INEQUALITIES AND SEPARATION FOR COVARIANT SCHröDINGER OPERATORS

OGNJEN MILATOVIĆ AND HEMANTH SARATCHANDRAN

Abstract. We consider a differential expression $L = \nabla^\dagger \nabla + V$, where $\nabla$ is a metric covariant derivative on a Hermitian bundle $E$ over a geodesically complete Riemannian manifold $(M, g)$ with metric $g$, and $V$ is a linear self-adjoint bundle map on $E$. In the language of Everitt and Giertz, the differential expression $L$ is said to be separated in $L^p(E)$ if for all $u \in L^p(E)$ such that $L u \in L^p(E)$, we have $V u \in L^p(E)$. We give sufficient conditions for $L$ to be separated in $L^2(E)$. We then study the problem of separation of $L$ in the more general $L^p$-spaces, and give sufficient conditions for $L$ to be separated in $L^p(E)$, when $1 < p < \infty$.

1. Introduction

The study of the separation property for Schrödinger operators on $\mathbb{R}^n$ was initiated through the work of Everitt and Giertz in [8]. We recall that the expression $-\Delta + V$ in $L^p(\mathbb{R}^n)$ is separated if the following property is satisfied: For all $u \in L^p(\mathbb{R}^n)$ such that $(-\Delta + V)u \in L^p(\mathbb{R}^n)$, we have that $-\Delta u \in L^p(\mathbb{R}^n)$ and $Vu \in L^p(\mathbb{R}^n)$. After the work of Everitt and Giertz, various authors took up the study of separation problems for (second and higher order) differential operators; see [4, 5, 21, 22] and references therein. The paper [17] then studied the separation property of the operator $\Delta_M + v$ on $L^2(M)$, where $M$ is a non-compact Riemannian manifold, $\Delta_M$ is the scalar Laplacian, and $v \in C^1(M)$. The separation problem for the differential expression $\Delta_M + v$ in $L^p(M)$ was first considered, in the bounded geometry setting, in [19]. The work in [20] gives another proof of the main theorem in [19], crucially without any bounded geometry hypothesis. For a study of separation in the context of a perturbation of the (magnetic) Bi-Laplacian on $L^2(M)$, see the papers [1] [20]. A closer look at the works mentioned in this paragraph reveals that the separation property is linked to the self-adjointness in $L^2$ (or $m$-accretivity in $L^p$) of the underlying operator. In the context of a Riemannian manifold $M$, the latter problem has been studied quite a bit over the past two decades. For recent references see, for instance, the papers [2, 9, 15, 23] and chapter XI in [12].

In this article we consider the separation problem for the differential expression $\nabla^\dagger \nabla + V$, where $\nabla$ is a metric covariant derivative on a Hermitian bundle $E$ over a Riemannian manifold $M$, $\nabla^\dagger$ its formal adjoint, and $V$ is a self-adjoint endomorphism of $E$. We start with the separation problem on $L^2(E)$, obtaining a result (see Theorem 2.1) that can be seen as an extension of the work carried out in [17]. The condition (2.6) on the endomorphism $V$ that guarantees the separation property is analogous to the one in the scalar case. We then move on to consider the
separation problem of the above operator in $L^p(E)$, $1 < p < \infty$, obtaining a result (see Theorem 2.2) that generalizes the work [19]. We do this (see Proposition 5.3 and Corollary 5.3 below for precise statements) by exploiting a coercive estimate for $\Delta_M + v$ in $L^p(M)$ from [19] alongside the following property from [11]: if $V \geq v$, where $v \geq 0$ is a real-valued function on $M$, then the $L^p$-semigroup corresponding to $\nabla^\dagger \nabla + V$ is dominated by the $L^p$-semigroup corresponding to $\Delta_M + v$. In the case $p \neq 2$, we assumed, in addition to geodesic completeness of $M$, that the Ricci curvature of $M$ is bounded from below by a constant. One reason is that, as far as we know, the only available proof of the coercive estimate (5.3) in the case $p \neq 2$ uses the Kato inequality approach, which leads one to apply, in the language of section XIII.5 of [12], the $L^p$-positivity preservation property of $M$. The latter property, whose proof is based on the construction of a sequence of Laplacian cut-off functions (see section III.1 in [12] for details), is known to hold under the aforementioned assumption on the Ricci curvature. Actually, as shown in [3], this hypothesis on Ricci curvature can be further weakened to assume boundedness below by a (possibly unbounded) non-positive function depending on the distance from a reference point. We mention in passing that the $L^p$-positivity preservation property of $M$ is related to the so-called BMS-conjecture, the details of which are explained in [6] and [13]. Another reason for the hypothesis on Ricci curvature is that this assumption is used (see section 3 below for details) for the $m$-accretivity of $\nabla^\dagger \nabla + V$ in $L^p(E)$ in the case $p \geq 3$.

