Gain and phase type multipliers for structured feedback robustness
Axel Ringh, Xin Mao, Wei Chen, Li Qiu, and Sei Zhen Khong

Abstract—It is known that the stability of a feedback interconnection of two linear time-invariant systems implies that the graphs of the open-loop systems are quadratically separated. This separation is defined by an object known as the multiplier. The theory of integral quadratic constraints shows that the converse holds under certain conditions. This paper establishes that if the feedback is robustly stable against certain structured uncertainty, then there always exists a multiplier that takes a corresponding form. In particular, if the feedback is robustly stable to certain gain-type uncertainty, then there exists a corresponding multiplier that is of phase-type, i.e., its diagonal blocks are zeros. These results build on the notion of phases of matrices and systems, which was recently introduced in the field of control. Similarly, if the feedback is robustly stable to certain phase-type uncertainty, then there exists a gain-type multiplier, i.e., its off-diagonal blocks are zeros. The results are meaningfully instructive in the search for a valid multiplier for establishing robust closed-loop stability, and cover the well-known small-gain and the recent small-phase theorems.

Index Terms—Feedback robustness, structured uncertainty, multipliers, quadratic graph separation.

I. INTRODUCTION

It is well known that topological graph separation is both necessary and sufficient for the stability of a well-posed feedback configuration [1], [2]. Such topological graph separation is required to hold in the hard (a.k.a. unconditional [3]) manner, i.e., the integrals involved are taken over $[0, T]$ for all $T > 0$. A specific type of separation, called quadratic graph separation with linear multipliers, has been studied extensively in the nonlinear [4], [5], [6], [7], [8], [9] and linear [10], [11], [12] literatures. Quadratic graph separation has often been employed in the soft (a.k.a. conditional) manner, where the integrals are taken over $[0, \infty)$ in conjunction with homotopies that are continuous in the graph topology.

In the linear time-invariant (LTI) setting, soft quadratic graph separation is equivalent, via the Parseval-Plancherel theorem, to two complementary frequency-domain inequalities [13]. Such inequalities are the main object of study in this paper. The quadratic separation of interest is a function of an LTI object known as a multiplier, and the search for a suitable multiplier for characterising the uncertainty in a feedback interconnection and separating the graphs of the open-loop subsystems is a common theme in the vast literature on robust control [4], [14], [6], [15], [16], [17], [18].

The chief focus of this paper is on the necessity of quadratic graph separation for the feedback stability of matrices and LTI systems. Some elegant results along this direction have been obtained in [19], where robust stability/well-posedness against matricial uncertainties has been carefully investigated. In particular, it is shown in [19] that the well-posedness or closed-loop stability of two matrices is equivalent to the existence of a multiplier by which quadratic separation of these matrices holds. In other words, quadratic graph separation is necessary and sufficient for the closed-loop stability of two matrices, and results along this direction generalise to LTI systems.

This paper further strengthens the existing results by revealing a number of intricate relationships between the type of feedback robustness and the structure of any multiplier needed to establish such a robustness. Specifically, multipliers of the gain type (a.k.a. magnitudinal multipliers) are those whose off-diagonal blocks are 0, and the existence of a gain type multiplier implies that the closed-loop system is robust against phasal uncertainties, i.e., multiplication by arbitrary stable unitary (i.e., all-pass) transfer functions. On the other hand, multipliers of the phase type (a.k.a. phasal multipliers) are 0 on the diagonal blocks, and they characterise feedback systems that are robust to magnitudinal uncertainties, i.e., arbitrary positive scalings. The necessity results in this paper show the converse: if a feedback system is robust against phasal (resp. magnitudinal) uncertainties, then its robust stability can always be established using a multiplier of the gain (resp. phase) type. Importantly, these results imply that if a feedback system is expected to be robust against a certain form of uncertainties, then the search for a suitable multiplier to establish its robust stability can be restricted to one that admits a prescribed structure, and vice versa.

It is noteworthy that there exist relevant converse quadratic separation results that are different from those examined in this paper. Such results typically state that a feedback system is robustly stable against an arbitrary uncertainty characterised by a quadratic constraint if and only if the other open-loop...
subsystem satisfies the reverse quadratic constraint \[20\], \[21\]. However, in \[20\], \[21\] the multiplier defining the quadratic constraint is explicitly specified, whereas in this work certain forms of feedback stability are shown to imply the existence of a multiplier by which quadratic graph separation of the open-loop systems is defined.

The outline of the paper is as follows: in Section II we introduce necessary background material, in particular that related to sectorial matrices, and to quadratic graph separation and its use in stability analysis for multiple-input–multiple-output (MIMO) LTI systems. In Section III we analyze the form of multipliers needed in order to guarantee robust stability with respect to certain types of gain perturbations. The conclusion is that the existence of certain types of phasal multipliers is a both necessary and sufficient condition. Similarly, Section IV is devoted to stability against certain types of phase perturbations, and the existence of certain types of magnitudinal multipliers turns out to be a both necessary and sufficient condition. The main part of the paper ends with Section V where we draw some conclusions. Finally, in order to improve the readability, some of the lengthier proofs are deferred to appendices in the end of the paper.

II. BACKGROUND AND NOTATION

In this section we present some background material on sectorial matrices and matrix phases, on quadratic graph separation, and on transfer matrices and multipliers for stability of LTI systems. Moreover, the section is also used to set up the notation. To this end, let \( \mathbb{R}, \mathbb{C}, \mathbb{R}^n \) and \( \mathbb{C}^n \) denote the real and complex numbers and real and complex vectors of length \( n \), respectively, \( \mathbb{R}_+ := [0, \infty) \), \( \mathbb{R}_- := (-\infty, 0) \), and \( \mathbb{R}_{-\infty} := \mathbb{R}_- \setminus \{0\} \) the positive, negative, and strictly negative real numbers, respectively, \( \mathbb{C}_+ := \{ z \in \mathbb{C} \mid z = a + jb, a > 0 \} \) the open right-half complex plane, \( \mathbb{T} := \{ z \in \mathbb{C} \mid |z| = 1 \} \) the unit circle, and \( \mathbb{D} := \{ z \in \mathbb{C} \mid |z| < 1 \} \) the open unit disc.

Next, \( \mathbb{M}_{n,m} \) denotes the set of complex matrices with \( n \) rows and \( m \) columns; for square matrices we simply write \( \mathbb{M}_n \). Subsets of \( \mathbb{M}_n \) that will be used are the set of invertible matrices, denoted \( \mathbb{GL}_n \), the set of Hermitian matrices, denoted \( \mathbb{H}_n \), the set of (Hermitian) positive definite matrices, denoted \( \mathbb{P}_n \), and the set of unitary matrices, denoted \( \mathbb{U}_n \). For the corresponding sets of matrices with real entries, we will use the symbol \( \mathbb{M}_{n,m}(\mathbb{R}) \), etc. Moreover, on the set of Hermitian matrices we use \( \succeq \) to denote the Loewner partial order, i.e., for \( A, B \in \mathbb{H}_n \), \( A \succeq B \) if and only if \( A - B \) is positive definite and positive semi-definite, respectively; see, e.g., \[22\], Sec. 7.7]. Furthermore, by \( T^* \) and \( * \) we denote the transpose and the conjugate transpose of a matrix, respectively, and two matrices \( A, B \in \mathbb{M}_n \) are said to be congruent if there exists a \( C \in \mathbb{GL}_n \) such that \( A = C^*BC \). By \( I_n \) we denote the identity matrix of size \( n \times n \); sometimes the subscript \( n \) is omitted when the dimension is clear from the context. Finally, \( \lambda(\cdot) \) denotes the set of eigenvalues, and \( \sigma(\cdot) \) denotes the set of singular values of a matrix, i.e., for a matrix \( A \in \mathbb{M}_{n,m} \), \( \sigma_1(A) = \sqrt{\lambda_1(A^*A)} \) and hence \( A \) has \( m \) singular values. By convention, the singular values are sorted in a nonincreasing order, and if \( m > n \) this means that \( \sigma_{n+\ell}(A) = 0 \) for \( \ell = 1, \ldots, m - n \).

A. Sectorial matrices and matrix phases

The numerical range, also called the field of values, of a matrix \( A \in \mathbb{M}_n \) is defined as
\[ W(A) := \{ z \in \mathbb{C} \mid z = x^*Ax, \ x \in \mathbb{C}^n, \ ||x||^2 := x^*x = 1 \}. \]

By the Toeplitz-Hausdorff theorem, for any \( A \in \mathbb{M}_n \) the numerical range \( W(A) \) is a compact convex subset of \( \mathbb{C} \), see, e.g., \[23\] Property 1.2.1 and 1.2.2], \[24\] Thm. 4.1], or \[25\] Thm. 1.1-2]. Moreover, the numerical range of a matrix always contains its eigenvalues \[23\], Property 1.2.6]. Next, the conic hull of \( W(A) \), i.e., the smallest convex cone that contains the numerical range, is given by the set
\[ W'(A) := \{ z \in \mathbb{C} \mid z = x^*Ax, \ x \in \mathbb{C}^n, \ x \neq 0 \}, \]
which is called the angular numerical range. In particular, by the convexity of \( W(A) \) it follows that if \( 0 \notin W(A) \), then \( W(A) \) is contained in an open half-plane and hence the opening angle of \( W'(A) \) is strictly less than \( \pi \) — such matrices are called sectorial. If \( 0 \notin \text{int} W(A) \), i.e., not in the interior, then \( W(A) \) is contained in a closed half-plane and hence the opening angle of \( W'(A) \) is less than or equal to \( \pi \) — such matrices are called semi-sectorial. If \( 0 \in \text{int} W(A) \), then the opening angle of \( W'(A) \) is defined to be \( 2\pi \). Clearly, all sectorial matrices are also semi-sectorial. However, there are matrices that are not sectorial but for which the opening angle of the angular numerical range is strictly less than \( \pi \); see Remark 1. Therefore, we also introduce the set of quasi-sectorial matrices: a semi-sectorial matrix \( A \) with opening angle of \( W'(A) \) strictly less than \( \pi \) is called quasi-sectorial. This definition gives the (strict) inclusions
\[ \text{sectorial} \subset \text{quasi-sectorial} \subset \text{semi-sectorial}. \]

Finally, an important subset of sectorial matrices is the set of strictly accretive matrices, which is defined as
\[ A_n := \{ A \in \mathbb{M}_n \mid A + A^* > 0 \}. \]

The closure of this set is the set of accretive matrices, i.e., the set of all matrices \( A \in \mathbb{M}_n \) such that \( A + A^* \geq 0 \), which is a subset of the semi-sectorial matrices. In relation to this, we also define a matrix to be quasi-strictly accretive if it is accretive and quasi-sectorial (cf. Remark 1).

