Sim-width and induced minors

Dong Yeap Kang *1, O-joung Kwon†2, Torstein J. F. Strømme3, and Jan Arne Telle3

1Department of Mathematical Sciences, KAIST, 291 Daehak-ro Yuseong-gu Daejeon, 305-701 South Korea
2Institute for Computer Science and Control, Hungarian Academy of Sciences, Budapest, Hungary
3Department of Informatics, University of Bergen, Norway

June 28, 2016

Abstract

We introduce a new graph width parameter, called special induced matching width, shortly sim-width, which does not increase when taking induced minors. For a vertex partition \((A, B)\) of a graph \(G\), this parameter is based on the maximum size of an induced matching \(\{a_1b_1, \ldots, a_mb_m\}\) in \(G\) where \(a_1, \ldots, a_m \in A\) and \(b_1, \ldots, b_m \in B\). Classes of graphs of bounded sim-width are much wider than classes of bounded tree-width, rank-width, or mim-width. As examples, we show that chordal graphs and co-comparability graphs have sim-width at most 1, while they have unbounded value for the other three parameters. In this paper, we obtain general algorithmic results on graphs of bounded sim-width by further excluding certain graphs as induced minors.

A t-matching complete graph is a graph that consists of two vertex sets \(\{v_1, \ldots, v_t\}\) and \(\{w_1, \ldots, w_t\}\) such that \(\{v_1, \ldots, v_t\}\) is a clique, and between the two sets, \(\{v_1w_1, \ldots, v_tw_t\}\) is an induced matching. We prove that for positive integers \(w\) and \(t\), a large class of domination and partitioning problems, including the Minimum Dominating Set problem, can be solved in time \(n^{O(w^2)}\) on \(n\)-vertex graphs of sim-width at most \(w\) and having no induced minor isomorphic to a t-matching complete graph, when the decomposition tree is given. To prove it, we show that such graphs have mim-width at most \((4w + 2)t^2\). In this way, we generate infinite non-trivial classes of graphs that are closed under induced minors, but not under minors, and have general algorithmic applications. For chordal graphs and co-comparability graphs, we provide polynomial-time algorithms to obtain decomposition trees certifying that their sim-width are at most 1. Note that Minimum Dominating Set is NP-complete on chordal graphs.

1 Introduction

It is well known that a graph has tree-width at most \(k\) if and only if it is a subgraph of a chordal graph with maximum clique size \(k + 1\). The algorithmic usefulness of tree-width came from the
Figure 1: Inclusion diagram of some well-known graph classes. (I) Classes where clique-width and rank-width are constant. (II) Classes where mim-width is constant. (III) Classes where sim-width is constant. (IV) Classes where it is unknown if sim-width is constant. (V) Classes where sim-width is unbounded.

property of chordal graphs that minimal separators are cliques and form a tree-like structure. Since the tree-width of a graph restricts the number of vertices in each minimal separator of its minimal chordal supergraphs, we can naturally design efficient (FPT) dynamic programming algorithms for many problems on graphs of bounded tree-width. On the other hand, tree-width is unbounded on some very simple classes of graphs, such as complete graphs and complete bipartite graphs. To deal with such classes, several other width parameters based on restricting the number of neighborhood types have been introduced, and they are mostly equivalent to clique-width [18, 15, 5]. Nowadays, most people believe tree-width and clique-width are the best width parameters for FPT algorithms in this jungle of width parameters.

In this paper we focus on XP algorithms and introduce a new graph width parameter using the maximum size of an induced matching between parts of a vertex partition. For a graph $G$ and its vertex subset $A$, we denote by $\text{simval}_G(A)$ the maximum size of an induced matching $\{a_1b_1, a_2b_2, \ldots, a_mb_m\}$ where $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in V(G) \setminus A$. We take a sim-decomposition $(T, L)$ of a graph where $T$ is a subcubic tree and $L$ is a bijection from the vertex set of $G$ to the leaves of $T$. For each edge $e$ in $T$ inducing the vertex partition $(A_e, B_e)$ of $G$, the width of $e$ is defined as $\text{simval}_G(A_e)$. As usual, the width of $(T, L)$ is the maximum width among all edges in $T$, and the sim-width of the graph is defined as the minimum width among all sim-decompositions of it. The linear variant of sim-width will be called linear sim-width. We will define these more carefully in Section 2. An important point is that when we consider the induced matching, we also care about edges in $A_e$ and $B_e$. If one ignores the edges in $A_e$ and $B_e$, then it creates another parameter called mim-width, introduced formally by Vatshelle in [23], but
implicitly used in a paper by Belmonte and Vatshelle [3] to give a common explanation for the existence of efficient XP algorithms for many well-known graph classes of unbounded clique-width and constant mim-width. We show that the modelling power of sim-width is strictly stronger than mim-width. See Figure 1 for an inclusion diagram of some well-known graph classes.

**Theorem 1.1.**

1. Chordal graphs have sim-width at most 1 but unbounded mim-width, and a sim-decomposition of width at most 1 can be found in polynomial time.

2. Co-comparability graphs have linear sim-width at most 1 but unbounded mim-width, and a linear sim-decomposition of width at most 1 can be found in polynomial time.

We conjecture that circle graphs also have constant sim-width. Note that for problems like **Minimum Dominating Set** which are NP-complete on chordal graphs [4], we cannot expect an XP algorithm parameterized by sim-width, i.e. with runtime $|V(G)|^{f(\text{simw}(G))}$, even if we are given a sim-decomposition. In this paper, we nevertheless obtain a general algorithmic result of this type for a large class of such problems, on graphs of bounded sim-width, by further excluding certain graphs as induced minors.

A **t-matching complete graph** is a graph that consists of two vertex sets $\{v_1, \ldots, v_t\}$ and $\{w_1, \ldots, w_t\}$ such that $\{v_1, \ldots, v_t\}$ is a clique, and between the two sets, $\{v_1w_1, \ldots, v_tw_t\}$ is an induced matching. See Figure 2 for an example. The class of **Locally Checkable Vertex Subset and Vertex Partitioning problems**, shortly LC-VSVP problems, is a subclass of MSO$_1$ problems that generalize **Maximum Independent Set**, **Minimum Dominating Set**, and **q-Coloring** problems [22]. We define this class of problems in Section 6. We show the following.

**Theorem 1.2.** Given an $n$-vertex graph having no $t$-matching complete graph as an induced minor and its sim-decomposition of width $w$, every fixed LC-VSVP problem on $G$ can be solved in time $n^{O(w^2)}$. For instance, the **Minimum Dominating Set** and **q-Coloring** problems can be solved in time $O(n^{6(2w+1)t^2+4})$ and $O(qn^{6q(2w+1)t^2+4})$, respectively.

We explain why both conditions in Theorem 1.2 on $t$-matching complete graphs and on sim-width, are necessary. First, the class of graphs having no 5-matching complete graph as an induced minor contains all planar graphs, because if a graph contains a 5-matching complete graph as an induced minor then it contains $K_5$ as a minor and is thus not planar. Moreover, it is known that the **Maximum Independent Set** problem is NP-complete on planar graphs [13]. Second, we show in Theorem 1.1 that chordal graphs admit a sim-decomposition of width at most 1, and moreover the **Minimum Dominating Set** problem is NP-complete on chordal graphs [4]. Therefore, if we remove one of the two conditions, we cannot hope to obtain the general result in Theorem 1.2.
Vatshelle [23], in his Ph.D Thesis, developed a way of obtaining XP-algorithms on graphs of bounded mim-width for LC-VSVP problems. Combining his result and the following relation between mim-width and sim-width, we can obtain Theorem 1.2.

**Proposition 1.3.** Every graph with sim-width \( w \) and no induced minor isomorphic to a \( t \)-matching complete graph has mim-width at most \( (4w + 2)t^2 \).

Fomin, Oum, and Thilikos [11] showed that if \( G \) is \( K_r \)-minor free, then the tree-width of a graph is bounded by \( c \cdot \text{rw}(G) \) where \( c \) is a constant depending on \( r \). Proposition 1.3 can be seen as a similar relation for mim-width and sim-width. We further prove in Section 5 that the class of graphs of sim-width 1 and having no induced minor isomorphic to a 3-matching complete graph has unbounded rank-width, and thus our restricted graph classes do not fall into a class of graphs of bounded tree-width or rank-width.

