Hamiltonian reduction of free particle motion
on group SL(2, \mathbb{R})

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Abstract

The structure of the reduced phase space arising in the Hamiltonian reduction of the phase space corresponding to a free particle motion on the group SL(2, \mathbb{R}) is investigated. The considered reduction is based on the constraints similar to those used in the Hamiltonian reduction of the Wess–Zumino–Novikov–Witten model to Toda systems. It is shown that the reduced phase space is diffeomorphic either to the union of two two-dimensional planes, or to the cylinder $S^1 \times \mathbb{R}$. Canonical coordinates are constructed for the both cases, and it is shown that in the first case the reduced phase space is symplectomorphic to the union of two cotangent bundles $T^*(\mathbb{R})$ endowed with the canonical symplectic structure, while in the second case it is symplectomorphic to the cotangent bundle $T^*(S^1)$ also endowed with the canonical symplectic structure.

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1 Introduction

Presently there is the well known method of obtaining various Toda systems by Hamiltonian reduction of the Wess–Zumino–Novikov–Witten (WZNW) model. In particular, the WZNW model for the case of the Lie group $\text{SL}(2, \mathbb{R})$ gives the simplest Toda system, the Liouville equation. The main technical tool used in the reduction under discussion is the Gauss decomposition of the group element describing the configuration of the WZNW model (see, for example, [1]). Already in an early paper on the Hamiltonian reduction of the WZNW model [2], it was emphasised that the Gauss decomposition is local. Therefore, the reduced phase space coincides with the phase space of the corresponding Toda model only locally and may have more complicated global structure. Unfortunately, one has not yet succeeded to obtain an exhaustive information on the topology of the reduced phase space for the case of the total two dimensional WZNW model.

In this connection, in work [3] a one dimensional model obtained from the WZNW model by the restriction to configuration which do not depend on the space variable, was considered. Actually, this is the model describing the motion of a particle on the corresponding group manifold. The authors of work [3] demonstrated that for the case of the Lie group $\text{SL}(2, \mathbb{R})$, in accordance with the value of the parameters characterising the reduction, there are two different cases. In the first case, the reduced phase space consists of the union of two phase spaces of the Liouville model. In the second one, the phase space has a more complicated topology. A detailed, although not exhaustive, investigation of the reduced phase spaces for various Lie groups was performed in [5].

A quantisation of the reduced system for the case of the Lie group $\text{SL}(2, \mathbb{R})$ was considered in paper [4]. The reduced phase space was considered there as a surface glued from two patches. The quantisation was performed separately for each, then a procedure of sewing the corresponding wave functions was performed. Another variant of a quantisation procedure, which is also based on a local representation of the phase space, is given in [6].

From our point of view, a quantisation based on the global description of the phase space would be most convincing. In the present work we make a first step in such a direction by demonstrating that the reduced phase space of the model based on the Lie group $\text{SL}(2, \mathbb{R})$ is diffeomorphic either to the union of two planes, or to the cylinder $S^1 \times \mathbb{R}$. For both cases we introduce canonical coordinates and show that in the first case the reduced phase space is symplectomorphic to the union of two cotangent bundles $T^*(\mathbb{R})$ endowed with the canonical symplectic structure, while in the second case it is symplectomorphic to the cotangent bundle $T^*(S^1)$ also endowed with the canonical symplectic structure. Note that in work [6] it was already observed that the reduced phase space has the topology described above, however the differential geometric equivalence was not proved.

2 Matrix Lie groups

Let $G$ be a real matrix Lie group, in other words, some Lie subgroup of the Lie group $\text{GL}(m, \mathbb{R})$. Identify the Lie algebra $\mathfrak{g}$ of the Lie group $G$ with the corresponding subalgebra of the Lie algebra $\mathfrak{gl}(m, \mathbb{R})$. Denote by $g = \|g_{ij}\|$ the matrix–valued function on $G$, defined
as
\[ g_{ij}(a) = a_{ij}, \]
for any \( a = \|a_{ij}\| \in G. \)

The Maurer–Cartan form of \( G \) is the matrix–valued 1–form, given by the relation
\[ \theta = g^{-1} dg. \] (2.1)

