Bishop-Runge approximations and inversion of a Riemann-Klein theorem

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Abstract. In this paper we give results about projective embeddings of Riemann surfaces, smooth or nodal, which we apply to the inverse Dirichlet-to-Neumann problem and to the inversion of a Riemann-Klein theorem. To produce useful embeddings, we adapt a technique of Bishop in the open bordered case and use a Runge-type harmonic approximation theorem in the compact case.

Bibliography: 37 titles.

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§ 1. Introduction

Applied to an open bordered Riemann surface, the works of Bishop [1] and Narasimhan [2] on the embeddability of Stein manifolds imply the following.

Theorem (Bishop-Narasimhan). Let \( Z \) be an open bordered Riemann surface, \( n \geq 2 \), and \( f: Z \to \mathbb{C}^n \) be a holomorphic map that is smooth up to the boundary. Then \( f \) can be uniformly approximated on \( Z \) by embeddings if \( n \geq 3 \), and, if \( n = 2 \), by immersions that are injective outside of a finite set.

Whether this holds when \( \mathbb{C}^n \) is replaced by the complex projective space \( \mathbb{CP}_n \) of the same dimension is a question arising in applications but one that seems to have not yet been addressed. A natural way to construct projective maps of a Riemann surface \( Z \) is to use global sections of its canonical bundle \( K(Z) \). Indeed, if \( \omega = (\omega_0, \ldots, \omega_n) \in K(Z)^{n+1} \) never vanishes, \( \omega \) induces a canonical map, denoted \([\omega]\) or \([\omega_0: \cdots: \omega_n]\), from \( Z \) to \( \mathbb{CP}_n \), defined by the formulae

\[
[\omega] = \left[ \frac{\omega_0}{\omega_j} : \cdots : \frac{\omega_{j-1}}{\omega_j} : 1 : \frac{\omega_{j+1}}{\omega_j} : \cdots : \frac{\omega_n}{\omega_j} \right] \quad (1.1)
\]

on \( \{\omega_j \neq 0\} \), \( 0 \leq j \leq n \). As in the affine case, there is an approximation statement when the target space is projective.

Theorem A. Let \( Z \) be an open bordered Riemann surface, \( n \geq 2 \), and \( f: Z \to \mathbb{CP}_n \) be a holomorphic map that is smooth up to the boundary. Then \( f \) can be uniformly approximated on \( Z \) by canonical embeddings if \( n \geq 3 \), and, if \( n = 2 \), by canonical immersions that are injective outside a finite set.

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Unexpectedly, this theorem has not been noticed before. In this paper, we view it as a consequence of Theorem 4, which is a variation of the density Theorem 1 needed in [3], [4] for our constructive solution of the inverse Dirichlet-to-Neumann problem for Riemann surfaces.

We prove Theorem 1 by adapting directly to canonical maps the constructive approximation method of Bishop [1] using the perturbations and $\bar{\partial}$-techniques applied to Riemann surfaces by Koppelman [5]; this yields an explicit approximation process. After prepublication of our paper, Forstneric indicated to us that Theorem A can be deduced from a result of Kaliman and Zaïdenberg [6] or Forstneric [7], Theorem 7.9.1, and a Bishop-Mergelyan-type theorem of [8].

Canonical maps and Theorem A appear naturally in the following inverse problem. Consider in $\mathbb{R}^3$ an open bordered real surface $Z$ endowed with the complex structure induced by the standard Euclidean structure of $\mathbb{R}^3$; every Riemann surface admits such a presentation according to Garsia [9] for the compact case and to Rüedy [10] for the bordered case. Assume that $Z$ has constant conductivity. Then the Dirichlet-to-Neumann operator of $Z$ can be seen as the operator $N$ which associates to $u \in C^1(bZ)$ the boundary current induced by the electrical potential $\tilde{u}$ created by $u$ in $Z$, that is, $Nu = (d^c\tilde{u})|_{bZ}$, where $\tilde{u}$ is the harmonic extension of $u$ to $Z$ and $d^c = i(\bar{\partial} - \partial)$, $\bar{\partial}$ being the Cauchy-Riemann operator of $Z$. The inverse Dirichlet-to-Neumann problem in this case is to reconstruct $Z$ from a finite number of measurements of boundary currents, that is, from $(Nu_\ell)_{0 \leq \ell \leq n}$, where the boundary potentials $u_\ell$, $0 \leq \ell \leq n$, are known. Canonical maps appear here naturally since the data

$$(u_\ell, Nu_\ell)_{0 \leq \ell \leq n} \quad \text{and} \quad (u_\ell, (\partial\tilde{u}_\ell)|_{bZ})_{0 \leq \ell \leq n}$$

are equivalent. A recent interesting survey on this topic is contained in [11].

In [3], we study this problem under its three aspects: uniqueness, reconstruction and characterization. We prove in particular that if the boundary of $Z$ is known, the knowledge of only three boundary potentials $u_\ell$, $0 \leq \ell \leq 2$, and their associated boundary currents $(\partial\tilde{u}_\ell)|_{bZ}$, $0 \leq \ell \leq 2$, is sufficient to recover $Z$ with Cauchy type integral formulae from the canonical boundary map $[(\partial\tilde{u})|_{bZ}]$ when $(u_\ell)_{0 \leq \ell \leq 2}$ fulfills a generic hypothesis which was not quite correctly formulated in [3], but correctly in [4], where some of our results are extended to open bordered nodal curves with electrically charged nodes.

In the present paper, we prove that data satisfying the assumptions of [4] are generic; these assumptions essentially mean that the canonical map $|\partial\tilde{u}|$ is an immersion which embeds $bX$ into $\mathbb{C}P_2$ and is proper in the sense that the pull-back of the image of $bX$ reduces to $bX$; such data are sufficient for the results of [3], [4] to hold.

Our results in the inverse Dirichlet-to-Neumann problem are also applied in [4] to obtain an inversion of the following Theorem of Riemann and Klein.

**Theorem** (Riemann 1857, Klein 1882). Let $Z$ be a compact Riemann surface, $(a^\pm_j)_{1 \leq j \leq \nu}$ be a family of mutually distinct points of $Z$, and $(c_j)_{1 \leq j \leq \nu} \in \mathbb{R}^\nu$. Then, there exists a unique (up to an additive constant) harmonic function $U$ on $Z \setminus \{a^\pm_j; 1 \leq j \leq \nu\}$, with at most logarithmic singularities such that the residue

$$\text{Res}_{a^\pm_j}(d^cU) \overset{\text{def}}{=} \frac{1}{2\pi} \int_{\text{dist}(\cdot, a^\pm_j) = \varepsilon} d^cU$$
\( (\varepsilon > 0 \text{ small enough}) \) of
\[
\partial U = i(\overline{\partial} - \partial) U
\]
at \( a_j^\pm \) is \( \pm cj_1 \), \( 1 \leq j \leq \nu \).

In this statement, \( Z \) should be seen as a real compact surface of \( \mathbb{R}^3 \) endowed with the complex structure induced by the standard Euclidean structure of \( \mathbb{R}^3 \) with constant conductivity, and the couples \( (a_j^-, a_j^+) \) as dipoles electrically charged with \( (-c_j, +c_j) \). In this setting, an electrical potential is a harmonic function having only a finite number of logarithmic isolated singularities whose charges, that is, residues in the above sense, have a vanishing sum; the sum of their modulus is called the \( L^1 \) norm of this distribution of charges.

Combining the formulations of Proposition 1 and Theorem 2 of [4], we obtain an inversion of a theorem of Riemann and Klein which can be formulated as the reconstruction of a compact Riemann surface from data collected on a (small) known subdomain.

**Theorem B.** Let \( S \) be an open subdomain of a compact Riemann surface \( Z \) considered as a submanifold of \( \mathbb{R}^3 \) equipped with the conformal structure induced by the standard Euclidean structure on \( \mathbb{R}^3 \). Let \( (U_\ell)_{0 \leq \ell \leq 2} \) be a family of potentials induced on \( Z \) by electrically charged dipoles in \( S \). Then \( Z \setminus S \) can be reconstructed from \( (U_\ell|_S)_{0 \leq \ell \leq 2} \) or \( (V_\ell|_S)_{0 \leq \ell \leq 2} \), where \( (V_\ell)_{0 \leq \ell \leq 2} \) is a family of electric potentials such that \( (U_\ell - V_\ell)|_S \) is arbitrarily close to 0 in \( C^\infty(Z \setminus S)^3 \) and is induced by a distribution of electrical charges confined in \( S \) with arbitrarily small \( L^1 \)-norm.

This result is a consequence of [4], Theorem 2, and Theorem 6 below, which implies the approximation statement announced by [4], Proposition 1. Note that our reconstruction process can be carried out constructively when the canonical map associated to the given family of electrical potentials sends \( Z \setminus S \) to a nonalgebraic curve of \( \mathbb{CP}_2 \). This can be achieved, for example, by perturbing the potentials \( U_\ell \) induced by the charges \( \pm c_\ell \) placed at \( a_\ell^\pm \) with charges uniformly distributed in a small neighbourhood of \( a_\ell^\pm \).

It is interesting to compare Theorem B with the results of Stahl [12] and of Gonchar, Rakhmanov and Suetin [13] on Padé approximations of algebraic functions. In spite of the fact that our technique is very different from the method of [13] and [12], our purpose is quite similar: constructive reconstruction of an algebraic Riemann surface \( Z \) from optimal data collected in a neighbourhood \( S \) of one point in \( Z \).

To prove the perturbation statement of Theorem B, we need a quantitative version of known qualitative approximation theorems for harmonic functions [14], [15]. This is done with Theorem 7; its proof uses a discretization of the Hodge-de Rham decomposition formula (4.2).

The next section is devoted to the statement of our main results, including Theorems A and B. Proofs are in the last two sections.

### § 2. Statements of theorems

An open bordered Riemann surface is the interior of a one-dimensional compact complex manifold with boundary all of whose connected components have nontrivial one real dimensional smooth boundary.
An open (bordered) nodal curve is a quotient of an open (bordered) Riemann surface \( Z \) by an equivalence relation which, given finite subsets \( N_1, \ldots, N_k \) of interior points of \( X \), identifies for each \( j \) points of \( N_j \). If \( X \) is a nodal curve, a branch of \( X \) is any connected Riemann surface contained in \( \overline{X} \). A node of \( X \) is a point of the singular locus \( \text{Sing} \, X \) of \( X \); the number of germs of branches of \( X \) passing through \( a \) is denoted by \( \nu(a) \).

If \( G \) is an open subset of \( X \) and \( r \in [0, +\infty] \), a function \( u \) on \( G \) is said to be of class \( C^r \) if it is continuous and if for any branch \( B \) of \( G \), \( u|_B \in C^r(B) \); the space of such functions is denoted by \( C^r(G) \). If \( p, q \in \{0, 1\} \), a \((p, q)\)-form \( \omega \) in \( C^r_{p,q}(G \cap \text{Reg} \, X) \) is said to be of class \( C^r \) on \( G \) if for any branch \( B \) of \( G \), \( \omega|_{B \cap \text{Reg} \, X} \) extends to an element of \( C^r_{p,q}(B) \). The space of such forms is denoted by \( C^r_{p,q}(G) \); note that \( C^r(G) \) and \( C^r_{0,0}(G) \) need not be equal.

