Use of the generating function to generalize the sum formula for quadruple zeta values

MACHIDE, Tomoya∗ †

Abstract

In the present paper, we prove an identity for the generating function of the quadruple zeta values with the action of the matrix ring $M_4(\mathbb{Z})$. Taking homogeneous parts on both sides of the identity and substituting appropriate values for the variables, we obtain the sum formula for quadruple zeta values. We also obtain its weighted analogues, which include the formulas for this case proved by Guo and Xie (2009, J. Number Theory 129, 2747–2765) and by Ong, Eie, and Liaw (2013, Int. J. Number Theory 9, 1185–1198).

1 Introduction and statement of results

A multiple zeta value (MZV) is defined by the convergent series

$$\zeta(l_r) = \zeta(l_1, l_2, \ldots, l_r) = \sum_{m_1 > m_2 > \cdots > m_r > 0} \frac{1}{m_1^{l_1} m_2^{l_2} \cdots m_r^{l_r}},$$

where $l_r = (l_1, l_2, \ldots, l_r)$ is an admissible index set, that is, it is a sequence of positive integers with $l_1 \geq 2$. The condition $l_1 \geq 2$ ensures convergence. The integers $d(l_r) = r$ and $w(l_r) = l_1 + \cdots + l_r$ are called the depth and weight, respectively. The single, double, triple, and quadruple zeta values (SZVs, DZVs, TZVs, and QZVs, respectively) are the MZVs of depth 1, 2, 3, and 4, respectively.

It is known that, among the MZVs, there are many linear relations over $\mathbb{Z}$. One notable example is the sum formula, which was conjectured by Moen and Markett (see [6] and [17], respectively):

$$\sum_{l_r \text{ adm} \ (w(l_r)=l)} \zeta(l_1, \ldots, l_r) = \zeta(l), \quad (1.1)$$

where the summation ranges over all admissible index sets of depth $r$ and weight $l$. Formula (1.1) was proved for DZVs by Euler [2], for TZVs by Hoffman and Moen [8], and for general MZVs by Granville [4]. Zagier also proved (1.1) independently in an unpublished manuscript (see [4]).

Formula (1.1) has been generalized and extended in various directions [1, 5, 7, 9, 11, 13, 18, 19, 20, 21, 22]. Recently, generalizations for DZVs and TZVs, from the point of view of the generating functions, were given in [3] and [14], respectively (see Appendix A for details). In the present paper, we give such a generalization of (1.1) for QZVs.

∗National Institute of Informatics, 2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo 101-8430, Japan
†JST, ERATO, Kawarabayashi Large Graph Project, Japan
e-mail : machide@nii.ac.jp
MSC-class: 11M32(Primary), 16S34,20C07(Secondary)
Key words: multiple zeta value, sum formula, generating function, group ring of general linear group
We will begin by introducing the notation and terminology that will be used to state our results. We define the generating function $Q$ of the QZVs as the formal power series,

$$Q(x_1, x_2, x_3, x_4) := \sum_{l_4: \text{adm}} \zeta(1_4)x_1^{l_1-1}x_2^{l_2-1}x_3^{l_3-1}x_4^{l_4-1}. \tag{1.2}$$

Replacing $\zeta(1_4)$ with $\zeta(w(1_4)) = \zeta(l_1 + l_2 + l_3 + l_4)$, we also define

$$q(x_1, x_2, x_3, x_4) := \sum_{l_4 : (w(1_4)) > (1.3)} \zeta(w(1_4))x_1^{l_1-1}x_2^{l_2-1}x_3^{l_3-1}x_4^{l_4-1}, \tag{1.3}$$

where the summation rules are different: the rule of (1.2) ranges over all admissible index sets whose depth is 4, but that of (1.3) ranges over all sequences of positive integers whose depth is 4 and whose weight is greater than 4.

We denote by $S_r$ and $e = e_r$ the symmetric group of degree $r$ and its identity element, respectively, and also denote by $GL_r(\mathbb{Z})$ and $I = I_r$ the general linear group of degree $r$ over $\mathbb{Z}$ and its identity element, respectively. We identify a permutation $\sigma$ in $S_r$ via the non-standard definition: we use the transpose $\sigma^T$ instead of the inverse $\sigma^{-1}$ to prove our main result, so we adopt the non-standard definition: we use the transpose $M^t$ instead of the inverse $M^{-1}$. Note that the action of any permutation $\sigma$ on $f$ is same in both definitions, that is,

$$(f|M)(x_1, \ldots, x_r) := f((x_1, \ldots, x_r)M^t). \tag{1.4}$$

This is a right action (i.e., $f|(M_1M_2) =$ $(f|M_1)|M_2$), and extends to an action of the group ring $\mathbb{Z}[GL_r(\mathbb{Z})]$ of $GL_r(\mathbb{Z})$ over $\mathbb{Z}$ in a natural way by $f|\sum a_jM_j = \sum a_j(f|M_j)$. We will need to consider actions of matrices of determinant 0 to prove our main result, so we adopt the non-standard definition: we use the transpose $M^t$ instead of the inverse $M^{-1}$. Note that the action of any permutation $\sigma$ on $f$ is same in both definitions, that is,

$$(f|\sigma)(x_1, \ldots, x_r) = f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(r)}),$$

because $\sigma^t = \sigma^{-1}$ in $GL_r(\mathbb{Z})$.

Let $p, q, r, s$ be the matrices in $GL_4(\mathbb{Z})$ given by

$$p = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad r = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{1.5}$$

and let $\Phi, \Psi$ be the elements in $\mathbb{Z}[GL_4(\mathbb{Z})]$ given by

$$\Phi = I + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\Psi = I + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.6}$$
For any subset \( H \) of \( S_4 \), we define an element \( \Sigma_H \) in \( \mathbb{Z}[GL_4(\mathbb{Z})] \) as
\[
\Sigma_H := \sum_{\sigma \in H} \sigma.
\]
For example,
\[
\Sigma_{C_4} = I + \begin{pmatrix}
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{pmatrix},
\]
where \( C_4 = \{e, (1234), (13)(24), (1432)\} \) is the cyclic subgroup in \( S_4 \) generated by \( (1234) \).

Our main result is stated as follows.

**THEOREM 1.1.** We have
\[
\mathcal{Q}|\Omega = q, \tag{1.7}
\]
where
\[
\Omega = p\Sigma_{\mathcal{E}_4} - (\Psi \Phi q - \Phi r - \Psi s + I)\Sigma_{\mathcal{E}_4}. \tag{1.8}
\]

Taking the homogeneous parts on both sides of (1.7) and substituting appropriate values for the variables \( x_1, x_2, x_3, \) and \( x_4 \), we can obtain the following formulas:

**COROLLARY 1.2.** For any integer \( l > 4 \), we have
\[
\sum_{l_4: \text{adm}} \zeta(l_4) = \zeta(l), \tag{1.9}
\]
\[
\sum_{l_4: \text{adm}} (2^{l_1+l_2+l_3-2} + 2^{l_1+l_2-2} + 2^{l_1-1} - 2^{l_2+l_3-1} - 2^{l_2-1}) \zeta(l_4) = l\zeta(l), \tag{1.10}
\]
\[
\sum_{l_4: \text{adm}} (2^{l_1} + 2^{l_3+1}) \zeta(l_4) = (l + 3)\zeta(l), \tag{1.11}
\]
\[
\sum_{l_4: \text{adm}} (3^{l_2+2l_1-1} - 3^{l_2} - 1) 2^{l_1+l_2} \zeta(l_4) = \left( \frac{(l + 7)(l + 2)(l - 3) + 2}{12} \right) \zeta(l). \tag{1.12}
\]

Formula (1.9) is, needless to say, the sum formula (1.1) for QZVs. Formulas (1.10) and (1.11) are the weighted sum formulas for QZVs that were proved in [5, Theorem 1.1] and [21, main theorem], respectively. These facts guarantee that, for QZVs, (1.7) is a natural generalization of (1.1). It appears that (1.12) is new, and we note that in its coefficients, it has not only powers of 2 but also powers of 3.

