Ising field theory on a pseudosphere

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\textbf{Abstract.} We show how the symmetries of the Ising field theory on a pseudosphere can be exploited to derive the form factors of the spin fields as well as the non-linear differential equations satisfied by the corresponding two-point correlation functions. The latter are studied in detail and, in particular, we present a solution to the so-called connection problem relating two of the singular points of the associated Painlevé VI equation. A brief discussion of the thermodynamic properties is also presented.

\textbf{Keywords:} correlation functions, integrable quantum field theory, Painlevé equations

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1. Introduction

The study of the behaviour of statistical systems on a curved space is a problem with much room for development. A natural question concerns the modification of the critical properties of a statistical system due to a non-zero curvature, which introduces an additional scale. This question is of interest both for its relations to quantum gravity, and for understanding the thermodynamics of classical systems on a curved space, a point of view which we adopt in this paper. It is natural to expect that the study of quantum field theories on a curved Euclidean-signature space can lead to a better understanding of this problem. In this paper we are interested in studying the Ising field theory at zero magnetic field on a two-dimensional curved space of constant negative curvature (the pseudosphere). The interest in this space is in part technical: since a space of constant curvature is maximally symmetric, known techniques developed for the case of flat space can be extended to the Ising field theory on such a space. The pseudosphere also has unusual characteristics, for instance it has an infinite (two-dimensional) volume while providing an infrared regularization, and it can be expected to have non-trivial effects on...
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the thermodynamics. The study of these effects should in fact throw light on the main properties of the thermodynamics on negatively curved spaces.

We will be mainly interested in studying the two-point correlation functions of the order and disorder fields $\sigma(x)$ and $\mu(x)$ in the Ising field theory on the pseudosphere. Although this theory possesses stable regimes where symmetries associated with some of the isometries of the pseudosphere are broken (as we briefly explain in section 2), we will only consider regimes with unbroken symmetries. Using the parametrization of the pseudosphere on the unit disc (the Poincaré disc), the theory can be described in terms of the boundary Ising conformal field theory [1] deformed by the energy field, with action

$$A = A_{BI} - \frac{m}{2\pi} \int_{\text{disc}} d^2x \, e^\phi(x) \varepsilon(x).$$

(1.1)

Here, $A_{BI}$ stands for the action of the Ising conformal field theory on the unit disc and $d^2x \, e^\phi(x)$ is the volume element in the metric of the Poincaré disc (written explicitly in (2.2)). The energy field $\varepsilon(x)$ is normalized by

$$d(x, x')^2 \langle \varepsilon(x)\varepsilon(x') \rangle \to 1 \quad \text{as} \quad d(x, x') \to 0,$$

(1.2)

where $d(x, x')$ is the geodesic distance between the points $x$ and $x'$. Because we have in mind an Ising statistical system on a curved space (in contrast to a system on a space with boundaries), our definitions of the spin and disorder fields differ from the usual ones defined in the Ising conformal field theory on the disc, in the sense that the fields $\varepsilon(x)$, $\sigma(x)$ and $\mu(x)$ are chosen scalar under the isometry group of the pseudosphere.

The boundary Ising conformal field theory $A_{BI}$ admits ‘free’ and ‘fixed’ boundary conditions [1], whereby the order field has, respectively, zero and non-zero vacuum expectation value (and vice versa for the disorder field). Likewise, introducing the parameter $R$ related to the Gaussian curvature $\hat{R}$ by

$$\hat{R} = -\frac{1}{R^2},$$

(1.3)

it is possible to see that when the energy perturbation is turned on ($m \neq 0$), the resulting theory (1.1) on the pseudosphere possesses in the region $-1/2 < mR < 1/2$ stable asymptotic conditions corresponding to ‘free’ and ‘fixed’ conditions: the order field has, respectively, zero and non-zero vacuum expectation value, and vice versa for the disorder field. In the domain $mR > 1/2$ only the ‘fixed’ asymptotic condition is stable, whereas in the domain $mR < -1/2$ only the ‘free’ asymptotic condition is stable (this description is close connected with some results of, for instance, [2]). Situations with ‘free’ and ‘fixed’ asymptotic conditions can be obtained from one another by duality transformation, which interchanges the order and disorder fields $\sigma(x) \leftrightarrow \mu(x)$ and reverses the sign of the energy field $\varepsilon(x) \mapsto -\varepsilon(x)$. In the following, we will assume ‘fixed’ asymptotic conditions and $mR > -1/2$.

Our main results are expressions for the order and disorder correlation functions $\langle \sigma(x)\sigma(x') \rangle$ and $\langle \mu(x)\mu(x') \rangle$, which admit a description in terms of a

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3 With the ‘fixed’ boundary condition, the order field can be fixed to a positive or a negative value at the boundary; we will choose to fix it to a positive value throughout the paper.

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Painlevé VI transcendent $w(\eta)$,

$$w'' = \frac{1}{2} \left( \frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-\eta} \right) w^2 - \left( \frac{1}{\eta} + \frac{1}{\eta-1} + \frac{1}{w-\eta} \right) w' + \left( \frac{1}{2} - 2r^2 \right) \frac{w(w-1)}{\eta(\eta-1)(w-\eta)}.$$  \hfill (1.4)

Here, primes denote derivative with respect to the projective invariant $\eta$, related to the geodesic distance by

$$\eta = \tanh^2 \left( \frac{d(x,x')}{2R} \right),$$  \hfill (1.5)

and we have introduced the notation

$$r = mR.$$  \hfill (1.6)

From the usual parametrization in terms of auxiliary functions $\chi(\eta)$ and $\varphi(\eta)$,

$$(2R)^{1/4} \langle \sigma(x) \sigma(x') \rangle = e^{\chi(\eta)/2} \cosh(\varphi(\eta)/2),$$
$$(2R)^{1/4} \langle \mu(x) \mu(x') \rangle = e^{\chi(\eta)/2} \sinh(\varphi(\eta)/2),$$  \hfill (1.7)

we have found

$$\cosh^2 \varphi = \frac{1}{w},$$

$$\chi' = \frac{\eta(\eta-1)}{4w(w-\eta)(w-1)} w^2 - \frac{\eta}{2w(w-\eta)} w' + \left( \frac{1}{4} - r^2 \right) \frac{w-1}{(\eta-1)(w-\eta)}. \hfill (1.8)$$

As expected, the Painlevé transcendent involved in this description of two-point correlation functions is the same as that involved in the description of the tau function of the Dirac operator on the Poincaré disc (when specialized to appropriate monodromies), found in [3] and studied in [4]. However, it is a complicated matter to derive the former description from the latter. We used very different and simpler methods, following ideas developed in [5].

The appropriate solution to the Painlevé equation (1.4) can be fixed, for instance, by the short distance $\eta \to 0$ behaviour

$$w = r^2 \eta \ln^2 \left( k(r)^2 \eta \right) + O(\eta^2 \ln^4 \eta).$$  \hfill (1.9)

The constant $k(r)$, given by

$$\ln k(r) = \psi(r) + \frac{1}{2r} + \gamma - \ln 4$$  \hfill (1.10)

($\psi(x) = d \ln \Gamma(x)/dx$ and $\gamma$ is Euler’s constant), can be obtained from the vacuum expectation value of the energy field $\langle \varepsilon \rangle$ by applying conformal perturbation theory (see appendix A). The power law in (1.9), as well as the behaviour $\chi(\eta) = (1/4) \ln(\eta) + O(\eta^{1/2})$ fixing the integration constant for $\chi(\eta)$ in (1.7), (1.8), are specified by the leading short distance behaviours

$$d(x,x')^{1/4} \langle \sigma(x) \sigma(x') \rangle \to 1, \quad d(x,x')^{1/4} \langle \mu(x) \mu(x') \rangle \to 1 \quad \text{as} \quad d(x,x') \to 0.$$  \hfill (1.11)
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This solution has the property that at large distances $\eta \to 1$ it behaves as

$$1 - w = A(r)^2 (1 - \eta)^{1+2r} + O((1 - \eta)^{2+2r}, (1 - \eta)^{2+4r}), \quad (1.12)$$

with the coefficient

$$A(r) = \frac{\Gamma((1/2) + r)}{4r \sqrt{\pi} \Gamma(1 + r)}. \quad (1.13)$$

Furthermore, $\chi(\eta)$ approaches the constant $4 \ln \bar{s}$, related to the magnetization \cite{4, 6}

$$\langle \sigma \rangle = (2R)^{-1/8} \bar{s} \quad \text{with} \, \bar{s}^2 = \sqrt{2} \prod_{n=1}^{\infty} \left( \frac{1 - (1/4(r + n)^2)}{1 - (1/4n^2)} \right)^n. \quad (1.14)$$

The leading behaviour (1.12) is in agreement with the form factors of the order and disorder fields which were calculated in \cite{6} as specializations of form factors of more general fields in the Dirac theory on the Poincaré disc \cite{4}; in section 4 we verify these results in a simpler way using ideas of \cite{7}. The behaviour (1.12) also provides an alternative description of the solution to the Painlevé equation (1.4) describing the correlation functions. Together with (1.9), it gives a solution to the connection problem relating the singular points $\eta = 0$ and 1.

Our results lay strong support, both analytical and numerical, for the validity of the one-point function (1.14) and of the large distance asymptotic of the two-point functions (1.7). We briefly analysed these quantities, along with the free energy, from the point of view of the thermodynamics of the model. We found that, under natural assumptions on an underlying lattice theory, there exist effective ‘critical’ temperatures where the leading scaling behaviour changes drastically, with ‘critical’ exponents closely related to those of mean field theory.

The paper is organized as follows. In section 2, some technical details regarding the field content and symmetries of the model (1.1) are presented and a space of asymptotic states is introduced. The algebra of local integrals of motion of the doubled model is introduced in section 3 and are then used in section 4 to derive the non-linear differential equations satisfied by the two-point correlation functions $\langle \sigma(x)\sigma(x') \rangle$ and $\langle \mu(x)\mu(x') \rangle$, as well as the associated form factors. Lastly, in section 5, we discuss some thermodynamical quantities in the Ising model on the pseudosphere.

2. Ising field theory on a pseudosphere

The Ising field theory on the pseudosphere (1.1) can be written in terms of a free massive Majorana fermion ($\psi, \bar{\psi}$) as

$$\mathcal{A} = \frac{1}{2\pi} \int_{|z| < 1} d^2x \left[ \psi \bar{\partial} \psi + \bar{\psi} \partial \psi + \frac{2ir}{1 - z\bar{z}} \bar{\psi} \psi \right], \quad (2.1)$$

where we have used, for the volume element $d^2x e^{\phi(x)}$ on the Poincaré disc,

$$e^{\phi(x)} = \frac{4R^2}{(1 - z\bar{z})^2}, \quad (2.2)$$
and for the energy field,
\[ \varepsilon(x) = i(2R)^{-1}(1 - z\bar{z})(\psi\bar{\psi})(x). \] (2.3)

In (2.1), the parameter \( r \) is related to the mass parameter \( m \) and Gaussian curvature (1.3) as in (1.6), \((z, \bar{z})\) are complex coordinates on the unit disc \(|z| < 1\), \( \partial \equiv \partial_z = (1/2)(\partial_x - i\partial_y) \), \( \bar{\partial} \equiv \partial_{\bar{z}} = (1/2)(\partial_x + i\partial_y) \) and \( d^2x = dz\,dy \); \((x, y)\) are Cartesian coordinates on the disc related in the usual way to the complex coordinates, \( z = x + iy, \bar{z} = x - iy \). In terms of complex coordinates, the geodesic distance \( d(x, x') \) between \( x \) and \( x' \) is given by
\[ \tanh^2 \frac{d(x, x')}{2R} = \frac{(z - z')(\bar{z} - \bar{z}')(1 - z\bar{z})(1 - z\bar{z}')}{(1 - z\bar{z})(1 - z\bar{z}')} = \frac{1}{2} \frac{d(x, x')}{2R}. \] (2.4)

The chiral components \( \psi \) and \( \bar{\psi} \) obey the linear field equations
\[ \bar{\partial} \psi(x) = \frac{ir}{1 - z\bar{z}} \bar{\psi}(x), \quad \partial \bar{\psi}(x) = \frac{-ir}{1 - z\bar{z}} \psi(x), \] (2.5)
and are normalized in (2.1) in accordance with the short distance limit
\[ (z - z') \psi(x)\psi(x') \rightarrow 1, \quad (\bar{z} - \bar{z}') \bar{\psi}(x)\bar{\psi}(x') \rightarrow 1 \quad \text{as} \quad |x - x'| \rightarrow 0. \] (2.6)

The zero-curvature limit \( R \rightarrow \infty \) corresponds to the familiar theory of free massive Majorana fermion in flat space (and mass \( m \)) after the rescaling
\[ z \mapsto z/(2R) \quad \text{and} \quad \psi(x) \mapsto (2R)^{1/2} \psi(x), \] (2.7)
with similar rescaling for \( \bar{z} \) and \( \bar{\psi} \).

