Effects of phonon interaction on the pairing in the high-$T_c$ superconductors

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We study the effects of phonon interaction on the superconducting pairing in the background of a d-wave gap, mediated by antiferromagnetic (AFM) spin fluctuations, using coupled BCS gap equations. We found that phonon interaction can induce a s-wave component to the d-wave gap in the (D+S) form with an interaction anisotropy and in the (D+iS) form without anisotropy, respectively. In either case, however, $T_c$ is not enhanced compared to the pure d-wave pairing without phonon interaction. On the other hand, anisotropic phonon interaction can dramatically enhance the d-wave pairing itself and therefore $T_c$, together with the AFM spin fluctuation interaction. This $(D_{AFM} + D_{ph})$ type pairing exhibits strongly reduced isotope coefficient despite the large enhancement of $T_c$ by phonon interaction. Finally, we study the combined type of $(D_{AFM} + D_{ph} + iS))$ gap and calculate the penetration depth and specific heat to be compared with the experiments.

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I. INTRODUCTION

The problems of the high $T_c$ superconductors (HTSC) have been mostly focused on the electronic correlations and phonons were usually considered as a secondary player at best. However, recent Angle Resolved Photoemission Spectroscopy (ARPES) experiments revived the interest of the possibly important role of phonons in the high-$T_c$ cuprates (HTC). In particular, the systematic measurements and analysis of the kink structures in the quasiparticle dispersion near the Fermi surface (FS) proved that: (1) the origin of the kink is electron-phonon coupling and its coupling is very strong, (2) the typical energy of the phonon(s) is $\sim 40 - 70$ meV, and (3) the coupling matrix is quite anisotropic. Therefore, it is currently a pressing question what possible roles and effects, in particular, for the superconducting (SC) pairing, the phonon(s) can do in the HTC.

Although the SC gap symmetry in the high-$T_c$ cuprates is well established as a d-wave by most experiments, there are also continuous experimental reports such as tunnelling conductance, penetration depth measurements, etc that provide rather convincing evidences for a small magnitupe of a s-wave component in addition to the dominant d-wave SC gap in some of HTC compounds. And it is well known that phonons mediate an attractive interaction between electrons and lead to an isotropic s-wave gap. Therefore, more sharpened question will be: is it possible to reconcile a d-wave gap and a s-wave gap together? and in that case, what is the role of phonon(s)?

While there is no consensus yet on the pairing mechanism for the d-wave gap in HTC, in this paper we assume that the AFM spin fluctuations is the mediating glue for d-wave gap[3]. The key ingredient of this mechanism is that the sign changing character of the d-wave gap turns the all positive definite potential of the AFM spin fluctuations mediated interaction (we will call it "AFM interaction" for short from now), in momentum space and in spin singlet channel, into an effectively attractive pairing interaction. From this point of view, it is clear that the phonon interaction and AFM interaction are antagonistic and difficult to work together for making a pairing because they tend to promote a different symmetry of the SC gap each other.

An easy way to reconcile the d-wave and s-wave gap with two different pairing interactions would be to invoke two separate bands for each gap[8]. The similar idea for the two s-wave gaps was very successful for MgB$_2$ because there two well separated bands exist. However, in HTSC, numerous experiments, in particular ARPES measurements[8], show that there is only one main band crossing FS. Then any idea of having s-wave and d-wave gaps in the single band, induced by two very different pairing interactions, seems too naive. However, if the coupling matrix of phonon interaction possesses a strong anisotropy, this combined pairing problem is not a trivial one and needs systematic investigation with a traceable formulation. This problem has been already studied by many authors with different techniques and aims[8-13]. In this paper, we took a simplest approach and studied the coupled gap equations for multiple gaps of a single band with two pairing interactions, i.e., AFM interaction and anisotropic phonon interaction, within the BCS framework. More justification for this approach will be discussed in the next section. By numerical solutions, we extensively investigated the necessary conditions of the strength and the degree of anisotropy of phonon interaction for various multigap solutions.

We found that the (D+S) and (D+iS) type solutions are indeed possible with a proper degree of anisotropy of phonon interaction for each case. However, $T_c$ is not enhanced at all in these cases even though an additional s-wave component is formed due to the phonon interaction. Other interesting possibility is that the anisotropic phonon interaction can induce an additional d-wave component $D_{ph}$ to the AFM interaction induced d-wave component $D_{AFM}$. As a result, $T_c$ is dramatically enhanced due to the phonon interaction. We then derived an analytic $T_c$ equation for this $(D_{AFM} + D_{ph})$ case and showed that phonon isotope effect can be strongly reduced due to the interplay between the AFM and phonon interactions despite the large increase of $T_c$ by phonon. This result can explain a long standing puzzle of the small phonon isotope effect in HTSC. In view of experiments[8,10], the best anisotropic phonon can be the B$_{1g}$ buckling phonon mode which fulfills
all necessary qualifications for our model. Similar conclusion
was obtained by other authors using different approaches.
We emphasize that the final effect of phonons in HTC com-
ounds should be judged by an average effect of all important
phonons together, which is beyond the scope of this paper.