For chordal and co-comparability graphs, Theorem 1.1 provides a polynomial-time algorithm to compute a sim-decomposition of width at most 1. One might ask if testing whether a given graph has a \( t \)-matching complete graph as an induced minor can be done efficiently, because of two reasons; testing \( H \)-induced minor for a fixed graph \( H \) is NP-complete in general [9], and the number of possible \( t \)-matching complete graphs are exponential in \( t \). It has been recently shown that for fixed graph \( H \), testing \( H \)-induced minor can be solved in polynomial time on chordal graphs [2], and AT-free graphs [14] which contain all co-comparability graphs. We can use those algorithms. Furthermore, in Section 7 we prove that there is a unique \( t \)-matching complete graph that is a chordal graph, and there are only \( t \) possible \( t \)-matching complete graphs that are co-comparability graphs. Therefore, we can efficiently check the conditions of Theorem 1.2 for chordal graphs and co-comparability graphs. However, we do not know whether \( H \)-induced minor testing can be solved in polynomial time on graphs of sim-width at most \( w \) in general.

The paper is organized as follows. Section 2 contains the necessary notions required for our results, including other width parameters tree-width, rank-width, and mim-width. In Section 3 we show that the class of graphs of sim-width at most \( w \) is closed under taking induced minors. In Section 4 we prove that chordal graphs have sim-width at most 1, and co-comparability graphs have linear sim-width at most 1, and provide polynomial-time algorithms to find such decompositions. We also provide lower bounds on mim-width of those classes, which were not previously known. Towards our main algorithmic result, we show in Section 5 that every graph of sim-width at most \( w \) and having no \( t \)-matching complete graph as an induced minor has mim-width at most \( (4w + 2)t^2 \) (Proposition 5.1). Then in Section 6 we introduce LC-VSVP problems and obtain the general algorithmic results (Corollary 6.2). We give the results on \( t \)-matching complete graphs in chordal graphs and co-comparability graphs in Section 7. We list some questions on sim-width in Section 8.

**2 Preliminaries**

We denote the vertex set and edge set of a graph \( G \) by \( V(G) \) and \( E(G) \), respectively. We denote by \( N_G(v) \) the set of neighbors of a vertex \( v \) in \( G \), and let \( N_G[v] := N_G(v) \cup \{v\} \). For two graphs \( G_1 \) and \( G_2 \) on disjoint vertex sets, the union of \( G_1 \) and \( G_2 \) is the graph \( (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)) \). For \( v \in V(G) \) and \( X \subseteq V(G) \), we denote by \( G - v \) the graph obtained from \( G \) by removing \( v \), and denote by \( G - X \) the graph obtained from \( G \) by removing all vertices in \( X \). For \( e \in E(G) \), we denote by \( G - e \) the graph obtained from \( G \) by removing \( e \), and denote by \( G/e \) the graph obtained from \( G \) by contracting \( e \). For a vertex \( v \) of \( G \) with exactly two neighbors \( v_1 \) and \( v_2 \) that are non-adjacent,
the operation of removing \(v\) and adding the edge between its neighbors is called smoothing a vertex \(v\). For \(X \subseteq V(G)\), we denote by \(G[X]\) the subgraph of \(G\) induced on \(X\). A clique is a set of vertices that are pairwise adjacent, and an independent set is a set of vertices that are pairwise non-adjacent. A set of edges \(\{v_1w_1, v_2w_2, \ldots, v_mw_m\}\) of \(G\) is called an induced matching in \(G\) if there are no other edges in \(G[\{v_1, \ldots, v_n, w_1, \ldots, w_m\}]\). For a vertex partition \((A, B)\) of \(G\) and an induced matching \(\{v_1w_1, v_2w_2, \ldots, v_mw_m\}\) in \(G\) where \(v_1, \ldots, v_m \in A\) and \(w_1, \ldots, w_m \in B\), we say that it is an induced matching in \(G\) between \(A\) and \(B\).

For two graphs \(H\) and \(G\), \(H\) is a subgraph of \(G\) if \(H\) can be obtained from \(G\) by removing some vertices and edges, and \(H\) is an induced subgraph of \(G\) if \(H = G[X]\) for some \(X \subseteq V(G)\), and \(H\) is an induced minor of \(G\) if \(H\) can be obtained from \(G\) by a sequence of removing vertices and contracting edges, and \(H\) is a minor of \(G\) if \(H\) can be obtained from \(G\) by a sequence of removing vertices, removing edges, and contracting edges. We note that it is not allowed to remove an edge in the induced minor relation; for instance, the complete graph on \(4\) vertices cannot contain the cycle of length \(4\) as an induced minor.

A pair of vertex subsets \((A, B)\) of a graph \(G\) is called a vertex partition if \(A \cap B = \emptyset\) and \(A \cup B = V(G)\). For a vertex partition \((A, B)\) of a graph \(G\), we denote by \(G[A, B]\) the bipartite graph on the bipartition \((A, B)\) where for \(a \in A, b \in B\), \(a\) and \(b\) are adjacent in \(G[A, B]\) if and only if they are adjacent in \(G\). For a bipartite graph \(G\) with a bipartition \((A, B)\), we say that a matrix \(M\) is a bipartite-adjacency matrix of \(G\), if the rows of \(M\) are indexed by \(A\), the columns of \(M\) are indexed by \(B\), and for \(a \in A, b \in B\), \(M_{a,b} = 1\) if \(a\) is adjacent to \(b\) in \(G\), and \(M_{a,b} = 0\) otherwise.

A tree is called subcubic if every internal node has exactly \(3\) neighbors. A tree \(T\) is called a caterpillar if contains a path \(P\) where for every vertex in \(T\) either it is in \(P\) or has a neighbor on \(P\). A graph is called chordal if it contains no induced subgraph isomorphic to a cycle of length \(4\) or more. For a graph \(G\), an ordering \(v_1, \ldots, v_n\) of the vertex set of \(G\) is called a co-comparability ordering if for every triple \(i, j, k\) with \(i < j < k\), \(v_j\) has a neighbor in each path from \(v_i\) to \(v_k\) avoiding \(v_j\). A graph is called a co-comparability graph if it admits a co-comparability ordering. The complete graph on \(n\) vertices will be denoted by \(K_n\).

### 2.1 Sim-width

For a graph \(G\), let \(\text{simval}_G : 2^{V(G)} \to \mathbb{N}\) be the function such that for \(A \subseteq V(G)\), \(\text{simval}_G(A)\) is the maximum size of an induced matching \(\{a_1b_1, a_2b_2, \ldots, a_mb_m\}\) in \(G\) where \(a_1, \ldots, a_m \in A\) and \(b_1, \ldots, b_m \in V(G) \setminus A\). For a graph \(G\), a pair \((T, L)\) of a subcubic tree \(T\) and a function \(L\) from \(V(G)\) to the set of leaves of \(T\) is called a branch-decomposition. For each edge \(e\) of \(T\), let \((A^e_1, A^e_2)\) be the vertex partition of \(G\) where \(T^e_1, T^e_2\) are the two connected components of \(T - e\), and for each \(i \in \{1, 2\}\), \(A^e_i\) is the set of all vertices in \(G\) mapped to leaves contained in \(T^e_i\). We call it the vertex partition of \(G\) associated with \(e\). For a branch-decomposition \((T, L)\) of a graph \(G\) and an edge \(e\) in \(T\), the width of \(e\) with respect to the \(\text{simval}_G\) function, denote by \(\text{simval}_{(T, L)}(e)\), is defined as \(\text{simval}_G(A^e_1)\) where \((A^e_1, A^e_2)\) is the vertex partition associated with \(e\). The width of \((T, L)\) with respect to the \(\text{simval}_G\) function is the maximum width over all edges in \(T\). The sim-width of a graph \(G\) is the minimum width over all its branch-decompositions, and we denote it by \(\text{simw}(G)\). If \(T\) is a subcubic caterpillar tree, then \((T, L)\) is called a linear branch-decomposition. The linear sim-width of a graph \(G\) is the minimum width over all its linear branch-decompositions, and we denote it by \(\text{lsimw}(G)\). When we consider sim-width, branch-decompositions are also called as sim-decompositions.
2.2 Other width parameters

A tree-decomposition of a graph \( G \) is a pair \((T, \mathcal{B} = \{B_t\}_{t \in V(T)})\) such that (1) \( \bigcup_{t \in V(T)} B_t = V(G) \), (2) for every edge in \( G \), there exists \( B_t \) containing both end vertices, and (3) for \( t_1, t_2, t_3 \in V(T) \), \( B_{t_1} \cap B_{t_3} \subseteq B_{t_2} \) whenever \( t_2 \) is on the path from \( t_1 \) to \( t_3 \). Each vertex subset \( B_t \) is called a bag of the tree-decomposition. The width of a tree-decomposition is \( w - 1 \) where \( w \) is the maximum size of bags in the decomposition, and the tree-width of a graph is the minimum width over all tree-decompositions of the graph.