2.1. Symmetries

The Ising field theory (2.1) possesses an \( SU(1, 1) \) symmetry group induced by the isometry group of the metric (2.2). In particular, the action is invariant under the coordinate transformation
\[ z \mapsto z' = f(z) = \frac{az + b}{bz + a}, \quad \bar{z} \mapsto \bar{z}' = \bar{f}(\bar{z}) = \frac{\bar{a}\bar{z} + b}{\bar{b}\bar{z} + a}. \] (2.8)
where we can choose \( a\bar{a} - b\bar{b} = 1 \). Under this transformation, the Fermi fields transform as
\[ \psi(x) \mapsto \psi'(x') = (\partial f)^{-1/2} \psi(x), \quad \bar{\psi}(x) \mapsto \bar{\psi}'(x') = (\bar{\partial} \bar{f})^{-1/2} \bar{\psi}(x). \] (2.9)

In order for the full quantum theory to be \( SU(1, 1) \) invariant, we have to impose appropriate \( SU(1, 1) \) invariant asymptotic conditions at the disc boundary \(|z| \rightarrow 1\), as will be discussed later in this section.

It will be convenient to classify the local fields in the theory (2.1) by their properties under \( SU(1, 1) \) transformations. A local field \( \mathcal{O}(x) \) will be said to have \( SU(1, 1) \)-dimension \((h, \bar{h})\) if it transforms under (2.8) as
\[ \mathcal{O}(x) \mapsto \mathcal{O}'(x') = (\partial f)^{-h}(\bar{\partial} \bar{f})^{-\bar{h}} \mathcal{O}(x). \] (2.10)

Thus, Fermi fields \( \psi \) and \( \bar{\psi} \) have dimensions \((1/2, 0)\) and \((0, 1/2)\), respectively, and the energy field \( \varepsilon \), as defined in (2.3), has dimension \((0, 0)\).
One can construct local fields of higher $SU(1, 1)$-dimension by using the covariant derivatives
\[
\mathcal{D} \mathcal{O}(x) = \left( \partial - \frac{2h\bar{z}}{1 - z\bar{z}} \right) \mathcal{O}(x), \quad \bar{\mathcal{D}} \mathcal{O}(x) = \left( \bar{\partial} - \frac{2\bar{h}z}{1 - z\bar{z}} \right) \mathcal{O}(x),
\] (2.11)
where the holomorphic covariant derivative $\mathcal{D}$ takes a field of $SU(1, 1)$-dimension $(h, \bar{h})$ to a field of dimension $(h + 1, \bar{h})$, and the anti-holomorphic covariant derivative $\bar{\mathcal{D}}$ to a field of dimension $(h, \bar{h} + 1)$.

A basis for the isometry algebra can be taken as the generators of the coordinate transformations
\[
z \mapsto z + \epsilon (1 - z^2), \quad z \mapsto z + i\epsilon (1 + z^2), \quad z \mapsto z + i\epsilon z,
\] (2.12)
with conjugate transformations for $\bar{z}$ and where $\epsilon$ is a real infinitesimal parameter. These give rise to Lie derivatives on the local fields (2.10), denoted respectively by $P, \bar{P}$ and $R$.

Introducing the notation
\[
P = \frac{1}{2} (P_x - iP_y), \quad \bar{P} = \frac{1}{2} (P_x + iP_y),
\]
(2.13)
(2.15)
where $L_l' = P_l^1 \bar{P}_l^1 R_l^3$, $L_n = P_n^0 \bar{P}_n^0 R_n^1$, $l, n$ being some non-negative integers.

As usual, integrals of motion can be written as integrals of the currents over appropriate lines on the pseudosphere. It will be convenient in what follows to choose for these lines a particular family of geodesics on the pseudosphere parametrized by elements of a non-compact subgroup of the $SU(1, 1)$ isometry group. More precisely, consider the system of ‘isometric’ coordinates $(\xi, \bar{\xi})$ related to coordinates $(z, \bar{z})$ on the Poincaré disc by
\[
z = \tan \xi, \quad \bar{z} = \tan \bar{\xi},
\] (2.16)
where $\xi = \xi_x + i\xi_y$ and $\bar{\xi} = \xi_x - i\xi_y$ (see figure 1).

The transformations
\[
\xi_y \mapsto \xi_y + q, \quad \text{for arbitrary real } q,
\] (2.17)
are the integrated versions of the second infinitesimal transformation of (2.12) and form a non-compact subgroup of $SU(1, 1)$. It will be convenient to write the integrals of motion
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Figure 1. In the $\xi$-plane, related to the unit disc by the map (2.16), the pseudosphere lies on the strip $-\pi/4 < \xi_x < \pi/4$ and $-\infty < \xi_y < \infty$; the lines $\xi_y = \text{const}$ correspond to a particular class of geodesics.

associated with the conserved currents (2.14) as integrals over lines of constant $\xi_y$,

\[
\mathbf{P} = \frac{1}{4\pi} \int_{\mathcal{C}} [\psi \mathcal{P} \psi \, dz - \bar{\psi} \mathcal{P} \bar{\psi} \, d\bar{z}],
\]

\[
\bar{\mathbf{P}} = \frac{1}{4\pi} \int_{\mathcal{C}} [\bar{\psi} \bar{\mathcal{P}} \bar{\psi} \, d\bar{z} - \psi \bar{\mathcal{P}} \psi \, dz],
\]

\[
\mathbf{R} = \frac{1}{4\pi i} \int_{\mathcal{C}} [\psi \mathcal{R} \psi \, dz - \bar{\psi} \mathcal{R} \bar{\psi} \, d\bar{z}],
\]

where

\[
\mathcal{C} = \{(z, \bar{z}) \mid -\pi/4 < \xi_x < \pi/4, \xi_y = \text{const}\}. \tag{2.19}
\]

Inside correlation functions with some local fields, these integrals are independent of the value of $\xi_y$ associated with the path $\mathcal{C}$, except for contributions at the positions of the fields, given by the corresponding Lie derivatives,

\[
[\mathbf{P}, \mathcal{O}(x)] = i \mathcal{P} \mathcal{O}(x), \quad [\bar{\mathbf{P}}, \mathcal{O}(x)] = -i \bar{\mathcal{P}} \mathcal{O}(x), \quad [\mathbf{R}, \mathcal{O}(x)] = \mathcal{R} \mathcal{O}(x), \tag{2.20}
\]

where, for instance,

\[
[\mathbf{P}, \mathcal{O}(x_0)] = -\frac{1}{4\pi} \oint_{x_0} [\psi \mathcal{P} \psi \, dz - \bar{\psi} \mathcal{P} \bar{\psi} \, d\bar{z}] \mathcal{O}(x_0), \tag{2.21}
\]

the contour being taken around the point $x_0$ in anticlockwise direction. The integrals of motion (2.18) satisfy the $SU(1, 1)$ algebra

\[
[\mathbf{P}, \mathbf{R}] = -i \mathbf{P}, \quad [\bar{\mathbf{P}}, \mathbf{R}] = i \bar{\mathbf{P}}, \quad [\mathbf{P}, \bar{\mathbf{P}}] = -2i \mathbf{R}, \tag{2.22}
\]

and the field equations (2.5) specify the Casimir of the representation generated by the Fermi fields,

\[
\frac{1}{2}([\mathbf{P}, [\mathbf{P}, \psi]] + [\bar{\mathbf{P}}, [\bar{\mathbf{P}}, \psi]]) - [\mathbf{R}, [\mathbf{R}, \psi]] = (r^2 - \frac{1}{4})\psi, \tag{2.23}
\]

with a similar equation for $\bar{\psi}$. 

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2.2. Hilbert space

Here, we present a space of asymptotic states for the Ising field theory (2.1). Details regarding its construction can be found in appendix B.

A particularly convenient Hilbert space is the one obtained by associating time translations to isometry transformations (2.17), so that $\xi_x$ represents the space coordinate and $\xi_y$ the Euclidean time coordinate. A basis can be obtained by diagonalizing the corresponding Hamiltonian

$$H = P + \bar{P}.$$  

(2.24)

As we will see, the corresponding states can be interpreted as ‘particle states’, since they form an irreducible representation of the $SU(1,1)$ symmetry group.

The Hilbert space is defined as a module for the canonical equal-time anticommutation relations of the fermion operators $\psi^{(iso)}(\xi_x, \xi_y)$, $\bar{\psi}^{(iso)}(\xi_x, \xi_y)$ in the isometric system of coordinates$^4$:

$$\{\psi^{(iso)}(\xi_x, \xi_y), \psi^{(iso)}(\xi'_x, \xi'_y)\} = -2\pi i \delta(\xi_x - \xi'_x),$$

$$\{\bar{\psi}^{(iso)}(\xi_x, \xi_y), \bar{\psi}^{(iso)}(\xi'_x, \xi'_y)\} = 2\pi i \delta(\xi_x - \xi'_x).$$  

(2.25)

Invariance under the subgroup described by (2.17) imposes that the Fermi fields vanish as $\xi_y \to \pm \infty$, which gives the following conditions on the vacuum state:

$$\lim_{\xi_y \to -\infty} \psi^{(iso)}(\xi_x, \xi_y)|\text{vac}\rangle = 0, \quad \lim_{\xi_y \to +\infty} \langle \text{vac}|\psi^{(iso)}(\xi_x, \xi_y) = 0,$$

(2.26)

with similar conditions for $\bar{\psi}^{(iso)}$. Correlation functions of local fields are then expressed as time-ordered vacuum expectation values of corresponding operators; the time ordering puts operators from left to right in decreasing values of their variable $\xi_y$.

The Hilbert space is further specified by the asymptotic conditions imposed on the Fermi fields at the disc boundary. In order to guarantee their stability, one has to choose such conditions giving a Hilbert space on which the Hamiltonian has real eigenvalues bounded from below. Among these, we shall only consider those which respect the $SU(1,1)$ symmetry. We note however that for $-1/2 < r < 1/2$ there are stable asymptotic conditions which are not $SU(1,1)$ invariant; we intend to report on these regimes in a future publication.

For $r > 1/2$, finiteness of matrix elements of the Hamiltonian gives the asymptotic conditions imposing that Fermi fields vanish as $\sim e^{-md}$ when $d$, the geodesic distance between the origin, say, and the position of the Fermi field, goes to infinity. These asymptotic conditions correspond to ‘fixed’ asymptotic conditions on the order field $\sigma$. For $0 < r < 1/2$, the above asymptotic conditions are also stable, and there is an additional set of stable asymptotic conditions, by which the Fermi fields diverge as $\sim e^{md} (1 + O(e^{-d/R}))$ as the geodesic distance to the origin goes to infinity. This second set corresponds to ‘free’ asymptotic conditions on the order field $\sigma$. The field theory with this second set of asymptotic conditions can be obtained by analytically continuing the field theory with the first set of asymptotic conditions from the region $0 < r < 1/2$ to the region $-1/2 < r < 0$.

$^4$ The Fermi fields in isometric coordinates are related to the Fermi fields on the Poincaré disc by (2.9), that is, $\psi^{(iso)}(\xi_x, \xi_y) = \psi(x)/\cos(\xi)$, $\bar{\psi}^{(iso)}(\xi_x, \xi_y) = \bar{\psi}(x)/\cos(\bar{\xi})$. 

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and vice versa. Hence it will be sufficient in what follows to consider only the conditions specifying the asymptotic behaviour \( \sim e^{-md} \) of the Fermi fields for \( r > 0 \).

With these asymptotic conditions, the Fermi fields, obeying equations (2.5), admit expansions in partial waves as

\[
\psi(\xi_x, \xi_y) = \cos \xi \sum_{n=0}^{\infty} \left[ e^{i(\pi/2)n} A_n^\dagger e^{i\omega_n \xi} g_n(\xi_x) - ie^{-i(\pi/2)n} A_n e^{-i\omega_n \xi} \bar{g}_n(\xi_x) \right],
\]

(2.27)

\[
\bar{\psi}(\xi_x, \xi_y) = \cos \xi \sum_{n=0}^{\infty} \left[ e^{i(\pi/2)n} A_n^\dagger e^{i\omega_n \xi} \bar{g}_n(\xi_x) + ie^{-i(\pi/2)n} A_n e^{-i\omega_n \xi} g_n(\xi_x) \right],
\]

with discrete energy spectrum

\[
\omega_n = 2r + 2n + 1 \quad (n \geq 0).
\]

(2.28)

Partial waves are given by

\[
g_n(\xi_x) = \frac{2^{1-2r} \sqrt{\pi}}{\Gamma(r + (1/2))} \frac{\Gamma(2r + n + 1)^{1/2}}{\Gamma(n + 1)^{1/2}} e^{-i\omega_n \xi} e^{i(\pi/2)n} (1 + e^{4i\xi})^r F(-n, r; 1 + 2r; 1 + e^{4i\xi}),
\]

(2.29)

with \( \bar{g}_n(\xi_x) \) denoting its complex conjugate, and \( F(a, b; c; x) \) stands for the Gauss hypergeometric function, here specialized to polynomials; phases in the decomposition (2.27) were chosen for later convenience when introducing particle states. The functions \( g_n(\xi_x) \), \( \bar{g}_n(\xi_x) \) satisfy the orthogonality relations

\[
\int_{-\pi/4}^{\pi/4} d\xi [g_n(\xi_x)\bar{g}_{n'}(\xi_x) + \bar{g}_n(\xi_x)g_{n'}(\xi_x)] = 4\pi \delta_{n,n'},
\]

(2.30)

\[
\int_{-\pi/4}^{\pi/4} d\xi [g_n(\xi_x)g_{n'}(\xi_x) - \bar{g}_n(\xi_x)\bar{g}_{n'}(\xi_x)] = 0,
\]

as well as the relations

\[
g_n(\xi_x)|_{r=1/2} = -ig_{n-1}(\xi_x)|_{r=1/2}; \quad \bar{g}_n(\xi_x)|_{r=1/2} = i\bar{g}_{n-1}(\xi_x)|_{r=1/2}.
\]

(2.31)

The creation and annihilation operators \( A_n^\dagger \) and \( A_n \) \((n \geq 0)\) in (2.27) satisfy canonical anti-commutation relations as a consequence of (2.25):

\[
\{A_n^\dagger, A_{n'}\} = \delta_{n,n'}, \quad \{A_n^\dagger, A_n^\dagger\} = \{A_n, A_{n'}\} = 0,
\]

(2.32)

with the vacuum state \( |\text{vac}\rangle \) obeying, from (2.26),

\[
A_n |\text{vac}\rangle = 0 \quad \text{for all } n \geq 0.
\]

(2.33)

A basis of \( N \)-particle states is obtained from the set of states

\[
|n_1 \cdots n_N\rangle \equiv A_{n_1}^\dagger \cdots A_{n_N}^\dagger |\text{vac}\rangle,
\]

(2.34)

which diagonalize the Hamiltonian,

\[
H |n_1 \cdots n_N\rangle = \left( \sum_{i=1}^{N} \omega_{n_i} \right) |n_1 \cdots n_N\rangle.
\]

(2.35)
with energy eigenvalues $\omega_n$, equation (2.28). The discretization of the energy spectrum is essentially a consequence of requiring trivial monodromy of the hypergeometric functions involved in the partial waves (2.29) as $\xi_x \rightarrow \xi_x + \pi/2$, a necessary condition to ensure the proper vanishing asymptotic behaviour at the boundary of the disc.