Lastly, we should point out that although the separation property for $\Delta_M + v$ in $L^p(M)$, with $v \geq 0$, was obtained in [20] under the geodesic completeness assumption on $M$ only, it was done so without explicitly establishing (5.3). Instead, assuming (2.0) with $v$ replaced by the Yosida approximation $v_\varepsilon := v(1 + \varepsilon v)^{-1}$, $\varepsilon > 0$, and with a certain condition on the constant $\gamma$, the work [20] establishes an estimate involving the operator $\Delta_M$ and the multiplication operator by $v_\varepsilon$. Using the abstract framework of [22], one concludes the $m$-accretivity of the (operator) sum of “maximal” operators corresponding to $\Delta_M$ and $v$, which, due to the fact (see section 3 below) that the “maximal” operator corresponding to $\Delta_M + v$ is $m$-accretive in $L^p(M)$, $1 < p < \infty$, leads to the separation property. The approach from [20] does not seem to carry to covariant Schrödinger operators.

2. Main Results

2.1. The setting. Let $M$ be a smooth connected Riemannian manifold without boundary, with metric $g$, and with Riemannian volume element $d\mu$. Let $E$ be a vector bundle over $M$ with Hermitian structure $\langle \cdot, \cdot \rangle_x$ and the corresponding norms $|\cdot|_x$ on fibers $E_x$. Throughout the paper, the symbols $C^\infty(E)$ and $C_c^\infty(E)$ denote smooth sections of $E$ and smooth compactly supported sections of $E$, respectively. The notation $L^p(E)$, $1 \leq p < \infty$, indicates the space of $p$-integrable sections of $E$ with the norm

$$
\|u\|_p := \int_M |u(x)|^p \, d\mu.
$$

In the special case $p = 2$, we have a Hilbert space $L^2(E)$ and we use $(\cdot, \cdot)$ to denote the corresponding inner product. For local Sobolev spaces of sections we use the notation $W^{k,p}_0(E)$, with $k$ and $p$ indicating the highest order of derivatives and the corresponding $L^p$-space, respectively. For $k = 0$ we use the simpler notation
$L^p_{\text{loc}}(E)$. In the case $E = M \times \mathbb{C}$, we denote the corresponding function spaces by $C^\infty(M)$, $C^\infty_0(M)$, $L^p(M)$, $W^{k,p}_0(M)$, and $L^p_{\text{loc}}(M)$.

In the remainder of the paper, $\nabla : C^\infty(E) \to C^\infty(T^*M \otimes E)$ stands for a smooth metric covariant derivative on $E$, and $\nabla^\dagger : C^\infty(T^*M \otimes E) \to C^\infty(E)$ indicates the formal adjoint of $\nabla$ with respect to $(\cdot, \cdot)$. The covariant derivative $\nabla$ on $E$ induces the covariant derivative $\nabla^\text{End}$ on the bundle of endomorphisms $\text{End} E$, making $\nabla^\text{End} V$ a section of the bundle $T^*M \otimes (\text{End} E)$.

We study a covariant Schrödinger differential expression
\begin{equation}
L^\nabla_V := \nabla^\dagger \nabla + V,
\end{equation}
where $V$ is a linear self-adjoint bundle map $V \in L^\infty_{\text{loc}}(\text{End} E)$. To help us describe the separation property, we define
\begin{equation}
D^\nabla_p := \{ u \in L^p(E) : L^\nabla_V u \in L^p(E) \}, \quad 1 < p < \infty,
\end{equation}
where $L^\nabla_V u$ is understood in the sense of distributions. In the case of a real valued function $v \in L^\infty_0(M)$, trivial bundle $E = M \times \mathbb{C}$ and $\nabla = d$, where $d$ is the standard differential, we will use the notations $L^\nabla_v := \Delta_M + v$, with $\Delta_M := d^\dagger d$ indicating the scalar Laplacian.

In general, it is not true that for all $u \in D^\nabla_p$ we have $\nabla^\dagger \nabla u \in L^p(E)$ and $V u \in L^p(E)$ separately. Using the language of Everitt and Gierz (see [3]), we say that the differential expression $L^\nabla_V = \nabla^\dagger \nabla + V$ is separated in $L^p(E)$ when the following statement holds true: for all $u \in D^\nabla_p$ we have $V u \in L^p(E)$.

### 2.2. Statements of the Results

Our first result concerns the separation property for $L^\nabla_V$ in $L^2(E)$. Before giving its exact statement, we describe the assumptions on $V$.

**Assumption (A1)** Assume that

(i) $V \in C^1(\text{End} E)$ and $V(x) \geq 0$, for all $x \in M$, where the inequality is understood in the sense of linear operators $E_x \to E_x$;

(ii) $V$ satisfies the inequality
\begin{equation}
|\nabla^\text{End} V(x)| \leq \beta (\nabla V(x))^{3/2}, \quad \text{for all } x \in M,
\end{equation}
where $0 \leq \beta < 1$ is a constant, $| \cdot |$ is the norm of a linear operator $E_x \to (T^*M \otimes E)_x$, and $\nabla : M \to \mathbb{R}$ is defined by
\begin{equation}
\nabla(x) = \min(\sigma(V(x))),
\end{equation}
where $\sigma(V(x))$ is the spectrum of the operator $V(x) : E_x \to E_x$.

