Interfacial elastic J integral for indentation test

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Abstract

Recently, the indentation test has drawn attention as a new method for evaluating the interfacial fracture toughness, since its application enables the in-situ measurement using a Vickers hardness tester. An empirical formula for the stress intensity factor or J integral has been utilized to evaluate the interfacial fracture toughness from the indentation test. However, the method’s applicability was restricted to a certain range of material combinations, which motivated this study. The object of this study is to find a general form of the interfacial J integral that can be applied to a wide range of material combinations. To accomplish this, the J integral was formulated by solving the interfacial fracture problem of the dissimilar material plate-shaped model with semi-elliptical crack located at the interface that is subjected to an indentation load modeled by point loads. Arbitrary material combinations and crack geometries were considered here. The validity of the J integral introduced here was confirmed by comparing to the numerical J integral evaluated by three-dimensional finite element analysis.

Keywords: Dissimilar material, Interfacial fracture toughness, Indentation test, Interfacial elastic J integral, Stress intensity factor, Finite element analysis

1. Introduction

Interface strength in dissimilar materials is an essential issue in multilayered structures and composite materials. Thus, when these advanced materials are developed, the associated interface strength is generally evaluated by tensile and bending tests (Cao, et al., 1989). However, those testing methods are also known to involve the following challenges: a specimen machined into a special shape from the multilayered structure and the composite material must be prepared prior to conducting the tests. Additionally, a loading jig must be specially designed owing to the required application of a tensile load normal to the interface or a shear load parallel to the interface of the specimen. However, it is difficult to precisely control a fracture plane along the interface in the tests.

There are interfacial fracture toughness tests based on fracture mechanics serving as alternative interfacial strength tests that could overcome such difficulties (Charalambides, et al., 1989; Reeder, et al. 1990). For instance, a double cantilever beam (DCB) test is a common method that consists of pushing a wedge into the pre-crack tip as a crack opening. The associated fracture toughness is evaluated from the relationship between the indentation load and the displacement (Valoroso, et al., 2013). The brazil-nut-sandwich (BNS) test is another method, where a specimen is fabricated by joining half disks of different materials and inserting a pre-crack in the bonded plane. Then, the specimen is subjected to a tensile or compressive loading until a fracture along the interface occurs (Yuuki, et al., 1994). The former method can simply evaluate the interfacial fracture toughness, while the latter method can evaluate the fracture toughness under mixed-mode conditions as well by changing the angle between the loading line and the interfacial crack plane. However, these methods are still limited by the common issue of not enabling in-situ measurements of interfacial fracture toughness.

Recently, the indentation test method has drawn attention providing a new approach (Liu, et al., 2006) as in-situ measurements of the interfacial fracture toughness can be carried out using a Vickers hardness tester. The principles of this test method are illustrated in Fig. 1 (Yamazaki, et al., 2013). The test procedure is as follows: the indenter is indented directly at the interface without preparing a specimen with specially machined geometry or special jig and the interfacial
crack is induced naturally by impressing a pyramid-type indenter. The indentation load, impression size, crack length and Vickers hardness, the elastic modulus of the material, and their combinations are essential parameters for identifying the interfacial fracture toughness based on the stress intensity factor $K$ or elastic $J$ integral.

Lesage and Chicot proposed the following empirical formula (Lesage, et al., 2002):

$$K_{IC} = 0.015 \frac{P_c}{a_c^{3/2}} \left( \frac{E}{H} \right)^{1/2},$$

where

$$\left( \frac{E}{H} \right)^{1/2} = \left( \frac{E}{H} \right)_{sub}^{1/2} + \left( \frac{E}{H} \right)_{coat}^{1/2},$$

$K_{IC}$ is the interfacial fracture toughness based on the stress intensity factor, $E$ is the Young’s modulus, $H$ is the Vickers hardness and the lower subscripts $sub$ and $coat$ correspond to the physical properties of the substrate and coating, respectively, $P_c$ is the critical indentation load for which the interfacial crack is initiated, and $a_c$ is the half-length of the crack observed from the surface associated with the indentation critical load. This is an empirical formula introduced under the assumption of linear fracture mechanics. The applicability of this formula was verified for the material combination of metal coating/metal substrate. It should be noted that the Vickers hardness of the coating is very close to that of the metal substrate (Dexarecaux, et al., 1996). However, it has been pointed out that the combination of ceramic coating/metal substrate—characterized by a large difference in hardness—is not suitable to the application of Eq. (1) (Yamazaki, et al., 2013).

