Nonexistence of stable $F$-stationary maps of a functional related to pullback metrics

Jing Li, Fang Liu and Peibiao Zhao

Abstract
Let $M^m$ be a compact convex hypersurface in $R^{m+1}$. In this paper, we prove that if the principal curvatures $\lambda_i$ of $M^m$ satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_m$ and $3\lambda_m < \sum_{i=1}^{m-1} \lambda_i$, then there exists no nonconstant stable $F$-stationary map between $M$ and a compact Riemannian manifold when (6) or (7) holds.

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1 Introduction
Let $u : (M^m, g) \rightarrow (N^n, h)$ be a smooth map between Riemannian manifolds $(M^m, g)$ and $(N^n, h)$. Recently, Kawai and Nakauchi [1] introduced a functional related to the pullback metric $u^*h$ as follows:

$$\Phi(u) = \frac{1}{4} \int_M \|u^*h\|^2 dv_g,$$

(1)

(see [2–4]), where $u^*h$ is the symmetric 2-tensor defined by

$$(u^*h)(X, Y) = h(du(X), du(Y))$$

for any vector fields $X, Y$ on $M$ and $\|u^*h\|$ is given by

$$\|u^*h\|^2 = \sum_{i,j=1}^m [h(du(e_i), du(e_j))]^2,$$

with respect to a local orthonormal frame $(e_1, \ldots, e_m)$ on $(M, g)$. The map $u$ is stationary for $\Phi$ if it is a critical point of $\Phi(u)$ with respect to any compact supported variation of $u$, and $u$ is stable if the second variation for the functional $\Phi(u)$ is nonnegative. They showed the nonexistence of a nonconstant stable stationary map for $\Phi$, either from $S^m$ ($m \geq 5$) to any manifold, or from any compact Riemannian manifold to $S^n$ ($n \geq 5$). In this paper, for a smooth function $F : [0, \infty) \rightarrow [0, \infty)$ such that $F(0) = 0$ and $F'(t) > 0$ on $t \in (0, \infty)$, we are concerned with the instability of $F$-stationary maps which is the generalization of a stationary map for $\Phi$ introduced by Asserda in [4]. In [4], they obtained some monotonicity.
formulas for $F$-stationary maps via the coarea formula and the comparison theorem. Also, by using monotonicity formulas, they got some Liouville type results for these maps.

The authors in [5] obtained the first and second variation formula for $F$-stationary maps. By using the second variation formula, they proved that every stable $F$-stationary map from $S^m(1)$ to any Riemannian manifold is constant if

$$\int_{S^m} \|u^*h\|^2 \left\{ F'' \left( \frac{\|u^*h\|^2}{4} \right) \|u^*h\|^2 + (4 - m) F \left( \frac{\|u^*h\|^2}{4} \right) \right\} d\nu_g < 0,$$

or every $F$-stationary map from any compact Riemannian manifold $N^n$ to $S^m$ is constant if

$$\int_{N^n} \|u^*h\|^2 \left\{ F'' \left( \frac{\|u^*h\|^2}{4} \right) \|u^*h\|^2 + (4 - m) F \left( \frac{\|u^*h\|^2}{4} \right) \right\} d\nu_g < 0.$$ 

In this paper, we obtain the results on the instability of $F$-stationary maps which are from or into the compact convex hypersurfaces in the Euclidean space.

## 2 Preliminaries

Let $F : [0, \infty) \to [0, \infty)$ be a $C^2$-function such that $F(0) = 0$ and $F'(t) > 0$ on $t \in (0, \infty)$. For a smooth map $u : (M, g) \to (N, h)$ between compact Riemannian manifolds $(M, g)$ and $(N, h)$ with Riemannian metrics $g$ and $h$, respectively, following Ara [6] for an $F$-harmonic map (also see [7–10]), Asserda in [4] gave the following definition.

**Definition 2.1** We call $u$ an $F$-stationary map for $\Phi_F$ if

$$\frac{d}{dt} \Phi_F(u_t)|_{t=0} = 0$$

for any compactly supported variation $u_t : M \to N$ with $u_0 = u$, where

$$\Phi_F(u) = \int_{M^n} F \left( \frac{\|u^*h\|^2}{4} \right) d\nu_g.$$

Let $\nabla$ and $\nabla^N$ always denote the Levi-Civita connections of $M$ and $N$, respectively. Let $\tilde{\nabla}$ be the induced connection on $u^*TN$ defined by $\tilde{\nabla}_X W = \nabla_X W$, where $X$ is a tangent vector of $M$ and $W$ is a section of $u^*TN$. We choose a local orthonormal frame field $\{e_i\}$ on $M$. We define the $F$-tension field $\tau_{\Phi_F}(u)$ of $u$ by

$$\tau_{\Phi_F}(u) = -\delta \left( F' \left( \frac{\|u^*h\|^2}{4} \right) \sigma_u \right)$$

$$= F' \left( \frac{\|u^*h\|^2}{4} \right) \nabla_g (\sigma_u) + \sigma_u \left( \nabla \left( F' \left( \frac{\|u^*h\|^2}{4} \right) \right) \right),$$

where $\sigma_u = \sum_j h(du(\cdot), du(e_j))du(e_j)$, which was defined in [1].

