Effective lower bounds for class numbers of CM fields and nonvanishing of class group L-functions

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Abstract. We show that the central value of class group L-functions of CM fields can be expressed in terms of derivatives of real-analytic Hilbert Eisenstein series at CM points. Then, following the idea of Iwaniec and Kowalski [27] we obtain a conditional explicit lower bound of class numbers of CM fields under a weaker assumption (3). This lower bound is then used to count CM fields with relative class number one. Moreover, our formulas provide an asymptotic estimate for the mean value of these twisted class group L-functions at s = 1/2, via which one obtains a better lower bound for nonvanishing class group L-functions and their derivatives at the central value. We also obtain some effective (both conditional and unconditional) nonvanishing results.

1. Introduction

1.1. A Lower Bound for the Class Number of CM Fields. For imaginary quadratic fields \( K = \mathbb{Q}(\sqrt{-D}) \), Gauss’ class number problem has for a long time inspired the study of lower bounds of \( h(-D) \), the class number of \( K \). Also, the magnitude of \( h(-D) \) is closely related to the exceptional characters, i.e., those characters \( \chi \) such that \( L(s, \chi) \) has a real zero near \( s = 1 \) (ref. [30], [13], [16] and [17]).

A repelling property of the exceptional zero gives the result \( h(-D) \to \infty \) as \( D \to \infty \) (ref. [9] and [19]). Landau then obtained the lower bound \( h(-D) \gg D^{1/8-\epsilon} \) (ref. [31]) by a quantitative analysis of the repelling effects. In the same year, Siegel got a stronger result: \( h(-D) \gg D^{1/2-\epsilon} \) (ref. [44]). See [24] for a more concrete historical introduction. However, all these results suffer from the serious defect of being ineffective. Hence one can not use them to determine the fields of class number one. Also, there are many other situations requiring an effective lower bound for \( h(-D) \), for example, by genus theory, the Euler idoneal number problem calls for an effective lower bound

\[
h(-D) \gg D^{c'/\log\log D}, \text{ with } c' > \log 2.
\]

Generally one hopes to show that \( h_K \geq CD_K^c \) for some positive (absolute) constant \( c \) and \( C \), where \( h_K \) is the class number and \( D_K \) is the absolute discriminant.

Unconditionally, Hoffstein (ref. [21]) showed that there exists some constant \( c_0 \in (0, 1/4) \) such that for all Galois CM-fields \( K \) of degree \( 2n > 4 \), (1) holds for \( c = c_0 \). The constant \( c_0 \) can be made explicitly, e.g. one can take \( c_0 = \frac{1}{4} - \frac{1}{2n} \) (ref. [35]). Also, for certain family of CM fields \( K \), \( h_K \) has effective lower bound in terms of the degree of \( K \), see [22] and [23] for details. For \( n = 1 \) the known unconditional lower bound is \( \prod_{p|D_K} (1 - 2\sqrt{\pi/(p + 1)})^{-1} \log D_K \) due to D. Goldfeld, B. Gross and D. Zagier (ref. [14] and [18]). On the other hand, if we assume the Grand Riemann Hypothesis or the existence of Siegel zeros, the exponent \( c \) can be taken to be \( 1/2 - \epsilon \) for any \( \epsilon > 0 \).
It is well known that the class group $L$-functions of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ can be expressed in terms of values of the real-analytic Eisenstein series for $SL_2(\mathbb{Z})$ at Heegner points (ref. [11]). Based on this fact, Iwaniec and Kowalski [27] obtained an effective lower bound for the class number $h_K \gg D^{1/4} \log D$ assuming that $L(1/2, \chi_{K/\mathbb{Q}}) \geq 0$, where $\chi_{K/\mathbb{Q}}$ is the quadratic character corresponding the extension $K/\mathbb{Q}$ (ref. Section 2). In this paper we generalize Iwaniec and Kowalski’s result to arbitrary CM fields and obtain an expression of class group $L$-functions in terms of derivatives of real-analytic Hilbert Eisenstein series at CM points. Due to the estimates on the Fourier expansions we show that for any CM field $K$, if $\zeta_K(1/2) \leq 0$, then $c = 1/4$ is admissible, and the implied constant is effective.

A precise statement is our Theorem 1. To achieve it, we shall introduce some analytic objects with respect to $F$. We let $\rho_F = \text{Res}_{s=1} \zeta_F(s)$ be the residue of the Dedekind zeta function $\zeta_F$ at $s = 1$, and $\gamma_F^*$ be the normalized Euler-Kronecker constant, i.e.,

$$\gamma_F^* = \lim_{s \to 1} \left( \frac{\rho_F^{-1} \zeta_F(s) - \frac{1}{s-1}}{s-1} \right).$$

For any CM extension $K/F$. We will always denote by $D_F$ (resp. $D_K$) the absolute discriminant of $F$ (resp. $K$). Let $h_F$ (resp. $h_K$) be the class number of $F$ (resp. $K$). Let $h_F^*$ be the relative class number of $F$, i.e. $h_F^* = #(\text{Cl}(F))/#(\text{Cl}(F)^+)$, where $\text{Cl}(F)^+$ is the narrow class group of $F$. Then we have

**Theorem 1.** Let $F/\mathbb{Q}$ be a totally real field of degree $n$. Let $K/F$ be a CM extension. Assume that

$$\zeta_K(1/2) \leq \frac{\rho_F}{4[\mathcal{O}_K^* : \mathcal{O}_F^*] \cdot h_F} \log \frac{\sqrt{D_K}}{D_F}. \leqno{(3)}$$

Then we have

$$h_K \geq \Psi_F^{-1} \cdot D_K^{1/4} \log \frac{\sqrt{D_K}}{D_F}, \leqno{(4)}$$

where

$$\Psi_F := \pi^{-n} \rho_F^{-1} h_F h_F^* D_F^2 + e^{2\gamma_F^* + 2n + 2} \Upsilon_F^+ D_F^{3/2},$$

and $\Upsilon_F^+ = 4\gamma_F^* + 3 \log D_F + 4n + \sqrt{n} + 8$.

**Remark.** Noting that $\zeta_K(s)$ is continuous on $(1/2, 1)$ and the fact that $\lim_{s \to 1^-} \zeta_K(s) = -\infty$, the Grand Riemann Hypothesis then gives $\zeta_K(1/2) \leq 0$, which is stronger than the assumption (3). Unconditionally, people only know that $\zeta_Q(1/2) < 0$ currently. Also, we can keep the normalized Euler-Kronecker constant $\gamma_F^*$ here as an invariant of $F$. One can refer Theorem 7 in [38] for an elementary upper bound for $\gamma_F^*$ and Theorem 1 in [26] for a conditional upper bound.

The outline of the proof to Theorem 1 is described in Section 2.2 below, with the new ingredients involved. See the rest sections for lengthy details.

Also, we have the following corollary to go beyond [27] by plugging the upper bound for $\gamma_F^*$ given in Lemma 29 into (4). It is of course weaker than the conditional result in [26] under GRH; however, it is simple enough compared to the elementary bound in [38].

**Corollary 2.** Let notation be as before, then there exist absolute constants $c_1$, $c_2 > 0$ such that if $\zeta_K(1/2)$ satisfies (3), then we have

$$h_K^- \geq c_1 D_F^{-c_2} \cdot D_K^4 \log D_{K/F}, \leqno{(5)}$$

where, $h_K^-$ is the relative class number of $K/F$ and $D_{K/F}$ the relative discriminant of $K/F$. 

Remark. According to the conditional estimate \( \gamma_F^* \ll \log \log D_F \) (ref. [26]), it is expected that one can take \( c_2 = 3 + \varepsilon \) in (5) for any \( \varepsilon > 0 \). Also, by Hermite’s theorem (ref. [20]) one sees that almost all \( h_K \) is a positive power of \( D_K/F \).

Remark. It can be seen that the inequality (4) can be naturally generalized to a conditional lower bound for \( h_D \), where \( \mathcal{O} \) is an order of \( K \) and \( h_\mathcal{O} \) denotes the number of proper \( \mathcal{O} \)–ideal classes of \( \mathcal{O} \), since everything in this paper has a counterpart in the order case.

According to a more precise formulation of Hermite’s theorem (ref. [40]), for any given \( D > 0 \), any totally real field \( F \), there are at most \( c'D \) quadratic extensions \( K/F \) such that \( D_K = D \), where \( c' \) is an absolute constant. The estimate (5) then shows that given a fixed totally real field \( F \), there are only finitely many CM extensions \( K/F \) such that \( h_K = 1 \). This is an analogue of the Class Number One problem of Gauss in CM extension case.

If one assumes the Generalized Riemann Hypothesis, the Gauss Class Number One Problem was solved for Galois CM-fields (ref. [33]). In the following, we give an answer to this problem for general CM-fields with a fixed common maximal totally real subfield without assuming GRH but under a weaker assumption.

Let \( F \) be a totally real field and \( C \) be a positive integer. Denote by
\[
\mathcal{S}_F(N) = \{ K : K/F \text{ is a CM extension such that } h_K \leq N \}.
\]
Then combining with the computation given in [7], Corollary 2 implies the following:

**Corollary 3.** Let the notations be as in Theorem 1. Then there exist absolute constants \( c \) and \( c_0 \) such that for any totally real field \( F \) and any \( N \in \mathbb{N}_{\geq 1} \), one has
\[
\# \{ K \in \mathcal{S}_F(N) : \text{ the assumption (3) holds for } K/F \} \leq c_0 N \cdot D_F^c.
\]

In particular, if we take \( N = 1 \) and assume (3) for any CM extension \( K \) over a fixed totally real field \( F \), then there are only finitely many such CM fields with relative class number 1.

1.2. Mean Values of Class Group \( L \)-functions. In this subsection, we compute a weighted mean value and the second moment of class group \( L \)-functions. The results here are parallel to those in [11] in general situations. For example, when \( F = \mathbb{Q} \), \( \chi \) is taken to be the trivial character of \( \tilde{\mathcal{O}}(K) \), (8) becomes Theorem 1 of [11], with a weaker error term \( \mathcal{E}(F, K) \ll D_K^{-1/4} \).

1.2.1. A Weighted Mean Value for \( L(\chi, 1/2) \). Let \( F/\mathbb{Q} \) be a fixed totally real field. For any CM extension \( K/F \), let \( \Phi = \Phi_K = (\sigma_1, \cdots, \sigma_n) \) be a CM type of \( K \). Then one can choose some \( \varepsilon_0 = \varphi_0^r \in K^\times \) such that \( \varepsilon_0 \tau = \tau \varepsilon_0 = -\varepsilon_0 \) and \( \Phi(\varepsilon_0) \in \mathbb{H}^n \), where \( \tau \) is the nontrivial element in \( \text{Gal}(K/F) \) and \( \mathbb{H}^n \) is a product of \( n \) copies of the upper half plane \( \mathbb{H} \). For convenience, we fix this CM type \( \Phi \) and the element \( \varepsilon_0 \) once and for all and denote by \( (K, \Phi, \varepsilon_0) \) this data. Then one sees, as in Section 3.1, that any fractional ideal \( a \) of \( K \) corresponds to a CM point \( z_a \in \mathbb{H}^n \). Denote by \( y_a \) the imaginary part of \( z_a \). Write \( N_\Phi(y_a) \) for the \( \Phi \) norm of \( y_a \). The precise definition of \( N_\Phi(y_a) \) is given in Lemma 19 and an explicit expression for this norm is given in (23), from which one sees that \( N_\Phi(y_a) \) depends only on the ideal class \( [a] \) of \( a \). For any \( (K, \Phi, \varepsilon_0) \) as above, set
\[
S(K, \Phi, \varepsilon_0) := \left\{ D' \in F^{\times,+} / (F^2 \cap F^{\times,+}) : K = F(\sqrt{-D'}) \right\}.
\]
Then clearly \( S(K, \Phi, \varepsilon_0) \) is nonempty. Moreover, \( S(K, \Phi, \varepsilon_0) \cap \mathcal{O}_F \) is nonempty. Take \( D \in S(K, \Phi, \varepsilon_0) \cap \mathcal{O}_F \) such that
\[
N_{F/\mathbb{Q}}(D) = \min_{D' \in S(K, \Phi, \varepsilon_0) \cap \mathcal{O}_F} N_{F/\mathbb{Q}}(D') \in \mathbb{N}_{\geq 1}.
\]
There may be several choices for such a \( D \), we take an arbitrary one and fix this choice for \( K/F \). Let \( \mathfrak{o}_{K/F} \) be the relative different of \( K/F \) and \( \mathfrak{q} \) be the index-ideal \( \left[ \mathcal{O}_K : \mathcal{O}_F[\sqrt{-D}] \right] \). It shows up naturally in the pseudo-basis of \( \mathcal{O}_K \) over \( \mathcal{O}_F \). Then clearly \( \mathfrak{q} \) is an integral ideal of \( F \) and

\[
\mathfrak{o}_{K/F} = -4D\mathfrak{q}^{-2}.
\]

According to (7), \( N_{F/Q}(\mathfrak{q}) \) only depends on the field extension \( K/F \). Throughout this paper, given any CM extension \( K/F \), we will always take \( D \) and \( \mathfrak{q} \) as above. Also, for simplicity, we shall always write \( \tilde{q} \) for \( q\mathcal{O}_K \).

**Theorem 4.** Let \( F/Q \) be a totally real field. For any CM extension \( K/F \) and data \((K, \Phi, \epsilon_0)\), we can write \( K = F(\sqrt{-D}) \) such that \( D \) satisfies (7). Let \( \mathfrak{q} \) be the index-ideal \( \left[ \mathcal{O}_K : \mathcal{O}_F[\sqrt{-D}] \right] \). Assume that \( a \) is any fractional ideal of \( K \), then one has

\[
\frac{1}{h_K} \sum_{\chi \in \hat{\mathcal{O}(K)}} \chi(a)L_K(\chi, 1/2) = N_{\tilde{q}}(a) \cdot \left\{ \log N\Phi(y_a) + \varepsilon_1(F, K; a) \right\},
\]

where

\[
N_{\tilde{q}}(a) = \frac{\rho_F \cdot N_{F/Q}(\mathfrak{q})}{[\mathcal{O}_K : \mathcal{O}_F] : h_F} \cdot \sqrt{\frac{N_{K/Q}(\epsilon_a)}{N_{K/Q}(\mathfrak{a})}},
\]

here \( \epsilon_a \) is given by (23), and \( f_a \) is determined by \( a \) according to (21). The error term \( \varepsilon_1(F, K; a) \) above satisfies that

\[
\varepsilon_1(F, K; a) \ll \log D_F + \frac{h_F^2 D_F^{1/4} N(f_a)}{\rho_F h_F} N\Phi(y_a)^{-3/2}.
\]

where the implied constant in is absolute.

Moreover, \( N_{F/Q}(\epsilon_a) = 1 \) if and only if \( a\tilde{q}^{-1} \) is primitive. Also, one can take \( f_a = \mathcal{O}_F \) if the narrow class group of the totally real field \( F \) is trivial.

**Remark.** Note that the right hand side of (8) does not depend on the choice of \( c_a \), \( f_a \) and \( \mathfrak{q} \), since \( N_{F/Q}(\epsilon_a) \), \( N_{F/Q}(f_a) \) and \( N_{F/Q}(\mathfrak{q}) \) only depend on the field extension \( K/F \) and the fractional ideal \( a \).

In fact by definition \( N_{\tilde{q}}(a) \) is independent of the choice of a particular representative of the class \([a]\), this is because, by definitions (21) and (23) the factors \( N_{F/Q}(f_a) \) and \( N_{K/Q}(\epsilon_a a^{-1}) \) are both invariant under scalar product by \( K^\times \). And if the narrow class group of \( F \) is trivial, then one can take \( N_{F/Q}(f_a) \equiv 1 \), which is independent of the ideal \( a \).

Moreover, if one assumes GRH, using the same method as in [4] it can be shown that

\[
|L_K(\chi, 1/2)| \leq c_1 \exp \left( \frac{c_2 \log D_K}{\log \log D_K} + \frac{c_2 \log D_K}{\log \log D_K} \right), \quad \forall \chi \in \hat{\mathcal{O}(K)},
\]

where \( c_1 \) and \( c_2 \) are absolute constants (e.g. we can take \( c_1 = 3 \)). Note that this conditional upper bound is just "slightly" better than the above asymptotic bound, so the cancellations among these \( L_K(\chi, 1/2) \)'s are not too significant.

One may find that the factor \( N_{F/Q}(\epsilon_a) \) in (9) is not easy to compute. However, one can get rid of this "annoying" term by choosing primitive representatives of \( \hat{\mathcal{O}(K)} \). More precisely, we have the following result.

**Corollary 5.** Let \( F/Q \) be a fixed totally real field. For any CM extension \( K/F \) and data \((K, \Phi, \epsilon_0)\), we can write \( K = F(\sqrt{-D}) \) such that \( D \) satisfies (7). Let \( \mathfrak{q} \) be
the index-ideal \( \mathcal{O}_K : \mathcal{O}_F[\sqrt{-D}] \). Then there is a set of integral ideals \( S \) consisting of representatives of \( \text{Cl}(K) \) such that as \( D_K \to \infty \) one has

\[
\frac{1}{h_K} \sum_{\chi \in \mathcal{C}(K)} \chi(a) L_K(\chi, 1/2) = (1 + o(1)) \mathcal{N}_q^*(a) \cdot \log \frac{\sqrt{D_K}}{D_F},
\]

for any \( a \in S \); here

\[
\mathcal{N}_q^*(a) = \frac{\rho_F \cdot N_{F/Q}(q)}{[\mathcal{O}_K^* : \mathcal{O}_F]^2} \cdot \frac{1}{h_F} \cdot \frac{1}{\sqrt{N_{K/Q}(a)}},
\]

and \( f_a \) is defined as in Theorem 4. In particular, \( q \in S \).

1.2.2. The Average of \( |L(\chi, 1/2)|^2 \). The following formula is a higher-degree analogue of Theorem 2 in [11].

**Theorem 6.** Let \( F/Q \) be a totally real field. For any CM extension \( K/F \) and data \( (K, \Phi, \epsilon_0) \), we can write \( K = F(\sqrt{-D}) \) such that \( D \) satisfies (7). Let \( q \) be the index-ideal \( \mathcal{O}_K : \mathcal{O}_F[\sqrt{-D}] \). Then one has

\[
\frac{1}{h_K} \sum_{\chi \in \mathcal{C}(K)} \left| L_K(\chi, 1/2) \right|^2 = \sum_{a \in \mathcal{C}(K)} \mathcal{N}_q(a)^2 \cdot \left\{ \left[ \log N_{F}(y_a) \right]^2 + \mathcal{E}_2(F, K; a) \right\},
\]

where \( \mathcal{N}_q(a) \) is given by (9) and \( \mathcal{E}_2(F, K; a) \) is the error term satisfying

\[
\mathcal{E}_2(F, K; a) \ll \rho_F^{-1} h_F^{-1} h_F D_F^{3/2} \log N_{F}(y_a) + \rho_F^{-2} h_F^2 D_F^3.
\]

where the implied constant is absolute, independent of \( F \).

Moreover, \( N_{F/Q}(e_a) = 1 \) if and only if \( a\bar{a}^{-1} \) is primitive. Also, one can take \( f_a = \mathcal{O}_F \) if the narrow class group of the totally real field \( F \) is trivial.

1.3. **Nonvanishing of Class Group L-functions.** It is an important problem in Number Theory to determine whether an automorphic L-function is nonvanishing at the central value \( s = 1/2 \). However, it is usually pretty difficult to check individually. A common strategy is to consider instead the average of L-functions over a family. One fruitful way for dealing with such averages combines periods relations of Waldspurger type with the equidistribution of special points on varieties (see e.g. [39], [36]). However, in this paper, we will use purely analytic methods to obtain asymptotic expressions for some averages of weighted class group L-functions. This way has several advantages. For instance, one can make the implied constant explicit. Moreover, we only use the first moment, which actually gives us a nontrivial unconditional explicit lower bound for nonvanishing class group L-functions of CM field \( K \) when the degree of \( K/Q \) is larger than 16. By (69) below one sees that the nonvanishing of derivatives of class group L-functions admits the same lower bound.

Although one can obtain the nonvanishing results on the critical line \( \text{Re}(s) = 1/2 \) by the same way, we just focus ourselves here on the central value \( s = 1/2 \). Precisely, we will use Theorem 4 to obtain a lower bound of the number of nonvanishing class group L-functions as follows. This bound, as can be seen, is more general and much sharper than that in [36]. Moreover, as can be seen in the proof, subconvexity bounds are not essential for us to get nontrivial ineffective nonvanishing results. Therefore, one might be able to handle higher derivative case by general convexic bounds for Dirichlet series via our methods here.

**Theorem 7.** Let \( F/Q \) be a totally real field and \( K/F \) is a CM extension. Denote by \( L_K^{(0)}(\chi, 1/2) = L_K(\chi, 1/2) \) and \( L_K^{(1)}(\chi, 1/2) = L_K'(\chi, 1/2) \), the derivative of \( L_K(\chi, s) \)
at \( s = 1/2 \). Then for any \( \varepsilon > 0 \) we have
\[
\# \left\{ \chi \in \text{Cl}(K) : L_k^{(k)}(\chi, 1/2) \neq 0 \right\} \gg_{F, \varepsilon} h_K D_K^{1 + \frac{1 - 2\theta}{4\varepsilon}}, \quad k = 0, 1,
\]
where \( \theta \) is any constant towards the Ramanujan-Petersson conjecture for cusp forms (not necessarily holomorphic) on \( \text{GL}_2(F) \), and the implied constant in (14) is computable.

