On quantum topology, hypergraphs and flag vectors

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Abstract
Each rule $f$ that assigns a vector $f(G)$ to an $(n+1)$-graph $G$ determines a class (or property) of $n$-manifold invariants. An invariant $v = v(M)$ is in this class if, for any triangulated manifold $|G| = M$, one has that $v(M)$ is a linear function of $f(G)$. This paper defines a flag vector $f(G)$ for $i$-graphs, whose associated invariants might be quantum, and which is of interest in its own right. The definition (via the concept of shelling, and a ‘disjoint pair of optional cells’ rule for the link) seems to apply to any finite combinatorial object, and so to any compact topological object that can be triangulated. It also applies to finite groups.

1 Introduction
The purpose of this paper is to formulate some combinatorial questions related to the theory of quantum manifold invariants, and make progress towards their solution. Briefly, the problem is this. Suppose $v = v(M)$ assigns a number, or more generally a vector, to every compact topological manifold $M$ of some fixed dimension $n$. In particular, if $T$ is a suitable simplicial complex then its geometric realisation $|T|$ will be such a manifold. Thus, the quantity $v(T) = v(|T|)$ is defined for certain simplicial complexes $T$. The problem is to extend $v$ in a useful and instructive way, so that it applies to more general simplicial complexes.

Quantum topology is not yet fully developed. For this reason, it may be best in the above to treat the topological invariant $v$ as an unknown. Instead, suppose that some rule $f = f(T)$ that assigns a vector to each simplicial complex $T$ has been provided. This rule determines a class (or property) of manifold invariants, in the following way.

Each manifold $M$ can be triangulated in many ways. Each triangulation is a solution to the equation $|T| = M$. For certain linear functions $v = v(T) = v(f(T))$ of the $f$-vector of $T$, the value of $v(T)$ will depend only on

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$M = |T|$, and not on the triangulation chosen. In other words, certain linear functions $v$ of $f$ are, on triangulated manifolds, topological invariants.

Each rule $f = f(T)$ that assigns a $f$-vector to a simplicial complex $T$ thus determines a class of topological invariants of manifolds, namely those that are linear functions of $f$. The problem is to formulate a rule that will produce the presently loosely defined class known as the quantum invariants. Note that it can be far easier to define a whole class of invariants in this way, than it is to produce a single invariant that satisfies the membership property. This paper will suggest a solution, but first some general remarks are in order.

Topologists, under the influence of homotopy and homology theory, have developed the concept of a simplicial complex. For the present purpose, however, the concept of a hypergraph is more appropriate. Suppose that some finite set $V$ of vertices is given. A simple $i$-uniform hypergraph on $V$, or $i$-graph for short [4], is simply a collection $C$ of $i$-element subsets of the vertex set $V$. Each element of $C$ will be called a cell (or edge) of the hypergraph. (Thus, a 2-graph is an ordinary graph, while a triangulation of the $n$-manifold $M$ produces an $(n + 1)$-graph. The cells are the vertex sets of the triangulation. A 1-graph is a collection of 1-element subsets, or what amounts to much the same thing, a subset of $V$. A cell of a 1-graph is then a vertex, that has been selected to appear in $C$.) In §2 the flag vector $fG$ of an $i$-graph will be defined. It may be that it corresponds to quantum topology. Some care is taken to justify the definition given. To do this, some arguments to appear in [6] will be summarized.

For this paper it is not so much the flag vector as the concept of a linear function of the flag vector that is of interest. This can be expressed in the following way. Let $\mathcal{G}$ be the module of all formal sums of $i$-graphs on some vertex set $V$. (Sometimes, a formal sum of $i$-graphs will be called an $i$-graph. The prefix ‘actual’ will be used to recover the original concept. A formal sum of two graphs is not the same as their disjoint union.) Now define the nullspace $\mathcal{G}_0$ to consist of all formal sums

$$G = \lambda_1 G_1 + \ldots + \lambda_r G_r$$

in $\mathcal{G}$, such that

$$f(G) = \lambda_1 f(G_1) + \ldots + \lambda_r f(G_r)$$

is zero. A rule $v = v(G)$ that assigns a vector to each $i$-graph is then a linear function of the flag vector $fG$ just in case

$$v(G) = \lambda_1 v(G_1) + \ldots + \lambda_r v(G_r)$$

is zero for every $G$ in $\mathcal{G}_0$. The flag vector is a means of defining $\mathcal{G}_0$. Conversely, the natural map that takes a $i$-graph $G$ to its residue $\mathcal{G}/\mathcal{G}_0$ assigns a vector $fG$ to each $i$-graph, and has $\mathcal{G}_0$ as its null space. This provides the abstract form of the flag vector.
To be useful, the nullspace should be neither too large or too small. If $G_0 = \{0\}$ then any rule $v = v(G)$ will be a linear function of the flag vector, and the concept ceases to have any significance. Similarly, if $G_0 = G$ then zero is the only linear function of the flag vector, and again the concept is without value. The flag vector defined in this paper produces a nullspace with several attractive properties. It may also be that some simple geometric operations will produce a spanning set of nullvectors. This is evidence, independent of the details of quantum topology, that the choice of $G_0$ is correct.

