Spinors for spinning $p$-branes

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Abstract
The group of the $p$-brane worldvolume preserving diffeomorphism is considered. The infinite-dimensional spinors of this group are related, by the nonlinear realization techniques, to the corresponding spinors of its linear subgroup, which are constructed explicitly. These two sets of spinors are mutually related by the infinite-component pseudo-frames parametrized by the nonlinear symmetry realizers. An algebraic construction of the Virasoro and Neveu–Schwarz–Ramond algebras, based on these infinite-dimensional spinors and tensors, is demonstrated.

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1. Introduction

The subject of extended objects was initiated in the particle/field theory framework by the Dirac action for a closed relativistic membrane as the $(2+1)$-dimensional worldvolume swept out in spacetime [1]. It evolved and became one of the central topics following the Nambu–Goto action for a closed relativistic string, as the $(1+1)$-dimensional worldsheet area swept out in spacetime [2, 3]. An important step was the Polyakov action for a closed relativistic string, with auxiliary metric [4], that enabled consequent formulations of the Green–Schwarz superstring [5], and the bosonic, and super $p$-branes with manifest spacetime supersymmetry [6, 7]. In this work, we follow the original path of the Nambu–Goto-like formulation of the bosonic $p$-brane and address the question of spinors of the brane worldvolume symmetries. For $p = 1$, these spinors are well known, and represent an important ingredient of the spinning string formulation and the Neveu–Schwarz–Ramond infinite algebras [8, 9].

It is interesting to point out that there is a direct analogy between the spinors of the $p$-brane action symmetry, which are considered in this work, and the so-called world spinors that describe the spinorial matter fields of the metric-affine [10] and gauge-affine [11] theories of gravity formulated in a generic (non-)Riemannian spacetime of arbitrary torsion and curvature. This is due to common geometric and group-theoretic structures of $p$-brane theories and affine generalizations of Einstein’s gravity theory. The global symmetry of these matter fields in four dimensions is the affine $\mathbb{A}(4, R)$ group, which generalizes the $\mathbb{P}(4)$ Poincaré group of
the conventional gauge approach to the theory of gravity. When gauging the affine group, one has a complete parallel of both anholonomic (local) and holonomic (world) descriptions of bosonic and fermionic matter fields. In contradistinction to the gauge Poincaré theory, where spinors are scalars of the general coordinate transformations group, in the affine case there are both local (tangent) and world (curved) spinors. Analogously, the spinors of the spinning string theory are just the faithful worldsheet spinors.

It was shown by Ogievetsky that the infinite algebra of the general coordinate transformations group in four dimensions arises upon a Lie algebra closure of the finite algebras of the $SL(3, R)$ group and the four-dimensional conformal group [12]. This result paved the way for various approaches to gravity theory, especially those that utilized the nonlinear representation techniques. In particular, it was proven that Einstein’s theory of gravity is obtained by simultaneous nonlinear realizations of the affine and conformal symmetries. The affine and its linear subgroup are nonlinearly realized w.r.t. the Poincaré and Lorentz subgroups, respectively [13]. Thus, the general coordinate transformations group is realized over its Poincaré and/or Lorentz subgroup in a nonlinear manner, resulting in a loss of the world (curved space) spinors, which exist only when the affine and/or linear subgroup are represented linearly. The infinite-dimensional spinorial representations of the infinite algebras of the general coordinate transformations in three and four dimensions were studied in [14] and [15], respectively. It is important to note that the general coordinate transformations are realized in $(p+1)$-dimensional spaces for $p$-branes, and thereafter the spinorial representations of its infinite algebras are not of direct relevance here. There has been recently a revival of the nonlinear realization techniques in studying the extended object theories, in particular the spinning $p$-branes [16]. The $p$-brane symmetry is, in these applications, nonlinearly realized over its $SO(1, p) \otimes SO(D − p − 1)$ subgroup. In the present work, we consider the whole symmetry of the general coordinate transformations covering group and study their spinorial nonlinear realizations over its linear subgroup $SL(p + 1, R) \supset SO(1, p)$. In this manner we set up a framework that has reach enough to accommodate a generic spinning $p$-brane.

In this work we study the topological, group-theoretical and unitarity features that are of relevance for a description of the $p$-brane worldvolume symmetries, especially those of importance for a description of spinorial quantities. It turns out that there are quite a number of subtle features of the extended object theories ($p \geq 1$), as compared to the point-particle ($p = 0$) case, that burden the description of the $p$-brane theory spinorial sector. In order to cope with these difficulties, as well as to study the generic case, we utilize the nonlinear realization techniques. In this way we relate the spinors of the $p$-brane worldvolume symmetry to the corresponding (infinite dimensional) spinors of its linear transformations subgroup. Furthermore, we provide an explicit construction of the relevant linear subgroup spinorial representations. This result, combined with the above nonlinear realization relation, provides an explicit expression of the $p$-brane general coordinate transformations spinors. Moreover, we introduce pseudo-frames that relate the spinning brane spinors to the linear transformations group spinors. These pseudo-frames are expressed in terms of certain group parameters that determine nonlinear realizations and behave as the Goldstone-like fields. Finally, we demonstrate, in the simplest case of a spinning string, a group-theoretical derivation of the infinite Virasoro and Neveu–Schwarz–Ramond algebras. This derivation is solely based on the algebraic properties of the infinite-dimensional tensorial and spinorial representations.

2. $p$-brane worldvolume symmetries

Consider a bosonic $p$-brane embedded in a $D$-dimensional flat Minkowski spacetime $M^{1, D−1}$. The classical Dirac–Nambu–Goto-like action for a $p$-brane is given by the volume of the
worldvolume swept out by the extended object in the course of its evolution from some initial to some final configuration:

$$S = -\frac{1}{\kappa} \int d^{p+1}x \sqrt{-\det \gamma_{\mu\nu} \gamma_{\mu\nu}}.$$  \hfill (1)

where \( i = 0, 1, \ldots, p \) labels the coordinates \( \xi^i = (x, \sigma_1, \sigma_2, \ldots) \) of the brane worldvolume with metric \( \gamma_{ij}(\xi) \), and \( \gamma = \det(\gamma_{ij}); \) \( m = 0, 1, \ldots, D - 1 \) labels the target space coordinates \( X^m(\xi^i) \) with metric \( \eta_{mn} \). The worldvolume metric \( \gamma_{ij} = \partial_i X^m \partial_j X^m \eta_{mn} \) is induced from the spacetime metric \( \eta_{mn} \).

The Poincaré \( P(1, D - 1) \) group, i.e. its homogeneous Lorentz subgroup \( SO(1, D - 1) \), is the physically relevant spacetime symmetries, while the \((p + 1)\)-dimensional brane worldvolume is preserved by the homogeneous volume preserving subgroup \( SDiff_0(p + 1, R) \) of the general coordinate transformation (GCT) group \( Diff(p + 1, R) \).

