Parton physics of the large-$N_c$ mesons

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We initiate the studies on the structural physics of the tower of stable large-$N_c$ mesons through a first computation of the collinear quark-structure of a large-$N_c$ pion using lattice Monte-Carlo methods. We adapt the large-$N_c$ continuum reduction for the determination of meson correlation functions involving the spatially-extended quasi-PDF operators as a perfect strategy to concentrate only on the short perturbative length scales. We find the internal structures of pion in the large-$N_c$ and $N_c = 3$ theories to be quite similar. Interestingly, we find hints that even the observed differences could arise to a large extent via the different perturbative QCD evolution in the two theories from similar initial conditions at low factorization scales.

I. INTRODUCTION

Quantum Chromodynamics (QCD) in the limit of large number of colors, $N_c$, at a fixed 't Hooft coupling $\lambda = N_c\alpha_s$, is greatly simplified by being a planar model in which quarks are naturally quenched, and it is well known to be a realistic QCD-like theory that approximately reproduces many features in the real-world, such as the ratios of low-lying meson masses [3–8]. The next frontier in QCD-physics is to understand the structural aspects of hadrons in more detail, so as to relate the emergent properties of hadrons, such as their masses and spins, to those of the short-distance quark-gluon (parton) degrees of freedom and their interactions (e.g., see [9–11]). In this respect, the large-$N_c$ limit motivates and crystallizes concepts in parton phenomenology, such as the linear Regge trajectories (proven in two-dimensions [2]), the dipole approach to BFKL formalism [12], and the concept of quark-hadron duality [13, 14] to name a few.

The large-$N_c$ baryons [15] are $\mathcal{O}(N_c)$ heavier degrees of freedom that can be described as a chiral soliton [16–19]. Such an identification has lead to mean-field theory studies of the parton distributions inside a nucleon (for initial works, see [20–23].) In contrast, the large-$N_c$ mesons are the leading lighter degrees of freedom, and nonperturbative methods (e.g., lattice simulations) are the only way to study them. With the access to an infinite tower of completely stable large-$N_c$ mesons of different $J^{PC}$, it is an ideal realization of QCD that is conducive to investigate the partonic origin of hadron physics. Development of such realistic models of mesons as hard-scatterers is especially important due to the reinvigorated experimental [11, 24, 25] and theoretical efforts towards the meson structures, especially of the pion, the Goldstone mode of Chiral symmetry-breaking (refer [26] for a review, and [27–36] for recent numerical works). Quite surprisingly, despite the continued effort to understand large-$N_c$ QCD over the years, the partonic nature of the large-$N_c$ mesons is to a large extent unknown. To our knowledge, the study in Ref [37] of the distribution amplitude of pion within a large-$N_c$ Regge model is a singular work towards this direction.

Through the present work, we bridge this persisting gap in our understanding of the canonical toy-model of QCD through a first computation of quark distribution function of the large-$N_c$ pion, and thereby, lay the framework for comparative studies of internal structures of different stable species of mesons. As an important feature of the large-$N_c$ theory, we present the large-$N_c$ continuum reduction as a novel tailor-made approach for the operator product expansion (OPE) based strategies [38–41] to perform parton physics on the lattice. We display the schematic of the central idea of the calculation in Fig. 1(A), and we elaborate on it in the following discussion. As an initial work in this direction, we keep the discussion simple by summarizing the main techniques and results in the main text, and by referring the reader to various appendices for the elaborate details.

II. BASICS OF CONTINUUM REDUCTION

Owing to the absence of a center-symmetry breaking deconfinement phase transition in two Euclidean space-time dimensions, the large-$N_c$ QCD$_2$ is well known to be reducible to a single-site matrix model [42, 43]. We

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can extend the Eguchi-Kawai reduction [43] to dimensions $d > 2$ [44–48], provided we preserve the $U^d(1)$ center symmetry by reducing the theory not to a point, but instead to a small box of volume $\ell^d$, the deconfinement temperature. The quarks hopping on such crystalline configuration are labeled by their positions in periodic $\ell^d$ box and their Bloch momenta. The correlation functions in the larger box can be obtained using lattice implementation of momentum space Feynman diagrams that use quark propagators in $\ell^d$ box. (B and C) The momentum space Feynman diagrams implemented directly on the lattice. The lines are quark propagators. The arrows show the off-shell 4-momentum injected at the vertices. The 2-point function of pion is shown in B. The 3-point function of quasi-PDF operator (double line) with pion creation and annihilation operators is shown in C.

FIG. 1. (A) Schematic of large-$N_c$ continuum reduction for quasi-PDF operator evaluated within a pion. The gauge fields on $\approx N_c,\ell$ sized box are obtained as replicas of gauge fields within a $\ell$ sized box, with $\ell > T_c^{-1}$, the deconfinement temperature. The quarks hopping on such crystalline configuration are labeled by their positions in periodic $\ell^d$ box and their Bloch momenta. The correlation functions in the larger box can be obtained using lattice implementation of momentum space Feynman diagrams that use quark propagators in $\ell^d$ box. (B and C) The momentum space Feynman diagrams implemented directly on the lattice. The lines are quark propagators. The arrows show the off-shell 4-momentum injected at the vertices. The 2-point function of pion is shown in B. The 3-point function of quasi-PDF operator (double line) with pion creation and annihilation operators is shown in C.

We discuss the details behind such a construction in Appendix A 1. By using such a relation, along with the global $U^d(1)$ center symmetry, we can reduce all $n$-point functions of quark bilinears in $\mathbb{R}^d$ to computations of $n$-point functions on $L^d$ periodic lattice. For example, as derived in Appendix A 2, we can write the 2-point function, $\tilde{C}_{2pt}(p) = \langle \pi(p|\pi(\mathbf{p})) \rangle$, for a pion ($\pi = d\gamma_5u$) in momentum space as

$$F_{x,y}(U_\mu) = \int \frac{d^d q}{(2\pi)^d} e^{i\frac{2\pi(x-y)}{L}} F^L_{x,y}(U_\mu e^{i\frac{2\pi q}{L}}).$$

We show the Feynman diagram for Eq. (2) in Fig. 1(B). From the spectral decomposition, $\tilde{C}_{2pt}(p_0, \mathbf{p}) = \sum_{i=0}^{2A_i E_i(\mathbf{p}) + E_i(\mathbf{p})^2}^{-1}$, we can obtain the long-distance energy spectrum, $E_i(\mathbf{p})$, and amplitudes, $A_i$. Alternatively, we can access the spectrum from the multi-exponential, $A_i e^{-E_i/\gamma_5 T_c}$, decay of $C_{2pt}(t_*; \mathbf{p}) = \int \frac{d^d q}{2\pi} \tilde{C}_{2pt}(p_0, \mathbf{p}) e^{i\mathbf{p} \cdot \mathbf{x}}$, in the Euclidean time $t_*$. We see that the long-distance hadronic spectral physics is trivialized by the ability to capture $|x|, t_* \gg \Lambda_{QCD}^{-1}$ using only simulation of a box of size $\ell \approx \Lambda_{QCD}^{-1}$. 
III. ZOOMING IN ON PARTON SCALES WITH CONTINUUM REDUCTION

This large-$N_c$ continuum reduction leads to a key simplification in lattice QCD computations of parton distributions. Many recent developments [38–41] in the ab initio computations of the Bjorken $x$-dependent parton distribution functions (PDFs), $f(x,\mu)$ at a $\overline{\text{MS}}$ factorization scale $\mu$, and related quantities, rely on the leading-twist expansion of certain equal-time renormalized invariant amplitudes, $\mathcal{M}(\nu,z^2)$ with $\nu = -z \cdot P$, involving an operator-pair [38, 41] or a bilocal extended operator [39, 40] with a spatial separation $z_\mu = z_3 \delta_{\mu,3}$ that is evaluated within a state $|P\rangle$ of an on-shell hadron moving with momentum $P = (E(p), p)$. Through lattice Monte Carlo determination of $\mathcal{M}$, we can relate it to $f(x,\mu)$ through an OPE truncated at leading-twist terms (see [49]),

$$
\mathcal{M}(\nu,z^2) = \sum_{n=0} (\frac{i\nu}{n!})^n C_n(\mu^2 z^2) \int_{-1}^{1} x^n f(x,\mu) dx.
$$

(3)

In the absence of higher-twist corrections, the Wilson coefficients $C_n$ capture $\ln(-z^2\mu^2)$-type QCD contributions to $\mathcal{M}$ using perturbation theory and leads to $f(x,\mu)$ at a chosen scale $\mu$. Thus, along with the necessity of non-zero $P_3$, the short-distance $|z|$ is crucial for the validities of OPE, the perturbation theory and for ignoring higher-twist terms. On the other hand, the leading-twist expansion is performed within hadronic in- and out-states, and therefore, having control of the long-distance aspects of QCD is equally important. Applying the above formalism to the large-$N_c$ theory is much simpler – we can capture the long-distance hadronic states easily by the virtue of continuum reduction, leaving only the relevant partonic scales for $z$ below the inverse deconfinement transition temperature, $T_c^{-1}$, to be captured by Monte Carlo sampling of gauge fields within $T_c^{-1}$ extent.

