QED Energy-Momentum Trace as a Force in Astrophysics

Lance Labun and Johann Rafelski

Department of Physics, University of Arizona, Tucson, Arizona, 85721 USA, and Department für Physik der Ludwig-Maximilians-Universität München und Maier-Leibniz-Laboratory, Am Coulombwall 1, 85748 Garching, Germany

(Dated: 15 February, 2010)

We study the properties of the trace $T$ of the QED energy-momentum tensor in the presence of quasi-constant external electromagnetic fields. We exhibit the origin of $T$ in the quantum nonlinearity of the electromagnetic theory. We obtain the quantum vacuum fluctuation-induced interaction of a particle with the field of a strongly magnetized compact stellar object.

PACS numbers: 11.15.Tk,12.20.Ds,95.36.+x,97.60.Bw

Introduction

Quantum electrodynamics (QED) in (quasi-)constant, homogeneous external electromagnetic (EM) fields provides an opportunity to study the properties of the vacuum state structure under the conditions of extreme external fields. In the presence of an electromagnetic field that varies negligibly on the space-time scale of the electron-positron fluctuations in the vacuum $\lambda_e = \hbar/m_e c$ leads to an effective nonlinear electromagnetic theory via the Euler-Heisenberg (EH) effective action $[1–10]$. The physical observables and effective action induced by quasi-constant external electromagnetic fields are well-defined, because QED is an infrared-stable theory in which the electron mass $m_e$ is defined, because QED is an infrared-stable theory in the presence of external fields. In the presence of an electromagnetic field structure under the conditions of extreme external fields. We exhibit the origin of $T$ in the quantum nonlinearity of the electromagnetic theory. We obtain the quantum vacuum fluctuation-induced interaction of a particle with the field of a strongly magnetized compact stellar object.

\[ T_{\mu\nu} = \frac{2}{\sqrt{-g}} \delta g_{\mu\nu} \int d^4x \sqrt{-g} V_{\text{eff}} \]

\[ = \varepsilon T_{\text{Max}}^{\mu\nu} - g_{\mu\nu} \left( V_{\text{eff}} - S \frac{\partial V_{\text{eff}}}{\partial S} - P \frac{\partial V_{\text{eff}}}{\partial P} \right) \]

The form of Eq. (5) agrees with Eq. (A3) in [17] and Eq. (4.17) in [18].

The first term in Eq. (5) is traceless. The second term in Eq. (5) provides as noted the only possible covariant extension and is identically the trace of energy momentum tensor. Using Eq. (1)

\[ T_{\mu} = -4 \left( V_{\text{eff}} - S \frac{\partial V_{\text{eff}}}{\partial S} - P \frac{\partial V_{\text{eff}}}{\partial P} \right) = -m \frac{dV_{\text{eff}}}{dm} \]

Our separation of the trace $T$ from the off-diagonal Maxwell-like part of the energy-momentum tensor isolates the field induced, gravitating energy-momentum of the vacuum.

Before exploring the physical consequences, we must pause and clarify the precise meaning of $T$. Terms linear in the invariant $S$ do not contribute to the right side of Eq. (6) since they cancel explicitly in the middle parentheses. Only nonlinear (in $S$, $P$) EM-theories can have an
energy-momentum trace and the bar above $V_{\text{eff}}$ reminds us of this. This is another way to say that QED with massive electrons is ab initio not conformally symmetric. Massive QED does not share the more challenging issues, such as conformal symmetry breaking, surrounding parallel efforts in quantum chromodynamics (QCD).

A better understanding of these remarks is achieved in QED by connecting $\mathcal{T}$ to the Dirac (electron-positron) condensate induced in the vacuum, which is directly related to the effective action

$$-m\langle\bar{\psi}\psi\rangle = im\text{tr}(S_F - S_F^0) = m\frac{dV_{\text{eff}}}{dm}. \quad (7)$$

The middle expression of Eq. (7) exhibits the condensate as the difference between normal ordering of operators and the trace of the energy-momentum trace and the bar above $V_{\text{eff}}$.