For a graph \( G \), we define two functions \( \text{mimval}_G : 2^{V(G)} \to \mathbb{N} \) and \( \text{cutrk}_G : 2^{V(G)} \to \mathbb{N} \) such that \( \text{mimval}_G(A) \) is the maximum size of an induced matching of \( G[A, V(G) \setminus A] \), and \( \text{cutrk}_G(A) \) is the rank of the bipartite-adjacency matrix of \( G[A, V(G) \setminus A] \) where the rank is computed over the binary field. For a branch-decomposition \((T, L)\) of a graph \( G \) and \( e \in E(T) \) and the vertex partition \((A, B)\) of \( G \) associated with \( e \), we define \( \text{cutrk}_{(T, L)}(e) := \text{cutrk}_G(A) \), and \( \text{mimval}_{(T, L)}(e) := \text{mimval}_G(A) \). The rank-width, and mim-width of a graph are defined in the same way as sim-width, with \( \text{cutrk}_G \) and \( \text{mimval}_G \) functions, respectively. The tree-width, rank-width, mim-width of a graph \( G \) are denoted by \( \text{tw}(G) \), \( \text{rw}(G) \), \( \text{mimw}(G) \), respectively.

**Lemma 2.1.** For a graph \( G \), we have \( \text{simw}(G) \leq \text{mimw}(G) \leq \text{rw}(G) \leq \text{tw}(G) + 1 \).

**Proof.** Oum [19] proved that \( \text{rw}(G) \leq \text{tw}(G) + 1 \). To show \( \text{mimw}(G) \leq \text{rw}(G) \), it is enough to show that for every branch-decomposition \((T, L)\) of \( G \) and \( e \in E(T) \), \( \text{mimval}_{(T, L)}(e) \leq \text{cutrk}_{(T, L)}(e) \). This holds since the size of an induced matching in \( G[A_1, A_2] \) gives a lower bound on the rank of the bipartite-adjacency matrix of \( G[A_1, A_2] \) where \((A_1, A_2)\) is the vertex partition associated with \( e \). By definition \( \text{simw}(G) \leq \text{mimw}(G) \) is clear. \( \square \)

2.3 \text{t-matching complete graphs}

A \text{t-matching complete graph} is a graph that consists of two vertex sets \( \{v_1, \ldots, v_t\} \) and \( \{w_1, \ldots, w_t\} \) such that

1. \( \{v_1, \ldots, v_t\} \) is a clique,
2. \( \{v_1 w_1, \ldots, v_t w_t\} \) is an induced matching in \( G[\{v_1, \ldots, v_t\}, \{w_1, \ldots, w_t\}] \).

We say that the ordered pair \((\{v_1, \ldots, v_t\}, \{w_1, \ldots, w_t\})\) is the canonical bipartition of the \text{t-matching complete graph}. Notice that every \text{t-matching complete graph} contains \( K_t \) as a minor. We will use this graph to arrive at an algorithmically useful subclass of graphs of bounded sim-width, still of unbounded rank-width and still closed under induced minors.

3 Sim-width and contraction

Let us start by showing that the sim-width of a graph does not increase when taking an induced minor. This is one of the main motivations to consider this parameter.

**Lemma 3.1.** The sim-width of a graph does not increase when taking an induced minor.

**Proof.** Clearly, the sim-width of a graph does not increase when removing a vertex. We prove for contractions.
Let $G$ be a graph, $v_1v_2 \in E(G)$, and let $(T, L)$ be a sim-decomposition of $G$ of width $w$. For convenience, let the contracted vertex in $G/v_1v_2$ be called $v_1$. We claim that $G/v_1v_2$ admits a sim-decomposition of $G$ of width at most $w$. We may assume that $G$ has at least 3 vertices. For $G/v_1v_2$, we obtain a sim-decomposition $(T', L')$ as follows:

- Let $T'$ be the tree obtained from $T$ by removing $L(v_2)$, and smoothing its neighbor. This neighbor of $L(v_2)$ has degree $3$ in $T$ because $T$ is a subcubic tree and $G$ has at least 3 vertices.

- Let $L'$ be the function from $V(G/v_1v_2)$ to the set of leaves of $T'$ such that $L'(w) = L(w)$ for $w \in V(G/v_1v_2) \setminus \{v_1\}$ and $L'(v_1) = L(v_1)$.

Let $e_1$ and $e_2$ be the two edges of $T$ incident with the neighbor of $L(v_2)$, but not incident with $L(v_2)$. Let $e_{cont}$ be the edge of $T'$ obtained by smoothing.

**Claim 1.** For each $e \in E(T')$, $\text{simval}_{(T',L')}(e) \leq \text{simval}_{(T,L)}(e)$ if $e \in E(T) \setminus \{e_1, e_2\}$, and $\text{simval}_{(T',L')}(e_{cont}) \leq \min(\text{simval}_{(T,L)}(e_1), \text{simval}_{(T,L)}(e_2))$.

**Proof.** Let $e \in E(T')$, and first assume that $e \in E(T) \setminus \{e_1, e_2\}$. Let $(A, B)$ be the vertex partition of $G/v_1v_2$ associated with $e$. Without loss of generality, we may assume that $v_1 \in A$. Suppose there exists an induced matching $\{a_1b_1, \ldots, a_mb_m\}$ in $G/v_1v_2$ with $a_1, \ldots, a_m \in A$ and $b_1, \ldots, b_m \in B$. Let $(A', B')$ be the vertex partition of $G$ associated with $e$. We will show that there is also an induced matching in $G$ of same size between $A'$ and $B'$.

We have either $A \cup \{v_2\} = A'$ and $B = B'$, or $A = A'$ and $B \cup \{v_2\} = B'$. If $v_1 \notin \{a_1, \ldots, a_m\}$, then $\{a_1b_1, \ldots, a_mb_m\}$ is also an induced matching between $A'$ and $B'$ in $G$. Without loss of generality, we may assume that $v_1 = a_1$.

**Case 1.** $A \cup \{v_2\} = A'$ and $B = B'$.

**Proof.** Note that in $G$, one of $v_1$ and $v_2$, say $v'$, is adjacent to $b_2$. And also, $v_1$ and $v_2$ are not adjacent to any of $\{a_2, \ldots, a_m, b_1, \ldots, b_m\}$. Therefore, $\{v'b_1, a_2b_2, \ldots, a_mb_m\}$ is an induced matching in $G$ between $A'$ and $B'$, as required. □

**Case 2.** $A = A'$ and $B \cup \{v_2\} = B'$.

**Proof.** If $v_1$ is adjacent to $b_1$ in $G$, then we have the same induced matching in $G$ between $A'$ and $B'$. However, $v_1$ is not necessary adjacent to $b_1$ in $G$. In this case, $v_2$ should be adjacent to $b_1$ in $G$. We now assume that $v_1$ is not adjacent to $b_1$ in $G$. In this case, $\{v_1v_2, a_2b_2, \ldots, a_mb_m\}$ is an induced matching between $B_1$ and $B_2$, because $v_1$ and $v_2$ are not adjacent to any of $\{a_2, \ldots, a_m, b_2, \ldots, b_m\}$.

It shows that $\text{simval}_{(T',L')}(e) \leq \text{simval}_{(T,L)}(e)$ if $e \in E(T) \setminus \{e_1, e_2\}$. We can follow the same procedure to show that the same holds for $e_{cont}$ as well. □

Claim [1] implies that the width of $(T', L')$ is at most the width of $(T, L)$. We conclude that $\text{simw}(G/v_1v_2) \leq \text{simw}(G)$. □
Figure 3: Constructing a sim-decomposition \((T, L)\) of a chordal graph \(G\) of width at most 1 from its tree-decomposition. We denote the path \(P\) where the order of the assigned vertices in this part can be freely changed.

