Ising Model on periodic and quasi-periodic chains in presence of magnetic field: some exact results

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Abstract
We present a general procedure for calculating the exact partition function of an Ising model on a periodic chain in presence of magnetic field considering both open and closed boundary conditions. Using same procedure on a quasiperiodic (Fibonacci) chain we have established a recurrence relation among partition functions of different Fibonacci generations from \( n \) \textsuperscript{th} to \( (n+6) \)-\textsuperscript{th}. In the large \( N \) limit we find \((2\tau+1)F_{n+1} = F_{n-2}\); where \( \tau \) is the golden mean and \( F_n \) stands for free energy/spin for the \( n \)-\textsuperscript{th} generation. We have also studied chemical potential in both cases.

PACS No: 05.50.+q

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I Introduction
In this paper we have calculated the exact partition function of an Ising model on a periodic chain in presence of magnetic field with open boundary conditions by splitting the transfer matrix into two particular non-commuting matrices. The result is quite nontrivial in contrast to the expression for the partition function of the closed one\cite{1,2}. Since there has been an enormous amount of work on Ising model it is difficult to do justice to all references, however rigorous results and developments can be found in \cite{1,3} and in the review article \cite{4}. We have studied chemical potentials for periodic chain in presence of magnetic field. Studying Ising model on a Fibonacci chain in presence of magnetic field is interesting since the discovery of quasicrystals \cite{5}. Scaling forms of thermodynamic functions for such system have been studied using renormalisation group technique \cite{6} through one step decimation. The ground state and thermodynamic properties of such a system have been studied using renormalization group technique \cite{6,7}. Our formulation helps us to express the partition function of Ising model on Fibonacci chain in presence of magnetic field as a sum of partition functions of usual Ising open chains with coefficients containing Fibonacci symmetry. We have also studied some symmetry properties of the Fibonacci chain. Using a special symmetry property ("Mirror Symmetry") and the usual trace map relation \cite{8,9} ( trace maps and invariants relating to two-letter substitution lattices have been studied in \cite{10} and references therein ) we have established a recurrence relation among the partition functions of different Fibonacci generations. This includes all the partition functions starting from \( n \)-\textsuperscript{th} up to \( (n+6) \)-\textsuperscript{th} generations. We observe that mirror symmetry is a characteristic property of each Fibonacci generation with \( n \)-\textsuperscript{th} and \( (n+6) \)-\textsuperscript{th} generations having same topology \cite{11}. Assuming that the free energy/spin for both open and closed chains for a particular generation are equal in the large \( N \) limit.
II Exact partition function for open Ising chain with magnetic field

The one dimensional Ising model consists of a chain of \( N \) spins \( S_i = \pm 1 \), \( i = 1, 2, \ldots, N \) with nearest neighbour interactions \( \epsilon_{i,i+1} \). The Hamiltonian is given by:

\[
\mathcal{H} = -\sum_{i=1}^{N-1} \epsilon_{i,i+1} S_i S_{i+1} - H \sum_{i=1}^{N} S_i \tag{1}
\]

For a uniform lattice \( \epsilon_{i,i+1} = \epsilon \), the partition function is given by:

\[
Z^o_N(T, H) = \sum_{S_1, S_2, \ldots, S_N = -1}^{+1} f(S_1, S_2)f(S_2, S_3) \cdots f(S_{N-1}, S_N)f_0(S_N, S_1) \tag{2}
\]

with \( f(S_i, S_{i+1}) = \exp[\beta \epsilon S_i S_{i+1} + \frac{1}{2} \beta H(S_i + S_{i+1})] \); \( f_0(S_N, S_1) = [f(S_N, S_1)]_{\epsilon=0} \). Here the superscript \( o \) stands for the chain with open boundary condition. Therefore the partition function (2) can be written in terms of transfer matrix as:

\[
Z^o_N(T, H) = Tr P^{N-1} P_0 \tag{3}
\]

where

\[
P = \sqrt{r} \left( 1 + \frac{\lambda}{r} \right) \sigma_1 = \sqrt{r} \sigma_1 \left( 1 + \frac{\lambda^T}{r} \right) \tag{4}
\]

\[
P_0 = [P]_{\epsilon=0} = (1 + \lambda) \sigma_1 = \sigma_1 (1 + \lambda^T) \tag{5}
\]

with \( r = \exp(-2\beta \epsilon) \), \( \lambda = \left( \begin{array}{cc} 0 & e^{\beta H} \\ e^{-\beta H} & 0 \end{array} \right) \), \( \lambda^T = \left( \begin{array}{cc} 0 & e^{-\beta H} \\ e^{\beta H} & 0 \end{array} \right) \) and \( \sigma_1 = \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \).

The general formula of the partition function for even and odd number of spins (i.e., odd and even number of bonds) can be derived by using equations (3) and (4) as:

\[
Z^o_{2N}(T, H) = r^{N-\frac{1}{2}} Tr(1 + x_1)(1 + x_2) \cdots (1 + x_{2N-1})(1 + \lambda^T) \tag{6}
\]

and

\[
Z^o_{2N+1}(T, H) = r^N Tr(1 + x_1)(1 + x_2) \cdots (1 + x_{2N})(1 + \lambda) \sigma_1 \tag{7}
\]

where

\[
x_{2i+1} = \frac{\lambda}{r}; x_{2i} = \frac{\lambda^T}{r}; i = \text{integer} \tag{8}
\]