Remark. One can take the above \( \theta = 7/64 \), which is the best known result (ref. [29]). When \( F = \mathbb{Q} \), i.e. \( K \) is imaginary quadratic, Blomer followed [11] to prove a better lower bound (ref. [2])
\[
\# \left\{ \chi \in \text{Cl}(K) : L_k^{(k)}(\chi, 1/2) \neq 0 \right\} \geq c \cdot h_K \prod_{p \mid D_K} \left( 1 - \frac{1}{p} \right)
\]
for some constant \( c > 0 \) and all sufficiently large \( D_K \). Once \( D_K \) is chosen, the constant \( c \) can be taken explicitly. However, one does not know how large \( D_K \) must be chosen for this lower bound to be valid because of an application of Siegel’s lower bound for \( L(1, \chi_{K/F}) \).

Corollary 8. Let \( F/\mathbb{Q} \) be a totally real field of degree \( n \) and \( K/F \) a CM extension with \( K/\mathbb{Q} \) Galois. Then for any \( \varepsilon > 0 \) we have
\[
\# \left\{ \chi \in \text{Cl}(K) : L_k^{(k)}(\chi, 1/2) \neq 0 \right\} \gg_{F, \varepsilon} D_K^{\frac{1 - 2\theta}{16\varepsilon}}, \quad k = 0, 1,
\]
where \( \theta \) is any constant towards the Ramanujan-Petersson conjecture and the implied constant is computable.

Remark. Note that (15) is unconditional. Since \( \theta = 7/64 \) is admissible, \( \frac{1 - 2\theta}{16\varepsilon} > 0 \) whenever \( n \geq 9 \). In this case we have nontrivial results.

By (14) and Theorem 1, if we assume (3), in particular, if we assume \( \zeta_K(1/2) \leq 0 \), then one obtains
\[
\# \left\{ \chi \in \text{Cl}(K) : L_k^{(k)}(\chi, 1/2) \neq 0 \right\} \gg_{F, \varepsilon} D_K^{1 + \frac{1 - 2\theta}{16\varepsilon}}, \quad k = 0, 1,
\]
where the implied constant is computable. Moreover, Siegel’s theorem gives that
\[
\# \left\{ \chi \in \text{Cl}(K) : L_k^{(k)}(\chi, 1/2) \neq 0 \right\} \gg_{F, \varepsilon} D_K^{1/4 + \frac{1 - 2\theta}{16\varepsilon}}, \quad k = 0, 1,
\]
where the implied constant is ineffective. Substituting \( \theta = 7/64 \) into (17) one then gets a lower bound \( D_K^{\frac{153}{512} - \varepsilon} \). It is clearly much better than that in [36], where the bound is \( D_K^{1/100 - \varepsilon} \).

1.4. An Explicit Chowla-Selberg Formula for General Abelian CM Fields.

In this section, we point out that our results will also provide a generalization of the Chowla-Selberg formula for CM fields as another application of class group \( L \)-functions via Proposition 24. In fact, Yoshida obtained such a formula for arbitrary CM fields in [47]. Here, due to the decomposition (25) and our expression for \( N_{\mathbb{K}/(\mathbb{Q}_a)} \) in (23), we are able to make Yoshida’s result more explicit. As a consequence, combining with Lerch’s identity (ref. [32]) and the results in [10], one obtains an explicit Chowla-Selberg formula for any given abelian CM fields in terms of (generalized) gamma functions, paralleling the original Chowla-Selberg formula, without the constraining assumption from [1]. Clearly, this is a direct generalization of Theorem 1.1 in [1].

Let \( K \) be an abelian CM field and \( F \) is its maximal totally real subfield. Then there is some \( N \in \mathbb{N}_{\geq 2} \) such that \( K \subset \mathbb{Q}(\zeta_N) \) with \( \zeta_N = e^{2\pi i/N} \). Let \( H_K \) (resp. \( H_F \)) be the subgroup of \( \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \) which fixes \( K \) (resp. \( F \)). Fix an isomorphism \( \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times \). Set \( X_K = \{ \chi \in (\mathbb{Z}/N\mathbb{Z})^\times : \chi |_{X_K} = 1 \} \). Let \( X_F \)
be defined similarly. For any \( \chi \in X_K \), it can be written as \( \chi = \chi^*\chi_0 \), where \( \chi^* \) is primitive and \( \chi_0 \) is trivial. Write \( c_\chi \) the conductor of \( \chi^* \). Since \( \chi^* \) is uniquely determined by \( \chi \), then \( c_\chi \) is well-defined. Define the Gauss sum associated with \( \chi \in X_F \) to be

\[
\tau(\chi) = \sum_{k=1}^{c_\chi} \chi(k)e^{2\pi ki/c_\chi}.
\]

Define the function \( \Gamma_2(w) = e^R(w) \), \( \text{Re}(w) > 0 \), where

\[
R(x) = \lim_{n \to \infty} \left( -\zeta''(0) + x \log^2 n - \log^2 x - \sum_{k=1}^{n-1} \left( \log^2(x+k) - \log^2 k \right) \right).
\]

This function \( \Gamma_2(w) \) is defined in [10] and it is analogous to \( \Gamma(s)/\sqrt{2\pi} \). Now we can state our version of the Chowla-Selberg formula for all abelian CM fields.

**Theorem 9.** Let \( F/\mathbb{Q} \) be a totally real field of degree \( n \) and \( K/F \) a CM extension with \( K/\mathbb{Q} \) abelian. Let \( \Phi \) be the fixed CM type for \( K \) as before. For any fractional ideal \( a \) of \( K \), denote by \( \mathfrak{f}_a \) a fixed integral ideal in the Steinitz class of \( \mathfrak{o}_a \) with minimal absolute norm, and write \( z_a \) for the corresponding CM points of type \((K, \Phi)\). Then one has

\[
\prod_{[a] \in Cl(K)} \left| H(z_a, \mathfrak{f}_a) \right| = C_F^K \prod_{\chi \in X_K \backslash X_F} \prod_{k=1}^{c_\chi} \prod_{\chi \neq 1} \Gamma \left( \frac{k}{c_\chi} \right)^{h_K \chi(k)/2} \prod_{\chi \neq 1} \left( \frac{k}{c_\chi} \right)^{\frac{h_K \tau(\chi)\mathfrak{R}(\chi^*)}{2\pi k} \mathfrak{C}(\chi^*)},
\]

where for any fractional ideal \( b \) of \( F \), \( H(z; b) = \left[ N_F(\mathfrak{f}(z))/N_F(b)^{-1} \right]^{1/h_F} \), and

\[
\log \phi(z; b) = -\frac{2\pi^n D_F^{1/2}y^s}{\rho_F N_F(b)} \cdot \zeta_F(2, [\mathcal{O}_K]) - \frac{2\rho_F^{-1}}{D_F N_F(b)} \sum_{b \in \mathcal{O}_F^*} |N(b)|^{-1} \lambda(b, 0)e(bz),
\]

and \( \lambda(b, s) \) is defined in (29) with the weight \( k = 0 \) and \( C := \mathcal{O}_F^*/\mathcal{O}_F^{\times} \). Also, the constant \( C_F^K = \left( 2^{-n-1} \pi^{-1} D_K^{-1/2} D_F \right)^{h_K/2} \).

**Remark.** It is clear that by definition \( H(z_a, \mathfrak{f}_a) \) is independent of the choice of \( \mathfrak{f}_a \).

If \( h_F = 1 \), we may take \( \mathfrak{f}_a = \mathcal{O}_F \) to recover Theorem 1.1 in [1]. Combining with Colmez’s theorem (ref. [8]), the formula above can be used to compute the average of Faltings heights of certain CM abelian varieties.

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2. Proof of Theorem 1 in Imaginary Quadratic Case

2.1. Review of the Imaginary Quadratic Case. We start with reviewing the case of Theorem 1 in the imaginary quadratic case. This is Iwaniec-Kowalski’s original idea. For the sake of illustration, we give a brief proof following [27].

Let \( K = \mathbb{Q}(\sqrt{-D}) \) be an imaginary quadratic field. Since \( \mathbb{Q} \) has class number 1, we can often factor a non-zero integral ideal uniquely as \((l)a \) where \( l \in \mathbb{Z}_{\geq 0} \) and \( a \) is a primitive ideal, i.e., \( a \) has no rational integer factors other than \( \pm 1 \).

If \( a \) is primitive, then it is generated by \( a = \left[ a, b+\sqrt{-D}/2 \right] \), where \( a = Na \) and \( b \) solves the congruence \( b^2 + D \equiv 0 \pmod{4a} \), and is determined modulo \( 2a \).
Conversely, given such \( a \) and \( b \) we get a primitive ideal \( a = \left[a, \frac{b + i\sqrt{D}}{2a}\right] \). Thus there exists a one-to-one correspondence between the primitive ideals and the points

\[
z_a := \frac{b + i\sqrt{D}}{2a} \in \mathcal{H}
\]
determined by modulo 1.

These will be called the Heegner points. Moreover, we have \( a^{-1} = [1, \bar{z}_a] \). Then according to [11] one has the following formula, which is a special case of (39).

**Lemma 10.** Let \( \mathcal{O}_K \) be the ring of integers of \( K \), \( h = h_K \) be the class number and \( w \) be the root of units of \( K \). Then we have

\[
\frac{1}{h} \sum_{\chi \in Cl(K)} \chi(a)L_K(s, \chi) = w^{-1} \left(\frac{\sqrt{D}}{2}\right)^{-s} \zeta(2s)E(z_a, s),
\]

where \( a \) is any primitive ideal, \( z_a \) is the Heegner point, and \( E(z, s) \) is the real analytic Eisenstein series of weight 0 for the modular group.

**Proof.** One has

\[
\zeta(2s)E(z, s) = y^s \sum_{(m, n) \neq (0, 0)} |m + nz|^{-2s}.
\]

On the other hand, By \( a^{-1} = [1, \bar{z}_a] \) we have

\[
\sum_{b \sim a} (Nb)^{-s} = w^{-1} \left(NA\right)^{-s} \sum_{0 \neq a \in \mathcal{A}^{-1}} |a|^{-2s} = w^{-1} a^{-s} \sum_{(m, n) \neq (0, 0)} |m + nz|^{-2s}.
\]

Thus summing over characters gives (1.1). \( \square \)

**Lemma 11** (Fourier Expansion of Eisenstein series). [25] The Eisenstein series at the Heegner point \( z_a \) admits the Fourier expansion:

\[
\Theta(s)E(z, s) = \Theta(s)y^s + \Theta(1-s)y^{1-s} + 4y^{\frac{s}{2}} \sum_{k=1}^{\infty} \sum_{m=n=k} \left(\frac{m}{n}\right)^{it} K_{it}(2\pi ky) \cos(2\pi kx),
\]

where \( \Theta(s) := \pi^{-s} \Gamma(s) \zeta(2s) \).

Apply Fourier inversion we get from (1.1) that

\[
L_K(s, \chi) = w^{-1} \left(\frac{\sqrt{D}}{2}\right)^{-s} \zeta(2s) \sum_{z_a \in \Lambda_D} \chi(a)E(z_a, s),
\]

where \( \Lambda_D := \{z_a \in \mathcal{F} : a \text{ primitive}\} \), and \( \mathcal{F} \subset \mathcal{H} \) is the fundamental domain for \( SL_2(\mathbb{Z}) \).

Clearly from the Fourier expansion in Lemma 11 we have \( E(z, 1/2) \equiv 0 \), since \( \zeta(2s) \sim \frac{1}{2\pi s} \) as \( s \to \frac{1}{2} \) and the right hand side is well defined.

Thus take the derivative of (1.2) at \( s = \frac{1}{2} \) and note \( K_0(y) \ll y^{-\frac{3}{2}} e^{-y} \) we have

\[
L_K\left(\frac{1}{2}, \chi\right) = \frac{\sqrt{2}}{w} |D|^{-\frac{3}{4}} \sum_{a} \frac{\hat{\chi}(a)E'(z_a, 1/2)}{a}
\]

\[
= \frac{1}{w} \sum_{a} \frac{\hat{\chi}(a)}{\sqrt{a}} \left\{ \log \frac{\sqrt{D}}{2a} + 4 \sum_{n=1}^{\infty} \pi(n)K_0\left(\frac{\pi n \sqrt{D}}{a}\right) \cos\left(\frac{\pi nb}{a}\right) \right\}
\]

\[
= \frac{1}{2} \sum_{a} \frac{\hat{\chi}(a) \log \frac{\sqrt{D}}{2a}}{a} + O\left(h(D)|D|^{-\frac{3}{4}}\right),
\]

Since \( \# \Lambda_D = h(D) \).
Assume $L(1/2, \chi_D) \geq 0$, i.e. $L_K(1/2, \chi_0) = \zeta(1/2) L(1/2, \chi_D) \leq 0$, we will derive

$$h(D) |D|^{-\frac{3}{4}} \gg \sum_a \frac{1}{\sqrt{a}} \log \frac{|D|}{2a} = \sum_a \frac{1}{\sqrt{a}} \log \frac{|D|}{a} + O \left( h(D) |D|^{-\frac{3}{4}} \right).$$

Thus we have $h(D) |D|^{-\frac{3}{4}} \gg \sum_a \frac{1}{\sqrt{a}} \log \frac{|D|}{a} \gg \log |D|$, which implies

**Theorem 12 (Iwaniec-Kowalski), [27]** Let notation be as before. Assume $L(1/2, \chi_D) \geq 0$, then we have

$$h(D) \gg |D|^{1/4} \log |D|.$$  \hspace{1cm} (1.3)

**Remark.** The implied constant in (1.3) is absolute. Actually, estimate everything explicitly we can get an explicit lower bound:

**Theorem 13 ([12]).** Let notation be as before. Assume $L(1/2, \chi_D) \geq 0$, then for any $\varepsilon \in (0, \frac{1}{4})$ we have

$$h(D) \geq 0.1265\varepsilon |D|^{1/4} \log |D|, \quad \forall D \geq 200 \frac{\varepsilon}{\log D}.$$  

### 2.2. Sketch of Proof of Theorem 1 in General CM Case.

To generalize Iwaniec-Kowalski’s results to CM fields case, one has to establish a general form of the Eisenstein period (1.1) (see below). The analogue of (1.1) in CM case is easy to build if $F$, the totally real subfield, has trivial narrow class group (e.g. ref. [36]). In general, one needs to compute CM points associated to each Steinitz class. However, the situation is quite different from the imaginary quadratic case since generally, integral representatives in $Cl(K)$, where generally, integral representatives in $Cl(F)$ bounded by Minkowski bounds may not necessarily be primitive. In addition, the CM type norm of the imaginary parts of CM points should be computed explicitly in order to compute the constant terms of Fourier coefficients of Eisenstein series. These problems are solved by the crucial Lemma 19 in Section 3.2 below. Roughly speaking, fix a CM type $\Phi$ on $K$, then a fractional ideal $a \subset K$ corresponds to a CM point $z_a$ in a Hilbert modular variety (ref. Section 3); we calculate the CM type norm $N_\Phi(y_a)$ explicitly, where $y_a$ the imaginary part of $z_a$. Also, unlike the cases in [27] or [36], generally a single Hilbert Eisenstein series does not has a functional equation, but it turns out that it does vanish at the central point $s = 1/2$. So we get an expression for $L_K(1/2, \chi)$ in terms of derivatives of Eisenstein series similar to (1.1) in section 3.

Then following Iwaniec-Kowalski’s idea, an explicit lower bound for $L_K(1/2, \chi_0)$ is given in Proposition 27, Section 4.1. Furthermore, several effective estimates such as bounds for $L_K(1, \chi)$ and the normalized Euler-Kronecker constant $\gamma_K$ are established in Section 4. With all these preparations, we eventually complete the proof of Theorem 1 in Section 4.2.

### 3. Generalization to CM case

#### 3.1. Hilbert Modular Varieties and CM Zero-Cycles.

**3.1.1. The Basic Correspondence.** Let $F/\mathbb{Q}$ be a totally real extension of degree $n$. For any $S \subset F$, let $S^+$ be the subset of $S$ consisting of totally positive elements. Given a fractional ideal $\mathfrak{f} \subset F$, define

$$\Gamma(\mathfrak{f}) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, F) : \quad a, d \in \mathcal{O}_F, \quad b \in \mathfrak{f}, \quad c \in \mathfrak{f}^{-1} \right\}.$$  

Let $\mathbb{H}$ be the upper half plane. Then $\Gamma(\mathfrak{f})$ acts on $\mathbb{H}^n$ via

$$\gamma \cdot z = (\sigma_1(\gamma) z_1, \cdots, \sigma_n(\gamma) z_n), \quad \forall z = (z_1, \cdots, z_n) \in \mathbb{H}^n.$$
Recall that the quotient
\[ X(f) := \Gamma(f) \backslash \mathbb{H}^n \]
is the open Hilbert modular variety associated to \( f \). It’s known ([15], Theorem 2.17) that \( X(f) \) parameterizes isomorphism classes of triples \((A, i, m)\) where \((A, i)\) is an abelian variety with real multiplication and \(K\) is an \( \mathbb{O}_F \)-isomorphism \( \mathfrak{M}_A \cong (\sigma_F)^{-1} \) which maps \( \mathfrak{M}_A^+ \) to \((\sigma_F)^{-1} \).

For any fractional ideal \( \mathfrak{a} \) of \( \mathbb{O}_F \), let \( \mathfrak{a}^\mathfrak{a} = \mathfrak{a} \), and let \( \Phi(\mathfrak{a}) \in \mathbb{H}^n \). Let \( \Phi(\mathfrak{a}) \) be a CM extension and let \( \Phi = (\sigma_1, \cdots, \sigma_n) \) be a CM type of \( K \). Then \( z = (\mathfrak{a}, i, m) \in X(f) \) is a CM point of type \((K, \Phi)\).

To relate CM points with ideals of \( K \), recall that we have fixed \( \varepsilon_0 \in K^\times \) such that \( \varepsilon_0 = -\varepsilon_0 \) and \( \Phi(\varepsilon_0) \in \mathbb{H}^n \). Let \( \mathfrak{a} \) be a fractional ideal of \( K \) and \( f^0 := \varepsilon_0 \mathfrak{a}_K/F \mathfrak{a} \cap F \).

Then by Lemma 3.1 in [3], the ideal class of \( f^0 \) is the Steinitz class of \( \mathfrak{a} \subset K \) as a projective \( \mathbb{O}_F \)-module. Then it is clear that the CM abelian variety \((A_{\mathfrak{a}} = \mathbb{C}^n/\Phi(\mathfrak{a}), i)\) has the polarization module
\[ (\mathfrak{M}_A, \mathfrak{M}_A^+ \mathfrak{a}) \rightarrow (\mathfrak{a} f^0, (\mathfrak{a} f^0)^{-1} + 1). \]

To give an \( \mathbb{O}_F \)-isomorphism between the above pair and \((\mathfrak{a} f^0, (\mathfrak{a} f^0)^{-1} + 1)\) amounts to giving some \( r \in F^+ \) such that \( f^0 = rf \). Therefore, to give a CM point \((A, i, m) \in X(f) \) is the same as to give a pair \((\mathfrak{a}, r)\), where \( \mathfrak{a} \) is a fractional ideal of \( K \) and \( f^0 = rf \) for some \( r \in F^+ \). Two such pairs \((\mathfrak{a}_1, r_1)\) and \((\mathfrak{a}_2, r_2)\) are equivalent if there exists an \( \gamma \in K^\times \) such that \( \mathfrak{a}_2 = \gamma \mathfrak{a}_1 \) and \( r_2 = r_1 \gamma \). We write \([\mathfrak{a}, r]\) for the class of \((\mathfrak{a}, r)\) and identify it with its associated CM point \((A_{\mathfrak{a}}, i, m) \in X(f)\). Note that for any fractional ideal \( \mathfrak{f} \subset K \) and any \( r \in F^+ \) we have the natural isomorphism of varieties:
\[
\tau : X(f) \longrightarrow X(\mathfrak{f}) \\
z = (z) \longmapsto rz = (\sigma_i(r) z).
\]

3.1.2. Upper Bounds of Minkowski Type and Steinitz Class. Let \( L/\mathbb{Q} \) be a number field of signature \((r_1, r_2)\). Let \( n \) be the degree of \( L/\mathbb{Q} \), then \( n = r_1 + 2r_2 \). Minkowski showed that there is a constant \( M(r_1, r_2) \) only depending on the signature such that for any \( \mathcal{C} \in \text{Cl}(L) \), there exists an integral ideal \( \mathfrak{a}_\mathcal{C} \subset \mathfrak{C} \) satisfying \( N_{L/\mathbb{Q}}(\mathfrak{a}_\mathcal{C}) \leq M(r_1, r_2) \sqrt{D_L} \), where \( D_L \) is the absolute discriminant of \( L \).