4 Sim-width of chordal graphs and co-comparability graphs

In this section, we show that chordal graphs and co-comparability graphs have sim-width at most 1, but have unbounded mim-width. Belmonte and Vatshelle [3] showed that chordal graphs either do not have constant mim-width or it is NP-complete to find such a decomposition. We strengthen their result.

4.1 Chordal graphs

For chordal graphs, we recursively construct a sim-decomposition of width at most 1. We use the fact that chordal graphs admit a tree-decomposition whose bags are maximal cliques.

Proposition 4.1. Chordal graphs have sim-width at most 1. Given a chordal graph, one can output a sim-decomposition of width at most 1 in polynomial time.

Proof. Let \(G\) be a chordal graph. We may assume that \(G\) is connected. We compute a tree-decomposition \((F, \mathcal{B} = \{B_t\}_{t \in V(F)})\) of \(G\) where every bag induces a maximal clique of \(G\). It is known that such a decomposition can be computed in polynomial time; for instance, see [17]. Let us choose a root node \(r\) of \(F\), and for each node \(t\), let \(F_t\) be the subtree of \(F\) induced on the union of all descendant nodes of \(t\), and let \(G_t\) be the subgraph of \(G\) induced by the union of bags \(B_{t'}\) where \(t' \in V(F_t)\). We remark that for each node \(t\), \((F_t, \{B_x\}_{x \in V(F_t)})\) can be regarded as a rooted tree-decomposition of \(G_t\) with root node \(t\).

We recursively compute a sim-decomposition \((T_t, L_t)\) of \(G_t\) of width at most 1 satisfying the following property:

- there are two internal nodes \(p\) and \(q\) of \(T_t\) where every node on the path \(P\) from \(p\) to \(q\) in \(T_t\) is adjacent to at least one leaf, and \(q\) is a node incident with two leaves,
- all vertices in \(B_t\) are assigned to leaves that are attached in a linear way on \(P\), and
- any reordering of vertices of \(B_t\) also gives a sim-decomposition of \(G_t\) of width at most 1.

We describe such a sim-decomposition in Figure 3. If \(t\) is a leaf node, then \(G_t\) is a complete graph, and thus, we can take any linear sim-decomposition of \(G_t\). We may assume that \(t\) is not a leaf node.
Let \( t_1, \ldots, t_m \) be the set of children of \( t \) in \( T \). Note that for each \( i \in \{1, \ldots, m\} \), \( V(B_t) \cap V(B_i) \neq \emptyset \) as \( G \) is connected.

By induction hypothesis, for each \( i \in \{1, \ldots, m\} \), there exists a sim-decomposition \((T_i, L_i)\) of \( G_{t_i} \), of width at most 1 satisfying the required property. We reorder the vertices of \( B_t \) so that the vertices in \( B_t \cap B_i \) appear at the last part of the ordering of \( B_t \). We obtain a new tree \( T' \) by contracting the minimal subtree connecting vertices of \( B_t \cap B_i \) into one vertex \( x_i \), and call it a modified tree.

Now, we obtain a sim-decomposition \((T, L)\) as follows.

1. Let \( Q := q_1q_2 \cdots q_{m+|V(B_t)|} \) be a path. Let \( T \) be the tree obtained from \( Q \) by adding a leaf \( r_i \) to \( q_i \) for each \( i \in \{1, \ldots, |V(B_i)|\} \), and adding a modified tree \( T'_i \) and identifying \( x_i \) with \( q_{|B_i|+i} \) for each \( i \in \{1, \ldots, m\} \), and smoothing degree 2 nodes \( q_1 \) and \( q_{m+|V(B_t)|} \).

2. Let \( L \) be the function from the set of leaves of \( T \) to \( V(G_t) \) such that \( L|_{B_t} \) is a bijective function from \( \{r_1, \ldots, r_{|B_t|}\} \) to \( B_t \), and \( L(v) = L_i(v) \) for all vertices \( v \) in \( T'_i \). We can easily check that \((T, L)\) has width 1 as the root bag induces a clique, and thus it forbid having two induced matchings. The second statement also holds as we take any ordering when constructing the decomposition. Note that we can update the sim-decomposition \((T, L)\) of \( G_t \) in linear time. Therefore, we can construct a sim-decomposition of width at most 1 for a chordal graph in polynomial time.

We now prove the lower bound on the mim-width of chordal graphs. We in fact show this for the class of split graphs that is a subclass of chordal graphs. A split graph is a graph that can be partitioned into two vertex sets \( C \) and \( I \) where \( C \) is a clique and \( I \) is an independent set. The Sauer-Shelah lemma \([20, 21]\) is essential in the proof.

**Theorem 4.2** (Sauer-Shelah lemma \([20, 21]\)). Let \( t \) be a positive integer and let \( M \) be an \( X \times Y \) \((0,1)\)-matrix such that any two row vectors of \( M \) are distinct. If \( |X| \geq |Y| \), then there are \( X' \subset X \), \( Y' \subset Y \) such that \( |X'| = 2^t \), \( |Y'| = t \), and all possible row vectors of length \( t \) appear in \( M[X', Y'] \).

**Proposition 4.3.** For every large enough \( n \), there is a split graph on \( n \) vertices having mim-width at least \( \sqrt{\log_2 \frac{n}{2}} \).

**Proof.** Let \( m \geq 10000 \) be an integer and let \( n := m + (2^m - 1) \). Let \( G \) be a split graph on the vertex partition \((C, I)\) where \( C \) is a clique of size \( m \), \( I \) is an independent set of size \( 2^m - 1 \), and all vertices in \( I \) have pairwise distinct non-empty neighborhoods on \( C \). We claim that every branch-decomposition of \( G \) has width at least \( \sqrt{\log_2 \frac{n}{2}} \) with respect to the mimvalc function.

Let \((T, L)\) be a branch-decomposition of \( G \). It is well known that there is an edge of \( T \) inducing a balanced vertex partition, but we add a short proof for it. We subdivide an edge of \( T \), and regard the new vertex as a root node. For each node \( t \in V(T) \), let \( \mu(t) \) be the number of leaves of \( T \) that are descendants of \( t \). Now, we choose a node \( t \) that is farthest from the root node such that \( \mu(t) > \frac{n}{3} \). By the choice of \( t \), for each child \( t' \) of \( t \), \( \mu(t') \leq \frac{n}{3} \). Therefore, \( \frac{n}{3} < \mu(t) \leq \frac{2n}{3} \). Let \( e \) be the edge connecting the node \( t \) and its parent. Then clearly, the vertex partition \((A_1, A_2)\) of \( G \) induced by the edge \( e \) satisfies that for each \( i \in \{1, 2\} \), \( \frac{n}{3} < |A_i| \leq \frac{2n}{3} \). Without loss of generality, we may assume that \( |A_1 \cap C| \geq |A_2 \cap C| \), and thus we have \( \frac{m}{2} \leq |A_1 \cap C| \leq m \).

Note that \( |A_2 \cap I| > \frac{n}{3} - m \geq \frac{2^m - 2m - 1}{3} \geq 2^{m-3} \). Since \( |A_2 \cap C| < \frac{m}{2} \) and \( m \geq 8 \), there are at least
\[
\frac{2^{m-3}}{2^\frac{m}{2}} \geq \frac{2^{m-3}}{2^8} \]
vertices in $A_2 \cap I$ that have pairwise distinct neighbors on $A_1 \cap C$. Let $I' \subseteq A_2 \cap I$ be the set of such vertices.

Now, by the Sauer-Shelah lemma, if $|I'| \geq |A_1 \cap C|^k$ for some positive integer $k$, then there will be an induced matching of size $k$ between $A_1 \cap C$ and $I'$ in $G[A_1, A_2]$. We choose $k := \sqrt{m}$. As $m \geq 10000$, we can deduce that $\frac{m}{2} - 3 \geq \sqrt{m} \log_2 m$. Therefore, we have

$$|I'| \geq \frac{m^{2/3}}{2} \geq m^{\sqrt{m}} \geq |A_1 \cap C|^{\sqrt{m}} ,$$

and there is an induced matching of size $\sqrt{m}$ between $A_1 \cap C$ and $I'$ in $G[A_1, A_2]$. It implies that $\text{minval}_{(T,L)(e)} \geq \sqrt{m}$. As $(T, L)$ was chosen arbitrary, the mim-width of $G$ is at least $\sqrt{\log_2 \frac{m}{2}}$.

