CONNES INTEGRATION FORMULA WITHOUT SINGULAR TRACES

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Abstract. A version of Connes Integration Formula which provides concrete asymptotics of the eigenvalues is given. This radically extending the class of quantum-integrable functions on compact Riemannian manifolds.

1. Introduction

Traditional noncommutative integration theory is based on normal linear functionals on von Neumann algebras, see [26] and the monographs [5], [23], [32] (among many). So it is somewhat surprising, and a disparity, that the formula (for some trace \( \varphi \) on the ideal \( \mathcal{L}_{1,\infty} \) and for some fixed \( T \in \mathcal{L}_{1,\infty} \))

\[
\varphi(A) = \varphi(AT), \quad A \in B(H),
\]

with its obscured normality, and not the formula (for some fixed \( T \in \mathcal{L}_1 \))

\[
\varphi(A) = \text{Tr}(AT), \quad A \in B(H),
\]

appears as the analogue of integration in noncommutative geometry. That it does is due to numerous results of A. Connes achieved with the Dixmier trace, see [7], §IV in [6] and [8] (as a sample). In Connes’ noncommutative geometry the formula (1) has been termed the noncommutative integral, see e.g. p.297 in [13] or p.478 in [14], due to the link to noncommutative residues in differential geometry described by the theorem of Connes, see Theorem 1 in [7] or Theorem 7.18 on p.293 in [13]. Below, we describe the special case of Connes theorem, known as Connes Integration Formula.

Let \((X, g)\) be a compact \(d\)-dimensional Riemannian manifold (see e.g. p.11 in [24]) and let vol\(_g\) be the Riemannian volume (see e.g. p.15 in [24]). Let \(\Delta_g\) be the Laplace-Beltrami operator on \(X\) (see Section 2.4 in [33]). An integration formula due to Connes\(^1\) reads as follows.

\[
\varphi(M_f(1 - \Delta_g)^{-\frac{d}{2}}) = \frac{\text{Vol}(S^{d-1})}{d(2\pi)^d} \int_X f \text{dvol}_g, \quad f \in C^\infty(X),
\]

where \(\text{vol}_g\) is the volume form corresponding to the Riemannian metric \(g\) and where \(\varphi\) is a positive normalised trace on the ideal \(\mathcal{L}_{1,\infty}\). By the tracial property, one can rewrite the preceding formula as

\[
\varphi((1 - \Delta_g)^{-\frac{d}{2}}M_f(1 - \Delta_g)^{-\frac{d}{2}}) = \frac{\text{Vol}(S^{d-1})}{d(2\pi)^d} \int_X f \text{dvol}_g, \quad f \in C^\infty(X).
\]

The following question was asked by Connes during his 70-th anniversary conference.

\(^1\)Connes used a special class of traces known as Dixmier traces. The formula, as stated, appears in Theorem 11.7.10 in [21].
Question 1.1. Is it possible to prove directly an asymptotic formula for the eigenvalues of the operator
\[ M_f(1 - \Delta_g)^{-\frac{d}{2}} \text{ or } (1 - \Delta_g)^{-\frac{d}{2}} M_f(1 - \Delta_g)^{-\frac{d}{2}} \]
which allows to deduce the Integration Formula without involving ultrafilters (extended limits, Banach limits or similar tools)?

In this paper, we answer the Question 1.1 in the affirmative. In fact, the following result radically extends the class of functions which admit a non-commutative integration formulae from [19] and [17] to the realm where the latter formulae are false.

Consider the finite measure space \((X, \nu_0)\). Let \(M(t) = t \log(e + t), t > 0\), and let \(L_M(X, \nu_0)\) be the Orlicz space associated with this function.

**Theorem 1.2.** Let \((X, g)\) be a compact \(d\)-dimensional Riemannian manifold and let \(\Delta_g\) be the Laplace-Beltrami operator. For every real-valued \(f \in L_M(X, \nu_0)\), we have
\[
\lim_{t \to \infty} t\mu(t, \left((1 - \Delta_g)^{-\frac{d}{2}} M_f(1 - \Delta_g)^{-\frac{d}{2}}\right)_+) = \frac{\Vol(S^{d-1})}{d(2\pi)^d} \int_X f_+ d\nu_0,
\]
\[
\lim_{t \to \infty} t\mu(t, \left((1 - \Delta_g)^{-\frac{d}{2}} M_f(1 - \Delta_g)^{-\frac{d}{2}}\right)_-) = \frac{\Vol(S^{d-1})}{d(2\pi)^d} \int_X f_- d\nu_0.
\]

Here, \(t \to \mu(t, T)\) is the singular value function of the operator \(T\) (see the definition in Subsection 2.1).

Theorem 1.2 strengthens/complements a number of earlier results in the literature (e.g. Theorems 2.8 and 5.9 in [19], Theorem 1.7 in [17], Theorem 11.7.10 in [21], Theorems 1.1 and 1.2 in [20]).

We caution the reader that the symmetric form of Connes Integration Formula is necessary if one wants to integrate the functions which do not belong to \(L_2(X, \nu_0)\). For functions from \(L_2(X, \nu_0)\), the (asymmetric) integration formula appeared in Corollary 7.24 in [17] (see also Theorem 11.7.10 in [21]). It is established in Theorem 2.5 in [19] that, for \(f \notin L_2(X, \nu_0)\), the operator \(M_f(1 - \Delta_g)^{-\frac{d}{2}}\) cannot belong to \(L_2\) (and, therefore, to \(L_{1, \infty}\)). Thus, Theorem 1.2 yields a non-trivial extension of integration formulae from [19] and [17] to the realm where the latter formulae are false.

The proof of Theorem 1.2 is based on a number of recent advances and discovery of a connection of old Birman and Solomyak results with our theme.

In [29], Solomyak proved the following specific Cwikel-type estimate in even dimension:
\[
\left\| M_f(1 - \Delta_{T_d})^{-\frac{d}{2}} \right\|_{L^\infty_{\nu_0}} \leq c_d \| f \|_{L_{M}^{(2)}}, \quad f \in L_{M}^{(2)}(T^d).
\]

Here, \(L_{2, \infty}\) is the weak Hilbert-Schmidt class and \(L_{M}^{(2)}\) is the 2-convexification of the Orlicz space \(L_{M}\). This result was extended to an arbitrary dimension in [31].

In this paper, we extend the estimate (2) to an arbitrary compact Riemannian manifold.

**Theorem 1.3.** Let \((X, g)\) be a compact \(d\)-dimensional Riemannian manifold and let \(\Delta_g\) be the Laplace-Beltrami operator. For every \(f \in L_{M}^{(2)}(X, \nu_0)\), we have
\[
\left\| M_f(1 - \Delta_g)^{-\frac{d}{2}} \right\|_{L^\infty_{\nu_0}} \leq C_{X,g} \| f \|_{L_{M}^{(2)}}.
\]
In Section 5 we prove Theorem 1.2 and compare it with the work of Birman and Solomyak [2].

For Euclidean space \( \mathbb{R}^d \), the following asymptotic formula was established by Birman and Solomyak [2]. If \( f \in L_p(\mathbb{R}^d), p > 2q, q > 1 \), is a positive compactly supported function, then Theorem 1 in [24] (taken with \( \Phi = 1 \) and \( \alpha = -\frac{d}{q} \)) yields

\[
\lim_{n \to \infty} n^{\frac{1}{q}} \mu \left( n, M_{f^\frac{1}{q}} \left( -\Delta_{\mathbb{R}^d} \right)^{-\frac{d}{2q}} M_{f^\frac{1}{q}} \right) = c_q \| f \|_q.
\]

Should such a formula hold for \( q = 1 \), it would imply a version of Theorem 1.2 for compactly supported functions on Euclidean spaces.

However, we are working on a compact manifold, not on Euclidean space and we do not have the Birman-Solomyak asymptotic formula for \( p = 1 \). For these reasons, our approach below is very different to that in [2].

