Holomorphic cuves in Exploded Torus Fibrations: Regularity

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Abstract

The category of exploded torus fibrations is an extension of the smooth category related to tropical geometry in which some adiabatic limits appear as smooth families. This paper contains regularity results for families of holomorphic curves in this category. The main result is a local model for the moduli space of holomorphic curves, which in the case of transversality of the $\bar{\partial}$ equation implies that the moduli space of holomorphic curves has the appropriate regularity. (This includes regularity of families of holomorphic curves in the smooth category which exhibit bubbling behavior.) A sketch of one method for constructing a ‘virtual class’ for the moduli stack of holomorphic curves using these local regularity results is included.

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1 Introduction

This paper does not include an introduction to the exploded category. Basic definitions can be found in [12], and an informal motivation with pictures can be found in the introduction of [11], available on the mathematics arXiv.

Most results will be stated using regularity $C^{\infty,\delta}$, defined below. The regularity $C^{\infty,\delta}$ should be thought of as a generalization of smooth functions with an exponential decay condition of weight $\delta$ at cylindrical ends. This regularity was defined in [12] and is briefly discussed again on page 15.
Definition 1.1.

\[ C^\infty_{\Delta} := \bigcap_{\delta'} < \delta \ C^\infty_{\Delta}, \delta' \]

The following theorem will be proved on page 55. The notation \( \bar{\mathcal{M}} \) refers to the usual Deligne Mumford space considered as a complex orbifold with normal crossing divisors given by its boundary components, and \( \text{Expl} \mathcal{M} \) is the explosion of this discussed in [12]. The map \( \text{Expl} \mathcal{M}^{+1} \rightarrow \text{Expl} \mathcal{M} \) is the explosion of the usual forgetful map from the moduli space of curves with one extra marked point to the moduli space of curves forgetting that extra marked point. The following theorem implies that \( \text{Expl} \bar{\mathcal{M}} \) represents the moduli stack of stable \( C^\infty_{\Delta} \) curves, and that the map \( \text{Expl} \mathcal{M}^{+1} \rightarrow \text{Expl} \mathcal{M} \) is a universal family of stable \( C^\infty_{\Delta} \) curves.

Theorem 1.2. Consider a \( C^\infty_{\Delta} \) family of exploded curves \( (\hat{\mathcal{C}}, j) \rightarrow \hat{\mathcal{F}} \) where \( 0 < \delta < 1 \) so that each exploded curve has \( 2g + n \geq 3 \) where \( g \) is the genus and \( n \) is the number of punctures. Then there exists a unique fiber-wise holomorphic map

\[
(\hat{\mathcal{C}}, j) \rightarrow (\text{Expl} \mathcal{M}^{+1}, j) \\
\downarrow \\
\hat{\mathcal{F}} \rightarrow \text{Expl} \mathcal{M}
\]

so that the map on each fiber \( \mathcal{C} \) factors into a degree one holomorphic map to a stable exploded curve \( \mathcal{C}' \) and a holomorphic map from \( \mathcal{C}' \) to a fiber of \( \text{Expl} \mathcal{M}^{+1} \) given by quotienting \( \mathcal{C}' \) by its automorphism group.

The above maps all have regularity \( C^\infty_{\Delta} \).

This paper studies the regularity of families of holomorphic curves in a smooth family of targets in the exploded category, \( \pi_{\mathcal{G}} : (\hat{\mathcal{B}}, J) \rightarrow \mathcal{G} \), where each fiber \( \mathcal{B} \) is a complete, basic exploded torus fibration with a civilized almost complex structure \( J \) (in the terminology of [12]). We will often talk about families of curves \( \hat{f} \) in \( \hat{\mathcal{B}} \rightarrow \mathcal{G} \) with various regularity which will correspond to commutative diagrams

\[
\mathcal{C}(\hat{f}) \xrightarrow{\hat{f}} \hat{\mathcal{B}} \\
\downarrow \pi_{\hat{\mathcal{B}}(\hat{f})} \\
\hat{\mathcal{F}}(\hat{f}) \rightarrow \mathcal{G}
\]

where \( \pi_{\hat{\mathcal{B}}(\hat{f})} : \mathcal{C}(\hat{f}) \rightarrow \hat{\mathcal{F}}(\hat{f}) \) is a family of curves (as defined in [12]). Where no ambiguity is present, this family is just referred to as \( \pi_{\hat{\mathcal{B}}} : \hat{\mathcal{C}} \rightarrow \hat{\mathcal{F}} \).

In what follows, we define an ‘evaluation map’ for a family of curves using a functorial construction of a family of curves \( f^{+n} \) with \( n \) extra punctures from a given family of curves \( \hat{f} \).

Definition 1.3. Given a submersion \( f : \mathcal{D} \rightarrow \mathcal{E} \), use the following notation for the fiber product of \( \mathcal{D} \) over \( \mathcal{E} \) with itself \( n \) times:

\[ (\mathcal{D})_\mathcal{E}^n := \mathcal{D} f \times_f \mathcal{D} f \times_f \cdots f \times_f \mathcal{D} \]
Definition 1.4. Given a family of curves $\hat{f}$ in $\hat{B} \to \mathcal{O}$ and $n \in \mathbb{N}$, the family $\hat{f}^+n$ is a family of curves with $n$ extra punctures

$$
\begin{array}{ccc}
\mathcal{C}(\hat{f}^+n) & \overset{\hat{f}^+n}{\longrightarrow} & (\hat{B})^n_{\mathcal{O}} \\
\downarrow \pi_{\mathcal{O}}(\hat{f}^+n) & & \downarrow \\
\mathcal{F}(\hat{f}^+n) & \overset{\hat{f}^+(n-1)}{\longrightarrow} & (\hat{B})^n_{\mathcal{O}}
\end{array}
$$

satisfying the following conditions

1. The family of curves $\hat{f}^+0$ is $\hat{f}$.
2. The base of the family $\hat{f}^+n$ is the total space of the family $\hat{f}^{+(n-1)}$.

$$
\mathcal{F}(\hat{f}^+n) = \mathcal{C}(\hat{f}^{+(n-1)})
$$

3. The fiber of $\pi_{\mathcal{O}}(\hat{f}^+1) : \mathcal{C}(\hat{f}^+1) \to \mathcal{F}(\hat{f}^+1)$ over a point $p \in \mathcal{F}(\hat{f}^+1) = \mathcal{C}(\hat{f})$ is equal to the fiber of $\pi_{\mathcal{O}}(\hat{f}) : \mathcal{C}(\hat{f}) \to \mathcal{F}$ containing $p$ with an extra puncture at the point $p$.

4. The family of curves $\hat{f}^+n$ is $(\hat{f}^{+(n-1)})^{+1}$

5. There exists a fiberwise holomorphic degree 1 map

$$
\begin{array}{ccc}
\mathcal{C}(\hat{f}^+1) & \longrightarrow & \mathcal{C}(\hat{f}) \\
\downarrow & & \downarrow \\
\mathcal{C}(\hat{f}) & \overset{id}{\longrightarrow} & \mathcal{C}(\hat{f})
\end{array}
$$

so that the following diagram commutes.

$$
\begin{array}{ccc}
\mathcal{C}(\hat{f}^+1) & \overset{\hat{f}^+1}{\longrightarrow} & (\hat{B})^n_{\mathcal{O}} \\
\downarrow & \hat{f} \times \hat{f} & \downarrow \\
\mathcal{C}(\hat{f}) & \overset{\hat{f}}{\longrightarrow} & \hat{B} \\
\downarrow \pi_{\mathcal{O}} & \hat{f} & \downarrow \pi_{\mathcal{O}} \\
\mathcal{F} & \longrightarrow & \mathcal{O}
\end{array}
$$

It is shown in appendix A on page 80 that $\hat{f}^+n$ exists and is smooth or $C^\infty_A$ if $\hat{f}$ is. The above conditions imply that the map $\hat{f}^+n : \mathcal{C}(\hat{f}^+n) \to (\hat{B})^n_{\mathcal{O}}$ factors as

$$
\begin{array}{ccc}
\mathcal{C}(\hat{f}^+n) & \longrightarrow & (\mathcal{C}(\hat{f}))^n_{\mathcal{O}} \\
& & \longrightarrow (\hat{B})^n_{\mathcal{O}}
\end{array}
$$

where the second map $(\mathcal{C}(\hat{f}))^n_{\mathcal{O}} \longrightarrow (\hat{B})^n_{\mathcal{O}}$ is given by the $(n+1)$-fold product of $\hat{f}$, and the first map $\mathcal{C}(\hat{f}^+n) \longrightarrow (\mathcal{C}(\hat{f}))^n_{\mathcal{O}}$ is a degree one map. The construction of $\hat{f}^+n$ is functorial, so given a map of families of curves...
\( \hat{f} \rightarrow \hat{g} \), there is an induced map \( \hat{f}^{+n} \rightarrow \hat{g}^{+n} \). This map \( \hat{f}^{+n} \rightarrow \hat{g}^{+n} \) is compatible with the naturally induced map \( (\mathcal{C}(\hat{f}))^{n+1}_{\mathfrak{B}(\hat{f})} \rightarrow (\mathcal{C}(\hat{g}))^{n+1}_{\mathfrak{B}(\hat{g})} \).

Combining \( \hat{f}^{+(n-1)} \) with the map \( \mathfrak{M}^{0} : \mathfrak{M}(\hat{f}^{+n}) \rightarrow \text{Expl} \mathfrak{M} \) given by Theorem 1.2 when \( n \) is large enough, we get the evaluation map

\[
\mathfrak{M}^{+n}(\hat{f}) := (\mathfrak{M}^{0}, \hat{f}^{+n-1}) : \mathfrak{M}(\hat{f}^{+n}) \rightarrow \text{Expl} \times (\mathfrak{B})^{n}_{\mathfrak{G}}
\]

The following notion of a core family gives a way of locally describing the moduli stack of \( C^\infty_1 \) curves. A notion such as this is necessary, as the ‘space’ of curves with in \( \mathfrak{B} \rightarrow \mathfrak{G} \) of a given regularity can not in general be locally modeled on even an orbifold version of a Banach space - this is because the domain of curves that we study are not fixed, and because of phenomena which would be called bubble and node formation in the setting of smooth manifolds.

(One analytic setup in which the moduli stack of curves could be described as a ‘space’ is an adaption to the exploded setting of the theory of polyfolds being developed by Hofer, Wysocki and Zehnder in a series of papers including [3]. An adaption of the theory of polyfolds to the exploded setting is a worthwhile direction for further research which is not explored in this paper.)

**Definition 1.5.** A core family of curves, \((\hat{f}/G, \{s_i\}, F)\) for an open substack \( \mathcal{O} \) of the moduli stack of \( C^\infty_1 \) curves in \( \mathfrak{B} \rightarrow \mathfrak{G} \) is:

- a basic \( C^\infty_1 \) family \( \hat{f} \) of curves with automorphism group \( G \),

\[
\begin{array}{ccc}
\mathcal{C}(\hat{f}) & \xrightarrow{\hat{f}} & \mathfrak{B} \\
\downarrow & & \downarrow \pi_{\mathfrak{G}} \\
\mathfrak{M}(\hat{f}) & \xrightarrow{} & \mathfrak{G}
\end{array}
\]

- a finite collection of \( C^\infty_1 \) ‘marked point’ sections \( s_i : \mathfrak{M}(\hat{f}) \rightarrow \mathcal{C}(\hat{f}) \) which do not intersect each other, and which do not intersect the edges of the curves in \( \mathcal{C}(\hat{f}) \).

- a \( C^\infty_1 \) map,

\[
\begin{array}{ccc}
\hat{f}^* T_{\text{vert}} \mathfrak{B} & \xrightarrow{F} & \mathfrak{B} \\
\downarrow & & \downarrow \\
\mathfrak{M}(\hat{f}) & \xrightarrow{} & \mathfrak{G}
\end{array}
\]

where \( T_{\text{vert}} \mathfrak{B} \) indicates the vertical tangent space of the family \( \mathfrak{B} \rightarrow \mathfrak{G} \).

- For all curves \( f \) in \( \hat{f} \), a neighborhood \( F^{-1}(\mathcal{O})|_f \) of 0 in the space of \( C^\infty_1 \) sections of \( f^* T_{\text{vert}} \mathfrak{B} \) which vanish on the image of all the marked point sections \( s_i \).

so that

1. For all curves \( f \) in \( \hat{f} \), there are exactly \( |G| \) maps \( f \rightarrow \hat{f} \).

2. For all curves \( f \) in \( \hat{f} \), the smooth part of the domain \( \mathcal{C}(f) \) with the extra marked points from \( \{s_i\} \) has no automorphisms.
3. The action of $G$ preserves the set of sections $\{s_i\}$, so there is some action of $G$ as a permutation group on the set of indices $\{i\}$ so that for all $g \in G$ and $s_i$,

$$s_i \circ g = g \circ s_{g(i)}$$

where $g$ indicates the action of $g$ on $\mathfrak{g}(\hat{f})$, $\mathcal{C}(\hat{f})$ or the set of indices $\{i\}$ as appropriate.

4. There exists a neighborhood $U$ of the image of the section

$$s : \mathfrak{g}(\hat{f}) \rightarrow \mathfrak{g}(\hat{f}^+)$$

defined by the $n$ sections $\{s_i\}$ so that

$$\text{ev}^{+n}(\hat{f}) : \mathfrak{g}(\hat{f}^+) \rightarrow \text{Exp} \mathcal{M} \times \left( \hat{\mathcal{B}} \right)^n$$

is an equi-dimensional embedding when restricted to $U$.

5. The map $F$ restricted to the zero section of $\hat{f}^*T_{\text{vert}}\hat{\mathcal{B}}$ is equal to $\hat{f}$, and $TF$ restricted to the inclusion of $f^*T_{\text{vert}}\mathcal{B}$ into the tangent space of $f^*T_{\text{vert}}\mathcal{B}$ over the zero section is the is the natural inclusion $f^*T_{\text{vert}}\mathcal{B} \rightarrow T_{\text{vert}}\mathcal{B}$.

6. Given any curve $f$ in $\hat{f}$ and section $\nu$ of $f^*T_{\text{vert}}\mathcal{B}$ in $F^{-1}(\mathcal{O})|_{\hat{f}}$, the curve

$$\mathcal{C}(f) \xrightarrow{F(\nu)} \mathcal{B}$$

is in $\mathcal{O}$.

7. Given any $C^\infty_{\mathbb{R}}$ family $\hat{f}'$ in $\mathcal{O}$, there exists a unique $C^\infty_{\mathbb{R}}$ fiber-wise holomorphic map

$$(\mathcal{C}(\hat{f}'), j) \xrightarrow{\Phi_{\hat{f}'}} (\mathcal{C}(\hat{f}), j)/G$$

and unique $C^\infty_{\mathbb{R}}$ section

$$\psi_{\hat{f}'} : \mathcal{C}(\hat{f}') \rightarrow \Phi_{\hat{f}'} \left( \hat{f}^*T_{\text{vert}}\mathcal{B} \right)$$

so that $\psi_{\hat{f}'}$ restricted to any curve $f'$ is equal to the pullback of a section in $F^{-1}(\mathcal{O})|_{\Phi_{\hat{f}', (f')}}$ (in particular $\psi$ vanishes on the pullback of marked points), and

$$\hat{f}' = F \circ \psi_{\hat{f}}$$

The last two conditions can be roughly summarized by saying that a $C^\infty_{\mathbb{R}}$ family in $\mathcal{O}$ is equivalent to a $C^\infty_{\mathbb{R}}$ map to $\mathfrak{g}(\hat{f})/G$ and a $C^\infty_{\mathbb{R}}$ section of some vector bundle associated to this map. Theorem 4.2 on page 60 states that if the first five conditions hold, there exists some $\mathcal{O}$ so that the last two conditions hold. Proposition 4.3 stated on page 63 constructs a core family containing any given $C^\infty_{\mathbb{R}}$ curve with at least one smooth component.
**Theorem 1.6.** Given any stable $C^\infty$ curve $f$ with at least one smooth component in a basic family of targets, $\mathcal{B} \xrightarrow{\pi_\mathcal{B}} \mathcal{G}$, there exists a $C^{1,\delta}$ open neighborhood $\mathcal{O}$ of $f$ in the moduli stack of $C^\infty$ curves and a core family $(\hat{f}/G, \{s_i\}, F)$ for $\mathcal{O}$ containing $f$.

Given any core family $(\hat{f}/G, \{s_i\}, F)$ there is a canonical orientation of $\hat{f}$ given as follows. The sections $\{s_i\}$ define a section $s : \mathcal{F}(\hat{f}) \longrightarrow \mathcal{F}(\hat{f}^{+n})$ so that in a neighborhood of the image of this section $s$, the map $ev^{+n}(\hat{f}) : \mathcal{F}(\hat{f}^{+n}) \longrightarrow \text{Expl} \times (\mathcal{B})$ is an equi-dimensional embedding. There is a canonical orientation of $\mathcal{F}(\hat{f}^{+n})$ relative to $G$ given by the orientation from the complex structure on $\text{Expl}$ and the almost complex structure $J$ on the fibers of $\hat{B} \longrightarrow \mathcal{G}$. Therefore, there is a canonical orientation of $\mathcal{F}(\hat{f})$ relative to $\mathcal{G}$, because the fibers of each of the maps $\mathcal{F}(\hat{f}^{+k}) \longrightarrow \mathcal{F}(\hat{f}^{+(k-1)})$ are complex.

Now we shall start describing the $\bar{\partial}$ equation on the moduli stack of $C^{\infty}$ curves.

**Definition 1.7.** Given a smooth (or $C^\infty$) family,

$$(\hat{C}, j) \xrightarrow{\hat{f}} (\hat{B}, J) \quad \xrightarrow{\pi_\hat{B}} \quad (\mathcal{F}, \pi_\mathcal{F})$$

1. Use the notation $T_{\text{vert}}\hat{C}$ to denote $\ker d\pi_\hat{B} \subset T\hat{C}$ and $T_{\text{vert}}\hat{B}$ to denote $\ker d\pi_\mathcal{B} \subset T\mathcal{B}$.

2. Define

$$d_{\text{vert}}\hat{f} : T_{\text{vert}}\hat{C} \longrightarrow T_{\text{vert}}\hat{B}$$

to be $df$ restricted to the vertical tangent space, $T_{\text{vert}}\hat{C} \subset T\hat{C}$.

3. Define

$$\bar{\partial}\hat{f} : T_{\text{vert}}\hat{C} \longrightarrow T_{\text{vert}}\hat{B}$$

$$\bar{\partial}\hat{f} := \frac{1}{2} (d_{\text{vert}}f + J \circ d_{\text{vert}}f \circ j)$$

Consider

$$\bar{\partial}\hat{f} \in \Gamma \left( (T^*\hat{C}/\pi_\mathcal{B}^*T^*\mathcal{G}) \otimes \left( \hat{f}^*T_{\text{vert}}\hat{B} \right) \right)^{0,1}$$

As the above bundle is cumbersome to write out in full, and can be considered as the pull back of a vector bundle $Y$ over the moduli stack of $C^{\infty}$ curves, use the following notation:
**Definition 1.8.** Use the notation $Y(\hat{f})$ to denote $\left((T^*\mathcal{C}/\pi_3^*T^*\mathfrak{F}) \otimes \left(f^*T_{vert}\mathfrak{B}\right)\right)^{(0,1)}$, which is the sub vector bundle of $(T^*\mathcal{C}/\pi_3^*T^*\mathfrak{F}) \otimes\mathbb{R}\left(f^*T_{vert}\mathfrak{B}\right)$ consisting of vectors so that the action of $J$ on the second factor is equal to $-1$ times the action of $j$ on the first factor.

Note that given any map of families of curves $\hat{f} \to \hat{g}$, there is a corresponding map of vector bundles $Y(\hat{f}) \to Y(\hat{g})$.

In what follows, we define obstruction models $(\hat{f}/G, V)$ which can be regarded as giving a kind of $C^{\infty, 1}$ Kuranishi structure to the moduli stack of holomorphic curves. Roughly speaking, an obstruction model is a $C^{\infty, 1}$ core family $\hat{f}/G$ with a finite dimensional vector bundle $V$ over $\mathfrak{F}(\hat{f})$ so that $\partial \hat{f}$ can be regarded as a $C^{\infty, 1}$ section $\mathfrak{F}(\hat{f}) \to V$, and so that an open subset of the moduli stack of holomorphic curves is equal to the intersection of this $\partial : \mathfrak{F}(\hat{f}) \to V$ with the zero section. In the case that this section $\partial$ is transverse to the zero section, this implies that the moduli stack of holomorphic curves locally has regularity $C^{\infty, 1}$.

**Definition 1.9.** A simple perturbation parametrized by a family

\[
\begin{array}{ccc}
\hat{C} & \xrightarrow{j} & \hat{B} \\
\downarrow \pi_3 & & \downarrow \pi_\mathfrak{G} \\
\mathfrak{F} & \to & \mathfrak{G}
\end{array}
\]

is a section $P$ of the bundle $\left((T^*\hat{C}/\pi_3^*T^*\mathfrak{F}) \otimes (T_{vert}\mathfrak{B})\right)^{(0,1)}$ over $\hat{C} \times \hat{B}$ with the same regularity as $\hat{f}$ which vanishes on all edges of the curves which are the fibers of $\hat{C}$.

Say that a simple perturbation is topologically compactly supported if the section $P$ vanishes outside of a topologically compact subset of $\hat{C} \times \hat{B}$. (For the use of the word ‘topologically’ see page 10 of [12].) Say that two simple perturbations parametrized by $\hat{f}$ are $C^{k,\delta}$ close if the corresponding sections are $C^{k,\delta}$ close.

**Definition 1.10.** A $C^{\infty, 1}$ obstruction model $(\hat{f}/G, V)$ for a substack $\mathcal{O}$ is given by

1. a $C^{\infty, 1}$ core family $(\hat{f}/G, \{s_i\}, F)$ for $\mathcal{O}$

\[
\begin{array}{ccc}
(\hat{C}, j) & \xrightarrow{\hat{j}} & (\hat{B}, J) \\
\downarrow \pi_3 & & \downarrow \pi_{\mathfrak{G}} \\
\mathfrak{F} & \to & \mathfrak{G}
\end{array}
\]

2. a vector bundle $V$ over $\mathfrak{F}$ with an action of $G$ compatible with the action of $G$ on $\mathfrak{F}$ and a $G$ equivariant $C^{\infty, 1}$ section called $\tilde{\partial}$

\[
\begin{array}{c}
V \\
\downarrow \uparrow \tilde{\partial} \\
\mathfrak{F}
\end{array}
\]
3. a $C^\infty_{\mathcal{L}}$ map of vector bundles over $\hat{\mathcal{C}}$

$$\pi_\hat{\mathcal{F}}^*(V) \rightarrow Y(\hat{f})$$

which vanishes on the edges of the curves in $\hat{\mathcal{C}} \rightarrow \hat{\mathfrak{F}}$, and so that the following diagram commutes

$$\begin{array}{ccc}
\pi_\hat{\mathcal{F}}^*(V) & \rightarrow & Y(\hat{f}) \\
\downarrow \pi_\hat{\mathcal{F}}^* \circ \bar{\partial} \circ \pi_\mathcal{F} & & \downarrow \bar{\partial} \\
\hat{\mathcal{C}} & \rightarrow & \hat{\mathcal{C}}
\end{array}$$

The $\pi_\hat{\mathcal{F}}^*$ from the upwards map on the left hand side needs to be interpreted as something which takes sections over $V$ over $\mathfrak{F}$ and spits out sections over $\mathcal{C}$.

The above map $\pi_\hat{\mathcal{F}}^*(V) \rightarrow Y(\hat{f})$ must be non trivial in the sense that for any nonzero vector in $V$, there exists a choice of lift to a vector in $\pi_\hat{\mathcal{F}}^*(V)$ which is not sent to $0$. (Said differently, a point $(p, v) \in V$ corresponds under the above map to a section of $\pi_\hat{\mathcal{F}}^*(V)$ over the curve which is the inverse image of $p \in \mathfrak{F}$. This section is the zero section if and only if $v$ is $0$.)

Note that a section $\mathfrak{F} \rightarrow V$ corresponds to a section of $\mathcal{C} \rightarrow Y(\hat{f})$. Call such a section of $\mathcal{C} \rightarrow Y(\hat{f})$ ‘a section of $V$’.

4. An identification of $Y(F(\nu))$ with $Y(\hat{f})$ for any section $\nu$ of $f^*T_{\text{vert}}\mathcal{B}$ induced from a $C^\infty_{\mathcal{L}}$ isomorphism from the bundle $F^*T_{\text{vert}}\mathcal{B}$ to the vertical tangent bundle of $f^*T_{\text{vert}}\mathcal{B}$ which preserves $J$. In other words, if $\pi : f^*T_{\text{vert}}\mathcal{B} \rightarrow \hat{\mathcal{C}}$ denotes the vector bundle projection, a $C^\infty_{\mathcal{L}}$ isomorphism between the two vector bundles $F^*T_{\text{vert}}\mathcal{B}$ and $\pi^*f^*T_{\text{vert}}\mathcal{B}$ over $f^*T_{\text{vert}}\mathcal{B}$ which preserves the almost complex structure $J$ on $T_{\text{vert}}\mathcal{B}$. This can also be written as a $C^\infty_{\mathcal{L}}$ vector bundle map

$$\begin{array}{ccc}
F^*T_{\text{vert}}\mathcal{B} & \rightarrow & \hat{f}^*T_{\text{vert}}\mathcal{B} \\
\downarrow & & \downarrow \\
\hat{f}^*T_{\text{vert}}\mathcal{B} & \rightarrow & \hat{\mathcal{C}}
\end{array}$$

One way such an identification arises is by parallel transport given by a $C^\infty_{\mathcal{L}}$ connection.

5. A $C^{1,\delta}$ neighborhood $U$ of $0$ in the space of $C^\infty_{\mathcal{L}}$ simple perturbations parametrized by $f$ so that given any simple perturbation $P \in U$,

(a) There exists a unique $C^\infty_{\mathcal{L}}$ section $\nu$ of $\hat{f}^*T_{\text{vert}}\mathcal{B}$ so that $\nu$ restricted to any curve $f$ is in $F^{-1}(\mathcal{O})|f$, and so that $\bar{\partial}F(\nu) = P(F(\nu)) + \nu$ where $\nu$ is a section of $V$ considered as a section of $Y(F(\nu))$ using the identification of $Y(\hat{f})$ with $Y(F(\nu))$. (In the above, $P(F(\nu))$ is shorthand for the section of the bundle $Y(F(\nu))$ over $\mathcal{C}(F(\nu)) = \hat{\mathcal{C}}$ given by $(\text{id}, F(\nu))^*P$.)
(b) Given any curve \( f \in \hat{f} \), there exists a unique curve \( f' \) in \( O \) so that \( f' = F(\nu) \) for some section \( \nu \) of \( f^*T_{vert} B \) in \( F^{-1}(O)|_f \), and \( \bar{\partial}_f' = P(\hat{f}') + v \) where \( v \) is some vector in \( V \) over \( f \) considered as a \( C^{\infty,1} \) section of \( Y(f') \) using the identification of \( Y(f') \) with \( Y(f) \).

An obstruction model \((\hat{f}/G, V)\) is said to be extendible if \( \hat{f}/G \) and \( V \) are the restriction of the corresponding objects from some other obstruction model to a topologically compactly contained subset.

For any obstruction model \((\hat{f}/G, V)\), the vector bundle \( V \) over \( \mathfrak{g} \) is given a canonical orientation relative to \( \mathfrak{g} \) as follows: For a given curve \( f \) in \( \hat{f} \), the space \( X \) of \( C^{\infty,1} \) sections of \( f^*T_{vert} B \) vanishing at the marked points corresponding to \( \{ s_i \} \) is a complex vector space, as is the space of \( C^{\infty,1} \) sections of \( Y(f) \), which we shall call \( Y \). Consider the map \( \bar{\partial} : X \rightarrow Y \) given by defining \( \bar{\partial} v \) as \( \bar{\partial} \) of the map \( F(\nu) \) considered as a section of \( Y(f) \) using item 4 above. There exist Banach spaces \( X_\delta \) and \( Y_\delta \) with \( X \subset X_\delta \) and \( Y \subset Y_\delta \) dense so that \( \bar{\partial} : X \rightarrow Y \) extends to a \( C^1 \) map \( \bar{\partial} : X_\delta \rightarrow Y_\delta \) so that \( D\bar{\partial}(0) : X_\delta \rightarrow Y_\delta \) is injective and Fredholm, and so that the restriction of \( V \) to \( f \in \mathfrak{g} \), \( V(f) \) is complementary to the image of \( D\bar{\partial}(0) \). The operator \( D\bar{\partial}(0) \) is homotopic through Fredholm operators to a complex Fredholm map \( \phi : X_\delta \rightarrow Y_\delta \) (which corresponds to the usual linear \( \bar{\partial} \) operator). As \( \phi \) is complex, the kernel and cokernel of \( \phi \) are canonically oriented. The spectral flow of \( D\bar{\partial}(0) \) to \( \phi \) then gives an orientation of the sum of the cokernel of \( D\bar{\partial}(0) \) and the kernel of \( D\bar{\partial}(0) \). As \( D\bar{\partial}(0) \) is injective, this gives an orientation for the cokernel of \( D\bar{\partial}(0) \), which gives an orientation for \( V(f) \). This orientation is the canonical orientation of \( V \) relative to \( \mathfrak{g} \).

Therefore, any obstruction model \((\hat{f}/G, V)\) has a canonical orientation which is an orientation of \( \hat{f} \) relative to \( \mathfrak{g} \) and an orientation of \( V \) relative to \( \hat{f} \). This gives an orientation relative to \( \mathfrak{g} \) for the transverse intersection of any two sections of \( V \).

The existence of obstruction models is proved on page \( \text{[69]} \). It follows from the existence of core families and Theorem \( \text{5.17} \) on page \( \text{[53]} \).

**Theorem 1.11.** Given any holomorphic curve \( f \) with at least one smooth component in a basic family of targets \( B \rightarrow \mathfrak{g} \), there exists an obstruction model \((\hat{f}/G, V)\) for a \( C^{1,\delta} \) open substack \( O \) of the moduli stack of \( C^{\infty,1} \) curves so that \( f \) is isomorphic to a member of the family \( \hat{f} \).

Obstruction models give a local model for the behavior of \( \bar{\partial} \) on the moduli stack of curves. For the construction of the virtual moduli space of holomorphic curves, we need some way of dealing with the usual orbifold issues that arise when dealing with moduli spaces of holomorphic curves. I think that the best way of defining the virtual moduli space probably involves the use of Kuranishi structures, first defined by Fukaya and Ono in \( \text{[2]} \). A generalization of the Kuranishi homology developed by Joyce in \( \text{[5]} \) should extend to the exploded setting, however this is not done in this paper. Instead, we shall work with weighted branched objects. There are a few approaches to weighed branched manifolds - our definition below only allows the definition of weighted branched sub objects, and is subtly different from the definition given by Cieliebak, Mundet i Rivera and Salamon in \( \text{[1]} \) or the intrinsic definition is given by McDuff in \( \text{[10]} \), because our definition has the notion of a ‘total weight’ and allows for the possibility...
of an empty submanifold being given a positive weight. I do not know which approach is better.

**Definition 1.12.** The following is a construction of a ‘weighted branched’ version of any sheaf of sets or vector spaces.

Given a vector space $V$, consider the group ring of $V$ over $\mathbb{R}$. This is the free commutative ring generated as a $\mathbb{R}$-module by elements of the form $tv$ where $v \in V$ and $t$ is a dummy variable used to write addition on $V$ multiplicatively. Multiplication on this group ring is given by

$$w_1tv_1 \times w_2tv_2 = (w_1w_2)t^{v_1+v_2}$$

where $w_i \in \mathbb{R}$, $t$ is a dummy variable, and $v_i \in V$. Denote by $wb(V)$ the sub semiring of the group ring of $V$ over $\mathbb{R}$ consisting of elements of the form $\sum_{i=1}^n w_it^{v_i}$ where $w_i \geq 0$.

There is a homomorphism

$$\text{Weight} : wb(V) \rightarrow \mathbb{R}$$

$$\text{Weight} \left( \sum_{i=1}^n w_it^{v_i} \right) := \sum_{i=1}^n w_i$$

Similarly, if $X$ is a set, consider the free $\mathbb{R}$ module generated by elements of the form $tx$ for $x \in X$. Define $wb(X)$ to be the $\mathbb{R}^+$ submodule consisting of elements in the form of finite sums $\sum_{i=1}^n w_it^{x_i}$ where $w_i \geq 0$. The homomorphism $\text{Weight} : wb(X) \rightarrow \mathbb{R}$ is defined similarly to the case of vector spaces: $\text{Weight}(\sum w_it^{x_i}) = \sum w_i$.

Given a sheaf $S$ with stalks $S_x$, define the corresponding weighted branched sheaf $wb(S)$ to be the sheaf with stalks $wb(S_x)$. Call a section of $wb(S)$ a weighted branched section of $S$. The Weight homomorphism gives a sheaf homomorphism of $wb(S)$ onto the locally constant sheaf with stalks equal to $[0, \infty)$. (The weight of a section of $wb(S)$ is a locally constant, $[0, \infty)$ valued section.) We shall usually just be interested in weighted branched sections of $S$ with weight 1.

This construction allows us to talk of the following weighted branched objects:

1. A smooth weighted branched section of a vector bundle $X$ over a manifold $M$ is a global section of $wb(C^\infty(X))$ where $C^\infty(X)$ indicates the sheaf on $M$ of smooth sections of $X$. In particular, such a weighted branched section is locally of the form

$$\sum_{i=1}^n w_it^{\nu_i}$$

where $\nu_i$ is a smooth section. This section has weight 1 if $\sum w_i = 1$.

2. Given a vector bundle $X$ over the total space of a family of curves $\mathcal{E} \rightarrow \mathfrak{F}$, a $C^{k,\delta}$ weighted section of $X$ branched over $[\mathfrak{F}]$ is defined as follows: consider the sheaf $C^{k,\delta}(X)$ over the topological space $[\mathfrak{F}]$ which assigns to each open set $U \subset [\mathfrak{F}]$, the vector space of $C^{k,\delta}$ sections of the vector bundle $X$ restricted to the inverse image of $U$ in $\mathcal{E}$. Then a $C^{k,\delta}$ weighted
section of $X$ branched over $[\mathcal{F}]$ is a global section of $wb(C^{k,\delta}(X))$. Such a weighted branched section is equal to $\sum w_i t^{\nu_i}$ restricted to subsets of our family sufficiently small in $[\mathcal{F}]$ where $\nu_i$ indicates a $C^{k,\delta}$ section of $X$. Note that considering $C^{k,\delta}$ sections of $X$ as a sheaf over different topological spaces allows different branching behavior.

3. For any topological space $X$, consider the sheaf of sets $S(X)$, where $S(U)$ is the set of subsets of $U$. A weighted branched subset of $X$ is a global section of $wb(S(X))$. (An example of what is meant by ‘branching’ in this context if $X = \mathbb{R}$, the global section $t^{(-1,2)} + t^{(0,1)} + t^{(-1,1)} + t^{(0,2)}$, $t^{(-2,-1)} + t^{(0,1)} = t^{(-2,-1)} + t^{(0,1)} + t^0$, but $t^{(-1,0)} + t^{(0,1)} \neq t^{(-1,0)} + t^{(0,1)} + t^0$.)

4. For any smooth manifold $M$, consider the sheaf of sets $S(M)$ where $S(U)$ is the set of smooth submanifolds of $U$. Then a smooth weighted branched submanifold of $M$ is a global section of $wb(S(M))$. Locally such a weighted branched submanifold is equal to

$$\sum_{i=1}^n w_i t^{N_i}$$

where each $N_i$ is a submanifold. Note that $N_i$ might intersect $N_j$, and $N_i$ might be equal to an empty submanifold.

Similarly, one can talk of weighted branched submanifolds which are proper, oriented, or have a particular dimension.

5. An $n$-dimensional substack of the stack of $C^{\infty,1}$ curves is a substack equal to $\hat{f}$ where $f$ is an $n$-dimensional $C^{\infty,1}$ family of curves. (In other words, a $C^{\infty,1}$ family $f'$ in this substack is equivalent to a $C^{\infty,1}$ map $f' \to f$.) Define a sheaf of sets $\mathcal{X}$ over the stack of $C^{\infty,1}$ curves $\mathcal{M}$ by setting $\mathcal{X}(U)$ to be the set of complete oriented $n$-dimensional $C^{\infty,1}$ substacks of $U \subset \mathcal{M}$. A complete oriented weighted branched $n$-dimensional $C^{\infty,1}$ substack of $\mathcal{M}$ is a global section of $wb(\mathcal{X})$ with weight 1. The support of such a weighted branched substack is the set of curves $f$ so that there is no neighborhood of $f$ in which our weighted branched substack is equal to the empty substack with weight 1.

**Definition 1.13.** A multi-perturbation on a substack of the moduli stack of $C^{\infty,1}$ curves $\mathcal{O} \subset \mathcal{M}$ is an assignment to each $C^{\infty,1}$ family of curves in $\mathcal{O}$, $\hat{f} \to \mathcal{O}$ a choice of $C^{\infty,1}$ weighted section of $Y(\hat{f})$ branched over $[\mathcal{F}]$ with weight 1 so that given any map of families of curves $\hat{f} \to \hat{g}$, the weighted branched section of $Y(\hat{f})$ is the pull back of the weighted branched section of $Y(\hat{g})$.

**Example 1.14.**

A simple perturbation $P$ parametrized by $\hat{f}$ where $\hat{f}/G$ is a core family for $\mathcal{O}$ defines a multi-perturbation on $\mathcal{O}$ which is a weighted branched section $(\hat{f}')^*P$ of $Y(\hat{f}')$ for all families of curves $\hat{f}'$ in $\mathcal{O}$ as follows:
Recall from the definition of the core family \( \hat{f}/G \) that given any \( C^\infty \) family \( \hat{f}' \) in \( \mathcal{O} \), there exists a unique map satisfying certain conditions

\[
\mathcal{C}(\hat{f}') \xrightarrow{\Phi_{\mathfrak{f}}} \mathcal{C}(\hat{f})/G \\
\mathfrak{F}(\hat{f}') \rightarrow \mathfrak{F}(\hat{f})/G
\]

Given such a map, around any point \( p \in \mathfrak{F}(\hat{f}) \), there exists a topological neighborhood \( U \) of \( p \) so that \( \Phi_{\mathfrak{f}} \) restricted to the lift \( \tilde{U} \) of \( U \) to \( \mathcal{C}(\hat{f}') \) lifts to exactly \(|G|\) maps \( \Phi_{U,g} : \tilde{U} \longrightarrow \mathcal{C}(\hat{f}) \)

\[
\mathcal{C}(\hat{f}) \xrightarrow{\Phi_{U,g}} \mathcal{C}(\hat{f})/G
\]

Recall that the simple perturbation \( P \) is some \( C^\infty \) section of the bundle

\[
\left( T^*\mathcal{C}(\hat{f})/\pi^*_\mathfrak{F}(\hat{f}) \otimes \left( T_{\text{vert}}\mathfrak{B}\right) \right)^{(0,1)} \text{ over } \mathcal{C}(\hat{f}) \times \mathfrak{B}.
\]

We can pull this section \( P \) back over the maps \( (\Phi_{U,g}, \hat{f}') \) to obtain \(|G|\) sections \( (\Phi_{U,g}, \hat{f}')^*P \) of the bundle \( Y(\hat{f}') \) restricted to \( U \). Define our weighted branched section \( (\hat{f}')^*P \) of \( Y(\hat{f}') \) over \( U \) to be

\[
(\hat{f}')^*(P)|_U := \sum_{g \in G} \frac{1}{|G|}(\Phi_{U,g}^*f)^*P
\]

This construction is clearly compatible with any map of families \( \hat{f}'' \longrightarrow \hat{f}' \) in \( \mathcal{O} \), so the above local construction glues together to a global weighted branched section \( (\hat{f}')^*P \) of \( Y(\hat{f}') \), and we get a multi-perturbation defined over all families in \( \mathcal{O} \). Note that \( \hat{f}'(P) \) is not just \((id, \hat{f})^*P \), but the weighted branched section which is \( \sum_{g \in G} \frac{1}{|G|}(g, \hat{f})^*P \) where \( g \) indicates the action of the group element \( g : \mathcal{C}(\hat{f}) \longrightarrow \mathcal{C}(\hat{f}) \).

