1. Introduction

Correlations in thermodynamic systems can arise for a variety of reasons. Examples include systems with intrinsic length or time scales (e.g. colloidal dispersions, polymers or biological fluids) or those in which large-scale fluctuations emerge due to the occurrence of second order phase transitions of various nature. Spatial confinement of these systems can lead to novel physical phenomena affecting the behavior of soft matter, ranging from entropic or depletion forces in polymer systems to thermodynamic (or critical) Casimir forces mediated by near-critical fluids. These forces are the classical analogue of the quantum Casimir forces which occur when the fluctuations of the quantized electromagnetic field are spatially confined [1–4].

Properties of confined systems have been studied extensively, primarily at thermodynamic equilibrium. In this setting, the possible emergence of collective behaviours, with the associated universality and scaling laws, allows the introduction of simplified minimal models which form the basis of theoretical descriptions applicable to a large class of physical systems, belonging to the same so-called universality class. Examples of these studies include finite-size scaling, wetting phenomena [5], effective forces in critical films with various homogeneous [6], rough [7] or chemically patterned surfaces [8–11], as well as many-body effects [12].
These effective forces have recently been the subject of very detailed experimental investigations, both in the case of quantum [1] and critical [4, 13] confined fluctuations. Such fluctuation-induced interactions facilitate the experimental manipulation of colloidal aggregations through correlated solvents, as they provide an exquisite experimental control over the range and magnitude of colloidal interactions through temperature changes [14–19], which may find practical applications in soft matter and beyond [20].

Correlated systems also exhibit non-trivial non-equilibrium properties. In the bulk, various transport coefficients of fluid media, such as the viscosity, are sensitive to the occurrence of phase transitions [21]. Phase separation, in turn, can be affected by the presence of strong shear [22, 23]. Due to the subtle interplay between fluctuations at different time- and length scales, spatially confined systems are expected to display a wealth of additional phenomena out of equilibrium in addition to interesting dynamical properties at equilibrium. Some of them have been investigated for film geometries, where time-dependent linear response and correlation functions were studied for a fluctuating medium with purely dissipative dynamics (the so-called Model A [24]) in [25, 26], while the behavior of critical Casimir forces away from equilibrium was investigated in [26–29]; drag forces, instead, were studied for inclusions moving within a medium [29, 30]. Non-equilibrium fluctuations arising from conservation laws have also been demonstrated to lead to Casimir forces far from criticality [31–33]. In this context the role of the stress tensor has been discussed in detail [27, 34, 35]. State-of-the-art experimental methods now allow extremely fine and accurate measurements of the dynamic properties of fluid systems not only in the bulk (e.g. the viscosity of helium near the critical point [36]), but also in confinement, e.g. the frictional parameters for confined complex fluids [37] and the dynamic shear viscosity of colloidal suspensions with varying shear [38, 39].

Theoretically, the behavior of shear fluids in confinement away from phase transitions (i.e. in the absence of long-range correlations within the fluid solvent) has been studied with various approaches [40–43]. However, to our knowledge, the combined effects of shear and correlated fluctuations on the physical properties of a confined system have not yet been investigated, in spite of the fact that they are within the reach of current experimental techniques.

In this work we study the viscosity of a confined, correlated fluctuating medium, such as a fluid, via linear response theory. In order to highlight the relevant effects of correlated fluctuations, we focus on a simple dynamical model, which is a modified version of the so-called dynamical Model H [24] for binary liquid mixtures. We analyze the effective viscosity $\eta_{\text{eff}}$ of the fluid as a function of the bulk correlation length $\xi$ of the fluctuations of its order parameter and as a function of the separation $L$ of the parallel surfaces which confine the medium within a film. Specifically, we shall assume that the dynamics of the order parameter is dissipative, coupled to a hydrodynamic flow field obeying Stokes equation. While the resulting stress and viscosity from this model are cutoff-dependent, i.e. they depend on microscopic parameters of the fluid, their dependence on the correlation length $\xi$ is to some extent universal. The viscosity $\eta_{\text{eff}}$ turns out to depend on the ratio between two length scales, i.e. the one determined by the dynamical parameters of the model and the film thickness $L$. Further scaling appears through a dimensionless function of $L/\xi$.

The rest of the presentation is organized as follows: in section 2 we introduce the fluid system under investigation. In section 3 we set out the dynamical model we focus on, which accounts for the coupling to hydrodynamics as in Model H. We show that the model satisfies the so-called potential conditions, which ensure that, in the absence of shear, the system relaxes to the correct equilibrium distribution. We then discuss the use of the stress tensor in non-equilibrium conditions. In section 4, we derive Green–Kubo relations for the system in confinement, which allow us to define the viscosity of the confined fluid. Then we show that a non-local viscosity naturally arises as a consequence of the correlated fluctuations. Section 5 presents explicit expressions of the viscosity for a Gaussian Hamiltonian, with the main prediction given in equation (68). We close with a discussion of our results and an outlook in section 6.

2. The system

We consider here a soft system, such as a fluid, with a correlation length $\xi$ characterizing the spatial correlation of fluctuations [21, 44]. The system is spatially confined along the $z$-direction by two planar plates positioned at $z = \pm L/2$, so that the medium occupies the domain $z \in D = [-L/2, L/2]$, being infinitely extended along the $x$-$y$ plane, as shown in figure 1. We assume that any microscopic or molecular length scale $a$ of the fluid is small compared to both $\xi$ and $L$. These conditions are met, for example, in fluids close to second-order phase transitions. In order to investigate the effect of shear, the confining plates are set into relative motion along the $x$ direction with velocities $\pm v^*$. Here we assume $v^*$ to be time-independent, and the system to have reached a steady state. The bare shear rate is denoted by $\dot{\gamma}_0 \equiv 2|v^*|/L$. It is also assumed that $v^*$ is sufficiently small so that non-linear effects in the fluid response to shear are

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**Figure 1.** Sketch of a fluid characterized by fluctuations (represented by wiggly lines) with bulk correlation length $\xi$, confined within a film of thickness $L$ (and large transverse area $A$) subject to the shear induced by the velocity $v^*$ imposed to the boundaries, as indicated by the red arrows. The force necessary to keep these surfaces in steady motion defines a distance-dependent effective viscosity, which we investigate here.
negligible. In other words, we focus on the linear response regime, in order to keep the discussion as simple as possible. Within this setting, the main quantity of interest is the force (per area $A$) which has to act on the plates in order to keep them in steady motion at the given velocities $\pm v^*$, and the associated effective viscosity

$$\eta_{\text{eff}} = \frac{|\Delta F_t|}{A v^*},$$

where $\Delta F_t$ is the net force difference between the two plates. $\Delta F_t$ may also be seen as the force necessary to move the upper plate with velocity $2v^*$, if the lower is at rest. In case the correlation length is small compared to $L$, $\eta_{\text{eff}}$ is expected to approach the bulk viscosity of the fluid. For larger $\xi$, it is natural to expect that $\eta_{\text{eff}}$ will eventually develop a dependence on $L$. In order to study the dependence of $\eta_{\text{eff}}$ on $L$ and $\xi$, the first step is to determine the (coarse-grained) velocity profile $v_c(z)$ for a given $\xi$, which will be shown to be the solution of a non-local Stokes equation in section 4.4. From it, $\eta_{\text{eff}}$ can finally be determined.

### 3. A coarse-grained model for correlated fluids

This section outlines the model and discusses its various ingredients. First we discuss the coarse-grained description of the fluid which accounts for the relevant fluctuation phenomena in terms of an order parameter, introducing a suitable (simplified) dynamics and commenting on the role of the dynamical conservation laws. We then address how this dynamics may be coupled to a hydrodynamic velocity field, which is an essential ingredient of fluid systems and is necessary in order to describe shear forces acting on the plates in figure 1 as detailed in section 3.4.

#### 3.1. Models for statics and dynamics

In order to explore how correlated fluctuations influence the viscosity under confinement, we consider a minimal model in which the relevant physical properties of the system can be effectively described in terms of an order parameter $\phi(r,t)$ and of the corresponding Landau–Ginzburg effective free energy $\mathcal{H}[\phi]$. The latter controls the equilibrium distribution $P[\phi] \propto \exp(-\mathcal{H}[\phi]/k_b T)$ of the fluctuations of $\phi$ at temperature $T$, where $k_b$ is Boltzmann’s constant. The effective free energy $\mathcal{H}[\phi]$ arises from coarse-graining to length scales which are large compared to the microscopic scale $a$, which yields a continuum description in terms of the field $\phi(r,t)$. The form of $\mathcal{H}[\phi]$ is obtained from an expansion in powers of the order parameter and its derivatives, ordered according to their relevance (in the sense of renormalization-group theory, see, e.g. [44]), which describes the relevant physical properties of the system if the spatial correlation length $\xi$ is macroscopically large.

$$\mathcal{H}[\phi] = \int \mathrm{d}r \ H[\phi],$$

with

$$H[\phi] = \frac{1}{2} (\nabla \phi)^2 + \frac{r}{2} \phi^2(r) + \frac{u}{4!} \phi^4(r) + \ldots,$$  \hspace{1cm} (3)

where $u > 0$. The parameter $r$ controls the correlation length $\xi$, which in the absence of confinement, formally diverges as $r$ approaches a critical value $r_c$. This kind of effective Hamiltonian $\mathcal{H}[\phi]$ captures universal phenomena associated with second-order phase transitions in a vast class of physical systems, see, e.g. [21, 24, 44]. In these cases, one assumes a simple temperature dependence of $r$ so that $r$ approaches $r_c$ when $T$ approaches the critical temperature $T_c$ of the phase transition. An example is the demixing transition of binary mixtures of fluids, in which the order parameter field $\phi$ is proportional to the deviation of the local concentration of one of the two components from the bulk concentration [21].

The quantitative description of critical phenomena for $T \to T_c$ requires considering the effects of non-Gaussian fluctuations of $\phi$, i.e. $\eta \to 0$ in equation (3). However, for our purposes, interesting qualitative effects emerge already in the simpler case of Gaussian fluctuations with $\eta = 0$, which we focus on below. Correspondingly, $r_c = 0$ and the (bulk) correlation length $\xi$ is related to $r = 0$ in equation (3) as $\xi = r^{-1/2}$. Boundaries like the two plates in figure 1 can either be described by introducing surface terms in the Hamiltonian $\mathcal{H}[\phi]$ or by imposing boundary conditions on $\phi$ [45]. As we focus below on the case $\eta = 0$, the latter approach is more convenient, and therefore we assume that these boundary conditions are of the Dirichlet type, i.e. $\phi = 0$ at the surfaces. For a binary liquid mixture, for instance, these can be effectively realized by considering chemically patterned substrates [10].