Let \( Z_r(x_1, \ldots, x_r) \) be the generating function of MZVs of depth \( r \):
\[
Z_r(x_1, \ldots, x_r) := \sum_{l_r: \text{adm}} \zeta(l_r)x_1^{l_1-1} \cdots x_r^{l_r-1}. \tag{1.13}
\]
Let \( S, D, \) and \( T \) denote \( Z_1, Z_2, \) and \( Z_3, \) respectively. Note that \( \mathcal{Q} = Z_4 \). Let \( \zeta_\mu(l_r) \) be regularized shuffle-type multiple zeta values (RMZVs), which were introduced in [10]. RMZVs are MZVs if \( l_r \) are admissible, but RMZVs are defined for non-admissible index sets,
unlike MZVs (see [10] for details). We thus define the generating function of RMZVs of depth $r$ by

$$Z_{m,r}(x_1, \ldots, x_r) := \sum_{l_r} \zeta_m(l_r)x_1^{l_1-1} \cdots x_r^{l_r-1},$$

(1.14)

where the summation includes not only admissible index sets but also non-admissible ones. Let $S_m$, $D_m$, $T_m$, and $Q_m$ denote $Z_{m,1}$, $Z_{m,2}$, $Z_{m,3}$, and $Z_{m,4}$, respectively.

For a square matrix $M$ of order 4 and an integer $k \in \{1, 2, 3, 4\}$, we define a row operation $r_k$ as follows: $r_k(M)$ is the matrix produced from $M$ by multiplying $k$-th row by $-1$ and then adding $(k-1)$-th and $(k+1)$-th rows to $k$-th row, where 0-th and 5-th rows mean the zero row vector. Then, $\Phi$ and $\Psi$ are expressed as

$$\begin{align*}
\Phi &= I + r_3(I) + r_2 \circ r_3(I), \\
\Psi &= I + r_4(I) + r_3 \circ r_4(I) + r_2 \circ r_3 \circ r_4(I),
\end{align*}$$

(1.15)

and any two adjacent matrices are transitive by some $r_k$. (Note that $r_k \circ r_k$ is the identity operation.) These expressions will help us to show the equations in $Z[GL_4(\mathbb{Z})]$ through the present paper. It may be worth noting that the matrices $r_4(I)$, $r_3 \circ r_1(I)$, and $r_2 \circ r_3 \circ r_4(I)$ in $\Psi$ can be obtained from the matrices $I$, $r_3(I)$, and $r_2 \circ r_3(I)$ in $\Phi$, respectively, by adding 4-th column to 3-th column and then multiplying 4-th column by $-1$.

The present paper is organized as follows. Sections 2 and 3 each have two subsections. In Section 2.1, we give some identities for $S_m$, $D_m$, $T_m$, and $Q_m$, and in Section 2.2, we discuss a relation between $Q_m$ and $Q$.

We prove Theorem 1.1 in Section 3.1, by using the results obtained in Section 2 and the identity for RMZVs that was proved in [16, Theorem 1.1]. We derive Corollary 1.2 from Theorem 1.1 in Section 3.2. We attach an appendix at the end of the paper, in which the identities for $D_m$ and $T_m$ proved in [3] and [14] are restated in terms of the actions of $Z[GL_2(\mathbb{Z})]$ and $Z[GL_3(\mathbb{Z})]$, respectively: they are lower depth versions of (1.7).

**REMARK** 1.3. Theorem 1.1 and Corollary 1.2 in the present paper are expansions of Theorems 1.1 and 1.2 in [12], respectively; the results that will be stated below are expansions of the results following Section 2.1 in [12]. The results of Section 2.1 have been amplified in [16].

## 2 Preliminaries

### 2.1 Identities for $S_m$, $D_m$, $T_m$, and $Q_m$

For functions $f_{r_1}, \ldots, f_{r_j}$ such that each $f_{r_i}$ has $r_i$ variables, we define a function of $r = r_1 + \cdots + r_j$ variables $x_r = (x_1, \ldots, x_r)$ by

$$f_{r_1} \otimes f_{r_2} \otimes \cdots \otimes f_{r_j}(x_r) := f_{r_1}(x_1, \ldots, x_{r_1})f_{r_2}(x_{r_1+1}, \ldots, x_{r_1+r_2}) \cdots f_{r_j}(x_{r_1+r_2+\cdots+r_{j-1}+1}, \ldots, x_r).$$

For example,

$$S_m \otimes S_m(x_2) = S_m(x_1)S_m(x_2) \quad \text{and} \quad D_m \otimes S_m(x_3) = D_m(x_1, x_2)S_m(x_3).$$

Let $\langle \sigma \rangle = \{ \sigma^n \mid n \in \mathbb{Z} \}$ denote the cyclic subgroup generated by a permutation $\sigma$.

The purpose of this subsection is to prove the following identities.
PROPOSITION 2.1. We have

\[ \mathcal{T}_m \otimes \mathcal{S}_m = \mathcal{Q}_m | \Psi s, \]
\[ \mathcal{D}_m \otimes \mathcal{D}_m = \mathcal{Q}_m | \Phi r \Sigma_{((13)(24))}, \]
\[ \mathcal{D}_m \otimes \mathcal{S}_m \otimes \mathcal{S}_m = \mathcal{Q}_m | \Psi \Phi q, \]
\[ \mathcal{S}_m \otimes \mathcal{S}_m \otimes \mathcal{S}_m \otimes \mathcal{S}_m = \mathcal{Q}_m | p \Sigma q. \]

We will need Lemmas 2.2 and 2.3 below to prove Proposition 2.1. We prepare some notation to state the lemmas, by referring to [10].

Let \( p_r \) be the matrix in \( GL_r(\mathbb{Z}) \) given by

\[ p_r = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\ \vdots & \cdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}, \quad p^t_r = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & \cdots & \cdots & 1 \\ \vdots & \cdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}. \]

For a function \( f \) of \( r \) variables, we define \( f^\sharp := f|p_r \), that is,

\[ f^\sharp(x_1, x_2, \ldots, x_r) = f(x_1 + x_2 + \cdots + x_r, x_2 + \cdots + x_r, \ldots, x_r). \]

We note that \( p_4 \) is \( p \) in (1.5), and \( f^\sharp = f \) if \( r = 1 \). Let \( sh_j^{(r)} \) be the shuffle elements in \( \mathbb{Z}[\mathcal{S}_r] \) defined by

\[ sh_j^{(r)} := \sum_{\sigma \in \mathcal{S}_r} \sigma, \]

where \( j \) and \( r \) are integers with \( 1 \leq j \leq r - 1 \). We set \( sh_j = sh_j^{(4)} \) for brevity.

LEMMA 2.2. The following identities hold.

\[ \mathcal{T}_m^t \otimes \mathcal{S}_m = \mathcal{Q}_m^t | sh_3, \]
\[ \mathcal{D}_m^t \otimes \mathcal{D}_m^t = \mathcal{Q}_m^t | sh_2, \]
\[ \mathcal{D}_m^t \otimes \mathcal{S}_m \otimes \mathcal{S}_m = \mathcal{Q}_m^t | sh_2 \Sigma_{((34))}, \]
\[ \mathcal{S}_m \otimes \mathcal{S}_m \otimes \mathcal{S}_m \otimes \mathcal{S}_m = \mathcal{Q}_m^t | \Sigma q. \]

where the concrete expressions of \( sh_3, sh_2, sh_2 \Sigma_{((34))} \) are

\[ sh_3 = e + (34) + (234) + (1234), \]
\[ sh_2 = e + (23) + (13)(24) + (123) + (243) + (1243), \]
\[ sh_2 \Sigma_{((34))} = e + (23) + (24) + (34) + (13)(24) + (123) + (124) + (234) + (243) + (1234) + (1243) + (1324). \]

LEMMA 2.3. The following equations in \( \mathbb{Z}[GL_4(\mathbb{Z})] \) hold.

\[ p sh_3 = \Psi p, \]
\[ p \ sh_2^{(3)} = \Phi p, \]  
\[ p \ sh_2 \Sigma_{((34))} = \Psi \Phi p, \]

where \( sh_2^{(3)} \) in (2.16) is regarded as an element in \( \mathbb{Z}[GL_4(\mathbb{Z})] \) by identifying \( \mathfrak{S}_3 \) as the subgroup of \( \mathfrak{S}_4 \) consisting of elements that fix 4, that is,

\[ sh_2^{(3)} = e + (23) + (123). \]  

Let \( M_r(\mathbb{Z}) \) be the ring of square matrices of order \( r \) over \( \mathbb{Z} \). For any pair \((A, B)\) in \( M_i(\mathbb{Z}) \times M_j(\mathbb{Z}) \), we define a block diagonal matrix \( A \oplus B \) in \( M_{i+j}(\mathbb{Z}) \) by

\[ A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}. \] 

Since \((A \oplus B)^t = A^t \oplus B^t\), we see from the definitions of \( \otimes \) and \( \oplus \) that

\[ f \otimes g | A \oplus B = f | A \otimes g | B, \]

where \( f \) and \( g \) are functions of \( i \) and \( j \) variables, respectively.