Combining Eqs. (4), (7), and (8), we obtain the well-known relation

$$\mathcal{T} = \frac{2\alpha}{3\pi}\mathcal{S} + m\langle\bar{\psi}\psi\rangle \quad (9)$$

$$= \frac{2\alpha^2}{4\beta}\left(7P^2 + 4S^2\right) + O(\alpha^3). \quad (10)$$

Eq. (9) displays two contributions to the trace of the energy-momentum tensor: gauge field and matter field fluctuations. There is an important exact cancellation between these two terms. Were the first term in Eq. (9) (erroneously) omitted, the trace $\mathcal{T}$ would be that much greater, and for any applied magnetic field the 00-energy-density-component of the energy-momentum tensor would be negative. The QED vacuum would be unstable, and the naive perturbative QED vacuum could reduce its energy by spontaneously generating a state with magnetic field. For all practical purposes, the form of Eq. (9) is confirmed by the observed stability of the QED vacuum and work claiming otherwise will need to address that important issue.

In fact, the relative sign in Eq. (9) agrees with Eq. (5) of Ref. [10] and Eqs.(35) in [22] with the recognition that the (fermion) Gell-Mann-Low $\beta$-function in QED is positive definite. Our Eq. (9) (and the more explicit form of $\mathcal{T}$, Eq. (11) below), agrees with [22], which result is a bit surprising since it follows from the clearly contradictory supposition that $\mathcal{T} = m\frac{dV_{\text{eff}}}{dm}$. Moreover, there are quite a few other instances in literature where the first term in Eq. (9) is omitted.

### Strong Fields Simulacrum of Dark Energy

Applying Eq. (9) with the Euler-Heisenberg-Schwinger effective action (using Schwinger’s notation and units in which $\alpha = e^2/4\pi$) gives an explicit formula for the trace,

$$\mathcal{T} = \frac{2\alpha}{3\pi}\mathcal{S} - \frac{m^2}{4\pi^2}\int_0^\infty ds e^{-ms^2}\left(e^2\frac{\text{coth}(ebs)}{\text{tan}(eas)} - \frac{1}{s^2}\right) \quad (11)$$

wherein the invariant magnetic- and electric-like field strengths are

$$b^2 = \sqrt{S^2 + P^2} \leftrightarrow B^2, \quad a^2 = \sqrt{S^2 + P^2} \rightarrow E^2,$$

reducing as indicated to the classical magnetic and electric fields when one invariant vanishes.

In numerical evaluation of the energy-momentum tensor for arbitrarily strong fields we employ the method developed in [11]. Consider first the stable field configuration $B \neq 0, E = 0$. The subtracted meromorphic (i.e. residue) expansions of the function

$$x \text{ coth } x - 1 = \sum_{k=1}^\infty \frac{2x^{2k}}{x^2 + k^2\pi^2} = \frac{x^2}{3} - \sum_{k=1}^\infty \frac{1}{(k\pi)^2} \frac{2x^4}{x^2 + k^2\pi^2}. \quad (12)$$

display the stabilizing change in sign following the second subtraction. The sums and integrals are absolutely convergent, so we may resum the resulting series, obtaining

$$V_{\text{eff}}(B) = \frac{m^4}{16\pi^2}\int_0^\infty dz \ln(z^2 + 1) \ln(1 - e^{-\beta'z}), \quad (13)$$

$$-m\langle\bar{\psi}\psi\rangle_B = -\frac{m^4}{2\pi^2}\int_0^\infty \frac{\ln(1 - e^{-\beta'z})}{1 + z^2} \frac{dz}{1 + z^2} > 0, \quad (14)$$

$$\mathcal{T}_B = -\frac{m^4}{2\pi^2}\int_0^\infty \frac{\ln(1 - e^{-\beta'z})}{1 + z^2} \frac{dz}{1 + z^2} > 0, \quad (15)$$

in which $\beta' = \pi m^2/eB = \pi/(B/E_0)$. Eq. (13) presents the (renormalized) effective action also seen in Ref. [6]. Numerical evaluations of the condensate Eq. (14) and trace Eq. (15) in the presence of magnetic and electrical field (real part only) are shown in figure [11]. We discuss elsewhere [10] the case that both $E, B$ are non zero, how real and imaginary particles contribute together for electric fields, and the case of spin-0 matter fields. For fields way above critical the results presented in figure [11] are not accessible in practice by direct integration of the proper time representation Eq. (11).

