Second-Order Radiative Corrections to the Axial Vector Anomaly

Walter Dittrich
Institut für theoretische Physik, Universität Tübingen,
72076 Tübingen, Germany

March 2, 2022

Abstract

We re-examine the historically important decay of the neutral pion into two photons. Schwinger’s Equivalence Theorem is confirmed. We then consider radiative corrections to the famous Adler-Bell-Jackiw (ABJ) anomaly. The result depends crucially on a physically motivated regularization scheme. Our approach is largely based on Schwinger’s source (dispersion) method.

1 Introduction

From time to time people get excited about the question as to whether the classic one-loop triangle ABJ anomaly \cite{1} obtains higher-order loop corrections. If so, what are the consequences of this modified anomaly for the $\pi^0 \rightarrow 2\gamma$ decay mode which is stated to one-loop order in form of a local action term \cite{2}

$$W_{\pi^0\rightarrow 2\gamma} = \frac{g}{m} \left( \frac{\alpha}{\pi} \right) \int d^4x \phi(x) \mathbf{E} \cdot \mathbf{B}(x).$$

It is almost thirty years ago that Lester L. DeRaad, Kim Milton, and Wu-yang Tsai \cite{3} challenged the prevalent view that the ABJ triangle anomaly is an exact statement. That no further corrections are expected is claimed by the so-called Adler-Bardeen theorem \cite{4}. Adler and Bardeen considered ultraviolet regularization and showed that three-particle exchange processes are not divergent and therefore do not contribute to the anomaly. But there is also the two-particle intermediate state with the pseudoscalar form factor to be taken into account. This contribution cannot be regularized in a chiral invariant way and so allows for a certain freedom in choosing a normalization point. Hence the occurrence of an anomaly correction depends crucially on the way the infrared regularization is performed. The anomaly as a short-distance or high-energy phenomenon with cutoff $\Lambda$ has
to be supplemented with the lower end of the momentum scale, \( \mu \). Source theoretical calculations show explicitly that it is indeed not sufficient to merely consider an ultraviolet regularization but an infrared regularization as well. Incidentally, this fact was known a long time ago to the late J. Schwinger who showed in the last chapter of his monograph, Ref. [5], how the Equivalence Theorem on the next dynamical level becomes modified by the replacement \( \frac{\alpha}{\pi} \rightarrow \frac{\alpha}{\pi} (1 + \frac{\alpha^2}{2\pi}) \) in the original one-loop triangle diagram.

More recently corrections to the one-loop chiral anomaly were also discussed by V.I. Zakharov [6] and by the authors in Ref. [7]. The greatest impetus on the matter came, however, from a new approach via the so-called average-effective action which M. Reuter published in Ref. [8]. Although we do not have the time to enter this subject, the reader is invited to consult that paper (especially the appendix) for further details.

2 The Equivalence Theorem, Prehistory

In his seminal work, Schwinger [2] proved the “Equivalence Theorem” which states that in the low-energy regime, a pseudoscalar interaction between a spinless neutral meson and a fermion leads to the same result for the meson decay into two photons as a pseudovector interaction.

For the pseudoscalar interaction between a neutral meson field \( \phi \) and a fermion field \( \psi \), the Lagrangian is simply given by

\[
L_{PS} = -ig \frac{\phi(x)}{2} \left[ \bar{\psi}(x), \gamma_5 \psi(x) \right].
\]  

(1)

In the fifties the fermion was identified with the proton; nowadays, \( \psi \) should be associated with a quark appearing in three colors. In our naive model the emphasis is still on electrodynamics. The only explicit assumption of the \( \psi \) and \( \phi \) particles enters through the restriction \( m \gg m_\pi \).

In order to describe the decay of the pion into two photons, we replace the fermion fields by their vacuum expectation value in the presence of an external electromagnetic field:

\[
L_{PS} \rightarrow L_{PS}^{\text{eff}} = -ig \phi(x) \frac{1}{2} \left[ \bar{\psi}(x), \gamma_5 \psi(x) \right] A = g \phi(x) \text{tr} \gamma_5 G(x, x|A).
\]  

(2)

This equation is diagrammatically represented in Fig. 1.