Another formula to evaluate the interfacial fracture toughness has been proposed by Arai (Kurihara and Arai, 2014), as follows:

$$K_{IC} = g \frac{P_c}{c \sqrt{2\pi a_c}} \cosh(\pi \epsilon),$$

where the bi-material constant $\epsilon$ is expressed as follows:

$$\epsilon = \frac{1}{2\pi} \ln \left[ \frac{\kappa_{coat} G_{sub} + G_{coat}}{\kappa_{sub} G_{coat} + G_{sub}} \right],$$

where $\kappa_I$ is the Kolosov constant, which can be expressed as follows under plane stress conditions:

$$\kappa_I = \frac{(3 - \nu_I)}{(1 + \nu_I)},$$

where $c$ is the half of the diagonal length of the impression, $G_i$ is the modulus of rigidity, and $\nu_I$ is the Poisson’s ratio, and the lower subscripts $sub$ and $coat$ correspond to the physical properties of the substrate and coating, respectively. It was found that Eq. (3) is applicable for a wide range of material combinations involving ceramic coating/metal substrate. However, to determine the correction factor $g$ included in Eq. (3), an additional separate test must be conducted by using a DCB specimen, e.g., which introduces additional complications when Eq. (3) is used to assess the interfacial fracture toughness by indentation tests.

The elastic $J$ integral is another important interfacial fracture parameter and is given as:

$$J = \frac{1}{16} \left( \frac{1 + \kappa_1}{G_1} + \frac{1 + \kappa_2}{G_2} \right) K K$$

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where \( K \) is a complex expression of the mixed-mode stress intensity factor: \( K_I + iK_{II} \) and \( (\overline{\cdot}) \) is a complex conjugate. This elastic \( J \) integral effectively satisfies the small-scale conditions in the plastic region at the tip of the interfacial crack. The benefit of using the elastic \( J \) integral is the removal of mode dependency included in the \( K \)-based interfacial fracture toughness, which facilitates the evaluation of the interfacial fracture toughness from fracture tests.

The object of this study is to provide a more general form of the interfacial elastic \( J \) integral—which does not include the semi-empirical factor—for an arbitrary shape of interfacial semi-elliptical crack, arbitrary material combination, and specimen size utilized in indentation tests. The interfacial fracture toughness for a multilayered structure and composite materials with arbitrary combinations could be precisely evaluated from both surface crack size and indentation load obtained from a Vickers indentation test by using the interfacial elastic \( J \) integral introduced in this study.

### 2. Formulation of the indentation test

#### 2.1 Indentation mechanics

In this study, an indentation load generated by the indenter was modeled by a point load vector in order to simplify the indentation mechanics.

In the indentation test, the impression shown in Fig. 2 (a) was assumed to remain on the surface of the target. The impression is marked with the diagonal half-length \( c \) by being subjected to the indentation load \( P \). The Cartesian coordinate \( (x_1, x_2) \) was set such that each axis is parallel to the diagonal of the impression and the \( x_3 \) axis was extended to the depth direction of the target. As the other local coordinate, \( (\xi, \eta) \) was considered as \( (x_1, x_2) \) rotated counterclockwise with an angle of 45°. Fig. 2 (b) indicates an \( A-A' \) plane cut along the \( \xi \) axis shown in Fig. 2 (a). The impression depth was assumed to be \( \delta \), and the ratio \( (\delta/c) \) was a parameter characterizing the tip shape of the indenter.

