leq \|g\|_{X'} \leq 1.$$  

In particular for $s = \psi(t)$, we have

$$g^*(\psi(t)) \leq \frac{1}{\varphi_{X'}(\psi(t))} = \frac{1}{\varphi_{X'} \varphi^{-1}_{X'}(\frac{1}{\varphi(t)})} = \varphi(t),$$

thus $h$ is non-increasing. Now thanks to the subadditivity of $f \mapsto f^{**}$ we have

$$\int_0^u (f_1 + f_2)^* \leq \int_0^u f_1^* + f_2^*, \quad u \in (0, \infty).$$

Using Hardy’s Lemma (2.5), we obtain

$$\int_0^\infty (f_1 + f_2)^* h \leq \int_0^\infty f_1^* h + f_2^* h. \tag{4.7}$$

From the definition of $h$, it is clear that

$$\int_0^{\psi(t)} (f_1 + f_2)^* + \frac{1}{\varphi(t)} \int_0^\infty (f_1 + f_2)^* g^* = \int_0^\infty (f_1 + f_2)^* h,$$
which, in combination with inequality (4.7), gives
\[
\int_0^\psi(t) (f_1 + f_2) \varphi(t) \int_0^\psi(t) (f_1 + f_2)^* g^* + \int_0^\psi(t) f_1^* g^* + \int_0^\psi(t) f_2^* g^*.
\]
\[
\leq \int_0^\psi(t) f_1^* (f_1 + f_2)^* + 1 \varphi(t) \int_0^\psi(t) f_1^* g^* + \int_0^\psi(t) f_2^* (f_1 + f_2)^* g^*.
\]
Since this holds for all \( g \in X' \) with \( \|g\|_{X'} \leq 1 \), we have
\[
R_X^X(f_1 + f_2)(t) \leq R_X^X(f_1)(t) + R_X^X(f_2)(t) \quad \text{for } t \in (0, \infty),
\]
which, using the (P2) property of the norm in \( W \), gives the triangle inequality.

The fact that \( \rho(f) = 0 \iff f = 0 \) holds trivially. Positive homogeneity is trivial. We have shown that (P1) holds. Next, (P6) holds obviously and (P2) and (P3) are direct consequences of the corresponding properties of \( f \mapsto f^* \) and of \( \| \cdot \|_X \) and \( \| \cdot \|_W \).

To show (P4), we only need to show that \( \rho(\chi_{(0,u)}) < \infty \) for any \( u \in (0, \infty) \) because \( \rho \) is defined in terms of the non-increasing rearrangement and we know, that for a measurable set \( E \subset (0, \infty) \) it holds that \( (\chi_E)^* = \chi_{(0,|E|)} \). Moreover, due to the property (P2), we may assume that \( u > 1 \). By the definition of \( \alpha \) and \( \beta \) we have for \( t \in (0, \infty) \)
\[
\alpha(\chi_{(0,u)})(t) = \min\{\psi(t), u\} \leq u \min\{\psi(t), 1\}
\]
and
\[
\beta(\chi_{(0,u)})(t) = \frac{1}{\varphi(t)} \|\chi_{(0,u)}\|_{(\psi(t), \infty)} \|_X \leq \frac{1}{\varphi(t)} \varphi_X(\max\{u - \psi(t), 0\}) \leq \frac{1}{\varphi(t)} \varphi_X(u),
\]
where \( \varphi_X \) denotes the fundamental function of \( X \). Furthermore, since \( \lim_{t \to 0^+} \psi(t) = \infty \), there exists \( \varepsilon > 0 \) such that \( \beta(\chi_{(0,u)})(t) = 0 \), for \( t < \varepsilon \), and \( \frac{1}{\varphi(\varepsilon)} \geq 1 \). Since \( \frac{1}{\varphi(\varepsilon)} \leq \frac{1}{\varphi(\varepsilon)} \) on \( (\varepsilon, \infty) \) we have
\[
\beta(\chi_{(0,u)}) \leq \varphi_X(u) \min\left\{ \frac{1}{\varphi(\varepsilon)}, \frac{1}{\varphi(u)} \right\} \leq \varphi_X(u) \frac{1}{\varphi(\varepsilon)} \min\left\{ \frac{1}{\varphi}, 1 \right\}.
\]
Thus, according to condition (4.6), \( \beta(\chi_{(0,u)}) \in W \). To show that \( \alpha(\chi_{(0,u)}) \in W \) we need to only show that there exists a constant \( C \) such that \( \min\{\psi, 1\} \leq C \min\{\frac{1}{\varphi}, 1\} \), for which it is sufficient to show that there exists \( t > 0 \) and \( C > 0 \) such that for all \( s > t \)
\[
\frac{1}{\varphi(s)} C \geq \psi(s) = \varphi_X^{-1}(\frac{1}{\varphi(s)}),
\]
which is equivalent to \( \varphi_X(\tau C) \geq \tau \), for \( \tau \leq \frac{1}{\varphi(t)} \), which follows from quasi-concavity of \( \varphi_X \). Indeed, each quasi-concave function dominates the function \( \min\{1, \tau\} \), \( \tau \in (0, \infty) \), up to a multiplicative constant, and so we have
\[
\varphi_X(\tau C) \geq C_1 \tau,
\]
for all \( C \) and \( \tau \) such that \( C \tau \leq 1 \) and for some \( C_1 \). If we set \( C = \frac{1}{C_1} \), then we have
\[
\varphi_X(\tau C) \geq \tau \quad \text{for } \tau \leq \frac{1}{C}.
\]
We only need to find \( t \) such that \( \frac{1}{\varphi(t)} \leq \frac{1}{C} \), which we can do since \( \varphi \) is unbounded. Now, we have \( t \) and \( C \) as we wanted, therefore we have just proven that \( \min\{1, \psi\} \) is dominated by \( \min\{1, \frac{1}{\varphi}\} \) up to a constant. Now since both \( \beta(\chi_{(0,u)}) \in W \) and \( \alpha(\chi_{(0,u)}) \in W \), it obviously holds that \( \rho(\chi_{(0,u)}) < \infty \) and thus (P4) holds.
It remains to show (P5). Let \( E \subset (0, \infty) \) be of finite measure, let \( f \in \mathcal{M}_+ \) and choose \( u \in \psi^{-1}(|E|) \). Then, since \( \psi \) is non-increasing, one has

\[
\rho(f) \geq \| \alpha(f) \|_W \\
\geq \| \chi_{(0,u)}(t) \int_0^{\psi(t)} f^+(s) ds \|_W \\
\geq \| \chi_{(0,u)} \|_W \int_0^{|E|} f^+(s) ds \\
\geq \| \chi_{(0,u)} \|_W \int_E f(s) ds,
\]

which establishes (P5).