We are ready to state the first result.

**Theorem 2.1.** Assume that $(M, g)$ is a smooth geodesically complete connected Riemannian manifold without boundary. Let $E$ be a Hermitian vector bundle over $M$ with a metric covariant derivative $\nabla$. Assume that $V$ satisfies the assumption (A1). Then
\begin{equation}
\|\nabla^\dagger \nabla u\|_2 + \|Vu\|_2 \leq C(\|L^\nabla_V u\|_2 + \|u\|_2),
\end{equation}
for all $u \in D^\nabla_p$, where $C \geq 0$ is a constant (independent of $u$). In particular, $L^\nabla_V$ is separated in $L^2(E)$.

The second result concerns the separation for $L^\nabla_V$ in $L^p(E)$.
Theorem 2.2. Let $1 < p < \infty$. Assume that $(M, g)$ is a geodesically complete connected Riemannian manifold without boundary. In the case $p \neq 2$, assume additionally that the Ricci curvature of $M$ is bounded from below by a constant. Furthermore, assume that there exists a function $0 \leq v \in C^1(M)$ such that
\begin{equation}
 v(x) I \leq V(x) \leq \delta v(x) I
\end{equation}
and
\begin{equation}
 |dv(x)| \leq \gamma v^{3/2}(x), \quad \text{for all } x \in M,
\end{equation}
where $\delta \geq 1$ and $0 \leq \gamma < 2$ are constants, and $I: E_x \to E_x$ is the identity operator. Then, the differential expression $L^\nabla_v$ is separated in $L^p(E)$.

Remark 2.1. From (2.5) it follows that $0 \leq V \in L^\infty_{\text{loc}}(\text{End } E)$.

3. Preliminaries on Operators

We start by briefly recalling some abstract terminology concerning $m$-accretive operators on Banach spaces. A linear operator $T$ on a Banach space $\mathcal{B}$ is called accretive if
\[ \| (\xi + T) u \|_\mathcal{B} \geq \xi \| u \|_\mathcal{B}, \]
for all $\xi > 0$ and all $u \in \text{Dom}(T)$. By Proposition II.3.14 in [7], a densely defined accretive operator $T$ is closable and its closure $T^\sim$ is also accretive. A (densely defined) operator $T$ on $\mathcal{B}$ is called $m$-accretive if it is accretive and $\xi + T$ is surjective for all $\xi > 0$. A (densely defined) operator $T$ on $\mathcal{B}$ is called essentially $m$-accretive if it is accretive and $T^\sim$ is $m$-accretive. As the proof of our first result uses the notion of self-adjointness, we recall a link between $m$-accretivity and self-adjointness of operators on Hilbert spaces: $T$ is a self-adjoint and non-negative operator if and only if $T$ is symmetric, closed, and $m$-accretive; see Problem V.3.32 in [16].

We now describe some known results on the (essential) $m$-accretivity of operators in $L^p(E)$ used in this paper. With $L^\nabla_v$ and $L^d_v$ as in section 2.1 and with $0 \leq V \in L^\infty_{\text{loc}}(\text{End } E)$ and $0 \leq v \in L^\infty_{\text{loc}}(M)$, define an operator $H^\nabla_{p,V}$ as $H^\nabla_{p,V} u := L^\nabla_v u$ with the domain $D^\nabla_p$ as in (2.2) and an operator $H^d_{p,V}$ as $H^d_{p,V} u := L^d_v u$ for all $u \in D^d_p$, where
\[ D^d_p := \{ u \in L^p(M) : L^d_v u \in L^p(M) \}. \]

For a geodesically complete manifold $M$ it is known that $(L^d_{\gamma} |_{C^\infty(M)})^\sim$ in $L^p(M)$, $1 < p < \infty$, is $m$-accretive and it coincides with $H^d_{p,V}$. Moreover, under the same assumption on $M$ and for $1 < p < 3$, the operator $(L^\nabla_{\gamma} |_{C^\infty(E)})^\sim$ in $L^p(E)$, is $m$-accretive and it coincides with $H^\nabla_{p,V}$. Both of these statements are proven in [24] for $V = 0$ and $v = 0$, but the arguments there work for $0 \leq V \in L^\infty_{\text{loc}}$ and $0 \leq v \in L^\infty_{\text{loc}}$ without any change, as the non-negativity assumption makes $V$ and $v$ “disappear” from the inequalities. It turns out that the $m$-accretivity result holds for $(L^\nabla_{\gamma} |_{C^\infty(E)})^\sim$ in $L^p(E)$ in the case $p \geq 3$ as well if, in addition to geodesic completeness, we assume that the Ricci curvature of $M$ is bounded from below by a constant. The latter statement was proven for manifolds of bounded geometry in Theorem 1.3 of [18], and it was observed in [14] that the statement holds if we just assume that $M$ is geodesically complete and with Ricci curvature bounded from below by a constant. For the explanation of why $(L^\nabla_{\gamma} |_{C^\infty(E)})^\sim$ coincides with $H^\nabla_{p,V}$, we again point the reader to [14]. As indicated above, in the case $p = 2$, the
term “$m$-accretivity” in the above statements has the same meaning as the term “self-adjointness.”