Since the effective free energy in equation (2) results from integrating out microscopic degrees of freedom corresponding to a certain mesoscopic configuration of $\phi$, there is no direct recipe (such as Hamilton’s equations of motion of classical mechanics) to obtain the system’s dynamics. Accordingly, an effective dynamical description is typically formulated in terms of Langevin equations for $\phi$ [21, 24, 44],

$$\partial_t \phi(r,t) = -\mu \frac{\delta \mathcal{H}}{\delta \phi} + \theta(r,t),$$  \hspace{1cm} (4)

where $\mu$ is a mobility (operator), which relates the field force $\delta \mathcal{H}/\delta \phi$ to the time derivative of $\phi$. The stochastic force $\theta(r,t)$ is chosen such that equation (4) fulfills the fluctuation-dissipation theorem, and the equilibrium fluctuations resulting from equation (4) are distributed according to $P[\phi]$, as discussed above. In the simplest case, $\theta(r,t)$ is expected to have short-ranged correlations in both time and space, i.e.

$$\langle \theta(r,t) \theta(r',t') \rangle = 2 \mu k_b T \delta^{(3)}(r-r') \delta(t-t').$$  \hspace{1cm} (5)

Depending on the conservation laws for the field $\phi$ in equation (4), one distinguishes two important classes of dynamical models: (a) in the absence of a local conservation of $\phi$, i.e. with purely dissipative dynamics (Model A in the notion of [24]), $\mu = \mu = \text{const}$; (b) if, instead, $\phi$ satisfies a local continuity equation $\partial_t \phi(r,t) + \nabla \cdot J_\phi(r,t) = 0$ with a suitable stochastic current $J_\phi(r,t)$, then $\phi$ is conserved (Model B in the notion of [24]) and $\mu = -\mu \nabla^2$. These two models have
been studied extensively for bulk systems in the literature, and, to a lesser extent, in the presence of surfaces. In particular, Model B dynamics in a semi-infinite geometry is discussed in [46–49]. Investigation of dynamical properties near criticality (e.g. of the relaxation towards equilibrium) within a film geometry appears to have been limited to dissipative dynamics (see, e.g. [25–27, 34]), due to the additional complexity of boundary conditions arising from the higher derivatives in Model B. The dynamics specified by equations (4) and (5) does not account for the presence of velocity fields, i.e. hydrodynamics, which is an essential ingredient when describing shear. We will introduce it in the next section.

3.2. Model H: coupling to hydrodynamics

A minimal model which captures the coupling of the order parameter \( \phi(\mathbf{r}, t) \) to hydrodynamics is the so-called Model H [21, 24, 44]. In particular, \( \phi \) is coupled with the velocity field \( \mathbf{v}(\mathbf{r}, t) \) of the fluid medium, such that equation (4) is supplemented by a typical advection term,

\[
\partial_t \phi(\mathbf{r}, t) = -\nabla \cdot (\phi \mathbf{v}) - \frac{\mu}{\sigma} \frac{\delta H}{\delta \phi} + \theta(\mathbf{r}, t). \tag{6}
\]

In turn, the dynamics of the velocity field \( \mathbf{v}(\mathbf{r}, t) \) follows the Stokes equation of a Newtonian, incompressible fluid with hydrodynamic viscosity \( \eta_h \) which is affected by the presence of \( \phi \),

\[
\rho \partial_t \mathbf{v}(\mathbf{r}, t) = -\nabla \cdot \left( \rho \mathbf{v} - \frac{\delta H}{\delta \phi} \right)_{\perp} + \eta_h \nabla^2 \mathbf{v} + \mathbf{\zeta}(\mathbf{r}, t). \tag{7}
\]

In equations (6) and (7), \( \mu / \sigma \) is the mobility coefficient from section 3.1 and \( \rho \) is the mass density of the fluid. As the fluid is assumed to be incompressible, \( \nabla \cdot \mathbf{v}(\mathbf{r}, t) = 0 \), \( \rho \) is spatially homogeneous. The subscript \( \perp \) in equation (7) indicates a projection onto the transverse component of the wave vector in Fourier space [21], which renders all the terms in equation (7) divergence-free. In the geometrical setup of shear, as considered further below, and within the approximations we shall introduce in section 4, this projection will be automatically implemented.

As before, the stochastic forces in equations (6) and (7) are chosen to satisfy the fluctuation-dissipation theorem, i.e. additionally to equation (5),

\[
\langle \mathbf{\zeta}(\mathbf{r}, t) \mathbf{\zeta}(\mathbf{r}', t') \rangle = -2 \eta_h k_B T \delta_{\mathbf{r} \mathbf{r}'} \times \nabla^2 \delta \Omega(\mathbf{r} - \mathbf{r}') \delta(t - t'). \tag{8}
\]

This choice ensures that the probability distribution of the variables \( \mathbf{v} \) and \( \phi \) relaxes to the Boltzmann equilibrium distribution \( \mathcal{P}_b[\phi, \mathbf{v}] \), where the Hamiltonian in equation (2) picks up an additional kinetic term from \( \mathbf{v} \), i.e.

\[
\mathcal{P}_b[\phi, \mathbf{v}] \propto \exp\left\{ -\beta \left( H[\phi] + \frac{1}{2} \int d^d \mathbf{v} v^2 \right) \right\}. \tag{9}
\]

This non-trivial fact follows from the so-called potential conditions [21] which fix the form of equations (6) and (7). Indeed, upon adding to equation (4) the advection term \( -\nabla \cdot (\phi \mathbf{v}) \) as in equation (6), the first term on the rhs of equation (7) has to be added in order to fulfil these potential conditions; Additional details are provided in appendix A.

Equations (6) and (7) thus describe an order parameter \( \phi \) with either conserved \((\bar{\mu} = -\mu \nabla^2)\) or non-conserved \((\bar{\mu} = \mu = \text{const})\) dynamics which is additionally coupled to a hydrodynamic velocity field \( \mathbf{v} \) and which relaxes towards the equilibrium state in equation (9). It is important to note that the potential conditions hold in both these cases—see appendix A. The former provides, for instance, a suitable description of a binary liquid mixture, since both the order parameter and the fluid momentum are conserved, and is termed Model H in the notion of [24]. The latter, instead, describes a field with purely dissipative dynamics such as colloidal particles carrying Ising-spins (see, e.g. [50–52]), because the corresponding ‘magnetization’ is not necessarily conserved. Equation (7) then describes advection by a flow, mimicking the motion of the solvent. A timely example may be given by ferrofluids, i.e. suspensions of magnetized colloids. Here, the nature of possible phase transitions is still debated [53–56], and driving or shearing might add additional insights. In what follows we refer to the latter model as Model H(A).

Despite the fact that the applicability of this model (as compared to Model H) for describing actual physical systems may be more limited, Model H(A) is amenable to simpler analytic calculations in confined systems, as it does not introduce the additional complications at boundaries (which make Model B dynamics a particularly challenging problem). This is the reason why we focus below on Model H(A).

3.3. Dimensional analysis of model H(A)

Having in mind phenomena appearing on large length scales, it is instructive to investigate the behaviour of the terms in equations (6) and (7) upon coarse-graining space and time. We therefore consider a rescaling transformation [44]

\[
r \to b r, \ t \to b^2 t, \ \phi \to b^\beta \phi, \ \mathbf{v} \to b^\gamma \mathbf{v}. \tag{10}
\]

In \( d \) spatial dimensions, standard dimensional analysis (see, e.g. [44]) yields the exponents \( z = 2, \chi = 1 - d/2 \) and \( \psi = -d/2 \). It follows that the coupling terms \( \nabla \cdot (\phi \mathbf{v}) \) and \( (\phi \nabla \delta H/\delta \phi) \), in equations (6) and (7), respectively, scale as \( \propto b^z \), with the exponent \( z = 1 - d/2 \). Under such coarse-graining transformations, the coupling terms therefore become increasingly irrelevant for \( d > 2 \). However, as the effect investigated below depends on the presence of such couplings, they cannot be set to zero from the outset.

In the considered setup with external shear-driving, we can estimate the importance of the mean value \( \propto s_0 z \) of the velocity, see figure 1 and the corresponding text. Assuming the shear strain not to be affected by the rescaling, we find

\[
s_0 z \to b^{1-z} s_0 z = b^{1-z} s_0 z, \tag{11}
\]

because \( z = b z \) and \( s_0 \) is the time derivative of strain. Accordingly, under rescaling, \( s_0 z \) will be more relevant than the fluctuations of \( \mathbf{v} \) (recall \( \psi = -z/2 \) in equation (10)) in \( d = 3 \), and remains relevant in equation (6). This insight will be used below in section 4 in order to simplify our analysis.
3.4. Stress tensor and forces

The effective Hamiltonian in equation (2) is designed to yield the (free) energy of a certain configuration of the mesoscopic field \( \phi \) in equilibrium. Away from equilibrium, instead, additional assumptions have to be made, e.g. that the microscopic degrees of freedom remain (instantaneously) equilibrated and that they exert the same force as they would do in equilibrium (e.g. on boundaries or inclusions). In equilibrium, forces such as pressure or fluctuation-induced forces acting on immersed objects, can equivalently be calculated from the free energy or via the stress tensor. However, out of equilibrium [26] additional conceptual issues arise regarding use of the stress tensor [27].

The effective viscosity in the setup described in figure 1 is given in terms of \( \Delta F_s \). As the surfaces in figure 1 are smooth so that the boundary conditions are translationally invariant, the Hamiltonian in equation (2) is invariant under translations of either plate along \( x \). Accordingly, the derivative with respect to a (marked) position \( x_p \) along one of the two surfaces vanishes identically at all times, i.e. \( \partial H(\phi)/\partial x_p = 0 \) and the order parameter field \( \phi \), together with its Hamiltonian (2), can not exert any force along \( x \) on the plates.

Shear forces are therefore solely transported through the coupled fluid in Model H and H(A). The term \( \phi \nabla \delta H/\delta \phi \) in equation (7) (due to the potential conditions, see appendix A) can be identified with the stress tensor \( T^\phi \) corresponding to the field \( \phi \), as set out in [21], i.e.

\[
-\phi \nabla \frac{\delta H}{\delta \phi} = - \nabla \cdot T^\phi.
\]

(12)

At equilibrium, \( T^\phi = 0 \). However, this has no bearing on the off-diagonal components \( T_{xz} \) which we require here in order to study the effective viscosity [21]. Therefore the possible issues mentioned at the beginning of the section are inconsequential. We may thus reformulate equation (7) as

\[
\rho \partial_t \mathbf{v}(r, t) = - (\nabla \cdot T^\phi)_z + \eta_0 \nabla \mathbf{v} + \zeta(r, t).
\]

(13)

The physical interpretation of this equation is now apparent, as the evolution of the velocity field is due to both hydrodynamic viscous (stress-) forces \( \sim \nabla^2 \mathbf{v} \) and (stress-) forces arizing from the order parameter field \( \phi \) via \( T^\phi \). The influence of the moving plates will therefore be transported by the velocity field \( \mathbf{v} \), which we assume to obey stick boundary conditions at the walls, which thereby play the role of an anchor for the field \( \phi \) with respect to the moving plates. In a steady state, equation (13) represents a force balance between viscous forces from the fluid and stresses from the order parameter field. The force acting on the moving walls can thus be determined unambiguously.

Next, in section 4, we provide the general solution of equations (6) and (7) for small shear velocities within linear response theory.

