Permanent confinement in compact QED$_3$ with fermionic matter

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We argue that the compact three dimensional electrodynamics with massless relativistic fermions is always in the confined phase, in spite of the bare interaction between the magnetic monopoles being rendered logarithmic by fermions. The effect is caused by screening by other dipoles, which transforms the logarithmic back into the Coulomb interaction at large distances. Possible implications for the chiral symmetry breaking for fermions are discussed, and the global phase diagram of the theory is proposed.

It is well known that the pure compact electrodynamics in three dimensions (3D) is always in the confined phase, i. e. in the phase where magnetic monopoles form a neutral plasma of free magnetic charges [1]. When the gauge field is coupled to matter, however, new possibilities emerge. In particular, Abelian gauge fields coupled to massless relativistic fermions turns the theory into a valuable toy model for the QCD. It has been argued that coupling to massless relativistic fermions turns the interaction between monopoles from $1/x$ to $-\ln(x)$ at large distances $x$ [1], [2]. cQED$_3$ also represents maybe the simplest theory that should contain the physics of confinement and chiral symmetry breaking, and as such it may be used as a valuable toy model for the QCD. It has been argued that coupling to massless relativistic fermions turns the interaction between monopoles from $1/x$ to $-\ln(x)$ at large distances $x$ [3], [4], [5], [6], [7], so that the deconfined phase with bound monopole-antimonopole pairs may become stable at low (effective) temperatures, in close analogy to the Kosterlitz-Thouless (KT) transition. If so, this would suggest that, at least for some values of coupling constants, compactness of the gauge field may be neglected at large distances, and that the results pertaining to the continuum QED$_3$ are likely to remain valid. In particular, the chiral symmetry breaking instability of the continuum QED$_3$ that has been linked to antiferromagnetism in cuprates might then be expected to survive essentially intact.

In this paper we consider the effects of finite density of monopole-antimonopole pairs (dipoles) on the putative, KT-like, confinement-deconfinement transition in the cQED$_3$. We find that, in stark contrast to two dimensions (2D), screening by logarithmically interacting magnetic charges bound to dipoles in 3D always alters the form of the interaction between the charges, from $-\ln(x)$ at large distance, to $1/x$. We demonstrate this first by an elementary electrostatic argument, and then support it by a controlled momentum shell renormalization group on the equivalent sine-Gordon theory. As a result, largest dipoles are always unbound, and would be expected to provide linear confinement of electric charges. Finally, we argue that the absence of the phase transition in the cQED$_3$ in terms of magnetic monopoles should imply the same in terms of fermions. Relying on the existing numerical results and qualitative arguments, we suggest that the phase of free monopoles corresponds to the (confined) phase of broken chiral symmetry for fermions, and that the (deconfined) chirally symmetric phase exists only in the continuum limit, and above a certain phase of fermion species [8]. The conjectured phase diagram is presented in Fig. 1.

We are interested in the cQED$_3$ defined on a cubic lattice:

\[
S = S_F(\chi, \theta_{\mu}) - \beta \sum_{x, \mu, \nu} \cos(F_{\mu\nu}),
\]

where $F_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma} \Delta_\mu \theta_\rho$ is the lattice version of the electromagnetic tensor, and the coupling to fermions on a lattice may be written as

\[
S_F = \frac{1}{2} \sum_{x, \nu} \sum_{i=1}^{N/2} \eta_i(x) [\bar{\chi}_i(x) e^{i\theta_\nu(x)} \chi_i(x + \hat{\nu}) - \bar{\chi}_i(x + \hat{\nu}) e^{-i\theta_\nu(x)} \chi_i(x)],
\]

using the standard staggered fermions. Here, $x$ denotes the lattice sites and $\hat{\nu}$ the direction of the links. In the continuum, $S_F$ corresponds to $N$ species of four component Dirac fermions [9].