With this realization, we extend the continuum reduction approach to $n$-point functions involving an extended operator, such as the $u$-quark quasi-PSD operator $O(z;q) \equiv \sum_x e^{iq\cdot x} \bar{u}_x \gamma_0 W_{x,x+z} u_{x+z}$, for purely-spatial $z = (0,0,0,z_3)$, and $W_{x,x+z}$ is a straight Wilson-line connecting $x$ to $x + z$. The spatial part $q = 0$ for the PDF we want to study. Following our discussion of the 2-point function and the method of folding Wilson loops of any size on an $L^4$ lattice, we can similarly write the 3-point function $\tilde{C}_{3pt}(z;p,q) \equiv \langle \pi(p+q)O(z;q)\pi^\dagger(p) \rangle$, as

$$
\tilde{C}_{3pt} = \sum_{x} \langle \text{tr} \left( \gamma_0 W^L \right)_{x,x+z} \left[ G^L \gamma_5 G^L \gamma_5 G^L \right]_{x+z,x} \rangle,
$$

(4)

where the gauge-links $U_{\mu}$ entering the propagators from left to right are multiplied by phases $1, e^{-ip\cdot x}$ and $e^{iq\cdot x}$, respectively. The trace is over color and spin, and $W^L$ is the folded Wilson line obtained by wrapping around the periodic lattice if $|z_3| \geq L$. In Appendix A.3, we present a detailed derivation of the above equation. We show the Feynman diagram for Eq. (4) in Fig. 1(C). Note that the quark-line disconnected piece in $\tilde{C}_{3pt}$ is $N_c^{-1}$ suppressed, and therefore as another large-$N_c$ advantage, we have ignored it in the above equation. As in the 2-point function, we can obtain the required bare quasi-PSD matrix element,

$$
2P_0 h^B(z,P) \equiv \langle \pi; P | O(z) | \pi; P \rangle,
$$

(5)

through the spectral analysis of Eq. (4) either in momentum space or in the real-space $t_s$ after Fourier transforming $\tilde{C}$ with respect to $p_0$ to form $C_{3pt}(z,t_s,p,q)$. A convenient choice $q = (0,0)$ gives the so-called summation method [50], wherein, $C_{3pt}(z,t_s,p,q = 0)/C_{2pt}(t_s,p) = t_s h^B(z,p) + \text{constant}$, up to $O(e^{-(E_1-E_0)t_s})$ excited-state corrections.

IV. COMPUTATIONAL DETAILS

We implemented the continuum reduction approach to determine the $u$-quark PDF of the large-$N_c$ pion in $d = 4$. As a first exploratory study, we performed our computation at a fixed simulation point at a large but finite value of $N_c = 17$ on an $L = 8$ lattice using a coupling $b = 0.355$, which is in the confined phase [48]. The lattice spacing in units of string tension [51, 52] is $\sqrt{\sigma a} = 0.254(2)$. Due to the finite large $N_c$, the $U(1)$ center symmetry reduces to $Z_{N_c}^d$ discrete symmetry, and therefore, we quantized the lattice momenta in units of $2\pi/(L N_c)$ and multiples thereof, to leave the above results intact. In this way, we effectively enlarged the $8^4$ lattice into a $68 \times 136^3$ lattice. We used Wilson-Dirac operator coupled to smeared gauge-links for $\mathcal{D}^L$ and tuned the quark mass to produce a pion of mass $m_\pi = 0.86\sqrt{\sigma}$. We stochastically computed $\tilde{C}_{2pt}(p)$ and $\tilde{C}_{3pt}(z,p,q = 0)$ at all values of $p_0$ at each given $p = (0,0,P_3)$, using 15K-32K configurations, and Fourier transformed them into functions of $t_s$. We studied nine different spatial momentum $P_3/\sqrt{\sigma} \in [0,5.82]$. We elaborate further on the lattice setup in Appendix B. We determined $h^B$ using summation type fits to
FIG. 2. The real and imaginary parts of the pseudo-ITD of pion $M(\nu, z^2)$ in the large-$N_c$ limit are shown. The lattice data from different values of quark-antiquark separation $z_3$ are shown using different colored symbols. The bands are fits to the leading-twist OPE with the large-$N_c$ NLO Wilson coefficients at MS scale $\mu = 4.55\sqrt{\sigma}$. The fit parameters are the Mellin moments.

$C_{3pt}/C_{2pt}$ ratio. For further details on the spectral analysis of 2-point and 3-point functions, the reader can refer to Appendix C and Appendix D respectively. Since $h^\xi(z, P)$ is multiplicatively renormalizable [53, 54], we took the renormalization group invariant ratio [55] of quasi-PDF matrix elements at $P_3 = 0$ with respect to $P_3 \neq 0$ to form the pseudo Ioffe-time distribution (pseudo-ITD), $M(\nu, z^2)$.

V. COLLINEAR QUARK STRUCTURE OF THE LARGE-$N_c$ PION

We show the real and imaginary parts of the $u$-quark pseudo-ITD, $M(\nu, z^2)$ as a function of $\nu$ in Fig. 2. As seen from Eq. (3), the two are governed by $u - \bar{u}$ and $u + \bar{u}$ PDFs respectively. In $SU(3)$ theory, $u + \bar{u}$ PDF mixes with gluon PDF, however this mixing is $N_c^{-1}$ suppressed and hence ignored here. The data points are the result of our lattice computation from different $(z_3, P_3)$ put together. The near-continuous set of momenta we were able to use, helped us pack the range of $\nu$ with data points. In the large-$N_c$ limit, the string tension $\sqrt{\sigma}$ sets a fiducial scale that distinguishes perturbative and nonperturbative length scales; therefore, we restricted the data for $M$ to only those up to the border-line $\sqrt{\sigma}z_3 \leq 1.27$. At the same time, a cautious use of $|P_3| < a^{-1}$ only let us scan a range of $\nu < 3.5$. The near universality of the data with respect to the scaling variable $\nu$ points to the viability of perturbative OPE methods in the large-$N_c$ theory. By fitting the lattice data using the leading-twist OPE in Eq. (3), we extracted the Mellin moments $\langle x^n \rangle_{u\pm \bar{u}} = \int_0^1 x^n f_{u\pm \bar{u}}(x, \mu)dx$ at a scale $\mu = 4.55\sqrt{\sigma}$ using 1-loop result [49] for the Wilson coefficients $C_n(\mu^2 z^2)$ in the large-$N_c$ limit; for this we used leading-order value, $\lim_{N_c \rightarrow \infty} C_F(N_c)\alpha_s(\mu) = 0.39$ using $\Lambda_{\overline{MS}}/\sqrt{\sigma} = 0.503$ [56, 57]. We chose a scale $\mu \approx a^{-1}$ so that it is characteristic of the typical small $z_3$ used in this work. We gather the technical details for the OPE fits and on the perturbative factors in Appendix E and Appendix F1 respectively. We find for the first few moments

\[
\langle x \rangle_{u+\bar{u}}, \langle x^3 \rangle_{u+\bar{u}} = [0.25(1), 0.10(2)] \text{ (via Im}M) \text{),}
\langle x^2 \rangle_{u-\bar{u}}, \langle x^4 \rangle_{u-\bar{u}} = [0.13(2), 0.10(2)] \text{ (via Re}M) ,
\]

(6)
FIG. 3. The comparison of \( \overline{\text{MS}} \) ITDs in large-\( N_c \) QCD (SU(\( \infty \))) with the global fit results (JAM20) for the ITDs in SU(3) QCD at scale \( \mu = 2 \) GeV. The top and bottom panels are \( u - \bar{u} \) and \( u + \bar{u} \) ITDs respectively. The red bands (moments fit) are results from OPE analysis by fitting Mellin moments. The green and blue bands are expectations for ITDs in SU(3) QCD based on the assumption of a nearly similar large-\( N_c \) PDF at a lower factorization scale \( \mu_0 = 0.8 \) and 0.6 GeV respectively. The results for \( u - \bar{u} \) ITD and PDF assuming an ansatz \( f_{u - \bar{u}}(x, \alpha, \beta, s) \) are shown as purple bands (Ansatz fit) in the top-panel and its inset respectively.

with correlated \( \chi^2/df \sim 39/26 \) in the two cases. As a cross-check that perturbative OPE framework is working for the chosen range in \( z_3 \), we used the fixed-\( z^2 \) moments analysis [79] as a diagnostic tool [30, 80] to detect any corrections – as discussed in Appendix G, we found the method to work well within statistical errors. In addition to the above Mellin moments analysis, we also performed fits to the valence \( u - \bar{u} \) data assuming a phenomenologically motivated functional form [58], \( f_{u - \bar{u}}(x; \alpha, \beta, s) = N x^\alpha (1 - x)^\beta (1 + sx^2) \). Since our access to the range of \( \nu \) is limited in this work, and the small-\( x \) region is believed to be harder to access on the lattice, we imposed a prior that \( \alpha \in [-0.6, -0.4] \) motivated by the Regge phenomenology. With the caveat of using an Ansatz, we found the data to be best described by a large-\( x \) exponent \( \beta = 0.7(3) \), similar to what is seen in recent lattice SU(3) QCD results [27–30] as well as by global fits [58, 59]. From an indirect estimation of the valence momentum fraction, \( \langle x \rangle_{u - \bar{u}} = 0.23(2) \) from the PDF Ansatz fit, we find it to be the same as \( \langle x \rangle_{u + \bar{u}} \) within errors; thus, there might only be negligible amount of anti-\( u \) in the large-\( N_c \) pion wavefunction.
VI. PHENOMENOLOGY