We now prove Proposition 2.1. We will then discuss proofs of Lemmas 2.2 and 2.3.

**Proof of Proposition 2.1.** By definition,

\[ f = f | p_r p_r^{-1} = f^t | p_r^{-1} \]

for any function \( f \) of \( r \) variables, and so

\[ T_m \otimes S_m = T_m^x \otimes S_m | p_3^{-1} \oplus I_1, \]  
\[ D_m \otimes D_m = D_m^x \otimes D_m^x | p_2^{-1} \oplus p_2^{-1}, \]
\[ D_m \otimes S_m \otimes S_m = D_m^x \otimes S_m \otimes S_m | p_2^{-1} \oplus I_2. \]

Thus we may show that

\[ T_m^x \otimes S_m | p_3^{-1} \oplus I_1 = Q_m | \Psi s, \]  
\[ D_m^x \otimes D_m^x | p_2^{-1} \oplus p_2^{-1} = Q_m | \Phi \Sigma((13)(24)), \]
\[ D_m^x \otimes S_m \otimes S_m | p_2^{-1} \oplus I_2 = Q_m | \Psi q. \]

In fact, equating (2.19) and (2.22) proves (2.1), equating (2.20) and (2.23) proves (2.2), and so on. We easily see that (2.11) yields (2.4) since \( Q_m^x = Q_m | p \).

By the definition (2.5), we have

\[ p_2^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad p_3^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}, \quad p_4^{-1} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \]

and

\[ p (p_3^{-1} \oplus I_1) = s, \]
\[ p (p_2^{-1} \oplus p_2^{-1}) = r. \]
We see from (2.8) that
\[ T^\#_m \otimes S_m | p_3^{-1} \oplus I_1 = Q^\dagger_m | sh_3(p_3^{-1} \oplus I_1) = Q_m | p \cdot sh_3(p_3^{-1} \oplus I_1), \]
which, together with (2.15) and (2.25), verifies (2.22). In a similar way, (2.24) is obtained from (2.10), (2.17), and (2.27). It follows from (2.13) and (2.18) that
\[ sh_2 = sh_2^{(3)} \Sigma_{((13)(24))}. \quad (2.28) \]
Since (13)(24) = (I_2 I_2), we have
\[ (13)(24)(p^{-1} \oplus p^{-1}) = (p^{-1} \oplus p^{-1})(13)(24). \quad (2.29) \]
Thus, we prove Lemma 2.2.

Proof of Lemma 2.2. The following identity was given in the proof of [10, Theorem 6] (see also Remark 2.4 below). For integers \( j \) and \( r \) with \( 1 \leq j \leq r - 1 \),
\[ (Z_{m,j})^\dagger \otimes (Z_{m,r-j})^\dagger = (Z_{m,r})^\dagger | sh_j^{(r)}. \quad (2.30) \]

Identities (2.8) and (2.9) immediately follow from (2.30) with \((j, r) = (3, 4)\) and \((2, 4)\), respectively. We see from (2.30) with \((j, r) = (1, 2)\) that
\[ S_{i,m}^\dagger \otimes S_{i,m}^\dagger = D_{m}^\dagger(x_3, x_4) + D_{m}^\dagger(x_4, x_3) = (D_{m}^\dagger | \Sigma_{((34))})(x_3, x_4), \]
and so we obtain
\[ D_{m}^\dagger \otimes S_{i,m}^\dagger \otimes S_{i,m}^\dagger = D_{m}^\dagger \otimes (D_{m}^\dagger | \Sigma_{((34))}) = D_{m}^\dagger \otimes D_{m}^\dagger | \Sigma_{((34))}. \quad (2.31) \]
Identity (2.10) is verified by combining (2.9) and (2.31). By (2.30) with \((j, r) = (1, 2)\), we have
\[ S_{i,m} \otimes S_{i,m} \otimes S_{i,m} \otimes S_{i,m} = D_{m}^\dagger \otimes S_{i,m}^\dagger \otimes S_{i,m}^\dagger | \Sigma_{((12))}, \quad (2.32) \]
and combining (2.10) and (2.32) we obtain
\[ S_{i,m} \otimes S_{i,m} \otimes S_{i,m} \otimes S_{i,m} = Q^\dagger_m | sh_2 \Sigma_{((34))} \Sigma_{((12))}. \]
Identity (2.11) holds since we see from (2.14) that
\[ sh_2 \Sigma_{((34))} \Sigma_{((12))} = \Sigma_{(4)}, \]
and we complete the proof. \( \square \)
REMARK 2.4. The notation $F^u_r$ in [10] is equivalent to $\mathcal{Z}_{u,r}$ in the present paper. The proof of (2.30) in [10] is summarized in the proof of [15, Lemma 3.1] (see [15, (3.11)]), where note that the inverse $M^{-1}$ are used in [10, 15] for the action of $GL_r(\mathbb{Z})$, unlike the definition (1.4).

Let $\sigma$ and $M = (m_{ij})$ be elements in $\mathfrak{S}_4$ and $GL_4(\mathbb{Z})$, respectively. By the definition of the embedding of $\mathfrak{S}_4$ into $GL_4(\mathbb{Z})$, the multiplications of $\sigma$ from left and right act on the rows and columns of $M$, respectively, as follows.

$$\sigma M = (m_{\sigma^{-1}(i)j}) \quad \text{and} \quad M\sigma = (m_{i\sigma(j)}). \quad (2.33)$$

That is, $\sigma M$ and $M\sigma$ are the matrices produced from $M$ by replacing every $i$-th row and $j$-th column with $\sigma^{-1}(i)$-th row and $\sigma(j)$-th column, respectively. Equation (2.33) will be used repeatedly below.

We are now in a position to prove Lemma 2.3.

Proof of Lemma 2.3. We see from (2.33) that

$$p(34) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad p(234) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad p(1234) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

and so

$$p(34) = r_4(p), \quad p(234) = r_3(p(34)) = r_3 \circ r_4(p), \quad p(1234) = r_2(p(234)) = r_2 \circ r_3 \circ r_4(p),$$

which, together with (1.15) and (2.12), proves (2.15). We can show (2.16) in a similar way, and we omit the proof. We have by (2.15) and (2.16)

$$p s h_3 s h_2^{(3)} = \Psi p s h_2^{(3)} = \Psi \Phi p.$$  

Since $s h_3 s h_2^{(3)} = s h_2 \Sigma_{(34)}$, we obtain (2.17). \qed

2.2 A relation between $Q_m$ and $Q$

It holds that $Q_m \neq Q$, because RMZVs for non-admissible index sets are not zero in general. For example, $\zeta_m(1,1,1,2) = -4\zeta(2,1,1,1)$. (We can make more examples from [10, (5.2)].)

The purpose of this subsection is to prove Proposition 2.5.

PROPOSITION 2.5. We have

$$Q_m|\Omega = Q|\Omega. \quad (2.34)$$

REMARK 2.6. Actual values of RMZVs do not affect (2.34) as we can see in its proof. That is, (2.34) holds if we replace RMZVs with formal variables.

\footnote{It may be worth noting that (34)(24) = (234) and (34)(24)(14) = (234)(14) = (1234).}
We define a block diagonal matrix \( j \) in \( M_4(\mathbb{Z}) \) composed of \((0)\) and \( I_3 \) by
\[
j := (0) \oplus I_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]

We also define four block diagonal matrices as
\[
t_i := (0) \oplus t'_i \quad (i = 1, 2, 3, 4),
\]
where
\[
t'_1 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad t'_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad t'_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad t'_4 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.
\]

We require Lemma 2.7 to prove Proposition 2.5.