The action of the operators $P$, $\bar{P}$ and $R$, defined in (2.18), can be easily determined from the fact that the above Hilbert space provides a lowest weight module for $SU(1, 1)$. The raising and lowering operators, $J_+$ and $J_-$ respectively, are given by

$$J_\pm = P - \bar{P} \pm 2iR,$$

and are related by Hermitian conjugation, $J_+^\dagger = J_-$. Together with the Hamiltonian (2.24), they satisfy the algebra

$$[H, J_\pm] = \pm 2J_\pm, \quad [J_-, J_+] = 4H,$$

from which the action of $J_\pm$ on eigenstates of the Hamiltonian follows:

$$J_+|n\rangle = \alpha_n |n + 1\rangle, \quad J_-|n\rangle = \alpha_{n-1} |n - 1\rangle,$$

with

$$\alpha_n = 2\sqrt{(n + 1)(2r + n + 1)}.$$

We thus have the action of the rotation operator $R$,

$$4iR|n\rangle = \alpha_n |n + 1\rangle - \alpha_{n-1} |n - 1\rangle,$$

that we will use in section 4 when deriving form factors.

### 2.3. Local fields

Besides the Fermi and the energy fields, other local fields are present in the theory. Two spin fields associated with the $\mathbb{Z}_2$ symmetry $(\psi, \bar{\psi}) \mapsto (-\psi, -\bar{\psi})$ of the action (2.1) can be defined, the order field $\sigma(x)$ and the disorder field $\mu(x)$. They are not mutually local with respect to the Fermi fields, since the products

$$\psi(x)\sigma(x'), \quad \bar{\psi}(x)\sigma(x'), \quad \psi(x)\mu(x'), \quad \bar{\psi}(x)\mu(x')$$

acquire negative signs when the point $x$ is brought around $x'$. This property does not define the fields $\sigma$ and $\mu$ uniquely. Besides having this property, they are required to be ‘primary’ with respect to the action of the Fermi fields in the operator algebra. This fixes operator product expansions (OPE) of the form

$$\psi(x)\sigma(x') = \sum_{n=0}^\infty c_n \left[ \sqrt{\frac{1}{2}} s_n(x, x') u_n(\eta) D^n\mu(x') + \sqrt{-\frac{1}{2}} \bar{t}_n(x, x') v_n(\eta) \bar{D}^n\mu(x') \right],$$

$$\bar{\psi}(x)\sigma(x') = \sum_{n=0}^\infty c_n \left[ -\sqrt{\frac{1}{2}} t_n(x, x') v_n(\eta) D^n\mu(x') + \sqrt{-\frac{1}{2}} \bar{s}_n(x, x') u_n(\eta) \bar{D}^n\mu(x') \right].$$

(2.42)
The factors $s_n$ and $t_n$ are given by
\begin{align}
    s_n(x, x') &= \left( \frac{1 - z'\bar{z}'}{1 - z\bar{z}} \right)^{n+(1/2)} (z - z')^{n-(1/2)}, \\
    t_n(x, x') &= \left( \frac{1 - z'\bar{z}'}{1 - z\bar{z}} \right)^{n+(1/2)} (z - z')^{n+(1/2)}.
\end{align}
(2.43)

$s_n$ transforms under the representation $(1/2, 0)$ in $x$ and $(-n, 0)$ in $x'$, and $t_n$ under $(0, 1/2)$ in $x$ and $(-n, 0)$ in $x'$; the factors $\bar{s}_n$, $\bar{t}_n$ are their complex conjugates; the projective invariant $\eta$ is given by (1.5), (2.4), i.e.
\begin{equation}
    \eta = \frac{(z - z')(\bar{z} - \bar{z}')}{(1 - \bar{z}z')(1 - \bar{z}'\bar{z}')};
\end{equation}
(2.44)
the functions $u_n(\eta)$ and $v_n(\eta)$,
\begin{align}
    u_n(\eta) &= (1 - \eta)^r F\left( r, r + \frac{1}{2} + n; \frac{1}{2} + n; \eta \right), \\
    v_n(\eta) &= \frac{ir}{n + 1/2} (1 - \eta)^r F\left( r + 1, r + \frac{1}{2} + n; \frac{3}{2} + n; \eta \right),
\end{align}
(2.45)
are determined by the field equations (2.5); and $\mathcal{D}$, $\bar{\mathcal{D}}$ are the covariant derivatives introduced in (2.11).

The constants $c_n$ can be determined, say, from requiring associativity of the operator algebra on $\psi(x)\psi(x')d\sigma(0)$,
\begin{equation}
    c_0 = 1, \quad c_n = 2/(\frac{1}{2})_n \quad (n \geq 1),
\end{equation}
(2.46)
with $(1/2)_n = \Gamma((1/2) + n)/\Gamma(1/2)$.

There are similar expressions for the products $\psi(x)\mu(x')$, $\bar{\psi}(x)\mu(x')$, obtained from the above OPE (2.42) by interchanging $\sigma \leftrightarrow \mu$ and $\sqrt{i} \leftrightarrow \sqrt{-i}$. These completely define the fields $\sigma$ and $\mu$, together with the normalization (1.11) and the convention that they are $SU(1, 1)$ scalars (so that their $SU(1, 1)$-dimensions are $h = \bar{h} = 0$).

3. Conserved charges of the doubled Ising field theory

In order to derive Ward identities for correlation functions and form factors, we shall follow closely the method presented in [5]. In particular, we start by introducing a system of two identical copies of the Ising field theory (2.1), here referred to as ‘copy a’ and ‘copy b’, a trick known to simplify many aspects of the Ising theory [8, 9]. The fields $\psi_a$, $\bar{\psi}_a$, $\sigma_a$, $\mu_a$ will denote respectively the two fermionic and the order and disorder fields belonging to copy a, while $\psi_b$, $\bar{\psi}_b$, $\sigma_b$, $\mu_b$ will denote those belonging to copy b. In addition, it is convenient to require Fermi fields from different copies to anti-commute. This can be achieved by introducing two auxiliary Klein factors, $\eta_a$ and $\bar{\eta}_b$, defined by
\begin{equation}
    \eta_a^2 = 1, \quad \bar{\eta}_b^2 = 1, \quad \eta_a \bar{\eta}_b = -\bar{\eta}_b \eta_a,
\end{equation}
(3.1)
and modifying original fields according to
\begin{align}
    \psi_a(x) \mapsto \eta_a \psi_a(x), \quad \bar{\psi}_a(x) \mapsto \eta_a \bar{\psi}_a(x), \quad \sigma_a(x) \mapsto \sigma_a(x), \quad \mu_a(x) \mapsto \eta_a \mu_a(x),
\end{align}
(3.2)
with similar redefinition for copy $b$. These redefinitions do not change any of the correlation functions involving only fields belonging to a given copy, as long as we assume ‘fixed’ asymptotic condition, so that $\langle \mu \rangle = 0$. Correlation functions involving fields belonging to both copies factorize into correlation functions in the model with a single copy in a simple manner. For instance, $\langle \sigma_a(x)\sigma_b(x') \sigma_a(x') \sigma_b(x) \rangle = \langle \sigma(x)\sigma(x') \rangle^2$ and $\langle \mu_a(x)\mu_b(x) \mu_a(x') \mu_b(x') \rangle = -\langle \mu(x)\mu(x') \rangle^2$, the minus sign in the latter coming from the redefinitions (3.2).

This doubled model is nothing but a theory of the free Dirac field on a pseudosphere. The $U(1)$ invariance of the Dirac theory is translated into an invariance under ‘rotations’ among the two copies, with the associated charge being

$$Z_0 = \frac{1}{2\pi} \int_c [\psi_a \psi_b \, dz - \bar{\psi}_a \bar{\psi}_b \, d\bar{z}], \quad \text{(3.3)}$$

where the contour is taken on an equal-time slice, given by (2.19). It acts on the Fermi fields as

$$[Z_0, \psi_a(x)] = i\psi_b(x), \quad [Z_0, \psi_b(x)] = -i\psi_a(x), \quad \text{(3.4)}$$

with similar relations for $\bar{\psi}$s.

Additional local integrals of motion for the doubled model can be directly obtained from the charges $P$, $\bar{P}$ and $R$, defined in (2.18), of each individual copy,

$$X_1 = P_a - P_b, \quad X_{-1} = \bar{P}_a - \bar{P}_b, \quad X_0 = R_a - R_b, \quad \text{(3.5)}$$

and from the commutators of these with the $U(1)$ charge (3.3), such as

$$Y_1 = \frac{i}{2} [X_1, Z_0], \quad Y_{-1} = \frac{i}{2} [X_{-1}, Z_0], \quad Y_0 = \frac{i}{2} [X_0, Z_0]. \quad \text{(3.6)}$$

In terms of integrals of local currents,

$$Y_1 = \frac{1}{2\pi} \int_c (\bar{\psi}_a \bar{P} \psi_b \, dz - \bar{\psi}_a \bar{P} \psi_b \, d\bar{z}),$$

$$Y_{-1} = \frac{1}{2\pi} \int_c (\bar{\psi}_a \bar{P} \psi_b \, dz - \bar{\psi}_a \bar{P} \psi_b \, d\bar{z}), \quad \text{(3.7)}$$

$$Y_0 = \frac{1}{2\pi i} \int_c (\bar{\psi}_a \bar{R} \psi_b \, dz - \bar{\psi}_a \bar{R} \psi_b \, d\bar{z}).$$

Yet other integrals of motion obtained from the above are

$$Z_2 = -\frac{i}{2} [X_1, Y_1], \quad Z_{-2} = \frac{i}{2} [X_1, Y_{-1}],$$

$$Z_1 = -\frac{i}{2} [X_1, Y_0] = -\frac{i}{2} [X_0, Y_1], \quad Z_{-1} = \frac{i}{2} [X_{-1}, Y_0] = \frac{i}{2} [X_0, Y_{-1}], \quad \text{(3.8)}$$

$$\bar{Z}_0 = -\frac{i}{2} [X_0, Y_0].$$

All these can be straightforwardly written as integrals of local densities, quadratic in the Fermi fields and their Lie derivatives. We present here the expressions for $Z_2$ and $Z_{-2},$

$$Z_2 = \frac{1}{2\pi} \int_c (\bar{P} \psi_a \bar{P} \psi_b \, dz - \bar{P} \psi_a \bar{P} \psi_b \, d\bar{z}),$$

$$Z_{-2} = \frac{1}{2\pi} \int_c (\bar{P} \psi_a \bar{P} \psi_b \, dz - \bar{P} \psi_a \bar{P} \psi_b \, d\bar{z}). \quad \text{(3.9)}$$
Some additional commutation relations among these integrals of motion are
\[
[Y_1, Z_0] = 2iX_1, \quad [Y_{-1}, Z_0] = 2iX_{-1}, \quad [Y_0, Z_0] = 2iX_0.
\]
A straightforward way to obtain these relations is to calculate them on the module provided by the Fermi fields (and their descendants) of both copies, since on them the action of the charge (3.3) is simple, and to use the equations of motion in the form (2.23).