4. Linear response relations for shear in confined systems

Within the film represented in figure 1, the stationary mean velocity field \( \langle \mathbf{v}(r) \rangle \) is expected to depend only on \( z \), to point in \( \hat{x} \) direction, and to vanish for \( z = 0 \), due to the symmetries of the problem,

\[
\langle \mathbf{v}(r) \rangle = \delta \langle \mathbf{v}(z) \rangle,
\]

(14)

where \( \langle \cdot \rangle _0 \) indicates the average calculated in the steady state. We also introduce the shear rate,

\[
s(z) = d \langle \mathbf{v}(z) \rangle /dz,
\]

(15)

which, in contrast to bulk shear, may depend on \( z \) in this inhomogeneous system. The boundary conditions for \( \mathbf{v} \) can be expressed as

\[
\mathbf{v}(z = \pm L/2) = \int \frac{dL/2}{d z'} s(z') = \pm v^*.
\]

(16)

where the reference frame is chosen such that \( \langle \mathbf{v}(0) \rangle = 0 \), with symmetry \( \langle \mathbf{v}(z) \rangle = - \langle \mathbf{v}(z) \rangle \). Therefore the stress tensor

\[
-\partial_t \left\langle T^\phi_z(z) \right\rangle + \eta_0 \partial_z^2 \left\langle \mathbf{v}(z) \right\rangle = 0.
\]

(17)

Note that equations (6) and (13) introduce a correlation between the fluctuations of \( \phi \) and those of \( \mathbf{v} \). The fluctuations of \( \mathbf{v} \) are the subject of fluctuating hydrodynamics (see, e.g. [57]), and follow \( \langle (\mathbf{v} - \langle \mathbf{v} \rangle)_z \rangle \sim k_B T/(2 \rho d^3) \), where \( d \) is the considered length scale. It is thus an accepted observation that the velocity fluctuations become increasingly irrelevant upon coarse-graining, relative to macroscopic driving such as shear. As stated in section 3.3, this may be seen also upon coarse-graining equations (6) and (7), including external shear. The remnant of the fluctuations of \( \mathbf{v} \) upon coarse-graining are—via the fluctuation dissipation theorem—hydrodynamic interactions, e.g. acting between the colloidal particles mentioned above [58]. Neglecting these, we thus omit in the following the first term in equation (7) (the time derivative) as well as the last term due to the noise.

Accordingly, the stress tensor \( T^\phi \), a functional \( T^\phi_0(z)\mathbf{v} \) of \( \mathbf{v} \) (via equation (6)), can be found from treating \( \mathbf{v} \) as a given input to equation (6),

\[
\left\langle T^\phi_0(z) \right\rangle = \left\langle T^\phi_0(z)\mathbf{v} \right\rangle = \left\langle T^\phi_0(z) \right\rangle \left\langle \mathbf{v} \right\rangle.
\]

(18)

where the square brackets indicate a functional dependence: the stress tensor at position \( z \) depends on the velocity profile. We will in the following omit the use of \( \langle \cdot \rangle _0 \) when referring to the velocity field for ease of notation. Also note that the transverse projection indicated by the subscript \( \perp \) in equation (13) is unnecessary for discussing the mean shear velocity profile.

In the remaining part of this section, we aim at determining \( \left\langle T^\phi_0(z) \right\rangle _{\perp} \) for a given velocity field. We start with a brief review
of linear response theory for a Fokker Planck equation in section 4.1. In section 4.2, as a reference, we apply it to the case of Brownian particles. Then, in section 4.3, we adapt this formalism to the case of the fluctuating fields as in equation (6).

4.1. Linear response for a Fokker–Planck equation

Consider a generic system characterized by phase-space variables \( \Gamma \) (below, \( \Gamma \) will stand for the positions of Brownian particles, or for the values of the field \( \phi(r) \)), and a time-dependent probability distribution \( \mathcal{P}(\Gamma, t) \). This probability obeys a Fokker–Planck equation

\[
\partial_t \mathcal{P} = \mathcal{L}(\Gamma, t) \mathcal{P},
\]

(19)

with a Fokker–Planck operator \( \mathcal{L}(\Gamma, t) = \mathcal{L}_e(\Gamma) + \delta \mathcal{L}(\Gamma, t) \), containing an equilibrium part \( \mathcal{L}_e \) and a perturbation \( \delta \mathcal{L} \), which may be time-dependent. In the absence of perturbations (i.e. with \( \delta \mathcal{L} = 0 \)), the equilibrium distribution \( \mathcal{P}_e \) obeys \( \mathcal{L}_e \mathcal{P}_e = 0 \), and the time-dependent correlation function of two phase space observables \( f(\Gamma) \) and \( g(\Gamma) \) can be written (see, e.g. [59] for the derivation),

\[
\langle f(t) g(0) \rangle_e = \int \! \! \! d\Gamma' e^{\mathcal{L}_e \tau} g(\Gamma') \mathcal{P}_e(\Gamma),
\]

(20)

Here, \( \langle \ldots \rangle \) indicates the average over the equilibrium distribution. Adding a time independent perturbation \( \delta \mathcal{L} \), switched on at \( t = 0 \), the solution of equation (19) is \( \mathcal{P}(\Gamma, t) \), which can rewritten [60] using \( e^{\mathcal{L} \tau} = 1 + \int_0^\infty \! \! \! dt' e^{\mathcal{L} \tau} \mathcal{L} \mathcal{P} \) (where we obtain the steady state \( \mathcal{P}_e(\Gamma) \) by letting \( t \to \infty \), and using \( \mathcal{L}_e \mathcal{P}_e = 0 \),

\[
\mathcal{P}(\Gamma) = \mathcal{P}_e + \int_0^\infty \! \! \! dt' e^{\mathcal{L} \tau} \delta \mathcal{L} \mathcal{P}_e.
\]

(21)

Equation (21) contains the perturbation to any order, and we obtain the linear response, valid for weak perturbations, by approximating \( \mathcal{L} \) with \( \mathcal{L}_e \) in the exponential,

\[
\mathcal{P}(\Gamma) = \mathcal{P}_e + \int_0^\infty \! \! \! dt' e^{\mathcal{L}_e \tau} \delta \mathcal{L} \mathcal{P}_e.
\]

(22)

Accordingly, the steady-state expectation value

\[
\langle \Delta f \rangle_e = \int \! \! \! d\Gamma' \mathcal{L}_e \Delta f(\Gamma') \mathcal{P}_e
\]

(23)

of the induced change \( \Delta f = f - \langle f \rangle_e \) of observable \( f(\Gamma) \) compared to its equilibrium expectation value \( \langle f \rangle_e = \int \! \! \! d\Gamma' f(\Gamma') \mathcal{P}_e \) is straightforwardly written,

\[
\langle \Delta f \rangle_e = \int_0^\infty \! \! \! dt' \int \! \! \! d\Gamma' \mathcal{L}_e \delta \mathcal{L} \mathcal{P}_e.
\]

(24)

The linear response is thus determined by the action of the perturbation \( \delta \mathcal{L} \) on the equilibrium distribution \( \mathcal{P}_e \). Often (e.g. in the case of shear as considered below, or for potential perturbations [61]) the equilibrium distribution is an eigenfunction of \( \delta \mathcal{L} \), i.e. \( \delta \mathcal{L} \mathcal{P}_e = -\partial_{\Gamma} \delta \mathcal{F} \mathcal{P}_e \), and equation (24) reduces to the time-integral of an equilibrium correlation function (see also equation (20)),

\[
\langle \Delta f \rangle_e = \int_0^\infty \! \! \! dt' \int \! \! \! d\Gamma' \mathcal{L}_e \delta \mathcal{F} \mathcal{P}_e.
\]

(25)

where the last equality follows as \( \langle \Delta f(t) \rangle_e = 0 \), and this relation eventually involves only the fluctuations of \( f \) and \( g \).

4.2. Viscosity from a microscopic model

As a reference, we first apply the linear response theory recalled in the previous section to \( N \) over-damped Brownian particles located at the positions \( r_1, \ldots, r_N \) in space, i.e. \( \Gamma = \{ r_i \}, i = 1, \ldots, N \). Here, the equilibrium Fokker–Planck operator \( \mathcal{L}(\Gamma, t) \), reads [59]

\[
\mathcal{L}_e = D_0 \sum_{i=1}^N \nabla_i \cdot (\nabla_i \beta \mathcal{F}^i),
\]

(26)

where \( D_0 \) is the bare diffusion coefficient and \( \mathcal{F}^i = (\mathcal{F}_r^1, \mathcal{F}_r^2, \mathcal{F}_r^3) \) indicates the force acting on particle \( i \). The force is given by \( \mathcal{F}^i = -\nabla U(\Gamma) \), where the potential \( U \) results from both the interparticle interaction and the possible external potentials. (The equilibrium distribution is accordingly given by \( \mathcal{P}_e \mathcal{e}^{-\beta U} \)). The presence of an imposed velocity field \( \mathbf{v}(r) \) of the solvent gives rise to the perturbation [58, 59],

\[
\delta \mathcal{L} = -\sum_{i=1}^N \nabla_i \cdot \mathbf{v}(r_i).
\]

(27)

For a system under shear with \( \mathbf{v} = v(z) \hat{x} \) (as in equation (14)), the linearized steady state distribution is determined by equation (22), i.e. (here we used \( \delta \mathcal{L} \mathcal{P}_e = -\sum_i \nabla_i v(z_i) \mathcal{F}_r^i \mathcal{P}_e \)),

\[
\mathcal{P}_e(\Gamma) = \mathcal{P}_e - \beta \sum_{i=1}^N \int_0^\infty \! \! \! dt' e^{\mathcal{L}_e \tau} v(z_i) \mathcal{F}_r^i \mathcal{P}_e.
\]

(28)

The induced change of observable \( f \) in the steady state, due to the presence of the fluid flow, is therefore, up to linear order, given by equation (25),

\[
\Delta f_e = -\beta \int_0^\infty \! \! \! dt' \left\{ \Delta f(\tau) \sum_{i=1}^N v(z_i) \mathcal{F}_r^i \right\}_e.
\]

(29)

In the case of homogeneous shear [60], with \( v(z) = s_0 z \) and constant shear rate \( s_0 \), the above relation can be written in terms of the potential part (omitting kinetic contributions in the overdamped regime) of the zero wave-vector microscopic stress tensor, \( \mathcal{T}_{zz}^{(0)} = -s_0 \sum_i \zeta_i F_r^3 \). In the present case of inhomogeneous shear with a position-dependent shear rate \( s(z) \), it is natural to define a local stress tensor \( \mathcal{T}_{zz}^{(z)} \) density (compare the text around equation (19) in [60]) involving only particles at height \( z \), i.e.

\[
\mathcal{T}_{zz}^{(z)} = -\frac{1}{2} \sum_{i,j=1}^N (z_i - z_j) F_r^3 \delta(z - z_i).
\]

(30)

where the forces \( F_{zz}^r \) arise from the pair potential \( u \), and where \( k_i = (k_x, k_y) \to 0 \) implies averaging
over co-ordinates \( x \) and \( y \). (We omit the superscript in what follows.) By expanding the \( \tau(z) \) for slowly varying shear rates \( s(z) \), we find from equation (29) to leading order

\[
\Delta f = \beta \int_0^\infty dt \int dz' s(z') (\Delta f(t)T_{xz}(z', 0), 0). \tag{31}
\]

In particular, since \( T_{xz}(z) = 0 \), which follows directly from spatial symmetries, the steady state expectation value of the local stress tensor density at height \( z \) is

\[
T_{xz}(z) = \beta \Lambda \int_0^\infty dt \int dz' s(z') (T_{xz}(z, t) T_{xz}(z', 0), 0), \tag{32}
\]

up to the linear order in the perturbation. \( \Lambda \) is the area arising from averaging over \( x \) and \( y \). Note that the integrand on the rhs of this equation involves the two-time correlation function of the stress tensor. Equation (32) is a generalization of the well-known Green–Kubo relation to the case of systems with spatially dependent shear rate \( s(z) \) (see equation (15)). A similar relationship has been derived in the context of Liouville dynamics [21, 62, 63] for systems subject to slowly varying velocity gradients. This allows one to express the shear viscosity \( \eta \) in terms of correlation functions. In fact, for a constant \( s(z) = s_0 \), on the basis of the definition in equation (1) and since \( T_{xz}^{(0)} = \int dzT_{xz}(z) \), we recover the well-known result [60, 61]

\[
\eta = \frac{T_{xz}^{(0)}}{s_0V} = \frac{\beta}{V} \int_0^\infty dt \int dz' \left( T_{xz}^{(0)}(t) T_{xz}^{(0)}(0) \right), \tag{33}
\]

where \( V \) is the system volume. In the bulk, this is indeed the definition of viscosity, which corresponds exactly to our effective viscosity \( \eta_{eff} \) in equation (1). This is because the force per area acting on the upper plate is given by \( \langle T_{xz}(z = L/2) \rangle_\phi \), so that for a bulk, equation (1) becomes equal to equation (33).