We can make the corollary in [49] (Ref. P374) more explicit. In fact, an elementary estimate gives that
\[
\log \frac{D_L}{N_{L/\mathbb{Q}}(\mathfrak{a}_\mathcal{C})} \geq (2 \log 2 + \gamma)r_1 + (\log 2 + 2\gamma)r_2 - \sqrt{m}.
\]
Then for \( n \geq 6 \), the right hand side is always positive since \( \sqrt{n} \leq \sqrt{r_1} + \sqrt{2r_2} \). Let \( M(n) := \frac{r_1^n}{r_2^n} \). Then \( M(n) \cdot \sqrt{D_L} \) gives the Minkowski constant of \( L/\mathbb{Q} \). Combining \( M(n) \) with (18) one can take

\[
M(r_1, r_2) := \min \{e^{-(2\log 2 + \gamma)r_1 - (\log 2 + 2\gamma)r_2 + \sqrt{\pi n}, n \geq 7 + M(6) \cdot 1_{n \leq 6}, M(n)} \}.
\]

In particular, when \( n \) is large, we have \( M(r_1, r_2) \leq 50.7^{-r_1/2} \cdot 19.9^{-r_2} \). From now on, we shall fix \( M(r_1, r_2) \) in (19). For totally real extension \( F/\mathbb{Q} \), each ideal class \( C \in Cl(F) \) contains an integral ideal \( f_C \) satisfying \( N_{F/\mathbb{Q}}(f_C) \leq M(n, 0) \cdot \sqrt{D_F} \).

Now we fix a set of fractional ideals

\[
(\mathcal{I}_F) := \{ f : f \in \mathcal{O}_F \text{ and } N(f) \leq M(n, 0) \cdot \sqrt{D_F} \},
\]

such that

\[
Cl(F)^+ = \{ [f] : f \in \mathcal{I}_F^+ \},
\]

where \( Cl(F)^+ \) denotes the narrow ideal class group of \( F \). For simplicity, we assume \( \mathcal{O}_F \in \mathcal{I}_F^+ \). Then for any fractional ideal \( a \subset K \), there exists a unique \( f \in \mathcal{I}_F^+ \) such that

\[
f^o := \epsilon_o a_{K/F}(a) \cap F \in [f],
\]

i.e., we can find some \( r \in F^+ \) such that \( f^o = r f \). Then by the above discussion, \( [a, r] \) gives a CM point in \( X(f) \). Actually we can construct the CM point more explicitly. To achieve, let’s recall the standard result:

**Proposition 14** ([47], Proposition 2.1. P179). Let \( F \) be an arbitrary algebraic number field and \( K/F \) be an algebraic extension of degree \( n \). Let \( a \subset K \) be a fractional ideal. Then there exist \( \alpha_1, \ldots, \alpha_n \in F \) and a fractional ideal \( f \subset F \) such that

\[
a = \mathcal{O}_F \alpha_1 \oplus \cdots \oplus \mathcal{O}_F \alpha_{n-1} \oplus f \alpha_n.
\]

Moreover, we have

\[
\mathcal{O}_K = \mathcal{O}_F \alpha_1 \oplus \cdots \oplus \mathcal{O}_F \alpha_{n-1} \oplus c \alpha_n;
\]

where \( c \) is a fractional ideal of \( F \), independent of \( a \), such that

\[
[c^2] = [D_{K/F}], \quad \text{where } D_{K/F} := N_{K/F}(a_{K/F}).
\]

**Proof.** The first part of the assertion comes from the structure theorem for a finitely generated torsion free module over a Dedekind domain. Then we can write

\[
\mathcal{O}_K = \mathcal{O}_F \alpha_1 \oplus \cdots \oplus \mathcal{O}_F \alpha_{n-1} \oplus c \alpha_n;
\]

where \( \{\alpha_1, \ldots, \alpha_n\} \) and \( \{\beta_1, \ldots, \beta_n\} \) are basis of \( K \) over \( F \) and \( f \) and \( b \) are fractional ideals of \( F \). Then there exists some \( \gamma \in GL(n, F) \) such that \( \gamma \alpha_i = \beta_i, 1 \leq i \leq n \). Take \( x \in K^F \) such that \( \text{div}(x) = c^{-1} f \) and set \( y = \text{diag}[x, 1, \ldots, 1] \). Then clearly \( y \gamma \mathcal{O}_K = a \). On the other hand we have \( a = a_{K/F} \), where \( a \in K^F \) such that \( \text{div}(a) = a \). Therefore we have

\[
a^{-1} y \gamma \mathcal{O}_K = \mathcal{O}_K,
\]

which gives that \( N_{K/F}(a^{-1} c^{-1} f) = \det(\gamma)^{-1} N_{K/F}(a^{-1} y \gamma) \in \mathcal{O}_F^\times \), i.e.,

\[
[N_{K/F}(a)] = [N_{K/F}(c^{-1} f)] \in Cl(F).
\]

Therefore the last assertion is reduced to the case \( a = \mathcal{O}_K \).

Let \( \{\alpha'_1, \ldots, \alpha'_n\} \) be the dual basis of \( K/F \) with respect to the relative trace \( T_{K/F} \). Then we have

\[
\alpha'^{-1}_n = \mathcal{O}_F \alpha'_1 \oplus \cdots \oplus \mathcal{O}_F \alpha'_{n-1} \oplus c^{-1} \alpha'_n.
\]
where $\mathfrak{d}_{K/F}$ is the relative different with respect to $K/F$. Then by the above discussion (i.e. taking $a = \mathfrak{d}_{K/F}^{-1}$) we have

$$[N_{K/F}(\mathfrak{d}_{K/F}^{-1})] = [e^{-1} \cdot c^{-1}] \in \text{Cl}(K).$$

Hence we have $[D_{K/F}] = [N_{K/F}(\mathfrak{d}_{K/F}^{-1})] = [e^2] \in \text{Cl}(K)$. □

Let $a$ be any fractional ideal of $K$, let $[\mathfrak{z}]$ be the Steinitz class of $a$ as before. Denote by $St_a$ the set of integral ideals in $[\mathfrak{z}]$ of $a$ that are of the minimal absolute norms, namely,

$$St_a = \{ f_a : N_{F/Q}(f_a) = \min_{f \in [\mathfrak{z}]} N_{F/Q}(f) \}.$$

Given a fractional ideal $a \subset K$, taking an $f_a \in St_a$. Without loss of generality, we may assume that $f_a \in \mathcal{O}_K$. We then fix this choice for any fractional ideal $a \in K$ once and for all. Then by Proposition 37 there is a decomposition

(21)

$$a = \mathcal{O}_F \alpha \oplus f_a \beta.$$

By the above proposition and the definition of $[\mathfrak{z}]$ we can take a appropriate $\beta$ such that there exists some $r \in F^+$ such that $[\mathfrak{z}] = rf_a$.

Define $z_a := \frac{\alpha}{\beta}$. Then we have as in the proof of Lemma 3.2 of [3] that

$$(\overline{\alpha \beta} - \alpha \overline{\beta})f_{K} = \mathfrak{d}_{K/F} \mathfrak{z}.$$

Then we have

$$0 = (\overline{\alpha \beta} - \alpha \overline{\beta}) = re \quad \text{for some } e \in \mathcal{O}_F^\times.$$

Replacing $\beta$ by $\beta^{-1}$ if necessary, we can assume $e = 1$. This implies that

$$\varepsilon_0(z - \overline{z}) = \frac{e}{\beta \overline{\beta}} \in F^\times,$$

and thus $z_a \in K^\times \cap \mathbb{H} = \{ z \in K^\times : \Phi(z) \in \mathbb{H} \}$. Moreover, $z$ represents the CM point $[a, r] \in X(f_a)$.

Let $CM(K, \Phi, \mathfrak{f})$ be the set of CM points $[a, r] \in X(\mathfrak{f})$ which we regard as a CM 0-cycle in $X(\mathfrak{f})$. Let

$$CM(K, \Phi) := \sum_{[\mathfrak{f}] \in \mathcal{O}(F)^+} CM(K, \Phi, \mathfrak{f}).$$

We have the natural surjective map

$$CM(K, \Phi) \rightarrow \text{Cl}(K),$$

$$[a, r] \mapsto [a].$$

The fiber is indexed by $e \in \mathcal{O}_F^{x,+}/N_{K/F}\mathcal{O}_K^{x,+}$, since every element in the fiber of $a$ is of the form $[a, r\epsilon]$ with $r$ fixed and $\epsilon \in \mathcal{O}_F^{x,+}$ a totally positive unit. Note that $\sharp (\mathcal{O}_F^{x,+}/N_{K/F}\mathcal{O}_K^{x,+}) \leq 2$.

3.2. Representation of Ideals. Let $z_a$ be the CM point corresponding to the fractional ideal $a$. Write $z_a = x_a + y_a$. To prove our main results, we need to compute $y_a$ explicitly. We start with recalling some definition.

Definition 15 (Primitive Ideals). Let $a$ be a fractional ideal of $\mathcal{O}_K$. We say that $a$ is primitive if $a$ is an integral ideal of $\mathcal{O}_K$ and if for any nontrivial integral ideal $\mathfrak{n}$ of $\mathcal{O}_F$, $n^{-1}a$ is not an integral ideal.

Fact 16. For any fractional ideal $a$ of $\mathcal{O}_K$, there exists a unique fractional ideal $\mathfrak{n}$ of $F$ such that $n^{-1}a$ is a primitive ideal. The ideal $\mathfrak{n}$ will be called the content of the ideal $a$. 

Since $K/F$ is a CM extension, there exists some $D \in F^{x,+}/(F^2 \cap F^{x,+})$ such that $K = F(\sqrt{-D})$. We may assume $D \in \mathcal{O}_F$ and fix this choice once for all. Then combining results in Section 2.6 of [6] we have

**Proposition 17.** Let $a$ be a fractional ideal of $K$. There exist unique ideals $n$ and $m$ and an element $b \in \mathcal{O}_F$ such that

$$a = n \left( m \oplus q^{-1}(-b + \sqrt{-D}) \right),$$

where $q$ is the index-ideal $[\mathcal{O}_K : \mathcal{O}_F[\sqrt{-D}]]$. In addition, we have the following:

1. $n$ is the content of $a$.
2. $a$ is an integral ideal of $\mathcal{O}_K$ if and only if $n$ is an integral ideal of $\mathcal{O}_F$.
3. $a$ is primitive in $K/F$ if and only if $n = \mathcal{O}_F$.
4. $m$ is an integral ideal and $d \mathfrak{a} = mn^2$.

**Remark.** $b$ is determined by the modulo relation

$$\begin{cases} \delta - b \in q, \\ b^2 + D \in mq^2, \end{cases}$$

where $\delta \in \mathcal{O}_F$ comes from the corresponding pseudo-matrix on the basis $(1, \sqrt{-D})$ (ref. [6], Corollary 2.2.9).

The equations (21) and (22) give us two decompositions of a fractional ideal $a$ of $K$. However, the main obstacle comes from the factor $n$ in (22). We may not easily get rid of $n$ unless the ideal class group $\text{Cl}(F)$ is trivial. Noting that $n$ is a content, one natural way is to use the decompositions to construct a group of primitive representatives of the ideal class group $\text{Cl}(K)$ such that the CM norms of the imaginary part of the corresponding CM points can be computed explicitly. In fact, It can be seen from the definition that an integral ideal $a$ of $\mathcal{O}_K$ is primitive if and only if its primary decomposition is of the following form:

$$a = \prod_j \mathfrak{P}_j^{\alpha_j} \prod_i \mathfrak{P}_i^{\beta_i},$$

where $\mathfrak{P}_j$ are ramified primes and $\mathfrak{P}_i$ are splitting primes with $\alpha_i \cdot \beta_i = 0$. In particular, every split prime ideal of $\mathcal{O}_K$ is primitive. On the other hand, by Cebotarev density theorem, there exist a group of representatives of $\text{Cl}(K)$ consisting of split prime ideals. This gives us a set of primitive representatives of $\text{Cl}(K)$. However, since we have to bound these representatives uniformly (as can be seen in the last section) and it is not easy to give such a bound for splitting ideals in each ideal class, we move on in another way.

It’s well known that, for any fractional $\mathcal{O}_F$–ideals $a$ and $b$, we have the isomorphism $a \oplus b \simeq \mathcal{O}_F \oplus ab$. But this is not enough, to make (22) into the form of (21), we need to make the isomorphism into an identity.

**Lemma 18.** Suppose $K/F$ is a finite extension of number fields. Let $a$ and $b$ be fractional ideals of $\mathcal{O}_F$. Let $\alpha$, $\beta$ be two elements in $K^\times$. Assume that $a \in a$, $b \in b$, $c \in b^{-1}$ and $d \in a^{-1}$ such that $ad - bc = 1 \in F$. Set

$$(\alpha', \beta') := (\alpha, \beta) \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

then we have

$$a \alpha + b \beta = \mathcal{O}_F a' + a b \beta'.$$

**Proof.** We have $a' = a \alpha + b \beta$ and $\beta' = c \alpha + d \beta$. Hence

$$\mathcal{O}_F a' + a b \beta' \subset (\mathcal{O}_F \cdot a + a b \cdot c) \alpha + (\mathcal{O}_F \cdot b a b \cdot d) \beta \subset a \alpha + b \beta.$$
Conversely, we have $\alpha = d \alpha' - b \beta'$ and $\beta = -c \alpha' + a \beta'$. Hence
\[ a \alpha + b \beta \subset O_F \alpha' + ab \beta'. \]

\[ \square \]

Then we can compute, for any fractional ideal $a$, the associated CM point $z_a$ and see that the CM type norms will be bounded from below by that of primitive ones. This will be essentially used in the following sections. Precisely, we have the following:

**Lemma 19.** Let $K = F(\sqrt{-D})$ be a CM extension of a totally real field $F$. Fix the CM type $\Phi$ as before. Let $a$ be a fractional ideal in $K$. Denote by $z_a$ the CM point corresponding to $a$. Set $y_a$ to be the imaginary part of $z_a$. Define the CM type norm of $y_a$ as $N_\Phi(y_a) := \prod_{\sigma \in \Phi} \sigma(y_a)$. Then we have
\[ N_\Phi(y_a) = \frac{N_{K/Q}(c_a)N_{F/Q}(f_a)N_{F/Q}(q)^2}{2^n N_{K/Q}(a)} \cdot \frac{\sqrt{D_K}}{D_F}, \]
where $c_a$ is an element in the content of $a\tilde{q}^{-1}$ that is of the minimal absolute norm, and $D_K$ (resp. $D_F$) is the absolute discriminant of $K/Q$ (resp. $F/Q$).

**Remark.** Clearly the term $N_{K/Q}(c_a)$ in the right hand side of (23) does not depend on a particular choice of $c_a$. In fact (23) shows that $N_\Phi(y_a)$ is independent of the choice of a particular representative of the class $[a]$, this is because the factors $N_{F/Q}(f_a)$ and $N_{K/Q}(c_a^{-1})$ are both invariant under scalar by $K^\times$.

**Proof.** By the argument in the above remark, we may assume $a$ is an integral ideal of $O_K$ such that $a\tilde{q}^{-1}$ is integral. Let $n$ be the content of $a\tilde{q}^{-1}$, then $n$ is integral. Noting that $q \subset O_F$, hence by (22) we have the decomposition
\[ a = n \cdot (-b + \sqrt{-D}) \oplus n^{-1} \prod a\tilde{a} = n \cdot (-b + \sqrt{-D}) \oplus n^{-1} q^{-1} a\tilde{a}, \]
where $b \in O_F$ and $q$ is the index-ideal $[O_K : O_F(\sqrt{-D})]$. Let $c_a \in n$ be an element of the minimal absolute norm, and we fix one such choice for each $a$ once and for all. Then by Lemma 18 we have
\[ a = n \cdot (-b + \sqrt{-D}) \oplus n^{-1} q^{-1} a\tilde{a} = O_F \cdot c_a \cdot (-b + \sqrt{-D}) \oplus q^{-1} a\tilde{a} \cdot c_a^{-1}. \]
The direct sum in the right hand side of the above identity can be verified easily from the proof of Lemma 18. Also noting that by the definition of $q$ we have $q_{K/F} = -Dq^{-2}$, where $q_{K/F}$ is the relative ideal-discriminant, then $[q^{-1} a\tilde{a}]$ is the Steinitz class of $a$.

Combining the decomposition (24) with (21), i.e. $a = O_F \alpha + f_a \beta$, we have, by the uniqueness of Steinitz class, that
\[ \alpha = c_a \cdot (-b + \sqrt{-D}) \varepsilon \quad \text{and} \quad f_a \beta = q^{-1} a\tilde{a} \cdot c_a^{-1}, \]
for some unit $\varepsilon \in O_F^\times$. So we have
\[ y_a = \Im(z_a) = \frac{c_a \cdot \sqrt{D}}{\beta}. \]

Noting that $q_{K/F} = N_{K/F}(q_{K/F})$ and $D_K = D_F^2 N_{F/Q}(q_{K/F})$, one thus obtains
\[ N_\Phi(y_a) = \prod_{\sigma \in \Phi} \left( \frac{c_a \cdot \sqrt{D}}{\beta} \right) = \frac{N_{K/Q}(c_a)N_{F/Q}(f_a)N_{F/Q}(q)^2}{2^n N_{K/Q}(a)} \cdot \frac{\sqrt{D_K}}{D_F}. \]

\[ \square \]

When $a\tilde{q}^{-1}$ is primitive, the content $n$ is trivial. So we can take $c_a = 1$ in the above proof. This gives:
Corollary 20. Let notation be as that in Lemma 19, and let $a$ be an integral ideal of $\mathcal{O}_K$ such that $a\mathcal{q}^{-1}$ is integral. Then one has

\begin{equation}
N_\Phi(y_a) \geq \frac{N_{F/Q}(I_a)N_{F/Q}(q)^2}{2^n N_{K/Q}(a)} \sqrt{D_K/D_F}.
\end{equation}

Moreover, (26) becomes an identity if and only if $a\mathcal{q}^{-1}$ is primitive.

Proof. By definition, that $a\mathcal{q}^{-1}$ is primitive means that the content of $a\mathcal{q}^{-1}$ is trivial, which amounts to the condition that $N_{K/Q}(c_a) = 1$.

Remark. From the above expression, it is clear that $N_\Phi(y_a)$ is independent of a particular choice of $I_a \in \mathcal{S}_\alpha$.

We will always fix the CM type $\Phi$ in this paper. For the sake of simplicity, we will write $g_a^\Phi$ for the CM type norm $N_\Phi(y_a)$ in computations in the following parts.

3.3. Hilbert Eisenstein Series.

3.3.1. Eisenstein Series over Totally Real Field. In this subsection, let’s review basic theory of Eisenstein series defined over totally real fields.

As before, let $F$ be a totally real algebraic number field of degree $n$. We set $J_F^n := \{\sigma_1, \ldots, \sigma_n\}$ to be the non-isomorphic real embeddings

$$F \hookrightarrow \mathbb{R}^n, \quad x \mapsto (x^{\sigma_1}, \ldots, x^{\sigma_n}).$$

Let $J_f$ (resp. $J_\infty$) denote the set of all finite (resp. infinite) places of $F$. For simplicity, we identify $J_\infty$ with $J_F^n$. Then for any $S \subset F$, $(s_1, \ldots, s_n) \in S^n$ can be identified with $(s_v)_{v \in J_\infty} \in \mathbb{R}^n$ by setting $s_v = s^{\sigma_v}$ if $v$ corresponds to $\sigma_v$.

Let $\mathcal{A}_F$ denote the finite part of $\mathcal{A}_F$ and let $S(\mathcal{A}_F)$ denote the space of Schwartz-Bruhat functions (i.e. locally constant and with compactly supported) on $\mathcal{A}_F$. Take the weight $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and a test vector $\phi \in S(\mathcal{A}_F)$.

For $x = (x_1, \ldots, x_n) \in \mathbb{C}^n$, write $x^k = \prod_{j=1}^n x_j^{k_j}$. For $\forall x \in \mathbb{R}^{n+}$ and $a \in \mathbb{C}^n$ we define $x^a$ similarly. Also, for any $c, d \in F$ and $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, we define

$$(cz + d)^a := \prod_{j=1}^n (c z_j + d)^{a_j}.$$ 

By definition of $\phi \in S(\mathcal{A}_F)$, we can find a subgroup $U_\phi \subset \mathcal{O}_F^{*+}$ of finite index such that

$$\phi(ux) = \phi(x) \text{ and } u^{-k}|u|^k = 1, \quad \forall u \in U_\phi, \forall x \in F.$$

We regard $GL(2)$ as an algebraic group over $F$, then clearly $GL(2, \mathbb{R})^{n+}$ acts on $\mathbb{H}^n$, where $GL(2, \mathbb{R})^{n+} := \{g \in GL(2, \mathbb{R}) : \det g > 0 \}$. For $g = (g_v)_{v \in J_\infty} \in GL(2, \mathbb{R})^{n+}$ and $z = (z_v)_{v \in J_\infty} \in \mathbb{H}^n$, we define the factor of automorphy by

$$j(g, z) := (c_v z_v + d_v)_{v \in J_\infty}, \quad g_v := \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix}.$$ 

Take $k \in \mathbb{Z}^n$ and $\varphi \in S(\mathcal{A}_F \oplus \mathcal{A}_F)$. Let $U_\varphi \subset \mathcal{O}_F^{*+}$ be a subgroup of finite index such that

$$\varphi(ux) = \varphi(x) \text{ and } u^{-k}|u|^k = 1, \quad \forall u \in U_\varphi, \forall x \in F \oplus F.$$

Let $F^{2,\times} := F \oplus F \setminus \{(0,0)\}$. We can define an Eisenstein series by

$$G_k(z, s; \varphi) := \frac{1}{[\mathcal{O}_F^{*+} : U_\varphi]} \sum_{(c, d) \in F^{2,\times} / U_\varphi} \varphi(c, d)(cz + d)^{-k} y^{s \sigma - \frac{1}{2} |cz + d|^{-2s\sigma}},$$

where $z \in \mathbb{H}^n$, $s \in \mathbb{C}$, $y = \Re(z)$, and $cz + d = (c z_1 + d z_1, \ldots, c z_n + d z_n) \in \mathbb{C}^n$.

