Static axisymmetric spacetimes with non-generic world-line SUSY

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ABSTRACT

The conditions for the existence of Killing-Yano tensors, which are closely related to the appearance of non-generic world-line SUSY, are presented for static axisymmetric spacetimes. Imposing the vacuum Einstein equation, the set of solutions admitting Killing-Yano tensors is considered. In particular, it is shown that static, axisymmetric and asymptotically flat vacuum solutions admitting Killing-Yano tensors are only the Schwarzschild solution.

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One of the most remarkable properties of the Kerr black hole is that, in this background, particle motion is completely integrable. From the point of view of canonical analysis, this is a direct consequence of the existence of a non-trivial Killing tensor $K_{\mu\nu}[1-3]$, which gives rise to the associated constant of motion, $K^{\mu\nu}p_\mu p_\nu$, quadratic in the four-momentum $p_\mu$. Namely, this constant of motion completes the maximal number of constants of motion in conjunction with the other three well-known constants of motion: the energy coming from the time translation invariance of the Kerr geometry, the angular momentum coming from the axial symmetry of it, and the Hamiltonian. More surprisingly, various field equations, particularly the Dirac equation [4], separate in the Kerr geometry, and this fact is related to the existence of the Killing-Yano tensor $f_{\mu\nu}$, which is defined as an antisymmetric second rank tensor satisfying the following Penrose-Floyd equation [5]:

$$\{\mu\nu\lambda\} \equiv 2D_{(\mu}f_{\nu)\lambda} = 0. \tag{1}$$

Here, $D_\mu$ represents the usual covariant derivative and, for later convenience, we defined the braces. Contracting repeated Killing-Yano tensors, we obtain the Killing tensor associated with the Killing-Yano tensor: $K_{\mu\nu} = f_{\mu}{}^{\lambda}f_{\lambda\nu}$, $D_{(\mu}K_{\nu)\lambda} = 0$. So, we might say that the Killing-Yano tensor is a square root of the Killing tensor.

Recently, it has been shown by considering supersymmetric particle mechanics that the Killing-Yano tensor can be understood as an object belonging to a larger class of possible structures which generate generalized supersymmetry algebras [7]. This novel aspect has renewed people’s interest in the Killing-Yano tensor which has long been known to relativists.

It may be of some significance to ask conversely what kind of background gravitational field admits the Killing-Yano tensor. Are there any spacetime admitting a Killing-Yano tensor other than the Kerr spacetimes? While we already know that the existence of Killing-Yano tensor is directly related to the appearance of non-generic supersymmetry for the motion of spinning particle, such an investigation will give another facet of the structural properties of spacetimes.

A formal answer to this question has been already given by Dietz and Rüdiger [8], where all metrics admitting Killing-Yano tensors were presented with no further assumptions restricting, say, the Ricci tensor. However, interpretation of those metrics seems far from trivial. In particular, the relation between the existence of Killing-Yano tensor and
isometries is not known.

In this paper, we take another approach to the same problem by restricting from the beginning the metric to be static and axisymmetric. Using the prolate spheroidal coordinates \((x, y, \phi)\), such metrics can be written in the form \[d s^2 = -e^{2\psi} d t^2 + \sigma^2 e^{-2\psi} \left[ e^{2\gamma} \left( x^2 - y^2 \right) \left( \frac{d x^2}{x^2 - 1} + \frac{d y^2}{1 - y^2} \right) + \left( x^2 - 1 \right) \left( 1 - y^2 \right) d \phi^2 \right], \tag{2}\]

where

\[\psi = \psi(x, y), \quad \gamma = \gamma(x, y). \tag{3}\]

