Lectures on Gromov invariants for symplectic 4-manifolds

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Introduction

Taubes's recent spectacular work setting up a correspondence between $J$-holomorphic curves in symplectic 4-manifolds and solutions of the Seiberg-Witten equations counts $J$-holomorphic curves in a somewhat new way. The “standard” theory concerns itself with moduli spaces of connected curves, and gives rise to Gromov-Witten invariants: see for example, McDuff–Salamon [15], Ruan–Tian [21, 22]. However, Taubes's curves arise as zero sets of sections and so need not be connected. These notes are in the main expository. We first discuss the invariants as Taubes defined them, and then discuss some alternatives, showing, for example, a way of dealing with multiply-covered exceptional spheres. We also calculate some examples, in particular finding the Gromov invariant of the fiber class of an elliptic surface by counting $J$-holomorphic curves, rather than going via Seiberg–Witten theory.

For background material on symplectic manifolds and $J$-curves the reader can consult [15, 16] as well as the article by F. Lalonde in this volume. We will make passing references to Seiberg–Witten theory, but the reader need know nothing about it to understand most of this article.

These notes are loosely based on the lectures which I gave in Montreal. The treatment of Gromov invariants has been expanded, and the material on the classification of ruled surfaces has been written up elsewhere (in [4, 5]).

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is the plan. A more detailed description of the contents appears at the end of Lecture 1. I wish to thank R. Stern and T. Parker for some helpful comments, and W. Lorek for taking the notes, for useful discussions concerning the material in Lecture 5, and for a careful reading of an earlier version of this manuscript.

Lecture 1 Gromov invariants: definition and examples. .....................2
Lecture 2 Proof of the main structure theorem.................................7
Lecture 3 Gromov invariants: further discussion. ......................... 13
  3.1 Multiply-covered exceptional spheres 13
  3.2 Components of Σ 17
  3.3 Examples with disconnected K 21
  3.4 Structure of the Gromov invariants when $b^+_2 > 1$ 23
Lecture 4 Spherical Gromov invariants...............................27
Lecture 5 Calculating Gromov invariants of tori....................32
  5.1 Tori in $S^2 \times T^2$ 32
  5.2 Taubes’s method for counting tori 33
  5.3 Elliptic surfaces 35
  5.4 The Gromov invariant of a fiber sum 37

1 Gromov invariants: definition and examples.

1.1 Basic ideas

Let $(M, \omega)$ be a compact symplectic 4-manifold with a compatible almost-complex structure $J$. Given a sequence of solutions to the perturbed Seiberg-Witten equations for some Spin$^c$ structure $\Gamma$, Taubes \cite{27} constructs a regular $J$-holomorphic curve $C$. The curve $C$ passes through $k$ generic points, where

$$k = k(A) = \frac{1}{2}(c_1(A) + A \cdot A).$$

Here $A \in H_2(M, \mathbb{Z})$ is the homology class of $C$ and is determined by the Spin$^c$ structure $\Gamma$, and $c_1$ is the first Chern class of the complex rank 2 bundle $(TM, J)$.

Because it appears as the zero section of a certain complex line bundle, the curve $C$ can be disconnected and can have multiply-covered components. It might have also components which are cusp-curves or have singularities. A natural question that arises is: what can be said about the geometry of such a curve?

To analyse $C$ we will parametrise it by a $J$-holomorphic map

$$\phi : \Sigma \to M$$
from a possibly disconnected Riemann surface $\Sigma = \bigsqcup \Sigma_i$ to the manifold $M$ chosen so that $[\phi_\ast(\Sigma)] = A$. The multiplicity of $\phi|_{\Sigma_i}$ on the component $\Sigma_i$ is an integer $m_i$ such that $\phi|_{\Sigma_i}$ may be written as a composite

$$\phi|_{\Sigma_i} : \Sigma_i \xrightarrow{\psi} \Sigma'_i \xrightarrow{\phi'_i} M,$$

where $\psi : \Sigma_i \to \Sigma'_i$ is a branched covering map of degree $m_i$ and the $J$-holomorphic map $\phi' : \Sigma'_i \to M$ is somewhere injective. We will assume that the images $\phi(\Sigma_i)$ of the different components $\Sigma_i$ are distinct. (This may be arranged by replacing several coincident components by a single multiply-covered component.) Thus the image curve $C = \phi(\Sigma)$ is a finite union of distinct connected curves $C_i = \phi_i(\Sigma_i)$, each with a multiplicity $m_i \geq 1$, such that

$$A = [C] = \sum_i m_i[C_i].$$

Two parametrizations $\phi, \Sigma$ and $\phi', \Sigma'$ are equivalent if their images and assigned multiplicities are equal, and we denote the equivalence class containing $\phi, \Sigma$ by $(\phi, \Sigma)$. The pair $(\phi, \Sigma)$ belongs to a moduli space $H(A)$ which is defined as follows.

**Definition 1.1** Given $A \in H^2(M, \mathbb{Z})$ let $\Omega_k$ be a set of $k(A) = \frac{1}{2}(c_1(A) + A \cdot A)$ distinct points on $M$. The moduli space $H(A)$ is the set of equivalence classes $(\phi, \Sigma)$ as above such that the image $\phi(\Sigma)$ contains $\Omega_k$:

$$H(A) = \{(\phi, \Sigma) : \Omega_k \subset \phi(\Sigma)\}$$

Moreover a pair $(\phi, \Sigma)$ will be called **good** if $\phi|_{\Sigma_i}$ has multiplicity $m_i = 1$ whenever $\phi(\Sigma_i)^2 = \phi(\Sigma_i) \cdot \phi(\Sigma_i) < 0$.

Note that the elements of $H(A)$ are unparametrized rather than parametrized curves. The following theorem is due to Taubes [27]. Intuitively, it says that $k(A)$ is the maximal dimension of a stratum in the space of all (possibly disconnected) $J$-holomorphic $A$-curves.

**Theorem 1.2** Suppose that $J$ is a generic $\omega$-tame almost-complex structure on $M$, and $A \in H^2(M, \mathbb{Z})$ a homology class.

(i) Suppose the moduli space $H(A)$ contains a good pair. Then every pair $(\phi, \Sigma) \in H(A)$ is good. Moreover:

(a) For every component $\Sigma_i$, $\phi(\Sigma_i)$ is an embedded curve, disjoint from all other curves $\phi(\Sigma_j)$.

---

1 A $(J$-holomorphic) map $\phi : \Sigma \to M$ is said to be somewhere injective if there is a point $z \in \Sigma$ at which the derivative $d\phi(z)$ has maximal rank and also is such that $\phi^{-1}(\phi(z)) = \{z\}$. 

3
The multiplicity $m_i$ of $\phi|_{\Sigma_i}$ is one, unless the genus $g(\Sigma_i) = 1$ and $\phi(\Sigma_i)$ has zero self-intersection.

The moduli space $\mathcal{H}(A)$ is 0-dimensional, and finite.

If $(\phi, \Sigma) \in \mathcal{H}(A)$, then the image $\phi(\Sigma_i)$ of every $\Sigma_i$ such that $\phi(\Sigma_i) \cdot \phi(\Sigma_i) < 0$ is an embedded exceptional sphere. However its multiplicity may be > 1.

The proof is deferred to the next lecture. If the elements of $\mathcal{H}(A)$ are not good, many of the statements made in (i) above still hold. The situation is fully explained in §3.1.

We continue here with a brief discussion of the Gromov invariant $\text{Gr}(A)$ and the calculation of some easy examples. The basic idea is that $\text{Gr}(A)$ counts the number of elements in $\mathcal{H}(A)$ with appropriate sign. It is quite easy to make this precise when no elements of $\mathcal{H}(A)$ have components which are multiply-covered. However, multiply-covered tori are very difficult to count, and we will postpone further discussion of this case to Lecture 5.

For the time being, let us suppose that for every $(\phi, \Sigma) \in \mathcal{H}(A)$ the components $\Sigma_i$ are mapped with multiplicity 1. Then, if $\Sigma_i$ has genus $g_i$, and its image has homology class $A_i$ and contains $k_i$ of the points of $\Omega_k$, there is an evaluation map of the form

$$\text{ev} : \prod_i \mathcal{M}(A_i, J, g_i) \times G_i(\Sigma_i)^{k_i} \to M^{k_i}(A),$$

where $\mathcal{M}(A_i, J, g_i)$ is the moduli space of (connected) $J$-holomorphic curves of genus $g_i$ in class $A_i$ and $G_i$ is an appropriate reparametrization group. (See Lecture 2.) Note that $\mathcal{M}(A_i, J, g_i)$ has a canonical orientation even in the case that it is zero-dimensional. It follows from the proof of Theorem 1.2 that the domain and range of $\text{ev}$ have the same dimension. Thus there is a bijection between the subset of $\mathcal{H}(A)$ corresponding to the given decomposition $A_i, k_i$ and the set $\text{ev}^{-1}(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k$ are the points of $\Omega_k$ listed in appropriate order. Since $\text{ev}$ maps between oriented manifolds, one can therefore assign a sign $\varepsilon(\phi, \Sigma) = \pm 1$ to each such element of $\mathcal{H}(A)$. Observe that this sign is simply the product of signs which are attached to each component via the evaluation map

$$\text{ev}_i : \mathcal{M}(A_i, J, g_i) \times G_i(\Sigma_i)^{k_i} \to M^{k_i}.$$
Definition 1.3 Given a homology class $A \in H_2(M, \mathbb{Z})$ such that $\mathcal{H}(A)$ only contains elements with components of multiplicity 1, we define the Gromov invariant $\text{Gr}(M, A) = \text{Gr}(A)$ by:

$$\text{Gr}(A) = \sum_{\{(\phi, \Sigma) \in \mathcal{H}(A)\}} \varepsilon(\phi, \Sigma)$$

This number is independent of the choice of generic $\omega$-tame $J$.

1.2 Examples

There are several basic examples where the Gromov invariants can be rather easily computed.

Example 1.4 Let $M = S^2 \times S^2$ with its standard integrable complex structure, and the standard product symplectic form. Let $A_1 = [S^2 \times pt]$ and $A_2 = [pt \times S^2]$. Then $k(A_1) = \frac{1}{2}(c_1(A_1) + A_1 \cdot A_1) = \frac{1}{2}(2 + 0) = 1$, i.e. we are counting $J$-curves in class $A_1$ passing through one generic point $z_0$. There clearly is a unique $J$-holomorphic sphere $S$ in class $A_1$ passing through the point $z_0$ and it is not hard to show that it is regular. Moreover there cannot be another $J$-holomorphic $A_1$-curve $S'$ through $z_0$ by positivity of intersections: if there were we would have $A_1^2 = S \cdot S' > 0$ which is absurd. Hence $\text{Gr}(A_1) = 1$.

A similar argument shows that $\text{Gr}(2A_1) = 1$. In this case $k(2A_1) = 2$ so that we are counting curves through 2 generic points. Because $J$ is a product, Theorem 1.2 implies that the only elements in $\mathcal{H}(2A_1)$ are doubly covered $A_1$-spheres and disconnected curves consisting of 2 disjoint $A_1$-spheres. Since curves of the former type only go through 1 generic point, we just have to count the number of pairs of $A_1$-spheres through a given pair of points. Since these points are generic, they do not lie on the same $A_1$-sphere and so (by positivity of intersections again) there is exactly one such pair.

Next consider the class $A_1 + A_2$. Because $J$ is a product, $J$-holomorphic spheres in class $A_1 + A_2$ are graphs of holomorphic maps $S^2 \to S^2$ and so there is a unique such graph through 3 generic points. This is consistent with the fact that $k(A_1 + A_2) = \frac{1}{2}(4 + 2) = 3$, and implies that $\text{Gr}(A_1 + A_2) = 1$.

In fact, it follows from Taubes’ results and the wallcrossing formula of Li-Liu that $\text{Gr}(A) = 1$ for all nonzero $A = pA_1 + qA_2$ with $p, q \geq 0$: see [6, 7, 8].

Example 1.5 This time let $M = \mathbb{C}P^2$ with its standard complex structure. Let $L = [\mathbb{C}P^1]$ and $A = 3L$. Then $k(A) = \frac{1}{2}(9 + 9) = 9$, and we are counting curves through 9 generic points. The curves in class $A$ are the cubic curves – either (embedded) tori or rational curves with a double point or a cusp. Recall that there exists a unique holomorphic torus through 9 generic points, hence $\# \mathcal{H}(A) \geq 1$. (We are in the integrable case here so that all signs are +1.) On the other hand, the complex dimension of the moduli space of holomorphic
curves of genus $g$ in class $A$ is $(c_1(A) + g - 1)$. It follows that there is a finite number of rational curves through 8 generic points, hence there are no rational curves through 9 points in generic position. Since all curves in class $A$ are either rational or tori we can conclude that $\text{Gr}(A) = \#\mathcal{H}(A) = 1$. For further discussion see Example 4.7.

Here we have given an independent argument to show that the elements of $\mathcal{H}(A)$ are embedded curves. However, this is part of Theorem 1.2. In fact, the proof of this part of the theorem is just a more elaborate version of the argument presented above.

Example 1.6 Let $M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$, and let $E$ be the homology class of the exceptional divisor, so that $E \cdot E = -1$. For any $J$ there exists a unique $J$-holomorphic representative $C_E$ of $E$. Consider now $A = L + E$. There is no connected $J$-holomorphic curve $C$ in that class. For if $[C] = L + E$, then the intersection index $C \cdot C_E = -1$ so that (by positivity of intersections) $C_E$ is a component of $C$. Now $k(A) = \frac{1}{2}(c_1(A) + A \cdot A) = \frac{1}{2}(4 + 1 - 1) = 2$. It is easily seen that there is only one curve in class $A$ through two generic points: it has two components, $C_E$ and an $L$-sphere $\Sigma$. The latter contains the two points of $\Omega_2$ and is disjoint from $C_E$.

Similarly, the class $L + 2E$ is represented by the disjoint union of a sphere in class $L$ through 2 generic points and a double cover of $C_E$. Thus $\text{Gr}(L + 2E)$ should be 1. But this element $(\phi, \Sigma)$ is not good, and so it does not appear in $\mathcal{H}(A)$. Instead, observe that $k(L + 2E) = 1$, so that there is a whole family of curves through $k(A)$ generic points. Thus part (i)(c) of Theorem 1.2 fails. In fact there also is an isolated representative of $A$ consisting of one $L - E$ curve together with a triple cover of the $E$ curve, but now the different components intersect so this should not contribute to $\text{Gr}(A)$.

