ON VISIBILITY AND BLOCKERS

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Abstract. This expository paper discusses some conjectures related to visibility and blockers for sets of points in the plane.

1. Visibility Graphs

Let \( P \) be a finite set of points in the plane. Two distinct points \( v \) and \( w \) in the plane are visible with respect to \( P \) if no point in \( P \) is in the open line segment \( vw \). The visibility graph \( \mathcal{V}(P) \) of \( P \) has vertex set \( P \), where two distinct points \( v, w \in P \) are adjacent if and only if they are visible with respect to \( P \). So \( \mathcal{V}(P) \) is obtained by drawing lines through each pair of points in \( P \), where two points are adjacent if they are consecutive on a such a line. Visibility graphs have many interesting properties. For example, they have diameter at most 2 (assuming \( P \) is not collinear). Consider the following Ramsey-theoretic conjecture by Kára et al. [18], which has recently received considerable attention [1, 2, 21].

Conjecture 1 (Big-Line-Big-Clique Conjecture [18]). For all positive integers \( k \) and \( \ell \) there is an integer \( n \) such that for every finite set \( P \) of at least \( n \) points in the plane:

- \( P \) contains \( \ell \) collinear points, or
- \( P \) contains \( k \) pairwise visible points (that is, \( \mathcal{V}(P) \) contains a \( k \)-clique).

Conjecture 1 is true for \( k \leq 5 \) or \( \ell \leq 3 \) [1, 2, 18], and is open for \( k = 6 \) or \( \ell = 4 \).

Note that the natural approach for attacking the Big-Line-Big-Clique Conjecture using extremal graph theory fails. Turán [35] proved that every \( n \)-vertex graph with more edges than the Turán graph \( T_{n,k} \) contains \( K_{k+1} \) as a subgraph. Thus the Big-Line-Big-Clique Conjecture would be proved if every sufficiently large visibility graph with no \( \ell \) collinear points has more edges than \( T_{n,k-1} \). However, Burr et al. [5] and Füredi and Palásti [15] constructed sets \( P \) of \( n \) points with no four collinear, such that \( P \) determines \( \frac{n^2}{6} - O(n) \) lines each containing three points. Thus \( \mathcal{V}(P) \) has \( \frac{n^2}{2} + O(n) \) edges, which is less than the number of edges in \( T_{n,k-1} \) for all \( k \geq 5 \) and large \( n \). These examples show that

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1 Let \( T_{n,k} \) be the \( k \)-coloured graph with \( n_i \) vertices in the \( i \)-th colour class, where two vertices are adjacent if and only if they have distinct colours, and \( n = \sum n_i \) and \( |n_i - n_j| \leq 1 \) for all \( i, j \in [k] \).
the number of edges in a visibility graph with no four collinear points is not enough to necessarily imply the existence of a large clique via Turán’s Theorem.

Consider the following weakening of Conjecture 1 due to Jan Kara [private communication, 2005].

**Conjecture 2.** For all integers \( k \geq 2 \) and \( \ell \geq 1 \) there is an integer \( n \) such that if \( P \) is a finite set of at least \( n \) points in the plane, and each point in \( P \) is assigned one of \( k - 1 \) colours, then:

- \( P \) contains \( \ell \) collinear points, or
- some pair of visible points in \( P \) receive the same colour (that is, the visibility graph \( \mathcal{V}(P) \) has chromatic number \( \chi(\mathcal{V}(P)) \geq k \)).

Conjecture 1 implies Conjecture 2 since the chromatic number of any graph containing a \( k \)-clique is at least \( k \). Thus Conjecture 2 is true for \( k \leq 5 \) or \( \ell \leq 3 \). Consider a proper colouring of a visibility graph \( \mathcal{V}(P) \). That is, visible points are coloured differently. In each colour class \( C \), no two vertices are visible. So the vertices not in \( C \) ‘block’ the lines of visibility amongst vertices in \( C \). This idea leads to the following definitions that were independently introduced by Matoušek [21] amongst others.