Under the assumption that the indentation load is indented onto the surface of the target along the \( x_3 \) axis, the reaction force normal to the contact plane of the indenter is \( N \) and the friction force is \( \mu N \). The balance of the force along the \( x_3 \) axis has to satisfy the following relationship:

\[
\sum Z = P - 4 \times N \sin \theta - 4 \times \mu N \cos \theta = 0
\]  

(7)

where \( \mu \) is the friction coefficient and \( \theta \) is the half angle at the tip of the indenter, which can be expressed as follows:

\[
\tan \theta = \frac{c}{\sqrt{2} \delta}
\]  

(8)

The reaction force at the contact plane is obtained by solving Eq. (7):

\[
N = \frac{P}{4(\sin \theta + \mu \cos \theta)}
\]  

(9)

The component along the \( \xi \) axis of this reaction force can be expressed as follows:

\[
N_{\xi} = N(\cos \theta - \mu \sin \theta)
\]  

(10)
Substituting Eq. (9) into Eq. (10) leads to:

\[ N_\xi = \frac{P}{4} \frac{1 - \mu \tan \theta}{\tan \theta + \mu} \tag{11} \]

Thus, the component along the \( x_2 \) axis of this force \( N_\xi \) becomes:

\[ N_2 = N_\xi \frac{1}{\sqrt{2}} = \frac{P}{4\sqrt{2}} \left( \frac{1 - \mu \tan \theta}{\tan \theta + \mu} \right) \tag{12} \]

and the resultant force \( F_2 \) along the \( x_2 \) axis, which can be generated by the indentation load \( P \), is:

\[ F_2 = 2N_2 = \frac{\sqrt{2} P}{4} \left( \frac{1 - \mu \tan \theta}{\tan \theta + \mu} \right) \tag{13} \]

By substituting Eq. (8) for the angle \( \theta \) in Eq. (13), we obtain the following equation:

\[ F_2 = \frac{1}{2} P \left( \frac{\delta}{c} \right) \left( \frac{1 - \frac{1}{\sqrt{2}} \left( \frac{c}{\delta} \right)}{1 + \frac{\mu}{\sqrt{2}} \left( \frac{\delta}{c} \right)} \right) \tag{14} \]

The resultant force \( F_3 \) along the \( x_3 \) axis can be easily derived:

\[ F_3 = P \tag{15} \]

### 2.2 Two-dimensional interfacial crack solutions

Firstly, let us consider the two-dimensional interfacial crack problem shown in Fig. 3. A semi-infinite plate \( (x_2 \geq 0, |x_1| < \infty) \) labeled as #1 with a modulus of rigidity of \( G_1 \) and Kolosov constant of \( \kappa_1 \) is perfectly bonded with another semi-infinite plate \( (x_2 \leq 0, |x_1| < \infty) \) labeled as #2 with \( G_2 \) and \( \kappa_2 \) along the \( (x_2 = 0, |x_1| < \infty) \) plane. Here, the Kolosov constant is defined as follows:

\[ \kappa_i = \begin{cases} 
(3 - 4\nu_i) & \text{for plane the strain condition} \\
\frac{3 - \nu_i}{1 + \nu_i} & \text{for plane the stress condition} 
\end{cases}, (i = 1, 2) \tag{16} \]

It is assumed that the interfacial crack is laid on the interval \( (|x_1| \leq a, x_2 = 0) \) and the concentrated force pair \((p, q)\) is acting at \( (x_1 = s(< |a|), x_2 = 0) \). The complex stress intensity factor for this problem is well known (Rice and Sih, 1965):

\[ K = K_i - iK_{ii} = (p - iq)A\exp(iB), \tag{17} \]

where

\[
\begin{align*}
A &= \sqrt{\frac{a + s}{\pi a(a - s)}} \\
B &= \varepsilon \ln \left[ \frac{2a(a - s)}{a + s} \right] \\
\varepsilon &= \frac{1}{2\pi} \ln \frac{G_2 \kappa_1 + G_1}{G_1 \kappa_2 + G_2}
\end{align*}
\tag{18} \]