The \( sDiff_0(p + 1, R) \) algebra operators, which generate the \( SDiff_0(p + 1, R) \) group, are given as follows:

$$sDiff_0(p + 1, R) = \left\{ L^{im}_{(n)k} = \xi^i \xi^j \cdots \xi^{j-n} \frac{\partial}{\partial \xi^j} \mid n = 2, 3, \ldots, \infty \right\}. \hfill (2)$$

Preservation of the worldvolume requires the \( L_{(2)} \) operator to be traceless as achieved by subtracting the dilation operator, i.e. \( L_{(2)k} = \frac{\partial}{\partial \xi^k} - \frac{1}{\alpha^2} \delta^i_k \xi^i \frac{\partial}{\partial \xi^i} \). The \( L_{(n)}, n = 2, 3, \ldots, \infty \), operators are irreducible tensor operators of the \( SL(p + 1, R) \) subgroup, and therefore naturally labeled by the \( SL(p + 1, R) \) irreducible representations given by the Young tableaux \( [\lambda_1, \lambda_2, \ldots, \lambda_p] \) with \( \lambda_1 = 2, 3, \ldots, \infty \), and \( \lambda_2 = \lambda_3 = \ldots = \lambda_p = 1 \).

The \( SDiff_0(p + 1, R) \) commutation relations read

$$\left[ L^{i_1 \cdots i_{m-1}}_{(m)k}, L^{j_1 \cdots j_{m-1}}_{(m)l} \right] = \delta^i_l L^{i_1 \cdots i_{m-1} j_1 \cdots j_{m-1}}_{(m+n-2)k} + \delta^j_k L^{i_1 \cdots i_{m-1} j_1 \cdots j_{m-1}}_{(m+n-2)l} + \cdots + \delta^{i_{m-1}}_l L^{j_1 \cdots j_{m-1} i_1 \cdots i_{m-1}}_{(m+n-2)k} - \delta^{j_{m-1}}_k L^{i_1 \cdots i_{m-1} j_1 \cdots j_{m-1}}_{(m+n-2)l} - \cdots - \delta^{i_{m-1}}_k L^{j_1 \cdots j_{m-1} i_1 \cdots i_{m-1}}_{(m+n-2)l}.$$

\hfill (3)

The above symmetry considerations are purely classical. In the quantum case, the corresponding classical symmetry is modified, up to eventual anomalies, in two ways: (i) the classical group is replaced by its universal covering group, and (ii) the group is minimally extended by the \( U(1) \) group of phase factors. The corresponding Lie algebra remains unchanged in the first case, while in the second one, it can have additional central charges.

The feasible ways of how to extend the Dirac–Nambu–Goto bosonic \( p \)-brane action by the fermionic degrees of freedom are determined by the universal covering group \( SDiff_0(p + 1, R) \) of the \( SDiff_0(p + 1, R) \) group and the form of its spinorial representations. In the following we address first the topological issues that define the type of the universal covering of the \( SDiff_0(p + 1, R) \) group, and subsequently, we face the problem of the \( SDiff_0(p + 1, R) \) group spinorial representations construction.

3. Existence of the double-covering \( SDiff_0(p, R) \)

It will be shown below that, when spinors are taken into account as well, the topological, algebraic and unitarity properties of the groups of linear and general coordinate transformations are, as a rule, more subtle than in the case of the orthogonal type of groups. Therefore, in order to address these questions properly, a more powerful theoretical machinery is to be involved in the present case of the \( SDiff_0(p + 1, R) \) groups.
Let us state first some relevant mathematical results.

Let \( g = k + a + n \) be an Iwasawa decomposition of a semisimple Lie algebra \( g \) over \( R \). Let \( G \) be any connected Lie group with Lie algebra \( g \), and let \( K, A, N \) be the analytic subgroups of \( G \) with Lie algebras \( k, a \) and \( n \) respectively. The mapping \( (k, a, n) \rightarrow kan, k \in K, a \in A, n \in N \), is an analytic diffeomorphism of the product manifold \( K \times A \times N \) onto \( G \), and the groups \( A \) and \( N \) are simply connected.

Any semisimple Lie group can be decomposed into the product of the maximal compact subgroup \( K \), an Abelian group \( A \) and a nilpotent group \( N \). As a result of the above statement, only \( K \) is not guaranteed to be simply-connected. There exists a universal covering group \( \overline{K} \) of \( K \), and thus also a universal covering of \( G: \overline{G} \simeq \overline{K} \times A \times N \).

For the group of volume-preserving diffeomorphisms, let \( Diff(n, R) \) be the group of all homeomorphisms \( f \) of \( R^n \) such that \( f \) and \( f^{-1} \) are of class \( C^1 \). Stewart proved the decomposition \( Diff(n, R) = GL(n, R) \times E \times R^n \), where the subgroup \( H \) is contractible to a point. In our case the relevant decomposition is \( SDiff_0(p+1, R) = SL(p+1, R) \times E \times R^{n+1} \). Thus, as \( SO(p+1) \) is the maximal compact subgroup of \( SL(p+1, R) \), we find that \( SO(p+1) \) is a deformation retract of \( SDiff_0(p+1, R) \).

As a result, there exists a universal covering of the diffeomorphism group \( \overline{SDiff}_0(p+1, R) \simeq \overline{SL}(p+1, R) \times H \times R^{n+1} \).

Summing up, we note that both \( SL(p+1, R) \) and \( SDiff_0(p+1, R) \) have double coverings, defined by \( \overline{SO}(p+1) \simeq Spin(p+1) \) the double-coverings of the \( SO(p+1) \) maximal compact subgroup.

Let us consider now the question of the universal, i.e. double, covering of the \( SL(p+1, R) \) and \( SDiff_0(p+1, R) \) groups themselves. The universal covering group \( \overline{G} \) of a given group \( G \) is a group with the same Lie algebra and with simply-connected group manifold. A finite-dimensional covering \( \overline{SL}(p+1, R) \), i.e. \( \overline{SDiff}_0(p+1, R) \), exists provided one can embed \( \overline{SL}(p+1, R) \) into a group of finite complex matrices that contain \( Spin(p+1) \) as a subgroup. A scan of the semi-simple classical algebras, as given by the Cartan classification, points first to the \( SL(p+1, C) \) groups as a natural candidate for the \( SL(p+1, R) \) groups coverings. However, there is no match whatsoever of the defining dimensionalities of the \( SL(p+1, C) \) and \( Spin(p+1) \) groups for \( p \geq 2 \),

\[
\dim(SL(p+1, C)) = p + 1 < 2^{\lfloor \frac{p+1}{2} \rfloor} = \dim(Spin(p+1)),
\]

except for \( p + 1 = 8 \). In the \( p + 1 = 8 \) case, one finds that the orthogonal subgroup of the \( SL(8, R) \) and \( SL(8, C) \) groups is \( SO(8) \) and not \( Spin(8) \). For a detailed account of the \( D = 4 \) case cf [17]. Thus, we conclude that there are no covering groups of the \( SL(p+1, R) \), i.e. \( \overline{SDiff}_0(p+1, R) \) groups for any \( p \geq 2 \) that are given by finite matrices (defined in finite-dimensional complex spaces). An explicit construction of all spinorial, unitary and nonunitary multiplicity-free [18] and unitary non-multiplicity-free [19], \( SL(3, R) \) representations shows that they are indeed all defined in infinite-dimensional spaces.