\[\square\]

**Theorem 4.11.** Assume that \( X \) and \( \varphi \) are as in Definition 4.8. Let \( Y \subset X + L^1 \) be a rearrangement-invariant space. Let \( a \) be a non-increasing non-negative function on \( (0, \infty) \) such that

\[
S_a : X \to m_\varphi, \\
S_a : L^1 \to L^\infty.
\]  

(4.8)

Assume that Equation (4.6) holds for \( W = Y' \) and let \( Z \) be the rearrangement-invariant space satisfying

\[
\| f \|_{Z'} = \| R_{\infty}^X f \|_{Y'}, f \in \mathcal{M}.
\]

Then

\[
S_a : Y \to Z.
\]

**Proof.** The fact that such \( Z \) is necessarily a rearrangement-invariant space was proved in the preceding lemma, so we only need to show \( S_a : Y \to Z \). To this end, we will need to calculate the \( K \)-functionals of spaces \((X, L^1)\) and spaces \((m_\varphi, L^\infty)\). By Theorem 3.3, we have

\[
K(f, t; m_\varphi, L^\infty) \approx \| \chi_{(0,}\varphi^{-1}(1/t)} f^* \|_{m_\varphi},
\]

and, by [23, Theorem 5.1], we have

\[
K(f, t, L^1, X) \leq C \int \varphi^{-1}_X(t) f^*(s) ds + t \| f^* \chi_{(\varphi^{-1}_X(t), \infty)} \|_X,
\]

for some constant \( C > 0 \). Fix arbitrary \( f \in L^1 + L^\infty \) and \( t \in (0, \infty) \). From endpoint estimates (4.8) and the definition of the \( K \)-functional, we have

\[
K(S_a f, t, L^\infty, m_\varphi) \leq CK(f, t, L^1, X)
\]

for some constant \( C > 0 \). Combining that with the equality

\[
\frac{1}{t} K(f, t, L^\infty, m_\varphi) = K \left( f, \frac{1}{t}, m_\varphi, L^\infty \right),
\]

we obtain

\[
\sup_{0 < u < \varphi^{-1}(\frac{1}{t})} (Tf)^*(u) \varphi(u) \leq C \left( \int \varphi^{-1}_X(t) f^*(s) ds + t \| f^* \chi_{(\varphi^{-1}_X(t), \infty)} \|_X \right).
\]
In particular, since \( \varphi \) is quasi-concave and therefore continuous and \( (S_a f)^+ \) is non-increasing, we can take \( u = \varphi^{-1}(\frac{1}{t}) \) and obtain

\[
\frac{1}{t}(S_a f)^+(\varphi^{-1}(\frac{1}{t})) \leq \frac{C}{t} \left( \int_0^{\varphi^{-1}(\frac{1}{t})} f^+(s) ds + t \| f^+ \chi_{(\varphi^{-1}(\frac{1}{t}), \infty)} \|_X \right).
\]

Now since \( \varphi \) is a one-to-one mapping on \((0, \infty)\) and the above holds for every \( t \in (0, \infty) \), substituting \( \varphi^{-1}(\frac{1}{t}) = u \) we obtain, for all \( u \in (0, \infty) \),

\[
\varphi(u)(S_a f)^+(u) \leq C\varphi(u) \int_0^{\psi(u)} f^+(s) ds + C \| f^+ \chi_{(\psi(u), \infty)} \|_X.
\]

Dividing by \( \varphi(u) \) and changing the variable \( u \) to \( t \) yields the following result. There exists \( C > 0 \) such that for all \( f \in X + L^1 \) and \( t > 0 \)

\[
(S_a f)^+(t) \leq CR_X f(t),
\]

which is the property (i) in Remark 4.4, therefore, \( S_a : Y \to Z \). \( \square \)

In conjunction with Theorem 4.7, we can now formulate a corollary of the preceding theorem, which makes the result more manageable.

**Corollary 4.12.** Let \( X \) be such that \( \varphi = E(X) \) is a quasi-concave unbounded strictly increasing function with \( \varphi(0+) = 0 \). Let \( Y \subset X + L^1 \) be a rearrangement-invariant space and let \( a \in X' \cap L^\infty \) be non-increasing. Assume that condition (4.6) holds for \( W = Y' \) and let \( Z \) be the rearrangement-invariant space satisfying

\[
\| f \|_{Z'} = \| R_{Z} f \|_{Y'}, f \in \mathcal{M}.
\]

Then, \( S_a : Y \to Z \).

**Proof.** Since \( a \in L^\infty = (L^1)' \), \( E(L^1) \equiv 1 \) and \( m_1 = L^\infty \), we have \( S_a : L^1 \to L^\infty \), and since \( a \in X' \) and \( E(X) = \varphi \), we have \( S_a : X \to m_\varphi \) thanks to Theorem 4.7. Now, we can apply Theorem 4.11. \( \square \)

**Lemma 4.13.** Assume that \( X \) and \( \varphi \) are as in Definition 4.9. Let \( W \) be a rearrangement-invariant space and set

\[
\rho(f) = \| R_X f \|_W, f \in \mathcal{M}.
\]

If the condition

\[
\min \left\{ \frac{1}{\varphi(t)}, \frac{1}{t} \right\} \in W
\]

holds, then \( \rho \) is a rearrangement-invariant Banach function norm.