Note that the parameter \(\sigma\) can absorb possible constants added to \(\psi\) and \(\gamma\), so that the additional constants can always be taken zero without loss of generality. We will first describe, in terms of \(\psi(x, y)\) and \(\gamma(x, y)\), the conditions for the existence of Killing-Yano tensor. Then we will see that the function \(\psi(x, y)\) which satisfies the vacuum Einstein equation and admits a Killing-Yano tensor is determined by its derivatives, \(\psi_x(p)\) and \(\psi_y(p)\), at an arbitrary point \(p = (x_0, y_0)\) (see \(\ref{eq:psi_y} \)). Since at that time \(\gamma(x, y)\) is determined in terms of \(\psi(x, y)\) through the Einstein equation, the static axisymmetric vacuum solutions admitting Killing-Yano tensors form a three dimensional set parametrized by \(\psi_x(p), \psi_y(p)\) and \(\sigma\). However, when contracting this set by all possible diffeomorphisms, we obtain, as the set of distinct solutions, a complicated one dimensional set. We then further restrict interest to asymptotically flat cases, and finally prove uniqueness of the Schwarzschild solution.

Before studying eq. \(\ref{eq:metric} \) in terms of the metric \(\ref{eq:metric} \), we comment upon three properties of the metric \(\ref{eq:metric} \), which will be frequently used later.

First, we can immediately observe that the metric \(\ref{eq:metric} \) is manifestly invariant under the interchange \(x \leftrightarrow y\). This implies, for example, that if there is a correct expression for \(\psi\) and/or \(\gamma\), the expression obtained by interchanging \(x \leftrightarrow y\) with \(\psi_x \leftrightarrow \psi_y, \gamma_x \leftrightarrow \gamma_y, \psi_{xx} \leftrightarrow \psi_{yy}, \) etc. will also be a correct expression. We call this interchange the “conjugation”.

Second, the following formal transformation

\[(t, \phi) \rightarrow (i\sigma \phi, -i\sigma^{-1} t) \tag{4}\]

preserves the characteristic form of the metric \(\ref{eq:metric} \), i.e., the metric obtained after the transformation is still designated by the (appropriately transformed) set of two functions
and one parameter, \((\psi(x, y), \gamma(x, y), \sigma)\). In fact, this transformation is equivalent to the following transformation for \(\psi\) and \(\gamma\):

\[
(\psi, \gamma, \sigma) \rightarrow (\psi, \gamma, \sigma)' = (\psi', \gamma', \sigma),
\]

\[
\psi' \equiv -\psi + \frac{1}{2} \log \left( x^2 - 1 \right) \left( 1 - y^2 \right),
\]

\[
\gamma' \equiv \gamma - 2\psi + \frac{1}{2} \log \left( x^2 - 1 \right) \left( 1 - y^2 \right).
\]

(5)

We call this transformation the \(t-\phi\) transformation. It is obvious from (4) that if \((\psi, \gamma, \sigma)\) is a solution of Einstein’s equation, then \((\psi', \gamma', \sigma)\) is also a solution, which generally represents another spacetime. This also implies that Einstein’s equation is invariant under the \(t-\phi\) transformation (5). Note that these transformations form a group isomorphic to \(\mathbb{Z}_2\), i.e. \((\psi, \gamma, \sigma)' = (\psi, \gamma, \sigma)\).

Finally, there exists a 2-parameter diffeomorphism which also preserves the characteristic form of (2):

\[
(x, y) \rightarrow (x', y'),
\]

\[
x' \equiv \frac{1}{2} \left( \sqrt{s^2(x^2 + y^2) + 2s(a + 1)xy + (a + 1)^2 - s^2} \right.
\]

\[
+ \sqrt{s^2(x^2 + y^2) + 2s(a - 1)xy + (a - 1)^2 - s^2} \right),
\]

\[
y' \equiv \frac{1}{2} \left( \sqrt{s^2(x^2 + y^2) + 2s(a + 1)xy + (a + 1)^2 - s^2} \right.
\]

\[
- \sqrt{s^2(x^2 + y^2) + 2s(a - 1)xy + (a - 1)^2 - s^2} \right),
\]

(6)

where \(a\) and \(s\) are arbitrary parameters. Under this transformation, the metric (2) simply transforms as

\[
(\psi(x, y), \gamma(x, y), \sigma) \rightarrow (\psi(x', y'), \gamma(x', y'), s\sigma).
\]

(7)

We call these diffeomorphisms the characteristic-form-preserving diffeomorphisms (CPDs) \(\dagger\).