Now suppose that $v = v(M)$ is a vector valued invariant of topological manifolds (and so also a function on hypergraphs whose realisation is a manifold). Even without extending $v$ to all hypergraphs, one can talk of $v$ being a linear function of the flag vector. Inside $G$ form $M$, the span of $n$-graphs whose realisation is a manifold. Define the manifold nullspace $M_0$ to be $M \cap G_0$. If $v$ vanishes on $M_0$, then say that it is a linear function of the flag vector. Such a function can of course be extended in many ways to the space $G$ of all hypergraphs.

The polynomial quantum knot invariants can be computed by means of crossing change rules. Such rules are easy to state and apply. To find the rules is harder, and harder yet is to show that the same answer will result, however a calculation is done. When expressed in Vassiliev form (substitute $q = e^x$ and let $v_i$ be the coefficient of $x^i$), it follows immediately from the form of the crossing change rule that each $v_i$ vanishes on a certain subspace $K_{i+1}$ of the space $K$ of all formal sums of knots, namely that spanned by knots with $i + 1$ double points. (This is because

$$(\sqrt{q} - 1/\sqrt{q}) = x + x^3/24 + \text{higher order}$$

and so each double point contributes a factor of $x$ to a product. This argument appears in [2, Thms. 2 and 3].)

One might wish for a similar theory, applying to manifolds. For knots, both the crossing change formulae and the definition of the nullspaces $K_i$ are easy to state, if not discover. For manifolds the situation is not so clear. In this context, see [11, Prop. 4.6]. The flag vector $fG$ of a hypergraph can be ‘filtered by complexity’ to produce ‘components’ $f_iG$, and this can be used to define subspaces $G_i$ and $M_i$ of $G$ and $M$ respectively. From here one may be able to work backwards, and find first a geometric characterisation of $M_i$, and then a ‘crossing change’ formula for some given quantum topological invariant. This interplay between crossing change formulae and nullspaces seems to be particularly important.
2 Hypergraphs

It is now time to turn to the definition of the flag vector \( f_G \) of a hypergraph \( G \). This will be done in two stages. First, the shelling vector \( \tilde{f}_G \) will be defined. Next, the flag vector \( f_G \) is defined, as a variant on the construction of \( \tilde{f}_G \). Suppose that \( G \) is an \( i \)-graph on \( N \) vertices. Now remove the vertices from \( G \), one at a time, until none are left. This, together with a record of the changes that occur at each step, is a shelling of \( G \). It is analogous to Morse theory in differential topology.

When a vertex \( v \) is removed from an \( i \)-graph, the cells that contain that vertex must also be removed. Thus, a shelling is an ordering of the vertices of \( G \), and for each vertex \( v \) a record of the cells which have \( v \) as their first vertex, for the given ordering. One can also reverse the process, and think of a shelling as a way of building up a graph out of nothing.

When a vertex \( v \) is removed from an \( i \)-graph \( G \), the cells that are removed can be described via an \((i-1)\)-graph on one fewer vertices. (Each removed cell contains \( v \). It will also contain \((i-1)\) of the remaining vertices.) Call this \((i-1)\)-graph the link \( L_v \) at the vertex \( v \) of \( G \). Locally, about \( v \), the graph \( G \) looks like the cone \( CL_v \) on the link \( L_v \). (The cone on a \( j \)-graph is formed by adding a new vertex to the vertex set, and also adding this vertex to all the cells of the \( j \)-graph. The result is a \((j+1)\)-graph.)

The shelling vector \( \tilde{f}_G \) is a sum over the \( N! \) possible shellings of \( G \). For each shelling sequence \( \sigma \) one obtains a sequence \( L_1, L_2, \ldots, L_N \) of links. The link on the \( l \)-th vertex is an \((i-1)\)-graph on \( N-l \) vertices. Loosely speaking, the contribution made by \( \sigma \) to \( \tilde{f}_G \) is this sequence of links, as a formal and noncommutative product. However, each link \( L_l \) is an \((i-1)\)-graph which, by induction, can be supposed to have a shelling vector \( \tilde{f}_{L_l} \).