The OPEs of Fermi and spin fields can be used to calculate the actions of the conserved charges on the order and disorder fields. A particularly simple case is that when the fields are located at the centre of the unit disc. For the operator \( Z_0 \), we have the commutators
\[
[Z_0, \sigma_a(0) \sigma_b(0)] = 0, \quad [Z_0, \mu_a(0) \mu_b(0)] = 0,
\]
\[
[Z_0, \sigma_a(0) \mu_b(0)] = -i \mu_a(0) \sigma_b(0), \quad [Z_0, \mu_a(0) \sigma_b(0)] = i \sigma_a(0) \mu_b(0).
\]

Another useful commutator is the one involving the combination \( Y_1 - Y_{-1} \),
\[
[Y_1 - Y_{-1}, \mu_a(0) \mu_b(0)] = i[H_a - H_b, \sigma_a(0) \sigma_b(0)],
\]
where \( H_a, H_b \) are the Hamiltonians (2.24) of each individual copy.

At an arbitrary point can be obtained either directly from the OPE (2.42) or from applying appropriate \( SU(1,1) \) transformations to their action on fields at the centre of the disc, as shown in appendix C. We shall need the results
\[
i[Z_0, \partial \bar{\sigma}_a(x) \mu_b(x)] = \partial \mu_a(x) \bar{\sigma}_a(x) + \partial \bar{\sigma}_a(x) \mu_b(x) - \partial \bar{\sigma}_a(x) \sigma_b(x) + \partial \bar{\sigma}_a(x) \sigma_b(x)
\]
\[
+ \frac{r^2 - 1/4}{(1 - z \bar{z})^2} \mu_a(x) \sigma_b(x),
\]
and
\[
[Z_2, \sigma_a(x) \sigma_b(x)] = 2 \partial \mu_a(x) \partial \mu_b(x) - \partial^2 \mu_a(x) \mu_b(x) - \mu_a(x) \partial^2 \mu_b(x)
\]
\[
- \bar{z}^4 (2 \partial \bar{\mu}_a(x) \partial \bar{\mu}_b(x) - \partial^2 \bar{\mu}_a(x) \mu_b(x) - \mu_a(x) \partial^2 \bar{\mu}_b(x))
\]
\[
+ 2 \bar{z}^3 (\partial \bar{\mu}_a(x) \mu_b(x) + \mu_a(x) \partial \mu_b(x)),
\]
plus a similar commutator \([Z_2, \mu_a(x) \mu_b(x)]\), given by (3.14) with \( \mu \) replaced by \( \sigma \).

Modes \( A_n, A_n^\dagger \) as introduced in (2.27) will be considered to enter the partial wave expansion of \( \psi_a, \psi_a \), whereas modes \( B_n, B_n^\dagger \) with the same commutation relations (and anti-commuting with the \( A \)) enter the partial wave expansion of \( \bar{\psi}_b, \bar{\psi}_b \) in the same manner. The Hilbert space of the doubled model is isomorphic to the tensor product of two copies of the Hilbert space described in section 2.2, with particle states denoted by
\[
|n_1 \cdots n_N; m_1 \cdots m_M \rangle \equiv A_{n_1}^\dagger \cdots A_{n_N}^\dagger A_{m_1}^\dagger \cdots A_{m_M}^\dagger |\text{vac}\rangle.
\]

The action of conserved charges of the doubled model on these asymptotic states can be easily deduced from the action of the conserved charges \( P, P, R \) on the states described in section 2.2, along with
\[
Z_0 |\text{vac}\rangle = 0, \quad [Z_0, A_n^\dagger] = i B_n^\dagger, \quad [Z_0, B_n^\dagger] = -i A_n^\dagger.
\]

\(^5\) In the zero-curvature limit \( R \to \infty \) the algebra specified by (3.6), (3.10) corresponds to the well-known \( \hat{sl}(2) \) affine algebra of level zero \([10]\) after appropriate rescaling dictated by (2.7); otherwise, the symmetry algebra of the doubled model fails to qualify as a Kac–Moody algebra due to non-trivial commutation relations of \( Z_0 \), in equation (3.8), with other charges.
which follows directly from the decomposition (2.27) and (3.3). The fact that all local integrals of motion annihilate the vacuum state will then give rise to useful Ward identities.

4. Correlation functions and form factors

In this section we use the Ward identities associated with the symmetries of the single and the doubled models in order to derive the differential equations satisfied by the spin fields as well as the corresponding form factors.

4.1. The correlation functions

The representation (1.4), (1.8) of the two-point functions of order and disorder fields

\[ G(x, x') = \langle \sigma(x) \sigma(x') \rangle, \quad \tilde{G}(x, x') = \langle \mu(x) \mu(x') \rangle, \]

(4.1)
can be derived using the symmetries of the doubled model described in the previous section. In particular, using (3.14) we can rewrite the Ward identity

\[ \langle \text{vac} | [Z_2, \sigma_a(x) \sigma_b(x) \mu_a(x') \mu_b(x')] | \text{vac} \rangle = 0, \]

(4.2)
as the quadratic differential equation

\[
\begin{align*}
\partial \tilde{G} \partial \tilde{G} - \partial^2 \tilde{G} \tilde{G} - z^4 \left( \partial \tilde{G} \partial \tilde{G} - \partial^2 \tilde{G} \tilde{G} \right) + 2z^3 \partial \tilde{G} \tilde{G} \\
= \partial' \tilde{G} \partial \tilde{G} - \partial^2 \tilde{G} \tilde{G} - z^4 \left( \partial' \tilde{G} \partial \tilde{G} - \partial^2 \tilde{G} \tilde{G} \right) + 2z^3 \partial' \tilde{G} \tilde{G},
\end{align*}
\]

(4.3)
where \( \partial = \partial_z, \partial' = \partial_{x'}, \partial = \partial_z \) and \( \partial' = \partial_{x'} \); it is important to keep track of the Klein factors in the definition of the fields \( \mu_a, \mu_b \) in order to get the signs in the above equation.

Also, using (3.13), we find that the Ward identity

\[ \langle \text{vac} | [Z_0, \partial \partial' \sigma_a(x) \mu_a(x') \mu_b(x')] | \text{vac} \rangle = 0 \]

(4.4)
gives

\[(1 - z^2)^2 \left[ \partial \tilde{G} \tilde{G} - \tilde{G} \partial \tilde{G} - \tilde{G} \partial \tilde{G} - \tilde{G} \partial \tilde{G} \right] = \left( r^2 - \frac{1}{4} \right) \tilde{G} \tilde{G}. \]

(4.5)

We are interested in a solution that respects the SU(1,1) symmetry, in which case the two-point correlation functions are simply functions of the projective invariant (2.44).

Then, equations (4.3) and (4.5) above imply the set of equations\(^6\)

\[
\begin{align*}
\eta (1 - \eta) \left( G'G' - G''G + \tilde{G}'\tilde{G}' - \tilde{G}''\tilde{G} \right) + (2\eta - 1) \left( G'G + \tilde{G}'\tilde{G} \right) &= 0, \\
(\eta - 1) \left( G'G' - G''G - \tilde{G}'\tilde{G}' + \tilde{G}''\tilde{G} \right) - \left( G'G - \tilde{G}'\tilde{G} \right) &= 0, \\
\eta \left( G''G + \tilde{G}''G - 2\tilde{G}'G' \right) + \left( \tilde{G}'G + \tilde{G}G' \right) &= \frac{r^2 - 1/4}{(1 - \eta)^2} \tilde{G}G,
\end{align*}
\]

(4.6a, 4.6b, 4.6c)
where primes denote derivatives with respect to \( \eta \) and the first two equations (4.6a), (4.6b) are consequences of equation (4.3).

It is easy to see that for the auxiliary functions \( \chi(\eta), \varphi(\eta) \) introduced in (1.7), equations (4.6) become

\[
\begin{align*}
\varphi'' \left( 1 - \eta \cosh^2 \varphi \right) + \frac{1}{2} \varphi'^2 \eta \sinh 2\varphi + \varphi' \left( \frac{1 - 2\eta + \eta^2 \cosh^2 \varphi}{\eta(1 - \eta)} \right) &= \frac{r^2 - 1/4}{2\eta(1 - \eta)} \sinh 2\varphi, \\
\chi' \left( 1 - \eta \cosh^2 \varphi \right) = \varphi'^2 \eta (1 - \eta) + \frac{1}{2} \varphi' \eta \sinh 2\varphi - \frac{1 - 1/4}{1 - \eta} \sinh^2 \varphi, \quad (4.7a, 4.7b)
\end{align*}
\]

\(^6\) Similar calculations can be easily performed for the Ising field theory on a sphere, as described in appendix E.
which immediately imply the representation in terms of the Painlevé VI transcendent in (1.4), (1.8). In the flat-space limit $R \to \infty$ these equations reduce to the well-known sinh–Gordon form.

The short distance asymptotic behaviour $\eta \to 0$ of the appropriate solution can be obtained by iteratively solving equations (4.7) with prescribed leading behaviour (1.9), (1.11). In terms of the combinations $F_-(\eta) = (2R)^{1/4} \eta^{1/8} G(\eta)$ and $F_+(\eta) = (2R)^{1/4} \eta^{1/8} \tilde{G}(\eta)$, we obtain

$$F_{\pm}(\eta) = 1 \pm r \eta^{1/2} + \frac{7}{8} (2r^2 - 1) \eta \pm \frac{1}{8} r \eta^{3/2} \left[2 + \Omega + 2r^2 \Omega \right]$$

$$+ \frac{1}{128} r \eta^2 \left[-7 + r^2 (15 + 8 \Omega^2) + r^4 (1 + 8 \Omega - 8 \Omega^2) \right]$$

$$\pm \frac{1}{128} r \eta^{5/2} \left[17 + 7 \Omega + 3r^2(3 + 7 \Omega) + r^4(-1 + 5 \Omega) \right] + O(\eta^3 \Omega^2),$$

where the term $\Omega$ contains the logarithmic dependence in $\eta$.

$$\Omega = \ln \left(k \eta^{1/2}\right).$$

The coefficient $k$ above, given by equation (1.10), is related to the expectation value of the energy field $\langle \sigma \rangle$ as described in appendix A.

Alternatively, this solution can be specified by the large distance asymptotic behaviour $\eta \to 1$ with leading behaviour (1.12) and $\chi(\eta) \sim 4 \ln \bar{s}$. Equations (4.7) then yield

$$\frac{G(\eta)}{\langle \sigma \rangle^2} = 1 + A^2 (1 - \eta)^{1+2r} g_2(1 - \eta) + A^4 (1 - \eta)^{2+4r} g_4(1 - \eta) + O((1 - \eta)^{18+6r}),$$

$$\frac{\tilde{G}(\eta)}{\langle \sigma \rangle^2} = A (1 - \eta)^{1/2+r} \tilde{g}_1(1 - \eta) + A^3 (1 - \eta)^{3/2+3r} \tilde{g}_3(1 - \eta)$$

$$+ A^5 (1 - \eta)^{5/2+5r} \tilde{g}_5(1 - \eta) + O((1 - \eta)^{49/2+7r}),$$

where the constant $A$, given in (1.13), can be obtained from the one-particle form factors; the magnetization $\langle \sigma \rangle$ is given in (1.14);

$$g_2(x) = \frac{1 + 2r}{2^{6}(1 + r)} \left[ \frac{x}{1 + r} + x^2 + \frac{(3 + 2r)(39 + 34r + 8r^2)}{32(2 + r)^2} x^3 + O(x^4) \right],$$

$$g_4(x) = \frac{3(1 + 2r)^3 (3 + 2r)}{2^{24}(1 + r)^4 (2 + r)^3 (3 + r)^2} \left[ \frac{x^6}{2 + r} + 2x^7 + O(x^8) \right];$$

and

$$\tilde{g}_1(x) = \frac{1}{2} F \left( \frac{5}{2} + r, \frac{7}{2} + r; 1 + 2r; x \right),$$

$$\tilde{g}_3(x) = \frac{(1 + 2r)^2 (3 + 2r)}{2^{10} (1 + r)^3 (2 + r)^2} \left[ \frac{x^3}{3 + 2r} + \frac{3}{4} x^4 + \frac{3(221 + 255r + 96r^2 + 12r^3)}{64(3 + r)^2} x^5 + O(x^6) \right],$$

$$\tilde{g}_5(x) = \frac{9(1 + 2r)^4 (3 + 2r)^2 (5 + 2r)}{2^{30} (1 + r)^5 (2 + r)^3 (3 + r)^4 (4 + r)^2} \left[ \frac{x^{10}}{5 + 2r} + \frac{5}{4} x^{11} + O(x^{12}) \right].$$

The functions $g_6(1 - \eta)$ and $\tilde{g}_7(1 - \eta)$ that enter in the terms $A^6 (1 - \eta)^{3+6r}$ and $A^7 (1 - \eta)^{7+7r}$ of $G$ and $\tilde{G}$ are of order $(1 - \eta)^{15}$ and $(1 - \eta)^{21}$, respectively. These powers are simple consequences of the Pauli exclusion principle on the particle states of
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Figure 2. Plots of the spin–spin correlation functions $G(\eta)$, $\tilde{G}(\eta)$, equations (4.1), against the distance $\eta^{1/2}$ for some values of the parameter $r$. These were obtained from numerical integration of equations (4.6) using the short distance expansion (4.8); agreement was found when comparing to the large distance behaviour (4.10); (a) $r = 0.5$; (b) $r = -0.1$; (c) $r = -0.3$; (d) $r \approx -0.5$.

the Hilbert space, as will be apparent in the following subsections, and are just the sums of the first integers up to 5 and up to 6, respectively.