Finally, we also considered the combined type of \((D_{AFM} +
D_{ph} + iS)\) gap which is a very natural solution of the cou-
pled gap equations and shows features such as (1) large en-
hanced \(T_c\), and (2) appearance of a s-wave component at
low temperatures. This type of gap will be a possible solu-
tion for the tunnelling conductance experiment in YBCO as
Calculated superfluidity density also demonstrated that
\((D_{AFM} + D_{ph} + iS)\) gap can explain the non-monotonic tem-
perature dependence of the penetration depth measurements
in YBCO and LSCO compounds.

II. FORMALISM

The interplay between phonons and electronic correlations
in HTSC has been studied by numerous authors using differ-
ent theoretical approaches and models. In this paper, we
took a simple minded approach with a specific question for
this problem. Experiments tell us that, after all correlation
effects taken place, the quasiparticles (q.p.) remain to form a
fermi surface (FS). And quite detailed properties of the AFM
spin fluctuations and symmetry allowed phonons are exper-
imentally measured. These measured bosonic fluctuations are
final outcomes after all interplays between them and correla-
tion effects. Now on the phenomenological basis, we consider
a q.p. band and the AFM spin fluctuations as a dominant pair-
ing interaction. In addition to that we add anisotropic phonon
interactions and we study the pairing instability of the model
for several possible types of multi-gap solution using a gen-
eralized BCS gap equations. In this approach, still remaining
interplay between spin fluctuations and phonons is ignored.
The results of this paper should be taken with this caveat. The
Hamiltonian is written as

\[
H = \sum_{k\sigma} \epsilon(k)c_{k\sigma}^\dagger c_{k\sigma} + \sum_{kk'\|} V_{AFM}(k,k')c_{k1\uparrow}^\dagger c_{k'1\downarrow} + \sum_{kk'\perp} V_{ph}(k,k')c_{k1\uparrow}^\dagger c_{k'1\downarrow} \tag{1}
\]

where \(\epsilon(k)\) is the dispersion of the quasiparticles created by
c_{k\sigma}^\dagger as standard notation. \(V_{AFM}(k,k')\) and \(V_{ph}(k,k')\) are
the effective interactions, for the singlet superconducting pair-
ing channel, originating from the AFM spin fluctuations and
phonon(s), respectively. For traceable numerical calculations,
we further simplify the above Hamiltonian as follows. The
real two dimensional FS is simplified as a circular FS and the
interactions are also modelled accordingly as follows.

\[
V_{AFM}(\Delta\phi) = V_M \frac{\phi_0^2}{(\Delta\phi \pm \phi_{AFM})^2 + \phi_0^2} \tag{2}
\]

and

\[
V_{ph}(\Delta\phi) = \begin{cases} -V_P & \text{for } |\Delta\phi| < \phi_{AN} \\ 0 & \text{for } |\Delta\phi| > \phi_{AN} \end{cases} \tag{3}
\]

where \(\Delta\phi = \phi - \phi'\) and \(\phi_{AFM} = \pi/2\) representing the
exchanged momentum \(k - k'\) and the AFM ordering vector \(Q\)
in the circular FS, respectively. \(V_M\) and \(V_P\) are chosen to be
positive, so that \(V_{AFM}\) is all repulsive and \(V_{ph}\) is all attractive
in momentum space.

The property of the AFM interaction with a short range
correlation is simulated with the inverse correlation length
parameter \(\phi_0\) \((\lambda = (\pi a/\sqrt{2})\phi_0)\). In this paper, we chose
\(\phi_0 = 1\) \((\lambda \sim 2a)\) for all numerical calculations, which is quite
a short range AFM correlation. The degree of anisotropy of
the phonon interaction is controlled by the anisotropy angle
parameter \(\phi_{AN}\) which restricts the scattering angle between
incoming and outgoing q.p momenta; for example, \(\phi_{AN} = \pi\)
would allow a perfectly isotropic interaction. The phonon in-
teraction \(V_{ph}(\phi - \phi')\), however, does not restrict the incom-
ing \((\phi)\) and outgoing momenta \((\phi')\). Now the reduced BCS
Hamiltonian in the mean field theory can be written as

\[
H = \sum_{\phi\xi\sigma} c_{\phi\xi\sigma}^\dagger c_{\phi\xi\sigma} + \sum_{\phi\xi} \Delta_{AFM}^*(\phi)c_{\phi\xi\downarrow}c_{\phi\xi\downarrow} + \sum_{\phi\xi} \Delta_{ph}^*(\phi)c_{\phi\xi\downarrow}c_{\phi\xi\downarrow} \tag{4}
\]

