SPECTRAL MULTIPLIERS VIA RESOLVENT TYPE ESTIMATES ON NON-HOMOGENEOUS METRIC MEASURE SPACES

PENG CHEN, ADAM SIKORA, AND LIXIN YAN

Abstract. We describe a simple but surprisingly effective technique of obtaining spectral multiplier results for abstract operators which satisfy the finite propagation speed property for the corresponding wave equation propagator. We show that, in this setting, spectral multipliers follow from resolvent type estimates. The most notable point of the paper is that our approach is very flexible and can be applied even if the corresponding ambient space does not satisfy the doubling condition or if the semigroup generated by an operator is not uniformly bounded. As a corollary we obtain the $L^p$ spectrum independence for several second order differential operators and recover some known results. Our examples include the Laplace-Belltrami operator on manifolds with ends and Schrödinger operators with strongly subcritical potentials.

1. Introduction

This paper is devoted to spectral multiplier theory, which is one of most significant areas of Harmonic Analysis. In the most general framework one considers a self-adjoint, non-negative usually unbounded operator $L$ acting on space $L^2(X,\mu)$ where $(X,d,\mu)$ is a metric measure space with metric $d$ and measure $\mu$.

By spectral theory for any bounded Borel function $F$ one can define the operator

$$F(L) = \int_0^\infty F(\lambda)dE_L(\lambda)$$

where $dE_L$ is a spectral resolution of the operator $L$. The $L^2(X)$ norm of the operator $F(L)$ is bounded by the $L^\infty$ norm of $F$. Then spectral multiplier theory asks what sufficient conditions are required to ensure that the operator $F(L)$ extends to a bounded operator acting on spaces $L^p(X)$ for all $1 \leq p \leq \infty$ or for some smaller range of exponents $p$. The question is motivated by the problem of convergence of eigenfunction expansion and includes the Bochner-Riesz means analysis. This area was at first inspired by celebrated Fourier multiplier results of Mikhlin and Hörmander, see [15, 16], which initiated stating sufficient conditions for function $F$ in terms of its differentiability.

The spectral multiplier theory is well developed and understood. However almost all results use the standard assumptions which include the doubling condition and uniform estimates for the corresponding semigroup. These two types of conditions are not completely natural and in fact there are many significant examples which do not satisfy these assumptions and we would like to investigate them. These two situations are main points of interest of our approach. Firstly we are able to study the ambient spaces $(X,d,\mu)$ which do not...
satisfy the doubling condition, see (2.1) and (2.2) below. In particular, we obtain a spectral multiplier result for Laplace-Beltrami operator acting on the class of manifolds with ends described by Grigor’yan and Saloff-Coste in [14]. The doubling condition usually fails for the manifolds with ends. A second point is that we are able to treat the case of operators for which the corresponding semigroup does not satisfy uniform bounds for large time in the full range of $L^p$ spaces. Examples of such situation come from investigation of strongly subcritical Schrödinger semigroups discussed by Davies and Simon in [11] and by Murata in [18]. Investigation of Hodge Laplacian operators also often leads to semigroups which are not uniformly bounded on $L^p$ spaces, see [7] and Remark 5.6 below. To the best of our knowledge the spectral multiplier results have not been studied before in the setting of manifolds with ends and strongly subcritical Schrödinger operators and our approach leads to first examples of such results.

We also would like to mention some other interesting features of the spectral multiplier techniques which we develop here. Our approach yields an alternative proof of the classical spectral multiplier result for operators which generate semigroups with the standard Gaussian bounds, see e.g. [4, 8, 13] and Remark 5.12 below. Our discussion provides also alternative proofs of most of Davies’ results from [10] and improves some of estimates stated there. Related to this point is the issue of $L^p$ spectral independence of the operators which we study here. There are well-known and important examples of operators of a similar nature to the ones which we investigate here, where their spectra depend upon the space $L^p(X,\mu)$ (see [1, 12]). It is natural to ask whether the spectra of operators which we consider here also depend upon $p$. Examples of results concerning the $L^p$ independence are described in [9, 10]. See also the references therein. As a corollary of our main result we prove the $L^p$ spectral independence in all settings which we consider here including the manifolds with ends.

One of main assumptions of our results is the finite propagation speed for the corresponding wave equation propagator. We describe this standard notion in the next section. To be able to state our main result we define the volume function $V_r$ by the formula

$$V_r(x) = V(x,r) = \mu(B(x,r)),$$

where $B(x,r)$ is a ball with radius $r$ and center at $x$. Next, for a function $W : X \to \mathbb{R}$ we set $M_W$ to be operator of multiplication by $W$, that is

$$M_Wf(x) = W(x)f(x).$$

To simplify notion, in what follows, we identify $W$ and the operator $M_W$ that is we denote $M_WT$ by $WT$ for any linear operator $T$.

Now our main result can be formulated in the following way.  

**Theorem 1.1.** Suppose that $L$ a self-adjoint, non-negative operator which satisfies the finite propagation speed property for the corresponding wave equation. We describe this standard notion in the next section. To be able to state our main result we define the volume function $V_r$ by the formula

$$V_r(x) = V(x,r) = \mu(B(x,r)),$$

where $B(x,r)$ is a ball with radius $r$ and center at $x$. Next, for a function $W : X \to \mathbb{R}$ we set $M_W$ to be operator of multiplication by $W$, that is

$$M_Wf(x) = W(x)f(x).$$

To simplify notion, in what follows, we identify $W$ and the operator $M_W$ that is we denote $M_WT$ by $WT$ for any linear operator $T$.

Now our main result can be formulated in the following way.  

**Theorem 1.1.** Suppose that $L$ a self-adjoint, non-negative operator which satisfies the finite propagation speed property for the corresponding wave equation. Suppose next that for some exponents $\sigma > 0$ and $\kappa \geq 0$

$$(R_{\sigma,\kappa}) \quad \|V_1^{1/2}(I + t^2L)^{-\sigma}\|_{2 \to \infty} \leq C(1 + t^2)^\kappa$$

for all $t > 0$. Then

i) There exists a constant $C > 0$ such that

$$\|e^{ktL}e^{-tL}\|_{1 \to 1} \leq C(1 + \xi^2)^{\sigma + \kappa + 1/4}(1 + t)^\kappa$$

for all $t > 0$ and $\xi \in \mathbb{R}$;
ii) For a bounded Borel function $F$ such that $\text{supp} F \subset [-1,1]$ and $F \in H^s(\mathbb{R})$ for some $s > 2\sigma + 2\kappa + 1$, the operator $F(L)$ is bounded on $L^p(X)$ for all $1 \leq p \leq \infty$, and there exists constant $C = C(s) > 0$ such that 
\begin{equation}
\|F(tL)\|_{p \to p} \leq C(1 + t)^\kappa \|F\|_{H^s}
\end{equation}
for all $t > 0$.