The above equations show that \( \lambda, \lambda^T \) are the signatures for the transfer matrices corresponding to bonds in odd and even positions. In the case of a chain with closed boundary condition the last factor in eqn. (2) is \( f(S_N, S_1) \) and consequently the partition function takes the form:

\[
Z^e_{2N}(T, H) = r^N Tr(1 + x_1)(1 + x_2) \cdots (1 + x_{2N}) \tag{9}
\]
\[ Z_{2N+1} = r^{N+\frac{1}{2}} Tr(1 + x_1)(1 + x_2)\ldots(1 + x_{2N+1}) \sigma_1 \] (10)

Here the superscript \( c \) indicates closed chain. One can show by elementary calculation that eqns. (9) and (10) reduce to the well known form[12]

\[ Z_N^c(T, H) = \lambda_+^N + \lambda_-^N \] (11)

where

\[ \lambda_\pm = r^{-\frac{1}{2}}[\cosh(\beta H) \pm \sqrt{\sinh^2(\beta H) + r^2}] \] (12)

are the eigenvalues of the transfer matrix \( P \). The expression for the partition function in the case of an open chain with even number of spins can be derived from eqn. (6) as follows:

\[ Z_{2N}^o(T, H) = r^{N-\frac{1}{2}} Tr(1 + x_1)(1 + x_2)\ldots(1 + x_{2N-1})(1 + \lambda T) \]
\[ = \sqrt{r} Z_{2N}^c(T, H) + r^{N-\frac{1}{2}}(1 - r) Tr(1 + x_1)(1 + x_2)\ldots(1 + x_{2N-1}) \]
\[ = \sqrt{r} Z_{2N}^c(T, H) + \sqrt{r}(1 - r) Z_{2(N-1)}^c(T, H) + r^{N-\frac{1}{2}}(1 - r) \]
\[ \times Tr(1 + x_1)(1 + x_2)\ldots(1 + x_{2N-2})x_{2N-1} \] (13)

The last term in the above expression can be written in terms of the eigenvalues of the transfer matrix \( P \) viz. \( \lambda_\pm \). By following the method of induction:

\[ r^{N-\frac{1}{2}}(1 - r) Tr(1 + x_1)(1 + x_2)\ldots(1 + x_{2N-2})x_{2N-1} \]
\[ = (1 - r)r^{N-\frac{1}{2}} \frac{4}{r^2} \cosh^2(\beta H) \sum_{i=0}^{N-2} \left( \frac{\lambda_+^2}{r} \right)^{N-2-i} \left( \frac{\lambda_-^2}{r^i} \right) \]
\[ = 4(1 - r)r^{-\frac{1}{2}} \cosh^2(\beta H) \frac{\lambda_+^{2(N-1)} - \lambda_-^{2(N-1)}}{\lambda_+^2 - \lambda_-^2} \] (14)

So eqn. (13) becomes:

\[ Z_{2N}^o(T, H) = \sqrt{r} Z_{2N}^c(T, H) + \sqrt{r}(1 - r) Z_{2(N-1)}^c(T, H) \]
\[ + 4(1 - r)r^{-\frac{1}{2}} \cosh^2(\beta H) \frac{\lambda_+^{2(N-1)} - \lambda_-^{2(N-1)}}{\lambda_+^2 - \lambda_-^2} \] (15)

Similarly the expression (7) for the open chain partition function with odd number of spins takes the form:

\[ Z_{2N+1}^o(T, H) = \sqrt{r} Z_{2N+1}^c + 4(1 - r) \cosh(\beta H) \]
\[ \times \frac{\lambda_+^{2N} - \lambda_-^{2N}}{\lambda_+^2 - \lambda_-^2} \] (16)

It can be shown easily that the free energy per spin calculated from equations (11) , (15) and (16) are all same. Thus the forms of the thermodynamic functions are same for both open and closed chains. However as the expressions of the partition function
are different for closed and open chain it will be highly instructive to study the chemical potential of such systems.

IIa Chemical Potential

The chemical potential for an open chain with \( N \) number of spins is given by:

\[
\mu^o_N = \frac{d \log Z^o_N}{d N} = \log \frac{Z^o_{N+1}}{Z^o_N}
\] (17)

The expression for the chemical potential for a system with even number of spin - say \( 2N \); is obtained by adding a spin to the spin to the system and substituting from equations (15) and (16) in the above relation (17) in the large \( N \) limit:

\[
e^{\mu_{\text{even}}} = \nu_{\text{even}}^o = \frac{Z^o_{2N+1}}{Z^o_{2N}} = (\nu_c)^2 \times \frac{\sqrt{r}(\nu_c) + 4(1 - r)cosh(\beta H) \times \frac{1}{\lambda_+^2 - \lambda_-^2}}{\sqrt{r}(\nu_c)^2 + \sqrt{r}(1 - r) + 4r^{\frac{1}{2}}(1 - r)cosh^2 \beta H \times \frac{1}{\lambda_+^2 - \lambda_-^2}}
\] (18)

where \( \nu_c = e^{\mu_c} = \frac{Z^c_{N+1}}{Z^c_N} = \lambda_+ \), \( \mu_c \) being the chemical potential for the closed chain. For \( H = 0 \), \( \lambda_+ = 2cosh \beta \epsilon \) and consequently expression (18) becomes

\[
\nu_{\text{even}}^o = 2cosh \beta \epsilon
\] (19)

Similarly for a system with odd number of spins we can write

\[
e^{\mu_{\text{odd}}} = \nu_{\text{odd}}^o = \frac{Z^o_{2N}}{Z^o_{2N-1}} = (\nu_c)^2 \times \frac{1}{\nu_{\text{even}}^o}
\] (20)

Therefore, \( \nu_{\text{even}}^o \times \nu_{\text{odd}}^o = (\nu_c)^2 \) which implies

\[
\mu_{\text{even}}^o + \mu_{\text{odd}}^o = 2\mu_c
\] (21)

From the above expressions we conclude that the chemical potentials \( \mu_c, \mu_{\text{even}}^o, \mu_{\text{odd}}^o \) are all different. However, for \( H = 0 \)

\[
\mu_c = \mu_{\text{even}}^o = \mu_{\text{odd}}^o = log(2cosh \beta \epsilon)
\] (22)

which shows that the chemical energies for three different conditions of the chain mentioned above are degenerate and it is removed when magnetic field is applied [equation (21)].