The form \( \theta \) is invariant with respect to the right shifts in \( G \), takes values in the Lie algebra \( \mathfrak{g} \), and satisfies the equality
\[ d\theta + \theta \wedge \theta = 0. \] (2.2)

Introduce on \( G \) local coordinates \( y^\mu \) and represent the Maurer–Cartan form as
\[ \theta = (g^{-1}\partial_\mu g) dy^\mu, \] (2.3)

where \( \partial_\mu = \partial/\partial y^\mu \). The matrix–valued functions \( g^{-1}\partial_\mu g \) take values in the Lie algebra \( \mathfrak{g} \). Fix some basis \( e_\alpha \) of \( \mathfrak{g} \). The corresponding structure constants \( c_{\alpha\beta}^\gamma \) of the Lie algebra \( \mathfrak{g} \) are defined by the relations
\[ [e_\alpha, e_\beta] = c_{\alpha\beta}^\gamma e_\gamma. \]

Expanding \( g^{-1}\partial_\mu g \) over the basis \( e_\alpha \), we get
\[ \theta = e_\alpha \theta^\alpha_\mu dy^\mu. \] (2.4)

It is not difficult to get convinced that relation (2.2) is equivalent to the equalities
\[ \partial_\mu \theta^\alpha_\nu - \partial_\nu \theta^\alpha_\mu + c_{\beta\gamma}^\alpha \theta^\beta_\mu \theta^\gamma_\nu = 0. \] (2.5)

It can be shown that the matrix \( \|\theta^\alpha_\mu\| \) is nondegenerate. Denote the matrix elements of the inverse matrix by \( \xi^\mu_\alpha \). Thus, we have
\[ \xi^\mu_\alpha \theta^\alpha_\mu = \delta^\mu_\alpha. \]

Using this relation and equalities (2.5), we obtain
\[ \xi^\mu_\alpha \partial_\nu \xi^\nu_\beta - \xi^\nu_\beta \partial_\nu \xi^\mu_\alpha - c_{\alpha\beta}^\gamma \xi^\mu_\gamma = 0. \] (2.6)

There exists also the left–invariant Maurer–Cartan form \( \bar{\theta} \), defined by the formula
\[ \bar{\theta} = dgg^{-1}. \] (2.7)

The form \( \bar{\theta} \) takes values in the Lie algebra \( \mathfrak{g} \) and satisfies the relation
\[ d\bar{\theta} - \bar{\theta} \wedge \bar{\theta} = 0. \] (2.8)

Using the local coordinates \( y^\mu \) and the basis \( e_\alpha \), represent the form \( \bar{\theta} \) as
\[ \bar{\theta} = \partial_\mu gg^{-1} dy^\mu = e_\alpha \bar{\theta}^\alpha_\mu dy^\mu. \] (2.9)
From relation (2.8) it follows that
\[ \partial_{\mu} \bar{\theta}^\alpha_{\nu} - \partial_{\nu} \bar{\theta}^\alpha_{\mu} - c^\alpha_{\beta\gamma} \bar{\theta}^\beta_{\mu} \bar{\theta}^\gamma_{\nu} = 0. \tag{2.10} \]

The matrix \( \| \bar{\theta}^\alpha_{\mu} \| \) is nondegenerate. Therefore, there exist the matrix \( \| \bar{\xi}^\mu_{\alpha} \| \) with the functions \( \bar{\xi}^\mu_{\alpha} \) determined by the relations
\[ \bar{\xi}^\mu_{\alpha} \partial_{\mu} \bar{\theta}^\beta_{\nu} = \delta^\beta_{\alpha}. \]

From equalities (2.10) we get
\[ \bar{\xi}^\mu_{\alpha} \partial_{\mu} \bar{\theta}^\beta_{\nu} - \bar{\xi}^\mu_{\nu} \partial_{\nu} \bar{\theta}^\beta_{\mu} + c^\gamma_{\alpha\beta} \bar{\xi}^\mu_{\gamma} = 0. \tag{2.11} \]