For the electric field case, the corresponding meromorphic expansions of $x \cot x$ show poles on the real $s$-axis. We assign to the mass a small imaginary component $m^2 \rightarrow m^2 + i\epsilon$, replacing in Eq. (14) and Eq. (15) the denominator $z^2 + 1 \rightarrow z^2 - 1 + i\epsilon$. Thus, in the presence of an electric field, there is also nonperturbative imaginary contribution to $\mathcal{T}$.

From now on in this work we address strong magnetic fields. $\mathcal{T}$ in the presence of a magnetic field is positive for any given field $B$, in contrast to the negative of the condensate $m\langle\bar{\psi}\psi\rangle$. The manifest signs of the two expressions Eqs. (14) and (15), which determine the physics
outcome of this investigation justify the time and effort spent showing how \( \mathcal{T} \) does not include the term linear in \( S \), while \( m(\bar{\psi}\psi) \) does. Clarification of this exclusion is necessary since as noted \( \mathcal{T} \) and \( m(\bar{\psi}\psi) \) are often conflated in literature.

The trace \( \mathcal{T} \) gravitates, just as the Casimir energy does \cite{24}. Because in the Euler-Heisenberg-Schwinger calculation the ‘constant’ external field is global in extent, this energy-momentum is manifested in the form of a cosmological constant. In contrast to matter, for which the particle pressure acts outwards, the pressure part of energy-momentum tensor described by \(-\mathcal{T}/4\) acts inwards. This is a general feature of any ‘false’ vacuum state: the outside true vacuum the squeezes the false, higher energy density vacuum out of existence.

The sign reversal of \( \mathcal{T} \) pressure (compared to pressure of regular matter) overwhelms the gravity of the positive energy density, providing the anti-gravity effect associated with dark energy. The similarity of \( \mathcal{T} \) to the cosmological constant was noted before by Schützhold \cite{19} and can be made explicit in the Einstein equation by separating the trace, \( T_{\mu\nu} = T_{\mu\nu} - g_{\mu\nu} T/4 \)

\[
\frac{1}{8\pi G} \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = -T_{\mu\nu} - g_{\mu\nu} \left( \frac{T}{4} + \frac{\lambda}{2} \right),
\]

(16)

With a sign like that of dark energy \( \Lambda/4\pi G \equiv \lambda \simeq (2.3\text{meV})^4 = 4.1 \times 10^{-34} m_4^4 \), \( T \) is the dominant contribution in a domain of space with strong fields and is naively expected to generate a pressure that sweeps out matter, in analogy with the cosmological acting constant at large scales and pushing the universe apart. For comparison, we note that the magnetic field-induced vacuum energy rivals cosmological dark energy \( \lambda \) at \( B_d = 108 T \), the scale of the largest static laboratory fields.

Energy-momentum sources in Eq. (16) involving local dark energy-like contributions have only recently been studied \cite{24,27}. To see that the effects of local \( \mathcal{T} \) are anti-gravitational just like cosmological \( \lambda \), we inspect the Oppenheimer-Volkov equation

\[
\frac{dp}{dr} = -\frac{G}{c^2} \left( T_0^0 + T_4^i \right) \frac{M + 4\pi r^3 T_i^i}{r(r - 2GM)}
\]

(17)

where \( M(r) = \int_0^r T_0^0 4\pi r^2 dr' \) and \( T_i^i = \rho - \lambda \) is assumed isotropic. \( \lambda \) does not contribute to the first term which is always non-negative: \( T_0^0 + T_4^i = (\rho + \lambda) + (\rho - \lambda) \), but \( \lambda \) can make the second term \( (M + 4\pi r^3 T_i^i) \) change sign. Contributions to \( T_4^i \) proportional to \( g_4^i \) (like \( \lambda \)) thus weaken the pressure gradient and support heavier stars than otherwise expected.

