Stable conjugacy and epipelagic $L$-packets for Brylinski–Deligne covers of $\text{Sp}(2n)$

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Abstract

Let $F$ be a local field of characteristic not $2$. We propose a definition of stable conjugacy for all the covering groups of $\text{Sp}(2n, F)$ constructed by Brylinski and Deligne, whose degree we denote by $m$. To support this notion, we follow Kaletha’s approach to construct genuine epipelagic $L$-packets for such covers in the non-archimedean case with $p \nmid 2m$, or some weaker variant when $4 \mid m$; we also prove the stability of packets when $F \supset \mathbb{Q}_p$ with $p$ large. When $m = 2$, the stable conjugacy reduces to that defined by J. Adams, and the epipelagic $L$-packets coincide with those obtained by $\Theta$-correspondence. This fits within Weissman’s formalism of $L$-groups. For $n = 1$ and $m$ even, it is also compatible with the transfer factors proposed by K. Hiraga and T. Ikeda.

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Denote by $G$ internal structure of these conjectural packets is related to the following issues. Let $\tilde{G}$ be a reductive group over a local field $F$. The covering groups in question are topological central extensions $1 \to \mu_m \to \tilde{G} \xrightarrow{\phi} G(F) \to 1$ where $m \in \mathbb{Z}_{\geq 1}$ and $\mu_m := \{z \in \mathbb{C}^\times : z^m = 1\}$. Harmonic analysis on $\tilde{G}$ aims at studying its genuine representations, that is, the representations $(\pi,\pi)$ of $\tilde{G}$ satisfying $\pi(\epsilon) = \epsilon \cdot \text{id}_{V_\pi}$ for all $\epsilon \in \mu_m$. In contrast with the case of reductive groups ($m = 1$), an important issue here is to single out a wide class of coverings admitting extra algebraic structures, and then try to formulate the local Langlands correspondence, etc.

Denote by $\mu(F)$ (resp. $\mu_m(F)$) the group of roots of unity (resp. $m$-th roots of unity) in $F$, and set $N_F := |\mu(F)|$ when $F \neq \mathbb{C}$, otherwise $N_F := 1$. Brylinski and Deligne built a functorial framework in [7] that produces central extensions of $G(F)$ by $K_2(F)$, or more precisely central extensions of $G$ by Quillen’s $K_2$ as Zariski sheaves. This includes the $K_2(F)$-extensions constructed by Matsumoto [37] for simply connected split groups. To obtain a cover, take $m | N_F$ and write $\mu_m = \mu_{m(F)}$. Pushing-out first by the Hilbert symbol $(\cdot,\cdot)_{F,m} : K_2(F) \to \mu_m$ and then by a chosen $\epsilon : \mu_m \to \mu_m \subset \mathbb{C}^\times$, we obtain a topological central extension $\mu_m \to \tilde{G} \to G(F)$ as above. Covers arising in this way are called $BD$-covers in this article, and one has the notion of $\epsilon$-genuine representations. This theory has a global counterpart and works over more general bases than $\text{Spec}(F)$.

Variants of Langlands’ program in this perspective have been proposed and studied in [14, 57], among others. In particular, Weissman [57] proposed a definition of $L$-group $L^{\Box} \tilde{G}$ for $BD$-covers as an extension $\tilde{G}^\vee \to L^{\Box} \tilde{G} \to \mathcal{W}_F$ of locally compact groups. One of his insights is that $L^{\Box} \tilde{G}$ is not always splittable into $\tilde{G}^\vee \times \mathcal{W}_F$; even when splittings exist, they often depend on auxiliary data.

One expects that the genuine irreducible admissible representations are organized into $L$-packets. Denote by $G_{\text{reg}}$ the open dense subset of strongly regular semisimple elements in $G$. It seems that the internal structure of these conjectural packets is related to the following issues.

(A) Of particular importance in harmonic analysis on $\tilde{G}$ are the good elements, i.e. those $\delta \in \tilde{G}$ with image $\delta$ satisfying $Z_{\tilde{G}}(\delta) = \mathcal{P}^{-1}(Z_{G(F)}(\delta))$; this property depends only on $\delta$. For $BD$-covers of a torus $T$, there is a canonical isogeny $\iota_{Q,m} : T_{Q,m} \to T$; it is known that all elements of $\text{im}(\iota_{Q,m})$ are good, and the converse holds when $T$ is split; this leads to a description of good elements in $T_{\text{reg}} := T \cap G_{\text{reg}}$ when $T \subset G$ is a split maximal torus. This is no longer true for non-split tori.
(B) For reductive groups, the internal structure of \( L \)-packets is elucidated by endoscopy, which originates from the difference between ordinary and stable conjugacy, i.e. conjugacy over the separable closure \( \bar{F} \), for elements of \( G_{\text{reg}}(F) \) at least. It is unclear if this can be lifted to BD-covers. It is more reasonable to do this on pull-backs \( T_{Q,m} \) of the BD-cover via \( \iota_{Q,m} : T_{Q,m} \to T \), for various maximal tori \( T \), or some translates thereof in order to include all good elements.

(C) One should be able to form the stable character attached to an \( L \)-packet: it is a sum of characters therein with certain multiplicities (often one). Desideratum: the stable character should be a genuine stable distribution, in an appropriate sense for the BD-cover in question.

Cf. [14, §§14—15] for further discussions. To the author’s knowledge, only two non-trivial cases have been discovered.

- Let \( G = \text{Sp}(W) \) where \( (W, \langle \cdot | \cdot \rangle) \) is a symplectic \( F \)-vector space of dimension \( 2n, n \in \mathbb{Z}_{\geq 1} \). Fix an additive character \( \psi \), then we have Weil’s metaplectic covering \( \mathcal{U}^{(2)}(\psi) \to G(F) \) with \( m = 2 \). It is a distinguished instance of BD-covers, namely (a) it carries the Weil representation \( \omega_{\psi} = \omega_{\psi}^{\pm} \oplus \omega_{\psi}^{-} \), and (b) all elements are good; in fact \( \iota_{Q,2} = \iota_{Q,1} \) in this case, for all \( T \subset G \). Thus the issue \((A)\) disappears, and the structure of packets should be explicated solely in terms of stable conjugacy.

J. Adams defined stable conjugacy on \( \mathcal{U}^{(2)}(\psi) \) in terms of the characters of \( \omega_{\psi}^{\pm} \). This is then developed by D. Renard and the author into a fully-fledged theory of endoscopy, as summarized in [35]. Due to the usage of Weil representations, it cannot be ported to other BD-covers.

The endoscopic character relations for this BD-cover are the topic of ongoing works of Caihua Luo,

- In an unpublished note by K. Hiraga and T. Ikeda [22], they defined the transfer factors for BD-covers of \( \text{SL}(2) \) with \( m \in \mathbb{Z} \) and established the transfer of orbital integrals; one still has to choose \( \psi \) when \( m \equiv 2 \pmod{4} \). They also classified the good elements in \( \text{SL}_{\text{reg}}(2, F) \) and stabilized the regular elliptic part of the trace formula. This is ultimately based on Flicker’s theory [13] for \( \text{GL}(2, F) \) and makes use of Kubota’s cocycles; both are unavailable in higher ranks.

They offer a testing ground for notions of stable conjugacy, since the transfer factors should transform under stable conjugacy by some explicit character, as in the case of reductive groups [32].

The aim of this article is to explore these issues for BD-covers of \( G = \text{Sp}(W) \), for general \( n \) and \( m \mid N_F \).

Note that the representation theory for these covers has also been considered in the recent work [33].

Main results

Henceforth we let \( G := \text{Sp}(W) \) for a symplectic \( F \)-vector space \( (W, \langle \cdot | \cdot \rangle) \) of dimension \( 2n \).

Stable conjugacy

Assume \( \text{char}(F) \neq 2 \) and let \( G = \text{Sp}(W) \) as above. The first goal of this article is to address \((A)\) and \((B)\) for all BD-covers of \( G \). This is inspired, and in turn generalizes some aspects of the formalism of Adams and Hiraga–Ikeda. By [7], the \( K_2 \)-extensions of \( G \) are classified by the Weyl-invariant quadratic forms \( Q : Y \to \mathbb{Z} \) where \( Y = X_*(T_0) \) for some maximal torus \( T_0 \subset G \); here we take the split one. There is a generator \( Q \) that corresponds to Matsumoto’s central extension \( K_2(F) \hookrightarrow E_G(F) \to G(F) \), and we shall stick to the BD-covers of degree \( m \mid N_F \) arising from \( E_G(F) \). For harmonic analysis, this is not a real restriction as explained in Remark 2.4.4: let \( k \in \mathbb{Z} \setminus \{0\} \), rescaling \( Q \) to \( kQ \) amounts to reduce \( m \) to \( m' \) and replacing \( \epsilon \) by \( \epsilon' := \epsilon^k \), where \( m' = m \gcd(k, m) \) and \( k' = k \gcd(k, m) \). The same reduction applies to Weissman’s \( L \)-groups, as explained in Theorem 5.3.5.

The definition of stable conjugacy is sketched as follows. Let \( \delta \in G_{\text{reg}}(F) \) so that \( T := Z_G(\delta) \) is a maximal torus. When \( n = 1 \) and \( G \simeq \text{SL}(2) \), stable conjugacy can always be realized by \( G_{\text{ad}}(F) \)-actions. This is no longer true when \( n > 1 \). Nonetheless, we may construct an intermediate group \( T \subset G^T \subset G \) such that

\[
\prod_{i \in I} R_{K_i^\dagger/F}(K_i^\dagger) \simeq T \hookrightarrow G^T \simeq \prod_{i \in I} R_{K_i^\dagger/F}(\text{SL}(2))
\]

where

- \( K_i^\dagger \) are finite separable extensions of \( F \),
- \( K_i \) is a quadratic étale \( K_i^\dagger \)-algebra, whose norm-one torus is denoted by \( K_i^1 \), and
asserts that when \( \delta \) is the stabilizer of \( g \) with \( \delta \in \mu_{\text{reg}}(2,m) \) into stable conjugacy, and the resulting operation is denoted by \( \text{CAd}(g) \). The price is that we need to prescribe \( \delta_0 \in i_Q^1(\delta) \); that is, we have to work in \( T_{Q,m} \). Here \( \delta = p(\delta) \) as usual.

More generally, for \( \sigma \in \{ \pm 1 \} \), we choose a symplectic basis for \( \mathcal{C} \) and \( \mathcal{D} \) so that \( \sigma = \pm 1 \) is the stabilizer of \( g \) with \( \sigma \in \mu_{\text{reg}}(2,m) \) into stable conjugacy, and the resulting operation is denoted by \( \text{CAd}^\sigma(g) \). The price is that we need to prescribe \( \sigma \in \mu_{\text{reg}}(2,m) \) and \( \delta_0 \in i_Q^1(\delta) \); that is, we have to work in \( T_{Q,m} \). Here \( \delta = p(\delta) \) as usual.

\[ \text{CAd}^\sigma(g)(\epsilon) = \epsilon \text{CAd}^{p(\delta)}(g) \] for all \( \epsilon \in T_{Q,m}^\sigma \).

AD.2. \( \text{CAd}(g) : T_{Q,m} \hookrightarrow S_{Q,m} \) is an isomorphism of topological groups. In general, \( \text{CAd}^\sigma(g) \) is a \( \text{CAd}(g) \)-equivariant map between torsors.

AD.3. If \( g \in G(\mathbb{F}) \) then \( \text{CAd}^\sigma(g) = \text{Ad}(g) \); if \( g \in T(\mathbb{F}) \), it equals id.

AD.4. Transitivity under composition of stable conjugations of maximal \( F \)-tori.

We have arrived at a possible answer to (B). As for (A), Proposition 4.3.1 characterizes the good elements in \( G_{\text{reg}}(F) \) completely: \( \delta \in T_{\text{reg}}(F) \) is good if and only if \( \delta = \sigma \cdot i_{Q,m}(\delta_0) \) for some \( \delta_0 \in T_{Q,m}(F) \), with \( \sigma \) needed only when \( 4 \mid m \). This boils down to the case of \( SL(2) \) by a similar reduction, which has been solved by Hiraga–Ikeda. It also gives some evidence for our formalism of stable conjugacy.

For potential applications to the trace formula, we show in Theorem 4.5.6 that when \( F \) is a global field, the local \( \text{CAd}(g) \) patch into an adélic one, and coincides with the usual \( \text{Ad}(g) \) on the image of \( G(\mathbb{F}) \) in the adèlic BD-cover. A similar characterization of adèlic good elements is obtained in Proposition 4.5.5.

Theorem 9.2.3 asserts that when \( m = 2 \), the notion of stable conjugacy coincides with that of Adams via Weil representations. When \( F = \mathbb{R} \), the only nontrivial case is \( m = 2 \); Adams gave in [1, Definition 3.4] a Lie-theoretic definition of stable conjugacy for real metaplectic groups, namely by lifting the action of stable Weyl groups. His recipe can still be reduced to \( SL(2,\mathbb{R}) \), and is seen to coincide with our recipe by Proposition 4.4.1 since the calibration factor \( C_{\mathcal{G}}(\cdots) \) disappears.

We also show in Theorem 9.4.7 that when \( n = 1 \) and \( m = 2 \), the Hiraga–Ikeda transfer factors satisfy a transformation property with respect to \( \text{CAd}^\sigma(g) \), which justifies our formalism in that case.

\section*{L-groups}

Fix \( \epsilon : \mu_m \rightarrow \mu_m \). It will be explained in \$5 \) that \( \tilde{G}^\epsilon \simeq \text{Sp}(2n,\mathbb{C}) \) (resp. \( \text{SO}(2n+1,\mathbb{C}) \)) when \( m = 2 \mathbb{Z} \) (resp. \( m \notin \mathbb{Z} \)), with trivial Galois action; furthermore, there is an \( L \)-isomorphism \( \tilde{L} \rightarrow \tilde{G}^\epsilon \times W_F \). Such a splitting of \( L \)-group is canonical when \( m \notin 2 \mathbb{Z} \), otherwise it depends on:

- an additive character \( \psi \) when \( m \equiv 2 \mod 4 \);
- a \( G(F) \)-conjugacy class of \( F \)-pinnings of \( G \), or more concretely the symplectic form \( \langle \cdot | \cdot \rangle \) on \( W \) as explained below.

Given \( \langle \cdot | \cdot \rangle \), one chooses a symplectic basis for \( W \) to produce a standard \( F \)-pinning, which is well-defined up to \( G(F) \)-conjugacy. The \( F \)-pinnings form a torsor under \( G_{\text{ad}}(F) \), and the homomorphism \( G_1 := \text{GSp}(W) \rightarrow G_{\text{ad}} \) induces

\[ G_1(F)/\langle Z_{G_1}(F)G(F) \rangle \simeq G_{\text{ad}}(F)/\text{im}[G(F) \rightarrow G_{\text{ad}}(F)]. \]

Note that \( Z_{G_1}(F)G_1(F) \) is the stabilizer of \( \langle \cdot | \cdot \rangle \) mod \( F^{\times 2} \) under \( G_1(F) \). Summing up, one may identify the \( G(F) \)-conjugacy classes of \( F \)-pinnings with the dilated symplectic forms \( c(\cdot | \cdot) \) where \( c \) is taken up to \( F^{\times 2} \). In this article, symplectic forms will usually be preferred over \( F \)-pinnings.
When $m \in 2\mathbb{Z}$, the $L$-isomorphism $^L\tilde{G} \simeq \tilde{G}^\vee \times \mathcal{W}_L$ undergoes an explicit quadratic twist $\chi_c : \Gamma_F \rightarrow \mu_2$ when $\psi$ or $\langle \cdot , \cdot \rangle$ is rescaled by $c \in F^\times$; see (5.5).

**Epipelagic $L$-packets** In order to justify the formalism of stable conjugacy, we construct *epipelagic supercuspidal $L$-packets* for $\tilde{G}$ when $F$ is non-archimedean with residual characteristic $p \nmid 2m$. This is a tractable class of supercuspidal representations, yet sufficiently rich to probe the genuine spectrum of $\tilde{G}$. The original definition is due to Reeder–Yu [45], but the approach here follows Kaletha [24, 25]. We shall take Yu’s construction [61] for BD-covers with $p \nmid m$ as granted, in the epipelagic case at least.

On the dual side, there is a well-defined notion of epipelagic supercuspidal $L$-parameters $\phi$. Given $\phi$, Kaletha’s recipe produces a stable class of embeddings $j : S \hookrightarrow G$ of certain elliptic maximal tori, which we call of type (ER). For reductive groups we also obtain a character $\theta$ of $S(F)$. For BD-covers, there is an isogeny $\iota_{Q,m} : S_{Q,m} \rightarrow S$ and what can be expected is just a character $\theta^\circ$ of $S_{Q,m}(F)$. The first task is to lift it to genuine character(s) $\theta_j$ of $\tilde{j}S := p^{-1}(jS(F))$ for various $j$. Since the genuine representation theory of BD-covers of anisotropic tori is not yet fully developed, we do this in an *ad hoc* way, namely

- we give explicit splittings of $p$ over $jS(F)$ upon enlarging the cover (see §7.1.8), thereby showing its commutativity;
- when $4 \mid m$, we take all $\theta_j$ with pro-$p$ component determined by $\theta^\circ$;
- when $4 \nmid m$, the desiderata on $\theta_j$ are encapsulated into the notion of *stable system* (Definition 7.3.3). Again, the pro-$p$ component of $\theta_j$ is pinned down by $\theta^\circ$.

By varying $j$, we construct the packet $\Pi_{\phi} := \Pi(S, \theta^\circ)$ by taking compact-induction of $\theta_j$ from $\tilde{j}S \cdot G(F)_{x,1/e}$ where $x \in B(G, F)$ is determined by $jS$ and $\frac{1}{e}$ stands for the depth of $\theta^\circ$. Specifically:

(i) Kaletha’s construction involves certain quadratic characters $\epsilon_{jS}$ of $jS(F)$ needed for stability (see [24, §4.6]), and here we multiply $\theta_j$ by the same character before inducing.

(ii) When $4 \mid m$, some packets might be empty and $\Pi_{\phi}$ depends on the splitting of $^L\tilde{G}$. As a workaround, we consider all $\chi_c$-twists of $\phi$ at once, for various $c \in F^\times$, to obtain the pre-$L$-packet $\Pi_{[\phi]}$. By Remark 7.4.5, there is at most one datum in the orbit $[\phi]$ under twists that yields a nonempty $\Pi(S, \theta^\circ)$.

(iii) When $m \not\equiv 2 \pmod{4}$, Proposition 7.3.5 furnishes a standard stable system, whilst for $m \equiv 2 \pmod{4}$, the stability and the canonicity (i.e. independence of the splitting of $^L\tilde{G}$) of $\Pi_{\phi}$ impose stringent constraints on the stable system. In Theorem 8.2.6, a stable system for $m \equiv 2 \pmod{4}$ will be constructed by reverse-engineering and extrapolation from the case $m = 2$, with the help from the description of $\Theta$-lifting by Loke–Ma–Savin [36, 60]. The main features of this stable system include

- the use of *moment maps* that relate $G$ and the pure inner forms $SO(V, q)$ of the split $SO(2n+1)$, which already appeared in [1];
- a sign character measuring the ratio between Kaletha’s quadratic characters for $G$ and $SO(V, q)$;
- it turns out that the ratio, as interpreted by Theorem 8.3.1, depends not only on the embedded tori, but also on the pro-$p$ part of inducing data, which is related to the unrefined minimal $K$-type [41, 40].

Theorem 7.4.10 shows that the elements in a packet need not share the same central character. Theorem 7.4.7 lists a few properties of $\Pi_{\phi}$ and $\Pi_{[\phi]}$, including cardinality and disjointness. We do not pursue the further issues such as basepoints (genericity) or formal degrees in this article.

By a straightforward comparison with [36], we show in Theorem 9.3.3 that when $\text{char}(F) = 0$ and $m = 2$, the $L$-packets $\Pi_{\phi}$ are obtained via $\Theta$-lifting from the epipelagic Vogan $L$-packets of $SO(2n + 1)$ constructed by Kaletha [24]. Our formalism is thus compatible with the local Langlands correspondence for $p$-adic metaplectic groups set forth by Gan–Savin [15].

The construction hinges on certain properties of maximal tori of type (ER), eg. Theorem 6.2.2. It would be beneficial to develop a theory for toral supercuspidals, or more generally for the regular supercuspidal representations considered in [25].

**Stability of packets** Let $\Xi$ be an invariant genuine distribution on $\tilde{G}$ represented by a locally $L^1$ function that is smooth over $\tilde{G}_{\text{reg}}$. In Definition 4.4.4, we say that $\Xi$ is *stable* if for any maximal torus
where \( \text{Ad}(g) : \delta \mapsto \eta \) is a stable conjugacy. If \( 4 \nmid m \), the choice of \( \delta_0 \) is immaterial. If \( 4 \mid m \), the dependence of this condition on \( \delta_0 \) is quantified by Proposition 4.4.2.

Now assume \( \text{char}(F) = 0 \) with large residual characteristic \( p \) as in Hypothesis 6.4.1. Let \( \phi \) be an epipelagic supercuspidal \( L \)-parameter for \( \tilde{G} \). We show in Theorems 7.6.3, 7.6.5 that the character-sum \( S\Theta_{\phi} \) (resp. \( S\Theta_{|\phi|} \)) of \( \Pi_{\phi} \) when \( 4 \nmid m \) (resp. \( \Pi_{|\phi|} \) when \( 4 \mid m \)) is a stable distribution in the sense above. As in [24], the basic ingredients are (a) the Adler–Spice character formula [4] for epipelagic supercuspidal representations of \( \tilde{G} \), whose validity is taken for granted, and (b) Waldspurger’s results [53] relating transfer and Fourier transform on Lie algebras, now completed by B. C. Ngô’s proof of the Fundamental Lemma.

We remark that when \( 4 \nmid m \), the SS.2 in Definition 7.3.3 is the key to stability. It stipulates that for stably conjugate embeddings \( j : S \hookrightarrow G \) and \( j' = \text{Ad}(g)j \) in the given stable class, the stable system must satisfy

\[
\theta_{j'}(\text{CAd}(g)(\tilde{\gamma})) = \theta_j(\tilde{\gamma}), \quad \tilde{\gamma} \in j\tilde{S};
\]

the reference to \( j\tilde{S}_{G,m} \) being dropped according to Corollary 4.4.4. This gives a preliminary answer to the issue (C) alluded to above.

Organization

The assumptions on \( F \) will be summarized in the beginning of each section and subsection. We caution the reader that they are not necessarily optimal.

In §2, we review the basic formalism of Brylinski–Deligne theory and collect several results concerning Weil restrictions, which are not entirely trivial and seem to be missing in the literature.

In §3, we recall some structural results on symplectic groups, including the parameterization of regular semisimple classes and the formation of the intermediate group \( T \subset G^T \subset G \). The parameterization of classes originates from [48] and has been used in [51, 35]; here we allow \( F \) of any characteristic \( \neq 2 \).

In §4 we introduce Matsumoto’s central extensions for \( G = \text{Sp}(W) \), and define stable conjugacy for the corresponding BD-covers. We also collect some results from Hiraga–Ikeda in the case \( \dim_F W = 2 \), with complete proofs. The properties studied in §4.4 will play a crucial role in the subsequent sections.

Weissman’s theory of \( L \)-groups is summarized in §5. We provide explicit splittings for \( ^1\tilde{G} \), cf. [14], and clarify their dependence on auxiliary data. In §5.3 we study the \( L \)-groups attached to Baer multiples of Matsumoto’s central extension. The conclusion is that on the dual side, it is legitimate to confine ourselves to Matsumoto’s central extension; this is unsurprising, but seem to be undocumented hitherto.

We fix the notation for Moy–Prasad filtrations, etc. in §6. The construction of genuine epipelagic supercuspidal representations is also reviewed there. The only new ingredient is a description of maximal tori of type (ER) in \( G \) and their preimages in \( \tilde{G} \).

In §7 we present the notions of inducing data and stable systems, then construct the \( L \)-packets and pre-\( L \)-packets and establish their basic properties (Theorems 7.4.7, 7.4.10). These packets are shown to be independent of the splitting of \( ^1\tilde{G} \) in §7.5. Stability is proven in §7.6.

In §8 we construct a stable system for \( m \equiv 2 \mod 4 \) using moment map correspondences. Auxiliary results in §8.1 and §8.3 will be used in the later comparison with \( \Theta \)-lifting.

The compatibility of the foregoing constructions with existing theories (Weil’s metaplectic groups, Hiraga–Ikeda theory) is resolved in §9. The necessary backgrounds are also reviewed there.

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Conventions

Fields For any field \( F \), we denote by \( \tilde{F} \) a chosen separable closure and denote \( \Gamma_F := \text{Gal}(\tilde{F}/F) \). Let \( \mu_m \) be the group functor \( R \mapsto \mu_m(R) := \{ r \in R^\times : r^m = 1 \} \) for commutative rings \( R \), and put \( \mu_m := \mu_m(\mathbb{C}) \).

The norm and trace in a finite extension \( K/F \) is denoted by \( N_{K/F} \) and \( \text{tr}_{K/F} \), respectively.
Suppose $F$ is a non-archimedean local field. The normalized valuation on $F$ is denoted by $v_F$. For a finite extension $L/F$ of non-archimedean local fields, the ramification degree is denoted by $e(L/F)$. The residue field of $F$ is written as $k_F = \mathfrak{o}_F/\mathfrak{p}_F = \mathfrak{o}_F/(\mathfrak{c}_F)$. Let $\{\Gamma'_F\}_{r \geq -1}$ be the upper ramification filtration, and set $\Gamma'_F := \bigcap_{r > 0} \Gamma'_F$. In particular $\Gamma'_F = \Gamma_F$, $\Gamma'_0 = I_F$ (inertia) and $\Gamma'_p = P_F$ (wild inertia). Denote by $F^m$ the maximal unramified extension of $F$ inside $\bar{F}$.

The Weil group of a local or global field $F$ is written as $W_F$. By an additive character over a local or finite field $F$ (resp. the adele ring $A_F$ of a global field), we mean a non-trivial continuous unitary homomorphism $\psi : F \to C^\times$ (resp. $\psi : A_F/F \to C^\times$). For $c \in F^\times$ we write $\psi_c : x \mapsto \psi(cx)$.

Denote by $\mu(F)$ the group of all roots of unity in $F^\times$, for any field $F$. When $F$ is a local or global field, we denote

$$N_F := \begin{cases} \mu(F), & F \neq \mathbb{C} \\ 1, & F = \mathbb{C}. \end{cases}$$

If $F \neq \mathbb{C}$ then $\mu(F) = \mu_{N_F}(F)$, and $N_F$ is always invertible in $F$. For $m \mid N_F$, we write $\mu_m = \mu_{m}(F)$ and denote by $(\cdot, \cdot)_{F,m} : F^\times \times F^\times \to \mu_m$ the Hilbert symbol of degree $m$.

**Cohomology** For a topological group $G$, we denote by $H^\bullet(G, \cdot)$ the continuous cohomology groups. The Galois cohomology over a field $F$ is written as $H^\bullet(F, \cdot)$. If $E/F$ is a separable quadratic extension of local fields, we denote by $\text{sgn}_{E/F} : F^\times \to \mu_2 \cong \mu_2$ the corresponding quadratic character; $\text{sgn}_{E/F}(x) = 1$ if and only if $x \in N_{E/F}(E^\times)$.

**Quadratic spaces** Let $F$ be a field of characteristic $\neq 2$. By a quadratic $F$-vector space we mean a pair $(V, q)$ where $V$ is a finite-dimensional $F$-vector space and $q : V \times V \to F$ is a non-degenerate symmetric form; $q$ is determined by the function $q(v) := q(v)v$. We write $(a)$ for the quadratic $F$-vector space on $V = F$ with $q : x \mapsto ax^2$, and put $(a_1, \ldots, a_n)$ for the orthogonal direct sum of the $(a_i)$.

The (signed) **discriminant** of $(V, q)$ is defined as $d(V, q) := (-1)^{(n-1)/2}d_1 \cdots d_n \in F^\times/F^\times 2$ whenever $(V, q) \simeq (d_1, \ldots, d_n)$. We have

$$d^\pm((V, q) \oplus (V', q')) = d^\pm(V, q)d^\pm(V', q').$$

The meaning of $O(V, q)$ and $SO(V, q)$ is then clear. By $SO(2n + 1)$ we always mean the split $SO(V, q)$ with $\dim_F V = 2n + 1$.

Let $F$ be local with an additive character $\psi$. Weil's constant associated to the character of second degree $\psi \circ \frac{1}{2} : V \to C^\times$ is denoted by $\gamma_\psi(q) \in \mu_k$, see [55, §24] or [42, §1.3]; this extends to a homomorphism into $\mu_k$ of the Witt group of quadratic $F$-vector spaces. For $a \in F^\times$, we write $\gamma_\psi(a) := \gamma_\psi(\langle a \rangle)$. The Hasse invariant of $(V, q)$ is denoted by $e(V, q) \in \mu_2$.

In a similar vein, a symplectic $F$-vector space means a pair $(W, (\langle \cdot, \cdot \rangle))$, where $\dim_F W = 2n$ is finite and $(\langle \cdot, \cdot \rangle)$ is non-degenerate anti-symmetric. Dropping $(\langle \cdot, \cdot \rangle)$ from the notation, we have the groups $\text{Sp}(W) \subset \text{GSp}(W)$; we will occasionally write $\text{Sp}(2n)$. For both quadratic and symplectic vector spaces, $\oplus$ stands for the orthogonal direct sum.

More generally, for a commutative ring $R$, a quadratic form or space, possibly singular, is the datum $(M, Q)$ where (a) $M$ is an $R$-module, (b) the map $Q : M \to R$ satisfies $Q(tx) = t^2Q(x)$, for all $t \in R$ and $x \in M$, and (c) $B_Q(x, y) := Q(x + y) - Q(x) - Q(y)$ is a symmetric $R$-bilinear form on $M$. This general notion will mainly be applied for $R = \mathbb{Z}$.

**Group schemes** For any $S$-scheme $X$ and a morphism $U \to S$, we write $X_U := X \times U$ for its base-change to $T$. When $S = \text{Spec}(F)$ and $U = \text{Spec}(A)$, we write $X_A = X_{\text{Spec}(A)}$ and $X(B) = X(\text{Spec}(B))$. Denote by $S_{\text{zar}}$ the big Zariski-site and by $S_{\text{ét}}$ the small étale site over $S$. Then $\mu_m$ is a sheaf for both sites.

The group $S$-schemes will be designated by letters $G$, $H$, etc.; for Lie algebras we use gothic letters. The dual of $\mathfrak{g}$ is denoted as $\mathfrak{g}^\ast$. The identity connected component of a smooth $S$-group scheme $G$ (see [16, Exposé VIB Théorème 3.10]) is denoted by $G^\circ$. We use $N_G(\cdot)$, $Z_G(\cdot)$ to indicate the normalizers and centralizers in $G$, respectively. The center of $G$ is written as $Z_G$. The same notation also pertains to abstract and topological groups.

When $F$ is a local field, $G(F)$ is endowed with the topology from $F$, and all the representations of $G(F)$ are over $\mathbb{C}$.
Reductive groups The reductive groups $G$ over a field $F$ are assumed to be connected. The simply connected covering of the derived subgroup is $G_{\text{sc}} \to G_{\text{der}} \subset G$. The adjoint group (resp. derived subgroup) of $G$ is $G_{\text{ad}} := G/Z_G$ (resp. $G_{\text{det}}$). The adjoint action of $G$ or $G_{\text{ad}}$ on $G$ is written as $\text{Ad}(g)(\delta) = g\delta g^{-1}$; on Lie algebras we have $\text{ad}(X)(Y) = [X,Y]$. The coroot corresponding to root $\alpha$ is denoted by $\alpha^\vee$.

We say a regular semisimple element $\delta \in G$ strongly regular if $Z_G(\delta) = Z_G(\delta)^o$. Denote by $G_{\text{reg}} \subset G$ the strongly regular locus, which is Zariski-open. For any maximal torus $T \subset G$, we write $T_{\text{reg}} := T \cap G_{\text{reg}}$. The same notation pertains to Lie algebras: we have $\mathfrak{g}_{\text{reg}}$, etc.

For any $\gamma \in G(F)$, denote $G_{\gamma} := Z_G(\gamma)^o$; the same notation pertains to $X \in \mathfrak{g}(F)$. When $\gamma \in G(F)$ or $X \in \mathfrak{g}(F)$ is semi-simple, define the Weyl discriminants

$$D^G(\gamma) := \det (1 - \text{Ad}(\gamma)|\mathfrak{g}/\mathfrak{g}_{\gamma}),$$
$$D^G(X) := \det (\text{ad}(X)|\mathfrak{g}/\mathfrak{g}_{X}).$$

The Weyl group of a maximal $F$-torus $T \subset G$ is $\Omega(G,T) := N_G(T)/T$. This is to be regarded as a sheaf over $\text{Spec}(F)_{\text{et}}$; thus $\Omega(G,T)(F) \supseteq N_G(T)(F)/T(F)$ in general. For a diagonalizable $F$-group $D$, we write $X^*(D) = \text{Hom}(D, G_m)$ and $X_*(D) = \text{Hom}(G_m, D)$, again considered as sheaves over $\text{Spec}(F)_{\text{et}}$.

We denote the Langlands dual group of $G$ by $G^{\vee}$ or $G^{\vee}$ as a pinned $\mathbb{C}$-group with $G_{\text{der}}$-action. The $L$-group is denoted by $^LC^G$: it usually means the Weil form unless otherwise specified. The relevant notation for coverings will be introduced in §5.1.

2 Theory of Brylinksi–Deligne

2.1 Multiplicative $K_2$-torsors

The main reference is [7]. Fix a base scheme $S$. Let $G,A$ be sheaves of groups over $S_{\text{Zar}}$ with $(A,+)$ commutative. Consider the central extensions of $G$ by $A$

$$0 \to A \to E \xrightarrow{\tilde{p}} G \to 1.$$

This means that $p$ is an epimorphism between sheaves and $A \simeq \ker(p)$. It is known [18, Exp VII, 1.1.2] that $E$ is an $A$-torsor over $G$ in this context; in particular, $E \to G$ is Zariski-locally trivial. We will make frequent use of the shorthand $A \Rightarrow E \to G$.

As in the set-theoretical case, the adjoint action of $G$ on itself lifts to $E$, which we still denote as $\tilde{x} \mapsto g\tilde{x}g^{-1}$ where $g \in G$ and $\tilde{x} \in E$: it is the conjugation by any preimage of $g$ in $E$. To any central extension we may associate the commutator pairing $G \times G \to A$. Set-theoretically, it is simply

$$[x,y] := \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1}$$ (2.1)

where $\tilde{x}, \tilde{y}$ are preimages of $x,y$ in $E$.

We have the following basic operations on $E$:

- the pull-back $f^*E$ by a homomorphism $f : G_1 \to G$,
- the push-out $h_*E$ by a homomorphism $h : A \to A_1$, as torsors this amounts to $A_1^{A,h} E$.

Up to a canonical isomorphism, the order of pull-back and push-out can be changed.

- Given two central extensions $E_1 \to G_1$ and $E_2 \to G_2$ by the same sheaf $A$, we have the contracted product $A \hookrightarrow E_1^A \times E_2^A \to G_1 \times G_2$: it is the push-out of central extension $E_1 \times E_2 \to G_1 \times G_2$ (by $A \times A$) by $+ : A \times A \to A$;
- when $G_1 = G_2$, the Baer sum $E_1 + E_2$ of $E_1, E_2$ can be realized by pulling $E_1 \times E_2^A$ back via the diagonal $G \hookrightarrow G \times G$.
Given \( f : G_1 \rightarrow G \), a homomorphism \( \varphi : E_1 \rightarrow E \) covering (or lifting) \( f \) is a commutative diagram of sheaves of groups
\[
\begin{array}{cccccc}
0 & \longrightarrow & A & \longrightarrow & E_1 & \longrightarrow & G_1 & \longrightarrow & 1 \\
& & \downarrow \text{id} & & \downarrow \varphi & & \downarrow f & & \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1
\end{array}
\]
Denote by \( \mathcal{CExt}(G, A) \) the category of central extensions of \( G \) by \( A \), with morphisms being the homomorphisms \( E \rightarrow E_1 \) covering \( \text{id}_G \). This makes \( \mathcal{CExt}(G, A) \) a groupoid equipped with the “addition” given by Baer sum, subject to the usual functorial constraints; such a structure is called a Picard groupoid. In general, giving \( \varphi : E_1 \rightarrow E \) covering \( f : G_1 \rightarrow G \) is the same as giving a morphism \( E_1 \rightarrow f^*E \) in \( \mathcal{CExt}(G_1, A) \).

In [7], the framework of central extensions is reformulated in terms of multiplicative \( A \)-torsors as in [18, Exp VII]. These are \( A \)-torsors \( p : E \rightarrow G \) equipped with a “multiplication” \( m : \text{pr}_1^*E + \text{pr}_2^*E \rightarrow \mu^*E \), where

- \( \mu : G \times G \rightarrow G \) is the multiplication,
- \( G \xrightarrow{\text{pr}_1} G \xrightarrow{\text{pr}_2} G \) are the projections,
- \(+\) signifies the Baer sum of \( A \)-torsors over \( G \times G \).

Furthermore, \( m \) is required to render the following diagram of \( A \)-torsors over \( G \times G \times G \) commutative
\[
\begin{array}{ccc}
\text{pr}_1^*E + \text{pr}_2^*E + \text{pr}_3^*E & \xrightarrow{id \times m} & \text{pr}_1^*E + \text{pr}_2^*E + \text{pr}_3^*E \\
\downarrow \cong & & \downarrow \cong \\
\text{pr}_1^*E + \mu^*_2 E & \xrightarrow{(id \times \mu)^*(m)} & \mu^*_1 E + \text{pr}_3^*E \\
\downarrow \cong & & \downarrow (\mu \times \text{id})^*(m) \\
(id \times \mu)^* \mu^* E & \cong & \mu^*_1 E \cong (\mu \times \text{id})^* \mu^* E
\end{array}
\]
where \( \mu_{123} = \mu \circ \mu_{13} : G \times G \times G \rightarrow G \) are the morphisms that multiply the slots in the subscript. In forming the diagram we used the compatibility between pull-back and Baer sum. The resulting groupoid of multiplicative torsors is denoted by \( \text{MultTors}(G, A) \).

Given a central extension \( A \hookrightarrow E \rightarrow G \), taking \( m \) to be the group law of \( E \) gives rise to a multiplicative \( A \)-torsor \( E \) over \( G \). By [18, Exp VII, 1.6.6], this establishes an equivalence \( \mathcal{CExt}(G, A) 
\rightarrow \text{MultTors}(G, A) \).

Let \( K_n \) be the Zariski sheaves associated to Quillen’s \( K \)-groups \( K_n \), for \( n \in \mathbb{Z}_{\geq 0} \); note that \( K_0 \simeq \mathbb{Z} \) and \( K_1 \simeq K_m \) canonically. Several observations are in order.

1. By [54, IV.6.4], \( K_n \) transforms products of rings to products of abelian groups. In parallel, \( K_n(U_1 \sqcup U_2) = K_n(U_1) \times K_n(U_2) \) for disjoint union of schemes; in fact, the latter holds for any sheaf.
2. The stalk of \( K_n \) at any point \( s \) of \( S \) equals \( K_n(O_{S,s}) \), where \( O_{S,s} \) stands for the local ring. Indeed, this is readily reduced to the case of affine \( S \). Then one can use the fact that \( K_n \) commutes with filtered \( \lim \), see [54, IV. 6.4].
3. Consider a multiplicative \( K_2 \)-torsor \( E \rightarrow G \). When \( S = \text{Spec}(R) \) where \( R \) is a field or a discrete valuation ring, there is a central extension
\[
1 \rightarrow K_2(S) \rightarrow E(S) \rightarrow G(S) \rightarrow 1
\]
of groups. Indeed, \( K_2(S) = K_2(R) \) by the previous observation; when \( R \) is a field we have \( H^1(S, K_2) = 0 \) for dimension reasons (true if \( K_2 \) is replaced by any \( A \)), and the same holds if \( R \) is a discrete valuation ring by [58, Corollary 3.5].

### 2.2 Classification of Brylinski–Deligne

In this subsection, \( S \) is assumed to be regular of finite type over a field. Fix a reductive group \( G \) over \( S \) and let
\[
0 \rightarrow K_2 \rightarrow E \rightarrow G \rightarrow 1
\]
be an object of $C\text{Ext}(G, K_2)$. We set out to review the classification in [7] of such objects in terms of triplets $(Q, D, \varphi)$ with étale descent data, formulated as an equivalence of categories. Note that the classification below has been extended to some other rings in [58], such as the spectrum of discrete valuation rings with finite residue fields.

(A) Let $G = T$ be a split torus. Set $Y := X_*(T)$, a constant sheaf over $S_{\text{Zar}}$. One constructs in [7, §3] a central extension

\[ 1 \to \mathbb{G}_m \to D \to Y \to 0 \quad (2.2) \]

as follows (in loc. cit. one writes $E$ instead of $D$). First, we introduce a variable $t$ and consider the base-change of $E$ to $\mathbb{G}_m, S = S[t, t^{-1}]$. It can be regarded as a central extension over $S$ by Sherman’s theorem [7, (3.1.2)] that asserts for all $n$,

\[ R^n(\mathbb{G}_m, S) \cdot K_n = \begin{cases} K_n \oplus K_{n-1}, & i = 0 \\ 0, & i > 0. \end{cases} \]

Pull the resulting extension back via $Y \to T[t, t^{-1}]$ that maps $\chi \in Y$ to $\chi(t) \in T[t, t^{-1}]$, and push it out by $K_2(\cdots [t, t^{-1}]) \to K_1 = \mathbb{G}_m$ using Sherman’s theorem, we obtain the required $\mathbb{G}_m \hookrightarrow D \to Y$. Alternatively, we can also base-change to $S((t))$, and push-out via the tame symbol [54, III. Lemma 6.3] $K_2(\cdots ((t))) \to K_1 = \mathbb{G}_m$ instead.

Moreover, one obtains a quadratic form $Q : Y \to \mathbb{Z}$ such that

\[ B_Q(y_1, y_2) := Q(y_1 + y_2) - Q(y_1) - Q(y_2) \]

satisfies $[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}$ for all $y_1, y_2 \in Y$, where $[,]$ is the commutator pairing of $\mathbb{G}_m \hookrightarrow D \to Y$.

The classification in this case [7, Proposition 3.11] says that $C\text{Ext}(T, K_2)$ is equivalent to the groupoid of pairs $(Q, D)$ satisfying $[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}$; by stipulation, $\text{Aut}(D, Q) = \text{Hom}(Y, \mathbb{G}_m)$.

It is important to note that for central extensions of $Y$ by $\mathbb{G}_m$, working over $S_{\text{Zar}}$ is the same as over $S_{\text{et}}$.

(B) Let $G$ be simply connected and split. Fix any split maximal torus $T \subset G$ and put $Y = X_*(T)$. In [7, §4] one obtains a Weyl-invariant quadratic form

\[ Q : Y \to \mathbb{Z} \]

from $E$. Furthermore, there are no non-trivial automorphisms of $E$. The classification in this case [7, Theorem 4.7] says that $C\text{Ext}(G, K_2)$ is equivalent to the groupoid of Weyl-invariant quadratic forms $Q : Y \to \mathbb{Z}$; by stipulation, $\text{Aut}(Q) = \{\text{id}\}$.

We may also pull-back $E$ by $T \hookrightarrow G$, obtaining a central extension of $T$ by $K_2$. The resulting quadratic form $Y \to \mathbb{Z}$ furnished by the case of tori is the same as the $Q$ above: this fact has been established in [7, 4.9].

(C) The case of a split reductive group $G$ is obtained by patching the cases above. Fix a split maximal torus $T$ with $Y = X_*(T)$. Then $C\text{Ext}(G, K_2)$ is equivalent to the groupoid of triplets $(Q, D, \varphi)$ such that

- $Q : Y \to \mathbb{Z}$ is a Weyl-invariant quadratic form, whose restriction to $Y_{\text{sc}} = X_*(T_{\text{sc}}) \hookrightarrow Y$ defines a central extension of $G_{\text{sc}}$, hence a central extension

\[ 1 \to \mathbb{G}_m \to D_{\text{sc}} \to Y_{\text{sc}} \to 0 \]

by the theory over $T_{\text{sc}}$;

- $D$ is a central extension of $Y$ by $\mathbb{G}_m$ satisfying $[y_1, y_2] = (-1)^{B_Q(y_1, y_2)}$;

- $\varphi : D_{\text{sc}} \to D$ covers $Y_{\text{sc}} \to Y$, i.e. there is a commutative diagram with exact rows

\[
\begin{array}{cccccc}
1 & \to & \mathbb{G}_m & \to & D_{\text{sc}} & \to & Y_{\text{sc}} & \to & 1 \\
\downarrow{id} & & \downarrow{\varphi} & & \downarrow{} & & \downarrow{} & & \downarrow{} \\
1 & \to & \mathbb{G}_m & \to & D & \to & Y & \to & 1.
\end{array}
\]

the morphisms are of the form $f : (Q, D', \varphi') \to (Q, D, \varphi'')$, where $f : D' \to D''$ is an isomorphism in $C\text{Ext}(Y, \mathbb{G}_m)$ and renders

\[ D' \xrightarrow{f} D'' \]

\[ D' \xrightarrow{\varphi'} D_{\text{sc}} \]

\[ D'' \xrightarrow{\varphi''} \]

\[ f \]

\[ \varphi' \]

\[ \varphi'' \]

\[ \]
(D) The case of general reductive groups $G$ is obtained in [7, Theorem 7.2] by rephrasing the description above étale-locally, by interpreting $D, Y, \text{etc.}$ as sheaves over $S_d$. This is legitimate since split maximal tori exist locally over the finite étale site of $S$, by [17, Exp XXII. Proposition 2.2]. For example, when $S$ is the spectrum of a field $F$, we may fix a maximal $F$-torus $T$ and $\text{CExt}(G, \mathbf{K}_2)$ is equivalent to the groupoid of triples $(Q, D, \varphi)$ such that $Q : Y \to Z$ is Weyl- and Galois-invariant, and so forth. Here $Y$ is viewed as a free $Z$-module endowed with Galois action.

(E) The Baer sum $E_1 + E_2$ gives rise to $(Q_1 + Q_2, D_1 + D_2, \varphi_1 + \varphi_2)$ with obvious notations. This is implicit in [7] and has been explicitly stated in [58, Theorem 2.2].

The classification immediately leads to the following statement. The assignment $T \mapsto \text{CExt}(G_T, \mathbf{K}_2)$ becomes naturally a stack fibered in Picard groupoids for the finite étale site over $S$.

**Remark 2.2.1.** For a split simple and simply connected group $G$ over a field $F$, the space of Weyl-invariant quadratic forms $Y \otimes R \to R$ is one-dimensional. By [7, Proposition 4.15], Matsumoto’s *central extension* [37] of $G$ by $\mathbf{K}_2$ corresponds to the unique Weyl-invariant $Y \to Z$ taking value 1 one short coroots. The multiplicative $\mathbf{K}_2$-torsors over $G$ are therefore classified by integers.

**Remark 2.2.2.** Let $H$ be a subgroup of $G$. Assume that $G, H$ are both simply connected and share a maximal torus $T$. For any object $E$ of $\text{CExt}(G, \mathbf{K}_2)$, its pull-back to $H$ and $E$ itself are classified by the same datum $Q : Y \to Z$. Indeed, both pull back to the same object of $\text{CExt}(T, \mathbf{K}_2)$, classified by some $(Q, D)$.

We record a result of Brylinski and Deligne for later use.

**Proposition 2.2.3.** Suppose that $G = G_{sc}$, and let $E$ be an object of $\text{CExt}(G, \mathbf{K}_2)$. Then the adjoint action of $Z_G$ on $E$ is trivial, and the action of $G_{ad}$ on $G$ lifts uniquely to an action on $E$.

**Proof.** When $G$ is split, this is stated in [7, 4.10]; see also [37, Proposition 5.13]. The general case follows by étale descent in view of the uniqueness of the lift. Note that the triviality of $Z_G$-action on $E$ is straightforward: it covers $id_G$, hence trivial since $G = G_{sc}$. $\square$

### 2.3 Weil restriction

Given a morphism $f : T \to S$, we have the functor $f^* : \text{Shv}(S_{\text{Zar}}) \to \text{Shv}(T_{\text{Zar}})$ given by $f^*\mathcal{F}(X) \simeq T = \mathcal{F}(X \to T, f)$. The functor of *Weil restriction* $f_* \mathcal{F}(Y \to S) = \mathcal{F}(Y \times_S T \to T)$ is initially defined for presheaves, but turns out to yield a functor $\text{Shv}(T_{\text{Zar}}) \to \text{Shv}(S_{\text{Zar}})$ by [6, p.194, Proposition 3]. Furthermore, $f_*$ is the right adjoint of $f^*$. In particular $f_*$ preserves $\varprojlim$, thus maps group objects to group objects.

**Remark 2.3.1.** Here is an easy case of Weil restriction. Suppose that $T = S^{\text{op}}$ for some set $T$ (disjoint union of $I$ copies). One readily checks that for a family of sheaves $(\mathcal{F}_i)_{i \in I}$ on $S$, the sheaf $\mathcal{G} := (\mathcal{F}_i)_{i \in I}$ on $T$ satisfies $f_*(\mathcal{G}) = \prod_{i \in I} \mathcal{F}_i$.

**Notation 2.3.2.** When $T = \text{Spec}(B)$, $S = \text{Spec}(A)$ and $f : T \to S$ corresponds to a ring homomorphism $A \to B$, it is customary to write $f_* = R_{B/A}$.

In what follows, the morphism $f : T \to S$ is assumed to be finite and locally free of constant rank $d$. The treatment below is inspired by [21].

**Definition 2.3.3.** Let $Y$ be a $T$-scheme and $E \to Y$ be an $A$-torsor, where $E, A$ are objects of $\text{Shv}(T_{\text{Zar}})$ and $A$ is a sheaf of abelian groups. We say $E \to Y$ is *well-behaved* if there exists an open covering $U = \{U_i\}_{i \in I}$ of $Y$ such that
- every $U_i$ is affine,
- every $d$-tuple $(x_1, \ldots, x_d)$ of closed points of $Y$ lies in some $U_i$;
- there exists a section $s_i : U_i \to E$ of $E \to Y$ for every $i \in I$.

For any $T$-group scheme $G$, we denote by $\text{CExt}(G, A)_0 \subset \text{CExt}(G, A)$ the full subcategory of $A \otimes E \otimes G$ such that $E \to G$ is a well-behaved $A$-torsor. When $G$ is affine, it is known that $f_*G$ is represented by an affine group $S$-scheme by [6, p.194, Theorem 4].

**Proposition 2.3.4.** Let $G$ be an affine $T$-group scheme and let $A$ be a sheaf of abelian groups over $T_{\text{Zar}}$. Then $f_*$ induces a functor $\text{CExt}(G, A)_0 \to \text{CExt}(f_*G, f_*A)$. 

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Proof. Let \( p : E \to G \) be an object of \( \mathsf{CExt}(G, A)_0 \). Since \( f_* \) preserves \( \lim \),

\[
1 \to f_*A \to f_*E \xrightarrow{f_*(p)} f_*G
\]

is exact. Furthermore,

\[
f_*E \times_{f_*G} f_*A \longrightarrow f_*E \times_{f_*G} f_*E
\]

\[
(x, a) \longmapsto (xa, a)
\]

is an isomorphism. To get a central extension, it suffices to show that \( f_*(p) : f_*E \to f_*G \) locally admits sections. Take an open covering \( U = \{U_i\}_{i \in I} \) of \( G \) as in Definition 2.3.3. Now [21] or the proof of [6, p.194, Theorem 4] implies that \( \{f(U_i)\}_{i \in I} \) with the evident Zariski gluing data yields \( f_*G \). Furthermore, for each \( i \) we deduce a section \( f_*(s_i) : f_*U_i \to f_*E \). This shows the local triviality of \( f_*E \to f_*G \).

\[ \square \]

**Example 2.3.5.** Suppose \( T = \mathsf{Spec}(L) \) where \( K \) is an infinite field, and \( Y = G \) is a reductive group over \( L \). It is well-known that \( G(L) \) is Zariski-dense in \( G \). For every Zariski \( A \)-torsor \( E \to G \), there exists an open affine subscheme \( U \subset G \), \( U \neq \emptyset \), together with a section \( s : U \to E \). For every \( g \in G(L) \), we may choose a preimage \( \bar{g} \in E(L) \) as \( E(L) \to G(L) \), and obtain a section \( s : gU \to E \) as the composite \( gU \xrightarrow{s^{-1}} U \xrightarrow{s} E \xrightarrow{\lambda_*} E \) where \( \lambda_* \) means left translation. We claim that

\[
U := \{gU : g \in G(L)\}
\]

forms an open covering satisfying the requirements of Definition 2.3.3. Let \( g_1, \ldots, g_d \) be closed points of \( G \). Observe that \( g^{-1}g_k \in U \) is an open condition on \( g \) for any \( k \), therefore those \( g \) satisfying \( g_k \in gU \) for \( k = 1, \ldots, d \) form a nonempty open subset of \( G \). Since \( G(L) \) is Zariski-dense, we can choose \( g \in G(L) \). The required section is given by \( s_g : gU \to E \). In particular, we see \( \mathsf{CExt}(G, A) = \mathsf{CExt}(G, A)_0 \).