In order to use large-$N_c$ theory as a model system to compare and contrast the $SU(3)$ QCD with, we first set the GeV scale in the $SU(\infty)$ world through a choice $\sqrt{\sigma} = 0.44$ GeV that is known [3, 7, 60] to result in a low-energy meson spectrum that is numerically similar to the real-world; this choice implies, $[\tilde{u}^{-1}, \mu, m_{\pi}] = [1.73, 2.00, 0.38]$ GeV in our computation. We use the $\overline{\text{MS}}$ Ioffe-time distribution (ITD), $\mathcal{M}^{\overline{\text{MS}}}_{u\bar{u}}(\nu, \mu)$, which are the cosine (for $u - \bar{u}$) and sine (for $u + \bar{u}$) Fourier transforms of PDFs from $x$ to $\nu$ space, to justifiably perform this comparison within the range of $\nu$ spanned by our lattice data. In Fig. 3, we compare $\mathcal{M}^{\overline{\text{MS}}}_{u\bar{u}}$ for the large-$N_c$ pion (red band), as inferred from the model-independent fits to Mellin moments, with the JAM20 global fit result [58] (gray band) for the real-world pion at $\mu = 2$ GeV. We find a good agreement between the two theories in the case of the valence $u - \bar{u}$ ITD. We suspect that the observed tendency for $SU(\infty)$ data to peel off at $\nu \approx 3$ could be a systematic effect due to the absence of constraint from data beyond $\nu = 3.5$, and in fact, such a feature is absent in the ITD reconstructed from $f_{u-a}(x; \alpha, \beta, s)$ (purple band). In the inset of Fig. 3, we also see a nearly similar $x$-dependencies of our Ansatz-based reconstruction of $f_{u-a}(x)$ (purple band) and the JAM20 result. Thus, the valence structure of pion is likely to be weakly dependent on $N_c$. It appears that features like the valence quarks that carry $\approx 50\%$ of the pion momentum at few GeV resolutions, could be typical in $SU(N_c)$ theories.

In the bottom panel of Fig. 3, we show a similar comparison between $u + \bar{u}$ ITDs at $\mu = 2$ GeV. Here, we see a visible difference between the two theories. Based on a better agreement seen in the valence sector, we ask if the difference seen in the singlet $u + \bar{u}$ distribution could originate from the perturbative radiative processes in the large-$N_c$ and $SU(3)$ QCD; as a main difference, the $g \rightarrow qq$ splitting is absent when $N_c \rightarrow \infty$. In Appendix F2, we discuss the perturbative evolution aspects in the large-$N_c$ limit. Working under a premise that the large-$N_c$ and $SU(3)$ theories have similar $u + \bar{u}$ PDFs at a low factorization scale $\mu_0$, we first evolved the pairs, $\{2\langle x_n \rangle_{u+\bar{u}}, \langle x_n \rangle_{\bar{u}}\}$ at $\mu = 2$ GeV in the large-$N_c$ theory to a scale $\mu_0 = 0.8$ to 0.6 GeV, and evolved that result back to $\mu = 2$ GeV using 3 flavor $SU(3)$ QCD DGLAP evolution. Since we have not explicitly calculated the gluon moments for large-$N_c$ theory, we used the sum-rule $\langle x \rangle_g = 1 - 2 \langle x \rangle_{u+\bar{u}}$ and simply set the other higher moments of the small-$x$ dominant gluon to be negligible. As we are looking only for qualitative tendencies, we performed the evolution at leading-logarithmic order using the same $\Lambda_{QCD}$ in both theories. We show the resulting ITDs based on evolutions from $\mu_0 = 0.8$ and 0.6 GeV as the green and the blue bands in Fig. 3 lower panel. Remarkably, the QCD evolution pulls the large-$N_c$ result closer to the JAM20 result when successively smaller $\mu_0$ are used. Thus, large-$N_c$ QCD presents itself as an interesting model system for singlet parton physics where $g \rightarrow qq$ splitting is switched off, with all other splitting remaining intact. Such a procedure only lead to a negligible effect in valence $u - \bar{u}$ ITD as seen in Fig. 3 top panel.

VII. DISCUSSION

We presented the large-$N_c$ mesons as an interesting uncharted model-system for understanding partonic physics, using the continuum reduction. Our first lattice computation of large-$N_c$ pion structure shows indeed that the structural properties in the large-$N_c$ theory are likely to be similar to our real-world, as has been seen in the meson spectrum; their differences seem to be even more interesting as it gives us a version of QCD where the sea is not radiatively proliferated with quark-antiquark pairs, and hence could help understand the role of sea quarks in real-world QCD. Not to be mistaken, the method needs to be improved by going to finer $\Lambda_{QCD}$ advantage could be the exponential suppression [66–68], of small instantons in large-$N_c$ limit that might suppress instanton-induced power corrections [69–71] to the OPE at typical short-distances reached in contemporary lattice calculations. It would be interesting to extend this work to probe the differences in gluon structures of the radial and angular stable-excitations [31, 72] of the ground-state mesons, and perform $x$-dependent spin physics of stable higher-spin large-$N_c$ mesons, such as the $\rho$.

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Appendix A: Details on continuum reduction

We consider a $L^d$ periodic lattice. The gauge action in terms of plaquettes $U_p$ is

$$S_g = bN \sum_p \text{Tr}(U_p + U_p^\dagger),$$

and gauge fields on an infinite $d$ dimensional lattice obey the periodic condition,

$$U_\mu(x) = U_\mu(x + L\hat{\nu}); \quad \forall \, \mu, \nu.$$  \hfill (A2)

As long as the lattice coupling $b < b_1(L)$ the theory is in the confined phase and infinite volume results can be computed exactly with finite lattice spacing effects. Below, we derive expressions for certain $n$-point functions of quark bilinears (mesons) using the continuum reduction framework. We specify the $n$-point functions as $G^{(n)}$.

We derive results using continuum reduction for a general dimension $d$, but we finally used only $d = 4$ in this work.

1. Quark propagator in infinite lattice from finite periodic lattice: Bloch wavefunctions and $U^d(1)$ global symmetry

Consider an operator $F(U)$ on an infinite lattice obtained by copying the gauge fields from the $L^d$ lattice using periodicity. As per Bloch’s theorem, the eigenvalue problem takes the form

$$\sum_{y} F_{x,y}(U) q_{y}^i = \lambda_i(p) q_{y}^i(p); \quad q_{x+nL}^i = e^{ip\cdot n} q_{x}^i(p),$$

where $n$ a tuple of integers. Under a gauge transformation $g$,

$$U^g_\mu(x) = g_x U_\mu(x) g_{x+\hat{\mu}}^\dagger; \quad F_{x,y}(U^g) = g_x F_{x,y}(U) g_y^\dagger.$$  \hfill (A4)

Of particular interest to us will be Abelian gauge transformations of the form $g_x = e^{-i\frac{h}{L}}$ on the infinite lattice. Under these gauge transformations,

$$U^g_\mu(x) = U_\mu(x) e^{-i\frac{\pi}{L}}; \quad \text{(written in short as } U e^{i\frac{\pi}{L}}),$$

and we can rewrite the eigenvalue problem as

$$\sum_{y} F_{x,y}(U e^{i\frac{\pi}{L}}) q_{y}^g(p) = \lambda_i(p) q_{x}^g(p); \quad q_{x+nL}^g(p) = e^{-i\frac{\pi}{L}} q_{x}^g(p); \quad q_{x+nL}^i(p) = q_{x}^i(p).$$

One can use the periodicity of $q_{x}^g(p)$ to further rewrite the eigenvalue problem as

$$\sum_{y} F_{x,y}^{L}(U e^{i\frac{\pi}{L}}) q_{y}^g(p) = \lambda_i(p) q_{x}^g(p); \quad F_{x,y}^{L}(U) \equiv \left[ \sum_{n=-\infty}^{\infty} F_{x,y+nL}(U) \right].$$  \hfill (A7)

and the induced operator $F^L$ on the finite periodic lattice satisfies

$$F_{x,y}^L(U) = F_{x,y+nL}^L(U) = F_{x+nL,y}^L(U)$$

for any vector $n$ with integer entries. The above eigenvalue equation is for a finite size matrix on a finite periodic lattice. We can write the operator and its inverse on the infinite lattice using their finite volume counterparts as

$$F_{x,y}(U) = \int \frac{d^d p}{(2\pi)^d} \sum_i \lambda_i(p) q_x^i(p) [q_y^i(p)]^\dagger = \int \frac{d^d p}{(2\pi)^d} e^{i\frac{p(x-y)}{L}} F_{x,y}^L(U e^{i\frac{\pi}{L}});$$
\[
F^{-1}_{x,y}(U) = \int \frac{d^dp}{(2\pi)^d} \, e^{i\frac{p(x-y)}{L}} \left[ F^L \right]^{-1}_{x,y}(U e^{i\pi}).
\] (A9)

