Geometrical Obstruction to the Integrability of Geodesic Flows: an Application to the Anisotropic Kepler Problem

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Abstract. Resorting to classical techniques of Riemannian geometry we develop a geometrical method suitable to investigate the nonintegrability of geodesic flows and of natural Hamiltonian systems. Then we apply such method to the Anisotropic Kepler Problem (AKP) and we prove that it is not analytically integrable.

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1. Introduction

Let $M$ be a manifold and let $TM$ and $T^*M$ be the tangent and cotangent bundles of $M$, respectively. Moreover let $g_{ij}(x)$ be a Riemannian metric on $M$. The corresponding flow on $TM$ is given by the Lagrangian $L = \frac{1}{2} g_{ij}(x) x^i x^j$. The resulting Hamiltonian system on $T^*M$ has Hamiltonian $H = \frac{1}{2} g^{ij}(x) p_i p_j$.

Such systems are very important examples of Hamiltonian systems, since many problems in mechanics reduce to the study of geodesic flows on Riemann and Finsler metrics [1]. For example, with a suitable transformation, the Kepler problem can be described as the geodesic flow on a sphere [2, 3].

In this paper we are interested in developing a geometrical method to investigate whether systems that can be reduced to geodesic flows on a manifold are integrable.

We recall that an Hamiltonian system with $n$ degrees of freedom is integrable if there exist $n$ first integrals $I_1, \ldots, I_n$ with the property that $\{I_i, I_j\} = 0$ for all $i$ and $j$, and such that they are functionally independent. In particular the definition above can be used when the Hamiltonian describes the geodesic flow on a manifold.

If $M$ is an analytic manifold and the metric $g_{i,j}$ is analytic then the flow is said to be analytically integrable if the indicated integrals $I_1, \cdots, I_n$ exist and can be chosen to be analytic.

In [1, 4] it was proved that the geodesic flow of any analytic metric on an analytic compact two dimensional manifold of genus $g > 1$ is not analytically integrable and in [5] the discussion was generalized to $n$ dimensional manifolds. However it is clear that the obstructions to integrability are not only of topological origin. This is because different Riemannian metrics can be associated with the same manifold, even if the topology (of the manifold) imposes restrictions to the possible metrics. Indeed Donnay [6] and Burns and Gerber [7] have constructed real analytic metrics on the sphere whose geodesic flows are Bernoulli. Knieper and Weiss [8] constructed real analytic metrics with positive curvature and positive topological entropy. Thus, even if the sphere has genus $g = 0$ they found that there are metrics on the sphere such that the geodesic flow is not analytically integrable. Hence to refine the criterion of Kozlov (and Taimanov) it is necessary to find a geometrical approach that takes into account the peculiar metric defined on the manifold. Moreover the topological considerations of Kozlov and Taimanov can be applied only to compact manifolds and it would be useful to find a technique applicable in the non compact case. The main purpose of this paper is to consider such geometrical approach that both takes into account the peculiar metric and can be applied to non-compact manifolds.

The approach we consider lies on the simple idea that if you have an integrable system and you add a perturbation that destroys the initial symmetry then, unless a new symmetry appears, you obtain a non integrable system. Examples of this situations are discussed in [9] and [10], where suitable generalizations of the Poicaré-Melnikov method (see for example [11, 12] and references therein) are applied to the perturbed systems. In [9] the Anisotropic Manev problem is discussed for weak anisotropies, and
resorting to the Melnikov method the occurrence of chaos on the zero energy manifold is proven. In [9] the broken symmetry is the rotational invariance. Similarly in [10], where we discussed black holes immersed in uniform electric and magnetic fields, the broken symmetry was the rotational invariance.

To formalize this intuitive idea, in the next section some results of Riemannian geometry are recalled, in particular the isometries, the infinitesimal isometries and some of their properties are discussed. Then we observe that the equation of Killing can be used to find the symmetries that preserve the equation of motion and hence if those equation have only the zero solution no first integral can be found other than the Hamiltonian.