Notice that given central extensions \( E_j \) of \( G \) by \( A_j \) with \( j = 1, \ldots, n \), one can find an open affine \( U \) as above (hence the covering \( U \)) that works for all \( E_j \). It follows that the functor in Proposition 2.3.4 preserves Baer sums. Indeed, given objects \( E_1, E_2 \) of \( \mathsf{CExt}(G, A) \), the natural arrow \( f_*(E_1 \times f_*E_2) \to f_*(E_1 + E_2) \) induces \( f_*(E_1 + f_*E_2) \to f_*(E_1 + E_2) \). To show the latter is an isomorphism, we recall from the proof of Proposition 2.3.4 that if \( E_j \) is glued from \( \{A \times U_i\}_{i \in I} \) with a system of transition functions \( a_{ij}^{(1)} \) (\( j = 1, 2 \)), then \( f_*(E_1 + E_2) \) is glued from the transition functions \( f_*(a_{ij}^{(1)} + a_{ij}^{(2)}) = f_*(a_{ij}^{(1)}) + f_*(a_{ij}^{(2)}) \). The same gluing datum defines \( f_*(E_1 + f_*E_2) \).

**Example 2.3.6.** Suppose that \( T \) is a local scheme and \( Y \) is isomorphic to the constant sheaf over \( T_{\text{Zar}} \) associated to some set (such as \( \mathbb{Z}^n \)). The requirements in Definition 2.3.3 are easily verified.

Still assume that \( f : T \to S \) is locally free of finite constant rank \( d \). This property is stable under base change, therefore the transfer/norm maps in \( K \)-theory [54, V.3.3.2] yields an arrow

\[
f_*K_n \to K_n, \quad n = 0, 1, 2, \ldots
\]

**Proposition 2.3.7.** Let \( L/F \) be a field extension of finite degree and write \( f : \mathsf{Spec}(L) \to \mathsf{Spec}(F) \). Let \( G \) be a reductive group over \( L \). Then \( E \to f_*E \) followed by push-out by \( f_*K_2 \to K_2 \) gives a functor

\[
\mathsf{CExt}(G, K_2) \to \mathsf{CExt}(f_*G, K_2)
\]

which is compatible with Baer sums, i.e. it is a monoidal functor.

**Proof.** The first part is a combination of Proposition 2.3.4 and Example 2.3.5. The preservation of Baer sums follows from the same property for \( f_* : \mathsf{CExt}(G, A) \to \mathsf{CExt}(f_*G, f_*A) \) as explained in Example 2.3.5. \( \square \)

In addition, if \( L/F \) is separable then \( f_*G \) is a reductive group over \( F \), otherwise it is pseudo-reductive; we refer to [9, A.5] for the relevant generalities.

Before discussing the effect of Weil restriction on the Brylinski–Deligne classification, we have to review the procedure of Galois descent. Let \( L/F \) be a finite extension of fields, with maximal separable subextension \( L_s/F \). Choose a finite Galois extension \( M/F \) and set

\[
\mathcal{I} := \text{Hom}_F(L_s, M), \quad M_i := L \otimes_{L_s} M, \quad (i \in \mathcal{I}).
\]
By taking sufficiently large $M$ to split $L_s/F$, we have an isomorphism
\[
L \otimes M \xrightarrow{\sim} L \otimes M_{\otimes \mathcal{I}} = \bigoplus_{i \in \mathcal{I}} M_i,
\]
where $\otimes \mathcal{I}$ denotes the $\mathcal{I}$-fold tensor product. The result follows immediately from the definitions.

(2.4)

We deduce from $L_s \subset L$ a morphism $g_i : \text{Spec}(M_i) \to \text{Spec}(M)$. The diagram
\[
\begin{array}{ccc}
\bigsqcup_{i \in \mathcal{I}} \text{Spec}(M_i) & \xrightarrow{\sim} & \text{Spec}(L) \times_{\text{Spec}(F)} \text{Spec}(M) \\
\downarrow \quad g_i & & \downarrow f \\
\bigsqcup_{i \in \mathcal{I}} \text{Spec}(M) & \xrightarrow{\text{natural}} & \text{Spec}(M) \quad \xrightarrow{f} \text{Spec}(F).
\end{array}
\]

is commutative and the rightmost square is Cartesian. We have a functor $X \mapsto X_i := X \times_M Z$ from $\text{Shv}(\text{Spec}(L)_{\text{Zar}})$ to $\text{Shv}(\text{Spec}(M)_{\text{Zar}})$. The canonical isomorphisms [6, p.192], (2.4) give
\[
f_* (X) \times_{\text{Spec}(F)} \text{Spec}(M) \xrightarrow{\sim} f'_* \left( X \times_{\text{Spec}(F)} \text{Spec}(M) \right) \\
\cong f'_* \left( X \times_{\text{Spec}(L)} \left( \text{Spec}(L) \times_{\text{Spec}(F)} \text{Spec}(M) \right) \right) \\
\cong f'_* \left( \bigsqcup_{i \in \mathcal{I}} X_i \right) \cong \prod_{i \in \mathcal{I}} (g_i)_* X_i. \quad (\because \text{Remark 2.3.1})
\]

Every $\sigma \in \text{Gal}(M/F)$ gives rise to isomorphisms $M_i \xrightarrow{\sim} M_{\sigma i}$, so $X_i \xrightarrow{\sim} X_{\sigma i}$. The $\text{Gal}(M/F)$-action on $\prod_{i \in \mathcal{I}} (g_i)_* X_i$ can be determined from (2.4), namely
\[
(x_i)_{i \in \mathcal{I}} \xrightarrow{\sigma \in \text{Gal}(M/F)} (\sigma(x_{i_{\sigma^{-1}}}))_{i \in \mathcal{I}}.
\]
(2.5)

**Remark 2.3.8.** If the sheaf $X$ in the formalism above is defined over $\text{Spec}(F)$, such as $G_m$ and $K_2$, then $X_i = X \times_M Z \xrightarrow{\sim} M_{\sigma i}$, and $X_{\sigma i}$ can be determined from (2.4). Namely, (2.5)

Next, let $G$ be a reductive group over $L$. Upon enlarging $M$, there exists a maximal $L$-torus $T$ of $G$ that splits over $LM$. For each $i \in \mathcal{I}$ there is the $\mathcal{I}$-module $Y_i := X_i (T \times_M M_i)$, with $G(M/F)$ acting via $\sigma : Y_i \to Y_{\sigma i}$. At the cost of neglecting Galois actions, one may identify $Y_i$ and $Y$. By the discussion above, $f_* T \subset f_* G$ is a maximal $F$-torus (see [9, Proposition A.5.15]) that splits over $M$ and $X_* (f_* T) = \bigoplus_{i \in \mathcal{I}} Y_i$ with Galois action given by the recipe (2.5). On the other hand, the Weyl group action on $\bigoplus_{i \in \mathcal{I}} Y_i$ is just the direct sum over $\mathcal{I}$ of the individual ones.

- Note that $f_* (G_\text{ad})$ can be identified with $(f_* G)_\text{ad}$. In view of the foregoing discussions, it suffices to notice that $(G_1 \times \cdots \times G_n)_\text{ad} = G_{1,\text{ad}} \times \cdots \times G_{n,\text{ad}}$ for any tuple of reductive groups $G_1, \ldots, G_n$.
- We also have $f_* (G_\text{ad}) = (f_* G)_\text{ad}$. To see this, note the identification between $f_* (Z_{G})$ and $Z_{f_* G}$ by [9, Proposition A.5.15]; it remains to apply [9, Corollary A.5.4 (3)] to see $f_* G / f_* (Z_{G}) \cong f_* (G/Z_{G})$ canonically.

**Theorem 2.3.10.** In the situation above, suppose that an object $E$ of $\text{CExt}(G, K_2)$ corresponds to the triplet $(Q, D, \varphi)$ with $\text{Gal}(M/L)$-equivariance, as in §2.2. Then $f_* T$ is split over $M$, and the image of $E$ in $\text{CExt}(f_* G, K_2)$ by Proposition 2.3.7 corresponds to the triplet $(f_* Q, f_* D, f_* \varphi)$ where

- $f_* Q : \bigoplus_{i \in \mathcal{I}} Y_i \to Z$ is the direct sum of $Q : Y_i = Y \to Z$, which is automatically Weyl- and Galois-invariant;
- $f_* D$ is the contracted product of the central extensions $G_m \hookrightarrow D_i \hookrightarrow Y_i$, which carries the evident Galois action/descent datum with respect to $M/F$;
• \( f_*\varphi : f_*D \to (f_*D)_G \cong f_*\mathcal{D} \) is obtained by first taking \( \varphi_\pi := \varphi \times M \) to get

\[
\begin{array}{cccc}
1 & \longrightarrow & G_m & \longrightarrow \bigoplus_{i \in I} D_{sc,i} \longrightarrow \bigoplus_{i \in I} Y_{sc,i} & \longrightarrow & 1 \\
1 & \longrightarrow & G_m & \longrightarrow \bigoplus_{i \in I} D_i & \longrightarrow & \bigoplus_{i \in I} Y_i & \longrightarrow & 1
\end{array}
\]

then push-out by product \( G_m \to G_m \); it makes the analogue of (2.3) commutative.

Proof. Since \( f_*T \subset f_*G \) is a maximal \( F \)-torus in a reductive \( F \)-group. We know that \( f_*E \) corresponds to some triplet \( (f_*Q, f_*D, f_*\varphi) \) attached to \( f_*T \), together with descent data with respect to \( M/F \). By the foregoing discussions,

\[
f_*E \times M = \left[ K_2^T \leftarrow \bigoplus_{i \in I} E_i \rightarrow \bigoplus_{i \in I} G_i \right] + \text{Gal}(M/F) \text{-action.}
\]

The norm map in \( K \)-theory induced by \( M \xrightarrow{\text{diag}} M^2 \) (cf. (2.4)) is \( K_2(\cdots)^T \xrightarrow{\text{sum}} K_2(\cdots) \); a further push-out yields the corresponding \( K_2 \)-extension. The corresponding descriptions of \( f_*D \) and \( f_*Q \) are then clear: the principle is the same as that for describing Baer sums under the Brylinski–Deligne classification; we refer to [7, §3] for the precise constructions of \( f_*D \) and \( f_*Q \).

The same construction applies to \( G_{sc} \), the description of \( f_*\varphi \) is thus evident. It is also evident by the constructions above that the descent data come from (2.3).

2.4 The case over local fields: BD-covers

Matsumoto’s theorem [54, III.6.1] says that \( K_2(F) \) is the abelian group generated by symbols \( \{x, y\}_F \) with \( x, y \in F^\times \), subject to the relations

\[
\{xx', y\}_F = \{x, y\}_F + \{x', y\}_F, \quad \{x, yy'\}_F = \{x, y\}_F + \{x, y'\}_F,
\]

\[
x \neq 0, 1 \implies \{x, 1 - x\}_F = 1.
\]

Therefore \( K_2(F) \) is a quotient of \( F^\times \otimes_\mathbb{Z} F^\times \). In other words \( K_2(F) \) equals the Milnor \( K \)-group \( K_2^M(F) \). As is customary, we call \( \{x, y\}_F \) the Steinberg symbol; they are actually anti-symmetric and satisfy \( \{x, -x\}_F = 0 \) or \( \{x, x\}_F = \{x, -1\}_F \) [54, p.246]. Bi-multiplicative maps \( F^\times \times F^\times \to A \) factorizing through \( F^\times \times F^\times \to K_2(F) \) are called \( A \)-valued symbols, where \( A \) is any abelian group.

Henceforth, we assume that \( F \) is a local field.

Following [7, 10.1], we consider the symbols that are locally constant with respect to the topology on \( F^\times \). By a result of Moore, there is an initial object \( F^\times \times F^\times \to K_2^\text{cont}(F) \) in the category of locally constant symbols: in fact

\[
K_2^\text{cont}(F) = \left\{ \begin{array}{ll}
\mu(F), & \mu \notin \mathbb{C} \\
1, & \mu \in \mathbb{C}.
\end{array} \right.
\]

The corresponding homomorphism \( K_2(F) \to K_2^\text{cont}(F) = \mu(F) \) is just the Hilbert symbol \( (\cdot, \cdot)_F, \mathbb{N}_p \).

Let \( G \) be a reductive group over \( F \), so \( G(F) \) inherits the topology from \( F \). For any object \( E \) of \( \text{CExt}(G, K_2) \), we first take \( E \)-points to obtain a central extension \( 0 \to K_2(F) \to E(F) \to G(F) \to 1 \), then push-out by \( K_2(F) \to K_2^\text{cont}(F) \) to obtain

\[
1 \to K_2^\text{cont}(F) \to \tilde{G} \to G(F) \to 1.
\]

Hence \( \tilde{G} \to G(F) \) inherits local sections and transition maps from those of \( E \to G \). It becomes a topological \( K_2^\text{cont}(F) \)-torsor over \( G(F) \) by the following fact [7, Lemma 10.2]: for any scheme \( X \to \text{Spec}(F) \) of finite type and \( s \in H^0(X, K_2) \), evaluation gives a locally constant function \( s_1 : X(F) \to K_2(F) \to K_2^\text{cont}(F) \). Consequently, \( \tilde{G} \) is a topological central extension of locally compact groups, also known as a covering; here \( K_2^\text{cont}(F) \) carries the discrete topology. This is interesting only when \( F \neq \mathbb{C} \).

If \( m \in \mathbb{Z}_{\geq 1} \) and \( m \mid N_F \), a further push-out by

\[
\mu(F) = \mu_{N_F}(F) \longrightarrow \mu_m(F) \quad \text{with } z \longmapsto z^{N_F/m}
\]

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furnishes a topological central extension of $G(F)$ by $\mu_m(F)$. It is the same as the push-out via $(\cdot, \cdot)_{F,m}$ by the following (2.6).

Remark 2.4.1. Suppose $m \mid N_F$. Below is a review of the cohomological interpretation of $(\cdot, \cdot)_{F,m}$. The Kummer map gives $\partial : F^\times \to H^1(F, \mu_m)$. Then $(x, y) \mapsto \partial x \cup \partial y$ factors through $K_2(F) \to H^2(F, \mu_m^{\otimes 2})$; this is called the norm-residue symbol, see [54, III.6.10]. Since the $\Gamma_F$ acts trivially on $\mu_m = \mu_m(F)$,

$$H^2(F, \mu_m^{\otimes 2}) = H^2(F, \mu_m) \otimes \mu_m(F) = m \text{Br}(F) \otimes \mu_m(F)$$

where $m \text{Br}(F)$ stands for the $m$-torsion part of $\text{Br}(F)$. The composite $K_2(F) \to \mu_m(F)$ turns out to be $(\cdot, \cdot)_{F,m}$. Moreover, if $d \mid m$, the interpretation above and the commutative diagram with exact rows

$$\begin{array}{cccc}
1 & \to & \mu_m & \to & G_m & \to & 1 \\
1 & \to & \mu_m/d & \to & G_m & \to & 1 \\
\downarrow d & & \downarrow d & & \downarrow m/d & & \\
\end{array}$$

immediately lead to

$$(x, y)^d_{F,m} = (x, y)_{F,m/d}, \quad x, y \in F^\times. \quad (2.6)$$

Finally, it is known (Merkurjev—Suslin) that $K_2(F)/m \cong \mu_m(F)$ under $(\cdot, \cdot)_{F,m}$; see [54, III.6.9.3].

Definition 2.4.2 (M. Weissman). By a BD-cover of degree $m$, we mean a covering $p : \tilde{G} \to G(F)$ with $\ker(p) = \mu_m(F)$ arising from the procedure above. It consists of the data $m \mid N_F$ together with a multiplicative $K_2$-torsor $E \to G$ over $F$. By convention, when $F = \mathbb{C}$ the covering splits.

The next result concerns Weil restrictions. Let $L/F$ be a finite separable extension of local fields, corresponding to $f : \text{Spec}(L) \to \text{Spec}(F)$. Let $G$ be a reductive group over $L$ and $E$ be an object of $\text{CExt}(G, K_2)$. Denote by $E'$ the image in of $\text{CExt}(f_*G, K_2)$ of $E$ furnished by Proposition 2.3.7. Note that $f_*G(F) = G(L)$. If $m \in \mathbb{Z}_{\geq 1}$ and $m \mid N_F$, then $\mu_m(F) = \mu_m(L)$ and the procedure above will yield

$$\begin{array}{c}
[K_2(L) \hookrightarrow E(L) \twoheadrightarrow G(L)] \\
\xrightarrow{\text{push-out}} [\mu_m(F) \hookleftarrow \tilde{G} \twoheadrightarrow G(L)]
\end{array}$$

$$\begin{array}{c}
[K_2(F) \hookleftarrow E'(F) \twoheadrightarrow G(L)] \\
\xrightarrow{\text{push-out}} [\mu_m(F) \hookleftarrow \tilde{G}' \twoheadrightarrow G(L)].
\end{array}$$

In the degenerate case $L = \mathbb{C}$, we interpret $(\cdot, \cdot)_{L,m}$ as the trivial homomorphism so that $\tilde{G} = G(L) \times \mu_m$.

Proposition 2.4.3. Under the assumptions above, there is a canonical isomorphism $\tilde{G} \cong \tilde{G}'$ between topological central extensions of $G(L)$ by $\mu_m(F)$.

Proof. By construction, the central extension $\tilde{G}'$ is obtained by first pushing-out $E(L)$ via the norm map $K_2(L) \to K_2(F)$ in $K$-theory, followed by $(\cdot, \cdot)_{F,m} : K_2(F) \to \mu_m(F)$. On the other hand, $\tilde{G}$ is obtained by pushing-out $E(L)$ via $(\cdot, \cdot)_{L,m} : K_2(L) \to \mu_m(L) = \mu_m(F)$. In view of the cohomological interpretation of Hilbert symbols (Remark 2.4.1) and the functoriality of push-out, the existence of a canonical $\tilde{G} \cong \tilde{G}'$ is reduced to the commutativity of

$$\begin{array}{ccc}
K_2(L) & \to & H^2(L, \mu_m^{\otimes 2}) \\
\downarrow \text{norm} & & \downarrow \text{cor} \\
K_2(F) & \to & H^2(F, \mu_m^{\otimes 2}) \\
\downarrow \cong & & \downarrow \cong \\
& & \mu_m(F).
\end{array}$$

Indeed, the commutativity of the rightmost square is [11, Proposition 5.5]. As regards the leftmost square, see [49, Lemma 18.2]. To show $\tilde{G} \cong \tilde{G}'$ is a homeomorphism, we may argue using [7, Lemma 10.2] as before. Note that all these make sense even when $L = \mathbb{C}$.  

In summary, the Weil restriction of multiplicative $K_2$-torsors (Proposition 2.3.7) does not affect harmonic analysis.
BD-covers intervene in harmonic analysis in the following way: let \( \tilde{G} \to G(F) \) be a BD-cover of degree \( m \mid N_F \). Upon choosing an embedding \( \epsilon : \mu_m(F) \xrightarrow{\sim} \mu_m \subset C^\times \), we obtain a topological central extension of locally compact groups
\[
1 \to \mu_m \to \tilde{G} \to G(F) \to 1.
\]

**Remark 2.4.4.** Let \( k \in \mathbb{Z} \setminus \{0\} \) and denote by \( \tilde{G}[kQ,m,\epsilon] \) the central extension by \( \mu_m \) obtained from the \( k \)-fold Baer sum of the \( E \to G \) in \( \text{CExt}(G, K_2) \). The notation stems from the fact that \( kE \) corresponds to the datum \( (kQ, \ldots) \), as noted in §2.2. Put \( m = n \epsilon d \), \( k = k'd \) with \( d := \gcd(k, m) \). In view of (2.6), \( \tilde{G}[kQ,m,\epsilon] \) the push-out of \( K_2(F) \to E(F) \to G(F) \) in two equivalent ways:
\[
\begin{bmatrix}
K_2(F) & (\cdot)\epsilon_m \to \mu_m \xrightarrow{k} \mu_m \xrightarrow{\epsilon} C^\times \\
\end{bmatrix} = \begin{bmatrix}
K_2(F) & (\cdot)\epsilon_m' \to \mu_m' \xrightarrow{\epsilon'} \mu_m' \xrightarrow{\epsilon} C^\times \\
\end{bmatrix}.
\]
Hence \( \tilde{G}[kQ,m,\epsilon] \) admits a canonical reduction to \( \tilde{G}[Q,m',\epsilon'] \), where \( \epsilon' := \epsilon k' : \mu_m' \xrightarrow{\sim} \mu_m' \).

We proceed to review a few generalities on topological central extensions. It is sometimes convenient to allow central extensions by \( C^\times \), and the results below will carry over.

**Notation 2.4.5.** Let \( \mu_m \hookrightarrow \tilde{G} \to G(F) \) be any topological central extension of \( G(F) \). For a subset \( C \subset G(F) \), it is customary to denote by \( \tilde{G} \) its preimage in \( \tilde{G} \). In this manner we define \( \tilde{G}_{\text{reg}} \), etc.

**Definition 2.4.6.** For \( \tilde{G} \) as above, we say that a smooth representation \( \pi \) of \( \tilde{G} \) on a \( C \)-vector space \( V \) is **genuine** if \( \pi(z) = z \cdot \text{id}_V \) for all \( z \in \mu_m \). In a similar vein, we have the notion of \( \epsilon \)-**genuine representations** of a BD-cover of degree \( m \) for any embedding \( \epsilon : \mu_m(F) \to C^\times \).

Similarly, let \( C \subset G(F) \) be a subset, we say a function \( f : \tilde{C} \to C^\times \) is genuine if \( f(\epsilon \tilde{z}) = \epsilon f(\tilde{z}) \) for all \( \tilde{z} \in \tilde{C} \) and \( \epsilon \in \mu_m \).

As in (2.1), we define the commutators \( [x,y] = \tilde{x} \tilde{y}^{-1} \tilde{y}^{-1} \) for topological central extensions.

**Definition 2.4.7.** Call an element \( \gamma \in \tilde{G}(F) \) **good** if
\[
[\gamma, \eta] = 1, \quad \eta \in Z_G(\gamma)(F).
\]
This amounts to \( \eta \tilde{\gamma} \eta^{-1} = \tilde{\gamma} \) whenever \( \eta \in Z_G(\gamma)(F) \) and \( \tilde{G} \ni \tilde{\gamma} \to \gamma \). Call an element of \( \tilde{G} \) good if its image in \( G(F) \) is.

Every genuine \( G(F) \)-invariant function on \( \tilde{G}_{\text{reg}} \) vanishes off the good locus.

**Proposition 2.4.8.** Suppose that \( G(F) \) is perfect, i.e. equals its own commutator subgroup. Let \( \sigma \) be an automorphism of the topological group \( G(F) \). Then there exists at most one automorphism \( \tilde{\sigma} : \tilde{G} \to \tilde{G} \) of coverings lifting \( \sigma \).

**Proof.** Let \( \tilde{\sigma}_1, \tilde{\sigma}_2 \) be two liftings, then \( \chi := \tilde{\sigma}_1^{-1} \circ \tilde{\sigma}_2 \) is an automorphism of \( \tilde{G} \) lifting \( \text{id}_{G(F)} \). Hence \( \chi \) is a homomorphism \( G(F) \to \mu_m \), which must be trivial. \( \square \)

Finally, we record a useful observation on splittings.

**Lemma 2.4.9.** Let \( A \subset G(F) \) be a pro-\( p \)-subgroup, where \( p \nmid m \) is a prime number. Then \( \tilde{G} \to G(F) \) admits a unique splitting over \( A \), and all elements in \( A \) are good.

**Proof.** Write \( A \simeq \varprojlim A_n \) where \( A_n \) is a finite \( p \)-group for each \( n \). The obstructions to the existence and uniqueness of splittings live in \( H^2(A, \mu_m) \) and \( H^1(A, \mu_m) \), respectively. Continuous cohomology satisfies \( H^*(A, \mu_m) = \lim_{\to} H^*(A_n, \mu_m) \). Evidently, \( H^*(A_n, \mu_n) = 0 \) for all \( n \).

Let \( a \in A \) and \( g \in Z_G(a)(F) \), then \( [a^p, g] = [a,g]^p \) in \( \mu_m \). We have \( \lim_{k \to \infty} a^{p^k} = 1 \) whereas \( \{p^k\}_{k=1}^\infty \) is periodic mod \( m \), hence \( [a, g] = 1 \). \( \square \)
2.5 The isogeny $T_{Q,m} \to T$

Retain the notation from §2.4. In particular, we fix a reductive $F$-group $G$ and $m \mid N_F$. Take an object

$$1 \to K_2 \to E \to G \to 1$$

of $\text{CExt}(G, K_2)$, and suppose $T \subset G$ is a maximal $F$-torus. Thus $\Gamma_F$ acts on $Y := X_*(T_{\bar F})$. The Brylinski–Deligne classification in §2.2 attaches to $E$ a $\Gamma_F$-invariant quadratic form $Q : Y \to \mathbb{Z}$. Define

$$B_{Q}(y_1, y_2) := Q(y_1 + y_2) - Q(y_1) - Q(y_2),$$

$$Y_{Q,m} := \{ y \in Y : \forall y' \in Y, B_{Q}(y, y') \in m\mathbb{Z} \} \supset mY,$$

$$X_{Q,m} := \{ x \in X \otimes \mathbb{Q} : \forall y \in Y_{Q,m}, \langle x, y \rangle \in \mathbb{Z} \} \subset \frac{1}{m}X.$$

One should view them as local systems over $\text{Spec}(F)_\text{et}$. From $Y_{Q,m} \hookrightarrow Y$ we deduce an isogeny

$$\iota_{Q,m} : T_{Q,m} \to T.$$

Note that $\iota_{Q,m}$ is an étale morphism and the scheme-theoretic $\ker(\iota_{Q,m})$ is a finite étale, since $Y_{Q,m} \supset mY$ and $m$ is invertible in $F$.

The invariance properties allow use to view $Q, B_Q$ as living on $\lim_{\leftarrow T} X_*(T_{\bar F})$, where $T$ ranges over maximal $F$-tori and the arrows are $G(\bar F)$-conjugation. In particular, we have the commutative diagram below.

$$\begin{array}{ccc}
T_{Q,m,\bar F} \overset{\text{Ad}(g)}{\longrightarrow} S_{Q,m,\bar F} \\
\iota_{Q,m,\bar F} \downarrow \quad \downarrow \iota_{Q,m} \\
T_{\bar F} \overset{\text{Ad}(g)}{\longrightarrow} S
\end{array} \tag{2.7}$$

Hence the absolute Weyl group $\Omega(G, T)(\bar F)$ acts on $T_{Q,m,\bar F}$. By taking $\text{Ad}(g)$ defined over $F$, we see that stable conjugacy $\text{Ad}(g) : T \to S$ (see §3.2) induces $T_{Q,m} \to S_{Q,m}$.

Now we take a finite separable extension $L$ of $F$, and write $f : \text{Spec}(L) \to \text{Spec}(F)$. Take $G$ to be a reductive group over $L$ and $T \subset G$ be a maximal $L$-torus. Let $E$ be an object of $\text{CExt}(G, K_2)$ (thus defined over $L$). We turn to the Weil restriction $f_*E$ afforded by Proposition 2.3.7.

**Proposition 2.5.1.** Suppose $m \mid N_F$, so that $m \mid N_E$ and consider the isogeny $\iota_{Q,m} : T_{Q,m} \to T$ between tori over $L$. Then the isogeny attached to $f_*T \subset f_*G$ and $f_*E$ can be identified with

$$f_*(\iota_{Q,m}) : f_*T_{Q,m} \to f_*T.$$

**Proof.** This is clear in view of the effect of $f_*$ on the datum $Q$, see Theorem 2.3.10. \qed

The discussions above work for both local and global fields. Henceforth we shall assume $F$ local. Consider the BD-cover obtained from $E$, denoted by $\tilde G \to G(F)$ as usual.

**Proposition 2.5.2** (M. Weissman [59]). Suppose $G = T$ is a torus, then all elements in $\text{im}(\iota_{Q,m})$ are good; the converse is true when $T$ is split.

In general, we consider the pull-back $\tilde T \to T(F)$ of $\tilde G \to G(F)$. It is also a BD-cover.

**Corollary 2.5.3.** Let $\gamma \in T_{\text{reg}}(F)$. If $\gamma$ lies in the image of $\iota_{Q,m}$, then it is good as an element of $G(F)$; the converse is true when $T$ is split.

**Proof.** Since $Z_G(\gamma) = T$, it suffices to notice that the isogeny $\iota_{Q,m}$ is the same for both $\tilde G$ and $\tilde T$ and apply Proposition 2.5.2. \qed

3 The symplectic group

Unless otherwise specified, $F$ is a local field of characteristic $\neq 2$. 
3.1 Conjugacy classes and maximal tori

Let \((W, \langle \cdot | \cdot \rangle)\) be a symplectic \(F\)-vector space with \(\dim_F W = 2n > 0\). Every orthogonal decomposition \(W = \bigoplus_{i=1}^{k} W_i\) yields to a natural embedding \(\prod_{i=1}^{k} \text{Sp}(W_i) \hookrightarrow \text{Sp}(W)\).

There always exists a basis \(e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}\) satisfying \(\langle e_i, e_j \rangle = 0\) and \(\langle e_i, e_{-j} \rangle = 1\) for all \(1 \leq i, j \leq n\), called a symplectic basis of \(W\). It gives rise to the Borel pair
\[
B := \text{Stab}_{\text{Sp}(W)}(\langle e_1 \rangle) \subset \langle e_1, e_2 \rangle \subset \cdots \subset \langle e_1, \ldots, e_{-1} \rangle),
T := \{\text{diag}(a_1, \ldots, a_n, a_{-1}, \ldots, a_{-1}) : (a_1, \ldots, a_n) \in \mathbb{G}_m^n \} \cong \mathbb{G}_m^n.
\]

Write
\[
X := X^*(T), \quad Y := X_*(T).
\]

Denote by \(\epsilon_i \in X\) the character \(\text{diag}(a_1, \ldots, a_i, 1, \ldots, 1) \mapsto a_i\). The \(B\)-positive roots are
\[
\epsilon_i \pm \epsilon_j (n \geq i > j \geq 1), \quad 2\epsilon_i (n \geq i \geq 1)
\]
and the simple ones are \(\{\epsilon_i - \epsilon_{i+1} : i = 1, \ldots, n-1\} \cup \{2\epsilon_i : i = 1, \ldots, n\}\). Writing \(\check{\epsilon}_1, \ldots, \check{\epsilon}_n\) for the dual basis, the corresponding coroots are
\[
\check{\epsilon}_i \pm \check{\epsilon}_j, \quad \check{\epsilon}_i.
\]
The long positive roots are \(2\check{\epsilon}_i\) whilst the short positive coroots are \(\check{\epsilon}_i\). This is a simply connected root datum of type \(C_n := A_1\) when \(n = 1\). The Weyl group attached to \(T \subset \text{Sp}(W)\) is
\[
\Omega(G, T) = \{ \pm 1 \}^n \rtimes \mathfrak{S}_n
\]
acting on \(Y = \bigoplus \mathbb{Z}\check{\epsilon}_i\), where the \(\{ \pm 1 \}^n\) acts by component-wise multiplication and \(\mathfrak{S}_n\) permutes. The Weyl-invariant quadratic forms \(Y \rightarrow \mathbb{Z}\) are readily seen to be integer multiples of
\[
Q : (y_1, \ldots, y_n) \mapsto y_1^2 + \cdots + y_n^2.
\]

(3.1)

Also note that \(Q\) takes value 1 on short coroots. In this case we have
\[
Y = \bigoplus_{\alpha > 0 \text{ simple root}} \mathbb{Z}\check{\alpha} = \bigoplus_{\alpha > 0 \text{ long root}} \mathbb{Z}\check{\alpha}.
\]

Next, we recall the parametrization of regular semisimple conjugacy classes in \(\text{Sp}(W)\) following [48, IV.2], [51] or [35, §3]. They are described by the data \((K, K^+, x, c)\) in which

- \(K\) is an \(\text{étale} F\)-algebra of dimension \(2n\) equipped with an involution \(\tau\);
- \(K^+\) is the subalgebra \(\{t \in K : \tau(t) = t\}\);
- \(x \in K^\times\) satisfies \(\tau(x) = x^{-1}\) and \(K = F[a]\);
- \(c \in K^\times\) satisfies \(\tau(c) = -c\).

In the general recipe of [48, IV.1], one sets \(K\) to be the subalgebra of \(\text{End}_F(W)\) generated by the element \(\gamma\) in question; since \(\gamma \in \text{Sp}(W)_{\text{reg}}\), the \(\text{étale}\) property of \(K\) can be seen over the algebraic closure of \(F\). The notation is justified since the extension \(K \supset K^2\) determines \(\tau\). Indeed, \(K^2\) is also an \(\text{étale} F\)-algebra, thus decomposes into \(\prod_{i \in I} K_i^2\) where each \(K_i^2\) is a field. Accordingly, \(K = \prod_{i \in I} K_i\) and for each \(i \in I\), either

(a) \(K_i\) is a quadratic field extension of \(K_i^2\) with \(\text{Gal}(K_i/K_i^2) = \{\text{id}, \tau\}_{K_i}\), or
(b) \(K_i \simeq K_i^2 \times K_i^2\) as \(K_i^2\)-algebras and \(\tau(u, v) = (v, u)\).

In both cases \(K_i/K_i^2\) determines \(\tau|_{K_i}\).

Notation 3.1.1. Given an \(\text{étale} F\)-algebra \(K\) with involution \(\tau\) and fixed subalgebra \(K^2\), we define the norm map
\[
N_{K/K^2} : K \longrightarrow K^2 \quad t \mapsto tr(t)
\]
and the norm-one \(F\)-torus
\[
K^1 := \{y \in K^\times : N_{K/K^2}(y) = 1\}.
\]
We say \((K, K^2, x, c)\) and \((L, L^2, y, d)\) are equivalent if there exists an isomorphism \(\varphi : K \overset{\sim}{\rightarrow} K^2\) of \(F\)-algebras such that

- \(\varphi\) preserves involutions,
- \(\varphi(x) = y,\)
- \(\varphi(c) d^{-1} \in N_{L/L^2}(K^x)\).

To each datum \((K, K^2, x, c)\), we deduce a symplectic form on the \(F\)-vector space \(K\):

\[
h(u|v) = \text{tr}_{K/F}(\tau(u)vc); \tag{3.2}
\]

the trace here is non-degenerate since \(K\) is étale over \(F\). It can also be described by breaking \(K\) into \(\prod_{i \in I} K_i\). For every \(x \in K^1\), the automorphism \(m_x : t \mapsto y t\) preserves \(h\). Every symplectic \(F\)-vector space \((W, \langle \cdot | \cdot \rangle)\) of dimension \(2n\) is isomorphic to \((K, h)\) via some \(\iota : W \overset{\sim}{\rightarrow} K\): therefore \(m_x\) transports to \(\delta := \iota^{-1}m_y \iota \in \text{Sp}(W)\). The conjugacy class \(\mathcal{O}(K/K^2, x, c)\) of \(\delta\) in \(\text{Sp}(W)\) is regular semisimple and is independent of the choice of \(\iota\). The following result is standard.

**Proposition 3.1.2.** The mapping \(\mathcal{O}\) is a bijection between the equivalence classes of data \((K, K^2, x, c)\) and the regular semisimple conjugacy classes in \(\text{Sp}(W)\). Furthermore, by choosing \(\iota : W \overset{\sim}{\rightarrow} K\) as above, there is an isomorphism of \(F\)-tori

\[
j : K^1 \overset{\sim}{\rightarrow} Z_{\text{Sp}(W)}(\delta) \quad y \mapsto \iota^{-1}m_y \iota.
\]

In particular \(Z_{\text{Sp}(W)}(\delta)\) is always connected; this is also known from the fact that \(\text{Sp}(W)\) is simply connected.

The datum \(x\) is not used in constructing \((K, h)\). There is a similar notion of equivalence among data \((K, K^2, c)\). Given such a datum, \(j : y \mapsto \iota^{-1}m_y \iota\) embeds \(K^1\) as a maximal \(F\)-torus \(j(K^1) \subset \text{Sp}(W)\); its conjugacy class is independent of the choice of \(\iota\).

**Proposition 3.1.3.** The assignment \((K, K^2, c) \mapsto j\) is a bijection between the equivalence classes of data \((K, K^2, c)\) and the conjugacy classes of embeddings of maximal \(F\)-tori \(j : T \mapsto \text{Sp}(W)\).

**Proof.** To show surjectivity, given \(j\) there exists \(t \in T(F)\) such that \(j(t) \in \text{Sp}(W)_{\text{reg}}\) by Zariski density. By Proposition 3.1.2, the class of \(j(t)\) is parameterized by some \((K, K^2, x, c)\) that comes with an embedding \(j_K : K^1 \rightarrow \text{Sp}(W)\); upon conjugating \(j\), we may assume \(j_K(x) = j(t)\). Taking centralizers yields \(\text{im}(j) = \text{im}(j_K)\), hence we deduce \(j_K^{-1}j : T \overset{\sim}{\rightarrow} K^1\) that implements the equivalence between \(j\) and \(j_K\).

As for injectivity, let \(j, j'\) be embeddings arising from data \((K, K^2, c)\) and \((K', K'^2, c')\). If \(j, j'\) are conjugate, there will exist \(x \in K^1\) and \(x' \in K'^1\) such that \(j(x)\) and \(j'(x')\) are conjugate in \(\text{Sp}(W)_{\text{reg}}\), which implies the equivalence of \((K, K^2, x, c)\) and \((K', K'^2, x', c')\) by Proposition 3.1.2.

Note that the classification of conjugacy classes of maximal tori \(T \subset \text{Sp}(W)\) is coarser than that of embeddings.

**Remark 3.1.4.** If \(\langle \cdot | \cdot \rangle\) becomes \(a \langle \cdot | \cdot \rangle\), then for the \(\iota : W \overset{\sim}{\rightarrow} K\) above to be an isometry, the datum \((K, K^2, x, c)\) must be replaced by \((K, K^2, x, ac)\), i.e. replace \(h\) by \(ah\). In a similar vein, suppose that the \(\iota^{-1}m_y \iota \in \text{Sp}(W)\) parameterized by \(y \in K^1\) becomes \(g_1 \iota^{-1}m_y \iota g_1^{-1}\), with \(g_1 \in \text{GSp}(W)\) having similitude factor \(a\); to make \(g_1^{-1} : W \rightarrow K\) an isometry, we must pass to \((K, K^2, x, ac)\).

Finally, the action of \(\Omega(G, T)(F) := (\text{NG}(T)/T)(F)\) on \(T(F) \simeq K^1\) has the algebraic description below.

**Proposition 3.1.5.** The action of \(\Omega(G, T)(F)\) on \(T\) is the same as that of \(\text{Aut}(K, \tau)\) on \(K^1\), where \((K, \tau)\) is seen as an étale \(F\)-algebra with involution.

**Proof.** We have \(K \otimes_F \bar{F} \simeq (F \times \bar{F})^n\), so the description of Weyl groups in §3.1 ensures that \(\Omega(G, T)(F) = \text{Aut}((K, \tau) \otimes_F \bar{F})\). This identification respects \(\Gamma_F\)-actions, so we conclude by Galois descent.

### 3.2 Stable conjugacy: parameters

Let \(G\) be a reductive \(F\)-group. We say \(\delta, \eta \in G_{\text{reg}}(F)\) are **stably conjugate**, written as \(\delta \overset{st}{\sim} \eta\), if they are conjugate in \(G(F)\). Define the variety \(\mathcal{T}(\delta, \eta) := \{y \in G : g \delta g^{-1} = \eta\}\), which admits an \(F\)-structure. It is nonempty (resp. has an \(F\)-point) if and only if \(\delta\) and \(\eta\) are stably conjugate (resp. conjugate).
Proposition 3.2.1. Let $\delta, \eta \in G_{\text{reg}}(F)$. Define the $F$-tori of centralizers

$$R := G_S, \quad S := G_\eta.$$ 

If $\delta$ and $\eta$ are stably conjugate, then there is a canonical isomorphism $\text{Ad}(g) : R \sim S$ between $F$-tori, where $g \in \mathcal{T}(\delta, \eta)(F)$ is arbitrary.

**Proof.** Indeed, $\mathcal{T}(\delta, \eta)$ is an $(S, R)$-bitorsor, meaning that it is simultaneously a left (resp. right) torsor under $S$ (resp. $R$) satisfying $s(gr) = (sg)r$ for all $g$ in $\mathcal{T}(x, y)$. Therefore $g$ defines an isomorphism $\sigma : S \sim R$ such that $sg = ge(r)$, and this is seen to be independent of $g$ as $S, R$ are both commutative. In our case $\sigma = \text{Ad}(g) : S \sim R$ for any $g \in \mathcal{T}(\delta, \eta)(F)$, and this isomorphism descends to $F$. □

We say two embeddings of maximal $F$-tori $j, j' : S \hookrightarrow G$ are **stably conjugate** if there exists $g \in G(\bar{F})$ inducing $j' = \text{Ad}(g) \circ j$. The foregoing Proposition entails that stable conjugacy of strongly regular semisimple elements is realized by stable conjugacy of embeddings of maximal tori, and the converse is evidently true.

**Definition 3.2.2.** For any subgroup $S \subset G$, define the pointed set

$$\mathcal{D}(S, G; F) := \ker [H^1(F, S) \rightarrow H^1(F, G)] .$$

Consider the situation of stable conjugacy $\eta = g\delta g^{-1}$ as before, where $g \in G(\bar{F})$. Put $T := G_\delta$. The element $g$ induces a $T$-valued $1$-cocycle $c_g : \Gamma_F \ni \tau \mapsto g^{-1}\tau(g)$. It is well-known (eg. [5, §27]) that the recipe induces a bijection

$$\text{inv}(\delta, \cdot) \{ \eta \in G(F) : \delta \sim \eta \} \bigg/ \text{conj} \sim \mathcal{D}(T, G; F),$$

$$\eta \mapsto g\delta g^{-1} \mapsto \text{inv}(\delta, \eta) := [c_g].$$

(3.3)

In fact, $[c_g] \in H^1(F, T)$ parameterizes the right $T$-torsor $\mathcal{T}(\delta, \eta)$. By fixing a stable conjugacy class $\mathcal{O} \subset G(F)$, we may work inside $\varprojlim_s H^1(F, G_\delta)$ for $\delta$ ranging over $\mathcal{O}$, at least as a device to simplify expressions. In this setting, one verifies readily that for $\delta, \delta', \delta'' \in \mathcal{O}$

$$\text{inv}(\delta, \delta') = \text{inv}(\delta, \delta'') + \text{inv}(\delta', \delta''), \quad \text{and} \quad \text{inv}(\delta, \delta') = -\text{inv}(\delta', \delta).$$

(3.4)

Likewise, we can measure the relative position of stably conjugate embeddings $j, j' : S \hookrightarrow G$ by $\text{inv}(j, j') \in H^1(F, S)$: it is still given by $[c_g]$ when $j' = \text{Ad}(g) \circ j$. It has the same properties above.

Let us turn to a special instance of stable conjugacy. The adjoint group $G_{\text{ad}}(F)$ acts on $G(F)$ and $g\delta g^{-1}$ is stably conjugate to $\delta$ whenever $g \in G_{\text{ad}}(F)$. On the other hand, the short exact sequence

$$1 \rightarrow Z_G \rightarrow G \rightarrow G_{\text{ad}} \rightarrow 1$$

induces $G_{\text{ad}}(F)/\pi(G(F)) \sim \mathcal{D}(Z_G, G; F)$.

**Proposition 3.2.3.** For all $\delta \in G_{\text{reg}}(F)$ with $T = G_\delta$ and all $g \in G_{\text{ad}}(F)$, we have

$$\text{inv}(\delta, g\delta g^{-1}) = \text{the image of } g \text{ under } G_{\text{ad}}(F) \rightarrow \mathcal{D}(Z_G, G; F) \rightarrow \mathcal{D}(T, G; F).$$

**Proof.** Unravel the definition of $G_{\text{ad}}(F) \rightarrow H^1(F, Z_G)$. □

Now specialize to $G := \text{Sp}(W)$. The following result is also standard, see for instance [35, §3].

**Proposition 3.2.4.** In terms of the parameterization of 3.1.2, two regular semisimple conjugacy classes $\mathcal{O}(K, K^t, x, c)$ and $\mathcal{O}(L, L^t, y, d)$ in $G = \text{Sp}(W)$ are stably conjugate if and only if there exists an isomorphism of $F$-algebras $K \sim L$ that

- respects the involutions and
- maps $x$ to $y$.

In other words, passing to stable conjugacy amounts to discarding the datum $c$. Choose $\delta \in \mathcal{O}(K, K^t, x, c)$ and set $T := G_\delta$, decompose $K^t = \prod_{i \in I} K_i^t$ and put $I_0 := \{ i \in I : K_i \text{ is a field} \}$. Then there are canonical isomorphisms of groups

$$\mathcal{D}(T, G; F) = H^1(F, T) \sim K^{t, x}/N_{K/K_1}(K^x) \sim \{ \pm 1 \}^{I_0}.$$ 

Finally, forgetting the datum $x$ gives a parameterization of stable conjugacy classes of embeddings of maximal $F$-tori.
Proof. The algebra $K = F[x]$ depends only on the conjugacy class placed in $GL(W)$, whilst the involution is determined by $x \mapsto x^{-1}$; the invariance of $(K, K^\perp, x)$ follows. Conversely, given data $(K, K^\perp, x, c)$ parameterizing conjugacy classes $\mathcal{O}$ and $\mathcal{O}'$, we define the corresponding symplectic forms $h, h^t : K \to F$. If there exists $a \in K^\perp$ such that $c' = cN_{K/K^1}(a)$, then $t \mapsto at$ will define an isomorphism $(K, h) \cong (K, h')$ between symplectic $F$-vector spaces. The existence of such a $a$ can be guaranteed upon base-change to a finite extension $E/F$, therefore $\mathcal{O}, \mathcal{O}'$ become conjugate in $Sp(W \otimes_F E)$.

As $H^1(F, G)$ is trivial, $\mathcal{D}(T, G; F) = H^1(F, T)$ is an abelian group. Define the $K_1^1$-torus $K_1^1$ (Notation 3.1.1) and use Notation 2.3.2 for Weil restriction. Shapiro’s lemma [17, Exp XXIV. Proposition 8.4] (which also works for non-separable extensions, see the Remarque 8.5 therein) implies that

$$H^1(F, T) \cong H^1(F, K^1) = \prod_{i \in I} H^1(F, R_{K^1/F}(K^1_i)) = \prod_{i \in I} H^1(K^1_i, K^1_i).$$

It remains to show $H^1(F, K^1) = F^x/N_{K/F}(K^x)$ when $K$ is a 2-dimensional étale $F$-algebra. If $K$ is a field, use the short exact sequence

$$1 \to K^1 \to R_{K/F}(G_{m, K}) \xrightarrow{N_{K/F}} G_{m, F} \to 1$$

of $F$-tori to see $F^x/N_{K/F}(K^x) \cong H^1(F, K^1)$; the case $K = F \times F$ is trivial. Local class field gives the description of $F^x/N_{K/F}(K^x) \cong \{ \pm 1 \}$.

For the final assertion, see the proof of Proposition 3.1.3. □

Definition 3.2.5. Let $T \subset G$ be parameterized by $(K, K^\perp, c)$. Denote by $\kappa_- = \kappa_-$ the homomorphism $H^1(F, T) \to \mu_2$ corresponding to

$$\prod_{i \in I_0} K^1_i \cong N_{K/F}(K^1_i) \ni (t_i) \mapsto \prod_{i \in I_0} \text{sgn}_{K^1_i/F}(t_i).$$

Pairings of the form $(\kappa_- \circ \text{inv}(\delta, \delta'))$ will play an important role, where $\delta' \overset{st}{=} \delta \in T_{reg}(F)$. One can show that $\kappa_-$ is canonically defined, cf. the interpretation via long roots in §3.3, but this will not be needed.

Lemma 3.2.6. Conserve the notation of Proposition 3.2.4. Then $T$ is anisotropic if and only if $I_0 = 1$.

Proof. For any finite extension $L/F$ and $L$-torus $S$, the adjunction

$$\text{Hom}_{F, \text{grp}}(G_{m, F}, R_{L/F}(S)) \simeq \text{Hom}_{L, \text{grp}}(G_{m, L}, S)$$

implies that $R_{L/F}(S)$ is anisotropic over $F$ if and only if $S$ is over $L$. Our assertion thus reduces to the case $\dim_F W = 2$, which is easy. □

Turn to the rank-one case $\dim_F W = 2$. By choosing a symplectic basis $\{e_1, e_2\}$, we may identify $G$ (resp. $G_{ad}$) with $SL(2)$ (resp. $PGL(2)$). The $PGL(2, F)$-action on $SL(2, F)$ is realizable in $GL(2, F)$.

From $\mathcal{D}(\mu_2, SL(2); F) = H^1(F, \mu_2) \hookrightarrow F^x/F^{x^2}$, we obtain a canonical homomorphism

$$\nu : G_{ad}(F)/\text{im}[G(F) \to G_{ad}(F)] \cong F^{x^2}/F^{x^2}.$$ (3.5)

Let $x \in SL(2, F)$ be semisimple regular with parameter $(K, F, \ldots)$ for its conjugacy class, so that $T := Z_{SL(2)}(x) \simeq K^1$. The following has been recorded in [31, p.728].

Proposition 3.2.7. The stable conjugacy class of $\delta \in SL(2, F)$ consists of elements of the form

$$g \delta g^{-1}, \quad g \in PGL(2, F).$$

Choose any $g_1 \in GL(2, F)$ that maps to $g$, then $\text{inv}(\delta, g \delta^{-1} g_1)$ equals

$$\text{image of } g \text{ under } PGL(2, F) \to \mathcal{D}(\mu_2, SL(2); F) \to \mathcal{D}(T, SL(2); F) = \text{image of } \det(g_1) \text{ or } \nu(g) \text{ under } F^x/F^{x^2} \hookrightarrow F^x/N_{K/F}(K^x) \cong \mathcal{D}(T, SL(2); F).$$

Consequently, stable conjugacy in $SL(2, F)$ is realizable by $PGL(2, F)$-action.
Proof. The first equality for \( \inv(\delta, g \delta g^{-1}) \) is Proposition 3.2.3. The second one amounts to the commutativity of

\[
\begin{align*}
PGL(2, F) & \longrightarrow \mathcal{D}(\mu_2, SL(2); F) \longrightarrow \mathcal{D}(T, SL(2); F) \\
\nu & \downarrow \quad \iso \quad \downarrow \iso \\
GL(2, F) & \underset{\text{det}}{\longrightarrow} F^x/F^x \times 2 \longrightarrow F^x/N_{K/F}(K^x)
\end{align*}
\]

The right square commutes because so does the following diagram.

\[
\begin{align*}
\mu_2 & \longrightarrow \mathbb{G}_m \overset{2}{\longrightarrow} \mathbb{G}_m \\
\downarrow & \quad \quad \downarrow \\
K^1 & \longrightarrow R_{K/L}\mathbb{G}_{m,K} \overset{1}{\longrightarrow} \mathbb{G}_m.
\end{align*}
\]

The triangle \( \Box \) commutes by the definition of \( \nu \). As for the \( \Box \), note that given \( g_1 \rightarrow g \), every \( \tau \in \Gamma_F \) multiplies \( (\det g_1)^{-1/2}g_1 \in SL(2, F) \) by a sign \( c(\tau) \), and \( \tau \mapsto c(\tau) \) represents the image of \( g \) in \( H^1(F, \mu_2) \). But the same cocycle represents the image of \( \det(g_1) \) in \( F^x/F^x \times 2 \iso H^1(F, \mu_2) \). \( \square \)

3.3 Stable conjugacy: reduction to \( SL(2) \)

Fix a maximal \( F \)-torus \( T \) in \( G := \text{Sp}(W) \). The lattices \( X := X^* (T_F) \) and \( Y := X_+ (T_F) \) are endowed with \( \Gamma_F \)-actions. We choose an isomorphism \( T_F \iso \mathbb{G}_m^2 \) as in §3.1, and enumerate the long roots (resp. short coroots) as \( \pm 2\epsilon_1, \ldots, \pm 2\epsilon_n \) (resp. \( \pm \hat{\epsilon}_1, \ldots, \pm \hat{\epsilon}_n \)). Then \( \Gamma_F \) acts on both sets and commutes with the bijection between roots and coroots. If \( \mathcal{O} \) is a \( \Gamma_F \)-orbit, so is \( -\mathcal{O} \). Now recall some definitions from [32, §2].

Definition 3.3.1. Let \( \mathcal{O} \) be a \( \Gamma_F \)-orbit of roots.

(i) If \( \mathcal{O} = -\mathcal{O} \), we say \( \mathcal{O} \) is symmetric.

(ii) If \( \mathcal{O} \cap (-\mathcal{O}) = \emptyset \), we say \( \mathcal{O} \) is asymmetric.

The same terminology pertains to \( \Gamma_F \)-orbits of coroots.

For any root \( \alpha \), set

\[
\begin{align*}
\Gamma_\alpha := \text{Stab}_{\Gamma_F} (\alpha) & \subset \text{Stab}_{\Gamma_F} (\{ \pm \alpha \}) =: \Gamma_{\pm \alpha}, \\
F_{\alpha} & \supset F_{\pm \alpha} \supset F : \quad \text{their fixed fields in} \ F.
\end{align*}
\]

These extensions are all separable since the splitting field of \( T \) is.

Lemma 3.3.2. The torus \( T \) is anisotropic if and only if every orbit \( \mathcal{O} \) of long roots (resp. short coroots) is symmetric.

Proof. For any orbit \( \mathcal{O} \) of long roots, \( \sum_{\alpha \in \mathcal{O}} \alpha \in X^\Gamma_F = 0 \) if and only if \( \mathcal{O} \) is symmetric. The existence of asymmetric \( \mathcal{O} \) implies \( X^\Gamma_F \neq 0 \), thus \( T \) is isotropic. Conversely, assume every \( \mathcal{O} \) is symmetric and consider \( v = \sum_{i=1}^n a_i \epsilon_i \in X^\Gamma_F \setminus \{0\} \). For each \( i \), there exists \( \sigma \in \Gamma_F \) such that \( \sigma(\pm \epsilon_i) = \mp \epsilon_i \) since \( 2\epsilon_i \) belongs to a symmetric orbit; from \( \sigma(v) = v \) we deduce \( a_i = 0 \). This implies \( v = 0 \) so \( T \) is anisotropic. \( \square \)

Hereafter we only consider the orbits of long roots or short coroots.

