Exact scaling solutions and fixed points for general scalar field

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Abstract

We show that the most general dark energy model that possesses a scaling solution $\rho_\phi \propto a^n$ is the k-essence model, which includes both of the quintessence and tachyon models. The exact scaling solutions are then derived. The potential that gives the tracking solution in which dark energy exactly tracks the background matter field is the inverse squared potential. The quintessence field with exponential potential can be obtained from the k-essence field with the inverse squared potential. We also find the fixed points and study their main properties, whereby the scalar field dominant fixed point is identified.

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1. Introduction

Recent cosmological observations of large-scale structure, type Ia supernova and the cosmic microwave background anisotropy suggests that the universe is currently experiencing an accelerated expansion [1]. To produce such an acceleration in general relativity, we have to introduce an “unseen” component with a large negative pressure to the matter contents of the universe, which is usually referred to as dark energy. This component accounts for about seventy percents of the total matter in the universe. The simplest candidate of dark energy is a very small cosmological constant. Due to its unknown origin, many other dynamical dark energy models have also been pursued, such as string/M-theory inspired models [2] and brane-world models [3]. For a review of dark energy models, please see [4].

One class of models of particular interest is the dynamical models that have scaling attractor solutions. It is well known that for the scalar field with the canonical kinetic term, exponential potentials have scaling attractor solutions [5,6]. For tachyon field, the potential $V(\phi) = V_0 \phi^{-2}$ gives rise to scaling attractor solutions [7,8]. For a general scalar field that interacts with the matter by a specific form, it was found that the general dark energy model which gives rise to scaling attractor solutions has the Lagrangian density $L(X, \phi) = X g(X e^{\lambda \phi})$ [9], where $X = \dot{\phi}^2 / 2$. After redefining the scalar field, this Lagrangian density is equivalent to the k-essence one $L(X, \phi) = V(\phi) f(X)$ with $V(\phi) = V_0 \phi^{-2}$. This k-essence model gives the scaling attractor solution when there is no interaction between the scalar field and the matter.

It is interesting to find the most general dark energy model that gives the scaling solution $\rho_\phi \propto a^n$ when the specific interaction between the scalar field and the matter is absent. Will the most general Lagrangian density still be $L(X, \phi) = X g(X e^{\lambda \phi})$ or $L(X, \phi) = V(\phi) f(X)$ with $V(\phi) = V_0 \phi^{-2}$? The answer to this question is not clear from the discussions in [9]. The question is ad-
dressed in this Letter. We find that the most general model that gives the scaling solution is the k-essence one. To better understand the scaling solution, it is necessary to study the dynamical properties of the k-essence solutions. In this Letter, we are particularly interested in finding the potential for a general scalar field with constant equation of state \( w_\phi \) and studying the properties of the fixed points of the solutions.

The Letter is organized as follows. In Section 2, we review the derivation of the general dark energy model \( L(X, \phi) = X g(X e^{\lambda \phi}) \) that gives rise to the scaling attractor solution with interaction and prove that the k-essence model is the most general dark energy model that gives rise to the scaling solutions with and without the interaction between the scalar field and the matter. In Section 3, we first review the exact solutions for a canonical scalar field with constant \( w_\phi \), then we derive the exact solutions for the k-essence field with constant \( w_\phi \). In Section 4, we discuss the properties of the fixed points for the potential \( V(\phi) = V_0 \phi^{-2} \). We conclude our paper in Section 5.

2. General dark energy model with tracking solutions

For a general dark energy model with the Lagrangian density \( L(X, \phi) = P(X, \phi) \), we get the energy density \( \rho_\phi \) and the pressure \( p_\phi \) of the scalar field \( \phi \), where \( P_X \equiv \partial P/\partial X \). By assuming a particular form of interaction between the dark energy and the matter, we have the following Friedman equations:

\[
\dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = -Q\rho_m \dot{\phi},
\]

\[
\dot{\rho}_m + 3H(1 + w_m)\rho_m = Q\rho_m \dot{\phi},
\]

\[
\Omega_m + \Omega_\phi = 1,
\]

where \( Q \) is assumed to be a constant, \( \Omega_m = \rho_m/(3m^2_{\text{pl}} H^2) \), \( \Omega_\phi = \rho_\phi/(3m^2_{\text{pl}} H^2) \) and the reduced Planck mass \( m^2_{\text{pl}} = 8\pi G \). We are interested in the tracking solution in which the scalar field tracks the matter field so that \( \Omega_\phi/\Omega_m \) is a constant, therefore we require

\[
\Omega_m = 1 - \Omega_\phi = \text{const}, \quad w_\phi = \text{const}.
\]