The universal (double) covering groups of the group \( \overline{SDiff}_0(p+1, R) \) and its \( \overline{SL}(p+1, R) \) subgroup are, for \( p \geq 2 \), the groups of infinite complex matrices. All their spinorial representations are necessarily infinite dimensional. In the reduction of these representations w.r.t. either \( Spin(p+1) \) subgroup, with a trivial metric tensor \( \delta \), or \( Spin(1, p) \), with a Minkowski-like metric tensor \( \eta \), one has representations of unbounded spin values.

4. The deunitarizing automorphism

Unitarity is one of the key stones of quantum theory. It imposes certain constraints on the Poincaré symmetry representations of the relativistic point-like object theories. The
unitarity of the $S$ matrix is assured by requiring that the Poincaré representations on states (on shell) are unitary. These representations are, due to semidirect group structure of the Poincaré group, i.e. $\overline{SDiff}_0(p+1, R)$ and/or $\overline{SL}(p + 1, R)$ symmetries have to be given (as in the point-particle Poincaré case) by the nonunitary representations on fields that are unitary over the relevant little groups.

The matching of the corresponding Poincaré representations on states and fields is provided by the relativistic field equations. In the case of physical particles, the matching is achieved by making use of the nonunitary, finite-dimensional Lorentz subgroup representations that are unitary over the relevant little group. The action of the non-hermitean boost-like Lorentz generators (generators beyond the little group algebra) is taken care of by the Lagrangian and/or Hamiltonian hermiticity requirement.

The unitarity requirement of the $p$-brane quantum theory implies that the $\overline{SDiff}_0(p+1, R)$ and/or $\overline{SL}(p + 1, R)$ symmetries have to be given (as in the point-particle Poincaré case) by the nonunitary representations on fields that are unitary over the relevant little groups.

The unitarity properties that ensure correct physical description of the relevant representations of the $\overline{SDiff}_0(p+1, R)$ and $\overline{SL}(p + 1, R)$ groups on quantum states and fields can be achieved by making use of the unitary (irreducible) representations construction of these groups and the so-called deunitarizing automorphism of the $\overline{SL}(n, R)$ group. This procedure ensures that in the special relativity limit (Lorentz invariance) all physical objects have the usual properties (i.e. boosted electron and/or quark retain their Poincaré properties).

The commutation relations of the $\overline{SL}(p + 1, R)$ generators

$$Q_{jk} = i\eta_{jl}L_{(2)jk}, \quad j, k, l = 0, 1, \ldots, p, \quad \eta_{jl} = \text{diag}(+1, -1, \ldots, -1),$$

are

$$[Q_{ij}, Q_{kl}] = i(\eta_{jk}Q_{il} - \eta_{il}Q_{jk}).$$

The important subalgebras are as follows.

(i) $so(1, p)$: The $M_{ij} = Q_{[ij]}$ operators generate the Lorentz-like subgroup $SO(1, p) \simeq Spin(1, p)$ with $J_{mn} = M_{mn}$ (angular momentum) and $K_m = M_{0m}$ (the boosts) $m, n = 1, 2, \ldots, p$.

(ii) $so(p + 1)$: The $R_{ij}$ operators, $i, j = 1, 2, \ldots, p + 1$, i.e. $J_{mn}$ and $N_m = Q_{[0m]}$ operators generate the maximal compact subgroup $SO(p + 1) \simeq Spin(p + 1)$.

(iii) $sl(p)$: The $J_{mn}$ and $T_{mn} = Q_{[mn]}$ operators generate the subgroup $\overline{SL}(p, R)$—an analog of the ‘little’ group of the massive particle states in Poincaré theory.

The $\overline{SL}(p + 1, R)$ commutation relations are invariant under the ‘deunitarizing’ automorphism (originally introduced for the $p = 3$ case [17]),

$$J'_{mn} = J_{mn}, \quad K'_m = iN_m, \quad N'_m = iK_m, \quad T'_m = T_{mn}, \quad T'_{00} = T_{00} = Q_{00},$$

so that $(J_{mn}, iK_m)$ generate the new compact $\overline{SO}(p + 1)'$ and $(J_{mn}, iN_m)$ generate $\overline{SO}(1, p)'$.

The above deunitarizing automorphism generalizes to the arbitrary signature case. Let $\overline{SL}(n, R)$ group act on $R^{r+s}, r + s = n$ with metric $\eta = \text{diag}(+1, \ldots, +1, -1, \ldots, -1)$ having $r$ times $+1$ and $s$ times $-1$ on the diagonal. The group generators $Q_{ij}$ split accordingly to

$$Q_{ab} = Q_{ab}, \quad Q'_{mn} = Q_{ab}, \quad Q_{am} = iQ_{am}, \quad Q'_{ma} = -iQ_{ma},$$

where $a, b = 1, 2, \ldots, r, m, n = 1, 2, \ldots, s$. The deunitarizing automorphism that leaves the $sl(n, R)$ algebra invariant is given as follows:
The construction of physically relevant representations is achieved by the following two-step procedure: (1) one constructs, utilizing the appropriate mathematical theorems and methods, the unitary irreducible spinorial, as well as tensorial, representations of the $\mathcal{SDiff}_{0}(p + 1, R)$ and $\mathcal{SL}(p + 1, R)$ groups in the basis of the maximal compact $Spin(p + 1)$ subgroup representations, and (2) one converts these representations, by making use of the deunitarizing automorphism, to representations that are finite and nonunitary for the physical $Spin(1, p)$ subgroup.

5. Nonlinear $\mathcal{SDiff}_{0}(p + 1, R)$ representations

The GCT group $\mathcal{SDiff}_{0}(p + 1, R)$ is an infinite parameter Lie group with the corresponding infinite algebra that acts linearly, e.g. as infinite matrices, on an infinite-dimensional vector space. However, its defining representation is given by the group of volume preserving nonlinear transformations of the $R^{*+1}$ spacetime. The $\mathcal{SDiff}_{0}(p + 1, R)$ group is nonlinearly realized over its $\mathcal{SL}(p + 1, R)$ subgroup.

The defining representation of the $\mathcal{SDiff}_{0}(p + 1, R)$ universal (i.e. double) covering group, as well as of its $\mathcal{SL}(p + 1, R)$ subgroup, is given, as demonstrated above, by the infinite-dimensional matrices. In other words, there is no group of finite complex matrices that is isomorphic to $\mathcal{SDiff}_{0}(p + 1, R)$.