The cQED$_3$ has been previously studied numerically in [10]. Here, to make progress analytically and following [11], we approximate [10] with

\[
S \approx \frac{1}{2} \sum_{x, \nu, \mu} [F_{\mu\nu}(x) - 2\pi n_{\mu\nu}(x)](\beta + \frac{N}{16|\Delta|}) \quad \text{for } |\Delta| \gg 1
\]

where $1/|\Delta|$ should be understood as an inverse of the square root of the lattice gradient squared. The term proportional to the coupling $\beta$ in Eq. (3) is the standard Villain approximation to the second term in (1). The form of the term proportional to $N$ in (3) is motivated by the known continuum limit of the fermion polarization: neglecting compactness of $a_\mu$ by setting $n_{\mu\nu}(x) \equiv 0$ would give the standard one-loop Gaussian action in the continuum, with the correct coefficient $\frac{N}{16|\Delta|}$. Just like in the $\sim \beta^3$ term, here we also retained only the leading power of fields and their lattice derivatives. Eq. (3) may be understood as the quadratic approximation to cQED$_3$ in Eq. (1). It leads to the expected logarithmic interaction between the magnetic monopoles, and therefore appears to contain the essential physics of cQED$_3$ we wish to address.
the original gauge coupling $\beta$

fermions ($\vec{x}$) in the continuum. In presence of

phase at all $\beta < 0$. This corresponds to a “critical” number

of fermion components $N = 24$ [3, 4, 5]. This argument neglects the effect of other dipoles, however. In the

standard KT transition their presence does not change the

critical temperature obtained by the simple energy-

entropy argument, since the KT fixed point lies at zero

temperature always dominates, and the largest dipoles should



\[ \Delta F = \frac{1}{2} \ln L - T \ln L^3, \quad (6) \]

so that for $T < 1/6$ monopoles would be expected to be

bound in pairs. This gives the critical temperature of

fermions as

\[ T_c = \frac{\beta}{\ln \left( \frac{L}{\Lambda} \right)}, \]

\[ \Lambda = \sqrt{\frac{\hbar}{m}}, \]

\[ N = 24. \]

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FIG. 1: The proposed phase diagram of the cQED. True

phase transition between the chirally symmetric (CS) and chiral

symmetry broken (CSB) phases of fermions occurs only in

the continuum limit ($\beta \to \infty$). Monopoles are in the plasma

phase at all $\beta < \infty$ and $N \to \infty$. Dashed line marks a
crossover that corresponds to the $N_{\text{cross}}(\beta)$, as discussed in

the text.

Standard duality transformations 12 on 13 lead to
the sine-Gordon action in the continuum

\[ S_{SG} = \int d^3\vec{x} \left[ \frac{\left( \vec{\nabla} \Phi(\vec{x}) \right) \left( \vec{\nabla} \left( \frac{\Phi(\vec{x})}{\pi^2(N + 16)} \right) \right)}{2} - 2y \cos \Phi(\vec{x}) \right], \quad (4) \]

where $y \ll 1$ is the fugacity of monopoles, and $\vec{x}$ is the

position vector in the continuum. In presence of

fermions ($N \neq 0$), at small momenta one can expand in

the original gauge coupling $\beta$ in Eq. (4), which becomes

a coefficient of the term quartic in derivatives, and thus

irrelevant by power counting. We may therefore safely set

it to zero, and return to its (non-universal) effects on the

phase diagram later. We also find it useful to have an alter-
native representation of the partition function defined by $S_{SG}$,

\[ Z = \sum_{n=0}^{\infty} \frac{y^n}{n!} \int d^3\vec{x} \exp \left[ -\frac{1}{2T} \sum_{i,j} q_i q_j V(|\vec{x}_i - \vec{x}_j|) \right], \quad (5) \]

in terms of unit magnetic charges $q_i = \pm 1$ at an effective

temperature $T = 4/N + O(\beta)$, interacting at large
distance via $V(x) = -\alpha (x\Lambda)$. $\Lambda$ is the ultraviolet cutoff

implicit in Eq. (4). The sine-Gordon field in Eq. (4) is then

\[ \langle \Phi(\vec{x}) \rangle = \langle i/T \rangle \sum_i q_i \langle V(|\vec{x}_i - \vec{x}|) \rangle \]

where $\langle \ldots \rangle$ denotes averaging with respect to $Z$.