Select a \( \Gamma_B \)-orbit \( \mathcal{O} \) in \( X \) together with a long \( \alpha \in \mathcal{O} \). After base-change to \( F_{\pm \alpha} \), we see that \( \{ \alpha, -\alpha \} \) generates a copy of \( SL(2) \) in \( G_{F_{\pm \alpha}} \). This \( SL(2) \) contains the subtorus \( T_{\pm \alpha} \) of \( T_F \) with \( X_+(T_{\pm \alpha}, F) = \mathbb{Z} \hat{\alpha} \subset Y \); as a shorthand, we say \( T_{\pm \alpha} \) is generated by \( \alpha \). Hence \( T_{\pm \alpha} \), \( SL(2) \) are both defined over \( F_{\pm \alpha} \).

1. First assume \( \mathcal{O} \) is symmetric so that \( (\Gamma_{\pm \alpha} : \Gamma_\alpha) = [F_\alpha : F_{\pm \alpha}] = 2 \). In this case \( T_{\pm \alpha} \) is anisotropic and it splits over \( F_\alpha \), and

\[
T_{\pm \alpha} \iso F_\alpha^1 := \ker [N_{F_\alpha/F_{\pm \alpha}} : F_\alpha \times F_{\pm \alpha}].
\]

2. Next, assume \( \mathcal{O} \) is asymmetric. Then \( \Gamma_{\pm \alpha} = \Gamma_\alpha, F_\alpha = F_{\pm \alpha}, \) and \( T_{\pm \alpha} \) is a split.
Identify $\mathcal{O}$ with $\text{Hom}_F(F_\alpha, \tilde{F}) = \Gamma_F/\Gamma_{F_\alpha}$. Let $T_\mathcal{O}$ be the subtorus of $T_F$ generated by $\mathcal{O}$ in the sense above, which is now defined over $F$. By the generalities on reductive groups and their Weil restrictions (cf. (2.4)), we see that there is an embedding (see §3.2 for discussions on $\text{SL}(2)$):

$$-1 \in \text{SL}(2, F_{\pm\alpha}) \quad \text{max. torus} \quad R_{F_{\pm\alpha}/F}(\text{SL}(2)) \longrightarrow G$$

$$R_{F_{\pm\alpha}/F}(T_{\pm\alpha}) \quad \text{max. torus} \quad T_\mathcal{O} \longrightarrow T$$

Denote the image of $R_{F_{\pm\alpha}/F}(\text{SL}(2))$ as $G_\mathcal{O}$. It is a canonically defined subgroup of $G$ relative to $\mathcal{O}$. Observe that $(G_\mathcal{O}, T_\mathcal{O}) = (G_{-\mathcal{O}}, T_{-\mathcal{O}})$. Furthermore,

$$\prod_{\pm\alpha} T_\mathcal{O} \simeq T.$$

This construction can be described in terms of the parameter $(K, K^2, c)$ of $T \hookrightarrow G$ supplied by Proposition 3.1.3. We shall use the familiar decomposition $K = \prod_{i \in I} K_i$.

- When $T$ is split, we may assume $I = \{1, \ldots, n\}$, $K_i^2 = F$, and $K_i = F \times F$ for all $i$. Identify $T$ with $\{(x_i, y_i)_{i=1}^n \in (F^\times \times F^\times)^n : x_i y_i = 1\}$. Now $X = \bigoplus_{i=1}^n \mathbb{Z} \varepsilon_i$ where $\varepsilon_i$ (resp. $-\varepsilon_i$) corresponds to $(x_i, y_i) \mapsto x_i$ (resp. $(x_i, y_i) \mapsto y_i$). The $\Gamma_F$-orbits of long roots are singletons $\{\pm 2\varepsilon_i\}$; they are all asymmetric.

- The general case is obtained by a twist as follows. The set $I$ is in bijection with the sets $\{\mathcal{O}, -\mathcal{O}\}$ of $\Gamma_F$-orbits. We have $i \in I_0$ (i.e. $K_i$ is a field) if and only if $\pm \mathcal{O}$ are symmetric. By choosing an homomorphism $K_i \rightarrow F$ of $F$-algebras, we pick out $\alpha \in \mathcal{O}$ and

$$K_i^2 \simeq F_{\pm\alpha}, \quad K_i \simeq \begin{cases} F_\alpha, & i \in I_0 \\ F_{\pm\alpha} \times F_{\pm\alpha}, & i \notin I_0. \end{cases} \quad (3.8)$$

Decompose $c = (c_i)_{i \in I}$ and define a symplectic form $h^i$ (resp. $h_i$) on the $K_i^2$-vector space (resp. $F$-vector space) $K_i$ à la (3.2) as

$$h^i(u|v):= \text{tr}_{K_i/K_i^2} (\tau(u)v\varepsilon_i), \quad h_i(u|v):= \text{tr}_{K_i/F} (\tau(u)v\varepsilon_i) = \text{tr}_{K_i^2/F} (h^i(u|v)). \quad (3.9)$$

Under this correspondence, (3.7) coincides with

$$-1 \in \text{SL}(2, K_i^2) \quad \text{max. torus} \quad R_{K_i^2/F}(\text{Sp}(K_i, h^i)) \longrightarrow \text{Sp}(K_i, h_i) \longrightarrow \text{Sp}(W) = G$$

$$-1 \in K_i^1 \quad T_\mathcal{O} = K_i^1 \longrightarrow K^1 \simeq T \quad (3.10)$$

- The isomorphism $(W, \langle \cdot | \cdot \rangle) \simeq \bigoplus_{i \in I}(K_i, h_i)$ of symplectic $F$-vector spaces gives rise to

$$\prod_{\pm\alpha} G_\mathcal{O} \simeq \prod_{i \in I} R_{K_i^1/F}(\text{Sp}(K_i, h^i)) \subset \prod_{i \in I} \text{Sp}(K_i, h_i) \hookrightarrow \text{Sp}(W) = G,$$

$$\prod_{\pm\alpha} T_\mathcal{O} \simeq \prod_{i \in I} R_{K_i^2/F}(K_i^1) = K^1 \simeq T.$$

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Definition 3.3.3. Given $T$ as above, we set $G^T := \prod_{i \in I} G\mathcal{O}$. It is a canonically defined subgroup of $G$ containing $T$, and can be described as $\prod_{i \in I} R_{K_i^1/F}(\text{SL}(2))$ in terms of the parameter $(K, K^\perp, c)$ for $T \hookrightarrow G$.

By Shapiro’s lemma, $H^1(F, G^T) = \prod_{i \in I} H^1(K_i^1, \text{SL}(2))$ is trivial. By the discussions in §2.3 on Weil restrictions, we see

- $G^T$ is simply connected since $\text{SL}(2)$ is;
- $G^T_{\text{ad}} = \prod_{i \in I} R_{F, K_i/F}(\text{PGL}(2))$ since Weil restriction commutes with the formation of adjoint groups, as noted in §2.3;
- if $\text{Ad}(g) : T \rightarrow T'$ is a stable conjugation inside $G^T$, then we have $G^T = G'^T$. In fact one can take $g \in G^T_{\text{ad}}(F)$ by Proposition 3.2.7.

Proposition 3.3.4. Conserve the notations above and suppose $\delta \in T_{\text{reg}}(F)$. Then every conjugacy class in the stable conjugacy class of $\delta$ contains an element of the form $g\delta g^{-1}$ where $g \in G^T_{\text{ad}}(F)$, and

$$\text{inv}(\delta, g\delta g^{-1}) = \text{image of } g \text{ under } G^T_{\text{ad}}(F) \rightarrow \mathfrak{D}(Z_{G^T}, G^T; F) \rightarrow \mathfrak{D}(T, G; F).$$

Furthermore, $\mathfrak{D}(Z_{G^T}, G^T; F) \rightarrow \mathfrak{D}(T, G; F)$ is bijective and in terms of parameters, $\text{inv}(x, g\delta g^{-1})$ can be identified with

$$\left(\det(g_{i,1})N_{K_i/K_1}(K_i^1)\right)_{i \in I} \in K^2_{x, K} / N_{K_i/K_1}(K_i^1).$$

Here $g_{i,1} \in \text{GL}(2, K_i^1)$ is any representative of $g_i \in \text{PGL}(2, K_i^1)$.

Note that the indexes $i \notin I_0$ have no contribution, and $I = I_0$ when $T$ is anisotropic by Lemma 3.2.6.

Proof. The first equality for $\text{inv}(\delta, g\delta g^{-1})$ reduces essentially to Proposition 3.2.3. The next step is to describe the image of $g$ in terms of $(g_{i,1})_{i \in I}$. The map $\mathfrak{D}(Z_{G^T}, G^T; F) \rightarrow \mathfrak{D}(T, G; F)$ factors into $\mathfrak{D}(Z_{G^T}, G^T; F) \rightarrow \mathfrak{D}(T, G^T; F) \rightarrow \mathfrak{D}(T, G; F)$, whilst $\mathfrak{D}(T, G^T; F) = H^1(F, T) = \mathfrak{D}(T, G; F)$. Proposition 3.2.4 asserts that

$$H^1(F, T) \cong \prod_{i \in I} N_{K_i/K_1}(K_i^1) = K^2_{x, K} / N_{K_i/K_1}(K_i^1).$$

This is compatible with the decomposition $T = \prod_{i \in I} R_{K_i^1/F}(K_i^1)$, so we may work separately for each $i \in I$. Shapiro’s lemma affords the commutative diagram

$$\begin{array}{ccc}
(R_{K_i^1/F}\text{PGL}(2))(F) & \longrightarrow & H^1(F, R_{K_i^1/F}(K_i^1)) \\
\downarrow & \cong & \downarrow \cong \\
PGL(2, K_i^1) & \longrightarrow & H^1(K_i^1, K_i^1) \longrightarrow K_i^1 / N_{K_i/K_1}(K_i^1). 
\end{array}$$

We are now reduced to the rank-one case of Proposition 3.2.7 and the description of $\text{inv}(\delta, g\delta g^{-1})$ follows. Finally, this description implies the surjectivity of $G^T_{\text{ad}}(F) \rightarrow \mathfrak{D}(T, G; F)$, therefore every $\eta \cong \delta$ is conjugate to some $g\delta g^{-1}$ with $g \in G^T_{\text{ad}}(F)$.

Summarizing, we obtain all stable conjugates of elements in $T_{\text{reg}} \subset G$ (up to ordinary conjugacy) by working inside $G^T$. Modulo Weil restrictions, the description of stable conjugacy boils down to the $\text{SL}(2)$ case. There is also an obvious version for $g$.

4 BD-covers of symplectic groups

Except in §4.5, the assumptions in §3 on the field $F$ remain in force. For the study of harmonic analysis, we can and do confine ourselves to the BD-covers of $\text{Sp}(2n)$ arising from Matsumoto’s central extension; see the Remarks 2.2.1 and 2.4.4.
4.1 The covers

Let $(W, \langle \cdot | \cdot \rangle)$ be a symplectic $F$-vector space of dimension $2n$. Fix a maximal $F$-torus $T$ of $G := \text{Sp}(W)$ and set $Y = X_1(T_F)$. Write $G := \text{Sp}(W)$ and denote by $E_G \to G$ the multiplicative $K_2$-torsor constructed by Matsumoto (Remark 2.2.1). It corresponds to the quadratic form $Q : Y \to \mathbb{Z}$ in (3.1). We will also write $E_{G,F}$ when the base field is to be stressed. When dim $W = 2$ with chosen symplectic basis, we adopt the shorthand $E_G = E_{\text{SL}(2)}$.

To $Q$ is associated the symmetric bilinear form on $Y$

$$B_Q(y, y') := Q(y + y') - Q(y) - Q(y') = 2 \sum_{i=1}^{n} y_i y_i'.$$

Suppose $m \mid N_F$ as in §2.5. Since $Y_{Q,m} = \frac{m}{\gcd(2, m)} Y \subset Y$, the isogeny $\iota_{Q,m} : T_{Q,m} \to T$, together with its compatibility under $G$-conjugation, can be identified with the endomorphism of $T \subset G$

$$\iota_{Q,m} : T_{Q,m} \to T \longrightarrow T \longmapsto t^{m/\gcd(2,m)} \quad (4.1)$$

Caveat: one must be careful when identifying $T_{Q,m}$ and $T$, as they will play different roles in our latter applications.

**Lemma 4.1.1.** Suppose that $(W, \langle \cdot | \cdot \rangle)$ is the orthogonal direct sum $\bigoplus_{i=1}^{r} W_i$ of symplectic vector subspaces. Write $G_i := \text{Sp}(W_i) \to G$. There is a natural morphism $\iota_i : E_{G_i} \to E_G|_{G_i}$ for all $1 \leq i \leq r$. They realize the pull-back of $E_G \to G$ to $G_1 \times \cdots \times G_r$, as the contracted product of multiplicative $K_2$-torsors $E_{G_1} \times \cdots \times E_{G_r}$.

In particular, elements lying over different components $G_i$ commute in $E_G$.

**Proof.** Choose symplectic bases of $W_1, \ldots, W_r$; their union is a symplectic basis of $W$. The corresponding split maximal $F$-torus of $G$ is $T = T_1 \times \cdots \times T_r$; in parallel $(Y, Q) = (Y_1, Q_1) \oplus \cdots \oplus (Y_r, Q_r)$ where $Q_i$ is the quadratic form associated to Matsumoto’s $E_{G_i}$, by (3.1).

By Remark 2.2.2, $E_{G_i} |_{G_j \times \cdots \times G_r}$ is also classified by the quadratic form $(Y, Q)$. On the other hand, $\bigoplus_{i=1}^{r} (Y_i, Q_i)$ corresponds to $E_{G_1} \times \cdots \times E_{G_r}$. This gives the required isomorphism. The required $\iota_i$ comes from composing with $E_{G_i} \hookrightarrow E_{G_1} \times \cdots \times E_{G_r}$. \hfill $\square$

Now consider the constructions in §3.3. We have a maximal $F$-torus $T \subset G$. Form the canonical subgroup $T \subset G_T \subset G$ of Definition 3.3.3. By choosing a parameter $(K, K^2, \ldots)$ for $T \to G$ (Proposition 3.1.3), we may identify $G_T$ with $\prod_{i \in I} R_{K_i/F} \text{SL}(2)$. To reconcile with the notations in §2.3, denote by $f_i : \text{Spec}(K_i^1) \to \text{Spec}(F)$ the structure morphisms for each $i \in I$.

**Theorem 4.1.2.** The restriction of $E_G$ to $G_T$ is isomorphic to the contracted product of the multiplicative $K_2$-torsors $f_i^* \left( E_{\text{SL}(2), K_i^1} \right)$.

**Proof.** Using the notation from §3.3, we restrict $E_G$ in stages

$$G_T = \prod_{i \in I} R_{K_i/F} \text{Sp}(K_i, h^i) \hookrightarrow \prod_{i \in I} \text{Sp}(K_i, h_i) \hookrightarrow \text{Sp}(W) = G.$$

With $G_i := \text{Sp}(K_i, h_i)$, Lemma 4.1.1 reduces the problem to the case $|I| = 1$ and $K^2$ is a field. Write $h = h^i$ and $f = f_i$. Since $R_{K^2/F} \text{Sp}(K, h)$ is simply connected and contains $T$, Remark 2.2.2 implies that $E_{G_T}|_{G_T}$ is classified by the $Q : Y \to \mathbb{Z}$ in (3.1).

Now turn to the quadratic form $Q' : Y \to \mathbb{Z}$ associated to $f_* E_{\text{Sp}(K, h)}$. Choose a coroot $\alpha$ of $K^1 \subset \text{Sp}(K, h)$, which is defined over $F_0$ and recall $K^2 = F_{\pm \alpha}$ in the notation of §3.3. Then $Y_{\text{Sp}(K, h)} = \mathbb{Z} \alpha$ and the $Q_{\text{Sp}(K, h)}$ associated to $E_{\text{Sp}(K, h)}$ is simply $y \alpha \mapsto y^2$ by (3.1). If $K$ is a field then $K = F_0$ and $\text{Gal}(F_\alpha/F_{\pm \alpha})$ acts on $Y_{\text{Sp}(K, h)}$ by $\alpha \mapsto \pm \alpha$, thus stabilizes $Q_{\text{Sp}(K, h)}$.

If $K \cong K^1 \times K^2$ then $T$ splits over
We conclude by applying Theorem 4.1.2. □

4.2 Kubota’s cover of GL(2)

We review Kubota’s description [30] of a covering of GL(2, F), cf. [28, 0.1] and [14, §16.2]. It is a multiplicative $K_2$-torsor $E_{Ku} \to GL(2)$ such that

- $E_{Ku}$ restricts to Matsumoto’s $E_{SL(2)} \to SL(2)$, see [28, 0.1];
- by [28, p.41], there is a preferred section $s : GL(2, F) \to E_{Ku}(F)$ with $s(x)s(y) = c(x, y)s(xy)$ in terms of an explicit 2-cocycle $c$.

Using the notations of §2.4, we describe $c$ by

$$c(g_1, g_2) := - \frac{\det g_1 \cdot x(g_2)}{x(g_1 g_2)} \in K_2(F).$$

Remark 4.2.1. We follow [37, Corollaire 5.12] to take the negative of the usual Kubota’s cocycle found in [30, 13, 28], otherwise $E_{Ku}|_{SL(2)}$ would be the negative of Matsumoto’s central extension. For the relation between $s$ and Steinberg’s presentation for $E_{SL(2)}(F)$, we refer to the discussions preceding [37, Corollaire 5.12]. After pushing-out to $\mu_2$, the difference disappears.

Lemma 4.2.2. The adjoint action of $GL(2)$ on $E_{Ku}$ induces the canonical $PGL(2)$-action on $E_{SL(2)}$ given by Proposition 2.2.3.

Proof. The $GL(2)$-action leaves $SL(2)$ and $E_{Ku}|_{SL(2)} = E_{SL(2)}$ invariant. The center of $GL(2)$ acts trivially on $SL(2)$, thereby giving rise to an automorphism of $E_{SL(2)} \to SL(2)$; this action must be trivial as $SL(2)$ is simply connected. We conclude by the uniqueness part of Proposition 2.2.3. □

Lemma 4.2.3. Let $g \in PGL(2, F)$ with preimage $g_1 \in GL(2, F)$. For any preimage $-1 \in E_{Ku}(F)$ of $-1$, we have $Ad(g)(-1) = g_1(-1)g^{-1}_1 = \xi(-1)$ where

$$\xi = \{-1, \det g_1\} \in K_2(F).$$
Proof. By Lemma 4.2.2 we have \( \text{Ad}(g)(-1) = g_1(-1)g_1^{-1} \). As \(-1\) is central in \( \text{GL}(2, F) \), we have \( \xi = c(g_1, -1) - c(-1, g_1) \). Write \( g_1 = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \). First suppose \( c \neq 0 \), then \( c(g_1, -1) = -\{1, (\text{det} g_1)c^{-1}\}_F \) whereas \( c(-1, g_1) = -\{1, -c^{-1}\}_F = -\{1, c^{-1}\}_F \) by the anti-symmetry of Steinberg symbols. Hence \( c(g_1, -1) - c(-1, g_1) = -\{1, \text{det} g_1\}_F \).

If \( c = 0 \), replacing \( c \) by \( d \) in the arguments above gives the same result. \( \square \)

By fixing \( m \mid N_F \) and pushing \( E_{Ku}(F) \) out via \( (\cdot, \cdot)_{F,m} : K_2(F) \to \mu_m \), we obtain a topological central extension

\[
1 \to \mu_m \to \widetilde{\text{GL}}(2, F) \to \text{GL}(2, F) \to 1.
\]

The resulting preferred section and cocycle are still denoted by \( s \) and \( c \), now with \( (\cdot, \cdot)_F \) replaced by \( (\cdot, \cdot)_{F,m} \).

Hence

- \( \text{GL}(2, F) \) restricts to \( \widetilde{\text{SL}}(2, F) \to \text{SL}(2, F) \);
- by Lemma 4.2.2, the adjoint action of \( \text{GL}(2, F) \) on \( \widetilde{\text{SL}}(2, F) \) induces the canonical \( \text{PGL}(2, F) \)-action on \( \widetilde{\text{SL}}(2, F) \) from Proposition 2.2.3;
- the statements in Lemma 4.2.3 hold for \( \widetilde{\text{SL}}(2, F) \), with \( (\cdot, \cdot)_F \) replaced by \( (\cdot, \cdot)_{F,m} \).

Let \( K \) be an étale \( F \)-algebra of dimension 2, therefore comes equipped with an involution \( \tau \neq \text{id} \). The \( F \)-torus \( K^\times \) embeds into \( \text{GL}(2) \) with \( \text{det}|K^\times = N_{K/F} \); it restricts to \( K^1 \times \to \text{SL}(2) \). As the \( n = 1 \) case of Proposition 3.2.4, this parameterize stable conjugacy classes of embeddings of maximal tori in \( \text{SL}(2) \).

**Notation 4.2.4.** When \( K \simeq F \times F \), the elements are expressed as \( x = (x_1, x_2) \) and we have \( \tau(x_1, x_2) = (x_2, x_1) \). The Hilbert symbols for such \( K \) can be conveniently defined as

\[
(x, y)_{F,m} := (x_1, y_1)_{F,m}(x_2, y_2)_{F,m}.
\]

The result below quantifies the non-commutativity of the preimage of \( K^\times \) in \( \widetilde{\text{GL}}(2, F) \).

**Proposition 4.2.5** (Flicker). Suppose that \( \gamma, g \in \text{GL}(2, F) \) arise from \( x, u \in K^\times \). Let \( \tilde{\gamma} \) be any preimage of \( \gamma \) in \( \widetilde{\text{GL}}(2, F) \). The factor \( [g, \gamma] \in \mu_m \) in (2.1) determined by \( g\tilde{\gamma}g^{-1} = [g, \gamma]\tilde{\gamma} \) has the form

\[
[g, \gamma] = (x, \tau(u))^{-1}_{K,m}.
\]

**Proof.** This is done in the calculations in [13, p.128] using the cocycle \(-c\). \( \square \)

In the situation above, we define

\[
t_Q,m : K^1 \to K^1 \quad x_0 \mapsto x_0^{m/gcd(2,m)}
\]

following (4.1). Given \( x = t_Q,m(x_0) \), there exists \( \omega \in K^\times \) with \( \omega/\tau(\omega) = x_0 \) by Hilbert’s theorem 90. Then \( N_{K/F}(\omega) \) is well-defined modulo \( F^{\times 2} \). When \( K \simeq F \times F \) and \( x_0 = (a, a^{-1}) \) with \( a \in F^\times \), we may take \( \omega = (a, 1) \) to see that \( N_{K/F}(\omega) \) represents the class of \( a \) inside \( F^{\times}/F^{\times 2} \).

We will need the projection formula for the next proof

\[
(a, b)_{K,m} = (a, N_{K/F}(b))_{F,m}, \quad a \in F^\times, \ b \in K^\times;
\]

it is standard when \( K \) is a field, and the case \( K \simeq F \times F \) is straightforward.

**Lemma 4.2.6.** Suppose \( x = t_Q,m(x_0) \in K^1 \) and choose \( \omega \) be as above. Then in the setting of Proposition 4.2.5 we have

\[
[g, \gamma] = (\omega, N_{K/F}(\omega))_{K,gcd(2,m)} = (\omega, \text{det} \gamma)_{K,gcd(2,m)}
\]

\[
= (N_{K/F}(\omega), \text{det} \gamma)_{F,gcd(2,m)}.
\]

In particular, in this case \([g, \gamma] = 1 \) whenever \( m \not\in 2\mathbb{Z} \). It also follows that \( (N_{K/F}(\omega), \text{det} \gamma)_{F,gcd(2,m)} \) is independent of the choice of \( \omega \), although this can also be verified directly.
Proof. From (2.6) and the Proposition 4.2.5 we infer
\[ [g, \gamma] = (x_0^{m/\gcd(2, m)}, \tau(u))_{K,m}^{-1} = (x_0, \tau(u))_{K, \gcd(2, m)} \]
\[ = (\omega, \tau(u))_{K, \gcd(2, m)} (\tau(\omega)^{-1}, \tau(u))_{K, \gcd(2, m)} \]
\[ = (\omega, \tau(u))_{K, \gcd(2, m)} (\omega, u)_{K, \gcd(2, m)} \]
\[ = (\omega, N_{K/F}(u))_{K, \gcd(2, m)} = (\omega, \det \gamma)_{K, \gcd(2, m)}. \]

Now apply (4.5) to obtain the remaining equality in the assertion. \( \square \)

We will need further invariance properties for this factor.

**Definition–Proposition 4.2.7.** Take \( x_0 = \omega/\tau(\omega) \in K^1 \) as before, and let \( \nu \in F^\times/F^\times 2 \). Let \( \gamma_0 \in \text{SL}(2, F) \) be associated to \( x_0 \) via \( K^1 \hookrightarrow \text{SL}(2) \) and put
\[ C_m(\nu, \gamma_0) := (N_{K/F}(\omega), \nu)_{F, \gcd(2, m)}. \]
Then \( C_m(\nu, \gamma_0) \) depends only on the stable conjugacy class of the element \( \gamma_0 \in \text{SL}(2, F) \) associated to \( x \). Furthermore, \( C_m(\nu, \gamma_0) \) is invariant under any automorphism of the \( F \)-group \( \text{SL}(2) \).

Proof. Given a stable conjugacy class \( \gamma_0 \), the datum \((K, x_0)\) is determined up to isomorphisms of étale \( F \)-algebras, thus \( N_{K/F}(\omega) \) is determined up to \( F^\times 2 \). Hence \( C_m(\nu, \gamma_0) \) is invariant under \( \text{PGL}(2, F) \), and \( \text{PGL}(2) \) equals the \( F \)-scheme of automorphisms of \( \text{SL}(2) \) by [17, Exp XXIV, Théorème 1.3]. \( \square \)

Observe that \( C_m(\nu, \gamma_0) \) is bi-multiplicative in \( \nu \) and \( \gamma_0 \). In applications, \( \nu \) will arise from the \( \nu(g) \) in (3.5).

We record the classification by K. Hiraga and T. Ikeda of good regular semisimple elements in \( \text{SL}(2, F) \). Write \( T \subset \text{SL}(2) \) for the maximal \( F \)-torus parameterized by a two-dimensional étale \( F \)-algebra \( K \).

**Theorem 4.2.8 (Hiraga–Ikeda).** The projection of \( Z_\varphi \) to \( T(F) \) equals \( \{ \pm 1 \} \cdot T(F)^{m/\gcd(2, m)} \). In particular, \( \gamma \in T^\text{reg}(F) \) is good if and only if \( \gamma \in \{ \pm 1 \} \cdot T(F)^{m/\gcd(2, m)} \).

Proof. Their original proof is reproduced below. We begin by showing that the preimage of \( \{ \pm 1 \} \cdot T(F)^{m/\gcd(2, m)} \) is central. In view of Proposition 2.5.2 and (4.1), it suffices to show \( \{ \pm 1, \eta \} = \{ \pm 1 \} \cdot T(F)^{m/\gcd(2, m)} \) for all \( \eta \in \text{SL}(2, F) \): this is already true on the \( K^1(F) \)-level by Proposition 2.2.3.

Now suppose \( \gamma \in T(F) \) is projected from \( Z_\varphi \). Set \( m_0 := m/\gcd(2, m) \). When \( T \) is split, by Proposition 2.5.2 we have \( \gamma \in T(F)^{m_0} \). Thus we can assume \( T \) is associated with a quadratic extension of fields \( K \) of \( F \), and \( \gamma \) corresponds to \( x \in K^1 \). Proposition 4.2.5 implies that for all \( v \in K^\times \),
\[ 1 = (x, \tau(v)/v)_{K,m} \]
\[ = (x, \tau(v))_{K,m} (x, v^{-1})_{K,m} = (\tau(x), x)_{K,m} (x, v^{-1})_{K,m} \]
\[ = (x^{-1}, x)_{K,m} (x, v^{-1})_{K,m} = (x^2, v^{-1})_{K,m}. \]
This shows \( \pm x \in K^1 \cap K^{x m_0} \) and it remains to show \( K^1 \cap K^{x m_0} = \pm (K^{1})^{m_0} \).

We have \(-1 \in K^{x m_0} \), because \((-1)^{m_0} = -1 \) when \( 4 \nmid m \) whilst \( \zeta^{m_0} = -1 \) when \( 4 \mid m \) and \( \mu_m = 1 \). Therefore \( K^1 \cap K^{x m_0} \geq \pm (K^{1})^{m_0} \). Next, suppose \( y_0 \in K^\times \) satisfies \( N_{K/F}(y_0^{m_0}) = 1 \), then there exists \( \xi \in \mu_m, \xi^{m_0} = \pm 1 \) such that \( N_{K/F}(y_0) = \xi^2 \). This leads to \( y_1 := \xi^{-1} y_0 \in K^1 \) satisfying \( (y_1)^{m_0} = \xi^{-m_0} y_0^{m_0} = \mp y_0^{m_0} \); we conclude that \( K^1 \cap K^{x m_0} \subset \pm (K^{1})^{m_0} \).

The last assertion results from Corollary 2.5.3. \( \square \)

Below is a supplement to the classification above.

**Proposition 4.2.9.** Let \( T \simeq K^1 \) as above, then
\[ T(F)^{m/\gcd(2, m)} = (-1) \cdot T(F)^{m/\gcd(2, m)}, \quad 4 \nmid m \] or \( T \) : split,
\[ T(F)^{m/\gcd(2, m)} \cap (-1) \cdot T(F)^{m/\gcd(2, m)} = \emptyset, \quad 4 \mid m \] and \( T \) : anisotropic.
On the other hand,
\[
\ker(t_{Q,m}) = \begin{cases} 
\mu_{m/\gcd(2,m)}, & T : \text{split} \\
1, & 4 \nmid m, \quad T : \text{anisotropic} \\
\{\pm 1\}, & 4 \mid m, \quad T : \text{anisotropic};
\end{cases}
\]
(4.6)
in the split case, \(K \cong F \times F\) and we embed \(\mu_m\) into \(K^1\) via \(z \mapsto (z, z^{-1})\).

**Proof.** For the first part, note that \((\pm 1) \cdot T(F)^{m/\gcd(2,m)}\) are either identical or disjoint. They are identical if and only if \(-1 = \xi^{m/\gcd(2,m)}\) for some \(\xi \in K^1\). When \(4 \nmid m\), taking \(\xi = -1\) suffices. When \(T\) splits (so \(K \cong F \times F\) and \(4 \mid m\)), we may take \(\xi = (\zeta, \zeta^{-1})\) where \(\mu_m = (\zeta)\), since \(\xi^{m/2} = -1\).

When \(4 \mid m\) and \(T\) is anisotropic, we know \(K\) is a field; by \(\xi^m = 1\) and \(m \mid N_F\) we obtain \(\xi \in F^\times\). Moreover, \(N_{K/F}(\xi) = 1\) implies \(\xi = \pm 1\); but \((-1)^{m/2} = 1\).

For the second part, the case \(K = F \times F\) is straightforward. When \(K\) is a field, \(t_{Q,m}(x_0) = 1 \implies x_0 \in F^\times \cap K^1\) since \(m \mid N_F\), hence \(x_0 = \pm 1\) as before; it remains to observe that \((-1)^{m/\gcd(2,m)} = 1\) if and only if \(4 \mid m\). \(\square\)

### 4.3 Stable conjugacy for BD-covers

Revert to the notation of \(\S 3.1\) and \(\S 4.1\). Fix a maximal \(F\)-torus \(T\). The subgroup \(T \subset G^T \subset G\) (Definition 3.3.3) together with the decomposition \(G^T = \prod_{i \in I} G_O\) are canonically defined by \(T\). Once a parameter \((K, K^2, \ldots)\) for \(T\) is chosen, \(G^T\) together with its decomposition can be identified with \(\prod_{i \in I} R_{K^1_i/F}(\SL(2))\), and \(T\) is identified with \(\prod_{i \in I} R_{K^1_i/F}(K^1_i)\).

Given \(\delta \in T_{\reg}(F)\) and \(\eta \in G_{\reg}(F)\), recall that \(T(\delta, \eta) := \{g : g\delta g^{-1} = \eta\}\) is nonempty if and only if \(\delta \not\sim \eta\), in which case it is a right \(T\)-torsor. As a first step, we assume \(\eta \in G^T(F)\) so that Proposition 3.3.4 entails \(T(\delta, \eta) \subset G^T\). In parallel with the decomposition of \(G^T\), we have \(T(\delta, \eta) = \prod_{i \in I} T_i(\delta_i, \eta_i)\), where each \(T_i(\delta, \eta)\) is defined inside \(R_{K_i^1/F}(\SL(2))\).

Now suppose \(\eta \in G_{\reg}(F)\) is stably conjugate to \(\delta \in T_{\reg}(F)\). Denote \(S := G_\eta\). By Proposition 3.3.4, the canonical isomorphism of pointed tori \(\Ad(g) : (T, \delta) \simto (S, \eta)\) induced by any \(g \in T(\delta, \eta)(F)\) can be decomposed as

\[
(T, \delta) \xrightarrow{\Ad(g)} (T', \delta') \xrightarrow{\Ad(g')} (S, \eta)
\]

for some \(\delta' \in G_{\reg}(F)\), \(g' \in T^{G^T}(\delta, \delta')(F)\) and \(g'' \in T(\delta', \eta)(F)\). In particular \(\Ad(g')(\delta') = \eta\) is ordinary conjugacy and

\[
\inv(\delta, \eta) = \inv(\delta, \delta').
\]

This equality also determines \(\delta'\) up to \(G^T(F)\)-conjugacy. The goal of this subsection is to adapt these to good elements in \(G_{\reg}(F)\) for the BD-cover \(\hat{G} \to G(F)\).

Fix \(m \mid N_F\). Denote by \(\hat{G}^T\) the pull-back of \(\mu_m \hookrightarrow \hat{G} \to G(F)\) to \(G^T(F)\). Theorem 4.1.3 says that as topological central extensions \(\hat{G}^T\) is isomorphic to the contracted product of \(\mu_m \hookrightarrow \hat{\text{SL}}(2, K^2) \to \text{SL}(2, K^2)\). Here it is convenient to identify \(t_{Q,m}\) with the endomorphism \(t_0 \mapsto t_{0}^{m/\gcd(2,m)}\) of \(T\) by (4.1); this decomposes into \(t_{Q,i,m} : K_i^1 \to K_i^1\) for each \(i \in I\).

**Proposition 4.3.1.** The projection of \(Z_\hat{F}\) to \(T(F)\) equals

\[
\prod_{i \in I} \{\pm 1\} \cdot \im(t_{Q,i,m}).
\]

In particular, its intersection with \(T_{\reg}(F)\) equals the set of good elements in \(T_{\reg}(F)\).

**Proof.** By the decomposition of \(\hat{G}^T\), we reduce immediately to the case \(\hat{G} = \hat{G}^T = \hat{\text{SL}}(2, K^2)\) treated in Theorem 4.2.8, upon passing to a finite extension of \(F\). \(\square\)

**Remark 4.3.2.** Every element \(\eta = (\eta_i)_{i \in I} \in \prod_{i \in I} \{\pm 1\}\) is good with respect to \(\hat{G} \to G(F)\). Indeed, Lemma 4.1.1 reduces the problem to the case \(\eta \in \{\pm 1\}\), and one concludes by Proposition 2.2.3.

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Notation 4.3.3. Recall that $I$ can be identified with $\{\text{long roots}\}/(T_F, \pm)$, thus canonical for the given $T$. The embedding $\{\pm\}_I^T \hookrightarrow T(F)$ is also definable in terms of long roots by (3.7), hence canonical and can be transported under stable conjugacy. Let $\{\pm\}_I^T$ act on $T$ by coordinate-wise multiplication. Consider $\sigma = (\sigma_i)_{i \in I} \in \{\pm\}_I^T$. Pull-back of $p : T \twoheadrightarrow T(F)$ along $\sigma \cdot \iota_{Q,m}$ yields

$$\begin{align*}
\tilde{T}_{Q,m}^\sigma &:= \{(\tilde{t}, t_0) \in \tilde{T} \times T_{Q,m}(F) : p(\tilde{t}) = \sigma \cdot \iota_{Q,m}(t_0)\}, \\
\tilde{T}_{Q,m} &:= \tilde{T}_{Q,m}^{(\pm, \ldots, \pm)}.
\end{align*}$$

There are natural maps $\tilde{T} \xrightarrow{\text{pr}_1} \tilde{T}_{Q,m}^\sigma \xrightarrow{\text{pr}_2} T_{Q,m}(F)$. Proposition 4.3.1 implies that every good regular element in $\tilde{G}_{\text{reg}}$ lies in some $\text{pr}_1(\tilde{T}_{Q,m}^\sigma)$; when $4 \nmid m$, it suffices to use $\sigma = (\pm, \ldots, \pm)$.

Note that $\tilde{T}_{Q,m}$ is a group, whilst $\tilde{G}_{Q,m}$ is only a $\tilde{T}_{Q,m}$-torsor for general $\sigma$. Consider a stable conjugation $\text{Ad}(g) : T \twoheadrightarrow S$ between maximal $F$-tori in $G$, then

- as explicated above, $\{\pm\}_I^T \hookrightarrow T(F)$ can be transported to $S$ by $\text{Ad}(g)$, thus $\tilde{S}_{Q,m}^\sigma$ makes sense;
- under the identification, $\sigma \cdot \text{Ad}(g)(t) = \text{Ad}(g)(\sigma \cdot t)$ for all $t \in T(F)$ and $\sigma \in \{\pm\}_I^T$;
- if moreover $g \in G(F)$, from (2.7) we have

$$\text{Ad}(g) : \tilde{T}_{Q,m}^\sigma \xrightarrow{\sim} \tilde{S}_{Q,m}^\sigma$$

$$(\tilde{t}, t_0) \mapsto (\text{Ad}(g)(\tilde{t}), \text{Ad}(g)(t_0))$$

which is a group isomorphism for $\sigma = (\pm, \ldots, \pm)$, and is an equivariant map between torsors for general $\sigma$. Proposition 4.3.1 implies that $\text{Ad}(g) = \text{id}$ when $g \in T(F)$.

Now comes the stable conjugacy in the $G \simeq \text{SL}(2)$ case; the notation above reduces to $\tilde{T}_{Q,m}^\pm$. We will employ systematically the $G_{\text{ad}}(F)$-action on $\tilde{G}$ from Proposition 2.2.3, again denoted by $\text{Ad}$.

Definition–Proposition 4.3.4. Suppose that $\dim_F W = \dim_F K = 2$, so that $K^G = F$ and $G^T = G$. Let $\text{Ad}(g) : T \twoheadrightarrow S$ be a stable conjugation of maximal $F$-tori; here we may assume $g \in G_{\text{ad}}(F)$ by Proposition 3.2.7. Set $\nu := \nu(g) \in F^\times/F^\times 2$ by (3.5).

1. Define an isomorphism $\text{CAd}(g) = \text{CAd}^+(g) : \tilde{T}_{Q,m}^+ \xrightarrow{\sim} \tilde{S}_{Q,m}^+$ that fits into the commutative diagram

2. When $4 \mid m$, define $\text{CAd}^-(g) : \tilde{T}_{Q,m}^- \xrightarrow{\sim} \tilde{S}_{Q,m}^-$ that fits into the commutative diagram

for any $\tilde{-1} \mapsto -1$ and $\tilde{p} \mapsto \iota_{Q,m}(t_0)$.

These constructions are independent of all choices. In particular it is independent of the identification $G \simeq \text{SL}(2)$.  

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**Proof.** We may choose a preimage \(g_1 \in \text{GL}(2, F)\) of \(g \in \text{PGL}(2, F)\) by identifying \(G\) and \(\text{SL}(2)\), and then apply the constructions in §4.2. By Definition–Proposition 4.2.7, the factor \(\mathbf{C}_m(\nu, t_0)\) depends only on \(g \mod \text{im}[G(F) \to G_{\text{ad}}(F)]\) and on the stable conjugacy class of \(t_0\); it is invariant under any re-parameterization or automorphisms of \(\text{SL}(2)\).

The following properties will also hold for stable conjugacy in general. We begin with the \(\text{SL}(2)\) case above.

**Proposition 4.3.5.** For any sign \(\sigma\) that is allowed in our situation, the maps \(\text{CAD}^{\sigma}(g)\) satisfy the following properties.

1. **AD.1.** \(\text{CAD}^{\sigma}(g)(\varepsilon t) = \varepsilon \text{CAD}^{\sigma}(g)(t)\) whenever \(\varepsilon \in \mu_m\).

2. **AD.2.** \(\text{CAD}(g) : \tilde{T}_{Q,m} \simeq \tilde{S}_{Q,m}\) is an isomorphism of topological groups. In general, \(\text{CAD}^{\sigma}(g)\) is a \(\text{CAD}(g)\)-equivariant map between torsors with respect to \(\tilde{T}_{Q,m} \xrightarrow{\text{CAD}(g)} \tilde{S}_{Q,m}\).

3. **AD.3.** If \(g \in G(F)\), then \(\text{CAD}^{\sigma}(g)\) reduces to ordinary conjugation. If \(g \in T(\tilde{F})\), it equals id.

4. **AD.4.** Given stable conjugations of maximal \(F\)-tori \(T \xrightarrow{\text{Ad}(h)} S \xrightarrow{\text{Ad}(g)} R\), we have
\[
\text{CAD}^{\sigma}(g) \circ \text{CAD}^{\sigma}(h) = \text{CAD}^{\sigma}(gh) : \tilde{T}_{Q,m}^\sigma \to R_{Q,m}^\sigma.
\]

**Proof.** The setting under consideration is \(G = \text{SL}(2), g, h \in G_{\text{ad}}(F)\).

**AD.1.** It follows from the analogous property of the \(G_{\text{ad}}(F)\)-action on \(G\).

**AD.2.** Since \(\mathbf{C}_m(\nu, t_0) = \mathbf{C}_m(\nu, t_0)\mathbf{C}_m(\nu, t_0')\) and \(\mathbf{C}_m(\nu, \cdot)\) is locally constant, \(\text{CAD}(g)\) is a continuous homomorphism. It will follow from **AD.4** that \(\text{CAD}(g)\text{CAD}(g^{-1}) = \text{CAD}(1) = \text{id}\).

For \((\tilde{t}, t_0) \in \tilde{T}_{Q,m}^\sigma\), **AD.1** implies that \(\text{CAD}^{\sigma}(g)(\tilde{t}, t_0)\) is independent of how we decompose \(\tilde{t} = \tilde{\nu} \cdot \tilde{t}'\).

Given \((\tilde{s}, s_0) \in \tilde{T}_{Q,m}^\sigma\), since \(-1\) is central by Proposition 2.2.3, \((\tilde{s}, s_0)(\tilde{t}, t_0)\) is mapped to
\[
\text{CAD}^{\sigma}(g)\left(\left(\tilde{s}\tilde{\nu}^{-1}\tilde{t}', s_0 t_0\right)\right) = \text{CAD}^{\sigma}(g)\left(\left(-1\right)s\tilde{\nu}^{-1}, s_0 t_0\right)
\]
\[
= \left(\mathbf{C}_m(\nu, t_0)\mathbf{C}_m(\nu, s_0)\left(-1\right)\text{Ad}(g)(\tilde{s})\text{Ad}(g)(\tilde{t})\right)\left(\text{Ad}(g)(s_0)\cdot\text{Ad}(g)(t_0)\right)
\]
\[
= \left(\mathbf{C}_m(\nu, s_0)\text{Ad}(g)(\tilde{s})\right.\left.\text{Ad}(g)(s_0)\right)\left(\mathbf{C}_m(\nu, t_0)\left(-1\right)\text{Ad}(g)(\tilde{t})\right)\left(\text{Ad}(g)(t_0)\right)
\]
\[
= \text{CAD}(g)(\tilde{s}, s_0)\text{CAD}^{\sigma}(g)(\tilde{t}, t_0);
\]

the equivariance in the case \(\sigma = -\) follows.

**AD.3.** If \(g\) comes from \(G(F)\), then \(\nu \in F^\times\) so \(\mathbf{C}_m(\nu, \cdot) = 1\); also note that \(\text{Ad}(g)(-1) = -1\) by Proposition 2.2.3. This shows the first assertion.

As for the second assertion, the premise implies that every \(g \in G(\tilde{F})\) realizing the given \(T \simeq S\) belongs to \(T(\tilde{F})\); in particular, we may assume that \(g \in (T/\Gamma_F)(F)\). The case \(\sigma = +\) follows from Lemma 4.2.6 which says that \(\text{Ad}(g)(\tilde{t}) = \mathbf{C}_m(\nu, t_0)^{-\tilde{t}}\).

For the same reason, in the case \(\sigma = -\) we have \(\text{Ad}(g)(\tilde{t}) = \mathbf{C}_m(\nu, t_0)^{-\tilde{t}}\) and the result follows.

**AD.4.** This equality follows from the fact that \(\mathbf{C}_m(\nu, \gamma_0)\) is multiplicative in \(\nu\) and depends on \(\gamma_0\) only through its stable class.

Proceed to define stable conjugacy in higher rank inside \(\tilde{G}^\sigma\). The allowable signs in the constructions below will be taken from
\[
\text{Sgn}_m(T) := \left\{ \begin{array}{ll}
1, & 4 \mid m \\
\pm 1, & 4 \mid m.
\end{array} \right.
\]

(4.8)

Write \(T = \prod_{i \in I} T_i\) with \(T_i = R_{K_i/F}(K_i^+)\). Multiplication induces the epimorphism
\[
\prod_{i \in I}^{\sigma}(\tilde{T}_i)_{Q,m} \rightarrow \tilde{T}_{Q,m}, \quad \text{kernel} = \left\{ (\varepsilon_i) : \varepsilon_i \in \mu_m^\sigma : \prod_{i} \varepsilon_i = 1 \right\}.
\]

(4.9)
Remark 4.3.6. By Proposition 2.2.3, \( G^T_{\text{ad}}(F) \) acts on \( \tilde{G}^T \); when restricted to the component corresponding to \( i \in I \), this is the same as the \( \text{PGL}(2, K_1^\delta) \)-action on \( \text{SL}(2, K_1^\delta) \) (working over \( K_1^\delta \)). One way to see this is to invoke Proposition 4.2.8.

Definition–Proposition 4.3.7. Let \( S \subset G^T \) be a maximal \( F \)-torus stably conjugate to \( T \) via \( \text{Ad}(g) : T \xrightarrow{\sim} S \), where \( g = (g_i)_{i \in I} \in (G^T)_{\text{ad}}(F) \). For \( \sigma \in \text{Sgn}_m(T) \), define

\[
\begin{align*}
T(F) & \xrightarrow{\text{Ad}(g)} S(F) \\
\uparrow & \\
\tilde{T}^\sigma_{Q,m} & \xrightarrow{\text{CAd}^\sigma(g)} \tilde{S}^\sigma_{Q,m} \\
\downarrow & \\
T_{Q,m}(F) & \xrightarrow{\text{Ad}(g)} S_{Q,m}(F)
\end{align*}
\]

\[\text{CAd}^\sigma(g)(\tilde{t}, t_0) = \prod_{i \in I} \text{CAd}^{\sigma_i}(g_i)(\tilde{t}_i, t_{0,i})\]

for \( (\tilde{t}_i, (t_{0,i})) \mapsto (\tilde{t}, t_0) \) under \( \text{Ad}(g) \). We shall abbreviate \( \text{CAd}(g) := \text{CAd}^{(+\ldots+)}(g) \).

These constructions are independent of all choices and satisfy the properties in Proposition 4.3.5, with \( \sigma \in \text{Sgn}_m(T) \).

Proof. This is just a multi-component version of Definition–Proposition 4.3.4 modulo Weil restrictions. It does not depend on the identification \( G_\sigma \cong R_{K_1^\sigma/F}(\text{SL}(2)) \). Indeed, the field \( K_1^\delta \cong F_{\pm \alpha} \) is uniquely determined by \( (G, T) \), and all \( F \)-automorphisms of \( R_{K_1^\sigma/F}(\text{SL}(2)) \) arise from \( K_1^\delta \)-automorphisms of \( \text{SL}(2) \) by [9, Proposition A.5.14]; this does not alter \( \text{CAd}^\sigma \) by Definition–Proposition 4.3.4. \( \square \)

We are ready to state the general recipe.

Definition 4.3.8. Let \( \text{Ad}(g) : T \xrightarrow{\sim} S \) be a stable conjugacy of maximal \( F \)-tori in \( G \). Decompose \( \text{Ad}(g) \) into

\[
T \xrightarrow{\text{Ad}(g')} T' \xrightarrow{\text{Ad}(g'')} S \quad g' \in G^T_{\text{ad}}(F), \; g'' \in G(F).
\]

In particular \( T' = \text{Ad}(g')T \subset G^T \). Call such a \( (g', g'') \) a factorization pair for \( \text{Ad}(g) \). Given \( \sigma \in \text{Sgn}_m(T) \), define the map

\[\text{CAd}^\sigma(g) := \text{Ad}(g'') \circ \text{CAd}^\sigma(g') : \tilde{T}^\sigma_{Q,m} \longrightarrow \tilde{S}^\sigma_{Q,m}.
\]

Recall that for all \( \delta \in T_{\text{reg}}(F) \), we have \( \text{inv}(\delta, g\delta g^{-1}) = \text{inv}(\delta, g'\delta g'^{-1}) \) in \( H^1(F, T) \).

Theorem 4.3.9. The map \( \text{CAd}^\sigma(g) \) in Definition 4.3.8 is independent of all choices and satisfy the properties in Proposition 4.3.5, with \( \sigma \in \text{Sgn}_m(T) \). In particular, it depends on the \( \text{Ad}(g) : T \xrightarrow{\sim} S \) but not on the choice of \( g \).

Proof. First we show that \( \text{CAd}^\sigma(g) \) is independent of the factorization pair. Choose \( \delta \in T_{\text{reg}}(F) \). Let \( \delta' := g'\delta g'^{-1} \in G^T(F), \; T' := g'Tg'^{-1} \subset G^T = G^T, \) and let \( (h', h'') \) be another factorization pair. Then \( \text{inv}(\delta, \delta') = \text{inv}(\delta, h'\delta h'^{-1}) \). Setting \( k := h'g'^{-1} \in G^T_{\text{ad}}(F) \), the formalism of (3.4) yields

\[
\text{inv}(\delta', k\delta'k^{-1}) = \text{inv}(\delta', h'\delta h'^{-1}) = \text{inv}(\delta', \delta) + \text{inv}(\delta, h'\delta h'^{-1})
\]

\[
= -\text{inv}(\delta, \delta') + \text{inv}(\delta, h'\delta h'^{-1}) = 0.
\]

Hence \( T^G(\delta', k\delta'k^{-1}) \) has an \( F \)-point \( r \in G^T(F) \). Since \( r \) also yields an \( F \)-point of the quotient \( T^G_{\text{ad}}(\delta', k\delta'k^{-1}) \) by \( Z_{G^T} \) which contains \( k \), the torsor structure entails \( k \in r \cdot (T' / Z_{G^T})(F) \). Property AD.3—4 for \( T^G \) entail \( \text{CAd}^\sigma(k) = \text{Ad}(r) \), and

\[\text{CAd}^\sigma(h') = \text{CAd}^\sigma(k)\text{CAd}^\sigma(g') = \text{Ad}(r)\text{CAd}^\sigma(g').\]

Also, as isomorphisms \( T' \xrightarrow{\sim} S \) we have

\[\text{Ad}(h'')\text{Ad}(r) = \text{Ad}(h'')\text{Ad}(k) = \text{Ad}(h'g'^{-1}) = \text{Ad}(g''), \quad h', r, g'' \in G(F)\]
that is, \( h''r = sg'' \) for some \( s \in S(F) \). Since \( \tilde{S}_{Q,m}^* \to \tilde{S} \) has central image, on \( \tilde{S}_{Q,m}^* \) acts \( \text{Ad}(s) \) as id hence we arrive at

\[
\text{Ad}(h'') \text{CA} \text{d}^\sigma(h') = \text{Ad}(h'') \text{Ad}(r) \text{CA} \text{d}^\sigma(g') = \text{Ad}(h''r) \text{CA} \text{d}^\sigma(g') = \text{Ad}(s) \text{Ad}(g'') \text{CA} \text{d}^\sigma(g') = \text{Ad}(g'') \text{CA} \text{d}^\sigma(g').
\]

The independence of \( \text{CA} \text{d}^\sigma(g) \) on parameters, identifications with \( \text{SL}(2) \), etc. result immediately.

The properties \( \text{AD.1} \text{—} 2 \) for \( G \) in Proposition 4.3.5 are inherited from \( G^T \). For \( \text{AD.3} \), if \( g \) comes from \( G(F) \) (resp. from \( T(F) \)), the factorization pair for \( \text{Ad}(g) \) may be taken as \((1, g)\) (resp. \((g, 1)\)) by adjusting \( g \) as in the proof of Proposition 4.3.5: the case of \((1, g)\) is ordinary conjugation, whereas the case of \((g, 1)\) is handled by \( \text{AD.3} \) for \( G^T \) (Definition—Proposition 4.3.7).