All sectorial matrices can be diagonalized by congruence \[26\], \[27\], \[28\], \[29\]. More specifically, any sectorial matrix \( A \) can be written as \( A = T^*DT \), where \( T \in \mathbb{GL}_n \) and where \( D \in \mathbb{U}_n \) is diagonal. This is called the sectorial factorization \[29\], and the matrix \( D \) is unique up to ordering of the diagonal elements \[28\], \[29\]. Based on this factorization, we define the phases of a sectorial matrix to be the phases of the eigenvalues of \( D \), and denote them by
\[ \phi(A) = [\phi_1(A), \phi_2(A), \ldots, \phi_n(A)]^T. \]

Each phase is only defined modulo \( 2\pi \), but by convention we sort them nonincreasingly, i.e., as
\[ \overline{\phi}(A) := \phi_1(A) \geq \phi_2(A) \geq \cdots \geq \phi_n(A) := \phi_n(A), \]
and define them so that \( \overline{\phi}(A) - \phi(A) < \pi \). With this convention, we can for example see that strictly accretive matrices
are sectorial matrices with phases contained in \((-\pi/2, \pi/2)\) modulo \(2\pi\). The phases of a sectorial matrix have many nice properties, and can for example be used to guarantee that a matrix of the form \(I + AB\) is of full rank; for an in-depth treatment of matrix phases we refer the reader to [30]. Moreover, the definition of phases can be extended to all semi-sectorial matrices; for the extension to quasi-sectorial matrices see Remark 1 below, and for the extension in the general case see [31, 32] for details. In any case, we still use \(\Phi(A)\) and \(\phi(A)\) to denote the largest and smallest phase, respectively.

Remark 1: Since the eigenvalues of a matrix are contained in its numerical range, any sectorial matrix must be full rank. The set of quasi-sectorial matrices extends the sectorial matrices to the set of matrices with opening angle of \(W'(A)\) is strictly less than \(\pi\) but that are not necessarily of full rank. In particular, let \(A \in M_n\) be a quasi-sectorial but not sectorial matrix. Then the origin must be a sharp point on \(\partial W(A)\), i.e., the boundary. This implies that \(0\) is a normal eigenvalue of \(A\) and that there exists a \(U \in \mathbb{U}_n\) such that

\[
A = U \begin{bmatrix} 0 & 0 \\ 0 & \tilde{A} \end{bmatrix} U^* \]

where \(\tilde{A}\) is sectorial and \(\text{rank}(\tilde{A}) = \text{rank}(A)\) [23 Thm. 1.6.6]. The phases of a quasi-sectorial matrix is hence defined as the phases of \(\tilde{A}\), and quasi-sectorial matrix thus have between 1 and \(n\) phases.

The use of phases in MIMO LTI systems: The concepts of magnitude and phase are well-established in the context of single-input-single-output LTI systems, and they both constitute highly useful and complementary tools. However, while the concept of system gain has a generally accepted and useful generalization to MIMO LTI systems, including small-gain theorems for robust stability, the concept of phase has attracted much less attention. Early works trying to establish definitions of phases with useful properties in the MIMO setting can be found in, e.g., [33, 34, 35, 36, 37]. Recently, there has been a renewed interest in the concept of phases for MIMO systems, both for LTI systems [38, 39, 40] and for nonlinear systems [40], with small-phase theorems for robust stability as a result. This concept of phases for MIMO LTI systems builds on the concept of matrix phases [40], as introduced above, and as will be seen below it is also connected to quadratic graph separation. In fact, this notion of phase turns out to be, in some sense, the correct notion in order to guarantee robust stability against certain types of magnitudinal uncertainties (see Section III).

B. Graph separation and multipliers for stability

For a matrix \(C \in M_{m,n}\), the graph is defined as all pairs \((x_1, x_2) \in \mathbb{C}^{n+m}\) such that \(Cx_1 = x_2\), and the inverse graph is defined as all pairs \((x_2, x_1)\). To understand how graph separation plays a role in stability of interconnected systems, first note that for two matrices \(A \in M_{m,n}\) and \(B \in M_{n,m}\) we have that \(\det(I + AB) \neq 0\) is equivalent to that \(\det(I - AB) \neq 0\). The latter determinant is zero if and only if there exists a nonzero \((x_1, x_2) \in \mathbb{C}^{n+m}\) such that

\[
0 = \begin{bmatrix} I_n & -B \\ A & I_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - Bx_2 \\ Ax_1 + x_2 \end{bmatrix}.
\]

Now, we can identify the above equality as \((x_1, x_2)\) being a nontrivial element in both the graph of \(-A\) and in the inverse graph of \(B\). Therefore, \(\det(I_m + AB) \neq 0\) if and only if the graph of \(-A\) and the inverse graph of \(B\) only intersect in the origin, i.e., if and only if

\[
\text{range} \left( \begin{bmatrix} I_n \\ -A \end{bmatrix} \right) \cap \text{range} \left( \begin{bmatrix} B \\ I_m \end{bmatrix} \right) = \{0\},
\]

where \(\text{range}(\cdot)\) denotes the range of a matrix. A similar condition holds for the stability of a well-defined interconnection of dynamical systems, see, e.g., [41, 42].

In [19], it was shown that a necessary and sufficient condition for (1) to hold is that there exists a multiplier which achieves quadratic separation. More precisely, (1) holds if and only if there exists a \(P \in \mathbb{H}_{n+m}\) such that

\[
(I - A^*) P \begin{bmatrix} I \\ -A \end{bmatrix} < 0 \quad (2a)
\]

\[
(B^* I) P \begin{bmatrix} B \\ I \end{bmatrix} \geq 0. \quad (2b)
\]

There are several equivalent forms of this condition. For example, if there exists a \(P \in \mathbb{H}_{n+m}\) such that (2) holds, due to the strict inequality in (2a) this \(P\) also satisfies

\[
(I - A^*) P \begin{bmatrix} I \\ -A \end{bmatrix} \leq -\varepsilon A^* A \quad (3a)
\]

\[
(B^* I) P \begin{bmatrix} B \\ I \end{bmatrix} \geq 0, \quad (3b)
\]

for some \(\varepsilon > 0\). Moreover, by rewriting the \((2,2)\)-block of the block-matrix \(P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\) as \(P_{22} = P_{22} - \varepsilon I_m\) in (3), with a slight abuse of notation we see that there exists another matrix \(P\) such that

\[
(I - A^*) P \begin{bmatrix} I \\ -A \end{bmatrix} \leq 0 \quad (4a)
\]

\[
(B^* I) P \begin{bmatrix} B \\ I \end{bmatrix} > 0, \quad (4b)
\]

which in turn implies that

\[
(I - A^*) P \begin{bmatrix} I \\ -A \end{bmatrix} \leq 0 \quad (5a)
\]

\[
(B^* I) P \begin{bmatrix} B \\ I \end{bmatrix} \geq \varepsilon B^* B, \quad (5b)
\]

for some \(\varepsilon > 0\). Finally, by rewriting the \((1,1)\)-block of \(P\) in (5) as \(P_{11} = P_{11} + \varepsilon I_n\), we have that the existence of a multiplier fulfilling (5) implies that there exists a multiplier fulfilling (2). Nevertheless, in the step from (3) to (4), and from (5) to (2), the actual multiplier (and hence also potentially the structure) changes. Since the results in this paper are concerned with necessary conditions for existence of multipliers of certain forms, we state all these cases explicitly. Moreover, for convenience we summarize the results in the following lemma.

**Lemma 2 ([19]):** For \(A \in M_{m,n}\) and \(B \in M_{n,m}\), the following statements are equivalent:

(i) \(\det(I_m + AB) \neq 0\);

(ii) condition (1) holds;

(iii) there exists a matrix \(P\) such that one of the conditions (2)-(5) holds.
C. LTI systems and multipliers for stability

Next, we consider extensions of the aforementioned results to LTI systems. To this end, let us first introduce the function spaces needed. Let \( \| \cdot \|_2 \) denote the matrix 2-norm. Define the Lebesgue space

\[
\mathbf{L}^m_{\infty} := \left\{ \phi: \mathbb{R} \to \mathbb{M}_{m,n} \left| \| \phi \|_\infty := \operatorname{ess \ sup}_{\omega \in \mathbb{R}} \| \phi(\omega) \|_2 < \infty \right. \right\}
\]

and the Hardy space

\[
\mathbf{H}^m_{\infty} := \left\{ \begin{array}{c} \phi \in \mathbf{L}^m_{\infty} \mid \phi \text{ has analytic continuation into } \mathbb{C} \text{ with } \operatorname{ess \ sup}_{s \in \mathbb{C}} \| \phi(s) \|_2 < \infty \end{array} \right\}.
\]

Denote by \( \mathbf{R}^m \times n \) the set of \( m \times n \) real/rational proper transfer function matrices, and let \( \mathbf{RH}^m_{\infty} := \mathbf{R}^m \times n \cap \mathbf{H}^m_{\infty} \), i.e., the subset of \( \mathbf{R}^m \times n \) with no poles in the closed right-half complex plane. A \( G \in \mathbf{RH}^m_{\infty} \) is said to be passive if \( G(\omega) + G(\omega)^* \geq 0 \) for all \( \omega \in \mathbb{R} \), and it is said to be output strictly passive if there exists \( \epsilon > 0 \) such that \( G(\omega) + G(\omega)^* \geq \epsilon G(\omega)^* G(\omega) \) for all \( \omega \in \mathbb{R} \).

