SOLVABILITY AND BLOW-UP CRITERION OF THE THERMOMECHANICAL CUCKER-SMALE-NAVIER-STOKES EQUATIONS IN THE WHOLE DOMAIN

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Abstract. We study local existence and uniqueness of a strong solution to the kinetic thermomechanical Cucker–Smale (in short TCS) model coupled with incompressible Navier–Stokes (NS) equations in the whole space. The coupled system consists of the kinetic TCS equation for particle ensemble and the incompressible NS equations for a fluid via a drag force. For the strong solution, we investigate the blow-up mechanism for the coupled system, and we also study the global existence of a weak solution in the whole space.

1. Introduction. Collective behaviors of many-body systems such as flocking of birds or schooling of fishes, draw a lot of attention over the past decades. Numerous mathematical models have been introduced to describe these phenomena, to name a few, such as the Vicsek model [27], the Cucker-Smale (CS) model [12], and the thermomechanical CS (TCS) model [20]. Among them, our main interest lies in the TCS ensemble. The TCS model was first introduced in [20] as a generalization of the CS flocking model in a self-consistent temperature field, since the TCS model reduces to the CS model when temperature field is constant. Until now, various topics on the TCS model have been studied, including the emergent behavior of microscopic TCS ensemble [18, 19], the mean-field limit [18], well-posedness and emergent dynamics of hydrodynamic model [17], discrete time model [15], etc.

However, the dynamical systems in real world do not form a closed system, when it interacts with surrounding environment. Thus, it is natural to consider the particle ensemble that is influenced by the dynamics of various kinds of fluid. Recently, the particle-fluid coupled models have been extensively studied due to its possible applications on physics or biochemical engineering [5, 6, 22, 24–26]. In this paper, we consider particle-fluid coupled model for TCS ensemble. Specifically,
we consider the TCS ensemble immersed in incompressible viscous fluids. In this situation, the dynamics is described by a kinetic TCS equation coupled with the incompressible Navier-Stokes equations, constituting the TCS–NS system. More precisely, let \( f = f(t, x, v, \theta) \) be a particle distribution function in phase space \((x, v, \theta)\) at time \( t \), with position \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), velocity \( v = (v_1, v_2, v_3) \in \mathbb{R}^3 \) and temperature \( \theta \in \mathbb{R}_+ \) respectively. Moreover, we denote the bulk velocity of incompressible flow by \( u = (u_1, u_2, u_3) \). Then, the TCS–NS system reads as follows:

\[
\begin{cases}
\partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (F[f, u]) + \partial_\theta(G[f, f]) = 0, \\ \partial_t u + (u \cdot \nabla) u + \nabla p - \Delta u = - \int_{\mathbb{R}^3 \times \mathbb{R}_+} F_d(u) f \, dv \, d\theta, \\ \nabla \cdot u = 0,
\end{cases}
\]

subject to the initial data:

\[
f(x, v, \theta, 0) = f_0(x, v, \theta), \quad u(x, 0) = u_0(x).
\]

Here, the nonlocal forces \( F[f, u] \) and \( G[f] \) are defined as

\[
F[f](t, x, v, \theta) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+} \phi(x - x_*) \left( \frac{v_*}{\theta_*} - \frac{v}{\theta} \right) f(t, x_*, v_*, \theta_*) \, dx_* \, dv_* \, d\theta_*,
\]

\[
G[f](t, x, \theta) = \int_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+} \zeta(x - x_*) \left( \frac{1}{\theta_*} - \frac{1}{\theta} \right) f(t, x_*, v_*, \theta_*) \, dx_* \, dv_* \, d\theta_*,
\]

\[
F_d(t, x, v) = u(t, x) - v, \quad F[f, u] = F[f] + F_d,
\]

where communication weights \( \phi, \zeta \in W^{1,\infty}(\mathbb{R}^3) \) are symmetric and positive functions satisfying \( 0 \leq \phi(x) \leq \phi_M, \quad 0 \leq \zeta(x) \leq \zeta_M, \quad x \in \mathbb{R}^3 \).

Below, we briefly review the history of particle-fluid coupled models in collective dynamics. In the series of papers [1–4], the authors studied the well-posedness and large-time behavior of the Cucker–Smale–Navier–Stokes (CS–NS) systems in the three-dimensional periodic domain \( \mathbb{T}^3 \). For the other literature on the particle-fluid coupled model, we refer to [8, 9, 11, 16, 21, 23, 28, 29]. In the same context, the first author and his collaborators presented the global existence of a weak and strong solution to (1.1) in [13], and the kinetic TCS equation coupled with compressible fluid [14], when the spatial domain is still the periodic box \( \mathbb{T}^3 \), as well as the emergent dynamics of it. Those results are restricted to the periodical for Poincaré inequality holds in the bounded domain. In [10], however, it was shown that the global existence of a weak solution to CS–NS system when the domain is two-dimensional whole space \( \mathbb{R}^2 \). Inspired by those works, we study existence and uniqueness of the local smooth solution, blow-up criterion and global existence of weak solution to TCS–NS system in the whole spatial domain \( \mathbb{R}^3 \).

Thus, the main goal of this paper is three-fold. First, we will present the local existence of a unique strong solution to (1.1). We mainly use the Gagliardo Nirenberg interpolation inequality, instead of the Poincaré inequality, to provide energy estimate of the fluid part (Theorem 2.1). Then, we provide a blow-up criterion for the strong solution (Theorem 2.2). If the lifetime of strong solution is finite, say \( T^* \), then we showed that

\[
\lim_{T \to T^*} \int_0^T \| \nabla u(t) \|_{L^\infty} \, dt = \infty.
\]
Finally, we prove the global existence of weak solution of (1.1) (Theorem 2.3). Again, the interpolation theorem plays an important role to derive an appropriate energy estimate, which guarantees the existence of a weak solution.

The rest of the paper is organized as follows. In Section 2, we present the preliminaries and the main theorems of this paper. In Section 3, we prove the local existence and uniqueness of strong solution to (1.1). Section 4 provides the blow-up criterion for the strong solution of TCS-NS system (1.1). In Section 5, we present the global existence of weak solution of (1.1) (Theorem 2.3).

Notation. Throughout the paper, we use $\Omega$ as an abbreviation of state space $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^+$. Moreover, we use the variable $z$ as an abbreviation of the total phase variable $(x, v, \theta)$. For two vectors $u \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$, we denote their tensor product as $u \otimes v := uv^T$. For two matrices $A = (a_{ij})_{n \times n}$ and $B = (b_{ij})_{n \times n}$, we define the matrix inner product as $A : B := \sum_{i,j} a_{ij} \cdot b_{ij}$. We also omit the $x$-dependence of partial derivatives, i.e., $\nabla_x = \nabla$. The generic constant $C$ may differ from line to line.

2. Preliminaries and main theorems. In this section, we first review the basic properties of the TCS-NS system and then present the main result of the paper.

2.1. Basic estimates. In this subsection, we investigate the basic properties of the TCS-NS system (1.1), including temperature support estimate and conservation laws. The most basic, yet important property of the TCS-NS system is the monotonicity of temperature support of $f$. We define the projections of the support of $f$ onto each state variables as follows:

$$
\text{supp}_x f(t) := \{x : \exists (v, \theta) \in \mathbb{R}^3 \times \mathbb{R}_+ \text{ such that } f(t, x, v, \theta) \neq 0\},
$$

$$
\text{supp}_v f(t) := \{v : \exists (x, \theta) \in \mathbb{R}^3 \times \mathbb{R}_+ \text{ such that } f(t, x, v, \theta) \neq 0\},
$$

$$
\text{supp}_\theta f(t) := \{\theta : \exists (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \text{ such that } f(t, x, v, \theta) \neq 0\}.
$$

Moreover, we also consider the characteristic curve:

$$(x(t), v(t), \theta(t)) := (x(t; 0, z_0), v(t; 0, z_0), \theta(t; 0, z_0)), \quad z_0 = (x_0, v_0, \theta_0)$$

of system (1.1) issued from $(x_0, v_0, \theta_0)$ defined as follows:

$$
\frac{dx(t)}{dt} = v(t), \quad t \geq 0,
$$

$$
\frac{dv(t)}{dt} = \int_{\Omega} \phi(x(t) - x_*) f(t, z_*) \left( \frac{v(t)}{\theta(t)} - \frac{v(t)}{\theta(t)} \right) dz_* + u(t, x(t)) - v(t),
$$

$$
\frac{d\theta(t)}{dt} = \int_{\Omega} \zeta(x(t) - x_*) f(t, z_*) \left( \frac{1}{\theta(t)} - \frac{1}{\theta(t)} \right) dz_*, \quad (x(0), v(0), \theta(0)) = (x_0, v_0, \theta_0).
$$

(2.1)
Moreover, for simplicity, we introduce the following notation:

\[
a(t, x) := \int_{\Omega} \phi(x - x_*) f(t, z_*) \, dz_* ,
\]

\[
b(t, x) := \int_{\Omega} \phi(x - x_*) \frac{v_*}{\theta_*} f(t, z_*) \, dz_* ,
\]

\[
c(t, x) := \int_{\Omega} \zeta(x - x_*) f(t, z_*) \, dz_* .
\]

Then, we present the monotonicity of \( \text{supp} \theta f(t) \) in the following lemma.

**Lemma 2.1.** Let \( (f, u) \) be a solution to TCS-NS system (1.1) and suppose that the initial data \( f_0 \) has a strictly positive compact support in temperature variable, i.e., there exist \( 0 < \theta_m < \theta_M \) such that

\[
\text{supp} \theta f_0 \subset [\theta_m, \theta_M].
\]

Then, the temperature support of \( f \) monotonically shrinks. More precisely, for any \( 0 \leq s \leq t \), we have

\[
\text{supp} \theta f(t) \subset \text{supp} \theta f(s).
\]

In particular, the temperature support of \( f(t) \) is contained in the same initial compact set \( [\theta_m, \theta_M] \):

\[
\text{supp} \theta f(t) \subset [\theta_m, \theta_M].
\]

**Proof.** Consider the characteristic curve \((x(t), v(t), \theta(t))\) in (2.1) that provides the maximal temperature, i.e., \( \theta(t) = \sup_{\theta} \text{supp} \theta f(t) \). Then, it follows from (2.1) that

\[
\frac{d \theta(t)}{dt} = \int_{\Omega} \zeta(x(t) - x_*) f(t, z_*) \left( \frac{1}{\theta(t)} - \frac{1}{\theta_*} \right) \, dz_* ,
\]

\[
= \int_{\Omega} \zeta(x(t) - x_*) f(t, z_*) \left( \frac{\theta_*}{\theta(t)} \theta_\ast \right) \, dz_* \leq 0,
\]

where the last inequality comes from the maximality of \( \theta(t) \). Therefore, we obtain

\[
\sup_{\theta} \text{supp} \theta f(t) \leq \sup_{\theta} \text{supp} \theta f(s), \quad \text{for} \quad 0 \leq s \leq t. \quad (2.2)
\]

For the minimal temperature, we can do the exactly same argument as in the maximal temperature case and we have

\[
\inf_{\theta} \text{supp} \theta f(t) \geq \inf_{\theta} \text{supp} \theta f(s), \quad \text{for} \quad 0 \leq s \leq t. \quad (2.3)
\]

Thus, we combine (2.2) and (2.3) to obtain the desired estimate on the temperature support.

The other basic property of the TCS-NS system (1.1) is that the total mass and momentum are conserved along with the dynamics.

**Lemma 2.2.** Let \( (f, u) \) be a solution to TCS-NS system (1.1). Then, the following conservation laws hold:

1. The total mass of TCS ensemble is conserved:

\[
\frac{d}{dt} \int_{\Omega} f(t, z) \, dz = 0.
\]

2. The total momentum of the coupled TCS-NS system is conserved:

\[
\frac{d}{dt} \left( \int_{\Omega} vf(t, z) \, dz + \int_{\mathbb{R}^3} u(t, x) \, dx \right) = 0.
\]
Proof. (1) The conservation of mass can be directly obtained after we integrate the first equation of TCS-NS system (1.1) over whole domain Ω. (2) We multiply (1.1)_1 by \( v \) and integrate over Ω to obtain

\[
\frac{d}{dt} \int_{\Omega} vf(t, z) \, dz = \int_{\Omega} F[f, u] \, dz = \int_{\Omega} F_d[u] \, dz,
\]

where we use

\[
\int_{\Omega} F[f] \, dz = \int_{\Omega} \phi(x - x_s) \left( \frac{v_s}{\theta_s} - \frac{v}{\theta} \right) ff_s \, dz \, dz_s = 0,
\]

in the last equality. Moreover, we integrate (1.1)_2 to have

\[
\frac{d}{dt} \int_{\Omega} u(t, x) \, dx = - \int_{\Omega} F_d[u] \, dz.
\]

Thus, we combine two estimates to obtain the conservation of total momentum. \( \square \)

**Remark 2.1.** Since the total mass of TCS ensemble is conserved, we will assume that the total mass is equal to 1:

\[
\int_{\Omega} f(t, z) \, dz = 1.
\]

Next, we define the local velocity moment \( m_{\alpha} \) and the global velocity moment \( M_{\alpha} \) of function \( g \) as

\[
m_{\alpha} g(x, t) := \int_{\mathbb{R}^3 \times \mathbb{R}^+} |v|^\alpha g(z, t) \, dv \, d\theta, \quad \text{and} \quad M_{\alpha} g(t) := \int_{\Omega} |v|^\alpha g(z, t) \, dz.
\]

We review the technical lemma comparing the low-order local moment with respect to high-order local moment.