In Section 5 we describe a number of applications of Theorem 1.1, including the manifolds with ends and strongly subcritical Schrödinger operators, which explain significance of the above statement and rationale for assuming resolvent type estimate $(R_{\sigma,\kappa})$. Additional rationale for condition $(R_{\sigma,\kappa})$ can be found in [2, Proposition 2.3.4].

The comprehensive list of relevant literature concerning spectral multipliers is enormous and too long to be included in this note so we just refer readers to [4, 13] and references within as a possible starting point for gathering complete bibliography of the subject.

2. Doubling condition and finite propagation speed property

Doubling condition. In our approach the doubling condition does not play essential role. However, because we use this notion in the discussion or in our results we recall it here. We say that metric measure space $(X, d, \mu)$ satisfies the doubling condition if there exists constant $C$ such that 
\begin{equation}
V(x, 2r) \leq CV(x, r)
\end{equation}
for all $x \in X$ and $r > 0$. As a consequence, there exist constants $C, n > 0$ such that 
\begin{equation}
V(x, sr) \leq Cs^n V(x, r)
\end{equation}
for all $s \geq 1, r > 0$.

Note that the doubling condition fails in the example which we consider in Section 5.1 and it may or may not be satisfied by $(X, d, \mu)$ in Section 5.3. Metric measure spaces which satisfy the doubling condition are often called homogenous spaces.

The following statement provides another rationale for condition $(R_{\sigma,\kappa})$ and will be used in Sections 5.2 and 5.4.

Lemma 2.1. Assume that space $(X, d, \mu)$ satisfies the doubling condition (2.2) with exponent $n$. Then for all $\sigma > n/4$ and $\kappa \geq 0$, condition $(R_{\sigma,\kappa})$ is equivalent with the following estimate 
\begin{equation}
\|V_{1/2}^s e^{-t^2 L}\|_{2 \to \infty} \leq C(1 + t^2)^\kappa.
\end{equation}

Proof. The proof is very simple. For example it is just a minor modification of the proof of Proposition 2.3.4 of [2]. \hfill \Box

Finite propagation speed. Let $L$ be a self-adjoint non-negative operator acting on $L^2(X)$. Following [6], we say that $(X, d, \mu, L)$, or in short $L$, satisfies the finite propagation speed property for the corresponding wave equation propagator if 
\begin{equation}
\langle \cos(r\sqrt{L}) f_1, f_2 \rangle = 0
\end{equation}
for all $f_i \in L^2(B_i, \mu)$, $i = 1, 2$, where $B_i$ are open balls in $X$ such that $d(B_1, B_2) > r > 0$.

We shall use the following notational convention, see [20]. For $r > 0$, set 
\[ D_r = \{(x, y) \in X \times X : d(x, y) \leq r \}. \]

Given a linear operator $T$ from $L^2(X, \mu)$ to $L^2(X, \mu)$, we write 
\begin{equation}
\text{supp } T \subseteq D_r
\end{equation}
if $\langle Tf_1, f_2 \rangle = 0$ whenever $f_1 \in L^2(B_1, \mu)$, $f_2 \in L^2(B_2, \mu)$, and $B_1, B_2$ are balls such that $d(B_1, B_2) > r$. Note that if $T$ is an integral operator with kernel $K_T$, then (2.5) coincides with the standard meaning of $\text{supp } K_T \subseteq D_r$, that is $K_T(x, y) = 0$ for all $(x, y) \notin D_r$. Now we can state the finite propagation speed property (2.4) in the following way

\[(\text{FS}) \quad \text{supp } \cos(r \sqrt{L}) \subseteq D_r, \quad \forall r > 0.\]

Property (FS) holds for most of second order self-adjoint operators and it is equivalent to Davies-Gaffney estimates, see estimate (DG) below and [20, 6]. See also [3] for earlier examples of the finite propagation speed property technique. Examples of application of the finite speed propagation techniques to the spectral multiplier theory can be found in [8, 4].

In what follows we will need the following straightforward observation, see [4, 20].

Lemma 2.2. Assume that $L$ satisfies the finite propagation speed property (FS) for the corresponding wave equation. Let $\Phi \in L^1(\mathbb{R})$ be an even function such that $\text{supp } \hat{\Phi} \subseteq [-1, 1]$. Then

\[(2.6) \quad \text{supp } \Phi(r \sqrt{L}) \subseteq D_r\]

for all $r > 0$.

Proof. If $\Phi$ is an even function, then by the Fourier inversion formula,

\[
\Phi(r \sqrt{L}) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\Phi}(s) \cos(rs \sqrt{L}) \, ds.
\]

However, $\text{supp } \hat{\Phi} \subseteq [-1, 1]$ so the lemma follows from (FS).

3. Spectral multiplier theorems - the proof of the main result

This section is devoted to the proof of Theorem 1.1, which is based on the wave equation technique.

Proof of Theorem 1.1. First, we prove estimate (1.1). A direct calculation as in [2, page 360] shows that for all $a > 0$, $x \in \mathbb{R}$,

\[
\frac{1}{\Gamma(a)} \int_0^\infty (s - x^2)_+^a e^{-s} \, ds = e^{-x^2},
\]

where

\[(t)_+ = t \quad \text{if} \quad t \geq 0 \quad \text{and} \quad (t)_+ = 0 \quad \text{if} \quad t < 0.
\]

Hence

\[
C_a \int_0^\infty \left(1 - \frac{x^2}{s}\right)_+^a e^{-\frac{s}{4}x^2} \, ds = e^{-\frac{x^2}{4}}
\]

for some suitable $C_a > 0$. Taking the Fourier transform of both sides of the above inequality yields

\[
\int_0^\infty F_a(\sqrt{s} \xi)s^{a+\frac{1}{2}}e^{-\frac{\xi^2}{4}} \, ds = e^{-\xi^2},
\]

where $F_a$ is the Fourier transform of the function $t \rightarrow (1 - t^2)_+^a$ multiplied by the appropriate constant. Hence, by spectral theory for $\nu > 0$

\[
e^{-\nu L} = \int_0^\infty F_a(\sqrt{s \nu} L)s^{a+\frac{1}{2}}e^{-\frac{\xi^2}{4}} \, ds.
\]
Thus we can rewrite $e^{ixtL}e^{-tL}$ as

\begin{equation}
(3.1) \quad e^{ixtL}e^{-tL} = \int_0^\infty F_a(\sqrt{s}L)s^{a+\frac{1}{2}}(t-ix)\frac{1}{t} \exp \left( -\frac{s}{4(t-ix)} \right) ds.
\end{equation}