Next, consider \( G \in \mathbf{RH}^m_{\infty} \) and \( K \in \mathbf{RH}^s_{\infty} \). The (negative) feedback interconnection of \( G \) and \( K \) is said to be stable if \( (I + GK)^{-1} \in \mathbf{RH}^m_{\infty} \).

**Proposition 3:** Given \( G \in \mathbf{RH}^m_{\infty} \) and \( K \in \mathbf{RH}^s_{\infty} \), then \( (I + GK)^{-1} \in \mathbf{RH}^m_{\infty} \) if there exists \( \Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{21} & \Pi_{22} \end{bmatrix} \in \mathbf{L}^{(n+m) \times (n+m)} \) such that for all \( \omega \in [0, \infty) \), \( \Pi(\omega) = \Pi(\omega)^* = \Pi(\omega) \), \( \Pi_{11}(\omega) \in [0, \infty) \), \( \Pi_{22}(\omega) \geq 0 \),

\[
\begin{pmatrix} I & -G(\omega)^* \end{pmatrix} \Pi(\omega) \begin{pmatrix} I \\ -G(\omega) \end{pmatrix} > 0,
\]

or equivalently,

\[
\begin{pmatrix} I & -G(\omega)^* \end{pmatrix} \Pi(\omega) \begin{pmatrix} I \\ -G(\omega) \end{pmatrix} \leq 0
\]

\[
(K(\omega)^*)^T \Pi(\omega) \begin{pmatrix} K(\omega) \\ I \end{pmatrix} \geq \epsilon K(\omega)^* K(\omega)
\]

for some \( \epsilon > 0 \). Furthermore, if \( m = n \) and \( G^{-1}, K^{-1} \in \mathbf{RH}^m_{\infty} \), then \( (I + GK)^{-1} \in \mathbf{RH}^m_{\infty} \) if there exists \( \Pi \in \mathbf{L}^{(2m) \times (2m)} \) such that for all \( \omega \in [0, \infty) \), \( \Pi(\omega)^* = \Pi(\omega) \), \( \Pi_{11}(\omega) \geq 0 \), \( \Pi_{22}(\omega) \geq 0 \), and (6) holds.

**Proof:** If \( \Pi_{11}(\omega) \geq 0 \) and \( \Pi_{22}(\omega) \geq 0 \), then

\[
(K(\omega)^*)^T \Pi(\omega) \begin{pmatrix} K(\omega) \\ I \end{pmatrix} \geq 0
\]

is equivalent to

\[
(\alpha K(\omega)^*)^T \Pi(\omega) \begin{pmatrix} \alpha K(\omega) \\ I \end{pmatrix} \geq 0
\]

for all \( \alpha \in [0, 1] \). Feedback stability can then be established using the Parseval-Plancherel theorem as in [13 Thm. 3.1] and the theory of integral quadratic constraints [6 Thm. 1] or [9 Cor. IV.3], where the proofs are written purely in the time domain. An alternative, more direct frequency-domain proof is provided below for completeness.

By applying Lemma 2 frequency-wise, it holds that \( \det(I + \alpha G(\omega)K(\omega)) \neq 0 \) for all \( \omega \in [0, \infty) \), \( \alpha \in [0, 1] \). It remains to show that \( \det(I + \alpha G(s)K(s)) \neq 0 \) for all \( s \in \mathbb{C}_+ \), from which \( (I + G)^{-1} \in \mathbf{RH}^m_{\infty} \) follows. To this end, observe that since \( G \in \mathbf{RH}^m_{\infty} \), \( \det(I + \alpha G(s)K(s)) \neq 0 \) for all \( s \in \mathbb{C}_+ \) for sufficiently small \( \alpha > 0 \). Suppose to the contrary that \( \det(I + G(s)K(s)) = 0 \) for some \( s \in \mathbb{C}_+ \). Then, by the continuity of the locations of the zeros of \( \det(I + \alpha G(s)K(s)) \) in \( \alpha \), there must exist a \( \alpha \in (0, 1) \) and an \( \omega \in [0, \infty) \) such that \( \det(I + \alpha G(j\omega)K(j\omega)) = 0 \), leading to a contradiction. Therefore, it must be true that \( \det(I + \alpha G(s)K(s)) \neq 0 \) for all \( s \in \mathbb{C}_+ \) and \( \alpha \in [0, 1] \).

On the other hand, if \( \Pi_{11}(\omega) \geq 0 \) and \( \Pi_{22}(\omega) \leq 0 \), then

\[
(K(\omega)^*)^T \Pi(\omega) \begin{pmatrix} K(\omega) \\ I \end{pmatrix} \geq 0
\]

is equivalent to

\[
(\alpha K(\omega)^*)^T \Pi(\omega) \begin{pmatrix} \alpha K(\omega) \\ I \end{pmatrix} \geq 0
\]

for all \( \alpha \geq 1 \). Since \( G^{-1}, K^{-1} \in \mathbf{RH}^m_{\infty} \) by the large gain theorem [43 Thm. 4.1], \( (I + \alpha G)^{-1} \in \mathbf{RH}^m_{\infty} \) for sufficiently large \( \alpha > 1 \). By repeating the preceding arguments, one may then establish that \( \det(I + \alpha G(s)K(s)) \neq 0 \) for all \( s \in \mathbb{C}_+ \) and \( \alpha \geq 1 \), from which \( (I + G)^{-1} \in \mathbf{RH}^m_{\infty} \) follows.

**Remark 4:** Proposition 3 remains true when all the inequality signs therein are flipped.

The following necessary condition for feedback stability, complementing the sufficient condition in Proposition 3, can be proved by using a construction from [19].

**Proposition 5:** Given \( G \in \mathbf{RH}^m_{\infty} \) and \( K \in \mathbf{RH}^s_{\infty} \), \( (I + GK)^{-1} \in \mathbf{RH}^m_{\infty} \) only if there exists \( \Pi \in \mathbf{L}^{(n+m) \times (n+m)} \) such that for all \( \omega \in [0, \infty) \), \( \Pi(\omega)^* = \Pi(\omega) \),

\[
\begin{pmatrix} I & -G(\omega)^* \end{pmatrix} \Pi(\omega) \begin{pmatrix} I \\ -G(\omega) \end{pmatrix} < 0,
\]

\[
(K(\omega)^*)^T \Pi(\omega) \begin{pmatrix} K(\omega) \\ I \end{pmatrix} \geq 0.
\]

**Proof:** That \( (I + G)^{-1} \in \mathbf{RH}^m_{\infty} \) implies that \( \inf_{\omega \in \mathbb{R}} |\det(I + K(\omega)G(\omega))|^2 > 0 \) for all \( \omega \in [0, \infty] \). Following the proof in [19 Cor. 1] frequency-wise, define

\[
\Pi(\omega) := \begin{pmatrix} G(\omega)^* \\ G(\omega) \end{pmatrix} \begin{pmatrix} G(\omega) \\ I \end{pmatrix} - \epsilon I.
\]

The claim may then be verified to hold for sufficiently small \( \epsilon > 0 \).

III. MULTIPLIERS OF PHASE TYPE

In this section we investigate the necessity of certain multipliers of phase type for robust stability of feedback interconnections with respect to magnitudinal uncertainties. In particular, we first show that \( I + AB \) is nonsingular for magnitude scaling and certain congruence transformations, respectively, only if there exists certain types of phaseal multipliers. The results are then extended to MIMO LTI systems.
A. Multipliers for scaling uncertainty

One of the simplest forms of uncertainty is an uncertainty in the scaling of one of the matrices. In order to guarantee that the interconnection is stable for all scalings, it would therefore be desirable to show that $I + \tau AB$ is nonsingular for all $\tau \in \mathbb{R}_+$. For $A, B \in \mathbb{GL}_n$, that is equivalent to that $\lambda(AB) \cap \mathbb{R}_- = \emptyset$, i.e., that the intersection is empty, and necessary and sufficient conditions for the latter is given in the following proposition.

**Proposition 6:** Given $A, B \in \mathbb{GL}_n$, there exists an $H \in \mathbb{GL}_n$ such that $HA$ and $H^*B$ are strictly accretive if and only if $\lambda(AB) \cap \mathbb{R}_- = \emptyset$.

**Proof:** The proof follows by using results in [44], [30]. More precisely, first assume $\lambda(AB) \cap \mathbb{R}_- = \emptyset$. Then, by [44] Thm. 1] we have that the matrix $AB$ can be factored as $AB$, where $A, B \in \mathbb{A}_n$. Let $H^* = B^*$, then $H^*B = B \in \mathbb{A}_n$.

Moreover, by congruence we have that $HA$ is accretive if and only if $H^{-1}(HA)H^{-*} = AH^{-*}$ is accretive. For the latter, we have that $AH^{-*} = ABB^{-1} = A \in \mathbb{A}_n$, and hence there exists an $H \in \mathbb{GL}_n$ so that $HA, H^*B \in \mathbb{A}_n$. This proves the “if”-statement. To show the “only if”-statement, assume that there exists an $H$ so that $HA, H^*B \in \mathbb{A}_n$. Again, by congruence $HA \in \mathbb{A}_n$ if and only if $AH^{-*} \in \mathbb{A}_n$. By [30] Thm. 6.2) it follows that $AB = AH^{-*}H^*$ have no eigenvalues along $\mathbb{R}_-$.

The result in Proposition 6 can be understood in terms of the existence of a phasal multiplier $P \in \mathbb{H}_n$ that fulfills (2), i.e., a multiplier $P$ where only the off-diagonal blocks are nonzero and where in fact both inequalities in (2) are strict. In particular, this formally confirms the intuition that in order to show that the interconnection is stable under an arbitrary positive scaling uncertainty, a certain type of “phase information” is the only thing that is needed. Moreover, these results can be strengthened to (certain) matrices which are not of full rank as follows.

**Theorem 7:** Given $A, B \in \mathbb{M}_n$, assume that a potential zero-eigenvalue of $AB$ is semi-simple. Then the following statements are equivalent:

(i) $\det(I + \tau AB) \neq 0$ for all $\tau \geq 0$;

(ii) there exists a $P \in \mathbb{H}_n$, fulfilling (3), and $P$ takes the form

\[ P = \begin{bmatrix} 0 & H \\ H^* & 0 \end{bmatrix} \tag{7} \]

for some $H \in \mathbb{M}_n$;

(iii) for the eigenvalues of $AB$, it holds that

$\lambda(AB) \cap \mathbb{R}_- = \emptyset$. \tag{8}

**Proof:** See Appendix A.

If the matrix A in Theorem 7 is full rank, then the statement in Theorem 7(ii) can be strengthened and a number of other equivalent conditions can also be derived.

