Geometric and deformation quantization

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Abstract

We present a simple geometric construction linking geometric to deformation quantization. Both theories depend on some apparently arbitrary parameters, most importantly a polarization and a symplectic connection, and for real polarizations we find a compatibility condition restricting the set of admissible connections. In the special case when phase space is a cotangent bundle this compatibility condition has many solutions, and the resulting quantum theory not only reproduces the well-known geometric quantization scheme, but also allows to quantize all interesting observables. For Kähler manifolds there is no compatibility condition, but a canonical choice for the parameters. The explicit form of the observables however remains undetermined.
1 Introduction

It is often stated that the problem of how to quantize a symplectic or Poisson manifold $M$ (phase space in physical terms) has been solved by deformation quantization, in particular by the work of Fedosov (for symplectic manifolds) [4, 5] and Kontsevich (for the more general Poisson manifolds) [10]. These constructions give the most general method to deform the pointwise product on $C^\infty(M)$ into a noncommutative product $\ast$, such that certain physical requirements are satisfied. The product $\ast$ corresponds to composition of operators in the ordinary framework of quantum theory on a flat phase space, and is called (Groenewold-)Moyal product there [8, 13].

What is missing in this formalism is the construction of a Hilbert space on which the resulting algebra acts, i.e. the space of states. This is not really a problem from a conceptual point of view, because one can define states in a purely algebraic framework as positive linear functionals on the algebra. In the present situation this is not completely straightforward, as deformation quantization usually does not produce well-behaved algebras, like $C^*$ algebras, but it might be possible to circumvent this problem. Once one has the states defined as functionals, one can also define representations of the algebra on Hilbert spaces, using the GNS construction [20].

However, classical quantum mechanics on flat space features one more physical property, which cannot easily be incorporated into the deformation framework. The wave functions there have a probability interpretation, e.g. in the position space representation $|\psi(q)|^2$ is a density in space, whose integral over some region $\Omega \subset \mathbb{R}^n$ gives the probability to find the particle in $\Omega$. Another example is the Fock space representation, where $|\psi_n|^2$ gives the probability that $n$ quanta or particles will be found in the state $\psi$. In principle this information can be extracted from the expectation values of suitable observables, e.g. $\int_\Omega |\psi(q)|^2 dq = \langle \hat{\chi}_\Omega \rangle_\psi$, where $\chi_\Omega$ is the characteristic function of $\Omega \times \mathbb{R}^n$ ($\mathbb{R}^n = $ momentum space), and $\hat{\chi}_\Omega$ its associated operator. The advantage of the classical formalism is really that the calculation of these expectation values becomes simple for a large class of observables.

For a long time the Hilbert space had therefore been considered as an essential ingredient of the theory, and the attempts to define quantization rigorously on curved phase spaces focused on its construction. These ideas culminated in the theory of geometric quantization, which provides a method for the construction of a Hilbert space consisting of wave functions with the same probability interpretation as in the flat case [21][1]. This works for symplectic phase spaces, to which we restrict our attention in this article. It is however not possible to give a general construction of all the observables in this framework, despite some promising ideas.
It would therefore be desirable to unify these two constructions such that one obtains a representation of the deformed algebra $\mathcal{A}$ on the Hilbert space $\mathcal{H}_P$ of geometric quantization. We will show in the paper that the geometrical structures appearing in the two theories are indeed closely related, that $\mathcal{A}$ has a canonical representation on a Hilbert space different from but closely related to $\mathcal{H}_P$, and that this representation can be carried over to $\mathcal{H}_P$ in an obvious way. However, one assumption has to be made, which is related to the parameters occurring in the separate theories.

This is not surprising; neither geometric nor deformation quantization are uniquely determined by the symplectic manifold $(M, \omega)$, but depend on further structures. In geometric quantization these are essentially a polarization of the tangent bundle, a metaplectic structure, and a so called prequantum line bundle. Deformation quantization enforces the choice of a symplectic connection on $M$, although it can be shown that different connections lead to isomorphic algebras. We will see that deformation quantization is also closely related to metaplectic structures, which gives the first hint at some relation between the two theories. Our assumption essentially gives a compatibility condition between the polarization and symplectic connection.

We will study in detail the case of cotangent bundles $T^*Q$, equipped with their canonical symplectic form, and show that the compatibility condition has many solutions in this case, even after fixing the polarization. Note that (at least at first sight) it would be much more desirable to find a unique solution, in order to decrease the number of apparently arbitrary parameters occurring in the quantization process. For manifolds with a totally complex polarization the compatibility condition does not even impose any restrictions on the connection.

There are two more advantages of geometric over deformation quantization. The latter often produces an infinite series for the observables, whose convergence properties are poorly understood, whereas the geometric operators are quite simple. What we will find in the case of cotangent bundles is that although our final quantum operators are constructed from the formal deformation operators, they are actually perfectly well defined for a large class of functions.

Further, there may exist inequivalent irreducible representations of the quantum algebra, and the different physical content of these representations often becomes at least partly obvious in the geometric quantization scheme, whereas it may be hard to extract this information from a state defined as a functional on the algebra.

Our notation will be a bit sloppy at times. Instead of dealing with triples $\mathcal{S}(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$, where $\mathcal{S}$ denotes Schwartz functions, and $\mathcal{S}'$ the dual space consisting of tempered distributions, we only work with the Hilbert space $L^2(\mathbb{R}^n)$, and pretend e.g. that it contains a delta-function. This is in order to keep the notation simple, but it should not be difficult to translate the results into rigorous statements.

In the first chapter we present a construction of the metaplectic group as a subgroup of the Weyl algebra. The conventional approach is to work in a fixed representation of the latter, such that the metaplectic group action becomes unitary. This is not really a restriction, as all irreducible representations are equivalent, by the Stone-von Neumann theorem. In order to demonstrate the importance of polarizations in the representation theory of the Weyl algebra and metaplectic group (thus giving a purely mathematical argument why geometric and deformation quantization belong together) we choose to work in a somewhat more formal setting here, and construct the representations only at the end of the section.
2 The Metaplectic Group

2.1 Formal construction

Let \( (V, \omega) \) be a real, 2n dimensional, symplectic vector space, and

\[
\text{Sp}(V, \omega) = \{ S \in \text{Aut}(V) \mid \omega(Sv, Sw) = \omega(v, w) \ \forall v, w \in V \}
\]

the symplectic group of \( V \), with Lie algebra

\[
\mathfrak{sp}(V, \omega) = \{ A \in \text{End}(V) \mid \omega(Av, w) + \omega(v, Aw) = 0 \ \forall v, w \in V \}. \tag{2.1}
\]

Consider the symmetric tensor algebra \( \text{Sym}(V) \) defined abstractly as

\[
\text{Sym}(V) = \bigoplus_{k \geq 0} \text{Sym}_k(V) \quad \text{on the dual } V^*.
\]

In the following we will simply write \( vw = wv \) for the symmetrized tensor product \( v \otimes_S w \), if \( v, w \in \text{Sym}(V) \). Symmetrization is understood with factors, e.g. \( vw = \frac{1}{2}(v \otimes w + w \otimes v) \) for \( v, w \in V^* \). Now we introduce a new product \( \circ \) on \( \text{Sym}(V) \) by dividing out from the full tensor algebra \( T(V) \) the ideal generated by elements \( v \otimes w - w \otimes v - ih\omega^{-1}(v, w) \), where \( v, w \in V^* \). For the time being \( h \) can be considered as a positive real number. The resulting space is indeed isomorphic as a vector space to \( \text{Sym}(V^*) \), as any of its elements can be brought to a symmetric form by reordering. In particular we get

\[
v \circ w = vw + \frac{ih}{2} \omega^{-1}(v, w), \quad \forall v, w \in V^*, \tag{2.2}
\]

as well as the 'canonical commutation relation'

\[
[v, w]_\circ = ih\omega^{-1}(v, w), \quad \forall v, w \in V^*. \tag{2.3}
\]

**Definition 2.1.** The Weyl algebra associated to \( (V, \omega) \) is \( W(V^*) := \text{Sym}(V^*, \circ) \).

Choose a basis \( \{e_1, \ldots, e_{2n}\} \) of \( V \), and let \( \{y^1, \ldots, y^{2n}\} \) be the dual basis of \( V^* \subset W(V^*) \). The Weyl operators \( W(a) \in W(V^*) \) are defined as

\[
W(a) = \exp \left\{ \frac{i}{h} \omega_{ij} a^i y^j \right\}, \quad a = a^i e_i \in V, \tag{2.4}
\]

and a formal application of the Baker-Campbell-Hausdorff formula shows that

\[
W(a) \circ W(b) = W(a + b) \exp \left[ \frac{i}{2h} \omega(a, b) \right]. \tag{2.5}
\]

Therefore the Weyl operators \( W(a) \) form a group \( H(V, \omega) \), called Heisenberg group, which can be defined abstractly as \( V \times \mathbb{R} \) with composition

\[
(v, s)(w, t) = (v + w, s + t + \frac{1}{2} \omega(v, w)). \tag{2.6}
\]

It is a central extension of the translation group \( V \). We define a map

\[
dU : \mathfrak{sp}(V, \omega) \to \text{Sym}_2(V^*), \quad A \mapsto -\frac{i}{2h} \omega_{ij} A^{kj} y^j y^k, \tag{2.7}
\]

and observe that for \( A \in \mathfrak{sp}(V, \omega) \) the expression \( \omega_{ij} A^{kj} \) is symmetric in \( i \) and \( k \), so that \( \omega_{ij} A^{kj} \) can also be written as \( dU(A) = -\frac{i}{2h} \omega_{ij} A^{kj} y^j y^k \), which simplifies the proof (that we skip) of

**Lemma 2.2.** \( dU : \mathfrak{sp}(V, \omega) \to W(V^*) \) is a Lie algebra homomorphism, i.e. for any \( A, B \in \mathfrak{sp}(V, \omega) \) the relation

\[
[dU(A), dU(B)]_\circ = dU([A, B]) \tag{2.8}
\]

holds.
**Definition 2.3.** The metaplectic Lie algebra is \( \mathfrak{mp}(V, \omega) := dU(\mathfrak{sp}(V, \omega)) \), and the metaplectic group \( \text{Mp}(V, \omega) = \exp(\mathfrak{mp}(V, \omega)) \).

As \( dU \) is injective, \( \mathfrak{mp}(V, \omega) \) is isomorphic to \( \mathfrak{sp}(V, \omega) \), but the associated Lie groups are not. It is known that the metaplectic group forms a two-fold covering group of \( \text{Sp}(V, \omega) \), but it is not the universal cover, as \( \pi_1(\text{Sp}(V, \omega)) = \mathbb{Z} \) \(^9\). Any faithful representation of \( \text{Mp}(V, \omega) \) must be infinite-dimensional, although it is a finite-dimensional Lie group. To \( T = e^A \in \text{Sp}(V, \omega) \) we associate \( U(T) \in \text{Mp}(V, \omega) \) through \( U(T) = e^{dU(A)} \), and note that \( U(T) \) is only defined up to a sign by \( T \).

**Lemma 2.4.** For \( A \in \mathfrak{sp}(V, \omega) \) and \( S \in \text{Sp}(V, \omega) \) the following relations hold

1. \( [dU(A), y^i]_o = -A^i{}_j y^j \)
2. \( U(S) \circ y^i \circ U(S)^{-1} = (S^{-1})^i{}_j y^j \)
3. \( W(Sa) = U(S) \circ W(a) \circ U(S)^{-1}, \quad \forall a \in V \)

**Proof.**

1. is a direct consequence of the definition of \( dU \) and the commutation relation (2.3).
2. follows from 1, and
3. from 2:

\[
W(Sa) = \exp \left\{ \frac{i}{\hbar} \omega_{ij}(Sa)^i y^j \right\} = \exp \left\{ \frac{i}{\hbar} \omega_{ij} a^i (S^{-1})^j_k y^k \right\} = U(S) \circ \exp \left\{ \frac{i}{\hbar} \omega_{ij} a^i y^j \right\} \circ U(S)^{-1}.
\]

\[\square\]

### 2.2 Representations

In an irreducible unitary representation of \( H(V, \omega) \) the central elements \((0, t)\) act as scalars \( e^{\frac{it}{\hbar}} \). The Stone-von Neumann theorem (see e.g. \([9]\)) states that up to unitary equivalence there is only one unitary representation of \( H(V, \omega) \) for fixed \( \hbar \). Such a representation \( \rho \) gives rise to one of the Weyl algebra, where the action of the generators \( y^i \in V^* \) is determined by

\[
\rho(y^i)\psi = -ih\omega_{ij} \frac{\partial}{\partial t} \bigg|_{t=0} \rho(te_j, 0)\psi,
\]

cf. (2.4). The induced representation of \( \text{Mp}(V, \omega) \) can be obtained more directly as follows. For \( S \in \text{Sp}(V, \omega) \) define another representation of \( H(V, \omega) \) on the same Hilbert space through

\[
\rho^S(v, t) = \rho(Sv, t).
\]

According to Stone-von Neumann \( \rho \) and \( \rho^S \) are unitarily equivalent, thus we can find \( V(S) \in \text{U(H)} \) s.t.

\[
\rho(Sv, t) = V(S)\rho(v, t)V(S)^{-1}.
\]

(2.9)

Obviously \( V(S) \) is not uniquely determined by \( S \). We are going to show that it is unique up to a phase. Suppose that \( \tilde{V}(S) \) satisfies equation (2.9) as well, then

\[
[\tilde{V}(S)^{-1}V(S), W(a)] = 0 \quad \forall a \in V,
\]
and Schur’s lemma implies \( \tilde{V}(S)^{-1}V(S) \in \mathbb{C}I \). We have
\[
V(S)V(T)W(a)V(T)^{-1}V(S)^{-1} = V(ST)W(a)V(ST)^{-1},
\]
and uniqueness of \( V(ST) \) up to a phase implies
\[
V(S)V(T) = c(S, T)V(ST), \quad c(S, T) \in S^1.
\] (2.10)

It is not possible to eliminate the phase \( c(S, T) \) completely, but it can always be chosen to be \( \pm 1 \) [6], and comparison with equation 3 of Lemma 2.4 leads us to conclude that \( V(S) = \rho(U(S)) \).