It is possible to check order by order in both of these expansions that this solution satisfies the ‘duality’ relation

$$\langle \sigma(x)\sigma(x') \rangle |_{r = \pm 1/2} = \langle \mu(x)\mu(x') \rangle |_{r = \mp 1/2},$$

(4.13)
as depicted in figure 2. The relation (4.13) should not come as a surprise given that, as described in the introduction, for $r < -1/2$ we indeed have to trade $\sigma \leftrightarrow \mu$ due to the fact that only the ‘free’ asymptotic condition $\langle \sigma \rangle = 0$ is stable. Nevertheless, while (4.13) is verified by the short distance expansion (4.8) essentially because of the property
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$k(1/2) = k(-1/2)$, for $k(r)$ in (1.10), its validity from the large distance expansion (4.10) point of view is far less trivial.

Asymptotics for the auxiliary functions $\chi(\eta), \varphi(\eta)$ can be found in appendix D.

4.2. The form factors

For our purpose it is enough to consider form factors of fields placed at the centre of the Poincaré disc. And because we are considering the ‘ordered’ case $\langle \sigma \rangle \neq 0$, form factors of the form $\langle \text{vac} | \sigma(0) | n_1 \cdots n_{2N+1} \rangle$, $\langle \text{vac} | \mu(0) | n_1 \cdots n_{2N} \rangle$ trivially vanish.

One-particle form factors can be obtained from the Ward identity associated with the invariance of a single Ising system under rotations around the origin,

$$\langle \text{vac} | [R, \mu(0)] | n \rangle = 0. \quad (4.14)$$

Using the action of $R$ on particle states, equation (2.40), this yields the recursion relation

$$\langle \text{vac} | \mu(0) | n + 2 \rangle = \frac{\alpha_n}{\alpha_{n+1}} \langle \text{vac} | \mu(0) | n \rangle, \quad (4.15)$$

with coefficients $\alpha_n$ given in (2.39). Hence, $\langle \text{vac} | \mu(0) | 2n + 1 \rangle = 0$ and

$$\langle \text{vac} | \mu(0) | 2n \rangle = \langle \text{vac} | \mu(0) | 0 \rangle \left( \frac{\Gamma(n + 1/2)(r + 1/2)}{\sqrt{\pi} \Gamma(n + 1)(r + 1)n} \right)^{1/2}, \quad (4.16)$$

where $n = 0, 1, 2, \ldots$ and $(a)_n = \Gamma(a+n)/\Gamma(a)$. The one-particle form factor $\langle \text{vac} | \mu(0) | n = 0 \rangle$ is determined from the short distance behaviour of

$$\langle \psi(x) \mu(0) \rangle \sim \langle \sigma \rangle \sqrt{\frac{-i}{2z}} \quad \text{as } |x| \to 0. \quad (4.17)$$

Using the expansion for $\psi(x)$ in (2.27) and re-summing the leading large $n$ behaviour of (4.16), we find

$$\langle \text{vac} | \mu(0) | 0 \rangle = \langle \sigma \rangle \left( \frac{\Gamma(r + 1/2)}{\sqrt{\pi} \Gamma(r + 1)} \right)^{1/2}, \quad (4.18)$$

which agrees with the result obtained in [6].

Two-particle form factors can be obtained from the Ward identity for the doubled system,

$$\langle \text{vac} | [Y_1 - Y_{-1}, \mu_a(0)\mu_b(0)] | n_1 n_2 ; \text{vac} \rangle = i \langle \text{vac} | [H_a - H_b, \sigma_a(0)\sigma_b(0)] | n_1 n_2 ; \text{vac} \rangle, \quad (4.19)$$

coming from (3.12). The combination $Y_1 - Y_{-1}$ can be expressed in terms of $X_1 - X_{-1}$ and $Z_0$ as in (3.6), so that it is straightforward to use the results from section 2 to evaluate its action on the particle state. Using (4.15) as well, the equation above can be written as

$$-i (\omega_{n_1} + \omega_{n_2}) \langle \sigma \rangle \langle \text{vac} | \sigma(0) | n_1 n_2 \rangle = \alpha_{n_2} \langle \text{vac} | \mu(0) | n_1 \rangle \langle \text{vac} | \mu(0) | n_2 + 1 \rangle$$

$$- \alpha_{n_1} \langle \text{vac} | \mu(0) | n_2 \rangle \langle \text{vac} | \mu(0) | n_1 + 1 \rangle, \quad (4.20)$$
and hence the non-zero elements are
\[
\langle \sigma \rangle \langle \text{vac}|\sigma(0)|2n_1,2n_2+1 \rangle = \frac{i\pi n_2}{\omega_{2n_1} + \omega_{2n_2+1}} \langle \text{vac}|\mu(0)|2n_1\rangle \langle \text{vac}|\mu(0)|2n_2\rangle,
\]
where \( \omega_n \) and \( \alpha_n \) are given, respectively, by (2.28) and (2.39). This formula agrees with the results found in [6].

Form factors involving three or more particles can be written in terms of the one-particle and two-particle form factors using Wick’s theorem, which can in fact be seen as a consequence of Ward identities as above. For instance, in the case of the three- and four-particle form factors, we can use equations (3.11) and the action of the \( U(1) \) charge \( Z_0 \) on particle states (3.16). The Ward identities
\[
\langle \text{vac} | Z_0, \sigma_a(0) \sigma_b(0) | n_1 n_2 n_3 \rangle = -i \langle \text{vac} | \mu_a(0) \sigma_b(0) | n_1 n_2 n_3 \rangle \text{ vac} \,
\]
give, respectively,
\[
\langle \sigma \rangle \langle \text{vac} | \mu(0) | n_1 n_2 n_3 \rangle = \langle \text{vac} | \mu(0) | n_1 \rangle \langle \text{vac} | \sigma(0) | n_2 n_3 \rangle + \langle \text{vac} | \mu(0) | n_3 \rangle \langle \text{vac} | \sigma(0) | n_1 n_2 \rangle
+ \langle \text{vac} | \mu(0) | n_2 \rangle \langle \text{vac} | \sigma(0) | n_3 n_1 \rangle
\]
and
\[
\langle \sigma \rangle \langle \text{vac} | \sigma(0) | n_1 n_2 n_3 n_4 \rangle = \langle \text{vac} | \sigma(0) | n_1 n_2 \rangle \langle \text{vac} | \sigma(0) | n_3 n_4 \rangle
- \langle \text{vac} | \sigma(0) | n_1 n_3 \rangle \langle \text{vac} | \sigma(0) | n_2 n_4 \rangle + \langle \text{vac} | \sigma(0) | n_1 n_4 \rangle \langle \text{vac} | \sigma(0) | n_2 n_3 \rangle.
\]

The large distance expansion for the two-point correlation functions (4.1) can be obtained by inserting a complete set of states between the two operators and by expressing them as summations over form factor contributions,
\[
G(\eta) = \langle \sigma \rangle^2 + \sum_{n_1,n_2 \geq 0} \left| \langle \text{vac} | \sigma(0) | 2n_1,2n_2+1 \rangle \right|^2 e^{-\omega_{2n_1} - \omega_{2n_2+1}(d/2R)} \text{ vac}
\]
\[
+ (2l - \text{ particle contributions}) \quad (l \geq 2),
\]
\[
\tilde{G}(\eta) = \sum_{n \geq 0} \left| \langle \text{vac} | \mu(0) | 2n \rangle \right|^2 e^{-\omega_{2n}(d/2R)}
\]
\[
+ (2l + 1 - \text{ particle contributions}) \quad (l \geq 1),
\]
where \( d \) denotes the geodesic distance, related to \( \eta \) by (1.5), and \( \omega_n \) is the energy eigenvalue (2.28). As noted in [4], these expressions are obtained using \( SU(1,1) \) transformations in order to bring one of the spin operators to the origin of the disc, the other to the imaginary axis \( z = iq, \overline{z} = -iq \), and by using the representation
\[
\sigma(iq, -iq) = e^{H(d/2R)} \sigma(0) e^{-H(d/2R)},
\]
where \( H \) is the Hamiltonian (2.24) and \( d \) is the geodesic distance from the point \( (iq, -iq) \) to the origin. The expansions (4.25) agree with the asymptotics (4.10) obtained from the analysis of the differential equations, with the coefficient \( A \) being related to the one-particle form factor \( \langle \text{vac} | \mu(0) | 0 \rangle \) in (4.18) by
\[
A \langle \sigma \rangle^2 = 4^{-r} \langle \text{vac} | \mu(0) | 0 \rangle^2,
\]
which gives (1.13).
5. Thermodynamics and discussion

It is natural to assume that the Ising quantum field theory on the pseudosphere represents the scaling limit of an Ising-like statistical system on a lattice embedded into the pseudosphere. Although we do not yet have a precise construction of this statistical system and of its scaling limit, it is interesting to interpret our results assuming that most of the basic concepts underlying the scaling limit in the flat-space situation carry over to the pseudosphere. In particular, one can assume that in the process of taking the scaling limit, the curvature of the pseudosphere is brought towards zero in microscopic units, and that the lattice embedded into the pseudosphere resembles more and more, in regions of radius much smaller than $R$, a prescribed (regular) flat-space lattice. At the same time, the temperature is brought towards the critical temperature of this prescribed flat-space lattice, with a prescribed ratio between $R$ and the resulting flat-space correlation length. This ratio is given by the parameter $r = mR$ that we introduced for the Ising quantum field theory on the pseudosphere.

5.1. The free energy

An interesting quantity to study is the specific free energy $f(m, R)$ as a function of the mass $m$ and curvature radius $R$, defined through the partition function $Z$ by $f = -\lim_{V \to \infty} \ln Z^{1/V}$ where $V$ is the (two-dimensional) volume of space.

A particularly simple case is the massless one, $m = 0$, where the free energy $f(m = 0, R)$ can be computed using the defining relation between the trace of the energy–momentum tensor $T^\mu_\mu$ and the variation of the action $S$ under a scale transformation of the metric $g_{\mu\nu}$, $\sqrt{g} T^\mu_\mu = 2g_{\mu\nu} \delta S / \delta g^{\mu\nu}$. This gives

$$R \frac{d}{dR} [V(R) f(0, R)] = V(R) \langle T^\mu_\mu \rangle,$$

where the volume $V(R)$ must be taken finite (but large) for this equation to make sense and must vary like $R^2$ under a scale transformation, $R dV(R)/dR = 2V(R)$. For the pseudosphere, the trace anomaly is related to the central charge $c$ by $\langle T^\mu_\mu \rangle = c/(12\pi R^2)$, where we have set to zero the constant piece related to the vacuum energy density. With $c = 1/2$, this yields

$$f(m = 0, R) = \frac{1}{24\pi R^2} \ln \left( \frac{2R}{L} \right),$$

where $L$ is an integration constant not determined by the quantum field theory but only fixed after specifying the microscopic theory, i.e. the theory with an explicit ultra-violet cut-off. In the scaling limit of the corresponding microscopic theory, that is, setting the temperature equal to the flat-space critical temperature and making $R$ very large in microscopic units, the corresponding free energy per unit volume is expected to have the leading behaviour (5.2).

Such a geometrical contribution to the specific free energy, and in particular the presence of the non-universal distance $L$, is expected for theories on a space that is not asymptotically flat. In the case of the pseudosphere, the logarithmic increase of $R^2 f(0, R)$ as $R$ increases is related to the decrease of the ‘space available’ around every site, which decreases the interaction energy and increases the free energy (as opposed to the case of
a sphere of radius $R$, where one observes a decrease of $R^2 f(0, R)$ [11]). In comparison, there is no such contribution to the specific free energy in asymptotically flat spaces without singularities, for instance. There is only a finite and universal contribution to the total, volume-integrated free energy; for a conformal-to-flat metric $g_{\mu\nu} = e^{2\phi} \delta_{\mu\nu}$, this contribution is given by the well-known formula

$$-\frac{c}{24\pi} \lim_{V \to \infty} \int_V d^2x \partial_\mu \phi \partial^\mu \phi. \quad (5.3)$$

The free energy $f(m, R)$ of the Ising field theory at arbitrary mass can be obtained from the vacuum expectation value of the energy field at finite volume $V$, here denoted by $\langle \phi \rangle_V(x)$, by taking the infinite-volume limit:

$$\frac{d}{dm} f(m, R) = -\frac{1}{2\pi} \lim_{V \to \infty} \frac{1}{V} \int_V d^2x \left[ \langle \phi \rangle_V(x) + m \ln \left( \frac{2R}{\epsilon} \right) \right], \quad (5.4)$$

where $\epsilon$ is another non-universal microscopic distance.

In (5.4), it is tempting to take the limit of infinite volume $V$ by simply replacing $\langle \phi \rangle_V(x)$ by its infinite-volume, position-independent expression (A.6). However, because on the pseudosphere the surface enclosing a finite region increases as fast as its volume for large volumes, it is possible that contributions proportional to the surface, arising from integration of $\langle \phi \rangle_V(x)$ at positions $x$ near the boundary (where it is significantly different from (A.6)), give in $f(m, R)$ additional finite terms. We have not yet evaluated these contributions, but expect to come back to this problem in a future work.