Note that $\text{supp } \hat{F}_a \subset [-1, 1]$. By Lemma 2.2

$$\text{supp } F_a(s\sqrt{L}) \subseteq D_s, \forall s > 0.$$ 

Thus

$$\|F_a(\sqrt{s}L)\|_{1 \to 1} = \sup_y \int_X |K_{F_a(\sqrt{s}L)}(x,y)|d\mu(x)$$

$$= \sup_y \int_{B(y, \sqrt{s})} |K_{F_a(\sqrt{s}L)}(x,y)|d\mu(x)$$

$$\leq \sup_y \mu(B(y, \sqrt{s}))^{1/2} \left( \int_X |K_{F_a(\sqrt{s}L)}(x,y)|^2 d\mu(x) \right)^{1/2}$$

$$= \|V_{\sqrt{s}}^{1/2} F_a(\sqrt{s}L)\|_{2 \to \infty}$$

$$\leq \|V_{\sqrt{s}}^{1/2} (1 + sL)^{-\sigma}\|_{2 \to \infty} \|(1 + sL)^\sigma F_a(\sqrt{s}L)\|_{L^\infty}$$

(3.2)

Next we estimate the second term of the last inequality above that is $L^\infty$ norm of $(1 + \lambda^2)^\sigma F_a(\lambda)$. For this purpose we recall the following well-known asymptotic for Bessel type functions, often used in discussion of the kernel of the standard Bochner-Riesz operators, see e.g. page 391 of [21]: $F_a(\lambda)$ is bounded and has the following asymptotic expansion

\begin{equation}
(3.3) \quad F_a(\lambda) \sim |\lambda|^{-1-a} \left[ e^{2\pi i|\lambda|} \sum_{j=0}^\infty \alpha_j |\lambda|^{-j} + e^{-2\pi i|\lambda|} \sum_{j=0}^\infty \beta_j |\lambda|^{-j} \right]
\end{equation}

as $|\lambda| \to \infty$, for suitable constants $\alpha_j$ and $\beta_j$.

Hence $\|(1 + \lambda^2)^\sigma F_a(\lambda)\|_{\infty} < \infty$ as soon as $2\sigma - 1 \leq a$. Now if $\kappa$ is an exponent from $(R_{\sigma, \kappa})$ and $2\sigma - 1 \leq a$ then by (3.2)

\begin{equation}
(3.4) \quad \|F_a(\sqrt{s}L)\|_{1 \to 1} \leq C(1 + s)^\kappa.
\end{equation}

By (3.1) and (3.4)

$$\|e^{ix\xi L}e^{-tL}\|_{1 \to 1} \leq \int_0^\infty (1 + s)^\kappa s^{a+\frac{1}{2}}(t^2 + (\xi t)^2)^{-\frac{a+\frac{3}{2}}{2}} \exp \left( -\frac{s}{4(t + t\xi^2)} \right) ds$$

$$\leq C(1 + t)^\kappa (1 + \xi^2)^{\frac{a+\kappa+\frac{3}{2}}{2}}.$$ 

Finally, noting that $2\sigma - 1 \leq a$, we obtain (1.1).

To prove (1.2) we write $G(\lambda) = F(\lambda)e^\lambda$, and then $F(tL) = G(tL)e^{-tL}$. Hence,

$$F(tL) = G(tL)e^{-tL} = \int_{\mathbb{R}} \hat{G}(\xi)e^{it\xi L}e^{-tL} d\xi.$$ 

Note that $s > 2\sigma + 2\kappa + 1$ so by (1.1)

$$\|F(tL)\|_{1 \to 1} \leq \int_{\mathbb{R}} \|\hat{G}(\xi)||e^{it\xi L}e^{-tL}\|_{1 \to 1} d\xi.$$
\[ \leq C \int_{\mathbb{R}} |\hat{G}(\xi)| (1 + \xi^2)^{\sigma + \kappa + 1/4} (1 + t)^{\kappa} d\xi \]

\[ \leq C(1 + t)^{\kappa}\|G\|_{H^s} \left( \int_{\mathbb{R}} (1 + \xi^2)^{\sigma + \kappa + 1/4 - s/2} d\xi \right)^{1/2} \]

\[ \leq C(1 + t)^{\kappa}\|G\|_{H^s}. \]

Observe that \( \text{supp } F \subset [-1, 1] \) so \( \|G\|_{H^s} \leq C\|F\|_{H^s} \) and

\[ \|F(tL)\|_{1 \rightarrow 1} \leq C_s(1 + t)^{\kappa}\|G\|_{H^s} \leq C_s(1 + t)^{\kappa}\|F\|_{H^s}. \]

Since \( F \) is a bounded Borel function, \( F(L) \) is bounded on \( L^2(X) \). By interpolation and duality, \( F(L) \) is bounded on \( L^p(X) \) for \( 1 \leq p \leq \infty \). This ends the proof of Theorem 1.1. \( \square \)

In the next section, in our discussion of spectral independence of operator \( L \) we shall use the following corollary of Theorem 1.1 which states the spectral multiplier result for non-compactly supported functions. Recall that \( [x] \) stands for the integer part of \( x \in \mathbb{R} \).

**Corollary 3.1.** Suppose that \( L \) satisfies the finite propagation speed property for the corresponding wave equation and that for some \( \sigma > 0, \kappa \geq 0 \) resolvent type estimate \((R_{\sigma, \kappa})\) holds. Next assume that for function \( F \) on \( \mathbb{R} \), there exists a constant \( C > 0 \) such that for \( m = 0, 1, \ldots, [2\sigma + 2\kappa + 1] + 1, \)

\[ \sup_{0 \leq \lambda < 1} \left| \frac{d^m}{d\lambda^m} F(\lambda) \right| \leq C \]

and for some \( \epsilon > 0 \)

\[ \sup_{\lambda \geq 1} \left| \lambda^{m+\epsilon} \frac{d^m}{d\lambda^m} F(\lambda) \right| \leq C. \]

Then the operator \( F(L) \) is bounded on \( L^p(X) \) for all \( 1 \leq p \leq \infty \).