Let us consider now the spinorial representations of the $\mathcal{SDiff}_{0}(p + 1, R)$ group. There are genuine linear spinorial representations of the $\mathcal{SDiff}_{0}(p + 1, R)$ group that are infinite dimensional. Moreover, all of its infinitely many Lie algebra generators are likewise represented linearly by infinite matrices. Besides, there are two distinct classes of $\mathcal{SDiff}_{0}(p + 1, R)$ nonlinear spinorial realizations characterized by

(i) $\mathcal{SDiff}_{0}(p + 1, R)$ group is nonlinearly realized over its maximal linear subgroup $\mathcal{SL}(p + 1, R)$; $\mathcal{SL}(p + 1, R)$ and $Spin(1, p)$ are represented linearly;
(ii) both $\mathcal{SDiff}_{0}(p + 1, R)$ and its $\mathcal{SL}(p + 1, R)$ subgroup are realized nonlinearly over the orthogonal subgroup $Spin(1, p)$.

We recall now a few basic notions from the nonlinear representations theory [20, 21] and set up required notation. Let $G$ be an $n_G$ parameter Lie group, and let $H$ be an $n_H$ parameter subgroup of $G$. Let $\mathcal{M}$ be a real analytic manifold of dimension $d$. The mappings $R$ from $g \times \mathcal{M}$ into $\mathcal{M}$ form a representation of $G$ if, for each $g \in G$, $p \in \mathcal{M}$, there is an element $R(g)[p] \in \mathcal{M}$ such that (i) $R : (g, p) \rightarrow R(g)[p]$ is analytic, (ii) $R(e)[p] = p$, for all $p \in \mathcal{M}$, $e$ is the identity in $G$, and (iii) $R(g_1)R(g_2)[p] = R(g_1g_2)[p]$, for all $g_1, g_2 \in G$, all $p \in \mathcal{M}$.

At each point $p \in \mathcal{M}$, local coordinates can be introduced by mapping an open neighborhood of $p$ into an open neighborhood of $R^d$. Let $q$ denote the coordinates of a general point $p \in \mathcal{M}$, and let $a$ be the group parameters of an element $g \in G$ in a neighborhood of $e$. Then $R(g)[p]$ can be expressed as an analytic function $r(q, a)$ of both $q$ and $a$, which is in general nonlinear.

An equivalence of two representations is naturally expressed through an independence of the choice of coordinates. Usually, there exists a special point, base point, on $\mathcal{M}$ which must be represented by the origin $q_0$ in all coordinates. Thus, one defines a concept of local equivalence. Two representations $R_1(g)$ and $R_2(g)$ are locally equivalent if there exists an (in general nonlinear) operator $S$ from $R^a \rightarrow R^a$ such that (i) $S : q \rightarrow S[q]$ is analytic and has an analytic inverse at $q_0$, (ii) $S[R_1(g)][q] = R_2(g)S[q]$, for all $g \in G$ in a suitable neighborhood of the identity, and all $q$ in a neighborhood of $q_0$, and (iii) $S[q_0] = q_0$. Representation is said linearizable if it is locally equivalent to a linear representation.
Let $H$ be a subgroup of $G$ such that for each $h \in H$, $R(h)[q_0] = q_0$, i.e. let $H$ be the isotropy subgroup of the origin $q_0$. Now, it turns out that a restriction $R(h), h \in H$ of the representation $R(g)$ is locally equivalent to a linear representation. In the expansion $R(g)[q], g = h \in H$ in power series $R(h) = D(h)q + O(q^2)$, one finds a linear representation $D(h)$ of $H$. The change of coordinates defined by $S : q \rightarrow \tilde{q} = S[q] = \int_H dhD^{-1}(h)R(h)[q]$, where $dh$ is the right invariant measure on $H$, establishes a local equivalence between $D(h)$ and the restriction of $R(g)$ to $H$, i.e. $R(h)[\tilde{q}] = D(h)\tilde{q}$.

An arbitrary element $g$ in $G$ can be written as $g = ch$, where $h$ belongs to $H$ and $c$ belongs to the left coset space $C = G/H$. Furthermore, an arbitrary point $q$ of the orbit can be written as $q = R(g)[q_0] = R(c)R(h)[q_0] = R(c)[q_0]$. Thus, the elements of the orbit are in one-to-one correspondence with the elements of the coset space $G/H$. They form a homogeneous space on which $G$ can be represented.

An action of an arbitrary element $g_1$ on $c$ is as follows: $g_1c = c_1h_1c = c'h'$. The parameters of the group element $h'$ depend both on the group element $g_1$ and on $c$, i.e. $h' = h'(c, g_1)$. The transformation $h \rightarrow h'$ is in general nonlinear, and it becomes linear when $g_1$ is restricted to $H$.

Let us choose the generators $X_a, a = 1, 2, \ldots, n_H$ of $H$ and the remaining generators $Y_b, b = 1, 2, \ldots, n G - n_H$ of $G$ such that they form together a complete set of generators of $G$ that is orthonormal with respect to the Cartan inner product. In some neighborhood of the identity of $G$, every element $g \in G$ can be decomposed uniquely as follows:

$$g = ch = e^{-i\xi Y}e^{-i\omega X}, \quad \xi \cdot Y = c^bY_b, \quad \omega \cdot X = \omega^aX_a, \quad \xi^b, \omega^a \in R. \quad (7)$$

The $c^b$ and $\omega^a$ parameters form a real $n_G$-component vector $(\xi, \omega)$. Now, owing to the fact that $H$ leaves the origin $q_0$ fixed, the orbit $\mathcal{N}$ of $q_0$ under $G$ separates the $G/H$ cosets defined by $L_\xi = e^{-i\xi Y}$. One has

$$R(g)[q_0] = R(e^{-i\xi Y})R(e^{-i\omega X})[q_0] = e^{-i\xi R(Y)}[q_0],$$

and the dimension of the orbit $\mathcal{N}$ is given by the number of $\xi^b$ parameters, i.e. it is equal to $n_G - n_H$. The simplest choice is to represent the orbit elements by $L_\xi$. We split now the manifold $\mathcal{M}$ into $\mathcal{N}$ and its orthogonal complement $\mathcal{V}$, which is $d = (n_G - n_H)$ dimensional, i.e. $\mathcal{M} = \mathcal{N} + \mathcal{V}$. Finally, for the coordinates of $\mathcal{M}$ we write $q = (L_\xi, \psi), L_\xi \in \mathcal{N}, \psi \in \mathcal{V}$. According to the linearization procedure, we can choose the coordinates $(L_\xi, \psi)$ so that $H$ acts linearly, and in particular the coordinates $\psi$ span a space of a linear representation $D(h)$ of $H$.