An internally consistent definition of the Gromov invariants for classes whose representation involves multiply-covered exceptional spheres is presented in Lecture 3 below. We will see that it suffices to alter the definition of $k(A)$. At this writing it is not clear whether this definition is appropriate in the context of Taubes’ identification of the Gromov invariants with the Seiberg–Witten invariants. However, in the above example, we know that the Seiberg–Witten invariant of the class $L + 2E$ is 1, and so the evidence points to it being the correct definition.

1.3 Further Contents

To finish, we briefly describe the contents of the remaining lectures.

- Lecture 2 gives the proof of Theorem 1.2. The argument is basically straightforward even though it is somewhat long: it is yet another indication that in dimension 4 homology determines geometry.
In Lecture 3 we take up two important questions concerning the Gromov invariants. The first (how to deal with multiply-covered exceptional curves) arose in Example 1.6 above. We propose a definition of a modified invariant \( \text{Gr}'(A) \) which takes care of this problem. The second question also appeared there, albeit indirectly. It is the question of how one knows that one has found all the elements of \( \mathcal{H}(A) \). The decomposition of each \((\phi, \Sigma) \in \mathcal{H}(A)\) into its components \((\phi|_{\Sigma_i}, \Sigma_i)\) gives rise to a corresponding decomposition \( A = \sum_j B_j \). (If there are no toral components, the set of \( B_j \) is simply the set of homology classes represented by the components of \( \Sigma \).) We discuss cases in which only one such decomposition occurs. To what extent this is true in general is an open problem.

In Lecture 4 we define an analog \( \text{Gr}_s(A) \) of \( \text{Gr}(A) \) which only counts spheres, and discuss its relation to \( \text{Gr}(A) \).

In Lecture 5 we discuss the calculation of \( \text{Gr}(A) \) in the case when \( A \) is represented by tori. We also give examples to show why problems arise when counting multiply-covered tori, and outline the method of counting them that Taubes developed in [28]. Finally, using a J-holomorphic analog of Gompf summing, we calculate \( \text{Gr}(A) \) for the fiber class in an elliptic surface.

2 Proof of the main structure theorem.

This lecture is devoted to a proof of Theorem 1.2. We will work in the following set-up:

(1) \( \Sigma_g \) denotes a connected a 2-dimensional manifold of genus \( g \).

(2) \( \mathcal{T}_g \) denotes Teichmüller space. Thus \( \dim_{\mathbb{R}} \mathcal{T}_g = 6g - 6 \) when \( g > 1 \), and there is a smooth mapping

\[
\begin{align*}
  j : \mathcal{T}_g & \to \mathcal{J}(\Sigma) \\
  \tau & \to j(\tau)
\end{align*}
\]

where \( \mathcal{J}(\Sigma) \) denotes the space of almost-complex structures on \( \Sigma \).

(3) \( G_g \) denotes the reparametrisation group. \( G_0 = \text{PSL}(2, \mathbb{C}) \), \( G_1 \) is an extension of \( \text{SL}(2, \mathbb{Z}) \) by the torus \( T^2 \). For \( g \geq 2 \) the group \( G_g \) is the mapping class group, isomorphic to \( \pi_0(\text{Diff}(\Sigma)) \). Thus, \( \dim G_0 = 6, \dim G_1 = 2, \dim G_g = 0 \) for \( g > 1 \).

Note that \( G_g \) is the full group of automorphisms for a generic element of Teichmüller space, but there is a singular set (of complex codimension \( \geq 1 \)) of elements that have larger automorphism groups.
Lemma 2.1  Let \( \phi : \Sigma \to M \) be a \((J,j(\tau))\)-holomorphic curve i.e
\[
d\phi \circ j(\tau) = J \circ d\phi
\]
Then for every \( \gamma \in G_g \) the composition \( \phi \circ \gamma \) is \((J,j(\gamma^{-1} \circ \tau))\)-holomorphic.

Proof:  This is obvious. \( \square \)

Definition 2.2  For an almost-complex structure \( J \) on \( M \), and a homology class \( A \in H_2(M, \mathbb{Z}) \) let \( \mathcal{M}(A,J,g) \) denote the space of \( J \)-holomorphic curves of genus \( g \) in class \( A \). More precisely,
\[
\mathcal{M}(A,J,g) = \left\{ (\phi, \tau) \in \text{Maps}(\Sigma, M) \times T_g : \text{the curve } \phi \text{ is} \right. \\
\left. (j(\tau), J) \text{-holomorphic, somewhere injective, and represents the homology class } A \right\}
\]
We have the following basic theorem:

Theorem 2.3  For a generic \( \omega \)-tame \( J \) the moduli space \( \mathcal{M}(A,J,g) \) is an oriented manifold of (real) dimension \( 2(c_1(A) + g - 1) + \dim G_g \). Further, if \( \ell = \ell_g(A) = c_1(A) + g - 1 \), there is a well defined evaluation mapping:
\[
ev : \mathcal{M}(A,J,g) \times_{G_g} \Sigma^\ell \to M^\ell \\
(\phi, \tau, z_1, \cdots, z_\ell) \mapsto (\phi(z_1), \cdots, \phi(z_\ell))
\]
between manifolds of equal dimension \( 4\ell \).

The moduli space \( \mathcal{M}(A,J,g) \) is used to count \( J \)-holomorphic curves. Roughly speaking, the number of \( J \)-holomorphic curves through \( \ell \) generic points is equal to the degree of \( ev \). (Note that \( ev \) maps between manifolds of dimension \( 4\ell \).) A precise statement requires compactification of \( \mathcal{M}(A,J,g) \), hence introduction of cusp-curves. When \( M \) has dimension 4 (or 6) the set of points of \( M \) which lie on \( A \)-cusp-curves always has codimension 2, and it follows by a standard argument that, except possibly in the case \( A \cdot A = 0, g = 1 \), the map \( ev \) represents a homology class. (It is a pseudocycle in the language of \([12]\).) For reasons of dimension, this homology class is a multiple \( p[M^\ell] \) of the fundamental class \([M^\ell]\), and we can figure out what \( p \) is by counting the points in the inverse image of any point \( (x_1, \cdots, x_\ell) \in M^\ell \). The case \( A \cdot A = 0, A = mB, m > 1 \) and \( g = 1 \) must be treated separately since, although the \( A \)-curves themselves are embedded tori (by definition the elements of \( \mathcal{M}(A,J,1) \) are somewhere injective), it is possible for these tori to converge to multiply-covered tori in some class \( kB \). As Ruan pointed out, this does not happen for generic \( J \). However, as Taubes realised in \([28]\), there are generic 1-parameter deformations of \( J \) along which embedded tori in class \( A = 2B \) are absorbed by tori in class \( B \). Hence the number of tori in such a class \( A \) is not globally constant, although it is locally constant. We will discuss this more in Lecture 5, contenting ourselves for now with the following theorem.
Theorem 2.4  In the situation of the previous theorem, given a generic set of points \((x_1, \cdots, x_\ell) \in M^\ell\) the inverse image \(ev^{-1}(x_1, \cdots, x_\ell)\) is finite. Moreover, except possibly when \(A^2 = 0, A = mB\) and \(g = 1\), the number of points in this inverse image (counted with sign) is independent of the choice of generic \(J\).

Let now \(C\) be a connected \(J\)-holomorphic \(A\)-curve of genus \(g\), and multiplicity 1:

\[
\phi : (\Sigma_g, j) \to M, \\
\phi_* [\Sigma_g] = [C] = A
\]

If \(J\) is generic and \(\mathcal{M}(A, J, g)\) is non-empty then necessarily \(c_1(A) + g - 1 \geq 0\). Moreover, the above theorems imply that there exist finitely many such curves \(C\) through \(\ell_g(A) = c_1(A) + g - 1\) distinct points. We will need a version of the adjunction formula for such curves \(C\).

Proposition 2.5  If a (connected) \(J\)-holomorphic curve \(C\) has genus \(g\) and is in the class \(A\) then:

\[
A \cdot A \geq c_1(A) + 2(g - 1)
\]

with equality if and only if \(C\) is embedded.

Proof:  (Sketch) Suppose first that \(C\) is immersed, with simple double points. Then \(c_1(A) = c_1(TM, J)(C) = c_1(TC)(C) + c_1(\nu C)(C) = 2 - 2g + C \cdot C - 2m\), where \(m\) is the number of double points, and \(\nu\) is the normal bundle. Hence:

\[
A \cdot A = C \cdot C = c_1(A) + 2(g - 1) + 2m
\]

If \(C\) is singular, use [12] to perturb \(C\) to an immersed curve with double points. Every singularity contributes a non-zero number of double points, and the proposition follows easily from the immersed case. \(\square\)

For later reference, we recap the properties of \(C\).

Corollary 2.6  Let \(C\) be a (connected) \(J\)-holomorphic curve for some generic \(J\).

(i) \(C\) exists only if \(\ell_g(A) = c_1(A) + g - 1 \geq 0\).

(ii) There is only a finite number curves of of genus \(g = g(C)\) and in the class \([C]\) through \(\ell_g(A)\) generic points. In particular if \(\ell > \ell_g(A)\) there are no curves of this kind through \(\ell\) generic points. Thus we will say that \(\ell_g(A)\) is the maximum number of generic points which can lie on \(C\).

(iii) The adjunction formula holds: \(A \cdot A \geq c_1(A) + 2(g - 1)\), with equality if and only if \(C\) is embedded.
(iv) If $k(A) = \frac{1}{2}(A \cdot A + c_1(A))$, then $k(A) \geq \ell_g(A)$, with equality if and only if $C$ is embedded.

The above results hold for curves of multiplicity 1. Suppose now that $C$ is a curve with multiplicity $m$, but still connected. Thus a parametrization $(\phi, \Sigma)$ of $C$ factors through a degree 1 mapping $\phi': \Sigma' \to M$, and we define $g = g(C)$ to be the genus of the underlying simply-covered curve $\Sigma'$. Further if $|C| = mB$ we set

$$\ell_{g,m}(mB) = \ell_g(B) = c_1(B) + (g - 1).$$

As with the number $\ell_g(B)$ defined above, this number $\ell_{g,m}(mB)$ is the maximum number of generic points which lie on a curve such as $C$.

**Lemma 2.7** Let $C$ be an $m$-fold cover of a $J$-curve in class $B$ where $J$ is generic.

(i) If $B \cdot B \geq 0$ then $k(mB) \geq \ell_{g,m}(mB)$ with equality if and only if either $m = 1$ and $C$ is embedded, or $C$ is an $m$-fold cover of an embedded torus of self-intersection 0.

(ii) If $B \cdot B < 0$ then $C$ is a (possibly multiply-covered) exceptional sphere.

**Proof:** Part (i) follows from a computation:

$$k(mB) = \frac{1}{2}(m^2B \cdot B + mc_1(B)) \geq \frac{m}{2}(B \cdot B + c_1(B)) \geq m \ell_g(B) \geq \ell_g(B).$$

If the first inequality holds with $m > 1$ we must have $B \cdot B = 0$, and if the second holds we need $\ell_g(B) = 0$. Hence $g = 1$. Moreover, since $k(B) = \ell_g(B)$ $C$ is an $m$-fold cover of an embedded curve.

As for the second part, we only need to observe that two inequalities hold:

$$\ell_g(B) = c_1(B) + g - 1 \geq 0$$
$$c_1(B) + 2(g - 1) \leq B \cdot B < 0.$$

It follows that $g - 1 < 0$, hence $g = 0$. Then $c_1(B) \geq 1$, and from the second inequality $c_1(B) = 1$. Finally it follows that $B \cdot B = -1$, and so $C$ is an $m$-fold cover of an (embedded) exceptional curve.

Observe that the only time that $k(mB)$ is less than $\ell_{g,m}(mB)$ is when $B$ is represented by an exceptional sphere and $m > 1$. This is why multiply-covered exceptional spheres have to be treated separately. Recall from Lecture 1 that a pair $(\phi, \Sigma)$ is called “good” if $\phi|_{\Sigma_i}$ has multiplicity $m_i = 1$ whenever $\phi(\Sigma_i) \cdot \phi(\Sigma_i) < 0$. 

10
Corollary 2.8 The pair $(\phi, \Sigma)$ is good if and only if it contains no component which is a multiply-covered exceptional sphere.

Proof: This is immediate from Lemma 2.7. □

Proposition 2.9 Suppose that all of the elements $(\phi, \Sigma) \in \mathcal{H}(A)$ are good. Then for every curve $(\phi, \Sigma)$:

(i) $\phi(\Sigma_i)$ is embedded, and disjoint from all other components $\phi(\Sigma_j)$, except possibly if $\phi(\Sigma_i)$ is a torus with zero self-intersection.

(ii) The multiplicity $m_i$ of $\phi|_{\Sigma_i}$ is one, except possibly if $\phi(\Sigma_i)$ is a torus with zero self-intersection.

Proof: Decompose \{1, \ldots , k\} into disjoint sets $I_p, p = 1, \ldots , r$, such that the images

$$C_p = \phi(\coprod_{i \in I_p} \Sigma_i)$$

are connected and mutually disjoint. If $A_i = \phi_*(\Sigma_i)$ and $A'_i = \sum_{i \in I_p} A_i$ then, because $A'_p \cdot A'_q = 0$, we have

$$k(A) = k(A'_1) + \cdots + k(A'_r).$$

Further, because $A_i \cdot A_j \geq 0$, for each $p$ we have

$$k(A'_p) \geq \sum_{i \in I_p} k(A_i)$$

with equality if and only if $I_p$ has cardinality 1.

Now let us look at the numbers $\ell_i$. As above, $\ell_i$ should be the maximum number of generic points on a curve of type $\phi(\Sigma)$. Thus, if $\phi|_{\Sigma_i}$ is an $m_i$-fold cover of a curve in class $B_i$ and of genus $g_i$, we set

$$\ell_i = \ell_{g_i,m_i}(mB_i) = \ell_{g_i}(B_i) = c_1(B_i) + g_i - 1.$$ 

Then, because $\phi(\Sigma)$ goes through $k(A)$ generic points by assumption, we must have $\sum_i \ell_i \geq k(A)$. On the other hand, by Lemma 2.7, we know that $k(A_i) \geq \ell_i$ for all $i$. Hence

$$k(A) = \sum_p k(A'_p) \geq \sum_i k(A_i) \geq \sum_i \ell_i \geq k(A).$$

Therefore we must have equality everywhere. In particular, each $I_p$ has cardinality 1 which implies that all the curves $\phi(\Sigma_i)$ are disjoint, and $k(A_i) = \ell_i$ for all $i$. The result now follows immediately from Lemma 2.7. □

To complete the proof of Theorem 1.2 we need the following lemma. We write $\mathcal{E}$ for the set of classes in $H_2(X)$ which are represented by exceptional spheres.
Lemma 2.10 The following statements are equivalent.