where \(\Delta_{AFM}(\phi)\) is the SC gap function induced by \(V_{AFM}\)
and \(\Delta_{ph}(\phi)\) is the one induced by \(V_{ph}\). The two gap functions
\(\Delta_{AFM}(\phi)\) and \(\Delta_{ph}(\phi)\) may or may not have the same sym-
metry. After diagonalizing the above Hamiltonian we obtain
two self-consistent equations as

\[
\Delta_{AFM}(\phi) = \sum_{\phi'\xi} V_{AFM}(\phi - \phi') < c_{\phi'\xi\downarrow}c_{\phi'\xi\downarrow} >, \tag{5}
\]

\[
\Delta_{ph}(\phi) = \sum_{\phi'\xi} V_{ph}(\phi - \phi') < c_{\phi'\xi\downarrow}c_{\phi'\xi\downarrow} >. \tag{6}
\]

Similar coupled gap equations were studied in previous
studies. The key difference is that our model has only one
band. Then because the same band electrons should form the
multigaps, there is severe competition between different gaps
and more constraint to allow multigap solutions. In the next
sections, we will consider several possible multigap solutions
of the above coupled gap equations.

A. (D+S) case

In this case, we assume \(\Delta_{AFM}(\phi) = \Delta_d \cos(2\phi)\) and
\(\Delta_{ph}(\phi) = \Delta_s\). This leads to the two coupled gap equations
as follows.

\[
\Delta_d(\phi) = -\sum_{\phi'} V_{AFM}(\phi - \phi') \Delta_d(\phi') \chi(\phi', \omega_{AFM} \mathcal{F}) \tag{7}
\]
with a SC gap function. These values which is defined below as a projected average dimensionless coupling constants $\lambda$. The total gap function ($\Delta$) is the total gap function. ($\Delta_D$ and $\Delta_S$) are the BCS energy cutoffs of each component ($\Delta_D$ for d-wave and $\Delta_S$ for s-wave interaction), solution is either a pure d-wave or a pure s-wave gap primarily depending on the strength of $\lambda_{AFM}$ and $\lambda_{ph}$; the final dominant instability is determined not only by the interaction strengths but also with the cutoff energy scales $\omega_{AFM}$ and $\omega_{ph}$. In this case, $\lambda_{ph,S} = 0.450$ is much bigger than $\lambda_{AFM,D} = 0.332$, therefore a pure s-wave gap becomes a solution. Only when $\phi_{AN} \leq \pi/2$ and $\lambda_{ph,S}$ is not dominant over $\lambda_{AFM,D}$, a s-wave gap component can coexist with a d-wave component. When these conditions satisfied, no matter how weak the phonon interaction is, a finite s-wave component coexists over the entire temperatures and shares the same transition temperature $T_c$. Also this $T_c$ remains the same as the pure d-wave $T_{c,0}$ without the phonon interaction, regardless of the presence of $\Delta_S$ and its magnitude. This somewhat unexpected result can be understood from the coupled gap equations Eq.(7-8). As far as the d-wave component is nonzero, that is the dominant pairing gap and $T_c$ is determined solely by the d-wave gap equation Eq.(7) in the limit of $\Delta_D, \Delta_S \rightarrow 0$. In the opposite limit, namely, when $\lambda_{ph,S}$ is dominant over $\lambda_{AFM,D}$, then even if $\phi_{AN} \leq \pi/2$, the d-wave gap is entirely suppressed and $T_c$ is determined solely by the s-wave gap equation Eq.(8).

In Fig.2 we fixed the anisotropy angle $\phi_{AN} (= \pi/4)$ and changed the coupling strength of phonon $N(0)V_p = 0.7, 0.9, 1.1$. The variation of the magnitude of the s-wave gap $\Delta_S$ can be understood with the effective coupling strength of $\lambda_{ph,S} (=0.31, 0.39, 0.48)$ respectively compared to $\lambda_{AFM,D} (=0.332)$ as in Fig.1. One peculiar feature occurs when $N(0)V_p = 0.9$ ($\lambda_{ph,S}=0.39$) (down triangles). The higher transition temperature $T_{c,high}$ is understood as the coexistence cases of Fig.1; although $\lambda_{ph,S} (=0.39) > \lambda_{AFM,D} (=0.332)$, the fact $\omega_{AFM} > \omega_{ph}$ makes the d-wave pairing still slightly dominant at high temperatures. But at lower temperature one more transition occurs where the d-wave gap collapses to zero and the s-wave gap component makes a sudden increase. This abrupt change of gaps is a first order transition and indicates that there are two competing local minima in the free energy. With temperature the global minimum changes from one local minimum to the other local minimum. This peculiar behavior is due to the closeness between $\lambda_{AFM,D}$ and $\lambda_{ph,S}$, and yet the distance of the energy scale between $\omega_{AFM} = 1$ and $\omega_{ph} = 0.5$ as we have chosen in our calculations (all energy scales are normalized by $\omega_{AFM}$ in this paper) in this particular case. This is a rather artificial result of our model; nevertheless, it is an interesting behavior of the (D+S) type gap equations of Eq.(7-8).