A similar situation was found in the study of the Ising model on hyperlattices [12]. As those authors did, we focus our attention on a ‘bulk’ free energy, defined by taking for $\langle \phi \rangle_V(x)$ in (5.4) the expression (A.6), valid at positions $x$ far from the boundary. This gives

$$2\pi R^2 f(m, R) = \ln (1 + r) - \frac{r}{2} \ln 2\pi - \frac{r^2}{2} \ln \left( \frac{2R}{\epsilon} \right) + \frac{1}{12} \ln \left( \frac{2R}{L} \right), \quad (5.5)$$

where $G(z)$ is Barnes’ $G$-function

$$G(z + 1) = (2\pi)^{z/2} e^{-z(z+1)/2} \prod_{n=1}^{\infty} \left( 1 + \frac{z}{n} \right)^{n} e^{-z^2/(2n)} \left[ \left( 1 + \frac{z}{n} \right)^n e^{-z^2/(2n)} \right], \quad (5.6)$$

and $\gamma$ is Euler’s constant. This expression has the small $r$ convergent expansion

$$2\pi R^2 f(m, R) = \left[ \frac{1}{12} \ln \left( \frac{2R}{L} \right) - \frac{r}{2} - \frac{r^2}{2} \ln \left( \frac{2R}{\epsilon} \right) + \sum_{n=3}^{\infty} (-1)^{n-1} \zeta(n-1) \frac{r^n}{n}, \quad (5.7)$$

where $\zeta(n)$ is Riemann’s zeta function, as well as the following asymptotic expansion at large $r$:

$$2\pi R^2 f(m, R) = \frac{r^2}{2} \ln \left( \frac{m \epsilon^{1/2}}{2} \right) - \frac{1}{12} \ln \left( \frac{mL}{2} \right) + \zeta'(-1) + O(r^{-2}), \quad (5.8)$$

with the first term corresponding to the specific free energy of the massive Majorana fermion theory in flat space (the $R \to \infty$ limit). As on the sphere [11], there is no logarithmic term in $R$ in this large $r$ expansion.
From the analytic properties of Barnes’ $G$-function, one can see that the free energy (5.5) has logarithmic singularities located at the negative integers $r = -1, -2, \ldots$. In particular, it is regular at $r = 0$, that is, the flat-space critical temperature does not correspond to a singularity of the free energy.

It is interesting to note that when we fix $r = 1/\sqrt{6}$, the free energy (5.5) does not depend on $R$ any longer, and only the ratio $L/\epsilon$ appears. In this case, the logarithmic increase of $R^2 f(m, R)$ as $R$ increases due to effects of the geometry as explained above is exactly cancelled out by the logarithmic decrease due to the increase of the interaction energy as the correlation length grows.

The ‘bulk’ free energy defined above still depends on the asymptotic conditions of the quantum field theory. Specifically, the expression (5.5) is valid for ‘fixed’ asymptotic conditions, whilst the replacement $r \mapsto -r$ gives the expression for ‘free’ asymptotic conditions. Both asymptotic conditions, or regimes, are stable in the region $-1/2 < r < 1/2$ and we intend to discuss the possible transitions between these regimes in a future work. A full treatment of the thermodynamics of the model, in fact, seems to require a better understanding of the nature and importance of the neglected surface terms as well as of the other stable asymptotic conditions that break part or all of the symmetries associated with the isometry of the pseudosphere, as described in section 2.2.

5.2. The magnetization

The expression (1.14) for the magnetization $\langle \sigma \rangle$ in the Ising field theory is expected to determine the coefficient of the leading asymptotic behaviour of the magnetization in the microscopic theory as the scaling limit is taken. As depicted in figure 3, it does not vanish at the flat-space critical temperature $m = 0$, but rather at a higher temperature, corresponding to the value $m = -1/2R$ of the mass parameter. That is, at $r = -1/2$, there is a change in the power law of the leading asymptotic behaviour of the magnetization in the microscopic theory as the scaling limit is taken. From this only, we cannot conclude that there exists an $R$-dependent temperature at which the magnetization vanishes identically in the microscopic theory for any finite $R$. However, the vanishing of the magnetization occurs at the value of $m$ below which the ordered regime is unstable and the disordered regime, where the magnetization is zero, is stable. It is plausible that there be a similar point joining an ordered and a disordered regime at finite $R$ in the microscopic theory at a temperature higher than the flat-space critical temperature (higher by an amount that has the power law behaviour $\sim R^{-1}$ as the scaling limit is taken). The magnetization would vanish at the turning point between the two regimes, as has been suggested for the regular lattice theory studied in [12]. We note, though, that our expression (5.5) for the free energy is regular at $r = -1/2$. Of course, as we have pointed out, the expression (5.5) probably does not give the full free energy; hence no serious conclusion can be drawn from it yet.

Near the effective ‘critical’ temperature, the magnetization in the Ising field theory vanishes as

$$\langle \sigma \rangle^2 = (2R)^{-1/4} \sqrt{\pi} \bar{s}_{\text{flat}} (r + \frac{1}{2}) + O((r + \frac{1}{2})^2),$$

(5.9)

where $\bar{s}_{\text{flat}} = 2^{1/12} e^{-1/8} A^{3/2}$ ($A$ being Glaisher’s constant). The exponent $1/2$ can be explained by recalling that a space of constant negative curvature is effectively infinite dimensional at large distances due to the fact that the volume grows exponentially [13].
Figure 3. A plot of the magnetization \((2R)^{1/4}\langle \sigma \rangle^2\), in equation (1.14). The full curve corresponds to the choice of ‘fixed’ boundary condition for the Ising field theory (1.1) in the region \(r > -1/2\), whereas the dotted curve corresponds to the choice of ‘free’ boundary condition for \(r < 1/2\).

In fact, a theory on the pseudosphere should essentially show, in some sense, a crossover behaviour going from a two-dimensional theory to an infinite-dimensional theory. Hence mean field theory could be used to predict the exponent ruling the vanishing of the magnetization in the Ising field theory, giving 1/2 as above. Assuming that the magnetization in the microscopic theory vanishes similarly at a ‘critical’ temperature, the exponent ruling its vanishing should then be 1/2, which agrees with the results of [12]. In the flat-space limit \(R \to \infty\), the magnetization takes the usual form \(\langle \sigma \rangle \to \bar{s}_{\text{flat}} m^{1/8}\), and the exponent 1/8 is recovered.

The fact that the effective ‘critical’ temperature is higher than the flat-space critical temperature is expected: the asymptotic conditions have a greater effect on the pseudosphere than they have on flat space; hence the ‘fixed’ asymptotic conditions will render the establishing of disorder more difficult, increasing the effective ‘critical’ temperature. Similar considerations apply in the disordered regime: there the effective temperature at which the average of the disorder variable vanishes is lower than the flat-space critical temperature, because ‘free’ asymptotic conditions make it more difficult to establish order.

5.3. Two-point correlation functions and susceptibility

An interesting characteristic of the two-point functions is their exponential decay at large geodesic distances,

\[
\frac{\langle \sigma(x)\sigma(x') \rangle}{\langle \sigma \rangle^2} - 1 \sim \frac{\Gamma((1/2) + r)\Gamma((3/2) + r)}{2\pi \Gamma^2(2 + r)} e^{-2(1+r)d(x,x')/R} \text{ as } d(x,x') \to \infty, \quad (5.10)
\]

and

\[
\frac{\langle \mu(x)\mu(x') \rangle}{\langle \sigma \rangle^2} \sim \frac{\Gamma((1/2) + r)}{\sqrt{\pi} \Gamma(1 + r)} e^{-((1/2)+r)d(x,x')/R} \text{ as } d(x,x') \to \infty. \quad (5.11)
\]

As expected, the leading exponential decay is different for order–order and disorder–disorder two-point functions; in the former it comes from two-particle contributions,
whereas in the latter it comes from one-particle contributions. However, contrary to the flat-space case, the vanishing of the exponent ruling the leading exponential decay occurs at different values of $m$ in order–order and in disorder–disorder two-point functions. This is simply due to the discreteness of the energy levels, and to Pauli’s exclusion principle that forces two particles to be in different energy levels. Hence one cannot define, in this way, a unique correlation length valid for describing the long distance behaviour of both correlation functions. It is natural, however, to choose the exponential decay of the disorder–disorder two-point function as the one defining an effective correlation length,

$$\xi = \frac{2R}{1 + 2r}. \quad (5.12)$$

We do indeed expect this correlation length to diverge at the point $r = -1/2$ in the ordered regime where the magnetization $\langle \sigma \rangle$ vanishes, since at this point, the disorder field acquires a non-zero expectation value and the large distance asymptotic behaviour of its two-point function changes. This correlation length is also in accordance with the usual definition, in finite-size systems, as the inverse of the gap between the ground state and the first excited state. It diverges as the inverse power of the difference of the temperature from the effective ‘critical’ temperature, as is the case for the Ising model on flat space, but it is defined here only for the behaviour from above the point $r = -1/2$, since below this point the system is necessarily in its disordered regime. A corresponding definition of the correlation length in the disordered regime leads to a divergence at the point $r = 1/2$ from below. Following considerations similar to those of the previous subsection, we expect to have the same power law behaviour of the correlation length in the lattice theory in the vicinity of the critical point. Note that a naive application of general results from mean field theory would predict the power law $\sim (1 + 2r)^{-1/2}$.

In the ordered regime ($r > -1/2$), and as the point $r = -1/2$ is approached, the two-point function of disorder fields goes at large distances to an almost constant value, before vanishing at larger and larger distances. This almost constant value approaches the value that $\langle \mu \rangle^2$ takes in the disordered regime at $r = -1/2$. More precisely, as $r = -1/2$ is approached in the ordered regime, both order–order and disorder–disorder two-point functions tend to the exact form that they have in the disordered regime at $r = -1/2$ (see figure 2). This is a consequence of the duality relating the point $r = -1/2$ to the point $r = 1/2$, which yields equation (4.13). A similar duality has also been observed in the study of the statistical Ising model on a hyperlattice [12].

It is also interesting to consider the general case where an external magnetic field $h$ is added to the Ising field theory (1.1), by adding the perturbation $h \int d^2x \ e^{\phi(x)} \sigma(x)$. The corresponding susceptibility $\chi$ giving the linear response of the magnetization is given by

$$\chi = \frac{1}{2} \int \frac{d^2x}{4R^2} \ e^{\phi(x)} \left( \langle \sigma(x)\sigma(0) \rangle - \langle \sigma \rangle^2 \right)$$

$$= \frac{\pi}{4R} \int_0^{\infty} ds \ \sinh \left( \frac{s}{R} \right) \left( \langle \sigma(x)\sigma(0) \rangle \big|_{d(x,0)=s} - \langle \sigma \rangle^2 \right). \quad (5.13)$$

Using the asymptotic behaviour (5.10) for the ordered regime, it is straightforward to see that the integral above is convergent for any $r > -1/2$, with a divergence at $r = -1/2$,

$$\chi \sim \frac{s^4_{\text{flat}}}{4\sqrt{\pi}} \frac{(2R)^{-1/4}}{1 + 2r} \quad \text{as } r \to -\frac{1}{2}. \quad (5.14)$$

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The susceptibility again shows a divergence at the effective ‘critical’ value \( m = -1/2R \), with a mean field power law behaviour. A similar phenomenon was observed for the model studied in [12]. In the disordered regime, using the asymptotic behaviour (5.11) with \( r \mapsto -r \) for the order–order two-point function, one can see that the susceptibility is finite for \( r < -1/2 \) and diverges with a mean field power law at \( r = -1/2 \), even though the regime is stable above \(-1/2\). Hence in the whole range \(-1/2 < r < 1/2\) in the disordered regime, the response of the magnetization to a magnetic field is not linear at small magnetic field. Note also that the susceptibility diverges at \( r = -1/2 \) from both directions with the same exponent.

Relating the susceptibility to the expansion of the free energy in powers of the magnetic field in the usual fashion, one could conclude from this analysis that the free energy possesses a singular behaviour at small magnetic field in the region \(-1/2 < r < 1/2\) of the disordered regime. However, from considerations similar to those above, it is possible that one needs to take into account surface terms in order to obtain the correct coefficients in the expansion of the free energy in powers of magnetic field. We hope to carry out this analysis in a future work.

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Appendix A

The vacuum expectation value of the energy field \( \varepsilon(x) = i(2R)^{-1}(1 - \bar{z}z)(\psi\bar{\psi})(x) \) can be obtained from the propagator

\[
\langle \psi(x) \bar{\psi}(x') \rangle = (1 - z\bar{z})^{-1}\mathcal{G}(\eta),
\]

where the piece

\[
\mathcal{G}(\eta) = \frac{\Gamma(r)\Gamma(1 + r)}{2i\Gamma(2r)}(1 - \eta)^r F(r, 1 + r; 1 + 2r; 1 - \eta),
\]

is a function of the projective invariant \( \eta \), equation (2.44). This propagator is determined by the equations of motion (2.5)

\[
(1 - z\bar{z}) \partial_z ((1 - z\bar{z})\partial_z \langle \psi(x)\bar{\psi}(x') \rangle) = r^2 \langle \psi(x)\bar{\psi}(x') \rangle,
\]

and the normalization condition

\[
\langle \psi(x)\bar{\psi}(x') \rangle \sim \frac{ir}{1 - z\bar{z}} \ln |z - z'|^2 \quad \text{as} \quad |z - z'| \to 0.
\]
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as well as the condition that it vanishes in the limit of large geodesic distance $\eta \to 1$.