**Lemma 2.7.** We have the following equations in \( \mathbb{Z}[M_4(\mathbb{Z})] \):
\[
j p \Sigma \xi_4 = t_1 \Sigma \xi_4, \quad (2.37)
\]
\[
j \Psi q \Sigma \xi_4 = t_1 \Sigma \xi_4 + (t_2 + t_3 + t_4)(e + (243)) \Sigma \xi_4, \quad (2.38)
\]
\[
j \Phi r \Sigma \xi_4 = (t_2 + t_3 + t_4)(243) \Sigma \xi_4, \quad (2.39)
\]
\[
j \Psi s \Sigma \xi_4 = (t_2 + t_3 + t_4 + j) \Sigma \xi_4. \quad (2.40)
\]

We now prove Proposition 2.5. We will then prove Lemma 2.7.

**Proof of Proposition 2.5.** Let \( \tilde{Q}_m \) denoted the difference between \( Q_m \) and \( Q \), that is,
\[
\tilde{Q}_m := Q_m - Q.
\]

Since an index set \( \mathbf{l} = (l_1, l_2, l_3, l_4) \) is not admissible if and only if \( l_1 = 1 \), we see from (1.13) and (1.14) that
\[
Q_m(x) - Q(x) = Q_m(0, x_2, x_3, x_4) = Q_m(x^{j'})
\]
or
\[
\tilde{Q}_m = Q_m|j,
\]
which, together with (2.37), (2.38), (2.39), and (2.40), yield
\[
\tilde{Q}_m|p \Sigma \xi_4 = Q_m|t_1 \Sigma \xi_4,
\]
\[
\tilde{Q}_m|\Psi q \Sigma \xi_4 = Q_m|(t_1 \Sigma \xi_4 + (t_2 + t_3 + t_4)(e + (243)) \Sigma \xi_4),
\]
\[
\tilde{Q}_m|\Phi r \Sigma \xi_4 = Q_m|((t_2 + t_3 + t_4)(243) \Sigma \xi_4),
\]
\[
\tilde{Q}_m|\Psi s \Sigma \xi_4 = Q_m|((t_2 + t_3 + t_4 + j) \Sigma \xi_4),
\]
respectively. Thus,
\[
\tilde{Q}_m|\Sigma \xi_4 = Q_m|j \Sigma \xi_4
\]
\[
= \tilde{Q}_m|p \Sigma \xi_4 - \tilde{Q}_m|\Psi q \Sigma \xi_4 + \tilde{Q}_m|\Phi r \Sigma \xi_4 + \tilde{Q}_m|\Psi s \Sigma \xi_4,
\]
and
\[ \widetilde{Q}_m |\Omega = 0, \]
which verifies (2.34).

Proof of Lemma 2.7. We have
\[ jp = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix} = t_1, \]
which proves (2.37).

We will next show (2.39) and (2.40). Direct calculations similar to (2.42) yield
\[ jr = t_4(243), \]
\[ jr_3(r) = t_3(243), \]
\[ jr_2 \circ r_3(r) = t_2(123). \]
Since
\[ (123) = (243)(13)(24), \]
we see from (1.15) and (2.43) that
\[ j\Phi r = t_2(243)(13)(24) + (t_3 + t_4)(243). \]
Multiplying both sides of (2.44) by \( \Sigma_{\xi_4} \) from the right proves (2.39), because \( (13)(24) \in \xi_4 \) and
\[ (13)(24)\Sigma_{\xi_4} = \Sigma_{\xi_4}. \]
In a similar way to (2.44), we can obtain
\[ j\Psi s = t_2 + t_3 + t_4 + j(1234), \]
which, together with \( (1234) \in \xi_4 \), gives (2.40).

By a straightforward calculation with (1.15), we have
\[ \Psi \Phi q = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \]
\[ + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \]
Thus,

\[ j \Psi \Phi q = t_1 + t_1(34) + t_1(234) + t_2(1234) + t_1(23) + t_1(243) + t_1(24) + t_3(123) + t_3(123) + t_4(123) + t_4(1234) = t_1 \Sigma_{\mathcal{S}_3'} + (t_2 + t_3 + t_4)((1234) + (123)), \quad (2.45) \]

where \( \mathcal{S}_3' \) is the permutation group on the set \( \{2, 3, 4\} \). Multiplying the first and last lines of (2.45) by \( \Sigma_{\mathcal{C}_4} \) from the right, we obtain (2.38), and complete the proof. \( \square \)

3 Proofs

3.1 Proof of Theorem 1.1

We begin by showing Lemma 3.1.

**Lemma 3.1.** Let \( \mathcal{S}_4 \) be the subset \( \{e, (1234)\} \) in \( \mathcal{C}_4 \). We have

\[ S_m^{\otimes 4} - D_m \otimes S_m^{\otimes 2}|\Sigma_{\mathcal{E}_4} + D_m^{\otimes 2}|\Sigma_{\mathcal{E}_4} + T_m \otimes S_m|\Sigma_{\mathcal{E}_4} - Q_m|\Sigma_{\mathcal{E}_4} = q, \quad (3.1) \]

where \( f^{\otimes r} \) means \( f \otimes \cdots \otimes f \).

**Proof.** We denote by \( S_m, D_m, T_m \), and \( Q_m \) the functions of one, two, three, and four variables given by the restrictions of \( \zeta_m \) to the domains \( \mathbb{N}, \mathbb{N}^2, \mathbb{N}^3, \) and \( \mathbb{N}^4 \), respectively. The following identity was shown in [16, Theorem 1.1]:

\[ S_m^{\otimes 4} - D_m \otimes S_m^{\otimes 2}|\Sigma_{\mathcal{E}_4} + D_m^{\otimes 2}|\Sigma_{\mathcal{E}_4} + T_m \otimes S_m|\Sigma_{\mathcal{E}_4} - Q_m|\Sigma_{\mathcal{E}_4} = q_m, \quad (3.2) \]

where \( q_m(l_1, l_2, l_3, l_4) \) is equal to 0 if \( l_1 = l_2 = l_3 = l_4 = 1 \) and \( \zeta(l_1 + l_2 + l_3 + l_4) \) otherwise. Multiplying both sides of (3.2) by \( x_1^{l_1} x_2^{l_2} x_3^{l_3} x_4^{l_4} \) and summing up over all index sets \( (l_1, l_2, l_3, l_4) \in \mathbb{N}^4 \), we can obtain (3.1). We omit the detail since the calculation is straightforward. \( \square \)

We are now able to prove Theorem 1.1.

**Proof of Theorem 1.1.** Since

\[ \Sigma_{\mathcal{E}_4} = (e + (13)(24))(e + (1234)) = \Sigma_{(13)(24)}|\Sigma_{\mathcal{E}_4}, \]

we see from Proposition 2.1 and Lemma 3.1 that

\[ Q_m|\Omega = S_m^{\otimes 4} - D_m \otimes S_m^{\otimes 2}|\Sigma_{\mathcal{E}_4} + D_m^{\otimes 2}|\Sigma_{\mathcal{E}_4} + T_m \otimes S_m|\Sigma_{\mathcal{E}_4} - Q_m|\Sigma_{\mathcal{E}_4} = q, \]

which, together with Proposition 2.5, proves (1.7). \( \square \)
3.2 Proof of Corollary 1.2

We will prove the formulas in Corollary 1.2 by taking the homogeneous parts on both sides of (1.7) and substituting appropriate values for the variables. Although the method of the proof is simple, it requires many calculations, and so we begin by introducing some notation and terminology, which will be useful for presenting the calculations.

We begin by defining the notation and terminology that we will use to show the formulas.