In terms of the variable \( N = \ln a \), we get \( \dot{\phi} = H d\phi/dN \). Eqs. (4) and (3) tell us that

\[
H^2 = \frac{\rho_\phi}{3\Omega_\phi m^2_{\text{pl}}} = \frac{\rho_m}{3\Omega_m m^2_{\text{pl}}},
\]

\[
\frac{d\ln \rho_\phi}{dN} = \frac{d\ln \rho_m}{dN} = \frac{d\ln H^2}{dN},
\]

and Eqs. (1) and (2) become

\[
\frac{d\rho_\phi}{dN} + 3(1 + w_\phi)\rho_\phi = -Q\rho_m \frac{d\phi}{dN},
\]

\[
\frac{d\rho_m}{dN} + 3(1 + w_m)\rho_m = Q\rho_m \frac{d\phi}{dN}.
\]

Substituting Eq. (6) into Eqs. (7) and (8), we get

\[
\frac{d\phi}{dN} = \frac{1}{Q} \left[ \frac{d\ln \rho_m}{dN} + 3(1 + w_m) \right] = -\frac{\Omega_\phi}{\Omega_m} \left[ \frac{d\ln \rho_\phi}{dN} + 3(1 + w_\phi) \right] = \frac{3\Omega_\phi}{Q}(w_m - w_\phi),
\]

\[
\frac{d\ln \rho_\phi}{dN} = \frac{d\ln \rho_m}{dN} = \frac{d\ln H^2}{dN} = -3(1 + w_{\text{eff}}),
\]

where

\[
w_{\text{eff}} = \Omega_m w_m + \Omega_\phi w_\phi.
\]

From the definition \( 2X = H^2(d\phi/dN)^2 \), we see that \( X \propto H^2 \). So, we get

\[
\frac{d\ln X}{dN} = \frac{d\ln P}{dN} = \frac{d\ln p_\phi}{dN} = \frac{d\ln \rho_m}{dN} = \frac{d\ln \rho_\phi}{dN} = -3(1 + w_{\text{eff}}).
\]

Since

\[
\frac{d\ln P}{dN} = \frac{\partial \ln P}{\partial \ln X} \frac{d\ln X}{dN} + \frac{\partial \ln P}{\partial \phi} \frac{d\phi}{dN},
\]

\[
\frac{d\ln p_\phi}{dN} = \frac{\partial \ln p_\phi}{\partial \ln X} \frac{d\ln X}{dN} + \frac{\partial \ln p_\phi}{\partial \phi} \frac{d\phi}{dN},
\]

\[
\frac{d\ln \rho_m}{dN} = \frac{\partial \ln \rho_m}{\partial \ln X} \frac{d\ln X}{dN} + \frac{\partial \ln \rho_m}{\partial \phi} \frac{d\phi}{dN},
\]

\[
\frac{d\ln \rho_\phi}{dN} = \frac{\partial \ln \rho_\phi}{\partial \ln X} \frac{d\ln X}{dN} + \frac{\partial \ln \rho_\phi}{\partial \phi} \frac{d\phi}{dN},
\]

\[
\frac{d\ln H^2}{dN} = \frac{\partial \ln H^2}{\partial \ln X} \frac{d\ln X}{dN} + \frac{\partial \ln H^2}{\partial \phi} \frac{d\phi}{dN}.
\]
so,
\[
\frac{\partial \ln P}{\partial \ln X} - \frac{1}{\lambda} \frac{\partial \ln P}{\partial \phi} = 1, \quad \lambda = Q \frac{1 + \Omega_m w_m + \Omega \phi w_\phi}{\Omega \phi (w_m - w_\phi)}.
\] (13)

Therefore, the general form of the Lagrangian is [9]
\[
P(X, \phi) = X g(X e^{\lambda \phi}).
\] (14)

where \(g(Y)\) is an arbitrary function.

