Reducible Connections in Massless Topological QCD and 4-manifolds

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ABSTRACT

A role of reducible connections in Non-Abelian Seiberg-Witten invariants is analyzed with massless Topological QCD where monopole is extended to non-Abelian groups version. By giving small external fields, we found that vacuum expectation value can be separated into a part from Donaldson theory, a part from Abelian Monopole theory and a part from non-Abelian monopole theory. As a by-product, we find identities of U(1) topological invariants. In our proof, the duality relation and Higgs mechanism are not necessary.
1 Introduction

Recently in differential topology of four manifolds there have been new developments by Seiberg-Witten monopole theory [1]. They conquered a difficulty of Donaldson theory. Donaldson theory solved many problems of differential topology of 4-manifolds about intersection form, polynomial invariants and so on [3][4]. Donaldson theory is described by non-Abelian gauge theory, hence calculations are difficult. Seiberg-Witten theory is easy for its computation since it is Abelian gauge theory. Donaldson invariants which is written by Kronheimer-Mrowka structure formula are related with Seiberg-Witten invariants, and the relation has been proved in several ways [5][6]. Hyun-J.Park-J.S.Park show the relation in path-integral formalism using massive Topological QCD [7]. Both Donaldson and Seiberg-Witten theory are understood as Topological field theory [8][9]. Hyun-J.Park-J.S.Park computed path-integrals of massive Topological QCD and they found the way of separating it into two branches that is Donaldson part and Seiberg-Witten part. Symbolically the result of their computation is

$$\langle \text{massive Topological } QCD \rangle = \frac{a}{m^k} \left\{ \langle \text{Donaldson} \rangle - \frac{b}{m^l} \langle \text{Seiberg-Witten} \rangle \right\} \quad (1)$$

Where $a$ and $b$ are some suitable constants and $m$ is mass of hyper-multiplets. $k$ and $l$ are determined by indices of some elliptic operator. $\langle \text{massive Topological } QCD \rangle$ is a vacuum expectation value of an observable with the action of massless Topological QCD. $\langle \text{Donaldson} \rangle$ stands for vacuum expectation value of an observable with the action of Donaldson-Witten theory [8], and is called Donaldson invariants. $\langle \text{Seiberg-Witten} \rangle$ stands for vacuum expectation value of an observable with the action of Abelian Seiberg-Witten topological field theory [9], and is called Seiberg-Witten invariants. The left hand side of the above equation is regular in the massless limit, $m \to 0$, so the Donaldson part and Seiberg-Witten part in the right hand side have to cancel each other. In this theory, mass terms lead a relation between vacuum expectation value of Higgs and matter fields. All computations in this paper are done in a weak coupling limit (or large scaling limit). If the weak-strong duality relation is necessary for understanding the relation of Donaldson invariants and Seiberg-Witten invariants, it is natural to think that massive particles decouple in the weak coupling limit as Witten mentioned in [1][2]. But, in the proof of Hyun-J.Park-J.S.Park mass terms do not decouple, and the duality relation is not used. Mathematicians did not use the duality relation similarly in their proofs [3][4]. This fact implies that mass terms of matter fields do not play essential roles in the relation between Seiberg-Witten and Donaldson invariants. The only important thing is to separate the path-integral into the Donaldson’s irreducible part and Seiberg-Witten’s reducible part in their theory.

In this paper, we investigate reducibility of the gauge connections in massless Topological QCD. There are two purposes. The first one is that we cull the Abelian
Seiberg-Witten part from massless Topological QCD without Higgs mechanism and weak-strong duality relation. The Abelian Seiberg-Witten part appear in massless Topological QCD as reducible connection part. The second purpose is to obtain the new relations of massless Topological QCD and their topological invariants. Especially, we will get a result that insists some topological invariants which contain the Abelian Seiberg-Witten invariants. This topological invariants is provided as reducible connection part of this Topological QCD. As a result of these relations, we find that the path-integral from non-Abelian extended monopoles is separated into Donaldson parts and non-Abelian Seiberg-Witten parts. We use regularization for zero mode of a scalar field, which do not remove zero-modes but shift them to infinitesimal eigenstates without BRS-like SUSY breaking with giving perturbation by external fields. As a result of this perturbation, we obtain new relations between Seiberg-Witten invariants extended to non-Abelian gauge and Donaldson Seiberg-Witten invariants. And some identities of U(1) topological invariants from the relation are given from the relation of SUSY symmetry of the Topological QCD and external fields.

This paper is organized as follows. We set up massless topological QCD whose action do not have Higgs potential, in section 2. Separation of vacuum expectation value of observables into reducible connection part and irreducible connection part is done in section 3. We get identities of Abelian Seiberg-Witten invariants and obtain the relation of massless Topological QCD and reducible connection part in section 4. We will find new formulas in there. In the last section, we summarize and discuss our conclusions.

2 Massless Topological QCD

In this section we set up massless Topological QCD modified slightly to separate a correlation function into a reducible connection part and a irreducible connection part and study the relations between non-Abelian and Abelian Seiberg-Witten theory with no Higgs mechanism. Hence Higgs potential like $[\bar{\phi}, \phi]^2$ do not appear here. We will find later in this section how the Donaldson invariants are embedded in massless Topological QCD.

Topological QCD were already constructed by Hyun-J.Park-J.S.Park J.M.F. Labastida and Mariño by twisting N=2 SUSY QCD. Donaldson theory and Seiberg-Witten theory are analyzed as topological field theory in references. Basically we use the Hyun-J.Park-J.S.Park theory and notation in . In the following, we only consider SU(2) gauge group and 4-dimensional compact Rieman manifolds with $b_2^+ > 2$ as a back ground manifold.

The action is

\[ S_{QCD} = -\delta V \]  \hspace{1cm} (2)
where

\[
V = \int d^4x g^2 \left[ \chi^{\mu\nu}_a (H^a_{\mu\nu} - i(F^{a+}_{\mu\nu} + q^i \sigma_{\mu\nu} T^a q)) \right]
\]

\[
- \frac{1}{2} g^{\mu\nu} (D^a_{\mu} \bar{\phi})_a \lambda^a_{\nu} + (X^a_{\bar{q}} \psi_q \alpha - \psi^2_{\bar{q}} X_{qa})
\]

and under SUSY(BRS like) transformations, Yang-Mills fields are transformed as

\[
\begin{align*}
\delta A_\mu &= i \lambda_\mu, \quad \delta X_{\mu\nu} = H_{\mu\nu}, \quad \delta \phi = i \eta, \\
\delta \lambda_\mu &= - D_\mu \phi, \quad \delta H_{\mu\nu} = i [\phi, X_{\mu\nu}], \quad \delta \eta = [\phi, \bar{\phi}],
\end{align*}
\]

and the matter fields are transformed as,

\[
\begin{align*}
\hat{\delta} q^\alpha &= - \bar{\psi}_{\bar{q}}^\alpha, \quad \hat{\delta} \bar{\psi}_{\bar{q}}^\alpha = -i \phi^a T_a q^\alpha, \\
\hat{\delta} q^\dagger_a &= - \bar{\psi}_{\bar{q}} a, \quad \hat{\delta} \bar{\psi}_{\bar{q}} a = i q^1 a \phi^a T_a, \\
\hat{\delta} \psi_{qa} &= - i \sigma^\dagger_{\alpha a} D_\mu \phi^a + X_{qa}, \\
\hat{\delta} X_{qa} &= i \phi^a T_a \psi_{qa} - i \sigma^\dagger_{\alpha a} D_\mu \bar{\psi}_{\bar{q}}^\alpha + \sigma^\mu_{\alpha a} \lambda^a_\mu T_q a \lambda^\dagger a T_a, \\
\hat{\delta} \bar{\psi}_{\bar{q}}^\dagger a &= i D_\mu q^\dagger a \bar{\sigma}^{\mu\dagger a} - X^a_{\bar{q}}, \\
\hat{\delta} X^a_{\bar{q}} &= i \psi^a_{\bar{q}} \phi^a T_a - i D_\mu \bar{\psi}_{\bar{q}} \bar{\sigma}^{\mu\dagger a} + q^\dagger a \bar{\sigma}^{\mu\dagger a} \lambda^a_\mu T_a.
\end{align*}
\]

These transformation laws are obtained by the usual way of twisting N=2 SUSY QCD. The matter fields \( q \) are sections of \( W_c^+ \otimes E \) where \( W_c^+ \) is spin\(^c \) bundle and \( E \) is a vector bundle whose fiber is a vector space of a representation of the gauge group \( SU(2) \). The topological action is given after integrating out the auxiliary fields \( H_{\mu\nu}, X_q \) and \( X_{\bar{q}} \) as

\[
S_{QCD} = \int d^4x g^2 \left( \frac{1}{4} |F^{a+}_{\mu\nu} + q^i \sigma_{\mu\nu} T^a q|^2 + \frac{1}{2} |\sigma^\mu D_\mu q|^2 - \frac{1}{2} g^{\mu\nu} (D_\mu \bar{\phi})_a (D_\nu \phi)^a \
- i g^{\mu\nu} (\lambda_\mu, \bar{\phi})_a \lambda^a_{\nu} - i \chi^{\mu\nu}_{a} (\phi, X_{\mu\nu})_a + \chi^{\mu\nu}_{a} (d\lambda^a_{\mu\nu} + i a (d\lambda) + a)_{\mu\nu} + \frac{i}{2} g^{\mu\nu} (D_\mu \eta) a \lambda^a_{\nu} \
- i \chi^{\mu\nu}_{a} \bar{\psi}_{\bar{q}} \sigma_{\mu\nu} T^a q + i \chi^{\mu\nu}_{a} q^\dagger \bar{\sigma}_{\mu\nu} T^a \bar{\psi}_{\bar{q}} - i D_\mu \bar{\psi}_{\bar{q}} \bar{\sigma}^{\mu\dagger a} \psi_q - i \psi^\dagger a \sigma^\mu_{\alpha a} D_\mu \bar{\psi}_{\bar{q}} + \bar{\psi}_{\bar{q}}^\dagger \phi_a T^a \psi_{qa} + q^\dagger a \lambda_{\mu a} T^a \bar{\sigma}^{\mu\dagger a} \psi_q + \psi^\dagger a \sigma_{\alpha a} D_\mu \bar{\psi}_{\bar{q}} \right).
\]

Where the indices \( \alpha \) and \( \dot{\alpha} \) are omitted and we do not change the position of these indices to keep the sign of each terms in the following. This action is constructed in order to lead the most important fixed points,

\[
F^{a+}_{\mu\nu} + q^\dagger \sigma_{\mu\nu} T^a q = 0, \quad \sigma^\mu D_\mu q = 0.
\]
The Eqs.(7) are monopole equations extended to non-abelian gauge group and often they are called non-abelian Seiberg-Witten monopoles [9][10].