One can extend the above relation to a product of operators:
\[
\sum_z A_{x,z}(U e^{i\phi}) B_{z,y}(U e^{i\chi}) = \sum_z \int \frac{d^dp}{(2\pi)^d} \frac{d^dq}{(2\pi)^d} e^{i\frac{p(x-z)}{L}} e^{i\frac{q(z-x)}{L}} A^L_{x,z}(U e^{i\phi}) B^L_{z,y}(U e^{i\chi}).
\] (A10)

One application of the above reduction we will use involves the quark propagator, \(G_{x,y}(U)\), in a fixed gauge field background and a smearing operator, \(S^L_{x,y}(U)\), in a fixed gauge field background where \(\phi\) labels the type of smearing. Specific to this work, \(S(U_{\mu})\) is the Wuppertal smearing kernel, and \(S^L_{x,y}(U_{\mu}) = S(e^{i\phi U_{\mu}})\) for a phase \(\phi = (0, \phi_1, \phi_2, \phi_3)\) with non-zero spatial components in general, which we also set to 0. Hence, what follows is for a more general case than actually used in the present computation. The smearing operator is typically diagonal in spinor space. We will assume \(S^L(0)\) is Hermitian as is true in most cases that are typically functions of the covariant Laplacian operator. A form of reduction we will need is
\[
\sum_{x',z'} S^{\phi_1}_{x',z'}(U) G_{x',z'}(U) S^{\phi_2}_{z',y}(U) = \int \frac{d^dp}{(2\pi)^d} \, e^{i\frac{p(x-y)}{L}} G^L_{x,y}(U e^{i\pi}),
\] (A11)
where we have used different smearing operators on either side of the unsmeared quark propagator and we define the smeared propagator on the finite periodic lattice by
\[
G^L_{x,y}(U) = \sum_{x',z'} S^{\phi_1}_{x',z'}(U) G^L_{x',z'}(U) S^{\phi_2}_{z',y}(U),
\] (A12)
which satisfies
\[
G^L_{x,y}(U) = G^L_{x,y+nL}(U) = G^L_{x+y,0}(U)
\] (A13)
for any vector \(n\) with integer entries as expected of a propagator on a periodic lattice.

2. Two point function of mesons

Let
\[
M^\Gamma_{ij}(x) = \sum_{x',z'} \bar{q}^{(i)}_{x'} S^{\phi_1}_{x',z'}(U) \Gamma S^{\phi_2}_{z',y'}(U_{\mu}) q^{(j)}_{y'}
\] (A14)
be a gauge invariant meson operator located at \(x\). Since \(S_{x,y} \propto \delta_{x,y}\), the sums above are actually restricted to time-slice containing \(x\) by construction, but written as a sum over the entire space-time. The indices \(i, j\) provide the quark flavor indices and \(\Gamma\) specifies the type of fermion in a spinor space. For the case of pion, considered in the paper,
\[
\pi(x) = M^\gamma_{du}(x) = \sum_{x',z'} \bar{d}_{x'} S^\gamma_{x',z'}(U e^{i\phi})\gamma_5 S^\phi_{z',y'}(U e^{-i\phi}) u_{y'}.
\] (A15)
Using Eq. (A11), the two point function of a meson of a type \(\Gamma_1\) with type \(\Gamma_2\) in the infinite lattice is
\[
G^{(2)}(x,y) = \langle M^\Gamma_{ij}(x) \bar{M}^{\Gamma}_{ij}(y) \rangle,
\] (A16)
which after Wick contraction yields
\[
G^{(2)}(x,y;U) = \int \frac{d^dq}{(2\pi)^d} \frac{d^dq'}{(2\pi)^d} \, e^{i\frac{q(x-y)+q'(x'-y')}{L}} \text{tr} \left[ \Gamma_1 G^L_{x,y}(U e^{i\pi}) \Gamma_2 G^L_{y,x}(U e^{i\pi}) \right].
\] (A17)
If we write down the two point function in momentum space using

$$G^{(2)}(p', p; U) = \delta(p' + p) \int \frac{d^4q}{(2\pi)^4} Tr \left[ \Gamma_1 G^{L,\phi_2}(U e^{i\frac{q}{2}}) \Gamma_2 G^{L,\phi_1}(U e^{i\frac{q}{2} + p'}) \right] ,$$  \hspace{1cm} (A19)

and split the infinite sum over \( x \) and \( y \) into blocks of finite sums over finite periodic lattice (i.e., \( x \to x + n \) mod \( L \), and replace sum to be over the periodic \( x \) and \( n \)) and invoke the periodicity property in Eq. (A13), we will arrive at condition \( Lp + q' - q = 0 \) along with momentum conservation \( p + p' = 0 \). Using these, we can write the 2-point function as

$$G^{(2)}(p', p; U) = \delta(p' + p) \int \frac{d^4q}{(2\pi)^4} Tr \left[ \Gamma_1 G^{L,\phi_2}(U) \Gamma_2 G^{L,\phi_1}(U e^{-ip}) \right] .$$  \hspace{1cm} (A20)

For the pion, for which \( \phi_1 = -\phi_2 = \phi \) so as to preserve isospin symmetry during quark-smearing, we defined the 2-point function in the main text as

$$\tilde{G}_{2pt}(p) = Tr \left[ \gamma_5 G^{L,-\phi,-\phi}(U) \gamma_5 G^{L,\phi,\phi}(U e^{-ip}) \right] ,$$  \hspace{1cm} (A21)

with explicit smearing factors included in the detailed expression.

3. QuasiPDF-pion-pion three-point function

Let the fermion bilinear connected by a spatial Wilson line from \( w \) to \( w + z \) for \( z = (0,0,0,z_3) \), i.e., the quasi-PDF operator, is given by

$$O(w; z) = \bar{u}_w \gamma_0 W_{w;w+z} u_w$$  \hspace{1cm} (A22)

Our focus will be on the three point function

$$G^{(3)}(x, y, w; z) \equiv \langle \pi(x) O(w; z) \pi^+(y) \rangle.$$  \hspace{1cm} (A23)

Strictly in the large \( N_c \) limit, we can ignore the quark-line disconnected diagrams and write

$$G^{(3)}(x, y, w; z) = \sum_{x',x''} Tr \left( \gamma_5 S_{x,x'}(U e^{-i\phi}) G_{x',w}(U) \gamma_5 W_{w,w+z}(U) G_{w+z,x''}(U) \right)$$  \hspace{1cm} (A24)

Replacing infinite lattice propagators by propagators on periodic lattice as we have done before,

$$G^{(3)}(x, y, w; z) = \int \frac{d^4q}{(2\pi)^4} Tr \left( \gamma_5 G^{L,-\phi,-\phi}(U e^{iq/L}) \gamma_5 W_{w,w+z}(U) G^{L,\phi,\phi}(U e^{iq/L}) \right) ,$$  \hspace{1cm} (A25)

where we have used the following notation,

$$G^{L,-\phi,-\phi}_{x',w}(U) = \sum_{x''} S^{L}_{x',w}(U e^{-i\phi}) G^{L}_{x'',y}(U)$$

$$G^{L,\phi,\phi}_{x',w}(U) = \sum_{x''} G^{L}_{x',w}(U) S^{L}_{x'',y}(U e^{-i\phi})$$

$$G^{L,\phi,\phi}_{x,y}(U) = \sum_{x''} S^{L}_{x,x''}(U e^{i\phi}) G^{L}_{x',y''}(U) S^{L}_{x'',y}(U e^{i\phi}) .$$  \hspace{1cm} (A26)
Table I. The table lists the momenta $P_3 = \frac{2\pi n_3}{L^2 N_c}$, and the amount of statistics at each momentum. The statistics comes from two sources; namely, independent number of gauge field configurations (second column) and the number of $Z_2$ stochastic vectors in each configuration. We used $3$ $Z_2$ random vectors which are diluted in chirality and in even-odd lattice sites, which comes out to 12 set of inversions over the components of the noise vectors. We fixed this for all momenta. To convert $P_3$ to GeV, $P_3 = 0.081 n_3$ GeV. All the momenta used in this work are below the lattice $a^{-1}$ scale.

| $n_3$ | $P_{3a}$ | $P_3/\sqrt{2}$ configurations |
|-------|----------|-------------------------------|
| 0     | 0        | 15353                         |
| 2     | 0.092    | 16320                         |
| 4     | 0.185    | 11520                         |
| 6     | 0.277    | 19200                         |
| 8     | 0.370    | 30720                         |
| 10    | 0.462    | 27552                         |
| 12    | 0.554    | 30720                         |
| 14    | 0.647    | 30720                         |
| 16    | 0.739    | 30504                         |

We can split the infinite sum over $Z$ momenta. To convert $P_{3a}$ that it is close to being the largest possible on $L = 8$ and keep the lattice gauge theory is in the confined phase.