From the above derivation, we see that \(d\phi/dN\) is not a constant in general if the interaction is absent, \(Q = 0\). So, when \(Q = 0\), the above derivation is not applicable. However, the dark energy model (14) gives tracking solutions even when the interaction is absent. From Eq. (13), we see that the tracking solutions give \(w_m = w_\phi\) if \(Q = 0\). In fact, if we make field transformation \(\phi = 2 \exp(\lambda \phi/2)/\lambda\), then we get \(X(\phi)g(X(\phi)e^{\lambda \phi}) = V(\phi)F(X(\phi))\) with \(V(\phi) = 4/(\lambda^2 \phi^2)\) and \(F(Y) = Y g(Y)\). So, the general dark energy model with tracking solutions is actually a particular k-essence model.

From the definition, we get \(w_\phi = P/(2XPX - P)\). So, a constant \(w_\phi\) tells us that \(\partial \ln P/\partial \ln X = (1 + w_\phi)/w_\phi\) is a constant.

Therefore, we obtain
\[
\frac{\partial^2 \ln P}{\partial \phi \partial \ln X} = 0.
\] (15)

The general solution to Eq. (15) is \(P(X, \phi) = V(\phi)F(X)\), which is the k-essence Lagrangian density. Therefore, we conclude that the k-essence is the general scalar field that gives rise to the scaling solution \(w_\phi = \text{const.}\)

3. The k-essence field

In this section, we look for the potential of the k-essence field that gives the scaling solutions. If we take \(g(Y) = 1 - c/Y\), then we have \(X g(X e^{\lambda \phi}) = X - c \exp(-\lambda \phi)\). Because the model \(L(X, \phi) = X g(X e^{\lambda \phi})\) is equivalent to the k-essence model with the inverse squared potential, so the quintessence model with the exponential potential is a particular case of the k-essence with inverse squared potential. Therefore, we first review the exact solutions for the quintessence field with constant equation of state.

3.1. Exact scaling solutions for quintessence field

If \(w_\phi\) is a constant, then the solution to Eq. (1) with \(Q = 0\) is
\[
\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) = \rho_{\phi 0} \left(\frac{a_0}{a}\right)^{3(1+w_\phi)},
\] (16)

where the subscript 0 means the current value of the variable. From the definitions of the energy density and pressure, we find
\[
\dot{\phi}^2 = \frac{2(1 + w_\phi)}{1 - w_\phi} V(\phi) = (1 + w_\phi) \rho_\phi.
\] (17)

So, Eq. (3) becomes
\[
H^2 = \frac{1}{3m_{\text{pl}}^2} \left[ \rho_{m 0} \left(\frac{a_0}{a}\right)^{3(1+w_m)} + \rho_{\phi 0} \left(\frac{a_0}{a}\right)^{3(1+w_\phi)} \right].
\] (18)

Take \(dt = a^{3(1+w_\phi)/2} d\tau\), we get
\[
\sqrt{\Omega_{\phi 0} H_0} d\tau = \frac{da}{a \sqrt{1 + \Omega_{m 0}/\Omega_{\phi 0} a^{3(w_\phi - w_m)}}}.
\] (19)

Thus the solutions are [11]
\[
a(\tau) = \begin{cases} \left(\frac{\Omega_{m 0}}{\Omega_{\phi 0}}\right)^{1/3(w_m - w_\phi)} \left[\sinh \left(\frac{3(w_m - w_\phi)}{2} \sqrt{\Omega_{\phi 0} H_0} \tau\right)\right]^{2/3(w_m - w_\phi)} & \text{for } w_m \neq w_\phi, \\
\exp \left[H_0(\tau - \tau_0)\right] & \text{for } w_m = w_\phi.
\end{cases}
\] (20)