If \( K \) is a compact subset of \( X \) and \( p, q \in \{0, 1\} \), the space \( C^\infty_{p,q}(K) \) of smooth \((p, q)\)-forms supported in \( K \) is equipped with the topology induced by the seminorms \( \|\omega\|_{m,K,B} = \sup_{B\cap K} |D^m(\omega)|_B \) where \( m \) is any integer, \( B \) is any branch of \( X \), and \( D \) is the total differential acting on coefficients; if \( K \subset \text{Reg} \, X \), the index \( B \) is omitted.

The space \( D^r_{p,q}(G) \) of smooth \((p, q)\)-forms compactly supported in \( G \) is equipped with the inductive limit topology of the spaces \( C^\infty_{p,q}(K) \) where \( K \) is any compact of \( G \). The space \( D'_{p,q}(G) \) of currents on \( G \) of bidegree \((p, q)\) is the topological dual of \( D^r_{p,q}(G) \); the elements of \( D'_{1,1}(G) \) are distributions on \( G \). The exterior differentiation \( d \) of smooth forms is well defined along branches of \( X \); similarly for \( \partial \) and \( \bar{\partial} \). These operators extend to currents by duality.

A harmonic distribution is an element \( U \) of \( D'_{1,1}(G) \) which is (weakly) harmonic in the sense that the current \( i\partial \bar{\partial} U \) vanishes, that is, \( \langle i\partial \bar{\partial} U, \varphi \rangle = 0 \) for all \( \varphi \in C^\infty(G) \). Thus, \( U \) is harmonic if and only if \( \partial U \) is a \( \bar{\partial} \)-closed current. Note that \( \omega \in D'_{0,1}(G) \) is \( \bar{\partial} \)-closed if and only if \( \omega \) is a weakly holomorphic \((1, 0)\)-form in the sense of Rosenlicht [16].

According to [4], Proposition 2, a harmonic distribution \( U \) on \( G \) is an ordinary harmonic function on \( \text{Reg} \, G \) such that for any node \( a \) of \( X \) lying in \( G \) and any branch \( B \) of \( X \) at \( a \), \( U|_{B \cap C} \) has at most a logarithmic isolated singularity, that is,

\[
U|_{B \cap C} = \text{Res}(U, a) \ln \text{dist} (\cdot, a) + R,
\]

where \( \text{Res}_B(U, a) \in \mathbb{C} \), the distance is computed in any hermitian metric of \( B \), and the remainder \( R \) is smooth near \( a \) in \( B \); in addition, the sum of the complex numbers

\[
\text{Res}_B(U, a) = \text{Res}((\partial U)|_B, a) = \frac{1}{2\pi i} \int_{B \cap \{\text{dist}(\cdot, a) = \varepsilon\}} \partial U
\]

(\( \varepsilon > 0 \) small enough) taken over the branches \( B \) of \( X \) at \( a \) is 0.

For any given \( u \in C^\infty(bX) \) and admissible family \( c \), that is, a family \((c_{a,j})_{a \in \text{Sing} \, X, 1 \leq j \leq \nu(a)} \) of complex numbers such that \( \sum_{1 \leq j \leq \nu(a)} c_{a,j} = 0 \) for each \( a \in \text{Sing} \, X \), there is a unique harmonic distribution extension \( \tilde{u}^c \) of \( u \) to \( X \) with \( c \) as family of residues (see, for instance, [4], Proposition 2); when \( X \) is smooth, the only admissible family is the empty one and \( \tilde{u}^{'\emptyset} = \tilde{u} \) is the usual harmonic extension of \( u \) to \( X \).

Weakly holomorphic \((1, 0)\)-forms on \( G \) are ordinary holomorphic forms on \( \text{Reg} \, G \), meromorphic on branches of \( G \) and the sum of residues along branches passing
through each pole is equal to 0. In particular, such forms admit canonical maps. It is
worth noting that the singularities of weakly holomorphic \((1,0)\)-forms on a nodal
curve are at most simple poles and that at each pole the sum of the residues is 0.
These facts are described in algebraic language in [17] for general curves; in the
case of nodal curves, [4], Proposition 2, gives an elementary justification.

An immersion of an open bordered nodal curve \(X\) in some complex manifold \(M\)
is a multivalued map from \(X\) to \(M\) such that its restriction to any branch of \(X\) is an
ordinary single-valued immersion; thus, such a map is an ordinary \(C^1\)-immersion
of \(\text{Reg} X\) which extends on each branch of \(X\) as a \(C^1\)-immersion. For the sake of
simplicity, and because it is sufficient for our purposes, all immersions are considered
to be of class \(C^1\) near the boundary. The possible lack of properness of an immersion
is avoided in the definition below.

An almost embedding is an immersion \(\varphi: X \to M\) with the following properties:
\(X' = \varphi(X)\) is an analytic subset of \(M \setminus bX'\); there exists a finite subset \(E\) of \(X\)
such that \(\varphi(E) \subset \varphi(X) \setminus \varphi(bX)\) and \(\varphi\) is an isomorphism of Riemann surfaces from
\((\text{Reg} X) \setminus E\) onto \((\text{Reg} X' ) \setminus \varphi(E)\) which extends as a diffeomorphism of manifolds
with boundary between some open neighbourhoods of \(bX\) and \(bX'\) in \(X\) and \(X'\).
In particular, \(\varphi|_{bX}\) embeds \(bX\) into \(M\) and \(\varphi^{-1}(\varphi(bX)) = bX\). Note that \(\varphi\) may
not preserve nodes.

An embedding is an almost embedding \(\varphi: X \to M\) which is a homeomorphism
from \(X\) to \(\varphi(X)\), or, equivalently, which is single-valued and such that for each
node \(a\) of \(X\), the (germs of) branches of \(X'\) at \(\varphi(a)\) are the images by \(\varphi\) of the
(germs of) branches of \(X\) at \(a\). When \(X\) is smooth, these definitions match the
usual ones.

Our first theorem is about boundary data coming from generic almost embed-
dings of the interior.

**Theorem 1** (generic boundary data). Let \(X\) be an open bordered nodal curve,
\(c = (c_\ell)_{0 \leq \ell \leq 2}\) be a 3-tuple of admissible families, and \(G^{ae}_c(bX)\) be the set of
\((u_\ell)_{0 \leq \ell \leq 2} \subset C^\infty(bX)^3\) such that \(0 \notin (\partial U_0)(bX)\) and \([\partial U_0 : \partial U_1 : \partial U_2]\) is an almost
embedding of \(X\) in \(\mathbb{CP}_2\), where \(U_\ell\) is the unique harmonic distribution extension of
\(u_\ell\) with \(c_\ell\) as family of residues. Then \(G^{ae}_c(bX)\) is a dense open subset of \(C^\infty(bX)^3\).
If \(c\) is real, the same result holds for spaces of real-valued functions.

Such generic data are closely related to DN-data, used in [4] to solve the inverse
Dirichlet-to-Neumann problem; these DN-data are boundary data of the form
\((\gamma, u, (\partial \bar{u})|_{\gamma})\), where \(\gamma\) is the oriented boundary of \(X\) and \(u = (u_\ell)_{0 \leq \ell \leq 2} \in G^{ae}_c(\gamma)\).
Hence, Theorem 1 tells us in a certain sense that random data are good. However,
it is particularly relevant for our inverse problem that one can a priori check that
a given \(u \in C^\infty(bX)^3\) is generic, that is, in \(G^{ae}_c(bX)\). With [18] and [3], Theorem 3a,
we get a criterion for this genericity question in terms of shock-wave decomposition
of a boundary integral. This theorem will be proved in another paper.

**Theorem** (genericity criterion). Let \(X\) be an open bordered Riemann surface with
connected boundary and \(u = (u_\ell)_{0 \leq \ell \leq 2} \subset C^\infty(bX)^3\) such that \(0 \notin (\partial \bar{u}_0)(bX)\) and
\[
(f_\ell)_{1 \leq \ell \leq 2} = \begin{pmatrix}
(\partial \bar{u}_1)|_{bX} & (\partial \bar{u}_2)|_{bX} \\
(\partial \bar{u}_0)|_{bX} 
\end{pmatrix}
\]
is an embedding. Consider the Cauchy-Fantappié indicatrix of the form

\[ G : \mathbb{C}^2 \ni (\xi_0, \xi_1) \mapsto \frac{1}{2\pi i} \int_{\partial X} f_1 \frac{d(\xi_0 + \xi_1 f_1 + f_2)}{\xi_0 + \xi_1 f_1 + f_2}. \]

Then \( u \in \mathcal{C}^{\text{ce}}_0(bX) \) and \( [\partial u](X) \) is not algebraic if and only if there are a nonempty open subset \( W \) of \( \mathbb{C}^2 \) and mutually distinct holomorphic functions \( h_1, \ldots, h_p \) on \( W \) satisfying the shock-wave equation \( h_1 \frac{\partial h}{\partial \xi_1} = \frac{\partial h}{\partial \xi_2} \) such that on \( W \)

\[ \frac{\partial^2}{\partial \xi_0^2} \left( G - \sum_{1 \leq j \leq p} h_j \right) = 0, \quad \frac{\partial^2 G}{\partial s_0^2} \neq 0. \tag{2.1} \]

When \( (u_\ell) \in \mathcal{C}^{\text{ce}}_0(bX), \ Y = [\partial \bar{u}_0^j : \partial \bar{u}_1^j : \partial \bar{u}_2^j](X) \) can be seen as a concrete presentation of \( X \). If \( Y \) can be explicitly computed by boundary data, one has a tool to solve the reconstruction problem which is of essential interest for applications. This is the spirit of the Cauchy-type integral formulae presented in [4], Theorem 2, and [4], Theorem 5. Note however that serious effort is still to be made to make these effective; [18], Proposition 3.33, [19], Theorem 8.3, [3], Theorem 4, [20], Theorem 1.2, and [21], Theorem 3.1 could be clues for this goal. The natural question of what to do with non- or less generic data is open.

The first step in the proof of Theorem 1 is to establish a weak version of it by an adaptation of Bishop’s technique for producing an affine embedding of a Stein manifold.

**Theorem 2** (weak approximation). Let \( \Sigma \) be an open nodal curve and \( X \subset \Sigma \) be open and smoothly bordered such that \( \Sigma \setminus X \subset \text{Reg } \Sigma \). For \( 0 \leq \ell \leq 2 \), let \( U_\ell \) be a harmonic distribution that is smooth in a neighbourhood of \( \Sigma \). Then, there is a triple \( V = (V_0)_{0 \leq \ell \leq 2} \) of harmonic distributions on \( \Sigma \) that are smooth near \( \Sigma \setminus X \) such that \( U - V \) is arbitrarily close to 0 in \( C^\infty(\Sigma)^3, \sigma = [\partial V_0 : \partial V_1 : \partial V_2] \) is an immersion of \( \Sigma \) into \( \mathbb{C} \) which embeds \( \gamma = bX \), and \( (\sigma|\Sigma)^{-1}(\delta) = \gamma \), where \( \delta = \sigma(\gamma) \). If \( U \) is real-valued, \( V \) can be chosen to be so.