\(\dagger\) The geometrical meaning of the CPDs becomes clear if one views them in the cylindrical coordinates defined by \((\rho, z, \phi) = (\sigma(x^2 - 1)^{1/2}(1 - y^2)^{1/2}, \sigma xy, \phi)\). In these coordinates, the spatial part of the metric (2) takes the form \(e^{-2\psi}[e^{2\gamma}(d\rho^2 + dz^2) + \rho^2d\phi^2]\), and the CPDs correspond to the diffeomorphisms which map \((\rho, z) \rightarrow (s\rho, sz + a)\). Therefore the parameter \(a\) is considered to represent the translation along the axis of symmetry, while \(s\) represents the similarity transformation.
We are now in a position to study static axisymmetric spacetimes which admit Killing-Yano tensors. Our strategy is quite straightforward—we simply write down all the components of the equation \( \{ \mu \nu \lambda \} \equiv D(\mu f_{\nu \lambda}) = 0 \) explicitly. So, it would be worth noticing how many independent components of eq. (1) exist. For this purpose, note first that, for the braces \( \{ \mu \nu \lambda \} \) defined in (1), we can read the following symmetries:

\[
\{ \mu \mu \mu \} = 0 \quad \text{(all indices coincide)},
\]

\[
\{ \mu \mu \nu \} = - \{ \nu \mu \mu \} = - \{ \mu \nu \mu \} \quad \text{(repeated indices exist)},
\]

\[
\{ \mu \nu \lambda \} = \{ \nu \mu \lambda \} \quad \text{(no repeated indices exist)}.
\]

Now, it is easy to see that, for the braces of the types both (9) and (10), there are twelve independent components. Thus, the total number of independent ones of eq. (1) is 24, while that of \( f_{\mu \nu} \) is six.

By explicit calculations of eq. (1), we can see that the equations for components \( \{ 331 \} \), \( \{ 332 \} \), \( \{ 001 \} \) and \( \{ 002 \} \) immediately imply that \( f_{12} \) and \( f_{21} \) must vanish, and therefore the equations for components \( \{ 112 \} \) and \( \{ 221 \} \) are identically satisfied. Similarly, the equations for components \( \{ 013 \}, \{ 310 \}, \{ 023 \}, \{ 320 \}, \{ 301 \}, \) and \( \{ 302 \} \) demand that \( f_{30} \) and \( f_{03} \) vanish.

The remaining twelve components of eq. (1) are found to be divided into two sets, \( \{ 113 \}, \{ 312 \}, \{ 223 \}, \{ 123 \}, \{ 231 \} \) and \( \{ 003 \} \) which contain \( f_{13}(= -f_{31}) \) and \( f_{23}(= -f_{32}) \) only, and \( \{ 110 \}, \{ 012 \}, \{ 220 \}, \{ 120 \}, \{ 201 \} \) and \( \{ 330 \} \) which contain \( f_{10}(= -f_{01}) \) and \( f_{20}(= -f_{02}) \) only. For convenience we label these two sets “T” and “R” respectively. It is easy to check that T and R interchange under the \( t-\phi \) transformation (5) with \( f_{13} \leftrightarrow f_{10} \) and \( f_{23} \leftrightarrow f_{20} \). This means that, if we solve T completely, the solution of R is automatically given. For this reason, we shall mainly consider T, since it is simpler than R.

Equation \( \{ 003 \} = 2D_0 f_{03} = 0 \) obviously contains no derivatives of \( f_{\mu \nu} \) due to the static nature of the spacetime. Its explicit form is

\[4\] As we shall see later, T and R are characterized by “constraints”, which originate from, respectively, the “Time” translation invariance and the “Rotational” invariance of our metric.
The other equations in T contain derivatives with respect to $x$ and/or $y$;

\{113\} = 0 \iff

\begin{align*}
f_{13,x} = & \left( \frac{x}{x^2 - y^2} - 2 \psi_x + \gamma_x \right) f_{13} + \frac{1 - y^2}{x^2 - 1} \left( \frac{y}{x^2 - y^2} + \psi_y - \gamma_y \right) f_{23} \\
(12) \end{align*}

