An iterative method for including Doppler shift in global wave solvers using FEM decomposition

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Abstract. A method for calculating the wave field for spatial dispersive media is proposed suitable for FEM. The method is based on operator splitting by separating the induced current and wave field calculations, and solving the system by means of iterations. In order to take into account several coexisting waves with different poloidal mode numbers when calculating the induced current the wave field is decomposed into wavelets, for which the current is calculated assuming the plasma to be weakly non-uniform.

1. Introduction
Modelling of the wave field for RF-heating is a challenging task because of the spatial dispersive nature of magnetised plasmas and the co-existence of different waves. The spatial dispersion requires that the different waves have their own response. The issue is often handled by Fourier decomposing the wave equation [1]. Hybrid methods using Fourier decomposition in the toroidal and the poloidal directions and FEM (finite element methods) decomposition across the flux surfaces are commonly used. The Fourier decomposition leads to dense matrixes that become time consuming to invert. FEM methods have the advantages of producing local decompositions which, in general, are faster to invert, but for which it is more difficult to include spatial dispersive effects. Lately a method to include or correct for spatial dispersive effects was developed applicable for strong single pass damping for which the local wave field is described by a single wave [2]. Another method is to use operator splitting and solve an inhomogeneous wave equation by means of iterations with the induced current as a source term calculated from the wave field [3, 4]. Instead of calculating the current by tracing the particle orbits we use here a higher order quasi-homogeneous approximation consistent with energy conservation. In order to take into account the non-local response from several coexisting waves to include the Doppler shift the wave field is locally decomposed into poloidal wavelets, for which the quasi-homogeneous approximation is applicable.

2. Wave-field
The wave equation for a spatial dispersive medium can be written as

\[ \nabla \times \nabla \times E - \left( \frac{\omega}{c} \right)^2 E = i\omega\mu_0 J_{\text{ext}} + i\omega\mu_0 J_{\text{ind}}. \]  

(1)

Here we assume the induced current

\[ J_{\text{ind}}(r,t) = J_{\text{ind}}(E), \]  

(2)
to be a linear function of the wave field $E$. The conductivity tensor, $\sigma_{j\ell}(\omega, k)$, for a fluctuating electric field can straightforwardly be defined in Fourier space ($\omega, k$), for which the induced current can be calculated

$$J_{\text{ind},j}(\omega, k) = \sigma_{j\ell}(\omega, k)E_{\ell}.$$  \hspace{1cm} (3)

Transforming the response into the real space gives

$$J_{\text{ind},j}(t, x) = \hat{\sigma}_{j\ell}(t, x)E_{\ell}(t, x),$$  \hspace{1cm} (4)

where * denotes the convolution between the electric field and $\hat{\sigma}_{j\ell}(t, x)$, the inverse Fourier transform of $\sigma_{j\ell}(\omega, k)$. The convolution describes a non-local dependence of the induced current on the electric field; the local induced current depends on the wave-particle interactions the particles experienced earlier. Multiplying Eq. (3) with the electric field $E$ and transforming it to the real space gives

$$P(t, x) = E_{\ell}(t, x)\left(\hat{\sigma}_{j\ell}(t, x)E_{\ell}(t, x)\right),$$  \hspace{1cm} (5)

which represents the time derivative of the free energy density available for dissipation. This quantity differs from the work done locally by the wave field obtained by multiplying Eq. (4) with $E$ in real space. If one instead calculates the work from the power theorem, one obtains a relation between the total transferred energies from the waves to the particles in the whole Fourier space to the whole real space from $t = -\infty$ to $t = +\infty$. Note that the work done by the wave field on the particles is not the same as heating, which is dissipation of the wave field into thermal energy of the particles. Dissipation requires decorrelation of the changes in energy due to wave-particle interactions e.g. by collisions. In absence of decorrelation, the energy the particles receives from wave-particle interaction is transferred to a coherent motion of the particles, which independent of the slope of the distribution function may result in energy transferred from the particles to the wave or vice versa.

