1. Introduction

1.1. Abstract. Object descriptors are generalizations of the ‘object’ aspect of a category, and have much less structure than sets. This paper gives the technical development of the theory, and the set theory located within it. The sequel, [Quinn], describes usage in everyday settings and in category theory, and discusses its use as a formal foundation for deductive mathematics. These replace an earlier version, [Quinn old].

The main result is that in this set theory there is an (almost) well-founded pairing that is universal in the sense that any (almost) well-founded pairing is isomorphic to a transitive subobject of this one. In particular, if an axiom system includes the axiom of foundation then all models of it are equivalent to such subobjects. The reason this object is the largest possible is that its “universe” does not support quantification, so cannot be a set in any theory.

The universal well-founded pairing satisfies the Zermello-Fraenkel-Choice axioms. We apply universality to show that a ZFC theory is either a truncation of the relaxed theory, or there is a ZFC set with a bijection to a subdomain of its universe that is not a set. This is a consequence of the role of first-order logic in ZFC, and is problematic for some applications. In fact, traditional axiomatic set theory has a vast and subtle literature, but much of it depends on first-order logic and is not relevant either to the development here or to mainstream mathematics. This is explained in more detail in [Quinn].

1.2. Some ideas. The larger context uses “assertion” logic. We might, for instance, assert that “a is known to be the same as b”. The main difference from standard binary logic is that assertions cannot be usefully negated. The negation of the example is “a is not known to be the same as b”, which has no logical force.

A “logical function” is a function that takes values in the two-element collection \{yes, no\}. These are key ingredients of binary logic, though they are mostly implicit in traditional treatments. A binary function determines a subcollection: the elements on which the function has value ‘yes’. Conversely, if \( A \supset B \) then \( a \mapsto (a \in B?) \) is supposed to define a logical function with support \( B \). Traditional treatments emphasize the subcollection. Here, however, we find subcollections that do not correspond to logical functions, so we focus on the functions. A lot of traditional material is clarified by this change in focus.

A logical domain is a collection of elements \( A \) with a logical pairing \( A \times A \to \text{y/n} \) such that applying it to \( (a, b) \) returns ‘yes’ if and only if \( a \) can be asserted to be the same as \( b \). Logical domains are the appropriate setting for binary logic. The powerset of a logical domain, denoted \( P[A] \), is the collection of all logical functions on \( A \). We say that a logical domain supports quantification if \( P[A] \) is again a logical domain. A logical domain supports \( n \)th order quantification if \( P^j[A] \) is a logical domain for \( j \leq n \). Finally, relaxed sets are logical domains that support infinite-order quantification.

1.3. Outline. Section 2 describes the primitives of descriptor theory (undefined objects, core logic, and assumed hypotheses). Primitive objects are essentially the ‘object’ and ‘morphism’ primitives of category theory. These emerged from a great deal of trial and error with set theory as the goal: their suitability for categories is a bonus rather than a design objective. The primitive logic is weaker than standard binary logic, and in particular does not include the “law of excluded middle”. Most
of it uses assertions rather than binary (yes/no) logic. The primitive hypotheses are essentially standard, including the axiom of Choice.

Section 3 describes logical functions, logical domains, and quantification.

The next two sections concern traditional topics (well-orders and cardinals), but reverse some of the logic. Usually they are developed within a set theory. Here we develop them in a more general context, and then use them to show the context is indeed a set theory. We briefly repeat arguments that have been well-known for a century to be sure the logic reversal does not cause problems, and to squeeze out a bit more information.

Section 6 gives the description of the universal well-founded pairing. This uses Cantor’s Beth function (§5.4) as a template, and the proof of universality is a version of Mostowski collapsing. This satisfies all the Zermello-Fraenkel-Choice axioms, though the Separation axiom is not restricted to first-order logic. As mentioned above, we use the universality to find, for most ZFC implementations, a ZFC set with a bijection to a subdomain of the associated universe that is not a ZFC set.

Finally, section 7 discusses the “quantification gap”: domains that support finite-order quantification but not infinite-order. This is the main unsettled structural issue in the theory.

2. Primitives

There are three types of irreducible ingredients: primitive objects, primitive logic, and primitive hypotheses. We particularly focus on the logic, since it is weaker than the binary logic used in mainstream mathematics and embedded in our language.

2.1. Primitive objects. Standard practice in mathematics is to define new things in terms of old, and use the definition to infer properties from properties of the old things. The old things are typically defined in terms of yet more basic things. But to get started there must be objects that are not defined. Properties and usage of primitive objects must be specified directly, since they cannot be inferred from a definition. Reliability of secondary objects is inherited from that of the primitives. Reliability of primitives is an experimental conclusion, based on extensive downstream development that has not revealed contradictions.

The primitive objects here are essentially the “object” and “morphism” primitives of category theory. This was not a deliberate choice: set theory was the goal, but a great deal of experimentation led to category theory anyway.

Object descriptors, usually shortened to just “descriptors”, describe things, and are indicated by the symbol $\in$. Usage takes the form $x \in A$, which we read as “$x$ is an output of the descriptor $\in A$”, or “$x$ is an object in $A$”.

The term “descriptor” is supposed to suggest that these describe things but, unlike the “element of” primitive in set theory, they have no logical ability to identify outputs. In more detail, if $x$ is already specified then in standard set theory the expression “$x \in A$” may be expected (by excluded middle) to be ‘true’ or ‘false’. $x \in A$ may be an assignment of a symbol to an object in $A$, but if $x$ has already been specified then it is an assertion that must be proved. Assuming that it is either true or false is a usage error that invalidates arguments. See the next section for more about assertion logic.
Syntax for defining descriptors takes the form “\( x \in A \) means ‘…””. For example, the descriptor whose objects are themselves descriptors is defined by:

\[ A \in \Omega \text{D} \text{ means “} A \text{ is an object descriptor.”} \]

A more standard example is the descriptor for groups. “\((G, m)\) is a group”, or \((G, m) \in \Omega \text{(groups)}\) means “\( G \) is a set, \( m \) is a binary operation that is associative and has a unit and inverses”.

So, for example, if \((G, m)\) is a set with a binary operation, then the assertion \((G, m) \in \Omega \text{(groups)}\) means that we have verified that \( m \) is associative and has a unit and inverses.

