Engineering multipartite entanglement in doubly pumped parametric down-conversion processes

Alessandra Gatti$^{1,2}$, Enrico Brambilla$^2$ and Ottavia Jedrkiewicz$^{3,2}$

$^1$Istituto di Fotonica e Nanotecnologie (IFN-CNR), Piazza Leonardo Da Vinci 32, Milano, Italy; $^2$Dipartimento di Scienza e Alta Tecnologia dell’ Università dell’Insubria, Via Valleggio 11, Como, Italy; $^3$Istituto di Fotonica e Nanotecnologie (IFN-CNR), Via Valleggio 11, Como, Italy

We investigate the quantum state generated by optical parametric down-conversion in a $\chi^{(2)}$ medium driven by two noncollinear light modes. The analysis shows the emergence of multipartite, namely 3- or 4-partite, entangled states in a subset of the spatio-temporal modes generated by the process. These appear as bright spots against the background fluorescence, providing an interesting analogy with the phenomenology recently observed in two-dimensional nonlinear photonic crystals. We study two realistic setups: i) Non-critical phase-matching in a periodically poled Lithium Tantalate slab, characterized by a 3-mode entangled state. ii) A type I setup in a Beta-Barium Borate crystal, where the spatial walk-off between the two pumps can be exploited to make a transition to a quadripartite entangled state. In both cases we show that the properties of the state can be controlled by modulating the relative intensity of two pump waves, making the device a versatile tool for quantum state engineering.

PACS numbers: 42.65.Lm, 42.50.Ar, 42.50.Dv
INTRODUCTION

Modern quantum technologies would greatly benefit from an efficient tool able to prepare and engineer multipartite entangled states of light. Such a tool would find for example applications in the generation of cluster states [12], which are the multipartite entangled states needed to perform the so-called one-way quantum computation protocol [3, 4], but also in quantum metrological protocols [5–6]. In the continuous variable regime of optics, a well-established scheme for producing multipartite entanglement consists of: i) Generating several single-mode squeezed states, where the typical source is the nonlinear process of optical parametric down-conversion (PDC), then ii) Interfering them by means of a passive linear network (see e.g. [7–9]).

On the other side, manipulations of the squeezed states, or of the two-mode squeezed states, generated by PDC sources are often necessary to introduce non-Gaussianity and to enable entanglement distillation, as in protocols of photon-subtraction where a small fraction of the light is redirected towards a photon counting detector, and the remaining state is conditioned upon detection of photons (see e.g. [10, 11]).

In this work, we instead propose to manipulate the nonlinear process which is the source of squeezing itself, by modulating the transverse profile of the pump beam driving the process. In a sense, we propose to invert the order of the above mentioned steps, by performing the interference before the squeezing by acting on the spatial modes of the classical laser pump beam. Ideally, we aim at engineering the nonlinear process to directly produce the desired state, or at least to implement some operations of interest for quantum technologies. The advantages would be that of avoiding as much as possible manipulations of the fragile quantum states, operating instead on the more robust classical pump, and the possibility of reconfiguring the state by modulating the properties of the pump. We consider in particular injecting two (or more) pump beams slightly tilted in the transverse direction.

The idea is not completely new: an ideal scheme was explored in [12], were a tripartite entanglement was theoretically predicted. The use of a spatially structured pump with a TEM$_{01}$ modal profile, to produce peculiar spatial correlation between twin photons was also explored in [13]. More recently, a scheme for engineering the quantum and classical properties of parametric generation by dual pumping a 2D nonlinear photonic crystal was proposed by some of us [14, 15]. Several four-wave mixing schemes, exploiting the $\chi^{(3)}$ nonlinearity, with dual spatial pump modes have been recently studied and experimentally realized [16–17].

The problem with down-conversion, intrinsically more efficient than four-wave mixing, is that both the phase matching and the effective nonlinearity depend critically on the direction of propagation of each pump beam, making the scheme more complex. In this work we explore two concrete setups suitable for generating multipartite entanglement by means of a structured pump.

The first and simpler scheme considers a type 0 process, taking place in a periodically poled Lithium Tantalate (PPLT) slab, where the pumps are tilted in the plane perpendicular to the optical axis of the crystal. In such configuration, neither the phase matching nor the nonlinearity depend on the direction of propagation of the modes. We show that this is the ideal framework to realize the proposal in [12]. With respect to [12], we analyse the more general case of arbitrary pump amplitudes. We find that that the tripartite entanglement thereby realized is formally equivalent to dividing one of the parties of the bipartite entangled state generated by standard PDC on a beam-splitter, whose reflection and transmission coefficients are controlled by the relative intensity of the two pumps. This result may be relevant for photon-subtraction protocols, because it shows that the doubly-pumped scheme direct implements an arbitrary beam-splitter, without the need of external alignments potentially detrimental for the quantum state. On a different perspective, we highlight an analogy with the phenomenology recently observed in a 2-dimensional nonlinear photonic crystal [18, 19], in particular the emergence of hot spots in correspondence with the spatio-temporal modes shared by both pumps.

The second scheme considers a type I process taking place in a standard Beta-Barium-Borate (BBO) crystal, where the pumps are tilted in a direction that is not perpendicular to the optical axis. The analysis here is strictly connected to a parallel experimental work [20]. In this configuration, we show that the strong birefringence of the BBO crystal, responsible for spatial walk-off effects, can be exploited to find a peculiar transition from a tripartite to a quadripartite entangled state. Somehow surprisingly, this transition turns out analogous to the transition to the Golden Ratio Entanglement recently predicted in a nonlinear photonic crystal (NPC) [19, 20]. As for the NPC source, at the transition point the parametric gain of the hot-spots undergoes a sudden enhancement [20]. From a quantum point of view, the Gaussian quadripartite state generated at resonance can be formally described as the interference of a pair of two-mode squeezed states. Remarkably, at difference with the NPC case, where the properties of the state were fixed by the geometry of the crystal, we show that in the doubly pumped BBO scheme the squeezing and the mixing parameters are again controlled by the relative intensity of the pumps, giving access to a potential control over the state.

The paper is organized as follows: Section 1 introduces the general theoretical framework and discusses the analogy between the doubly pumped PDC scheme and parametric generation in nonlinear photonic crystals. Section 2 analyses the PPLT case and the tripartite entanglement associated with it, with a blend of analytical calculations, performed
in the parametric limit, and numerical simulations. See [III] analyzes the BBO case, the transition to resonance and the 4-mode entanglement. Numerical and experimental data for this part are presented in the related work [20].

I. GENERAL FRAMEWORK

This section introduces the general theoretical framework, formulated in terms of 3D+1 propagation equations inside the nonlinear $\chi^{(2)}$ material for the quantum field operators associated with the interacting light fields.

Our work focuses on a degenerate type 0 or type I process, in which the down-converted light is described by a single field envelope centered around half of the pump frequency. Thus, we consider the two slowly varying field operators associated with the high-frequency pump and the low-frequency down-converted signal, which in the Fourier domain read: $\hat{A}_j(\vec{q}, \Omega, z) = \int \frac{d^3 \vec{r}}{2\pi} \int \frac{dt}{\sqrt{2\pi}} e^{i(\omega_j + \Omega)t} e^{-i[k_jz + \vec{q}\cdot\vec{r}]t} \hat{E}_j(\vec{r}, z, t), (j = p, s)$ (see [21] [22] for details), where: $z$ is the mean direction of propagation of the fields, assumed to be paraxial waves; $\Omega_j$ is the frequency shift from the carriers $\omega_p$ and $\omega_s = \frac{\omega_p}{2}$; $\vec{q} = q_x \vec{e}_x + q_y \vec{e}_y$ is the transverse component of the wave-vector; $k_{jz}(\vec{q}, \Omega) = \sqrt{k_j^2(\vec{q}, \Omega) - q_z^2}$ is its $z$-component, where $k_j(\vec{q}, \Omega) = n_j(\vec{q}, \omega_j + \Omega) \frac{\omega_j + \Omega}{\epsilon_0 c n_j}$ is the wave-number, $n_j(\vec{q}, \omega)$ being the index of refraction of the $j$-th wave. For the extraordinary wave, the index depends both on the frequency and on the direction of propagation, implicitly identified by the transverse wave-vector component $\vec{q}$. Finally, $\hat{E}_j(\vec{r}, z, t)$ is the full field operator in the direct space, such that $\hat{E}_j^\dagger \hat{E}_j$ has the dimensions of a photon number per unit area and unit time. By using the shorthand notation $\vec{w} = (\vec{q}, \Omega) \in \mathbb{R}^3$, the coupled propagation equations have the form:

$$\frac{\partial}{\partial z} \hat{A}_s(\vec{w}_s, z) = \int \frac{d^3 \vec{w}_p}{(2\pi)^2} \chi(\vec{w}_p; \vec{w}_s) \hat{A}_p(\vec{w}_p, z) \hat{A}_s^\dagger(\vec{w}_p - \vec{w}_s, z) e^{-i\delta(\vec{w}_p; \vec{w}_s)z}$$

$$\frac{\partial}{\partial z} \hat{A}_p(\vec{w}_p, z) = \frac{1}{2} \int \frac{d^3 \vec{w}_s}{(2\pi)^2} \chi(\vec{w}_p; \vec{w}_s) \hat{A}_s(\vec{w}_s, z) \hat{A}_s^\dagger(\vec{w}_p - \vec{w}_s, z) e^{i\delta(\vec{w}_p; \vec{w}_s)z}$$

The two equations describe all the possible down- and up-conversion processes between a pump photon in mode $\vec{w}_p$ and a pair of signal and idler photons in modes $\vec{w}_s$ and $\vec{w}_s = \vec{w}_p - \vec{w}_s$, satisfying the energy and transverse momentum conservation (for simplicity, we assumed that the crystal is infinite in the transverse directions). The conservation of longitudinal momentum is less stringent because of the finite longitudinal size of the medium, and is expressed by the phase-mismatch function

$$\delta(\vec{w}_s; \vec{w}_p) = k_{sz}(\vec{w}_s) + k_{sz}(\vec{w}_p - \vec{w}_s) - k_{pz}(\vec{w}_p) + G_z$$

where we allow for the possibility of a longitudinal 1D poling of the material, such that the reciprocal vector of the nonlinear grating $\vec{G}_z = \frac{2\pi}{\lambda_{pol}}$ contributes to the momentum balance. For more generality, we also leave the possibility for the effective nonlinearity to depend on the direction of propagation of the three waves, through $\chi(\vec{w}_p, \vec{w}_s) \propto d_{eff} \frac{\omega_0}{2} \sqrt{\sum_{m \neq 0} \frac{\alpha_m}{n_m^2 n_{s, m}^2}}$. In standard configurations, where the pump is a weakly focused Gaussian beam propagating around a single direction, this dependence can be neglected. When the pump transverse profile is structured, in particular when it is formed by several waves propagating at different angles, the effective nonlinearity can significantly differ in each direction.