In Section 3 we introduce our main example: the Anisotropic Kepler Problem (AKP) and we show that with a suitable canonical transformation it can be reduced to the geodesic flow on a Riemannian surface with non constant curvature. The Anisotropic Kepler Problem describes the motion of one body orbiting around another considered fixed at the origin in an anisotropic space (i.e. such that the force acts differently in each direction). The AKP was carefully studied in a series of paper by various authors (see [13, 14] for an overview of the problem and of the main results), but not much is known for weak anisotropies. Indeed even if there was evidence of a chaotic behaviour no proof of this fact can be found in the literature (except for the occurrence of chaos on the zero energy level that was proved in [4]). Moreover the AKP is an example of geodesic flow on a non-compact surface and hence it is particularly suitable to illustrate some of the peculiarities of the technique under discussion in this work. Hence showing the nonintegrability of the AKP for weak anisotropies seems an interesting application.

The last section is devoted to show that the Equations of Killing do not have any nontrivial solutions for the AKP and hence that, by the results of Section 3, the Anisotropic Kepler Problem has no analytic first integral independent of the Hamiltonian.

2. Isometries and Infinitesimal Isometries: The Killing’s Field

We will now recall some results and known facts from Riemannian geometry that we need in order to prove the main result of the paper; for a more detailed discussion see, for example, [13, 16, 17].

Let $M$ be a $n$-dimensional manifold with a Riemannian metric $g_{ij}$. An isometry of $M$ is a transformation of $M$ which leaves the metric invariant. More precisely, given the coordinates $x_1, \ldots, x_n$ on $M$, a transformation $x_i = x_i(x_1, \ldots, x_n)$ is an isometry (or motion of the metric) if

$$\bar{g}_{ij}(\bar{x}_1, \ldots, \bar{x}_n) = g_{ij}(x_1, \ldots, x_n).$$

It is clear that such motions form a group, that is often called group of motion. Moreover a vector field $\xi^i$ is called an infinitesimal isometry (or a Killing vector field) if the local 1-parameter group of local transformations generated by $\xi^i$ in a neighbourhood of each point of $M$ consists of local isometries. The set of infinitesimal isometries of $M$, denoted
by \( i(M) \), forms a Lie algebra. Now a transformation is an infinitesimal motions of \( M \) into itself, i.e. a transformation that preserves the metric if the Lie derivative vanishes, that is, if the equations of Killing

\[
L_\xi g_{ij} = \xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{ik} \frac{\partial \xi^k}{\partial x^j} + g_{jk} \frac{\partial \xi^k}{\partial x^i} = 0
\]

are satisfied [7]. In order that there may be a non-null motion we must have the additional condition

\[
g_{ij} \xi^i \xi^j \neq 0. \tag{3}
\]

The first fact to verify is that the infinitesimal isometries are all the transformations that preserve the variational problem defined by the Lagrangian \( \mathcal{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \), so that we can reduce the study of the symmetries preserving the equation of motion to the study of the isometries. This problem is solved by the following

**Lemma 1** An infinitesimal transformation is an isometry for \( M \) with the metric \( g_{ij} \) if and only if it preserves the variational problem defined by the Lagrangian \( \mathcal{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \) on \( TM \).

**PROOF** Let us remark that a transformation is an infinitesimal isometry if and only if the Lie derivative of the metric is zero, i.e. if the Killing’s equations (2) are satisfied.

On the other hand a transformation preserves the variational problem defined by the Lagrangian (13) when the Lie derivative with respect to the field \( \xi^i \) is zero,

\[
L_\xi (\mathcal{L}) = \frac{1}{2} \left\{ \xi^s \frac{\partial g_{ij}}{\partial x^s} + \frac{\partial \xi^i}{\partial x^j} g_{kj} + \frac{\partial \xi^k}{\partial x^j} g_{ik} \right\} \dot{x}^j \dot{x}^i = 0. \tag{5}
\]

It is trivial to see that (2) implies (3). The converse is also true since \( x^i \) is arbitrary and the tensor in the braces is symmetric. This concludes the proof.