III Ising model on Fibonacci chain with magnetic field

A Fibonacci chain can be inflated by two bonds \( L(\text{large}) \) and \( S(\text{small}) \) by the inflation rule \( L \rightarrow LS, S \rightarrow L \). The chain can be represented by the sequence:

\[
L \rightarrow LS \rightarrow LSL \rightarrow LSLLS \rightarrow LSLLSLSL \rightarrow .......
\] (23)

In this case the interaction strengths in the Hamiltonian (1) \( \epsilon_{i,i+1} = \epsilon \) for long bonds and \( \epsilon_{i,i+1} = \bar{\epsilon} \) for the short ones where the bonds are arranged according to the Fibonacci sequence (23). The corresponding partition function of the \( nth \) generation Fibonacci chain having \( N \) spins with \( N - 1 \) bonds is given by:

\[
Z^o_N(F) = Tr\left(PPPPP.....P_0\right)
\] (24)
where for long bonds the transfer matrix $P$ is given by eqn.(4) and for short bonds the transfer matrix $\bar{P}$ is given by eqn.(4) with $r$ replaced by $\bar{r} = r_{\bar{r}}$. Henceforth $Z^o_N(F)$ and $Z^o_N(I)$ will represent partition functions for Ising models on an open Fibonacci chain and on an open regular lattice respectively. The expressions for the partition functions with odd and even number of bonds take the same forms as shown in eqns.(6) and (7) with $x_i$’s given in eqn.(8) for long bonds whereas for short bonds we replace $r$ by $\bar{r}$ in eqn.(8). The explicit expressions for the partition functions for open and closed chains are:

$$Z^o_{2N}(F) = r^{N_L} \bar{r}^{N_S} \text{Tr}(1 + x_1)(1 + x_2)\ldots(1 + x_{2N-1})(1 + \lambda^T) \quad (25)$$

$$Z^o_{2N+1}(F) = r^{N_L} \bar{r}^{N_S} \text{Tr}(1 + x_1)(1 + x_2)\ldots(1 + x_{2N})(1 + \lambda)\sigma_1 \quad (26)$$

Note that the subscript $2N$ in the left hand side of eq.(25) stands for number of spins so that number of bonds is $2N - 1$ which is odd. In eq.(26) number of spins is $2N + 1$ and number of bonds is $2N$ which is even.

Similarly for closed chain eqs. (24),(25) and (26) take the following forms:

$$Z^c_N(F) = \text{Tr}\bar{P}PPPP\ldots\bar{P} \quad (27)$$

$$Z^c_{2N}(F) = r^{N_L+1} \bar{r}^{N_S} \text{Tr}(1 + x_1)(1 + x_2)\ldots(1 + x_{2N})(1 + x_{2N+1}) \quad (28)$$

$$Z^c_{2N+1}(F) = r^{N_L+1} \bar{r}^{N_S} \text{Tr}(1 + x_1)(1 + x_2)\ldots(1 + x_{2N+1})(1 + \lambda)\sigma_1 \quad (29)$$

where $N_L, N_S$ are number of long and short bonds in a particular sequence.

Now $\bar{P}$ is related to $P$ through the following equation:

$$\bar{P} = \sqrt{r}(1 - \frac{r}{\bar{r}})\sigma_1 + \sqrt{\frac{r}{\bar{r}}}P \quad (30)$$

Using eqn.(30) in eqns.(25) and (26) the Fibonacci partition function for any generation can be written in terms of open Ising partition functions as follows:

$$Z^o_{2N}(F) = h_0(\epsilon, \bar{\epsilon}) + \sum_{i=1}^{N} h_{2i}(\epsilon, \bar{\epsilon})Z^o_{2i}(I) \quad (31)$$

$$Z^o_{2N-1}(F) = l_0(\epsilon, \bar{\epsilon}) + \sum_{i=1}^{N} l_{2i-1}(\epsilon, \bar{\epsilon})Z^o_{2i-1}(I) \quad (32)$$

where $Z^o_{2i}(I)$ and $Z^o_{2i-1}(I)$ are given by eqns.(15) and (16) respectively. We observe that the quasiperiodic nature of the Fibonacci chain is encoded in the functions $h(\epsilon, \bar{\epsilon})$ and $l(\epsilon, \bar{\epsilon})$. Though for small generations these functions can be derived exactly still we could not find out their general forms.

### IIIa Recurrence relation among partition functions

To circumvent the above difficulty we study the recurrence relations among the partition functions of different Fibonacci generations. A survey of different Fibonacci generations depicted by eqn.(23) shows a symmetric pattern in terms of the number of bonds, viz.,
The trace map relation [8,9] was introduced to study the spectrum of 1D Schrodinger equation in a discontinuous quasiperiodic potential. The use of trace map in different kind of substitution lattices have been studied in the review article [4] with many relevant references therein.