Recall that the adjoint representation for the case of a matrix Lie group is described by the formula
\[ \text{Ad}(a)u = au a^{-1}, \quad a \in G, \quad u \in \mathfrak{g}. \]
Comparing definitions (2.1) and (2.7) of the forms \( \theta \) and \( \bar{\theta} \), we obtain
\[ \bar{\theta} = g \theta g^{-1} = \text{Ad}(g) \circ \theta. \]

The matrix elements \( \text{Ad}\beta_{\alpha}(a) \) of the adjoint representation with respect to the basis \( e_\alpha \), are determined by the equality
\[ \text{Ad}(a)e_\alpha = ae_\alpha a^{-1} = e_\beta \text{Ad}\beta_{\alpha}(a). \tag{2.12} \]
Using this equality and relations (2.4) and (2.9), we get
\[ \bar{\theta}^\alpha_{\mu} = \text{Ad}\beta_{\alpha}(g) \theta^\beta_{\mu}, \]
that can be also written as
\[ \text{Ad}\beta_{\alpha}(g) = \bar{\theta}^\alpha_{\mu} \xi^\mu_{\beta}. \tag{2.13} \]

Concluding this section, let us obtain a system of partial differential equations satisfied by the matrix elements of the adjoint representation of the group \( G \). From equality (2.12) it follows that
\[ \partial_{\mu} (ge_\beta g^{-1}) = e_\gamma \partial_{\mu} (\text{Ad}\beta_{\alpha}(g)). \tag{2.14} \]
It is easy to get convinced that
\[ \partial_{\mu} (ge_\beta g^{-1}) = [\partial_{\mu} g g^{-1}, ge_\beta g^{-1}]. \]
Using the equality
\[ \bar{\xi}^\mu_{\alpha} \partial_{\mu} g g^{-1} = e_\alpha, \]
which follows from (2.3), we get
\[ \bar{\xi}^\mu_{\alpha} \partial_{\mu} (ge_\beta g^{-1}) = e_\gamma c^\gamma_{\alpha\delta} \text{Ad}\beta_{\delta}(g). \]
Relation (2.14) now gives
\[ \bar{\xi}^\mu_{\alpha} \partial_{\mu} (\text{Ad}\gamma_{\beta}(g)) = c^\gamma_{\alpha\delta} \text{Ad}\gamma_{\delta}(g). \tag{2.15} \]
3 Free motion on matrix Lie group

The motion of a point particle on a matrix Lie group $G$ is described by a mapping sending each instant of time to an element of $G$. Suppose that the scalar product on $\mathfrak{g}$ defined by the relation

$$ (u, v) = \text{tr}(uv) $$

for any $u, v \in \mathfrak{g}$, is nondegenerate. This supposition is equivalent to the requirement of nondegeneracy of the matrix $c = \|c_{\alpha\beta}\|$, where

$$ c_{\alpha\beta} = \text{tr}(e_{\alpha}e_{\beta}). $$

Scalar product (3.1) is invariant with respect to the action of the group $G$ in $\mathfrak{g}$, defined by the adjoint representation. This invariance leads to the equalities

$$ \text{Ad}_{\gamma}(a) \text{Ad}_{\delta}(a)c_{\gamma\delta} = c_{\alpha\beta}. $$

The free motion on the group $G$ is described by the Lagrangian

$$ L = \frac{1}{2} \text{tr}(g^{-1}\dot{g}g^{-1}\dot{g}), $$

where dot means the derivative over $t$. Lagrangian (3.4) is invariant with respect to the right and left shifts in the group $G$. Using equality (2.4), we get

$$ g^{-1}\dot{g} = (g^{-1}\partial_{\mu}g)\dot{y}^\mu = e_{\alpha}\theta_{\mu}^\alpha\dot{y}^\mu. $$

This equality allows to write the expression for the Lagrangian $L$ in terms of the coordinates $y^\mu$ and the velocities $\dot{y}^\mu$:

$$ L = \frac{1}{2} \dot{y}^\mu G_{\mu\nu}\dot{y}^\nu, $$

where

$$ G_{\mu\nu} = \theta_{\mu}^\alpha c_{\alpha\beta}\theta_{\nu}^\beta = \tilde{\theta}_{\mu}^\alpha c_{\alpha\beta}\tilde{\theta}_{\nu}^\beta $$

are the components of the bi–invariant metric on $G$. The last equality in (3.5) follows from (3.3).