Note that the series (27) is defined independent of $U_\varphi$, and the sum converges when $\Re(z) > 1$. Let

$$\Gamma_\varphi := \{ \gamma \in GL(2, F) : \varphi(x \gamma) = \varphi(x), \forall x \in F \oplus F, \det \gamma \in \mathcal{O}_F^{*+} \}.$$
Then $\Gamma_\varphi$ is a congruence subgroup of $GL(2, F)$ and one can verify that

$$G_k(\gamma z, s; \varphi) = (\text{det} \gamma)^{-\frac{k}{2}} j(\gamma, z)^k G_k(z, s; \varphi), \quad \forall \gamma \in \Gamma_\varphi.$$  

3.3.2. **Fourier Expansion of Hilbert Eisenstein Series.** Now we are going to derive the Fourier expansion of the Eisenstein series $G_k(z, s; \varphi)$.

Let $\mathbb{C}^+ := \{ z \in \mathbb{C} : \Re(z) > 0 \}$. For any $z \in \mathbb{C}^+$ and $\alpha \in \mathbb{C}$, put $z^\alpha = e^{\alpha \log z}$ choosing the branch of log $z$ such that it is real on $\mathbb{R}^+$. For $z \in \mathbb{H}$ we write $z^\alpha := i^\alpha (-iz)^\alpha$ and $\bar{z}^\alpha := i^{-\alpha} (-i\bar{z})^\alpha$, where $i^\alpha := e^{\frac{\alpha \pi i}{2}}$.

For $z \in \mathbb{C}^+$ and $(\alpha, \beta) \in \mathbb{C}^2$, define the confluent hypergeometric function

$$\zeta(z; \alpha, \beta) := \int_0^\infty (x + 1)^{\alpha-1} x^\beta e^{-xz} dx.$$  

Clearly the integral converges when $\Re(\beta) > 0$. Put $\omega(z; \alpha, \beta) := \Gamma(\beta)^{-1} z^\beta \zeta(z; \alpha, \beta)$. Then according to [41] (Page 85) and [42], $\omega(z; \alpha, \beta)$ can be continued to a holomorphic function on $\mathbb{C}^+ \times \mathbb{C}^2$ and satisfies the functional equation:

$$\omega(z; \alpha, \beta) = \omega(z; 1 - \beta, 1 - \alpha).$$

For $a \in \mathbb{R}^+$, $b \in \mathbb{R}$ and $(\alpha, \beta) \in \mathbb{C}^2$, let’s define

$$\zeta(a, b; \alpha, \beta) := \int_{-\infty}^\infty (x + ai)^{-\alpha} (x - ai)^\beta e^{-2\pi i bx} dx.$$  

The integral converges when $\Re(\alpha + \beta) > 0$. By (1.29) and (3.3) of [42], we have

$$\zeta(a, b; \alpha, \beta) = \frac{i^{\beta-\alpha}(2\pi)^{\alpha+\beta} b^{\alpha+\beta-1}}{\Gamma(\alpha)\Gamma(\beta)e^{2\pi ab}} \zeta(4\pi ab; \alpha, \beta), \quad b > 0.$$  

For $z = (z_\nu)_{\nu \in J_\infty} \in \mathbb{C}^\infty$, we set $e(z) := e^{2\pi i \sum_{\nu \in J_\infty} z_\nu}$. Applying the Poisson summation formula to $f(x) := (x + iy)^{-\alpha} (x - iy)^{-\beta}$ with a fixed $y \in \mathbb{R}^{n_+}$ we then get (ref. [43], Lemma 18.4):

**Lemma 21.** Let $m$ be a fractional ideal of $F$ and $z = x + iy \in \mathbb{H}^n$, $\alpha, \beta \in \mathbb{C}^n$, $b \in F$. Let

$$\Xi(y, b; \alpha, \beta) := \prod_{\nu \in J_\infty} \zeta(y_\nu, b_\nu; \alpha_\nu, \beta_\nu).$$  

Then if $\Re(\alpha_\nu + \beta_\nu) > 1$, $\forall \nu \in J_\infty$, we have

$$D_F^\sigma N(m) \sum_{a \in m} (z + a)^{-\alpha} (z + a)^{-\beta} = \sum_{b \in (m)^{-a}} \Xi(y, b; \alpha, \beta).$$  

For any $\phi \in S(A_{F,F})$, let

$$D_k(s, \phi) := \frac{1}{|O_{F,F}^\times : U_\phi|} \sum_{d \in \mathcal{D}} \phi(d) d^{-k} |d|^{k-s \sigma},$$

where $s \sigma := (s, \cdots, s) \in \mathbb{C}^n$. Clearly this series converges when $\Re(s) > 1$.

Let $a$ and $b$ be fractional ideals of $F$. Take $\varphi$ to be the characteristic function of the closure of $a b \oplus b$. Then $U_\varphi = O_{F,F}^\times$. Let $\varphi_{a b}$ be the characteristic function of the closure of $a b$, and $\varphi_b$ be the characteristic function of the closure of $b$. Then we have

$$y^{\frac{1}{2s \sigma}} G_k(z, s; \varphi) = \sum_{(c, d) \in \mathcal{D} \cap \mathcal{D}^F \cap /O_{F,F}^\times} \varphi(c, d) (cz + d)^{-k} |cz + d|^{k-2s \sigma}$$

$$= \sum_{d \in \mathcal{D}} \varphi(0, d) d^{-k} |d|^{k-2s \sigma} + \sum_{c \in \mathcal{C}} \sum_{d \in \mathcal{D}} (cz + d)^{-k} |cz + d|^{k-2s \sigma}$$

$$= D_k(2s, \varphi_b) + G_k,$$

where $\mathcal{C} := (a b \cap F^\times) /O_{F,F}^\times$ and $\mathcal{D} := (b \cap F^\times) /O_{F,F}^\times$. 
Lemma 22. Note that for $\Re(s) > 1$, we have
\[
G_k^* = \sum_{c \in \mathcal{C}} \sum_{d \in \mathbb{Z}} (cz + d)^{-k}|cz + d|^{k - 2s\sigma}
= \sum_{c \in \mathcal{C}} c^{-k}|c|^{k - 2s\sigma} \sum_{d \in c^{-1}\mathbb{Z}} (z + d)^{-k}|z + d|^{k - 2s\sigma}
= D_F^{-\frac{1}{2}} N(b)^{-1} \sum_{c \in \mathcal{C}} c^{-k}|c|^{k - 2s\sigma} |N(c)| \sum_{b \in a^{-1} \times \mathbb{Z}} e(bx) \Xi(y, b; s\sigma + \frac{k}{2}, s\sigma - \frac{k}{2}),
\]
where the last equality comes from Lemma 21. To summary, we have the Fourier expansion:
\[
G_k(z, s; \varphi) = y^{s\sigma - \frac{k}{2}} D_k(2s, \varphi_b)
+ D_F^{-\frac{1}{2}} N(b)^{-1} y^{s\sigma - \frac{k}{2}} \Xi(y, 0; s\sigma + \frac{k}{2}, s\sigma - \frac{k}{2}) D_k(2s - 1, \varphi_{ab})
+ \lambda_F^{-1} D_F^{-\frac{1}{2}} N(b)^{-1} y^{s\sigma - \frac{k}{2}} \sum_{b \in F} e(bx) \Xi(y, b; s\sigma + \frac{k}{2}, s\sigma - \frac{k}{2}) \lambda(b, s),
\]
where
\[
(29) \quad \lambda(b, s) = \sum_{(a, c) \in b^{-1} \times \mathcal{C}} c^{-k}|c|^{k + (2s - 1)\sigma}.
\]
Define
\[
\Gamma_a := \left\{ \gamma \in \left( \begin{array}{ccc} a & b \\ c & d \end{array} \right) \in GL(2, \mathbb{F}) : a, d \in \mathcal{O}_F, b \in a^{-1}, c \in a, \det \gamma \in \mathcal{O}_F^{\times} \right\}.
\]
Then clearly $\varphi(x\gamma) = \varphi(x)$ for $x \in F \otimes F$, $\gamma \in \Gamma_a$. Form now on, we assume $k = 0$, then by (28) we have
\[
G_k(\gamma z, s; a, b) = G_k(z, s; a, b), \quad \forall \gamma \in \Gamma_a.
\]
Also by definition we have
\[
D_0(s, \varphi_{ab}) = N(ab)^{-\frac{s}{2}} \zeta_F(s, [ab]^{-1}), \quad D_0(s, \varphi_b) = N(b)^{-s} \zeta_F(s, [b]^{-1}),
\]
where for a fractional ideal $\mathfrak{a} \subset F$, we denote by $[\mathfrak{a}]$ the class of $\mathfrak{a}$ in $\text{Cl}(F)$, and $\zeta_F(s, [\mathfrak{a}])$ the partial zeta function of the class $[\mathfrak{a}]$.

Recall the $K$–Bessel function
\[
K_s(z) := \frac{1}{2} \int_0^\infty t^{s - 1} e^{-\frac{z}{2}(t + t^{-1})} dt.
\]
It is well known that the integral converges locally uniformly on the domain $(z, s) \in \mathbb{C}^2 : \Re(z) > 0$, and thus defines an analytic function there.

Note that by definition we have $K_s(z) = K_{-s}(z)$.

Lemma 22. For $a > 0$, $\Re(s) > \frac{1}{2}$, we have
\[
\xi(a, b; s, s) = \begin{cases} 2\pi^s a^{1/2 - s}|b|^{-1/2} \Gamma(s)^{-1} K_{s - 1/2}(2\pi a |b|) & \text{if } b \in \mathbb{R} \setminus \{0\}; \\ \sqrt{\pi} a^{1/2 - s} \Gamma(s - 1/2) \Gamma(s)^{-1} & \text{if } b = 0. \end{cases}
\]

Proof. By changing of variables we have
\[
\int_0^\infty t^{s - 1} e^{-\pi(a^2 + x^2)t} dt = \pi^{-s} (a^2 + x^2)^{-s} \Gamma(s), \quad \Re(s) > 0.
\]
Hence we have
\[
\int_{-\infty}^\infty \int_0^\infty t^{s - 1} e^{-\pi(a^2 + x^2)t - 2\pi i bx} dt dx = \pi^{-s} \Gamma(s) \xi(a, b; s, s).
\]
On the other hand, one sees easily that
\[ \int_{-\infty}^{\infty} e^{-\pi(a^2+x^2)t-2\pi ibx} \, dx = \frac{1}{\sqrt{t}} e^{-\pi(a^2 t + \frac{b^2}{4})} \, dt. \]
Hence we have
\[ \int_{-\infty}^{\infty} \int_{0}^{\infty} t^{s-1} e^{-\pi(a^2+x^2)t-2\pi ibx} \, dt \, dx = \int_{0}^{\infty} \int_{0}^{\infty} t^{s-\frac{1}{2}} e^{-\pi(a^2 t + \frac{b^2}{4})} \, dt \, dx. \]
If \( b = 0 \), then the right hand side above is equal to \( \pi^{-s+\frac{3}{2}} a^{1-2s} \Gamma(s - \frac{1}{2}) \). This gives the second identity in the lemma. If \( b \neq 0 \), then changing the variable \( t \) to \( \frac{a^2}{t} \) we see that
\[ \int_{0}^{\infty} t^{s-\frac{1}{2}} e^{-\pi(a^2 t + \frac{b^2}{4})} \, dt = 2 \left( \frac{|b|}{a} \right)^{s-\frac{1}{2}} K_{s-\frac{1}{2}}(2\pi a |b|). \]
This gives the first formula in the lemma. \( \square \)

Let \( G(z, s; a, b) := G_0(z, s; a, b) \), define the regularized Eisenstein series as
\[ E(z, s; a, b) := \frac{G(z, s; a, b)}{\zeta_F(2s)}. \]
Then combining the Fourier expansion of \( G(z, s; a, b) \) and the Lemma 22 we have the explicit Fourier expansion:
\[
E(z, s; a, b) = N(b)^{-2s} y^s \sigma \frac{\zeta_F(2s, [b]^{-1})}{\zeta_F(2s)} \\
+ \left( \frac{\sqrt{-1}(s - 1/2)}{\Gamma(s)} \right)^n D_F^{1/2} N(b)^{-1} y^{(1-s)\sigma} N(ab)^{1-2s} \zeta_F(2s - 1, [ab]^{-1}) \\
+ \left( \frac{2\pi}{\Gamma(s)} \right)^n \frac{2\pi}{D_F^{1/2} N(b) \zeta_F(2s)} \sum_{b \in F^*} |N(b)|^{s-1/2} \lambda(b, s) e(bz) \\
\times \prod_{v \in \mathcal{O}_{F}} K_{s-\frac{1}{2}}(2\pi y_v |b_v|). 
\]
Since we have the following Laurent expansion of (partial) Dedekind zeta function around \( s = 1 \):
\[
(30) \quad \zeta_F(s, \mathcal{C}) = \frac{h_F^{-1} \rho_F}{s - 1} + \gamma_{F, [\mathcal{C}]} + O(s - 1),
\]
where \( \mathcal{C} \) is an ideal class in \( \text{Cl}(F) \), and
\[
\rho_F := \frac{2^n h_F R_F}{w_F \sqrt{D_F}}.
\]
In particular, around \( s = 1 \) we have
\[
(31) \quad \zeta_F(s) = \frac{\rho_F}{s - 1} + \gamma_F + O(s - 1),
\]
where
\[
\gamma_{F, \mathcal{C}} := \lim_{s \to 1} \left\{ \zeta_F(s, \mathcal{C}) - \frac{h_F^{-1} \rho_F}{s - 1} \right\};
\]
and by \( \zeta_F(s) = \sum_{\mathcal{C} \in \text{Cl}(F)} \zeta_F(s, \mathcal{C}) \) we have
\[
\gamma_F = \sum_{\mathcal{C} \in \text{Cl}(F)} \gamma_{F, \mathcal{C}} = \rho_F \gamma^*_F,
\]
where \( \gamma^*_F \) is defined in (2). \( \gamma_F \) and \( \gamma_{F, \mathcal{C}} \) are called unnormalized Euler-Kronecker constants with respect to \( F/\mathbb{Q} \), which we will deal with later.
From the Fourier expansion above we see that $E(z, s; a, b)$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ with residue

$$\text{Res}_{s=1} E(z, s; a, b) = \frac{2^{n-1}r^n R_F}{w_F D_F N(b)N(ab)\zeta_F(2)}.$$ 

Note that we have the Taylor expansion around $s = 0$ that $\Gamma(s) = s^{-1} + O(1)$ and

$$\zeta_F(s) = -\frac{h_F R_F}{w_F}s^{-1} + O(s^n),$$

where $R_F$ is the regulator and $w_F$ is the number of roots of unity. Then $E(z, s; a, b)$ is holomorphic at $s = 1/2$. Moreover, we can show actually $E(z, s; a, b)$ vanishes at $s = 1/2, \forall z \in \mathbb{C}$.

**Lemma 23.** Let notation be as above, then we have

$$E \left( z, \frac{1}{2}; a, b \right) \equiv 0, \quad \forall z \in \mathbb{H}^n.$$

**Proof.** Let $f \subset F$ be a fractional ideal, and $\tilde{f}$ be its dual, i.e. $[f] \cdot [\tilde{f}] = [\mathcal{O}]$, where $\mathcal{O}$ is the different of $F/\mathbb{Q}$. Let

$$Z_F(s, [f]) := Z_{F, \infty}(s)\zeta_F(s, [f]),$$

and $Z_F(s) := Z_{F, \infty}(s)\zeta_F(s)$.

where $Z_{F, \infty}(s) := D_F^{s/2}n^{-s/2}\Gamma(s/2)^n$.

It is well known that we have the following functional equation for the partial completed zeta function:

$$Z_F(s, [f]) = Z_F(1 - s, [\tilde{f}]).$$

An immediate consequence of this is the functional equation for the completed Dedekind zeta function (obtained adding the partial ones), which has exactly the same form. Also, all the partial zeta functions have a simple pole at $s = 1$ with the same residue $2^n R_F w_F^{-1} D_F^{-1/2}$.

Let’s introduce some functions to simplify the notations. Define

$$M_1(s) := Z_F(2s) N(b)^{-2s} y^{\sigma} \zeta_F(2s, [b]^{-1}) \zeta_F(2s);$$

$$M_2^*(s) := \left( \frac{\sqrt{\pi} \Gamma(s - 1/2)}{\Gamma(s)} \right)^n D_F^{-\frac{s}{2}} N(b)^{-y^{(1-s)\sigma}} N(ab)^{1-2s} F(2s - 1, [ab]^{-1}) \zeta_F(2s);$$

$$M_2(s) := Z_F(2s) M_2^*(s);$$

$$= Z_{F, \infty}(2s - 1) N(b)^{-y^{(1-s)\sigma}} N(ab)^{1-2s} \zeta_F(2s - 1, [ab]^{-1}).$$

Then by (33) (taking $f = ab$) we see that $M_2(1 - s) = H(s) M_1(s)$, where

$$H(s) := N(b)^{2s-1} N(ab)^{2s-1} \zeta_F(2s, [ab]^{-1}) \zeta_F(2s, [b]^{-1}).$$

Since the Dirichlet series $\zeta_F(2s, [b]^{-1})$ absolutely converges when $\Re(s) > \frac{1}{2}$, $H(s)$ is holomorphic when $\Re(s) > \frac{1}{2}$, and can be continued to a meromorphic function on $\mathbb{C}$. Since all the partial zeta functions have a simple pole at $s = 1$ with the same residue, we see $H(s)$ is holomorphic at $s = 1/2$ and $H(1/2) = 1$.

Let $L(s)^{-1} := H(s) H(1 - s)$, then we have

$$L(s)^{-1} = \frac{\zeta_F(2s, [ab]^{-1}) \zeta_F(2 - 2s, [ab]^{-1})}{\zeta_F(2s, [b]^{-1}) \zeta_F(2 - 2s, [b]^{-1})}.$$
Likewise, \( L(s) \) is a meromorphic function on \( \mathbb{C} \) and is analytic at \( s = 1/2 \), with \( L(1/2) = 1 \). So we have

\[
M(1-s) = H(s)M_1(S) + H(s)L(s)M_2(s).
\]

Now let

\[
E_1(s) : = \mathcal{Z}_F(2s) \left( \frac{2\pi^s}{\Gamma(s)} \right)^n \frac{y^{\frac{s}{2}}}{D_F^{1/2} N(b) \zeta_F(2s)} = 2^n D_F^{1/2} \frac{y^{\frac{s}{2}}}{D_F^{1/2} N(b)}; \\
E_2(s) : = \sum_{b \in \mathcal{F}} |N(b)|^{s-1/2} \lambda(b, s)e(bz) \prod_{v \in J_\infty} K_{s-\frac{1}{2}} (2\pi y_v |b_v|) \\
= \sum_{b \in \mathcal{F}} |N(b)|^{s-1/2} \sum_{(a,c) \in \mathbb{Z} \times \mathbb{C}} |N(c)|^{1-2s} e(bz) \prod_{v \in J_\infty} K_{s-\frac{1}{2}} (2\pi y_v |b_v|) \\
= \sum_{b \in \mathcal{F}} \sum_{(a,c) \in \mathbb{Z} \times \mathbb{C}} \left( \frac{|N(a)|}{|N(c)|} \right)^{s-\frac{1}{2}} e(bz) \prod_{v \in J_\infty} K_{s-\frac{1}{2}} (2\pi y_v |b_v|); \\
E(s) : = E_1(s)E_2(s).
\]

Then both \( E_1(s) \) and \( E_1(s) \) are entire functions. Let’s briefly explain why \( E_2(s) \) is entire: by Turan’s inequality for Bessel functions, \( \log K_0(x) \) is convex. Also note the fact that

\[
\lim_{x \to +\infty} \frac{\log K_0(x)}{x} = -1,
\]

hence we have \( K_0(x) \leq (eK_0(1))e^{-x} \). Namely, the Bessel K-function has exponentially decay, which forces the sum in \( E_2(s) \) to converge absolutely. Also one sees easily that \( E_1(s) = D_F^{1/2} E_1(1-s) \). So we have

\[
E(1-s) = D_F^{1-2s} E_1(s)E_2(1-s).
\]

Let \( \mathcal{E}(z, s; a, b) := \mathcal{Z}_F(2s)E(z, s; a, b) \) be completed Eisenstein series, then by the Fourier expansion of \( \mathcal{E}(z, s; a, b) \) we have

\[
\mathcal{E}(z, s; a, b) = M(s) + E(s) = M_1(s) + M_2(s) + E_1(s)E_2(s).
\]

By the above computation one has

\[
\mathcal{E}(z, 1-s; a, b) = H(s)M_1(s) + L(s)H(s)M_2(s) + D_F^{1-2s} E_1(s)E_2(1-s).
\]

Therefore, we have (noting that \( K_0(x) = K_{-\nu}(x) \))

\[
E(z, \frac{1}{2}; a, b) = \lim_{s \to \frac{1}{2}} \frac{\mathcal{E}(z, s; a, b)}{\mathcal{Z}_F(2s)\zeta_F(2s)} \\
= \lim_{s \to \frac{1}{2}} \frac{\mathcal{E}(z, 1-s; a, b)}{\mathcal{Z}_F(2s)\zeta_F(2s)} \\
= - \lim_{s \to \frac{1}{2}} \frac{H(s)M_1(s) + L(s)H(s)M_2(s) + D_F^{1-2s} E_1(s)E_2(1-s)}{\mathcal{Z}_F(2s)\zeta_F(2s)} \\
= - \lim_{s \to \frac{1}{2}} \frac{H(\frac{1}{2})M_1(s) + L(\frac{1}{2})H(\frac{1}{2})M_2(s) + E_1(\frac{1}{2})E_2(\frac{1}{2})}{\mathcal{Z}_F(2s)\zeta_F(2s)} \\
= - \lim_{s \to \frac{1}{2}} \frac{\mathcal{E}(z, s; a, b)}{\mathcal{Z}_F(2s)\zeta_F(2s)} \\
= - E(z, \frac{1}{2}; a, b).
\]

Thus we have \( E(z, \frac{1}{2}; a, b) = 0. \) □
Remark. Note that \( E(z; s; a, b) \) may not have a functional equation, since the Hilbert modular variety may have several cusps. The Eisenstein matrix will always have a functional equation. However, when \( a = b = \mathcal{O}_F \), there is only one cusp. In fact, we see that in this situation \( H(s) = L(s) = 1 \) and \( E(s) = E(1 - s) \), then we have the functional equation

\[
E(z, s; a, b) = E(z, 1 - s; a, b),
\]

which gives directly that \( E(z, \frac{1}{2}; a, b) = 0 \). This is the case in [27].