**Lemma 2.3.** [13, Lemma 4.2] Let \( 0 < \alpha < \beta \) be positive constants and \( g \) be a non-negative function in \( L^\infty(\Omega \times (0, T)) \) such that \( \text{supp}_t g(t) \subset [\theta_m, \theta_M] \) with the finite \( \beta \)-local velocity moment, i.e., \( m_{\beta} g(x, t) < \infty \). Then, the following estimate holds:

\[
m_{\alpha} g(x, t) \leq (|B(1)|(\theta_M - \theta_m)||g||_\infty + 1)m_{\beta} g(x, t)^{\frac{\alpha}{\beta + 2}}, \quad \text{a.e.} \quad (x, t) \in \mathbb{R}^3 \times (0, T),
\]

where \( |B(1)| \) is the volume of \( d \)-dimensional unit ball.

We close this subsection with the useful interpolation inequalities in \( \mathbb{R}^3 \).

**Lemma 2.4** (Interpolation inequality). For any \( f \in H^1_0(\mathbb{R}^3) \), we have

\[
\|f\|_{L^3} \leq C\|f\|_{L^2}^{\frac{3}{2}}\|\nabla f\|_{L^2}^{\frac{1}{2}}, \quad \|f\|_{L^5} \leq C\|f\|_{L^2}^{\frac{5}{3}}\|\nabla f\|_{L^2}^{\frac{2}{3}}.
\]

### 2.2. Main results

In this subsection, we present the detailed statements of the main results. First of all, we provide the definition of a strong solution to (1.1).

**Definition 2.1.** For a fixed time \( 0 < T < \infty \), the pair \((f, u)\) is said to be a strong solution to (1.1) in the time interval \([0, T)\) if this pair satisfies the following conditions:

1. \( f \in C([0, T]; W^{1, \infty}(\Omega)) \),
2. \( u \in C([0, T]; H^2(\mathbb{R}^3)) \cap L^2(0, T; H^3(\mathbb{R}^3)) \),
3. \((f, u)\) satisfies equation (1.1) in the sense of distributions.

Then, we have the following local existence of a strong solution.
Theorem 2.1 (Local existence). Assume that the initial data $f_0 \geq 0$ and $u_0$ satisfy the following conditions:
1. $f_0 \in W^{1,\infty}(\Omega)$, $u_0 \in H^2(\mathbb{R}^3)$,
2. $f_0$ has compact support in each variables, i.e.,
$$\text{supp } f_0 \subseteq B(R_1) \times B(R_2) \times [\theta_m, \theta_M],$$
where $B(R)$ denotes the ball centered at the origin with a radius $R$. Then, the Cauchy problem (1.1)-(1.2) admits a unique local strong solution in the sense of Definition 2.1. In other words, there exists a positive time $T^*$ such that there exists a unique strong solution up to $T^*$ in the sense of Definition 2.1.

Together with the existence and uniqueness of the strong solution, we also provide the following blow-up criterion.

Theorem 2.2 (Blowup criterion). Under the conditions in Theorem 2.1, let $(f, u)$ be the strong solutions to (1.1)-(1.2) in the sense of Definition 2.1. If the life span $T^*$ of the solution $(f, u)$ is finite, we have the following blow-up criterion:
$$\lim_{T \to T^*} \int_0^T \|\nabla u(t)\|_{L^\infty} \, dt = \infty.$$

We now provide the global existence of weak solution. We consider the following function spaces
$$\mathcal{H} := \left\{ w \in (L^2(\mathbb{R}^3))^3 \mid \nabla \cdot w = 0 \right\}, \quad \mathcal{V} := \left\{ w \in (H^1(\mathbb{R}^3))^3 \mid \nabla \cdot w = 0 \right\}.$$
Then, we define the weak solution of (1.1) as follows.

Definition 2.2. For a fixed time $0 < T < \infty$, the pair $(f, u)$ is a weak solution to (1.1) on the time interval $[0, T)$ if this pair satisfies the following conditions:
1. $f \in L^\infty(0, T; (L^1 \cap L^\infty)(\Omega))$, $|v|^2 f \in L^\infty(0, T; L^1(\Omega))$,
2. $u \in L^\infty(0, T; \mathcal{H}) \cap L^2(0, T; \mathcal{V}) \cap C^0([0, T), \mathcal{V})$,
3. $(f, u)$ satisfies equation (1.1) in the sense of distributions.

We present the last main result of this paper regarding the global existence of a weak solution of system (1.1).

Theorem 2.3. Assume that the initial data $(f_0, u_0)$ satisfy the following conditions:
$$f_0 \in (L^1 \cap L^\infty)(\Omega), \quad (|x|^2 + |v|^2)f_0 \in L^1(\Omega) \quad u_0 \in \mathcal{H}.$$
Then, for any given positive constant $T > 0$, there exist at least one weak solution $(f, u)$ to (1.1) with initial data $(f_0, u_0)$ satisfying the following energy estimates:
1. $\|f\|_{L^\infty(\Omega \times [0, T])} \leq C(T) \|f_0\|_{L^\infty}$,
2. $\frac{1}{2} M_2 f(t) + \frac{1}{2} \|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds + \int_0^t \int_{\Omega} |u - v|^2 f \, dz \, ds$
$$\leq (1 + C(T)) \left( \frac{1}{2} \|u_0\|_{L^2}^2 + \frac{1}{2} M_2 f_0 \right).$$

Remark 2.2. We compare our results with the other previous results on the particle-fluid coupled model in collective dynamics. As we mentioned briefly in the introduction, the well-posedness problems for TCS-NS system (1.1) with the spatially periodic domain $\mathbb{T}^3$ were studied in the recent literature [13]. We extend the previous results to the whole space $\mathbb{R}^3$. Regarding the existence of a weak solution, they use the Sobolev embedding $H^1(\mathbb{T}^3) \hookrightarrow L^6(\mathbb{T}^3) \hookrightarrow L^5(\mathbb{T}^3)$, where the boundedness
of the domain is critical in the second embedding. For the two-dimensional whole space domain \( \mathbb{R}^2 \), the existence of a weak solution for Cucker-Smale-Navier-Stokes system was proved in [10] by using Ladyzhenskaya inequality. As we will see later, we use the interpolation inequality Lemma 2.4 to extend the result to \( \mathbb{R}^3 \) for TCS-NS system case. For the existence of a strong solution, a similar issue occurs. In [13], the inclusion \( L^6(\mathbb{T}^3) \hookrightarrow L^3(\mathbb{T}^3) \) is used, which is not true for \( \mathbb{R}^3 \). Instead, we use the Gagliardo-Nirenberg interpolation inequality in the estimates.

3. A local existence and uniqueness of strong solution. In this section, we provide the detailed proof of Theorem 2.1. We construct the following sequence of approximated solutions \((f^n, u^n)\) as a solution of the following iterations:

\[
\begin{align*}
\partial_t f^{n+1} + v \cdot \nabla f^{n+1} + \nabla \cdot (F[f^{n+1}, u^n] f^{n+1}) + \partial_\theta (G[f^{n+1}] f^{n+1}) &= 0, \\
\partial_t u^{n+1} + (u \cdot \nabla) u^{n+1} + \nabla p^{n+1} - \Delta u^{n+1} &= \int_{\mathbb{R}^3 \times \mathbb{R}^3} f^{n+1}(v - u^n) \, dv \, d\theta, \\
\nabla \cdot u^{n+1} &= 0,
\end{align*}
\]

where the functions at the initial step \( f^0 \) and \( u^0 \) are defined as

\[
f^0(t, z) \equiv f_0(z), \quad \text{for all } t \geq 0, \quad \text{and} \quad \begin{cases} 
\partial_t u^0 - \Delta u^0 = 0, \\
u^0(0, x) = u_0(x) \in H^2(\mathbb{R}^3).
\end{cases}
\]

In order to guarantee the sequence of the approximated solution is well-defined, we first need to prove the solvability of (3.1). We will first show the solvability of the first equation of (3.1).

3.1. Solvability of the kinetic TCS equation (3.1). In this subsection, we present the solvability of the kinetic TCS equation by the iterative scheme (3.1). For simplicity, we suppress all the \( n \)-dependence and consider the following equation:

\[
\begin{align*}
\partial_t f + v \cdot \nabla f + \nabla \cdot (F[f] f + (u - v) f) + \partial_\theta (G[f] f) &= 0, \\
f(0, z) &= f_0(z),
\end{align*}
\]

when the fluid part \( u \in C([0, T]; H^2(\mathbb{R}^3)) \cap L^2(0, T; H^3(\mathbb{R}^3)) \) is already known. We define the radius of \( x \)-support and \( v \)-support of \( f \) at the time \( t \) as

\[
R_1(t) := \sup \left\{ |x| : (x, v, \theta) \in \text{supp} f(t, \cdot, \cdot, \cdot) \right\}
\]

and

\[
R_2(t) := \sup \left\{ |v| : (x, v, \theta) \in \text{supp} f(t, \cdot, \cdot, \cdot) \right\}.
\]

In the following proposition, we prove the well-posedness of (3.2).

**Proposition 3.1.** Let \( R_2(0), R_3(0) > 0, \ T > 0. \ Assume \ 0 \leq f_0 \in W^{1,\infty}(\Omega) \), and \( \text{supp} f_0 \subseteq B(R_1(0)) \times B(R_2(0)) \times [\theta_m, \theta_M] \). Then, for given \( u \in C([0, T]; H^2(\mathbb{R}^3)) \cap L^2(0, T; H^3(\mathbb{R}^3)) \), there exists a unique non-negative strong solution \( f \in C([0, T]; W^{1,\infty}(\Omega)) \) to (3.2). Moreover, we have

1. The temperature support of \( f \) is contained in \([\theta_m, \theta_M] \):

\[
\text{supp}_\theta f(t) \subseteq [\theta_m, \theta_M].
\]
2. The size of $x$- and $v$-support of $f$ are estimated as follows:

$$R_1(t) \leq R_1(0) + R_2(0)t + Ct^2 \left( R_2(0) + \sup_{0 \leq s \leq T} \|u(s)\|_{H^2} \right),$$

$$R_2(t) \leq R_2(0) + Ct \left( R_2(0) + \sup_{0 \leq s \leq T} \|u(s)\|_{H^2} \right), \quad 0 \leq t \leq T,$$

where $C$ is a positive constant.

3. For $1 \leq p \leq \infty$, the $W^{1,p}$-norm of $f$ is bounded:

$$\|f(t)\|_{W^{1,p}} \leq C(T)\|f_0\|_{W^{1,p}}.$$

4. Let $\tilde{f}$ be a solution to (3.2) with $\tilde{u}, \tilde{f}_0$ instead of $u, f_0$ respectively. Then, we have the following stability

$$\sup_{0 \leq t \leq T} \|f(t) - \tilde{f}(t)\|_{L^p} \leq C \left( \|f_0 - \tilde{f}_0\|_{L^\infty} + \sup_{0 \leq s \leq T} \|u(s) - \tilde{u}(s)\|_{L^\infty} \right).$$

**Remark 3.1.** Since $f_0$ is contained in $W^{1,\infty}(\Omega)$ and has a compact support in each variables, we deduce that

$$\|f_0\|_{L^1} = \int_\Omega f_0(\mathbf{z})\,d\mathbf{z} \leq C(\theta_M - \theta_m)|R_1(0)|^3|R_2(0)|^3\|f_0\|_{L^\infty}.$$

Thus, the initial data $f_0$ has a finite mass. Moreover, we integrate (3.2) over $[0,t] \times \Omega$ to obtain the conservation of mass (See also Remark 2.1):

$$\|f(t)\|_{L^1} = \|f_0\|_{L^1} = 1. \quad (3.3)$$

3.1.1. *A priori estimate.* In order to prove Proposition 3.1, we need the following *a priori* estimates.

**Lemma 3.1 (A priori estimate).** Under the same assumptions on $f_0$ and $u$ in Proposition 3.1, suppose that there exists a smooth solution $f$ to (3.2). Then, all the assertions (1)–(4) for $f$ in Proposition 3.1 hold.

**Proof.** (1) Since the estimate for the temperature support of $f$ is exactly the same as Lemma 2.1 by considering the characteristic curves, we omit the proof.

