COUPLING LÉVY MEASURES AND COMPARISON PRINCIPLES
FOR VISCOSITY SOLUTIONS

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Abstract. We prove new comparison principles for viscosity solutions of non-linear
integro-differential equations. The operators to which the method applies include but
are not limited to those of Lévy-Itô type. The main idea is to use an optimal transport
map to couple two different Lévy measures, and use the resulting coupling in a doubling
of variables argument.

1. Introduction

In this paper we study comparison principles for viscosity subsolutions and supersolu-
tions of integro-differential equations of the form
\[
I(u, x) = \sup_{\alpha \in A} \inf_{\beta \in B} \{-L_{\alpha\beta}(u, x) + c_{\alpha\beta}(x)u(x) + f_{\alpha\beta}(x)\} = 0 \quad {\text{in}} \ \mathcal{O},
\]
where \(\mathcal{O}\) is a bounded domain of \(\mathbb{R}^d\), \(c_{\alpha\beta}(x) \geq \lambda > 0\), and
\[
L_{\alpha\beta}(u, x) = \int_{\mathbb{R}^d} \left[ u(x + z) - u(x) - \chi_{B_1(0)}(z)Du(x) \cdot z \right]d\mu_{\alpha\beta}^x(z),
\]
where \(\mu_{\alpha\beta}^x\) are the respective Lévy measures. Equations of the form (1.1) arise in sto-
chastic optimal control and stochastic differential games where the operators are the
generators of pure jump processes. In a work by one of the authors and Schwab [12] it
is proved that (roughly speaking) that the class of operators given by a min-max as in
(1.1) is the same as the class of operators satisfying the global comparison property.

Comparison principles for viscosity solutions of such equations are now well under-
stood in two broad cases. The first case is when the operators admit a Lévy-Itô form.
This means that all of the measures \(\mu_{\alpha\beta}^x\) are push-forward measures of a single reference
measure \(\mu\), so that \(\mu_{\alpha\beta}^x = (T_{\alpha\beta}^x)_#\mu\), where \(T_{\alpha\beta}^x : U \to \mathbb{R}^d\) is a family of Borel measurable
maps defined on some separable Hilbert space and \(\mu\) is a Lévy measure on \(U \setminus \{0\}\) (see
(5.15) and (5.17)). First comparison principles were obtained by Soner in [19, 20]. Fur-
ther results, including results for equations with second order PDE terms were obtained
subsequently, see [5, 6, 7, 14]. The second case is that of equations of order less than or
equal to 1. Here we mention the works of Soner [19, 20], and the paper of Sayah [17],
where a comparison principle is proved for very general operators in the class where the
operators \(L_{\alpha\beta}\) are all such that the function \(|z|\) is uniformly integrable with respect to the
measures $\mu_{\alpha\beta}^x$. Also Alvarez and Tourin [2] and Alibaud [1] considered various parabolic equations with non-local terms of order zero, that is with $\mu_x$ of finite mass.

Little is known when the Lévy measures arising in (1.1)-(1.2) are neither integrable with respect to $|z|$ nor of Lévy-Ito form. Two of the authors proved in [16] several comparison results for viscosity solutions which have some regularity. Proving comparison in general is an important question as many operators of interest are not covered by the two situations discussed above, such as the Dirichlet-to-Neumann maps for nonlinear elliptic equations or control/game problems where the processes are not classical Lévy-Ito diffusions.

In this paper we introduce optimal transport techniques in an attempt to understand this question. We obtain a comparison for non-local equations (1.1)-(1.2) that cover the previous two instances without requiring a Lévy-Ito structure nor a restriction on the order of the operators. The idea is to use an optimal coupling for the Lévy measures arising in the non-local terms. Then, the continuity of the Lévy measures with respect to the base point $x$ is estimated with respect to an optimal-transport based metric.

The condition we impose is Lipschitz continuity with respect to an $L^p$-transport metric. The exponent $p \in [1,2]$ is related to the order of the singularity at $z = 0$ for the Lévy measures. In the case of operators of order smaller than 1, it is possible to use the metric corresponding to $p = 1$ in which case our condition is (essentially) a dual formulation of the condition used by Sayah [17]. Likewise, in the Lévy-Ito case our condition reduces to the one typically imposed in the literature [7, 14].

Unfortunately, it is rather difficult to check the Lipschitz regularity of $\mu_x$ with respect to our $L^p$ transport metric when $p > 1$ and $\mu_x$ is not in Lévy-Ito form (this is precisely the case where comparison is still unknown). Such Lipschitz estimates are even non-trivial to check for Lévy measures of finite mass and fail to hold\(^1\). It is our hope that this paper will spur further research that will expand the class of families of measures $\{\mu_{x}^{\alpha\beta}\}_{x,\alpha,\beta}$ where this new approach can be applied.

1.1. The basic idea. Let us illustrate the main idea of the paper in a simple situation. Consider the linear equation

$$\lambda u(x) - L(u,x) = 0 \quad \text{in } O,$$

where $\lambda > 0$ and $L(u,x)$ is an operator of the form (1.2) where we make the following simplifying assumption on the Lévy measures $\mu_x$: $\mu_x$ is a probability measure with finite second moments for every $x$, and there is some $C > 0$ such that for any $x, y$

$$d_2(\mu_x, \mu_y) \leq C|x - y|.$$  

Here $d_2$ denotes the optimal transport distance with respect to the square distance (the so called Wasserstein distance). Suppose that $u$ is a bounded viscosity subsolution of (1.3) and $v$ is a bounded viscosity supersolution of (1.3) such that $u \leq v$ on $\mathbb{R}^d \setminus O^c$. We start with the typical comparison proof. We assume that $u \not\leq v$. We double the variables

\(^1\)The authors would like to thank Alessio Figalli for helpful comments regarding this question.
and penalize the doubling considering for \( \varepsilon > 0 \) the function

\[
u(x) - v(y) - \frac{1}{\varepsilon}|x - y|^2.
\]

Suppose that for all sufficiently small \( \varepsilon \) the global maximum is attained at \((x_\varepsilon, y_\varepsilon)\) where \( u(x_\varepsilon) - v(y_\varepsilon) \geq \ell > 0 \) for some \( \ell > 0 \). In such circumstances it is well known that

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2 = 0,
\]

so for small \( \varepsilon \) we must have \((x_\varepsilon, y_\varepsilon) \in \mathcal{O} \times \mathcal{O}\). Because of the global maximum, we have

\[
\left(u(x_\varepsilon + x) - u(x_\varepsilon) - \frac{2x \cdot (x_\varepsilon - y_\varepsilon)}{\varepsilon}\right) - \left(v(y_\varepsilon + y) - v(y_\varepsilon) - \frac{2y \cdot (x_\varepsilon - y_\varepsilon)}{\varepsilon}\right) \leq \frac{1}{\varepsilon}|x - y|^2, \ \forall \ x, y.
\]

For \( x \) and \( y \) let \( \pi_{x,y} \) denote a probability measure on \( \mathbb{R}^d \times \mathbb{R}^d \) with marginals \( \mu_x \) and \( \mu_y \) achieving the optimal (quadratic) transport cost between them. Then, we integrate the above inequality with respect to the measure \( \pi_{x_\varepsilon, y_\varepsilon} \) to obtain

\[
L(u, x_\varepsilon) - L(v, y_\varepsilon) \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\pi_{x_\varepsilon, y_\varepsilon}(x, y).
\]

Thus by the definition of viscosity solution and the definition of \( \pi_{x,y} \) we get

\[
\lambda(u(x_\varepsilon) - v(y_\varepsilon)) \leq \frac{1}{\varepsilon} \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2 \, d\pi_{x_\varepsilon, y_\varepsilon}(x, y) = \frac{1}{\varepsilon} d_2(\mu_{x_\varepsilon}, \mu_{y_\varepsilon})^2;
\]

where the last equality follows from the optimality of \( \pi_{x_\varepsilon, y_\varepsilon} \). Then, using (1.4), we obtain

\[
0 < \lambda \ell \leq \lambda(u(x_\varepsilon) - v(y_\varepsilon)) \leq C' \frac{1}{\varepsilon}|x_\varepsilon - y_\varepsilon|^2.
\]

Since the right hand side goes to zero as \( \varepsilon \to 0 \) we obtain a contradiction. Thus in this model case the proof of comparison reduces to checking if the measures \( \mu_x \) satisfy the Lipschitz condition (1.4) with respect of the (quadratic) optimal transport distance.

Of course, it is atypical for a Lévy measure to also be a probability or even a finite measure of constant total mass. To deal with this issue, we will make use of an optimal transport problem featuring “an infinite mass reservoir” at 0 (after all, the mass of the Lévy measure at 0 is immaterial). This means in particular that one can consider transport between measures which may have unequal or infinite masses. This problem was studied by Figalli and Gigli in [11], motivated by questions of gradient flows with Dirichlet boundary conditions, and their work is aptly suited for our purposes.

1.2. Outline of the paper. The notation and definitions are explained in Section 2. The transport metric is explained in Section 3. Section 4 contains the assumptions and the statement of the main result. In Section 5 we prove the main comparison principle using the above technique. We then show how the result covers comparison principles for non-local equations involving non-local terms either of Lévy form of order \( \sigma < 1 \) (see Example 5.12) or of Lévy-Itô form (see Example 5.13). Finally in Section 6 we derive
various comparison principles for equations which have more regular viscosity solutions. They lead to uniqueness of viscosity solutions for a class of uniformly elliptic non-local equations (see Example 6.6).

2. Notation and definitions

In the whole paper we will consider equation (1.1) where the operators $L^{\alpha\beta}$ are assumed to be of the form (1.2) and $\{\mu_x^{\alpha\beta}\}_{x,\alpha,\beta}$ is a family of Lévy measures (see Definition 2.1). Denoting

$$\delta u(x, z) := u(x + z) - u(x) - \chi_{B_1(0)}(z) Du(x) \cdot z,$$

we will write

$$L^{\alpha\beta}(u, x) = \int_{\mathbb{R}^d} \delta u(x, z) d\mu_x^{\alpha\beta}(z). \quad (2.1)$$

We will denote by $B_r$ the open ball in $\mathbb{R}^d$ centered at 0 with radius $r > 0$. Given an open set $\Omega \subset \mathbb{R}^d$ and $h > 0$ we define

$$\Omega_h = \{x \in \Omega \mid d(x, \partial \Omega) > h\}.$$

For a subset $A \subset \mathbb{R}^d$ we denote by $A^c$ its complement, i.e. $A^c = \mathbb{R}^d \setminus A$.

For $0 < \alpha \leq 1$ and a domain $\mathcal{O}$ in $\mathbb{R}^d$, we denote by $C^{0,\alpha}(\mathcal{O})$ the space of $\alpha$-Hölder continuous functions in $\mathcal{O}$.

We write $C^k(\mathcal{O})$, $k = 1, 2, ..., \text{for the usual spaces of k-times continuously differentiable functions in } \mathcal{O}$. For $1 < p < 2$ we denote by $C^p(\mathcal{O})$ the space of functions in $C^1(\mathcal{O})$ whose partial derivatives are in $C^{0,p-1}(\mathcal{O})$ and by $C^{1,1}(\mathcal{O})$ the space of functions in $C^1(\mathcal{O})$ whose partial derivatives are in $C^{0,1}(\mathcal{O})$. The space $C^k_b(\mathcal{O})$ (respectively, $C^p_b(\mathcal{O})$) consists of functions in $C^k(\mathcal{O})$ (respectively, $C^p(\mathcal{O})$) which are bounded. We write $\text{BUC}(\mathbb{R}^d)$ for the set of bounded and uniformly continuous functions in $\mathbb{R}^d$. For two bounded measures $\mu, \nu$, we will write $dTV(\mu, \nu)$ to denote the total variation of $\mu - \nu$.

Let $\mu$ be a Borel measure in $\mathbb{R}^d \setminus \{0\}$ and $1 \leq p \leq 2$. We define

$$\mathcal{N}_p(\mu) := \int_{\mathbb{R}^d \setminus \{0\}} \min\{1, |z|^p\} d\mu(z). \quad (2.2)$$

**Definition 2.1.** Let $1 \leq p \leq 2$. We define

$$\mathbb{L}_p(\mathbb{R}^d) := \left\{ \mu \text{ positive Borel measure in } \mathbb{R}^d \setminus \{0\} \mid \mathcal{N}_p(\mu) < \infty \right\}.$$

The set of all Lévy measures is $\mathbb{L}_2(\mathbb{R}^d)$. If $\Omega$ is an open subset of $\mathbb{R}^d$, we will consider the set

$$\mathbb{L}_p(\Omega) := \left\{ \mu \in \mathbb{L}_p(\mathbb{R}^d) \mid \text{spt}(\mu) \subset \Omega \right\}.$$

Note that

$$\mathbb{L}_p(\Omega) \subset \mathbb{L}_q(\Omega), \text{ whenever } p \leq q.$$
In other words, measures $\mu$ in $L_p(\Omega)$ are measures in $L_p(\mathbb{R}^d)$ such that $\mu(\mathbb{R}^d \setminus \Omega) = 0$. We decompose every measure $\mu \in L_p(\mathbb{R}^d)$ as

$$\mu = \hat{\mu} + \check{\mu},$$

(2.3)

where $\hat{\mu}(\cdot) := \mu(\cdot \cap B_1) \in L_p(B_1)$ and $\check{\mu}(\cdot) := \mu(\cdot \cap (\mathbb{R}^d \setminus B_1))$. We note that $\check{\mu}$ is a bounded measure.

Consider the Lévy operator given by some measure $\mu$,

$$L(u, x) = \int_{\mathbb{R}^d} [u(x + z) - u(x) - \chi_{B_1(0)}(z)Du(x) \cdot z] d\mu(z).$$

(2.4)

Let us decompose this operator as the sum of two operators, corresponding to the Lévy measure decomposition in (2.3),

$$L(u, x) = \hat{L}(u, x) + \check{L}(u, x),$$

where

$$\hat{L}(u, x) = \int_{\mathbb{R}^d} [u(x + z) - u(x) - \chi_{B_1(0)}(z)Du(x) \cdot z] d\hat{\mu}(z),$$

(2.5)

$$\check{L}(u, x) = \int_{\mathbb{R}^d} [u(x + z) - u(x)] d\check{\mu}(z).$$

(2.6)

**Definition 2.2.** Given a Lévy measure $\mu$ in $\mathbb{R}^d$, we define

$$L_\mu(u, x) := \int_{\mathbb{R}^d} [u(x + z) - u(x) - \chi_{B_1(0)}(z)Du(x) \cdot z] d\mu(z).$$

**Definition 2.3.** For $p \in (1, 2)$, a function $u$ is said to be pointwise-$C^p$ at a point $x_0$ if $u$ is differentiable at $x_0$ and if there exists a constant $C > 0$ such that for all $x$ in a neighborhood of $x_0$,

$$|u(x) - u(x_0) - Du(x_0) \cdot (x - x_0)| \leq C|x - x_0|^p.$$  

(2.7)

If $u$ is differentiable at $x_0$ and (2.7) is satisfied with $p = 2$ we say that $u$ is pointwise-$C^{1,1}$ at $x_0$. For $p \in (0, 1]$, a function is said to be pointwise-$C^p$ at a point $x_0$ if there is a constant $C > 0$ such that for all $x$ in a neighborhood of $x_0$,

$$|u(x) - u(x_0)| \leq C|x - x_0|^p.$$  

3. **A transportation metric for Lévy measures**

We will use a transportation metric on the space of Lévy measures. This metric takes advantage of an “infinite reservoir” of mass which allows one to handle measures which may not have equal (or finite) total mass. Such a metric was considered by Figalli and Gigli [11], where they studied the basic properties of such a metric, and used it to analyze gradient flows with Dirichlet boundary conditions. Our presentation here generally follows that of [11]. This is not the only possible extension of the transport metric to the case of unequal masses, other notions have been considered by Kantorovich and Rubinstein. Another notion of distance for Lévy measures is considered in [13].
We consider the following set of measures
\[ \mathcal{M}_p(\mathbb{R}^d) := \{ \mu \in L^p(\mathbb{R}^d) \mid \int_{\mathbb{R}^d \setminus \{0\}} |z|^p d\mu(z) < \infty \}. \]
That is, the set of Lévy measures with finite $p$-moment. Note in particular that $L^p(B_1) \subset \mathcal{M}_p(\mathbb{R}^d)$ due to the measures being supported in $B_1$.

First we define the notion of admissible couplings between Lévy measures (see also Definition A.3 for an analogous definition in a more general setting).

**Definition 3.1.** Let $\mu_1, \mu_2 \in \mathcal{M}_p(\mathbb{R}^d)$. An admissible transport plan between $\mu_1$ and $\mu_2$ is any positive Borel measure on $\mathbb{R}^d \times \mathbb{R}^d$ such that
\[ \pi_1^\# \gamma |_{\mathbb{R}^d \setminus \{0\}} = \mu_1, \quad \pi_2^\# \gamma |_{\mathbb{R}^d \setminus \{0\}} = \mu_2, \]
where for $i = 1, 2$, $\pi^i : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ is defined by $\pi^i(x_1, x_2) = x_i$. The set of admissible transport plans will be denoted by $\text{Adm}(\mu_1, \mu_2)$.

In particular, if $\gamma$ is admissible then for a Borel set $A$ compactly supported in $\mathbb{R}^d \setminus \{0\}$,
\[ \gamma(A \times \mathbb{R}^d) = \mu_1(A), \quad \gamma(\mathbb{R}^d \times A) = \mu_2(A). \]

The key point in Definition 3.1 which distinguishes it from the notion of optimal transport plans is that the marginals of $\gamma$ only coincide with $\mu_1$ and $\mu_2$ away from the origin. In particular, the marginals of $\gamma$ may assign any amount of mass to the origin.

**Definition 3.2.** Let $1 \leq p \leq 2$. For a positive Borel measure $\gamma$ on $\mathbb{R}^d \times \mathbb{R}^d$, we define
\[ J_p(\gamma) := \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\gamma(x, y). \]

In the Appendix we study the problem of minimizing $J_p(\gamma)$ over $\gamma \in \text{Adm}(\mu_1, \mu_2)$ in greater generality. In this section we limit ourselves to stating a few further definitions and a few results needed in latter sections.