Consider now the Hamiltonian formulation of the model. The phase space in this case is the cotangent bundle $T^*(G)$, endowed with the canonical symplectic structure. The local coordinates $y^\mu$ on $G$ generate the local canonical coordinates $y^\mu$, $p_\mu$ on $T^*(G)$, and the canonical symplectic 2–form in terms of these coordinates has the form

$$ \Omega = d(p_\mu dy^\mu). $$

This symplectic form leads to the following Poisson brackets for the coordinate functions $y^\mu$ and $p_\mu$:

$$ \{y^\mu, y^\nu\} = 0, \quad \{p_\mu, p_\nu\} = 0, \quad \{y^\mu, p_\nu\} = \delta_\mu^\nu. $$
The Legendre transformation is described in the case under consideration by the relations
\[ p_\mu = \frac{\partial L}{\partial \dot{y}_\mu} = G_{\mu\nu} \dot{y}_\nu, \]
and we have the following expression for the Hamiltonian of the system:
\[ H = \frac{1}{2} p_\mu G^{\mu\nu} p_\nu, \]
where \( G^{\mu\nu} \) are the matrix elements of the matrix inverse to the matrix \( \|G_{\mu\nu}\| \). An explicit expression for \( G^{\mu\nu} \) has the form
\[ G^{\mu\nu} = \xi^\mu_\alpha c^{\alpha\beta} \xi^\nu_\beta = \bar{\xi}^\mu_\alpha c^{\alpha\beta} \bar{\xi}^\nu_\beta, \quad (3.6) \]
where \( c^{\alpha\beta} \) are the matrix elements of the matrix inverse to the matrix \( \|c_{\alpha\beta}\| \).

Define the functions
\[ j_\alpha = -\xi^\mu_\alpha p_\mu, \quad \bar{j}_\alpha = -\bar{\xi}^\mu_\alpha p_\mu. \]
Taking into account (2.13), we obtain
\[ j_\alpha = Ad^\alpha_\beta(g) \bar{j}_\beta. \quad (3.7) \]
From (2.6) and (2.11) one gets the following expressions for the Poisson brackets of the functions \( j_\alpha \) and \( \bar{j}_\alpha \):
\[ \{j_\alpha, j_\beta\} = c^{\gamma\delta}_{\alpha\beta} j_\gamma, \quad \{\bar{j}_\alpha, \bar{j}_\beta\} = -c^{\gamma\delta}_{\alpha\beta} \bar{j}_\gamma, \]
while equations (2.13) and relation (3.7) give
\[ \{j_\alpha, \bar{j}_\beta\} = 0. \]
The functions \( j_\alpha \) and \( \bar{j}_\alpha \) are the generators of the left and right shifts in the group \( G \). Indeed, from relations (2.3) and (2.4) we get
\[ \xi^\mu_\alpha \partial_\mu g = ge_\alpha. \]
This equality allows to show that
\[ \{j_\alpha, g\} = ge_\alpha. \]
Similarly, using (2.3), we obtain
\[ \{\bar{j}_\alpha, g\} = e_\alpha g. \]
As it follows from (3.6), the Hamiltonian of the system in terms of the functions \( j_\alpha \) or \( \bar{j}_\alpha \) has the following form
\[ H = \frac{1}{2} j_\alpha c^{\alpha\beta} j_\alpha = \frac{1}{2} \bar{j}_\alpha c^{\alpha\beta} \bar{j}_\beta. \quad (3.8) \]
Using (2.4), we get
\[ dy^\mu = \xi^\mu_\alpha c^{\alpha\beta} \text{tr}(e_\beta \theta). \]
Hence, there is valid the equality
\[ \Omega = - \text{tr}(j \theta). \quad (3.9) \]
where we have introduced the notation
\[ j = j_\alpha e^{\alpha \beta} e_\beta. \]
Similarly, we obtain
\[ \Omega = -\text{tr}(\bar{j}\bar{\theta}), \]
where
\[ \bar{j} = \bar{j}_\alpha e^{\alpha \beta} e_\beta. \]
From formulas (3.7) and (3.3) it follows that the matrix valued functions \( j \) and \( \bar{j} \) are related by the equality
\[ \bar{j} = g j g^{-1} = \text{Ad}(g) \circ j. \quad (3.10) \]