Finally, the description of descriptors is vague, and there are frivolous examples. For example, define \( t \in X \) to mean “\( t \) is a Tuesday in September 1984”. The later use of logical functions filter these out. Logical functions are not time-dependent, for instance, so they cannot recognize anything in the physical world.

Morphisms of descriptors are essentially the primitives behind functors of categories. “\( f : \in A \rightarrow \in B \) is a morphism” means that every object \( x \in A \) specifies an object \( f[x] \in B \).

“Specifies” can be made more precise, but this form seems to work well enough in practice that we forego complications. Morphisms have some of the structure expected of functors: for instance morphisms \( A \xrightarrow{f} B \) and \( B \xrightarrow{g} C \) can be composed to get a morphism \( A \xrightarrow{g \circ f} C \).

This composition is associative, for the usual reason, but stating this requires use of the assertion form of \( = \), see below.

The definition of ‘morphism’ is implicitly a descriptor. Given descriptors \( \in A \), \( \in B \), the morphism descriptor is defined by: \( f \in \text{morph}[A, B] \) means “\( f \) is a morphism \( A \rightarrow B \)”.

2.2. Assertion logic. Logic provides methods of reasoning with primitive objects and hypotheses. Traditional set theory uses binary (ie yes/no valued) logic. The core logic here is weaker in that it is based on assertions. We describe primitive logical terms, and provide examples to illustrate usage.

There is a language problem: binary logic is deeply embedded in both our natural language and our mathematical thinking. If we say “\( A \) is \( B \)”, for instance, it is usually implicit that this is a logical function that returns ‘yes’ if it is true, and ‘no’ if not. In particular we can formally negate to get “\( A \) is not \( B \)” simply by interchanging ‘yes’ and ‘no’. To avoid this we use the word ‘known’ in natural-language formulations, as for example, “\( A \) is known to be \( B \)”.

This is sometimes awkward, but it does prevent unwarranted negation. The formal negation of “\( A \) is known to be \( B \)” is “\( A \) is not known to be \( B \)”, which has no logical force.

Assertions. An assertion is a statement that is known to be correct. \( a \in B \), for instance, is implicitly an assertion because it means “\( a \) is (known to be) an output of \( B \)”.

Assertions are often indicated by ‘!’

For clarity, statements implemented by logical functions are often indicated by ‘?’. For example, suppose \( a \in A \) and \( A \supset B \). Then “\( a ? \in B \)” means “there is a logical function \( (#? \in B) : A \rightarrow y/n \), and \( a ? \in B \) is the value of that function on
a”. Officially the statement itself does not include an assertion about its value, but it is common to omit “is ‘yes’ ” when the context doesn’t make sense otherwise. For example if \( h: B \to y/n \) is a logical function on a set then the collection of elements it identifies is traditionally written \( \{ a \in B \mid h[a] = \text{yes} \} \), rather than \( \{ a \in B \mid h[a] \text{ is ‘yes’} \} \).

Examples. Examples relevant to set and descriptor theory include:

1. “\( \#A \) is read as “it is known that \( a \) is an output of the descriptor \( \in A \)”.
2. “\( \#b \) is read as “\( a \) is known to be the same as \( b \)” (see below for “same as”);
3. “\( \#A \mid (\ldots) \)” is read as “it is known that there exist an element \( a \) such that (\ldots) holds”. Beware that this is not quantification in the traditional sense, because it is not a logical function of (\ldots).
4. “\( \#A \mid \ldots \)” is read as “it is known that for all \( a \), (\ldots) holds”.
5. (negative assertions) we can assert that something is not the case. For instance \( \#a \neq b \) means “\( x \) is known not to be the same as \( b \)”.
6. \( \# \) is the denial modifier, read as “it cannot be that”. For example \( \#a \neq b \) translates as “it cannot be that \( a \) is known to be the same as \( b \)”.

We also use the common notation := for “defined by”. Note (:=) ⇒ (\#).

Example: Russell’s paradox. This gives a statement that cannot be negated, so cannot be implemented by a logical function. There are mathematically more interesting examples, for instance the failure of quantification on \( \mathcal{W} \), but this one needs essentially no preparation.

Suppose \( \in A \) is a descriptor, and consider the statement \( \in A \in A \). This makes sense as an assertion, ie. “consider \( A \) such that \( A \) is known to be an object of itself”. Suppose for a moment that it is implemented by a logical function defined on descriptors. In that case we could define a descriptor \( \in \# \in \# \) by: \( \in \# \in \# \) means “the logical function \( \#? \in \# \) has value ‘no’ on \( A \)”. This leads to a contradiction: it is easily seen that \( \# \) is not equivalent to either ‘yes’ or ‘no’. The argument is valid except possibly for the assumption about a logical function. Since it leads to a contradiction, such a logical function cannot exist.

More about ‘\( \# \)’. Officially, \( \# \neq \) means that \( a, b \) are symbols representing a single output of a descriptor.

Example: This formulation of “same as” makes the usual proof of associativity of composition work. Explicitly, suppose

\[
A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4
\]

are morphisms of descriptors. Then we can see \( f_3 \circ (f_2 \circ f_1) \neq (f_3 \circ f_2) \circ f_1 \).

Example: Sometimes “same as” can be expressed in terms of “lower-level” information. For example, suppose \( f: \in A \to \in B \) are morphisms of descriptors. Then \( f \neq g \) is equivalent to \( \forall a \in A, f[a] \neq g[a] \). Thus sameness of morphisms reduces to sameness of images in the range descriptor and an assertion of quantification in the domain.
2.3. Primitive hypotheses. Primitive hypotheses are assertions that we believe are consistent, but cannot justify by reasoning with other primitives. Instead we regard these as experimental hypotheses. Extremely heavy use suggests all of these should be reliable. It seems likely that a hypothesis about quantification will be added, but alternatives must be explored and tested first; see the Note on unions, below.

Hypotheses.

**Two:** There is a descriptor $\in \{\text{yes}, \text{no}\}$ such that $\text{yes} \in \{\text{yes}, \text{no}\}$ and $\text{no} \in \{\text{yes}, \text{no}\}$ and if $a \in \{\text{yes}, \text{no}\}$ then either $a \neq \text{yes}$ or $a \neq \text{no}$, but not both.