The basis for the induction are the 0-graphs. These are collections of zero element subsets of the vertex set. There is only one zero element subset, namely the empty set, and so there are only two possible 0-graphs, namely \( \emptyset \) and \( \{\emptyset\} \). (The first is the empty collection of subsets, the second a non-empty collection, whose only element is the empty set.) Call these 0-graphs \([a]\) and \([b]\) respectively.

Now let \( [\ldots, \ldots, \ldots] \) denote a noncommutative but linear formal product. The equation

\[
\tilde{f}_G = \sum_{\sigma} [\tilde{f}_{L_1}, \tilde{f}_{L_2}, \ldots, \tilde{f}_{L_N}]
\]

provides a recursive definition of the shelling vector \( \tilde{f}_G \), in terms of \([\ldots, \ldots, \ldots]\) and the basic 0-graphs \([a]\) and \([b]\). (During calculations, the brackets help keep track of where one is in the recursion. Each term in \( \tilde{f}_G \) is an expression in \( a, b \) and \([\ldots, \ldots, \ldots]\).)

There is an analogy between shelling vectors and Feynman path integrals. In both cases one considers the totality of all ways of achieving something, in the one case a particular hypergraph, in the other some outward form of an
interaction between particles. The final value is a sum of the contributions made by the various ways. In the one case the contribution is considered as a formal entity, in the other it is a number depending on the paths and inner interactions of the particles.

The shelling vector $fG$ is extremely large. It may even be that if

$$G = \lambda_1 G_1 + \ldots + \lambda_r G_r$$

is a formal sum of $i$-graphs, then the associated shelling vector

$$\tilde{f}(G) = \lambda_1 \tilde{f} G_1 + \ldots + \lambda_r \tilde{f} G_r$$

is zero only when $G$ itself is zero. If true, say that the shelling vector distinguishes formal sums (of $i$-graphs). It would be useful to know whether or not this statement is true. It is considerably stronger than saying that if two $i$-graphs $G_1$ and $G_2$ have the same shelling vector, then they are isomorphic. This is the property of distinguishing individual $i$-graphs.

The flag vector $fG$ is obtained by using only part of the information available in each link $L_i$. Indeed, if $\tilde{f} G$ does distinguish formal sums of $n$-graphs, some data in the link will have to be discarded to obtain the flag vector, if not all manifold invariants are to be linear functions of the flag vector.

Here is an example. The shelling vector of the 1-graph that consists of $m$ vertices chosen to be cells out of $n$ vertices is the sum of all ways of multiplying $m$ copies of $b$ and $n-m$ copies of $a$. If $n=2$ then $2aa$, $ab+ba$, and $2bb$ are the possible shelling vectors. (Here, the square brackets are redundant.) Notice that the 1-graphs on $n$ vertices have linearly independent shelling vectors.

How many is an important question to ask of a 1-graph (or subset of the vertices). For $G$ a 1-graph, define $f'G$ to be $a+mb$, where $m$ is the number of cells in the graph $G$. Here $a$ and $b$ are not quite the same symbols are were used for the shelling vector. (The $a$ denotes the 1-graph itself. In a formal sum of graphs, one would like to know how many graphs, which is the sum of the coefficients $\lambda_i$.)

The quantity $f'G$ is a linear function of $\tilde{f} G$. Its kernel is rather interesting. Consider 1-graphs on 2 vertices. One has $0+2 = 1+1$, or $(0+2) - (1+1) = 0$. For shelling vectors the corresponding expression

$$2aa + 2bb - 2(ab + ba) = 2(a-b)(a-b)$$

lies in the kernel. More generally, the symmetric products with two occurrences of $(a-b)$ span the kernel. This may be important later.

The flag vector of a 1-graph can now be defined. It is the sum

$$fG = \sum_{\sigma} f' L_1 \cdot f' L_2 \cdot \ldots \cdot f' L_N$$

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over all shellings of the formal product of the corresponding link contributions $f'L_l$, where $f'$ is as above. In other words, as each shelling removes a vertex, contribute an ‘a’ for the vertex being removed, and a ‘b’ for each edge that is removed. Clearly $fG$ is a formal sum of words in $a$ and $b$, of length $N$. It thus has $2^N$ components.

The components of $fG$ are not independent. For example, if $A$, $B$, $C$ and $D$ are the graphs on three vertices with 0, 1, 2 and 3 edges respectively, then the equation $fA - 3fB + 3fC - fD = 0$ holds. In fact [6], the flag vectors of 2-graphs on $n$ vertices have $p(n)$, the number of partitions of $n$, as the number of independent components. (Thus, any given linear function $v$ of graph flag vectors $fG$ can be written in many ways, as a linear function of words in $a$ and $b$.)