Owing to $g_1c = c'h' = c'h(c, g_1)$, and $c = L_\xi$, one finds for $L_\xi$ the following transformation law,

$$g : L_\xi \rightarrow L_{\xi'} = gL_\xi h^{-1}(\xi, g), \quad g \in G, h \in H, \quad (8)$$

while $\psi$ transforms according to

$$g : \psi \rightarrow \psi' = D(h(\xi, g))\psi = D(L_{\xi'}^{-1}gL_\xi)\psi = D(e^{-i\omega(X-\xi)\cdot Y})\psi. \quad (9)$$

When $g = h$,

$$L_{\xi'} = hL_\xi h^{-1} = D(\xi)(h)L_\xi, \quad h \in H$$

where $D(\xi)$ is a linear representation of $H$ in the $\xi^b$ space, while

$$\psi' = D(h)\psi = D(e^{-i\omega X})\psi.$$

For a linear representation $D(g), g \in G$, one has

$$D(L_\xi) \rightarrow D(L_{\xi'}) = D(gL_\xi h^{-1}(\xi, g)) = D(g)D(L_\xi)D(h^{-1}(\xi, g)).$$
Let $\Psi$ be a basis of this linear representation, i.e. $\Psi' = D(g)\Psi$, $g \in G$. By defining
\[ \psi = D(L^{-1}_\xi)\Psi, \]
(10)
one relates the linear and nonlinear representations, i.e. one projects the linear representation into the corresponding nonlinear one. Indeed, one has
\[ \psi \rightarrow \psi' = D(h)\psi, \quad h = h(\xi, g) = L^{-1}_\xi gL_\xi \in H. \]
(11)Moreover, one can express the basis $\Psi$ of a linear representation $D(g)$ in terms of the corresponding basis $\psi$ of its nonlinear representation $D(h(\xi, g))$ as follows:
\[ \Psi = D(L_\xi)\psi. \]
(12)

5.1. Nonlinear representations over $\text{SL}(p + 1, R)$

Let us consider the case where $\text{SDiff}_0(p + 1, R)$ group is nonlinearly realized over its maximal linear subgroup $\text{SL}(p + 1, R)$. This is a natural extension of $\text{SDiff}_0(p + 1, R)$ being linearly realized over $\text{SL}(p + 1, R)$.

As stated above, $\text{SDiff}_0(p + 1, R) = \text{SL}(p + 1, R) \times E \times R^{p+1}$, and thus we have now $g \in G = \text{SDiff}_0(p + 1, R)$, $h \in H = \text{SL}(p + 1, R)$ and $c = L_\xi \in G/H = E \times R^{p+1}$.

Let $\Psi$ transform w.r.t. a spinorial representation of the $\text{SL}(p + 1, R)$ group, i.e.
\[ \psi' = (D(\text{SDiff}_0(p+1,R))\Psi_B, \quad h \in \text{SL}(p + 1, R)A, B = 1, 2, \ldots, \infty \]
(13)where the index that enumerates the components of $\Psi$ runs over an infinite range due to the fact that the spinorial representations of the $\text{SL}(p + 1, R)$ group are for $p + 1 \geq 3$ necessarily infinite dimensional. The $\text{SDiff}_0(p + 1, R)$ spinor $\Psi$ transforms as follows:
\[ \Psi' = (D(\text{SDiff}_0(p+1,R)(g))\Psi_B, \quad g \in \text{SDiff}_0(p + 1, R), A, B = 1, 2, \ldots, \infty. \]
(14)The $D(\text{SDiff}_0(p+1,R))$ representations can be reduced to direct sum of infinite-dimensional $D(\text{SL}(p + 1, R))$ representations. We consider here those representations of $\text{Diff}_0(p + 1, R)$ that are nonlinearly realized over the maximal linear subgroup $\text{SL}(p + 1, R)$.

Provided the relevant $D(\text{SL}(p + 1, R))$ spinorial representations are known, one can first define the corresponding spinors, $\Psi_A$, and then make use of the infinite-component pseudo-frames
\[ E^A_A = (D(L_\xi))_A \]
(15)to achieve the required linear-to-nonlinear mapping [22]
\[ \Psi_A = E^A_A(x)\Psi_A, \quad E^A_A \sim \text{Diff}_0(p + 1, R)/\text{SL}(p + 1, R). \]
(16)The pseudo-frames $E^A_A$ infinitesimal transformations are given by
\[ \delta(\text{SL}(p+1,R))E^A_A = i\epsilon^A_A(Q^A_B)E^B_A \]
(17)where $\epsilon^A_A$ and $Q^A_B$ are the group parameters and generators of $\text{SL}(p + 1, R)$, respectively.

The above-outlined construction allows one to define a $\text{Diff}(p + 1, R)$ covariant Dirac-like wave equation for the corresponding spinor $\Psi$ provided a Dirac-like wave equation for the $\text{SL}(p + 1, R)$ group is known. In other words, one can lift up an $\text{SL}(p + 1, R)$ covariant equation of the form
\[ (i(\Gamma^k_{\text{SL}(p+1)})^A_B \partial_k - m)\psi_B = 0, \quad k = 0, 1, \ldots, p \]
(18)to a $\text{Diff}(p + 1, R)$ covariant equation
\[ (iE^A_A(\Gamma^k_{\text{SL}(p+1)})^A_B \partial_k - m)\psi_B = 0, \quad k = 0, 1, \ldots, p \]
(19)
where the former equation exists provided a spinorial $\mathcal{SL}(p + 1, R)$ representation for $\psi$ is given, such that the corresponding representation Hilbert space is invariant w.r.t. $\Gamma^a_{\mathcal{SL}(p+1)}$ action. The crucial step toward a Dirac-like GCT spinor equation is a construction of the vector operator $\Gamma^a_{\mathcal{SL}(p+1)}$ in the space of $\mathcal{SL}(p + 1, R)$ spinorial representations. We have recently presented an explicit construction of the diffeomorphism covariant Dirac-like equation in the $p + 1 = 3$-dimensional case [23].

5.2. Nonlinear representations over Spin(1, p)

Let us consider now the case where $\mathcal{SDiff}_0(p + 1, R)$ group is nonlinearly realized over its maximal compact subgroup $\text{Spin}(p + 1)$ or over the related, physically more interesting, Lorentz-like group $\text{Spin}(1, p)$.

The relevant group decompositions are $\mathcal{SDiff}_0(p + 1, R) = \mathcal{SL}(p + 1, R) \times E \times R^{p+1}$, and the Iwasawa decomposition $\mathcal{SL}(p + 1, R) = \text{Spin}(1, p) \times A_{p+1} \times N_{p+1}$, where $A_{p+1}$ and $N_{p+1}$ are the groups of $(p + 1) \times (p + 1)$ Abelian and nilpotent matrices, respectively. Therefore, $g \in G = \mathcal{SDiff}_0(p + 1, R)$, $h \in H = \text{Spin}(1, p)$ and $c = L_\varepsilon \in G/H = E \times R^{p+1} \times A_{p+1} \times N_{p+1}$.

Here, $\psi$ transforms w.r.t. a spinorial representation of the $\text{Spin}(1, p)$ group, i.e.

$$\psi' = (\mathcal{D}_{\text{Spin}(1,p)}(h))_\alpha^\beta \psi_\beta, \quad h \in \text{Spin}(1, p) \alpha, \beta = 1, 2, \ldots \dim(\mathcal{D}_{\text{Spin}(1,p)}),$$

where the indices $\alpha, \beta$ enumerate the finite-dimensional nonunitary or infinite-dimensional unitary $\text{Spin}(1, p)$ representation spaces.