\{231\} = 0 \iff

\begin{align*}
f_{13,y} = & - \left( \frac{2y}{1 - y^2} + \frac{y}{x^2 - y^2} + 3 \psi_y - \gamma_y \right) f_{13} - \left( \frac{x}{x^2 - 1} - \frac{x}{x^2 - y^2} - \gamma_x \right) f_{23} \\
(13) \end{align*}

\{312\} = 0 \iff

\begin{align*}
f_{23,x} = & \left( \frac{y}{1 - y^2} - \frac{y}{x^2 - y^2} + \gamma_y \right) f_{13} + \left( \frac{2x}{x^2 - 1} + \frac{x}{x^2 - y^2} - 3 \psi_x + \gamma_x \right) f_{23} \\
(14) \end{align*}

\{223\} = 0 \iff

\begin{align*}
f_{23,y} = & - \frac{x^2 - 1}{1 - y^2} \left( \frac{x}{x^2 - y^2} - \psi_x + \gamma_x \right) f_{13} - \left( \frac{y}{x^2 - y^2} + 2 \psi_y - \gamma_y \right) f_{23}. \\
(15) \end{align*}

The remaining equation \{123\} = 0 becomes trivial if eqs. (13) and (14) are satisfied. Note that T is a set of simultaneous first order partial differential equations with a "constraint" (11). For consistency, the derivative of (11) with respect to $x$ must vanish. This yields

\begin{align*}
\Delta \gamma_y = & \left( x^2 - 1 \right) \left[ \psi_x \psi_{xy} - \psi_y \psi_{xx} + \frac{y}{x^2 - y^2} \left( \psi_x^2 - \psi_y^2 \right) \right] \\
(16) \end{align*}

with

\begin{align*}
\Delta \equiv & \left( x^2 - 1 \right) \psi_x^2 + \left( 1 - y^2 \right) \psi_y^2. \\
(17) \end{align*}

The derivative of (11) with respect to $y$ yields the conjugation of (16), which reads

\begin{align*}
\Delta \gamma_x = & \left( 1 - y^2 \right) \left[ \psi_y \psi_{xy} - \psi_x \psi_{yy} + \frac{x}{x^2 - y^2} \left( \psi_x^2 - \psi_y^2 \right) \right]. \\
(18) \end{align*}

We have also to impose the integrability condition for $\gamma$, which gives rise to a non-trivial condition:

\begin{align*}
\frac{\partial}{\partial x} \left\{ \text{(the r.h.s. of (16))}/\Delta \right\} = \frac{\partial}{\partial y} \left\{ \text{(the r.h.s. of (18))}/\Delta \right\}. \\
(19) \end{align*}

Moreover, the integrability of $f_{13}$ must also be imposed. Using (11) and (18) to eliminate derivatives of $\gamma$, this implies
\[2\psi_x^2\psi_y + \psi_{xx}\psi_y \left[ x \left( x^2 - 1 \right) \psi_x^2 - x \left( 1 - y^2 \right) \psi_y^2 - 2y \left( x^2 - 1 \right) \psi_x\psi_y \right] - \psi_{xy} \left[ x \left( x^2 - 1 \right) \psi_x^3 - 3y \left( x^2 - 1 \right) \psi_x\psi_y \right] = \text{the conjugation of the l.h.s.} \tag{20}\]

Here, we have supposed \(\psi_x \neq 0, \psi_y \neq 0\). When imposing (20), we can see that the integrability of \(f_{23}\) is trivially satisfied.

Now, we can summarize the results of the above calculations as follows. Once given a metric of the form (2), one can examine through (16) \sim (20) whether it admits non-trivial components \(f_{13} = -f_{31}\) and \(f_{23} = -f_{32}\) of a Killing-Yano tensor. Moreover, transforming the metric by (5) and reexamining (16) \sim (20) for the new metric \(\S\), one can also know whether the original metric admits non-trivial components \(f_{10} = -f_{01}\) and \(f_{20} = -f_{02}\). As regards the other components of \(f_{\mu\nu}\), they always vanish.