### 4.3. Viscosity of fluctuating fields

Let us now consider equations (6) and (5) for the dynamics of the field \( \phi \) (coupled to \( \nu \)). The advection term in equation (6) with an assigned velocity profile \( \nu \) is treated as a small perturbation to the equilibrium dynamics of \( \phi \).

We first find the Fokker–Planck equation for the probability functional \( P[\phi] \), associated with equations (6) and (5). It takes the form

\[
\partial_t P[\phi] = L^0 P[\phi] = (\hat{L}^0 + \delta L^0) P[\phi], \tag{34}
\]

where the Fokker–Planck operator \( \hat{L}^0 \) is naturally split into a term \( \hat{L}^0_\phi \) corresponding to the dynamics in the absence of advection (see appendix A for details, in particular equation (A.4)), i.e.

\[
\hat{L}^0_\phi P = \int d\nu \frac{\delta}{\delta \nu} \left[ \frac{1}{\beta} \frac{\delta}{\delta \phi} + \frac{\delta}{\delta \phi} \mathcal{H} \right] P, \tag{35}
\]

and a contribution due to advection (see equation (A.5)) related to the non-equilibrium perturbation,

\[
\delta L^0_\phi P = \int d\nu \left[ \nabla \cdot (\nu \phi) \right] \frac{\delta}{\delta \phi} P. \tag{36}
\]

Here, the Boltzmann distribution \( P[\phi] \propto e^{-\beta \mathcal{H}[\phi]} \) associated with the Hamiltonian in equation (2) is the equilibrium distribution of the system with \( \nu = 0 \), and therefore it satisfies the condition \( \hat{L}^0_\phi P[\phi] = 0 \).

Switching on the perturbation due to \( \nu \) at \( t = 0 \), the resulting distribution \( P[\phi](t) \) at a later time is formally determined by \( P[\phi](t) = e^{t \hat{L}^0_\phi + \delta L^0_\phi} \) and a linearization of the evolution operator in the perturbation \( \nu \) renders an expression analogous to equation (22),

\[
P[\phi](t) = P[\phi] + \int_0^t dt' e^{t' \hat{L}^0_\phi + \delta L^0_\phi} \mathcal{O}(\nu^2). \tag{37}
\]

The perturbation acts on \( P_\nu \) as (see appendix A)

\[
\delta L^0_\phi P[\phi] = -\beta \int d\nu \left[ \nabla \cdot (\nu \phi) \right] \frac{\delta \mathcal{H}}{\delta \phi} P[\phi] = \beta \int d\nu \left[ \nabla \cdot (\nu v) \right] P[\phi]. \tag{38}
\]

We identify the stress tensor in equation (12), \( \phi \nabla (\delta \mathcal{H}[\phi]) = \nabla \cdot T^\phi \). Since \( \nu \cdot (\nabla \cdot T^\phi) = 0 \), integration by parts gives

\[
\delta L^0 P_\nu = -\beta \left[ \sum_i \int d\nu \left( \partial_i v_i T^\phi \right) \right] P_\nu = -\beta \int d\nu \left( \nabla \cdot T^\phi \right) P_\nu. \tag{39}
\]

As before, this allows us to calculate the average of an observable \( f \) in the steady state \( (t \to \infty) \),

\[
\langle \Delta f(r) \rangle = -\beta \int_0^\infty dr \int d\nu' \langle \nabla_r \nu(r') \rangle : \langle \Delta f(r, t) T^\phi(r', 0) \rangle_{\nu}, \tag{40}
\]

which is analogous to equation (31). Equation (40) generalizes to the case of a fluctuating field. In the case of a shear velocity profile such as the one in equation (14), the previous equation becomes

\[
\langle \Delta f(r) \rangle = -\beta \int_0^\infty dr' \langle \delta s(z') \rangle \int_0^\infty dt \langle \Delta f(r, t) T^\phi_z(r', 0) \rangle_{\nu}, \tag{41}
\]

The local shear stress required in equation (17) finally follows by replacing \( f \) by \( T^\phi_z \) (we recall that \( T^\phi_z(z) = 0 \)),

\[
\langle T^\phi_z(z) \rangle = -\beta \int dr' \int_0^\infty dz' \times \int_0^\infty dr \langle \delta s(z') \rangle \langle T^\phi_z(r, t) T^\phi_z(r', 0) \rangle_{\nu}, \tag{42}
\]

where \( r = (r, z) \) and analogous decomposition for \( r' \). Note that on the lhs we have omitted the dependence on \( r \), as it disappears in the steady state due to its translational invariance—see section 3.4. Analogously, the correlation function on the rhs depends on \( r = r', z \) and \( z' \). Accordingly, in order to determine the expectation value of the stress tensor in the non-equilibrium stationary state in the presence of a weak fluid flow—which we need in order to compute the effective
viscosity according to equation (17)—we calculate below correlation functions of the stress tensor in equilibrium. Note also that, compared to equation (32), equation (42) involves an additional integral over $r_1$ which comes from the Fokker–Planck operator (see appendix A).

4.4. Non-local Stokes equation

As an interesting observation, we note that equation (7) in the stationary state (or, equivalently, equation (17)) for the geometrical setting of figure 1 may be cast in the form

$$0 = \frac{d}{dz} \int_D dz' \eta(z, z') \frac{d}{dz'} \nu(z'),$$

(43)

where a non-local viscosity kernel $\eta(z, z')$ appears, once the linear response relation in equation (42) has been used together with the definition in equation (15). This equation may be viewed as a non-local Stokes equation; it is a consequence of the fact that equation (7) for $\nu$ has the form of a continuity equation. In turn, equation (43) amounts to requiring that the shear stress $\int_0^\infty dz' \eta(z, z') \nu(z')$ is independent of the coordinate $z$, i.e. that it is constant across the film. In particular, $\eta(z, z')$ takes the explicit form

$$\eta(z, z') = \delta(z - z') \eta_0 + \beta \int_0^\infty dr \int Dz' \langle T_\phi^{zz}(z, t) T_\phi^{zz}(z', 0) \rangle,$$

(44)

which shows that the local contribution $\eta_0$ arising from the pure solvent is effectively modified by a potentially non-local correction due to the coupling to the velocity field in equation (6) and to the correlations of the field $\phi$. In fact, $\eta(z, z')$ in equation (44) does not necessarily vanish for $z \neq z'$. Accordingly, correlated fluctuations of the order parameter are expected to cause non-local effects in the hydrodynamics of the solvent.

5. Effective viscosity of model H(A) in confined geometry

In this section we apply the linear response formalism derived in section 4 to Model H(A) within the Gaussian approximation, as discussed in section 3. This leads to a set of self-consistent equations for the shear rate, as set out in section 5.1. We discuss the exact autocorrelation functions which arise in the model in section 5.2 and present our predictions for the dependence of the effective viscosity $\eta_{\text{eff}}$ on the correlation length $\xi$ in section 5.3.

5.1. Self-consistent equation for the shear rate

The condition of homogeneous, $z$-independent stress imposed by equation (17) (or, equivalently, equation (43) after having used the linear response relation in equation (42)) provides a self-consistent equation for the shear rate $s(z)$ in equation (15), after integration of the equation along $z$ (we already point to the dependence on correlation length $\xi$),

$$s(z) + \frac{1}{\eta_0} \beta \int_D dz' s(z') C(z, z', \xi) = \text{const.},$$

(45)

where we introduced

$$C(z, z', \xi) = \int_0^\infty dr \int Dz' \langle T_\phi^{zz}(z, t) T_\phi^{zz}(z', 0) \rangle.$$  

(46)

Equations (45) and (46) can be solved self-consistently subject to the boundary conditions for $v$ and $\phi$ at the confining surfaces of the film. Note that the constant in equation (45) can be readily identified with $T_{\text{tot}} / \eta_0$, where $T_{\text{tot}}$ is the total stress. It contains the part due to shearing the solvent with viscosity $\eta_0$, and the stress arising from the order parameter field. In terms of it, the effective viscosity $\eta_{\text{eff}}$ defined in equation (1) is eventually given by

$$\eta_{\text{eff}} = \frac{T_{\text{tot}}}{\eta_0}.$$  

(47)

We now discuss the specific model and solve equation (45) for that case.

5.2. Model and stress-tensor autocorrelation function

As anticipated in section 1, our aim is to investigate how confined and correlated fluctuations affect the effective viscosity. We consider a simple model within which correlations occur, namely Model H(A) discussed in section 3.2 with a Gaussian effective Hamiltonian corresponding to equation (3) with $\mu = 0$, i.e.

$$\mathcal{H}[\phi] = \frac{1}{2} \int dr \left[ (\nabla \phi)^2 + z^{-2} \phi^2(r) \right].$$  

(48)

As noted above, we use Dirichlet boundary conditions for $\phi$ at the two walls in figure 1, i.e. $\phi(z = \pm L/2) = 0$ and the stick boundary conditions for $v$ given by equation (16).

In order to calculate $\langle T_\phi^{zz}(z) \rangle$, the linear response relation in equation (42) requires the knowledge of the correlation function of the $xz$-component of the field stress tensor from section 3.4, which, for $\mathcal{H}$ in equation (48), is given by

$$T_\phi^{zz}(r) = \partial_z \phi \partial_z \phi.$$  

(49)

The equilibrium stress-tensor autocorrelation function $C(z, z', \xi)$ in equation (46) required for the calculation of the total stress $T_{\text{tot}}$ from equation (45) thus comprises suitable derivatives of a four-point correlation function of the order parameter field $\phi$. For a Gaussian Hamiltonian such as (48), the latter can be calculated by using Wick’s theorem, as detailed in appendix B.3. In turn, this requires the knowledge of the two-point and two-time correlation function $\langle \phi(r, t) \phi(r', t') \rangle$, of the order parameter in equilibrium within the film of thickness $L$, with Dirichlet boundary conditions at the confining surfaces.