**Corollary 8:** Let $A, B \in \mathbb{M}_n$ be as in Theorem 7. If $A \in \mathbb{GL}_n$, then the statements in Theorem 7 are also equivalent to

(iii) there exists an $H \in \mathbb{GL}_n$ such that $HA$ is strictly accretive and $H^*B$ is quasi-strictly accretive.

(v) there exists an $H \in \mathbb{GL}_n$ such that $HA$ is strictly accretive and $H^*B$ is accretive.

Moreover, the multiplier $P$ in Theorem 7(ii) can be selected so that it fulfills (3).

**Proof:** See Appendix A.

In many applications, we would be interested in corresponding results for real-valued matrices. By just slightly modifying the proof of the theorem, we have the following corollary.

**Corollary 9:** Under the assumptions in Theorem 7 if $A, B \in \mathbb{M}_n(\mathbb{R})$, the same conclusion is true where we can restrict $H$ to also be real.

**Proof:** See Appendix A.

Observe that, in general, the assumption in Theorem 7 that a zero-eigenvalue in $AB$ is semi-simple cannot be relaxed. This can be seen by the following counterexample for $3 \times 3$ matrices, where the zero-eigenvalue of $AB$ has a Jordan block of size $2 \times 2$.

**Example 10:** Let

\[ A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \quad B = I_3, \quad H = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \]

and note that $\det(I + \tau AB) = 1 + \tau \neq 0$ for all $\tau \geq 0$. Moreover, $A^*A = \text{diag}(1, 0, 1)$. A direct calculation gives that

\[ HA + A^*H^* = \begin{bmatrix} h_{11} + h_{11}^* & h_{12} + h_{31}^* & h_{13} \\ h_{31} + h_{12}^* & h_{32} + h_{32}^* & h_{33} \\ h_{13}^* & h_{32}^* & h_{33}^* \end{bmatrix}, \]

and for this to be positive semidefinite, all principle minors need to be nonnegative [22, Cor. 7.1.5.]. In particular, this means that we must have $h_{13} = h_{33} = 0$. Therefore, $HA + A^*H^*$ has at most rank 2, and can hence only be positive semidefinite. Moreover, it is easily seen that $HA + A^*H^* \not\in \varepsilon A \mathbb{A}_n$ for all $\varepsilon > 0$. Therefore, there is no $H$ so that a multiplier of the form $H A + A^*H^*$ satisfies (2) or (3). Next, note that

\[ H^*B + B^*H = H + H^* = \begin{bmatrix} h_{11} + h_{11}^* & h_{12} + h_{21}^* & h_{31} \\ h_{21} + h_{12}^* & h_{22} + h_{22}^* & h_{23} + h_{32}^* \\ h_{31} & h_{32} + h_{32}^* & h_{33} \end{bmatrix}, \]

which, similar to above, can only be positive semidefinite if $h_{31} = 0$ and $h_{32} = -h_{23}$. However, that means that $H + H^*$ has rank 2, and hence can only be positive semidefinite. In particular, for all $\varepsilon > 0$ we therefore have $H^*B + B^*H \not\in \varepsilon B \mathbb{B}_n$. Therefore, there is no $H$ so that a multiplier of the form $H A + A^*H^*$ satisfies (4) or (5).

Nevertheless, while the above counterexample shows that the condition on the semi-simple zero-eigenvalue can in general not be relaxed, the case for matrices of size $2 \times 2$ is still open. In fact, the following gives an example of where there exists a multiplier of the form $H A + A^*H^*$ that fulfills (3).

**Example 11:** Let

\[ A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = I_2, \quad H = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \]

and note that $\det(I + \tau AB) = 1 \neq 0$ for all $\tau \geq 0$. A direct calculation gives that $A^*A = \text{diag}(0, 1)$, that $HA = \text{diag}(0, 1)$, and that $H^*B + HB^* = H^* + H = \text{diag}(0, 2)$. Therefore, for $\varepsilon = 1$ we have that $P$ as in (7) fulfills (3).
B. Multipliers for stability under congruence

Theorem 7 shows that stability under magnitude scaling is equivalent to the existence of a phasal multiplier. However, due to the fact that the set of magnitudinal perturbations was limited (in some sense minimal), the set of possible multipliers of phase type to which we could restrict our attention, and still have a necessary and sufficient condition, was large (in some sense maximal). Next, we investigate the type of perturbations against which a minimal set of phasal multipliers can guarantee robust stability. In this case, we have the following result.

**Theorem 12:** Given $A, B \in M_n \setminus \{0\}$, the following statements are equivalent:

(i) $\det(I + T^*AT^*BS) \neq 0$ for all $T, S \in \mathbb{GL}_n$;

(ii) there exists a $P \in H_{2n}$ fulfilling (3) or (4), and which takes the form

$$P = \begin{bmatrix} 0 & z^*I \\ z^* & 0 \end{bmatrix}$$

for some $z \in \mathbb{T}$;

(iii) one matrix is quasi-sectorial, the other is semi-sectorial, $\phi(A) + \phi(B) < \pi$, and $\phi(A) + \phi(B) > -\pi$.

Finally, if the quasi-sectorial matrix in (iii) is of full rank, then the multiplier $P$ in (ii) fulfills (4) or (5).

**Proof:** See Appendix B.

The above results is a type of small-phase theorem, akin to [32, Lem. 4]. The difference is that Theorem 12 considers robust stability against congruence of two given matrices, while [32, Lem. 4] considers robust stability with respect to a matrix cone of semi-sectorial matrices. Moreover, similar to before we get the following real-valued version of the theorem as a corollary.

**Corollary 13:** Theorem 12 remains true when $A, B, T, S$, and $z$ are all real.

**Proof:** This can be established by noting that a real matrix $A$ is semi-sectorial if and only if either $A + AT \geq 0$ or $A + A^T \leq 0$.

C. Phasal multipliers for LTI systems

Next, we extend the above results to LTI systems. In particular, in Section 11 it was shown how quadratic graph-separation results for matrices can be extended to LTI systems. Here, we follow along the same line. In particular, for magnitudinal perturbations we have the following necessary and sufficient condition for stability.

**Theorem 14:** Given $G \in RH_{\infty \times n}$ and $K \in RH_{\infty \times n}$ for which any potential zero-eigenvalue of $G(j\omega)K(j\omega)$, for $\omega \in [0, \infty)$, is semi-simple, then $(I + \tau K)^{-1} \in RH_{\infty \times n}$ for all $\tau > 0$ if and only if there exists an $H \in L_{\infty \times n}^n$ such that for all $\omega \in [0, \infty)$,

$$\left( I - G(j\omega)^* \right) \Pi(j\omega) \begin{bmatrix} I & -G(j\omega) \\ -G(j\omega) & K(j\omega) \end{bmatrix} \geq 0,$n

where

$$\Pi(j\omega) := \begin{bmatrix} 0 & H(j\omega) \\ H(j\omega)^* & 0 \end{bmatrix}.$$n

**Proof:** Sufficiency follows from Proposition 3 and Remark 4 and the fact that the inequality

$$(K(j\omega)^* I) \Pi(j\omega) \begin{bmatrix} K(j\omega) & I \end{bmatrix} \geq 0$$

implies that

$$(\tau K(j\omega)^* I) \Pi(j\omega) \begin{bmatrix} \tau K(j\omega) & I \end{bmatrix} \geq 0$$

for all $\tau > 0$. Necessity can be established by applying Theorem 7 and Corollary 9 frequency-wise in a similar fashion to the proof of Proposition 5. In particular, since $G$ and $K$ are continuous on the imaginary axis, $\Pi$ may also be chosen to be continuous on the imaginary axis.

The result above shows that if the feedback interconnection is robustly stable against arbitrary positive scaling, then only phasal properties of the open-loop components are required to establish its stability, i.e., any corresponding multiplier $\Pi$ has its diagonal blocks being $0$.

Analogously, the following two results, which establish sufficiency and necessity for stability under real congruence transformations, may be readily derived.

**Theorem 15:** Given $G \in RH_{\infty \times n}$ and $K \in RH_{\infty \times n}$, then $(I + T^*GTS^*KS)^{-1} \in RH_{\infty \times n}$ for all $T, S \in \mathbb{GL}_n(\mathbb{R})$ if there exists $z \in \mathbb{L}_\infty$ such that for all $\omega \in [0, \infty)$,

$$\begin{bmatrix} I - G(j\omega)^* \\ I - G(j\omega) \end{bmatrix} \Pi(j\omega) \begin{bmatrix} I & -G(j\omega) \\ -G(j\omega) & K(j\omega) \end{bmatrix} \geq 0$$

or

$$\begin{bmatrix} I - G(j\omega)^* \\ I - G(j\omega) \end{bmatrix} \Pi(j\omega) \begin{bmatrix} I & K(j\omega) \\ K(j\omega)^* & I \end{bmatrix} \geq 0$$

where

$$\Pi(j\omega) := \begin{bmatrix} 0 & z(j\omega)^*I \\ z(j\omega)I & 0 \end{bmatrix}.$$n

**Proof:** Observe that (9) implies that for all $T, S \in \mathbb{GL}_n(\mathbb{R})$, there exists $\epsilon_T$ such that

$$\left( I - TT^*G(j\omega)^*T \right) \Pi(j\omega) \begin{bmatrix} I & -TT^*G(j\omega)T \\ -TT^*G(j\omega)^*T & K(j\omega)^*S \end{bmatrix} \geq 0,$n

and similarly for (10). The claim then follows from Proposition 3 and Remark 4.