(2) In order to get the boundness of $\text{supp}_x f$, we first need to obtain the estimate of $M_2 f$. We multiply (3.2) by $|v|^2$ and integrate over the state space $\Omega$ to yield

$$\frac{d}{dt} \int_\Omega f|v|^2\,d\mathbf{z} = \int_\Omega \phi(\mathbf{x} - \mathbf{x}_*) f(t, \mathbf{z}) f(t, \mathbf{z}_*) (\mathbf{v} - \mathbf{v}_*) \cdot \left( \frac{\mathbf{v}_*}{\theta_*} - \frac{\mathbf{v}}{\theta} \right) d\mathbf{z}_* \, d\mathbf{z}$$

$$- 2 \int_\Omega f|v|^2\,d\mathbf{z} + 2 \int_\Omega f\mathbf{v} \cdot \mathbf{u}\,d\mathbf{z}$$

$$= - \int_\Omega \phi(\mathbf{x} - \mathbf{x}_*) \frac{(\mathbf{v} - \mathbf{v}_*)^2}{\theta_*} f(t, \mathbf{z}) f(t, \mathbf{z}_*) d\mathbf{z}_* \, d\mathbf{z}. \quad (3.4)$$
This together with (3.7) gives the following control of the size of velocity support:

\[ + \int_{\Omega} \phi(x - x_s) \nabla \cdot (v - v_s) \left( \frac{1}{\theta_s} - \frac{1}{\theta} \right) f(t, z) f(t, z_s) \, dz \, dz_s \leq 2M_2 f + 2\|u(t)\|_{L^\infty}(M_2 f)^{\frac{1}{2}}(M_0 f)^{\frac{1}{2}} \]

We use Cauchy-Schwartz inequality, (3.3) and (3.6) to estimate \( |v| \) can be written as

\[ \leq \frac{1}{4} \int_{\Omega} \phi(x - x_s) (\theta - \theta_s)^2 |v|^2 f(t, z) f(t, z_s) \, dz \, dz_s \leq 2M_2 f + 2\|u(t)\|_{L^\infty}(M_2 f)^{\frac{1}{2}} \]

\[ \leq \left( \frac{\phi_M (\theta_M - \theta_m)^2}{4\theta_m^3} - 2 \right) M_2 f + 2\|u(t)\|_{L^\infty}(M_2 f)^{\frac{1}{2}}. \]

We now set

\[ \alpha := \frac{\phi_M (\theta_M - \theta_m)^2}{4\theta_m^3} - 2, \quad \beta := 2 \sup_{0 \leq t \leq T} \|u(t)\|_{L^\infty}. \]

Note that the Sobolev embedding \( H^2(\mathbb{R}^3) \hookrightarrow C^{0, \frac{1}{2}}(\mathbb{R}^3) \) implies \( \beta < \infty \). Then, (3.4) can be written as

\[ \frac{dM_2 f}{dt} \leq \alpha M_2 f + \beta(M_2 f)^{\frac{1}{2}}. \tag{3.5} \]

We apply Grönwall’s lemma to (3.5) to yield

\[ (M_2 f(t))^{\frac{1}{2}} \leq (M_2 f(0))^{\frac{1}{2}} e^{\alpha t} + \frac{\beta}{\alpha} \left( e^{\alpha t} - 1 \right) \leq R_2(0)\|f_0\|_{L^1}e^{\alpha t} + \frac{\beta}{\alpha} \left( e^{\alpha t} - 1 \right). \]

Then, there exists a constant \( C = C(T) \) such that

\[ (M_2 f(t))^{\frac{1}{2}} \leq C \left( R_2(0) + \sup_{0 \leq t \leq T} \|u(t)\|_{L^\infty} \right), \quad 0 \leq t \leq T. \tag{3.6} \]

With the estimate of \( M_2 f \), we are able to control the velocity support. It follows from the characteristic equation (2.1) that

\[ v(t) = V_0 e^{-\int_0^t \frac{1 + \frac{a(t, x(t))}{\theta(t)}}{\theta(t)} \, dt} \]

\[ + e^{-\int_0^t \frac{1 + \frac{a(s, x(s))}{\theta(s)}}{\theta(s)} \, ds} \int_0^t \left[ b(\tau, x(\tau)) + u(\tau, x(\tau)) \right] e^{\int_\tau^t \frac{1 + \frac{a(s, x(s))}{\theta(s)}}{\theta(s)} \, ds} \, d\tau. \tag{3.7} \]

We use Cauchy-Schwartz inequality, (3.3) and (3.6) to estimate \( b(t, x) \) as

\[ |b(t, x)| \leq \left( \int_{\Omega} f(t, z_s) \, dz \right)^{\frac{1}{2}} \left( \int_{\Omega} \left| \frac{v_s}{\theta_s} \right|^2 f(t, z_s) \, dz \right)^{\frac{1}{2}} \leq \|f_0\|_{L^1}^{\frac{1}{2}} \frac{1}{\theta_m} (M_2 f(t))^{\frac{1}{2}} \]

\[ \leq \frac{C}{\theta_m} \left( R_2(0) + \sup_{0 \leq t \leq T} \|u(t)\|_{L^\infty} \right), \quad 0 \leq t \leq T. \]

This together with (3.7) gives the following control of the size of velocity support:

\[ R_2(t) \leq R_2(0) + Ct \left( R_2(0) + \sup_{0 \leq t \leq T} \|u(t)\|_{H^2} \right), \quad 0 \leq t \leq T, \]
where \( C \) is a constant depends on \( T \). The boundness of \( \text{supp}_x f \) directly follows from the characteristic equation (2.1)_1 that

\[
x(t) = x_0 + \int_0^t \mathbf{v}(s) \, ds \leq R_1(0) + \int_0^t R_2(s) \, ds
\]

\[
\leq R_1(0) + R_2(0) t + C t^2 \left( R_2(0) + \sup_{0 \leq t \leq T} \| u(t) \|_{H^2} \right), \quad 0 \leq t \leq T.
\]

(3) Since the support of \( f \) is bounded with respect to \( x, \mathbf{v}, \theta \), we only need to estimate the \( L^\infty \) norm to estimate the other \( L^p \) norms. Below, we split the proof for \( W^{1, \infty} \)-estimate of \( f \) into zeroth order and first order derivatives of \( f \).

\[\bullet \text{Step A (Zeroth order estimate)}: \text{We use the method of characteristic by considering the characteristic equations (2.1) to obtain}\]

\[
f(t, x(t; z_0), \mathbf{v}(t; z_0), \theta(t; z_0))
\]

\[
= f_0(z_0) \exp \left( \int_0^t \left[ 3 + \frac{3}{\theta(\tau; z_0)} a(\tau, x(\tau; z_0)) + \frac{1}{\theta^2(\tau; z_0)} c(\tau, x(\tau; z_0)) \right] \, d\tau \right).
\]

\[\text{(3.8)}\]

However, since we have the uniform upper and lower bounds for temperature and the conservation of mass, we deduce

\[
\frac{3}{\theta(\tau; z_0)} a(\tau, x(\tau; z_0)) \leq \frac{3\phi_M}{\theta_m}, \quad \frac{1}{\theta^2(\tau; z_0)} c(\tau, x(\tau; z_0)) \leq \frac{\zeta_M}{\theta_m^2}.
\]

We substitute these estimates into (3.8) to obtain the following \( L^\infty \) estimate on \( f \):

\[
\| f(t) \|_{L^\infty} \leq C(T) \| f_0 \|_{L^\infty}.
\]

\[\bullet \text{Step B (First order estimate): We introduce the following transport operator}\]

\[
\mathcal{T} := \partial_t + \mathbf{v} \cdot \nabla + F[f, \mathbf{u}] \cdot \nabla \mathbf{v} + G[f] \partial_\theta.
\]

Then, we use the similar estimates in [13, Lemma 5.4] to have the following estimates on derivatives of \( f \):

\[
\mathcal{T} \left( \nabla^\alpha \nabla_\mathbf{v}^\beta \partial_\theta^\gamma f(t) \right) \leq C (1 + \| \nabla \mathbf{u} \|_{L^\infty} ) \left( \nabla^\alpha \nabla_\mathbf{v}^\beta \partial_\theta^\gamma f(t) \right), \quad 0 \leq \alpha + \beta + \gamma \leq 1.
\]

Thus, we use Grönwall’s inequality and combine with the estimate of \( \| f \|_{L^\infty} \) to obtain

\[
\| f(t) \|_{W^{1, \infty}} \leq \| f_0 \|_{W^{1, \infty}} \exp \left( C \int_0^t (1 + \| \nabla \mathbf{u}(s) \|_{L^\infty}) \, ds \right).
\]

However, since \( \mathbf{u} \in L^2(0, T; H^3(\mathbb{R}^3)) \), we have

\[
\| \nabla \mathbf{u} \|_{L^1(0, T; L^\infty)} = \int_0^T \| \nabla \mathbf{u}(t) \|_{L^\infty} \, dt \leq C \int_0^T \| \mathbf{u}(t) \|_{H^3} \, dt
\]

\[
\leq C \sqrt{T} \| \mathbf{u} \|_{L^2(0, T; H^3(\mathbb{R}^3))} < C.
\]

This implies the boundedness of \( \| f(t) \|_{W^{1, \infty}} \) for \( 0 \leq t \leq T \).

(4) Similarly, since the supports of \( f, \tilde{f} \) are bounded with respect to \( x, \mathbf{v}, \theta \), we only need to estimate the \( L^\infty \) norm to estimate the \( L^p \) norm. We define the difference
between two solutions as \( h := f - \tilde{f} \). Then, it satisfies
\[
\partial_t h + v \cdot \nabla h + \nabla v \cdot (F[f, u]h + \partial_0 (G[f]h)) = -\nabla v \cdot \left( (F[f, u] - F[\tilde{f}, \tilde{u}]) \tilde{f} \right) - \partial_0 \left( (G[f] - G[\tilde{f}]) \tilde{f} \right).
\]

Thus, as in the estimates of \( f \) in (3.8), along the characteristic curves, \( h \) satisfies
\[
\frac{d|h(t, z(t))|}{dt} \leq C|\tilde{h}(t, z(t))| + C(\|f - \tilde{f}\|_{L^\infty} + \|u - \tilde{u}\|_{L^\infty}) \|\tilde{f}(t)\|_{W^{1, \infty}}.
\]

Since we already estimate \( \|\tilde{f}\|_{W^{1, \infty}} \), the above estimate implies
\[
\frac{d\|h(t)\|_{L^\infty}}{dt} \leq C(T) \left( \|h(0)\|_{L^\infty} + \sup_{0 \leq s \leq T} \|u(s) - \tilde{u}(s)\|_{L^\infty} \right).
\]

We integrate the above estimate to obtain
\[
\|h(t)\|_{L^\infty} \leq e^{C(T)} \left( \|h(0)\|_{L^\infty} + \sup_{0 \leq s \leq T} \|u(s) - \tilde{u}(s)\|_{L^\infty} \right), \quad 0 \leq t \leq T.
\]

\[\square\]

3.1.2. Proof of proposition 3.1. We now provide the proof of Proposition 3.1. For the reader's convenience, we split the proof into two steps.

- Step A (Construction of \( f \)): We consider the mollifiers \( j(x, v, \theta) \) and \( k(x) \) satisfying
\[
\begin{align*}
\mathcal{J} &\geq 0, \quad j \in C^\infty_c(\Omega), \quad \text{supp } j \subset B_1(0) \times B_1(0) \times [-1, 1], \quad \int_{\Omega} j \, dz = 1, \\
\mathcal{K} &\geq 0, \quad k \in C^\infty_c(\mathbb{R}^3), \quad \text{supp } k \subset B_1(0), \quad \int_{\mathbb{R}^3} k \, dx = 1,
\end{align*}
\]

and thier scalings
\[
\begin{align*}
\mathcal{J}_\varepsilon(x, v, \theta) &:= \frac{1}{\varepsilon^3} j \left( \frac{x}{\varepsilon}, \frac{v}{\varepsilon}, \frac{\theta}{\varepsilon} \right), \\
\mathcal{K}_\varepsilon(x) &:= \frac{1}{\varepsilon^3} k \left( \frac{x}{\varepsilon} \right).
\end{align*}
\]

Then, we mollify the initial data \( f_0 \) and \( u \) by convolution, i.e.,
\[
\begin{align*}
f^{\varepsilon}_0(z) &= (f_0 * \mathcal{J}_\varepsilon)(z) \quad \text{and} \quad u^{\varepsilon}(t, x) = (u * \mathcal{K}_\varepsilon)(t, x).
\end{align*}
\]

Then, by the standard procedure, we have a unique smooth solution to the following Cauchy problem:
\[
\begin{align*}
\partial_t f^{\varepsilon} + v \cdot \nabla f^{\varepsilon} + \nabla v \cdot (F[f^{\varepsilon}] f^{\varepsilon} + (u^{\varepsilon} - v) f^{\varepsilon}) + \partial_0 (G[f^{\varepsilon}] f^{\varepsilon}) &= 0, \\
f^{\varepsilon}_{|t=0} &= f^{\varepsilon}_0.
\end{align*}
\]

Moreover, we use a priori estimates Lemma 3.1 (1)-(3) to extend the local smooth solution to the whole time interval \([0, T]\). Moreover, it follows from Lemma 3.1 (4) that
\[
\sup_{0 \leq t \leq T} \|f^{\varepsilon}(t) - f^{\varepsilon}_0(t)\|_{L^\infty} \leq C \left( \|f^{\varepsilon}_0 - f^{\varepsilon}_0\|_{L^\infty} + \sup_{0 \leq t \leq T} \|u^{\varepsilon}(t) - u^{\varepsilon}(t)\|_{L^\infty} \right).
\]

Thus, there exists a unique \( f \in C([0, T]; L^\infty(\Omega)) \) such that
\[
f^{\varepsilon} \to f \quad \text{in} \quad C([0, T]; L^\infty(\Omega)), \quad \text{as} \quad \varepsilon \to 0.
\]
Therefore, the limiting function \( f \) satisfies Proposition 3.1 (1)-(2) and it satisfies (1.1) in the sense of distribution, which is
\[
\partial_t f + \mathbf{v} \cdot \nabla f + \nabla \mathbf{v} \cdot (F[f]f + (\mathbf{u} - \mathbf{v})f) + \partial_\theta (G[f]f) = 0 \quad \text{in } \mathcal{D}'((0,T) \times \Omega).
\]

\begin{itemize}
\item Step B (Regularity of \( f \)): Next, we prove \( f \in \mathcal{C}([0,T];W^{1,\infty}(\Omega)) \). We use Lemma 3.1 (3) to deduce that \( f^{i+1} \) constructed in the previous step are uniformly bounded in \( L^\infty(0,T;W^{1,p}(\Omega)) \), and together with (3.11), we conclude that \( f \in L^\infty(0,T;W^{1,\infty}(\Omega)) \).