3.4. Periods of Eisenstein Series. In this section we combine the discussion in last two subsections to show the class group \( L \)-function \( L_K(\chi, s) \) can be expressed as a weighted period of the Eisenstein series \( E(z, s; f) \) with respect to the CM 0-cycles \( \mathcal{CM}(K, \Phi, f) \), where \([j]\) \( \in \mathcal{C}(F)_+ \).

Recall that we have the natural surjective map

\[
\mathcal{CM}(K, \Phi) \twoheadrightarrow \text{Cl}(K), \quad [a, r] \mapsto [a].
\]

And the fiber is indexed by \( \epsilon \in \mathcal{O}^{\pm}_F / N_{K/F} \mathcal{O}_K^\times \) with order at most 2.

Recall that by Lemma 19. Since \( K/F \) is a CM extension of number fields of degree 2n, we have that (ref. [49]) for each ideal class \( C \in \text{Cl}(K) \), there exists an integral ideal \( ac \in C \) such that

\[
N_{K/Q}(ac) \leq M(0, n)\sqrt{D_K}.
\]

Clearly, we may assume \( a_{[\mathcal{O}_K]} = \mathcal{O}_K \). Thus we can define a set of representatives of \( \text{Cl}(K) \) as

\[
\mathcal{I}_K := \left\{ a : a = acq, \quad \forall C \in \text{Cl}(K) \right\},
\]

where \( q \) is the index ideal in (22).

For convenience, let us fix \( \mathcal{I}_K \) once for all. Clearly we have

\[
\text{Cl}(K) = \left\{ [a] : a \in \mathcal{I}_K \right\}.
\]

For any \( a \in \mathcal{I}_K \), let \( y_a \) be the imaginary part of \( z_a \), the associated CM point. Then by (23) we have

\[
y_a^\sigma = N_F(y_a) \geq \frac{M(0, n)^{-1}N_{F/Q}(f_a)}{2^nD_F}, \quad \forall a \in \mathcal{I}_K.
\]

Also, for any fractional \( a \) of \( K \), there exists a unique \( f_a \in \mathcal{I}_F^+ \) and a CM point \([a, r] \in X(f_a) := \Gamma(f_a) \setminus \mathbb{H}^n \) mapping to \([a]\). Note that there are at most 2 pre-images of \([a]\) \( \in \text{Cl}(K) \). From now on, we fix one of them \([a, r] \in \mathcal{CM}(K, \Phi, f_a), \forall a \).

Then we have a decomposition (21), i.e.

\[
a = \mathcal{O}_F \alpha + f_a \beta_a
\]

with \( z_a := \frac{\alpha}{\beta_a} \in K^\times \cap \mathbb{H}^n = \{ z \in K^\times : \Phi(z) \in \mathbb{H}^n \} \). Moreover, \( z_a \) represents the CM point \([a, r]\).

Proposition 24. Let \( K \) be a CM extension of a totally real number field \( F \) of degree \( n \), and \( \Phi \) be a CM type of \( K \). Then we have

\[
L_K(\chi, s) = \frac{(2^nD_F)^s}{D_K^{1/2}\mathcal{O}_K^\times : \mathcal{O}_F^\times} \sum_{[a^{-1}] \in \text{Cl}(K)} \mathcal{N}(f_a)^s \zeta_F(2s)E(z_a, s; f_a^{-1}, f_a),
\]

where \( f_a \in \mathcal{I}_F^+ \) is defined as above, \( z_a \) is the corresponding CM points of \( a \) via the map \( \mathcal{CM}(K, \Phi) \twoheadrightarrow \text{Cl}(K) \).
Proof. Let \( C \in \text{Cl}(K) \) be an ideal class. Then there exist a unique primitive ideal \( a \in \mathcal{I}_K \) such that \([a] = C^{-1}\). Hence as \( b \) runs over integral ideals in \( C \), \( ab = (w) \) runs over principal ideals \( (w) \) with \( w \in a/\mathcal{O}_K^{\times} \). Let \( \sum' \) denote that the summation is taken over nonzero integral variables (e.g. \( \sum'_a \) means the summation is taken over all nonzero integral ideals \( a \subset K \)), then the partial Dedekind zeta function can be written

\[
\zeta_K(s, C) = \sum_{b \in C} N_{K/Q}(b)^{-s} = N_{K/Q}(a)^s \sum'_{u \in a/\mathcal{O}_K^{\times}} N_{K/Q}((u))^{-s}
\]

\[
= N_{K/Q}(a)^s \sum'_{[\mathcal{O}_K^{\times} : \mathcal{O}_F^{\times}]_{(c,d) \in \mathcal{O}_F/F, c,d \in \mathcal{O}_F^{\times}}} N_{K/Q}((\alpha c + d \beta))^{-s}
\]

\[
= N_{K/Q}(a)^s N_{K/Q}((\beta))^{-s} \sum'_{[\mathcal{O}_K^{\times} : \mathcal{O}_F^{\times}]_{(c,d) \in \mathcal{O}_F/F, c,d \in \mathcal{O}_F^{\times}}} N_{K/Q}((c a + d))^{-s}
\]

\[
= N_{K/Q}(O_F z a + f a)^s \sum'_{[\mathcal{O}_K^{\times} : \mathcal{O}_F^{\times}]_{(c,d) \in \mathcal{O}_F/F, c,d \in \mathcal{O}_F^{\times}}} N_{K/Q}((c a + d))^{-s}.
\]

Write \( z_a = x_a + iy_a \), then a calculation with determinants (ref. P237 of [37]) yields

\[N_{K/Q}(O_F z a + f a) = y_a^s N_{F/Q}(f a) \cdot \frac{2^n D_F}{\sqrt{D_K}}\]

By a calculation with the CM type \( \Phi \) we have

\[N_{K/Q}((c a + d)) = |c a + d|^{2\sigma},\]

where we have identified \( z_a \) with \( \Phi(z_a) \in \mathbb{H}^n \). Thus by combining the preceding computations we obtain

\[\zeta_K(s, C) = \frac{(2^n D_F N_{F/Q}(f a))^s}{D_K^{\frac{s}{2}} [\mathcal{O}_K^{\times} : \mathcal{O}_F^{\times}]} \sum'_{(c,d) \in \mathcal{O}_F/F, c,d \in \mathcal{O}_F^{\times}} y_a^s |c a + d|^{-2s\sigma} \]

\[= \frac{(2^n D_F N_{F/Q}(f a))^s}{D_K^{\frac{s}{2}} [\mathcal{O}_K^{\times} : \mathcal{O}_F^{\times}]} G(z_a, s; f a^{-1}, f a) \]

Finally, using that

\[L_K(\chi, s) = \sum_{C \in \text{Cl}(K)} \chi(C) \zeta_K(s, C)\]

to obtain the formula

\[L_K(\chi, s) = \frac{(2^n D_F)^s}{D_K^{\frac{s}{2}} [\mathcal{O}_K^{\times} : \mathcal{O}_F^{\times}]} \sum_{[a] \in \text{Cl}(K)} \chi([a]) N_{F/Q}(f a)^s G(z_a, s; f a^{-1}, f a) \]

\[\square\]

Remark. One can also refer [47] Lemma 2.3 (P. 181-182) for a proof of (37) via comparing residues of twisted Eisenstein series and partial Dedekind Zeta functions at \( s = 1 \).

Corollary 25. Let notations be as above, then we have

\[L_K \left( \chi, \frac{1}{2} \right) = \frac{2^{\frac{3}{2}} \rho_{\mathcal{F}} \sqrt{D_F}}{2 D_K^{\frac{1}{2}} [\mathcal{O}_K^{\times} : \mathcal{O}_F^{\times}]} \sum_{[a] \in \text{Cl}(K)} \chi([a]) \sqrt{N(f a)} \mathcal{E} \left( z_a, \frac{1}{2}; f a^{-1}, f a \right).\]
and
\[ \frac{1}{h_K} \sum_{\chi \in \text{Cl}(K)} \left| L_K(\chi, 1/2) \right|^2 = \frac{2^{n-2} D_F \rho_F}{\sqrt{\rho_F}} \left( \frac{\rho_F}{D_F} \right)^2 \times \sum_{[a^{-1}] \in \text{Cl}(K)} N_{F/Q}(f_a) \left| E' \left( z_a, \frac{1}{2} f_a^{-1}, f_a \right) \right|^2. \]

**Proof.** By the Laurent expansion of \( \zeta_F(s) \) at \( s = 1 \) (i.e. (31)) we have
\[ \zeta_F(2s) = \frac{\rho_F}{2(s-\frac{1}{2})} + \gamma_F + O(s - \frac{1}{2}), \]
and because \( E \left( z_a, s; f_a^{-1}, f_a \right) \) vanishes identically at \( s = 1/2 \) (ref. Lemma 23),
\[ E \left( z_a, s; f_a^{-1}, f_a \right) = E' \left( z_a, s; f_a^{-1}, f_a \right) (s - 1/2) + O ((s - 1/2)^2). \]
Thus we have
\[ \zeta_F(2s) E \left( z_a, s; f_a^{-1}, f_a \right) = \frac{\rho_F}{2} E' \left( z_a, s; f_a^{-1}, f_a \right) + O(s - 1/2). \]
Then the first formula comes Proposition 24 with \( s = 1/2 \), and thus the second formula comes from the first and the orthogonality of characters. \( \square \)

**Remark.** Note that \( \sqrt{N_{F/Q}(f_a)} E' \left( z_a, \frac{1}{2}, f_a^{-1}, f_a \right) \) is independent of the choice of \( f_a \), for any fractional ideal \( a \) in \( K \).

Also, orthogonality on (38) gives an expression of an average of twisted class group \( L \)-functions in terms of derivatives of Hilbert Eisenstein series:
\[ \frac{1}{h_K} \sum_{\chi \in \text{Cl}(K)} \chi([a]) L_K(\chi, 1/2) = \frac{2\pi \rho_F \sqrt{D_F}}{2 D_F^{1/2} [O_K^+: O_F^+]} \sqrt{N(f_a)} E' \left( z_a, \frac{1}{2}, f_a^{-1}, f_a \right). \]
Note that this is a generalization of Theorem 1 in [11].

### 3.5. Derivatives of Eisenstein Series at the Central Point.

In this subsection the derivative of the Eisenstein series and its Fourier expansion will be investigated. While further estimates will be provided in the next section. We start from the vanishing property of \( E' \left( z, \frac{1}{2}; a, b \right) \) as follows.

**Lemma 26.** Let \( a \) and \( b \) be fractional ideals of \( F \) as before. Then we have
\[ E' \left( z, \frac{1}{2}; a, b \right) = \frac{\gamma_F}{N_{F/Q}(b)} \left\{ 2 h_F^{-1} \log y \sigma + \frac{4(\gamma_F[b], 1 - h_F^{-1} \gamma_F)}{h_F \rho_F} \right. \]
\[ + h_F^{-1} \left[ \log N_{F/Q}(ab^{-1}) - n(\gamma + 2 \log 2) \right] \]
\[ + \frac{2^{n+1}}{\rho_F \sqrt{D_F}} \sum_{b \in F^+} \sum_{a^{-1} \in C} \sum_{e(bx) \prod_{v \in J_{\infty}} K_0(2\pi y_v | b_v)} \left\{ \right. \]
where the Euler-Kronecker constants \( \gamma_F[b], 1 \) and \( \gamma_F \) are defined in (30) and (31) respectively; \( \gamma = 0.57721 \ldots \) is the Euler-Mascheroni constant and
\[ C := (ab \cap F^+) / O_F^+. \]
Proof. To simplify the computation, let’s introduce some notation. Set
\[ M_1(s) := N_F/Q(b)^{-2s}y^{\sigma_F(2s, [b]^{-1})}/\zeta_F(2s); \]
\[ M_2(s) := \left(\frac{\sqrt{\pi \Gamma(s - 1/2)}}{\Gamma(s)}\right)^n D_F^{-\frac{s}{2}} N(b)^{-1} y^{(1-s)\sigma} N(ab)^{1-2s} \zeta_F(2s - 1, [ab]^{-1})/\zeta_F(2s); \]
\[ E(s) := \left(\frac{2\pi^s}{\Gamma(s)}\right)^n D_F^{1/2} N(b)\zeta_F(2s) \sum_{b \in F^*} |N(b)|^{-1/2} \lambda(b) e(bx) \]
\[ \times \prod_{v \in A_F} K_{s - \frac{1}{2}}(2\pi y_v |b_v|) \].

Note that for convenience, we short the notation of the norm $N_{F/Q}$ just for $N$ occasionally. Then clearly we have
\[ E'(z, 1/2; a, b) = M'_1(1/2) + M'_2(1/2) + E'(1/2). \]

We have
\[ M'_1(s) = \left[ -2 \log N(b) + \log y^{\sigma}\right] \cdot M_1(s) + N(b)^{-2s} y^{\sigma} \left(\frac{\zeta_F(2s, [b]^{-1})}{\zeta_F(2s)}\right) \]
\[ \left(\frac{\zeta_F(2s, [b]^{-1})}{\zeta_F(2s)}\right)' \Bigg|_{s = \frac{1}{2}} = \lim_{s \to \frac{1}{2}^\pm} \frac{2\zeta_F'(2s, [b]^{-1})\zeta_F(2s) - 2\zeta_F'(2s)\zeta_F(2s, [b]^{-1})}{\zeta_F(2s)} \]
\[ = \lim_{s \to \frac{1}{2}^\pm} \frac{4\rho_F}{(2s-1)^2} \left( \frac{h_F^{-1} \rho_F}{2s-1} + \gamma_F, [b]^{-1} \right) - \frac{4h_F^{-1} \rho_F}{(2s-1)^2} \left( \frac{\rho_F}{2s-1} + \gamma_F \right) \]
\[ = \frac{4(\gamma_F, [b]^{-1} - h_F^{-1} \gamma_F)}{\rho_F}. \]

Note that by (30) and (31) we have
\[ M_1\left(\frac{1}{2}\right) = \frac{y_F}{N(b)} \lim_{s \to \frac{1}{2}^\pm} \frac{\zeta_F(2s, [b]^{-1})}{\zeta_F(2s)} = \frac{y_F}{h_F N(b)}. \]

Thus we have
\[ (40) \quad M'_1(1/2) = \frac{y_F}{h_F N(b)} \left\{ \log \frac{y^{\sigma}}{N(b)} + \frac{4(\gamma_F, [b]^{-1} - h_F^{-1} \gamma_F)}{\rho_F} \right\}. \]

For the $M'_2(1/2)$-term, by definition we have
\[ \log M_2(s) = C + n \log \Gamma(s - 1/2) - n \log \Gamma(s) + (1-s) \log y^{\sigma} \]
\[ + (1-2s) \log N(ab) + \log \zeta_F(2s - 1, [ab]^{-1}) - \log \zeta_F(2s), \]

where $C := n \log \sqrt{\pi} - 1/2 \log D_F - \log N(b)$. From this identity we obtain
\[ M'_2(s) = M_2(s) \left\{ \frac{n\Gamma'(s - \frac{1}{2})}{\Gamma(s - \frac{1}{2})} - \frac{n\Gamma'(s)}{\Gamma(s)} - \log y^{\sigma} - 2s \log N(ab) \right. \]
\[ + \left. \frac{2\zeta_F'(2s - 1, [ab]^{-1})}{\zeta_F(2s - 1, [ab]^{-1})} - \frac{2\zeta_F'(2s - 1)}{\zeta_F(2s - 1)} \right\}. \]
Since $\Gamma(s - 1/2) \sim (s - 1/2)^{-1}$ around $s = 1/2$ and noting (31) and (32), we thus have

\[
M'_2(1/2) = M_2(1/2) \lim_{s \to 1/2} \left\{ -\frac{n}{s - 1/2} - \frac{n\Gamma'(1/2)}{\Gamma(1/2)} - \log y^\sigma \\
- 2s \log N(ab) + \frac{n - 1}{s - 1/2} + \frac{1}{s - 1/2} \right\}
\]

\[= -M_2(1/2) \cdot \left\{ \frac{n\Gamma'(1/2)}{\Gamma(1/2)} + \log y^\sigma + \log N(ab) \right\}.
\]

By (31) and (32) and the functional equation (33) one can easily deduce that

\[
M_2(1/2) = -h_F^{-1} N(b)^{-1} y^{\sigma/2}.
\]

Hence we have

\[
(41) \quad M_2'(1/2) = h_F^{-1} N(b)^{-1} y^{\sigma/2} \cdot \left\{ \frac{n\Gamma'(1/2)}{\sqrt{\pi}} + \log y^\sigma + \log N(ab) \right\}.
\]

Now let's compute $\Gamma'(1/2)$: recall that $\Gamma(s)^{-1}$ is an entire function with the following Hadamard decomposition:

\[
\frac{1}{\Gamma(s)} = se^{\gamma s} \prod_{n=1}^\infty \left( 1 + \frac{s}{n} \right) e^{-\frac{s}{n}}.
\]

Differentiating it logarithmically at $s = 1$ we see $\Gamma'(1) = -\gamma$ by the definition of $\gamma$. Since $\Gamma(s + 1) = s\Gamma(s)$, we have $\Gamma'(2) = 1 - \gamma$. Now consider the duplication formula:

\[
\Gamma(2s) = \pi^{-1/2} n^{2s-1} \Gamma(s) \Gamma(s + 1/2).
\]

Differentiating it at $s = 1/2$ we thus obtain $\Gamma'(1/2) = -\sqrt{\pi}(\gamma + 2 \log 2)$. Plug this into (41) to obtain

\[
(42) \quad M_2'(1/2) = h_F^{-1} N(b)^{-1} y^{\sigma/2} \cdot \left\{ \log y^\sigma + \log N(ab) - n(\gamma + 2 \log 2) \right\}.
\]

Finally we deal with $E'(1/2)$-term. Noting that $\lim_{s \to 1/2} \zeta_F^{-1}(2s) = 0$, we have

\[
E'(1/2) = -\frac{2n y^\sigma}{D_F^{1/2} N(b) \rho_F} \lim_{s \to 1/2} \frac{\zeta_F'(2s)}{\zeta_F(2s)} \sum_{b \in F \times} |N(b)|^{s - 1/2} \lambda(b, s) e(bx)
\]

\[
\times \prod_{v \in J_\infty} K_{s - \frac{1}{2}}(2\pi y_v | b_v|)
\]

\[
= \frac{2n + 1 y^\sigma}{D_F^{1/2} N(b) \rho_F} \sum_{b \in F \times} \lambda \left( h, \frac{1}{2} \right) e(bx) \prod_{v \in J_\infty} K_{s - \frac{1}{2}}(2\pi y_v | b_v|)
\]

\[
= \frac{2n + 1 y^\sigma}{D_F^{1/2} N(b) \rho_F} \sum_{b \in F \times} \sum_{a \in b^{-1} \times \mathcal{C}} e(bx) \prod_{v \in J_\infty} K_0(2\pi y_v | b_v|).
\]

Combining this formula with (40) and (42) we thus obtain the conclusion. \[\Box\]

4. PROOF OF THE MAIN THEOREMS

4.1. Estimates Related to L-functions. Let $F$ be a totally real number field of degree $n$. Let $K/F$ be a CM extension and $\hat{\mathcal{O}}(K)$ be the dual group of the ideal

\[
\text{...}
\]
Proposition 27. Let notation be as above, and \( \chi_0 \) be the trivial character in \( \Cl(K) \), then we have

\[
L_K \left( \chi_0, \frac{1}{2} \right) \geq \frac{\rho_F}{|O_K^\times : O_F^\times|} \left( \frac{1}{2} \log \frac{D_K}{D_F} - \Phi_F^0 \cdot h \right)
\]

where

\[
\Phi_F^0 := \frac{2^{m/2}M(0,n)D_F^{7/4}h_F^2}{\pi^n \rho_F h_F^2} + e^{2\rho_F^2 L_F + n \log D_F + (3 \log 2 - \log \pi)n + \sqrt{\frac{n}{2}} + 4}
\]

and \( L_F = 4\rho_F^{-1}L_F + \log D_F + (3 \log 2 - \log \pi)n + \sqrt{\frac{n}{2}} + 4 \).