The second and last step of the proof of Theorem 1 is the theorem below, named after the formula, from which it follows that a holomorphic function on an open set of \( \mathbb{C} \) that is continuous up to the boundary is injective if its boundary restriction is such.

**Theorem 3** (argument principle). Let \( X \) be an open bordered nodal curve, \( F : \Sigma \to \mathbb{C} \) be an immersion such that \( F|\gamma \) is injective, and \( F^{-1}(F(bX)) = bX \). Then \( Y = F(X) \) is a complex curve, \( \text{Sing } Y \) is a possibly empty finite set, and \( F \) is an isomorphism from \( X \setminus F^{-1}(\text{Sing } Y) \) onto \( \text{Reg } Y \).

A slight modification of the proof of Theorem 1 enables us to establish Theorem 4 below which concerns projective embeddings or immersions of an open bordered Riemann surface. It can be seen as a variation of the Bishop-Narasimhan Theorem for the first case and of a result of Bishop [1] for the second. Theorem 4 shows that it is not necessary to make some complicated construction to realize a projective embedding of an open bordered surface since in a certain sense, canonical maps chosen at random are embeddings. Another interesting feature of Theorem 4 is that, in contrast to some embedding theorems, it does not use the genus which, in general, is not so easy to find, especially in the inverse problems we are interested in.
Theorem 4 (approximation by almost embeddings). Let $X$ be an open bordered Riemann surface and $K(X)$ be the space of holomorphic $(1,0)$-forms on $X$ which are smooth up to $bX$. If $n \in \mathbb{N}^*$, we denote by $G_{n}^{ae}(X)$ the set of $(\omega_{\ell})_{0 \leq \ell \leq n} \in K(X)^{n+1}$ such that $[\omega]$ is an almost embedding of $X$ into $\mathbb{C}\mathbb{P}_{n}$ and by $G_{n}^{e}(X)$ the set of those whose associated canonical map is actually an embedding. Then $G_{n}^{e}(X)$ is a dense open subset of $K(X)^{n+1}$ when $n \geq 3$, and $G_{2}^{ae}(X)$ is a dense open subset of $K(X)^{3}$.

This theorem is a particular case of Theorem 5, for which additional notation is needed. Let $X$ be a nodal curve. If $c$ is an admissible family, $K_{c}(X)$ is the set of weakly holomorphic $(1,0)$-forms on $X$ which are smooth up to $bX$ near $bX$ and have $c$ as family of residues. The set $K_{c}(\bar{X})$ is equipped with the distance $(\alpha, \beta) \mapsto \text{dist}(\alpha - \beta, 0)$, where dist is any distance defining the natural topology of the canonical bundle $K(X)$ of smooth $(1,0)$-forms on $\bar{X}$. Let $c = (c_{\ell})_{0 \leq \ell \leq n}$ be a triple of admissible families and write

$$c_{\ell} = (c_{p,j}^{\ell})_{p \in \text{Sing}X, 1 \leq j \leq \nu(p)}, \quad 0 \leq \ell \leq n.$$ 

If the $c_{p,j} = (c_{p,j}^{\ell})_{0 \leq \ell \leq n}$ are nonzero for any $(p,j)$, $c$ is called a nodal family, and a true nodal family if, in addition, for any $p \in \text{Sing}X$, the $c_{p,j}$, $1 \leq j \leq \nu(p)$, are on the same complex line of $\mathbb{C}^{3}$. If $p \in \text{Sing}X$ is obtained by identification of some points $p_{1}, \ldots, p_{\nu(p)}$ of some Riemann surface, the map $j \mapsto [c_{p,j}^{0} : \cdots : c_{p,j}^{n}]$ is well defined in the first case and constant in the second case. A true nodal family $c$ is said to be injective if $\text{Sing}X \ni p \mapsto [c_{0}^{0} : \cdots : c_{n}^{0}]$ is so.

Theorem 5 (approximation by almost embeddings, nodal case). Let $X$ be an open bordered nodal curve and $c = (c_{\ell})_{0 \leq \ell \leq n}$ be an injective nodal family. If $n \in \mathbb{N}^*$, we denote by $G_{n,c}^{e}(\bar{X})$ (resp., $G_{n,c}^{ae}(\bar{X})$) the set of $\omega \in K_{n,c}(\bar{X}) = K_{0,c}(\bar{X}) \times \cdots \times K_{n,c}(\bar{X})$ such that $[\omega]$ is an embedding (resp., an almost embedding) of $\bar{X}$ into $\mathbb{C}\mathbb{P}_{n}$. Then $G_{n,c}^{e}(\bar{X})$ is a dense open subset of $K_{n,c}(\bar{X})$ when $n \geq 3$, and $G_{2,c}^{ae}(\bar{X})$ is a dense open subset of $K_{2,c}(\bar{X})$.

That every open bordered nodal curve can be embedded in $\mathbb{C}^{3}$ is a consequence of a general result of Wiegmann [22] and Schürmann [23]. A nodal version of Theorem A is obtained by using Theorem 5 in its proof instead of Theorem 4.

We now turn our attention to the problem of reconstructing a compact Riemann surface $Z$ from data collected in a small subdomain $S$. More precisely, we consider the space of functions $U$ which are harmonic outside a finite subset $P(U)$ of $Z$, have isolated logarithmic singularities at each point of $P(U)$, and are such that $\sum_{p \in P(U)} \text{Res}(U, p) = 0$. This last condition ensures that $U$ is a harmonic distribution on the nodal surface $X$ where the points of $P(U)$ have been identified. In the sequel, we speak of such special function as harmonic distributions on $Z$.

We denote by $D_{Z}$ the set of $(a, c)$ in $Z^{6} \times \mathbb{C}^{3}$ such that $a = (a_{-}, a_{+})_{0 \leq \ell \leq 2}$ is a family of six mutually distinct points of $Z$ and $c = (c_{\ell})_{0 \leq \ell \leq 2} \in \mathbb{C}^{3}$. If $(a, c) \in D_{Z}$ and $0 \leq \ell \leq 2$, $U_{Z,\ell}^{a,c}$ is a harmonic distribution whose singular support is $\{a_{-}, a_{+}\}$ and has residues $\pm c_{\ell}$ at $a_{-}$, as a matter of fact, $U_{Z,\ell}^{a,c}$ is a standard Green bipolar function and, while it is determined only up to an additive constant, $\partial U_{Z,\ell}^{a,c}$ is unique.

For $n \in \mathbb{N}^*$, $D_{Z,n}$ is the set of $(a, c, p, z)$ in $Z^{6} \times \mathbb{C}^{3} \times (Z^{n})^{3} \times (\mathbb{C}^{n})^{3}$ such that $(a, c) \in D_{Z}$, and for any $\ell \in \{0, 1, 2\}$, $p_{\ell} = (p_{\ell,j})_{1 \leq j \leq n}$ is a family of mutually distinct points of $Z \setminus \{a_{-}, a_{+}, a_{1}, a_{1}^{+}, a_{2}, a_{2}^{+}\}$ and $z_{\ell} = (z_{\ell,j})_{1 \leq j \leq n} \in \mathbb{C}^{n}$ satisfies $\sum_{1 \leq j \leq n} z_{\ell,j} = 0$. 

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If \((a, c, p, \varkappa) \in D_{Z,n}\), we denote by \(V_{Z,\ell}^{p,\varkappa}\) a harmonic distribution with \(\{p_{\ell,j}; 1 \leq j \leq n\}\) as singular support and residue \(\varkappa_{\ell,j}\) at \(p_{\ell,j}\), \(1 \leq j \leq n\); \(V_{Z,\ell}^{p,\varkappa}\) is unique up to an additive constant; we set \(U_{Z,\ell}^{a,c,p,\varkappa} = U_{Z,\ell}^{a,c} + V_{Z,\ell}^{p,\varkappa}\), \(0 \leq \ell \leq 2\). We denote by \(E_{Z,n}(S)\) the set of \((a, c, p, \varkappa) \in D_{Z,n}\) such that

\[
F_{Z}^{a,c,p,\varkappa} = [\partial U_{Z,0}^{a,c,p,\varkappa}; \partial U_{Z,1}^{a,c,p,\varkappa}; \partial U_{Z,2}^{a,c,p,\varkappa}]
\]

is well defined and injective outside some finite subset of \(Z \setminus S\); \(E_{Z,n}(S)\) is open in \(D_{Z,n}\).

Note that since \(F_{Z}^{a,c,p,\varkappa}\) takes the value \((1, 0, 0)\) at each pole of \(\partial U_{Z,0}^{a,c,p,\varkappa}\), the map \(F_{Z}^{a,c,p,\varkappa}\) cannot be an embedding even when it is defined on the whole of \(Z\). It cannot be an almost embedding since its degree would be at least 2. Hence, the result below which establishes the genericity assumption claimed in [4], Proposition 1, is somehow delicate. In this theorem, \(S\) should be considered as a known subdomain in which, in essence, the noninjectivity of the maps is concentrated.

**Theorem 6** (almost embedding by dipole perturbation). Let \(Z\) be a compact Riemann surface and \(S\) be a nonempty open subset of \(Z\). Consider \((a, c)\) in \(D_{Z}\) with \(a \in S_0^*\). Then for any \(\varepsilon \in \mathbb{R}_+\) there exist \(n \in \mathbb{N}^*\) and \((p, \varkappa) \in (S^n)^3 \times (\mathbb{C}^{n})^3\) such that

\[
(a, c, p, \varkappa) \in E_{Z,n}(S), \quad |\varkappa|_1 \overset{\text{def}}{=} \sum_{0 \leq \ell \leq 2, 1 \leq j \leq n} |\varkappa_{\ell,j}| \leq \varepsilon.
\]

This theorem is proved by applying Theorem 1 to \(Z \setminus S\) and uses Theorem 7 below. Note that this result and others of [15], [14] can be seen as a development or improvement of a theorem of Runge-Behnke-Stein [24], [25] for holomorphic functions on an open Riemann surface which is quoted by Remmert in his book [26] as follows.

**Theorem C** (Behnke-Stein). For every subdomain \(D\) of a noncompact Riemann surface \(Z\) there exists a set \(T\) of boundary points of \(D\) (in \(Z\)) that it is at most countable and has the following property: Every function in \(\mathcal{O}(D)\) can be approximated compactly in \(D\) by functions meromorphic in \(Z\) that each have finitely many poles, all of which lie in \(T\).

In the statement below, \(L_{m}^{1}(Z \setminus S), m \in \mathbb{N},\) is the Sobolev space of distributions on \(Z \setminus \overline{S}\) whose total differentials up to order \(m\) are integrable on \(Z \setminus S\), \(Z\) being equipped with any hermitian metric.

