Jancar’s formal system for deciding bisimulation of first-order grammars and its non-soundness.
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Abstract: We construct an example of proof within the main formal system from [Jan10], which is intended to capture the bisimulation equivalence for non-deterministic first-order grammars, and show that its conclusion is semantically false. We then locate and analyze the flawed argument in the soundness (meta)-proof of [Jan10].

Keywords: first-order grammars; bisimulation problem; formal proof systems.

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1 The grammar

We consider the alphabet of actions $\mathcal{A}$, an intermediate alphabet of labels $\mathcal{T}$ and a map $\text{LAB}_\mathcal{A}: \mathcal{T} \to \mathcal{A}$ defined by:

$$\mathcal{T} := \{x, y, z, \ell_1\}, \quad \mathcal{A} := \{a, b, \ell_1\}, \quad \text{and} \quad \text{LAB}_\mathcal{A}: x \mapsto a, \quad y \mapsto a, \quad z \mapsto b, \quad \ell_1 \mapsto \ell_1.$$

(These intermediate objects $\mathcal{T}$, $\text{LAB}_\mathcal{A}$ will ease the definition of $\text{ACT}$ below). We define a first-order grammar $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{R})$ by:

$$\mathcal{N} := \{A, A', A'', B, B', B'', C, D, E, L_1\}$$

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and the set of rules $\mathcal{R}$ consists of the following:

\begin{align*}
A(v) & \xrightarrow{y} C(v) \quad (1) \\
A(v) & \xrightarrow{x} A'(v) \quad (2) \\
B(v) & \xrightarrow{x} C(v) \quad (3) \\
B(v) & \xrightarrow{y} B'(v) \quad (4) \\
C(v) & \xrightarrow{x} D(v) \quad (5) \\
C(v) & \xrightarrow{y} E(v) \quad (6) \\
A'(v) & \xrightarrow{x} A''(v) \quad (7) \\
B'(v) & \xrightarrow{x} B''(v) \quad (8) \\
A''(v) & \xrightarrow{x} D(v) \quad (9) \\
B''(v) & \xrightarrow{x} E(v) \quad (10) \\
D(v) & \xrightarrow{x} v \quad (11) \\
E(v) & \xrightarrow{x} v \quad (12) \\
E(v) & \xrightarrow{z} v \quad (13) \\
L \xrightarrow{\ell_1} \bot \quad (14)
\end{align*}

Let us name rule $r_i$ (for $1 \leq i \leq 14$), the rule appearing in order $i$ in the above list. We define a map $\text{LAB}_T : \mathcal{R} \rightarrow T$ by: $\text{LAB}_T(r_i)$ is the terminal letter used by the given rule $r_i$. Subsequently we define $\text{ACT}(r_i) := \text{LAB}_A(\text{LAB}_T(r_i))$. Namely, $\text{ACT}$ maps all the rules $r_1, \ldots, r_{14}$ onto $a$, $r_{13}$ on $b$ and $r_{14}$ on $\ell_1$.

2 The formal system

We consider the formal systems $\mathcal{J}(T_0, T'_0, S_0, B)$ defined in page 22 of [Jan10], which are intended to be sound and complete for the bisimulation-problem for non-deterministic first-order grammars. Let us denote by $\mathcal{T}$ the set of all terms over the ranked alphabet $\mathcal{N} \cup \{L_i \mid i \in \mathbb{N}\} \cup \{\bot\}$ (here the symbols $L_i$ have arity 0).

2.1 Prefixes of strategies

The notion of finite prefix of a D-strategy is mentioned p. 23, line 11. We assume it has the following meaning

**Definition 1.** Let $T, T' \in \mathcal{T}$. A finite prefix of a D-strategy w.r.t. $(T, T')$ is a subset $S \subseteq (\mathcal{R} \times \mathcal{R})^*$ of the form

$$S = S' \cap (\mathcal{R} \times \mathcal{R})^{\leq n}$$

for some $n \in \mathbb{N}$ and some D-strategy $S'$ w.r.t. $(T, T')$.