**Definition 1.15.** Say that a family \( \hat{g} \) of curves meets a substack \( \mathcal{O} \) with a core family \( \hat{f}/G \) properly if the following holds. Let \( \hat{g}|_{\mathcal{O}} \) indicate the sub family of \( \hat{g} \) consisting of all curves contained in \( \mathcal{O} \), let \( \iota : \mathcal{C}(\hat{g}|_{\mathcal{O}}) \longrightarrow \mathcal{C}(\hat{g}) \) indicate the natural inclusion, and let \( \Phi_{\hat{g}|_{\mathcal{O}}} : \mathcal{C}(\hat{g}|_{\mathcal{O}}) \longrightarrow \mathcal{C}(\hat{f})/G \) indicate the projection from the definition of a core family. Then the map

\[
(\iota, \Phi_{\hat{g}|_{\mathcal{O}}}) : \mathcal{C}(\hat{g}|_{\mathcal{O}}) \longrightarrow \mathcal{C}(\hat{g}) \times \mathcal{C}(\hat{f})/G
\]

is topologically proper.

Say that a substack \( \mathcal{U} \) meets \( \mathcal{O} \) properly if every family \( \hat{g} \) in \( \mathcal{U} \) meets \( \mathcal{O} \) properly.

**Example 1.16.**

Let \( P \) be a compactly supported simple perturbation parametrized by a core family \( \hat{f}/G \) for \( \mathcal{O} \), and let \( \mathcal{U} \) meet \( \mathcal{O} \) properly. Define a multi-perturbation on \( \mathcal{U} \) as follows: If \( \hat{g} \) is a family in \( \mathcal{U} \), and \( \hat{g}|_{\mathcal{O}} \) indicates the sub family of all curves contained in \( \mathcal{O} \), let \( \hat{g}^*P \) indicate the weighted branched section of \( Y(\hat{g}) \) which
when restricted to \( \hat{g}|_O \) equals the multi perturbation \((\hat{g}|_O)^*P\) from example 1.14 and which is equal to the zero section with weight 1 everywhere else. As \( \hat{g} \) meets \( O \) properly and \( P \) is compactly supported, \( \hat{g}^*P \) is a \( C^{\infty,1} \) weighted branched section if \( \hat{g} \) is \( C^{\infty,1} \).

The following theorem discussed beginning on page 75 should be thought of as outlining the construction of a ‘virtual class’ for a component of the moduli stack of holomorphic curves. This virtual class is a cobordism class of finite dimensional \( C^{\infty,1} \) weighted branched substacks, oriented relative to \( \mathcal{G} \). Other approaches such as those used in [2], [5], [4], [6], [7], [13], [9] and [14] should also generalize to the exploded setting. The existence of obstruction models and the compactness results proved in [12] imply that given any basic family of targets \( \mathcal{B} \to \mathcal{G} \) with a family of strict tamings in the sense of [12] and a topologically compact subset \( \mathcal{G}' \subset \mathcal{G} \), the stack of holomorphic curves in any connected component of the moduli stack of \( C^{\infty,1} \) curves over \( \mathcal{G}' \) may be covered by a finite number of extendible obstruction models. The following theorem constructs a ‘virtual class’ for this component of the moduli stack of holomorphic curves using these obstruction models.

**Theorem 1.17.** Given

- a basic family of targets \( \mathcal{B} \to \mathcal{G} \) with a family of strict tamings in the sense of [12],
- a connected component of the moduli stack of \( C^{\infty,1} \) curves in \( \mathcal{B} \to \mathcal{G} \),
- a topologically compact subset \( \mathcal{G}' \subset \mathcal{G} \),
- and any finite collection of extendible obstruction models covering the moduli stack of holomorphic curves over \( \mathcal{G}' \) in our connected component of the moduli stack of \( C^{\infty,1} \) curves,

each obstruction model may be modified by restricting to a topologically open subset covering the same set of holomorphic curves, and satisfying the following: There exists an open \( C^{\infty,1} \) neighborhood \( U \) of 0 in the space of collections of compactly supported simple perturbations parametrized by these obstruction models so that for any collection \( \{P_i\} \in U \) of such perturbations, the following is true

1. There is some \( C^{1,\delta} \) neighborhood \( O \) of the set of holomorphic curves under study which meets each of the obstruction models properly. On \( O \) there is a \( C^{\infty,1} \) multi-perturbation \( \theta \) defined by

\[
\theta(f) := \prod_i \hat{f}^*P_i
\]

where each multi-perturbation \( \hat{f}^*P_i \) is as defined in example 1.14 (Note that the notation of a product of the weighted branched sections involves adding sections and multiplying weights. See definition 1.12 on page 10).

2. For each of our obstruction models \( (\hat{f}/G,\{s_i\},F,V) \) there exists a unique \( C^{\infty,1} \) weighted section \( \nu \) of \( \hat{f}^*T_{vert}\mathcal{B} \) branched over \( |\hat{f}(\mathcal{F})| \) with weight 1 which vanishes on the image of the marked point sections \( s_i : \hat{\mathcal{F}} \to \hat{\mathcal{C}} \) so that

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(a) locally on $\mathfrak{F}$,

$$\nu = \sum_{k=1}^{n} \frac{1}{n} \nu_k$$

where the sections $\nu_k$ correspond to families $F(\nu_k)$ in $O$

(b)

$$\theta(F(\nu_k)) = \sum_{j=1}^{n} \frac{1}{n} P_{k,j}$$

and $\bar{\partial}F(\nu_k) - P_{k,k}$ is a section of $V$.

(c) Given any curve $f$ in the above subset of $\hat{f}$ where $\nu_k$ is defined, if $\nu'$ is a section of $\ast T_{\text{vert}} \mathfrak{B}$ vanishing on $\{s_i\}$ so that $F(\nu')$ is in $O$ and the multi-perturbation $\theta(F(\nu')) = \sum_{j} w_j t^{Q_j}$, then $\frac{1}{n}$ times the number of the above $\nu_k$ so that $\nu'$ is the restriction of $\nu_k$ is equal to the sum of $w_j$ so that $\bar{\partial}F(\nu') - Q_j$ is in $V$.

3. Say that the multi perturbation $\theta$ is transverse to $\bar{\partial}$ on a sub family $C \subset \hat{f}$ if the sections $\bar{\partial}F(\nu_k) - P_{k,k}$ of $V$ are transverse to the zero section on $C$.

Given any topologically compact subfamily of one of our obstruction models $C \subset \hat{f}$, the subset of the space $U$ of collections of simple perturbations $\{P_i\}$ discussed above so that the corresponding multi-perturbation $\theta$ is transverse to $\bar{\partial}$ is open and dense in the $C^{\infty,1}$ topology.

4. Say that the multi-perturbation $\theta$ is fixed point free on a sub family $C \subset \hat{f}$ if none of the curves in $F(\nu_k)$ restricted to $C$ have smooth part with nontrivial automorphism group.

If the relative dimension of $\mathfrak{B} \to \mathfrak{S}$ is greater than 0, then given any topologically compact subfamily of one of our obstruction models $C \subset \hat{f}$, the subset of the space $U$ of collections of simple perturbations $\{P_i\}$ discussed above so that $\theta$ is fixed point free is open and dense in the $C^{\infty,1}$ topology.

5. There exists an open substack $O^o \subset O$ which contains the component of the moduli stack of holomorphic curves over $\mathfrak{S}'$ under study and a collection of topologically compact sub families of our obstruction models $C_i \subset \hat{f}_i$ so that

(a) if $f$ is a curve in $O^o$, then there exists some curve $f'$ in one of these sub families $C_i$ and section $\nu$ of $(f')_{*}T_{\text{vert}} \mathfrak{B}$ vanishing on marked points so that $f = F(\nu)$,

(b) if $f$ is any curve in $O$ over $\mathfrak{S}'$, so that for any collection of perturbations in $U$, $\theta(f) = wt^f + \ldots$ where $w > 0$, then $f$ is in $O^o$.

6. Say that $\theta$ is transverse to $\bar{\partial}$ and fixed point free if it is transverse to $\bar{\partial}$ and fixed point free on each $C_i$ from item (a) above. Given any such $\theta$, there is a unique complete weighted branched finite dimensional $C^{\infty,1}$ substack $M_{\theta} \subset O^o$, oriented relative to $\mathfrak{S}$ which is the solution to $\bar{\partial} = \theta$ in the following sense:
Definition 2.2. If use the notation
\[ e \]
For an empty collection of strata, define \( f \) such that \( f \) is equal to the sum of weights \( w'_k \). Then \( \partial f = Q_k \).

7. The support of \( \mathcal{M}_\theta \) restricted to the moduli stack of curves over \( \mathcal{G}' \) is topologically compact.

8. Given any other small enough multi-perturbation \( \theta' \) which is fixed point free and transverse to \( \tilde{\partial} \), defined on some other open neighborhood of the same component of the moduli stack of holomorphic curves over \( \mathcal{G}' \) using different choices of obstruction models, \( \mathcal{M}_\theta \) is cobordant to \( \mathcal{M}_{\theta'} \) in the sense that there exists a finite dimensional \( C^\infty \) complete weighted branched substack \( \mathcal{M}_{\tilde{\theta}} \) of an open substack of the stack of \( C^\infty \) families of curves in \( \mathcal{B} \times \mathbb{R} \rightarrow \mathcal{G} \times \mathbb{R} \) so that

- \( \mathcal{M}_{\tilde{\theta}} \) is oriented relative to \( \mathcal{G} \times \mathbb{R} \);
- \( \mathcal{M}_{\tilde{\theta}} \) is topologically compact restricted to curves over \( \mathcal{G}' \times [0,1] \);
- \( \mathcal{M}_{\tilde{\theta}} \) is locally equal to \( \sum w_i \tilde{t}_i \), where the maps \( h_i : \mathcal{G}(\tilde{h}_i) \rightarrow \mathcal{B} \times \mathbb{R} \) are transverse to \( \mathcal{G} \times 0 \) and \( \mathcal{B} \times 1 \);
- \( \mathcal{M}_{\tilde{\theta}} \) restricts to \( \mathcal{M}_\theta \) over \( \mathcal{G} \times 0 \) and \( \mathcal{M}_{\theta'} \) over \( \mathcal{G} \times 1 \).

2 Some norms on sections of vector bundles

In what follows, we define some regularity for sections of a real vector bundle \( V \rightarrow \mathcal{E} \) over a family of curves \( \mathcal{E} \rightarrow S \). Recall the following definitions from \([12]\).

Definition 2.1. Given any \( C^0 \) function \( f \) defined on a strata \( S \subset A \) in \( \mathbb{R}^n \times \mathcal{T}_A^m \), define
\[ e_S(f)(x, \tilde{z}) := f(x, \tilde{z}t^a) \]
where \( t^a := (t^{a_1}, \ldots, t^{a_m}) \) is any point in \( S \), and \( \tilde{z}t^a \) means \( (\tilde{z}_1t^{a_1}, \ldots, \tilde{z}_mt^{a_m}) \).

For example, \( \mathcal{T}_2^2 := \mathcal{T}_{[0,\infty)^2} \) has two one dimensional strata
\[ S_1 := \{ [\tilde{z}_2] = t^0, [\tilde{z}_1] \neq t^0 \} \quad S_2 := \{ [\tilde{z}_1] = t^0, [\tilde{z}_2] \neq t^0 \} \]
If we have a function \( f \in C^0(\mathcal{T}_2^2) \), then
\[ e_{S_1}f(z_1, z_2) = f(0, z_2) \quad e_{S_2}f(z_1, z_2) = f(z_1, 0) \]

Note that the operations \( e_{S_i} \) commute and \( e_{S_i}e_{S_i} = e_{S_i} \).

Definition 2.2. If \( I \) denotes any non empty collection of strata \( \{S_1, \ldots, S_n\} \), use the notation
\[ e_I f := e_{S_1}(e_{S_2}(\cdots e_{S_n} f)) \]
\( \Delta_I f := \left( \prod_{S_i \in I} (\text{id} - e_{S_i}) \right) f \)
For an empty collection of strata, define \( e_{\emptyset} f = f \) and \( \Delta_{\emptyset} f = f \).
For example,

$$\Delta_{S_1,S_2}f(z_1,z_2) := (1 - e^{S_1})(1 - e^{S_2})(f(z_1,z_2))$$

$$:= f(z_1,z_2) - f(0,z_2) - f(z_1,0) + f(0,0)$$

Note that if $S \in I$, $e^{S\Delta I} = 0$. In the above example, this corresponds to $\Delta_{S_1,S_2}f(z_1,0) = 0$ and $\Delta_{S_1,S_2}f(0,z_2) = 0$.

Below, we shall define norms in a class of allowable coordinate charts.

**Definition 2.3.** An allowable coordinate chart on a family of exploded curves $\hat{C}^{\pi_F} \rightarrow F$ is a coordinate chart on $\hat{C}^{\pi_F} \rightarrow F$ satisfying the following requirements:

1. The coordinate chart on $F$ is some open subset $U \subset \mathbb{R}^k \times T_A$ contained in a topologically compact subset of $\mathbb{R}^k \times T_A$ so that if $f$ is any continuous function defined on $U$, and $S$ any strata of $A$, if $f(\tilde{z})$ is defined, $e^{S\Delta}f(\tilde{z})$ is also defined.

   This coordinate chart also has the property that if $p$ and $p' \in \tilde{F}$ are topologically equivalent, then $p \in U$ if and only if $p' \in U$.

2. The coordinate chart on $\hat{C}$ and the map $\pi_F$ is some restriction of a standard projection $\mathbb{R}^k \times T_{B}^{m+1} \rightarrow \mathbb{R}^k \times T_A$

   so that

   (a) In standard coordinates, $\pi$ is given by

   $$(x,\tilde{z}_1,\ldots,\tilde{z}_{m+1}) = (x,\tilde{z}_2,\ldots,\tilde{z}_{m+1})$$

   (b) The cone $B = [\mathbb{R}^k \times T_{B}^{m+1}]$ is defined by the equations

   $$([\tilde{z}_2],\ldots,[\tilde{z}_{m+1}]) \in A$$

   $$[\tilde{z}_1] \leq t^0$$

   and possibly

   $$[\tilde{z}^{-1,\beta_2,\ldots,\beta_{m+1}}] \leq t^0$$

   Use the notation $\tilde{z}^\beta := \tilde{z}^{-1,\beta_2,\ldots,\beta_{m+1}}$

3. The coordinate chart $\hat{U}$ on $\hat{C}$ is in one of the following forms:

   (a) If the cone $B$ is not a product of $A$ with $[0,\infty)$, then

   $$\hat{U} := \pi^{-1}U \cap \{[\tilde{z}_1] < c, [\tilde{z}^\beta] < c\}$$

   and $[\tilde{z}_1 \tilde{z}^\beta] < c^2/16$ on $U$.

   (b) If the cone $B$ is equal to the product of $A$ with $[0,\infty)$, $\hat{U}$ is a subset of the product of $U$ with some open subset of $T_A$ which is in one of the following forms:
i. \( \tilde{U} \) is equal to the intersection of \( \pi^{-1}U \) with the set where the coordinate \( \tilde{z}_1 \) on \( \mathbb{T}^1 \) takes values in some open ball compactly contained in \( \mathbb{C}^* \).

ii. \( \tilde{U} \) is equal to the intersection of \( \pi^{-1}U \) with the set where \( |\tilde{z}_1| < c \) and possibly \( |\tilde{z}_1| > t^0[\tilde{z}^\alpha] \), where \( \tilde{z}^\alpha \) is some coordinate function on \( U \) so that \( t^0[\tilde{z}^\alpha] < t^0 \) on \( U \).

iii. \( \tilde{U} \) is equal to the intersection of \( \pi^{-1}U \) with the set where \( |\tilde{z}_1| < t^0 \) and possibly \( |\tilde{z}_1| > t^0[\tilde{z}^\alpha] \) where \( \tilde{z}^\alpha \) is as above.

**Definition 2.4.** Given any allowable coordinate chart \( \tilde{U} \), and any collection of strata \( S := \{ S_j \subset [\tilde{U}] \subset B \} \), define a weight function \( w_S \) as follows: choose some collection of generators \( \{ \tilde{z}^\alpha \} \) for the ideal of coordinate functions \( \tilde{z}^\alpha \in +\mathcal{E}^R(\mathbb{T}_B^r) \) so that \( e_S[\tilde{z}^\alpha] = 0 \) for all \( S_j \in S \). Another way of saying this is \( [\tilde{z}^\alpha] < t^0 \) on the interior of all \( S_j \), or these are coordinate functions so that the smooth part vanishes on all of these strata. Choose these generators for our ideal using the following algorithm: Choose some set of generators for our coordinate functions on \( \mathbb{T}_B^{m+1} \) which consists of \( \tilde{z}_1, \tilde{z}^\beta \) and lifts of coordinate functions on \( \mathbb{T}_A^m \). Include in our set \( \{ \tilde{z}^\alpha \} \) all the generators which are in our ideal, then all products of two of the unused generators which are in our ideal. Continue this, adding all products of \( k \) generators which do not have a sub product that has appeared earlier.

Now define

\[
    w_S := \left( \sum |[\tilde{z}^\alpha]| \right)
\]

This weight function \( w_S \) vanishes on all the strata in \( S \), and has the following property: If \( f \) is any smooth function, then for any \( \delta < 1 \), the function \( w^{-\delta} e_S f \) extends to a continuous function that vanishes on the strata in \( S \).

A special case of this weight function is given by the collection of strata \( S_0 \) which do not project homeomorphically onto strata of \( [U] \). These are the strata which correspond to the edges of the exploded curves in our family. We shall use the notation

\[
    w_0 := w_{S_0}
\]

For example, in the case of coordinate charts in the form of 3a above, if

\[
    \tilde{z}_1 \tilde{z}^\beta = z^{k\alpha}
\]

where \( \tilde{z}^\alpha \) is not a power of any other coordinate function, then we can define

\[
    w_0 := \left( |[\tilde{z}_1]| + |[\tilde{z}^\beta]| + |[\tilde{z}^\alpha]| \right)
\]

In the case of coordinate charts in the form of 3b we can simply define \( w_0 = |[\tilde{z}_1]| \). This weighting function has the property that if \( f \) is a smooth function on \( \mathbb{T}_B^m \) which vanishes when \( [\tilde{z}_1] = [\tilde{z}^\beta] = 0 \), then for all \( \delta < 1 \), \( f w_0^{-\delta} \) extends to a continuous function on \( \mathbb{T}_B^m \) which vanishes whenever \( [\tilde{z}_1] = [\tilde{z}^\beta] = 0 \). Another way to say this is as follows: If \( S_0 \) indicates the set of strata in \( B \) which do not project homeomorphically onto some strata of \( A \), then if \( e_S f = 0 \) for all \( S \) in \( S_0 \), then \( f w_0^{-\delta} \) extends to a continuous function so that \( e_S(f w_0^{-\delta}) = 0 \).

Now define a series of norms on vector valued functions defined on an allowable coordinate chart \( U \subset \mathbb{R}^k \times \mathbb{T}_B^{m+1} \). In all that follows, use the standard metric in which \( \frac{1}{t^0} \) and the real and imaginary parts of \( \frac{\partial}{\partial x_i} \) are orthonormal.

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1. Choose some exponent $p > 2$.

$$\|\nu\|_\delta := \sup_{x \in \pi(U)} \left( \int_{\pi^{-1}(x) \cap \Delta} |w_0^{-\delta} \nu|^p \right)^{1/p}$$

Of course, this is only defined when $\Delta_0 \nu = \nu$, in other words, when $e_S \nu = 0$ for all $S \in S_0$, or equivalently $\nu$ vanishes on all the edges of the curves in our family.

2. Choose some small exponent $\delta' > 0$ so that $\delta + \delta' < 1$, then define

$$\|\nu\|_{0, \delta} := \|\nu\|_{\delta' + \delta} + \max_S \left( \|w_S^{-\delta} \Delta_S \nu\|_{\delta'} \right)$$

The maximum is taken over all collections $S$ of substrata of $B$ which do not include the smallest strata.

3. $\|\nu\|_{k, \delta} := \|\nu\|_{k-1, \delta} + \|d\nu\|_{k-1, \delta}$

These norms should be thought of as suitable generalizations of $L^p_k$ with exponential weights.

4. For this last collection of norms, we shall use the notation $d_{\text{vert}} \nu$ to refer to $d\nu$ restricted to the real and imaginary parts of $\tilde{z}_1 \frac{\partial}{\partial \tilde{z}}$.

$$\|\nu\|_1^{\delta} := \sup |\nu| + \|d_{\text{vert}} \nu\|_\delta$$

$$\|\nu\|_{0, \delta}^{1, \delta} := \|\nu\|_{\delta' + \delta}^{1, \delta} + \max_S \left( \|w_S^{-\delta} \Delta_S \nu\|_{\delta'} + \|w_S^{-\delta} \Delta_S d_{\text{vert}} \nu\|_{\delta'} \right)$$

$$\|\nu\|_{k, \delta}^{1, \delta} := \|\nu\|_{k-1, \delta}^{1, \delta} + \|d\nu\|_{k-1, \delta}^{1, \delta}$$

Recall the following definition of $C^{k, \delta}$ for any $0 < \delta < 1$:

**Definition 2.5.** Define $C^{0, \delta}$ to be the same as $C^0$. A sequence of smooth functions $f_i \in C^\infty(\mathbb{R}^n \times \mathbb{T}_\Lambda)$ converge to a continuous function $f$ in $C^{k, \delta}(\mathbb{R}^n \times \mathbb{T}_\Lambda)$ if the following conditions hold:

1. Given any collection of at most $k$ nonzero strata $I$,

$$|w_I^{-\delta} \Delta_I (f_i - f)|$$

converges to 0 uniformly on compact subsets of $[\mathbb{R}^n \times \mathbb{T}_\Lambda]$ as $i \to \infty$.

(This includes the case where our collection of strata is empty and $f_i \to f$ uniformly on compact subsets.)

2. For any smooth vectorfield $v$, $v(f_i)$ converges to some function $vf$ in $C^{k-1, \delta}$.

Define $C^{k, \delta}(\mathbb{R}^n \times \mathbb{T}_\Lambda)$ to be the closure of $C^\infty$ in $C^0$ with this topology. Define $C^{\infty, \delta}$ to be the intersection of $C^{k, \delta}$ for all $k$. 

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Note that if $\delta'' > \delta + \delta'$, any $C^{\infty,\delta''}$ function $\nu$ restricted to any topologically compact subset will have $\|\nu\|_{1,k,\delta}$ finite. Standard Sobolev estimates imply that if $\|\nu\|_{k,\delta}$ or $\|\nu\|_{1,k,\delta}$ is bounded for all $k$, then $\nu$ is $C^{\infty,\delta}$.

We shall use the adjective extendible in many contexts to get around the difficulties introduced by not working with topologically compact sets. In particular:

**Definition 2.6.**
1. An extendible neighborhood is a neighborhood which is contained inside a topologically compact subset.

2. An extendible allowable coordinate chart is an allowable coordinate chart which is the restriction of some larger allowable coordinate chart to an extendible neighborhood in the larger coordinate chart.

3. An extendible function is the restriction of some function on a larger domain to an extendible neighborhood inside that larger domain.

4. An extendible vector bundle is a vector bundle which is the restriction of some vector bundle to an extendible neighborhood or coordinate chart.

5. An extendible function on an extendible vector bundle is the restriction of some function to an extendible vector bundle.

We shall often need the following observations:

**Lemma 2.7.**

1. On any allowable coordinate chart, $w_{I_1}w_{I_2}w^{-1}_{I_1 \cup I_2}$ is bounded

2. $\Delta_S(\phi \psi) = \sum_{I \subset S} (e_{S-I} \Delta_I \phi) \Delta_{S-I} \psi$

(Note that throughout this paper, the symbol $\subset$ means $\subseteq$.)

3. On any allowable coordinate chart, there exists some constant $c$ depending only on $k$ and our choices defining the above norms so that if $\phi$ and $\psi$ are two real valued functions,

   $\|\phi \psi\|_{k,\delta} \leq c \|\phi\|_{k,\delta} \|\psi\|_{1,k,\delta}$

4. On any allowable coordinate chart, there exists some constant $c$ depending only on $k$ and our choices defining the above norms so that if $\phi$ and $\psi$ are two real valued functions,

   $\|\phi \psi\|^{1}_{k,\delta} \leq c \|\phi\|^{1}_{k,\delta} \|\psi\|^{1}_{k,\delta}$

**Proof:** To see item 1, note that the $w_{I_1}$ is a finite sum of absolute values of smooth parts coordinate functions which vanish on strata in $I_1$, so $w_{I_1}w_{I_2}$ is a finite sum of absolute values of smooth parts of coordinate functions $\mathcal{E}^\alpha$ that vanish on strata in $I_1 \cup I_2$. The weight function $w_{I_1 \cup I_2}$ is a sum of absolute values of smooth parts of generators for the ideal of coordinate functions of this type, so on $\mathbb{R}^n \times \mathbb{T}^m_B$, every item in the former sum is a continuous function.
times an element of the latter sum. As allowable coordinate charts are always topologically compactly contained inside $\mathbb{R}^n \times \mathbb{T}^m$, it follows that \( w_{l_1} w_{l_2}^{-1} \cap l_2 \) is bounded.

To prove item 2, note first that if \( S \) denotes a single strata,

\[
\Delta_S \phi \psi = \phi \psi - (e_S \phi)(e_S \psi) = (\Delta_S \phi) \psi + (e_S \phi) \Delta_S \psi
\]

so the required identity holds if \( S \) consists of a single strata. Suppose now that \( S = S' \cup \{ S \} \), and the required identity holds for \( S' \). Then

\[
\Delta_S \phi \psi = \Delta_S \left( \sum_{I \subset S'} (e_{S'} \Delta_I \phi) \Delta_{S'} \psi \right)
\]

\[
= \sum_{I \subset S'} (e_{S'} \Delta_S \Delta_I \phi) \Delta_{S'} \psi + (e_S e_{S'} \Delta_I \phi) \Delta_S \Delta_{S'} \psi
\]

\[
= \sum_{I \subset S} (e_{S-I} \Delta_I \phi) \Delta_{S-I} \psi
\]

So by induction, the required identity holds for any set of strata \( S \). (Note that we need only consider finite sets of strata, as there are only a finite number of strata, and \( \Delta_S \Delta_S = \Delta_S \).)

To prove item 3, note

\[
\| \phi \psi \|_\delta \leq \| \phi \|_\delta \sup | \psi | \leq \| \phi \|_\delta \| \psi \|_\delta^1
\]

Next, we use item 2, then item 1, then the above observation to show

\[
\| w_{S}^{-\delta} \Delta_S (\phi \psi) \|_{\delta'} \leq \sum_{I \subset S} \| w_{S-I}^{-\delta} (e_{S-I} \Delta_I \phi) \Delta_{S-I} \psi \|_{\delta'}
\]

\[
\leq c \sum_{I \subset S} \| w_{S-I}^{-\delta} (e_{S-I} \Delta_I \phi) \Delta_{S-I} \psi \|_{\delta'}
\]

\[
\leq c \sum_{I \subset S} \| w_{S-I}^{-\delta} (e_{S-I} \Delta_I \phi) \|_{\delta'} \sup \| w_{S-I}^{-\delta} \Delta_{S-I} \psi \|
\]

The constant \( c \) above depends on the collection of strata \( S \), but as there are only a finite number of strata, and \( \Delta_S \Delta_S = \Delta_S \), we get for a different constant \( c \),

\[
\| \phi \psi \|_{0,\delta} \leq c \| \phi \|_{0,\delta} \| \psi \|_{0,\delta}^1
\]

Now we can use induction on the number of derivatives: Suppose that the required inequality holds for \( k - 1 \) derivatives,

\[
\| \phi \psi \|_{k,\delta} = \| \phi \psi \|_{k-1,\delta} + \| d(\phi \psi) \|_{k-1,\delta}
\]

\[
\leq c \| \phi \|_{k-1,\delta} \| \psi \|_{k-1,\delta}^1 + c \| d\phi \|_{k-1,\delta} \| \psi \|_{k-1,\delta}^1 + c \| \phi \|_{k-1,\delta} \| d\psi \|_{k-1,\delta}^1
\]

\[
\leq 3c \| \phi \|_{k,\delta} \| \psi \|_{k,\delta}^1
\]

Therefore, by induction the required inequality holds for all \( k \).

We prove item 4 similarly as follows:

\[
\sup | \phi \psi | \leq \sup | \phi | \sup | \psi |
\]
so as above, we may estimate using item 2 and item 1

\[
\sup |w_i^{−δ} \Delta_S(\phi \psi)| \leq \sum_{i \in S} \sup |w_i^{−δ} (e_S \Delta_i \phi) \Delta_S \psi|
\]

\[
\leq c \sum_{i \in S} \sup |w_i^{−δ} (e_S \Delta_i \phi) w_i^{−δ} \Delta_S \psi|
\]

\[
\leq c \sum_{i \in S} \sup |w_i^{−δ} e_S \Delta_i \phi| \sup |w_i^{−δ} \Delta_S \psi|
\]

We also have from item 3,

\[
\|d_{vert}(\phi \psi)\|_{k,\delta} \leq \|d_{vert}(\phi)\|_{k,\delta} + \|\phi d_{vert} \psi\|_{k,\delta}
\]

\[
\leq c \|d_{vert} \phi\|_{k,\delta} \|\psi\|_{k,\delta} + c \|\phi\|_{k,\delta} \|d_{vert} \psi\|_{k,\delta}
\]

\[
\leq 2c \|\phi\|_{k,\delta} \|\psi\|_{k,\delta}
\]

Therefore,

\[
\|\phi \psi\|_{0,\delta} \leq c \|\phi\|_{0,\delta} \|\psi\|_{0,\delta}
\]

The general case now follows by induction because if it holds for \(k - 1\),

\[
\|\phi \psi\|_{k,\delta} = \|\phi \psi\|_{k-1,\delta} + \|d(\phi \psi)\|_{k-1,\delta}
\]

\[
\leq c \|\phi\|_{k-1,\delta} \|\psi\|_{k-1,\delta} + c \||d\phi\|_{k-1,\delta} \|\psi\|_{k-1,\delta} + c \|\phi\|_{k-1,\delta} \|d\psi\|_{k-1,\delta}
\]

\[
\leq 3c \|\phi\|_{k,\delta} \|\psi\|_{k,\delta}
\]

\(\square\)

**Lemma 2.8.** If on some extendible allowable coordinate chart \(\hat{U}\), \(\nu\) is a \(\mathbb{R}^n\) valued function with \(\|\nu\|_{k,\delta}\) finite and \(E\) is a \(C^{∞,0}\) extendible function on the extendible vector bundle \(\mathbb{R}^n \times \hat{U}\), where \(\delta_0 > \delta + \delta'\), then \(E(\nu)\|_{k,\delta}\) is bounded. If \(\Delta_S E = E\), then \(\|E(\nu)\|_{k,\delta}\) is bounded. These bounds can be chosen to depend continuously on \(\nu\) in the \(\|\cdot\|_{k,\delta}\) topology.

**Proof:**

Note that for a single strata, \(\Delta_S E(\nu) = (\Delta_S E)(\nu) + \Delta_S ((e_S E)(\nu))\), so if \(S\) is a collection of strata,

\[
\Delta_S(\nu) = \sum_{i \in S} \Delta_i ((e_I \Delta \nu))
\] (1)

Because \(\|\nu\|_{k,\delta}\) is bounded, \(\sup \|\nu\|\) is bounded, so we may restrict attention on \(E\) to a subset of \(\mathbb{R}^n \times \hat{U}\) which is fiberwise bounded. Here we will have appropriate bounds on \(E\) and all its derivatives because \(E\) and \(\hat{U}\) are extendible. In the following, let \(D_I\) indicate the derivative with respect to \(\frac{\partial}{\partial t_i}\) for all \(i \in I\).

\[
\Delta_I(e_I E)(\nu) = \int_0^1 \cdots \int_0^1 D_I(e_I E) \left( \prod_{i \in I} (e_S, t_i, \Delta_S) \right) \nu \prod_{i \in I} dt_i
\] (2)
To estimate $\Delta_I(e_I E)(\nu)$, we shall estimate this integrand. Use the notation $\phi_I := \prod_{i \in I} (e_i s_{i+1} \Delta s_i)^{\nu}$.

\[
D_I(e_I E)(\phi_I) = \sum_{\prod_{j=1}^n I_j = I} (D^n e_I E)(D_{I_1} \phi_I) \cdots (D_{I_n} \phi_I) = \sum_{\prod_{j=1}^n I_j = I} (D^n e_I E)(\Delta_{I_1} \phi_{I-I_1}) \cdots (\Delta_{I_n} \phi_{I-I_n})
\]

(3)

The sum above is over all partitions of $I$. Using the equations (1), (2) and Lemma 2.7 part 1, we get that $\sup |w_\delta^\frac{\nu}{\delta} \Delta \mathcal{S}(E(\nu))|$ is bounded by a constant times

\[
\sum_{I \subset S} \sum_{\prod_{j=1}^n I_j = I} \sup |w_\delta^{-\frac{\nu}{\delta}} \Delta_{S-I} e_I (D^n E)(\phi_I)| \prod_{j=1}^n \sup |w_\delta^{-\frac{\nu}{\delta}} \Delta_{I_j} \phi_{I-I_j}|
\]

(4)

The first term in each of the above summands is finite because $E$ is extendible and in $C^{\infty, \delta_0}$ where $\delta_0 > \delta$, and the other terms are bounded by $\|\nu\|_{0, \delta}^1$. Note that our estimate for each of these terms can be chosen to depend continuously on $\nu$ in the $\|\cdot\|_{0, \delta}$ topology. (The estimate of the first term can be chosen continuous in the supremum topology, which is weaker than the $\|\cdot\|_{0, \delta}$ topology.)

Similar to the above, if $\Delta_{S_0} E = E$, we may bound $\|w_\delta^\frac{\nu}{\delta} \Delta \mathcal{S}(E(\nu))\|_{0, \delta}$ by a constant times

\[
\sum_{I \subset S} \sum_{\prod_{j=1}^n I_j = I} \sup |w_\delta^{-\frac{\nu}{\delta}} \Delta_{S-I} e_I (D^n E)(\phi_I)| \prod_{j=1}^n \sup |w_\delta^{-\frac{\nu}{\delta}} \Delta_{I_j} \phi_{I-I_j}|
\]

for some $\epsilon > 0$. The first term is bounded because $\Delta_{S_0} E = E$, and $E$ is $C^{\infty, \delta_0}$ where $\delta_0 > \delta + \delta'$, and the other terms are bounded by $\|\nu\|_{0, \delta}^1$. Again, the bounds can be chosen to be continuous in the $\|\cdot\|_{0, \delta}$ topology. Therefore, we have that in this case $|E(\nu)|_{0, \delta}^1$ is bounded, and the bound can be chosen to depend continuously on $\nu$ in the $\|\cdot\|_{0, \delta}$ topology.

In the case that $\Delta_{S_0} E$ is not necessarily equal to $E$, we already know that $\sup |w_\delta^\frac{\nu}{\delta} \Delta \mathcal{S}(E(\nu))|$ is bounded, and we must show that $\|d_{\text{vert}}(E(\nu))\|_{0, \delta}$ is bounded. To this end, note that $d_{\text{vert}}(E(\nu)) = DE(\nu)(d_{\text{vert}} \nu) + (d_{\text{vert}} E)(\nu)$. The second term can be dealt with by observing that $d_{\text{vert}} E$ is in $C^{\infty, \delta_0}$ and $\Delta_{S_0} d_{\text{vert}} E = d_{\text{vert}} E$, therefore, as argued above, $\|d_{\text{vert}}(E)(\nu)\|_{0, \delta}$ is bounded. The first term can be dealt with in the same way as the product was dealt with in Lemma 2.7 part 3. Note that in that argument, the only part of the norm $\|\cdot\|_{0, \delta}$ used was the part involving the supremum. Following this argument, we can bound $\|DE(\nu)(d_{\text{vert}} \nu)\|_{0, \delta}$ by the product of $\|d_{\text{vert}} \nu\|_{0, \delta}$ with $\sum \sup |w_\delta^\frac{\nu}{\delta} \Delta \mathcal{S}(DE(\nu))|$, which as argued above is bounded. Again, our bounds may be chosen to depend on $\nu$ continuously in the $\|\cdot\|_{0, \delta}$ topology.

We have now shown that if $\|\nu\|_{0, \delta}$ is bounded, $\|E(\nu)\|_{0, \delta}$ is bounded, and if $\Delta_{S_0} E = E$, then $\|E(\nu)\|_{0, \delta}$ is bounded, and these bounds can be chosen to depend continuously on $\nu$ in the $\|\cdot\|_{0, \delta}$ topology. Suppose now that the equivalent statement holds for $\|\cdot\|_{k, \delta}$ and $\|\cdot\|_{k, \delta}$.
If \( \|\nu\|_{k+1,\delta}^1 \) is bounded, the first term is a composition of the \( C^{\infty,\delta_0} \) function \( DE(\cdot)(0,\cdot) \) and \( (\nu,d\nu) \) which has \( \| (\nu,d\nu) \|_{k,\delta}^1 \) bounded, and the second term is a composition of the \( C^{\infty,\delta_0} \) function \( DE(\cdot)(0,\cdot) \) with \( \nu \). Therefore, the \( \|\cdot\|_{k,\delta}^1 \) norm of both these terms is bounded, and \( \|dE(\nu)\|_{k+1,\delta} \) is bounded. Similarly, \( \|d(E(\nu))\|_{k+1,\delta} \) is bounded if \( \|\nu\|_{k+1,\delta}^1 \) is bounded and \( \Delta_{S_0}E = E \), as in that case \( \Delta_{S_0}DE = DE \). All these bounds can be chosen to depend continuously on \( \nu \) in the \( \|\cdot\|_{k,\delta}^1 \) topology. By induction, we have proved the lemma for all \( k \).

\[ \square \]

**Corollary 2.9.** If \( E \) is an extendible function on \( \mathbb{R}^n \times \hat{U} \) which is \( C^{\infty,\delta_0} \) where \( \delta_0 > \delta + \delta' \), then \( E(\nu) \) in the \( \|\cdot\|_{k,\delta}^1 \) topology depends continuously on \( \nu \) in the \( \|\cdot\|_{k,\delta}^1 \) topology. If \( \Delta_{S_0}E = E \), then \( E(\nu) \) in the \( \|\cdot\|_{k,\delta}^1 \) topology depends continuously on \( \nu \) in the \( \|\cdot\|_{k,\delta}^1 \) topology.

**Proof:**

\[
E(\nu_1) - E(\nu_2) = \int_0^1 DE(\nu_1 + t(\nu_2 - \nu_1))(\nu_2 - \nu_1)dt
\]

so using lemma 2.7 part 3,

\[
\|E(\nu_1) - E(\nu_2)\|_{k,\delta}^1 \leq \|\nu_1 - \nu_2\|_{k,\delta} \int_0^1 \|DE(\nu_1 + t(\nu_2 - \nu_1))\|_{k,\delta} dt
\]

As \( DE \) is \( C^{\infty,\delta_0} \), Lemma 2.8 tells us that \( \|DE(\nu')\|_{k,\delta}^1 \) can be bounded uniformly for \( \nu' \) in a \( \|\cdot\|_{k,\delta}^1 \) neighborhood of \( \nu_1 \), therefore, if \( \nu_2 \) is in this neighborhood,

\[
\|E(\nu_1) - E(\nu_2)\|_{k,\delta}^1 \leq c \|\nu_1 - \nu_2\|_{k,\delta}^1
\]

So \( E(\nu) \) in the \( \|\cdot\|_{k,\delta}^1 \) topology depends continuously on \( \nu \) in the \( \|\cdot\|_{k,\delta}^1 \) topology. Similarly, using lemma 2.7 part 3,

\[
\|E(\nu_1) - E(\nu_2)\|_{k,\delta} \leq \|\nu_1 - \nu_2\|_{k,\delta} \int_0^1 \|DE(\nu_1 + t(\nu_2 - \nu_1))\|_{k,\delta} dt
\]

As \( DE \) is \( C^{\infty,\delta_0} \) and \( \Delta_{S_0}DE = DE \) if \( \Delta_{S_0}E = E \), \( \|DE(\nu')\|_{k,\delta} \) can be bounded uniformly for \( \nu' \) in a \( \|\cdot\|_{k,\delta}^1 \) neighborhood of \( \nu_1 \) if \( \Delta_{S_0}E = E \). Therefore, if \( \nu_2 \) is in this neighborhood,

\[
\|E(\nu_1) - E(\nu_2)\|_{k,\delta} \leq c \|\nu_1 - \nu_2\|_{k,\delta}^1
\]

So \( E(\nu) \) in the \( \|\cdot\|_{k,\delta} \) topology depends continuously on \( \nu \) in the \( \|\cdot\|_{k,\delta}^1 \) topology if \( \Delta_{S_0}E = E \). \[ \square \]

We shall now give equivalent norms for \( \|\cdot\|_{0,\delta}^1 \) and \( \|\cdot\|_{0,\delta}^1 \). The advantage of using these norms is they only involve weighting functions \( w_S \) for which \( d_{vert}w_S = 0 \). For this we shall need the following concepts:

**Definition 2.10.** Given an allowable coordinate chart \( \hat{U} \xrightarrow{\pi} U \) and a set of strata \( S \) of \( \{U\} \), the lift of \( S \) is a set of strata of \( \{\hat{U}\} \) denoted by \( \hat{S} \) defined by

\[
\hat{S} := \{ \text{S so that } [\pi](S) \in S \}
\]
For our purposes, two sets of strata $\mathcal{S}$ and $\mathcal{S}'$ will act identically if $\Delta_{\mathcal{S}} = \Delta_{\mathcal{S}'}$, so define a lifted set of strata $\mathcal{S}$ to be a set of strata of $[\tilde{U}]$ with the property that if for some strata $\mathcal{S}$, the projection $[\pi](\mathcal{S}) \in [\pi](\mathcal{S})$, then $\Delta_{\mathcal{S}} \Delta_{\mathcal{S}} = \Delta_{\mathcal{S}}$.