We need the following second variation formula for $F$-stationary maps (cf. [5]). Let $u : (M, g) \to (N, h)$ be an $F$-stationary map. Let $u_{st} : M \to N (-\varepsilon < s, t < \varepsilon)$ be a compactly supported two-parameter variation such that $u_{0,0} = u$, and set $V = \frac{\partial}{\partial s} u_{s,t}|_{s=0, t=0}$, $W = \frac{\partial}{\partial t} u_{s,t}|_{s=0, t=0}$.
Then
\[
\frac{\partial^2}{\partial s \partial t} \Phi_F(u_{st})|_{s,t=0} = \int_M F'' \left( \frac{\|u^*h\|^2}{4} \right) \langle \tilde{\nabla} V, \sigma_u \rangle \langle \tilde{\nabla} W, \sigma_u \rangle \, dv_g \\
+ \int_M F'' \left( \frac{\|u^*h\|^2}{4} \right) \sum_{i,j=1}^m h(\tilde{\nabla}_{e_i} V, \tilde{\nabla}_{e_j} W) h(du(e_i), du(e_j)) \, dv_g \\
+ \int_M F'' \left( \frac{\|u^*h\|^2}{4} \right) \sum_{i,j=1}^m h(\tilde{\nabla}_{e_i} V, du(e_j)) h(\tilde{\nabla}_{e_j} W, du(e_j)) \, dv_g \\
+ \int_M F'' \left( \frac{\|u^*h\|^2}{4} \right) h(R^N(V, du(e_j)) W, du(e_j)) h(du(e_i), du(e_j)) \, dv_g,
\]
where \( \langle \cdot, \cdot \rangle \) is the inner product on \( T^*M \otimes u^{-1}TN \) and \( R^N \) is the curvature tensor of \( N \).
We put
\[
I(V, W) = \frac{\partial^2}{\partial s \partial t} \Phi_F(u_{st})|_{s,t=0}.
\]
An \( F \)-stationary map \( u \) is called stable if \( I(V, V) \geq 0 \) for any compactly supported vector field \( V \) along \( u \).

3 \( F \)-stationary maps from compact convex hypersurfaces

In this section, we obtain the following result.

**Theorem 3.1** Let \( M \subset R^{m+1} \) be a compact convex hypersurface. Assume that the principal curvatures \( \lambda_i \) of \( M^m \) satisfy \( 0 < \lambda_1 \leq \cdots \leq \lambda_m \) and \( 3 \lambda_m < \sum_{i=1}^{m-1} \lambda_i \). Then every nonconstant \( F \)-stationary map from \( M \) to any compact Riemannian manifold \( N \) is unstable if there exists a constant \( c_F = \inf \{ c \geq 0 | F(t)/t^c \text{ is nonincreasing} \} \) such that
\[
c_F < \frac{1}{4 \lambda_m^2} \min_{1 \leq i \leq m} \left\{ \lambda_i \left( \sum_{k=1}^m \lambda_k - 2 \lambda_i - 2 \lambda_m \right) \right\},
\]
or when \( F''(t) = F(t) \) (for example, \( F(t) = \exp(t) \))
\[
\|u^*h\|^2 < \frac{1}{4 \lambda_m^2} \min_{1 \leq i \leq m} \left\{ \lambda_i \left( \sum_{k=1}^m \lambda_k - 2 \lambda_i - 2 \lambda_m \right) \right\}.
\]

**Proof** In order to prove the instability of \( u : M^m \rightarrow N \), we need to consider some special variational vector fields along \( u \). To do this, we choose an orthonormal field \( \{ e_i, e_{m+1} \} \), \( i = 1, \ldots, m \), of \( R^{m+1} \) such that \( \{ e_i \} \) are tangent to \( M^m \subset R^{m+1} \), \( e_{m+1} \) is normal to \( M^m \) and
\( \nabla_{e_i} e_j |_p = 0 \). Meanwhile, we take a fixed orthonormal basis \( E_A, A = 1, \ldots, m + 1 \), of \( R^{m+1} \) and set
\[
V_A = \sum_{i=1}^m v_A^i e_i, \quad v_A^i = \langle E_A, e_i \rangle, v_A^{m+1} = \langle E_A, e_{m+1} \rangle,
\]
where \((\cdot, \cdot)\) denotes the canonical Euclidean inner product. Then \(du(V_A) \in \Gamma(\mu^{-1}TN)\) and