**Proof.** Let \( \phi \in C_c^\infty(\mathbb{R}) \) be a function such that \( \text{supp } \phi \subseteq \{ \lambda : 1/4 \leq |\lambda| \leq 1 \} \) and \( \sum_{\ell \in \mathbb{Z}} \phi(2^{-\ell} \lambda) = 1 \) for all \( \lambda > 0 \). Set \( \phi_0(\lambda) = 1 - \sum_{\ell=1}^{\infty} \phi(2^{-\ell} \lambda) \) and

\[ F(\lambda) = F(\lambda)\phi_0(\lambda) + \sum_{\ell=1}^{\infty} F(\lambda)\phi(2^{-\ell} \lambda). \]

Set \( \delta_{\ell} F(\lambda) = F(\lambda t) \). It follows from ii) of Theorem 1.1 that for \( s > [2\sigma + 2\kappa + 1] + 1, \)

\[ \|F(L)\phi(2^{-\ell}L)\|_{1 \rightarrow 1} \leq (1 + 2^{-\ell})^{\kappa}\|\phi\delta_{2^\ell} F\|_{H^s}, \quad \ell = 1, 2, \ldots, \]

which yields

\[ \|F(L)\|_{1 \rightarrow 1} \leq C\|F\phi_0\|_{H^s} + C \sum_{\ell=1}^{\infty} (1 + 2^{-\ell})^{\kappa}\|\phi\delta_{2^\ell} F\|_{H^s} \leq C \]

when \( F \) satisfies conditions (3.5) and (3.6).

Since \( F \) is a bounded Borel function, \( F(L) \) is bounded on \( L^2(X) \). By interpolation and duality, \( F(L) \) is bounded on \( L^p(X) \) for \( 1 \leq p \leq \infty \). \( \square \)
4. \(L^p\)-SPECTRAL INDEPENDENCE

In this section, we assume that operator \(L\) is a non-negative self-adjoint operator on \(L^2(X)\) such that \(e^{-tL}\) can be extended to a strongly continuous one-parameter semigroup on \(L^p(X)\) for all \(1 \leq p < \infty\). For \(1 \leq p < \infty\) we denote by \(L_p\) the generator of the considered semigroup acting on \(L^p(X)\) space and by \(\text{Spec}(L_p)\) its spectrum. This means that \(L_p\) and \(L_q\) coincide on \(L^p \cap L^q\) and so if \(z \notin \text{Spec}(L_p)\) and \((z - L_p)^{-1}\) extends to a bounded operator on \(L^q(X)\) then \(z \notin \text{Spec}(L_q)\). To simply notation we use the convention \(L = L_2\) consistent with the rest of this note.

We start by proving the following lemma.

**Lemma 4.1.** Suppose that \(L\) satisfies the finite propagation speed property for the corresponding wave equation and that for some \(\sigma > 0\), \(\kappa \geq 0\) the resolvent type estimate \((R_{\sigma,\kappa})\) holds. Then

\[
\|(z^2 + L_p)^{-1}\|_{p \to p} \leq C \left( \frac{|z|}{\text{Re} z} \right)^{2\sigma + 2\kappa + 3/2} \frac{1}{|z|^2} \left( 1 + \frac{1}{|z|^2} \right)^\kappa
\]

for all \(\text{Re} z > 0\) and \(1 \leq p < \infty\).

In addition if \(z \notin \text{Spec}(L_p)\) and \(z \notin \text{Spec}(L_q)\) then the resolvents \((z - L_p)^{-1}\) and \((z - L_q)^{-1}\) are consistent that is they coincide on \(L^p(X) \cap L^q(X)\).

**Proof.** Write \(z = re^{i\theta}\) for \(r > 0\) and \(-\pi/2 < \theta < \pi/2\). By a standard formula, see e.g. \([10]\) or \([19, \text{Proposition} 7.9]\)

\[
(L_p + z^2)^{-1} = (L_p + r^2 e^{2i\theta})^{-1} = e^{-i\theta} \int_0^\infty \exp[-L_p se^{-i\theta} - sr^2 e^{i\theta}] ds.
\]

Note that the above relation shows consistency of the resolvents. Next we set \(t = r \cos \theta\) and \(\xi = \tan \theta\). Then by the above formula and \((1.1)\),

\[
\|(L_p + z^2)^{-1}\|_{p \to p} \leq \int_0^\infty \|e^{i\xi \sin \theta L_p} e^{-s \cos \theta L_p}\|_{p \to p} e^{-sr^2 \cos \theta} ds = \int_0^\infty \|e^{i\xi \theta L_p} e^{-L_p}\|_{p \to p} e^{-sr^2 \cos \theta} ds \leq C \int_0^\infty (1 + \xi^2)^{\sigma + \kappa + 1/4} (1 + t)^\kappa e^{-sr^2 \cos \theta} ds \leq C(1 + \tan^2 \theta)^{\sigma + \kappa + 1/4} \frac{1}{r^2 \cos \theta} \left( 1 + \frac{1}{r^{2\kappa}} \right),
\]

which implies \((4.1)\).

Our spectral independence result can be stated now in the following way.

**Theorem 4.2.** Suppose that \(L\) satisfies the finite propagation speed property for the corresponding wave equation and that for some \(\sigma > 0\), \(\kappa \geq 0\) resolvent type estimate \((R_{\sigma,\kappa})\) holds for \(L\). Then \(\text{Spec}(L_p)\), the spectrum of the operator \(L_p\), is independent of the space \(L^p(X)\) for all \(1 \leq p < \infty\).

**Proof.** By \((4.1)\), for \(z \notin [0, \infty)\) the resolvent \((z - L)^{-1}\) is bounded as an operator acting on any \(L^p(X)\) space for \(1 \leq p < \infty\) and so \(\text{Spec}(L_p) \subset \mathbb{R}_+\).
Now assume that $\rho \geq 0$ is not contained in $\text{Spec}(L_2)$. Then there exists $\epsilon > 0$ such that 
$(\rho - \epsilon, \rho + \epsilon) \cap \text{Spec}(L_2) = \emptyset$. Now consider a smooth function $\psi \in C^\infty_c(\rho - \epsilon, \rho + \epsilon)$ such that 
$\psi(x) = 1$ for $x \in (\rho - \epsilon/2, \rho + \epsilon/2)$. Then $\psi(L) = 0$ so 
$$(I - \psi(L))(\rho - L)^{-1} = (\rho - L)^{-1}.$$ 
Notice that function $g(\lambda) = (1 - \psi(\lambda))(\rho - \lambda)^{-1}$ satisfies conditions (3.5) and (3.6) of Corollary 3.1. Hence the operator $(I - \psi(L))(\rho - L)^{-1} = (\rho - L)^{-1}$ extends to a bounded operator acting on $L^p(X)$. This shows that $\text{Spec}(L_p) \subseteq \text{Spec}(L_2)$.