A point \( x \) in the plane blocks two points \( v \) and \( w \) if \( x \in vw \). Let \( P \) be a finite set of points in the plane. A set \( B \) of points in the plane blocks \( P \) if \( P \cap B = \emptyset \) and for all distinct \( v, w \in P \) there is a point in \( B \) that blocks \( v \) and \( w \). That is, no two points in \( P \) are visible with respect to \( P \cup B \), or alternatively, \( P \) is an independent set in \( \mathcal{V}(P \cup B) \).

The purpose of this expository paper is to discuss some conjectures related to blocking sets. We remark that in the last few years, a number of researchers have started studying blocking sets around the same time (see [9, 21, 25] and the named researchers therein). So we expect that some of the observations in this paper have been independently discovered by others.

### 2. The Blocking Conjecture

If \( P \) is a set of collinear points then there is a set of \( |P| - 1 \) points that block \( P \). At the other extreme, how small can a blocking set be if \( P \) is in general position (that is, no three points are collinear)? Let \( b(P) \) be the minimum size of a set of points that block \( P \). Let \( b(n) \) be the minimum of \( b(P) \), where \( P \) is a set of \( n \) points in general position in the plane. We conjecture that every set of points in general position requires a super-linear number of blockers.

**Conjecture 3.** \( \frac{b(n)}{n} \to \infty \) as \( n \to \infty \).

In fact, Pinchasi [25] conjectured that \( b(n) \in \Omega(n \log n) \). Linear lower bounds on \( b(n) \) are known [9, 21]. Let \( P \) be a set of \( n \) points in the plane in general position with \( t \) vertices on the boundary of the convex hull. Each edge of a triangulation of \( P \) requires
a distinct blocker, and every triangulation of $P$ has $3n - 3 - t$ edges. So every blocking set of $P$ has at least $3n - 3 - t \geq 2n - 3$ vertices, and $b(n) \geq 2n - 3$. Dumitrescu et al. [9] improved this bound to $b(n) \geq \left(\frac{22}{8} - o(1)\right)n$.

3. Blocking Graph Drawings

A *drawing* of a graph $G$ represents each vertex of $G$ by a distinct point in the plane, and represents each edge of $G$ by a simple closed curve between its endpoints, such that a vertex $v$ intersects an edge $e$ only if $v$ is an endpoint of $e$. We do not distinguish between graph elements and their representation in a drawing. Note that multiple edges may intersect at a common point. A drawing is simple if any two edges intersect at most once, at a common endpoint or as a proper crossing (“kissing” edges are not allowed). A drawing is geometric if each edge is a straight line-segment. Obviously, every geometric drawing is simple.

Blockers for point sets generalise for graph drawings as follows. A set of points $B$ blocks a drawing of a graph $G$ if no vertex of $G$ is in $B$ and every edge of $G$ contains some point in $B$. Observe that if $P$ is a set of points in general position, then $B$ blocks $P$ if and only if $B$ blocks the geometric drawing of the complete graph with vertices drawn at $P$.

Some geometry is needed in Conjecture 3, in the sense that $K_n$ has a simple (non-geometric) drawing that can be blocked by $2n - 3$ blockers. As illustrated in Figure 1 if $V(K_n) = \{v_1, \ldots, v_n\}$ then place $v_i$ at $(i, 0)$ and draw each edge $v_iv_j$ with $i < j$ by a curve from $v_i$ into the upper half-plane, through the point $(-i - j, 0)$, into the lower half-plane, and across to $v_j$. As illustrated in Figure 1 the edges can be drawn so that two edges intersect at most once. Each edge is blocked by one of the $2n - 3$ points in \{\((-k, 0) : k \in [3, 2n - 1]\)\}. This observation improves upon a $O(n \log n)$ upper bound on the number of blockers in a simple drawing of $K_n$, due to Dumitrescu et al. [9]. A similar construction is due to Harborth and Mengersen [17]; see Pach et al. [24]. Note that at least $n - 1$ blockers are needed for every simple drawing of $K_n$ (since each point can block at most $\frac{n}{2}$ edges).