(i) One element \((\phi, \Sigma) \in \mathcal{H}(A)\) is good.

(ii) Every element in \(\mathcal{H}(A)\) is good.

(iii) \(E \cdot A \geq -1\) for every \(E \in \mathcal{E}\).

Proof: We will show that \((i) \implies (iii) \implies (ii)\). Suppose first that there is a pair \((\hat{\phi}, \hat{\Sigma})\) which has no components that are multiply-covered exceptional spheres. Then, if \(\hat{A}_j\) are the homology classes of the components of \(\hat{\phi}(\hat{\Sigma})\), for each \(E \in \mathcal{E}\) we have \(E \cdot \hat{A}_j \geq 0\) unless \(E = \hat{A}_j\). Hence there is at most one \(j\) for which \(E \cdot \hat{A}_j < 0\), and for this \(j\) we have \(E \cdot \hat{A}_j = -1\). Therefore \(E \cdot A \geq -1\) for all \(E \in \mathcal{E}\). Thus \((i) \implies (iii)\).

We prove that \((iii) \implies (ii)\) by contradiction. Therefore, let us suppose that \((\phi, \Sigma)\) does contain components which are multiply-covered exceptional spheres. By reordering the components, we may suppose that these components are \(\Sigma_i, i = 1, \ldots, s\), and that they have multiplicities \(m_i > 1\). Note that they occur in distinct classes \(E_i \in \mathcal{E}\), because, by assumption, all components of \(\Sigma\) have distinct images under \(\phi\) and, by positivity of intersections, there is a unique \(J\)-holomorphic representative of each class in \(\mathcal{E}\). Thus we may write

\[
A = \sum_{i=1}^{s} m_i E_i + B,
\]

where \(B\) is represented by the pair \((\phi, \Sigma' = \bigsqcup_{i \geq s} \Sigma)\). By construction, no component of \(B\) is a multiply-covered exceptional sphere or an \(E_i\)-curve. Further, all the \(k(A)\) generic points on \((\phi, \Sigma)\) must lie on the the \(B\)-curve. By the previous theorem (which applies because \((\phi, \Sigma')\) is good), this implies that \(k(B) \geq k(A)\). Moreover, we must have \(E_i \cdot B \geq 0\) for all \(i \leq s\). For, if \(E_i \cdot B < 0\) it follows from positivity of intersections that every representative of \(B\) includes an \(E_i\)-curve of multiplicity at least 1, contradicting the definition of \(B\).

Our hypothesis on \(E \cdot A\) implies in particular, that for \(i = 1, \ldots, s\),

\[
E_i \cdot A = -m_i + \sum_{j \neq i} E_i \cdot m_j E_j + E_i \cdot B \geq -1,
\]

so that

\[
\sum_{j \neq i} E_i \cdot m_j E_j + E_i \cdot B \geq -1 + m_i.
\]

Therefore, since \(E_i \cdot B \geq 0\),

\[
2k(A) = c_1(\sum_i m_i E_i + B) + \sum_i (m_i E_i)^2 + B^2
\]

12
+ \sum_i m_i \left( \sum_{j \neq i} E_i \cdot m_j E_j + E_i \cdot B \right) + \sum_i m_i E_i \cdot B \\
\geq \sum_i (m_i - m_i^2) + 2k(B) + \sum_i m_i(-1 + m_i) \\
= 2k(B).

Therefore, we must have equality everywhere. So, for each \( i \),

\[ A \cdot E_i = -1, \quad B \cdot E_i = 0, \]

and

\[ -m_i + \sum_{j \neq i} E_i \cdot m_j E_j = -1. \]

If \( E_i \cdot E_j \geq 1 \) for some \( i \neq j \), we may suppose (by interchanging \( i, j \) if necessary), that \( m_j \geq m_i \). But then

\[ -m_i + \sum_{j \neq i} E_i \cdot m_j E_j \geq -m_i + m_j \geq 0 \]

which is impossible. Therefore \( E_i \cdot E_j = 0 \) for all \( i \neq j \), which implies that \( m_i = -1 \) for all \( i \), again contradicting our choice of \( m_i \). Thus the lemma must hold.

\[ \square \]

3 Gromov invariants: further discussion

We first show how to take into account multiply-covered exceptional curves: see Definition 3.16. Next we discuss conditions under which the surface \( \Sigma \) in \((\phi, \Sigma) \in H(A)\) is connected, and give some examples (in minimal manifolds) where it is not. Finally, we discuss the question of the uniqueness of the decomposition of \( \Sigma \) into its components.

3.1 Multiply-covered exceptional spheres

We saw in Example 1.6 that Taubes’s definition does not give the expected answer when the class \( A \) is represented by a curve which has a multiply-covered exceptional sphere as one component. Moreover Theorem 1.2 fails in this case. By Lemma 2.10, this happens if and only if \( A \cdot E < -1 \) for the class \( E \) of some exceptional sphere. In fact, Taubes shows in [27] that this problem arises only for manifolds with \( b_2^+ = 1 \) since otherwise \( \text{Gr}(A) = 0 \) when some \( E \cdot A < -1 \). Nevertheless, it is worth attempting a better definition.

\[ ^2 \text{Recall that } b_2^+ \text{ is the maximum dimension of a subspace of } H^2(M, \mathbb{Q}) \text{ on which the quadratic form } a \cdot b = \langle a \cup b, [M] \rangle \text{ is positive definite.} \]
It is not hard to deal with the problem. The solution is to redefine the number $k(A)$. (This amounts to looking at a different stratum of the moduli space of all $J$-curves in class $A$.) Before, we set

$$k(A) = \frac{1}{2}(c_1(A) + A \cdot A).$$

Now we set

$$k'(A) = \frac{1}{2} \left( c_1(A) + A \cdot A + \sum_{E \in E} (m_E(A)^2 - m_E(A)) \right),$$

where $E$ is the set of classes $E$ which are represented by exceptional spheres and where

$$m_E(A) = \max(-A \cdot E, 0).$$

(Think of $m_E(A)$ as the algebraic multiplicity of $E$ in $A$.) We will look at the set $H'(A)$ of pairs $(\phi, \Sigma)$ which are defined as before, except now we require that $\phi(\Sigma)$ meets a set $\Omega'$ of $k'(A)$ generic points.

**Proposition 3.1** Suppose that $J$ is a generic almost-complex structure on $M$, and $A \in H_2(M, \mathbb{Z})$ a homology class. Then for any pair $(\phi, \Sigma) \in H'(A)$:

(a) For every component $\Sigma_i$, $\phi(\Sigma_i)$ is an embedded curve, disjoint from all other curves $\phi(\Sigma_j)$.

(b) The multiplicity $m_i$ of $\phi|_{\Sigma_i}$ is one, unless $\phi(\Sigma_i)$ is a torus of zero self-intersection or an exceptional sphere.

(c) The moduli space $H'(A)$ is 0-dimensional, and finite.

**Proof:** As in Lemma 2.10, write

$$A = \sum_{i=1}^{s} m_i E_i + \sum_{j=1}^{\ell} k_j F_j + B,$$

where the $E_i, F_j$ are the classes of the exceptional spheres in the image $\phi(\Sigma)$ with multiplicities $m_i, k_j \geq 2$ and where $B$ is good. The $E_i$ are chosen so that

$$A \cdot E_i = -m_{E_i}(A) = -n_i < 0, \quad i = 1, \ldots, s,$$

and the $F_j$ are chosen so that

$$A \cdot F_j \geq 0.$$

Moreover, we choose the $m_i, k_j$ as large as possible so that $B$ contains no components in the classes $E_i, F_j$. Hence

$$B \cdot E_i \geq 0, \quad B \cdot F_j \geq 0.$$
We aim to show that $\ell = 0$ (i.e. there are no classes $F_j$), that $m_i = n_i$ for all $i$ and that $k'(A) = k(B)$. This will easily imply that the $E_i$ are mutually disjoint and also disjoint from $B$. Since $B$ is good, the result will now follow from Theorem 1.2.

Observe that, by definition, $\phi(\Sigma)$ goes through $k'(A)$ generic points. These must lie on the $B$ curve since exceptional spheres do not move. Hence $k(B) \geq k'(A)$. To prove the converse, note first that for each $i \leq s$

$$E_i \cdot A = -m_i + \sum_{i' \neq i} E_i \cdot m_{i'} E_{i'} + \sum_j E_i \cdot k_j F_j + E_i \cdot B = -n_i < 0. \quad (2)$$

This implies that $m_i > m_{i'}$ for any $i, i' \leq s$ such that $E_i \cdot E_i' \neq 0$. Hence by symmetry we must have

$$E_i \cdot E_i' = 0, \quad i, i' \leq s.$$

Further,

$$\sum_{\{j : E_i \cdot F_j \neq 0\}} k_j \leq m_i - n_i,$$

so that

$$\sum_{\{j : E_i \cdot F_j \neq 0\}} k_j^2 \leq (m_i - n_i)^2.$$

Therefore, if $L = \{j : F_j \cdot E_i \neq 0 \text{ for some } i\}$,

$$\sum_{j \in L} k_j^2 \leq \sum_i (m_i - n_i)^2. \quad (3)$$

Equation (3) also implies

$$E_i \cdot \left( \sum_j k_j F_j + B \right) = m_i - n_i.$$

Similarly, the fact that $A \cdot F_j \geq 0$ implies

$$F_j \cdot (A - k_j F_j) = F_j \cdot \left( \sum_i m_i E_i + \sum_{j' \neq j} k_{j'} F_{j'} + B \right) \geq k_j,$$

so that, when $j \not\in L$ we have

$$F_j \cdot (A - k_j F_j) = F_j \cdot \left( \sum_{j' \neq j} k_{j'} F_{j'} + B \right) \geq k_j. \quad (4)$$

Thus, using equation (4), we find

$$A^2 = \left( \sum_i m_i E_i + \sum_j k_j F_j + B \right) \cdot \left( \sum_i m_i E_i + \sum_j k_j F_j + B \right).$$
\[ \begin{align*}
\geq & \sum_i (m_i E_i)^2 + \sum_j (k_j F_j)^2 + B^2 + 2 \sum_i m_i E_i \cdot (\sum_j k_j F_j + B) \\
& + \sum_{j \not\in L} k_j F_j \cdot (\sum_{j' \not= j} k_{j'} F_{j'} + B) \\
\geq & \quad B^2 - \sum_i m_i^2 - \sum_j k_j^2 + 2 \sum_i m_i (m_i - n_i) + \sum_{j \in L} k_j^2 \\
= & \quad B^2 + \sum_i (m_i^2 - 2m_i n_i) - \sum_{j \in L} k_j^2.
\end{align*} \]

Hence
\[ 2k'(A) = c_1(A) + A^2 + \sum_i (n_i^2 - n_i) \geq 2k(B) + \sum_i (m_i - n_i) \geq 2k(B) \]

where the penultimate inequality uses equation (3). But, as we observed earlier, \( k(B) \geq k'(A) \). Therefore we must have equality everywhere. This gives \( m_i = n_i \) for all \( i \), which, by equation (2), implies that \( E_i \cdot F_j = E_i \cdot B = 0 \) for all \( i, j \). Using equation (3) we see also that \( L = \emptyset \). Therefore, if \( B' = \sum_j k_j F_j + B \), we have that \( E_i \cdot B' = 0 \) for all \( i \), which easily implies that
\[ E \cdot B' \geq -1 \]

for all \( E \in \mathcal{E} \). The result now follows from Lemma 2.11. \( \square \)

We can now define modified Gromov invariants.

**Definition 3.2** Given a homology class \( A \in H_2(M, \mathbb{Z}) \) such that \( \mathcal{H}'(A) \) contains no multiply-covered tori, we define the Gromov invariant \( \text{Gr}'(A) \) by:
\[ \text{Gr}'(A) = \sum_{\{ (\phi, \Sigma) \in \mathcal{H}'(A) \} } \varepsilon(\phi, \Sigma) \]

Here we assign the sign +1 to each multiply-covered exceptional sphere and then define the sign \( \varepsilon(\phi, \Sigma) \) as before. This number \( \text{Gr}'(A) \) is independent of the choice of generic \( \omega \)-tame \( J \).

**Lemma 3.3** \( \text{Gr}'(A) = \text{Gr}(A) \) unless there is an \( E \in \mathcal{E} \) such that \( E \cdot A < -1 \), in which case \( \text{Gr}'(A) = \text{Gr}(B) \) where
\[ B = A - \sum_{E : E \cdot A < -1} (E \cdot A) E. \]
Proof: By Proposition 3.1, each element \((\phi, \Sigma)\) in \(H'(A)\) consists of a good representative of the class \(B\) together with a collection of disjoint multiply-covered exceptional spheres, lying in the classes \(E\) such that \(E \cdot A < -1\). The result follows immediately. \(\square\)

Remark 3.4 As we shall explain in more detail in Lecture 4, somewhere injective spheres are always assigned the sign +1. Hence it is consistent also to assign +1 to all exceptional spheres, including the multiply-covered ones.

Example 3.5 (Example 1.6 revisited) Consider \(M = \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}\). Then it follows immediately from the above lemma that \(\text{Gr}^'(L + 2E) = 1\).

3.2 The components of \(\Sigma\).

The decomposition of \((\phi, \Sigma) \in H(A)\) into its \(\ell\) components \((\phi|_{\Sigma_i}, \Sigma_i)\), \(i = 1, \ldots, \ell\), gives rise to a corresponding decomposition \(A = \sum_{i=1}^{\ell} A_i\) where \(A_i\) is the class represented by \(\phi|_{\Sigma_i}\). We now look at what we can say about the \(A_i\). Are there any conditions under which \(\ell = 1\)? Are the \(A_i\) uniquely determined by \(A\)?