An often used approximation of the conductivity tensor in an inhomogenous medium is that for a local homogenous medium with $k$ as the local wave vector:

$$J_{\text{ind},j}(x, t) = \sigma_{j\ell}(x; \omega, k)E_{\ell}(x, t; \omega, k).$$  \hspace{1cm} (6)

The local wave vector $k$, which may be chosen in different ways, should be consistent with the local dispersion relation. Assuming that the wave field can be decomposed in orthogonal modes one would then obtain an energy relation of the form

$$\frac{1}{2\mu_0} \frac{\partial}{\partial t}(\vec{B} \cdot \vec{B}) + \frac{\varepsilon_0}{2} \frac{\partial}{\partial t} \sum_k \left(\vec{E}_k \cdot \vec{K}_k \cdot \vec{E}_k\right) + \frac{1}{\mu_0} \nabla \cdot \left(\vec{E} \times \vec{B}\right) - i\omega\varepsilon_0 \sum_k \vec{E}_k \cdot \vec{K}_k \cdot \vec{E}_k = -\vec{E} \cdot \vec{J}_{\text{out}},$$  \hspace{1cm} (7)

where $\vec{E} = \sum_k \vec{E}_k$, $\vec{B}$ is the magnetic induction related to the wave field $E$, $\vec{K}_H$ and $\vec{K}_A$ are the Hermitian and antiHermitian part of the dielectric tensor, respectively. The conductivity and dielectric tensors are related through $i\omega\varepsilon_0 \left(\vec{K}_H - \vec{K}_A\right) = \sigma_{j\ell}$. The two first terms in Eq. (7) represent the change in the local wave energy density, the third term the divergence of the Poynting flux, the fourth term the dissipation of wave energy and the left hand side the work done by the antenna, the emitted wave energy. In this approximation the total energy flow is carried by the Poynting flux, thus the energy flux carried by particles is not included. If one instead starts with the energy conservation law including the kinetic energy flux,

$$\frac{1}{2\mu_0} \frac{\partial}{\partial t}(\vec{B} \cdot \vec{B}) + \frac{\varepsilon_0}{2} \frac{\partial}{\partial t} \sum_k \left(\vec{E}_k \cdot \vec{K}_k \cdot \vec{E}_k\right) + \frac{1}{\mu_0} \nabla \cdot \left(\vec{E} \times \vec{B}\right) - \omega\varepsilon_0 \sum_k \nabla \cdot \left(\vec{E}_k \cdot \vec{K}_k \cdot \vec{E}_k\right)$$

$$+ i\omega\varepsilon_0 \sum_k \vec{E}_k \cdot \vec{K}_A \cdot \vec{E}_k = -\vec{E} \cdot \vec{J}_{\text{out}},$$  \hspace{1cm} (8)
and assumes that the electric field of the modes $E_k$ can be decomposed into slowly varying amplitude functions and rapidly oscillating phases with a slowly varying wave vectors $k_k$ of the form

$$E_k(x,t) = E_{k,0}(x,t) \exp \left( i \int k_k \cdot dx - \omega t \right),$$

the induced current for a single mode, $J_{ind,k}^s(x,t)$, then becomes

$$J_{ind,k}^s(x,t) = \exp \left( i \int k_k \cdot dx - \omega t \right) \times \left\{ \sigma_k E_{k,0}(x,t) + i \frac{\partial \sigma_k}{\partial k} \frac{\partial E_{k,0}}{\partial x} \right\}^{-1} \right\}$$

Or written in component form in a Cartesian coordinate system

$$J_{ind,k}^s(x,t) = \exp \left( i \int k_k \cdot dx - \omega t \right) \times \left\{ \sigma_{k,ij} E_{k,0,ij}(x,t) + i \frac{\partial \sigma_{k,ij}}{\partial k_i} \frac{\partial E_{k,0,ij}}{\partial x_j} \right\}^{-1} \right\}$$

where $\omega_k = \omega_k(k, \omega, x)$ is the local dispersion relation for the mode $k$.