\section{4.2 Co-comparability graphs}

We observe the same properties for co-comparability graphs. We recall that co-comparability graphs are exactly graphs that admit a co-comparability ordering.

\textbf{Theorem 4.4} (McConnell and Spinrad [16]). \textit{Given a co-comparability graph $G$, one can output a co-comparability ordering in polynomial time.}

\textbf{Proposition 4.5.} Co-comparability graphs have linear sim-width at most 1. Given a co-comparability graph, one can output a linear sim-decomposition of width at most 1 in polynomial time.

\textbf{Proof.} Let $G$ be a co-comparability graph. Using Theorem 4.4 we can obtain its co-comparability ordering $v_1, \ldots, v_n$. From this, we take a linear branch-decomposition $(T, L)$ following the sequence. We claim that for each $i \in \{2, \ldots, n-1\}$, there is no induced matching of size 2 between $\{v_1, \ldots, v_i\}$ and $\{v_{i+1}, \ldots, v_n\}$. Suppose there are $i_1, i_2 \in \{1, \ldots, i\}$ and $j_1, j_2 \in \{i+1, \ldots, n\}$ such that $\{v_{i_1}v_{j_1}, v_{i_2}v_{j_2}\}$ is an induced matching. Without loss of generality we may assume that $i_1 < i_2$. Then we have $i_1 < i_2 < j_1$, and thus by the definition of the co-comparability ordering, $v_{i_2}$ should be adjacent to one of $v_{i_1}$ and $v_{j_1}$, which contradicts to our assumption. Therefore, there is no induced matching of size 2. It implies that $(T, L)$ has width at most 1.

To show that co-comparability graphs have unbounded mim-width, we provide a grid-like structure. For positive integers $p, q$, the $(p \times q)$ \textit{column-clique grid} is the graph on the vertex set $\{v_{i,j} : 1 \leq i \leq p, 1 \leq j \leq q\}$ where

- for every $i \in \{1, \ldots, q\}$, $\{v_{1,j}, \ldots, v_{p,j}\}$ is a clique,
- for every $i \in \{1, \ldots, p\}$ and $j_1, j_2 \in \{1, \ldots, q\}$, $v_{i,j_1}$ is adjacent to $v_{i,j_2}$ if and only if $|j_2 - j_1| = 1$,
• for \(i_1, i_2 \in \{1, \ldots, p\}\) and \(j_1, j_2 \in \{1, \ldots, q\}\), \(v_{i_1, j_1}\) is not adjacent to \(v_{i_2, j_2}\) if neither \(i_1 \neq i_2\) nor \(j_1 \neq j_2\).

We depict an example in Figure 4. For each \(1 \leq i \leq p\), we call \(\{v_{i,1}, \ldots, v_{i,q}\}\) the \(i\)-th row of \(G\), and define its columns similarly.

**Lemma 4.6.** For integers \(p, q \geq 12\), the \((p \times q)\) column-clique grid has mim-width at least \(\min(\frac{p}{4}, \frac{q}{4})\).

**Proof.** Let \(G\) be the \((p \times q)\) column-clique grid. Suppose that \(G\) has a branch-decomposition of width at most \(d\) with respect to the mimval\(_G\) function, for some positive integer \(d\). It is enough to show that \(d \geq \min(\frac{p}{4}, \frac{q}{4})\).

Firstly, assume that for each row \(R\) of \(G\), \(R \cap A \neq \emptyset\) and \(R \cap B \neq \emptyset\). Then there is an edge between \(R \cap A\) and \(R \cap B\), as \(G[R]\) is connected. For each \(i\)-th row \(R_i\), we choose a pair of vertices \(v_{i_1, a_i} \in R \cap A\) and \(v_{i_2, b_i} \in R \cap B\) that are adjacent. We know that there is an index subset \(X \subseteq \{1, \ldots, p\}\) such that \(|X| \geq \frac{p}{4}\) and every pair \((v_{i_1, a_i}, v_{i_2, b_i}): i \in X\) satisfies that \(a_i + 1 = b_i\). By taking the same parity of \(a_i\)'s, we know there is an index subset \(Y \subseteq \{1, \ldots, p\}\) such that \(|Y| \geq \frac{q}{4}\), all integers in \(\{a_i: i \in Y\}\) have the same parity, and every pair \((v_{i_1, a_i}, v_{i_2, b_i}): i \in Y\) satisfies that \(a_i + 1 = b_i\), we observe that \(\{v_{i_1, a_i}, v_{i_2, b_i}: i \in Y\}\) is an induced matching in \(G[A, B]\). If it is not, then there are distinct integers \(y, z \in Y\) such that either \(v_{y,a_y}\) is adjacent to \(v_{z,b_z}\), or \(v_{y,b_y}\) is adjacent to \(v_{z,a_z}\). But this is not possible; for instance, if \(v_{y,a_y}\) is adjacent to \(v_{z,b_z}\), then \(a_y = b_z\), and we have \(a_y = a_z + 1\) as \(z \in Y\). However, it contradicts to our assumption that all integers in \(\{a_i: i \in Y\}\) have the same parity. Therefore, we conclude that \(G[A, B]\) contains an induced matching of size at least \(\frac{p}{4}\).

Now, we assume that there exists a row \(R\) such that \(R\) is fully contained in one of \(A\) and \(B\). Without loss of generality, we may assume that \(R\) is contained in \(A\). Since \(|B| > \frac{|V(G)|}{3}\), we can choose an index set \(X \subseteq \{1, \ldots, q\}\) such that \(|X| > \frac{q}{3}\) and for each \(i \in X\), the \(i\)-th column contains a vertex of \(B\). For each \(i\)-th column where \(i \in X\), we choose a vertex \(v_{a,i}\) in \(B\). It is not hard to verify that the edges between \(\{v_{a,i}: i \in X\}\) and the rows in \(R\) form an induced matching of size \(\frac{p}{4}\) in \(G[A, B]\).

Therefore, we have \(d \geq \min(\frac{p}{4}, \frac{q}{4})\). \(\square\

**Corollary 4.7.** For every large enough \(n\), there is a co-comparability graph on \(n\) vertices having mim-width at least \(\sqrt{\frac{n}{12}}\).

**Proof.** Let \(p \geq 4\) be an integer, and let \(n := 12p^2\). Let \(G\) be the \((4p \times 3p)\) clique-grid graph. It is not hard to see that

\[v_{1,1}, v_{2,1}, \ldots, v_{4p,1}, v_{1,2}, v_{2,2}, \ldots, v_{4p-1,3p}, v_{4p,3p}\]

is a co-comparability ordering. Thus, \(G\) is a co-comparability graph. By Lemma 4.6, \(\text{mimw}(G) \geq p = \sqrt{\frac{n}{12}}\). \(\square\

### 5 Excluding \(t\)-matching complete graphs

In the previous section, we proved that graphs of sim-width at most 1 contain all chordal graphs. A classical result on chordal graphs is that the problem of finding a minimum dominating set in a chordal graph is NP-complete \([4]\). So, even for this kind of locally-checkable problem, we cannot expect efficient algorithms on graphs of sim-width at most \(w\). Therefore, to obtain a meta-algorithm for graphs of bounded sim-width encompassing many locally-checkable problems, we must impose
some restrictions. We approach this problem in a way analogous to what has previously been done in the realm of rank-width [11].

It is well known that complete graphs have rank-width at most 1, but they have unbounded tree-width. Fomin, Oum, and Thilikos [11] showed that if $G$ is $K_r$-minor free, then the tree-width of a graph is bounded by $c \cdot rw(G)$ where $c$ is a constant depending on $r$. This can be utilized algorithmically, to get a result for graphs of bounded rank-width when excluding a fixed minor, as the class of problems solvable in FPT time is strictly larger when parameterized by tree-width than rank-width [15].

We will do something similar by focusing on the distinction between mim-width and sim-width. However, $K_r$-minor free graphs are too strong, as one can show that on $K_r$-minor free graphs, the mim-width of a graph is also bounded by some constant factor of its sim-width. To see this, one can use Lemma 3.1 and the result on contraction obstructions for graphs of bounded tree-width [10].

Vatshelle [23], in his Ph.D Thesis, developed a way of obtaining XP-algorithms on graphs of bounded mim-width for locally checkable problems such as the Minimum Dominating Set problem. We will recall such problems formally. In the same spirit of work as Fomin, Oum, and Thilikos [11], we try to bound the mim-width of a graph in terms of a function of sim-width by excluding a certain configuration. On the other hand, we do not want to destroy, by excluding this certain configuration, the property that the class is closed under taking induced minors. Based on this idea, $t$-matching complete graphs naturally came up.

