The $A_2$ Rogers–Ramanujan Identities Revisited

To George Andrews on his 80th birthday

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Abstract. In this note, we show how to use cylindric partitions to rederive the four $A_2$ Rogers–Ramanujan identities originally proven by Andrews, Schilling and Warnaar, and provide a proof of a similar fifth identity.

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1. Introduction

The Rogers–Ramanujan identities were first proved in 1894 by Rogers and rediscovered in the 1910s by Ramanujan [14]. They are

$$\sum_{n \geq 0} q^{n(n+i)} \frac{(q; q)_n}{(q^{1+i}; q^5)_\infty(q^{4-i}; q^5)_\infty} = 1$$

with $i = 0, 1$, where $(a, q)_\infty = \prod_{i \geq 0} (1 - aq^i)$ and $(a; q)_n = (a; q)_\infty/(aq^n; q)_\infty$.

There have been many attempts to give combinatorial proofs of these identities and the first one is due to Garsia and Milne [11]. Unfortunately, it is not simple, and no simple combinatorial proof is known. Recently in [6], the first author presented a new bijective approach to the proofs of the Rogers–Ramanujan identities via the Robinson–Schensted–Knuth correspondence as presented in [13]. The bijection does not give the Rogers–Ramanujan identities but the Rogers–Ramanujan identities divided by $(q; q)_\infty$, namely

$$\frac{1}{(q; q)_\infty} \sum_{n \geq 0} q^{n(n+1)} \frac{(q; q)_n}{(q; q)_\infty} = 1$$
where \((a_1, \ldots, a_k; q)_{\infty} = \prod_{i=1}^{k} (a_i; q)_{\infty}\). This proof uses the combinatorics of cylindric partitions \([10]\). We interpret both sides as the generating function of cylindric partitions of profile \((3,0)\) and the bijection is a polynomial algorithm in the size of the cylindric partition. The idea to use cylindric partitions is due to Foda and the second author \([12]\) in the more general setting of the Andrews–Gordon identities \([2]\). For \(k > 0\) and \(0 \leq i \leq k\), these identities divided by \((q; q)_{\infty}\) are

\[
\frac{1}{(q; q)_{\infty}} \sum_{n_1, \ldots, n_k} \frac{q^{\sum_{j=1}^{k} n_j^2 + \sum_{j=i}^{k} n_j}}{(q)_{n_1-n_2-\cdots-(q)_{n_{k-1}-n_k}}(q)_{n_k}} = \frac{(q^i, q^{2k+3-i}, q^{2k+3}; q^{2k+3})_{\infty}}{(q; q)_2^\infty}.
\]

In \([12]\), the sum side is interpreted as a generating function for (what the authors call) decorated Bressoud paths, and the product side is interpreted as the generating function of cylindric partitions of profile \((2k+1-i, i)\), and a bijection between these two objects is provided. See \([12]\) for more details.

In this note, we take the idea of applying cylindric partitions to Rogers–Ramanujan type identities a step further, using them to give an alternative proof of the \(A_2\) Rogers–Ramanujan identities due to Andrews et al. \([3]\).

**Theorem 1.1.** We have

\[
\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2+n_1+n_2}}{(q; q)_{n_1}} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix} = \frac{1}{(q^2, q^3, q^4, q^5, q^6, q^7)_{\infty}},
\]

\[
\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2+n_1+n_2}}{(q; q)_{n_1}} \begin{bmatrix} 2n_1+1 \\ n_2 \end{bmatrix} = \frac{1}{(q, q^2, q^3, q^4, q^5, q^6, q^7)_{\infty}},
\]

\[
\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{2n_1+1} \frac{q^{n_1^2+n_2^2-n_1n_2+n_1+1}}{(q; q)_{n_1}} \begin{bmatrix} 2n_1+1 \\ n_2 \end{bmatrix} = \frac{1}{(q, q^2, q^3, q^4, q^5, q^6, q^7)_{\infty}},
\]

\[
\sum_{n_1=0}^{\infty} \sum_{n_2=0}^{2n_1+1} \frac{q^{n_1^2+n_2^2-n_1n_2+n_1+1}}{(q; q)_{n_1}} \begin{bmatrix} 2n_1+1 \\ n_2 \end{bmatrix} = \frac{1}{(q, q^2, q^3, q^4, q^5, q^6, q^7)_{\infty}},
\]

where the Gaussian polynomial \([n \atop k]\) is defined by

\[
[n \atop k] = \frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}}.
\]