Fourier transforming over $(x, w, y) \rightarrow (p', Q, p)$ on the infinite lattice keeping $z$ fixed,

$$\tilde{G}^{(3)}(p', Q, p; z, U) = \sum_{x,w,y} \int \frac{d^2q}{(2\pi)^2} \frac{d^2q'}{(2\pi)^2} \frac{d^2q''}{(2\pi)^2} e^{-iq((w-x)/L)} e^{-iq'(y-w-z)/L} e^{-iq''(x-y)/L} \left( p' + Q + p \right)$$

and use the folded property of Wilson lines. This will result in

$$q - q'' + p' L = 0; \quad q' - q + Q L = 0; \quad q'' - q' + p L = 0,$$

which includes the momentum conservation, $p' + Q + p = 0$. We arrive at

$$\tilde{G}^{(3)}(p', Q, p; z, U) = \delta(p' + Q + p) \int \frac{d^2q'}{(2\pi)^2} \sum_{x,w,y} \text{tr} \left( \gamma_5 G_{x,w}^{L,-,\phi}(U e^{i/2}) \gamma_0 W_{w,w+z}^L(U) G_{w+z,y}^{L,\phi,-}(U e^{i/2}) \gamma_5 G_{y,x}^{L,\phi,\phi}(U e^{i/2}) \right).$$

Using $U^d(1)$ symmetry,

$$\tilde{G}^{(3)}(p', Q, p; z, U) = \delta(p' + Q + p) \sum_{x,w,y} \text{tr} \left( \gamma_0 W_{w,w+z}^L(U) G_{w+z,y}^{L,\phi,-}(U) \gamma_5 G_{y,x}^{L,\phi,\phi}(U e^{-i/2}) \gamma_5 G_{x,w}^{L,-,\phi}(U e^{i/2}) \right).$$

We referred to the above equation in the main text, now with an explicit specification of quark smearing factors, as

$$\tilde{C}_{3pt}(z, p, Q) = \sum_{x,w,y} \text{tr} \left( \gamma_0 W_{w,w+z}^L(U) G_{w+z,y}^{L,\phi,-}(U) \gamma_5 G_{y,x}^{L,\phi,\phi}(U e^{-i/2}) \gamma_5 G_{x,w}^{L,-,\phi}(U e^{i/2}) \right).$$

We used $Q = 0$ in this work.

Appendix B: Details of the lattice calculation

In the present work, we used a fixed large value of $N_c = 17$, since it is the smallest value of $N_c$ beyond which the $1/N_c$ corrections are typically found to small in previous works. We used $L^4$ lattices in this paper with $L = 8$. We used the standard single plaquette Wilson gauge action and set the lattice coupling $b = 0.355$, such that it is close to being the largest $b$ possible on $L = 8$ and keep the lattice gauge theory is in the confined phase.
(phase 0c). The critical \( L_c(b) = 6.6 \) for the value of \( b \) we used. Since fermion loops are \( 1/N_c \) suppressed, the quenched lattice computation of fermionic quantities is exact in the large-\( N_c \) limit. Each update of the gauge fields on the entire lattice was made up of \( N_c(N_c-1) \) SU(2) heat-bath updates on every link followed by one SU(N) over-relaxation update on every link [48]. We performed 100 such updates between measurements to avoid autocorrelation. To make sure the configurations thermalized to the 0c phase, we successively decreased the value of \( b \) from a higher value of \( b = 0.365 \). By monitoring the gap in the Polyakov loop eigenvalues [48] in all four directions, we ensured that the configurations were in the correct phase. We computed the pion-pion two-point and pion-quasi\( DF \)-pion three-point functions (Equations 2 and 4 in the main text) on every configuration at 9 different values of momentum, with \( \xi \sim \phi \) ratio at higher momenta in the SU(3) QCD at some value of \( aE \). We tuned to \( w_{\text{up}} = 36 \) to realize a pion mass that was feasible given the computational resource available to us. We implemented the Wilson-Dirac inversion using BiCG-Stab algorithm [74].

We evaluated the two-point and three-point functions (Equations 2 and 4 in the main text) stochastically. For the two-point function, the stochastic estimator using noise vectors \( \xi \), namely, for the two-point function, the stochastic estimator using noise vectors \( \xi \), that is\( \xi^{\alpha,a,x} = \frac{1}{\sqrt{2}} \) for spin, color and position indices \( \alpha, a, x \) respectively. We further diluted the noise-vectors over even-odd lattice sites and over the two chiral fields on the entire lattice was made up of \( N_c(N_c-1) \) SU(2) heat-bath updates on every link followed by one SU(N) over-relaxation update on every link [48]. We performed 100 such updates between measurements to avoid autocorrelation. To make sure the configurations thermalized to the 0c phase, we successively decreased the value of \( b \) from a higher value of \( b = 0.365 \). By monitoring the gap in the Polyakov loop eigenvalues [48] in all four directions, we ensured that the configurations were in the correct phase. We computed the pion-pion two-point and pion-quasi_DF-pion three-point functions (Equations 2 and 4 in the main text) on every configuration at 9 different values of momentum, with \( \xi \sim \phi \) ratio at higher momenta in the SU(3) QCD at some value of \( aE \). We tuned to \( w_{\text{up}} = 36 \) to realize a pion mass that was feasible given the computational resource available to us. We implemented the Wilson-Dirac inversion using BiCG-Stab algorithm [74].

We used \( N_{\text{vec}} = 3 \) number of \( Z_2 \) noise vectors for \( \xi \); that is \( \xi^{\alpha,a,x} = \frac{1}{\sqrt{2}} \) for spin, color and position indices \( \alpha, a, x \) respectively. We further diluted the noise-vectors over even-odd lattice sites and over the two chiral projections. For the three-point function, we used the stochastic estimator as

\[
\begin{align*}
\hat{\mathcal{C}}_{3\text{pt}}(p, q) &= \sum_{x} \xi^{5}_{x} \gamma_{0} W_{x,x+q} \phi_{x+q}(p), \quad \phi(p) = G^{J}(U) \gamma_{0} G^{L}(U \epsilon_{x}^{-ip}) \gamma_{0} G^{L}(U \epsilon_{x}^{ip}) \xi.
\end{align*}
\]

In this work, we only used \( q = 0 \) above. We used 2-steps of Stout smearing for the gauge-links that are used to construct the Wilson line \( W \).

We used Wilson-Dirac operator \( \mathcal{D}^{L}(m_{w}) \) to compute the propagators \( G^{L} = [\mathcal{D}^{L}]^{-1} \). We improved the Dirac operator by using gauge-links that are smeared by two steps of the large-\( N_c \) version of the Stout smearing [73]. With smearing, we expect the zero quark mass to be in the region of the Wilson mass \( m_{w} = [-0.38, -0.39] \). We tuned to \( m_{w} = -0.36 \) to realize a pion mass that was feasible given the computational resource available to us. We implemented the Wilson-Dirac inversion using BiCG-Stab algorithm [74].

We used smeared quark sources in the construction of two-point and three-point functions using a smearing kernel \( S(U; N_{\text{wup}}, \delta) \) for the Wuppertal smearing [75]. We implemented Wuppertal smearing using \( (N_{\text{wup}}, \delta) = (40, 0.6) \), which we chose to be optimal through a set of initial tuning runs. We kept the radius of Wuppertal smearing fixed at all pion momenta. A puzzling experience during the tuning process at non-zero momenta was the negligible effect of phased momentum smearing [76], \( S(c^{i\phi} U) \), which typically improves the signal to noise ratio at higher momenta in the SU(3) QCD at some value of \( \phi \); we did not find any such improvement within statistical errors during the tuning phase in which we used only about \( \sim 1000 \) configurations. Therefore, we simply used unphased (\( \phi = 0 \)) Wuppertal kernel for quark smearing at all momenta.

### Appendix C: Spectral content of pion two-point functions

1. Construction

We determined the two-point function \( \hat{\mathcal{C}}_{2\text{pt}}(p) \) with \( p = (p_0, 0, 0, P_3) \) for \( p_0 = 2\pi n_0/L_{0\text{eff}}^{m} \) using \( n_0 \in [0, \frac{L_{0\text{eff}}^{m}}{2}] \) and the effective temporal extent of \( L_{0\text{eff}}^{m} = 68 = N_c L / 2 \). We could have used an effective temporal extent of up to 136, but used a smaller one to make the computation easier, and an effective temporal extent of 68 is quite comparable to what is being used in present structural computations in SU(3) QCD. One possibility to investigate the spectral content in the 2-point function is to fit the data at different fixed spatial \( P_3 \) to

\[
\begin{align*}
\hat{\mathcal{C}}_{2\text{pt}}(p_0, P_3) &= B + \sum_{i=0}^{N_a-1} \frac{|A_i|^2 \sinh(a E_i(P_3))}{\cosh(a E_i(P_3)) - \cos(a p_0)}; \quad A_i = \langle E_i | \pi^+ | 0 \rangle,
\end{align*}
\]
where one can truncate the momentum space spectral decomposition at $N_{st}$ number of states. It is to be remembered that, even if higher excited states might not contribute to the $p_0$ dependence of the correlator in the range of smaller $p_0$, they can still contribute a momentum independent constant to the above correlator. Therefore, we corrected such a truncated series by a constant term, $B$, to account for such effects of all other higher excited states.