**Choice:** Suppose $f: \in A \to \in B$ is a morphism of object descriptors and $f$ is known to be onto. Then there is a morphism $s: \in B \to \in A$ so that $f \circ s$ is (known to be) the identity. We refer to such $s$ as sections of $f$.

**Infinity:** The natural numbers are a set (ie. a logical domain that supports infinite-order quantification).

**Union:** The disjoint union of a family of sets, indexed by a set, is a set.

(Automatic if the Quantification Hypothesis holds.)

Discussion: About Two: The force of this hypothesis is that, unlike general descriptors, we can tell the objects apart. This is an excluded-middle property that needs to be made explicit because we do not require the general principle.

The names 'yes', 'no' are chosen to make it easy to remember operations ('and', 'or', etc.). One might prefer '1' and '0' for indexing or connections to Boolean algebra. We avoid 'true' and 'false' because philosophers have attached a lot of baggage to them.

About Choice: The term “choice” comes from the idea that if a morphism is onto, then we can “choose” an element in each preimage to get a morphism $s$. Note that in general there is no logical-function way to determine if a morphism is onto, or if the composition is the identity. These must be assertions, as above.

The axiom of choice in traditional settings has strong consequences that have been extensively tested for more than a century. No contradictions have been found, and it is now generally accepted. The above form extends the well-established version to contexts without quantification. This extension has implicitly been used in category theory, again without difficulty.

About Infinity: Using primitive objects and hypotheses other than Infinity, we can construct the natural numbers $\mathbb{N}$ as a logical domain. However we cannot show that it supports quantification. The Infinity hypothesis asserts it supports infinite-order quantification, ie. is a (relaxed) set. This is essentially the same as the ZFC axiom.

About Union: This is essentially the union axiom of ZFC. It plays no role in the development, only in the properties of the final theory.

The Quantification Hypothesis. Quantification is built into the language of traditional set theory, so the traditional development gives no insight into how it fails in more generality. Qualitatively there are two possibilities: QH holds, or QH fails.

QH asserts that if a domain supports quantification then so does its powerset. By induction this implies that the domain supports infinite-order quantification; ie. is a relaxed set. In this case the Union hypothesis holds automatically. If QH is false then there is a maximal strong-limit cardinal, and a version of the von
Neumann size axiom holds: a logical domain that supports quantification is a set if and only if it has cardinality less than the maximal strong-limit cardinal. The Union hypothesis asserts that this cardinal is regular. See §7 for further details.

3. Logical domains, and quantification

We provide formal definitions and basic properties of logical functions, domains and quantification. Only first-order quantification is used in this section.

3.1. Images and the set-builder notation. The set-builder notation is often described as follows: suppose \( A \) is a descriptor and \( P \) is a “property” that outputs of \( A \) might have. Then

\[
\{ a \in A \mid a \text{ has } P \}
\]

is the collection of elements of \( A \) that have \( P \).

In the context here this notation has two cases. The first is unproblematic: if \( P \) is implemented by a logical function \( p: A \to \{\text{y/n}\} \) (i.e. “\( a \) has \( P \)” means \( p[a] = \text{yes} \)), then \( \{b \in \{a \in A \mid a \text{ has } P \} \) is just an inefficient way to write \( p[a] = \text{yes} \). If a logical function is given then we often use the same notation for the subdomain, eg. write \( a \in p \).

The other case is when \( P \) is not known to be logical. For instance, if \( f: \in \in \in A \to \in \in \in B \) is a morphism of descriptors, \( P \) could be “is in the image of \( f \)”. More precisely, the image of \( f \) is the descriptor

\[
a \in \text{im}[f] \text{ means } \exists \exists b \in B \mid f[b] \neq a
\]

together with the projection to \( a \in A \). This is almost the standard set-builder notation for images. Writing it that way, and writing the element of \( B \) explicitly as a dummy variable gives

\[
a \in \text{im}[f] := \{ \exists \exists \# \in B \mid f[\#] \neq a \}
\]

Caution: the use of ‘\( \exists \exists \)’ means this expression does not automatically define a logical function of \( a \). Fortunately, we will see (in §4.5) that this is true in cases of interest, but until then careful distinctions are necessary.

3.2. Logical domains. A descriptor \( \in \in A \) is a logical domain, or simply “domain”, if there is a logical function of two variables that detects equality. Explicitly, there is \( ?= : A \times A \to \{\text{y/n}\} \) such that \( (a ?= b) \Rightarrow (a \neq b) \). We reserve the notation ‘\( ?= \)’ for this use, i.e. for equality-detecting pairings on domains.

Logical domain are first approximations to traditional sets, so we use traditional terms. Objects in a domain are referred to as elements, and we use \( a \in D \) instead of \( a \in D \).

Domains of equivalence classes. A pre-domain consists of \( \in \in D, \text{eqv} \) where

1. \( \in \in D \) is an object descriptor,
2. \( \text{eqv}[x, y] \) is a logical pairing \( D \times D \to \{\text{y/n}\} \);
3. \( \text{eqv} \) satisfies the standard requirements for an equivalence relation:
   - reflexive: \( \text{eqv}[x, x] = \text{yes} \);
   - symmetric: \( \text{eqv}[x, y] \Rightarrow \text{eqv}[y, x] \);
   - and transitive: \( \text{eqv}[x, y] \& \text{eqv}[y, z] \Rightarrow \text{eqv}[x, z] \).
The hypothesis that \( \text{eqv} \) pairing, and does not depend on the representatives \( x, y \). Thus (Cantor-Bernstein) Theorem.

We pose (logical function) \( J \) to emphasize it does not require quantification. It is put here to illustrate that we review to clarify quantification and binary-logic requirements. It is put here exactly those for which the quotient morphism \( D \to D/\text{eqv} \) is a bijection.

3.3. Cantor-Bernstein theorem. This is the first of a number of classical results that we review to clarify quantification and binary-logic requirements. It is put here to emphasize it does not require quantification.

Theorem. (Cantor-Bernstein) Suppose \( A, B \) are logical domains and \( A \to B \to A \) are injections with logical image. Then there is a bijection \( A \simeq B \).