The $\mathcal{SDiff}_0(p + 1, R)$ spinor $\Psi$ transforms as in the previous case. The $\mathcal{D}_{\text{Spin}(1,p)}$ representations can be reduced to a direct sum of finite-dimensional or infinite-dimensional $\mathcal{D}_{\text{Spin}(1,p)}$ representations.

Owing to the fact that, in this case, both $\mathcal{SDiff}_0(p + 1, R)$ and $\mathcal{SL}(p + 1, R)$ groups are represented nonlinearly over $\text{Spin}(1, p)$, one has that both $\mathcal{SDiff}_0(p + 1, R)$ and $\mathcal{SL}(p + 1, R)$ groups are represented nonlinearly over $SO(1, p)$ as well. Therefore, in this case there are no usual, linearly transforming, $\mathcal{SL}(p + 1, R)$ tensorial quantities. Therefore, this case seems to be of no importance for description of a generic spinning $p$-brane because it fails to provide for a group-theoretical formulation of the bosonic theory sector built over the representations of the $\mathcal{SL}(p + 1, R)$ group.

6. $\mathcal{SL}(p + 1, R)$ representations construction

We now face the problem of constructing the (unitary) infinite-dimensional spinorial and tensorial representations of the $\mathcal{SL}(p + 1, R)$ group. The $\mathcal{SL}(p + 1, R)$ group can be contracted (a la Wigner–Inönü) w.r.t. its $\text{Spin}(p + 1)$ subgroup to yield the semidirect-product group $\hat{T} \wedge \text{Spin}(p + 1)$. $\hat{T}$ is a $\frac{1}{2}(p + 3)$ parameter Abelian group generated by operators $U_{ij} = \lim_{\varepsilon \to 0} (\varepsilon T_{ij})$, which form a $\text{Spin}(p + 1)$ second rank symmetric operator obeying the following commutation relations,

$$[J_{ij}, J_{kl}] = -i \delta_{ij} J_{kl} + i \delta_{kl} J_{ij} + i \delta_{jk} J_{il} - i \delta_{il} J_{jk},$$
$$[J_{ij}, U_{kl}] = -i \delta_{ij} U_{kl} - i \delta_{kl} U_{ij} + i \delta_{jk} U_{il} + i \delta_{il} U_{jk},$$
$$[U_{ij}, U_{kl}] = 0.$$  (20)
An efficient way of constructing explicitly the $\mathfrak{SL}(p + 1, R)$ unitary infinite-dimensional representations is given by the so-called decontraction formula, which is an inverse of the Wigner–Inönü contraction. According to the decontraction formula, the following operators,

$$T_{ij} = rU_{ij} + \frac{1}{2\sqrt{U\cdot U}}[C_2(\text{Spin}(p + 1)), U_{ij}],$$

(21)
together with $J^r_{ij}$ form the $\mathfrak{SL}(p + 1, R)$ algebra. The parameter $r$ is an arbitrary complex number, $r \in C$, and $C_2(\text{Spin}(p + 1))$ is the $\text{Spin}(p+1)$ second-rank Casimir operator.

For the representation Hilbert space we take the homogeneous space of $L^2$ functions of the maximal compact subgroup $\text{Spin}(p + 1)$ parameters. The $\text{Spin}(p + 1)$ representation labels are given either by the Dynkin labels $(\lambda_1, \lambda_2, \ldots, \lambda_q)$ or by the highest weight vector which we denote by $|j\rangle = \{j_1, j_2, \ldots, j_q\}, q = \left\lceil \frac{p+1}{2} \right\rceil$. The $\mathfrak{SL}(p + 1, R)$ commutation relations are invariant w.r.t. an automorphism defined by

$$s(J) = +J, \quad s(T) = -T.$$  

(22)

This allows us to associate an ’s-parity’ to each $\text{Spin}(p + 1)$ representation contained in an $\mathfrak{SL}(p + 1, R)$ representation. In terms of the Dynkin labels we find

$$s(D_2) = (-)^{\frac{1}{2}(\lambda_1 + \lambda_2 + \lambda_3 + \epsilon)},$$

$$s(D_{n\geq3}) = (-)^{\lambda_1 + \lambda_2 + \ldots + \lambda_n + \frac{n+1}{2}(\delta_{n,1} - \epsilon)}$$

(23)

$$s(B_1) = (-)^{\frac{1}{2}(\lambda_1 + \epsilon)},$$

$$s(B_{n\geq2}) = (-)^{\frac{1}{2}(\lambda_1 + \lambda_2 + \ldots + \lambda_n + \frac{n+1}{2}(\delta_{n,1} - \epsilon)}$$

where $\epsilon = 0$ and $\epsilon = 1$ for $\lambda$ even and odd, respectively, and $D$ and $B$ refer to Cartan’s Lie algebra notation.

The s-parity of the $\frac{1}{2}(p)(p+3)$-dimension representation $(20 \cdots 0) = \square$ of $\text{Spin}(p + 1)$ is: $s(\square) = +1$. A basis of a $\text{Spin}(p + 1)$ irreducible representation is provided by the Gel’fand–Zetlin pattern characterized by the maximal weight vectors of the subgroup chain $\text{Spin}(p + 1) \supset \text{Spin}(p) \supset \cdots \supset Spin(2)$. We write the basic vectors as $|j\rangle$, where $|j\rangle$ are the $\text{Spin}(p+1)$ group labels, and the additional labels $|m\rangle$ correspond to $\text{Spin}(p) \supset \text{Spin}(p - 1) \supset \cdots \supset Spin(2)$ subgroup chain weight vectors.

The Abelian group generators $\{U\} = \{U_{[\mu]}|\mu\}$, $|\mu\rangle = 1, 2, \ldots, \frac{1}{2}(p)(p+3)$, can be, in the case of multiplicity free representations, written in terms of the $\text{Spin}(p+1)$-Wigner functions as follows,

$$U_{[\mu]}|\phi\rangle = D_{[\mu]}(|\phi\rangle).$$  

(24)

$\phi$ being $\text{Spin}(p + 1)$ group parameters (e.g. Euler angles).