A coordinate $z$ on $\tilde{U}$ is lifted if it is the composition of $\pi$ with some coordinate on $U$.

For any set of strata $\mathcal{S}$ in $\tilde{U}$, define the complement $\mathcal{S}^c$ to be the set of strata $\mathcal{S}'$ so that $\Delta_{\mathcal{S}} \Delta_{\mathcal{S}^c} \neq \Delta_{\mathcal{S}}$, and $[\pi](\mathcal{S}) = [\pi](\mathcal{S})$ for some $S \in \mathcal{S}$.

We shall use the shorthand $\mathcal{S}^c = \emptyset$ to indicate that $\mathcal{S}$ is a lifted set of strata.

Note that if $\mathcal{S}^c = \emptyset$, the vertical derivative $d_{vert}w_S = 0$ because $w_S$ is constructed using lifted coordinate functions. We shall use that $(\mathcal{S} \cup \mathcal{S}^c) = \emptyset$.

**Lemma 2.11.** On any allowable coordinate chart, given a strata $\mathcal{S}_j$ and $\tilde{\mathcal{S}}_j \in S^c_j$, there exists a constant $c > 0$ so that

$$\left| e_{\mathcal{S}_j} \Delta_{\mathcal{S}_j} \phi \right| \leq c \left\| e_{\mathcal{S}_j} d_{vert} \phi \right\|_{\delta} e_{\mathcal{S}_j} w_{\delta}$$

Proof:

The left hand side of the above inequality is $\left| e_{\mathcal{S}_j} \phi - e_{\mathcal{S}_j} e_{\mathcal{S}_j} \phi \right|$. This is equal to the difference between the value on $\phi$ on some fiber of the coordinate chart with the value of $\phi$ on the edge of the same fiber. We shall bound this using a standard Sobolev estimate on this fiber. The fact that we get a uniform bound will follow from the bounded geometry of allowable coordinate charts.

Without losing generality, we may assume that the part of this fiber we are interested in has coordinate $\tilde{z}_1$, and the smooth part of the coordinate chart of interest is equal to $\{ \| \tilde{z}_1 \| < c \} \subset \mathbb{C}$, where $c$ is a constant depending only on the coordinate chart, and not the particular fiber. Use cylindrical coordinates $[\tilde{z}_1] = e^{t+i\theta}$. We shall denote $\phi$ restricted to this smooth part of this fiber simply as $\phi$.

We are interested in bounding $|\phi(t, \theta) - \phi(-\infty, \theta)|$ in terms of $\left\| e_{\mathcal{S}_j} d_{vert} \phi \right\|_{\delta} e_{\mathcal{S}_j} w_{\delta}$.

In the case we have restricted ourselves to, $e_{\mathcal{S}_j} w_{\delta} = \| \tilde{z}_1 \| = e^t$. Using this observation and the definition of $\| \cdot \|_\delta$, we have

$$\left\| e_{\mathcal{S}_j} d_{vert} \phi \right\|_{\delta} \geq \left( \int_{\{t < \log c\}} (e^{-\delta t} |d\phi|)^p dtd\theta \right)^{\frac{1}{p}}$$

(The inequality sign above is there because the right hand side is considering only one fiber. The $d$ on the right hand side is $\phi$ restricted to this fiber, so $d_{vert} \phi = d\phi$.) So long as $p > 2$, a Sobolev estimate implies that there exists some constant $c_1 > 0$ so that

$$\left( \int_{\{x-1 < t < x\}} |d\phi|^p dtd\theta \right)^{\frac{1}{p}} \geq c_1 \sup_{\{x-1 < t, < x\}} |\phi(t_1, \theta_1) - \phi(t_2, \theta_2)|$$

Therefore,

$$\sup_{\{x-1 < t, < x\}} |\phi(t_1, \theta_1) - \phi(t_2, \theta_2)| \leq c_1 e^{\delta x} \left\| e_{\mathcal{S}_j} d_{vert} \phi \right\|_{\delta}$$
so
\[
\sup_{\{t_i \leq x\}} |\phi(t_1, \theta_1) - \phi(t_2, \theta_2)| \leq c_1 \frac{e^{\delta x}}{1 - e^{-\delta}} \|e_{S_j} d_{\text{vert}} \phi\|_\delta
\]
The above implies the required estimate:
\[
|\phi(t, \theta) - \phi(-\infty, \theta)| \leq c_1 \frac{e^{\delta t}}{1 - e^{-\delta}} \|e_{S_j} d_{\text{vert}} \phi\|_\delta = \frac{c_1}{1 - e^{-\delta}} \|e_{S_j} d_{\text{vert}} \phi\|_\delta \|e_{S_j} w_0\|
\]
\[\square\]

**Lemma 2.12.** Define the norm \(\|\nu\|_{\delta+k,\delta'}\) by using the \(\|\cdot\|_{\delta'}\) norm on smooth manifold fibers of \(\pi : \tilde{U} \rightarrow U\) and the norm \(\|\cdot\|_{\delta+k}\) on fibers with tropical part not equal to a point:
\[
\|\nu\|_{\delta+k,\delta'} := \max \left\{ \sup_{\pi^{-1}(x)}=\text{point} \left( \int_{\pi^{-1}(x)} \mu U \right) \|w_0^{-\delta'} \nu\|^{\frac{1}{p}} \right\}
\]

Define the norm \(\|\nu\|_{\delta+k,\delta'}^1 = \sup \|\nu\| + \|d_{\text{vert}} \nu\|_{\delta+k,\delta'}\).

1. On any allowable coordinate chart, the norm \(\|\nu\|_{0,\delta}\) is equivalent to the following norm using the lifts \(\tilde{S}\) of sets of strata \(S\) in \(U\):
\[
\|\nu\|_{\delta+k,\delta'} + \max_S \left\| w^{-\delta} \Delta_S \nu \right\|_{\delta+k,\delta'}
\]

2. On any allowable coordinate chart, the norm \(\|\nu\|_{0,\delta}^1\) is equivalent to the following norm:
\[
\|\nu\|_{\delta+k,\delta'} + \max_S \left\| w^{-\delta} \Delta_S \nu \right\|_{\delta+k,\delta'}^1
\]

**Proof:** The strata on which the higher weight \(\delta + \delta'\) is used for the \(\|\cdot\|_{\delta+k,\delta'}\) norm are the strata \(T\) so that \(T^c \neq \emptyset\). Therefore, the norm \(\|\nu\|_{\delta+k,\delta'}\) is equivalent to the following norm
\[
\|\nu\|_{\delta'} + \max_{T^c \neq \emptyset} \|e_T \nu\|_{\delta'+\delta'}
\]

Therefore, to show that \(\|\nu\|_{0,\delta}\) and \(\|\nu\|_{0,\delta}^1\) dominate the two new norms above, it suffices to show that \(\|\nu\|_{0,\delta}\) and \(\|\nu\|_{0,\delta}^1\) dominate \(\|e_T w^{-\delta} \Delta_S \nu\|_{\delta'+\delta}\) and \(\|e_T w^{-\delta} \Delta_S \nu\|_{\delta'+\delta}^1\) respectively.

**Claim:** if \((T)^c \neq \emptyset\) and \(e_T \Delta_S \neq 0\), then
\[
eq_T w_{S_0} w_0 \leq e_T w_{\tilde{S}^c} w_0
\]
(5)
(Here \(\tilde{S}\) is a lifted set of strata and \(S_0\) indicates the set of strata on which \(w_0\) disappears.)

The above claim holds trivially if \(T \in S_0\), as then both \(e_T w_{\tilde{S}^c} w_0\) and \(e_T w_{\tilde{S}^c} w_0\) are 0. Therefore, we may assume without losing generality that \(e_T w_0 = [(\tilde{z}_1)]\).

Suppose now that \(\tilde{z}_0\) is one of the coordinate functions used in the definition of \(w_{\tilde{S}^c} w_0\). For any coordinate function, either \(e_S [\tilde{z}_0] = 0\) or \(e_S [\tilde{z}_0] = [\tilde{z}_0]\).
Because $[\bar{z}^0] = 0$ on $S_0$, if $e_T[\bar{z}^0] \neq 0$, then $\bar{z}^0$ must be equal to $\bar{z}_1\bar{z}^{\alpha'}$ where $\bar{z}^{\alpha'}$ is some lifted coordinate function. More than this, $[\bar{z}^{\alpha'}]$ must disappear on $\tilde{S}$; this is because both $\tilde{S}$ and $\bar{z}^{\alpha'}$ are lifted, so if $[\bar{z}^{\alpha'}]$ is nonzero on some strata $S \in \tilde{S}$, it is nonzero on some strata $S' \in \tilde{S}$ with the same projection so that $[\bar{z}_1] \neq 0$ on $S$, which would contradict the fact that $[\bar{z}^0]$ disappears on $\tilde{S} \cup S_0$. Therefore, $[\bar{z}^{\alpha'}]$ is bounded by some constant times $w_{\tilde{S}}$, which proves the above claim that $e_T w_{\tilde{S} \cup S_0}$ is bounded by a constant times $e_T w_{\bar{S}} w_0$.

Therefore $\|e_T w_{\tilde{S}}^{-\Delta} \Delta_S \nu\|_{\delta' + \delta}$ is dominated by $\|w_{\tilde{S} \cup S_0}^{-\Delta} \Delta_S \nu\|_{\delta'}$ and $\|e_T w_{\tilde{S}}^{-\Delta} \Delta_S \nu\|_{\delta' + \delta}^1$ is dominated by $\|w_{\tilde{S} \cup S_0}^{-\Delta} \Delta_S \nu\|_{\delta'} + \sup |w_{\tilde{S}}^{-\Delta} \Delta_S \nu|$ so our new norms are dominated by $\|\nu\|_{0, \delta}$ and $\|\nu\|_{0, \delta}$ respectively.

We now must show that $\|\nu\|_{0, \delta}$ and $\|\nu\|_{0, \delta}$ are dominated by the new norms above. For $I$ an arbitrary collection of strata, we need to estimate $\|w_I^{-\Delta} \Delta I \nu\|_{\delta}$ with these new norms. Let $\tilde{S}$ be the largest lifted collection of strata so that $\Delta_S \Delta I = \Delta I$. Then $\Delta I = \Delta_S \bigcup S_i$ where $S_i$ are strata in $I$ not contained in $\tilde{S}$, which implies that $S_i \neq \emptyset$.

$$\Delta S_i = \Delta S_{i \cup S_i^c} + \sum_{S_j \in S_i^c} e_{S_j} \Delta S_j$$

Therefore,

$$\Delta I = \sum_{I' \subset (I - \tilde{S})} \left( \prod_{j \in I - \tilde{S} - I'} \sum_{S_j \in S_j^c} e_{S_j} \Delta S_j \right) \Delta_{S \cup I' \cup (I')^c}$$

(6)

Note that if $S_j \in S_j^c$, then $e_{S_j} \Delta S_j \Delta S_0 = e_{S_j} \Delta S_0$. If $\|\nu\|_{\delta'}$ is finite, $\Delta_S \nu = \nu$, so if $\|\nu\|_{\delta'}$ is finite we can replace $e_{S_j} \Delta S_j$ with $e_{S_j}$ in the above equation (6), getting

$$\Delta_I \nu = \sum_{I' \subset (I - \tilde{S})} \left( \prod_{j \in I - \tilde{S} - I'} \sum_{S_j \in S_j^c} e_{S_j} \right) \Delta_{S \cup I' \cup (I')^c} \nu \quad \text{if} \quad \Delta_S \nu = \nu$$

(7)

To bound $\|w_I^{-\Delta} \Delta I \nu\|_{\delta'}$ and $\|w_I^{-\Delta} \Delta I \nu\|_{\delta'}^1$ using the above decomposition of $\Delta_I$, we shall also need the following estimate:

$$cw_I \geq \left( \prod_{j \in I - \tilde{S} - I'} e_{S_j} \right) w_{S \cup I' \cup (I')^c} w_0$$

(8)

To prove the above inequality, suppose that $z^\alpha$ is one of the coordinate functions involved in the right hand side, so $[z^\alpha]$ vanishes on all the strata in $S_0 \cup \tilde{S} \cup I' \cup (I')^c$, but does not vanish on $S_j$. In particular, $[z^\alpha] = 0^0$ on some $S_j \in S_j^c$ and $[z^\alpha] < 0^0$ on $S_j^0 \subset S_0$, where $[\pi](S_j) = [\pi](S_j^c)$, therefore, $[z^\alpha] < 0^0$ on $S_j$. This is valid for all $j \in I - \tilde{S} - I'$. As all other strata in $I$ appear in $\tilde{S} \cup I' \cup (I')^c$, it follows that $[\bar{z}^0]$ vanishes on $I$, so $[\bar{z}^0] \leq cw_I$. The above inequality (8)
follows. Using the decomposition (7), the above inequality (8) and the fact that \( w_I \geq c w_{I_0} \), we get that if \( \Delta_{S_0} \nu = \nu \),

\[
\| w_I^\Delta \Delta_I \nu \|_{\delta'} \leq c \| w_I^\Delta \Delta_{I_0} \nu \|_{\delta'} + c \sum_{I' \subset (I - \delta)} \left\| \left( \prod_{j \in I - \delta - I'}^\text{c} \sum_{S_j \in S_j^I} e_{S_j} \right) \right\| w_I^\delta \Delta_{I' \cup (I')} c w_{I_0}^\delta \Delta_{I_0' \cup (I')} \nu \|_{\delta' + \delta}
\]

All the terms above are dominated by the first new norm, and we have proved that \( \| \nu \|_{0, \delta} \) is dominated by the first new norm.

To dominate \( \| w_I^\Delta \Delta_{\text{vert}} \nu \|_{\delta'} \) by the second new norm, we can use the above inequality. Now dominate \( \sup | w_I^\Delta \Delta | \) using (6) and (8) as follows:

\[
\sup | w_I^\Delta \Delta_I | \leq c \sup | w_I^\Delta \Delta_{I_0} | + c \sum_{I' \subset (I - \delta)} \left\| \left( \prod_{j \in I - \delta - I'}^\text{c} \sum_{S_j \in S_j^I} e_{S_j} \right) \right\| w_I^\delta \Delta_{I' \cup (I')} c w_{I_0}^\delta \Delta_{I_0' \cup (I')} \nu \|_{\delta' + \delta}
\]

We must bound the terms appearing in the sum above in (9). Lemma 2.11 gives that that \( e_{S_j} \Delta S_j \phi \leq c \| e_{S_j} d_{\text{vert}} \phi \|_{\delta' + \delta} \), so

\[
\sup \left\| \left( \prod_{j \in I - \delta - I'} \sum_{S_j \in S_j^I} e_{S_j} \right) \right\| w_I^\delta \Delta_{I' \cup (I')} c w_{I_0}^\delta \Delta_{I_0' \cup (I')} \nu \|_{\delta' + \delta}
\]

This completes the proof that \( \| \nu \|_{0, \delta} \) is dominated by second new norm as required.

We are now ready to define these norms on a vector bundle \( V \) over a family \( \pi_{\mathcal{F}} : \mathcal{C} \rightarrow \mathcal{F} \). To avoid the problems that arise if \( \mathcal{F} \) is not topologically compact, we shall define the notion of an ‘allowable’ family, which is extendable and can be covered by an ‘allowable’ collection of extendable allowable coordinate charts. We shall also need a version of this definition when there is a collection of marked points on our family.

**Definition 2.13.** An allowable collection of coordinate charts on a vector bundle \( V \) over a family \( \pi_{\mathcal{F}} : \mathcal{C} \rightarrow \mathcal{F} \) is a finite collection of extendable allowable coordinate charts \( \pi : U_{\alpha, i} \rightarrow U_\alpha \) satisfying the following conditions:

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1. \[ \pi^{-1}_3(U_\alpha) = \bigcup_i U_{\alpha,i} \]

2. The restriction of the vector bundle \( V \) to \( U_{\alpha,i} \) is \( \mathbb{R}^n \times U_{\alpha,i} \) with the obvious projection.

3. Coordinate change maps and intersections between \( U_{\alpha,i} \) and \( U_{\alpha,j} \) satisfy the following:

   (a) If \( U_{\alpha,i} \) and \( U_{\alpha,j} \) are coordinate charts of type 3a from the definition on page 16 (these are the charts that cover a region where an edge collapses), and \( i \neq j \), then they do not intersect.

   (b) If \( U_{\alpha,i} \) and \( U_{\alpha,j} \) are coordinate charts of type 3b, they are a subset of the product of \( \mathbb{T}_1 \) with \( U_\alpha \), and their smooth part is equal to the product of some subset of \( \lceil \mathbb{T}_1 \rceil \) with \( \lceil U_\alpha \rceil \). In this case, the smooth part of their intersection in either of these coordinate charts is also equal to the product of a subset of \( \lceil \mathbb{T}_1 \rceil \) with \( \lceil U_\alpha \rceil \), and the smooth part of coordinate change map is a product of some map between these subsets of \( \lceil \mathbb{T}_1 \rceil \) with the identity map on \( \lceil U_\alpha \rceil \).

   (c) If \( U_{\alpha,i} \) is a coordinate chart of type 3a and \( U_{\alpha,j} \) is a coordinate chart of type 3b, their intersection is as follows: In \( U_{\alpha,j} \), it is equal to the product of a subset \( O \subset \mathbb{T}_1 \) with \( U_\alpha \). If \( U_{\alpha,i} \) is given by \( \{ |\tilde{z}_1| < c, |\tilde{z}_\beta| < c \} \), then the intersection with \( U_\alpha \) is equal to a subset of \( \{ \tilde{z}_1 \in O' \} \) or \( \{ \tilde{z}_\beta \in O' \} \) where \( O' \subset \{ \frac{1}{2} < |z| < c \} \subset \mathbb{C} \). In either case, we identify this subset with the product \( O' \times U_\alpha \). The transition map in this case is given by the product of a diffeomorphism between \( O \) and \( O' \) and the identity on \( U_\alpha \).

If our family also has a collection of marked points which correspond to smooth nonintersecting sections \( \mathfrak{F} \rightarrow \mathcal{C} \) which do not intersect the edges of any of the curves in our family, then an allowable collection of coordinate chart is an allowable collection as above with the extra conditions that no marked points are inside coordinate charts of type 3a, and in the coordinate charts of type 3b containing marked points, the sections corresponding to these marked points are constant sections \( U_\alpha \rightarrow \mathbb{T}_1 \times U_\alpha \).

It is easy to verify that any single curve in a family of curves is covered by an allowable collection of coordinate charts.

**Definition 2.14.** An allowable family \( \tilde{\mathcal{C}} \xrightarrow{\pi_3} \mathfrak{F} \) is a subset of a family \( \tilde{\mathcal{C}} \xrightarrow{\pi_3} \mathfrak{F}' \) which is the image of an allowable collection of coordinate charts. An allowable family with a vector bundle is the restriction of a vector bundle \( V \) over \( \tilde{\mathcal{C}} \xrightarrow{\pi_2} \mathfrak{F} \) to the image of an allowable collection of coordinate charts.

An allowable family of curves

\[ \begin{array}{ccc}
\tilde{\mathcal{C}} & \xrightarrow{f} & \mathcal{B} \\
\downarrow \pi_3 & & \downarrow \pi_3 \\
\mathfrak{F} & \rightarrow & \mathfrak{G}
\end{array} \]

is a family so that \( \tilde{\mathcal{C}} \rightarrow \mathfrak{F} \) with the vector bundle \( f^* T_{\text{vert}} \mathcal{B} \) is allowable.
**Definition 2.15.** On an allowable family $\hat{f}$ with vector bundle $V$ covered by the allowable collection of coordinate charts $U_{\alpha,i} \rightarrow U_{\alpha}$, define the norms

$$\|\nu\|_{\text{Blah}} := \sum_{U_{\alpha,i}} \|\nu|_{U_{\alpha,i}}\|_{\text{Blah}}$$

for example, $\|\nu\|_{k,\delta} := \sum_{U_{\alpha,i}} \|\nu|_{U_{\alpha,i}}\|_{k,\delta}^1$.

Given a choice of weight $\delta_{\alpha,i}$ for each coordinate chart $U_{\alpha,i}$, defined the norms

$$\|\nu\|_{\text{mixed}}^{\delta} := \sum_{U_{\alpha,i}} \|\nu|_{U_{\alpha,i}}\|_{\delta_{\alpha,i}}$$

$$\|\nu\|_{\text{mixed}}^{1,\delta} := \sum_{U_{\alpha,i}} \|\nu|_{U_{\alpha,i}}\|_{\delta_{\alpha,i}}^1$$

One problem with the norms $\|\cdot\|_{k,\delta}$ which will come up in inductive arguments, is that $w_S \Delta_S \nu$ is only defined on coordinate patches and not globally defined on a fiber. The following definition provides a means of remedying this.

**Definition 2.16.** Given an allowable collection of coordinate charts $U_{\alpha,i} \rightarrow U_{\alpha}$, and a collection $S$ of strata in $[U_{\alpha}]$, define $\tilde{\Delta}_S \nu$ on $\bigcup_i U_{\alpha,i}$ as follows:

1. Choose a smooth cut off function $\rho : \mathbb{C}^* t^k \rightarrow [0,1]$ so that

   $$\rho(z) = \begin{cases} 1 & \text{if } |z| \leq \frac{1}{2} \\ 0 & \text{if } |z| \geq 1 \end{cases}$$

2. On coordinate charts $U_{\alpha,i}$ of type $\mathbb{A}$ we have coordinates $\tilde{z}_1$ and $\tilde{z}_\beta$ so that $|\tilde{z}_1| < c$, $|\tilde{z}_\beta| < c$ and $|\tilde{z}_1 \tilde{z}_\beta^\beta| < \frac{c}{16}$. Use the notation $\hat{S}$ for the lift of $S$ to this chart, $S^+$ for the collection of strata in $\hat{S}$ so that $[\tilde{z}_1] = t^0$, and $S^-$ for the collection of strata in $\hat{S}$ so that $[\tilde{z}_\beta] = t^0$. Define $\hat{\Delta}_S \nu$ on this chart by:

   $$\hat{\Delta}_S \nu := \rho \left( \frac{2\tilde{z}_1}{c} \right) \rho \left( \frac{2\tilde{z}_\beta}{c} \right) \Delta_{\hat{S}} \nu$$

   $$+ \left( 1 - \rho \left( \frac{2\tilde{z}_1}{c} \right) \right) \Delta_{S^+} \nu$$

   $$+ \left( 1 - \rho \left( \frac{2\tilde{z}_\beta}{c} \right) \right) \Delta_{S^-} \nu$$

3. On all other coordinate charts $U_{\alpha,i}$ of type $\mathbb{B}$ let $\hat{S}$ denote the lift of $S$ to this coordinate chart, and define

   $$\hat{\Delta}_S \nu := \Delta_{\hat{S}} \nu$$

It follows from the types of transition functions allowed for allowable collections of coordinate charts that $\hat{\Delta}_S \nu$ is well defined on $\bigcup_i U_{\alpha,i}$ and smooth if $\nu$ is smooth. We can now state a version of Lemma 2.12 which is global in the fiber.
Lemma 2.17. Restricted to a collection of allowable coordinate charts $U_{\alpha,i}$ over a single chart $U_\alpha$, the norm $\|\nu\|_{\delta+\delta'}$ is equivalent to the norm

$$\|\nu\|_{\delta+\delta'} + \max_S \left\| w_S^{-\delta} \Delta_S \nu \right\|_{\delta+\delta'}$$

and the norm $\|\nu\|_{0,\delta}$ is equivalent to the norm

$$\|\nu\|_{0,\delta} + \max_S \left\| w_S^{-\delta} \Delta_S \nu \right\|_{0,\delta}$$

In the above, the maximum is taken over all collections of strata $S$ in $[U_\alpha]$, and $w_S$ indicates the lift of the weighting function $w_S$ on $U_\alpha$, (which is equal to $w_T$ in each of the coordinate charts $U_{\alpha,i}$). The norm $\|\cdot\|_{\delta+\delta'}$ is defined in Lemma 2.12 on page 25.

Proof:

On all coordinate charts of type $3b$ this lemma follows immediately from Lemma 2.12 as in this case $\Delta_S = \tilde{\Delta}_S$.

On a coordinate chart of type $3a$ where $|\tilde{z}| < c$ and $|\tilde{z}\beta| < c$, using the notation from Definition 2.16 we have that

$$\Delta_S \nu - \tilde{\Delta}_S \nu = \left( 1 - \rho \left( \frac{2\tilde{z}}{c} \right) \right) \Delta_S \nu - \left( 1 - \rho \left( \frac{2\tilde{z}}{c} \right) \right) \tilde{\Delta}_S \nu$$

Note that given any strata $T$ for which $T^c \neq \emptyset$, $e_T [\tilde{z} \beta \tilde{z} \beta] = 0$. If $e_T [\tilde{z} \tilde{z}] = 0$, then $e_T \tilde{\rho}^{T} = 0$ and $e_T \tilde{\Delta}_S = e_T \Delta_S$. Therefore, for any strata $T$ so that $T^c \neq \emptyset$, in our coordinate chart

$$e_T \tilde{\Delta}_S = e_T \Delta_S$$

The above equation (11) holds on any fiber of our coordinate chart $U_{\alpha,i} \rightarrow U_\alpha$ with non trivial tropical part, so (11) implies that

$$\left\| w_S^{-\delta} \Delta_S \nu - w_S^{-\delta} \tilde{\Delta}_S \nu \right\|_{\delta+\delta'} = \left\| w_S^{-\delta} \Delta_S \nu - w_S^{-\delta} \Delta_S \nu \right\|_{\delta+\delta'}$$

and

$$\left\| w_S^{-\delta} \Delta_S \nu - w_S^{-\delta} \tilde{\Delta}_S \nu \right\|_{0,\delta} = \left\| w_S^{-\delta} \Delta_S \nu - w_S^{-\delta} \Delta_S \nu \right\|_{0,\delta}$$
Equation (10) and the observation that implies that there exists some constant $c'$ so that on this coordinate chart,
\[ \left\| \hat{w}^{-\delta}_S \hat{\Delta} S \nu - \hat{w}^{-\delta}_S \Delta_S \nu \right\|_{\delta'} \leq c' \sum_{T \in \hat{S} - \hat{S}^+} \left\| \hat{w}^{-\delta}_S \hat{\rho}^+ e_T \Delta_{S^+S} \nu \right\|_{\delta'} + \sum_{T \in \hat{S} - \hat{S}^-} \left\| \hat{w}^{-\delta}_S \hat{\rho}^- e_T \Delta_{S^-S} \nu \right\|_{\delta'}.
\]

Use the notation $\hat{S}$ to indicate the largest lifted set of strata in $S^+$. If $T \in \hat{S} - \hat{S}^+$, then $e_T \Delta_{S^+} = e_T \Delta_{\hat{S}} \Delta_{S^0}$, where $S_0$ is the set of strata where $[\check{z}_1] = 0 = [\check{z}^\beta]$. To see this, note that if $T' \in S^+$, then either $T' \in \hat{S}$ or $e_T e_T \Delta_{S_0} = 0$ because $e_T [\check{z}_1] = 0$ and $e_T [\check{z}^\beta] = 0$. Therefore, $e_T \Delta_{S^+} e_T \Delta_{\hat{S}} \Delta_{S_0} = e_T \Delta_{\hat{S}} \Delta_{S_0}$. Similarly, if $S \in S_0$, then $e_S e_T \Delta_{S^+} = 0$, because if $T^+$ indicates the strata in $\hat{S}^+$ with the same projection as $T$, then $e_S e_T = e_S e_T$. Therefore, $e_T \Delta_{S^+} e_T \Delta_{\hat{S}} \Delta_{S_0} = e_T \Delta_{S^+}$, so $e_T \Delta_{S^+} = e_T \Delta_{\hat{S}} \Delta_{S_0}$. Similarly, if $T \in \hat{S} - \hat{S}^+$, then $e_T \Delta_{S^-} = e_T \Delta_{\hat{S}} \Delta_{S_0}$. Therefore, we get
\[ \left\| \hat{w}^{-\delta}_S \hat{\Delta} S \nu - \hat{w}^{-\delta}_S \Delta_S \nu \right\|_{\delta'} \leq c' \sum_{T \in \hat{S} - \hat{S}^+} \left\| \hat{w}^{-\delta}_S \hat{\rho}^+ e_T \Delta_{S^+S_0} \nu \right\|_{\delta'} + \sum_{T \in \hat{S} - \hat{S}^-} \left\| \hat{w}^{-\delta}_S \hat{\rho}^- e_T \Delta_{S^-S_0} \nu \right\|_{\delta'}.
\]

Claim: if $T \in \hat{S} - \hat{S}^+$, then $\hat{w}^{-\delta}_S \hat{\rho}^+$ is bounded by a constant times $e_T \hat{w}^{-\delta}_S w_0^{-\delta}$. To prove this claim, it suffices to show that $\hat{\rho}^+ e_T \hat{w} \nu w_0$ vanishes on all the strata in $\hat{S}$. If $S$ is a strata in $\hat{S}$, then either $S \in \hat{S}$, $S \in \hat{S}_0$, $e_S [\check{z}_1] = 0$, or $e_S [\check{z}^\beta] = 0$. In the first two cases, $e_S \hat{w} \nu w_0 = 0$. If $e_S [\check{z}_1] = 0$, then $e_S \hat{\rho}^+ = 0$. If $e_S [\check{z}^\beta] = 0$, then as $e_T [\check{z}_1] = 0$, we get $e_S e_T \nu w_0 = 0$. Therefore $e_S \hat{\rho}^+ e_T \hat{w} \nu w_0 = 0$, and the above claim follows. Similarly, if $T \in \hat{S} - \hat{S}^-$, then $\hat{w}^{-\delta}_S \hat{\rho}^-$ is bounded by a constant times $e_T \hat{w}^{-\delta}_S w_0^{-\delta}$. Therefore,
\[ \left\| \hat{w}^{-\delta}_S \hat{\Delta} S \nu - \hat{w}^{-\delta}_S \Delta_S \nu \right\|_{\delta'} \leq c'' \sum_{T \in \hat{S} - \hat{S}} \left\| e_T \hat{w}^{-\delta}_S w_0^{-\delta} \Delta_{S^0S^0} \nu \right\|_{\delta'}.
\]

As all the strata $T$ in the above inequality satisfy $T^c \neq \emptyset$, on the strata $T$ the norm $\left\| \cdot \right\|_{\delta', \delta'}$ always uses the higher weight $w_0^{-\delta - \delta'}$. Therefore, we get that
\[ \left\| \hat{w}^{-\delta}_S \hat{\Delta} S \nu - \hat{w}^{-\delta}_S \Delta_S \nu \right\|_{\delta'} \leq c''' \left\| \hat{w}^{-\delta}_S \hat{\Delta} S \nu \right\|_{\delta', \delta'}.
\]

The set of strata $\hat{S}$ is the lift of some set of strata which we shall again call $\check{S}$, because each of these strata lifts to a unique strata. With this slight abuse of notation $\hat{S}^+ = \hat{S} = \check{S}^-$, so $\hat{\Delta} \check{S} = \Delta_{\hat{S}}$, and we get
\[ \left\| \hat{w}^{-\delta}_S \hat{\Delta} S \nu - \hat{w}^{-\delta}_S \Delta_S \nu \right\|_{\delta'} \leq c''' \left\| \hat{w}^{-\delta}_S \hat{\Delta} S \nu \right\|_{\delta', \delta'}.
\]

This together with Lemma 2.12 proves that the norm $\left\| \nu \right\|_{0, \delta}$ is equivalent to the norm
As we already have the required estimates for the part of the $\|\nu\|_\delta^1$ norm involving $\|d_{\text{vert}}\nu\|_\delta$, it remains to estimate
\[
\sup \left| w_S^{-\delta} \tilde{\Delta} S \nu - w_S^{-\delta} \Delta S \nu \right|
\]
on our coordinate chart. As argued for the $\|\cdot\|_\delta$ norm above, we get

\[
\sup \left| w_S^{-\delta} \tilde{\Delta} S \nu - w_S^{-\delta} \Delta S \nu \right| \leq c' \sum_{T \in \tilde{S} - S} \sup \left| e_T w_S^{-\delta} \Delta S_{\text{dist}} \nu \right|
\]

We can estimate the right hand side of the above inequality with Lemma 2.11 as $S_0$ will contain some strata who’s complement contains $T$. Therefore,

\[
\sup \left| w_S^{-\delta} \tilde{\Delta} S \nu - w_S^{-\delta} \Delta S \nu \right| \leq c' \sum_{T \in \tilde{S} - S} \left\| e_T w_S^{-\delta} \Delta S_{\text{dist}} \nu \right\|_\delta
\]

So, as argued above, we get the two inequalities

\[
\sup \left| w_S^{-\delta} \tilde{\Delta} S \nu - w_S^{-\delta} \Delta S \nu \right| \leq c'' \left\| w_S^{-\delta} \tilde{\Delta} S_{\text{dist}} \nu \right\|_\delta
\]  \hspace{1cm} (16)

and

\[
\sup \left| w_S^{-\delta} \tilde{\Delta} S \nu - w_S^{-\delta} \Delta S \nu \right| \leq c'' \left\| w_S^{-\delta} \Delta S_{\text{dist}} \nu \right\|_\delta
\]  \hspace{1cm} (17)

This together with Lemma 2.12 completes the proof of our lemma.

The above proof contains the inequalities (14), (15), (16) and (17), which imply the following estimates which will be useful later:

**Lemma 2.18.** Let $\hat{\mathcal{S}}$ denote the subset of $\tilde{\mathcal{S}}$ consisting of all strata $T \in \tilde{\mathcal{S}}$ so that $T^c = \emptyset$, and also let $\check{\mathcal{S}}$ denote the corresponding subset of $\mathcal{S}$. Then

\[
\left\| w_S^{-\delta} \tilde{\Delta} S \nu - w_S^{-\delta} \Delta S \nu \right\|_{\delta \& \delta'} \leq c \left\| w_S^{-\delta} \tilde{\Delta} S \nu \right\|_{\delta \& \delta'} = c \left\| w_S^{-\delta} \Delta S \nu \right\|_{\delta \& \delta'}
\]

and

\[
\left\| w_S^{-\delta} \tilde{\Delta} S \nu - w_S^{-\delta} \Delta S \nu \right\|_{\delta \& \delta'}^{1} \leq c \left\| w_S^{-\delta} \tilde{\Delta} S \nu \right\|_{\delta \& \delta'}^{1} = c \left\| w_S^{-\delta} \Delta S \nu \right\|_{\delta \& \delta'}^{1}
\]

**3 Analysis of $\tilde{\partial}$ equation in families**

**Definition 3.1.** Consider an allowable smooth or $C^\infty_{\tilde{\Delta}}$ family $\hat{f}$ with some (possibly empty) disjoint collection of marked points on $\mathcal{C} \rightarrow \mathcal{F}$, corresponding to nonintersecting smooth or $C^\infty_{\tilde{\Delta}}$ sections $\mathcal{F} \rightarrow \mathcal{C}$ which stay away from the edges of the curves in $\mathcal{F}$. (In other words, the marked points are in the interior of the smooth part of the curves in our family.)

\[
\begin{array}{ccc}
\mathcal{C} & \hat{f} & \mathcal{F} \\
\downarrow \pi_\mathcal{F} & & \downarrow \pi_\mathcal{C} \\
\mathcal{F} & \rightarrow & \mathcal{C}
\end{array}
\]

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Define the Banach space $X_{\text{blah}}$ to be the space of sections of $\hat{f}^\ast T_{\text{vert}}\hat{\mathcal{B}}$ with norm $\| \cdot \|_{\text{blah}}^1$ bounded which vanish at the marked points.

Define $Y_{\text{blah}}$ to be the space of sections of the vector bundle $Y(\hat{f})$ with norm $\| \cdot \|_{\text{blah}}$. We can similarly define $Y_\delta(p)$.

Given a point $p$ in $\mathfrak{F}$, we can restrict our family $\hat{f}$ and the other data to the fiber of $\mathfrak{C}$ over $p$. Define $X_\delta(p)$ to be the corresponding Banach space $X_\delta$ with this restricted data.

Note that if $\nu \in X_{k,\delta}$, then $\nu(p) \in X_{\delta,\delta'}(p)$.

**Definition 3.2.** Given a smooth or $C^{\infty,\delta}$ family,

\[
\begin{array}{ccc}
\mathfrak{C} & \xrightarrow{\hat{f}} & \mathfrak{B} \\
\downarrow & & \downarrow \\
\mathfrak{F} & \longrightarrow & \mathfrak{G}
\end{array}
\]

a choice of trivialization is

1. a smooth or $C^{\infty,\delta}$ map

\[
\begin{array}{ccc}
\hat{f}^\ast T_{\text{vert}}\mathfrak{B} & \xrightarrow{\Phi} & \hat{\mathfrak{B}} \\
\downarrow & & \downarrow \\
\mathfrak{F} & \longrightarrow & \mathfrak{G}
\end{array}
\]

so that $F$ restricted to the zero section is equal to $\hat{f}$, and $TF$ restricted to the natural inclusion of $\hat{f}^\ast T_{\text{vert}}\mathfrak{B}$ over the zero section is equal to the identity.

2. A smooth or $C^{\infty,\delta}$ isomorphism from the bundle $F^\ast T_{\text{vert}}\mathfrak{B}$ to the vertical tangent bundle of $\hat{f}^\ast T_{\text{vert}}\mathfrak{B}$ which preserves $J$. In other words, if $\pi : \hat{f}^\ast T_{\text{vert}}\mathfrak{B} \longrightarrow \mathfrak{C}$ denotes the vector bundle projection, a smooth or $C^{\infty,\delta}$ isomorphism between $F^\ast T_{\text{vert}}\mathfrak{B}$ and $\pi^\ast \hat{f}^\ast T_{\text{vert}}\mathfrak{B}$ which preserves the almost complex structure $J$ on $T_{\text{vert}}\mathfrak{B}$. This can be written as a smooth or $C^{\infty,\delta}$ vector bundle map

\[
\begin{array}{ccc}
F^\ast T_{\text{vert}}\mathfrak{B} & \xrightarrow{\Phi} & \hat{f}^\ast T_{\text{vert}}\mathfrak{B} \\
\downarrow & & \downarrow \\
\hat{f}^\ast T_{\text{vert}}\mathfrak{B} & \longrightarrow & \mathfrak{C}
\end{array}
\]

A trivialization allows us to define $\bar{\partial}$ of a section $\nu : \mathfrak{C} \longrightarrow \hat{f}^\ast T_{\text{vert}}\mathfrak{B}$ as follows: $F \circ \nu$ is a family of maps $\hat{\mathfrak{C}} \longrightarrow \mathfrak{B}$, so $\bar{\partial}(F \circ \nu)$ is a section of $Y(F \circ \nu) = \left(\left(T^\ast \hat{\mathfrak{C}}/\pi_0^\ast T^\ast \hat{\mathfrak{F}}\right) \otimes \left((F \circ \nu)^\ast T_{\text{vert}}\mathfrak{B}\right)\right)^{(0,1)}$. Applying the map $\Phi$ to the second component of this tensor product gives an identification of $Y(F \circ \nu)$ with $Y(\hat{f})$, so we may consider $\bar{\partial}(F \circ \nu)$ to be a section of $Y(\hat{f})$. Define $\partial \nu$ to be this section of $Y(\hat{f})$.

A trivialization on an extendible family is always assumed to be extendible.

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For example, a choice of smooth $J$ preserving and fiber preserving connection defines a trivialization where $F$ is given by exponentiation, and $\Phi$ is given by parallel transport along a linear path to the zero section. In the theorems that follow, a choice of trivialization will be assumed.