To verify \( \text{AD.4} \), suppose that to \( \text{Ad}(g) \) (resp. \( \text{Ad}(h) \)) is associated a factorization pair \((g', g'')\) (resp. \((h', h'')\)), and accordingly

\[
\begin{array}{c}
\xymatrix{ T \ar[r]^{\text{Ad}(g')} & T' \ar[r]^{\text{Ad}(g'')} & S \ar[r]^{\text{Ad}(h')} & S' \ar[r]^{\text{Ad}(h'')} & R. }
\end{array}
\]

Observe that \( G^T = G^T', G^S = G^{S'} \). Set \( k' = g''^{-1}h'g'' \) and \( p := g''^{-1}S'g'' \), then transport \( \sigma \) from \( \text{Sgn}_m(S) \) to \( \text{Sgn}_m(P) \) via \( \text{Ad}(g'')^{-1} \). Since \( h' \in G^S(F) \) (resp. \( S' \subset G^S \)), we see \( k' \in G^T_{ad}(F) = G^T_{ad}(F) \) (resp. \( P \subset G^T = G^T \)). The composite above equals \( \text{Ad}(h''g'')\text{Ad}(k'g') : T \to P \to R \), therefore \((k', h''g'')\) is a factorization pair for \( \text{Ad}(hg) \). Now

\[
\text{Ad}(h'') \text{CA} \text{d}^\sigma(h') \text{Ad}(g'') \text{CA} \text{d}^\sigma(g') = \text{Ad}(h'') \text{Ad}(g'')^{-1} \text{CA} \text{d}^\sigma(h') \text{Ad}(g'') \text{CA} \text{d}^\sigma(g').
\]

We contend that

\[
\text{Ad}(g'')^{-1} \text{CA} \text{d}^\sigma(h') \text{Ad}(g'') = \text{CA} \text{d}^\sigma(k'); \tag{4.10}
\]

if so, we will obtain \( \text{Ad}(h''g'') \text{CA} \text{d}^\sigma(k'g') \) by \( \text{AD.4} \) inside \( G^T = G^T' \), which equals \( \text{Ad}(hg) \) via the factorization pair \((k'g', h''g'')\). In view of the invariance of \( \text{CA} \text{d}^\sigma \) afforded by Definition—Proposition 4.3.7, the (4.10) is a straightforward transport of structure.

**Definition 4.3.10.** Let \( \delta, \eta \in G_{\text{reg}}(F) \) and set \( T := G_\delta, S := G_\eta \). For \( \sigma \in \text{Sgn}_m(T) \), we say \( (\tilde{\delta}, \tilde{\eta}) \in \tilde{T}_{Q,m}^* \) and \((\tilde{\eta}, \tilde{\eta}) \in \tilde{S}_{Q,m}^* \) are stably conjugate if

- there exists \( g \in G(F) \) such that \( g\delta g^{-1} = \eta \);
- \( \text{CA} \text{d}^\sigma(g)(\tilde{\delta}, \tilde{\eta}) = (\tilde{\eta}, \tilde{\eta}) \).

The reference to \( \delta_0, \eta_0 \) can be dropped when \( 4 \mid m \); see Corollary 4.4.4.

### 4.4 Further properties and stability

Let \( T \subset G \) be any maximal \( F \)-torus, parameterized by the datum \((K, K^\perp, \ldots) \) with \( K = \prod_{i \in K} K_i \), etc.

**Proposition 4.4.1.** Let \( \text{Ad}(g) : T \to S \) be a stable conjugacy of maximal \( F \)-tori with factorization pair \((g', g'')\) (Definition 4.3.8). Suppose either \((a) \) \( F \) is archimedean, or \((b) \) \( m \not\equiv 2 \mathbb{Z} \). Then \( \text{CA} \text{d}^T(g)(\tilde{\delta}, \tilde{\delta}_0) = (\text{Ad}(g'')\text{Ad}(g')\tilde{\delta}, \text{Ad}(g)\tilde{\delta}_0) \) for all \((\tilde{\delta}, \tilde{\delta}_0) \in \tilde{T}_{Q,m}^* \).

**Proof.** It suffices to treat the \( \text{SL}(2) \) case. This boils down to show that the \( C_m(\nu, \delta_0) \) from Definition—Proposition 4.2.6 is trivial. When \( m \not\equiv 2 \mathbb{Z} \) this is evident. When \( F = \mathbb{R} \) and \( T \) splits, we may take \( g' = 1 \) in factorization pair since \( H^1(F, T) = 0 \), accordingly \( \nu = 1 \). When \( F = \mathbb{R} \) and \( T \) anisotropic, this follows from \( N_{C/F}(\mathbb{C}^*) = \mathbb{R}_{>0} \).

Below are some useful results for the basic building block: the \( \text{SL}(2) \) case.
Proposition 4.4.2. Assume $G = \text{SL}(2)$. Choose any preimage $\tilde{1} \in p^{-1}(-1)$. Let $g \in G_{ad}(F)$ with $\nu = \nu(g) \in F^\times/F^\times_2$ via (3.5). Suppose $(\delta, \delta_0) \in T_{Q,m}$.

1. When $m \not\equiv 2\mathbb{Z}$, we have $\text{CAD}(g) \left(\tilde{1} \cdot \delta, -\delta_0\right) = \left(-\tilde{1}, -1\right) \cdot \text{CAD}(g) \left(\delta, \delta_0\right)$.

2. When $m \equiv 2 \pmod{4}$,

$$\text{CAD}(g) \left(\tilde{1} \cdot \delta, -\delta_0\right) = \text{sgn}_{K/F}(\nu) \cdot \left(-\tilde{1}, -1\right) \cdot \text{CAD}(g)(\delta, \delta_0)$$

with $\text{sgn}_{K/F}(\nu) \in \mu_2 \subset \mu_m = \ker(p)$. Note that $\text{sgn}_{K/F}(\nu) = (\kappa_-, \text{inv}(\delta, \text{Ad}(g)\delta))$ (Definition 3.2.5).

Proof. In the case $m \not\equiv 2\mathbb{Z}$, the factor $C_m(\cdot, \cdot) = 1$ by Proposition 4.4.1. It remains to show that $\tilde{1}$ is central in $\text{GL}(2, F)$. By Lemma 4.2.3, this amounts to $(-1, x)_{F,m} = 1$ for all $x \in F^\times$. Indeed, $(-1)^2 = 1$ implies that $(-1, x)_{F,m} \in \mu_2 \cap \mu_m$, hence trivial.

In the case $m \equiv 2 \pmod{4}$, suppose $\delta_0$ is parameterized by $x_0 = \omega/\tau(\omega) \in K^1$ for some $\omega \in K^\times$. Then $-x = (-x_0)^{m/2}$ and

$$-x_0 = \frac{c_\omega}{\gamma(c_\omega)} , \quad c := \frac{\sqrt{D}}{\tau(\omega)}, \quad K = F(\sqrt{D}) : \text{field} \begin{cases} (1,-1), & K = F \times F. \end{cases}$$

By Lemma 4.2.3,

$$\text{CAD}(g) \left(\tilde{1} \cdot \delta, -\delta_0\right) = (N_{K/F}(c_\omega), \nu)_{F,2} \cdot \left(\text{Ad}(g) \left(\tilde{1} \cdot \delta\right), -\text{Ad}(g)(\delta_0)\right)$$

$$= (N_{K/F}(c), \nu)_{F,2} \cdot (\text{Ad}(g)(\tilde{1}), -\text{Ad}(g)(\delta_0))$$

$$= (N_{K/F}(c), \nu)_{F,2} \cdot \left(-\tilde{1}, -1\right) \cdot \text{CAD}(g)(\delta, \delta_0). \tag{4.11}$$

Notice that $(-1, \nu)_{F,m} = ((-1)^{m/2}, \nu)_{F,m} = (-1, \nu)_{F,2}$ by (2.6). Suppose $K = F(\sqrt{D})$, then

$$(N_{K/F}(c), \nu)_{F,2} = (-D, \nu)_{F,2},$$

hence (4.11) is $(-\tilde{1}, -1) \cdot \text{CAD}(g)(\delta, \delta_0)$ times $(D, \nu)_{F,2} = \text{sgn}_{K/F}(\nu)$. Next, suppose $K = F \times F$ so that $\text{sgn}_{K/F}(\nu) = 1$. Then $(N_{K/F}(c), \nu)_{F,2} = (-1, \nu)_{F,2}$ and (4.11) reduces to $(-\tilde{1}, -1) \cdot \text{CAD}(g)(\delta, \delta_0)$. \hfill \Box

Proposition 4.4.3. Assume $G = \text{SL}(2)$ and $y_0 \in \ker_{(Q,m)}$ corresponds to $y_0 \in K^1$. Let $g \in G_{ad}(F)$ with $\nu = \nu(g) \in F^\times/F^\times_2$ via (3.5). For all $(\delta, \delta_0) \in T_{Q,m}$ and $\sigma \in \text{Sgn}_{m}(T)$,

1. when $T$ splits, we have $y_0 \in \mu_m/\gcd(2, m)$ and the diagram

$$\begin{array}{c}
\hat{T}_{Q,m} \\
\text{CAD}(g) \downarrow \\
\hat{T}_{Q,m}
\end{array} \xrightarrow{(1,m)} \begin{array}{c}
\hat{T}_{Q,m} \\
\text{CAD}^*(g) \downarrow \\
\hat{T}_{Q,m}
\end{array}$$

commutes: both composites send $(\delta, \delta_0)$ to $(\text{Ad}(g)\tilde{\delta}, \text{Ad}(g)(\eta_0\delta_0))$;

2. when $T$ is anisotropic, only for $4 \mid m$ can $y_0$ be nontrivial, in which case $y_0 = -1$ and

$$\begin{array}{c}
\hat{T}_{Q,m} \\
\text{CAD}(g) \downarrow \\
\hat{T}_{Q,m}
\end{array} \xrightarrow{(1,m)} \begin{array}{c}
\hat{T}_{Q,m} \\
\text{CAD}^*(g) \downarrow \\
\hat{T}_{Q,m}
\end{array}$$

commutes, where $\text{sgn}_{K/F}(\nu)$ is viewed as an element of $\mu_2 \subset \mu_m$. 

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Theorem 4.5.1 ([29, Proposition 7.1]). Suppose that for every root \( \alpha \) of \( T \times \hat{F} \) in \( G \times \hat{F} \), either \( \alpha(\delta) = 1 \) or \( v(\alpha(\delta)) - 1 = 0 \). If \( \eta \in G(\sigma_F) \) is stably conjugate to \( \delta \), then \( T(\delta, \eta) \) contains a point in \( G(\sigma_F) \); in particular \( \eta \) is conjugate to \( \delta \).
Proposition 4.5.2. In the circumstance above, write \( \text{Ad}(g) : T \rightarrow S := G_n \) for the corresponding stable conjugacy between maximal tori, and suppose that

\[(\delta, \delta_0) \in \tilde{T}_{Q,m}, \quad \sigma \in \text{Sgn}_m(T).\]

Then we can take \( g \in G(\sigma_F) \cap T(\delta, \eta)(F) \) and \( \text{CAD}^p(g)(\delta, \delta_0) = (\text{Ad}(g)\delta, \text{Ad}(g)\delta_0) \), i.e. the usual conjugacy.

Proof. One may take \( g \in G(\sigma_F) \cap T(\delta, \eta)(F) \) by Theorem 4.5.1. It remains to apply AD.3 in Proposition 4.3.5.

Next, let \( F \) be a global field with characteristic \( \not\equiv 2 \) and consider Matsumoto’s \( E_G \rightarrow G \) over \( F \). Fix \( m \mid N_F \) and take a large finite subset \( S \) of places of \( F \), verifying

- \( S \supset \{ v : v \mid \infty \} \cup \{ v : v \nmid \text{res.char}(F_v) \} \};
- \( G \) is defined over the ring \( \sigma_S \) of \( S \)-integers as a connected smooth group scheme, and \( E_G \rightarrow G \) is defined over \( \sigma_S \) as well;
- the earlier conditions in the unramified case hold at every \( v \notin S \).

At each place \( v \), we construct the BD-cover \( \mu_m \rightarrow \tilde{G}_v \rightarrow G(F_v) \), except that for complex places we set \( \tilde{G}_v = G(F_v) \). Following [7, 10.4], the adélic BD-cover \( \tilde{G} \) is the limit of \( \prod_{V} \)

\[
\tilde{G}_V := \prod_{v \in V} \tilde{G}_v / N_V \xrightarrow{p_V} \prod_{v \in V} G(F_v),
\]

\[
N_V := \left\{ (\varepsilon_v)_{v} \in \prod_{v \in V \setminus V_{\text{non-complex}}} \mu_m : \prod_{v} \varepsilon_v = 1 \right\}. \tag{4.12}
\]

over finite subsets of places \( V \supseteq S \), i.e. the limit of the contracted product \( p_V \) of the local BD-covers \( \tilde{G}_v \rightarrow G(F_v) \). Using the aforementioned section \( G(\sigma_v) \hookrightarrow \tilde{G}_v \), the transition map for \( V' \supset V \) is

\[
\tilde{G}_V \hookrightarrow \tilde{G}_{V'} \times \prod_{v \in V' \setminus V} G(\sigma_v) \subset \tilde{G}_{V'}. \]

By [7, 10.4.3], we obtain a central extension of locally compact groups equipped with a section \( s \) over \( G(F) \)

\[
1 \xrightarrow{\mu_m} \tilde{G} \xrightarrow{p} G(\sigma_F) \xrightarrow{\exists s} G(F) \]

Remark 4.5.3. The same construction applies to all multiplicative \( K_2 \)-torsors over a reductive \( F \)-group, but the \( s \) here is unique since \( G(F) \) equals its own commutator subgroup.

Furthermore, the formation of adélic BD-covers is compatible with Weil restriction by Proposition 2.4.3: if \( E_G \rightarrow G \) is over a separable extension \( L/F \), then the adélic BD-cover obtained from \( R_{L/K}(E_G) \rightarrow R_{L/K}(G) \) is the same as the one from \( E_G \rightarrow G \).

Definition 4.5.4. Call an element \( (\delta_v)_{v} \) of \( G(\mathbb{A}_F) \) good in \( G \) if \( \delta_v \) is good in \( \tilde{G}_v \) for all \( v \). Call \( \delta \in G(F) \) good in \( G \) if \( s(\delta) \in \tilde{G} \) is.

We consider only the case of \( \delta \in G(\mathbb{A}_F) \). The local classification of good elements in Proposition 4.3.1 can be adapted to the present setting. Notice that the regular semisimple classes and maximal tori in \( G \) can still be parameterized by étale \( F \)-algebras with involution, together with other data. The construction \( T \twoheadrightarrow G^T \subset G \) (Definition 3.3.3) also works here.

For any closed subvariety \( H \subset G \), denote \( H := p^{-1}(H(k_F)) \subset \tilde{G} \).
Proposition 4.5.5. Let $T \subset G$ be a maximal $F$-torus parameterized by $(K,K^2,\ldots)$. An element $\delta \in T(F)$ has central preimages in $\tilde{T}$ if and only if

$$\delta \in \prod_{i \in I} \{ \pm 1 \} \cdot \im [i_{Q,m} : K^1_i \to K^1_i].$$

In particular, $\delta \in T_{\text{reg}}(F)$ is good if and only if the property above holds.

Proof. By theorem 4.1.3 applied to each place $v$, together with Remark 4.5.3, this reduces immediately to the case $\dim_{\mathcal{F}} K = 2$ and $K^2 = F$. Let $x \in K^1$ be an element corresponding to the class of $\delta$, and choose any $\hat{\delta} \in \mathbb{P}^{-1}(\delta)$.

- If $m \not\equiv 2 \mathbb{Z}$, then $1$ clearly belongs to $\im (i_{Q,m})$. The goal is to show $\hat{\delta} \in Z_{\tilde{\mathcal{F}}} \iff x \in (K^1)^m$.

- If $m \equiv 2 \mathbb{Z}$, the goal is to show $\hat{\delta} \in Z_{\tilde{\mathcal{F}}} \iff x^2 \in (K^1)^m$, as the 2-torsion subgroup of $K^1$ is clearly $\{ \pm 1 \}$.

The same recipe applies to each non-complex place $v$. Set $\mathcal{K}_v := K \otimes_F F_v$. By Theorem 4.2.8, we see

$$\hat{\delta} \in Z_{\tilde{\mathcal{F}}} \iff \begin{cases} \forall v, x_v \in (K^1_v)^m, & m \not\equiv 2 \mathbb{Z} \\ \forall v, (x^2)_v \in (K^1_v)^m, & m \equiv 2 \mathbb{Z} \end{cases}$$

where $v$ ranges over the non-complex places; this is immaterial since the conditions are clearly satisfied when $F_v = \mathbb{C}$. Hence we are reduced to show that $y \in K^1$ is an $m$-th power in $K^1$ if and only if it is so locally everywhere.

Write $Z := \ker [K^1 \twoheadrightarrow K^1]$ and set $\ker (F,Z) := \ker [H^1(F,Z) \to \prod_v H^1(F_v,Z)]$. The required local-global principle amounts to $\ker(F,Z) = 0$. When $K \simeq F \times F$, we have $K^1 \simeq F^2 \times Z \simeq \mu_m$ and this is covered by the Grunwald–Wang theorem since $m | N_F$. When $K$ is a field, the vanishing of $\ker (F,Z)$ has been shown in [38, Proposition 2.1] (taking $S = \mathcal{S}$, $2^m||m$), which also works over function fields. $\Box$

Next, denote by $\tilde{T}$ the preimage of $T(\mathbb{A}_F)$ in $\tilde{G}$. Given $\sigma \in \text{Sgn}_{m}(T)$, define

$$T^\sigma_{Q,m} := \{(t,t_0) \in T \times T_{Q,m} : t = \sigma \cdot t_{Q,m}(t_0)\} \quad \text{(fibered product of varieties)},$$

$$\tilde{T}^\sigma_{Q,m} := \{(\tilde{t},t_0) \in \tilde{T} \times T_{Q,m}(\mathbb{A}_F) : \mathcal{P}(l) = \sigma \cdot t_{Q,m}(t_0)\}.$$}

Here the action of $\sigma$, etc. are defined in the same manner as in §4.3. As in (4.12), observe that

$$\tilde{T}^\sigma_{Q,m} = \varprojlim \left( \prod_{v \in \mathbb{V}} \tilde{T}^\sigma_{Q,m,v} / \mathcal{N}_V \right),$$

the transition maps are again defined using integral models of BD-covers off $V$, for $V$ sufficiently large. Using the section $s$, one embeds $T^\sigma_{Q,m}(F)$ into $\tilde{T}^\sigma_{Q,m}$.

Given Proposition 4.5.5, it is natural to study the effect of adélic stable conjugacy on $T^\sigma_{Q,m}$ and $T^\sigma_{Q,m}(F)$. This is based on the local avatars $\text{CAD}^\sigma(g)_v$ at each place $v$; it reduces to the usual one when $F_v = \mathbb{C}$.

Theorem 4.5.6. Let $\text{Ad}(g) : T \to S$ be stable conjugacy between maximal $F$-tori of $G$. Then $\text{CAD}^\sigma(v) := \prod_v \text{CAD}^\sigma(g)_v$ defines a map $\tilde{T}^\sigma_{Q,m} \to \tilde{S}^\sigma_{Q,m}$. It satisfies the properties enounced in Proposition 4.3.5, and restricts to a map $T^\sigma_{Q,m}(F) \to S^\sigma_{Q,m}(F)$; when $g \in G^\mathcal{S}_ad(F)$, this restriction comes from the usual $\text{Ad}(g)$.

Proof. Pick $\delta \in T_{\text{reg}}(F)$ and $\eta = \text{Ad}(g)\delta \in S_{\text{reg}}(F)$. There is a large finite set $S$ of places such that for all $v \not\in S$, the Theorem 4.5.1 applies to $\text{Ad}(g) : \delta_v \to \eta_v$. At such places, Proposition 4.5.2 implies that $\text{CAD}^\sigma(g)$ reduces to ordinary conjugacy by $G(\mathcal{O}_v)$. Together with $\text{AD}.1$ of Proposition 4.3.5, this implies $\prod_v \text{CAD}^\sigma(g)_v$ is well-defined. The properties in Proposition 4.3.5 are inherited from $\text{CAD}^\sigma(g)_v$, for all non-complex place $v$.

Now move to the restriction of $\text{CAD}^\sigma(g)$ to $T^\sigma_{Q,m}(F)$. As in the local setting, we may choose a factorization pair $(g',g'')$ for $\text{Ad}(g)$ over $F$, with $g' \in G^\mathcal{S}_ad(F)$ and $g'' \in G(F)$. Hence it suffices to consider the case $G \simeq \text{SL}(2)$ and $g \in G_{ad}(F)$ (see Remark 4.5.3). The relevant factors $C_m(\cdots)$ in
4.2.7
3.1
4.5.1
5.3
57
57
4.1
57
72x338
and that of the simple ones; similarly comes with a maximal torus $C$
There is a homomorphism $\hat{F}$ or later use, the data in §5.1 Definitions
unramified integral cases. Except in §5.1 Definitions, we call $\tilde{\Phi}\subset \Phi \subset X_{Q,m}$ be the sets of
modified roots and coroots. Let $\tilde{\Delta}\subset \Phi \subset X_{Q,m}$ be the simple ones; similarly $\tilde{\Delta}'\subset \Phi' \subset Y_{Q,m}$. In loc. cit., the dual group $\tilde{G}'$ is defined as the
pinned $\mathbb{C}$-group with based root datum $(Y_{Q,m},\Delta',X_{Q,m},\Delta)$, here with trivial $\Gamma_F$-action. Thus $\tilde{G}'$ comes with a maximal torus $\tilde{T}'$ with
$X^*(\tilde{T}') = Y_{Q,m}$, $Z_{\tilde{G}'} = \mathrm{Hom}(Y_{Q,m}/Y_{Q,m}^{sc},\mathbb{C}^\times) \subset \tilde{T}'$.
There is a homomorphism $\tau_{Q,m}: \mu_2 \to Z_{\tilde{G}'}$ that is Cartier-dual to
$\phi Q,\phi M : Y_{Q,m}/Y_{Q,m}^{\phi} \to (Y_{Q,m}^{\phi} + mY_{Q,m}) \xrightarrow{y^\phi + m^{-1}Q(y)} \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. As the roots/coroots are modified by rescaling, $\tilde{G}'$ and $G$ share the same Weyl group. In fact
$\tilde{G}' = \begin{cases} \mathrm{SO}(2n+1,\mathbb{C}), & m \notin 2\mathbb{Z} \\ \mathrm{Sp}(2n,\mathbb{C}), & m \in 2\mathbb{Z}. \end{cases}$
Quoting [57, §2.7.4], we have $\tau_{Q,m}(-1) \neq 1$ if and only if $m \equiv 2 \pmod{4}$.
To define the Galois form of the $L$-group of $\tilde{G}$, consider the following two extensions of groups.
1. In [57, §4.1] the metaGalois group is defined as the central extension
\begin{equation}
1 \to \mu_2 \to \tilde{\Gamma}_F \to \Gamma_F \to 1.
\end{equation}
It is just $\Gamma_F \times \mu_2$ with multiplication given by the cocycle $(\tau_1,\tau_2) \mapsto (\mathrm{rec}_F(\tau_1),\mathrm{rec}_F(\tau_2))_{F,2}$, where the reciprocity homomorphism $\mathrm{rec}_F : \Gamma_F \to \mathbb{F}^\times$ is normalized to send a geometric Frobenius to a uniformizer in $\mathfrak{p}_F$. We obtain the push-out $Z_{\tilde{G}'} \xrightarrow{\tau_{Q,m}} Z_{\tilde{G}'}(\Gamma_F) \to \Gamma_F$.
2. The constructions in [57, §3.2] yield a gerbe $E_r(\tilde{G})$ over $\text{Spec}(F)_\text{et}$ banded by $Z_{\tilde{G}^\vee}$. Its fundamental group sits in an extension

$$1 \to Z_{\tilde{G}^\vee} \to \pi_1^\text{et}(E_r(\tilde{G})) \to \Gamma_F \to 1. \tag{5.2}$$

The relevant definitions will be recalled in §§5.2—5.3 whenever needed.

Their Baer sum is an extension $Z_{\tilde{G}^\vee} \hookrightarrow \Gamma_F$; a further $\Gamma_F$-equivariant push-out via $Z_{\tilde{G}^\vee} \hookrightarrow \tilde{G}^\vee$. To obtain a continuous section $\Gamma_F \to \tilde{G}$, we inspect the metaGalois group first.

When $m \not\equiv 2 \pmod{4}$, the central extension $\tau_{Q,m,*}(\Gamma_F)$ is already trivial. Now suppose $m \equiv 2 \pmod{4}$. Fix an additive character $\psi$. According to [57, Proposition 4.5], upon enlarging $\Gamma_F$ to $\Gamma_F^\sim(4)$ by $\mu_2 \hookrightarrow \mu_4$, there is a splitting $s(\psi) : \Gamma_F \to \Gamma_F^\sim(4)$ given by

$$s(\psi)(\tau) := \left(\tau, \frac{\gamma_\psi(\text{rec}_F(\tau))}{\gamma_\psi(1)}\right) \in \Gamma_F \times \mu_4.$$

This is based on the standard identity [42, Proposition 1.3.3] for Weil’s constants

$$(a, b)_{F, 2} = \frac{\gamma_\psi(a b)}{\gamma_\psi(a) \gamma_\psi(b)}. \tag{5.3}$$

Notice that the enlargement to $\mu_4$ is realizable inside $\tilde{G}^\vee = \text{Sp}(2n, \mathbb{C})$: consider the subgroup

$$C := \{\text{diag}(\zeta, \ldots, \zeta, \zeta^{-1}, \ldots, \zeta^{-1}) \in \text{Sp}(2n, \mathbb{C}) : \zeta \in \mu_4.\} \tag{5.4}$$

We have $\tau_{Q,m}(-1) = -1 \in C$, and such a subgroup $C \simeq \mu_4$ is unique up to conjugacy.

**Lemma 5.1.1** ([57, Proposition 4.5]). If $\psi$ is replaced by $x \mapsto \psi(c x)$ where $c \in F^\times$, then $s(\psi)$ will be twisted by the character

$$\chi_c : \Gamma_F \to \mu_2, \quad \chi_c(\tau) = (\text{rec}_F(\tau), c)_{F, 2}. \tag{5.5}$$

Clearly, $\chi_{c_1 c_2} = \chi_{c_1} \chi_{c_2}$ for all $c_1, c_2 \in F^\times$.

Later on, by using the form $\langle \cdot | \cdot \rangle$ or the associated $F$-pinning [17, Exp XXIII] of $G$,

- it will be shown in Lemma 5.2.1 that (5.2) also splits after a push-out via $Z_{\tilde{G}^\vee} \hookrightarrow C$;
- in Lemma 5.2.2, it will be shown that changing $\langle \cdot | \cdot \rangle$ or the pinning will twist that splitting by the $\chi_c$ in (5.5).

With these auxiliary data, we can switch to the Weil form of $^{\perp}G$ and identify

$$\begin{array}{ccc}
^{\perp}G & \simeq & \tilde{G}^\vee \times W_F \\
\downarrow \sim & & \downarrow & \leftarrow & \\
W_F & & W_F
\end{array}$$

**Definition 5.1.2.** An $L$-parameter for $\tilde{G}$ is a continuous homomorphism $\phi : W_F \times SU(2) \to ^{\perp}G$ commuting with the projections to $W_F$, such that the $\tilde{G}^\vee$-component of $\phi(w)$ is semi-simple for all $w \in W_F$. Equivalence between $L$-parameters is given by $\tilde{G}^\vee$-conjugacy.

By the foregoing discussions, the choice of splitting $^{\perp}G \simeq \tilde{G}^\vee \times W_F$ does not affect the definitions above. The $L$-parameters that we will encounter are all trivial on $SU(2)$.

**5.2 On the second twist**

For the next result, we recall from [57, Remark 19.8] that $\pi_1^\text{et}(E_r(\tilde{G}))$ is realized as $\text{lim}_{\leftarrow} \pi_1^\text{et}(E_r(\tilde{G}), \tilde{z})$ with respect to unique transition isomorphisms, where the “geometric basepoints” $\tilde{z}$ ranges over objects of $E_r(\tilde{G})(F)$. We will employ the concrete description of (5.2) in [56] or [14, §5.2] for pinned split groups, called the second twist in [14].

Take a symplectic basis, which gives rise to a standard $F$-pinning for $G$; note that the symplectic bases form a single $G(F)$-orbit. Let $T \subset G$ be the corresponding split maximal torus adapted to the chosen symplectic basis and consider the $\mathcal{D}$ from (2.2). Taking pull-backs, we obtain the objects $\mathcal{D}_{Q,m}$,
A further push-out yields an object $D$ such isomorphisms are shown to be compatible with the transition isomorphisms when $\chi$ splits as well. These splittings depend on $\chi$ which is legitimate by the foregoing computation. Then $\chi$ splits after a push-out by Hilbert’s Satz 90 admits a splitting $s_0$ over $Y^s_{Q,m}$, by sending each $\alpha^\vee \in \Delta^\vee$ to $s(n_\alpha \alpha^\vee)$.

Now assume $m \in 2\mathbb{Z}$ so that $[57, Assumption 3.1]$ is in force. Choose a convenient basepoint $\bar{z}_0$ as in $[56, §1.1]$ for the pinning, which arises from a splitting $s_0 : Y_{Q,m} \to D_{Q,m}(\bar{F})$ over $\bar{F}$ extending $s_0$. Define the group $E_0 := \{ \chi \in \text{Hom}(D_{Q,m}(F), \mathbb{C}^\times) : \chi|_{F^\times} = \chi|_{\text{im}(s_0)} = 1 \}$. Then $E_0 \to \text{Hom}(F^x/F^{x,m}, \mu_m)$ with kernel $Z_{\bar{G}^\vee}$. Consider the surjective homomorphism

$$q_m : \Gamma_F \to \text{Hom}(F^x/F^{x,m}, \mu_m)$$

$$\gamma \mapsto \left[ u \mapsto \epsilon \left( \frac{\gamma^{-1} u^{1/m}}{u^{1/m}} \right) \right].$$

By $[56, §§2.1–2.3]$, one can describe $\pi^f_1(E, (\bar{G}), \bar{z}_0)$ via the comparison isomorphism of central extensions in loc. cit.

$$C_0 : \pi^f_1(E, (\bar{G}), \bar{z}_0) \to (q_m)^* E_0 \to \text{pull-back by } q_m.$$ 

Such isomorphisms are shown to be compatible with the transition isomorphisms when $\bar{z}_0$ varies.

**Lemma 5.2.1.** When $m \not\in 2\mathbb{Z}$, the central extension (5.2) is trivial. When $4 | m$, (5.2) splits; when $m \equiv 2 \pmod{4}$, it splits after a push-out $Z_{\bar{G}^\vee} \to C$, see (5.4). These splittings for $m \in 2\mathbb{Z}$ are canonical for the chosen $(W, (\cdot|\cdot))$.

**Proof.** When $m \not\in 2\mathbb{Z}$ we have $Z_{\bar{G}^\vee} = \{ 1 \}$, so assume $m \in 2\mathbb{Z}$ in what follows.

Let us take the quotient of $D_{Q,m}(F)$ by $s_0(Y^s_{Q,m})$, which gives an object $D_1$ of $\text{CExt}(Y^s_{Q,m}/Y^s_{Q,m}, F^x)$. A further push-out yields an object $D_2$ of $\text{CExt}(Y^s_{Q,m}/Y^s_{Q,m}, F^x/F^{x,m})$. The discussions above imply that (5.2) is obtained from $q_m^\ast \text{Hom}(D_2, \mathbb{C}^\times)$, the multiplication in $\text{Hom}(D_2, \mathbb{C}^\times)$ being pointwise.

To find splittings, we first reduce $D_1$ to a $\mu_2$-torsor $D_1^\mu \to Y^s_{Q,m}/Y^s_{Q,m}$. As $\frac{m_1}{2} \mu_1$ generates $Y^s_{Q,m}/Y^s_{Q,m}$ using the pinning, the coset of $s(m_1/2)$ affords a canonical preimage in $D_1$. By $[7, (11.1.4)–(11.1.5)]$ and $\text{Res}(\{ t, t \}) = -1$, we have

$$s \frac{m_1^2}{2} = \text{Res}(t^{m/2}, t^{m/2} Q(\xi_1)) \to s(m_1^2) \equiv 1 \in D_1$$

$$= (-1)^{m/2} \in \mu_2.$$ 

This furnishes the $D_1^\mu$. Accordingly $\text{Hom}(D_2, \mathbb{C}^\times)$ is pulled back from $\text{Hom}(D_1^\mu, \mathbb{C}^\times)$ by $\text{Hom}(\mu_2, \mathbb{C}^\times)$ and both splits canonically when $4 | m$.

Hereafter suppose $m \equiv 2 \pmod{4}$. Take $\zeta \in \mathbb{C}$ with $\zeta^2 = -1$. Prescribe $\chi_\zeta \in \text{Hom}(D_1^\mu, \mathbb{C}^\times)$ lying over $\epsilon_{\mu_2} \in \text{Hom}(\mu_2, \mathbb{C}^\times)$ by

$$\chi_\zeta|_{\mu_2} = \epsilon_{\mu_2}, \quad \chi_\zeta(s(m_1/2)) = \zeta,$$

which is legitimate by the foregoing computation. Then $\chi_\zeta^2 \in \text{Hom}(Y^s_{Q,m}/Y^s_{Q,m}, \mathbb{C}^\times) = Z_{\bar{G}^\vee}$ is nontrivial, so it is not a section for $\text{Hom}(D_1^\mu, \mathbb{C}^\times)$ but $\text{Hom}(\mu_2, \mathbb{C}^\times)$. However, upon pushing out by $Z_{\bar{G}^\vee} \to C \simeq \mu_4$, we have $(\zeta^{-1} \wedge \chi_\zeta)^2 = 1$ and this does afford a splitting.

Finally, $\chi_\zeta^{-1}/\chi_\zeta$ is trivial on $\mu_2 \subset D_1^\mu$ and maps $s(m_1/2)$ to $-1$; this cancels with the ratio $\zeta/\zeta^{-1} = -1$ in the push-out to $C$. Hence the splitting does not depend on the choice of $\zeta$. 

The upshot is that both (5.1) and (5.2) split after pushed out to $C$, therefore the Galois form of $\bar{G}$ splits as well. These splittings depend on $\psi$ and $(W, (\cdot|\cdot))$ (or the $F$-pinning). Our ultimate goal is an
ambiguity-free formalism of $L$-packets. The dependence on $\psi$ is quantified by Lemma 5.1.1 so we shall concentrate on $\langle \cdot | \cdot \rangle$. Set $G_1 := $GSp$(W)$ and note that

$$
\frac{[F \text{- pinnings}]}{G(F) \text{- cong}} \xrightarrow{\text{torsor}} \frac{\text{im}[G(F) \to G_{\text{ad}}(F)]}{G_{\text{ad}}(F)} \xrightarrow{\sim} \frac{G_1(F)}{Z_{G_1}(F)G(F)} \quad \text{commutes.}
$$

Hence $G_{\text{ad}}(F)$ has three effects: (a) dilation of $\langle \cdot | \cdot \rangle$ up to $F^{\times 2}$, (b) change of $F$-pinnings, (c) action on $E_G$ via Proposition 2.2.3.

Define the groups $\hat{T} := \text{Hom}(Y_{Q,m}, \hat{F}^{\times})$, $\hat{T}_{\text{sc}} := \text{Hom}(Y_{Q,m}^{\text{sc}}, \hat{F}^{\times})$ and $\hat{Z} := \text{Hom}(Y_{Q,m}/Y_{Q,m}^{\text{sc}}, \hat{F}^{\times})$.

**Lemma 5.2.2.** Suppose that $m \in 2\mathbb{Z}$ and let $g_1 \in G_1(F)$ be of similitude factor $c \in F^{\times}$. Then changing pinnings by $\text{Ad}(g_1)$ induces a twist by the $\chi_c$ of (5.5) on the splitting constructed in Lemma 5.2.1.

**Proof.** Let $T_1 := Z_{G_1}(T)$. Upon a translation by $G(F)$, we may assume $g_1 \in T_1(F)$. In fact, in the symplectic base we take $g_1 = t := \text{diag}(e_1, \ldots, e_1, 1, 1, 1, 1)$.

For any two choices of $s_0, \hat{s}_0$ and $s_1, \hat{s}_1$ as before, which give rise convenient base points $\hat{z}_0, \hat{z}_1$, we have comparison isomorphisms

$$C_i : \pi_1^{\text{et}}(E_i(\hat{G}), \hat{z}_i) \xrightarrow{\sim} (g_m)^*E_i, \quad i = 0, 1.$$

The prior constructions are based on the pinning and canonical splittings of multiplicative $K_2$-torsors over unipotent groups; therefore they are transported by the $\text{Ad}(t) : E_G \to E_G$ of Proposition 2.2.3. Now regard $s_0, \hat{s}_0$ as given and set

$$s_1 := \text{Ad}(t) \circ s_0, \quad \hat{s}_1 := \text{Ad}(t) \circ \hat{s}_0.$$

Define $\varphi : E_0 \xrightarrow{\sim} E_1$ by $\chi \mapsto \chi \circ \text{Ad}(t)^{-1}$ and observe that $\varphi_\tau$ transports the splittings (after a push-out) afforded by Lemma 5.2.1 for similar reasons. It suffices to show that the commutativity up to $\chi_c$-twist of

$$\begin{align*}
&\pi_1^{\text{et}}(E_i(\hat{G}), \hat{z}_0) \xrightarrow{\sim} \pi_1^{\text{et}}(E_i(\hat{G}), \hat{z}_1) \\
&\quad \downarrow \quad \downarrow \quad \downarrow \\
&(g_m)^*E_0 \xrightarrow{\sim} (g_m)^*E_1
\end{align*}
$$

is transition isomorphism.

(5.6)

We shall make free use of the computations in [56] below. Let $b \in \hat{T}$ with $b^m = \hat{s}_1/\hat{s}_0$. Represent the elements of $\pi_1^{\text{et}}(E_i(\hat{G}), \hat{z}_0)$ as $(\tau, \zeta) \in \hat{T} \times Z_{G_1}$. To simplify matters, we fix $\gamma \in \Gamma_F$ and look only at the fibers in $E_0, E_1$ over $g_m(\gamma)$; it is shown in [56, §§2.2–2.3] that

$$\iota(\tau, \zeta) = \left( \tau \cdot \frac{b}{\gamma^{-1}b}, \zeta \right), \quad C_i(\tau, \zeta) = \left[ \hat{s}_i(y)u \mapsto \epsilon \left( \frac{\gamma^{-1}u^{1/m}}{u^{1/m}} \cdot \tau(y) \right) \zeta(y) \right] \quad (i = 0, 1)
$$

where $(y, u) \in Y_{Q,m} \times \hat{F}^{\times}$ satisfies $\hat{s}_i(y)u \in D_{Q,m}(F) = \text{D}_{Q,m}(F)$. Note that in §2.3 of loc. cit. one has $s_0 = s_1$ in describing $\iota$; we cannot assume this, so $b \notin \hat{Z}$ in our case.

Suppose $\zeta = 1$ for simplicity. Then $\varphi_\tau \circ C_0(\tau, 1)$ is the map

$$\hat{s}_1(y)u \xrightarrow{\text{Ad}(t)^{-1}} (g_m)^*E_0 : \hat{s}_0(y)u \mapsto \epsilon \left( \frac{\gamma^{-1}u^{1/m}}{u^{1/m}} \cdot \tau(y) \right).$$

On the other hand, $C_1 \circ \iota(\tau, 1)$ maps $\hat{s}_1(y)u \in D_{Q,m}(F)$ to

$$\epsilon \left( \frac{\gamma^{-1}u^{1/m}}{u^{1/m}} \cdot (\tau b)(y) \right) = \epsilon \left( \frac{\gamma^{-1}u^{1/m}}{u^{1/m}} \cdot \tau(y) \right) \epsilon \left( \frac{b(y)}{\gamma^{-1}b(y)} \right).$$

So the twist to make (5.6) commutative is $y \mapsto \epsilon(b(y)/\gamma^{-1}b(y))$. 41
Note that \( \text{Ad}(t) \) induces an automorphism of the central extension \( D_{Q,m} \), i.e. an element of \( \text{Hom}(Y_{Q,m}, \mathbb{G}_m) \).
By Lemma 4.1.1 and our choice of \( T, t \), it suffices to study \( \text{Ad}(t) \) inside copies of \( E_{SL(2)} \rightarrow SL(2) \), one for each \( \ell_i \). For any preimage \( \delta \in E_{SL(2)}(F(t)) \) of \( \ell_1(t) \), Kubota’s cocycle (4.4) yields \( \text{Ad}(e_1)(\delta) = (-\{t, c\}_F) \cdot \delta \). From \( \text{Res}\{t, c\}_F = c \) we deduce
\[
\frac{b^m}{s_0} = \frac{\text{Ad}(t) \circ \delta_0}{s_0} : Y_{Q,m} \rightarrow F^\times
\]
and our choice of \( s_i \), \( Y_{Q,m} \rightarrow F^\times \)

Recall \( Y_{Q,m} = \mathbb{Z}[Y] \); thus the definition of Hilbert symbols and (2.6) imply
\[
e(\frac{b(y)}{\gamma^{-1}b(y)}) = \epsilon(\text{rec}_F(\gamma), e^{a_1+\ldots+a_n})^{-1}_{F,m} = \epsilon(\text{rec}_F(\gamma), e^{2(a_1+\ldots+a_n)/m})^{-1}_{F,2}.
\]
As \( Y_{Q,m}^{sc} = \mathbb{Z}[Y_0] \), it yields a character of \( Y_{Q,m}/Y_{Q,m}^{sc} \) mapping any generator to \( \epsilon(\text{rec}_F(\gamma), c)_{F,2} = \chi_{e}(\gamma) \).

See also [14, §7] for a different construction of splittings.

5.3 Rescaling

In this subsection we work with a pinned quasisplit \( F \)-group \( G \) and \( E \rightarrow G \) in \( C\text{Ext}(G, K_2) \), classified by some \( (Q, D, \varphi) \). We only retain the datum \( Q \) in the notation; for instance \( kQ \) really means the \( k \)-fold Baer sum of \( E \) or the corresponding triple. Consider
\[
m \in \mathbb{Z}_{\geq 1}, \quad k \in \mathbb{Z} \setminus \{0\}, \quad d := \gcd(k, m), \quad m = dm', \quad k = dk'.
\]
Fix \( \epsilon : \mu_m \hookrightarrow \mathbb{C}^\times \) and set \( \epsilon' := \epsilon^{k'} : \mu_{m'} \hookrightarrow \mathbb{C}^\times \). Denote by \( \tilde{G}[kQ, m] \), \( \tilde{G}[Q, m'] \) the BD-covers attached to these data, and affix \( \epsilon, \epsilon' \) to denote their push-outs; the same formalism also applies to other constructions. Our goal is to identify the \( L \)-groups \( \tilde{G}_c[kQ, m] \) and \( \tilde{G}_c[Q, m'] \), which should reflect Remark 2.4.4. Unexplained notations can be found in [57, §3].

Note that \( B_{kQ} = kB_Q \). We claim that
\[
Y_{kQ,m} := \{ y \in Y : kB_Q(y, Y) \subset m\mathbb{Z} \} = \{ y \in Y : k'B_Q(y, Y) \subset m'\mathbb{Z} \} = Y_{Q,m'}.
\]
Only the last equality is nontrivial, for which \( \subset \) is clear. As for \( \supset \), take \( a, b \in \mathbb{Z} \) such that \( k'a + m'b = 1 \), then \( y \in Y_{kQ,m} \) implies that \( B_Q(y, Y) = (k'a + m'b)B_Q(y, Y) \subset m'\mathbb{Z} \), thus \( y \in Y_{Q,m'} \).

Accordingly, \( X_{kQ,m} := \{ x \in X \otimes \mathbb{Q} : \langle x, Y_{kQ,m} \rangle \subset \mathbb{Z} \} \) equals \( X_{Q,m'} \). Next, for every root \( \phi \) we have
\[
n_{\phi}[kQ, m] := \frac{m}{\gcd(m, kQ(\phi))} = \frac{m'}{\gcd(m', k'Q(\phi))} = n_{\phi}[Q, m'].
\]
Hence the modified roots/corets for \( kQ, m \) are the same as those for \( Q, m' \). It also follows that \( Y_{kQ,m}^{sc} = Y_{Q,m'}^{sc} \). Write \( \tilde{Z}^\vee := \text{Hom}(Y_{Q,m'}/Y_{Q,m'}^{sc}, \mathbb{C}^\times) \).

Lemma 5.3.1. The dual groups \( \tilde{G}_c[kQ, m] \) and \( \tilde{G}_c[Q, m'] \) are defined by the same based root datum with \( \Gamma_F \)-action; the corresponding \( \tau_{kQ,m}, \tau_{Q,m'} : \mu_2 \rightarrow \tilde{Z}^\vee \) are also equal.

Proof. We only need to show \( \tau_{kQ,m} = \tau_{Q,m'} \). By construction, they are dual to the rows of the commutative diagram below (see [57, §2.2]).
\[
\begin{array}{ccc}
Y_{kQ,m}/Y_{kQ,m}^{sc} & \rightarrow & (Y_{Q,m}/Y_{Q,m}^{sc})/m(\cdots) \rightarrow \frac{1}{m^{-1}Q, \mathbb{Z}/\mathbb{Z}} \rightarrow Y_{Q,m}/Y_{Q,m}^{sc} \\
Y_{Q,m'}/Y_{Q,m'}^{sc} & \rightarrow & (Y_{Q,m}/Y_{Q,m}^{sc})/m'(\cdots) \rightarrow \frac{1}{m'^{-1}Q, \mathbb{Z}/\mathbb{Z}} \rightarrow Y_{Q,m'}/Y_{Q,m'}^{sc}
\end{array}
\]
We contend that the two arrows \( Y_{..}/Y_{..}^{sc} \rightarrow \frac{1}{m, \mathbb{Z}/\mathbb{Z}} = \frac{1}{m', \mathbb{Z}/\mathbb{Z}} \) are equal. We may assume \( k' \) even; \( Y_{kQ,m}/Y_{kQ,m}^{sc} \rightarrow \frac{1}{m, \mathbb{Z}/\mathbb{Z}} \) is then trivial. In this case \( m' \) must be odd, so \( Y_{Q,m'}/Y_{Q,m}^{sc} \rightarrow \frac{1}{m', \mathbb{Z}/\mathbb{Z}} \) is trivial as well since it factorizes through an \( m' \)-torsion group.

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As a consequence, the extensions \( \tau_{\mathcal{Q},m}^A(\Gamma_F) \) and \( \tau_{\mathcal{Q},m'}^A(\Gamma_F) \) of \( \Gamma_F \) by \( \mathcal{Z} \) (see (5.1)) are also equal. The next step is to compare the gerbes \( \mathcal{E}_r(\mathcal{G}(\mathcal{Q},m)) \) and \( \mathcal{E}_r(\mathcal{G}(\mathcal{Q},m')) \). To \( E \) and \( kE \) are associated \( \mathcal{D}(Q), \mathcal{D}(kQ) \) from (2.2), respectively; their pull-backs to \( Y_{Q,m'} = Y_{\mathcal{Q},m} \) yield \( \mathcal{D}_{Q,m'} \) and \( \mathcal{D}_{kQ,m} \). The following is a quick replacement of [57, Assumption 3.1]. It enables us to use the constructions in [57, §§3.1—3.3].

**Hypothesis 5.3.2.** The group \( \mathcal{D}_{Q,m'} \) is commutative.

By construction, \( \mathcal{D}(kQ) \) (resp. \( \mathcal{D}_{kQ,m} \)) is the \( k \)-fold Baer sum of \( \mathcal{D}(Q) \) (resp. of \( \mathcal{D}_{Q,m'} \)), thus is commutative as well.

**Example 5.3.3.** When \( G = \text{Sp}(W) \), we know \( BQ \) is even-valued on \( Y \); it suffices to consider the \( Q \) in (3.1). Hence the commutator formula of [7, Proposition 3.11] implies the commutativity of \( \mathcal{D}(Q) \).

We tabulate some functorial operations from [57, §19.3]. In what follows, \( A, B \), etc. stand for sheaves of commutative groups over \( S_{\mathcal{Q}} \) where \( S \) is some scheme; more generally one can work in a 1-topos. For all \( m \in \mathbb{Z} \setminus \{0\} \) in view, we assume \( A \xrightarrow{m} A \) is an epimorphism and set \( A_{\|m\|} := \ker(\widetilde{m}) \). Torsors will be manipulated “set-theoretically” below, as justified by the usual formal apparatus, eg. [18, Exp VII, I.1.2.1]. Likewise, we shall forget topology when talking about gerbes (generally in a (2,1)-topos) and treat them as groupoids.

- Let \( \varphi : A \to B \) be a homomorphism of sheaves of groups. The push-out of an \( A \)-torsor \( \mathcal{P} \) under \( \varphi \) will be written as \( \varphi_* \mathcal{P} = B \wedge_A \varphi_* \mathcal{P} \), whose elements are expressed as \( b \wedge p \), with \( b \wedge ap = b \varphi(a) \wedge p \) as usual. Let \( \mathcal{P} \) be a torsor and \( Q \) be a \( B \)-torsor; an equivariant morphism \( \mathcal{P} \to Q \) relative to \( \varphi \) is the same as a morphism \( B \wedge_A \varphi_\mathcal{P} \to Q \) of \( B \)-torsors (covering \( \text{id}_B \)), which sends \( b \wedge p \mapsto b \varphi(p) \).

- Let \( m \in \mathbb{Z}_{\geq 1} \). The gerbe \( \sqrt{\mathcal{P}} \) banded by \( A_{\|m\|} \) is defined as follows. Its objects are morphisms \( \mathcal{H} \xrightarrow{g} \mathcal{P} \) of torsors, equivariant relative to \( m : A \to A \). Its morphisms are commutative diagrams \( \mathcal{H} \xrightarrow{g} \mathcal{H}' \xrightarrow{f} \mathcal{P} \) where \( g \) is a morphism of \( A \)-torsors, in particular \( \text{Aut}(\mathcal{H}, f) = A_{\|m\|} \).

- Consider a commutative central extension \( C \xrightarrow{\varepsilon} \mathcal{E} \to Z \) over \( S_{\mathcal{Q}} \). Take \( A := \text{Hom}(Z, C) \) (internal Hom). The splittings of \( \mathcal{E} \to Z \) form an \( A \)-torsor, denoted by \( \text{Spl}(\mathcal{E}) \). Let \( k \in \mathbb{Z} \). Every splitting \( s \) for \( \mathcal{E} \) gives rise to a splitting \( 1 \wedge s \) for \( k_* \mathcal{E} \). Thus we obtain an isomorphism of \( A \)-torsors \( k_* \text{Spl}(\mathcal{E}) \to \text{Spl}(k_* \mathcal{E}) \), \( s \mapsto 1 \wedge s \).

The \( k_* \) on the left-hand side is required to obtain a morphism covering \( \text{id}_A \).

- Suppose \( m = dm' \). There is a functor of gerbes \( \sqrt{\mathcal{P}} \to \sqrt{\mathcal{P}'} \) lying over \( A_{\|m'\|} \to A_{\|m\|} \), which maps an object \( \mathcal{H} \xrightarrow{d} \mathcal{P} \) to \( \mathcal{H} \xrightarrow{1 \wedge f} d_* \mathcal{P} \). Here \( 1 \wedge f : h \mapsto 1 \wedge f(h) \) is \( m \)-equivariant since \( 1 \wedge f(th) = 1 \wedge f^t(h) = t^m \wedge f(h) \) for all \( t \in A, h \in \mathcal{H} \). The definition on morphisms is clear, and it is readily seen to be \( A_{\|m'\|} \to A_{\|m\|} \) on automorphisms.

- Suppose \( k', m' \) are coprime, \( m' \in \mathbb{Z}_{\geq 1} \). There is a functor between gerbes \( \sqrt{\mathcal{P}} \to \sqrt{k'_* \mathcal{P}} \) lying over \( k' : A_{\|m'\|} \to A_{\|m\|} \). Indeed, on objects we send \( \mathcal{H} \xrightarrow{f} \mathcal{P} \) to \( k'_* \mathcal{H} \xrightarrow{k'_* f} k'_* \mathcal{P} \), where \( k'_* f(t \wedge h) = t^{m'} \wedge f(h) \). One readily verifies that \( k'_* f \) is well-defined and \( m' \)-equivariant. The definition on morphisms is clear, and it sends every automorphism \( t \in A_{\|m'\|} \) to \( t^{k'} \).

Reverting to our original problem, define

\[
\hat{T} := \text{Hom}(Y_{Q,m'}, \mathbb{G}_m) \to \text{Hom}(Y_{Q,m'}, \mathbb{G}_m) =: \hat{T}_{sc} \quad (F \text{ - tori}),
\]

\[
\hat{T}^\vee := \text{Hom}(Y_{Q,m'}, \mathbb{C}^\times) \to \text{Hom}(Y_{Q,m'}, \mathbb{C}^\times) =: \hat{T}^\vee_{sc}, \quad (\mathbb{C} \text{- tori}).
\]

By taking \( A = \hat{T}, C = \mathbb{G}_m \) and \( Z = Y_{Q,m'} \), we deduce \( k_* \text{Spl}(\mathcal{D}_{Q,m'}) \xrightarrow{\sim} \text{Spl}(\mathcal{D}_{kQ,m}) \) and obtain functors of gerbes

\[
\begin{align*}
\mathcal{D}(Q,m') &\xrightarrow{k' \wedge} \mathcal{D}(Q,m) \xrightarrow{\varepsilon} \mathcal{D}(Q,m') \\
\hat{T}_{\|m'\|} &\xrightarrow{k' \wedge} \hat{T}_{\|m\|} \xrightarrow{\text{id}} \hat{T}_{\|m\|}.
\end{align*}
\]
In parallel, $k_*\text{Spl}(\mathcal{D}_Q^{sc},m) \xrightarrow{\cong} \text{Spl}(\mathcal{D}_Q^{sc},m')$ and there is a functor $\sqrt[n]{\text{Spl}(\mathcal{D}_Q^{sc},m')} \to \sqrt[n]{\text{Spl}(\mathcal{D}_Q^{sc},m)}$ lying over $\tilde{T}_{sc,m}' \xrightarrow{\pi} \tilde{T}_{sc,m} \xleftarrow{\epsilon} \tilde{T}_{sc,m}$.