**Proof.** To show that (P1) holds, we need only to show the triangle inequality, since the other two assertions clearly hold. To that end take \( f_1, f_2 \in \mathcal{M} \) and \( t \in (0, \infty) \). We have

\[
R_X(f_1 + f_2)(t) = \frac{1}{\varphi(t)} \| (f_1 + f_2)^+ \chi_{(0, \psi(t))} \|_X
\]

\[
= \frac{1}{\varphi(t)} \sup_{\| g \|_Y \leq 1} \int_0^{\psi(t)} (f_1 + f_2)^+ g^+ ds
\]

\[
\leq \frac{1}{\varphi(t)} \sup_{\| g \|_Y \leq 1} \int_0^{\psi(t)} f_1^+ g^+ + \int_0^{\psi(t)} f_2^+ g^+
\]

\[
= R_X^1(f_1)(t) + R_X^1(f_2)(t).
\]
where the only inequality is a consequence of the Hardy lemma (2.5). Now the (P2) property of the norm in $W$ gives the triangle inequality. Properties (P2) and (P3) obviously hold for $\rho$. To show (P4) take $u \in (0, \infty)$. We have for all $t > 0$
\[
\frac{1}{\varphi(t)} \|X(0,u)X(0,\hat{\varphi}(t))\|_X = \frac{1}{\varphi(t)} \varphi_X(\min\{u, \hat{\varphi}(t)\})
= \frac{1}{\varphi(t)} \min\{\varphi_X(u), \varphi_X(\hat{\varphi}(t))\}
= \frac{1}{\varphi(t)} \min\left\{ \varphi_X(u), \frac{1}{\varphi(t)} \right\},
\]
where we have used the fact that $\varphi_X$ is non-decreasing and the definition of $\hat{\varphi}$. If we move $\frac{1}{\varphi(t)}$ inside the argument of the minimum, we get from the definition of $\tilde{\varphi}$ that
\[
\frac{1}{\varphi(t)} \min\left\{ \varphi_X(u), \frac{1}{\varphi(t)} \right\} = \min\left\{ \frac{\varphi_X(u)}{\varphi(t)}, \frac{1}{t} \right\} \leq C \min\left\{ \frac{1}{\varphi(t)}, \frac{1}{t} \right\}
\]
for some $C > 0$ depending on $u$ but not on $t$. Now condition (4.9) gives (P4) property of $\rho$. It remains to show (P5). To that end take $E \subset (0, \infty)$ with positive finite measure, $f \in M_+$ and choose $u \in \hat{\varphi}^{-1}(|E|)$. Since $\hat{\varphi}$ is non-increasing, we have
\[
\rho(f) = \left\| \frac{1}{\varphi(t)} \right\| f^*X(0,\hat{\varphi}(t)) \|_X \|_W
\geq \left\| X(0,u) \right\| \frac{1}{\varphi(t)} \left\| f^*X(0,\hat{\varphi}(t)) \right\|_X \|_W
\geq \left\| X(0,u) \right\| \frac{1}{\varphi(u)} \left\| f^*X(0,|E|) \right\|_X
\geq C \left\| X(0,u) \right\| \frac{1}{\varphi(u)} \int_0^{|E|} f^*X(0,|E|)
\geq C \left\| X(0,u) \right\| \frac{1}{\varphi(u)} \int_E f
\]
for some $C > 0$ only depending on $E$ and the space $X$, where the second to last inequality is the (P5) property of the norm in $X$. Since $u$ does not depend on $f$, we have obtained the (P5) property of $\rho$. \hfill \Box

**Theorem 4.14.** Assume that $X$ and $\varphi$ are as in Definition 4.9. Let $Y \subset X + L^\infty$ be a rearrangement-invariant space, assume that $\varphi$ is strictly increasing. Let $a$ be a non-increasing non-negative function on $(0, \infty)$ such that
\[
S_a : X \rightarrow m_\varphi,
S_a : L^\infty \rightarrow L^{1,\infty}.
\]
Assume condition (4.9) holds for $W = Y'$ and let $Z$ be the rearrangement-invariant space for which
\[
\|f\|_Z = \|R^X_1 f\|_{Y'}, f \in M.
\]
Then
\[
S_a : Y \rightarrow Z.
\]

**Proof.** Easy modifications of Theorem 3.2 and [23, Theorem 4.1] give respectively
\[
\sup_{u < \varphi^{-1}(t)} (S_a f)^*(2u)u + t \sup_{u \geq \varphi^{-1}(t)} (S_a f)^*(2u)\varphi(u) \leq C_1 K(S_a f, t, L^{1,\infty}, m_\varphi)
\]
(4.11)
and
\[ K(f, t, L^\infty, X) \leq C_2 \left( f^* \left( \varphi_X^{-1} \left( \frac{1}{t} \right) \right) + t \| f^* \chi_{(0, \varphi_X^{-1}(\frac{1}{t}))} \|_X \right), \]  
\[ (4.12) \]
for all \( t \in (0, \infty), f \in M \) and some constants \( C_1, C_2 > 0 \). Positive homogeneity and the definition of the fundamental function immediately gives
\[ \| f^* \chi_{(0, \varphi_X^{-1}(\frac{1}{t}))} \|_X \geq f^* \left( \varphi_X^{-1} \left( \frac{1}{t} \right) \right) \varphi_X \left( \varphi_X^{-1} \left( \frac{1}{t} \right) \right), \]
therefore we obtain
\[ K(f, t, L^\infty, X) \leq (C_2 + 1)(t \| f^* \chi_{(0, \varphi_X^{-1}(\frac{1}{t}))} \|_X). \]

From the definition of the \( K \)-functional and endpoint estimates (4.10), we obtain
\[ K(S_a f, t, L^1, L^\infty, \mu) \leq C K(f, t, L^\infty, X), \]
for some constant \( C > 0 \) and all \( f \in M \) and \( t \in (0, \infty) \). Using Equations (4.11) and (4.12), we have a possibly different constant \( C > 0 \) such that, for all \( f \in M \) and \( t \in (0, \infty) \),
\[ \sup_{u < (\varphi)^{-1}(t)} (S_a f)^*(2u)u + t \sup_{u \geq (\varphi)^{-1}(t)} (S_a f)^*(2u)u \leq C t \| f^* \chi_{(0, \varphi_X^{-1}(\frac{1}{t}))} \|_X. \]