2. Preliminaries

2.1. Trace ideals. The following material is standard; for more details we refer the reader to [21, 28]. Let \( H \) be a complex separable infinite dimensional Hilbert space, and let \( B(H) \) denote the set of all bounded operators on \( H \), and let \( K(H) \) denote the ideal of compact operators on \( H \). Given \( T \in K(H) \), the sequence of singular values \( \mu(T) = \{ \mu(k, T) \}_{k=0}^\infty \) is defined as:

\[
\mu(k, T) = \inf \{ \| T - R \|_\infty : \text{rank}(R) \leq k \}.
\]

It is often convenient to identify the sequence \( (\mu(k, T))_{k=0}^\infty \) with a step function \( \sum_{k \geq 0} \mu(k, T) \chi_{(k,k+1)} \).

Let \( p \in (0, \infty) \). The weak Schatten class \( L_{p, \infty} \) is the set of operators \( T \) such that \( \mu(T) \) is in the weak \( L_p \)-space \( l_{p, \infty} \), with the quasi-norm:

\[
\| T \|_{p, \infty} = \sup_{k \geq 0} (k + 1)^{\frac{1}{p}} \mu(k, T) < \infty.
\]

Obviously, \( L_{p, \infty} \) is an ideal in \( B(H) \). We also have the following form of Hölder’s inequality,

\[
\| TS \|_{r, \infty} \leq c_{p,q} \| T \|_{p, \infty} \| S \|_{q, \infty}
\]

where \( \frac{1}{r} = \frac{1}{p} + \frac{1}{q} \), for some constant \( c_{p,q} \). Indeed, this follows from the definition of these quasi-norms and the inequality (see e.g. [11] Proposition 1.6, [12] Corollary 2.2)

\[
\mu(2n, TS) \leq \mu(n, T) \mu(n, S), \quad n \geq 0.
\]

The closure of the set of all finite rank operators in \( L_{p, \infty} \) is called the separable part of \( L_{p, \infty} \) and is denoted by \( (L_{p, \infty})_0 \).

The ideal of particular interest is \( L_{1, \infty} \), and we are concerned with traces on this ideal. For more details, see [21, Section 5.7] and [27]. A linear functional \( \varphi : L_{1, \infty} \to \mathbb{C} \) is called a trace if it is unitarily invariant. That is, for all unitary operators \( U \) and for all \( T \in L_{1, \infty} \) we have that \( \varphi(U^* TU) = \varphi(T) \). It follows that for all bounded operators \( B \) we have \( \varphi(BT) = \varphi(TB) \).

Every trace \( \varphi : L_{1, \infty} \to \mathbb{C} \) vanishes on the ideal of finite rank operators (such traces are called singular). In fact, \( \varphi \) vanishes on the ideal \( L_1 \) (see [10] or [21]). For the state of the art in the theory of singular traces and their applications in Non-commutative Geometry, we refer the reader to the survey [22].
2.2. Sobolev spaces on compact manifolds. In our definition of Sobolev space on compact manifolds, we follow [33].

Let \((X, g)\) be a compact \(d\)-dimensional Riemannian manifold and let \(\text{vol}_g\) be the Riemannian volume. If \(u \in L^2(X, \text{vol}_g)\), we say \(u \in W^{s, 2}(X, \text{vol}_g)\) provided that, on each chart \(U \subset X\), every \(\phi \in C_0^\infty(U)\), the element \(\phi u\) belongs to \(W^{s, 2}(U)\) (if \(U\) is identified with its image in \(\mathbb{R}^d\)). By the invariance under coordinate changes derived in Section 4.2 in [33], it suffices to work with any single coordinate cover of \(X\). If \(s = m\), a nonnegative integer, then \(W^{m, 2}(X, \text{vol}_g)\) is equal to the set of all \(u \in L^2(X, \text{vol}_g)\) such that, for any smooth vector fields \((X_l)_{l=1}^m\), we have \(X_1 \cdots X_m u \in L^2(X, \text{vol}_g)\).

The following theorem is stated in Section 4.3 in [33]. For the details of complex interpolation, we refer to the book [18].

**Theorem 2.1.** For every \(m \in \mathbb{Z}_+\), for every \(0 < \theta < 1\), we have

\[
[L^2(X, \text{vol}_g), W^{m, 2}(X, \text{vol}_g)]_\theta = W^{m \theta, 2}(X, \text{vol}_g).
\]

3. Proof of Theorem 1.3

Having Theorem 1.1 in [31] at hand, we derive Theorem 1.3. This part is based on and simultaneously improves upon the similar argument in [20].

We begin with the simplest case of Theorem 1.3 when the manifold \(X\) is given by \(\mathbb{T}^d\) with the flat metric.

**Lemma 3.1.** If \(f \in L^\infty(\mathbb{T}^d)\), then

\[
\|M_f\|_{\mathcal{L}_{2, \infty}(L^2(\mathbb{T}^d) \to L_2(\mathbb{T}^d))} \leq c_d \|f\|_{L^2_\infty(\mathbb{T}^d)}.
\]

**Proof.** We view the operator

\[
(M_f)_{W^{\frac{d-2}{2}, 2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)}
\]
as a combination

\[
(1 - \Delta)^{-\frac{d}{4}}_{L_2(\mathbb{T}^d) \to L_2(\mathbb{T}^d)} \circ (1 - \Delta)^{-\frac{d}{4}}_{W^{\frac{d-2}{2}, 2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)}.
\]

By definition, the operator

\[
(1 - \Delta)^{-\frac{d}{4}}_{W^{\frac{d-2}{2}, 2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d)}
\]
is an isometry between Hilbert spaces, and therefore

\[
\|M_f\|_{\mathcal{L}_{2, \infty}(W^{\frac{d-2}{2}, 2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))} = \left\|M_f(1 - \Delta)^{-\frac{d}{4}}\right\|_{\mathcal{L}_{2, \infty}(L_2(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}.
\]

Obviously,

\[
\left\|M_f(1 - \Delta)^{-\frac{d}{4}}\right\|_{2, \infty}^2 = \left\|M_f(1 - \Delta)^{-\frac{d}{4}}\right\|_{1, \infty}^2 = \left\|(1 - \Delta)^{-\frac{d}{4}} M_f M_f(1 - \Delta)^{-\frac{d}{4}}\right\|_{1, \infty}^2.
\]

Appealing now to Theorem 1.1 in [31], we obtain

\[
\left\|M_f(1 - \Delta)^{-\frac{d}{4}}\right\|_{2, \infty}^2 \leq c_d \|f\|_{L_\infty}^2 = c_d \|f\|_{L^2_\infty}^2.
\]

Combining three preceding displays, we complete the proof. \(\square\)
Since $X$ is a compact manifold, we may assume without loss of generality that our atlas consists of finitely many charts. Furthermore, we assume that, for each chart $(U, \gamma)$ in our atlas, the set $\gamma(U)$ is bounded. Thus, $\gamma(U)$ is compactly supported in a sufficiently large open box $(-N, N)^d$. By applying a dilation if necessary, we may assume without loss of generality that $\gamma(U)$ is compactly supported in $(-\pi, \pi)^d$. By identifying the edges of $(-\pi, \pi)^d$, we may view $\gamma$ as a continuous function $\gamma : U \to \mathbb{T}^d$.

Further, we assume that, in every chart $(U, \gamma)$ from our atlas, the metric tensor is bounded from above and from below. Hence, the measure $\operatorname{vol}_g \circ \gamma^{-1}$ is equivalent to the Haar measure in the following sense: those measures are mutually absolutely continuous and Radon-Nikodym derivatives are bounded. These considerations yield the following two lemmas.

**Lemma 3.2.** Let $(U, \gamma)$ be a chart.

(i) A linear mapping $U_\gamma : L^2(X, \operatorname{vol}_g) \to L^2(\mathbb{T}^d)$ defined by the formula

$$U_\gamma \xi = \chi_U(\gamma) \cdot (\xi \circ \gamma^{-1}), \quad \xi \in L^2(X, g),$$

is bounded;

(ii) A linear mapping $V_\gamma : L^2(\mathbb{T}^d) \to L^2(X, \operatorname{vol}_g)$ defined by the formula

$$V_\gamma \xi = \chi_U \cdot (\xi \circ \gamma), \quad \xi \in L^2(\mathbb{T}^d),$$

is bounded.