\[
\sum_{A} v^i_A v_j^A = \sum_{A} \langle E_{A_i}, e_i \rangle \langle E_{A_j}, e_j \rangle = \delta_{ij},
\]

\[
\nabla_{e_i} V_A = v_{A_i}^{m+1} B_{ij} e_j,
\]

\[
\nabla_{e_i} (\nabla_{e_j} V_A) = -v_{A_i}^{k} B_{ik} B_{ij} e_j + v_{A_i}^{m+1} (\nabla_{e_i} h) e_j,
\]

\[
\tilde{\nabla}_{e_i} (du(\nabla_{e_j} V_A)) = -v_{A_i}^{k} B_{ik} B_{ij} du(e_j) + v_{A_i}^{m+1} B_{ij} \tilde{\nabla}_{e_i} du(e_j),
\]

where \(B_{ij}\) denotes the components of the second fundamental form of \(M^m\) in \(\mathbb{R}^{m+1}\). Suppose that \(u : M^m \to N\) is a nonconstant \(F\)-stationary map. Then the condition \(\tau_{F}(u) = -\delta(F'(\|u^* h\|^2/4) \sigma_u) = 0\) implies that

\[
\sum_{A} \int_{M^m} F\left(\frac{\|u^* h\|^2}{4}\right) \langle \Delta du(V_A), \sigma_u(V_A) \rangle dv_g
\]

\[
= \sum_{A} \int_{M^m} F\left(\frac{\|u^* h\|^2}{4}\right) v_{A_i}^{m+1} B_{ij} \langle \Delta du(e_i), \sigma_u(e_j) \rangle dv_g
\]

\[
= \sum_{i} \int_{M^m} F\left(\frac{\|u^* h\|^2}{4}\right) \langle \Delta du(e_i), \sigma_u(e_i) \rangle dv_g
\]

\[
= \int_{M^m} F\left(\frac{\|u^* h\|^2}{4}\right) \langle \Delta du, \sigma_u \rangle dv_g
\]

\[
= \int_{M^m} \delta du, \delta \left(\frac{\|u^* h\|^2}{4}\right) \sigma_u \right) dv_g = 0.
\]

It follows from the Weitzenböck formula that

\[
- \sum_{h=1}^{m} R^{N}(du(X), du(e_h)) du(e_h) + du(Ric^{M}(X)) = \Delta du(X) + \bar{\nabla}^2 du(X),
\]

where \(X\) is any smooth vector field on \(M^m\). With respect to the variational vector field \(du(V_A)\) along \(u\), it follows from (13) and (14) that

\[
\sum_{A} I(du(V_A), du(V_A))
\]

\[
= \int_{M} F\left(\frac{\|u^* h\|^2}{4}\right) \sum_{A} \langle \tilde{\nabla} du(V_A), \sigma_u \rangle^2 dv_g
\]

\[
+ \int_{M} F\left(\frac{\|u^* h\|^2}{4}\right) \sum_{i,j,A} h(\tilde{\nabla}_{e_i} du(V_A), \tilde{\nabla}_{e_j} du(V_A)) h(du(e_i), du(e_j)) dv_g
\]

\[
+ \int_{M} F\left(\frac{\|u^* h\|^2}{4}\right) \sum_{i,j,A} h(\tilde{\nabla}_{e_i} du(V_A), du(e_j)) h(\tilde{\nabla}_{e_i} du(V_A), du(e_j)) dv_g
\]

\[
+ \int_{M} F\left(\frac{\|u^* h\|^2}{4}\right) \sum_{i,j,A} h(\tilde{\nabla}_{e_i} du(V_A), du(e_j)) h(du(e_i), \tilde{\nabla}_{e_j} du(V_A)) dv_g
\]

\[
= \sum_{i,j,A} h(\tilde{\nabla}_{e_i} du(V_A), du(e_j)) h(du(e_i), \tilde{\nabla}_{e_j} du(V_A)) dv_g
\]
For any fixed point \( P \in M \), choose \( \{e_i\} \) such that \( \nabla e_i|_P = 0 \). We have

\[
\nabla^2 du(V_A) = \nabla_{e_i} \nabla_{e_i} (du(V_A)) - 2 \nabla_{e_i} (du(V_A)) + du(\nabla e_i \nabla e_i V_A)
\]  