The flag vector is defined not only for graphs, but also for formal sums of graphs. Certain formal sums are useful in the study of flag vectors. Recall that an $i$-graph is a pair $(C, V)$, where $V$ is the vertex set, and $C$ is a collection of $i$-element subsets of $V$. An $i$-graph $G$ with optional cells is defined in the following way. Let $A$ and $B$ be disjoint collections of $i$-element subsets of $V$, or in other words, collections of potential cells. Form the alternating sum

$$G = \sum_{C \subseteq B} (-1)^{\#B - \#C} (A \cup C, V)$$

of the $2^\#B$ graphs $(A \cup C, V)$ whose cell set lies between $A$ and $A \cup B$. (Here, $\#B$ means the number of elements in the set $B$.) To denote an optional edge on a picture of a 2-graph, use a dotted line. The options are to either join (+) or erase (−) the dots.

Now suppose that a 2-graph contains an optional cycle, or in other words a cycle consisting of optional edges. It then follows that its flag vector is zero. (For each shelling, $f'L_l$ will be zero for the first vertex that lies on the cycle of options.) This result, applied to a triangle of optional edges, gives the flag vector equation just recently stated.

For $i$-graphs, the rule for the link contribution $f'L_l$ will be chosen by elevating to a general principle a particular aspect of the case already considered. Let $G_0$, $G_1$ and $G_2$ denote the 1-graphs on 2 vertices, with 0, 1, and 2 cells (selected vertices) respectively. The equation

$$f'G_0 + f'G_2 = f'G_1 + f'G_1$$

(3)

can be interpreted in the following way. Label the vertices $L$ and $R$ (for left and right). Let $G_L$ and $G_R$ be the 1-graphs in which $\{L\}$ and $\{R\}$ are the only cells. Now consider the 1-graph in which both $L$ and $R$ are optional cells. This is a formal sum

$$G_2 - G_L - G_R + G_0$$

whose link contribution $f'$ is, by (3), zero.
Now suppose $G = G_+ - G_-$ is a 1-graph with one optional cell. Clearly, when formally considered, the number of cells in $G$ (the number in $G_+$ less that in $G_-$) is equal to one. Similarly if
\[ G = G_{++} - G_{+-} - G_{-+} + G_{--} \]  
(4)
is a 1-graph with two optional cells, then formally considered $G$ has zero as its number of cells. Clearly, any $m$ celled 1-graph on $n$ vertices can be written as $G_0 + mG_1$, plus various terms of type [1]. No other value for $m$ is possible. (Here, $G_0$ and $G_1$ are 1-graphs on $n$ vertices, with 0 and 1 cells respectively.)

As already noted, in this paper the flag vector is of interest only as a means of defining the concept of a linear function of the flag vector. (When it comes to calculations and proofs, a well chosen system of coordinates may of course be useful.) The same applies to the link contributions. With this in mind, the definition of $fG$ for 2-graphs will be reformulated.

It is as before a sum
\[ fG = \sum \sigma f'_{L_1} \otimes \ldots \otimes f'_{L_N} \]  
(5)
over all shellings. Each link $L_l$ is a 1-graph on $N - l$ vertices. Let $V_l$ be the vector space of formal sums of such graphs (up to isomorphism), modulo the graphs with two optional cells. Let the link contribution $f'_{L_l}$ be the residue of $L_l$ in $V_l$. Each $V_l$ has dimension 2 (except $V_N$, for there the vertex set is empty), and the previous value of $a + mb$ provides a system of coordinates on $V_l$. This new definition differs from the old by an invertible linear transformation.

The concept of a graph with two optional cells applies not only to 1-graphs, but also to $i$-graphs. In the general case, however, a restriction will be placed on the pairs of cells that can be used. Say that two cells are disjoint if they do not share a common vertex. It means that for any shelling, one or the other of the pair of optional cells will appear first. They will not both be removed at the same time. The options are independent in that they do not directly interact. This concept is trivial for 1-graphs.

The flag vector $fG$ of an $i$-graph, in its tensor form, is defined by the recursive formula (3), where now each $f'_{L_l}$ is the residue of $fL_l$, modulo the graphs with a disjoint pair of optional cells. Thus, $f'_{L_l}$ is defined by the $(i - 1)$-graph flag vector nullspace, augmented by the disjoint pairs of optional cells.

This defines a flag vector for $i$-graphs, and thus, as described already, a class (or property) of invariants of compact topological manifolds.