The nonlinear equations [1] have been numerically simulated, by means of fully 3D +1 simulations (see [23] and methods of [18]), which take into account a broad frequency bandwidth, typically on the order 200-400 nm. The input pump modes are modelled by two Gaussian pump pulses, of duration $\sim 1$ ps and transverse waist $\sim 400\mu$m, which propagate close to the $z$ axis tilted one with respect to the other by few degrees.

A. Multiple pump waves, analogy with Nonlinear Photonic Crystals

Although numerical simulations can fully account for pump depletion effects, in the remaining of this work we shall largely exploit the undepleted pump limit. Thus we focus on Eq.[1a] only, with the pump field operator $\hat{A}_p(\vec{w}_p, z)$ replaced by the classical envelope $A_p(\vec{w}_p)$ describing the profile of the injected pump. In particular, we shall consider the injection of multiple plane-wave modes, propagating at slightly different directions around the $z$-axis, i.e.

$$A_p(\vec{q}, \Omega) = (2\pi)^2 \delta(\Omega = 0) \sum_m \alpha_m \delta(\vec{q} - \vec{Q}_m)$$

(3)
Neglecting for simplicity in Eq (1a) the dependence of the effective nonlinearity on the propagation direction, we can then build a straightforward analogy with the process of parametric generation in 2D nonlinear photonic crystals [24, 25]. In such materials, the nonlinear response of the medium is artificially modulated, typically via ferroelectric poling, according to a 2D periodic pattern, the pattern lying in a plane \((x,z)\) perpendicular to the optical axis, including thus one transverse direction. The transverse modulation of the nonlinear response can be often reduced to \(\chi(q_x) \rightarrow \sum_m \chi_m \delta(q_x - \vec{G}_m)\), where \(\vec{G}_m\) are the transverse components of the reciprocal vectors of the nonlinear lattice participating to quasi phase-matching [26]. For example, for a hexagonally poled crystal \(\vec{G}_m = \pm \vec{G}e_x\) are the transverse components of the two fundamental reciprocal lattice vectors (see [15, 18, 19]). According to Eq.(1a), it is therefore equivalent to inject a single plane-wave pump into a photonic crystal equipped with several non-collinear reciprocal lattice vectors, or to inject several non-collinear pumps into a standard crystal (or into a 1D poled crystal).

In a less formal way, the down-conversion process from an undepleted pump beam is ruled by the product of the medium nonlinear response and the pump profile: thus it is equivalent to structure the transverse profile of either one or the other. In the following of this work we shall indeed show that the behaviour of a nonlinear photonic crystal can be fully mimicked by injecting two non-collinear pumps, with the additional benefit that the dual pump scheme enables a reconfigurable control over the properties of the process, by modulating the pump amplitudes.

II. TYPE 0 PROCESS: TRIPARTITE ENTANGLEMENT

This section studies the simplest configuration: a type O process where all the waves are extraordinarily polarized, pumped by two beams that propagate noncollinearly in the plane perpendicular to the optical axis.

For definiteness, we consider a periodically poled LiTaO\(_3\) slab \(^1\) with a poling period \(\Lambda_{\text{pol}} \simeq 7.9\mu\text{m}\), suitable to phase-match the type O interaction \(\lambda_p = 532\text{nm} \rightarrow \lambda_s = \lambda_i = 1064\text{nm}\) at a temperature of \(T \approx 75^\circ\). Fig.1 shows the geometry of the scheme: \(O_3 \equiv y\) is the optical axis of the crystal; the crystalline \(O_2 \equiv z\) axis represents the mean propagation direction of all waves; the two injected pump waves are slightly tilted in the \(O_1 \equiv x\) direction, and thus propagate in the plane perpendicular to the optical axis. In these conditions, also known as non-critical phase matching, their wave numbers do not depend on the tilt angle, and, assuming a paraxial propagation, also the nonlinear coefficient does not depend to a good approximation on their propagation directions.

We approximate the two pumps as classical plane-waves of complex amplitudes \(\alpha_1\) and \(\alpha_2\), characterized by trans-

---

\(^1\) The analysis can be straightforwardly extended to PPLN, we choose LiTaO\(_3\) as an active material because of its very small birefringence.
verse wave vectors $\mathbf{Q}_1 = Q_1 \hat{e}_x$, $\mathbf{Q}_2 = Q_2 \hat{e}_x$, where $|Q_m| \ll \frac{2\pi}{\lambda_p}$. By substituting into Eq. (1a), we get:

$$\frac{\partial}{\partial z} \hat{A}_s(\mathbf{w}, z) = \chi \left[ \alpha_1 \hat{A}_s^\dagger(\mathbf{Q}_1 - \mathbf{w}, z)e^{-iD(\mathbf{w}; \mathbf{Q}_1)z} + \alpha_2 \hat{A}_s^\dagger(\mathbf{Q}_2 - \mathbf{w}, z)e^{-iD(\mathbf{w}; \mathbf{Q}_2)z} \right],$$

where $\chi \simeq \chi(\mathbf{w}_{p1}; \mathbf{w}) = \chi(\mathbf{w}_{p2}; \mathbf{w})$ is the common value of the nonlinear coefficient. The r.h.s of Eq. (4) shows the contribution of the two processes originating from each pump. For the large majority of modes, only one of the two processes is phase-matched, giving rise to two noncollinear branches of down-converted modes (examples are shown in Fig. 2), corresponding to the standard conical emission around each pump taken separately. In the Fourier space $(\mathbf{q}, \Omega)$ photon pairs down-converted from each pump populate surfaces of equation

$$\Sigma_1 : D(\mathbf{w}; \mathbf{Q}_1) = 0, \quad \text{pump 1}$$
$$\Sigma_2 : D(\mathbf{w}; \mathbf{Q}_2) = 0, \quad \text{pump 2}$$

A light mode $\mathbf{w}$ belonging to the branch $\Sigma_1$ ($\Sigma_2$), but not to the intersection $\Sigma_1 \cap \Sigma_2$, hosts signal photons down-converted from pump 1 (2), whose twin idler photon is generated in a single coupled mode $\mathbf{Q}_1 - \mathbf{w}$ ($\mathbf{Q}_2 - \mathbf{w}$), giving rise to the standard two-mode coupling of PDC. Conversely, the modes lying at the geometrical intersection $\Sigma_1 \cap \Sigma_2$ are special, because here phase-matching is simultaneously realized for both pumps. Therefore, a photon appearing in one of these shared modes has been down-converted from either pump 1 or 2, indistinguishably. Its twin photon
appears in either one of two coupled modes, which evolve according to:

\[
\frac{\partial}{\partial z} \hat{A}_s(\bar{Q}_1 - \bar{w}, z) = \chi \left[ \alpha_1 \hat{A}^\dagger_s(\bar{w}, z) e^{-iD(\bar{w}; \bar{Q}_1)z} + \alpha_2 \hat{A}^\dagger_s(\bar{Q}_2 - \bar{Q}_1 + \bar{w}, z) e^{-iD(\bar{Q}_1 - \bar{w}; \bar{Q}_2)z} \right],
\]

\[
\frac{\partial}{\partial z} \hat{A}_s(\bar{Q}_2 - \bar{w}, z) = \chi \left[ \alpha_2 \hat{A}^\dagger_s(\bar{w}, z) e^{-iD(\bar{w}; \bar{Q}_2)z} + \alpha_1 \hat{A}^\dagger_s(\bar{Q}_1 - \bar{Q}_2 + \bar{w}, z) e^{-iD(\bar{Q}_2 - \bar{w}; \bar{Q}_1)z} \right],
\]

(6)

(7)

where we used the fact that \(D(\bar{Q}_m - \bar{w}; \bar{Q}_m) = D(\bar{w}; \bar{Q}_m)\), \((m = 1, 2)\), implicit in the definition \(\hat{w}\) of the phase-mismatch function. In the present configuration, as shown in Sec. II B, if the shared mode condition

\[
\frac{\partial}{\partial z} \hat{A}_s(\bar{Q}_1) = \frac{\partial}{\partial z} \hat{A}_s(\bar{Q}_2) \approx 0
\]

(8)

is satisfied for a mode \(\bar{w}_0\), then the second of the two processes appearing at r.h.s of Eqs. (6) and (7) is not phase matched, that is, \(\frac{\partial}{\partial z} \hat{A}_s(\bar{Q}_2 - \bar{w}_0; \bar{Q}_1)\) and \(\frac{\partial}{\partial z} \hat{A}_s(\bar{Q}_1 - \bar{w}_0; \bar{Q}_2)\) are significantly different from zero. In other words, if the mode \(\bar{w}_0\) is shared, then its two coupled modes cannot be themselves shared. This leads to the tripartite entangled state that will be described in the next Sec. II A.