The excess of free energy due to a single isolated

monopole in the sample of a linear size $L$ is therefore

\[ \Delta F = \frac{1}{2} \ln L - T \ln L^3, \quad (6) \]

in spirit of multipole expansion 14. Thus, for a medium

with a monopole density $\rho(x)$ and a dipole moment den-
sity $\vec{P}(\vec{x}) = \rho(x) \vec{x}$, the potential is given by

\[ V(\vec{x}) = \int d^3\vec{x}' \left( -\ln |\vec{x} - \vec{x}'| \right) \rho(\vec{x}') - \vec{x} \cdot \vec{P}(\vec{x}) \right]. \quad (8) \]

The energy of a small dipole with a moment $\vec{P} = q\vec{r}$

($q = 1$) at $\vec{x}$ in a weak potential $V(\vec{x})$ is still $\vec{P} \cdot \nabla V(\vec{x})$. Assuming non-interacting dilute dipoles at a temperature

$T$ then gives the average dipole moment at $\vec{x}$

\[ \langle \vec{P}(\vec{x}) \rangle = \frac{\langle r^2 \rangle}{3T} (-\nabla V(\vec{x})). \quad (9) \]

Since the density of dipoles is proportional to $y^2$, the polarizability of the medium is thus

\[ \chi \sim \frac{\langle r^2 \rangle}{3T} y^2, \quad (10) \]

where the constant of proportionality depends on the pre-
cise form of the short distance regularization of the log-
arithmetic interaction.

Writing Eq. (8) in Fourier space and combining Eqs.

(9) and (10) into $\vec{P}(\vec{q}) = -i\chi q V(\vec{q})$, we finally obtain the

interaction due to an external point charge $Q$ in presence

of a finite density of dipoles

\[ V(\vec{q}) = \frac{Q}{|\vec{q}|^3 + \chi q^2}, \quad (11) \]

or, in real space

\[ V(\vec{x}) = \frac{1}{4\pi \chi} \frac{1}{x^2} + O \left( \frac{1}{x^4} \right). \quad (12) \]

This implies that in presence of other dipoles the first

(energy) term in Eq. (11) scales like $1/L$, so that the en-
tropy always dominates, and the largest dipoles should
with momenta $\Lambda/b < q \ll \Lambda$ one finds that the original coupling is renormalized as $y$ to the second order in $y$, generated at lower cutoffs, and 2) $N$ is marginal to the standard sine-Gordon problem in 2D, our regularization scheme gives the correct anomalous dimension at the KT transition. With this regularization we find

$$G_{\geq}(\bar{x}) = \frac{\pi^2 N \ln(b)}{2 \pi^2 N a} e^{-\pi H} + O((\ln b)^2),$$

and therefore

$$\hat{y} = \left[3 - \frac{\pi N}{8(2 + \pi^2 N \hat{a})}\right] y + O(y^3),$$

$$\hat{a} = \hat{a} + \frac{8\pi^2 N}{2 + \pi^2 N} y^2 + O(y^3),$$

where $\hat{y} = y/\Lambda$ and $\hat{a} = a/\Lambda$, are the dimensionless couplings, and $\hat{b} \equiv dz/d\ln b|_{b=1}$.

Two important features of the flow equations (19)-(21) should be noted: 1) the relevant coupling $a$, even if absent initially, becomes generated at lower cutoffs, and 2) $N$ is marginal to $O(y^3)$. We suspect $N$ to be an exactly marginal coupling, since the action for slow modes should be analytic in $(q/\Lambda)^2$, so the coefficient of the non-analytic $|q|^3$ term cannot get renormalized to any order $O(y^3)$.