3.2.1 Components of negative self-intersection

The question of whether there are components with \(\phi(\Sigma_i)^2 < 0\) and of how they appear is completely answered by the structure theorems. If we are dealing with the original invariant \(\text{Gr}(A)\) and if \((\phi, \Sigma)\) is good then it follows from Theorem 1.2 that the only negative components are exceptional spheres. Moreover \((\phi, \Sigma) \in H(A)\) has a component which is an exceptional curve in class \(E\) if and only if \(E \cdot A = -1\), and all \(E\) which appear in this way are disjoint. Therefore, if

\[
B = A - \sum_{\{E \in E: E \cdot A = -1\}} E,
\]

there is a bijective correspondence between the elements of \(H(A)\) and of \(H(B)\). Further, no components of negative self-intersection appear in \(H(B)\). Therefore, we can replace the study of the structure of elements of \(H(A)\) by that of elements of \(H(B)\).

Similarly, if we are dealing with \(\text{Gr}'(A)\), \((\phi, \Sigma)\) has a component which is an \(m\)-fold cover of an exceptional curve in class \(E\) if and only if \(E \cdot A = -m\). Again, all \(E\) which appear in this way are disjoint and all the other components of \(\phi(\Sigma)\) have nonnegative self-intersection. Hence, as before, the structure of the components of negative self-intersection is determined by homological information.

Since the only difference between \(\text{Gr}(A)\) and \(\text{Gr}'(A)\) is in the negative components, from now on we consider only \(\text{Gr}(A)\).
3.2.2 Components of zero self-intersection

These are either tori or spheres, since when the genus \( g \) is \( > 0 \) the moduli space of embedded \( J \)-curves of genus \( g \) and zero self-intersection has negative dimension. Moreover, these can give rise to disconnected \((\phi, \Sigma)\). We saw this in Example 1.4 with spheres. In §5.1 we give a similar example (on \( T^2 \times S^2 \)) with tori. The next lemma shows that it is impossible for both spheres and tori to occur.

**Lemma 3.6** Suppose that \((\phi, \Sigma) \in \mathcal{H}(A)\) contains a component which is a sphere \( C \) of zero self-intersection. Then \( M \) is a blow-up of a ruled surface with \( C \) as one of the fibers. Moreover any other components in \((\phi, \Sigma)\) of nonnegative self-intersection are also fibers.

**Proof:** The first statement follows from the basic structure theorem in [11]. It is easy to see that \([C]\) has nonempty intersection with every other class \( B \) with \( B^2 \geq 0 \) that could have a \( J \)-holomorphic representative. (Use the Light Cone lemma (Lemma 3.7) stated below.) Hence if there are any more components in \((\phi, \Sigma)\) with nonnegative self-intersection they must also be fibers. \(\square\)

3.2.3 Components of positive self-intersection

We will consider the cases \( b_2^+ = 1 \) and \( b_2^+ > 1 \) separately, since they are rather different.

**The case** \( b_2^+ = 1 \)

The most relevant fact when considering the components of \( K \) is the light cone lemma. It is useful to consider the positive cone

\[
P = \{ B \in H_2(M, \mathbb{R}) : B^2 > 0 \}.
\]

Since \( b_2^+ = 1 \) this has two components which are separated by the hyperplane where \( \omega = 0 \). The component on which \( \omega \) is positive is called the forward positive cone and is denoted by \( P^+ \). Its closure is

\[
P^+ = \{ B \in H_2(M, \mathbb{R}) : B^2 \geq 0, \omega(B) \geq 0 \}.
\]

**Lemma 3.7 (Light Cone lemma)** Suppose that \((M, \omega)\) is a symplectic 4-manifold with \( b_2^+ = 1 \) and let \( B_1, B_2 \in \overline{P^+} \). Then \( B_1 \cdot B_2 > 0 \) unless \( B_1 = \lambda B_2 \) and \( B_1^2 = B_2^2 = 0 \).

**Proof:** There is a basis \( L, E_1, \ldots, E_\ell \) for \( H_2(M, \mathbb{R}) \) which is orthogonal with respect to the intersection pairing and is such that \( L^2 = 1, E_j^2 = -1 \) for all \( j \). Moreover, by changing the sign of \( L \) if necessary we may suppose that \( \omega(L) > 0 \),
i.e. that \( L \in \mathcal{P}^+ \). Then the elements of \( \mathcal{P} \) have the form \( mL + \sum \lambda_i E_i \) where \( \sum \lambda_i^2 < m^2 \). Since \( L \in \mathcal{P}^+ \), this element is in \( \mathcal{P}^+ \) exactly when \( m > 0 \). Hence we may write the \( B_i \) as:

\[
B_1 = mL + \sum \lambda_j E_j, \quad B_2 = nL + \sum \mu_j E_j,
\]

where

\[
m, n > 0, \quad m^2 \geq \sum \lambda_j^2, \quad n^2 \geq \sum \mu_j^2.
\]

Therefore

\[
B_1 \cdot B_2 = mn - \sum \lambda_j \mu_j 
\geq mn - (\sum \lambda_j^2)^{\frac{1}{2}}(\sum \mu_j^2)^{\frac{1}{2}}
\geq 0
\]

as claimed. Moreover, equality occurs only if all the \( \lambda_j \) are equal, all the \( \mu_j \) are equal and if \( B_1^2 = B_2^2 = 0 \). The conclusion readily follows.

The next proposition shows that if \( (\phi, \Sigma) \) has a component of positive self-intersection, then this is the only one other than exceptional curves.

**Proposition 3.8** Suppose that \( b_1^+ = 1 \) and consider \((\phi, \Sigma) \in \mathcal{H}(A)\). Let \( \Sigma_i, i = 1, \ldots, p \) be the components for which \( \phi(\Sigma_i)^2 \geq 0 \).

(i) If some \( \phi(\Sigma_i)^2 > 0 \) then \( p \) is at most 1 and this component contains all the \( k(A) \) generic points.

(ii) If some \( \phi(\Sigma_i)^2 = 0 \) then \( p \) can be \( > 1 \) but the classes \([\phi(\Sigma_i)], i = 1, \ldots, p\) differ by at most a constant factor. Moreover, either they are all represented by spheres (in which case \( k(A) = p \) and all the classes \([\phi(\Sigma_i)], i = 1, \ldots, p\), are equal) or they are all represented by tori (in which case \( k(A) = 0 \) and the classes \([\phi(\Sigma_i)], i = 1, \ldots, p\), all lie on the same ray in \( H_2 \)).

(iii) If \( \Sigma_i \) is a sphere for some \( i \leq p \), then \( M \) is a blow-up of a rational or ruled surface. Moreover \( Gr(A) = 1 \).

**Proof:** Parts (i) and (ii) follows immediately from Lemmas 3.6 and 3.7, and the fact that components with negative self-intersection are rigid so that they do not go through any generic points. Recall also that the genus of the representing curves is determined homologically through the adjunction formula. The first statement in (iii) follows immediately from the main theorem of [13] and holds without the assumption that \( b_1^+ = 1 \). The second may either be proved using Seiberg–Witten theory or by direct calculation. See Propositions 4.1 and 4.5 below.
Remark 3.9 In fact, we have not yet defined \( \text{Gr}(A) \) in the case when some components in \((\phi, \Sigma) \in \mathcal{H}(A)\) are multiply covered tori. The above proposition shows that in this case the components of \((\phi, \Sigma)\) are either exceptional spheres or are tori whose homology classes lie in some ray in \(H_2\). The exceptional spheres do not affect the value of \(\text{Gr}(A)\), and so we can suppose that there are none. Then, we define \(\text{Gr}(A)\) to be \(\text{Gr}_0(A)\) as given in Definition 5.2.

The Gromov invariants for symplectic manifolds with \(b^+_2 = 1\) can be completely calculated thanks to the wall-crossing formula in Seiberg–Witten theory: see Li-Liu \([6, 7]\). When \(H_1(M, \mathbb{R}) = 0\), \(\text{Gr}(A)\) is either 0 or 1, but if \(b_1(M) \neq 0\) the invariant can take different values. This leads to many interesting results. For example, Liu showed in \([8]\) that a minimal symplectic 4-manifold with \(K^2 < 0\) is ruled. However, there are still several open questions about their structure: see the survey article \([17]\).

The case \(b^+_2 > 1\)

When \(b^+_2 > 1\) the situation is more complicated. For simplicity, we will restrict attention to the minimal case.\(^3\) Using Seiberg–Witten theory, Taubes \([27]\) has proved the following important structure theorem for Gromov invariants. Recall that the canonical class \(K \in H_2(M)\) is the Poincaré dual of minus the first Chern class of \(M\), ie

\[
K = -PD(c_1(TM, J)).
\]

In particular, \(2k(K) = c_1(K) + K^2 = -K^2 + K^2 = 0\).

**Theorem 3.10 (Taubes)** Let \(M\) be a minimal symplectic manifold with \(b^+_2 > 1\). Then

(i) \(\text{Gr}(A) = 0\) except possibly if \(k(A) = 0\).

(ii) \(|\text{Gr}(K)| = 1\).

(iii) For all \(A \in H_2(M)\), \(\text{Gr}(A) = \pm \text{Gr}(K - A)\).

(iv) If \(K^2 = 0\) and \(\text{Gr}(A) \neq 0\) then \(A^2 = 0\).

(v) \([\text{Witten} 29]\) If \(M\) is Kähler and \(K^2 > 0\) then \(\text{Gr}(A) \neq 0\) only in the case \(A = 0, K\).

I know no way of proving the above results just in the context of holomorphic curves: at present one has to go via Seiberg–Witten theory. Note also that (ii) implies that \(K^2 \geq 0\) and \(\omega(K) > 0\), i.e. \(K\) is in the closure of the forward

\(^3\) In fact, if \(M\) is a symplectic 4-manifold with \(b^+_2 > 1\), \(M\) has a unique minimal reduction \(M'\), ie there is a unique maximal set of exceptional curves in \(M\) (see \([13]\)). Moreover, there is a sum formula which allows one to recover the Seiberg–Witten (or Gromov) invariants of \(M\) from those of \(M'\). Hence we do not lose any information by restricting to the minimal case.
positive cone \( P^+ \). This follows from Theorem 1.2 on the structure of elements of \( \mathcal{H}(A) \), which states that the only components of \( \phi(\Sigma) \) with negative self-intersection are exceptional spheres.

### 3.3 Examples with disconnected \( K \)

Before going further, we look at some examples in which \( K \) is realised by a disconnected curve. The easiest example is that of elliptic surfaces. In this case, \( K^2 = 0 \) and \( K \) is realised by a disjoint union of parallel tori: see Lecture 5. Here is another example in which \( K^2 > 0 \).

**Example 3.11** We construct a symplectic manifold with a disconnected representative of \( K \) by the process of the Gompf sum. Recall from [2] that if \((M_i, X_i)\) are two manifold/submanifold pairs such that the \( X_i \) are symplectically embedded surfaces of the same genus but opposite self-intersection number, one can form their connected sum

\[
M = M_1 \#_{X_1 = X_2} M_2,
\]

by cutting out suitable neighborhoods of the \( X_i \) and gluing their complements together. This is particularly easy when the \( X_i \) are tori of zero self-intersection: see §5.4 below. In this case we also have

\[
K_{M_i} \cdot X_i = 0, \quad i = 1, 2,
\]

so that \( K_{M_i} \) may be represented by a cycle which is disjoint from \( X_i \). It is then not hard to check that the canonical class \( K_M \) of \( M \) is given by the formula

\[
K_M = K_{M_1} + K_{M_2} + X_1 + X_2.
\]

(Note that this formula makes no sense when \( X_i^2 \neq 0 \) since none of the classes on the RHS can be identified in the homology of the glued manifold \( M \).)

As an example, consider \( T^4 \) with the symplectic form \( \omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 + dx_1 \wedge dx_3 \). Then \( T^4 \) contains disjoint nonparallel symplectically embedded tori \( X, Y \). (For example, take \( X = \{(x_1, x_2, 0, 0)\} \) and \( Y = \{(x_1, 0, x_3, 1/3)\} \).) As in Lecture 5, let \( V = V(1) \) denote the rational elliptic surface \( \mathbb{C}P^2 \) with 9 points blown up and fiber \( F \), and consider the triple sum

\[
M = V \#_{F=\infty} T^4 \#_{Y=F'} V',
\]

where \( V' \) is another copy of \( V(1) \). Then, because \( K_{T^4} = 0 \) and \( K_V = -F \), the above formula shows that

\[
K_M = F + F' = X + Y.
\]

To get an example with \( K^2 > 0 \), consider the manifold

\[
M = V(4) \#_{S(-4)} = Q \mathbb{C}P^2.
\]
Here $Q$ is the quadric in $\mathbb{CP}^2$ and $S(-4)$ is a sphere of self-intersection $-4$ in the elliptic surface $V(4)$. (The manifold $V(4)$ is described in more detail in Lecture 5. The sphere $S(-4)$ is a section of the map $V(4) \to \mathbb{CP}^1$, and $M$ is called a rational blowdown of $V(4)$: see, for example, [2].) The canonical class for $V(4)$ is $2F$ where $F$ is the fiber class (represented by a torus with zero self-intersection) and the canonical class for $\mathbb{CP}^2$ is, of course, $-3L$, where $L = [\mathbb{CP}^1]$. Consider the curve $C$ of genus $g_C = 2$ which is obtained by gluing a sphere in class $L$ to the fibers $F$ through the two points where $L$ meets $Q$. Thus $C$ is made from two copies of $T^2 - (\text{disc})$, each with trivial normal bundle, plus a copy of $S^2 - (2 \text{discs})$ which has self-intersection $+1$. Thus $C^2 = 1$. It is not hard to verify that $K_M = C$. For example the adjunction formula for $C$ works out:

$$-C^2 = -K_M \cdot C = 2 - 2g_C + C^2 = -1.$$ 

To get a manifold with disconnected $K$, observe that $V(4)$ contains many Lagrangian tori $Y$ which are disjoint from $F$. To see this, think of $V(4)$ as the fiber sum $V(2) \# F V(2)$ of two copies of the K3 surface $V(2)$, and realise $V(2)$ as the Kummer surface, which is obtained from $T^4$ by identifying $(x_1, x_2, x_3, x_4)$ with $(-x_1, -x_2, -x_3, -x_4)$ after having blown up the 16 fixed points of this involution. The torus $Y = \{(x_1, 0, x_3, 1/3)\}$ (which is Lagrangian for the usual symplectic form) descends to a torus in $V(2)$ which is disjoint from a generic fiber $F$ of the projection $V(2) \to \mathbb{CP}^1$ given by $(x_1, x_2, x_3, x_4) \mapsto (x_3, x_4)$. Hence $Y$ also embeds in $V(4) = V(2) \# F V(2)$. Observe also that because $Y$ when considered as a subset of $T^4$ does not go through the 16 fixed points of the involution, the image of $Y$ in $V(2)$ is disjoint from the sections $S(-2)$ of self-intersection $-2$ (which are the images of the blown-up points). Hence we may assume that $Y$ in $V(4)$ is disjoint from the section $S(-4)$. Since the homology class of $Y$ in $V(4)$ is nonzero, we may slightly perturb the symplectic form on $V(4)$ to make $Y$ a symplectic torus. (This is Gompf’s trick: see [3].) Then we can form the triple sum

$$M' = V(1) \#_{F = Y} V(4) \#_{S(-4) = Q} \mathbb{CP}^2,$$

which has canonical class $K' = Y + K_M$ with $(K')^2 = 1$.