A major problem when solving the wave equation for a spatial dispersive medium for which $K_k$ differ for the different modes, $k$, is that the composition of the modes is not known in advance. An alternative to calculate the wave field by first substituting the expression for the induced current into the wave equation and then solving the equation is to solve the wave equation (1) and the equation for the induced current (2) by means of iterations, so called operator splitting. Green and Berry proposed to solve the system by calculating the induced current by integrating the acceleration due to the wave field along the unperturbed orbits [3]. Here the idea is to separate the wave field locally into wavelets and use a quasi-homogeneous approximation based on Eq. (9) for the induced current in weakly inhomogeneous plasma.

When solving the wave equation by means of iterations it is an advantage, if the initial guess is close to the solution. An initial guess of the wave field can be obtained by using an approximation of the induced current with $J_{0,ind}(x,t)$, which does not depend on how the waves are separated

$$\nabla \times \nabla \times E^{(n+1)} - \left( \frac{\omega}{c} \right)^2 K_\nu E^{(n+1)} = i \omega \mu_0 J_{ant} + i \omega \mu_0 \left( J_{ind}^{(n)} - J_{0,ind}^{(n)} \right),$$

where the induced current $J_{ind}^{(n)}$ is calculated from Eq. (9) with $E_{k,0} = E_{k,0}^{(n)}$ and $J_{0,ind}^{(n)} = \sigma_0 E_{k,0}^{(n)}$, where the upper index $(n)$ represent the $n$:th iteration. Here we are interested to correct for the upshift of $k_n = n_\nu R / \theta B + m_\nu B_\phi / \rho B$ when the fast wave has different poloidal mode numbers, where $R$ denotes the major radius, $\rho$ the minor radius $\theta$ the poloidal angle and $\phi$ the toroidal angle with $m_\nu$ and $n_\nu$ as the poloidal and toroidal mode numbers, respectively. As the zeroth order approximation $\sigma_0$ we use the conductivity tensor in homogeneous plasma with $|k_n R | = 0$.

The boundary condition, in addition to the standard boundary conditions on continuity across the boundary between the plasma and the vacuum regions, is that there should be not net emission or damping at the plasma boundary i. e.

$$E^* \cdot J_{ind} = 0.$$  

(11)
3. The induced current by wave-particle interactions

Calculating the induced current at a point \( x_0 \) and at a time \( t_0 \) by wave-particle interactions from the Vlasov equation gives

\[
J_{\text{ind}}(x_0, t_0, E) = -\int d^3 \nu Z e \nu \left( \delta v(x_0, t_0, \nu, E) \cdot \frac{\partial f_0}{\partial \nu} \right),
\]

where \( f_0 \) is the unperturbed distribution function at \( x_0 \) and \( \delta v(x_0, t_0, \nu, E) \) the change in velocity caused by the wave field \( E \) including interactions all along the orbit for \( t \leq t_0 \). How far back in time the interactions have to be taken into account depends on the decorrelation time, which is determined by the decorrelation of the particle phase relative the wave phase. When the change in the particle energy caused by the wave field is averaged over the difference between the wave and particle phase the linear contribution vanishes. The second order contribution is given by

\[
\Delta W(z_0, t_0, \nu, E) = \sum_{k_1} \sum_{k_2} \int d^3 \nu \left\{ (m \nu^* \cdot \delta v_k^*) \left( \delta v_{k_1} \cdot \frac{\partial f_0}{\partial \nu} \right) + \frac{m}{2} \delta v_k \cdot \delta v_{k_1} f_0 \right\},
\]

where the sum includes interactions \( k_1 \) and \( k_2 \) before \( t = t_0 \) and \( \delta v_k = \delta v_k( z_0, t_0, \nu, E_k ) \). That the change in \( \delta v \) arises from interactions all along the orbit before the considered time, is consistent with that the relation between the current and field appear as a convolution of the electric field and the conductivity tensor as in Eq. (4) and that the time derivative of the free energy density due to the wave field is given by a double convolution as in Eq. (5).