**Theorem 16:** Given $G \in RH_{\infty \times n}$ and $K \in RH_{\infty \times n}$, $(I + T^*GTS^*KS)^{-1} \in RH_{\infty \times n}$ for all $T, S \in \mathbb{GL}_n(\mathbb{R})$ only if for $\omega \in \{0, \infty\}$,

$$\begin{bmatrix} I - G(j\omega)^* \\ I - G(j\omega) \end{bmatrix} \Pi(j\omega) \begin{bmatrix} I & -G(j\omega) \\ -G(j\omega) & K(j\omega) \end{bmatrix} \geq 0$$

or

$$\begin{bmatrix} I - G(j\omega)^* \\ I - G(j\omega) \end{bmatrix} \Pi(j\omega) \begin{bmatrix} I & K(j\omega) \\ K(j\omega)^* & I \end{bmatrix} \geq 0$$

where $\Pi(j\omega)$ is a type of small-phase theorem, akin to [32, Lem. 4].
statement is trivial if any of the two matrices $A$, $B$ are straightforward. We therefore restrict our attention to the
This is a well-known passivity theorem.

phasal multipliers in order to guarantee robust stability with respect to certain magnitudinal perturbations. In this section, following statements are equivalent:

$$\begin{align*}
(A) & \quad \det(A) \neq 0, \\
(B) & \quad \text{there exists a } P \in \mathbb{H}_{n+m} \text{ fulfilling (2), with both inequalities strict, which takes the form} \\
\quad & \quad P = \begin{bmatrix} -N & 0 \\ 0 & M \end{bmatrix} \\
(C) & \quad \text{for the eigenvalues of } AB, \text{ it holds that} \\
\quad & \quad \lambda(AB) \cap \mathbb{T} = \emptyset; \\
(D) & \quad \text{there exists } M \in \mathbb{H}_m \text{ and } N \in \mathbb{H}_n \text{ such } A^*MA \prec N \text{ and } B^*NB \prec M.
\end{align*}$$

Proof: The equivalences “(i) $\iff$ (iii)” and “(ii) $\iff$ (iv)” are straightforward. We therefore restrict our attention to the equivalence “(iii) $\iff$ (iv)”. To this end, first note that the statement is trivial if any of the two matrices $A$, $B$ is the zero matrix. Therefore, in the remaining we will, without loss of generality, assume that both are nonzero.

To show “(iv) $\Rightarrow$ (iii)”: assume that there exist $M \in \mathbb{H}_m$ and $N \in \mathbb{H}_n$ such that $A^*MA \prec N$ and $B^*NB \prec M$. Together with [22] Obs. 7.1.8, the former inequality implies that

$$B^*A^*MAB \preceq B^*NB.$$

In particular, this means that

$$B^*A^*MAB \preceq B^*NB \prec M$$

which means that the Stein equation

$$M - B^*A^*MAB = V$$

has a solution for some $V \in \mathbb{P}_m$. By [46] Thm. 13.2.2 we therefore have that $\lambda(AB) \cap \mathbb{T} = \emptyset$. To show the “(iii) $\Rightarrow$ (iv)” assume that (12) holds, and let $AB = XJX^{-1}$ be a Jordan decomposition of $AB$. By [46] we can, without loss of generality, assume that

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix},$$

where $J_1 \in \mathbb{M}_{m_1}$ has all eigenvalues in $\mathbb{D}$ and $J_2 \in \mathbb{M}_{m_2}$ has all eigenvalues in $(\mathbb{D})^c$, i.e., outside of the close unit disc, and where $m = m_1 + m_2$. Next, using [46] Sec. 13.2 and [47] Exer. 4.9.30 we have that for any $P_1 \in \mathbb{P}_{m_1}$ and $P_2 \in \mathbb{P}_{m_2}$, there is at least one solution to the Stein equation

$$M - B^*A^*MAB = X^{-\epsilon} \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} X^{-1} := Q > 0.$$ 

In particular, (13) means that $B^*A^*MAB \prec M$. Now, define $\tilde{N} := A^*MA$ and note that this implies that

$$A^*MA = \tilde{N} \preceq \tilde{N},$$

$$B^*\tilde{N}B = B^*A^*MAB \prec M.$$

To prove that there exist $M \in \mathbb{H}_m$ and $N \in \mathbb{H}_n$ with both inequalities above strict, consider $\tilde{N} := \tilde{N} + \epsilon I$ for some $\epsilon > 0$. In particular,

$$A^*MA = \tilde{N} \prec \tilde{N} + \epsilon I = N$$

for all $\epsilon > 0$. Moreover, since $M - B^*\tilde{N}B = M - B^*A^*MAB = Q > 0$, we have that

$$M - B^*NB = M - B^*\tilde{N}B - \epsilon B^*B = Q - \epsilon B^*B \succ 0,$$

for $\epsilon$ small enough. This completes the proof.

Proof: This follows by applying Corollary [13] to the pairs of real matrices $\{G(j0), K(j0)\}$ and $\{G(j\infty), K(j\infty)\}$. The separation condition in the theorem above holds for sufficiently small and large frequencies by the continuity of the transfer functions $G$ and $K$. Such properties are useful, for instance, in the study of negative imaginary systems [45], where an example of open-loop systems being passive on sufficiently small and large frequencies and negative imaginary elsewhere can be found.

Example 17: Let $G \in \mathbb{RH}_\infty^{n \times n}$ be output strictly passive and $K \in \mathbb{RH}_\infty^{n \times n}$ be passive. Then they satisfy the separation conditions in all three of the theorems above with

$$\Pi : = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.$$ This is a well-known passivity theorem.

IV. MULTIPLIERS OF GAIN TYPE

In the previous section, we investigated the necessity of phasal multipliers in order to guarantee robust stability with respect to certain magnitudinal perturbations. In this section, we turn to the necessity of magnitudinal multipliers in order to guarantee robust stability with respect to certain phasal perturbations.

A. Multiplier for rotation uncertainty

In analogy with Section III where we first investigated pure scaling uncertainties, we here first consider pure rotational uncertainties. In this case, we have the following result.

Theorem 18: Given $A \in \mathbb{M}_{m,n}$ and $B \in \mathbb{M}_{n,m}$, the following statements are equivalent:

(i) $\det(I + e^{i\theta}AB) \neq 0$ for all $\theta \in [0, 2\pi]$;

(ii) there exists a $P \in \mathbb{H}_{n+m}$ fulfilling (2), with both inequalities strict, which takes the form

$$P = \begin{bmatrix} -N & 0 \\ 0 & M \end{bmatrix}$$

for some $N \in \mathbb{H}_n$ and $M \in \mathbb{H}_m$;

(iii) for the eigenvalues of $AB$, it holds that

$$\lambda(AB) \cap \mathbb{T} = \emptyset;$$

(iv) there exists $M \in \mathbb{H}_m$ and $N \in \mathbb{H}_n$ such $A^*MA \prec N$ and $B^*NB \prec M$.

Proof: The equivalences “(i) $\iff$ (iii)” and “(ii) $\iff$ (iv)” are straightforward. We therefore restrict our attention to the equivalence “(iii) $\iff$ (iv)”. To this end, first note that the statement is trivial if any of the two matrices $A$, $B$ is the zero
B. Multiplier for stability under unitary perturbation

Next, having established that the existence of a magnitudinal multiplier is necessary and sufficient for stability in case of a pure rotational uncertainty, we now consider for which type of phasal uncertainties a much smaller set of magnitudinal multipliers can guarantee robust stability. The set of magnitudinal multipliers considered are diagonal and completely parametrized by a nonnegative number and an element that is either 1 or -1, i.e., an element whose only information is the sign.

We start by establishing a lemma. To state the results, recall the convention we use that a matrix \( A \in \mathbb{M}_{m,n} \) has \( n \) singular values, which are given by \( \sigma(A) = \sqrt{\lambda(A^*A)} \), and hence if \( n > m \), then \( \sigma_{m+1}(A) = \cdots = \sigma_n(A) = 0 \).

Lemma 19: Given \( A \in \mathbb{M}_{m,n} \) and \( B \in \mathbb{M}_{n,m} \),

(i) there exists a \( P \in \mathbb{H}_{n+m} \) of the form \( P = \text{diag}(\gamma^2 I, I) \) for \( \gamma \in \mathbb{R} \) fulfilling (12), with both inequalities strict, if and only if \( \sigma_1(A)\sigma_1(B) < 1 \);

(ii) there exists a \( P \in \mathbb{H}_{n+m} \) of the form \( P = \text{diag}(\gamma^2 I, -I) \) for \( \gamma \in \mathbb{R} \) fulfilling (12), with both inequalities strict, if and only if \( \sigma_n(A)\sigma_m(B) > 1 \).

Before we proceed, note that the conditions in Lemma 19(ii) can only ever be fulfilled if \( n = m \) and both matrices are full rank, since otherwise at least one of the two singular values \( \sigma_n(A), \sigma_m(B) \) equals zero.

Proof: We start with proving (i). To this end, note that a direct calculation in (12) (with both inequalities strict) gives that a multiplier of the prescribed form exists if and only if

\[
A^*A < \gamma^2 I \quad \text{and} \quad \gamma^2 B^*B < I, \tag{14}
\]

which is the case if and only if there exists a \( \gamma \in \mathbb{R} \) such that all singular values of \( A \) are strictly smaller than \( |\gamma| \), and all singular values of \( B \) are strictly smaller than or equal to \( 1/|\gamma| \).

Therefore, the existence of such a multiplier clearly implies that \( \sigma_1(A)\sigma_1(B) < 1 \). Conversely, if \( \sigma_1(A)\sigma_1(B) < 1 \), then a direct calculation shows that \( 1/\sigma_1^2(A) - \sigma_1^2(B) > 0 \), and for any \( 0 < \epsilon < 1/\sigma_1^2(A) - \sigma_1^2(B) \), we take \( \gamma^2 = 1/(\sigma_1^2(B)+\epsilon) \) and that \( \gamma^2 > 1/(\sigma_1^2(A)+1/\sigma_1^2(B)) = \sigma_1^2(A) \), and that \( 1/\gamma^2 < \sigma_1^2(B) \), and hence such \( \gamma \) fulfills (14).