In order to make clear that the above notion is effective, we consider the following notion of D-q-strategy (Defender’s quasi-strategy).

**Definition 2.** Let $T, T' \in \mathcal{T}$. A D-q-strategy w.r.t. $(T, T')$ is a subset $S \subseteq (\mathcal{R} \times \mathcal{R})^*$ such that:

$DQ1$: $(\varepsilon, \varepsilon) \in S$
One can check that, if $P \subseteq E$ has some upper-bound.

Proof

Lemma 2. Let $T, T' \in T$. The extension ordering over the set of all D-q-strategies w.r.t. $(T, T')$, is inductive.

Proof: We recall that a partial order $\leq$ over a set $E$ is inductive iff, every totally ordered subset of $E$ has some upper-bound.

One can check that, if $P$ is a set of D-q-strategies w.r.t. $(T, T')$, which is totally ordered by $\subseteq$, then the set

$$S := \bigcup_{s \in P} s$$
is still a D-q-strategy and fulfills:

$$\forall s \in P, s \sqsubseteq S.$$  

Hence the extension ordering over the set of D-q-strategies w.r.t. \((T, T')\) is inductive. \(\square\)

**Lemma 3.** Let \(S \subseteq (\mathcal{R} \times \mathcal{R})^*\) be finite and let \(n := \max\{|\alpha| \mid \alpha \in S\}\).

\(S\) is a finite prefix of a D-strategy w.r.t. \((T, T')\) iff

1. \(S\) is a D-q-strategy w.r.t. \((T, T')\)
2. \(\forall \beta \in S, [\beta \setminus S = \{(\varepsilon, \varepsilon)\} \Rightarrow (|\beta| = n \text{ or } NEXT((T, T'), \beta) \notin \sim_1)].\)

**Proof:** Direct implication: Let \(S'\) be a D-strategy w.r.t. \((T, T')\) and

\[S = S' \cap (\mathcal{R} \times \mathcal{R})^{\leq n}\]

for some \(n \in \mathbb{N}\) and some \(S'\) which is a D-strategy w.r.t. \((T, T')\).

1. By Lemma \([1]\) \(S\) is a D-q-strategy w.r.t. \((T, T')\).
2. Suppose that \(\beta \in S, \beta \setminus S = \{(\varepsilon, \varepsilon)\}\) and \(|\beta| < n\). Then \(\beta \setminus S' = \{(\varepsilon, \varepsilon)\}\) too. Since \(S'\) is a D-strategy w.r.t. \((T, T')\), this implies that \(NEXT((T, T'), \beta) \notin \sim_1\).

Converse: Suppose that \(S\) fulfills conditions (1)(2). By Lemma \([2]\) Zorn’s lemma applies on the set of D-q-strategies w.r.t. \((T, T')\): there exists a maximal D-q-strategy \(S'\) (for the extension ordering) such that \(S \subseteq S'\). Since \(S'\) is maximal, if \(\alpha \in S'\) and \(\alpha \setminus S = \{(\varepsilon, \varepsilon)\}\), \(NEXT((T, T'), \alpha) \notin \sim_1\). Thus, instead of the weak property DQ4, \(S'\) fulfills the property:

\[\forall \alpha \in S', NEXT((T, T'), \alpha) \notin \sim_1 \text{ or } \]

\([NEXT((T, T'), \alpha) \in \sim_1 \text{ and } \{(\pi, \pi') \in \mathcal{R} \times \mathcal{R} \mid \alpha \cdot (\pi, \pi') \in S\} \text{ is full for } NEXT((T, T'), \alpha)].\]

Hence \(S'\) is a strategy w.r.t. \((T, T')\).

Clearly

\[S \subseteq S' \cap (\mathcal{R} \times \mathcal{R})^{\leq n}.\]

Let us prove the reverse inclusion.