In [57, §19.3.2], the push-out of a gerbe is defined by pushing out the Hom-torsors, thus we obtain $k'_* \sqrt[n]{\text{Spl}(\mathcal{D}_Q^{sc},m')} \to \sqrt[n]{\text{Spl}(\mathcal{D}_Q^{sc},m)}$ lying over $\tilde{T}_{m}' \to \tilde{T}_{m}$; same for the sc-case. Pushing out by $\epsilon_* : \tilde{T}_{m} \to \tilde{T}_{m}'$ and its sc-variant yield a diagram of gerbes and functors (cf. [57, §3.1])

$$\begin{array}{c}
E_{\epsilon'}(\tilde{T}(Q,m')) & \to & E_{\epsilon}(\tilde{T}(kQ,m')) \\
\pi & \downarrow & \pi' \downarrow \\
E_{\epsilon'}^e(\tilde{T}(Q,m')) & \to & E_{\epsilon}^e(\tilde{T}(kQ,m')) \\
\end{array} \hspace{1cm} \text{lying over} \hspace{1cm} \begin{array}{c}
\tilde{T}_{m}' \to \tilde{T}_{m} \\
\tilde{T}_{sc,m}' \to \tilde{T}_{sc,m} \end{array} \hspace{1cm} (5.7)$$

which is 2-commutative. Here $\pi$ is induced by the natural arrows $\text{Spl}(\mathcal{D}_Q,m') \to \text{Spl}(\mathcal{D}_Q^{sc},m')$ and $\text{Spl}(\mathcal{D}_Q,m') \to \text{Spl}(\mathcal{D}_Q^{sc},m')$.

**Lemma 5.3.4.** Under Hypothesis 5.3.2, the extensions (5.2) attached to $(kQ,m')$ and $(Q,m',\epsilon')$ are canonically isomorphic.

**Proof.** Every functor $E_1 \to E_2$ of gerbes induces a morphism $\pi|_e(E_1) \to \pi|_e(E_2)$ of group extensions by [57, §19.4]. By definition, $E_{\epsilon'}(\tilde{G}(Q,m'))$ is the “gerbe of liftings” $\pi^{-1}(w(Q,m'))$ (see [57, §19.3.3]) for the “Whittaker object” $w(Q,m')$ in $E_{\epsilon'}(\tilde{T}(Q,m'))$. Similarly for $E_{\epsilon}(\tilde{G}(kQ,m))$ and $w(kQ,m)$. Both gerbes are banded by $Z'$ and we want a functor in between. For reasons of functoriality, it suffices to show that $w(Q,m') \to w(kQ,m)$ under (5.7). 

By the identification made in Lemma 5.3.1, the $\tilde{T}_{sc}$-torsor Whit of generic characters in [57, §3.3] is the same for both $(kQ,m')$ and $(Q,m')$. Given each $\eta \in \text{Whit}$ (living possibly in some étale covering), we have $\omega(Q,m'|\eta) \in \text{Spl}(\mathcal{D}_Q^{sc},m)$ defined using the pinning, as reviewed in §5.2. By construction, $\omega(kQ,m)|\eta$ is nothing but its image $1 \wedge \omega(Q,m'|\eta)$ in $\text{Spl}(\mathcal{D}_k^{sc},m) \cong k_*\text{Spl}(\mathcal{D}_k^{sc},m)$. Next, for each root $\phi$,

$$m_\phi|kQ,m| := \frac{kQ(\phi)}{\gcd(m,kQ(\phi))} = \frac{k'Q(\phi)}{\gcd(m',Q(\phi))} = k'm_\phi|kQ,m|.$$ 

Hence the endomorphism $\mu|kQ,m| \in \tilde{T}_{sc}$ in [57, §3.3] equals $\mu|Q,m'| \gamma'$. It is shown in loc. cit. that $1 \wedge \eta \to \omega(Q,m'|\eta)$ yields an $m'$-equivariant morphism $\mu|Q,m'| \text{Whit} \to \text{Spl}(\mathcal{D}_Q^{sc},m)$ of $\tilde{T}_{sc}$-torsors. This furnishes the Whittaker object $w(Q,m')$ of $\sqrt[n]{\text{Spl}(\mathcal{D}_Q^{sc},m)}$; the same holds for $w(kQ,m)$. Let us study the image of $w(Q,m)$ under the lower row of (5.7) in stages.

$$\begin{array}{c}
\mu|Q,m'| \text{Whit} \xrightarrow{m'} \text{Spl}(\mathcal{D}_Q^{sc},m') \\
\xrightarrow{\gamma'} \text{Spl}(\mathcal{D}_Q^{sc},m') \\
\xrightarrow{1 \wedge \eta \frac{\mu|Q,m'|}{\gamma'}} 1 \wedge \gamma' \omega(Q,m'|\eta) \\
\xrightarrow{1 \wedge \eta \frac{\gamma'}{\gamma}} 1 \wedge \eta \omega(Q,m'|\eta) \\
\xrightarrow{\mu|kQ,m| \text{Whit} \xrightarrow{m'} \text{Spl}(\mathcal{D}_Q^{sc},m')} 1 \wedge \eta \omega(Q,m'|\eta) \\
\end{array}$$

Here $m'$, etc. over the arrows between torsors record the equivariance. This completes the proof. 

**Theorem 5.3.5.** Under Hypothesis 5.3.2, the $L$-groups $\gamma[Q,m]$ and $\gamma[G(Q,m')$ are canonically isomorphic as extensions of $\Gamma_F$ by $\tilde{G}^{\hat{\gamma}}$.

**Proof.** Combine Lemma 5.3.1 and 5.3.4. 

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6 Construction of epipelagic supercuspidals for $\widehat{\text{Sp}(W)}$

Throughout this section, $F$ stands for a non-archimedean local field of residual characteristic $p$. From §6.3 onwards, we assume $p \neq 2$ and work with $G = \widehat{\text{Sp}(W)}$; this excludes the bad and torsion primes [48, I.4] for $G$. The BD-covers $\overline{G} \rightarrow G(F)$ will have degree $m$ with $p \nmid m$, which amounts to the tameness of the cover, see [14, §4].

In §6.4, stronger constraints on $F$ and $p$ will be imposed (Hypothesis 6.4.1).

6.1 Generalities

For any reductive $F$-group $G$, denote by $B(G,F)$ (resp. $B^{\text{red}}(G,F)$) the enlarged Bruhat–Tits building (resp. the reduced one) of $G$. In what follows we assume $G$ splits over a tame ramified extension.

Given $x \in B^{\text{red}}(G,F)$, the Moy–Prasad filtration on $G(F)$ (resp. $\mathfrak{g}(F)$) is an increasing family of open compact subgroups (resp. of $\mathfrak{o}_F$-lattices), written as

$$G(F)_{x,r} := \bigcup_{s \geq r} G(F)_{x,s} \quad (r \geq 0),$$

$$\mathfrak{g}(F)_{x,r} := \bigcup_{s \geq r} \mathfrak{g}(F)_{x,s} \quad (r \in \mathbb{R}),$$

respectively. Conjugation by $g \in G(F)$ carries $G(F)_{x,r}$, $\mathfrak{g}(F)_{x,r}$ to $G(F)_{y,x,r}$, $\mathfrak{g}(F)_{y,x,r}$. Also set

$$G(F)_{x,r,s} := G(F)_{x,r}/G(F)_{x,s}, \quad \mathfrak{g}(F)_{x,r,s} := \mathfrak{g}(F)_{x,r}/\mathfrak{g}(F)_{x,s}$$

whenever $s > r$. The meanings of $G(F)_{x,0}$ and $\mathfrak{g}(F)_{x,0}$ are clear. Also recall that $G(F)_{x,0}$ is the parahoric subgroup of $G(F)$ determined by $x$, and $G(F)_{x,0}$ is its pro-unipotent radical; they are both contained in the stabilizer $G(x) := \text{Stab}_{G(F)}(x)$. On the dual side, define the filtration

$$\mathfrak{g}^*(F)_{x,-r} := \{ X \in \mathfrak{g}^*(F) : \forall Y \in \mathfrak{g}(F)_{x,r}, \quad v([X,Y]) > 0 \},$$

$$\mathfrak{g}^*(F)_{x,(r)} := \bigcup_{s < r} \mathfrak{g}^*(F)_{x,s}.$$
Assume henceforth that $S$ is tame. It admits the ft-Néron model satisfying $S(\sigma_F) = S(F)_0$. This Néron model coincides with all the other versions when $S$ is inertially anisotropic, see the explanations in [27, p.50]; in that case

$$S(F)_0 = S^0(\sigma_F).$$

In what follows, $S$ will be a maximal $F$-torus of $G$ that is anisotropic modulo $Z_G$. It determines a point $x \in \mathcal{B}^{\text{red}}(G, F)$ by [43, Remark 3]; this point being fixed by $N_G(S)(F)$, we deduce $S(F) \subset G(F)_x$. From [3, Proposition 1.9.1] we have $s(F)_x = g(F)_{x,r} \cap s(F);$ same for the dual avatars $s^*(F)_{x,r}$ and $g^*(F)_{x,r}$.

Let $\Omega(G, S)(\hat{F})$ act on $Y := X_\ast(S_F)$ and $X := X^\ast(S_F)$. Following Springer [47], we say an element $w \in \Omega(G, S)(\hat{F})$ is

- regular, if some eigenvector of $w$ acting on $Y \otimes \mathbb{C}$ has trivial stabilizer under $\Omega(G, S)(\hat{F})$;
- elliptic, if $\det (w - 1) Y \otimes \mathbb{R} \neq 0$.

The torus $S$ gives rise to continuous $\Gamma_F$-actions on the $\mathbb{Z}$-modules $Y$ and $X$.

**Definition 6.1.1** ([24, Conditions 3.3]). Assume $Z_F^2$ is anisotropic. We say that a maximal $F$-torus $S \subset G$ is of type (ER) if

(a) $S$ is tame, and

(b) the $I_F$-action on $Y$ is generated by an elliptic regular element of $\Omega(G, S)(\hat{F})$.

It follows that $S$ is inertially anisotropic, and the foregoing discussions are applicable. When the $I_F$-action is generated by a Coxeter element, such $S$ is said to be of type (C) in [27].

**Lemma 6.1.2.** Suppose $S$ is an inertially anisotropic $F$-torus. We have

$$S(F)_0 = S(F)_{0+} = S(F)_{1/e} = S(F)_{tu}.$$  

Consequently, (6.1) becomes $S(F) = S(F)_{0'} \times S(F)_{1/e}$ and $S(F)_{1/e}$ is the pro-$p$ part of $S(F)$.

**Proof.** Clearly we have $S(F)_0 \supset S(F)_{0+} \supset S(F)_{1/e}$. If $\gamma \in S(F)_{tu}$ then $v(\chi(\gamma) - 1) \geq \frac{1}{2}$ for all $\chi$, as one sees over the splitting field, thus by [25, (3.1)] we have $S(F)_{1/e} \supset S(F)_{tu}$ as well. It remains to show that $S(F)_0 \subset S(F)_{tu}$.

By [25, Lemma 3.1.3] we have $S(F)_0 = S(F)_{0+}$, since $Y^{I_F} = \{0\}$. Hence every element in $S(F)_0$ is pro-$p$ and must belong to $S(F)_{tu}$ as desired. 

### 6.2 Tori in covers

Assume $\text{char}(F) \neq 2$, $n \in \mathbb{Z}_{\geq 1}$ and specialize the formalism in §6.1 to $G := \text{Sp}(W)$ where $\dim_F W = 2n$. Let $S \subset G$ be a maximal $F$-torus and label the short coroots of $S_F \subset G_F$ by $\pm \epsilon_1, \ldots, \pm \epsilon_n \in Y$ as in §3.1. Given $w \in \Omega(G, S)(\hat{F})$, we decompose $\{\pm \epsilon_i : 1 \leq i \leq n\}$ into $\langle w \rangle$-orbits. In parallel with Definition 3.3.1, we say a $\langle w \rangle$-orbit $O$ of short coroots is symmetric if $O = -O$, otherwise asymmetric.

**Lemma 6.2.1.** Assume that $S$ is a tame and inertially anisotropic, for example when $S$ is of type (ER) in $G$. Then $x^2 = 1$ for all $x \in S(F)_{\ast}$.

**Proof.** It follows from Lemma 6.1.2 that $S(F)_{\ast} \cong S(F)/S(F)_0$. It suffices to show that every continuous homomorphism $\theta : S(F) \to \mathbb{C}^\times$ that is trivial on $S(F)_0$ satisfies $\theta^2 = 1$. Let $\hat{S}$ be the $\mathbb{C}$-torus dual to $S$. By [25, Lemma 3.1.5], the $L$-parameter $\phi \in H^1(W_F, \hat{S})$ of $\theta$ is inflated from $H^1(W_F/I_F, \hat{S}^{I_F})$. Hence it suffices to show that $\hat{S}^{I_F}$ is a 2-torsion group.

We contend that the action of $I_F$ on $\hat{S}$ contains an elliptic element $w$ from $\Omega(G, S)(\hat{F})$. Indeed, by writing $S_F = gT_Fg^{-1}$ with $g \in G(\hat{F})$, where $T$ is a split maximal $F$-torus, we see that $\Gamma_F$ acts through $\Omega(G, S) = g\Omega(G, T)g^{-1}$. Since $\hat{S} = \hat{S}^{I_F}$ and $I_F/P_F$ is pro-cyclic, the action of $I_F$ is generated by some $w \in \Omega(G, S)(\hat{F})$, which must be elliptic since $S$ is inertially anisotropic.

It remains to show that $\hat{S}^{w=1}$ is a 2-torsion group. By decomposing the short coroots into $\langle w \rangle$-orbits, ellipticity is equivalent to that every orbit is symmetric; cf. the proof of Lemma 3.3.2. It is now clear that $\hat{S}^{w=1} = \mu_2$. 

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**Theorem 6.2.2.** Assume \( p \neq 2 \). Choose a parameter \((K,K^2,\ldots)\) for a maximal \( F\)-torus \( S \subset G = \text{Sp}(2n)\) of type \((\text{ER})\) (see Definition 6.1.1), \( K = \prod_{i \in I} K_i \). Identify \( S \) with \( \prod_{i \in I} R_{K_i^1/F}(K_i^1) \). Then

\[
S(F)_{\mu'} = \prod_{i \in I} \{ \pm 1 \} \subset \prod_{i \in I} K_i^1.
\]

Furthermore, each \( K_i/K_i^1 \) is a tamely ramified quadratic extension of fields.

**Proof.** Since \( p \neq 2 \), we have \( \{ \pm 1 \}^I \subset S(F)_{\mu'} \). The reverse inclusion follows from Lemma 6.2.1 since the 2-torsion part of each \( K_i^1 \) is \( \{ \pm 1 \} \).

Each \( K_i \) is a field since \( S \) is anisotropic (Lemma 3.2.6). Let \( q_i \) be the cardinality of the residue field of \( K_i^1 \). If \( K_i \) is unramified over \( K_i^1 \), there will be an embedding \( \mathbb{Z}/(q_i - 1) \mathbb{Z} \hookrightarrow K_i^1 \) with image in \( S(F)_{\mu'} \) since \( p \nmid q_i \). This contradicts the first part.

In what follows, we fix \( m \mid N_F \) with \( p \nmid m \). Consider the BD-cover constructed in §4.1

\[
1 \to \mu_m \to \tilde{G} \xrightarrow{\mu} G(F) \to 1.
\]

We fix an embedding \( \epsilon : \mu_m \xrightarrow{\sim} \mu_m \subset \mathbb{C}^\times \) and identify \( \tilde{G} \) with a topological central extension of \( G(F) \) by \( \mu_m \).

Let \( S \subset G \) be a maximal \( F\)-torus of type \((\text{ER})\). Fix a parameter \((K,K^2,\ldots)\) for \( S \), with the usual decomposition \( K_i = \prod_{l \in I} K_i \), etc. Write \( p : \tilde{S} \to S(F) \) for the pull-back of \( p \) to \( S(F) \). Recall from Lemma 6.1.2 that \( S(F) = S(F)_{\mu'} \times S(F)_{\mu} \), and \( S(F)_{\mu} = S(F)_{\mu} \).

**Lemma 6.2.3.** The covering \( \tilde{S} \to S(F) \) splits uniquely over \( S(F)_{\mu} \), and all elements from \( S(F)_{\mu} \) is good in \( \tilde{S} \).

**Proof.** This follows from Lemma 2.4.9 and \( p \mid m \), since \( S(F)_{\mu} = S(F)_{\mu} \) is a \( p \)-group.

It remains to study the covering over \( S(F)_{\mu'} \), whose structure is described by Theorem 6.2.2.

**Proposition 6.2.4.** When \( m \notin 2\mathbb{Z} \), the covering \( \tilde{S} \to S(F) \) splits uniquely.

**Proof.** In view of Lemma 6.2.3, the assertion follows from that \( S(F)_{\mu'} \) is of 2-torsion whilst \( 2 \nmid m \). An explicit section of \( p \) over \( S(F)_{\mu'} \) is given by \( \gamma \mapsto \bar{\eta}^m \), where we take the \( \eta \in S(F)_{\mu'} \) with \( \bar{\eta}^m = \gamma \), and \( \bar{\eta} \in p^{-1}(\eta) \) is arbitrary.

The case \( m \in 2\mathbb{Z} \) requires more sophisticated constructions. As a first step, we prove the commutativity of \( \tilde{S} \). Refined results will be given in §7.1.

**Proposition 6.2.5.** The group \( \tilde{S} \) is commutative. In particular, all elements from \( S_{\text{reg}}(F) \) are good in \( \tilde{G} \).

**Proof.** Decompose \( S = \prod_{i \in I} R_{K_i^1/F}(K_i^1) \) using parameters. Recall from Theorem 4.1.3 that \( S \subset G^S \) and \( p : \tilde{G}^S \to G^S(F) \) is isomorphic to the contracted product of topological central extensions \( \mu_m \hookrightarrow \text{SL}(2,K_i^1) \xrightarrow{\sim} \text{SL}(2,K_i^2) \). We are thus reduced the case \( G = \text{SL}(2) \), \( F = K^2 \). By Lemma 6.2.3, it suffices to show that the preimage of \( S(F)_{\mu'} = \{ \pm 1 \} \) is commutative.

To show this, one may push-out the covering by \( \mu_m \hookrightarrow \mathbb{C}^\times \). The result follows since any topological central extension \( 1 \to \mathbb{C}^\times \to C \to \{ \pm 1 \} \to 1 \) splits.

We conclude that the irreducible \( \epsilon \)-genuine representations of \( \tilde{S} \) are continuous characters. As \( \epsilon \) is fixed throughout, we shall abbreviate \( \epsilon \)-genuine as genuine.
6.3 Compact induction

Assume $p \neq 2$. Fix an additive character $\Lambda : \kappa_F \to \mathbb{C}^\times$. Let $G = \text{Sp}(W)$ and $p : \tilde{G} \to G(F)$ be as in §6.2; thus $p \nmid m$. We begin by reviewing [24, §3.2].

Let $S \subset G$ be a tame anisotropic maximal $F$-torus, and let $x \in B^{\text{red}}(G, F)$ be the point its determines. Let $L$ be the splitting field of $S_{\text{red}}$, with residue field $\kappa_L$.

Take $r \in \mathbb{R}_{>0}$. To a continuous character $\theta : S(F) \to \mathbb{C}^\times$ such that $\theta|_{S(F)_r} = 1$, we attach a $\kappa_L$-line $\ell_\theta \subset \mathfrak{g}^*(L)_{x,0:0+}$ as follows. The MP : $S(F)_{r,r+} \cong \mathfrak{g}(F)_{r,r+}$ and $\theta|_{S(F)_r}$ gives rise to a character $\theta_r$ of the $\kappa_F$-vector space $\mathfrak{g}(F)_{r,r+}$. Identify $\theta_r$ with the $Y^* \in \mathfrak{s}^*(F)_{r,-r}^{(+)}$ such that $\Lambda(Y^*, -) = \theta_r$. As $S_L$ splits, there exists $z \in L^\times$ such that $zY^* \in \mathfrak{s}^*(L)_{0:0+}$, and the $\kappa_L$-line obtained from by $zY^*$ depends only on $\theta_r$. The line $\ell_\theta$ is obtained via the inclusion $s^* = (\mathfrak{g}^*)^0 \hookrightarrow \mathfrak{g}^*$, which induces $s^*(L)_{0:0+} \hookrightarrow \mathfrak{g}^*(L)_{x,0:0+}$.

Following [24], call a continuous character $\theta : S(F) \to \mathbb{C}^\times$ generic of depth $r$ if

1. the restriction of $\theta$ to $S(F)_{r,+}$ (resp. to $S(F)_r$) is trivial (resp. non-trivial);
2. the $\kappa_L$-line $\ell_\theta$ attached to $\theta|_{S(F)_r}$ is strongly regular semisimple.

This notion depends solely on $\theta|_{S(F)_r}$, and $r$ must be a jump for the filtration on $S(F)$. We proceed to adapt it to the cover $p_m : \tilde{G} \to G(F)$ and $\tilde{S} \to S$, where $S \subset G$ is of type (ER). By Lemma 6.3 Compact induction, the covering splits uniquely over $S(F)_0 = S(F)_{1/e}$, where $e$ stands for the ramification degree of the splitting extension of $S$.

**Definition 6.3.1.** Let $S \subset G$ be a maximal $F$-torus of type (ER). Call a continuous genuine character $\theta : \tilde{S} \to \mathbb{C}^\times$ generic of depth $r > 0$ if

1. the restriction of $\theta$ to $S(F)_{r,+}$ (resp. to $S(F)_r$) via the unique splitting over $S(F)_0$ is trivial (resp. non-trivial);
2. the $\kappa_L$-line $\ell_\theta$ attached to $\theta|_{S(F)_r}$ is strongly regular semisimple.

Call a continuous genuine character $\theta : \tilde{S} \to \mathbb{C}^\times$ epipelagic if it is generic of depth $1/e$.

By Lemma 6.1.2, the first jump of the Moy–Prasad filtration on $S(F)$ occurs at $r = 1/e$. Hence the minimal possible depth is attained by epipelagic characters. Assume $\theta : \tilde{S} \to \mathbb{C}^\times$ to be genuine epipelagic. We proceed to construct supercuspidals as follows.

1. Decompose $\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{n}$, where $\mathfrak{n}$ is the direct sum of nontrivial isotypic components under $S$. Then for all $r \geq 0$ we have

   $$g(F)_{x,r} = g(F)_r \oplus n(F)_{x,r},$$

where $n(F)_{x,r} := g(F)_{x,r} \cap n(F)$. Cf. [3, Proposition 1.9.3]. From now onwards $r := 1/e$. The isomorphisms MP : $g(F)_{r,r+} \cong S(F)_{r,r+}$ and $g(F)_{x,r,r+} \cong G(F)_{x,r,r+}$ allow us to extend $\theta|_{S(F)_r}$ to a character $\hat{\theta}_0$ of the group $G(F)_{x,r,r+}$.

2. Let $V_{x,r} := g(F^{nr})_{x,r,r+}$, which is a $\varpi_F$-vector space with a descent datum to $\kappa_F$ such that $V_{x,r}(\kappa_F) = g(F)_{x,r,r+}$. Let $\lambda \in V_{x,r}$ be the linear functional determined by

   $$\Lambda((\lambda, -)) = \text{the composite } V_{x,r}(\kappa_F) \xrightarrow{\text{MP}} G(F)_{x,r,r+} \xrightarrow{\hat{\theta}_0} \mathbb{C}^\times.$$  

Recall that $S(F) \subset G(F)_x$. By [24, Proposition 3.4],

$$\text{Stab}_{G(F)_x}(\lambda) = S(F)G(F)_{x,r},$$

and

$$\lambda \in V_{x,r} \text{ is a stable vector for the } G(F^{nr})_{x,0:0+}\text{-action}. \quad (6.2)$$

We infer that $(G(F)_{x,r}, \lambda)$ affords an unrefined minimal K-type [40, 3.4] for $\pi_{S,g}$; stability here is understood in the sense of geometric invariant theory.

By Lemma 2.4.9, $p$ splits uniquely over $G(F)_{x,r}$. As $S(F)$ normalizes $G(F)_{x,r}$, one can define the genuine continuous character

$$\hat{\theta} : \tilde{S} \cdot G(F)_{x,r} = p^{-1}(\text{Stab}_{G(F)_x}(\lambda)) \hookrightarrow \mathbb{C}^\times$$

$$\hat{\gamma} \cdot g \mapsto \theta(\hat{\gamma})\hat{\theta}_0(g)$$
where \( \hat{\gamma} \in \hat{S}, g \in G(F)_{x,r} \). By Lemma 6.1.2 we may also write \( \hat{\theta} \) as
\[
\hat{\theta} : \overline{S(F)}_{p'} \times G(F)_{x,r} \longrightarrow \mathbb{C}^{	imes}
\]
where \( \overline{S(F)}_{p'} := p^{-1}(S(F))_{p'} \). Notice that \( \hat{S} \cdot G(F)_{x,r} \) is a compact open subgroup of \( \hat{G} \).

4. Now we can build a smooth representation from \( \overline{S(F)}_{p'} \) via compact induction
\[
\pi_{\hat{S},\hat{\theta}} := c\text{-Ind}_{\overline{S(F)}_{p'}}^{\overline{S(F)}_{1/e}}(\hat{\theta}).
\]

(6.3)

It is clear that \( \pi_{\hat{S},\hat{\theta}} \) contains the unrefined minimal \( K \)-type \( (G(F))_{x,r} \).

The construction of \( \pi_{\hat{S},\hat{\theta}} \) is a covering version of [45], therefore can be baptized as a genuine epipelagic supercuspidal representation for the following reason.

**Theorem 6.3.2.** Let \( \theta : \hat{S} \rightarrow \mathbb{C}^{	imes} \) be an epipelagic genuine character. The representation \( \pi := \pi_{\hat{S},\hat{\theta}} \) in (6.3) is irreducible and supercuspidal. When \( S \) and \( \theta|_{S(F)_{1/e}} \) are fixed, different choices of \( \theta|_{S(F)} \) lead to non-isomorphic \( \pi \).

**Proof.** Since \( \hat{\theta} \) is genuine, so is \( \pi \). Irreducibility and the determination of \( \theta|_{S(F)} \) follow by plugging these data into [45, §2.1] with \( H := \overline{G(F)}_{x}, J := G(F)_{x,1/e} \) and \( A_{\theta} \simeq \hat{S(F)}_{p'} \); see also the Remark 1 in loc. cit. More precisely, let \( \theta_{\lambda} \) be the character of \( S(F)_{1/e} \) attached to \( \lambda \in \mathbb{V}_{x,r} \); the key ingredient [45, (2.6)] is the property that for all \( g \in G(F) \),
\[
\theta_{\lambda}|_{Jg^{-1}Jg} = \theta_{\lambda} \circ \text{Ad}(g)|_{Jg^{-1}Jg} \implies g \in G(F)_{x},
\]

which has nothing to do with coverings, since the splittings of \( p \) over subgroups of \( J \) are unique.

The cuspidality of \( \pi \) is well-known, see for instance [8, 11.4]. □

**Remark 6.3.3.** The construction of \( \pi = \pi_{\hat{S},\hat{\theta}} \) is also a special case of Yu’s construction [61] of tame supercuspidals, adapted to the case of coverings with \( p \nmid m \), attached to the following quintuple (with length \( d = 1 \))
\[
\hat{G} := (G_{0} = S, G_{1} = G), \text{ or rather their preimages in } \hat{G}, \quad \text{(i.e. a toral datum)}
\]
\[
x \in B(S, F) \hookrightarrow \mathbb{B}^{\text{ed}}(G, F), \text{ cf. [43]},
\]
\[
\gamma := \left( r_{0} = \frac{1}{\varepsilon}, r_{1} = \frac{1}{\varepsilon} \right),
\]
\[
\rho := \theta|_{S(F)_{p'}}, \text{ inflated to a genuine character of } \hat{S} = \overline{S(F)}_{p'} \times S(F)_{1/e},
\]
\[
\tilde{\phi} := (\phi_{0}, \phi_{1} = 1), \quad \phi_{0} := \theta|_{S(F)_{1/e}}, \text{ inflated to a character of } S(F) = S(F)_{p'} \times S(F)_{1/e}.
\]

See [61, §3] as well as the explanations in [19, §3.1]; in the notation of the latter reference we have \( \pi_{-1} = \rho \) and \( \pi_{0} = \rho \circ \rho_{0} = \theta \). The choice of \( \tilde{\phi} \) and \( \rho \) is not unique, however.

**Remark 6.3.4.** We say \( (\hat{S}_{1}, \hat{\theta}_{1}) \) and \( (\hat{S}_{2}, \hat{\theta}_{2}) \) are conjugate if there exists \( g \in G(F) \) such that \( \hat{S}_{2} = g \hat{S}_{1} g^{-1} \) and \( \hat{\theta}_{2} \circ \text{Ad}(g) = \hat{\theta}_{1} \), where \( (\hat{S}_{i}, \hat{\theta}_{i}) \) stands for maximal tori and epipelagic genuine characters \( i = 1, 2 \).

In view of the case of reductive groups [24, Fact 3.8], one expects that \( \pi_{\hat{S}_{1}, \hat{\theta}_{1}} \simeq \pi_{\hat{S}_{2}, \hat{\theta}_{2}} \) if and only if \( (\hat{S}_{1}, \hat{\theta}_{1}) \) and \( (\hat{S}_{2}, \hat{\theta}_{2}) \) are conjugate. The “if” direction is evident. To establish the other direction, one has to address the issue of uniqueness of Yu’s data (Remark 6.3.3) for \( \pi \). I For reductive groups, this uniqueness follows from Hakim–Murnaghan theory [19, Corollary 6.10]. Their results should carry over to \( \hat{G} \) and its maximal tori of type (ER), and the “only if” part would follow. Cf. the discussions in [60, §3.5], especially for the case \( m = 2 \). This is expected to follow from an ongoing project of Ju-Lee Kim and Wee Teck Gan.

### 6.4 The Adler–Spice character formula

Keep the formalism of 6.3. We also fix an additive character \( \xi : F \rightarrow \mathbb{C}^{	imes} \) which restricts to an additive character \( \Lambda : \kappa_{F} \rightarrow \mathbb{C}^{	imes} \). The assumptions below are to be imposed.
Hypothesis 6.4.1. Assume that

P.1 char\((F) = 0\).

P.2 \(p \geq (2 + e(F/\mathbb{Q}_p))n(G)\), where \(n(G)\) is the dimension of some faithful \(F\)-rational representation of \(G\); in particular \(p > 2\).

P.3 \(p \nmid m\) as usual, where \(\mu_m = \ker(p: \tilde{G} \rightarrow G(F))\).

We remark that P.2 is required for the exponential and logarithm maps in the character formula, see [10, Appendix B]. The assumptions above make sense for any group \(G\).

Let \(\pi = \pi_{\tilde{G}, \theta}\) be the genuine irreducible supercuspidal representation constructed in Theorem 6.3.2. Denote its character by \(\Theta\). We state the Adler–Spice character formula (see the discussions in §6.1) also applies to \(\tilde{G}\): it yields a unique expression

\[\tilde{\gamma} = \tilde{\gamma}_0 \tilde{\gamma} > 0 = \tilde{\gamma}_0 \tilde{\gamma}_0\]

with \(\tilde{\gamma}_0\) of finite order prime to \(p\), and \(\tilde{\gamma} > 0\) is topologically unipotent. Taking images by \(p\) yields the topological Jordan decomposition \(\gamma = \gamma_0 \gamma > 0 = \gamma_0 \gamma_0\) in \(G(F)\). The following easy result says that we can lift \(\gamma > 0\) uniquely to \(\tilde{G}\), which is exactly \(\tilde{\gamma}_0 \tilde{\gamma}\). Hence we can simply write \(\tilde{\gamma} = \tilde{\gamma}_0 \tilde{\gamma} = \gamma_0 \gamma\) for the decomposition of \(\tilde{\gamma}\).

Lemma 6.4.3. Let \(\tilde{G} \rightarrow G(F)\) be any topological covering with \(\ker(p) = \mu_m\) and \(p \nmid m\). Given a topologically unipotent element \(\gamma > 0\) of \(G(F)\), there exists a unique \(\tilde{\gamma}_0 \in p^{-1}(\gamma > 0)\) that is topologically unipotent.

Proof. For the uniqueness, note that if \(\tilde{\gamma}_0\) and \(\tilde{\gamma} > 0\) are both topologically unipotent, then \(\tilde{\gamma}^n \rightarrow 1\), which implies \(\epsilon = 1\) since \(p \nmid m\).

To show existence, take the topological Jordan decomposition of any preimage of \(\gamma > 0\), say \(\tilde{\gamma}_0 \tilde{\gamma} > 0 = \gamma_0 \gamma_0\). Since its image is topologically unipotent, we must have \(p(\tilde{\gamma}) = 1\) and \(\tilde{\gamma}_0 \tilde{\gamma}\) is the required lifting of \(\gamma > 0\).

Also note that \(\gamma_0\) is semi-simple and \(\gamma_0 \in J_{\mathrm{reg}}(F)\), where \(J := G_{\mathrm{nr}}\).

It is shown in [12] that the character of any supercuspidal representation of \(\tilde{G}\) is supported on compact elements. Moreover, \(\Theta_{\pi|_{\tilde{G}_{\mathrm{reg}}}}\) vanishes off the good locus since it is an invariant genuine function. Let us state the Adler–Spice character formula [4, Theorem 7.1] for \(\pi\), rephrased as in [24, (6.1)]; the only difference is that we work with covering groups. Several definitions are in order.

- Condition P.2 implies that we have the exponential map \(g(F)^{nr}_{0+} \rightarrow G(F)^{nr}_{0+}\) that is a \(G(F)^{nr}\) and \(F\)-equivariant homeomorphism, whose inverse we denote by \(\log\); see [27, p.57] or [10, Appendix B].

- Now take \(Y \in \mathfrak{s}^*(F)_{-1/\epsilon}\) satisfying \(\theta \circ \exp = \xi((Y, \cdot)) : \mathfrak{s}(F)_{1/\epsilon} \rightarrow \mathbb{C}^\times\). Only the coset \(Y + \mathfrak{s}^*(F)_{0}\) is well-defined.

- It is sometimes useful to identify \(g, g^*, \mathfrak{s}\), and \(\mathfrak{s}^*\). To achieve this, we use a non-degenerate invariant bilinear form \(B_\mathfrak{s}\) on \(\mathfrak{s}\) such that for some (thus for all) maximal \(F\)-torus \(T\) and every \(H_\alpha := \text{d} \alpha(1)\) over \(F\), we have \(v(B_\mathfrak{s}(H_\alpha, H_\alpha)) = 0\). For classical groups \(G\) inside \(GL(W)\), such as \(G = \text{Sp}(W)\), we follow [60, §2.1.1] to take

\[
\mathbb{B}(X_1, X_2) := \frac{1}{2} \text{tr}(X_1 \cdot X_2|W) , \quad X_1, X_2 \in \mathfrak{s}.
\] (6.4)

This corrects the earlier [36, Definition 6.3.1.1] by a sign, as kindly explained to the author by H. Y. Loke (private communication).

Using \(\mathbb{B}\), one can identify the dual of \(g(F)^{x,t,t+}\) with \(g(F)^{x,(-t)(-t)+}\) for all \(x\) and \(t \in \mathbb{R}\), same for \(\mathfrak{s}\); see [10, Lemma A.1.1]. In particular, we may view \(Y\) as an element of \(\mathfrak{s}(F)^{-1/\epsilon}\). The following result guarantees the regularity of \(Y\) in \(\mathfrak{g}\).
Proposition 6.4.4. Let \( v_F \) be the valuation of \( \tilde{F} \) extending \( v_F \). For \( Y \) as above, we have \( v_F(\, \text{d}\alpha(Y)) = \frac{-1}{2} \) for any root \( \alpha \) of \( S_F \subset G_F \). In particular \( Y \in \mathfrak{g}(F)_{-1/\epsilon} \cap \mathfrak{g}_{\text{reg}}(F) \), and each eigenvalue \( \lambda \) of \( Y \) (as an element of \( \mathfrak{sp}(W) \subset \mathfrak{gl}(W) \)) satisfies \( v_F(\lambda) = -\frac{1}{2} \).

Proof. In view of (6.2), this is just [36, Lemma 7.3.1] or [24, Lemma 3.2]. To deduce the assertion on \( v_F(\lambda) \), consider the long roots \( \alpha \).

Notation 6.4.5. For any reductive \( F \)-group \( J \), a maximal torus \( S \subset J \) together with \( Z \in \mathfrak{g}_{\text{reg}}(F) \) (i.e. \( J_Z = S \)), we define the unnormalized orbital integral

\[
\mu^J_Z : J^* \longmapsto \int_{S(F) \backslash J(F)} f^*(g^{-1}Zg) \, dg, \quad f^* \in C_c^\infty(J^*(F))
\]

and its Fourier transform

\[
\hat{\mu}^J_Z(f) = \hat{\mu}^J_Z(f), \quad \hat{f}(X) = \int_{j(F)} f(X')\xi(X',X) \, dX'
\]

for all \( f \in C_c^\infty(j(F)) \); here we adopt the Haar measures in [25, §4.2]. Set

\[
D^J(Z) := \prod_{\alpha \text{ root} \atop \text{d}\alpha(Z) \neq 0} \text{d}\alpha(Z),
\]

\[
i^J(Z,X) := |D^J(Z)|^{\frac{1}{2}} \cdot |D^J(X)|^{\frac{1}{2}} \cdot \hat{\mu}^J_Z(X), \quad (Z, X) \in j^*_\text{reg}(F) \times j^*_\text{reg}(F).
\]

This is well-defined since \( \hat{\mu}^J_Z \) is representable by a locally integrable function over \( j(F) \), smooth over \( j^*_\text{reg}(F) \) by [20]. It is \( J(F) \)-invariant in both variables.

Theorem 6.4.6 (Cf. [24, (6.1)] or [25, §4.3]). Let \( \bar{\gamma} \in \bar{G}_{\text{reg}} \) be a compact element with topological Jordan decomposition \( \bar{\gamma} = \bar{\gamma}_0\bar{\gamma}_{>0} \). Write \( \gamma, \gamma_0 \) for their images in \( G(F) \) and put \( J := G_{\gamma_0} \). Then

\[
|D^G(\gamma)|^{\frac{1}{2}} \Theta_\pi(\bar{\gamma}) = \sum_{g \in J(F) \backslash G(F) / S(F)} \theta(g^{-1}\gamma_0g)\hat{i}^J(gYg^{-1}, \log \gamma_{>0}).
\]

Proof. It is unrealistic to reproduce [4] here, so we only indicate the following ingredients.

1. Since \( S \) is a maximal torus,

\[
g^{-1}\gamma_0g \in S(F) \iff \text{Ad}(g)(S) \subset G_{\gamma_0} = J.
\]

Thus \( gYg^{-1} \in j^*_\text{reg}(F) \) and \( \hat{i}^J(gYg^{-1}, \cdot) \) makes sense.

2. There are no Weil representations in Yu’s construction [61] for the epipelagic case (cf. Remark 6.3.3), which simplifies enormously the character formula. In fact, the complex fourth roots of unity (summarized in [25, §4.3]) in the formula of Adler–Spice disappear here.

3. Since we are in the case of positive depth \( r \), by Lemma 2.4.9 \( G(F)_{x,r} \) splits uniquely in \( \bar{G} \).

4. In view of the description of Yu’s data in Remark 6.3.3, the normal approximation in [4, 25] reduces to the topological Jordan decomposition, which works for \( \bar{G} \) as explained above.

5. Basic results in harmonic analysis on coverings such as (a) semisimple descent, (b) local integrability, smoothness and local expansions of \( \Theta_\pi \), and (c) properties of orbital integrals, have been established in [34].

6. All elements from \( S(F)_{x,r} \) are good by Theorem 6.2.2 + Remark 4.3.2.

The arguments should actually be simpler since the mock-exponential maps in [4] are avoided by P.2.
7 The L-packets

We shall work with a local field $F$ with char($F$) ≠ 2, and a symplectic $F$-vector space $(W, ⟨·|·⟩)$ with dim$_F W = 2n$. Set $G = \text{Sp}(W)$ and let $p : \hat{G} \to G(F)$ be the BD-cover defined in §4.1 with kernel $\mu_m$, where $m \mid N_F$. We also fix $\varepsilon : \mu_m \to \mu_m \subset \mathbb{C}^\times$ in order to talk about genuine representations of $\hat{G}$, etc. When $m = 2 \pmod{4}$, we also fix an additive character $\psi : F \to \mathbb{C}$. The class of multiplicative $K_2$-torsors over $G$ under consideration is the same as in §4. For the dual side, this limitation is justified by Theorem 5.3.5.

From §7.3 onwards, $F$ will be non-archimedean of residual characteristic $p \nmid 2m$. In §7.6, the conditions from §6.4 will be in force.

7.1 Convenient splittings

The first task is to prescribe a preimage of $-1$ in the $K_2(F)$-torsor $E_G(F) \to G(F)$ using the symplectic form $⟨·|·⟩$. Pick a symplectic basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}$ of $W$ and define $B = TU$ as in §3.1, with opposite $B^- = TU^-$. For every root $\alpha$ of $T$, let $U_{\alpha}$ be the root subgroup. The symplectic basis gives rise to a standard pinning for $G$: for each simple $\alpha$ we are given an isomorphism $x_{\alpha} : \mathbb{G}_{a} \to U_{\alpha}$; concurrently we have $x_{-\alpha} : \mathbb{G}_{a} \to U_{-\alpha}$, see [17, Exp XXIII, 1.2]. They correspond to $X_{\pm \alpha} := dx_{\pm \alpha}(1)$ and $x_{-\alpha}$ is characterized by

$$[X_{\alpha}, X_{-\alpha}] = H_{\alpha} := d\alpha(1);$$

see [17, Exp XX, Corollaire 2.11]. Now set

$$w_{\alpha}(t) := x_{\alpha}(t)x_{-\alpha}(-t^{-1})x_{\alpha}(t), \quad t \in \mathbb{G}_{a};$$

this gives a representative of the root reflection with respect to $\alpha$. Recall that $E_G \to G$ splits canonically over $U$ and $U^-$ by [7, Proposition 3.2]. Hence one may view $w_{\alpha}(t)$ as elements of $E_G(F)$ for all $t \in F$. So it makes sense on the level of $E_G$ to define

$$h_{\alpha}(t) := w_{\alpha}(t)w_{\alpha}(-1). \quad (7.1)$$

We will only need to deal with positive long roots $\alpha = 2e_i$. Since it involves only $e_{\pm i}$ in the symplectic basis, one can calculate inside $\text{SL}(2)$ to see that $H_{\alpha} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and

$$x_{\alpha}(t) = \begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}, \quad x_{-\alpha}(t) = \begin{pmatrix} 1 & t \\ 1 & 1 \end{pmatrix}, \quad w_{\alpha}(t) = \begin{pmatrix} -1 & t \\ -t & 1 \end{pmatrix}, \quad h_{\alpha}(t) = \begin{pmatrix} t \\ -t \end{pmatrix}. $$

Note that if we pass to the basis $e_{-1}, \ldots, e_{-n}, e_{n+1}, \ldots, e_{2n}$ by conjugation in $G(F)$, namely by $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ in copies of $\text{SL}(2)$, then $x_{\alpha}(t) = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ will become $x_{-\alpha}(-t) = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Lemma 7.1.1. Denote by $\alpha$ a positive long root of $T$.

(i) We have $w_{\alpha}(1) = w_{-\alpha}(-1), w_{\alpha}(1)w_{\alpha}(-1) = 1$ and $h_{\alpha}(-1) = w_{\alpha}(-1)^2 = w_{\alpha}(1)^{-2}$ in $E_G(F)$.

(ii) When $\alpha$ ranges over the positive long roots of $T$, the elements $h_{\alpha}(-1)$ form a commuting family in $E_G(F)$. Their product

$$\prod_{\alpha \text{long root} \atop \alpha > 0} h_{\alpha}(-1)$$

is a preimage of $-1 \in G(F)$.

(iii) The element $\prod$ depends only on $(W, ⟨·|·⟩)$, not on the choice of symplectic bases.

Proof. For (i), notice that the elements $e^{+} := x_{\alpha}(1), -e^{-} := x_{-\alpha}(-1)$ and $v := e^{+}e^{-}e^{+}$ of $G(F)$ are in Tits’ trijection in the sense of [7, 11.1]. It is shown in [7, p.73] that $e^{+}e^{-}e^{+} = e^{-}e^{+}e^{-}$ holds on the level of $E_G(F)$, which amounts to $w_{\alpha}(1) = w_{-\alpha}(-1)$; the equation $w_{\alpha}(1)w_{\alpha}(-1) = 1$ results from the definitions. The third equality in (i) follows immediately.
For (ii), apply Lemma 4.1.1 to \( W = \bigoplus_{i=1}^{n} (e_i, e_{-i}) \) to deduce commutativity. It follows that \( -1 \) is well-defined and maps to \( -1 \in G(F) \).

The symplectic bases of \( W \) form a single \( G(F) \)-orbit; changing them amounts to replacing \( -1 \) by a \( G(F) \)-conjugate. Proposition 2.2.3 asserts that \( -1 \) is central in \( E_{G}(F) \). This proves (iii).

Lemma 7.1.2. The element \( -1 \) in \( E_{G}(F) \) constructed in Lemma 7.1.1 satisfies

\[
(-1)^2 = (-1, -1)^n_F.
\]

Proof. In view of Lemma 7.1.1, it suffices to deal with the case \( n = 1 \) and calculate inside \( SL(2) \). By the construction of \( E_{G}(F) \) in [37, §5], we know that \( h_\alpha(t)h_\alpha(t') = \{ t, t' \}_F \) for all \( t, t' \in F^\times \).

Henceforth we consider the BD-cover \( \mu_{m} \hookrightarrow \tilde{G} \to G(F) \) deduced from \( E_{G}(F) \) and \( \epsilon \). Denote again by \( -1 \) the image of the element in Lemma 7.1.1 in the push-out \( \tilde{G} \). It satisfies \( (-1)^2 = (-1, -1)_{F,m} \) (omitting \( \epsilon \)), which is \( \pm 1 \) by what follows.

Lemma 7.1.3. For any \( c \in F^\times \), we have

\[
(-1, c)_{F,m} = \begin{cases} 
1, & m \not\equiv 2 \pmod{4} \\
(-1, c)_{F,2}, & m \equiv 2 \pmod{4}
\end{cases}
\]

where \( \zeta \) is any generator of \( \mu_{m} \). It belongs to \( \mu_2 \) and depends solely on \( cF^\times \).

Proof. Bi-multiplicativity implies \( (-1, c)_{F,m} \in \mu_{2} \cap \mu_{m} \), hence \( (-1, c)_{F,m} = 1 \) for \( m \not\equiv 2 \pmod{4} \). When \( m \equiv 2 \pmod{4} \) (resp. \( 4 \mid m \)), use \( (-1)^{m/2} = -1 \) (resp. \( \zeta^{m/2} = -1 \)) and (4.5) to reduce to \( (-1, c)_{F,2} \).

Define 7.1.4. Set \( \tilde{G} \to G(F) \) to be the push-out of \( \tilde{G} \) via \( \mu_{m} \hookrightarrow \mu_{\operatorname{ad}(4, m)} \) if \( m \not\equiv 2 \pmod{4} \), otherwise set \( \tilde{G} = \tilde{G} \). The next step is to define a preimage \( \tilde{1} = 1 \in \tilde{G} \) such that \( (-1)^2 = 1 \), using Lemma 7.1.3.

- The case \( m \equiv 2 \pmod{4} \). Take \( -1 := -1 \). This works when \( m \not\equiv 2 \pmod{4} \) by Lemma 7.1.3. When \( 4 \mid m \), the same Lemma yields \( (-1, -1)_{F,m} = (\zeta, -1)_{F,2} = (\zeta, \zeta)_{F,2}^{m/2} \), which still equals 1.

- The case \( m \equiv 2 \pmod{4} \). We have a quadratic character \( (-1, \cdot)_{F,m} : F^\times \to \mu_2 \). The local root number \( \epsilon (\frac{1}{2}, (-1, \cdot)_{F,m}; \psi) \) lies in \( \mu_4 \) and satisfies

\[
\epsilon \left( \frac{1}{2}, (-1, \cdot)_{F,m}; \psi \right)^2 = (-1, -1)_{F,m} = (-1, -1)_{F,2}.
\]

Thus we can take

\[
\tilde{1} := \epsilon \left( \frac{1}{2}, (-1, \cdot)_{F,m}; \psi \right)^{-n} \cdot -1 \in \tilde{G}.
\]

Lemma 7.1.5. The formation of \( \tilde{1} \) depends on \( (W, \cdot | \cdot) \) and \( \psi \) in the following manner. Suppose (i) all data are acted upon by \( \operatorname{Ad}(g) \) with \( g \in \operatorname{Gad}(F) \) (Proposition 2.2.3), and \( g \) comes from \( g_1 \in \operatorname{GSp}(W) \) with similitude factor \( N(g_1) = c \);

(ii) \( \cdot | \cdot \) is replaced by \( c \cdot | c \cdot \); (iii) \( c \equiv 2 \pmod{4} \) and \( \psi \) is replaced by \( \psi_c \)

where \( c \in F^\times \). Then \( -1 \) will be replaced by \( (-1, c)_{F,m} \cdot \tilde{1} \) in each case.

Proof. The case (iii) follows from the dependence of local root numbers on \( \psi \). (i) and (ii) are equivalent, so it suffices to address (i).

Decompose \( W \) as \( \bigoplus_{i=1}^{n} W_i \) with \( W_i := F e_i \oplus F e_{-i} \). Note that if \( h_i \in \operatorname{GSp}(W_i) \) satisfies \( N(h_i) = c \), then \( g_1 := \operatorname{diag}(h_1, \ldots, h_n) \in \operatorname{GSp}(W) \) satisfies \( N(g_1) = c \) as well. It remains to show \( \operatorname{Ad}(g)(-1) = (-1, c)_{F,m} \) in the case \( n = 1 \), which is just Lemma 4.2.3 joint with Lemma 7.1.3. \( \square \)
Finally, consider an orthogonal decomposition of symplectic $F$-vector spaces

$$W = \bigoplus_{i=1}^{r} W_i$$

and let $A$ be the commutative group $\prod_{i=1}^{r} \{\pm 1\} \subset \prod_{i=1}^{r} \text{Sp}(W_i)$. Let $G_i := \text{Sp}(W_i)$. In each $G_i$ we have the element $\bar{1}_i$ from Definition 7.1.4.

**Proposition 7.1.6.** The topological central extension $\tilde{G} \to G(F)$ admits a section over $A$ given by

$$(1, \ldots, -1, \ldots, 1) \mapsto \iota_i(\bar{1}_i)$$

for every $1 \leq i \leq r$, where $\iota_i : \tilde{G}_i \to \tilde{G}$ is the natural homomorphism furnished by Lemma 4.1.1.

**Proof.** Using Lemma 4.1.1, the problem is reduced to the case $r = 1$. But to say $-1 \mapsto \bar{1}$ gives a section is the same as requiring $(-1)^2 = 1$. \hfill $\square$

**Remark 7.1.7.** In the situation above, if we transport everything by applying $g \in G(F)$, the resulting splitting over $gAg^{-1}$ only differs by $\text{Ad}(g) : \tilde{G} \to \tilde{G}$.

**Example 7.1.8.** Consider the setting of §6.2, assume that $S \subset G$ is of type (ER) and $p \neq 2$. We have $S(\bar{F}) = S(\bar{F})_{\text{der}} \times S(\bar{F})_{\text{ad}}$ by (6.1), and Theorem 6.2.2 asserts $S(\bar{F})_{\text{der}} = \{\pm 1\}$. The parameters $K = \prod_{i \in I} K_i$ for $S \subset G$ decompose $W$ into $\bigotimes_{i \in I} W_i$, and accordingly $A = S(\bar{F})_{\text{der}} = \{\pm 1\}$ in Proposition 7.1.6. In fact $W = \bigoplus_{i \in I} W_i$ is exactly the decomposition into joint eigenspaces under $S(\bar{F})_{\text{der}}$. In this way we obtain a section over $S(\bar{F})_{\text{der}}$. In view of Lemma 6.2.3, it extends uniquely to a section

$$\tilde{S}^2 \xrightarrow{\sigma} S(F)$$

where $\tilde{S}^2$ is the preimage of $S(F)$ in $\tilde{G}^2$. Writing $\Pi(-)(\cdot)$ (resp. $\Pi(\cdot)$) to denote the set of equivalence classes of genuine representations (resp. usual representations), we obtain bijections

$$\Pi(-)(\tilde{S}) \overset{1:1}{\longleftrightarrow} \Pi(-)(\tilde{S}^2) \overset{1:1}{\longleftrightarrow} \Pi(S(F)).$$

Their dependence on $(W, \langle \cdot | \cdot \rangle)$ and $\psi$ (when $m \equiv 2 \pmod{4}$) is described by Lemma 7.1.5.

### 7.2 Toral invariants

Consider a maximal $F$-torus $S$ in a reductive $F$-group $G$, and let $\alpha$ be an absolute root with $[F_{\pm}\alpha : F_{\alpha}] = 2$, i.e. $\alpha$ is a symmetric root, cf. (3.6). We shall denote by $R(G, S)(\bar{F})$ the set of absolute roots of $S$, and by $R(G, S)(\bar{F})_{\text{sym}}$ the subset of symmetric roots; both are $\Gamma_{\text{F}}$-stable. Below is a review of the toral invariant of $\alpha \in R(G, S)(\bar{F})_{\text{sym}}$ in [24, §4.1], which will enter into our construction of $L$-packets. Define

$$f_{(G, S)}(\alpha) := \text{sgn}_{F_{\alpha}/F_{\pm\alpha}}(\frac{[X_{\alpha}, \tau X_{\alpha}]}{H_{\alpha}}),$$

where $X_{\alpha} \in g_{\alpha}(F_{\alpha}) \setminus \{0\}$ is arbitrary, and $H_{\alpha} := \text{d}\alpha(1)$ is the infinitesimal coroot.