The aim of this section is to calculate the change in \( \delta v \). Wave-particle interactions can be separated in resonant and non-resonant interactions. Resonant high frequency interactions in inhomogeneous plasma differ conceptually from that in homogeneous plasmas. In homogeneous plasma the resonant particles are continuously accelerated. In inhomogeneous plasma the particle orbits are characterised by different nearly periodic cycles. If one defines resonances as large constructive changes for nearly periodic cycles, the definition of a resonance will then depend on the decorrelation time. When the decorrelation time is shorter than the bounce time, the resonances are characterised by acceleration over several wave periods for which the wave phase and particle phase coincides. The rapid variation of the phase difference between the wave and the particle position for high frequency waves results in a localisation of the effective resonant interaction into small regions in real space. The Doppler shifted cyclotron and Landau resonances become then the resonances. The net increase in velocity can for these be calculated with the stationary phase method, which for a single wave when gyro averaged gives

\[
\delta v_{\perp} = \frac{Z e}{m} \left( 1 - \frac{k_{\perp} v_{\parallel}}{\omega} \right) \left( E_{+} J_{n+1}(k_{\perp} \rho) + E_{-} J_{n-1}(k_{\perp} \rho) + \frac{v_{\parallel}}{v_{\perp}} E_{k} J_{n}(k_{\perp} \rho) \right) \exp \left( i \frac{\Theta + \pi}{4} \right) \frac{2\pi}{\sqrt{|J|}},
\]

\[
\delta v_{\parallel} = \frac{Z e}{m \omega} \left( v_{\perp} k_{\parallel} E_{+} J_{n+1}(k_{\perp} \rho) + v_{\perp} k_{\parallel} E_{-} J_{n+1}(k_{\perp} \rho) + v_{\parallel} k_{\parallel} E_{k} J_{n}(k_{\perp} \rho) \right) \exp \left( i \frac{\Theta + \pi}{4} \right) \frac{2\pi}{\sqrt{|J|}},
\]

where \( \Theta \) is the difference between the gyro phase and wave phase at the stationary phase point where \( \dot{\theta} = \omega - n \omega_k (x) - k_{\perp} v_{\parallel} = 0 \) for which \( \dot{\theta} \approx n \omega_k r v_{\parallel} \sin \theta / q R^2 \). \( \omega_k \) denotes the cyclotron frequency, \( n \) the harmonic number, \( k_{\parallel} \) the parallel wave number and \( v_{\parallel} \) the parallel velocity. A local Cartesian coordinate system \((x, y, z)\) is used, where the \( z \)-direction is parallel with the magnetic field and rotated such that \( k_{\parallel} = 0 \). \( E_{+} = 0.5(E_x + iE_y) \) and \( E_{-} = 0.5(E_x - iE_y) \) are the electric field components rotating co and counter to the ions. Note for \( n = 0 \), i.e. for the Landau resonances \( \delta v_{\perp} = 0 \) because \( \omega - k_{\parallel} v_{\parallel} = 0 \). In the limit of small \( k_{\perp} \rho \) we have for these resonances
\begin{align}
\delta v_{||} &= \left(-\frac{\mu}{m} k_{||} B_{||} + \frac{Z e}{m} E_{||}\right) \exp i \left(\Theta + \pi / 4\right) \frac{2\pi}{\sqrt{|\mathcal{J}|}},
\end{align}

where \(\mu\) is the magnetic moment. The first term is the acceleration by the TTMP and the second term the Landau damping. A more detailed exposition of how to add and calculate the coherent interactions can be found in Ref. \[5\].

Although the wave-particle interactions are global interactions the net changes in velocity can be described by local quantities. In order to estimate the effective localisation in space of the wave-particle interaction we estimate the interaction time with twice the time it takes for the phase difference \(\mathcal{J}\) to increase with \(\pi/2\). Assuming \(v_{||}\) to be constant gives \(2\delta t = 2\sqrt{\pi q R^2 / r n_0 e v_{||} \sin \theta}\).