Proof: Composing the injections reduces the hypotheses to the following. Suppose \( J : A \to A \) is an injection, \( j : A \to y/n \) detects the image of \( J \), and \( k \) is a logical function with \( j \subset k \subset A \), \( k \) detects \( B \). Then there is a bijection \( A \simeq k \).

Composing with iterates of \( J \) gives a sequence

\[
\cdots k \circ J^{n+1} \subset j \circ J^n \subset k \circ J^n \subset j \circ J^0 \subset k \circ J^0 \subset A.
\]

Define a new function \( \tilde{J} : A \to A \) by:

\[
\tilde{J}[a] := \begin{cases} 
\text{if } \exists n \in \mathbb{N} \text{ s.t. } a \in k \circ J^n - j \circ J^n, & \text{then } a \\
\text{if not} & \text{then } J[a]
\end{cases}
\]

Then it is straightforward to see that \( \tilde{J} \) is a bijection \( A \to k \simeq B \), as required. \( \square \)

The first case in the definition of \( \tilde{J} \) uses quantification over the natural numbers, but the Axiom of Infinity asserts that this is valid. In the second case “if not” makes sense because we can apply \( \text{not}[*] \) to a logical function.

3.4. Quantification. First, if \( A \) is a descriptor then the ‘powerset’ \( P[A] \) is the descriptor \(" h \in P[A] \) means ‘p is a morphism \( A \to y/n \)”\). The empty function on \( A \) is \( \emptyset[a] := \emptyset \), for all \( a \in A \).

Definition. A logical domain \( (A, \text{?=} ) \) supports quantification if there is a logical function \( P[A] \to y/n \) that detects the empty function.

We use “\( A \) is \( Q^1 \)” as shorthand for “\( (A, \text{?=} ) \) supports quantification”.

The traditional quantification notation for the empty-set detecting function is \( (\forall a \in A, h[a] := \emptyset) \). The logic here does not imply that such expressions define logical functions. However if there is a logical function that implements this particular expression, then all expressions using quantification over \( A \) will define logical functions. This is illustrated by:

Lemma. A domain supports quantification if and only if \( P[A] \) is a domain.

Suppose \( \text{?=} \) is a logical pairing on \( P[A] \) such that \( \text{?=} \Rightarrow \text{!=} \). Then \( \# \neq \emptyset \) is a logical function that detects the empty function. Conversely, suppose \( \phi : P[A] \to y/n \) detects \( \emptyset \), ie \( \phi[h] = \text{yes} \iff (h !\neq \emptyset) \). Define ‘\( \text{?=} \)” by

\[
(h \text{?=} g) := \phi[\text{not}[\text{#} \Rightarrow (h[#] = g[#])]]
\]
Then
\[(h \neq g) \implies (\forall \# \in A, h[\#] = g[\#]) \implies (h \neq g).\]

\[\square\]

**Operations on \(P[A]\).** The logical operations on \(y/n\) induce operations on logical functions \(A \to y/n\), and these correspond to traditional operations on subsets. Explicitly:

1. **intersection:** \(h_1 \cap h_2 := h_1 \& h_2\)
2. **union:** \(h_1 \cup h_2 := h_1 \or h_2\)
3. **complement:** \(D - h_1 := \not h_1\)

We caution that defining these for subdomains that are not logical requires assertion logic. Fortunately the Proposition in §4.5 shows that these do not occur in situations we consider.

### 3.5. Domains of functions.

This is a \(Q^1\) version of the Union hypothesis, and implies this hypothesis if \(QH\) holds.

**Proposition.** Suppose \(f: A \to B\) is a function of logical domains. If the image of \(f\) is \(Q^1\), and all point preimages \(f^{-1}[b]\) are \(Q^1\). Then \(A\) is \(Q^1\).

**Proof:** A logical function on \(A\) is empty if and only if the restriction to every \(f^{-1}[b]\) is empty. Since each \(f^{-1}[b]\) is \(Q^1\), there is a logical function that implements “is the restriction to \(f^{-1}[b]\) empty?”. The outputs of these define a logical function on \(im[f]\), and the function on \(A\) is empty if and only if this function is always ‘yes’. Or equivalently, if the negation is empty. But since \(im[f]\) is \(Q^1\) there is a logical function on \(P[im[f]]\) that detects this. Putting these together gives an empty-detecting function on \(P[A]\). \(\square\)

### 4. Well-orders

We briefly recall the properties of well orders, and show that there is a universal almost well-ordered domain. Basic properties of this domain are described. Only first-order quantification is used until the last subsection, §4.7.

#### 4.1. Definitions.

Suppose \((A, =)\) is a logical domain.

1. A **linear order** is a pairing \((\#1 \geq \#2) : A \times A \to y/n\) that is transitive; any two elements are related: \((a \geq b) \or (b \geq a)\); and elements related both ways are the same: \((a \geq b) \& (b \geq a) \iff (a = b)\).
2. Suppose \((A, \geq)\) is a linear order. A subdomain \(B \subset A\) is said to be **transitive** if \((a \in B) \& (a \geq b) \implies b \in B\).
3. \((A, \geq)\) is a **well-order** if it is a linear order, \(A\) is \(Q^1\), and if \(A \supset B\) is transitive then it is logical and either \(B = A\) or the complement \(A - B\) has a least element.
4. \((A, \geq)\) is an **almost** well-order if subdomains of the form \((\# < x)\) are well-ordered.

**Notes.**

1. In (3), note that ‘logical’ is a conclusion, not a hypothesis, but it is only required for transitive subdomains. Among other things, this ensures that ‘complement’ is defined. The quantification \((Q^1)\) requirement is needed for “\(B = A\)” to have value ‘yes’ or ‘no’, and to define “least element”.
(2) In a linear order “logical and least element in the complement” is equivalent to “is of the form \((\# < x)\) for some \(x\)”. The latter is easier to use, the former generalizes to well-founded pairings.

(3) The definition of ‘well-order’ is different from the usual one (cf. Jech [Jech] definition 2.3), but it is essentially equivalent and is better for the development here.

(4) As usual, well-orders are hereditary in the sense that if \((A, \geq)\) is well-ordered, and \(B \subset A\) is a logical subdomain, then the induced order on \(B\) is a well-order. Similarly for almost well-orders. If \(B\) is nonempty then it has a unique minimal element denoted by \(\min[B]\).