Note that the second and third expressions are equal. All but the fourth of these identities were obtained in Theorem 5.2 of \([3]\), while the fourth was conjectured in Section 2.4 of \([8]\) (and proved here for the first time). In \([16]\), Warnaar gave another approach to proving these identities, making use of Hall–Littlewood functions.

In this note, we prove the following theorem, giving the generating functions \(F_{c,n}(q)\) of cylindric partitions indexed by compositions \(c\) of 4 into 3 parts, with largest entry at most \(n\):
Theorem 1.2.

\[
F_{(4,0),n}(q) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2+n_1+n_2}}{(q; q)_{n-n_1} (q; q)_{n_1}} \left[ \begin{array}{c} 2n_1 \\ n_2 \end{array} \right],
\]

\[
F_{(3,1),n}(q) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2+n_2}}{(q; q)_{n-n_1} (q; q)_{n_1}} \left[ \begin{array}{c} 2n_1 \\ n_2 \end{array} \right],
\]

\[
F_{(3,0),n}(q) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2} q^{n_1}}{(q; q)_{n-n_1} (q; q)_{n_1}} \left[ \begin{array}{c} 2n_1 \\ n_2 \end{array} \right]
\]

\[
+ \sum_{n_1=1}^{n} \sum_{n_2=0}^{2n_1-2} \frac{q^{n_1^2+n_2^2-n_1n_2} q^{2n_2}}{(q; q)_{n-n_1} (q; q)_{n_1-1}} \left[ \begin{array}{c} 2n_1-2 \\ n_2 \end{array} \right],
\]

\[
F_{(2,2),n}(q) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2} q^{n_1}}{(q; q)_{n-n_1} (q; q)_{n_1}} \left[ \begin{array}{c} 2n_1 \\ n_2 \end{array} \right]
\]

\[
+ \sum_{n_1=1}^{n} \sum_{n_2=0}^{2n_1-2} \frac{q^{n_1^2+n_2^2-n_1n_2} q^{2n_2} (1+q^{n_1+n_2})}{(q; q)_{n-n_1} (q; q)_{n_1-1}} \left[ \begin{array}{c} 2n_1-2 \\ n_2 \end{array} \right],
\]

\[
F_{(2,1),n}(q) = \sum_{n_1=0}^{n} \sum_{n_2=0}^{2n_1} \frac{q^{n_1^2+n_2^2-n_1n_2}}{(q; q)_{n-n_1} (q; q)_{n_1}} \left[ \begin{array}{c} 2n_1 \\ n_2 \end{array} \right].
\]

This result gives finite versions of the sum sides of the $A_2$ Rogers–Ramanujan identities.

In the $n \to \infty$ limit, we recover the sum sides of the identities of Theorem 1.1 divided by $(q; q)_{\infty}$. On the other hand, the product sides are obtained using a result of Borodin [4] on the generating functions of cylindric partitions. In Sect. 2, we start by defining cylindric partitions and then obtain the product sides of particular cylindric partitions. These yield the right-hand sides of the expressions in Theorem 1.1. The sum expressions on the left-hand sides are computed in Sect. 3.

2. Cylindric Partitions and the Product Side

Cylindric partitions were introduced by Gessel and Krattenthaler [10] and appeared naturally in different contexts [4, 5, 7, 9, 12, 15]. Let $\ell$ and $k$ be two positive integers. In this note, we choose to index cylindric partitions by compositions of $\ell$ into $k$ non-negative parts.