Let \(\alpha \in S' \cap (\mathcal{R} \times \mathcal{R})^{\leq n}\). Let \(\beta\) be the longest word in \(\text{PREF}(\alpha) \cap S\).

If \(\beta = \alpha\), then \(\alpha \in S\), as required.

Otherwise \(\alpha \in S' - S\). By condition E2 of definition \([3]\) there exists some \(\beta \in S\), which is maximal in \(S\) for the prefix ordering and such that

\[\beta \prec \alpha.\]

Maximality of \(\beta\) implies, by condition (2) of the lemma, that

\[|\beta| = n \text{ or } NEXT((T, T'), \beta) \notin \sim_1.\]

Since \(\beta \prec \alpha\) we are sure that \(|\beta| < n\) so that

\[NEXT((T, T'), \beta) \notin \sim_1.\]

This last statement contradicts the fact that \(\beta \setminus S'\) is a D-strategy, w.r.t \(NEXT((T, T'), \beta)\) which is non-reduced to \(\{(\varepsilon, \varepsilon)\}\) (since it possess \(\beta^{-1} \alpha\)).

We can conclude that \(\alpha \in S\). Finally:

\[S = S' \cap (\mathcal{R} \times \mathcal{R})^{\leq n}.\]
Lemma 4. Let \( T, T' \in \mathbb{T} \) and let \( S \subseteq (\mathcal{R} \times \mathcal{R})^* \) be finite. One can check whether \( S \) is a finite prefix of a D-strategy w.r.t. \((T, T')\)

This follows immediately from the characterisation given by Lemma 3.

2.2 Formal systems

For every \( T_0, T'_0 \in \mathbb{T}, S_0 \) finite prefix of strategy w.r.t \((T_0, T_0)\) and finite \( B \subseteq \mathbb{T} \times \mathbb{T}, \) is defined a formal system

\[ \mathcal{J}(T_0, T'_0, S_0, B) \]

The set of judgments of all the systems are the same. But the axiom and one rule (namely R7), is depending on the parameters \((T_0, T'_0, S_0, B)\).

2.3 Judgments

A judgment has one of the three forms:

\textbf{FORM 1:}

\[ m \models (T, T', S) \]

where \( m \in \mathbb{N}, \) and \( T, T' \in \mathbb{T} \) are regular terms and \( S \) is a finite prefix of a strategy. w.r.t. \((T, T')\) (D-strategies are defined p.20, lines 27-30; finite prefixes are mentionned, though in a fuzzy way. at p. 23, line 11; we shall apply here Definition 1).

\textbf{FORM 2:}

\[ m \models (T, T', S) \leadsto \alpha \models (T_1, T'_1, S_1) \]

where \( m \in \mathbb{N}, (T, T', S), (T_1, T'_1, S_1) \) fulfilling the above conditions, \( \alpha \in S \) and \( \alpha \setminus S = S_1. \)

\textbf{FORM 3:}

\[ m \models (T, T', S) \leadsto \alpha \models SUCC \]

where \( m \in \mathbb{N}, (T, T', S) \) fulfill the above conditions and \( \alpha \in S. \)

For all systems \( \mathcal{J}(T_0, T'_0, S_0, B) \) the set of judgments is the same and consists of all the items of one of the three above forms.

2.4 Basis

We call basis every finite set

\[ B \subseteq \mathbb{T} \times \mathbb{T}. \]

2.5 Axioms

\( \mathcal{J}(T_0, T'_0, S_0, B) \) has a single axiom:

\[ 0 \models (T_0, T'_0, S_0) \]
2.6 Deduction rules

All the systems \( \mathcal{J}(T_0, T'_0, S_0, \mathcal{B}) \) have the set of rules described page 22 of [Jan10]. We name them \( R1, R2, \ldots, R10 \), the number corresponding to the one in the text. Note that \( R7 \) depends on the basis \( \mathcal{B} \).