4. Proof of Theorem 2.1

Working in the $L^2$-context only, we find it convenient to indicate by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and the inner product in the spaces $L^2(E)$ and $L^2(T^*M \otimes E)$. In subsequent discussion, we adapt the approach from [4, 8] to our setting.

Lemma 4.1. Under the hypotheses of Theorem 2.1, the following inequalities hold for all $u \in C_c^\infty (E)$:

\[
(4.1) \quad \| \nabla^1 \nabla u \| + \| Vu \| \leq \tilde{C} \| L_V u \|
\]

and

\[
(4.2) \quad \| V^{1/2} \nabla u \| \leq \tilde{C} \| L_V u \|
\]

where $L_V$ is as in (2.7), the notation $V^{1/2}$ means square root of the operator $V(x): E_x \to E_x$, and $\tilde{C}$ is a constant depending on $n = \dim M$, $m = \dim E_x$, and $\beta$.

Proof. By the definition of $L_V^\nu$, for all $\nu > 0$ and all $u \in C_c^\infty (E)$ we have

\[
\| L_V^\nu u \|^2 = \| Vu \|^2 + \| \nabla^1 \nabla u \|^2 + 2 \text{Re}(\nabla^1 \nabla u, Vu) = \| Vu \|^2 + \nu \| \nabla^1 \nabla u \|^2 + (1 - \nu) \| \nabla^1 \nabla u \|^2 + 2 \text{Re}(\nabla^1 \nabla u, Vu)
\]

\[
= \| Vu \|^2 + \nu \| \nabla^1 \nabla u \|^2 + (1 - \nu) \text{Re}(\nabla^1 \nabla u, L_V^\nu u - Vu) + 2 \text{Re}(\nabla^1 \nabla u, Vu)
\]

Using integration by parts and the “product rule”

\[
\nabla (Vu) = (\nabla^\text{End} V)u + V \nabla u,
\]

for all $u \in C_c^\infty (E)$ we have

\[
\text{Re}(\nabla^1 \nabla u, Vu) = \text{Re}(\nabla u, \nabla (Vu)) = \text{Re}(\nabla u, (\nabla^\text{End} V)u + V \nabla u)
\]

\[
= \text{Re}(\nabla u, (\nabla^\text{End} V)u) + (\nabla u, V \nabla u) = (\text{Re} Z) + W,
\]

where

\[
Z := (\nabla u, (\nabla^\text{End} V)u)
\]

and

\[
W := (\nabla u, V \nabla u) = (V^{1/2} \nabla u, V^{1/2} \nabla u).
\]

From (4.4) we get

\[
(1 + \nu) \text{Re}(\nabla^1 \nabla u, Vu) = (1 + \nu) \text{Re} Z + (1 + \nu)W \geq -(1 + \nu)|Z| + (1 + \nu)W.
\]

Using (2.6) and

\[
2ab \leq ka^2 + k^{-1}b^2,
\]

where $a$, $b$ and $k$ are positive real numbers, we obtain

\[
|Z| \leq (\beta + 1) \int_M |V^{1/2} \nabla u|_{(T^*M \otimes E)_x} |Vu|_{E_x} \, d\mu
\]

\[
\leq \frac{\nu \delta}{2} \| V^{1/2} \nabla u \|^2 + \frac{(\beta + 1)^2}{2\nu \delta} \| Vu \|^2,
\]

where $\nu \delta$ and $\delta$ are positive real numbers.
for all \( \delta > 0 \). Using (4.6) again, we get

\[
(4.8) \quad | \operatorname{Re}(\nabla^2 \nabla u, L^\nu V u)| \leq \frac{\alpha}{2} \| \nabla^2 \nabla u \|^2 + \frac{1}{2\alpha} \| L^\nu V u \|^2,
\]

for all \( \alpha > 0 \). Combining (4.3), (4.5), (4.7) and (4.8) we obtain

\[
\| L^\nu V u \|^2 \geq \| V u \|^2 + \nu \| \nabla^2 \nabla u \|^2 - \frac{(1 + \nu)(\beta + 1)^2}{2\nu\delta} \| V u \|^2 + (1 + \nu)\| V^{1/2} \nabla u \|^2 - \frac{|1 - \nu\alpha|}{2\alpha} \| \nabla^2 \nabla u \|^2 - \frac{1 - \nu\alpha}{2\alpha} \| L^\nu V u \|^2,
\]

which upon rearranging leads to

\[
\left( 1 + \frac{[1 - \nu\beta]}{2\alpha} \right) \| L^\nu V u \|^2 \geq \left( 1 - \frac{(1 + \nu)(\beta + 1)^2}{2\nu\delta} \right) \| V u \|^2 + \left( \nu - \frac{|1 - \nu\beta|}{2\alpha} \right) \| \nabla^2 \nabla u \|^2 + \left( (1 + \nu) - \frac{(1 + \nu)(\beta + 1)^2}{2\nu\delta} \right) \| V^{1/2} \nabla u \|^2.
\]

