On nearly Kähler and Kähler-Codazzi type manifolds

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Abstract

Nearly Kähler and Kähler-Codazzi type manifolds are defined in a very similar way. We prove that nearly Kähler type manifolds have sense just in Hermitian and para-Hermitian contexts, and that Kähler-Codazzi type manifolds reduce to Kähler type manifolds in all the four Hermitian, para-Hermitian, Norden and product Riemannian geometries.

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1 Introduction

Nearly Kähler manifolds were studied by many authors (see, for instance, the classical work [6] of Gray). They form a class between those of almost Hermitian and Kähler manifolds. The six sphere is a nice example of a nearly Kähler manifold which is not a Kähler manifold. In the almost para-Hermitian case there also exist strict nearly para-Kähler manifolds. The Libermann’s quadric $S_6^3$ is also a nice example of nearly para-Kähler non para-Kähler manifold (see [1, Ex. 3.7]).

In this paper we will prove that the condition defining nearly Kähler type manifolds when applied in almost Norden or almost product Riemannian manifolds leads to a Kähler type condition. The same study will be done for the condition defining Codazzi-Kähler type manifolds in the Norden and product Riemannian cases, proving that it reduces to a Kähler type condition in all the four geometries, thus there not being strict Codazzi-Kähler type manifolds.

The above result is the primary goal of the present paper. Now we point out other achieved results throughout the paper and we show its organization at a time. Section 2 contains the definitions and known results necessary to fulfill the objectives set out. In particular, we will recall the notion of ($J^2 = \pm 1$)-metric manifold which is a common framework for the four aforementioned geometries. We will also recall the definition of the first canonical connection of a such manifold. In Section 3 we will show how the torsion tensor of the first canonical connection of a ($J^2 = \pm 1$)-metric manifold characterizes Kähler type and integrable manifolds (see Theorem 3.1). In Section 4 we will prove that the class of nearly Kähler type manifolds in the almost Norden and almost product Riemannian cases is the class of the Kähler type manifolds (see Theorem 4.1). We will finish this section showing how the torsion tensor

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of the first canonical connection characterizes the class of nearly Kähler type manifolds in the almost Hermitian and almost para-Hermitian cases (see Theorem 4.2). Section 5 is devoted to study the class of Kähler-Codazzi type manifolds, previously introduced in Section 2, and to achieve the primary goal of this paper (see Theorem 5.1). As a direct consequence of this theorem we will conclude that there are no strict Codazzi-Kähler type manifolds in the Norden case. Finally, in Section 6, we will recall the notion of Codazzi-coupled connection. In [3], the authors introduce the notion of Codazzi-coupled connection on almost Hermitian or almost para-Hermitian manifolds as a connection, not necessary torsion free, that fullfills the Codazzi-coupled conditions (13). We will use one of the main results of the quoted paper to show another demonstration of Theorem 5.1 on almost Hermitian and almost para-Hermitian manifolds.

2 Preliminaries

A manifold will be called to have an \((\alpha, \varepsilon)\)-structure \((J, g)\) if \(J\) is an almost complex \((\alpha = -1)\) or almost product \((\alpha = 1)\) structure and \(J\) is an isometry \((\varepsilon = 1)\) or anti-isometry \((\varepsilon = -1)\) of a semi-Riemannian metric \(g\). The metric \(g\) will be a Riemannian metric if \(\varepsilon = 1\). It is also said that \((M, J, g)\) is a \((J^2 = \pm 1)\)-metric manifold. Thus, there exist four kinds of \((\alpha, \varepsilon)\)-structures according to the values \(\alpha, \varepsilon \in \{-1, 1\}\), where

\[
J^2 = \alpha \text{Id}, \quad g(JX, JY) = \varepsilon g(X, Y), \quad \forall X, Y \in \mathfrak{X}(M),
\]

\(\text{Id}\) denotes the identity tensor field and \(\mathfrak{X}(M)\) denotes the set of vector fields on \(M\).

In the above conditions, it is easy to prove the following equivalence

\[
g(JX, JY) = \varepsilon g(X, Y) \iff g(JX, Y) = \alpha \varepsilon g(X, JY), \quad \forall X, Y \in \mathfrak{X}(M).
\]

The corresponding manifolds are known as:

i) Almost-Hermitian manifold if it has a \((-1, 1)\)-structure.

ii) Almost product Riemannian manifolds if it has an \((1, 1)\)-structure. We shall consider through this paper that the trace of \(J\) vanishing, which in particular means these manifolds have even dimension. Almost para-Norden manifolds is another denomination for this kind of metric manifolds having even dimension (see, e.g., [8, Defin. 2.1]).

iii) Almost anti-Hermitian or almost Norden manifolds if it has a \((-1, -1)\)-structure.

iv) Almost para-Hermitian manifolds if it has an \((1, -1)\)-structure.