One can calculate the vacuum expectation value of the energy field by the point-splitting technique. Due to resonance between the energy field $\varepsilon(x)$ and the identity field multiplied by the mass parameter, $m \mathbb{1}$, one needs one more condition to define the energy field. This condition can be taken as

$$\frac{d}{dm} \langle \varepsilon \rangle \bigg|_{m=0} = 0,$$

which gives

$$2R \langle \varepsilon \rangle = -2r (\psi(r) + \gamma) - 1,$$

where $\psi(x) = d \ln \Gamma(x)/dx$ and $\gamma$ is Euler’s constant. The vacuum expectation value of the energy field is related to the constant $k$, equation (1.10), appearing in the short distance expansion of the spin–spin correlation function. This can be seen from the spin–spin operator product expansion (OPE) in the massive Majorana theory on the pseudosphere, which has the form

$$\sigma(x)\sigma(x') = (2R)^{-1/4} \eta^{-1/8} \left( C_{\mathbb{1}}(\eta, r) \mathbb{1} + 2R \eta^{1/2} C_{\varepsilon}(\eta, r) \varepsilon(x') + \cdots \right),$$

where $C_{\mathbb{1}}(\eta, r)$ and $C_{\varepsilon}(\eta, r)$ are structure functions and the dots represent contributions coming from descendent fields. In the Majorana theory of mass $m$ on infinite flat space, the main property of such OPEs is that the structure functions involved are analytic in the perturbing parameter $m$ in some region around $m = 0$ (in fact, they are entire functions of $m$). All non-analyticities around $m = 0$ of correlation functions come from the vacuum expectation values of the fields appearing in OPEs. In the massive Majorana theory on the pseudosphere, analyticity of structure functions is a trivial statement, since all correlation functions are expected to be analytic in some region around $m = 0$ (expected to be finite). This comes from the fact that the negative curvature plays the role of an infrared regulator. A more useful statement is that the flat-space limit $R \to \infty$ (that is, the limit where the infrared regulator disappears) should be well defined independently on every term in the expansion in $m$ of the structure functions. This gives, for the structure functions, the form

$$C_{\mathbb{1}}(\eta, r) = 1 + C r^{1/2} + O(\eta) + r^\eta \left( a \ln \eta + b + O(\eta^{1/2} \ln \eta) \right) + O(r^2 \eta \ln^2 \eta),$$

$$C_{\varepsilon}(\eta, r) = \frac{1}{2} + O(\eta^{1/2} \ln \eta).$$

Logarithmic terms appear in the part proportional to $r$ in $C_{\mathbb{1}}(\eta, r)$ because of the resonance between $\varepsilon$ and $m \mathbb{1}$; from (4.8) the coefficient $a$ is equal to $-1/2$.

Clearly, the OPE (A.7) shows that the constant $-r \ln k$ involved in the short distance expansion (4.8) of the spin–spin correlation function is given by

$$-r \ln k = R \langle \varepsilon \rangle + rb + C$$

in terms of the vacuum expectation value of the energy field and of the constants $b$ and $C$ appearing in the structure function $C_{\mathbb{1}}(\eta, r)$. These constants can be obtained from the conformal limit $m \to 0$ [1],

$$\langle \sigma(x)\sigma(x') \rangle_{m=0} = (2R)^{-1/4} \sqrt{\eta^{1/4} + \eta^{-1/4}},$$
which gives \( C = 0 \), and from the known flat-space limit \( k(r) \to r^{7/4} \) as \( r \to \infty \) [14], so \( b = \ln 4 \). This gives equation (1.10).

It is instructive to explicitly evaluate the constant \( b \) by conformal perturbation theory, thereby giving a simple derivation of the known flat-space limit. This illustrates the use of a negative curvature as an infrared regulator. The form (A.8) of the structure functions implies that the constant \( b \) can be calculated by perturbation theory for the two-point function \( \langle \sigma(x)\sigma(x') \rangle \) about the boundary Ising CFT, expanding to first order in \( m \):

\[
\langle \sigma(x) \sigma(x') \rangle = \langle \sigma(x) \sigma(x') \rangle_{m=0} + \frac{2R^2m}{\pi} \int_{|\zeta|<1} \frac{dx'' dy''}{(1 - \zeta^2)^2} \left( \langle \sigma(x)\sigma(x')\varepsilon(x'') \rangle_{m=0} - (2R)^{-1} \langle \sigma(x)\sigma(x') \rangle_{m=0} \right) + O(m^2),
\]

(A.11)

where \( \zeta = x'' + iy'' \) and \( \bar{\zeta} = x'' - iy'' \). By simple considerations of large geodesic distance asymptotics, the integral above is infrared (\( |\zeta| \to 1 \)) convergent. From the result (A.10) and from

\[
\langle \sigma(x) \sigma(x') \rangle = \langle \sigma(x) \sigma(x') \rangle_{m=0} - \frac{(2R)^{-5/4}}{2\sqrt{\eta^{1/4} + \eta^{-1/4}}} \left\{ \eta^{1/4} \left( \frac{|\zeta||\zeta - \eta^{1/2}|}{|1 - \eta^{1/2}\zeta|} + \frac{|1 - \eta^{1/2}|}{|\zeta||\zeta - \eta^{1/2}|} \right) \right\},
\]

(A.12)

which, as expected, gives \( C = 0, b = \ln 4 \).

**Appendix B**

In this appendix we briefly sketch the steps for obtaining the mode decomposition (2.27). The Hamiltonian (2.24) in the isometric system of coordinates (2.16) reads

\[
H = \int_{-\pi/4}^{\pi/4} \frac{d\xi_x}{4\pi} \left( -i\psi_i^{(iso)}, i\bar{\psi}_j^{(iso)} \right) \mathcal{H} \left( \begin{array}{c} \psi_i^{(iso)} \\ \bar{\psi}_j^{(iso)} \end{array} \right),
\]

(B.1)

where the Hamiltonian density is

\[
\mathcal{H} = \begin{pmatrix} \frac{d}{d\xi_x} & 2r \\ 2r & \frac{\cos(2\xi_x)}{\cos(2\xi_x)} & -i\frac{d}{d\xi_x} \end{pmatrix}.
\]

(B.2)

The Hamiltonian is just, in the language of first quantization, the diagonal matrix element of \( \mathcal{H} \) in the state represented by the spinor wavefunction \( \Psi = \begin{pmatrix} \psi_i^{(iso)} \\ \bar{\psi}_j^{(iso)} \end{pmatrix} \),

\[
H = \langle \Psi, \mathcal{H} \Psi \rangle,
\]

(B.3)
where the inner product between two spinor wavefunctions $\Psi_1$ and $\Psi_2$ is
\[
(\Psi_1, \Psi_2) = \int_{-\pi/4}^{\pi/4} \frac{d\xi}{4\pi} \Psi^*_1(\xi) \Psi_2(\xi).
\] (B.4)

From the condition on the phases of the fermion operators $\psi^{(iso)}(\xi_x, \xi_y)^\dagger = i\psi^{(iso)}(\xi_x, \xi_y)$ and $\psi^{(iso)}(\xi_x, \xi_y)^\dagger = -i\psi^{(iso)}(\xi_x, \xi_y)$, and from the condition that charge conjugation symmetry $i\psi(\xi_x, \xi_y) \leftrightarrow \bar{\psi}(\xi_x, -\xi_y)$ be implemented on modes by $A_\omega^\dagger \leftrightarrow A_\omega$, the mode decomposition of the fields has the form
\[
\psi^{(iso)}(\xi_x, \xi_y) = \sum_{\omega > 0} (e^{i\omega \xi_x} G_\omega(\xi_x) A_\omega^\dagger - ie^{-i\omega \xi_y} \bar{G}_\omega(\xi_x) A_\omega),
\]
\[
\bar{\psi}^{(iso)}(\xi_x, \xi_y) = \sum_{\omega > 0} (e^{i\omega \xi_x} \bar{G}_\omega(\xi_x) A_\omega^\dagger + ie^{-i\omega \xi_y} G_\omega(\xi_x) A_\omega). \tag{B.5}
\]

The spinor wavefunctions
\[
S_\omega(\xi_x) = \begin{pmatrix} G_\omega(\xi_x) \\ \bar{G}_\omega(\xi_x) \end{pmatrix}, \tag{B.6}
\]
for all real values of $\omega$ and with $G_{-\omega}(\xi_x) = -i\bar{G}_\omega(\xi_x)$, should form a complete orthogonal set of wavefunctions diagonalizing the Hamiltonian density (B.2)
\[
\mathcal{H} S_\omega = \omega S_\omega. \tag{B.7}
\]

A set of independent spinors of the form (B.6) diagonalizing the Hamiltonian density with eigenvalue $\omega$ is given by
\[
s^+_{\omega} = \begin{pmatrix} g^+_{\omega} \\ \bar{g}^+_{\omega} \end{pmatrix}, \quad s^-_{\omega} = \begin{pmatrix} ig^-_{\omega} \\ -i\bar{g}^-_{\omega} \end{pmatrix}, \tag{B.8}
\]
where
\[
g^\pm_{\omega}(\xi_x) = e^{-i\omega \xi_x - i(\pi/4)(\omega - 1 + 2r)(1 + e^{i\xi_x})_{\pm r} \left( -\frac{\omega}{2} + \frac{1}{2} \pm r; 1 \pm 2r; 1 + e^{i\xi_x} \right)}, \tag{B.9}
\]
and
\[
\bar{g}^\pm_{\omega}(\xi_x) = -ig^\pm_{-\omega}(\xi_x). \tag{B.10}
\]
The branch cut of the hypergeometric function is taken from $-\infty$ to 1, and the hypergeometric function is chosen to be unity in the limit $\xi_x \to -\pi/4$.

The (not normalized) spinors (B.6) can then be expressed as
\[
S_\omega = \begin{cases} s^+_{\omega} + C_\omega s^-_{\omega} & (\omega > 0), \\
-s^+_{\omega} + C_\omega s^-_{\omega} & (\omega < 0), \end{cases} \tag{B.11}
\]
for real constants $C_\omega$ satisfying $C_{-\omega} = C_\omega$. For a given set of $\omega$ and given associated constants $C_\omega$, they will form a complete orthogonal set if the inner product $(S_\omega, S_\omega')$ is well defined for all $\omega$ and $\omega'$ in this set; and if the Hamiltonian is Hermitian, $(S_\omega, H S_\omega) = (H S_\omega, S_\omega)$. These lead respectively to the conditions
\[
\lim_{\epsilon \to 0} \epsilon \left\{ \text{Re} (\bar{G}_\omega G_\omega')_{\xi_x = \pi/4 - \epsilon} + \text{Re} (\bar{G}_\omega G_\omega')_{\xi_x = -\pi/4 + \epsilon} \right\} = 0, \tag{B.12}
\]
and
\[ \lim_{\epsilon \to 0} \left\{ \text{Im} \left( \bar{G}_\omega G_{\omega'} \right)_{\xi_0 = \pi/4 - \epsilon} - \text{Im} \left( \bar{G}_\omega G_{\omega'} \right)_{\xi_0 = -\pi/4 + \epsilon} \right\} = 0. \] (B.13)

In the case \( r > 1/2 \), condition (B.12) is satisfied only with \( C_\omega = 0 \) and by taking the hypergeometric function in (B.9) to have trivial monodromy: \( \omega = \pm(1 + 2r + 2n), \) \( n = 0, 1, 2, \ldots \). The function \( G_\omega \) then vanishes at the boundaries \( \xi_x = \pm \pi/4 \). With this wave decomposition, the Fermi fields vanish as \( e^{-md} \) as the geodesic distance \( d \) from the centre of the disc goes to infinity.

In the case \( 0 < r < 1/2 \), condition (B.12) is always satisfied. Condition (B.13) is then satisfied for many sets \( \{ \omega; C\} \). They correspond to many sets of stable asymptotic conditions on the fields, and hence to many stable regimes of the quantum field theory with different thermodynamic properties. In this paper we concentrate our attention on the regimes which preserve the \( SU(1, 1) \) symmetry; we intend to analyse other regimes in a future paper. The charges (2.18) must then be well defined (and the Hilbert space must form a lowest weight module for the \( SU(1, 1) \) algebra that they generate), which imposes again that the hypergeometric function in (B.9) has trivial monodromy, but not that the function \( G_\omega \) be vanishing at the boundaries \( \xi_x = \pm \pi/4 \). Hence there are two possible sets: \( \omega = \pm(1 + 2r + 2n), \) \( n = 0, 1, 2, \ldots \) with \( C_\omega = 0 \); and \( \omega = \pm(1 - 2r + 2n), \) \( n = 0, 1, 2, \ldots \) with \( C_\omega \to \infty \). In the first set, the Fermi fields vanish as \( e^{-md} \) as the geodesic distance \( d \) to the centre of the disc goes to infinity, whereas in the second set, they diverge as \( e^{md}(1 + O(e^{-d/R})) \). These correspond respectively to the ‘fixed’ and ‘free’ asymptotic conditions on the order field \( \sigma \).

The decomposition (2.27) follows from these considerations, with the identification \( A_\omega^\dagger \to e^{i(\pi/2)n} A_n^\dagger \) and \( A_\omega \to e^{-i(\pi/2)n} A_n \); this choice of phases ensures Hermiticity of the conserved charges (2.18) on the Hilbert space.

