HELICITY BASIS AND PARITY*

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Abstract

Abstract. We study the theory of the (1/2, 0) ⊕ (0, 1/2) representation
in helicity basis. Helicity eigenstates are not the parity eigenstates. This is
in accordance with the consideration of Berestetski˘ı, Lifshitz and Pitaevski˘ı.
Relations to the Gelfand-Tsetlin-Sokolik-type quantum field theory are dis-
cussed. Finally, a new form of the parity operator is proposed. It commutes
with the Hamiltonian.

First of all, I would like to congratulate Professor J. Plebanski with his 75th birthday.
Thank you for your hard work in theoretical physics, which we all admire.

What are scientific motivations for my talk? Recently we generalized the Dirac formal-
ism [1–4] and the Bargmann-Wigner formalism [5,7], and on this basis we proposed a set
of twelve equations for antisymmetric tensor (AST) field; some of them may lead to parity-
vioeating transitions. In this paper we are going to study somewhat related matter, the
transformation from the standard basis to the helicity basis in the Dirac theory. The spin
basis rotation changes the properties of corresponding states with respect to parity. The
parity is a physical quantum number; so, we try to extract corresponding physical contents
from considerations of the various spin bases.

Briefly, I repeat the results of ref. [6,7]. One can find solutions of the 2(2J + 1)-theory
with different parity properties [6]. They can be related to the polarization vectors obtained
by Ruck and Greiner [8], who found the helicity states of the 4-vector potential on the basis
of the Jackob and Wick paper [9]. Next, I used the generalized Bargmann-Wigner formalism
based on the equations

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1The parity-violating Dirac equation has been derived in [4]. The method of the derivation refers
to the van der Waerden, Sakurai and Gersten works, see references in the previous papers of mine.
\[
\begin{align*}
[i\gamma_\mu \partial_\mu + \epsilon_1 m_1 + \epsilon_2 m_2 \gamma_5]_{\alpha\beta} \Psi_{\beta\gamma} &= 0, \\
[i\gamma_\mu \partial_\mu + \epsilon_3 m_1 + \epsilon_4 m_2 \gamma_5]_{\alpha\beta} \Psi_{\gamma\beta} &= 0,
\end{align*}
\]

(1a)

(1b)

Different equations for the antisymmetric tensor field follow from this set by means of the standard procedure [10]. We concluded in [7] in part that: 1) in the \((1/2, 0) \oplus (0, 1/2)\) representation it is possible to introduce the \textit{parity-violating} frameworks; 2) the mappings between the Weinberg-Tucker-Hammer formalism for \(J = 1\) and the AST fields of the 2nd rank of, at least, eight types exist; Four of them include both \(F_{\mu\nu}\) and \(\tilde{F}_{\mu\nu}\), which tells us that the parity violation may occur during the study of the corresponding dynamics; 3) if we want to take into account the \(J = 1\) solutions with different parity properties, the Bargmann-Wigner (BW) formalism is to be generalized; 4) the 4-potentials and the fields in the helicity basis can be constructed; they have different parity properties comparing with the standard ("parity") basis; 5) generalizing the BW formalism in such a way, twelve equations for the AST fields have been obtained; 6) finally, a hypothesis was proposed therein that the obtained results are related to the spin basis rotations and to the choice of normalization.

Beginning the consideration of the helicity basis, we observe that it is well known that the operator \(\hat{S}_3 = \sigma_3/2 \otimes I_2\) does not commute with the Dirac Hamiltonian unless the 3-momentum is aligned along with the third axis and the plane-wave expansion is used:

\[
[\hat{\mathcal{H}}, \hat{S}_3] = (\gamma^0 \gamma^k \times \nabla_i) \gamma_3
\]

Moreover, Berestetskiĭ, Lifshitz and Pitaevskiĭ wrote [11]: "... the orbital angular momentum \(l\) and the spin \(s\) of a moving particle are not separately conserved. Only the total angular momentum \(j = l + s\) is conserved. The component of the spin in any fixed direction (taken as \(z\)-axis is therefore also not conserved, and cannot be used to enumerate the polarization (spin) states of moving particle.” The similar conclusion has been given by Novozhilov in his book [12]. On the other hand, the helicity operator \(\sigma \cdot \hat{p}/2 \otimes I, \hat{p} = p/|p|\), commutes with the Hamiltonian (more precisely, the commutator is equal to zero when acting the one-particle plane-wave solutions).