Here is another example with a disconnected $K$ which contains no toroidal components. It was suggested to me by Ron Stern.

**Example 3.12** Take two surfaces $X, Y$ of general type which contain the Gompf nucleus of the K3 surface. (This nucleus is the union of a symplectic torus of square 0 with a symplectic sphere of square $-2$, and its regular neighborhood is just the trace of 0-framed surgery on the right-handed trefoil and $-2$ surgery on a meridional curve.) There are plenty of such surfaces in, for example, complete intersections. Now take the fiber sum

$$Z = X \#_{T_X = T_Y} Y$$

22
of these two surfaces along the tori $T_X$ and $T_Y$ of square 0. Then $K_Z = K_X + K_Y + T_X + T_Y$. The sum of the two $-2$-spheres in the Gompf nuclei is a sphere of square $-4$ that intersects each of $T_X$ and $T_Y$ once. Therefore, one can form the connected sum $W$ of $Z$ with $\mathbb{CP}^2$ by identifying the complement of this $-4$ sphere with the complement of the quadric surface. (This is the rational blow-down process of [2] and Fintushel–Stern [3].) Then it is not hard to check that

$$K_W = K_X + K_Y + C$$

where $C^2 = 1$: for more details see [3].

As we shall see in Lemma 3.17 below, this phenomenon of disconnected $K$ with $K^2 > 0$ cannot occur for minimal Kähler surfaces of general type. These manifolds satisfy the Noether inequality $c^2_1 \geq 2\xi - 6$. (Here $\xi$ denotes the holomorphic Euler characteristic $\frac{1}{2} - b_1^+ + b_2^+$. ) Study of the known examples of symplectic manifolds with $K^2 > 0$ has led Fintushel and Stern to suggest that all minimal symplectic manifolds with $K$ connected must satisfy the inequality $c^2_1 \geq \xi - 3$.

### 3.4 Structure of the Gromov invariants when $b_2^+ > 1$

In this section we show how the invariant $\text{Gr}(A)$ is built up from a simpler invariant which I will call $\text{Gr}_0(A)$. Roughly speaking, $\text{Gr}_0$ counts connected curves. We will suppose that we are working on a minimal manifold $M$ with $b_2^+ > 1$, so that the only classes with nonzero Gromov invariants are those with

$$k(A) = \frac{-K \cdot A + A^2}{2} = 0.$$ 

Of course, similar definitions can be made in the case $b_2^+ = 1$. However, the situation there is fully described in Proposition 3.8 and the remarks that follow it.

**Lemma 3.13** Consider $(\phi, \Sigma) \in \mathcal{H}(A)$ and let $A_i = \phi_*[\Sigma_i]$. If $k(A) = 0$ and $\text{Gr}(A) \neq 0$ then $k(A_i) = 0$ for all $i$. Further, the genus $g_i$ of $\Sigma_i$ is $1 + A_i^2$.

**Proof:** Observe that $A_i \cdot A_j = 0$ when $i \neq j$ since distinct components are disjoint. Hence $k(A) = \sum_i k(A_i)$. Since $k(A_i) \geq 0$ for all $i$ in order to have a nontrivial Gromov invariant, this shows that $k(A_i) = 0$. The last statement follows from the adjunction formula. We already know that all components of $\Sigma$ are embedded, and so

$$g_i = 1 + \frac{1}{2}(K \cdot A_i + A_i^2) = 1 + A_i^2 - k(A) = 1 + A_i^2,$$

as claimed.
**Definition 3.14** Each element \((\phi, \Sigma) \in \mathcal{H}(A)\) determines a decomposition \(D = \{B_1, \ldots, B_\ell\}\) of \(A\) in the following way. If \(\Sigma_i\) is a component of \(\Sigma\) of genus \(\neq 1\), then the corresponding homology class \(\phi_*[\Sigma_i]\) is in \(D\), but if \(\Sigma_i\) has genus 1 then we group together all the components with homology class on the ray \(\{\lambda \phi_*[\Sigma_i] : \lambda > 0\}\) into one element in \(D\). Note that by Proposition 3.8 there are no components of genus 0. Moreover, by the previous lemma, any component of genus 1 must have self-intersection 0.

Thus the elements of \(D\) are characterised by the following properties:

- \(\sum_i B_i = A\) and \(B_i \cdot B_j = 0, i \neq j\);
- If \(i \neq j\) then \(B_i \neq \lambda B_j\) for any \(\lambda > 0\);
- if \(B_i^2 > 0\), \(B_i\) is represented by a connected and embedded \(J\)-holomorphic submanifold;
- if \(B_i^2 = 0\), \(B_i\) is represented by a union of coverings of embedded \(J\)-holomorphic tori whose homology classes all lie on the ray \(\{\lambda B_i : \lambda > 0\}\).

For each such decomposition \(D\) of \(A\) we can add up (with signs) the \(J\)-holomorphic representatives of \(A\) with components in these classes, getting an invariant which we will call \(\text{Gr}_D(A)\). To be more precise, consider the following definitions.

**Definition 3.15** (i) If \(A^2 > 0\) then \(\text{Gr}_0(A)\) is defined to be the number of connected, embedded \(J\)-holomorphic curves of genus 1 + \(A^2\) in the class \(A\) counted with appropriate sign. (This sign is determined by the evaluation map as described in Lecture 1.) Thus in this case the invariant coincides with the one considered by Ruan in [20].

(ii) If \(A^2 = 0\) then \(\text{Gr}_0(A)\) counts the number of representatives of \(A\) by disjoint unions of possibly multiply-covered tori with homology classes on the ray \(\{\lambda A : \lambda > 0\}\). (This is the invariant which is called \(\text{Qu}(A)\) in [28].) In order to get a number which is invariant under symplectic deformation it is necessary to weight each component torus by a number which depends on certain twisted Cauchy-Riemann operators in the normal bundle of the torus. This weighting is described in more detail in Section 5.2. Its possible values are 0, \(\pm 2\), and \(\pm 2k + 1, k \geq 0\). One of Taubes’s interesting discoveries in [28] is that it is impossible to get a well-defined invariant if one restricts attention just to connected toral representatives in a fixed homology class.

A class \(B\) with \(\text{Gr}(B) = \text{Gr}_0(B)\) will be called **indecomposable**. Observe also that part (i) of Theorem 3.10 implies that if \(\text{Gr}_0(A) \neq 0\) then \(K \cdot A = A^2\).

We can now give a more precise definition of the Gromov invariant.

**Definition 3.16** Given a decomposition \(D = \{B_1, \ldots, B_\ell\}\) of \(A\) define

\[
\text{Gr}_D(A) = \prod_{i=1}^{\ell} \text{Gr}_0(B_i),
\]
and set
\[ \text{Gr}(A) = \sum_D \text{Gr}_D(A). \]

It is not known in general whether there is a unique decomposition \( D \) such that \( \text{Gr}_D(K) \neq 0 \). However, this does hold for minimal Kähler surfaces of general type, i.e., surfaces with \( b_2^+ > 1 \) and \( K^2 > 0 \).

**Lemma 3.17** If \( M \) is a minimal Kähler surface of general type then \( \text{Gr}_D(K) = 0 \) if \( D \neq \{K\} \). Hence \( \text{Gr}(K) = \text{Gr}_0(K) \).

**Proof:** One way to prove the first statement is to use Witten’s result in part (v) of Theorem 3.10. If \( \text{Gr}_D(K) \neq 0 \) for some decomposition \( D = \{B_1, \ldots, B_\ell\} \) then, by the very definition of \( \text{Gr}_D(K) \) we must have \( \text{Gr}_0(B_j) \neq 0 \) for all \( j \). Hence \( \ell = 1 \) by Witten’s result. Another way to see this is to use the Hodge index theorem. The classes \( B_j \) would have to lie in \( H^{1,1}(M) \) (because they can be represented by holomorphic curves), and also must have \( B_j^2 \geq 0 \) and \( K \cdot B_j \geq 0 \). The result now follows from the Light Cone Lemma 3.7 because the intersection pairing on \( H^{1,1}(M) \) has type \( 1 \oplus -1 \oplus \ldots \oplus -1 \).

As the next proposition shows we know much less about the general situation. Here the possible presence of \( J \)-holomorphic tori causes extra problems.

**Proposition 3.18** Let \( M \) be a minimal symplectic manifold with \( b_2^+ > 1 \).

(i) Suppose that \( \text{Gr}_D(K) \neq 0 \) for a unique decomposition \( D = \{B_1, \ldots, B_\ell\} \). If there is a class \( A \) such that \( A^2 > 0 \) and \( \text{Gr}_0(A) \neq 0 \), then \( A \) must equal some \( B_i \) and satisfy
\[ \text{Gr}_0(A) = \text{Gr}(A) = \pm 1. \]
In particular, if \( \text{Gr}_0(T) = 0 \) for all classes \( T \) with \( T^2 = 0 \), then \( \text{Gr}(A) = 0 \) unless \( A \) is a union of some of the \( B_i \) in which case \( \text{Gr}(A) = \pm 1 \).

(ii) If there are distinct classes \( A_1, A_2 \) such that
\[ A_1 \cdot A_2 > 0, \quad \text{Gr}(A_i) \neq 0, \quad i = 1, 2, \]
then \( A_i^2 > 0 \) for \( i = 1, 2 \) and \( \text{Gr}_D(K) \neq 0 \) for at least two different decompositions. The converse holds if \( K^2 > 0 \).

**Proof:** Suppose first that \( \text{Gr}_0(A) \neq 0 \). Because \( \text{Gr}(K - A) \neq 0 \) by Theorem 3.10 (iii), Definition 3.16 implies that there is a decomposition \( D' = \{B_1', \ldots, B_j'\} \) of \( K - A \) such that \( \text{Gr}_{D'}(K - A) \neq 0 \). Observe that because
\[ 2k(A) = -K \cdot A + A^2 = 0, \]
\[ A \cdot (K - A) = 0. \]
Therefore the union \( \{A\} \cup D' \) is a decomposition of \( K \), except possibly if \( A^2 = 0 \). In the latter case there may be a component of \( D' \) in the ray \( \lambda A \), and if there is
it must be amalgamated with \( A \). We will suppose that done, if necessary, and call the resulting decomposition \( \overline{D} \). It follows immediately that

\[
\text{Gr}_{\overline{D}}(K) \neq 0,
\]

unless we had to amalgamate some \( B'_i = \lambda A \) with \( A \) and it happens that \( \text{Gr}(A + B'_i) = 0 \). This argument shows that if \( \text{Gr}_0(A) \neq 0 \) and if \( A^2 > 0 \) then there is a decomposition \( \overline{D} \) of \( K \) with \( \text{Gr}_{\overline{D}}(K) \neq 0 \) which contains \( A \) as one of its elements.

Now suppose that we are in the situation of (i) and that \( A^2 > 0, \text{Gr}_0(A) \neq 0 \). Then \( \text{Gr}_{\overline{D}}(K) \neq 0 \) and so \( \overline{D} \) must equal \( D \), which implies that \( A \) must be one of the \( B_i \). To see that \( \text{Gr}(A) = \text{Gr}_0(A) \) we argue by contradiction. If this is not true we must have \( \text{Gr}_{D''}(A) \neq 0 \) for some non-trivial decomposition \( D'' \) of \( A \).

Because (by the compactness theorem) there are only finitely many classes \( B \) with bounded symplectic area \( \omega(B) \) which have \( J \)-holomorphic representatives, we may assume that \( D'' = \{B''_1, \ldots, B''_p\} \) consists of indecomposable elements. Further, since \( A^2 > 0 \), one of these elements, say \( B''_1 \), must have positive self-intersection number. Therefore, our previous argument shows that there is a decomposition \( D'' \) of \( K \) which contains \( B''_1 \) such that \( \text{Gr}_{\overline{D}}(K) \neq 0 \). Since \( A \neq B''_1 \) and \( A \cdot B''_1 = (B''_1)^2 > 0 \) by construction, \( \overline{D''} \) cannot equal \( D \); a contradiction. This proves the first statement in (i). The other statements in (i) are now obvious.

Now suppose that there are classes \( A_1, A_2 \) as in (ii). We first claim that \( A_i^2 > 0 \). To see this, observe that if \( A_1^2 = 0 \) for example, then \( K \cdot A_1 = 0 \) (since \( k(A_1) = 0 \)). But \( \text{Gr}(A_2) \neq 0 \) implies \( \text{Gr}(K - A_2) \neq 0 \), and so, by positivity of intersections we have

\[
K \cdot A_1 = A_2 \cdot A_1 + (K - A_2) \cdot A_1 \geq A_2 \cdot A_1 > 0,
\]

a contradiction. Hence \( A_i^2 > 0 \) for both \( i \) and so as above, one can create two different decompositions \( D \) of \( K \) (one containing each \( A_i \)) with \( \text{Gr}_D(K) \). The converse is obvious.

**Remark 3.19** If there were classes \( A_1, A_2 \) as in (ii) above, then, by Ruan–Tian’s composition law, there would be a corresponding nonzero Gromov–Witten invariant in class \( A = A_1 + A_2 \). Note that \( k(A) > 0 \). However, this does not contradict part (i) of Taubes’s Structure Theorem [3.11] since the complex structures on the \( A \)-curves that we count are not allowed to vary freely but are restricted to be in a certain cycle in the moduli space arising from the decomposition of \( A \) into \( A_1 + A_2 \); see [22].