Proof. By Lemma 26 we have \( E'(\zeta; \frac{1}{2}, f_a) = I_M(z; f_a) + I_E(z; f_a) \), where

\[
I_M(z; f_a) := \frac{\gamma_n}{N_{F/Q}(f_a)h_F} \left\{ 2\log y_{\sigma} + \frac{4\gamma_n f_a^{-1}}{\rho_F} - 2\log N_{F/Q}(f_a) - (\gamma + 2\log 2)n \right\}
\]

\[
I_E(z; f_a) := \frac{\gamma_n}{N_{F/Q}(f_a)} \frac{2^{n+1}}{\rho_F \sqrt{D_F}} \sum_{b \in F_{\infty}} \sum_{c \in F_{\infty}} \text{e}(bx) \prod_{v \in J_{\infty}} K_0(2\pi y_v|b_v)),
\]

where \( C := O_F^\times / O_F^{\times, +} \), and

\[
L_{F,[f_a]^{-1}} := \gamma_{F,[f_a]^{-1}} - h_{F}^{-1} \gamma_F = \frac{1}{h} \sum_{\chi \in \Cl(F)} \chi([f_a])L_F(\chi, 1).
\]

By Corollary 25 we can write \( L_K(\chi, 1/2) = L_{M, \chi} + L_{E, \chi} \), where

\[
L_{M, \chi} = \frac{2^{\frac{n}{2}} \sqrt{D_F}}{2D_K} \frac{\rho_F}{|O_K^\times : O_F^\times|} \sum_{[a^{-1}] \in \Cl(K)} \chi([a])N_{F/Q}(f_a)^{-1} \cdot I_M(z; f_a),
\]

\[
L_{E, \chi} = \frac{2^{\frac{n}{2}} \sqrt{D_F}}{2D_K} \frac{\rho_F}{|O_K^\times : O_F^\times|} \sum_{[a^{-1}] \in \Cl(K)} \chi([a])N_{F/Q}(f_a)^{-1} \cdot I_E(z; f_a).
\]

We will start with bounding \( I_E(z; f_a) \) and further estimating \( L_{E, \chi} \):

\[
I_E(z; f_a) \leq \frac{\gamma_n}{N(f_a)} \frac{2^{n+1}}{\rho_F \sqrt{D_F}} \sum_{b \in F_{\infty}} \sum_{c \in F_{\infty}} \prod_{v \in J_{\infty}} K_0(2\pi y_v|b_v))
\]

\[
= \frac{\gamma_n}{N(f_a)} \frac{2^{n+1}|O_F^\times : O_F^{\times, +}|}{\rho_F \sqrt{D_F}} \sum_{b \in F_{\infty}} \prod_{v \in J_{\infty}} K_0(2\pi y_v|b_v)).
\]

To compute \( |O_F^\times : O_F^{\times, +}| \), let’s fix an ordering \( (\phi_1, \ldots, \phi_n) \) of \( Hom(F, \mathbb{Q}) \) and consider the homomorphism

\[
\tau : O_F^\times \to \{\pm 1\}^n, \quad x \mapsto (\phi_1(x)/|\phi_1(x)|, \ldots, \phi_n(x)/|\phi_n(x)|).
\]

Then clearly, \( \ker(\tau) = O_F^{\times, +} \). Hence \( \text{Im}(\tau) \simeq O_F^\times / O_F^{\times, +} \). In fact, by Lemma 11.2 of [5] we have

\[
\text{coker}(\tau) \simeq \text{Gal}(H_F^+/H_F) \simeq \ker(\Cl(F)^+) \to \Cl(F),
\]
where $H_F$ is the Hilbert class field of $F$ and $H_F^+$ is the narrow Hilbert class field of $F$. By the above isomorphism, we have
\[
[\mathcal{O}_F^\times : \mathcal{O}_F^{\times, +}] = \frac{2^n h_F}{h_F^+}.
\]
Noting that $F$ is totally real, consider the canonical embedding
\[
j : F \rightarrow \mathbb{R} := \prod_{v \in J_{\infty}} F_v \cong \mathbb{R}^n.
\]
Since $f_{a}^{-1} \varphi^{-1} \neq 0$, the lattice $\Gamma := j(f_{a}^{-1} \varphi^{-1})$ is complete in $F_{\mathbb{R}}$. Let $\alpha_1, \ldots, \alpha_n$ be a $\mathbb{Z}$-basis of $f_{a}^{-1} \varphi^{-1}$. Let $\beta_v = j(\alpha_v)$, then we may assume that $\beta_v > 0$, $\forall 1 \leq v \leq n$. Since $\mathbb{Z}$ is a PID, we have
\[
\Gamma = \mathbb{Z} \beta_1 \oplus \cdots \oplus \mathbb{Z} \beta_n.
\]
Then a computation with determinants gives
\[
\prod_{v \in J_{\infty}} \beta_v^{-1} = \text{vol}(\Gamma)^{-1} = \frac{1}{\sqrt{D_F}} \cdot N_{F/Q}(f_a) = N_{F/Q}(f_a) \sqrt{D_F}.
\]
For any $b \in f_{a}^{-1} \varphi^{-1} \cap F^{\times}$, we may write
\[
j(b) = (m_1 \beta_1, \ldots, m_n \beta_n), \text{ where } m_i \neq 0, \forall 1 \leq v \leq n.
\]
Otherwise, we may assume $m_1 = 0$. Then there exists some $v \in J_{\infty}$ such that $b_v = 0$. Then the minimal polynomial of $b$ has 0 as its root. This is impossible unless $b = 0$.

On the other hand, note that $K_0(x) < K_{1/2}(x) = \sqrt{\frac{2}{\pi}} e^{-x}, \forall x > 0$. Combining these results we have
\[
\left| I_{E}(z; f_a) \right| \leq \frac{y^\frac{p}{2}}{N_{F/Q}(f_a)} \cdot \frac{2^{n+1} h_F}{\rho_{F} h_F^+ \sqrt{D_F}} \prod_{v \in J_{\infty}} \left( \sum_{m=1}^{\infty} \sqrt{\frac{1}{m}} \beta_v^{-1/2} \prod_{v \in J_{\infty}} \left( \sum_{m=1}^{\infty} \sqrt{\frac{1}{m}} e^{-2\pi y_v \beta_v} \right) \right)
\leq \frac{1}{N_{F/Q}(f_a)} \cdot \frac{2^{n+1} h_F}{\rho_{F} h_F^+ \sqrt{D_F}} \prod_{v \in J_{\infty}} \beta_v^{-1/2} \prod_{v \in J_{\infty}} \frac{1}{e^{2\pi y_v \beta_v} - 1}
\leq \frac{1}{N_{F/Q}(f_a)} \cdot \frac{2^{n+1} h_F D_F^{1/4}}{\pi^n \rho_{F} h_F^+} \cdot \frac{y^{-\sigma}}{\sqrt{D_K}}.
\]

Therefore, by Lemma 19 and the definition of $I_K$, we have
\[
\left| I_{E}(z; f_a) \right| \leq \frac{2^{n+1} h_F D_F^{1/4}}{\pi^n \rho_{F} h_F^+} \cdot \frac{2^n D_F N_{K/Q}(a)}{N_{F/Q}(f_a) N_{F/Q}(Q)^2 \sqrt{D_K}} \leq \frac{2^{n+1} M(0, n) h_F D_F^{5/4}}{\pi^n \rho_{F} h_F^+ \sqrt{N_{F/Q}(f_a)}}, \quad \forall a \in I_K.
\]
Note that by definition, each $f_a \in \mathcal{I}_K^+$ defined in (20). Thus we have

$$|L_{E,\gamma}| \leq \frac{2^{2\gamma}D_F}{2D_K^{1/2}} \frac{\rho_F}{[O_K^* : O_F^*]} \sum_{a \in \mathcal{I}_K} \frac{2^{2n+1}M(0,n)h_F D_F^{5/4}}{\pi^n \rho_F \eta_F}$$

$$\leq \frac{1}{[O_K^* : O_F^*]} \frac{2^{n/2}M(0,n)D_F^{1/4} h_F}{\pi^n \eta_F} \cdot h_K D_{K}^{-1/4},$$

where $\mathcal{I}_K$ is defined in (34), so that (35) is available here.

On the other hand, we will give a lower bound for $I_M(z; f_a)$ and $L_{M,\gamma}$. Recall the definition of $L_F$ given in (43), then clearly for any $a \in \mathcal{I}_K$, one has

$$L_F \geq \frac{1}{h_F} \sum_{\chi \in \mathcal{C}(\mathbb{F}) \setminus \{\chi_0\}} |L_F(\chi, 1)| \geq |L_{F,[f_a]}|^{-1}|.$$  

Hence by the expression for $y^\gamma$ given in Lemma 19 we obtain

$$I_M(z; f_a) \geq \sqrt{N_{K/F}(a)} N_{F/F}(q)^{2/3} \left\{ 2\log \frac{\sqrt{D_K}}{2^n D_F} - C_{F,a} \right\} - \frac{D_K^{1/4}}{\sqrt{2^n D_F} h_F},$$

where the tail $C_{F,a}$ is defined as

$$C_{F,a} = 2\log \frac{N_{K/F}(a)}{N_{K/F}(q)^2} + \frac{4L_F}{\rho_F} + (\gamma + 2 \log 2)n.$$

Combining (45) with (35) yields

$$L_{M,\gamma_0} \geq \frac{\rho_F}{2[O_K^* : O_F^*] \cdot h_F} \sum_{a \in \mathcal{I}_K} \sqrt{N_{K/F}(c_a) N_{F/F}(q)^2}$$

$$\times \left\{ 2\log \frac{\sqrt{D_K}}{2^n D_F} - 2\log \frac{N_{K/F}(a)}{N_{K/F}(c_a) N_{F/F}(q)^2} - \frac{4L_F}{\rho_F} - (\gamma + 2 \log 2)n \right\}$$

$$= \frac{1}{2} \sum_{a \in \mathcal{I}_K} N_q(a) \left\{ 2\log \frac{\sqrt{D_K}}{D_F} - 2\log \frac{2^n N_{K/F}(a)}{N_{K/F}(c_a) N_{F/F}(q)^2} - C_F \right\},$$

where $N_q(a)$ is defined in (9) and $C_F = 4\rho_F L_F + (\gamma + 2 \log 2)n$.

Write $N_q(a)^* = 2^n N_{K/F}(c_a)^{-1} N_{K/F}(q)^{-2} N_{K/F}(a)$, then we can introduce an undetermined parameter $T$ satisfying $0 < T \leq M(0,n)\sqrt{D_K}$ such that

$$L_{M,\gamma_0} \geq \sum_{a \in \mathcal{I}_K} N_q(a) \cdot \left\{ \log \frac{\sqrt{D_K}}{D_F \cdot N_q(a)^*} - \frac{1}{2} C_F \right\} - L_{M,\gamma_0}^T,$$

where the truncation term $L_{M,\gamma_0}^T$ is defined as

$$L_{M,\gamma_0}^T = \sum_{a \in \mathcal{I}_K} N_q(a) \cdot \left\{ \left| \log \frac{\sqrt{D_K}}{D_F \cdot N_q(a)^*} \right| + \frac{1}{2} C_F \right\}.$$  

We can take $T = T_{K/F}$ to be

$$T_{K/F} = \min \left\{ e^{-C_F} \sqrt{D_K} / D_F, 2^n M(0,n) \sqrt{D_K} \right\}.$$  

Then due to (19) and (20) one has $T_{K/F} = e^{-C_F} D_F^{-1} \sqrt{D_K}$ if $n \geq 9$.

Note that the choice of $T_{K/F}$ above implies that

$$\log \frac{\sqrt{D_K}}{D_F \cdot N_q(a)^*} \geq C_F, \quad \forall a \in \mathcal{I}_K \text{ such that } N_q(a)^* \leq T_{K/F}.$$
Noting that $N_q(a)^* \leq M(0,n)\sqrt{D_K}$, one then define

$$\Phi_F = \max_{T_{K/F} \leq N_q(a)^* \leq 2^n M(0,n)\sqrt{D_K}} \log \frac{\sqrt{D_K}}{D_F \cdot N_q(a)^*} + \frac{1}{2} C_F.$$ 

Then by monotonicity we obtain

$$(46) \quad \Phi_F = \max \left\{ \frac{3}{2} C_F, \log 2^n M(0,n)D_F + \frac{1}{2} C_F \right\} \leq \log M(0,n)D_F + 2C_F.$$  

When $e^{-C_F}D_F^{-1}\sqrt{D_K} > 2^n M(0,n)\sqrt{D_K}$ (e.g. when $n \geq 9$), this implies that

$$L_{K/F}^{T_{K/F}} \leq \frac{2^{n/2}\rho_F}{|O_K^* : O_F^*| \cdot h_F} \sum_{n \in T_{K/F}} \frac{1}{\sqrt{N_q(a)^*}} \Phi_F$$

and when $e^{-C_F}D_F^{-1}\sqrt{D_K} > 2^n M(0,n)\sqrt{D_K}$ (according to our discussion before, this might happen only when $n \leq 8$), we just take $L_{K/F}^{T_{K/F}} = 0$. Hence, we have

$$L_{M,\chi_0} \geq \frac{1}{2} \sum_{a \in \mathbb{Z}_K} N_q(a) \cdot \log \frac{\sqrt{D_K}}{D_F \cdot N_q(a)^*} - L_{K/F}^{T_{K/F}}$$

$$\geq \frac{1}{2} N_q(q) \cdot \log \frac{\sqrt{D_K}}{D_F} \geq \frac{\rho_F \Phi_F e^{C_F/2}}{|O_K^* : O_F^*| \cdot h_F} \sqrt{D_F \cdot h_K D_K^{-1/4}}.$$  

Since $N_q(q) = |O_K^* : O_F^*| \cdot \rho_F h_F^{-1}$, then involving the upper bound of $|L_E,\chi|$ developed as before we have

$$(47) \quad L_K \left( \chi, 0, \frac{1}{2} \right) \geq \frac{\rho_F \Phi_F e^{C_F/2}}{\pi^{n/2} h_F} \left( \frac{1}{2} \log \frac{\sqrt{D_K}}{D_F} - \Phi_F \cdot h_K D_K^{-1/4} \right).$$  

where

$$\Phi_F := 2 \left[ \frac{n}{2} \log D_F + 3 \log 2 - \log \pi n + \sqrt{\pi} n + 4 \right].$$

By (19) and the usual Minkowski constant $M(n)$ one can obtain an elementary computation of $M(0,n)$, substituting this bound into (46) leads to the inequality

$$\Phi_F \leq 2C_F + \log e^{-2\gamma + \log 2\pi n + (\sqrt{\pi} n + 4)D_F},$$

from which one then has

$$(48) \quad \Phi_F \leq 4\rho_F \cdot L_F + \log D_F + (3 \log 2 - \log \pi n + \sqrt{\pi} n + 4).$$

Then the proof follows from the estimate (47) and inequalities (48). \hfill \square

From Proposition 27 one sees naturally that we need an upper bound for $|L_F(\chi, 1)|$ to make the inequality (44) more explicit. The desired estimate is provided in the following lemma.

**Lemma 28.** Let $F/\mathbb{Q}$ be any field extension of degree $n < +\infty$. Let $\chi$ be any nontrivial primitive Grossencharacter of modulus $m$, then we have

$$(49) \quad \left| L_F(\chi, 1) \right| \leq \left[ \frac{e^{2n} \log (D_F N_{F/\mathbb{Q}}(m))}{2n} \right]^n.$$  

Moreover, if for any $\chi \in \text{Cl}(F) \setminus \{ \chi_0 \}$, we have

$$(50) \quad \left| L_F(\chi, 1) \right| \leq \left( 1 + \frac{3 + 2\log 2 - \log \pi n + \frac{1}{2} \log D_F + \rho_F \cdot \gamma_F }{2} \right) \cdot \rho_F,$$

where $\rho_F := \text{Res}_{s=1}^{F}(s)$ and $\gamma_F$ is the Euler-Kronecker constant of $F/\mathbb{Q}$. \hfill \Box
Proof. Denote by \((r_1, r_2)\) the signature of \(F/\mathbb{Q}\). Let \(b\) be the number of real places of \(F\) dividing the infinite part of the conductor of \(\chi\). Let \(a := r_1 - b\). Then \(a \geq 0\) and \(b \geq 0\). Consider the completed Hecke L-function associated with \(\chi\):

\[
\Lambda_F(\chi, s) := \left(DF, N_{F/\mathbb{Q}}(m)\right)^{s/2} L_{F, \infty}(\chi, s)L_F(\chi, s),
\]

where \(L_{F, \infty}(\chi, s)\) is the infinite part of \(L_F(\chi, s)\), i.e.,

\[
L_{F, \infty}(\chi, s) := 2^{-r_2}s^{-\alpha s/2}\Gamma(s/2)\Gamma((s + 1)/2)^b \Gamma(s)^{r_2}.
\]

Then \(\Lambda_F(\chi, s)\) is an entire function satisfying the functional equation:

\[
\Lambda_F(\chi, s) = W_\chi \Lambda_F(\overline{\chi}, 1 - s),
\]

where \(W_\chi \in \mathbb{C}\) is the root number with \(|W_\chi| = 1\).

Let \(\Theta_\chi(x)\) be the inverse Mellin transform of \(\Lambda_F(\chi, s)\). Then one can verify easily that (53) yields the functional equation of \(\Theta_\chi:\)

\[
\Theta_\chi(x) = x^{-1}W_\chi \Theta_\chi(x^{-1}).
\]

Then by Mellin transform and (54) we have the integral representation of \(\Lambda_F(\chi, s)\):

\[
\Lambda_F(\chi, s) = \int_1^\infty x^{-s} \Theta_\chi(x) dx = \int_1^\infty x^{s-1} \Theta_\chi(x) dx + W_\chi \int_1^\infty x^{-s} \Theta_\chi(x) dx.
\]

Since \(\Gamma(s)\) is the Mellin transform of the function \(f(x) = e^{-x}\), \((x > 0)\), the inverse Mellin transform of the \(g_{\lambda, \mu}(s) := \Gamma(\lambda s + \mu), (\forall \lambda > 0, \mu \in \mathbb{R})\) is

\[
(\mathcal{M}^{-1}g_{\lambda, \mu})(x) = \lambda^{-1}x^{\mu/\lambda}e^{-x^{1/\lambda}}, \quad \lambda > 0, \mu \in \mathbb{R}.
\]

Clearly, \(\mathcal{M}^{-1}g_{\lambda, \mu}\) is positive. Since the parameters \(a, b\) and \(n\) and \(m\) are nonnegative integers, so we can regard \(\Gamma(s/2)^a \Gamma((s + 1)/2)^b \Gamma(s)^{r_2}\) as a product of Gamma functions of the form \(\Gamma(\lambda s)\), \(\lambda > 0\). Let \(\Theta_{\chi, \infty}\) denote the inverse Mellin transform of \(2^{-r_2}s^{-\alpha s/2} L_{F, \infty}(\chi, s) = \Gamma(s/2)^a \Gamma((s + 1)/2)^b \Gamma(s)^{r_2}\), then we have

\[
\Theta_{\chi, \infty}(x) = \left(\mathcal{M}^{-1,a}g_{1/2,0} * \mathcal{M}^{-1,b}g_{1,2,1/2} * \mathcal{M}^{-1,r_2}g_{1,0}\right)(x), \quad \forall x > 0.
\]

Note that by definition one has

\[
\Theta_\chi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \Lambda_F(\chi, s) ds = \sum_{\mathfrak{p} \neq \mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a}) \Theta_{\chi, \infty} \left(\frac{2^{r_2}s^{n/2}N_{F/\mathbb{Q}}(\mathfrak{a})}{\sqrt{DF, N_{F/\mathbb{Q}}(\mathfrak{m})}} \cdot x\right).
\]

By the definition of \(a\) and \(b\) we see \(\Theta_{\chi, \infty} = \Theta_{\chi, \infty}\). Since \(\Theta_{\chi, \infty}\) is positive, we have \(\Theta_\chi = \overline{\Theta}_\chi\). Thus for any \(s \geq \frac{1}{2}\), one obtains

\[
\left|\Lambda_F(\chi, 1)\right| \leq \int_1^{\infty} \Theta_\chi(x) dx + \int_1^{\infty} x^{-1} \Theta_\chi(x) dx
\]

\[
\leq 2 \int_1^{\infty} \sum_{\mathfrak{p} \neq \mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a}) \Theta_{\chi, \infty} \left(\frac{2^{r_2}s^{n/2}N_{F/\mathbb{Q}}(\mathfrak{a})}{\sqrt{DF, N_{F/\mathbb{Q}}(\mathfrak{m})}} \cdot x\right) dx
\]

\[
\leq 2 \sum_{\mathfrak{p} \neq \mathfrak{a} \subset \mathcal{O}_F} \int_0^{\infty} x^{s-1} \Theta_{\chi, \infty} \left(\frac{2^{r_2}s^{n/2}N_{F/\mathbb{Q}}(\mathfrak{a})}{\sqrt{DF, N_{F/\mathbb{Q}}(\mathfrak{m})}} \cdot x\right) dx
\]

\[
= 2 \left(DF, N_{F/\mathbb{Q}}(\mathfrak{m})\right)^{s/2} L_{F, \infty}(\chi, s) \zeta_F(s).
\]

From this inequality we obtains the upper bound for \(L_F(\chi, 1)\):

\[
\left|L_F(\chi, 1)\right| \leq 2 \left(DF, N_{F/\mathbb{Q}}(\mathfrak{m})\right)^{s/2} \frac{L_{F, \infty}(\chi, s)}{L_{F, \infty}(\chi, 1)} \cdot \zeta(s)^n, \quad \forall s > 1.
\]
Let \( H(s) := s^{a+r_2}(s-1)^n \frac{L_F(s \chi)}{L_F(s \chi_1)} \cdot \zeta(s)^n = \zeta(s)^n G(s)^{b+r_2} \), where
\[
\xi(s) := s(s-1)\pi^{-s/2} \Gamma(s/2) \zeta(s), \quad \text{and} \quad G(s) := \frac{\sqrt{\pi} \Gamma((s+1)/2)}{s \Gamma(s/2)}.
\]
Then by (56) we have
\[
\left| L_F(\chi, 1) \right| \leq \frac{2 (D_F N_F/Q(m))^{\frac{a}{n-1}}}{{s_0}^{a-r_2}} H(s_0),
\]
where \( s_0 := 1 + 2n \lfloor \log (D_F N_F/Q(m)) \rfloor^{-1} \).