Let \(P_{n-2}\) be the \((n-2)th\) Fibonacci generation with even number of bonds. This automatically ensures that the previous as well as the next two consecutive generations will have odd number of bonds. The recurrence relation for Fibonacci generations is given by:

\[
P_n = P_{n-1}P_{n-2}
\]

(34)

Now adding a term \(D_{n-2}P_{n-3}\) in the above equation gives

\[
P_n + D_{n-2}P_{n-3} = P_{n-1}P_{n-2} + D_{n-2}P_{n-3}
\]

(35)

where \(D_{n-2} = Det(P_{n-2})\). The following operations are applied sequentially on eqn.(35):

1. substitute \(P_{n-2}^{-1}P_{n-1}\) in place of \(P_{n-3}\) on the right hand side
2. finally use Cayley-Hamilton theorem to get the usual trace map relation [8] on the Fibonacci lattice:

\[
TrP_n = TrP_{n-1}TrP_{n-2} - D_{n-2}TrP_{n-3}
\]

(36)

The above equation will be necessary for calculating recurrence relation among different Fibonacci generations. For this purpose we must understand symmetry properties of Fibonacci chain. Inspecting different generations of the Fibonacci chain it reveals that if the total number of bonds \(N\) of a particular generation is odd then there is a mirror reflection symmetry around the \(\left(\frac{N-1}{2}\right)th\) bond; except the last two bonds. If the special bond around which mirror symmetry occurs is a short(long) one the Fibonacci generation will have equal number of odd and even short(long) bonds. However if the total number of bonds \(N\) is even, the mirror reflection symmetry is around a cluster of two successive long bonds at the \(\left(\frac{N}{2}\right)th\) and \(\left(\frac{N-2}{2}\right)th\) positions of the chain. So ”Mirror reflection symmetry” is a characteristic property of a Fibonacci chain.

The \(n\)th and \((n \pm 3)th\) generations have the mirror reflection symmetry property around the same kind of bond with last two bonds interchanged, while the \((n \pm 6)th\) generations are topologically same as the \(n\)th one.

Using recurrence relation (34) we can write

\[
D_{n-2}P_{n-3} = D_{n-2}P_{n-2}^{-1}P_{n-1}
\]

(37)

Using Cayley-Hamilton theorem on the right hand side of eq.(37) we get:
Multiplying eq.(38) by $P$ from the right and taking trace we obtain:

$$D_{n-2}Z_{n-3}^c = (Tr P_{n-2})Z_{n-1}^c - Tr(P_{n-2}P_{n-1}P)$$

(39)

In a similar fashion we obtain:

$$D_{n-2}Z_{n-3}^o = (Tr P_{n-2})Z_{n-1}^o - Tr(P_{n-2}P_{n-1}P_0)$$

(40)

The above relations clearly show that one cannot obtain a recurrence relation among the partition functions through the trace map relation alone. To circumvent this difficulty we take recourse to mirror symmetry property. The expression $P_{n-2}P_{n-1}$ in eqns.(39) and (40) is similar to $P_n = P_{n-1}P_{n-2}$ with last two bonds interchanged, i.e., both of them have the same mirror symmetric part $\Omega_n$. Therefore eqns.(39) and (40) can be written as:

$$D_{n-2}Z_{n-3}^c = Z_{n-1}^c(Tr P_{n-2}) - Tr(\Omega_{n-1}P\bar{P}P)$$

(41)

$$D_{n-2}Z_{n-3}^o = Z_{n-1}^o(Tr P_{n-2}) - Tr(\Omega_{n-1}P\bar{P}P_0)$$

(42)

The transfer matrix has the property that $P = P^T$ and $\bar{P} = \bar{P}^T$. If such transfer matrices are arranged in a mirror symmetric fashion then the resulting matrix $(\Omega_n)$ will have the following properties:

i) Off diagonal elements are same i.e., $(\omega_n)_{12} = (\omega_n)_{21}$

ii) Diagonal elements are not same but satisfy the condition:

$$(\omega_n)_{11}(p, q) = (\omega_n)_{22}(q, p); \text{ where } p = e^{\beta H}, q = e^{-\beta H}.$$\

Thus the matrix $\Omega_n$ in eqns. (41) and (42) is of the form:

$$\Omega_n = \begin{pmatrix}
(\omega_n)_{11} & (\omega_n)_{12} \\
(\omega_n)_{21} & (\omega_n)_{22}
\end{pmatrix}$$\

Equns.(41) and (42) can be written explicitly in the following way:

$$D_{n-2}Z_{n-3}^c = Z_{n-1}^c(Tr P_{n-2}) - r\sqrt{t}[((\omega_n)_{11}(y + \frac{pu}{r}) + (\omega_n)_{12}(u + v + \frac{px + qy}{r}) + (\omega_n)_{22}(x + \frac{qv}{r})]$$

(43)

$$D_{n-2}Z_{n-3}^o = Z_{n-1}^o(Tr P_{n-2}) - \sqrt{t}\sqrt{r}[((\omega_n)_{11}(y + pu) + (\omega_n)_{12}(u + v + px + qy) + (\omega_n)_{22}(x + qv)]$$