The summary for the (D+S) type gap is: (1) The mixed type gap solution is possible if the phonon interaction has a
proper anisotropy and its coupling strength is subdominant to the d-wave pairing strength; (2) $T_c$ is not enhanced as far as the d-wave gap remains finite regardless of the presence and magnitude of the s-wave gap component.

**B. (D+S) case**

In this case, we assume $\Delta_{AM}(k) = \Delta_d \cos(2\phi)$ and $\Delta_{ph} = i\Delta_s$. Accordingly, $\Delta_d(\phi) = \Delta_d \cos(2\phi) + i\Delta_s$ and $E(\phi) = \sqrt{\xi^2 + \Delta_d^2(\phi) + \Delta_s^2}$. The gap equations are

$$\Delta_d(\phi) = -\sum_{\phi'} V_{AFM}(\phi - \phi') \Delta_d(\phi') \chi(\phi', \omega_{AFM})$$

$$i\Delta_s(\phi) = -\sum_{\phi'} V_{ph}(\phi - \phi') \Delta_s(\phi') \chi(\phi', \omega_{ph})$$

The main difference from the (D+S) case is that because of the phase $i$ between $\Delta_d(\phi)$ and $\Delta_s$, two gaps are only indirectly coupled through the quasiparticle energy $E(\phi)$ in the pair susceptibility $\chi(\phi, \omega_{AFM}, \omega_{ph})$. Therefore the coupling of two gaps is much smoother than the (D+S) case. One consequence of it is that the gap equations can allow two transition temperatures $T_{c,D}$ and $T_{c,S}$ for each gap $\Delta_d$ and $\Delta_s$, respectively. This feature is shown in Fig. 3 where $N(0)V_M = 2.0$ ($\lambda_{AFM,D} = 0.332$) and $N(0)V_P = 0.55$ are fixed and the anisotropy angle of phonon interaction $\phi_{AN}$ is varied as $\pi/4$, $\pi/2$, and $\pi$ (correspondingly $\lambda_{ph,S} = 0.243, 0.414$ and 0.55).

Main features are summarized as: (1) when $\lambda_{ph,S}$ is too weak, the effect of phonon interaction is totally ignored and the gap and $T_c$ is solely determined by the AFM interaction (see $\phi_{AN} = \pi/4$ case). (2) when $\lambda_{ph,S}$ is much stronger than $\lambda_{AFM,D}$, the d-wave gap is totally suppressed and the gap and $T_c$ is determined solely by phonon interaction (see $\phi_{AN} = \pi$ case). This behavior is similar to the (D+S) case. (3) when the effective coupling strengths are comparable, namely $\lambda_{AFM,D} \approx \lambda_{ph,S}$, both d-wave and s-wave gaps have separate transition temperatures $T_{c,D}$ and $T_{c,S}$ and two gaps can coexist at low temperatures (see $\phi_{AN} = \pi/2$ case). In this case, $T_{c,D}$ is always higher transition temperature and this transition temperature is just the same as the pure d-wave $T_{c,0}$ without phonon interaction. At the lower transition temperature $T_{c,S}$, the s-wave gap $i\Delta_s$ appears and the magnitude of d-wave gap $\Delta_D$ accordingly decreases.

To clarify the roles of the anisotropy and the strength of phonon interaction, in Fig.4 we fixed anisotropy angle as $\phi_{AN} = \pi$, which simulates a perfectly isotropic phonon interaction, and varied the interaction strength. Fig.4 shows the results of these calculations. With $N(0)V_M = 2.0$ ($\lambda_{AFM,D} = 0.332$), we varied $N(0)V_P = 0.3, 0.4, 0.5$; accordingly $\lambda_s = 0.3, 0.4, 0.5$). This plot demonstrates that the anisotropy of phonon interaction doesn’t play a particular role in the case of the (D+iS) type gap. The mixed gap solution of the (D+iS) type is still possible when $\lambda_{AFM,D}$ and $\lambda_{ph,S}$ are of the comparable strength regardless of the anisotropy of phonon interaction.
The summary for the (D+iS) type gap is: (1) The mixed type gap solution is possible if the phonon interaction $\lambda_{ph,S}$ is comparable but still subdominant to the AFM interaction strength $\lambda_{AFM,D}$. (2) In contrast to the (D+S) case, separate two transition temperatures $T_{c,D}$ and $T_{c,S}$ exist for each gap $\Delta_D$ and $\Delta_S$; $T_{c,D}$ is always higher than $T_{c,S}$ and only below $T_{c,S}$ two gaps coexist. (3) As in the (D+S) case, $T_c$ is not enhanced as far as the d-wave gap remains finite regardless of the presence and magnitude of the s-wave gap component. (4) Anisotropy of the phonon interaction is not particularly necessary.