**Definition 3.3.** Let $1 \leq p \leq 2$. The $p$-distance between measures $\mu, \nu \in \mathcal{M}_p(\mathbb{R}^d)$ is defined by
\[ d_{L^p}(\mu, \nu) := \left( \inf_{\gamma \in \text{Adm}(\mu, \nu)} J_p(\gamma) \right)^{\frac{1}{p}}. \]

The optimization problem used in the definition of $d_{L^p}(\mu_1, \mu_2)$ shares many properties with the usual optimal transportation problem.

**Theorem 3.4.** For $\mu_1, \mu_2 \in \mathcal{M}_p(\mathbb{R}^d)$ there is at least one $\gamma \in \text{Adm}(\mu_1, \mu_2)$ that achieves the minimum value of $J_p$.

**Proof.** The theorem is a special case of Theorem A.5 (see the Appendix). \qed

The fact that $d_{L^2}$ defines a distance was proved in [11, Theorem 2.2, Proposition 2.7]. We will need this result for any $p$.

**Theorem 3.5.** $d_{L^p}$ defines a metric in $\mathcal{M}_p(\mathbb{R}^d)$. 

Lemma 3.6. For $1 \leq p \leq 2$ and $\mu_1, \mu_2 \in \mathcal{M}_p(\mathbb{R}^d)$, we have

$$d_{L_p}(\mu_1, \mu_2)^p = \sup \left\{ \int_{\mathbb{R}^d} \phi(x) \, d\mu_1(x) + \int_{\mathbb{R}^d} \psi(y) \, d\mu_2(y) \mid (\phi, \psi) \in \text{Adm}^p \right\}.$$ 

Here, $\text{Adm}^p$ denotes the set

$$\text{Adm}^p := \left\{ (\phi, \psi) \mid \phi, \psi \text{ are continuous in } \mathbb{R}^d, \phi(0) = \psi(0) = 0 \text{ and } \phi(x) + \psi(y) \leq |x - y|^p \forall x, y \in \mathbb{R}^d \right\}.$$

Proof. The lemma is a special case of Lemma A.11. □

Remark 3.7. In most of the paper we only need to take $d_{L_p}(\mu, \nu)$ for $\mu, \nu \in L_p(B_1)$. In this case we could equivalently define the distance by considering the transport problem in $\mathbb{R}^d$, i.e. taking $\Omega = B_1$ instead of $\Omega = \mathbb{R}^d$ (see the Appendix). We note that if $\mu, \nu \in L_p(B_1)$ and $\gamma \in \text{Adm}(\mu, \nu)$ then $\gamma((B_1 \times B_1)^c) = 0$.

The following proposition (proved in the Appendix) will be used in Section 5.

Proposition 3.8. Let $\psi$ be a Lipschitz continuous function with compact support in $\overline{B}_1(0) \setminus \{0\}$. If $\mu, \nu \in L_p(B_1)$ then

$$\left| \int_{B_1} \psi \, d\mu - \int_{B_1} \psi \, d\nu \right| \leq (\mu(\text{spt}(\psi)) + \nu(\text{spt}(\psi)))^{\frac{p-1}{p}} [\psi]_{\text{Lip}} d_{L_p}(\mu, \nu),$$

where $[\psi]_{\text{Lip}}$ is the Lipschitz constant of $\psi$.

4. Assumptions and Main Results

In this section we make the necessary assumptions about the measures and various functions appearing in the operator $I(u, x)$ in (1.1). We recall that throughout the whole paper $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain. The measures $\mu_{x}^{\alpha \beta} \in L_p(\mathbb{R}^n)$ for all $x \in \mathcal{O}, \alpha \in \mathcal{A}, \beta \in \mathcal{B}$ for some index sets $\mathcal{A}, \mathcal{B}$.

Assumption A. There are $p \in [1, 2]$ and a constant $C \geq 0$ such that

$$d_{L_p}(\hat{\mu}_{x}^{\alpha \beta}, \hat{\mu}_{y}^{\alpha \beta}) \leq C|x - y|, \quad \forall x, y \in \mathcal{O}, \forall \alpha, \beta.$$ (4.1)

Assumption B. There is a modulus of continuity $\theta$ such that

$$d_{TV}(\hat{\mu}_{x}^{\alpha \beta}, \hat{\mu}_{y}^{\alpha \beta}) \leq \theta(|x - y|), \quad \forall x, y \in \mathcal{O}, \forall \alpha, \beta.$$ (4.2)

Assumption C. There are a modulus of continuity $\theta$ and a constant $C \geq 0$ such that

$$|f_{\alpha \beta}(x) - f_{\alpha \beta}(y)| \leq \theta(|x - y|), \quad \forall x, y \in \mathcal{O}, \forall \alpha, \beta,$$ (4.3)

$$|f_{\alpha \beta}(x)| \leq C, \quad \forall x \in \mathcal{O}, \forall \alpha, \beta.$$ (4.4)

Assumption D. There are constants $0 < \lambda \leq \lambda_1$ such that

$$\lambda \leq \inf_{x \in \mathcal{O}} \inf_{\alpha, \beta} c_{\alpha \beta}(x) \leq \sup_{x \in \mathcal{O}} \sup_{\alpha, \beta} c_{\alpha \beta}(x) \leq \lambda_1$$ (4.5)
and there is a modulus $\theta$ such that
\[ |c_{\alpha\beta}(x) - c_{\alpha\beta}(y)| \leq \theta(|x - y|) \quad \forall x \in \mathcal{O}, \forall \alpha, \beta. \]

**Assumption E.** Let $p$ be from Assumption A. There exist a modulus of continuity $\theta$ and a constant $\Lambda \geq 0$ such that
\[ \sup_{x \in \mathcal{O}} \sup_{\alpha, \beta} \int_{B_r} |z|^p \, d\mu^\alpha_x(z) \leq \theta(r), \quad (4.6) \]
\[ \sup_{x \in \mathcal{O}} \sup_{\alpha, \beta} \mathcal{N}_p(\mu^\alpha_x) \leq \Lambda. \quad (4.7) \]

Assumption B can be weakened, however we want to keep its simpler form to focus on the main difficulty of dealing with the singular part of the Lévy measures. We leave such generalizations to the interested reader.

We recall two definitions of viscosity solutions of (1.1) which will be used in this paper. To minimize the technicalities we will assume that viscosity sub/supersolutions are in $\text{BUC}(\mathbb{R}^d)$. The same results could be obtained assuming that they are just bounded and continuous in $\mathbb{R}^d$.

**Definition 4.1.** Let $p \in [1, 2]$. A function $u \in \text{BUC}(\mathbb{R}^d)$ is a viscosity subsolution of (1.1) if whenever $u - \varphi$ has a global maximum over $\mathbb{R}^d$ at $x \in \mathcal{O}$ for some $\varphi \in C^2_{\alpha\beta}(\mathbb{R}^d)$ and $\varphi(x) = u(x)$, then $I(\varphi, x) \leq 0$. A function $u \in \text{BUC}(\mathbb{R}^d)$ is a viscosity supersolution of (1.1) if whenever $u - \varphi$ has a global minimum over $\mathbb{R}^d$ at $x \in \mathcal{O}$ for some $\varphi \in C^2_{\alpha\beta}(\mathbb{R}^d)$ and $\varphi(x) = u(x)$, then $I(\varphi, x) \geq 0$. A function $u$ is a viscosity solution of (1.1) if it is both a viscosity subsolution and viscosity supersolution of (1.1).

**Definition 4.2.** Let $p \in [1, 2]$. A function $u \in \text{BUC}(\mathbb{R}^d)$ is a viscosity subsolution of (1.1) if whenever $u - \varphi$ has a global maximum over $\mathbb{R}^d$ at $x \in \mathcal{O}$ for some $\varphi \in C^2(\mathbb{R}^d)$, then for every $0 < \delta < 1$
\[
\sup_{\alpha \in A} \inf_{\beta \in \mathbb{R}} \left\{ - \int_{|z|<\delta} \delta \varphi(x, z) d\mu^\alpha_x(z) \right. \\
- \int_{|z|>\delta} [u(x + z) - u(x) - \chi_{B_1(0)}(z)D\varphi(x) \cdot z] d\mu^\alpha_x(z) + c_{\alpha\beta}(x)u(x) + f_{\alpha\beta}(x) \left. \right\} \leq 0.
\]

A function $u \in \text{BUC}(\mathbb{R}^d)$ is a viscosity supersolution of (1.1) if whenever $u - \varphi$ has a global minimum over $\mathbb{R}^d$ at $x \in \mathcal{O}$ for some $\varphi \in C^2(\mathbb{R}^d)$, then for every $0 < \delta < 1$
\[
\sup_{\alpha \in A} \inf_{\beta \in \mathbb{R}} \left\{ - \int_{|z|<\delta} \delta \varphi(x, z) d\mu^\alpha_x(z) \right. \\
- \int_{|z|>\delta} [u(x + z) - u(x) - \chi_{B_1(0)}(z)D\varphi(x) \cdot z] d\mu^\alpha_x(z) + c_{\alpha\beta}(x)u(x) + f_{\alpha\beta}(x) \left. \right\} \geq 0.
\]

A function $u$ is a viscosity solution of (1.1) if it is both a viscosity subsolution and viscosity supersolution of (1.1).
We remark that, since the Lévy measures $\mu^{\alpha \beta}_{x}$ are in $L^p(\mathbb{R}^d)$, we could use test functions in $C^p_b(\mathbb{R}^d)$ and $C^p(\mathbb{R}^d)$ instead of test functions in $C^2_b(\mathbb{R}^d)$ and $C^2(\mathbb{R}^d)$. However it is not clear if such definitions and the standard definitions provided above are equivalent under general assumptions. It is easy to see however that they are equivalent for the most common measures considered in Example 5.12.

Below we show that Definitions 4.1 and 4.2 are equivalent to each other.

**Proposition 4.3.** Under the assumptions of this paper Definitions 4.1 and 4.2 are equivalent.

**Proof.** We only consider the case of subsolutions. It is obvious that if $u$ is a viscosity subsolution in the sense of Definition 4.2 then it is a viscosity subsolution in the sense of Definition 4.1. Let now $u$ be a viscosity subsolution in the sense of Definition 4.1. It is easy to see that without loss of generality all maxima/minima in both definitions can be assumed to be strict. So let $u - \varphi$ have a strict global maximum over $\mathbb{R}^d$ at $x \in O$ for some $\varphi \in C^2_b(\mathbb{R}^d)$ and we can obviously require that $\varphi(x) = u(x)$. Let $\varphi_n \in C^2_b(\mathbb{R}^d)$ be functions such that $u \leq \varphi_n \leq \varphi$ on $\mathbb{R}^d$, $\varphi_n(x) = u(x), D\varphi_n(x) = D\varphi(x)$ and $\varphi_n \to u$ as $n \to +\infty$ uniformly on $\mathbb{R}^d$. Then

\[
\sup_{\alpha \in A} \inf_{\beta \in B} \left\{ -\int_{|z|<\delta} \delta \varphi(x,z) d\mu^{\alpha \beta}_{x}(z) \right. \\
- \left. \int_{|z|\geq\delta} \left[u(x+z) - u(x) - \chi_{B_1(0)}(z) D\varphi(x) \cdot z\right] d\mu^{\alpha \beta}_{x}(z) + c_{\alpha \beta}(x) u(x) + f_{\alpha \beta}(x) \right\}
\]

\[
= \lim_{n \to +\infty} \sup_{\alpha \in A} \inf_{\beta \in B} \left\{ -\int_{|z|<\delta} \delta \varphi(x,z) d\mu^{\alpha \beta}_{x}(z) \right. \\
- \left. \int_{|z|\geq\delta} \left[\varphi_n(x+z) - \varphi_n(x) - \chi_{B_1(0)}(z) D\varphi_n(x) \cdot z\right] d\mu^{\alpha \beta}_{x}(z) + c_{\alpha \beta}(x) \varphi_n(x) + f_{\alpha \beta}(x) \right\}
\]

\[
\leq I(\varphi_n, x) \leq 0.
\]

The main result of the paper is the following theorem.

**Theorem 4.4.** Let Assumptions A-E hold for $p \in [1, 2]$. Then the comparison principle holds for equation (1.1). That is, if $u$ and $v$ are respectively a viscosity subsolution and a viscosity supersolution of (1.1) and $u(x) \leq v(x)$ for all $x \notin O$, then

\[ u(x) \leq v(x) \quad \forall \ x \in O. \]

The following is a special case of Theorem 4.4, which we highlight to illustrate its scope.

**Corollary 4.5.** Let Assumptions C and D be satisfied. Suppose that the measures $\mu^{\alpha \beta}_{x}(z)$ are of the form

\[ d\mu^{\alpha \beta}_{x}(z) = K_{\alpha \beta}(x,z) dz \]
and that, for some $\sigma \in (0, 1)$,

$$0 \leq K_{\alpha \beta}(x, z) \leq K(z) := \Lambda_1 |z|^{-(d+\sigma)},$$

$$|K_{\alpha \beta}(x, z) - K_{\alpha \beta}(y, z)| \leq C|x - y|K(z).$$

Then, in this case Assumption B holds and Assumptions A and E hold with $p = 1$. In particular, the comparison principle holds for equation (1.1) in this case.

Theorem 4.4 and Corollary 4.5 will be proved in the next section.

5. Comparison Principle

In this section we prove Theorem 4.4. A well known property of sup/inf- convolutions is that they produce approximations of viscosity sub- and supersolutions which enjoy one-sided regularity (semi-convexity and semi-concavity), which makes it easier - under the right circumstances - to evaluate the operator $I(\cdot, x)$ in the classical sense.

Remark 5.1. An approach to Theorem 4.4 that does not rely on such approximations can be found in Section 6, where we prove a comparison result (Theorem 6.4) under a different set of assumptions that are not amicable to such approximations. A posteriori, it became clear that the approach in Section 6 leads to a simpler proof of Theorem 4.4, however we have decided to keep both approaches as the tools developed in this section are of interest in many other situations. See Remark 6.2 for further comments.

Definition 5.2. Given $u, v \in \text{BUC}(\mathbb{R}^d)$ and $0 < \delta < 1$ we define the sup-convolution $u^\delta$ of $u$ and the inf-convolution $v^\delta$ of $v$ by

$$u^\delta(x) = \sup_{y \in \mathbb{R}^d} \left\{ u(y) - \frac{1}{\delta} |x - y|^2 \right\},$$

$$v^\delta(x) = \inf_{y \in \mathbb{R}^d} \left\{ v(y) + \frac{1}{\delta} |x - y|^2 \right\}.$$

For the reader’s convenience, we review some well known properties of the sup/inf-convolutions in the following proposition.

Proposition 5.3. The sup-convolutions and the inf-convolutions have the following properties.

1. If $\delta_1 \leq \delta_2$ then $u^{\delta_1} \leq u^{\delta_2}$ and $v^{\delta_1} \geq v^{\delta_2}$. Moreover $\|u^\delta\|_\infty \leq \|u\|_\infty$, $\|v^\delta\|_\infty \leq \|v\|_\infty$.
2. $u^\delta(x) \geq u(x)$ and $v^\delta(x) \leq v(x)$ for all $x \in \mathbb{R}^d$.
3. $u^\delta \to u$ and $v^\delta \to v$ uniformly on $\mathbb{R}^d$ as $\delta \to 0$.
4. The function $u^\delta$ is semi-convex and for any $x_0 \in \mathbb{R}^d$, $u^\delta$ is touched from below at $x_0$ by a function of the form

$$u(x_0^*) - \frac{1}{\delta} |x - x_0^*|^2,$$

for some $x_0^* \in \mathbb{R}^d$.

The function $v^\delta$ is semi-concave and for any $x_0 \in \mathbb{R}^d$, $v^\delta$ is touched from above at $x_0$ by a function of the form

$$v(x_0^*) + \frac{1}{\delta} |x - x_0^*|^2,$$

for some $x_0^* \in \mathbb{R}^d$. 
(5) Let \( \omega \) be a modulus of continuity of \( u \). For any \( x_0 \in \mathbb{R}^d \) and \( x_0^* \in \mathbb{R}^d \) such that
\[
u (\delta) = u(x_0^*) - \frac{1}{\delta} |x_0 - x_0^*|^2,
\]
we have
\[
|x_0 - x_0^*| \leq (2\delta \|u\|_\infty)^{1/2}.
\]
and
\[
\frac{1}{\delta} |x_0 - x_0^*|^2 \leq \omega((2\delta \|u\|_\infty)^{1/2}).
\]

The analogous property holds for \( \nu_\delta \).

(6) Let \( \Omega \subset \mathbb{R}^d \) and let \( h > 0 \). If \( \omega \) is a modulus of continuity for \( u \) in \( \Omega \) then for sufficiently small \( \delta \), \( \omega (s) = \max (\omega (s), \frac{2}{h} \|u\|_\infty s) \) is a modulus of continuity for \( \nu (\delta) \) in \( \Omega_{2h} \). Similar property holds for \( \nu_\delta \).

Proof. To prove (1), note that if \( \delta_1 \leq \delta_2 \) then \( \frac{1}{\delta_1} |x - y|^2 \geq \frac{1}{\delta_2} |x - y|^2 \) for all \( x \) and \( y \), and thus \( \nu^{\delta_1} (x) \leq \nu^{\delta_2} (x) \) for all \( x \). The respective statement for \( \nu_\delta \) and \( \nu_{\delta_2} \) is proved in the same way. Property (2) is obvious from the definitions. Property (3) follows from (2) and (5).

Regarding (4) we note that the semi-convexity follows from the fact that \( \nu (\delta) + \frac{1}{\delta} |x|^2 \) is the supremum of affine functions and is hence convex. If we fix \( x_0 \) and if \( x_0^* \) is such that
\[
u (\delta) = u(x_0^*) - \frac{1}{\delta} |x_0 - x_0^*|^2,
\]
then for all other \( x \) we have \( \nu (x) \geq P(x) := u(x_0^*) - \frac{1}{\delta} |x - x_0^*|^2 \) by the definition of \( \nu (\delta) \), so \( P \) is the desired paraboloid. To prove (5), let \( x_0 \) and \( x_0^* \) be as above. Then
\[
\frac{1}{\delta} |x_0 - x_0^*|^2 = u(x_0^*) - u(x_0) \leq u(x_0^*) - u(x_0) \leq 2 \|u\|_\infty
\]
so
\[
|x_0 - x_0^*| \leq (2\delta \|u\|_\infty)^{1/2}.
\]
This means that \( u(x_0^*) - u(x_0) \) is in fact bounded from above by \( \omega((2\delta \|u\|_\infty)^{1/2}) \) which gives (5).