Definition 2.1. Given a composition $c = (c_1, \ldots, c_k)$, a cylindric partition of profile $c$ is a sequence of $k$ partitions $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ such that

- $\lambda^{(i)}_{j} \geq \lambda^{(i+1)}_{j+c_{i+1}}$,
- $\lambda^{(k)}_{j} \geq \lambda^{(1)}_{j+c_{1}}$,

for all $i$ and $j$. 

For example, the sequence $\Lambda = ((3, 2, 1, 1), (4, 3, 3, 1), (4, 1, 1))$ is a cylindric partition of profile $(2, 2, 0)$. One can check that for all $j$, $\lambda_j^{(1)} \geq \lambda_j^{(2)}$, $\lambda_j^{(2)} \geq \lambda_j^{(3)}$ and $\lambda_j^{(3)} \geq \lambda_{j+2}^{(1)}$ for all $j$. Note that this definition implies that cylindric partitions of profile $(c_1, \ldots, c_k)$ are in bijection with cylindric partitions of profile $(c_k, c_1, \ldots, c_{k-1})$.

Our goal is to compute generating functions of cylindric partitions of a given profile $c$ according to two statistics. Given a $\Lambda = (\lambda_1^{(1)}, \ldots, \lambda_k^{(k)})$, let

- $|\Lambda| = \sum_{i=1}^k \sum_{j \geq 1} \lambda_j^{(i)}$, the sum of the entries of the cylindric plane partition, and
- $\max(\Lambda) = \max(\lambda_1^{(1)}, \ldots, \lambda_k^{(k)})$, the largest entry of the cylindric plane partition.

Going back to our example, we have $|\Lambda| = 24$, and $\max(\Lambda) = 4$.

Let $C_c$ be the set of cylindric partitions of profile $c$ and let $C_{c,n}$ be the set of cylindric partitions of profile $c$ and such that the largest entry is at most $n$. We are interested in the following generating functions:

\begin{align*}
F_c(q) &= \sum_{\Lambda \in C_c} q^{|\Lambda|}, \\
F_c(y, q) &= \sum_{\Lambda \in C_c} q^{|\Lambda|} y^{\max(\Lambda)}, \\
F_{c,n}(q) &= \sum_{\Lambda \in C_{c,n}} q^{|\Lambda|}.
\end{align*}

A surprising and beautiful result is that for any $c$, the generating function $F_c(q)$ can be written as a product. Namely, with $t = k + \ell$,

\begin{align*}
\frac{1}{(q^t; q^t) \prod_{i=1}^k \prod_{j=i+1}^k \prod_{m=1}^{c_i} (q^{m+d_{i,j+1,j}+j-i}; q^t)_\infty \prod_{i=2}^k \prod_{j=2}^{i-1} \prod_{m=1}^{c_i} (q^{t-(m+d_{j,i-1+i-j})}; q^t)_\infty}
\end{align*}

where $d_{i,j} = c_i + c_{i+1} + \cdots + c_j$.

The original result is written in a different but equivalent form.

For what follows, we restrict attention to the case $\ell = 4$ and $k = 3$. As cylindric partitions of profile $(c_1, \ldots, c_k)$ are in bijection with partitions of profile $(c_k, c_1, \ldots, c_{k-1})$, we need only compute the generating functions for the compositions $(4, 0, 0)$, $(3, 1, 0)$, $(3, 0, 1)$, $(2, 2, 0)$, and $(2, 1, 1)$. We now apply the previous theorem:
Corollary 2.3.

\[
F_{(4,0,0)}(q) = \frac{1}{(q;q)_\infty(q^2, q^3, q^4, q^5; q^7)_\infty},
\]
\[
F_{(3,1,0)}(q) = \frac{1}{(q;q)_\infty(q, q^2, q^3, q^4, q^5; q^7)_\infty},
\]
\[
F_{(3,0,1)}(q) = \frac{1}{(q;q)_\infty(q, q^2, q^3, q^4, q^5; q^7)_\infty},
\]
\[
F_{(2,2,0)}(q) = \frac{1}{(q;q)_\infty(q, q^2, q^5, q^5, q^6; q^7)_\infty},
\]
\[
F_{(2,1,1)}(q) = \frac{1}{(q;q)_\infty(q, q, q^3, q^4, q^6; q^7)_\infty}.
\]

Note that these five products are precisely those in Theorem 1.1 divided by \((q;q)_\infty\).