Usually, Seiberg-Witten theory has a Higgs potential $[\phi, \bar{\phi}]^2$. Spontaneous symmetry breakdown occur if the vacuum expectation value of Higgs fields $\langle \phi \rangle$ is non-zero. Then we get $U(1)$ monopole(Seiberg-Witten) equations. But one of our purposes is to clarify whether relations of topological invariants can be understood without Higgs mechanism and weak-strong duality relations. So the Higgs potential is not included in our theory. (But if we add these potential, they do not disturb following discussion.)

Later in this section, we find how the Donaldson invariants are embedded in this theory. We study three kind of fixed points, which give important contribution to vacuum expectation value. The path-integral is expressed as a sum of three parts. We call a part of the path-integral from fixed points determined by $F^+ = 0$ and $q = 0$ as Donaldson part. We denote a part whose gauge connections $A$ of fixed points determined by Eqs.(7) are reducible as Abelian Seiberg-Witten part and a part which has irreducible connections at the fixed point as Non-Abelian Seiberg-Witten part. The observable is

$$\exp\left(\frac{1}{4\pi^2} \int \frac{1}{2} \lambda \wedge \lambda - \frac{1}{8\pi^2} Tr \phi^2 \right),$$

where $\gamma$ is in a 2-dimensional homology class i.e. $\gamma \in H_2(M; \mathbb{Z})$. Now let us separates vacuum expectation value with Donaldson invariants from non-Abelian Seiberg-Witten invariants. If fixed points $\langle q \rangle$ from Eqs.(7) is zero then the contribution to the expectation value of (8) is from only Donaldson theory because now our fixed point equation is simply as

$$F^+_{\mu\nu} = 0.$$  

We know that we can estimate exactly the vacuum expectation values of this observables by one-loop approximation around the fixed point determined by the Eqs.(7) [11][12]. We can decompose the action of Eq.(6) into two parts [7],as

$$S_{QCD} = S_D + S_M,$$

where $S_D$ is action of Donaldson-Witten theory [8],

$$S_D = \int d^4x g^{\frac{1}{2}} Tr \left[ \frac{1}{4} F^\mu_{\nu} F^{+\mu\nu} - \frac{1}{2} g^{\mu\nu} D_\nu \bar{\phi} D_\mu \phi - i \chi^{\mu\nu} [\phi, \chi_{\mu\nu}] 
+ \chi^{\mu\nu} (d_A \lambda)^+ + \frac{i}{2} g^{\mu\nu} (D_\mu \eta) \lambda_\nu - \frac{i}{2} g^{\mu\nu} [\lambda, \bar{\phi}] \lambda_\nu \right]$$  

and $S_M$ is matter part,
We find that the quadratic part of the matter part action $S_M$ is given by

$$
S_M^{(2)} = \int d^4x g^{\frac{1}{2}} \left[ -3X_q^\alpha X_{q\alpha} + iX_q^\alpha \sigma_{\alpha\hat{\alpha}} D_\mu q^{\hat{\alpha}} + iD_\mu q^{\dagger} \tilde{\sigma}^{\mu\hat{\alpha}\alpha} X_{q\alpha} 
- iD_\mu \tilde{\psi}_q \tilde{\sigma}^{\mu\hat{\alpha}\alpha} \psi_{q\alpha} - i\psi_q^\alpha \tilde{\sigma}^{\mu\hat{\alpha}} D_\mu \tilde{\psi}_q^{\hat{\alpha}} \right].
$$

(12)

Note that the gauge field $A_\mu$ used in $D_\mu$ is an external field. We expand the gauge field around the solution of Eqs.(7) denoted as $A_c$, as

$$
A = A_c + A_q.
$$

(13)

where $A_q$ is a quantum fluctuation around the $A_c$. So the covariant derivative in the $S_M^{(2)}$ is written by external fields $A_c$ as $d_A = d + A_c$. Note that $A_c$ is a irreducible connection because we set $b_{2}^+ \geq 1$. After the Gaussian integrals of $X, q, \psi$ and $\bar{\psi}$, from Eq.(12) we get

$$
det (-\pi)_{W^-} \det (-4\pi)_{W^+}.
$$

(14)

The indices $W^-$ and $W^+$ of determinants (14) show that the determinants are defined in the subspaces of $W^-, W^+$, and we use the similar notations also in the following. Therefore we can understand that the Donaldson invariants appear with the determinants (14) in massless Topological QCD and this fact is used in section 4.

In the next step, we want to evaluate the other part which correspond to the Seiberg-Witten theory and separate the path-integral into a reducible connection part and an irreducible part. But, as we will see it soon, it is impossible to separate the Seiberg-Witten part into the non-Abelian Seiberg-Witten brunch and the Abelian Seiberg-Witten brunch in the same manner as above. In this case fixed points value $\langle q \rangle$ is non zero. We consider the SU(2) gauge group. Our massless theory has no spontaneous symmetry break down, so we have to treat separately reducible gauge connections and irreducible connection by some way. Judgments whether the connections are irreducible or not can be done by examination of the existence of $D_\mu$ zero-mode. (See appendix A.) Strictly speaking, the following two propositions are same.

- $d_A : Ad \xi \otimes \Lambda_0 \rightarrow Ad \xi \otimes \Lambda_1$ is injection.
- $A$ is a irreducible connection.

Where we represent a vector bundle with a structure group SU(2) as $\xi$. So we use this condition to divide the contribution to vacuum expectations into one from Abelian Seiberg-Witten theory and another from non-abelian Seiberg-Witten theory. At first, we pay attention to $\bar{\phi}$ equation,

$$
- \frac{1}{2} D^\mu D_\mu \phi - \frac{i}{2} [\lambda_\mu, \lambda^\mu] = 0.
$$

(15)
\( \lambda \) in this equation is determined by \( \eta \) and \( \chi \) equations like appendix B or references [11]. When there is no zero mode solution of \( \lambda \) then this equation is a distinction formula of reducibility i.e. \(-\frac{1}{2}D^\mu D_\mu \phi = 0\). So we can conclude that the connection is reducible and the contribution to vacuum expectation value is from the U(1) abelian Seiberg-Witten theory when \( \langle \phi \rangle \neq 0 \). But unfortunately we can not distinguish connection by this method when there are \( \lambda \) zero mode solutions. In the next section we find new approach to separate the contribution of abelian Seiberg-Witten branch from non-Abelian Seiberg-Witten part.

### 3 Separation of reducible connection part

In this section we construct a new theory in which vacuum expectation value are separated into three parts i.e. Donaldson, Abelian Seiberg-Witten and non-Abelian Seiberg-Witten part. We use the determinant obtained by integration of \( \phi \) and \( \bar{\phi} \) to find whether connections are reducible or not. Therefore we consider the case when the determinant vanishes. Usually we avoid this case and remove zero eigenvalue states with various ways. For example in [20], zero-modes give a symmetry to whole action and they are removed by BRS method. In our case this method can not be used because zero-mode dose not give the existence of a local symmetry. But in this section we take zero-modes into account, and we find that this zero-modes play a essential role to distinguish between reducible connections and irreducible connections.