- By [24, Fact 4.1], $\alpha \mapsto f_{(G, S)}(\alpha)$ is well-defined and is $\Gamma_{\text{F}}$-invariant. It is even $N_{G}(S)(\bar{F})$-invariant, cf. the explanation of [25, Fact 4.7.4].
- Due to the infinitesimal nature of this definition, one can also compute $f_{(G, S)}(\alpha)$ in $G_{\text{det}}(\bar{F})$ or $G_{\text{ad}}(\bar{F})$. We compute two cases below.

**The case** $G = \text{Sp}(W)$. Here $W$ is a $2n$-dimensional symplectic $F$-vector space. Take a finite separable extension $L/F$ to split $S$, such that $S_L$ is associated to a symplectic basis $e_{\pm 1}, \ldots, e_{\pm n}$ of $W \otimes_F L$.

1. Suppose $\alpha$ is a long root. Reasoning as in §3.3, one may calculate inside $\text{SL}(2)$ over $F_{\pm \alpha}$. We deduce $f_{(G, S)}(\alpha) = 1$ by [24, Lemma 7.3].
2. Suppose $\alpha$ is a short root. Without loss of generality, assume $\alpha = e_1 - e_2$ in this basis, so that $\bar{\alpha} = \bar{e}_1 - \bar{e}_2$. Take the data

$$
X_\alpha : \begin{array}{ccc}
e_2 & \mapsto & e_1 \\
e_{-1} & \mapsto & -e_{-2}
\end{array}, \quad H_\alpha : \begin{array}{ccc}e_1 & \mapsto & e_1 \\
e_2 & \mapsto & -e_2 \\
e_{-1} & \mapsto & -e_{-1}
\end{array}
$$

(the other basis vectors $\mapsto 0$).

Recall that $\text{Gal}(L/F)$ acts on $\{ \pm \epsilon_1, \ldots, \pm \epsilon_n \} \subset X^*(S_L)$ by permutation since it does so on long roots. Let $\sigma$ be the nontrivial element of $\text{Gal}(F_{\alpha}/F_{\pm \alpha})$.

(i) Suppose that $\epsilon_1, \epsilon_2$ are in the same $\text{Gal}(L/F)$-orbit, so that $\sigma : \epsilon_1 \leftrightarrow \epsilon_2$. Calculate the Lie bracket as

$$
[X_\alpha, \sigma X_\alpha] = \begin{array}{ccc}
e_1 & \mapsto & e_1 \\
e_2 & \mapsto & -e_2 \\
e_{-1} & \mapsto & -e_{-1}
\end{array} = H_\alpha.
$$

Hence $f_{(G,S)}(\alpha) = 1$ in this case. Alternatively, we may also calculate $f_{(G,S)}(\alpha)$ in a twisted Levi subgroup $R_{F_{\pm \alpha}/F}(\text{GL}(2)) \times \cdots$, and argue as in the case of long roots that $f_{(G,S)}(\alpha) = 1$.

(ii) Suppose that $\epsilon_1, \epsilon_2$ are not in the same $\text{Gal}(L/F)$-orbit, then $\sigma : \epsilon_i \leftrightarrow -\epsilon_i$ ($i = 1, 2$). This time

$$
[X_\alpha, \sigma X_\alpha] = \begin{array}{ccc}
e_1 & \mapsto & -e_1 \\
e_2 & \mapsto & e_2 \\
e_{-1} & \mapsto & e_{-1}
\end{array} = -H_\alpha.
$$

Hence $f_{(G,S)}(\alpha) = \text{sgn} F_{\alpha}/F_{\pm \alpha}(-1)$ in this case.

**Definition 7.2.1.** By stipulation, if a quadratic $F$-vector space $(L, h)$ of dimension $2n$ can be written as the orthogonal direct sum of non-degenerate subspaces $Fe_i \oplus Fe_{-i}$, where $1 \leq i \leq n$, such that

$$
h(\epsilon_i | e_{-i}) \neq 0, \quad h(\epsilon_i | \epsilon_i) = 0 = h(e_{-i} | e_{-i}), \quad 1 \leq i \leq n
$$

then we say $\{ \epsilon_{\pm i} \}_{i=1}^n$ is a hyperbolic basis for $(L, h)$.

Suppose that $(L, h)$ admits a hyperbolic basis, there is then a split maximal torus $T \subset \text{SO}(L, h)$ consisting of matrices in the basis $e_1, \ldots, e_n, e_{-n}, \ldots, e_{-1}$:

$$
\gamma = \text{diag}(a_1, \ldots, a_n, a_n^{-1}, \ldots, a_1^{-1}), \quad a_1, \ldots, a_n \in \mathbb{G}_m.
$$

Therefore $X^*(T) = \bigoplus_{i=1}^n \mathbb{Z} \epsilon_i$ by setting $\epsilon_i(\gamma) = a_i$. Note that we do not require $h(\epsilon_i | e_{-i}) = 1$.

**The case $G = \text{SO}(V, q)$**. Here $(V, q)$ is a $(2n + 1)$-dimensional quadratic $F$-vector space. Take a finite separable extension $L/F$ to split $S$, such that $S_L$ is the maximal torus associated to a hyperbolic basis $\{ \epsilon_{\pm i} \}_{i=1}^n$ of a $2n$-dimensional non-degenerate $L$-subspace of $V \otimes_F V$; define $\{ \epsilon_{\pm i} \}_{i=1}^n$ as above. Its orthogonal complement is an anisotropic line $\ell$ which is the weight-zero subspace of $V \otimes_F L$ under $S_L$, hence $\ell$ is generated by an anisotropic $F$-vector $e_0$.

1. Suppose $\alpha$ is a short root. Without loss of generality, assume $\alpha = e_1$. Consider the subspace of $S$-weights $\{ \alpha, 0, -\alpha \}$ of $V \otimes_F F_\alpha$, namely

$$
U := F_\alpha e_1 \oplus F_\alpha e_0 \oplus F_\alpha e_{-1}.
$$

The restriction $q_U := (q \otimes F_\alpha)|_U$ is non-degenerate, and $f_{(G,S)}(\alpha)$ can be calculated inside $\text{SO}(U, q_U)$. Since $F_{\pm \alpha}$ permutes $\{ \alpha, 0, -\alpha \}$, we see that $(U, q_L)$ descends to a quadratic $F_{\pm \alpha}$-vector space $(U_0, q_0)$ of dimension $3$.

(i) If $\text{SO}(U_0, q_0)$ is split, it will be isomorphic to $\text{PGL}(2)$ and we have $f_{(G,S)}(\alpha) = 1$ by [24, Lemma 7.13] as before.

(ii) If $\text{SO}(U_0, q_0)$ is not split, it will be an inner form of $\text{PGL}(2)$ of Kottwitz sign $-1$. By [24, Proposition 4.3] we have $f_{(G,S)}(\alpha) = -1$ in this case.
Note that $\text{SO}(U_0, q_0)$ is split if and only if $(U_0, q_0)$ is isotropic, hence by [46, 12.7 + 14.3]
\[
f_{G, S}(\alpha) = \epsilon(U_0, q_0)(-1, d^\pm(U_0, q_0))_{F_{\pm}, 2}.
\]

(7.3)

2. Suppose $\alpha$ is a long root. We may assume $\alpha = \epsilon_1 - \epsilon_2$. The formulas for $X_{\alpha}$ and $H_{\alpha}$ are exactly the same as the case for $\text{Sp}(W)$, the same calculation thus leads to
\[
f_{G, S}(\alpha) = \begin{cases} 
1, & \text{sgn}_{F_{\pm}, 2}(-1), \\
\epsilon_1, \epsilon_2 \in \text{the same Gal}(L/F)-\text{orbit}, & \epsilon_1, \epsilon_2 \notin \text{the same Gal}(L/F)-\text{orbit}.
\end{cases}
\]

Henceforth assume $F$ non-archimedean with residual characteristic $\neq 2$. For $S \subset G$ as above, Kaletha constructed in [24, §4.6] a character $\epsilon_S : S(F) \to \{\pm 1\}$ using various $f_{G, S}(\alpha)$. By construction, $\epsilon_T$ is $N_G(T)(F)$-invariant since $f_{G, T}(\alpha)$ is, cf. [25, Fact 4.7.4].

**Lemma 7.2.2.** Let $G = \text{Sp}(W)$ where $(W, \langle \cdot, \cdot \rangle)$ is a $2n$-dimensional symplectic $F$-vector space. Let $j, j' : S \to G$ be two embeddings of maximal $F$-tori, related by stable conjugacy $\text{Ad}(g) : jS \sim j'S$. For any $\gamma \in S(F)$ with topological Jordan decomposition $\gamma = \gamma_0 \gamma_0^*$, we have
\[
\epsilon_j(\gamma_0) = \prod_{\alpha \in R(G, jS)_{\text{sym}}(F)} f_{G, jS}(\alpha),
\]
and similarly for $\epsilon_j'(\gamma_0')$. Now choose a Galois extension $L/F$ to split $S$ and a symplectic basis $\{e_{\pm 1}\}_{i=1}^n$ for $W \otimes_F L$ to calculate $f_{G, jS}(\gamma)$ as before. Transport it to $e'_{\pm i} := ge_{\pm i}$. Define $\epsilon_{\pm i} \in X^*(jS_L)$ (resp. $\epsilon'_{\pm i} \in X^*(j'S_L)$) using the basis $\{e_{\pm 1}\}_{i=1}^n$ (resp. $\{e'_{\pm 1}\}_{i=1}^n$); these data will be used to calculate $f_{G, jS}(\gamma)$ and $f_{G, j'S}(\gamma')$.

From $\text{Ad}(g)$ we deduce a $\Gamma_F$-equivariant isomorphism $X^*(jS_L) \sim X^*(j'S_L)$. It restricts to a bijection $R(G, jS)(F) \sim R(G, j'S)(F)$, written as $\alpha \mapsto \alpha' := \alpha \circ \text{Ad}(g)^{-1}$. Concurrently, $\epsilon_{\pm i}$ are mapped to $\epsilon'_{\pm i} = \epsilon_{\pm i} \circ \text{Ad}(g)^{-1}$. We also have $\text{Ad}(g)\gamma_0 = \gamma_0'$, hence $\alpha'(\gamma_0') \neq 1 \iff \alpha(\gamma_0) \neq 1$.

It follows from the previous calculations for $\text{Sp}(W)$ that $f_{G, jS}(\alpha) = f_{G, j'S}(\alpha')$, because these invariants depend solely on the $\Gamma_F$-action on $\{\epsilon_{\pm 1}\}_{i=1}^n$.

\[\square\]

Note that the argument fails for $\text{SO}(V, q)$: already in the case $n = 1$ and $S$ anisotropic, we saw that $f_{G, S}(\alpha)$ depends on finer invariants of quadratic forms.

### 7.3 Inducing data

In this subsection, $F$ is non-archimedean of residual characteristic $p \neq 2$, and for the BD-covers we assume $p \nmid m$. The characters in question are all continuous, and Hom($\cdots$) denotes the continuous Hom.

Fix a nonempty stable conjugacy class $E$ of embeddings of tame maximal $F$-tori $j : S \sim jS \subset G$ of type (ER) (Definition 6.1.1). Here we leave $S$ fixed and vary $j$. As usual, $e$ will stand for the ramification degree of the splitting field extension of $S$. Stable conjugacy preserves Moy–Prasad filtrations. These embeddings together with stable conjugacy form a groupoid with trivial automorphism groups. Furthermore,

- for any $j, j' \in E$, stable conjugacy furnishes a canonical $F$-isomorphism $\Omega(G, jS) \sim \Omega(G, j'S)$;
- in view of the absence of nontrivial automorphisms, one may form the universal absolute root system $R(G, S) := \varprojlim jR(G, jS)$ living on $X^*(\bar{S}_F)$, and similarly for $\Omega(G, S)$;
- regard $R(G, S), R(G, jS)$ and $\Omega(G, S), \Omega(G, jS)$ as sheaves over $\text{Spec}(F)_{\text{et}}$, so that $\Omega(G, S)(F)$ acts on $S$ and on $R(G, S)$;
- since the formation of $\iota_{Q, m} : (jS)_{Q, m} \to jS$ from §2.5 respects conjugacy, we have the universal isogeny $\iota_{Q, m} : S_{Q, m} \to S$.

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Next, consider a character $\theta^p : S_{Q,m}(F) \to \mathbb{C}^\times$, or equivalently a family of characters $\theta_j^p : jS_{Q,m}(F) \to \mathbb{C}^\times$ compatibly with the groupoid. Our requirement is that any character $\theta : S(F) \to \mathbb{C}^\times$ with $\theta \circ \iota_{Q,m} = \theta^p$ is epipelagic (Definition 6.3.1). This is a condition on $\theta^p|_{S_{Q,m}(F)_{0+}}$ since $\iota_{Q,m}$ is an isomorphism on the pro-$p$ parts, as one infers from (4.1).

**Definition 7.3.1.** Abbreviate the data above as $(\mathcal{E}, S, \theta^p)$. An isomorphism $(\mathcal{E}, S, \theta^p) \overset{j}{\to} (\mathcal{E}_1, S_1, \theta_1^p)$ consists of a commutative diagram of $F$-tori

$$
\begin{array}{ccc}
S_{Q,m} & \xrightarrow{\varphi_{Q,m}} & S_{1,Q,m} \\
\downarrow & & \downarrow \\
S & \xrightarrow{\varphi} & S_1
\end{array}
$$

such that $\varphi$, $\varphi_{Q,m}$ are isomorphisms and

- $\theta_1^p = \theta^p \circ \varphi_{Q,m}$,
- $j \mapsto j \circ \varphi^{-1}$ is a bijection from $\mathcal{E}$ onto $\mathcal{E}_1$.

Note that $\varphi_{Q,m}$ determines $\varphi$; in view of (4.1) we can even write $\varphi = \varphi_{Q,m}$. Using automorphisms we may modify $\theta^p$ by $\Omega(G, S)$.

**Lemma 7.3.2.** We have $\text{Hom}(S(F)_{p'}, \mathbb{C}) = \text{Hom}(S(F)/S(F)_{0+}, \mathbb{C}^\times) \simeq (\mathbb{Z}/2\mathbb{Z})^1$. Given $\theta^p$, the set

$$
\text{Fiber}(S, \theta^p) := \left\{ \theta \in \text{Hom}(S(F), \mathbb{C}^\times) : \theta \circ \iota_{Q,m} = \theta^p \right\}
$$

is a singleton when $4 \mid m$. When $4 \nmid m$, we have

$$
\text{Fiber}(S, \theta^p) = \left\{ \begin{array}{ll}
a \text{torsor under } \text{Hom}(S(F)_{p'}, \mathbb{C}), & \theta^p|_{S_{Q,m}(F)_{p'}} = 1 \\
\varnothing, & \text{otherwise.}
\end{array} \right.
$$

Furthermore, all the $\theta \in \text{Fiber}(S, \theta^p)$ share the same pro-$p$ part $\theta|_{S(F)_{0+}}$.

**Proof.** Theorem 6.2.2 gives the structure of $\text{Hom}(S(F)_{p'}, \mathbb{C})$. Observe from this and (4.1) that

- $\iota_{Q,m} : S_{Q,m}(F)_{0+} \to S(F)_{0+}$ is an isomorphism,
- $\iota_{Q,m}$ induces $S_{Q,m}(F)_{p'} \overset{\sim}{\to} S(F)_{p'}$ when $4 \mid m$, and $\iota_{Q,m}|_{S_{Q,m}(F)_{p'}} = 1$ when $4 \nmid m$.

For $4 \mid m$ we have complete freedom to choose $\theta|_{S(F)_{p'}}$, whence the torsor structure.

Given $(\mathcal{E}, S, \theta^p)$, the next step is to prescribe genuine characters $\theta_j : \widetilde{jS} \to \mathbb{C}^\times$ to each $j \in \mathcal{E}$. The isogeny $\iota_{Q,m} : jS_{Q,m} \to jS$ pulls back to

$$
\begin{array}{ccc}
\widetilde{jS}_{Q,m} & \xrightarrow{\iota_{Q,m}} & \widetilde{jS} \\
\downarrow & & \downarrow \\
jS_{Q,m}(F) & \xrightarrow{\iota_{Q,m}} & jS(F)
\end{array}
$$

Cf. (4.7). Using (7.2), it suffices to prescribe a character $\theta_j$ of $jS(F)$, and it can be further transported to $S(F)$ via $j$. The precise recipe will depend on $m$ mod 4.

**The case** $4 \mid m$. We take all the possible $\theta_j \in \text{Fiber}(S, \theta^p)$ described by Lemma 7.3.2, viewed as a genuine character of $\widetilde{jS}$. By Lemma 7.3.2 they have the same pro-$p$ part. This construction is non-vacuous only when $\theta_j^p|_{S_{Q,m}(F)_{p'}} = 1$. Cf. Remark 7.4.8.

**The case** $4 \nmid m$. Lemma 7.3.2 asserts that $\text{Fiber}(S, \theta^p)$ is a singleton; the unique element therein transports to a genuine character $\theta_j^p : \widetilde{jS} \to \mathbb{C}^\times$ by (7.2) for all $j$. Further modifications on the prime-to-$p$ part of $\theta_j^p$ are needed to achieve canonicity and stability, at least when $m \equiv 2 \pmod{4}$. The required modification is encapsulated into the following axioms.

**Definition 7.3.3.** Suppose $4 \nmid m$. A **stable system** is a rule assigning a family of genuine characters $\theta_j : \widetilde{jS} \to \mathbb{C}^\times$ to a triple $(\mathcal{E}, S, \theta^p)$, relative to any given $(W, \langle \cdot | \cdot \rangle)$ (and $\psi$ when $m \equiv 2 \pmod{4}$). It must satisfy the following properties.
SS.1 We require that \( \theta_j = \theta_j^\vee \theta_j^\dagger \), where \( \theta_j^\dagger \) is a character of \( j(S(F)/S(F)_{0^+}) \). When \( m \equiv 2 \) (mod 4), we require further that if \( c \in F^\times \) and
- \( \psi \) is replaced by \( \psi_c \), or
- \( \langle \cdot | \cdot \rangle \) is replaced by \( c(\langle \cdot | \cdot \rangle) \)
then \( \theta_j^\dagger \) will be twisted by the quadratic character of \( jS(F)_{0^+} \) that maps
\[
-1 \in K_i^1 \mapsto \prod_{i' \in I} K_{i'}^1 \simeq S(F) \mapsto \text{sgn}_{K_i/K_i^1}(c)
\]
(notation of Theorem 6.2.2) for all \( i \in I \).

SS.2 For any stable conjugacy \( j' = \text{Ad}(g) \circ j \in \mathcal{E} \), we require that
\[
\theta_{j'}(\text{CAd}(g)(\bar{\gamma})) = \theta_j(\bar{\gamma}), \quad \bar{\gamma} \in \bar{J}S
\]
where \( \text{CAd}(g) \) is as in Definition 4.3.8.

SS.3 The character \( \theta_j^\dagger \) depends only on the maximal torus \( jS \subset G \) and the \( \theta^\circ|_{S_{Q,m}(F)_{0^+}} \) transported to \( jS_{Q,m}(F)_{0^+} \). It follows that if \( (\varphi, \varphi_{Q,m}) : (\mathcal{E}, S, \theta^\circ) \xrightarrow{\sim} (\mathcal{E}_1, S_1, \theta_1^\dagger) \) is an isomorphism, then \( \theta_j^\dagger = \theta_{j'}^\dagger \); indeed, \( \text{im}(j) = \text{im}(j\varphi^{-1}) =: R \) and the transportations to \( R_{Q,m}(F) \) of \( \theta^\circ \) and \( \theta_1^\dagger \) coincide since \( \theta^\circ = \theta_1^\dagger \varphi_{Q,m} \)

Several quick observations are in order.
- **SS.1** and Lemma 7.3.2 determine the pro-\( p \) component of \( \theta_j \); the pro-\( p \) version of all the other conditions follow.
- The dependence of \( \theta_j^\dagger \) on \( (\psi, \langle \cdot | \cdot \rangle) \) in **SS.1** is imposed by the canonicity of \( L \)-packets for BD-covers, see Theorem 7.5.1.
- In view of AD.3 of Proposition 4.3.5, taking \( g \in G(F) \) in **SS.2** implies that \( \theta_j \) is compatible with \( G(F) \)-conjugacy of \( j \).
- For each \( j \in \mathcal{E} \), the genuine character \( \theta_j \) determines \( \theta^\circ \) as follows: \( \theta_j|_{S(F)_{0^+}} \) determines \( \theta^\circ|_{S(F)_{0^+}} \).
- By **SS.3**, this in turn determines \( \theta_j^\dagger \) as well as \( \theta_j^\circ = \theta_j(\theta_j^\dagger)^{-1} \); we conclude that \( \theta^\circ \) is also determined.

**Lemma 7.3.4.** In each case, \( \theta_j \) is an epipelagic genuine character of \( \bar{J}S \) in the sense of Definition 6.3.1.

**Proof.** This concerns only the pro-\( p \) part of \( \theta_j \), thus unaffected by \( \theta_j^\dagger \). The required property is thus built into the definition of triples \((\mathcal{E}, S, \theta^\circ)\).

**Proposition 7.3.5** (Standard stable system for odd \( m \)). Suppose that \( m \notin 2\mathbb{Z} \). There is a standard stable system given by \( \theta_j := \theta_j^\circ \) for all \( j \).

**Proof.** The properties **SS.1** and **SS.3** are automatic. Since \( m \notin 2\mathbb{Z} \), by Proposition 4.4.1 \( \text{CAd}(g) \) is defined by the natural actions of \( G_{\text{ad}}^\circ(F) \) and \( G(F) \). On the other hand, (7.2) is realized by the unique splitting over \( S(F)_{0^+} \) (Proposition 6.2.4). This entails **SS.2**.

**Definition 7.3.6.** Suppose we are given
- a triple \((\mathcal{E}, S, \theta^\circ)\) as above;
- a stable system (Definition 7.3.3) when \( 4 \nmid m \).

Write \( \epsilon_j := \epsilon_jS \) for the character of \( jS(F) \) constructed from toral invariants, as reviewed in §7.2. Define the set
\[
\Pi(S, \theta^\circ) := \{ \pi_{jS, \epsilon_jS(\theta_j)} \}_{j, \theta_j}
\]
where
- \( j \) ranges over all \( G(F) \)-conjugacy classes of embeddings \( S \hookrightarrow G \) in \( \mathcal{E} \);
- for each \( j \), let \( \theta_j \) be the genuine epipelagic character(s) of \( jS \) specified as follows:
(4 \mid m): \ \theta_j \text{ ranges over all the elements of Fiber}(S, \theta^0) \text{ transported to } \tilde{j}S, \text{ which could be empty,}

(4 \nmid m): \ \theta_j \text{ is uniquely specified by the stable system } (E, S, \theta^0);

• \ \pi_{(jS),\iota_j,\theta_j} \text{ is the representation of } \tilde{G} \text{ constructed in Theorem 6.3.2.}

**Lemma 7.3.7.** The set \( \Pi(S, \theta^0) \) depends only on the isomorphism class of \((E, S, \theta^0)\).

**Proof.** Indeed, given \((\varphi, \varphi_{Q,m}) : (E, S, \theta^0) \xrightarrow{\sim} (E_1, S_1, \theta_1^0)\), pull-back by \(\varphi\) induces bijections

\[
\text{Fiber}(S, \theta^0) \rightarrow \text{Fiber}(S_1, \theta_1^0) \quad \text{and} \quad E \rightarrow E_1.
\]

Suppose \( j \mapsto j_1 \), the characters of \( jS(F) = j_1S_1(F) \) specified from \( \theta^0 \) and \( \theta_1^0 \) are therefore equal. Since the identification (7.2) depends only on \( jS = j_1S_1 \), the inducing genuine characters in the case \( 4 \mid m \) also coincide. For the case \( 4 \nmid m \), we invoke SS.3. \qed

**Proposition 7.3.8.** The set \( \Pi(S, \theta^0) \) consists of genuine epipelagic supercuspidal representations of \( \tilde{G} \).

Conversely every genuine epipelagic supercuspidal representation belongs to some \( \Pi(S, \theta^0) \).

Granting the basic properties of Yu’s construction for \( \tilde{G} \) in Remark 6.3.4, we have

• when \( 4 \nmid m \), \( \Pi(S, \theta^0) \) is a torsor under \( H^1(F, S) \), of cardinality \( 2^{|I|} \), where \( I \) is as in Theorem 6.2.2;

• suppose \( 4 \mid m \), \( \Pi(S, \theta^0) \) is

\[
- \text{a torsor under } \text{Hom}(S(F)_{\varphi'}, \mathbb{C}) \times H^1(F, S) \text{ of cardinality } 2^{|I|}, \text{ if } \theta^0|_{S_Q,m(F)_{\varphi'}} = 1, \\
- \text{empty if } \theta^0|_{S_Q,m(F)_{\varphi'}} \neq 1.
\]

Under the same premises, any two \( \Pi(S_1, \theta_1^0) \) and \( \Pi(S_2, \theta_2^0) \) are either disjoint or equal, and the latter case occurs if and only if both are empty, or \( (E_1, S_1, \theta_1^0) \) is isomorphic to \( (E_2, S_2, \theta_2^0) \).

**Proof.** The first part follows from the construction and Theorem 6.3.2. For the second part, in view of (6.2) and Remark 6.3.4, different \( G(F) \)-conjugacy classes of \( j \) give non-isomorphic supercuspidals of \( \tilde{G} \). The group \( H^1(F, S) \) alters the embeddings \( j \) via \( H^1(F, S) \cong H^1(F, jS) = O(jS, G; F) \) and makes a torsor. Proposition 3.2.4 implies \( |H^1(F, S)| = 2^{|I|} \). This settles the case \( 4 \nmid m \), and for \( 4 \mid m \) we appeal to Lemma 7.3.2.

The last part is clear by Lemma 7.3.2 when \( 4 \mid m \) (only the pro-\( p \) part matters). Suppose \( 4 \nmid m \) and that there exist \( j_i \in E_i' \) with \( (j_iS_i, \theta_{i,j_i}) \) conjugate for \( i = 1, 2 \); upon modifying \((E_1, S_1, \theta_1^0)\) by an isomorphism \((\varphi, \varphi_{Q,m})\) we may assume \( S_1 = S_2, E_i = E_2 \) and \( j_1S_1 = j_2S_2 \). Then \( \theta_{i,j_i} = \theta_{2,j_2}Ad(w) \) for some \( w \in \Omega(G, S_2)(F) \); by a further modification we may assume \( \theta_{1,j_1} = \theta_{2,j_2} \). As remarked below Definition 7.3.3, this implies \( \theta_1^0 = \theta_2^0 \) by using SS.3. \qed

### 7.4 Epipelagic L-parameters

We continue the thread of the \( L \)-group \( \tilde{G} \) for \( \tilde{G} \) has been introduced in §5.1. Given the symplectic \( F \)-vector space \((W, \langle \cdot | \cdot \rangle)\), together with \( \psi : F \rightarrow \mathbb{C}^\times \) when \( m \equiv 2 \pmod{4} \), we deduce an identification \( \tilde{G} = \tilde{G}^{\psi} \times W_F \) that respects the projections to \( W_F \).

**Definition 7.4.1** ([24, Conditions 5.1]). An \( L \)-parameter \( \phi : W_F \rightarrow \tilde{G} \) is called epipelagic if

• \( Z_{\tilde{G}^{\psi}}(\phi(P_F)) \) is a maximal torus \( \tilde{T}^{\psi} \) that fits into a \( \Gamma_F \)-pinning of \( \tilde{G}^{\psi} \);

• the image of \( \phi(I_F) \) in \( \Omega(\tilde{G}^{\psi}, T^{\psi}) \) is generated by a regular elliptic element \( t \);

• \( \text{let } \ell(t) \text{ be the order of } t, \text{ then } w \in \Gamma_F^{\psi} \implies \phi(w) = (1, w). \)

Upon conjugating by \( \tilde{G}^{\psi} \), we may assume that \( \tilde{T}^{\psi} \) is the maximal torus in the given pinning for \( \tilde{G}^{\psi} \). As in loc. cit., \( \tilde{T}^{\psi} \) can be endowed with the continuous \( \Gamma_F \)-action induced from

\[
W_F \xrightarrow{\phi} N_{\tilde{G}^{\psi}}(T^{\psi}) \times W_F \implies \Omega(\tilde{G}^{\psi}, T^{\psi}) \times W_F,
\]

which factors through a finite quotient. Name the resulting torus with Galois action as \( S_{Q,m}^{\psi} \); recall that \( X^*(S_{Q,m}^{\psi}) = Y_{Q,m} \).
This is a special case of toral supercuspidal $L$-parameters defined in [25, Definition 6.1.1]. These conditions are independent of the splitting for $^\ell \tilde{G}$, since different splittings differ by an element from $H^1(W_F, Z_{\tilde{G}^\vee})$.

The stable conjugacy classes of maximal $F$-tori in $G = \text{Sp}(W)$ are parameterized by $H^1(F, \Omega)$, where $\Omega$ is the absolute Weyl group of $\text{Sp}(2n)$. Now comes the type map for maximal tori, let us choose a Borel subgroup $B$ with a reductive quotient $T$, thereby obtaining the Weyl group $\Omega$. For any maximal $F$-torus $S \subset G$ there exists $g \in G(F)$ such that $S^g = gT_\ell g^{-1}$. Then $\sigma \mapsto g^{-1}\sigma(g)$ is the required 1-cocycle, whose class in $H^1(F, \Omega)$ is independent of the choice of $g$. Facts: (a) Two maximal tori are stably conjugate if and only if they have the same class in $H^1(F, \Omega)$, (b) Every class in $H^1(F, \Omega)$ comes from some $S \subset G$; see [25, §3.2] or [44, Theorem 1.1].

Recall from §5.1 that $\Omega(\tilde{G}^\vee, \tilde{T}^\vee)$ and $\Omega(G, T)$ can be identified: $Y_{Q,m}$ and $Y$ span the same $\mathbb{Q}$-vector space on which $\Omega$ acts by reflections. Hence the $\Gamma_F$-action on $S^\vee_{Q,m}$ gives rise to (i) a class $c \in H^1(F, \Omega)$; (ii) whence a stable class $\mathcal{E}$ of embeddings $j : S \hookrightarrow G$; (iii) by construction, $\Gamma_F$ acts on $X_S(S_F)$ via a 1-cocycle in the class $c$. We may assume that $\Gamma_F$ acts on $X_S(S_F)$ and $X^*(S^\vee_{Q,m})$ by the same cocycle. Let $S^\vee_{Q,m}$ be the $F$-torus dual to $S^\vee_{Q,m}$, so we deduce $\Gamma_F$-equivariant isomorphisms

$$X_S(S_F) \cong Y_{Q,m} \twoheadrightarrow X^*(S^\vee_{Q,m}) = X_S(S^\vee_{Q,m}) = \tilde{T}^\vee.$$  

The naming is thus justified: the isomorphisms above induces $S^\vee_{Q,m} \rightarrow S$ that is exactly the $\iota_{Q,m}$ in §2.5 for the BD-cover $\tilde{G}$.

To proceed, we follow [24, §5.2] to construct an $L$-embedding $^\ell j : ^\ell S_{Q,m} \rightarrow \tilde{G}$ up to $\tilde{G}^\vee$-conjugacy, an $L$-parameter $\phi_{S^\vee_{Q,m}, ^\ell j}$ together with factorization

$$\begin{array}{ccc}
\mathcal{W}_F & \xrightarrow{\phi} & \tilde{G} \\
\phi_{S^\vee_{Q,m}, ^\ell j} & \longmapsto & ^\ell S^\vee_{Q,m}
\end{array} \quad \left( ^\ell j|_{S^\vee_{Q,m}} : S^\vee_{Q,m} \twoheadrightarrow \tilde{T}^\vee \right) = \text{the given one.}
$$

The resulting $L$-parameter $\phi_{S^\vee_{Q,m}, ^\ell j}$ will be canonical up to $\Omega(G, S)(F)$-action by [24, Lemma 5.3]. In loc. cit. such an $^\ell j$ comes from a carefully chosen $\chi$-datum of $(G, S)$ in the sense of Langlands–Shelstad [32, (2.5), (2.6)]. A $\chi$-datum consists of characters $\chi_\alpha : F^\times_\alpha \rightarrow \mathbb{C}^\times$ (see (3.6)) where $\alpha \in R(G, S)(F)$, such that

$$\begin{align*}
\chi_{-\alpha} &= \chi_\alpha^{-1}, \\
\sigma \in \Gamma_F &\implies \chi_{\sigma\alpha} = \chi_\alpha \circ \sigma^{-1}, \quad [F_\alpha : F_{\pm\alpha}] = 2 \implies \chi_\alpha|_{F_{\pm\alpha}} = \text{sgn}_{F_\alpha / F_{\pm\alpha}}.
\end{align*}$$

Rigorously speaking, here we must work with the $S^\vee_{Q,m}$ embedded into the split $F$-group $G_{Q,m}$ dual to $\tilde{G}^\vee$, since the roots/coroots are rescaled in constructing $\tilde{G}^\vee$. However, the $\Gamma_F$-action on the roots are unaffected by such rescaling. All in all, the construction of $^\ell j$ carries over to our setting, and [24, Lemma 5.4] implies that

- $jS$ is a maximal torus of type (ER) in $G$, for every $j \in \mathcal{E}$;
- local Langlands correspondence for $S_{Q,m}$ yields a character $\theta^\vee : S_{Q,m}(F) \rightarrow \mathbb{C}^\times$, such that $(\mathcal{E}, S, \theta^\vee)$ satisfies the requirements of Definition 7.3.1.

Indeed, both conditions can be phrased in terms of Weyl group actions, thus the arguments in loc. cit. carry over verbatim. The triple $(\mathcal{E}, S, \theta^\vee)$ is well-defined only up to isomorphism: recall the ambiguity by $\Omega(G, S)(F)$.

Remark 7.4.2. Different choices of $\chi$-data lead to different assignments $\phi \rightsquigarrow (\mathcal{E}, S, \theta^\vee)$. This choice will not be used in the proof of stability, but it will intervene in our later comparison with $\Theta$-correspondence in §9.3. Also note that the $\chi$-data is not fixed in the approach of [25].

Recall that the identification $^\ell \tilde{G} = \tilde{G}^\vee \times \mathcal{W}_F$ can be twisted by $H^1(W_F, Z_{\tilde{G}^\vee})$. All elements therein take the form $\chi_\epsilon$ as in (5.5). Its effect on $(\mathcal{E}, S, \theta^\vee)$ is to twist $\theta^\vee$ by the quadratic character associated to the image of $\chi_\epsilon$ in $H^1(W_F, S^\vee_{Q,m})$.  

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Lemma 7.4.3. Suppose \( 4 \mid m \). The \( H^1(W_F, Z_{G'}) \)-orbit of \( (E, S, \theta') \) contains at most one element \( \theta'_1 \) satisfying \( \theta'_1|_{S_{Q,m}(F)_{p'}} = 1 \).

Proof. Recall that \( S_{Q,m}(F) = S_{Q,m}(F)_{p'} \times S_{Q,m}(F)_{0,p} \), as \( S \) satisfies the same property and \( \cong S_{Q,m} \). Since \( p \neq 2 \), the quadratic twists do not affect the pro-\( p \) part of \( \theta' \).

Definition 7.4.4 (\( L \)-packets). Choose a stable system (Definition 7.3.3) when \( 4 \nmid m \). Let \( \phi : W_F \to \hat{\mathcal{G}} \) be an epipelagic \( L \)-parameter and let \( (E, S, \theta') \) be the resulting isomorphism class of inducing data. Using the notation from Lemma 7.4.3 for \( 4 \mid m \), define

\[
\Pi_\phi := \begin{cases}
\Pi(S, \theta'), & 4 \nmid m \\
\Pi(S, \theta'_1), & 4 \mid m, \exists \theta'_1 \\
\emptyset, & 4 \mid m, \nexists \theta'_1
\end{cases}
\]

where \( \Pi(S, \cdot) \) is that of Definition 7.3.6. Call it the \( L \)-packet (resp. pre-\( L \)-packet) attached to \( \phi \) when \( 4 \nmid m \) (resp. to the \( H^1(W_F, Z_{G'}) \)-orbit \([\phi]\) of \( \phi \) when \( 4 \mid m \)). We also define the centralizer group

\[ C_\phi := Z_{G'}(\text{im}(\phi)). \]

Remark 7.4.5. Assume \( 4 \mid m \). Then \( \Pi_\phi \) and \( C_\phi \) depend only on the \( H^1(W_F, Z_{G'}) \)-orbit \([\phi]\), hence we may write \( \Pi_{[\phi]}, C_{[\phi]} \) instead. Note that among the twists of \( \theta' \), Lemma 7.4.3 specified the unique one (if exists) such that \( \Pi(S, \theta'_1) \neq \emptyset \), by Proposition 7.3.8.

Lemma 7.4.6. The \( \mathbb{C} \)-group \( C_\phi \) is finite and diagonalizable. There is an isomorphism

\[ (S_{Q,m}^\dagger)_{/\mathbb{C}} \cong C_\phi. \]

Proof. This is just [24, (5.2)].

Let \( i_{Q,m} : S^\vee \to S_{Q,m}^\vee \) be the dual of \( i_{Q,m} \). As \( i_{Q,m} \) is identifiable with the endomorphism \( t_0 \mapsto t_0^{m/gcd(2,m)} \) of \( S \), from Theorem 6.2.2 and Proposition 3.2.4 we infer that

\[
H^1(W_F, S^\vee) \xrightarrow{(i_{Q,m})_*} H^1(W_F, S_{Q,m}^\vee) : \quad \ker \cong \text{coker} \cong \begin{cases} 0, & 4 \nmid m \\ \mu_2, & 4 \mid m \end{cases};
\]

\[
H^1(F, S_{Q,m}) \xrightarrow{i_{Q,m}^*} H^1(F, S) : \quad \begin{cases} \text{bijective}, & 4 \nmid m \\ \text{trivial}, & 4 \mid m. \end{cases}
\]

Theorem 7.4.7. Every genuine epipelagic irreducible representation of \( \hat{\mathcal{G}} \) belongs to some \( \Pi_\phi \). Granting the premises in Proposition 7.3.8, we have the following properties.

- When \( 4 \nmid m \), all \( \Pi_\phi \) are nonempty. When \( 4 \mid m \), denote by \([\phi_{S_{Q,m}, i_{Q,m}}]\) the \( H^1(W_F, Z_{G'}) \)-orbit of \( \phi_{S_{Q,m}, i_{Q,m}} \); we have
  \[ \Pi_{[\phi]} = \emptyset \iff [\phi_{S_{Q,m}, i_{Q,m}}] \cap \text{im}(i_{Q,m}) = \emptyset. \]

- When \( \Pi_\phi \neq \emptyset \), it is canonically a torsor under \( \ker ([i_{Q,m}]_\ast) \times H^1(F, S) \); for \( 4 \nmid m \), it is canonically a torsor under \( \pi_0(C_\phi, 1)^D \) where \( (\cdots)^D \) means the Pontryagin dual.

- Any two \( \Pi_\phi, \Pi_{\phi'} \) are either disjoint or equal, and the latter case occurs exactly when
  - both are empty,
  - \( 4 \nmid m \) and \( \phi \) is equivalent to \( \phi' \), or
  - \( 4 \mid m \) and \([\phi] = [\phi']\).

Proof. The descriptions involving \( i_{Q,m} \) and orbits just reinterpret Proposition 7.3.8 in terms of local Langlands correspondence for tori.

In view of Lemma 7.4.6 and the discussion on \( i_{Q,m} \), the refinement for \( 4 \nmid m \) is reduced to the existence of a natural \( H^1(F, S_{Q,m}) \to \pi_0(C_\phi, 1)^D \). This follows from Kottwitz’s isomorphism \( H^1(F, S_{Q,m}) \cong \pi_0((S_{Q,m}^\dagger)_{/\mathbb{C}})^D \).

It remains to show that non-equivalent \( L \)-parameters \( \phi_1, \phi_2 \) correspond to non-isomorphic \( (E_1, S_1, \phi_1) \), \( (E_2, S_2, \phi_2) \), provided that both \( L \)-packets are non-empty. This follows from the broader framework in [25, Proposition 5.2.4], since the \( \chi \)-data here are prescribed.
Remark 7.4.8. The recipe here can probably be compared with the local Langlands for metaplectic tori discussed in [14, §§8.1–8.3].

Remark 7.4.9. The description of $\Pi_\phi$ here is modeled upon [24]. The extended pure inner forms in loc. cit. reduce to $G$ or $\tilde{G}$ itself in our situation, since the set of basic $G$-isocrystals $B(G)_{bas}$ is trivial. This also amounts to taking $Z = \{1\}$ in the general framework of [26].

We compare the central characters $\omega_\pi$ for $\pi \in \Pi_\phi$ next. Proposition 2.2.3 ensures $Z_{\tilde{G}} = p^{-1}(\{\pm 1\})$ and $\omega_\pi$ is genuine. Recall that when $4 \nmid m$, the members of $\Pi_\phi$ are parameterized by conjugacy classes in a given stable $\mathcal{E}$.

Theorem 7.4.10. Assume $4 \nmid m$. Let $j, j' : S \hookrightarrow G$ correspond to $\pi, \pi' \in \Pi_\phi$.

- If $m \not\equiv 2Z$, then $\omega_\pi = \omega_{\pi'}$.
- If $m \equiv 2 (mod 4)$, then $\omega_\pi = \omega_{\pi'} \otimes \eta$ where $\eta : \{\pm 1\} \to \mathbb{C}^\times$ maps $-1$ to $(\kappa_S, inv(j, j'))$.

Here $\kappa_S$ is as in Definition 3.2.5.

Proof. We have $\omega_\pi = \epsilon_j \theta_j|_{Z_{\tilde{G}}}$ and $\omega_{\pi'} = \epsilon_{j'} \theta_{j'}|_{Z_{\tilde{G}}}$. Apply SS2 of Definition 7.3.3 with Proposition 4.4.2 (resp. Lemma 7.2.2) to compare the restrictions to $Z_{\tilde{G}}$ of $\theta_j, \theta_{j'}$ (resp. of $\epsilon_j, \epsilon_{j'}$).

This is in clear contrast with the case of reductive groups and conforms to the prediction in [14, §12.1].

7.5 Independencies

In what follows, $\phi$ always denote an epipelagic $L$-parameter into $\tilde{\mathbb{G}}$. Recall from §5.1 that when $m \not\equiv 2Z$, the identification $\tilde{\mathbb{G}} = \tilde{\mathbb{G}}^\vee \times W_F$ is canonical, and so is our construction of $\Pi_\phi$.

When $m \in 2Z$, the identification $\tilde{\mathbb{G}} = \tilde{\mathbb{G}}^\vee \times W_F$ depends on the choices of $(W, \langle \cdot | \cdot \rangle)$, as well as $\psi : F \to \mathbb{C}^\times$ when $m \equiv 2 (mod 4)$; the same choices enter in (7.2). The next result should be compared with [14, Proposition 11.1].

Theorem 7.5.1. Assume $m \equiv 2 (mod 4)$. For $\phi : W_F \to \tilde{\mathbb{G}}$, the $L$-packet $\Pi_\phi$ is independent of the choice of $\psi$; it is also invariant under dilation of $\langle \cdot | \cdot \rangle$.

Proof. First consider the effect of changing $\psi$. Let $c \in F^\times$.

1. Changing $\psi$ to $\psi_c$ amounts to twist $\phi : W_F \to \tilde{\mathbb{G}}^\vee \times W_F$ by the $\chi_c$ in (5.5). Accordingly, $\theta^i$ in the inducing datum will be twisted by the $\chi_{S_{Q,m}} : S_{Q,m}(F) \to \mu_2$ corresponding to the image of $\chi_c$ under $H^i(W_F, Z_{\tilde{G}}^\vee) \to H^i(W_F, S_{Q,m}^\vee)$.

2. The genuine character $\theta^i_j$ in Definition 7.3.3 is also controlled by the splitting over $S(F)_{\psi'}$; Lemma 7.1.5 describes its dependence on $\psi$.

3. The SS1 of Definition 7.3.3 describes the dependence of $\theta^i_j$ on $\psi$.

By construction (cf. §5.1), there are $\Gamma_F$-equivariant commutative diagrams

\[
\begin{array}{ccc}
Y & \overset{\varphi}{\longrightarrow} & Y_{Q,m} \\
Y & \overset{\varphi}{\longrightarrow} & Y_{Q,m} \\
& \overset{\varphi}{\longrightarrow} & \mathbb{Z}/2\mathbb{Z} \\
\end{array}
\]

related by $\text{Hom}(-, \mathbb{C}^\times)$. We see that $\chi_{S_{Q,m}} = \chi_S \circ i_{Q,m}$ where $\chi_S : S(F) \to \mathbb{C}^\times$ corresponds to the composite of $i \circ \chi_c$ under the Langlands correspondence. If we identify $S^\vee$ with $(\mathbb{C}^\times)^n$ through the usual basis $\{\xi_n\}_{n=1}^i$ of $Y$, then $i(-1) = (-1, \ldots, -1)$.

To simplify notations, let us assume $S \simeq R_{K^1/F}(K^1)$ in terms of parameters in §3.2, where $K, K^1$ are fields and $[K^1 : F] = n$. We must show

\[
\text{sgn}_{K^1/F}(c)(-1, c)_{F,m} = \chi_S(-1), \quad -1 \in K^1.
\]

By Lemma 7.1.3 and (4.5), this is equivalent to $\text{sgn}_{K/K^1}(c)(-1, c)_{K^1,2} = \chi_S(-1)$.

Express $S$ as the quotient of $S_1 := R_{K/F}(G_{m,K})$ via $\omega \mapsto \omega/\tau(\omega)$, where $\text{Gal}(K/K^1) = \{1, \tau\}$; dually $i_1 : S^\vee \hookrightarrow S^1$ is the anti-diagonal embedding into $(\mathbb{C}^\times \times \mathbb{C}^\times)^n = (\mathbb{C}^\times)^{2n}$ (cf. §2.3). The composite
i_1 \circ i \circ \chi_e : \Gamma_F \to S^1$ corresponds, by Shapiro’s isomorphism (“restriction composed with evaluation at 1”), to $\chi_e|_{\pi_\infty} : \Gamma_K \to \mu_2$; it corresponds via local class field theory to the character
\[ \chi_S : K^\times \to \mu_2 \]
\[ \omega \mapsto (c, N_{K/F}(\omega))_{K,2} = (c, N_{K/K^2}(\omega))_{K,2}. \] (4.5).

Take $\omega = D \in K^{1,\times}$ such that $K = K^2(\sqrt{D})$, we conclude that
\[ \chi_S(-1) = \chi_S(\sqrt{D}) = (c, -D)_{K,2} = (c, D)_{K,2}(-1)_{K,2} = \text{sgn}_{K/K^2}(c)(-1)_{K,2}. \]

Now keep $\psi$ fixed and replace $\langle \cdot | \cdot \rangle$ by $c^\langle \cdot | \cdot \rangle$. Upon replacing Lemma 5.1.1 by Lemma 5.2.2, the argument here is verbatim.

Remark 7.5.2. As a consequence, $\Pi_{[\phi]}$ depends only on $\psi \circ \langle \cdot | \cdot \rangle : W \to \C^\times$ when $m \equiv 2 \pmod{4}$. This principle is familiar in the case $m = 2$.

Next, assume that $4 \mid m$. The first observation is that the splitting over $S(F)_{\psi}$ in §7.1 is irrelevant. Indeed, when $\Pi_{[\phi]} \neq \emptyset$, the inducing datum $\theta_j : jS \to \C^\times$ ranges over all genuine characters with prescribed $\text{pro-} p$ component. On the other hand, there is still an ambiguity by $H^1(W_F, \Z_{\psi})$ for identifying $\tilde{G}$ and $\tilde{G} \times \text{W}_F$.

Theorem 7.5.3. The $L$-packet $\Pi_{[\phi]}$ is unaltered by $H^1(W_F, \Z_{\psi})$-twists, therefore invariant under dilatation of $\langle \cdot | \cdot \rangle$.

Proof. This is built into the construction, since we worked with the $H^1(W_F, \Z_{\psi})$-orbit $[\phi]$.

6.4 Proof of stability

Let $F$ and the additive characters $\xi, \lambda$ be as in §6.4. Choose a stable system (Definition 7.3.3) when $4 \mid m$. Given any epipelagic $L$-parameter $\phi : \text{W}_F \to \tilde{G}$, in §7.4 we have obtained
- the triple $(\mathcal{E}, S, \theta^\mathcal{E})$ up to isomorphism;
- the $L$-packet $\Pi_{[\phi]}$ (or the pre-$L$-packet $\Pi_{\phi}$) when $4 \mid m$.

Definition 7.6.1. The stable character associated to $\phi$ is
\[ S\Theta_{[\phi]} := \sum_{\pi \in \Pi_{\phi}} \Theta_\pi \]
where $\Theta_\pi$ is as in §6.4. It is an invariant distribution represented by a locally integrable genuine function on $\tilde{G}$, smooth over $\tilde{G}_{\text{reg}}$. As in Remark 7.4.5, when $4 \mid m$ it is reasonable to write $S\Theta_{[\phi]}$ instead.

In view of the prescription $\theta^\mathcal{E} \twoheadrightarrow \theta_j$ on the $\text{pro-}p$ part and the Adler–Spice character formula, let us take $Y = Y_\xi \in \mathfrak{s}^*(F)_{1/e}$ satisfying
\[ \theta \circ \iota_{Q,m} = \theta^\mathcal{E} \implies \theta \circ \exp = \xi((Y, \cdot)) : \mathfrak{s}(F)_{1/e} \to \C^\times, \] (7.4)
where $j : S \to G$ is any embedding in $\mathcal{E}$. We may also view $Y$ as in $\mathfrak{s}(F)_{1/e}$ using (6.4), i.e. $\theta \circ \exp = \xi(B_{\phi}(jY, j(\cdot)))$ for any $j \in \mathcal{E}$. Proposition 6.4.4 asserts that $Y$ is regular.

In what follows, $\gamma \in \tilde{G}_{\text{reg}}$ will stand for a compact element with topological Jordan decomposition $\gamma = \gamma_0 \gamma_0$, and we write $\gamma, \gamma_0$ for their images in $G(F)$. Also put $J := G_{\gamma_0} = Z_G(\gamma_0)$.

Lemma 7.6.2. For $\gamma$ as above, we have
\[ |D^G(\gamma)|^{\frac{1}{2}} S\Theta_{[\phi]}(\gamma) = \sum_{[j] : S \to J} \sum_{k \in [j]} \sum_{\theta_k} (\theta_k \epsilon_k)(\gamma_0)^{\frac{1}{2}}(kY, \log \gamma_0), \]
where
- $[j]$ ranges over the stable $J$-conjugacy classes of embeddings $j : S \to J$, whose composite with $J \to G$ lies in $\mathcal{E}$;
• $k$ ranges over the conjugacy classes of embeddings $S \hookrightarrow J$ within $[j]$;
• $\theta_k$ ranges over the genuine characters $kS \to \mathbb{C}^*$ prescribed in §7.3, which is a singleton unless $4 \mid m$.

On the other hand, $\Theta_\phi$ vanishes at non-compact elements in $\tilde{G}_{\text{reg}}$.

Note that $\gamma_0 \in Z_J(F)$ implies that $\gamma_0 \in kS(F)$ for all $k \in [j]$.

**Proof.** The vanishing at non-compact $\tilde{\gamma}$ follows from the same property for each $\Theta_\pi$. Apply Theorem 6.4.6 to express $|D^G(\gamma)|^{1/2} S\Theta_\phi(\tilde{\gamma})$ as the sum of

$$
\sum_{g \in J(F) \setminus G(F)/jS(F)} (\epsilon_j \theta_j)(g^{-1}\gamma_0 g) \hat{\nu}^j (gj(Y)g^{-1}, \log \gamma_{>0})
$$

over conjugacy classes of $j : S \hookrightarrow G$ and those $\theta_j$ prescribed in §7.3. As remarked in the proof of Theorem 6.4.6,

$$
g^{-1}\gamma_0 g \in jS(F) \iff \text{Ad}(g)(jS) \subset G_{\gamma_0} = J.
$$

One verifies readily the bijection

$$
\begin{cases}
(j,g) : & g \in J(F) \setminus G(F)/jS(F) \\
\text{s.t. } \gamma_0 \in \text{Ad}(g)jS(F) & \forall h \in G(F) \Rightarrow (j,g) \sim (\text{Ad}(h)j, gh^{-1})
\end{cases}
$$

follows from the same property for each $\Theta_\pi$, for $\gamma_0$ and those $\theta_j$ prescribed in §7.3. As remarked in the proof of Theorem 6.4.6,

$$
g^{-1}\gamma_0 g \in jS(F) \iff \text{Ad}(g)(jS) \subset G_{\gamma_0} = J.
$$

One verifies readily the bijection

$$
\begin{cases}
(j,g) : & g \in J(F) \setminus G(F)/jS(F) \\
\text{s.t. } \gamma_0 \in \text{Ad}(g)jS(F) & \forall h \in G(F) \Rightarrow (j,g) \sim (\text{Ad}(h)j, gh^{-1})
\end{cases}
$$

By SS.2 in Definition 7.3.3 and the naturality of toral invariants, for $k = \text{Ad}(g) \circ j$ as above we have

$$
\theta_k = \theta_j \circ \text{Ad}(g)^{-1}, \quad \epsilon_k = \epsilon_j \circ \text{Ad}(g)^{-1}.
$$

All in all, the formula for $|D^G(\gamma)|^{1/2} S\Theta_\phi(\tilde{\gamma})$ can be written as a sum over $k$ modulo $J(F)$-conjugacy, followed by a sum over $\theta_k$. Furthermore, the $k$-sum can be partitioned according to the stable $J$-conjugacy classes $[j]$ of embeddings $S \hookrightarrow J$, whose composite with $J \hookrightarrow G$ lies in $\mathcal{E}$. This leads to the required formula $\sum_j \sum_k \theta_k \epsilon_k(\gamma_0) \hat{\nu}^j(kY, \log \gamma_{>0})$.