We have checked using Eq. (17) that direct gravitational modifications to the mass-radius relation for compact stellar objects remain negligible, as might be expected. At 60\( B_d \), \( T = 0.8 m_4^4 = 1.14 \times 10^{25}\text{erg/cm}^3 \), 8 orders of magnitude smaller than the pressures expected in the high density nuclear matter in the core of a post-main sequence star \cite{13}. However, this energy density is 2-4 orders of magnitude larger than the gravitational potential energy density of infalling stellar plasma. Thus while the insignificance of the QED \( \mathcal{T} \) in gravity is a consequence of its objectively small energy density, the anti-gravitational effect of \( \mathcal{T} \) suggests a closer study of the force experienced by individual particles in nonlinear electromagnetism is necessary.

**Particles in Overcritical Quasi-Constant Fields**

Forces present in the dynamical case can be much greater than those observed when the interacting bodies are studied in the hydrostatic equilibrium of Eq. (17). The electromagnetic force determining individual particle dynamics does not make relevant contributions, and the vacuum fluctuation-induced force contributes even less to Eq. (17). Though very much smaller than Maxwell’s force this force can be stronger than gravity and at times more relevant than the linear order force of Maxwellian electromagnetism. We will describe its features relevant to the charged particle dynamics within collapsing stellar objects.
Consider that the total (‘t’) electromagnetic energy-momentum tensor $T^{t\mu\nu}$ due to both an external field (‘e’) and a probe charge (‘p’) includes also an interaction energy-momentum $T_{\text{int}}^{\mu\nu}$,

$$T^{t\mu\nu} = T^{e\mu\nu} + T_{p\mu}^{\nu} + T_{\text{int}}^{\mu\nu} \quad (18)$$

with tensors $T_{e\mu\nu}, T_{p\mu}^{\nu}$ defined by the forms they take in isolation from each other. When the external field is much larger than the field of the probe particle, the electromagnetic energy-momentum tensor is expanded in the displacement tensor $K^{\mu\nu}$

$$K^{\mu\nu} = \frac{\partial V_{\text{eff}}}{\partial F^{\mu\nu}} = F^{\mu\nu} - \frac{\partial f_{\text{eff}}}{\partial F^{\mu\nu}} \quad (19)$$

around the dominant contribution of the external field,

$$T^{t\mu\nu} = T_{e\mu\nu} + \frac{\partial T^{e\mu\nu}}{\partial K^{\alpha\beta}} \bigg|_{e} \frac{K^{\alpha\beta}}{2} + ... \quad (20)$$

with the subscript $e$ reminding that derivatives are to be evaluated at the external field.

The energy-momentum tensor $T^{t\mu\nu}$ is expressed in Eq. (3) in terms of the field tensor $F^{\mu\nu}$, but only the displacement fields of the probe particle are known explicitly by solving Maxwell’s equations with source

$$\partial_{\rho}K_{p}^{\mu\nu} = j_{p}^{\nu} \quad (21)$$

By inverting Eq. (19) we obtain

$$\frac{\partial F^{\alpha\beta}}{\partial K^{\mu\nu}} = (\delta_{\mu}^{\alpha} \delta_{\nu}^{\beta} - \delta_{\mu}^{\beta} \delta_{\nu}^{\alpha}) + \frac{\partial^{2} f_{\text{eff}}}{\partial F^{\mu\nu} \partial F_{\alpha\beta}} \quad (22)$$

$$+ \frac{\partial^{2} f_{\text{eff}}}{\partial F^{\mu\nu} \partial F_{\gamma\delta}} \frac{\partial^{2} f_{\text{eff}}}{\partial F^{\gamma\delta} \partial F_{\alpha\beta}} + ...$$

This rank 4 tensor transforms the functional dependence from the field tensor to the displacement tensor. We checked the validity of truncation by numerical evaluation of the derivatives of the action, which shows that (normalized) higher derivatives are suppressed even when the field is supercritical. A separable contribution of the Maxwell self-energy $T_{p}^{\mu\nu} \approx T_{\text{Max},p}^{\mu\nu}$ of the probe particle is indeed found at next order $\partial^{2}/\partial K^{2}$ and would be subtracted. However, we find the effects of the terms from the second derivative are many orders of magnitude smaller than those from the first derivative and do not discuss them further here.