In the same way as section 2, we pay attention to Eq.(15). From Eq.(15), we get the vacuum expectation value of the scalar fields \( \phi \) in 1-loop order as

\[
\langle \phi \rangle = -\frac{i}{D^\mu D_\mu} [\lambda_\nu, \chi^\nu] 
\]

Now we have to recall that if \( b_2^+ > 0 \), anti-selfdual connections which satisfy the equation (9) do not contain the reducible connections with U(1) isotropy group [3][13], but the connections which satisfy the monopole equations (7) contain such reducible connections in general. When the connection is reducible then \( D_\mu \) has zero-mode on 0-form, (see the appendix A). Then the Green function \( \frac{1}{D^\mu D_\mu} \) has singularities. Normally we avoid this kind of singularity to define a meaningful theory. However, in the present case, this singularity pays the important role. Reducibility of the gauge connection is judged by the zero-mode. The theory should keep holding characteristic properties of zero-modes while it is regularized. Such problem doesn’t exist in the Donaldson theory. Or we can say that we set the condition \( b_2^+ > 2 \) to avoid the complication of reducible connections in Donaldson theory. But in the Seiberg-Witten theory it is the most important merit that the gauge group is U(1) Abelian group. So, if we take away such reducible connections, then this Topological QCD has almost no value. Therefore we have to manage the singularities in the Green function. Usually we dispose of singularities of this kind by removing zero-modes, inducing mass terms and so on. The following way makes it possible.
Before considering regularization we ascertain that these singularities make topological symmetry break. To see it concretely, we introduce a BRS exact observable, \(\hat{\delta}(\bar{\phi}\lambda_\mu)\). If there are no singularities, topological symmetry is not broken and vacuum expectation value of this observable vanishes. But as we saw before, the propagator \(\langle \bar{\phi}\phi \rangle \sim \frac{1}{D\mu D\mu}\) is singular at least in tree level. To avoid these singularities we add regularization term \(i\epsilon\bar{\phi}\phi\) to our Lagrangian and the propagator is changed to \(\frac{1}{D\mu D\mu - i\epsilon}\), naively. Because the \(\bar{\phi}\phi\) is not invariant under the BRS-like SUSY transformation (4)(5), the vacuum expectation value is

\[
\langle \hat{\delta}(\bar{\phi}\lambda_\mu) \rangle = \int D\bar{X} \hat{\delta}(\bar{\phi}\lambda_\mu) e^{-S} = \epsilon \int D\bar{X} \hat{\delta}(\bar{\phi}\lambda_\mu) e^{-S} = \epsilon \langle \bar{\phi}\lambda_\mu \eta \phi \rangle = \epsilon \langle \langle \bar{\phi}\phi \rangle \langle \lambda_\mu \eta \rangle + \cdots \rangle. \tag{17}
\]

(See references [14].) The last equality is from the Wick’s theorem. It make clear that the singularities of the propagator \(\langle \bar{\phi}\phi \rangle \simeq \frac{1}{\epsilon}\) cause the right hand side of Eq.(7) non-zero in a limit as \(\epsilon \to 0\). This fact means the vacuum expectation value of the BRS exact observable is non-zero. This is topological symmetry breaking. It is equivalent to a phenomenon observed in [14] [15] [16]. This is seen after integration by \(\bar{\phi}\). We get a delta function,

\[
\delta \left( -\frac{1}{2} D\mu D\mu \phi - \frac{i}{2} [\lambda_\mu, \lambda^\mu] + i\epsilon \phi \right) = \sum_{\text{all } \hat{\phi}_\epsilon} \frac{\delta \left( \phi - \hat{\phi}_\epsilon \right)}{|\det (D\mu D\mu - i\epsilon)|}, \tag{18}
\]

where \(\hat{\phi}_\epsilon\) is the solutions of \(-\frac{1}{2} D\mu D\mu \phi - \frac{i}{2} [\lambda_\mu, \lambda^\mu] + i\epsilon \phi = 0\) and \(\lambda\) is a zero-mode which is determined by \(\eta\) and \(\chi\) equations. (see the Appendix B). We know it from seeing the delta function of (18) that the determinant \(|\det (D\mu D\mu - i\epsilon)|\) is zero in a limit as \(\epsilon\) approaches zero when the connection is reducible, and these singularities break the topological symmetry. From this consideration, it seems that the regularization term is not suitable. For our purpose we do not hope topological symmetry breaking, so we have to choose the regularization term to be invariant under BRS-like SUSY transformations.

Here we define the determinant by adding infinitesimal sift terms to our Lagrangian. Our problem is that we could not separate vacuum expectation value into a reducible connection part and a irreducible connection part with the way in the previous section. In this section, the method used in [14] adapt to the separation. We pay attention to the determinant \(\det (D\mu D\mu)\) which is obtained by integration of \(\phi\) and \(\bar{\phi}\). We saw above that vanishing of this determinant cause topological symmetry breaking or at least cause some singularity. Usually we used to modify
determinants by removing zero eigenvalues. But these zero-modes are necessary to judge the reducibility of the gauge connection. So we do not remove but shift them by infinitesimal perturbation. This infinitesimal perturbation is given by adding some shift terms to our Lagrangian. We denote this shift term as $\delta f_\epsilon$ where $f_\epsilon$ is a some functional and $f_\epsilon = 0$ as the limit of $\epsilon \to 0$. Similarly we also shift the determinant which is obtained by $\eta, \chi$ and $\lambda$ integrations. (See the appendix B.) This shift term is represented to $\eta \zeta_\epsilon$ in the following where $\zeta_\epsilon$ is a some fermionic functional and $\zeta_\epsilon$ vanishes in the limit, $\epsilon \to 0$. We mention a little more about this shift. The $\eta$ integral of $g^{\mu\nu}(D_\mu \eta) \lambda_\nu$ in action (6) vanishes, if the covariant derivative $D_\mu$ acting to the $\eta$ has zero-modes. (Note that $\lambda_\mu$ zero-modes define the dimension of the moduli space, but $\eta$ zero-modes do not have such a roles in usual case.) Usually we take away this zero-modes, but now instead of doing so we add infinitesimal shift terms $\zeta_\epsilon$ to the Lagrangian, in order to shift the fermionic zero-eigenstates into infinitesimal eigenvalue states. Total Lagrangian is now written as

$$S = S_{QCD} + \int d^4x g^{\frac{1}{2}}(\bar{\phi} f_\epsilon + \eta \zeta_\epsilon)$$

Here let us enumerate conditions for the infinitesimal terms.

(a) Shift terms are invariant under SUSY transformations (4) and (5).

This condition is necessary to avoid topological symmetry breaking as we saw it above.

(b) It is desirable that non-zero solutions, $\phi$ and $\lambda$, of the following $\bar{\phi}$ and $\eta$ equations do not vanish by adding shift terms:

$$-\frac{1}{2} D^\mu D_\mu \phi - \frac{i}{2} [\lambda_\mu, \lambda^\mu] - f_\epsilon = 0$$

$$\frac{i}{2} D_\mu \lambda^\mu - \zeta_\epsilon = 0.$$  

From this condition, observable contain non-trivial one. The $\lambda$ in Eq.(20) is determined by Eq.(21) and fermionic field equations, (refer [11] and see appendix B). When $f_\epsilon$ contains the field $\phi$ like

$$f_\epsilon = \epsilon g \phi + \epsilon h$$

where $g$ and $h$ are some functionals which do not contain scalar field $\phi$, $\langle \phi \rangle$ is represented as

$$\langle \phi \rangle = -\frac{1}{D^\mu D_\mu + \epsilon g} (i [\lambda_\nu, \lambda^\nu] + 2 \epsilon h).$$

Hence $\langle \phi \rangle$ becomes order $\epsilon$ i.e.$\langle \phi \rangle = -\frac{1}{D^\mu D_\mu + \epsilon g} (+2 \epsilon h) = O(\epsilon)$ if there is only $\lambda = 0$ solution of (21) and $D_\mu$ has no zero-mode. Then the vacuum expectation of observable
(8) is mere $1 + O(\epsilon)$ and this is not suitable. Note that it is not desirable that $f$ is independent from $\phi$. This reason is made clear by next condition (c). So we can not put $g$ zero in Eq.(22). The righthand side of Eq.(22) has at most linear in the field $\phi$, however $f_{\epsilon}$ may contain higher power terms on $\phi$. For simplicity, we treat only case Eq.(22). The 3rd condition for infinitesimal shift terms is

(c) following two operators have no zero-modes regardless of whether gauge connections are reducible or not.

$$\frac{1}{2} D^\mu D_\mu + \frac{\delta f}{\delta \phi} \equiv \frac{1}{2} D^2_\epsilon. \quad (24)$$

$$\frac{i}{2} D_\mu - \epsilon' \frac{\delta \zeta}{\delta \lambda^\mu} \equiv i D_{\mu\epsilon}. \quad (25)$$

Where the operators (24) and (25) operate on $Ad(E)$ valued 0-form. This condition is just to shift determinants from zero to infinitesimal finiteness when gauge connections are reducible and $D_\mu$ has zero-modes. From these conditions, we can take count of $\eta$ and $\phi$ zero-modes when calculate $det D^2_\mu$ obtained by $\phi$ and $\bar{\phi}$ integrations and $det T_\epsilon$ (see appendix.B) whose $T_\epsilon$ has $D_{\mu\epsilon}$ in first row and is obtained by fermionic fields integrations.

Example of the additional terms $\bar{\phi} f_\epsilon$ and $\eta_a \zeta^a_\epsilon$ which satisfy the conditions (a)(b)(c) are

$$\bar{\phi} f_\epsilon + \eta_a \zeta^a_\epsilon = \left. \delta \left( \epsilon' \bar{\phi}_a \lambda^a_\mu m^\mu + \epsilon n (q^\dagger \bar{\phi} \bar{\psi} q - \bar{\psi} q \bar{\phi} q) \right) \right|_{\epsilon' \Delta \phi}$$

$$= \epsilon' \bar{\phi}_a (D_\mu \phi)^a m^\mu + \epsilon \bar{\phi}_a \left( n q^\dagger (\phi T^a + T^a \phi) q - 2 i n \bar{\psi} q T^a \bar{\psi} q \right)$$

$$+ \epsilon' \eta_a \left( \lambda^a_\mu m^\mu \right) - \epsilon \eta_a \left( n \bar{\psi} q T^a q + n q^\dagger T^a \bar{\psi} q \right), \quad (26)$$

where $m_\mu$ and $n$ are some back ground fields which are chosen to satisfy the conditions (b) and (c) and are gauge singlet. These external fields do not break SU(2) gauge symmetry and topological symmetry because they are gauge singlet fields. Infinitesimal constant number $\epsilon$ and $\epsilon'$ are independent each other. We can add shift terms (26) to our action since (26) are BRS-like SUSY invariant and have 0 U-number. Note that sometime the determinant $det D_\mu D^\mu$ obtained by $\phi$ and $\bar{\phi}$ integrations is ignored in other papers because this determinant can be countervailed by a Fadeev-Popov determinant of the SU(2) gauge fixing. But in our theory, this determinant plays a essential role for separating reducible connections.