It is now rather straightforward to determine explicitly the non-compact operators matrix elements, which are given by the following expression:

$$\begin{align*}
\begin{pmatrix}
|j\rangle \cr |m\rangle
\end{pmatrix}^T
\begin{pmatrix}
\{\square\} & \{j\} \\
\{m\} & \{\mu\}
\end{pmatrix}
\begin{pmatrix}
|j\rangle \\
|m\rangle
\end{pmatrix}
&= \left\lceil \begin{array}{c}
\begin{pmatrix}
\{\square\} & \{j\} \\
\{m\} & \{\mu\}
\end{pmatrix} \langle j||T_{[\mu]}|\square\rangle
\end{array}\right\rceil
\langle j||T_{[\mu]}|\square\rangle
\end{align*}$$

(25)

$$\langle j||T_{[\mu]}|\square\rangle \langle \square||j\rangle = \sqrt{\text{dim}(|j\rangle)\text{dim}(|\rangle)} r + \frac{1}{2} (C_2(|j\rangle) - C_2(|\rangle))
\begin{pmatrix}
|\rangle & |\square\rangle \\
|0\rangle & |0\rangle
\end{pmatrix}$$

(26)

where $\langle \rangle$ is the appropriate ‘3j’-like symbol for the $\text{Spin}(p + 1)$ group, and $r$ denotes a label of the $\mathfrak{SL}(p + 1, R)$ principal series representation. The (unitary) infinite-dimensional
representations of the $SL(p+1, R)$ algebra are given by these expressions of the non-compact generators together with the well-known representation expressions for the maximal compact $Spin(p+1)$ algebra generators. Finally, we apply the deunitalizing automorphism for a correct physical interpretation.

The very fact that the $SL(p+1, R)$ generators are constructed in the basis of the maximal compact subgroup $Spin(p+1)$, i.e. in the Hilbert space of square integrable functions, guarantees that they can be exponentiated to the corresponding $SL(p+1, R)$ group representations,

$$D_{\mathfrak{SL}(p,R)}(e^{-i\xi^\mu T^\mu} e^{-i\omega^\mu J^\mu}) = e^{-i\xi^\mu D_{\mathfrak{SL}(p,R)}(T^\mu)} e^{-i\omega^\mu D_{\mathfrak{SL}(p,R)}(J^\mu)}.$$  \hspace{1cm} (27)

In the case of the multiplicity free $SL(p+1, R)$ representations, each $Spin(p+1)$ subrepresentation appears at most once and has the same $s$-parity. This feature is especially useful for the task of reducing infinite-dimensional spinorial and tensorial representations of the $SL(p+1, R)$ group to the corresponding $SL(p, R)$ subgroup representations.

We now present just a few examples of the simplest $SL(p+1, R)$ spinorial representations in terms of the corresponding $Spin(p+1)$ subgroup representations,

- $p = 2$ : $D_{\mathfrak{SL}(3,R)} \supset D_{Spin(3)}^2 \oplus D_{Spin(3)}^6 \oplus D_{Spin(3)}^{10} \oplus \cdots$
- $p = 3$ : $D_{\mathfrak{SL}(4,R)} \supset D_{Spin(4)}^2 \oplus D_{Spin(4)}^6 \oplus D_{Spin(4)}^{12} \oplus \cdots$
- $p = 4$ : $D_{\mathfrak{SL}(5,R)} \supset D_{Spin(5)}^4 \oplus D_{Spin(5)}^{10} \oplus D_{Spin(5)}^{16} \oplus \cdots$
- $p = 7$ : $D_{\mathfrak{SL}(8,R)} \supset D_{Spin(8)}^8 \oplus D_{Spin(8)}^{56} \oplus D_{Spin(8)}^{224} \oplus \cdots$
- $p = 9$ : $D_{\mathfrak{SL}(10,R)} \supset D_{Spin(10)}^{16} \oplus D_{Spin(10)}^{144} \oplus D_{Spin(10)}^{720} \oplus \cdots$

where the $Spin(p+1)$ representation superscript denotes its dimensionality.

7. The spinning string case

Let us finally address the question of a group-theoretical approach to construction of spinning $p$-brane infinite-dimensional Lie algebras that generalize the Virasoro, and Neveu–Schwarz–Ramond algebras, and superalgebras, respectively.

Fradkin and Linetsky [24] proposed a method of constructing infinite-dimensional Lie algebras (of the Virasoro type) by analytic continuation of the finite classical algebras in the space of weight diagrams. This method fails for $\mathcal{Diff}_0(p+1, R)$ and/or $SL(p+1, R)$ algebras, since in these cases there are no finite-dimensional weight diagrams to be continued to an infinite system.

We have explicitly constructed above the infinite-dimensional spinorial and tensorial representations of the $SL(p+1, R)$ group, over which the full $p$-brane invariance $\mathcal{SDiff}_0(p+1, R)$ is realized nonlinearly. There are two relevant facts: (i) an action of the $\mathcal{SDiff}_0(p+1, R)$ generators leaves the $\mathfrak{SL}(p+1, R)$ group representation space $V_{\mathfrak{SL}(p+1, R)}$ invariant, and (ii) the $\mathcal{SDiff}_0(p+1, R)$ generators $L_{nk}^{(1\ldots n-1, n = 2, \ldots, \infty}$ transform w.r.t. $\mathfrak{SL}(p+1, R)$ subalgebra generators $L_{(2)k}$ as components of an irreducible tensor operator.

On the basis of these two facts, we propose the following procedure to construct the infinite $p$-brane Lie algebras/superalgebras:

(a) introduce an infinite set of operators characterized by the $\mathfrak{SL}(p+1, R)$ group representation labels;
(b) require these operators to have commutation relations with the $L_{(2)k}$ generators as components of an irreducible tensor operator;
We demand that these operators satisfy mutually, as well as with the $\mathfrak{SL}(p+1, R)$ generators, the (graded) Jacobi relations.

We demonstrate now this three-step procedure in the well-known, $p = 1$, case of the spinning string Virasoro and Neveu–Schwarz–Ramond algebras.

7.1. Irreducible representations of the $\mathfrak{SL}(2, R)$ group

We present here a concise derivation of a new form of the infinite-dimensional $\mathfrak{SL}(2, R)$ algebra representations. The representation expressions are given in terms of complex numbers, and appear as square roots of the usual expressions. This form of the $\mathfrak{SL}(2, R)$ group representations simplifies the construction of the spinning string infinite (graded) algebras considerably.

The commutation relations of the $\mathfrak{SL}(2, R)$ algebra \{\(J_0, T_\pm\)\} read

\[
[J_0, T_\pm] = \pm T_\pm [T_+, T_-] = -2J_0.
\]

According to the Iwasawa decomposition, \(G = NAK\), where \(N, A, K\) are nilpotent, Abelian and maximal compact subgroups, respectively. Any group element \(g \in G\) can be written as

\[
g = n(v)a(\lambda)k(\gamma) = \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \exp \left(\frac{v}{2}\right) & 0 \\ 0 & \exp \left(-\frac{v}{2}\right) \end{pmatrix} \begin{pmatrix} \cos \left(\frac{\gamma}{2}\right) & -\sin \left(\frac{\gamma}{2}\right) \\ \sin \left(\frac{\gamma}{2}\right) & \cos \left(\frac{\gamma}{2}\right) \end{pmatrix}.
\]

The differential forms of the group generators and the Casimir operator, in terms of the above parameters, are

\[
J_0 = i \frac{\partial}{\partial \gamma}, \quad T_\pm = e^{\pm i \gamma} \left(\frac{i}{1 - a} \mp \frac{\partial}{\partial \gamma}\right); \quad C^2 = \frac{\partial}{\partial \lambda} \left(\frac{\partial}{\partial \lambda} - 1\right).
\]

The generators matrix elements, in the \(J_0\) eigenstate basis \(f_m(\gamma) = \langle \gamma | m \rangle, m = 0, \pm \frac{1}{2}, \ldots\) \((\frac{\partial}{\partial \gamma} \rightarrow a)\) are as follows:

\(J_0|m\rangle = m|m\rangle, \quad T_\pm|m\rangle = i(a \pm m)|m \pm 1\rangle; \quad C^2|m\rangle = a(a - 1)|m\rangle \quad \forall a.