**Conjecture 4.** The minimum number of blockers in a simple drawing of $K_n$ equals $2n - 3$.

While this example suggests that geometry is needed in Conjecture 3, Stefan Langerman [personal communication, 2009] proposed an alternative. A drawing of a graph is extendable if the edges are contained in a pseudoline arrangement; that is, for each edge $e$ there is a simple unbounded curve $C_e$ containing $e$, such that for all distinct edges $e$ and $e'$, the curves $C_e$ and $C_{e'}$ intersect at most once. Observe that the above simple drawing that can be blocked by $O(n)$ blockers is not extendable. We conjecture that every extendible simple drawing of $K_n$ needs a super-linear number of blockers.
Conjecture 3 is related to results by Pach [23] about midpoints. For a set $P$ of points in the plane, let $m(P)$ be the number of midpoints determined by distinct points in $P$; that is, $m(P) := |\{ \frac{1}{2}(x+y) : x, y \in P, x \neq y \}|$. Let $m(n)$ be the minimum of $m(P)$, where $P$ is a set of $n$ points in general position in the plane. Since midpoints are also blockers, $b(n) \leq m(n)$. Pach [23] (and later Matoušek [21]) constructed a set of $n$ points in general position in the plane that determine at most $nc^{\sqrt{\log n}}$ midpoints for some constant $c$. Thus

$$b(n) \leq m(n) \leq nc^{\sqrt{\log n}}.$$  

(This function is between $n \log n$ and $n^{1+\epsilon}$.) Moreover, Pach [23] proved that $\frac{m(n)}{n} \to \infty$ as $n \to \infty$. Thus Conjecture 3 would strengthen this lower bound on $m(n)$. 

4. Midpoints and Freiman’s Theorem

Figure 1. A drawing of $K_7$ blocked by 11 blockers.
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Pach’s proof of this lower bound is based on Freiman’s Theorem\(^2\) which implies that if \(m(P) = \alpha n\) for some set \(P\) of \(n\) points in the plane (not necessarily in general position), then \(P\) is a subset of a \(d\)-dimensional progression of size at most \(\beta n\), for some \(d\) and \(\beta\) depending only on \(\alpha\). Pach concluded that at least \(\frac{1}{\beta} n^{1/d}\) points in \(P\) are collinear. Since no three points in \(P\) are collinear, \(n\) is bounded by a function of \(\alpha\). It follows that \(\frac{m(n)}{n} \to \infty\). We can obtain a more precise lower bound on \(m(n)\) as follows. Chang \(6\) proved that in Freiman’s Theorem one can take \(d = \lfloor \alpha - 1 \rfloor\) and \(\beta = 2^{2^{\alpha^2 \log^3 \alpha}}\), for some absolute constant \(c > 0\). Applying this result in Pach’s proof, it follows that \(\alpha^3 \log^3 \alpha \geq \frac{1}{c} \log n\). Hence for all \(\epsilon > 0\) and sufficiently large \(n\), for some absolute constant \(c > 0\),

\[
(1) \quad m(n) \geq cn(\log n)^{1/(3+\epsilon)}.
\]

Analogously, the following conjectured ‘convex combination’ version of Freiman’s Theorem would establish Conjecture \(5\).

**Conjecture 5.** Let \(P\) be a set of points in the plane with at most \(\frac{1}{2}|P|\) points collinear. Suppose that \(P\) can be blocked by some set \(B\) with \(|B| \leq \alpha|P|\). That is, for all distinct \(x, y \in P\) there is a real number \(\gamma \in (0, 1)\), such that \(\gamma x + (1-\gamma)y \in B\). Then \(P\) is a subset of a \(d\)-dimensional progression of size at most \(\beta|P|\), for some \(d\) and \(\beta\) depending only on \(\alpha\).

Note that some assumption on the number of collinear points is needed in Conjecture \(5\). For example, a set of \(n\) random collinear points can be blocked by \(n - 1\) points, but is not a subset of a progression of bounded dimension and linear size. This conjecture generalises Freiman’s Theorem for the plane, which assumes \(\alpha = \frac{1}{2}\) for all \(x, y \in P\).