Recall from the definition on page 7 that a simple perturbation parametrized by a family $\hat{f}$ is a $C^\infty$ section $P$ of the bundle $\left( T^*\hat{\mathcal{C}}/\pi^*_2T^*\hat{\mathcal{S}} \right) \otimes \left( T_{\text{vert}}\hat{\mathcal{B}} \right)$ over $\hat{\mathcal{C}} \times \hat{\mathcal{B}}$ with the same regularity as $\hat{f}$ which vanishes on all edges of the curves which are the fibers of $\hat{\mathcal{C}}$.

**Definition 3.3.** Given a trivialization for $\hat{f}$, a simple perturbation of $\bar{\partial}$ is a map $\bar{\partial}'$ from sections of $\hat{\mathcal{C}} \rightarrow \hat{f}^*T_{\text{vert}}\hat{\mathcal{B}}$ to sections of $Y(\hat{f})$ so that

$$\bar{\partial}'\nu = \bar{\partial}(\nu) + \Phi(P(\text{id}, F(\nu)))$$

Where $P$ is a simple perturbation parametrized by $\hat{f}$, and $\Phi$ and $F$ are maps from the trivialization of $\hat{f}$ in the notation of Definition 3.2 above.

Whenever $\hat{f}$ is extendible, the section $P$ is always chosen to be extendible.

Use the following topology on the space of simple perturbations of $\bar{\partial}$ for a given family and trivialization: The perturbation $\bar{\partial}'_1$ is $C^{k,\delta}$ close to $\bar{\partial}'_2$ if the associated sections $P_1$ and $P_2$ of $\left( T^*\hat{\mathcal{C}}/\pi^*_2T^*\hat{\mathcal{S}} \right) \otimes \left( T_{\text{vert}}\hat{\mathcal{B}} \right)$ over $\hat{\mathcal{C}} \times \hat{\mathcal{B}}$ are $C^{k,\delta}$ close.

We shall prove regularity theorems for any simple perturbation $\bar{\partial}'$ of $\bar{\partial}$ below. Note that this of course includes the case that $\bar{\partial}' = \bar{\partial}$.

**Lemma 3.4.** If $\hat{f}$ is an allowable family of curves with a choice of trivialization, then in any one of the allowable collection of coordinate charts on $\hat{f}^*T_{\text{vert}}\hat{\mathcal{B}}$ in the form $\mathbb{R}^n \times U$, if we identify a section with a map $\nu : U \rightarrow \mathbb{R}^n$, there is the following formula for $\bar{\partial}'$:

$$\bar{\partial}'\nu(u) = E(\nu(u), u) + H(\nu(u), u)(d_{\text{vert}}\nu)$$

where $E$ is an extendible function on $\hat{f}^*T_{\text{vert}}\hat{\mathcal{B}}$ with values in $Y(\hat{f})$ and $H$ is an extendible tensor on $\hat{f}^*T_{\text{vert}}\hat{\mathcal{B}}$ valued in the pullback of $\hat{f}^*T_{\text{vert}}\hat{\mathcal{B}} \otimes Y(\hat{f})$. If $f$ is $C^{\infty,\delta}$ or smooth, then $E$ and $H$ are $C^{\infty,\delta}$ or smooth, and $E$ vanishes on the edges of curves which are the fibers of $\hat{\mathcal{C}} \rightarrow \hat{\mathcal{S}}$. At the zero section, $H(\theta) = \frac{1}{2}(\theta + J \circ \theta \circ j)$.

**Proof:** The function $E(x,u)$ is given by $\bar{\partial}'$ of the section $u \mapsto (x,u)$ which is constant in the coordinate patch. This vanishes on the edges of fibers of $\hat{\mathcal{C}} \rightarrow \hat{\mathcal{S}}$, and is smooth or $C^{\infty,\delta}$ if $\hat{f}$ is. $E$ is also extendible because it has an extension with the same definition to any extension of our coordinate chart.

The tensor $H$ is given by the formula

$$H(x,u)(\theta) := \frac{1}{2}\left( \Phi(dF(x,u)(\theta)) + J \circ \Phi(dF(x,u)(\theta \circ j)) \right)$$

where $\Phi$ and $F$ are as in Definition 3.2. Because $\Phi$ and $F$ are smooth or $C^{\infty,\delta}$ if $\hat{f}$ is, $H$ is smooth or $C^{\infty,\delta}$ too. As noted above, $H$ is extendible. The fact that $\bar{\partial}'\nu$ obeys the above formula follows by direct computation.
Theorem 3.5. If \( \hat{f} \) is an allowable \( C^{\infty, \delta_0} \) family where \( \delta_0 > \delta + \delta' \), then \( \partial' \) defines a \( C^1 \) map from \( X_{k, \delta} \) to \( Y_{k, \delta} \).

Proof: Lemma 3.4 tells us that in local coordinates,
\[
\partial' \nu(u) = E(\nu(u), u) + H(\nu(u), u)(d_{vert} \nu)
\]

By using Lemma 2.7 part 3 from page 19 we may estimate in this coordinate chart
\[
\| \partial' \nu \|_{k, \delta} \leq \| E(\nu(u), u) \|_{k, \delta} + c \| H(\nu(u), u) \|_{k, \delta}^{1/2} \| d_{vert} \nu \|_{k, \delta}
\]

Lemma 2.8 together with Lemma 3.4 from pages 21 and 34 imply that the terms above \( \| E(\nu) \|_{k, \delta} \) and \( \| H(\nu) \|_{k, \delta} \) are bounded if \( \| \nu \|_{k, \delta} \) is bounded. Therefore, \( \| \partial' \nu \|_{k, \delta} \) is bounded, and \( \partial' \) gives a well defined map from \( X_{k, \delta} \) to \( Y_{k, \delta} \).

The derivative is given by the formula
\[
D\partial'(\nu)(\psi)(u) = DE(\nu, u)(\psi(u)) + DH(\nu, u)(\psi(u))(d_{vert} \nu) + H(\nu, u)(d_{vert} \psi)
\]

Using Lemma 2.7 part 3 estimate
\[
\| D\partial'(\nu)(\psi) \|_{k, \delta} \leq c \left( \| DE(\nu) \|_{k, \delta} + \| DH(\nu) \|_{k, \delta}^{1/2} \| d_{vert} \nu \|_{k, \delta} + \| H(\nu) \|_{k, \delta}^{1/2} \right) \| \psi \|_{k, \delta}
\]
The terms \( \| H(\nu) \|_{k, \delta}^{1/2} \), \( \| DH(\nu) \|_{k, \delta} \) and \( \| DE(\nu) \|_{k, \delta} \) are bounded by Lemma 2.8, therefore \( D\partial'(\nu) \) defines a bounded map from \( X_{k, \delta} \) to \( Y_{k, \delta} \). We now must prove that \( D\partial'(\nu) \) is continuous in \( \nu \).

\[
D\partial'(\nu_1)(\psi) - D\partial'(\nu_2)(\psi) = (DE(\nu_1) - DE(\nu_2))(\psi) + (H(\nu_1) - H(\nu_2))(d_{vert} \psi) + DH(\nu_1)(d_{vert} \nu_1 - d_{vert} \nu_2) + (DH(\nu_1)(d_{vert} \nu_1 - d_{vert} \nu_2))\]

so using Lemma 2.7 part 3
\[
\| D\partial'(\nu_1)(\psi) - D\partial'(\nu_2)(\psi) \|_{k, \delta} \leq \| DE(\nu_1) - DE(\nu_2) \|_{k, \delta} \| \psi \|_{k, \delta}^{1/2} + \| H(\nu_1) - H(\nu_2) \|_{k, \delta}^{1/2} \| d_{vert} \psi \|_{k, \delta} + \| DH(\nu_1) \|_{k, \delta} \| \psi \|_{k, \delta}^{1/2} \| d_{vert} \nu_1 - d_{vert} \nu_2 \|_{k, \delta} + \| DH(\nu_1) - DH(\nu_2) \|_{k, \delta}^{1/2} \| d_{vert} \nu_2 \|_{k, \delta} \| \psi \|_{k, \delta}^{1/2}
\]

The term \( DE \) is \( C^{\infty, \delta_0} \) and \( \Delta_S, DE = DE \), so Corollary 2.9 implies that for a fixed \( \nu_1 \), the term \( \| DE(\nu_1) - DE(\nu_2) \|_{k, \delta} \) converges to zero as \( \| \nu_1 - \nu_2 \|_{k, \delta} \rightarrow 0 \), similarly, Corollary 2.7 tells us that \( \| H(\nu_1) - H(\nu_2) \|_{k, \delta} \) and \( \| DH(\nu_1) - DH(\nu_2) \|_{k, \delta} \) converge to zero as \( \nu_2 \rightarrow \nu_1 \) in \( \| \cdot \|_{k, \delta} \). Lemma 2.8 implies that \( \| DH(\nu_1) \|_{k, \delta} \) is bounded. Therefore, \( D\partial'(\nu_2) \rightarrow D\partial'(\nu_1) \) as \( \nu_2 \rightarrow \nu_1 \) in \( \| \cdot \|_{k, \delta} \), so \( \partial' : X_{k, \delta} \rightarrow Y_{k, \delta} \) is \( C^1 \) as required.

The formula 1.18 for \( D\partial' \) in the proof above shows that it induces a map on \( X_{\delta}(p) \). In other words, the restriction of \( D\partial'(\nu) \) to the curve which is the fiber over \( p \), \( D\partial'(\nu)(p) \) only depends on the restriction of \( \nu \) to this curve, \( \nu(p) \). Use the notation \( D\partial'(p) \) to refer to the restriction of \( D\partial' \) at the zero section to the curve over \( p \).
Lemma 3.6. For any allowable $C^\infty, \delta_0$ family of curves, and a choice of weight $0 < \delta < \delta_0 \leq 1$, the linearization of $\bar{\partial}'$ at the zero section restricted to any curve

$$D\bar{\partial}'(p) : X_\delta(p) \to Y_\delta(p)$$

is Fredholm.

More generally, given a choice of weight $0 < \delta < \delta_0$ for every coordinate chart in some allowable collection of coordinate charts on our family of curves, we may use the norms with mixed weights defined on page 26 and

$$D\bar{\partial}'(p) : X_{\text{mixed}}\delta(p) \to Y_{\text{mixed}}\delta(p)$$

is Fredholm.

Proof:

Note that the norm $\|\cdot\|_\delta$ restricted to a smooth component of an exploded curve is equivalent to the norm $L^p$ with exponential weight on the cylindrical ends given by $\delta$, and $\|\cdot\|_1\delta$ is equivalent to the $L^p$ norm with the same exponential weight on derivatives plus the sup norm. The mixed version of these norms is similar, but used different weights at different punctures. So $X_\delta(p)$ or $X_{\text{mixed}}\delta(p)$ restricted to a smooth component is equal to the corresponding Sobolev space $L^2_p$ with exponential weights $\delta$ at punctures plus a finite dimensional space allowing sections which are asymptotic to constants instead of 0 at cylindrical ends (restricted to the subspace which vanishes on our marked points if appropriate). From the proof of theorem 3.5, we have the following formula for $D\bar{\partial}'(p)$ in local coordinates

$$D\bar{\partial}'(\nu_0(p))(\psi) = DE(\nu_0(p))(\psi) + DH(\nu_0(p))(\psi)(d_{\text{vert}}\nu_0(p)) + H(\nu_0(p))(d_{\text{vert}}\psi)$$

Here $\nu_0$ indicates the zero section, so the middle term is 0. The first term is a compact operator if $\delta < \delta_0$, because restricted to smooth components $DE(\nu_0)$ is smooth and decays exponentially on cylindrical ends with weight $\delta_0 > \delta$. As $\nu_0(p) = 0$, Lemma 3.4 states that the last term is equal to the linear $\bar{\partial}\psi$. It is well known (and proved in [3]) that for $0 < \delta < 1$, $\bar{\partial}$ is a Fredholm operator on the above weighted Sobolev spaces restricted to any smooth component. It follows that $\bar{\partial}$ is Fredholm from $X_\delta(p)$ to $Y_\delta(p)$, and $X_{\text{mixed}}\delta(p)$ to $Y_{\text{mixed}}\delta(p)$. Therefore $D\bar{\partial}'(p)$ is the Fredholm operator $\bar{\partial}$ plus some compact operator, so it is Fredholm.

□

Lemma 3.7. The index of $D\bar{\partial}'(p) : X_\delta(p) \to Y_\delta(p)$ or $D\bar{\partial}'(p) : X_{\text{mixed}}\delta \to Y_{\text{mixed}}\delta$ does not change in connected families, and does not depend on the choice of weights $0 < \delta < \delta_0 \leq 1$.

Proof: This is just the index of the usual $\bar{\partial}$ operator. If the relative dimension of the target family $\hat{B} \to \mathfrak{g}$ is $2n$, the index is equal to the sum of the indexes of each smooth component minus $2n$ times the number of internal edges. The index for each smooth component is equal to $2c_1 - 2n(g_s - 1)$ where $c_1$ is the first Chern number of the pullback of $(T_{\text{vert}}\hat{B}, J)$ to that component, $g$ is the genus of that component, and $s$ is the number of marked points on that

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component where sections of $X_\delta(p)$ are required to vanish. The sum over all components of $c_1$ and $s$ does not change in connected families. The sum over all components of $(g - 1)$ plus the number of internal edges is equal to the total genus minus 1. This does not change in connected families. Therefore, the total index is invariant in smooth families and equal to

$$2c_1 - 2n(g + s - 1)$$

where $c_1$ is the sum of the first Chern number of the pull back of $(T_{vert} \hat{B}, J)$ to each smooth component, $g$ is the total genus, and $s$ is the number of marked points where sections of $X_\delta$ are required to vanish.

$\square$

**Definition 3.8.** A smooth or $C^\infty_\delta$ pre obstruction model $(\hat{f}, V)$ is given by

1. an allowable smooth or $C^\infty_\delta$ family

\[
\begin{array}{ccc}
(\hat{C}, j) & \xrightarrow{\hat{f}} & (\hat{B}, J) \\
\downarrow \pi_{\hat{B}} & & \downarrow \pi_{\hat{B}} \\
\hat{\mathcal{G}} & \longrightarrow & \mathcal{G}
\end{array}
\]

2. A choice of trivialization for $\hat{f}$ in the sense of definition

3. a vector bundle $V$ over $\hat{\mathcal{G}}$

\[
\begin{array}{c}
V \\
\downarrow \\
\hat{\mathcal{G}}
\end{array}
\]

4. a smooth or $C^\infty_\delta$ map of vector bundles over $\hat{C}$

\[
\pi_{\hat{\mathcal{G}}}^*(V) \longrightarrow Y(\hat{f}) := \left( \left( T^* \hat{C} / \pi_{\hat{\mathcal{G}}}^* T^* \hat{\mathcal{G}} \right) \otimes \left( \hat{f}^* T_{vert} \hat{B} \right) \right)^{0,1}
\]

which vanishes on the edges of curves in $\hat{C} \to \hat{\mathcal{G}}$. The above map must be non trivial in the sense that for any nonzero vector in $V$, there exists a choice of lift to a vector in $\pi_{\hat{\mathcal{G}}}^*(V)$ which is not sent to 0. (Said differently, a point $(p, v) \in V$ corresponds to a section of $\pi_{\hat{\mathcal{G}}}^*(V)$ over the curve $\hat{f}$ which is the inverse image of $p$. This is sent by the above map to a section of the bundle $Y(f)$. This section is the zero section if and only if $v$ is 0.)

**Definition 3.9.** Any $C^\infty_\delta$ pre obstruction model $(\hat{f}, V)$ defines a closed sub space $Y_{k,\delta} \subset Y_{k,\delta}$ consisting of all sections in $Y_{k,\delta}$ which are contained in $V$. Define the natural projection

\[
\pi_V : Y_{k,\delta} \rightarrow \frac{Y_{k,\delta}}{V_{k,\delta}} := Y/V_{k,\delta}
\]

Similarly define the restriction of $\pi_V$ to $Y_{\delta}(p)$. Note that in this case, the projection $\pi_V$ has finite dimensional kernel equal to $V$ restricted to $p$, so

\[
\pi_V \circ D\bar{\partial}'(p) : X(p)_{\delta} \longrightarrow Y/V_{\delta}(p)
\]
is still Fredholm.

We shall now prove a standard gluing theorem. First, we shall describe a 'gluing' and 'cutting' map.

**Lemma 3.10.** Given a $C^{\infty, \delta_0}$ pre obstruction model $(\hat{f}, V)$, any point $p \in \mathfrak{F}(\hat{f})$ and any finite collection of weights $\delta < \delta_0$, there exists a topological neighborhood $U$ of $p$ and for all $p' \in U$ bounded linear maps

$$G : X_\delta(p) \rightarrow X_\delta(p')$$
$$G : Y_\delta(p) \rightarrow Y_\delta(p')$$
$$C : Y_\delta(p') \rightarrow Y_\delta(p)$$

so that

1. The maps $G$ and $C$ are well defined restricted to some allowable sub collection of coordinate charts covering the total space of our family $\hat{f}$ restricted to $U$.

2. If $\delta_1 \leq \delta_2$, the map $G : X_{\delta_2}(p) \rightarrow X_{\delta_2}(p')$ is the restriction of the map $G : X_{\delta_1}(p) \rightarrow X_{\delta_1}(p')$, the map $G : Y_{\delta_2}(p) \rightarrow Y_{\delta_2}(p')$ is the restriction of the map $G : Y_{\delta_1}(p) \rightarrow Y_{\delta_1}(p')$, and the map $C : Y_{\delta_2}(p') \rightarrow Y_{\delta_1}(p)$ is the restriction of the map $C : Y_{\delta_1}(p') \rightarrow Y_{\delta_1}(p)$.

3. The norms of $G$ and $C$ are bounded uniformly independent of $p' \in U$.

4. $G$ is a left inverse to $C$. In other words, $G \circ C : Y_{\delta}(p') \rightarrow Y_{\delta}(p) \rightarrow Y_{\delta}(p')$ is the identity.

5. For any $\epsilon > 0$, there exists a topologically open neighborhood of $p$ so that for all $p'$ in this neighborhood:

   (a) If $\bar{D}\partial'$ indicates the linearization at the zero section,

   $$\| \bar{D}\partial' \circ G\psi - G \circ \bar{D}\partial' \psi \|_\delta \leq \epsilon \|\psi\|_\delta^1$$

   (b) For all $v \in V(p)$,

   $$\|\pi_{V}Gv\|_\delta \leq \epsilon \|v\|_\delta$$

   Note that the fact that these estimates hold for any finite choice of weights $\delta < \delta_0$ implies that the above estimates also hold for the mixed norms $\|\cdot\|_{\text{mixed} \; \delta}$ and $\|\cdot\|_{\text{mixed} \; \delta}^1$.

**Proof:**

Note that if $[p'] = [p] \in [\mathfrak{F}]$, then the smooth parts of the corresponding curves are identical and covered by the same collection of allowable coordinate charts, so there is an identification of $X_\delta(p)$ with $X_\delta(p')$ and $Y_\delta(p)$ with $Y_\delta(p')$ which preserves $V$ and $\bar{D}\partial'$. We may therefore restrict to the case of proving the lemma for an open set in a coordinate chart of $\mathfrak{F}$, as the general case follows from this case applied to some finite number of coordinate charts that when projected to $[\mathfrak{F}]$ contain $[p]$. 

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Consider a sub collection of our allowable coordinate charts on \( \hat{f} \) consisting of a coordinate chart \( U \) on \( \hat{f} \) containing \( p \) and the corresponding allowable coordinate charts \( U_i \) on \( \hat{f} \) so that \( U_i \xrightarrow{\pi_3} U \) are allowable coordinate charts. This sub collection is itself an allowable collection of coordinate charts. If we define \( X_{\delta}(p') \) or \( Y_{\delta}(p') \) with this smaller collection, we get an equivalent norm to the norm using the full collection of coordinate charts. This equivalence of norms is because the fact that these coordinate charts are extendible means that the difference between the weighting functions and metrics between coordinate charts will be uniformly bounded independent of \( p' \). Therefore it is sufficient to prove our estimates with this equivalent norm involving only this smaller collection of coordinate charts. We may assume, by restricting \( U \) to a smaller subset if necessary, that \( p \) is in the largest strata of \( |U| \).

Now define the maps \( G \) and \( C \) in these coordinate charts \( \pi_3 : U_i \rightarrow U \) as follows. Suppose that \( f \) is a map from \( \pi_3^{-1}(p) \cap U_i \) to some vector space. If \( U_i \) is of type 3a from the definition of allowable coordinate charts on page 16, \( U_i \) is some subset of \( \mathbb{T}_1 \times U \) with smooth part which is a product of a subset of \( \mathbb{T}_1 \) with \( |U| \). The coordinate \( \tilde{z}_1 \) on \( \mathbb{T}_1 \) is a coordinate for both \( \pi_3^{-1}(p) \cap U_i \) and for \( \pi_3^{-1}(p') \cap U_i \). In this case, we can define \( G(f)(\tilde{z}_1) = f(\tilde{z}_1) \) and \( C(g)(\tilde{z}_1) = g(\tilde{z}_1) \), extending both \( G(f) \) and \( C(g) \) to be constant on edges of the curve that appear in our coordinate chart if necessary. Note that if \( f \) vanishes at marked points, then \( G(f) \) does too as we chose our coordinates so that the position of marked points is independent of \( U \). (This is a condition of being an allowable collection of coordinate charts.)

If, on the other hand, \( U_i \) is a chart of type 3a, then \( U_i \) is of the form \( U_i := \pi_3^{-1}(U) \cap \{ |\tilde{z}_1| < c, |\tilde{z}^\beta| < c \} \). Choose some smooth cut off function \( \rho : \mathbb{T}_1 \rightarrow \mathbb{R} \) so that

\[
\rho(\tilde{z}) = 0 \text{ if } |\tilde{z}| > \frac{c}{2} \\
\rho(\tilde{z}) = 1 \text{ if } |\tilde{z}| \leq \frac{c}{4}
\]

Recall that \( \tilde{z}_1 \tilde{z}^\beta \) is a coordinate function on \( U \), and coordinates for \( \pi_3^{-1}(p) \) are given by \( \tilde{z}_1 \in \mathbb{T}_1 \) and \( \tilde{z}^\beta \in \mathbb{T}_1 \) so that \( \tilde{z}_1 \tilde{z}^\beta = \tilde{z}_1 \tilde{z}^\beta(p) \), and \( |\tilde{z}_1| < c, |\tilde{z}^\beta| < c \). We have assumed that \( |\tilde{z}_1 \tilde{z}^\beta(p)| \leq 4^0 \) by assuming that \( |p| \) is in the largest strata of \( |U| \). Note that \( f \) must be equal to some constant \( x \) on the tropical part of \( \pi_3^{-1}(p) \). Define \( f_+(\tilde{z}_1) := f(\tilde{z}_1) \) when \( |\tilde{z}_1| > |\tilde{z}^\beta| \), and extend \( f_+ \) to be \( x \) elsewhere. Similarly define \( f_-(\tilde{z}^\beta) = f(\tilde{z}^\beta) \) for \( |\tilde{z}^\beta| > |\tilde{z}_1| \), and extend \( f_- \) to be \( x \) everywhere else.

Note that \( \pi_3^{-1}(p') \cap U_i \) has the same coordinates \( \tilde{z}_1 \) and \( \tilde{z}^\beta \), except \( \tilde{z}_1 \tilde{z}^\beta = \tilde{z}_1 \tilde{z}^\beta(p') \). Define \( G \) in these coordinate charts by

\[
G(f)(\tilde{z}_1, \tilde{z}^\beta) := \rho(\tilde{z}^\beta)(f_+(\tilde{z}_1) - x) + \rho(\tilde{z}_1)(f_-(\tilde{z}^\beta) - x) + x
\]

Note that our assumptions about the transition maps between allowable collections of coordinate charts on page 27 ensure that on the intersection with other coordinate charts, this definition of \( G \) coincides with the definition of \( G \) given there, so we have a globally defined map \( G \). It is not hard to see that the norm of \( G \) as a map from \( X_{\delta}(p) \) to \( X_{\delta}(p') \) or \( Y_{\delta}(p) \) to \( Y_{\delta}(p') \) is bounded independent of \( p' \in U \).
Define the cutting map $C$ in these coordinates by

$$C(g)(\tilde{z}_1, \tilde{z}^\beta) = g(\tilde{z}_1) \text{ if } |\tilde{z}_1| > \sqrt{|\tilde{z}_1 \tilde{z}^\beta(p')|}$$

$$C(g)(\tilde{z}_1, \tilde{z}^\beta) = g(\tilde{z}^\beta) \text{ if } |\tilde{z}^\beta| \geq \sqrt{|\tilde{z}_1 \tilde{z}^\beta(p')|}$$

$$C(g)(\tilde{z}_1, \tilde{z}^\beta) = 0 \text{ everywhere else}$$

Note that our assumption on page 16 that on coordinate charts of type $3a$ $|\tilde{z}_1 \tilde{z}^\beta| < \frac{\rho}{16}$ tells us that $G \circ C$ is the identity (this is because the cutoff functions involved will simply be equal to 1 in the relevant regions). Our assumptions on transition maps within allowable collections of coordinate charts ensure that $C$ is defined independent of coordinates. Observe also that $C : Y_\delta(p') \rightarrow Y_\delta(p)$ has norm bounded by 1. We have now verified the first three items of our lemma.

Recall that the formula (18) from the proof of Theorem 3.5 combined with Lemma 3.4 tell us that $D \tilde{\partial}'$ is in the following form:

$$D \tilde{\partial}'(p)(\psi)(\tilde{z}_1, \tilde{z}^\beta) = DE(\tilde{z}_1, \tilde{z}^\beta, p)(\psi(\tilde{z}_1, \tilde{z}^\beta)) + \frac{1}{2} (d_{\text{vert}} \psi + J \circ d_{\text{vert}} \psi \circ j)(\tilde{z}_1, \tilde{z}^\beta, p)$$

In the above, by $D \tilde{\partial}'(p)$ we mean the restriction of $D \tilde{\partial}'$ to the curve over $p$. We also write things as depending on $(\tilde{z}_1, \tilde{z}^\beta)$ even though these coordinates are related at $p$. Below, we shall sometimes use only one of these coordinates when we wish to emphasize dependence on that coordinate. In the above expression, $DE$ is some linear map which depends in a $C^{\infty, \delta_0}$ way on position in $U_1$ and disappears on the edges of fibers of $\pi_3$, and $J$ and $j$ also depend in a $C^{\infty, \delta_0}$ way on position in $U_1$. Therefore, in the interesting case that $U_i$ is a chart of type $3a$ the expression $D \tilde{\partial}' \circ G \psi - G \circ D \tilde{\partial}' \psi$ which we must bound will have the following terms:

1. Terms involving $DE(\psi)$. These include the following: a term involving

$$\rho(\tilde{z}^\beta)(DE(\tilde{z}_1, p) - DE(\tilde{z}_1, p'))(\psi_+(\tilde{z}_1) - x) \tag{19}$$

a similar term swapping the roles of $\tilde{z}^\beta$ and $\tilde{z}_1$ and the remaining terms involving $DE$ which are equal to

$$DE(\tilde{z}_1, \tilde{z}^\beta, p')(x) - \rho(\tilde{z}_1)(DE(\tilde{z}^\beta, p)(x))_- - \rho(\tilde{z}^\beta)(DE(\tilde{z}_1, p)x)_+ \tag{20}$$

As $\rho(\tilde{z}_1)$ and $\rho(\tilde{z}_2)$ are smooth and $DE$ is $C^{\infty, \delta_0}$, the expression (20) is a bounded $C^{\infty, \delta_0}$ function of $(\tilde{z}_1, \tilde{z}^\beta, p')$, vanishes when $p' = p$, and also vanishes on edges of fibers of $\pi_3$. We therefore have that the $Y_\delta(p')$ norm of (20) can be made as small as we like compared to $|x|$ by choosing $p'$ topologically close to $p$.

Because $DE$ is $C^{\infty, \delta_0}$, the $Y_\delta(p')$ norm of the terms of the form (19) can be made as small as we like compared to $||\psi - x||_\delta$ by choosing $p'$ topologically close to $p$.

Therefore, the $Y_\delta(p')$ norm of all these terms can be made as small as we like compared to $||\psi||^1_\delta$ by choosing $p'$ topologically close to $p$.  

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2. Terms involving the difference between \( j \) or \( J \) at \((\hat{z}, p)\) and \((\hat{z}, p')\) or at \((\hat{z}, p)\) and \((\hat{z}, p')\) composed with \( d_{\text{vert}} \). As \( J \) and \( j \) are \( C^{\infty, \delta_0} \), the \( Y_{\delta} \) norms of these can be made as small as we like by choosing \( p' \) topologically close to \( p \).

3. Terms involving \( d_{\text{vert}}(\hat{z}) \) multiplied by \( \psi(\hat{z}) - c \) (perhaps composed with \( J \) and \( j \)), and similar terms reversing the roles of \( \hat{z} \) and \( \hat{z}' \). Note \( |d_{\text{vert}}(\hat{z})| \) is bounded, and is supported on the region \( \{|\hat{z}| < \frac{1}{4} \} \).

Therefore, by choosing our neighborhood of \( p \) small enough, we can achieve item 3b from our lemma on charts of type 3a. On the other charts of type 3b proving the same thing is similar, but easier as it involves no cut off functions and just analysis of how \( DE, j \) and \( J \) depend on position.

We now just need to prove item 3b from our lemma. To see this, consider \( C^{\infty, \delta_0} \) sections of \( V \), \( v_1, \ldots, v_n \) so that \( \{v_i(p)\} \) is a basis for \( V(p) \). Then \( G(v_i(p)) - v_i \) is \( C^{\infty, \delta_0} \) and vanishes on \( \pi_{\delta}^{-1} p \). Therefore, we can get \( \|G(v_i(p))(p') - v_i(p')\|_{\delta} \) as small as we like by choosing \( p' \) topologically close to \( p \). This in turn bounds \( \|\pi_V G(v_i(p))\|_{\delta} \), and item 3b follows from the linearity of \( G \):

\[
\|\pi_V G(v)\|_{\delta} \leq \epsilon \|v\|_{\delta}
\]

\[\square\]

**Theorem 3.11.** Suppose that \((\hat{f}, V)\) is a \( C^{\infty, \delta_0} \) pre obstruction model, \( 0 < \delta < \delta_0 \leq 1 \), and

\[
\pi_V \circ D\hat{\partial}'(p) : X_{\delta}(p) \longrightarrow Y/V_{\delta}(p)
\]

is invertible.

Then there exists a topologically open neighborhood \( U \) of \( p \) so that for all \( p' \in U \),

\[
\pi_V \circ D\hat{\partial}'(p') : X_{\delta}(p) \longrightarrow Y/V_{\delta}(p')
\]

is invertible, and the norm of the inverse is uniformly bounded independent of \( p' \in U \).

More generally, given a choice of weight \( 0 < \delta < \delta_0 \) for each coordinate chart in an allowable collection of coordinate charts, the same result holds using the mixed norm: there exists some topologically open neighborhood \( U \) of \( p \) so that for all \( p' \in U \),

\[
\pi_V \circ D\hat{\partial}'(p') : X_{\text{mixed}}(p) \longrightarrow Y/V_{\text{mixed}}(p')
\]

is invertible, and the norm of the inverse is uniformly bounded independent of \( p' \in U \).
Proof: Note that if \( \pi_V \circ D\bar{\partial}' \) is invertible with one weight \( \delta \), it is invertible for any choice of weights \( 0 < \delta < \delta_0 \). This is because the index of \( \pi_V \circ D\bar{\partial}' \) is independent of the choice of weights, surjectivity using a stronger norm implies injectivity using a weaker norm, and injectivity using a weaker norm implies surjectivity using a stronger norm.

Use the notation

\[
Q := (\pi_V \circ D\bar{\partial}'(p))^{-1} \circ \pi_V
\]

The fact that \( \pi_V \circ D\bar{\partial}'(p) \) is Fredholm implies that \( Q \) is bounded (using any of the norms under consideration). Now, consider a small topologically open neighborhood \( O \) of \( p \) and define cutting maps \( G \) and \( C \) satisfying the conditions of Lemma 3.10 on \( O \) for the mixed norm. Suppose that \( \|D\bar{\partial}'(p)\| \) and \( \|Q\| \) are bounded by \( M \), and \( \|G\| \) and \( \|C\| \) are also bounded by \( M \) on \( O \). Choose a topological open neighborhood \( U \) of \( p \) contained in \( O \) so that for all \( p' \in U \),

\[
\|D\bar{\partial}'(p') \circ G\psi - G \circ D\bar{\partial}'(p)\psi\| \leq \epsilon \|\psi\|_{\text{mixed} \delta} \quad (21)
\]

and for all \( v \in V(p) \),

\[
\|\pi_V Gv\| \leq \epsilon \|v\|_{\text{mixed} \delta} \quad (22)
\]

Now consider the map \( Q(p') : Y_{\text{mixed} \delta}(p') \rightarrow X_{\text{mixed} \delta}(p') \) defined by

\[
Q(p') := G \circ Q \circ C
\]

By exchanging \( D\bar{\partial}'(p') \circ G \) with \( G \circ D\bar{\partial}'(p) \) using the inequality (21), and using that \( G \circ C \) is the identity, and then using that \( D\bar{\partial}'(p) \circ Q(C\nu - C\nu) \in V(p) \) and the inequality (22), we get the following:

\[
\|\pi_V (D\bar{\partial}'(p') \circ Q(p') \nu - \nu)\| \leq \epsilon \|Q(C\nu)\|_{\text{mixed} \delta} + \epsilon \|Q(C\nu - C\nu)\|_{\text{mixed} \delta} \\
\leq \epsilon M^2 \|\nu\|_{\text{mixed} \delta} + \epsilon (M^3 + M) \|\nu\|_{\text{mixed} \delta} \quad (23)
\]

As \( V(p') \) is \( n \) dimensional, there exists a linear inclusion

\[
i_V : Y/V_{\text{mixed} \delta}(p') \rightarrow Y_{\text{mixed} \delta}(p')
\]

with norm bounded by \( 2^n \) so that \( \pi_V \circ i_V \) is the identity. To see this for the case \( n = 1 \), use the Hahn Banach theorem to choose a linear functional \( L \) on \( Y_{\text{mixed} \delta}(p') \) so that \( |Lu| = \|v\|_{\text{mixed} \delta} \) for \( v \in V(p) \), and \( \|L\| = 1 \). Then the obvious inclusion with image \( \ker L \) is bounded by \( 2 \). Repeating this argument \( n \) times gives the \( n \) dimensional case.

In particular, if we choose \( \epsilon \) small enough, the above inequality (23) tells us that

\[
\|\pi_V D\bar{\partial}'(p') \circ (Q(p') \circ i_V) - Id\| < \frac{1}{2}
\]

so a right inverse to \( \pi_V D\bar{\partial}'(p') \) is given by

\[
(Q(p') \circ i_V) \left( \pi_V D\bar{\partial}'(p') \circ (Q(p') \circ i_V) \right)^{-1}
\]
which is bounded by $M_2^{n+1}$.

The fact that the index of $D\bar{\nabla}'$ does not change in connected families tells us that this right inverse must be a genuine inverse to $\pi_V D\bar{\nabla}'$.

We can extend the results of Theorem 3.11 to include the norm $\| \cdot \|_{\delta, \delta'}$ which occurs in the equivalent form of the $\| \cdot \|_{0, \delta}$ norm found in Lemmas 2.12 and 2.17 on pages 25 and 30.

Corollary 3.12. If $(\hat{f}, V)$ is a $C^{\infty, \delta_0}$ pre obstruction model, $0 < \delta < \delta_0 \leq 1$ and,

$$\pi_V \circ D\bar{\nabla}' : X_{\delta}(p) \to Y/V_{\delta}(p)$$

is invertible,

then given any $\delta' > 0$ so that $\delta + \delta' < \delta_0$, there exists some topologically open neighborhood $U$ of $p$ so that restricted to this neighborhood,

$$\pi_V \circ D\bar{\nabla}' : X_{\delta+\delta'} \to Y/V_{\delta+\delta'}$$

and

$$\pi_V \circ D\bar{\nabla}' : Y_{\delta+\delta'} \to Y/V_{\delta+\delta'}$$

both have a bounded inverse.

Proof:

The statement of this corollary for $\pi_V \circ D\bar{\nabla}' : X_{\delta+\delta'} \to Y/V_{\delta+\delta'}$ follows immediately from Theorem 3.11. We must see why the map

$$\pi_V \circ D\bar{\nabla}' : X_{\delta+\delta'} \to Y/V_{\delta+\delta'}$$

has a bounded inverse. This is equivalent to getting a bound on the inverse restricted to any point $p' \in U$ for the map

$$\pi_V \circ D\bar{\nabla}'(p') : X_{\delta+\delta'}(p') \to Y/V_{\delta+\delta'}(p')$$

which is uniform in $p'$.

Restricted to $p'$, $X_{\delta+\delta'}(p')$ is equal to $X_{\text{mixed}}(\delta'(p'))$, and $Y/V_{\delta+\delta'}(p')$ is equal to $Y/V_{\text{mixed}}(\delta'(p'))$ for some choice of weights for each coordinate chart out of the set $\{\delta', \delta' + \delta\}$. As we get a uniform bound on the inverse in this mixed norm restricted to a small enough topologically open neighborhood, and there are only a finite number of ways to choose weights for each chart from the set $\{\delta', \delta' + \delta\}$, it follows that restricted to a small enough topologically open neighborhood, there is a uniform bound for the inverse of

$$\pi_V \circ D\bar{\nabla}'(p') : X_{\delta+\delta'}(p') \to Y/V_{\delta+\delta'}(p')$$

Proposition 3.13. If $(\hat{f}, V)$ is a $C^{\infty, \delta_0}$ pre obstruction model, $\delta + \delta' < \delta_0$, $\hat{f}$ an allowable family and

$$\pi_V \circ D\bar{\nabla}' : X_{\delta+\delta'} \to Y/V_{\delta+\delta'}$$

$$\pi_V \circ D\bar{\nabla}' : X_{\delta+\delta'} \to Y/V_{\delta+\delta'}$$

both have a bounded inverse, then

$$\pi_V \circ D\bar{\nabla}' : X_{0, \delta} \to Y_{0, \delta}$$

has a bounded inverse.
Proof:

We need to bound \( \| \phi \|_{0, \delta}^1 \) in terms of \( \| \pi_V \circ D \bar{\partial}' \phi \|_{0, \delta} \). Note first that our assumptions tell us that

\[
\| \phi \|_{\delta + \delta'}^1 \leq c \| \pi_V \circ D \bar{\partial}'(\nu) \phi \|_{\delta + \delta'}
\]

(24)

\[
\| \phi \|_{\delta \& \delta'}^1 \leq c \| \pi_V \circ D \bar{\partial}'(\nu) \phi \|_{\delta \& \delta'}
\]

(25)

We shall be using the equivalent form of \( \| \cdot \|_{0, \delta}^1 \) from Lemma 2.12 and Lemma 2.17 which involves only weights \( w \) and \( w \) would be adequate to prove our lemma. The main part of the following proof shall be involved with estimating the extent to which this fails to hold.

Restrict attention to the set coordinate charts \( U_i \) over single chart \( U \) in our allowable collection of coordinate charts. Let \( S \) indicate a set of strata of \( [U] \), and \( \tilde{S} \) indicate the lift of \( S \) to a set of strata in \( [U_i] \). Lemma 2.17 on page 30 tells us that the norm \( \| \phi \|_{0, \delta}^1 \) is equivalent to the norm

\[
\| \phi \|_{\delta + \delta'}^1 + \max_S \| w_S \tilde{\Delta} \phi \|_{\delta \& \delta'}
\]

Assume for induction that for any collection of strata \( I \) in \( [U] \) with less than \( |S| \) members,

\[
\| w_I^{-\delta} \tilde{\Delta}_I \phi \|_{\delta \& \delta'}^1 \leq c \| \pi_V D \bar{\partial}' \phi \|_{0, \delta}
\]

(26)

The case when \( I \) has no members is satisfied because of the inequality (25) and Lemma 2.17. In what follows, we shall attempt to bound \( \| w_S^{-\delta} \tilde{\Delta} \phi \|_{\delta \& \delta'}^1 \) by bounding \( \| D \bar{\partial}' w_S^{-\delta} \tilde{\Delta} \phi \|_{\delta \& \delta'} \).