The map \( \text{Sym}(V^*) \rightarrow W(V^*) \) assigning to a polynomial on \( V \) its element in the Weyl algebra is the quantization map on the flat phase space \( (V, \omega) \) and Heisenberg group is that the symmetry group \( \text{Sym}(V^*) \) and \( \text{Mp}(V, \omega) \) acts by the adjoint action: \( U(S) \ast W(a) = U(S)cW(a)cU(S)^{-1} = W(Sa) \).

**Example 2.5** (Schrödinger representation). We construct a representation of the Weyl algebra, following the well-known construction of spin representations of the Clifford algebra. Let \( P \) be a Lagrangian subspace of \( V \), meaning that its symplectic complement \( P^\perp = \{ v \in V \mid \omega(v, p) = 0 \ \forall p \in P \} \) equals \( P \). Define \( P^0 = \{ \omega(p, \cdot) \mid p \in P \} \subset V^* \). One can now introduce Darboux coordinates \( q^i, p_j \) on \( V^* \), such that \( P^0 \) is spanned by the \( q^i \). The representation space is \( \text{Sym}(P^0_C) \), i.e. the space of complex polynomials in the \( q^i \), and the action is determined by
\[
\begin{align*}
\sigma(q)\psi & = q\psi \\
\sigma(p)\psi & = [p, \psi]_o,
\end{align*}
\] (2.11)
for \( q \in P^0_C, p \in Q^0_C := \text{span}\{p_1, \ldots, p_n\} \), and \( \psi \in \text{Sym}(P^0_C) \). Note that choosing different coordinates \( p_j' = p_j + A_{jk}q^k \) does not change the representation because \([q^k, \psi]_o = 0\), so it really depends only on \( P^0 \), and not on the complementary subspace \( Q^0 \).

(2.11) is then extended to the whole Weyl algebra by requiring \( \sigma \) to be a homomorphism w.r.t. the Weyl product [2,2]. Observe that for a basis element \( p_i \) one obtains the well-known Schrödinger operator \( \frac{i\hbar}{\gamma}\partial_q^i \).

In our Darboux coordinates adapted to the decomposition \( V = Q \oplus P \) the symplectic form is \( \omega = dp_j \wedge dq^j \). An element \( X \in \mathfrak{sp}(V, \omega) \) can then be represented as \( X = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \), where \( A, B, C \in \mathfrak{gl}(n, \mathbb{R}), B^T = B, C^T = C \). The subset of \( \mathfrak{sp}(V, \omega) \) respecting the polarization \( P \) consists of matrices
\[
X = \begin{pmatrix} A & 0 \\ C & -A^T \end{pmatrix}.
\] (2.12)

From the definition of \( dU \) (2.7) we obtain the metaplectic representation in the Schrödinger picture:
\[
\sigma(dU(X)) = \frac{i\hbar}{2} B_{jk}^{\prime} \frac{\partial^2}{\partial q^i \partial q^k} - A_{jk}^{\prime} q^j \frac{\partial}{\partial q^k} - \frac{1}{2} \text{tr} A + \frac{i}{2\hbar} C_{jk} q^j q^k.
\] (2.13)

**Example 2.6** (Fock-Bargmann representation). Let \( P \subset V_C \) be a Lagrangian subspace again, w.r.t. the complexified symplectic form, such that \( P \cap \overline{P} = \{ 0 \} \). Then we can introduce complex coordinates \( z^j, \overline{z}^j \) on \( V_C^* \) such that \( P^0 \) is spanned by the \( \overline{z}^j \), and \( \omega \) assumes the complex standard form \( idz^j \wedge d\overline{z}^j \). The representation space is \( \text{Sym}(P^0) \), i.e. the space of holomorphic polynomials. The action of \( W(V^*) \) is determined by
\[
\begin{align*}
\beta(z)\psi & = z\psi, \\
\beta(\overline{z})\psi & = [\overline{z}, \psi]_o,
\end{align*}
\] (2.14)
leading to $\beta(\mathbb{R}) = h \frac{\partial}{\partial x}$. Elements of the symplectic Lie algebra again take the form $X = (A B \quad C -A^T)$ w.r.t. the splitting $V_C = P \oplus P$, where $A,B,C \in \mathfrak{gl}(n,\mathbb{R})$, $B = B^T, C = C^T$. The ones respecting also the complex structure are

$$\beta(u) = \Phi(\alpha,\beta) \quad \text{leading to} \quad \Phi(u) = \Phi(A,B,C),$$

which should be compared to the Weyl product (2.2). The isomorphism from $V$ of $\mathfrak{gl}(n,\mathbb{R})$ to the spin algebra is given by

$$X = \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}, \quad A \in \mathfrak{gl}(n,\mathbb{R}), \quad (2.15)$$

and form a $\mathfrak{u}(n)$-algebra. The Fock space representation of the metaplectic algebra becomes

$$\beta(dU(X)) = -\frac{h}{2}B_{ab}\frac{\partial^2}{\partial z^a \partial \bar{z}^b} - A_{ab}z^b \frac{\partial}{\partial z^a} - \frac{1}{2} \text{tr} A + \frac{1}{2h}C_{ab}z^a z^b. \quad (2.16)$$

The representation spaces $\text{Sym}(\mathfrak{P})$ in these examples consist of polynomial functions on a subset of $V$. It is now possible to let $W(V^*)$ act on more general functions, e.g. Schwartz functions $\mathcal{S}(Q) \subset L^2(Q)$ in the real case, $Q \cong \mathbb{R}^n$. We would like to find a suitable function space endowed with an inner product, such that the representations of $H(V,\omega)$ and $\text{Mp}(V,\omega) \subset W(V^*)$ become unitary. This will certainly be the case if the elements of $V^*$ act as self-adjoint operators on a common invariant, dense domain, which can be accomplished by the choice $L^2(\mathbb{D},d^nq)$ in the real case, and $L^2_{\text{hol}}(V,e^{-|z|^2/h}d^n zd^n\bar{z})$ in the complex one, where the index $\text{hol}$ denotes restriction to holomorphic functions. Is there a systematic way to find the inner product? The answer is yes, and the method is geometric quantization, where the inner product plays the role of a Hermitian structure on the prequantum bundle $B$, determined by the condition that a given connection on $B$ has to be metric. See section 4. It is interesting to note that in the holomorphic case the Hilbert space is indeed spanned by polynomials in $z$, whereas in the real one it is not spanned by polynomials in $q$.

### 2.3 Weyl bundle and metaplectic structures

On a symplectic manifold $(M,\omega)$ one has a symplectic vector space $(T_m M,\omega_m)$ over every point $m \in M$. One can then form the infinite-rank Weyl bundle $\mathcal{W}$, consisting of the collection of $W_m := W(T_m M)$. Associated to $\mathcal{W}$ (or $TM$) is a principal $\mathbb{R}^{2n} \times \text{Sp}(2n)$ bundle ($\text{Sp}(2n)$ denotes $\text{Sp}(\mathbb{R}^{2n}, dp_i \wedge dq_i)$), which can be lifted to a principal $H(2n) \times \text{Mp}(2n)$ bundle if the second Stiefel-Whitney class $w_2(M)$ vanishes. Then a unitary representation $\rho$ of $H(2n)$ induces one of $H(2n) \times \text{Mp}(2n)$ and gives rise to an associated Hilbert bundle $\mathcal{H}$ on $M$, such that $\mathcal{W} \subset \text{End}(\mathcal{H})$.

### 2.4 Analogy to the spin group

The metaplectic group is the symplectic analogue of the spin group. Let $(V,g)$ be a metric vector space, then its associated Clifford algebra $\text{Cl}(V^*)$ is, as a vector space, the antisymmetric tensor algebra $AV_C^*$. The Clifford product is defined by

$$v \circ w = \frac{1}{2}v \wedge w + \frac{1}{2}g^{-1}(v,w), \quad \forall v, w \in V^*,$$

which should be compared to the Weyl product (2.2). The isomorphism from

$$\mathfrak{so}(V,g) = \{ A \in \text{End}(V) \mid g(Av,w) + g(v,Aw) = 0 \forall v, w \in V \}$$

to the spin algebra is given by

$$dS : \mathfrak{so}(V,g) \rightarrow \text{spin}(V,g), \quad A \mapsto \frac{1}{2}g_{ab}A^b c \gamma^a \circ \gamma^c,$$
where the $\gamma^i$ are basis elements of $V^*$, and thus satisfy $\{\gamma^i, \gamma^j\} := \gamma^i \circ \gamma^j + \gamma^j \circ \gamma^i = g^{ij}$. The analogue of Lemma (2.4) is given by

$$\left[ dS(A), \gamma^i \right]_\circ = -A^i_j \gamma^j$$

and its exponentiated form. The representations of $\text{Cl}(V^*)$ can be obtained in a manner similar to the Fock representation of $W(V^*)$.

### 3 Deformation Quantization

We describe deformation quantization of symplectic manifolds according to Fedosov [4, 5]. Let $(M, \omega)$ be a $2n$-dimensional symplectic manifold. Our goal is the construction of a star product $\ast$ on $C^\infty(M)$, i.e. a deformation of the ordinary point-wise product of functions on $M$. It turns out that we will have to consider $\hbar$ as a formal deformation parameter here. E.g. instead of $C^\infty(M)$ we have to deal with $\left[C^\infty(M), \hbar \right] = \sum_{k \geq 0} \hbar^k a_k \mid a_k \in C^\infty(M)$, where no sort of convergence property is imposed on the series $\sum_k \hbar^k a_k$. The results of section 2 will have to be interpreted in this sense as well then. In the following we will simply denote $C^\infty(M)[[\hbar]]$ by $C^\infty(M)$, and analogously for other structures.

The following conditions are to be satisfied by $\ast$:

1. the coefficients $c_k$ of the product of $f = \sum_k \hbar^k f_k$ and $g = \sum_k \hbar^k g_k$:

   $$f \ast g = \sum_{k=0}^\infty \hbar^k c_k(f, g)$$

   are bi-differential operators (of finite order).

2. $c_0(x) = f_0(x)g_0(x)$

3. the correspondence principle

   $$[f, g]_\ast = f \ast g - g \ast f = i\hbar \{f_0, g_0\} + O(\hbar^2),$$

   holds, where $\{f, g\} = \omega^{-1}(df, dg) = \omega^{ab} \partial_a f \partial_b g$ denotes the poisson bracket defined by $\omega$.