A. Tripartite entanglement

Let us concentrate on a specific triplet of modes whose coordinates \(\bar{w}_0\) (shared mode) and \(\bar{w}_{b,c} = \bar{Q}_{1,2} - \bar{w}_0\) (modes coupled to \(\bar{w}_0\) via pump 1 and 2, respectively) are a solution of Eq.\(8\), as for example the modes shown by the dots in Fig.2c,d. Indicating by \(\hat{a}_{0s} := \hat{A}_s(\bar{w}_0)\), \(\hat{b}_i := \hat{A}_s(\bar{Q}_1 - \bar{w}_0)\), \(\hat{c}_i := \hat{A}_s(\bar{Q}_2 - \bar{w}_0)\) the three field operators involved, Eqs.\(4\),\(6\) and \(7\) lead to the 3-mode evolution:

\[
\frac{d}{dz} \hat{a}_{0s}(z) = \chi \left[ \alpha_1 \hat{b}_1^\dagger(z) + \alpha_2 \hat{c}_1^\dagger(z) \right] e^{-iD(\bar{w}_0)z}
\]

\[
\frac{d}{dz} \hat{b}_1(z) = \chi \left[ \alpha_1 \hat{a}_{0s}(z) \right] e^{-iD(\bar{w}_0)z}
\]

\[
\frac{d}{dz} \hat{c}_1(z) = \chi \left[ \alpha_2 \hat{a}_{0s}(z) \right] e^{-iD(\bar{w}_0)z}
\]

(9a)

(9b)

(9c)

where \(D(\bar{w}_0) = D(\bar{w}_0; \bar{Q}_1) = D(\bar{w}_0; \bar{Q}_2)\) is the common value of the phase-mismatch. Eqs.\(9\) can be readily solved by means of a linear transformation acting on the 2 side modes:

\[
\begin{pmatrix}
\hat{d}_{i+} \\
\hat{d}_{i-}
\end{pmatrix} =
\begin{pmatrix}
\alpha_p^* & \alpha_p \\
-\alpha_p^* & \alpha_p
\end{pmatrix}
\begin{pmatrix}
\hat{b}_i \\
\hat{c}_i
\end{pmatrix}
\]

(10)

where

\[
\alpha_p = e^{i\phi_1 + i\phi_2} \sqrt{|\alpha_1|^2 + |\alpha_2|^2},
\]

(11)

can be seen as the complex amplitude of a single pump carrying the sum of the energies of the two pumps (\(\phi_1, \phi_2\) being the phases of each wave). As it can be immediately verified, the new modes evolve according to:

\[
\frac{d}{dz} \hat{d}_{0s}(z) = \chi \alpha_p \hat{d}_{i+}^\dagger(z) e^{-iDz}
\]

\[
\frac{d}{dz} \hat{d}_{i+}(z) = \chi \alpha_p \hat{d}_{0s}^\dagger(z) e^{-iDz},
\]

(12a)

(12b)

while

\[
\frac{d}{dz} \hat{d}_{i-}(z) = 0.
\]

(13)

Eqs.\(12\) describe a standard PDC process involving modes \(\hat{a}_{0s}\) and \(\hat{d}_{i+}\), pumped by a single wave of amplitude \(\alpha_p\), with energy \(|\alpha_p|^2 = |\alpha_1|^2 + |\alpha_2|^2\). As well known, the solution of Eq.\(12\), starting from
initial conditions $\hat{a}^{in}_{0a}, \hat{d}^{in}_{i+}$ at the crystal entrance face, are Bogoliubov transformations, which in the general case can be found in the Appendix A of Ref.[19], substituting the parameter $\gamma_{g0}$, appearing there with $g^2 = \chi |\tilde{\alpha}_p|^2 z$. For phase-matched modes such that $D(\tilde{w}_0) = 0$ they take the simple form (not dependent on the specific modes chosen):

$$\begin{align*}
\hat{a}_{0a}(z) &= \cosh(\tilde{g} z) \hat{a}^{in}_{0a} + e^{i\phi_p} \sinh(\tilde{g} z) \hat{d}^{in}_{i+}, \\
\hat{d}_{i+}(z) &= \cosh(\tilde{g} z) \hat{d}^{in}_{i+} + e^{i\phi_p} \sinh(\tilde{g} z) \hat{a}^{in}_{0a}.
\end{align*}$$

(14)

If instead of the fields, the quantum state is evolved along the medium, the joint state of modes $\hat{a}_{0a}, \hat{d}_{i+}$ is the Two-Mode Squeezed State (TMSS, see e.g. [28]), with squeeze parameter $\chi \tilde{\alpha}_p z$. Conversely, mode $\hat{d}_{i-}$ does not evolve along the crystal, and its state remains the same it had at the crystal input (e.g. vacuum or a coherent state): $\hat{d}_{i-}(z) = \hat{d}^{in}_{i-}$.

On the other hand, the unitary transformation (10) can be easily inverted, giving:

$$
\begin{pmatrix}
\hat{b}_i \\
\hat{c}_i
\end{pmatrix} =
\begin{pmatrix}
\frac{\alpha_i}{|\alpha_p|} - \frac{\alpha_p}{|\alpha_i|} & \frac{\alpha_p}{|\alpha_i|} \\
\frac{\alpha_p}{|\alpha_i|} & \frac{\alpha_i}{|\alpha_p|}
\end{pmatrix}
\begin{pmatrix}
\hat{d}_{i+} \\
\hat{d}_{i-}
\end{pmatrix}
$$

(15)

This transformation is equivalent to the action of a lossless beam-splitter with transmission and reflection coefficients $\cos(\theta) = |\alpha_i|/|\alpha_p|$, $\sin(\theta) = |\alpha_p|/|\alpha_i|$. Precisely, by defining $\phi_\pm = \frac{2 \phi_i - \phi_p}{2}$, one has:

$$
\begin{pmatrix}
\hat{b}_i \\
\hat{c}_i
\end{pmatrix} =
\begin{pmatrix}
e^{i\phi_-} & 0 \\
0 & e^{-i\phi_-}
\end{pmatrix}
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\hat{d}_{i+} \\
\hat{d}_{i-}
\end{pmatrix}
$$

(16)

Thus, for each triplet of entangled modes, the doubly pumped PDC scheme can be considered formally equivalent to the sequence shown in Fig.3, i.e. to:

i) A standard parametric process, pumped by a single pump of amplitude $\tilde{\alpha}_p$, carrying the same total energy of the two pumps, generating a two-mode squeezed state in modes $\hat{a}_{0a}$ and $\hat{d}_{i+}$;

ii) A beam-splitter mixing one of the twin beams generated in step i) with an independent input beam $\hat{d}_{i-} = \hat{d}^{in}_{i-}$. The intensity transmission and reflection coefficients of the beam-splitter are in the same ratio as the intensities of the two pumps: $T/R = \cos^2 \theta / \sin^2 \theta = |\alpha_i|^2/|\alpha_p|^2$;

iii) Phase rotations on the two outputs by the half-phase difference: $\hat{b}_i, \hat{c}_i \rightarrow e^{i\phi_-} \hat{b}_i, e^{-i\phi_-} \hat{c}_i$.

We notice that the two pumps can be thought as derived from a single pump of complex amplitude $\tilde{\alpha}_p$, through the same linear transformation described by Eq.(16): $(\alpha_2) =
\begin{pmatrix}
e^{i\phi_-} & 0 \\
0 & e^{-i\phi_-}
\end{pmatrix}
\begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}
(\alpha_1)$. Notice that this represents also in practice a method to obtain the two pump modes [20]. We have shown that the doubly pumped source formally implements the same linear transformation on one of the two parties of a two-modes squeezed state, where the other mode $\hat{d}_{i-}$ can be in principle externally supplied in any arbitrary state. Such a splitting-mixing can be of relevant practical applications, in protocols of photon-subtracted Gaussian states [10, 11]), the device gives the possibility of redirecting a portion of one party of the TMSS in a separate spatial mode without the need of external alignments, potentially detrimental for the quantum state. The same operations are instead performed on the less fragile classical background. This is especially true when the two pumps are balanced, as shown by the simulations in the upper row of Fig.4 performed with two Gaussian pump pulses of equal peak amplitudes. Notice that in these plots all the spectral components within a rather large bandwidth are superimposed, resulting in three continuous branches of hot-spots in the source far-field. According to the results of the plane-wave model, their exponential growth rate along the medium is $\sqrt{2}$ times larger than that of the background conical emission from each pump, in complete analogy with what observed in NPC sources [18, 29, 50]. The case of two strongly unbalanced pumps is illustrated by the second row of Fig.4, where $g_2 = 4g_1$: then, the fluorescence from pump 1 is basically not visible on the scale...
FIG. 3. (a) Schematics of the dual pump source, where the pump modes $\alpha_1, \alpha_2$ are obtained by dividing a pump $\bar{\alpha}_p$ on the beam-splitter BS. Equivalent scheme (b), where the same beam $\bar{\alpha}_p$ pumps the $\chi^{(2)}$ medium, and one of the two parties of the two-mode squeezed state is mixed on the same BS with an input mode $\hat{d}_- \text{ in an arbitrary state.}$. The description holds for each triplet of shared-coupled modes (see text).