Next we consider the zero limit of $\epsilon$ and $\epsilon'$. We take this limit after functional integrals. When we calculate around fixed points of irreducible connections which have no zero modes of $D_\mu$, additional terms like (26) play no role and there is no modification in Donaldson and non-Abelian Seiberg-Witten theory in the limit as $\epsilon$ approaches zero. But if a connection of a fixed point is reducible some different
points appear. At first, we can estimate the order of the delta function obtained by \( \phi \) integrations as

\[
\delta \left( -\frac{1}{2} D^\mu D_\mu \phi - \frac{i}{2} [\lambda, \lambda^\mu] + f_\epsilon \right) = \sum_{\phi_c} \frac{\delta \left( \phi - \phi_c \right)}{\left| \text{det} (D^\mu D_\mu - \frac{\delta f}{\delta \phi}) \right|}
\]

where \( l \) is a dimension of 0-cohomology \( H^0_A \) i.e. \( l = \text{dim ker}(D_\mu) \). We denoted \( \phi \) that do not make delta function vanish as \( \hat{\phi}_c \). Especially it is important to notice that this determinant can be chosen to depend on not \( \epsilon' \) but \( \epsilon \) like the case (26). Since the \( \epsilon' \hat{\phi}_c (D_\mu \phi)^a m^\mu \) term in shift terms (26) vanishes under \( D_\mu \) zero-modes integral. For infinitesimal \( \epsilon \) and \( \epsilon' \), \( \epsilon' \) terms in the determinant in Eqs.(27) are negligible. Note that the shift terms which break the balance of bosonic and fermionic shift terms in zero-mode integral like Eq.(26) have less variations. In general, \( \epsilon \) or \( \epsilon' \) appear in the both coefficients of \( \phi \) term and \( \eta \) term since \( \hat{\phi} \) is a super partner of \( \eta \).

The equations to obtain vacuum expectation value of \( \phi \) change also as

\[
\lim_{\epsilon \to 0} \langle \phi \rangle = -\lim_{\epsilon \to 0} \frac{i}{D^\mu D_\mu - \epsilon g} ([\lambda^\nu, \lambda^\nu] + \epsilon h) = \left\{ \begin{array}{ll}
\phi_c & \text{(for irreducible connections } A_\mu) \\
\phi_c + \epsilon \phi' & \text{(for reducible connections } A_\mu) 
\end{array} \right.
\]

\[
\epsilon \phi' \equiv \lim_{\epsilon \to 0} \frac{i \epsilon}{D^\mu D_\mu - \epsilon g} (g \phi_c + h). \tag{28}
\]

Where \( \phi_c \) is a solution of the equation (15) of each connections \( A_\mu \). Note that \( \epsilon \phi' \) may be finite in the 0-limit of \( \epsilon \) since \( D_\mu \) has zero-modes.

In a similar manner, we estimate a delta function which obtained by fermionic field \( \eta \) integration as

\[
\prod_{\eta_0} \zeta_\eta \prod_{\eta'} \delta \left( \frac{i}{2} D_\mu \lambda^\mu - \zeta_\eta \right) \sim \left\{ \begin{array}{ll}
\Pi_{\eta} \delta \left( \frac{i}{2} D_\mu \lambda^\mu - \zeta_\eta \right) & \text{(for irreducible connections } A_\mu) \\
O(\epsilon, \epsilon')^l \prod_{\eta'} \delta \left( \frac{i}{2} D_\mu \lambda^\mu - \zeta_\eta \right) & \text{(for reducible connections } A_\mu) 
\end{array} \right. \tag{29}
\]

Where \( \eta_0 \) are \( D_\mu \) zero-modes and \( \eta' \) are other non-zero modes and \( O(\epsilon, \epsilon')^l \) is order \( \epsilon^n \epsilon^m \) where \( n + m = l \). Or we can say it as follows. The first row of \( T_\epsilon \) that is \( D_\mu \) goes back to \( D_\mu \) in the zero limit of \( \epsilon \) and \( \epsilon' \) when connections of fixed points are irreducible. On the other hand when connections of fixed points are reducible, \( \eta \) and \( \lambda \) integral can be written by separation \( D_\mu \) zero-modes \( \eta_0 \)

\[
\int \mathcal{D} \eta' \mathcal{D} \lambda \exp \left( -\int d^4 x g^\frac{2}{n} ((D_\mu \eta') \lambda^\mu - \zeta_\epsilon \eta') \right) \int \mathcal{D} \eta_0 \exp \left( -\int d^4 x g^\frac{2}{n} \eta_0 \zeta_\epsilon \right). \tag{30}
\]
So we can estimate \( \det |T| \) as

\[
\lim_{\epsilon, \epsilon' \to 0} \det |T_\epsilon| = \begin{cases} 
\det |T| \quad \text{(for irreducible connections } A_\mu) \\
\lim_{\epsilon, \epsilon' \to 0} (O(\epsilon, \epsilon'))^t \quad \text{(for reducible connections } A_\mu) 
\end{cases}
\]

(31)

Where \( O(\epsilon, \epsilon')^t \) is a determinant of a matrix in which the \( \eta_0 \) row \( \lambda_\mu \) column of \( T \) is replaced with \( \delta \zeta^{a}_\epsilon \delta \lambda_m^{\mu} \). In the example (26), the \( \eta_0 \) row \( \lambda_\mu \) column element in \( T_\epsilon \) is

\[
\frac{\delta \zeta^a}{\delta \lambda^n_{\mu}} = \epsilon' \delta^{ab} m^\mu. 
\]

(32)

From Eqs.(27) and (29), we conclude an order of vacuum expectation values is given by,

\[
\begin{cases} 
1 \quad \text{(for irreducible connections } A_\mu) \\
\left( \frac{O(\epsilon, \epsilon')}{\epsilon} \right)^t \quad \text{(for reducible connections } A_\mu) 
\end{cases}
\]

(33)

We can conclude from (33) that if we set \( \epsilon \) and \( \epsilon' \) as same order, then our path-integral are sums over reducible and irreducible connection parts with an equal weight.

Let us consider changing the ratio \( \epsilon'/\epsilon \) and changing the contributions from reducible connection part in our path-integral. The ratio of \( \epsilon \) and \( \epsilon' \) can be changed without changing of vacuum expectation value. This fact is seen as follows. We put \( \epsilon' = k \epsilon \) and \( k \) is some positive real number. When an action is given as

\[
S = \int d^4x g^\frac{1}{2} \hat{\delta}V + \epsilon \hat{\delta}F + \epsilon' \hat{\delta}G 
\]

(34)

where \( V, F \) and \( G \) are any functionals, then the change in a vacuum expectation value of any observable \( O \) which satisfy \( \hat{\delta}O = 0 \) under an infinitesimal deformation of \( k \) is

\[
\frac{\delta}{\delta k} \langle O \rangle = \int DX \frac{\delta}{\delta k} \left( \epsilon' \hat{\delta}G \right) e^{-S} = \int DX \hat{\delta} \left( O(\epsilon G) \right) e^{-S} = 0. 
\]

(35)

So we can change \( k \) without changing vacuum expectation value. In our case, only reducible connection part depend on \( k \). We denote \( \langle O \rangle_{IR} \) as a irreducible connection (non-abelian connection) part of \( \langle O \rangle \) and denote \( \langle O \rangle_R \) as a reducible connection (Abelian connection) part, then the only \( \langle O \rangle_R \) depend on the ratio \( k \). When the power expansion of \( k \) of \( \langle O \rangle_R \) is written as \( \langle O \rangle_R = \sum_{n=0}^{l} \langle O \rangle_{R,n} k^n \), vacuum expectation value \( \langle O \rangle \) is expressed as

\[
\langle O \rangle = \langle O \rangle_{IR} + \langle O \rangle_R \\
= \langle O \rangle_{IR} + \sum_{n=0}^{l} \langle O \rangle_{R,n} k^n 
\]

(36)
Since \( \langle O \rangle \) is \( k \) independent, we obtain

\[
\langle O \rangle_{R,n} = \left( \frac{\delta}{\delta k} \right)^n \langle O \rangle_R \big|_{k=0} = 0, \quad (n = 1, \ldots, l). \tag{37}
\]

This fact means that we can remove the contribution to vacuum expectation value from reducible connection without a \( k \) proportional term \( \langle O \rangle_{R,0} \). Note that \( \langle O \rangle_{R,0} = \langle O \rangle_R \big|_{\epsilon' = 0} \). This is the most important fact to derive the relation of Abelian Seiberg-Witten invariants in the next section. Note that the vacuum expectation values \( \langle O \rangle \) is invariant under changing of \( \epsilon \). This is easily ascertained by a similar way of (35).