### 4 Spherical Gromov invariants

We now develop a theory of “spherical” Gromov invariants for symplectic 4-manifolds which count the number of ways in which a class \( A \) can be represented
by a union of (possibly singular) \(J\)-holomorphic spheres. It is the natural generalization to disconnected curves of the genus 0 invariants which were considered in [13, 21] and which arise in quantum cohomology. However, these spherical invariants are much more limited in scope than the Gromov invariant considered by Taubes since they vanish on all minimal symplectic manifolds except for those which are rational and ruled. Since the modified invariant \(Gr'\) (developed in §3.1 above) is more appropriate here than the original invariant \(Gr\), we will generalize \(Gr'\).

Let \((M,\omega)\) be a symplectic 4-manifold. Given \(A \in H_2(M,\mathbb{Z})\) and \(J \in \mathcal{J}(M,\omega)\), the space \(\mathcal{M}(A,J)\) of all somewhere injective \(J\)-spheres is an oriented manifold of dimension \(2(c_1(A)+2)\). Hence, if \(k = c_1(A) - 1\), the evaluation map

\[
e_k : \mathcal{M}(A,J) \times G_0(S^2)^k \to M^k
\]

is a map between manifolds of equal dimension. It is shown in [14] that even though the domain of \(e_k\) may not be compact, the map \(e_k\) has the structure of a pseudocycle and hence represents a well-defined element of \(H_{4k}(M^k)\). (The point is that, by Gromov’s compactness theorem, the image of \(e_k\) can be compactified by adding pieces corresponding to \(A\)-cusp-curves. These pieces have to have codimension at least 2 and so do not contribute to the homological boundary of \(e_k\).) Moreover, the class \([e_k]\) represented by \(e_k\) is independent of the choice of \(J \in \mathcal{J}(M,\omega)\). Thus we get a well-defined number by taking the intersection of \([e_k]\) with a point \((x_1,\ldots,x_k) \in M^k\), or, informally, by counting the number of (unparametrized) \(J\)-spheres in class \(A\) which go through a fixed generic set of \(k\) points in \(M\). This number is the correct number to be called the \textbf{spherical Gromov invariant} \(Gr_s(A)\) provided that \(A\) can be represented by a somewhere injective \(J\)-holomorphic immersion of a 2-sphere.

Many useful results about \(J\)-spheres can be reformulated in terms of this invariant. For example:

**Proposition 4.1** If the class \(A\) can be represented by a symplectically embedded sphere of self-intersection number \(\geq -1\) then \(Gr_s(A) = 1\).

**Proof:** The hypothesis implies that \(A\) can be represented by an embedded \(J\)-sphere for some regular \(J\). By the adjunction formula, this implies that \(c_1(A) = 2 + A \cdot A\). Then \(k = c_1(A) - 1 > A \cdot A\) and so there can be at most one \(J\)-sphere through \(k\) distinct points. But there is at least one by hypothesis. Hence result. For more details see [13]. □

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4 If the class \(A\) represented by the \(J\)-sphere is such that \(c_1(A) = -K \cdot A > 1\), then this follows by [13], where it is shown that the only symplectic manifolds that contain such spheres are blow-ups of rational or ruled manifolds. Since blowing down spheres increases \(-K \cdot A\), the only case not covered is that when \(M\) is minimal and \(-K \cdot A = 1\). Here we appeal to Theorem 3.10 which shows that \(b_2 = 1\). The results of Liu [8] now show that \(M\) is rational or ruled.

5 Recall from [12] that any somewhere injective singular \(J\)-holomorphic map can be perturbed to a somewhere injective \(J'\)-holomorphic immersion for a nearby \(J'\).
The following result is proved in [13].

**Proposition 4.2** If the moduli space $M(A, J)$ is non-empty for some regular $J$ then, when $k = c_1(A) - 1 \geq 1$, there are a finite number of $J$ curves through a generic set of $k$ points in $M$ and each of these curves contributes $+1$ to $\text{Gr}_s(A)$. In particular, $\text{Gr}_s(A) \geq 1$.

It is not hard to check that the above hypothesis is satisfied whenever $M$ contains a symplectically immersed 2-sphere $C$ in class $A$ with $c_1(C) \geq 2$ whose only singularities are double points at which the two sheets intersect positively.

The above definition is fine as far as it goes. However, as before, some classes which should have a nontrivial Gromov invariant do not have connected representatives. For example, if $A = 2B$ where $B = [S^2 \times \text{pt}] \in H^2(S^2 \times S^2)$, then it follows from positivity of intersections that, because $B$ always has a $J$-holomorphic representative and $B \cdot B = 0$, the only $J$-curves in class $2B$ are 2-fold coverings of $B$-curves. Therefore, we must count curves which may be disconnected.

The only problem in extending our invariant to this case is to get the correct formula for the number of points $k$. Taubes allowed his curves to have arbitrary genus and so used the number $k(A) = \frac{1}{2}(c_1(A) + A \cdot A)$. However this is not appropriate for spheres, since the dimension of the space of (unparametrized) immersed spheres with $n$ double points is $2n$ less than the corresponding space of embedded curves of genus $n$. This is why we set $k = c_1(A) - 1$ above. (The adjunction formula for connected curves implies that $k(A) - k$ is exactly the genus of embedded $A$-curves.) Now the genus of a disjoint union of $p$ spheres is $-p + 1$ (because the Euler characteristic is $2p$), and so when $p > 1$ we must change $k$ appropriately.

With this understood, the spherical version of Taubes’s definition is as follows.

**Definition 4.3** Given a class $A \in H_2(M)$ consider all pairs $(k, p)$ where

$$0 \leq k \leq c_1(A) - 1, \quad p + k = c_1(A).$$

For each such $k$ fix a generic set $\Omega_k$ of $k$ points in $M$ (thus $\Omega_0 = \emptyset$) and consider the set $\mathcal{H}_s(A, J)$ of all equivalence classes $(\phi, \Sigma)$ such that

(a) $\Sigma = \cup_i \Sigma_i$ is a Riemann surface which is a disjoint union of spheres $\Sigma_i$;

(b) the map $\phi$ is $J$-holomorphic on $\Sigma$ and maps its components to disjoint curves $\phi(\Sigma_i)$ in $M$, unless $\phi(\Sigma_i)^2 < 0$ in which case the images of components are allowed to coincide. Moreover $[\phi_*(\Sigma)] = A$;

(c) elements $\phi, \Sigma$ and $\phi', \Sigma'$ are equivalent if $\Sigma = \Sigma'$ and $\phi'$ is a reparametrization of $\phi$;
(d) there is some pair \((k, p)\) as above such that \(\Sigma\) has \(p\) components and 
\(\Omega_k \subset \phi(\Sigma)\).

The **spherical Gromov invariant** \(\text{Gr}_s(A)\) is simply the number of elements in \(\mathcal{H}_s(A, J)\). (Because of Proposition 4.2 we count each pair \((\phi, \Sigma)\) with sign +1.)

**Remark 4.4**

(i) Observe that by condition (b) above exceptional spheres may occur with multiplicity, so that this is the spherical analog of the invariant \(\text{Gr}'\) rather than \(\text{Gr}\). This multiplicity is handled somewhat differently than before: we will see below that the restriction of \(\phi\) to each component of \(\Sigma\) has multiplicity one, however now components may coincide. To get the spherical analog of \(\text{Gr}\) it is enough to insist that the images of all components are disjoint. (ii) Taubes does not specify in his definition that the images of the different components of \(\Sigma\) should be disjoint because this is a consequence of his dimension formula. However we need to do this because we have allowed a choice of \((k, p)\). For example, when \(A = 2L\) in \(CP^2\) we wish only to count the unique conic through 5 generic points, and not the unique pair of lines through 4 generic points: see Example 4.7.

Part (iii) of the following proposition shows that the new definition of \(\text{Gr}_s(A)\) agrees with old one (ie the one obtained by counting connected curves) in all cases when that was nonzero. Recall that \(\mathcal{E}\) denotes the set of classes represented by exceptional spheres.

**Proposition 4.5**

(i) The number \(\text{Gr}_s(A)\) is finite and independent of the choice of generic \(J\) and \(\Omega\).

(ii) If \((\phi, \Sigma) \in \mathcal{H}_s(A, J)\), \(\phi\) is somewhere injective on every component of \(\Sigma\). Moreover, if \(\phi(\Sigma_i)^2 \leq 0\), \(\phi|_{\Sigma_i}\) is an embedding.

(iii) If the class \(A\) can be represented by a somewhere injective \(J\)-sphere for some generic \(J\), then \(\mathcal{H}_s(A, J)\) only contains elements with \(p = 1\), and so \(\text{Gr}_s(A)\) agrees with the previously defined invariant. This is the case if \(A^2 > 0\) and \(E \cdot A \geq 0\) for all \(E \in \mathcal{E}\).

(iv) \(\text{Gr}_s(A) = 1\) if all the components of \(\mathcal{H}_s(A, J)\) are embedded.

**Proof:** Suppose first that \(\Sigma\) has \(p\) components and that the restriction of \(\phi\) to \(\Sigma_i\) is an \(m_i\)-fold covering of a \(B_i\)-curve. Then \(A = \sum m_i B_i\), and so there is an associated element \((\phi', \Sigma)\) in \(\mathcal{H}_s(\sum_i B_i, J)\). Since \(\phi'\) is somewhere injective on \(\Sigma_i\) the curve \(\phi_i(\Sigma_i)\) can go through at most \(k_i = c_1(B_i) - 1\) generic points. Thus

\[
k = c_1(A) - p = \sum_i k_i = \sum_i c_1(B_i) - p,
\]
which is possible only if all $m_i = 1$. (Notice that $c_1(B_i) \geq 1$ because, by assumption, the moduli space $\mathcal{M}(B_i, J)/G_0$ of unparametrized $B_i$-spheres is non empty and so has dimension $2c_1(B_i) - 2 \geq 0$.) This proves the first statement in (ii). The second follows in the usual way from the adjunction formula and the fact that $c_1(B_i) \geq 1$.

Suppose now that that the class $A$ may be represented both by a somewhere injective $A$-sphere and by a union of nonmultiply-covered spheres in classes $B_1, \ldots , B_p$ where $p > 1$, which may coincide if they are exceptional spheres but otherwise are disjoint. We first claim that there cannot be any exceptional spheres among the $B_i$. For if there were some, in class $E$ say, we would have $E \cdot A < 0$ which contradicts positivity of intersections unless $A = E$. But this is impossible because $p > 1$. Therefore, we may assume that all the $B_i$ are disjoint. Therefore,

$$B_i \cdot B_j = 0 \text{ if } i \neq j, \quad A \cdot B_i = B_i \cdot B_i \geq 0 \text{ for all } i.$$  

Suppose further that $c_1(B_i) > 1$ for some $i$. Then, by Proposition 4.2, every generic point of $M$ lies on a $B_i$-curve. Therefore, it is possible to have $B_i \cdot B_j = 0$ only if $B_i = B_j$ and $B_i \cdot B_i = 0$. By the adjunction formula, $B_i \cdot B_i \geq c_1(B_i) - 2$ with equality only if $B_i$ is embedded. It follows that the $B_i$ must be parallel copies of an embedded curve of self-intersection 0. But then $A = pB_i$ has no $J$-holomorphic somewhere injective representative. Therefore, we must have $c_1(B_i) = 1$ for all $i$.

Since $B_i \cdot B_i - c_1(B_i)$ is even (the adjunction formula again), we must have $B_i \cdot B_i \geq 1$ so that $A \cdot A \geq p \geq 2$. By [13] this means that $M$ must be a blow up of $CP^2$ or $S^2 \times S^2$. The proof of (ii) will be finished by showing that this manifold does not contain two distinct curves in classes $B_1, B_2$ satisfying

$$B_1 \cdot B_1 \geq 1, \quad B_2 \cdot B_2 \geq 1, \quad B_1 \cdot B_2 = 0.$$  

But this follows from the Light Cone Lemma [3.7] which holds on all 4-manifolds with $b_+^2 = 1$.

This proves (iii), and (i) is clear. Finally (iv) is proved by arguing as in Proposition [14]. Further details are left to the reader. □

The following examples illustrate part (iv) of the above proposition.

**Example 4.6** (i) Let $X$ be a Riemann surface, and set $A = 2F$, where $F$ is the class of the fiber in $X \times S^2$. Then $A$ is represented by a disjoint union of 2 fibers and $Gr_s(A) = 1$.

(ii) Let $M$ be $CP^2$ blown up at 2 points, with $L = [CP^1]$ and $E_1, E_2$ the two exceptional classes. Then $A = L + E_1 + E_2$ is represented by the disjoint union of 3 spheres in classes $L, E_1$, and $E_2$ and again it is easy to check that $Gr_s(A) = 1$ Similarly, $A = L - E_1 + E_2$ is represented by the disjoint union of 2 spheres in classes $L - E_1$ and $E_2$ and $Gr_s(A) = 1$.
Here are some examples which illustrate the difference between $Gr_s(A)$ and the Gromov invariant $Gr(A)$ (or $Gr'(A)$).

**Example 4.7**  
(i) If $A = 3L \in H_2(\mathbb{C}P^2)$, then $Gr_s(A) = 12$ is the number of (immersed) $J$-holomorphic rational cubics through 8 generic points, while $Gr(A) = 1$ is the number of embedded $J$-holomorphic tori through 9 generic points. If one blows up a point in $\mathbb{C}P^2$ and considers $A = 3L + E$, then we may take $(k, p) = (8, 2)$ to obtain $Gr_s(A) = Gr_s(3L) = 12$. (Note that $A$ itself has no connected $J$-holomorphic representative because $E$ does and $A \cdot E < 0$.) In this case, $Gr(A) = Gr'(A) = 1$ is represented by the union of the unique torus through 9 generic points with the exceptional curve.

(ii) If $A$ is the class of $T^2 \times \text{pt}$ in $T^2 \times S^2$ then $Gr_s(A)$ is obviously 0 since $A$ has no spherical representatives, while $Gr(A) = Gr'(A) = 2$: see Lecture 5.

However it is easy to check that the two invariants do agree in the following situation.