Finally, we observe that (4.1) and (4.2) will follow from the last inequality if

\[
|1 - \nu\beta| < \frac{2\nu}{\alpha}, \quad \nu\delta < 2, \quad \text{and} \quad (1 + \nu)(\beta + 1)^2 < 2\nu\delta.
\]

Since, by hypothesis, \( 0 \leq \beta < 1 \), there exist numbers \( \nu > 0, \alpha > 0 \) and \( \delta > 0 \) such that the inequalities (4.9) hold.

\[\square\]

**Continuation of the Proof of Theorem 2.1**

As indicated in section 3, the operator \( L^\nu V |_{C^\infty(E)} \) is essentially self-adjoint and \( (L^\nu V |_{C^\infty(E)})^* = H^\nu V \). We will show that (4.1) and (4.2) hold for all \( u \in D^\nu_2 = \text{Dom}(H^\nu V) \), from which (2.4) follows directly.

As \( H^\nu V \) is a closed operator, there exists a sequence \( \{ u_k \} \) in \( C^\infty_c(E) \) such that \( u_k \to u \) and \( L^\nu V u_k \to H^\nu V \) in \( L^2(E) \). By Lemma 4.1 the sequence \( \{ u_k \} \) satisfies (4.1) and (4.2); hence, \( \{ V u_k \}, \{ \nabla^2 \nabla u_k \}, \) and \( \{ V^{1/2} \nabla u_k \} \) are Cauchy sequences in the appropriate \( L^2 \)-space (corresponding to \( E \) or \( T^* M \otimes E \)). Furthermore, \( \{ \nabla u_k \} \) is a Cauchy sequence in \( L^2(T^* M \otimes E) \) because

\[
\| \nabla u_k \|^2 = \langle \nabla u_k, \nabla u_k \rangle = \langle \nabla \nabla u_k, u_k \rangle \leq \| \nabla^2 \nabla u_k \| \| u_k \|.
\]

It remains to show that \( V u_k \to V u \), \( V^{1/2} \nabla u_k \to V^{1/2} \nabla u \), and \( \nabla u_k \to \nabla u \) in the appropriate \( L^2 \)-space. As the proofs of these three convergence relations follow the same pattern, we will only show the details for the third one. We start by observing that from the essential self-adjointness of \( \nabla^2 \nabla |_{C^\infty_c(E)} \) we get \( \nabla^2 \nabla u_k \to \nabla^2 \nabla u \) in \( L^2(E) \). Since \( \{ \nabla u_k \} \) is a Cauchy sequence in \( L^2(T^* M \otimes E) \), it follows that \( \nabla u_k \) converges to some element \( \omega \in L^2(T^* M \otimes E) \). Then, for all \( \psi \in C^\infty_c(T^* M \otimes E) \) we have

\[
0 = \langle \nabla u_k, \psi \rangle - \langle u_k, \nabla^2 \psi \rangle - \langle \omega, \psi \rangle - \langle u, \nabla^2 \psi \rangle = \langle \omega, \psi \rangle - \langle \nabla u, \psi \rangle,
\]

where in the second equality we used integration by parts (see, for instance, Lemma 8.8 in [6]), which is applicable because elliptic regularity tells us that \( \text{Dom}(H^\nu V) \subset W^{2,2}_0(E) \).

With the three convergence relations at our disposal, taking the limit as \( k \to \infty \) in all terms in (4.1) and (4.2) (with \( u \) replaced by \( u_k \)) shows that (4.1) and (4.2) hold for all \( u \in D^\nu_2 = \text{Dom}(H^\nu V) \).

\[\square\]
5. Proof of Theorem 2.2

5.1. Semigroup Representation Formula. Assuming that $M$ is geodesically complete (with Ricci curvature bounded from below by a constant in the case $p \geq 3$) and $0 \leq V \in L^\infty_{loc}(\text{End } E)$, the operator $H_{p,V}$ is $m$-accretive (see section 3), and its negative, $-H_{p,V}$, generates a strongly continuous contraction semigroup $S_t$ on $L^p(E)$, $1 < p < \infty$; see abstract Theorem II.3.15 in [7].

Before stating a crucial proposition for the proof of Theorem 2.2, we describe a probabilistic setting. In the subsequent discussion, we assume that the underlying filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_s, \mathbb{P})$, where $\mathcal{F}_s$ is right-continuous and the pair $(\mathbb{P}, \mathcal{F}_t)$ is complete in measure theoretic sense for all $t \geq 0$, carries a Brownian motion $W$ on $\mathbb{R}^l$ sped up by 2, that is, $d[W_t^j, W_k^l] = 2\delta_{jk} dt$, where $\delta_{jk}$ is the Kronecker delta and $l \geq n = \dim M$ is sufficiently large. We will also assume $\mathcal{F}_s = \mathcal{F}_s(W)$. Let $B_t(x): \Omega \times [0, \zeta(x)) \to M$ be a Brownian motion starting at $x \in M$ with lifetime $\zeta(x)$. It is well known that this process can be constructed as the maximally defined solution of the Stratonovich equation

$$dB_t(x) = \sum_{j=1}^l A_j(B_t(x))dB_t^j, \quad B_0(x) = x,$$

where $A_j$ are smooth vector fields on $M$ such that $\sum_{j=1}^l A_j^2 = \Delta_M$.