In the last two cases the metric \(g\) is semi-Riemannian having signature \((n, n)\).

Kähler type and integrable manifolds are the most studied classes in each of the four geometries aforementioned. Let \((M, J, g)\) be a manifold endowed with an \((\alpha, \varepsilon)\)-structure. Let us denote by \(\nabla^g\) the Levi Civita connection of \(g\). The manifold \((M, J, g)\) is said to be a Kähler type manifold if \(\nabla^g J = 0\). The manifold \((M, J, g)\) is said to be integrable if the Nijenhuis tensor of the tensor field \(J\) vanishes.

It is well-known that the Nijenhuis tensor of \(J\), \(N_J\), and the Levi Civita connection of \(g\) satisfy the next relation

\[
N_J(X, Y) = (\nabla^g_X J)Y + (\nabla^g_Y J)X - (\nabla^g_J J)Y - (\nabla^g_J J)X, \quad \forall X, Y \in \mathfrak{X}(M).
\]
Then, the Levi Civita also plays an important role in the characterization of integrable manifolds. A \((J^2 = \pm 1)\)-metric manifold \((M, J, g)\) is an integrable manifold if and only if equation \((3)\) vanishes.

Nevertheless, there exist other connections that allow to characterize the above two classes. One of them is the first canonical connection, firstly introduced in the Hermitian geometry as follows

\[
\nabla^0_X Y = \nabla_X^g Y + \frac{1}{2}(\nabla_X^g J)JY, \quad X, Y \in \mathfrak{X}(M),
\]

(see \([5, 9]\)), and later extended to the other three \((\alpha, \varepsilon)\) geometries as

\[
\nabla^0_X Y = \nabla_X^g Y + \frac{1}{2}(-\alpha)(\nabla_X^g J)JY, \quad \forall X, Y \in \mathfrak{X}(M),
\]

(see \([2]\)).

If \(\alpha \varepsilon = -1\), i.e., in the almost Hermitian and the almost para-Hermitian cases, there is another class of manifolds carefully studied as well: nearly Kähler type manifolds. This class can be characterized with the help of the Levi Civita connection by one of the two equivalent conditions

\[
(\nabla_X^g J)X = 0, \quad (\nabla_X^g J)Y + (\nabla_X^g J)X = 0, \quad \forall X, Y \in \mathfrak{X}(M).
\]

(5)

This class of manifolds can be introduced on \((J^2 = \pm 1)\)-metric manifolds in the same way, i.e., a \((J^2 = \pm 1)\)-metric manifold \((M, J, g)\) will be called a nearly Kähler type manifold if the Levi Civita connection \(\nabla^g\) and the tensor field \(J\) satisfy the equivalent conditions \((5)\).

The common technique that allows to classify the classes of manifolds in the four geometries corresponding with the notion of \((J^2 = \pm 1)\)-metric manifold is based in the study of the vectorial subspaces of the next subspace of \(V^* \otimes V^* \otimes V^*\)

\[
W = \{ \varphi \in V^* \otimes V^* \otimes V^*: \varphi(x, y, z) = \alpha \varepsilon \varphi(x, z, y), \varphi(x, Jy, z) = -\alpha \varepsilon \varphi(x, y, Jz), \forall x, y, z \in V \},
\]

where \((V, J, <, >)\) is \(2n\)-dimensional real vectorial space, \(J\) is an endomorphism of \(V\) and \(<, >\) is an inner product satisfying the same identities that \(J\) and \(g\) described in \((1)\) (see \([11, 7, 4, 13]\)). Note that the elements of the subspace \(W\) have the same symmetries that the tensor \(g((\nabla_X^g J)Y, Z)\), \(X, Y, Z \in \mathfrak{X}(M)\) (see equations \((11)\) and \((12)\)). In the case \(\alpha \varepsilon = -1\), the subspace \(W_1\) of \(W\) defined as follows

\[
W_1 = \{ \varphi \in W: \varphi(x, x, y) = 0, \forall x, y \in V \},
\]

allows to introduce the class of nearly Kähler type manifolds. It is easy to prove that the subspace \(W_1\) also can be defined in the next way

\[
W_1 = \{ \varphi \in W: \varphi(x, y, z) + \varphi(y, x, z) = 0, \forall x, y, z \in V \}.
\]