So, it is a bit surprising, why the 4-spinors have been studied so well when the basis was chosen in such a way that they are eigenstates of the \(\hat{S}_3\) operator:

\[
u_{\frac{1}{2},-\frac{1}{2}} = N_+^{\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \nu_{\frac{1}{2},\frac{1}{2}} = N_-^{\frac{1}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \nu_{-\frac{1}{2},\frac{1}{2}} = N_-^{-\frac{1}{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \quad \nu_{-\frac{1}{2},-\frac{1}{2}} = N_+^{-\frac{1}{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix},
\]

(3)

and, oppositely, the helicity basis case has not been studied almost at all (see, however, refs. [12,9]). Let me remind that the boosted 4-spinors in the ‘common-used’ basis are

\[
u_{\frac{1}{2},-\frac{3}{2}} = \frac{N_+^{\frac{1}{2}}}{\sqrt{2m(E + m)}} \begin{pmatrix} p^+ + m \\ pr \\ p^- + m \\ -pr \end{pmatrix}, \quad 
\nu_{\frac{3}{2},-\frac{3}{2}} = \frac{N_-^{\frac{1}{2}}}{\sqrt{2m(E + m)}} \begin{pmatrix} pl \\ p^- + m \\ -pl \\ p^+ + m \end{pmatrix},
\]

(4a)

\[
u_{\frac{3}{2},\frac{1}{2}} = \frac{N_-^{\frac{1}{2}}}{\sqrt{2m(E + m)}} \begin{pmatrix} p^+ + m \\ pr \\ -p^- - m \\ pr \end{pmatrix}, \quad \nu_{\frac{1}{2},\frac{1}{2}} = \frac{N_-^{-\frac{1}{2}}}{\sqrt{2m(E + m)}} \begin{pmatrix} pl \\ p^- + m \\ pl \\ -p^+ - m \end{pmatrix},
\]

(4b)
\( p^\pm = E \pm p_z, \ p_{r,t} = p_x \pm ip_y \). These are the parity eigenstates with eigenvalues of \( \pm 1 \). In the parity operator the matrix \( \gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \) is used.

Let me turn now your attention to the helicity spin basis. The 2-eigen spinors of the helicity operator

\[
\frac{1}{2} \sigma \cdot \hat{p} = \frac{1}{2} \begin{pmatrix} \cos \theta & \sin \theta e^{-i\phi} \\ \sin \theta e^{-i\phi} & -\cos \theta \end{pmatrix}
\]

(5)
can be defined as follows [13,14]:

\[
\phi_\uparrow = \begin{pmatrix} \cos \frac{\theta}{2} e^{-i\phi/2} \\ \sin \frac{\theta}{2} e^{i\phi/2} \end{pmatrix}, \quad \phi_\downarrow = \begin{pmatrix} \sin \frac{\theta}{2} e^{-i\phi/2} \\ -\cos \frac{\theta}{2} e^{i\phi/2} \end{pmatrix},
\]

(6)
for \( \pm 1/2 \) eigenvalues, respectively.

We start from the Klein-Gordon equation, generalized for describing the spin-1/2 particles (i.e., two degrees of freedom); \( c = \hbar = 1 \):

\[ (E + \sigma \cdot p)(E - \sigma \cdot p)\phi = m^2 \phi. \]

(7)

It can be re-written in the form of the set of two first-order equations for 2-spinors. Simultaneously, we observe that they may be chosen as eigenstates of the helicity operator which present in (7):\(^2\)

\[
\begin{align*}
(E - (\sigma \cdot p))\phi_\uparrow &= (E - p)\phi_\uparrow = m\chi_\uparrow, \\
(E + (\sigma \cdot p))\chi_\uparrow &= (E + p)\chi_\uparrow = m\phi_\uparrow,
\end{align*}
\]

(8a)

\[
\begin{align*}
(E - (\sigma \cdot p))\phi_\downarrow &= (E + p)\phi_\downarrow = m\chi_\downarrow, \\
(E + (\sigma \cdot p))\chi_\downarrow &= (E - p)\chi_\downarrow = m\phi_\downarrow.
\end{align*}
\]

(8b)

If the \( \phi \) spinors are defined by the equation (6) then we can construct the corresponding \( u \)– and \( v \)– 4-spinors\(^3\)

---

\(^2\)This opposes to the choice of the basis (3), where 4-spinors are the eigenstates of the parity operator.