(5) We will see that an almost well-order fails to be a well-order if and only if the domain is not \(Q^1\), §4.3. It will turn out that there is only one of these, up to order-isomorphism.

Well-ordered equivalence classes. This is a variation on ‘Domains of equivalence classes’ in §3.2. We will use this construction in the definition of the universal almost well-order.

A linear pre-order consists of an object descriptor \(\in A\) and a logical pairing \(\geq[\#1, \#2]\) defined on pairs of outputs from the object descriptor. The pairing satisfies:

1. (transitive) \(\geq[a, b] \& \geq[b, c] \Rightarrow \geq[a, c]\);
2. (reflexive) \(\geq[a, a]\); and
3. (pre-linear) for all \(a, b \in A\), either \(\geq[a, b]\) or \(\geq[b, a]\) (or both).

Given this structure, define \(\text{eqv}[a, b] := (\geq[a, b] \& \geq[b, a])\). The quotient \(A/\text{eqv}\) is a logical domain, and \(\geq\) induces a linear order in the ordinary sense on elements (ie. on equivalence classes of objects). The additional conditions used to define well-orders in this context are the same as those on the element level.

4.2. Recursion. Our version is slightly different from the standard one (cf. [Jech], Theorem 2.15) in part because we do not find the standard one to be completely clear. There is a version for well-founded pairings in §6.2.

First we need a notation for restrictions. Suppose \(f: A \to B\) is a partially-defined function and \(D \subset A\) is \(Q^1\). Then \(f \upharpoonright D\) is the restriction to \(\text{dom}[f] \cap D\).

Now, suppose \((A, \geq)\) is an almost well-order, \(B\) is a domain, and \(R\) is a partially-defined function

\[ R: \text{tfn}[A, B] \times A \to B. \]

Here ‘tfn’ denotes functions with transitive domains. We refer to such an \(R\) as a recursion condition.

A (partially-defined) function \(f: A \to B\) is said to be \(R\)-recursive if:

1. \(\text{dom}[f]\) is transitive; and
2. for every \(c \in \text{dom}[f]\), \(f[c] = R[f \upharpoonright (\# < c), c]\).

Note that this is hereditary in the sense that if \(D \subset \text{dom}[f]\) is transitive then the restriction \(f \upharpoonright D\) is also recursive.

Proposition. (Recursion) If \((A, \geq), B, R\) are as above, then there is a unique maximal \(R\)-recursive (partially-defined) function \(r: A \to B\).

“Maximal” refers to domains: \(r\) is maximal if there is no recursive function with larger domain. We describe a criterion for maximality below. Note that the domain of \(r\) may not be \(Q^1\) (ie. may be all of \(A\)); the recursion condition applies
to restrictions to subdomains of the form $(# < a)$, and according to the definition of ‘almost’ well-order these are all $Q^1$.

The proof is essentially the same as the classical one; we sketch it to illustrate the slightly non-standard definitions. First, if $f, g$ are recursive and have the same domain, then they are equal. Suppose not and let $a$ be the least element on which they differ. Minimality of $a$ implies the restrictions to $(# < a)$ are equal. But then

$$f[a] = R[f \upharpoonright ( # < a)] = R[g \upharpoonright ( # < a)] = g[a]$$

a contradiction.

The maximal $r$ has domain the union $\cup (\text{dom}[f] : f \text{ is } R - \text{recursive})$. If $x$ is in this union then $x \in \text{dom}[f]$ for some recursive $f$. Define $r[x] := f[x]$. Uniqueness implies this is well-defined, and is recursive. □

A recursive $r$ is maximal if $\text{dom}[r] = A$ or, if $\text{dom}[r] \neq A$ and $a$ is the least element not in $\text{dom}[r]$, then $(r, a)$ is not in the domain of $R$.

As an aside, a recursion condition can be thought of as a sort of “vector field” on functions with transitive domain, and recursion gives a “flow” following this vector field. Specifically, a function $F$ is a ‘recursive extension’ of a function $f$ if $F \upharpoonright \text{dom}[f] = f$, and for each $c \in \text{dom}[F] - \text{dom}[f]$, $F[c] = R[F \upharpoonright ( # < c), c]$. The recursive extensions of $f$ trace out the flow line beginning at $f$. The proof above shows there is a unique maximal element in each such flow line. The traditional result concerns the flow line beginning with the empty function.

Order isomorphisms. The first application of recursion is a key result in traditional set theory. It is extended to well-founded pairings in §6.2.

**Proposition.** Suppose $(A, \geq), (B, \geq)$ are almost well-orders. Then there is a unique maximal order-isomorphism, from a transitive subdomain of $A$ to a transitive subdomain of $B$. Maximality is characterized by either full domain or full image.

This results from the recursion condition $R[f, a] := \min[B - f((# < a))]$. □

4.3. **Universal almost well-order.** In this section the domain $W$ is defined, and shown to be universal for almost well-orders. $W$ corresponds to the “ordinal numbers” of classical set theory, cf. Jech [Jech], §2.

**Definition of $W$.** $W$ is the quotient of an equivalence relation on a descriptor $WO$.

1. The object descriptor is defined by: $(A, \geq) \in WO$ means “$(A, \geq)$ is a well-order”;
2. $\text{geq}[(A, \geq), (B, \geq)] := \text{im}[r] \models B$, where $r : A \to B$ denotes the maximal order-isomorphism of transitive subdomains described just above.

We expand on (2). Since $A$ is $Q^1$, the image $\text{im}[r]$ is a logical function on $B$. Since $B$ is $Q^1$ the comparison of logical functions $\text{im}[r] \models B$ is a logical function. Finally, since $r$ is uniquely determined by $A, B$ and their well-orders, $\text{im}[r] \models A$ is a logical function of $A, B$.

$(WO, \text{geq})$ is a linear pre-order; let $(W, \geq)$ denote the quotient domain with its induced linear order. The elements of the domain $W$ are order-isomorphism classes, and we denote the class represented by $(A, \geq)$ by $\langle A, \geq \rangle$ (angle brackets).
Canonical embeddings. Recall that if \((A, \geq)\) is almost well-ordered and \(x \in A\) then \((\# < x) \subset A\), with the induced order, is well-ordered. It therefore determines an equivalence class \(((\# < x), \geq) \in W\). This defines the canonical embedding \(\omega : A \to W\). Explicitly, \(\omega[x] = ((\# < x), \geq)\).