To obtain the time continuity, we define the functional \( F \) as
\[
F := \sum_{0 \leq \alpha + \beta + \gamma \leq 1} \| \nabla^\alpha \nabla^\beta \mathbf{v} \partial_\theta^\gamma f \|_{L^\infty}.
\]

Then, as in the proof of Lemma 3.1, we have
\[
\frac{dF}{dt} \leq C(1 + \| \nabla \mathbf{u} \|_{L^\infty})F, \quad (3.12)
\]
which implies that the \( W^{1,\infty} \)-norm of \( f \) can be controlled by that of initial data:
\[
F(t) \leq e^{C(1 + \| \nabla \mathbf{u} \|_{L^\infty})t}F(0) < C.
\]

It follows from the boundedness of \( F \) and (3.12) that
\[
\frac{dF}{dt} \leq C(1 + \| \nabla \mathbf{u} \|_{L^\infty}),
\]
and thus, for any \( 0 \leq t_1 < t_2 \leq T \),
\[
F(t_2) - F(t_1) \leq C(t_2 - t_1) + C \int_{t_1}^{t_2} \| \nabla \mathbf{u}(t) \|_{L^\infty} \, dt \leq C(t_2 - t_1) + C\sqrt{t_2 - t_1} \| \mathbf{u} \|_{L^2(0,T;H^3(\mathbb{R}^3))}.
\]

This estimate implies \( f \in C([0,T];W^{1,\infty}) \), and thus it is a strong solution to (3.2).

Finally, the stability estimate in Proposition 3.1 (4) can be obtained by using the exactly same argument as in the proof of Lemma 3.1 (4) and the uniqueness of a strong solution directly comes from this stability estimate. This completes the proof of Proposition 3.1. \( \square \)

### 3.2. Local existence of strong solution

In this subsection, we present the local existence of strong solution to (1.1) and complete the proof of Theorem 2.1. Recall that we construct the sequence of approximated solutions \((f^n, \mathbf{u}^n)\) in (3.1) defined as
\[
\partial_t f^{n+1} + \mathbf{v} \cdot \nabla f^{n+1} + \nabla \mathbf{v} \cdot (F[f^{n+1}, \mathbf{u}^n]f^{n+1}) + \partial_\theta (G[f^{n+1}]f^{n+1}) = 0, \quad (x, \mathbf{v}, \theta, t) \in \Omega \times \mathbb{R}^+, \]
\[
\partial_t \mathbf{u}^{n+1} + (\mathbf{u} \cdot \nabla)\mathbf{u}^{n+1} + \nabla p^{n+1} - \Delta \mathbf{u}^{n+1} = \int_{\mathbb{R}^3 \times \mathbb{R}^+} f^{n+1}(\mathbf{v} - \mathbf{u}^n) \, dv \, d\theta, \]
\[
\nabla \cdot \mathbf{u}^{n+1} = 0,
\]
where the initial step \( \mathbf{u}^0 \) was given as
\[
\partial_t \mathbf{u}^0 - \Delta \mathbf{u}^0 = 0, \quad \mathbf{u}^0(t, x)|_{t=0} = \mathbf{u}_0(x) \in H^2(\mathbb{R}^3). \quad (3.13)
\]
We multiply (3.13) by $u$ and integrate over $\mathbb{R}^3$ to obtain the following estimate:

$$\frac{1}{2} \frac{d}{dt} \|u^0\|_{L^2}^2 + \|\nabla u^0\|_{L^2}^2 = 0, \text{ or } \frac{1}{2} \|u^0(t)\|_{L^2}^2 + \int_0^t \|\nabla u^0(s)\|_{L^2}^2 \, ds = \frac{1}{2} \|u_0\|_{L^2}^2.$$ 

Since $\|u^0(t)\|_{L^2} \leq \|u_0\|_{L^2}$, we further have

$$\|u^0(t)\|_{L^2}^2 + \int_0^t \|u^0(s)\|_{H^1}^2 \, ds \leq C(T)\|u_0\|_{L^2}^2.$$ 

Thus, whenever, $u_0 \in H^2(\mathbb{R}^3)$, we have the following estimate for $u^0$:

$$\|u^0\|_{L^\infty(0,T;H^2(\mathbb{R}^3))}^2 + \int_0^T \|u^0(t)\|_{H^3}^2 \, dt \leq C(T)\|u_0\|_{H^2}^2. (3.14)$$

We now provide the following uniform boundedness of norms of $f^n$ and $u^n$ up to small time interval.

**Proposition 3.2.** Let $(f^n, u^n)$ be a sequence of approximated solutions constructed as in (3.1). Then, there exists $0 < T^* < \infty$ such that

$$\|f^n\|_{L^\infty(0,T^*;W^{1,\infty})} < C_1, \quad \|u^n\|_{L^\infty(0,T^*;H^2)} + \int_0^{T^*} \|u^n(s)\|_{H^3}^2 \, ds < C_2, \quad n = 1, 2, \ldots,$$

where $C_1$ and $C_2$ are constants independent of $n$.

**Proof.** Let $C_0 = (1 + \sqrt{C})(\|f_0\|_{W^{1,\infty}} + \|u_0\|_{H^2})$, where $C$ is the same constant $C$ in (3.14). Then, for any $T$, by the definition of $C_0$ and the estimate (3.14), we have

$$\|f^n\|_{L^\infty(0,T;W^{1,\infty})} < C_0, \quad \|u^n(t)\|_{H^2}^2 + \int_0^t \|u^n(s)\|_{H^3}^2 \, ds < C^n_2, \quad 0 \leq t \leq T.$$ 

Now, we choose $C_1 := 2C_0$ and $C_2 := 2C^n_2$ and assume that

$$\|f^n\|_{L^\infty(0,T;W^{1,\infty})} < C_1, \quad \sup_{0 \leq t \leq T} \|u^n(t)\|_{H^2} + \int_0^T \|u^n(s)\|_{H^3}^2 \, ds < C_2.$$ 

Note that the above estimates when $n = 0$ automatically hold by the choice of $C_1$ and $C_2$.

\*\*\* Step A (Estimate of $f^{n+1}$): We use the *a priori* estimate in Lemma 3.1 to obtain

$$\sup_{0 \leq t \leq T} \|f^{n+1}(t)\|_{W^{1,\infty}} \leq \|f_0\|_{W^{1,\infty}} \exp \left( C \int_0^T (1 + \|u^n(s)\|_{L^\infty}) \, ds \right) 
\leq C_0 \exp \left( CT + C\sqrt{T} \|u^n\|_{L^2(0,T;H^3)} \right) 
< C_0 \exp \left( C(T + \sqrt{T}C_2) \right).$$

Thus, from the choice of $C_1 = 2C_0$, there exists sufficiently small $T^*$ such that

$$\sup_{0 \leq t \leq T^*} \|f^{n+1}(t)\|_{W^{1,\infty}} \leq C_0 \exp \left( C(T^* + \sqrt{T}C_2) \right) < 2C_0 = C_1.$$ 

Moreover, Lemma 3.1 implies that, for $0 \leq t \leq T^*$, the following estimates hold:

$$\sup_{0 \leq t \leq T^*} \|f^{n+1}(t, x, v, \cdot)\|_{[\theta_m, \theta_M]} \leq R^{n+1}_2(t) \leq C(T^*), \quad R^{n+1}_1(t) \leq C(T^*),$$

where $C(T^*)$ is a constant that does not depend on $n$. 
Step B (Estimate of $u^{n+1}$): We now provide the energy estimate of $u^{n+1}$. We start from the lowest order energy estimate.

- (Zeroth-order energy estimate): We multiply (3.1) by $u^{n+1}$ and integrate over $\mathbb{R}^3$ to obtain the following estimate:

$$
\frac{1}{2} \frac{d}{dt} \|u^{n+1}\|^2_{L^2} + \|\nabla u^{n+1}\|^2_{L^2} \\
= - \int_{\mathbb{R}^3} (u^n \cdot \nabla)u^{n+1} \cdot u^{n+1} \, dx - \int_{\Omega} u^{n+1} \cdot (u^n - v) f^{n+1} \, dz \\
= \frac{1}{2} \int_{\mathbb{R}^3} |u^{n+1}|^2 (\nabla \cdot u^n) \, dx - \int_{\Omega} u^{n+1} \cdot u^n f^{n+1} \, dz + \int_{\Omega} u^{n+1} \cdot v f^{n+1} \, dz \\
\leq C \|f^{n+1}\|_{L^\infty} |R_{n+1}^{2}(t)|^3 (\theta_{M} - \theta_{m}) \|u^{n+1}\|_{L^2} \|u^n\|_{L^2} \\
+ C \|f^{n+1}\|_{L^\infty} |R_{n+1}^{2}(t)|^4 |R_{n+1}^{2}(t)|^{3/2} (\theta_{M} - \theta_{m}) \|u^{n+1}\|_{L^2} \\
\leq C_1 C(T^*) \|u^{n+1}\|_{L^2} (1 + \|u^n\|_{L^2}).
$$

(3.15)

- (First-order energy estimate): We take any first order derivative $\partial = \partial_x$, to (3.1)$_2$, multiply $\partial u^{n+1}$ and then integrate over $\mathbb{R}^3$ to derive

$$
\frac{1}{2} \frac{d}{dt} \|\partial u^{n+1}\|^2_{L^2} + \|\nabla \partial u^{n+1}\|^2_{L^2} \\
= - \int_{\mathbb{R}^3} \partial[(u^n \cdot \nabla)u^{n+1}] \cdot \partial u^{n+1} \, dx - \int_{\Omega} \partial u^{n+1} \cdot \partial[(u^n - v) f^{n+1}] \, dz \\
=: I_{21} + I_{22}.
$$

- (Estimate of $I_{21}$): The integration by parts yields

$$
I_{21} = - \int_{\mathbb{R}^3} \partial u^{n+1} \cdot [(u^n \cdot \nabla)\partial u^{n+1}] \, dx - \int_{\mathbb{R}^3} \partial u^{n+1} \cdot [(\partial u^n \cdot \nabla)u^{n+1}] \, dx \\
= - \int_{\mathbb{R}^3} u^n \cdot \frac{1}{2} \nabla |\partial u^{n+1}|^2 \, dx - \int_{\mathbb{R}^3} \partial u^{n+1} \cdot [(\partial u^n \cdot \nabla)u^{n+1}] \, dx \\
\leq \|\partial u^{n+1}\|_{L^2} \|\nabla u^{n+1}\|_{L^6} \|\partial u^n\|_{L^6} \leq C \|\partial^2 u^n\|_{L^6} \|\partial u^{n+1}\|^{3/2}_{L^2} \|\nabla u^{n+1}\|^{3/2}_{L^2} \\
\leq \frac{1}{4} \|\nabla \partial u^{n+1}\|^2_{L^2} + \frac{3}{4} C^2 \|\partial^2 u^n\|^{3/2}_{L^2} \|\partial u^{n+1}\|^{3/2}_{L^2},
$$

where we used Gagliardo-Nirenberg interpolation inequalities

$$
\|\partial u^n\|_{L^6} \leq C \|\partial^2 u^n\|_{L^2}, \quad \|\nabla u^{n+1}\|_{L^6} \leq \|\partial u^{n+1}\|^{3/2}_{L^2} \|\nabla \partial u^{n+1}\|^{3/2}_{L^2}
$$

and Young’s inequality.
(Estimate of $I_{22}$): We directly estimate $I_{22}$ as

$$I_{22} \leq - \int_\Omega \partial u^{n+1} \cdot (\partial u^n f^{n+1} + (u^n - \nu) \partial f^{n+1}) \, dz$$

$$\leq C \|f^{n+1}\|_{W^{1,\infty}} |R_2^{n+1}(t)|^3 (\theta_M - \theta_m) \|\partial u^{n+1}\|_{L^2} (\|\partial u^n\|_{L^2} + \|u^n\|_{L^2})$$

$$+ C \|f^{n+1}\|_{W^{1,\infty}} |R_2^{n+1}(t)|^4 R_1^{n+1}(t)^{\frac{2}{3}} (\theta_M - \theta_m) \|\partial u^{n+1}\|_{L^2}$$

$$\leq C_1 C(T^*) \|\partial u^{n+1}\|_{L^2} (\|\partial u^n\|_{L^2} + \|u^n\|_{L^2}) + C_1 C(T^*) \|\partial u^{n+1}\|_{L^2}$$

$$\leq C_1 C(T^*) \|\partial u^{n+1}\|_{L^2} (1 + \|u^n\|_{H^1})$$

$$\leq \frac{1}{4} \|\partial u^{n+1}\|_{L^2}^2 + (C_1 C(T^*))^2 (1 + \|u^n\|_{H^1})^2.$$  

(3.16)