2.7 Proofs

Let \( T_0, T'_0 \in \mathbb{T} \). A proof of \( T_0 \sim T'_0 \) within the family of formal systems defined above is a finite basis \( \mathcal{B} \), together with, for each \( (T, T') \in \mathcal{B} \cup \{ (T_0, T'_0) \} \) a finite prefix of D-strategy \( S \) w.r.t. \( (T, T') \) and a proof, within system \( \mathcal{J}(T, T', S, \mathcal{B}) \) of the judgment

\[
0 \models (T, T', S) \vdash (\varepsilon, \varepsilon) \models \text{Succ}.
\]

3 The Equivalence proof

We exhibit here a proof of

\[
A(\bot) \sim B(\bot).
\]

According to the above notion of proof, it consists of the following items.

Basis:

\[
\mathcal{B} := \{(C(L_1), C(L_1)), (D(L_1), D(L_1)), (E(L_1), E(L_1))\}.
\]

Proofs:

- a proof of the judgment \( 0 \models A(\bot), B(\bot), S \vdash (\varepsilon, \varepsilon) \models \text{Succ} \) in the formal system \( \mathcal{J}(A(\bot), B(\bot), S, \mathcal{B}) \) (see \( \pi_3 \)).
- a proof of the judgment \( 0 \models C(L_1), C(L_1), \text{Id}_{C,1} \vdash (\varepsilon, \varepsilon) \models \text{Succ} \) in the formal system \( \mathcal{J}(C(L_1), C(L_1), \text{Id}_{C,1}, \mathcal{B}) \) (see \( \pi_4 \)).
- a proof of the judgment \( 0 \models D(L_1), D(L_1), \text{Id}_{D,2} \vdash (\varepsilon, \varepsilon) \models \text{Succ} \) in the formal system \( \mathcal{J}(D(L_1), D(L_1), \text{Id}_{D,2}, \mathcal{B}) \) (see \( \pi_5 \)).
- a proof of the judgment \( 0 \models E(L_1), E(L_1), \text{Id}_{E,2} \vdash (\varepsilon, \varepsilon) \models \text{Succ} \) in the formal system \( \mathcal{J}(E(L_1), E(L_1), \text{Id}_{D,2}, \mathcal{B}) \) (see \( \pi_6 \)).

\[
\begin{align*}
0 \models A(\bot), B(\bot), S &\xrightarrow{\text{ax}} \\
H(A, B) \rightsquigarrow (y, x) &\models C(\bot), C(\bot), S_1 &\xrightarrow{R1} \\
H(A, B) \rightsquigarrow (yx, x) &\models D(\bot), E(\bot), S_2 &\xrightarrow{R1} \\
H(A, B) \rightsquigarrow (x^3, yx^2) &\models E(\bot), E(\bot), S_5 &\xrightarrow{R2} \\
H(A, B) \rightsquigarrow (x^3, yx^2) &\models \text{Succ} &\xrightarrow{R7} \\
H(A, B) \rightsquigarrow (x^2, yx) &\models \text{Succ} &\xrightarrow{R8} \end{align*}
\]

\[0 \models A(\bot), B(\bot), S \vdash (x, y) \models \text{Succ} \]

Figure 1. The proof \( \pi_1 \)
Finally, we define $S \equiv \{ (y, x), (yy, xx), (xxx, yxx) \}$.

For every subset $Z$ of $(\mathcal{A} \times \mathcal{A})^*$, by $\text{PREF}(Z)$ we denote its set of prefixes.

We define

$\mathcal{P} := \text{PREF}(S)$

namely:

$\mathcal{P} = \{ (\varepsilon, \varepsilon), (y, x), (yy, xx), (x, y), (xx, yx), (xxx, yxx) \}$

Finally, we define $S$ as the subset of $(\mathcal{R} \times \mathcal{R})^*$ obtained by replacing, in $\mathcal{P}$, every 2-tuple $(u, v) \in (\mathcal{A} \times \mathcal{A})^*$ by the unique 2-tuple $(r_u, r_v) \in (\mathcal{R} \times \mathcal{R})^*$, such that $r_u$ (resp. $r_v$) is applicable on $A$ (resp. on $B$), $\text{LAB}_T(r_u) = u$ and $\text{LAB}_T(r_v) = v$. Namely:

$S = \{ (\varepsilon, \varepsilon), (r_1, r_2), (r_1 r_5, r_2 r_6), (r_1 r_6, r_2 r_5), (r_2, r_1), (r_2 r_7, r_1 r_8), (r_2 r_7 r_9, r_1 r_8 r_{10}) \}.$

**Figure 2.** The proof $\pi_2$

\[
\begin{align*}
\pi_1 & \\ \vdots & \\ H(A, B) \leadsto (x, y) & \models \text{Succ} \\
\pi_2 & \\ \vdots & \\ H(A, B) \leadsto (y, x) & \models \text{Succ} \\
0 & \models A(\bot), B(\bot), S \models \text{Succ}
\end{align*}
\]

**Figure 3.** The proof $\pi_3$

\[
\begin{align*}
0 & \models C(L_1), C(L_1), \text{Id}_{C, 1}^{\text{ax}} \\
H(C, C) & \leadsto (x, x) \models D(L_1), D(L_1), \text{Id}_{C, 0}^{\text{ax}} \\
H(C, C) & \leadsto (x, x) \models \text{Succ} \\
0 & \models C(L_1), C(L_1), \text{Id}_{C, 1}^{\text{ax}} \\
H(C, C) & \leadsto (y, y) \models E(L_1), E(L_1), \text{Id}_{C, 0}^{\text{ax}} \\
H(C, C) & \leadsto (x, x) \models \text{Succ} \\
0 & \models C(L_1), C(L_1), \text{Id}_{C, 1} \leadsto (\varepsilon, \varepsilon) \models \text{Succ}
\end{align*}
\]

**Figure 4.** The proof $\pi_4$

\[
\begin{align*}
0 & \models A(\bot), B(\bot), S^{\text{ax}} \\
0 & \models A(\bot), B(\bot), S \leadsto (y, x) \models C(\bot), C(\bot), S_1^{\text{ax}} \\
0 & \models A(\bot), B(\bot), S \leadsto (y, x) \models \text{Succ}
\end{align*}
\]
Lemma 5. S is a prefix of D-strategy w.r.t. \((A(\bot), B(\bot))\).

Proof: Let us check that S fulfills the criterium given by Lemma 3. Here \(n = 3\). Point (1) is easily checked.

Let \(\beta \in (\mathcal{R} \times \mathcal{R})^*\) such that \(\beta \setminus S = \{(\varepsilon, \varepsilon)\}\). Either \((\text{NEXT}((A, B, \beta)) \in \{(E, D), (D, E)\}\), while \(D \not\dashv_1 E\) or \(|\beta| = 3\). Hence Point (2) holds. \(\square\)

For proving the equivalences of the members of the basis we shall use the “trivial” prefixes of strategies, consisting of 2-tuples of identical rules on both sides:

\[
\begin{align*}
\text{Id}_{C,1} &:= \{(\varepsilon, \varepsilon), (r_5, r_5), (r_6, r_6)\} \\
\text{Id}_{D,2} &:= \{(\varepsilon, \varepsilon), (r_{11}, r_{11}), (r_{11}r_{14}, r_{11}r_{14})\} \\
\text{Id}_{E,2} &:= \{(\varepsilon, \varepsilon), (r_{12}, r_{12}), (r_{13}, r_{13}), (r_{12}r_{14}, r_{12}r_{14}), (r_{13}r_{14}, r_{13}r_{14})\}.
\end{align*}
\]

(See figures 78).