FIG. 4. Numerical simulations of Eqs.(1) for a doubly pumped PPLT, showing the Fourier ($q_x, q_y$) and angular ($\theta_x, \theta_y$) intensity distributions of light downconverted in the bandwidth 950 – 1210 nm. Gaussian pump pulses, of 1ps duration and 400 $\mu$m waist, tilted at $\theta_{p1,p2} = \pm 1.2^\circ$. In (a,b,c) $\alpha_2 = \alpha_1$ ($\alpha_j$ being the peak amplitude of each pump), with the 3 hot-spot branches becoming progressively brighter for increasing propagation length. In (d,e,f) $\alpha_2 = 4 \alpha_1$, and the left hot-spot branch is much weaker than the right one. The overall peak gain is $\bar{g} = \sqrt{g_1^2 + g_2^2} = 1.2 \text{mm}^{-1}$, other parameters as in Fig. 2.
We find that shared and coupled modes at the frequency spatial mode, having the smallest spatial superposition to the pump of transverse momentum conservation, the two side modes have Fourier coordinates

\[ \tilde{d}_{\pm} = Q_{1} \hat{e}_{x} + \tilde{Q}_{2} \hat{e}_{x}. \]  

Then, as shown in App.A, shared modes propagate in the symmetry plane between the two pumps at \( \theta_{\text{p}} = 0 \). Because of transverse momentum conservation, the two side modes have Fourier coordinates \( \tilde{q}_{b} = Q_{1} \hat{e}_{x} \) and \( \tilde{q}_{e} = Q_{2} \hat{e}_{x} \). They generate a transverse field distribution of the form:

\[ \tilde{b}_{\pm} e^{iQ_{1}x} + \tilde{c}_{\pm} e^{iQ_{2}x} = \left[ \tilde{d}_{\pm} + f_{p}(x) \right] \]  

where we used Eq. (15) and \( f_{p}(x) = \frac{\alpha_{1} e^{iQ_{1}x} + \alpha_{2} e^{iQ_{2}x}}{\alpha_{p}} \) can be recognized as the pump spatial mode, such that the pump envelope is \( \tilde{A}_{p}(x) = \alpha_{p} f_{p}(x) \) [see Eq. (17)]. On the other side, \( f_{s}^\pm(x) = -\frac{\alpha_{s} e^{iQ_{1}x} + \alpha_{e} e^{iQ_{2}x}}{\alpha_{s}} \) is the orthogonal spatial mode, having the smallest spatial superposition to the pump \( \int dx f_{s}^\pm(x) f_{p}^\pm(x) = 0 \). This makes clear the decomposition in Eq. (10): \( \tilde{d}_{\pm} \) is the spatial mode of the pump, and it is the only one to be parametrically amplified, while \( \tilde{d}_{\mp} \) is the spatial mode orthogonal to the pump and it is not affected by the parametric generation. Notice that the result is less trivial than it might appear: if both the pump modes were not simultaneously phase matched, then it wouldn’t hold true.

### B. Position of shared and coupled modes

The tripartite entangled state studied in the previous section concerns all the triplets of shared and coupled modes that are solutions of Eq. (5). Their Fourier coordinates are studied in App.A by using the paraxial approximation, and are for example shown by the numerical simulations of Fig. 4. These results can be mapped into angles of propagation that are solutions of Eq. (8). Their Fourier coordinates are studied in App.A by using the paraxial approximation, and (shared modes) and (coupled modes) can be recognized as the pump spatial mode, such that the pump envelope is \( \tilde{A}_{p}(x) = \alpha_{p} f_{p}(x) \) [see Eq. (17)]. On the other side, \( f_{s}^\pm(x) = -\frac{\alpha_{s} e^{iQ_{1}x} + \alpha_{e} e^{iQ_{2}x}}{\alpha_{s}} \) is the orthogonal spatial mode, having the smallest spatial superposition to the pump \( \int dx f_{s}^\pm(x) f_{p}^\pm(x) = 0 \). This makes clear the decomposition in Eq. (10): \( \tilde{d}_{\pm} \) is the spatial mode of the pump, and it is the only one to be parametrically amplified, while \( \tilde{d}_{\mp} \) is the spatial mode orthogonal to the pump and it is not affected by the parametric generation. Notice that the result is less trivial than it might appear: if both the pump modes were not simultaneously phase matched, then it wouldn’t hold true.

\[ \theta_{0z}(\Omega) = \frac{\theta_{p1} + \theta_{p2}}{2} \left( 1 + \frac{G - D_{0}(\Omega)}{k_{s}(\Omega)} \right) \approx \frac{\theta_{p1} + \theta_{p2}}{2} \]  

\[ \theta_{b, cz}(\Omega) = \frac{\theta_{p1} + \theta_{p2}}{2} \left( 1 + \frac{k_{p}}{k_{s}(\Omega)} \right) \approx \frac{\theta_{p1} + \theta_{p2}}{2} \]  

where \( \theta_{p1}, \theta_{p2} = \frac{\Omega}{k_{s}} \) are the (internal) angles formed by the two pumps with the \( z \)-axis, and \( D_{0}(\Omega) = k_{s}(\Omega) + k_{s}(-\Omega) - k_{p} + G \) is the collinear phase-mismatch parameter (i.e. the mismatch one would have if the 3 waves propagated collinearly along the \( z \)-axis). As it could be expected, shared modes are approximately emitted in the symmetry plane between the two pumps. This is exactly true when \( \theta_{p1} + \theta_{p2} = 0 \), i.e. the career of the pump field propagates along \( z \), but approximately holds also when the tilts are not symmetric, because \( \frac{\theta_{s}}{k_{s}} \approx \frac{\lambda_{s}}{n_{c}(\lambda_{s}) A_{pol}} \approx 0.06 \), and \( D_{0} \ll k_{s}^{2} \). The side modes are approximately displaced by \( \pm (\theta_{p1} - \theta_{p2}) \) with respect to the shared ones. Examples of triplets of modes are shown in Fig. 2 which plots the phase-matching surfaces \( \Sigma_{1} \) and \( \Sigma_{2} \) in Eq. (5), with shared modes at their intersections. The green dots in Fig. 2 show the three entangled modes at the degenerate wavelength, while Figs. 2 and 2 illustrate the case of two conjugate wavelengths out of degeneracy, showing two independent triplets of modes, labelled by dots and stars (notice that at any two conjugate wavelengths there are actually 4 independent triplets of modes).

If the emission frequency is not resolved, the various spectral components of the shared and coupled modes form in the far-field of the source three continuous branches at approximately \( \theta_{x} \approx \frac{\theta_{p1} + \theta_{p2}}{2} \) (shared modes) and \( \theta_{x} \approx \frac{\theta_{p1} - \theta_{p2}}{2} \) (coupled modes).
\[ \frac{\theta_{p1} + \theta_{p2}}{2} \pm (\theta_{p1} - \theta_{p2}) \] (coupled modes). These are shown by the numerical simulation in Fig. 4, where shared and coupled modes appear as bright bands of hot spots against the less intense background of the 2-mode fluorescence. Notice that these simulations encompass a rather large bandwidth \( \Delta \lambda = 260 \text{ nm} \), so that the angular positions of high and low frequency spectral components split as predicted by Eq. (20).

Most important for our discussion, we notice that for a given finite tilt \( \pm (\theta_{p1} - \theta_{p2}) \) between the two pumps, the pattern of shared and coupled modes translates rigidly with the angle of propagation \( \frac{\theta_{p1} + \theta_{p2}}{2} \) of the career. As a consequence, the position of coupled modes never superimpose to shared ones (stars never superimpose to dots in Fig. 2c, d. We will see in Sec. III a different phase-matching configuration, where such a superposition may take place, originating a transition to a 4-mode coupling.

### III. TYPE I PROCESS IN BBO: TRANSITION TO A QUADRIPARTITE ENTANGLEMENT

This section studies a second configuration, where the two pump modes propagate inside a standard BBO crystal, forming in general different angles with the optical axis. We shall see that the presence of strong walk-off effects enables a peculiar resonance condition, with a transition to a 4-mode entangled state, analogous to the one observed in nonlinear photonic crystals [18, 19].

![Diagram](image)

**FIG. 5.** Geometry of the scheme, for the \( e \rightarrow oo \) process in a BBO crystal, cut at \( \gamma_0 = 33.44^\circ \). \( O_3 \) is the optical axis. The two pumps propagate mainly along \( z \), with a slight tilt in the \( x \)-direction. In the configuration (A) the pumps form roughly the same angle with the optical axis. In (B) and (C) the two pumps propagate at different angles with \( O_4 \), and have different wave-numbers.

We consider the same setup as in the experiment of Ref. [20]. The active material is a BBO crystal, cut for the collinear type I process \( e \rightarrow oo \) from \( \lambda_p = 352 \text{ nm} \) to \( \lambda_s = \lambda_i = 704 \text{ nm} \). Fig. 5 shows the basic geometry: the optical axis \( O_3 \) forms an angle \( \gamma_0 \approx 33.44^\circ \) with the mean propagation direction \( z \). Unlike the noncritical phase-matching of Fig. 4, two pump modes slightly tilted with respect to \( z \) experience in general different refraction indices, because they propagate at different angles \( \gamma_1 \) and \( \gamma_2 \) with the optical axis, and have different wave numbers \( k_{p1} \) and \( k_{p2} \), with \( k_{pj} = n_e(\omega_p, \gamma_j) \frac{\omega_p}{c} \). The difference \( k_{p2} - k_{p1} \) depends not only on the tilt angle, but also on the transverse direction of the tilt. As we shall see in the following, the ability to tune this parameter enables the possibility to achieve the resonance associated to the 4-mode entanglement. Fig. 5 A, B and C schematically depict the different
geometries, where we associate the direction of the relative tilt between the pumps to the $x$ axis of a reference frame \{x, y, z\} which is allowed to rotate by an angle $\beta$ in the input facet of the crystal. Notice that in practice the various configurations are realized by implementing a $-\beta$ rotation of the crystal around the $z$-axis \[20\].

Then, as shown in App. A [see in particular Eqs. (A14)-(A16)], for a given transverse rotation of the crystal.

By translating these results into \{x,y,z\} geometries, where we associate the direction of the relative tilt between the pumps to the transverse rotation of the crystal.

\[
\frac{\Delta k_p}{\Delta Q_p} = \frac{k_{y2} - k_{p1}}{Q_2 - Q_1} \simeq \rho_\gamma \left( \sin \beta \cos \theta_p \frac{\sin \gamma_0}{\sin \gamma} - \sin \theta_p \frac{\cos \gamma_0}{\sin \gamma} \right) \left|_{\theta_p = \frac{\theta_{p1} + \theta_{p2}}{2}} \right.
\]

\[
\rightarrow \begin{cases} 
\pm \rho_\gamma & \text{for } \beta = \pm \frac{\pi}{2} \\
\rho_0 \left( \sin \beta - \frac{\theta_{p1} + \theta_{p2}}{2} \frac{1}{|\tan \gamma_0|} \right) & \text{for } |\beta| \ll \frac{\pi}{2}
\end{cases}
\]

where $\rho_\gamma = -\frac{1}{k_p} \frac{dk_p}{d\gamma}$ is the \textit{walk-off angle} between the wave-vector of the extraordinary wave and its Poynting vector, representing the direction of the energy flux of the wave inside the medium \[31\] \[32\]. Here it is calculated at the angle $\gamma$ formed by the carrier wave with the optical axis, but we make a small error in taking it at the cut angle $\gamma_0$, $\rho_\gamma \rightarrow \rho_0 \simeq 0.0744$ rad = 4.26$^\circ$, according to the Sellmeier relations in Ref. \[33\]. Clearly $\Delta k_p/\Delta Q_p$ is minimal in the configuration labelled as A in Fig. 6 (\(\beta = 0\)), and it is maximal in the configuration C (\(\beta = \pm \frac{\pi}{2}\)), where it coincides with the walk-off angle between the carrier wave and its Poynting vector.