(16)

and

\[
\int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_{A,i} h(\nabla_{e_i} du(V_A), \sigma_u(V_A)) \, d\nu_g
\]

\[
= - \int_M \sum_{A,i} h(\nabla_{e_i} du(V_A), \nabla_{e_i} F' \left( \frac{\|u^* h\|^2}{4} \right) \sigma_u(V_A)) \, d\nu_g
\]

\[
= - \int_M \sum_{A,i} h(\nabla_{e_i} du(V_A), F' \left( \frac{\|u^* h\|^2}{4} \right) \nabla_{e_i} \sigma_u(V_A)) \, d\nu_g
\]

\[
= - \int_M \sum_{A,i} h(\nabla_{e_i} du(V_A), F' \left( \frac{\|u^* h\|^2}{4} \right) \sigma_u(V_A)) \, d\nu_g
\]

\[
= - \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_{A,i} h(\nabla_{e_i} du(V_A), du(e_i)) h(du(V_A), du(e_i)) \, d\nu_g
\]

\[
= - \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_{A,i} h(\nabla_{e_i} du(V_A), du(e_i)) h(\nabla_{e_i} du(V_A), du(e_i)) \, d\nu_g
\]

\[
= - \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_{A,i} h(\nabla_{e_i} du(V_A), du(e_i)) h(du(V_A), \nabla_{e_i} du(e_i)) \, d\nu_g.
\]  

(17)

Substituting (16) and (17) into (15), we have

\[
\sum_{A} l(du(V_A), du(V_A))
\]

\[
= \int_M \left\{ F'' \left( \frac{\|u^* h\|^2}{4} \right) \sum_{A} (\nabla du(V_A), \sigma_u)^2 - h(\nabla_{e_i} du(V_A), \nabla_{e_i} F' \left( \frac{\|u^* h\|^2}{4} \right) \sigma_u(V_A)) \right\} \, d\nu_g
\]

\[
+ \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) h(-2 \nabla_{e_i} (du(V_A)))
\]

\[
+ du(\nabla_{e_i} \nabla_{e_i} V_A) - du(\text{Ric}^M(V_A), \sigma_u(V_A)) \, d\nu_g
\]

\[
+ \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_{A,i,j} h(\nabla_{e_i} du(V_A), \nabla_{e_j} du(V_A)) h(du(e_i), du(e_j)) \, d\nu_g.
\]
\[ + \int_M F' \left( \frac{\|u^*h\|^2}{4} \right) \sum_{ij,A} h(\bar{\nabla}_{e_i} du(V_A), du(e_j)) h(du(e_i), \bar{\nabla}_{e_j} du(V_A)) dv_g \]
\[ - \int_M F' \left( \frac{\|u^*h\|^2}{4} \right) \sum_{A,ij} h(\bar{\nabla}_{e_i} du(V_A), \bar{\nabla}_{e_j} du(V_A)) h(du(V_A), du(e_j)) dv_g \]
\[ - \int_M F' \left( \frac{\|u^*h\|^2}{4} \right) \sum_{A,ij} h(\bar{\nabla}_{e_i} du(V_A), du(e_j)) h(du(V_A), \bar{\nabla}_{e_j} du(e_j)) dv_g. \] (18)

In the following, we shall estimate each term in (18). Because trace is independent of the choice of orthonormal basis, we can take pointwisely \{e_i, e_{m+1}\} such that \(B_{ij} = \lambda_i \delta_{ij}\).

A straightforward computation shows
\[
\sum_A h(\bar{\nabla}_{e_i} du(V_A), \bar{\nabla}_{e_i} \left[ F' \left( \frac{\|u^*h\|^2}{4} \right) \right] \sigma_u(V_A))
\]
\[ = \sum_A F'' \left( \frac{\|u^*h\|^2}{4} \right) \bar{\nabla}_{e_i} \left( \frac{\|u^*h\|^2}{4} \right) h(v^{m+1}_A B_{ij} du(e_k) + v^k_A \bar{\nabla}_{e_i} du(e_k), v'_A \sigma_u(\sigma_u))
\]
\[ = F'' \left( \frac{\|u^*h\|^2}{4} \right) \bar{\nabla}_{e_i} \left( \frac{\|u^*h\|^2}{4} \right) \sigma_u(\sigma_u)
\]
\[ = F'' \left( \frac{\|u^*h\|^2}{4} \right) (\bar{\nabla}_{e_i} du, \sigma_u)^2 \] (19)

and
\[
\sum_A F'' \left( \frac{\|u^*h\|^2}{4} \right) \bar{\nabla} du(V_A), \sigma_u)^2
\]
\[ = \sum_A F'' \left( \frac{\|u^*h\|^2}{4} \right) (v^{m+1}_A B_{ij} du(e_k) + v^k_A \bar{\nabla}_{e_i} du(e_k), \sigma_u)^2
\]
\[ = \sum_A F'' \left( \frac{\|u^*h\|^2}{4} \right) \{B_{ijk} h(du(e_k), \sigma_u(\sigma_u)) h(du(e_i), \sigma_u(\sigma_u))
\]
\[ + h(\bar{\nabla}_{e_i} du(e_k), \sigma_u(\sigma_u)) h(\bar{\nabla}_{e_i} du(e_k), \sigma_u(\sigma_u)) \}
\[ = \sum_A F'' \left( \frac{\|u^*h\|^2}{4} \right) \{\lambda_i \lambda_j h(du(e_i), \sigma_u(\sigma_u)) h(du(e_j), \sigma_u(\sigma_u)) + (\bar{\nabla}_{e_i} du, \sigma_u)^2 \}. \] (20)