Remark 5.1. In the setting of Theorem 2.2 for $p \neq 2$ our assumptions $M$ imply that $M$ is stochastically complete; hence, in this case we have $\zeta(x) = \infty$.

In the sequel, $//_t^x: E_x \to E_{B_t(x)}$ stands for the stochastic parallel transport corresponding to the covariant derivative $\nabla$ on $E$. Additionally, the symbol $\mathcal{V}_t^x$ stands for the End $E_x$-valued process (with lifetime $\zeta(x)$) defined as the unique pathwise solution to

$$d\mathcal{V}_t^x = -\nabla_t^x (//_t^x^{-1} V(B_t(x))//_t^x) dt, \quad \mathcal{V}_0^x = I,$$

where $//_t^x^{-1}$ is the inverse of $//_t^x$ and $I$ is the identity endomorphism.

We now state the proposition, which in the $p = 2$ context is a special case of Theorem 1.3 in [11]. For an extension to possibly negative $V$, in the case $p = 2$, see Theorem 2.11 in [11]. The proof of Theorem 1.3 in [10] is almost entirely applicable to the proposition below. Thus, we will only explain those parts in need of small changes to accommodate the general $1 < p < \infty$.

Proposition 5.2. Let $1 < p < \infty$, let $M$ be a (smooth) connected Riemannian manifold without boundary, and let $E$, $\nabla$ be as in Theorem 2.2. Assume that $M$ is geodesically complete. In the case $p \geq 3$, assume additionally that the Ricci curvature of $M$ is bounded from below by a constant. Assume that $V \in L^\infty_{loc}(\text{End } E)$ satisfies the inequality $V(x) \geq 0$, for all $x \in M$. Let $S_t$ be the semigroup defined in section 5.1. Then, we have the representation

$$(S_t f)(x) = \mathbb{E} \left[ \mathcal{V}_t^x//_t^x^{-1} f(B_t(x)) 1_{\{t < \zeta(x)\}} \right],$$

for all $f \in L^p(E)$.

Proof. We first assume that $0 \leq V \in C(\text{End } E) \cap L^\infty(\text{End } E)$. Denoting by $L^0(E)$ Borel measurable sections, define a family of operators $Q_t: L^p(E) \to L^0(E)$, $t \geq 0,$
as
\[(Q_t h)(x) := \mathbb{E} \left[ \frac{\partial^\tau}{\partial t^\tau} h(B_t(x)) 1_{t<\zeta(x)} \right].\]

We will show that \(Q_t\) are bounded operators \(L^p(E) \to L^p(E)\). Using Hölder’s inequality, on letting \(q\) be the dual exponent to \(p\), we can estimate
\[
\|Q_t h\|_p \leq e^{p\|V\|_\infty} \int_M \mathbb{E}[|h(B_t(x))|] d\mu(x)
\]
\[
= e^{p\|V\|_\infty} \int_M \left( \int_M |h(y)| \rho_t(x, y) d\mu(y) \right)^p d\mu(x)
\]
\[
\leq e^{p\|V\|_\infty} \int_M \int_M |h(y)|^p \rho_t(x, y) d\mu(y) \left( \int_M \rho_t(x, z) d\mu(z) \right)^{p/q} d\mu(x)
\]
\[
\leq e^{p\|V\|_\infty} \|h\|_p^p
\]
where \(\rho_t(x, y)\) denotes the minimal heat kernel of \(M\). It follows that \(Q_t : L^p(E) \to L^p(E)\) are bounded operators for all \(t \geq 0\).

As shown in the discussion following equation (17) in [10], the operator \(Q_t\) satisfies the equation
\[
(Q_t \psi)(x) = \psi(x) - \int_0^t Q_s H_{p,V}^\nabla \psi(x) ds
\]
for all \(\psi \in C_c^\infty(E)\).

Therefore \(Q_t\) solves the following differential equation
\[
\frac{dQ_t}{dt} \psi = -Q_t H_{p,V}^\nabla \psi, \quad Q_0 \psi = \psi,
\]
for all \(\psi \in C_c^\infty(E)\).

On the other hand, by Lemma II.1.3 (ii) in [7], the semi-group \(S_t\) satisfies the same equation
\[
\frac{dS_t}{dt} \psi = -S_t H_{p,V}^\nabla \psi, \quad S_0 \psi = \psi,
\]
for all \(\psi \in C_c^\infty(E)\). Hence, \(Q_t \psi = S_t \psi\) for all \(\psi \in C_c^\infty(E)\), and thus \(Q_t f = S_t f\) for all \(f \in L^p(E)\). This proves the proposition in the case that \(0 \leq V \in C(\text{End } E) \cap L^\infty(\text{End } E)\).