Finally to show (6) we observe that if \( x, y \in \Omega_{2h} \) and \( \nu (\delta) = u(x^*) - \frac{1}{\delta} |x - x^*|^2 \) then for small \( \delta \), \( x, y \in \Omega_h \). Now if \( |y - x| < h \), we have \( u^{\delta} (y) \geq u(x^* + y - x) - \frac{1}{\delta} |x - x^*|^2 \) so
\[
u^{\delta} (x) - 
u^{\delta} (y) \leq u(x^*) - u(x^* + y - x) \leq \omega (|x - y|)
\]
If \( |y - x| \geq h \) then obviously \( \nu^{\delta} (x) - \nu^{\delta} (y) \leq \frac{2}{\delta} \|u\|_\infty |y - x| \).

\[\Box\]

**Definition 5.4.** Given \( y \in \mathcal{O} \), and the operator \( I(\cdot, x) \) from (1.1), we define
\[
I^{(y)} (\phi, x) = \sup_{\alpha} \sup_{\beta} \inf \left\{ -L_{\mu_y}^{\alpha\beta} (\phi, x) + c_{\alpha\beta} (y) \phi (x) + f_{\alpha\beta} (y) \right\}, \tag{5.1}
\]
where
\[
L_{\mu_y}^{\alpha\beta} (\phi, x) = \int_{\mathbb{R}^d} [\phi(x + z) - \phi(x) - \chi_{B^1(0)} D\phi(x) \cdot z] \, d\mu_y^{\alpha\beta}(z).
\]
Note that this last expression is almost identical to $L^{\alpha\beta}(\phi, x)$, except that the Lévy measure used is the one corresponding to the point $y$. Moreover the coefficients in (5.1) are evaluated at $y$.

In the rest of this section, unless stated otherwise, we will always assume that Assumptions A-E are satisfied.

**Proposition 5.5.** If $u$ is a viscosity subsolution of $I(u, x) = 0$ in $\mathcal{O}$, then $u^\delta$ is a viscosity subsolution of $I_\delta(u^\delta, x) = 0$ in $\mathcal{O}_h$, $h = (2\delta\|u\|_\infty)^{1/2}$, where

$$I_\delta(\phi, x) := \inf \{ I(y)(\phi, x) : |y - x| \leq h \}.$$ 

If $v$ is a viscosity supersolution of $I(v, x) = 0$ in $\mathcal{O}$, then $v_\delta$ is a viscosity supersolution of $I_\delta(v_\delta, x) = 0$ in $\mathcal{O}_h$, $h = (2\delta\|v\|_\infty)^{1/2}$, where

$$I_\delta(\phi, x) := \sup \{ I(y)(\phi, x) : |y - x| \leq h \}.$$ 

**Proof.** Let us prove the statement for $u$ and $I_\delta$ (the corresponding one for $v$ and $I^\delta$ is entirely analogous and we omit it). Let $\phi$ touch $u^\delta$ from above at some $x_0 \in \mathcal{O}_h$. Let $x_0^* \in \mathbb{R}^d$ be such that

$$u^\delta(x_0) = u(x_0^*) - \frac{1}{\delta}|x_0 - x_0^*|^2.$$ 

It follows from part (5) of Proposition 5.3 that $x_0^* \in \mathcal{O}$. Then, by the definition of $u^\delta$, for any $x$ and $y$ we have

$$u^\delta(x + x_0 - x_0^*) \geq u(y) - \frac{1}{\delta}|x + x_0 - x_0^* - y|^2.$$ 

Choosing $y = x$ it follows that for every $x$ we have

$$u^\delta(x + x_0 - x_0^*) \geq u(x) - \frac{1}{\delta}|x_0 - x_0^*|^2,$$

with equality for $x = x_0^*$. It follows that if define a new test function $\phi^*(x)$ by

$$\phi^*(x) = \phi(x + x_0 - x_0^*) + \frac{1}{\delta}|x_0 - x_0^*|^2,$$

then $\phi^*$ touches $u$ from above at $x_0^*$. Since $u$ is a subsolution, it follows that

$$I(\phi^*, x_0^*) \leq 0.$$ 

Let us rewrite the expression on the left. First, recall

$$I(\phi^*, x_0^*) = \sup_\alpha \inf_\beta \{-L^{\alpha\beta}(\phi^*, x_0^*) + c_{\alpha\beta}(x_0^*)\phi^*(x_0^*) + f_{\alpha\beta}(x_0^*)\}.$$ 

Next, note that

$$L^{\alpha\beta}(\phi^*, x_0^*) = \int_{\mathbb{R}^d} [\phi^*(x_0^* + z) - \phi^*(x_0^*) - \chi_{B_1(0)}D\phi^*(x_0^*) \cdot z] d\mu_{x_0^*}^{\alpha\beta}(z).$$ 

Since,

$$\phi^*(x_0^* + z) - \phi^*(x_0^*) - \chi_{B_1(0)}D\phi^*(x_0^*) \cdot z = \phi(x_0 + z) - \phi(x_0) - \chi_{B_1(0)}D\phi(x_0) \cdot z$$

we have

$$L^{\alpha\beta}(\phi^*, x_0^*) = \int_{\mathbb{R}^d} [\phi(x_0 + z) - \phi(x_0) - \chi_{B_1(0)}D\phi(x_0) \cdot z] d\mu_{x_0^*}^{\alpha\beta}(z).$$
it follows that
\[ L^{\alpha \beta} (\phi^*, x_0^*) = \int_{\mathbb{R}^d} [\phi(x_0 + z) - \phi(x_0) - \chi_{B_1(0)}D\phi(x_0) \cdot z] \, d\mu^{\alpha \beta}_{x_0^*}(z) = L^{\alpha \beta}_{\mu^{\alpha \beta}_{x_0^*}} (\phi, x_0). \]

In conclusion
\[ 0 \geq I(\phi^*, x_0^*) = \sup_{\alpha \beta} \inf \{ -L^{\alpha \beta}_{\mu^\alpha_{x_0}(\phi, x_0)} + c_{\alpha \beta}(x_0^*) (\phi(x_0) + \frac{1}{\delta} |x_0 - x_0^*|^2) + f_{\alpha \beta}(x_0^*) \} \]
\[ \geq \sup_{\alpha \beta} \inf \{ -L^{\alpha \beta}_{\mu^\alpha_{x_0}(\phi, x_0)} + c_{\alpha \beta}(x_0^*) (\phi(x_0) + f_{\alpha \beta}(x_0^*)) \} \]
\[ \geq I_\delta(\phi, x_0). \]

Using part (5) of Proposition 5.3 in this last inequality, the proposition follows.

\[ \square \]

Let us also state in a single lemma two basic facts about classical evaluation of Lévy operators and viscosity solutions. The proof of the lemma goes along lines similar to those of the proofs of [8][Lemma 4.3 and Lemma 5.7].

**Lemma 5.6.** For any function \( u \in \text{BUC}(\mathbb{R}^d) \) that is pointwise-C\(^1,1\) at a point \( x_0 \in \mathcal{O} \) (respectively, \( x_0 \in \mathcal{O}_h \)) the operator \( I(u, x_0) \) (respectively, \( I_\delta(u, x_0) \)) is classically defined. If furthermore \( u \) is a viscosity subsolution of \( I(u, x) = 0 \) in \( \mathcal{O} \) (respectively, \( I_\delta(u, x) \leq 0 \) in \( \mathcal{O}_h \)), then also \( I(u, x_0) \leq 0 \) (respectively, \( I_\delta(u, x_0) \leq 0 \)) pointwise. Similar statement is true for viscosity supersolutions.

**Proof.** We will only prove the statement for \( I(u, x_0) \) as the other statements are proved similarly. Recall that from Assumption E,
\[ \Lambda = \sup_{x, \alpha, \beta} \{ \mathcal{N}_2(\hat{\mu}_{x}^{\alpha \beta}) + \hat{\mu}_{x}^{\alpha \beta}(B_1) \}. \]

From the pointwise-C\(^1,1\) assumption at \( x_0 \), we have
\[ \int_{B_1} |u(x_0 + z) - u(x_0) - \chi_{B_1(0)}(z)Du(x_0) \cdot z| \, d\mu^{\alpha \beta}_{x_0}(z) \leq \int_{B_1} C_{u, x_0} |z|^2 \, d\mu^{\alpha \beta}_{x_0}(z), \]
\[ \int_{B_1^c} |u(x_0 + z) - u(x_0)| \, d\mu^{\alpha \beta}_{x_0}(z) \leq 2\|u\|_\infty \int_{B_1^c} \mu^{\alpha \beta}_{x_0}(z) \],
where \( C_{u, x_0} \) is from Definition 2.3. It thus follows that each integral defining \( L^{\alpha \beta}(u, x_0) \) converges and
\[ \sup_{\alpha \beta} |L^{\alpha \beta}(u, x_0)| \leq (C_{u, x_0} + 2\|u\|_\infty)\Lambda < \infty. \]

From here, it is immediate that \( I(u, x_0) \) is classically defined. As for the second assertion, define
\[ u_r := \begin{cases} \phi & \text{in } B_r(x_0), \\ u & \text{outside of } B_r(x_0), \end{cases} \]
where \( \phi(x) = u(x_0) + Du(x_0) \cdot (x - x_0) + C_{u, x_0} |x - x_0|^2 \). The function \( \phi \) is touching \( u \) from above in a neighborhood of \( x_0 \). From Definition 4.2 we have \( I(u_r, x_0) \leq 0 \) for every
For $r > 0$, on the other hand,

$$I(u, x_0) \leq I(u_r, x_0) + \mathcal{M}^+_I(u - u_r, x_0),$$

where the operator $\mathcal{M}^+_I$ is given by

$$\mathcal{M}^+_I(u - u_r, x_0) = \sup_{\alpha, \beta} \{-L^\alpha\beta(u - u_r, x_0)\}.$$

Using the special form of $u_r$, particularly that $u_r = u$ outside of $B_r$, we have

$$\mathcal{M}^+_I(u - u_r, x_0) = \sup_{\alpha, \beta} \left\{ \int_{B_r} \left[ \partial(x + z) - u(x + z) \right] d\mu^\alpha_{x_0}(z) \right\} \leq 2C_{u,x_0} \sup_{\alpha, \beta} \left\{ \int_{B_r} |z|^2 d\mu^\alpha_{x_0}(z) \right\} \leq \theta(r),$$

where the last inequality follows from (4.6). Taking the limit as $r \to 0$, we conclude that

$$I(u, x_0) \leq 0.$$  

\[ \Box \]

We will need smooth approximations of functions $|x - y|^p$ for $p \in [1, 2]$. For $\kappa > 0$ we define a function $\tilde{\psi}_\kappa : [0, +\infty) \to [0, +\infty)$ by

$$\tilde{\psi}_\kappa(r) = (\kappa + r^2)^{\frac{p}{2}} - \kappa^\frac{p}{2}.$$  

Then the function

$$\psi_\kappa(x) := \tilde{\psi}_\kappa(|x|)$$

is smooth and converges as $\kappa \to 0$ to $|x|^p$ uniformly on $\mathbb{R}^d$. We will be using the following lemma.

**Lemma 5.7.** Let $p \in [1, 2]$. For every $R > 0$ the function $\psi_\kappa(x)$ is uniformly pointwise-C$^p$ on $B_R$, i.e. there exists a constant $C_{p,R}$ such that for every $0 < \kappa < 1$ and every $x_0, x \in B_R$

$$|\psi_\kappa(x) - \psi_\kappa(x_0) - D\psi_\kappa(x_0) \cdot (x - x_0)| \leq C_{p,R} |x - x_0|^p \quad \text{if } 1 < p \leq 2,$$

$$|\psi_\kappa(x) - \psi_\kappa(x_0)| \leq C_{1,R} |x - x_0| \quad \text{if } p = 1.$$

The following is the main lemma of the paper. We refer the reader to Definition 2.2 for the definition of $L_\mu$.

**Lemma 5.8.** Let $u, v \in \text{BUC}(\mathbb{R}^d)$. Let $\alpha > 0, p \geq 1, 0 < \kappa < 1$ and suppose that $(x_*, y_*) \in \mathcal{O} \times \mathcal{O}$ is a global maximum point of the function

$$w(x, y) := u(x) - v(y) - \alpha \psi_\kappa(x - y).$$

Furthermore, suppose that $u$ and $v$ are pointwise-C$^{1,1}$ at $x_*$ and $y_*$, respectively. Then, for any two Lévy measures $\mu, \nu \in \mathbb{L}_p(B_1)$, we have the inequality

$$L_\mu(u, x_*) - L_\nu(v, y_*) \leq C_p \alpha d_{\mathbb{L}_p}(\mu, \nu)^p,$$

where $C_p$ is independent of $\kappa$. 

Proof. First, note that as \((x_*, y_*)\) is a maximum point of \(w\), we have
\[
    u(x) \leq \alpha \psi_\kappa(x - y_*) + v(y_*) + (u(x_*) - v(y_*) - \alpha \psi_\kappa(x_* - y_*))
\]
\[
    v(y) \geq -\alpha \psi_\kappa(x_* - y) + u(x_*) - (u(x_*) - v(y_*) - \alpha \psi_\kappa(x_* - y_*))
\]
with equalities at \(x_*\) and \(y_*\) respectively. Second, for any \((x, y) \in \mathbb{R}^d \times \mathbb{R}^d\)
\[
    w(x_* + x, y_* + y) - w(x_*, y_*) \leq 0.
\]
Let \(\gamma \in \text{Adm}(\mu, \nu)\). Using that \(\gamma((B_1 \times B_1)^c) = 0\), and since \(\delta u(x_*, 0) = 0\) and \(\delta v(y_*, 0) = 0\), we thus have
\[
    L_\mu(u, x_*) - L_\nu(v, y_*)
    = \int_{B_1 \times B_1} \left( u(x_*) + x - v(y_*) + y - (u(x_*) - v(y_*)) \right)
    - \alpha D\psi_\kappa(x_* - y_*) \cdot (x - y) \, d\gamma(x, y).
\]
On the other hand, if \(x, y \in B_1\), using Lemma 5.7, we also have
\[
    u(x_* + x) - v(y_* + y) - (u(x_*) - v(y_*)) - \alpha D\psi_\kappa(x_* - y_*) \cdot (x - y)
    \leq \alpha \psi_\kappa(x_* + x - y - y) - \alpha \psi_\kappa(x_* - y) - \alpha D\psi_\kappa(x_* - y_*) \cdot (x - y)
    \leq C_\rho \alpha |x - y|^p.
\]
Therefore,
\[
    L_\mu(u, x_*) - L_\nu(v, y_*) \leq C_\rho \alpha \int_{B_1 \times B_1} |x - y|^p \, d\gamma(x, y).
\]
Taking the infimum over all \(\gamma \in \text{Adm}(\mu, \nu)\), it thus follows that
\[
    L_\mu(u, x_*) - L_\nu(v, y_*) \leq C_\rho \alpha d_{\text{Ep}}(\mu, \nu)^p.
\]
\(\square\)

Corollary 5.9. Let \(u, v, x_*,\) and \(y_*\) be as in Lemma 5.8, and let \(\mu, \nu \in \mathbb{L}_p(\mathbb{R}^d)\). Then,
\[
    L_\mu(u, x_*) - L_\nu(v, y_*) \leq C_\rho \alpha d_{\text{Ep}}(\hat{\mu}, \hat{\nu})^p + 2\|v\|_{\text{TV}} d_{\text{TV}}(\hat{\mu}, \hat{\nu}).
\]

Proof. Let us write the difference as follows
\[
    L_\mu(u, x_*) - L_\nu(v, y_*) = L_{\hat{\mu}}(u, x_*) - L_{\hat{\nu}}(v, y_*) + L_{\hat{\mu}}(u, x_*) - L_{\hat{\nu}}(v, y_*).
\]
Thanks to Lemma 5.8, the first difference in the right-hand side above is less than or equal to \(C_\rho \alpha d_{\text{Ep}}(\hat{\mu}, \hat{\nu})^p\). For the second one, note that
\[
    L_{\hat{\mu}}(u, x_*) - L_{\hat{\nu}}(v, y_*) = \int_{B_1^c} [u(x_* + z) - u(x_*)] \, d\mu(z) - \int_{B_1^c} [v(y_* + z) - v(y_*)] \, d\nu(z)
\]
\[
    = \int_{B_1^c} [u(x_* + z) - v(x_* + z) - (u(x_*) - v(y_*))] \, d\mu(z)
    \]
\[
    + \int_{B_1^c} [v(y_* + z) - v(y_*)] \, d(\mu - \nu)(z).
\]
Since \( w \) achieves its global maximum at \((x_*,y_*)\), it follows that \( u(x_* + z) - v(y_* + z) - (u(x_*) - v(y_*)) \leq 0 \). Hence we obtain

\[
L_{\mu}(u,x_*) - L_{\nu}(v,y_*) \leq \int_{B_R^c} [v(y_* + z) - v(y_*)] \, d(\mu - \nu)(z).
\]

\[
\leq 2\|v\|_\infty d_{TV}(\mu,\nu).
\]

\( \square \)

We need a variant of a well known doubling lemma (see e.g. [10, Lemma 3.1]).

**Lemma 5.10.** Let \( u, v \in \text{BUC}(\mathbb{R}^d) \) be such that \( M = \text{sup}(u - v) > \tau > 0 \) and \( u(x) - v(x) \leq 0 \) for \( x \in B_R^c \) for some \( R > 0 \). For any \( \varepsilon, \delta, \kappa > 0 \), set

\[
w(x, y) := u^\delta(x) - v^\delta(y) - \frac{1}{\varepsilon}\psi_\kappa(x - y),
\]

\[
M_{\varepsilon, \delta, \kappa} := \sup_{\mathbb{R}^d \times \mathbb{R}^d} w(x, y).
\]

Then, for sufficiently small \( \delta, \varepsilon, \kappa \), there exist \((x_\varepsilon, y_\varepsilon)\) such that

\[
M_{\varepsilon, \delta, \kappa} = w(x_\varepsilon, y_\varepsilon).
\]

Then, we have

\[
\limsup_{\kappa \to 0} \frac{|x_\varepsilon - y_\varepsilon|^p}{\varepsilon} \leq \omega(C \varepsilon^{\frac{1}{p}}),
\]

\[
\lim_{\varepsilon \to 0} \lim_{\delta \to 0} \lim_{\kappa \to 0} M_{\varepsilon, \delta, \kappa} = M,
\]

where above \( C = \|u\|_\infty + \|v\|_\infty \) and \( \omega \) is a modulus of continuity of \( u \).