To show the opposite inclusion, assume that $\rho \geq 0$ is not contained in $\text{Spec}(L_p)$. By duality 
$(\rho - L)^{-1}$ is bounded on $L^p$. Hence by consistency and interpolation $(\rho - L)^{-1}$ is bounded 
on $L^2$. The proof of Theorem 4.2 is complete. \hfill \qed 

5. Applications

In this section we describe a number of applications of our main results. Note that in 
Section 5.2 below estimate $(R_{\sigma,\kappa})$ cannot hold for $\kappa = 0$. This is an important motivation for 
the type of assumptions which we consider in our main result.

5.1. Manifolds with ends. Firstly we want to show one can apply Theorem 1.1 to the 
setting of manifolds with ends studied in details by Grigor'yan and Saloff-Coste in [14]. The 
precise description of the notion of manifolds with ends in full generality is complex and 
technical. These technical details are not relevant for the spectral multiplier technique which 
we describe in this note. Therefore we consider here only simple model case from [14]. We 
leave it to the interested readers to check that the approach described below can be applied 
to the whole class of manifolds with ends discussed there.

Let $M$ be a complete non-compact Riemannian manifold. Let $K \subset M$ be a compact set 
with non-empty interior and smooth boundary such that $M \setminus K$ has $k$ connected components 
$E_1, \ldots, E_k$ and each $E_i$ is non-compact. We say in such a case that $M$ has $k$ ends with respect 
to $K$. Here we are going to consider only the case $k = 2$ that is two different ends. We are 
also going to assume that each $E_i$ for $i = 1, 2$ is isometric to the exterior of a compact set 
in another manifold $M_i$ described below. In such case we write $M = M_1 \sharp M_2$. Next, fix an 
teger $m \geq 3$, which will be the topological dimension of $M$, and for any integer $2 < n \leq m$, 
define the manifold $\mathcal{R}^n$ by 
$$ \mathcal{R}^n = \mathbb{R}^n \times S^{m-n}. $$ 
The manifold $\mathcal{R}^n$ has topological dimension $m$ but its “dimension at infinity” is $n$ in the 
sense that $V(x, r) \approx r^n$ for $r \geq 1$, see [14, (1.3)]. Thus, for different values of $n$, the manifold 
$\mathcal{R}^n$ have different dimension at infinity but the same topological dimension $m$. This enables 
us to consider finite connected sums of the $\mathcal{R}^n$ and $\mathbb{R}^m$. Namely consider integers $2 < n \leq m$ 
and define $M$ as 
$$ M = \mathcal{R}^n \sharp \mathbb{R}^m. $$ 
Next for any $x \in M$ we put 
$$ |x| := \sup_{z \in K} d(x, z) $$ 
and we recall that $V(x, r) = \mu(B(x, r))$. From the construction of the manifold $M$, we can see that 
(a) $V(x, r) \approx r^m$ for all $x \in M$, when $r \leq 1$; 
(b) $V(x, r) \approx r^n$ for $B(x, r) \subset \mathcal{R}^n$, when $r > 1$; and
(c) \( V(x, r) \approx r^n \) for \( x \in \mathbb{R}^n \setminus K, r > 2|x|, \) or \( x \in \mathbb{R}^m, r > 1. \)

It is not difficult to check that if \( n < m \) then \( M \) does not satisfy the doubling condition. Indeed, consider a family of balls \( B(x_r, r) \subset \mathbb{R}^n \) such that \( r = |x_r| \to \infty. \) Then \( V(x_r, r) \approx r^n \) but \( V(x_r, 2r) \approx r^m \) and the doubling condition fails.

Let \( \Delta \) be the Laplace-Belltrami operator acting on \( M \) and let \( p_t(x, y) \) be the heat kernel corresponding to the heat propagator \( e^{-t\Delta}. \) In \cite{14} Grigor’yan and Saloff-Coste establish both the global upper bound and lower bound for the heat kernel of the semigroup \( e^{-t\Delta} \) acting on this model class. The following lemma is an obvious consequence of their results.

**Lemma 5.1.** Let \( M = \mathbb{R}^n_+ \mathbb{R}^m \) with \( 2 < n \leq m \) and \( \Delta \) be the Laplace-Belltrami operator acting on \( M. \) Then the kernel \( p_t(x, y) \) of the semigroup \( e^{-t\Delta} \) satisfies the following on-diagonal estimate

\[
\sup_{x \in M} p_t(x, x) \leq C \left( t^{-n/2} + t^{-m/2} \right).
\]

**Proof.** The proof is a straightforward consequence of \cite[Corollary 4.16 and Section 4.5]{14}. In fact the estimate in Lemma 5.1 is essentially more elementary than the quoted results and can be verified directly. \( \square \)

Lemma 5.1 implies the following result.

**Lemma 5.2.** Let \( M = \mathbb{R}^n_+ \mathbb{R}^m \) with \( 2 < n \leq m \) and \( \Delta \) be the Laplace-Belltrami operator acting on \( M. \) Then for any \( \sigma > m/4 \) and \( \kappa = (m - n)/4 \) estimate \( R_{\sigma, \kappa} \) holds for \( \Delta, \) that is there exists constant \( C \) such that

\[
\| V^{1/2}_t(I + t^2\Delta)^{-\sigma} \|_{2 \to \infty} \leq C(1 + t^2)^{(m-n)/4}
\]

for all \( t > 0. \)

**Proof.** For every \( \sigma > 0 \)

\[
(I + t^2\Delta)^{-2\sigma} = \frac{1}{\Gamma(2\sigma)} \int_0^\infty e^{-s} s^{2\sigma-1} \exp(-st^2\Delta) \, ds.
\]

Hence if \( \sigma > m/4, \) then by Lemma 5.1

\[
|K_{(I+t^2\Delta)^{-2\sigma}}(x, x)| \leq C \int_0^\infty e^{-s} s^{2\sigma-1} \left( (st^2)^{-n/2} + (st^2)^{-m/2} \right) \, ds
\]

\[
\leq C \left( t^{-n} + t^{-m} \right)
\]

for all \( x \in M \) and \( t > 0. \)

Recall that \( V(x, t) \leq Ct^m \) for all \( x \in M \) and \( t > 0 \) so by duality and the above inequality

\[
\| V^{1/2}_t(I + t^2\Delta)^{-\sigma} \|_{2 \to \infty}^2 = \sup_{x \in M} V(x, t) \int_M |K_{(I+t^2\Delta)^{-\sigma}}(x, y)|^2 \, d\mu(y)
\]

\[
\leq C \sup_{x \in M} V(x, t) |K_{(I+t^2\Delta)^{-\sigma}}(y, y)|
\]

\[
\leq C(1 + t^2)^{(m-n)/2}.
\]

This proves Lemma 5.2. \( \square \)

Theorems 1.1, 4.2 and Lemma 5.2 yield the following theorem.
Theorem 5.3. Let $M = \mathbb{R}^n \setminus \mathbb{R}^m$ with $2 < n \leq m$. Suppose that $\Delta$ is the Laplace-Belltrami operator acting on $M$. Then estimate (1.1) holds for any exponent $\sigma > m/4$ and $\kappa = (m-n)/4$ and $\text{Spec}(\Delta_p)$ - the spectrum of the operator $\Delta_p$ is independent of $p$ for all $1 \leq p < \infty$.