Our will show the stability of $\Theta_\phi$ (Definition 4.4.5) by analyzing the formula of Lemma 7.6.2. Consider compact elements $\tilde{\gamma} = \gamma_0 \gamma_{>0} \in \tilde{G}_{\text{reg}}$ as before, with $\gamma_0 = \mathbf{p}(\gamma_0)$, etc. For the sum $\sum_k$ to be non-vacuous, we can further assume that $\gamma_0$ lies in some $kS$. It follows from Theorem 6.2.2 that $\gamma_0^2 = 1$.

Some notational preparations are in order. Let $X \in \mathfrak{g}_{\text{reg}}(F)$, whose conjugacy class is parameterized by the datum $(L, L^2, y, d)$ with $L = \prod_{h \in H} L_h$, $L_h^2 = \prod_{h \in H} L_h^2$ as in §3.2 on the Lie algebra level, cf. the end of [35, §3.1]. Write $H_0 := \{ h \in H : L_h \text{ is a field} \}$ and $T := G_X$. As in §3.2 stable conjugacy $\text{Ad}(g) : X \to X'$ modulo conjugacy is parameterized by $\text{inv}(\text{Ad}(g)) = \text{inv}(X, X') \in H^1(F, T)$. Similarly, the stable conjugacy $\text{Ad}(h) : kS \to k'^{S}$ between embeddings $k, k' : S \hookrightarrow G$ will be parameterized by $\text{inv}(\text{Ad}(h)) = \text{inv}(k, k') \in H^1(F, S)$.

**Theorem 7.6.3.** When $4 \mid m$, the distribution $\Theta_\phi$ on $\tilde{G}$ is stable.

**Proof.** Recall that stable conjugacy operates on the level of $\tilde{G}_{\text{reg}}$ since $4 \mid m$. Consider a stable conjugacy $\text{Ad}(g) : \gamma \to \gamma'$ of compact elements of $\tilde{G}_{\text{reg}}(F)$, which lifts to $\text{CA}(g) : \gamma \to \gamma'$, Uniqueness of topological Jordan decompositions implies $\text{CA}(g)(\gamma_0) = \gamma_0'$, since $\text{CA}(g) : G_\gamma \to G_{\gamma'}$ is a group isomorphism (Proposition 4.3.5, AD.2).

Observe that upon adjusting $g$ by $G(F)$, we may assume $\gamma_0 = \gamma_0'$. Indeed, for $2$-torsion elements of $G(F)$, conjugacy in $G(\tilde{F})$ is the same as ordinary conjugacy since $H^1(F, \text{Sp}(W_+) \times \text{Sp}(W_-)) = 0$ where $W_\pm$ are the $\pm 1$-eigenspaces of $\gamma_0$; it remains to apply AD.3–4 of Proposition 4.3.5. After this adjustment, we have $J := Z_G(\gamma_0) = Z_G(\gamma_0')$ and $g \in J(\tilde{F})$.
Fix $[j]$. It remains to establish the stability of the piece
\[ \sum_{k \in [j]} (\epsilon_k \theta_k)(\tilde{\gamma}_0)^l(kY, \log \gamma > 0) \]
under $CAd(g)$ where $g \in J(\tilde{F})$. By Lemma 7.2.2, the character $\epsilon_k$ on $kS(F)$ is not affected by stable conjugacy. Therefore it suffices to look at
\[ \sum_k \theta_k(\tilde{\gamma}_0)^l(kY, \log \gamma > 0). \]

By writing $W = W_+ \oplus W_-$ according to the eigenvalues of $\gamma_0$, we may write $S = S_+ \times S_-$ and decompose $\theta_k = \theta_{k,+} \times \theta_{k,-}$ in parallel. The sum breaks into
\[ \sum_{k_+: S_+ \to Sp(W_+)} \theta_{k,+}(\tilde{\gamma}_{0,+})^lSp(W_+)(k_+Y_+, \log \gamma > 0, +) \]
\[ \sum_{k_-: S_- \to Sp(W_-)} \theta_{k,-}(\tilde{\gamma}_{0,-})^lSp(W_-)(k_-Y_-, \log \gamma > 0, -). \]

Caution: the characters $\theta_{k,\pm}$ are not necessarily the ones associated to some stable system for $p^{-1}(Sp(W_\pm))$, whence the different notation. Nonetheless, $g = (g_+, g_-) \in J(F)$ and $CAd(g) = CAd((1, g_-))CAd((g_+, 1))$ operates separately in $p^{-1}(Sp(W_\pm))$ as $CAd(g_\pm)$, a property that can be traced back to Theorem 4.1.3. Hence both of $\theta_{k,\pm}$ inherit the property SS.2 of Definition 7.3.3 under $CAd(g_\pm)$.

Our problem is thus reduced to the case $\gamma_0 = \pm 1$ and $J = G$, at the cost of replacing $\theta_k$ by some genuine character $\vartheta_k$ satisfying only SS.2. Given the AD.1 of Proposition 4.3.5, we may and do assume that $\tilde{\gamma}_0 = \pm 1 = \gamma_0$ via the splitting in Definition 7.1.4. Put
\[ X := \log \gamma > 0, \quad X' := \log \gamma' > 0 = Ad(g)X, \quad X, X' \in g_{reg}(F). \]

We have to show the constancy of $\sum_k \vartheta_k(\tilde{\gamma}_0)^lG(kY, X')$ when $g$ (thus $\tilde{\gamma}_0$, $X'$) varies in $G(F) \setminus (G/G_\gamma)(F)$. Observe that the sum over $k$ also varies in a stable conjugacy class.

In what follows, we shall regard $Y$ as an element of $g_{reg}(F)$, and $\vartheta^G$ as a function on $g_{reg}(F) \times g_{reg}(F)$, by using the $B$ from (6.4).

**Case A:** $\gamma_0 = 1$. We also have $\tilde{\gamma}_0 = CAd(g)(1) = 1$, therefore $\vartheta_k(\tilde{\gamma}_0) = 1 = \vartheta_k(\tilde{\gamma}_0)$. The required stability amounts to
\[ \sum_k \vartheta^G(kY, X) = \sum_k \vartheta^G(kY, X'). \tag{7.5} \]
This is assured by Waldspurger’s result [53, 1.6 Corollaire] with $G = Sp(W) = H$, or its non-standard version [52, §1.8].

**Case B:** $\gamma_0 = -1$. Treat the easier case $m \not\equiv 2 \pmod{4}$ first. Using the Proposition 4.4.2 with $\delta_0 = 1$, we see that $\tilde{\gamma}_0 = CAd(g)(-1) = -1$ for all $g$. Moreover, when $k$ gets replaced by a stable conjugate $k' = Ad(h)k$, SS.2 of Definition 7.3.3 and the previous step imply
\[ \vartheta_k(-1) = \vartheta_k(CAd(h)(-1)) = \vartheta_k(-1). \]
Hence $\vartheta_k(\tilde{\gamma}_0)$ is independent of $(k, g)$. The stability can thus be established as in Case A.

Henceforth assume $m \equiv 2 \pmod{4}$ in Case B. Recall the description of endoscopic data in [51, Chapitre X]. Choose an endoscopic datum $(SO(V_1, q_1), \ldots)$ of $G$ (regarding only the endoscopic group), where $dim_F V_1 = 2n$, such that there is a matching $Y \leftrightarrow Y_{SO} \in so(V_1, q_1)_{reg}$ between stable conjugacy classes. Such an endoscopic datum is necessarily elliptic since $Y \in s(F)_{reg}$ and $S/Z_G$ is anisotropic. Pick a transfer factor $\Delta^1$ on Lie algebras for this datum, which is canonical up to $C^\times$.

Claim: there exists a function $\Delta^2 : \{Z \in g_{reg}(F) : Z \overset{s}{\sim} X\} \to C^\times$ such that
\[ Z_1, Z_2 \overset{s}{\sim} X \implies \Delta^2(Z_2) = (\kappa, \text{inv}(Z_1, Z_2))\Delta^2(Z_1), \]
\[ \frac{\vartheta_k(\tilde{\gamma}_0)\Delta^2(X')}{\Delta^1(Y_{SO}, kY)} = \text{constant, when } k, g \text{ vary;} \]

recall Definition 3.2.5 for $\kappa$. By picking basepoints for $X'$ and $k$, this reduces to the observations below.
• The variation of \( g \) can be realized in various \( \text{SL}(2, K^p_{\mathbb{F}}) \) with \( h \in H_0 \), by the Lie algebra version of Proposition 3.3.4. Since \( \text{CAd}(g) \) is also realized in this manner, Proposition 4.4.2 implies that \( \gamma_0 = \text{CAd}(g)(-1) \) equals \( \langle \kappa_-, \text{Ad}(g) \rangle \cdot (-1) \). The same holds for \( \theta_k(\gamma_0) \) because \( \theta_k \) is genuine.

• when \( k \) is replaced by a stable conjugate \( \text{Ad}(h)k \); SS.2 and Proposition 4.4.2 imply

\[
(\kappa_-, \text{inv}(\text{Ad}(h))) \cdot \theta_k(\gamma_0) = \theta_k(\text{CAd}(h)(\gamma_0)) = \theta_k(\gamma_0).
\]

These match the behavior of \( \Delta^1(Y_{\text{SO}}, \cdot) \) (resp. \( \Delta^2 \)) when \( k \) (resp. \( g \)) varies; for \( \Delta^1 \) we invoke the description in [51, X.8]. This proves the claim and we are reduced to show that

\[
\Delta^2(X')^{-1} \sum_k \Delta^1(Y_{\text{SO}}, kY) j^G(kY, X')
\]

is independent of \( g \) or of the conjugacy class of \( X' \). Set \( i^G(Z, X) := |D^G(X)|^{-1} i(Z, X) \). Then

\[
\Delta^2(X')^{-1} \sum_k \Delta^1(Y_{\text{SO}}, kY) j^G(kY, X') = \gamma_0^{-1} \gamma_0(\text{so}(V_1, q_1))
\]

\[
\times \sum_{Y_1 \leq Y_{\text{SO}}} \sum_{Z_1 \in X'} w(Z_1)^{-1} \Delta^2(X')^{-1} \Delta^1(Z_1, X') i^{SO(V_1, q_1)}(Y_1, Z_1)
\]

by virtue of [53, p.155], where

• \( \gamma_0^{-1} \gamma_0(\text{so}(V_1, q_1)) \) are as in [53, p.154],

• \( Y_1, Z_1 \) range over the conjugacy classes in \( \text{so}(V_1, q_1) \),

• \( w(Z_1) \) is the number of conjugacy classes in the stable class of \( Z_1 \).

Next, note that \( Z_1 \leftrightarrow X' \iff Z_1 \leftrightarrow X \) since \( X \cong X' \) in \( g \). Assume that such a \( Z_1 \) exists, otherwise (7.6) reduces to 0 for all \( X' \cong X \), and there is nothing to prove.

All in all, we are reduced to show the constancy of \( \Delta^2(X')^{-1} \Delta^1(Z_1, X') \) when \( Z_1 \) is kept fixed. Again, this is because both factors undergo a sign change \( \langle \kappa_-, \text{inv}(X'', X') \rangle \) when \( X' \) is replaced by a stable conjugate \( X'' \); for \( \Delta^1(Z_1, \cdot) \) this is again a consequence of [51, X.8].

The case \( 4 \mid m \) requires different arguments.

**Lemma 7.6.4.** Suppose \( 4 \mid m \). For every \( \tilde{\gamma} \in \hat{G}_{\text{reg}} \), we have \( \Theta_{\theta_0}(\tilde{\gamma}) = 0 \) unless the image \( \gamma \in G_{\text{reg}}(F) \) of \( \tilde{\gamma} \) is topologically unipotent, in which case the formula of Lemma 7.6.2 reduces to

\[
|D^G(\gamma)||\tilde{\gamma}| \Theta_{\theta_0}(\tilde{\gamma}) = |S(F)_{p^r}| \sum_{\ell \in \mu_m, k \in E} i^G(kY, \log \gamma_{\geq 0}).
\]

**Proof.** We may assume \( \tilde{\gamma} \) to be a compact element. The formula in Lemma 7.6.2 contains a sum

\[
\left( \sum_{\theta_k} \theta_k \right)(\gamma_0)
\]

where \( \theta_k \) ranges over all genuine characters of \( \tilde{jS} \) that has a prescribed pro-\( p \) component. Since \( \tilde{jS} = jS(F)_{p^r} \times S(F)_{0^+} \), it remains to apply Fourier inversion.

**Theorem 7.6.5.** When \( 4 \mid m \), the distribution \( \Theta_{\theta_0} \) on \( \hat{G} \) is stable.

**Proof.** Following the paradigm of Definition 4.4.5, we consider a maximal \( F \)-torus \( T \), take \( \sigma \in \text{Sgn}_m(T) \) as in (4.8) and form the homomorphism \( \hat{T}^\sigma_{Q, m} \to \hat{T} \subset \hat{G} \) of (4.7). By Lemma 7.6.4, it suffices to consider \( \delta \in \hat{T}_{\text{reg}} \) of topologically unipotent image, or equivalently \( \delta_0 \in \mu_m \).

Consider an element \( (\delta, \delta_{Q, m}) \in \hat{T}^\sigma_{Q, m} \); we have to adjust \( \delta_{Q, m} \) to some \( \delta_{Q, m} \) to verify the requirement of Definition 4.4.5. For this purpose, we may translate \( \delta \) by \( \mu_m \) so that \( \delta = \delta_{Q, m} \).

1. Parameterize \( T \) by a datum \( (L, L', \ldots) \) as usual, with \( L = \prod_{h \in H} L_h \), etc. Recall that \( \sigma \in \text{Sgn}_m(T) = \{ \pm 1 \}^H \). Decompose \( \delta \) into \( (\delta_h)_{h \in H} \); each \( \delta_h \in L_h^\mu \) is still topologically unipotent. The decomposition also applies to \( \sigma, \delta_{Q, m} \) and it respects \( i_{Q, m} \). Hereafter we fix \( h \in H \) and work inside \( \text{SL}(2, L_h^\mu) \). In other words, we reduce to the case \( n = 1 \) modulo Weil restriction.
2. Assume \( n = 1 \). By the topological unipotence of \( \delta \) and \( p \mid m \), there exists a topologically unipotent \( \mu \in T_{Q,m}(F) \) such that \( \delta = \iota_{Q,m}(\mu) \). We contend that \( \sigma = 1 \) when \( T \) is anisotropic: otherwise we would have \( \delta \in \text{im}(\iota_{Q,m}) \cap ((-1) \cdot \text{im}(\iota_{Q,m})) \) that contradicts Proposition 4.2.9.
   - If \( T \) is split, we take \( \delta_{Q,m} = \delta_{Q,m}^0 \).
   - If \( T \) is anisotropic, then \( \sigma = 1 \) and we take \( \delta_{Q,m} = \mu \).

3. Reassembling these rank-one pieces, we obtain \( \langle \tilde{\delta}, \delta_{Q,m} \rangle \in T_{Q,m}^\sigma \).

Using Lemma 7.6.4, the stability of \( S\Theta_{[\rho]} \) amounts to

\[
\text{CAd}^\sigma(g)(\delta_{>0}, \delta_{Q,m}) = \langle \tilde{\eta}, \text{Ad}(g)(\delta_{Q,m}) \rangle \implies \sum_{k: \mathbb{N} \to G, \ k \in \mathcal{E}} e^G(kY, \log \gamma_{>0}) = \tilde{\eta}_0 \sum_{k: \mathbb{N} \to G, \ k \in \mathcal{E}} e^G(kY, \log \eta_{>0})
\]

for any stable conjugation \( \text{Ad}(g)(\delta) = \eta \), which also implies \( \text{Ad}(\delta_{>0}) = \eta_{>0} \). As seen in (7.5), the two sums \( \sum_k \) are equal, thus it suffices to show \( \tilde{\eta}_0 = 1 \). Now recall that \( \text{CAd}^\sigma(g) \) is built upon

- stable conjugacy in the case of \( \text{SL}(2, L^1_\mathbb{A}) \) (Definition 4.3.4), which uses \( g \in G_{ad}(F) \) and incorporates a factor \( C_m(\nu, t_0) \);
- the harmless \( G(F) \)-conjugacy.

Hence the calculation of \( \tilde{\eta}_0 \) to the case \( n = 1 \), which is dealt with as follows.

- When \( T \) is split, we have \( H^1(F, T) = 0 \) so \( \text{CAd}^\sigma(g) \) reduces to ordinary conjugacy by AD.3 of Proposition 4.3.5. Thus \( \tilde{\eta} = \text{Ad}(g)(\delta_{>0}) = \eta_{>0} \).

- When \( T \) is anisotropic, \( \sigma = 1 \), we may assume \( g \in G_{ad}(F) \) and \( \tilde{\eta} = C_m(\nu, t_0)\text{Ad}(g)(\delta_{>0}) \), where \( t_0 \in L^1 \) corresponds to \( \delta_{Q,m} \). The factor \( C_m(\nu, \cdot) \) is multiplicative and \( \mu_2 \)-valued. On the other hand \( p \neq 2 \) as \( p \mid m \), and \( t_0 \) is topologically unipotent since \( \delta_0 \) is. Hence \( C_m(\nu, t_0) = 1 \) and \( \tilde{\eta} = \eta_{>0} \).

Reassembling matters, we conclude that \( \tilde{\eta}_0 = 1 \) as desired. \( \square \)

8 A stable system for \( m \equiv 2 \mod 4 \)

For the definition of stable systems, see Definition 7.3.3. The case of \( m \in 2\mathbb{Z} \) has been discussed in Proposition 7.3.5. Now we address the case of \( m \equiv 2 \mod 4 \). Note that the only external evidence comes from the \( m = 2 \) case, see Theorem 9.3.3.

8.1 Moment maps

Let \( F \) be a field with \( \text{char}(F) \neq 2 \). Let \( (V, q) \) be a quadratic \( F \)-vector space with \( \dim_F V = 2n + 1 \), \( d^\delta(V, q) = 1 \). The corresponding special orthogonal group is \( \text{SO}(V, q) \).

For any given \( F \)-linear map \( T: W \to V \), define its adjoint \( ^*T \) by

\[
(^*T)v|w = q(v|Tw), \quad v \in V, \ w \in W.
\]

Note that this differs from [36, §6.1] by a sign. We say \( Y \in \mathfrak{sp}(W) \) corresponds to \( Y' \in \mathfrak{so}(V, q) \) if they are related by a \( T \in \text{Hom}_F(W, V) \) by the diagram below.

The arrows \( M_W, M_V \) are the moment maps. Note that \( \text{Sp}(W) \times O(V, q) \) acts on the left of \( \text{Hom}_F(W, V) \) as

\[
T \mapsto (g, h)T = hTg^{-1}, \quad (g, h) \in \text{Sp}(W) \times O(V, q).
\]

It is routine to verify that \( ^*((g, h)T) = g \cdot ^*T \cdot h^{-1} \). Hence

\[
M_W((g, h)T) = gM_W(T)g^{-1}, \quad M_V((g, h)T) = hM_V(T)h^{-1}, \quad (g, h) \in \text{Sp}(W) \times O(V, q).
\]

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Theorem 8.1.1. Suppose $F$ is a local field. The correspondence above yields a bijection

$$\mathfrak{sp}(W)_{\text{reg}} \overset{1:1}{\leftrightarrow} \bigsqcup_{(V,q)} \mathfrak{so}(V,q)_{\text{reg}}$$

where $(V,q)$ ranges over isomorphism classes of quadratic $F$-vector spaces with $\dim_F V = 2n + 1$ and $d^2(V,q) = 1$. Two elements match if and only if they have the same nonzero eigenvalues (counting multiplicities).

Proof. The archimedean case is [1, Proposition 2.5, Lemma 2.8], and the arguments therein work in general.

Remark 8.1.2. Suppose that $Y \in \mathfrak{sp}(W)_{\text{reg}}$ lands in $\mathfrak{so}(V,q)$. In the proof cited above, the quadratic $F$-spaces $(V,q)$ is obtained as follows. Define the quadratic form

$$q(Y) : w \mapsto \langle Yw|w \rangle, \quad w \in W.$$ 

There exists a class $a \in F^\times/F^\times 2$ such that $(V,q) := q(Y) \oplus \langle a \rangle$ satisfies $d^2(V,q) = 1$. To determine $a$, take $d^s$ on both sides to conclude that $(-1)^n \det Y = 1$ modulo $F^\times 2$, i.e., $a = (-1)^n \det Y$ mod $F^\times 2$.

Write $V = W \oplus F \mathfrak{pr}$ for the evident projection and inclusion. We can construct $T \in M_W^{-1}(Y)$ by taking $T = I$. Indeed, by definition $T = Y \circ \mathfrak{pr}$, hence $T \cdot T = Y$ and $Y' := T \cdot *T = I \circ Y \circ \mathfrak{pr} \in \mathfrak{so}(V,q)$ corresponds to $Y$.

Consider the maximal $F$-torus $S := Z_{\mathfrak{sp}(W)}(Y)$. It naturally sits in $\mathbf{SO}(W,q(Y)) \subset \mathbf{SO}(V,q)$ as a maximal torus, since for all $g \in S(F)$ we have

$$q(Y)(gw) = \langle Ygw|gw \rangle = q(Y)(g)w, \quad w \in W.$$ 

Remark 8.1.3. In the explicit construction of Remark 8.1.2, assume that $S$ is the split maximal torus associated to a symplectic basis $\{e_{\pm i}\}_{i=1}^n$ of $W$, thus $Y$ is diagonalizable. Then $\{e_{\pm 1}\}_{i=1}^n$ will become a hyperbolic basis (Definition 7.2.1) for the quadratic $F$-vector space $(W,q(Y)) \subset (V,q)$, as a quick computation in $\mathbf{SL}(2)$ shows. Consequently, $Y'$ belongs to the split maximal torus of $\mathbf{SO}(V,q)$ associated to the latter basis. This also implies that the Weyl groups of $S$ inside $\mathbf{Sp}(W)$ and $\mathbf{SO}(V,q)$ are naturally identified; the same holds for the roots. Caution: long roots of $\mathbf{Sp}(W)$ go to short roots of $\mathbf{SO}(V,q)$.

The correspondence via moment maps is stable under change of base fields, thus the conclusions above extend to non-split $S$: the absolute Weyl groups and roots are in bijection in a $\Gamma_F$-equivariant manner.

Let $F$ be local with residual characteristic $\neq 2$. To prove the next result, we shall adopt the language of lattice functions to describe the Bruhat–Tits buildings for classical groups, as summarized in [36, §4].

Definition 8.1.4. For an $F$-vector space of finite dimension, a lattice function is an assignment $s \mapsto \mathcal{L}_s$, where $\mathcal{L}_s$ are $\mathfrak{o}_F$-lattices in $V$, satisfying

- $s^l \geq s \iff \mathcal{L}_s \subset \mathcal{L}_s$;
- $\mathcal{L}_{s+s'}(\mathfrak{pr}) = \mathfrak{pr}\mathcal{L}_s$;
- $\mathcal{L}_s = \bigcap_{s'<s} \mathcal{L}_{s'}$.

Denote by $\text{Latt}_V^+$ the set of lattice functions. For any $\mathcal{L} \in \text{Latt}_V^+$ set $\mathcal{L}[r] : s \mapsto \mathcal{L}_{s+r}$. We also set $\mathcal{L}_{s+} := \bigcup_{s'>s} \mathcal{L}_{s'}$. Next, let $h$ be a quadratic or symplectic form on $V$. For every $\mathcal{L} \in \text{Latt}_V^+$, set

$$\mathcal{L}^l : s \mapsto \{v \in V : h(v|\mathcal{L}_{(-)}(s)) \subset \mathfrak{p}_F\}, \quad \mathcal{L}^l \in \text{Latt}_V^+.$$ 

We say $\mathcal{L}$ is self-dual if $\mathcal{L} = \mathcal{L}^l$; the self-dual lattice functions form a subset $\text{Latt}_V^{\text{self-dual}} \subset \text{Latt}_V^+$.

It is known that there exists an equivariant affine bijection $\text{Latt}_V^+ \to \mathcal{B}(U(V,h),F)$, where $U(V,h)$ stands for the isometry group of $(V,h)$, possibly disconnected. Also, $\text{Latt}_V$ is in equivariant affine bijection with $\mathcal{B}(\mathbf{GL}(V),F)$; furthermore $\mathfrak{sl}(V)_{\mathcal{L}^r} = \{A \in \mathfrak{sl}(V) : \forall s, \mathcal{L}_s \subset \mathcal{L}_{s+}\}$ under this identification.

Now consider the circumstance of Remark 8.1.2, write $G := \mathbf{Sp}(W)$, $H := \mathbf{SO}(V,q)$ and assume that

- $S := Z_G(Y)$ is a tame anisotropic maximal $F$-torus;
\[ v_F(\alpha(Y)) = r \] for all absolute roots \( \alpha \) of \( S \subseteq G \), for some \( r \in \mathbb{Q} \).

Denote by \( G \xrightarrow{\iota} S \xrightarrow{\pi} H \) the natural embeddings, and let \( x \in B(G, F) \), \( x' \in B(H, F) \) be determined by \( j, k \) as in [43]. Denote by \( L \in \text{Latt}_{W}^{(\varepsilon)} \) and \( L' \in \text{Latt}_{V}^{(\varepsilon)} \) the self-dual lattice functions corresponding to \( x \) and \( x' \), respectively.

**Lemma 8.1.5.** For \( L \) as above, we have \( YL_{s} = L_{s+r} \) for all \( s \in \mathbb{R} \).

**Proof.** The goal is to identify \( s \mapsto YL_{s} \) and \( L[r] \) in \( \text{Latt}_{W} \). Take a tamely ramified finite Galois extension \( L/F \) to split \( S \), and consider the base change map \( BC : B(GL(W), F) \to B(GL(W), L)^{\text{Gal}(L/F)} \). Note that \( BC[L[r]] = BC[L][r] \), provided that the simplicial structures are defined by the valuation \( v \) of \( L \) extending \( v_F \). Since \( BC \) is \( GL(W) \)-equivariant, \( BC(YL) = Y \cdot BC(L) \). Put \( L^2 : = BC(L) \). By tamely ramified descent on Bruhat–Tits buildings, \( BC \) is bijective and it suffices to check \( L^2[r] = YL^3 \).

What remains is easy: by the recipe of [43, Remark 3] there exists a symplectic basis of \( W \otimes_F L \) that diagonalizes \( S \), such that \( L^3 \) lies in the corresponding apartment (equivalently, \( L^3 \) corresponds to a splittable norm under this basis, see [36, Definition 4.1.2]). Since all eigenvalues of \( Y \) satisfy \( v(\cdot) = r \), an easy computation in the split case yields \( L^2[r] = YL^3 \) as asserted.

**Proposition 8.1.6.** In the circumstance above, the \( T = v \in \text{Hom}_F(W, V) \) in Remark 8.1.2 satisfies \( TL_s \subseteq L'_{s+\frac{r}{2}} \) for all \( s \in \mathbb{R} \).

**Proof.** Recall that \( (V, q) = (W, q(Y)) \oplus ((-1)^n \det Y) \). Take any self-dual lattice function \( M \) for \( ((-1)^n \det Y) \). Take

\[ L'' := L \left[ -\frac{r}{2} \right] \oplus M \subseteq \text{Latt}_V. \]

We contend that \( L'' \in \text{Latt}^{(\varepsilon)}_V \). Since the \( \varepsilon \)-operator is readily seen to commute with \( \oplus \), it suffices to show \( L' \left[ -\frac{r}{2} \right] \) is self-dual relative to \( q(Y) \). Lemma 8.1.5 implies that

\[
\left( L \left[ -\frac{r}{2} \right] \right)^{s} = \left\{ w \in W : (Yw)L_{(s-\frac{r}{2}^+)} \subseteq \mathfrak{p}_F \right\} \\
= \left\{ w \in W : (w)L_{(s+\frac{r}{2}^+)} \subseteq \mathfrak{p}_F \right\} \\
= (L')_{-s-\frac{r}{2}} = L'_{-s},
\]

thus \( L' \left[ -\frac{r}{2} \right] \) is self-dual. Lemma 8.1.5 also implies \( YL_s = L'_{s+\frac{r}{2}} \). Hence it remains to show that \( L'' = L' \).

Make a tame base change to a Galois extension \( L/F \) that splits \( S \). As seen in the proof of Lemma 8.1.5, the fact that \( L \) is determined by \( j \) means that \( L \) lies in the apartment \( A_j \subseteq B(G, L) \) associated to a symplectic basis \( \{ e_{\pm i} \} \), diagonalizing \( jS_i \). By Remark 8.1.3, \( \{ e_{\pm i} \} \) is also a hyperbolic basis for \( (W, q(W)) \otimes_F L \), diagonalizing \( kS_i \). We conclude that \( L'' = L' \left[ -\frac{r}{2} \right] \oplus M \) corresponds to a point of \( B(H, L) \) lying in the apartment determined by \( kS_i \), after base change.

The aforementioned point is \( \text{Gal}(L/F) \)-fixed as it is defined over \( F \). By the characterization [43] of \( x' \) via tamely ramified descent, we infer that \( x' \) is the point corresponding to \( L'' \). See also [36, §5.1].

### 8.2 The construction

The assumptions on \( F, \tilde{G}, \) etc. from §7.3 are in force. Assume \( m \equiv 2 \) (mod 4) and choose an additive character \( \psi \) of \( F \). According to Definition 7.3.3, given

- an inducing datum \( (E, S, \theta') \) (Definition 7.3.1),
- a \( G(F) \)-conjugacy class \( j : S \hookrightarrow G \) in \( E \),

we have to define a character \( \theta_j^M : jS(F)_\psi \to \{ \pm 1 \} \). To begin with, define

\[
a := a(\psi) = \min \left\{ b \in \mathbb{Z} : \psi|_{\mathfrak{p}_F^b} \text{ non-trivial} \right\}. \tag{8.2}
\]

Pick \( Y = Y_{\psi} \in \mathfrak{s}(F) \) such that for any \( j \in E \),

\[
\theta \circ t_{Q, m} = \theta^\psi \implies \theta \circ \exp = \psi(B(jY_{\psi}, j(\cdot))) : \mathfrak{s}(F)_{1/e} \to \mathbb{C}^\times. \tag{8.3}
\]

This is reminiscent of (7.4), but here we pass to \( s \) using \( B \) and make no assumption on \( a(\psi) \).
Lemma 8.2.1. We have $Y_\psi \in \mathfrak{s}(F)_{a(\psi) - \frac{1}{2}}$; only the coset $Y_\psi + \mathfrak{s}(\bar{F})_{a(\psi)}$ is canonical. Furthermore, $Y_\psi \in \mathfrak{s}_{\text{reg}}(F)$ and all eigenvalues $\lambda \in \bar{F}$ of $Y_\psi$ satisfy $v_F(\lambda) = a(\psi) - \frac{1}{2}$.

Proof. Set $\xi := \psi_\nabla(\psi)$. Then $Y_\psi = \varepsilon_F^a(\psi)Y_\xi$ and $a(\xi) = 0$. Our problem is thus reduced to the case $a(\psi) = 0$, which has been addressed in Proposition 6.4.4. \hfill \qedsymbol

Definition 8.2.2. For each $j \in K = \prod_{i=1}^n K_i$, $K_j^1 = \prod_{i=1}^n K_{j_i}^1$, $\bar{y} = (y_i)_{i \in I}$ and $\vec{c} = (c_i)_{i \in I} \in K^\times$, with $\tau(\bar{y}) = -\bar{y}$, $\tau(\vec{c}) = \vec{c}$ as usual. All $K_i$ are fields since $S$ is anisotropic. We caution the reader that the parametrization depends on $|\cdot|$, which will be rescaled later on. Forgetting $Y$ gives rise to the datum $(K, K^2, \bar{y})$ that parameterizes the conjugacy class of $j$.

Lemma 8.2.3. Let $\psi_1$ be any additive character of $K_{j_i}^1$, for a given $i \in I$. The value of $\theta_j^1$ at $-1 \in K_{j_i}^1$ equals

$$\gamma_{\psi_1}(q_{j_i}^0, (-1)^n \det Y) \gamma_{\psi_1}(1)^{-2} \gamma_{\psi_1}(d_{\psi_1}(-1)^n \det Y)^{-1} \cdot (-1, d_{\psi_1}(-1)^n \det Y)_{K_{j_i}^1, 2},$$

where $d_{\psi_1}$ is the class in $F^\times / F_1^\times$ represented by $D_i$, if $K_i = K_{j_i}^1(\sqrt{D_i})$.

Proof. The relation between Weil’s constant, discriminant and Hasse invariant over $K_{j_i}^1$ (see eg. [42, 1.3.4]) says that

$$e(V_{i,Y}, q_{i,Y}) = \gamma_{\psi_1}(q_{i,Y}) \gamma_{\psi_1}(1)^{-2} \gamma_{\psi_1}(d_{\psi_1}(-1)^n \det Y).$$

We have $\gamma_{\psi_1}(q_{i,Y}) = \gamma_{\psi_1}(q_{j_i}^0) \gamma_{\psi_1}((-1)^n \det Y)$ by the additivity of $\gamma_{\psi_1}$. It remains to calculate

$$d_{\psi_1} = d_{\psi_1}(V_{i,Y}, q_{i,Y}) = d_{\psi_1}(K_{j_i}^1, q_{j_i}^0, (-1)^n \det Y).$$

Note that $q_{j_i}^0(x|x) = 2y_i c_i N_{K_i/K_{j_i}^1}(x)$, therefore its discriminant is the same as that of the norm form $N_{K_i/K_{j_i}^1}(\cdot)$, which equals $d_i$ mod $(K_{j_i}^{1, x})^2$. \hfill \qedsymbol

Lemma 8.2.4. Suppose that the datum $\bar{y}$ or $\vec{c}$ is multiplied by $\vec{d} = (d_i)_{i \in I} \in (K^\times)^I$, then the value of $\theta_j^1$ at $-1 \in K_{j_i}^1$ is multiplied by $\text{sgn}_{K_i/K_{j_i}^1}(d_i)$.

Proof. Note that multiplying $\bar{y}$ by $\vec{d}$ will multiply det $Y$ by $N_{K/F}(\vec{d}) = N_{K_1/F}(\vec{d})^2$. Given Lemma 8.2.3, it suffices to apply [35, Lemma 4.13] to $K_{j_i}^1$ and $\psi_i$. \hfill \qedsymbol

By taking $\vec{d} \in N_{K/K_1}(K^\times)$, we infer that $\theta_j^1$ depends only on the equivalence class of $(K, K^2, \bar{y}, \vec{c})$. This is not enough: we need a homomorphism that depends only on the coset $Y + \mathfrak{s}(\bar{F})_0$. Moreover, $\theta^j$ intervenes only through its pro-$p$ component.
Proof. Recall that prescribing \( Y + s(F)_0 \) amounts to prescribing the pro-\( p \) component of \( \theta^o \). In view of Lemma 8.2.3, it suffices to fix \( i \in I \) and argue that

\[
\gamma_{\psi_i}((-1)^n \det Y), \quad \gamma_{\psi_i}(\delta_i(-1)^n \det Y), \quad (-1, \delta_i(-1)^n \det Y)_{K'_i,2}, \quad \gamma_{\psi_i}(q^0_{i,Y})
\]

are all determined by the coset. The following standard fact will be used: consider

- \( M \): a non-archimedean local field of residual characteristic \( \neq 2 \),
- \( \eta : M \to \mathbb{C}^\times \) is an additive character with \( a(\eta) \) defined as in (8.2),
- \( t \in M^\times \),
- \( \mathcal{L} := p_M^r \subset p_M^l : = \mathcal{L}' \), where \( r := \frac{a(\eta) + 1 - v_M(t)}{2} \).

Then \( \gamma_\eta(t) = g_\eta(t, \mathcal{L}')/[g_\eta(t, \mathcal{L})] \), with

\[
g_\eta(t, \mathcal{L}') := \sum_{x \in \mathcal{L}' / \mathcal{L}} \eta(tx^2).
\]

Here \( \eta(tx^2) \) depends only on \( x + \mathcal{L} \subset \mathcal{L}' \). Indeed, using \( [r] + [r] = a(\eta) + 1 - v_M(t) \), it is routine to check that \( f(x) := \eta(tx^2) \) equals 1 on \( \mathcal{L} \), and \( \mathcal{L}' = \{ y \in M : \eta(ty\mathcal{L}) = 1 \} \). Thus the required formula follows from [55, §16 and §27].

As a consequence, suppose \( t_1 \in M^\times \) satisfies \( v_M(t_1 - t) > v_M(t) \), then the \( r \) associated to \( t, t_1 \) are the same, whilst for all \( x \in \mathcal{L}', \)

\[
v_M((t-t_1)x^2) \geq v_M(t) + 1 + 2[r] \geq a(\eta) + 1 \implies \eta(tx^2) = \eta(t_1x^2).
\]

Hence \( \gamma_\eta(t) = \gamma_\eta(t_1) \).

Now apply this to \( M = K^+_i, \eta = \psi_i \) and \( t = (-1)^n \det Y, t_1 = (-1)^n \det(Y + Z) \), where \( Z \in s(F)_{a(\psi)} \). By Proposition 8.2.1, the eigenvalues \( \lambda \in F \) of \( Y \) satisfy \( v_F(\lambda) = a(\psi) - 1/e \). On the other hand, the eigenvalues of \( Z \) satisfy \( v_F(\lambda) \geq a(\psi) \); this is a consequence of the concrete description of Moy–Prasad filtrations inside \( gl(W) \), see [36, §4.1]. Since \( [Y, Z] = 0 \), we infer that \( v_F(\det(Y + Z) - \det Y) > v_F(\det Y) \), and same for \( v_M(\cdot \cdot \cdot) \). Therefore the previous result implies \( \gamma_{\psi_i}((-1)^n \det Y) = \gamma_{\psi_i}((-1)^n \det(Y + Z)) \).

Multiplying \( t, t_1 \) by \( d := d(K_i/K^+_i) \), the same argument gives

\[
\gamma_{\psi_i}(\delta_i(-1)^n \det Y) = \gamma_{\psi_i}(\delta_i(-1)^n \det(Y + Z)).
\]

Next, we contend that \( \gamma_{\psi_i}(q^0_{i,Y}) \) and \( (-1, \delta_i(-1)^n \det Y)_{K'_i,2} \) are both unaltered under \( Y \sim Y + Z \). Parameterize \( Y, Z \) by \( y, z \in K^+ \) as usual. The \( yi, z_i \) are actually eigenvalues of \( Y, Z \), thus \( \frac{y_i + z_i}{y_i} \in 1 + p_{K^+_i} \) by the foregoing discussions. Since the group \( 1 + p_{K^+_i} \) is pro-\( p \), we have

\[
\left( -1, \frac{\det(Y + Z)}{\det Y} \right)_{K'_i,2} = 1, \quad \text{sgn}_{K_i/K^+_i} \left( \frac{y_i + z_i}{y_i} c_i \right) = 1.
\]

The latter implies \( \gamma_{\psi_i}(q^0_{i,Y}) = \gamma_{\psi_i}(q^0_{i,Y + Z}) \) by Lemma 8.2.4. \( \square \)

One can also regard \( \theta^j \) as a character of \( j(S(F)/S(F)_0+) \). All the foregoing constructions hinge on \( \psi \) and \( \{\cdot\} \).

Theorem 8.2.6. The characters \( \theta^j \) in Definition 8.2.2 depend only on \( (E, S, \theta^o) \) and \( j \), and they form a stable system in the sense of Definition 7.3.3 by putting \( \theta j = \theta^j \).

Proof. The remarks after Lemma 8.2.4 entail that \( \theta^j \) is \( G(F) \)-invariant in \( j \); the same invariance holds for \( \theta^o \) by Remark 7.1.7. Lemma 8.2.5 shows that only the pro-\( p \) part of \( \theta^o \) matters for \( \theta^j \).

SS.1 The first part follows by construction. For the second part, Remark 3.1.4 asserts that multiplying \( \{\cdot\} \) by \( a \in F^\times \) amounts to replacing \( (K, K^2, \mathcal{C}) \) by \( (K, K^2, a\mathcal{C}) \). Similarly, replacing \( \psi \) by \( \psi_a \) is the same as replacing \( Y_\psi \) by \( Y_{\psi_a} = a^{-1}Y_\psi \) by (8.3); this is in turn equivalent to replacing \( \bar{y} \) by \( a^{-1}\bar{y} \). The required behavior of \( \theta^j \) is then ensured by Lemma 8.2.4.
SS.2 The invariance under $G(F)$-conjugacy has been observed above. In general, consider $j' = \text{Ad}(g) \circ j$ and $\tilde{g} \in jS$ as in Definition 7.3.3. Take the topological Jordan decomposition $\tilde{g} = g_0\gamma_{>0}$. The covering splits uniquely over pro-$p$ subgroups, thus

$$\text{CAd}(g)(\tilde{g}) = \text{CAd}(g_0)\text{CAd}(g)(\gamma_{>0}) = \text{CAd}(g)(\tilde{g}_0)\text{Ad}(g)(\gamma_{>0}).$$

On the other hand, the pro-$p$ parts of $\theta_j$, $\theta_{j'}$ coincides with those of $\theta_{j'}$, $\theta_{j}$ by SS.1, which are respected by $\text{Ad}(g)$. Therefore it remains to show that

$$\theta_{j'}(\text{CAd}(g)(\tilde{g}_0)) = \theta_j(\tilde{g}_0).$$

Recall that $\text{CAd}(g) = \text{CAd}(g')\text{CAd}(g'')$ where $(g', g'')$ is a factorization pair for $\text{Ad}(g)$ (Definition 4.3.8). The verification thus reduces to the case $g \in G_{\text{ad}}^{\flat}(F)$, and then to $\dim_F W = 2$ upon Weil restriction; recall §4.3 for this procedure. By Theorem 6.2.2 we have $\gamma_0 := p(\tilde{g}_0) \in \{\pm 1\}$, thus it suffices to deal with the case $\gamma_0 = -1$. Choose a parameter $(K, K^1 = F, \ldots)$ for $j$. Proposition 4.4.2 asserts that $\text{CAd}(g)(\gamma_0) = \text{sgn}_{K/F}(\nu(g))\tilde{g}_0$, with $\nu(g)$ coming from (3.5).

On the other hand, $\theta_{j'}(\tilde{g}_0) = \theta_{j'}(\tilde{g}_0)_{\gamma}$ since they rely on $\theta_j$ and the $-1$ prescribed in §7.1, which is insensitive to $j, j'$ when $\dim_F W = 2$. Compare $\theta_{j'}(\gamma_0)$ and $\theta_j(\gamma_0)$ next. Let $(K, K^1, c')$ and $(K, F, c^\circ)$ the parameters of $j$ and $j'$. Lemma 8.2.4 implies $\theta_{j'}(\gamma_0) = \text{sgn}_{K/F}(c'/c')(\tilde{g}_0)_{\gamma_0}$. On the other hand, Proposition 3.2.7 implies $\text{sgn}_{K/F}(c'/c') = \text{sgn}_{K/F}(\nu(g))$: both are identifiable with $\text{inv}(j, j')$.

SS.3 First fix $j \in E$. Lemma 8.2.5 asserts that $\theta_{j'}$ is determined by the pro-$p$ part of $\theta_j$. Next, fix $\theta_j$ and suppose that $jS = j'S$ where $j, j' \in E$; denote this common image by $R$. Then $j' = \text{Ad}(w)j$ for some $w \in \Omega(G, R)(F)$. By Proposition 3.1.5, $\text{Ad}(w)$ corresponds to the action by some $\varphi \in \text{Aut}(K, \tau)$ on parameters. An automorphism $\varphi$ induces a permutation $f$ on $I$ characterized by $\varphi|_K, K_i \cong K_{f(i)}$, thus the identification $\mu_{ij}^\circ \cong R(F)_{\text{pr}}$ changes by $f$. As remarked after Definition 8.2.2, $q_{i, Y} \simeq q_{f(i), Y}$ as quadratic vector spaces over $K_i^2 \simeq K_{f(i)}^2$. It follows that $\theta_{j'} = \theta_{j \text{Ad}(w)j}$ as functions on $R(F)_{\text{pr}}$.

\[ \square \]

8.3 Interplay

Keep the assumptions of §8.2. Let $(E, S, \theta_j)$ be as in Definition 7.3.1 and consider a $j \in E$. Define $Y = Y_\psi$ by (8.3). By Theorem 8.1.1, there exists a unique $(V, q)$ with $\dim_F V = 2n + 1$ and $d(q(V, q)) = 1$ such that $jS$ corresponds to some $Y' \in \mathfrak{h}_{\text{reg}}(F)$, where $H := \text{SO}(V, q)$; such a $Y'$ is unique up to conjugacy. Furthermore, Remark 8.1.2 says that we can take

$$\langle V, q \rangle := q(Y) \oplus \langle (-1)^n \det Y \rangle$$

and there is a natural embedding $k : S \hookrightarrow H$ determined by $j$.

Also recall Kaletha’s quadratic character $\epsilon_{S}$ (resp. $\epsilon_{kS}$) of $jS(F) \subset G$ (resp. $kS(F) \subset H$) from §7.2. Denote their pull-backs to $S(F)$ as $\epsilon_j$ and $\epsilon_k$, respectively.

**Theorem 8.3.1.** For every $\gamma \in S(F)$, let $\gamma = \gamma_0\gamma_{>0}$ be its topological Jordan decomposition. Then

$$\frac{\epsilon_k(\gamma)}{\epsilon_j(\gamma)} = \theta_{j'}(\gamma_{0})$$

where $\theta_{j'}$ is as in Definition 8.2.2, pulled back to $S(F)_{\text{pr}}$.

**Proof.** First observe that $kS \subset H$ satisfies Definition 6.1.1, since $G$ and $H$ share the same Weyl group whose actions on $jS$ and $kS$ match. This follows immediately from Remark 8.1.3.

By [24, Lemma 4.12] we have $\epsilon_j(\gamma) = \epsilon_j(\gamma_0)$ and $\epsilon_k(\gamma) = \epsilon_k(\gamma_0)$. It also implies

$$\frac{\epsilon_j(\gamma_0)}{\epsilon_k(\gamma_0)} = \frac{\prod_{\alpha \in R(G, jS)_{\text{sym}}/T_{\text{pr}}} f_{(G, jS)}(\alpha)}{\prod_{\beta \in R(H, kS)_{\text{sym}}/T_{\text{pr}}} f_{(H, kS)}(\beta)} \in \{\pm 1\}.$$
\( \alpha \) is a short symmetric root, \( \alpha(\gamma_0) = -1 \), then \( \beta(k\gamma_0) = -1 \) as well; in fact both roots take the form \( \epsilon r \pm \epsilon s \) or its negative, where \( 1 \leq r, s \leq n \). Since \( F_\alpha = F_\beta, F_\pm \alpha = F_\pm \beta \), the calculations in §7.2 lead to \( f_{G,j,S}(\alpha) = f_{H,KS}(\beta) \).

\( \alpha \) is a long root, say of the form \( \pm 2\epsilon_r \) with \( 1 \leq r \leq n \), then \( \beta = \pm \epsilon_r \) and \( \alpha(\gamma_0) = 1 \) always holds. In this case \( \alpha, \beta \) are both symmetric as \( S \) is anisotropic (Lemma 3.3.2). Let \((U, q_U)\) be the 3-dimensional \( F_\alpha \)-vector subspace with \( KS \)-weights \( \{\epsilon_r, 0, -\epsilon_r\} \). The calculations in §7.2 show that \((U, q_U)\) descends to a quadratic \( F_\pm \alpha \)-vector space \((U_\beta, q_\beta)\), and by (7.3)

\[
\tilde{f}_{H,KS}(\beta) = \epsilon(\text{SO}(U_\beta, q_\beta))(-1, d^k(U_\beta, q_\beta))_{F_\pm \beta, 2}.
\]

We shall determine \((U_\beta, q_\beta)\) in steps. First, we parameterize the conjugacy class of \( jY \) by a datum \((K, K^2, \bar{y}, \bar{c})\) as in Definition 8.2.2, and identify \( W \) with \( K \) to simplify notation. Let \( i \in I \).

**Step 1.** By the general recipe (2.4),

\[
K_i \otimes \bar{F} = K_i \otimes K_i^1 \left( K_i^2 \otimes \bar{F} \right) \sim K_i \otimes K_i^1 \otimes \bar{F} \otimes \text{Hom}_F(K_i^2, \bar{F}); \tag{8.4}
\]

the \( \Gamma_F \)-action on the right-hand side can be described by (2.5): it operates only on the second slot, and permutes the summands transitively.

Take \( \iota \in \text{Hom}_F(K_i^2, \bar{F}) \); there are exactly two \( \iota', \iota : K_i \rightarrow \bar{F} \) extending \( \iota \). Identifying \( \text{Hom}_F(K_i, \bar{F}) \) and \( \text{Hom}_F \left( K_i \otimes \bar{F}, \bar{F} \right) \), we obtain \( \pm \epsilon_i \in X^*(S_F) \) where \( \epsilon_i := \iota'|(K_i \otimes_F \bar{F}) \). They are the \( S \)-weights of the \( \iota \)-th component of \( K_i \otimes \bar{F} \) in the decomposition above; denote this space as \( K_i[i] \).

Also note that \( \text{Stab}_F \left( \{\pm \epsilon_i \} \right) = \text{Stab}_F \left( \iota \right) \) (resp. \( \text{Stab}_F \left( \epsilon_i \right) = \text{Stab}_F \left( \iota' \right) \)) corresponds to the intermediate field \( \iota(K_i) \simeq K_i \) (resp. \( \iota'(K_i) \simeq K_i \)).

**Step 2.** Define \( h_i, h^i \) as in (3.9). The involution on \( K_i \) transports to the right-hand side of (8.4), acting only on \( K_i \). Hence \( h_i \otimes \bar{F} \) equals \( h^i \otimes \bar{F} \otimes \text{Hom}_F(K_i^2, \bar{F}) \).

Let \( h_i[i] \) denote the \( i \)-th component of \( h_i \otimes \bar{F} \); it lives on the subspace \( K_i[i] = K_i \otimes_{K_i^1} K_i \) with \( S \)-weights \( \{\pm \epsilon_i \} \). Hence \( h_i[i] \) descends to the symplectic form \( h^i \) on the \( K_i^2 \)-vector space \( K_i \). The same descent works if we consider the symmetric forms \( (u, v) \mapsto h_i(y_i u | v), h^i(y_i u | v) \) instead, which yield the quadratic form \( q^i_{Y,Y} \) on \( K_i \).

**Step 3.** Now we may choose a symplectic basis for \( h^i \otimes K_i^2 \) with associated characters \( \pm \epsilon_i \in X^*(S_K) \) in the notation above; this is easily done by reducing to \( n = 1 \). By varying \( (i, \iota) \) and \( i \in I \), we obtain a symplectic basis \( \{\epsilon_{i,r} \}_{r=1}^n \) for \( K \otimes F \), as well as the adapted characters \( \pm \epsilon_1, \ldots, \pm \epsilon_n \). The procedure in Remark 8.1.2, 8.1.3 renders \( \{\epsilon_{i,r} \}_{r=1}^n \) into a hyperbolic basis for \( q(Y) \otimes \bar{F} \).

Now consider \( \beta \in R(H, kS)(\bar{F}) \) and let \( R(G, jS)(\bar{F}) \supseteq \beta \), so that \( \alpha = 2\epsilon_r \) for some \( r \) as in Step 3.

As remarked in Step 1 (cf. (3.8)),

\[
F_\beta = F_\alpha \simeq K_i, \quad F_{\pm \beta} = F_{\pm \alpha} \simeq K_i^1,
\]

so that \( \beta \in R(H, kS)(\bar{F})_{\text{sym}} \). Comparing the step 2 with §7.2 yields

\[
(U_\beta, q_\beta) \simeq \left( K_i, q_{0, Y}^i \right) \oplus \left( (-1)^n \det Y \right) = (V_i, q_i, Y).
\]

Thus for \( \beta \) as above, \( \tilde{f}_{H,KS}(\beta) \) equals the value of \( \theta_{j}^i \) at \(-1 \in K_i^1 \). Recall from the first part of our proof that

\[
\frac{\epsilon_j(\gamma)}{\epsilon_k(\gamma)} = \prod_{\beta \in R(H,KS)/\text{pr}_F \text{short} \beta(\gamma_0) = -1} \tilde{f}_{H,KS}(\beta);
\]

the product can be equivalently taken over \( i \in I \) by the construction of \( \{\epsilon_{i,r} \}_{r=1}^n \). If we write \( \gamma_0 = (\gamma_{0,i})_{i \in I} \) with \( \gamma_{0,i} \in \{\pm 1\} \subset K_i^1 \), then \( \beta(\gamma_0) = -1 \iff \gamma_{0,i} = -1 \) and the product is exactly \( \theta_{j}^i(\gamma_0) \). \( \square \)
9 Compatibilities

Fix a local field $F$ with $\text{char}(F) \neq 2$, and consider a symplectic $F$-vector space $(W, \langle \cdot | \cdot \rangle)$ of dimension $2n$. Set $G := \text{Sp}(W)$. To rule out trivial cases, we assume $F \neq \mathbb{C}$.