Using Eq. (22) in Eq. (20) gives

$$T_{\text{int}}^{\mu\nu} = T_{e\mu\nu} + (\varepsilon - 1)^{2}T_{e\mu\nu} + \frac{g^{\nu\mu}}{4} \frac{\partial^{2} F^{e\nu}}{\partial S_{A\beta}} K_{p}^{\alpha\beta} \quad (23)$$

where

$$T_{e\mu\nu} = -(F_{e}^{\mu\lambda} K_{e}^{\nu\lambda} + F_{e}^{\nu\lambda} K_{p}^{\mu\lambda})g_{\alpha\lambda} + g^{\nu\mu} \frac{1}{2} F_{e\alpha\beta} K_{p}^{\alpha\beta} \quad (24)$$

is the Maxwellian interaction with $F_{e\alpha\beta} K_{p}^{\alpha\beta} = 2(\vec{B}_{e} \cdot \vec{H}_{p} - \vec{E}_{e} \cdot \vec{D}_{p})$. The latter two terms of Eq. (23) remain after cancellation among the order $\alpha$ terms, and despite being order $\alpha^{3}$ the last term $(1 - \varepsilon)(\partial T/\partial S) = (\partial T_{\text{eff}}/\partial S)(\partial T/\partial S)$ is kept for now.

We view the net force (density) acting on the charged probe particle entering the domain of the external field in the usual way, requiring that inertial resistance balance any breach of the conservation of the field energy-momentum,

$$f^{\mu} \equiv -\partial_{\nu}T_{\text{int}}^{\mu\nu} = j_{p}^{\nu} F_{e\nu}^{\mu} + \delta f^{\mu} \quad (25)$$

using that $-\partial_{\nu}T_{e\nu}^{\mu\nu} = j_{p}^{\nu} F_{e\nu}^{\mu}$ is the force obtained within Maxwell’s linear electromagnetism. Here

$$\delta f^{\mu} \approx (\varepsilon_{e} - 1)^{2} j_{p}^{\nu} F_{e\nu}^{\mu} - T_{e\mu\nu}^{\nu}_{p} \partial_{\nu}(\varepsilon_{e} - 1)^{2}$$

$$- \partial_{\nu} \frac{2 - \varepsilon}{4} \frac{\partial T}{\partial S} \bigg|_{e} F_{e\alpha\beta}^{\nu} K_{p}^{\alpha\beta} \quad (26)$$

with equality only approximate on account of the finite order expansions in $T_{e\mu\nu}$ Eq. (20) and $\alpha$ Eq. (22).

The conventional contributions of $T_{e\mu\nu}$ and $F_{e\alpha\beta} K_{p}^{\alpha\beta}$, i.e. the first term of Eq. (23), to the force on the particle are obtained by integrating over a covariant hypersurface, and all frames being equivalent this integration is done most conveniently in the rest frame of the particle,

$$\int d^{3}x E_{e}^{i} D_{p}^{i} = - \int d^{3}x F_{e}^{i} \nabla^{i} \nabla^{j} F_{p}^{j} = \delta^{j} \int d^{3}x F_{e\rho \rho} \quad (27)$$

using that $\vec{D}_{p} = (Ze/r^{2})\hat{r}$ (for a spherical charge) has only one component when choosing spherical coordinates.

A physical charged particle has also a magnetic moment $\vec{\mu}_{p}$, and thus a corresponding dipole magnetic field. As is well known this part of the force cannot usually compete with the effect of electrical particle charge. However, in the present context we reach beyond the usual Lorentz force to the effect of vacuum fluctuations and it is necessary to check if it is still justified to neglect the magnetic dipole in the strong magnetic field environment of a collapsing star. From the magnetic dipole interaction energy $\mu_{p} \cdot \vec{B}$, a force due to the gradient of the external magnetic field arises. $B_{e}$ changes on macroscopic scale though, and the parameter characterizing smallness of the effect is $1/mL \approx 10^{-16}$ when $B_{e}$ varies on the scale $L \approx 4 \text{ km}$. For comparison, the smallness parameter of the vacuum fluctuations arising from Euler-Heisenberg action is $(B_{e}/B_{c})^{3}/4 \pi r$. Seeing that vacuum fluctuation effects should dominate the magnetic dipole interaction for a stellar magnetic field $B_{c} > 10^{-5} B_{e}$, we explore this domain further.