We shall work in a local coordinate chart where we can use the following formula which is formula (18) from the proof of Theorem 3.5

\[
D \bar{\partial}' \phi = DE(\phi) + H(d_{\text{vert}} \phi)
\]

The important facts are that \( DE \) is a \( C^{\infty, \delta_0} \) tensor so that \( \tilde{\Delta} \phi = DE, \) and \( H \) is a \( C^{\infty, \delta_0} \) tensor.

First, note that for any strata \( S, \)

\[
\tilde{\Delta}_S(D \bar{\partial}' \phi) = D \bar{\partial}' \tilde{\Delta}_S \phi + (\tilde{\Delta}_S DE)(e_S \phi) + (\tilde{\Delta}_S H)(e_S d_{\text{vert}} \phi)
\]

Similarly, for the lift, \( \tilde{S} \) of our set of strata \( S \) in \( [U] \) to the coordinate chart under consideration,

\[
D \bar{\partial}' \tilde{\Delta}_S \phi = \tilde{\Delta}_S(D \bar{\partial}' \phi) - \sum_{I \subseteq \tilde{S}} \Delta_I DE \left( e_I \tilde{\Delta}_{(\tilde{S} - I)} \phi \right) + \left( \Delta_I H \right) \left( e_I \Delta_{(\tilde{S} - I)} d_{\text{vert}} \phi \right)
\]

(27)

In what follows we shall bound the \( \| w_S^{-\delta} \|_{\delta \& \delta'} \) norm of terms in the above sum. Observe that each of the terms in (27) above vanish on all strata in \( \tilde{S}, \)
so if \( \nu \) indicates one of the above terms, \( \Delta \tilde{s} \nu = \nu \). For such \( \nu \), the following inequality holds:

\[
\left\| w_\delta^- \Delta \tilde{s} \nu \right\|_{\delta \& \delta'} \leq c \left\| w_\delta^{-\delta} \Delta \tilde{s} \nu \right\|_{\delta'}
\]

(28)

In the above, \( S_0 \) indicates the set of strata on which \( w_0 \) vanishes. To see why the inequality (28) holds, note that it holds on strata \( T \) so that \( T^c = \emptyset \) because then the norm on the left hand side is just the \( \| \cdot \|_{\delta'} \) norm. The inequality (28) also holds trivially on strata in \( \tilde{S} \), because both sides are 0 restricted to strata in \( \tilde{S} \).

It remains to show (28) holds on strata \( T \) so that \( T \notin \tilde{S} \) and \( T^c \neq \emptyset \). It suffices to show that in this case \( e_T w_\delta^- w_0 \) is bounded by a constant times \( e_T w_\delta^- w_0 \), which is inequality (5) from the proof of Lemma 2.12 on page 25 Therefore the inequality (28) holds.

We shall now bound terms involving \( H \) in (27).

\[
\left\| w_\delta^- (\Delta H) \left( e_T \Delta (\tilde{S} - I) d_{vert} \phi \right) \right\|_{\delta \& \delta'} \leq c \left\| w_\delta^{-\delta} (\Delta H) \left( e_T \Delta (\tilde{S} - I) d_{vert} \phi \right) \right\|_{\delta'}
\]

\[
\leq c' \left\| (w_\delta^{-\delta} (\Delta H) \left( e_T w_\delta^- (\tilde{S} - I) d_{vert} \phi \right) \right\|_{\delta'}
\]

\[
\leq c'' \left\| e_T w_\delta^- (\tilde{S} - I) d_{vert} \phi \right\|_{\delta'}
\]

(29)

We wish to use our inductive hypothesis to estimate this last term, but \( \tilde{S} - I \) may not be a lifted set of strata. To remedy this, note that

\[
e_T \Delta (\tilde{S} - I) d_{vert} \phi = e_T \Delta \tilde{S} d_{vert} \phi
\]

where \( \tilde{S}' \) is the largest lifted collection of strata which is a subset of \( \tilde{S} - I \). Therefore \( e_T \Delta \tilde{S} d_{vert} \phi = e_T \Delta \tilde{S} d_{vert} \phi \). We also have that

\[
e_T w_\delta^- (\tilde{S} - I) \leq c e_T w_\delta^- w_0
\]

(30)

This is because if \( e_T [\tilde{s}^\alpha] \) vanishes on \( \tilde{S}' \) and \( S_0 \), it must vanish on \( \tilde{T}^c \), and therefore, it must also vanish on \( \tilde{S} - I \) (which should be thought of as equal to \( \tilde{S}' \cup \tilde{T}^c \)). Therefore,

\[
\left\| e_T w_\delta^- (\tilde{S} - I) d_{vert} \phi \right\|_{\delta'} \leq c \left\| e_T w_\delta^- \Delta \tilde{S} d_{vert} \phi \right\|_{\delta + \delta'}
\]

If \( I \) is not a lifted set of strata, then the right hand side of the above inequality is bounded by a constant times \( \left\| w_\delta^- \Delta (\tilde{S} - I) d_{vert} \phi \right\|_{\delta \& \delta'} \). So if \( I \) is not a lifted set of strata,

\[
\left\| w_\delta^- (\Delta H) \left( e_T \Delta (\tilde{S} - I) d_{vert} \phi \right) \right\|_{\delta \& \delta'} \leq c \left\| w_\delta^- \Delta \tilde{S} d_{vert} \phi \right\|_{\delta \& \delta'}
\]

(31)

On the other hand, if \( I \) is a lifted set of strata, then \( \tilde{S} - I \) is lifted, so \( \tilde{S}' = \tilde{S} - I \), and we can derive the above inequality (31) as follows:

\[
\left\| w_\delta^- (\Delta H) \left( e_T \Delta (\tilde{S} - I) d_{vert} \phi \right) \right\|_{\delta \& \delta'} \leq c \left\| (w_\delta^- (\Delta H) \left( e_T w_\delta^- \Delta (\tilde{S} - I) d_{vert} \phi \right) \right\|_{\delta \& \delta'}
\]

\[
\leq c' \left\| e_T w_\delta^- \Delta \tilde{S} d_{vert} \phi \right\|_{\delta \& \delta'}
\]

\[
\leq c' \left\| w_\delta^- \Delta \tilde{S} d_{vert} \phi \right\|_{\delta \& \delta'}
\]

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Now, we need to bound the terms involving $DE$ in (27).

$$\| w^{-\delta}_S (\Delta I DE) \left( e_I \Delta (S-I) \phi \right) \|_{\delta \& \delta'} \leq c \| w^{-\delta}_S \Delta (\Delta I DE) \left( e_I \Delta (S-I) \phi \right) \|_{\delta'}$$

using (25)

$$\leq c' \| w^{-\delta}_S \Delta (\Delta I (S-I) \phi) \|_{\delta'}$$

$$\leq c'' \sup \| e_I w^{-\delta}_S (\Delta (S-I) \phi) \|$$

Now we must bound $\sup \| e_I w^{-\delta}_S (\Delta (S-I) \phi) \|$. If $I$ is a lifted set of strata and $S' = \tilde{S} - I$, then the above inequality immediately gives

$$\| w^{-\delta}_S (\Delta I DE) \left( e_I \Delta (S-I) \phi \right) \|_{\delta \& \delta'} \leq c \| w^{-\delta}_S \Delta (\Delta I \phi) \|_{\delta \& \delta'} \tag{32}$$

On the other hand, if $I$ is not a lifted set of strata, there exists some strata $S_i \subset \tilde{S} - I$ and $\tilde{S}_i \in S_i$ so that $\tilde{S}_i \in I$. Then we can use Lemma 2.11 and the observation that $e_{I} w^{-\delta}_S \Delta (\Delta I \phi) \leq \| e_{I} w^{-\delta}_S \Delta (\Delta I \phi) \|_{\delta \& \delta'}$ (which follows from (30)) to get the following:

$$\sup \| e_I w^{-\delta}_S (\Delta (S-I) \phi) \| \leq c \sup \| e_I w^{-\delta}_S (\Delta (S-I) \phi) \|$$

$$= c \sup \| e_I w^{-\delta}_S (\Delta (S-I) \phi) \|$$

$$\leq c' \| e_I w^{-\delta}_S (\Delta (S-I) \phi) \|_{\delta \& \delta'}$$

$$= c' \| e_I w^{-\delta}_S (\Delta (S-I) \phi) \|_{\delta \& \delta'}$$

$$\leq c' \| e_I w^{-\delta}_S (\Delta (S-I) \phi) \|_{\delta \& \delta'}$$

$$\leq c' \| e_I w^{-\delta}_S (\Delta (S-I) \phi) \|_{\delta \& \delta'}$$

as $I^c \neq \emptyset$

Therefore, inequality (32) holds for all sets of strata $I \subset \tilde{S}$. Therefore, using the inequalities (31) and (32) along with equation (27) gives

$$\| D\tilde{\phi} w^{-\delta}_S \Delta \phi - w^{-\delta}_S \Delta \tilde{\phi} D\tilde{\phi} \|_{\delta \& \delta'} \leq c \sum_{S' \subset \tilde{S}} \| w^{-\delta}_S \Delta \tilde{\phi} \|_{\delta \& \delta'} \tag{33}$$

If $\tilde{S}$ consists entirely of strata $S$ so that $S^c = \emptyset$, then $\Delta \tilde{\phi} = \Delta \phi$. Therefore, we may apply the estimates of Lemma 2.18 on page 32 to the above inequality to exchange $\Delta \phi$ with $\Delta \tilde{\phi}$ and get error terms involving $\Delta \tilde{\phi}$ where $S' \subset S$, and also apply the estimates of Lemma 2.18 to the right hand side of (33) to get

$$\| D\tilde{\phi} w^{-\delta}_S \Delta \phi - w^{-\delta}_S \Delta \tilde{\phi} D\tilde{\phi} \|_{\delta \& \delta'} \leq c \sum_{S' \subset \tilde{S}} \| w^{-\delta}_S \Delta \tilde{\phi} \|_{\delta \& \delta'} \tag{34}$$

The right hand side of (34) is bounded by $c \| \pi_{V} D\tilde{\phi} \|_{\delta \& \delta}$ by our inductive assumption (26). As (34) can now be regarded as a statement which applies on the all the coordinate charts $U_i$ over $U$, we can apply $\pi_{V}$ to the left hand term in (34) to get

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\[
\left\| \pi_V D \bar{\partial}' w_S^{-\delta} \Delta S \phi \right\|_{\delta \& \delta'} \leq \left\| \pi_V w_S^{-\delta} \Delta_S \bar{\partial}' \phi \right\|_{\delta \& \delta'} + c \left\| \pi_V D \bar{\partial}' \phi \right\|_{0, \delta} \tag{35}
\]

We now estimate this right hand side of this inequality by a constant times \(\|\pi_V D \bar{\partial}' \phi\|_{0, \delta}\). (This estimation is trivial if \(V\) is 0 dimensional, but it will take us over two pages.) After that, the inequality (25) can be applied to complete the inductive argument.

\[
\left\| \pi_V w_S^{-\delta} \Delta_S \bar{\partial}' \phi \right\|_{\delta \& \delta'} := \inf_v \left\| w_S^{-\delta} \Delta_S \bar{\partial}' \phi - \right\|_{\delta \& \delta'} 
\leq c \left\| \pi_V D \bar{\partial}' \phi \right\|_{0, \delta} + \inf_v \left\| w_S^{-\delta} \Delta_S v' - v \right\|_{\delta \& \delta'} \tag{36}
\]

\[
= c \left\| \pi_V D \bar{\partial}' \phi \right\|_{0, \delta} + \left\| \pi_V w_S^{-\delta} \Delta_S v' \right\|_{\delta \& \delta'}
\]

where \(v'\) is any section of \(V\) with \(\|v'\|_{0, \delta}\) finite so that

\[
2 \left\| \pi_V D \bar{\partial}' \phi \right\|_{0, \delta} \geq \left\| D \bar{\partial}' \phi - v' \right\|_{0, \delta} \tag{37}
\]

We must show that \(\left\| \pi_V w_S^{-\delta} \Delta_S v' \right\|_{\delta \& \delta'}\) can be bounded in terms of \(\left\| \pi_V D \bar{\partial}' \phi \right\|_{0, \delta}\).

In our coordinate chart \(U\), we can choose a trivialization of \(V\) using a basis of \(C^{\infty, \delta_0}\) sections \(\{v_i\}\). We then have

\[
v' = \sum_i f_i v_i
\]

where \(\{f_i\}\) is some collection of real valued functions which are constant on fibers of our coordinate charts \(U_i \rightarrow U\). We can choose \(v_i\) so that for any collection of functions \(g_i\) on \(U\),

\[
|g_i| \leq c \left\| \sum_i g_i v_i \right\|_{\delta'} \tag{38}
\]

Using the identity \(\Delta_S fg = (\Delta_S f)g + (e_S f)\Delta_S g\) repeatedly, we get

\[
\Delta_S v' = \sum_{i \in S, i} \left( \Delta_I e_{(\tilde{S}-1)} f_i \right) \left( \Delta_{(\tilde{S}-1)} v_i \right) \tag{39}
\]

As \(f_i\) is constant on fibers of our coordinate chart, if \(\tilde{S} \in S^c\), then \(\Delta_S e_{\tilde{S}} f_i = 0\). Therefore, we can rewrite the above expression as

\[
\Delta_S v' = \sum_{i \in S, i} \left( \Delta_I e_{(\tilde{S}-1)} f_i \right) \left( \Delta_{(\tilde{S}-1)} v_i \right) \tag{40}
\]

We need to estimate \(\Delta_S v'\). This is equal to \(\Delta_S v'\) unless we’re in a coordinate chart of type \(\mathbb{R}^4\) with coordinates including \(|\tilde{z}_1| < c\) and \(|\tilde{z}^\beta| < c\). Using the notation introduced in Definition 2.16 on page 29, write \(I^+\) for the subset of \(\tilde{I}\) on which \(|\tilde{z}_1| = 0\), and \(I^-\) for the subset on which \(|\tilde{z}^\beta| = 0\), then

\[
\tilde{\Delta}_S = \Delta_{\tilde{S}} - \hat{\rho}^+(\Delta_{\tilde{S}} - \Delta_{S^+}) - \hat{\rho}^-(\Delta_{\tilde{S}} - \Delta_{S^-})
\]

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where \( \tilde{\rho}^+ \) and \( \tilde{\rho}^- \) are some cut off functions. Note that as \( f_i \) is constant on fibers, \( e_{I^+} f = e_{I^-} f = e_{I} f_i \) where in the right most expression, we consider \( f_i \) as a function on \( U \). Similarly, \( \Delta_{I^-} f_i = \Delta_{I^+} = \Delta_{I^-} f_i = \Delta_{I} f_i \), so \( \Delta_{I^-} f = \Delta_{I} f \). As every strata in \( S \) has a unique lift to \( S^+ \), equation (39) implies that

\[
\Delta_{S^+} v' = \sum_{I \subseteq S, i} (\Delta_{I} e_{(S-I)} f_i) (\Delta_{(S-I)^+} v_i)
\]  

(41)

and similarly,

\[
\Delta_{S^-} v' = \sum_{I \subseteq S, i} (\Delta_{I} e_{(S-I)} f_i) (\Delta_{(S-I)^-} v_i)
\]  

(42)

so using equations (40), (41) and (42) on a coordinate chart of type 3a, we get

\[
\Delta_{S} v' = \sum_{I \subseteq S, i} (\Delta_{I} e_{(S-I)} f_i) (\Delta_{(S-I)} v_i)
\]  

(43)

We’ve proved the above equation (43) for coordinate charts of type 3a, and on all other coordinate charts, equation (43) is equivalent to equation (44), so (43) is valid for all our coordinate charts over \( U \).

We can apply equation (43) along with the inequality (38) to bound \( |w_{S^+}^\delta \Delta_{S} f_i| \) as follows:

\[
\sup |w_{S^+}^\delta \Delta_{S} f_i| \leq c \sum_{I \subseteq S} |w_{S^+}^\delta \Delta_{S} f_i| v_i
\]

\[
\leq c \left| w_{S^+}^\delta \Delta_{S} v' \right|_{\delta'} + c \sum_{I \subseteq S, j} \left| w_{S^+}^\delta (\Delta_{I} e_{(S-I)} f_i) (\Delta_{(S-I)} v_j) \right|_{\delta'}
\]

\[
\leq c \left| w_{S^+}^\delta \Delta_{S} v' \right|_{\delta'} + c' \sum_{I \subseteq S, j} \left| (w_{I}^\delta \Delta_{I} e_{(S-I)} f_j) (w_{S-I}^\delta \Delta_{(S-I)} v_j) \right|_{\delta'}
\]

\[
\leq c \left| w_{S^+}^\delta \Delta_{S} v' \right|_{\delta'} + c'' \sum_{I \subseteq S, j} \sup |w_{I}^\delta \Delta_{I} e_{(S-I)} f_j|
\]

Using the above inequality again on the terms with fewer strata gives

\[
\sup |w_{S^+}^\delta \Delta_{S} f_i| \leq c \sum_{I \subseteq S} \left| \left| w_{I}^\delta \Delta_{I} v' \right| \right|_{\delta'}
\]

(44)

As the sections \( v_i \) are \( C^{\infty, \delta_0} \), the norm \( \left| \left| w_{I}^\delta \Delta_{I} v_i \right| \right|_{\delta, \delta'} \) is bounded. Now bound terms on the right hand side of equation (43) as follows:

\[
\left| \left| \pi_V w_{S^+}^\delta (\Delta_{I} e_{(S-I)} f_i) (\Delta_{(S-I)} v_i) \right| \right|_{\delta, \delta'}
\]

\[
\leq c \left| \left| \pi_V (w_{I}^\delta \Delta_{I} e_{(S-I)} f_i) (w_{S-I}^\delta \Delta_{(S-I)} v_i) \right| \right|_{\delta, \delta'}
\]

\[
\leq c \sup |w_{I}^\delta \Delta_{I} e_{(S-I)} f_i| \left| \left| \pi_V w_{S-I}^\delta \Delta_{(S-I)} v_i \right| \right|_{\delta, \delta'}
\]

\[
\leq c' \sum_{I \subseteq I} \left| \left| w_{I}^\delta \Delta_{I} v' \right| \right|_{\delta'} \text{ using (44)}
\]

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When considering \( \pi_V \) of equation (43), we can remove the term where \( I = S \), because \((\Delta_S f_i) \nu_i \in V \). In the remaining terms, \( f_i \) is always being acted on by \( \Delta_I \) where \(|I| < |S|\). Therefore, using equation (43) and the above inequality,

\[
\| \pi_V w_S^{-\delta} v' \|_{\delta \& \delta'} \leq c \sum_{I \subseteq S} \left \| \pi_I^* \Delta_I v' \right \|_{\delta'}
\]

\[
\leq c' \left ( \sum_{I \subseteq S} \left \| \pi_I^* \Delta_I D \partial' \phi \right \|_{\delta'} + \| \pi_V D \partial' \phi \|_{0, \delta} \right ) \text{ using (37)}
\]

\[
\leq c'' \left ( \sum_{I \subseteq S} \left \| \pi_I^* \Delta_I \phi \right \|_{\delta \& \delta'} + \| \pi_V D \partial' \phi \|_{0, \delta} \right ) \text{ using (39)}
\]

\[
\leq c''' \| \pi_V D \partial' \phi \|_{0, \delta} \text{ using (26)}
\]

Using the above inequality with equations (38) and (39), we get

\[
\| \pi_V D \partial' w_S^{-\delta} \Delta_S \phi \|_{\delta \& \delta'} \leq c \| \pi_V D \partial' \phi \|_{0, \delta}
\]

Applying inequality (26) to the left hand side of the above gives

\[
\| \pi_V D \partial' w_S^{-\delta} \Delta_S \phi \|_{\delta \& \delta'}^1 \leq c \| \pi_V D \partial' \phi \|_{0, \delta}
\]

This completes the inductive argument that the above expression holds for all sets of strata \( S \) in \([U]\). Therefore, using the equivalent norm from Lemma 2.14, we’ve proved the required inequality on the set of coordinate charts over \( U \),

\[
\| \phi \|_{0, \delta}^1 \leq c \| \pi_V D \partial' \phi \|_{0, \delta}
\]  \hspace{1cm} (45)

It follows that inequality (43) holds on our entire family, as our entire family is covered by a finite collection of allowable coordinate charts. We shall use the above inequality (43) to see that \( \pi_V D \partial' : X_{0, \delta} \to Y/V_{0, \delta} \) has a bounded inverse, determined by the individual inverses to \( \pi_V D \partial'(p) : X_{0, \delta}(p) \to Y/V_{0, \delta}(p) \) by

\[
\left((\pi_V D \partial')^{-1}(\nu)\right)(p) := (\pi_V D \partial'(p))^{-1}(\nu(p))
\]

For each \( \nu \in Y/V_{0, \delta} \), this determines a unique \( \phi := (\pi_V D \partial')^{-1}(\nu) \in X_{\delta + \delta'} \), so that \( \pi_V D \partial' \phi = \nu \). The smooth part of this inverse is the same on topologically equivalent curves, so for each strata \( S \), \( e_S \phi \) makes sense, which is all that is needed to prove (43), therefore \( \phi \in X_{0, \delta} \), and \( \pi_V D \partial' : X_{0, \delta} \to Y/V_{0, \delta} \) has an inverse bounded by the constant \( c \) in the above inequality (43). 

\[\square\]

**Proposition 3.14.** If \((\tilde{f}, V)\) is a \( C^{\infty, \delta_0} \) pre obstruction model with \( \delta + \delta' < \delta_0 \), then there exists a neighborhood \( U \) of \( 0 \in X_{0, \delta} \), so that for any \( \nu \in U \) which is \( C^{\infty, \delta_0} \), if \( \pi_V D \partial'(\nu) : X_{0, \delta} \to Y/V_{0, \delta} \) has a bounded inverse, then

\[
\pi_V D \partial'(\nu) : X_{k, \delta} \to Y/V_{k, \delta}
\]

has a bounded inverse.
Proof: We shall prove this proposition for \( \nu \) being the zero section first. In this case just use the notation \( D\bar{\partial}'( \text{zero section}) = D\bar{\partial}' \). This proposition is trivially true for \( k = 0 \). Assume for induction that this proposition is true for \( k - 1 \). Suppose that \( w \) is an extendible \( C^{\infty,\delta_0} \) vector field on the total space of our family, \( \mathcal{E} \) which is the lift of some vector field on the base \( \mathfrak{F} \). In a coordinate chart, we have the following expression:

\[
\nabla_w(D\bar{\partial}' \phi) = \nabla_w \left( DE(\phi) + \frac{1}{2} (d_{\text{vert}} \phi + Jd_{\text{vert}} \phi j) \right)
\]

\[
= D\bar{\partial}'(\nabla_w \phi) + \nabla_w(DE)(\phi) + \frac{1}{2}([d_{\text{vert}}, \nabla_w] \phi + J[d_{\text{vert}}, \nabla_w] \phi j)
\]

\[
+ (\nabla_w J)d_{\text{vert}} \phi j + Jd_{\text{vert}} \phi \nabla_w j.
\]

(46)

Using the fact that \( J, j \) and \( DE \) are all extendible and \( C^{\infty,\delta_0} \), and \([d_{\text{vert}}, \nabla_w]\) is an extendible \( C^{\infty,\delta_0} \) first order operator which involves derivatives only in the vertical direction as \( w \) is the lift of a vector field on \( \mathfrak{F} \), the above equation gives the following estimate with a constant \( c \) which depends on \( w \):

\[
\| D\bar{\partial}' \nabla_w \phi - \nabla_w(D\bar{\partial}' \phi) \|_{k-1,\delta} \leq c \| \phi \|^1_{k-1,\delta}
\]

We have proved the above inequality for a single coordinate chart. As our family is covered by a finite collection of such coordinate charts, the above inequality holds globally. Therefore, taking \( \pi_V \) of the left hand side gives the following inequality where the constant \( c \) depends on \( w \):

\[
\| \pi_V D\bar{\partial}' \nabla_w \phi \|_{k-1,\delta} \leq \| \pi_V \nabla_w(D\bar{\partial}' \phi) \|_{k-1,\delta} + c \| \phi \|^1_{k-1,\delta}
\]

(47)

We wish now to bound \( \| \pi_V \nabla_w(D\bar{\partial}' \phi) \|_{k-1,\delta} \) by something depending on \( w \) times \( \| \pi_V D\bar{\partial}' \phi \|_{k,\delta} \) plus \( \| \phi \|^1_{k-1,\delta} \) (a particularly easy task if \( V \) is a zero dimensional vector bundle). Choose some section \( v \) of \( V \) with \( \| v \|_{k,\delta} \) finite so that

\[
2 \| \pi_V D\bar{\partial}' \phi \|_{k,\delta} \geq \| D\bar{\partial}' \phi - v \|_{k,\delta}
\]

(48)

Therefore there exists some constant \( c \) depending on \( w \) so that

\[
c \| \pi_V D\bar{\partial}' \phi \|_{k,\delta} \geq \| \nabla_w(D\bar{\partial}' \phi - v) \|_{k-1,\delta}
\]

\[
\geq \| \pi_V \nabla_w(D\bar{\partial}' \phi) \|_{k-1,\delta} - \| \pi_V \nabla_w v \|_{k-1,\delta}
\]

(49)

Work in some finite collection of extendible coordinate patches on \( \mathfrak{F} \) on which \( V \) is a trivial vector bundle, and choose some basis \( v_1 \ldots v_n \) of \( C^{\infty,\delta_0} \) sections of \( V \) so that we can write any section of \( V \) as \( v = \sum f_i v_i \) where \( \| f_i \|_{k-1,\delta} \leq \| v \|_{k-1,\delta} \). Note that

\[
\pi_V \nabla_w (f_i v_i) = \pi_V f_i \nabla_w v_i
\]

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Therefore we can bound the right most term in [49] as follows:

\[ \| \pi_V \nabla_w v \|_{k-1, \delta} \leq \left\| \sum_i f_i \nabla_w v_i \right\|_{k-1, \delta} \]
\[ \leq \sum_i \| f_i \|_{k-1, \delta} \| \nabla_w v_i \|_{k-1, \delta} \]
\[ \leq c \| v \|_{k-1, \delta} \]
\[ \leq c (\| D\bar{\partial} \phi \|_{k-1, \delta} + 2 \| \pi_V D\bar{\partial} \phi \|_{k, \delta}) \text{ using } [48] \]
\[ \leq c'(\| \phi \|_{k-1, \delta} + \| \pi_V D\bar{\partial} \phi \|_{k, \delta}) \]

Using the inequality [50] for the right most term in [49] and rearranging gives

\[ \| \pi_V D\bar{\partial} \nabla_w \phi \|_{k-1, \delta} \leq c (\| \pi_V D\bar{\partial} \phi \|_{k, \delta} + \| \phi \|_{k-1, \delta}) \]

Then, using our inductive hypothesis,

\[ \| \nabla_w \phi \|_{k-1, \delta} \leq c \| \pi_V D\bar{\partial} \phi \|_{k, \delta} \]

In the above inequality \( c \) depends on \( w \), which is any extendible \( C^{\infty, \delta_0} \) vector field which is the lift of a vector field on \( \mathfrak{F} \). As we can span all vector fields on \( \mathfrak{C} \) by real functions times a finite collection of such \( C \) field which is the lift of a vector field on \( \mathfrak{F} \), we need \( \| \phi \|_{k, \delta} \) so that the above conclusion holds for the restriction of this neighbor-

hood to any of the curves in \( \mathfrak{C} \). Therefore we can bound the right most term in [49] as follows:

\[ \| \pi_V \nabla_w v \|_{k-1, \delta} \leq \left\| \sum_i f_i \nabla_w v_i \right\|_{k-1, \delta} \]
\[ \leq \sum_i \| f_i \|_{k-1, \delta} \| \nabla_w v_i \|_{k-1, \delta} \]
\[ \leq c \| v \|_{k-1, \delta} \]
\[ \leq c (\| D\bar{\partial} \phi \|_{k-1, \delta} + 2 \| \pi_V D\bar{\partial} \phi \|_{k, \delta}) \text{ using } [48] \]
\[ \leq c'(\| \phi \|_{k-1, \delta} + \| \pi_V D\bar{\partial} \phi \|_{k, \delta}) \]

Using the inequality [50] for the right most term in [49] and rearranging gives

\[ \| \pi_V D\bar{\partial} \nabla_w \phi \|_{k-1, \delta} \leq c (\| \pi_V D\bar{\partial} \phi \|_{k, \delta} + \| \phi \|_{k-1, \delta}) \]

Then, using our inductive hypothesis,

\[ \| \nabla_w \phi \|_{k-1, \delta} \leq c \| \pi_V D\bar{\partial} \phi \|_{k, \delta} \]

In the above inequality \( c \) depends on \( w \). For the bound [51] to imply the restriction of the inverse \( (\pi_V D\bar{\partial})^{-1} : Y/V_{0, \delta} \rightarrow X_{0, \delta} \) to \( Y/V_{k, \delta} \) is the inverse to \( \pi_V D\bar{\partial} : X_{k, \delta} \rightarrow Y/V_{k, \delta} \), we need \( \nabla_w \left( (\pi_V D\bar{\partial})^{-1} \nu \right) \) to exist for \( \nu \in Y/V_{k, \delta} \) so that the above argument can prove that \( (\pi_V D\bar{\partial})^{-1} \nu \in X_{k, \delta} \). This follows from the formula [49] and the bounded inverse to \( \pi_V D\bar{\partial} \) restricted to any fiber. Therefore, \( \pi_V D\bar{\partial} \) has a bounded inverse, and we have proved our proposition for the case that \( \nu \) is the zero section.

If \( \nu \) is a small enough in \( \| \phi \|_{k, \delta} \), then using the notation \( F \) and \( \Phi \) the trivialization from Definition 3.2 on page 33 \( \text{d}F(\nu)(\cdot) \) defines a \( C^{\infty, \delta_0} \) isomorphism between \( f^* T_{\text{vert}} \mathfrak{B} \) and \( (F(\nu))^* T_{\text{vert}} \mathfrak{B} \). We can repeat the above argument for the \( C^{\infty, \delta_0} \) family \( F(\nu) \) using a trivialization induced from \( F \) and \( \Phi \). Invertability of \( \pi_V D\bar{\partial}(\nu) \) in any norm weaker than \( C^{\infty, \delta_0} \) is equivalent to invertability of the corresponding operator at the zero section of the family \( F(\nu) \). Therefore, our proposition follows from the argument for the case of the zero section.

\[ \Box \]

**Proposition 3.15.** If \( f \) is a \( C^{\infty, \delta_0} \) curve where \( 0 < \delta_0 \leq 1 \), then for any \( 0 < \delta < \delta_0 \), there exists a neighborhood of 0 in \( X_{\delta} \) so that if \( \nu \) is in this neighborhood and \( \bar{\partial}' \nu \in C^{\infty, \delta_0} \) for any simple perturbation \( \bar{\partial}' \) of \( \bar{\partial} \), then \( \nu \in C^{\infty, \delta_0} \).

If \( f \) is part of a \( C^{\infty, \delta_0} \) family \( \hat{f} \), there is a neighborhood of 0 in \( X_{\delta} \) for this family \( \hat{f} \) so that the above conclusion holds for the restriction of this neighborhood to any of the curves in \( \hat{f} \).
Proof: Recall that Lemma 3.4 tells us that in a local coordinate chart,

$$\tilde{\partial}^* \nu = E(\nu) + H(\nu)(d_{\text{vert}} \nu)$$

where $E$, but not $H$ depends on the choice of simple perturbation of $\tilde{\partial}$. (Simple perturbations are defined on page 34.) The $d_{\text{vert}} \nu$ in this formula is dependent on our coordinate chart as it uses the connection given by a trivialization of our coordinate chart, so it (and $E$) are not globally defined. We can remedy this by choosing a linear $C^\infty$ connection $\nabla$. Then

$$\tilde{\partial}^* \nu = E'(\nu) + H(\nu)(\nabla \nu)$$

where $E'$ and $H$ are $C^\infty$ globally defined, and $H(0)(\nabla \theta) = \frac{1}{2}(\nabla \theta + J \circ \nabla \theta \circ J)$. The tensor $H$ is independent of the perturbation of $\tilde{\partial}$ which is chosen. As argued in Lemma 5.6, $\theta \mapsto H(\nu)(\nabla \theta)$ is Fredholm from $X_\delta \to Y_\delta$ for any $\delta < 1$. If we now choose $\delta < \delta_0$, then $H(\nu)(\nabla \cdot): X_\delta \to Y_\delta$ depends continuously on $\nu \in X_\delta$, so we have that there exists some neighborhood $U$ of 0 in $X_\delta$ so that $H(\nu)(\nabla \cdot): X_\delta \to Y_\delta$ is Fredholm. We can also choose this neighborhood $U$ so that $H(\nu)(\nabla \cdot)$ is a first order elliptic operator.

Now suppose that $\delta_1 + \delta' = \delta$ so in the case of a single curve $X_{0,\delta_1} = X_\delta$. If $\nu$ is in this neighborhood $U$, then there exists some constant $c$ depending on $\nu$ so that

$$\|\theta\|_{0,\delta_1}^1 \leq c(\sup |\theta| + \|H(\nu)(\nabla \theta)\|_{0,\delta_1})$$

Now let us try induction on the number of derivatives involved. Suppose that if $\nu$ is in the above neighborhood $U$, and $\nu \in X_{k,\delta_1}$, then

$$\|\theta\|_{k,\delta_1}^1 \leq c(\sup |\theta| + \|H(\nu)(\nabla \theta)\|_{k,\delta_1})$$

where $c$ depends on $k$ and $\nu$. Now, let $v$ be a $C^\infty$ vector field.

$$\|\nabla_v \theta\|_{k,\delta_1} \leq c(\sup |\nabla_v \theta| + \|H(\nu)(\nabla_v \nabla \theta)\|_{k,\delta_1})$$

where $c$ depends on $k$ and $\nu$. Now, let $v$ be a $C^\infty$ vector field.

$$\|\nabla_v \theta\|_{k,\delta_1} \leq c(\sup |\nabla_v \theta| + \|H(\nu)(\nabla_v \nabla \theta)\|_{k,\delta_1})$$

where $c$ depends on $k$ and $\nu$. Now, let $v$ be a $C^\infty$ vector field.

$$\|\nabla_v \theta\|_{k,\delta_1} \leq c(\sup |\nabla_v \theta| + \|H(\nu)(\nabla_v \nabla \theta)\|_{k,\delta_1})$$

We wish to estimate the term $\|\nabla_v H(\nu)\|_{k,\delta_1}$ above. We can estimate this by $\|\nabla_v H(\nu)\|_{k,\delta_1}$ times the supremum of the first $k$ derivatives of $\nabla \theta$. Standard Sobolev estimates imply that the map $\theta \mapsto \nabla \theta : X_{k+1,\delta_1} \to C^k$ is compact. (Recall that $X_{k+1,\delta_1}$ involves $k+2$ derivatives.) Our assumption that $\nu \in X_{k,\delta_1}$ implies that $\|\nabla_v H(\nu)\|_{k,\delta_1}$ is bounded, so the term $\|\nabla_v H(\nu)\|_{k,\delta_1}$ above is a compact operator $K(\theta)$.

$$\|\theta\|_{k+1,\delta_1}^1 \leq c\|H(\nu)(\nabla \theta)\|_{k+1,\delta_1} + \|K(\theta)\|$$

(52)

where $K$ is some compact linear operator. It follows that there exists some $c'$ so that

$$\|\theta\|_{k+1,\delta_1}^1 \leq c'(\sup |\theta| + \|H(\nu)(\nabla \theta)\|_{k+1,\delta_1})$$

(52)
converges, which contradicts the assumption that \( \|\theta_i\|_{k+1,\delta_i} = 1 \) and \( \sup |\theta_i| \to 0 \).

By induction, if \( \nu \in X_{k,\delta_1} \) is in the above neighborhood \( U \), then

\[
\|\theta\|_{k+1,\delta_1}^1 \leq c(\sup |\theta| + \|H(\nu)(\nabla \theta)\|_{k+1,\delta_1}) \tag{53}
\]

Now suppose that \( \nu \in X_{k,\delta_1} \) and \( \partial'\nu \in Y_{k+1,\delta_1} \). The assumption that \( \nu \in X_{k,\delta_1} \) implies that \( E'(\nu) \in Y_{k+1,\delta_1} \). Therefore \( H(\nu)(\nabla \nu) \in Y_{k+1,\delta_1} \). Our above estimate \( (53) \) then implies that \( \nu \in X_{k+1,\delta_1} \). We have therefore proved by induction that if \( \nu \) is in our neighborhood \( U \) chosen at the start of the proof, \( \partial'\nu \in Y_{k,\delta_1} \) implies that \( \nu \in X_{k,\delta_1} \).

In particular, suppose that \( \nu \) is in the above neighborhood \( U \) and \( \partial'\nu \in C^{\infty,\delta_1} \). Then we have \( \nu \in C^{\infty,\delta_1} \). Now apply \([8]\). The operator \( H(\nu)(\nabla \cdot) \) is elliptic and smooth (in the sense required for \([8]\)). The asymptotic operator corresponding to \( H(\nu)(\nabla \cdot) \) at any cylindrical end of a smooth component has spectrum equal to the integers. This is because \( H(\nu)(\nabla \theta) = \frac{1}{\nu}(\Phi \circ dF(\nabla \theta) + J\Phi \circ dF(\nabla \theta)j) \) where \( F \) and \( \Phi \) are as in Definition \([3.12]\) on page \([3.35]\). In particular, on edges of the domain \( \mathcal{C} \), the map \( \Phi \circ dF \) is just a constant linear isomorphism, because the smooth part of a section \( \nu \) is always constant along edges of the domain \( \mathcal{C} \). Therefore the spectrum of the asymptotic operator corresponding to \( H(\nu)(\nabla \theta) \) does not depend on \( \nu \), and is easily seen to be equal to the integers in the case that \( \nu = 0 \). Therefore, as \( \delta_0 \leq 1 \), we may apply \([8]\) to see that \( H(\nu)(\nabla \cdot) : X_\delta \to Y_\delta \) is Fredholm for all \( 0 < \delta < \delta_0 \). Noting that \( E'(\nu) \in C^{\infty,\delta_0} \), we then have that \( H(\nu)(\nabla \nu) \in C^{\infty,\delta_0} \), and we may repeat the above argument to see that \( \nu \in C^{\infty,\delta_1} \) for all \( 0 < \delta_0' < \delta_0 \). In other words, \( \nu \in C^{\infty,\delta_1} \).

Similarly, if \( f \) is part of a family \( \hat{f} \), there exists a neighborhood of 0 in \( X_\delta \) for this family \( \hat{f} \) so that for any \( \nu \) in this neighborhood, restricting to any fiber we have \( H(\nu(\nu))(\nabla \cdot) : X_\delta(p) \to Y_\delta(p) \) is Fredholm and elliptic. This was all that was required for the above argument to work. Therefore, restricting this neighborhood to any curve in our family \( \hat{f} \), the conclusions of the lemma hold as required.

\[ \square \]

**Definition 3.16.** If \( (\hat{f}, V) \) is a \( C^{\infty,\delta} \) pre obstruction model and \( \psi \) is a section of \( Y(\hat{f}) \), then say that \( \pi_V \psi \) is \( C^{\infty,\delta} \) if \( \psi \) is.

**Theorem 3.17.** Suppose that \( (\hat{f}, V) \) is a \( C^{\infty,\delta_1} \) pre obstruction model, so that on the curve over \( p \), \( \partial'f \in V \), and

\[
\pi_V \circ D\partial'(p) : X_\delta(p) \to Y/V_\delta(p)
\]

is invertible for some \( 0 < \delta < \delta_0 \leq 1 \). Then the restriction of \( (\hat{f}, V) \) to some topologically open neighborhood of \( p \), \( (\hat{f}', V) \) satisfies the following:

- There exists a neighborhood \( O \) of \( \partial' \) in the \( C^{1,\delta} \) topology, and a neighborhood \( O \) of 0 in \( X_{0,\delta} \) so that for any \( \delta'' \in O \), the map \( \pi_V \partial'' : O \to Y/V_0,\delta \) is a homeomorphism onto a neighborhood of 0 in \( Y/V_{0,\delta} \). For any \( \nu \in O \), \( \nu \) has regularity \( C^{\infty,\delta_1} \) if \( \pi_V \partial'' \nu \) has regularity \( C^{\infty,\delta_1} \). In particular, there is a unique \( C^{\infty,\delta_1} \) solution to the equation \( \pi_V \partial'' \nu = 0 \) over \( \hat{f}' \).