**Proposition 5.1.** Every graph with sim-width $w$ and no induced minor isomorphic to a $t$-matching complete graph has mim-width at most $(4w + 2)t^2$.

We use the following result. Notice that the optimal bound of Theorem 5.2 has been slightly improved by Fox [12], and then by Balogh and Kostochka [1].

**Theorem 5.2** (Duchet and Meyniel [8]). For positive integers $k$ and $n$, every $n$-vertex graph contains either an independent set of size $k$ or a $K_1$-minor where $t \geq \frac{n}{2k-1}$.

**Proof of Proposition 5.1.** Let $G$ be a graph with sim-width $w$ and no induced minor isomorphic to a $t$-matching complete graph. Let $(T, L)$ be a branch-decomposition of $G$ of width $w$ with respect to the simval$_G$ function. We claim that for each edge $e$ of $T$, simval$_{(T, L)}(e) \leq 2(2w + 1)t^2 - t - 1$. It implies that $G$ has mim-width at most $2(2w + 1)t^2 - t - 1 \leq (4w + 2)t^2$.

Let $e \in E(T)$, and let $(A, B)$ be the vertex partition of $G$ associated with $e$. Suppose for contradiction that there is an induced matching $\{v_1w_1, \ldots, v_mw_m\}$ in $G[A, B]$ where $v_1, \ldots, v_m \in A$, $w_1, \ldots, w_m \in B$, and $m \geq 2(2w + 1)t^2 - t$. Let $f$ be the function from $\{v_1, \ldots, v_m\}$ to $\{w_1, \ldots, w_m\}$ such that $f(v_i) = w_i$ for each $i \in \{1, \ldots, m\}$. As $m \geq 2(2w + 1)t^2 - t$, by Theorem 5.2, the subgraph $G[\{v_1, \ldots, v_m\}]$ contains either an independent set of size $(2w + 1)t$, or a $K_t$-minor.

Suppose that $G[\{v_1, \ldots, v_m\}]$ contains a $K_t$-minor. Thus, there exist pairwise disjoint subsets $S_1, \ldots, S_t$ of $\{v_1, \ldots, v_m\}$ such that

- for each $i \in \{1, \ldots, t\}$, $G[S_i]$ is connected,
- for two distinct integers $i, j \in \{1, \ldots, t\}$, there is an edge between $S_i$ and $S_j$.

From this, we can obtain a $t$-matching complete graph by contracting each set $S_i$ to a vertex, and taking one vertex among vertices in each set $f(S_i)$. It contradicts to the assumption that $G$ contains no $t$-matching complete graphs as an induced minor. Thus, $G[\{v_1, \ldots, v_m\}]$ contains an independent set of size $(2w + 1)t$. Let $\{v_{i_1}, v_{i_2}, \ldots, v_{i(2w+1)t}\}$ be an independent set in $G[\{v_1, \ldots, v_m\}]$. 

Figure 5: The $(4 \times 4)$ Hsu-clique chain graph.

Now, by applying Theorem 5.2 again, $G[\{w_{i1}, w_{i2}, \ldots, w_{i(2w+1)t}\}]$ contains an independent set of size $w+1$ or a $K_t$-minor. If it has an independent set of size $w+1$, then we have $\sinval_{(T,L)}(e) \geq w+1$ which contradicts to our assumption. If it contains a $K_t$-minor, then by the same argument in the previous paragraph, we can find a $t$-matching complete graph as an induced minor, which also contradicts to our assumption. Thus, we conclude that $\minval_{(T,L)}(e) \leq 2(2w+1)t^2 - t - 1$, as required.

One might wonder whether the class of graphs of sim-width at most $k$ and having no induced minor isomorphic to a $3$-matching complete graph falls into a class of graphs of bounded rank-width or tree-width. We confirm that this is not true, by showing that Hsu-clique chain graphs in Figure 5 are chordal, but do not contain any $3$-matching complete graph as an induced minor. Belmonte and Vatshelle showed that a $(p \times q)$ Hsu-clique chain graph has rank-width at least $\frac{p}{3}$ [3 Lemma 16] when $q = 3p+1$. So, our algorithmic applications based on Proposition 5.1 are beyond algorithmic applications of graphs of bounded tree-width or rank-width.

We formally define Hsu-clique chain graphs. For positive integers $p,q$, the $(p \times q)$ Hsu-clique chain grid is the graph on the vertex set $\{v_{i,j} : 1 \leq i \leq p, 1 \leq j \leq q\}$ where

- for every $i \in \{1, \ldots, q\}$, $\{v_{1,j}, \ldots, v_{p,j}\}$ is a clique
- for every $i_1, i_2 \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q-1\}$, $v_{i_1,j}$ is adjacent to $v_{i_2,j+1}$ if and only if $i_1 \leq i_2$,
- for $i_1, i_2 \in \{1, \ldots, p\}$ and $j_1, j_2 \in \{1, \ldots, q\}$, $v_{i_1,j_1}$ is not adjacent to $v_{i_2,j_2}$ if $|j_1 - j_2| > 1$.

Proposition 5.3. The class of graphs of sim-width 1 and having no induced minor isomorphic to a $3$-matching complete graph has unbounded rank-width.

Proof. Let $p$ be a positive integer and $q := 3p+1$. Let $G$ be a $(p \times q)$ Hsu-clique chain graph. Belmonte and Vatshelle showed that a $(p \times q)$ Hsu-clique chain graph has rank-width at least $\frac{p}{3}$ [3 Lemma 16]. It is not hard to see that this graph is chordal, and thus it has sim-width at most 1 by Proposition 4.1. Now, we claim that $G$ has no induced minor isomorphic to a $3$-matching complete graph. Let $H$ be a $3$-matching complete graph with canonical bipartition $(\{v_1, v_2, v_3\}, \{w_1, w_2, w_3\})$ where $\{v_1w_1, v_2w_2, v_3w_3\}$ is the induced matching in $H[\{v_1, v_2, v_3\}, \{w_1, w_2, w_3\}]$. For contradiction, suppose that $G$ contains an induced minor isomorphic to $H$.

We observe that the class of chordal graphs is closed under taking induced minors. And later, we will show in Proposition 7.1 that every chordal $t$-matching complete graph is a $t$-matching complete graph with the canonical bipartition $(A, B)$ where $A$ is a clique and $B$ is an independent set.
Therefore, without loss of generality, we may assume that \( \{v_1, v_2, v_3\} \) is a clique and \( \{w_1, w_2, w_2\} \) is an independent set in \( H \).

Since \( G \) contains \( H \) as an induced minor, there is a mapping \( \mu \) from \( V(H) \) to \( 2^{V(G)} \) where

- \( \{\mu(v) : v \in V(H)\} \) are pairwise disjoint vertex subsets of \( G \), and each set in \( \{\mu(v) : v \in V(H)\} \) induces a connected subgraph of \( G \),

- for two distinct vertices \( v, w \in V(H), vw \in E(H) \) if and only if there is an edge between \( \mu(v) \) and \( \mu(w) \).

For each \( v \in V(H) \), let \( I_v = \{i : v_{i,j} \in \mu(v)\} \). For convenience, we say that a finite set \( I \) of consecutive integers is an interval. Let \( I := I_{v_1} \cup I_{v_2} \cup I_{v_3} \), and let \( \ell, r \) be the least and greatest integers in \( I \), respectively. Let \( x, y \in \{v_1, v_2, v_3\} \) such that

1. \( \mu(x) \) contains a vertex in the \( \ell \)-th column, but for \( z \in \{v_1, v_2, v_3\} \setminus \{x\}, \mu(z) \) has no vertex whose row index is higher than all vertices in \( \mu(x) \),

2. similarly, \( \mu(y) \) contains a vertex in the \( r \)-th column, but for \( z \in \{v_1, v_2, v_3\} \setminus \{x, y\}, \mu(z) \) has no vertex whose row index is lower than all vertices in \( \mu(y) \).

As \( \{v_1, v_2, v_3\} \) is a clique, it is easy to observe that \( I_x \cup I_y = I \). In other words, \( \mu(x) \cup \mu(y) \) contains a vertex in each column from the \( \ell \)-th column to the \( r \)-th column. Now, let \( z \in \{v_1, v_2, v_3\} \setminus \{x, y\} \).