Turning now to the latter two terms of Eq. (23) that represent the additional vacuum fluctuation-induced force, we observe that the gradient in the corresponding last term of Eq. (20) generates two contributions: the first as the gradient of Eq. (22) and the second as the net change of the slowly-varying coefficient $\partial T/\partial S$ over the domain of particle’s field. The integrals over the particle’s field Eq. (27) computed in the particle’s rest frame,
the spatial components of the force Eq. (28) on a point charge \( \rho_p = Ze \delta(\vec{x}) \) in its rest frame are

\[
\frac{1}{Ze} \delta\vec{f} \approx \left( (\varepsilon - 1)^2 + \varepsilon - 2 \frac{\partial T}{\partial S} \right) E_e - C_e \Phi_e \nabla S, \quad (28)
\]

where

\[
C_e B_e^2 = - \left( 2(\varepsilon - 1) \frac{\partial \varepsilon}{\partial S} + \varepsilon - 2 \frac{\partial^2 T}{\partial S^2} + 1 \frac{\partial \varepsilon}{\partial S} \frac{\partial T}{\partial S} \right) \quad (29)
\]

depends only on the scalar invariant of the external field \( S_e \). On the right side of Eq. (28) the gradient applied to the invariant \( S \) preserves the correct Lorentz transformation property: although the potential \( \Phi_e \) appears, Eq. (28) is the gauge invariant correction to the linear force \( \vec{f} = e(\vec{E_e} + \vec{v} \times \vec{B_e}) \) computed in the particle’s rest frame.

The component of Eq. (28) proportional to \( C_e \) is qualitatively different because, being proportional the gradient \( \nabla S \), it allows the transfer of energy from the magnetic field to in-falling particles. This property is in contrast to the first term in Eq. (28) which produces a tiny change in the effective linear force (per mille at \( B_e = B_c \)).

The weak-field expansion of \( C_e \) Eq. (29)

\[
C_e B_e^2 \approx \frac{8\alpha}{45\pi} \left( 1 - \left( \frac{B_e}{B_c} \right)^2 \left( \frac{6}{7} + \frac{2\alpha}{45\pi} + \ldots \right) + \ldots \right) \quad (30)
\]

obtained from the Euler-Heisenberg effective action, is usable up to \( B_e \approx 1B_c \). The expansion shows that the predominant contribution to the gradient force is \( T \), as the leading constant in Eq. (30) is traced to \( \partial^2 T / \partial S^2 \). Non-perturbative computation requiring employment of the nonperturbative numerical methods presented of the coefficients \( \varepsilon - 1, \partial T / \partial S, \) etc. shows that the force persists in the considered high magnetic field domain despite what the perturbative expansion suggests.

Particle Dynamics in a Stellar Magnetic Field

As an application, we consider the dipole field of a strongly magnetized star. We are studying the force \( f_e \) in the rest frame of the particle and in order to allow that the particle has a velocity relative to the star, we need to Lorentz-transform the field in the star’s rest frame \( B' \) oriented at an angle \( \psi \) from the direction of the particle’s motion. As a consequence of the transformation, the field of the star is seen by the particle to have an electric component. Specifically, \( v^\mu = (\cosh y, 0, 0, \sinh y) \) and the Lorentz transformed fields are \( \vec{B_e} = B'(\cos \psi \hat{z} - \gamma \sin \psi \hat{x}) \) and \( \vec{E_e} = -B' \gamma \beta \sin \psi \hat{y} \). Using this in Eq. (28) we obtain the force of the star’s field on the moving particle. Note that \( \beta, y \) can be positive or negative.