While Freiman’s Theorem applies in some sense for sum sets along the edges of any dense graph \(11\), it is worth noting that there is a geometric drawing of \(K_{n,n}\) that can be blocked by \(O(n)\) blockers. Say the colour classes of \(K_{n,n}\) are \(\{v_1, \ldots, v_n\}\) and \(\{w_1, \ldots, w_n\}\). Position \(v_i\) at \((2i, 0)\), and \(w_j\) at \((2j, 2)\). Thus \(v_iw_j\) is blocked by \((i+j, 1)\), and \(\{(i, 1) : i \in [2, 2n]\}\) is a set of \(2n - 1\) points blocking every edge. In fact, there is a geometric drawing of \(K_{n,n}\) with its vertices in general position that can be similarly blocked. Position \(v_i\) at \((-2^i, 2^i)\) and \(w_j\) at \((2^j, 2^2j)\). These points lie on opposite sides of the parabola \(y = x^2\). The edge \(v_iw_j\) is blocked by \((0, 2^{i+j})\), and \(\{(0, 2^i) : i \in [2, 2n]\}\) is a set of \(2n - 1\) points blocking every edge.

In general, say \(S = \{s_1, \ldots, s_n\}\) is a set of \(n\) positive integers. Draw \(K_{n,n}\) by positioning each \(v_i\) at \((-s_i, s_i^2)\) and each \(w_j\) at \((s_j, s_j^2)\) (again on opposite sides of the parabola

\(^2\)A \(d\)-dimensional progression in the plane is a set \(\{v_0 + x_1v_1 + \cdots + x_dv_d : x_i \in [1, n]\}\) for some vectors \(v_0, \ldots, v_d \in \mathbb{R}^2\). Freiman’s Theorem is usually stated in terms of the sum set \(P + P := \{x + y : x, y \in P\}\). Clearly \(m(P) \leq |P + P| \leq m(P) + |P|\). Freiman’s Theorem actually applies in any abelian group; see \(34\). See \(14, 31, 32, 33\) for more on Freiman’s Theorem in the plane.
y = x^2). Say we block every edge by a point on the y-axis. The edge \(v_iw_j\) crosses the y-axis at \((0, s_is_j)\). Thus to have few blockers, \(S\) should be chosen so that the product set \(S \cdot S := \{ab : a, b \in S\}\) is small. Geometric progressions, such as \(2^1, 2^2, \ldots, 2^n\), minimise the size of the product set (leading to the construction of \(K_{n,n}\) above). It is interesting that both sum sets (that is, midpoints) and product sets appear to be related to blocking sets. There is a known trade-off between the sizes of sum sets and product sets (so-called sum-product estimates). In particular, \(|S + S|\) or \(|S \cdot S|\) is at least \(c|S|^{1+\epsilon}\) for some \(c > 0\) and \(\epsilon > 0\); see [7, 8, 10, 13, 30]. Especially given that geometric methods based on the Szemerédi-Trotter theorem can be used to prove such a result [10], it is plausible that sum-product estimates might shed some light on Conjecture 3.

5. Colouring Edges

Now consider edge-colourings of graph drawings, such that if two edges have the same colour, then they cross. This idea is related to blockers, since if a graph drawing can be blocked by \(b\) blockers, then it can be coloured with \(b\) colours. Let \(t(n)\) be the minimum integer such that the edges in some geometric drawing of \(K_n\) can be coloured with \(t(n)\) colours such that every monochromatic pair of edges cross. Each colour class is called a crossing family [3]. Hence \(t(n) \leq b(n)\). We conjecture the following strengthening of Conjecture 3.

**Conjecture 6.** \(\frac{t(n)}{n} \to \infty\) as \(n \to \infty\).

The analogous conjecture could be made for extendible simple drawings of \(K_n\).

6. Convex Position

For point sets in convex position, the above edge-colouring problem is equivalent to covering a circle graph by cliques. It follows from a result by Kostochka [20] (see [19]) that the minimum number of colours is at least \(n \ln n - c\) and at most \(n \ln n + cn\) for some constant \(c\). Thus the number of blockers for a point set in convex position is at least \(n \log n - c\). We conjecture the answer is quadratic.

**Conjecture 7.** Every set of \(n\) points in convex position require \(\Omega(n^2)\) blockers.

For \(n\) equally spaced points around a circle, at least \(\frac{n^2}{14} - O(n)\) blockers are required, since except for the point in the centre, at most 7 edges intersect at a common interior point [27]. This property does not hold for arbitrary points in convex position, since as described in Section 4 for the point set \(P = \{(-2^i, 2^{2i}), (2^i, 2^{2i}) : i \in [1, n]\}\), the point \((0, 2^k)\) blocks each edge \((-2^i, 2^{2i})(2^j, 2^{2j})\) for which \(k = i + j\). Thus \(\Omega(n)\) points on the y-axis each block \(\Omega(n)\) edges.