4 Lie group \( \text{SL}(2, \mathbb{R}) \)

The Lie group \( \text{SL}(2, \mathbb{R}) \) is formed by all real \( 2 \times 2 \) matrices
\[ a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \]
satisfying the relation
\[ \det a = a_{11}a_{22} - a_{12}a_{21} = 1. \]

The Lie algebra of the group \( \text{SL}(2, \mathbb{R}) \) is the Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \), which consists of all real traceless \( 2 \times 2 \) matrices. Consider the canonical basis of \( \mathfrak{sl}(2, \mathbb{R}) \):
\[ e_1 = x_{-} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad e_2 = h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_3 = x_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \]

With such a choice of a basis, in accordance with (3.2) we have
\[ \| e_{\alpha \beta} \| = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \| e^{\alpha \beta} \| = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1/2 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (4.1) \]

Introducing the notation
\[ j_- = j_1, \quad j_0 = j_2, \quad j_+ = j_3, \]
\[ \bar{j}_- = \bar{j}_1, \quad \bar{j}_0 = \bar{j}_2, \quad \bar{j}_+ = \bar{j}_3, \]
we get for the matrix valued functions \( j \) and \( \bar{j} \) the expressions
\[ j = \begin{pmatrix} j_0/2 & j_- \\ j_+ & -j_0/2 \end{pmatrix}, \quad \bar{j} = \begin{pmatrix} \bar{j}_0/2 & \bar{j}_- \\ \bar{j}_+ & -\bar{j}_0/2 \end{pmatrix}. \quad (4.2) \]

Using now relation (3.10), we obtain
\begin{align*}
\bar{j}_- &= (g_{11})^2 j_- - g_{11}g_{12}j_0 - (g_{12})^2 j_+, \\
\bar{j}_0 &= -2g_{11}g_{21}j_- + (g_{11}g_{22} + g_{12}g_{21})j_0 + 2g_{22}g_{12}j_+, \\
\bar{j}_+ &= -(g_{21})^2 j_- + g_{22}g_{21}j_0 + (g_{22})^2 j_+.
\end{align*}
For the case under consideration we have the following expression for the inverse element:
\[ a^{-1} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}. \]

From this equality we get the explicit expression for the Maurer–Cartan form:
\[ \theta = \begin{pmatrix} g_{22}dg_{11} - g_{12}dg_{21} & g_{22}dg_{12} - g_{12}dg_{22} \\ -g_{12}dg_{11} + g_{11}dg_{12} & -g_{21}dg_{12} + g_{11}dg_{22} \end{pmatrix}. \]

Now, taking into account (3.9) and (4.2), we obtain
\[ \Omega = d[j- (g_{21}dg_{11} - g_{11}dg_{21}) + j_0(g_{11}dg_{22} - g_{22}dg_{11}) \\
+ g_{12}dg_{21} - g_{21}dg_{12})/2 + j+(g_{12}dg_{22} - g_{22}dg_{12})]. \quad (4.3) \]

5 Reduced phase space

The phase space reduction in question is performed by imposing two first class constraints:
\[ j_+ = \mu, \quad j_0 = \nu, \quad (5.1) \]

where \( \mu \) and \( \nu \) are two nonzero real constants. Constraints (5.1) generate on the surface defined by them an action of a two dimensional Lie group. This action is free and gives the foliation of the constraint surface into two dimensional orbits. As it follows from the general theory of the Hamiltonian reduction, the space of orbits is a symplectic manifold with the symplectic structure inherited from the initial phase space. This phase space is usually called the reduced phase space. In the case under consideration the reduced phase space can be realised as the surface obtained as the intersection of the constraint surface with the surface defined by the relations called gauge conditions. Note that the projection of the considered group action to the configuration space is not a free action, therefore, the gauge conditions must be imposed on the coordinates \( j_0 \) and \( j_0 \). It is not difficult to get convinced that as admissible gauge conditions we can take, for example, the relations
\[ j_0 = 0, \quad j_0 = 0. \]