If \( \Omega \) is an open subset of \( \mathbb{R}^d \) and in addition \( u \in C^r(\Omega), 0 < r \leq 1 \), and all the points \( x_\varepsilon, y_\varepsilon \in \Omega \), then

\[
\limsup_{\kappa \to 0} \frac{|x_\varepsilon - y_\varepsilon|^{p-r}}{\varepsilon} \leq C_1
\]

for some constant \( C_1 \) independent of \( \delta, \varepsilon, \kappa \).

**Proof.** It is easy to see that the uniform convergence of the \( u^\delta, v^\delta \) to \( u, v \), the uniform convergence of \( \psi_\kappa(x-y) \) to \( |x-y|^p \) and the uniform continuity of \( u, v \) (and hence of \( u^\delta, v^\delta \), uniform in \( \delta \)) implies that for sufficiently small \( \delta, \varepsilon, \kappa \) we must have \( w(x, y) \leq \tau/2 \) when either \( x \) or \( y \) is in \( B_R^c \). Thus \( w \) must attain maximum at some point \((x_\varepsilon, y_\varepsilon) \in B_R \times B_R \).

Denote

\[
M_{\varepsilon, \delta} := \sup_{\mathbb{R}^d \times \mathbb{R}^d} (u^\delta(x) - v^\delta(y) - \frac{1}{\varepsilon}|x-y|^p),
\]

\[
M_\varepsilon := \sup_{\mathbb{R}^d \times \mathbb{R}^d} (u(x) - v(y) - \frac{1}{\varepsilon}|x-y|^p).
\]

Again, using the uniform convergence of \( u^\delta, v^\delta, \psi_\kappa(x-y) \) and the uniform continuity of \( u, v \) we easily find (see also the proof of [10, Lemma 3.1]) that

\[
\lim_{\kappa \to 0} M_{\varepsilon, \delta, \kappa} = M_{\varepsilon, \delta}, \quad \lim_{\delta \to 0} M_{\varepsilon, \delta} = M_\varepsilon, \quad \lim_{\varepsilon \to 0} M_\varepsilon = M.
\]
We obviously have (for $\varepsilon, \delta$ fixed)
\[
\limsup_{\kappa \to 0} \frac{1}{\varepsilon} \psi_\kappa(x_\varepsilon - y_\varepsilon) = \limsup_{\kappa \to 0} \frac{1}{\varepsilon} |x_\varepsilon - y_\varepsilon|^p.
\] \hspace{1cm} (5.4)

Now
\[
u_\delta(y_\varepsilon) - v_\delta(y_\varepsilon) \leq u_\delta(x_\varepsilon) - v_\delta(y_\varepsilon) - \frac{1}{\varepsilon} \psi_\kappa(x_\varepsilon - y_\varepsilon)
\]
which, by (5.4), implies
\[
\limsup_{\kappa \to 0} \frac{|x_\varepsilon - y_\varepsilon|^p}{\varepsilon} \leq \limsup_{\kappa \to 0} (u_\delta(x_\varepsilon) - u_\delta(y_\varepsilon)) \leq \limsup_{\kappa \to 0} \omega(|x_\varepsilon - y_\varepsilon|).
\]

This, together with the fact that we must have
\[
\limsup_{\kappa \to 0} \frac{|x_\varepsilon - y_\varepsilon|^p}{\varepsilon} \leq \|u\|_\infty + \|v\|_\infty,
\]
gives (5.2). The last claim (5.3) follows by a similar argument since now
\[
\frac{1}{\varepsilon} \left(\limsup_{\kappa \to 0} |x_\varepsilon - y_\varepsilon|^p\right) \leq \limsup_{\kappa \to 0} C|x_\varepsilon - y_\varepsilon|^r = C(\limsup_{\kappa \to 0} |x_\varepsilon - y_\varepsilon|)^r.
\]

\[
\blacksquare
\]

Proof of Theorem 4.4. Arguing by contradiction, assume there is some $\ell > 0$ such that
\[
\sup_{x \in \mathbb{R}^d} \{u(x) - v(x)\} = \ell > 0.
\]

Step 1. (Taking inf/sup-convolutions)

Let $u_\delta$ and $v_\delta$ denote the sup- and inf-convolutions of $u$ and $v$ for $\delta > 0$. Then,
\[
\sup_{x \in \mathbb{R}^d} \{u_\delta(x) - v_\delta(x)\} \geq \ell.
\]

We may make $\delta_0$ small enough so that for $\delta < \delta_0$ we have
\[
\sup_{x \in \mathcal{O}} \{u_\delta(x) - v_\delta(x)\} \leq \frac{1}{4} \ell.
\]

Recall that if $\omega$ is a modulus of continuity of $u$ and $v$, then it is also a modulus of continuity of $u_\delta$ and $v_\delta$. Therefore, reducing $\delta_0$ if necessary, we have
\[
u_\delta(x) - v_\delta(x) \leq \frac{1}{2} \ell \quad \text{for } x \in \mathcal{O} \setminus \mathcal{O}_{2h_0},
\]
as long as $\delta < \delta_0$, where $\mathcal{O}_{h_0} = \{x \in \mathcal{O} \mid d(x, \partial \mathcal{O}) > h_0\}$ and $h_0 > 0$ is some constant. In particular, for such $\delta$ the supremum of $u_\delta - v_\delta$ in $\mathbb{R}^d$ can only be achieved within $\mathcal{O}_{2h_0}$.

Step 2. (Doubling of variables)

For $\varepsilon, \delta, \kappa > 0$, we let $w$ be as in Lemma 5.10 and let $(x_\varepsilon, y_\varepsilon) \in \mathbb{R}^d \times \mathbb{R}^d$ be such that
\[
w(x_\varepsilon, y_\varepsilon) = \max_{\mathbb{R}^d \times \mathbb{R}^d} w(x, y).
\]

From Step 1, we know that $u_\delta - v_\delta \leq \ell/2$ in $\mathcal{O} \setminus \mathcal{O}_{2h}$ and $u_\delta - v_\delta \geq \ell$ somewhere in $\mathcal{O}$. Furthermore, we know $u_\delta$ and $v_\delta$ are uniformly continuous in $\mathcal{O}$, and uniformly so with
respect to $\delta < 1$. From these facts, and (5.2), it follows that $(x_\varepsilon, y_\varepsilon)$ must belong to $O_h$ for all sufficiently small $\varepsilon, \delta$ and $\kappa$ or else it cannot be the maximum point of $u^\varepsilon$.

On the other hand, Proposition 5.5 says that $u^\delta$ is a viscosity subsolution of $I_\delta(u^\delta, x) = 0$ and $v_\delta$ is a viscosity supersolution of $I_\delta(v_\delta, x) = 0$ in $O_h$ for sufficiently small $\delta$. The function $u^\delta$ is touched from above by a smooth function at $x_\varepsilon$ and $v_\delta$ is touched from below at $y_\varepsilon$. It follows that $u^\delta$ and $v_\delta$ are pointwise-$C^{1,1}$ at $x_\varepsilon$ and $y_\varepsilon$, respectively (see Definition 2.3). Applying Lemma 5.6, we conclude that $I_\delta(u^\delta, x_\varepsilon)$ and $I_\delta(v_\delta, y_\varepsilon)$ are well defined in the classical sense, with $I_\delta(u^\delta, x_\varepsilon) \leq 0$ and $I_\delta(v_\delta, x_\varepsilon) \leq 0$. It follows from Proposition 5.5 that there are points $x_\delta^*$ and $y_\delta^*$ such that

$$I(x_\delta^*), (u^\delta, x_\varepsilon) \leq \delta, \quad I(y_\delta^*), (v_\delta, y_\varepsilon) \geq -\delta,$$

and

$$|x_\varepsilon - x_\delta^*|, |y_\varepsilon - y_\delta^*| \leq h, \quad (5.5)$$

where $h = (2\delta(\|u\|_{\infty} + \|v\|_{\infty}))^{\frac{1}{2}}$.

Step 3. (Equation structure)

Let us use the structure of $I(\cdot, x)$ to bound $I(x_\delta^*), (u^\delta, x_\varepsilon) - I(y_\delta^*), (u^\delta, y_\varepsilon)$ from below. Using the expression in (5.1), we have

$$I(x_\delta^*), (u^\delta, x_\varepsilon) = \sup_{\alpha} \inf_{\beta} \left\{-L_{\mu^\alpha \mu^\beta}(u^\delta, x_\varepsilon) + c_{\alpha \beta}(x_\delta^*)u^\delta(x_\varepsilon) + f_{\alpha \beta}(x_\delta^*)\right\},$$

$$I(y_\delta^*), (v_\delta, y_\varepsilon) = \sup_{\alpha} \inf_{\beta} \left\{-L_{\mu^\alpha \mu^\beta}(v_\delta, y_\varepsilon) + c_{\alpha \beta}(y_\delta^*)v_\delta(y_\varepsilon) + f_{\alpha \beta}(y_\delta^*)\right\}.$$

Therefore, for our purposes it suffices to compare the expressions appearing on the right hand side for each fixed $\alpha, \beta$. Let us write

$$(I)_{\alpha \beta} = -L_{\mu^\alpha \mu^\beta}(u^\delta, x_\varepsilon) + c_{\alpha \beta}(x_\delta^*)u^\delta(x_\varepsilon) + f_{\alpha \beta}(x_\delta^*),$$

$$(II)_{\alpha \beta} = -L_{\mu^\alpha \mu^\beta}(v_\delta, y_\varepsilon) + c_{\alpha \beta}(y_\delta^*)v_\delta(y_\varepsilon) + f_{\alpha \beta}(y_\delta^*).$$

We now look for an upper bound for $(I)_{\alpha \beta} - (II)_{\alpha \beta}$ which is independent of $\alpha$ and $\beta$ by breaking this difference into parts. First, recall that the function $u^\delta(x) - v_\delta(y) - \frac{1}{\theta}|x - y|$ achieves its global maximum at $(x_\varepsilon, y_\varepsilon)$, in which case Corollary 5.9 guarantees that

$$L_{\mu^\alpha \mu^\beta}(u^\delta, x_\varepsilon) - L_{\mu^\alpha \mu^\beta}(v_\delta, y_\varepsilon) \leq \frac{C}{\varepsilon}d_{\rho}(\mu_{\delta}^\alpha, \mu_{\delta}^\beta)p + 2\|v_\delta\|_{\infty}d_{TV}(\mu_{\delta}^\alpha, \mu_{\delta}^\beta)$$

Then, thanks to Assumptions A and B, and (5.5), we have

$$L_{\mu^\alpha \mu^\beta}(u^\delta, x_\varepsilon) - L_{\mu^\alpha \mu^\beta}(v_\delta, y_\varepsilon) \leq \frac{C}{\varepsilon}|x_\delta^* - y_\delta^*|^p + 2\|v\|_{\infty}\theta(|x_\delta^* - y_\delta^*|)$$

$$\leq \frac{C}{\varepsilon}|x_\varepsilon - y_\varepsilon|^p + 2\|v\|_{\infty}\theta(|x_\varepsilon - y_\varepsilon|) + \rho_\varepsilon(\delta), \quad (5.6)$$

where for a fixed $\varepsilon$, $\lim_{\delta \to 0} \rho(\delta) = 0$. 

Next, we have the elementary inequality
\[
c_{\alpha\beta}(x_\delta^*)u^\delta(x_\varepsilon) - c_{\alpha\beta}(y_\delta^*)v_\delta(y_\varepsilon) \geq c_{\alpha\beta}(x_\delta^*)(u^\delta(x_\varepsilon) - v_\delta(y_\varepsilon)) - |c_{\alpha\beta}(x_\delta^*) - c_{\alpha\beta}(y_\delta^*)||v_\delta(y_\varepsilon)| \\
\geq \lambda \ell - \theta(|x_\varepsilon - y_\varepsilon|)||v||_{\infty} - \rho_\varepsilon(\delta),
\]  
(5.7)
where \(\rho_\varepsilon(\delta)\) is a function as before and we used that \(u^\delta(x_\varepsilon) - v_\delta(y_\varepsilon) \geq \ell\).

Finally, by Assumption C
\[
|f_{\alpha\beta}(x_\delta^*) - f_{\alpha\beta}(y_\delta^*)| \leq \theta(|x_\varepsilon - y_\varepsilon|) + \rho_\varepsilon(\delta).
\]
(5.8)

Now, combining (5.6), (5.7), (5.8), we have the estimate
\[
(I)_{\alpha\beta} - (II)_{\alpha\beta} \geq \lambda \ell - \frac{C}{\varepsilon}|x_\varepsilon - y_\varepsilon|^p - C\theta(|x_\varepsilon - y_\varepsilon|) - \rho_\varepsilon(\delta),
\]
where \(C\) above is some absolute constant. Therefore we conclude that
\[
\lambda \ell \leq 2\delta + \frac{C}{\varepsilon}|x_\varepsilon - y_\varepsilon|^p + C\theta(|x_\varepsilon - y_\varepsilon|) + \rho_\varepsilon(\delta).
\]

Step 4. (Using the subsolution and supersolution property)

Recalling the way \(x_\delta^*\) and \(y_\delta^*\) were selected, we have \(I(x_\delta^*)(u^\delta, x_\varepsilon) - I(y_\delta^*)(v_\delta, y_\varepsilon) \leq 2\delta\), and therefore
\[
\lambda \ell \leq 2\delta + \frac{C}{\varepsilon}|x_\varepsilon - y_\varepsilon|^p + C\theta(|x_\varepsilon - y_\varepsilon|) + \rho_\varepsilon(\delta).
\]

It now remains to take \(\lim_{\varepsilon \to 0}\lim_{\delta \to 0}\limsup_{\kappa \to 0}\) on both sides of the above inequality and use (5.2) to obtain a contradiction. \(\square\)

5.1. Estimating \(d_{L_p}\) in special cases.

**Proposition 5.11.** Let \(p \in [1, 2]\).

(i) Let \(\phi, \psi\) be continuous functions on \(\overline{B}_1\) such that \(\phi(0) = \psi(0) = 0\) and \(\phi(x) + \psi(y) \leq |x - y|^p\) for all \(x, y \in \overline{B}_1\). Let \(\mu, \nu \in \mathbb{L}_p(B_1)\). If \(\mu - \nu\) is a positive measure then
\[
\int_{B_1} \phi(x)d\mu(x) + \int_{B_1} \psi(y)d\nu(y) \leq \int_{B_1} |x|^p d(\mu - \nu)(x).
\]
(5.9)

(ii) For any \(\mu, \nu \in \mathbb{L}_p(B_1)\) we have
\[
d_{L_p}(\mu, \nu) \leq 2^{\frac{p+1}{2}} d_{TV}(\mu_p, \nu_p)^{\frac{1}{2}},
\]
(5.10)
where \(d_{\mu_p} = |x|^p d\mu, d_{\mu_p} = |x|^p d\mu\).
\textbf{Proof.} (i): We have $\phi(x) \leq |x|^p$. If $\int_{B_1} \phi(x)d\mu(x) = -\infty$ there is nothing to prove. If $\int_{B_1} \phi(x)d\mu(x) > -\infty$ then also $\int_{B_1} \phi(x)d\nu(x) > -\infty$. Then we may write

$$
\int_{B_1} \phi(x)d\mu(x) = \int_{B_1} \phi(x)d\mu(x) - \int_{B_1} \phi(x)d\nu(x) + \int_{B_1} \phi(x)d\nu(x) + \int_{B_1} \psi(y)d\nu(y)
$$

$$
= \int_{B_1} \phi(x)d\mu(x) - \int_{B_1} \phi(x)d\nu(x) + \int_{B_1} \phi(y)d\nu(y) + \int_{B_1} \psi(y)d\nu(y)
$$

$$
\leq \int_{B_1} \phi(x)d(\mu - \nu)(x) \leq \int_{B_1} |x|^pd(\mu - \nu)(x),
$$

where in the last line we used $\phi(y) + \psi(y) \leq |y - y|^p = 0 \forall y \in \overline{B_1}$ and $\phi(x) \leq |x|^p \forall x \in \overline{B_1}$.

(ii): Denoting by $(\mu - \nu)^+$ and $(\mu - \nu)^-$, the positive and negative parts of $\mu - \nu$, we have $|\mu - \nu| = (\mu - \nu)^+ + (\mu - \nu)^-$. We also notice that $\mu - (\mu - \nu)^+ = \nu - (\mu - \nu)^-$. It thus follows from (5.9) and Lemma A.11 that

$$
d_{L_p}(\mu, \mu - (\mu - \nu)^+) \leq \int_{B_1} |x|^p d(\mu - \nu)^+(x),
$$

$$
d_{L_p}(\nu, \mu - (\mu - \nu)^+) \leq \int_{B_1} |x|^p d(\mu - \nu)^-(x).
$$

Moreover it is obvious that

$$
d_{\text{TV}}(\mu_p, \nu_p) = \int_{B_1} |x|^p d(\mu - \nu)^+(x) + \int_{B_1} |x|^p d(\mu - \nu)^-(x).
$$

Therefore, using the triangle inequality for the distance and the inequality $a + b \leq 2^{\frac{p-1}{p}}(a^p + b^p)^{\frac{1}{p}}$ for $a, b \geq 0$, we obtain

$$
d_{L_p}(\mu, \nu) \leq d_{L_p}(\mu, \mu - (\mu - \nu)^+) + d_{L_p}(\nu, \mu - (\mu - \nu)^+)
$$

$$
\leq 2^{\frac{p-1}{p}} \left( \int_{B_1} |x|^p d(\mu - \nu)^+(x) + \int_{B_1} |x|^p d(\mu - \nu)^-(x) \right)^{\frac{1}{p}}
$$

$$
= 2^{\frac{p-1}{p}} d_{\text{TV}}(\mu_p, \nu_p)^{\frac{1}{p}}.
$$

\hfill \Box

Let us now discuss the case when the Lévy measures $\mu_{x}^{\alpha \beta}$ are absolutely continuous with respect to the Lebesgue measure.