In order to determine the latter, it is convenient to carry out the analysis in Fourier space, where $\mathbf{p}$ is the spatial wave vector conjugate to $\mathbf{r}_1 = (x, y)$, and the frequency $\omega$ results from the Fourier transform in time (which can be introduced because equilibrium correlation functions are time-translational.
invariant). The integral $C$ of the stress correlation function defined in equation (46) turns out to be given by (see appendix B)

$$C^i(z, z', \xi) = \frac{(k_B T)^2}{\mu (2 \pi)^2} \int_0^{2 \pi} \int_0^L d \xi \sqrt{f(\xi, z, \xi', \tilde{\omega}, \tilde{\beta}, \tilde{\xi})}$$

$$= \frac{(k_B T)^2}{\mu (2 \pi)^2} C^i(z, z', \xi),$$

(50)

where $C^i(z, z', \xi)$ is a dimensionless function and $\mu$ is the mobility appearing in the Langevin equation of Model H(A); see the discussion after equation (4). We have introduced a cutoff $\Lambda = 1/a$, set by the microscopic length scale $a$, and have rescaled to dimensionless variables

$$\tilde{\omega} = \omega / \mu, \quad \tilde{\beta} = L \beta, \quad \tilde{\xi} = z / L,$$

(51)

and similarly for the remaining length scales, i.e. $z' = z' / L$, $\tilde{\xi} = \xi / L$, and $\tilde{\Lambda} = \Lambda L$. Additional details as well as the explicit expression of the integrand $f$ are provided in appendix B.4.

The integrals in equation (50) can now be computed numerically as functions of $z$, $z' \in D = [-1/2, 1/2]$ for various values of the dimensionless cutoff $\tilde{\Lambda}$, indicated by the superscript. The function $C^i(z, z', \xi)$, which essentially determines the non-local contribution to the viscosity in equation (43), is indeed non-local for finite $\Lambda$ (and therefore $\tilde{\Lambda}$). However, upon increasing $\tilde{\Lambda}$, $C^i(z, z', \xi)$ becomes increasingly local in that at $\tilde{\xi} = \tilde{\xi}'$ the $\tilde{\beta}$-integral in equation (50) turns out to be ultraviolet divergent, as shown in appendix B.5 (see, in particular, figure B1 therein). A consequence of this divergence is that integrals involving $C^i(z, z', \xi)$ can be expressed in terms of an envelope function for asymptotically large $\tilde{\Lambda}$. In particular, for a smooth function $\psi(\xi)$, we find for $\tilde{\Lambda} \gg 1$,

$$\int_{-1/2}^{1/2} d \xi' \psi(\xi') C^i(z, z', \xi) \overset{\tilde{\Lambda} \gg 1}{=} g^i(\xi, \tilde{\eta}),$$

(52)

where

$$g^i(\xi, \tilde{\eta}) = \int_{-1/2}^{1/2} d \xi' C^i(z, z', \xi).$$

(53)

The envelope function $g^i(\xi, \tilde{\eta})$ takes two different values depending on whether $\xi$ is at the boundaries $\xi = \pm 1/2$ or not, with the latter value scaling linearly with the cutoff $\tilde{\Lambda}$,

$$g^i(\xi, \tilde{\eta}) = \left\{ \begin{array}{ll} \approx 0.39 \tilde{\Lambda} & \text{for } \xi \in (-1/2, 1/2), \\ 0 & \text{for } \xi = \pm 1/2. \end{array} \right.$$

(54)

Equations (52) and (54) hold as long as the correlation length is large compared to the microscopic cutoff, i.e. $\xi \gg \tilde{\Lambda}^{-1}$. Hence we drop the dependence of $g^i$ on $\xi$. (Note that the limit $\tilde{\Lambda} \gg 1$ actually corresponds to considering the fluid confined within a film of large thickness $L \gg a$.) Indeed, this suggests that $\tilde{C}^i(z, z', \xi) \to g^i(\xi, \tilde{\eta}) \delta(z' - z)$ upon increasing $\tilde{\Lambda}$.

A plot of $g^i(\xi, \tilde{\xi} = \infty)$ for various values of $\tilde{\Lambda}$ is reported in figure B2 of appendix B.5, where we also discuss in detail equation (52). We conclude that, in our model, bulk effects dominate the computed viscosity kernel $\eta(z, z')$. Next, we analyze the resulting velocity profile.

Using equation (52), the self-consistent equation (45) has a simple algebraic solution $s^i(\xi)$ for $\tilde{\Lambda} \gg 1$,

$$s^i(\xi) = \tilde{T}_{\text{tot}} / \tilde{\eta}_0 \left[ 1 + \alpha g^i(\xi) \right].$$

(55)

For convenience we introduced here

$$\alpha = \frac{1}{2 \pi^2 \beta \mu \eta_L},$$

(56)

which is a dimensionless parameter arising from rescaling, noise correlators and prefactors in the linear response relation. In fact, from the Langevin equations (6) and (7), combined with equation (48), one infers that $\mu$ has dimensions of $(\text{length})^2 / \text{(time)}$, $\eta_0$ of $(\text{energy} \times \text{time}) / (\text{length})^3$, and therefore

$$\ell \equiv \frac{1}{\beta \mu \eta_0}$$

(57)

is a length scale which makes $\alpha \propto \ell / L$ in equation (56) dimensionless. $\ell$ is related to the mobility of the order parameter field, and can be seen as the effective hydrodynamic radius with respect to $\eta_0$. The total stress $T_{\text{tot}}$ (yet undetermined by this analysis) is fixed by imposing the velocity $v(\xi)$ resulting from the integration of the shear rate $s(\xi)$ in equation (55) (see equation (15)) to be equal to $\pm v^\ast$ at the boundaries (equation (16)), which implies

$$\frac{T_{\text{tot}}}{\eta_0} = \frac{v^\ast}{L} \left[ \int_0^{1/2} d \xi' \left( \frac{1}{1 + \alpha g^i(\xi')} \right) \right]^{-1}.$$  

(58)

For $\tilde{\Lambda} \gg 1$, $T_{\text{tot}} \sim \alpha \tilde{\Lambda}$ due to equation (54), while equation (55) implies that a linear velocity profile is recovered with shear rate

$$s^i(\xi) \overset{\tilde{\Lambda} \gg 1}{=} 2 v^\ast / L = s_0.$$  

(59)

Here $s_0$ is the shear rate for a linear velocity profile with velocities $\pm v^\ast$ at the surfaces; see section 2. This is a consequence of the fact that $\eta(\xi, z')$ is eventually dominated by (local) bulk contributions. The linear increase of $T_{\text{tot}}$ as a function of $\tilde{\Lambda} \gg 1$ carries over to the effective viscosity determined from equation (47), which formally diverges. As stated, this holds also for finite correlation lengths, as long as $\xi \gg \tilde{\Lambda}^{-1}$.

The above analysis of model H(A) under confinement reveals a dependence of the effective viscosity $\eta_{\text{eff}}$ on the cutoff $\Lambda$ for macroscopic correlation lengths $\xi$. Within this simple model, introducing this cutoff is just a convenient way of taking into account the microscopic structure of the system under study. We expand on the cut-off dependence in appendix C. The physical interpretation of this strong dependence of $\eta_{\text{eff}}$ on $\Lambda$ is that the contributions of
fluctuations to this quantity are essentially determined by the material properties of the system under investigation, which the simplified mesoscopic description investigated here is unable to capture in detail.

However, $\eta_{\text{eff}}$ might still display a dependence on the correlation length $\xi$ and eventually on the thickness $L$ of the film within which fluctuations are constrained. In other words, universal (and non-local) aspects may appear in the interplay of the film thickness $L$ and the correlation length $\xi$, as we do further below.

5.3. Dependence of the effective viscosity on the correlation length

In order to distill the effects of large-scale fluctuations, we differentiate equation (58) with respect to the (rescaled) correlation length $\tilde{\xi} = \xi/L$,

$$
\frac{1}{s_0 h_0} \partial_{\tilde{\xi}} T_{\text{tot}} = \frac{\alpha}{\int_0^{1/2} dz [1 + \alpha g^2(\tilde{z})]^{-2}} \partial_{\tilde{\xi}} g^2(\tilde{z})
$$

Since $g^2(\tilde{z})$ becomes spatially constant for large cutoffs—see equation (52)—we are left with computing

$$
\partial_{\tilde{\xi}} g^2(\tilde{z}) = \int_{D} d\tilde{z} \partial_{\tilde{\xi}} C^2(\tilde{z}, z', \tilde{\xi}) \equiv -\frac{1}{\tilde{\xi}^3} h(\tilde{z}, \tilde{\xi}).
$$

This form arises since $\partial_{\tilde{\xi}} C^2(\tilde{z}, z', \tilde{\xi}) = \int_0^{\tilde{L}} d\bar{\varphi} \int d\varphi \partial_{\tilde{\xi}} \bar{f}$, where $\bar{f}$ is the same as in equation (50). (See also equation (B.19) in appendix B.3.) The derivative of $\bar{f}$ with respect to $\tilde{\xi}$ removes the large-$\tilde{\varphi}$ divergence of the corresponding integral, mentioned before equation (52), and therefore this expression has a well-defined limit as $\tilde{L} \to \infty$. In other words, the corresponding functional dependence on $\tilde{L}$ is independent of the microscopic details of the system. Physically, the function $h(\tilde{z}, \tilde{\xi})$ is the integral across the system of the derivative with respect to correlation length $\tilde{\xi}$, as we do further below.

Figure 2. Dependence of the function $h(\tilde{z}, \tilde{\xi})$ (see equation (61)) on the reduced coordinate $\tilde{z}$ across the film (with $|\tilde{z}| \leq 1/2$) at criticality, i.e. for $\tilde{\xi} = \infty$, $h(\tilde{z})$ is related to the derivative of the non-local viscosity $\eta(z, z')$ with respect to correlation length $\xi$ (see, e.g. figure B1) after integration over $z'$ (up to prefactors of $\tilde{\xi}$) and it turns out to become independent of the cutoff $\tilde{L}$ for $\tilde{\xi} \gg 1$.

Figure 3. Dependence of the function $A(\tilde{\xi})$ (see equation (62)), calculated for $\tilde{\xi} = \infty$, on $\tilde{L}$. We show the relative deviation $[0.113 - A(\tilde{\xi} = \infty)]/0.113$ from the asymptotic numerical value 0.113 in equation (63), which is approached as $\tilde{L} \to \infty$.

Figure 4. Dependence of the amplitude $A(\tilde{\xi})$ on $\tilde{\xi}$, calculated for $\tilde{L} = \infty$. Symbols represent numerical data points, and the blue line is a guide to the eye. The function is linear for $\tilde{\xi} \ll 1$ (as indicated by the dashed red line), and approaches a finite value for $\tilde{\xi} \gg 1$. These limits correspond to the case of correlation lengths much shorter and longer than the thickness of the film $L$, respectively.

respect to $\tilde{\xi}$ of $\eta(z, z')$ in the Stokes equation, and is shown in figure 2 as a function of $\tilde{z} \in [-1/2, 1/2]$ for $\tilde{\xi} = \infty$. We have therefore identified the contribution in $\eta(z, z')$, which is sensitive to confinement. Due to the Dirichlet boundary conditions, $\phi(\tilde{z} = \pm 1/2) = 0$, it follows that $h(\tilde{z} = \pm 1/2, \tilde{\xi}) = 0$.