Then it follows from (19) and (20) that
\[
\int_M \left\{ F'' \left( \frac{\|u^*h\|^2}{4} \right) \sum_A (\bar{\nabla} du(V_A), \sigma_u)^2
\right\} dv_g
\]
\[ - h(\bar{\nabla}_{e_i} du(V_A), \bar{\nabla}_{e_j} \left[ F' \left( \frac{\|u^*h\|^2}{4} \right) \right] \sigma_u(V_A)) \right\} dv_g
\[ = \int_M F'' \left( \frac{\|u^*h\|^2}{4} \right) \lambda_i \lambda_j h(du(e_i), \sigma_u(\sigma_u)) h(du(e_j), \sigma_u(\sigma_u)) dv_g. \] (21)

From the Gauss equation it follows that
\[
\text{Ric}^M(V_A) = \nu_A^i (B_{ijk} - B_{ijk}) e_j. \] (22)
Using (10), (11), (12) and (22), we have

\[
\int_M F' \left( \frac{\|u^* h\|^2}{4} \right) h\left( -2\tilde{\nabla}_e (du_e(V_A)) \right) \\
+ du(\nabla_e \nabla_e V_A) - du(\text{Ric}^m(V_A), \sigma_u(V_A)) \, dv_g
\]

= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \left[ h(2v^1_A B_{ik} B_{ij} du(e_k) - v^m_{A1} \nabla_{e_i} (B_{ij}) du(e_j)) \\
- v^m_{A1} B_{ij} \tilde{\nabla}_{e_i} du(e_j), v^m_{A1} \sigma_u(e_i)) \right] \\
+ h(-v^k_A B_{ik} B_{ij} du(e_k) + v^m_{A1} (\nabla_e B_{ij}) du(e_j), v^m_{A1} \sigma_u(e_i)) \\
+ h(v^k_A B_{ik} B_{ij} du(e_k) - v^m_{A1} B_{ik} B_{ij} du(e_j), v^m_{A1} \sigma_u(e_i)) \right] \, dv_g
\]

= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \left\{ h(2v^1_A B_{ik} B_{ij} du(e_k) - v^m_{A1} B_{ik} B_{ij} du(e_j)) \right\} dv_g
\]

= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_i \left[ \left[ 2\lambda_i - \left( \sum_k \lambda_k \right) \right] \lambda_i h(du(e_i), \sigma_u(e_i)) \right] \, dv_g. \quad (23)

A straightforward computation shows

\[
\int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_{i,j \in A} h(\tilde{\nabla}_{e_i} du(V_A), \tilde{\nabla}_{e_j} du(V_A)) h(du(e_i), du(e_j)) \, dv_g
\]

= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) h(v^m_{A1} B_{ik} du(e_k) + v^k_A \tilde{\nabla}_{e_k} du(e_k), \\
v^m_{A1} B_{ij} du(e_j) + v^k_A \tilde{\nabla}_{e_j} du(e_j)) \, dv_g
\]

= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \left\{ B_{ik} B_{ij} h(du(e_k), du(e_j)) \right\} dv_g
\]

= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \left\{ \lambda_i \lambda_j h(du(e_i), du(e_j)) \right\} dv_g
\]

and

\[
\int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_{i,j \in A} h(\tilde{\nabla}_{e_i} du(V_A), du(e_j)) h(du(e_i), \tilde{\nabla}_{e_j} du(V_A)) \, dv_g
\]

= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \left\{ h(v^m_{A1} B_{ik} du(e_k) + v^k_A \tilde{\nabla}_{e_k} du(e_k), du(e_j)) \right\}
\]

\times h(du(e_i), v^m_{A1} B_{ik} du(e_k) + v^k_A \tilde{\nabla}_{e_k} du(e_k)) \, dv_g
\]

= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \left\{ \lambda_i \lambda_j h(du(e_i), du(e_j)) \right\} dv_g
\]

+ h(\tilde{\nabla}_{e_i} du(e_k), du(e_j)) h(\tilde{\nabla}_{e_k} du(e_j), du(e_j)) \, dv_g. \quad (25)
and

\[
\int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_{A,j} h(\nabla_{e_j} du(V_A), \nabla_{e_j} du(e_j)) \, du(V_A), du(e_j) \, dv_g
\]

\[
= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \left\{ h(v_A^{m+1} B_{ik} du(e_k) + v_A^{k} \nabla_{e_j} du(e_k), \nabla_{e_j} du(e_j)) \right\} \, dv_g
\]

\[
= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) h(\nabla_{e_j} du(e_k), \nabla_{e_j} du(e_j)) \, du(e_k), du(e_j) \, dv_g.
\]