\(^3\)One can also try to construct yet another theory differing from the ordinary Dirac theory. The 4-spinors might be \textit{not} the eigenspinors of the helicity operator of the \((1/2, 0) \oplus (0, 1/2)\) representation space, cf. [2]. They might be the eigenstates of the \textit{chiral} helicity operator introduced in [2a]. In this case, the momentum-space Dirac equations can be written (cf. [2c], [3])

\[
\begin{align*}
p_{\mu} \gamma^\mu U_\uparrow - mU_\uparrow &= 0, \\
p_{\mu} \gamma^\mu U_\downarrow - mU_\downarrow &= 0, \\
p_{\mu} \gamma^\mu V_\uparrow + mV_\uparrow &= 0, \\
p_{\mu} \gamma^\mu V_\downarrow + mV_\downarrow &= 0.
\end{align*}
\]

(9a)

(9b)

(9c)

(9d)

Here \( \uparrow \downarrow \) refers already to the chiral helicity eigenstates, e.g. \( u_\eta = \frac{1}{\sqrt{2}} \left( \begin{pmatrix} N\phi_\eta \\ N^{-1} \phi_{-\eta} \end{pmatrix} \right) \).
where the normalization to the unit (±1) was used:

\[\bar{u}_\lambda u_{\lambda'} = \delta_{\lambda\lambda'}, \quad \bar{v}_\lambda v_{\lambda'} = -\delta_{\lambda\lambda'}, \quad \bar{u}_\lambda v_{\lambda'} = 0 = \bar{v}_\lambda u_{\lambda'}\]  

(11a)

One can prove that the matrix

\[P = \gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}\]  

(12)

can be used in the parity operator as well as in the original Dirac basis. Indeed, the 4-spinors (10a,10b) satisfy the Dirac equation in the spinorial representation of the \(\gamma\)-matrices (see straightforwardly from (7)). Hence, the parity-transformed function \(\Psi'(t, -x) = P\Psi(t, x)\) must satisfy

\[[i\gamma^\mu \partial'_\mu - m]\Psi'(t, -x) = 0,\]  

(13)

with \(\partial'_\mu = (\partial/\partial t, -\nabla_i)\). This is possible when \(P^{-1}\gamma^0 P = \gamma^0\) and \(P^{-1}\gamma^i P = -\gamma^i\). The matrix (12) satisfies these requirements, as in the textbook case.

Next, it is easy to prove that one can form the projection operators

\[P_+ = \sum_\lambda u_\lambda(p)\bar{u}_\lambda(p) = \frac{p_\mu \gamma^\mu + m}{2m},\]  

(14a)

\[P_- = -\sum_\lambda v_\lambda(p)\bar{v}_\lambda(p) = \frac{m - p_\mu \gamma^\mu}{2m},\]  

(14b)

with the properties \(P_+ + P_- = 1\) and \(P_+^2 = P_+\). This permits us to expand the 4-spinors defined in the basis (3) in linear superpositions of the helicity basis 4-spinors and to find corresponding coefficients of the expansion:

\[u_\sigma(p) = A_{\sigma\lambda} u_\lambda(p) + B_{\sigma\lambda} v_\lambda(p),\]  

(15a)

\[v_\sigma(p) = C_{\sigma\lambda} u_\lambda(p) + D_{\sigma\lambda} v_\lambda(p).\]  

(15b)

Multiplying the above equations by \(\bar{u}_\lambda, \bar{v}_\lambda\) and using the normalization conditions, we obtain \(A_{\sigma\lambda} = D_{\sigma\lambda} = \bar{u}_\lambda u_\sigma, B_{\sigma\lambda} = C_{\sigma\lambda} = -\bar{v}_\lambda u_\sigma\). Thus, the transformation matrix from the common-used basis to the helicity basis is

\[4\text{Of course, there are no any mathematical difficulties to change it to the normalization to ±m, which may be more convenient for our study of the massless limit.}\]
\[
\begin{pmatrix} u_\sigma \\ v_\sigma \end{pmatrix} = U \begin{pmatrix} u_\lambda \\ v_\lambda \end{pmatrix}, \quad U = \begin{pmatrix} A & B \\ B & A \end{pmatrix}
\] (16)