**Proof:** Apply Theorem \([3.11]\) to see that there exists a topologically open neighborhood of \( p \) so that for \( p' \) in this neighborhood \( \pi_V \circ D\partial'(p') : X_\delta(p') \to Y/V_\delta(p') \)
$Y/V_k(p')$ is invertible with a uniformly bounded inverse. Note that this implies that $\pi_V \circ D\bar{\vartheta}'(p') : X_\delta(p') \to Y/V_\delta(p')$ is also invertible for any $0 < \delta' < \delta_0$. If we restrict to any topologically compactly contained topologically open subset of this, we can use Theorem 3.11 again to tell us that for each $0 < \delta' < \delta_0$, the above map will have a uniformly bounded inverse.

Restrict so that resulting family $f'$ is an allowable family, then Proposition 3.13 states that for any $\delta + \delta' < \delta_0$, the map $\pi_V \circ D\bar{\vartheta}' : X_0,\delta \to Y/V_{0,\delta}$ is invertible. Call the norm of the inverse $M$. Theorem 3.5 implies that $\pi_V \circ \bar{\vartheta}' : X_{0,\delta} \to Y/V_{0,\delta}$ is $C^1$, so we may choose a neighborhood $O$ of $0 \in X_{0,\delta}$ so that for $\nu \in X_{0,\delta}$,

$$\left\| \left( \pi_V \circ D\bar{\vartheta}' \right)^{-1} \circ \pi_V \circ D\bar{\vartheta}'(\nu) \circ \text{id} \right\| < \frac{1}{2}$$

We may also choose a neighborhood $O'$ of $\delta'$, open in the $C^{1,\delta}$ topology so that for any $\delta'' \in O'$ and $\nu \in O$,

$$\left\| \left( \pi_V \circ D\bar{\vartheta}' \right)^{-1} \circ \pi_V \circ D\bar{\vartheta}''(\nu) \circ \text{id} \right\| < \frac{3}{4}$$

(To see the above inequality, note that the definition of the $C^{1,\delta}$ topology on page 40 implies that $\delta'(\nu) - \delta''(\nu) = E(\nu)$ where $E$ is $C^{\infty,\delta_0}$, vanishes on edges of curves in our family, and is small in the $C^{1,\delta}$ topology. Therefore $D\bar{\vartheta}(\nu)(\phi) - D\bar{\vartheta}'(\nu)(\phi) = DE(\nu)(\phi)$, so $D\bar{\vartheta}(\nu) - D\bar{\vartheta}'(\nu) : \nu \in Y/V_{0,\delta}$ may be made as small as we like independent of $\nu \in O$ by restricting $\bar{\vartheta}''$ to a $C^{1,\delta}$ small neighborhood $O$ of $\delta'$.)

Therefore, $\pi_V \circ \bar{\vartheta}''$ is a homeomorphism from $O$ to an open subset of $Y/V_{0,\delta}$. Choose $O$ small enough so that Proposition 3.15 holds. As the curve over $p$ satisfies $\pi_V \bar{\vartheta}' = 0$, by restricting to a small enough topological neighborhood of $p$, the image under $\pi_V \bar{\vartheta}'$ of $O$ will contain $0 \in Y/V_{0,\delta}$. By restricting $O$ to a neighborhood small enough in the $C^{1,\delta}$ topology, the same will hold for $\delta'' \in O$: the open set $\pi_V \bar{\vartheta}''(O)$ will contain $0 \in Y/V_{0,\delta}$.

It remains to prove that if $\nu \in O$ and $\pi_V \bar{\vartheta}'' \nu = \theta$ then $\nu$ is $C^{\infty,\delta_0}$. Do this locally around any point $q$ as follows: First, as $\pi_V \bar{\vartheta}''(q)$ is $C^{\infty,\delta_0}$, $\bar{\vartheta}''(q) \in C^{\infty,\delta_0}$ and Proposition 3.15 implies that $\nu(q) \in C^{\infty,\delta_0}$. Choose any $C^{\infty,\delta_0}$ extension of $\nu(q)$ to a topological neighborhood of $q$, and call this extension $\nu_q$.

Consider the Newton iteration scheme

$$\nu_{k+1} := \nu_k - \left( \pi_V \circ D\bar{\vartheta}''(\nu_k) \right)^{-1} \pi_V \circ D\bar{\vartheta}''(\nu_k) - \theta$$

Restricted to a small enough topological neighborhood of $q$, the above Newton iteration scheme will converge in $X_{0,\delta}$ and remain in (the restriction of) our chosen neighborhood $O$. The section $\nu_k$ is $C^{\infty,\delta_0}$. Now we shall see that this Newton iteration scheme will converge in $X_{k,\delta'}$ for any $k$ and $\delta' < \delta_0$ when restricted to a small enough topologically open neighborhood of $q$.

Claim: The size of $\| \pi_V \bar{\vartheta}''(\nu_k) \|_{k,\delta'}$ can be made arbitrarily small by restricting to a suitably small topologically open neighborhood of the curve over $q$. More precisely, $\| \pi_V \bar{\vartheta}''(\nu_k) \|_{k,\delta'}$ can be made arbitrarily small by restricting to a small enough neighborhood of the curve over $q$ while using the same metric and coordinate chart choices in the definition of $\| \|_{k,\delta'}$. Consider $\bar{\vartheta}''(\nu_k)$ as a section of the bundle $Y(f)$, As $\nu_k$ and $\theta$ are $C^{\infty,\delta_0}$, the above claim is equivalent to $\bar{\vartheta}''(\nu_k)$
being tangent to order \( k \) at the curve over \( q \) to some \( C^\infty_\theta \) section \( \theta' \) so that \( \pi_V \theta' = \theta \).

We shall prove the above claim by induction. This claim is true by assumption for \( k = 0 \), now assume that this claim is true for some \( k \). Therefore, \( \| \nu_{k+1} - \nu_k \|_{k,\delta'} \) can be made arbitrarily small by restricting to a suitably small open neighborhood of the curve over \( q \), i.e. \( \nu_k \) and \( \nu_{k+1} \) are tangent to order \( k \) at the curve over \( q \). Define an operator \((D^{k+1})\bar{\partial}'\) as follows: The domain of \((D^{k+1})\bar{\partial}'\) consists of \( \nu_\infty, \delta' \) so that \( \nu \) is tangent to \( \nu_k \) to order \( k \) over \( q \). Define \((D^{k+1})\bar{\partial}'\nu \) to be the derivative to order \( k + 1 \) of the section \( \bar{\partial}'\nu \), restricted to the curve over \( q \). With this domain, \((D^{k+1})\bar{\partial}'\) is an affine operator. In other words, the property of \( \bar{\partial}'\) that this operator measures is affine restricted to this domain. Therefore the next step in our Newton iteration will give an exact solution from the perspective of this operator. In other words, \( \nu_{k+1} \) will be tangent to order \( k + 1 \) at the curve over \( q \) to some \( \theta' \) so that \( \pi_V \theta' = \theta \). The claim of the above paragraph has now been proven.

Proposition 3.14 and Theorem 3.5 combined with the above claim implies that for any \( \delta' < \delta_0 \), restricted to a small enough topological neighborhood of the curve over \( q \), the above Newton iteration scheme converges in \( X_{k,\delta'} \) to our solution \( \nu \) to \( \pi_V \theta' = \theta \). This implies that for all \( k \) and \( \delta' < \delta_0 \), there exists some topologically open neighborhood of \( q \) so that our solution \( \nu \) is in \( C^k,\delta' \) restricted to this neighborhood. Repeating the argument around any point gives that \( \nu \) is \( C^\infty_\theta \).

4 Results independent of analytic choices

The following theorem states roughly that the explosion of Deligne Mumford space, Expl\( \mathcal{M} \) (discussed in \cite{12}) represents the moduli stack of \( C^\infty_\Delta \) families of stable exploded curves. A similar theorem probably holds over the complex version of the exploded category with ‘smooth and holomorphic’ replacing ‘\( C^\infty_\Delta \)’.

Theorem 4.1. Consider any \( C^\infty_\Delta \) family of exploded curves \((\hat{\mathcal{C}},j) \to \mathfrak{F})\) so that each exploded curve is connected and has \( 2g + n \geq 3 \) where \( g \) is the genus and \( n \) is the number of punctures. Then for \( \delta \leq 1 \), there exists a unique fiber wise holomorphic map

\[
\begin{array}{ccc}
(\hat{\mathcal{C}},j) & \to & (\text{Expl} \mathcal{M}^{+1},j) \\
\downarrow & & \downarrow \\
\mathfrak{F} & \to & \text{Expl} \mathcal{M}
\end{array}
\]

so that the map on each fiber \( \mathcal{C} \) factors into a degree one holomorphic map to a stable exploded curve \( \mathcal{C}' \) and a map from \( \mathcal{C}' \) to a fiber of \( \text{Expl} \mathcal{M}^{+1} \) given by quotienting \( \mathcal{C}' \) by its automorphism group.

The above maps all have regularity \( C^\infty_\Delta \).

Proof:

We first construct this map for the fiber \( \mathcal{C} \) over a single point of \( \mathfrak{F} \). The first stage of this is to construct a stable curve \( \mathcal{C}' \) with a holomorphic degree one
map \( \mathcal{C} \to \mathcal{C}' \). The idea is to ‘remove’ all unstable components using a series of maps of the following two types:

1. If a smooth component of \( \mathcal{C} \) is a sphere attached to only one edge, the other end of the edge is attached to some other smooth component in a coordinate chart modeled on an open subset of \( \mathbb{T}_1 \) with coordinate \( \hat{z} \). Replace this coordinate chart with the corresponding open subset of \( \mathcal{C} \) with coordinate \( z = [\hat{z}] \). There is an obvious degree one holomorphic map from our old curve to this new one that is given in this coordinate chart by \( \hat{z} \mapsto [\hat{z}] \), and sends our unstable sphere and the edge attached to it to the point \( p \) where \( z(p) = 0 \). (This map is the identity everywhere else.)

2. If a smooth component of \( \mathcal{C} \) is a sphere attached to two edges, there exists a holomorphic identification of a neighborhood of this smooth component with a refinement of an open subset of \( \mathbb{T} \). Replace this open set with the corresponding open subset of \( \mathcal{C} \). The degree one holomorphic map from the old exploded curve to the new one is this refinement map. (Refer to [12] for the definition of refinements.)

Each of the above types of maps removes one smooth component, so after applying maps of the above type a finite number of times, we obtain a connected exploded curve with no smooth components which are spheres with one or two punctures. Our theorem’s hypotheses then imply that the resulting exploded curve \( \mathcal{C}' \) must be stable. It is not difficult to see that the stable curve obtained this way is unique.

The smooth part of this stable exploded curve \( \mathcal{C}' \) is a stable nodal Riemann surface with punctures, \([\mathcal{C}']\). This nodal Riemann surface determines a point in Deligne Mumford space \( p_{[\mathcal{C}']} \in \mathcal{M} \), and a corresponding map of \([\mathcal{C}']\) to the fiber of \( \mathcal{M}^{+1} \) over this point \( p_{[\mathcal{C}']} \). Note that \( \mathcal{M} \) is the smooth part of \( \text{Expl}\mathcal{M} \) and \( \mathcal{M}^{+1} \) is the smooth part of \( \text{Expl}\mathcal{M}^{+1} \). The smooth part of our map \( \mathcal{C}' \to \text{Expl}\mathcal{M}^{+1} \) must correspond with this map \([\mathcal{C}']\to\mathcal{M}^{+1} \). We must now determine the remaining information.

If we give Deligne Mumford space its usual holomorphic structure, there is a holomorphic uniformising chart \((U,G)\) containing this point \( p_{[\mathcal{C}']} \), where \( U \) is some subset of \( \mathbb{C}^n \) so that the boundary strata of \( \mathcal{M} \) correspond to where coordinates \( z_i = 0 \), and \( G \) is a finite group with a holomorphic action on \( U \) which preserves the boundary strata. \( \text{Expl}\mathcal{M} \) is constructed so that it has a corresponding uniformising coordinate chart \((\hat{U}, G)\) where \( \hat{U} \) is an open subset of \( \mathbb{T}_n \) which corresponds to the set where \([\hat{z}] \in U\), and the action of \( G \) on \( \hat{U} \) induces the action of \( G \) on the smooth part \([\hat{U}] = U\). The inverse image of \( \hat{U} \) in \( \text{Expl}\mathcal{M}^{+1} \) is some exploded object \( U^{+1} \) quotiented by \( G \), and the smooth part of this is the inverse image of \( U \) in \( \mathcal{M}^{+1} \), which is equal to some smooth complex manifold \( U^{+1} \) quotiented by \( G \). There are \([G]\) identifications of \([\mathcal{C}]\) with a fiber of \( U^{+1} \to U \), which are permuted by the action of \( G \), (so together they correspond to a unique map to \( \mathcal{M}^{+1} \)). Choose one of these maps.

Each of the nodes of \([\mathcal{C}]\) now correspond to some coordinate \( z_i \) on \( U \) which is equal to 0. We must determine the value of these \( \hat{z}_i \). (All other coordinates are nonzero so \( \hat{z}_k \) is given by \( \hat{z}_k = z_k \).) There is a chart \( U_i^{+1} \) on \( U^{+1} \) containing this node which is equal to a convex open subset of \( \mathbb{C}^{n+1} \) with coordinates \( z_j \), \( j \neq i \) and \( z_i^+ \), \( z_i^- \), so that the map \( U_i^{+1} \to U \) is given by \( z_i = z_i^+z_i^- \) and \( z_j = z_j \). The
identification of a neighborhood of this node with a fiber of $U_i^{+1}$ means that we can use $z_i^+$ and $z_i^-$ respectively to parametrize the two discs that make up the neighborhood of the node. The open subset of $\mathcal{C}'$ with smooth part equal to this neighborhood can then be covered by two open subsets of $\mathbb{T}_i^1$ with coordinates $\tilde{z}_i^+$ and $\tilde{z}_i^-$ so that $[\tilde{z}_i^+] = z_i^+$. The transition between these coordinates charts is given by an equation

$$\tilde{z}_i^+ \tilde{z}_i^- = ct^i$$

Note that the constant $ct^i$ is canonically determined by our choice of coordinate chart. Our coordinate $\tilde{z}_i$ must be equal to $ct^i$. To see this consider the corresponding coordinate chart $\tilde{U}_i^{+1}$ with coordinates $\tilde{z}_j$ and $\tilde{z}_i^+$ so that $[\tilde{z}_j] = z_j$ and $[\tilde{z}_i^+] = z_i^+$. The map $\tilde{U}_i^{+1} \rightarrow \tilde{U}$ is given by $\tilde{z}_i = \tilde{z}_i^+ \tilde{z}_i^-$ and $\tilde{z}_j = \tilde{z}_j$.

The smooth part of the intersection of our curve with $\tilde{U}_i^{+1}$ must be as described above, and the parametrization of the smooth part by $[\tilde{z}_i^+]$ must also be as above. The fiber is over a point where $\tilde{z}_i = ct^i$ is given by gluing two coordinate charts in $\mathbb{T}_i^1$ by the equation $\tilde{z}_i^+ \tilde{z}_i^- = ct^i$. This fiber is equal to the corresponding open subset of our curve $\mathcal{C}'$ and parametrized correctly if and only if $\tilde{z}_i = ct^i$.

We have shown that after choosing any one of the $|G|$ holomorphic maps \([\mathcal{C}'] \rightarrow U^{+1}\) there is a unique holomorphic map $\mathcal{C}' \rightarrow \tilde{U}^{+1}$ onto a fiber of $U^{+1}$ with smooth part equal to this. Therefore, there is a unique holomorphic map $\mathcal{C}' \rightarrow \text{Expl} \mathcal{M}^{+1}$ which factors as an inclusion as a fiber of $\tilde{U}^{+1}$ followed by quotienting by the action of the group $G$. In particular, there is a unique holomorphic map $\mathcal{C}' \rightarrow \text{Expl} \mathcal{M}^{+1}$ satisfying the required conditions of our theorem. This completes the construction of our map for each individual fiber. We must now verify that the resulting map on the total space has regularity $C^{\infty, \Delta}$.

To verify the regularity of the map we've constructed, we need only to work locally around a fiber. As this is local, we may assume that $\mathfrak{F}$ is covered by a single standard coordinate chart. Start with the map on a single fiber $\mathcal{C} \rightarrow \tilde{U}^{+1}$ constructed above. We shall prove that this extends to a $C^{\infty, \Delta}$ fiberwise holomorphic map from a topological neighborhood of the fiber. The uniqueness of our map on fibers shall then imply that this map must agree with the map constructed above, proving the required regularity. We shall consider $\tilde{U}^{+1} \rightarrow \tilde{U}$ to give a family of targets, to which we shall first construct a smooth map from a topological neighborhood of the fiber, and then apply Theorem 3.17 to correct this to a fiberwise holomorphic map.

Construct a smooth extension of $\mathcal{C} \rightarrow \mathcal{C}' \rightarrow \tilde{U}^{+1}$ using local coordinate charts as follows. Cover $\tilde{U}^{+1}$ with a finite number of charts of the following three types: the charts $\tilde{U}_{i}^{+1}$ mentioned earlier which cover edges of the image of $\mathcal{C}'$; charts covering punctures of the image of $\mathcal{C}'$, which are all of the form of some open subset of $\mathbb{T}_i^1 \times \tilde{U}$; and charts containing only smooth parts of the image of $\mathcal{C}'$, which can all be identified with some open subset of $\mathcal{C}$ times $\tilde{U}$. On $\mathfrak{F} \rightarrow \mathfrak{F}$, consider a single coordinate chart on $\mathfrak{F}$ which we may assume (without losing generality in this part of the construction) is equal to some subset of $\mathbb{T}_i^1$ containing all strata of $A$. (If this is not the case, and our coordinate chart on $\mathfrak{F}$ is a subset of $\mathbb{R}^n \times \mathbb{T}_i^1$, we may construct our smooth map to be independent of the $\mathbb{R}^n$ coordinates.) Cover the inverse image of this coordinate chart in $\mathfrak{F}$ with a finite number of coordinate charts $\mathfrak{V}$ which project to $\mathbb{T}_i^1$ in one of the standard forms for coordinate charts on families discussed in [12]. Construct these charts $\mathfrak{V}$ small enough so that the portion of $\mathcal{C}$ contained in any one of
these coordinate charts is contained well inside one of the coordinate charts on $\tilde{U}^{+1}$. In this case say that $\mathcal{Y}$ is ‘sent to’ the corresponding chart on $\tilde{U}^{+1}$.

Consider a chart $\mathcal{Y}$ on $\mathcal{C}$ which is sent to a chart $\tilde{U}^{+1}_i$ corresponding to the $i$th node of $\mathcal{C}$. We may assume by extending the tropical part of this chart $\mathcal{Y}$ as much as possible, and throwing away unneeded charts, that:

- this chart $\mathcal{Y}$ has an integral affine polygon, $\mathcal{Y}$ in $\tilde{U}^{+1}_i$.
- the tropical part of this chart $\mathcal{Y}$ intersects at most one other chart in
  a set of dimension larger than the dimension of $A = [\Sigma_n]$, and further
  more that when such an intersection does occur, the intersection of the
two charts is dense inside each of their tropical parts.

Define integral affine functions $h_i, h_{\mathcal{Y}}$ and $h_{i,\mathcal{Y}}^\pm$ on $[\mathcal{Y}]$ as follows: If the tropical part of $\mathcal{C}$ intersecting this chart is sent to a single point in $\tilde{U}^{+1}_i$, then set all three of these functions equal to $t^0$. If not, define $h_i, h_{\mathcal{Y}}$ on $A$ to be the size of the inverse image of $A$ in $[\mathcal{Y}]$, and define $h_{i,\mathcal{Y}}^\pm$ on $[\mathcal{Y}]$ to be the distance to either end of the fibers of $[\mathcal{Y}] \to A$, choosing the relevant ‘ends’ so that on the intersection with $\mathcal{C}$ these $h_{i,\mathcal{Y}}^\pm$ are equal to the pull back of $|\tilde{z}_i^\pm|$ times some constant. We may define similar functions on all intersections of charts $\mathcal{Y}$. Now define the function $h_i$ on $A$ as follows: multiply together all the functions $h_{i,\mathcal{Y}}$ from each of the coordinate charts above, dividing by the corresponding functions for intersections. Note that on the intersection with $\mathcal{C}$, this integral affine function is equal to $|\tilde{z}_i|$. We now define integral affine functions $h_i^\pm$ which correspond to $|\tilde{z}_i^\pm|$. Define a partial order on these charts as follows: if $|\tilde{z}_i^\pm|$ is greater on the part of $\mathcal{C}$ in chart 1 than on the part of $\mathcal{C}$ in chart 2, and at some point strictly greater, then say that chart 1 is greater than chart 2. Define $h_i^+$ on $\mathcal{C}$ to be equal to the product of $h_i^+\mathcal{Y}$ with $h_{i,\mathcal{Y}}^+$ for all $\mathcal{Y}$ greater that $\mathcal{Y}$ divided by the corresponding functions for intersections which are greater than $\mathcal{Y}$. Similarly define $h_i^-$. Define the function $\tilde{z}_i$ on $\Sigma_n$ to be equal to the unique monomial so that $|\tilde{z}_i^\pm| = h_i$ and $\tilde{z}_i$ restricted to $\mathcal{C}$ is equal to the pull back of $\tilde{z}_i$ from $\tilde{U}$. Doing the same for all other nodes and setting the other coordinates constant gives our smooth map from our subset of $\Sigma_n$ to $\tilde{U}$.

Now choose functions $\tilde{z}_i^\pm$ on each coordinate chart $\mathcal{Y}$ so that

1. $\tilde{z}_i^+\tilde{z}_i^- = \tilde{z}_i$
2. $|\tilde{z}_i^\pm| = h_i^\pm$
3. Restricted to $\mathcal{C}$, $\tilde{z}_i^\pm$ is equal to the pull back of $\tilde{z}_i^\pm$ from $\tilde{U}_i$.

Because the tropical part $|\tilde{z}_i^\pm|$ is compatible with coordinate changes, and because $\tilde{z}_i^\pm$ is compatible with coordinate changes, these functions are compatible with coordinate changes on any fiber topologically equivalent to $\mathcal{C}$, and are almost compatible with coordinate changes in a small topological neighborhood of $\mathcal{C}$. We can therefore modify them to obey all the above conditions, and be compatible with coordinate changes, defining smooth exploded functions $\tilde{z}_i^\pm$ on the union of all coordinate charts $V$ which are sent to $\tilde{U}_i^{+1}$. These together with the map from our subset of $\Sigma_n$ to $\tilde{U}$ defined above define smooth maps from these coordinate charts to $\tilde{U}_i^{+1}$ which are compatible with coordinate changes.
We must also define our map on coordinate charts which are sent to coordinates charts on $\tilde{U}^+$ which are a product of $\tilde{U}$ with some topologically open subset of $\mathbb{T}_1^1$. As we already have our map to $\tilde{U}$, this amounts to constructing a map into $\mathbb{T}_1^1$, which we shall give a coordinate $\tilde{w}$. The construction of this map is entirely analogous to the construction of the function $\tilde{z}_i^+$ above. Once we have done this, we have smooth maps from each of our coordinate charts into $\tilde{U}^+$ which are compatible with coordinate changes on any fiber topologically equivalent to $\mathcal{C}$, and which agree with the map constructed earlier on $\mathcal{C}$. There is no obstruction to modifying these maps to give a smooth map which is compatible with all coordinate changes and satisfies the required conditions.

We have now shown that there exists a smooth map

$$
\begin{array}{ccc}
\mathcal{C} & \to & \tilde{U}^+ \\
\downarrow & & \downarrow \\
\tilde{F} & \to & \tilde{U}
\end{array}
$$

so that the restriction to the fiber $\mathcal{C}$ is holomorphic. (We proved this under the assumption that $\tilde{F}$ is covered by a single coordinate chart.) We now wish to show that this can be modified to a fiber wise holomorphic $C^\infty$ map on a topological neighborhood of $\mathcal{C}$. We shall show below that if the above is considered as a map into a family of targets $\tilde{U}^+ \to \tilde{U}$, the cokernel of the relevant linearized $\bar{\partial}$ operator is naturally identified with the cotangent space of $\tilde{U}$. To deal with this cokernel within the framework of this paper, we shall extend our map to a smooth map

$$
\begin{array}{ccc}
\mathcal{C} \times \mathbb{R}^n & \to & \tilde{U}^+ \\
\downarrow & & \downarrow \\
\tilde{F} \times \mathbb{R}^n & \to & \tilde{U}
\end{array}
$$

where the tangent space of $\tilde{U}$ is identified with $\mathbb{R}^n \times \tilde{U}$, and the derivative of this map on the $\mathbb{R}^n$ factor at 0 is the identity.

Consider the corresponding linearized operator $D\bar{\partial}$ at $\dot{\tilde{\gamma}}$ restricted to our curve $\mathcal{C}$. This is just the standard $\bar{\partial}$ operator on sections of the pullback of $T\mathcal{C}'$ to $\mathcal{C}$. Standard complex analysis tells us that as $\mathcal{C}'$ is stable, this operator is injective, and has a cokernel which we may identify with ‘quadratic differentials’, which are holomorphic sections $\theta$ of the pull back to $\mathcal{C}$ of the symmetric square of the holomorphic cotangent bundle of $\mathcal{C}'$ which vanish at punctures. (This actually corresponds to allowing a simple pole at punctures viewed from the smooth perspective as quadratic differentials on $\mathbb{T}_1^1$ look like holomorphic functions times $(\tilde{z}^{-1}d\tilde{z})^2$). This is proved by showing that the quadratic differentials are the kernel of the adjoint of $\bar{\partial}$. All we shall use is that the relationship is as follows: the wedge product of $\bar{\partial}v$ with $\theta$ gives a two form which is equal to $d(\theta(v))$. This vanishes at all edges and punctures because $\bar{\partial}v$ does, so the integral is well defined. Therefore, as it equals $d(\theta(v))$, and $\theta(v)$ is a one form which vanishes on punctures, the integral of the wedge product of $\bar{\partial}v$ with $\theta$ over $\mathcal{C}$ must vanish. As holomorphic sections of the pullback to $\mathcal{C}$ of any bundle on $\mathcal{C}'$ can be identified with holomorphic sections of the bundle on $\mathcal{C}'$, we may identify the cokernel of our $D\bar{\partial}$ with the quadratic differentials on $\mathcal{C}'$.

We can also identify the holomorphic cotangent space to $\tilde{U}$ at the image of the curve $\mathcal{C}'$ with the space of quadratic differentials as follows: Refine $\tilde{U}$ so that $\mathcal{C}'$ is the fiber over a smooth point, and trivialize a small neighborhood of
in the corresponding refinement of $\hat{U}^{+1}$. Given a tangent vector $v$ to $\hat{U}$, by differentiating the almost complex structures on fibers using our trivialization, we then obtain a tensor $v(j)$ which is a section of $T^*\mathcal{C} \otimes T\mathcal{C}'$. The derivative of $\tilde{\partial}g$ using this trivialization at the curve $\mathcal{C}$ in the direction corresponding to $v$ is $\frac{1}{2}v(j) \circ j$. Then taking the wedge product of $\frac{1}{2}v(j) \circ j$ with a quadratic differential gives a two form on $\mathcal{C}'$ which we can then integrate over $\mathcal{C}'$. The result of this integral does not depend on the choice of trivialization because of the above discussion identifying the cokernel of the restriction of $\bar{D}\tilde{\partial}$ to vertical vector fields with the quadratic differentials. It is a standard fact from Teichmuller theory that this will give an identification of quadratic differentials with the holomorphic cotangent space to $U$ when $\mathcal{C}'$ has no internal edges. It follows from this fact that restricting to quadratic differentials that vanish on edges, we get an identification of the space of quadratic differentials with the holomorphic cotangent space to the appropriate strata of the smooth part of $\hat{U}$. Using this fact, it is not difficult to prove directly that the above gives an identification of the space of quadratic differentials with the holomorphic cotangent space to $\hat{U}$ in general.

Add a bundle $V$ to our smooth map to get a smooth pre obstruction model $(\hat{g}, V)$ so that the fibers of $V$ are dual to the space of quadratic differentials, and $\pi_V \bar{D}\tilde{\partial}$ is an isomorphism. Theorem 5.17 implies that we can modify $(\hat{g}, V)$ on a topological neighborhood of $\mathcal{C}$ to a $C^\infty$ pre obstruction model $(\hat{f}, V)$ with $\bar{\partial}\hat{f}$ a section of $V$. Referring to this topological neighborhood as $\mathbb{R}^n \times \mathfrak{g}$, we have

$\begin{array}{c}
V \\
\downarrow \bar{\partial} \\
\mathbb{R}^n \times \mathfrak{g}
\end{array}$

The differential of $\bar{\partial}$ restricted to the $\mathbb{R}^n$ factor at 0 is surjective due to the identification of the cotangent space of $U$ with the space of quadratic differentials. Therefore, there is a $C^\infty$ map from a topological neighborhood of $\mathcal{C}$ in $\mathfrak{g}$ to $\mathbb{R}^n \times \mathfrak{g}$ so that the composition with $\bar{\partial}$ is 0. This constructs a $C^\infty$ map from a topological neighborhood of $\mathcal{C}$ to $\hat{U}^{+1}$ which is fiber wise holomorphic, and which is equal to our chosen map on $\mathcal{C}$. The uniqueness proved above gives that this must agree with our map to $\text{Exp} \mathcal{M}^{+1}$, therefore this map must therefore actually be $C^\infty$.

Recall the definition of a core family given on page 4. The following theorem gives criteria for when a given family with a collection of marked point sections is a core family:

**Theorem 4.2.** A $C^{\infty,1}$ family of curves $\hat{f}$ with automorphism group $G$, set of disjoint sections $s_i : \mathfrak{g}(\hat{f}) \rightarrow \mathcal{C}(\hat{f})$ which do not intersect the edges of the curves in $\mathcal{C}(\hat{f})$ and a $C^{\infty,1}$ map

$\begin{array}{c}
\hat{f}^*T_{\text{vert}}\mathcal{B} \\
\downarrow \\
\mathfrak{g}(\hat{f}) \rightarrow \mathcal{E}
\end{array}$

is a core family $(\hat{f}/G, \{s_i\}, F)$ for some $C^{1,1}$ neighborhood $O$ of $\hat{f}$ in the moduli stack of $C^{\infty,1}$ curves if and only if the following criteria are satisfied:

1. For all curves $f$ in $\hat{f}$, there are exactly $|G|$ maps $f \rightarrow \hat{f}$.
2. For all curves \( \vec{f} \) in \( \hat{\mathfrak{F}} \), the smooth part of the domain \( \mathcal{C}(f) \) with the extra marked points from \( \{s_i\} \) has no automorphisms.

3. The action of \( G \) preserves the set of sections \( \{s_i\} \), so there is some action of \( G \) as a permutation group on the set of indices \( \{i\} \) so that for all \( g \in G \) and \( s_i \),

\[
s_i \circ g = g \circ s_{g(i)}
\]

where \( g \) indicates the action of \( g \) on \( \mathfrak{F}(\hat{f}) \), \( \mathcal{C}(\hat{f}) \) or the set of indices \( \{i\} \) as appropriate.

4. There exists a neighborhood \( U \) of the image of the section

\[
s : \mathfrak{F}(\hat{f}) \rightarrow \mathfrak{F}(\hat{f}^{-n})
\]

defined by the \( n \) sections \( \{s_i\} \) so that

\[
ev^{+n}(\hat{f}) : \mathfrak{F}(\hat{f}^{-n}) \rightarrow \text{Expl}_\mathcal{C} \times \left( \mathfrak{F} \right)_\mathfrak{F}
\]

is an equi-dimensional embedding when restricted to \( U \).

5. The map \( F \) restricted to the zero section of \( f^*T_{vert} \mathfrak{B} \) is equal to \( \hat{f} \), and \( TF \) restricted to the inclusion of \( f^*T_{vert} \mathfrak{B} \) into the tangent space of \( f^*T_{vert} \mathfrak{B} \) over the zero section is the natural inclusion \( f^*T_{vert} \mathfrak{B} \rightarrow T_{vert} \mathfrak{B} \).

Proof: Throughout this proof, use \( \mathfrak{F} \) to refer to \( \mathfrak{F}(\hat{f}) \). Consider the pullback of the family of curves \( \hat{f}^{+n} \) under the map

\[
s : \mathfrak{F} \rightarrow \mathfrak{F}(\hat{f}^{+n})
\]

This gives a family of curves \( s^*(\hat{f}^{+n}) \) over \( \mathfrak{F} \) with \( n \) extra punctures. For any individual curve, \( \vec{f} \in \hat{f} \), criteria \[ and \[ imply that the family \( \hat{f}^{+n} \) contains exactly \( |G| \) curves which are contained in \( s^*(\hat{f}^{+n}) \), and criterion \[ implies that \( ev^{+n}(\hat{f})(\mathfrak{F}(\hat{f}^{+n})) \) intersects the image of the section \( s : \mathfrak{F} \rightarrow \mathfrak{F}(\hat{f}^{+n}) \) under \( ev^{+n}(\hat{f}) \) transversely at each of the \( |G| \) points in \( \mathfrak{F}(\hat{f}^{+n}) \) corresponding to these curves. Therefore, for any curve \( \vec{f}' \) sufficiently topologically close to \( \vec{f} \) in \( C^{1,\delta} \), \( ev^{+n}(\vec{f}')(\mathfrak{F}(\hat{f}^{+n})) \) intersects the image of \( ev^{+n}(\vec{f})(s(\mathfrak{F})) \) transversely \( |G| \) times so that the corresponding \( |G| \) curves in \( f^{+n} \) are topologically close in \( C^{1,\delta} \) to curves in \( s^*(\hat{f}^{+n}) \). In other words, there exists a \( C^{1,\delta} \) neighbourhood \( \mathcal{O} \) of \( \vec{f} \) and a \( C^{1,\delta} \) neighbourhood \( \mathcal{O}_s \) of \( s^*(\hat{f}^{+n}) \) so that for any curve \( f' \) in \( \mathcal{O} \), \( ev^{+n}(f')(\mathfrak{F}(\hat{f}^{+n}|_{\mathcal{O}_s})) \) intersects the image of \( ev^{+n}(\vec{f})(s(\mathfrak{F})) \) transversely exactly \( |G| \) times, where \( f^{+n}|_{\mathcal{O}_s} \) indicates the restriction of \( f^{+n} \) to curves in \( \mathcal{O}_s \).

It follows that for any family \( \hat{f} \) of curves in \( \mathcal{O} \), the following fiber product is a \( |G| \)-fold multisection of \( \mathfrak{F}(\hat{f}^{+n}) \rightarrow \mathfrak{F}(\hat{f}') \)

\[
\mathfrak{F}(\hat{f}^{+n}) \leftarrow \mathfrak{F}(\hat{f}^{+n}|_{\mathcal{O}_s})_{ev^{+n}(\hat{f})} \times_{ev^{+n}(\hat{f})_{\mathcal{O}_s}} \mathfrak{F}(\hat{f}) \rightarrow \mathfrak{F}(\hat{f})
\]

\[ (54) \]

We therefore get a map from this \( |G| \)-fold cover of \( \mathfrak{F}(\hat{f}') \) to \( \mathfrak{F}(\hat{f}) \). Criterion \[ implies that the action of \( G \) on \( \mathfrak{F}(\hat{f}) \) gives an action of \( G \) on \( ev^{+n}(\hat{f})(s(\mathfrak{F})) \)
which does nothing apart from permuting the marked points. As the image of \(ev^{+n}(\hat{f}')\) automatically contains all the results of a permutation of marked points, this \(G\) action gives an action of \(G\) on the above fiber product in (54). This makes the above \(|G|\)-fold cover of \(\mathfrak{g}(\hat{f}')\) into a \(G\)-bundle because the action on \(ev^{+n}(\hat{f})(s(\mathfrak{g}))\) simply permutes the marked points, so each \(G\)-orbit is contained within the same fiber of \(\mathfrak{g}(\hat{f}')\). Therefore, the above map from our \(G\)-bundle to \(\mathfrak{g}\) is equivalent to a map from \(\mathfrak{g}(\hat{f}')\) to \(\mathfrak{g}/G\). It follows from Theorem 4.1 that if \(\hat{f}' \in C^\infty,\mathfrak{l}\), this map is actually \(C^\infty,\mathfrak{l}\) map.

There is a unique lift of this map \(\mathfrak{g}(\hat{f}') \rightarrow \mathfrak{g}(\hat{f})/G\) to a fiberwise holomorphic map

\[
\begin{align*}
\mathcal{C}(\hat{f}') & \xrightarrow{\Phi} \mathcal{C}(\hat{f})/G \\
\downarrow & \downarrow \\
\mathfrak{g}(\hat{f}') & \rightarrow \mathfrak{g}(\hat{f})/G
\end{align*}
\]

so that \(\hat{f}'\) is equal to \(\hat{f} \circ \Phi\) when restricted to the pullback under \(\Phi\) of each of the sections \(\hat{s}_i\). This map \(\Phi\) is constructed as follows: Consider the map

\[
ev^{+n}(\hat{f}) : \mathcal{C}(\hat{f}^{+n}) \rightarrow \text{Expl}_\mathfrak{M} \times \left(\mathfrak{B}\right)^n
\]

which is equal to \(ev^{+n+1}(\hat{f})\) composed with a projection \(\text{Expl}_\mathfrak{M} \times \left(\mathfrak{B}\right)^{n+1} \rightarrow \text{Expl}_\mathfrak{M} \times \left(\mathfrak{B}\right)^n\) forgetting the image of the \((n + 1)\)st marked point and also equal on the second component to the composition of the projection \(\mathcal{C}(\hat{f}^{+n}) \rightarrow \mathfrak{g}(\hat{f}^{+n})\) with the map \(\hat{f}^{+n-1}\). Criteria 2 and 4 imply that this evaluation map \(\hat{f}^{+n}(\hat{f})\) is an equidimensional embedding in a neighborhood of \(\mathcal{C}(s^*\hat{f}^{+n}) \subset \mathcal{C}(\hat{f}^{+n})\), and the following is a pullback diagram of families of curves

\[
\begin{align*}
\mathcal{C}(\hat{f}^{+n}) & \xrightarrow{\hat{f}^{+n}(\hat{f})} \text{Expl}_\mathfrak{M}^{+1} \times \left(\mathfrak{B}\right)^n \\
\downarrow & \downarrow \\
\mathfrak{g}(\hat{f}^{+n}) & \xrightarrow{\hat{f}^{+n}(\hat{f})} \text{Expl}_\mathfrak{M} \times \left(\mathfrak{B}\right)^n
\end{align*}
\]

Use the notation \(\mathfrak{g}'\) for the \(G\)-bundle over \(\mathfrak{g}(\hat{f}')\) featured above in (54)

\[
\mathfrak{g}' := \mathfrak{g}(\hat{f}'^{+n}|_{\mathfrak{g}_0}) \times_{ev^{+n}(\hat{f})}\mathfrak{g}(\hat{f}) \subset \mathfrak{g}(\hat{f}'^{+n})
\]

The action of \(G\) on \(\mathfrak{g}'\) is some permutation of marked points. This \(G\) action extends to a \(G\) action on \(\mathfrak{g}(\hat{f}'^{+n})\) permuting these marked points, and lifts to a \(G\) action on \(\mathcal{C}(\hat{f}'^{+n})\) which just permutes the same marked points. Let \(\mathcal{C}'\) be the subset of \(\mathcal{C}(\hat{f}'^{+n})\) over \(\mathfrak{g}'\). The above mentioned \(G\) action makes \(\mathcal{C}'\) a \(G\)-bundle over \(\mathcal{C}(\hat{f}')\). There is a unique \(C^\infty,\mathfrak{l}\) map \(\hat{\Phi}\) from \(\mathcal{C}'\) to \(\mathcal{C}(\hat{f}^{+n})\) so that \(\hat{f}^{+n}(\hat{f}) \circ \hat{\Phi} = \hat{f}^{+n}(\hat{f})\) on \(\mathcal{C}'\). This map \(\hat{\Phi}\) composed with the projection \(\mathcal{C}(\hat{f}^{+n}) \rightarrow \mathcal{C}\) which forgets the marked points is \(G\) equivariant, and corresponds to a \(C^\infty,\mathfrak{l}\) map \(\Phi : \mathcal{C}(\hat{f}') \rightarrow \mathcal{C}(\hat{f})/G\).