In the next section, we explicitly investigate a relation of the non-Abelian Seiberg-Witten, Donaldson and the Abelian Seiberg-Witten invariants with the information obtained in this section.

### 4 The relation of Topological Invariants

In the previous sections, we prepared the tools for investigation of relations of the topological invariants i.e. Donaldson, Abelian Seiberg-Witten and non-Abelian Seiberg-Witten invariants. Now we actually construct the formulas of these invariants. We treat three parts of vacuum expectation value of observable (8) separately. The first part is Donaldson part which is defined as fixed point \( q = 0 \) and gauge connections of the fixed points are irreducible connections. There is no solution of the instanton equation when \( b_2^+ \geq 1 \) and connections are reducible. The second part is Abelian part whose fixed points \( q \neq 0 \) and connections are reducible. The third part is non-Abelian part that has irreducible connections and \( q \neq 0 \). In our theory, vacuum expectation value of \( O \) is defined by

\[
\langle O \rangle = \lim_{\epsilon, \epsilon' \to 0} \int DX \ O \ \exp(-S) \tag{38}
\]

\[
= \left\{ \begin{array}{ll}
\int DX \ O \ \exp(-S_{QCD}) & \text{(for Donaldson and non--Abelian Seiberg--Witten part)} \\
\lim_{\epsilon, \epsilon' \to 0} \int DX \ O \ \exp(-S_{QCD} - \int d^4x g^\frac{4}{3} (\bar{\phi} f_\epsilon + \eta \zeta_\epsilon)) & \text{(for Abelian Seiberg--Witten part).}
\end{array} \right.
\]

The fact that the \( \epsilon \) terms do not influence irreducible connection part was seen in previous section. In noting this point we advance analysis in the following.

**Donaldson part**

We already saw the transvers path-integral of this part in section 2. We represent common factors from \( H_{\mu\nu} \) and \( X \) integrations as \( \mathcal{N} \). Then we can write Donaldson part as,

\[
\mathcal{N} \det(-4\pi) \langle \exp(\bar{\nu} + \tau \bar{u}) \rangle_D \tag{39}
\]
where
\[ \tilde{v} = \frac{1}{4\pi^2} \int_\gamma Tr \left( i\phi F + \frac{1}{2} \lambda \wedge \lambda \right) \]
\[ \tilde{u} = -\frac{1}{8\pi^2} Tr \left( \phi^2 \right) \]  
(40)
and \( \langle O \rangle_D \) means vacuum expectation value of \( O \) with the action (11) of Donaldson-Witten theory. \( \tau \) is a parameter.

**Abelian Seiberg-Witten part**

In this part, \( A_\mu \) on fixed points are reducible connections. When we calculate in the large scaling limit it is possible that we choose back ground fields \( A_c \) in (13) as \( U(1) \) connections. The vector bundle \( E \) reduces to the sum of line bundle i.e. \( E = \zeta \oplus \zeta^{-1} \). (See the Appendix A.) We set the direction of back ground gauge fields to \( T_3 \) i.e. \( A_{\mu T_3} = A_{3\mu T_3} \). Since we take \( q \) as fundamental representation of \( SU(2) \), we can write \( q \) as

\[ q^\hat{\alpha} = \begin{pmatrix} q_1^\hat{\alpha} \\ q_2^\hat{\alpha} \end{pmatrix}. \]  
(41)

We can reduce line bundle \( \partial_\mu \zeta \) to \( +\zeta \) and \( q_2^\hat{\alpha} = 0 \) because there is a symmetry defined by \( A_3 \rightarrow -A_3 \) and \( q_1 \rightarrow q_2 \) in this part. So we can estimate the vacuum expectation value of (38) around fixed points \( A_c \) and \( q_c \) by considering quadratic action of quantum fields \( A, q \) and others. We expand \( A \) and \( q \) as

\[ A = A_c + A, \quad q = q_c + q \]  
(42)
where the each second term of right hand side is a quantum field and \( A_c \) and \( q_c \) is chosen as

\[ A_{\mu c} = A_{\mu 3} T_3, \quad q_{\hat{\alpha}c} = \begin{pmatrix} q_{1\hat{\alpha}} \\ 0 \end{pmatrix}. \]  
(43)

Next we decompose the action \( S_{QCD} \) into two parts as we did in section 2 as

\[ S_{QCD} \approx S_c + S_t, \]  
(44)
where \( S_c \) is the action which consists of the Calman part of adjoint fields and the first component of fundamental representation fields. The \( S_c \) is written as

\[ S_c = \int d^4 x g^{\frac{1}{2}} \left\{ \frac{1}{4} \left| F_{\mu\nu}^+ \right|^2 + \frac{1}{2} q_1^\dagger \sigma_\mu q_1 \right| \frac{1}{2} \left| \bar{\psi} q_1 \right|^2 - \frac{1}{2} (\partial^\mu \bar{\psi}_3 \partial_\mu \psi_3 + \lambda_3^{\mu\nu}(\partial^\mu \lambda_3)_{\nu})^+ \\
+ \frac{i}{2} (\partial^\mu \eta_3) \lambda_3^\mu - \frac{i}{2} \lambda_3^{\mu\nu} \bar{\psi} q_1 \sigma_\mu q_1 + \frac{i}{2} \lambda_3^{\mu\nu} q_1^\dagger \sigma_\mu \bar{\psi} q_1 - i(\bar{\psi} q_1) \psi q_1 \\
- i \bar{\psi} q_1 \bar{\psi} q_1 + \frac{1}{2} q_1^\dagger \lambda_{3\mu \nu} \bar{\psi} q_1 + \frac{1}{2} \bar{\psi} q_1 \sigma_\mu \lambda_3^{\mu\nu} q_1 \right\}, \]  
(45)
where $\tilde{D}_\mu = \partial^\mu - i\frac{1}{2} A_\mu \gamma^5$ and spinor indices $\alpha$ and $\dot{\alpha}$ are omitted. This action $S_c$ is surely the action of topological Abelian Seiberg-Witten theory [7][9]. This Caltan part action is Witten type topological action as follows

$$S_c = \hat{c}_c \left[ \frac{1}{2} \chi^{\mu\nu}((F^+_{3\mu\nu} + \frac{1}{2} q_1^1 \tilde{\sigma} q_1) - H_{3\mu\nu}) \right. $$

$$\left. - \frac{1}{2} (\partial_\mu \tilde{\phi}_3) \lambda_3^\mu + (X_{q_1} \psi_{q_1} - \bar{\psi}_{q_1} X_{q_1}) \right], \quad (46)$$

where we define the BRS-like SUSY as

$$\hat{c}_c A_{3\mu} = i \lambda_{3\mu}, \quad \hat{c}_c \chi_{3\mu\nu} = H_{3\mu\nu}, \quad \hat{c}_c \phi_3 = i \eta_3, \quad \hat{c}_c \chi = 0, \quad \hat{c}_c \eta = 0,$$

$$\hat{c}_c \phi = 0.$$

Another terms in $S_{QCD}$ have to be expanded around fixed points $A_{3c}$ and $q_{1c}$. The quadratic action of $S_t$ is obtained as

$$S_t^{(2)} = \int d^4x g^2 \left[ \frac{1}{4} (D_\mu A_{\nu+} - D_\nu A_{\mu+})(D^*_\mu A_{\nu-} - D^*_\nu A_{\mu-})^+ ight. $$

$$+ \frac{1}{8} (D_\mu A_{\nu+} - D_\nu A_{\mu+})^+ q_1^1 \tilde{\sigma} q_1^c + \frac{1}{8} (D^*_\mu A_{\nu-} - D^*_\nu A_{\mu-})^+ q_1 \tilde{\sigma} q_2 $$

$$+ \frac{1}{4} |q_{1c} \tilde{\sigma} q_{2c}|^2 + \frac{i}{2} [\tilde{D}^* q_2]^2 - \frac{i}{4} (\tilde{D}^* q_2)^+ \sigma^\mu A_{\mu+} q_{1c} - \frac{1}{4} (\sigma^\mu A_{\mu+} q_{1c})^+ (\tilde{D}^* q_2) $$

$$+ \frac{1}{8} \chi^\mu_{-}(D_\mu \lambda^\nu_-)^+ + \frac{i}{2} (D_\mu \eta_+)(D^*_\mu \phi_-) - \frac{1}{2} (D^*_\mu \phi_-)(D^*_\mu \phi_-) + \chi^\mu_+ (D^*_\mu \lambda^\nu_-)^+ $$

$$+ \frac{i}{2} \chi^\mu_{-} q_{1c} \tilde{\sigma} q_{2c} - i(\tilde{D}^* q_{2c}) q_{2c} - i \bar{\psi}_{q_2} (\tilde{D}^* q_{2c}) $$

$$+ \frac{i}{2} q_{1c} \lambda_{\alpha+} \tilde{\sigma} q_{2c} + \frac{1}{2} \psi_{q_2} \tilde{\sigma} \lambda_{\alpha+} q_{1c}]. \quad (48)$$
Where $\mathcal{D}_\mu = \partial_\mu - iA_{3\mu}$ and we put $T^\pm$ as $T^\pm = T_1 \pm iT_2$. $S^{(2)}_t$ is transformed with matrices in the same way of the appendix B as

\[
S^{(2)}_t = (A_{\rho-} q_2^\dagger) \begin{pmatrix} M_{A,A} & M_{A,q} \\ M_{q,A} & M_{q,q} \end{pmatrix} \begin{pmatrix} A_{\tau+} \\ q_2 \end{pmatrix} + \phi_+ \left( \frac{1}{2} \mathcal{D}_{\mu}^* \mathcal{D}_{\mu} \right) \phi_+ + \phi_- \left( \frac{1}{2} \mathcal{D}_{\mu} \mathcal{D}_{\mu}^* \right) \phi_+ \\
+ \left( \eta_- \eta_+ \chi_{0i} - \chi_{0i} \psi_{q2} \bar{\psi}_{q2} \right) (T_t).
\]