Since \(\dot{\phi} = a H \, d\phi/da\), taking \(a_0 = 1\) and combining Eqs. (17) with (18), we get
\[
\left(\frac{d\phi}{da}\right)^2 = \frac{3m_{\text{pl}}^2 (1 + w_\phi) \Omega_{\phi 0}}{\Omega_{m 0} a^{2 + 3(w_\phi - w_m)} + \Omega_{\phi 0} a^2}.
\] (21)
The solution to the above equation is

\[ \phi(a) = \begin{cases} \frac{\pm \sqrt{3m^2_{\text{pl}}(1 + w_{\phi})}}{3(w_m - w_{\phi})} \text{Arsh} \left( \sqrt{\frac{\Omega_{\phi0}}{\Omega_{m0}}} a^{3(w_m - w_{\phi})/2} \right) + \phi_{in}, & w_m \neq w_{\phi} \text{ and } \Omega_{m0} \neq 0, \\ \pm \sqrt{3m^2_{\text{pl}}(1 + w_{\phi})} \Omega_{\phi0} \ln a + \phi_{in}, & w_m = w_{\phi} \text{ or } \Omega_{m0} = 0, \end{cases} \quad (22) \]

where \( \phi_{in} \) is an arbitrary constant determined by the initial condition. One interesting case is when \( \Omega_{\phi0} \ll \Omega_{m0} \) and \( w_m \neq w_{\phi} \). In this case, we have

\[ \phi(a) = \pm \frac{2}{3(w_m - w_{\phi})} \sqrt{\frac{3m^2_{\text{pl}}(1 + w_{\phi})\Omega_{\phi0}}{\Omega_{m0}}} a^{-3(w_m - w_{\phi})/2} + \phi_{in}, \quad \text{if } \Omega_{\phi0} \ll \Omega_{m0}. \quad (23) \]

Then, combining Eqs. (16), (17) and (22), we obtain the potential of the scalar field \[10\]

\[ V(\phi) = \begin{cases} \frac{1 - w_{\phi}}{2} \rho_{\phi0} \left[ \sqrt{\frac{\Omega_{\phi0}}{\Omega_{m0}}} \sinh \left( \frac{3(w_m - w_{\phi})}{2\sqrt{3(1 + w_{\phi})}} a \right) \right]^{-2(1 + w_{\phi})/(w_m - w_{\phi})}, & w_m \neq w_{\phi} \text{ and } \Omega_{m0} \neq 0, \\ \frac{1 - w_{\phi}}{2} \rho_{\phi0} \exp \left( - \sqrt{\frac{3(1 + w_{\phi})}{2\sqrt{3(1 + w_{\phi})}}} a \right), & w_m = w_{\phi} \text{ or } \Omega_{m0} = 0, \\ \frac{1 - w_{\phi}}{2} \rho_{\phi0} \left[ \frac{3(w_m - w_{\phi})}{2} \sqrt{\frac{\Omega_{\phi0}}{\Omega_{m0}}} \sinh \left( \frac{3(w_m - w_{\phi})}{2\sqrt{3(1 + w_{\phi})}} a \right) \right]^{2(1 + w_{\phi})/(w_m - w_{\phi})}, & \Omega_{\phi0} \ll \Omega_{m0} \text{ and } w_m \neq w_{\phi}. \quad (24) \end{cases} \]

3.2. Exact scaling solutions for the k-essence field

The Lagrangian density for the k-essence field is

\[ L = -V(\phi) F(X). \]

From the above Lagrangian density, we find

\[ \rho_\phi = V(\phi) [F - 2X F_X], \quad p_\phi = -V(\phi) F(X), \quad (25) \]

where \( F_X \equiv dF/dX \). Then, from the definition of \( w_\phi \) we find

\[ \frac{d \ln F(X)}{d \ln X} = \frac{X_c F_X(X_c)}{F(X_c)} = \frac{1 + w_\phi}{2w_\phi}. \quad (26) \]