Taking these gauge conditions into account, we see that the reduced phase space is defined by the equations
\[ (g_{11})^2j_- - (g_{12})^2\mu = \nu, \quad (5.2) \]
\[ g_{22}g_{12}\mu - g_{11}g_{21}j_- = 0, \quad (5.3) \]
\[ g_{11}g_{22} - g_{12}g_{21} = 1. \quad (5.4) \]

Multiplying (5.3) by \( g_{11} \) and using (5.2) and (5.4), we get
\[ g_{12}\mu - g_{21}\nu = 0. \quad (5.5) \]

Substituting this equality into (5.3), we come to the equality
\[ g_{12}(g_{22}\nu - g_{11}j_-) = 0. \quad (5.6) \]
From the other hand, taking into account (5.5), we can rewrite (5.2) in the form

$$(g_{11})^2 - g_{12}g_{21} \nu = \nu,$$

that gives after the usage of (5.4) the relation

$$g_{11}(g_{22} \nu - g_{11} j-) = 0.$$  \hspace{1cm} (5.7)

Since the functions $g_{12}$ and $g_{11}$ cannot take zero value simultaneously, then it follows from (5.6) and (5.7) that

$$g_{22} \nu - g_{11} j- = 0.$$  \hspace{1cm} (5.8)

Using relations (5.3) and (5.8) to exclude the functions $g_{21}$ and $g_{22}$, it is easy to get convinced that system of equations (5.2)–(5.4) is equivalent to the single equation (5.2). In other words, the reduced phase space can be considered as a two dimensional surface in the three dimensional space with the coordinates $g_{11}$, $g_{12}$ and $j-$, defined by equation (5.2). As it follows from (4.3) the symplectic form on the reduced phase space is given by the relation

$$\Omega = \frac{\mu}{\nu} d[2j-(g_{12}dg_{11} - g_{11}dg_{12}) + g_{11}g_{12}dj].$$  \hspace{1cm} (5.9)

The reduced phase space has different topologies depending on the value of the parameters $\mu$ and $\nu$. Actually, there are two essentially different variants, determined by the relative sign of these parameters.

Suppose first that the parameters $\mu$ and $\nu$ have the same sign. Without loss of generality one can put $\mu = \nu = 1$. From (5.2) it follows that in the case under consideration $g_{11}$ cannot take zero value, and the coordinate $j-$ can be expressed through $g_{11}$ and $g_{12}$:

$$j- = \frac{(g_{12})^2 + 1}{(g_{11})^2}.$$  \hspace{1cm} (5.10)

Thus, the reduced phase space is topologically the union of two nonintersecting two dimensional surfaces. It is clear also that these surfaces can be realised as the open subsets of the plane described by the coordinates $g_{11}$ and $g_{12}$, singled out by the conditions $g_{11} > 0$ and $g_{11} < 0$.

From (5.9), using (5.10), we get for the symplectic form on the reduced phase space the following expression

$$\Omega = -2 \frac{dg_{12} \wedge dq_{11}}{(g_{11})^2}.$$  \hspace{1cm} (5.11)

Now it is not difficult to introduce canonical coordinates assuming, for example,

$$Q = \ln g_{11}, \quad P = -2 \frac{g_{12}}{g_{11}}.$$  \hspace{1cm} (5.12)

for the case $g_{11} > 0$. From (5.8) and (4.1) we conclude that the Hamiltonian of the reduced system coincides with $j-$. Taking into account (5.10) and the equalities

$$g_{11} = \exp Q, \quad g_{12} = -\frac{1}{2} P \exp Q,$$
we get
\[ H = \frac{1}{4}P^2 + \exp(-2Q). \]

The solution of the equations of motions for the system with this Hamiltonian is well known, therefore we will not write it here.