Let \(Q[\mathbb{R}^4]\) be the free module on \(\mathbb{R}^4\) over \(Q\), where we consider elements in \(\mathbb{R}^4\) as row vectors such that \(\mathbf{x} = (x_1, x_2, x_3, x_4)\). For a function \(f(\mathbf{x})\) with the domain \(\mathbb{R}^4\), we extend it to a homomorphism with the domain \(Q[\mathbb{R}^4]\) in a natural way by

\[
f[\alpha] := \sum_i a_i f(x_i) \quad (\alpha = \sum_i a_i \mathbf{x}_i \in Q[\mathbb{R}^4]).
\]

(3.3)

For the extension, we assign the same symbol \(f\), but we use square brackets \([\ ]\) instead of parentheses \((\ )\). The difference between \(f[\alpha]\) and \(f(\alpha)\) is demonstrated as follows:

\[
f[\mathbf{u} - 2\mathbf{v}] = f(\mathbf{u}) - 2f(\mathbf{v}) \quad \text{and} \quad f(\mathbf{u} - 2\mathbf{v}) = f(\mathbf{w}),
\]

where \(\mathbf{u} = (1, 1, 1, 1), \mathbf{v} = (0, 1, 0, 1),\) and \(\mathbf{w} = (1, -1, 1, -1)\). The matrix ring \(M_4(\mathbb{Z})\) acts on \(Q[\mathbb{R}^4]\) by the right multiplication, and so it acts on the extended function (3.3) as

\[
(fM)[\alpha] = f[\alpha M^t].
\]

(3.4)

This action is a generalization of (1.4) since \((fM)[\alpha] = (fM)(\alpha)\) if \(\alpha \in \mathbb{R}^4\), and we will use the same notation, \(fM\). For convenience, we extend (3.4) to an action of the free module \(Q[M_4(\mathbb{Z})]\) in the usual way by \(f(\sum b_j M_j) = \sum b_j f(M_j)\).

For any integer \(l > 4\), let \(Q_l\) and \(q_l\) be the homogeneous parts of degree \(l - 4\) of \(Q\) and \(q\), respectively. Equivalently, these are defined as

\[
Q_l(\mathbf{x}_4) := \sum_{l_4 \text{adm}} \zeta(l_1, l_2, l_3, l_4) x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1},
\]

(3.5)

\[
q_l(\mathbf{x}_4) := \zeta(l) \sum_{l_4 \text{adm}} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1}.
\]

(3.6)

We obtain from (1.7) that

\[
Q_l[\Omega] = q_l.
\]

(3.7)

Let \(\langle E_0 \rangle = \langle E_0 \rangle_Q\) be the submodule in \(Q[Z^4]\) defined by

\[
\langle E_0 \rangle := \left\{ \sum r_i k_i \right\} \text{ where } r_i \in Q \text{ and } k_i = (0, b_i, c_i, d_i) \in Z^4,\}
\]

We define a congruence relation \(\equiv\) such that \(\alpha \equiv \beta\) if and only if \(\alpha - \beta \in \langle E_0 \rangle\), where \(\alpha, \beta \in Q[Z^4]\). We have

\[
Q_l[\alpha] = Q_l[\beta]
\]

when \(\alpha \equiv \beta\), since \(Q_l(0, b, c, d) = 0\) for integers \(b, c, d \in Z\). For short, a row vector \((a, b, c, d) \in Z^4\) will be expressed as

\[
e_{abcd} := (a, b, c, d).
\]

We require Proposition 3.2 to prove Corollary 1.2.
PROPOSITION 3.2. We have the following congruence equations in $\mathbb{Q}[\mathbb{Z}^4]$:  

$$ e_{1000} \Omega^l \equiv e_{1111}, \quad (3.8) $$

$$ e_{1100} \Omega^l \equiv 2(e_{2221} - e_{1221}) + (e_{2211} + e_{2111} - e_{1211}) - 3e_{1111}, \quad (3.9) $$

$$ (e_{1100} - \frac{1}{2}e_{1010}) \Omega^l \equiv e_{2111} + 2e_{1212} - 3e_{1111}, \quad (3.10) $$

$$ (e_{1100} + \frac{1}{2}e_{1010} + e_{1110} + \frac{1}{4}e_{1111}) \Omega^l \equiv 6(e_{4321} - e_{2321}) - 2e_{2121} - e_{1111}. \quad (3.11) $$

We now prove Corollary 1.2. We will then discuss a proof of Proposition 3.2.

Proof of Corollary 1.2. Let $\mathbb{N}_0$ denote the set of non-negative integers, and let $|X|$ denote the number of the elements of a set $X$. Considering the correspondence

$$ k_1 + k_2 + \cdots + k_s \leftrightarrow \circ \cdots \circ | \cdots | \circ \cdots \circ $$

for integers $k_1, k_2, \ldots, k_s \in \mathbb{N}_0$, we see that

$$ |\{(k_1, \ldots, k_s) \in \mathbb{N}_0^s | k_1 + \cdots + k_s = k\}| = \binom{k + s - 1}{s - 1}, \quad (3.12) $$

where $k \geq 0$ and $s \geq 1$.

Let $x_4 = (x_1, x_2, x_3, x_4)$ be a vector in $\{0, 1\}^4$ with $x_4 \neq (0, 0, 0, 0)$, and let $i_1, \ldots, i_r$ be the distinct indices such that $x_{i_1} = \cdots = x_{i_r} = 0$; note that $0 \leq r \leq 3$. We have

$$ \sum_{(w(l_4))=l} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} $$

$$ = |\{l_4 \in \mathbb{N}^4 | w(l_4) = l \text{ and } l_{i_1} = \cdots = l_{i_r} = 1\}| $$

$$ = |\{k_4-r \in \mathbb{N}_0^{1-r} | w(k_{4-r}) = l-4\}|, $$

which, together with (3.12), gives

$$ \sum_{(w(l_4))=l} x_1^{l_1-1} x_2^{l_2-1} x_3^{l_3-1} x_4^{l_4-1} = \binom{l - r - 1}{3 - r}. \quad (3.13) $$

By the definition (3.6), we thus obtain

$$ q_l(x) = \begin{cases} 
\zeta(l) & (x = e_{1000}), \\
(l - 3)\zeta(l) & (x = e_{1100}, e_{1010}), \\
\frac{(l - 2)(l - 3)}{2}\zeta(l) & (x = e_{1110}), \\
\frac{(l - 1)(l - 2)(l - 3)}{6}\zeta(l) & (x = e_{1111}).
\end{cases} \quad (3.13) $$
We now prove the desired formulas. We see from (3.8) and (3.13) that

\[ Q_l(e_{1000} \Omega^t) = Q_l(e_{1111}) \quad \text{and} \quad q_l(e_{1000}) = \zeta(l), \]

respectively, which, together with (3.7), yields

\[ Q_l(e_{1111}) = \zeta(l). \tag{3.14} \]

Rewriting the left-hand side of (3.14), we obtain (1.9).

In a similar way, it follows from (3.9) and (3.13) that

\[ Q_l(e_{1100} \Omega^t) = 2(Q_l(e_{2221}) - Q_l(e_{1221})) + Q_l(e_{2111}) - Q_l(e_{1211}) - 3Q_l(e_{1111}) \]

and

\[ q_l(e_{1100}) = (l - 3)\zeta(l), \]

respectively, and so

\[ 2(Q_l(e_{2221}) - Q_l(e_{1221})) + Q_l(e_{2211}) + Q_l(e_{2111}) - Q_l(e_{1211}) = l\zeta(l). \tag{3.15} \]

which shows (1.10).

Combining (3.10) and (3.13), together with (3.7), yields

\[ Q_l(e_{2111}) + 2Q_l(e_{1121}) = \frac{l + 3}{2} \zeta(l). \tag{3.16} \]

Multiplying both sides of (3.16) by 2, we prove (1.11).

We can deduce from (3.11) and (3.13) that

\[ 6(Q_l(e_{4321}) - Q_l(e_{2321})) - 2Q_l(e_{2121}) = \left( (l - 3) + \frac{l - 3}{2} + \frac{(l - 2)(l - 3)}{2} + \frac{(l - 1)(l - 2)(l - 3)}{24} + 1 \right) \zeta(l). \tag{3.17} \]

We see that

\[ \text{(LHS of (3.17))} = \sum_{l_4: \text{adm}} \left( 4^{l_1-1}3^{l_2}2^{l_3} - 3^{l_2}2^{l_1-l_3-1} - 2^{l_1+l_3-1} \right) \zeta(l) \]

\[ = \sum_{l_4: \text{adm}} \left( 3^{l_2}2^{l_1-1} - 3^{l_2} - 1 \right) 2^{l_1+l_3-1} \zeta(l), \]

and

\[ \text{(RHS of (3.17))} = \left( \frac{36 + 12(l - 2) + (l - 1)(l - 2)(l - 3)}{24} + 1 \right) \zeta(l) \]

\[ = \left( \frac{(l + 7)(l + 2)(l - 3)}{24} + 1 \right) \zeta(l). \]

Thus, multiplying both sides of (3.17) by 2, we obtain (1.12), which completes the proof. \( \square \)