3. The Sum Side

We first prove a general functional equation for \(F_c(y, q)\) for any profile \(c\). Suppose that \(k > 1\) and \(c = (c_1, \ldots, c_k)\). Let \(I_c\) be the subset of \(\{1, \ldots, k\}\) such that \(i \in I_c\) if and only if \(c_i > 0\). For example if \(c = (2, 2, 0)\) then \(I_c = \{1, 2\}\). Given a subset \(J\) of \(I_c\), we define the composition \(c(J) = (c_1(J), \ldots, c_k(J))\) by

\[
c_i(J) = \begin{cases} 
  c_i - 1 & \text{if } i \in J \text{ and } (i - 1) \not\in J, \\
  c_i + 1 & \text{if } i \not\in J \text{ and } (i - 1) \in J, \\
  c_i & \text{otherwise}.
\end{cases}
\]

Here, we set \(c_0 = c_k\).

Proposition 3.1. For any composition \(c = (c_1, \ldots, c_k)\),

\[
F_c(y, q) = \sum_{\emptyset \subset J \subseteq I_c} (-1)^{|J| - 1} \frac{F_c(J)(yy^{|J|}, q)}{1 - yq^{|J|}}. \tag{3.1}
\]

with the conditions \(F_c(0, q) = 1\) and \(F_c(y, 0) = 1\).

Proof. The proof makes use of an inclusion–exclusion argument.

First, for fixed \(J\) such that \(\emptyset \subset J \subseteq I_c\), we require the generating function of cylindric partitions \(\Lambda\) of profile \(c\) such that \(\lambda_1^{(j)} = \max(\Lambda)\) for all \(j \in J\).

Let \(M = (\mu^{(1)}, \ldots, \mu^{(k)})\) be a cylindric partition of profile \(c(J)\), and set \(n = \max(M)\). Then, for a fixed integer \(m \geq 0\), create a cylindric partition \(\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})\) using the following recipe:

\[
\lambda^{(j)} = \begin{cases} 
  (m + n, \mu_1^{(j)}, \mu_2^{(j)}, \ldots) & \text{if } j \in J, \\
  \mu^{(j)} & \text{if } j \not\in J.
\end{cases}
\]
It is easily checked that $\Lambda$ is a cylindric partition of profile $c$ and that $\max(\Lambda) = m + n$. Moreover, $\lambda^{(j)}_i = \max(\Lambda)$ for all $j \in J$. The generating function for all cylindric partitions $\Lambda$ obtained from $M$ in this way is

$$\sum_{m=0}^{\infty} y^{n+m} q^{|J|(m+n)} q^{|M|} = y^n q^{|J|n+|M|} \sum_{m=0}^{\infty} (yq^{|J|})^m = \frac{y^n q^{|J|n+|M|}}{1 - yq^{|J|}}. \quad (3.2)$$

Then, the generating function for all cylindric partitions $\Lambda$ obtained in this way from any cylindric partition $M$ of profile $c(J)$ is

$$\sum_{M \in \mathcal{C}_{c(J)}} \frac{y^{\max(M)} q^{|J| \max(M)+|M|}}{1 - yq^{|J|}} = \frac{F_{c(J)}(yq^{|J|}, q)}{1 - yq^{|J|}}, \quad (3.3)$$

making use of the definition (2.2).

Let $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(k)})$ be an arbitrary cylindric partition of profile $c$, and let $p = \max(\Lambda)$. Because $\lambda^{(i-1)}_1 \geq \lambda^{(i)}_1$ whenever $i \notin I_c$, it must be the case that $p = \lambda^{(j)}_1$ for some $j \in I_c$. Then, if $J \neq \emptyset$ is such that $p = \lambda^{(j)}_1$ for each $j \in J$ (this $J$ might not be unique), we see that $\Lambda$ is one of the cylindric partitions enumerated by (3.3). However, because $\Lambda$ can arise from various different $J$, the generating function for cylindric partitions of profile $c$ is obtained via the inclusion–exclusion process. This immediately gives (3.1).