#### 3.1 The Fedosov connection

In every point $m \in M$ we can form the Weyl algebra $W_m = W(T^*_m M)$, whose elements have the form

$$\sum_{k, r \geq 0} \hbar^k f_{k_1, \ldots, k_r} y^{l_1} \cdots y^{l_r}$$

(including a sum over the $l_i$), with multiplication determined by \((2.2)\). The collection of local Weyl algebras $W_m$ forms the Weyl bundle $W$, whereas in the flat case $M = V$ there was only one Weyl algebra and representation. We want to get rid of these extra algebras by identifying 'neighboring' Weyl algebras, which can be done by introducing a flat connection on $W$. Choose a symplectic connection $\nabla$ on $TM$, i.e. one satisfying $\nabla \omega = 0$ for the induced connection on $T^*M \otimes T^*M$, with vanishing torsion. Symplectic connections always exist, but contrary to the Riemannian case they
are not unique. It induces a connection on the whole tensor algebra of \( M \), in particular one on \( W = \text{Sym}(T^*M) \), through the Leibniz formula. In local Darboux coordinates \( \nabla \) can be decomposed as \( \nabla = d + \Gamma \), where \( \Gamma \in \Omega^1(U; \mathfrak{sp}(2n)) \) is a Lie algebra valued 1-form. Its components are defined by \( \Gamma^k e_k = \Gamma(e_i) \cdot e_j \), in the local Darboux basis \( \{ e_i \} \) of \( TM \). Further we define \( \Gamma_{ijk} = \omega_{il} \Gamma^l_{jk} \), which is symmetric in all indices due to \( \nabla \) being torsion-free and symplectic. The curvature of \( \nabla \) is defined as \( R = \nabla^2 \in \Omega^2(M; T^*M \otimes TM) \), with components

\[
R_{ijkl} = \partial_k \Gamma^i_{lj} - \partial_l \Gamma^i_{kj} + \Gamma^i_{km} \Gamma^m_{lj} - \Gamma^i_{lm} \Gamma^m_{kj},
\]

and \( R_{ijkl} = \omega_{im} R_{mijkl} \) is symmetric in the first two (Lie algebra) indices, and antisymmetric in the last two (differential form) indices. We denote the local basis elements of \( T^*_m M \) by \( \gamma^i \) when considered as elements of the Weyl algebra \( W_m \), and by \( dx^i \) when considered as elements of the exterior algebra \( \Lambda T^*_m M \). Differential forms with values in the Weyl bundle are sections of \( \Omega(W) = \Gamma(W \otimes \Lambda T^*M) \). Note that, as a vector bundle, \( W \otimes \Lambda T^*M \) is just the full tensor bundle of \( M \). On \( \Omega(W) \) we write \( \circ \) for \( \circ \otimes \wedge \), and use the graded commutator

\[
[\xi, \eta] = \xi \circ \eta - (-1)^{pq} \eta \circ \xi,
\]

for \( \xi, \eta \) differential forms of degree \( p, q \) respectively, with values in \( W \). The connection is extended to \( \Omega(W) \) through

\[
\nabla(\xi \circ \eta) = \nabla \xi \circ \eta + (-1)^q \xi \circ \nabla \eta, \quad \xi \in \Gamma(W \otimes \Lambda^q)
\]

\[
\nabla(\phi \wedge \eta) = d\phi \wedge \eta + (-1)^q \phi \wedge \nabla \eta, \quad \phi \in \Omega^q(M).
\]

Explicitly we have

\[
\nabla y^i_1 \cdots y^i_k = - \sum_j \Gamma^j_{ab} y^i_1 \cdots \hat{y}^i_j \cdots y^i_k y^a dx^b,
\]

where \( \hat{y}^i_j \) means omitting the element, and which can be written in the form

\[
\nabla = d + [dU(\Gamma), \cdot] = d - \frac{i}{2\hbar} \Gamma_{ijk} [y^i y^j, \cdot] dx^k.
\]

Here \( dU \) is the isomorphism between the symplectic and metaplectic Lie algebras (2.7). Now we introduce two further operators on \( \Omega(W) \):

\[
\delta = dx^k \wedge \frac{\partial}{\partial y^k}, \quad \delta^* = y^k \frac{\partial}{\partial x^k},
\]

where \( y^k \) denotes multiplication with \( y^k \) (w.r.t. the symmetric tensor product), and the contraction \( \iota \left( \frac{\partial}{\partial x^i} \right) \) acts only on the form part. In brief, \( \delta \) replaces one of the \( y^i \) by \( dx^i \), whereas \( \delta^* \) replaces \( dx^i \) by \( y^i \). One easily checks

**Lemma 3.1.**

1. \( \delta^2 = (\delta^*)^2 = 0 \)

2. Applied to \( y^i_1 \cdots y^i_l dx^{j_1} \wedge \cdots \wedge dx^{j_p} \) the following identity holds:

\[
\delta \delta^* + \delta^* \delta = (l + p)id.
\]
We also define $\delta^{-1}$ by

$$\delta^{-1} = \frac{1}{l+p}\delta^*$$

for $l + p > 0$, and $\delta^{-1} = 0$ otherwise. If the projection of an element $\xi \in \Omega(W)$ to its part in $C^\infty(M) \subset \Omega(W)$ is denoted by $\xi_{00}$, then the following decomposition holds, analogously to the Hodge-de Rahm decomposition of forms:

$$\xi = \delta\delta^{-1}\xi + \delta^{-1}\delta\xi + \xi_{00}.$$  

(3.6)

**Definition 3.2.** The $\hbar$-degree of a homogeneous element

$$\sum_m h^k \xi_{k;\mathbf{i}_1,\ldots,\mathbf{i}_l;\mathbf{j}_1,\ldots,\mathbf{j}_m} y^{\mathbf{i}_1} \ldots y^{\mathbf{i}_l} dx^{\mathbf{j}_1} \wedge \cdots \wedge dx^{\mathbf{j}_m} \in \Omega(W)$$

(no sum over $k,l$) is defined as $k + l/2$.

From its definition it follows that $\circ$ respects the gradation on $W$ defined by the $\hbar$-degree. The subspaces of homogeneous elements of $\hbar$-degree $j$ are denoted by $W_j$. Now $W$ carries two gradations, the other one being defined by the $\hbar$-degree in $\text{Sym}(T^*_C M)$, which is respected by the symmetric tensor product, but not by $\circ$. The corresponding projections are

$$\pi_j^\hbar : W \to W_j,$$

and we also adopt the convention to denote $\pi_0^\hbar(\xi)$ by $\xi_0$. It is important to note that $\delta$ decreases the $\hbar$-degree by $1/2$, whereas $\delta^*$ increases it.

Now we come to the construction of a flat connection on $W$, with curvature $\Omega = \frac{i}{\hbar} [\omega, \cdot] = 0$. The ansatz

$$D = \nabla + \frac{i}{\hbar} [\gamma, \cdot] = d + [dU(\Gamma) + \frac{i}{\hbar} \gamma, \cdot],$$

(3.9)

with an as yet undetermined 1-form $\gamma \in \Omega^1(W)$, leads to the curvature $\Omega = D^2 = \frac{i}{\hbar} [\tilde{\Omega}, \cdot]$, where

$$\tilde{\Omega} = \tilde{R} + \nabla\gamma + \frac{i}{\hbar} \gamma^2.$$  

(3.10)

Here $R$ denotes the curvature 2-form of $\nabla$, $\tilde{R} = \frac{h}{2} dU(R) = -\frac{1}{4} R_{ijkl} y^i y^j dx^k \wedge dx^l$, and $\gamma^2 = \gamma \circ \gamma$. Of course, $\gamma$ is only determined up to addition of a scalar form. We require the normalization $\gamma_0 = 0$. In the flat case $M = \mathbb{R}^{2n}$ with standard symplectic form $\omega = dp_i \wedge dq^i$ we can choose $\Gamma = 0$, and $\gamma = \omega_{ab} y^b dx^a$ leads to $\tilde{\Omega} = \omega$. Therefore, in the general case we split $\gamma$ as

$$\gamma = \omega_{ab} y^b dx^a + r.$$

(3.11)

Observing that $\delta$ can be written in the form $\delta \xi = -\frac{i}{\hbar} \omega_{ab} dx^a [y^b, \xi]$, we obtain for the curvature

$$\tilde{\Omega} = \omega + \tilde{R} - \delta r + \nabla r + \frac{i}{\hbar} r^2,$$

so that $\tilde{\Omega} = \omega$ becomes equivalent to

$$\delta r = \tilde{R} + \nabla r + \frac{i}{\hbar} r^2.$$  

(3.12)

**Theorem 3.3** (Fedosov). Under the condition $\delta^{-1}r = 0$, eq. (3.12) has exactly one solution $r$. 

9
Sketch of proof. For a 1-form we have \( r_{00} = 0 \), which together with the condition \( \delta^{-1}r = 0 \) implies that the decomposition (3.6) takes the form \( r = \delta^{-1} \hat{R} + \delta^{-1} \left( \nabla r + \frac{i}{\hbar} r^2 \right) \). (3.13)

As \( \nabla \) preserves the \( \hbar \)-filtration, whereas \( \delta^{-1} \) increases the degree by \( 1/2 \), it follows by iteration that (3.13) has exactly one solution. The iteration steps are

\[
 r^{(3)} = \delta^{-1} \hat{R}, \quad r^{(n+1)} = \delta^{-1} \hat{R} + \delta^{-1} \left( \nabla r^{(n)} + \frac{i}{\hbar} (r^{(n)})^2 \right) \mod \hbar^{n/2+1}, \quad \text{(3.14)}
\]

and \( r = \lim_{n \to \infty} r^{(n)} \). Here \( \mod \hbar^{n/2+1} \) means discarding all terms of \( \hbar \)-degree at least \( n/2 + 1 \). These terms are not stable under iteration yet, and it is convenient, although not necessary, to ignore them. The condition \( \delta^{-1} r = 0 \) is fulfilled due to \( (\delta^{-1})^2 = 0 \). For the proof that \( r \) indeed solves (3.12) we refer to Fedosov’s texts \[4, 5\].

The first iteration gives

\[
 r^{(4)} = -\frac{1}{8} R_{ijkl} y^i y^j y^k dx^l - \frac{1}{40} \nabla_m R_{ijkl} y^i y^j y^k y^m dx^l \quad \text{(3.15)}
\]

where \( \nabla \) is the product connection on \( \mathcal{W} \otimes \Lambda \), thus acts the same way on \( y^i \) and \( dx^i \).

3.2 Observables and star product

Having constructed a flat connection on \( \mathcal{W} \) we want to identify the quantum operators with the set of flat sections of \( \mathcal{W} \) with respect to \( D \), i.e. those satisfying \( D \hat{f} = 0 \). As \( D = \nabla - \delta + \frac{i}{\hbar} [r, \cdot] \), this equation can be written in the form

\[
 \delta \hat{f} = \nabla \hat{f} + \frac{i}{\hbar} [r, \hat{f}] \quad \text{(3.16)}
\]

We denote the set of flat (or parallel) sections by \( \Gamma_D(\mathcal{W}) \).

Theorem 3.4 (Fedosov). To every \( f \in C^\infty(M) \) there exists exactly one \( \hat{f} \in \Gamma_D(\mathcal{W}) \) such that \( \hat{f}_0 = f \).

Sketch of proof. For a 0-form \( \hat{f} \) we have \( \delta^{-1} \hat{f} = 0 \) and \( \hat{f}_{00} = \hat{f}_0 \), so that the decomposition (3.6) becomes \( \hat{f} = \hat{f}_0 + \delta^{-1} \delta \hat{f} \). Then equation (3.16) implies

\[
 \hat{f} = \hat{f}_0 + \delta^{-1} \left( \nabla \hat{f} + \frac{i}{\hbar} [r, \hat{f}] \right) \quad \text{(3.17)}
\]

Again this equation has a unique solution, which can be determined by iteration:

\[
 \hat{f}^{(0)} = \hat{f}_0 = f, \quad \hat{f}^{(n+1)} = \hat{f}_0 + \delta^{-1} \left( \nabla \hat{f}^{(n)} + \frac{i}{\hbar} [r, \hat{f}^{(n)}] \right) \mod \hbar^{n/2+1} \quad \text{(3.18)}
\]

For the proof that \( \hat{f} \) indeed solves (3.16) we again refer to \[4, 5\].

The first iterations give:

\[
 \hat{f}^{(3)} = f + y^k \nabla_k f + \frac{1}{2} y^i \nabla_j y^k \nabla_k f + \frac{1}{6} y^j y^i \nabla_j y^k \nabla_k f + \frac{1}{24} R_{abcd} \omega^e_{\cdot k} \partial_k f y^a y^b y^d \quad \text{(3.19)}
\]
An explicit calculation of the covariant derivatives shows that in third order one has

\[ \hat{f}^{(3)} = f + \partial_k f y^k + \frac{1}{2} \omega_{jil}(\nabla_k X_f)^l y^i y^l + \frac{1}{6} \omega_{jil}(\nabla_i \nabla_k X_f)^l y^i y^l + \Gamma_{ijl}(\nabla_k X_f)^l y^i y^l - \frac{1}{4} R_{ijkl} X_f^l y^i y^j y^k, \] (3.20)

where \( X_f \) is the Hamiltonian vector field of \( f \), defined by \( X_f(g) = \omega^{-1}(dg, df) \). Note that a solution of the equation \( D \hat{f} = 0 \) is not uniquely determined by its value at a point \( m \in M \). This value is determined by the Taylor series of \( f \) at \( m \), which does not fix \( f \).