We prepare Lemma 3.3 to prove Proposition 3.2.
LEMMA 3.3. We have the following congruence equations in $\mathbb{Q} \mathbb{Z}_4^4$:

(i) 
\[
x (p \Sigma e_4)^t \\
\equiv \begin{cases} 
6(e_{1111} + e_{1110} + e_{1100} + e_{1000}) & (x = e_{1000}), \\
4(e_{2221} + e_{2211} + e_{2210} + e_{2111} + e_{2110} + e_{2100}) & (x = e_{1100}, e_{1010}), \\
6(e_{3321} + e_{3221} + e_{3211} + e_{3210}) & (x = e_{1110}), \\
24e_{4321} & (x = e_{1111}).
\end{cases}
\]

(ii) 
\[
x (\Psi \Phi q \Sigma e_4)^t \\
\equiv \begin{cases} 
12e_{1000} + 10e_{1100} + 8e_{1110} + 6e_{1111} & (x = e_{1000}), \\
6(e_{2100} + e_{1111}) + 5e_{2110} + 4(e_{2210} + e_{2111} + e_{1110}) + 3e_{2211} + 2(e_{2221} + e_{1221} + e_{1211} + e_{1210} + e_{1121} + e_{1100}) + e_{1101} + e_{1011} + e_{1010} + e_{1001} & (x = e_{1100}), \\
8e_{3210} + 6e_{2110} + 4(e_{2210} + e_{2111} + e_{1221} + e_{1210} + e_{1211}) + 2(e_{2211} + e_{1211} + e_{1011} + e_{1010} + e_{1001}) & (x = e_{1010}), \\
8e_{2221} + 6(e_{3210} + e_{2321} + e_{2211}) + 4(e_{3211} + e_{2210} + e_{2121} + e_{2111}) + 2(e_{3221} + e_{2110} + e_{2101}) & (x = e_{1110}), \\
24e_{3321} + 16e_{3221} + 8e_{3211} & (x = e_{1111}).
\end{cases}
\]

(iii) 
\[
x (\Phi r \Sigma e_4)^t \\
\equiv \begin{cases} 
3e_{1000} + 2e_{1100} + e_{1110} & (x = e_{1000}), \\
3e_{1111} + 2(e_{1210} + e_{1110}) + e_{1211} + e_{1100} + e_{1011} + e_{1010} + e_{1001} & (x = e_{1100}), \\
4e_{2100} + 2e_{2110} & (x = e_{1010}),
\end{cases}
\]

(iv) 
\[
x (\Psi s \Sigma e_4)^t \\
\]

\[\text{(3.18)}\]
which proves (3.8).

By (3.19) and (3.20) for \( x = e_{1100} \), we have
\[
\begin{align*}
\text{e}_{1100} \left\{ (\Psi q \Sigma \xi_4)^t - (\Phi r \Sigma \xi_4)^t \right\} \\
\equiv 6(\text{e}_{2100} + 5\text{e}_{2110} + 4(\text{e}_{2210} + 4\text{e}_{2211} + 3(\text{e}_{2221} + \text{e}_{1211} + \text{e}_{1110} + \text{e}_{1111}) \\
+ 2(\text{e}_{2221} + \text{e}_{1221} + \text{e}_{1121} + \text{e}_{1110} + \text{e}_{1111})) - \text{e}_{1000} \\
\equiv \text{e}_{1111}.
\end{align*}
\]
which proves (3.8).

Before discussing a proof of Lemma 3.3, we will show Proposition 3.2 by substituting the congruence equations in Lemma 3.3 into
\[
\begin{align*}
x \Omega^t = x \left\{ (p \Sigma \xi_4)^t - (\Psi q \Sigma \xi_4)^t + (\Phi r \Sigma \xi_4)^t + (\Psi s \Sigma \xi_4)^t - \Sigma^t \xi_4 \right\}
\end{align*}
\]
for \( x \in \{ e_{1000}, e_{1100}, e_{1010}, e_{1110}, e_{1111} \} \). We note that the right-hand sides of (3.18), (3.19), (3.20), (3.21), and (3.22) in Lemma 3.3 include vectors in
\[ V = \{ e_{i1234} | i_j = 0 (\exists j) \}; \]
however, those of (3.8), (3.9), (3.10), and (3.11) in Proposition 3.2 do not include such vectors. That is, in calculating (3.23), the vectors in \( V \) cancel each other. This fact will help us with the proof of Proposition 3.2.

**Proof of Proposition 3.2.** Substituting equations from (3.18) through (3.22) for \( x = e_{1000} \) into the right-hand side of (3.23), we obtain
\[
\begin{align*}
\text{e}_{1000} \Omega^t \\
\equiv 6(\text{e}_{1111} + \text{e}_{1110} + \text{e}_{1100} + \text{e}_{1000}) - (12\text{e}_{1000} + 10\text{e}_{1100} + 8\text{e}_{1110} + 6\text{e}_{1111}) \\
+ (3\text{e}_{1000} + 2\text{e}_{1100} + \text{e}_{1110}) + (4\text{e}_{1000} + 2\text{e}_{1100} + \text{e}_{1110} + \text{e}_{1110}) - \text{e}_{1000} \\
\equiv \text{e}_{1111},
\end{align*}
\]
which proves (3.8).

**Proof of Proposition 3.2.** Substituting equations from (3.18) through (3.22) for \( x = e_{1100} \), we have
\[
\begin{align*}
\text{e}_{1100} \left\{ (\Psi q \Sigma \xi_4)^t - (\Phi r \Sigma \xi_4)^t \right\} \\
\equiv 6(\text{e}_{2100} + 5\text{e}_{2110} + 4(\text{e}_{2210} + 4\text{e}_{2211} + 3(\text{e}_{2221} + \text{e}_{1211} + \text{e}_{1110} + \text{e}_{1111}) \\
+ 2(\text{e}_{2221} + \text{e}_{1221} + \text{e}_{1121} + \text{e}_{1110} + \text{e}_{1111}) + \text{e}_{1211} + \text{e}_{1101} + \text{e}_{1100},
\end{align*}
\]
and by (3.21) and (3.22) for \( x = e_{1100} \),
\[
\begin{align*}
\text{e}_{1100} \left\{ (\Psi s \Sigma \xi_4)^t - \Sigma^t \xi_4 \right\}
\end{align*}
\]
\begin{align*}
\Rightarrow 2(e_{2100} + e_{1110} + e_{1110}) &+ e_{2111} + e_{2110} + e_{1101} + e_{1100}. & \quad (3.25) \\

\text{Noting (3.24) and (3.25), we see that substituting equations from (3.18) through (3.22) for } x = e_{1100} \text{ into the right-hand side of (3.23) gives}
\end{align*}

\begin{align*}
e_{1100} \Omega^t &\equiv e_{1100} (p \Sigma e_i)^t \\
&- e_{1100} \{(\Psi \Phi q \Sigma e_i)^t - (\Psi \Phi r \Sigma e_i)^t\} + e_{1100} \{(\Psi s \Sigma e_i)^t - \Sigma e_i^t\} \\
&\equiv 4(e_{2221} + e_{2211} + e_{2210} + e_{2111} + e_{2110} + e_{1210}) \\
&- 6e_{2100} - 5e_{2110} - 4(e_{2210} + e_{2111}) - 3(e_{2211} + e_{1111}) \\
&- 2(e_{2221} + e_{1221} + e_{1121} + e_{1110}) - e_{2111} - e_{1101} - e_{1100} \\
&+ 2(e_{2110} + e_{1121} + e_{1110}) + e_{2111} + e_{2110} + e_{1101} + e_{1100} \\
&\equiv -3e_{1111} + 2(e_{2221} - e_{1221}) + e_{2211} + e_{2111} - e_{1211},
\end{align*}

which proves (3.9).