Now, for each composition $c$, define

$$G_c(y, q) = (yq; q)_{\infty} F_c(y, q). \quad (3.4)$$

In terms of this, the previous result translates to

$$G_c(y, q) = \sum_{\emptyset \subset J \subseteq I} (-1)^{|J|-1} (yq; q)_{|J|-1} G_{c(J)}(yq^{|J|}, q) \quad (3.5)$$

with $G_c(0, q) = G_c(y, 0) = 1$. 

Theorem 3.2.

\[
G_{(4,0,0)}(y, q) = \sum_{n_1=0}^{2n_1} \sum_{n_2=0}^{2n_1} y^{n_1} \frac{q^{n_1^2 + n_2^2 - n_1 n_2 + n_1 + n_2}}{(q; q)_{n_1}} \left[ \frac{2n_1}{n_2} \right],
\]

\[
G_{(3,1,0)}(y, q) = \sum_{n_1=0}^{2n_1} \sum_{n_2=0}^{2n_1} y^{n_1} \frac{q^{n_1^2 + n_2^2 - n_1 n_2 + n_2}}{(q; q)_{n_1}} \left[ \frac{2n_1}{n_2} \right],
\]

\[
G_{(3,0,1)}(y, q) = \sum_{n_1=0}^{2n_1} \sum_{n_2=0}^{2n_1} y^{n_1} q^{n_1^2 + n_2^2 - n_1 n_2} \frac{q^{n_1}}{(q; q)_{n_1}} \left[ \frac{2n_1}{n_2} \right]
+ \sum_{n_1=1}^{2n_1-2} \sum_{n_2=0}^{2n_1} y^{n_1} q^{n_1^2 + n_2^2 - n_1 n_2} \frac{q^{n_2}}{(q; q)_{n_1-1}} \left[ \frac{2n_1 - 2}{n_2} \right],
\]

\[
G_{(2,2,0)}(y, q) = \sum_{n_1=0}^{2n_1} \sum_{n_2=0}^{2n_1} y^{n_1} q^{n_1^2 + n_2^2 - n_1 n_2} \frac{q^{n_1}}{(q; q)_{n_1}} \left[ \frac{2n_1}{n_2} \right]
+ \sum_{n_1=1}^{2n_1-2} \sum_{n_2=0}^{2n_1} y^{n_1} q^{n_1^2 + n_2^2 - n_1 n_2} q^{n_2} (1 + q^{n_1 + n_2}) \frac{q^{n_2}}{(q; q)_{n_1-1}} \left[ \frac{2n_1 - 2}{n_2} \right],
\]

\[
G_{(2,1,1)}(y, q) = \sum_{n_1=0}^{2n_1} \sum_{n_2=0}^{2n_1} y^{n_1} q^{n_1^2 + n_2^2 - n_1 n_2} \frac{q^{n_1}}{(q; q)_{n_1}} \left[ \frac{2n_1}{n_2} \right].
\]

Proof. In this proof, we abbreviate \( G_c(y, q) \) to \( G_c(y) \) for convenience. Applying the form (3.5) of Proposition 3.1 to the case \( \ell = 4 \) and \( k = 3 \) yields

\[
G_{(4,0,0)}(y) = G_{(3,1,0)}(yq),
\]

\[
G_{(3,1,0)}(y) = G_{(3,0,1)}(yq) + G_{(2,2,0)}(yq) - (1 - yq) G_{(2,1,1)}(yq^2),
\]

\[
G_{(3,0,1)}(y) = G_{(4,0,0)}(yq) + G_{(2,1,1)}(yq) - (1 - yq) G_{(3,1,0)}(yq^2),
\]

\[
G_{(2,2,0)}(y) = G_{(3,0,1)}(yq) + G_{(2,1,1)}(yq) - (1 - yq) G_{(2,1,1)}(yq^2),
\]

\[
G_{(2,1,1)}(y) = G_{(2,1,1)}(yq) + G_{(2,2,0)}(yq) + G_{(3,1,0)}(yq)
- (1 - yq)(G_{(2,2,0)}(yq^2) + G_{(2,1,1)}(yq^2) + G_{(3,0,1)}(yq^2))
+ (1 - yq)(1 - yq^2) G_{(2,1,1)}(yq^3).
\]