**Theorem.** (Universality of \(W\))

1. \((W, \geq)\) is almost well-ordered, but not well-ordered because it does not support quantification;
2. If \((A, \geq)\) is almost well-ordered then the canonical embedding \(\omega : A \to W\) defined above is an order-isomorphism to a transitive subdomain, and \(\omega\) is uniquely determined by this property;
3. If \(A\) is \(Q^1\) then the image of \(\omega\) is \((\# < (A, \geq))\); if \(A\) is not \(Q^1\) then the image is all of \(W\).

**Proof of the theorem.** First, the properties of maximal order-isomorphisms described in §4.2 imply that \((W, \geq)\) is a linear pre-order. We next see that it is almost well-ordered. If \(a \in W\) then \(a\) is an equivalence class of well-ordered domains \((A, \geq)\). As explained in “canonical embeddings”, there is an order-preserving bijection, from \((A, \geq)\) to the subdomain \((\# < (A, \geq)) \subset W\).

Since \(A\) is well-ordered, so is the indicated subdomain of \(W\). But this is the definition of “almost” well-order.

This proves all but the second halves of (1) and (3). We take these up next.

**Quantification fails in \(W\).** To begin, we make explicit that quantification is the issue.

First, an almost well-order is a well-order if and only if the domain supports quantification. One direction is clear: a well-order is \(Q^1\) by definition. For the converse suppose \((A, \geq)\) is an almost well-order with \(A \not\equiv Q^1\), and suppose \(h\) is a transitive logical subdomain of \(A\). Since \(A\) is \(Q^1\), \(\operatorname{not}[h] \equiv 0\) is a logical function that returns either \(\text{yes}\), in which case \(h = A\), or \(\text{no}\), in which case \(\exists x \mid h[x] = \text{no}\). Transitive implies that \(h[y] = \text{no}\) for \(y \geq x\), so \(h\) has domain contained in \((\# < x)\). But \((\# < x)\) is well-ordered, so either \(h := (\# < x)\) or there is \(x > z\) so that \(h := (\# < z)\). In either case \(h\) has the form required to show \(A\) is well-ordered.

Next we use a form of the Burali-Forti paradox to show that \((W, \geq)\) cannot be a well-order. If it were, then by definition of \(W\) there would be \(x \in W\) and an isomorphism \(\omega : W \simeq (\# < x)\). But then, \(\omega^2\) gives an isomorphism with \((\# < \omega[x])\).

Since \(\omega[x] < x\), this contradicts the fact that isomorphism classes of well-orders correspond uniquely to elements of \(W\). Thus \((W, \geq)\) cannot be a well-order, and therefore does not support quantification.

The final part of the theorem is to see that if \(A\) is an almost well-order and not \(Q^1\) then the canonical embedding is an isomorphism to \(W\). Suppose there is \(x\) not in the image. The image is transitive, so must be contained in \((\# < x)\). But this is well-ordered, so a transitive subdomain of it must be \(Q^1\). This contradicts the hypothesis that \(A\) is not \(Q^1\). We conclude that there cannot be any such \(x\), and therefore \(\omega\) is onto. It follows that it is an order-isomorphism.

This completes the proof of the theorem.)

4.4. Existence of well-orders. We review a crucial classical theorem, and get some information about \(W\).
Proposition. Suppose \( A \) is a logical domain.

1. If \( A \) is \( Q^1 \) then it has a well-order.
2. If \( A \) is not \( Q^1 \) then there is an injection \( \mathcal{W} \to A \).
3. There is no logical function on domains that detects which of these alternatives holds.

Proof: As above, \( Q^1 \mathcal{P}[A] \) denotes the logical functions on \( A \) whose supports are \( Q^1 \), and \( Q^1 \mathcal{P}[A] \& (\# \neq A) \) the ones whose support is not all of \( A \). Comment on the logic: if \( A \) is \( Q^1 \) then \( Q^1 \mathcal{P}[A] = \mathcal{P}[A] \) is a logical domain and \( \# \neq A \) is a logical function on it. If \( A \) is not \( Q^1 \) then \( \mathcal{P}[A] \) is not a logical domain. But \( (\# \neq A) \) is still a logical function because it is always ‘yes’.

Define a choice function for \( A \) to be \( ch : (Q^1 \mathcal{P}[A] \& (\# \neq A)) \to A \), satisfying \( h[ch[h]] = \text{no} \) (ie. \( ch[h] \) is in the complement of \( h \)). The axiom of Choice implies that any \( A \) has a choice function, as follows: Define \( c : (Q^1 \mathcal{P}[A] \& (\# \neq A)) \times A \to y/n \) by \( c[h, a] := \text{not}[h[a]] \). The projection of (the support of) \( c \) to \( Q^1 \mathcal{P}[A] \) is known to be onto, due to the “not all of \( A \)” condition. According to Choice, there is a section of this, and sections are exactly choice functions.

We now set up for recursion. Fix a choice function and define a condition \( R: \text{tfn}[^A] \times A \to A \), where \( \text{tfn}[^A] \) denotes partially-defined functions with transitive domain, by:

\[
R[f, a] := ch[\text{im}[f \restriction (\# < a)]]
\]

In words, \( f[a] \) is the chosen element in the complement of the image of the restriction \( f \restriction (\# < a) \).

This is clearly a recursive condition. We conclude that partially-defined functions \( r: \mathcal{W} \to A \) satisfying:

1. \( \text{dom}[r] \subset \mathcal{W} \) is transitive;
2. \( f[a] = R[r \restriction (\# < a), a] \) holds for all \( a \in \text{dom}[r] \).

form a linear-ordered domain with a maximal element, and \( r \) is maximal if and only if either \( \text{im}[r] = A \) or \( \text{dom}[r] = \mathcal{W} \).

If \( \text{dom}[r] = \mathcal{W} \) then \( r \) gives an injective function \( \mathcal{W} \to A \). We will see, just below, that this implies \( A \) cannot be \( Q^1 \). If \( \text{dom}[r] \neq \mathcal{W} \) then \( r \) gives a bijection from a transitive proper subdomain of \( \mathcal{W} \) to \( A \). But a transitive proper subdomain has a well-order, so \( A \) has one also.