With the preparations above we are well on our way to establishing the following

**Theorem 1** Let \( M \) be a analytic \( n \)-manifold with Riemannian metric \( g_{ij} \) and \( \mathcal{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j \) be the Lagrangian defined on \( TM \). If the Killing’s equations for \( M \) have less then \( n - 1 \) independent solutions the geodesic flow is not analytically integrable.

To complete the proof of the theorem we can remark that to each integral of motion there corresponds a one parameter group of transformation that preserve the equation of motion [13]. Hence, since the equation of Killing admit less than \( n - 1 \) independent solutions, by Lemma [1], there are less than \( n - 1 \) transformations that preserve the variational problem. The remark above concludes the proof.

From the previous theorem we easily obtain the following

**Corollary 1** If the only solution of the Killing’s equations is \( \xi^i \equiv 0 \), the corresponding geodesic flow is not analytically integrable.
We now want to consider some more properties of the infinitesimal isometries. A classical result \[16, 17\] is the following

**Proposition 1** The Lie algebra of infinitesimal isometries $i(M)$ of a connected Riemannian manifold $M$ is of dimension at most $\frac{1}{2}n(n+1)$, where $n = \dim M$. If $\dim i(M) = \frac{1}{2}n(n+1)$, then $M$ is a space of constant curvature.

This Proposition gives the maximum number of infinitesimal isometries and hence also the maximum number of first integrals for a geodesic flow on a manifold of dimension $n$. In particular in the two dimensional case we have

**Corollary 2** If $M$ is a two dimensional connected Riemannian manifold of non constant curvature then the Lie algebra of infinitesimal isometries $i(M)$ is of dimension at most 1.

One example that illustrates the results above is the Kepler problem on the plane. Such problem can be described as the geodesic flow on a sphere $S^2$, hence the isometries form a Lie algebra isomorphic to $SO(3)$, and there are three first integrals other than the Hamiltonian (one component of the angular momentum and two components of the Runge-Lenz vector). On the other hand, if we consider, for instance, an homogeneous potential, other than the harmonic oscillator and the Kepler one, we can describe the motion only as a geodesic flow on a surface that has a $SO(2)$ symmetry.

3. The Anisotropic Problems as Geodesic Flow on a Surface

The Hamiltonian function for the Anisotropic Kepler Problem on the plane can be written as

$$H(q, p) = \frac{p^2}{2} - \frac{a}{\sqrt{q_1^2 + \mu q_2^2}}$$

where $\mu > 1$ is a constant and $q = (q_1, q_2)$ is the position of one body with respect to the other, considered fixed at the origin of the coordinate system, and $p = (p_1, p_2)$ is the momentum of the moving particle. The constant measures the strength of the anisotropy and for $\mu = 1$ we recover the classical Kepler Problem.

With some canonical transformations we want to write the problem as the geodesic flow on a surface. To accomplish this task we need the following Lemma

**Lemma 2** Let $(q(t), p(t))$ be a solution of the Hamiltonian system with Hamiltonian $H(q, p)$ which lies on the level surface $H = 0$. Let us change the time variable $t$ to $\tau$ along this trajectory according to the formula $d\tau/dt = 1/G(q(t), p(t)) \neq 0$. Then $(q(\tau), p(\tau)) = (q(t(\tau)), p(t(\tau)))$ is a solution of the Hamiltonian system (with respect to the same symplectic structure) with Hamiltonian $\tilde{H} = HG$. If $G = 2(H + \alpha)$ with $\alpha = \text{const}$, one can also put $\tilde{H} = (H + \alpha)^2$. 
The statement above can be found in [18] and the proof is based on the idea of extended phase space (see [15]). If we perform the change of time \( \frac{d\tau}{dt} = (q_1^2 + \mu q_2^2)^{1/2} \) on the submanifold \( H = h \), according to Lemma 2, we obtain the new Hamiltonian

\[
(q_1^2 + \mu q_2^2)^{1/2}(H - h) = (q_1^2 + \mu q_2^2)^{1/2} \left( \frac{p^2 - 2h}{2} \right) - a.
\] (7)