In this paper, we used the computed momentum space 2-point functions to Fourier transform them into real space, so that we can perform a rather traditional lattice QCD analysis via effective masses and multi-exponential fits. That is, the real space correlator is

$$C_{2pt}(t_s, P_3) = \sum_{n_0=0}^{N_{st}-1} \tilde{C}_{2pt}(P_0, P_3)e^{i p_0 \frac{t_s}{a}}; \quad \tilde{C}_{2pt}(L_0^{\text{eff}} - n_0, P_3) = \tilde{C}_{2pt}(n_0, P_3). \quad (C2)$$

Note that we have used the momenta $p$ and their integer quanta $n$ interchangeably as arguments above, and we will do so in the rest of the text without any obvious confusion. After the above Fourier transformation, we performed the usual $N_{st}$-state fits to study their spectral content,

$$C_{2pt}(t_s, P_3) = \sum_{i=0}^{N_{st}-1} |A_i|^2 \left( e^{-E_i \frac{t_s}{a}} + e^{-E_i (L_0^{\text{eff}} - \frac{t_s}{a})} \right), \quad (C3)$$

and obtained their effective masses by solving

$$\frac{\cosh(E_i (L_0^{\text{eff}}/2 - (t_s + a)/a))}{\cosh(E_i (L_0^{\text{eff}}/2 - t_s/a))} = \frac{C_{2pt}(t_s + a, P_3)}{C_{2pt}(t_s, P_3)}. \quad (C4)$$

2. An issue with long-tailed distributions in the zero-momentum case

First, we discuss the case of zero spatial momentum $P_3 = 0$ which we found to be challenging within the stochastic approach of constructing trace along with the Fourier transform to real space. In the top panels of Fig. 4, we show the momentum space correlator $\tilde{C}(p_0, P_3)$ as a function of temporal momentum $p_0$; the top-left and top-right panels show the results at $P_3 = 0$ and $P_3/(2\pi/136) = n_3 = 10$ respectively. The black curves are the best fit curves using Eq. (C1) truncated at $N_{st} = 3$ and fitted over a range of $n_0 \in [1, 20]$. The fits work well with $\chi^2/\text{dof} \approx 1$. However, we note that the curve for $P_3 = 0$ when extrapolated to $n_0 = 0$ is slightly, but in a statistically significant manner, below the actual stochastically evaluated data point for $\tilde{C}(p_0 = 0, P_3 = 0)$. Such a problem existed only at $n_0 = 0$ cases, and at other non-zero $P_3$ (such as $n_3 = 10$ case on the top-right) the fitted curve automatically passed through $n_0 = 0$ data point as well. While the problem is easy to fix by avoiding the $n_0 = 0$ data point while performing fits, it causes problem when reconstructing real-space correlators via Fourier transform; namely, if $n_0 = 0$ is not evaluated very accurately, then its effect is to add a spurious low-mass state into the real-space correlator. The origin of the problem is easy to understand. In the bottom left and bottom right panels of Fig. 4, we show the Monte Carlo histogram of the stochastic estimator in Eq. (B2) for the two spatial momenta. For each of them, we have shown the histograms at three values of $n_0$. We see that for $n_0 = 0$, $n_3 = 0$, which is nothing but pion susceptibility, the distribution is very long-tailed, and hence, it is likely that the difficulty we are finding is due to the inability to robustly estimate the mean and the statistical error of such a long-tailed distribution. At non-zero $n_0$, the distribution gets narrower. Also, the distribution at $n_0 = 0$ gets narrower at non-zero $n_3$, and hence we were able to reconstruct real-space correlators well. Having understood the problem, we found the following procedure to correct the $n_0 = 0$ data points to solve the issue; we took the lowest three non-zero $n_0 = 1, 2, 3$ data points that is dominated by the ground-state $E_0$, and solved a system of equations,

$$\frac{A}{\cosh(a E_0)} + B = \tilde{C}_{2pt}(p_0, P_3). \quad (C4)$$

to find the unknown parameters $A, B, E_0$. Using them, we corrected the $n_0 = 0$ point with the estimated value $\frac{A}{1 - \cosh(a E_0)} + B$. Using this corrected $n_0 = 0$ data point, we used Eq. (C2) to perform the Fourier transform and obtained the correlator as a function of $t_s$. With this procedure, the spurious low mode disappeared at $P_3 = 0$. At non-zero $P_3$, such a procedure did not have any significant effect at all.

3. Analysis of 2-point functions in real-space

In the left panel of Fig. 5, we show our “raw” data for $\tilde{C}_{2pt}(p_0, P_3)$ that we directly computed on the lattice as a function of $p_0$ for the 9 different spatial $P_3$. As such, we find our determination of $\tilde{C}_{2pt}(p_0, P_3)$ to be
FIG. 4. (top panels) Sample pion-pion two point functions $\tilde{C}_{2pt}(n_0; n_3)$ at spatial momenta $n_3 = 0$ (left) and $n_3 = 10$ (right) as a function of temporal momentum $n_0$. The black curves are fits to the momentum space correlator to Eq. (C1) with $N_{st} = 3$ to data from $n_0 \in [1, 20]$. For $n_3 = 10$, the fit automatically passes through the $n_0 = 0$ data point. For $n_3 = 0$, the $n_0 = 0$ data point is slightly above the expectation. This causes a problem for $n_3 = 0$ real space correlator construction and shows up as a pathological state with near zero mass. The possible issue is purely numerical and has to do with $n_3 = 0, n_0 = 0$ being the pion susceptibility that is hard to evaluate stochastically due to the long tailed nature of its Monte Carlo histogram. (Bottom panels) Such Monte Carlo histograms for $\tilde{C}_{2pt}(n_0, n_3)$ are shown in the bottom two panels for $n_3 = 0$ and $n_3 = 10$. Indeed the $n_3 = 0, n_0 = 0$ case is long tailed, and it might require even larger statistics to evaluate it robustly. The histograms immediately get narrower at non-zero $n_3$ and $n_0$ and thereby, does not cause any issues.

FIG. 5. Pion two point function correlator as a function of temporal Euclidean momentum $n_0$ (left) and as a function of temporal separation $t_s/a$ (right) as constructed from the momentum space correlations via Fourier transformation. The different symbols from top to bottom in the two panels are the data points at different spatial momentum along the $z$-direction, $n_3$, from $n_3 = 0$ to 16 in steps of 2.
smoothly varying in both $p_0$ and $P_3$, up to an issue noted above for $P_3 = 0$. The well-determined nature of $\tilde{C}$ in Fourier space is deceiving, as the long-distance exponential fall-off in the Fourier transformed $C_{2pt}(t_s)$ comes from delicate cancellations between different $\tilde{C}_N(p_0)$, resulting in noise at larger $t_s$. This can be seen in the right panel of Fig. 5, where we show such an inferred real-space two-point function, $C_{2pt}(t_s, P_3)$ using Eq. (C2). We have displayed the results at various spatial momenta as a function of $t_s/a$.

In the determination of the quasi-PDF matrix element, the spectral data of two-point function does not enter in the summation type analysis we performed (as we discuss in the next section). We present our analysis of the spectral content of the two-point function now for the sake of completion. In Fig. 6, we show the effective mass $E_{eff}(t_s)$ determined from $\tilde{C}_{2pt}(t_s)$. We were able to perform stable two-state fits ($N_{st} = 2$ in Eq. (C3)) to $C_{2pt}(t_s)$. For this, we used a fit range $t_s \in [2a, 15a]$ for $n_3 = 0, 2, 4, 6, 8$, $t_s \in [2a, 10a]$ for $n_3 = 10, 12$, and $t_s \in [2a, 6a]$ for $n_3 = 14, 16$. We used a smaller minimum of $2a$ so as to be sensitive to excited states, and at the same time make the fits stable. We changed the maximum range of $t_s$ so as to avoid the noisier data points as well as those that are not well-determined after all the intricate cancellations in the Fourier transform from $\tilde{C}_{2pt}(p_0)$ resulting in orders of magnitude smaller values for $C_{2pt}(t_s)$ as seen in Fig. 5 (for example, the $t_s > 6a$ data points in Fig. 6 for $n_3 = 14, 16$ that are suddenly pulled to smaller values than expected, and it is clear that they are not well-determined numerically and might need more precise data for $\tilde{C}$.) In this way, we found the pion mass in our calculation to be $m_{\pi a} = 0.219(2)$ in lattice units. We show the resulting effective mass curves from the two-state fits at different momentum $n_3$ as the bands in Fig. 6. For comparison, the expected values for $E(P_3)$ from one-particle dispersion relation based on the value of $m_{\pi a} = 0.219$ are shown as the dot-dashed horizontal lines in the figure. We see that the resulting ground-state energies agree with the continuum dispersion within errors. As a curious observation that is unrelated to the results in the paper, we found the first excited state energy at $n_3 = 0$ to be $aE_1(P_3 = 0) = 0.79(4)$ from the two-state fits. Using a string-tension value of $\sqrt{\sigma} = 440$ MeV, we find this value to be about 1.3 GeV which seems to agree quite nicely with the pion radial excitation pole-mass in SU(3) QCD. Thus, the usage of string tension to set the large-$N_c$ GeV scale has its advantage as noted in the main text. However, we only found a poor agreement of the momentum dependence of $E_1(P_3)$ with a single particle dispersion curve and hence we cannot rule out the possibility of the agreement with pion(1300) at $P_3 = 0$ to be a numerical coincidence in our calculation, and the $E_1$ could simply be effectively capturing the tower of excited states.