Neither \( A \) nor \( B \) are unitary:

\[
A = (a_{++} + a_{+-})(\sigma_\mu a^\mu) + (a_{-+} + a_{--})(\sigma_\mu a^\mu)\sigma_3, \\
B = (a_{++} + a_{+ -})(\sigma_\mu a^\mu) + (a_{-+} + a_{--})(\sigma_\mu a^\mu)\sigma_3,
\]

where

\[
a^0 = -i \cos(\theta/2) \sin(\phi/2) \in \mathbb{I}_m, \quad a^1 = \sin(\theta/2) \cos(\phi/2) \in \mathbb{R}, \\
a^2 = \sin(\theta/2) \sin(\phi/2) \in \mathbb{R}, \quad a^3 = \cos(\theta/2) \cos(\phi/2) \in \mathbb{R},
\]

and

\[
a_{++} = \sqrt{(E + m)(E + p)} \sqrt{2m}, \quad a_{+-} = \sqrt{(E + m)(E - p)} \sqrt{2m}, \\
a_{-+} = \sqrt{(E - m)(E + p)} \sqrt{2m}, \quad a_{--} = \sqrt{(E - m)(E - p)} \sqrt{2m}.
\]

However, \( A^\dagger A + B^\dagger B = \mathbb{1} \), so the matrix \( U \) is unitary. Please note that this matrix acts on the spin indices \( (\sigma, \lambda) \), and not on the spinorial indices; it is \( 4 \times 4 \) matrix. Alternatively, the transformation can be written:

\[
u_\sigma = [A_{\sigma\lambda} \otimes I_{\alpha\beta} + B_{\sigma\lambda} \otimes \gamma_5^5_{\alpha\beta}]u_\lambda, \\
\nu_\sigma = [A_{\sigma\lambda} \otimes I_{\alpha\beta} + B_{\sigma\lambda} \otimes \gamma_5^5_{\alpha\beta}]v_\lambda.
\]

We now investigate the properties of the helicity-basis 4-spinors with respect to the discrete symmetry operations \( P, C \) and \( T \). It is expected that \( \lambda \to -\lambda \) under parity, as Berestetskiǐ, Lifshitz and Pitaevskii claimed [11]. With respect to \( p \to -p \) (i.e., the spherical system angles \( \theta \to \pi - \theta, \varphi \to \pi + \varphi \)) the helicity 2-eigenspinors transform as follows: \( \phi_{\uparrow\downarrow} \to -i\phi_{\downarrow\uparrow} \), ref. [14]. Hence,

\[
P u_{\uparrow}(-p) = -iu_{\downarrow}(p), \quad P v_{\uparrow}(-p) = +iv_{\downarrow}(p), \\
P u_{\downarrow}(-p) = -iu_{\uparrow}(p), \quad P v_{\downarrow}(-p) = +iv_{\uparrow}(p).
\]

Thus, on the level of classical fields, we observe that the helicity 4-spinors transform to the 4-spinors of the opposite helicity.

Under the charge conjugation operation we have:

\[
C = \begin{pmatrix} 0 & \Theta \\ -\Theta & 0 \end{pmatrix} \mathcal{K}.
\]

\[\text{---}\]

\[5\]Indeed, if \( x \to -x \), then the vector \( p \to -p \), but the axial vector \( S \to S \), that implies the above statement.
Hence, we observe

\[ Cu_\uparrow(p) = -v_\downarrow(p), \quad Cu_\downarrow(p) = +u_\downarrow(p), \quad (23a) \]
\[ Cu_\uparrow(p) = +v_\uparrow(p), \quad Cu_\downarrow(p) = -u_\uparrow(p). \quad (23b) \]
due to the properties of the Wigner operator \( \Theta \phi_\uparrow^* = -\phi_\downarrow \) and \( \Theta \phi_\downarrow^* = +\phi_\uparrow \). For the \( CP \) (and \( PC \)) operation we get:

\[ \begin{align*}
CPu_\uparrow(-p) &= -PCu_\uparrow(-p) = +iv_\downarrow(p), \\
CPu_\downarrow(-p) &= -PCu_\downarrow(-p) = -iv_\uparrow(p), \\
CPv_\uparrow(-p) &= -PCv_\uparrow(-p) = +iu_\uparrow(p), \\
CPv_\downarrow(-p) &= -PCv_\downarrow(-p) = -iu_\downarrow(p). 
\end{align*} \quad (24a-d) \]

Similar conclusions can be drawn in the Fock space. We define the field operator as follows:

\[ \Psi(x^\mu) = \sum_\lambda \int \frac{d^3 p}{(2\pi)^3} \frac{\sqrt{m}}{2E} [u_\lambda a_\lambda e^{-ip_\mu x^\mu} + v_\lambda b_\lambda^\dagger e^{+ip_\mu x^\mu}]. \quad (25) \]

The commutation relations are assumed to be the standard ones \(^{[15–18]}\) (compare with \([2,3]\))

\[ \begin{align*}
[a_\lambda(p), a_{\lambda'}^\dagger(k)]_+ &= 2E \delta^{(3)}(p - k) \delta_{\lambda\lambda'}, [a_\lambda(p), a_\lambda(k)]_+ = 0 = [a_{\lambda'}(p), a_{\lambda'}^\dagger(k)]_+, \\
[b_\lambda(p), a_{\lambda'}^\dagger(k)]_+ &= 0 = [b_\lambda(p), a_{\lambda'}(k)]_+, \\
[b_\lambda(p), b_{\lambda'}^\dagger(k)]_+ &= 2E \delta^{(3)}(p - k) \delta_{\lambda\lambda'}, [b_\lambda(p), b_\lambda(k)]_+ = 0 = [b_{\lambda'}(p), b_{\lambda'}^\dagger(k)]_+. \quad (26a-d) 
\end{align*} \]

If one defines \( U_P \Psi(x^\mu) U_P^{-1} = \gamma^0 \Psi(x^\mu), U_C \Psi(x^\mu) U_C^{-1} = \tilde{C} \Psi(x^\mu) \) and the anti-unitary operator of time reversal \( (V_T \Psi(x^\mu) V_T^{-1})^\dagger = T \Psi(x^\mu) \), then it is easy to obtain the corresponding transformations of the creation/annihilation operators (cf. the cited textbooks).

\[ \begin{align*}
U_P a_\lambda U_P^{-1} &= -ia_{-\lambda}(-p), \\
U_P b_{\lambda} U_P^{-1} &= ib_{-\lambda}(-p), \\
U_C a_\lambda U_C^{-1} &= (-1)^{\frac{1}{2} + \lambda} b_{-\lambda}(-p), \\
U_C b_{\lambda} U_C^{-1} &= (-1)^{\frac{1}{2} - \lambda} a_{-\lambda}(-p). \quad (27a-b) 
\end{align*} \]

As a consequence, we obtain (provided that \( U_P |0> = |0>, U_C |0> = |0> \))

\[ \begin{align*}
U_P a_\lambda^\dagger(p)|0> &= U_P a_\lambda^\dagger U_P^{-1}|0> = i a_{-\lambda}^\dagger(-p)|0> = i| - p, -\lambda >^+, \\
U_P b_{\lambda}^\dagger(p)|0> &= U_P b_{\lambda}^\dagger U_P^{-1}|0> = ib_{-\lambda}^\dagger(-p)|0> = i| - p, -\lambda >^-; \quad (28a-b) 
\end{align*} \]

and

\[ \begin{align*}
U_C a_\lambda^\dagger(p)|0> &= U_C a_\lambda^\dagger U_C^{-1}|0> = (-1)^{\frac{1}{2} + \lambda} b_{-\lambda}^\dagger(-p)|0> = (-1)^{\frac{1}{2} + \lambda}|p, -\lambda >^-, \\
U_C b_{\lambda}^\dagger(p)|0> &= U_C b_{\lambda}^\dagger U_C^{-1}|0> = (-1)^{\frac{1}{2} - \lambda} a_{-\lambda}^\dagger(-p)|0> = (-1)^{\frac{1}{2} - \lambda}|p, -\lambda >^+. \quad (29a-b) 
\end{align*} \]

\(^6\)The only possible changes may be related to a different form of normalization of 4-spinors, which would have influence on the factor before \( \delta \)-function.