**Lemma 3.3.** Let $(U, \gamma)$ be a chart and and let $U_\gamma$ be as in Lemma 3.2. For every $f \in L^\infty(X, g)$, we have

$$\|U_\gamma f\|_{L^2(\mathbb{T}^d)} \leq C_U \|f\|_{L^2(X, \operatorname{vol}_g)}.$$

Our next lemma is a variant of Lemma 3.2 for Sobolev spaces.

**Lemma 3.4.** Let $(U, \gamma)$ be a chart and let $U_\gamma$ be as in Lemma 3.2. If $\phi \in C^\infty_c(U)$, then

$$U_\gamma M_\phi : W^{1, 2}(X, \operatorname{vol}_g) \to W^{1, 2}(\mathbb{T}^d)$$

is everywhere defined bounded mapping.

**Proof.** This is, essentially, a definition of Sobolev space $W^{1, 2}(X, \operatorname{vol}_g)$ (see Subsection 2.2 above).

**Lemma 3.5.** Let $(U, \gamma)$ be a chart and let $K \subset U$ be compact. For every $f \in L^\infty(X, \operatorname{vol}_g)$ supported in $K$, we have

$$\left\|M_f\right\|_{L^2_\infty(W^{1, 2}(X, \operatorname{vol}_g) \to L^2(X, \operatorname{vol}_g))} \leq C_{K, U, \gamma, X, g} \|f\|_{L^2(X, \operatorname{vol}_g)}.$$

**Proof.** Let $U_\gamma$ and $V_\gamma$ be as in Lemma 3.2. Note that

$$V_\gamma U_\gamma = M_{\chi_U}.$$

Choose $\phi \in C^\infty_c(U)$ such that $\phi = 1$ on $K$. We write

$$M_f = M_f M_\phi, \quad M_f = M_f V_\gamma, \quad M_f = V_\gamma U_\gamma M_f.$$

Thus,

$$M_f = V_\gamma U_\gamma M_f V_\gamma U_\gamma M_\phi = V_\gamma \cdot U_\gamma M_f V_\gamma \cdot U_\gamma M_\phi = V_\gamma \cdot M_{\chi_f} U_\gamma M_\phi.$$
Hence,
\[
\left\| M_f \right\|_{L_{2, \infty}(W^{2,2}(X, \text{vol}_g) \to L_2(X, \text{vol}_g))} \leq ABC,
\]
where
\[
A = \left\| V_\gamma \right\|_{L_2(\mathbb{T}^d) \to L_2(X, \text{vol}_g)}, \quad B = \left\| M_{U, f} \right\|_{L_{2, \infty}(W^{2,2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))}, \quad \text{and} \quad C = \left\| U_\gamma M_\phi \right\|_{W^{2,2}(X, \text{vol}_g) \to W^{2,2}(\mathbb{T}^d)}.
\]
The first factor \( A \) is finite by Lemma 3.5 (it depends on \( X, U \) and \( \gamma \)). The third factor \( C \) is finite by Lemma 3.6 (it depends not only on \( X, U \) and \( \gamma \), but also on \( \phi \) and, hence, on \( K \)). It follows from Lemma 3.1 and Lemma 3.3 that
\[
B = \left\| M_{U, f} \right\|_{L_{2, \infty}(W^{2,2}(\mathbb{T}^d) \to L_2(\mathbb{T}^d))} \leq c_2 ||U_\gamma f||_{L_2(\mathbb{T}^d)} \leq c_4 C_{U, \gamma} \| f \|_{L_M^{(2)}(X, \text{vol}_g)}.
\]
Combining these estimates, we complete the proof. \( \square \)

The next lemma extends the result of Lemma 3.5 by removing the assumption that \( f \) is supported in a chart.

**Lemma 3.6.** If \( f \in L_\infty(X, g) \), then
\[
\left\| M_f \right\|_{L_{2, \infty}(W^{2,2}(X, \text{vol}_g) \to L_2(X, \text{vol}_g))} \leq C_{X, g} \| f \|_{L_M^{(2)}(X, \text{vol}_g)}.
\]

**Proof.** Let \( \{ \phi_k \}_{k=1}^N \) be a partition of unity subordinate to an atlas of bounded charts \((U_k, \gamma_k)\). We write
\[
M_f = \sum_{i=k}^N M_f \phi_k.
\]
Recall the triangle inequality in \( L_{2, \infty} \) (it can be found e.g. in [21]):
\[
\left\| \sum_{k=1}^N A_k \right\|_{2, \infty} \leq 2 \sum_{k=1}^n \| A_k \|_{2, \infty}.
\]
Thus,
\[
\left\| M_f \right\|_{L_{2, \infty}(W^{2,2}(X, \text{vol}_g) \to L_2(X, \text{vol}_g))} \leq 2 \sum_{k=1}^N \left\| M_f \phi_k \right\|_{L_{2, \infty}(W^{2,2}(X, \text{vol}_g) \to L_2(X, \text{vol}_g))}.
\]
By Lemma 3.5 we have
\[
\left\| M_f \phi_k \right\|_{L_{2, \infty}(W^{2,2}(X, \text{vol}_g) \to L_2(X, \text{vol}_g))} \leq C_{\text{supp}(\phi_k), U_k, \gamma_k, X, g} \| f \phi_k \|_{L_M^{(2)}(X, \text{vol}_g)}.
\]
Thus,
\[
\left\| M_f \right\|_{L_{2, \infty}(W^{2,2}(X, \text{vol}_g) \to L_2(X, \text{vol}_g))} \leq 2 \sum_{k=1}^N C_{\text{supp}(\phi_k), U_k, \gamma_k, X, g} \| f \phi_k \|_{L_M^{(2)}(X, \text{vol}_g)} \leq 2 \left( \sum_{k=1}^N C_{E, \text{supp}(\phi_k), U_k, \gamma_k, X, g} \| \phi_k \|_{2, \infty} \right) \| f \|_{L_M^{(2)}(X, \text{vol}_g)}.
\]
\( \square \)

Not every author defines Sobolev space on compact manifolds as in [33]. An equally important definition is via powers of Laplace-Beltrami operator \( \Delta_g \). Those definitions are known to be equivalent. We only need one side of this equivalence as established in the following lemma.
Lemma 3.7. For every $s > 0$, the operator $(1 - \Delta_g)^{-\frac{s}{2}}$ is a well defined and bounded mapping from $L_2(X, \text{vol}_g)$ to $W^{s,2}(X, \text{vol}_g)$.

Proof. Let $H$ be a Hilbert space and let $P : \text{dom}(P) \to H$ be self-adjoint operator. Suppose, in addition, that $P \geq 1$. The space $H_s = \text{dom}(P^s)$ becomes a Hilbert space when equipped with the norm $\xi \to \|P^s\xi\|$, $\xi \in H^s$. It is immediate from this definition that $P^{-s} : H \to H_s$. Note the complex interpolation:

$$[H_{s_1}, H_{s_2}]_\theta = H_{(1-\theta)s_1 + \theta s_2}, \quad s_1 \neq s_2 \in \mathbb{R}_+.$$ 

In particular,

$$[H_0, H_m]_\theta = H_{\theta m}, \quad m \in \mathbb{Z}_+.$$ 

Now, let $H = L_2(X, \text{vol}_g)$ and $P = (1 - \Delta_g)^{\frac{s}{2}}$. Obviously, $H_{2m} = \text{dom}((1 - \Delta_g)^m)$ for $m \in \mathbb{Z}_+$. By elliptic regularity, $\text{dom}((1 - \Delta_g)^m) \subset W^{2m,2}(X, \text{vol}_g)$ (see e.g. Theorem 19.5.1 in [14]). By Gårding inequality (see e.g. Theorem 2.44 in [14]), we have

$$\|(1 - \Delta_g)^m u\|_{L_2(X, \text{vol}_g)} \approx_m \|u\|_{W^{2m,2}(X, \text{vol}_g)}, \quad u \in W^{2m,2}(X, \text{vol}_g).$$

Thus, $H_{2m} = W^{2m,2}(X, \text{vol}_g)$. By Theorem 3.7, we have

$$[L_2(X, \text{vol}_g), W^{2m,2}(X, \text{vol}_g)]_\theta = W^{2\theta m,2}(X, \text{vol}_g), \quad m \in \mathbb{Z}_+.$$ 