(26)

and

\[
\int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_{A,j} h(\nabla_{e_j} du(V_A), du(e_j)) \, du(V_A), \nabla_{e_j} du(e_j) \, dv_g
\]

\[
= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \left\{ h(v_A^{m+1} B_{ik} du(e_k) + v_A^{k} \nabla_{e_j} du(e_k), du(e_j)) \right\} \, dv_g
\]

\[
= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) h(\nabla_{e_j} du(e_k), du(e_j)) \, du(e_k), \nabla_{e_j} du(e_j) \, dv_g.
\]

(27)

From (18), (21), (23), (24), (25), (26), (27) and \(\nabla_{e_j} du(e_j) = \nabla_{e_j} du(e_j)\), we obtain

\[
\sum_A I(du(V_A), du(V_A))
\]

\[
= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \lambda_i, \lambda_j h(du(e_i), \sigma(u(e_j))) \, dv_g
\]

\[
+ \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_i \left[ 2\lambda_i - \left( \sum_k \lambda_k \right) \right] \lambda_j h(du(e_i), \sigma(u(e_j))) \, dv_g
\]

\[
+ 2 \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \lambda_i, \lambda_j h(du(e_i), du(e_j)) \, dv_g
\]

\[
\leq \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \lambda_i, \lambda_j h(du(e_i), \sigma(u(e_j))) \, dv_g
\]

\[
+ \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_i \left[ 2\lambda_i - \left( \sum_k \lambda_k \right) \right] \lambda_j h(du(e_i), \sigma(u(e_j))) \, dv_g
\]

\[
+ 2 \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \lambda_i, \lambda_j h(du(e_i), u(e_j)) \, dv_g
\]

\[
= \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \lambda_i, \lambda_j h(du(e_i), \sigma(u(e_j))) \, dv_g
\]

\[
+ \int_M F' \left( \frac{\|u^* h\|^2}{4} \right) \sum_i \left[ 2\lambda_i + 2\lambda_m - \left( \sum_k \lambda_k \right) \right] \lambda_j h(du(e_i), \sigma(u(e_j))) \, dv_g.
\]

(28)
If $F''(t) = F'(t)$, then (28) leads to the following inequality:

$$\sum A I(\alpha u, \beta u) \leq \int_M F'(\frac{\|u'h\|^2}{4}) \lambda_m^2 \|u'h\|^2 d\nu_g$$

$$+ \int_M F'(\frac{\|u'h\|^2}{4}) \max_{1 \leq i \leq m} \left\{ 2\lambda_i + 2\lambda_m - \left( \sum_k \lambda_k \right) \lambda_i \right\} \|u'h\|^2 d\nu_g$$

$$= \int_M F'(\frac{\|u'h\|^2}{4}) \|u'h\|^2 \left\{ \lambda_m^2 \|u'h\|^2 \right\}$$

$$+ \max_{1 \leq i \leq m} \left\{ 2\lambda_i + 2\lambda_m - \left( \sum_k \lambda_k \right) \lambda_i \right\} d\nu_g.$$  \hspace{1cm} (29)

If there exists a constant $c_F$ such that $\frac{F'(t)}{t}$ is nonincreasing, it follows that $F''(t) \leq c_F F'(t)$ on $t \in (0, \infty)$, thus (28) implies

$$\sum A I(\alpha u, \beta u) \leq \int_M 4c_F F'(\frac{\|u'h\|^2}{4}) \lambda_m^2 \|u'h\|^2 d\nu_g$$

$$+ \int_M F'(\frac{\|u'h\|^2}{4}) \max_{1 \leq i \leq m} \left\{ 2\lambda_i + 2\lambda_m - \left( \sum_k \lambda_k \right) \lambda_i \right\} \|u'h\|^2 d\nu_g$$

$$= \int_M F'(\frac{\|u'h\|^2}{4}) \|u'h\|^2 \left\{ 4c_F \lambda_m^2 \right\}$$

$$+ \max_{1 \leq i \leq m} \left\{ 2\lambda_i + 2\lambda_m - \left( \sum_k \lambda_k \right) \lambda_i \right\} d\nu_g.$$ \hspace{1cm} (30)

If $u$ is nonconstant and (6) or (7) holds, we have

$$\sum A I(\alpha u, \beta u) < 0$$ \hspace{1cm} (31)

and $u$ is unstable.

\[\square\]

**Corollary 3.2** Let $u : S^m \to N$ be a nonconstant $F$-stationary map and $m > 4$. If $c_F < \frac{m}{4} - 1$ or $\|u'h\|^2 < m - 4$, then $u$ is unstable.

### 4 $F$-stationary maps into compact convex hypersurfaces

In this section, we obtain the following result.