Subsequently:

\[
\begin{align*}
S_1 &:= \{(\varepsilon, \varepsilon), (r_5, r_6), (r_6, r_5)\} \\
S_2 &:= \{(\varepsilon, \varepsilon)\} \\
S_3 &:= \{(\varepsilon, \varepsilon), (r_7, r_8), (r_7r_9, r_8r_{10})\} \\
S_4 &:= \{(\varepsilon, \varepsilon), (r_9, r_{10})\} \\
S_5 &:= \{(\varepsilon, \varepsilon)\} \\
S_6 &:= \text{INDSTR}(S_2, S_5) = S_2^{-1} \circ S_5 = \{(\varepsilon, \varepsilon)\}
\end{align*}
\]
One can check that $\text{Id}_{C,1}$ is a prefix of the strategy, for the game with initial position $(C, C)$,

$$\text{Id}_{C,\infty} := \{(u, u) \mid u \in \mathcal{R}^*, C(L_1) \xrightarrow{u}\}.$$  

The set $\text{Id}_{D,2}$ (resp. $\text{Id}_{E,2}$) is really a strategy for the game with initial position $(D, D)$ (resp. $(E, E)$) since no rule $r_i$ is applicable on $\bot$. For every $N \in \{C, D, E\}$, the symbol $\text{Id}_{N,i}$ will denote a residual of length $i$ of the strategy $\text{Id}_{N,n}$:

$$\text{Id}_{C,0} = \text{Id}_{D,0} = \text{Id}_{E,0} = \{\varepsilon, \varepsilon\},$$

$$\text{Id}_{D,1} = \text{Id}_{E,1} = \{\varepsilon, \varepsilon, (r_{14}, r_{14})\}.$$

### 4 The Non-equivalence (meta-) proof

**Lemma 6.** $A(\bot) \not\sim B(\bot)$

**Proof:**

$$\forall u \in \mathcal{R}^*, ACT(u) = aaab \Rightarrow A(\bot) \xrightarrow{u}$$

while

$$\exists u \in \mathcal{R}^*, ACT(u) = aaab \text{ and } B(\bot) \xrightarrow{u}$$

hence $A(\bot) \not\sim B(\bot)$. $\square$

From section 3 and Lemma 6 we conclude

**Theorem 1.** The family of formal systems $(J(T_0, T'_0, S_0, B))$ is not sound.

### 5 Variations

Let us describe variations around this example.
Description of the proofs
We chose to write the proofs with judgments of the form \( m \models (T, T', S) \) or \( m \models (T, T', S) \sim \alpha \models (T_1, T'_1, S_1) \) or \( m \models (T, T', S) \sim \alpha \models \text{SUCC} \), where, in the case of forms 2, 3, the prefix \( \alpha \) is given by its image under the map \( \text{LAB}_T \) (its image is enough to determine \( \alpha \in (\mathcal{R} \times \mathcal{R})^* \) just because the grammar is deterministic). Of course the proofs can be rewritten with prefixes \( \alpha \in (\mathcal{R} \times \mathcal{R})^* \).

Strategies
The formal systems \( \mathcal{J}(T_0, T'_0, S_0, \mathcal{B}) \) described in subsection 2.2 were devised so that their set of judgments is recursive. Let us consider now the formal systems \( \hat{\mathcal{J}}(T_0, T'_0, S_0, \mathcal{B}) \) really considered in pages 21-24. Their judgments are also of the forms
\[
m \models (T, T', S), \quad m \models (T, T', S) \sim \alpha \models (T_1, T'_1, S_1), \quad m \models (T, T', S) \sim \alpha \models \text{SUCC}
\]
but where \( S, S_1 \) are D-strategies (instead of finite prefixes of strategies), “except when a judgment is obtained by rule R2”: see the fuzzy remark on page 23, line 11, followed by the enigmatic remark that “we could complete the definition anyhow for such cases”. Since \( S, S_1, S_2, S_3, S_4, S_5, \text{Id}_{D,2}, \text{Id}_{E,2} \) are really D-strategies and \( S_6 \) is obtained by an application of rule R2, it seems that our proofs \( \pi_3, \pi_5, \pi_6 \) are also proofs in the systems \( \hat{\mathcal{J}}(T_0, T'_0, S_0, \mathcal{B}) \). As well, replacing \( \text{Id}_{C,1} \) by \( \text{Id}_{C,\infty} \) in \( \pi_4 \), we obtain a proof of judgment \( 0 \models (C(L_1), C(L_1), \text{Id}_{C,\infty}) \sim (\varepsilon, \varepsilon) \models \text{SUCC} \) in the system \( \hat{\mathcal{J}}(C(L_1), C(L_1), \text{Id}_{C,\infty}, \mathcal{B}) \).