Take $s \in \mathbb{R}_+$, choose integer $2m > s$ and let $\theta = \frac{s}{2m}$. We have

$$H_s = W^{s,2}(X, \text{vol}_g), \quad s \in \mathbb{R}_+.$$ 

It follows immediately that

$$(1 - \Delta_g)^{-\frac{s}{2}} = P^{-s} : L_2(X, \text{vol}_g) = H \to H_s = W^{s,2}(X, \text{vol}_g), \quad s \in \mathbb{R}_+.$$ 

Proof of Theorem 1.3. Step 1: Suppose first that $f \in L_\infty(X, g)$. Obviously,

$$\left( M_f (1 - \Delta_g)^{-\frac{s}{2}} \right)_{L_2(X, \text{vol}_g) \to L_2(X, \text{vol}_g)} = (M_f)_{W^{\frac{s}{2},2}(X, \text{vol}_g) \to L_2} \circ \left( (1 - \Delta_g)^{-\frac{s}{2}} \right)_{L_2(X, \text{vol}_g) \to W^{\frac{s}{2},2}(X, \text{vol}_g)}.$$ 

By Lemma 3.7,

$$\left( (1 - \Delta_g)^{-\frac{s}{2}} \right)_{L_2(X, \text{vol}_g) \to W^{\frac{s}{2},2}(X, \text{vol}_g)}$$

is bounded. It follows that

$$\left\| M_f (1 - \Delta_g)^{-\frac{s}{2}} \right\|_{L_2, \infty(L_2(X, \text{vol}_g))} \leq AB,$$

where

$$A = \left\| M_f \right\|_{L_2, \infty(W^{\frac{s}{2},2}(X, \text{vol}_g) \to L_2(X, \text{vol}_g))}, \quad B = \left\| (1 - \Delta_g)^{-\frac{s}{2}} \right\|_{L_2(X, \text{vol}_g) \to W^{\frac{s}{2},2}(X, \text{vol}_g)}.$$ 

The assertion follows now from Lemma 3.6.

Step 2: Suppose now that $f \in L^{(2)}_M(X, \text{vol}_g)$. We claim that the operator

$$M_f (1 - \Delta_g)^{-\frac{s}{2}}$$

is bounded. It follows that

$$\left\| M_f (1 - \Delta_g)^{-\frac{s}{2}} \right\|_{L_2, \infty(L_2(X, \text{vol}_g))} \leq AB,$$

where

$$A = \left\| M_f \right\|_{L_2, \infty(W^{\frac{s}{2},2}(X, \text{vol}_g) \to L_2(X, \text{vol}_g))}, \quad B = \left\| (1 - \Delta_g)^{-\frac{s}{2}} \right\|_{L_2(X, \text{vol}_g) \to W^{\frac{s}{2},2}(X, \text{vol}_g)}.$$ 

The assertion follows now from Lemma 3.6.

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$^2$By the spectral theorem, it suffices to check it when $H = L_2(\Omega, \nu)$ and when $P$ is a multiplication operator. In this case, it is a standard result about complex interpolation of weighted $L_2$-spaces (see Theorem 5.4.1 in [14]).
is everywhere defined and bounded on $L_2(X, \text{vol}_g)$. Moreover, we claim that
\[
\left\| M_f (1 - \Delta_g)^{-\frac{4}{n}} \right\|_{\infty} \leq C_{X,g} \| f \|_{L_M^{(2)}(X, \text{vol}_g)}.
\]

For every $\xi \in L_2(X, \text{vol}_g)$, the function
\[
f \cdot (1 - \Delta_g)^{-\frac{4}{n}} \xi
\]
is well-defined and measurable (as a product of two measurable functions). We show that this function belongs to $L_2(X, \text{vol}_g)$ (and also establish the estimate for its $L_2$-norm from the above). Set
\[
f_n = \min\{|f|, n\}, \quad n \in \mathbb{N}.
\]

If follows from the Fatou Theorem that
\[
\| f \cdot (1 - \Delta_g)^{-\frac{4}{n}} \xi \|_{L_2(X, \text{vol}_g)} = \sup_{n \in \mathbb{N}} \| f_n \cdot (1 - \Delta_g)^{-\frac{4}{n}} \xi \|_{L_2(X, \text{vol}_g)}.
\]

On the other hand, it follows from Step 1 that
\[
\| f_n \cdot (1 - \Delta_g)^{-\frac{4}{n}} \xi \|_{L_2(X, \text{vol}_g)} \leq \| M_{f_n} (1 - \Delta_g)^{-\frac{4}{n}} \|_{\infty} \| \xi \|_{L_2(X, \text{vol}_g)} \leq
\]
\[
\leq \| M_{f_n} (1 - \Delta_g)^{-\frac{4}{n}} \|_{2, \infty} \| \xi \|_{L_2(X, \text{vol}_g)} \leq
\]
\[
\leq C_{X,g} \| f_n \|_{L_M^{(2)}(X, \text{vol}_g)} \| \xi \|_{L_2(X, \text{vol}_g)} \leq C_{X,g} \| f \|_{L_M^{(2)}(X, \text{vol}_g)} \| \xi \|_{L_2(X, \text{vol}_g)}.
\]

It follows that
\[
\| f \cdot (1 - \Delta_g)^{-\frac{4}{n}} \xi \|_{L_2(X, \text{vol}_g)} \leq C_{X,g} \| f \|_{L_M^{(2)}(X, \text{vol}_g)} \| \xi \|_{L_2(X, \text{vol}_g)}.
\]

This proves the claim.

**Step 3:** It follows from Step 1 that
\[
\left\| M_{f_n} (1 - \Delta_g)^{-\frac{4}{n}} \right\|_{L_2, \infty(L_2(X, \text{vol}_g))} \leq C_{X,g} \| f_n \|_{L_M^{(2)}(X, \text{vol}_g)} \leq C_{X,g} \| f \|_{L_M^{(2)}(X, \text{vol}_g)}.
\]

Using Step 2, we obtain
\[
\left\| M_{f_n} (1 - \Delta_g)^{-\frac{4}{n}} - M_{|f|} (1 - \Delta_g)^{-\frac{4}{n}} \right\|_{\infty} \leq C_{E,X,g} \| f_n - |f| \|_{L_M^{(2)}(X, \text{vol}_g)}.
\]

Thus,
\[
M_{f_n} (1 - \Delta_g)^{-\frac{4}{n}} \to M_{|f|} (1 - \Delta_g)^{-\frac{4}{n}}
\]
in the uniform norm as $n \to \infty$. By the Fatou property (see [28] and [21]) for the space $L_2, \infty$, it follows that
\[
\left\| M_{|f|} (1 - \Delta_g)^{-\frac{4}{n}} \right\|_{L_2, \infty(L_2(X, \text{vol}_g))} \leq C_{X,g} \| f \|_{L_M^{(2)}(X, \text{vol}_g)}.
\]

This completes the proof.
4. Abstract lemmas on the asymptotics of eigenvalues

The starting point is the assertion known (in various forms) since at least 1940’s.

**Lemma 4.1.** Let \((T_n)_{n \geq 0} \subset L_{1,\infty}\) be such that

\[
\lim_{t \to \infty} t\mu(t, T_n) = \alpha_n, \quad n \geq 0.
\]

If \(T_n \to T\) in \(L_{1,\infty}\), then \(\alpha_n \to \alpha\) and

\[
\lim_{t \to \infty} t\mu(t, T) = \alpha.
\]

**Proof.** To lighten the notations, we assume without loss of generality that \(\|T_n\|_{1,\infty} \leq 1\) for \(n \geq 0\). This also means that \(\|T\|_{1,\infty} \leq 1\).