Both definitions correspond with the equivalent conditions \((5)\). In the case \(\alpha \varepsilon = 1\), the subspace \(W_1\) defined as above is the zero subspace. Indeed, given \(\varphi \in W_1\) and \(x, y, z \in V\), one has

\[
\varphi(x, y, z) = \varphi(x, z, y) = -\varphi(z, x, y), \quad \varphi(x, y, z) = -\varphi(y, x, z) = -\varphi(y, z, x) = \varphi(z, y, x) = \varphi(z, x, y),
\]

then \(\varphi(x, y, z) = 0\), thus \(W_1 = \{0\}\). This fact explains why nearly Kähler type manifolds in the case \(\alpha \varepsilon = 1\) are Kähler type manifolds.
In the Mathematical Literature one can find other attempts to introduce in the case $\alpha \varepsilon = 1$ an analogous class to the nearly Kähler one if $\alpha \varepsilon = -1$. Given an almost Hermitian or almost para-Hermitian manifold $(M, J, g)$, one can define the fundamental two form

$$\omega(X, Y) = g(JX, Y), \ \forall X, Y \in \mathfrak{X}(M).$$

Thus, nearly Kähler type manifolds can also be characterized with the help of the fundamental two form by the condition

$$(\nabla^g_X \omega)(Y, Z) + (\nabla^g_Y \omega)(X, Z) = 0, \ \forall X, Y, Z \in \mathfrak{X}(M).$$

Almost anti-Hermitian and almost product Riemannian cases are quite different. Given a manifold $(M, J, g)$ in the previous conditions, the tensor

$$\tilde{g}(X, Y) = g(JX, Y), \ \forall X, Y \in \mathfrak{X}(M),$$

defines another (semi)-Riemannian metric instead of a two form like in the $\alpha \varepsilon = -1$ case, called the twin metric of $g$. The following equality

$$(\nabla^\tilde{g}_X \tilde{g})(Y, Z) - (\nabla^\tilde{g}_Y \tilde{g})(X, Z) = 0, \ \forall X, Y, Z \in \mathfrak{X}(M),$$

is called the Codazzi equation. In the case $\alpha = \varepsilon = -1$, manifolds satisfying the Codazzi equation are called anti-Kähler-Codazzi manifolds, while in the case $\alpha = \varepsilon = 1$, this kind of manifolds are called para-Kähler-Norden-Codazzi manifolds. First, in the Norden case, they were introduced in [12] and have been intensively studied (see also [10, 11]). Afterward, the class of manifolds characterized by the Codazzi equation were extended without changes in [8] to the other $\alpha \varepsilon = 1$ case, the product Riemannian case.

It is easy to prove that Codazzi equation (6) is equivalent to the next one

$$(\nabla^g_X J)Y - (\nabla^g_Y J)X = 0, \ \forall X, Y \in \mathfrak{X}(M).$$

For the sake of simplicity a $(J^2 = \pm 1)$-metric manifold such that $\alpha \varepsilon = 1$ satisfying equation (7) will be named a Kähler-Codazzi type manifold.

3 Characterizations of Kähler type and integrable manifolds by means of the first canonical connection

The first canonical connection of a $(J^2 = \pm 1)$-metric manifold $(M, J, g)$ parallelizes both $J$ and $g$, i.e., $\nabla^0 J = 0$ and $\nabla^0 g = 0$ (see [2, Lemma 3.10]), but in general has torsion. As direct consequence of identity (1) one can prove that the torsion tensor $T^0$ of $\nabla^0$ satisfies the following one

$$T^0(X, Y) = \frac{(-\alpha)}{2}((\nabla^g_X J)Y - (\nabla^g_Y J)X), \ \forall X, Y \in \mathfrak{X}(M).$$

Therefore, straightforward calculations allow to conclude that the tensors $T^0$ and $N_J$ are related by the next equality

$$-\frac{1}{2}N_J(X, Y) = T^0(JX, JY) + \alpha T^0(X, Y), \ \forall X, Y \in \mathfrak{X}(M).$$

The above properties and identities allow to characterize Kähler type and integrable manifolds as follows.
**Theorem 3.1.** Let \((M, J, g)\) be a \((J^2 = \pm 1)\)-metric manifold.

i) The manifold \((M, J, g)\) is a Kähler type manifold if and only if the torsion tensor of the first canonical connection vanishes.