3 Summary and Further Problems

Much remains to be done, before a clear connection can be established between the hypergraph flag vector on the one side and quantum topology on
the other, assuming indeed that there is such a connection. This section provides a summary of what is already known, and a description of some untouched problems. It is an essay that outlines possible future developments.

To begin with what is known: for ordinary or 2-graphs, the flag vector has attractive properties \[6\]. The space spanned by the flag vectors on \(n\) vertices has dimension \(p(n)\), the number of partitions of \(n\). The nullspace of the flag vector can be given a geometric description. For \(n = 4\), the flag vector distinguishes graphs. The flag vectors of the 11 distinct such graphs are the vertices of a convex polytope in an affine 4 (\(= p(4) - 1\)) dimensional space. (Thus, the affine linear functions that define the polytope are non-negative on the flag vectors of actual graphs.) Methods similar to those used for convex polytopes may make similar subtle linear inequalities accessible, for larger \(n\). Finally, if the realisation \(|G|\) is a 1-manifold (\(|G|\) is a disjoint union of polygons), the number of components in \(|G|\) is a linear function of the flag vector \(fG\). (This linear function is quite complicated, when expressed in \(a\) and \(b\) form. This is because \(fG\) has only a restricted view of the connectivity of \(G\).)

For topology, quantum or otherwise, the three basic questions are these. First, \textit{does the shelling vector encode significant topological information?} If it distinguishes formal sums of \(i\)-graphs then the answer is, of course, yes. The second question is this: \textit{does the flag vector make this information available in a useful form?} For example, can the dimension of the span of the flag vectors (and, in coordinates, the span itself) be easily described. The third question is: \textit{what part of the flag vector is a topological invariant of the realisation?}

Suppose now that there is significant information in the shelling vector. A major problem in using it is this. Recall that the shelling vector \(fG\) of an \(n\)-graph is a formal sum of words, where each word is a product of the shelling vectors \(fL_l\) of the links \(L_l\). However the shelling (and flag) vectors of \((n - 1)\)-graphs on an unlimited number of vertices have a span with unlimited dimension. Thus, \(fG\) as a sum of words must be constructed out of an infinite alphabet of letters. This is most likely extremely inconvenient. The passage from \(fL_l\) (or \(fL_l\)) to the vertex contribution \(f'\) reduces this alphabet. Loosely speaking, it corresponds to taking the tail part of the shelling (or flag) vector of the link. It is a truncation.

Clearly, there are many possible ways of truncating \(fL_l\), so that the result is finite dimensional. Each such rule gives rise to a possibly different nullspace or abstract flag vector. Finding the correct rule is an important problem. The \textit{disjoint pair of options} rule presented in §2 ensures that a property useful in the study of 2-graphs continues to hold. (It is something like taking a linear approximation.) For each \(i\), the properties of \((i - 1)\)-graphs must be well understood, before the flag vector of an \(i\)-graph can be given an explicit form.
For a 3-graph on $N$ vertices, its flag vector $fG$ can be written as a sum of length $N$ words in $a$, $b$ and $c$. For each shelling, as each vertex is removed, record the following. First, an ‘$a$’ for the vertex itself. Second, a ‘$b$’ for each cell that is removed. Third, for each ‘pair of cells meeting along an edge’ that is removed, record a ‘$c$’. This follows from properties of 1-graphs. As noted, until 3-graphs are investigated, the flag vector of 4-graphs cannot be given an explicit form.

Topology also studies manifolds that possess additional structure. Suppose, for example, that $M$ is an oriented topological manifold. In that case, a triangulation of $M$ will produce an oriented hypergraph $G$. This means that each cell is given an orientation (a means of attaching a sign to each ordering of its vertices, that is invariant under even permutations). As before, one can define a shelling vector $\tilde{f}G$ for oriented $i$-graphs. This is because at each stage in the shelling, the change can be expressed as the cone on something simpler, namely an oriented $(i - 1)$-graph. In the oriented case, however, the induction will be based on the oriented 0-graphs. These can be denoted by $[a]$, $[b_+]$ and $[b_-]$. (Here $[b_+]$ is $[b]$, which has $\emptyset$ as its only cell, with ‘$+$’ attached as sign to the only ordering of its vertices.)

One can now define the flag vector of an oriented hypergraph. As before, use the recursive formula (5). As before, the link contribution $fL_l$ is $fL_l$ modulo disjoint pairs of options. (When, as here, a set of vertices is the support for several different cells, a basic optional cell is where one has either some cell (+) or no cell (−) supported on these vertices. Other options can be built up using addition.)