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6The only possible changes may be related to a different form of normalization of 4-spinors, which would have influence on the factor before \( \delta \)-function.
Finally, for the \( CP \) operation one should obtain:

\[
U_P U_C a^\dagger_\lambda(p)|0\rangle = -U_C U_P a^\dagger_\lambda(p)|0\rangle = (-1)^{\frac{1}{2} + \lambda} U_P b^\dagger_{-\lambda}(p)|0\rangle = \nonumber
\]

\[
i(-1)^{\frac{1}{2} + \lambda} b^\dagger_\lambda(-p)|0\rangle = i(-1)^{\frac{1}{2} + \lambda} | -p, \lambda > ,
\]

\[
U_P U_C b_\lambda(p)|0\rangle = -U_C U_P b_\lambda(p) = (-1)^{\frac{1}{2} - \lambda} U_P a^\dagger_{-\lambda}(p)|0\rangle = \nonumber
\]

\[
i(-1)^{\frac{1}{2} - \lambda} a^\dagger_\lambda(-p)|0\rangle = i(-1)^{\frac{1}{2} - \lambda} | -p, -\lambda > .
\]

(30a)

(30b)

As in the classical case, the \( P \) and \( C \) operations anticommutes in the \((\frac{1}{2}, 0) \oplus (0, \frac{1}{2})\) quantized case. This opposes to the theory based on 4-spinor eigenstates of chiral helicity (cf. [3]).

Since the \( V_T \) is an anti-unitary operator the problem must be solved after taking into account that in this case the \( c \)-numbers should be put outside the hermitian conjugation without complex conjugation:

\[
[V_T \lambda AV_T^{-1}]^\dagger = [\lambda^* V_T AV_T^{-1}]^\dagger = \lambda [V_T A^\dagger V_T^{-1}] .
\]

(31)

With this definition we obtain:

\[
V_T a^\dagger_\lambda V_T^{-1} = +i(-1)^{\frac{1}{2} - \lambda} a^\dagger_\lambda(-p) ,
\]

(32a)

\[
V_T b_\lambda V_T^{-1} = +i(-1)^{\frac{1}{2} - \lambda} b_\lambda(-p) .
\]

(32b)

Furthermore, we observed that the question of whether a particle and an antiparticle have the same or opposite parities depend on a phase factor in the following definition:

\[
U_P \Psi(t, x) U_P^{-1} = e^{io} \gamma^0 \Psi(t, -x) .
\]

(33)

Indeed, if we repeat the textbook procedure [18]:

\[
U_P \left[ \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \sqrt{m} \left( u_\lambda(p) a_\lambda(p)e^{-ip \cdot x} + v_\lambda(p) b^\dagger_\lambda(p)e^{+ip \cdot x} \right) \right] U_P^{-1} = \nonumber
\]

\[
e^{io} \left[ \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \sqrt{m} \left( \gamma^0 u_\lambda(-p) a_\lambda(-p)e^{-ip \cdot x} + \gamma^0 v_\lambda(-p) b^\dagger_\lambda(-p)e^{+ip \cdot x} \right) \right] = \nonumber
\]

\[
e^{io} \left[ \sum_\lambda \int \frac{d^3p}{(2\pi)^3} \sqrt{m} \left( -iu_{-\lambda}(p) a_{-\lambda}(-p)e^{-ip \cdot x} + iv_{-\lambda}(p) b^\dagger_{-\lambda}(-p)e^{+ip \cdot x} \right) \right] .
\]

(34)

Multiplying by \( u_\lambda(p) \) and \( v_\lambda(p) \) consequently, and using the normalization conditions we obtain

\[
U_P a_\lambda U_P^{-1} = -ie^{io} a_{-\lambda}(-p) ,
\]

(35a)

\[
U_P b^\dagger_\lambda U_P^{-1} = +ie^{io} b^\dagger_{-\lambda}(-p) .
\]