The propagator \( G \) satisfies the Green’s function equation of a Dirac particle

\[
[m + \gamma^\mu \Pi_\mu] G(x, x'|A) = \delta(x - x'),
\]  

(3)
where $\Pi_\mu = -i\partial_\mu - eA_\mu$. At this point we introduce the proper-time representation for the operator $G[A]$:

$$G[A] = \frac{1}{m + \gamma\Pi} = \frac{m - \gamma\Pi}{m^2 - (\gamma\Pi)^2} = (m - \gamma\Pi)i \int_0^\infty ds \ e^{-is[m^2 - \epsilon - (\gamma\Pi)^2]}.$$  \hspace{1cm} (4)

Inserting this representation into Eq. (2), we obtain

$$L_{PS}^{\text{eff}} = g\phi(x) \text{ tr } \left\{ \gamma_5 \langle x| (m - \gamma\Pi)i \int_0^\infty ds \ e^{-is[m^2 - (\gamma\Pi)^2]} |x\rangle \right\}$$

$$= gm\phi(x) i \text{ tr } \left\{ \gamma_5 \int_0^\infty ds \ e^{-ism^2} \langle x| e^{is(\gamma\Pi)^2} |x\rangle \right\}. \hspace{1cm} (5)$$

Now, while identifying the loop fermion with protons, Schwinger argued that the momentum of the outgoing photons of the pion decay is much smaller than the mass of the proton. Therefore, the electromagnetic fields associated with the photons vary slowly compared to the length scale set by the Compton wavelength of the proton. As a consequence, the constant-field approximation for the proper-time transition amplitude appears to be appropriate in the present situation.

So assuming that a heavy fermion (proton) runs in the loop we simply proceed with the constant-field/low-photon-energy approximation. For this limiting situation Eq. (5) yields

$$L_{PS}^{\text{eff}}(x) = -\frac{1}{4} \frac{\alpha}{\pi m} \phi(x) F_{\mu\nu}^*F^{\mu\nu} = \frac{\alpha}{\pi} \frac{g}{m} \phi(x) \mathbf{E} \cdot \mathbf{B}. \hspace{1cm} (6)$$

This is the famous formula (5.25) in Schwinger’s paper [2] of 1951.

Note that although we included the coupling of the loop-fermion to all orders to the external field, the final result (5) is only of second order in the electromagnetic field strength.
Hence, if we had expanded the loop perturbatively in $\alpha$, then only the graph with two external photons would have contributed to the final result. Note also that we did not encounter any singular terms while calculating $G(x,x|A)$; the dangerous terms vanished by Dirac $\gamma$-algebraic arguments. This is not true for the pseudovector interaction which we treat next. Its Lagrangian is given by

$$L_{PV} = -i\frac{g}{2m} \partial_{\mu} \phi(x) \frac{1}{2i} \left[ \bar{\psi}(x), \gamma_5 \gamma^\mu \psi(x) \right].$$

(7)

Classically, this pseudovector interaction Lagrangian is formally equivalent to the pseudoscalar counterpart as defined in Eq. (1), since

$$L_{PV} = i\frac{g}{2m} \phi(x) \frac{1}{2i} \left[ \partial_{\mu} \bar{\psi}, \gamma_5 \gamma^\mu \psi \right] + \text{surface terms}$$

$$= -ig \phi(x) \frac{1}{2} \left[ \bar{\psi}(x), \gamma_5 \psi(x) \right] + \text{s.t.}$$

(8)

In the second step, we employed the equations of motion $\gamma^\mu \partial_{\mu} \psi = -im \psi$, $\partial_{\mu} \bar{\psi} \gamma^\mu = im \bar{\psi}$.