Returning to equation (60), we obtain in the limit $\tilde{L} \to \infty$,

$$
\frac{\partial_{\tilde{\xi}} T_{\text{tot}}}{s_0 h_0} = -\alpha \tilde{\xi}^{-3} \int_0^{1/2} d\tilde{z} h(\tilde{z}, \tilde{\xi}) = \alpha \tilde{\xi}^{-3} A(\tilde{\xi}).
$$

For sufficiently large $\tilde{L}$, $h(\tilde{z}, \tilde{\xi})$ and therefore the amplitude $A(\tilde{\xi})$ is independent of $\tilde{L}$, i.e. of the microscopic details of the system. This is shown in figure 3 for $\tilde{\xi} = \infty$, but holds for all values of the correlation length. The dependence of $A$ on $\tilde{\xi}$, reported in figure 4, is characterized by a marked crossover between the following two limiting behaviors:

$$
A(\tilde{\xi}) \approx 0.113 \begin{cases} 
\tilde{\xi} & \text{for } \tilde{\xi} \ll 1, \\
1 & \text{for } \tilde{\xi} \gg 1. 
\end{cases}
$$

(63)
Through equations (47) and (62), one now finds the derivative of the effective viscosity, defined in equation (1) (not to be confused with the non-local viscosity entering equation (43)),

$$\frac{\partial \eta_{\text{eff}}}{\partial \xi} = \frac{1}{(2\pi)^2} \int_{\tilde{L}}^L d\tilde{L} \frac{\partial \eta_{\text{eff}}}{\partial \xi} = \left\{ \begin{array}{ll} 0.37(\xi_1^{-1} - \xi_2^{-1}), & \xi_1, \xi_2 \ll 1, \\ -1.36 + 0.37\xi_1^{-1} - 0.06\xi_2^{-2}, & \xi_1 \ll 1, \xi_2 \gg 1, \\ 0.06(\xi_1^{-2} - \xi_2^{-3}), & \xi_1, \xi_2 \gg 1. \end{array} \right. \quad (64)$$

This is shown in figure 5 from the full numerical solution of equation (60), where the limits of equation (64) are apparent.

Integrating this derivative between two arbitrarily chosen limits, $\xi_1$ and $\xi_2$, yields

$$\eta_{\text{eff}}(\xi_2) - \eta_{\text{eff}}(\xi_1) = \int_{\xi_1}^{\xi_2} d\xi \frac{\partial \eta_{\text{eff}}}{\partial \xi} \equiv \int_{\xi_1}^{\xi_2} \frac{0.37\xi^{-2}}{(2\pi)^2} d\xi \approx 0.12\xi^{-3}$$

The constant term in the second line comes from integrating across the crossover regime around $\xi = 1$ where the power-law dependence on $\xi$ changes. Choosing an arbitrary lower bound $\xi_0$, at which $\xi^{-1} \ll \xi_0 \ll 1$, we obtain

$$\frac{\eta_{\text{eff}}(L, \xi) - \eta_{\text{eff}}(\xi_0)}{\eta_0} = \frac{\xi}{(2\pi)^2}\left( \begin{array}{ll} 0.37(L/\xi_0 - L/\xi), & \xi \ll L, \\ -1.359 + 0.37L/\xi_0 - 0.06L^2/\xi_0^2, & \xi \gg L, \end{array} \right) \quad (65)$$

from which we can also extract the bulk viscosity $\eta_{\text{eff}}^{\text{bulk}}$ at finite $\xi$.

$$\eta_{\text{eff}}^{\text{bulk}}(\xi) \equiv \eta_{\text{eff}}(L \to \infty, \xi) = \eta_{\text{eff}}(\xi_0) + \eta_0 \frac{0.37\xi}{(2\pi)^2}(\xi_0^{-1} - 1) \quad (67)$$

In particular, the difference between the bulk viscosity and that of a system where $\xi \gg L$ can now be expressed as

$$\frac{\eta_{\text{eff}}^{\text{bulk}}(\xi) - \eta_{\text{eff}}(\xi_0 \gg L)}{\eta_0} = 0.034 \tilde{\xi} \tilde{Y}(\xi) \quad (68)$$

where $\tilde{\xi} = \xi/L$, and

$$\tilde{Y}(\xi) = 1 - 0.27\xi^{-1} + 0.04\xi^{-2} + O(\xi^{-3}). \quad (69)$$

As long as $\xi \ll 0.034\tilde{\xi}$, contributions from the lower integration bound $\xi_0$ cancel in the subtraction. Equation (68) is our main result: The difference considered in equation (68), normalized by $\eta_0$, is proportional to the ratio of the dynamical length scale $\tilde{\xi}$ and the film thickness $L$, and diverges in the limit $L \to 0$ (bearing in mind that $L$ should anyhow be much larger than the microscopic length scale $\lambda^{-1}$). These prefactors are multiplied by a scaling function $Y(\xi)$, which is a function of $\xi$, thus displaying universal properties.

Equation (68) can be tested experimentally or in non-equilibrium molecular dynamics simulations such as those in [64], if the viscosity can be measured at different values of $\xi/L$.

6. Discussion and outlook

We have studied a confined, correlated fluid system driven out of equilibrium. Through linear response theory for inhomogeneous shear, the self-consistent equations for fluid velocity were shown to take the form of a Stokes equation with a non-local viscosity. The effective viscosity of the system, defined in terms of the total stress in the steady state (or, equivalently, of the force necessary to shear the system), is found to be cutoff-dependent in general, as in the case of the corresponding bulk system. Relevant (finite) quantities were identified by taking the derivative of this effective viscosity with respect to correlation length $\xi$. This shows that the change of the viscosity with $\xi$ is independent of microscopic details, and as such is expected to be displayed universally.

Specifically, we studied the case of a non-conserved order parameter coupled to a flowing medium, which could for instance describe systems with magnetic colloids where criticality is triggered magnetically. An interesting future perspective is the case of a conserved order parameter coupled to a flowing medium (Model H), which describes a binary fluid mixture. This analysis, left for future work, is more involved due to the added complexity in Model B dynamics in confined geometries [34, 47, 49]. Such an analysis would provide further insight into the relevant dynamical universality classes. Other avenues to explore include the incorporation of an external field and mixed or inhomogeneous boundary conditions for the film. Regarding response theory, an interesting question is whether an extension beyond the linear regime is possible. Such studies have been done for strong shear in bulk systems [60] using a transient dynamics approach. Future work may also study the present model H(A) via renormalization group theory [21, 24], or consider a more microscopic theory [65].
Appendix A. Potential conditions in model H

In this appendix we show that the potential conditions fix the form of the reversible force density induced by correlations of \( \phi \), and demonstrate that these conditions hold irrespective of whether the dynamics of \( \phi \) is purely dissipative or conserved.

We condense the coupled Langevin equations (6) and (7) as follows:

\[
\begin{aligned}
\partial_t \phi (\mathbf{w}) &= \left( f_0 (\mathbf{w})_t \right) + \left( \frac{- \beta \partial \mathcal{H}}{\partial (\mathbf{w}, \mathbf{r})} \right) + \left( \frac{\mu}{\beta} \right), \\
\end{aligned}
\]  

(A.1)

where \( \mathcal{H} = \mathcal{H}[\phi] + \frac{1}{2} \int d \mathbf{r} \mathbf{w}^2 \) is the full Hamiltonian, and we write the coupling terms as

\[
\begin{aligned}
f_0 &= - \nabla \cdot (\phi \mathbf{w}) = - \mathbf{v} \cdot \nabla \phi \\
f_r &= - \mathbf{v} \cdot T^\phi.
\end{aligned}
\]  

(A.2)

The corresponding Fokker–Planck equation for the joint probability functional \( \mathcal{P}[\phi, \mathbf{v}(t)] \) is then [21, 66]

\[
\partial_t \mathcal{P} = \hat{L} \mathcal{P} \equiv \left( \hat{L}^0 + \hat{L}_e^* + \hat{L}_e^r + \hat{L}_r^* \right) \mathcal{P}.
\]  

(A.3)

The ‘diffusion’ terms (with irreversible contributions) are

\[
\begin{aligned}
\hat{L}_e^0 \mathcal{P} &= \int d \mathbf{r} \frac{\delta}{\delta \phi} \left[ \frac{1}{\beta} \frac{\partial}{\partial \phi} + \frac{\delta \mathcal{H}}{\delta \phi} \right] \mathcal{P}, \\
\end{aligned}
\]  

\[
\begin{aligned}
\hat{L}_e^r \mathcal{P} &= - \int d \mathbf{r} \frac{\delta}{\delta \mathbf{v}} \partial_0 \left[ \frac{1}{\beta} \frac{\partial}{\partial \mathbf{v}} + \frac{\delta \mathcal{H}}{\delta \mathbf{v}} \right] \mathcal{P},
\end{aligned}
\]  

(A.4)

and the ‘streaming’ terms (with reversible contributions) are

\[
\begin{aligned}
\hat{L}_r^0 \mathcal{P} &= - \int d \mathbf{r} \frac{\delta}{\delta \phi} (f_0 \mathcal{P}), \\
\hat{L}_r^r \mathcal{P} &= - \int d \mathbf{r} \frac{\delta}{\delta \mathbf{v}} (f_r \mathcal{P}).
\end{aligned}
\]  

(A.5)

The equilibrium probability distribution (9) must lie in the null space of \( \hat{L} \). (Note that \( \hat{L}_e^0 \mathcal{P} = \hat{L}_e^r \mathcal{P} = 0 \) is trivially satisfied.) This gives rise to the potential conditions on the streaming terms, i.e.

\[
\int d \mathbf{r} \left[ f_0 \frac{\delta}{\delta \phi} \mathcal{H} + f_r \frac{\delta}{\delta \mathbf{v}} \mathcal{H} \right] = \frac{1}{\beta} \int d \mathbf{r} \left[ \frac{\delta}{\delta \phi} f_0 + \frac{\delta}{\delta \mathbf{v}} f_r \right].
\]  

(A.6)

The second term on the rhs of equation (A.6) vanishes since the reversible force density induced by \( \phi \) is independent of \( \mathbf{v} \). The first term, instead, is

\[
\int d \mathbf{r} \frac{\delta}{\delta \phi} f_0 = - \int d \mathbf{r} \frac{\delta}{\delta \phi} [\mathbf{v} \cdot \nabla \phi].
\]

In general,

\[
\frac{\delta}{\delta \phi} \nabla \phi (\mathbf{r}) = - \nabla \delta (\mathbf{r} - \mathbf{r}').
\]

Therefore, since \( - \nabla \delta (0) = 0 \) (justified by considering the delta function as a limit of a Gaussian [27]), the first term is also zero. Accordingly, equation (A.6) implies that \( \int d \mathbf{r} [\mathbf{v} \cdot \nabla \phi] f_0 \cdot \mathbf{v} = 0 \). Since \( \nabla \cdot \mathbf{v} = 0 \), after integration by parts, this condition can be written as \( \int d \mathbf{v} [\mathbf{v} \cdot \nabla \phi] f_0 = 0 \), which must hold for any \( \mathbf{v} \). We conclude that the advection term \( f_0 \) therefore fixes the reversible force density \( f_r \) to be

\[
f_r = - \mathbf{v} \cdot T^\phi. \quad \text{(A.7)}
\]

As discussed in section 3.4, \( f_r \) may also be viewed as the divergence of the stress tensor \( T^\phi \) induced by \( \phi \), i.e.

\[
- \phi \nabla^2 \mathcal{H} = - \nabla \cdot T^\phi.
\]

Importantly, the potential conditions are satisfied for both Model H and our Model H(A), since compliance of the diffusion terms with the fluctuation-dissipation theorem ensures correct relaxation irrespective of conservation laws.