The quantization map

\[ (\pi_0^0)^{-1} : C^\infty(M) \rightarrow \Gamma_D(W), \quad f \mapsto \hat{f} \]

allows for the definition of a star product on \( C^\infty(M) \), namely

\[ f \ast g := \pi_0^0(\hat{f} \circ \hat{g}). \] (3.21)

Using the explicit expression (3.20), as well as the relation

\[ \pi_0^0(y^{a_1} y^{a_2} \ldots y^{a_k} \circ y^{i_1} y^{i_2} \ldots y^{i_k}) = \left( \frac{i\hbar}{2} \right)^k \sum_{\pi \in S_k} \omega^{a_1 i_{\pi(1)}} \ldots \omega^{a_k i_{\pi(k)}}, \] (3.22)

we arrive at

\[ f \ast g = fg - \frac{i\hbar}{2} \omega(X_f, X_g) + \frac{\hbar^2}{4} (\nabla_j X_f) (\nabla_k X_g) + O(\hbar^3). \] (3.23)

The conditions for a star product mentioned at the beginning of the section are easily checked, using the fact that the Poisson bracket can be expressed as \( \{ f, g \} = \omega(X_g, X_f) \). For \( M = \mathbb{R}^{2n} \) with \( \Gamma = 0 \) one obtains

\[ \hat{f} = \sum_{k=0}^{\infty} \frac{1}{k!} (\partial_{i_1} \ldots \partial_{i_k} f) y^{i_1} \ldots y^{i_k}, \] (3.24)

and thus the Groenewold-Moyal product [8, 13]

\[ f \ast g(x) = \exp \left( \frac{i\hbar}{2} \omega^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} \right) f(x) g(y) \bigg|_{y=x}, \] (3.25)

or \( f \ast g = \mu \circ \exp \{ \frac{i\hbar}{2} \omega^{-1} \} f \otimes g \), recalling that \( \omega^{-1} \in \Gamma(TM \otimes TM) \) acts naturally on \( C^\infty(M) \otimes C^\infty(M) \); \( \mu(f \otimes g)(x) := f(x) g(x) \).

If \( M \) admits a metaplectic structure \( \mathcal{H} \rightarrow M \), then we know that \( \mathcal{W} \subset \text{End}(\mathcal{H}) \). Now \( \Gamma_D(W) \) is our algebra of quantum observables, and it is natural to conjecture that the physical states are sections of \( \mathcal{H} \). But again \( \Gamma(\mathcal{H}) \) is too large, and we need a way to identify different fibres of \( \mathcal{H} \). The most obvious way to do this would be by means of a flat connection, as in the case of the observables [16]. The connection form of \( D \) on \( \mathcal{W} \) is given by a commutator, i.e. \( D = d + [A, \cdot] \), where

\[ A = dU(\Gamma) + \frac{i}{\hbar} \omega_{ij} y^i dx^j - \frac{i}{2\hbar} \omega_{ij} \Gamma_{kl} y^i y^j dx^k - \frac{i}{8\hbar} R_{ijkl} y^i y^j y^k dx^l + \ldots, \] (3.26)

from (3.3), implying that \( D \) is the induced connection of \( D_H = d + A \) on \( \mathcal{H} \). However, \( D_H \) is not flat, as \( D_H^2 = \frac{i}{\hbar} \omega \neq 0 \), and therefore \( \Gamma_D(\mathcal{H}) = \{ \psi \in \Gamma(\mathcal{H}) \mid D_H \psi = 0 \} \) consists only of the zero section. We will find out how to get a flat connection in section [5].
4 Geometric Quantization

Geometric quantization is a prescription for constructing a Hilbert space $\mathcal{H}_P$ (the space of states) on a symplectic manifold, as well as a few quantum operators acting on it. Our main reference for the material presented here is Woodhouse’s book [21].

4.1 The Hilbert space

$\mathcal{H}_P$ is not uniquely determined by $(M,\omega)$ but depends on further structure:

- a 'prequantum bundle' $\mathcal{B} \to M$, i.e. a Hermitian line bundle over $M$, with metric connection $\nabla^\mathcal{B}$ of curvature $-\frac{i}{\hbar}\omega$.
- a metaplectic structure $\mathcal{H} \to M$.
- a polarization, i.e. a strongly integrable ([21], p.92) Lagrangian subbundle $P \subset T_C M$, Lagrangian meaning that $P_m^\perp := \{v \in T_m C M \mid \omega(v,p) = 0 \forall p \in P_m\}$ equals $P_m$. In particular this implies $\text{rk}_C P = \frac{1}{2} \dim_R M$.

Existence of $\mathcal{B}$ implies the condition $[\omega/2\pi \hbar] \in H^2(M,\mathbb{Z})$, whereas the metaplectic structure exists iff the second Stiefel-Whitney class $w_2(M)$ of $M$ vanishes. We will assume these conditions to be satisfied, although they can be slightly weakened for pure geometric quantization, see [18], or [21], p. 233. The existence problem for polarizations is quite subtle, and not completely solved as it seems. In particular there are symplectic manifolds that do not admit any polarization at all [7].

Different prequantum bundles correspond to distinct physical situations, e.g. different magnetic monopole charges, or different vacuum $\theta$-angles [14]. The same might be true for the metaplectic structure, but it is not obvious how to interpret the polarization; it is usually claimed that 'most' polarizations can be considered as equivalent, but this seems not to be possible for all of them, and is definitely not in the infinite-dimensional case, as demonstrated by Shale’s theorem [19, 17].

Suppose a choice of the above mentioned structures $(\mathcal{B} \to M, \mathcal{H} \to M, P)$ has been made. The polarization can be transferred to the cotangent bundle through

$$\flat : T_C M \to T_C^* M, \quad X^\flat(Y) = \omega(Y, X) = \omega_{ab} Y^a X^b.$$  

Example 4.1 (Real Polarizations). Suppose $P$ is the complexification of a subbundle of $TM$. In this case we have $P \cap \overline{P} = P$, the polarization is called real, and one can locally find Darboux coordinates $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ on $M$ (i.e. $\omega = dp_i \wedge dq^i$) such that

$$P = \text{span}\{\partial_{p_i} \mid i = 1, \ldots, n\}, \quad P^b = \text{span}\{dq^j \mid j = 1, \ldots, n\}.$$  

These coordinates are said to be adapted to $P$. Define $Q_m = \text{span}\{\partial_{q^i} \mid i = 1, \ldots, n\}$, then the local Hilbert space $\mathcal{H}_m$ from the metaplectic structure is the Schrödinger representation space $L^2(Q_m)$ on which $dq^j|_m, dp_i|_m \in W_m$ act as the standard Schrödinger operators $\hat{q}^j, \hat{p}_i$.

Example 4.2 (Complex Polarizations). Suppose $P \cap P = \{0\}$, or equivalently $P \oplus \overline{P} = T_C M$. In this case the polarization is called totally complex, and defines a complex structure $J$ on $M$ in the following way. Split $X \in TM$ as $X = Z + \overline{Z}$, where $Z \in P$, and define $JX = iZ - i\overline{Z}$.
We assume that the associated symmetric bilinear form \( g(X, Y) := \omega(X, JY) \) is positive definite, and therefore gives \((M, \omega)\) the structure of a Kähler manifold. One can then choose complex coordinates \( z^a \) on \( M \), s.t.

\[
P = \text{span}\{\partial_a | a = 1, \ldots, n\}, \quad P^a = \text{span}\{dz^a | a = 1, \ldots, n\},
\]
and \( \omega \) assumes the standard form \( \omega = i \partial \bar{\partial} K \) for some real valued function \( K \in C^\infty(M) \), the so-called Kähler potential. If \( \omega_m = iK_{ab} dz^a \wedge d\bar{z}^b \), then the local Hilbert space \( \mathcal{H}_m \) is the Fock space \( L^2_{\text{hol}}(T_m M, e^{-K}dz^a \wedge d\bar{z}^b / \hbar dw d\bar{w}) \), where \( w^a \) are the induced coordinates on \( T_m M \). \( dz^a |_m \) and \( d\bar{z}^b |_m \) act on \( \mathcal{H}_m \) as multiplication with \( w^a \) and differentiation \( \bar{w}^b = \hbar K_{\alpha\beta} \delta_{\alpha\beta} \) respectively, where as usual \( K_{\alpha\beta} \) are the matrix elements of the inverse matrix. In particular we have \((\bar{w}^a)^\dagger = \bar{w}^a \).

The so-called canonical line bundle \( K^P \) on \( M \) associated to \( P \) has fibre

\[
K^P_m = \{ \alpha \in \Lambda^n T^*_m M | \iota_X \alpha = 0 \ \forall X \in \mathcal{P}_m \}, \quad \text{(4.1)}
\]
where \( \iota_X \) denotes contraction with \( X \). In the real case \( K^P \) is spanned (locally) by \( d^n q \), in the complex one by \( d^n z \). Its definition requires a reduction of the structure group \( \text{Sp}(2n) \) of \( M \) to the subgroup preserving the polarization. For real polarizations the corresponding Lie algebra elements have the form \((2.12)\), where \( A \) can be restricted to \( \mathfrak{so}(n) \), as it is always possible to reduce to a maximal compact subgroup. Then \( K^P \) has structure group \( \mathbb{Z}_2 \), which we assume to be further reducible to the trivial group. In the complex case the Lie algebra of the structure group is \( \mathfrak{u}(n) \), spanned by elements \((2.15)\), and \( K^P \) has structure group \( U(1) \).

Further we need a square root of \( K^P \), i.e. a line bundle \( \Delta^P \) such that \( \Delta^P \otimes \Delta^P = K^P \). Such a square root exists if \( M \) admits a metaplectic structure; we will describe how this comes about at the end of this section. Elements of \( \Delta^P \) will be denoted by square roots of \( n \)-forms. In the real case \( \Delta^P \) is spanned locally by \( \sqrt{d^n q} \), in the complex one by \( \sqrt{d^n z} \).

Now we introduce the partial connection on \( \Delta^P \), which only allows differentiation in \( \mathcal{P} \)-direction. It is just the exterior derivative restricted to \( \mathcal{P} \)-vectors:

\[
\nabla_{\mathcal{P} \nu} = \mathcal{P} \nu, \quad X \in \Gamma(P), \nu \in \Gamma(\Delta^P). \quad \text{(4.2)}
\]

That the r.h.s. is well-defined can be seen from the fact that \( \nabla \) is induced by a partial connection on \( K^P \), which coincides with the exterior derivative acting on \( \Omega^\bullet(M) \) there. Consult \( \text{[21]} \) for details.

**Definition 4.3.** Let \( E \to M \) be a vector bundle with (partial) connection. A section \( \psi \) of \( E \) is called polarized if \( \nabla_{\mathcal{P} \nu} \psi = 0 \) for any \( X \in \Gamma(P) \), and the set of polarized sections is denoted \( \Gamma_P(E) \). If we want to emphasize the dependence on \( \nabla \) we also write \( \Gamma_{P,\nabla}(E) \). Further we define polarized functions to be elements of

\[
C^\infty_P(M) = \{ f \in C^\infty(M) | \mathcal{P} f = 0 \ \forall X \in \Gamma(P) \}.
\]

\( \Gamma_P(E) \) is a \( C^\infty_P(M) \)-module. Let \( P, P' \) be two polarizations, and \( E, F \) vector bundles with connections. Then one can take the tensor product \( \Gamma_P(E) \otimes \Gamma_P(F) \) in the category of \( C^\infty_P(M) \)-modules. However, the resulting space can be made into a \( C^\infty_{P \cap P'}(M) \) module, which we also denote by the tensor product. One easily checks that

\[
\Gamma_P(E) \otimes \Gamma_P'(F) \subset \Gamma_{P \cap P'}(E \otimes F). \quad \text{(4.3)}
\]

We only cite here the result that there is an isomorphism \( \Delta^P \otimes \mathcal{P}^\bullet \cong \pi^* \Delta(M/D) \), where \( D = (P \cap \mathcal{P}) \cap TM \), the space of maximal connected integral manifolds of \( D \) is denoted by \( M/D \), and
\( \Delta(M/D) \) is the density bundle on \( M/D \). One should think of its sections (called densities) as absolute values of differential forms of top degree, which can be integrated over \( M/D \). \( \pi : M \to M/D \) is the projection. We assume here and in the following that \( D \) has constant rank, and \( M/D \) is an orientable manifold. In the real case \( D \) has rank \( n \), for totally complex polarizations \( D \) is trivial and \( M/D = M \). The isomorphism of line bundles induces one on the level of sections:

\[
\Gamma_{P \cap \bar{P}}(\Delta^P \otimes \bar{\Delta}^P) \to \Gamma(\Delta(M/D)),
\]

which, due to (4.3), gives rise to a sesquilinear pairing

\[
(\cdot, \cdot) : \Gamma_P(\Delta^P) \times \Gamma_P(\Delta^P) \to \Gamma(\Delta(M/D)).
\]

The restriction to polarized sections is needed to ensure that the resulting section of \( \pi^* \Delta(M/D) \) is constant on the integral manifolds of \( D \), and so defines a section of \( \Delta(M/D) \). In the real case we have \( (\sqrt{d^nq}, \sqrt{d^nq}) = |d^nq| \), in the complex situation we get \( (\sqrt{d^n\bar{z}}, \sqrt{d^n\bar{z}}) = (\det g)^{1/4} \theta \), where the factor \( (\det g)^{1/4} \) comes from the canonical trivialization of \( \Delta^P \otimes \bar{\Delta}^P \), determined by the global, nonvanishing section \( (\det g)^{1/4} \theta \). For the general case see [21], p.230-234.