However, at the quantum level, things become more complicated. Proceeding in the same way as in the pseudoscalar case, we naively arrive at

$$L_{PV}^{\text{eff}} = -i\frac{g}{2m} \partial_{\mu} \phi(x) \frac{1}{2i} \left[ \bar{\psi}(x), \gamma_5 \gamma^\mu \psi(x) \right] \gamma_{\mu} G(x,x|A)$$

$$\xrightarrow{\text{i.b.p.}} i\frac{g}{2m} \phi(x) \partial_{\mu} \text{tr} \gamma_5 \gamma^\mu G(x,x|A) + \text{s.t.}$$

(9)

Now we are in trouble! Not only do we have to face the problem of singularities in $G(x,x|A)$, but we also have to give a meaning to the derivative at this singular coincidence point. Schwinger solved this problem by writing

$$\partial_{\mu} \text{tr} \gamma_5 \gamma^\mu G(x,x|A) \to \lim_{x',x'' \to x} \left\{ \left[ \partial_{\mu} - ie A_{\mu}(x') \right] + \left[ \partial_{\mu} + ie A_{\mu}(x'') \right] \right\} \text{tr} \gamma_5 \gamma^\mu G(x',x''|A).$$

(10)

Now we could follow Schwinger and evaluate the right-hand side of Eq. (10) in the weak-field limit, i.e., up to second order in the field strength. This would again correspond to the triangle graph, which is known to contribute solely to the axial-vector anomaly to any finite order of perturbation theory.

Instead, we will try to maintain the coupling to the external field to all orders in order to pursue the question of possible non-perturbative contributions to the meson-photon interaction. Of course, the price we have to pay is that we are strictly tied to the slowly varying (constant) field approximation.
So, let us employ the representation of the fermionic Green’s function in an arbitrary constant electromagnetic field:

\[ G(x, x'|A) = \Phi(x, x'|A) \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} \left[ m - \frac{1}{2} \gamma^\mu [f(s) + eF_{\mu\nu}(x-x')^\nu] \right] \]

\[ \times \exp \left[ -im^2 s - L(s) + \frac{i}{4}(x-x')f(s)(x-x') \right] \exp \left( i\frac{e}{2} \sigma Fs \right), \tag{11} \]

where

\[ f(s) = eF \coth(eFs), \]

\[ L(s) = \frac{1}{2} \text{tr} \ln \frac{\sinh(eFs)}{eFs} \Rightarrow e^{-L(s)} = \frac{e^{as} \cosh bs}{\sin e^{as} \sinh e^{bs}}, \tag{12} \]

\[ a = \left( \sqrt{F^2 + G^2 + F} \right)^{1/2}, \quad b = \left( \sqrt{F^2 + G^2 - F} \right)^{1/2}, \]

\[ F = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}(B^2 - E^2), \tag{13} \]

\[ G = \frac{1}{4} F_{\mu\nu} \ast F^{\mu\nu} = -E \cdot B, \]

and

\[ \Phi(x, x'|A) = \exp \left[ \int_{x'}^x d\xi \left( A^\mu(\xi) + \frac{1}{2} F^{\mu\nu}(\xi - x')^\nu \right) \right]. \tag{14} \]

completely carries the gauge dependence of the propagator. Having separated the gauge dependence in this way, we may also write

\[ G(x', x''|A) = \Phi(x', x''|A) G(x', x''|A_{SF}), \tag{15} \]

where \( G(x', x''|A_{SF}) \) is the Green’s function evaluated in the Schwinger–Fock gauge and depends only on the field strength: \( A_{SF}^\mu = \frac{1}{2} F^{\mu\nu}(x - x')^\nu. \)

Substituting all these results back into the starting point, i.e., into the effective Lagrangian in Eq. (9), we find

\[ L_{PV}^{\text{eff}} = -\frac{1}{4} \frac{\alpha}{\pi} \frac{g}{m} \phi(x) F_{\mu\nu} F^{\mu\nu} \]

\[ \times \lim_{x', x'' \to x} \left\{ \Phi(x', x''|A) \int_0^\infty ds \frac{d}{ds} \left[ \exp \left( i\frac{e}{2} \sigma Fs \right) \right] \right\}. \tag{16} \]

Comparing this with our result for the pseudoscalar interaction in (6), it is obvious that an equivalence exists between the two different interactions on the quantum level if the
limiting expression in (16) finally reduces to 1 for any kind of constant electromagnetic field. By construction, the proper-time integration has to be performed before the limit \( x', x'' \rightarrow x \) can be taken. E.g., if we interchanged these processes in (16), then we would find a zero result, since \( (d/ds)e^0 = 0 \).