By manipulating these equations, we obtain

\[
G_{(4,0,0)}(y) = G_{(3,1,0)}(yq),
\]

\[
G_{(3,1,0)}(y) = G_{(2,2,0)}(yq) + yq^2 G_{(3,1,0)}(yq^3) + yq G_{(2,1,1)}(yq^2),
\]

\[
G_{(3,0,1)}(y) = G_{(2,1,1)}(yq) + yq G_{(3,1,0)}(yq^2),
\]

\[
G_{(2,2,0)}(y) = G_{(2,1,1)}(yq) + yq G_{(2,1,1)}(yq^2) + yq^2 G_{(3,1,0)}(yq^3),
\]

\[
G_{(2,1,1)}(y) = G_{(2,1,1)}(yq) + yq G_{(2,2,0)}(yq) + yq G_{(2,2,0)}(yq^2)
+ yq^3 G_{(3,1,0)}(yq^4) + yq^2 G_{(2,1,1)}(yq^3).
\]

We claim that this system of Eq. (3.6) together with the boundary conditions \( G_c(0, q) = G_c(y, 0) = 1 \) for each composition \( c \) is uniquely solved by the
expressions stated in the theorem. This is proved using an induction argument involving all five expressions.

   We use induction on the exponents of \( y \). For each composition \( c \), let \( g_c(n) \) denote the coefficient of \( y^n \) in the solution \( G_c(y) \) of (3.6). The boundary conditions \( G_c(0, q) = 1 \) imply that each \( g_c(0) = 1 \). This holds for the expressions of the theorem. So now, for \( n > 0 \), assume that \( g_c(n_1) \) agrees with the coefficient of \( y^{n_1} \) in the statement of the theorem for each \( n_1 < n \) and each composition \( c \). We must check that \( g_c(n) \), as determined by the expressions (3.6), is equal to the coefficient of \( y^n \) in the statement of the theorem for each \( c \).

   The fifth expression in (3.6) implies that
   
   \[
   (1 - q^n)g_{(2,1,1)}(n) = (q^n + q^{2n-1})g_{(2,2,0)}(n - 1) + q^{4n-1}g_{(3,1,0)}(n - 1) + q^{3n-1}g_{(2,1,1)}(n - 1).
   \]

   Using the expressions for \( g_c(n) \) implied by the induction hypothesis then yields
   
   \[
   g_{(2,1,1)}(n) = \sum_{n_2=0}^{2n} \frac{q^{n_2+n_2-n n_2}}{(q; q)_n} \left[ \frac{2n}{n_2} \right].
   \]

   This expression, along with the other expressions for \( g_c(n) \) implied by (3.6), enables us to compute, in turn, \( g_{(2,2,0)}(n) \), \( g_{(3,0,1)}(n) \), \( g_{(3,1,0)}(n) \) and finally \( g_{(4,0,0)}(n) \). Because the expressions that result agree with the corresponding coefficients of \( y^n \) in the statement of the theorem, the induction argument is complete. \( \square \)

   **Proof of Theorem 1.2.** Comparing (2.2) and (2.3) leads to
   
   \[
   F_{c,n}(q) = [y^0]F_c(y, q) + [y^1]F_c(y, q) + \cdots + [y^n]F_c(y, q)
   \]

   \[
   = [y^n]F_c(y, q) = [y^n] \frac{G_c(y, q)}{(y; q)_n}
   \]

   using (3.4). By the \( q \)-binomial theorem [1], we have
   
   \[
   \frac{1}{(y; q)_n} = \sum_{i=0}^{n} \frac{y^i}{(q; q)_i},
   \]

   and, therefore, it follows that
   
   \[
   F_{c,n}(q) = \sum_{n_1=0}^{n} \frac{1}{(q; q)_{n-n_1}} [y^{n_1}]G_c(y, q).
   \]