To compare with the gravitational force \( f_g \), we must also Lorentz transform it to the rest frame of the moving particle. Although general relativistic corrections to the Newtonian potential are significant near the stellar objects where such strong magnetic fields have been inferred, the Newtonian force is a reasonable first estimate modified only by multiplicative numerical factors of order unity down to a few times the radius of the future neutron star.

We obtain the transformation property of the force \( f_g \) considering the geodesic in the Schwarzschild metric (see e.g. Eq. 9.32 in [28]): transforming to the rest frame of a relativistic particle dilates the proper time, multiplying kinetic and total energies by \( \gamma^2 \). The energy equation for the geodesic therefore preserves its form if the same factor \( \gamma^2 \) is included also in the ‘potential’ terms, thus giving the transformation \( f_g \rightarrow \gamma^2 f_g \). The ratio of the radial vacuum fluctuation force to the Newtonian gravitational force for a transversely moving \( (\psi = \pi/2) \) electron is

\[
f_e / f_g = 3C_e Ze \beta \gamma B_{\text{surf}} \left( \frac{R_{\text{surf}}}{r} \right)^9 \left( \frac{e B_{\text{surf}}}{m_e^2} \right)^2 \frac{\gamma^2 GM_\odot m_p}{r^3}. \quad (31)
\]

illustrated in figure 2.

The ratio \( f_e / f_g \) Eq. (31) can be large as shown in figure 2 due to the weakness of gravity, particularly for \( B_e \gg 10^{-3} B_c \). The stellar magnetic field contribution to the gravitating energy density remains relatively small, yet it affects particle dynamics through the coupling to moving charge, through vacuum fluctuation nonlinearity suppressed by \( \alpha^2 / m_e^4 \). Regarding sign in Eq. (31), CPT symmetry of the vacuum assures that matter and antimatter are expelled to the same degree: the effect of the force is the same for a left moving electron as for a right moving positron, as seen by simultaneously flipping the signs of \( Z \) and \( \beta \). Allowing for the distribution of charges and velocities with respect to the orientation of the field, we recognize that in a random medium (plasma) half of
the charged particles of each polarity at any given time is expelled.

The interesting feature of the force Eq. (28) and Eq. (31) is that a magnetic field gradient correlates velocity and charge. As the magnetic field curves trajectories of both left moving electrons and right moving positrons in the same direction, the noted symmetry in the effect of the vacuum fluctuation-induced force generates rotation in a net neutral plasma even while net current remains zero. Therefore, in a plasma made of negatively charged electrons and positively charged light nucleons, the matter which is ejected has opposite net momentum compared to the matter which is attracted and thus angular momentum is imparted to the magnetic source due to the mass asymmetry between positively and negatively charged particles.

In summary, we have evaluated the trace $T$ of the QED energy-momentum tensor and demonstrated that its gradient entails a significant and often repulsive force, which can be large compared to gravity, even while the relativistic energy density of $T$ remains small. Although the magnitude of the usual magnetic force $\vec{v} \times \vec{B}$ is much larger than the vacuum-fluctuation induced correction $\vec{v}$, only the latter is relevant in consideration of energy transfer to the particle and escape from the gravitational potential well. The requisite energy exchange with a magnetic field, seen in the gradient $\nabla S$, is a consequence of the nonlinearity of the induced vacuum fluctuations, absent in classical Maxwellian electromagnetism.

The quantitative study we present in Fig. 2 for the ratio $\frac{\vec{v}}{B}$ indicates the force Eq. (26) and Eq. (28), should have an impact on matter accretion and stellar collapse dynamics in astrophysical situations where strong magnetic fields in excess of $B \gg 10^{-4} B_c = 10^5 T$ are known to exist. While treatment of the complete dynamical situation is beyond the scope of this work, it is easy to imagine that this force could help the neutrino based transport phenomena [20] to propel the supernova bounce.

Acknowledgments

This work in part supported by the DFG Cluster of Excellence MAP (Munich Centre of Advanced Photonics), we thank Prof. D. Habs for hospitality. Supported by a grant from the U.S. Department of Energy, DE-FG02-04ER41318.