To further study the integral in Eq. (16) we give an explicit expression of the function \( f(s)_{\mu \nu} \):

\[
 f(s)_{\mu \nu} = \frac{1}{a^2 + b^2} \left( a^2 g_{\mu \nu} + F_{\mu \nu}^2 \right) eb \coth eb s + \frac{1}{a^2 + b^2} \left( b^2 g_{\mu \nu} - F_{\mu \nu}^2 \right) ea \cot eas. 
\]  

(17)

For small values of \( s \) we obtain

\[
 f(s)_{\mu \nu} = \frac{1}{s} g_{\mu \nu} + \frac{e^2}{3} s F_{\mu \nu}^2 + O(s^3).
\]  

(18)

and since the weak-field expansion of \( f(s)_{\mu \nu} \) coincides with the small-\( s \) expansion it is here that we can make contact with Schwinger’s original calculation and so produce an effective Lagrangian of a pseudovector interaction between a spinless meson and a heavy fermion field in the presence of a slowly varying and weak external electromagnetic field:

\[
 L_{PV}^{\text{eff}} = -\frac{1}{4} \frac{\alpha}{\pi} \frac{g}{m} \phi(x) F_{\mu \nu} \ast F^{\mu \nu}. 
\]  

(19)

This is identical to the outcome for the pseudoscalar interaction and constitutes the essence of Schwinger’s Equivalence Theorem for the low-energy regime. In this sense, the terminology “low energy” refers to the energy of the outgoing photons (variation of the field strength) as well as the strength of the field.

Now we could go one step further and prove the validity of the Equivalence Theorem without the weak-field assumption. Details of the proof can be found in our monograph [9]. Our result is that Eq. (19) holds for arbitrary electromagnetic field strengths as long as the fields vary slowly compared to the Compton wavelength of the fermionic loop particle.

It has often been emphasized in the original literature [1] that the discovery of the ABJ axial-vector anomaly has its roots in Schwinger’s work. The ABJ anomaly states that the axial-vector current is not conserved, not only because of an explicit breaking of the axial symmetry by a mass term, but also due to the appearance of the \( F_{\mu \nu} \ast F^{\mu \nu} \) term induced by quantum vacuum effects. The celebrated result reads:

\[
 \partial_{\mu} \langle j_{5}^{\mu} \rangle = -2i m \langle j_{5} \rangle + \frac{\alpha}{2\pi} F_{\mu \nu} \ast F^{\mu \nu}.
\]  

(20)

Employing the Equivalence Theorem, we can prove Eq. (20), but only for constant external fields. Hence Schwinger’s work on the constant-field case is only capable of deriving the anomaly in a certain energy regime, namely, the low-energy domain. Within the usual diagrammatic approach, Adler [1] and Zumino [10] arrive at Eq. (20), but this time
without assuming that the electromagnetic field has to be constant. This restriction has also been given up in the source approach for the one-loop anomaly as presented in Ref. [3] and [5].

Let us conclude this chapter with an interesting observation for the constant-field case. Inserting the expression for \( \langle j_5 \rangle^A \) for constant fields,

\[
\langle j_5 \rangle^A = i \text{tr} \gamma_5 G(x, x|A) = -\frac{i}{4} \frac{\alpha}{\pi m} F_{\mu\nu}^* F^{\mu\nu} \quad \Leftrightarrow \quad \mathcal{L}_{\text{eff}}^{PS} = -ig \phi(x) \langle j_5 \rangle^A,
\]

into Eq. (20), we find that the divergence of the axial-vector current vanishes: \( \partial_\mu \langle j_5^\mu \rangle = 0 \).