(44)

where we have used

$$P\bar{P} = \sqrt{\frac{t}{rt}}\begin{pmatrix} u & y \\ x & v \end{pmatrix}$$

with $x = \frac{p}{r} + \frac{q}{r}, y = \frac{p}{r} + \frac{q}{r}, u = 1 + \frac{p^2}{rt}$ and $v = 1 + \frac{q^2}{rt}$. Eliminating $Tr P_{n-2}$ from eqns.(43) and (44) we have:
are obtained from the usual formulae:

\[
\frac{yV_{n-1} + puV'_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}}(\omega_n)_{11} + \frac{xV_{n-1} + qvV'_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}}(\omega_n)_{22} + (\omega_n)_{12}
\]

\[
= \frac{D_{n-2}}{r \sqrt{r}} \times \frac{Z^c_{n-3}Z^c_{n-3} - Z^o_{n-1}Z^o_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}}
\]

(45)

where

\[
V_{n-1} = \frac{Z^c_{n-1}}{\sqrt{r}} - Z^o_{n-1} \quad \text{and} \quad V'_{n-1} = \frac{Z^c_{n-1}}{r} - \frac{Z^o_{n-1}}{r}.
\]

To solve for \((\omega_n)_{11}, (\omega_n)_{12}\) and \((\omega_n)_{22}\) we need another two equations. These equations are obtained from the usual formulae:

\[
Z^c_n = Tr(\Omega_n \bar{P}PP), \quad Z^o_n = Tr(\Omega_n \bar{P}PP_0).
\]

The explicit forms of these two relations are:

\[
\frac{x + pu}{u + v + \frac{qx + py}{r}}(\omega_n)_{11} + \frac{y + qu}{u + v + \frac{qx + py}{r}}(\omega_n)_{22} + (\omega_n)_{12} = \frac{Z^c}{r \sqrt{r}} \times \frac{1}{u + v + \frac{qx + py}{r}}
\]

(46)

\[
\frac{x + pu}{u + v + qx + py}(\omega_n)_{11} + \frac{y + qv}{u + v + qx + py}(\omega_n)_{22} + (\omega_n)_{12} = \frac{Z^o_n}{\sqrt{r}} \times \frac{1}{u + v + qx + py}
\]

(47)

By elementary calculation one obtains:

\[
(\omega_n)_{11}(x, y) = \frac{\Gamma_{n-1} - \Gamma'_{n-1}}{\Gamma_{n-1}\Lambda_{n-1} - \Gamma'_{n-1}\Lambda_{n-1}} \times \frac{D_{n-2}}{r \sqrt{r}} \times \Delta_{n-1} - \frac{1}{\sqrt{r}} \times \frac{1}{\Gamma_{n-1}\Lambda_{n-1} - \Gamma'_{n-1}\Lambda_{n-1}} \times (\Gamma_{n-1}Z^cK' - \Gamma'_{n-1}Z^cK)
\]

(48)

\[
(\omega_n)_{22}(x, y) = \frac{-\Lambda_{n-1} - \Lambda'_{n-1}}{\Gamma_{n-1}\Lambda_{n-1} - \Gamma'_{n-1}\Lambda_{n-1}} \times \frac{D_{n-2}}{r \sqrt{r}} \times \Delta_{n-1} + \frac{1}{\sqrt{r}} \times \frac{1}{\Gamma_{n-1}\Lambda_{n-1} - \Gamma'_{n-1}\Lambda_{n-1}} \times (\Lambda_{n-1}Z^oK' - \Lambda'_{n-1}Z^oK)
\]

(49)

and

\[
(\omega_n)_{12}(x, y) = \frac{1}{(qx + py)(1 - \frac{1}{r})} \left[ - \frac{1}{\sqrt{r}} \times V_n - (1 - \frac{1}{r}) \left( pu(\omega_n)_{11} + qv(\omega_n)_{22} \right) \right]
\]

(50)

where

\[
\Lambda_{n-1}(x, y) = \frac{yV_{n-1} + puV'_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}} - \frac{x + pu}{u + v + \frac{qx + py}{r}}
\]

(51)
\[
\begin{align*}
\Lambda'_{n-1}(x, y) &= \frac{y V_{n-1} + pu V'_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}} - \frac{x + pu}{u + v + qx + py} \\
\Gamma_{n-1}(x, y) &= \frac{x V_{n-1} + qv V'_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}} - \frac{y + qv}{u + v + qx + py} \\
\Gamma'_{n-1}(x, y) &= \frac{x V_{n-1} + qv V'_{n-1}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}} - \frac{y + qv}{u + v + qx + py} \\
\Delta_{n-1}(x, y) &= \frac{Z^o_{n-1} Z^c_{n-3} - Z^c_{n-1} Z^o_{n-3}}{(u + v)V_{n-1} + (px + qy)V'_{n-1}} \\
K(x, y) &= \frac{1}{u + v + \frac{qx + py}{r}} \\
K'(x, y) &= \frac{1}{u + v + qx + py}
\end{align*}
\]

Eliminating \( D_{n-2} \) from eqns.(43) and (44) we obtain:

\[
\begin{align*}
Tr P_{n-2} &= \frac{r \sqrt{r} \times V_{n-3}}{Z^o_{n-1} Z^c_{n-3} - Z^c_{n-1} Z^o_{n-3}} \left[ \left( \alpha_{yx} + \beta_{yx} \frac{V'_{n-3}}{V_{n-3}} \right) \left( \omega_n \right)_{11}(x, y) \right. \\
&+ \left( \alpha'_{yx} + \beta'_{yx} \frac{V'_{n-3}}{V_{n-3}} \right) \left( \omega_n \right)_{22}(x, y) \\
&- \left. \frac{1}{\sqrt{r} \sqrt{1 - \frac{1}{r}} (px + qy)} \left( (u + v) + (px + qy) \frac{V'_{n-3}}{V_{n-3}} \right) V_n \right]
\end{align*}
\]

where
\[
\begin{align*}
\alpha_{xy} &= y - \frac{u + v}{py + qx} \times pu, & \beta_{xy} &= \frac{(x - y)(q - p)}{py + qx} \times pu \\
\alpha'_{xy} &= x - \frac{u + v}{py + qx} \times qv, & \beta'_{xy} &= \frac{(x - y)(q - p)}{py + qx} \times qv
\end{align*}
\]