**Step 1:** We show that \((\alpha_n)_{n \geq 0}\) converges (its limit will be denoted by \(\alpha\)).

Fix \(\epsilon \in (0, 1)\) and choose \(N\) such that \(\|T_n - T_m\|_{1,\infty} \leq \epsilon^2\) for \(n, m \geq N\). We have

\[
\mu(t, T_n) = \mu(t, T_m) + (T_n - T_m) \leq \mu\left(\frac{t}{1 + \epsilon}, T_m\right) + \mu\left(\frac{t\epsilon}{1 + \epsilon}, T_n - T_m\right) \leq \mu\left(\frac{t}{1 + \epsilon}, T_m\right) + 1 + \epsilon \cdot \|T_n - T_m\|_{1,\infty} \leq \mu\left(\frac{t}{1 + \epsilon}, T_m\right) + 2\epsilon.
\]

Thus,

\[
\lim_{t \to \infty} t\mu(t, T) \leq \lim_{t \to \infty} t\mu\left(\frac{t}{1 + \epsilon}, T_m\right) + 2\epsilon.
\]

In other words,

\[
\alpha_n \leq (1 + \epsilon)\alpha_m + 2\epsilon.
\]

Consequently,

\[
\alpha_n - \alpha_m \leq 2\epsilon + \epsilon\alpha_m \leq \epsilon + \epsilon\|T_m\|_{1,\infty} \leq 3\epsilon.
\]

Similarly,

\[
\alpha_m - \alpha_n \leq 3\epsilon.
\]

Finally,

\[
|\alpha_m - \alpha_n| \leq 3\epsilon, \quad m, n \geq N.
\]

Thus, \((\alpha_n)_{n \geq 0}\) is a Cauchy sequence and the claim in Step 1 follows.

**Step 2:** We show that

\[
\limsup_{t \to \infty} t\mu(t, T) \leq \alpha.
\]

Fix \(\epsilon \in (0, 1)\) and choose \(N\) such that \(\|T_n - T\|_{1,\infty} \leq \epsilon^2\) for \(n \geq N\). We have

\[
\mu(t, T) = \mu(t, T_n + (T_n - T_n)) \leq \mu\left(\frac{t}{1 + \epsilon}, T_n\right) + \mu\left(\frac{t\epsilon}{1 + \epsilon}, T_n - T_n\right) \leq \mu\left(\frac{t}{1 + \epsilon}, T_n\right) + 1 + \epsilon \cdot \|T_n - T_n\|_{1,\infty} \leq \mu\left(\frac{t}{1 + \epsilon}, T_n\right) + 2\epsilon t^{-1}.
\]

Thus,

\[
\limsup_{t \to \infty} t\mu(t, T) \leq \lim_{t \to \infty} t\mu\left(\frac{t}{1 + \epsilon}, T_n\right) + 2\epsilon = (1 + \epsilon)\alpha_n + 2\epsilon \leq \alpha_n + 3\epsilon.
\]

Passing \(n \to \infty\), we obtain

\[
\limsup_{t \to \infty} t\mu(t, T) \leq \alpha + 3\epsilon.
\]

Since \(\epsilon\) is arbitrarily small, the claim in Step 2 follows.

**Step 3:** We show that

\[
\alpha \leq \liminf_{t \to \infty} t\mu(t, T).
\]
Fix \( \epsilon \in (0, 1) \) and choose \( N \) such that \( \|T_n - T\|_{1, \infty} \leq \epsilon^2 \) for \( n \geq N \). We have
\[
\mu(t, T_n) = \mu(t, T_n + (T_n - T)) \leq \mu\left(\frac{t}{1 + \epsilon}, T\right) + \mu\left(\frac{t\epsilon}{1 + \epsilon}, T_n - T\right) \leq \mu\left(\frac{t}{1 + \epsilon}, T\right) + \frac{1 + \epsilon}{\epsilon t} \cdot \|T_n - T\|_{1, \infty} \leq \mu\left(\frac{t}{1 + \epsilon}, T\right) + 2\epsilon t^{-1}.
\]
Thus,
\[
\alpha_n = \lim_{t \to \infty} t \mu(t, T_n) \leq \liminf_{t \to \infty} t \mu\left(\frac{t}{1 + \epsilon}, T\right) + 2\epsilon = (1 + \epsilon) \liminf_{t \to \infty} t \mu(t, T) + 2\epsilon \leq \liminf_{t \to \infty} t \mu(t, T) + 3\epsilon.
\]
Passing \( n \to \infty \), we obtain
\[
\alpha \leq \liminf_{t \to \infty} t \mu(t, T) + 3\epsilon.
\]
Since \( \epsilon \) is arbitrarily small, the claim in Step 3 follows.

**Step 4:** Combining the results of Steps 2 and 3, we write
\[
\limsup_{t \to \infty} t \mu(t, T) \leq \alpha \leq \liminf_{t \to \infty} t \mu(t, T).
\]
Thus,
\[
\limsup_{t \to \infty} t \mu(t, T) = \alpha = \liminf_{t \to \infty} t \mu(t, T).
\]
The assertion follows immediately. \( \square \)

The following lemma is a by-product of recent studies of the authors jointly with J. Huang of operator \( \theta \)-Holder functions for quasi-norms [16].

**Lemma 4.2.** If \( T, S \in \mathcal{L}_{1, \infty} \) are self-adjoint operators, then
\[
\|T_+ - S_+\|_{1, \infty} \leq \varsigma_{abs}\|T - S\|_{1, \infty} 1/2 (\|T\|_{1, \infty} + \|S\|_{1, \infty}) 1/4.
\]

**Proof.** Equation (7) in [16] taken with \( f(t) = t^{\frac{1}{2}} \), \( t \in \mathbb{R} \), and with \( p = \theta = \frac{1}{2} \) reads as
\[
\|T_+^{\frac{1}{2}} - S_+^{\frac{1}{2}}\|_{2, \infty} \prec \varsigma_{abs}\|T - S\|_{1, \infty}^{\frac{1}{4}}.
\]
In particular,
\[
\|T_+^{\frac{1}{2}} - S_+^{\frac{1}{2}}\|_{2, \infty} \leq \varsigma_{abs}\|T - S\|_{1, \infty}^{\frac{1}{4}}.
\]
It is clear that
\[
T_+ - S_+ = T_+^{\frac{1}{2}}(T_+^{\frac{1}{2}} - S_+^{\frac{1}{2}}) + (T_+^{\frac{1}{2}} - S_+^{\frac{1}{2}})S_+^{\frac{1}{2}}.
\]
The assertion follows now from Hölder inequality. \( \square \)

**Corollary 4.3.** Let \( (T_n)_{n \geq 0} \subset \mathcal{L}_{1, \infty} \) be self-adjoint operators such that
\[
\lim_{k \to \infty} k \mu(k, (T_n)_+) = \alpha_n, \quad \lim_{k \to \infty} k \mu(k, (T_n)_-) = \beta_n, \quad n \geq 0.
\]
If \( T_n \to T \) in \( \mathcal{L}_{1, \infty} \), then \( \alpha_n \to \alpha, \beta_n \to \beta \) and
\[
\lim_{k \to \infty} k \mu(k, T_+) = \alpha, \quad \lim_{k \to \infty} k \mu(k, T_-) = \beta.
\]

**Proof.** By Lemma 4.2, we have \( (T_n)_+ \to T_+ \) and \( (T_n)_- \to T_- \) as \( n \to \infty \). The assertion follows from Lemma 4.1. \( \square \)

**Lemma 4.4.** Let \( T, S \in \mathcal{L}_{1, \infty} \) be self-adjoint elements such that \( T - S \in (\mathcal{L}_{1, \infty})^0 \). We have that \( T_+ - S_+ \in (\mathcal{L}_{1, \infty})^0 \).
Proof. Equation (7) in [16] taken with \( f(t) = t^{\frac{1}{2}}, t \in \mathbb{R} \), and with \( p = \theta = \frac{1}{2} \) reads as

\[
|T_+^{\frac{1}{2}} - S_+^{\frac{1}{2}}|^{\frac{1}{2}} \ll c_{abs}|T - S|^{\frac{1}{2}}.
\]

That is,

\[
(n + 1)\mu^\frac{1}{4}(n, T_+^{\frac{1}{2}} - S_+^{\frac{1}{2}}) \leq \sum_{k=0}^{n} \mu^\frac{1}{4}(k, T_+^{\frac{1}{2}} - S_+^{\frac{1}{2}}) \leq c_{abs} \sum_{k=0}^{n} \mu^\frac{1}{4}(k, T - S) = o(n^\frac{1}{2})
\]
as \( n \to \infty \). In other words,

\[
T_+^{\frac{1}{2}} - S_+^{\frac{1}{2}} \in (L_{2,\infty})_0.
\]