(Second-order energy estimate): We differentiate (3.1) twice, multiply $\partial^2 u^{n+1}$ and integrate over $\mathbb{R}^3$ to get

$$\frac{1}{2} \frac{d}{dt} \|\partial^2 u^{n+1}\|_{L^2}^2 + \|\nabla \partial^2 u^{n+1}\|_{L^2}^2$$

$$= - \int_{\mathbb{R}^3} \partial^2 [(u^n \cdot \nabla) u^{n+1}] \cdot \partial^2 u^{n+1} \, dx - \int_{\Omega} \partial^2 u^{n+1} \cdot \partial^2 [(u^n - \nu) f^{n+1}] \, dz$$

$$= I_{31} + I_{32}.$$ 

(Estimate of $I_{31}$): We again use Gagliardo-Nirenberg interpolation theorem Lemma 2.4 and Young’s inequality to obtain

$$I_{31} = - \int_{\mathbb{R}^3} \partial^2 u^{n+1} \cdot [(u^n \cdot \nabla) \partial^2 u^{n+1}] \, dx - \int_{\mathbb{R}^3} 2 \partial^2 u^{n+1} \cdot [(\partial u^n \cdot \nabla) \partial u^{n+1}] \, dx$$

$$- \int_{\mathbb{R}^3} \partial^2 u^{n+1} \cdot [(\partial^2 u^n \cdot \nabla) u^{n+1}] \, dx$$

$$= - \int_{\mathbb{R}^3} u^n \cdot \frac{1}{2} \nabla |\partial^2 u^{n+1}|^2 \, dx - \int_{\mathbb{R}^3} 2 \partial^2 u^{n+1} \cdot [(\partial u^n \cdot \nabla) \partial u^{n+1}] \, dx$$

$$- \int_{\mathbb{R}^3} \partial^2 u^{n+1} \cdot [(\partial^2 u^n \cdot \nabla) u^{n+1}] \, dx$$

$$\leq 2 \|\partial u^n\|_{L^6} \|\partial^2 u^{n+1}\|_{L^2} \|\nabla \partial u^{n+1}\|_{L^3} + \|\partial^2 u^n\|_{L^2} \|\partial^2 u^{n+1}\|_{L^2} \|\nabla u^{n+1}\|_{L^6}$$

$$\leq C \|\partial^2 u^n\|_{L^2} \|\partial^2 u^{n+1}\|_{L^2}^2 \|\nabla \partial^2 u^{n+1}\|_{L^2} + C \|\partial^2 u^n\|_{L^2} \|\partial^2 u^{n+1}\|_{L^2}^2 \|\nabla \partial^2 u^{n+1}\|_{L^2}^2$$

$$\leq \frac{1}{4} \|\nabla \partial^2 u^{n+1}\|_{L^2}^2 + \frac{3}{4} C \|\partial^2 u^n\|_{L^2}^2 \|\partial^2 u^{n+1}\|_{L^2}^2.$$
\(\diamond\) (Estimate of \(I_{32}\)): We directly estimate \(I_{32}\) as
\[
I_{32} \leq -\int_{\Omega} \partial^2 u^{n+1} \cdot [\partial u^n f^{n+1} + (u^n - v)\partial f^{n+1}] \, dz \\
\leq \|f^{n+1}\|_{W^{1,\infty}} |R_2^{n+1}(t)|^3 (\theta_M - \theta_m) \|\nabla \partial^2 u^{n+1}\|_{L^2} (\|\partial u^n\|_{L^2} + \|u^n\|_{L^2}) \\
+ \|f^{n+1}\|_{W^{1,\infty}} |R_2^{n+1}(t)|^4 |R_1^{n+1}(t)|^2 (\theta_M - \theta_m) \|\nabla \partial^2 u^{n+1}\|_{L^2} \\
\leq C_1 C(T^*) \|\nabla \partial^2 u^{n+1}\|_{L^2} (1 + \|\partial u^n\|_{L^2} + \|u^n\|_{L^2}) \\
\leq C_1 C(T^*) \|\nabla \partial^2 u^{n+1}\|_{L^2} (1 + \|u^n\|_{H^1}) \\
\leq \frac{1}{4} \|\nabla \partial^2 u^{n+1}\|_{L^2}^2 + (C_1 C(T^*))^2 (1 + \|u^n\|_{H^1}^2).
\]

We also combine the estimates of \(I_{31}\) and \(I_{32}\) to derive the following second order energy estimate:
\[
\frac{1}{2} \frac{d}{dt} \|u^{n+1}\|_{H^2}^2 + \frac{1}{2} \|\nabla u^{n+1}\|_{H^2}^2 \\
\leq \frac{1}{2} \|\nabla u^{n+1}\|_{L^2}^2 + C\|\partial^2 u^n\|_{L^2}^2 \|\partial u^{n+1}\|_{H^1}^2 + (C_1 C(T^*))^2 (1 + \|u^n\|_{H^1}^2). \\
(3.17)
\]

Then, it follows from all the estimates for each order of energy (3.15)–(3.17) that
\[
\frac{1}{2} \frac{d}{dt} \|u^{n+1}\|_{H^2}^2 + \frac{1}{2} \|\nabla u^{n+1}\|_{H^2}^2 \\
\leq C_1 C(1 + \|u^n\|_{H^1}) \|u^{n+1}\|_{H^1}^2 + C\|\partial^2 u^n\|_{L^2}^2 \|\partial u^{n+1}\|_{H^1}^2 + CC_1^2 (1 + \|u^n\|_{H^1}^2) \\
\leq \frac{1}{2} \|u^{n+1}\|_{H^1}^2 + C\|\partial^2 u^n\|_{L^2}^2 \|\partial u^{n+1}\|_{H^1}^2 + CC_1^2 (1 + \|u^n\|_{H^1}^2) \\
\leq C \left(1 + \|\partial^2 u^n\|_{L^2}^2\right) \|u^{n+1}\|_{H^2}^2 + CC_1^2 (1 + \|u^n\|_{H^1}^2).
\]

We use Grönwall’s inequality to obtain
\[
\|u^{n+1}\|_{H^2}^2 + \int_0^t \|\nabla u^{n+1}(s)\|_{H^2}^2 \, ds \\
\leq \left(\|u_0\|_{H^2}^2 + \int_0^t CC_1^2 (1 + \|u^n(s)\|_{H^1}^2) \, ds \right) \exp \left[\int_0^t \left(1 + \|\partial^2 u^n(s)\|_{L^2}^2\right) \, ds\right] \\
\leq (C_0^2 + CC_1^2) (1 + C_2 t) \exp \left[t \left(1 + C_2^\frac{2}{3}\right)\right].
\]

Therefore, we have
\[
\|u^{n+1}\|_{H^2}^2 + \int_0^t \|u^{n+1}(s)\|_{H^3}^2 \, ds \leq (1 + t)(C_0^2 + CC_1^2) (1 + C_2 t) \exp \left(t \left(1 + C_2^\frac{2}{3}\right)\right).
\]

Since we choose \(C_2 = 2C_0^2\), we are able to find sufficiently small \(T^*\) such that
\[
(1 + t)(C_0^2 + CC_1^2) (1 + C_2 t) \exp \left(t \left(1 + C_2^\frac{2}{3}\right)\right) \leq C_2, \quad 0 \leq t \leq T^*.
\]

Thus we prove
\[
\sup_{0 \leq t \leq T^*} \|u^{n+1}(t)\|_{H^2}^2 + \int_0^{T^*} \|u^{n+1}(s)\|_{H^3} \, ds < C_2.
\]

\(\square\)
In the following proposition, we will show that the sequence of approximated solutions \((f^n, u^n)\) is a Cauchy sequence in lower regularity function space to show the convergence of it.

**Proposition 3.3.** Let \((f^n, u^n)\) be a sequence of approximated solutions constructed as in (3.1). Then \((f^n, u^n)\) is a Cauchy sequence in \(L^\infty(0; L^\infty(\Omega)) \times L^\infty(0; H^1(\Omega)) \cap L^2(0; T; H^2(\Omega))\).

**Proof.** We define the following functional increments \(\Delta_{n+1}(t)\) as

\[
\Delta_{n+1}(t) := \|(f^{n+1} - f^n)(t)\|_{L^\infty}^2 + \|(u^{n+1} - u^n)(t)\|_{H^1}^2 + \int_0^t \|\nabla (u^{n+1} - u^n)(s)\|_{H^1}^2 \, ds.
\]

Below, we estimate the kinetic part \((f^{n+1} - f^n)\) and fluid part \((u^{n+1} - u^n)\) separately.

- **Step A (Estimate of \((f^{n+1} - f^n)\):** We take a difference between two equations for \((f^{n+1} + f^n)\) to obtain

\[
\partial_t (f^{n+1} - f^n) + \mathbf{v} \cdot \nabla (f^{n+1} - f^n) + F[f^n, u^n] \cdot \nabla (f^{n+1} - f^n)
+ G[f^n] \partial_\theta (f^{n+1} - f^n)
= -(F[f^n, u^n] - F[f^{n-1}, u^{n-1}]) \cdot \nabla (f^{n-1} - f^n) \cdot \nabla (f^{n+1} - f^n) - G[f^n] \partial_\theta (f^{n+1} - f^n)
- (f^{n+1} - f^n) \partial_\theta (G[f^{n-1}]) - f^{n+1} \partial_\theta (G[f^n] - G[f^{n-1}]) =: \sum_{i=1}^6 I_{4i}.
\]

- \((\text{Estimate of } I_{41}):\) We use the boundedness of support of \(f^n\) and \(f^{n-1}\) to have

\[
|F[f^n, u^n] - F[f^{n-1}, u^{n-1}]| \leq |F[f^n, u^n] - F[f^{n-1}, u^n]| + |F[f^n, u^n] - F[f^n, u^{n-1}]|
= \left| \int_\Omega \phi(x - x_*) \frac{\mathbf{v} \cdot \nabla}{\partial_x} (f^n - f^{n-1}) \, dz_\ast \right| + |u^n - u^{n-1}|
\leq C \|\phi\|_{L^\infty} \frac{(R_1(t))^3 (R_2(t))^4}{\theta_m} (\theta_M - \theta_m) \|f^n - f^{n-1}\|_{L^\infty} + \|u^n - u^{n-1}\|_{L^\infty}.
\]

This implies the following estimate for \(I_{41} \leq C(\|f^n - f^{n-1}\|_{L^\infty} + \|u^n - u^{n-1}\|_{L^\infty}) \|f^n\|_{W^{1,\infty}}\)

\[
\leq C(\|f^n - f^{n-1}\|_{L^\infty} + \|u^n - u^{n-1}\|_{H^1}) \|f^n\|_{W^{1,\infty}}.
\]

- \((\text{Estimate of } I_{42}):\) Since \(|\nabla \cdot \mathbf{F} \mid \leq \frac{3(||\phi||_{L^\infty} + 1)}{\theta_m} \), we have

\[
I_{42} \leq \frac{3(||\phi||_{L^\infty} + 1)}{\theta_m} \|f^n - f^{n-1}\|_{L^\infty}.
\]

- \((\text{Estimate of } I_{43}):\) Note that

\[
|\nabla \cdot (F[f^n, u^n] - F[f^{n-1}, u^{n-1}].SP)| = |\nabla \cdot (F[f^n] - F[f^{n-1}])|
\leq 3 \left| \int_\Omega \phi(x - x_*) \frac{1}{\theta_m} (f^n - f^{n-1}) \, dz_\ast \right|
\leq 3 ||\phi||_{L^\infty} \frac{(R_1(t))^3 (R_2(t))^4}{\theta_m} (\theta_M - \theta_m) \|f^n - f^{n-1}\|_{L^\infty}.
\]

These estimates complete the proof of the proposition.
Thus, $I_{43}$ can be estimated as
\[ I_{43} \leq C \| f^{n+1} \|_{L^\infty} \| f^n - f^{n-1} \|_{L^\infty}. \]

\(\diamond\) (Estimate of $I_{44}$): We estimate $I_{44}$ similar to the estimate of $I_{41}$:
\[ |G[f^n] - G[f^{n-1}]| = \left| \int_{\Omega} \zeta(x - x_s) \left( \frac{1}{\theta} - \frac{1}{\theta_s} \right) (f^n - f^{n-1}) \, dz_s \right| \leq C \| \zeta \|_{L^\infty} \frac{(R_1(t))^3(R_2(t))^3}{\theta_m} (\theta_M - \theta_m) \| f^n - f^{n-1} \|_{L^\infty}. \]

This yields
\[ I_{44} \leq C \| f^n - f^{n-1} \|_{L^\infty} \| f^n \|_{W^{1,\infty}}. \]

\(\diamond\) (Estimate of $I_{45}$): Since the derivative of $G$ can be estimated as
\[ |\partial_\theta G[f^{n-1}]| = \left| \int_{\Omega} \zeta(x - x_s) \frac{1}{\theta^2} f_s^{n-1} \, dz_s \right| \leq \frac{\| \zeta \|_{L^\infty}}{\theta_m^2}, \]
we have
\[ I_{45} \leq C \| f^n - f^{n-1} \|_{L^\infty}. \]

\(\diamond\) (Estimate of $I_{46}$): Finally, we estimate $I_{46}$ as
\[ I_{46} \leq |\partial_\theta(G[f^n] - G[f^{n-1}])| \| f^{n+1} \|_{L^\infty} \]
\[ \leq C (R_1(t))^3(R_2(t))^3(\theta_M - \theta_m) \| f^n - f^{n-1} \|_{L^\infty} \| f^{n+1} \|_{L^\infty} \]
\[ \leq C \| f^{n+1} \|_{L^\infty} \| f^n - f^{n-1} \|_{L^\infty}. \]

In conclusion, we collect all the estimates of $I_{4i}$ for $i = 1, 2, \ldots, 6$ together to get
\[ \partial_t |f^{n+1} - f^n| + \nabla |f^{n+1} - f^n| + F[f^n, u^n] \cdot \nabla_v |f^{n+1} - f^n| \]
\[ + G[f^n] \partial_\theta |f^{n+1} - f^n| \]
\[ \leq C (\| f^n - f^{n-1} \|_{L^\infty} + \| u^n - u^{n-1} \|_{H^2}) (1 + \| f^n \|_{W^{1,\infty}} + \| f^{n+1} \|_{L^\infty}) \]
\[ \leq C (\| f^n - f^{n-1} \|_{L^\infty} + \| u^n - u^{n-1} \|_{H^2}). \]