In particular, we can clearly replace the second supremum with the value of its argument in \((\varphi)^{-1}(t)\). Furthermore, we can do the same in the first supremum since \( S_a(f^*) \) is non-increasing and \( u \mapsto 2u \) is continuous. This results in the following inequality:
\[ (S_a f)^*(2(\varphi)^{-1}(t))(\varphi)^{-1}(t) + t(S_a f)^*(2(\varphi)^{-1}(t))\varphi((\varphi)^{-1}(t)) \leq C t \| f^* \chi_{(0, \varphi_X^{-1}(\frac{1}{t}))} \|_X \]
for all \( f \in M \) and \( t \in (0, \infty) \). Now, we can substitute \( t = \varphi(s) \), since \( \varphi \) is a one-to-one mapping of \((0, \infty)\), which results in
\[ (S_a f)^*(2\varphi(s))(S_a f)^*(2s)\varphi(s) \leq C \varphi(s) \| f^* \chi_{(0, \varphi(\varphi^{-1}(s)))} \|_X \]
for all \( f \in M \) and \( s \in (0, \infty) \). A simple calculation now gives
\[ (S_a f)^*(2s) \leq C \frac{1}{\varphi(s)} \| f^* \chi_{(0, \varphi(\varphi^{-1}(s)))} \|_X \]
for all \( f \in M \) and \( s \in (0, \infty) \), which implies the following statement. There exists \( C > 0 \) such that for all \( f \in X + L^\infty \) and \( t > 0 \)
\[ (S_a f)^*(t) \leq CR^X_1 f \left( \frac{1}{2t} \right). \]

Now, we need only to use Remark 4.4 the fact that the dilation operator is bounded on every rearrangement-invariant space. \( \square \)

**Corollary 4.15.** Let \( X \) be such that if \( \varphi = E(X) \), then \( \varphi \) is a quasi-concave unbounded strictly increasing function with \( \varphi(0_+) = 0 \). Let \( Y \subset X + L^\infty \) be a rearrangement-invariant space and let \( a \in X' \cap L^1 \) be non-increasing and non-negative. Assume Equation (4.9) holds for \( W = Y' \) and let \( Z \) be the rearrangement-invariant space for which
\[ \| f \|_Z' = \| R^X_1 f \|_{Y'}, f \in M. \]
Then, \( S_\alpha : Y \to Z \).

Now, we explore the properties of \( S_\alpha \) when the kernel \( \alpha \) is both integrable and bounded. This is indeed the strongest possible condition, at least in the context of rearrangement-invariant spaces, as \( L^1 \cap L^\infty \) is the smallest rearrangement-invariant space. The chosen approach uses the operator \( R_\infty^X \) and the space \( X = L^\infty \). Working with \( R_1^X \) and setting \( X = L^1 \), unincidentally, yields the same result.

**Lemma 4.16.** Let \( \alpha : (0, \infty) \to (0, \infty) \) be non-increasing and non-zero on at least some set of non-zero measure. Then, there is a constant \( C \) such that, for all \( f \in M \),

\[
S_\alpha(f^*)(t) \geq C \int_0^1 f^*(s)ds \quad \text{for all } t \in (0, \infty).
\]

**Proof.** Since \( \alpha \) is non-increasing and not zero on at least some set of non-zero measure, there exists \( u \in (0, 1) \) such that \( \inf_{s < u} \alpha(s) = C_1 > 0 \). Now if \( f \in M \) and \( t \in (0, \infty) \), then

\[
S_\alpha(f^*)(t) = \int_0^\infty \alpha(st)f^*(s)ds \geq \int_0^u \alpha(st)f^*(s)ds \geq C_1 \int_0^1 f^*(s)ds \geq C_1 u \int_0^1 f^*(s)ds,
\]

where the last inequality follows from the fact that \( f^{**} \) is non-increasing. \( \square \)

**Lemma 4.17.** Let \( X = L^\infty \) and \( \varphi = E(L^\infty) \), that is, \( \varphi(t) = t \), \( t \in (0, \infty) \). Then, we have, for all \( f \in L^1 + L^\infty \) and \( t > 0 \),

\[
\int_0^1 f^* \leq R_\infty^X f(t) \leq 2 \int_0^1 f^*.
\]

**Proof.** Take \( f \in L^1 + L^\infty \) and \( t > 0 \). First, we shall observe that \( \varphi_X(t) = t \), since \( X' = L^1 \), therefore \( \psi(t) = \frac{1}{t} \). This means that

\[
\alpha(f)(t) = \int_0^1 f^*
\]

and, since \( \beta(f) \geq 0 \), we can easily obtain the first inequality from the definition of \( R \). Now, a simple calculation shows that

\[
\beta(f)(t) = \frac{1}{t} \|f^* X_{(1, \infty)}\|_\infty = \frac{1}{t} f^* \left( \frac{1}{t} \right) \leq \frac{1}{t} f^{**} \left( \frac{1}{t} \right) = \alpha(f)(t).
\]

Therefore,

\[
R_\infty^X(t) = \alpha(f)(t) + \beta(f)(t) \leq \alpha(f)(t) + \alpha(f)(t) = 2 \int_0^1 f^*,
\]

which establishes the second inequality. \( \square \)

What follows is a result of independent interest in the spirit of [7, Theorem 3.4].

**Theorem 4.18.** Let \( \alpha \in L^\infty \cap L^1 \) be non-trivial, non-negative and non-increasing. Let \( Y \) be a rearrangement-invariant space such that

\[
\min \left\{ \frac{1}{t}, \frac{1}{k} \right\} \in Y'. \quad (4.13)
\]
For $f \in \mathcal{M}$ and $t > 0$, set
\[
\alpha(f)(t) = \int_0^t f^*
\]
and
\[
\rho(f) = \|\alpha(f)\|_{Y'}.
\]
Then, $\rho$ is a rearrangement-invariant norm such that if we set $Z$ to be the rearrangement-invariant space given by $\rho'$, then
\[
S_a : Y \to Z,
\]
and $Z$ is optimal for $S_a$ and $Y$. Furthermore, if the condition (4.13) does not hold, then there is no rearrangement-invariant space $Z$ such that $S_a : Y \to Z$.

**Proof.** The proof of the fact that $\rho$ is a rearrangement-invariant norm is almost identical to that of Theorem 4.11, see also [7, Proposition 3.3], and therefore omitted. We thus know from Corollary 4.12 and Lemma 4.17 that $S_a : Y \to Z$.