   Applying this to the expressions of Theorem 3.2 then yields those of Theorem 1.2. \( \square \)

   **Proof of Theorem 1.1.** The product sides of the five identities result from multiplying each of the expressions in Corollary 2.3 by \( (q; q)_\infty \). Because \( F_c(q) = \lim_{n \to \infty} F_{c,n}(q) \), the sum sides of the identities arise by, for each expression in Theorem 1.2, taking the \( n \to \infty \) limit and then multiplying by \( (q; q)_\infty \). We get the sum side of the first, second and fifth identities directly in this way.
The other two identities require a bit more effort. They will require use of the Gaussian polynomial recurrence relations [1]:

\[
\binom{n}{k} = \binom{n - 1}{k} + q^{n-k} \binom{n - 1}{k - 1} = q^{k} \binom{n - 1}{k} + \binom{n - 1}{k - 1}.
\] (3.7)

For convenience, for \(j, n \geq 0\) define

\[
U^{(j)}(n) = \sum_{m \geq 0} q^{n^2 + m^2 - nm + jm} \binom{2n}{m}
\]

and

\[
V^{(j)}(n) = \sum_{m \geq 0} q^{n^2 + m^2 - nm + jm} \binom{2n + 1}{m}.
\]

Using the first identity in (3.7) gives

\[
V^{(j)}(n) = \sum_{m \geq 0} q^{n^2 + m^2 - nm + jm} \binom{2n}{m} + \sum_{m \geq 0} q^{n^2 + m^2 - nm + 2n+1+jm-m} \binom{2n}{m-1} = U^{(j)}(n) + \sum_{m \geq 0} q^{n^2+(m-1)^2-n(m-1)+n+(j+1)(m-1)+j+1} \binom{2n}{m-1}.
\]

After replacing \(m\) by \(m+1\) in the second term (and noting that the original \(m = 0\) summand is zero), we thus obtain

\[
V^{(j)}(n) = U^{(j)}(n) + q^{n+j+1} U^{(j+1)}(n).
\] (3.8)

The sum side of the third identity in Theorem 1.1 is, via Theorem 1.2, given by

\[
(q; q)_{\infty} \lim_{n \to \infty} F_{(3,0,1),n}(q)
\]

\[
= \sum_{n_1,n_2} \frac{q^{n_1^2+n_2^2-n_1n_2+n_1}}{(q; q)_{n_1}} \binom{2n_1}{n_2} + \sum_{n_1,n_2} \frac{q^{n_1^2+n_2^2-n_1n_2+2n_2}}{(q; q)_{n_1-1}} \binom{2n_1-2}{n_2}
\]

\[
= \sum_{n_1} \frac{q^{n_1}}{(q; q)_{n_1}} U^{(0)}(n_1) + \sum_{n_1,n_2} \frac{q^{(n_1-1)^2+n_2^2-(n_1-1)n_2+2n_1+n_2-1}}{(q; q)_{n_1-1}} \binom{2n_1-2}{n_2}
\]

\[
= \sum_{n_1} \frac{q^{n_1}}{(q; q)_{n_1}} U^{(0)}(n_1) + \sum_{n_1} \frac{q^{2n_1+1}}{(q; q)_{n_1}} U^{(1)}(n_1)
\]

after replacing \(n_1\) by \(n_1 + 1\) in the second term. Via the \(j = 0\) case of (3.8), this gives the sum side of the third identity in Theorem 1.1, as required.