\textbf{Example 5.12.} Let us consider operators whose Lévy measures $d\mu_x^{\alpha \beta}(z)$ are all of the form $K_{\alpha \beta}(x, z)dz$, where the kernels $K_{\alpha \beta}(x, z)$ satisfy two properties. First,

$$
0 \leq K_{\alpha \beta}(x, z) \leq K(z),
$$

(5.11)
where $K(z)$ is such that for some $p \in [1, 2]$

$$\int_{\mathbb{R}^d} \min(1, |z|^p)K(z)dz < +\infty.$$  \hspace{1cm} (5.12)

Second, there is some $\gamma \in (0, 1]$ such that

$$|K_{\alpha\beta}(x, z) - K_{\alpha\beta}(y, z)| \leq |x - y|^{\gamma}K(z) \quad \forall x, y \in \mathcal{O}, \forall z \in B_1, \forall \alpha, \beta.$$  \hspace{1cm} (5.13)

Let $x, y \in \mathcal{O}$. To estimate $d_{L^p}(\hat{\mu}_x^{\alpha\beta}, \hat{\mu}_y^{\alpha\beta})$, we use Proposition 5.11. It follows from (5.10), (5.12) and (5.13), that

$$d_{L^p}(\hat{\mu}_x^{\alpha\beta}, \hat{\mu}_y^{\alpha\beta}) \leq 2^{\frac{1}{p-1}} \left(\int_{B_1} |z|^p |K_{\alpha\beta}(x, z) - K_{\alpha\beta}(y, z)| dz\right)^{\frac{1}{p}} \leq C|x - y|^\gamma. \hspace{1cm} (5.14)$$

In particular, (4.1) is satisfied for these measures when $p = 1$ and $\gamma = 1$.

We can now prove Corollary 4.5.

**Proof of Corollary 4.5.** We notice that the measures $\mu_x^{\alpha\beta}$ satisfy (5.11), (5.12) and (5.13) with $\gamma = p = 1$ and $K(z) = \Lambda_1|x|^{-d-\sigma}$ and hence they satisfy Assumptions A and E. It is also easy to see that they satisfy Assumption B. Thus the result follows from Theorem 4.4. \hfill \Box

**Example 5.13.** A well studied subclass of operators which arise in zero-sum two-player stochastic differential games are those of Lévy-Itô form. This corresponds to the situation where the $L^{\alpha\beta}$ appearing in (1.1) have the form

$$L^{\alpha\beta}(u, x) = \int_{U \setminus \{0\}} [u(x + T^{\alpha\beta}_x(z)) - u(x) - Du(x) \cdot T^{\alpha\beta}_x(z)] d\mu(z). \hspace{1cm} (5.15)$$

Here $U$ is a separable Hilbert space and $\mu$ is a fixed reference Lévy measure on $U \setminus \{0\}$. The maps $T^{\alpha\beta}_x : U \to \mathbb{R}^d$ are Borel measurable and such that for all $\alpha \in \mathcal{A}, \beta \in \mathcal{B}, x, y \in \mathcal{O}, z \in U \setminus \{0\}$,

$$|T^{\alpha\beta}_x(z) - T^{\alpha\beta}_y(z)| \leq C \rho(z)|x - y|, \quad |T^{\alpha\beta}_x(z)| \leq C \rho(z),$$

for some positive Borel function $\rho : U \setminus \{0\} \to \mathbb{R}$ which is bounded on bounded sets, $\inf_{|z|>r} \rho(z) > 0$ for every $r > 0$, and

$$\int_{U \setminus \{0\}} \rho(z)^2d\mu(z) \leq C. \hspace{1cm} (5.16)$$

Under these conditions the measures $\mu_x^{\alpha\beta} = (T^{\alpha\beta}_x)_#\mu$ are Lévy measures. The comparison principle for sub/super solutions of (1.1) with $L^{\alpha\beta}$ as in (5.15) is known to hold, as discussed in the introduction. Let us revisit it using the transport metric. For every $x_0, y_0 \in \mathcal{O}, \alpha \in \mathcal{A}, \beta \in \mathcal{B}$, we have $\gamma = (T^{\alpha\beta}_{x_0} \times T^{\alpha\beta}_{y_0})_#\mu \in \text{Adm}(\mu_{x_0}^{\alpha\beta}, \mu_{y_0}^{\alpha\beta})$ and therefore,

$$d_{L^2}(\mu_{x_0}^{\alpha\beta}, \mu_{y_0}^{\alpha\beta}) \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^2d\gamma(x, y) = \int_{U \setminus \{0\}} |T^{\alpha\beta}_{x_0}(z) - T^{\alpha\beta}_{y_0}(z)|^2 d\mu(z) \leq C|x_0 - y_0|^2.$$

In this case the whole measures $\mu_{x_0}$ and $\mu_{y_0}$ satisfy Assumption A for $p = 2$ and our approach can be applied without the decomposition of the measures $\mu_x^{\alpha\beta}$ into $\hat{\mu}_x^{\alpha\beta}$ and $\hat{\mu}_x^{\alpha\beta}$.
If the operators $L^{\alpha\beta}$ in (1.1) have a more common Lévy-Itô form

$$L^{\alpha\beta}(u, x) = \int_{U \setminus \{0\}} [u(x + T^{\alpha\beta}_x(z)) - u(x) - \chi_{B_1(0)} Du(x) \cdot T^{\alpha\beta}_x(z)] d\mu(z),$$

where instead of (5.16) we now only assume

$$\int_{U \setminus \{0\}} (\rho(z)^2 \chi_{|z|<1} + \chi_{|z|\geq 1}) d\mu(z) \leq C,$$

we need to modify this approach. We now do the decomposition

$$\mu^{\alpha\beta}_x = \hat{\mu}_x^{\alpha\beta} + \tilde{\mu}_x^{\alpha\beta},$$

where $\hat{\mu}_x^{\alpha\beta} = (T^{\alpha\beta}_x \# \hat{\mu})$, $\tilde{\mu}_x^{\alpha\beta} = (T^{\alpha\beta}_x \# \tilde{\mu})$, and consider the measures $\hat{\mu}_x^{\alpha\beta} + \tilde{\mu}_x^{\alpha\beta}$ as measures on $\mathbb{R}^d$ by the usual extension. Then the measures $\hat{\mu}_x^{\alpha\beta} \in \mathbb{L}_p(\mathbb{R}^d)$ and they satisfy Assumption A for $p = 2$. Unfortunately the measures $\tilde{\mu}_x^{\alpha\beta}$ may not satisfy Assumption B now, however the terms containing them can be handled in a standard way (see e.g. [14]) and thus our approach can still be implemented.

6. Comparison Principles under Additional Assumptions

As in the previous section, throughout this section we consider a fixed bounded domain $O \subset \mathbb{R}^d$. In this section we prove a few comparison results for more regular viscosity sub/supersolutions. In return, we are allowed to replace Assumption A by a weaker assumption.

**Assumption A1.** Let $p \in [1, 2]$. There exist $C > 0$ and $s \in (0, 1)$ such that

$$d_{L_p}(\hat{\mu}_x^{\alpha\beta}, \tilde{\mu}_y^{\alpha\beta}) \leq C|x - y|^s, \quad \forall x, y \in O, \forall \alpha, \beta.$$

**Remark 6.1.** Consider a Lévy measure $\mu \in \mathbb{L}_p(B_1)$. For $r \in (0, 1)$ we define

$$\mu_r(\cdot) := \mu(\cdot \cap B_r^c).$$

Then, we have the estimate

$$d_{L_p}(\mu, \mu_r)^p \leq \int_{B_r} |x|^p \, d\mu(x).$$

To see why this is so, simply note that among the admissible plans we have the one that sends all of the mass of $\mu$ in $B_r \setminus \{0\}$ to 0, and leaves the rest of the mass fixed in place. To be more precise, define

$$T(x) = \begin{cases} 0 & \text{for } x \in B_r \setminus \{0\}, \\ x & \text{otherwise.} \end{cases}$$

Then $\gamma = (T \times \text{Id}) \# \mu \in \text{Adm}(\mu_r, \mu)$ and

$$d_{L_p}(\mu, \mu_r)^p \leq \int_{\mathbb{R}^d \times \mathbb{R}^d} |x - y|^p d\gamma(x, y) = \int_{\mathbb{R}^d \times \mathbb{R}^d} |T(x) - x|^2 \, d\mu(x) = \int_{B_r} |x|^p \, d\mu(x).$$
**Remark 6.2.** Estimating the distance between \( \mu \) and \( \mu_r \) is of interest to us since it can be used to bound the difference between the operators

\[
L_\mu(u, x) := \int_{B_1} \delta u(x, z) \, d\mu(z) \quad \text{and} \quad L_{\mu_r}(u, x) := \int_{B_1 \setminus B_r} \delta u(x, z) \, d\mu_r(z).
\]

The operator on the right can be classically evaluated for any continuous function, while the one on the left in general is not. Being able to estimate the difference between them will be an important step in the proof of Theorem 6.4, removing the need for the use of the sup/inf-convolutions (as mentioned in Remark 5.1).

**Theorem 6.3.** Let Assumptions A1 and B-E be satisfied. Let \( u \) be a viscosity subsolution and \( v \) be a viscosity supersolution of \((1.1)\) and let \( u(x) \leq v(x) \) for all \( x \notin \mathcal{O} \). If either \( u \) or \( v \) is in \( C^r(\mathcal{O}) \), and we have \( 1 - \frac{r}{p} < s \) (where \( s \) is from Assumption A1), then

\[
u(x) \leq v(x) \quad \forall \ x \in \mathcal{O}.
\]

**Proof.** The proof follows the lines of the proof of Theorem 4.4. The only difference is that when either \( u \) or \( v \) is \( C^r \), instead of \((5.2)\) we now have \((5.3)\), i.e.

\[
\limsup_{\kappa \to 0} \frac{|x - y|^{p-r}}{\varepsilon} \leq C
\]

and in this case, following the original proof we obtain

\[
\lambda \leq 2\delta + \frac{C}{\varepsilon} |x - y|^p + C \theta(|x - y|) + \rho_c(\delta).
\]

which produces a contradiction after taking \( \lim_{\varepsilon \to 0} \lim_{\delta \to 0} \limsup_{\kappa \to 0} \) if \( sp > p - r \), which is the case precisely when \( 1 - \frac{r}{p} < s \). \( \square \)

The previous Theorem does not cover the situation where \( r = 1 \), however, with extra work one can show that if \( u \) or \( v \) is of class \( C^1 \) then we can choose \( s \) as small as \( 1 - \frac{1}{p} \) and we still have comparison. The proof is different from that of Theorem 4.4 since we do not use the sup/inf-convolutions.

**Theorem 6.4.** Let Assumptions B-F hold, and Assumption A1 hold with \( p > 1 \) and \( s = 1 - \frac{1}{p} \). Suppose that \( u \) and \( v \) are respectively a viscosity subsolution and viscosity supersolution of \((1.1)\) and \( u(x) \leq v(x) \) for all \( x \notin \mathcal{O} \). If either \( u \) or \( v \) is in \( C^1(\mathcal{O}) \) then

\[
u(x) \leq v(x) \quad \forall \ x \in \mathcal{O}.
\]

**Remark 6.5.** If we allowed \( C^1 \) functions to be test functions in the case \( p = 1 \) then Theorem 6.4 would trivially hold for \( p = 1 \) without the need for Assumptions A, B and the continuity of the coefficients, since then either \( u \) or \( v \) would be a classical sub/supersolution of \((1.1)\) and could thus be used as a test function.

**Proof.** Without loss of generality, let us say that \( u \in C^1(\mathcal{O}) \). As before we argue by contradiction, in which case there is some \( \ell > 0 \) such that

\[
\sup_{x \in \mathbb{R}^d} \{u(x) - v(x)\} = \ell.
\]

Step 1. (Doubling of variables and perturbation)
Let $K \subset \mathcal{O}$ be a compact neighborhood of the set of maximum points of $u - v$ in $\mathcal{O}$. There exists a sequence of $C^3(\mathbb{R}^d) \cap \text{BUC}(\mathbb{R}^d)$ functions $\{\phi_n\}_n$ each of which has second and third derivatives bounded in $\mathbb{R}^d$ and such that $|u - \phi_n| \to 0$ uniformly in $\mathbb{R}^d$ and
\[
\lim_{n \to \infty} \sup_K |Du - D\phi_n| = 0.
\]

Now, let $(x_{n,\varepsilon}, y_{n,\varepsilon}) \in \mathbb{R}^d \times \mathbb{R}^d$ be a global maximum point of $w_{n,\varepsilon}$ over $\mathbb{R}^d \times \mathbb{R}^d$, where
\[
w_{n,\varepsilon}(x, y) := (u(x) - \phi_n(x)) - (v(y) - \phi_n(y)) - \frac{\psi_\kappa(x - y)}{\varepsilon}.
\]

Similarly to Lemma 5.10, one can show that for any $n$
\[
\lim_{\varepsilon \to 0} \limsup_{\kappa \to 0} \frac{|x_{n,\varepsilon} - y_{n,\varepsilon}|^p}{\varepsilon} = 0,
\]
\[
\lim_{\varepsilon \to 0} \limsup_{\kappa \to 0} \frac{(u(x_{n,\varepsilon}) - v(y_{n,\varepsilon}))}{\varepsilon} = \ell.
\]

Observe that $u$ is touched from above at $x_{n,\varepsilon}$ by
\[
\bar{\phi}(x) := u(x_{n,\varepsilon}) - \phi_n(x_{n,\varepsilon}) + \phi_n(x) + \frac{1}{\varepsilon} (\psi_\kappa(x - y_{n,\varepsilon}) - \psi_\kappa(x_{n,\varepsilon} - y_{n,\varepsilon}))
\]
while $v$ is touched from below at $y_{n,\varepsilon}$ by
\[
\underline{\phi}(y) := v(y_{n,\varepsilon}) - \phi_n(y_{n,\varepsilon}) + \phi_n(y) - \frac{1}{\varepsilon} (\psi_\kappa(x_{n,\varepsilon} - y) - \psi_\kappa(x_{n,\varepsilon} - y_{n,\varepsilon})).
\]

Since $u$ is $C^1$ this means first that
\[
Du(x_{n,\varepsilon}) - D\phi_n(x_{n,\varepsilon}) = p(\kappa + |x_{n,\varepsilon} - y_{n,\varepsilon}|^2) \frac{1}{\varepsilon} \frac{x_{n,\varepsilon} - y_{n,\varepsilon}}{\varepsilon}.
\]

There is some small $0 < c$ such that $B_c(x_{n,\varepsilon}) \cup B_c(y_{n,\varepsilon}) \subset K$ if $\varepsilon$ and $\kappa$ are sufficiently small. Therefore,
\[
\lim_{\varepsilon \to 0} \limsup_{\kappa \to 0} \frac{|x_{n,\varepsilon} - y_{n,\varepsilon}|^{p-1}}{\varepsilon} = o_\varepsilon(1).
\]

On the other hand, since $u$ is a viscosity subsolution and $v$ a viscosity supersolution, for any $0 < r < 1$ we have
\[
I(u_r, x_{n,\varepsilon}) \leq 0 \text{ and } I(v_r, y_{n,\varepsilon}) \geq 0,
\]
where (recall Definition 4.1)
\[
u_r(x) := \begin{cases} \bar{\phi}(x) & \text{in } B_r(x_{n,\varepsilon}) \\ u(x) & \text{in } B_r^c(x_{n,\varepsilon}) \end{cases} \quad \text{and} \quad v_r(x) := \begin{cases} \underline{\phi}(x) & \text{in } B_r(y_{n,\varepsilon}) \\ v(x) & \text{in } B_r^c(y_{n,\varepsilon}) \end{cases}.
\]

Step 2. (Equation structure, main term)
Using that $I(\cdot, x)$ has the inf-sup representation in (1.1), it follows that

$$I(v_r, y_{n, \varepsilon}) - I(u_r, x_{n, \varepsilon}) \leq \sup_{\alpha, \beta} \left\{ \hat{L}^{\alpha \beta}(u_r, x_{n, \varepsilon}) - \hat{L}^{\alpha \beta}(v_r, y_{n, \varepsilon}) \right\}$$

$$+ \sup_{\alpha, \beta} \left\{ \hat{L}^{\alpha \beta}(u_r, x_{n, \varepsilon}) - \hat{L}^{\alpha \beta}(v_r, y_{n, \varepsilon}) \right\}$$

$$+ \sup_{\alpha, \beta} \left\{ c_{\alpha \beta}(y_{n, \varepsilon})v(y_{n, \varepsilon}) - c_{\alpha \beta}(x_{n, \varepsilon})u(x_{n, \varepsilon}) \right\}$$

$$+ \sup_{\alpha, \beta} \left\{ f_{\alpha \beta}(y_{n, \varepsilon}) - f_{\alpha \beta}(x_{n, \varepsilon}) \right\}.$$  \hspace{1cm} (6.2)

Let us bound each of the terms on the right hand side of (6.2). As before, the most delicate term is the first one. Fix $\alpha$ and $\beta$, we note that

$$\hat{L}^{\alpha \beta}(u_r, x_{n, \varepsilon}) = \int_{B_r} \delta u_r(x_{n, \varepsilon}, x) \, \hat{d} \mu^{\alpha \beta}_{x_{n, \varepsilon}}(x) + \int_{B_r^c} \delta u(x_{n, \varepsilon}, x) \, \hat{d} \hat{\mu}^{\alpha \beta}_{x_{n, \varepsilon}}(x),$$

$$\hat{L}^{\alpha \beta}(v_r, y_{n, \varepsilon}) = \int_{B_r} \delta v_r(y_{n, \varepsilon}, y) \, \hat{d} \hat{\mu}^{\alpha \beta}_{y_{n, \varepsilon}}(y) + \int_{B_r^c} \delta v(y_{n, \varepsilon}, y) \, \hat{d} \hat{\mu}^{\alpha \beta}_{y_{n, \varepsilon}}(y).$$

Let us choose $\gamma_r \in \text{Adm} (\hat{\mu}^{\alpha \beta}_{x_{n, \varepsilon}, r}, \hat{\mu}^{\alpha \beta}_{y_{n, \varepsilon}, r})$ (using the notation introduced in Remark 6.1) which minimizes the $p$-cost. Denote

$$A_r := (\{0\} \times (B_1 \setminus B_r)) \cup ((B_1 \setminus B_r) \times \{0\}) \cup ((B_1 \setminus B_r) \times (B_1 \setminus B_r)).$$