Similarly, we obtain
\begin{align*}
e_{1010} \Omega^t &\equiv 4(e_{2221} + e_{2211} + e_{2210} + e_{2111} + e_{2110} + e_{1210}) \\
&- 6e_{2100} + 6e_{2110} + 4(e_{2210} + e_{2111} + e_{1221} + e_{1211} + e_{1210} + e_{1121}) \\
&+ 2(e_{2211} + e_{1101} + e_{1111} + e_{1101} + e_{1100}) \\
&+ 4e_{2110} + 2e_{2110} + 4(e_{1210} + e_{1101}) \\
&+ 2(e_{1221} + e_{1101} + e_{1111} + e_{1001}) - 2e_{1010},
\end{align*}

which can be summarized as
\begin{align*}
e_{1010} \Omega^t &\equiv 4(e_{2221} - e_{1221} - e_{1121}) + 2(e_{2211} - e_{1211}). & \quad (3.26)
\end{align*}

Combining (3.9) and (3.26) yields
\begin{align*}
\left(e_{1100} - \frac{1}{2}e_{1100}\right) \Omega^t &\equiv 2(e_{2221} - e_{1221}) + e_{2211} + e_{1121} + 3e_{1111} \\
&- 2(e_{2221} - e_{1221} - e_{1121}) - e_{2211} + e_{1211} \\
&\equiv e_{2111} + 2e_{1211} - 3e_{1111},
\end{align*}

which proves (3.10).

We see from (3.18) and (3.19) for \( x = e_{1110} \) that
\begin{align*}
e_{1110} \{(p \Sigma e_i)^t - (\Psi \Phi q \Sigma e_i)^t\} \\
&\equiv -8e_{2221} + 6(e_{3231} - e_{2321} - e_{2221}) \\
&+ 4(e_{3221} - e_{2210} - e_{2121} - e_{2111}) + 2(e_{3211} - e_{2110} - e_{2101}), & \quad (3.27)
\end{align*}

and from (3.20), (3.21), and (3.22) for \( x = e_{1100} \) that
\begin{align*}
e_{1110} \{(\Phi r \Sigma e_i)^t + (\Psi s \Sigma e_i)^t - \Sigma e_i^t\} \\
&\equiv 4(e_{2210} + e_{1211}) + 3e_{1111} \\
&+ 2(e_{2211} + e_{2121} + e_{2111} + e_{2110} + e_{2101} + e_{1211} + e_{1121}). & \quad (3.28)
\end{align*}

Substituting (3.27) and (3.28) into the right-hand side of (3.23), we obtain
\begin{align*}
e_{1110} \Omega^t &\equiv -8e_{2221} + 6(e_{3231} - e_{2321}) + 4(e_{3221} + e_{1211} - e_{2211}) \\

\text{17}
Combining (3.9), (3.26), (3.29), and (3.30) yields
\[ e_{1111} \Omega^t = 24(e_{3211} - e_{1111}) + 16(e_{2221} - e_{3221}) + 8(e_{2221} - e_{3221}) + 4(e_{2111} - e_{1111}). \] (3.30)

Substituting equations from (3.18) through (3.22) for \( x = e_{1111} \) into the right-hand side of (3.23), we also have
\[ e_{1111} \Omega^t = 24(e_{3211} - e_{1111}) + 16(e_{2221} - e_{3221}) + 8(e_{2221} - e_{3221}) + 4(e_{2111} - e_{1111}). \] (3.30)

Combining (3.9), (3.26), (3.29), and (3.30) yields
\[
\begin{align*}
\left(e_{1100} + \frac{1}{2} e_{1010} + e_{1110} + \frac{1}{4} e_{1111}\right) \Omega^t \\
\equiv \{2(e_{2221} - e_{1221}) + (e_{2211} + e_{2111} - e_{1211}) - 3e_{1111}\} \\
+ \{2(e_{2221} - e_{1221} - e_{1211}) + e_{2211} - e_{1211}\} \\
+ \{-8e_{2221} + 6(e_{3321} - e_{2321}) + 4(e_{3221} - e_{2211} - e_{2211})\} \\
+ \{3e_{1111} + 2(e_{2311} + e_{1211} + e_{1211} - e_{2211} - e_{1111})\} \\
+ \{6(e_{4321} - e_{3321}) + 4(e_{2221} - e_{3221}) + 2(e_{2211} - e_{3211} + e_{2111} - e_{1111})\} \\
\equiv 6(e_{4321} - e_{2321}) - 2e_{2211} - e_{1111},
\end{align*}
\]

which proves (3.11), and this completes the proof. \(\square\)

For a matrix \( M = (m_{ij}) \) in \( M_4(\mathbb{Z}) \) and distinct integers \( i_1, \ldots, i_r \) in \( \{1, 2, 3, 4\} \) (\( r \leq 4 \)), we define a row vector in \( \mathbb{Z}^4 \) by
\[ M_{[i_1 \ldots i_r]} := e_{\delta_{i_1}, \delta_{i_2}, \delta_{i_3}, \delta_{i_4}}M, \]
where \( \delta_i \) is 1 if \( i \in \{i_1, \ldots, i_r\} \) and 0 otherwise. Equivalently, \( M_{[i_1 \ldots i_r]} \) is determined by
\[ M_{[i_1 \ldots i_r]} = \left( \sum_{k=1}^r m_{i_k1}, \sum_{k=1}^r m_{i_k2}, \sum_{k=1}^r m_{i_k3}, \sum_{k=1}^r m_{i_k4} \right). \]

For example, if
\[ M = p^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \]

then
\[ M_{[2]} = e_{0100}M = e_{1100} \quad \text{and} \quad M_{[12]} = e_{1101}M = e_{3211}. \]

Recall that a permutation \( \sigma \) in \( \mathfrak{S}_4 \) is identified with the matrix \((\delta_{\sigma(j)})_{1 \leq j \leq 4} \). Thus, \( x\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}) \), and we can deduce the following equations by direct calculations:
\[
\begin{align*}
x \Sigma_{\mathfrak{S}_4} & = \begin{cases} 
6(e_{1000} + e_{0100} + e_{0010} + e_{0001}) & (x = e_{1000}), \\
4(e_{1100} + e_{1010} + e_{1001}) & (x = e_{1010}), \\
+ e_{0110} + e_{0101} + e_{0011}) & (x = e_{1100}), \\
6(e_{1110} + e_{1101} + e_{1011} + e_{0111}) & (x = e_{1110}), \\
24e_{1111} & (x = e_{1111}), 
\end{cases}
\end{align*}
\]
\[ x \Sigma_{c_4} = \begin{cases} e_{1000} + e_{0100} + e_{0010} + e_{0001} & (x = e_{1000}), \\ e_{1100} + e_{0110} + e_{0011} + e_{1001} & (x = e_{1100}), \\ 2(e_{1010} + e_{0101}) & (x = e_{1010}), \\ e_{1110} + e_{1101} + e_{1011} + e_{0111} & (x = e_{1110}), \\ 4e_{1111} & (x = e_{1111}). \end{cases} \tag{3.32} \]

For any \( M \in M_4(\mathbb{Z}) \), we thus have
\[ x \Sigma_{c_4}M = \begin{cases} 6(M_{[1]} + M_{[2]} + M_{[3]} + M_{[4]}) & (x = e_{1000}), \\ 4(M_{[12]} + M_{[13]} + M_{[14]}) \\ + M_{[23]} + M_{[24]} + M_{[34]}) & (x = e_{1100}, e_{1010}), \\ 6(M_{[123]} + M_{[124]} + M_{[134]} + M_{[234]}) & (x = e_{1110}), \\ 24M_{[1234]} & (x = e_{1111}), \end{cases} \tag{3.33} \]

and
\[ x \Sigma_{d_4}M = \begin{cases} M_{[1]} + M_{[2]} + M_{[3]} + M_{[4]} & (x = e_{1000}), \\ M_{[12]} + M_{[23]} + M_{[34]} + M_{[14]} & (x = e_{1100}), \\ 2(M_{[13]} + M_{[24]}) & (x = e_{1010}), \\ M_{[123]} + M_{[124]} + M_{[134]} + M_{[234]} & (x = e_{1110}), \\ 4M_{[1234]} & (x = e_{1111}). \end{cases} \tag{3.34} \]

Using (3.33) and (3.34), we now prove Lemma 3.3 for the completeness of the proof of Proposition 3.2, or for that of Corollary 1.2.

**Proof of Lemma 3.3.** We obtain by (3.33)
\[ x (p \Sigma_{d_4})^t = x \Sigma_{d_4}p^t = \begin{cases} 6(e_{1000} + e_{1100} + e_{1110} + e_{1111}) & (x = e_{1000}), \\ 4(e_{2100} + e_{2110} + e_{2111} + e_{2210} + e_{2211} + e_{2221}) & (x = e_{1100}, e_{1010}), \\ 6(e_{3210} + e_{3211} + e_{3321} + e_{3321}) & (x = e_{1110}), \\ 24e_{4321} & (x = e_{1111}), \end{cases} \]
which proves (3.18).