Appendix B. Details of stress tensor calculation

B.1. Green’s functions

For the Gaussian Hamiltonian (48),

\[
\frac{\delta \mathcal{H}}{\delta \phi} = - \nabla^2 \phi + \frac{1}{\xi \phi} \phi (\mathbf{r}, t).
\]  

(B.1)

Correspondingly, for \( \mathbf{v} = 0 \) in the Langevin equation (6) (the equilibrium case), \( \phi \) can be written in terms of Green’s functions,

\[
\phi (\mathbf{r}, t) = \int_V d \mathbf{r}' \int_0^t d t' G (\mathbf{r}, t; \mathbf{r}', t') \phi (\mathbf{r}', t').
\]  

(B.2)

where \( V \) is the volume of the system, and

\[
\int_0^t d t' \left( \nabla^2 - \frac{1}{\xi \phi} \right) G (\mathbf{r}, t; \mathbf{r}', t') = \delta (\mathbf{r} - \mathbf{r}') \delta (t - t').
\]  

(B.3)

\( G \) must then be determined subject to boundary conditions and geometry, allowing for computation of two-point correlation functions of \( \phi \) as required for the linear response calculation. This is discussed in appendix B.2.

B.2. Equilibrium field correlators

According to equations (10)–(12) of [28], the general correlator can be written as

\[
\langle \phi (\mathbf{r}, t) \phi (\mathbf{r}', t') \rangle = 2 \mu k_B T \int d \mathbf{v} e^{-\beta \mathcal{H}} \psi (\mathbf{r}, \mathbf{r}', \mathbf{v}) I (\mathbf{z}, \mathbf{z}'),
\]

(B.4)
where \( \int \int \equiv \int_\Gamma \equiv \pi \omega \int_{-\infty}^{\infty} d\omega d\Omega \). (Recall here the decomposition \( \omega = r r \zeta \), and similarly for \( \omega' \).) Here we define
\[
D \int \omega \zeta \equiv \int_{-\infty}^{\infty} d\omega d\Omega.
\] (B.5) in terms of the dynamic Green’s function \( g \) for Model A with Dirichlet boundary conditions in a film geometry \( z \in \mathcal{D} = [-L/2, L/2] \). (Here \( g \) is the Green’s function in temporal Fourier space as well as spatial Fourier space for the parallel coordinates—see equations (B.4) and (B.5).) With the Hamiltonian (48), we see from the Green’s function equation (B.3) that [28]
\[
g(z, z', \omega, p) = \begin{cases} 
\sinh[Q(z + \frac{L}{2})] \sinh[Q(L - z')] / \mu Q \sinh(QL), & \text{if } z < z', \\
\sinh[Q(z + \frac{L}{2})] \sinh[Q(L - z')] / \mu Q \sinh(QL), & \text{if } z \geq z'.
\end{cases}
\]
(B.6)
with
\[
Q = \sqrt{p^2 + \frac{1}{\xi^2} + i \frac{\omega}{\mu}}.
\]
(B.7)

In [28] the case \( \xi = \infty \) is considered; the modification to finite correlation length is trivially obtained by considering equations (48), (B.1) and (B.3) by replacing \( p^2 \) with \( p^2 + 1/\xi^2 \).

Since \( g \) is an even function of \( p = |p| \), complex conjugation in equation (B.5) implies replacing \( Q \) with its complex conjugate, \( P = Q^* = \sqrt{p^2 + 1/\xi^2} + i \omega / \mu}. \) Explicitly,
\[
I(z, z') = \begin{cases} 
\frac{\text{Pesch}(LQ) \sinh(Qz) \sinh(QL - z')}{\mu^2 PQ(p^2 - Q^2)} - \text{c.c.,} & \text{if } z < z', \\
\frac{\text{Pesch}(LQ) \sinh(Qz') \sinh(QL - z)}{\mu^2 PQ(p^2 - Q^2)} - \text{c.c.,} & \text{if } z \geq z'.
\end{cases}
\]
(B.8)
The result is proportional to the imaginary part of \( g \)—this essentially encodes the fluctuation-dissipation theorem [25].

B.3. Stress tensor autocorrelator

To compute \( C(z, z', \xi) \) as defined in equation (46) we need to determine the correlator
\begin{align}
\langle \Delta T_{x_{z}}^\phi(z, t) \Delta T_{x_{z}}^\phi(z', t') \rangle &= \int \langle \Delta T_{x_{z}}^\phi(r, t) \Delta T_{x_{z}}^\phi(r', t') \rangle \\
&= i \mathcal{L} \hat{D} \left( \phi(r, t) \phi(r', t') \right)
\end{align}
where we have introduced the operators
\begin{align}
\hat{I} &\equiv \int_0^\infty \text{d}(t - t') \int \text{d} \eta_1, \\
\hat{L} &\equiv \lim_{r_1 - r - r_1' - r'} \text{lim}_{r_2 - r - r_2' - r'} \\
\hat{D} &\equiv \partial_2 \partial_2 z_1 \partial_2 z_2.
\end{align}

The four-point correlator in equation (B.9) can be Wick-contracted; we use the short-hand notation
\begin{equation}
(1234) = (12)(34) + (13)(24) + (14)(23).
\end{equation}

The contribution
\begin{equation}
\mathcal{L} \hat{D} (12)(34) = \langle T_{x_{z}}^\phi \rangle
\end{equation}
cancels with those that are subtracted in \( \Delta T_{x_{z}}^\phi \). From the remaining two contributions we obtain
\begin{equation}
C(z, z', \xi) = i \mathcal{L} \hat{I}\left[ \langle \partial_z \phi_x(z, t) \partial_z \phi_x(z', t') \rangle \langle \partial_z \phi_x(r, t) \partial_z \phi_x(r', t') \rangle + \langle \partial_z \phi_x(z, t) \partial_z \phi_x(z', t') \rangle \langle \partial_z \phi_x(r, t) \partial_z \phi_x(r', t') \rangle \right].
\end{equation}

For the first term of equation (B.13) we find from equation (B.4) that
\begin{align}
\langle \phi_x(r, t) \phi_x(z, t) \rangle &= \left( 2 \mu k_B T \right)^2 \int \text{d} \Gamma \int \text{d} \Gamma' e^{\xi z + \xi r} e^{\xi z + \xi r'} \\
&= \left( 2 \mu k_B T \right)^2 \int \text{d} \Gamma \int \text{d} \Gamma' e^{\xi z + \xi r} e^{\xi z + \xi r'}
\end{align}
The differential operator \( \partial_z \partial_z \xi \) brings down (\( p_{1_1} - p_{1_2} \)) \( = p_{2_2} \). The limits \( \lim_{r_1 - r - r_1' - r'} \) change the term \( e^{\xi z_1 + \xi r_1} \) and set \( z_1 \to z', z_1' \to z \). The limits \( \lim_{z_2 - r - r_2' - z_2'} \) change \( e^{\xi z_2 + \xi r_2} \) and change where the \( \partial_z \partial_z \xi \) derivatives are evaluated. \( \int \text{d} \eta_1 \) gives
\begin{equation}
(2 \pi)^2 \delta (p_1 + p_2) \int \frac{\text{d} p_2}{(2 \pi)^2}
\end{equation}
Then sets \( p_2 = -p_1 \) throughout. Lastly, it suffices to consider the real part of the time integral,
\begin{equation}
9 \left[ \int_0^\infty \text{d}(t - t') e^{\xi z + \xi r} \cos(\omega_1 + \omega_2) \right] = \frac{1}{2} (2 \pi) \delta (\omega_1 + \omega_2),
\end{equation}
What remains for this term is
\begin{align}
\mathcal{I} \hat{L} \hat{D} \left[ \langle \partial_z \phi_x(z, t) \partial_z \phi_x(z', t') \rangle \langle \partial_z \phi_x(r, t) \partial_z \phi_x(r', t') \rangle \right] \\
= \frac{1}{2} \left( 2 \mu k_B T \right)^2 \int \text{d} \Gamma \int \text{d} \Gamma' L(z, z') \partial_z \partial_z I(z, z').
\end{align}

For the second term of equation (B.13) the analysis is similar, except that a relative minus sign arises since the derivatives act on different terms. The result is
\begin{align}
\mathcal{I} \hat{L} \hat{D} \left[ \langle \partial_z \phi_x(z, t) \partial_z \phi_x(z', t') \rangle \langle \partial_z \phi_x(r, t) \partial_z \phi_x(r', t') \rangle \right] \\
= \frac{1}{2} \left( 2 \mu k_B T \right)^2 \int \text{d} \Gamma \int \text{d} \Gamma' L(z, z') \partial_z \partial_z I(z, z').
\end{align}

Note the relative minus sign between the integrands in the first and second term of equation (B.13). Combining these expressions gives
\begin{align}
C(z, z', \xi) &= \left( \frac{k_B T}{2 \pi} \right)^2 \int_0^\infty \text{d} p f(z, z', p, L, \xi).
\end{align}

where the integrand \( f \) is
\begin{align}
f(z, z', p, L, \xi) &= \int \frac{\text{d}^3 \omega}{(2 \pi)^3} \frac{\delta \cosh (LP) \cosh (LQ)}{PQ(P^2 - Q^2)^2} \\
&= \left[ P \cosh (P + \frac{L}{2}) \sinh (Q + \frac{L}{2}) \right. \\
&\quad - Q \sinh (P + \frac{L}{2}) \cosh (Q + \frac{L}{2}) \right]
\end{align}

\begin{equation}
\left[ P \cosh (\frac{L}{2} - z') \sinh (Q - \frac{L}{2} - z') \right. \\
&\quad - Q \sinh (\frac{L}{2} - z') \cosh (Q - \frac{L}{2} - z') \right]
\end{equation}

with
\begin{equation}
Q(p, \omega, \xi) = \sqrt{p^2 + 1 \xi^2 - i \omega \mu} \quad \text{and} \\
P(p, \omega, \xi) = \sqrt{p^2 + 1 \xi^2 + i \omega \mu} = Q'.
\end{equation}

To obtain a finite result, the \( P \) integral in equation (B.17) requires an upper cutoff \( \Lambda = 1/a \), where \( a \) is a molecular microscopic length scale.

B.4. Rescaling for film

Consider equation (B.17) with a cutoff \( \Lambda \). We define rescaled coordinates \( \tilde{z} = z/L, \tilde{z}' = z'/L, \xi = \xi L, \tilde{p} = pL, \tilde{\omega} = \omega L^2 \mu \).