![Fig. 3 Interfacial crack subjected to a concentrated load pair](image-url)
In the case of a concentrated load located at the center of the interfacial crack, Eq. (17) becomes:

\[ K = K_I - iK_{II} = (p - iq) \frac{1}{\sqrt{\pi a}} \exp(i \varepsilon \ln(2a)) \]  

(19)

Thus, the substitution of Eq. (19) into Eq. (6) leads to the following expression:

\[ J = \frac{p^2 + q^2}{16\pi a} \left( \frac{1 + \kappa_1}{G_1} + \frac{1 + \kappa_2}{G_2} \right) \]  

(20)

Here, it should be noticed that the elastic \( J \) integral was termed the interfacial elastic \( J \) integral. The assumption of a plane stress gives us the following equation:

\[ J = \frac{p^2}{2\pi a} \left( \frac{1}{E_1} + \frac{1}{E_2} \right) \]  

(21)

which naturally leads to the following proportional relation:

\[ J \propto \frac{1}{a}, (\frac{1}{E_1} + \frac{1}{E_2}), p^2 \]  

(22)

Consequently, let us consider a two-dimensional semi-infinite crack problem, as shown in Fig. 4. A quarter plate \( (x_2 \geq 0, x_1 \geq 0) \) labeled as \#1 with a Young’s modulus of \( E_1 \) and Poisson’s ratio of \( \nu_1 \) is bonded perfectly with another quarter plate \( (x_2 \leq 0, x_1 \geq 0) \) labeled as \#2 with \( E_2 \) and \( \nu_2 \) along the \( (x_2 = 0, x_1 \geq 0) \) plane. It is assumed that the interfacial crack is laid on the interval \( (0 \leq x_1 \leq b, x_2 = 0) \), and the concentrated force pair of \( (Q, P) \) is acting at the edge \( (x_1 = 0, x_2 = 0) \). The interfacial elastic \( J \) integral for this problem is known to be expressed as follows (Nachman and Walton, 1980):

\[ J_0 = \frac{16(1 - \nu_1^2)P^2}{\pi(\pi^2 - 4)E_1b} \]  

(26)

The combination of Eqs. (24), (25), and (26) leads to the following relationship:

\[ \frac{J}{J_0} = \frac{8P^2}{\pi(\pi^2 - 4)b} \left( \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \right) \]  

(27)
Consequently, Eq. (23) can be rewritten as follows:

\[
J = \frac{8}{\pi(\pi^2 - 4)b}\left(\frac{1}{E'_1} + \frac{1}{E'_2}\right)\left[\left(\frac{\alpha^2(\pi^2 - 4)^2}{4(\pi^2 - 4\alpha^2)} + 1\right)P^2 - \pi PQ + \left(\frac{\pi}{2}\right)^2 Q^2\right]
\]  

(28)

where

\[
E'_i = \frac{E_i}{1 - \nu_i} \quad (i = 1, 2)
\]

which also leads to the following proportional relation:

\[
J \propto \frac{1}{b}\left(\frac{1}{E'_1} + \frac{1}{E'_2}\right)P^2, PQ, Q^2
\]

(29)

2.3 Interfacial elastic J integral for indentation tests

The J integral for the indentation test is formulated based on Eqs. (22) and (29), which are the solutions provided to the two-dimensional interfacial crack problem. In order to determine the J integral suitable for the indentation test, the dimensional considerations lead to the following proportional relationship:

\[
J \propto \frac{1}{b}\left(\frac{1}{E'_1} + \frac{1}{E'_2}\right)
\]

(22)