We multiply $|f^{n+1} - f^n|$ to above inequality and use Cauchy-Schwartz inequality to get
\[ \partial_t |f^{n+1} - f^n|^2 + \nabla |f^{n+1} - f^n|^2 + F[f^n, u^n] \cdot \nabla_v |f^{n+1} - f^n|^2 \]
\[ + G[f^n] \partial_\theta |f^{n+1} - f^n|^2 \]
\[ \leq C (\| f^{n+1} - f^n \|_{L^\infty}^2 + \| f^n - f^{n-1} \|_{L^\infty}^2 + \| u^n - u^{n-1} \|_{H^2}^2). \]

We now Integrate (3.18) along the characteristic curve to get
\[ \| f^{n+1}(t) - f^n(t) \|_{L^\infty}^2 \leq C \int_0^t \| (f^{n+1} - f^n)(s) \|_{L^\infty}^2 + \| (f^n - f^{n-1})(s) \|_{L^\infty}^2 \, ds \]
\[ + C \int_0^t \| (u^n - u^{n-1})(s) \|_{H^2}^2 \, ds \]
and the Grönwall’s inequality implies that
\[
\| f^{n+1}(t) - f^n(t) \|_{L^\infty}^2 \\
\leq C \int_0^t \| (f^n - f^{n-1})(s) \|_{L^\infty}^2 \, ds + C \int_0^t \int_0^t \| (u^n - u^{n-1})(\tau) \|_{H^2}^2 \, d\tau \, ds \\
\leq C \int_0^t \| (f^n - f^{n-1})(s) \|_{L^\infty}^2 \, ds + C(T) \int_0^t \| (u^n - u^{n-1})(s) \|_{H^2}^2 \, ds.
\] (3.19)

- Step B (Estimate of \( u^{n+1} - u^n \)): The estimate of \( \| u^{n+1} - u^n \|_{H^1}^2 \) is similar to the estimate of \( \| u^n \|_{H^1}^2 \). We consider the following equations for \( u^{n+1} - u^n \):
\[
\partial_t (u^{n+1} - u^n) - \Delta (u^{n+1} - u^n) + \nabla(p^{n+1} - p^n) \\
= -(u^n \cdot \nabla)(u^{n+1} - u^n) - [(u^n - u^{n-1}) \cdot \nabla]u^n - \int_{\mathbb{R}^3 \times \mathbb{R}^+} (u^n - u^{n-1}) f^{n+1} \, dv \, d\theta \\
- \int_{\mathbb{R}^3 \times \mathbb{R}^+} u^{n+1} (f^{n+1} - f^n) \, dv \, d\theta + \int_{\mathbb{R}^3 \times \mathbb{R}^+} v (f^{n+1} - f^n) \, dv \, d\theta, \\
\nabla \cdot (u^{n+1} - u^n) = 0.
\] (3.20)

- (Zeroth order estimate): We take an inner product (3.20) with \( u^{n+1} - u^n \) and integrate over \( \mathbb{R}^3 \) to obtain
\[
\frac{1}{2} \frac{d}{dt} \| u^{n+1} - u^n \|_{L^2}^2 + \| \nabla (u^{n+1} - u^n) \|_{L^2}^2 \\
\leq C(\| \nabla u^n \|_{L^2} \| u^n - u^{n-1} \|_{H^1} + \| u^{n+1} - u^n \|_{L^6} \\
+ \| f^{n+1} \|_{L^\infty} \| u^n - u^{n-1} \|_{L^2} + \| u^{n+1} - u^n \|_{H^1} + \| f^n \|_{L^\infty} \| u^{n+1} - u^n \|_{L^2}) \\
\leq C(\| u^n - u^{n-1} \|_{H^1}^2 + \| u^n - u^{n-1} \|_{L^2}^2) + C(\| u^{n+1} - u^n \|_{L^2}^2 + \| u^{n+1} - u^n \|_{H^1}^2) \\
+ C(\| f^{n+1} \|_{L^\infty}^2 + \| u^{n+1} - u^n \|_{L^2}^2) + C(\| f^n \|_{L^\infty}^2 + \| u^{n+1} - u^n \|_{L^2}^2) \\
\leq C(\| f^{n+1} \|_{L^\infty}^2 + \| u^{n+1} - u^n \|_{H^1}^2 + \| u^n - u^{n-1} \|_{H^1}^2).
\] (3.21)

- (First order estimate): We derivate (3.20), take an inner product with \( \nabla (u^{n+1} - u^n) \), and integrate over the space domain to get
\[
\frac{1}{2} \frac{d}{dt} \| \nabla (u^{n+1} - u^n) \|_{L^2}^2 + \| \nabla^2 (u^{n+1} - u^n) \|_{L^2}^2 \\
= -\int_{\mathbb{R}^3} \nabla (u^{n+1} - u^n) \cdot \nabla [(u^n \cdot \nabla) (u^{n+1} - u^n)] \, dx \\
- \int_{\mathbb{R}^3} \nabla (u^{n+1} - u^n) \cdot \nabla [(u^n - u^{n-1}) \cdot \nabla u^n] \, dx \\
- \int_{\Omega} \nabla (u^{n+1} - u^n) \cdot \nabla [(u^n - u^{n-1}) f^{n+1}] \, dz \\
- \int_{\Omega} \nabla (u^{n+1} - u^n) \cdot \nabla (u^{n-1} (f^{n+1} - f^n)) \, dz \\
+ \int_{\Omega} \nabla (u^{n+1} - u^n) \cdot \nabla (f^{n+1} - f^n) \, dz =: \sum_{i=1}^5 I_{5i}.
\]
We estimate each $I_{5i}$ using Sobolev inequality as follows:

$$I_{51} \leq \|\nabla(u^{n+1} - u^n)\|_{L^2}(\|\nabla u^n\|_{L^2} + \|\nabla u^{n+1} - u^n\|_{L^2}) + \|u^n\|_{L^6}\|\nabla^2(u^{n+1} - u^n)\|_{L^2}$$

$$\leq C\|u^n\|_{H^1}\|\nabla(u^{n+1} - u^n)\|^2_{H^1},$$

$$I_{52} \leq \|\nabla(u^{n+1} - u^n)\|_{L^2}(\|\nabla u^n\|_{L^2} + \|\nabla^2 u^n\|_{L^2} + \|\nabla^2 u^{n+1} - u^n\|_{L^2})$$

$$\leq C\|u^n\|_{H^2}\|u^n - u^{n-1}\|_{H^1}\|\nabla(u^{n+1} - u^n)\|_{H^1},$$

$$I_{53} \leq C\|\nabla(u^{n+1} - u^n)\|_{L^2}(\|f^{n+1}\|_{L^\infty}\|\nabla(u^n - u^{n-1})\|_{L^2})$$

$$+ \|\nabla f^{n+1}\|_{L^\infty}\|u^n - u^{n-1}\|_{L^2})$$

$$\leq C\|f^{n+1}\|_{W^{1,\infty}}\|u^n - u^{n-1}\|_{H^1}\|\nabla(u^{n+1} - u^n)\|_{L^2},$$

$$I_{54} \leq C\|f^{n+1} - f^n\|_{L^\infty}\|\nabla^2(u^{n+1} - u^n)\|_{L^2}\|u^n - u^{n-1}\|_{L^2},$$

$$I_{55} \leq C\|f^{n+1} - f^n\|_{L^\infty}\|\nabla^2(u^{n+1} - u^n)\|_{L^2}. $$

Here, we use the following interpolation inequality in the estimate of $I_{52}$:

$$\|\nabla(u^{n+1} - u^n)\|_{L^2} \leq \|\nabla(u^{n+1} - u^n)\|^\frac{1}{2}\|\nabla^2(u^{n+1} - u^n)\|^\frac{1}{2} \leq \|\nabla(u^{n+1} - u^n)\|_{H^1}.$$  

We combine all the estimates of $I_{5i}$ for $i = 1, 2, \ldots, 5$ to deduce

$$\frac{1}{2} \frac{d}{dt}\|u^{n+1} - u^n\|^2_{L^2} + \|\nabla^2(u^{n+1} - u^n)\|^2_{L^2}$$

$$\leq \frac{1}{4}\|\nabla^2(u^{n+1} - u^n)\|^2_{L^2} + \frac{1}{4}\|\nabla(u^{n+1} - u^n)\|^2_{H^1} + C\|u^n - u^{n-1}\|^2_{H^1} + C\|f^{n+1} - f^n\|^2_{L^\infty}.$$  

We combine (3.21) and (3.22) to obtain the following estimate on $u^{n+1} - u^n$:

$$\frac{1}{2} \frac{d}{dt}\|u^{n+1} - u^n\|^2_{H^1} + \frac{1}{2}\|\nabla(u^{n+1} - u^n)\|^2_{H^1}$$

$$\leq C\|u^{n+1} - u^n\|^2_{H^1} + C\|u^n - u^{n-1}\|^2_{H^1} + C\|f^{n+1} - f^n\|^2_{L^\infty}.$$  

Then, we combine the estimate (3.19) in Step A and (3.23) to obtain the following inequality for $\Delta_n$:

$$\Delta_{n+1}(t) \leq C\left(\int_0^t \Delta_n(s) \, ds + \int_0^t \Delta_{n+1}(s) \, ds\right).$$  

Then, we use the Grönwall type inequality in [7] to derive

$$\Delta_{n+1}(t) \leq \frac{(CT^*)_n}{(n + 1)!}, \quad \text{for} \quad t \leq T^*.$$  

This implies that $(f^n, u^n)$ is a Cauchy sequence in $L^{\infty}(0, T^*; \Omega) \times L^{\infty}(0, T^*; H^1(\mathbb{R}^3)) \cap L^2(0, T^*; H^2(\mathbb{R}^3)).$

3.3. **Proof of Theorem 2.1.** Now, we complete the proof of Theorem 2.1. Since the sequence of approximated solutions $(f^n, u^n)$ is a Cauchy sequence, there exists a pair of functions $(f, u) \in L^{\infty}(0, T^*; L^{\infty}(\Omega)) \times L^{\infty}(0, T^*; H^1(\mathbb{R}^3)) \cap L^2(0, T^*; H^2(\mathbb{R}^3))$ so that

$$f^n \to f \quad \text{in} \quad L^{\infty}(0, T^*; L^{\infty}(\Omega)), \quad u^n \to u \quad \text{in} \quad L^{\infty}(0, T^*; H^1(\mathbb{R}^3)) \cap L^2(0, T^*; H^2(\mathbb{R}^3)).$$

Since we already have the uniform-in-n bound in Proposition 3.2 up to small time interval $T^*$, the regularity of limit function $(f, u)$ can be recovered as in [3, 13] and this completes the proof of local existence. For the uniqueness of strong solution, suppose we have two strong solutions $(f, u)$ and $(\tilde{f}, \tilde{u})$ with the same initial data.
Moreover, we multiply (1.1) those two strong solutions:

$$\Delta(t) := \|(f - \tilde{f})(t)\|_{L^2}^2 + \|(u - \tilde{u})(t)\|_{H^1}^2 + \int_0^t \|\nabla (u - \tilde{u})(s)\|_{H^1}^2 \, ds.$$  

Then, after the almost same estimation in Proposition 3.3 we obtain the following Grönwall inequality:

$$\Delta(t) \leq C \int_0^t \Delta(s) \, ds, \quad \Delta(0) = 0,$$

which implies $\Delta(t) \equiv 0$. Therefore, we have

$$f \equiv \tilde{f} \text{ in } L^\infty(0,T^*;L^\infty(\Omega)), \quad u \equiv \tilde{u} \text{ in } L^\infty(0,T^*;H^1(\mathbb{R}^3)) \cap L^2(0,T^*;H^2(\mathbb{R}^3)).$$

This completes the proof for the uniqueness part of Theorem 2.1.

4. Blow-up criterion. In this section, we provide a blow-up criterion for the TCS-NS system (1.1). More precisely, we provide the condition on the blow-up time of the local strong solution constructed in the previous section, in terms of $\|\nabla u\|_{L^1(0,T^*,L^\infty)}$.