Next, to prove (ii) we follow along the same lines. However, first note that \( \sigma_n(A)\sigma_m(B) > 1 \) only if \( n = m \) and both \( A \) and \( B \) are invertible, since otherwise at least one of the two singular values equals zero. Now, a multiplier of the prescribed form exists if and only if

\[
A^*A > \gamma^2 I \quad \text{and} \quad \gamma^2 B^*B > I, \tag{15}
\]

which, similarly, can only hold if \( n = m \) and both \( A \) and \( B \) are invertible. Henceforth, we can therefore restrict our attention to that case. Now, (15) holds if and only if there exists a \( \gamma \in \mathbb{R} \) such that all singular values of \( A \) are strictly larger than \( |\gamma| \), and all singular values of \( B \) are strictly larger than \( 1/|\gamma| \).

The preceding lemma considers two different domains in which stability of \( I + AB \) can be guaranteed: when both \( A \) and \( B \) have either small gain or large gain. In both cases, we expect that stability should be preserved under a suitable notion of rotation. We can now formalize this as follows.

Theorem 20: Given \( A \in \mathbb{M}_{m,n} \) and \( B \in \mathbb{M}_{n,m} \), the following statements are equivalent:

(i) \( \det(I + uGK) \neq 0 \) for all \( u \in \mathbb{U}_m \) and all \( V \in \mathbb{U}_n \);

(ii) there exists a \( P \in \mathbb{H}_{n+m} \) fulfilling (12), with both inequalities strict, which takes the form

\[
P = \begin{bmatrix} -\xi & 0 \\ 0 & \xi I \end{bmatrix} \tag{16}
\]

for some \( \xi \in \mathbb{R} \) and \( \gamma \in \{-1, 1\} \);

(iii) either \( \sigma_1(A)\sigma_1(B) < 1 \) or \( \sigma_n(A)\sigma_m(B) > 1 \).

Proof: See Appendix 3.

Remark 21: Note that in a numerical implementation searching for multipliers to guarantee stability, the conditions in Theorem 20(ii) can be relaxed to searching for multipliers of the form \( P = \text{diag}(\eta_1 I, \eta_2 I) \), for \( \eta_1, \eta_2 \in \mathbb{R} \). This means that the search for multipliers fulfilling (12) can either be formulated as two LMIIs of the form (16), each of which has one unknown \( \gamma^2 \geq 0 \), or it can be solved as one LMI in the two unknowns \( \eta_1, \eta_2 \in \mathbb{R} \).

C. Magnitudinal multipliers for LTI systems

In order to extend the above results to LTI systems, we first need the following definitions and results: a transfer function \( G \in \mathbb{RH}_{\infty}^{m \times n} \) is said to be unitary if \( G(j\omega) \in \mathbb{U}_n \) for all \( \omega \in [0, \infty) \). Moreover, by the proof of [48, Lem. 1.14], it holds that for every \( \omega > 0 \) and \( X \in \mathbb{U}_n \), there exists unitary \( Q \in \mathbb{RH}_{\infty}^{n \times n} \) such that \( Q(j\omega)X = X \). Next, the following expression will be used in the forthcoming theorems:

\[
(I - G(j\omega)^*) \Pi(j\omega) \begin{bmatrix} I \\ -G(j\omega) \end{bmatrix} \geq 0. \tag{17}
\]

First, a sufficiency condition for robust stability against phasal uncertainties is stated.

Theorem 22: Given \( G \in \mathbb{RH}_{\infty}^{m \times n} \) and \( K \in \mathbb{RH}_{\infty}^{n \times m} \), then \( (I + uGK)^{-1} \) is \( \mathbb{RH}_{\infty}^{m \times m} \) for all unitary \( u \in \mathbb{RH}_{\infty}^{1 \times 1} \).

For the proof, we consider the block diagonal form of \( G \) and use the results from [48].

Remark 21: Note that in a numerical implementation searching for multipliers to guarantee stability, the conditions in Theorem 20(ii) can be relaxed to searching for multipliers of the form \( P = \text{diag}(\eta_1 I, \eta_2 I) \), for \( \eta_1, \eta_2 \in \mathbb{R} \). This means that the search for multipliers fulfilling (12) can either be formulated as two LMIIs of the form (16), each of which has one unknown \( \gamma^2 \geq 0 \), or it can be solved as one LMI in the two unknowns \( \eta_1, \eta_2 \in \mathbb{R} \).

Next, a necessary condition for robust stability to phasal uncertainties is provided.

Theorem 23: Given \( G \in \mathbb{RH}_{\infty}^{m \times n} \) and \( K \in \mathbb{RH}_{\infty}^{n \times m} \), then \( (I + uGK)^{-1} \) is \( \mathbb{RH}_{\infty}^{m \times m} \) for all unitary \( u \in \mathbb{RH}_{\infty}^{1 \times 1} \) only if there exist \( N \in \mathbb{L}_{\infty}^{n \times n} \) and \( M \in \mathbb{L}_{\infty}^{m \times m} \) such that for all
$\omega \in [0, \infty], N(j\omega) = N(j\omega)^*, M(j\omega) = M(j\omega)^*$, and (17) holds with
\[
\Pi := \begin{bmatrix} N & 0 \\ 0 & M \end{bmatrix}.
\]

Proof: The claim can be established by applying Theorem 18 frequency-wise as in the proof for Proposition 5.

Theorem 24: Given $G, G^{-1}, K, K^{-1} \in \mathbb{RH}_{\infty}^{m \times m}$, then $(I + UVG K)^{-1} \in \mathbb{RH}_{\infty}^{m \times m}$ for all unitary $U \in \mathbb{RH}_{\infty}^{m \times m}$ and $V \in \mathbb{RH}_{\infty}^{m \times m}$ if and only if there exists $\gamma \in L_1^\infty$, such that for all $\omega \in [0, \infty], |\gamma(j\omega)| > 0$ and (17) holds with
\[
\Pi(j\omega) := \begin{bmatrix} -\xi |\gamma(j\omega)|^2 I & 0 \\ 0 & \xi I \end{bmatrix},
\]
where $\xi \in \{-1, 1\}$.

Proof: Necessity can be established by applying Theorem 20 frequency-wise as in the proof for Proposition 5.

In particular, continuity of $G$ and $K$ on the imaginary axis guarantees the uniqueness of $\xi$ for all $\omega \in [0, \infty]$. Sufficiency follows from Proposition 5 and the fact that (17) implies
\[
(I - G(j\omega)^*U(j\omega)^*) \Pi(j\omega) \begin{pmatrix} I \\ -U(j\omega)G(j\omega) \end{pmatrix} < 0
\]
and
\[
(K(j\omega)^*V(j\omega)^* I) \Pi(j\omega) \begin{pmatrix} I \\ V(j\omega)K(j\omega) \end{pmatrix} \gtrsim 0
\]
for all unitary $U \in \mathbb{RH}_{\infty}^{m \times m}$ and $V \in \mathbb{RH}_{\infty}^{m \times m}$.

Example 25: Consider $G \in \mathbb{RH}_{\infty}^{m \times n}$ and $K \in \mathbb{RH}_{\infty}^{m \times m}$ for which $\|G\|_\infty < \gamma$ and $\|K\|_\infty \leq \frac{1}{\gamma}$. Then $G$ and $K$ satisfy the quadratic separation condition in the theorems above, i.e., (17), with
\[
\Pi := \begin{bmatrix} -\gamma^2 I & 0 \\ 0 & I \end{bmatrix}.
\]
This is the celebrated small-gain theorem.

V. CONCLUSIONS

We have shown that robustness of feedback interconnections against certain structured uncertainty corresponds to specific forms of quadratic separation of the open-loop systems. Specifically, gain-type multipliers define quadratic separation needed in a feedback that is robust against all phase-type uncertainty. Analogously, a robustly stable feedback against all gain-type uncertainty can always be established via the existence of phase-type multipliers. These results are importantly informative when using multiplier-based methods for establishing robust feedback stability. Future research directions of interest include the consideration of block-diagonal structured uncertainty as in the $\mu$-analysis. The exploration of a possibly unifying description of the structures of uncertainties and the corresponding multipliers beyond those examined in this paper is also desirable.

ACKNOWLEDGEMENT

The authors would like to thank Chao Chen, Dan Wang, Di Zhao, and Ding Zhang for valuable discussions.

APPENDIX

A. Proof of Theorem 7, Corollary 8, and Corollary 9

Proof of Theorem 7: The equivalence between (i) and (iii) is clear: the determinant is nonzero for all nonnegative scaling if and only if $AB$ have no eigenvalue along the strictly negative real axis.

Next, we prove that “(ii) $\Rightarrow$ (i)”. To this end, by Lemma 2 the existence of a multiplier $P$ that fulfills (3) means that $\det(I + AB) \neq 0$. Now, for any $\tau \geq 0$ consider the matrices $A = A$ and $\bar{B} = \tau B$. For these matrices, it is easily verified that $P$ fulfills (3). Therefore, by Lemma 2 we have that $0 \neq \det(I + \bar{A}B) = \det(I + \tau AB)$.

We complete the proof by showing that “(iii) $\Rightarrow$ (ii)”. To this end, assume that (3) holds. This means that the principal part of the matrix square root $(AB)^{1/2}$ is well-defined and that all eigenvalues of $(AB)^{1/2}$ lie in the open right half-plane or at the origin [49]. Moreover, $(AB)^{1/2}$ has as many zero-eigenvalues as $AB$, and since a potential zero-eigenvalue of $AB$ is assumed to be semi-simple, so will the potential zero-eigenvalue of $(AB)^{1/2}$. Next, let $(AB)^{1/2} = PJP^{-1}$ be a Jordan normal form, where $J = J_1 \oplus \cdots \oplus J_{k_1} \oplus 0$ and each block $J_k$ is of size $n_k$ and have the nonzero eigenvalue $\lambda_k$ on the diagonal, as per usual. However, let the Jordan normal form be such that the elements on the sup-diagonal of each $J_k$ take the value
\[
\epsilon = \min_{k \in \{1, \ldots, k_1\}} \text{real}(\lambda_k(AB)^{1/2}) > 0,
\]
which is always possible [22].