Moreover, if $F$ is a bounded Borel function such that $\text{supp} F \subset [-1, 1]$ and $F \in H^s(\mathbb{R})$ for some $s > m/2 + (m-n)/2 + 1$, then the operator $F(\Delta)$ is bounded on $L^p(M)$ and (1.2) holds with $\kappa = (m-n)/4$ that is

$$\|F(t\Delta)\|_{p \rightarrow p} \leq C(1 + t)^{(m-n)/4}\|F\|_{H^s}$$

for all $t > 0$.

Remark 5.4. Using the estimates from [14] one can show that $\|\exp(-t\Delta)\|_{p \rightarrow p} \leq C$ uniformly in $t$ and $p$. Therefore we conjecture that one can strengthen all the above results concerning the manifolds with ends by taking $\kappa = 0$ instead of $(m-n)/4$.

5.2. Semigroups without uniform $L^p$ bounds - Schrödinger operators with strongly subcritical potentials. In this subsection, we consider Schrödinger operators $L = -\Delta + V$ on $\mathbb{R}^n$, $n \geq 3$ and we assume that $L \geq 0$. Let $V = V_+ - V_-$ be the decomposition of $V$ into its positive and negative parts. We say $L$ (or $V$) is strongly subcritical, if there exists small enough $\varepsilon > 0$ such that

$$L - \varepsilon V_- \geq 0.$$ 

Following Davies and Simon [11], we say that $L$ has a resonance $\eta$ if there exists a non-zero function $\eta$ such that $L\eta = 0$ and then we say that it is slowly varying with index $\alpha$ for some $0 < \alpha < (n-2)/2$ if there exists a constant $C > 0$ such that

$$\frac{\eta(x)}{\eta(y)} \leq C(1 + |x - y|)^\alpha$$

for all $x, y \in \mathbb{R}^n$.

The following example of Schrödinger operators with a slowly varying resonance comes from Murata [17, 18]. Let $n \geq 3$ and put $L = -\Delta + V$ where $V(x) = 0$ if $|x| \leq 1$ and $V(x) = -c/|x|^2$ if $|x| > 1$ with constant $0 < c \leq ((n-2)/2)^2$. Then $L \geq 0$ and $V$ is strongly subcritical. Also $L$ has one positive radial resonance $\eta$ satisfying $\eta(x) \sim |x|^{-\alpha}$ as $|x| \to \infty$. Here

$$0 < \alpha = \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 - c} < \frac{n-2}{2}.$$ 

The following theorem is a consequence of Theorems 1.1 and 4.2 applied to the setting considered in [11].

Theorem 5.5. Let $L = -\Delta + V$ be Schrödinger operator on $\mathbb{R}^n$, $n \geq 3$. Assume that $L \geq 0$, that $V$ is strongly subcritical and that $L$ has a resonance $\eta \geq 0$ in $L^w_{n/2}^{n/\alpha}$ which is slowly varying with index $\alpha$ where $0 < \alpha < (n-2)/2$. Then there exists a constant $C = C(\sigma, \varepsilon) > 0$ such that estimate (1.1) holds for any $\sigma > n/4$ and $\kappa = \alpha/2 + \varepsilon$ and $\text{Spec}(L_p)$ the spectrum of the operator $L_p$ is independent of the space $L^p(\mathbb{R}^n)$ for all $1 \leq p < \infty$.

Moreover, if $F$ is a bounded Borel function such that $\text{supp} F \subset [-1, 1]$ and $F \in H^s(\mathbb{R})$ for some $s > n/2 + 1$, then the operator $F(L)$ is bounded on $L^p(\mathbb{R}^n)$ and (1.2) holds with $\kappa = \alpha/2 + \varepsilon$ that is

$$\|F(tL)\|_{p \rightarrow p} \leq C(1 + t)^{\alpha/2 + \varepsilon}\|F\|_{H^s}$$

for all $t > 0$. 

Let $\eta$ be the decomposition of $\eta(x) \sim |x|^{-\alpha}$ as $|x| \to \infty$. Here

$$0 < \alpha = \frac{n-2}{2} - \sqrt{\left(\frac{n-2}{2}\right)^2 - c} < \frac{n-2}{2}.$$ 

The following theorem is a consequence of Theorems 1.1 and 4.2 applied to the setting considered in [11].
for all \( t > 0 \).

**Proof.** By [6, Theorem 3.3] the semigroup generated by operator \( L \) satisfies Davies-Gaffney estimates \((DG)\) and so \( L \) satisfies the finite speed propagation property. Next by the result obtained by Davies and Simon, see [11, Theorem 14], for any \( \varepsilon > 0 \)

\[
(5.2) \quad c_1 e^{-n/4} (1 + t)^{\alpha/2 + \varepsilon} \leq \|e^{-tL}\|_2 \leq c_2 e^{-n/4} (1 + t)^{\alpha/2 + \varepsilon}.
\]

Now it follows from the above estimate and Lemma 2.1 that for any \( \sigma > n/4, \varepsilon > 0 \) and \( \kappa = \alpha/2 + \varepsilon \) resolvent estimate \((R_{\sigma, \kappa})\) holds. That is there exists a constant \( C = C(\sigma, \varepsilon) > 0 \) such that

\[
\|V_{t^{1/2}} (I + t^2L)^{-\sigma}\|_2 \leq C (1 + t^2)^{\alpha/2 + \varepsilon}.
\]

Thus the theorem follows from Theorems 1.1 and 4.2. \( \square \)

**Remark 5.6.** (A). In contrast to Remark 5.4 we expect that one cannot remove the additional term \((1 + t)^{\alpha/2 + \varepsilon}\) especially for \( p \) close to 1 or \( \infty \). Otherwise the estimate from Theorem 5.5 with \( \kappa = 0 \) would imply the uniform bounds for the semigroup which contradicts lower estimates for the semigroup in [11, Theorem 15]. However, for some range of \( p \) close to 2, the uniform version with \( \kappa = 0 \) holds, see [6, Example 4.17]. We do not discuss it here.