Then again change the time \( \tau \) to \( s \) by \( \frac{ds}{d\tau} = 2((q_1^2 + \mu q_2^2)^{1/2}(H - h) + a) \) on the same level surface \( H = h \). By the second part of Lemma 2 we obtain the Hamiltonian system with Hamiltonian

\[
\tilde{H} = (q_1^2 + \mu q_2^2)\left( \frac{p^2 - 2h}{4} \right).
\] (8)

Finally, applying Legendre’s transformation, regarding \( p \) as coordinates and \( q \) as the canonically conjugate momenta, we can write

\[
\begin{aligned}
q_1' &= \frac{\partial \tilde{H}}{\partial q_1} = \frac{q_1}{2}(p^2 - 2h)^2 \\
n_2' &= \frac{\partial \tilde{H}}{\partial q_2} = \frac{q_2 \mu}{2}(p^2 - 2h)^2
\end{aligned}
\] (9)

where the prime indicates the derivative with respect to \( s \). Consequently

\[
\begin{aligned}
q_1 &= \frac{2p_1'}{(p^2 - 2h)^2} \\
n_2 &= \frac{2p_2'}{\mu(p^2 - 2h)^2}
\end{aligned}
\] (10)

Moreover since we have that

\[
\mathcal{L} = p_1'q_1(p, p_1') + p_2'q_2(p, p_2') - \tilde{H}(q_1(p, p_1'), q_2(p, p_2'), p_1, p_2) =
\]

we obtain a natural Lagrangian system with Lagrangian

\[
\mathcal{L} = \frac{1}{(p^2 - 2h)^2} \left( p_1'^2 + \frac{p_2'^2}{\mu} \right)
\] (12)

and if we let \( p_1 = x, p_2 = y \) and \( x^i = (x, y) \) we obtain

\[
\mathcal{L} = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j
\] (13)

where

\[
g_{ij} = \frac{2}{(x^2 + y^2 - 2h)^2} \begin{pmatrix}
1 & 0 \\
0 & 1/\mu
\end{pmatrix}.
\] (14)

The Lagrangian above defines a Riemann metric of Gaussian curvature given by

\[
K = -(1 - \mu)(x^2 - y^2) - 2h(1 + \mu)
\] (15)

that for \( \mu = 1 \) describes a metric of constant Gaussian curvature (positive for \( h < 0 \) and negative for \( h > 0 \)) [2, 3, 18]. It is interesting to remark that, by Corollary 2, for \( \mu \neq 1 \) we have at most one infinitesimal isometry (and hence also at most one first integral), since the curvature is not constant.
4. Nonexistence of Solutions of the Killing’s Equations And
Nonintegrability in the Anisotropic Problem

In the previous section we reduced the Anisotropic Kepler Problem to a geodesic flow
on a two dimensional surface, and hence we can apply the results obtained in Section 2.
This section is devoted to prove the nonintegrability of such system. Thus we are well
on our way to proving the following

**Theorem 2** The Anisotropic Kepler Problem is not analytically integrable, i.e. it does
not have any analytic first integral independent of the Hamiltonian.