The corresponding distance over which the interaction takes place is given by \(2\delta l = 2v_{||} \delta t = 2\sqrt{\pi q R^2 v_{||} / r n o e \sin \theta}\), where \(q\) is the safety factor. The cyclotron resonances with \(n \neq 0\) are rather well localised because of the high cyclotron frequencies. The interaction time \(2\delta t\) can be compared with the effective interaction time obtained from the stationary phase method \(\delta t_{res} = \sqrt{2\pi / |\mathcal{J}|}\), which differs with \(\sqrt{2}\) from the estimation. For Landau resonances, those with \(n = 0\), the characteristic time and distance of the resonant variation are determined by the variation of \(k_{||}\) and \(v_{||}\) along the orbit. To calculate the change in velocity due to resonant interactions, it is sufficient that the wave field is well described by a planar wave near the resonant region, which is typically shorter than the distance the particle makes under a decorrelation time.

In order to include interactions at earlier times, within a decorrelation time, we assume the wave field can be decomposed locally into different coherent planar waves with a single frequency for which the local phases of the waves are determine by the phase of the antenna current. A particle moving along a magnetic field line may interact with several planar waves. For simplicity we assume the interactions to be well localised and to be defined by the stationary phase points. A single particle will then only interact with one wave at the same location described by \((k_{||}, z_k)\) provided \(v_{||} \neq 0\) and the parallel wave numbers \(k_{||}\) differ for the different waves. Between the interactions the particle phase may be decorrelated or partly decorrelated. The change in the velocity at \((t_0, z_0)\) of a particle having a velocity \(v_0\) depends then on all previous interactions taking into account the decorrelation

\begin{align}
\delta v(z_0, t_0, v_0, E) &= \sum_k \delta v_k \exp i \left(\Theta_k + \pi / 4\right) \delta_k,
\end{align}

where \(\delta_k\) is the coherence between the interaction at the resonance at \(z_k\) and that remaining when the particle reaches the point \(z_k\) at the time \(t = t_0\).

In order to be able to add up the interactions we need to calculate the phase difference \(\Theta_k\) between the different interactions. The difference between the angular wave frequency and the harmonic of the gyro frequency at the point \(z_0\) and the resonance at \(z_k\) in the guiding centre frame of the particle is given by \(\Delta \Theta_k = \int_{k_0}^{t_0} \left(\omega - n \omega_e - k_{||,k_0} \cdot v_{||}\right) dt\). Expanding the harmonic cyclotron frequency around the point \(z_0\) yields \(z_k = \left(\omega - n \omega_e - k_{||,k_0} \cdot v_{||}\right) / \left(n \partial \omega_e / \partial z\right) + z_0\). Assuming \(v_{||} = \text{const.}\) yields

\begin{align}
\Delta \Theta_k &= \left[\omega - v_{||} k_{||,k_0} - n \omega_e - 0.5n \frac{\partial \omega_e}{\partial z} (z - z_0) \right] \frac{(z - z_0)}{v_{||}}.
\end{align}

The difference in gyro phase at the stationary phase points for a particle interacting with two waves \(k_1\) and \(k_2\) becomes
\[ \Theta_{k_1} - \Theta_{k_2} = 0.5 \nu_\parallel (k_1 - k_2)^2 \left( n \frac{\partial \omega}{\partial z} \right)^{-1}, \]  

where \( k_1 \) and \( k_2 \) denote the parallel wave number for the waves 1 and 2, respectively, typically \( |\Theta_{k_1} - \Theta_{k_2}| \ll \pi / 2 \) for thermal particles.

When treating the wave-particle interactions we have to distinguish between coherent interactions and decorrelated interactions. The interactions have to be regarded as coherent when the interactions for a single mode are represented as a superposition of several interactions related to different basis functions describing one mode. If the time between the interactions is not sufficient for decorrelation, coherent interactions may also take place by interactions with different modes e. g. toroidal Fourier modes, mode converted waves or reflected waves because their phases are all determined by the phase of the antenna current. For fully coherent interactions we have for the first term in Eq. (13)

\[
\sum \sum \int d^3 \mathbf{v} \left( m \mathbf{v} \cdot \delta \mathbf{v}_{k_1}^* \right) \left( \delta \mathbf{v}_{k_2} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \right) =
\]