In order to get more definite consequences, we here impose the vacuum Einstein equation given by

\[
\gamma_x = \frac{1 - y^2}{x^2 - y^2} \left[ x \left( x^2 - 1 \right) \psi_x^2 - x \left( 1 - y^2 \right) \psi_y^2 - 2y \left( x^2 - 1 \right) \psi_x\psi_y \right], \tag{21}\]

\[
\gamma_y = \frac{x^2 - 1}{x^2 - y^2} \left[ y \left( x^2 - 1 \right) \psi_x^2 - y \left( 1 - y^2 \right) \psi_y^2 + 2x \left( 1 - y^2 \right) \psi_x\psi_y \right], \tag{22}\]

\[
0 = 2x\psi_x + \left( x^2 - 1 \right) \psi_{xx} - 2y\psi_y + \left( 1 - y^2 \right) \psi_{yy}. \tag{23}\]

The integrability of \(\gamma\) in the Einstein equation is guaranteed by eq. (24), so that we do not have to consider another integrability condition (19). Moreover, by a direct calculation eq. (20) is found to be trivial if eqs. (16), (18) and (21) \sim (23) are satisfied. As a result, it turns out that, for the present purpose, we need only eqs. (16) and (18) over the vacuum Einstein equation (21) \sim (23).

Eliminating \(\gamma_x\) and \(\gamma_y\) in (13) and (18) by substituting (21) and (22), we obtain three equations for \(\psi_x, \psi_y, \psi_{xx}, \psi_{yy}, \text{and} \psi_{xy}\), and then we can algebraically solve these equations for \(\psi_{xx}, \psi_{yy}, \text{and} \psi_{xy}\) in terms of \(\psi_x\) and \(\psi_y\) (with \(x\) and \(y\));

\[
\psi_{xx} = F(\psi_x, \psi_y, x, y),
\psi_{yy} = F(\psi_y, \psi_x, y, x),
\psi_{xy} = G(\psi_x, \psi_y, x, y), \tag{24}\]

\(^5\)This operation is equivalent to transforming eqs. (13) \sim (21) by (5) and reexamining them for the original metric.
where $F$ and $G$ are functions with four arguments. Although we do not write down these lengthy expressions in explicit forms, it is evident that they do always exist (except for $\psi_x = \psi_y = 0$) and are unique because the three equations are linear in $\psi_{xx}, \psi_{yy},$ and $\psi_{xy}$.

Equation (24) implies that, given the first order derivatives $\psi_x$ and $\psi_y$ at an arbitrary point $p$, higher derivatives of $\psi$ at $p$ are all given. Thus, $\psi(x,y)$ and accordingly $\gamma(x,y)$ are determined at least in the neighbourhood of $p$ in terms of two parameters $\psi_x(p)$ and $\psi_y(p)$. This implies that the static axisymmetric vacuum solutions admitting Killing-Yano tensors form a three dimensional set parametrized by $\psi_x(p), \psi_y(p),$ and $\sigma$, which we denote as $Y_p$. We must, however, recall the existence of two dimensional diffeomorphisms (CPDs), which contract $Y_p$ and yield a connected one dimensional set, $\overline{Y}_p$, as the set of distinct solutions.

Although we cannot find all solutions belonging to $\overline{Y}_p$, some features of $\overline{Y}_p$ can be seen as follows. First, one can confirm by direct substitutions into (16), (18), (21) and (22) that the Schwarzschild solution

$$\psi_{\text{Sch}} = \frac{1}{2} \log \frac{x-1}{x+1}$$

(25)
certainly admits a Killing-Yano tensor. We can further look for similar solutions in the form

$$\psi = \log(x+1)^{\xi_1}(x-1)^{\xi_2}(1+y)^{\eta_1}(1-y)^{\eta_2},$$

(26)
where, $\xi_1, \xi_2, \eta_1$ and $\eta_2$ are parameters which satisfies $\xi_1 + \xi_2 = \eta_1 + \eta_2$ resulting from the vacuum Einstein equation (23). For example, we find

$$\psi = -\frac{1}{2} \log \left(x^2 - 1 \right) (1 - y^2),$$
$$\psi = \frac{1}{2} \log \frac{(x-1)(1+y)}{(x+1)(1-y)}, \quad \text{etc.}$$