Recall that we have the well known Hadamard decomposition for entire functions
\[
\frac{1}{\Gamma(s)} = s e^{-\gamma s} \prod_{n=1}^{\infty} \left( 1 + \frac{s}{n} \right) e^{-s/n}, \quad \text{and} \quad \xi(s) = e^{B s} \prod_{\rho \neq 0} \left( 1 - \frac{s}{\rho} \right) e^{s/\rho},
\]
where \( B \in \mathbb{C} \) is a constant and \( \rho \) runs through nontrivial zeros of \( \xi(s) \). Then
\[
G'(s)/G(s) = (\log G(s))' = \sum_{j=1}^{\infty} (-1)^j (j + s)^{-1} \leq 0, \quad \forall \ s > 0.
\]
This gives that \( \log G''(s) = \sum_{j=1}^{\infty} (-1)^j (j + s)^{-2} > 0 \) when \( s > 0 \). Hence \( \log G \) is convex when \( s > 0 \). Now we work out \( B \) and thus see for every nontrivial zero \( \rho = \sigma + i \tau \), we have \( |\tau| \geq 6 \). In fact, by definition, \( B = \xi(0)'/\xi(0) \). The functional equation \( \xi(s) = \xi(1-s) \) gives \( \xi(s)'/\xi(s) = -\xi(1-s)'/\xi(1-s) \). Thus \( B = -\xi(1)'/\xi(1) \). Therefore,
\[
B = \frac{1}{2} \log \pi - \frac{1}{2} \Gamma(\frac{3}{2}) - \lim_{s \to 1^+} \left( \frac{\xi'(s)}{\xi(s)} + \frac{1}{s-1} \right) = -1 - \gamma + \frac{1}{2} \log 4\pi.
\]
On the other hand, by \( B = -\xi(1)'/\xi(1) \) and symmetry of the nontrivial zeros, we have
\[
B = -\frac{1}{2} \sum_{\rho} \left( \frac{1}{1-\rho} + \frac{1}{\rho} \right) = -\sum_{\rho = \sigma + i \tau} \frac{1}{\rho} = -\sum_{\rho = \sigma + i \tau} \frac{2\sigma}{\sigma^2 + \tau^2}.
\]
Then for any \( \rho = \sigma + i \tau \) with \( 1/2 \leq \rho \leq 1 \), one has \( -B \geq 2\sigma \cdot (\sigma^2 + \tau^2)^{-1} \), which gives the lower bound
\[
|\tau| \geq \sqrt{\frac{2\sigma}{-B}} - \sigma^2 \geq \sqrt{\frac{1}{-B} - \frac{1}{4}} \geq 6.
\]
Thus the function
\[
h(s) := \sum_{\rho = \sigma + i \tau} \frac{1}{s-\rho} = \sum_{\rho = \sigma + i \tau} \frac{s-\sigma}{(s-\sigma)^2 + \tau^2}, \quad 1 \leq s \leq 6,
\]
is increasing. So \( \log \xi(s)'/\xi(s) = B + h(1) + h(s) \) is increasing when \( 1 \leq s \leq 6 \). Then \( \log \xi(s) \) is convex when \( 1 \leq s \leq 6 \). Therefore we have \( \log (s^{-a-r_2} H(s)) \) is convex when \( 1 \leq s \leq 6 \).

By [49] there exists an integral ideal \( \mathfrak{a} \) such that \( N_{F/Q}(\mathfrak{a}) \leq M(r_1, r_2) \sqrt{D_F} \). Then clearly
\[
\log (D_F N_{F/Q}(m)) \geq \log D_F \geq M(r_1, r_2)^{-1}.
\]
Recall that \( M(r_1, r_2) \leq M(n) := \frac{4^{r_2}n!}{\pi^{r_2} n^{r_2}} \). Hence
\[
s_0 = 1 + 2n \lfloor \log (D_F N_{F/Q}(m)) \rfloor^{-1} \leq 1 + 2n M(r_1, r_2)
\leq 1 + 2n \cdot \frac{4^{r_2}n!}{\pi^{r_2} n^{r_2}} \leq 1 + 2n \cdot \frac{4^n n!}{\pi n^n} \leq 6.
\]
By convexity we have \( H(s_0) \leq \max \{H(1), H(6)\} = 1 \).
When $\chi \in \widehat{Cl}(F)$ is a nontrivial Hilbert character, one has, by class field theory, that

$$\text{Cl}(F) \simeq \text{I}_F/\text{I}_F^\infty F^\times,$$

where

$$\text{I}_F^\infty := \prod_{p \mid \infty} F_p^\infty \times \prod_{p \mid \infty} \mathcal{O}_{F_p}^\times.$$

So in this case $b = 0$, and $a = r_1$, and $m = \mathcal{O}_F$. Then we have the completed L-function (51), where $N_{F/Q}(m) = 1$ and (52) becomes

$$L_{F,\infty}(\chi, s) := 2^{-r_2} \pi^{-ns/2} \Gamma(s/2)^r \Gamma(s)^{r^2}.$$

As before, noting that $\Theta_\chi = \overline{\Theta}$, by Mellin transform and functional equations we have

$$\Gamma(s/2)^r \Gamma(s)^{r^2} \zeta_F(s) \text{ where } \zeta_F(s) \text{ is the inverse Mellin transform of } \zeta(s).$$

Recall that combining the elementary bound and functional equation of $\Lambda(s)$, and by the Phragmén–Lindelöf theorem we have the fact that

$$\zeta_F(s) \ll |t|^{1/2-\sigma} \log |t|, \quad 0 \leq \sigma \leq 1, \quad |t| \geq 2.$$

By functional we have $\zeta_F(s) = \zeta(s)$. Note that in the area $S := \{s = \sigma + it : 1 - c \leq \sigma \leq c, t \geq 1\}$ we have uniformly that

$$\Gamma(s) = \sqrt{2\pi e^{-\frac{1}{2} t \log |t|}} \exp \left( \frac{\log |t| - 1}{2} \right) \left( 1 + O_c \left( |t|^{-1} \right) \right).$$

Therefore, $\Lambda_F(s)$ decays exponentially in $S$ when $|t| \to \infty$. Thus we can shift the contour to get

$$I_F = \text{Res}_{s=1} \Lambda_F(s) + \text{Res}_{s=1} \Lambda_F(s) + \frac{1}{2\pi i} \int_{c-i\infty}^{1-c+i\infty} \Lambda_F(s) ds$$

$$= 2 \text{Res}_{s=1} \Lambda_F(s) - \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Lambda_F(s) ds.$$

Hence $I_F = \text{Res}_{s=1} \Lambda_F(s) = \Lambda_{F,1} + \Lambda_{F,2}$, where

$$\Lambda_{F,1} := \lim_{s \to 1} (s - 1) \cdot \Lambda_F(s), \quad \text{and } \Lambda_{F,2} := \lim_{s \to 1} (s - 1) \cdot \Lambda_F(s)'.$$
By the Laurent expansion of $\zeta_F(s)$ at $s = 1$ we have $\lim_{s \to 1} \left( \frac{1}{s-1} + \zeta_F(s) \right) = \rho_F^{-1}\gamma_F$. Hence
\[
\Lambda_{F,2}/\Lambda_{F,1} = (\log \Lambda_F(s))' \big|_{s=1} = \left( \frac{1}{2} \log D_F - r_2 \log 2 - \frac{n}{2} \log \pi + \frac{r_1}{2} \Gamma' \left( \frac{1}{2} \right) + r_2 \Gamma' \left( \frac{1}{2} \right) \right) + \gamma + 2 \log 2 - \log \pi \leq \frac{\log \pi}{2} - \frac{\gamma}{2} + \rho_F^{-1}\gamma_F.
\]
Then by the inequality $|\Lambda_F(\chi, 1)| \leq I_F$ we obtain (50). \hfill \Box

Now we move on to handle the Euler-Kronecker constant $\gamma_F$. Clearly we need an upper bound for it. The known result on upper bounds for $\gamma_F$ is essentially $2\log \log \sqrt{D_F}$, which is established under GRH (ref. [26]). To prepare for the proof of Theorem 1, we give an elementary unconditional effective upper bound for $\gamma_F$.

**Lemma 29.** Let notation be as before, then there is an absolute constant $c > 0$ such that
\[
-\frac{1}{2} \log D_F - \frac{\gamma + 2 \log 2 - \log \pi}{2} \leq \gamma_F - n - 1 \leq c \log D_F.
\]

**Remark.** Note that the main term of this lower bound in (58) is $-\frac{1}{2} \log D_F$, which is slightly better than the general result (i.e. lower bound of main term $-\log D_F$) given in [26]. On the other hand, under GRH, one has $\gamma_F \ll \log \log D_F$ according to main theorems in [26].

**Proof.** The lower bound for $\gamma_F$ can be deduced simply from (50). We thus will focus on the upper bound here.

For $s = \sigma + it \in \mathbb{C}$ with $\frac{1}{2} \leq \sigma \leq 1$, one has
\[
(s - 1)\zeta_F(s) \ll \rho_F |sD_F|^{(1-\sigma)/2}.
\]
Note (59) is essentially Theorem 5.31 in [27] without $|sD_F|^{\varepsilon}$. We drop the $\varepsilon$-factor by using a subconvexity bound for $\zeta_F$ at $\sigma = \text{Re}(s) = 1/2$ as an endpoint rather than using the bound for $\zeta_F(it)$ via the functional equation before applying Phragmén–Lindelöf theorem. In fact, a subconvexity result (e.g. ref. [45]) shows that there exists some $\delta$ (e.g. one can take $\delta = 1/200$) such that $\zeta_F(s) \ll |sD_F|^{1/2-\delta}$, where $s = 1/2 + it$. Then Phragmén–Lindelöf theorem implies that
\[
(s - 1)\zeta_F(s) \ll |sD_F|^{(1/2-\delta)(1-\sigma)}, \quad \text{where } s = \sigma + it, 1/2 \leq \sigma < 1.
\]
Hence (59) comes from (60) and Theorem 5.31 in [27].

Let $C$ be the circle centered at $s = 1$ with radius $r = 1 - (2 \log D_F)^{-1}$. Since $\zeta_F$ is a meromorphic function with a simple pole at $s = 1$, then by Cauchy’s theorem we have
\[
\gamma_F = \frac{1}{2\pi i} \oint_C \frac{\zeta_F(s)}{s-1} ds \ll \oint_C \frac{\rho_F}{|s-1|^2} ds \ll \rho_F \log D_F.
\]
Then the proof follows. \hfill \Box

With these preparation we can eventually give a proof of our main theorems.

**4.2. Proof of Theorem 1.**

**Proof.** As before, let $K/F$ be a CM extension and $[F: \mathbb{Q}] = n$. Note that for every $\chi \in \mathcal{O}(K)$, the conductor of $\chi$ is $\mathcal{O}_K$. So we have, by Lemma 28, that
\[
\rho_F^{-1} L_F \leq 1 + \frac{\gamma + 2 \log 2 - \log \pi}{2} n + \frac{1}{2} \log D_F + \gamma_F.
\]
Then Lemma 29 implies that $\rho_F^{-1} \mathcal{L}_F \leq c'_1 \log D_F$ for some absolute constant $c'_1 > 0$, since a classical lower bound for $D_F$ implies that $\log D_F \gg n$.

Likewise, one has $\mathcal{L}_F = 4\rho_F^{-1} \mathcal{L}_F + \log D_F + (4 \log 2 - \pi)n + \sqrt{n} + 4 \leq c'_2 \log D_F$ for some positive absolute constant $c'_2$. Then (4) follows from Proposition 27 and thus (5) follows from (4) and elementary computations of $M(n,0)$ and $M(0, n)$. □

4.3. Asymptotic Behavior of $E'(z, \frac{1}{2}, f_a^{-1}, f_a)$. In this subsection, we will derive an explicit upper bound (under GRH) and an unconditional lower bound of the mean value of $L_K(\chi, 1/2)$. One can see the bounds are quite close. We start with a lower bound of the derivative of Eisenstein series at the central point $s = 1/2$ as follows.

**Lemma 30.** Let notations be as before. If $\alpha$ is any integral ideal in $K$, then we have that

\begin{equation}
\left| E'(z, \frac{1}{2}; f_a^{-1}, f_a) \right| = \frac{2y_a^2}{N(f_a)h_F} \log y_a^\sigma \leq \frac{y_a^2}{N(f_a)h_F} \frac{\log D_F}{h_F} + \frac{h_F D_F^{1/4} \sqrt{\langle f_a \rangle}}{\rho_F h_F^+} y_a^{-\sigma},
\end{equation}

where $z_a$ is the CM point corresponding to $\alpha$, and $y_a$ is the imaginary part of $z_a$ and $y_a^\sigma = \sqrt{\langle f_a \rangle}$ is the CM norm of $y_a$ associated the fixed CM type $\Phi$ given in (23). Moreover, the implied constant in (61) is absolute.

**Proof.** By Lemma 26 we have $E'(z, 1/2; f_a^{-1}, f_a) = I_M(z; f_a) + I_E(z; f_a)$, where $z = z_a$, and

\begin{align*}
I_M(z; f_a) &:= \frac{y_a^2}{N(f_a)h_F} \left\{ 2 \log \frac{y_a^\sigma}{N(f_a)^2} + \frac{4\gamma_F}{\rho_F} - (\gamma + 2 \log 2)n \right\}, \\
I_E(z; f_a) &:= \frac{2^{n+1} y_a^2}{\rho_F \sqrt{D_F N(f_a)}} \sum_{b \in \mathcal{O}_F^+} \sum_{c \in \mathcal{O}_{F/Q}^+} \sum_{\chi \in \mathcal{C}(\mathcal{F})} \epsilon(bz) \prod_{v \in \mathcal{A}_\infty} K_0(2\pi y_v | b_v|),
\end{align*}

where $\mathcal{O}_{F/Q}$ is the different, $\mathcal{C} := \mathcal{O}_{F^+}^2 / \mathcal{O}_{F^+}^2$, and

\[ \gamma_F := \gamma_{F[f_a]} + h_F^{-1} \gamma_F = \frac{1}{h_F} \sum_{\chi \in \mathcal{C}(\mathcal{F})} \chi([f_a]) L_F(\chi, 1). \]

As in the proof of Proposition 27 we have, for any $\alpha \in \mathcal{I}_K$, that

\begin{equation}
|E(z; f_a)| \leq \sqrt{\frac{\langle f_a \rangle}{N(f_a)^2}} \cdot \frac{2^{n+1} h_F D_F^{1/4}}{\pi^n \rho_F h_F^+} y_a^{-\sigma}.
\end{equation}

On the other hand, by definition of $I_M(z; f_a)$ we have

\[ I_M(z; f_a) = \frac{2y_a^2}{N(f_a)h_F} \log y_a^\sigma + C(F, K), \]

where the error term $C(F, K)$ is defined as

\[ |C(F, K)| = \left| \frac{2y_a^2}{N(f_a)h_F} \cdot \left( -2 \log N_{F/Q}(f_a) + \frac{2\gamma_F}{\rho_F} - (\gamma + 2 \log 2)n \right) \right| \leq \frac{2y_a^2}{N(f_a)h_F} \cdot \left\{ 2 \log M(n, 0) \sqrt{D_F} + \frac{2\gamma_F}{\rho_F} - (\gamma + 2 \log 2)n \right\} \leq \frac{4y_a^2}{N(f_a)h_F} \cdot (\log D_F + \gamma_F^2) \leq \frac{y_a^2}{N(f_a)h_F} \cdot \log D_F. \]
The last two inequalities comes from \((50)\) and \((58)\) respectively, and the implied constant in the last estimate is absolute. Then \((61)\) follows from this upper bound and \((62)\).

\[ \square \]

**Remark.** There is another explicit inequality version of \((61)\). To achieve that, instead of computing the constant \(c\) in \((58)\), we may use \((49)\) instead of \((50)\) to avoid it when bounding \(C(F, K)\):

\[
|C(F, K)| \leq \frac{2y_a^\zeta}{N(\{f_a\})h_F} \cdot \left\{ \log D_F + \frac{2\gamma_F}{\rho_F} + \frac{(\gamma + 2\log 2)n}{2} + 2\log M(n, 0) \right\}
\]

\[
\leq \left[ 4\rho_F^{-1} \left( \frac{e}{2n} \log D_F \right)^n + \log D_F + 3n \right] \cdot \frac{2y_a^\zeta}{N(\{f_a\})h_F}.
\]

Then likewise, combining this inequality with \((62)\) we will obtain a formula as \((61)\) with explicit implied constant.

**Corollary 31.** Let notations be as before. Then for any \(\alpha \in \mathcal{I}_K\), we have that

\[
(63) \quad \left| E' \left( z_a, \frac{1}{2}, f_a^{-1}, f_a \right) - \frac{2y_a^\zeta}{N(\{f_a\})h_F} \log y_a^\sigma \right| \ll \frac{y_a^\zeta}{N(\{f_a\})} \cdot \frac{h_F^{-D_F^{1/4}}}{\rho_F}.
\]

where \(z_a\) is the CM point corresponding to \(\alpha\), and \(y_a\) is the imaginary part of \(z_a\) and \(y_a^\sigma = N_{\Phi}(y_a)\) is the CM norm of \(y_a\) associated the fixed CM type \(\Phi\) given in \((23)\). Moreover, the implied constant in \((63)\) is absolute.

**Proof.** As in the proof of Proposition 27 we have, for any \(\alpha \in \mathcal{I}_K\), that

\[
|I_E(z; f_a)| \leq \sqrt{N_{F/Q}(\{f_a\})} \cdot \frac{2^{n+1}h_FD_F^{1/4}}{\pi^3 \rho_F h_F} \cdot \frac{D_F^{1/4}}{\pi^3 \rho_F} \cdot \left( \frac{M(0, n)^{-1}N_{F/Q}(\{f_a\})}{2^n D_F} \right)^{-3/2}.
\]

The last inequality comes from \((35)\). Hence we have

\[
(64) \quad |I_E(z; f_a)| \leq \frac{y_a^\zeta}{N(\{f_a\})} \cdot \frac{2^{n+1}M(0, n)^{-1/2}h_F^{-D_F^{1/4}}}{\pi^3 \rho_F} \ll \frac{y_a^\zeta}{N(\{f_a\})} \cdot \frac{h_F^{-D_F^{1/4}}}{\rho_F}.
\]

where the last estimate comes from bounds for \(M(0, n)\), and the implied constant is absolute. Combining this upper bound with \((62)\) implies

\[
\left| E' \left( z_a, \frac{1}{2}, f_a^{-1}, f_a \right) - \frac{2y_a^\zeta}{N(\{f_a\})h_F} \log y_a^\sigma \right| \ll \frac{y_a^\zeta}{N(\{f_a\})} \cdot \left( \frac{h_F^{-1} \log D_F + h_F^{-D_F^{1/4}}}{\rho_F} \right).
\]

According to Theorem 1 in \([34]\), one has \(\rho_F \leq 4^{n-1}n^{1-n} \log D_F^{n-1}\), if \(n = [F : Q] \geq 2\). Also, \(\rho_F = 1\) if \(n = 1\). Thus \(h_F^{-1} \log D_F \ll \rho_F^{-1}h_F^{-D_F^{1/4}}\) and the implied constant is absolute. Then \((63)\) follows. \(\square\)

Immediately, Corollary 31 and Corollary 20 implies that

**Corollary 32.** Let \(F/Q\) be a totally real field as before. Then there exist some computable \(N = N_{F/Q} \in \mathbb{N}_{\geq 1}\) such that for any CM extension \(K/F\) with \(D_K \geq N\) we have

\[
E' \left( \tilde{z}_q, \frac{1}{2}, f_q^{-1}, f_q \right) \geq \frac{1}{2\sqrt{\pi}} D_K^{1/4} \log D_{K/F},
\]

where \(\tilde{z}_q\) is the CM point corresponding to \(\tilde{q}\) and \(D_{K/F}\) is the relative discriminant.

**Remark.** The point is that when \(D_K\) is large, the derivative of Eisenstein is nonvanishing at the central point \(s = 1/2\). This fact will be used as an implicit important condition in the proof of Proposition \((34)\).
Also, combining Corollary 32 with the second part of Corollary 25 one sees easily that when $D_K$ is large, there is at least one class group $L$-function nonvanishing. In the following sections, we will show that there are actually "a lot of" nonvanishing class group $L$-functions.