So, \( X_c \) is a constant along the scaling solution and its value is determined by Eq. (26). Using this fact, we obtain

\[ \frac{d \phi}{da} = \pm \frac{\sqrt{2X_c}}{Ha} = \pm \frac{\sqrt{2X_c} a^{3(1 + w_{\phi})/2 - 1}}{H_0 \sqrt{\Omega_{\phi0} + \Omega_{m0}} a^{3(w_m - w_{\phi})}}, \quad (27) \]

\[ V(\phi) = -3m^2_{\text{pl}} \Omega_{\phi0} H_0^2 \frac{w_\phi}{F(X_c)} a^{-3(1 + w_{\phi})}. \quad (28) \]

(1) \( w_m = w_{\phi} \) or \( \Omega_m = 0 \). In this case, we have

\[ a(\phi) = \left[ \frac{3(1 + w_{\phi})}{2\sqrt{2X_c}} H_0 (\phi - \phi_{in}) \right]^{2(1 + w_{\phi})}, \quad (29) \]

\[ V(\phi) = -\frac{8\Omega_{\phi0} w_{\phi} X_c}{3(1 + w_{\phi})^2 F(X_c)} \left( \frac{m_{\text{pl}}}{a} \right)^2. \quad (30) \]

(2) \( w_m \neq w_{\phi} \) and \( \Omega_m \neq 0 \). In this case, letting \( d \phi = a^{3(1 + w_{\phi})/2} d \phi \), we find

\[ a(\phi) = \left( \frac{\Omega_{m0}}{\Omega_{\phi0}} \right)^{1/3(w_m - w_{\phi})} \left[ \sinh \left( \frac{3(w_m - w_{\phi})}{2} \sqrt{\frac{\Omega_{\phi0}}{2X_c}} H_0 (\phi - \phi_{in}) \right) \right]^{2(1 + w_{\phi})/(w_m - w_{\phi})}, \quad (31) \]

and the potential is

\[ V(\phi) = -3m^2_{\text{pl}} H_0^2 \Omega_{\phi0} \frac{w_\phi}{F(X)} \left( \frac{\Omega_{\phi0}}{\Omega_{m0}} \right)^{1(w_m - w_{\phi})/(w_m - w_{\phi})} \left[ \sinh \left( \frac{3(w_m - w_{\phi})}{2} \sqrt{\frac{\Omega_{\phi0}}{2X_c}} H_0 (\phi - \phi_{in}) \right) \right]^{2(1 + w_{\phi})/(w_m - w_{\phi})}. \quad (32) \]

The solution for the scale factor is also given by Eq. (20).
4. Fixed points of tracking solutions

In the previous sections, we show that the k-essence scalar field with the inverse squared potential \( V(\phi) \propto \phi^{-2} \) gives the most general tracking solutions in which \( w_\phi = w_m = \) constant. In this section, we discuss the fixed points for the general tracking solutions. Without loss of generality, we write the k-essence potential as \( V(\phi) = 4/\lambda^2 \phi^2 \), and set

\[
\begin{align*}
    x &= \dot{\phi} = \sqrt{2X}, \\
y &= \frac{V(\phi)}{3m^2_{pl}H^2} = \frac{4}{3\lambda^2 m^2_{pl} H^2 \phi^2}, \\
z &= \frac{\rho_m}{3m^2_{pl}H^2}
\end{align*}
\]

Then, the energy conservation equation for the scalar field \( \dot{\rho}_\phi + 3H(1 + w_\phi)\rho_\phi = 0 \) gives

\[
\frac{dx}{d\ln a} = -\frac{\sqrt{3}\lambda m_{pl}(F - x^2 F_X) y^{1/2} + 3x F_X}{F_X + x^2 F_{XX}},
\]

while the Friedman Eq. (18) yields

\[
\frac{dy}{d\ln a} = -xy\left[\sqrt{3}\lambda m_{pl} y^{1/2} + \frac{3x F_X}{F - x^2 F_X}\right] + 3yz\left[1 + w_m + \frac{x^2 F_X}{F - x^2 F_X}\right],
\]

and the energy conservation equation for the matter field \( \dot{\rho}_m + 3H(1 + w_m)\rho_m = 0 \) together with Eq. (18) give

\[
\frac{dz}{d\ln a} = 3z(z - 1)\left[1 + w_m + \frac{x^2 F_X}{F - x^2 F_X}\right].
\]

Eq. (18) gives the constraint equation

\[
y(F - x^2 F_X) + z = 1.
\]

For the tachyon, \( F(X) = \sqrt{1 - 2X} \), the above equations reduce to Eqs. (11)–(13) in Ref. [8].