The optimality of $Z$ is a direct consequence of Lemma 4.16 and Remark 4.4. Indeed, from Lemma 4.16 we have a constant $C > 0$ such that, for all $f \in \mathcal{M}$ and $t \in (0, \infty)$,
\[
S_a(f^*)(t) \geq C \int_0^t f^*(s) ds,
\]
whence Remark 4.4 gives the optimality of $Z$.

It remains to show that if Equation (4.13) does not hold, then there is no rearrangement-invariant space $Z$ such that $S_a : Y \to Z$. Since $a$ is integrable, bounded, non-zero on some set of non-zero measure and non-increasing, we can find a constant $C' > 0$ such that
\[
a^{**}(t) \geq C' \min\left\{1, \frac{1}{t} \right\} \quad \text{for all } t > 0.
\]
Indeed, suppose first that $t \leq 1$. Then, we find $u > 0$ such that $\inf_{s \in (0, u)} a(s) = C_1 > 0$. If $u \geq 1$, then simply
\[
\frac{1}{t} \int_0^t a(s) ds \geq \frac{1}{t} tC_1 = C_1.
\]
If $u < 1$, then we have
\[
\frac{1}{t} \int_0^t a(s) ds \geq \frac{1}{t} \int_0^{\min\{t, u\}} a(s) ds \geq \min\left\{\frac{1}{t} tC_1, \frac{1}{t} uC_1\right\} = uC_1.
\]
Now, let $t > 1$. Then, we set $C_2 = \int_0^1 a(s) ds$ and observe that
\[
\frac{1}{t} \int_0^t a(s) ds \geq \frac{1}{t} C_2.
\]
Combining these estimates, we arrive at
\[
a^{**}(t) \geq C' \min\left\{1, \frac{1}{t} \right\} \quad \text{for all } t > 0,
\]
where $C' = \min\{C_1, uC_1, C_2\}$. Therefore, $a^{**} \notin Y'$, whence Theorem 4.3 implies that there is no rearrangement-invariant space $Z$ such that $S_a : Y \to Z$. Indeed, this holds since $L^1 + L^\infty$ is the largest among all rearrangement-invariant spaces. \(\square\)
In this section, we shall formulate some sufficient conditions involving the kernel function \( a \) under which we can obtain the optimal range partner \( Z \) for \( S_a \) and a fixed domain Lorentz space \( Y \). It turns out that the space \( Z \) is also a Lorentz space and, perhaps aside from extremal cases, its exponents only depend on the exponents of the space \( Y \) and not on the kernel \( a \).

**Theorem 5.1.** Let \( p \in (1, \infty] \) and \( \xi \in (1, \infty) \). Define the operators \( S_p \) and \( T_p \) by

\[
(S_p f)(t) = t^{1 - \frac{1}{p}} \int_0^1 f(s) s^{-\frac{1}{p}} ds
\]

and

\[
(T_p f)(t) = \int_0^1 f(s) ds + t^{1 - \frac{1}{p}} \int_1^{\infty} f(s) s^{-\frac{1}{p}} ds,
\]

for \( t \in (0, \infty) \) and all \( f \in M \) for which the right sides make sense.

(i) It holds that

\[
S_p : L^{p',1} \to L^{p,\infty}; \quad S_p : L^{\infty} \to L^{1,\infty}.
\]  

(ii) If \( \xi < p \) and \( \eta \in [1, \infty] \), then

\[
S_p : L^{\xi',\eta} \to L^{\xi,\eta}.
\]

(iii) It holds that

\[
T_p : L^{p',1} \to L^{p,\infty}; \quad T_p : L^{1} \to L^{\infty}.
\]

(iv) If \( p < \xi \) and \( \eta \in [1, \infty] \), then

\[
T_p : L^{\xi',\eta} \to L^{\xi,\eta}.
\]

(v) The following inequality holds for all \( f \in M \), \( \xi \in (1, \infty) \) and \( \eta \in [1, \infty] \)

\[
\|f\|_{\xi',\eta} \leq \left\| \int_0^1 f(s) s^{-\frac{1}{p}} ds \right\|_{\xi,\eta}.
\]

**Proof.** We shall prove statements (i)–(iv) as the statement (v) is very simple and can be found in a more complete form in [7, Proposition 3.7]. Let \( f \in M \). It then holds that \( |S_p(f)| \leq S_p|f| \) whenever the function on the right is finite a.e. Hence, we may assume \( f \in M_+ \). Since \( S_p(f) \) is non-increasing it follows that

\[
\|S_p f\|_{L^{p,\infty}} = \sup_{t \in (0, \infty)} \int_0^1 f(s) s^{-\frac{1}{p}} ds = \|f\|_{L^{p',1}}.
\]

On the other hand, let \( f \in L^{\infty} \). Then, it holds that

\[
\|S_p f\|_{L^{1,\infty}} = \sup_{t \in (0, \infty)} t^{1 - \frac{1}{p}} \int_0^1 f(s) s^{-\frac{1}{p}} ds \leq \|f\|_{L^{\infty}} \sup_{t \in (0, \infty)} t^{1 - \frac{1}{p}} \int_0^t \left( \frac{1}{t} \right)^{1 - \frac{1}{p}} \frac{1}{1 - \frac{1}{p}} = p'\|f\|_{L^{\infty}}.
\]

We have shown that (i) holds. It follows immediately from the Marcinkiewicz interpolation theorem (see, e.g., [4, Theorem 4.4.13]) that (ii) holds.
We also have $|T_p(f)| \leq T_p(|f|)$ whenever the function on the right is finite a.e. Hence, we may once again assume $f \in M_+$ to show (iii). Note that $T_p(f)$ is non-increasing as it may be written in the form

$$T_p(f)(t) = \int_0^\infty \min \left\{ 1, (ts)^{-\frac{1}{p}} \right\} f^*(s)ds.$$  

From the definition and by sub-additivity of supremum, we have

$$\|T_p(f)\|_{L_{p,\infty}} \leq \sup_{t \in (0,\infty)} t^{\frac{1}{p}} \int_0^1 f(s)ds + \sup_{t \in (0,\infty)} t^{\frac{1}{p} - \frac{1}{p}} \int_t^\infty f(s)\frac{1}{s^{\frac{1}{p}}}ds = \|S_\infty f\|_{L_{p,\infty}} + \|f\|_{L_{p',1}}.$$  