By the choice of \( x \) and \( y \), every vertex in \( G \) having a neighbor in \( \mu(z) \) should have a neighbor in \( \mu(x) \cup \mu(y) \). Thus, it contradicts to the assumption that \( G \) contains \( H \) as an induced minor. \( \square \)

6 Algorithms for LC-VSVP problems

In this section, we describe algorithmic applications for graphs of bounded sim-width having no \( t \)-matching complete graph as an induced minor. Telle and Proskurowski [22] classified a class of problems called Locally Checkable Vertex Subset and Vertex Partitioning problems, which is a subclass of MSO\(_1\) problems. These problems generalize problems like Maximum Independent Set, Minimum Dominating Set, \( q \)-Coloring etc.

Let \( \sigma, \rho \) be finite or co-finite subsets of natural numbers. For a graph \( G \) and \( S \subseteq V(G) \), we call \( S \) a \((\sigma, \rho)\)-set of \( G \) if

- for every \( v \in S \), \(|N_G(v) \cap S| \in \sigma \), and

- for every \( v \in V(G) \setminus S \), \(|N_G(v) \cap S| \in \rho \).

For instance, a \((0, \mathbb{N})\)-set is an independent set as there are no edges inside of the set, and we do not care about adjacency between \( S \) and \( V(G) \setminus S \). Another example is that a \((\mathbb{N}, \mathbb{N}^+)\)-set is a dominating set as we require that for each vertex in \( V(G) \setminus S \), it has at least one neighbor in \( S \).

The class of locally checkable vertex subset problems consist of finding a minimum or maximum \((\sigma, \rho)\)-set in an input graph \( G \), and possibly on vertex-weighted graphs.

For a positive integer \( q \), a \((q \times q)\)-matrix \( D_q \) is called a degree constraint matrix if each element is either a finite or co-finite subset of natural numbers. A partition \( \{V_1, V_2, \ldots, V_q\} \) of the vertex set of a graph \( G \) is called a \( D_q\)-partition if

- for every \( i, j \in \{1, \ldots, q\} \) and \( v \in V_i \), \(|N_G(v) \cap V_j| \in D_q[i,j] \).
For instance, if we take a matrix $D_q$ where all diagonal entries are 0, and all other entries are $N$, then a $D_q$-partition is a partition into $q$ independent sets, which corresponds to a $q$-coloring of the graph. The class of *locally checkable vertex partitioning problems* consist of deciding if $G$ admits a $D_q$-partition.

All these problems will be called *Locally Checkable Vertex Subset and Vertex Partitioning problems*, shortly LC-VSVP problems. As shown in [6] the runtime solving an LC-VSVP problem by dynamic programming relates to the finite or co-finite subsets of natural numbers used in its definition. The following function $d$ is central.

1. Let $d(\mathbb{N}) = 0$.
2. For every finite or co-finite set $\mu \subseteq \mathbb{N}$, let $d(\mu) = 1 + \min(\max\{x \in \mathbb{N} : x \in \mu\}, \max\{x \in \mathbb{N} : x \notin \mu\})$.

**Theorem 6.1** (Belmonte and Vatshelle [3] and Bui-Xuan, Telle, and Vatshelle [6]). *Given an* $n$-*vertex graph and its branch-decomposition* $(T, L)$ *of mim-width* $w$ *we solve*

- any $(\sigma, \rho)$-vertex subset problem with $d = \max(d(\sigma), d(\rho))$ in time $O(n^{3dw+4})$,
- any $D_q$-vertex partitioning problem with $d = \max_{i,j} d(D_q[i,j])$ in time $O(qn^{3dwq+4})$.

Combining Theorem 6.1 with Proposition 5.1 we get the following.

**Corollary 6.2.** *Given an* $n$-*vertex graph having no* $t$-*matching complete graph as an induced minor and its branch-decomposition* $(T, L)$ *of sim-width* $w$, *we solve*

- any $(\sigma, \rho)$-vertex subset problem with $d = \max(d(\sigma), d(\rho))$ in time $O(n^{6d(2w+1)t^2+4})$,
- any $D_q$-vertex partitioning problem with $d = \max_{i,j} d(D_q[i,j])$ in time $O(qn^{6d(2w+1)qt^2+4})$.

For example, for Minimum Dominating Set and $q$-COLORING we plug in $d = 1$ since $\max(d(\mathbb{N}), d(\mathbb{N}^+)) = 1$ and $\max(d(0), d(\mathbb{N})) = 1$.

### 7 Finding $t$-matching complete graphs in chordal graphs and co-comparability graphs

We observe structural properties of $t$-matching complete graphs in chordal graphs or co-comparability graphs. We say that a $t$-matching complete graph with canonical bipartition $(A, B)$ is of *type 1* if $B$ is an independent set, and it is of *type 2* if $B$ is one clique or the disjoint union of two cliques.

**Proposition 7.1.** *Let* $t$ *be a positive integer.*

1. Every chordal $t$-matching complete graph is a $t$-matching complete graph of type 1.
2. Every co-comparability $t$-matching complete graph is a $t$-matching complete graph of type 2.

**Proof.** (1) Let $G$ be a chordal graph that is a $t$-matching complete graph with canonical bipartition $(\{v_1, \ldots, v_t\}, \{w_1, \ldots, w_t\})$. If $G[\{w_1, \ldots, w_t\}]$ contains an edge, then it contains a subgraph isomorphic to an induced cycle of length 4, contradicting to the assumption that $G$ is chordal.
(2) Let \( G \) be a co-comparability graph that is a \( t \)-matching complete graph with canonical bipartition \( (\{v_1, \ldots, v_t\}, \{w_1, \ldots, w_t\}) \). First assume that \( \{w_1, \ldots, w_t\} \) contains an independent set of size 3 in \( G \). Then they are matched with three vertices on \( A \) that form a clique. However, in this case, the independent set of size 3 is an asteroidal triple. So, it is contradiction. We know that \( \{w_1, \ldots, w_t\} \) contains no independent set of size 3.

Now, suppose that \( G[\{w_1, \ldots, w_t\}] \) has an induced path of length 3. Then two end vertices of this path will be matched with two vertices on the other part that are adjacent. Therefore, it creates an induced cycle of length 5, but co-comparability graphs have no induced subgraph isomorphic to \( C_5 \). So, we can conclude that \( G[\{w_1, \ldots, w_t\}] \) consists of one or two cliques.

**Lemma 7.2.** The class of all chordal graphs and the class of all co-comparability graphs are closed under taking induced minors.

**Proof.** Every chordal graph can be seen as an intersection graph of subtrees in a subtree. So, when we apply an edge contraction, we can obtain a new representation by merging two corresponding subtrees. It is clear that this class is closed under by removing a vertex.

Let \( G \) be a co-comparability graph with a co-comparability ordering \( v_1, \ldots, v_n \), and let \( v_pv_q \in E(G) \) for some \( p, q \in \{1, \ldots, n\} \). Without loss of generality, we may assume that \( p < q \). We call the contracted vertex in \( G/v_pv_q \) as \( v_p \), and let \( L \) be the linear ordering of \( V(G) \setminus \{v_q\} \) that is the restriction of the ordering \( v_1, \ldots, v_n \). We claim that \( L \) is a co-comparability ordering of \( G/v_pv_q \).

Let \( v_i, v_j, v_k \in V(G/v_pv_q) \) with \( i < j < k \), and suppose that there is a path \( P \) from \( v_i \) to \( v_k \) in \( G/v_pv_q \) avoiding \( v_j \). If \( P \) exists in \( G \), then \( v_j \) has a neighbor in \( P \), as \( L \) is a restriction of the given co-comparability ordering. We may assume that \( P \) was obtained by contraction. Then there exists a path \( P' \) in \( G \) where \( P'/v_pv_q = P \), and this path also avoids \( v_j \) in \( G \). Thus, \( v_j \) has at least one neighbor on \( P' \) in \( G \), and thus it has a neighbor on \( P \) in \( G/v_pv_q \). It shows that \( L \) is a co-comparability ordering of \( G/v_pv_q \), and thus \( G/v_pv_q \) is a co-comparability graph. Clearly, the class of all co-comparability graphs is closed under by removing a vertex.