ii) The manifold \((M, J, g)\) is an integrable manifold if and only if the next relation holds

\[ T^0(JX, JY) + \alpha T^0(X, Y) = 0, \ \forall X, Y \in \mathcal{X}(M). \]

**Remark 1.** We finish this section highlighting the next property of the tensor field \(\nabla^g J\) on a \((J^2 = \pm 1)\)-metric manifold:

\[ (\nabla^g_X J)Y = -J(\nabla^g_X J)Y, \ \forall X, Y \in \mathcal{X}(M). \quad (9) \]

Indeed, given \(X, Y\) vectors fields on \(M\) and taking into account that \(J^2 = \alpha Id\), one has

\[ (\nabla^g_X J)Y = \alpha \nabla^g_X Y - J\nabla^g_X JY = J^2(\nabla^g_X Y) - J\nabla^g_X JY = J(J\nabla^g_X Y - \nabla^g_X JY) = -J(\nabla^g_X J)Y. \]

Identities (8) and (9) allow us to write the torsion tensor of \(\nabla^0\) as follows

\[ T^0(X, Y) = \frac{\alpha}{2} J((\nabla^g_X J)Y - (\nabla^g_Y J)X), \ \forall X, Y \in \mathcal{X}(M). \quad (10) \]

**4 The class of nearly-Kähler type manifolds**

First we recall the following two properties of manifolds endowed with an \((\alpha, \varepsilon)\) structure.

**Remark 2.** Let \((M, J, g)\) be \((J^2 = \pm 1)\)-metric. Given \(X, Y, Z\) vector fields on \(M\), then taking into account the equivalence (2) one has the following identities

\[ 0 = (\nabla^g_X g)(JY, Z) = Xg(JY, Z) - g(\nabla^g_X JY, Z) - g(JY, \nabla^g_X Z) \]

\[ = Xg(JY, Z) - g(\nabla^g_X JY, \nabla^g_X Z) - \alpha \varepsilon g(Y, J\nabla^g_X Z), \]

\[ 0 = \alpha \varepsilon (\nabla^g_X g)(Y, JZ) = \alpha \varepsilon (Xg(Y, JZ) - g(\nabla^g_X Y, JZ) - g(Y, \nabla^g_X JZ)) \]

\[ = Xg(JY, Z) - g(J\nabla^g_X Y, Z) - \alpha \varepsilon g(Y, \nabla^g_X JZ), \]

then subtracting the above equalities one obtains

\[ g((\nabla^g_X J)Y, Z) = \alpha \varepsilon g((\nabla^g_X J)Z, Y). \quad (11) \]

Moreover, if one combines the properties (2) and (9) one also obtains the next identity

\[ g((\nabla^g_X J)Y, Z) = -g(J(\nabla^g_X J)Y, Z) = -\alpha \varepsilon g((\nabla^g_X J)Y, JZ), \ \forall X, Y, Z \in \mathcal{X}(M). \quad (12) \]

As we have recalled in the Preliminaries, nearly Kähler type manifolds in the almost Hermitian and the almost para-Hermitian cases can be characterized by one of the two equivalent conditions (5). Now, we will study the class of manifolds satisfying one of these two conditions in the case \(\alpha \varepsilon = 1\).

**Theorem 4.1.** Let \((M, J, g)\) be a \((J^2 = \pm 1)\)-metric manifold satisfying \(\alpha \varepsilon = 1\). If the Levi Civita connection \(\nabla^g\) and the tensor field \(J\) satisfy

\[ (\nabla^g_X J)Y + (\nabla^g_Y J)X = 0, \ \forall X, Y \in \mathcal{X}(M), \]

then \((M, J, g)\) is a Kähler type manifold.
Proof. Given \(X, Y, Z\) vector fields on \(M\), taking into account the above condition and property (11) in the case \(\alpha \varepsilon = 1\) one obtains
\[
g((\nabla^g_X J) Y, Z) = g((\nabla^g_X J) Z, Y) = -g((\nabla^g_Y J) X, Y),
\]
\[
g((\nabla^g_X J) Y, Z) = -g((\nabla^g_X J) X, Z) = g((\nabla^g_Y J) Z, X) = g((\nabla^g_Z J) X, Y),
\]
then \(g((\nabla^g_X J) Y, Z) = 0\), and thus, one can conclude that \((M, J, g)\) is a Kähler type manifold.