(B). The same approach can be used to study generalised Schrödinger operators

\[
\mathcal{L} = \nabla^* \nabla + \mathcal{R},
\]

acting on a finite-dimensional Riemannian bundle \( E \to M \). Here \( \nabla \) is a connection on \( E \to M \) which is compatible with the metric, and \( \nabla^* \nabla \) is the so-called “rough Laplacian”. Such operators include Hodge Laplacians \( \tilde{\Delta} = dd^* + d^*d \) in the case where \( \mathcal{R} \) is Ricci curvature, see e.g. [7, 5]. One can expect that in the subcritical case it leads to a similar discussion as described in this section. We do not study these operators here.

### 5.3. Non-uniform Gaussian upper bounds - Davies’ example.

The following example comes from [10]. In this section we assume that

\[
(5.3) \quad V_r(x) \leq \begin{cases} 
Cr^{n_1} & \text{if } 0 < r \leq 1 \\
Cr^{n_2} & \text{if } 1 \leq r < \infty,
\end{cases}
\]

where \( 0 < C < \infty, 0 < n_1 \leq n_2 < \infty \) and \( V_r(x) = \mu(B(x,r)) \). We want to stress here that we do not assume that the doubling condition \((2.1)\) holds. Let \( L \) be a non-negative self-adjoint operator acting on \( L^2(X) \) such that an integral kernel \( p_t(x,y) \) of the semigroup \( e^{-tL} \) satisfies

\[
(5.4) \quad |p_t(x,y)| \leq Ct^{-n_1/2} \exp \left\{ -c \frac{d(x,y)^2}{t} \right\}.
\]

Considering a new metric \( d' = 2\sqrt{cd} \) we can assume that \( c = 1/4 \) in the above estimate. Note that \((5.3)\) still holds possibly with a new constant \( C \). In what follows we always assume that \( c = 1/4 \) in \((5.4)\).

First we claim that \( L \) satisfies the finite propagation speed property for the corresponding wave equation or equivalently the semigroup \( e^{-tL} \) satisfies Davies-Gaffney estimates. Recall that the semigroup satisfies Davies-Gaffney estimates if for all \( f \in L^2(B_1, \mu) \) and \( g \in L^2(B_2, \mu) \)

\[
(DG) \quad |\langle e^{-tL} f, g \rangle| \leq e^{-\frac{r^2}{4t}} \|f\|_2 \|g\|_2,
\]
where $B_1$ and $B_2$ are open balls in $X$ and $d(B_1, B_2) > r > 0$.

Now we are able to state the following lemma.

**Lemma 5.7.** Suppose that the space $(X, d, \mu)$ and the operator $L$ satisfies conditions (5.3) and (5.4). Then $L$ satisfies Davies-Gaffney estimates (DG) and as a consequence the finite propagation speed property (FS) holds as well.

**Proof.** Let $f \in L^2(B_1, \mu)$ and $g \in L^2(B_2, \mu)$, where $B_1 = B(x_1, r_1)$ and $B_2 = B(x_2, r_2)$ are open balls in $X$ and $d(B_1, B_2) > r > 0$. Set $U_k(x_1, r_1) = B(x_1, 2^{k+1}(r_1 + r)) \setminus B(x_1, 2^k(r_1 + r))$. Decompose

$$X \setminus B(x_1, r_1 + r) = \bigcup_{k=0}^{\infty} U_k(x_1, r_1) \quad \text{and} \quad X \setminus B(x_2, r_1 + r) = \bigcup_{k=0}^{\infty} U_k(x_2, r_2).$$

Then by interpolation

$$\|e^{-tL}\|_{L^2(B_1) \to L^2(B_2)} \leq \|e^{-tL}\|_{L^1(B_1) \to L^1(B_2)} + \|e^{-tL}\|_{L^\infty(B_1) \to L^\infty(B_2)} \leq \sup_{x \in B_1} \int_{B_2} |p_t(x, y)|d\mu(x) + \sup_{x \in B_2} \int_{B_1} |p_t(x, y)|d\mu(y) \leq C(1 + t)^{(n_2 - n_1)/2} e^{-\frac{r^2}{4}},$$

which implies

$$|\langle e^{-tL}f, g \rangle| \leq C(1 + t)^{(n_2 - n_1)/2} e^{-\frac{r^2}{4}} \|f\|_2 \|g\|_2.$$

Then by [6, Lemma 3.2], the self-improving property for Davies-Gaffney estimates, the above estimate implies the Davies-Gaffney estimates (DG). \hfill \Box

Next we would like to make the following observation.

**Lemma 5.8.** Let $X$ and $L$ satisfy conditions (5.3) and (5.4). Then for any $\sigma > n_1/4$ and $\kappa = (n_2 - n_1)/4$ resolvent estimate $(R_{\sigma, \kappa})$ holds for $L$.

**Proof.** The proof is similar to that of Lemma 5.2. We omit the details. \hfill \Box

The following theorem follows from Theorem 1.1, 4.2 and Lemma 5.8.

**Theorem 5.9.** Let $X$ and $L$ satisfy conditions (5.3) and (5.4). Then there exists a constant $C = C(\sigma) > 0$ such that estimate (1.1) holds for any $\sigma > n_1/4$ and $\kappa = (n_2 - n_1)/4$ and Spec($L_p$) the spectrum of the operator $L_p$ is independent of the space $L^p(X)$ for all $1 \leq p < \infty$.

Moreover, if $F$ is a bounded Borel function such that supp $F \subset [-1, 1]$ and $F \in H^s(\mathbb{R})$ for some $s > n_2/2 + 1$, then the operator $F(L)$ is bounded on $L^p(X)$ and (1.2) holds with $\kappa = (n_2 - n_1)/4$, that is

$$\|F(t\Delta)\|_{p \to p} \leq C(1 + t)^{(n_2 - n_1)/4}\|F\|_{H^s}$$

for all $t > 0$.

**Remark 5.10.** In [10, Lemma 2] Davies proved that under assumptions (5.3) and (5.4) the following estimate holds

$$\|e^{tL}e^{-tL}\|_{1 \to 1} \leq C(1 + \xi^2)^{\frac{n_2}{2}} e^{\xi t}$$

(5.6)
for all $t > 0$ and $\xi \in \mathbb{R}$. Theorem 5.9 improves the above estimates and shows that

$$\|e^{i\xi t L} e^{-tL}\|_{1\to 1} \leq C(1 + \xi^2)^{s(1 + t)^{(n_2-n_1)/4}}$$

for any $s > n_2/4 + 1/4$. As a consequence the estimates from Lemma 4.1 improve the complex time resolvent estimates described in [10, Lemma 3].

5.4. Ambient spaces satisfying the doubling condition. Let $(X, d, \mu)$ be a metric measure space satisfying doubling condition with homogenous dimension $n$. The following statement is a direct consequence of Corollary 4.16 of [6].