In general \( n \)th and \( (n \pm 2) \)th generations have same arrangement of the last two bonds apart from their respective mirror symmetric parts. That is why \( Tr P_n \) and \( Tr P_{n \pm 2} \) will have similar expressions. Since we have assumed \( P_n = \Omega_n PP \) it follows from the expression (33) that \( P_{n-3} = \Omega_{n-3} PP \). Therefore proceeding in a similar way as above we get:

\[
\begin{align*}
Tr P_{n-3} &= \frac{r \sqrt{r} \times V_{n-4}}{Z^o_{n-2} Z^c_{n-4} - Z^c_{n-2} Z^o_{n-4}} \left[ \left( \alpha_{yx} + \beta_{yx} \frac{V'_{n-4}}{V_{n-4}} \right) \left( \omega_{n-1} \right)_{11}(y, x) \right. \\
&+ \left( \alpha'_{yx} + \beta'_{yx} \frac{V'_{n-4}}{V_{n-4}} \right) \left( \omega_{n-1} \right)_{11}(y, x) \\
&- \left. \frac{1}{\sqrt{r} \sqrt{1 - \frac{1}{r}} (px + qy)} \left( (u + v) + (qx + py) \frac{V'_{n-4}}{V_{n-4}} \right) V_{n-1} \right]
\end{align*}
\]

Using eqns. (58) and (59) and similar expressions for \( Tr P_n \), \( Tr P_{n-1} \) in the trace map relation (36) we obtain the following recurrence relation among partition functions of different Fibonacci generations as:
same for any large odd generation. Similarly, for large odd number of spins the partition functions can be written as:

\[ Z_n = \frac{1}{\sqrt{r}(1 - \frac{1}{r})(qx + py)} (u + v + (px + qy) V_{n-1}) V_{n+2} \]

\[ \times \frac{V_{n-3}}{Z_{n-1}^o Z_{n-3}^o - Z_{n-2}^o Z_{n-3}^c} \left[ (\alpha_{xy} + \beta_{yx} V'_{n-2}/V_{n-2}) (\omega_{n+1})_1 (y, x) \right. \]

\[ \left. + (\alpha'_{xy} + \beta'_{yx} V'_{n-3}/V_{n-3}) (\omega_{n+1})_2 (y, x) \right) \]

\[ - \frac{D_{n-2} V_{n-4}}{Z_{n-2}^o Z_{n-4}^o - Z_{n-3}^o Z_{n-4}^c} \left[ (\alpha_{xy} + \beta_{yx} V'_{n-4}/V_{n-4}) (\omega_{n+1})_1 (y, x) \right. \]

\[ \left. + (\alpha'_{xy} + \beta'_{yx} V'_{n-3}/V_{n-3}) (\omega_{n+1})_2 (y, x) \right) \]

(60)

The above equation reveals the recurrence relation among the partition functions of different Fibonacci generations from \((n - 4) - th\) to \((n + 2) - th\). The partition functions have entered in the above equation through the quantities given by equations from (48) to (57). This is in conformity with the symmetry properties of the Fibonacci chain. It is worth noticing that in the recurrence relation above, the partition functions for both closed and open chains have entered. The procedure we have presented here through symmetry properties of the chain is generic to any substitution lattice.

**IIIb Recurrence relation in the large \(N\) limit**

Recurrence relation (60) can be written in a much simpler form in the large \(N\) limit. To achieve this we notice that in equation (60) both open and closed partition functions of a particular generation occur simultaneously. From the physical consideration that thermodynamic quantities must be same in the large \(N\) limit for both open and closed partition functions of a particular generation, we assume that the free energy per spin is same in both cases in a particular generation. Therefore, for the \(n - th\) generation relation between the partition functions can be written as:

\[ Z^c_n(F) = f_n(r, \bar{r}, H) Z^o_n(F) \]  

(61)

where \(f_n(r, \bar{r}, H)\) is independent of the number of spins, \(N_n + 1\); \(N_n\) being the number of bonds in the \(n - th\) generation.

**Case I: Ising Limit \( r = \bar{r} \), \( H \neq 0 \)**

Considering equations (25) and (28) we can write equation (61) in the following form:

\[ Z^c_{2M}(F)|_{\epsilon = \bar{\epsilon}} = \left[ \sqrt{r} + \sqrt{r}(1 - r) \frac{1}{\lambda_+^2} + 4(1 - r)r^{1/2}\cosh^2 \beta H \times \frac{1}{\lambda_+^2 (\lambda_+^2 - \lambda_-^2)} \right]^{-1} Z^o_{2M}(F)|_{\epsilon = \bar{\epsilon}} \]  

(62)

which is equivalent to equation (15) in the large \(N\) limit where \(2M\) stands for the number of spins in a large odd generation. From (61) and (62) we conclude that \(f_n^{odd}(r = \bar{r}, H)\) is same for any large odd generation. Similarly, for large odd number of spins \(2N + 1\) in a
large even generation, equations (26) and (29) give:

$$Z^c_{2N+1}(F)|_{\epsilon=\bar{\epsilon}} = \left[ \sqrt{r} + 4(1 - r) \cosh \beta R \times \frac{1}{\lambda_+ (\lambda_+^2 - \lambda_-^2)} \right]^{-1} Z^o_{2N+1}(F)|_{\epsilon=\bar{\epsilon}}$$  \hspace{1cm} (63)

which is equivalent to equation (16) in the large \(N\) limit. From (61) and (63) we conclude that \(f^\text{even}_{n}(r = \bar{r}, H)\) is same for any large even generation.