The only fact that remains to be proved is that the Killing’s equations doesn’t have
any non-null solution. The equations of Killing for the metric (14) can be written as

\[ \begin{align*}
\frac{\partial \xi_1}{\partial y} + \frac{1}{\mu} \frac{\partial \xi_2}{\partial x} &= 0, \\
-4\frac{(x\xi_1 + y\xi_2)}{(x^2 + y^2 - 2h)} + 2\frac{\partial \xi_1}{\partial x} &= 0, \\
-4\frac{(x\xi_1 + y\xi_2)}{(x^2 + y^2 - 2h)} + 2\frac{\partial \xi_2}{\partial y} &= 0.
\end{align*} \]

(16)

Subtracting the third equation from the second we get

\[ \frac{\partial \xi_1}{\partial x} = \frac{\partial \xi_2}{\partial y}, \]

(17)

combining (17) with the first equation in system (16) we obtain

\[ \begin{align*}
\frac{\partial^2 \xi_1}{\partial x^2} &= -\mu \frac{\partial^2 \xi_1}{\partial y^2}, \\
\frac{\partial^2 \xi_2}{\partial y^2} &= -\frac{1}{\mu} \frac{\partial^2 \xi_2}{\partial x^2}.
\end{align*} \]

(18)

Now multiplying the second equation in system (16) by \((x^2 + y^2 - 2h)\) (assume
\(h < 0\)), differentiating with respect to \(x\) and using equation (17) and the first relation
in (16) we get the following second order differential equation

\[ 2\mu(x^2 + y^2 - 2h)\frac{\partial^2 \xi_1}{\partial y^2} - 4\mu y \frac{\partial \xi_1}{\partial y} + 4\xi_1 = 0. \]

(19)

Similarly from the third equation in (16) we get another second order differential equation

\[ \frac{2}{\mu}(x^2 + y^2 - 2h)\frac{\partial^2 \xi_2}{\partial x^2} - \frac{4}{\mu} x \frac{\partial \xi_2}{\partial x} + 4\xi_2 = 0. \]

(20)

It is easy to see that if \(h < 0\), \(y = 0\) is an ordinary point since the functions
\(-2y/(x^2 + y^2 - 2h)\) and \(2/(\mu(x^2 + y^2 - 2h))\) are analytical in \(y\) at \(y = 0\). Let each
of these functions be represented by its Taylor series at \(y = 0\) on the real line, then
every solution of (19) on \(\mathbb{R}\) is analytic at \(y = 0\) and can be represented on \(\mathbb{R}\) by its
Taylor series at \(y = 0\). Similarly every solution of (20) can be represented on \(\mathbb{R}\) by
its Taylor series at \(x = 0\) since \(x = 0\) is an ordinary point. Therefore we can look for
solutions of equations (19) of the form \(\xi_1 = \sum_{n=0}^{\infty} a_n(x)y^n\) and solutions of (20) of
the form \(\xi_2 = \sum_{n=0}^{\infty} b_n(y)x^n\). Routine calculations show that, from (16) we must have

\[ a_2 = -\frac{a_0}{\mu(x^2 - 2h)}, \quad a_3 = \frac{(\mu - 1)a_1}{3\mu(x^2 - 2h)}. \]

(21)
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and the following recurrence relation

$$a_{n+2} = -\frac{2 + \mu n(n - 3)}{\mu(x^2 - 2h)(n + 2)(n + 1)} a_n$$

holds for \( n > 1 \). From equation (20) we get

$$b_2 = -\frac{\mu b_0}{(y^2 - 2h)} , \quad b_3 = -\frac{(\mu - 1)b_1}{3(y^2 - 2h)}$$

and the recurrence relation

$$b_{n+2} = -\frac{2\mu + n(n - 3)}{(y^2 - 2h)(n + 2)(n + 1)} b_n$$

for \( n > 1 \). With the preparations above we can find the general solution of (19)

$$\xi^1 = a_0(x) \left( 1 - \frac{1}{\mu(x^2 - 2h)} y^2 + \frac{(1-\mu)}{6\mu^2(x^2 - 2h)^2} y^4 + \ldots \right) +$$