On \( \Delta^P \otimes B \) we have the product partial connection, and consider the set of polarized sections \( \Gamma_P(\Delta^P \otimes B) \). It carries a pairing \( \langle \cdot, \cdot \rangle_G \), given by

\[
\langle \mu \otimes a, \nu \otimes b \rangle_G = \int_{M/D} (\mu, \nu) \langle a, b \rangle_B.
\]

Our quantum Hilbert space is then of course the space of square-integrable elements

\[
\mathcal{H}_P = L^2_P(\Delta^P \otimes B) := \{ \psi \in \Gamma_P(\Delta^P \otimes B) \mid \langle \psi, \psi \rangle_G < \infty \}.
\]

### 4.2 Observables

We do not give the general construction of the observables here. Suffice it to say that only a very limited class of functions on \( M \) can be quantized in the traditional formalism of geometric quantization, in the real case these are the functions which are affine-linear in the momentum: \( f(q,p) = \alpha(q) + \beta^i(q)p_i \). The corresponding quantum observable is an operator on \( \mathcal{H}_P \) (unbounded in general). We can choose a local trivialization in which the connection on \( B \) takes the form \( \nabla^B = d - \frac{i}{\hbar} \theta \), where \( \theta = p_i dq^i \). Wave functions can then be represented as \( \psi \otimes \sqrt{d^nq} \), where \( \psi \) is a function of \( q \) alone, the operators are the canonical ones

\[
\frac{\partial}{\partial q^j} a(q) \otimes \sqrt{d^nq}, \quad \frac{\partial}{\partial q^j} a(q) \otimes \sqrt{d^nq} = -i\hbar \frac{\partial}{\partial a(q)} \sqrt{d^nq},
\]

and the function \( \beta^i(q)p_i \) is mapped to the symmetrized product of the operators \( \beta^i(\hat{q}) \) and \( \hat{p}_i \).

On a Kähler manifold we have \( \omega = i\partial\bar{\partial}K \), a symplectic potential is \( \theta = -i\partial\bar{\partial}K, B = \text{a holomorphic line bundle} \), and wave functions take the form \( \psi(z) \otimes \sqrt{d^nz} \). The Hermitian metric on \( B \) is nontrivial however, the condition for the connection \( \nabla^B \) to be compatible with \( \langle \cdot, \cdot \rangle_B \) implies

\[
\langle \psi, \phi \rangle_B = \psi(z)\phi(z)e^{-\frac{K}{\hbar}}.
\]

Holomorphic functions act as multiplication operators, and \( \partial_a K := \frac{\partial K}{\partial z^a} \) as

\[
\partial_a K(\psi \otimes \sqrt{d^nz})(z) = \hbar \frac{\partial}{\partial z^a} \psi(z) \otimes \sqrt{d^nz}.
\]
4.3 The role of $\mathcal{H}$

The line bundle $\Delta^P$ is closely related to the bundle $\mathcal{H}$ of symplectic spinors [11]. The situation turns out however to be essentially different for real and totally complex polarizations.

Real polarizations  In this case $(\Delta^P)^{-1}$ can be identified with the subbundle

$$\{ \psi \in \mathcal{H} \mid y\psi = 0 \ \forall y \in P^b \} \subset \mathcal{H},$$

where we consider $y \in T^*M$ as an element of the Weyl algebra $W \subset \text{End}(\mathcal{H})$. The space $P^b$ is spanned by the $\hat{q}^j$, $\mathcal{H}_m$ is the Schrödinger representation space $L^2(Q_m)$, where $Q_m \subset T_m M$ is spanned by the $\frac{\partial}{\partial q^j}$, and $(\Delta^P)^{-1}$ is generated by the $\delta$-function. The latter transforms as

$$\delta \mapsto \frac{1}{2 \text{tr}(A)} \delta$$

under a symplectic gauge transformation of $TM$ of the form (2.12), which can be seen from (2.13). Note that this is the right transformation behavior for $(\Delta^P)^{-1}$.

Kähler polarizations  In the complex case $\mathcal{H}_m$ is the Fock representation space consisting of holomorphic functions on $T_m M$, and $P^b$ is spanned by the $\hat{z}^j$, which act as differentiation operators. Therefore the (vacuum) line bundle

$$\{ \psi \in \mathcal{H} \mid y\psi = 0 \ \forall y \in P^b \},$$

is spanned by the constant functions on $T_m M$, which transform under an (infinitesimal) symplectic transformation of the form (2.15) as

$$c \mapsto -\frac{1}{2 \text{tr}(A)} c,$$

according to (2.16). But this is the transformation behavior of $\Delta^P$ itself, which we therefore identify with (4.11).

Note that the constant holomorphic functions act like multiples of a delta function in Fock space: if $\omega_m$ has standard form $idz^j \wedge d\overline{z}^j$, then the inner product on the local Fock space $\mathcal{H}_m$ also has standard form, and the inner product of a constant function $c$ with $\psi \in \mathcal{H}_m$ gives

$$\langle c, \psi \rangle_{\mathcal{H}_m} = c \int_{T^*M} \overline{\psi(z)} e^{-|z|^2/\hbar} d^m z d^n \overline{z} = \pi \hbar \overline{c \psi(0)}. \quad (4.12)$$

5 The Full Quantization

5.1 Pairing

Having constructed the quantum algebra $\mathcal{A} = (C^\infty(M), \ast)$ through deformation quantization, as well as the quantum Hilbert space $\mathcal{H}_P$ through geometric quantization, it is natural to ask whether one can define a representation of $\mathcal{A}$ on $\mathcal{H}_P$. If this is possible, the question arises whether one obtains the same operators for functions that are also quantizable in geometric quantization. As $\mathcal{A}$ and $\mathcal{H}_P$ depend on several parameters, the answer to the questions above might depend on them as well.

Recall that in the two quantization schemes the following structures occur:
| geometric quantization | deformation quantization |
|------------------------|--------------------------|
| metaplectic structure $\mathcal{H} \rightarrow M$ | metaplectic structure $\mathcal{H} \rightarrow M$ |
| deformation quantization | symplectic connection $\nabla$ |
| prequantum bundle $B \rightarrow M$; curvature $-\frac{i}{\hbar}\omega$ | connection $D_H(\nabla)$ on $\mathcal{H}$; curvature $\frac{i}{\pi}\omega$ |
| line bundle $\Delta^P$ | |

In the pure deformation case we did not know how to define states, because the connection on $\mathcal{H}$ was not flat. Now there is one obvious solution, to consider the product with the prequantum bundle: $\mathcal{H} \otimes B$. This is a Hermitian bundle with flat, metric connection, implying that

$$\Gamma_D(\mathcal{H} \otimes B) := \{ \psi \in \Gamma(\mathcal{H} \otimes B) \mid (D_H \otimes 1 + 1 \otimes \nabla_B)\psi = 0 \}$$

(5.1)

defines a Hilbert space, as the scalar product of two flat sections does not depend on the base point:

$$d \langle \psi, \phi \rangle = 0 \quad \forall \psi, \phi \in \Gamma_D(\mathcal{H} \otimes B).$$

The observables remain unchanged, and act as $\hat{f} \otimes 1$ on $\mathcal{H} \otimes B$, where $1$ is the identity on $B$. Here it is important that the induced connection on $\text{End}(\mathcal{H} \otimes B)$ is $D \otimes 1 + 1 \otimes d$ (because $B$ is a rank one bundle, and therefore its connection form is central in $\text{End}(B)$), so that $\hat{f} \otimes 1$ is parallel, and leaves the space (5.1) invariant.

However, the Hilbert space (5.1) is not the one from geometric quantization, given as a subset of $\Gamma_P(\Delta^P \otimes B)$. Suppose there was a sesquilinear pairing

$$\langle \cdot, \cdot \rangle : \Gamma_P(\Delta^P \otimes B) \times \Gamma_D(\mathcal{H} \otimes B) \rightarrow \mathbb{C}.\quad\quad(5.2)$$

Then we could define the quantum operator $\sigma(f)$ corresponding to $f \in C^\infty(M)$ through $\langle \sigma(f) \psi, \phi \rangle = \langle \psi, (\hat{f} \otimes 1)\phi \rangle$. Indeed there is a canonical pairing, its construction is most conveniently described separately for real and totally complex polarizations.

**Real polarizations**

We need the following definition, which has no counterpart in the complex situation:

**Definition 5.1.** A real polarization $P$ and a symplectic connection $\nabla$ are called compatible if

$$\Gamma_P((\Delta^P)^{-1} \otimes B) \times \Gamma_D(\mathcal{H} \otimes B) \xrightarrow{(\cdot, \cdot)_{\mathcal{H} \otimes B}} C_P^\infty(M).\quad\quad(5.2)$$

In general one expects the full space of functions $C^\infty(M)$ on the r.h.s.

We will discuss this condition at the end of the section, and assume it to be satisfied from now on. Then we can define a pairing

$$\langle \cdot, \cdot \rangle : \Gamma_P(\Delta^P \otimes B) \times \Gamma_D(\mathcal{H} \otimes B) \rightarrow \Gamma_P((\Delta^P)^{-1} \otimes K^P \otimes B) \times \Gamma_D(\mathcal{H} \otimes B) \rightarrow \mathbb{C},\quad\quad(5.3)$$

$$\left( \sqrt{d^n q} \otimes \mu, \psi \otimes \nu \right) \mapsto \langle \delta \otimes d^n q \otimes \mu, \psi \otimes \nu \rangle \mapsto \int_{M/D} \langle \mu, \nu \rangle_B \langle \delta, \psi \rangle_H d^n q$$

In the first step $\delta \otimes \sqrt{d^n q} \in (\Delta^P)^{-1} \otimes \Delta^P$ was inserted. We obtain

$$\langle \sqrt{d^n q} \otimes \mu, \psi \otimes \nu \rangle = \int_{M/D} \mu \overline{\psi}(0) d^n q.\quad\quad(5.4)$$

The condition (5.2) is necessary to ensure that $\langle \mu, \nu \rangle_B \overline{\psi}(0)$ is polarized (does not depend on $p$), and thus can be integrated over $M/D$. 

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**Complex polarizations** Here the situation appears to be simpler. We can identify $\sqrt{d^n}z$ with the locally defined constant section $c$ of $\Delta^P \subset \mathcal{H}$, whose value at any point is the constant wave function 1. Let $\pi$ be the orthogonal projection from $\mathcal{H}$ to $\Delta^P$, then we define

$$
\langle \cdot, \cdot \rangle : \Gamma_P(\Delta^P \otimes B) \times \Gamma_D(\mathcal{H} \otimes B) \to \Gamma_P(\Delta^P \otimes B) \times \Gamma(\Delta^P \otimes B) \to \mathbb{C},
$$

$$
(\sqrt{d^n}z \otimes \mu, \psi \otimes \nu) \mapsto (\sqrt{d^n}z \otimes \mu, \pi \psi \otimes \nu) \mapsto \int_M \langle \mu, \nu \rangle_B \psi, \pi \psi \rangle_{\mathcal{H}}(\det g)^{1/4}d^n z \wedge d^n \overline{z},
$$

using the pairing $(\sqrt{d^n}z, \sqrt{d^n}z) \mapsto (\det g)^{1/4}d^n z \wedge d^n \overline{z}$ from geometric quantization. Up to normalization this gives

$$
\langle \sqrt{d^n}z \otimes \mu, \psi \otimes \nu \rangle = \int_M \mu^D \overline{\psi}(0)e^{-\frac{K}{\hbar}}(\det g)^{1/4}d^n z \wedge d^n \overline{z}. \quad (5.6)
$$

In the following we will also assume the pairing between our Hilbert spaces to be non-degenerate, in the sense that the map $\mathcal{H}_P \to \Gamma_D(\mathcal{H} \otimes B)'$, $\psi \mapsto \langle \psi, \cdot \rangle$ is an isomorphism. This might impose a severe restriction on the admissible pairs of connections and polarizations, but it is difficult to find a more explicit formulation of this condition.

**Definition 5.2.** A representation $\sigma$ of $(C^\infty(M), \ast)$ on $\mathcal{H}_P$ is defined through

$$
\langle \sigma(f)\psi, \phi \rangle = \langle \psi, (f^\dagger \otimes 1)\phi \rangle \quad \forall \psi \in \mathcal{H}_P, \ \phi \in \Gamma_D(\mathcal{H} \otimes B). \quad (5.7)
$$

To be precise, some regularity conditions on $f$ will have to be imposed for the formula to make sense. In the flat case it would be enough to claim $f \in S'(\mathbb{R}^{2n})$ if one restricts the $\phi$s in (5.7) to sections of the Schwartz bundle $S \subset \mathcal{H}$, with fibre $S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$.

We will see that on cotangent bundles our definition reproduces the geometric quantization operators. In this particular example one has a distinguished polarization, and it is possible to find a symplectic connection satisfying the compatibility condition.