This result appears to be a bit unfamiliar, because it signals the conservation of the axial-vector current at the quantum level, although this current is not conserved at the classical level due to the breaking of the axial symmetry by the mass term. Therefore, the constant-field case is an exceptional situation which creates an “inverse anomaly”: a classically and explicitly broken symmetry is restored by quantum effects.

Our considerations so far are strictly at the one-loop level. Similarly to the Fujikawa [11] or any other method, photonic fluctuations have not been taken into account. This will be done in the next chapter, in which we want to challenge the correctness of the Adler-Bardeen theorem [4].

3 Radiative Corrections to \( \pi^0 \rightarrow 2\gamma \) Decay

Let us begin by looking again at the lowest-order triangle process. So consider a causal arrangement in which an extended pion source emits a pair of charged fermions that eventually annihilate to produce a pair of photons. The primitive interaction between the
neutral pseudoscalar particle (pion) and spin-$\frac{1}{2}$ fermions is given by
\[ \mathcal{L}^{PS} = g \phi(x) \frac{1}{2} \psi(x) \gamma^0 \gamma_5 \psi(x), \] (22)
so that the total Lagrangian reads
\[ \mathcal{L} = -\frac{1}{2} \psi \gamma^0 \left[ \gamma^\mu \Pi_\mu - g \gamma_5 \phi + m \right] \psi. \]

In the present chapter we have switched to a Majorana representation to make closer contact with the source literature.

The goal is to compute the vacuum persistence amplitude
\[ \langle 0_+ | 0_- \rangle = e^{iW^{PS}_{\pi^0 \rightarrow 2\gamma}} = \cdots + ig \int d^4x \phi(x) \frac{1}{2} \psi(x) \gamma^0 \gamma_5 \psi(x) + \cdots \]
or
\[ W^{PS}_{\pi^0 \rightarrow 2\gamma} = g \int d^4x \phi(x) \frac{1}{2} \psi(x) \gamma^0 \gamma_5 \psi(x). \] (23)

As for Feynman diagrams there are also standard techniques for computing causal diagrams. The result for the vacuum-to-vacuum amplitude corresponding to the causal process indicated in Fig. 2 is given by
\[ \langle 0_+ | 0_- \rangle = i \int d^4x d^4x' dM^2 \frac{\alpha g}{\pi m} \left( -\frac{1}{4} \right) \frac{1}{2} \epsilon_{\kappa\lambda\mu\nu} F^{\kappa\lambda}(x) F^{\mu\nu} \times \Delta_+(x - x'; M^2) \phi(x') \frac{2m^2}{M^2} \ln \frac{1 + \sqrt{1 - 4m^2 M^2}}{1 - \sqrt{1 - 4m^2 M^2}}. \] (24)

This yields for the contribution to the action
\[ W_{\pi^0 \rightarrow 2\gamma} = \frac{\alpha g}{\pi m} \int d^4x d^4x' (E \cdot B) F(x - x') \phi(x'), \] (25)
where the form factor has the momentum version
\[ F(k^2) = \int_{(2m)^2}^{\infty} dM^2 \frac{2m^2}{M^2} \ln \frac{1 + \sqrt{1 - 4m^2 M^2}}{1 - \sqrt{1 - 4m^2 M^2}} \frac{1}{k^2 + M^2 - i\epsilon}. \] (26)

It is normalized at $k^2 = 0$: $F(0) = 1$. In the situation under consideration $F(-m_n^2) = 1 + \frac{1}{12} \left( \frac{m_n}{m} \right)^2$, so that with $\frac{m_n}{m} \simeq \frac{1}{6.7}$ the correction is about 0.2%. Hence for $F(x - x') \simeq \delta(x - x')$
or \( F(0) = 1 \) we again obtain the low-energy result corresponding to a local effective-action term for the pion-photon coupling:

\[
\mathcal{W}_{\pi^0 \rightarrow \gamma \gamma} = \frac{\alpha g}{\pi m} \int d^4x (\mathbf{E} \cdot \mathbf{B})(x) \phi(x),
\]

which is Schwinger’s result from 1951 (with slowly varying fields) and is the anomaly.