4.5. Logicality of subdomains.

Proposition. Subdomains of \( Q^1 \) domains or of \( \mathcal{W} \) are logical.

The hereditary aspect of this property shows that the \( Q^1 \) case is implied by the \( \mathcal{W} \) case, but we include it in the statement for later reference.

Proof: Abbreviate “all subdomains of \( A \) are logical” to “\( A \) is all-logical”. The first observation is that this property is hereditary in the sense that if \( X \subset B \subset A \) and \( B \) is logical in \( A \), then \( X \) is logical in \( B \iff \) logical in \( A \). In particular if \( f: A \to y/n \) detects \( X \) in \( A \) then the restriction \( f \restriction B \) detects \( X \) in \( B \).

The next step is to contradict the existence of \( a \in \mathcal{W} \) such that \( (\# < a) \) contains an illogical domain. The observation above implies “(\( \# < b \)) is all-logical” is a transitive property, so if there is an illogical \( a \) then there is a least one. Denote it by \( a \). We rule out possibilities.
(1) \(a\) cannot be finite. The hypothesis of Two implies the two-element domain \(y/n\) is all logical, and finite domains follow from this.

(2) \(a\) must be a limit. If not then \(a - 1\) is defined, and since \(a\) is infinite there is a bijection \((# < a) \simeq (# < a - 1)\). By minimality, \((# < a - 1)\) is all-logical. But “all logical” is a bijection invariant, so \(a\) is also all-logical, which contradicts the hypothesis about \(a\).

(3) If \(a\) is the limit of all-logical elements then it is all-logical. Suppose \(X \subset (# < a)\). If \(b < a\) then \(X \cap (# < b)\) is logical in \# < b\). Let \(f_b : (# < b) \to y/n\) be a logical function that detects it. The observation about restrictions at the beginning of the proof implies that if \(y < c < b < a\) then \(f_b[y] = f_c[y]\). Now define \(f : (# < a) \to y/n\) by: \(f[y] = f_b[y]\) for any \(b\) with \(y < b < a\). Since \(a\) is a limit there always is such \(b\), so \(f\) is well-defined, and detects \(X\). This implies \(a\) is all-logical, which contradicts the supposition that there is \(a \in \mathbb{W}\) that is not all-logical.

(4) According to the above, for every \(a \in \mathbb{W}\) \((# < a)\) is all-logical. But \(\mathbb{W}\) is the limit of these, so the argument in (3) implies \(\mathbb{W}\) is also all-logical.

4.6. Identifying \(\mathbb{W}\). We will need criteria for existence of a bijection \(\mathbb{W} \simeq A\).

Proposition. Suppose \(f : A \to \mathbb{W}\) is a function that is onto and has \(Q^1\) point inverses. Then there is a bijection \(b : \mathbb{W} \to A\) so that \(f \circ b\) is nondecreasing.

Since the point inverses are \(Q^1\) they have well-orders. Choice enables us to choose a well-order \((f^{-1}[a], \geq_a)\), for every \(a\). Now define an order on \(A\) by:

\[(x \geq y) := (f[x] > f[y]) \text{ or } (f[x] = f[y] \& (x \geq_{f[x]} y)).\]

This is easily seen to be an almost well-order, and \(f\) is nondecreasing with respect to this order. Finally, by the characterization theorem, \((A, \geq)\) must be order-isomorphic to \(\mathbb{W}\).

A logical function \(h : \mathbb{W} \to y/n\) is said to be \textbf{cofinal} if for every \(a \in \mathbb{W}\) there is \(b \in h\) such that \(b \geq a\). See Jech [Jech] §3.6. Note that \(P[\mathbb{W}]\) is the disjoint union of bounded and cofinal functions, but there is no logical function that identifies the two pieces.

**Corollary.** A logical function \(h\) on \(\mathbb{W}\) is (known to be) cofinal if and only if \(h\) with the induced order is (known to be) order-isomorphic to \(\mathbb{W}\).

Note that this is essentially the traditional definition of “regular cardinal”.

Proof: we show that \(h\) is \(Q^1\) if and only if it is not cofinal. If it is not cofinal then there is \(m\) larger than any element of \(h\). This means \(h\) is a logical function on \((# < m)\). But this is \(Q^1\) and a logical subdomain of a \(Q^1\) domain is \(Q^1\).

For the converse, define a function \(p : \mathbb{W} \to h\) by \(p[a] := \min[h[#] \& (# < a)]\).

If \(b \in h\) then (since \(h\) is cofinal) the preimage \(p^{-1}[b]\) is bounded and therefore \(Q^1\). According to the Proposition above, this implies \(\mathbb{W}\) is \(Q^1\) if and only if \(h\) is \(Q^1\). But \(\mathbb{W}\) is not \(Q^1\) so neither is \(h\). The induced order on \(h\) is an almost well-order, so by the last part of the Universal almost well-order Theorem, it is order-isomorphic to \(\mathbb{W}\).

**Corollary.** If \(A\) is not \(Q^1\) and there is an injection \(A \to \mathbb{W}\) then there is a bijection \(A \simeq \mathbb{W}\).

According to the previous section, the image of \(A\) is logical in \(\mathbb{W}\). Since \(A\) is not \(Q^1\), this image must be cofinal, therefore bijective to \(\mathbb{W}\).
4.7. Higher-order quantification. We say a domain $A$ is $Q^j$ if $P^j[A]$ is defined and is a logical domain. Let $Q^j\mathcal{W}$ denote the well-orders whose underlying domains are $Q^j$. It is easy to see that $Q^j\mathcal{W}$ is transitive in $\mathcal{W}$ (ie. the $Q^j$ property is hereditary), and $Q^j \subset Q^{j-1}$.

**Proposition.** There is $\tau \geq 1$ (the quantification threshold) such that $Q^\infty = Q^\tau \neq Q^{\tau-1}$. For $\tau > j \geq 1$, the domain $Q^j\mathcal{W}$ is $Q^{j-1}$ but not $Q^j$.

Proof: First, the sequence $Q^1\mathcal{W} \supset \cdots Q^j\mathcal{W} \supset Q^{j+1}\mathcal{W} \supset \cdots$ is non-increasing. This means it must be eventually constant; otherwise we would get a sequence with no minimal element in a well-ordered domain. This implies that there is a quantification threshold $\tau$, as claimed.