C. $(D_{AFM}+D_{ph})$ case

This is the case that phonon interaction also supports the d-wave pairing. Because it is clear that a perfectly isotropic phonon interaction has null effect on the d-wave gap, anisotropy of the phonon interaction is crucial for this case. The total gap is, therefore, $\Delta_t(\phi) = (\Delta_{d1} + \Delta_{d2}) \cos(2\phi)$ and the gap equations are written as

$$\Delta_{d1}(\phi) = -\sum_{\phi'} V_{AFM}(\phi - \phi') \Delta_t(\phi') \chi(\phi', \omega_{AFM}), (13)$$

$$\Delta_{d2}(\phi) = -\sum_{\phi'} V_{ph}(\phi - \phi') \Delta_t(\phi') \chi(\phi', \omega_{ph}). (14)$$

The above two gap equations can be combined to a single

![Graph showing gap magnitudes for different phonon interaction strengths](image)

**FIG. 4:** (Color online) $(D+iS)$ case. Calculated magnitudes of d-wave component ($\Delta_d/\omega_{AFM}$) and s-wave component ($\Delta_s/\omega_{AFM}$) for different strengths of phonon interaction $N(0)V_p = 0.3, 0.4, 0.5$. For all cases $N(0)V_M = 2.0, \phi_{AN} = \pi, \omega_{ph}/\omega_{AFM}=0.5$.

![Graph showing gap magnitudes for different anisotropy angles](image)

**FIG. 5:** (Color online) $(D_{AFM}+D_{ph})$ case. Calculated magnitudes of $D_{AFM}$ component ($\Delta_{d1}/\omega_{AFM}$) and $D_{ph}$ component ($\Delta_{d2}/\omega_{AFM}$) for different anisotropy angles $\phi_{AN} = \pi/8, \pi/4, \pi/2, \pi$. For all cases $N(0)V_M = 2.0, N(0)V_p = 0.5$ and $\omega_{ph}/\omega_{AFM}=0.5$.

![Graph showing total d-wave gap](image)

**FIG. 6:** (Color online) $(D_{AFM}+D_{ph})$ case. Calculated magnitudes of the total d-wave gap ($\Delta_{d-tot}/\omega_{AFM}$) for different strength of phonon interaction $N(0)V_p = 0.0, 0.2, 0.4$, and $0.6$. For all cases $N(0)V_M = 2.0, \phi_{AN} = \pi/4$ and $\omega_{ph}/\omega_{AFM}=0.5$. 
gap equation but we keep the separate form in order to trace the effect of the phonon interaction. In contrast to the previous cases, $T_c$ can be dramatically enhanced when the phonon interaction $V_{ph}(\Delta \phi)$ has a proper degree of anisotropy. Other authors also obtained a similar result using different approaches. In Fig.5, we plot $\Delta_{d1}$ and $\Delta_{d2}$ separately: between the same symbols the smaller value is the phonon induced d-wave gap $\Delta_{d2}$ and the larger one is the AFM induced d-wave gap $\Delta_{d1}$. We change the anisotropy angle of $V_{ph}(\Delta \phi)$ with the fixed interaction strengths of $N(0)V_M = 2.0 (\lambda_{AFM,D} = 0.332)$ and $N(0)V_p = 0.5$. For the anisotropy angle $\phi_{AN} = \pi/2$ and $\pi$, the phonon interaction has absolutely no effect on the d-wave pairing; this is simply because the d-wave projected average Eq.(10) becomes zero for these two commensurate angles with our model potential Eq.(3). With a stronger anisotropy ($\phi_{AN} = \pi/4; \lambda_{ph,D} = 0.12$), the phonon scattering sees only a limited part of the d-wave gap, therefore the d-wave gap can be seen effectively as a s-wave gap for the anisotropic phonon. For a even stronger anisotropy ($\phi_{AN} = \pi/8; \lambda_{ph,D} = 0.098$), too narrow scattering angle reduces the effective coupling strength. In our model potential, $\phi_{AN} = \pi/4$ is the optimal anisotropy. In fact, even weaker anisotropy of $\pi/4 < \phi_{AN} < \pi$ except the special commensurate angles $\phi_{AN} = \pi/2$ and $\pi$ produces some uncompensated effective interaction $\lambda_{ph,D}$ and boosts the d-wave pairing. This is an artifact of the model potential Eq.(10) which always emphasizes the small angle scattering. Therefore, for some real phonons, it is possible to have a repulsive (destructive for d-wave pairing) $\lambda_{ph,D}$ if large angle (around $\Delta \phi = \pi/2$) scattering overweights small angle scattering.