**Case II: \(r \neq \bar{r}, H = 0\)**

Equations (25) and (28) give the exact expressions for open and closed partition functions with odd number of bonds in the following form:

$$Z^o_{2M}(F)|_{H=0} = 2^{2M}(\cosh \beta \epsilon)^N L(\cosh \beta \bar{\epsilon})^N S$$  \hspace{1cm} (64)

and

$$Z^c_{2M}(F)|_{H=0} = 2^{2M}[(\cosh \beta \epsilon)^N L + 1(\cosh \beta \bar{\epsilon})^N S + (\sinh \beta \epsilon)^N L + 1(\sinh \beta \bar{\epsilon})^N S]$$  \hspace{1cm} (65)

It follows from equations (64) and (65) that in the large \(N\) limit

$$Z^c_{2M}(F)|_{H=0} = (\cosh \beta \epsilon)Z^o_{2M}(F)|_{H=0}$$  \hspace{1cm} (66)

Equations (26), (29) give the exact expressions for open and closed partition functions with even number of bonds as:

$$Z^o_{2N+1}(F)|_{H=0} = 2^{2N+1}(\cosh \beta \epsilon)^N L(\cosh \beta \bar{\epsilon})^N S$$  \hspace{1cm} (67)

and

$$Z^c_{2N+1}(F)|_{H=0} = 2^{2N+1}[(\cosh \beta \epsilon)^N L + 1(\cosh \beta \bar{\epsilon})^N S - (\sinh \beta \epsilon)^N L + 1(\sinh \beta \bar{\epsilon})^N S]$$  \hspace{1cm} (68)

It follows from equations (67) and (68) that in the large \(N\) limit

$$Z^c_{2N+1}(F)|_{H=0} = (\cosh \beta \epsilon)Z^o_{2N+1}(F)|_{H=0}$$  \hspace{1cm} (69)

From equations (61), (66) and (69) we conclude that \(f^\text{odd}_{n}(r \neq \bar{r}, H = 0)\) \([f^\text{even}_{n}(r \neq \bar{r}, H = 0)\)] is same for any large odd [even] generation.

Considering the above cases we make the assumption that in the case of Fibonacci with \(H \neq 0\), all \(f^\text{even}_{n}(r, \bar{r}, H)\) in equation (61) for large odd generation number are equal and all \(f^\text{odd}_{n}(r, \bar{r}, H)\) in equation (61) for large even generation number are equal; but \(f^\text{even}_{n}(r, \bar{r}, H) \neq f^\text{odd}_{n}(r, \bar{r}, H)\).

Using the above properties it reveals that the expression \(Z^o_{n-1}Z^c_{n-3} - Z^c_{n-1}Z^o_{n-3}\) appearing in equation (60) vanishes. As a consequence equation (60) becomes:

$$\left(\omega_{n+1}\right)_{11}(y, x)(\alpha_{yx} + \beta_{yx} V'_{n-2} V_{n-2}) + \left(\omega_{n+1}\right)_{22}(y, x)(\alpha'_{yx} + \beta'_{yx} V'_{n-2} V_{n-2}) \times \frac{1}{\sqrt{r}(1 - \frac{1}{r})(px + qy)} (u + v + (qx + py) (V'_{n-2} V_{n-2}) f^\text{even}_{n+1}(x, y)) (\sqrt{r} - 1) Z^o_{n+1} = 0$$  \hspace{1cm} (70)

where
\[(\omega_{n+1})_{11}(y, x) = \left[ \frac{\Gamma_n - \Gamma'_n}{\Gamma_n n_n - \Gamma'_n n_n} \times \frac{D_{n-1}}{r^{2/\tau}} \times \frac{\Delta_n}{Z_{n-2}} \right](y, x) \times Z^0_{n-2} \]

\[-\frac{\Gamma_n K' - \Gamma'_n K'_{even}}{\sqrt{r^{2/\tau} (\Gamma_n n_n - \Gamma'_n n_n)}}(y, x) \times Z^0_{n+1} \]

\[= \Phi_{n-2}^1(y, x)Z^0_{n-2} - \Phi_{n+1}^1(y, x)Z^0_{n+1} \quad (71)\]

Here \(\Phi(r, \bar{r}, H)\)'s are independent of the partition functions. Similarly, one can write

\[(\omega_{n+1})_{22}(y, x) = \Phi_{n-2}^{22}(y, x)Z^0_{n-2} - \Phi_{n+1}^{22}(y, x)Z^0_{n+1} \]

In reducing \((\omega_{n+1})_{11}(y, x)\) and \((\omega_{n+1})_{22}(y, x)\) in the form \((71)\) and \((72)\) we have used the simplified expression

\[\frac{V_n'}{V_n} = \frac{f^{odd} - 1}{r} \quad (73)\]

Since we have assumed \((n - 2) - \text{th} \) generation to be even; therefore \((n - 1) - \text{th} \), \(n - \text{th}\) generations will be odd and \((n + 1) - \text{th} \) generation will be even.