Appendix D: Determination of bare pion quasi-PDF matrix elements

1. Construction of $t_s$ dependent 3-point function

We used the stochastic estimator in Eq. (B4) to determine the 3-point function in momentum space, $\hat{C}_{3pt}(z, p, q)$. We found it computationally simpler and cheaper to fix the momentum insertion $q$ and scan the entire set of $p_0$ for each choice of pion’s spatial momentum $P_3$. We chose $q_0 = 0$ as this choice is the summation method as noted in the main text. Further, we set the spatial part of $q$ also to zero, as we want the forward
matrix element in this work for the case of PDF; thus \( q = 0 \). We reconstructed the \( t_s \) dependence of the 3-point function as,

\[
C_{3pt}(z, t_s, P_3) = \sum_{n_0=0}^{t_s^{\text{eff}}-1} \tilde{C}_{3pt}(z, p_0, P_3, q = 0) e^{i p_0 \tau}; \quad \tilde{C}_{3pt}(z, L_0^{\text{eff}} - n_0, n_3, q = 0) = -\tilde{C}_{3pt}(z, n_0, n_3, q = 0).
\]

The second identity is simply due to the usage of \( \gamma_0 \) in the definition of quasi-PDF operator, which makes the three-point function to be proportional to \( p_0 \), and hence, antisymmetric with respect to \( n_0 \) and \( L_0^{\text{eff}} - n_0 \). As is usual, we used \( z = (0, 0, 0, z_3) \) along the \( z \)-axis. By using folded Wilson line, we scanned \( z_3/a \in [-16, 16] \). In the end, we only used \(|z_3|/a \leq 6\) so as to ensure the applicability of perturbation theory.

Let us make the connection of 3-point function with \( q_0 = 0 \) to summation method obvious. We suppress the arguments for \( z, p \) in the 2- and 3-point functions for the sake of brevity, and both of them should be at understood to be at the same \( p \) below. For the sake of argument, let \( C_{3pt}(t_s, \tau) \) be the 3-point function by Fourier transforming with respect to both \( p_0 \) and \( q_0 \); in that case, \( \tau \) is the insertion time of the quasi-PDF operator \( \tau \), and it could be both within and outside the pion source and sink locations. For the case \( \tau \leq t_s \), we can do a spectral decomposition to get

\[
C_{3pt}(t_s, \tau) = \sum_{i, j=0} A_i^* A_j \langle i | O | j \rangle e^{-E_i (t_s - \tau) - E_j \tau}; \quad A_i = \langle 0| \pi^\dagger | E_i \rangle,
\]

for \( \tau \leq t_s \). The sum within this region gives, \( \sum_{\tau=0}^{t_s} C_{3pt}(t_s, \tau) \sim |A_0|^2 \langle 0 | O | 0 \rangle t_s e^{-E_0 t_s} + \text{const.} \times e^{-E_0 t_s} \), up to \( O(e^{-E_1 t_s}) \) excited state corrections. When the operator is “outside” the source and sink, \( \tau > t_s \), then the terms which are not exponentially suppressed with the effective temporal extent, \( L_0^{\text{eff}} \), are of the form

\[
C_{3pt}(t_s, \tau) = \sum_{i, j} \langle 0 | O | i \rangle e^{-E_i \tau} e^{-E_j t_s} \langle i | \pi | j \rangle \langle j | \pi^\dagger \rangle | 0 \rangle,
\]
for \( \tau > t_s \). Since the state \( j \) cannot be the ground-state pion, and \( i \neq j \), the sum over \( \tau > t_s \) cannot have the linear piece and will contribute simply as yet another exponentially suppressed excited state contribution. Therefore, we can sum over \( \tau \) for all values from 0 to \( t_\text{off} \), and the linear piece in \( t_s \) gives the information on the ground state matrix element. The sum, \( \sum_{\tau=0}^{t_\text{off}} C_{3pt}(t_s, \tau) = C_{3pt}(t_s, q_0 = 0) \). To cancel off the amplitudes \( A_i \), we form the ratio

\[
R(t_s) = \frac{C_{3pt}(t_s, q_0 = 0)}{C_{2pt}(t_s)}.
\]

(D4)

From the arguments above, we see that

\[
R(t_s) = \langle E_0 \mid O \mid E_0 \rangle t_s + C + O(e^{-(E_1-E_0)t_s}).
\]

(D5)

By fitting the linear \( t_s \) dependence of \( R(t_s) \), we obtained the bare quasi-PDF matrix element from the slope.

2. Extraction of ground-state bare quasi-PDF matrix elements

We fit the functional form

\[
R(t_s; P_3, z_3) = t_s h^B(z_3, P_3) + C,
\]

(D6)

to the summed ratio \( R(t_s) \) at different \( z_3 \) and \( P_3 \), using \( h^B(z_3, P_3) \) and \( C \) as fit parameters over ranges \( t_s \in [t_{\text{min}}, t_{\text{max}}] \). At the precision allowed by our data, we restricted the value of \( t_{\text{min}} = 2a, 3a, 4a \), and finally used \( t_{\text{min}} = 3a \), which in physical units, \( t_{\text{min}} \sqrt{\sigma} = 0.76 \), is on the verge of the typical nonperturbative mass-gapped scales. By changing \( t_{\text{min}} \) to \( 2a \) and \( 4a \), we checked for the level of consistency as means to test for the residual presence of excited state contributions. We used \( t_{\text{max}} \) between \( 6a \) and \( 8a \), and avoided points beyond \( 8a \) to not use the noisy as well as stochastically not-so well determined points at even larger \( t_{\text{max}} \). In the various panels of
The real part of the bare quasi-PDF matrix element $h_B(z_3,n_3)$ is shown as a function of $z_3/a$ at different spatial momentum $s$ used in this work as separate panels. The extrapolated results from summation methods over different fit ranges in $s$ are shown together in the plots. We used extrapolations from $t_s \in [3a, 8a]$ for $n_3 \in [0,10]$, and $t_s \in [3a, 6a]$ for $n_3 \in [12,16]$ to avoid badly determined points beyond $t_s \geq 6a$. Using ranges with even larger minimum $t_s$ was not feasible and forms a limitation of this work.
Within the statistical errors, it is clear that the data nicely agrees with a linear dependence in the ranges of \( s \).

In each panel, the points are our lattice determination of \( R_{ts} \) used in this work as separate panels. The description is similar to Fig. 9.

We have shown the fits to Eq. (D6) for different fit ranges as the bands; the slopes of these lines are the needed values of \( h^B \).

Within the statistical errors, it is clear that the data nicely agrees with a linear \( ts \) dependence in the ranges of \( ts \) specified above. At the smaller \( n_3 \), where the data quality is better, we see that the fitted bands from the

Fig. 7 and Fig. 8, we show the data points and fitted straight-lines to determine Re\( h^B \) and Im\( h^B \) respectively. Each column in these figures show the fits at \( z_3 = 0, 2a, 4a, 6a \) at fixed momentum \( n_3 \). The different rows show them at \( n_3 = 0, 4, 8, 12, 16 \). In each panel, the points are our lattice determination of \( R(ts) \). We have shown the fits to Eq. (D6) for different fit ranges as the bands; the slopes of these lines are the needed values of \( h^B \). Within the statistical errors, it is clear that the data nicely agrees with a linear \( ts \) dependence in the ranges of \( ts \) specified above. At the smaller \( n_3 \), where the data quality is better, we see that the fitted bands from the
The extraction of matrix elements is done correctly and there is no momentum dependent lattice corrections, we should not find any $P_3$ dependence in $Z_V$. Indeed, we find that to be case in Fig. 11 up to statistical errors.

Appendix E: Implementation of leading-twist OPE and construction of $\overline{MS}$ ITD

We implemented the leading-twist OPE using the truncated form of Eq. (3) written explicitly as,

Re$M(\nu, z_3^2) = 1 + \sum_{n=1}^{N_{max}} (-1)^n \nu^{2n} C_{2n}(z_3^2 \mu^2) (x^{2n})_{u-\bar{u}},$

Im$M(\nu, z_3^2) = \sum_{n=1}^{N_{max}} (-1)^n \nu^{2n-1} C_{2n-1}(z_3^2 \mu^2) (x^{2n-1})_{u+\bar{u}}.$

(E1)

We used the truncation as $N_{max} = 4$ to fit the data up to $\nu = 3.5$, and we checked that the results do not change well within errors when $N_{max}$ is changed from 3 to 4. The Wilson coefficients $C_n$ are the isovector quark coefficients, usually written explicitly as $C_{qq}^{qq}$. For the imaginary part, which is not a isovector quantity, one would have to include the corresponding $C_{qq}^{qg}$ and $C_{qq}^{gq}$ which will cause mixing with quark and gluon PDFs [77]; here, in the large-$N_c$ limit, the $C_{qq}^{qq}$ which are proportional to $\alpha_s T_F(N_c)$ are $1/N_c$ suppressed and hence, we have considered only the $C_{qq}^{qg}$ Wilson coefficients above and in the main text.

Using the above OPE, we performed combined fits to the $z_3$ and $P_3$ dependencies of the lattice data. We performed two types of fits in the main text:

1. Moments fit: here, we used the Mellin moments $\langle x^n \rangle$ entering the OPE as the free fit parameters. Since, we are assuming no functional form for the $x$-dependence of the PDF, we referred to these types of fits as the model-independent fit analysis. Assuming the positivity of the underlying $u + \bar{u}$ and $u - \bar{u}$ PDFs
help impose additional constraint \cite{30} on their Mellin moments. We implemented such inequalities using a change of variable from moments to \(\lambda_i\),

\[
\langle x^{2n} \rangle_{u-\bar{u}} = \sum_{i=n}^{N_{\text{max}}} \sum_{j=1}^{N_{\text{max}}} e^{-\lambda_j},
\]

and similarly for odd-moments \(\langle x^{2n-1} \rangle_{u+\bar{u}}\).