It should now be clear that shelling and flag vectors exist more generally. Let $\mathcal{X}$ denote any class of objects that can be described as a union of cells (possibly with additional structure) on a finite set of vertices. Suppose also that the change that occurs at each stage of a shelling can be described as the cone on something. That something should be of some type $\mathcal{X}'$, for which a shelling vector has already been defined. In this situation, $\mathcal{X}$ will also have a shelling vector. If, in addition, $\mathcal{X}'$ has a flag vector, then the disjoint pair of options rule will define a flag vector for $\mathcal{X}$ also.

Thus, for example, each triangulated manifold with boundary will have both a shelling and flag vector. The 0-objects that are the basis for the induction can be represented as $[a]$, $[b]$ and $[b']$. Both $[a]$ and $[b]$ are as before. However, $[b']$ when coned twice will produce first a special sort of 1-object, and then a 2-object (representing a 1-cell) which has the second apex as its ‘boundary’. (Note that the concept of a 0-manifold with boundary is somewhat bizarre.) As before, impose ‘$+$’ and ‘−’ subscripts on the ‘$b$’ terms to obtain the oriented form.

Suppose $M$ is a triangulated differential manifold. In that case, an oriented matroid can be used to provide a combinatorial record of the differential structure. In this case also, surely it is true that shelling and flag vectors can be defined. This the author has not investigated. The Pontrja-
gin numbers, one would wish to be linear functions of the flag vector. By virtue of \([9]\), this problem may be accessible.

It is not necessary that the classes \(X\) and \(X'\) have the same general type. Manifolds with boundary provide a mild example of this. Here is another.

Let \(G\) be a finite group, considered as the ternary relation \(xy = z\) (or, if one prefers, \(xyz = 1\)). As before, a ternary relation can be shelled. The change at each stage of the shelling is once again the cone on something, but what that something is requires some thought. Indeed, for a ternary relation the very concept of a cell requires some thought. The following appears to be correct. Each solution \(S\) to the equation \(xy = z\) (or triple in the ternary relation) has a support, which are the vertices that appear in \(S\). A cell will consist of a (non-empty) set of solutions \(S\), each of which has the same support. A cell is prime if it contains a single solution. Thus, for example, if \(a\) and \(b\) commute and \(ab = c\), then \(\{ab = c, ba = c\}\) is a cell, supported on \(\{a, b, c\}\). For an arbitrary ternary relation, there are \(3! = 6\) prime cells supported on a triple \(\{a, b, c\}\), and so \(2^6 - 1 = 63\) possible cells altogether. Supported on a pair \(\{a, b\}\) there are again 6 prime cells, and so 63 possible cells. Supported on a singleton \(\{a\}\) there is of course only one cell, namely \(\{aa = a\}\).

Consider now the shelling of a ternary relation \(R\). A prime cell such as \(\{(a, b, c)\}\) can be thought of as the cone on the prime cell \(\{(? , b, c)\}\), where \(a\) is the apex of the cone. This then is an example of a prime cell, supported on the pair \(\{b, c\}\) of vertices, for the link (at \(a\)) of a ternary relation. Once again, there will be six possible prime cells for the link, supported on \(\{b, c\}\).

Similarly, there will be six prime cells for the link, supported on \(\{b\}\). To deal with \(\{(a, a, a)\}\) one will, for the link, need the concept of a cell supported on an empty set of vertices. Such will, in a shelling, be removed at step zero, before any other cell. Proceeding in this way one will obtain shelling and flag vectors for ternary relations.

(The prime 0-objects, that are the basis for the induction, are symbols such as \([123]\) or \([223]\), with the property that

\[
C_a C_b C_c [123] = \{(c, b, a)\}; \quad C_a C_b C_c [312] = \{(a, c, b)\};
\]

\[
C_a C_b C_c [232] = \{(b, a, b)\}; \quad C_a C_b C_c [333] = \{(a, a, a)\};
\]

where \(C_x\) is the operator that forms a cone, and labels its apex ‘\(x\)’. Applied to a sequence \([\ldots]\) of numbers and vertices, it replaces ‘1’ by ‘\(x\)’, and lowers the remaining numbers by one. Certain symbols, such as \([121]\), are not needed, and will not be used. Thus, \(C_c [232] = [121];\ C_b [121] = [b1b];\) and

\(C_a [b1b] = [bab]\), which is the prime cell as above. As usual, use the disjoint pair of optional (and not necessarily prime) cells rule to define the link contribution \(f'L_L\).)

The resulting flag vector will clearly be quite complex. This is probably in its favour, for a group is also quite complex, and so a simple flag vector
would not be able to take a good grasp of its structure. The author has not investigated this matter.