From equations \((70),(71)\) and \((72)\) we can write

\[\left\{ \Phi_{n+1}^1(y, x)(\alpha_{yx} + \beta_{yx} V_{n-2}') + \Phi_{n+1}^{22}(y, x)(\alpha'_{yx} + \beta'_{yx} V_{n-2}') \right\} \]

\[+ \frac{1}{r^{2/\tau}(1 - \frac{2}{\tau})} \left[ u + v + (q x + p y) V_{n-2}' \right] \left( \frac{f_{even}}{\sqrt{r}} - 1 \right) \}

\[= \left\{ \Phi_{n-2}^1(y, x) + \Phi_{n-2}^{22}(y, x) \right\} Z^0_{n-2} \]

or,

\[\Psi_{n+1}(r, \bar{r}, H) Z^0_{n+1} = \Psi_{n-2}(r, \bar{r}, H) Z^0_{n-2} \quad (74)\]

In terms of free energy per spin equation \((74)\) takes the form

\[(N_{n+1} + 1)F_{n+1} - (N_{n-2} + 1)F_{n+2} = \beta \log \frac{\Psi_{n-2}}{\Psi_{n+1}} \quad (75)\]

Using the Fibonacci recurrence relation among the number of bonds expression \((75)\) reduces to

\[(2\tau + 1)F_{n+1} = F_{n-2} \quad (76)\]

where \(\tau = \lim_{N_n \to \infty} \frac{N_L}{N_S} \), the golden mean.

Equation \((76)\) shows that the free energy per spin for consecutive even generations are scaled by a factor \((2\tau + 1)\) and therefore the thermodynamic quantities will be scaled in a similar fashion.

**IIIc Chemical Potential for H=0**

For calculating chemical potential we must know the exact form of the partition function. Since we do not know the exact partition function for finite \(H\) we calculate it for \(H = 0\).

Let \(N_{n-2}\) denote the number of bonds in the \((n - 2) - \text{th} \) generation which is even. We add another spin at the end with a long bond, say, to the open chain. Then using equation \((26)\) we get

\[Z^{0}_{N_{n-2}+2}(F)|_{H=0} = r^{\frac{N_{L+1}}{2}} r^{\frac{N_{S}}{2}} \frac{1}{Tr} (1 + \frac{\sigma_{L+1}}{r}) \left(1 + \frac{\sigma_{L+1}}{r} \right)^{N_{S}} (1 + \sigma_{1})\]

\[= 2^{N_{n-2}+2}(\cosh \beta \epsilon)^{N_{L+1}}(\cosh \beta \epsilon)^{N_{S}} \quad (77)\]
while the exact expression for the partition function with the open boundary condition in the $(n-2)$ -th generation is given in equation (67) as ,

$$Z_{N_n-2+1}^0|_{H=0} = 2^{N_n-2+1}(\cosh \beta \epsilon)^{NL}(\cosh \beta \bar{\epsilon})^{N_S}$$ (78)

For large generation the expression for chemical potential as defined in equation (17) is

$$\mu_{\text{even}}^o(F)|_{H=0} = \nu_{\text{even}}^o(F)|_{H=0} = \frac{Z_{N_n-2+2}^0(F)}{Z_{N_n-2+1}^0(F)} = 2\cosh \beta \epsilon (= 2\cosh \beta \bar{\epsilon})$$ (79)

The expression in the parenthesis stand for addition of a spin at the end of the chain with a short bond .

In a similar fashion , the chemical potential for closed chain is

$$\nu_{\text{even}}^c = 2\cosh \beta \epsilon (or 2\cosh \beta \bar{\epsilon})$$ (80)

We would have got the same result had we considered odd generation. Comparing expressions (79) and (80) with that of (22) we conclude that , in absence of external magnetic field the chemical potential do not depend upon the underlying lattice structure as long as we consider nearest neighbour interaction.

**IV Conclusion**

We have found an exact expression for the partition function of an Ising model on a regular lattice with open boundary conditions in presence of magnetic field .This always includes closed partition functions because of the fact that out of four different spin configurations at the end points of the chain , two configurations $\uparrow\uparrow$ and $\downarrow\downarrow$ satisfy closed boundary conditions.Free energy per spin are equal for both open and closed chains in the large $N$ limit , hence thermodynamic quantities are same in both cases ; but chemical potentials are different . For aperiodic chain we have established the recurrence relation among partition functions of different Fibonacci generations and we have also shown that this can only be obtained by using trace map relations and the characteristic symmetry property viz.,"Mirror Symmetry" of Fibonacci generations . This procedure is generic to all substitution lattices. In our case the recurrence relation among the partition functions correctly reflects the fact that $n - th$ and $(n \pm 6) - th$ generations are topologically same . As a consequence one must go through six times decimation renormalization group procedure to find scaling forms! of thermodynamic functions. The recurrence relation gets simplified in the large $N$ limit showing that free energy per spin in two consecutive large even generations are related through a scaling factor $2\tau + 1$ . The expression for chemical potential in absence of magnetic field reveals that it is independent of the underlying lattice structure with nearest neighbour interaction.

Acknowledgement: We are thankful to S.N.Karmakar of SINP,Calcutta and to S.M. Bhattacharjee of IOP,Bhubaneswar for illuminating discussions. One ofthe authors (SKP) thanks J.P.Chakraborty of IACS,Calcutta for lending him necessary books from his library.

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