Note that Erdős et al. [12] proved that the minimum number of midpoints for a set of \(n\) points in convex position is between \(0.8 \binom{n}{2}\) and \(0.9 \binom{n}{2}\).
7. Point Sets with Bounded Collinearities

Now consider midpoints and blocking sets for point sets with a bounded number of collinear points. Let \( m_\ell(n) \) be the minimum number of midpoints determined by some set of \( n \) points in the plane with no \( \ell \) collinear points. Thus \( m_3(n) = m(n) \). The proof of the lower bound on \( m(n) \) described in Section 4 generalises to show that for all \( \epsilon > 0 \) and sufficiently large \( n > n(\epsilon) \), for some absolute constant \( c \),

\[
m_\ell(n) \geq c \log_\ell n (\log n)^{1/(3+\epsilon)}.
\]

Similarly, let \( b_\ell(n) \) be the minimum integer such that every set of \( n \) points in the plane with no \( \ell \) collinear points is blocked by some set of \( b_\ell(n) \) points. Thus \( b_3(n) = b(n) \). We conjecture that \( b_\ell(n) \) is also super-linear in \( n \) for fixed \( \ell \).

**Conjecture 8.** For all fixed \( \ell \), we have \( \frac{b_\ell(n)}{n} \to \infty \) as \( n \to \infty \).

**Proposition 9.** Conjecture 8 implies Conjecture 2.

**Proof.** Suppose on the contrary that Conjecture 8 holds but Conjecture 2 does not. Thus there are constants \( \ell \) and \( k \), and there are arbitrarily large point sets \( P \) containing no \( \ell \) collinear points, and with \( \chi(\mathcal{V}(P)) \leq k \). Conjecture 8 implies that \( b_\ell(n) \geq n \cdot g_\ell(n) \) for some non-decreasing function \( g_\ell \) for which \( g_\ell(n) \to \infty \) as \( n \to \infty \). Thus there is an integer \( n' \) such that \( g_\ell(n') > k - 1 \). Let \( P \) be a set of \( n \geq kn' \) points, containing no \( \ell \) collinear points, and with \( \chi(\mathcal{V}(P)) \leq k \). Let \( S \) be the largest colour class in a \( k \)-colouring of \( \mathcal{V}(P) \). Thus \( S \) has no \( \ell \) collinear points and \( P - S \) blocks \( S \). That is, there is a set of \( s = \lceil \frac{n}{k} \rceil \) points blocked by a set of \( n - s \) points. Thus \( b_\ell(s) \leq n - s \leq n(1 - \frac{1}{k}) \). On the other hand, \( b_\ell(s) \geq s \cdot g_\ell(s) \geq \frac{s}{k} \cdot g_\ell(s) \). Hence \( \frac{s}{k} \cdot g_\ell(s) \leq n(1 - \frac{1}{k}) \) and \( g_\ell(s) \leq k - 1 \). Since \( n' \leq s \) and \( g \) is non-decreasing, \( g_\ell(n') \leq k - 1 \), which is the desired contradiction. \( \square \)

8. A Final Conjecture

We finish the paper with a strengthening of Conjecture 2.

**Conjecture 10.** For all positive integers \( k \) and \( \ell \) there is an integer \( n \) such that if \( P \) is a set of at least \( n \) points in the plane, and each point in \( P \) is assigned one of \( k \) colours, then:

- \( P \) contains \( \ell \) collinear points, or
- \( P \) contains a monochromatic line (that is, a maximal set of collinear points, all receiving the same colour).

Conjecture 10 is trivially true for \( k = 1 \) and \( n = 2 \), or \( \ell = 3 \) and \( n = k + 1 \). The Motzkin-Rabin Theorem says that it is true for \( k = 2 \) with \( n = \ell \); see [4, 22, 28]. Conjecture 10 is related to the Hales-Jewett Theorem [16, 26, 29], which states that for
sufficiently large $d$, every $k$-colouring of the grid $[1, \ell - 1]^d$ contains a monochromatic "combinatorial" line of length $\ell - 1$.

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