The flow in the $\hat{y}$-$\hat{a}$ plane for $N < N_{cross}$ and $N > N_{cross}$, $N_{cross} = 48/\pi$, is depicted in Fig. 2. Starting from $\hat{a} = 0, \hat{y} = \hat{y}_0$, the fugacity in the former case monotonically increases as $b \to \infty$, while in the latter it begins to increase only for $b > b^*$, with $b^*(\hat{y}_0) \approx 1/(96\pi \hat{y}_0)$ at $N \gg 1$. Nevertheless, since $\hat{y} = 0$ is an invariant line under the scale transformations, fugacity must increase at large enough $b$ for any initial value. We interpret this upward flow of the fugacity as an indication that monopoles are always in the plasma phase.

Interestingly, massless fermions still make themselves felt in such a plasma, since the non-analytic $|\nabla|^3$ term in Eq. (4) translates into a power-law behavior of the screened interaction at long distances. Instead of the expected Debye-Hückel exponential decay, in the quadratic approximation to (4) we find $V_{sc}(x) \approx -6/(\pi^2 |y|^2 N|x|^6)$ at large $x$. The reader may recall a similar phenomenon.
of Friedel oscillations in metals, where a non-analyticity due to the Fermi surface produces a power-law behavior of the screened interaction as well. In the present case the sign of the above power-law term indicates over-screening, presumably due to the extremely long range of the bare (logarithmic) interaction between monopoles.

Let us next turn to possible implications of our findings for fermions. First, the dimensionless coupling $\beta = 1/\langle e^2 l \rangle$, where $e$ is the dimensionful charge, and $l$ the lattice spacing. The lattice gauge field $\theta_\mu(x) = e l A_\mu(x)$, where $A_\mu(x)$ is the gauge field in the continuum. The continuum limit is achieved by keeping $e$ constant and taking $l \to 0$. In other words, $\beta \to \infty$ should naively correspond to the continuum QED$_3$. In this theory chiral symmetry for fermions is expected to be broken, and the condensate $\langle \bar{\Psi} \Psi \rangle \neq 0$, for $N < N_{\text{ch}}$, where $N_{\text{ch}}$ is of order unity [3, 11, 24]. On the other hand, for $\beta \to \infty$ we argued that once fermions are integrated out, the theory has no phase transition. If correct, this implies that, had we proceeded in reverse and integrated out everything but the fermions, we would have had to find them in a single phase at any finite $\beta$. Relying on numerical evidence that free monopoles seem to enhance chiral symmetry breaking [11], we propose that it is the phase with broken chiral symmetry that survives a finite $\beta$.

It was found previously [21] that the critical $N$ for chiral symmetry breaking at $\beta = 0$ is significantly larger than $N_{\text{ch}}$ in the continuum limit. We expect that for $\beta > 0$, there is no true phase transition in the cQED$_3$, just the crossover occurring at $N_{\text{cross}}$. Making the simplest assumption that $\beta$ always stays irrelevant, the same would hold at any $\beta < \infty$, but with the crossover line shifted to $N_{\text{cross}}(\beta) = N_{\text{cross}} - 16 \beta A_0 \langle \bar{\Psi} \Psi \rangle$. The conjectured phase diagram that summarizes this discussion is presented in Fig. 1 $\langle \bar{\Psi} \Psi \rangle \neq 0$ everywhere, except on the line $1/\beta = 0$, $N > N_{\text{ch}}$.

To conclude, we argued that magnetic monopoles are always free in the cQED$_3$ with gapless relativistic fermions, and the theory is thus expected to be permanently confining. We proposed the phase diagram, and conjectured that chiral symmetry for fermions is always spontaneously broken. Our results seem to be in qualitative agreement with the early numerical results of ref. 22. They can also be generalized to the case with non-relativistic fermions with Fermi surface [23]. Although we have studied only the $U(1)$ theory coupled to fermions in this paper, similar arguments should be applicable to the case of critical scalar field [6, 22].

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