4.4. Nonvanishing Class Group $L$-functions. Due to various applications in arithmetic geometry and number theory, people are interested in the number of nonvanishing class group $L$-functions. Traditionally, to compute the number, we consider the moments of these Hecke-Hilbert $L$-functions. Oftentimes, these kinds of moments can be expressed in terms of some kind of periods. For example, sometimes the first moment can be related to periods associated to some automorphic representation via some Waldspurger-type formulas. While for the class group $L$-functions, [11] used a second moment to bound the number of nonvanishing $L$-functions in the imaginary quadratic case. Here we start with a generalization to the CM case, using essentially the same idea.

4.4.1. The Second Moments and Nonvanishing of Class Group $L$-functions. Recall that, we have established second moment of the class group $L$-functions in terms of Eisenstein periods in Corollary 25. Combining it with estimates on $E'\left(\frac{z_1}{2}; f^{-1}_q, f_q\right)$ in Corollary 32 one obtains:

Theorem 33. Let $F/Q$ be a totally real field. Then for any CM extension $K/F$ and for any $\varepsilon > 0$, we have

$$\#\left\{\chi \in \mathcal{C}l(K) : L_K\left(\chi, \frac{1}{2}\right) \neq 0\right\} \gg \frac{h_K}{\log D_K} \cdot \frac{D_K^{1 - 2\theta - \varepsilon}}{D_K},$$

where $\theta$ is any constant towards the Ramanujan-Petersson conjecture and the implied constant is explicitly depended on $F/Q$.

Proof. By the second part of Corollary 25 and Corollary 32, there exist some computable $N = N(F/Q) \in \mathbb{N}_{\geq 1}$ such that for any CM extension $K/F$ with $D_K \geq N$ we have

$$\sum_{\chi \in \mathcal{C}l(K)} \left| L_K\left(\chi, \frac{1}{2}\right) \right|^2 \geq \frac{2^{n-2}D_F}{\sqrt{D_K}} \cdot \frac{h_K\rho_F^2}{\mathcal{O}_K^\times : \mathcal{O}_F^\times} \cdot E'\left(\frac{z_1}{2}; f^{-1}_q, f_q\right)^2 \geq \frac{h_K}{\mathcal{O}_K^\times : \mathcal{O}_F^\times} \cdot (\log D_K)^2.$$

Then (65) follows from the subconvex bound for level-aspect for $GL_2 \times GL_1$ in [46].

Remark. One may combine the second moment with subconvex bounds and some equidistribution laws of CM points to estimate $\#\left\{\chi \in \mathcal{C}l(K) : L_K\left(\chi, \frac{1}{2}\right) \neq 0\right\}$. Then we will obtain a somewhat weaker form of (65), which is a direct generalization of the main result in [36]. Actually we have:

Proposition 34. For any $[f] \in \mathcal{C}l(F)^+$, let $\mathcal{P}_f$ be a non-negative smooth, bounded from above by 1, compactly support function on the Hilbert modular variety of $X(f)$ such that $\text{supp}(\mathcal{P}_f)$ does not contain any cusps. Then there exist some $N = N_{F, \mathcal{P}} > 0$ such that for any $D_K \geq N$ we have

$$\#\left\{\chi \in \mathcal{C}l(K) : L_K\left(\chi, \frac{1}{2}\right) \neq 0\right\} \geq C(F, \mathcal{P}) \cdot \frac{D_K^{1/100}}{\mathcal{O}_K^\times} \cdot \frac{h_K^2}{D_K}.$$
where \( \theta \) is any constant towards the Ramanujan-Petersson conjecture, and

\[
C(F, \mathcal{P}) := \frac{2^{4n-4}h_F^2}{P_F(n)h_F} \sum_{j \in C_l(F)^+} N(j)^2 \int_{X(j)} |\mathcal{P}_j(z)|E'(z,1/2;\mathbf{f}^{-1},\mathbf{f})|^2 d\mu_X.
\]

\( P_F(n) \) is an effectively computable polynomial.

Since the proof is not long, we give it here.

**Proof.** Define the function

\[
f = g \circ \tau : \mathcal{CM}(K, \Phi) \rightarrow \mathbb{R}^+, \quad [a, r] \mapsto N(f_a)^2|E'(z_a, 1/2; f_a^{-1}, f_a)|^2.
\]

By Corollary 32 we see that clearly \( f \neq 0 \) on \( \mathcal{CM}(K, \Phi) \). In particular, \( \text{supp}(f) \) is nonempty. So by the same construction as in [36] (ref. Lemma 6.1), for any \( [\mathbf{f}] \in \mathcal{Cl}(F)^+ \), there exists a smooth, compactly supported function \( \mathcal{P}_j : X(j) \rightarrow [0, 1] \) such that

\[
\mu_{X(j)}(\text{supp}(\mathcal{P}_j) \cap \text{supp}(f)) > 0.
\]

Let \( h_j(z) := \mathcal{P}_j(z)f(z) \). Then clearly \( h_j \neq 0 \).

Also noting that all the fibers \( \tau^{-1}([a]) \) have the same cardinality of \( \mathcal{O}_F^\times/N_{K/F}\mathcal{O}_K^\times \) which is 1 or 2. Then according to [48] and [45] the CM points \( \mathcal{CM}(K, \Phi) \) are equidistributed as \( D_K \rightarrow \infty \). Combining this with Corollary 25 we will have

\[
\sum_{\chi \in \mathcal{Cl}(K)} \left| L_K(\chi, 1/2) \right|^2 = \frac{2^{n-2}D_F}{h_F \sqrt{D_K}} \cdot \frac{h_K^2 \rho_F^2}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]^2} \frac{1}{h_K} \sum_{[a^{-1}] \in \mathcal{Cl}(K)} f(z_a) \]

\[
\geq \frac{2^{n-2}D_F}{h_F \sqrt{D_K}} \cdot \frac{h_K^2 \rho_F^2}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]^2} \frac{1}{h_K} \sum_{[a^{-1}] \in \mathcal{Cl}(K)} \sum_{[\mathbf{f}] = [a]} h_j(z_a) \]

\[
= \frac{2^{n-2}D_F}{h_F \sqrt{D_K}} \cdot \frac{h_K^2 \rho_F^2}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]^2} \cdot \left( \int_{X(j)} h_j(z) d\mu_X + o(1) \right),
\]

as \( D_K \rightarrow \infty \). Then by the above discussion we obtain

\[
\sum_{\chi \in \mathcal{Cl}(K)} \left| L_K(\chi, 1/2) \right|^2 \geq \frac{2^{n-4}C_F, P, D_F}{h_F \sqrt{D_K}} \cdot \frac{h_K^2 \rho_F^2}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]^2}.
\]

By Theorem 6.1 in [45] we have

\[
L_K(\chi, 1/2) \leq P(n) \left( N_{K/Q}(\text{Cond}(\chi)) D_K \right)^{1 - \frac{m}{m'}} = P(n) D_K^{1 - \frac{m}{m'}};
\]

where \( P \) is an absolute computable polynomial and \( n \) is the degree of \( F/Q \) as before. Thus we have

\[
\sum_{\chi \in \mathcal{Cl}(K)} \left| L_K(\chi, 1/2) \right|^2 \leq P(n)^2 (D_K)^{1 - \frac{m}{m'}} \# \left\{ \chi \in \mathcal{Cl}(K) : L_K \left( \chi, \frac{1}{2} \right) \right\}.
\]

Combining (66) and (67) we obtain Theorem 33. \( \square \)
4.4.2. The first moment and Nonvanishing of Class Group $L$-functions.

Proof of Theorem 4. By (39) we have
\[
\frac{1}{h_K} \sum_{\chi \in \text{Cl}(K)} \chi(a)L_K(\chi, 1/2) = \frac{2^2 \sqrt{D_F}}{2D_K^2} \frac{\rho_F}{[O_K^* : O_F^*]} N_{F/Q}(f_a) E' \left( z_q, \frac{1}{2} f_q^{-1}, f_q \right).
\]
Hence, the theorem follows from Lemma 30. And the estimate for the error term (10) comes from the inequality (61) and a uniform upper bound for $N_{F/Q}(f_a)$, which can be deduced easily from the fact that $f_a \in \mathcal{I}_F$ and an upper bound for $M(n, 0)$ via (19). □

Proof of Corollary 5. Recall that every split prime ideal of $O_K$ is primitive. On the other hand, Čebogarev density theorem implies that there exist a set of representatives of $\text{Cl}(K)$ consisting of split prime ideals. This gives us a set of primitive representatives of $\text{Cl}(K)$, $S'$, say.

We thus can take $S$ to be
$$S = \{ a'q : a' \in S' \}.$$ Clearly, $S$ is a set of representatives of $\widehat{\text{Cl}}(K)$ such that for any $a \in S$, $aq^{-1}$ is primitive. Thus the formula (12) follows from Theorem 4. □

Proof of Theorem 6. The proof is parallel to that of Theorem 4 except Corollary 31 will be used to replace Lemma 30. The proof follows from plugging the square of (63) into (39). □

Proof of Theorem 7. Plug $a = \bar{q}$ into Theorem 4. Then (10) becomes
\[
E_1(F, K; \bar{q}) \ll \log D_F + 8^n \rho_F^{-1} h_F D_F^{7/4} D_K^{-3/8} \ll_F 1,
\]
since $N_F(y_{\bar{q}}) \geq 2^{-n} D_F^{-1} D_K^{1/4} N_{F/Q}(f_{\bar{q}})$. Also, note that
\[
N_{\bar{q}}(q) = \rho_F \frac{[O_K^* : O_F^*] \cdot h_F}{\sqrt{N_{F/Q}(f_{\bar{q}})}} \gg_F 1,
\]
and the estimate $\log N_F(y_{\bar{q}}) \gg_F \log D_K$. Hence (8) becomes
\[
\begin{align*}
\frac{1}{h_K} \sum_{\chi \in \text{Cl}(K)} |L_K(\chi, 1/2)| &\gg_F \log D_K.
\end{align*}
\]
Therefore, the $k = 0$ case comes from (68) and the subconvex bound in [46]. For the derivative case, noting that whenever $L_K(\chi, 1/2) \neq 0$, one can always take the logarithmic derivative on both sides of the functional equation
\[
\Lambda_K(\chi, s) := 2^{-ns} \pi^{-ns} D_K^{1/2} \Gamma(s)^n = \Lambda_K(\chi, 1 - s).
\]
and evaluate at $s = 1/2$ to get
\[
\begin{align*}
\frac{L_K'(\chi, 1/2)}{L_K(\chi, 1/2)} &= -\frac{1}{2} \log D_K + \left( \gamma + 3 \log 2 + \frac{1}{2} \log \pi \right) n.
\end{align*}
\]
Thus the $k = 1$ case follows from the $k = 0$ case and (69). □

Remark. Clearly, (68) can also be seen from (39) and Corollary 32.

Lemma 35 (Theorem 1. [35]). Let $K$ range over a sequence of normal CM-fields or over a sequence of (not necessarily normal) CM-fields of a given degree $2n$ such that $D_K^{1/2n} \to +\infty$, then we have
\[
h_K \gg D_K^{1/2} \chi^{-o(1)},
\]
where \( o(1) \) is an error term which tends to 0 as \( D_K^{1/2n} \to +\infty \), and the implied constant is effective and explicit.

**Proof of Corollary 8.** Clearly the \( k = 0 \) case in (15) follows from (14) and (70), and the \( k = 1 \) case follows from the \( k = 0 \) case and (69).

**Proof of Theorem 9.** Recall that by (36) we have, for any \( \chi \in \hat{C}(K) \), that

\[
\sum_{[a] \in \hat{C}(K)} \chi([a]) N(f_a)^n E(z_a; s; f_a^{-1}, f_a) = \frac{D_K^{n/2} |\mathcal{O}_K^f : \mathcal{O}_F^f|}{(2n D_F)^n} L_K(\chi, s) \frac{\zeta_f(2s)}{\zeta_f(2^n)}. \tag{71}
\]

It is known (ref. [47], page 170) that \( \zeta_f(s, [c]) = -w_F^{-1} R_F s^{-1} [1 + \delta_F([c]) s] + O(s^{n+1}) \), as \( s \to 0 \), where \([c]\) is any ideal class in \( \mathcal{C}(F) \), and

\[
\delta_F([c]) = n \gamma + n \log 2\pi - \log D_F - \frac{w_F D_F^{1/2} \zeta_f([c])}{2^n D_F}. \tag{72}
\]

Also, we have, as \( s \to 0 \), that

\[
\zeta_f(s) = \frac{h_F R_F}{w_F} s^{-1} \left[ 1 + \left( n \gamma + n \log 2\pi - \log D_F - \frac{w_F D_F^{1/2} \zeta_f([c])}{2^n D_F} \right) s \right] + O(s^{n+1}). \tag{73}
\]

Substituting (72) and (73) into the Fourier expansion of \( E(z, s; a, b) \) (ref. Section 3.3) to get

\[
E(z, s; b^{-1}, b) = A(s) + B(s) + C(s),
\]

where

\[
A(s) = N(b)^{-2s} y^s \xi_F(2s, [b]^{-1}) \cdot \zeta_f(2s)^{-1} = \frac{1}{h_F} \left[ 1 + \left( \log y^s - 2 \log N(b) - \frac{w_F D_F^{1/2} \zeta_f([c])}{2^n D_F} \right) s \right] + O(s^2)
\]

\[
B(s) = \frac{\sqrt{\pi} (s - 1/2)^n}{\Gamma(s)} D_F^{s} N_{F/Q}(b)^{-1} y^{(1-s)\sigma} \frac{\zeta_f(2s - 1, [\mathcal{O}_K])}{\zeta_f(2s)} = -2\rho_F^{-1} \left( \frac{\sqrt{\pi} (s - 1/2)^n}{\Gamma(s)} \right)^{n} D_F^{s} N_{F/Q}(b)^{-1} y^{s} \zeta_f(-1, [\mathcal{O}_K]) \cdot (s + O(s^2))
\]

\[
C(s) = \frac{2^{n} y_s}{\Gamma(s)} D_F^{s} \sum_{b \in F^x} |N(b)|^{-1/2} \lambda(b, s) e(bz) \prod_{v \in \mathcal{A}_\infty} K_{\mathbb{A}} \left( 2\pi y_v |b_v| \right) \cdot (s + O(s^2))
\]

\[
= -\frac{2^{n+1} y_s}{D_F N(b) \rho_F} \sum_{b \in F^x} |N(b)|^{-1/2} \lambda(b, 0) e(bz) \prod_{v \in \mathcal{A}_\infty} K_{\mathbb{A}} \left( 2\pi y_v |b_v| \right) \cdot (s + O(s^2))
\]

\[
= -\frac{2}{D_F N(b) \rho_F} \sum_{b \in F^x} |N(b)|^{-1} \lambda(b, 0) e(bz) \cdot (s + O(s^2)),
\]

where \( \lambda(b, s) \) is defined in (29) with the weight \( k = 0 \) and \( C := \mathcal{O}_F^* / \mathcal{O}_F^* \). Then reorganize the above expansions to get the Taylor series at \( s = 0 \):

\[
E(z, s; b^{-1}, b) = \frac{1}{h_F} + G(z; b)s + \log (\phi(z; b)) s + O(s^2),
\]

where \( G(z; b) = h_F^{-1} \left( \log y^s - 2 \log N(b) - \rho_F^{-1} \left[ \zeta_f - h_F \gamma_f([b]^{-1}) \right] \right) \), and

\[
\log \phi(z; b) = -\frac{2^{n} y_s}{\rho_F N_{F/Q}(b)} \cdot \zeta_f(2, [\mathcal{O}_K]) - \frac{2\rho_F^{-1}}{D_F N_{F/Q}(b)} \sum_{b \in F^x} |N(b)|^{-1} \lambda(b, 0) e(bz).
\]
Also we have the Taylor expansion
\begin{equation}
\frac{\zeta_F(s)}{\zeta_F(2s)} = \frac{1}{2^{n-1}} \left\{ 1 - \frac{1}{n} \zeta_F^{(n)}(0) s + O(s^2) \right\}.
\end{equation}

Let $\chi_{K/F}$ be the Hecke character of $\mathcal{A}_F$ which corresponds to the quadratic extension $K/F$. Taking $\chi$ to be trivial in (71), combining (74), (75) with the fact that $\zeta_k(s) = \zeta_F(s)L(s, \chi_{K/F})$, we have
\begin{equation}
\sum_{[a] \in Cl(K)} N([\mathfrak{a}]) \left\{ \frac{1}{h_F} + A([a]) s + O(s^2) \right\} = \frac{[\mathcal{O}_F^+: \mathcal{O}_F^+]}{2^n} \cdot I(s),
\end{equation}
where $A([a]) = G(z_a; \mathfrak{a}) s + \log (H(z_a; \mathfrak{a}))$, and
\begin{equation}
I(s) = 2 + \log \left( \frac{\sqrt{D_K}}{2^n D_F} \right)^s - \frac{1}{n} \zeta_F^{(n)}(0) s + \frac{L(0, \chi_{K/F})}{L(0, \chi_{K/F})} s + O(s^2).
\end{equation}

Let $s = 0$ in (76) to recover the class number formula
\begin{equation}
L(0, \chi_{K/F}) = \frac{2^{n-1} h_K}{[\mathcal{O}_K^+: \mathcal{O}_F^+] h_F}.
\end{equation}

Taking the derivative with respect to $s$ at $s = 0$ in (76) and plugging (77) into the equality to obtain
\begin{equation}
\sum_{[a] \in Cl(K)} \left\{ \log \frac{y^s}{N_F/Q([\mathfrak{a}])^{1/2}} - \frac{\gamma_F - h_F \gamma_F([\mathfrak{a}])}{\rho_F} + h_F \log H(z_a, \mathfrak{a}) \right\} = h_F \mathcal{J}_K,
\end{equation}
where
\begin{equation}
\mathcal{J}_K = \frac{h_K}{2h_F} \left\{ \log \left( \frac{\sqrt{D_K}}{2^n D_F} \right) - \frac{1}{n} \zeta_F^{(n)}(0) + \frac{L(0, \chi_{K/F})}{L(0, \chi_{K/F})} \right\}.
\end{equation}

Define $H(z; b) = [y^s N_{F/Q}(b)^{-1}]^{1/h_F} \varphi(z; b)$. Then by Lemma 19 we have
\begin{equation}
H(z_a; \mathfrak{a}) = \phi(z_a; \mathfrak{a}) \cdot \left( \frac{y^s}{N_F/Q([\mathfrak{a}])} \right)^{1/2} = \phi(z_a; \mathfrak{a}) \cdot \left( \frac{N_{K/Q}(z_a) N_{F/Q}(q)^2 \sqrt{D_K}}{2^n N_{K/Q}(a) D_F} \right)^{1/2},
\end{equation}
and the right hand side of (79) is independent of representatives of $[a]$.

On the other hand, class field theory implies that for any $C_0 \in Cl(F)^+$, there are exactly $h_K/h_F$ classes $[a] \in Cl(K)$ such that $[\mathfrak{a}] = C_0$. Thus we have
\begin{equation}
\sum_{[a] \in Cl(K)} \frac{\gamma_F - h_F \gamma_F([\mathfrak{a}])}{\rho_F} = 0.
\end{equation}

It then follows from (78), (79) and (80) that
\begin{equation}
\sum_{[a] \in Cl(K)} \log H(z_a, \mathfrak{a}) = \mathcal{J}_K.
\end{equation}

Thus Theorem 9 follows from (81) and a similar computation of $\mathcal{J}_K$ in [1].

Remark: An alternative way to compute the Taylor expansion of $E(z, s; a, b)$ at $s = 0$ starts with the Eisenstein period formula with respect to an ideal class $C = [a]^{-1} \in Cl(K)$:
\begin{equation}
\zeta_K(s, C) = \frac{(2^n D_F N_{F/Q}([\mathfrak{a}])^s}{D_K^{1/2} [\mathcal{O}_K^+: \mathcal{O}_F^+] \zeta_F(2s) E (z_a, s; \mathfrak{a}^{-1}, \mathfrak{a})}
\end{equation}
and the functional equation for the partial zeta function:
\begin{equation}
D_K^{(s)} \pi^{-n} \Gamma_K(s/2) \zeta_K(s, C) = D_K^{1-s}/2 \pi^{-n(1-s)} \Gamma_K((1-s)/2) \zeta_K(1-s, C'),
\end{equation}
where $\zeta_K(s, C) = \sum_{[a] \in Cl(K)} \log H(z_a, \mathfrak{a}) = \mathcal{J}_K$. 

where \( C' \) is the dual ideal class of \( C \), i.e. \( CC' = \mathfrak{o}_{K/Q} \). Then (82) and (83) implies

\[
E(z_a, s, \frac{1}{s}, f_a) = 2^{(s-1)n} n^{2(2s-1)n} \left( \frac{\Gamma(1-s)}{\Gamma(s)} \right)^n \cdot \frac{[\mathcal{O}^*_{K}: \mathcal{O}^*_{F}]D_{K}^{(1-s)/2}}{(D_F N_{F/Q}(f_a)) \zeta_F(2s)}
\]

Clearly, \((84)\) reduces the Taylor series of \( E(z_a, s, \frac{1}{s}, f_a) \) at \( s = 0 \) to the Taylor expansion of \( \zeta_K(1-s, C') \) at \( s = 1 \), where Theorem 2 of [28] applies.

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