It is clear that

\[
T_+ - S_+ = T_+^{\frac{1}{2}}(T_+^{\frac{1}{2}} - S_+^{\frac{1}{2}}) + (T_+^{\frac{1}{2}} - S_+^{\frac{1}{2}})S_+^{\frac{1}{2}}.
\]

The assertion follows now from Hölder inequality.

\[\Box\]

**Lemma 4.5.** Let \( T, S \in L_{1,\infty} \) be self-adjoint elements such that \( T - S \in (L_{1,\infty})_0 \).

If

\[
\lim_{t \to \infty} t\mu(t, T) = \alpha,
\]

then

\[
\lim_{t \to \infty} t\mu(t, S) = \alpha.
\]

**Proof.** Fix \( \epsilon > 0 \). We have

\[
\mu(t, S) \leq \mu(t, T) + \mu(t, T - S) \leq \mu(t, T) + o(t^{-1}), \quad t \to \infty.
\]

Thus,

\[
\limsup_{t \to \infty} t\mu(t, S) \leq (1 + \epsilon)\alpha.
\]

On the other hand, we have

\[
\mu(t, T) \leq \mu(t, S) + \mu(t, T - S) = \mu(t, S) + o(t^{-1}), \quad t \to \infty.
\]

Thus,

\[
\alpha \leq \liminf_{t \to \infty} t\mu(t, S) = (1 + \epsilon)\liminf_{t \to \infty} t\mu(t, S).
\]

Since \( \epsilon > 0 \) is arbitrary, it follows that

\[
\limsup_{t \to \infty} t\mu(t, S) \leq \alpha \leq \liminf_{t \to \infty} t\mu(t, S).
\]

In other words,

\[
\limsup_{t \to \infty} t\mu(t, S) = \alpha = \liminf_{t \to \infty} t\mu(t, S).
\]

This completes the proof. \[\Box\]

Below, tensor product of sequences \( \alpha \) and \( \beta \) is a double sequence given by the formula

\[
(\alpha \otimes \beta)(k, l) = \alpha(k)\beta(l), \quad k, l \in \mathbb{Z}_+.
\]

Tensor product of a sequence and a function (on \((0, \infty))\) is defined similarly

\[
(\alpha \otimes f)(k, s) = \alpha(k)f(s), \quad k \in \mathbb{Z}_+, \quad s \in (0, \infty).
\]

**Lemma 4.6.** Let \( z(n) = \frac{1}{n+1} \). For every finite sequence \( \alpha \), there exists a limit

\[
\lim_{t \to \infty} t\mu(t, z \otimes \alpha) = ||\alpha||_1.
\]
Proof. Let $Z(t) = t^{-1}, t > 0$. Note the key fact
$$
\mu(Z \otimes \alpha) = \|\alpha\|_1 Z.
$$

Clearly, $z \leq Z$. Thus,

(4) $$
t \mu(t, z \otimes \alpha) \leq t \mu(t, Z \otimes \alpha) = \|\alpha\|_1.
$$

Suppose $\alpha$ is a sequence of length $n$. We have
$$
\mu(t, Z \otimes \alpha) \leq \mu(n, Z \chi_{(0,1)} \otimes \alpha) + \mu(t - n, Z \chi_{(1,\infty)} \otimes \alpha), \quad t > n.
$$

Obviously, $Z \chi_{(0,1)} \otimes \alpha$ is supported on a set of measure $n$. Thus,
$$
\mu(n, Z \chi_{(0,1)} \otimes \alpha) = 0.
$$

Consequently,

(5) $$
\|\alpha\|_1 = t \mu(t, Z \otimes \alpha) \leq t \mu(t - n, z \otimes \alpha), \quad t > n.
$$

The assertion follows by combining (4) and (5). □

In the following lemma (and further below), the notation $\oplus_{k \in \mathbb{Z}} T_k$ is a shorthand for an element $\sum_{k \in \mathbb{Z}} T_k \otimes e_k$ in the von Neumann algebra $B(H) \bar{\otimes} l_\infty(\mathbb{Z})$. Here, $e_k$ is the unit vector having the only non-zero component on the $k$-th position.

Lemma 4.7. Let $(T_k)_{1 \leq k \leq K} \subset L_{1,\infty}$ be such that
$$
\lim_{t \to \infty} t \mu(t, T_k) = \alpha_k, \quad 1 \leq k \leq K.
$$

It follows that
$$
\lim_{t \to \infty} t \mu(t, \bigoplus_{1 \leq k \leq K} T_k) = \sum_{1 \leq k \leq K} \alpha_k.
$$

Proof. Let $z(n) = \frac{1}{n+1}$. For every $1 \leq k \leq K$, choose $S_k \in L_{1,\infty}$ such that
$$
\mu(S_k) = \alpha_k z,
$$
and such that $S_k - T_k \in (L_{1,\infty})_0$. We have
$$
\mu\left( \bigoplus_{1 \leq k \leq K} S_k \right) = \mu\left( z \otimes \{\alpha_k\}_{1 \leq k \leq K} \right).
$$

It follows from Lemma 4.6 that
$$
\lim_{t \to \infty} t \mu(t, \bigoplus_{1 \leq k \leq K} S_k) = \sum_{1 \leq k \leq K} \alpha_k.
$$

On the other hand, we have
$$
\bigoplus_{1 \leq k \leq K} S_k - \bigoplus_{1 \leq k \leq K} T_k \in (L_{1,\infty})_0.
$$

The assertion follows now from Lemma □

4.5
5. Proof of Theorem 1.2

In this section, we prove an asymptotic formula for singular values in Theorem 1.2. It is modeled after the proof of Lemma 1 in [4].

We refer the reader to [25] for the theory of pseudo-differential operators.

Definition 5.1. Pseudo-differential operator \( Q : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d) \) is called compactly supported if there exists \( \phi \in \mathcal{C}_c^{\infty}(\mathbb{R}^d) \) such that \( M_\phi Q M_\phi = Q \).

The definition below should be compared with the Definition 10.2.24 in [21].

Definition 5.2. Pseudo-differential operator \( Q : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d) \) with the symbol \( q \) of order \( \text{ord}(Q) \) is called classical if there exists a sequence \( (q_n)_{n \leq \text{ord}(Q)} \) of functions on \( \mathbb{R}^d \times \mathbb{R}^d \) and sequence \( (m_n)_{n \leq \text{ord}(Q)} \) of real numbers such that

(i) sequence \( (m_n)_{n \leq \text{ord}(Q)} \) is strictly increasing, \( m_{\text{ord}(Q)} = \text{ord}(Q) \) and \( m_n \to -\infty \) as \( n \to -\infty \);
(ii) for every \( n \leq \text{ord}(Q) \), \( q_n \) is homogeneous of degree \( m_n \) in the second variable;
(iii) for every \( n \leq \text{ord}(Q) \), \( q_n \in \mathcal{C}_c^{\infty}(\mathbb{R}^d \times S^{d-1}) \) (i.e., function and all its derivatives are bounded);
(iv) for every \( k \leq \text{ord}(Q) \),

\[
q - \sum_{n=k}^{\text{ord}(Q)} q_n \cdot (1 - \phi)
\]

is a symbol of order \( m_{k-1} \).

Here, \( \phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \) does not depend on the first variable, is compactly supported in the second variable and equals 1 near 0.

The key ingredient is a re-statement of Theorems 1 and 2 in [3]. In the notation of [3], \( m = d, \mu = 1; a \) and \( c \) are smooth functions on \( \mathbb{R}^d \) compactly supported in the cube; \( b \) smooth (except at 0) function on \( \mathbb{R}^d \times \mathbb{R}^d \) compactly supported in the first argument and homogeneous of degree \( -d \) in the second argument (so that \( \gamma = 1 \) and \( \tau_1 = \cdots = \tau_d = 1 \)). The fact that \( a_\gamma \) is a Schur multiplier on \( L^\infty_\beta \) follows by writing \( a_{|\mathbb{R}^d \times \mathbb{R}^d|} \) as Fourier series in spherical functions (see a similar argument in Lemma 8.1 in [20]). That is, we are in the conditions of the third part of Theorem 1 in [3]. That theorem together with Theorem 2 in [3] yields the assertion below.