As in the case of Kähler type and integrable \((J^2 = \pm 1)\)-metric manifolds, nearly-Kähler type manifolds in the \(\alpha \varepsilon = -1\) case can be easily characterized by means of the first canonical connection as follows.

Theorem 4.2. Let \((M, J, g)\) be a \((J^2 = \pm 1)\)-metric manifold satisfying \(\alpha \varepsilon = -1\). Then \((M, J, g)\) is a nearly Kähler type manifold if and only if
\[
g(T^0(X, Y), X) = 0, \quad X, Y \in \mathfrak{X}(M).
\]

Proof. Given \(X, Y\) vector fields on \(M\), if \(\alpha \varepsilon = -1\) then, taking into account (11) and (12), one has
\[
g((\nabla^g_Y J) X, X) = g((\nabla^g_Y J) X, JX) = -g((\nabla^g_Y J) JX, X),
\]
i.e., \(g((\nabla^g_Y J) X, X) = 0\). Taking into account (5), (11) and the last identity one obtains the next equality
\[
g(T^0(X, Y), X) = \alpha \varepsilon g((\nabla^g_X J) Y, X) = \alpha \varepsilon g((\nabla^g_X J) Y, X), \quad \forall X, Y \in \mathfrak{X}(M),
\]
thus, one can conclude that the equivalence of this statement is true.

5 The class of Kähler-Codazzi type manifolds in the \(\alpha \varepsilon = 1\) case

Obviously, in the \(\alpha \varepsilon = 1\) case, the Levi Civita connection of any Kähler type manifold satisfies equation (7), thus any Kähler type manifold being a Kähler-Codazzi type manifold. In [8, Prop. 2.1], the authors prove that any para-Kähler-Norden-Codazzi manifold in the almost product Riemannian case is a para-Kähler-Norden manifold, i.e., the Levi-Civita connection satisfies \(\nabla^g J = 0\). In the almost anti-Hermitian case there is no analogous result. However, examples of strict anti-Kähler-Codazzi manifolds in this case, i.e., \(\nabla^g J \neq 0\), are not yet shown. Now we prove that every anti-Kähler-Codazzi manifold is an anti-Kähler manifold. In general, we prove that every manifold having an \((\alpha, \varepsilon)\)-structure such that its Levi Civita connection satisfies identity (7) is a Kähler type manifold, which is a direct consequence of the last identity of Section 3 as we will see below.

Theorem 5.1. Let \((M, J, g)\) be a \((J^2 = \pm 1)\)-metric manifold. If the Levi Civita connection \(\nabla^g\) and the tensor field \(J\) satisfy identity (7) then \((M, J, g)\) is a Kähler type manifold.

Proof. If \(\nabla^g\) satisfies \((\nabla^g_X J) Y - (\nabla^g_Y J) X = 0\), for all vector fields \(X, Y\) on \(M\), then, taking into account (10), one concludes that the torsion tensor \(T^0\) vanishes, and therefore, \((M, J, g)\) is Kähler type manifold (see Theorem 5.1).

The above theorem shows that every Kähler-Codazzi type manifold in the case \(\alpha \varepsilon = 1\) is a Kähler type manifold. In particular, we obtain that anti-Kähler-Codazzi manifolds in the sense of [12] are Kähler type manifolds.
Corollary 5.2. Let \((M, J, g)\) be an anti-Hermitian manifold. Every anti-Kähler-Codazzi manifold is an anti-Kähler manifold, i.e., the Levi Civita connection \(\nabla^g\) and the tensor field \(J\) satisfy \(\nabla^g J = 0\). Therefore, there are no strict anti-Kähler-Codazzi manifolds.

Remark 3. Theorem 5.1 allows us to recover the analogous result for the almost product Riemannian case proved in [8, Prop. 2.1] and previously recalled: Every para-Kähler-Norden-Codazzi manifold \((M, J, g)\) is a para-Kähler-Norden manifold. Ida and Manea use the properties of tensor \(g((\nabla^g_X J)Y, Z)\), \(X, Y, Z \in \mathfrak{X}(M)\), in the case \(\alpha \varepsilon = 1\) (see equations (11) and (12)) and identity (7) to demonstrate this in a similar way to the proof of Theorem 4.1 of this paper. We propose a unified proof of both results in the \(\alpha \varepsilon = 1\) case taking into account that the equivalent condition to the Codazzi equation (6),
\[
(\nabla^g_X J)Y - (\nabla^g_Y J)X = 0, \quad \forall X, Y \in \mathfrak{X}(M),
\]
also allows to show that the torsion of the first canonical connection \(\nabla^0\) vanishes,
\[
T^0(X, Y) = \frac{(-\alpha)}{2}((\nabla^g_X J)Y - (\nabla^g_Y J)X) = \frac{\alpha}{2} J ((\nabla^g_X J)Y - (\nabla^g_Y J)X), \quad \forall X, Y \in \mathfrak{X}(M).\]