Had we treated instead of the pseudoscalar coupling the pseudovector coupling, i.e.,

\[ g\gamma_5 \phi \rightarrow \frac{g}{2m} i \gamma^\mu \gamma_5 \partial_\mu \phi, \]

our calculation would have again resulted in expression (27) – in accordance with the Equivalence Theorem.

In Eq. (26) we met the expression

\[
F(k^2) = \int_{4m^2}^\infty \frac{dM^2}{k^2 + M^2 - i\epsilon} J(M^2),
\]

where

\[
J(M^2) = \frac{2m^2}{M^2} \ln \frac{1 + \sqrt{1 - \frac{4m^2}{M^2}}}{1 - \sqrt{1 - \frac{4m^2}{M^2}}}. \tag{29}
\]

We also found that for the lowest-order result \( F(k^2 = 0) \equiv \tilde{I} = 1 \), and this is intimately related to the anomaly equation.

Now it is time to turn to the two- and three-particle exchange processes which we put side by side with the simple triangle: The radiative correction to the triangle process computed in Ref. [3] is obtained by adding the two- and three-particle exchange contributions so that together with the bare triangle graph we obtain

\[
\tilde{I} = \tilde{I}^{(1)} + \tilde{I}^{(2)} + \tilde{I}^{(3)} = 1 + \frac{\alpha}{2\pi}(1 + \delta), \tag{30}
\]

where \( \delta \) depends on the \( \Gamma_5 \) normalization point.
Miraculously, only $\delta$ is needed; all the other contributions either cancel or yield a very simple finite expression. Hence, it is indeed the on-shell pseudoscalar form factor that matters.

What, then, is the value of the quantity $\delta$? It must have something to do with the normalization of the pseudoscalar form factor $F(P^2)$. Causal analysis of diagram Fig. 5 yields

$$F(P^2) = 1 - \frac{\alpha}{2\pi} P^2 \int_0^1 dv \left(1 + v\right) \frac{\ln \left(\frac{4m^2}{\mu^2}, 1-v^2\right)}{4m^2 + (1-v^2)P^2}. \quad (31)$$

$$F(0) = 1, \quad \mu = \text{photon mass.}$$

Eq. (31) clearly shows that the correction to the simple triangle anomaly is infrared sensitive. To work out the spectral weight function that is involved in Eq. (31), let us
rewrite Eq. (31) slightly:

\[ F(P^2) = 1 + \alpha \int \frac{dM^2}{2\pi} \left( -\frac{P^2}{M^2} \right) \frac{a(M^2)}{P^2 + M^2 - i\epsilon}, \quad F(0) = 1, \quad (32) \]

with

\[ a(M^2) = \frac{(M^2 - 2m^2)}{M^2} \frac{1}{\sqrt{1 - \frac{4m^2}{M^2}}} \ln \frac{M^2 - 4m^2}{\mu^2}. \quad (33) \]

So far our result is expressed as a form factor multiplying the original primitive interaction:

\[ \langle 0_+|0_- \rangle = ig \int d^4x d^4x' \frac{1}{2} \psi(x) \gamma^0 \Gamma_5(x - x') \psi(x') \phi(x'). \quad (34) \]

In momentum space we have

\[ \Gamma_5(P^2) = \gamma_5 (1 + G(P^2)), \quad (35) \]

where \( G(P^2) \) is given by the second term in Eq. (32). Now, since the two-particle exchange contribution (with the form factor \( \Gamma_5 \) and massive photon) cannot be regularized in a chiral invariant way, it would appear that an arbitrary normalization point for \( G(P^2) \) is allowed. Is this permitted or is there a preferred normalization point? Let us start by introducing an arbitrary normalization point \( M_0 \neq 0 \) and write instead of \( -P^2 - M^2_0 \) in Eq. (32) the subtracted form

\[ -\frac{P^2}{M^2} \frac{1}{M^2 + P^2} - \frac{M_0^2}{M^2 - M_0^2}. \quad (36) \]