Now by (ii) in case $p < \infty$ or by (i) in case $p = \infty$, we have a constant $C > 0$ such that $\|S_\infty f\|_{L_{p,\infty}} \leq C\|f\|_{L_{p',1}}$ and (iii) follows. Moreover, due to Marcinkiewicz interpolation theorem (iv) holds. This completes the proof. \hfill \Box

Now that we have established some necessary estimates, it is time to use them to calculate the optimal range partners for $S_a$, in the case when the domain space is a Lorentz space. Let us first fix some notation. If it is stated what space $X$ is, we automatically assume that $\varphi = E(X)$ and that the operators $R_X^X$ and $R_X^1$ are defined as in Definitions 4.8 and 4.9. Furthermore by $Z_{\infty}$ and $Z_1$ we shall denote the spaces given by norms associated with $\|R_X^X f\|_{Y'}$ and $\|R_X^1 f\|_{Y'}$, respectively, where $Y$ is the fixed domain space.

**Lemma 5.2.** Let $\xi \in (1,\infty)$, $\eta \in [1,\infty]$ and $a \in M_+$ a non-increasing function. If $W$ is a rearrangement-invariant space such that $S_a : L^\xi,\eta \to W$, then $L^\xi,\eta \hookrightarrow W$.

**Proof.** Let

$$Tf(t) = \int_0^t f^*(s)ds \quad \text{for} \quad f \in M, \ t > 0,$$

then by Theorem 4.18 we know that the expression $\|Tf\|_{\xi',\eta'}$ defines a rearrangement-invariant norm since $\min\{1, \frac{1}{t}\} \in L^\xi,\eta$. Furthermore, from Lemma 4.16 we have a constant $C > 0$ such that $S_a f^* \geq CTf$ for all $f \in M$. If we now set $Z$ to be the rearrangement-invariant space determined by the norm associated with $\|Tf\|_{\xi',\eta'}$, then Remark 4.4 gives $Z \hookrightarrow W$. Furthermore, inequality (5.5) gives $Z' \hookrightarrow L^\eta,\eta$ or equivalently $L^\eta,\eta \hookrightarrow Z$, whence $L^\xi,\eta \hookrightarrow W$. \hfill \Box

**Theorem 5.3.** Let $p \in [1,\infty)$ and $a \in M_+$ a non-increasing function such that $a \in L^1 \cap L^p',\infty$. If $\xi \in (1,\infty)$, $p < \xi$, $\eta \in [1,\infty]$, then $S_a : L^\xi,\eta \to L^\xi,\eta$ and furthermore, $L^\xi,\eta$ is the optimal range partner for $S_a$ and $L^\xi,\eta$.

**Proof.** First, set $X = L^{p,1}$ and $Y = L^\xi,\eta$. Then, since $p < \infty$ we have up to a constant that $\varphi(t) = t^{1/p}$, $t > 0$ which is a quasi-concave, unbounded, strictly increasing function with $\varphi(0^+) = 0$. Furthermore, $Y = L^\xi,\eta \subset X + L^\infty$ since $p < \xi$, also $a \in X' \cap L^1$ and the condition (4.9) holds for $W = L^\xi,\eta$. Therefore, Corollary 4.15 gives $T : L^\xi,\eta \to Z_1$. It is easy to calculate that $\hat{\varphi}(t) = \frac{1}{t}$ and so by (5.2) we have a constant $C > 0$ such that for all $f \in M$

$$\|f\|_{Z_1'} = \left\| \frac{1}{t} \int_0^t f^*(s)\frac{1}{s^{\frac{1}{p}}}ds \right\|_{\xi',\eta'} \leq C\|f\|_{\xi',\eta'}.$$  

In other words, $L^\xi,\eta \hookrightarrow Z_1'$ or equivalently $Z_1 \hookrightarrow L^\xi,\eta$, therefore $S_a : L^\xi,\eta \to L^\xi,\eta$. The optimality of $L^\xi,\eta$ is now an immediate consequence of Lemma 5.2. \hfill \Box

**Theorem 5.4.** Let $q \in (1,\infty)$ and $a \in M_+$ a non-increasing function such that $a \in L^\infty \cap L^{q',\infty}$. If $\xi \in (1,\infty)$, $\xi < q$, $\eta \in [1,\infty]$ then $S_a : L^\xi,\eta \to L^\xi,\eta$ and furthermore, $L^\xi,\eta$ is the optimal range partner for $S_a$ and $L^\xi,\eta$.
Proof. First, set \( X = L^{q,1} \) and \( Y = L^{\xi,\eta} \). Then, since \( q > 1 \) we have up to a constant that \( \varphi(t) = t^{\frac{1}{q'}} \), \( t > 0 \), which is a quasi-concave, unbounded, strictly increasing function. Furthermore, \( Y = L^{\xi,\eta} \subset X + L^1 \) since \( \xi < q \), also \( a \in L^\infty \cap X' \) and the condition (3.1) holds for \( W = Y' = L^{\xi',\eta'} \). Therefore, Corollary 4.15 gives \( S_a : L^{\xi,\eta} \to Z_{\infty} \). It is easy to calculate that \( \varphi(t) = t\frac{1}{q'} \), \( t > 0 \), and by (5.4) we have a constant \( C > 0 \) such that for all \( f \in M \)

\[
\|f\|_{p',\infty} = \left\| \int_0^1 f^*(s)ds + t\frac{1}{q'} \int_1^\infty f^*(s)s\frac{1}{q'}ds \right\|_{\xi',\eta'} \leq C\|f\|_{\xi,\eta'}.
\]

In other words, \( L^{\xi,\eta} \hookrightarrow Z_{\infty} \) or equivalently \( Z_{\infty} \hookrightarrow L^{\xi',\eta'} \), therefore \( S_{\tilde{a}} : L^{\xi,\eta} \to L^{\xi',\eta'} \). The optimality of \( L^{\xi',\eta'} \) is now an immediate consequence of Lemma 5.2. \( \square \)

Now that we have these optimal results for \( a \in L^\infty \) and \( a \in L^1 \) it is time to apply them to a general \( a \in M_+ \). We recall that by Theorem 4.3 we require \( a \) to be in at least some rearrangement-invariant space, otherwise there is no hope of finding a range partner for \( S_{\tilde{a}} \) and any fixed rearrangement-invariant domain space. Since \( L^1 + L^\infty \) is the largest rearrangement-invariant space we can assume \( a \in L^1 + L^\infty \), which allows us to divide \( a \) into its integrable and bounded parts and then use the previous theorems and the fact that \( S_{\tilde{a}} \) is linear in \( a \).