**Theorem 7** (quantitative Runge-harmonic approximation). Let \(Z\) be a compact connected oriented smooth Riemann surface and \(S\) be a smoothly bordered open subset of \(Z\). Then, there exists \((C_{m}) \in \mathbb{R}_{+}^n\) depending only on \(S\) such that for every \(\varepsilon \in \mathbb{R}_+\) and \(\varphi \in C^{\infty}(Z \setminus S)\) harmonic in \(Z \setminus \overline{S}\), there is a finite subset \(P_{\varepsilon}\) of \(S\) and a function \(\varphi_{\varepsilon}\) harmonic in \(Z \setminus P_{\varepsilon}\) with isolated logarithmic singularities such that

\[
\|\varphi - \varphi_{\varepsilon}\|_{C^{m}(Z \setminus S)} \leq C_{m}\varepsilon\|\varphi\|_{L_{1}^{2}(Z \setminus S)}
\]

for any \(m \in \mathbb{N}\) and

\[
\sum_{a \in P_{\varepsilon}} |\text{Res}(\varphi_{\varepsilon}, a)| \leq C_{0}\|\varphi\|_{L_{1}^{2}(Z \setminus S)}.
\]

If \(\varphi\) is real-valued, \(\varphi_{\varepsilon}\) can also be chosen real-valued.
§ 3. Proofs for the open bordered case

All proofs, apart from those of Theorem 3 and Theorem A, which, as shown by Lemma 8 below, follow from Theorem 4, are given after several lemmas. Important notation and settings are stated after the proof of Lemma 8.

Proof of Theorem 3. As an open bordered nodal curve is an open bordered Riemann surface where a finite number of points have been identified, it is sufficient to prove this proposition when X is smooth. Since $F^{-1}(\delta) = \gamma$, $F|_X$ is proper and we know by a theorem of Remmert [27] that $Y$ is an analytic subset of $\mathbb{CP}_2 \setminus \delta$; $Y$ has pure dimension 1 because $F$ is an immersion. By reasoning with a connected component of $X$, we reduce the proof to the case when $X$ is connected. Then, $Y$ is connected and because $F$ is an immersion up to the boundary, $F$ has a degree $\nu$ over $\text{Reg} Y$ which is defined by

$$\nu = \max \{ \text{Card}(X \cap F^{-1}\{y\}); y \in \text{Reg} Y \}.$$  

As $F|_X$ is proper and smooth near $bX$ according to the definition of an immersion in this paper, $F_*[X]$ is well defined and equals $\nu[Y]$. Hence $\nu d[Y] = dF_*[X] = F_*d[X] = F_*[\gamma] = [\delta]$ because $F|_\gamma$ is injective. As $F$ is an immersion, $d[Y]$ is locally of the form $[\pm \delta]$ and $\nu = 1$.

Lemma 8 is a plain consequence of the fact that any holomorphic bundle on an open bordered Riemann surface is trivial, but we find it useful to prove it by elementary means.

Lemma 8. Every holomorphic projective map of an open or open bordered Riemann surface is canonical.

Proof. Let $R$ be a Riemann surface as above and $F: \overline{R} \to \mathbb{CP}_n$ be a holomorphic map, smooth up to the boundary if $R$ is bordered. We assume, without loss of generality, that $R$ is connected. Let $(t_0, \ldots, t_n)$ be the homogeneous coordinates of $\mathbb{CP}_n$ and, for $0 \leq j \leq n$, putting $T_j = \{ t_j \neq 0 \}$, we denote by $\zeta_j = (\zeta_{j,k})_{k \neq j} = (t_k/t_j)$ the natural affine coordinates for $\mathbb{CP}_n$ in $T_j$; we set $\zeta_{j,j} = 1$. Then the functions $f_{j,k} = \zeta_{j,k} \circ F$ form the data for a multiplicative Cousin problem on $Z$ associated with the covering $(R_j)_{0 \leq j \leq n} = (F^{-1}(T_j))_{0 \leq j \leq n}$. Original proofs that such a problem always has a solution on an open or bordered Riemann surface can be found in the paper of Behnke-Stein [24] for the first case and can be deduced from the results of Koppelman in [5] by $\partial$-resolution with regularity up to the boundary. So, we can find $(f_j) \in \mathcal{O}^*(R_0) \times \cdots \times \mathcal{O}^*(R_n)$ such that $f_{j,k} = f_j/f_k$ on each $R_j \cap R_k$. When $k$ is fixed, the relations $f_k = f_{j,k}f_j$ show that $f_k$ extends holomorphically to $R$ and smoothly to $\overline{R}$. Hence, the $f_{j,k}$ extend holomorphically to $R$ and the relations $f_{j,k} = f_k/f_j$ hold on $\overline{R}$. As a map on $\overline{R}$, $f = (f_k)$ has no zeros and $[f]$ is well defined. We have $F = [f]$ because on each $R_j$, $[(f_k/f_j)_{0 \leq k \leq n}] = [(f_{j,k})_{0 \leq k \leq n}]$.

Consider now $\alpha_0 \in K(\overline{R})$ not identically zero; if $R$ is bordered, we choose for $\alpha_0$ the restriction to $\overline{R}$ of a nonzero element of $K(R')$, where $R'$ is some open connected neighbourhood of $\overline{R}$ in its double. Then, using the Weierstrass theorem, we take $A \in \mathcal{O}(R) \cap C^\infty(\overline{R})$ whose divisor is the divisor of $\alpha_0$. The continuous extension $\alpha$ of $\frac{1}{A} \alpha_0$ is in $K(\overline{R})$ and never vanishes. Hence, $F = [f \alpha]$ is a canonical map. The proof is complete.

Subsequent statements and proofs are technically complicated by the existence of nodes, but one can easily isolate the smooth case. We now set the notation for
the remainder of the section. Let $X$ be an open bordered nodal curve. Since $X$ can be seen as a subdomain of its double, we assume that $X$ is a relatively compact open subset of an open nodal curve $\Sigma$, all of whose components meet $X$, and such that $\Sigma \setminus X \subset \text{Reg } \Sigma$ and $X \cap \text{Reg } \Sigma$ is smoothly bordered in $\text{Reg } \Sigma$. In the sequel, $\Sigma$ is replaced by a sufficiently small open neighborhood of $X$ when needed.

Let $R$ be the smooth Riemann surface of which $\Sigma$ is a quotient and $\pi: R \to \Sigma$ be the natural projection. We set $W = \pi^{-1}(X)$ and

$$S = \pi^{-1}(\text{Sing } \Sigma) = \{p_j; p \in \text{Sing } X, 1 \leq j \leq \nu(p)\},$$

where $\{p_1, \ldots, p_{\nu(p)}\} = \pi^{-1}(p)$ when $p \in \text{Sing } X$. As $\gamma \subset \text{Reg } \Sigma$, we identify $\pi^{-1}(\gamma)$ and $\gamma$.

Note that the pullback by $\pi$ of a harmonic distribution (resp., a weakly holomorphic $(1,0)$-form) on $X$ is a harmonic function (resp., a holomorphic $(1,0)$-form) on $W \setminus S$ with at most isolated logarithmic singularities (resp., simple poles) and such that for any $p \in \text{Sing } X$, the sum of its residues at $p$ vanishes. Conversely, such harmonic functions or holomorphic $(1,0)$-forms on $W \setminus S$ have a well-defined direct image by $\pi$ as a harmonic distribution or weakly holomorphic $(1,0)$-form.

We fix a triple $c = (c_\ell)_{0 \leq \ell \leq 2}$ of admissible families and write $c_\ell = (c_{p,j}^\ell)_{p \in \text{Sing } X, 1 \leq j \leq \nu(p)}$. In the sequel, $u = (u_\ell)_{0 \leq \ell \leq 2}$, where for $0 \leq \ell \leq 2$, $u_\ell$ is the restriction to $bX$ of a harmonic distribution $U_\ell$ on $\Sigma$ that is smooth near $\Sigma \setminus X$ with $c_\ell$ as a family of residues; we set

$$\omega = (\omega_\ell)_{0 \leq \ell \leq 2} = (\partial U_\ell)_{0 \leq \ell \leq 2}, \quad \theta = (\theta_\ell)_{0 \leq \ell \leq 2} = (\pi^* \omega_\ell)_{0 \leq \ell \leq 2},$$

and for $0 \leq \ell \leq 2$,

$$S_\ell = \{p \in S; \text{Res}(\theta_\ell, p) \neq 0\}, \quad Z_\ell = \{\theta_\ell = 0\}.$$

If $p \in \text{Sing } X$ and $1 \leq j \leq \nu(p)$, $p_j \in Z_\ell$ if and only if $c_{p,j}^\ell = 0$ and $\theta_\ell$ is continuously extended by 0 at $p$. When this occurs for any $\ell \in \{0,1,2\}$, our data correspond to a nodal curve where only the points of $\pi^{-1}(p) \setminus \{p_j\}$ have been identified. Hence, we can assume for the remainder of this section that the following holds:

- either $c = 0$, or $c$ is a nodal family.

The first case means that we are working on the smooth Riemann surfaces $W$ and $R$ with ordinary harmonic functions and holomorphic forms. In the second case, $[\theta]$ is well defined on each $p_j$, where it takes the value $[c_{p,j}^0 : c_{p,j}^1 : c_{p,j}^2]$. When $c$ is a true nodal family, $[\omega]$ is single-valued at the nodes of $X$ and cannot be an embedding if $c$ is not injective.

Our approximation process needs the following lemma whose statement first appeared in [28] for the holomorphic case. Our proof uses solutions with estimates for $\overline{\partial}$ and $\partial \overline{\partial}$ on Riemann surfaces.

**Lemma 9.** Let $Z$ be a smoothly bordered relatively compact open subset of an open Riemann surface $R$. Then there exists an open neighborhood $Z'$ of $Z$ in $R$ such that $(Z, Z')$ is a harmonic pair (resp., a Runge pair), which means that any element of $C^\infty(Z)$ (resp., $C^\infty_{1,0}(Z)$) that is harmonic (resp., holomorphic) on $Z$ can be arbitrarily approximated in $C^\infty(Z)$ (resp., $C^\infty_{1,0}(Z)$) by harmonic functions (resp., holomorphic $(1,0)$-forms) on $Z'$.
Proof. Since $\overline{Z}$ is compact, the harmonic statement reduces to the fact that each boundary point of $Z$ has an open relatively compact neighbourhood $D$ such that $Z' = Z \cup D$ is smoothly bordered and $(Z, Z')$ is a harmonic pair. Let $\omega \in \mathcal{B}Z$. As $Z$ is smoothly bordered, one can construct an open neighbourhood $D$ of $\omega$ that is relatively compact in a coordinate chart of $R$ such that $D \setminus Z \cap Z' \setminus D = \emptyset$ and $D \setminus Z$ has no compact component in $D$. Hence, the classical Runge Theorem (1885; see also [14], [15], [26]) in $C$ implies that $(Z \cap D, D)$ is a harmonic pair. Consider $\chi \in C^\infty_c(R, [0, 1])$ such that $\chi = 1$ in a neighbourhood of $D \setminus Z$ and $\chi = 0$ in a neighbourhood of $\overline{Z} \setminus D$. Let $f \in C^\infty(\overline{Z})$ be harmonic in $Z$, consider $g \in C^\infty(D)$ harmonic in $D$ and set $h = (g - f)|_{D \setminus Z}$. Denote by $h_D$ (resp., $h_D$) the trivial extension of $\chi h$ (resp., $(1 - \chi)h$) to $Z \setminus D$ (resp., $D \setminus Z$) and consider the well-defined smooth form $\Phi$ by setting $\Phi = i\partial \bar{\partial} h_Z$ on $\overline{Z}$ and $\Phi = -i\partial \bar{\partial} h_D$ on $D$. Because $D \setminus Z \cap D = \emptyset$, for any $m \in \mathbb{N}$ there is a constant $C_m$ independent of $f$ and $g$ such that $||\Phi||_{C^m(D)} \leq C_m ||h||_{C^{m+1}(D \setminus Z)}$; in what follows, such inequalities are denoted by $\prec_m$. According, for instance, to [29], Theorem 4.3, the equation $\Phi = i\partial \bar{\partial} \varphi$ can be solved with $\varphi \in C^\infty(\overline{Z'})$ such that for any integer $m$

$$||\varphi||_{C^m(\overline{Z'})} \prec_m ||\Phi||_{C^m(\overline{Z'})} \prec_m ||h||_{C^{m+1}(D \setminus Z)}.$$ 