7.2. Infinite bosonic algebra—Virasoro algebra

Let \{\(E_m|m = 0, \pm 1, \pm 2, \ldots\)\} be an infinite set of operators, such that \([E, E] \subset E\), which transform as components of $\mathfrak{SL}(2, R)$ irreducible tensor operator,

\[
[J_0, E_m] = mE_m, \quad [T_\pm, E_m] = i(a \pm m)E_{m \pm 1}.
\]

The Jacobi relation for \((J_0, E_m, E_n)\) implies

\[
[E_m, E_n] = A_{m,n}E_{m+n} + C_m \delta_{m+n,0},
\]

while the Jacobi relation for \((T_+, E_m, E_n)\) implies

\[
(a + m + n)A_{m,n} = (a + m)A_{m+1,n} + (a + n)A_{m,n+1}, \quad (a + m)C_{m+1} + (a + n)C_m = 0 \quad m + n + 1 = 0.
\]

There is a solution of these relations for \(a = -1\), and finally, we arrive at the Virasoro algebra, i.e.

\[
[E_m, E_n] = (m - n)E_{m+n} + dm(m^2 - 1)\delta_{m+n+1,0}, \quad d \in R.
\]
7.3. Infinite superalgebra—Neveu–Schwarz–Ramond superalgebra

Let \( \{ E_m | m = 0, \pm 1, \pm 2, \ldots \} \) and \( \{ S_\mu | \mu = \pm \frac{1}{2}, \pm \frac{3}{2}, \ldots \} \) be infinite sets of operators, such that \( [E, E] \subset E, [E, S] \subset S \) and \( [S, S] \subset E \), which transform as components of \( SL(2, R) \) irreducible tensor operators,

\[
[J_0, E_m] = mE_m, \quad \quad [T_{\hat{m}}, E_m] = i(a \pm m)E_{m \pm 1},
\]

\[
[J_0, S_\mu] = \mu S_\mu, \quad \quad [T_{\hat{\mu}}, S_\mu] = i(b \pm \mu)S_{\mu \pm 1}.
\]

The Jacobi relation for \( (J_0, S_\mu, S_\nu) \) implies

\[
\{ S_\mu, S_\nu \} = 2E_\mu + \nu + \frac{1}{4} d' \left( \mu^2 - \frac{1}{4} \right) \delta_{\mu \nu}, 0,
\]

while the Jacobi relation for \( (T_+, E_m, S_\mu) \) implies

\[
(b + m + \mu)F_{m, \mu} = (a + m)F_{m+1, \mu} + (b + \mu)F_{m, \mu+1},
\]

and for \( a = -1, b = -\frac{1}{2} \) one has

\[
F_{m, \mu} = \left( \frac{m}{2} - \mu \right) F.
\]

The Jacobi relation for \( (E_m, E_n, S_\mu) \) implies

\[
\left( \frac{n}{2} - \mu \right) \left( \frac{m}{2} - n - \mu \right) F^2 = (m - n) \left( \frac{m}{2} + \frac{n}{2} - \mu \right) F + \left( \frac{m}{2} - \mu \right) \left( \frac{n}{2} - m - \mu \right) F^2.
\]

For \( F = 1 \) one has

\[
[E_m, S_\mu] = \left( \frac{m}{2} - \mu \right) S_{m+\mu}.
\]

The Jacobi relation for \( (S_\mu, E_m, S_\nu) \) implies

\[
d' \left( \frac{m}{2} - \nu \right) \left( \mu^2 - \frac{1}{4} \right) = d' \left( \mu - \frac{m}{2} \right) \left( (m + \mu)^2 - \frac{1}{4} \right) + 2dm(m^2 - 1),
\]

i.e. \( d' = 4d \).

Finally, we obtain the Neveu–Schwarz–Ramond superalgebra:

\[
[E_m, E_n] = (m - n)E_{m+n} - dm(m^2 - 1)\delta_{m+n, 0},
\]

\[
[E_m, S_\mu] = \left( \frac{m}{2} - \mu \right) S_{m+\mu},
\]

\[
\{ S_\mu, S_\nu \} = 2E_{\mu+\nu} + 4d \left( \mu^2 - \frac{1}{4} \right) \delta_{\mu+\nu, 0} \quad d \in R.
\]
8. Discussion

In this paper we have constructed explicitly spinors of a spinning $p$-brane using a theory of nonlinear realizations of the generale coordinate transformations, and a theory of infinite-dimensional unitary irreducible representations of its linear subgroup. The quantum-mechanical group of the $p$-brane worldvolume preserving diffeomorphisms, $SDif f_0 (p + 1, R)$, is the universal double covering of the corresponding bosonic $SDif f_0 (p + 1, R)$ symmetry. A new proof of the fact that $SDif f_0 (p + 1, R)$ and/or $SL (p + 1, R)$, $p \geq 2$ groups are groups of infinite matrices is presented. The unitary irreducible infinite-dimensional spinorial $SL (p + 1, R)$ representations are presented. A generalization of a deunitarizing automorphism for the $SL (p + 1, R)$ algebras is given, allowing for physically correct unitarity features of the spinorial representations on fields (off-shell). Nonlinear realizations of the $SDif f_0 (p + 1, R)$ group over its maximal linear subgroup $SL (p + 1, R)$ are derived. Various relevant nonlinear realization objects are identified, including the coset space $SDif f_0 (p + 1, R) / SL (p + 1, R) = E \times R^{p+1}$. The parameters $\zeta$ of this coset space are nonlinear realizers, from the group-theoretical point of view, and represent the Goldstone-like fields of the corresponding $p$-brane theory. Indeed, the infinite $SDif f_0 (p + 1, R)$ algebra is replaced by the finite $SL (p + 1, R)$ algebra and an infinite set of Goldstone-like fields. These fields enter the theory via pseudo-frames $E_A^i = (D (L_{\zeta}))^i_A$, which play the role of the frame fields of the gravity theory.

The infinite-component spinors and tensors, presented in this paper, are of importance when infinite spinning $p$-brane algebras are considered. The generators of these (graded) algebras transform as irreducible tensors of the $Spin (1, p)$ subgroup, which are mutually organized according to the $SL (p + 1, R)$ spinorial and/or tensorial representations. Derivations of the infinite-dimensional linear group representations, as well as the corresponding infinite bosonic and fermionic algebras, in the simplest $p = 1$ case, are presented.

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