**Proposition 4.8**  If the class $A$ can be represented by a disjoint union of embedded $J$-spheres then $Gr_s(A) = Gr'(A)$. This is the case whenever $Gr'(A)$ is calculated using spheres. In particular $Gr_s(A) = Gr'(A)(= Gr(A))$ when $A$ is the class of the fiber in a ruled symplectic manifold or when $A = [\mathbb{C}P^1] \in H_2(\mathbb{C}P^2)$.

**Proof:** Suppose that $A$ can be represented by a disjoint union of embedded $J$-spheres. If all these components are exceptional spheres then $A$ is a sum of elements from $\mathcal{E}$ and the result follows from Proposition 3.1. Otherwise, $M$ must be a blow-up of a rational or ruled manifold and the result follows from Propositions 3.8 and 4.5. \qed

**Remark 4.9** According to the perspective of Ruan–Tian in [22], instead of counting immersed $J$-spheres with $d$ double points going through a set of $k$ points, one can resolve the singularities and count instead the number of genus $d$ curves through $k$ points whose complex structures are constrained to lie in an appropriate cycle in the moduli space (of complex structures on genus $d$ curves). Thus the spherical invariant $Gr_s$ is really a special case of their Gromov–Witten invariants. There are many open questions about these invariants. For example, given a minimal symplectic manifold with $b_2^+ > 1$ what is the minimal genus of a curve for which some Gromov–Witten invariant is nonzero? It is not even known whether there are non trivial examples of these invariants on manifolds with $b_2^+ > 1$, i.e. ones which cannot be reduced to Taubes’s Gromov invariants: cf Remark 3.11.

## 5 Calculating Gromov invariants of tori

We begin with an example illustrating what can happen with tori, and then outline Taubes’s method for counting them. Finally we discuss the Gromov
invariant of the fiber class in an elliptic surface without multiple fibers, and show how to calculate it using a sum formula. A good general reference for facts about complex surfaces is Griffiths and Harris [8].

5.1 Tori in $S^2 \times T^2$

Let $M = S^2 \times T^2$, and $B = \text{pt} \times T^2$. Suppose that $B$ is represented by an embedded torus. Since

$$\dim \mathcal{M}(B, J, 1) = 2(c_1(B) + g - 1) + \dim G_1 = \dim G_1,$$

the dimension of the unparametrized moduli space is 0. In other words, regular tori in class $B$ are isolated. Thus the product complex structure on $T^2 \times S^2$ is not regular. To find a regular $J$. realise $S^2 \times T^2$ as the projectivization $\mathbb{P}(L \oplus \mathbb{C})$ of the rank-2 bundle $L \oplus \mathbb{C}$, where $L \to T^2$ is a nontrivial holomorphic line bundle with $c_1 = 0$.

We claim that with this complex structure $J_L$ the manifold $M$ has exactly 2 $J_L$-tori in class $B$ which both count with +1. Hence $\text{Gr}(M, B) = 2$.

To see this, observe first that if $L$ had a holomorphic section, this section cannot vanish anywhere, since every intersection with the zero section counts positively by positivity of intersections. Hence our $L$, which is nontrivial by assumption, does not admit nonzero holomorphic sections. Moreover, the only holomorphic sections of $\mathbb{P}(L \oplus \mathbb{C})$ are $[L \oplus 0]$, the section at “infinity” and $[0 \oplus \mathbb{C}]$, the “zero section”. (To see this, observe that such sections are in bijective correspondence with line subbundles $E$ of $L \oplus \mathbb{C}$. But if $E \neq L \oplus \{0\}$ or $\{0\} \oplus \mathbb{C}$, $E$ gives rise to a nontrivial homomorphism from $L^*$ to $\mathbb{C}$, which does not exist. For more details on this kind of argument see [13].) Further it is easy to check that the normal bundle of $[L \oplus 0]$ is isomorphic to $L^*$, while that of $[0 \oplus \mathbb{C}]$ is isomorphic to $L$. This is obvious for the section $[0 \oplus \mathbb{C}]$, and follows for $[L \oplus 0]$ since the latter can also be identified with $[\mathbb{C} \oplus 0]$ in

$$\mathbb{P}(L \oplus \mathbb{C}) = \mathbb{P}(L^* \otimes (L \oplus \mathbb{C})) = \mathbb{P}(L \oplus \mathbb{C}).$$

Since these normal bundles $\nu$ are nontrivial, $H^1_{\nu}(T^2, \nu) = 0$ for both sections, which implies that these curves are regular. Hence $J_L$ is regular for the class $B$. Moreover, since $J_L$ is integrable both tori count with a + sign. Hence $\text{Gr}(B) = 2$.

For generic $L$, I next claim that there are no embedded $J_L$-holomorphic tori in the class $2B$. For if there were, there would be a double cover map $\psi : T^2 \to T^2$ such that this torus pulls back to a torus $T'$ say in class $\text{pt} \times [T^2]$ in the manifold $(S^2 \times T^2, J_{L'})$ where $L' = \phi^*(L)$. Since $L' \neq \mathbb{C}$ for generic $L$ and $T'$ is neither the section at zero nor at infinity, this is impossible. However, there are three representatives $(\phi, \Sigma)$ of the class $2B$, namely double covers of each of the $B$-curves and a disconnected curve with 2 distinct components. It is not hard to check that these are regular (for generic $L$). Again, adopting
the principle that (regular) holomorphic objects always count with +1. we find
that \( \text{Gr}(2B) = 3 \). More generally, we have

**Lemma 5.1** If \( B = [pt \times T^2] \in H_2(S^2 \times T^2) \) then \( \text{Gr}(kB) = k + 1 \) for \( k > 0 \).

**Proof:** (Sketch) Arguing as above we see that for generic \( L \) the only connected curves in class \( pB \) are \( p \)-fold covers of the sections at zero and infinity. Hence there are exactly \( k + 1 \) ways of representing the class \( kB \), and each counts with a +1.

**Note:** The above result can be fully justified using Taubes’s work in [28]. It is also compatible with the calculation via Seiberg–Witten invariants: see Li–Liu [6, 7].

Now Taubes shows in [28] how to define an almost complex structure \( J_1 \) on \( S^2 \times T^2 \) which is also regular for the class \( B \) but which admits 4 \( J \)-holomorphic tori in class \( B \): the two above plus a cancelling pair, one which occurs with a + sign and one with a − sign. Moreover he shows that this \( J_1 \) admits no embedded tori in class \( 2B \). (Such an example was worked out independently by Lorek in [9].) Suppose that the correct way to count multiply-covered tori is simply to assign a \( ±1 \) to each multiple covering according to some rule and otherwise to follow the scheme laid out in Definition 1.3. Then of the 6 disconnected representatives of \( 2B \) three occur with a +1 and three with a −1 and so the net contribution is 0. But there are 4 doubly covered curves, and there is no way to assign the numbers \( ±1 \) to these four curves to make them give 3.

### 5.2 Taubes’s method for counting tori

Looking at examples like this, Taubes realised that to take proper account of the way in which a multiply covered torus contributes to the Gromov invariant one has to look at more than just the orientation of the underlying embedded curve. To understand why this is so, consider a generic path \( J_t, 0 \leq t \leq 1 \), of \( \omega \)-tame almost complex structures. Any regular embedded \( J_0 \)-holomorphic torus \( C_0 \) in class \( A \) is the endpoint of a path \( C_t \) of \( J_t \)-holomorphic tori in class \( A \). As \( t \) increases, two kinds of bifurcations occur. One is the birth or death of a pair of tori (one + and one − as above). The other is more complicated: a torus \( T_t \) in class \( 2A \) can split off from the basic path \( C_t \). (A beautiful explicit model for this bifurcation is described by Lorek in §4 of [10] which exhibits it as a quotient of a standard birth bifurcation.) The double covering map \( T_t \to C_t \) is classified by one of the 4 elements of \( \tau \in H_1(C_t, \mathbb{Z}/2\mathbb{Z}) \). Taubes’s basic insight is to realise that in order to determine the weight to assign to a \( k \)-fold cover of the torus \( C_0 \) one must keep track of the sign of the determinants \( \text{Det } D_\tau \) of the

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6 For nonmultiply-covered curves, this principle is justified by the fact that in the case all evaluation maps are holomorphic maps between complex manifolds and so all intersection numbers are positive.
linearized Cauchy-Riemann operators $D_\tau$ on the normal bundle $\nu_{C_0}$ twisted by all four of the elements $\tau$. (Taubes defines these signs in terms of a suitable spectral flow.) When $\tau = 0$ this sign corresponds exactly to the sign $\pm 1$ that we were using above to weight a curve, but when $\tau \neq 0$ this is new data. Thus each regular curve gets one of eight possible labels

$$(\pm, i), \quad i = 0, 1, 2, 3$$

where the label $(\pm, i)$ means that $\text{Det} D_\tau = \pm 1$ when $\tau = 0$ and that exactly $i$ of the other three signs are equal to $-1$. Observe that if $J$ is integrable near the curve $C$ its labels are always $(+, 0)$; in other words, all determinants are $+1$ in accordance with our previous positivity principle.

Taubes establishes:

**Birth rule:** when two curves are born the pair has labels $(\pm, i)$ for some $i$.

**Bifurcation rules:** these describe how the labels for $C_t$ change as $C_t$ passes through a bifurcation in which a torus $T_t$ corresponding to $\tau$ splits off, and they also give a formula for the label of the new torus $T_t$.

Using this information, he proves that the following method for weighting tori gives rise to a way of counting tori which is invariant under symplectic deformation. For each label $(\pm, i)$, let

$$f(\pm, i)(t) = \sum_{k \geq 0} n_k t^k$$

be the generating function for its contribution to the Gromov invariant. This means that if a torus in class $A$ has label $(\pm, i)$ then its contribution to the count of tori in class $kA$ is $n_k$. Then

$$f(\pm, i)(t) = \frac{1}{f(\mp, i)}.$$  

$$f(\pm, 0) = \frac{1}{1 - t}, \quad f(\pm, 1) = 1 + t,$$

$$f(\pm, 2) = \frac{1 + t}{1 + t^2}, \quad f(\pm, 3) = \frac{(1 + t)(1 - t^2)}{1 + t^2}.$$  

We now complete the definition of $\text{Gr}_0(A)$ that was begun in Definition 3.16.

**Definition 5.2 (Taubes)** Let $A \in H_2(M, \mathbb{Z})$ be a primitive element such that $A^2 = 0$ and $K \cdot A = 0$, and , for generic $J$, let $\mathcal{T}(A, m)$ denote the set of embedded, regular $J$-holomorphic tori in class $mA$. Then the Gromov invariant $\text{Gr}(M, kA) = \text{Gr}(kA)$ for $kA$ in $M$ is the coefficient of $t^k$ in the power series

$$\prod_{m \leq k} \left( \sum_{C \in \mathcal{T}(A, m)} f_{\ell_C}(t^m) \right),$$

where $\ell_C$ is the label for $C$. Moreover, we set $\text{Gr}_0(kA) = \text{Gr}(kA)$ in this case.
To illustrate this, consider the extra ± pair of \(J_1\)-holomorphic tori which cause trouble in the example above. In the construction, it turns out that the + torus has type \( (+, 0) \), so that the − torus has type \( (−, 0) \). Hence all \( k \)-fold coverings of the + torus count with +1, while \( k \)-fold coverings of the − torus with \( k > 1 \) contribute 0 (because the above rules give \( f(−0) = 1 − t \)). To see how this gives rise to an invariant which is independent of \( J \), consider a generic path \( J_t \) from an integrable structure \( J_0 \) on \( T^2 \times S^2 \) to \( J_1 \), and suppose, for simplicity, that there is a single bifurcation point on this path at which the ± pair \( C_± \) is created. Then, the bifurcation creates 6 new elements that contribute to \( \text{Gr}(2A) \). These are the disjoint union of one of the new tori with one of the two old ones, together with two elements involving just the new tori \( C_± \), namely the double cover of \( C_+ \) and the disjoint union of the pair of them. It is now easy to check that the net contribution to \( \text{Gr}(2A) \) of these 6 new elements is zero. Observe, in particular, that it is impossible to get a well defined, consistent invariant, which counts only connected curves in class \( 2A \). This is why we lumped all parallel tori together when defining the decomposition of \( A \) in Definition 3.14.

5.3 Elliptic surfaces

Consider the rational elliptic surface \( V(1) = \mathbb{C}P^2 \# 9 \mathbb{C}P^2 \). This may be understood as follows. Take nine points in general position in \( \mathbb{C}P^2 \). There exists a unique torus (a cubic curve) through those nine points in the class \( 3L = 3 [\mathbb{C}P^1] \). (One can check this simply by looking at the corresponding system of 9 equations in the 9 unknown coefficients of the cubic.) Blow up the nine points to obtain a torus \( T \) in class \( F = 3L − E_1 − ⋯ − E_9 \), with zero self-intersection \( F \cdot F = 0 \). Since all elliptic surfaces do embed in \( \mathbb{C}P^2 \) we may suppose that the induced complex structure \( j \) on this torus \( T \) is generic. Then, if we choose the 9 points on this torus \( T \) so that they are also generic, it is not hard to check that \( T \) is regular, i.e. its holomorphic normal bundle satisfies \( H^1(T, ν) = 0 \). This shows that the Gromov invariant \( \text{Gr}(F) = 1 \). In fact, it follows from Seiberg–Witten theory that \( \text{Gr}(B) = 1 \) for every class \( B = nL − \sum_i m_i E_i \) such that \( n > 0 \) and \( B^2 \geq 0 \), as well as for the classes \( B = E_i \).

Now choose nine points which lie on two cubic curves \( C_i = \{ f_i = 0 \} \), for \( i = 1, 2 \). Then in fact the points lie on the one parameter family of cubics

\[
C_\lambda = \{ f_1 + \lambda f_2 = 0 \}.
\]

Moreover, each point of \( \mathbb{C}P^2 \) except for the 9 points of intersection lies on exactly one of these cubics (provided that we allow \( \lambda = \infty \) to include the cubic \( C_2 \)). Thus there is a well-defined map of the complement of the 9 points to the parameter space \( \mathbb{C}P^1 \). It is not hard to check that it extends to a smooth holomorphic map \( V(1) \to \mathbb{C}P^1 \). If \( C_1 \) and \( C_2 \) are generic, this is a singular
fibration in the sense that the fiber over all but finitely many of the points $\lambda$ is the nonsingular cubic $C_\lambda$.