Let $\tilde{f}$ be defined by $\tilde{f} = f + h_Z - \varphi$ on $\overline{Z}$ and $\tilde{f} = g - h_D - \varphi$ on $D$. It is smooth in $\overline{Z} \cup D$ and harmonic in $Z \cup D$ and by construction $\tilde{f} - f = -\varphi$ in $Z \setminus D$ while $\tilde{f} - f = \chi h - \varphi$ on $Z \cap D$. Thus, if $m$ is any integer,

$$||\tilde{f} - f||_{C^m(\overline{Z})} \prec_m ||g - f||_{C^{m+1}(D \setminus Z)}.$$ 

We get that $\tilde{f}$ is close to $f$ in $C^\infty(\overline{Z})$, as wished, provided that $g|_{Z \cap D}$ is close enough to $f|_{Z \cap D}$ in $C^\infty(\overline{Z} \cap D)$. This proves that $(Z, Z')$ is a harmonic pair. The statement for $(1,0)$-forms is obtained in the same way but with $\partial$-resolution of $(1,1)$-forms on an open Riemann surface for which, for example, the original work of Koppelman [5] can be used. The proof is complete.

When $A$ and $A'$ are open subsets of $\Sigma$ such that $A \subset A'$, $(A, A')$ is called a harmonic pair if for any harmonic distribution $U$ on $A$ that is smooth near $bA$ in $\overline{A}$, there is a harmonic distribution $V$ on $A'$ whose singularities and residues on $A'$ are those of $U$ on $A$ such that $V - U$ is arbitrarily close to 0 in $C^\infty(\overline{A})$. Runge pairs are defined likewise. Our approximation process starts with the following corollary of Lemma 9.

**Lemma 10.** Let $\Sigma$ be an open nodal curve and $X$ be a relatively compact open smoothly bordered subset of $\Sigma$ such that $(bX) \cap \text{Sing} \Sigma = \emptyset$. Then there exists an open neighbourhood $X'$ of $\overline{X}$ in $\Sigma$ such that $(X, X')$ is a harmonic pair and a Runge pair.

**Proof.** The proof of Lemma 9 may be applied step by step, except, the notation being the same, that the equation $\Phi = i\partial \bar{\partial} \varphi$ has to be solved with $\varphi \in C^\infty(\overline{\pi^{-1}(Z')})$, which is possible because $\Phi$ can be seen as a smooth form on $\overline{\pi^{-1}(Z')} \subset W$. Then $\tilde{f}|_Z$ is a perturbation of $f$ by the push forward of a smooth function and hence is still a harmonic distribution. This yields the harmonic case. The holomorphic case is handled in the same way.
Lemma 10 gives a triple \((V_\ell)_{0 \leq \ell \leq 2}\) of harmonic distributions near \(\bar{X}\) such that \((V_\ell - U_\ell)_{0 \leq \ell \leq 2}\) is arbitrarily small in \(C^\infty(\bar{X})^3\). Thus, it is sufficient to prove Theorem 5 when \(U \in \mathcal{O}\) is a triple of harmonic distributions on \(\Sigma\) that are smooth near \(\Sigma \setminus X\); a similar reduction holds for Theorem 5. We have now to prove Theorem 2. This is done with Lemmas 11–14, which rely mainly on the fact that as \(R\) is a Stein manifold, Oka-Cartan techniques or \(\bar{\partial}\)-techniques involving \(L^2\)-estimates with singular plurisubharmonic weights enable one to prove the following.

(RS1) If \(z \in R\), for any holomorphic \((1,0)\)-form \(\Omega\) not vanishing at \(z\), there is \(h \in \mathcal{O}(R)\) such that
\[
\partial_\Omega h \overset{\text{def}}{=} \frac{\partial h}{\Omega}
\]
is a coordinate for \(R\) near \(z\).

(RS2) If \(z, z' \in R\) and \(z \neq z'\), for any holomorphic \((1,0)\)-form \(\Omega\) not vanishing at \(z\) and \(z'\), there is \(h \in \mathcal{O}(R)\) such that
\[
(\partial_\Omega h)(z) \neq (\partial_\Omega h)(z').
\]

We first use this tool in Lemma 11 to approximate \(U_0\) and \(U_1\) by harmonic distributions on \(\Sigma\) whose derivative induces a canonical map.

Lemma 11. There exist harmonic distributions \(U'_0\) and \(U'_1\) on \(\Sigma\) such that for \(U' = (U'_0, U'_1, U_2)\), \(U' - U\) is arbitrarily close to 0 in \(C^\infty(\bar{X})^2\), \(0 \notin (\partial U'_0)(\gamma)\), and \((\partial U'_0, \partial U'_1)\) (resp., \(\partial U'\)) induces a well-defined canonical map on a neighbourhood of \(\text{Reg}\, \bar{X}\) (resp., \(\bar{X}\)) in \(\Sigma\). Moreover, \(U\) can also be chosen real-valued if \(u\) is real-valued.

Proof. As in the proof of Lemma 8, we use an auxiliary nowhere-vanishing form \(\alpha \in K(R)\). Since \(W\) is compact, (RS2) enables us to find \(\eta_1, \ldots, \eta_m \in \mathcal{O}(R)\) and a covering \(A_1, \ldots, A_m\) of \(W\) by open subsets of \(R\) such that \(0 \notin (\partial_\alpha \eta_j)(\overline{A_j})\), \(1 \leq j \leq m\). The Whitney-Sard lemma implies that
\[
\left( \frac{\theta_0/\alpha}{\partial_\alpha \eta_1}, \frac{\theta_1/\alpha}{\partial_\alpha \eta_1} \right)(\overline{A_1} \setminus S_\ell) \quad \text{and also} \quad \frac{\theta_0/\alpha}{\partial_\alpha \eta_1}(\gamma \cap A_1)
\]
(which is a subset of \(\mathbb{C}\) since \(S_0 \cap \gamma = \emptyset\)) have Lebesgue measure zero. Hence, for almost all \(\varepsilon_1 \in \mathbb{C}^2\), \((\theta_0^1, \theta_1^1) = (\theta_\ell - \varepsilon_1, \varepsilon_1 \partial \eta_1)_{0 \leq \ell \leq 1}\) and \(\theta_0^1\) never vanish on \(\overline{A_1}\) and \(\gamma \cap \overline{A_1}\), respectively. If \(S \neq \emptyset\) and \(p \in S\), \(\theta_1^1\) and \(\theta_\ell\), \(0 \leq \ell \leq 1\), have the same residues at \(p\) and, as \(c\) is nodal, \([\theta_0^1 : \theta_1^1 : \theta_2^1]\) is well defined at \(p\).

Iterating this argument we get that for arbitrarily small \(\varepsilon_1, \ldots, \varepsilon_m\) in \(\mathbb{C}^2\) with the property that \(\varepsilon_{j+1}\) is small enough with respect to \(\varepsilon_j\), \((\theta_m^m, \theta_1^m) = (\theta_\ell + \varepsilon_1, \varepsilon_1 \partial \eta_1 + \cdots + \varepsilon_m \partial \eta_m)\) induces a well-defined canonical map near \(W \setminus S\), and that \(0 \notin \theta_m^m(\gamma)\). As \(\theta_\ell^m = \partial(U_\ell + \Re B_\ell)\), where \(B_\ell = \sum_{1 \leq j \leq r} \varepsilon_j \eta_j \in \mathcal{O}(R)\), \(B_{\ell}|_W\) is arbitrarily small in \(C^\infty(W)\) and \((U_\ell') = (U_\ell + \pi_* \Re B_\ell)\) fulfils our demand, which completes the proof.

Applying Lemma 11 to \(U = 0\) in the smooth case and choosing for \(R\) a sufficiently small open neighborhood of \(W\), we get \((\eta_0, \eta_1) \in \mathcal{O}(R)^2\) such that \((\alpha_0, \alpha_1) = (\partial \eta_0, \partial \eta_1)\) never vanishes. This will be useful below for preserving properties of approximating forms.
In the general case, Lemma 11 enables us to assume that $[\theta]$ is well defined and that $0 \notin \theta_0(\gamma)$. Thus $\gamma \subset W_0 = \overline{W} \setminus Z_0$ and $\overline{W}$ is the disjoint union of $W_0$ and the finite sets $W_1 = (\overline{W} \cap Z_0) \setminus Z_1$ and $W_2 = \overline{W} \cap Z_0 \cap Z_1 \subset \overline{W} \setminus Z_2$. Note that at points of $W_2$, $\theta_2$ has a simple pole or is nonzero.

The next lemmas follow the path of Bishop’s proof that a Stein manifold of dimension $d$ can be embedded in $\mathbb{C}^{2d+1}$. Actually, as our goal is much simpler than the results of Bishop [1], we use the simplified lecture of [30]. Starting with $(\theta_\ell)_{0 \leq \ell \leq 2} = \pi_* \omega$, we use (RS1) and (RS2) to find sufficiently many $\theta_j$ so that $[\theta_0 : \cdots : \theta_{n+1}]$ essentially embeds $\overline{W} \setminus S$ into $\mathbb{C}P_{n+1}$. Then, we decrease $n$ as much as we can by a repeated use of a Morse-Whitney-Sard lemma [31], [32], [33], which implies that if $\varphi: M \to M'$ is a smooth map between Riemannian manifolds and $\dim R M < \dim R M'$, $\varphi(M)$ has measure 0.