Consider now the case when the parameters \( \mu \) and \( \nu \) have different signs and put \( \mu = -\nu = -1 \). Introduce for \( g_{11} \) and \( g_{12} \) the polar coordinates,
\[ g_{11} = R \sin \Phi, \quad g_{12} = R \cos \Phi. \]

From equation (5.2) it follows that
\[ R^2 = \frac{1}{j_- \sin^2 \Phi + \cos^2 \Phi}. \]

We can treat \( \Phi \) and \( j_- \) as coordinates on the reduced phase space. It is easy to see that these coordinated take only those values for which
\[ j_- \sin^2 \Phi + \cos^2 \Phi > 0. \]

The symplectic form in terms of the coordinates \( \Phi \) and \( j_- \) has the form
\[ \Omega = -\frac{1}{j_- \sin^2 \Phi + \cos^2 \Phi} dj_- \wedge d\Phi. \]

Thus, we see that the coordinates \( \Phi \) and \( j_- \) have two essential deficiencies. First, these coordinates do not take arbitrary values, and, second, they are not canonical. It is easy to get convinced that the general form of a coordinate conjugated to \( \Phi \) is given by the relation
\[ \Pi = -\frac{1}{\sin^2 \Phi} \ln \left( j_- \sin^2 \Phi + \cos^2 \Phi \right) + F(\Phi), \]
where \( F(\Phi) \) is an arbitrary periodic function. Choosing \( F(\Phi) = 0 \), we come to convenient canonical coordinates \( \Phi \) and \( \Pi \). Note that these coordinates take arbitrary values.

It is not difficult to show now that \( g_{11}, g_{12} \) and \( j_- \) are expressed via the coordinates \( \Phi \) and \( \Pi \) in the following way:
\[ g_{11} = \exp(\Pi \sin^2 \Phi/2) \sin \Phi, \quad g_{12} = \exp(\Pi \sin^2 \Phi/2) \cos \Phi, \quad j_- = \left( \exp(-\Pi \sin^2 \Phi) - \cos^2 \Phi \right) / \sin^2 \Phi. \]

Relations (5.11)–(5.13) give a parametric representation of the reduced phase space considered as the surface defined by equation (5.2). Here the function entering this representation are infinitely differentiable. Moreover, the corresponding tangent vectors are linearly independent. Therefore, the reduced phase space is diffeomorphic to the cylinder \( S^1 \times \mathbb{R} \). Taking into account the fact that the coordinates \( \Phi \) and \( \Pi \) are canonical coordinates, we see that the reduced phase space is symplectomorphic to the cotangent bundle \( T^*(S^1) \) endowed with the canonical symplectic structure.
The Hamiltonian of the system in terms of the coordinates Φ and Π is

\[ H = \frac{1}{\sin^2 \Phi} \left( \cos^2 \Phi - \exp(-\Pi \sin^2 \Phi) \right). \]

Hence, the Hamiltonian equations of motion have the form

\[ \dot{\Phi} = \exp(-\Pi \sin^2 \Phi) \quad \text{(5.14)} \]

\[ \dot{\Pi} = 2 \cot \Phi \left( \frac{1}{\sin^2 \Phi} - \exp(-\Pi \sin^2 \Phi) \left( \Pi + \frac{1}{\sin^2 \Phi} \right) \right). \quad \text{(5.15)} \]

Despite the fact that these equations have a rather complicated form, they can be easily solved. Indeed, the Hamiltonian of the system is a conserved quantity, which has the sense of energy. Let us look for the solutions of the equations of motion for which \( H = \epsilon \). Using (5.13) and (5.14), we come to the relation

\[ \epsilon = \frac{1}{\sin^2 \Phi} (\cos^2 \Phi - \dot{\Phi}). \]

Consider the function \( T = \tan \Phi \). For this function we get the equation

\[ \dot{T} = 1 - \epsilon T^2. \quad \text{(5.16)} \]