It should now be clear that one would expect more or less anything that can be shelled to have a flag vector. Suppose the realisation $|G|$ of $G$ is a (possibly oriented and/or differential) manifold. Now consider the question: what part of the flag vector $f_G$ is a topological invariant of $M = |G|$? To understand how barycentric subdivision will change $f_G$ is an obvious starting point, for topological (differential) invariants will vanish on such changes to $f_G$. Morally, although perhaps not in fact, this is all that is required, to produce the class of invariants that corresponds to the given $f$-vector.

Now note that a ‘$G$ with a barycentric subdivision’ is an object $\hat{G}$ that can be shelled. Presumably, there is a corresponding flag vector $f_{\hat{G}}$, from which both $f_G$ and $f_{G'}$ (the result of the subdivision) can be computed, and thus the change $f(\partial \hat{G}) = f_{G'} - f_G$ in $f_G$ also.

Objects $G$ whose realisation $|G|$ is a manifold are rather special, and in general their flag vectors $f_G$ will span a subspace of all flag vectors. (The precise determination of the subspace is an important problem, which may lead to crossing-change rules. The first step is to produce a spanning set for the manifold nullspace $\mathcal{M}_0$, defined in §1.) The same applies to $\hat{G}$, namely ‘$G$ with a barycentric subdivision’. If the properties of the flag vector allow these subspaces to be determined then, at least morally, $f_G$ modulo all possible $f(\partial \hat{G})$ is an invariant of $M = |G|$.

Before concluding, here are some comments that do not conveniently belong elsewhere. First, the domain of the theory. By using a suitable notion of a cell, one can triangulate and shell any of: a submanifold $N \subset M$; a singular stratified space $X$; the realisation of a simplicial complex. Thus, for example, one can obtain a class of invariants for knots in $\mathbb{R}^3$. For knots in a 3-manifold $M$ more work is required, for here homeomorphism for $N \subset M$ will in general be a weaker concept than isotopy. To allow a 4-manifold to have quantum topology, but not singular algebraic surfaces is, in the author’s view, not reasonable. Infinite objects can be accommodated, provided some factor is inserted into (5), so as to ensure convergence. In addition, only ‘finite partial shellings’ should be used.

Flag vectors are very important in the study of convex polytopes (and the associated algebraic varieties). Here, there are four steps. First, the definition of the flag vector. Second, the linear equations on the flag vector. Third, the linear inequalities and the associated homology theory. Fourth, the pseudopower (usually non-linear) inequalities and the ‘ring-like’ structure.

(For polytopes, the first step is all but trivial. The flag vectors of dimension $n$ polytopes span a space, whose dimension is $\left\lfloor \frac{3}{2} \right\rfloor$ the $(n + 1)$st Fibonacci number. Middle perversity intersection homology and, the author believes, the local-global extension [7] is the associated homology theory, while [8] may provide the ‘ring-like’ structure. For simple polytopes, the
entire theory has a satisfactory form. This possible analogy between convex
polytopes and $i$-graphs has been, for the author, an important motive.)

For $i$-graphs, this paper has considered only the first step. For topology,
there is a fifth step, which is to determine which linear functions $v$ of the
$f$-vector are topological (differential) invariants. As with the Kontsevich
knot integral, part of $f$ will vary with the representation of $M = |G|$. As
noted for 2-graphs, if there is a suitable homology theory associated to the
flag vector, it will produce subtle geometric inequalities.

The quantum 3-manifold invariants \cite{10,12} can be expressed as a for-
mal sum of trigraphs (with vertex orientations) modulo certain relations.
One would like to be able to express this theory in terms of flag vectors. A
first step is this. If $G$ is an oriented 4-graph (perhaps a triangulation of a
3-manifold), one can produce trigraphs from $G$ in the following way. The
trigraph will be realised in $|G|$. Each edge will pass through the centroid
of a facet of a cell. Within each cell (or rather its realisation) one might have
nothing, a segment linking two facet centroids, a pair of such segments, or a
trivalent vertex (oriented by the cell) connecting three facet centroids. De-
determining the coefficients is of course another matter. The relations should
be correspond, of course, to barycentric subdivision.

For an oriented compact and connected 2-manifold $M$, the Euler char-
acteristic $\chi(M)$ is the only topological invariant, and it can be computed by
integrating the curvature. If however $M$ is not connected, and such cases
should be considered, it has a $\chi$ for each connected component. Now tri-
angulate $M$, to solve $|G| = M$. The Euler characteristic is on $G$ a sum of
local contributions. It may be that ‘facet centroid to adjacent facet centroid’
loops will allow these local contributions to be glued together in a quantum
(or flag) way, so that they contribute ‘only to the same component’. Look-
ing the other way, inside a triangulated 4-manifold one can see embedded
3-graphs (singular surfaces).