A. \textbf{Shared-coupled modes and transition to resonance}

As in the former configuration of Sec [11] each pump generates its own branch of down-converted modes, laying on the phase matching surfaces $\Sigma_1$ and $\Sigma_2$ defined by Eq. (9). Examples are shown in Fig. 6 where the three columns correspond to three different rotation angles $\beta$. At difference with the PPLT case of Fig 2, we notice that now the surfaces $\Sigma_1$ and $\Sigma_2$ have quite different shapes, as discussed in Appendix A, and that their shape changes substantially with $\beta$. Actually, for the choice of parameters in this figure, the crystal rotation affects only the phase-matching surface $\Sigma_2$, which changes from non-collinear (Fig. 2a) for negative $\beta$, to non-degenerate for positive $\beta$ (Fig. 2b).

The geometrical intersections $\Sigma_1 \cap \Sigma_2$ determine the position of shared modes $\vec{w}_0 = (Q_{0x}, Q_{0y}, Q_{0z})$, each of them being coupled to the two modes $\vec{w}_b = (Q_1 - q_{0x}, -q_{0y}, -Q_{0z})$ and $\vec{w}_c = (Q_2 - q_{0x}, -q_{0y}, -Q_{0z})$. Their Fourier coordinates are determined by Eq. (8), and are studied in Appendix A [see Eqs. (A17)-(A10)]. By translating these results into propagation angles around the z axis, and neglecting infinitesimal terms $|\frac{\Delta k_p}{k_p}| \ll 1$, we find that the angular positions of shared and coupled modes are given by

\[
\theta_{0x}(\Omega) = \frac{\theta_{p1} + \theta_{p2}}{2} + \frac{\Delta k_p}{\Delta Q_p} \frac{k_x(\Omega)}{k_s(\Omega)}
\]

\[
\theta_{b,cx}(\Omega) = \frac{\theta_{p1} + \theta_{p2}}{2} \pm \frac{\theta_{p1} - \theta_{p2}}{2} \frac{k_p}{k_s(\Omega)} - \frac{\Delta k_p}{\Delta Q_p}
\]

while in the y-direction $\theta_{0y}(\Omega) = \theta_{b,cy}(\Omega)$. In comparison with the noncritical phase-matching of Sec 11 [see Eqs. (19) and (20)] we here see the presence of additional terms \(\propto \frac{\Delta k_p}{\Delta Q_p}\), which have the effect of shifting the angular positions of shared and coupled modes in opposite directions. Thus, by continuously varying this parameter, one of the side modes may arrive to superimpose to the central shared mode at the same frequency, to which it was originally uncoupled, thus becoming itself shared. As it can be easily verified, the condition $\theta_{b,c}(\Omega) = \theta_0(\Omega)$ takes place for

\[
\frac{\Delta k_p}{\Delta Q_p} \left(1 + \frac{D_0(\Omega)}{k_p} \right) \simeq \frac{\Delta k_p}{\Delta Q_p} = \begin{cases} 
\frac{\theta_{p1} - \theta_{p2}}{2} & \theta_0(\Omega) = \theta_b(\Omega) \\
\frac{\theta_{p2} - \theta_{p1}}{2} & \theta_0(\Omega) = \theta_c(\Omega)
\end{cases}
\]

where again we neglected $\frac{D_0(\Omega)}{k_p} \ll 1$. According to the results in Eq. (21), we notice that such conditions can be reached for any value of the tilt angle between the pumps smaller than the walk-off angle, by properly adjusting the transverse rotation of the crystal.

\footnote{\(\frac{D_0(\Omega)}{k_p} < 10^{-2}\) for wavelengths in the whole interval 0.43-2.1 \(\mu m\)}
We call the conditions in Eq. (25) **resonances**, because of their striking analogy with the resonance that was observed in nonlinear photonic crystals [18, 19] by tilting the direction of a single pump wave inside the nonlinear grating. As for the NPC, at resonance two triplets of modes, originally uncoupled, merge into a group of four modes, whose joint state is the quadripartite entangled state that will be described in Sec.III.B. Moreover, as demonstrated by the experiment of Ref.[20], at resonance the parametric gain of hot-spots undergoes a **Golden Ratio** enhancement, again in perfect analogy with what observed in a hexagonally poled NPC [18].

Fig.6 provides an example of the transition to resonance, for the two conjugate wavelengths $\lambda_s = 0.6\mu$m and $\lambda_i = 0.85\mu$m. In first column $\beta = 0$, and the configuration is analogue to the one studied in Sec.II: dots and the stars correspond to two independent triplets of modes, which evolve according to the 3-mode propagation equation (9), and whose state is the tripartite entangled state described in Sec.II.A. In the second and third columns $\sin(\beta) = \pm \frac{\theta_p^1 - \theta_p^2}{2\theta_0} + \frac{\theta_p^2 + \theta_p^1}{2}$, respectively, corresponding to the two resonance conditions in Eq.(25) [See also Eq. (21)]. At $\beta = -7.16^\circ$, all the shared modes merge with the left branch of coupled modes, generated by pump 1: $\hat{a}_0s, \hat{a}_0i \rightarrow \hat{b}_s, \hat{b}_i$. At $\beta = 8.98^\circ$ the merging takes place between shared modes and the right branch of coupled modes, generated by pump 2: $\hat{a}_0s, \hat{a}_0i \rightarrow \hat{c}_s, \hat{c}_i$. Remarkably, resonance is achieved simultaneously for all shared-coupled modes in a huge bandwidth around the degenerated wavelength. Indeed, even though the position of shared-coupled modes depends on the frequency, the
Thereon evolution equations read
\[
\frac{d}{dz} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \Omega} + \frac{\partial}{\partial \Omega'},
\]
where the coupling coefficients \( g_1 = \chi_1 \alpha_1 \) and \( g_2 = \chi_2 \alpha_2 \) are proportional to the pump amplitudes, but may also include a small effect due to the different nonlinear response of the medium in the two pump directions. The parameter \( D(\vec{w}_0) = D(\vec{w}_0, \vec{Q}_1) = D(\vec{w}_0, \vec{Q}_2) = D(\vec{w}_0, \vec{Q}_2 - \vec{Q}_1) \) is the common value of the phase-mismatch, that must be assumed small.

If one prefers the quantum state picture, then the evolution law of the state associated to a quadruplet of modes is easily found in the simplest case of perfect phase-matching. For \( D = 0 \), the propagation equations (28) can be recast as \( \frac{d}{dz} = \frac{1}{P} \left[ \hat{P}, \hat{O} \right] \), where \( \hat{O} = \hat{b}_s \ldots \hat{c}_i \), and the “momentum” operator is \( \hat{P} = -i\hbar \left[ g_1 \hat{b}_s^\dagger \hat{b}_i^\dagger + g_2 (\hat{b}_s^\dagger \hat{c}_i^\dagger + \hat{c}_s^\dagger \hat{b}_i^\dagger) - \text{h.c} \right] \). Then the state evolves according to
\[
|\psi\rangle_{\text{out}} = e^{\frac{i}{\hbar} \hat{P}} |\psi\rangle_{\text{in}} = e^{[g_1 \hat{b}_s^\dagger \hat{b}_i^\dagger + g_2 (\hat{b}_s^\dagger \hat{c}_i^\dagger + \hat{c}_s^\dagger \hat{b}_i^\dagger) - \text{h.c}] |\psi\rangle_{\text{in}}
\]
\[
\rightarrow |\psi\rangle_{\text{in}} + z \left[ g_1 \hat{b}_s^\dagger \hat{b}_i^\dagger + g_2 (\hat{b}_s^\dagger \hat{c}_i^\dagger + \hat{c}_s^\dagger \hat{b}_i^\dagger) \right] |\psi\rangle_{\text{in}}
\]
where in writing Eq. (29), we assumed some form of discretization of Fourier modes (details not relevant to our discussion). Eqs. (29) or (30) show the two-photon processes occurring in the quadruplet of modes: a photon pair may be down-converted from pump 1, with probability amplitude \( g_1 \propto \alpha_1 \), and appear in modes \( \hat{b}_s, \hat{b}_i \). Alternatively, paired photons may be generated from pump 2, with probability amplitude \( g_2 \propto \alpha_2 \) and appear in one of the two
couples of modes $\hat{b}_s, \hat{c}_i$ or $\hat{c}_s, \hat{b}_i$. Notice that when one of the two pumps is absent, the state reduces to the standard bipartite TMSS generated by each pump. For example, for $g_1 = 0$ the equations show the contribution of two independent couples of entangled modes over the many couples generated by down-conversion from pump 2.