We can obtain by (3.34) the following congruence equations:
\[ x \Sigma_{c_4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e_{1000} + e_{1110} & (x = e_{1000}), \\ e_{1100} + e_{1210} + e_{1111} + e_{1001} & (x = e_{1100}), \\ 2e_{2110} & (x = e_{1010}), \\ e_{2210} + e_{1101} + e_{2111} + e_{1211} & (x = e_{1110}), \\ 4e_{2211} & (x = e_{1111}), \end{pmatrix} \tag{3.35} \]

19
We see from (1.15) that
\[
(\Phi_r)^t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix},
\]
and so
\[
x(\Phi_r \Sigma \xi_4)^t = x \Sigma \xi_4 (\Phi_r)^t
\]
\[
= x \Sigma \xi_4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + x \Sigma \xi_4 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} + x \Sigma \xi_4 \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{pmatrix}.
\]
Thus, the sum of (3.35), (3.36), and (3.37) proves (3.20).

With the same method that we used for (3.20), we can prove (3.19) and (3.21). We omit these proofs because of space limitations.

Since \( e_{abcd} = 0 \) for \( a = 0 \), (3.22) immediately follows from (3.34) with \( M = I \), which completes the proof. 

\section*{Acknowledgements}

This work was supported by JST ERATO Grant Number JPMJER1201, Japan.

\section*{Appendix A}

Let \( \mathfrak{C}_3 \) denote the cyclic subgroup \( \langle (123) \rangle \) in \( \mathfrak{S}_3 \). We note that \( \mathfrak{C}_3 \) is the alternating group \( \mathfrak{A}_3 \) of degree 3. We define the formal power series \( \mathcal{d}(x_1, x_2) \) and \( t(x_1, x_2, x_3) \) as
\[
\mathcal{d}(x_1, x_2) := \sum_{(l_2) > 2} \zeta(l_1 + l_2)x_1^{l_1 - 1}x_2^{l_2 - 1},
\]
and
\[
t(x_1, x_2, x_3) := \sum_{(l_3) > 2} \zeta(l_1 + l_2 + l_3)x_1^{l_1 - 1}x_2^{l_2 - 1}x_3^{l_3 - 1}.
\]
\[ t(x_1, x_2, x_3) := \sum_{l_3 \text{ (s.t. } l_3 > 3)} \zeta(l_1 + l_2 + l_3)x_1^{l_1-1}x_2^{l_2-1}x_3^{l_3-1}, \]

respectively.

The generalizations of (1.1) for DZVs and TZVs from the viewpoint of the generating functions \( D(x_1, x_2) \) and \( T(x_1, x_2, x_3) \) can be stated as follows:

\[
\delta(x_1, x_2) = \sum_{\sigma \in \mathfrak{S}_2} (D(x_{\sigma(1)} + x_{\sigma(2)}, x_{\sigma(2)}) - D(x_{\sigma(1)}, x_{\sigma(2)})), \tag{3.38}
\]

\[
t(x_1, x_2, x_3) = \sum_{\sigma \in \mathfrak{S}_3} T(x_{\sigma(1)} + x_{\sigma(2)} + x_{\sigma(3)}, x_{\sigma(2)} + x_{\sigma(3)}, x_{\sigma(3)}) \\
- \sum_{\sigma \in \mathfrak{C}_3} \left( \sum_{\tau \in \langle (23) \rangle} T(x_{\sigma(1)} + x_{\sigma(2)}, x_{\sigma(3)} + x_{\sigma\tau(3)}, x_{\sigma\tau(3)}) + T(x_{\sigma(1)} + x_{\sigma(2)}, x_{\sigma(3)} - T(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})) \right). \tag{3.39}
\]

The above formulas were proved in [3, (27)] and [14, Theorem 1.2], respectively. Note that the original formula in [14, Theorem 1.2] was not written in terms of formal power series but in terms of homogeneous polynomials.

Let \( q_3, r_3 \) be the matrices in \( GL_3(\mathbb{Z}) \) given by

\[
q_3 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad r_3 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix},
\]

and let \( \Phi_3 \) be the element in \( \mathbb{Z}[GL_3(\mathbb{Z})] \) given by

\[
\Phi_3 = I_3 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.
\]

Recall from (2.5) that

\[
p_2 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad p_3 = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\]

We can now restate (3.38) and (3.39) as follows.

**Theorem** ([3, 14]). We have

\[
D|(p_2 - I_2)\Sigma_{\mathfrak{S}_2} = \delta, \tag{3.40}
\]

\[
T|(p_3\Sigma_{\mathfrak{S}_3} - (q_3 + \Phi_3 r_3 - I_3)\Sigma_{\mathfrak{C}_3}) = t. \tag{3.41}
\]

**References**

[1] M. Eie, W-C. Liaw, and Y. L. Ong, *A restricted sum formula among multiple zeta values*, J. Number Theory 129 (2009), 908–921.
[2] L. Euler, *Meditationes circa singulare serierum genus*, Novi Comm. Acad. Sci. Petropol. 20 (1776), 140–186; reprinted in Opera Omnia Ser. I, vol. 15, 217–267.

[3] H. Gangl, M. Kaneko, and D. Zagier, *Double zeta values and modular forms*, Automorphic forms and zeta functions, 71–106, World Sci. Publ., Hackensack, NJ, 2006.

[4] A. Granville, *A decomposition of Riemann’s zeta-function*, Analytic Number Theory (Kyoto, 1996), 95–101, London Math. Soc. Lecture Note Ser., 247, Cambridge Univ. Press, Cambridge, 1997.

[5] L. Guo and B. Xie, *Weighted sum formula for multiple zeta values*, J. Number Theory 129 (2009), 2747–2765.

[6] M. E. Hoffman, *Multiple harmonic series*, Pacific J. Math. 152 (1992), 275–290.

[7] M. E. Hoffman, *On multiple zeta values of even arguments*, preprint; arXiv:1205.7051v2 [math.NT], 2012.

[8] M. E. Hoffman and C. Moen, *Sums of triple harmonic series*, J. Number Theory 60 (1996), 329–331.

[9] M. E. Hoffman and Y. Ohno, *Relations of multiple zeta values and their algebraic expression*, J. Algebra 262 (2003), 332–347.

[10] K. Ihara, M. Kaneko, and D. Zagier, *Derivation and double shuffle relations for multiple zeta values*, Compositio Math. 142 (2006), 307–338.

[11] T. Machide, *Weighted sums with two parameters of multiple zeta values and their formulas*, Int. J. Number Theory 8 (2012), 1903–1921.

[12] T. Machide, *A parameterized generalization of the sum formula for quadruple zeta values*, preprint; arXiv:1210.8005 [math.NT], 2012.

[13] T. Machide, *Some restricted sum formulas for double zeta values*, Proc. Japan Acad. Ser. A Math. Sci. 89 (2013), 51–54.

[14] T. Machide, *Extended double shuffle relations and the generating function of triple zeta values of any fixed weight*, Kyushu J. Math. 67 (2013), 281–307.

[15] T. Machide, *Congruence identities of regularized multiple zeta values involving a pair of index sets*, to appear in Int. J. Number Theory.

[16] T. Machide, *Identities involving cyclic and symmetric sums of regularized multiple zeta values*, Pacific J. Math. 286 (2017), 307–359.

[17] C. Markett, *Triple sums and the Riemann zeta function*, J. Number Theory 48 (1994), 113–132.

[18] T. Nakamura, *Restricted and weighted sum formulas for double zeta values of even weight*, Šiauliai Math. Semin. 4(12) (2009), 151–155.

[19] Y. Ohno, *A generalization of the duality and sum formulas on the multiple zeta values*, J. Number Theory 74 (1999), 39–43.

[20] Y. Ohno and W. Zudilin, *Zeta stars*, Commun. Number Theory Phys. 2 (2008), 47–58.
[21] Y. L. Ong, M. Eie, and W-C. Liaw, *On generalizations of weighted sum formulas of multiple zeta values*, Int. J. Number Theory 9 (2013), 1185–1198.

[22] Z. Shen and T. Cai, *Some identities for multiple zeta values*, J. Number Theory 132 (2012), 314–323.