Definition 5.5. Given \( a \in L^1 + L^\infty \) we define the functions \( a_1, a_\infty \) with following formulas:

\[
a_1(t) = (a(t) - a(1))\chi_{(0,1)} \quad \text{for } t > 0
\]
\[
a_\infty(t) = \min\{a(1), a(t)\} \quad \text{for } t > 0.
\]

Note that \( a = a_1 + a_\infty \) and that by identity (2.10) we have \( a_1 \in L^p \), \( a_\infty \in L^\infty \).

Theorem 5.6. Let \( a \in L^1 + L^\infty \) be non-increasing. Let \( p \in [1, \infty), q \in (1, \infty) \) be such that \( a_1 \in L^{p',\infty}, a_\infty \in L^{q',\infty} \). If \( \xi \in (p, q), \eta \in [1, \infty) \), then \( S_a : L^{\xi,\eta} \to L^{\xi',\eta'} \) and furthermore, \( L^{\xi',\eta'} \) is the optimal range partner for \( S_a \) and \( L^{\xi,\eta} \).

Proof. From Theorems 5.3 and 5.4, we have

\[
S_{a_1} : L^{\xi,\eta} \to L^{\xi',\eta'}
\]
\[
S_{a_\infty} : L^{\xi,\eta} \to L^{\xi',\eta'}.
\]

respectively, with both of these range spaces being optimal. Now since \( a_1 + a_\infty = a \), we also have \( S_a = S_{a_1} + S_{a_\infty} = S_a \), which implies \( S_a : L^{\xi,\eta} \to L^{\xi',\eta'} \) with \( L^{\xi',\eta'} \) being the optimal range partner for \( S_a \) and \( L^{\xi,\eta} \). \( \square \)

If we set \( A = \{r \in [1, \infty), a_1 \in L^{r',\infty}\}, B = \{r \in [1, \infty), a_\infty \in L^{r',\infty}\} \), then it is easy to see that \( A, B \) are some intervals. Theorem 5.6 gives the optimality of \( S_a : L^{\xi,\eta} \to L^{\xi',\eta'} \) for all \( \xi \in (A \cap B)' \). It is also easy to see that if \( \xi \notin A \cap B \), then there is no range partner for \( S_a \) and \( L^{\xi,\eta} \) simply because there is no hope of \( a^{**} \in L^{\xi,\eta} \). The problematic case is when \( A \cap B = [p, q] \) for some \( p, q \in (1, \infty) \) and \( \xi = p \) or \( \xi = q \). It is not hard, however, to find an example that shows that a similar result in this case does not hold. If one considers \( a(t) = t\frac{1}{p'} \chi_{(0,1)}, t > 0 \), then we have \( A = [p, \infty), B = (1, \infty) \), thus \( A \cap B = [p, \infty) \). Setting \( \xi = p, \eta = 1 \) we have \( a^{**} \in L^{\xi',\eta'} = L^{p',\infty} \) and so by Corollary 4.5 we know that the optimal space is given by norm associated with

\[
\|S_a f^*\|_{p',\infty} = \sup_{t > 0} t^\frac{1}{p'} t^\frac{1}{p'} \int_0^1 f^*(s)s^\frac{1}{p'} = \|f\|_{p,1}.
\]

This means that for this specific \( a \) we have the optimal result \( S_a : L^{\xi,\eta} \to L^{\xi',\eta'} \). All we can say, using relationships (5.1), (5.3), and (5.5) and the same approach as in the preceding theorems, is the following. If \( A \cap B = [p, q] \), then the optimal range partner \( X_{\text{opt}}^p \) for \( S_a \) and \( L^{p,1} \) satisfies \( L^{p',1} \hookrightarrow X_{\text{opt}}^p \hookrightarrow L^{p',\infty} \). In that case, the optimal range partner \( X_{\text{opt}}^q \) for \( S_a \) and \( L^{q,1} \) satisfies \( L^{q',1} \hookrightarrow X_{\text{opt}}^q \hookrightarrow L^{q',\infty} \).
The preceding assertions sum up the boundedness and optimality properties of $S_a$ on the scale of Lorentz spaces $L^{\xi,\eta}$. It is a simple observation that as long as $a$ is non-trivial, $a^{**} \not\in L^1$ and therefore, by Theorem 4.3, there is no range partner for $S_a$ and $L^\infty$. Combined with the preceding results and denoting by $p = \inf A$, $q = \sup B$, we know the following.

(i) If $\xi \in (p, q)$, $\eta \in [1, \infty]$, then a range partner for $S_a$ and the domain space $L^{\xi,\eta}$ exists. If that is the case, then the optimal range partner is the space $L^{\xi',\eta}$.

(ii) If $\xi \in (1, \infty)$, $\xi \not\in [p, q]$, $\eta \in [1, \infty]$, then no range partner for $S_a$ and the domain space $L^{\xi,\eta}$ exists.

(iii) If $\xi \in [p, q]$, $\eta \in [1, \infty]$, then no range partner for $S_a$ and the domain space $L^{\xi,\eta}$ exists.

(iv) If $\xi = p$ or $\eta = q$, then no range partner for $S_a$ and the domain space $L^{\xi,\eta}$ exists.

(v) $S_a : L^1 \to L^\infty$ if and only if $a \in L^\infty$, if that is the case, then $L^\infty$ is the optimal range partner for $S_a$ and $L^1$.

(vi) If $\xi = \eta = \infty$ then no range partner for $S_a$ and the domain space $L^{\xi,\eta}$ exists.