The fact that this map \(\Phi : \mathcal{C}(\hat{f}') \rightarrow \mathcal{C}/G\) is \(C^\infty,\mathfrak{l}\) means the composition of it with \(\hat{f}\) is \(C^\infty,\mathfrak{l}\). This is by construction close to our other family of curves \(\hat{f}'\), and is equal to \(\hat{f}'\) on all the marked points coming from \(\{s_i\}\). Criterion 5 implies that there is some neighborhood \(U\) of the zero section in \(\hat{f}^*T_{vert}\mathfrak{B}\) so that \(F\)
restricted to fibers of $\hat{f}^*T_{\text{vert}} \mathfrak{B}$ in $U$ is an equidimensional embedding into fibers of $\mathfrak{B} \rightarrow \mathfrak{S}$. Therefore if we restrict $\mathcal{O}$ to be a small enough $C^1$ neighborhood of $\hat{f}$, there is a unique $C^{\infty, 1}$ section $v$ of $\Phi^* \hat{f}^*T_{\text{vert}} \mathfrak{B}$ which vanishes at all marked points, is contained in $\Phi^* U$, and so that $f' = F(\Phi_* (v))$. For any curve $g$ in $\hat{f}$, we can denote by $F^{-1}(\mathcal{O})|_g$ the set of sections $v$ of $g^*T_{\text{vert}} \mathfrak{B}$ which vanish on marked points, are contained in the restriction of $U$ to $\mathcal{E}(g)$ and so that $F(v)$ is a curve in $\mathcal{O}$. It follows that for every $C^{\infty, 1}$ family $\hat{f}'$ in $\mathcal{O}$, there is a unique $C^{\infty, 1}$ section $v$ of $\Phi^* T_{\text{vert}} \mathfrak{B}$ so that $F(\Phi_* v)$, and so that $v$ restricted to any individual curve $g$ in $\hat{f}'$ is contained in $F^{-1}(\mathcal{O})|_{\Phi(g)}$. In other words, $(\hat{f}/\mathcal{G}, \{s_i\}, F)$ is a core family for $\mathcal{O}$.

The following proposition constructs a core family containing a given $C^{\infty, 1}$ exploded curve which is stable and has at least one smooth component (so it isn’t $\mathfrak{T}$ or a quotient of $\mathfrak{T}$).

**Proposition 4.3.** Given a stable connected $C^{\infty, 1}$ exploded curve $f$ with at least one smooth component in a smooth basic family of targets $\hat{\mathfrak{B}}$ and a collection of marked points $\{p_j\}$ in the interior of the smooth components of $\mathcal{E}(f)$, there exists a $C^{\infty, 1}$ core family $(\hat{f}/\mathcal{G}, \{s_i\}, F)$ with $\hat{f}$ a basic family containing $f$ so that the restriction of $\{s_i\}$ to $f$ contains the given marked points $\{p_j\}$.

**Proof:** The automorphism group $G$ of $\hat{f}$ shall be equal to the group of automorphisms of $\lceil f \rceil$, the smooth part of $f$.

By restricting to a topologically open subset of our family $\hat{\mathfrak{B}} \rightarrow \mathfrak{S}$ containing the image of our curve, we may assume that $\lceil \mathfrak{S} \rceil$ is an integral affine polygon.

We shall enumerate the steps of this construction so that we can refer back to them.

1. Choose extra marked points on the smooth components of $\mathcal{E}$ so each smooth component of $\mathcal{E}$ contains at least one marked point, the smooth part of $\mathcal{E}$ has no automorphisms with these extra marked points, and so that we can divide the marked points on $\mathcal{E}$ into the following types:

   (a) On any smooth component of $\mathcal{E}$ which is unstable, choose enough extra marked points at which $d[\lceil f \rceil]$ is injective to stabilize the component. Note that $G$ has a well defined action on the smooth part of $\mathcal{E}$. Choose the set of marked points of this type to be preserved by the action of $G$. (Note that the stability of $f$ implies that each unstable smooth component of $\mathcal{E}$ must contain a nonempty open set where $d[\lceil f \rceil]$ is injective.)

   (b) Choose the set of remaining marked points to be preserved by the action of $G$.

2. The tropical part of the family $\mathfrak{F}$ will be an integral affine polygon $P = |\mathfrak{F}|$. Construct this polygon $P$ as follows:

   (a) Construct a polygon $\mathcal{P}$ by taking the fiber product over $|\mathfrak{S}|$ of the polygon $|\mathfrak{B}_i|$ for the strata $\mathfrak{B}_i$ that contains the image of each
marked point above, and taking the product of this with a copy of $t^{[0,\infty)}$ for every internal edge of $\mathcal{C}$. The coordinates on $\hat{P}$ record the tropical position of each marked point, and the length of each internal edge of the tropical curves in our family. Note that $\hat{P}$ is a convex integral affine polygon. (This uses the notation from [12] that $[\hat{\mathcal{B}}_i]$ is the integral affine polygon which is the closure of $[\mathcal{B}_i]$ in $[\hat{\mathcal{B}}]$)

(b) The requirement that two marked points in the same smooth component are sent to the same tropical point, and that the two vertices at the end of each internal edge are joined by an edge of the specified length with the same velocity as the original edge of $[f]$ gives a number of integral affine equations on $\hat{P}$. $P$ is the solution to these equations. As $\hat{P}$ was a convex integral affine polygon, $P \subset \hat{P}$ is too. $P$ must be nonempty because it contains a point corresponding to $[f]$.

3. Use the standard coordinates on $\mathcal{B}$ from Lemma 4.3 on page 57 of [12]. In particular, for each strata $\mathcal{B}_i \subset \mathcal{B}$, there exists some $U_i \subset \mathcal{B}_i$ containing $\mathcal{B}_i$ so that

(a) The image in the smooth part $[U_i] \subset [\mathcal{B}]$ is an open neighborhood of $[\mathcal{B}_i] \subset [\mathcal{B}]$.

(b) If $[\mathcal{B}_i]$ is $n$ dimensional, then there is an identification of $U_i$ with $[\mathcal{B}_i] \times \mathcal{T}^m_{[\mathcal{B}_i]}$. We can make this identification so that the (sometimes defined) free action of $\mathcal{T}^n$ given by considering the embedding $[\mathcal{B}_i] \times \mathcal{T}^m_{[\mathcal{B}_i]} \subset [\mathcal{B}_i] \times \mathcal{T}^n$ satisfies the following:

i. The $\mathcal{T}^n$ action preserves fibers of the family $\mathcal{B} \longrightarrow \mathcal{G}$, in the sense that if $p_1$ and $p_2$ have the same image in $\mathcal{G}$, then $\tilde{z} \ast p_1$ and $\tilde{z} \ast p_2$ also have the same image in $\mathcal{G}$.

ii. If $[\mathcal{B}_i]$ is in the closure of $[\mathcal{B}_j]$, then the action of $\mathcal{T}^n$ on $U_i \cap U_j$ is equal to the action of a subgroup of the $\mathcal{T}^m$ acting on $U_j$.

4. Construct $\mathcal{F}$ as follows: Coordinates on $\mathcal{F}$ shall be given by the position of marked points and the complex structure of our curves. $\mathcal{F}$ will be equal to some open subset of $\mathbb{R}^n \times \mathcal{T}_{\hat{P}}^{\mathcal{P}}$. First construct $\mathcal{F}$ in analogy to $\hat{P}$ which will have tropical part $\hat{P}$. $\mathcal{F}$ will be an open subset of a refinement of $\mathcal{F}$ corresponding to $P \subset \hat{P}$.

(a) Construct $\mathcal{F}$ using the following coordinates:

i. For each marked point $p$ from part 1a, $d(f)(p)$ is contained inside some strata $\mathcal{B}_i$, which is itself contained inside the coordinate chart $U_i = [\mathcal{B}_i] \times \mathcal{T}^m_{[\mathcal{B}_i]}$. As $d(f)(p)$ is injective, we can choose a smooth coordinate chart on $[\mathcal{B}_i]$ which identifies a neighborhood of $f(p)$ with $\mathbb{R}^2 \times \mathbb{R}^k \times \mathcal{T}^m_{[\mathcal{B}_i]}$, so that the restriction of $f$ to a neighborhood of $p$ is equal to an inclusion $x \in \mathbb{R}^2 \mapsto (x,c)$ where $c \in \mathbb{R}^k \times \mathcal{T}^m_{[\mathcal{B}_i]}$. We can also construct our coordinate chart above so that the slices $\mathbb{R}^2 \times c'$ are
all contained in a fiber of $\mathfrak{B} \to \mathfrak{G}$, so that there is a well defined submersion $\mathbb{R}^k \times T^m_{(B_i)} \to \mathfrak{G}$. Note that we can use the same coordinate chart for each marked point in an orbit of $G$. Include in our coordinates for $\hat{\mathfrak{F}}$ the fiber product of $\mathbb{R}^k \times T^m_{(B_i)} \to \mathfrak{G}$ for every marked point from part 1a.

ii. Take the fiber product over $\mathfrak{G}$ of the coordinates from part 4(a)i with a copy of $U_i \to \mathfrak{G}$ for each marked point from part 1b, where $U_i$ is the standard coordinate from item 3 corresponding to the strata containing the image of the marked point.

iii. Each smooth component of $\mathcal{C}$ can be regarded as a stable punctured Riemann surface with labeled punctures determined by the exploded structure of $\mathcal{C}$ plus the extra marked points from part 1a. Take the product of the above coordinates from part 4(a)i and 4(a)ii with uniformizing neighborhood of the corresponding point in Deligne Mumford space. Do this so that the obvious $G$ action is well defined.

iv. For each internal edge of $\mathcal{C}$, a copy of $T^1_{(0,l)}$.

Observe that the tropical part of $\hat{\mathfrak{F}}$, $[\hat{\mathfrak{F}}]$ is equal to $\hat{P}$. If $\hat{P}$ is $m'$ dimensional, there is a (sometimes defined) free $T^m_{m'}$ action on $\hat{\mathfrak{F}}$ given by multiplication on the correct coordinates from item 4(a)i, 4(a)ii, and 4(a)iv. The (sometimes defined) action of a subgroup $T^m \subset T^m_{m'}$ preserves $\hat{P} \subset \hat{P}$ where $P$ is $m$ dimensional. There is a corresponding action of $T^m$ on each of the coordinate charts $U_i$ referred to in item 4(a)ii (which of course is not necessarily free).

There is a distinguished point $p \to \hat{\mathfrak{F}}$ corresponding to our curve $f$, which is the point $f(p)$ in item 4(a)i and 4(a)ii, the point corresponding to the complex structure on the smooth components of $\mathcal{C}$ in item 4(a)iii, and for item 4(a)iv $1^t$ where the strata of $\mathfrak{C}$ corresponding to the internal edge in question is equal to $\mathfrak{T}^1_{(0,l)}$. Roughly speaking, our family $\hat{\mathfrak{F}}$ will be some neighborhood of orbit of this point under the above mentioned $T^m$ action.

5. Construct $\mathfrak{F}$ as follows:

(a) Take any refinement $\hat{\mathfrak{F}}'$ of $\hat{\mathfrak{F}}$ so that the $[\hat{\mathfrak{F}}']$ includes a strata with closure equal to $P \subset \hat{P}$.

(b) $\mathfrak{F}$ is given by a small topologically open neighborhood of the point corresponding to $f$ in $\hat{\mathfrak{F}}'$ so that the coordinates from item 4(a)iv have absolute value strictly less that some $\varepsilon_0$.

6. We shall now construct $(\hat{\mathcal{C}}, j) \xrightarrow{\pi} \hat{\mathfrak{F}}$. Roughly speaking, the coordinates 4(a)iii and 4(a)iv give a map from $\hat{\mathfrak{F}}$ to a neighborhood of $\text{Expl} \mathcal{M}$ which at $f$ corresponds to the complex structure on $\mathcal{C}$ with the extra punctures mentioned in 1a. Pulling back $\text{Expl} \mathcal{M} \to \text{Expl} \mathcal{M}$ gives $(\mathfrak{C}, j)$ with the sections corresponding to marked points from 1a and we just need to extend the other marked points to appropriate sections to define $(\mathfrak{C}, j) \to \hat{\mathfrak{F}}$ with all its sections $s_i$. We shall do this below in an explicit way to enable us to describe more explicitly the extension of the map $f$ to $\hat{f}$. 

65
(a) Choose holomorphic identifications of a neighborhood of each internal edge of $\mathcal{C}$ with

$$A_i := \{ c_i t^0 > |\bar{z}| > 1 t^1 \} \subset \mathbb{T}_{[0,1]}$$

so that these neighborhoods $A_i$ are disjoint, and all marked points are in the complement of these annuli $A_i$. Do this so that the set of images of $A_i$ in the smooth part of $\mathcal{C}$ are preserved by the action of $G$. Also choose $c_i > 8 \epsilon$ where $\epsilon$ is the constant mentioned in part 5 above. (Of course, to achieve this, we need to choose $\epsilon$ small enough.)

(b) Use the notation $\hat{A}_i$ to refer to the part of $\hat{\mathcal{C}}$ corresponding to $A_i$.

This is given as follows:

In the construction of $\mathfrak{g}$, replace the factor of $\mathbb{T}_1$ from item 4(a)iv corresponding to the edge $A_i$ with $\mathbb{T}_2$. If this $\mathbb{T}_2$ has coordinates $\hat{w}_1, \hat{w}_2$, then $A_i$ is the subset of this $\mathbb{T}_2$ so that $|\hat{w}_1| < c_i t^0$ and $|\hat{w}_2| < 1 t^0$. The map $A_i \rightarrow \mathfrak{g}$ is given by the map $\hat{w}_1 \hat{w}_2 : \mathbb{T}_2 \rightarrow \mathbb{T}_1$ and is the identity on all other coordinates.

There is a natural action of $G$ on the union of these $\hat{A}_i$ given as follows: On the coordinates corresponding to all coordinates on $\mathfrak{g}$ apart from part 4(a)iv there is an obvious action of $G$. If $g \in G$ sends the smooth part of $A_i$ to $A_j$, the pull back of smooth part of the coordinates $[\hat{w}_1], [\hat{w}_2]$ on $A_j$ is equal to some constant times the smooth part of the corresponding coordinates on $A_i$. Define the map from $A_i$ to $A_j$ by defining the pull back of $\hat{w}_1$ and $\hat{w}_2$ simply to be the corresponding coordinate multiplied by the above constant, and the pull back of other coordinate functions as given by the obvious $G$ action on coordinates from parts 4(a)ii and 4(a)iii. This induces an action of $G$ on $\mathfrak{g}$ so that the map $\bigcup_i A_i \rightarrow \mathfrak{g}$ is equivariant.

(c) Label by $C_k$ the connected components of the complement of the sets $\{ \frac{c}{2} t^0 > |\bar{z}| > 2 t^1 \} \subset A_i$. Again, use the notation $\hat{C}_k$ to refer to the corresponding part of $\hat{\mathcal{C}}$. This is simply given by the product

$$\hat{C}_k := C_k \times \mathfrak{g}$$

The map $C_k \rightarrow \mathfrak{g}$ is simply projection onto the second component.

Note that there is an action of $G$ on the union of these $C_k$ given by the action on $\mathfrak{g}$ defined at the end of item 5 above, and the action of $G$ on the union of $C_k$ as a subset of the smooth part of $\mathcal{C}$.

(d) The transition maps between $A_i$ and $C_k$ induce in an obvious way transition maps between $\hat{A}_i$ and $\hat{C}_k$, which defines the family $\hat{\mathcal{C}} \rightarrow \mathfrak{g}$. Note that the inverse image of our special point $p \rightarrow \mathfrak{g}$ is equal to $\mathcal{C}$.

Note also that these transition maps are compatible with the action of $G$ on the union of the $A_i$ and the union of the $\hat{C}_k$, so there is an action of $G$ on $(\mathcal{C}, j) \rightarrow \mathfrak{g}$.

Remembering the positions of our marked points in $C_k$ gives the sections $s_i : \mathfrak{g} \rightarrow \mathcal{C}$ referred to in the statement of this proposition.

(e) It remains to construct the complex structure $j$ on the fibers of $\hat{\mathcal{C}}$. Recall that the coordinates on $\mathfrak{g}$ from item 4(a)iii are intended to give the almost complex structure on smooth components of $\mathcal{C}$. Choose a
smooth family of complex structures $j$ on the smooth components of $C$ parameterised by these coordinates with the correct isomorphism class of complex structure, so that $j$ at our special point is the original complex structure on $C$, and $j$ restricted to the subsets $A_i$ is constant. Do this equivariantly with respect to the action of $G$ on the smooth part of $C$ and the action of $G$ on the coordinates from part [(a)iii]. This gives a family of complex structures on the fibers of $\hat{C}_k$. This is compatible with the standard holomorphic structure on $\hat{A}_i$, so using this gives us our globally defined $(\hat{C}, j)$. Note that the restriction of this to the curve our special point $p \to \mathcal{F}$ gives $C$ with the original complex structure.

7. Construct the family of maps $\hat{f} : \hat{C} \to \hat{\mathcal{B}}$. This will involve translating around pieces of the original map $f$, modifying this map near marked points as directed by the coordinates of $\mathcal{F}$, and gluing together the result. The last 'gluing' step only affects the map $\hat{f}$ on $\hat{A}_i$, so we shall now perform the first two steps to construct $\hat{f}$ on $\hat{C}_k$.

Construct $\hat{f}$ on $\hat{C}_k$ as follows:

(a) Suppose that $f(C_k) \subset B_i \subset U_i$. Recall that there is a (sometimes defined) action of $T^m$ on $\mathcal{F}$ and a corresponding (sometimes defined) action of $T^m$ on $U_i$. If $p' = \hat{c} \ast p$, where $p \to \mathcal{F}$ is the special point corresponding to $f$, then define

$$\hat{f}(z, p') := c \ast f(z) \text{ when } z \in C_k$$

This defines $\hat{f}$ on the part of $\hat{C}_k$ over the orbit of $p \to \mathcal{F}$ under the action of $T^m$. Note that the above orbit and this map are both preserved by the action of $G$.

(b) We must make sure that each of the individual smooth curves in $\hat{f}$ are contained in the correct fiber of $\hat{\mathcal{B}} \to \mathcal{B}$. Note that this is automatically true so far, because of the compatibility of our $T^m$ action with the map $\hat{\mathcal{B}} \to \mathcal{B}$. (In fact, there is a (sometimes defined) action of $T^m$ on $\mathcal{B}$ so that this map is equivariant.) We shall now extend the definition of $\hat{f}$ to a subset of $\hat{C}$ which is equal to $\mathbb{R}^n \times T^m' Q$ where it is already defined by 'translating in directions coming from $\mathcal{B}'$.

As constructed, the obvious map (trivial on all coordinates apart from those from items [(a)ii] and [(a)iii]), $\mathcal{F} \to \mathcal{B}$ is a submersion which is preserved by the action of $G$. The image of the tropical part $P$ under this map is some polygon $Q \subset \lfloor T^m \rfloor$. Therefore, the image of $\mathcal{F}$ under this map is an open subset of some refinement of $\mathcal{B}$ that has the interior of $Q$ as a strata. If $\mathcal{F}$ is chosen small enough, this open subset of the refinement of $\mathcal{B}$ is isomorphic to $\mathbb{R}^n \times T^m_Q$. We can pull our family $\hat{\mathcal{B}} \to \mathcal{B}$ back to to be a family over $\mathbb{R}^n \times T^m_Q$.

If $\mathcal{F}$ was chosen small enough, this family will split into a product $\mathbb{R}^n \times \hat{\mathcal{B}}' \to \mathbb{R}^n \times T^m_Q$ which is the identity on the $\mathbb{R}^n$ component, and some family $\hat{\mathcal{B}}' \to T^m_Q$ on the second component. We can choose this splitting so that it is compatible with our local actions of
$\mathbb{T}^m$ on $\mathcal{U}_i$, and the action of $G$. This also gives a splitting of $\mathfrak{F}$ into $\mathbb{R}^n \times \mathfrak{F}'$. We can choose this splitting so that the subset of $\mathfrak{F}$ where we have already defined $\hat{f}$ is contained inside $0 \times \mathfrak{F}'$. Now we can define $\hat{f}$ as a map to $\mathbb{R}^n \times \mathfrak{B}'$ follows:

$$\hat{f}(z, x, y) = (x, y) \text{ if } f(z, 0, y) \text{ is already defined, and } f(z, 0, y) = (0, y)$$

Here $(z, x, y)$ denotes coordinates on $\hat{C}_k = C_k \times \mathbb{R}^n \times \mathfrak{F}'$. This map is defined on a $G$ invariant subset, and is preserved by the action of $G$.

(c) Split $\mathfrak{F}$ further into an open subset of $\mathbb{R}^n \times \mathfrak{F}''$ where our map $\hat{f}$ is defined so far on the subset of $\hat{C}_k$ which is over $0 \times \mathfrak{F}''$, and the splitting is preserved by the action of $G$. Extend the map defined so far to a smooth map $\hat{f}$ defined on all of $\hat{C}_k$ so that

i. $\hat{f}$ fits into the commutative diagram

$$\begin{array}{ccc}
\hat{C}_k & \xrightarrow{\hat{f}} & \hat{\mathfrak{B}} \\
\downarrow \pi_{\hat{\mathfrak{F}}} & & \downarrow \pi_{\hat{\mathfrak{B}}} \\
\hat{\mathfrak{F}} & \longrightarrow & \mathfrak{B}
\end{array}$$

ii. $\hat{f}$ is preserved by the action of $G$ on $\hat{C}_k$.

iii. On the intersection of $\hat{A}_i$ with $\hat{C}_k$ and outside a small neighborhood of all marked points, $\hat{f}(z, x, y) = f(z, 0, y)$. (This uses coordinates $\hat{C}_k = C_k \times \mathbb{R}^n \times \mathfrak{F}''$.)

iv. For each marked point $q$ from part 1a, the value of $\hat{f}$ at the point $(q, x, y)$ is equal to the corresponding coordinate of $\hat{\mathfrak{F}}$ from part 4(a).

v. For each marked point $q$ from part 1b, $(x, y) \in \mathbb{R}^n \times \mathfrak{F}''$ determines a value for the coordinate on $\hat{\mathfrak{F}}$ from part 4(a) which is a point inside $\mathcal{U}_i \subset \hat{\mathfrak{B}}$. For such a marked point, $f(q, x, y)$ is equal to this point.

8. Define $\hat{f}$ on $\hat{A}_i$ by cutting $\hat{A}_i$ into two pieces, translating each piece the same way as the $C_k$ it is attached to, and then using a smooth gluing procedure to glue together the result which only modifies $\hat{f}$ on the region where $\hat{A}_i$ does not intersect $C_k$. (An example of such a smooth gluing procedure is contained in the proof of Lemma 3.10.) Do this so that $\hat{f}$ is compatible with $\hat{\mathfrak{B}} \longrightarrow \mathfrak{B}$, and $\hat{f}$ is preserved by the action of $G$. Note that modification is only necessary when the fiber of $\hat{A} \longrightarrow \hat{\mathfrak{F}}$ is an annulus with finite conformal modulus.

9. We have now constructed the required family.

$$\begin{array}{ccc}
(\hat{\mathfrak{C}}, j) & \xrightarrow{\hat{f}} & (\hat{\mathfrak{B}}, J) \\
\downarrow \pi_{\hat{\mathfrak{F}}} & & \downarrow \pi_{\hat{\mathfrak{B}}} \\
\hat{\mathfrak{F}} & \longrightarrow & \mathfrak{B}
\end{array}$$

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This map \( \hat{f} \) is smooth or \( C^\infty_\Lambda \) if \( f \) is. The family \( \hat{f} \) with the sections \( \{ s_i \} \) satisfies criteria 1, 2 and 3 from Theorem 4.2 on page 60, and if \( F \) is any map

\[
\hat{f}_* T_{\text{vert}} \hat{\mathcal{B}} \xrightarrow{F} \hat{\mathcal{B}} \xrightarrow{\hat{s}(\hat{f})} \mathcal{G}
\]

given by exponentiation using some smooth family of connections on \( \hat{\mathcal{B}} \to \mathcal{G} \), then \( F \) is \( C^\infty_\Lambda \) if \( \hat{f} \) is, and \( F \) satisfies the criterion 5 from Theorem 4.2. Therefore, it remains to check criterion 4 from Theorem 4.2.

In the remainder of this proof, let \( n \) denote the number of our sections, so \( n = |\{ s_i \}| \). We need to show that the evaluation map from \( \hat{\mathcal{G}}(\hat{f} + n) \) to \( \text{Expl} M \times \text{product of } \hat{\mathcal{B}}_{\mathcal{G}} \times \cdots \times \hat{\mathcal{B}}_{\mathcal{G}} \) is an equidimensional embedding when restricted to some neighborhood of the section \( s : \hat{\mathcal{G}} \to \hat{\mathcal{G}}(\hat{f} + n) \) given by our sections \( \{ s_i \} \). Coordinates on a neighborhood of the image of \( \hat{s} : \hat{\mathcal{G}} \to \hat{\mathcal{G}}(\hat{f} + n) \) are given by coordinates on \( \hat{s} \) and coordinates on a neighborhood of each marked point. The evaluation map from this neighborhood splits into two equidimensional embeddings as follows: By construction, the coordinates from 4(a)(iii) and 4(a)(iv) together with the coordinates around all marked point not of type 1a describe an equidimensional embedding into \( \text{Expl} \hat{\mathcal{M}} \). The coordinates from 4(a)(iii) and the coordinates from 4(a)(iv) plus the coordinates around marked points of type 1a describe an equidimensional embedding to \( \left( \hat{\mathcal{B}}_{\mathcal{G}} \right)^n \) restricted to a small enough neighborhood. (We should restrict to a suitably small \( G \)-equivariant subset so that this holds.)

We now prove the existence of obstruction models, defined on page 7.

**Theorem 4.4.** Any connected holomorphic curve \( f \) with at least one smooth component in a basic family of targets is contained inside some \( C^\infty_\Lambda \) obstruction model \( (\hat{g}/G, V) \).

**Proof:**

Proposition 4.3 tells us that the curve \( f \) must be contained in a \( C^\infty_\Lambda \) core family \( \hat{g}/G \). We may choose this core family to include any collection of marked points on \( f \), so after choosing a \( (G \text{ invariant}) \) trivialization, we may arrange that \( D\hat{\partial} : X_{\hat{g}}(f) \to Y_{\hat{g}}(f) \) is injective. We can therefore choose a finite dimensional complement \( V_0(f) \) to \( D\hat{\partial}(X_{\hat{g}}(f)) \) consisting of \( C^\infty_\Lambda \) sections of \( Y(f) \), and extend this on a topological neighborhood to a \( C^\infty_\Lambda \) pre obstruction model \( (\hat{g}, V_0) \), where \( V_0 \) is not necessarily \( G \)-invariant. As \( \pi_{V_0} D\hat{\partial} : X_{\hat{g}}(f) \to (Y/V_0)_{\hat{g}}(f) \) is an isomorphism, Theorem 3.11 tells us that \( V_0 \) is complementary to the image of \( D\hat{\partial} \) in some topological neighborhood of \( f \) in \( \hat{g} \). Then, in this neighborhood, considering \( V_0 \) as giving a projection onto the \( (G\text{-invariant}) \) image of \( D\hat{\partial} \). We may average this projection under the action of \( G \) to obtain a \( G \)-invariant projection corresponding to a \( C^\infty_\Lambda \) pre obstruction model \( (\hat{g}, V) \), invariant under the action of \( G \), so that \( V \) is complementary to the image of \( D\hat{\partial} \). Then Theorem
\[\text{Lemma 3.17}\] gives that restricted to a small enough topologically open neighborhood of \(f_0\), we may modify \(\hat{g}\) to a \(G\) equivariant \(C^\infty,1\) family \(\hat{f}\) so that \(\pi_V \partial \hat{f} = 0\), and \((\hat{f}/G,V)\) is a \(C^\infty,1\) obstruction model.

\[\square\]

The following theorem describes the ‘solution’ to the \(\bar{\partial}\) equation perturbed by multiple simple perturbations parametrized by different obstruction models.

**Theorem 4.5.** Given

- a finite collection of obstruction models \((\hat{f}_i/G_i,V_i)\) for the substacks \(O_i'\),
- topologically compactly contained \(G_i\) invariant sub families \(\hat{f}_i \subset \hat{f}_i'\),
- and an open substack \(O\) which contains \(f_0'\) and which meets \(O_i'\) properly for all \(i\) (definition 4.13),

then given any collection of \(C^\infty,1\) simple perturbations \(P_i\) parametrized by \(\hat{f}_i'\), which are compactly supported in \(\hat{f}_i\), and are small enough in \(C^\infty,1\), there exists a ‘solution mod \(V_0\) on \(f_0\)’ which is \(C^\infty,1\) weighted branched section \(\nu\) of \(f_0\) \(T_{\text{vert}} \mathfrak{B}\) (branched over \([\mathfrak{F}(f_0)]\)) with weight 1 so that the following holds:

Topologically locally on \(\mathfrak{F}(f_0)\),

\[\nu = \sum_{i=1}^{n} \frac{1}{n} t^{\nu_i}\]

where \(\nu_i\) are \(C^\infty,1\) sections of \(f_0\) \(T_{\text{vert}} \mathfrak{B}\) vanishing on the marked points in the definition of the obstruction model \((\hat{f}_0,V)\) so that \(F(\nu_i)\) are in \(O\) and in the notation of example 1.16 on page 12

\[\prod_i F(\nu_i)^* P_i = \sum_{j=1}^{n} \frac{1}{n} t^{P_j,i}\]

so that \(\bar{\partial} F(\nu_i) - P_{1,i}\) is a section of \(V_0\). The weighted branched section \(\nu\) is the unique weighted branched section of \(f_0\) \(T_{\text{vert}} \mathfrak{B}\) with weight 1 satisfying the following two conditions:

1. Given any curve \(f\) in \(O_0' \cap O\) which projects to curve in \(\hat{f}_0\), if \(\prod_i f^* P_i = \sum w_k t^{Q_k}\), and near \(f\), \(\nu = \sum w'_i t^{\nu_i}\) then the sum of the weights \(w_k\) so that \(\bar{\partial} f - Q_k\) is in \(V_0\) is equal to the sum of the weights \(w'_i\) so that \(f\) is contained in \(F(\nu_i)\).

2. For any locally defined section \(\psi\) of \(f_0\) \(T_{\text{vert}} \mathfrak{B}\), if the multi perturbation \(\prod_i F(\psi)^* P_i = wt^{Q_k} + \ldots\), and \(\bar{\partial} F(\psi) - Q\) is a section of \(V_0\), then locally, \(\nu = wt^{Q_k} + \ldots\).

If \(\{P'_i\}\) is another collection of simple perturbations satisfying the same assumptions as \(\{P_i\}\) then the sections \(\nu'_i\) corresponding to \(\nu_i\), with the correct choice of indexing can be forced to be as close to \(\nu_i\) as we like in \(C^\infty,1\) by choosing \(\{P'_i\}\) close to \(\{P_i\}\) in \(C^\infty,1\). If \(\{P_{1,i}\}\) is a \(C^\infty,1\) family of simple perturbations satisfying the same assumptions as \(\{P_i\}\), then the corresponding family of ‘solutions mod \(V_0\’, \(\nu_i\) form a \(C^\infty,1\) family of weighted branched sections.
Proof:

Use \( \mathcal{O}_i \) to denote the restriction of \( \mathcal{O}'_i \) to the subset with core \( f_i/G_i \). As \( f'_i \) meets \( \mathcal{O}'_i \) properly for all \( i \), and \( f_i \) is topologically compactly contained in \( f'_i \), there is some \( C^{1,\delta} \) neighborhood \( U \) of 0 in the space of sections of \( \nu_0^*T_{vert}\mathcal{B} \) and some finite covering of \((f_0, V_0)\) by allowable pre obstruction models \((\hat{f}, V)\) so that either

- for all \( \nu \) which are the restriction to \( f \) of sections in \( U \), \( F(\nu) \) is contained inside \( \mathcal{O}'_i \)

or

- \( F(\nu) \) does not intersect \( \mathcal{O}_i \) for any \( \nu \) which is the restriction to \( f \) of some section in \( U \).

Let \( I \) indicate the set of indices \( i \) so that the first option holds, so \( F(\nu) \) is contained inside \( \mathcal{O}'_i \) for \( \nu \) small enough.

The main problem that must be overcome in the rest of this proof is that the simple perturbations \( P_i \) are not parametrized by \( f \). We will extend \( f \) to a family \( \hat{h} \) which can be regarded as parametrizing the simple perturbations \( P_i \) for all \( i \in I \) and use the resulting unique solution \( \hat{\nu} \) to the corresponding perturbed \( \hat{\partial} \) equation over \( \hat{\nu} \) to construct the weighted branched section of \( f^*T_{vert}\mathcal{B} \) which is our ‘solution’ with the required properties. This will involve reexamination of ideas that came up in the proof of Theorem 4.2.

Use the notation

\[
s^i : \mathfrak{F}(f'_i) \to \mathfrak{F}(f'^{i+n_i})
\]

for the map coming from the extra marked points on the core family \( f'_i \).

The map \( ev^{i+n_i}(\hat{f}_i^i) : \mathfrak{F}(f'^{i+n_i}) \to \text{Exp} \mathcal{M} \times \mathfrak{B} \) has the property that it is an equidimensional embedding in a neighborhood of the section \( s^i \). There exists an open neighborhood \( \mathcal{O}_s \) of the family of curves \( s^i(f'^{i+n} \mid \mathcal{O}_s) \) so that given any curve \( f \) in \( \mathcal{O}_s \), if \( f'^{i+n} \mid \mathcal{O}_s \) indicates the restriction of the family \( f'^{i+n} \) to \( \mathcal{O}_s \), then \( ev^{i+n}(f)(\mathfrak{F}(f'^{i+n} \mid \mathcal{O}_s)) \) intersects \( ev^{i+n}(\hat{f}_i^i)(s^i(\mathfrak{F}(f'_i))) \) transversely exactly \( |G_i| \) times, corresponding to the \( |G_i| \) maps from \( c(f) \) into \( c(f'_i) \).

Consider the family \( \hat{f}^{i+(n-1)} : \mathfrak{F}(f^{i+n}) \to \mathfrak{B} \). Use the notation \( X^{i+(n-1)} \) to denote the vector bundle over \( \mathfrak{F}(f^{i+n}) \) which is the pullback under \( \hat{f}^{i+n-1} \) of the vertical tangent space of the family \( \mathfrak{B} \) maps from \( c(f) \) into \( c(f'_i) \). Any section \( \nu \) of \( f^{i+n} \) \( \mathcal{B} \) corresponds in an obvious way to a section \( \nu^{i+(n-1)} \) of \( X^{i+(n-1)} \), and the map \( F : \hat{f}^{i+n} \mathcal{B} \to \mathcal{B} \) corresponds to a \( C^{\infty,\Lambda} \) map

\[
F^{i+(n-1)} : X^{i+(n-1)} \to \mathfrak{B}
\]

so that

\[
F^{i+(n-1)}(\nu^{i+(n-1)}) = (F(\nu))^{i+(n-1)}
\]

Use the notation \( \nu^{i+(n-1)} \mid \mathcal{O}_s \) to denote the restriction of \( \nu^{i+(n-1)} \) to the subset \( \mathfrak{F}(F(\nu)^{i+n} \mid \mathcal{O}_s) \subset \mathfrak{F}(f'^{i+n}) \)

Define a map

\[
EV^{i+n} : X^{i+(n-1)} \to \text{Exp} \mathcal{M} \times \mathfrak{B}
\]
so that $EV^{+n}$ is equal to the natural map coming from the complex structure of curves in $\mathcal{C}(\hat{f}^{+n}) \rightarrow \mathfrak{F}(\hat{f}^{+n})$ on the first component, and $F^{+(n-1)}$ on the second component. So

$$EV^{+n}(\nu^{+(n-1)}(\cdot)) = ev^{+n}(F(\nu))(\cdot)$$

The map $EV^{+n}$ is $C^{\infty,1}$.

Use the notation $\nu(g)^{+(n-1)}|_{\mathcal{O}_s}$ for the restriction of $\nu^{+(n-1)}|_{\mathcal{O}_s}$ to the inverse image of a curve $g \in \hat{f}$. For any section $\nu$ small enough in $C^{1,\delta}$, the map $EV^{+n}$ restricted to $\nu(g)^{+(n-1)}|_{\mathcal{O}_s}$ intersects $ev^{+n_i}(\hat{f}_i^n)(\mathfrak{s}(\hat{f}_i^n))$ transversely in exactly $|G_i|$ points. Denote by $\hat{S}_i$ the subset of $X^{+(n-1)}$ which is the pullback of the image of the section $s_i$:

$$\hat{S}_i := (EV^{+n_i})^{-1} \left( ev^{+n_i}(\hat{f}_i^n)(\mathfrak{s}(\hat{f}_i^n)) \right) \subset X^{+(n-1)}$$

Close to the zero section in $X^{+(n-1)}$, $\hat{S}_i$ has regularity $C^{\infty,1}$, and for sections $\nu$ small enough in $C^{1,\delta}$, $\hat{S}_i$ is a $|G_i|$-fold multisection $\mathfrak{g}(\hat{f}) \rightarrow \hat{S}_i \subset X^{+(n-1)}$.

Use the notation $\tilde{X}^{+(n-1)}$ for the pullback along the map $\mathcal{C}(\hat{f}^{+n}) \rightarrow \mathfrak{F}(\hat{f}^{+n})$ of the vector bundle $X^{+(n-1)}$, $\tilde{S}_i$ for the inverse image of $S_i$ in $\tilde{X}^{+(n-1)}$, and $\nu_{\mathcal{O}_s}^{+(n-1)}$ for the pullback of $\nu^{+(n-1)}|_{\mathcal{O}_s} \subset X^{+(n-1)}$ to a section of $\tilde{X}^{+(n-1)}$.

Define a map

$$\tilde{EV}^{+n} : \tilde{X}^{+n-1} \rightarrow \mathcal{M} \times (\mathfrak{B}^n)_{\mathfrak{G}}$$

so that recalling the notation $\tilde{ev}^{+n}$ from [55] on page 62 in the proof of Theorem 4.2,

$$\tilde{EV}^{+n}(\nu^{+(n-1)}) = \tilde{ev}^{+n}(F(\nu))$$

Note that

$$\tilde{S}_i = \left( \tilde{EV}^{+n_i} \right)^{-1} \left( \tilde{ev}^{+n_i}(\hat{f}_i^n)(\mathfrak{C}(s)^* \hat{f}_i^{+n_i}) \right)$$

The fact that $ev^{+n_i}(\hat{f}_i^n)$ and $\tilde{ev}^{+n_i}(\hat{f}_i^n)$ are embeddings in a neighborhood of $s_i$ imply that there are natural maps

$$\begin{align*}
\hat{S}_i & \rightarrow \mathcal{C}(s)^* \hat{f}_i^{+n} \rightarrow \mathfrak{F}(\hat{f}_i^n) \\
\downarrow & \downarrow \\
S_i & \rightarrow \mathfrak{F}(s)^* \hat{f}_i^{+n} \rightarrow \mathfrak{F}(\hat{f}_i^n)
\end{align*} \quad (56)$$

so that the map on the left is an isomorphism on each fiber, and the map on the right is the map that forgets the extra marked points. We can define a family of curves $\hat{S}_i \rightarrow S_i$ by forgetting the extra marked points in the family $S_i \rightarrow S_i$ - so $\hat{S}_i$ is equal to the domain of the pullback of the family $\hat{f}_i^n$ over the map $S_i \rightarrow \mathfrak{F}(\hat{f}_i^n)$. The above map of families of curves [56] factors differently into the following diagram

$$\begin{align*}
\hat{S}_i & \rightarrow \hat{S}_i \xrightarrow{\Phi} \mathcal{C}(\hat{f}_i^n) \\
\downarrow & \downarrow \\
S_i & \rightarrow S_i \rightarrow \mathfrak{F}(\hat{f}_i^n)
\end{align*} \quad (57)$$

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As discussed in the proof of Theorem 1.22, the right hand map $\Phi_i$ above determines the maps $\Phi_{F(\nu)}$ from the definition of the core family $\tilde{f}_i/G_i$ in the following sense: For $\nu$ small enough, the intersection of $\nu^{+(n_i-1)}|_{\mathcal{S}_i\nu}$ with $S_i$ is transverse, and is a $|G_i|$-fold cover of $\mathfrak{F}(\tilde{f})$, which lifts to a $|G_i|$-fold cover of $\mathfrak{C}(\tilde{f})$ which is a subset of $S$. Then $\Phi_i$ gives a map of our $|G_i|$-fold cover of $\mathfrak{C}(\tilde{f})$ into $\mathfrak{C}(\tilde{f}_i)$, which corresponds to the map $\Phi_{F(\nu)}: \mathfrak{C}(\tilde{f}) \to \mathfrak{C}(\tilde{f}_i)/G$.