**Theorem 4.1** With the same assumption on $M^m$ as in Theorem 3.1, every nonconstant $F$-stationary map from any compact Riemannian manifold $N$ to $M^m$ is unstable if (6) or (7) holds.

**Proof** In order to prove the instability of $u : N^m \to M^m$, we need to consider some special variational vector fields along $u$. To do this, we choose an orthonormal field $\{\epsilon_{\alpha}, \epsilon_{m+1}\}$,
\[ \alpha = 1, \ldots, m, \] of \( R^{m+1} \) such that \( \{e_{\alpha}\} \) are tangent to \( M^m \subset R^{m+1} \), \( \epsilon_{m+1} \) is normal to \( M^m \), 
\[ M^m \nabla_{e_{\alpha}} \epsilon_{\beta} | \_P = 0 \] and \( B_{\alpha\beta} = \lambda_{\alpha} \delta_{\alpha\beta} \), where \( B_{\alpha\beta} \) denotes the components of the second fundamental form of \( M^m \) in \( R^{m+1} \). Meanwhile, take a fixed orthonormal basis \( E_A, A = 1, \ldots, m + 1, \) of \( R^{m+1} \) and set 

\[ V_A = \sum_{\alpha = 1}^{m} v_A^\alpha e_{\alpha}, \quad V_A^m = \langle E_A, e_{\alpha} \rangle, \quad \langle E_A, \epsilon_{m+1} \rangle, \quad (32) \]

where \( \langle \cdot, \cdot \rangle \) denotes the canonical Euclidean inner product. We shall consider the second variation

\[ \sum_A I(V_A, V_A) = \int_{N} F'' \left( \left( \frac{\|u^*h\|}{4} \right) \langle \nabla V_A, \sigma_u \rangle \langle \nabla V_A, \sigma_u \rangle dV_g \right) \]

\[ + \int_{N} F'' \left( \left( \frac{\|u^*h\|}{4} \right) \sum_{i,j=1}^{m} h(\nabla_{e_i} V_A, \nabla_{e_j} V_A) h(du(e_i), du(e_j)) dV_g \right) \]

\[ + \int_{N} F'' \left( \left( \frac{\|u^*h\|}{4} \right) \sum_{i,j=1}^{m} h(\nabla_{e_i} V_A, du(e_j)) h(\nabla_{e_j} V_A, du(e_j)) dV_g \right) \]

\[ + \int_{N} F'' \left( \left( \frac{\|u^*h\|}{4} \right) \sum_{i} h(\nabla^m_{e_i} (V_A, du(e_i))) V_A, \sigma_u(e_i)) dV_g \right), \quad (33) \]

where \( \{e_1, \ldots, e_m\} \) is the local orthonormal frame of \( N^n \).

Firstly, we compute the first term of (33)

\[ \sum_A \int_{N} F'' \left( \left( \frac{\|u^*h\|}{4} \right) \langle \nabla V_A, \sigma_u \rangle \langle \nabla V_A, \sigma_u \rangle dV_g \right) \]

\[ = \int_{N} F'' \left( \left( \frac{\|u^*h\|}{4} \right) \left[ \sum_{i} h(\nabla_{e_i} V_A, \sigma_u(e_i)) \right]^2 dV_g \right) \]

\[ = \int_{N} F'' \left( \left( \frac{\|u^*h\|}{4} \right) \left[ \sum_{i} h(M^m_{e_i} V_A, \sigma_u(e_i)) \right]^2 dV_g \right) \]

\[ = \int_{N} F'' \left( \left( \frac{\|u^*h\|}{4} \right) \left[ \sum_{i} v_A^{m+1} u_i^\alpha B_{\alpha\beta} h(\epsilon_{\beta}, \sigma_u(e_i)) \right]^2 dV_g \right) \]

\[ = \int_{N} F'' \left( \left( \frac{\|u^*h\|}{4} \right) \left[ \sum_{i} v_A^{m+1} u_i^\alpha \lambda_{\alpha\beta} h(\epsilon_{\beta}, \sigma_u(e_i)) \right]^2 dV_g \right) \]

\[ = \int_{N} F'' \left( \left( \frac{\|u^*h\|}{4} \right) \left[ \sum_{i} v_A^{m+1} u_i^\alpha \lambda_{\alpha\beta} h(\epsilon_{\beta}, \sigma_u(e_i)) \right]^2 dV_g \right), \quad (34) \]
The second term of (33)

\[
\sum_{A} \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) \sum_{i,j=1}^{m} h(\nabla_{i} V_{A}, \nabla_{j} V_{A}) h(du(e_{i}), du(e_{j})) \, dv_{g}
\]

\[
= \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) h^{(M_{A}, \nabla_{u} V_{A}, M_{A}, \nabla_{\epsilon} V_{A})} h(du(e_{i}), du(e_{j})) \, dv_{g}
\]