In terms of these, we find from equation (B.19) that
\begin{equation}
\tilde{Q} = Q(\tilde{p}, \tilde{\omega}, \xi), \tilde{P} = P(\tilde{p}, \tilde{\omega}, \xi).
\end{equation}

Therefore
\begin{equation}
f(z, z', p, L, \xi) = \tilde{L} \tilde{f}(\tilde{z}, \tilde{z}'),
\end{equation}

where \( \tilde{f} \) is obtained from \( f \) by replacing \( Q \to \tilde{Q}, P \to \tilde{P}, \tilde{p} \to \tilde{p}, \tilde{z} + L/2 \to \tilde{z} + 1/2 \text{ and } L/2 - \tilde{z}' \to 1/2 - \tilde{z}' \).

B.5. Cutoff dependence of the stress–stress autocorrelation function \( C(z, z', \xi) \)

In section 5.3 we remarked that the function \( C(z, z', \xi) \), which is the kernel of the linear response calculation and may be viewed as the non-local viscosity in the Stokes equation, becomes increasingly local as \( \Lambda \) is increased. This behaviour is shown in figure B1, and motivates the envelope ansatz in equations (52) and (54), as illustrated in figure B2. As an example of the accuracy of this ansatz, figure B3 compares the actual
Clearly, as seen in figure C1 for various values of the cutoff, $\Lambda = 80, 160, 700, 1300, 1900$ ($\xi = \infty$). As $\Lambda \to \infty$, the dependence on $\alpha$ disappears. (The latter also follows by considering equation (55) for large cutoffs.)

In section 5.3 we computed the self-consistent solution for $\partial_3 s(\xi)$; this is shown for various (large) $\Lambda$ in figure C2 for $\xi = \infty$. Clearly $\partial_3 s(\xi)$ diverges at the boundaries; this divergence comes closer to the boundaries as the cutoff is increased. In the limit $\Lambda \to \infty$, we have also checked that $\partial_3 s(0) \sim 1/\Lambda \to 0$, i.e. the derivative w.r.t. $\xi$ of the shear rate (and therefore also of the velocity profile) becomes independent of $\xi$ near the critical point for large $\Lambda$.

![Figure C1. Velocity profile $\nu^z(\bar{z})$ for the confined system, in units of the boundary velocity $v^z$, computed with $\alpha = 1$ for various values of the cutoff, $\Lambda = 80, 160, 700, 1300, 1900$ ($\xi = \infty$). As $\Lambda \to \infty$, the dependence on $\alpha$ disappears. (The latter also follows by considering equation (55) for large cutoffs.)](image)

![Figure C2. $\partial_3 s(\xi)$ in units $\omega_0 \xi^{-3}$ with $\alpha = 1$, shown on the range $\bar{\xi} \in [-0.48, 0.48]$. The integral of this function approaches zero as $\Lambda \to \infty$.](image)

The integral $\int_{-1/\xi}^{1/\xi} d\xi' \cos(2\pi \bar{z}') C^\Lambda(\xi, \xi', \xi = \infty)$ with the ansatz $g^\Lambda(\xi) \cos(2\pi \bar{z}')$, showing that their agreement improves as $\Lambda \to \infty$.

### Appendix C. Cutoff dependence of the shear rate, total stress and effective viscosity

For asymptotically large values of $\Lambda$, the non-local viscosity kernel $\eta(z, z')$ (recall equation (43)) becomes increasingly local, and the effective viscosity diverges and approaches the bulk result of the same model [21]. For a finite value of $\Lambda$, the velocity profile $\nu^z$, as seen in figure C1 for $\xi = \infty$, shows a deviation from the simple shear profile, and the effective viscosity scales as $\eta_{\text{eff}} \sim 1/\Lambda$. In order to get rid of this dependence on $\Lambda$, and to highlight those features of the confined system which depend primarily on the confinement but not on the microscopic details, we consider in section 5.3 the dependence of these quantities on the correlation length $\xi$, which is demonstrated to be independent of $\Lambda$, and is thus expected to describe universal behavior.

It is clear from figure C1 that the shear rate approaches simple shear, $s(\xi) \to s_0 = 2\nu/\bar{L}$, as $\Lambda \to \infty$. Correspondingly the velocity profile becomes asymptotically linear. (Note that the results for both these figures were obtained through the ansatz in equation (52), but agree well with actual iterative calculations using the full $\bar{z}$ integral with $C^\Lambda(\bar{z}, \bar{z}', \bar{z})$ for large $\bar{z}$.)

### References

[1] Bordag M, Klimchitskaya G, Mohideen U and Mostepanenko V 2009 Advances in the Casimir Effect (Oxford: Oxford University Press)
[2] Korch M 1994 The Casimir Effect in Critical Systems (Singapore: World Scientific)
[3] Brankov G, Tonchev N S and Danchev D M 2000 Theory of Critical Phenomena in Finite-Size Systems (Singapore: World Scientific)
[4] Gambassi A 2009 J. Phys.: Conf. Ser. 161 012037
[5] Nightingale M P and Indekeu J O 1985 Phys. Rev. Lett. 54 1824
[6] Krech M and Dietrich S 1991 Phys. Rev. Lett. 66 345
[7] Li H and Kardar M 1991 Phys. Rev. Lett. 67 3275
[8] Nellen U, Helden L and Bechinger C 2009 Europhys. Lett. 88 26001
[9] Soyka F, Zvyagolskaya O, Hertlein C, Helden L and Bechinger C 2008 Phys. Rev. Lett. 101 208301
[10] Gambassi A and Dietrich S 2011 Soft Matter 7 1247–53
[11] Tröndle M, Zvyagolskaya O, Gambassi A, Vogt D, Harnau L, Bechinger C and Dietrich S 2011 Mol. Phys. 109 1169
[12] Paladugu S, Callegari A, Tuna Y, Barth L, Dietrich S, Gambassi A and Volpe G 2016 Nat. Commun. 7 11403
[13] Hertlein C, Helden L, Gambassi A, Dietrich S and Bechinger C 2008 Nature 451 172
[14] Zvyagolskaya O, Archer A and Bechinger C 2011 Europhys. Lett. 96 28005
[15] Bonn D, Orwinowski J, Sacanna S, Guo H, Wegdam G and Schall P 2009 Phys. Rev. Lett. 103 156101
[16] Gambassi A and Dietrich S 2010 Phys. Rev. Lett. 105 059601
[17] Faber S et al 2013 Nat. Commun. 4 1584
[18] Shellek P, Nguyen V, Limaye A and Schall P 2013 Adv. Mater. 25 1499
[19] Veen S, Antoniok U, Weber B, Potenza M, Mazzoni S, Schall P and Wegdam G 2012 Phys. Rev. Lett. 109 248302
[20] Marino E, Kudler T E, ten Hoeve J B, Velders A H and Schall P 2016 Sol. Energy Mater. Sol. Cells 158 154
[21] Onuki A 2002 Phase Transition Dynamics (Cambridge: Cambridge University Press)
[22] Corberi F, Giovannella G and Lamura A 1998 Phys. Rev. Lett. 81 3852
[23] Corberi F, Giovannella G and Lamura A 1999 Phys. Rev. Lett. 83 4057
[24] Hohenberg P and Halperin B 1977 Rev. Mod. Phys. 49 435
[25] Gambassi A and Dietrich S 2006 J. Stat. Phys. 123 929–1005
[26] Gambassi A 2008 Eur. Phys. J. B 64 379
[27] Dean D S and Gopinathan A 2010 Phys. Rev. E 81 041126
[28] Hänke A 2013 PLoS One 8 e53228
[29] Funakawa A, Gambassi A, Dietrich S and Tanaka H 2013 Phys. Rev. Lett. 111 055701
[30] Démery V and Dean D 2010 Phys. Rev. Lett. 104 080601
[31] Kirkpatrick T R, Ortiz de Zárate J M and Sengers J V 2013 Phys. Rev. Lett. 110 235902
[32] Aminov A, Kafri Y and Kardar 2015 Phys. Rev. Lett. 114 230602
[33] Rohwer C M, Kardar M and Krüger M 2017 Phys. Rev. Lett. 118 015702
[34] Dean D S and Gopinathan A 2009 J. Stat. Mech. L08001
[35] Bitbol A-F and Fournier J-B 2011 Phys. Rev. E 83 061107
[36] Agosta C, Wang S, Cohen L and Meyer H 1987 J. Low Temp. Phys. 67 237
[37] Luengo G, Israelachvili J and Granick S 1996 Wear 200 328
[38] Cheng X, McCoy J, Israelachvili J and Cohen I 2011 Science 333 1276
[39] Brader J M, Siebenbürger M, Ballauff M, Reinheimer K, Wilhelm M, Frey S J, Weysser F and Fuchs M 2010 Phys. Rev. E 82 061401
[40] Bocquet L and Barrat J-L 1994 Phys. Rev. E 49 3079
[41] Petravic J and Harrowell P 2005 Phys. Rev. E 71 061201
[42] Aerov A A and Krüger M 2014 J. Chem. Phys. 140 094701
[43] Aerov A A and Krüger M 2015 Phys. Rev. E 92 042301
[44] Kardar M 2007 Statistical Physics of Fields (Cambridge: Cambridge University Press)
[45] Diehl H W 1986 Phase Transitions and Critical Phenomena vol 10, ed C Domb and J L Lebowitz (New York: Academic)
[46] Dietrich S and Diehl H 1983 Z. Phys. B 51 343
[47] Diehl H and Janssen H 1992 Phys. Rev. A 45 7145
[48] Diehl H 1994 Phys. Rev. B 49 2846
[49] Wichmann F and Diehl H 1995 Z. Phys. B 97 251
[50] Löwen H, Messina R, Hoffmann N, Likos C N, Eisenmann C, Keim P, Gasser U, Maret G, Goldberg R and Palberg T 2005 J. Phys.: Condens. Matter 17 S3379
[51] Ebert F, Dillmann P, Maret G and Keim P 2009 Rev. Sci. Instrum. 80 083902
[52] Assoud L, Ebert F, Keim P, Messina R, Maret G and Löwen H 2009 Phys. Rev. Lett. 102 238301
[53] Zhang H and Widom M 1994 Phys. Rev. E 49 R3591
[54] Mamiya H, Nakatani I and Furubayashi T 2000 Phys. Rev. Lett. 84 6106
[55] Hansen M F, Jönsson P E, Nordblad P and Svedlindh P 2002 J. Phys.: Condens. Matter 14 4901
[56] Soto-Aquino D and Rinaldi C 2010 Phys. Rev. E 82 046310
[57] Dünweg B and Ladd A 2009 Adv. Polym. Sci. 221 89
[58] Dhont J K G 1996 An Introduction to Dynamics of Colloids (Amsterdam: Elsevier)
[59] Risken H 1989 The Fokker–Planck Equation (Berlin: Springer)
[60] Fuchs M and Cates M E 2005 J. Phys.: Condens. Matter 17 S1681
[61] Kubo R, Toda M and Hashitsume N 2012 Statistical Physics II: Nonequilibrium Statistical Mechanics vol 31 (Berlin: Springer)
[62] Procaccia I, Ronis D, Collins M, Ross J and Oppenheim I 1979 Phys. Rev. A 19 1290
[63] Machta J, Oppenheim I and Procaccia I 1980 Phys. Rev. A 22 2809
[64] Roy S, Dietrich S and Höfling F 2016 J. Chem. Phys. 145 134505
[65] Krüger M and Dean D S 2017 J. Chem. Phys. 146 134507
[66] Zwanzig R 2001 Nonequilibrium Statistical Mechanics (Oxford: Oxford University Press)