Now, set $D = J_1 \oplus \cdots \oplus J_{k_1} \oplus I$ and note that $D$ is of full rank, that $D^{-1} = J_1^{-1} \oplus \cdots \oplus J_{k_1}^{-1} \oplus I$, and that $D^{-1}J = I \oplus \cdots \oplus I \oplus 0$. Moreover, $D$ is strictly accretive. To see the latter, first note that by [23] 1.2.10, p. 12] we have that $W(D) = W(J_1 \oplus \cdots \oplus J_{k_1} \oplus I) = \text{Co}(W(J_1) \cup \cdots \cup W(J_{k_1}) \cup 1)$. Moreover, $J_k = \lambda_k(AB)^{1/2} I_{n_k \times n_k} + c S$, where $S$ is the nilpotent matrix with zeros everywhere except the first sup-diagonal which is ones. By [23] 1.2.10, p. 12] $W(J_k) \subset \lambda_k + c W(S)$, and by [23] 29, pp. 45-46] the set $W(S)$ is contained in the unit disc. Since by construction $\epsilon \leq \text{real}(\lambda_k)/2$, we therefore have that $W(J_k) \subset C_+$ for all $k$, and hence $W(D) \subset C_+$, i.e., $D$ is strictly accretive. In particular, this means that $D + D^* \succ 0$.

Finally, take $H = A^* P^{*} D^{-1} P^{-1}$ and note that
\[
HA + A^* H^* = A^* P^{*} (D^{*} + D^{-1}P^{-1})A \succeq \epsilon A^* A
\]
for some $\epsilon > 0$ small enough, since $P^{-1} D^{-1} P^{-1}$ is small. By multiplying the above inequality with $-1$, (3a) follows. Moreover,
\[
H^* B = P^{-*} D^{-1} P^{-1} AB = P^{-*} D^{-1} P^{-1} PJP^{-1} P^{-1} = P^{-*} (I \oplus \cdots \oplus I \oplus 0) J P^{-1} = P^{-*} J P^{-1},
\]
which is congruent to $J$ and hence quasi-strictly accretive. A direct calculation in (3b) therefore verifies that last claim, and hence proves that “(iii) $\Rightarrow$ (ii)”. ■
Proof of Corollary 8: To prove the corollary, assume that $A \in \mathbb{GL}_n$. A direct calculation shows that if (iv) is fulfilled, then so is (ii). Moreover, observe that the fact that $A$ is of full rank implies that $H = A^* P^{-1} D^* P^{-1}$ constructed in the proof of Theorem 7 above, is of full rank. Furthermore, it also means that $HA + A^* H^* = A^* P^{-1} (D^{-1} + D^{-1}) P^{-1} A$ is congruent to $D^{-1} + D^{-1}$ and therefore positive definite, i.e., $HA$ is strictly accretive. In particular, that means that the corresponding multiplier $P$ fulfills \([4], \) and also that (iii) implies (iv) in this case.

Finally, clearly (iv) implies (v), since a quasi-strictly accretive matrix is accretive. What is left to show is thus that under the assumption that $A \in \mathbb{GL}_n$, (v) implies any of the statements (i)-(iv). To this end, note that if $H \in \mathbb{GL}_n$ and $HA$ is strictly accretive, then by congruence $HA$ and $AH^{-*}$ have the same phases. Therefore, using [32, Lem. 3] we have that

$$-\pi < \phi(HA) + \phi(H^* B) = \phi(AH^{-*}) + \phi(H^* B) \leq \angle \lambda_i(AH^{-*} HB) = \angle \lambda_i(AB) \leq \angle \lambda_i(AH^{-*} HB) \leq \phi(HA) + \phi(H^* B) < \pi$$

for $i = 1, \ldots, n$, which shows that (v) implies (iii).

Proof of Corollary 9: Reexamining the proof of Theorem 7 the proof of "(i) $\iff$ (iii)" and the proof of "(ii) $\Rightarrow$ (i)" hold directly also in the case of real matrices $A, B$ and $H$. Moreover, the remaining parts of the proof would also hold if the constructed $H$ is real. The latter is true if $P$ and $D$ are real, which is true if $(AB)^{1/2}$ is real. Thus, the conclusion follows if $AB$ has a real primary square root. Since a potential zero-eigenvalue is assumed to be semi-simple, by [33, Thm. 1.23] the matrix $AB$ has a real primary square root.

B. Proof of Theorem 7

The proof proceeds by showing that (i) and (iii) are equivalent, and that (ii) is equivalent to (iii). The former equivalence is the lengthier part, and for improved readability we hence separate the equivalence of (i) and (iii) into a separate proposition.

Proposition 26: Let $n \geq 2$, and let $A, B \in \mathbb{M}_n \setminus \{0\}$. Then the statement

$$\det(I + T^* AT^*SB) \neq 0$$

is equivalent to the statement

one matrix is quasi-sectorial, the other semi-sectorial, and

$$\phi(A) + \phi(B) < \pi, \quad \phi(A) + \phi(B) > -\pi.$$  \hfill (18b)

Proof: To show $\Leftarrow$: assume that (18b) holds. For any $T, S \in \mathbb{GL}_n$, by congruence invariance of phases of matrices we have that $\phi(T^* AT) = \phi(A)$ and that $\phi(S^* BS) = \phi(B)$. Therefore, by [32, Lem. 3] we have that

$$-\pi < \phi(A) + \phi(B) = \phi(T^* AT) + \phi(S^* BS) \leq \angle \lambda_i(T^* AT S^* BS) \leq \phi(T^* AT) + \phi(S^* BS) = \phi(A) + \phi(B) < \pi$$

for $i = 1, \ldots, n$. In particular, this means that there exists an $\epsilon > 0$ so that $\lambda(T^* AT S^* BS) \cap \{z \in \mathbb{C} | z = -r e^{i\theta}, r > 0, \theta \in [-\epsilon, \epsilon]\} = \emptyset$ for all $T, S \in \mathbb{GL}_n$. The latter implies that (18a) holds.

Next, to show $\Rightarrow$ we will show that the contraposition is true, namely that if (18b) is not true, then there exists $T, S \in \mathbb{GL}_n$ such that $\det(I + T^* AT^* BS) = 0$, i.e., such that $T^* AT^* BS$ has an eigenvalue in $-1$. The latter is shown by explicitly considering all possible cases using the results in [27], [31], and is also making heavy use of [50, Thm. 1] (see also [23, Thm. 1.7.9], cf. [27, Thm. 3]).

To this end, first assume that $B$ is arbitrary and with at least one nonzero eigenvalue, and $A$ has only the zero-eigenvalue. Since $A \neq 0$, the eigenvalue cannot be semisimple, and hence $A$ must have a Jordan block of size at least $2 \times 2$. The latter has a numerical range that is a circle centered around the origin [23, Prop. 9, pp. 25], and hence the angular numerical range is the entire complex plane. Now, let $B = V_B^* \Gamma_B V_B$ be a Schur decomposition of $B$, i.e., where $V_B$ is unitary and $\Gamma_B$ is upper triangular. Any such $\Gamma_B$ is called a Schur form of $B$. Moreover, note that at least on element of $\Gamma_B$ is nonzero: without loss of generality assume it is $(\Gamma_B)_{11}$. Next, by [50, Thm. 1] there exists a $C \in \mathbb{M}_n$ such that one of the eigenvalues of $C$ is $- (\Gamma_B)_{11}$ and such that $T^* AT = C$ for some $T \in \mathbb{GL}_n$. Moreover, let $C = V_C^* \Gamma_C V_C$ be a Schur decomposition of $C$ such that $(\Gamma_C)_{11} = - (\Gamma_B)_{11}$. By taking $S = V_B^*$ and $T = \tilde{T} V_C^*$ we have that

$$(T^* AT)(S^* BS) = (V_C T^* \tilde{T} V_C)(V_B^* V_B^* V_B^* V_B^*) = (V_C V_B^*) \Gamma_B = \Gamma_C \Gamma_B,$$

which is upper triangular and with $-1$ in the upper left corner, i.e., for these $T$ and $S$ we have that $T^* AT S^* BS$ has an eigenvalue in $-1$.

Note that the above procedure can be carried out, mutatis mutandis, if $B$ also only has the zero-eigenvalue. In particular, if $B$ only has the zero-eigenvalue it must also have a Jordan block of size at least $2 \times 2$. Using [50, Thm. 1], by an appropriate selection of $S = S_1 S_2$, we can thus make sure that $S_1^* B S_1$ has a nonzero eigenvalue, after which the above procedure can be repeated to select $T$ and $S_2$ so that $T^* AT(S^* BS)$ has an eigenvalue in $-1$. This means that in the following, we can always assume that both $A$ and $B$ have at least one nonzero eigenvalue. In fact, for any non-sectoral (an hence nonzero) matrix $A$ and arbitrary nonzero matrix $B$, a similar argument to the preceding one shows that $\det(I + T^* AT^* BS) = 0$ for some $S, T \in \mathbb{GL}_n$, since the angular numerical range of $A$ is the entire complex plane (see [50, Thm. 1]).

The above argument shows that a necessary condition for (18a) to hold is that both $A, B$ are semi-sectorial. By [31, Thm. 5] this means that, without loss of generality, we can restrict ourselves to consider matrices of the form

$$A = \begin{bmatrix} e^{i\theta_A} k_1^A \otimes & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ diag}(e^{i \phi_1(A)}, \ldots, e^{i \phi_k(A)}),$$

where $\theta_A + \pi / 2 \geq \phi_1(A) \geq \cdots \geq \phi_k(A) \geq \theta_A - \pi / 2$, $k_1^A \geq 0$, $k_2^A \geq 0$, and $n = 2k_2^A + k_1^A$, and analogously for $B$. Note also that $\phi(A) = \theta_A + \pi / 2$ if $k_2^A > 0$ and $\phi(A) = \phi_1(A)$.
if $k^2 = 0$; an analogous observation holds for $\phi(A)$. Moreover a matrix $A$ of the form (19) is quasi-sectorial if and only if $k^2 = 0$ and $\overline{\phi}(A) - \phi(A) < \pi$. Finally, note that by applying an appropriate permutation we can, without loss of generality, restrict our attention to matrices of size $2 \times 2$.