A manifold which fibers like this over $\mathbb{C}P^1$ is called an elliptic surface. $V(1)$ is the simplest elliptic surface. We can construct others by Gompf’s construction of the fiber connected sum. For example $V(2)$ is obtained from two copies of $V(1)$ by removing a neighborhood of a fiber in each copy and then gluing the boundaries by a suitable orientation reversing symplectomorphism. To be precise, recall that by the symplectic neighborhood theorem a small open neighborhood $N$ of a fiber $T^2$ in $V(1)$ is symplectomorphic to a product $T^2 \times D^2$. By making $N$ smaller if necessary, we may assume that this product structure extends to a neighborhood $W$ of the boundary of $V(1) - N$. Thus $W$ is symplectomorphic to $T^2 \times A$ where $A \subset \mathbb{R}^2$ is an annulus. The surface $V(2)$ is then defined by

$$V(2) = (V(1) - N) \cup_{W=W'} (V(1) - N'),$$

where the identification of $W$ with $W'$ is via a symplectomorphism of the form

$$\text{id} \times \phi : T^2 \times A \to T^2 \times A'.$$

Here $\phi : A \to A'$ is area preserving and turns the annulus $A$ inside out so that it maps the boundary of $V(1) - N$ into the interior of $V(1) - N'$. When repeated this construction yields a family of elliptic surfaces $V(n) = V(n-1)\#_T V(1)$.

The construction given above takes place in the symplectic rather than holomorphic category. However it is not hard to construct $V(n)$ (eg as a branched cover) in a way that makes clear that it does have a complex structure. Interestingly enough, when $n > 1$ all complex structures on $V(n)$ give it the structure of an elliptic surface. (This is not true when $n = 1$.) When $n = 2$ we get a $K3$ surface. This surface is analogous to the 4-torus $T^4$ in that generic complex structures on it admit no holomorphic curves at all. Therefore all its Gromov invariants vanish. Moreover, as in the case $n = 1$, a generic complex structure on $V(2)$ is regular in the Fredholm sense, i.e the moduli spaces of holomorphic curves are manifolds of the right dimension and so can be used to calculate Gromov invariants.

However, when $n > 2$ no complex structure on $V(n)$ is regular in the sense of Fredholm theory. For, regular $J$-holomorphic tori of zero self-intersection are isolated. However, because the moduli space of holomorphic fibers in $V(n)$ is a manifold (albeit of too high a dimension) one can still try to use it to calculate the Gromov invariant. In fact, Ruan shows in [15] that the contribution of a compact component of the moduli space to the Gromov invariant of a class with formal dimension (or index) 0 is precisely its oriented Euler characteristic. One cannot simply apply that result here though, since the moduli space of holomorphic tori in the class of the fiber is not compact. (Its ends are the singular fibers.) Nevertheless, as Ruan pointed out to me, there is a natural holomorphic compactification of this moduli space which has Euler characteristic $2-n$. Thus, this heuristic calculation suggests that $\text{Gr} (V(n), F)$ should be $2-n$. 

36
Note that this is a negative number. However, this does not contradict the positivity principle that we were using before because the holomorphic objects here are not regular in the Fredholm sense. In fact, one can think that this is a reason why no complex structure on $V(n)$ can be regular when $n > 2$: the only way that a negative number can be holomorphically represented is through a nonregular family with too large dimension and negative global twisting.

5.4 The Gromov invariants of a fiber sum

One can show that $\text{Gr}(V(n), F) = n - 2$ from Seiberg–Witten theory, using the fact that $V(n)$ is Kähler. Here we show how to calculate it directly from the construction of $V(n)$ as a fiber sum. The full details, together with a general statement valid for any connected sum along tori, have been worked out by Lorek [10]. I wish to thank him for useful discussions and in particular for pointing out the role of the so-called boundary classes.

The basic idea is to consider Gromov invariants for compact manifolds $(X, \omega)$ which have boundary components diffeomorphic to $S^1 \times T^2$. We will always suppose that the symplectic form restricts on the boundary to the pullback of the usual area form on the torus $T^2$. Hence, by the symplectic neighborhood theorem, we may identify a neighborhood of each boundary component with

$$(N \times T^2, \omega) = ([−1, 0] \times S^1 \times T^2, du \wedge d\theta + ds \wedge dt),$$

where the boundary is at $u = 0$. We will take $A = pt \times T^2$. In order for the Gromov invariant to be well-defined we must make sure that $J$-holomorphic tori cannot escape through the boundary of $X$. Therefore we will only consider almost complex structures $J$ on $X$ which have the following standard form on a neighborhood of each boundary component of $X$.

Given a function $\beta : [−1, 0] \to \mathbb{R}$ which satisfies the following conditions:

(i) $0 < |\beta(u)| < 1$ for $u \in [−1, 0)$ and $\beta(0) = 0$;

(ii) $\beta$ has isolated critical points, and all its derivatives vanish at $u = 0$;

we define $J_\beta$ by setting:

$$J_\beta(\partial_u) = \partial_\theta, \quad J_\beta(\partial_\theta) = -\partial_u,$$

$$J_\beta(\partial_s) = \partial_t + \beta(u)\partial_\theta, \quad J_\beta(\partial_t) = -\partial_s + \beta(u)\partial_u.$$

It is easy to check that condition (i) implies that $J_\beta$ is $\omega$-tame. Observe also that for those $c$ with $\beta(c)$ rational, the 3-torus $\{u = c\}$ is foliated by $J_\beta$-holomorphic tori of “slope” $\beta(c)$. More precisely, this foliation is the kernel of the 1-forms

$$du = 0, \quad d\theta + \beta(c)dt = 0,$$

**Definition 5.3** An almost complex structure $J$ on $(X, \omega)$ is said to be **compatible with the boundary** of $X$ if each boundary component of $X$ has a
neighbourhood $N$ such that the triple $(N, \omega, J)$ is symplectomorphic to $([-1,0] \times S^1 \times T^2, du \wedge d\theta + ds \wedge dt, J_{\beta})$ for some $J_{\beta}$ which satisfies the above conditions. Elements $B \in H_2(X, \mathbb{Z})$ which are in the image of some inclusion map $H_2(N) \to H_2(X)$ will be called boundary classes of $X$.

The next lemma shows that the Gromov invariant $\text{Gr}(X, A)$ is well-defined when $A \in H_2(X)$ is any class that has zero intersection with all boundary classes $B$.

**Lemma 5.4** Suppose that $X$ has boundary components $S^1 \times T^2$ as above, and suppose that $J_X$ is a generic $\omega$-tame almost complex structure on $X$ which is compatible with the boundary of $X$. Then, for any class $A$ such that $A \cdot B = 0$ for all boundary classes $B$, the number of $J$-holomorphic tori in class $A$ is independent of the choice of $J$.

**Proof:** We suppose for simplicity that $X$ has a single boundary component. The proof in the general case is similar.

Let $F = \text{pt} \times T^2$. We first show that $\text{Gr}(X, [F])$ is well-defined. Observe first that the boundary 3-torus $\{u = 0\}$ is foliated by $J_X$-holomorphic tori in class $[F]$, and so there is a corresponding circle in the moduli space $\mathcal{M}(J_X, [F])/G$. By Ruan’s result in [19] the contribution of this circle to the Gromov invariant is its Euler characteristic, namely 0. Because $|\beta(u)| < 1$ none of the other compact leaves in the 3-tori $u = c$ can represent $[F]$. The next important fact is that no other $J$-holomorphic torus can intersect the boundary region $N \times T^2$. For if it did, it would have to cross one of the 3-tori $\{u = c\}$ with $\beta(c)$ rational, and hence it would have to intersect one of the compact leaves of the $J$-holomorphic foliation of this 3-torus. Since these leaves lie in a boundary class $B$, this contradicts the fact that $F \cdot B = 0$. Therefore any other $J_X$-holomorphic torus in class $[F]$ has to be contained in the complement of $N \times T^2$. This means that the boundary region $N \times T^2$ functions somewhat like a pseudo-convex boundary, containing the $J$-holomorphic curves in class $[F]$. In particular, the moduli space of $[F]$-curves in $X$ is compact. Moreover, it follows from the usual transversality arguments that we can therefore calculate $\text{Gr}(X, [F])$ by counting the elements in the moduli space of $J$-holomorphic $[F]$-tori for a generic element $J$ of the set

$$\mathcal{J}_N = \{\omega\text{-tame } J \text{ on } X : J = J_X \text{ on } N \times T^2\}.$$  

(For example, if you look at the proof of Proposition 3.4.1 in [15], you see that in order to prove that the universal moduli space $\mathcal{M}(A, \mathcal{J}_N)$ is a manifold it suffices to be able to make $J \in \mathcal{J}_N$ generic somewhere on the image of every curve in $\mathcal{M}(A, \mathcal{J}_N)$, but not necessarily everywhere on $X$.)

This proves the result when $A$ is the fiber class $[F]$. The argument for other $A$ is similar.  

**Proposition 5.5** (i) $\text{Gr}(D^2 \times T^2, [F]) = 1$ where $F = \text{pt} \times T^2$.  

38
(ii) If $N(F)$ is a neighborhood of the fiber $F$ in the elliptic surface $V(n)$ then
\[ Gr(V(n) - \text{Int } N(F), [F]) = 1 - n, \quad Gr(V(n), [F]) = 2 - n. \]

**Proof:** In [10] Lorek calculates $Gr(D^2 \times T^2, [F])$ by explicit construction of a suitable $J$ for which one can count the tori. We will give a nonexplicit proof which uses the fact that we know that $Gr(S^2 \times T^2, [F]) = 2$.

First, let consider the situation when $F$ is a symplectic torus with $F^2 = 0$ in a closed manifold $Y$. Then $F$ has a neighborhood $N(F)$ which is symplectomorphic to the product $D^2 \times F$. Consider the decomposition of $Y$ into
\[ Y = Y_0 \cup N(F), \quad \text{where } Y_0 = Y - \text{Int } N(F), \]
and let $J$ be a generic almost complex structure on $Y$ formed by putting together almost complex structures on $Y_0$ and on $N(F)$ which are compatible with their boundaries. (Observe that $J$ is smooth because of condition (ii) on $\beta$.) Then, the moduli space of $J$-holomorphic tori in class $A = [F]$ splits into three parts: the circle of tori along the boundary, the set $\mathcal{M}_Y$ of tori in $Y_0$ and the set $\mathcal{M}_N$ of tori in $N(F)$. Clearly,
\[ Gr(Y, [F]) = \# \mathcal{M}_Y + \# \mathcal{M}_N, \]
where one counts the number $\#$ with appropriate signs.

One complicating factor that we have to consider here is that the inclusion $Y_0 \rightarrow Y$ need not induce an injection on $H_2$. For example, if we take $Y = V(1)$, the torus in the boundary $S^1 \times F$ which is the kernel of the 1-form $d\theta + dt = 0$ is homologous to $F$ in $Y$ but not in $Y_0$. Hence the set of tori in $\mathcal{M}_Y$ need not all lie in class $A$. Thus, although $\# \mathcal{M}_N = Gr(N(F), [F])$, the number $\# \mathcal{M}_Y$ is, in general, a sum of Gromov invariants. However, if $Y = S^2 \times T^2$, then $Y_0 = D^2 \times T^2 = N(F)$ and this problem does not arise. Thus we find
\[ 2 = Gr(S^2 \times T^2, [F]) = 2Gr(D^2 \times T^2, [F]). \]
This proves (i).

We prove (ii) by induction. Above we showed that there was a regular complex structure on $V(1)$ which had exactly one holomorphic torus in class $[F]$. Hence $Gr(V(1), [F]) = 1$. Decompose $Y = V(1)$ into $Y_0 \cup N(F)$ as above, and consider the set of tori $\mathcal{M}_Y$. It would theoretically be possible that these tori would give rise to nonzero Gromov invariants $Gr(Y_0, A)$ where $A$ is some boundary class, since $\mathcal{M}_Y$ might contain a plus torus in one class $B_+$ and a minus torus in another class $B_-$ which would cancel out in $Y$ but not in $Y_0$. However, if we double $Y_0$ we get the $K3$ surface $V(2)$ and we know that all invariants vanish for that. Since the inclusion of $Y_0$ into its double induces an injection on homology, we know that the invariants for the double are exactly twice those for $Y_0$. Hence the invariants for $Y_0$ must vanish. This shows that
\[ Gr(V(1) - \text{Int } N(F), B) = 0, \]

39
for all boundary classes $B$ including $B = [F]$.

Thus (ii) holds for $n = 1$. Here is the rough idea for the inductive step. Think of $V(n + 1)$ as the fiber sum $V(n) \# V(1)$. Thus we get $V(n + 1)$ by cutting out a neighborhood of a fiber in each of $V(n)$ and $V(1)$ and gluing the remaining pieces together. We saw above that if we make $J$ compatible with the boundaries of the pieces then the neighborhoods of the cut-out fibers each contain exactly one + torus. As we saw above, this uses up the + torus in $V(1)$ leaving it with no tori. Now $V(n)$ may not have any + tori, and so to do the cutting we must create a ± pair (without introducing any new tori) and then cut out the + one. This creates an extra − torus in $V(n + 1)$, which gives the result.

To make this precise we just have to see that it is possible to make a new ± pair of tori in the fiber class $[F]$ without creating any new tori in boundary classes. Inductively, we can suppose that $J_n$ is an almost complex structure on $V(n)$ which has the form $J_0$ on some set symplectomorphic to $N = [0, 1] \times S^1 \times T^2$, and which is generic outside $N$. (This means that $J_n$ can be used to calculate the Gromov invariant of $V(n)$.) It follows from Lemma 5.4 that $\text{Gr}(N, B) = 0$ for all boundary classes $B$. Let $J'_n$ be an almost complex structure which equals $J_n$ outside a compact set in $\text{Int} N$ and is such that it is compatible with the boundary of some subset $P$ of $\text{Int} N$ symplectomorphic to $D^2 \times T^2$. Then, we claim that

$$\text{Gr}(N - \text{Int} P, [F]) = -1, \quad \text{Gr}(N - \text{Int} P, B) = 0,$$

where $B$ is any boundary class other than $[F]$. The first statement holds because $\text{Gr}(P, [F]) = 1$, and the second holds because, as before, there would otherwise be nontrivial invariants for the $K3$ surface. Hence, when forming $V(n + 1)$ we can cut out $P$ and glue in $V(1) - N(F)$, a process which leaves us with one more − torus than there was in $V(n)$. □

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