Then, the form inferred from these relationships can be put as:

\[
J = \eta_1\left(\frac{F_2}{t_e}\right)^2, \eta_2\left(\frac{F_3}{t_e}\right)^2, \eta_3\left(\frac{F_3}{t_e}\right)^2\cdot \left[\frac{1}{E'_1} + \frac{1}{E'_2}\right]
\]

where \(G_i (i = 1, 2, 3)\) are coefficients depending on a crack geometry. Thus, if it is assumed that the polynomial could be resolved into factors, the former expression can be rearranged as follow:

\[
J = \eta_1\frac{F_2}{t_e} + \eta_2\frac{F_3}{t_e} + \eta_3\frac{F_3}{t_e}^2\cdot \left[\frac{1}{E'_1} + \frac{1}{E'_2}\right]
\]

(30)

where \(r\) is the equivalent semi-circle radius converted from the area of the semi-elliptical crack \(\pi ab\):

\[
r = \sqrt{ab},
\]

(31)

\(F_2\) is the component normal to the crack plane, and \(F_3\) is the component along the indentation direction of the indentation load \(P\). These components are represented by Eqs. (13) and (14). Moreover, \(\eta_i (i = 1, 2, 3)\) are the correction factors and \(t_e\) is the equivalent thickness that relates the two-dimensional interfacial crack model to the three-dimensional elliptical semi-crack model. These parameters will be identified from the numerical results obtained by finite element analysis, using a procedure that is described in the next section.

3. Evaluation of interfacial elastic J integral by FE calculations

3.1 Numerical procedure

Fig. 5 shows the half model where a point load pair \((F_2, F_3)\) modeling the indentation load is acting at the center of the interfacial elliptical semi-crack. The symbols for the geometry of the model and the semi-crack are shown in Fig. 5.
Fig. 6 shows an example of an FE mesh for the half model considered in this study. The model was divided into hexahedron elements. The region around the interfacial crack tip was divided into particularly fine elements compared to the rest of the model. The minimum mesh size was 1 μm. Typical material combinations and model geometry values are listed in Table 1. Here, the friction coefficient was assumed to be 0.5. This is a typical value for the actual friction coefficient. In the present analysis, the material combination and model geometry set were varied at the center of the range of values listed in Table 1. The boundary conditions were adapted such that the bottom surface \( y = t, (\forall x, \forall z) \) was fixed to the direction of the \( y \) axis, the symmetry plane \( x = 0, (\forall y, \forall z) \) was fixed to the direction of the \( x \) axis, and the point \( (W, t, h) \) was fixed to avoid the ridged motion of the FE model. Numerical analysis was conducted by using the commercial finite-element software MARC 2010 (MSC Software Corporation). The local \( J \) integral was evaluated on the basis of the domain integral method (Nahta and Moran, 1993). The accuracy of the \( J \) integral evaluation was verified via the comparison of the Raju-Newman solution (Newman and Raju, 1981) of the surface elliptical semi-crack in the three-dimensional homogeneous plate subjected to a uniform tensile stress.

| Table 1 Typical values of the model geometry and material combination |
|---------------------------------------------------------------|
| Model thickness \( t \) (mm)                      | 10 |
| Model height \( d_1, d_2 \) (mm)             | 30 |
| Model width \( W \) (mm)                    | 40 |
| Young's modulus (#1) \( E_1 \) (GPa)       | 2.15 |
| Young's modulus (#2) \( E_2 \) (GPa)       | 69.2 |
| Friction coefficient \( \mu \)              | 0.5 |
| Indenter tip geometry \( \delta/c \)    | \( \sqrt{1.5} \approx 1.23 \) |

3.2 Numerical results

3.2.1 Effect of the interfacial crack geometry

The influence of the interfacial crack geometry on the \( J \) integral is discussed first. The aspect of \( a/b \) was varied under the boundary conditions presented in Table 1. Fig. 7 indicates typical results for the variation in an average \( J \) integral (shown simply as the \( J \) integral in this graph) value along the crack front line with the indentation load. It was confirmed that the \( J \) integral increases with as the square value of the indentation load and that the aspect ratio of the crack shape strongly affects the averaged \( J \) integral value.
Fig. 7. Variation in the averaged $J$ integral as a function of the indentation load under various combinations of crack shapes