Fig.6 shows the results of total gap $\Delta_{tot} = \Delta_{d1} + \Delta_{d2}$ with the optimal anisotropy angle $\phi_{AN} = \pi/4$ and varying phonon coupling strength $N(0)V_p=0.0$, $0.2$, $0.4$, and $0.6$ $(\lambda_{ph,D} = 0.0, 0.047, 0.095,$ and $0.142$, respectively). The results demonstrate that the relatively weak phonon interaction $\lambda_{ph,D}$ can significantly increase $T_c$ from the pure AFM interaction induced $T_{c0}$.

Considering the experimental findings about phonons in the high-$T_c$ cuprates, the best candidate for the anisotropic pairing phonon is $B_{1g}$ buckling mode of the plane oxygen motion.

Devereaux and coworkers extensively analyzed the behaviors of $B_{1g}$ mode and found : (1) $B_{1g}$ mode has a strong anisotropic coupling matrix element concentrating around antinodal points; (2) the maximum coupling strength can reach as high as $\lambda \sim 3$. Compared to our model calculations, the optimal anisotropy for the d-wave pairing $\phi_{AN} = \pi/4$ is just about the same degree of anisotropy of the $B_{1g}$ mode. Regarding the coupling strength, for example, in Fig.6 the maximum coupling strength $\lambda_{max}$ is $N(0)V_p=0.6$ and the corresponding average strength is $\lambda_{ph,D}=0.142$. These values are much smaller than the ones extracted by Devereaux and coworkers. Larger values of coupling constants would be more realistic. However since our study is limited within the BCS gap equations we chose smaller values of coupling constants. With larger coupling strength ($\lambda > O(1)$), we should go for the strong coupling theory for the reliable analysis. Also for the same phonon, the coupling strength extracted from ARPES experiments and the coupling strength for the pairing problem can be very different due to the on-site Coulomb interaction.

Another important anisotropic phonon in Bi-2212 is the breathing mode which strongly interacts near the nodes. The effect of this phonon is partially included in our calculations because our model phonon interaction Eq.(3) only constraints the scattering angle by $\phi_{AN}$ but doesn’t constraint the incoming $\mathbf{k}$ and outgoing momenta $\mathbf{k}'$ ($\phi$ and $\phi'$ in our model). The anisotropic phonon scattering concentrating near nodes will be mostly averaged out by the sign changing d-wave gap and will be of no importance for the d-wave pairing. For realistic estimate of the phonon effects in HTC compounds, all important phonons should be included to calculate $\lambda_{ph,D}$ using full coupling matrix elements. This is beyond the scope of the current paper.

Since we found that an anisotropic phonon can dramatically enhance $T_c$ of d-wave pairing, we have to consider the isotope effect of the phonon, which is reported to be anomalously small, in particular, near optimal doping region by many experiments. For this purpose, Fig.7 shows the calculated $T_c$’s (symbols) from Eq.(13-14) as a function of the phonon energy cutoff $\omega_{ph}/\omega_{AFM}$ for various phonon coupling strength $N(0)V_p = 0.0, 0.2, 0.4$ and $0.6$, respectively. The BCS theory predicts $T_c \sim \omega_{ph}$ ($\alpha = 0.5$), which should show up as a linear lines in Fig.7. Our numerical results show much weaker power than the linear one.
indicating $\alpha < 0.5$. For more analytic investigation for the isotope effect and the origin of the $T_c$ enhancement by phonon in the (D$_{AFM}$+D$_{ph}$) case, we derived an analytic $T_c$ equation. The gap equations Eq.(13-14) can reduce to a single $T_c$ equation by adding two equations and taking a limit of $\Delta_{d1}, \Delta_{d2} \to 0$. The pair susceptibility $\chi(\omega_{AFM,ph})$ (Eq.(9)) is integrated out using BCS approximation which is \[ \int_0^{\omega_c} d\xi [\tanh \frac{\xi}{2T}] / \xi \approx \ln [2C \omega_c / \pi T] \] valid when $\omega_c / 2T_c \gg 1$. Finally we obtained the $T_c$ formula of the (D$_{AFM}$+D$_{ph}$) case as

$$ T_c \simeq 1.13 \frac{\tilde{\lambda}_{AFM}}{\omega_{AFM}} \frac{\lambda_{ph}}{e^{-1/\lambda_t}}. \quad (15) $$