4.1. Energy estimate. First we provide the following lemma for the energy estimate of TCS-NS system.

**Lemma 4.1.** Suppose that initial density $f_0$ has a compact support, and let $(f,u)$ be a strong solution to (1.1) with the initial data $(f_0,u_0)$. Then, the following estimate holds:

$$\begin{align*}
\frac{1}{2} \left( M_2 f(t) + \|u(t)\|_{L^2}^2 \right) + \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds + \int_0^t \int_{\Omega^2} |u - v|^2 f \, dz \, ds &+ \frac{1}{4} \int_0^t \int_{\Omega^2} \phi(x - x_*)(v - v_*)^2 \frac{\phi(x - x_*)}{\theta_*} f(t,z) f(t,z) \, dz, \, dz \\
&\leq \frac{1}{2} \left( M_2 f_0 + \|u_0\|_{L^2}^2 \right) e^{\beta t}, \quad \beta := \frac{\|\phi\|_{L^\infty}(\theta_M - \theta_m)^2}{\theta_m^3}.
\end{align*}$$

**Proof.** We multiply (1.1)$_1$ by $\frac{|\nabla u|^2}{2}$ and then integrate over $\Omega$ to obtain

$$\frac{1}{2} \frac{d}{dt} M_2 f(t) - \int_{\Omega} \nabla \cdot (u - v) f \, dz + \int_{\Omega^2} \phi(x - x_*) \nabla \cdot \left( \frac{v}{\theta} - \frac{v_*}{\theta_*} \right) f(t,z) f(t,z) \, dz, \, dz = 0. \tag{4.1}$$

Moreover, we multiply (1.1)$_2$ by $u$ and integrate over $\mathbb{R}^3$ to get

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^2 \, dx + \int_{\mathbb{R}^3} \nabla |u|^2 \, dx + \int_{\Omega} u \cdot (u - v) f \, dz = 0. \tag{4.2}$$

Two estimates (4.1) and (4.2) directly yields

$$\begin{align*}
\frac{1}{2} \frac{d}{dt} \left( M_2 f(t) + \|u\|_{L^2}^2 \right) + \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_{\Omega} |u - v|^2 f \, dz &+ \frac{1}{4} \int_{\Omega^2} \phi(x - x_*)(v - v_*) \cdot \left( \frac{v}{\theta} - \frac{v_*}{\theta_*} \right) f(t,z) f(t,z) \, dz, \, dz \\
&= 0.
\end{align*}$$
Therefore,
\[
\frac{1}{2} \frac{d}{dt} (M_2 f(t) + \|u\|_{L^2}^2) + \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_{\Omega} |u - v|^2 \, f \, dz \\
+ \frac{1}{2} \int_{\Omega^2} \phi(x - x_*) \frac{|v - v_*|^2}{\theta_*} f(t, z) f(t, z_*) \, dz_*, \, dz \\
= \frac{1}{2} \int_{\Omega^2} \phi(x - x_*) \frac{|v - v_*|^2}{\theta_*} f(t, z) f(t, z_*) \, dz_*, \, dz \\
\leq \frac{1}{4} \int_{\Omega^2} \phi(x - x_*) \frac{|v - v_*|^2}{\theta_*} f(t, z) f(t, z_*) \, dz_*, \, dz \\
+ \frac{1}{4} \int_{\Omega^2} \phi(x - x_*) \left| \frac{|v - v_*|}{\theta_*} \right|^2 f(t, z) f(t, z_*) \, dz_*, \, dz \\
\leq \frac{1}{4} \int_{\Omega^2} \phi(x - x_*) \frac{|v - v_*|^2}{\theta_*} f(t, z) f(t, z_*) \, dz_*, \, dz + \frac{\|\phi\|_{L^\infty}(\theta_m - \theta_m)^2}{\theta_m^3} M_2 f(t).
\]
Thus, we have the following estimate:
\[
\frac{1}{2} \frac{d}{dt} (M_2 f(t) + \|u\|_{L^2}^2) + \int_{\mathbb{R}^3} |\nabla u|^2 \, dx + \int_{\Omega} |u - v|^2 \, f \, dz \\
+ \frac{1}{2} \int_{\Omega^2} \phi(x - x_*) \frac{|v - v_*|^2}{\theta_*} f(t, z) f(t, z_*) \, dz_*, \, dz \leq \frac{\|\phi\|_{L^\infty}(\theta_m - \theta_m)^2}{\theta_m^3} M_2 f(t).
\]
Finally, we use Grönwall’s inequality to conclude the desired estimate. \(\square\)

4.2. Proof of Theorem 2.2. We now present the proof of Theorem 2.2 by using proof by contradiction. Suppose we have
\[
\int_0^{T^*} \|\nabla u(t)\|_{L^\infty} \, dt \leq C(T^*) < \infty.
\] (4.3)
Since we already proved the local existence of TCS-NS system, it suffices to show that \((f(T^*), u(T^*))\) satisfies the initial conditions in Theorem 2.1, i.e.,
\[f(T^*) \in W^{1,\infty}(\Omega), \quad u(T^*) \in H^2(\mathbb{R}^3),\]
and \(f(T^*)\) has a compact support with respect to each variable. It follows from the \textit{a priori} estimates in Proposition 3.1 (1) and (2) that \(f(T^*)\) has a compact support. Moreover, proof of Proposition 3.1 (3) implies
\[
\|f(T^*)\|_{W^{1,\infty}} \leq \|f_0\|_{W^{1,\infty}} \exp \left( C \int_0^{T^*} (1 + \|\nabla u(s)\|_{L^\infty}) \, ds \right) < \infty.
\]
Next, we will prove \(\|u(T^*)\|_{H^2}\) is bounded. Again, we separate the energy estimates for each order.

\textbullet \ Step A (Zeroth order energy estimate): We multiply \((1.1)_2\) by \(u\) and integrate over \(\mathbb{R}^3\) to obtain
\[
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \int_{\Omega} |u|^2 \, f \, dz \\
= - \int_{\mathbb{R}^3} (u \cdot \nabla) u \cdot u \, dx + \int_{\Omega} u \cdot v f \, dz = \int_{\mathbb{R}^3} |u|^2 \nabla \cdot u \, dx + \int_{\Omega} u \cdot v f \, dz \\
\leq \frac{1}{2} \int_{\Omega} |u|^2 f \, dz + \frac{1}{2} M_2 f(t) \leq C \|u\|_{L^2}^2 + C.
\] (4.4)
Here, we used the boundedness of $\|f\|_{W^{1,\infty}}$ and support of $f$.

- Step B (First order energy estimate): We take the first order derivative $\partial$ to (1.1), multiply $\partial u$ and then integrate over $\mathbb{R}^3$ to yield

$$\frac{1}{2} \frac{d}{dt} \|\partial u\|_{L^2}^2 + \|\nabla \partial u\|_{L^2}^2 = - \int_{\mathbb{R}^3} \partial [(u \cdot \nabla)u] \cdot \partial u \, dx - \int_{\Omega} \partial u \cdot \partial [(u - v)f] \, dz$$

$$=: I_{61} + I_{62}.$$  

- (Estimate of $I_{61}$): We use integration by parts and the divergence-free condition to obtain

$$I_{61} = - \int_{\mathbb{R}^3} \partial u \cdot [(u \cdot \nabla)\partial u] \, dx - \int_{\mathbb{R}^3} \partial u \cdot [(\partial u \cdot \nabla)u] \, dx$$

$$= - \int_{\mathbb{R}^3} u \cdot \nabla |\partial u|^2 \, dx - \int_{\mathbb{R}^3} \partial u \cdot [(\partial u \cdot \nabla)u] \, dx \leq \|\partial u\|_{L^\infty} \|\nabla u\|_{L^2}.$$  

- (Estimate of $I_{62}$): We use the boundedness of $\|f\|_{W^{1,\infty}}$ and support of $f$ to derive the following estimate of $I_{62}$:

$$I_{62} = - \int_{\Omega} \partial u \cdot [\partial uf + (u - v)\partial f] \, dz = - \int_{\Omega} |\partial u|^2 f \, dz - \int_{\Omega} \partial u \cdot u \partial f \, dz$$

$$+ \int_{\Omega} \partial u \cdot v \partial f \, dz$$

$$\leq C(\|\partial u\|_{L^2}^2 + \|u\|_{L^2}^2 + 1).$$  

Thus, we combine estimates for $I_{61}$ and $I_{62}$ to obtain

$$\frac{1}{2} \frac{d}{dt} \|\partial u\|_{L^2}^2 + \|\nabla \partial u\|_{L^2}^2 \leq \|\partial u\|_{L^\infty} \|\nabla u\|_{L^2} + C(\|\partial u\|_{L^2}^2 + \|u\|_{L^2}^2 + 1). \quad (4.5)$$  

- Step C (Second order energy estimate): As in the lower order case, we take the second order derivative $\partial^2$ to (1.1), take inner product with $\partial^2 u$ and integrate over $\mathbb{R}^3$ to derive

$$\frac{1}{2} \frac{d}{dt} \|\partial^2 u\|_{L^2}^2 + \|\nabla \partial^2 u\|_{L^2}^2 = - \int_{\mathbb{R}^3} \partial^2 [(u \cdot \nabla)u] \cdot \partial^2 u \, dx - \int_{\Omega} \partial^2 u \cdot \partial^2 [(u - v)f] \, dz$$

$$=: I_{71} + I_{72}.$$  

- (Estimate of $I_{71}$): We estimate $I_{71}$ similar to the estimate of $I_{61}$ as follows:

$$I_{71} = - \int_{\mathbb{R}^3} \partial^2 u \cdot [(u \cdot \nabla)\partial^2 u] \, dx - \int_{\mathbb{R}^3} 2\partial^2 u \cdot [(\partial u \cdot \nabla)\partial u] \, dx$$

$$- \int_{\mathbb{R}^3} \partial^2 u \cdot [(\partial^2 u \cdot \nabla)u] \, dx$$

$$= - \int_{\mathbb{R}^3} u \cdot \nabla |\partial^2 u|^2 \, dx - 2 \int_{\mathbb{R}^3} \partial^2 u \cdot [(\partial u \cdot \nabla)\partial u] \, dx - \int_{\mathbb{R}^3} \partial^2 u \cdot [(\partial^2 u \cdot \nabla)u] \, dx$$

$$\leq 3 \|\nabla u\|_{L^\infty} \|\partial^2 u\|_{L^2}^2.$$
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Then, we consider the following regularized TCS-CS system in

\[ \|f\|_{W^{1,\infty}} \] and support of \( f \) to get

\[ \mathcal{I}_{T^2} \leq \int_{\Omega} \partial^3 u \cdot [\partial u f + (u - v) \partial f] \, dz \]

\[ \leq C \|\nabla \partial^2 u\|_{L^2} \|\partial u\|_{L^2} + C \|\nabla \partial^2 u\|_{L^2} \|u\|_{L^2} + \int_{\Omega} |\nabla \partial^2 u| \|v\| \|\partial f\| \, dz \]

\[ \leq \frac{1}{2} \|\nabla \partial^2 u\|_{L^2}^2 + C(\|\partial u\|_{L^2}^2 + \|u\|_{L^2}^2 + 1). \] (4.6)

We collect estimates of \( \mathcal{I}_{T^1} \) and \( \mathcal{I}_{T^2} \) to derive the following inequality:

\[ \frac{1}{2} \frac{d}{dt} \|\partial^2 u\|_{L^2}^2 + \frac{1}{2} \|\nabla \partial^2 u\|_{L^2}^2 \leq 3\|\nabla u\|_{L^\infty} \|\partial^2 u\|_{L^2}^2 + C(\|\partial u\|_{L^2}^2 + \|u\|_{L^2}^2 + 1). \] (4.6)

We now collect all the estimates (4.4), (4.5) and (4.6) to derive the following estimate for \( \|u\|_{H^2} \):

\[ \frac{1}{2} \frac{d}{dt} \|u\|_{H^2}^2 + \frac{1}{2} \|\nabla u\|_{H^2}^2 \leq C(1 + \|\nabla u\|_{L^\infty}) \|u\|_{H^2}^2 + C. \] (4.7)

It follows from (4.7) and Grönwall's inequality that

\[ \|u(t)\|_{H^2}^2 \leq \|u_0\|_{H^2}^2 \exp \left( C \int_0^t (1 + \|\nabla u(s)\|_{L^\infty}) \, ds \right) \]

\[ + C \int_0^t \exp \left( \int_0^s (1 + \|\nabla u(\tau)\|_{L^\infty}) \, d\tau \right) \, ds \]

\[ \leq \|u_0\|_{H^2}^2 \exp \left( C \int_0^{T^*} (1 + \|\nabla u(s)\|_{L^\infty}) \, ds \right) \]

\[ + C T^* \exp \left( \int_0^{T^*} (1 + \|\nabla u(s)\|_{L^\infty}) \, ds \right) < \infty, \quad \text{for} \quad t \leq T^*. \]

Thus, we have \( \|u(T^*)\|_{H^2} < \infty \) and this completes the proof of Theorem 2.2.

5. An existence of weak solution. In this section, we provide the detailed proof of Theorem 2.3, which is a global existence of a weak solution of TCS-CS system (1.1). Basically, we follow the method introduced in [10].

5.1. Regularized system. As in [10, 13], we first consider the regularized version of system (1.1). Let \( \varepsilon > 0 \) and recall the standard mollifier \( k_\varepsilon \) in (3.9). We also define the cut-off function \( \gamma_\varepsilon \in C^\infty(\mathbb{R}^3) \) as

\[ \text{supp } \gamma_\varepsilon \subset B_{\frac{1}{2}}(0), \quad 0 \leq \gamma_\varepsilon \leq 1, \quad \gamma_\varepsilon = 1 \quad \text{on} \quad B_{\frac{1}{2\varepsilon}}(0), \quad \gamma_\varepsilon \to 1 \quad \text{as} \quad \varepsilon \to 0. \]

Then, we consider the following regularized TCS-CS system in \( \mathbb{R}^3 \):

\[ \partial_\varepsilon f^\varepsilon + v \cdot \nabla f^\varepsilon + \nabla \cdot (F[f^\varepsilon, k_\varepsilon * u^\varepsilon]f^\varepsilon) + \partial_0(G[f^\varepsilon]f^\varepsilon) = 0, \]

\[ \partial_\varepsilon u^\varepsilon + ((k_\varepsilon * u^\varepsilon) \cdot \nabla)u^\varepsilon + \nabla P^\varepsilon - \Delta u^\varepsilon = - \int_{\mathbb{R}^3 \times \mathbb{R}_+} (u^\varepsilon - v) f^\varepsilon \gamma_\varepsilon(v) \, dv \, d\theta, \] (5.1)

\[ \nabla \cdot u^\varepsilon = 0, \]

subjected to the initial data:

\[ (f_\varepsilon, u_\varepsilon)|_{t=0} = (f_0^\varepsilon, u_0^\varepsilon), \]
where \((f^\varepsilon_0, u^\varepsilon_0)\) is a family of pair of \(C^\infty\) functions approximating \((f_0, u_0)\) in the sense that

\[
\begin{align*}
&f^\varepsilon_0 \to f_0 \text{ in } L^p(\Omega), \quad \forall p < \infty, \\
&f^\varepsilon_0 \to f_0 \text{ in weak } - \ast L^\infty(\Omega), \quad \|f_0^\varepsilon\|_{L^\infty} \leq \|f_0\|_{L^\infty}, \\
&M_2f^\varepsilon_0 \to M_2f_0, \quad \|\nabla f^\varepsilon_0\|_{L^1} \to \|\nabla f_0\|_{L^1}, \\
&u^\varepsilon_0 \to u_0 \text{ in } L^2(\mathbb{R}^3).
\end{align*}
\] (5.2)

The existence of weak solution \((f^\varepsilon, u^\varepsilon) \in L^\infty(\Omega \times [0, T]) \times L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3))\) to the regularized system (5.1) can be shown by using exactly same procedure in \([10, \text{Section 3.2}]\) or \([13, \text{Proposition 4.1}]\). Hence, we omit the detailed proof and provide the result only.