Now assume \(0 \leq V \in L^\infty(\text{End } E)\). By Lemma 3.1 of [10], we can find a sequence \(0 \leq V_k \in C(\text{End } E) \cap L^\infty(\text{End } E)\) such that for all \(\psi \in C_c^\infty(E)\) we have
\[
\|H_{p,V_k}^\nabla \psi - H_{p,V}^\nabla \psi\|_p \to 0
\]
as \(k \to \infty\). Denote by \(S_t^k\) the (strongly continuous, contractive) semigroup in \(L^p(E)\) generated by \(-H_{p,V_k}^\nabla\). As \(C_c^\infty(E)\) is a common core for \(H_{p,V_k}^\nabla\) and \(H_{p,V}^\nabla\), it follows from the abstract Kato–Trotter theorem, see Theorem III.4.8 in [7], that \(S_t^k f \to S_t f\) in \(L^p(E)\), for all \(f \in L^p(E), 1 < p < \infty\). From here, the proof of Theorem 1.1 in [10] applies to obtain the formula (6.1) for \(0 \leq V \in L^\infty(\text{End } E)\).

The case of \(0 \leq V \in L^\infty_c(\text{End } E)\) proceeds in exactly the same way as Theorem 1.3 in [10].

Before stating a corollary concerning the resolvent domination, we introduce the resolvent notations:
\[
R_t^\nabla := (H_{p,V}^\nabla + 1)^{-1} : L^p(E) \to L^p(E),
\]
\[
R_t^d := (H_{p,V}^d + 1)^{-1} : L^p(M) \to L^p(M).
\]
With the formula (5.1) and the assumption $V \geq vI$ at our disposal, the proof of the following corollary is the same as that of property (iv) in Theorem 2.2 of [11].

**Corollary 5.3.** Let $M$, $\nabla$, and $E$ be as in Proposition 5.2. Assume that $V \in L^\infty_{\text{loc}}(\text{End} E)$ satisfies the inequality $V(x) \geq v(x)I$, for all $x \in M$, where $0 \leq v \in L^\infty_{\text{loc}}(M)$. Then, for all $f \in L^p(E)$, $1 < p < \infty$, we have

\begin{equation}
|R^\nabla_V f(x)|_{E_x} \leq R^{\delta v}_u |f(x)|.
\end{equation}

In the next proposition we state a coercive estimate for $L^d_u$. In the case $p = 2$, assuming just geodesic completeness on $M$, the inequality (5.3) below was proven in Lemma 8 in [17]. For the proof of (5.3) in the case $p \neq 2$ see Theorem 1.2 in [19]. Though stated under a bounded geometry hypothesis on $M$, the proof of the quoted result from [19], which uses a sequence of second order cut-off functions along with $L^p$-positivity preservation property mentioned in section 1 above, works without change if we assume, in addition to geodesic completeness, that the Ricci curvature of $M$ is bounded from below by a constant. We should also mention that the two cited results from [17, 19] use the assumption (2.6).

**Proposition 5.4.** Let $M$ be as in the hypotheses of Theorem 2.2. Assume that $0 \leq v \in C^1(M)$ satisfies (2.6). Then, the following estimate holds for all $u \in D^d_p$:

\begin{equation}
\|vu\|_p \leq C\|L^d_u u\|_p = C\|H^d_{p,v} u\|_p,
\end{equation}

where $C \geq 0$ is a constant.

**Continuation of the Proof of Theorem 2.2** In the following discussion, $C$ will indicate a non-negative constant, not necessarily the same as the one in (5.3). Let $v: L^p(M) \to L^p(M)$ denote the maximal multiplication operator corresponding to the function $v$. We first show that the operator $vR^d_u : L^p(M) \to L^p(M)$ is bounded. Letting $w \in L^p(M)$ be arbitrary, we have $R^d_u w \in \text{Dom}(H^d_{p,v}) = D^d_p$. Applying (5.3) with $u = R^d_u w$, we obtain

\[\|vR^d_u w\|_p \leq C\|H^d_{p,v} R^d_u w\|_p \leq C(\|w\|_p + \|R^d_u w\|_p) \leq C\|w\|_p.\]

This proves $vR^d_u : L^p(M) \to L^p(M)$ is a bounded operator. We then observe that by the boundedness of the operator $vR^d_u$, the assumption $v(x)I \leq V(x) \leq \delta v(x)I$, and the domination inequality (5.2), we have

\[\|VR^\nabla_V f\|_p \leq \delta \|vR^d_u f\|_p \leq C\|f\|_p,\]

for all $f \in L^p(E)$. This shows that the operator $VR^\nabla_V : L^p(E) \to L^p(E)$ is bounded. Let $h \in D^\nabla_V := \text{Dom}(H^\nabla_V)$ be arbitrary and write $Vh = VR^\nabla_V h$. Using the boundedness of the operator $VR^\nabla_V$, we obtain

\[\|Vh\|_p \leq C(\|h\|_p + \|L^\nabla_V h\|_p),\]

which shows that $L^\nabla_V$ is separated in $L^p(E)$.