In the case \( \epsilon < 0 \), this equation has the solution

\[ T(t) = \frac{1}{\sqrt{-\epsilon}} \tan \left( \sqrt{-\epsilon}(t - c) \right), \]

where \( c \) is the integration constant. From this we easily get

\[
\begin{align*}
\cos 2\Phi(t) &= \frac{\epsilon + \tan^2 \left( \sqrt{-\epsilon}(t - c) \right)}{\epsilon - \tan^2 \left( \sqrt{-\epsilon}(t - c) \right)}, \\
\sin 2\Phi(t) &= -\frac{2\sqrt{-\epsilon} \tan \left( \sqrt{-\epsilon}(t - c) \right)}{\epsilon - \tan^2 \left( \sqrt{-\epsilon}(t - c) \right)}, \\
\Pi(t) &= \frac{\tan^2 \left( \sqrt{-\epsilon}(t - c) \right) - \epsilon}{\tan^2 \left( \sqrt{-\epsilon}(t - c) \right)} \ln \left[ \frac{\epsilon + \epsilon \tan^2 \left( \sqrt{-\epsilon}(t - c) \right)}{\epsilon - \epsilon \tan^2 \left( \sqrt{-\epsilon}(t - c) \right)} \right].
\end{align*}
\]

In the case \( \epsilon > 0 \) equation (5.16) has the solution

\[ T(t) = \frac{1}{\sqrt{\epsilon}} \tanh \left( \sqrt{\epsilon}(t - c) \right), \]

and we come to the relations

\[
\begin{align*}
\cos 2\Phi(t) &= \frac{\epsilon - \tanh^2 \left( \sqrt{\epsilon}(t - c) \right)}{\epsilon + \tanh^2 \left( \sqrt{\epsilon}(t - c) \right)}, \\
\sin 2\Phi(t) &= \frac{2\sqrt{\epsilon} \tanh \left( \sqrt{\epsilon}(t - c) \right)}{\epsilon + \tanh^2 \left( \sqrt{\epsilon}(t - c) \right)}, \\
\Pi(t) &= -\frac{\tanh^2 \left( \sqrt{\epsilon}(t - c) \right) + \epsilon}{\tanh^2 \left( \sqrt{\epsilon}(t - c) \right)} \ln \left[ \frac{\epsilon - \epsilon \tanh^2 \left( \sqrt{\epsilon}(t - c) \right)}{\epsilon + \epsilon \tanh^2 \left( \sqrt{\epsilon}(t - c) \right)} \right].
\end{align*}
\]
Thus, we have the explicit solution of the equations of motions.

To construct the corresponding quantum theory, we have to solve the ordering problem for the quantum Hamiltonian. In any case, in the representation where the state space is realised as the space of square integrable functions on the circle, the corresponding operator is not local. However, the availability of the explicit general solution to the classical equations of motions gives us a hope that the eigenvalue problem for the quantum Hamiltonian can be solved.

6 Conclusion

In the present work we have studied in detail the structure of the reduced phase space arising as the result of the Hamiltonian reduction of the phase space corresponding to a free particle motion on the Lie group SL(2, C). We have shown that the reduced phase space is diffeomorphic either to the union of two cotangent bundles $T^*(\mathbb{R})$, or to the cotangent bundle $T^*(S^1)$. Moreover, for both cases we have constructed canonical coordinates and have shown that the arising phase spaces are symplectomorphic to the corresponding cotangent bundles endowed with the canonical symplectic structure.

Note that despite of a complicated structure of the Hamiltonian, arising in the second case, the classical equations of motions can be explicitly integrated. This fact allows us to hope that the quantisation problem has an explicit solution.

The authors are deeply grateful to G. L. Rcheulishvili, G. P. Pron’ko and M. V. Saveliev for the interesting and fruitful discussions. This work was partially supported by the Russian Foundation for Basic Research (project # 95–01–00125a).

References

[1] L. Fehér, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Phys. Rep. 222, 1 (1992).
[2] P. Forgács, A. Wipf, J. Balog, L. Fehér and L. O’Raifeartaigh, Phys. Lett. 227B, 214 (1989).
[3] I. Tsutsui and L. Fehér, Prog. Theor. Phys. Suppl. 118, 173 (1995).
[4] T. Fülöp, J. Math. Phys. 37, 1617 (1996).
[5] L. Fehér and I. Tsutsui, Regularization of Toda lattices by Hamiltonian reduction, preprint INS-1123 [hep-th/9511118].
[6] H. Kobayashi and I. Tsutsui, Quantum mechanical Liouville model with attractive potential, preprint INS-1124 [hep-th/9601111].