The fixed points are those points that satisfy \( dx = dy = dz = 0 \) in Eqs. (33)–(36). So, there are three fixed points:

(i) \( x = y = 0 \) and \( z = 1 \) which gives \( \Omega_\phi = 0 \).
(ii) \( z = 0 \) and \( \sqrt{3}\lambda m_{pl}(F - x^2 F_X) y^{1/2} = -3x F_X \). This is the scalar field dominated case \( \Omega_\phi = 1 \) and \( \lambda^2 = -3(1 + w_\phi) F_X/m^2_{pl} \).
(iii) \( \sqrt{3}\lambda m_{pl}(F - x^2 F_X) y^{1/2} = -3x F_X \) and \( 1 + w_m + x^2 F_X/(F - x^2 F_X) = 0 \). This is the case when the scalar field tracks the background field with \( w_m = w_\phi \) and \( \Omega_\phi = -3(1 + w_\phi) F_X/(m^2_{pl}\lambda^2) \).

Now let us discuss the stability of these critical points. We first consider the property of the fixed point \( x_c = y_c = 0 \). Let \( x = x_c + \delta x \) and \( y = y_c + \delta y \), then Eqs. (33), (34) and (36) give the following equations for small perturbations:

\[
\begin{align*}
    \frac{d\delta x}{d\ln a} &= -3\delta x - \frac{\sqrt{3}}{2} \lambda m_{pl} \frac{F}{F_X} y^{-1/2} \delta y, \\
    \frac{d\delta y}{d\ln a} &= 3(1 + w_m) \delta y.
\end{align*}
\]

So, the eigenvalues are \(-3\) and \(3(1 + w_m)\). When \( w_m > -1 \), the fixed point is an unstable saddle point, and when \( w_m = -1 \) the fixed point is a stable node.

For the fixed points (ii) and (iii), Eqs. (33), (34) and (36) give us the following equations for small perturbations:

\[
\begin{align*}
    \frac{d\delta x}{d\ln a} &= 3w_\phi \delta x + \frac{3}{2} c_s^2 x_c y_c^{-1} \delta y, \\
    \frac{d\delta y}{d\ln a} &= -3(1 + w_\phi) x_c^{-1} y_c \left[\frac{w_m \Omega_\phi}{c_s^2} + 1 - \Omega_\phi\right] \delta x - \left[3(w_m - w_\phi) \Omega_\phi + \frac{3}{2}(1 + w_\phi)\right] \delta y,
\end{align*}
\]

where the sound speed is defined as

\[
c_s^2 = \frac{dp_\phi}{d\rho_\phi} = \frac{F_X(X)}{F_X(X) + x^2_c F_{XX}(X)} \bigg|_{x = x_c^2/2},
\]

\( x_c \) satisfies

\[
w_\phi = \left. \frac{F(X)}{F(X) - x^2_c F_X(X)} \right|_{x = x_c^2/2},
\]
solution with includes a variety of interesting dark energy models, such as quintessence and tachyon. The potential that gives rise to the tracking

This result is applicable to the cases both with and without interactions between the dark energy and the matter. Moreover, it also

with the potentials given by Eqs. (30) and (32) are the most general dark energy models which possess a constant equation of state.

We need to find out the eigenvalues of the matrix

For the fixed point (ii), we have 3\frac{1}{2}\sqrt{\lambda_{m}y_{c}} \pm \frac{3}{2} \frac{w_{\phi}}{2} \leq -\frac{3}{2} \frac{w_{m}}{2} < 0. So, the fixed point is a stable node for \( w_{\phi} \leq w_{m} \) and an unstable saddle for \( w_{\phi} \geq w_{m} \).