where $\lambda_t = (\lambda_{AFM} + \lambda_{ph})$, $\tilde{\lambda}_{AFM} = \lambda_{AFM} / \lambda_t$ and $\lambda_{ph} = \lambda_{ph} / \lambda_t$, and $\lambda_{AFM}$ and $\lambda_{ph}$ are the dimensionless effective coupling constants obtained from Eq.(10) with d-wave gap average for both couplings. $\omega_{AFM}$ and $\omega_{ph}$ are the energy cutoffs of the AFM interaction and phonon interaction, respectively. As mentioned above, this $T_c$ formula is derived assuming $\omega_{M,ph}/2T_c \gg 1$ and this condition is well satisfied when $\lambda_t < O(1)$. In Fig.7, we also plot the results of $T_c$'s from Eq.(15) (lines) with the same parameters and compare them with the numerical calculations (symbols) from Eq.(13-14). It shows a reasonably good agreement between two results although the deviations increase with increasing the coupling constant $N(0)V_p$ (or $\lambda_{ph,D}$) as expected. However, the overall and quantitative behavior of $T_c$ is well captured by Eq.(15).

Now we are in position to read the phonon isotope coefficient $\alpha$ from Eq.(15), which is

$$ \alpha = \frac{1}{2} \frac{\lambda_{ph}}{\lambda_{AFM}} = \frac{1}{2} \frac{\lambda_{ph}}{\lambda_{AFM} + \lambda_{ph}}. \quad (16) $$

For example, with representative values of $\lambda_{AFM} = 0.33$, $\lambda_{ph} = 0.1$ and $\omega_{ph}/\omega_{AFM} = 0.5$, we obtain $\alpha \approx 0.116$, which is pretty small value compared to the standard BCS value of 0.5 while $T_c$ is enhanced by about 100% from $T_c$ at the transition temperature without phonon interaction (see Fig.7; $N(0)V_p = 0.6$ is $\lambda_{ph} = 0.142$). This isotope coefficient equation Eq.(16) provides a very plausible resolution why the isotope coefficient is so low near optimal doping region where $T_c$ is the highest and increases toward underdoped regime (decreasing $T_c$). The phonon coupling strength ($\lambda_{ph}$) is likely unchanged with doping but the effective coupling strength of the AFM mediated interaction ($\lambda_{AFM}$) will change sensitively with doping. If we assume that $\lambda_{AFM}$ increases with increasing doping but $\lambda_{ph}$ remains constant, Eq.(16) describes the general trend of the experimentally observed oxygen isotope effect $\alpha_Q$[13].

D. (D$_{AFM}$+D$_{ph}$+iS$_{ph}$) case

Combining the (D$_{AFM}$+D$_{ph}$) and (D+ iS) case, we can consider the (D$_{AFM}$+D$_{ph}$+iS$_{ph}$) type solution. This is indeed a natural solution of the combined gap equations (12), (13), and (14) and the total gap will be $\Delta_s(\phi) = (\Delta_{d1} + \Delta_{d2}) \cos(2\phi) + i\Delta_s$. In this case, phonon interaction(s) mediate both d-wave and s-wave pairings, which seems contradicting to a common knowledge. The fact is that phonon alone - no matter how anisotropic - cannot induce a d-wave gap but it can boost it if a d-wave gap is formed by other interaction. Then as we found in the (D+ iS) section, a phonon interaction can add iS component at lower temperatures. The interesting point is that the phonon interaction boosting d-wave gap and the phonon interaction inducing iS gap can originate from the same phonon if the anisotropy and the interaction strength are properly tuned. Of course more general case is that two different phonon modes play separate roles, respectively. For clearness, we rewrite the gap equations describing (D$_{AFM}$+D$_{ph}$+iS$_{ph}$) case below.

$$ \Delta_{d1}(\phi) = - \sum_{\phi'} V_{AFM}(\phi - \phi') \Delta_{i}(\phi') \chi(\phi', \omega_{AFM}) \quad (17) $$

$$ \Delta_{d2}(\phi) = - \sum_{\phi'} V_{1,ph}(\phi - \phi') \Delta_{i}(\phi') \chi(\phi', \omega_{ph}), \quad (18) $$

$$ i\Delta_s(\phi) = - \sum_{\phi'} V_{2,ph}(\phi - \phi') \Delta_{i}(\phi') \chi(\phi', \omega_{ph}) \quad (19) $$

where the pair susceptibilities $\chi(\phi, \omega)$ are defined in Eq.(9) and the quasiparticle energy is $E(\phi) = \sqrt{\Delta_s^2(\phi) + \Delta_s^2}$. For generality we
experiments should be able to discern the jump in the specific heat. The specific heat jump is partially cancelled. Nevertheless, careful experiments should be able to discern the jump in the specific heat at the second transition temperature $T_{c2}$ if this scenario is true.