Formally the non-degeneracy condition on the pairing implies equivalence of the two algebra representations, so that nothing seems to be gained by the transition to $\mathcal{H}_P$. One has to remember however that the representation on $\Gamma_D(\mathcal{H} \otimes B)$ is in terms of infinite series in $\hbar$ and the local Weyl operators $y^\mu$, whereas on $\mathcal{H}_P$ one might hope to get an extension of geometric quantization by well-defined operators.

### 5.2 Compatibility

We want to discuss the implications of compatibility of a real polarization with a symplectic connection, in the sense of definition 5.1. Choose sections $\psi \in \Gamma_P((\Delta^P)^{-1} \otimes B)$ and $\phi \in \Gamma_D(\mathcal{H} \otimes B)$. From now on we denote $D_\mathcal{H}$ simply by $D$ as well, and write $D \otimes \nabla^B := D \otimes 1 + 1 \otimes \nabla^B$. Condition (5.2) takes the form $(\forall X \in \Gamma(P))$

$$
0 = X(\langle \psi, \phi \rangle_{\mathcal{H} \otimes B}) = \langle (D \otimes \nabla^B)_X \psi, \phi \rangle_{\mathcal{H} \otimes B} + \langle \psi, (D \otimes \nabla^B)_X \phi \rangle_{\mathcal{H} \otimes B}.
$$

The last term vanishes, implying $(D \otimes \nabla^B)_X \psi = 0$. In other words

$$
\Gamma_P,\nabla((\Delta^P)^{-1} \otimes B) \subset \Gamma_{P,D \otimes \nabla^B}(\mathcal{H} \otimes B), \quad (5.8)
$$
where $\nabla$ is the partial connection. If $\psi = \delta \otimes \mu$ with $\delta \in \Gamma_{P,\nabla}((\Delta^P)^{-1})$ and $\mu \in \Gamma_P(B)$ this implies $D_X \delta = 0$ for $X \in \Gamma(P)$, and therefore

$$\Gamma_{P,\nabla}((\Delta^P)^{-1}) \subset \Gamma_{P,D}(\mathcal{H}).$$

Note that the 'integrability condition'

$$(D_X D_Y - D_Y D_X - D_{[X,Y]}) \psi = \Omega(X,Y) \psi = 0 \quad \forall X, Y \in \Gamma(P)$$

for $\psi \in \Gamma_{P,D}(\mathcal{H})$ is satisfied due to the Lagrangian property of $P$, otherwise $\Gamma_{P,D}(\mathcal{H})$ would contain only the zero section. Of course, the same applies to $\Gamma_{P}$. This implies $A(X) \delta = 0$, where $A$ is the connection form (3.26) of $D = d + A$. In Darboux coordinates adapted to $P$ our condition becomes:

$$\left(\omega_{ka} y^a - \frac{1}{2} \omega_{ab} \Gamma^k_{bc} y^a y^c - \frac{1}{8} R_{abc} y^a y^b y^c + \ldots \right) \delta = 0,$$

(5.10)

where $k$ denotes an index in $P$, and the other ones are arbitrary. Due to the $\hbar$-grading all terms must vanish separately, in particular we have $\omega_{ka} y^a \delta = 0$, which is always satisfied due to $y \delta = 0$ for $y \in P^\flat$, cf. (4.10). In the next higher order we get a condition on the symplectic connection:

$$\Gamma_{kab} y^a y^b \delta = 0$$

(recall that $\Gamma_{abc}$ is fully symmetric in its indices). This amounts to saying that $\Gamma_{kla} = 0$, and implies

$$\nabla \Gamma(P) \subset \Gamma(P \otimes T^*M).$$

(5.11)

The third order gives

$$R_{abc} y^a y^b y^c \delta = 0,$$

implying that $R_{(abc)c} \delta$ vanishes if two or more of $a, b, c$ are in $P$. Here $(abc)$ denotes symmetrization in the indices. This condition however is automatically satisfied if the second order condition holds; the latter implies the vanishing of $R_{abcd}$ whenever three or more indices are in $P$, which can be seen from (3.1). We must leave it as an open question whether the higher order conditions really impose further constraints on $\nabla$.

A result of Bressler and Donin should be mentioned here [3]. They use a somewhat different formalism for deformation quantization, based on normal instead of Weyl ordering. Under the conditions

$$\nabla \Gamma(P) \subset \Gamma(P \otimes T^*M)$$

and

$$R(X,Y) Z = 0 \quad \forall X, Y, Z \in \Gamma(P)$$

they show that the star product constructed from $\nabla$ satisfies the equation $f \star g = fg$ for $f \in C^\infty_P(M)$ and arbitrary $g$ (which certainly does not hold for our star product; compare in the flat case). This strong result suggests that the two conditions should suffice to guarantee compatibility of $P$ and $\nabla$. They are both satisfied if the second order condition $\Gamma_{kla} = 0$ holds, and also for the Kähler connection and holomorphic polarization, see below.
6 Examples

6.1 The flat case with real polarization: $M = \mathbb{R}^{2n}$.

We introduce linear coordinates $(q^i, p_j)$, $i, j = 1, \ldots, n$, choose the constant symplectic form $\omega = dp_j \wedge dq^j$, the symplectic connection $\nabla = d$, and a real polarization given by $P = \text{span}\{\partial_{p_j} \mid j = 1, \ldots, n\}$. The Fedosov connection on $H$ is then

$$D = d + \frac{i}{\hbar}(dp_j \partial_{q^j} - dq^j \partial_{p_j}),$$

and the operator $\hat{f} \in \Gamma_D(W)$ corresponding to $f \in C^\infty(M)$ is given by (3.24), which is often written as $\hat{f}_{(q,p)} = f(q + \tilde{q}, p + \tilde{p})$.

Let us assume that the connection on the trivial prequantum bundle $B$ has the form $\nabla^B = d - \frac{i}{\hbar}p_j dq^j$. Choose a constant section $\mu \neq 0$ of $B$. Then $\psi \otimes \mu$ is in $\Gamma_D(H \otimes B)$ iff $\psi \in \Gamma(H)$ satisfies the following equation:

$$\left(d - \frac{i}{\hbar}p_j dq^j + \frac{i}{\hbar}(dp_j \partial_{q^j} - dq^j \partial_{p_j})\right) \psi = 0.$$ (6.1)

The value $\psi_{(q,p)}$ of $\psi$ in a point $(q, p) \in \mathbb{R}^{2n}$ is a wave function on $Q_{(q,p)} \subset T_{(q,p)}\mathbb{R}^{2n}$; we denote the variables in $Q_{(q,p)} \simeq \mathbb{R}^n$ by $\tilde{q} = (\tilde{q}^1, \ldots, \tilde{q}^n)$. The operators $\hat{q}^j$ and $\hat{p}_j$ in (6.1) act as multiplication with $\tilde{q}^j$ and differentiation $\frac{1}{\hbar} \partial_{\tilde{q}^j}$ respectively, and the solutions of equation (6.1) are given by

$$\psi_{(q,p)}(\tilde{q}) = \chi(q + \tilde{q}) e^{-\frac{i}{\hbar} \tilde{q} \tilde{p}},$$ (6.2)

for arbitrary functions $\chi \in L^2(\mathbb{R}^n)$. We want to determine the operator $\sigma(f)$ acting on the physical Hilbert space $L^2_p(\Delta^P \otimes B)$, defined by (5.7). Assuming $\langle \mu, \mu \rangle \equiv 1$, insertion of (6.2) into the pairing (5.4) results in

$$\langle a \sqrt{d^n q} \otimes \mu, \psi \otimes \mu \rangle = \int_{\mathbb{R}^n} a(q) \chi(q) d^n q.$$

Observing that $q^j + \tilde{q}^j$ acts on $\psi(\tilde{q} = 0)$ as multiplication operator $q^j$, and $p_j + \tilde{p}_j$ as differentiation $\frac{1}{\hbar} \tilde{p}_j$ applied to $\chi$, it is then obvious that $\sigma(f)$ is given by the well-known Schrödinger operator in Weyl ordering. In particular we have

$$\sigma(q^j)(a \sqrt{d^n q} \otimes \mu)(q) = q^j(a \sqrt{d^n q} \otimes \mu)(q) \quad \text{and} \quad \sigma(p_j)(a \sqrt{d^n q} \otimes \mu)(q) = \frac{\hbar}{i} (\partial_j a \sqrt{d^n q} \otimes \mu)(q).$$

6.2 The flat case with Kähler polarization: $M = \mathbb{C}^n$.

The quantization data are $\omega = idz^l \wedge d\bar{z}^a$, corresponding to the Kähler potential $K(z, \bar{z}) = |z|^2$, $\nabla = d$, and $P = \text{span}\{\partial_{\bar{z}^a} \mid j = 1, \ldots, n\}$. We choose a gauge in which $\nabla^B = d - \frac{1}{\hbar} \bar{z}^a dz^a$. The Fedosov connection on $H$ is

$$D = d + \frac{1}{\hbar}(dz^a \partial_{\bar{z}^a} - d\bar{z}^a \partial_z).$$

If $\mu \in \Gamma(B)$ is constant, then $\psi \otimes \mu$ is in $\Gamma_D(H \otimes B)$ iff $\psi$ satisfies

$$\left(d - \frac{1}{\hbar} \bar{z}^a dz^a + \frac{1}{\hbar}(dz^a \partial_{\bar{z}^a} - d\bar{z}^a \partial_z)\right) \psi = 0.$$ (6.3)

The solutions to this equation are given by

$$\psi_{(z, \bar{z})}(\tilde{z}) = \chi(z + \tilde{z}) e^{-\frac{1}{\hbar} \bar{z} \tilde{z}},$$ (6.4)
for arbitrary \( \chi \in L^2_{hol}(\mathbb{C}^n, e^{-|z|^2/h}d^nzdz) \). Now one easily checks that \( \bar{\tau} + \tilde{\tau} \) acts on \( \psi \) as differentiation \( h\partial_{z^a} \) applied to \( \chi \), and from the pairing between \( \mathcal{H}_P = L^2_P(\Delta^P \otimes B) \) and \( \Gamma_D(\mathcal{H} \otimes B) \):

\[
\langle \sqrt{d^n z} \otimes \mu, \psi \otimes \mu \rangle \sim \int_{\mathbb{C}^n} a(z)\bar{\psi}(0)|\mu|^2e^{-K/h}d^nzdz = \int_{\mathbb{C}^n} a(z)\bar{\chi}(z)e^{-|z|^2/h}d^nzdz
\]

it follows that one obtains the standard Fock space operators.

### 6.3 Cotangent bundles

Let \((Q,g)\) be a (semi-)Riemannian manifold, and \( M = T^*Q \). Suppose one has chosen local coordinates \( q^i \) on \( Q \), then there is a set of canonical coordinates on \( M \), given by \((q^i,p_j)\), where \( p_i(q^i) = \delta^i_j \). \( M \) carries a canonical symplectic form \( \omega = dp_j \wedge dq^i \), as well as a canonical polarization, given by the vertical subbundle of \( TM \), i.e. \( P_{(q,p)} = \text{span}\{\partial_{p_1}, \ldots, \partial_{p_n}\} \subset T_{(q,p)}T^*Q \). The prequantum bundle \( B \) can be chosen to be trivial, as its curvature \( \omega = d(p_jdq^i) \) is exact. It is not so obvious how to choose the symplectic connection. As we assumed \( Q \) to be equipped with a metric, we have the Levi-Civita connection \( \nabla^{LC} \) on \( Q \). Bordemann et.al. have shown that there is a canonical lift of \( \nabla^{LC} \) to \( M \), which is symplectic and leads to a star product with nice properties \[2\]. In particular the product of two functions polynomial in \( p \) will be a finite polynomial in \( h \) and \( p \), so that \( h \) need not be considered as a formal element in the end, but can be given its physical value.