(49)

Where we chose space elements of self-dual field $\chi$, i.e. $\chi_{0i}$, as substantial elements. Matrix $M$ elements are given as follows.

\[
\begin{align*}
(A_{\rho-} \text{ row } A_{\tau+} \text{ column}) \\
M_{A,A} &= -\frac{1}{4} (\mathcal{D}_\mu \delta_{\nu\rho} - \mathcal{D}_\nu \delta_{\mu\rho})^+ (\mathcal{D}_\mu \delta_{\nu\tau} - \mathcal{D}_\nu \delta_{\mu\tau}) + \frac{1}{8} q_{1c}^\dagger \sigma^\mu \delta_{\mu\rho} \sigma_{\nu\tau} q_{1c} \\
(A_{\rho-} \text{ row } q_2 \text{ column}) \\
M_{A,q} &= -\frac{1}{8} (\mathcal{D}_\mu \delta_{\nu\rho} - \mathcal{D}_\nu \delta_{\mu\rho})^+ q_{1c}^\dagger \sigma_{\mu\nu} - \frac{1}{4} (q_{1c}^\dagger \sigma_{\mu\rho} \delta_{\nu\rho}) \bar{\mathcal{D}}^* \\
(q_2^\dagger \text{ row } A_{\tau+} \text{ column}) \\
M_{q,A} &= \frac{1}{8} \sigma_{\mu\nu} q_{1c} (\mathcal{D}_\mu \delta_{\nu\tau} - \mathcal{D}_\nu \delta_{\mu\tau})^+ + \frac{1}{4} (\bar{\mathcal{D}}^\dagger \sigma^\mu \delta_{\mu\tau} q_{1c} \\
(q_2^\dagger \text{ row } q_2 \text{ column}) \\
M_{q,q} &= -\frac{1}{2} \bar{\mathcal{D}}^\dagger \bar{\mathcal{D}}^* \frac{1}{4} \sigma_{\mu\nu} q_{1c} q_{1c}^\dagger \sigma^{\mu\nu}.
\end{align*}
\]

(50)

Where the index $+$ means anti-selfdual about indices $\mu$ and $\nu$. The elements of the matrix $T_t$ is obtained in the same process in appendix B. We write it as

\[
\begin{pmatrix}
-\frac{i}{2} D_0 & 0 & -\frac{i}{2} D_j & 0 & 0 & 0 \\
0 & -\frac{i}{2} D_0^* & 0 & -\frac{i}{2} D_j^* & 0 & 0 \\
-\frac{i}{2} D_i & 0 & \frac{1}{2} (PD)^+_{ij} & 0 & 0 & -\frac{i}{2} q_{1c} \sigma_{0i} \\
0 & -\frac{i}{2} D_i^* & 0 & \frac{1}{2} (PD^*)_ij & 0 & -\frac{i}{2} q_{1c} \sigma_{0i} \\
-\frac{i}{2} q_{1c} \sigma_0 & 0 & -\frac{i}{2} q_{1c} \sigma_j & 0 & 0 & i \bar{\mathcal{D}}^* \\
0 & \frac{1}{2} \sigma_0 q_{1c} & 0 & \frac{1}{2} \sigma_j q_{1c} & -i \bar{\mathcal{D}}^* & 0
\end{pmatrix},
\]

(51)

where $D_{\mu}$ is defined as $\partial_{\mu} - iA_{3\mu}$ and we defined $(PD)^+_{ij}$ as

\[
(PD)^+_{ij} = (D_0 \delta_{ij} - \frac{1}{2} \epsilon_{0ijk} D_i \delta_{jk}).
\]

(52)
Only in this Abelian part, we must not forget the $\epsilon$ terms. As we saw in section 3, $\frac{i}{2}D_\mu$ of $\eta$ rows $\lambda$ columns should be replaced by $D_i\mu$, and the components of $\eta$ rows $\psi$ columns are replaced by order $\epsilon$ operators $O(\epsilon)$ as we find the example (26). We introduce $D_\mu\nu$ as $D_\mu$ shifted by $\epsilon_i$ term like (25). Then we write down $T_{\epsilon\epsilon}$ which defined as the $T_i$ added shift terms from $\phi f_\epsilon + \eta \zeta_\epsilon$ in (19) as

$$
\begin{pmatrix}
-\frac{i}{2}D_\epsilon\epsilon' & 0 & -\frac{i}{2}D_{j\epsilon'} & 0 & O(\epsilon) & O(\epsilon) \\
0 & -\frac{i}{2}D_\epsilon\epsilon' & 0 & -\frac{i}{2}D_{j\epsilon'} & O(\epsilon) & O(\epsilon) \\
-\frac{i}{2}D_i & 0 & \frac{1}{2}(PD)_{ij} & 0 & 0 & -\frac{i}{2}q_{i\epsilon}\bar{\sigma}_{0i} \\
0 & -\frac{1}{2}D_i & 0 & \frac{1}{2}(PD)_{ij} & -\frac{i}{2}q_{i\epsilon}\bar{\sigma}_{0i} & 0 \\
-\frac{i}{2}q_{i\epsilon}\bar{\sigma}_{0i} & 0 & -\frac{1}{2}q_{i\epsilon}\bar{\sigma}_{ij} & 0 & 0 & i\bar{D}^* \\
0 & \frac{1}{2}\sigma_0q_{1c} & 0 & \frac{1}{2}\sigma_jq_{1c} & -i\bar{D}^* & 0
\end{pmatrix}
$$

(53)

Let us path-integrate out the transversal part. To carry out this Gaussian integration, we decide the determinant of $T_{\epsilon\epsilon}$ as

$$
det(T_{\epsilon\epsilon}) \equiv det(T_{\epsilon\epsilon}^*T_{\epsilon\epsilon})^{1/2}.
$$

(54)

(See the reference [17].) When we denote the $T_{\epsilon\epsilon}^*T_{\epsilon\epsilon}$ as

$$
T_{\epsilon\epsilon}^*T_{\epsilon\epsilon} = 
\begin{pmatrix}
T^*T_{\mu\mu}^+ & T^*T_{\mu\mu}^- & T^*T_{\mu\mu}^+ & T^*T_{\mu\mu}^- \\
T^*T_{\mu\nu}^+ & T^*T_{\mu\nu}^- & T^*T_{\mu\nu}^+ & T^*T_{\mu\nu}^- \\
T^*T_{\mu\bar{\nu}}^+ & T^*T_{\mu\bar{\nu}}^- & T^*T_{\mu\bar{\nu}}^+ & T^*T_{\mu\bar{\nu}}^- \\
T^*T_{\bar{\mu}\bar{\nu}}^+ & T^*T_{\bar{\mu}\bar{\nu}}^- & T^*T_{\bar{\mu}\bar{\nu}}^+ & T^*T_{\bar{\mu}\bar{\nu}}^-
\end{pmatrix}
$$

(55)

elements of $T_{\epsilon\epsilon}^*T_{\epsilon\epsilon}$ is obtained as follows.

$$
\begin{align*}
T^*T_{\mu\mu}^+ &= (\lambda_{\mu+} \text{ row } \lambda_{\mu+} \text{ column}) \\
&= (-\frac{1}{4}D^2\delta_{\mu\mu} - F_{3\mu\mu}) + \frac{1}{4}q_{i\epsilon}\sigma_{i\nu}q_{1c} + O(\epsilon') \\
T^*T_{\mu\mu}^- &= 0 \\
T^*T_{\mu\nu}^+ &= (\lambda_{\mu-} \text{ row } \lambda_{\nu+} \text{ column}) \\
&= 0 \\
T^*T_{\mu\nu}^- &= (\lambda_{\mu-} \text{ row } \lambda_{\nu-} \text{ column}) \\
&= (-\frac{1}{4}D^2\delta_{\mu\nu} - F_{3\mu\nu}) + \frac{1}{4}q_{i\epsilon}\sigma_{i\nu}q_{1c} + O(\epsilon') \\
T^*T_{\bar{\mu}\bar{\nu}}^+ &= (\bar{\psi}_{q\epsilon} \text{ row } \lambda_{\nu+} \text{ column}) \\
&= O(\epsilon) \\
T^*T_{\bar{\mu}\bar{\nu}}^- &= (\bar{\psi}_{q\epsilon} \text{ row } \lambda_{\nu-} \text{ column}) \\
&= 0 \\
T^*T_{\bar{\mu}\bar{\nu}}^- &= (-\frac{1}{4}\bar{\sigma}_{\nu\rho}q_{1c})^{+}D_{\rho} + \frac{i}{2}\bar{D}\sigma_{\nu}q_{1c} + O(\epsilon, \epsilon')
\end{align*}
$$

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When we integrate $\phi$ and $\bar{\phi}$ we have to pay attention to the shift from $\epsilon$ terms. We introduce $D_{\mu\epsilon}$ which include $D_{\mu}$ and shifts from $\epsilon$ terms, like (24). After these transverse fields path-integration of (48) we can write this part with matrices $M$ and $T_{\epsilon}$ as,

$$
\sum_{\text{reducible } A_{\mu}} \mathcal{N} \det(M)^{-1/2} \det(T_{\epsilon}) \det(\frac{1}{2} D_{\mu} D_{\mu}^*) \det(\frac{1}{2} D_{\mu} D_{\mu}^*) \langle \exp(\bar{\psi} + \tau \psi) \rangle_A.
$$

(57)

Where $\langle O \rangle_A$ means vacuum expectation value of $O$ with Abelian Seiberg-Witten theory whose action is composed by $S_c$ and Caltan part from $\bar{\phi} f_\epsilon + \eta \zeta_\epsilon$ i.e.

$$
S_c + \int d^4 x g^2 (\bar{\phi} f_\epsilon + \eta \zeta_\epsilon).
$$

(58)

Where we denote the Caltan part of $\bar{\phi} f_\epsilon + \eta \zeta_\epsilon$ as $(\bar{\phi} f_\epsilon + \eta \zeta_\epsilon)_c$. 