**References**

[1] Atia, H. A., Alsaedi, R. S., Ramady, A.: Separation of bi-harmonic differential operators on Riemannian manifolds. Forum Math. 26 (2014) 953–966.

[2] Bandara, L., Saratchandran, H.: Essential self-adjointness of powers of first-order differential operators on non-compact manifolds with low-regularity metrics. J. Funct. Anal. 273 (2017) 3719–3758.

[3] Bianchi, D., Setti, A. G.: Laplacian cut-offs, porous and fast diffusion on manifolds and other applications. Calc. Var. 57(4) (2018) https://doi.org/10.1007/s00526-017-1267-9
[4] Boimatov, K. Kh.: Coercive estimates and separation for second order elliptic differential equations, Soviet Math. Dokl. 38(1) (1989) 157–160.
[5] Boimatov, K. Kh.: On the Everitt and Gertz method for Banach spaces, Dokl. Akad. Nauk 356(1) (1997) 10–12 (Russian).
[6] Braverman, M., Milatovic, O., Shubin, M.: Essential self-adjointness of Schrödinger type operators on manifolds, Russian Math. Surveys 57(4) (2002) 641–692.
[7] Engel, K. J., Nagel, R.: One-Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics 194. Springer, Berlin (2000).
[8] Everitt, W. N., Gertz, M.: Inequalities and separation for Schrödinger type operators in $L^2(\mathbb{R}^n)$, Proc. Royal Soc. Edinburgh, 79A (1977) 257–265.
[9] Grummt, R., Kolb, M.: Essential selfadjointness of singular magnetic Schrödinger operators on Riemannian manifolds. J. Math. Anal. Appl. 388 (2012) 480–489.
[10] Güneysu, B.: The Feynman–Kac formula for Schrödinger operators on vector bundles over complete manifolds. J. Geom. Phys. 60 (2010) 1997–2010.
[11] Güneysu, B.: On generalized Schrödinger semigroups. J. Funct. Analysis 262 (2012) 4639–4674.
[12] Güneysu, B.: Covariant Schrödinger Semigroups on Riemannian Manifolds. Operator Theory: Advances and Applications 264. Birkhäuser, Basel (2017).
[13] Güneysu, B.: The BMS-conjecture. Ulmer Seminare. Preprint: arXiv:1709.07463 (to appear).
[14] Güneysu, B., Pigola, S.: $L^p$-interpolation inequalities and global Sobolev regularity results. Annali di Matematica Pura ed Applicata. Preprint: arXiv:1706.00591v2 (to appear).
[15] Güneysu, B., Post, O.: Path integrals and the essential self-adjointness of differential operators on noncompact manifolds. Math. Z. 275 (2013) 331–348.
[16] Kato, T.: Perturbation Theory for Linear Operators. Springer-Verlag, New York (1980).
[17] Milatovic, O.: Separation property for Schrödinger operators on Riemannian manifolds, J. Geom. Phys. 56 (2006) 1283–1293.
[18] Milatovic, O.: On $m$-accretivity of perturbed Bochner Laplacian in $L^p$-spaces on Riemannian manifolds. Integr. Equ. Oper. Theory 68 (2010) 243–254.
[19] Milatovic, O.: Separation property for Schrödinger operators $L^p$-spaces on non compact manifolds, Complex Variables and Elliptic Equations 58 (2013) 853–864.
[20] Milatovic, O.: Self-Adjointness, $m$-accretivity, and separability for perturbations of Laplacian and bi-Laplacian on Riemannian manifolds. Integr. Equ. Oper. Theory (2018) 90:22. https://doi.org/10.1007/s00020-018-2452-8
[21] Nguyen, X. D.: Essential selfadjointness and selfadjointness for even order elliptic operators. Proc. Roy. Soc. Edinburgh Sect. A 93 (1982/83) 161–179.
[22] Okazawa, N.: An $L^p$ theory for Schrödinger operators with nonnegative potentials. J. Math. Soc. Japan 36 (1984) 675-688.
[23] Prandi, D., Rizzi, L., Seri, M.: Quantum confinement on non-complete Riemannian manifolds. Journal of Spectral Theory. Preprint: arXiv:1609.01724 (to appear).
[24] Strichartz, R.: Analysis of the Laplacian on the complete Riemannian manifold. J. Funct. Anal. 52(1) (1983) 48–79.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NORTH FLORIDA, JACKSONVILLE, FL 32224, USA
E-mail address: omilatov@unf.edu

BEIJING INTERNATIONAL CENTER FOR MATHEMATICAL RESEARCH, NO. 5 YIHEYUAN ROAD, Haidian District, Beijing 100871, CHINA P.R.
E-mail address: hemanth.saratchandran@gmail.com