\[
\sum \sum \int d^3 \mathbf{v} \left( m \mathbf{v} \cdot \delta \mathbf{v}_{k_1}^* \right) \left( \delta \mathbf{v}_{k_2} \cdot \frac{\partial f_0}{\partial \mathbf{v}} \right) \cos \left( -\Theta_{k_1} - \alpha_{k_1} (z_{k_1}) + \Theta_{k_2} + \alpha_{k_2} (z_{k_2}) \right),
\]

where \( \mathbf{E}_k = |\mathbf{E}_k| \exp(i \alpha_k (z_k)) \). When averaged over all phases \( \mathcal{Q} \) for which the interactions with the modes \( k_1 \) and \( k_2 \) are given by the phases \( \mathcal{Q} \) and \( \mathcal{Q} - \Theta_{k_1} + \Theta_{k_2} \) where \( \mathcal{Q} \) is random number we have

\[ \Delta W_{k_1k_2} \approx \frac{|\mathbf{E}_{k_1}|^2}{|\delta_{k_1}|} + \frac{|\mathbf{E}_{k_2}|^2}{|\delta_{k_2}|} + \frac{2 |\mathbf{E}_{k_1}| |\mathbf{E}_{k_2}|}{\sqrt{|\delta_{k_1}| |\delta_{k_2}|}} \cos \left( -\Theta_{k_1} - \alpha_{k_1} (z_{k_1}) + \Theta_{k_2} + \alpha_{k_2} (z_{k_2}) \right). \]  

Which of the two interactions is the first interaction for particles with low \( \nu_\parallel \), thermal particles, depends only on the wave numbers \( k_1 \) and \( k_2 \) and the location of the resonance, but not on the parallel velocity. For the first interactions the incoming phases are random, the net energy taken from the first mode constitutes of those from the first interactions only. The induced current caused at the first resonance is given by Eq. (9). The second interactions are affected by the accelerations at the first interactions because the changes in energy depend on the velocity the particles have before the interactions. The net exchange of energy at the second interactions includes therefore both the cross term and the interactions with the second mode alone. The absorption described by the cross term depends on the phase difference \( -\Theta_{k_1} - \alpha_{k_1} (z_{k_1}) + \Theta_{k_2} + \alpha_{k_2} (z_{k_2}) \). Assuming \( \delta_{k_1} \approx \delta_{k_2} \) and \( |\Theta_{k_1} - \Theta_{k_2}| \ll \pi / 2 \) the induced current at the second resonance can be approximated by

\[
J_{ind,k_1} (x,t) \approx J_{ind} \left( \mathbf{E}_{k_1} \right) + \frac{2 |\mathbf{E}_{k_1}| (x_{k_1})}{|\mathbf{E}_{k_2} (x_{k_2})|} J_{ind} \left( \mathbf{E}_{k_2} \right) \cos \left( -\alpha_{k_1} (z_{k_1}) + \alpha_{k_2} (z_{k_2}) \right),
\]

A more practical approximation in the presence of several modes is to neglect the order of the interactions and assume for all modes \( k_2 \)

\[
J_{ind,k_2} (x,t) \approx \sum \left[ \frac{|\mathbf{E}_{k_1} (x_{k_1})|}{|\mathbf{E}_{k_2} (x_{k_2})|} J_{ind} \left( \mathbf{E}_{k_2} \right) \cos \left( -\alpha_{k_1} (z_{k_1}) + \alpha_{k_2} (z_{k_2}) \right) \right]. \]

Coherent interactions can also be approximated by representing the different interactions by a single \( k_1 \) with the electric field component added up linearly including the phase differences.
For completely decorrelated interactions the phase difference between the interactions are random, and the cross terms vanish after averaging. Then only the contributions from the wave resonating at $z_0$ remain which give $\delta_k = 0$ if $z_k \neq z_0$ and $\delta_k = 1$ if $z_k = z_0$. When the modes are added incoherently, which is a typical assumption for different toroidal modes, one obtains

$$\Delta W \propto \frac{|E_{k_0}|^2}{\partial_{k_0}^2} + \frac{|E_{k_1}|^2}{\partial_{k_1}}. \quad (22)$$

4. Calculation of induced current by decomposing the wave field into wavelets

Because of the semi-local character of high frequency wave-particle interactions a local decomposition of the wave field into planar waves is sufficient for calculating the response. The localisation is of the order of a gyro radius for the gyro motion and $2\delta l$ for the effective region of wave-particle interaction.