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If we put $\tilde{\phi} f_\epsilon + \eta \zeta_\epsilon$ as an example (26), this Caltan part is

\[
(\tilde{\phi} f_\epsilon + \eta \zeta_\epsilon)_c = \epsilon' \tilde{\phi}_3 \partial_\mu \phi_3 m^\mu + \epsilon' \eta_3 \lambda_3 m^\mu - \frac{1}{2} \epsilon n q^I_1 \eta_3 \bar{\psi}_1 q_1
\]

\[
- \frac{1}{2} \epsilon n \bar{\psi}_q \eta_3 q_1 + \frac{1}{2} \epsilon n q^I_1 \tilde{\phi}_3 \psi_3 q_1 - i \epsilon n \bar{\psi}_q \tilde{\phi}_3 \bar{\psi}_q.
\]

\[
= \epsilon' \delta_c (\tilde{\phi}_3 \lambda_3 m^\mu) + \epsilon' \delta_c (n q^I_1 \tilde{\phi}_3 \bar{\psi}_q - n \bar{\psi}_q \tilde{\phi}_3 q_1).
\]

This is the same case of Eqs. (35)(36). We find $\sum \epsilon$ the ratio of $\lambda$.

From Eq.(37), identities are obtained as

\[
\frac{1}{2} \epsilon n q^I_1 \tilde{\phi}_3 \psi_3 q_1 - i \epsilon n \bar{\psi}_q \tilde{\phi}_3 \bar{\psi}_q = 0.
\]

\[
+ \sum \epsilon q^I_1 \tilde{\phi}_3 \psi_3 q_1 - i \epsilon \bar{\psi}_q \tilde{\phi}_3 \bar{\psi}_q.
\]

As we saw in section 3 that this vacuum expectation value is invariant under changing $u$.

Therefore only $\sum \epsilon q^I_1 \tilde{\phi}_3 \psi_3 q_1 - i \epsilon \bar{\psi}_q \tilde{\phi}_3 \bar{\psi}_q$ term in (57) depend on $k$. As we saw in section 3, the coefficients of $k$ have to be zero. Then we get the non-trivial result,

\[
\langle \exp(\tilde{v} + \tau \bar{u}) \rangle_A = 0.
\]

But if there are several solutions of $A_\mu$ and $q_1$, it is unclear whether Eq.(60) is correct or not.

**non-Abelian Seiberg-Witten part**

Finally we denote non-Abelian part as

\[
\mathcal{N} \langle \exp(\tilde{v} + \tau \bar{u}) \rangle_{nA}.
\]

Where the connections of fixed point are restricted within irreducible connections. (61) is a pure non-Abelian Seiberg-Witten invariants.

Now the vacuum expectation value is separately written as

\[
\langle \exp(\tilde{v} + \tau \bar{u}) \rangle = \mathcal{N} \langle \exp(\tilde{v} + \tau \bar{u}) \rangle_D + \mathcal{N} \langle \exp(\tilde{v} + \tau \bar{u}) \rangle_{nA} + \sum \mathcal{N} \langle \exp(\tilde{v} + \tau \bar{u}) \rangle_{\text{reducible} A_\mu}
\]

As we saw in section 3 that this vacuum expectation value is invariant under changing the ratio of $\epsilon$ and $\epsilon'$. As we saw in section 3 that this vacuum expectation value is invariant under changing the ratio of $\epsilon$ and $\epsilon'$, so we found that the Abelian part vanishes without $\langle O \rangle_{R,0} = \langle O \rangle_R |_{\epsilon'=0}$ in Eq.(60). From this fact, we find following formulas,

\[
\langle \exp(\tilde{v} + \tau \bar{u}) \rangle = \mathcal{N} \langle \exp(\tilde{v} + \tau \bar{u}) \rangle_D + \mathcal{N} \langle \exp(\tilde{v} + \tau \bar{u}) \rangle_{nA}
\]

\[
+ \left[ \lim_{\epsilon \to 0} \sum \mathcal{N} \langle \exp(\tilde{v} + \tau \bar{u}) \rangle_{\text{reducible} A_\mu} \right]_{\epsilon'=0}.
\]

From Eq.(37), identities are obtained as

\[
\left( \frac{\delta}{\delta k} \right)^n \left[ \lim_{\epsilon \to 0} \sum \mathcal{N} \langle \exp(\tilde{v} + \tau \bar{u}) \rangle_{\text{reducible} A_\mu} \right] = 0.
\]

18
where \( n = 1, ..., \text{dim}(\text{ker} d_A) \). These formulas are non-trivial. These identities of U(1) topological invariants are obtained from SU(2) Topological QCD. Note that vacuum expectation value of \( \phi \) was changed as (28), but detail character of the shift terms did not need to get above formulas. We comment on the Eq.(60) little more. This formula may imply that Abelian Seiberg-Witten invariants vanish in general. Indeed it is possible to apply our methods for massive topological QCD with no obstacle. But there are some problems to identify the Abelian part and usual Seiberg-Witten invariants, for example Eq.(28) and there are problems to extend to the case which has plural monopole solutions. This subjects are discussed in [21].

5 Conclusions

We have studied massless topological QCD in detail and found new relations (63) and (64). One of the results of excluding mass terms is that our theory does not have spontaneous gauge symmetry breaking phase in usual meaning. Hence, we could not distinguish between reducible and irreducible connections without no modification. We gazed \( \det(D^\mu D_\mu) = 0 \) when connections are reducible. Infinitesimal shift terms are introduced to the Lagrangian to account zero-eigenvalue states of \( D^\mu D_\mu \). For this terms we could contain the Abelian part and treat separately reducible and irreducible connections. That determinant are obtained from \( \phi \) and \( \bar{\phi} \) integral. Fermionic integral of \( \eta \), which is super partner of \( \bar{\phi}, \lambda, \chi \) etc. cause the determinant of \( T \). This determinant offset \( \det(D^\mu D_\mu) = 0 \). Infinitesimal shift terms is added to the Lagrangian to shift the both zero-determinants. We could change the ratio of infinitesimal shift terms without changing vacuum expectation value. As a result of this, we got formulas (63) and (64) (and especially case Eq.(60)). These are non-trivial relations between topological invariants. The identities of U(1) topological invariants are obtained from BRS symmetry of Topological QCD like Ward-Takahashi identity.

When we interpret Topological Field Theory as a gauge fixing theory like [14], the zero-modes of \( D_\mu \) is interpreted as Gribov zero-modes. As we saw in section 4, Gribov zero-modes break BRS symmetry often and topological symmetry breaking occur. So the external fields in the section 4 avoid topological symmetry breaking from Gribov zero-modes.

The next subjects we have to investigate are to caluculate actually in some models and to ascertain these formulas. We studied only SU(2) gauge theries in the present paper, so we want to extend it more general case. To carry it out in our formalism, we have to construct the tool that embed the equations to classify the reducible connections in equations of motion from some topological action. This is one of our future problems.

The relation between the \( \langle \exp(\tilde{\nu} + \tau \tilde{u}) \rangle_A \) and usual Seiberg-Witten invariants have to be studied more carefully. \( \langle \exp(\tilde{\nu} + \tau \tilde{u}) \rangle_A \) is topological invariants but have some difference from usual Seiberg-Witten invariants. It is important problem to make the difference clear.
A Reducible connection

In this appendix, we summarize some basic of reducible connections for physicists who is unfamiliar with this words. For the convenience, we treat the only case of SU(2) gauge group. We consider a connection on a point $p$ on a back ground manifold $M$. A holonomy group is defined as subgroup of SU(2) which transform the connection as parallel transformations around any loop with the start point $p$. Therefore, holonomy groups are understood as groups which is needed actually to introduce each connection. We put $H$ as the holonomy group. We can define the reducible connection with the holonomy group which is classified in following two cases.

(1) $H \subseteq \{\pm 1\}$.

(2) $H$ is conjugate to U(1) .

In the first case connections are flat and this case is realized when the Chern number is zero. In our theory, case (1) is ignored. When connections do not satisfy above each conditions then we call them irreducible connections, and centralizer of $H$ is $\{\pm 1\}$.