Now, we first show that we cannot have $k^2 > 0$ and $k^2 > 0$. To this end, let $S = I$ and consider the unitary matrix

$$
T = \begin{bmatrix}
\cos \left( \frac{\pi}{2} + \frac{\theta_A + \theta_B}{2} \right) & -\sin \left( \frac{\pi}{2} + \frac{\theta_A + \theta_B}{2} \right) \\
\sin \left( \frac{\pi}{2} + \frac{\theta_A + \theta_B}{2} \right) & \cos \left( \frac{\pi}{2} + \frac{\theta_A + \theta_B}{2} \right)
\end{bmatrix}.
$$

Let $S_{AB} := \sin((\theta_A + \theta_B)/2)$ and $C_{AB} := \cos((\theta_A + \theta_B)/2)$. A direct (albeit somewhat cumbersome) calculation gives that

$$
T^* = \begin{bmatrix}
1 & 2 \\ 0 & 1
\end{bmatrix}
$$

and

$$
T = \begin{bmatrix}
1 & 2 \\ 0 & 1
\end{bmatrix}
$$

which has eigenvalues $-\cos(\theta_A + \theta_B) \pm i \sin(\theta_A + \theta_B) = -e^{i\theta_A + \theta_B}$. Therefore, taking $T$ as above and $S = I$, the matrix $T^* A T S^*$ has an eigenvalue in $-1$.

Next, we therefore assume that $k^2 > 0$ and $k^2 > 0$. In this case, first assume that $B$ only has one non-zero phase, in which case it suffices to consider

$$
A = e^{i\phi_A} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}
$$

and

$$
B = \text{diag}(e^{i\phi_B}, 0).
$$

If $\theta_A + \pi/2 + \phi(B) \geq \pi$ or $\theta_A - \pi/2 + \phi(B) \leq -\pi$, then we can write

$$
T^* A T = T^* \tilde{A} \text{diag}(1, 0)
$$

where $\tilde{A} = e^{i\phi(B)} A$. However, since $\theta_A + \pi/2 + \phi(B) \geq \pi$ or $\theta_A - \pi/2 + \phi(B) \leq -\pi$, $-1$ is in the numerical range of $\tilde{A}$. Therefore, by using (20) Thm. 1) we can make a construction similar to before in order to select an appropriate $T$ such that $T^* A T B$ has an eigenvalue in $-1$. On the other hand, if $\theta_A + \pi/2 + \phi(B) < \pi$ and $\theta_A - \pi/2 + \phi(B) > -\pi$, then (18b) is fulfilled and thus (18a) holds (see the proof of the implication “$\Rightarrow$”).

The next case we consider is when the diagonal unitary part of $B$ is of size at least $2 \times 2$. To this end, it suffices the consider

$$
A = e^{i\theta_A} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}
$$

and

$$
B = \text{diag}(e^{i\phi(B)}, e^{i\phi(B)}).
$$

We split this into two different subcases. In the first case, assume that $B$ is quasi-sectorial, which means that $\phi(B) - \phi(B) < \pi$. If $\theta_A + \pi/2 + \phi(B) \geq \pi$ or $\theta_A - \pi/2 + \phi(B) \leq -\pi$, then we can make constructions analogous to the above one, and if $\theta_A + \pi/2 + \phi(B) < \pi$ and $\theta_A - \pi/2 + \phi(B) > -\pi$, then (18b) is fulfilled and thus (18a) holds. Therefore, we next assume that $\phi(B) - \phi(B) = \pi$, in which case $B$ is rotation-Hermitian, i.e., $B = e^{i\phi_B} \text{diag}(1, -1)$. Moreover, that means that either $\theta_A + \pi/2 + \phi(B) \geq \pi$ or $\theta_A - \pi/2 + \phi(B) = \theta_A - \pi/2 + \phi(B) < -\pi$. In any case, let $S = I$ and let

$$
T = \begin{bmatrix}
\cos \left( \frac{\theta_A + \phi(B)}{2} \right) & j \sin \left( \frac{\theta_A + \phi(B)}{2} \right) \\
-j \sin \left( \frac{\theta_A + \phi(B)}{2} \right) & \cos \left( \frac{\theta_A + \phi(B)}{2} \right)
\end{bmatrix}.
$$

This $T$ is unitary, and a direct (albeit somewhat cumbersome) calculation verify that $T^* A T B$ has an eigenvalue in $-1$.

Now, consider the case where both $A$ and $B$ have a rotation-Hermitian $2 \times 2$ block, i.e., when

$$
A = e^{i\phi(A)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = e^{i\phi(B)} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.
$$

Similarly to the last case just above, that means that either $\phi(A) + \phi(B) \geq \pi$ or $\phi(A) - \pi + \phi(B) - \pi \leq -\pi$. In any case, let $S = I$ and let

$$
T = \begin{bmatrix}
\cos \left( \frac{\theta_A + \phi(B)}{2} \right) & -\sin \left( \frac{\theta_A + \phi(B)}{2} \right) \\
\sin \left( \frac{\theta_A + \phi(B)}{2} \right) & \cos \left( \frac{\theta_A + \phi(B)}{2} \right)
\end{bmatrix}.
$$

A calculation similar to before shows that $T^* A T B$ has an eigenvalue in $-1$.

The two final cases to consider is when either i) $A$ has a rotation-Hermitian $2 \times 2$ block and $B$ is quasi-sectorial, or ii) when both $A$ and $B$ are quasi-sectorial, but when the phase condition is not satisfied in either case. The two cases can be handled together, and we can, without loss of generality, assume that $\phi(A) + \phi(B) \geq \pi$. In this case, by (20) Thm. 1) there is a $T$ such that $C = T^* A T$ has an eigenvalue in $e^{i\phi(A)}$. Let $C = V_2 \Gamma e^{i\phi(D)}$ be a Schur decomposition, with $e^{i\phi(A)}$ as top-left element. Let $S = S_1 S_2$, and note that since $\phi(B) \geq \pi - \phi(A)$ and $B$ is quasi-sectorial we can in a similar way select $S_1$ so that $D = S_1^* B S_1$ has an eigenvalue in $e^{i(\pi - \phi(A))}$. Let $D = V_2 \Gamma D_2 V_2^*$ be a Schur decomposition with $e^{i(\pi - \phi(A))}$ as top-left element. By taking $S_2 = V_2$, we get

$$
(T^* A T)(S B S^*) = C(S_2^* D S_2)
$$

which by construction has an eigenvalue in $-1$.

In summary, this means that unless (18b) holds, then there exist $T, S \in \mathbb{GL}_{2n}$ such that $\det((I + T^* A T S^* B) S^*) = 0$. This shows the implication $\Rightarrow$, and hence the result follows.

Proof of Theorem 12] For $n = 1$ the matrices are scalar and hence commute. Therefore, in this case $\det((I + T^* A T S^* B) S^*) = \det(1 + \tau ab)$ for $\tau > 0$, and the conclusions follow almost trivially.

For $n \geq 2$, Proposition 26 shows that (i) and (ii) are equivalent. Next, we prove that “(iii) $\Rightarrow$ (ii)”. To this end, without loss of generality, assume that $A$ is quasi-sectorial of rank $n - k$, and that $B$ is semi-sectoral. The fact that the sum of the largest and smallest phases are bounded away from $\pm \pi$, respectively, implies that there exists a $z \in T$ such that $z A$ is quasi-sectorially accretive and $z^* B$ is accretive, and hence in particular that $-z A - z^* A^* \leq 0$ and $z^* B + z B^* \geq 0$. The latter means that for this $P$, (19b) holds. It remains to show that (19b) holds, i.e., that the former inequality above can be strengthened to $-z A - z^* A^* \leq -\varepsilon z A^*$ for some $\varepsilon > 0$. To do so, let $z A = T^* D T$ be a sectorial decomposition of the quasi-sectorially accretive $z A$. In particular, this means that

$$
D = \text{diag}(e^{i\phi_1}, \ldots, e^{i\phi_{n-k}}, 0, \ldots, 0),
$$

and $z A + z^* A^* = T^* (D + D^*) T$, which is positive semi-definite with the top-left block of $D + D^*$ containing the $n - k$
strictly positive eigenvalues. A direct calculation also gives that
\[ A^* A = A^* z_A z_A = T^* D^* T T^* D T = T^* \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} T =: T^* \Delta T, \]
where the block * is of dimension \( n - k \times n - k \) and is positive definite. In particular, this means that for \( \epsilon > 0 \) small enough we have that \( D + D^* > \epsilon \Delta \), and hence that
\[ z A + z^* A^* = T^* (D + D^*) T > \epsilon T^* \Delta T = \epsilon A^* A. \]
Multiplying the above inequality by \(-1\) gives the inequality \([5.3]\). This completes the proofs of the implication “(iii) \( \Rightarrow \) (ii)”.

To show that “(ii) \( \Rightarrow \) (iii)”, without loss of generality, assume that \( P \) fulfills \([5]\). The proof for the case where \( P \) fulfills \([5]\) is analogous. Now, note that \([5]\) implies that both \( z A \) and \( z^* B \) are accretive. What remains to be shown is thus that \( z A \) is in fact quasi-strictly accretive. To this end, let \( z A = T^* DT \) be the sectorial decomposition, with \( D \) of the form \([19]\). By an argument similar to the one above, we have that \([5.3]\) implies that
\[ -\begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} = -D - D^* \preceq -\epsilon D^* T^* D = -\epsilon \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix}, \]
where * is block-diagonal and positive semi-definite, and \( z A \) is positive definite. However, the inequality means that \( z A \) is quasi-strictly accretive. Therefore, \( D \) cannot contain any such blocks, which implies that \( z A \) is in fact be quasi-strictly accretive.

Finally, the last part of the theorem follows by simply reexaming the proofs for the equivalence of (ii) and (iii) under the additional assumption that \( A \) is of full rank. It is then easily seen that the same conclusion holds, but with \([5.3]\) replaced by \([2]\).

\[ \begin{align*}
\text{REFERENCES} \\
[1] & A. R. Teel, T. T. Georgiou, L. Praly, and E. D. Sontag, “Input-output stability,” in The Control Systems Handbook : Control System Advanced Methods, 2nd ed., W. S. Levine, Ed. Boca Raton, FL: CRC Press, 2011, pp. 44.1–44.23.
\end{align*} \]