Corollary 6.2 assumed that the given graph has no \( t \)-matching complete graph as an induced minor, and also its sim-decomposition of width at most \( w \) is given. For chordal graphs and co-comparability graphs, we can produce a sim-decomposition of width at most 1 in polynomial time using Propositions 4.4 and 4.5. To test whether a chordal or co-comparability graph contains a \( t \)-matching complete graph as an induced minor, we can use known XP algorithms [2, 14]. Notice that for fixed graph \( H \), testing \( H \)-induced minor in general graphs is known to be NP-complete [9].

Furthermore, by Lemma 7.2, if a chordal or co-comparability graph \( G \) contains a \( t \)-matching complete graph \( H \) as an induced minor, then \( H \) should be a chordal or co-comparability graph, respectively. By Proposition 7.1, there are few candidates for \( t \)-matching complete graphs; there is a unique chordal \( t \)-matching complete graph, and there are \( t \) pairwise non-isomorphic co-comparability \( t \)-matching complete graphs. Thus, the runtime will not be exponentially multiplied from the number of \( t \)-matching complete graphs to be tested.

### 8 Concluding remarks

In this paper, we show that every graph with sim-width at most \( w \) and having no induced minor isomorphic to a \( t \)-matching complete graph has mim-width at most \((4w+2)t^2\), and every LC-VSVP problem can be solved in time \( n^{O\left(w t^4\right)} \) on such \( n \)-vertex graphs, when its branch-decomposition is
given. Also, polynomial-time algorithms to find branch-decompositions certifying sim-width at most 1 for chordal graphs and co-comparability graphs are provided.

It is worth noting that there remains a number of interesting open problems about this new parameter. We would like to find more classes that have constant sim-width, but unbounded value for tree-width, rank-width, or mim-width. We propose some possible classes, that are also presented in Figure 1.

Question 1. Do weakly chordal graphs, AT-free graphs, or circle graphs have constant sim-width?

Two extreme cases of $t$-matching complete graphs with canonical bipartition $(A, B)$ is when $B$ is a clique or an independent set. We note that if we allow a slight worse bound in Proposition 5.1, then we can bound mim-width by excluding only two these extreme $t$-matching complete graphs. Thus, we naturally ask whether we can test whether a given graph contains such an extreme $t$-matching complete graph as an induced minor efficiently.

Question 2. Let $H$ be a fixed $t$-matching complete graph with canonical bipartition $(A, B)$ where $B$ is a clique or an independent set. Can we test whether a given graph contains an induced minor isomorphic to $H$ in polynomial time?

In the line of research on sim-width and mim-width, one of main problems is to find an efficient algorithm to find a branch-decomposition of relevant width. As we see, we obtain general XP algorithms, and thus it make sense to ask whether there is an XP-algorithm to find a branch-decomposition. We remark that $\text{simval}_G$ is not a submodular function.

Question 3. Is there a function $f$ such that, given a graph $G$, we can in XP time parameterized by $k = \text{simw}(G)$ (or $k = \text{mimw}(G)$) compute a branch-decomposition of width at most $f(k)$ with respect to the $\text{simval}_G$ function (or the $\text{mimval}_G$ function)?

Probably, the most interesting property of sim-width is that the class of graphs of sim-width at most $w$ is closed under taking induced minors. We could try to characterize classes of graphs of bounded sim-width in terms of forbidden induced minors. We remark that interval graphs are not well-quasi-ordered by the induced minor operation [7], and thus, graphs of bounded sim-width are not well-quasi-ordered by the induced minor operation in general. So, we even do not know whether there is a finite list of obstructions for graphs of bounded sim-width.

Question 4. What are the induced minor obstructions for the class of graphs of sim-width at most 1? For fixed $\ell$, what are the induced minor obstructions for the class of graphs of sim-width at most 1 and having no $\ell$-matching complete graphs as an induced minor?

The last question is about finding a certain induced minor in a graph of sufficiently large sim-width. For the minor operation, planar graphs satisfy that for fixed planar graph, every graph of sufficiently large tree-width contains it as a minor. It would be interesting to know whether we can get some grid-like structure from graphs of sufficiently large sim-width as an induced minor.

Question 5. Is there any non-trivial graph class $C$ such that every graph of sufficiently large sim-width contains a graph in $C$ of certain size as an induced minor? In particular, can $C$ be the class of planar graphs?
References

[1] J. Balogh and A. Kostochka. Large minors in graphs with given independence number. *Discrete Mathematics*, 311(20):2203 – 2215, 2011.

[2] R. Belmonte, P. A. Golovach, P. Heggernes, P. van ’t Hof, M. Kamiński, and D. Paulusma. Finding contractions and induced minors in chordal graphs via disjoint paths. In *Algorithms and computation*, volume 7074 of *Lecture Notes in Comput. Sci.*, pages 110–119. Springer, Heidelberg, 2011.

[3] R. Belmonte and M. Vatshelle. Graph classes with structured neighborhoods and algorithmic applications. *Theoret. Comput. Sci.*, 511:54–65, 2013.

[4] K. S. Booth and J. H. Johnson. Dominating sets in chordal graphs. *SIAM J. Comput.*, 11(1):191–199, 1982.

[5] B.-M. Bui-Xuan, J. A. Telle, and M. Vatshelle. Boolean-width of graphs. *Theoret. Comput. Sci.*, 412(39):5187–5204, 2011.

[6] B.-M. Bui-Xuan, J. A. Telle, and M. Vatshelle. Fast dynamic programming for locally checkable vertex subset and vertex partitioning problems. *Theoret. Comput. Sci.*, 511:66–76, 2013.

[7] G. Ding. Chordal graphs, interval graphs, and wqo. *Journal of Graph Theory*, 28(2):105–114, 1998.

[8] P. Duchet and H. Meyniel. On hadwiger’s number and the stability number. In B. Bollobs, editor, *Graph TheoryProceedings of the Conference on Graph Theory, Cambridge*, volume 62 of *North-Holland Mathematics Studies*, pages 71 – 73. North-Holland, 1982.

[9] M. R. Fellows, J. Kratochvıl, M. Middendorf, and F. Pfeiffer. The complexity of induced minors and related problems. *Algorithmica*, 13(3):266–282, 1995.

[10] F. V. Fomin, P. Golovach, and D. M. Thilikos. Contraction obstructions for treewidth. *J. Combin. Theory Ser. B*, 101(5):302–314, 2011.

[11] F. V. Fomin, S. Oum, and D. M. Thilikos. Rank-width and tree-width of H-minor-free graphs. *European J. Combin.*, 31(7):1617–1628, 2010.

[12] J. Fox. Complete minors and independence number. *SIAM Journal on Discrete Mathematics*, 24(4):1313–1321, 2010.

[13] M. Garey, D. Johnson, and L. Stockmeyer. Some simplified np-complete graph problems. *Theoretical Computer Science*, 1(3):237 – 267, 1976.

[14] P. A. Golovach, D. Kratsch, and D. Paulusma. Detecting induced minors in at-free graphs. *Theoretical Computer Science*, 482:20 – 32, 2013.

[15] P. Hliněný, S. Oum, D. Seese, and G. Gottlob. Width parameters beyond tree-width and their applications. *The Computer Journal*, 51(3):326–362, 2008.
[16] R. M. McConnell and J. P. Spinrad. Linear-time modular decomposition and efficient transitive orientation of comparability graphs. In Proceedings of the Fifth Annual ACM-SIAM Symposium on Discrete Algorithms, SODA ’94, pages 536–545, Philadelphia, PA, USA, 1994. Society for Industrial and Applied Mathematics.

[17] J. Naor, M. Naor, and A. A. Schffer. Fast parallel algorithms for chordal graphs. SIAM Journal on Computing, 18(2):327–349, 1989.

[18] S. Oum. Rank-width and vertex-minors. J. Comb. Theory, Ser. B, 95(1):79–100, 2005.

[19] S. Oum. Rank-width is less than or equal to branch-width. J. Graph Theory, 57(3):239–244, 2008.

[20] N. Sauer. On the density of families of sets. Journal of Combinatorial Theory, Series A, 13(1):145 – 147, 1972.

[21] S. Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. Pacific J. Math., 41(1):247–261, 1972.

[22] J. A. Telle and A. Proskurowski. Algorithms for vertex partitioning problems on partial k-trees. SIAM J. Discrete Math., 10(4):529–550, 1997.

[23] M. Vatshelle. New width parameters of graphs. Ph.D. thesis, University of Bergen, 2012.