2. PDF Ansatz fit: Here, we assumed a global fit analysis inspired ansatz for the x-dependence of the valence PDF, \(f_{u-\bar{u}}(x) = N x^\alpha (1-x)^\beta (1+sx^2)\), with \(\int_0^1 f_{u-\bar{u}}(x)dx = 1\). In practice, it results in Mellin moments \(\langle x^{2n} \rangle_{u-\bar{u}}(\alpha, \beta, s)\), that in turn enter Eq. (E1). We fit the parameters \(\alpha, \beta\) and \(s\) in this manner. We imposed a prior \(\alpha \in [-0.4,-0.6]\) based on Regge intercept expectation for valence PDF. We did not perform an equivalent analysis for \(u+\bar{u}\), as it was not clear if we should assume it to be a combination of ansatz for valence PDF and sea-quark PDF, and what prior to impose on small-x behavior of sea-quarks in large-\(N_c\) theory. Therefore, we avoided such issues here by performing only moments fit to \(u+\bar{u}\) case.

In the main text, we constructed the \(\overline{\text{MS}}\) ITD at \(\mu = 2\) GeV based on the analysis of pseudo-ITD lattice data above. These ITDs are defined as

\[
\mathcal{M}_{u-\bar{u}}^{\text{MS}}(\nu, \mu) \equiv \int_0^1 f_{u-\bar{u}}(x, \mu) \cos(x \nu)dx; \quad \mathcal{M}_{u+\bar{u}}^{\text{MS}}(\nu, \mu) \equiv \int_0^1 f_{u+\bar{u}}(x, \mu) \sin(x \nu)dx.
\]

In practice, the construction of \(\overline{\text{MS}}\) ITD using the limited range of \(\nu\) is simplified into a truncated series in \(\nu\) as

\[
\mathcal{M}_{u-\bar{u}}^{\text{MS}}(\nu, \mu) = 1 + \sum_{n=1}^{N_{\text{max}}} \frac{(-1)^n \nu^{2n}}{(2n)!} \langle x^{2n} \rangle_{u-\bar{u}}(\mu),
\]

\[
\mathcal{M}_{u+\bar{u}}^{\text{MS}}(\nu, \mu) = \sum_{n=1}^{N_{\text{max}}} \frac{(-1)^n \nu^{2n-1}}{(2n-1)!} \langle x^{2n-1} \rangle_{u+\bar{u}}(\mu),
\]

using the best fit estimates of the Mellin moments from the leading-twist analysis.

Appendix F: Details regarding the perturbative aspects

1. Coupling constant and Wilson coefficients

We borrowed various existing perturbative results computed for general \(N_c\), and we simply used the large-\(N_c\) values of the color factors, \(C_F(N_c) \rightarrow N_c/2\), \(C_A(N_c) \rightarrow N_c\), and \(T_F(N_c) \rightarrow 1/2\) in those expressions. In the absence of a nonperturbative running of the large-\(N_c\) \(\overline{\text{MS}}\) coupling, we simply used the LO ’t Hooft coupling in the large-\(N_c\) limit,

\[
\lambda(\mu) \equiv \lim_{N_c \rightarrow \infty} \alpha_s(\mu)N_c = \frac{1}{12\pi \ln \left( \frac{\mu^2}{\Lambda_{\overline{\text{MS}}}^2} \right)}.
\]

With \(\sqrt{s} = 0.44\) GeV to set the scale, we used \(\Lambda_{\overline{\text{MS}}} = 0.22\). At \(\mu = 2\) GeV, we get \(\lambda(2\text{GeV}) = 0.778\). For the Wilson coefficients that enter the leading-twist OPE in Eq. (3), we used the 1-loop expressions in Ref \[49\] with the replacement \(C_F(N_c)\alpha_s(\mu) \rightarrow \lambda(\mu)/2 = 0.389\). The Wilson coefficients for the \(u+\bar{u}\) PDF would differ in the SU(3) QCD due to it being a flavor singlet quantity. In the large-\(N_c\) limit, such differences due to the mixing terms \(\propto T_F(N_c)\) are sub-leading in \(1/N_c\), and hence, we simply used the non-singlet Wilson coefficients \(C_n\) for odd values of \(n\).

2. Large-\(N_c\) LO DGLAP evolution

In the main text, we checked whether a universal initial condition at a low factorization scale \(\mu_0\) could explain the observed differences between SU(3) QCD and in large-\(N_c\) theory. For this, we performed the DGLAP evolution of large-\(N_c\) PDF (or equivalently its ITD), from scale \(\mu\) to a lower-scale \(\mu_0\), that is then used as an initial condition for 3 flavor SU(3) QCD evolution back to scale \(\mu\) using corresponding DGLAP evolution in Mellin space. That is, taking

\[
X_n^{(N_c)}(\mu) = [2\langle x^n \rangle_{u-\bar{u}}(\mu), 2\langle x^n \rangle_{u+\bar{u}}(\mu), \langle x^n \rangle_q(\mu)],
\]
as the array of quark and gluon moments in $SU(N_c)$ QCD, we evolved them as,

$$X_n^{(N_c)}(\mu') = \left( \begin{array}{ccc} P_{NS}^{qg,N_c}(n,\mu,\mu') & 0 & 0 \\ 0 & P_{S}^{qq,N_c}(n,\mu,\mu') & P_{S}^{qg,N_c}(n,\mu,\mu') \\ 0 & P_{S}^{qg,N_c}(n,\mu,\mu') & P_{S}^{qg,N_c}(n,\mu,\mu') \end{array} \right) \cdot X_n^{(N_c)}(\mu),$$

where $P^{ij,N_c}(n,\mu,\mu')$ are the $SU(N_c)$ theory DGLAP factors from parton species $i$ to species $j$ in Mellin space (e.g., textbook such as [78]) that evolve the moments from scale $\mu$ to $\mu'$. The subscript $S$ and $NS$ specify singlet and non-singlet respectively. In this paper, we used a LO DGLAP evolution, at which order $\mu$ (e.g., textbook such as [78]) that evolve the moments from scale $\mu$ to $\mu'$.

In this work, we only computed the quark moments $\langle x^3 \rangle_{u+a}$ in the large-$N_c$ theory, and we did not explicitly compute the gluon PDF in the large-$N_c$ pion. Therefore, we deduced the leading moment $\langle x \rangle_g = 1 - 2 \langle x^2 \rangle_{u+a}$ from the momentum sum rule. Since we expect the gluon PDF to be contributing dominantly in the small-$z$ region, we assumed that the next moment $\langle x^5 \rangle_g$ (and all other higher odd moments) can be neglected. With these inputs from the large-$N_c$ theory, we followed the chain of evolution,

$$X_n^{(\infty)}(\mu) \rightarrow X_n^{(\infty)}(\mu_0) \rightarrow X_n^{(3)}(\mu).$$

Using such an expectation $X_n^{(3)}(\mu)$ for $SU(3)$ QCD moments at $\mu = 2$ GeV based on the above evolution, we constructed the corresponding NIS ITD by using Eq. (E4).

### Appendix G: Efficacy of 1-loop large-$N_c$ leading-twist OPE

The leading-twist OPE can be applied to the lattice data at fixed values of $z_3$ [79], so as to capture the $\nu$ dependence coming only via variation in the momentum $P_3$. Such an application has been found [30, 80] to be a nice diagnostic of the effectiveness of perturbative as well as leading-twist framework in a region of $z_3$, and as way to detect corrections to the framework. In the left panel of Fig. 12, we show such a $z_3$ dependent leading non-trivial moment $\langle x \rangle$, from the analysis of $\text{Re} \mathcal{M}^{(3)}(\nu, z_3^2)$. In the right panel, we show a similar $z_3$ dependence of $\langle x^2 \rangle$ from $\text{Re} \mathcal{M}^{(3)}(\nu, z_3^2)$. We used $\mu = 2$ GeV in the scale set by $\sqrt{\sigma} = 0.44$ GeV as explained above. The filled circles are the results using 1-loop Wilson coefficients in the OPE. If 1-loop is sufficient, and if there are no higher-twist corrections to the OPE and $z_3$-dependent lattice spacing corrections to the continuum OPE, then one should observe a plateau in the moments as a function of $z_3$. In the range $z_3 \in [2a, 5a]$ that we used, we see an approximate plateau in the 1-loop results in the two panels. Our determinations of the two moments via
a combined fits to the entire data in the range of $z_3 \in [2a, 5a]$ is shown as the gray bands, which are consistent with the plateau in the data. We skipped the $z_3 = a$ point to be cautious of avoiding any lattice corrections at those separations. Given the quality of our data, we did not add any lattice spacing and higher-twist corrections by hand to the leading-twist continuum OPE. To see the effect of 1-loop evolution in $z_3$ effected by the Wilson coefficients, we also plot the results using tree-level (i.e., set $\alpha = 0$) in the two panels in Fig. 12. The effect of 1-loop is rather small in comparison with typical statistical errors, but it is quite pronounced at shorter $z_3 = 1a – 3a$.

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