### Appendix C

In order to derive the \( SU(1, 1) \) transformation properties of the conserved charges introduced in section 3 (equations (2.18), (3.3), (3.5), (3.6) and (3.8)), it is convenient to use the fact that they can be written as integrals of local densities quadratic in Fermi fields and their Lie derivatives. Hence, it suffices to consider the properties of the Fermi fields specified in equation (2.9) under the \( SU(1, 1) \) coordinate transformation (2.8). The transformation of their Lie derivatives (2.13) follows immediately:

\[
\begin{align*}
P \psi(x) &\mapsto (\partial f)^{-1/2} \left( a^2 \mathcal{P} - b^2 \bar{\mathcal{P}} - 2iab\mathcal{R} \right) \psi(x), \\
\bar{P} \psi(x) &\mapsto (\partial f)^{-1/2} \left( a^2 \bar{\mathcal{P}} - b^2 \mathcal{P} + 2iab\bar{\mathcal{R}} \right) \psi(x), \\
\mathcal{R} \psi(x) &\mapsto (\partial f)^{-1/2} \left( i\bar{a}b\mathcal{P} - iab\bar{\mathcal{P}} + (a\bar{a} + \bar{b}b)\mathcal{R} \right) \psi(x),
\end{align*}
\] (C.1)

with similar expressions for \( \bar{\psi} \) obtained by replacing \( \psi \to \bar{\psi} \) and \( \partial f \to \bar{\partial} \bar{f} \).

It is then a straightforward exercise to derive the transformation properties of the integrals of motion and hence their action on local fields at an arbitrary point. Consider the charges \( P, \bar{P}, \mathcal{R} \) defined in (2.18). We have

\[
\begin{align*}
P &\mapsto a^2 \mathcal{P} + b^2 \bar{\mathcal{P}} + 2iab\mathcal{R}, \\
\bar{P} &\mapsto b^2 \mathcal{P} + a^2 \bar{\mathcal{P}} + 2iab\bar{\mathcal{R}}, \\
\mathcal{R} &\mapsto a\bar{a}b\mathcal{P} + ab\bar{\mathcal{P}} + (a\bar{a} + \bar{b}b)\mathcal{R}.
\end{align*}
\] (C.2)
Starting with their action on a local field placed at the origin, we change coordinates with \( \bar{b} = az_0 \) and \( \bar{a} = a = (1 - z_0\bar{z}_0)^{-1/2} \) in (2.8) so that the origin is taken to \( (z_0, \bar{z}_0) \). Equations (C.2), (2.10) allow us to express everything at the origin. Replacing ordinary coordinate derivatives by the covariant ones (2.11) one is left with fields of definite \( SU(1,1) \)-dimension, so it is a simple matter to express everything in the frame \( (z, \bar{z}) \).

We illustrate this procedure with \( P \) acting on a local field of \( SU(1,1) \)-dimension \( (h, \bar{h}) \), and use a prime to denote quantities in the frame \( (z, \bar{z}) \):

\[
l[P, \mathcal{O}(0)] \mapsto [P', \mathcal{O}'(x)] = i a^{2(h+\bar{h}+1)} (\partial - \bar{z}^2 \bar{\partial} + 2(h - \bar{h})\bar{z}) \mathcal{O}(0)
= i \left( \mathcal{D}' - \bar{z}^2 \bar{\mathcal{D}'} + \frac{2(h - \bar{h})\bar{z}}{1 - \bar{z}\bar{z}} \right) \mathcal{O}'(x),
\]

in agreement with (2.20).

The transformation properties of the remaining charges are

\[
Y_1 \mapsto \bar{a}^b Y_1 + b Y_{-1} + 2\bar{a}b Y_0, \\
Y_{-1} \mapsto \bar{b}^a Y_1 + a Y_{-1} + 2ab Y_0, \\
Y_0 \mapsto \bar{a}b Y_1 + ab Y_{-1} + (a\bar{a} + b\bar{b}) Y_0,
\]

and

\[
Z_0 \mapsto Z_0, \\
Z_1 \mapsto \bar{a}^b b Z_2 - ab^2 Z_{-2} + a^2 (a\bar{a} + 3\bar{b}b) Z_1 - b^3 (3a\bar{a} + b\bar{b}) Z_{-1} + \bar{a}\bar{b}(a\bar{a} + b\bar{b}) M, \\
Z_2 \mapsto \bar{a}^b Z_2 - b^1 Z_{-2} + 4\bar{a}^3 b Z_1 - 4\bar{a}b^2 Z_{-1} + 2a^2 b^2 M, \\
M \mapsto 3a^2 b^2 Z_2 - 3a^2 b^2 Z_{-2} + 6(a\bar{a} + b\bar{b}) [\bar{a}b Z_1 - ab Z_{-1}] + (1 + 6a\bar{a}b\bar{b}) M,
\]

where we have introduced the combination

\[
M = (r^2 - 1/4) Z_0 + 3\mathbf{Z}_0.
\]

The transformation properties of \( Z_{-2} \) and \( Z_{-1} \) can be obtained from those of \( Z_2 \) and \( Z_1 \) by making the interchanges \( Z_2 \leftrightarrow Z_{-2}, Z_1 \leftrightarrow Z_{-1}, a \leftrightarrow \bar{a} \) and \( b \leftrightarrow \bar{b} \). Basically, \( \mathbf{Y}_{\pm 1} \) and \( Y_0 \) transform in the three-dimensional adjoint representation of \( SU(1,1) \) whereas \( Z_{\pm 2}, Z_{\pm 1} \) and \( \mathbf{M} \) transform in the symmetric part of the tensor product of two three-dimensional adjoint representation.

Using the procedure described above one can check that (3.13) follows from

\[
i[Z_0, \partial \bar{\partial} \sigma_a(0) \mu_b(0, 0)] = \partial \mu_a(0) \bar{\partial} \sigma_b(0) + \bar{\partial} \mu_a(0) \partial \sigma_b(0) - \partial \bar{\partial} \mu_a(0) \sigma_b(0)
+ (r^2 - 1/4) \mu_a(0) \sigma_b(0),
\]

whereas (3.14) follows from

\[
[Z_2, \sigma_a(0) \sigma_b(0)] = 2\partial \mu_a(0) \partial \sigma_b(0) - \partial^2 \mu_a(0) \mu_b(0) - \mu_a(0) \partial^2 \sigma_b(0), \\
[Z_2, \mu_a(0) \mu_b(0)] = 2\bar{\partial} \sigma_a(0) \bar{\partial} \sigma_b(0) - \bar{\partial}^2 \sigma_a(0) \sigma_b(0) - \sigma_a(0) \bar{\partial}^2 \sigma_b(0), \\
[Z_1, \sigma_a(0) \sigma_b(0)] = -\frac{1}{2} (\partial \mu_a(0) \mu_b(0) + \mu_a(0) \partial \mu_b(0)),
\]

\[
[Z_1, \mu_a(0) \mu_b(0)] = -\frac{1}{2} (\bar{\partial} \sigma_a(0) \sigma_b(0) + \sigma_a(0) \bar{\partial} \sigma_b(0)), \\
[Z_0, \sigma_a(0) \sigma_b(0)] = [Z_0, \mu_a(0) \mu_b(0)] = 0.
\]
Appendix D

We present here a few terms of the asymptotic behaviour of the functions $\chi(\eta)$ and $\varphi(\eta)$, related to the correlation functions of spin fields by (1.7). The appropriate solution to equations (4.7) has the short distance $\eta \to 0$ behaviour

$$\varphi(\eta) = -\ln(\eta^{1/2}) - \ln(-\Omega) + \eta f_1(\Omega) + \eta^2 f_2(\Omega) + O(\eta^3 \Omega^5),$$

$$\chi(\eta) = \frac{1}{2} \ln(8\eta^{1/2}) + \ln(-\Omega) + \eta h_1(\Omega) + \eta^2 h_2(\Omega) + O(\eta^3 \Omega^5),$$  \hspace{1cm} (D.1)

where $\eta$ is given by (2.44), $\Omega$ by (4.9),

$$f_1(\Omega) = -\frac{1}{4\Omega} (1 + \Omega),$$

$$f_2(\Omega) = \frac{1}{24\Omega^2} (4 - 13\Omega - 14\Omega^2 + r^2\Omega(-1 + 2\Omega + 8\Omega^2) + r^4\Omega(1 - 4\Omega + 8\Omega^2 - 8\Omega^3)), \hspace{1cm} (D.2)$$

and

$$h_1(\Omega) = \frac{1}{4\Omega} (1 + 2r^2\Omega),$$

$$h_2(\Omega) = \frac{1}{24\Omega^2} (-4 + 13\Omega - 2\Omega^2 + r^2\Omega(1 + 36\Omega + 8\Omega^2))$$

$$+ r^4\Omega(-1 - 2\Omega + 8\Omega^2 - 8\Omega^3). \hspace{1cm} (D.3)$$

This solution has the large distance $\eta \to 1$ expansion

$$\varphi(\eta) = Ax^{1/2+r} F_1(x) + A^3 x^{3/2+3r} F_3(x) + A^5 x^{5/2+5r} F_5(x) + O(x^{7/2+7r}),$$

$$\chi(\eta) = 4 \ln \bar{s} - A^2 x^{1+2r} H_2(x) - A^4 x^{2+4r} H_4(x) - A^6 x^{3+6r} H_6(x) + O(x^{4+8r}), \hspace{1cm} (D.4)$$

where we have introduced the notation $1 - \eta = x$, $A$ is given by (1.13), $\bar{s}$ by (1.14),

$$F_1(x) = F \left( \frac{1}{2} + r, \frac{1}{2} + r; 1 + 2r; x \right),$$

$$F_3(x) = \frac{1}{12} + \frac{(1 + 2r)^2 (3 + 2r)}{64(1 + r)^2} x + \frac{(1 + 2r)^2 (8 + 15r + 6r^2)}{256(1 + r)^2} x^2 + O(x^3), \hspace{1cm} (D.5)$$

$$F_5(x) = \frac{1}{80} + \frac{(1 + 2r)^2 (3 + 2r)}{256(1 + r)^2} x + O(x^2)$$

and

$$H_2(x) = \frac{1}{4} + \frac{(1 + 2r)^2 (3 + 2r)}{32(1 + r)^2} x + \frac{(1 + 2r)^2 (7 + 4r)}{128(1 + r)} x^2 + O(x^3),$$

$$H_4(x) = \frac{1}{32} + \frac{(1 + 2r)^2 (3 + 2r)}{128(1 + r)^2} x + O(x^2), \hspace{1cm} (D.6)$$

$$H_6(x) = \frac{1}{192} + \frac{(1 + 2r)^2 (3 + 2r)}{512(1 + r)^2} x + O(x^2).$$
Appendix E

The procedure presented in section 4 for obtaining differential equations for the spin–spin correlation functions can be easily extended to the Ising field theory on a sphere with zero magnetic field:

\[ A = A_{CFT,c=1/2} - \frac{m}{2\pi} \int_{\mathbb{R}^2} d^2x \, e^{\phi(x)} \varepsilon(x), \quad (E.1) \]

where \( A_{CFT,c=1/2} \) denotes the free Majorana fermion conformal field theory on the plane, the conformal factor reads

\[ e^{\phi(x)} = \frac{4R^2}{(1 + z\bar{z})^2} \quad (E.2) \]

and the energy operator is \( \varepsilon(x) = i(2R)^{-1}(1 + z\bar{z})(\psi\bar{\psi})(x) \). Isometries of the sphere correspond to the coordinate transformation

\[ z \mapsto z' = f(z) = \frac{az + b}{-b\bar{z} + a}, \quad \bar{z} \mapsto \bar{z'} = \bar{f}(\bar{z}) = \frac{\bar{a}\bar{z} + b}{-b\bar{z} + a}, \quad (E.3) \]

where we can choose \( a\bar{a} + b\bar{b} = 1 \). Introducing the anharmonic ratio

\[ \eta = \frac{(z - z')(\bar{z} - \bar{z}')}{{(1 + z\bar{z}')(1 + \bar{z'}z')}} = \tan^2 \left( \frac{d(x, x')}{2R} \right), \quad (E.4) \]

related to the geodesic distance \( d(x, x') \) between points \( x \) and \( x' \), the correlation functions \( G(\eta) = \langle \sigma(x)\sigma(x') \rangle, \tilde{G}(\eta) = \langle \mu(x)\mu(x') \rangle \) are then found to satisfy

\[ \eta(1 + \eta) \left( G'G' - G''G + \tilde{G}'\tilde{G}' - \tilde{G}''\tilde{G} \right) - (1 + 2\eta) \left( G'G + \tilde{G}'\tilde{G} \right) = 0, \]

\[ (1 + \eta) \left( G''G - G'G' - \tilde{G}''\tilde{G}' + \tilde{G}'\tilde{G} \right) - \left( G'G - \tilde{G}'\tilde{G} \right) = 0, \quad (E.5) \]

where \( r = mR \) and the prime denotes the derivative with respect to \( \eta \) in \( E.4 \).

Appropriate solutions to the quadratic differential equations \( E.5 \) can be obtained using analysis similar to that presented here for the pseudosphere.

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