Depth of the examples
One can devise such proofs of non-bisimilar pairs, with an arbitrary long initial strategy: it suffices to add non-terminals \( D_1, D_2, \ldots, D_k, E_1, E_2, \ldots, E_k \) and to replace rules 11, 12, 13, 14.
by the sequence of rules:

\[ D(v) \xrightarrow{x} D_1(v) \]  
\[ E(v) \xrightarrow{x} E_1(v) \]  
\[ \vdots \]  
\[ D_1(v) \xrightarrow{x} D_2(v) \]  
\[ E_1(v) \xrightarrow{x} E_2(v) \]  
\[ \vdots \]  
\[ D_k(v) \xrightarrow{x} v \]  
\[ E_k(v) \xrightarrow{x} v \]  
\[ E_k(v) \xrightarrow{x} v \]  
\[ L_1 \xrightarrow{\ell_1} \bot \]  

A proof of \( 0 \models A(\bot), B(\bot), \hat{S} \sim (\varepsilon, \varepsilon) \models SUCC \) can still be written, but with a longer initial strategy \( \hat{S} \) where the maximal length of words is \( 3 + k \), and a prefix of strategy \( \hat{S}_6 \) of length \( k \). Note that the sizes of the proofs \( \pi_3, \pi_4, \pi_5, \pi_6 \) still remain the same.

6 The flawed argument

Let us locate precisely, in [Jan10], the crucial flawed argument in favor of soundness of the systems.

Page 24, line 4-7, the following assertion (FA) is written:

“The final rule in deriving \( m \models (U, U', S') \sim (\varepsilon, \varepsilon) \models SUCC \) could not be the Basis rule, due to the least eq-level assumption for \( T, T' \) (recall Prop. 17).”

In our example:

\( (T, T') = (A(\bot), B(\bot)), \quad EqLv((A(\bot), B(\bot))) = 3 \)

Let us take

\( (U, U', S') = (E(\bot), E(\bot), S_6) \)

We have:

\( EqLv(U, U', S') = 0 = EqLv(T, T', S) - 3 \)

And the judgment

\( 3 \models E(\bot), E(\bot), S_6 \sim (\varepsilon, \varepsilon) \models SUCC \)

can be derived by the proof \( \pi_7 \) below. Hence \( (T, T') \) has the least equivalence level, among the EqLevels of the elements of \( \{(T, T')\} \cup B \) while \( m, U, U' \) fulfills the maximality hypothesis of the text (line 6-7).

But the final rule used in this proof is the basis rule (R7), contradicting the assertion (FA).

The bug seems to be the following: by Proposition 17

\[ EqLv(E(L_1), E(L_1)) \leq EqLv(E(\bot), E(\bot)) \]  

(23)
BUT

\[ EqLv(E(L_1), E(L_1)) > EqLv(E(\bot), E(\bot), S_6) \]  \tag{24}  

A superficial look at the instance \[23\] of Proposition 17 can induce the idea that, for every D-strategy \(S\) (in particular for \(S_6\)), the inequality

\[ EqLv(E(L_1), E(L_1)) \leq EqLv(E(\bot), E(\bot), S) \]  \tag{25}  

holds. In fact, what shows Proposition 17, is that inequality \[25\] does hold but, only for strategies \(S\) which are optimal for the defender, hence realizing exactly the equivalence level of \((E(\bot), E(\bot))\).

References

Jan10. P. Jancar. Short decidability proofs for dpda language equivalence and 1st order grammar bisimilarity. \texttt{arXiv:1010.4760v3}, pages 1–35, 2010.