**Lemma 12.** There exist $h_3, \ldots, h_{n+1} \in \mathcal{O}(R)$ such that

$$S \subset \bigcap_{j \geq 3} \{ \partial h_j \neq 0 \}$$

and $[\partial h_3 : \cdots : \partial h_{n+1}]$ is an embedding of $\overline{W}$. Moreover, with $(\theta_j)_{j \geq 3} = (\partial h_j)_{j \geq 3}$, $[\theta] = [\theta_0 : \cdots : \theta_{n+1}]$ is an almost embedding of $\overline{W}$ such that

$$[\theta](\overline{W} \setminus S) \cap [\theta](S) = \emptyset.$$

**Proof.** As $\overline{W}$ is compact, (RS1) enables us to find an open covering of $\overline{W}$ by open subsets $V_2, \ldots, V_m$ of $R$, functions $\eta_2, \ldots, \eta_m \in \mathcal{O}(R)$ and $\lambda \in \{0, 1\}^{\{2, \ldots, m\}}$, such that each $\frac{\alpha_j}{\alpha\lambda(j)}$, where $\alpha_j = \partial \eta_j$, is a coordinate for $R$ in $V_j$. Hence $\Psi = [\alpha_0 : \cdots : \alpha_m]$ is an immersion of $\overline{W}$ such that $\Psi(z) = \Psi(z')$ implies $z = z'$ when $(z, z') \in V = \bigcup_{3 \leq j \leq m} V_j \times V_j$. As, for any $z, z' \in K = (\overline{W} \times \overline{W}) \setminus V$, at least one form $\beta = [\alpha_0, \alpha_1, \alpha_0 + \alpha_1]$ does not vanish at both $z$ and $z'$, (RS2) enables us to find families of open subsets $(V_j')_{m+1 \leq j \leq r}$ and $(V''_j)_{m+1 \leq j \leq r}$ of $R$ for which $(W_j)_{m+1 \leq j \leq r} = (V_j' \times V''_j)_{m+1 \leq j \leq r}$ is a covering of the compactum $K = (\overline{W} \times \overline{W}) \setminus V$, $\eta_{m+1}, \ldots, \eta_r \in \mathcal{O}(R)$, $\beta_{m+1}, \ldots, \beta_r \in \beta$ such that $(\partial \beta, \eta_j)(z) \neq (\partial \beta, \eta_j)(z')$ for any $(z, z')$ in $W_j$. By construction, $[\partial \eta_0 : \cdots : \partial \eta_{n+1}]$, where $\eta_{r+1} = \eta_0 + \eta_1$ is an injective immersion of $\overline{W}$, and in the smooth case the lemma is proved with

$$(h_j)_{3 \leq j \leq n+1} = (\eta_j)_{0 \leq j \leq r+1}.$$

Hence we can apply Lemmas 13 and 14 below in the smooth case and get that for any $k \in \{0, \ldots, r+1\}$ and almost all $a \in \mathbb{C}^{r+2}$ with $a_k = 0$, $[\theta^a_3 : \cdots : \theta^a_{n+1}]$, where $(\theta^a_j)_{j \geq 3} = (\partial \eta_{j-3} - a_{j-3} \partial \eta_k)_{j \geq 3}$, is an embedding of $\overline{W}$. Let

$$T = \bigcap_{j \geq 3} S \cap \{ \partial \eta_j \neq 0 \}.$$

If $q_\ast \in S \setminus T$ and $\partial \eta_k(q_\ast) \neq 0$, the $\theta^a_j$ do not vanish in $T' = T \cup \{ q_\ast \}$ when

$$a \notin \bigcup_{m \geq 3, q \in T'} \left\{ t \in \mathbb{C}^{r+2} : t = \frac{\partial \eta_m(q)}{\partial \eta_k(q)} \right\}.$$
Lemma 13. Consider \( \omega^a(0) \leq \ell \leq 2 \in K_{3, c}(\Sigma) \) smooth near \( \Sigma \setminus X \), and, for \( n \geq 2 \),

\[
(\omega_j)_{3 \leq j \leq n+1} \in K_{n-1,0}(\Sigma) \quad \text{and} \quad \theta = (\pi^* \omega_j)_{0 \leq j \leq n+1}.
\]

Assume that \([\omega^a(0)_{0 \leq \ell \leq 2}]\) is well defined, \([\theta] \) is an embedding of \( \overline{W} \setminus S \) and an immersion of \( \overline{W} \), \([\theta](\overline{W} \setminus S) \cap [\theta](S) = \emptyset \), and that \( S \subset \bigcap_{j \geq 3} \{ \theta_j \neq 0 \} \). For \( a \in \mathbb{C}^n \), set \( \theta^a = (\theta_0, \theta_1 - a_1 \theta_{n+1}, \ldots, \theta_n - a_n \theta_{n+1}) \).

Then for almost all \( a \),

\[
S \subset \bigcap_{j \geq 3} \{ \theta_j^a \neq 0 \}, \quad [\theta^a](W_0 \setminus S) \cap [\theta^a](S) = \emptyset,
\]

\( [\theta^a] \) is an immersion of \( W_0 \cup S \) which embeds \( \gamma \), and \( \delta_a = [\theta^a](\gamma) \), where \( ([\theta^a]|_{\overline{W}})^{-1}(\delta_a) = \gamma \). If \( n \geq 3 \), \([\theta^a] \) is also an embedding of \( W_0 \setminus S \).

Proof. Because \( c \) is nodal, \( \theta^a \) is well defined at \( p \in S \) and takes the value \([c_p^0 : c_p^1 : c_p^2 : 0 : \cdots : 0] \), where \( c_p^0 = \text{Res}(\theta_j, p), 0 \leq j \leq n+1 \). If \( z \in R \setminus S \) and \( \theta_{n+1}(z) = 0 \), \( \partial \) is well defined at \( z \) whatever \( a \) is. By the Morse-Sard-Whitney lemma, \( ([\omega_{n+1}^{-1}]_{0 \leq j \leq n+1}) \) has Lebesgue measure zero. Hence, \([\theta^a] \) is well defined for almost all \( a \in \mathbb{C}^n \).

Set \( \tau = (\theta_j/\theta_0)_{1 \leq j \leq n+1} \) and \( \bar{\tau} = (\tau_j)_{1 \leq j \leq n} \). In \( W_0 \), the natural affine coordinates of \( \theta \) and \( \theta^a \) are \( \tau \) and \( \bar{\tau} - \tau_{n+1} a \), where if \( p \in S_0 \), \( \tau(p) = (c_p^1/c_p^0)_{1 \leq j \leq n+1} \). The maps considered in [1, 30] can be rewritten for Riemann surfaces in the form

\[
T : \mathbb{C} \times W_0 \to \mathbb{C}^{n+1}, \quad (t, z) \mapsto t(\partial_{\omega_0} \tau)(z),
\]

\[
I : \mathbb{C} \times W_0 \times W_0 \to \mathbb{C}^{n+1}, \quad (t, z, z') \mapsto t[\tau(z) - \tau(z')].
\]
The mapping $[\theta^a]$ is an immersion of $W_0^r = W_0 \setminus S$ if and only if $(a, 1) \notin T(\mathbb{C} \times W_0^r)$, and is injective on $W_0^r$ (resp., on $\gamma$) if and only if $(a, 1) \notin I(\mathbb{C} \times W_0^r \times W_0^r)$ (resp., $(a, 1) \notin I(\mathbb{C} \times \gamma \times \gamma)$). As explained in [30], the Morse-Whitney-Sard lemma and a homogeneity argument give that for almost all $a \in \mathbb{C}^n$ the first and third properties above are true because $n \geq 2$, and that the second is true for almost all $a$ when $n \geq 3$.

Using the notation of Lemma 12 to write the $\theta_j$ in a coordinate $\zeta$ centred at $q \in S$, (3.1) may be applied and gives that for $j \in \{0, 1, 2\} \setminus \{\ell\}$,

$$\frac{\partial (\theta_j - \theta_{n+1} a_j) / \partial \zeta}{\partial \zeta}(q) = \frac{1}{c_q} \left( f_j(0) - \frac{c_j}{c_q} f_\ell(0) - f_{n+1}(0) a_j \right).$$

As $\omega_{n+1}(q) \neq 0$, it appears that $[\theta^a]$ is regular at $q$ when $a$ does not belong to a finite union of complex affine hyperplanes of $\mathbb{C}^n$.

The inclusion $S \subset \bigcap_{j \geq 3} \{\theta^a_j \neq 0\}$ is achieved by means of the trick in the proof of Lemma 12.

If $z \in W_0^r$ and $p \in S$ share the same image under $\theta_a$,

$p \in S_0, \quad \tau_{n+1}(p) = 0, \quad \bar{\tau}(z) - \tau_{n+1}(z) a = \bar{\tau}(p), \quad \tau_{n+1}(z) \neq 0,$

because $[\theta](z) \neq [\theta](p)$. Hence

$$[\theta^a](W_0^r) \cap [\theta^a](S) = \emptyset$$

if $(a, 1) \notin I(\mathbb{C} \times W_0^r \times S)$. For such $a$, $((\theta^a)|W_0^r)^{-1}(\delta_a) = (\theta_a|W_0^r)^{-1}(\delta_a)$ (because $\gamma \subset W_0$) and equals $\gamma$ if and only if $(a, 1) \notin I(\mathbb{C} \times W_0^r \times \gamma)$. Both conditions are satisfied for almost all $a$ because $n \geq 2$. Lemma 13 is proved.

**Lemma 14.** Let the hypothesis and notation be those of Lemma 13. Then for almost all $a \in \mathbb{C}^n$, the following also holds:

$$[\theta^a](\overline{W} \setminus S) \cap [\theta^a](S) = \emptyset,$$

if $n \geq 3$, $[\theta^a]$ is an embedding of $\overline{W} \setminus S$, which induces an embedding of $\overline{X}$ when $c$ is injective.

**Proof.** Consider the map $[\theta^a]$ obtained in Lemma 13, $\alpha \in K(R)$ never vanishing, $g = (\omega_j/\alpha)_{0 \leq j \leq n+1}$ and $\bar{g} = (g_j)_{0 \leq j \leq n}$.

Suppose $z \in \overline{W} \setminus S$ and $q \in S$ share the same image by $\theta^a$. Then $z, q \in Z_0$ (because $[\theta^a](W_0 \setminus S) \cap [\theta^a](S) = \emptyset$) and

$$\bar{g}(z) - a g_{n+1}(z) \in \mathbb{C}^*b_q,$$

where $b_q = [0 : \text{Res}(\theta_1, q) : \text{Res}(\theta_2, q) : \cdots : 0]$, a condition which forces $g_{n+1}(z) \neq 0$ since $[\theta](z) \neq [\theta](q)$. Hence

$$[\theta^a](\overline{W} \setminus S) \cap [\theta^a](S) = \emptyset$$

when $a$ is not in the union of the affine lines $\frac{\bar{g}(z)}{g_{n+1}(z)} + \mathbb{C}b_q$, where $(q, z)$ takes all values in the finite set $S \times (\overline{W} \cap Z_0 \cap \{\theta_{n+1} \neq 0\})$. 
Applying Lemmas 13 and 14 inductively, we get that 
\[ [g_j - a_jg_{n+1}]d[g_k - a_kg_{n+1}] - [g_k - a_kg_{n+1}]d[g_j - a_jg_{n+1}] = 0 \]
somewhere in \( Z_0 \setminus S \) so that \( a \in \bigcup_{z \in Z_0} \bigcap_{j \neq k} H_{z,j,k} \) with
\[
H_{z,j,k} = \{ t \in \mathbb{C}^n; \alpha_j(z) - t_j \beta_z(z) - t_k \gamma_j(z) = 0 \},
\]
where
\[
\alpha_j,k = g_j \partial \Omega g_k - g_k \partial \Omega g_j,
\beta_k = g_{n+1} \partial \Omega g_k - g_k \partial \Omega g_{n+1},
\gamma_j = g_j \partial \Omega g_{n+1} - g_n \partial \Omega g_j.
\]
If \( z \in \overline{W} \cap Z_0 \setminus S \) is a zero of \( (\alpha_j,k, \beta_k, \gamma_j) \) for all \( j, k \in \{1, \ldots, n\} \) with \( j \neq k \), then
\( (g_j \partial \Omega g_k - g_k \partial \Omega g_j)(z) = 0 \) for all \( j, k \in \{1, \ldots, n+1\} \) and \( [\theta] \) has rank 0 at \( z \). As this is not the case, \( \bigcup_{z \in Z_0} \bigcap_{j \neq k} H_{z,j,k} \) is a finite union of proper subspaces of \( \mathbb{C}^n \) and \( [\theta^a] \) is regular on \( \overline{W} \setminus S \) for almost all \( a \).