Fig. 8 indicates the variation in the localized $J$ integral as a function of the angle $\phi$ measured from the model surface. Since the $J$ integral value changed as the square of the indentation load without depending on the aspect ratio of the crack shape, the $J$ integral value calculated for the indentation load of 1.0 kN was regarded as a typical result henceforth. The graph indicates that the $J$ integral in case of a semi-circle crack shape ($a/b = 1.0$) takes its maximum value at the surface and then approaches an almost constant value as the angle increases. The increase in the depth in a semi-elliptical crack shape ($a/b < 1.0$) leads to the $J$ integral decreasing towards the deepest part of the crack shape. On the contrary, the maximum $J$ integral value shifts from the surface to the deepest point with the increasing aspect ratio ($a/b > 1.0$).

Fig. 8 Influence of aspect ratio of the crack geometry on the localized $J$ integral

### 3.2.2 Effect of the model geometry

The influence of the model thickness $t$ on the $J$ integral is discussed here. Fig. 9 (a) shows the variation in the localized $J$ integral with the angle increasing from the surface in the case of a crack geometry characterized by $a = 6.0$ mm and $b = 3.0$ mm. The localized $J$ integral increases with the angle without depending on the model thickness. It is also found that the localized $J$ integral increases as the model thickness becomes thinner and that the $J$ integral calculated in the case of $t = 4.0$ mm is especially affected in comparison with other cases. However, the results in the case of $a = 3.0$ mm and $b = 6.0$ mm indicate that the $J$ integral value is almost unaffected by the model thickness, as shown in Fig. 9 (b).
The alternative ratio \((r/t)\) was introduced in order to consider the combination of every analysis condition. This ratio parameter was related to the normalized \(J\) integral divided by the averaged \(J\) integral obtained for the thickest case \((t = 40 \text{ mm})\), which is shown in Fig. 10. Here, the normalized \(J\) integrals were averaged along the crack front line. It was confirmed that the normalized \(J\) integral increases with the ratio \((r/t)\)–which is smaller than 0.5–without depending on the crack shape. This relationship also suggests that the specimen geometry should be \((r/t) \leq 0.5\), which restricts the specimen thickness.

Next, the influence of the model height \(d_i\) on the interfacial elastic \(J\) integral is discussed. Fig. 11 (a) shows the relationship between the ratio \((d_1/r)\) and the normalized \(J\) integral. The normalized \(J\) integral decreases as the \((d_1/r)\) ratio increases without depending on the crack shape and consequently approaches 1.0 as the ratio becomes \((d_1/r) > 4.0\). Fig. 11 (b) shows the relationship between the \((d_2/r)\) ratio and the normalized \(J\) integral. In this case, the normalized \(J\) integral also decreases as the \((d_2/r)\) ratio increases. Thereby, the normalized \(J\) integral quickly approaches 1.0 for \((d_2/r) > 2.0\), which is strongly affected by the material combination. Thus, this result suggests that the specimen geometry should have at least \((d_i/r) > 4.0\), which restricts the specimen height.
3.2.3 Effect of the material combination

Here, the influence of the material combination on the interfacial elastic $J$ integral is discussed. The reciprocal relationship of the elastic modulus $\left(1/E_1 + 1/E_2\right)$ and the Dunders parameter $\alpha$ characterize a material combination.

Fig. 12 indicates the relationship between the reciprocal relationship of the elastic modulus and the averaged $J$ integral along the interfacial crack front line. It was confirmed that the $J$ integral is proportional to the reciprocal relationship, which means that the $J$ integral increases linearly as the mismatch of the material combination becomes larger. This relationship can be recognized for every crack shape. The relationship between the $J$ integral and the Dunders parameter $\alpha$ is shown in Fig. 13. It was found that both parameters could be described with a single quadratic curve and independent of the material combination and aspect ratio of the crack shape.