We can understand the relation, which is used many times in this paper, between reducibility and zero-modes of $d_A: Adj \otimes \Lambda_0 \rightarrow adj \otimes \Lambda_1$ as follows. If $d_A$ has a zero-mode $\phi_0$, we obtain an one parameter group $\{\exp(t\phi_0) \mid t \in \mathbb{R} \ \ d_A\phi_0 = 0\}$ whose elements transform the connection identically because

$$g^{-1}dg + g^{-1}Ag = td_A\phi_0 = 0. \quad (65)$$

This means that centralizer of $H$ is not $\{\pm 1\}$ and this connection is reducible. Oppositely if a connection is reducible, then there is an one parameter group whose elements satisfy

$$g^{-1}dg + g^{-1}Ag = 0 \quad (66)$$

and $g \neq \pm 1$. We obtain

$$d_A\left(\frac{dg}{dt}\right) = 0 \quad (67)$$

after differentiate (60) by the parameter $t$. Therefore we understand $\frac{dg}{dt}$ is a zero mode of $d_A$.

Next we will see the process of reducible connection defined as

$$A = \begin{pmatrix} A_L & 0 \\ 0 & A_L^* \end{pmatrix} \quad (68)$$
where $A_L$ is a connection on a complex line bundle $L$ and $A_L^*$ is its complex conjugate. We can put the orthogonal basis of 2-dimensional complex vector bundle,

$$q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix},$$  
(69)

as

$$e_1 = \begin{pmatrix} q_1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ q_2 \end{pmatrix}. $$
(70)

We introduce complex line bundle $L$ and $L'$ with a parallel transformation operator $P_l$ where $l$ is represented as some loop as

$$L = \{c P_l(e_1)|c \in \mathbb{C}\}$$
(71)

$$L' = \{c P_l(e_2)|c \in \mathbb{C}\}. $$
(72)

In our theory, reducible connections means U(1) connections and we can take $P_l$ as diagonal matrices. So we find that the definitions of $L$ and $L'$ are unrelated of loop $l$. We could established $L$ and $L'$ in this way. Therefore connection $A$ can be represented as

$$A = A_L \oplus A_{L'} $$
(73)

where $A_L$ and $A_{L'}$ are connections of $L$ and $L'$ respectively. Since $A$ have to be valued in $u(1)$ now, $A$ is pure imaginary i.e. $A_{L'} = A_L^* = -A_L$. We obtain (61) and

$$F_A = \begin{pmatrix} dA_L & 0 \\ 0 & -dA_L \end{pmatrix}. $$
(74)

In this paper, we used these results in section 4.

**B Gaussian Integral**

In the section 4, we integrate fermionic fields in the Abelian Seiberg-Witten part. The methods of integration used there are studied in reference [11][17]. It is possible to do this integration in not only Abelian case but also other case. For the non-Abelian parts, we can do it more general. i.e. we put classical background fields as $A^a_{c\mu} T_a$ and $q_c$ in generally and expand $S_{QCD}$ to second order of quantum fields. We write down the result of fermionic part as follows.

$$(\eta a \chi_{0ia} \bar{\psi}_q \psi_q) \begin{pmatrix} \lambda_{0b} \\ \lambda_{jb} \\ \bar{\psi}_q^t \psi_{\bar{q}}^t \end{pmatrix} (T),$$
and $T$ is concretely

$$
\begin{pmatrix}
-iTr(T_aD_0T_b) & -iTr(T_aD_jT_b) & 0 & 0 \\
-Tr(T_aD_iT_b) & Tr(T_a(PD)^+_iT_b) & iq^i_a\sigma_0T_a & -iq^i_a\sigma_0T^i_a \\
-T_a^{\dagger}\sigma_0(q^i_a)^t & -T_a^{\dagger}\sigma_j(q^i_c)^t & 0 & i\bar{\varphi} \\
\sigma_0T_bq_c & \sigma_jT_bq_c & -i\bar{\varphi} & 0
\end{pmatrix}.
$$

Where we denote $(D_0\delta_{ij} - \frac{1}{2}\epsilon_{ijk}\delta_{jk})$ as $(PD)^+_i$. We chose space elements of self dual field $\chi$ as substantial elements. When we take account of $\epsilon$ shift terms, then the first row of $T$ change in order $\epsilon$. We call this shifted $T$ as $T_\epsilon$. For example, we obtained $T_\epsilon$ of the case concretely as

$$
\begin{pmatrix}
-iTr(T_a(D_0 - \epsilon'm_0)T_b) & -iTr(T_a(D_j - \epsilon'm_j)T_b) & -\epsilon q^i_aT_a & -\epsilon q^i_aT^i_a \\
-Tr(T_aD_iT_b) & Tr(T_a(PD)^+_iT_b) & iq^i_a\sigma_0T_a & -iq^i_a\sigma_0T^i_a \\
-T^i_a\sigma_0(q^i_c)^t & -T^i_a\sigma_j(q^i_c)^t & 0 & i\bar{\varphi} \\
\sigma_0T_bq_c & \sigma_jT_bq_c & -i\bar{\varphi} & 0
\end{pmatrix}.
$$

$T$ is not a map from a space into itself. So we have to pay attention to define its determinant. We put adjoint operator of $T$ as $T^*$, and define $det(T)$ as

$$
det(T) \equiv det(T^*T)^{1/2}.
$$

(See the references [11] [17].) Now we can treat $(T^*T)$ with not space indices $i$ but space-time indices $\mu$. We name the elements of $(T^*T)$ by

$$
T^*T = \begin{pmatrix}
T^{*T}_{\mu\nu} & T^{*T}_{\mu\bar{q}} & T^{*T}_{\mu\bar{q}} \\
T^{*T}_{\bar{q}\nu} & T^{*T}_{\bar{q}\bar{q}} & T^{*T}_{\bar{q}\bar{q}} \\
T^{*T}_{\bar{q}\bar{q}} & T^{*T}_{\bar{q}\bar{q}} & T^{*T}_{\bar{q}\bar{q}}
\end{pmatrix}
$$

(76)

The elements of $(T^*T)$ is obtained as follows.

$$
T^{*T}_{\mu\nu} = -\frac{1}{2}Tr(T^a(D^2\delta_{\mu\nu} - F_{\mu\nu}^-)T_b) + q^\dagger_cT_b\sigma_{\mu}\sigma_{\nu}q_c + q^\dagger_cT_b\sigma_{\mu}\sigma_{\nu}q_c
$$

($\bar{\psi}_i$ row $\lambda^b_{\mu}$ column)

$$
T^{*T}_{\bar{q}\nu} = \frac{i}{2}(T_a\bar{\sigma}_{\nu\rho}q_c)^+ + (2TrT_aD^\rho T_b) + i\bar{\varphi}\sigma_{\nu}T_bq_c
$$

($\bar{\psi}_q^t$ row $\lambda^b_{\mu}$ column)

$$
T^{*T}_{\bar{q}\bar{q}} = \left(\frac{i}{2}\bar{\sigma}_{\nu\rho}T^{\dagger}_a(q^i_c)^t\right)^+ + D^\rho + i\bar{\varphi}T^{\dagger}_a\bar{\sigma}_{\nu}(q^i_c)^t
$$

($\lambda^c_{\mu}$ row $\bar{\psi}_q$ column)
\begin{align*}
T^* T_{\mu \bar{q}} & = i (\text{Tr} \ T_c D^\mu T_a) (q_c^\dagger \bar{\sigma}_\rho T_a)^+ - i q_c^\dagger T_c \sigma_\mu \mathcal{D} \\
 & \quad \left( \bar{\psi}_\bar{q} \text{ row } \bar{\psi}_\bar{q} \text{ column} \right) \\
T^* T_{\bar{q} \bar{q}} & = (T_a \bar{\sigma}_{\mu \nu} q_c)^+ (q_c^\dagger \bar{\sigma}^{\mu \nu} T_a)^+ + \mathcal{D}^2 \\
 & \quad \left( \bar{\psi}_\bar{q} \text{ row } \bar{\psi}_\bar{q} \text{ column} \right) \\
T^* T_{\bar{q} q} & = (T_a^t \bar{\sigma}_{\mu \nu} (q_c^\dagger)^t)^+ (q_c^\dagger T_a^t \bar{\sigma}_{\mu \nu})^+ \\
 & \quad \left( \bar{\psi}_\bar{q}^t \text{ row } \bar{\psi}_\bar{q}^t \text{ column} \right) \\
T^* T_{\mu q} & = -i (\text{Tr} \ T_c D^\rho T_a) (q_c^\dagger \bar{\sigma}_{\rho \mu} T_a)^+ - i (q_c^\dagger T_c \sigma_\mu \mathcal{D}) \\
 & \quad \left( \bar{\psi}_\bar{q} \text{ row } \bar{\psi}_\bar{q} \text{ column} \right) \\
T^* T_{\bar{q} q} & = -(T_a \sigma_{\mu \nu} q_c)^+ (q_c^\dagger T_a \sigma_{\mu \nu})^+ + \mathcal{D}^2 \\
 & \quad \left( \bar{\psi}_\bar{q}^t \text{ row } \bar{\psi}_\bar{q}^t \text{ column} \right) \\
T^* T_{q q} & = (\bar{\sigma}_{\mu \nu} T_a (q_c^\dagger)^t)^+ (q_c^\dagger T_a^t \bar{\sigma}_{\mu \nu})^+ + \mathcal{D}^2. \quad \text{(77)}
\end{align*}

When we compares this to the matrices of the section 4, we understand that differences are in only covariant derivative \( D_\mu \) and \( \bar{D}_\mu \). Note that matrix obtained from bosonic part is almost same as \( T^* T \).

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