If \( [\theta^a](z) = [\theta^a](z') \) with \( z, z' \in \overline{W} \setminus S \), \( z, z' \in Z_0 \) and there is \( \lambda \in \mathbb{C}^* \) such that
\[
\tilde{g}(z) - \lambda \tilde{g}(z') = [g_{n+1}(z) - \lambda g_{n+1}(z')]a.
\]
Then either \( g_{n+1}(z) = \lambda g_{n+1}(z') \), \( [\theta](z) = [\theta](z') \), and hence \( z = z' \), or \( g_{n+1}(z) \neq \lambda g_{n+1}(z') \) and \( a \) belongs to the image \( I_{z,z'} \) of \( \mathbb{C} \setminus \{g_{n+1}(z)/g_{n+1}(z')\} \) \((1/0 = \infty \) by convention) under
\[
H_{z,z'} : \lambda \mapsto [\tilde{g}(z) - \lambda \tilde{g}(z')]/[g_{n+1}(z) - \lambda g_{n+1}(z')].
\]
When \( (z, z') \) belongs to the finite set \( \Lambda = (Z_0 \cap \{g_{n+1} \neq 0\} \setminus S)^2 \) and \( H_{z,z'} \) is not constant, \( I_{z,z'} \) is a holomorphic smooth curve of \( \mathbb{C}^n \). Hence \( \bigcup_{(z,z') \in \Lambda} I_{z,z'} \) is of Lebesgue measure 0. Therefore, \( [\theta^a] \) is injective on \( \overline{W} \setminus S \) for almost all \( a \in \mathbb{C}^n \) if \( n \geq 3 \). The proof is complete.

**Proof of Theorem 1.** Lemma 10 reduces the proof to the case when the hypothesis of Theorem 2 holds. In that reduced case, Theorem 1 follows from Theorems 2 and 3.

**Proof of Theorem 2.** First, we apply Lemma 11 to approximate initial data by data inducing a well-defined canonical map \( \{\omega_0 : \omega_1 : \omega_2\} \) in a neighbourhood of \( X \) with the same singularities. Then, starting with \( (\theta_\ell)_{0 \leq \ell \leq 2} = \pi^*\omega \), we apply the adjunction Lemma 12 to get \( \theta_3, \ldots, \theta_{n+1} \in C_\infty(\overline{W}) \) such that the canonical map associated to \( \theta = (\theta_\ell)_{0 \leq \ell \leq n+1} \) almost embeds \( W \) into \( \mathbb{C}P_{n+1} \) and separates \( S \) from \( \overline{W} \setminus S \). Then, for \( a = ((a_\ell \nu)_{0 \leq \ell \leq n-\nu})_{1 \leq \nu \leq n-2} \in \mathbb{C}^n \times \mathbb{C}^{n-1} \times \cdots \times \mathbb{C}^3 \), we set
\[
(\theta_{0,\ell})_{1 \leq \ell \leq n+1} = (\theta_\ell)_{1 \leq \ell \leq n+1},
\]
and if \( 0 \leq \nu \leq n-2 \),
\[
(\theta_{\nu+1,\ell})_{1 \leq \ell \leq n-\nu} = (\theta_{\nu,\ell} - a_\ell \nu \theta_{\nu,n-\nu+1})_{1 \leq \ell \leq n-\nu}.
\]
Applying Lemmas 13 and 14 inductively, we get that \( \sigma = [\theta_0 : \theta_{n-1,1} : \theta_{n-1,2}] \) is an immersion of \( \overline{W} \) which embeds \( \gamma \) and satisfies \( (\sigma|_{\overline{W}})^{-1}(\sigma(\gamma)) = \gamma \). By construction,
$\sigma = [\theta_0 : \theta_1 + K_1 : \theta_2 + K_2]$, where $K_\ell = \sum_{j \geq 3} b_{\ell,j} \theta_j$, each $b_{\ell,j}$ being a universal polynomial in the coordinates of $a$. Hence, $\sigma$ has the same singularities and residues as $\pi^* \omega$ so that $\pi_* \sigma$ is well defined. Moreover, $K_\ell \mid_\gamma$ can be chosen arbitrarily small in $C^\infty(W)$. Moreover, using the notation of Lemma 11,

$$K_\ell = \sum_{j \geq 3} b_{\ell,j} dh_j = dk_\ell, \quad \text{where} \quad k_\ell = \sum_{j \geq 3} b_{\ell,j} h_j.$$  

As $K_\ell$ is holomorphic, the function $R_\ell = 2 \Re k_\ell$ is harmonic, is arbitrarily small on $C^\infty(W)$, and satisfies $\partial R_\ell = K_\ell$. Then $(V_\ell)_{0 \leq \ell \leq 2} = (U_\ell + \pi_* R_\ell)_{0 \leq \ell \leq 2}$ is a well-defined harmonic distribution which has the expected properties.

**Proof of Theorem 4.** Theorem 4 is the smooth case in Theorem 5.

**Proof of Theorem 5.** If in the above lines establishing Theorem 1 one forgets that the $(1,0)$-forms appearing are the image by $\partial$ of a harmonic distribution, one gets a proof for Theorem 5 — strictly speaking when $n = 3$, but the case $n \geq 4$ is obtained in the same way.

**Remark.** When $X$ is an open bordered nodal Riemann surface and $c$ an admissible family, a 4-DN-datum is defined as a triple $(\gamma, u, \theta u)$, where $\gamma$ is the oriented boundary of $X$, $u = (u_\ell)_{0 \leq \ell \leq 3} \in C^\infty(\gamma)^4$, $\theta u = (\partial \tilde{u}_\ell^c |_{\gamma})_{0 \leq \ell \leq 3}$, and $[(\partial \tilde{u}_\ell^c)]$ embeds $X$ in $\mathbb{C}P_3$. Then, a byproduct of the proof of Theorem 4 is that

$$\{u \in C^\infty(\gamma)^4, (\gamma, u, \theta u) \text{ is a 4-DN-datum}\}$$

is a dense open subset of $C^\infty(\gamma)^4$.

### § 4. Proofs for the compact case

Let $Z$ be a compact Riemann surface equipped with a Kähler form $\omega$ such that

$$\int_Z \omega = 1.$$  

Let $*$ be the Hodge operator on forms and $\delta = -\ast d \ast$ be the adjoint of the unbounded operator $d: L^2_{p,q}(Z) \rightarrow L^2_{p,q+1}(Z)$, where $L^2_{r,s}(Z)$ denotes the space of $(r,s)$-forms with coefficients in $L^2$. We denote by $\Delta = \delta d + d \delta$ the Laplace-Beltrami operator and by $G$ a Green function for it, that is, a smooth real-valued function defined on $Z \times Z$, outside its diagonal, such that for every $z \in Z$, the function $G_z = G(z, \cdot)$ satisfies

$$\Delta G_z = \delta_z - 1 \quad (4.1)$$

in the sense of currents, where $\delta_z$ is the Dirac measure. It is a classical result (see, for instance, [34]) that $G$ is symmetric, $|G(z, \zeta)| = O(\ln \text{dist}(z, \zeta))$ for distances associated to hermitian metrics on $Z$, and that $z \mapsto \int_Z G_z \omega$ is constant; as $G$ is unique up to an additive constant, we choose the one for which this constant is 0.

As $Z$ is compact, harmonic functions on $Z$ are constant and the Hodge-De Rham orthogonal decomposition (see [35], [36] or [34]) takes the form

$$\varphi = H \varphi + \Delta G \varphi = H \varphi + G \Delta \varphi,$$

where $H \varphi = \int_Z \varphi \omega$ and $G \varphi$ is the function $Z \ni z \mapsto \int_Z \varphi G_z \omega$.  


Lemma 15. Let $z$ be a point of $Z$ and $w$ be a coordinate for $Z$ centred at $z$. Then $G_z - \frac{1}{\pi} \ln |w|$ extends as a smooth function near $z$ and the residue of $G_z$ at $z$, that is
\[
\lim_{\varepsilon \to 0^+} \frac{1}{2\pi i} \int_{|w| = \varepsilon} \partial G_z,
\]
is equal to 1.

Proof. This is a plain consequence of the ellipticity of $\Delta$ and of the fact that $\Delta(G_z - \frac{1}{\pi} \ln |w|) = -1$ on a neighbourhood of $z$. The proof is complete.

We also need the following lemma which is a minor adaptation of classical results (see [37]) about $L^p_m(Z \setminus S)$, $(p, m) \in [1, \infty] \times \mathbb{N}$, which is the Sobolev space of distributions on $Z \setminus \overline{S}$ whose total differentials up to order $m$ are in $L^p(Z \setminus S)$, $Z$ being equipped with any hermitian metric.

Lemma 16. Let $S$ be a smooth open subset of $Z$. Then there exists an extension operator $E: C^\infty(Z \setminus S) \to C^\infty(Z)$ which is continuous from $L^p_m(Z \setminus S)$ to $L^p_m(Z)$ for any $(p, m) \in [1, \infty] \times \mathbb{N}$, sends $C^\infty(Z \setminus S, \mathbb{R})$ to $C^\infty(Z \setminus S, \mathbb{R})$, and is such that $\mathcal{H} \circ E = 0$.

Proof. From [37], we get an extension operator $E_0$, sending (real-valued) functions on $Z \setminus S$ to (real-valued) functions on $Z$, which is continuous from $L^p_m(Z \setminus \overline{S})$ to $L^p_m(Z \setminus S)$ for any $(p, m) \in [1, \infty] \times \mathbb{N}$. Choose open subsets $U_1, V_1$ of $Z$ and $\chi_1, \chi_2 \in C^\infty(Z, [0, 1])$ such that
\[
Z \setminus S \subset \text{Supp } \chi_1 \subset U_1 \subset \subset V_1, \quad \chi_1|_{U_1} = 1, \quad \text{Supp } \chi_2 \subset V_1 \setminus \overline{U_1}, \quad \int_Z \chi_2 \omega = 1.
\]
Then the extension operator $E$ defined by
\[
\forall f \in C^\infty(Z \setminus S) \quad Ef = \chi_1 E_0 f - \chi_2 \int_Z (E_0 f) \chi_1 \omega
\]
has the same continuousness as $E_0$, and for $f \in C^\infty(Z \setminus S)$,
\[
\mathcal{H}(E \varphi) = \int_Z \chi_1 E_0 f \omega - \left( \int_Z \chi_2 \omega \right) \int_Z (E_0 f) \omega = 0.
\]
The proof is complete.