[Graphs showing the relationship between the $J$ integral and the reciprocal elastic modulus, as well as the Dunders parameter $\alpha$.]
3.2.4 Determination of the correction factors

The correction factors \((t_e, \eta_1, r)\) involved in Eq. (30) are determined based on FE analysis. First, \(t_e\) with a length dimension was considered. This factor is a parameter relating the crack shape in a three-dimensional interfacial crack model to one in a two-dimensional model with a unit thickness shown in Figs. 3 and 4. In this study, the characteristic length \(t_e\) was defined by the representative crack radius \(r\), as follows:

\[
t_e \equiv r = \sqrt{ab}
\]

Next, let us consider the correction factor \(\eta_1\). From the FE results, it was found that the increasing mismatch of the material combination led to the increase in the \(J\) integral, which varied with the square of \(\alpha\) in the Dunders parameter. It was determined that \(\eta_1\) is a function of the square of \(\alpha\). This consideration and the best fit of the FE analysis results determined the following:

\[
\begin{align*}
\eta_1 &= (1 + 0.1467\alpha^2) \\
\eta_2 &= 0.5311 \\
\eta_3 &= -0.1103
\end{align*}
\]

Fig. 14 shows a comparison between the \(J\) integral assessed from the three-dimensional FE analysis and the \(J\) integral evaluated from Eq. (30). The comparison was carried out for various cases of the tip parameter of the indenter \((\delta/c = 2.45, 22.2,\) and 1.22, corresponding to 16.1°, 22.2°, and 30.0°, respectively). It was confirmed that the \(J\) integral could be precisely evaluated by Eq. (30) independently of the tip parameter of the indenter. This validity was verified for the model geometries of \(t = 4.0, 6.0, 7.0, 8.0, 10.0, 15.0,\) and 20.0 mm and \(d_i = 4.0, 6.0, 8.0, 10.0, 18.0,\) and 30.0 mm, and the material combination \(E_i = 1, 2, 4,\) and 10 GPa.
4 Extraction of the stress intensity factor based on three-dimensional FE analysis

4.1 Interfacial stress intensity factor

The treatment of the linear elastic fracture mechanics with regard to the interfacial crack in this study has been well established (Rice, 1988). The stress distribution in the vicinity of the interfacial crack tip \( (\theta = 0) \) exhibits a singularity as follows:

\[
\sigma_{22} + i\sigma_{12}|_{\theta=0} = \frac{K_I + iK_{II}}{\sqrt{2\pi r}} (\frac{r}{l})^i, \tag{34}
\]

where \( r \) and \( \theta \) are the polar coordinates centered at the crack tip and \( l \) is the representative length, which is typically taken as the crack length (Shih and Asaro, 1988).

The complex stress intensity in-plane factor \( (K_I, K_{II}) \), and the out-of-plane factor \( K_{III} \) can be extrapolated using the following asymptotic forms:

\[
\begin{aligned}
K_I &= \lim_{r \to 0} \sqrt{2\pi r}(\sigma_{22}\cos \gamma + \sigma_{12}\sin \gamma), \\
K_{II} &= \lim_{r \to 0} \sqrt{2\pi r}(\sigma_{12}\cos \gamma - \sigma_{22}\sin \gamma), \\
K_{III} &= \lim_{r \to 0} \sqrt{2\pi r}\sigma_{23},
\end{aligned} \tag{35}
\]

where

\[ Q = \varepsilon \ln \left( \frac{r}{l} \right) \]

Here, the interfacial stress intensity factor is evaluated by applying Eq. (35) to the FE analysis results.