The only points perhaps not explained above are (v) and (vi). Theorem 4.7 gives $S_a : L^1 \to L^\infty$ if $a \in L^\infty$, whereas the optimality of this result is a consequence of Lemma 5.2 and the fact that

$$\left\| \int_0^1 f^* \right\|_\infty = \| f \|_1.$$

If $a \not\in L^\infty$ then also $a^{**} \not\in L^\infty$, thus by Theorem 4.5 there is no range partner for $S_a$ and $L^1$. We have shown that (v) holds. To show (vi), it suffices to show that $a^{**} \not\in Y$, this is, however, a direct consequence of $q = -\infty$.

**ORCID**

Jakub Takáč https://orcid.org/0000-0003-2158-7456

**REFERENCES**

[1] K. F. Andersen, *On Hardy’s inequality and Laplace transforms in weighted rearrangement invariant spaces*, Proc. Amer. Math. Soc. **39** (1973), no. 2, 295–299.

[2] K. F. Andersen, *Weighted generalized Hardy inequalities for nonincreasing functions*, Canad. J. Math. **43** (1991), no. 6, 1121–1135. https://doi.org/10.4153/CJM-1991-065-9.

[3] C. Bennett, *Estimates for weak-type operators*, Bull. Amer. Math. Soc. **79** (1973), 933–935. https://doi.org/10.1090/S0002-9904-1973-13266-5.

[4] C. Bennett and R. Sharpley, *Interpolation of operators*, Pure and applied mathematics, vol. 129, Academic Press, Inc., Boston, MA, 1988, pp.xiv+469.

[5] A. Boumenir and A. Al-Shuaibi, *The inverse Laplace transform and analytic pseudo-differential operators*, J. Math. Anal. Appl. **228** (1998), 16–36.

[6] O. G. Bravo, *On the optimal domain of the Laplace transform*, Bull. Malays. Math. Sci. Soc. **40** (2017), no. 1, 389–408. https://doi.org/10.1007/s40840-016-0402-7.

[7] M. Cwikel and E. Pustylnik, *Sobolev type embeddings in the limiting case*, J. Fourier Anal. Appl. **4** (1998), nos. 4–5, 433–446. https://doi.org/10.1007/BF02498218.

[8] M. Cwikel, A. Kaminska, L. Maligranda, and L. Pick, *Are generalized Lorentz “spaces” really spaces?*, Proc. Amer. Math. Soc. **132** (2004), no. 12, 3615–3625. https://doi.org/10.1090/S0002-9939-04-07477-5.
[15] O. Delgado and J. Soria, *Optimal domain for the Hardy operator*, J. Funct. Anal. 244 (2007), no. 1, 119–133. https://doi.org/10.1016/j.jfa.2006.12.011.

[16] D. E. Edmunds, R. Kerman, and L. Pick, *Optimal Sobolev imbeddings involving rearrangement-invariant quasinorms*, J. Funct. Anal. 170 (2000), no. 2, 307–355. https://doi.org/10.1006/jfan.1999.3508.

[17] D. E. Edmunds, Z. Mihula, V. Musil, and L. Pick, *Boundedness of classical operators on rearrangement-invariant spaces*, J. Funct. Anal. 278 (2020), no. 4, 108341. https://doi.org/10.1016/j.jfa.2019.108341.

[18] S. Ericsson, *Exact descriptions of some K and E functionals*, J. Approx. Theory 90 (1997), no. 1, 75–87. https://doi.org/10.1006/jath.1996.3066.

[19] R. Kerman and L. Pick, *Optimal Sobolev imbedding spaces*, Studia Math. 192 (2009), no. 3, 195–217. https://doi.org/10.4064/sm192-3-1.

[20] R. A. Kerman, *Function spaces continuously paired by operators of convolution type*, Canad. Math. Bull. 22 (1979), no. 4, 499–507. https://doi.org/10.4153/CMB-1979-065-5.

[21] S. Lai, *Weighted norm inequalities for general operators on monotone functions*, Trans. Amer. Math. Soc. 340 (1993), no. 2, 811–836. https://doi.org/10.2307/2154678.

[22] J. G. McWhirter and E. R. Pike, *On the numerical inversion of the Laplace transform and similar Fredholm integral equations of the first kind*, J. Phys. A: Math. Gen. 11 (1978), no. 9, 1792–1745.

[23] M. Milman, *Interpolation of operators of mixed weak-strong type between rearrangement invariant spaces*, Indiana Univ. Math. J. 28 (1979), no. 6, 985–992. https://doi.org/10.1512/iumj.1979.28.28071.

[24] V. Musil, *Fractional maximal operator in Orlicz spaces*, J. Math. Anal. Appl. 474 (2019), no. 1, 94–115. https://doi.org/10.1016/j.jmaa.2019.01.034.

[25] V. Musil and R. Ol’hava, *Interpolation theorem for Marcinkiewicz spaces with applications to Lorentz gamma spaces*, Math. Nachr. 292 (2019), no. 5, 1106–1121. https://doi.org/10.1002/mana.201700452.

[26] R. O’Neil, *Integral transforms and tensor products on Orlicz spaces and $L^p,q$ spaces*, J. Analyse Math. 21 (1968), 1–276.

[27] D. Peša, *Reduction principle for a certain class of kernel-type operators*, Math. Nachr. 293 (2020), no. 4, 761–773. https://doi.org/10.1002/mana.201800510.

[28] L. Pick, et al., *Function spaces. Vol. I*, De Gruyter Series in Nonlinear Analysis and Applications, vol. 14, extended edition, Walter de Gruyter & Co., Berlin, 2013, pp. xvi+479.

[29] R. Sharpley, *Counterexamples for classical operators on Lorentz–Zygmund spaces*, Studia Math. 68 (1986), 141–158.

[30] F. Sukochev, K. Tulenov, and D. Zanin, *The optimal range of the Calderón operator and its applications*, J. Funct. Anal. 277 (2019), no. 10, 3513–3559. https://doi.org/10.1016/j.jfa.2019.05.012.

[31] A. Torchinsky, *The K-functional for rearrangement invariant spaces*, Studia Math. 64 (1979), no. 2, 175–190. https://doi.org/10.4064/sm-64-2-175-190.

How to cite this article: J. Takáč, *Optimality of function spaces for kernel integral operators*, Math. Nachr. 296 (2023), 4429–4453. https://doi.org/10.1002/mana.201900545