The continuous Morlet wavelets, which have been used for analyzing time sequences, are well suited for such a decomposition, for which the transform is defined by

$$S_{m,n} = \int_{-\infty}^{+\infty} s(x') \psi^*_{m,n}(x') dx', \quad (23)$$

where $^*$ denotes the complex conjugate and $\psi_{m,n}$ is related to the mother wavelet $\psi$ by

$$\psi_{m,n}(x) = a^{-m/2} \psi\left(\frac{x - nb}{a^m}\right), \quad (24)$$

where $\psi(x) = \left(e^{-i\kappa_0 x} - e^{-\frac{x^2}{2}}\right) e^{-\frac{x^2}{2}}$. The inverse transform is given by

$$s(x) = k_\psi \sum_{m=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \psi_{m,n}(x) S_{m,n}, \quad (25)$$

where $k_\psi$ is a normalisation constant, $\psi_{m,n}$ is called a voice. The effective number of oscillations of the wavelet can be defined arbitrarily by the parameter $\kappa_0$. The 1D-wavelet expresses the wave field in terms of translations (space) by $nb$ and dilatations (wave numbers) by $a^m$. Contrary to the Fourier expansion, which is an expansion in one variable $k$, wavelet transformation is in an expansion in two variables corresponding to $x$ and $k$. The most common form is $a = 2^{-j}$, where $j$ is an integer. Because of the finite width of the wavelets a reduced set voices can be used compared to a Fourier expansion.

For our purpose of resolving waves, we rewrite the wavelets in a more convenient form by making the substitution $a^{-m} \kappa_0 = 2\pi k$, which gives

$$\psi_{k,n}(x) = \frac{2\pi k}{\kappa_0} \left(e^{-2\pi i k(x - nb)} - e^{-\frac{x^2}{2}}\right) e^{-\frac{2\pi^2 k^2(x - nb)^2}{\kappa_0^2}}. \quad (26)$$

Note that $k$ is a countable set of numbers, which may not necessary be integers. The set of $k$ has to be chosen together with the points $x_i = nb$, which typically are the grid points in the FEM decomposition. A local mode, for which the wave vector changes slowly, will be composed of neighbouring voices i.e. neighbouring $k$-values. The derivative $\partial E_{0,i}/\partial x_i$ in Eq. (6b) is calculated by taking the derivative on the envelope function of the wavelet given by $\exp(- 2\pi^2 k^2 (x - nb)^2 / \kappa_0^2)$, weighted with the coefficient $S_{k,n}$. The responses from them have then to be added coherently, whereas the response from different modes could be added coherently or by assuming them to be fully decorrelated.
Modelling the upshift with FEM codes for the fast magnetosonic wave can be done by using 1D wavelets to decompose the wave in the poloidal direction on each flux surface.

5. Conclusions and discussions
Modelling of the wave field of spatial dispersive plasma is a challenging task because the co-existence of different waves requiring different responses. Here a method has been proposed suitable for FEM codes. The method is based on solving the wave equation by operator splitting, for which the induced current is calculated from an approximate solution and the inhomogeneous wave equation is solved by means of iterations. In order to separate the response for the different modes and include the resonant response for an inhomogeneous spatial dispersive medium the wave field is separated into wavelets. When calculating the response it is important to separate coherent interactions from decorrelated interactions.

Calculating the wave field by making operator splitting and solving an inhomogeneous wave equation by means of iterations put severe restrictions on the iteration method, since for each current distribution there exists a wave field solving Eq. (10). For the wave field to be a physical solution the local wave vector has to be consistent with the local dispersion function and the wave field satisfying energy conservation. To obtain converged results requires advanced iteration schemes such as the minimum polynomial extrapolation system, so called Anderson acceleration method [6], which has been found by Green et al to give good results [4].

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