4.2 FE analysis results for the stress intensity factor

![Stress intensity factor variation along the crack front line](image)

Fig. 15 Stress intensity factor variation along the crack front line

The stress intensity factors \( (K_I, K_{II}, K_{III}) \), which varied along the crack front line, are shown in Fig. 15, which shows the case of a homogeneous media \( (E_1 = E_2 = 2.0 \text{ GPa}) \) with a crack shape of \( a = 3.0 \) and \( b = 3.0 \). The indentation load geometry was \( \theta = 30^\circ \), and the indentation load components were \( F_2 = 612 \text{ N} \) and \( F_3 = 1,000 \text{ N} \), respectively. It is clear that the stress intensity factor \( K_I \) (pure crack opening mode) takes an approximately constant value, except near the surface, while the other stress intensity factors are zero. Thus, it was confirmed that the surface elliptical semi-crack in the homogeneous media—which was subjected to the indentation load—was under a condition known as pure Mode-I in linear elastic fracture mechanics. Moreover, Fig. 15 (b) shows the results for a dissimilar material combination of \( E_1 = 2.15 \text{ GPa}, E_2 = 69.2 \text{ GPa} \). The associated stress intensity factor \( K_I \) varies along the crack front line, i.e., it decreases from the surface to the deepest part of the crack plane, while \( K_{II} \) varies with the same trend as \( K_I \) along the negative axis. However, \( K_{III} \) is smaller than that of \( K_I \). The out-of-plane stress intensity factor \( K_{III} \) has the highest value near the surface and immediately approaches zero as it approaches the deepest part of the crack plane. Subsequently,
it was determined that the stress intensity factor near the surface has the highest value under the mixed condition of Mode-I and Mode-II. This mode mixture caused a variation in the interfacial elastic $J$ integral along the crack front line, as shown in Fig. 8.

Finally, the influence of the indenter tip angle on the stress intensity factors is shown in Fig. 16. The tip angle varied from 17.6° to 45.0° with a material combination of $E_1 = 2.15$ GPa and $E_2 = 69.2$ GPa. From this comparison, it was found that the stress intensity factor $K_I$ decreases with the increase in the indenter tip angle, while $K_{II}$ increases with the increase in the tip angle. This means that if the indentation test is conducted with the indenter with a sharp tip angle, the crack mainly initiates and propagates under the crack opening mode. On the other hand, the out-of-plane stress intensity factor $K_{III}$ appeared only near the surface, and then immediately disappeared.

5 Conclusion

The subject of this study was a fracture toughness testing method using a Vickers hardness tester to measure the interfacial strength of multilayered structures and composite materials. Several empirical formulas have been developed for evaluating the fracture toughness. Thus, in this study, the general form of the interfacial elastic $J$ integral—which does not include the semi-empirical factor—was formulated based on familiar solutions of the two-dimensional interfacial crack problem, applicable in indentation tests for arbitrary shapes of the interfacial semi-elliptical crack, arbitrary material combinations, and specimen size. The following complete formula was developed:

$$J = \frac{\eta_1}{\pi r} \left( \frac{\eta_2 F_2 + \eta_3 F_3}{t_e} \right)^2 \left( \frac{1}{E_1} + \frac{1}{E_2} \right),$$

where $r = \sqrt{ab}$, $t_e = r$,

$$\eta_1 = (1 + 0.1467 \alpha^2),$$
$$\eta_2 = 0.5311,$$
$$\eta_3 = -0.1103,$$

$$F_2 = \frac{1}{2} P \left( \frac{\delta}{c} \right) \left( \frac{1 - \mu \frac{1}{\sqrt{2}} \left( \frac{c}{\delta} \right)}{1 + \mu \frac{1}{\sqrt{2}} \left( \frac{\delta}{c} \right)} \right),$$

and $F_3 = P$,

where $\left( \frac{\delta}{c} \right)$ is the tip shape of the indenter and $P$ is the indentation load. FE analysis determined the following restriction for the specimen geometry applicable for the indentation test: $r/t \leq 0.5$ and $d/r \geq 4$.

The usage of this $J$ integral formula obtained in this study enables the evaluation of the interfacial strength of a wide range of material combinations to which the indentation test can be applied.

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