**Proposition 5.1.** [10, 13] For initial data \((f^\varepsilon_0, u^\varepsilon_0)\) satisfying (5.2), there exists \((f^\varepsilon, u^\varepsilon) \in L^\infty(\Omega \times [0, T]) \times (L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3)))\) satisfying regularized system (5.1) in the sense of distribution such that

\[
\sup_{0 \leq t \leq T} \|f^\varepsilon(t)\|_{L^\infty} \leq C(T) \|f_0^\varepsilon\|_{L^\infty}.
\]

With the constructed solution to the regularized system (5.1), we are able to pass the limit \(\varepsilon \to 0\) in the following subsection.

5.2. Uniform-in-\(\varepsilon\) bounds and limit in \(\varepsilon\). In this subsection, we derive the uniform-in-\(\varepsilon\) bounds in order to show the convergence of \((f^\varepsilon, u^\varepsilon)\) to a weak solution of TCS-NS system (1.1). Again, we basically follow the [10] to obtain the estimate on \(\mathbb{R}^3\).

**Proposition 5.2.** For fixed time \(0 < T < \infty\), let \((f^\varepsilon, u^\varepsilon)\) be a solution to (5.1). Then, there exist a finite time \(T^* \in (0, T]\) and a positive constant \(C\) independent of \(\varepsilon\) such that

\[
\begin{align*}
(i) & \quad \|f^\varepsilon\|_{L^\infty(\Omega \times [0, T])} \leq C(T) \|f_0\|_{L^\infty}, \\
(ii) & \quad \|u^\varepsilon(t)\|^2_{L^2} + \int_0^t \|\nabla u^\varepsilon(s)\|_{L^2}^2 \, ds + M_2f^\varepsilon(t) \leq C, \quad \text{a.e. } t \leq T^*.
\end{align*}
\]

**Proof.** We follow the mainstream of the proof in [10, Section 3.3].

(i) From the construction of \(f_0^\varepsilon\), we have \(\|f_0^\varepsilon\|_{L^\infty} \leq \|f_0\|_{L^\infty}.\) This implies

\[
\|f^\varepsilon\|_{L^\infty(\Omega \times (0,T))} \leq C(T) \|f_0^\varepsilon\|_{L^\infty} \leq C(T) \|f_0\|_{L^\infty}.
\]

(ii) We use Lemma 2.1 to estimate \(M_2f^\varepsilon\) as

\[
\begin{align*}
\frac{d}{dt} M_2f^\varepsilon &= 2 \int_{\Omega} \mathbf{v} \cdot F[f^\varepsilon, k_z \ast \mathbf{u}^\varepsilon] f^\varepsilon \, dz \\
&= 2 \int_{\Omega} \mathbf{v} \cdot (k_z \ast \mathbf{u}^\varepsilon) f^\varepsilon \, dz + 2 \int_{\Omega} \mathbf{v} \cdot F[f^\varepsilon] f^\varepsilon \, dz - 2M_2f^\varepsilon \\
&\leq 2\|k_z \ast \mathbf{u}^\varepsilon\|_{L^\infty} \|m_1 f^\varepsilon\|_{L^2}^2 - 2 \left( \frac{\phi_M}{\theta_m} \right) M_2 f^\varepsilon \\
&\leq C_1 \|\mathbf{u}^\varepsilon\|_{L^2}^2 (M_2 f^\varepsilon)^{\frac{4}{3}} - C_2 M_2 f^\varepsilon,
\end{align*}
\]

where we use Lemma 2.3 to obtain

\[
\|m_1 f^\varepsilon\|_{L^2}^2 \leq C_1(M_2 f^\varepsilon)^{\frac{4}{3}}.
\]
Therefore, we obtain
\[
\frac{d}{dt} M_2 f^\varepsilon + C_2 M_2 f^\varepsilon \leq C_1 \|u^\varepsilon\|_{L^5} (M_2 f^\varepsilon)^{\frac{2}{3}}.
\]
and together with Grönwall’s inequality, this yields
\[
M_2 f^\varepsilon(t) \leq C \left( 1 + \int_0^t \|u^\varepsilon\|_{L^5} \, ds \right)^5.
\] (5.3)

Now, we move to estimate the fluid part. We multiply \(u^\varepsilon\) to (5.1) and integrate over the domain to obtain
\[
\frac{1}{2} \frac{d}{dt} \|u^\varepsilon\|_{L^2}^2 + \|\nabla u^\varepsilon\|_{L^2}^2 = \int_{\Omega} f^\varepsilon u^\varepsilon \cdot (v - u^\varepsilon) \gamma^\varepsilon(v) \, dz \leq \int_{\mathbb{R}^3} |u^\varepsilon| \, m_1 f^\varepsilon \, dx
\]
\[
\leq \|u^\varepsilon\|_{L^6} \|m_1 f^\varepsilon\|_{L^\frac{2}{3}} \leq C \|u^\varepsilon\|_{L^5} \left( 1 + \int_0^t \|u^\varepsilon\|_{L^5} \, ds \right)^4.
\] (5.4)

Here, we use the following estimate on the last inequality
\[
\|m_1 f^\varepsilon\|_{L^\frac{2}{3}} \leq C (M_2 f^\varepsilon)^{\frac{1}{2}} \leq C \left( 1 + \int_0^t \|u^\varepsilon\|_{L^5} \, ds \right)^4.
\]

Now, we use the interpolation inequalities in Lemma 2.4 and Young’s inequality to obtain
\[
\frac{1}{2} \frac{d}{dt} \|u^\varepsilon\|_{L^2}^2 + \frac{1}{2} \|\nabla u^\varepsilon\|_{L^2}^2 \leq C \left( 1 + \|u^\varepsilon\|_{L^2}^2 + \left( \int_0^t \|u^\varepsilon\|_{L^5} \, ds \right)^8 \right).
\] (5.5)

More precisely, the right-hand side of (5.4) can be further estimated by using interpolation inequality and Young’s inequality as
\[
C \|u^\varepsilon\|_{L^5} \left( 1 + \int_0^t \|u^\varepsilon\|_{L^5} \, ds \right)^4 \leq C \|u^\varepsilon\|_{L^2}^{\frac{1}{2}} \|\nabla u^\varepsilon\|_{L^2}^{\frac{3}{2}} \left( 1 + \int_0^t \|u^\varepsilon\|_{L^5} \, ds \right)^4
\]
\[
\leq C \left( \|u^\varepsilon\|_{L^2}^{\frac{1}{2}} \|\nabla u^\varepsilon\|_{L^2}^{\frac{3}{2}} + 1 + \left( \int_0^t \|u^\varepsilon\|_{L^5} \, ds \right)^8 \right)
\]
\[
\leq C \|u^\varepsilon\|_{L^2}^2 + \frac{1}{2} \|\nabla u^\varepsilon\|_{L^2}^2 + C \left( 1 + \left( \int_0^t \|u^\varepsilon\|_{L^5} \, ds \right)^8 \right)
\]
\[
\leq C \|u^\varepsilon\|_{L^2}^2 + \frac{1}{2} \|\nabla u^\varepsilon\|_{L^2}^2 + C \left( 1 + \left( \int_0^t \|u^\varepsilon\|_{L^2} + \|\nabla u^\varepsilon\|_{L^2} \, ds \right)^8 \right).
\]

Then, we integrate (5.5) from the time 0 to \(t\) to obtain
\[
\frac{1}{2} \|u^\varepsilon\|_{L^2}^2 + \frac{1}{2} \int_0^t \|\nabla u^\varepsilon\|_{L^2}^2 \, ds \leq \frac{1}{2} \|u_0^\varepsilon\|_{L^2}^2 + C \int_0^t \|u^\varepsilon\|_{L^2}^2 \, ds
\]
\[
+ C \int_0^t \left( \int_0^\tau \|\nabla u^\varepsilon(\tau)\|_{L^2}^2 \, d\tau \right)^4 \, ds + C \int_0^t \|u^\varepsilon(s)\|_{L^2}^8 \, ds.
\]

Then, the nonlinear Grönwall’s inequality guarantees the existence of positive time \(T^* > 0\) such that
\[
\|u^\varepsilon\|_{L^2}^2 + \int_0^t \|\nabla u^\varepsilon\|_{L^2}^2 \, ds \leq C, \quad \text{a.e. } t \leq T^*.
\]
and from this and the estimation on \(M_2f^\varepsilon\) (5.3), together with interpolation inequality again, we conclude

\[ M_2f^\varepsilon \leq C, \quad \text{a.e.} \quad t \leq T^*. \]

Then, with the uniform-in-\(\varepsilon\) boundedness of \(f^\varepsilon\) and \(u^\varepsilon\) in Proposition 5.2 and interpolation inequalities in Lemma 2.4, we are able to pass the limit \(f^\varepsilon \to f\) and \(u^\varepsilon \to u\) as \(\varepsilon \to 0\) similar to [10], where the pair \((f, u)\) is the weak solution of (1.1) in the sense of Definition 2.2, up to finite time \(T^*\). To attain the global existence of weak solution and complete the proof of Theorem 2.3, we can follow the extension technique in [7]. In order to do so, we only need the following \textit{a priori} energy estimate.

**Lemma 5.1.** Let \((f, u)\) be a weak solution to (1.1). Then, we have the following energy estimate:

1. \[
\frac{1}{2} M_2f(t) + \frac{1}{2} \|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(s)\|_{L^2}^2 \, ds + \int_0^t \int_\Omega |f| |v|^2 \, dz \, ds
\leq C(T^*) \left( \frac{1}{2} \|u_0\|_{L^2}^2 + \frac{1}{2} M_2f_0 \right),
\]

2. \[
\sup_{0 \leq t \leq T^*} \|f(t)\|_{L^\infty} \leq e^{CT^*} \|f_0\|_{L^\infty}.
\]

**Proof.** The proof follows the calculation in [10, Lemma A.1]. We multiply (5.1)_1 by \(|v|^2\) and (5.1)_2 by \(u^\varepsilon\) and integrate over the domain to see

\[
\frac{1}{2} M_2f^\varepsilon(t) + \frac{1}{2} \|u^\varepsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla u^\varepsilon(s)\|_{L^2}^2 \, ds + \int_0^t \int_\Omega |u^\varepsilon - v| |f^\varepsilon| \, dz \, ds
\]

\[
= \frac{1}{2} M_2f_0^\varepsilon + \frac{1}{2} \|u_0^\varepsilon\|_{L^2}^2 + \int_0^t \int_\Omega v \cdot F[f^\varepsilon] f^\varepsilon \, dz \, ds + R_\varepsilon(t),
\]

where the residual term \(R_\varepsilon\) is given by

\[
R_\varepsilon := \int_0^t \int_\Omega |u^\varepsilon|^2 (1 - \gamma_\varepsilon(v)) f^\varepsilon \, dz \, ds - \int_0^t \int_\Omega u^\varepsilon \cdot v(1 - \gamma_\varepsilon(v)) f^\varepsilon \, dz \, ds
\]

\[
+ \int_0^t \int_\Omega (u^\varepsilon - k_\varepsilon \ast u^\varepsilon) \cdot v f^\varepsilon \, dz \, ds.
\]

We need to show that \(R_\varepsilon \to 0\) as \(\varepsilon \to 0\). The overall proof is similar to that of [10], with the help of standard Sobolev embedding \(H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)\). Thus, we are able to show that \(R_\varepsilon \to 0\) as \(\varepsilon\) decays to 0. The remaining procedure can be done as in [7] and [13].

The remaining step is to extend the local solution \((f, u)\) up to \(T^*\) to the global one. However, since the extension procedure is standard after we obtain the energy estimate Lemma 5.1, we refer [7, Section 3.6] and this completes the proof for the global existence of a weak solution.
6. Conclusion. In this paper, we studied the existence theory for the solutions to TCS-NS coupled system. We first proved the existence and uniqueness of the local strong solution to TCS-NS system. We construct the sequence of approximated solutions and derive a uniform boundedness of them up to small finite time. In order to avoid using Poincaré inequality, we used Gagliardo-Nirenberg interpolation inequality to estimate the energy of Navier-Stokes equations part. Then, we provide the blow-up criterion for the strong solution, in terms of the gradient of fluid. We show that if the solution blow-up at time $T^*$, then we have $\int_0^{T^*} \| \nabla u(t) \|_{L^\infty} dt = \infty$. Finally, we presented the global existence of the weak solution to TCS-NS system. We constructed a family of solutions parameterized by $\varepsilon$, using mollifiers and cut-off functions. Then, we derive the uniform-in-$\varepsilon$ estimate to guarantee the existence of a weak solution, in which we used interpolation theorem again to estimate the fluid part.

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