Since $\delta u(x_{n, \varepsilon}, 0) = \delta v(y_{n, \varepsilon}, 0) = 0$ and

$$\gamma_r ((((B_1 \setminus B_r) \cup \{0\})^c \times \mathbb{R}^d) \cup (\mathbb{R}^d \times ((B_1 \setminus B_r) \cup \{0\})^c)) = 0,$$  \hspace{1cm} (6.3)

we have

$$\int_{B_r} \delta u(x_{n, \varepsilon}, x) \, \hat{d} \hat{\mu}^{\alpha \beta}_{x_{n, \varepsilon}}(x) = \int_{B_r^c} \delta u(x_{n, \varepsilon}, x) \, \hat{d} \hat{\mu}^{\alpha \beta}_{x_{n, \varepsilon}, r}(x)$$

$$= \int_{\mathbb{R}^d \times \mathbb{R}^d} \delta u(x_{n, \varepsilon}, x) \, d\gamma_r(x, y) = \int_{A_r} \delta u(x_{n, \varepsilon}, x) \, d\gamma_r(x, y).$$

Similarly,

$$\int_{B_r^c} \delta v(y_{n, \varepsilon}, y) \, \hat{d} \hat{\mu}^{\alpha \beta}_{y_{n, \varepsilon}, r}(y) = \int_{A_r} \delta v(y_{n, \varepsilon}, y) \, d\gamma_r(x, y).$$

Therefore we obtain

$$\hat{L}^{\alpha \beta}(u_r, x_{n, \varepsilon}) - \hat{L}^{\alpha \beta}(v_r, y_{n, \varepsilon})$$

$$= \int_{B_r} \delta u_r(x_{n, \varepsilon}, x) \, \hat{d} \hat{\mu}^{\alpha \beta}_{x_{n, \varepsilon}}(x) - \int_{B_r} \delta v_r(y_{n, \varepsilon}, y) \, \hat{d} \hat{\mu}^{\alpha \beta}_{y_{n, \varepsilon}}(y)$$

$$+ \int_{A_r} [\delta u(x_{n, \varepsilon}, x) - \delta v(y_{n, \varepsilon}, y)] \, d\gamma_r(x, y).$$
Using that \((x_{n, \varepsilon}, y_{n, \varepsilon})\) is a maximum point of \(w^{n, \varepsilon}\), we have the following pointwise bound for pairs \((x, y) \in A_r\)

\[
\delta u(x_{n, \varepsilon}, x) - \delta u(y_{n, \varepsilon}, y) \leq \frac{C}{\varepsilon} |x - y|^p + \delta \phi(x_{n, \varepsilon}, x) - \delta \phi(y_{n, \varepsilon}, y),
\]

It thus follows (again using \(\delta \phi(x_{n, \varepsilon}, 0) = \delta \phi(y_{n, \varepsilon}, 0) = 0\) and (6.3)) that

\[
\int_{A_r} [\delta u(x_{n, \varepsilon}, x) - \delta u(y_{n, \varepsilon}, y)] d\gamma_r(x, y)
\]

\[
\leq \frac{C}{\varepsilon} \int_{A_r} |x - y|^p d\gamma_r(x, y)
\]

\[
+ \int_{B_r} \delta \phi(x_{n, \varepsilon}, x) d\mathcal{A}^{\alpha \beta}_{x_{n, \varepsilon}, x} - \int_{B_r} \delta \phi(y_{n, \varepsilon}, y) d\mathcal{A}^{\alpha \beta}_{y_{n, \varepsilon}, x},
\]

and since \(\gamma_r\) is the optimizer in \(\text{Adm}(\hat{\mu}_{x_{n, \varepsilon}}, \hat{\mu}_{y_{n, \varepsilon}})\),

\[
\int_{A_r} [\delta u(x_{n, \varepsilon}, x) - \delta u(y_{n, \varepsilon}, y)] d\gamma_r(x, y)
\]

\[
\leq \frac{C}{\varepsilon} d_{L_p}(\hat{\mu}_{x_{n, \varepsilon}, r}, \hat{\mu}_{y_{n, \varepsilon}, r}) + \int_{B_r} \delta \phi(x_{n, \varepsilon}, x) d\mathcal{A}^{\alpha \beta}_{x_{n, \varepsilon}, x} - \int_{B_r} \delta \phi(y_{n, \varepsilon}, y) d\mathcal{A}^{\alpha \beta}_{y_{n, \varepsilon}, x}.
\]

As for the integrals over \(B_r\), note that

\[
\int_{B_r} \delta u_r(x_{n, \varepsilon}, x) d\mathcal{A}^{\alpha \beta}_{x_{n, \varepsilon}, x} - \int_{B_r} \delta u_r(y_{n, \varepsilon}, y) d\mathcal{A}^{\alpha \beta}_{y_{n, \varepsilon}, x}
\]

\[
= \int_{B_r} \delta \phi_{n}(x_{n, \varepsilon}, x) + \frac{1}{\varepsilon} \delta \psi_{\kappa}(\cdot, - y_{n, \varepsilon})(x_{n, \varepsilon}, x) d\mathcal{A}^{\alpha \beta}_{x_{n, \varepsilon}, x}
\]

\[
- \int_{B_r} \delta \phi_{n}(y_{n, \varepsilon}, y) - \frac{1}{\varepsilon} \delta \psi_{\kappa}(x_{n, \varepsilon} - \cdot)(y_{n, \varepsilon}, y) d\mathcal{A}^{\alpha \beta}_{y_{n, \varepsilon}, x}.
\]

Putting the last inequality and last equality together, we have

\[
\hat{L}^{\alpha \beta}(u_r, x_{n, \varepsilon}) - \hat{L}^{\alpha \beta}(v_r, y_{n, \varepsilon})
\]

\[
\leq \frac{C}{\varepsilon} d_{L_p}(\hat{\mu}_{x_{n, \varepsilon}, r}, \hat{\mu}_{y_{n, \varepsilon}, r}) + \hat{L}^{\alpha \beta}(\phi_{n}, x_{n, \varepsilon}) - \hat{L}^{\alpha \beta}(\phi_{n}, y_{n, \varepsilon}) + \frac{C}{\varepsilon} \theta(r),
\]

where we used Assumption E and Lemma 5.7.

Next, we use Remark 6.1 to get \(d_{L_p}(\hat{\mu}_{x_{n, \varepsilon}, r}, \hat{\mu}_{y_{n, \varepsilon}, r}) \leq d_{L_p}(\hat{\mu}_{x_{n, \varepsilon}, r}, \hat{\mu}_{y_{n, \varepsilon}, r}) + \rho(r)\), where \(\rho(r) \to 0\) as \(r \to 0\). Then using Assumption A1 (recall \(s = 1 - 1/p\)) it follows that

\[
d_{L_p}(\hat{\mu}_{x_{n, \varepsilon}, r}, \hat{\mu}_{y_{n, \varepsilon}, r}) \leq C |x_{n, \varepsilon} - y_{n, \varepsilon}|^{p-1} + \rho(r).
\]

Thus

\[
\hat{L}^{\alpha \beta}(u_r, x_{n, \varepsilon}) - \hat{L}^{\alpha \beta}(v_r, y_{n, \varepsilon}) \leq \frac{C}{\varepsilon} |x_{n, \varepsilon} - y_{n, \varepsilon}|^p
\]

\[
+ \hat{L}^{\alpha \beta}(\phi_{n}, x_{n, \varepsilon}) - \hat{L}^{\alpha \beta}(\phi_{n}, y_{n, \varepsilon}) + \frac{C}{\varepsilon} (\theta(r) + \rho(r)).
\]
Letting $r \to 0$, it follows that for every $\kappa$, $n$, and $\varepsilon$,
\[
\limsup_{r \to 0} \sup_{\alpha, \beta} \left\{ \tilde{L}^{\alpha\beta}(u_r, x_{n,\varepsilon}) - \tilde{L}^{\alpha\beta}(v_r, y_{n,\varepsilon}) \right\} 
\leq \frac{C}{\varepsilon} |x_{n,\varepsilon} - y_{n,\varepsilon}|^{p-1} + \sup_{\alpha, \beta} \left\{ \tilde{L}^{\alpha\beta}(\phi_n, x_{n,\varepsilon}) - \tilde{L}^{\alpha\beta}(\phi_n, y_{n,\varepsilon}) \right\}.
\] (6.4)

Step 3. (Equation structure, remaining terms)
For any $r \in (0, 1)$ and any $\alpha, \beta$ we have
\[
\tilde{L}^{\alpha\beta}(u_r, x_{n,\varepsilon}) - \tilde{L}^{\alpha\beta}(v_r, y_{n,\varepsilon}) = \tilde{L}^{\alpha\beta}(u, x_{n,\varepsilon}) - \tilde{L}^{\alpha\beta}(v, y_{n,\varepsilon}).
\]
Furthermore, arguing as in Step 3 of the proof of Theorem 4.4
\[
c_{\alpha\beta}(x_{n,\varepsilon}) u(x_{n,\varepsilon}) - c_{\alpha\beta}(y_{n,\varepsilon}) v(y_{n,\varepsilon}) \geq \lambda(u(x_{n,\varepsilon}) - v(y_{n,\varepsilon})) - ||v||_{L^\infty} \theta(|x_{n,\varepsilon} - y_{n,\varepsilon}|),
\]
\[
|f_{\alpha\beta}(x_{n,\varepsilon}) - f_{\alpha\beta}(y_{n,\varepsilon})| \leq \theta(|x_{n,\varepsilon} - y_{n,\varepsilon}|).
\] (6.6)

Going back to (6.2) and combining it with (6.4)-(6.6), it follows that for any $\kappa, n$, and $\varepsilon$
\[
\limsup_{r \to 0} \left\{ I(v_r, y_{n,\varepsilon}) - I(u_r, x_{n,\varepsilon}) \right\} \leq \frac{C}{\varepsilon} |x_{n,\varepsilon} - y_{n,\varepsilon}|^{p-1} + (1 + ||v||_{L^\infty}) \theta(|x_{n,\varepsilon} - y_{n,\varepsilon}|)
+ \lambda(v(y_{n,\varepsilon}) - u(x_{n,\varepsilon}))
+ \sup_{\alpha, \beta} \left\{ \tilde{L}^{\alpha\beta}(u, x_{n,\varepsilon}) - \tilde{L}^{\alpha\beta}(v, y_{n,\varepsilon}) \right\}
+ \sup_{\alpha, \beta} \left\{ \tilde{L}^{\alpha\beta}(\phi_n, x_{n,\varepsilon}) - \tilde{L}^{\alpha\beta}(\phi_n, y_{n,\varepsilon}) \right\}.
\] (6.7)

Let us handle the last two terms on the right. Using Assumption B and the fact that
\[
\phi_n \in C_0^1(\mathbb{R}^d),
\]
we have for any $\alpha$ and $\beta$,
\[
\tilde{L}^{\alpha\beta}(u, x_{n,\varepsilon}) - \tilde{L}^{\alpha\beta}(v, y_{n,\varepsilon})
= \int_{B^c_1} \left[ \delta u(x_{n,\varepsilon}, x) - \delta v(y_{n,\varepsilon}, x) \right] d\tilde{\mu}_{x_{n,\varepsilon}}^{\alpha\beta}(x) + \int_{B^c_1} \delta v(y_{n,\varepsilon}, y) d\left[ \tilde{\mu}_{x_{n,\varepsilon}}^{\alpha\beta}(y) - \tilde{\mu}_{y_{n,\varepsilon}}^{\alpha\beta}(y) \right]
\leq \int_{B^c_1} \left( \delta \phi_n(x_{n,\varepsilon}, x) - \delta \phi_n(y_{n,\varepsilon}, x) \right) d\tilde{\mu}_{x_{n,\varepsilon}}^{\alpha\beta}(x) + 2 \|v\|_{L^\infty} \theta(\|x_{n,\varepsilon} - y_{n,\varepsilon}\|)
\leq C(n) |x_{n,\varepsilon} - y_{n,\varepsilon}| + 2 \|v\|_{L^\infty} \theta(\|x_{n,\varepsilon} - y_{n,\varepsilon}\|).
\]

Then, we have
\[
\sup_{\alpha, \beta} \left\{ \tilde{L}^{\alpha\beta}(u, x_{n,\varepsilon}) - \tilde{L}^{\alpha\beta}(v, y_{n,\varepsilon}) \right\} \leq C(n) |x_{n,\varepsilon} - y_{n,\varepsilon}| + 2 \|v\|_{L^\infty} \theta(\|x_{n,\varepsilon} - y_{n,\varepsilon}\|).
\] (6.8)

For the other remaining term, we note that
\[
\tilde{L}^{\alpha\beta}(\phi_n, x_{n,\varepsilon}) - \tilde{L}^{\alpha\beta}(\phi_n, y_{n,\varepsilon}) = \int_{B_1} \left[ \delta \phi_n(x_{n,\varepsilon}, x) - \delta \phi_n(y_{n,\varepsilon}, x) \right] d\tilde{\mu}_{x_{n,\varepsilon}}^{\alpha\beta}(x)
+ \int_{B_1} \delta \phi_n(y_{n,\varepsilon}, x) d\tilde{\mu}_{x_{n,\varepsilon}}^{\alpha\beta}(x) - \int_{B_1} \delta \phi_n(y_{n,\varepsilon}, y) d\tilde{\mu}_{y_{n,\varepsilon}}^{\alpha\beta}(y).
\]
Since the third derivatives of $\phi_n$ are bounded, we have

$$\left| \int_{B_1} \left[ \delta\phi_n(x_n,\varepsilon, x) - \delta\phi_n(y_n,\varepsilon, x) \right] d\hat{\mu}_{x_n,\varepsilon}^\alpha (x) \right|$$

$$\leq \int_{B_1} \int_0^1 |(D^2\phi_n(x_n,\varepsilon + sx) - D^2\phi_n(y_n,\varepsilon + sx))x, x(1-s)| ds \ d\hat{\mu}_{x_n,\varepsilon}^\alpha (x)$$

$$\leq C(n)|x_{n,\varepsilon} - y_{n,\varepsilon}|.$$

The remaining integrals are estimated as follows. For $\tau \in (0,1)$, let $\eta_\tau$ be a smooth function such that $0 \leq \eta_\tau \leq 1$, $\eta_\tau \equiv 1$ in $B_\tau(0)$ and $\eta_\tau \equiv 0$ outside of $B_{2\tau}$. Then, we may write

$$\int_{B_1} \delta\phi_n(y_n,\varepsilon, x) d\hat{\mu}_{x_n,\varepsilon}^\alpha (x) = \int_{B_1} (1 - \eta_\tau(x))\delta\phi_n(y_n,\varepsilon, x) d\hat{\mu}_{x_n,\varepsilon}^\alpha (x) + \int_{B_1} \eta_\tau(x)\delta\phi_n(y_n,\varepsilon, x) d\hat{\mu}_{x_n,\varepsilon}^\alpha (x),$$

$$\int_{B_1} \delta\phi_n(y_n,\varepsilon, y) d\hat{\mu}_{y_n,\varepsilon}^\alpha (y) = \int_{B_1} (1 - \eta_\tau(y))\delta\phi_n(y_n,\varepsilon, y) d\hat{\mu}_{y_n,\varepsilon}^\alpha (y) + \int_{B_1} \eta_\tau(y)\delta\phi_n(y_n,\varepsilon, y) d\hat{\mu}_{y_n,\varepsilon}^\alpha (y).$$

Applying Proposition 3.8, together with (4.7), and using again Assumption A1, it is straightforward to observe that for fixed $n$ and $\tau$,

$$\limsup_{\varepsilon \to 0} \sup_{\alpha,\beta} \left| \int_{B_1} (1 - \eta_\tau(x))\delta\phi_n(y_n,\varepsilon, x) d\hat{\mu}_{x_n,\varepsilon}^\alpha (x) - \int_{B_1} (1 - \eta_\tau(x))\delta\phi_n(y_n,\varepsilon, y) d\hat{\mu}_{y_n,\varepsilon}^\alpha (y) \right| = 0.$$

On the other hand, since each $\phi_n$ is $C^3$, we have $|\delta\phi_n(y_n,\varepsilon, x)| \leq C_n|\varepsilon|^2$ for all $x \in B_1$. Therefore

$$\left| \int_{B_1} \eta_\tau(x)\delta\phi_n(y_n,\varepsilon, x) d\hat{\mu}_{x_n,\varepsilon}^\alpha (x) \right| \leq \int_{B_1} |\eta_\tau(x)\delta\phi_n(y_n,\varepsilon, x)| d\hat{\mu}_{x_n,\varepsilon}^\alpha (x),$$

$$\leq C_n \int_{B_\tau} |x|^2 d\hat{\mu}_{x_n,\varepsilon}^\alpha (x),$$

$$\left| \int_{B_1} \eta_\tau(y)\delta\phi_n(y_n,\varepsilon, y) d\hat{\mu}_{y_n,\varepsilon}^\alpha (y) \right| \leq C_n \int_{B_\tau} |y|^2 d\hat{\mu}_{y_n,\varepsilon}^\alpha (y),$$

and we have

$$\limsup_{\tau \to 0} \sup_{\alpha,\beta} \int_{B_\tau} |x|^2 d\hat{\mu}_{x_n,\varepsilon}^\alpha (x) = \limsup_{\tau \to 0} \sup_{\alpha,\beta} \int_{B_\tau} |y|^2 d\hat{\mu}_{y_n,\varepsilon}^\alpha (y) = 0.$$

Gathering these estimates, we conclude that for every $n$,

$$\limsup_{\varepsilon \to 0} \sup_{\alpha,\beta} |\hat{L}_{x_n,\varepsilon}^\alpha (\phi_n) - \hat{L}_y^\alpha (\phi_n)| = 0. \quad (6.9)$$

Step 4. (Using the subsolution and supersolution property)
Using Definition 4.2 we obtain from (6.7) that
\[
\lambda(u(x_{n,\varepsilon}) - v(y_{n,\varepsilon})) \leq C \varepsilon |x_{n,\varepsilon} - y_{n,\varepsilon}|^{p-1} + (1 + \|v\|_{L^\infty}) \theta(|x_{n,\varepsilon} - y_{n,\varepsilon}|) \\
+ \sup_{\alpha, \beta} \{ \hat{L}^{\alpha, \beta}(u, x_{n,\varepsilon}) - \hat{L}^{\alpha, \beta}(v, y_{n,\varepsilon}) \} \\
+ \sup_{\alpha, \beta} \{ \hat{L}^{\alpha, \beta}(\phi_{n, x_{n,\varepsilon}}) - \hat{L}^{\alpha, \beta}(\phi_{n, y_{n,\varepsilon}}) \}.
\] (6.10)

Letting \( \kappa \to 0, \varepsilon \to 0 \) in (6.10) and using (6.1), (6.8) and (6.9), we now obtain
\[
0 < \lambda \leq o_{n}(1),
\]
which gives a contradiction. \( \Box \)

The following is an example of measures satisfying the assumptions of Theorem 6.4.