As for any multipartite Gaussian entangled state [1], the quadripartite state in Eq. (29) can be decomposed into 4 single-mode squeezed states mixed by passive linear transformations. In our case, we prefer a decomposition into a pair of bipartite TMSS (each TMSS can be in turn thought of as the balanced interference of two squeezed states). This decomposition is accomplished by the following linear transformation acting separately on the signal and idler modes

$$\begin{pmatrix} \hat{b}_j \\ \hat{c}_j \end{pmatrix} = \mathbf{U} \begin{pmatrix} \hat{\sigma}_j \\ \hat{\delta}_j \end{pmatrix} \quad (j = s, i),$$

(31)

$$\mathbf{U} = \begin{pmatrix} e^{i\frac{\varphi_2}{2}} & 0 \\ 0 & e^{i\frac{\varphi_2}{2}} e^{-i\varphi_1} \end{pmatrix} \cdot \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$$

(32)

where the mixing coefficients $\cos \theta$ and $\sin \theta$ are given by Eq. (37), while a plot is provided by Fig. 7b. Under this transformation the system of Eqs. (28) decouple into two independent standard parametric processes of the form

$$\frac{d}{dz} \hat{\delta}_s(z) = \Lambda_\delta \hat{\delta}_{s}^\dagger(z)e^{-iDz}$$

$$\frac{d}{dz} \hat{\delta}_i(z) = \Lambda_\delta \hat{\delta}_{i}^\dagger(z)e^{-iDz}$$

(33)

and

$$\frac{d}{dz} \hat{\sigma}_s(z) = \Lambda_\sigma \hat{\sigma}_{s}^\dagger(z)e^{-iDz}$$

$$\frac{d}{dz} \hat{\sigma}_i(z) = \Lambda_\sigma \hat{\sigma}_{i}^\dagger(z)e^{-iDz}$$

(34)

Both the squeeze parameters $\Lambda_\sigma, \Lambda_\delta$ and the mixing coefficients of the unitary $\mathbf{U}$ depend only on the ratio

$$\rho = \frac{|g_2|}{|g_1|} \propto \frac{\alpha_2}{\alpha_1}$$

(35)

\[\text{FIG. 7. a) Squeeze eigenvalues } \Lambda_\sigma \text{ and } \Lambda_\delta \text{ in Eq. (36), normalized to the reference squeeze parameter } \bar{g} \text{ of a single pump carrying all the energy. The inset shows the maximum of } \Lambda_\sigma, \text{ occurring at } |g_2| = \sqrt{2}|g_1|. \text{ b) Mixing coefficients of the unitary transformation in Eq. (32).}\]
Under the same transformation the output state reduces to the product of two independent TMSS in modes \( \hat{\sigma}^\dagger \) and \( \hat{\delta}^\dagger \): followed by ii) a beam splitter with transmission and reflection coefficients \( \cos \theta \) and \( \sin \theta \), respectively, which mixes the two TMSS, and iii) phase rotations in the two output arms, by \( \frac{\phi_+}{2} \) and \( \frac{\phi_-}{2} - \phi_- \), respectively.

We notice that for \( \rho = 1 \), i.e. when the two pump intensities are balanced, the squeeze eigenvalues reduce to \( \Lambda_{\sigma} = |g_1| \Phi \) and \( \Lambda_{\delta} = -\frac{|g_2|}{\Phi} \) where \( \Phi = \frac{1+\sqrt 3}{2} \) is the Golden Ratio: in this case the doubly pumped PDC scheme realizes a complete analogy with the "Golden Ratio Entanglement" realized in a hexagonally poled photonic crystal [19]. In addition, the doubly pumped scheme offers the possibility of engineering the 4-mode state by varying the relative pump intensities, according to Eqs. (36) and (37), respectively. The description holds for each quadruplet of shared-coupled modes (see text).

In these formulas \( \bar{g} = \sqrt{|g_1|^2 + |g_2|^2} \) is the reference squeeze parameter, corresponding to standard PDC pumped by a single beam carrying the total energy of the two modes (apart from minor corrections arising from different nonlinear coefficients).

Under the same transformation the output state reduces to the product of two independent TMSS in modes \( \hat{\sigma}^\dagger \) and \( \hat{\delta}^\dagger \), \( |\psi\rangle_{\text{out}} \rightarrow e^{[\Lambda_{\sigma} \hat{\sigma}^\dagger \hat{\sigma} - \text{h.c.}]^j} e^{[\Lambda_{\delta} \hat{\delta}^\dagger \hat{\delta} - \text{h.c.}]^j} |\psi\rangle_{\text{in}} \). Figure 8 shows the unfolding of the state: the 4-mode entangled state generated at resonance is formally equivalent to: i) two nonlinear processes, each generating a TMSS with squeeze parameters \( \Lambda_{\sigma} \) and \( \Lambda_{\delta} \); followed by ii) a beam splitter with transmission and reflection coefficients \( \cos \theta \) and \( \sin \theta \), respectively, which mixes the two TMSS, and iii) phase rotations in the two output arms, by \( \frac{\phi_+}{2} \) and \( \frac{\phi_-}{2} - \phi_- \), respectively.

We notice that for \( \rho = 1 \), i.e. when the two pump intensities are balanced, the squeeze eigenvalues reduce to \( \Lambda_{\sigma} = |g_1| \Phi \) and \( \Lambda_{\delta} = -\frac{|g_2|}{\Phi} \) where \( \Phi = \frac{1+\sqrt 3}{2} \) is the Golden Ratio: in this case the doubly pumped PDC scheme realizes a complete analogy with the "Golden Ratio Entanglement" realized in a hexagonally poled photonic crystal [19]. In addition, the doubly pumped scheme offers the possibility of engineering the 4-mode state by varying the relative pump intensities. As shown by Fig. 8, by modulating the relative pump intensities the mixing coefficients of the unitary \( U \) can be arbitrarily varied, which means that in this resonant case the scheme is potentially able to produce any arbitrary mixing of a pair of TMSS. Conversely, the squeeze eigenvalues that characterize the two TMSS have a more limited range of variation, and in particular always remain of opposite signs, meaning that squeezing takes place in orthogonal quadratures. These findings may be compared with the case of a doubly pumped nonlinear photonic crystal analysed in Ref. [15], where a quadrupartite entangled state is also generated: in this latter case, however, the squeeze eigenvalues are controlled not only by the relative intensity but also by the relative phase of the two pumps, which allows to access a larger variety of states.

Interestingly, the positive squeeze eigenvalue is always slightly larger than \( \bar{g} \), and presents a maximum at \( |g_2| = \sqrt{2}|g_1| \), i.e. when the pump 2 is approximately twice as intense as pump 1, where \( \Lambda_{\sigma} = \frac{2}{\sqrt{3}} \bar{g} \simeq 1.15 \bar{g} \). This means that at resonance the doubly pumped scheme achieves a larger amount of squeezing/gain in the auxiliary modes \( \hat{\delta}^\dagger \) with respect to a standard single-pump scheme, at the same level of injected energy. In this way squeezing/entanglement is concentrated in specific modes.
C. The resonance and the Poynting vectors

The resonance, as we called the transition from 3 to 4-mode entanglement, admits an interesting interpretation in terms of a superposition between the Poynting vector of the pump career, representing the mean direction of propagation of the energy flux, and one of the pump modes. This interpretation is particularly evident in the configuration C of Fig[5] and is illustrated in Fig[9]. In this case, the problem becomes 2-dimensional because the pump modes share the same principal plane, which includes the optical axis $O_3$. The wave-vector of the pump career lies at an angle $\theta_{p2}+\theta_{p1}$ from the z-axis, and its Poynting vector walks-off in the principal plane by an amount $\rho_\gamma$, away from the optical axis (BBO is a negative uniaxial crystal), i.e. it forms an angle $\theta_{p}\gamma = \frac{\theta_{p2}+\theta_{p1}}{2} - \rho_\gamma$ with the z-axis. For $\beta = \frac{\pi}{2}$, the resonance conditions, described by Eq.(25) and Eq.(21), reduce to $\rho_\gamma = \frac{\theta_{p2} - \theta_{p1}}{2}$. For our choice of parameters $\theta_{p2} > \theta_{p1} > 0$, and only the lower condition can be satisfied, leading to $\theta_{S} = \theta_{p1}$. For $\beta = -\frac{\pi}{2}$ the roles of p1 and p2 are exchanged (the x axis is reversed), leading to

$$\theta_{S} = \begin{cases} \theta_{p2} & \text{for } \beta = -\frac{\pi}{2} \\ \theta_{p1} & \text{for } \beta = \frac{\pi}{2} \end{cases}$$

(38)

i.e. to the result that the resonance condition exactly corresponds to the superposition between the direction of propagations of the Poynting vector of the career and one of the pump modes.

The general case is slightly more involved, because of the full 3-dimensional geometry of the problem. It becomes quite clear when the pump tilts are symmetric, i.e. the pump career propagates along z. Then its Poynting vector points as in Fig[9] and the resonance condition becomes $\rho_0 \sin \beta = \frac{\theta_{p2} - \theta_{p1}}{2}$. For $\beta = 0$ the plane ($\vec{p}_2$, $\vec{p}_1$) is perpendicular to the plane of the figure, and there is no possibility of superposition. For $\beta \neq 0$, the transverse component of the Poynting vector in the x direction of the tilt can superimpose to one of the pump modes, allowing thus a resonance.

IV. CONCLUSIONS

This work has analysed two doubly pumped schemes of parametric down-conversion, in realistic experimental configurations, which exploit standard and commercially available nonlinear media. It has highlighted a stringent analogy with the phenomena predicted and observed in 2-dimensional nonlinear photonic crystals, by using simpler sources which do not need lengthy poling procedures, and offering in addition the possibility of reconfiguring some properties of the state by simple modulation of the classical laser beam driving the process.

In the non-critical phase-matching case of the PPLT our analytical results, complemented by numerical simulation, may constitute a proposal for future experimental implementations. In our opinion the main outcome here concerns the possibility of implementing an arbitrary beam-splitter on one of two parties of the TMSS generated by standard parametric down-conversion by acting on the spatial structure of the classical laser beam rather then on the fragile quantum state.
The BBO case has already found an experimental demonstration for what concerns the classical properties of the process [20]. For the quantum properties, the highlight result is the possibility of directly generating quadripartite entangled states, and of modulating their properties by acting on the intensities of the two pump modes. This possibility is enabled by the walk-off effects present in such an anisotropic material, in a way that is in our opinion highly nontrivial. In particular, the 4-mode entanglement can be realized at any small tilt angles between the pumps (namely provided that the tilt angle is smaller than the walk-off angle in the central direction of light propagation). We offered also an interpretation of the resonance, as we called the transition from 3- to 4-mode entanglement, in terms of a superposition between the career Poynting vector, which identifies the direction of propagation of the energy flux, with either one pump mode or the other.