**Proposition 5.11.** Let $X$ satisfy the doubling condition (2.2) with exponent $n$. Suppose next the wave propagator corresponding to $L$ satisfies finite speed property (FS) and that condition $(R_{\sigma, \kappa})$ holds for some $\sigma > 0$ and $\kappa = 0$ that is

$$\|V_{t/2}^{1/2}(I + t^2L)^{-\sigma}\|_{2\to\infty} \leq C.$$

Then there exists a constant $C > 0$ such that

$$\|e^{\xi t L} e^{-tL}\|_{1\to 1} \leq C(1 + \xi^2)^{n/4}$$

for all $t > 0$ and $\xi \in \mathbb{R}$.

**Proof.** Estimates (5.8) are proved in [6, Corollary 4.16] under condition $\|V_{t/2}^{1/2} e^{-t^2L}\|_{2\to\infty}$. However

$$\|V_{t/2}^{1/2} e^{-t^2L}\|_{2\to\infty} \leq \|V_{t/2}^{1/2}(I + t^2L)^{-\sigma}\|_{2\to\infty} \leq C\|V_{t/2}^{1/2}(I + t^2L)^{-\sigma}\|_{2\to\infty}$$

compare Lemma 2.1. Hence Proposition 5.11 follows directly from [6, Corollary 4.16]. □

**Remark 5.12.** (A). The direct application of Theorem 1.1 yields the estimate (5.8) with exponent $\sigma + 1/4$ instead of $n/4$. Note that one usually expect that $\sigma > n/4$, see Lemma 2.1. However the proof of Theorem 1.1 is still valid if doubling condition fails and is essentially less complex than the proof of [6, Corollary 4.16]. See also [19, Theorem 7.3, Proposition 7.9].

(B). In the doubling setting the estimates $\|V_{t/2}^{1/2} e^{-t^2L}\|_{2\to\infty} \leq C$ and the finite speed propagation property (FS) are equivalent to the standard Gaussian upper bound for the heat kernel corresponding to the operator $L$

$$|p_t(x, y)| \leq C\frac{1}{V(y, \sqrt{t})} \exp\left\{ -c\frac{d(x, y)^2}{t} \right\}$$

see e.g. [20]. Therefore Proposition 5.11 can be reformulated in term of Gaussian upper bound assumption.

(C). Again in the doubling setting, the similar results can be obtained using the techniques from [8, 13, 4] but the proof is also more complex than our approach. Vice versa, Theorem 1.1 yields alternative proof of compactly supported version of the results described in [8, 13, 4]. However the number of required derivatives will be slightly bigger in this approach.

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References

[1] W. Arendt, Gaussian estimates and interpolation of the spectrum in $L^p$. Differential Integral Equations 7 (1994), no. 5–6, 1153–1168.

[2] S. Boutayeb, T. Coulhon and A. Sikora, A new approach to pointwise heat kernel upper bounds on doubling metric measure spaces. Adv. in Math. 270 (2015), 302–374.

[3] J. Cheeger, M. Gromov and M. Taylor, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds. J. Differential Geom. 17 (1982), no. 1, 15–53.

[4] P. Chen, E.M. Ouhabaz, A. Sikora and L.X. Yan, Restriction estimates, sharp spectral multipliers and endpoint estimates for Bochner-Riesz means. J. Anal. Math. 129 (2016), 219–283.

[5] T. Coulhon, B. Devyver, A. Sikora, Gaussian heat kernel estimate: from functions to forms, preprint, arXiv:1606.02423.

[6] T. Coulhon and A. Sikora, Gaussian heat kernel upper bounds via the Phragmén-Lindelöf theorem. Proc. Lond. Math. Soc. 96 (2008), no. 2, 507–544.

[7] T. Coulhon and Qi S. Zhang, Large time behavior of heat kernels on forms. J. Differential Geom. 77 (2007), no. 3, 353–384.

[8] M. Cowling and A. Sikora, A spectral multiplier theorem for a sublaplacian on SU(2). Math. Z. 238 (2001), no. 1, 1–36.

[9] E.B. Davies, The functional calculus. J. London Math. Soc. 52 (1995), 166–176.

[10] E.B. Davies, $L^p$ spectral independence and $L^1$ analyticity. J. London Math. Soc. 52 (1995), 177–184.

[11] E.B. Davies and B. Simon, $L^p$ norms of non-critical Schrödinger semigroups. J. Funct. Anal. 102 (1991), 95–115.

[12] E.B. Davies, B. Simon and M. Taylor, $L^p$ spectral theory of Kleinian groups. J. Funct. Anal. 78 (1988), 116–136.

[13] X.T. Duong, E.M. Ouhabaz and A. Sikora, Plancherel-type estimates and sharp spectral multipliers. J. Funct. Anal. 196 (2002), 443–485.

[14] A. Grigor’yan and L. Saloff-Coste, Heat kernel on manifolds with ends, Ann. Inst. Fourier (Grenoble) 59 (2009), no.5, 1917–1997.

[15] L. Hörmander, Estimates for translation invariant operators in $L^p$ spaces. Acta Math. 104 (1960), 93–140.

[16] S.G. Mikhlin, Multidimensional Singular Integrals and Integral Equations. Translated from the Russian by W. J. A. Whyte. Pergamon Press, Oxford, 1965.

[17] M. Murata, Positive solutions and large time behaviors of Schrödinger semigroups. J. Funct. Anal. 56 (1984), 300–310.

[18] M. Murata, Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbb{R}^n$. Duke Math. J. 53 (1986), 869–943.

[19] E.-M. Ouhabaz. Analysis of heat equation on domains, London Mathematical Society monographs, Princeton University Press, 2005.

[20] A. Sikora, Riesz transform, Gaussian bounds and the method of wave equation. Math. Z. 247 (2004), 643–662.

[21] E.M. Stein, Harmonic analysis: Real variable methods, orthogonality and oscillatory integrals. With the assistance of Timothy S. Murphy. Princeton Mathematical Series, 43. Monographs in Harmonic Analysis, III. Princeton University Press, Princeton, NJ, 1993

Peng Chen, Department of Mathematics, Sun Yat-sen (Zhongshan) University, Guangzhou, 510275, P.R. China

E-mail address: chenpeng3@mail.sysu.edu.cn
ADAM SIKORA, DEPARTMENT OF MATHEMATICS, MACQUARIE UNIVERSITY, NSW 2109, AUSTRALIA
E-mail address: adam.sikora@mq.edu.au

LIXIN YAN, DEPARTMENT OF MATHEMATICS, SUN YAT-SEN (ZHONGSHAN) UNIVERSITY, GUANGZHOU, 510275, P.R. CHINA
E-mail address: mcsylx@mail.sysu.edu.cn