**Example 6.6.** Let \( d\mu_{x}^{\alpha, \beta}(z) := K_{\alpha, \beta}(x, z)dz \) be such that (5.11) and (5.13) hold for some \( p \in (1, 2] \) and \( \gamma \in (0, 1] \), where \( K \) satisfies (5.12). Using (5.14), Assumption A1 is satisfied if \( \gamma = p - 1 \).

**Assumption F.** There is \( \sigma \in (1, 2) \) and positive constants \( \bar{\lambda} < \bar{\Lambda} \) such that the measures \( \mu_{x}^{\alpha, \beta} \) are all of the form \( d\mu_{x}^{\alpha, \beta}(z) = K_{\alpha, \beta}(x, z)dz \), with \( K_{\alpha, \beta}(x, z) = K_{\alpha, \beta}(x, -z) \), and
\[
\frac{\bar{\lambda}}{|x|^{d+\sigma}} \leq K_{\alpha, \beta}(x, z) \leq \frac{\bar{\Lambda}}{|x|^{d+\sigma}}
\] (6.11)
and
\[
|K_{\alpha, \beta}(x, z) - K_{\alpha, \beta}(y, z)| \leq \frac{\bar{\Lambda}|x - y|^\gamma}{|z|^{d+\sigma}}.
\]

**Corollary 6.7.** Let the measures \( \mu_{x}^{\alpha, \beta} \) be as above. Assume that Assumptions C, D and F hold with some \( \sigma \in (1, 2) \) and \( \gamma > \sigma - 1 \). Then, given a viscosity solution \( u \) and a viscosity subsolution (respectively, supersolution) \( v \) of (1.1) such that \( v \leq u \) (respectively, \( u \leq v \)) in \( \mathcal{O}^c \), we have
\[
v \leq u \text{ in } \mathcal{O} \text{ (respectively, } u \leq v \text{ in } \mathcal{O}).
\]

**Proof.** The proof is an immediate application of Theorem 6.4 with \( p = 1 + \gamma \), the computation in Example 5.12 and the fact that \( u \in C^{1}(\mathcal{O}) \) by Theorem 4.1 in [15]. \( \Box \)

**Remark 6.8.** The comparison result of Corollary 6.7 can be extended to the case \( \sigma = 1 \) if we use Theorem 6.3 instead of Corollary 6.7. The result of Corollary 6.7 can be also extended to the case \( \lambda = 0 \), see Theorem 4.1 in [16].

It is worth noting that in [16], two of the authors obtained uniqueness results under similar assumptions to those of Corollaries 4.5 and 6.7 including Lipschitz-type assumption on the continuity of the kernels with respect to \( x \). However, uniqueness results in [16] cover only \( \sigma \) in the range \((0, 3/2)\), whereas the combination of Corollaries 4.5 and 6.7 (see also Remark 6.8) covers all \( \sigma \) up to 2.
Remark 6.9. The assumption (6.11) used in Corollary 6.7 can be relaxed a great deal. This assumption was used merely in order to guarantee that the viscosity solution is $C^{1+\alpha}$ in the interior. Indeed, interior $C^\alpha$ and $C^{1+\alpha}$ regularity estimates are now available for non-local equations for a far larger class of kernels, including those $K(x,z)$ which may not be symmetric in $z$ or which vanish even for large sets of directions of $z$. See works of Schwab and Silvestre [18, Section 8] and Kriventsov [15].

Appendix A. A variant of the optimal transportation problem

In this appendix, which follows [11], we describe the optimal transport problem “with boundary”. Throughout we make the following assumptions: $\Omega$ is an open subset of $\mathbb{R}^d$ and $\Gamma$ is a compact subset of $\overline{\Omega}$. We are also given a function $c: \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}$, known as the cost. We impose several assumptions on $c(x,y)$ and $\Gamma$, recorded in (A.1), (A.2).

First of all, we assume $c$ satisfies

$$c(x,y) \text{ is continuous; } c(x,y) = c(y,x), \quad c(x,x) = 0, \quad c(x,y) > 0 \text{ if } x \neq y, \forall \ x, y \quad (A.1)$$

Secondly, $\Gamma$ and $c$ must be such that there is a measurable function $P: \overline{\Omega} \to \Gamma$ which plays the role of the “projection” onto $\Gamma$, in the sense that

$$c(x,P(x)) = \inf_{y \in \Gamma} c(x,y). \quad (A.2)$$

Definition A.1. Let $E$ be a Borel subset of $\overline{\Omega}$, we define the function

$$c(x,E) = \inf_{y \in E} c(x,y).$$

Lastly, the following auxiliary cost will be relevant in what follows

$$\tilde{c}(x,y) = \min\{c(x,y), c(x,\Gamma) + c(y,\Gamma)\}.$$   

We also consider the set

$$\mathcal{K} = \{(x,y) \in \overline{\Omega} \times \overline{\Omega} \mid c(x,y) \leq c(x,\Gamma) + c(y,\Gamma)\}. \quad (A.3)$$

Definition A.2. Given $\Omega$ and $\Gamma$ we let $\mathcal{M}_c(\overline{\Omega})$ be the set of positive Borel measures $\mu$ on $\overline{\Omega} \setminus \Gamma$ such that

$$\int_{\overline{\Omega} \setminus \Gamma} c(x,\Gamma) \, d\mu(x) < \infty.$$ 

Definition A.3. Let $\mu, \nu \in \mathcal{M}_c(\overline{\Omega})$. By an admissible coupling of $\mu$ and $\nu$, we mean a positive Borel measure $\gamma$ over $\overline{\Omega} \times \overline{\Omega}$, satisfying $\gamma(\Gamma \times \Gamma) = 0$ and

$$\pi_1^\# \gamma |_{\overline{\Omega} \setminus \Gamma} = \mu, \quad \pi_2^\# \gamma |_{\overline{\Omega} \setminus \Gamma} = \nu.$$ 

The set of admissible couplings will be denoted by $\text{Adm}_\Gamma(\mu, \nu)$.

Note that a measure in $\mathcal{M}_c(\overline{\Omega})$ may fail to have finite mass since $\inf c(x,\Gamma) = 0$. We are now ready to state the optimal transport problem “with boundary”.

Problem A.4. Consider two measures $\mu, \nu \in \mathcal{M}_c(\overline{\Omega})$. Among all admissible measures $\gamma \in \text{Adm}_\Gamma(\mu, \nu)$, find one that minimizes the functional

$$J_c(\gamma) := \int_{\Omega \times \Omega} c(x, y) d\gamma(x, y).$$

We make no claim as to whether all of the assumptions on the cost and $\Omega$ are necessary, but they are sufficiently general for our purposes and make most of the proofs relatively straightforward (for instance, the symmetry assumption on $c(x, y)$ is not necessary but makes the notation simpler). In any case, the costs we care about in the main body of the paper are

$$c_p(x, y) := |x - y|^p, \quad 1 \leq p \leq 2.$$ 

Since we are specially concerned with these costs, we shall write $J_p(\gamma)$ to refer to the above functional when the cost is $c_p(x, y)$. At the same time, the main $\Omega$ and $\Gamma$ we care about are

$$\Omega = \mathbb{R}^d \setminus \{0\}, \quad \text{and} \quad \Gamma = \{0\}.$$ 

Evidently, these sets, together with the costs $c_p$, comply with our requirements. The first basic fact about Problem A.4 is the existence of minimizers. The proof is essentially the same as in the optimal transport case (compactness of the measures and lower semi-continuity of $J_c(\gamma)$) (cf. [3, Theorem 1.5] and [11, Section 2]).

Theorem A.5. Let $\mu, \nu \in \mathcal{M}_c(\overline{\Omega})$. Then $J_c(\gamma) < \infty$ for at least one $\gamma^* \in \text{Adm}_\Gamma(\mu, \nu)$. Moreover, there exists at least one minimizer $\gamma$ for Problem A.4.

Proof. With the map $P$ as in (A.2), we define the measure

$$\gamma^* := (\text{Id} \times P)\# \mu + (P \times \text{Id})\# \nu.$$ 

It is clear that $\gamma^* \in \text{Adm}_\Gamma(\mu, \nu)$. At the same time,

$$J_c(\gamma^*) = \int_{\Omega \times \Omega} c(x, y) \, d\gamma^*(x, y) = \int_{\Omega} c(x, \Gamma) \, d\mu(x) + \int_{\Omega} c(y, \Gamma) \, d\nu(y),$$

and thus $J_c(\gamma^*) < \infty$ since $\mu, \nu \in \mathcal{M}_c(\overline{\Omega})$.

Let us now prove that the infimum is achieved. Let $K$ be any compact subset of $\overline{\Omega} \setminus \Gamma \times \Gamma$. Since $\Gamma \times \Gamma$ and $K$ are compact, we have $d(K, \Gamma \times \Gamma) > 0$. Then there exists a compact subset $\overline{K} \setminus \Gamma$ such that $K \subset \overline{K} \times \overline{K}$. Since $\Gamma$ is compact and (A.1) holds, there is an $\varepsilon_0 > 0$ such that $\inf_{x \in \overline{K}} c(x, \Gamma) > \varepsilon_0$. And thus $\mu(\overline{K}) < +\infty$ since $\mu \in \mathcal{M}_c(\overline{\Omega})$. Similarly, we have $\nu(\overline{K}) < +\infty$. Therefore, if $\gamma \in \text{Adm}_\Gamma(\mu, \nu)$, we have

$$\gamma(K) \leq \mu(\overline{K}) + \nu(\overline{K}) < \infty.$$ 

Since $\mu(\overline{K}) + \nu(\overline{K})$ is independent of $\gamma$, it follows that the set $\text{Adm}_\Gamma(\mu, \nu)$ is sequently weakly compact with respect to functions in $C_0^0(\overline{\Omega} \setminus \overline{\Omega} \setminus \Gamma \times \Gamma)$. Let $\gamma_n$ be a minimizing sequence in $\text{Adm}_\Gamma(\mu, \nu)$, that is a sequence such that $J_c(\gamma_n) \to \inf J_c$ as $n \to \infty$. At the same time, let $c_k$ be a monotone increasing sequence of continuous functions with compact support in $\overline{\Omega} \setminus \overline{\Omega} \setminus \Gamma \times \Gamma$ and such that $c_k(x, y) \to c(x, y)$ locally uniformly in
Using a diagonal argument, there exist a subsequence, still denoted by \( \gamma_n \), and \( \gamma_* \in \text{Adm}_\Gamma(\mu, \nu) \) such that for every fixed \( k \)

\[
\lim_{n \to \infty} \int_{\Omega \times \Omega} c_k(x, y) \, d\gamma_n(x, y) = \int_{\Omega \times \Omega} c_k(x, y) \, d\gamma_*(x, y).
\]

Now, by the monotonicity of the \( c_k \), we have

\[
J_c(\gamma_*) = \int_{\Omega \times \Omega} c(x, y) \, d\gamma_*(x, y) = \sup_k J_{c_k}(\gamma_*),
\]

while for any \( k \) we have

\[
J_{c_k}(\gamma_*) = \lim_{n \to \infty} \int_{\Omega \times \Omega} c_k(x, y) \, d\gamma_n(x, y) \leq \lim_{n \to \infty} J(\gamma_n) = \inf_{\gamma \in \text{Adm}_\Gamma(\mu, \nu)} J(\gamma).
\]

This proves that \( \gamma_* \) achieves the minimum value of \( J_c \) among all admissible plans. \( \square \)

We now characterize minimizers for Problem A.4 using \( c \)-concave functions and \( c \)-cyclical monotonicity.

**Definition A.6.** For a function \( \phi : \Omega \to \mathbb{R} \cup \{ \pm \infty \} \), its \( c \)-transform \( \phi^c : \Omega \to \mathbb{R} \cup \{-\infty\} \) is the function given by

\[
\phi^c(y) = \inf_{x \in \Omega} \{c(x, y) - \phi(x)\}.
\]

A function \( \phi \) is said to be \( c \)-concave if there is some \( \psi \) such that

\[
\phi = \psi^c.
\]

If \( \phi \) and \( \psi \) are two \( c \)-concave functions such that \( \phi = \psi^c \) and \( \psi = \phi^c \) then we say they are \( c \)-conjugate to one another. Just the same, we talk about \( \tilde{c} \)-transforms and \( \tilde{c} \)-concave functions.

**Definition A.7.** A subset of \( \Omega \times \Omega \) is said to be \( c \)-cyclically monotone if given a finite sequence \( \{(x_i, y_i)\}_{i=0}^{n} \) and any permutation \( \sigma \), we have

\[
\sum_{i=1}^{n} c(x_i, y_i) \leq \sum_{i=1}^{n} c(x_i, y_{\sigma(i)}).
\]

If \( c \) is replaced by \( \tilde{c} \), we have \( \tilde{c} \)-cyclical monotonicity.

The following Proposition is a (minor) modification of a well known convex analysis result of Rockafellar (previously extended for \( c \)-concave functions). This modification pertains the set \( \Gamma \) and the costs \( c(x, y) \) and \( \tilde{c}(x, y) \).

**Proposition A.8.** Let \( \gamma \) be a measure concentrated on \( K \) and such that \( \text{spt}(\gamma) \cup \Gamma \times \Gamma \) is \( \tilde{c} \)-cyclically monotone. Then, there are \( c \)-conjugate functions \( \phi \) and \( \psi \) such that

\[
\phi \equiv \psi \equiv 0 \text{ on } \Gamma, \text{ and } \text{spt}(\gamma) \subset \partial^c \phi.
\]
Proof. This follows from the standard optimal transport theory. Indeed, as shown in
the proof of [3, Theorem 1.13, (ii) ⇒ (iii)], since spt(γ) ∩ Γ × Γ is $\tilde{c}$-cyclically
monotone, there must be a $\tilde{c}$-concave function $\phi$ such that
\[ \text{spt}(\gamma) \cup \Gamma \times \Gamma \subset \partial^c \phi. \]
Since any pair $(x, y) \in \Gamma \times \Gamma$ belongs to $\partial^c \phi$, it follows that
\[ \phi(x) + \phi^c(y) = \tilde{c}(x, y) = 0 \quad \forall \ x, y \in \Gamma. \]
We emphasize that the above holds for any two points $x$ and $y$ in $\Gamma$, which in
particular means that $\phi$ and $\phi^c$ are constant on $\Gamma$. Adding a constant to $\phi$ we
can assume without loss of generality that $\phi = 0$ on $\Gamma$, which in turn guarantees
that $\phi^c = 0$ on $\Gamma$ as well.

The function $c(x, \Gamma)$ is $c$-concave since it is an infimum of $c$-concave functions.
In particular, this means that for every $y$, the function $\tilde{c}(x, y)$ is $c$-concave
given that it is the minimum between $c(x, y)$ and $c(x, \Gamma) + c(y, \Gamma)$. What
follows from this is that any $\tilde{c}$-concave function $\phi$ is automatically $c$-concave.

We claim that $\partial^c \phi \cap \mathcal{K} \subset \partial^c \phi$. Indeed, if $(x_0, y_0) \in \mathcal{K}$ is
such that $y_0 \in \partial^c \phi(x_0)$ then
\[ \phi(x) \leq \tilde{c}(x, y_0) - \phi^c(y_0) \leq c(x, y_0) - \phi^c(y_0), \]

since $\tilde{c}(x, y) \leq c(x, y)$ for all $x$ and $y$. Since $(x_0, y_0) \in \mathcal{K}$ we have
\[ \tilde{c}(x_0, y_0) = c(x_0, y_0), \]

so
\[ \phi(x_0) = \tilde{c}(x, y_0) - \phi^c(y_0) = c(x, y_0) - \phi^c(y_0). \]

It follows from this that $(x_0, y_0) \in \partial^c \phi(x)$, and the claim is proved. The same
argument also shows that if $y \in \Gamma$, then $\phi^c(y) = \phi^c(y) = 0$. Since $\phi$ was
chosen so that spt(γ) $\subset \partial^c \phi$ and $\gamma$ is supported in $\mathcal{K}$, it follows
that spt(γ) $\subset \partial^c \phi$. Therefore $\phi$ and $\phi^c$ are the desired
$c$-concjugate functions. \qed

As in the usual optimal transport problem, a basic tool for the analysis of Problem
A.4 is a dual problem. This problem deals with a family of admissible pairs of functions
\[ \text{Adm}^c := \left\{ (\phi, \psi) \mid \phi \text{ and } \psi \text{ are bounded and continuous in } \overline{\Omega}, \right. \]
\[ \phi \equiv \psi \equiv 0 \quad \text{on } \Gamma, \quad \phi(x) + \psi(y) \leq c(x, y) \text{ in } \overline{\Omega} \times \overline{\Omega}. \quad (A.4) \]

We now can state the problem dual to Problem (A.9).

Problem A.9. Among all pairs $(\phi, \psi) \in \text{Adm}^c$, find one that maximizes the functional
\[ J^*(\phi, \psi) := \int_{\overline{\Omega}} \phi(x) \, d\mu(x) + \int_{\overline{\Omega}} \psi(y) \, d\nu(y). \]

The characterization of minimizers in Problem A.4 and maximizers for Problem A.9
is the content of Theorem A.10 and Lemma A.11. In the proof we will make use of
Proposition A.8, together with the characterization of optimizers for the usual optimal
transportation problem [3, Theorem 1.13].