Appendix A: Analytical calculations in paraxial approximation

This Appendix summarizes some analytic results, derived by using the paraxial approximation. The dependence on the frequency $\Omega$ is maintained till the very end, because we are interested in large emission bandwidths. Precisely, the $z$-component of the signal wave-vector is approximated as:

$$k_{sx}(\vec{q}, \Omega) = \sqrt{k_s^2(\Omega) - q^2} \rightarrow k_s(\Omega) - \frac{q^2}{2k_s(\Omega)}$$ (A1)

valid for $q \ll k_s(\Omega)$ (small angles around the $z$). The wave-number $k_s(\Omega)$ does not depend on the propagation direction because a) in the PPLT case the down-converted light propagates close to $z$ (and the material has a very small birefringence), and b) in the BBO case the signal is an ordinary wave. For the extraordinary pump waves:

$$k_{p_{jz}} = \sqrt{\frac{k_p^2 - Q_j^2}{k_p}} \simeq k_p - \frac{Q_j^2}{2k_p} \quad (j = 1, 2)$$ (A2)

where $k_p = n_e(\omega_p, \gamma_0)\frac{2\pi}{\lambda}$, and: a) in the PPLT case $k_{p_{jz}} = k_p$; b) in the BBO case $k_{p_{jz}} = n_e(\omega_p, \gamma_j)\frac{2\pi}{\lambda}$ depends on the angle $\gamma_j$ formed by the wave with the optical axis $O_3$. Let us consider the geometry in Fig.5 where the transverse tilt of the pump takes place along the $x$-direction, inclined at an angle $\beta$ in the input facet of the crystal. In the reference frame $(x', y', z)$ parallel to the facets of the crystal [not to be confused with the crystalline reference frame $(O_1, O_2, O_3)$], the versors associated with the direction of propagation of a generic pump wave and with the optical axis $O_3$ are respectively:

$$\vec{k_p} = \begin{pmatrix} \sin \theta_p \cos \beta \\ \sin \theta_p \sin \beta \\ \cos \theta_p \end{pmatrix}, \quad \vec{e}_3 = \begin{pmatrix} 0 \\ \sin \gamma_0 \\ \cos \gamma_0 \end{pmatrix}$$ (A3)

The angle formed by the pump with the optical axis is thus determined by

$$\cos \gamma = \frac{\vec{k}_p \cdot \vec{e}_3}{k_p} = \cos \theta_p \cos \gamma_0 + \sin \theta_p \sin \gamma_0 \sin \beta. \quad \text{(A4)}$$

For small pump tilts, the variation of $\gamma$ with $\theta_p$ is minimal for $\beta = 0$ (as in Fig.5A), where $\cos \gamma \simeq \cos \gamma_0(1 - \frac{\theta_p^2}{2})$, while it is maximal for $\beta = \pm 90^\circ$, where $\gamma = \gamma_0 \mp \theta_p$.

Phase matching surfaces

By inserting the approximated expressions (A1) and (A2) into the definition of the phase matching function in Eq. (1), and performing some long but simple algebra, the equation for the phase matching surfaces $\Sigma_1$ and $\Sigma_2$ defined in Eq. (5) can be obtained as:

$$\left| \vec{q} - \vec{Q}_j \right|^2 \left[ k_s(\Omega) + k_s(-\Omega) \right]^2 = F_j(\Omega) \quad (j = 1, 2), \quad \text{(A5)}$$

$$F_j(\Omega) = \tilde{k}(\Omega) \left[ D_0(\Omega) - (k_{p_{jz}} - k_p) + \frac{Q_j^2}{k_p} \frac{D_0(\Omega) - G_z}{k_p + D_0(\Omega) - G_z} \right] \quad \text{(A6)}$$

where $\tilde{k}(\Omega) = \frac{2k_s(\Omega)k_s(-\Omega)}{k_s^2(\Omega) + k_s(-\Omega)}$, and $D_0(\Omega) = k_s(\Omega) + k_s(-\Omega) - k_p + G_z$ is the collinear phase-mismatch function. In Eq. (A6) one must take: a) $G_z \neq 0$ and $k_{p_{jz}} - k_p = 0$ for the PPLT; b) $G_z = 0$ for the BBO (poling is absent). For the frequencies such that $F_j(\Omega) > 0$, Eq. (A5) represents a family of circumferences, centered around
$q_{jx}(\Omega) = Q_j \frac{k_{s}(\Omega)}{k_j(\Omega)+k_{s}(\Omega)} \simeq \theta_{pj} k_s(\Omega)$. Thus the angular coordinate of the center is approximately $q_{jx}(\Omega)/k_s(\Omega) \simeq \theta_{pj}$. As expected, the two emission branches are conical surfaces roughly collinear with each pump, examples being shown in figures 2 and 6. The shape of each surface depends on the value of $F_j(\Omega = 0)$ in the standard way, i.e. it is an open tube for $F_j(0) > 0$, which collapses to a "hourglass" for $F_j(0) = 0$, while it presents two separate branches for $F_j(0) < 0$. Notice that in the PPLT case the shape changes slowly with the tilt angle, so that the two phase-matching branches look very similar (see Fig. 2), while in the BBO case it has a much faster variation due to the term $k_{pj} - k_p$, so that in general $\Sigma_1$ and $\Sigma_2$ look quite different (see Fig. 6).

### Shared and coupled modes.

The Fourier coordinates of shared modes and of their coupled ones is determined by Eq. (8). By imposing the shared mode condition $\mathcal{D}(\vec{w}_0; \vec{Q}_1) = \mathcal{D}(\vec{w}_0; \vec{Q}_2)$, using again Eqs (A1) and (A2), and reordering the various terms, one obtains the following condition on the x-component of the wave-vector:

$$q_{0x}(\Omega) = \frac{Q_1 + Q_2}{2} \left( 1 - \frac{k_s(\Omega)}{k_p} \right) + \frac{\Delta k_p}{\Delta Q_p} k_s(\Omega) \tag{A7}$$

where

$$\frac{\Delta k_p}{\Delta Q_p} = \frac{k_{p2} - k_{p1}}{Q_2 - Q_1} \tag{A8}$$

measures the rate of variation of the pump wave-numbers with their transverse tilts. Such a term is absent in the PPLT scheme, but plays a crucial role in the BBO case because of the strong birefringence of the material. The y-component of the wave-vector is obtained by requiring that phase matching is satisfied, i.e. that $\mathcal{D}(\vec{w}_0; \vec{Q}_1) = \mathcal{D}(\vec{w}_0; \vec{Q}_2) = 0$. Using Eq. (A5), one has

$$q_{0y}(\Omega) = \pm \sqrt{F_j(\Omega) - [q_{0x} - q_{jx}^c(\Omega)]^2} \tag{A9}$$

for $F_j(\Omega) - [q_{0x} - q_{jx}^c(\Omega)]^2 \geq 0$, i.e. provided that the intersection between $\Sigma_1$ and $\Sigma_2$ exists. The $\pm$ signs correspond to the two possible intersection points of two circumferences.

The modes coupled to each shared mode have equation $\vec{q}_0(\Omega) = \vec{Q}_1 - \vec{q}_0(\Omega)$ (via pump 1) and $\vec{q}_c(\Omega) = \vec{Q}_2 - \vec{q}_0(\Omega)$ (via pump 2). At a given frequency $\Omega$, their transverse coordinates are:

$$q_{bc,x}(\Omega) = Q_{1,2} - \frac{Q_1 + Q_2}{2} \left( 1 - \frac{k_s(\Omega)}{k_p} \right) - \frac{\Delta k_p}{\Delta Q_p} k_s(\Omega) \tag{A10}$$

$$q_{bc,y}(\Omega) = -q_{0y}(\Omega) = \pm q_{0y}(\Omega)$$

where the last equality follows from the symmetry of equations (A9) and (A6) with respect to the exchange $\Omega \rightarrow -\Omega$.

### The resonance.

We use here the resonance condition in Eq. (26) $\vec{q}_0(\Omega) + \vec{q}_0(\Omega) = \vec{Q}_{1,2}$. This equation can be always satisfied for the y-coordinate, since $F_j(\Omega)$ in Eq. (A9) is an even function of $\Omega$, so that one can choose $q_{0y}(\Omega) = -q_{0y}(\Omega)$. For the x-coordinate, using Eq. (A7), it requires that

$$Q_1 + Q_2 + [k_s(\Omega) + k_s(\Omega)] \left( \frac{\Delta k_p}{\Delta Q_p} - \frac{Q_1 + Q_2}{2k_p} \right) = Q_{1,2} \tag{A11}$$

$$\Delta k_p = \frac{\theta_{p1} + \theta_{p2}}{2} - \frac{\theta_{p2,p1} k_p}{k_s(\Omega) + k_s(\Omega)}$$

$$= \pm \frac{\theta_{p1} - \theta_{p2}}{2} + \frac{\theta_{p2,p1} D_0(\Omega) - G_z}{k_p + D_0(\Omega) - G_z} \tag{A12}$$

where, as usual, we approximated $\theta_{pj} \simeq \frac{Q_j}{k_j}$, and we used the identity $k_p = k_s(\Omega) + k_s(\Omega) - D_0(\Omega) + G_z$. First of all, we notice that the second term at r.h.s. of Eq. (A12) is a very small correction, because $|G_z - D_0(\Omega)| \ll k_p$. Thus, Eq. (A12) cannot be satisfied when $\Delta k_p = 0$ because it would require $|\theta_{p1} - \theta_{p2}| \ll |\theta_{p2,p1}|$ (in practice that the pump modes are collinear). Therefore, the resonance cannot take place in the PPLT configuration considered in Sec.II, and from now on we focus on the BBO case only, setting $G_z = 0$. 