Then \( G(P^2) \) in Eq. (32) can also be written as

\[ G(P^2) = \alpha \int \frac{dM^2}{2\pi} \left( -\frac{P^2 - M_0^2}{M^2 - M_0^2} \right) \frac{a(M^2)}{M^2 + P^2 - i\epsilon}, \quad (37) \]

or

\[ G(P^2) = \frac{\alpha}{2\pi} \delta + \alpha \int \frac{dM^2}{2\pi} \left( -\frac{P^2}{M^2} \right) \frac{a(M^2)}{M^2 + P^2 - i\epsilon}, \quad (38) \]

where

\[ \delta = -M_0^2 \int \frac{dM^2}{2\pi} \frac{a(M^2)}{M^2 - M_0^2}. \quad (39) \]
With $\mu := \lambda m$ and $a(M^2)$ given by Eq. (33) we have

$$a(M^2) = \ldots \ln \frac{M^2 - 4m^2}{\lambda^2 m^2} = \ldots \left( \ln \frac{M^2 - 4m^2}{m^2} - 2 \ln \lambda \right). \quad (40)$$

If $M_0$ were a finite number, $\delta$ would depend on $\ln \lambda$, which is not acceptable since $\Gamma(\pi^0 \to 2\gamma) = 1/\tau$ would depend on $\ln \lambda$. Hence, $M_0$ has to vanish, and then $\delta = 0$. However, if $M_0 \sim \ln \frac{\mu}{m}$, one can obtain finite $\delta$’s, e.g., $\delta = -1$, which produces no correction. But then we would have to normalize the pseudoscalar form factor at an infrared sensitive point. So we see that the result depends essentially on the way the infrared regularization is performed. For the above reason we consider the choice $\delta = 0$ as the more physically motivated regularization and this leads to the replacement

$$\frac{\alpha}{\pi} \to \frac{\alpha}{\pi} \left( 1 + \frac{\alpha}{2\pi} \right)$$

in Eq. (27):

$$W_{\pi^0 \to 2\gamma} = \frac{g}{m} \frac{\alpha}{\pi} \left( 1 + \frac{\alpha}{2\pi} \right) \int d^4x \phi(x)(E \cdot B)(x). \quad (41)$$

We have seen that there exist two mass scales in the theory, $\mu$ and $\Lambda$, the two ends of a momentum flow. So when renormalizing the theory we have to separate the renormalization constants into infrared- and ultraviolet-sensitive parts. Both ends enter with equal weight into the renormalization prescription. Furthermore, if two renormalization constants have the same singular behavior at one end, $\Lambda \to \infty$ say, then it is certainly not true that they are equal over the whole momentum range. There are jumps on both ends of the momentum flow of the renormalization constants whose difference gives rise to the finite anomaly correction $\frac{\alpha}{\pi}$.

To make contact with the work of Adler and Bardeen [4] we have to study the $\gamma_5$ vertex when the fermions are not on their mass shell. For this reason we write down a double spectral form for the pseudoscalar vertex function. Fig. 6 depicts the causal arrangement of the exchange, in the presence of an external pion field, of a fermion-photon pair between two extended fermion sources.

The source-theoretical calculation yields the result

$$\Gamma_5(p, p') = \gamma_5 \left( 1 - \frac{\alpha}{2\pi} (P^2 + m\gamma P) \int \frac{dM^2 dM'^2}{\sqrt{\Delta}} \frac{1}{(p^2 + M^2)(p'^2 + M'^2)} + \text{c.t.} \right), \quad (42)$$

where

$$\Delta = (P^2 + M^2 + M'^2)^2 - 4M^2M'^2$$
and the region of integration is bounded by

\[-\mu^2 P^4 + P^2 ((M^2 - m^2 + \mu^2)(M'^2 - m^2 + \mu^2) - 2\mu^2(M^2 + M'^2)) \geq m^2(M^2 - M'^2)^2.\]