We can now prove Theorems 7 and 6.

Proof of Theorem 7. Lemma 9 enables us to reduce the proof for functions which are restrictions on $Z \setminus S$ of harmonic functions on $Z \setminus \overline{S'}$, where $S'$ is a relatively compact smoothly bordered open subset of $S$. Let $\varphi: Z \to \mathbb{R}$ be harmonic in $Z \setminus S$, and $\tilde{\varphi} = E \varphi$, where $E: C^\infty(Z \setminus S') \to C^\infty(Z)$ is an extension operator, as in Lemma 16. Consider a family $(S_\nu')_{1 \leq \nu \leq N}$ of mutually disjoint open conformal discs of diameter at most $\varepsilon \in \mathbb{R}_+^*$ such that $\overline{S'} = \bigcup_{1 \leq \nu \leq N} \overline{S_\nu'}$. Using the mean value lemma, one can find for each $\nu$, $\zeta_\nu \in S_\nu'$ such that
\[
\int_{S_\nu'} (\Delta \tilde{\varphi}) \omega = I_\nu m_\nu,
\]
where \( I_\nu = \int_{S'_\nu} \omega \) and \( m_\nu = (\Delta \tilde{\varphi})(\zeta_\nu) \). Set \( P_\varepsilon = \{ \zeta_\nu, 1 \leq \nu \leq n \} \) and \( Z_\varepsilon = Z \setminus P_\varepsilon \).

The function \( \varphi_\varepsilon \) defined on \( Z_\varepsilon \) by the formula

\[
\varphi_\varepsilon(z) = \sum_\nu I_\nu m_\nu G(\zeta_\nu, z),
\]

is real analytic on \( Z_\varepsilon \), real-valued provided \( \varphi \) is so, and, according to (4.1) and to the choice of \( (\zeta_\nu) \), satisfies

\[
\Delta \varphi_\varepsilon = \sum_\nu I_\nu m_\nu (\delta \zeta_\nu - 1) = \sum_\nu I_\nu m_\nu \delta \zeta_\nu - \int_{S'_\nu} (\Delta \tilde{\varphi}) \omega.
\]

As \(^1 (\Delta \tilde{\varphi}) \omega = d^c d \tilde{\varphi} \), Stokes’ formula yields

\[
\int_{S'} (\Delta \tilde{\varphi}) \omega = - \int_{Z \setminus S'} d^c d \tilde{\varphi} = 0,
\]

and it appears that \( \varphi_\varepsilon \) is harmonic on \( Z_\varepsilon \). The singularity of \( \varphi_\varepsilon \) at \( \zeta_\nu \) is the same as that of \( G_{\zeta_\nu} \) at \( \zeta_\nu \), which is a logarithmic isolated one. In addition,

\[
\sum_\nu |\text{Res}(\partial \varphi_\varepsilon, \zeta_\nu)| \leq \sum_\nu \int_{S'_\nu} |\Delta \tilde{\varphi}| \omega \leq C \text{te} \| \tilde{\varphi} \|_{L^2(\delta S')} \leq C \text{te} \| \varphi \|_{L^2(Z \setminus S)}.
\]

It remains to estimate how \( \varphi_\varepsilon \) approaches \( \varphi \). As \( H^f \circ E = 0 \), Hodge’s identity (4.2) gives \( \tilde{\varphi} = \mathcal{G} \Delta \tilde{\varphi} \). As \( \tilde{\varphi} = \varphi \) is harmonic in \( Z \setminus S' \), the symmetry of \( G \) yields that for any \( z \) in \( Z \setminus S' \),

\[
\varphi(z) - \varphi_\varepsilon(z) = \int_{S'} (\Delta \tilde{\varphi}) G_z \omega - \sum_\nu \int_{S'_\nu} (\Delta \tilde{\varphi}) G(\zeta_\nu, z) \omega = \sum_\nu I_\nu(z),
\]

where

\[
I_\nu(z) = \int_{\xi \in S'_\nu} (\Delta \tilde{\varphi})(\zeta) [G_z(\zeta) - G_z(\zeta_\nu)] \omega(\zeta).
\]

As \( Z \setminus S \) is a compact subset of \( Z \setminus S' \) and each \( S'_\nu \) has diameter at most \( \varepsilon \), this implies

\[
\| \varphi - \varphi_\varepsilon \|_{0, Z \setminus S} \leq C \text{te} \| G \|_{1, (Z \setminus S) \times \overline{S}} \sum_\nu \int_{S'_\nu} |\Delta \tilde{\varphi}| \omega \leq C \text{te} \varepsilon \leq C_0 \| \varphi \|_{L^2(\delta Z \setminus S)} \tag{4.3}
\]

where \( C_0 \) depends only on \( (S, S') \), where \( \| G \|_{1, (Z \setminus S') \times \overline{S}} \) is the supremum norm on \( (Z \setminus S) \times \overline{S} \) of \( G \) and its full differential with respect to its second variable. As \( \varphi \) and \( \varphi_\varepsilon \) are harmonic in \( Z \setminus S' \), (4.3) implies that for any \( m, \) there is \( C_m \in \mathbb{R}_+ \) depending only on \( (S, S') \) such that

\[
\| \varphi - \varphi_\varepsilon \|_{m, Z \setminus S} \leq C_m \varepsilon \| \varphi \|_{L^2(\delta Z \setminus S)}.
\]

The proof is complete.

\(^1\)Let us fix \( z \) in \( Z \) and a geodesic coordinate \( w \) centred at \( z \). Then \( \omega = i dw \wedge dw + O(|w|^2) \). Let \( f \) be a function of class \( C^2 \) near \( z \). As \( * \) acts as multiplication by \(-i \) (resp., \( i \)) on forms of bidegree \((1, 0)\) (resp., \((0, 1)\)), and as \(*w = 1\), we get \(*df = -i \frac{\partial f}{\partial w} dw + i \frac{\partial f}{\partial \overline{w}} d\overline{w} \) and \( df = -d \ast df = -2 \frac{\partial^2 f}{\partial w \partial \overline{w}} i dw \wedge d\overline{w} \). On the other hand, \( dd^c f = 2 \frac{\partial^2 f}{\partial w \partial \overline{w}} i dw \wedge d\overline{w} \). Evaluation at \( z \) yields \( (dd^c f)_z = -(\Delta f)(z) \omega_z \).
Proof of Theorem 6. We assume without loss of generality that $S$ is smooth so that we can consider the oriented boundary $\gamma$ of $Z \setminus S$; set $u = (U_{Z,\ell}^{a,c,}\gamma)_{0 \leq \ell \leq 2}$ and $\theta u = ((\partial U_{Z,\ell}^{a,c,}\gamma)_{0 \leq \ell \leq 2}$. We apply Theorem 1 to find in $C^\infty(Z \setminus S)^3$ a triple $(V_1^\ell)$ of harmonic functions on $Z \setminus S$ such that $(V_1^\ell)$ is arbitrarily close to $(U_{Z,\ell}^{a,c,}\gamma)$ in $C^\infty(Z \setminus S)^3$ and the canonical map $\Phi$ associated to $(V_1^\ell)$ is an almost embedding from $Z \setminus S$ to $\mathbb{CP}_2$, which means in particular that $\Phi$ embeds $\gamma$ into $\mathbb{CP}_2$ and $\Phi^{-1}(\delta) = \gamma$, where $\delta = \Phi(\gamma)$. In fact, since $(U_{Z,\ell}^{a,c,}\gamma)$ is smooth in a neighbourhood of $Z \setminus S$, we are in the situation of Theorem 2, whose proof concludes that $(V_1^\ell - U_{Z,\ell}^{a,c,}\gamma) = (\text{Re } H_\ell)$, where $(H_\ell)$ is a triple of holomorphic functions on a neighbourhood of $Z \setminus S$.

Consider now $\varepsilon \in \mathbb{R}_+^*$. We apply Theorem 7 to get, for each $\ell \in \{0, 1, 2\}$, a real-valued function $R_\ell$ which is harmonic outside a finite subset $P_\ell$ of $S$, has only logarithmic isolated singularities at points of $P_\ell$, whose restriction to $Z \setminus S$ is arbitrarily close to $\text{Re } H_\ell$ in $C^\infty(Z \setminus S)$, and such that

$$|\varkappa|_1 \leq C \| \text{Re } H_\ell \|_{L^2_1(Z \setminus S)},$$

(4.4)

where $C$ is some constant and $\varkappa = (\varkappa_\ell)_{0 \leq \ell \leq 2}$, where $\kappa_\ell$ denotes the family of residues of $R_\ell$ at points of $P_\ell$, $0 \leq \ell \leq 2$. By construction, $(V_1^\ell) = (U_{Z,\ell}^{a,c} + R_\ell)$ differs from $U_{Z,\ell}^{a,c,p,\varkappa}$ only by an additive constant, is arbitrarily close to $(V_1^\ell)$ in $C^\infty(Z \setminus S)^3$, and, thanks to (4.4), $|\varkappa|_1 \leq \varepsilon$, provided $(H_\ell)$ is sufficiently small in $C^\infty(Z \setminus S)^3$.

When $(V_1^\ell)$ is sufficiently close to $(V_1^\ell)^\ell$, $(\partial V_1^\ell)$ induces a canonical map $\Psi$ from $Z \setminus S$ to $\mathbb{CP}_2$, which is an immersion and embeds $\gamma$ into $\mathbb{CP}_2$. Since $\Phi$ is an almost embedding, we can find smoothly bordered open neighbourhoods $\Gamma$ and $\Delta$ of $\gamma$ and $\delta$ in $X$ and $Y = \Phi(X)$, respectively, such that $\Phi|\overline{\Delta}$ is a diffeomorphism. Set $\Psi(X) = Y'$, $\Psi(\overline{\Delta}) = \Delta'$, $\delta' = \Psi(\delta)$, consider any hermitian metric on $\mathbb{CP}_2$ and denote by $h$ the associated Hausdorff distances between subsets of $Z$. If $(V_1^\ell)$ is close enough to $(V_1^\ell)^\ell$, $\Psi|\overline{\Delta}$ is also a diffeomorphism and $h(Y \setminus \Delta, Y' \setminus \Delta') + h(\delta, \delta')$ can be made arbitrarily small and, in particular, less that dist$(Y \setminus \Delta, \delta)$, which is a positive number since $\Phi^{-1}(\delta) = \gamma)$. This forbids $\Phi^{-1}(\delta') \cap \Delta' = \varnothing$. Hence $\Phi^{-1}(\delta') = \gamma'$ and we can apply Theorem 3 to conclude that $\Psi$ is an almost embedding of $Z \setminus S$. Thus, $(a,c,p,\varkappa) \in E_{Z,n}(S)$, where $n = \max_{0 \leq \ell \leq 2} \text{Card } p_\ell$. The proof is complete.

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