Theorem 5.3. Let \( Q \) be a classical compactly supported pseudo-differential operator of order \( -d \) on \( \mathbb{R}^d \). If \( Q \) is self-adjoint, then there exists a limit

\[
\lim_{t \to \infty} t \mu(t, Q).
\]

Proof. By Definition 5.2, we can find a smooth (except at 0) homogeneous of degree \( -d \) (in the second variable) function \( q_{-d} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C} \), a smooth compactly supported function \( \phi \) on \( \mathbb{R}^d \) such that \( \phi = 1 \) near 0 and a pseudo-differential operator \( R \) of order \( -d - \epsilon \) such that

\[
Q = T_{q_{-d}} \cdot (1 - \phi)(\nabla) + R.
\]

Since \( Q \) is compactly supported, it follows that there exists smooth compactly supported function \( \psi \) on \( \mathbb{R}^d \) such that

\[
Q = M_\psi Q M_\psi = M_\psi T_{q_{-d}} (1 - \phi)(\nabla) M_\psi + M_\psi RM_\psi.
\]
Since $Q$ is self-adjoint, it follows that
\[ Q = \Re(M\psi T_{q-d}(1 - \phi)(\nabla)M\psi) + \Re(M\psi RM\psi). \]
Note that $P = R(1 - \Delta)^{\frac{d}{2}}$ is a pseudo-differential operator of order 0. Hence, $M\psi P$ is bounded.

\[ M\psi RM\psi = M\psi P \cdot (1 - \Delta)^{-\frac{d}{2}} M\psi \in L^{\infty}_{\text{loc}} \subset (L^{1,\infty})_0. \]

Theorem 2 in [3] asserts the existence of the limit
\[ \lim_{t \to \infty} t\mu(t, \Re(M\psi T_{q-d}(1 - \phi)(\nabla)M\psi)). \]
The assertion follows now from Lemma 4.5.

Now, we want a similar result for compact manifolds.

**Definition 5.4.** Pseudo-differential operator $P : S(X,g) \to S(X,g)$ is called compactly supported in the chart $(U,\gamma)$ if there exists a smooth function $\phi$ compactly supported in this chart such that $M\phi PM\phi = P$.

**Definition 5.5.** Pseudo-differential operator $P : S(X,g) \to S(X,g)$ is called classical if, for every chart $(U,\gamma)$ and for every smooth function $\phi$ compactly supported in this chart, the operator $M\phi PM\phi$ becomes classical when expressed in local coordinates.

Let $(U,\gamma)$ be a chart. A linear mapping $W_\gamma : L^2(\mathbb{R}^d) \to L^2(X,\text{vol}_g)$ defined by the formula
\[ W_\gamma \xi = \chi_U \cdot ((\xi \cdot \det(g^{-\frac{1}{2}})) \circ \gamma), \quad \xi \in L^2(\mathbb{R}^d), \]
is an isometry.

**Lemma 5.6.** Let $(X,g)$ be a compact $d$-dimensional Riemannian manifold. Let $P$ be a classical pseudo-differential operator of order $-d$ on $(X,g)$. If $P$ is self-adjoint and compactly supported in the chart $(U,\gamma)$, then there exists a limit
\[ \lim_{t \to \infty} t\mu(t, P). \]

**Proof.** It is immediate that $\mu(P) = \mu(Q)$, where $Q = W_\gamma^* PW_\gamma$. Since $P$ is a pseudo-differential operator of order $-d$ compactly supported in the chart $(U,\gamma)$, it follows that $Q$ is a compactly supported pseudo-differential operator of order $-d$ on $\mathbb{R}^d$. $P$ is classical, hence so is $Q$. The assertion follows now from Theorem 5.3.

**Lemma 5.7.** Let $(X,g)$ be a compact $d$-dimensional Riemannian manifold. Let $P$ be a classical pseudo-differential operator of order $-d$ on $(X,g)$. If $P$ is self-adjoint, then there exists a limit
\[ \lim_{t \to \infty} t\mu(t, P). \]

**Proof.** Fix a finite atlas on $(X,g)$. Choose a sequence $(\phi_{n,k})_{1 \leq k \leq K} \subset C^\infty(X)$ such that
(1) we have
\[ \phi_n \overset{\text{def}}{=} \sum_{1 \leq k \leq K} \phi_{n,k} \to 1 \]
in $L^2(X,\text{vol}_g)$.
(2) for every $n \in \mathbb{Z}$ and for every $1 \leq k \leq K$, $\phi_{n,k}$ is compactly supported in some chart.
Let Lemma 5.8.

By Theorem 1.3, we have

\[ A, B \in \mathcal{L}_{2,\infty}. \]

In particular, it belongs to \( \mathcal{L}_{\frac{d+1}{d+1}, \infty} \). Thus, \( B_n \in (\mathcal{L}_{1, \infty})_0 \).

Recall that \( B_n \in (\mathcal{L}_{1, \infty})_0 \). Combining (3) and Corollary 4.3 we infer the existence of the limit

\[ \lim_{t \to \infty} t \mu(t, A_n). \]

By Fact 4.7 there exists a limit

\[ \lim_{t \to \infty} t \mu(t, A_n). \]

Recall that \( B_n \in (\mathcal{L}_{1, \infty})_0 \). Combining (3) and Corollary 4.3 we infer the existence of the limit

\[ \lim_{t \to \infty} t \mu(t, A_n). \]

Thus, \( B_n \in (\mathcal{L}_{1, \infty})_0 \). The assertion follows now by combining (7) and Corollary 4.7.

The proof of the next lemma is based on a deep result from [10].

Lemma 5.8. Let \((X, g)\) be a compact \(d\)-dimensional Riemannian manifold and let \(\Delta_g\) be the Laplace-Beltrami operator. For every \(f \in L_{\infty}(X, \text{vol}_g)\) and for every normalised continuous trace on \(\mathcal{L}_{1, \infty}\), we have

\[ \varphi((1 - \Delta_g)^{-\frac{d}{4}} M_f (1 - \Delta_g)^{-\frac{d}{4}}) = \frac{\text{Vol}(S^{d-1})}{d(2\pi)^d} \int_X f \text{dvol}_g. \]

Proof. Let

\[ A = (1 - \Delta_g)^{-\frac{d}{4}}, \quad B = M_f (1 - \Delta_g)^{-\frac{d}{4}}. \]

By Theorem 1.3 we have \(A, B \in \mathcal{L}_{2, \infty}\). In particular, \(AB, BA \in \mathcal{L}_{1, \infty}\) and

\[ [A, B] \overset{\text{def}}{=} AB - BA \in [\mathcal{L}_{2, \infty}, \mathcal{L}_{2, \infty}]. \]

Here, the notation \([\mathcal{I}, \mathcal{J}]\) stands for the linear span of all commutators \([X, Y]\), \(X \in \mathcal{I}, Y \in \mathcal{J}\). It is a deep result, proved in [10] (see e.g. p.3 there) that

\[ [\mathcal{L}_{2, \infty}, \mathcal{L}_{2, \infty}] = [\mathcal{L}_{1, \infty}, \mathcal{L}_{\infty}]. \]
Since \( \varphi \) vanishes on \([\mathcal{L}_{1,\infty}, \mathcal{L}_{\infty}]\), it follows that \( \varphi \) vanishes on \([\mathcal{L}_{2,\infty}, \mathcal{L}_{2,\infty}]\). In particular, \( \varphi([A, B]) = 0 \).

Hence,

\[
\varphi((1 - \Delta_g)^{-\frac{n}{2}} M_f (1 - \Delta_g)^{-\frac{n}{2}}) = \varphi(AB) = \varphi(BA) = \varphi(M_f (1 - \Delta_g)^{-\frac{n}{2}}).
\]

The assertion follows now from Theorem 11.7.10 in [21].

\(\square\)

Proof. Choose a sequence \((\phi_n)_{n \geq 0} \subset C^\infty(X, \mathbb{R})\) such that \(\phi_n \to f\) in \(L_M(X, \text{vol}_g)\).