(35b)

From this, if \( \alpha = \pi/2 \) we obtain opposite parity properties of creation/annihilation operators for particles and anti-particles:

\[\[\]

\[\]

\[7T \text{ is chosen to be } T = \begin{pmatrix} \Theta & 0 \\ 0 & -\Theta \end{pmatrix} \text{ in order to fulfill } T^{-1} \gamma_0^T T = \gamma_0; \ T^{-1} \gamma_i^T T = \gamma_i \text{ and } T^T = -T.\]
However, the difference with the Dirac case still preserves ($\lambda$ transforms to $-\lambda$). As a conclusion, the question of the same (opposite) relative intrinsic parity is intrinsically related to the phase factor in (33). We find somewhat similar situation with the question of constructing the neutrino field operator (cf. with the Goldhaber-Kayser creation phase factor).

Next, we find the explicit form of the parity operator $U_P$ and prove that it commutes with the Hamiltonian operator. We prefer to use the method described in [18, §10.2-10.3]. It is based on the anzatz that $U_P = \exp[i\alpha \hat{A}] \exp[i\hat{B}]$ with $\hat{A} = \sum_s \int d^3p [a_{p,s}^\dagger a_{-p,s} + b_{p,s}^\dagger b_{-p,s}]$ and $\hat{B} = \sum_s \int d^3p [\beta a_{p,s}^\dagger a_{p,s} + \gamma b_{p,s}^\dagger b_{p,s}]$. On using the known operator identity

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \ldots$$

and $[\hat{A}, \hat{B} \hat{C}]_+ = [\hat{A}, \hat{B}]_+ \hat{C} - \hat{B} [\hat{A}, \hat{C}]_+$ one can fix the parameters $\alpha, \beta, \gamma$ such that satisfy the physical requirements that a Dirac particle and its anti-particle have opposite intrinsic parities.

In our case, we need to satisfy (27a), i.e., the operator should invert not only the sign of the momentum, but the sign of the helicity too. We may achieve this goal by the analogous postulate $U_P = e^{i\alpha \hat{A}}$ with

$$\hat{A} = \sum_s \int \frac{d^3p}{2E} [a_{\lambda}^\dagger(p)a_{-\lambda}(-p) + b_{\lambda}^\dagger(p)b_{-\lambda}(-p)].$$

By direct verification, the equations (27a) are satisfied provided that $\alpha = \pi/2$. Cf. this parity operator with that given in [17,18] for Dirac fields:

$$U_P = \exp \left[ i\frac{\pi}{2} \int d^3p \sum_s \left( a(p, s)^\dagger a(\tilde{p}, s) + b(p, s)^\dagger b(\tilde{p}, s) - a(p, s)^\dagger a(\tilde{p}, s) + d(\tilde{p}, s)^\dagger d(p, s) \right) \right], \quad (10.69) \text{ of ref. [18].}$$

By direct verification one can also come to the conclusion that our new $U_P$ commutes with the Hamiltonian:

$$\mathcal{H} = \int d^3x \Theta^{00} = \int d^3k \sum_{\lambda} \left[ a_{\lambda}^\dagger(k)a_{\lambda}(k) - b_{\lambda}(k)b_{\lambda}^\dagger(k) \right],$$

i.e.

\[\text{(40)}\]

Greiner used the following commutation relations $[a(p, s), a^\dagger(p', s')]_+ = [b(p, s), b^\dagger(p', s')]_+ = \delta^3(p - p')\delta_{ss'}$. One should also note that the Greiner form of the parity operator is not the only one. Itzykson and Zuber [17] proposed another one differing by the phase factors from (10.69) of [18]. In order to find relations between those two forms of the parity operator one should apply additional rotation in the Fock space.
\[ [U_p, \mathcal{H}]_- = 0. \] (41)

Alternatively, we can try to choose another set of commutation relations [2b,3] (for the set of bi-orthonormal states), that will be the matter of future publications.