In Fig.10 we plot the solutions of $\Delta_d = \Delta_{c1} + \Delta_{c2}$ and $\Delta_i$ components separately. We fix the AFM interaction $N(0)V_M=2.0$ and the d-wave boosting phonon interaction $N(0)V_{P2}=0.6$ as a case of Fig.6 and vary the phonon interaction $N(0)V_{P2}$ inducing $iS$ gap component as 1.0, 1.1, and 1.2, respectively. $T_c$ is enhanced by $N(0)V_{P1}$ and doesn’t change for different $N(0)V_{P2}$ values because it is determined only by the d-wave gap component. We also chose the same anisotropy $\phi_{AN} = \pi/4$ for both phonons $V_{1,ph}$ and $V_{2,ph}$ for convenience. Anisotropy condition for the d-wave boosting phonon is important; optimal anisotropy is $\phi_{AN} = \pi/4$ for our model. However, the phonon interaction $V_{2,ph}$ inducing $iS$ gap doesn’t need to be anisotropic.

This type of gap solution can provide a possible resolution to the recent experiments of HTC which indicate the presence of a s-wave component at low temperatures. The anomalous tunnelling conductance in YBCO[13], non-monotonic behavior of the penetration depth at low temperatures and its isotope effect[14] etc indicate a mixed gap of the (D+iS) type at low temperatures and a phonon effect with it. Fig.9 show the normalized specific heat $C(T)/C(T_c)$ for the gap solutions of Fig.8. General feature is that it shows a d-wave type specific heat below $T_{c1}$ and then turns into a s-wave type behavior below $T_{c2}$ where $iS$ gap component opens. Since the second transition at $T_{c2}$ is a true second order phase transition, the specific heat exhibits a jump. But the size of jump is far smaller than the BCS value of $\Delta C = 1.43 C(T_{c2})$. The reason is because when $iS$ gap opens at $T_{c2}$, the magnitude of d-wave gap also abruptly decreases so that the amount of the specific heat jump is partially cancelled. Nevertheless, careful experiments should be able to discern the jump in the specific heat at the second transition temperature $T_{c2}$ if this scenario is true.

In this paper we studied the effects of phonon interaction on the superconducting pairing in the background of d-wave gap already formed by the AFM interaction. In particular, we studied the role of anisotropy of the phonon interaction and possible multigap type solutions within a generalized BCS theory. For many cases the anisotropy of the phonon interaction is a crucial condition for interesting interplay with the AFM interaction but not always; the (D+S) type gap need anisotropic phonon but (D+iS) type gap is more tolerable with the anisotropy of phonon interaction because two gaps are more gently coupled in the (D+iS) case. In both cases, $T_c$ is
not enhanced at all by phonon interaction as far as the d-wave gap component remains finite. Anisotropic phonon can boost $T_c$ together with the AFM interaction in the $(D_{AFM} + D_{ph})$ type solution. With numerical calculations and analytic $T_c$ equation, we showed that $T_c$ is enhanced dramatically following the modified BCS exponential form $T_c \sim \exp\left(-1/\lambda_c\right)$ with the total coupling strength $\lambda_c = \lambda_{AFM} + \lambda_{ph}$. This type of solution can also explain how and why the phonon isotope effect is strongly reduced despite the large enhancement of $T_c$ by phonon. Our isotope coefficient formula Eq.(16) provides a good description of the overall trend of oxygen isotope coefficient $\alpha_D$.

Also a combined type solution of $(D_{AFM} + D_{ph} + iS)$ gap is considered. This type of gap not only shows the features of the $(D_{AFM} + D_{ph})$ gap – enhanced $T_c$ and small isotope effect – but also shows the appearance of a $s$-wave component at low temperatures. The latter feature can provide a microscopic origin of the small $s$-wave component suggested from recent experiments of tunnelling conductance and penetration depth measurements. Calculations of the superfluidity density showed a qualitative agreement with experiments of the abrupt increase of $\lambda_c^{-2}(T)$ at lower temperatures. However, this scenario should be confirmed by experimental observation of a specific heat jump at the second low transition temperature $T_{c2}$ for the $s$-wave gap component as shown in Fig.9. We proposed a systematic impurity doping study to test this scenario.

Finally, we would like to restate the limitation of our work. First, the dynamical interplay between phonon(s) and AFM interaction is not properly treated. This problem was already studied by many authors. We bypassed this complicated problem on the phenomenological basis; after all correlation effects taken place we assumed that fermion quasiparticles remain, and AFM spin fluctuations and several phonon modes are considered as experimentally defined quantities after mutual screening and renormalization. Second, when we consider the $s$-wave pairing, we should have included the on-site Coulomb interaction in the HTC cuprates in addition to the phonon interaction. Therefore, $\lambda_{ph,S}$ in this paper should be considered as a renormalized quantity after including the on-site Coulomb interaction. Perhaps, this is the reason why our $\lambda_{ph,S} \sim 0.1$ is order of magnitude smaller than the experimental value $\lambda_{exp} \sim O(1)$.

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