Before proving the fourth identity, we apply the final expression in (3.7) to \(V^{(j)}(n)\) to give

\[
V^{(j)}(n) = \sum_{m \geq 0} q^{n^2 + m^2 - nm + (j+1)m} \binom{2n}{m} + \sum_{m \geq 0} q^{n^2 + m^2 - nm + jm} \binom{2n}{m-1} = U^{(j+1)}(n) + \sum_{m \geq 0} q^{n^2+(m-1)^2-n(m-1)-n+(j+2)(m-1)+j+1} \binom{2n}{m-1}.
\]
Then, replacing \( m \) by \( m + 1 \) in the second term gives

\[
V^{(j)}(n) = U^{(j+1)}(n) + q^{-n+j+1}U^{(j+2)}(n).
\]  

(3.9)

The sum side of the fourth identity in Theorem 1.1 is, via Theorem 1.2, given by

\[
(q; q) \lim_{n \to \infty} F_{(2,2,0),n}(q) \\
= \sum_{n_1, n_2} q^{n_1^2 + n_2^2 - n_1 n_2 + n_1} \begin{bmatrix} 2n_1 \\ n_2 \end{bmatrix} \\
+ \sum_{n_1, n_2} q^{n_1^2 + n_2^2 - n_1 n_2 + n_2} (1 + q^{n_1 + n_2}) \begin{bmatrix} 2n_1 - 2 \\ n_2 \end{bmatrix} \\
= \sum_{n_1} q^{n_1} \frac{U^{(0)}(n_1)}{(q; q)_{n_1}} \\
+ \sum_{n_1, n_2} q^{(n_1-1)^2 + n_2^2 - (n_1-1)n_2 + 2n_1 - 1} (1 + q^{n_1 + n_2}) \begin{bmatrix} 2n_1 - 2 \\ n_2 \end{bmatrix} \\
= \sum_{n_1} q^{n_1} \frac{U^{(0)}(n_1)}{(q; q)_{n_1}} + \sum_{n_1} q^{2n_1 + 1} \frac{U^{(0)}(n_1)}{(q; q)_{n_1}} + \sum_{n_1} q^{3n_1 + 2} \frac{U^{(1)}(n_1)}{(q; q)_{n_1}} \\
= \sum_{n_1} \frac{1}{(q; q)_{n_1}} \left( q^{n_1} (1 + q^{n_1 + 1}) U^{(0)}(n_1) + q^{3n_1 + 2} U^{(1)}(n_1) \right)
\]

where the third equality results from replacing \( n_1 \) by \( n_1 + 1 \) in the second term. Using first the \( j = 0 \) case of (3.8), then the \( j = 0 \) case of (3.9), then the \( j = 1 \) case of (3.8) shows that

\[
q^{n_1} (1 + q^{n_1 + 1}) U^{(0)}(n_1) + q^{3n_1 + 2} U^{(1)}(n_1) \\
= q^{n_1} (1 + q^{n_1 + 1}) \left( V^{(0)}(n_1) - q^{n_1 + 1} U^{(1)}(n_1) \right) + q^{3n_1 + 2} U^{(1)}(n_1) \\
= q^{n_1} (1 + q^{n_1 + 1}) V^{(0)}(n_1) - q^{2n_1 + 1} U^{(1)}(n_1) \\
= q^{n_1} (1 + q^{n_1 + 1}) V^{(0)}(n_1) - q^{2n_1 + 1} V^{(0)}(n_1) + q^{n_1 + 2} U^{(2)}(n_1) \\
= q^{n_1} V^{(0)}(n_1) + V^{(1)}(n_1) - U^{(1)}(n_1).
\]

Therefore

\[
(q; q) \lim_{n \to \infty} F_{(2,2,0),n}(q) \\
= \sum_{n_1} \frac{q^{n_1}}{(q; q)_{n_1}} V^{(0)}(n_1) + \sum_{n_1} \frac{1}{(q; q)_{n_1}} V^{(1)}(n_1) - \sum_{n_1} \frac{1}{(q; q)_{n_1}} U^{(1)}(n_1).
\]

The first and third terms here cancel by virtue of the equality of the second and third identities in Theorem 1.1. This leaves the sum side of the fourth expression and, thus, the Proof of Theorem 1.1 is complete.
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