We will use the following notational convention. A latin index \( i \) runs from 1 to \( n \) and stands for \( q^i \), whereas a barred latin index \( \bar{i} \) denotes \( p_i \). Greek indices run from 1 to \( 2n \), and include all coordinates. Generators of the local Weyl algebra \( W_m \) are denoted by \( y^\mu \) (as in section \[3\]), or by \( \bar{q}^i \) and \( \bar{p}_i \) (not to be confused with the elements of \( \Gamma_D(W) \) corresponding to the functions \( q^i, p_i \). We never use the symbols \( \bar{q}^i, \bar{p}_i \) in this latter sense, although it would be consistent with \( f \mapsto \hat{f} \in \Gamma_D(W) \)). One forms are \( dx^\mu \), or \( dq^i \) and \( dp_i \). The product \( y^\mu y^\nu \) still denotes the symmetric tensor product, whereas \( \bar{q}^i \bar{p}_i \) is composition of operators, or the Weyl product. The connection and curvature coefficients of the lifted connection on \( M \) are

\[
\Gamma_{ij}^k = -\Gamma_{ik}^j = -\Gamma_{jk}^i = \hat{\Gamma}_{ij}^k,
\]

\[
\Gamma_{ij}^\bar{k} = \frac{p^a}{3} \left( 2\Gamma_{ji}^a \hat{\Gamma}_k^a - \partial_j \hat{\Gamma}_{ik}^a + \text{cycl.}(ijk) \right),
\]

\[
R_{kij} = \hat{R}_{kij} = \hat{R}_{kij},
\]

\[
R_{kij} = \hat{R}_{kij} = \frac{1}{3}(\hat{R}_{kij} + \hat{R}_{kji} + \hat{R}_{jki}),
\]

\[
R_{kijl} = \frac{p^a}{3} \left( \nabla_i \hat{R}_{jkl} - 3\hat{R}_{im} \hat{R}_{jkl} - \hat{R}_{im} \hat{R}_{jkl} + \hat{R}_{km} \hat{R}_{ijl} + (i \leftrightarrow j) \right),
\]

where \( \hat{\Gamma}_{ij} \) and \( \hat{R}_{ijkl} \) are the Christoffel symbols and curvature tensor on \( Q \), and \( \Gamma_{ij}^\bar{k}, R_{kijl} \) the lifted objects on \( T^*Q \). Components not listed vanish. Notice in particular that our first order condition \( \Gamma_{klm} = \Gamma_{kml} = 0 \) from \( \[5.10\] \) is satisfied. Now we want to show that the lifted connection \( \nabla \) is compatible with the vertical polarization in every order in \( h \). We need the following

**Definition 6.1.** For a homogeneous element

\[
\eta = y^{\mu_1} \cdots y^{\mu_k} dx^{\nu_1} \wedge \cdots \wedge dx^{\nu_l} \in \Gamma(\text{Sym}_k(T^*M) \otimes \Lambda^l T^*M)
\]

\((f \in C^\infty(M))\) we define its \( Q \)-degree, \( q(\eta) \), as the difference of the number of \( \hat{q}^i \)'s minus the one of \( \hat{p}_j \), plus the number of \( dq^i \)'s minus the one of \( dp_j \). The \( Q \)-degree defines a (vector space) filtration on \( \Gamma(\mathcal{W} \otimes \Lambda T^*M) \); we write \( q(\eta) > m \) if this holds for every homogeneous component of \( \eta \in \Gamma(\mathcal{W} \otimes \Lambda T^*M) \).
Note that the $Q$-degree is not additive in general: $q(\eta \circ \chi) \neq q(\eta) + q(\chi)$, because of the relation $dq^i \wedge dq^j = 0$. It is additive only when restricted to suitable sections:

$$q(\eta \circ \chi) = q(\chi \circ \eta) = q(\eta) + q(\chi), \quad \forall \eta \in \Gamma(W), \chi \in \Gamma(W \otimes \Delta^* TM). \quad (6.7)$$

Let us investigate the $Q$-degree of the connection form $A$ of $D$, and the operator in $\Gamma_D(W)$ corresponding to a function. We use the notation of section [3]

**Lemma 6.2.** The following hold:

- $q(A) \geq 0$,
- $q(A - \frac{i}{\hbar} \omega_{\mu
u} y^\nu dx^\mu) > 0$, unless $\Gamma = 0$,
- $q(r - r^{(k)}) > k - 1$ for $k \geq 3$, unless $r - r^{(k)} = 0$.
- Let $f \in C^\infty(M)$ be a polynomial of degree $N$ in $p$, with arbitrary $q$-dependence. Then $q(\hat{f} - \hat{f}^{(k)}) > k - 2N$, unless $\hat{f} - \hat{f}^{(k)} = 0$.

**Proof.** The first two properties follow from the third, as the first terms in $A$ have $Q$-degrees $= 0$ and $> 0$ respectively. This is because $\Gamma_{\alpha\beta\gamma}$ vanishes if two or more indices are barred, as can be read off from (6.5). The same holds true for $R_{\alpha\beta\gamma\mu}$, so that the third term $r^{(3)}$ in $A$

$$r^{(3)} = -\frac{i}{8\hbar} R_{\alpha\beta\gamma\mu} y^\alpha y^\beta y^\gamma dx^\mu$$

has $q(r^{(3)}) > 1$. Now we make use of the iteration formula (3.14):

$$r^{(n+1)} = \delta^{-1} \hat{R} + \delta^{-1} \left(\nabla r^{(n)} + \frac{i}{\hbar} (r^{(n)})^2\right) \mod h^{(n+2)/2}$$

for the higher order terms in $r$. Note that $\delta^{-1}$ leaves the $Q$-degree invariant. We have $\nabla = d + [dU(\Gamma),.]$, where $d$ can only lower the $Q$-degree of the term proportional to $R_{ijkl} \hat{q}^i \hat{q}^j \hat{q}^k dq^l$ in $r^{(3)}$, which remains $> 2$. The commutator $-\frac{i}{\hbar} \Gamma_{\alpha\beta\gamma} dx^\gamma [y^\alpha y^\beta,.]$ increases the $Q$-degree by 1 (unless the result of its application vanishes), due again to $q(dU(\Gamma)) \geq 1$. In the higher order iteration steps $\nabla$ can only decrease the $Q$-degree of terms containing either $R_{ijkl}$ or $\Gamma_{ijk}$ (only unbarred indices!), as these are linear in $p$ (cf. (6.5)). These however must arise from terms $\Gamma_{ijk} \hat{q}^i \hat{q}^j dq^k$ or again the starting term $R_{ijkl} \hat{q}^i \hat{q}^j \hat{q}^k dq^l$, which both have a two units higher $Q$-degree as we would expect. Therefore we can infer that effectively every application of $\nabla$ increases the $Q$-degree by 1, where we have still ignored the $(r^{(n)})^2$-term in the iteration formula. It is obvious from the (almost) additivity of the $Q$-degree that its inclusion does not change the picture. The same reasoning then works for $\hat{f}$, using the iteration formula (3.15), and the fact that $\nabla$ can decrease the $q$-degree $N$ times.

**Proposition 6.3.** In the situation of this paragraph we have $\Gamma_{P,\nabla}((\Delta^F)^{-1}) \subset \Gamma_{P,D}(H)$ (cf. 5.7), thus the pairing (5.3) is well defined.

**Proof.** As discussed above equation (5.10) we must show that $A_{(q,p)}(X)\delta = 0$ for every $X \in P_{(q,p)}$, where $\delta$ is the delta distribution on $Q_{(q,p)} \cong T_qQ$. This is the case if every homogeneous term in $A_\delta = A(\partial_{q^k})$ contains more operators $q^j$ than $p_j$s, i.e. if $q(A_\delta) > 0$. But we have $q(A_\delta) \geq q(A) + 1 > 0$ from the lemma above. \[\square\]
We conclude that the symplectic connection (6.5) is compatible with the vertical polarization of $T^*Q$, and (5.7) defines an operator $\sigma(f)$ on the physical Hilbert space. Now we can compare our formalism to geometric quantization.

**Proposition 6.4.** If $f$ is quantizable in geometric quantization, then $\sigma(f)$ coincides with the geometric quantization operator.

**Proof.** Let $f(q,p) = f(q)$ be a polarized function, and $g_j(q,p) = p_j$. From Lemma 6.2 we have

$$q(\hat{f} - f) > 0, \quad q(\hat{g}_j - g_j^{(2)}) > 0. \quad \text{(6.8)}$$

It immediately follows that $(\hat{f}\psi)(0) = f\psi(0)$ for $\psi \in \Gamma(H)$. Recalling the explicit solution (3.19)

$$\hat{g}_j = g_j + \partial_\mu g_j y^\mu + \frac{1}{2}(\partial_\nu \partial_\mu - \Gamma^\kappa_{\mu\nu})g_j y^\mu y^\nu + O(h^{3/2})$$

to the iteration equation (3.18) for $\hat{g}_j$, we obtain

$$\langle \hat{g}_j \psi \rangle(0) = (p_j + \hat{p}_j - \frac{1}{2}\Gamma_{ab}^j \hat{p}_b q^a)\psi(0). \quad \text{(6.9)}$$

This looks rather complicated yet, but will simplify in a moment. We want to apply the operators $\hat{f} \otimes 1$, $\hat{g}_j \otimes 1$ to elements of $\Gamma_D(H \otimes B)$. Suppose that $\mu$ is a constant section of $B$. Then the condition for $\psi \otimes \mu$ to be in $\Gamma_D(H \otimes B)$ is $(d + A - \frac{i}{\hbar}\theta)\psi = 0$, where $\theta = p_j dq^j$ is the connection form on $B$. But we know from Lemma 6.2 that $q(A - \frac{i}{\hbar}\omega_{ab} y^b dx^a) > 0$, and $q(r) > 1$ (unless $r$ vanishes). Therefore the condition evaluated in $\hat{q} = 0$ takes the simple form

$$(d + A - \frac{i}{\hbar}\theta)(\partial_{q^j})\psi(0) = 0 \quad \text{(6.10)}$$

The first equation tells us that $\psi(0)$ does not depend on $p$, ensuring that the integral in our pairing (5.4) is well defined. Using $\Gamma_{ab} = -\Gamma_{ab}^j$ in Darboux coordinates, the second equation together with (6.9) gives

$$(\hat{g}_j \psi)(0) = \frac{\hbar}{i} \partial_{q^j} \psi(0). \quad \text{(6.11)}$$

From the pairing $\langle \cdot, \cdot \rangle$ between $\Gamma_P(\Delta^P \otimes B)$ and $\Gamma_D(H \otimes B)$

$$\langle a \sqrt{d^aq} \otimes \mu, \psi \otimes \mu \rangle = \int_Q a(q)\sqrt{\psi_q(0)}dx^a \quad \text{(6.12)}$$

(if $(\mu, \mu)_B \equiv 1$) we then obtain the usual geometric quantization operators, i.e.

$$\sigma(f)(a \sqrt{d^aq} \otimes \mu) = f(q)a(q)\sqrt{d^aq} \otimes \mu, \quad \text{(6.13)}$$

$$\sigma(p_j)(a \sqrt{d^aq} \otimes \mu) = \frac{\hbar}{i} \partial_{a} a(q)\sqrt{d^aq} \otimes \mu.$$

We still have to check that $\sigma(\alpha^j(q)p_j)$ gives the symmetrized product of $\alpha^j(\sigma(q))$ and $\sigma(p_j)$. But

$$\langle \alpha^j \psi \rangle(p_j) = \left[\alpha^j p_j + \partial_{k} \alpha^j p_j q^k + \frac{1}{2}(\partial_{l} \alpha^j - a^{k} \Gamma_{lj}^{k})(q^l \hat{p}_j + \hat{p}_j q^l)\right]\psi(0)$$

$$= \left[\alpha^j (p_j + \hat{p}_j - \frac{1}{2}\Gamma_{ab}^j \hat{p}_b q^a) + \frac{1}{2}(\partial_{l} \alpha^j \hat{p}_j q^l)\right]\psi(0)$$

$$= \left[\alpha^j \hat{g}_j - \frac{i\hbar}{2} \partial_{a} \alpha^j\right]\psi(0),$$

which leads to the symmetrized product. □
The next proposition is a pure deformation quantization result.

**Proposition 6.5.** Let \( f \) and \( g \) be polynomials of degree \( N \) and \( M \) in \( p \) respectively (with arbitrary dependence on \( q \)), and no \( h \)-dependence. Then \( f \ast g \) is a polynomial of degree \( \leq N + M \) in \( h \).

**Proof.** Lemma 6.2 and the additivity of the \( Q \)-degree imply that \( q(\hat{f} - \hat{f}^{(r)}) \circ (\hat{g} - \hat{g}^{(s)}) > r + s - 2(N + M) \). But we have \( \pi_0^0(\chi) = 0 \) if \( g(\chi) > 0 \) for \( \chi \in \Gamma(W) \) (\( \pi_0^0 \) projects onto the part containing no operators \( \hat{q}, \hat{p} \), definition (3.8)). Therefore only terms \( \hat{f}^{(r)} \circ \hat{g}^{(s)} \) with \( r + s \leq 2(N + M) \) contribute to \( f \ast g \) (defined in (3.21)), but these have \( h \)-degree \( \leq \frac{2(N + M)}{2} = N + M \). \( \Box \)

In fact a stronger result holds true [2]; define \( H = p_j \frac{\partial}{\partial p_j} + h \frac{\partial}{\partial h} \), then

\[
H(f \ast g) = (Hf) \ast g + f \ast (Hg),
\]
a property called homogeneity in [2]. Recall the expansion in \( h \) of the star product

\[
f \ast g = fg + \sum_{k=1}^{\infty} h^k c_k(f, g).
\]

We have shown that the series is indeed finite if \( f \) and \( g \) are polynomials in \( p \). A monomial \( h \) of degree \( N \) in \( p \) can be split as \( h = h^p p_j \), where \( h^p \) is a monomial of degree \( N - 1 \). As our quantization map \( \sigma \) is a homomorphism w.r.t. the star product, we obtain

\[
\sigma(h) = \sigma(h^p) \sigma(p_j) - \sum_{k=1}^{\infty} h^k \sigma(c_k(h^p, p_j)), \tag{6.14}
\]

enabling us to determine \( \sigma(h) \) recursively, because all the functions on the r.h.s. are polynomials of degree at most \( N - 1 \), and we know the operators corresponding to affine-linear functions. The calculations involved become very complex however, already for polynomials of degree 2! In this case the formula (for two affine-linear functions) reads

\[
\sigma(fg) = \sigma(f)\sigma(g) - \frac{i}{2} \sigma(\{f, g\}) - \frac{h^2}{4} \sigma((\nabla_\mu X f)^\nu (\nabla_\nu X g)^\mu). \tag{6.15}
\]

In [15] the operator corresponding to the kinetic energy term \( g^{ab}(q)p_a p_b \) has been determined this way, the result being

\[
\sigma(g^{ab} p_a p_b)(\psi \otimes \sqrt{\mu}) = -\hbar^2 \left[ (\Delta - \hat{R}) \psi \right] \otimes \sqrt{\mu}, \tag{6.16}
\]

for \( \psi \in \Gamma p(B) = C^\infty(Q) \), \( \mu = \sqrt{\det g} d^n q \), where \( \Delta \) is the Laplace-Beltrami operator, and \( \hat{R} \) the scalar curvature of \( Q \). The section \( \sqrt{\det g} d^n q \) instead of \( d^n q \) has been chosen to make the result independent of coordinates, so that Riemannian normal coordinates can be used, which simplify the calculation.