Theorem A.10. Let $\gamma \in \text{Adm}_c(\mu, \nu)$ for two measures $\mu, \nu \in \mathcal{M}_c(\Omega)$. Then $\gamma$ is a
minimizer for Problem A.4 if and only if $\gamma$ is concentrated on the set $\mathcal{K}$ defined in (A.3)
and spt(γ) $\cup \Gamma \times \Gamma$ is a $\tilde{c}$-cyclically monotone set.
Proof. Assume first that $\gamma \in \text{Adm}_\Gamma(\mu, \nu)$ is optimal. Consider $\tilde{\gamma}$, the plan given by

$$\tilde{\gamma} := \gamma|_K + (\pi_1, P \circ \pi_1)^\# \left( \gamma|_{\overline{\Omega} \times \overline{\Omega} \setminus K} \right) + (P \circ \pi_2, \pi_2)^\# \left( \gamma|_{\overline{\Omega} \times \overline{\Omega} \setminus K} \right),$$

where $P$ is as in (A.2). What this plan is meant to do is adjusting the original plan $\gamma$ by shifting the transport of some of the mass so that it is sent to $\Gamma$, whenever it is advantageous to do so (and only for points $(x, y)$ outside of $K$). It can be seen that $\tilde{\gamma} \in \text{Adm}(\mu, \nu)$ and that we have the formula

$$\int_{\overline{\Omega} \times \overline{\Omega}} c(x, y) \, d\tilde{\gamma}(x, y) = \int_K c(x, y) \, d\gamma(x, y) + \int_{\overline{\Omega} \times \overline{\Omega} \setminus K} [c(x, \Gamma) + c(\Gamma, y)] \, d\gamma(x, y).$$

From the definition of $K$, we have $c(x, y) \geq c(x, \Gamma) + c(\Gamma, y)$ outside of $K$, thus

$$\int_{\overline{\Omega} \times \overline{\Omega} \setminus K} [c(x, \Gamma) + c(\Gamma, y)] \, d\gamma(x, y) \leq \int_{\overline{\Omega} \times \overline{\Omega} \setminus K} c(x, y) \, d\gamma(x, y).$$

It follows that

$$\int_{\overline{\Omega} \times \overline{\Omega}} c(x, y) \, d\tilde{\gamma}(x, y) \leq \int_{\overline{\Omega} \times \overline{\Omega}} c(x, y) \, d\gamma(x, y),$$

with strict inequality if and only if $\gamma(\overline{\Omega} \times \overline{\Omega} \setminus K) > 0$. By the optimality of $\gamma$ we then conclude that $\gamma(\overline{\Omega} \times \overline{\Omega} \setminus K) = 0$, that is, $\gamma$ is supported in $K$.

Now, we must show that $\text{spt}(\gamma) \cup \Gamma \times \Gamma$ is $\tilde{c}$-cyclically monotone. We deal first with the case where $\gamma$ has finite mass. In this instance, let us write

$$\tilde{\mu} = \pi^1_{\#} \gamma, \quad \tilde{\nu} = \pi^2_{\#} \gamma. \quad (A.5)$$

Then, as $\tilde{\mu}$ and $\tilde{\nu}$ are the marginals of $\gamma$ (in all of $\overline{\Omega}$), they must have the same total mass which is finite since $\gamma$ has finite mass. Let $\gamma_0$ denote the optimal transport plan between $\tilde{\mu}$ and $\tilde{\nu}$ according to $\tilde{c}$, and let $\tilde{\gamma}_0$ be constructed from $\gamma_0$ in the same way as $\tilde{\gamma}$ was constructed from $\gamma$. Since $\mu_{\overline{\Omega}} = \mu$ and $\nu_{\overline{\Omega}} = \nu$ we have that $\tilde{\gamma}_0$ is a measure in $\text{Adm}_\Gamma(\mu, \nu)$, and the same argument applied to $\tilde{\gamma}$ shows that $\tilde{\gamma}_0$ is supported in $K$. Therefore, we have shown

$$J_{\tilde{c}}(\gamma_0) = J_{\tilde{c}}(\tilde{\gamma}_0) \quad \text{and} \quad J_{\tilde{c}}(\gamma) = J_{\tilde{c}}(\tilde{\gamma}).$$

From the optimality of $\gamma$, we must have

$$J_{\tilde{c}}(\gamma_0) = J_{\tilde{c}}(\gamma).$$

Thus $\gamma$ is an optimal plan for the usual transport problem with cost $\tilde{\gamma}_0$. By optimal transport theory, the support set $\text{spt}(\gamma)$ is $\tilde{c}$-cyclically monotone. To prove that $\text{spt}(\gamma) \cup \Gamma \times \Gamma$ is still $\tilde{c}$-cyclically monotone, simply note that if $\gamma_0$ is any measure supported in $\Gamma \times \Gamma$, then $\gamma + \gamma_0 \in \text{Adm}_\Gamma(\mu, \nu)$ and is optimal if $\gamma$ is optimal. The support of $\gamma + \gamma_0$ is simply $\text{spt}(\gamma) \cup \Gamma \times \Gamma$ so the $\tilde{c}$-cyclic monotonicity of the set follows.

This covers the case where $\gamma$ has finite mass. For the general case, we argue just as in [11, Proposition 2.3], that the one property from the classical optimal transport problem that we needed was that if the support of $\gamma$ is not $\tilde{c}$-cyclically monotone, then $\gamma$ cannot be optimal with respect to $\tilde{c}$. It is worth noting that that even if $\tilde{\mu}$ and $\tilde{\nu}$ do not have finite mass, they are still the marginals of $\gamma$ by definition (A.5), so the set of measures
with marginals $\bar{\mu}$ and $\bar{\nu}$ is non-empty, so one can proceed with the Kantorovich problem as in the standard optimal transport theory. Therefore, the above argument extends to the case of $\gamma$ with infinite mass and we conclude that $\text{spt}(\gamma) \cup \Gamma \times \Gamma$ is $\tilde{c}$-cyclically monotone in all cases.

Conversely, assume that $\gamma$ is supported in $K$ and that $\text{spt}(\gamma) \cup \Gamma \times \Gamma$ is a $\tilde{c}$-cyclically monotone set. Then Proposition A.8 says that there is a function $\phi$ which is $c$-concave, such that $\phi$ and $\phi^c$ both vanish on $\Gamma$, and

$$\text{spt}(\gamma) \subset \partial^c \phi.$$  

In particular, this means that $\phi(x) + \phi^c(y) = c(x, y)$ on $\text{spt}(\gamma)$, so

$$\int_{\Pi \times \Pi} c(x, y) \ d\gamma(x, y) = \int_{\Pi \times \Pi} [\phi(x) + \phi^c(y)] \ d\gamma(x, y),$$

$$= \int_{\Pi(\Gamma) \times \Pi} \phi(x) \ d\gamma(x, y) + \int_{\Pi \times (\Pi \Gamma)} \phi^c(y) \ d\gamma(x, y),$$

$$= \int_{\Pi(\Gamma)} \phi(x) \ d\mu(x) + \int_{\Pi(\Gamma)} \phi^c(y) \ d\nu(y).$$

This suffices to guarantee the optimality of $\gamma$. Indeed, take any $\tilde{\gamma} \in \text{Adm}_\Gamma(\mu, \nu)$, then

$$\int_{\Pi \times \Pi} c(x, y) \ d\tilde{\gamma}(x, y) \geq \int_{\Pi \times \Pi} [\phi(x) + \psi^c(y)] \ d\tilde{\gamma}(x, y)$$

$$= \int_{\Pi \setminus \Gamma} \phi(x) \ d\mu(x) + \int_{\Pi \setminus \Gamma} \phi^c(y) \ d\nu(y) = \int_{\Pi \times \Pi} c(x, y) \ d\gamma(x, y),$$

and we conclude that $\gamma$ achieves the minimum value. \qed

Just as in the usual optimal transport problem, a solution to Problem A.4 corresponds to a solution to Problem A.9, and the corresponding values coincide.

**Lemma A.11.** The problems (A.4) and (A.9) are dual, meaning that

$$\inf_{\gamma \in \text{Adm}_\Gamma(\mu, \nu)} J(\gamma) = \sup_{(\phi, \psi) \in \text{Adm}_c^e} J^*(\phi, \psi).$$

**Proof.** If $(\phi, \psi) \in \text{Adm}_c^e$, then $\phi(x) + \psi(y) \leq c(x, y)$ for all $x$ and $y$ and $\phi \equiv \psi \equiv 0$ on $\Gamma$. Therefore, for any $\gamma \in \text{Adm}_\Gamma(\mu, \nu)$ we have

$$\int_{\Pi \times \Pi} c(x, y) \ d\gamma(x, y) \geq \int_{\Pi \times \Pi} [\phi(x) + \psi(y)] \ d\gamma(x, y)$$

$$= \int_{\Pi \setminus \Gamma} \phi(x) \ d\mu(x) + \int_{\Pi \setminus \Gamma} \psi(y) \ d\nu(y)$$

$$= \int_{\Pi} \phi(x) \ d\mu(x) + \int_{\Pi} \psi(y) \ d\nu(y).$$

Since $(\phi, \psi) \in \text{Adm}_c^e$ and $\gamma \in \text{Adm}_\Gamma(\mu, \nu)$ were arbitrary, it follows that

$$\inf_{\gamma \in \text{Adm}_\Gamma(\mu, \nu)} \int_{\Pi \times \Pi} c(x, y) \ d\gamma(x, y) \geq \sup_{(\phi, \psi) \in \text{Adm}_c^e} \left\{ \int_{\Pi} \phi \ d\mu(x) + \int_{\Pi} \psi \ d\nu(y) \right\}.$$
The reverse inequality follows from Theorem A.10. To see why, let $\pi \in \text{Adm}_F(\mu, \nu)$ be the minimizer, then the theorem says that $\text{spt}(\gamma)$ is $c$-cyclically monotone and is contained in $K$, in which case Proposition A.8 says that there are functions $\phi$ and $\psi$ which are $c$-conjugate, vanish on $\Gamma$, and such that $\phi(x) + \psi(y) = c(x, y)$ for $\gamma$-almost every $(x, y)$. This means that both $(\phi, \psi) \in \text{Adm}_F$ and
\[
\int_{\Omega \times \Omega} c(x, y) \, d\gamma(x, y) = \int_{\Omega \times \Omega} [\phi(x) + \psi(y)] \, d\gamma(x, y) = \int_{\Omega} \phi(x) \, d\mu(x) + \int_{\Omega} \psi(y) \, d\nu(y),
\]
which shows the reverse inequality and completes the proof of the lemma. \hfill \Box

The following lemma is a minor modification of [11, Lemma 2.1] and we omit its proof. The lemma itself is a variant of a standard lemma in optimal transport theory [4, Lemma 5.3.2]. We recall that below $\mathcal{M}_p(\Omega) := \mathcal{M}_c(\Omega)$ for $c(x, y) = |x - y|^p$.

**Lemma A.12.** Let $p \geq 1$ and consider measures $\mu_1, \mu_2, \mu_3 \in \mathcal{M}_p(\Omega)$, $\gamma^{12} \in \text{Adm}_F(\mu_1, \mu_2)$, and $\gamma^{23} \in \text{Adm}_F(\mu_2, \mu_3)$. Then, there is a Borel measure in $\Omega \times \Omega \times \Omega$, denoted $\gamma^{123}$, whose 2-marginals satisfy
\[
\pi^{12}_#\gamma^{23} = \gamma^{12} + \sigma^{12}, \quad \pi^{23}_#\gamma^{12} = \gamma^{23} + \sigma^{23}, \quad (A.6)
\]
where $\sigma^{12}$ and $\sigma^{23}$ are measures concentrated on the set $\{(x, x) \mid x \in \Gamma\}$ and $\pi^{12}(x_1, x_2, x_3) = (x_1, x_2), \pi^{23}(x_1, x_2, x_3) = (x_2, x_3)$.

We can now prove that $d_{L_p}(\mu, \nu)$ is a metric in $\mathcal{M}_p(\Omega)$.

**Theorem A.13.** The quantity
\[
d_{L_p}(\mu, \nu) := \inf_{\gamma \in \text{Adm}_F(\mu, \nu)} \left( \int_{\Omega \times \Omega} |x - y|^p d\gamma(x, y) \right)^{\frac{1}{p}}
\]
defines a metric in $\mathcal{M}_p(\Omega)$.

**Proof.** It is clear that $d_{L_p}(\mu, \nu) = d_{L_p}(\nu, \mu)$ and that $d_{L_p}(\mu, \nu) \geq 0$ for all $\mu$ and $\nu$. Moreover, if $d_{L_p}(\mu, \nu) = 0$ then there is some $\gamma \in \text{Adm}_F(\mu, \nu)$ such that
\[
0 = \int_{\Omega \times \Omega} |x - y|^p \, d\gamma(x, y) \Rightarrow \text{spt}(\gamma) \subset \{(x, y) \in \Omega \times \Omega \mid x = y\}.
\]
This implies that for any $\phi \in C^0_c(\Omega \setminus \Gamma)$ we have
\[
\int_{\Omega \setminus \Gamma} \phi(x) \, d\mu(x) = \int_{\Omega \times \Omega} \phi(x) \, d\gamma(x, y) = \int_{\Omega \times \Omega} \phi(y) \, d\gamma(x, y) = \int_{\Omega \setminus \Gamma} \phi(y) \, d\nu(y),
\]
in other words, $\mu = \nu$. It remains to prove the triangle inequality. Consider measures $\mu_1, \mu_2, \mu_3$ in $\mathcal{M}_p(\Omega)$ and let the measures $\gamma^{12} \in \text{Adm}_F(\mu_1, \mu_2)$ and $\gamma^{23} \in \text{Adm}_F(\mu_2, \mu_3)$ be optimizers for the respective problems. Then Lemma A.12 guarantees there is a measure $\gamma^{123}$ satisfying (A.6).

It will be convenient to denote an element $\Omega \times \Omega \times \Omega$ as $(x_1, x_2, x_3)$. At the same time, the “coordinates” $x_1, x_2, x_3$ define three functions $\Omega \times \Omega \times \Omega \to \Omega \subset \mathbb{R}^d$. With
this in mind, we note that the function $|x_1 - x_3|^p$ is independent of $x_2$, so (denoting $\pi^{13}(x_1, x_2, x_3) = (x_1, x_3)$)

$$d_{L^p}(\mu_1, \mu_3)^p \leq \int_{\Omega} |x_1 - x_3|^p d\pi^{13}_{\#} \gamma_{123}^2(x_1, x_3) = \int_{\Omega} |x_1 - x_3|^p d\gamma_{123}^2(x_1, x_2, x_3)$$

(A.7)

On the other hand, applying the Minkowski’s inequality in $L^p(\Omega \times \Omega \times \Omega, d\gamma_{123}^2)$ for the functions $x_1 - x_2$, and $x_2 - x_3$, we have

$$\left( \int_{\Omega} |x_1 - x_3|^p d\gamma_{123}^2(x_1, x_2, x_3) \right)^{\frac{1}{p}} \leq \left( \int_{\Omega} |x_1 - x_2|^p d\gamma_{123}^2(x_1, x_2, x_3) \right)^{\frac{1}{p}} + \left( \int_{\Omega} |x_2 - x_3|^p d\gamma_{123}^2(x_1, x_2, x_3) \right)^{\frac{1}{p}}.$$

Then, using the optimality of $\gamma_{12}^2$ as well as (A.6),

$$\int_{\Omega} |x_1 - x_2|^p d\gamma_{123}^2(x_1, x_2, x_3) = \int_{\Omega} |x_1 - x_2|^p (\gamma_{12}^2 + \sigma_{12}^2)(x_1, x_2)$$

$$= \int_{\Omega} |x_1 - x_2|^p d\gamma_{12} = d_{L^p}(\mu_1, \mu_2)^p,$$

where the second to last inequality used the fact that $\sigma_{12}^2$ is supported on the diagonal, so that $\sigma_{12}^2$-a.e. we have $|x_1 - x_2| = 0$. Just the same, we can see that

$$\int_{\Omega} |x_2 - x_3|^p d\gamma_{123}^2(x_1, x_2, x_3) = d_{L^p}(\mu_2, \mu_3)^p.$$

Then, recalling (A.7), we conclude that

$$d_{L^p}(\mu_1, \mu_3) \leq d_{L^p}(\mu_1, \mu_2) + d_{L^p}(\mu_2, \mu_3),$$

which finishes the proof that $d_{L^p}(\mu, \nu)$ is a metric. $\square$

**Proof of Proposition 3.8.** For any $\gamma \in \text{Adm}_\Gamma(\mu, \nu)$ we have

$$\int_{B_1} \psi \, d\mu(x) - \int_{B_1} \psi \, d\nu(y) = \int_{\text{spt}(\psi) \times \mathbb{R}^d} \psi(x) \, d\gamma(x, y) - \int_{\mathbb{R}^d \times \text{spt}(\psi)} \psi(y) \, d\gamma(x, y)$$

$$= \int_{A_\psi} [\psi(x) - \psi(y)] \, d\gamma(x, y),$$

where $A_\psi := (\text{spt}(\psi) \times \mathbb{R}^d) \cup (\mathbb{R}^d \times \text{spt}(\psi))$. Then

$$\left| \int_{B_1} \psi \, d\mu(x) - \int_{B_1} \psi \, d\nu(y) \right| \leq \int_{A_\psi} |\psi(x) - \psi(y)| \, d\gamma(x, y) \leq \int_{A_\psi} [\psi]_{\text{Lip}} |x - y| \, d\gamma(x, y).$$
Since spt(ψ) is a positive distance away from Γ, for any admissible γ we have γ(A_ψ) ≤ \mu(spt(ψ)) + \nu(spt(ψ)) < +\infty. Thus, by Hölder’s inequality,
\[
\left| \int_{B_1} \psi \, d\mu - \int_{B_1} \psi \, d\nu \right| \leq \left[ \psi \right]_{Lip} \gamma(A_\psi)^{\frac{p-1}{p}} \left( \int_{A_\psi} |x-y|^p \, d\gamma(x,y) \right)^{\frac{1}{p}}.
\]
Taking infimum over all \gamma \in \text{Adm}(\mu, \nu), we thus obtain
\[
\left| \int_{B_1} \psi \, d\mu - \int_{B_1} \psi \, d\nu \right| \leq \left( \mu(spt(\psi)) + \nu(spt(\psi)) \right)^{\frac{p-1}{p}} \left[ \psi \right]_{Lip} d_{L^p}(\mu, \nu).
\]
\[\square\]

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