The contact terms (c.t.) stand for single spectral forms plus local functions, which have to be determined by imposing further physical restrictions such as gauge invariance which is stated in the form of a Ward identity:

\[2m\Gamma_5(p, p) = \{\gamma_5, G^{-1}_+ (p)\}. \quad (43)\]

As a further restriction to finally fix the physical \(\Gamma_5\) one finds that normalization to the on-shell result is necessary. Altogether this brings us to an on-shell version of \(\Gamma_5\):

\[\tilde{\Gamma}_5(p, p') \rightarrow \gamma_5\left(1 - \frac{\alpha}{2\pi}\right) - \gamma_5 \frac{\alpha}{2\pi} P^2 \int_{4m^2}^{\infty} \frac{dM^2}{M^2} \frac{\alpha(M^2)}{M^2 + P^2}, \quad (44)\]

which corresponds to \(G(P^2)\) [Eq. (37)] with

\[\delta = -1, \quad \left(\frac{m}{M_0}\right)^2 = \frac{2}{3} \ln \frac{m}{\mu} + \mathcal{O}(1). \quad (45)\]

This is identical to our former physically unacceptable result which leads to an infrared-divergent anomaly.

If instead we choose \(\delta = 0\) as a physical requirement, we can either retain the naive
Ward identity (43) and add a constant $\gamma_5 \gamma \cdot P$ renormalization,

$$\Gamma_5(p, p') = \tilde{\Gamma}_5(p, p') + \frac{\alpha}{2\pi} \gamma_5 \gamma \cdot P 2m,$$

or, alternatively, we can modify the renormalized Ward identity such that

$$2m \left(1 - \frac{\alpha}{2\pi}\right) \Gamma_5(p, p) = \{\gamma_5, G^{-1}_+(p)\},$$

which corresponds to choosing

$$m_0 Z_2 = \left(1 - \frac{\alpha}{2\pi}\right) m Z_D,$$

where $Z_D$ is the pseudoscalar vertex renormalization constant.

Acknowledgement

I thank H. Gies for useful discussions and carefully reading the manuscript.

References

[1] S.L. Adler, Phys. Rev. 177, 2426 (1969); S. Bell and R. Jackiw, Nouvo Cimento 60A, 47 (1969); R. Jackiw and K. Johnson, Phys. Rev. 182, 1459 (1969).

[2] J. Schwinger, Phys. Rev. 82, 664 (1951).

[3] L.L. DeRaad, Jr., K.A. Milton, and W.-y. Tsai, Phys. Rev. D 6, 1766 (1972); K.A. Milton, W.-y. Tsai, and L.L. DeRaad, Jr., Phys. Rev. D 6, 3491 (1972).

[4] S.L. Adler and W.A. Bardeen, Phys. Rev. 182, 1517 (1969).

[5] J. Schwinger, “Particles, Sources, and Fields” Vol. III, 5-9, Addison-Wesley, (1989); “Anomalies in quantum field theory”, in “Superworld III”, Proceedings of the 26th Course of the International School of Subnuclear Physics, Erice, Italy, 7-15 August 1988 (ed. A. Zichichi), Plenum, New York (1990).

[6] V.I. Zakharov, Phys. Rev. D 42, 1208 (1990).

[7] M.A. Shifman, A.I. Vainshtein, Nucl. Phys. B 365, 312 (1991); A.A. Johansen, Nucl. Phys. B 376, 432 (1992).

[8] M. Reuter, Mod. Phys. Lett. A 12, 2777 (1997).
[9] W. Dittrich, H. Gies, “Probing the Quantum Vacuum”, Springer Tracts in Modern Physics, Vol. 166 (2000).

[10] B. Zumino, in “Proceedings of the Topological Conference on Weak Interactions”, p. 361, CERN, Geneva, (1969).

[11] K. Fujikawa, Phys. Rev. Lett. 42, 1195 (1979); Phys. Rev. D 21, 2848 (1980).