By Theorem 1.3, we have

\[
(1 - \Delta_g)^{-\frac{n}{2}} M_f_n (1 - \Delta_g)^{-\frac{n}{2}} \to (1 - \Delta_g)^{-\frac{n}{2}} M_f (1 - \Delta_g)^{-\frac{n}{2}}
\]

in \(\mathcal{L}_{1,\infty}\) as \(n \to \infty\). Operators

\[
(1 - \Delta_g)^{-\frac{n}{2}} M_f_n (1 - \Delta_g)^{-\frac{n}{2}}, \quad n \geq 0,
\]

are self-adjoint and pseudo-differential. By Lemma 5.7, there exists a limit

\[
\lim_{n \to \infty} t \mu \left( t, (1 - \Delta_g)^{-\frac{n}{2}} M_f_n (1 - \Delta_g)^{-\frac{n}{2}} \right) = \alpha_n.
\]

By Lemma 4.1, there exists a limit

\[
\lim_{n \to \infty} t \mu \left( t, (1 - \Delta_g)^{-\frac{n}{2}} M_f (1 - \Delta_g)^{-\frac{n}{2}} \right) = \alpha.
\]

Hence, for every continuous normalised trace \( \varphi \) on \( \mathcal{L}_{1,\infty} \), we have

\[
\varphi \left( (1 - \Delta_g)^{-\frac{n}{2}} M_f (1 - \Delta_g)^{-\frac{n}{2}} \right) = \alpha.
\]

The assertion follows now from Lemma 5.8.

The proof of the next lemma exploits recent advances in Birman-Kolienko-Solomyak inequality for quasi-Banach ideals.

Lemma 5.10. Let \( (X, g) \) be a compact \( d \)-dimensional Riemannian manifold. Let \( 0 \leq f \in L_\infty(X, \text{vol}_g) \). We have

\[
\lim_{t \to \infty} t \mu \left( t, (1 - \Delta_g)^{-\frac{n}{2}} M_f (1 - \Delta_g)^{-\frac{n}{2}} \right) = \frac{\text{Vol}(\mathbb{S}^{d-1})}{d(2\pi)^d} \int_X f \text{dvol}_g.
\]

Proof. Choose a sequence \((f_n)_{n \geq 0} \subset C^\infty(X, \mathbb{R})\) such that \(f_n \to f\) in \(L_M(X, \text{vol}_g)\).

By Theorem 1.3, we have

\[
(1 - \Delta_g)^{-\frac{n}{2}} M_f_n (1 - \Delta_g)^{-\frac{n}{2}} \to (1 - \Delta_g)^{-\frac{n}{2}} M_f (1 - \Delta_g)^{-\frac{n}{2}}
\]

in \(\mathcal{L}_{1,\infty}\) as \(n \to \infty\). Operators

\[
(1 - \Delta_g)^{-\frac{n}{2}} M_f_n (1 - \Delta_g)^{-\frac{n}{2}}, \quad n \geq 0,
\]

are self-adjoint and pseudo-differential. By Lemma 5.7, there exists a limit

\[
\lim_{n \to \infty} t \mu \left( t, (1 - \Delta_g)^{-\frac{n}{2}} M_f_n (1 - \Delta_g)^{-\frac{n}{2}} \right) = \alpha_n.
\]

By Lemma 4.1, there exists a limit

\[
\lim_{n \to \infty} t \mu \left( t, (1 - \Delta_g)^{-\frac{n}{2}} M_f (1 - \Delta_g)^{-\frac{n}{2}} \right) = \alpha.
\]

Hence, for every continuous normalised trace \( \varphi \) on \( \mathcal{L}_{1,\infty} \), we have

\[
\varphi \left( (1 - \Delta_g)^{-\frac{n}{2}} M_f (1 - \Delta_g)^{-\frac{n}{2}} \right) = \alpha.
\]

The assertion follows now from Lemma 5.8. \(\square\)

The proof of the next lemma exploits recent advances in Birman-Kolienko-Solomyak inequality for quasi-Banach ideals.

Lemma 5.10. Let \( A \in \mathcal{L}_{2,\infty} \) and \( B \in \mathcal{L}_{\infty} \) be self-adjoint elements. If \([A, B] \in (\mathcal{L}_{2,\infty})_0\), then

\[
(ABA)_+ - AB + A \in (\mathcal{L}_{1,\infty})_0.
\]

Proof. We have

\[
|ABA|^2 - (A|B|A)^2 = (ABA)^2 - (A|B|A)^2
\]

\[
= A \cdot BA \cdot AB \cdot A - A \cdot |B|A \cdot A|B| \cdot A
\]

\[
= A \cdot [B, A] \cdot AB \cdot A + A \cdot AB \cdot [A, B] \cdot A
\]

\[
- A \cdot [|B|, A] \cdot A|B| \cdot A - A \cdot A|B| \cdot [A, |B|] \cdot A.
\]

It follows from the assumption \([A, B] \in (\mathcal{L}_{2,\infty})_0\) and from Theorem 3.4 in [9] that \([A, |B|] \in (\mathcal{L}_{2,\infty})_0\). By Hölder inequality, we have

\[
|ABA|^2 - (A|B|A)^2 = (\mathcal{L}_{\frac{3}{2},\infty})_0.
\]
Now recall the Birman-Koplienko-Solomyak inequality for quasi-Banach ideals (see Theorem 6.3 in [16]). Applying the latter theorem with $E = (L^2, \infty), f(t) = |t|^{\theta}, t \in \mathbb{R},$ and $p = \theta = \frac{1}{2},$ we obtain

$$|ABA| - A|B|A \in (L_{1,\infty}).$$

Thus,

$$(ABA)^+ - AB^+ A = \frac{ABA + |ABA|}{2} - \frac{ABA + A|B|A}{2} = \frac{1}{2}(|ABA| - A|B|A) \in (L_{1,\infty}).$$

\[ \square \]

**Lemma 5.11.** Let $(X, g)$ be a compact $d$-dimensional Riemannian manifold. Let $f \in L_{\infty}(X, \text{vol}_g)$. We have

$$[M_f, (1 - \Delta_g)^{-\frac{d}{4}}] \in (L_{2,\infty}).$$

**Proof.** Choose a sequence $(f_n)_{n \geq 0} \subset C^\infty(X, \mathbb{R})$ such that $f_n \to f$ in $L_M(X, \text{vol}_g)$. By Theorem 1.3 we have

$$[M_{f_n}, (1 - \Delta_g)^{-\frac{d}{4}}] \to [M_f, (1 - \Delta_g)^{-\frac{d}{4}}]$$

in $L_{2,\infty}$ as $n \to \infty$. The operators

$$[M_{f_n}, (1 - \Delta_g)^{-\frac{d}{4}}], \quad n \geq 0,$$

are pseudo-differential of order $-\frac{d}{4} - 1$. In particular, they belong to $(L_{2,\infty})$. The assertion follows immediately. \[ \square \]

**Proof of Theorem 1.2.** Suppose first $f$ is bounded. By Lemma 5.11 and Lemma 5.10 we have

$$\left( (1 - \Delta_g)^{-\frac{d}{4}} M_f (1 - \Delta_g)^{-\frac{d}{4}} \right)^+ - (1 - \Delta_g)^{-\frac{d}{4}} M_f (1 - \Delta_g)^{-\frac{d}{4}} \in (L_{1,\infty}).$$

By Lemma 5.9 and Lemma 4.5 we have

$$\lim_{t \to \infty} t \mu\left( t, \left( (1 - \Delta_g)^{-\frac{d}{4}} M_f (1 - \Delta_g)^{-\frac{d}{4}} \right)^+ \right) = \frac{\text{Vol}(S^{d-1})}{d(2\pi)^d} \int_X f_+ d\text{vol}_g.$$

Applying the latter formula to $-f$, we obtain

$$\lim_{t \to \infty} t \mu\left( t, \left( (1 - \Delta_g)^{-\frac{d}{4}} M_f (1 - \Delta_g)^{-\frac{d}{4}} \right)^- \right) = \frac{\text{Vol}(S^{d-1})}{d(2\pi)^d} \int_X f_- d\text{vol}_g.$$

This proves the assertion for bounded $f$.

Suppose now $f \in L_M(X, \text{vol}_g)$. Choose a sequence $(f_n)_{n \geq 0} \subset L_{\infty}(X, \text{vol}_g)$ such that $f_n \to f$ in $L_M(X, \text{vol}_g)$. The assertion follows from the preceding paragraph (applied to $f_n$) and Corollary 4.3. \[ \square \]
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