For the fixed point (iii), we have 3\frac{1}{2}\sqrt{\lambda_{m}y_{c}} \pm \frac{3}{2} \frac{w_{\phi}}{2} < 0. So, the fixed point is a stable node for \( w_{\phi} < w_{m} \) and an unstable saddle for \( w_{\phi} \geq w_{m} \).

The eigenvalues are

\( \Omega_{\phi} = y_{c}\left[F(X) - x_{c}^{2}F_{X}(X)\right]_{X = \frac{\phi}{\lambda_{m}y_{c}}} \) \hspace{1cm} (43)

We need to find out the eigenvalues of the matrix

\[
M = \left[\begin{array}{c}
3w_{\phi} \\
3w_{\phi} - 3\frac{1}{2}\sqrt{\lambda_{m}y_{c}} x_{c}^{-1} x_{c}^{2} + 1 - \Omega_{\phi} \\
3\frac{1}{2}\sqrt{\lambda_{m}y_{c}} x_{c}^{-1} x_{c}^{2} + 1 - \Omega_{\phi} + (1 + w_{\phi})/2 \end{array}\right] \hspace{1cm} (44)
\]

For the fixed point (ii), we have 3\frac{1}{2}\sqrt{\lambda_{m}y_{c}} \pm \frac{3}{2} \frac{w_{\phi}}{2} \leq -\frac{3}{2} \frac{w_{m}}{2} \leq -\frac{3}{2} \frac{w_{m}}{2} < 0. So, the fixed point is a stable node for \( w_{\phi} < w_{m} \) and an unstable saddle for \( w_{\phi} \geq w_{m} \).

For the fixed point (iii), we have 3\frac{1}{2}\sqrt{\lambda_{m}y_{c}} \pm \frac{3}{2} \frac{w_{\phi}}{2} < 0. So, the fixed point is a stable node for \( w_{\phi} < w_{m} \) and an unstable saddle for \( w_{\phi} \geq w_{m} \).

The eigenvalues are

\[
3\frac{1}{2}\sqrt{\lambda_{m}y_{c}} \pm \frac{3}{2} \frac{w_{\phi}}{2} \leq -\frac{3}{2} \frac{w_{m}}{2} \leq -\frac{3}{2} \frac{w_{m}}{2} < 0. So, the fixed point is a stable node for \( w_{\phi} < w_{m} \) and an unstable saddle for \( w_{\phi} \geq w_{m} \).
\]

5. Conclusion

In [9], it was found that when a specific interaction between the dark energy and the matter is present, the most general dark energy model that gives the tracking solution is the model \( L(X, \phi) = X_{g}(X e^{\lambda X}) \). In this Letter, we prove that the k-essence models with the potentials given by Eqs. (30) and (32) are the most general dark energy models which possess a constant equation of state. This result is applicable to the cases both with and without interactions between the dark energy and the matter. Moreover, it also includes a variety of interesting dark energy models, such as quintessence and tachyon. The potential that gives rise to the tracking solution with \( w_{m} = w_{\phi} \) is the inverse squared potential \( V(\phi) \propto \phi^{-2} \). This potential reduces to the exponential potential of the quintessence if we choose \( F(X) = X - c \). We also show that for a scalar field with constant equation of state, there are two fixed points relevant to dark energy: (a) The scalar field-dominant attractor \( \Omega_{\phi} = 1 \). (b) The tracking attractor \( w_{m} = w_{\phi} \). The properties of the fixed points are summarized in Table 1. For the first fixed point, it is a stable node if \( w_{\phi} < w_{m} \) and an unstable saddle if \( w_{\phi} \geq w_{m} \). For the dark energy model, we require \( w_{\phi} < 0 \) and \( w_{m} \geq 0 \), so it is an attractor. For the second fixed point, it is a stable node if \( c_{2}^{2} > w_{m} \) and an unstable saddle if \( c_{2}^{2} < w_{m} \). Considering the late time behavior of the dark energy, we set \( w_{m} = 0 \). We also require the sound speed to be positive, so again the fixed point is an attractor for relevant dark energy model.

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