$$+ a_1(x) \left( y - \frac{\mu - 1}{3\mu(x^2 - 2h)} y^3 + \frac{\mu - 1}{3\mu^2(x^2 - 2h)^2} y^5 + \ldots \right)$$

and the general solution of (20)

$$\xi^2 = b_0(y) \left( 1 - \frac{\mu}{(x^2 - 2h)} x^2 + \frac{(\mu - 1)\mu}{6(x^2 - 2h)x} x^4 + \ldots \right) +$$

$$+ b_1(y) \left( x - \frac{\mu - 1}{3(x^2 - 2h)} x^3 + \frac{(\mu - 1)\mu}{30(x^2 - 2h)^2} x^5 + \ldots \right).$$

We will not need the explicit form of the solutions, but we will retain the recurrence relations. Now we need to check that the solutions we found above satisfy the “consistency” relations (18). Consider a solution of (19) written as a power series

$$\sum_{n=0}^{\infty} a_n(x)y^2,$$

using the first of equations (18) we find the relations

$$\frac{d^2a_n}{dx^2} = -2\mu a_{n+2}$$

since, by analyticity, we can exchange the sums with the derivatives. It is easy to check that choosing \( n = 0 \) the previous relation reduces to

$$\frac{d^2a_0}{dx^2} = -2\mu a_2 = \frac{2a_0}{(x^2 - 2h)}$$

that is solved by \( a_0 = (x^2 - 2h) \). For \( n = 2 \) instead we have

$$\frac{d^2a_2}{dx^2} = -2\mu a_4 = \frac{1 - \mu}{3(x^2 - 2h)} a_2.$$

From the first of equations (21) with \( a_0 = (x^2 - 2h) \) we get that \( a_2 = -1/\mu \) doesn’t satisfy (23) if \( \mu \neq 1 \). Thus the consistency conditions impose that \( a_0 = 0 \) and consequently \( a_{2n} = 0 \) for every integer \( n \).

Now, to show that the odd terms of the series also vanish, we consider equation (27) with \( n = 1 \) and we get

$$\frac{d^2a_1}{dx^2} = 2\mu a_3 = \frac{2(\mu - 1)}{3(x^2 - 2h)} a_1$$

(30)
also for \( n = 3 \) we obtain
\[
\frac{d^2 a_3}{dx^2} = -2\mu a_5 = \frac{a_3}{5(x^2 - 2h)}.
\]  
(31)

Differentiating the second of equations (21) twice and combining the result with equation (30) and (21) we find the following differential equation
\[
3(x^2 - 2h)\frac{d^2 a_3}{dx^2} + 12x\frac{da_3}{dx} + 2(2 + \mu)a_3 = 0
\]
(32)

Combining (31) and (32) we can eliminate the second order derivatives and we obtain the following first order differential equation
\[
\frac{da_3}{dx} = -\frac{2(2 + \mu) + 3/5}{12x} a_3 = -\frac{\gamma}{x} a_3
\]
(33)

where \( \gamma = \frac{2(2+\mu)+3/5}{12} \) and the general solution is \( a_3 = C x^{-\gamma} \) and satisfies equation (21) if and only if \( C = 0 \). Consequently \( a_3 = 0 \) and hence \( a_{2n+1} = 0 \) for every integer \( n \). Therefore \( \xi^1 \equiv 0 \).

Similarly we can apply the same reasoning to prove that \( \xi^2 \equiv 0 \). Indeed it is enough to consider again equations (28-33) exchanging \( x \) and \( y \), replacing \( \mu \) with \( 1/\mu \) and \( a_n \) with \( b_n \). Therefore we can conclude that there is no non-null motion, i.e. \( g_{ij}\xi^i\xi^j = 0 \) consequently, by Corollary 1, the geodesic flow is not analytically integrable and thus the initial anisotropic problems does not have any analytic first integrals other than the Hamiltonian. This completes the proof of Theorem 2.

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