Denote by $X^+$ the fiber product of $X^{+(n_i-1)}$ over $\mathfrak{F}(\tilde{f})$ for all $i \in I$, and denote by $\mathfrak{F}^+$ the fiber product of $\mathfrak{F}(\tilde{f}_i)$ over $\mathfrak{F}(\tilde{f})$ for all $i \in I$ - so $X^+$ is a vector bundle over $\mathfrak{F}^+$. A $C^\infty_*$ section $\nu$ of $\tilde{f}^*\text{vert}^*\mathfrak{B}$ corresponds in the obvious way to a $C^\infty_*$ section $\nu^+$ of $X^+$ which is equal to $\nu^{+(n_i-1)}$ on each $X^{+(n_i-1)}$ factor. Similarly, denote by $\nu^+|_{\mathcal{S}_i\nu}$ the open subset of $\nu^+$ inside $\nu^{+(n_i-1)}|_{\mathcal{S}_i\nu}$ on each $X^{+(n_i-1)}$ factor. Denote by $S^+ \subset X^+$ the subset corresponding to all $S_i \subset X^{+(n_i-1)}$ restricted to a neighborhood of the zero section small enough that $S^+$ is $C^\infty_*$. We can choose $S^+$ so that pulling back $(\hat{f},\nu)$ over the map $S^+ \to \mathfrak{F}(\tilde{f})$ gives an allowable pre obstruction model $(\hat{h},\nu)$. Note that $\mathfrak{C}(\hat{h})$ is some open subset of the fiber product of $S_i$ over $\mathfrak{C}(\tilde{f})$ for all $i \in I$, so the maps $\Phi_i$ from 1.16 induce maps

\[
\begin{align*}
\mathfrak{C}(\hat{h}) & \xrightarrow{\Phi_i} \mathfrak{C}(\tilde{f}_i) \\
\downarrow & \downarrow \\
\mathfrak{F}(\hat{h}) & \xrightarrow{\Phi_i} \mathfrak{F}(\tilde{f}_i)
\end{align*}
\]

Pulling a simple perturbation $P_i$ parametrized by $\tilde{f}_i$ back over the map $\Phi_i$ gives a simple perturbation $\Phi_i^* P_i$ parametrized by $\hat{h}$. Use the notation

\[
P := \sum_{i \in I} \Phi_i^* P_i
\]

If $\nu$ is any small enough section of $\tilde{f}^*\text{vert}^*\mathfrak{B}$, then the multi perturbation $\prod_{i \in I} F(\nu)^* P_i$ defined as in example 1.16 on page 12 can be constructed as follows: If $\nu$ is small enough, then $\nu^+|_{\mathcal{S}_i\nu}$ is transverse to $S^+$, and the intersection of $\nu^+|_{\mathcal{S}_i\nu}$ with $S^+$ is a $n$-fold cover of $\mathfrak{F}(\tilde{f})$ in $\mathfrak{F}(\hat{h})$ which lifts to a multiple cover of $\mathfrak{C}(\tilde{f})$ inside $\mathfrak{C}(\hat{h})$ (where $n = \prod_{i \in I} |G_i|$). Together these give the domain for a family of curves $F(\nu)'$ which is a n-fold multiple cover of $F(\nu)$. Restricting $P$ to $F(\nu)'$ then gives a section of $Y(F(\nu))$, which corresponds to a n-fold multi section of $Y(F(\nu))$. Locally, giving each of these $n$ sections a weight $1/n$ gives a weighted branched section of $Y(F(\nu))$ with total weight 1 which is equal to the multi perturbation $\prod_{i \in I} F(\nu)^* P_i$ defined as in example 1.16.

As $(\hat{f},\nu)$ comes from an obstruction model, Theorem 3.17 applies to $(\hat{h},\nu)$ and implies that there is some neighborhood of 0 in the space of simple perturbations parametrized by $\hat{h}$ so that for such any $P$ in this neighborhood, there is a unique small $C^\infty_*$ section $\nu$ of $h^*\text{vert}^*\mathfrak{B}$ vanishing at the relevant marked points so that $\pi_Y(\partial - P)(\nu) = 0$. The fact that $(\hat{f},\nu)$ is part of an obstruction model for $\mathcal{O}_0$ implies the following uniqueness property for $\nu$ if $P$ is small enough: Given any curve $h$ in $\tilde{h}$ and section $\psi$ of $h^*\text{vert}^*\mathfrak{B}$ vanishing at the relevant marked points so that $F(\psi)$ is in $\mathcal{O}_0$, then $\pi_Y(\partial - P)(\psi) = 0$ if and only if $\psi$ is the restriction to $h$ of $\nu$. 

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Denote by $\hat{X}^+$ the pullback of $X^+$ over the map $\hat{\mathcal{F}}(\hat{h}) \to \mathcal{F}(\hat{f})$, and denote by $\hat{S}^+$ the pullback of $S^+$. This $\hat{S}^+$ comes with two maps into $S^+$, one the restriction of the map $\hat{X}^+ \to X^+$, and one the restriction of the map $\hat{X}^+ \to \hat{\mathcal{F}}(\hat{h}) = S^+$. Denote by $S^+_\Delta$ the subset of $\hat{S}^+$ on which these two maps agree. Because these two above maps agree when composed with the relevant maps to $\hat{\mathcal{F}}(\hat{f})$, $\hat{S}^+$ can be regarded as the fiber product of $S^+$ with itself over $\hat{\mathcal{F}}(\hat{f})$ and $S^+_\Delta$ is the diagonal in this fiber product $\hat{S}^+$. Therefore, $S^+_\Delta$ is $C^\infty$ and the map $S^+_\Delta \to S^+$ is an isomorphism. A section $\hat{v}$ of $h^*\mathcal{F} \vert \hat{\mathcal{F}} \to \mathcal{F}(\hat{f})$ defines a section $\hat{v}^+$ of the vector bundle $\hat{X}^+$ so that if $\tilde{v}$ is the pullback over the map $\mathcal{C}(\hat{h}) \to \mathcal{F}(\hat{f})$ of some section $\nu$ of $f^*\mathcal{F} \vert \mathcal{C}$, then $\tilde{v}^+$ is the pullback of $\nu^+$. We can define $\nu^+ \vert_{\mathcal{O}^+}$ similarly to the definition of $\nu^+ \vert_{\mathcal{O}^+}$.

As $\nu^+ \vert_{\mathcal{O}^+}$ is transverse to $S^+$ for $\nu$ small enough and $\nu^+ \vert_{\mathcal{O}^+} \cap S^+$ gives a $n$-fold cover of $\mathcal{F}(\hat{f})$, $\nu^+ \vert_{\mathcal{O}^+}$ is transverse to $S^+_\Delta$ for $\nu$ small enough, and $\nu^+ \vert_{\mathcal{O}^+} \cap S^+_\Delta$ also defines a $n$-fold cover of $\mathcal{F}(\hat{f})$ with regularity $C^\infty$. We may consider this multiple cover of $\mathcal{F}(\hat{f})$ as being a multi section $\mathcal{F}'$ of $S^+ = \mathcal{F}(\hat{h}) \to \mathcal{F}(\hat{f})$, which lifts to a multi section of $\mathcal{C}(\hat{h}) \to \mathcal{C}(\hat{f})$. Restricting $\hat{v}$ to this multi section gives locally $n$ sections $\nu_l$ of $f^*\mathcal{F} \vert \mathcal{C}$ with regularity $C^\infty$. Then

$$\nu = \sum_{i=1}^{n} \frac{1}{n} \nu_i$$

is the weighed branched solution which is our ‘solution mod $V$’. We shall now show that this weighted branched section has the required properties if $\{P_i\}$ is small enough. Note first that close by simple perturbations $\{P_i\}$ given close by solutions $\nu_l$. Also note that if we have a $C^\infty$ family of simple perturbations $\{P_{i,t}\}$, Theorem 3.17 implies that the corresponding family of solutions $\nu_{i,t}$ to $\nu(\partial - P_t)\nu_{i,t} = 0$ is a $C^\infty$ family, so the corresponding weighted branched sections $\nu_l$ form a $C^\infty$ family. If we choose $\{P_i\}$ small enough, then $F(\nu_l)$ will not intersect $\mathcal{O}_i$ for any $i \notin I$. Therefore, the multi perturbation under study is given by

$$\prod_i \nu_i^+ P_i = \sum_{j=1}^{n} \frac{1}{n} \prod_{j \neq l} P_{j,l}$$

(59)

where $P_{j,l}$ is constructed as follows: $\nu_i^+ \vert_{\mathcal{O}^+} \cap S^+$ is a $n$-fold cover of the open subset of $\mathcal{F}(\hat{f})$ where $\nu_l$ is defined. By working locally, this $n$-fold cover can be thought of as $n$ local sections of $S^+ = \mathcal{F}(\hat{h}) \to \mathcal{F}(\hat{f})$, which lift to $n$ local sections of $\mathcal{C}(\hat{h}) \to \mathcal{F}(\hat{f})$. The restriction of $P$ to these $n$ local sections gives the $n$ sections $P_{j,l}$ of $Y(F(\nu_l))$ in the formula (59) above. As one of these sections of $\mathcal{F}(\hat{h}) \to \mathcal{F}(\hat{f})$ coincides the multi section $\mathcal{F}'$ mentioned in the paragraph preceding equation (58) obtained using the solution $\hat{v}$ to the equation $\pi_V(\partial - P)\hat{v} = 0$, one of the sections $P_{j,l}$ of $Y(F(\nu_l))$ has the property that $\partial F(\nu_l) - P_{j,l}$ is a section of $V$.

Suppose that $f$ is some curve in $\mathcal{O}_0 \cap \mathcal{O}$ which projects to $f'$ in the region of $\hat{f}$ where these $\nu_l$ in formula (58) are defined. In other words, there is a section $\psi$ of $f'^*\mathcal{F} \vert \mathcal{C}$ vanishing at the relevant marked points so that $f = F(\psi)$. If the simple perturbations $P_i$ are chosen small enough, the fact that $(\hat{f}, V)$ comes from an obstruction model will imply that if $\prod_i f^* P_i = wt + \ldots$ where $w > 0$ and $\pi_V(\partial f - Q) = 0$, then $\psi$ must be small - choose $\{P_i\}$ small enough that such
f must have \( \psi^+|_{\mathcal{O}_{\nu_i}} \) intersecting \( S^+ \) transversely \( n \) times and \( f \) is not in \( \mathcal{O}_j \) for all \( j \notin I \). Then \( \prod_i f^*P_i = \sum_{i=1}^n \frac{1}{n} f^*(\nu_i) \) where the \( n \) sections \( \nu_i \) of \( Y(f) \) are obtained as follows: The \( n \) points of \( \psi^+|_{\mathcal{O}_{\nu_i}} \) correspond to \( n \) maps of \( \mathcal{C}(f) \) into \( \mathcal{C}(\bar{h}) \) - the \( n \) sections \( \nu_i \) are given by pulling back the simple perturbation \( P \) over these maps. Then \( \pi_V(\delta f - \nu_i) = 0 \) if and only if \( \psi \) is equal to the pullback under the relevant map of the solution \( \tilde{\nu} \) to \( \pi_V(\delta - P)\tilde{\nu} = 0 \). Therefore, if \( \{P_i\} \) is small enough, the number of \( \nu_i \) so that \( \pi_V(\delta f - \nu_i) = 0 \) is equal to the number of \( \nu_i \) from formula [58] so that \( f \) is in \( F(\nu_i) \).

Similarly, if \( \nu' \) is locally a section of \( f^*T_{vert}\mathcal{B} \) vanishing on the relevant marked points so that \( F(\nu') \in \mathcal{O}_0\cap \mathcal{O} \) and \( \prod_i F(\nu'^*P_i = wtQ + \ldots \) where \( w > 0 \) and \( \pi_V(\delta F(\nu') - Q) = 0 \), then so long as \( \{P_i\} \) is small enough, \( \nu'^+|_{\mathcal{O}_{\nu_i}} \cap S^+ \) is locally a \( n \)-fold cover of \( \mathfrak{S}(\bar{f}) \) corresponding to \( n \) sections of \( \mathfrak{S}(\bar{h}) \longrightarrow \mathfrak{S}(\bar{f}) \) which lift locally to \( n \) sections of \( \mathcal{C}(\bar{h}) \longrightarrow \mathcal{C}(\bar{f}) \). Then \( Q \) must locally correspond to the pullback of \( P \) under one of these local maps \( \mathcal{C}(f) \longrightarrow \mathcal{C}(\bar{h}) \), and \( \nu' \) must locally be the pullback of the solution \( \tilde{\nu} \) to \( \pi_V(\delta - P)\tilde{\nu} = 0 \). It follows that \( \nu' \) must coincide locally with one of these \( \nu_i \) from formula [58], and the weighted branched section locally equal to \( \sum_{i=1}^n \frac{1}{n} \psi^n_i \) is the unique weighted branched section with the required properties.

We now begin a construction of a ‘virtual class’ for the moduli stack of holomorphic curves which is a cobordism class of oriented \( C^\infty \) weighed branched substacks of the moduli stack of \( C^\infty \) curves. This will include the proof of Theorem 1.17 stated on page 13.

Any holomorphic curve with at least one smooth component in a basic family of targets \( \mathcal{B} \longrightarrow \mathfrak{S} \) is contained in some \( C^\infty \) obstruction model \( (f/G, V) \), and any obstruction model covers a \( C^{1,\delta} \) open neighborhood in the moduli stack of holomorphic curves. If our family of targets has a family of strict tamings in the sense of [12], then every stable holomorphic curve must have at least one smooth component. Therefore the compactness properties of the moduli stack of holomorphic curves proved in [12] imply that given any topologically compact subset \( \mathfrak{S}' \subset \mathfrak{S} \), the part of the moduli stack of holomorphic curves in any connected component of the moduli stack of \( C^\infty \) curves in \( \mathcal{B} \longrightarrow \mathfrak{S} \) over \( \mathfrak{S}' \) may be covered by a finite number of extendible obstruction models.

The rough idea of how the virtual class is constructed is that the \( \bar{\delta} \) equation is perturbed in some neighborhood of the holomorphic curves in the moduli stack of \( C^\infty \) curves to achieve ‘transversality’, and a \( C^\infty \) solution set. ‘Transversality’ is easy to achieve locally with a simple perturbation parametrized by an obstruction model. For such a simple perturbation to be defined independent of coordinate choices, it must be viewed as a multi-perturbation in the sense of example 1.16 on page 12. One problem is that for a simple perturbation to give a \( C^\infty \) multi-perturbation restricted to a particular family, that family must meet the domain of definition of the simple perturbation properly in the sense of Definition 1.15 on page 12.

Restrict our obstruction models to satisfy the requirements of Theorem 1.15 as follows. Each of the obstruction models we start off with has an extension \((f_i/G_i, V_i)\) on \( \mathcal{O}'_i \). This substack \( \mathcal{O}'_i \) can be viewed as corresponding to a neighborhood of \( 0 \) in the space of sections of \( f^i_{vert}\mathcal{B} \) which vanish at marked points. We may assume that this neighborhood is convex, and denote by \( c\mathcal{O}'_i \) the open
substack corresponding to the above neighborhood multiplied by \( c \) (i.e. sections \( \nu \) so that \( \frac{1}{c} \nu \) is in the above neighborhood). The fact that \( (\tilde{f}'_i/G_i, V_i) \) is an obstruction model implies that any holomorphic curves that are in \( O'_i \) are actually contained in the the family \( \tilde{f}'_i \); so all holomorphic curves in \( O'_i \) are contained inside \( \frac{1}{2} O'_i \). We may assume that \( (\tilde{f}'_i/G_i, V) \) itself is an extendible obstruction model, and that the closure \( \left( \frac{\nu}{4} O'_i - \frac{1}{2} O'_i \right) \) of \( \left( \frac{3}{4} O'_i - \frac{1}{2} O'_i \right) \) in the \( C^{1,\delta} \) topology contains no holomorphic curves. Define a \( C^{1,\delta} \) neighborhood of the part of the stack of holomorphic curves under study by

\[
O := \bigcup_i \frac{1}{2} O'_i - \bigcup_i \left( \frac{3}{4} O'_i - \frac{1}{2} O'_i \right)
\]

The substack \( O \) meets \( \frac{1}{2} O'_i \) with the core family \( \tilde{f}'_i/G_i \) properly in the sense of Definition \ref{def:extendible_obstruction}. We may restrict our original obstruction model family to a \( G_i \) invariant sub family \( f_i \), which is topologically compactly contained inside \( \tilde{f}'_i \) so that \( f_i \) still contains the same set of holomorphic curves as our original obstruction model. Use the notation \( O_i \) to refer to the restriction of \( \frac{1}{2} O'_i \) to the subset with core given by this new family \( f_i/G_i \). \( O \) meets all these new \( O_i \) properly, so item 1 from Theorem \ref{thm:transverse_perturbations} holds, and any compactly supported \( C^{\infty,\perp} \) simple perturbation parametrized by \( f_i \) defines a \( C^{\infty,\perp} \) multi-perturbation on \( O \) as in Example \ref{ex:multi_perturbation}.

Theorem \ref{thm:weighted_sections} holds for this collection of obstruction models when we use \( (\tilde{f}'_i/G_i, V_i) \) on the corresponding restriction of \( \frac{1}{2} O'_i \) for the extensions of our obstruction models \( (f_i/G_i, V_i) \) on \( O_i \). It follows that item 2 from Theorem \ref{thm:transverse_perturbations} holds.

In particular, for a collection of compactly supported \( C^{\infty,\perp} \) simple perturbations \( P_i \) parametrized by \( f_i \), let \( \theta \) denote the multi-perturbation on \( O \) so that

\[
\theta(\tilde{f}) := \prod_i \tilde{f}^* P_i
\]

where \( \tilde{f}^* P_i \) is as in example \ref{ex:multi_perturbation}. Then for some convex \( C^{\infty,\perp} \) neighborhood \( U \) of 0 in the space of collections of compactly supported \( C^{\infty,\perp} \) simple perturbations \( \{P_i\} \), for each of our obstruction models \( f_i/G_i, V_i \), there exists a unique \( C^{\infty,\perp} \) weighted branched section \( \nu \) of \( \tilde{f}^* T_{vert} \mathfrak{B} \) so that topologically locally on \( \tilde{f}(f_i) \),

\[
\nu = \sum_{k=1}^n \frac{1}{n} \nu_k
\]

where \( \nu_k \) vanishes on marked points, and \( F(\nu_k) \) is a family of curves in \( O_i \cap O \) so that

\[
\theta(F(\nu_k)) = \sum_{j=1}^n \frac{1}{n} P_{k,j}
\]

where \( \tilde{\partial} F(\nu_k) - P_{k,k} \) is a section of \( V_i \), and for any curve \( f \in O_i \cap O \), if \( \theta(f) = \sum_j w_j Q_j \), then the sum of the weights \( w_i \) so that \( \tilde{\partial} f - Q_i \) is in \( V_i \) is equal to \( \frac{1}{n} \) times the number of the above \( \nu_k \) so that \( f \) is contained in the family \( F(\nu_k) \).

Say that \( \theta \) is transverse to \( \tilde{\partial} \) on a sub family \( C \subset \tilde{f} \) if each of the sections of \( V_i \) given by \( \tilde{\partial} F(\nu_k) - P_{k,k} \) from \ref{eq:weighted_sections} and \ref{eq:weighted_sections_2} are transverse to the zero section.
of $V_i$ on $C$. If $\theta'$ indicates the multi perturbation corresponding to a collection $\{P'_i\}$ of simple perturbations close in $C^\infty\downarrow$ to $\{P_i\}$, then Theorem 1.3 implies that the sections $\nu'_k$ corresponding to $\nu_k$ will be $C^\infty\downarrow$ close to $\nu_k$, and therefore, the corresponding sections of $V_i$ will also be $C^\infty\downarrow$ close to the original sections. It follows that the subset of $U$ consisting of collections of perturbations $\{P_i\}$ so that $\partial$ is transverse to $\theta$ on any particular topologically compact sub family $C \subset \hat{f}_i$ is open in $C^\infty\downarrow$ topology. If we choose $P'_i - P_i$ to consist of multi perturbations which take values in $V_i$, then the sections $\nu'_k$ from (60) will be equal to the sections $\nu_k$. It follows that the subset of $U$ so that $\theta$ is transverse to $\partial$ on a topologically compact subset $C$ is dense and open in $C^\infty\downarrow$, so item 3 from Theorem 1.17 holds. Similarly, given any two $\theta$ and $\theta'$ transverse to $\partial$ on a topologically compact sub family $C \subset \hat{f}_i$, a generic $C^\infty\downarrow$ family of perturbations $\theta_i$ joining $\theta$ to $\theta'$ is transverse to $\partial$ in the sense that the corresponding $C^\infty\downarrow$ family of sections of $V_i$ analogous to $\partial F(\nu_k) - P_{k,k}$ is transverse to the zero section on $C$.

Say that $\theta$ is fixed point free on the sub family $C \subset \hat{f}_i$ if none of the curves in $F(\nu_k)$ over $C$ have smooth parts with a non trivial automorphism group. If $C$ is topologically compact, the set of such curves in some $F(\nu_k)$ over $C$ is topologically compact. If $\theta'$ is the multi perturbation corresponding to a close collection of simple perturbations, then Theorem 1.3 implies that the set of corresponding curves in $F(\nu'_k)$ over $C$ are close to the original set, so if $\theta$ is fixed point free on $C$, $\theta'$ is fixed point free on $C$ if the new simple perturbations are chosen close enough in $C^\infty\downarrow$. If $C$ is topologically compact, it is covered by a finite number of topologically compact subsets on which the sections $\nu_k$ from (60) are defined. Theorem 1.3 implies that for any close by modification $\nu'_k$ of $\nu_k$, there exists a small modification of $P_i$ to $P'_i$ so that $\nu'_k$ is the solution corresponding to the modified multi perturbation $\theta'$. If the relative dimension of $\mathcal{B} \rightarrow \mathcal{G}$ is greater than 0, $\nu'_k$ may be chosen so that $F(\nu'_k)$ contains no curves who’s smooth parts have non trivial automorphism group. Therefore, item 4 from Theorem 1.17 holds and the subset of our space of perturbations $U$ so that $\theta$ is fixed point free is open and dense. (This is of course not the case when the relative dimension of $\mathcal{B} \rightarrow \mathcal{G}$ is zero - then all our perturbations are trivial, and the nature of the moduli space of holomorphic curves is easily deduced from Theorem 1.1.) Similarly, if the relative dimension of $\mathcal{B} \rightarrow \mathcal{G}$ is not zero, then given any two perturbations $\theta$, $\theta'$ in the set under consideration which are fixed point free on $C$, a generic family $C^\infty\downarrow$ family of perturbations $\theta_i$ in the set under consideration joining $\theta$ to $\theta'$ is fixed point free on $C$.

The compactness properties of the moduli space of holomorphic curves proved in 12 imply that we may cover the component of the moduli stack of holomorphic curves over $\mathcal{G}'$ under study by $G_i$ invariant topologically open sub families $\hat{f}_i \subset \hat{f}_i$ with closures $\hat{f}_i \subset \hat{f}_i$ which are topologically compact. Let $\mathcal{O}_i'$ denote the subset of $\mathcal{O}_i$ with core $f_i'/G_i$. Let $\mathcal{O}'$ denote the union of $\mathcal{O}_i'$. This is some open substack which contains the component of the moduli stack of holomorphic curves over $\mathcal{G}'$ under study. Theorem 1.17 implies that if the collection of simple perturbations $\{P_i\}$ is small enough in $C^\infty\downarrow$, then if any curve $f$ in $\mathcal{O}$ over $\mathcal{G}'$ satisfies $\theta(f) = w\theta f + \ldots$ where $w > 0$, then $f$ is in $\mathcal{O}'$. So if we choose our open set of perturbations $U$ small enough, item 3 from Theorem 1.17 holds.

If we say that $\theta$ is fixed point free and transverse to $\partial$ if $\theta$ is fixed point free and transverse to $\partial$ on all of the above sub families $C_i$, it follows that $\theta$
is fixed point free and transverse to \( \bar{\partial} \) in this sense for an open dense subset of perturbations in \( U \). Suppose that \( \theta \) is fixed point free and transverse to \( \bar{\partial} \).

Then define a weighted branched substack \( \mathcal{M}_\theta \) of \( \mathcal{O}_g \) as follows: locally in \( \mathcal{O}_g \), denote by \( \hat{g}_k \) the family which is the subset of \( \tilde{F}(v_k) \) so that \( \bar{\partial} \hat{g}_k = P_{k,k} \) in the notation of (60) and (61) above. In other words, \( \hat{g}_k \) corresponds to the transverse intersection of \( \partial F(v_k) - P_{k,k} \) with the zero section. Then as \( \theta \) is fixed point free, restricting to small enough neighborhoods in the family \( \hat{f}_i \), the family \( \hat{g}_k \) is a substack of the moduli stack of curves. Define \( \mathcal{M}_\theta \) to be locally equal to \( \sum_{k=1}^n \frac{1}{n} t^{\nu_k} \).

The weighted branched substack \( \mathcal{M}_\theta \) has the following two properties which make it well defined:

1. If \( \sum_k w_k t^{\nu_k} \) and \( \theta(f) = \sum_j w_j t^{\nu_j} \), then the sum of the weights \( w_k \) so that \( f \) is in \( \hat{g}_k \) is equal to the sum of the weights \( w_j \) so that \( \bar{\partial} f = Q_j \).

2. If \( \mathcal{M}_\theta \) is locally \( \mathcal{O}_g \) as \( wt^\nu + \ldots \), then \( \theta(\hat{g}) = wt^\nu + \ldots \).

The first property follows from the analogous fact for the multi section \( \sum_{k=1}^n \frac{1}{n} t^{\nu_k} \).

That these properties make \( \mathcal{M}_\theta \) well defined as a weighted branched substack can be proved in a similar way to uniqueness part of the statement of Theorem 4.5.

We’ve seen that \( \mathcal{M}_\theta \) is a complete weighted branched substack of \( \mathcal{O}_g \) of some fixed dimension. \( \mathcal{M}_\theta \) also has a well defined orientation relative to \( \mathfrak{G} \) - this orientation is determined as follows:

The core family \( \hat{f}_i/G_i \) comes with a collection of sections corresponding to marked points which when taken together give a section \( s : \mathfrak{F}(\hat{f}_i) \longrightarrow \mathfrak{F}(\hat{f}_i^{i+1}) \) so that \( ev^{i+1}(\hat{f}_i) : \mathfrak{F}(\hat{f}_i^{i+1}) \longrightarrow \text{Expl} \mathcal{M} \times (\mathfrak{B})_\mathfrak{G} \) is an equidimensional embedding in a neighborhood of \( s \). The canonical orientation of \( \mathfrak{G} \) relative to \( \mathfrak{G} \) given by the almost complex structure, and the orientation of \( \text{Expl} \mathcal{M} \) given by the complex structure give an orientation to \( \text{Expl} \mathcal{M} \times (\mathfrak{B})_\mathfrak{G} \) relative to \( \mathfrak{G} \). Give \( \hat{f}_i^{i+1} \) the orientation relative to \( \mathfrak{G} \) so that this map \( ev^{i+1}(\hat{f}_i) \) is oriented in a neighborhood of the image of \( s \), and give \( \hat{f}_i \) the corresponding orientation relative to \( \mathfrak{G} \) so that the complex fibers of \( \mathfrak{F}(\hat{f}_i^{i+1}) \longrightarrow \mathfrak{F}(\hat{f}_i^{i+k}) \) are positively oriented.

Give the vector bundle \( V_i \) over \( \mathfrak{F}(\hat{f}_i) \) an orientation relative to \( \hat{f}_i \) as follows: restricted to a curve \( f \) in \( \hat{f}_i \), we may identify \( V_i(f) \) with the cokernel of the injective Fredholm operator \( D\bar{\partial} : X_\chi(f) \longrightarrow Y_\phi(f) \). As both \( X_\chi(f) \) and \( Y_\phi(f) \) have complex structures, and \( D\bar{\partial} \) is homotopic through Fredholm maps to a complex map, we may give an orientation of \( V_i(f) \) using the spectral flow to this complex map, for which the kernel and cokernel are canonically oriented by their complex structures.

The orientation of \( f_i \) relative to \( \mathfrak{G} \) and the orientation of \( V_i \) relative to \( \hat{f}_i \) gives an orientation to the \( \hat{g}_k \) relative to \( \mathfrak{G} \) by considering \( \hat{g}_k \) as the intersection of the section \( \bar{\partial}v_k - P_{k,k} \) of \( V_i \) with the zero section. (The order of intersection does not matter as \( V_i \) is always an even dimensional vector bundle because the index of \( D\bar{\partial} \) restricted to \( X_\chi(f) \) is even.) We must see why this construction gives a well defined orientation for \( \mathcal{M}_\theta \) relative to \( \mathfrak{G} \) - in other words why we will get the same orientation using a different obstruction model. As a first step,
we may replace the family $\tilde{f}_i$ with the family $F(\nu_k)$ which actually contains $\tilde{g}_k$, and do our calculation of orientations at a curve $f$ in $\tilde{g}_k$. This will not change the orientations constructed as above. We may add a collection of $l'$ extra marked points and extend $F(\nu_k)$ to a family $\tilde{h}$ with extra parameters corresponding to the image of these extra marked points so that $ev^{+(l+l')}(\tilde{h})$ is an equidimensional embedding in a neighborhood of the section $s' : \mathcal{G}(\tilde{h}) \to \mathcal{G}(\tilde{h}^{+(l+l')})$ corresponding to all of these marked points. Denote by $X'_i(f)$ the complex subspace of $X_h(f)$ consisting of sections which vanish at the extra marked points. The tangent space to the extra parameter space at the curve $f$ can also be identified with $X_h(f)/X'_h(f)$. The orientation of $\tilde{h}$ relative to $\mathcal{G}$ given by $ev^{+(l+l')}(\tilde{h})$ agrees with the orientation from $\tilde{f}_i$ and the orientation from the almost complex structure on the extra parameter space. Again, applying the spectral flow to a complex operator gives an orientation of the cokernel of $D\partial : X'_i(f) \to Y_\delta(f)$. This cokernel can be identified with $V_i(f)$ times the extra parameter space, and the orientation of this cokernel given by the spectral flow is equal to the orientation of $V_i(f)$ times the orientation given by the complex structure on the extra parameter space. Therefore, the orientation on $M_{\theta}$ we obtain does not depend on the choice of marked points in our obstruction model.

Theorem 3.17 implies that all obstruction models containing $f$ with the same set of marked points are homotopic in some neighborhood of $f$ as all other choices can be changed continuously. The orientation of $\tilde{f}_i$ and the orientation of $V_i(f)$ given above doesn’t change under homotopy, so the orientation we obtain on on $M_{\theta}$ is well defined. Therefore item (2) from Theorem 3.17 holds. (Another way to see that this orientation is well defined would be to define a tangent space to the moduli stack of $C^\infty$ curves using the existence of core families and then identify the tangent space to $\tilde{g}_k$ with the kernel of some perturbed $D\partial \delta$ operator, with an orientation given in a similar way to the orientation of $V_i$.)

It is clear from the construction of $\mathcal{M}_\theta$ that the restriction of the support of $\mathcal{M}_\theta$ to curves over $\mathcal{G}'$ is topologically compact, as it is a finite union of topologically compact subsets, so item (7) of Theorem 3.17 is true.

To see that $\mathcal{M}_\theta$ gives a well defined cobordism class of finite dimensional weighted branched substacks oriented relative to $\mathcal{G}$, first suppose that $\theta_1$ is another multi perturbation which is fixed point free and transverse to $\partial$, defined using a different choice of simple perturbation $\{P_{1,i}\}$ in $U$. Then we may choose a $C^\infty$ family of collections of simple perturbations $\{P_{1,i}\}$ connecting $\{P_1\}$ to $\{P_{1,i}\}$. As a generic such family will have the corresponding family of multi perturbations $\theta_i$ transverse to $\partial$ and fixed point free $\mathcal{M}_\theta$ will be cobordant to $\mathcal{M}_{\theta_1}$.

Now suppose that $\theta$ and $\theta'$ are multi perturbations defined using different choices (including different choices of obstruction models etc.). Then we can consider the family of targets with one extra parameter $\mathfrak{B} \times \mathbb{R} \to \mathcal{G} \times \mathbb{R}$. We may choose a finite collection of extendible obstruction models on $\mathfrak{B} \times \mathbb{R}$ which cover the relevant component of holomorphic curves over $\mathcal{G}' \times [0,1]$ and which restrict to be our two chosen collections on $\mathfrak{B} \times 0$ and $\mathfrak{B} \times 1$. If $\theta$ and $\theta'$ are small enough, all other choices in the above construction can also be made to restrict to our given choices when restricted to $\mathfrak{B} \times 0$ and $\mathfrak{B} \times 1$ respectively. Let $\tilde{\theta}$ be the resulting multi perturbation which restricts to be $\theta$ on $\mathfrak{B} \times 0$ and $\theta'$ on $\mathfrak{B} \times 1$. The multi perturbation $\tilde{\theta}$ is fixed point free and transverse to $\partial$ in some neighborhood of $\mathfrak{B} \times 0$ and $\mathfrak{B} \times 1$, and may be modified away from
these neighborhoods to be fixed point free and transverse to \( \partial \). The resulting \( M_{\hat{\theta}} \) is the required cobordism between \( M_\theta \) and \( M_{\theta'} \). In the case that \( \theta \) and \( \theta' \) are not small enough to extend to a multi perturbation \( \hat{\theta} \) for which the above construction of \( M_{\hat{\theta}} \) works, we may first choose smaller multi perturbations \( \theta_1 \) and \( \theta'_1 \) so that the above construction gives a cobordism of \( M_{\theta_1} \) to \( M_{\theta'_1} \) and use the fact pointed out in the previous paragraph that \( M_{\theta} \) is cobordant to \( M_{\theta_1} \) and \( M_{\theta'} \) is cobordant to \( M_{\theta'_1} \).

In the case of a single target \( \mathcal{B} \), \( M_\theta \) is a finite dimensional \( \mathcal{C}^\infty \) oriented weighted branched substack of the moduli stack of \( \mathcal{C}^\infty \) curves in \( \mathcal{B} \). This should be thought of as giving a virtual class for a component of the moduli space of holomorphic curves in \( \mathcal{B} \), which is a cobordism class of \( \mathcal{C}^\infty \) finite dimensional oriented weighted branched substacks of the moduli stack of \( \mathcal{C}^\infty \) curves in \( \mathcal{B} \). The above discussion implies that this virtual class behaves well in family of targets \( \hat{\mathcal{B}} \to \mathcal{G} \), so enumerative invariants of holomorphic curves such as Gromov Witten invariants behave well in connected families of targets in the exploded category.

### A Construction and properties of \( \hat{f} + n \)

In this section we fill in the details of Definition 1.4 from page 3 and construct the family of curves \( \hat{f} + n \) with \( n \) extra marked points from a given family of curves \( \hat{f} \). As the definition is inductive, with \( \hat{f} + n = (\hat{f} + n-1) + 1 \), we shall describe \( \hat{f} + 1 \). This is some family of curves

\[
\begin{array}{ccc}
\hat{\mathcal{C}}^+ & \xrightarrow{\hat{f}^+} & (\hat{\mathcal{B}})^2 \\
\downarrow & & \downarrow \\
\hat{\mathcal{C}} & \xrightarrow{\hat{f}} & \hat{\mathcal{B}}
\end{array}
\]

that fits into the following diagram

\[
\begin{array}{ccc}
\hat{\mathcal{C}}^+ & \xrightarrow{\hat{f}^+} & (\hat{\mathcal{B}})^2 \\
\downarrow & & \downarrow \\
\hat{\mathcal{C}} & \xrightarrow{\hat{f}} & \hat{\mathcal{B}} \\
\downarrow \pi_{\hat{\mathcal{B}}} & & \downarrow \pi_{\mathcal{B}} \\
\hat{\mathcal{B}} & \to & \mathcal{B}
\end{array}
\]

The total space of the domain, \( \hat{\mathcal{C}}^+ \) is constructed by ‘exploding’ the diagonal of \( (\hat{\mathcal{C}})^2 \) as follows:

Consider the diagonal map \( \Delta : \hat{\mathcal{C}} \to (\hat{\mathcal{C}})^2 \). The image of the tropical part of this map \( |\Delta| \) defines a subdivision of the tropical part of \( (\hat{\mathcal{C}})^2 \), which determines a unique refinement \( \hat{\mathcal{C}}' \to (\hat{\mathcal{C}})^2 \). Note that the diagonal map to
this refinement $\hat{C}'$ is still defined, 

$$
\hat{C} \xrightarrow{\Delta} (\hat{C})^2 \hat{\mathcal{F}}
$$

and a topological neighborhood of the image of the diagonal in $\hat{C}'$ is equal to a neighborhood of $0$ in a $\mathbb{C}$ bundle over $\hat{C}$.

Now ‘explode’ the image of the diagonal in $\hat{C}'$ to make $\hat{C}^{+1} \to \hat{C}'$ as follows:

We may choose coordinate charts on $\hat{C}'$ so that any coordinate chart intersecting the image of the diagonal is equal to some subset of $\mathbb{C} \times U$ where $U$ is a coordinate chart on $\hat{C}$, the projection to $\hat{C}$ is the obvious projection to $U$, the complex structure on the fibers of this projection is equal to the standard complex structure on $\mathbb{C}$, and the image of the diagonal is $0 \times U$. Replace these charts with the corresponding subsets of $\mathbb{T}^1 \times U$, and leave coordinate charts that do not intersect the image of the diagonal unchanged. Any transition map between coordinate charts of the above type is of the form $(z,u) \mapsto (g(z,u)z,\phi(u))$ where $g(z,u)$ is $\mathbb{C}^*$ valued. In the corresponding ‘exploded’ charts, the corresponding transition map is given by $(\hat{z},u) \mapsto (g([\hat{z}],u)\hat{z},\phi(u))$. The transition maps between other charts can remain unchanged. This defines $\hat{C}^{+1}$. The map $\hat{C}^{+1} \to \hat{C}'$ is given in the above coordinate charts by $(\hat{z},u) \mapsto ([\hat{z}],u)$. Composing this with the refinement map $\hat{C}' \to (\hat{C})^2$ then gives a degree one fiberwise holomorphic map

$$
\hat{C}^{+1} \to \left(\hat{C}\right)^2 \hat{\mathcal{F}}
$$

The map $\hat{f}^{+1} : \hat{C}^{+1} \to \left(\mathbb{B}\right)^2$ is given by the above constructed map $\hat{C}^{+1} \to \left(\hat{C}\right)^2 \hat{\mathcal{F}}$ composed with the map

$$
\left(\hat{C}\right)^2 \to \left(\mathbb{B}\right)^2
$$

which is $\hat{f}$ in each component. All the above maps are smooth or $C^\infty$ if $\hat{f}$ is.

The above construction is functorial. Given a map of families $\hat{f} \to \hat{g}$, there is an induced map $\hat{f}^{+1} \to \hat{g}^{+1}$. To see this, consider the naturally induced map

$$
\left(\hat{C}(\hat{f})\right)^2 \to \left(\hat{C}(\hat{g})\right)^2
$$

As $\phi \times \phi$ sends the diagonal to the diagonal, this map lifts to the refinement referred to in the above construction. As $\phi \times \phi$ is holomorphic on fibers and
sends the diagonal to the diagonal, in the special coordinates on the refinement used in the above construction of the form \((z, u)\) and \((w, v)\), the map \(\phi \times \phi\) is of the form
\[
\phi \times \phi(z, u) = (h(z, u)z, \phi(u))
\]
where \(h(z, u)\) is \(\mathbb{C}^*\) valued. Then the map \(\phi^{+1} : \mathcal{C}(\hat{f}^{+1}) \rightarrow \mathcal{C}(\hat{g}^{+1})\) is given in the corresponding exploded coordinates by
\[
\phi^{+1}(\tilde{z}, u) = (h(\lceil \tilde{z} \rceil), u, \tilde{z}, \phi(u))
\]
We then get a map
\[
\begin{array}{ccc}
\mathcal{C}(\hat{f}^{+1}) & \xrightarrow{\phi^{+1}} & \mathcal{C}(\hat{g}^{+1}) \\
\downarrow & & \downarrow \\
(\mathcal{C}(\hat{f}))^2 & \xrightarrow{\phi \times \phi} & (\mathcal{C}(\hat{g}))^2 \\
\downarrow & & \downarrow \\
\mathcal{C}(\hat{f}) & \xrightarrow{\phi} & \mathcal{C}(\hat{g}) \\
\downarrow & & \downarrow \\
\mathcal{F}(\hat{f}) & \longrightarrow & \mathcal{F}(\hat{g})
\end{array}
\]
The map \(\phi^{+1}\) is clearly compatible with the maps \(\hat{f}^{+1}\) and \(\hat{g}^{+1}\), so \(\phi^{+1}\) is a map of families \(\hat{f}^{+1} \longrightarrow \hat{g}^{+1}\). It follows that the construction of \(\hat{f}^{+n}\) is functorial for all \(n\).

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