The third term of (33)

\[
\sum_{A} \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) \sum_{i,j=1}^{m} h(\nabla_{i} V_{A}, du(e_{j})) h(\nabla_{j} V_{A}, du(e_{j})) \, dv_{g}
\]

\[
= \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) \lambda_{a} \lambda_{b} h(u^{a}_{E}, u^{b}_{E}, \epsilon_{E}) h(du(e_{i}), du(e_{j})) \, dv_{g}
\]

The fourth term of (33)

\[
\sum_{A} \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) \sum_{i,j=1}^{m} h(\nabla_{i} V_{A}, du(e_{j})) h(du(e_{i}), \nabla_{j} V_{A}) \, dv_{g}
\]

\[
= \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) \lambda_{a} \lambda_{b} h(u^{a}_{E}, du(e_{i})) h(du(e_{i}), u^{b}_{E}) \, dv_{g}
\]

The fifth term of (33)

\[
\sum_{A} \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) \sum_{i} h(R^{M_{A}}(V_{A}, du(e_{i}))) V_{A}, \sigma_{a}(e_{i})) \, dv_{g}
\]

\[
= \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) \nu^{i}_{A} h(R^{M_{A}}(V_{A}, du(e_{i}))) \epsilon_{i}, \sigma_{a}(e_{i})) \, dv_{g}
\]

\[
= \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) u^{i}_{a} h(R^{M_{A}}(V_{A}, du(e_{i}))) \epsilon_{a}, \epsilon_{E}) \, dv_{g}
\]

\[
= \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) u^{i}_{a} h(R^{M_{A}}(V_{A}, du(e_{i}))) \epsilon_{E}, \epsilon_{E}) \, dv_{g}
\]

\[
= \int_{N} F^{c} \left( \frac{\|u^{*} h\|^{2}}{4} \right) u^{i}_{a} u^{j}_{b} \left[ \lambda_{a} - \left( \sum_{\beta} \lambda_{\beta} \right) \lambda_{a} \right] h(du(e_{i}), du(e_{j})) \, dv_{g}
\]
From (33)-(38), we have

\[
\sum_A I(V_A, V_A) \leq \int \frac{F''(t)}{4} \left( \frac{\|u^*h\|^2}{4} \right) \left( \frac{\|u^*h\|^2}{4} \right) \left( \frac{\|u^*h\|^2}{4} \right) \left( \frac{\|u^*h\|^2}{4} \right) v_g.
\]

If \( F'(t) = F(t) \), then (39) leads to the following inequality:

\[
\sum_A I(V_A, V_A) \leq \int \frac{F''(t)}{4} \left( \frac{\|u^*h\|^2}{4} \right) \left( \frac{\|u^*h\|^2}{4} \right) \left( \frac{\|u^*h\|^2}{4} \right) \left( \frac{\|u^*h\|^2}{4} \right) v_g.
\]

If there exists a constant \( c_F \) such that \( F'(t) / t \) is nonincreasing, it follows that \( F''(t) / t \leq c_F F(t) \) on \( t \in (0, \infty) \), thus (39) implies

\[
\sum_A I(V_A, V_A) \leq \int \frac{F''(t)}{4} \left( \frac{\|u^*h\|^2}{4} \right) \left( \frac{\|u^*h\|^2}{4} \right) \left( \frac{\|u^*h\|^2}{4} \right) \left( \frac{\|u^*h\|^2}{4} \right) v_g.
\]
Now, if $u : N \to M^m$ is a nonconstant $F$-stationary map and (6) or (7) holds, then, from (41) or (40), we know that $\sum A I(V_A, V_A) < 0$ and $u$ is unstable. \hfill $\square$

**Corollary 4.2.** Let $u : N \to S^m$ be a nonconstant $F$-stationary map with $m > 4$, where $N$ is any compact Riemannian manifold. If $c_F < \frac{m}{4} - 1$ or $\|u^*h\|^2 < m - 4$, then $u$ is unstable.

**5 Conclusions**

In this paper, we investigate $F$-stationary maps between the compact convex hypersurface $M^m$ and any compact Riemannian manifold $N$. Assume that the principal curvatures $\lambda_i$ of $M^m$ satisfy $0 < \lambda_1 \leq \cdots \leq \lambda_m$ and $3\lambda_m < \sum_{i=1}^{m-1} \lambda_i$, then every nonconstant $F$-stationary map from $M^m$ to $N$ or from $N$ to $M^m$ is unstable if (6) or (7) holds. We mainly use the second variation formula for $F$-stationary maps (cf. [5]) to get the instability. In particular, we consider $S^m$ as a special case of compact convex hypersurfaces and obtain similar inferences.

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**Competing interests**
The authors declare that they have no competing interests.

**Authors’ contributions**
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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