Finally, due to the fact that my recent works are related to the so-called “Bargmann-Wightman-Wigner-type” quantum field theory, I want to clarify some misunderstandings in the recent discussions. This type of theories has been first proposed by Gel’fand and Tsetlin [19a]. In fact, it is based on the two-dimensional representation of the inversion group, which is used when someone needs to construct a theory where \( C \) and \( P \) anticommute. They indicated applicability of this theory to the description of the set of \( K \)-mesons and possible relations to the Lee-Yang result. The commutativity/anticommutativity of the discrete symmetry operations has also been investigated by Foldy and Nigam [20]. Relations of the Gel’fand-Tsetlin construct to the representations of the anti-de Sitter \( SO(3, 2) \) group and the general relativity theory (including continuous and discrete transformations) have been discussed in [19b] and in subsequent papers of Sokolik. E. Wigner [21] presented somewhat related results at the Istanbul School on Theoretical Physics in 1962. Later, Fushchich discussed corresponding wave equations. At last, in the paper [22] the authors called a theory where a boson and its antiboson have opposite intrinsic parities as the theory of “the Bargmann-Wightman-Wigner type”. Actually, the theory presented by Ahluwalia, Goldman and Johnson is the Dirac-like generalization of the Weinberg \( 2(2J + 1) \)-theory for the spin 1. It has already been presented in the Sankaranarayanan and Good paper of 1965, ref. [23]. In ref. [22b] (and in the previous IF-UNAM preprints of 1994) I presented a theory based on a set of 6-component Weinberg-like equations (I called them the “Weinberg doubles”). In ref. [2b] the theory in the \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) representation based on the chiral helicity 4-eigenspinors was proposed. The connection with the Foldy and Nigam consideration has been claimed. The corresponding equations have been obtained in [3] and in several less known articles. However, later we found the papers by Ziino and Barut [1] and the Markov papers [24], which also have connections with the subject under consideration.

A similar theory may be constructed from our consideration above if we define the field operators as follows:

\[
\Psi_1 = \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{m}}{2E} \left[ (u_{\uparrow} a_{\uparrow} + v_{\uparrow} b_{\downarrow}) e^{-ip_{\mu} x^{\mu}} + (u_{\uparrow} a_{\uparrow}^\dagger + v_{\uparrow} b_{\downarrow}^\dagger) e^{+ip_{\mu} x^{\mu}} \right],
\]

\[
\Psi_2 = \int \frac{d^3p}{(2\pi)^3} \frac{\sqrt{m}}{2E} \left[ (u_{\downarrow} a_{\downarrow} - v_{\downarrow} b_{\uparrow}) e^{-ip_{\mu} x^{\mu}} + (u_{\downarrow} a_{\downarrow}^\dagger - v_{\downarrow} b_{\uparrow}^\dagger) e^{+ip_{\mu} x^{\mu}} \right].
\]

The conclusions of my talk are:

- Similarly to the \( (\frac{1}{2}, \frac{1}{2}) \) representation, the \( (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) field functions in the helicity basis are not eigenstates of the common-used parity operator; \( |p, \lambda > \Rightarrow |-p, -\lambda > \) both on the classical and quantum levels. This is in accordance with the earlier consideration of Berestetskii, Lifshitz and Pitaevskii.

- Helicity field functions may satisfy the ordinary Dirac equation with \( \gamma \)'s to be in the spinorial representation. Meanwhile, the chiral helicity field functions satisfy the equations of the form \( \hat{\rho} \Psi_1 - m \Psi_2 = 0 \).
Helicity field functions can be expanded in the set of the Dirac 4-spinors by means of the matrix $\mathcal{U}^{-1}$ given in this paper. Neither $A$, nor $B$ are unitary, however $A^\dagger A + B^\dagger B = 1$.

$P$ and $C$ operations anticommute in this framework, both on the classical and quantum levels (this is opposite to the theory based on the chiral helicity eigenstates [3].

Particle and antiparticle may have either the same or the opposite properties with respect to parity. The answer depends on the choice of the phase factor in $U_P \Psi U_P^{-1} = e^{i\alpha \gamma^0} \Psi'$; alternatively, that can be made by additional rotation $U_{P_2}$.

Earlier confusions in the discussion of the Gelfand-Tsetlin-Sokolik-Nigam-Foldy-Bargmann-Wightman-Wigner-type (GTsS-NF-BWW) quantum field theory have been clarified.

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