9.1 Review of Adams’ stable conjugacy

Fix an additive character $\psi$ of $F$. Let $H(W)$ be the Heisenberg group of $(W, \langle \cdot | \cdot \rangle)$, which has a smooth irreducible representation $(\nu_\psi, S_\psi)$ with central character $\psi$, unique up to isomorphisms. Well’s \textit{metaplectic group} is a topological central extension

$$1 \to \mathbb{C}^c \to \overline{G}_\psi \to G(F) \to 1,$$

where $\rho^\psi_h(h) = \rho_\psi(xh)$ for all $h \in H(W)$, and $(x, M_x)(y, M_y) = (xy, M_x M_y)$. Specifically, we choose a Lagrangian $\ell \subset W$ and follow [35, §2.4.1] to construct $(\rho_\psi, S_\psi)$ using the \textit{Schrödinger model} attached to $\ell$; there is then a set-theoretic section $x \mapsto (x, M_x)$ of $\overline{G}_\psi \to G(F)$. The multiplication is described by the \textit{Maslov cocycle} [35, Théorème 2.6]:

$$M_x[x]M_y[y] = \gamma_\psi(\tau(\ell, y\ell, xy\ell)) M_{x\ell}[xy].$$

(9.1)

Here, for any Lagrangians $\ell_1, \ell_2, \ell_3$ in $W$, one has the quadratic $F$-vector space $\tau(\ell_1, \ell_2, \ell_3)$ canonically constructed by T. Thomas [50]. This reduces $\overline{G}_\psi$ to $\mu_\psi \hookrightarrow \overline{G}_\psi^{(2)} \to G(F)$, by setting $\mu_\psi = \{ (x, z M_{\psi[x]} : z \in \mathfrak{m}_\psi \}$. To get a Lagrangian-independent definition, one may use the canonical intertwining operators between Schrödinger models; see [42, §2.1].

Furthermore, taking the commutator subgroup yields a further reduction $\mu_\psi \hookrightarrow \overline{G}_\psi^{(2)} \to G(F)$. A direct description of $\overline{G}_\psi^{(2)}$ is given by Lion and Perrin [42]

$$\overline{G}_\psi^{(2)} = \{ (x, \pm m(x\ell, \ell) M_{x[x]} \in \overline{G}_\psi : x \in G(F) \},$$

$m(\ell_1, \ell_2) := \gamma_\psi(1)^{n - \dim \ell_1 \cap \ell_2} \gamma_\psi(A_{\ell_1, \ell_2})$.

The notation is explained as follows.

- We take $\ell_1, \ell_2$ to be Lagrangians in $W$ endowed with orientations $o_1, o_2$. An orientation on a finite-dimensional $F$-vector space $V$ means a nonzero element from $\Lambda^\text{max} V$ taken up to $F^\times$, with the convention $\bigwedge^\text{max} \{0\} = F$.
- Put $\ell'_i := \ell_i / \ell_1 \cap \ell_2$ for $i = 1, 2$. The restriction of $\langle \cdot | \cdot \rangle$ to $\ell'_1 \times \ell'_2$ is non-degenerate and induces a pairing

$$\alpha : \bigwedge^\text{max} \ell'_1 \otimes \bigwedge^\text{max} \ell'_2 \to F.$$

Specifically, if $v^{(1)}_1, v^{(1)}_2, \ldots$ and $v^{(2)}_1, v^{(2)}_2, \ldots$ are dual bases, then $\alpha(v^{(1)}_1 \wedge \ldots, v^{(2)}_1 \wedge \ldots) = 1$. Now set $A_{\ell_1, \ell_2} := \alpha(o'_1, o'_2)$ by writing $o_i = o'_i \otimes c$ where $c$ is any orientation on $\ell_1 \cap \ell_2$.
- In $m(x\ell, \ell)$ we choose any orientation $o$ on $\ell$ and transport it to $x\ell$ by $\bigwedge^\text{max}(x)$. It turns out that $\overline{G}_\psi^{(2)} \to G(F)$ is isomorphic to the BD-cover $\tilde{G} \to G(F)$ in §4.1 with $m = 2$. Denote by $\sigma_{\ell, \psi}$ the set-theoretic section $x \mapsto (x, m(x\ell, \ell) M_{x[x]}$. Multiplication on $\overline{G}_\psi^{(2)}$ is given by Rao’s cocycle in terms of $\sigma_{\ell, \psi}$. When $n = 1$, we get Kubota’s cocycle $c$ in (4.4) by identifying $\sigma_{\ell_1, \psi}$ with $s$; see [39, Chapitre 3, 1.3] or [42, 2.4.2].

Let $P_\ell := \text{Stab}_G(\ell)$. It is a Siegel parabolic with Levi component $\text{GL}(\ell)$ once transversal Lagrangians $W = \ell \oplus \ell'$ are chosen. The Schrödinger model furnishes a splitting $\sigma_\ell : x \mapsto (x, M_{x[x]}$ of $\overline{G}_\psi^{(8)}$ over $P_\ell(F)$; see [39, Chapitre 2, II.6]. The following result will be needed in §9.3.
Proposition 9.1.1. We have \( \sigma(-1) = \frac{\gamma(g_{(1)})}{\gamma(h_{(-1)})} \sigma_{\text{LP}}(-1) \). Furthermore, the \( -1 \) in Definition 7.1.4 coincides with \( \sigma(-1) \) via \( \mathcal{G}^2 \hookrightarrow \mathcal{G}_{\psi}^{(2)} \).

Proof. For all \( x \in \text{GL}(\ell) \) we have \( \sigma_{\ell}(x) = (m(x, \ell, \ell))^{-1} \sigma_{\text{LP}}(x) \). Put \( x = -1 \) in the definition of \( m(x, \ell, \ell) \) to deduce the first assertion.

Take a basis of \( \ell \) and extend it to a symplectic basis of \( W = \ell \oplus \ell' \), so that \( \sigma_{\ell}(-1) = \prod_{a} \sigma_{\ell}(\tilde{a}(-1)) \) where \( a \) ranges over the positive long roots. One can calculate \( \sigma_{\ell}(\tilde{a}(-1)) \) inside rank-one pieces, see [39, Chapitre 2, II.6]. Since \( -1 \) and \( \bar{-1} \) also decompose in this manner, we can assume \( n = 1 \).

Let \( t \in F^{\times} \). By the discussions preceding [37, Corollaire 5.12] or a direct computation via Kubota’s cocycle,
\[
\sigma_{\text{LP}}(\tilde{a}(t)) = s \left( t \right)^{-1} = (t, t)^{-1} h_{\alpha}(t) = (t, -1)_{F,2} h_{\alpha}(t) \in \text{SL}(2, F).
\]

Using (5.3), we see \( (-1,-1)_{F,2} = \gamma_{\psi}(1)^4 \) so that \( \sigma_{\text{LP}}(-1) = \gamma_{\psi}(1)^4 \cdot \bar{-1} \). It remains to prove
\[
\gamma_{\psi}(1)^6 = e \left( \frac{1}{2}, (-1, -1) \right) = ( (-1, -1)_{F,2} \psi)^{-1}, \quad \text{i.e.} \quad \gamma_{\psi}(1,1) = e \left( \frac{1}{2}, -1, -1 \right).
\]

As \( \langle 1, 1 \rangle \) is the norm form of the F-algebra \( \mathbb{A} \mathbb{F}^+(X)/\mathbb{F}^+(X+1) \), from [23, Lemma 1.2] we deduce \( \gamma_{\psi}(1,1) = e(\frac{1}{2}, -1, -1)_{F,2} \psi \), noting the different normalization of \( \gamma_{\psi} \) therein. As \( (-1,1)_{F,2} = (-1,-1)_{F,2} = 1 \), we may replace \( \psi_{1/2} \) by \( \psi \).

The group \( \mathcal{G}_{\psi} \) and its reductions carry the Weil representation \( \omega_{\psi} = \omega_{\psi}^+ \oplus \omega_{\psi}^- \) on \( S_{\psi} \), which is genuine and canonically defined with respect to \( \psi \circ \langle \cdot | \cdot \rangle \). Here \( \omega_{\psi}^\pm \) are irreducible genuine admissible representations. There is also a canonical element \( -1 \in \mathcal{G}_{\psi} \) lying over \( -1 \in \text{GL}(F) \), described as \( (-1,M_{[-1]}) \) in the Schrödinger model, such that \( \omega_{\psi}^+(1) = \pm i \) and \( (-1)^2 = 1 \). Write \( \Theta_{\psi} = \Theta_{\psi}^+ + \Theta_{\psi}^- \) for the corresponding characters. We are in a position to state the notion of stable conjugacy after Adams.

Definition 9.1.2 (J. Adams). Call two elements \( \gamma, \delta \in \mathcal{G}_{\psi, \text{reg}} \) stably conjugate, if

- the images \( \gamma, \delta \in G_{\text{reg}}(F) \) are stably conjugate, and
- \( (\Theta_{\psi}^+ - \Theta_{\psi}^-)(\delta) = (\Theta_{\psi}^+ - \Theta_{\psi}^-)(\gamma) \).

This definition does not rely on the choice of Lagrangians. For details of these constructions, we refer to [35] and the bibliography therein. We record the key property below. For any \( x \in G(F) \), set \( \Gamma_x := \{ (w, xw) : w \in W \} \); it is a Lagrangian in \( W^- \oplus W \) where \( W^- := (W, -\langle \cdot | \cdot \rangle) \). Define the genuine function \( \mathcal{G}_{\psi} \to \mathbb{C} \)
\[
\nabla : (x, z M_{[-1]}(x)) \mapsto z \gamma \tau(\Gamma_{-x}, \Gamma_1, \ell \oplus \ell).
\]

Theorem 9.1.3. The function \( \nabla_{\mathcal{G}_{\psi, \text{reg}}} \) is invariant under \( G(F) \)-conjugation. Two elements \( \gamma, \delta \in \mathcal{G}_{\psi, \text{reg}} \) are stably conjugate if and only if their images \( \gamma, \delta \) are stably conjugate in \( G_{\text{reg}}(F) \) and \( \nabla(\gamma) = \nabla(\delta) \).

If \( \delta \in \mathcal{G}_{\psi, \text{reg}}^{(2)} \) is stably conjugate to \( \gamma \in \mathcal{G}_{\psi, \text{reg}} \), then \( \delta \in \mathcal{G}_{\psi, \text{reg}}^{(2)} \) as well.

Proof. From (9.1) we infer that \( M_{[-1]}(x) M_{[-1]}(x) = M_{[-1]}(x) \) since \( \dim \tau(\ell, x, x) = 0 \) by [35, Proposition 2.5]. Therefore when \( \det(x + 1) \neq 0 \), Maktouf’s character formula [35, Corollaire 4.4] becomes
\[
(\Theta_{\psi}^+ - \Theta_{\psi}^-)((x, z M_{[-1]}(x))) = (\Theta_{\psi}^+ + \Theta_{\psi}^-)((-1, M_{[-1]}((x, z M_{[-1]}(x))))
\]
\[
= \Theta_{\psi}((x, z M_{[-1]}(x)))
\]
\[
= \frac{\det(x + 1)}{\tau} z \gamma \tau(\Gamma_{-x}, \Gamma_1, \ell \oplus \ell).
\]

Hence \( \nabla_{\mathcal{G}_{\psi, \text{reg}}} \) is \( G(F) \)-invariant, and the first part follows. The second part is a consequence of [35, Théorème 4.2 (iii)].
Next, let $T \subset G$ be a maximal $F$-torus. Construct $T \subset G_T \subset G$ following Definition 3.3.3. By choosing a parameter $(K, K^T, c)$ for $T \to G$ via Proposition 3.1.3, with $K = \prod_{i \in I} K_i$ etc., we may identify $G^T(F)$ with $\prod_{i \in I} \text{SL}(2, K_i^T)$. We may regard each $\text{SL}(2, K_i^T)$ as $G_i := \text{Sp}(W_i)$ for a symplectic $K_i^T$-vector space $W_i$ of dimension 2 with suitably chosen Lagrangian $\ell_i$, which will be specified anon. To each $i \in I$ we construct Weil’s metaplectic group

$$1 \to \mathbb{C}^\times \to \overline{G}_{\psi,i} \overset{p_i}{\to} G_i(F) \to 1, \quad \psi_i := \psi \circ \text{tr}_{K_i^T/F}.$$  

On each $\overline{G}_{\psi,i}$ we define the genuine function $\nabla_i$ using (9.2) for $W_i, \psi_i$.

**Lemma 9.1.4.** The pull-back of $\overline{G}_\psi$ (resp. of $\overline{G}_\psi^{(2)}$) to $G^T(F)$ is isomorphic to the contracted product of $\mathbb{C}^\times \to \overline{G}_{\psi,i} \to G_i(F)$ (resp. $\mu_2 \to \overline{G}_{\psi,i}^{(2)} \to G_i(F)$), for $i \in I$. Under such an isomorphism, the restriction of $\nabla$ to $p^{-1}(G^T(F))$ coincides with $\otimes_{i \in I} \nabla_i$.

Note that the asserted isomorphism must be unique, since $G_i(F)$ is a perfect group.

**Proof.** Identify $(W_i, \langle \cdot, \cdot \rangle)$ with $\bigoplus_{i \in I} (K_i, h_i)$ using the parameter, where $h_i = \text{tr}_{K_i^T/F} \circ h^i$ are defined in (3.9). Select a Lagrangian $\ell_i \subset K_i$ relative to $h^i$. Then $\ell_i$ is also a Lagrangian for $(K_i, h_i)$ since it is totally isotropic for $h_i$ and has the right $F$-dimension. We use the Lagrangian $\ell := \bigoplus_{i \in I} \ell_i$ (resp. $\ell_i$) to realize $\overline{G}_\psi$ (resp. $\overline{G}_{\psi,i}$).

To prove the first assertion for $\overline{G}_\psi$, denote by $\overline{G}_\psi^T$ the contracted product in question. Represent the elements of $G^T(F)$ as $x = (x_i)_{i \in I} \in G^T(F)$ with $x_i \in \text{Sp}(K_i, h^i) \simeq \text{SL}(2, K_i^T)$.

Consider the bijection

$$\text{Bij} : \overset{\cdot}{p^{-1}}(G^T(F)) \to \overline{G}_\psi^T,$$

$$(x, z M_{\ell}[x]) \mapsto z \cdot \prod_{i \in I} (x_i, M_{\ell_i}[x_i])_{/K_i^T};$$

the final subscript indicates that we work over symplectic $K_i^T$-vector spaces. Note that $x \mapsto (x, M_{\ell}[x])$ is a continuous section over the open cell $\{x : x \ell \cap \ell = 0\}$; the same applies to each $\ell_i$ as well. Therefore it suffices to show that Bij is a homomorphism, which amounts to matching the Maslov cocycles from both sides.

The symplectic additivity of $\tau(\cdot, \cdot)$ (see [35, p.532]) implies

$$\tau(\ell, y \ell, xy \ell) = \bigoplus_{i \in I} \tau(\ell_i, y_i \ell_i, x_i y_i \ell_i).$$

Since $x_i, y_i, \ell_i$ all come from “upstairs” by forgetting $K_i^T$-structures, which is an exact functor, the construction of $\tau(\cdot, \cdot)$ in [50, §2.2.3] immediately leads to

$$\tau(\ell_i, y_i \ell_i, x_i y_i \ell_i) = \left(\text{tr}_{K_i^T/F}\right)_\ast \left(\tau(\ell_i, y_i \ell_i, x_i y_i \ell_i)_{/K_i^T}\right).$$

Now invoke the additivity of $\gamma_\psi$ to obtain

$$\gamma_\psi(\tau(\ell, y \ell, xy \ell)) = \prod_{i \in I} \gamma_{\psi_i} \left(\tau(\ell_i, y_i \ell_i, x_i y_i \ell_i)_{/K_i^T}\right).$$

The right-hand side matches the Maslov cocycle for $\overline{G}_\psi^T$. This proves the case of $\overline{G}_\psi$. Now notice that the pullback of $\overline{G}_\psi^{(2)}$ to $G^T(F)$ is closed under commutators, thus contains the contracted product of $\overline{G}_{\psi,i}^{(2)}, i \in I$. Both are twofold coverings of $G^T(F)$, hence they coincide and the case of $\overline{G}_\psi^{(2)}$ follows.

As for the second assertion, apply the same reasoning to (9.2) for the symplectic $K_i^T$-vector spaces $W_i^{-1} \oplus W_i$.

[\text{End proof}]

Last but not least, the adjoint $G(F)$-action on $\overline{G}_\psi^{(2)}$ extends uniquely to $\text{GSp}(W)$ by [39, Chapitre 4, I.8].

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9.2 Stable conjugacy for \( m = 2 \)

We begin with the case \( n = 1 \). Fix a symplectic basis \( e_1, e_{-1} \) for \( W \) and take \( \ell := Fe_1 \). Use the standard orientation generated by \( e_1 \) of \( \ell \) in the Lion–Perrin construction. Identifying \( W \) with \( F^2 \) in this basis, we have
\[
(r, s|r', s') = rs' - r's, \quad \ell = \binom{0}{1}, \quad \text{Sp}(W) = \text{SL}(2), \quad \text{GSp}(W) = \text{GL}(2).
\]

The similitude factor becomes determinantal. As seen in §9.1, \( \text{SL}(2, F) \simeq \text{GL}(2) \) by a unique isomorphism such that \( \sigma_{\text{LP}} \) matches Kubota's \( s \) in §4.2.

**Lemma 9.2.1.** Let \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, F)_{\text{reg}} \). Suppose \( c \neq 0 \), then
\[
\nabla(\sigma_{\text{LP}}(\gamma)) = \gamma(v((-c, c(2 + \text{tr}(\gamma))))).
\]

**Proof.** We have \( \sigma_{\text{LP}}(\gamma) = (\gamma, m(\gamma\ell, \ell)M_{\ell}[\gamma]) \). First calculate
\[
A_{\gamma\ell, \ell} = \langle a, c|1, 0 \rangle = -c \mod F^{\times 2},
\]
\[
m(\gamma\ell, \ell) = \gamma(v(1)^{1-0}1\gamma(-c) = \gamma(-c).
\]

Next, consider the following Lagrangians of \( W^\square := W^- \oplus W^+ \):
\[
\ell_1 := \ell \oplus \ell, \quad \ell_2 := \Gamma_1, \quad \ell_3 := \Gamma_{-\gamma},
\]
noting that \( \ell_3 \) is transversal to both \( \ell_1, \ell_2 \) by the assumptions on \( \gamma \). By into [42, Lemma 1.4.2], \( \tau(\ell_1, \ell_2, \ell_3) \) is Witt-equivalent to the quadratic form on \( \Gamma_1 \) defined by \( q_{123} : v \mapsto \langle \pi_1 v, \pi_3 v \rangle_{W, \square} \) where \( \pi_1, \pi_3 \) are the projections attached to \( W^\square = \ell_1 \oplus \ell_3 \). This form is degenerate: we will soon see that its radical is \( \Gamma_1 \cap (\ell \oplus \ell) = \ell \). We contend that \( q_{123} \) is Witt-equivalent to \( \langle -c(2 + \text{tr}(\gamma)) \rangle \).

To see this, represent the elements of \( W^\square \) as \( (x, y; x', y') \). Let \( v = (\alpha, \beta; \alpha, \beta) \in \Gamma_1 \). There is a unique decomposition
\[
v = (r, 0; s, 0) + (w, -\gamma w) = (r, 0; s, 0) + (t, u; -at - bu, -ct - du)
\]
where \( w = (t, u) \in W \), so that \( q_{123}(v) = -ru - s(ct + du) \). The resulting linear system
\[
r + t = \alpha = s - at - bu
\]
\[
u = \beta = -ct - du
\]
entails \( u = \beta \) and \( -ru - s(ct + du) = -r\beta + s\beta = (t - \alpha + s)\beta \). Furthermore,
\[
(s, t) = (at + b\beta + \alpha, t) = \left( a\beta(d + 1) -c + b\beta + \alpha, \frac{(d + 1)\beta}{-c} \right).
\]

This leads to
\[
q_{123}(v) = \frac{(d + 1)\beta}{-c} + \frac{a\beta(d + 1)}{-c} + b\beta + \alpha \beta
\]
\[
= \left( \frac{(a + 1)(d + 1)\beta}{-c} + b\beta \right) \beta = \frac{2 + a + d}{-c} \cdot \beta^2;
\]
here we used \( ad - bc = 1 \). Therefore \( q_{123} \) factors through the \( \beta \)-coordinate, and is Witt-equivalent to \( \langle -(2 + a + d)/c \rangle \simeq \langle -c(2 + a + d) \rangle \) as asserted.

Now apply the dihedral symmetry [35, p.532] of \( \tau(\cdots) \) to deduce the Witt equivalences
\[
\tau(\Gamma_{-\gamma}, \Gamma_1) \sim (-\gamma (\ell \oplus \ell), \Gamma_1, \Gamma_{-\gamma}) \sim (\ell(2 + \text{tr}(\gamma))).
\]

We conclude by comparing with (9.2) and (9.3). \( \square \)

**Lemma 9.2.2.** Let \( \gamma \in \text{SL}(2, F)_{\text{reg}} \) and \( \bar{\gamma} \in p^{-1}(\gamma) \). For every \( g \in \text{GL}(2, F) \) with \( \nu := \det g \) we have
\[
\nabla(g\bar{\gamma}g^{-1}) = \nabla(\bar{\gamma})C_2(\nu, \gamma)
\]
where \( C_2 \) is defined as in Definition–Proposition 4.2.7.
Proof. We may also assume \( \tilde{\gamma} = \sigma_{LP}(\gamma) \) since \( \nabla \) is genuine. Since \( \nabla(\tilde{\gamma}) = |\det(\gamma + 1)|^{1/2}(\Theta^+ - \Theta^-)(\tilde{\gamma}) \) is locally constant, upon perturbation we may further assume \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \) with \( c \neq 0 \). Parameterize the stable class of \( \gamma \) by \((K, F, \lambda)\) as in §4.2, where \( K \) is an etale \( F \)-algebra of dimension 2 and \( \lambda \in K^1 \); denote by \( \tau \) the nontrivial \( F \)-involution on \( K \). There exists \( \omega \in K^* \) such that \( \lambda = \omega/\tau(\omega) \). Hence

\[
2 + \text{tr}(\gamma) = N_{K/F}(1 + \lambda) = (\omega + \tau(\omega))^2N_{K/F}(\omega)^{-1}.
\]

By Lemma 9.2.1 and [42, Proposition 1.3.4], we express \( \nabla(\tilde{\gamma}) \) as

\[
\gamma(\tilde{\gamma}) = \gamma(\tilde{\gamma})(\tilde{\gamma})(\tilde{\gamma})^{-1}N_{K/F}(\omega)^{-1} = (\omega + \tau(\omega))^2N_{K/F}(\omega)^{-1} = (\omega + \tau(\omega))^2N_{K/F}(\omega)^{-1} - (\omega + \tau(\omega))^2N_{K/F}(\omega)^{-1}.
\]

Using \( (-c, c)^F = 1 \). As \( \nabla \) is \( G(F) \)-invariant, we may assume \( g_1 = \left( \begin{smallmatrix} 1 & \nu \\ 0 & 1 \end{smallmatrix} \right) \). It follows that \( \nabla(g_1\tilde{\gamma}g_1^{-1}) = \nabla(\gamma)(N_{K/F}(\omega), \nu)^F \). Since \( N_{K/F}(\omega) \) mod \( F \times 2 \) does not change.

We switch back to the case of general \( n \), identifying \( \mathfrak{G}_v^{(2)} \) with \( \hat{\mathfrak{G}} \) by the unique isomorphism.

**Theorem 9.2.3.** Adams’ notion of stable conjugacy in Definition 9.1.2, when restricted to \( \mathfrak{G}_v^{(2)} \simeq \hat{\mathfrak{G}} \), coincides with Definition 4.3.10 for \( m = 2 \).

**Proof.** Let \( \tilde{\gamma} \in \mathfrak{G}_v^{(2)} \) with image \( \gamma \) and \( T := Z_G(\gamma) \). Let \( \text{Ad}(g) : \gamma \mapsto \delta \) be a stable conjugation in \( G \). By [35, Lemme 5.7], there exists a unique \( \hat{\delta} \) \( \delta \in \mathfrak{G}_v^{(2)} \) with \( \text{Ad}(\hat{\delta}) = \hat{\delta} \), or equivalently \( \nabla(\text{Ad}(g)(\tilde{\gamma})) = \nabla(\hat{\gamma}) \) by Theorem 9.1.3.

Take a factorization pair \( (g', g'') \) for \( \text{Ad}(g) \) (Definition 4.3.8) so that \( \text{CAd}(g) = \text{Ad}(g'')\text{CAd}(g') \). Since \( \nabla \) is \( G(F) \)-invariant, we may assume \( g' = \tilde{g}' \in G_{\text{ad}}^T(F) \) and \( g'' = 1 \). Take a parameter \( (K, K^2, x, c) \) for the conjugacy class of \( \gamma \), with \( K = \prod_{\mathfrak{c} \in T} K_{\mathfrak{c}} \), etc.; see §3.1.

Write \( g = (g_i)_{i \in I} \). Since \( \text{CAd}(g) \) is the composition of \( \text{CAd}(g_i) \) in any order by AD.4 of Proposition 4.3.5, Lemma 9.1.4 and AD.1 reduce the problem to the case \( n = 1 \) upon passing to a finite separable extension of \( F \). Hence we may write \( \gamma = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \) and represent \( g \) by \( g_1 \in \text{GL}(2, F) \) by choosing a symplectic basis. Note that the \( \text{GL}(2, F) = \text{GSp}(W) \) action on \( \text{SL}(2, F) \) mentioned in §9.1 coincides with the one from Proposition 2.2.3, say by Proposition 2.4.8.

Now Lemma 9.2.2 implies \( \nabla(\text{CAd}(g)(\tilde{\gamma})) = \nabla(\tilde{\gamma}) \) by the very definition of \( \text{CAd}(g) \), which involves the same factor \( C_2(\det g_1, \gamma) \).

**9.3 Relation with \( \Theta \)-lifting**

Suppose \( \text{char}(F) = 0 \) and fix \( \psi \) to form Weil’s metaplectic group \( \bar{\mathfrak{G}}_v^{(2)} \) as in §9.1. Denote by \( \Pi_-\left( \bar{\mathfrak{G}}_v^{(2)} \right) \) the genuine admissible dual of \( \bar{\mathfrak{G}}_v^{(2)} \), i.e., the set of isomorphism classes of genuine irreducible admissible representations. Also denote by \( \hat{\mathfrak{G}} \to G(F) \) the BD-cover with \( m = 2 \). As mentioned in §9.1, there is a unique topological isomorphism \( \hat{\mathfrak{G}} \simeq \bar{\mathfrak{G}}_v^{(2)} \). Note that \( Y_{Q, 2} = Y \), thus the isogenies \( \iota_{Q, 2} \) are identity maps.

Let \( (V, q) \) be a quadratic \( F \)-vector space with \( \dim_F V = 2n + 1 \) and \( d^\pm(V, q) = 1 \). Denote by \( \Pi(\text{SO}(V, q)) \) (resp. \( \Pi(\text{O}(V, q)) \)) the admissible dual of \( \text{SO}(V, q) \) (resp. of \( \text{O}(V, q) \)). A fundamental result of Adams–Barbasch [2] and Gan–Savin [15], for archimedean and non-archimedean \( F \) respectively, says that for every \( \pi_{SO} \in \Pi(\text{SO}(V, q)) \), there exists a unique extension \( \pi_O \) to \( \text{O}(V, q) = \text{SO}(V, q) \times \{ \pm 1 \} \) such that the \( \theta \)-lift \( \theta_{v}(\pi_O) \) to \( \bar{\mathfrak{G}}_v^{(2)} \) is nonzero. Furthermore, it asserts that \( \pi_{SO} \mapsto \theta_{v}(\pi_O) \) yields a bijection

\[
\Pi_-\left( \hat{\mathfrak{G}} \right) \simeq \Pi_-\left( \bar{\mathfrak{G}}_v^{(2)} \right) \leftrightarrow \bigcup_{\dim_F V = 2n + 1 \atop d^\pm(V, q) = 1 \mod 2} \Pi(\text{SO}(V, q))
\]

preserving discrete series, supercuspidal representations, etc. All these \( \text{SO}(V, q) \) are pure inner forms of each other.

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Hereafter we assume $F$ is non-archimedean of residual characteristic $p \neq 2$. Observe that $^1\text{SO}(V,q) = \text{Sp}(2n, \mathbb{C}) \times W_F$. As $\psi$, $(W, \langle \cdot | \cdot \rangle)$ are chosen, this can be identified with $^1\tilde{G}$ by §5.1. Also recall that $\text{SO}(2n+1)$ and $\text{Sp}(W)$ share the same Weyl group $\Omega$ relative to some split maximal torus.

Now consider an epipelagic $L$-parameter (Definition 7.4.1)

$$\phi : \mathcal{W}_F \rightarrow ^1\tilde{G},$$

which factorizes into

$$\mathcal{W}_F \xrightarrow{\phi_{S_{Q,2},^{1j}}} ^1S_{Q,2} \xrightarrow{^{1j}} ^1\tilde{G}.$$

As remarked in §7.4 (see also [24, §5]), this is done by choosing a $\chi$-datum and works in the exactly same way for both $\tilde{G}$ and $\text{SO}(2n+1) = G_{Q,2}$ on the dual side. The parameter $\phi_{S_{Q,2},^{1j}}$ yields

- the triple $(\mathcal{E}, S, \theta')$ as in Definition 7.3.1, now with $\theta' : S(F) \rightarrow \mathbb{C}^\times$ an epipelagic character;
- a stable class $\mathcal{E}$ of embeddings $j : S \hookrightarrow G$ of maximal tori of type (ER), corresponding to an element of $H^1(F, \Omega)$;
- the element in $H^1(F, \Omega)$ also determines a stable class $\mathcal{F}$ of embeddings $k : S \hookrightarrow \text{SO}(V,q)$, where $(V,q)$ varies as in (9.5); see [25, §3.2] for generalities on this extension across inner forms.

For each $k \in F$, we can transport $\theta'$ to a character $\theta_k$ of $kS(F)$. Kaletha’s epipelagic supercuspidal $L$-packet is defined as

$$\Pi^\text{SO}_\phi := \{ \pi_{kS,\theta_k,\epsilon_{k,S}} : k \in F \} \subset \bigcup_{d^2=\dim F=2n+1, \text{mod } 1} \Pi(\text{SO}(V,q)),$$

where

- $\epsilon_{k,S} : kS(F) \rightarrow \{ \pm 1 \}$ is the character in [24, §4.6], see also §7.2;
- $\pi_{kS,\theta_k,\epsilon_{k,S}}$ is the irreducible supercuspidal representation of $\text{SO}(V,q)$ constructed in [24, §3], see also Theorem 6.3.2.

Remark 9.3.1. By [24, Proposition 5.7], $|\Pi^\text{SO}_\phi|$ equals $|\pi_0(C_{\phi}, 1)^\wedge| = |\mathcal{B}(S)|$. More precisely, $H^1(F, S) \simeq \mathcal{B}(S)$ (Kottwitz’s isomorphism) acts on the conjugacy classes of embeddings in $\mathcal{E}$ and $\mathcal{F}$, and makes both sets into torsors.

On the other hand, using the stable system for $m=2$ of Theorem 8.2.6, we construct the packet $\Pi_\phi \subset \Pi_\Omega(G)$ of Definition 7.4.4. Our aim is to show that $\Pi_\phi = \Pi^\text{SO}_\phi$ under (9.5). The main tool will be the theory of Loke–Ma–Savin [36]. To this end, we follow loc. cit. to assume

$$\psi|_{\sigma_F} \neq 1, \quad \psi|_{\sigma_F} = 1.$$

and take $\xi = \psi$ in the constructions of §6.3.

Before applying their results, notice that the inducing data of depth $\frac{1}{2}$ in loc. cit. take the form $(x, \lambda, \chi)$, where

$$x \in \mathcal{B}(G,F), \quad \lambda \in \tilde{\mathcal{V}}_{x,1/e}, \quad \chi \in \text{Hom}(S_\lambda, \mathbb{C}^\times), \quad S_\lambda := \text{Stab}_{G(F)}(\lambda)/G(F)_{x,1/e}.$$

Given $(\mathcal{E}, S, \theta')$ as above and $j \in \mathcal{E}$, the point $x$ will be associated to $j$, and $\lambda$ arises from $\theta'_j|_{jS(F)_{x,1/e}}$ (depending on $\psi$). Before explicating $\chi$, we split the cover over $\text{Stab}_{G(F)}(\lambda)$ using the following facts:

- by [36, §3.4] there exists an $\mathfrak{s}_F$-lattice $\Lambda \subset W$ such that $\Lambda = \{ w \in W : \langle w|\Lambda \rangle \subset \mathfrak{p}_F \}$ (i.e. self-dual), and $G(F)_x \subset \text{Stab}_{G(F)}(\Lambda)$;
- the lattice model [39, Chapitre 2, II.8] associated to $\Lambda$ furnishes a splitting $\sigma_\lambda : \text{Stab}_{G(F)}(\Lambda) \hookrightarrow \mathbb{G}_m(\mathbb{A}) \simeq \tilde{G}$, and $\sigma_{\lambda_{G(F)_x}}$ is independent of $\lambda$ by [36, Lemma A.3.1].

The character produced from $(\lambda, \chi)$ of $\text{Stab}_{G(F)_x}(\lambda)$ can thus be lifted to a genuine one of its preimage, and the remaining construction is as in §6.3.

By Lemma 2.4.9, the splitting restricted to $G(F)_{x,1/e}$ coincides with the one in §6.3. From Lemma 6.1.2 and (6.2) we have $\text{Stab}_{G(F)_x}(\lambda) = jS(F)_{\psi^*} \cong G(F)_{x,1/e}$, thus the crux is to compare the splittings over $jS(F)_{\psi^*}$.
Lemma 9.3.2. The genuine epipelagic representation \( \pi_{jS,\theta_j,s} \) corresponds to the datum \((x, \lambda, \chi)\) in \([36]\), where \(x, \lambda\) are determined as above, and

\[
\chi = \theta^0 \otimes \theta^1 \otimes \epsilon_{jS} : jS(F)_{p'} \to \mathbb{C}^\times;
\]

here \(\theta^0\) are transported from \(S(F)\) to \(jS(F)\).

Proof. By inspecting the construction of \(\theta_j\) in §7.3, it boils down to identify the splittings over \(jS(F)_{p'}\) given by (a) the lattice models, and (b) the recipe of Example 7.1.8. Recall that in Example 7.1.8, we have \(W = \bigoplus_{i \in I} W_i\) as joint eigenspaces under \(jS(F)_{p'} = \langle \pm 1 \rangle^I\), and lift each \((-1)_{Sp(W_i)}\) into \(\tilde{G}^0 \to \tilde{G}^0\). Let \(\Lambda\) be a self-dual lattice stabilized by \(jS(F)\). Therefore we have a decomposition \(\Lambda = \bigoplus_{i \in I} \Lambda_i\) where \(\Lambda_i = \Lambda \cap W_i\) is the joint eigen-lattice since \(p > 2\); each \(\Lambda_i\) is still self-dual. Cf. [35, p.551].

For each \(i\), there exist transversal Lagrangians \(W_i = \ell_i \oplus \ell'_i\) such that \(\Lambda_i = (\Lambda_i \cap \ell_i) \oplus (\Lambda_i \cap \ell'_i)\). Let us compare the splittings \(\sigma_{\Lambda_i}\) and \(\sigma_{\ell_i}\), the latter being reviewed before Proposition 9.1.1. By [35, Proposition 2.13] they agree over \(\text{Stab}(\Lambda_i)\cap \text{GL}(\ell_i) \supseteq -1_{\text{Sp}(W_i)}\). Proposition 9.1.1 ensures that \(\sigma_{\ell_i}(-1)\) equals the \(-1_{\text{Sp}(W_i)}\) in Definition 7.1.4, thus completes the proof. \(\square\)

More notations: given \((V, q)\) with \(\dim_F V = 2n + 1\), we systematically write \(H := SO(V, q)\). For an inducing datum \((x', \lambda', \chi')\) for epipelagic supercuspidals for \(O(V, q)\) as in [36], where \(\lambda' \in \mathcal{V}_{x', r}\), define \(S_{\lambda'}\) as the stabilizer of \(\lambda'\) in \(O(V, q)_{x', r}\) modulo \(H(F)_{x', r}\). The version defined using \(H(F)_{x', r}\) is denoted by \(S_H\). To distinguish \(G\) and \(H\), we shall also write \(\mathcal{V}_{x, r}\), etc. For both \(G, H\) we use the invariant form (6.4) to identify Lie algebras and their duals.

Theorem 9.3.3. For every additive character \(\psi\) and every epipelagic \(L\)-parameter \(\phi: W_F \to \tilde{G}\), in the notation of (9.5) we have

\[
\Pi_\phi = \{ \theta_\phi(\pi_O) : \pi_{SO} \in \Pi_{SO}^s \},
\]

where \(\Pi_\phi\) is constructed using the stable system of Theorem 8.2.6.

Proof. To begin with, we reduce the problem to the case that \(\psi\) induces a nontrivial character of \(\sigma_F/\rho_F\). Indeed, \(\psi\) enters in the identification \(\tilde{G}^0 \simeq \tilde{G}^0 \times W_F\), which is not fixed at this moment. It remains to prove that the packets from both sides are independent of \(\psi\): for \(\Pi_\phi\) this is Theorem 7.5.1, whilst for \(\{ \theta_\phi(\pi_O) : \pi_{SO} \in \Pi_{SO}^s \}\) this is [14, Proposition 11.1] that stems from [15, Theorem 12.1 (i)].

Let \(Y \in \mathfrak{g}_{reg}(F) \cap \mathfrak{a}(F)_{-1/e}\) be defined by \(\theta^0 \circ \exp = \psi(\mathbb{B}(Y, \cdot))\) as in §6.4, now with \(\psi = \xi\). Apply Remarks 8.1.2, 8.1.3 and Proposition 8.1.6 to see that for any \(j : S \leftrightarrow G\) in the given stable class, we have

- a quadratic \(F\)-vector space \((V, q) = (W, q(Y)) \oplus \langle (-1)^n \det Y \rangle\),
- a natural embedding \(k : S \hookrightarrow SO(V, q) = : H\),
- \(\iota : W \hookrightarrow V\) standing for the natural inclusion,
- \(\mathcal{L}, \mathcal{L}'\): the self-dual lattice functions corresponding to the points \(x \in B(G, F), x' \in B(SO(V, q), F)\) arising from \(j, k\) respectively.

They satisfy

\[
j_s(F)_{-1/e} \ni jY \xleftrightarrow{M_W} (\iota \in \text{Hom}_F(W, V)) \xrightarrow{M_V} kY \in k_s(F)_{-1/e},
\]

\[
\mathcal{L}_s \subset \mathcal{L}_{s'} \iff \frac{Y}{\det Y} = 0.
\]

The construction in §6.3 or [24, §3.3] associates to \(jY, kY\) the stable linear functionals \(\lambda \in \mathcal{V}_{x, 1/e}^G\) and \(\lambda' \in \mathcal{V}_{x', 1/e}'\). The formulas above “witness” the matching condition (M) of [36, Proposition 1.2.1] for \((x, \lambda)\) and \((x', -\lambda')\). The \(H\)-version of (6.2) entails isomorphisms

\[
S_{\lambda} \simeq jS(F)/jS(F)_{1/e} \cong jS(F)_{p'} \cong kS(F)/kS(F)_{1/e} \simeq S_{\lambda}'.
\]

We construct a map \(\alpha : S_{\lambda} \to S_{\lambda}\) as follows. For every \(\gamma \in kS(F)\), Remark 8.1.2 guarantees that the action (8.1) satisfies

\[
(1, k(\gamma)) \iota = k(\gamma) \circ \iota = \iota \circ j(\gamma) = (j(\gamma)^{-1}, 1)\iota;
\]

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likewise $-1_O \in O(V, q)$ satisfies $(1, -1_O)\epsilon = -\epsilon = (-1_S, 1)\epsilon$. It is routine to check that the surjective homomorphism
\[
\alpha : S_N = S^H_N \times \{ \pm 1_O \} \longrightarrow S_N
\]
\[
k(\gamma) \mapsto j(\gamma), \quad (\gamma \in S(F)_{F'})
\]
\[
-1_O \longmapsto -1_{1_S}.
\]
coincides with the $\alpha$ constructed in [36, A.2 Proof of Lemma 8.1.1, (ii)]. Now transport $\theta_j^I$, $\epsilon_j s$, $\epsilon_k s$ to $S(F)_{F'}$ via $j, k$, and invoke Lemma 9.3.2 to write
\[
\chi' := \chi \circ \alpha = (\theta_j^I \cdot \epsilon_j / \epsilon_k) \cdot \epsilon_k s \cdot \theta^\sigma \boxtimes [-1_O \mapsto \chi(-1_{1_S})]
\]
modulo transportation among $S, jS$ an $kS$. Theorem 8.3.1 asserts that $\theta_j^I \cdot \epsilon_j / \epsilon_k = 1$, so $\chi' = \epsilon_k s \theta^\sigma \boxtimes \cdots$. Denote by $\pi(x, \lambda, \chi) \in \Pi_-(\hat{G})$ and $\pi(x', \lambda', \chi') \in \Pi(O(V, q))$ the epipelagic supercuspidals so obtained. We are ready to apply [36, Theorem 1.2.3 (i)]: keeping track of the sign and contragredient therein, we have
\[
\theta_\omega \left( (\pi(x, \lambda, \chi)) \right) = \pi \left( (x', -\lambda', \chi'^{-1}) \right)
\]
under the $\theta$-lifting backwards to $O(V, q)$.

In terms of Kaletha’s data, $\pi(x', -\lambda', \chi'^{-1})$ is associated to the epipelagic character $(\theta^\sigma)^{-1}$ transported to $kS(F)$. Now vary the $G(F)$-conjugacy class of $\gamma \in \mathcal{E}$, so that the $\pi(x, \lambda, \chi)$ exhaust $\Pi_\psi$ without repetition. On the other hand, Theorem 8.1.1 implies that the corresponding $k$ exhausts the set $\mathcal{F}$ (modulo conjugation) of embeddings into pure inner forms of $SO(2n + 1)$, without repetition. In view of Remark 9.3.1, we obtain Kaletha’s $L$-packet for $SO(2n + 1)$ induced from $(\theta^\sigma)^{-1}$ by discarding the $\{ \pm 1_O \}$-components.

Finally, $(\theta^\sigma)^{-1} = \theta^\sigma \circ Ad(u_0)$ where $u_0 \in \Omega(H, kS)(\hat{F})$ is the longest element, thus defined over $F$. The $\Omega(H, kS)(F)$-action leaves Kaletha’s $L$-packet intact. All in all, we obtain $\Pi_\psi^{SO}$ on the $SO(2n + 1)$-side.

### 9.4 Theory of Hiraga–Ikeda

Hereafter we assume $m \in 2\mathbb{Z}$. Fix $\epsilon : \mu_m \overset{\sim}{\to} \mu_m$, and fix an additive character $\psi$ when $m \equiv 2 \pmod{4}$.

Following Hiraga–Ikeda, define two morphisms $\tau^\pm : PGL(2) \to SL(2)$ by
\[
\tau^\pm(\gamma) = \pm 1 \cdot (\det \hat{\gamma})^{-m/2} \hat{\gamma}^m, \quad \hat{\gamma} \in GL(2) : \text{representative}.
\]

Given $\delta \in SL(2)_{reg}$, put $T := Z_{SL(2)}(\delta)$. Up to stable conjugacy, the pair $(T, \delta)$ is parameterized by a 2-dimensional étale $F$-algebra $K$ with nontrivial $F$-involution $\tau$, and $x \in K^1$ as done in §4.2. Set $T_1 := Z_{GL(2)}(T)$, then $T(F) \simeq K^1$ and $T_1(F) \simeq K^x$.

Note that if $\delta = \tau^\pm(\gamma) \in SL(2, F)_{reg}$, then $\gamma \in T_1 / \mathbb{G}_m \subset PGL(2)$. Since $\det |\gamma|$ corresponds to $N_{K/F}$, by choosing a representative $\omega \in K^x$ of $\gamma$, we have $\det \gamma = N_{K/F}(\omega)$ and the definition of $\tau^\pm$ translates into
\[
\pm x = \omega^m N_{K/F}(\omega)^{-m/2} = (\omega / \tau(\omega))^{m/2}.
\]

Now consider the BD-cover $\mu_m \mapsto \widehat{SL}(2, F) \overset{p}{\to} SL(2, F)$ constructed in §4.1 (with $n = 1$). We shall employ the preferred section $s$ and the cocycle $c$ of (4.4). By convention $\delta := p(\delta)$. Fix a maximal $F$-torus $T \subset SL(2)$. Recalling (4.1), the discussion above leads to a bijection
\[
\left\{ (\gamma, \tilde{\delta}) \in PGL(2, F) \times T_{reg} : \delta = \tau^\pm(\gamma) \right\} \overset{1:1}{\longleftrightarrow} \hat{T}_{Q, m, reg}^+.
\]

Here $\hat{T}_{Q, m, reg}^+ := \left\{ (\tilde{\delta}, \delta_0) \in T_{Q, m}^+ : \delta \in T_{reg}(F) \right\}$, see (4.7), and $\delta_0$ corresponds to $\omega / \tau(\omega) \in K^1$. We can rephrase the transfer factor of Hiraga–Ikeda as follows.

**Definition 9.4.1** (K. Hiraga, T. Ikeda). Let $\sigma \in \{ +, - \}$, $T \subset SL(2)$ be a maximal torus, and $(\tilde{\delta}, \delta_0) \in \hat{T}_{Q, m, reg}^+$. Write $\delta = \sigma(\delta)$ for some $\epsilon \in \mu_m$ and take $\omega \in K^x$ such that $\delta_0$ is parameterized by $\omega / \tau(\omega)$.
1. Suppose $m \equiv 2 \pmod{4}$. Set
\[
\Delta^\pm(\delta, \delta_0) := \varepsilon \cdot \frac{\gamma(1)}{\gamma(N_{K/F}(\omega))} \cdot (N_{K/F}(\omega), -x(\delta))_{F,2} \quad (\sigma = +),
\]
\[
\Delta^\pm(\delta, \delta_0) := \gamma(1)^2 \Delta^\pm(s(-1)\delta, \delta_0) \quad (\sigma = -).
\]

2. Suppose $4 \mid m$. Set
\[
\Delta^\pm(\delta, \delta_0) := \varepsilon \cdot (N_{K/F}(\omega), -x(\delta))_{F,2} \quad (\sigma = +),
\]
\[
\Delta^\pm(\delta, \delta_0) := \Delta^\pm(s(-1)\delta, \delta_0) \quad (\sigma = -).
\]

**Remark 9.4.2.** In view of the Remark 4.2.1, the original factor $\Delta^\pm$ of Hiraga–Ikeda lives on the “opposite” central extension of $\text{SL}(2, F)$ by $\mu_m$, which can be obtained from ours by changing $\varepsilon$ to $\varepsilon^{-1}$ by Remark 2.4.4. Thus there is no essential difference.

**Proposition 9.4.3** (Hiraga–Ikeda). The factors $\Delta^\pm$ are invariant under adjoint $\text{SL}(2, F)$-action.

**Proof.** As $s(-1)$ is central by Proposition 2.2.3, it suffices to consider $\Delta^+$. Let $N := N_{K/F}(\omega)$. Conjugation does not change $N \mod F^\times 2$, thus it remains to show that for $\delta' = \text{Ad}(g)(\delta)$ and $\varepsilon \sigma(\delta') = \text{Ad}(g)(\sigma(\delta))$ where $g \in \text{SL}(2, F)$, we have $\varepsilon(N, x(\delta))_{F,2} = (N, x(\delta))_{F,2}$. Note that $\varepsilon \in \mu_m$ is unique since $\delta$ is a good element.

Suppose that $\tau^+(\gamma) = (\det \gamma)^{-m/2} \gamma = \delta$, where $\gamma \in \text{PGL}(2)$ has representative $\hat{\gamma}$ parameterized by the chosen $\omega \in K^\times$. Put $\gamma' := \text{Ad}(g)(\gamma)$. Since $(N/m^2, \gamma^{-1})_{F,m} = (N, x(\delta))_{F,2}$ and similarly for $x(\delta')$, the following holds in $\text{GL}(2, F)$:
\[
(N, x(\delta))_{F,2} s(\gamma^m) = s(N/m^2) s(\delta), \quad (N, x(\delta'))_{F,2} s(\gamma^m) = s(N/m^2) s(\delta').
\]
By the easy fact that $\gamma \mapsto s(\gamma^m)$ respects conjugation and [13, p.130 + Lemma 1.2.1], we have $\text{Ad}(g)(s(\gamma^m)) = s(\text{Ad}(g)(\gamma^m))$. On the other hand, it is easily seen that $s(N/m^2)$ centralizes $\text{SL}(2, F)$. All these combine into
\[
(N, x(\delta))_{F,2} s(\gamma^m) = \text{Ad}(g)(s(N/m^2)) s(\delta) = \varepsilon s(N/m^2) s(\delta) = \varepsilon(N, x(\delta'))_{F,2} s(\gamma^m).
\]
This proves the required equality. \(\square\)

**Proposition 9.4.4.** When $m = 2$, we have $\Delta^+(\tilde{\delta}, \delta_0) = \nabla(\tilde{\delta})$.

**Proof.** Write $\delta = (a \ b \ c \ d)$. Both sides being $\text{SL}(2, F)$-invariant (Proposition 9.4.3) and genuine, we may adjust $\tilde{\delta}$ to assume $c \neq 0$ and $\delta = s(\delta)$. Now compare $\Delta^+(\tilde{\delta}, \delta_0)$ with (9.4), identifying $s(\delta)$ with $\sigma_{\ell}(\delta)$.

**Lemma 9.4.5.** When $m \equiv 2 \pmod{4}$, the factors $\Delta^+(\tilde{\delta}, \delta_0)$ depend only on $\tilde{\delta}$.

**Proof.** It suffices to show that $\Delta^+(\tilde{\delta}, \delta_0)$ is independent of $\delta_0$ or $\omega$. When $T$ is anisotropic (resp. split), Proposition 4.2.9 implies that $\delta$ determines $N_{K/F}(\omega)$ uniquely (resp. up to $\mu_m/2$). Since $m/2$ is odd, $(N_{K/F}(\omega), x(\delta))_{F,2}$ is unaffected. \(\square\)

**Remark 9.4.6.** When $m = 2$, in [35, Définition 5.9] are defined the two elliptic endoscopic data $(1, 0), (0, 1)$ of $\text{SL}(2, F)$ as well as the transfer factors $\Delta_{1, 0}, \Delta_{0, 1}$, both viewed as genuine functions on $\text{SL}(2, F)_{\text{reg}}$. Proposition 9.4.4 amounts to $\Delta^+ = \Delta_{(1, 0)}$. Consider $\Delta^-$ next. Embed $\text{SL}(2, F)$ into $\text{GL}^\times$, then $\gamma(1)^2 s(-1) \in \text{SL}(2, F)^\times$ corresponds to $\frac{\tau^-(1)}{\gamma(1)^2} \sigma_{\ell}(-1) = \sigma_{\ell}(-1)$ (Proposition 9.1.1). It follows from [35, Proposition 5.16] that $\Delta^- = \Delta_{(0, 1)}$.

For any maximal torus $T \subset \text{SL}(2)$ we have $\kappa_-$ from Definition 3.2.5; denote by $\kappa_+$ the trivial character of $H^1(F, T)$.

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Theorem 9.4.7. Suppose $\text{Ad}(g) : \delta \mapsto \eta$ is a stable conjugation in $\text{SL}(2, F)_{\text{reg}}$. Let $\tilde{\delta}, \tilde{\eta} \in \text{SL}(2, F)$ be their preimages.

- When $m \equiv 2 \pmod{4}$ and $\tilde{\eta} = \text{CAd}(g)(\tilde{\delta})$ (dropping references to $\delta_0, \eta_0$ by Corollary 4.4.4), we have
  \[ \Delta^\pm(\tilde{\eta}, \ldots) = \langle \kappa_{\pm}, \text{inv}(\delta, \eta) \rangle \Delta^\pm(\tilde{\delta}, \ldots) \]

- When $4 \mid m$ and $(\tilde{\eta}, \eta_0) = \text{CAd}^\pm(g)(\tilde{\delta}, \delta_0)$, we have
  \[ \Delta^\pm(\tilde{\eta}, \eta_0) = \Delta^\pm(\tilde{\delta}, \delta_0). \]

Proof. By the $\text{SL}(2, F)$-invariance of $\Delta^\pm$ (Proposition 9.4.3), it suffices to consider stable conjugacy realized by $\text{Ad}(g) : \delta \mapsto \eta$ such that $g = \left( \begin{smallmatrix} 1 & \nu \\ \nu & 1 \end{smallmatrix} \right) \in \text{PGL}(2, F)$, where $\nu \in F^\times$. If $T$ splits then $\text{Ad}(g)$ reduces to ordinary conjugacy and we conclude by AD.3 of Proposition 4.3.5. Hereafter assume $T$ anisotropic, so that $\delta, \eta$ take the form \( \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) \) with $c \neq 0$. A routine computation in $\text{GL}(2, F)$ using (4.4) shows
\[
\textbf{A}(g)s \left( \begin{smallmatrix} 1 & \nu \\ \nu & 1 \end{smallmatrix} \right)^{-1} = s \left( \begin{smallmatrix} 1 & \nu^{-1} \\ \nu^{-1} & 1 \end{smallmatrix} \right), \quad \text{Ad}(g)s \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) = s \left( \begin{smallmatrix} a & \nu^{-1}b \\ \nu c & d \end{smallmatrix} \right) \quad (c \neq 0).
\]
The cases of $\Delta^+$ follow because $\textbf{A}(\text{Ad}(g)\delta) = \nu \textbf{A}(\delta)$ and $C_m(\nu, \delta_0) = (N_{K/F}(\omega), \nu)_2$. The case of $\Delta^-$ for $4 \mid m$ follows by a comparison of the definitions of $\text{CAd}^-(g)$ and $\Delta^-$. The case of $\Delta^-$ for $m \equiv 2 \pmod{4}$ is accounted by Proposition 4.4.2, which says $s(-1)\text{CAd}(g)(\tilde{\delta}) = \text{sgn}_{K/F}(\nu)\text{CAd}(g)(s(-1)\tilde{\delta})$. \[ \square \]

When $m = 2$, Theorem 9.4.7 reduces to [35, Proposition 5.13].

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