18
We notice that in principle the r.h.s. of Eq. (A12) depends on the frequency. The only exception is when one of the pumps is not tilted, e.g. \( \theta_{p1} = 0 \). Then, by requiring that shared modes are generated by the other one, i.e. that \( q_{0x}(\Omega) + q_{0x}(-\Omega) = Q_2 \), for \( \Delta k_p/\Delta Q_p = \frac{\theta_{p1} + \theta_{p2}}{2} \) the resonance takes place simultaneously at all the frequencies. However, even when this "magic" configuration is not considered, the bandwidth of modes that enter into resonance is so huge that can be practically considered infinite. We assume that Eq. (A12) is satisfied at degeneracy where \( D_0(0) = 0 \), i.e. that

\[
\frac{\Delta k_p}{\Delta Q_p} = \left( \frac{\Delta k_p}{\Delta Q_p} \right)_{res} = \pm \frac{\theta_{p1} - \theta_{p2}}{2} \tag{A13}
\]

Then, at a frequency \( \Omega \neq 0 \) the relative correction to the resonance condition in Eq. (A12) is on the order \( \frac{D_0(\Omega)}{k_p} \approx 1/k_p k_s' \Omega^2 = \frac{\Omega^2}{\Omega_B} \), where \( \Omega_B = \sqrt{k_p k_s} \approx 2 \times 10^{16} \text{s}^{-1} \). Thus, for any practical purpose, condition (A13) can be taken as the resonance condition.

A further insight into the problem is gained by approximating the incremental ratio in Eq. (A8) with its Taylor expansion. It turns out that the lowest order approximation is not precise enough, therefore we choose to expand each \( k_{pj} \) around the middle point \( Q_p = \frac{Q_{1+Q_2}}{2} \) as \( k_{p2,1} = k_p(Q_p) \pm \frac{dk_p}{dQ_p} \frac{\Delta Q_p}{2} + \frac{1}{2} \frac{d^2k_p}{dQ_p^2} \Delta Q_p^2 + O(\Delta Q_p^3) \). In this way, \( \Delta k_p/\Delta Q_p = \frac{dk_p}{dQ_p} \bigg|_{Q_p} + O(\Delta Q_p^2) \). Therefore, up to first order in \( \Delta Q_p \) one has

\[
\frac{\Delta k_p}{\Delta Q_p} \approx \frac{dk_p}{dQ} \bigg|_{Q_p} = \frac{1}{k_p \rho_p} \frac{dQ}{d\theta} \bigg|_{\theta} = \frac{1}{k_p \rho_p} \frac{dQ}{d\gamma} \bigg|_{\theta} = \frac{dQ}{d\gamma} \bigg|_{\theta} \tag{A14}
\]

where \( \rho_p = \frac{\theta_{p1} + \theta_{p2}}{2} \), and we remind that \( \gamma \) is the angle formed by the pump propagation direction with the optical axis. In this expression we recognize that the quantity \( \frac{1}{k_p} \frac{dQ}{d\gamma} = -\rho_p \) is the \textit{walk-off angle} formed by the wave-vector of the extraordinary wave and its Poynting vector, representing the direction of the energy flux. \[31\] It depends on the angle \( \gamma \), but we make a small error in taking it at the cut angle \( \gamma_0 \). \( \rho_{\gamma} \rightarrow \rho_0 \approx 0.0744 \text{ radians} = 4.26^\circ \). Thus, with a precision up to first order in the small quantities the following expression holds:

\[
\frac{\Delta k_p}{\Delta Q_p} = -\rho_p \frac{d\gamma}{d\theta_p} \bigg|_{\theta} \tag{A15}
\]

On the other side, the functional dependence of the angle \( \gamma \) on the tilt angle \( \theta_p \) is provided by Eq. (A4). By differentiating this expression with respect to \( \theta_p \), one gets

\[
\frac{d\gamma}{d\theta_p} = -\sin \beta \cos \theta_p \frac{\sin \gamma_0}{\sin \gamma} + \sin \theta_p \cos \gamma_0 \frac{\cos \gamma}{\sin \gamma} \rightarrow \begin{cases} \frac{\pm 1}{\sin \beta + \sin \theta_p \frac{1}{\tan \gamma_0}} & \text{for } \beta = \pm \frac{\pi}{2} \\ -\sin \beta + \sin \theta_p \frac{1}{\tan \gamma_0} & \text{for } |\beta| \ll \frac{\pi}{2} \end{cases} \tag{A16}
\]

The resonance condition of Eq. (A13) can then be reformulated in terms of the tilt angles of the two pumps as

\[
\pm \frac{\theta_{p1} - \theta_{p2}}{2} = \rho_\gamma \left( \sin \beta \cos \theta_p \frac{\sin \gamma_0}{\sin \gamma} - \sin \theta_p \cos \gamma_0 \frac{\cos \gamma}{\sin \gamma} \right) \bigg|_{\theta_p = \hat{\theta}_p} \tag{A17}
\]

\[
\simeq \begin{cases} \frac{\pm \rho_0}{\sin \beta} & \beta = \pm \frac{\pi}{2} \\ -\rho_0 \frac{\sin \beta - \frac{\theta_{p1} + \theta_{p2}}{2}}{1/\tan \gamma_0} & |\beta| \ll \frac{\pi}{2} \end{cases} \tag{A18}
\]

This condition can be understood as a requirement on the pump tilt angles, for a fixed angle of rotation \( \beta \) of the crystal, or viceversa, for given pump tilts \( \theta_{p1}, \theta_{p2} \) as a receipt for the angle of rotation of the crystal at which resonance takes place.

\[
\sin(\beta_{res}) = \frac{\pm \theta_{p1} - \theta_{p2}}{2 \rho_0} + \theta_{p2} + \theta_{p1} \frac{1}{2 \tan \gamma_0} \tag{A19}
\]

[1] H. J. Briegel and R. Raussendorf, \textit{Phys. Rev. Lett.} \textbf{86}, 910 (2001)
[2] J. Zhang and S. L. Braunstein, Phys. Rev. A 73, 032318 (2006).
[3] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. 86, 5188 (2001).
[4] N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, Phys. Rev. Lett. 97, 110501 (2006).
[5] R. Demkowicz-Dobrzaski, M. Jarzyna, and J. Koodyski (Elsevier, 2015) pp. 345 – 435.
[6] P. A. Knott, New Journal of Physics 18, 073033 (2016).
[7] P. van Loock, C. Weedbrook, and M. Gu, Phys. Rev. A 76, 032321 (2007).
[8] S. Yukeyama, R. Ukai, S. Armstrong, C. Sornphiphatphong, T. Kaji, S. Suzuki, J.-I. Yoshikawa, H. Yonezawa, N. Menicucci, and A. Furusawa, Nature Photonics 7, 982 (2013) cited By 240.
[9] C. Navarrete-Benlloch, R. Garca-Patrn, J. Shapiro, and N. Cerf, Physical Review A - Atomic, Molecular, and Optical Physics 86 (2012), 10.1103/PhysRevA.86.012328 cited By 93.
[10] D. Daems, F. Bernard, N. Cerf, and M. Kolobov, Journal of the Optical Society of America B 27 (2010), 10.1364/JOSAB.27.000447.
[11] R. Menzel, A. Heuer, D. Puhlmann, K. Dechoum, M. Hillery, M. Sphin, and W. Schleich, Journal of Modern Optics 60, 86 (2013) https://doi.org/10.1080/09500340.2012.746400.
[12] E. Brambilla and A. Gatti, Opt. Express 27, 30233 (2019).
[13] A. Gatti, Phys. Rev. A 101, 053841 (2020).
[14] H. Wang, C. Fabre, and J. Jing, Phys. Rev. A 95, 051802 (2017).
[15] S. Liu, Y. Lou, and J. Jing, Opt. Express 27, 37999 (2019).
[16] O. Jedrkiewicz, A. Gatti, E. Brambilla, M. Levenius, G. Tamoauskas, and K. Gallo, Sci. Rep. 8 (2018), 10.1038/s41598-018-30014-7.
[17] A. Gatti, E. Brambilla, K. Gallo, and O. Jedrkiewicz, Physical Review A 98 (2018), 10.1103/PhysRevA.98.053827.
[18] A. Gatti, E. Invernizzi, E. Brambilla, and A. Gatti, “Hot-spots and gain enhancement in a doubly pumped parametric down-conversion process,” (2020), arXiv:2007.12429 [quant-ph].
[19] A. Gatti, R. Zambrini, M. San Miguel, and L. A. Lugiato, Phys. Rev. A 68, 053807 (2003).
[20] E. Brambilla, O. Jedrkiewicz, L. A. Lugiato, and A. Gatti, Phys. Rev. A 85, 063834 (2012).
[21] A. Gatti, H. Wiedemann, L. A. Lugiato, I. Marzoli, G.-L. Oppo, and S. M. Barnett, Phys. Rev. A 56, 877 (1997).
[22] V. Berger, Phys. Rev. Lett. 81, 4136 (1998).
[23] N. G. R. Broderick, G. W. Ross, H. L. Offerhaus, D. J. Richardson, and D. C. Hanna, Phys. Rev. Lett. 84, 4345 (2000).
[24] A. Arie, N. Habshoosh, and A. Bahabad, Opt. Quant. Electron. 39, 361 (2007).
[25] I. Dolev, A. Ganany-Padowicz, O. Gayer, A. Arie, J. Mangin, and G. Gadret, Applied Physics B 96, 423 (2009).
[26] C. C. Gerry and P. L. Knight, Introductory Quantum Optics (Cambridge University Press, 2005) Chap. 7, pp. 167–169,182–187.
[27] M. Levenius, V. Pasiskevicius, and K. Gallo, Appl. Phys. Lett. 101, 121114 (2012).
[28] L. Chen, P. Xu, Y. F. Bai, X. W. Luo, M. L. Zhong, M. Dai, M. H. Lu, and S. N. Zhu, Opt. Express 22, 13164 (2014).
[29] M. Born, E. Wolf, A. B. Bhatia, P. C. Clemmow, D. Gabor, A. R. Stokes, A. M. Taylor, P. A. Wayman, and W. L. Wilcock, “Optics of crystals,” in Principles of Optics: Electromagnetic Theory of Propagation, Interference and Diffraction of Light (Cambridge University Press, 1999) p. 790852, 7th ed.
[30] N. Boeuf, D. A. Branning, I. Chaperot, E. Dauler, S. Guerin, G. S. Jaeger, A. Muller, and A. L. Migdall, Optical Engineering 39, 1016 (2000).
[31] K. Kato, IEEE Journal of Quantum Electronics 22, 1013 (1986).