6 On Codazzi couplings on \((J^2 = \pm 1)\)-metric manifolds in the case \(\alpha \varepsilon = -1\)

In a more general setting, in [3], the authors introduce a relaxation of the parallelism conditions over a connection on a certain kind of manifolds. In the particular case of an almost Hermitian or almost para-Hermitian manifold \((M, J, g)\), i.e., in a manifold having an \((\alpha, \varepsilon)\)-structure satisfying \(\alpha \varepsilon = -1\), they introduce the notion of Codazzi-coupled connection as a connection \(\nabla\) on \(M\), not necessary torsion-free connection, that satisfies the below conditions, named the Codazzi-coupled conditions,
\[
(\nabla_X J)Y - (\nabla_Y J)X = 0, \quad (\nabla_Z g)(X, Y) - (\nabla_X g)(Z, Y) = 0, \quad \forall X, Y, Z \in \mathfrak{X}(M). \tag{13}
\]

They also prove the following theorem about this kind of manifolds having a Codazzi-coupled torsion-free connection.

Theorem 6.1 ([3, Theor. 3.2]). Let \((M, J, g)\) be an almost almost Hermitian or almost para-Hermitian manifold endowed with a torsion-free connection \(\nabla\) satisfying
\[
(\nabla_X J)Y - (\nabla_Y J)X = 0, \quad (\nabla_Z g)(X, Y) - (\nabla_X g)(Z, Y) = 0, \quad \forall X, Y, Z \in \mathfrak{X}(M).
\]

Then \((M, J, g)\) is a Kähler or para-Kähler manifold.

An almost Hermitian or almost para-Hermitian manifold endowed with a Codazzi-coupled torsion-free connection is called an Codazzi-Kähler or Codazzi-para-Kähler manifold (see [3, Defin. 3.8]).

The above statement is given just for the two geometries obtained with \(\alpha \varepsilon = -1\). In this conditions, if the Levi Civita connection of a manifold \((M, J, g)\) having an \((\alpha, \varepsilon)\)-structure satisfies the next condition
\[
(\nabla^g_X J)Y - (\nabla^g_Y J)X = 0, \quad \forall X, Y \in \mathfrak{X}(M),
\]
then it also satisfies Codazzi-coupled conditions (13), i.e., the Levi Civita connection is a Codazzi-coupled torsion-free connection. Thus, one can recover our Theorem 5.1 from the above one for the case \(\alpha \varepsilon = -1\).
Corollary 6.2. Let \((M, J, g)\) be a \((J^2 = \pm 1)\)-metric manifold such that \(\alpha \varepsilon = -1\). If the next condition is fulfilled,
\[
(\nabla^g_X J)Y - (\nabla^g_Y J)X = 0, \quad \forall X, Y \in \mathfrak{X}(M),
\]
then \((M, J, g)\) is a Kähler type manifold.

Therefore, Kähler type manifolds satisfying \(\alpha \varepsilon = -1\) are Codazzi-Kähler or Codazzi-para-Kähler manifolds such that the Levi Civita connection is a Codazzi-coupled connection.

Table 1 summarizes the results obtained about \((J^2 = \pm 1)\)-metric manifolds satisfying
\[
(\nabla^g_X J)Y + \alpha \varepsilon (\nabla^g_Y J)X = 0, \quad \forall X, Y \in \mathfrak{X}(M),
\]
according to the value of the product \(\alpha \varepsilon = \pm 1\) through the present paper.

| Condition | \(\alpha \varepsilon = -1\) | \(\alpha \varepsilon = 1\) |
|-----------|----------------|----------------|
| \((\nabla^g_X J)Y + (\nabla^g_Y J)X = 0 \iff (\nabla^g_X J)X = 0\) | nearly Kähler type manifolds | Kähler type manifolds |
| \((\nabla^g_X J)Y - (\nabla^g_Y J)X = 0\) | Kähler type manifolds | Kähler type manifolds |

Table 1: \((J^2 = \pm 1)\)-metric manifolds satisfying \((\nabla^g_X J)Y \pm (\nabla^g_Y J)X = 0\)

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