There are some points of contact between the flag vector and the concept
of a topological quantum field theory (TQFT) \cite{1} Chap. 2. Briefly, suppose
a rule $v = v(M)$ that assigns a vector to every (oriented) $n$-manifold has
been given. Now, for every (oriented) $(n-1)$-manifold $B$ define a space $Z(B)$
as follows. Take all formal sums of manifolds with boundary $B$, modulo the
following relation. A formal sum

$$M_1 = \lambda_1 M_{1,1} + \ldots + \lambda_r M_{1,r}$$

is treated as zero if

$$v(M_1 \cup_B M_2) = \lambda_1 v(M_{1,1} \cup_B M_2) + \ldots + \lambda_r v(M_{1,r} \cup_B M_2)$$

is zero for every manifold $M_2$ (or formal sum of such) with boundary $B^*$ (the
boundary $B$ of $M_1$, but with the reverse orientation). One can then ask of
$v = v(M)$, whether its associated $Z(B)$ spaces satisfy properties such as the
multiplicative and associative axioms of TQFT. A similar construction can
be defined, and thus similar questions asked, for the flag vector of \( i \)-graphs, and suitable functions thereof.

(The definition of \( G = G_1 \cup_B G_2 \) requires some thought. The following may work the best. Suppose the vertex set of \( G \) is partitioned into \( V_1 \) and \( V_2 \). Now define \( G_i \) to be the subgraph of \( G \), whose cells have support a subset of \( V_i \). Define \( B \) to be the \( i \)-graph consisting of the remaining cells. This corresponds to a ‘collared’ form of the manifold construction. Note that the cells of \( G_1 \) are disjoint from those of \( G_2 \).)

Finally, to conclude here is a concise summary of the whole paper. Any finite object \( G \) built out of ‘cells’ can be shelled. By induction, this gives rise to a shelling vector \( \tilde{f}G \). The same can be done for some infinite \( G \) also. The 0-objects, that are the basis of the induction, depend on the type of object that \( G \) is. For each rule \( L_l \mapsto f'L_l \) defining a link contribution \( f'L_l \), the recursive formula

\[
fG = \sum_{\text{shellings}} f'L_1 \otimes \ldots \otimes f'L_N
\]

defines a flag vector \( fG \). This paper, without investigating the matter closely, proposes that \( f'L_l \) should be \( fL_l \) modulo disjoint pairs of optional cells. In at least some cases the resulting spaces (of flag vectors and link vectors) can be given a pleasant and explicit description. Indeed, to do this, without discarding vital information, is perhaps the main purpose of the rule for the link contribution \( f'L_l \). When \( G \) has a topological (or differential) realisation \( |G| \), the topological (differential) invariants that are a linear function of the flag vector \( fG \) can be studied, particularly via barycentric subdivision. Whether or not these invariants are related to those of quantum topology requires further investigation.

References

[1] M.F. Atiyah, *The geometry and physics of knots*, CUP, (1990)

[2] D. Bar-Natan, *On the Vassiliev knot invariants*, Topology, 34, (1995), 423–472

[3] M.M. Bayer and L.J. Billera, *Generalized Dehn-Sommerville equations for polytopes, spheres, and Eulerian ordered sets*, Invent. Math., 79 (1985), 143–157

[4] P. Duchet, *Hypergraphs*, in “Handbook of Combinatorics, vol I”, ed. R. Graham, M. Grötschel and L. Lovász, Elsevier Science (1995), 381–432

[5] J. Fine, *The Mayer-Vietoris and IC equations for convex polytopes*, Discrete and Computational Geometry, 5 (1995), 177-188
[6] , Graphs, flags and partitions, (in preparation)

[7] , Local-global intersection homology, (in preparation)

[8] , Intersection homology and ring structure, (in preparation)

[9] I.M. Gelfand and R.D. MacPherson, A combinatorial formula for the Pontrjagin classes, Bull. Amer. Math. Soc. (N.S.) 26 (1992), 304–309

[10] T.T.Q. Le, J. Murakami and T. Ohtsuki, On a universal quantum invariant of 3-manifolds, preprint q-alg/9512002, (1995)

[11] T.T.Q. Le, An invariant of integral homology 3-spheres which is universal for all finite type invariants, preprint q-alg/9601002, (1996)

[12] E. Witten, Quantum field theory and the Jones polynomial, Comm. Math. Phys. 121 (1989), 351–399