**Comparison** A different quantization scheme for cotangent bundles exists, where the space of states is \( L^2(Q, \sqrt{\det g} d^n q) \) [12]. It starts from the observation, that in the flat case the Schrödinger quantization map can be written as

\[
\hat{f} \psi(x) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} e^{ip(x-q)} f(\frac{x+q}{2}, p) \psi(q) dq d^np
\]
for $f \in \mathcal{S}'(\mathbb{R}^{2n})$, $\psi \in \mathcal{S}(\mathbb{R}^n)$. In order to generalize this formula to the curved case, we rewrite it in terms of the exponential function in the sense of differential geometry. Recall that in the flat case we have $T_x^*\mathbb{R}^n \cong T_x\mathbb{R}^n \cong \mathbb{R}^n$ and

$$\exp_x : T_x\mathbb{R}^n \to \mathbb{R}^n, \quad v \mapsto x + v,$$

implying $\frac{x-q}{2} = \exp_{\frac{x}{2}}^{-1}(x)$. Replacing $\mathbb{R}^n$ by a Riemannian manifold $Q$, we also have to replace the midpoint $\frac{x+q}{2}$ by the geodesic midpoint

$$m(x, q) := \pi(\nu^{-1}(x, q)),$$

where

$$\nu : TQ \to Q \times Q, \quad X \mapsto (\exp_{\pi(X)}(\frac{1}{2}X), \exp_{\pi(X)}(-\frac{1}{2}X)).$$

Taking into account the measures $\sqrt{\det g(q)}d^nq$ on $Q$ and $\frac{dp}{\sqrt{\det g(q)}}$ on $T_q^*Q$, one arrives at the formula

$$\hat{f}\hat{\psi}(x) = \frac{1}{(2\pi\hbar)^n} \int_Q d^nq \int_{T(m(x), q)^*Q} d^n p \sqrt{\det g(q)} e^{\frac{2i}{\hbar}p \cdot (\exp^{-1}_m(x, q) \cdot \exp^{-1}_m(x, q) \cdot f(m(x, q), p) \psi(q)).} \quad (6.17)$$

This is well defined for functions $f$ that are polynomial in $p$, in general one has to include a cutoff function, as $\nu$ is not a global diffeomorphism $TQ \cong Q \times Q$. The details can be found in [12], where also the operators corresponding to functions $q^i, p_j$, and $g^{ab}(q)p_a p_b$ have been calculated in this formalism, with the results

$$\hat{q}^i = q^i, \quad \hat{p}_j = -i\hbar(\partial_j + \frac{1}{2} \Gamma^k_{kj}), \quad g^{-1}(p, p) = -\hbar^2(\Delta - \frac{\nabla^2}{\partial^2}). \quad (6.18)$$

The first two operators again coincide with the ones from geometric quantization if $\sqrt{\det gd^nq}$ is chosen as constant section of $\Delta^P$. The kinetic energy operator however is different from the one we obtained (6.16).

It appears to us as a drawback of the above quantization scheme that the metaplectic group, which is the symmetry group of quantum mechanics on a symplectic vector space, does not seem to play any role there. But there is no criterion to decide which symplectic quantization is the physical one.

### 6.4 A remark on Kähler manifolds

On a Kähler manifold one has the holomorphic polarization, and also a canonical symplectic connection, the Levi-Civita connection of the Kähler metric $g$. Its nonvanishing components are [9]

$$\Gamma_m^o = -i\partial_\pi \partial_\theta \partial_\tau K, \quad \Gamma_{\alpha\beta} = i\partial_\alpha \partial_\tau \partial_\sigma K.$$ 

We are interested in the operators corresponding to holomorphic functions, as well as the function $\frac{\partial}{\partial \tau^2} K$, in order to compare to the geometric quantization operators (6.19). In the case of cotangent bundles we made use of the fact that the covariant derivative $D$ on $\mathcal{H}$ contained only finitely many terms contributing to $D\psi(0)$, which made it possible to determine the operator corresponding to $p_\sigma$ in the proof of proposition 6.4.

There seems to be no analogous property of $D$ when constructed from a Kähler connection, and the method from above cannot be applied here. We take a look at the first terms appearing in the
corresponding expressions nevertheless. Let $\mu \in \Gamma(B)$ be constant (locally, in a trivialization with $\theta = -i\partial K$), then $\psi \otimes \mu$ is in $\Gamma_D(\mathcal{H} \otimes B)$ iff $(D - \frac{i}{\hbar}\theta)\psi = 0$. We calculate

$$(D - \frac{i}{\hbar}\theta)\frac{\partial}{\partial \bar{z} \bar{a}}\psi(0) = \left[\frac{\partial}{\partial \bar{z} \bar{a}} - \frac{1}{\hbar}\partial_a K - \frac{1}{\hbar}\partial_a \partial_c K \bar{z}^b - \frac{1}{4\hbar}\partial_a \partial_b \partial_c K \bar{z}^c \bar{z}^b + \mathcal{O}(\hbar^{1/2})\right]\psi(0),$$

$$(D - \frac{i}{\hbar}\theta)\frac{\partial}{\partial \bar{z} \bar{a}}\psi(0) = \left[\frac{\partial}{\partial \bar{z} \bar{a}} + \frac{1}{4\hbar}\partial_a \partial_b \partial_c K \bar{z}^c \bar{z}^b + \mathcal{O}(\hbar^{1/2})\right]\psi(0). \quad (6.19)$$

In the real case the second equation was $(D - \frac{i}{\hbar}\theta)\psi(0) = \frac{\partial}{\partial \bar{p}}\psi(0) = 0$, and told us that $\psi(0)$ was independent of $p$. One would expect to find $\psi(0)$ being holomorphic here, instead we have

$$\frac{\partial}{\partial \bar{p}}\psi(0) = -\frac{i}{4}\Gamma_{\bar{c}a}^\tau\psi(0) + \mathcal{O}(\hbar^{1/2}).$$

Further, the two equations (6.19) do not seem to imply any simplifications on the operators $\hat{f}^\dagger = \hat{f}$ and $\hat{\partial_a}K^\dagger = \hat{\partial_a}K$, for holomorphic $f$, which are needed to determine $\sigma(f)$ and $\sigma(\partial_a K)$. There is still an obvious necessary condition for the results $\sigma(f) = f$ and $\sigma(\partial_a K) = \hbar \frac{\partial}{\partial \bar{z} \bar{a}}$, as well as symmetrized operators for products $f\partial_a K$. For the star product these would imply the relations $f * g = fg$ for holomorphic $f, g$, and

$$f * \partial_a K = f\partial_a K + \frac{i\hbar}{2}\{f, \partial_a K\}, \quad (6.20)$$

which can be checked in pure deformation quantization. There is no obvious way however to calculate the star products exactly. A term by term calculation up to $\mathcal{O}(\hbar^4)$ suggests that they might indeed hold [15].

One should also keep in mind that it is not strictly necessary to work with the Kähler connection here, in principle any symplectic connection is possible. But the condition (6.20) will certainly not be satisfied for arbitrary connections, it is indeed very restrictive and might even single out the Kähler connection. It is also questionable whether the pairing we constructed is always non-degenerate.

After all there is not much we can say about Kähler manifolds, except in the trivial case $M = \mathbb{C}^n$. This might be an indication that the presented formalism needs some modification in general.

7 Conclusion

We have shown that it is possible to define a representation of the deformation quantization algebra on the Hilbert space of geometric quantization, if some conditions are satisfied. For real polarizations one has to find a compatible connection (definition 5.1), and in general the pairing of the different Hilbert spaces should induce an isomorphism between them.

The difficulties in geometric quantization to define operators of higher order in $p$ or $\bar{z}$ can be explained in this formalism by the fact that they depend on the chosen connection, and are therefore not uniquely determined by the geometric quantization data $(\omega, P, B, \mathcal{H})$.

An open problem is the determination of the quantum operators on a Kähler manifold. If this can be solved, the resulting theory should be compared to existing quantization schemes on Kähler manifolds, like Berezin, Berezin-Toeplitz (these are generalizations of normal and antinormal ordered quantization, and we do not expect our quantization to coincide with either of them) and of course geometric quantization. See [1] and references therein for an overview of different quantization schemes.
When should we consider two quantizations \((A_1, \mathcal{H}_1)\) and \((A_2, \mathcal{H}_2)\) of \((M, \omega)\) as equivalent? Of course, they must give the same expectation values for the observables. Suppose there is an algebra isomorphism \(\alpha : A_1 \rightarrow A_2\) and a unitary map \(U : \mathcal{H}_1 \rightarrow \mathcal{H}_2\) such that \(\alpha(f) = U f U^{-1}\). Then the two quantizations are equivalent if we are willing to identify \(\psi \in \mathcal{H}_1\) with \(U \psi \in \mathcal{H}_2\) and \(f \in A_1\) with \(\alpha(f) \in A_2\), because

\[
\langle \psi, f \psi \rangle_1 = \langle U \psi, \alpha(f) U \psi \rangle_2.
\]

(7.1)

This is an appropriate mathematical notion of equivalence, physically however the elements of the algebra are fixed observables (recall that as sets our algebras are contained in \(C^\infty(M)\)), and we cannot identify \(f\) with the different observable \(\alpha(f)\).

If an algebra is compatible with two different polarizations, we still want to identify the two systems, so if the corresponding representations are \(\sigma_i : \mathcal{A} \rightarrow \text{End}(\mathcal{H}_i), i = 1, 2\), and \(U : \mathcal{H}_1 \rightarrow \mathcal{H}_2\) is a unitary map satisfying

\[
\sigma_2(f) = U^{-1} \sigma_1(f) U
\]

(7.2)

then we consider \((\mathcal{A}, \mathcal{H}_1)\) and \((\mathcal{A}, \mathcal{H}_2)\) as equivalent, and identify \(\psi \in \mathcal{H}_1\) with \(U \psi \in \mathcal{H}_2\). Note that \(\sigma_i\) are really the particular representations constructed above, and determine \(U\) uniquely up to a phase.

Another way to see that not all quantizations satisfying (7.1) are equivalent, is to consider a fixed polarization with two different algebras. We know that \(|\psi|^2\) is supposed to have a physical meaning as a probability density on \(Q = M/D\), and unless \(U \in U(1)\) we cannot identify \(\psi\) with \(U \psi\), because they define different densities. Therefore we must consider any two quantum systems constructed from different symplectic connections as physically inequivalent.

What is then the moduli space of inequivalent quantizations of \((M, \omega)\)? Assuming that we can always find a unitary intertwiner \(U\) satisfying (7.2), the polarization is determined by the connection up to equivalence (however, there are symplectic manifolds not admitting any polarization at all, which are therefore not quantizable [7]). There is then still a dependence on the prequantum bundle \(B\), the metaplectic structure \(\mathcal{H}\), and the symplectic connection \(\nabla\). In the example of cotangent bundles we have seen that different symplectic connections correspond to physically inequivalent classical systems. The same is true for the prequantum bundle \(B\), but the interpretation of the metaplectic structure is not obvious. Neither do we understand the meaning of the symplectic connection in general. It is also very questionable whether one can always find a unitary intertwiner \(U\) as in (7.2) for different polarizations compatible with a given connection, because the set of compatible polarizations contains at least all totally complex polarizations. A possible way out might be to impose a compatibility condition on these as well, like the one obtained by Bressler and Donin [3], see our discussion at the end of section 5.2, or to require holomorphic functions to be quantized to multiplication operators. Further one needs the pairing between the different Hilbert spaces to be non-degenerate, which might also restrict the admissible polarizations. The construction of \(U\) is discussed in [21].

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