(27)
(and also their conjugations) as solutions admitting Killing-Yano tensors. Now, note that eqs. (23) and (27) give rise to representative metrics in $Y_p$ parametrized by $\sigma$ of which gauge orbits are generated by the CPDs. One can investigate by the aid of computers what portions of $Y_p$ the orbits share, and may find that the structure of $\overline{Y}_p$ is very complicated. In particular, it seems that $\overline{Y}_p$ has infinitely many branches. However, we do not go into further details on this digressive subject.

Our next step is to select out out of $\overline{Y}_p$ all asymptotically flat solutions admitting Killing-Yano tensors. To this end, we expand $\psi$ at spatial infinity ($x \to \infty$) as
\[ \psi = -\frac{1}{x} + \frac{f_3(y)}{x^3} + \frac{f_4(y)}{x^4} + \cdots, \]  
(28)

where the first term of the r.h.s. has been taken to coincide with that of \( \psi_{\text{Sch}} \). This can be always done by choosing the parameter \( s \) in the CPDs appropriately. The second order term in the expansion, which corresponds to the dipole component of the system, has been taken zero by using the remaining freedom of the CPDs parametrized by \( a \). Having restricted the form of the solutions to (28), \( \psi \) no longer has continuous freedom, because the remaining 1-parameter freedom of the metric is to be carried by the spatial conformal parameter \( \sigma \). Uniqueness of the expression (24) suggests that the Schwarzschild solution (25) is the unique solution of the form (28). In fact, this is confirmed by directly solving the expanded equations of (24). This fact implies more precisely that the solutions of the form (28) admitting non-trivial components \( f_{13} = -f_{31} \) and \( f_{23} = -f_{32} \) of Killing-Yano tensors are only the Schwarzschild solutions parametrized by \( \sigma \).

What we have to do next is to examine if there are asymptotically flat solutions which admit non-trivial components \( f_{10} = -f_{01} \) and \( f_{20} = -f_{02} \) of Killing-Yano tensors. However, it is found as follows that no such solutions exist. Namely, if there were such an asymptotically flat solution, the asymptotic form of the Schwarzschild solution \( \psi \to -1/x \) should satisfy the asymptotic equations of the \( t-\phi \) transformed version of equations (24). However, this is not the case, and we conclude that the claim is true.

Gathering all results, we have proved that, modulo diffeomorphisms, static, axisymmetric, and asymptotically flat vacuum solutions admitting Killing-Yano tensors are only the Schwarzschild solutions.

In summary, we first supposed spacetimes to be static and axisymmetric, then finally derived the Schwarzschild solutions as the unique asymptotically flat vacuum solutions admitting Killing-Yano tensors. Note that, as shown in [8], in order to admit a Killing-Yano tensor a spacetime must be of Petrov type D (or IV). Although for Petrov type D spacetimes all vacuum solutions are known [10], many of them are difficult to interpret. In particular, it is not trivial how many spacetimes which have isometries exist and which are the asymptotically flat spacetimes. Our approach has, contrary to this, an advantage that physical interpretation of the results are manifest.

If we consider the stationary cases instead of static ones, a new function \( \omega(x, y) \) must
be introduced in (2) [4]:
\[\begin{align*}
ds^2 &= -e^{2\psi}(dt - \omega d\phi)^2 + \sigma^2 e^{-2\psi} \left[ e^{2\gamma}(x^2 - y^2) \left( \frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2} \right) + (x^2 - 1)(1 - y^2) \right] d\phi^2.
\end{align*}\]

The function \( \omega \) glues the sets T and R in a so complicated way that we have not tried such cases here. Nevertheless, we believe that the extra conditions for the existence of Killing-Yano tensors would still be so restrictive that, as asymptotically flat vacuum solutions, only the Kerr solutions would admit Killing-Yano tensors.

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