If $Q^1 = Q^\infty$, ie. $\tau = 1$, then the proof is complete because we know $\mathcal{W} = Q^1\mathcal{W}$ is $Q^0$ and not $Q^1$.

Suppose, then, that $\tau > 1$. For $j > 1$ $Q^j\mathcal{W}$ is bounded in $\mathcal{W}$, so the induced order is a well-order. Let $q_j \in \mathcal{W}$ denote its equivalence class. Since $Q^j\mathcal{W}$ is transitive, $q_j$ is the minimal element in the complement. But $Q^j\mathcal{W}$ is defined to be equivalence classes whose underlying domain is $Q^j$. Since $q_j$ is in the complement, its underlying domain $Q^j\mathcal{W}$ is not $Q^j$.

It remains to show that for $\tau \geq j > 1$, $Q^j\mathcal{W}$ is $Q^{j-1}$. For this we need $Q^{j-1}\mathcal{W} \neq Q^j\mathcal{W}$. But it is easy to see that if $Q^{j-1}\mathcal{W} = Q^j\mathcal{W}$ then the sequence stabilizes there: $j - 1 \geq \tau$. This contradicts the constraints on $j$. Alternatively, we can exhibit an element in the complement, $Q^{j-1}\mathcal{W} - Q^j\mathcal{W}$, namely $P^n[Q^\tau\mathcal{W}]$ for $n = \tau - j$. \hfill $\square$

5. Cardinals

Cardinals in set theories have been studied for well over a century. Here we turn this around a bit by developing cardinality for $Q^1$ domains, and then using it to study set theories.

5.1. **Definitions.** An element $a \in \mathcal{W}$ is a **cardinal element** if $b < a$ implies that there is no injective function $(\# < a) \rightarrow (\# < b)$. A **cardinal well-order** is one whose equivalence class is a cardinal element. In particular if $a$ is a cardinal element then $(\# < a)$ with its induced order is a cardinal well-order.

For this definition to be legitimate in the logic used here we need:

**Lemma.** There is a logical function $?inj: \mathcal{W} \times \mathcal{W} \rightarrow y/n$ such that $?inj[a, b] = \text{yes}$ if and only if there is an injection $(\# < a) \rightarrow (\# < b)$.

Fix $b \in \mathcal{W}$ and define a subdomain of $\mathcal{W}$ by

$$\{ y \in \mathcal{W} \mid \text{there is an injective function } (\# < y) \rightarrow (\# < b) \}.$$  

If $y \geq z$ then the inclusion $(\# < z) \rightarrow (\# < y)$ is an injective function. It follows that the subdomain is transitive, and given by either the logical function ‘yes’ (ie. all of $\mathcal{W}$) or $(\# < c)$ for some $c$.

**Definition of $\text{card}[A]$.** Suppose $A$ is a $Q^1$ domain. Choose a well-order $(A, \geq)$, then $\text{card}[A]$ is defined to the the smallest element in $?inj[(A, \geq), \#]$.

Note $(A, \geq) \in ?inj[(A, \geq), \#]$, so it is nonempty, and a nonempty logical subdomain of $\mathcal{W}$ has a least element. Minimality implies that $\text{card}[A]$ is a cardinal element of $\mathcal{W}$. It also implies that $\text{card}[A]$ is well-defined (ie. doesn’t depend on the choice of well-order on $A$).
The global perspective is that this defines a morphism of descriptors
\[ \text{card}: \mathbb{Q}^1 \to \mathbb{W}, \]
where \( \mathbb{Q}^1 \) is the descriptor whose outputs are \( \mathbb{Q}^1 \) logical domains.

### 5.2. Bijectons, logicality, and Cantor’s theorem.

**Lemma.** Suppose \( A \) is \( \mathbb{Q}^1 \), and \( (B, \geq) \) is a representative of the equivalence class \( \text{card}[A] \). Then there is a bijection \( A \simeq B \).

**Proof:** Choose a well-order on \( A \). By minimality, \( (A, \geq) \geq (B, \geq) \), so there is an inclusion \( B \to A \). By definition of \( \text{card}[A] \), there is an injection \( A \to B \). But the Cantor-Bernstein theorem 3.3 then asserts that there is a bijection \( A \simeq B \). \( \Box \)

**Corollary.** There is an injection \( A \to B \) if and only if \( \text{card}[A] \leq \text{card}[B] \), and a bijection \( A \simeq B \) if and only if \( \text{card}[A] = \text{card}[B] \).

**Higher-order quantification.** If the Quantification hypothesis is false \((\mathbb{Q}^1 \neq \mathbb{Q}^2)\) then it is easy to see that the elements denoted by \( q_j \in \mathbb{W} \) in 4.7 are cardinals for \( j \geq 2 \). They have the property that a \( \mathbb{Q}^1 \) domain \( A \) is \( \mathbb{Q}^j \) for some \( j \geq 2 \), if and only if \( \text{card}[A] < q_j \). In particular \( A \) is a relaxed set \((\mathbb{Q}^\infty)\) if and only if \( \text{card}[A] < q_\tau \), where \( \tau \) is the quantification threshold of 4.7. This is a version of von Neumann’s “axiom of size”, which postulates that sets can be identified by their cardinalities.

Note that in the above we had to know that \( A = \mathbb{Q}^1 \) before we could use cardinality to identify higher-order properties. This does not extend to give a way to identify \( \mathbb{Q}^1 \) domains. We know, for instance, that a subdomain of \( \mathbb{W} \) is \( \mathbb{Q}^1 \) if and only if it is bounded. However, there is no logical function that detects whether or not a subdomain is bounded.

**Cantor’s theorem.**

**Theorem.** (Cantor) If \( A \) is a nonempty \( \mathbb{Q}^2 \) domain then \( \text{card}[\mathcal{P}[A]] > \text{card}[A] \).

Note that for \( \text{card}[\mathcal{P}[A]] \) to be defined, \( \mathcal{P}[A] \) must be \( \mathbb{Q}^1 \), and therefore \( A \) must be \( \mathbb{Q}^2 \).

**Proof:** The conclusion is equivalent to: there is no surjective function \( A \to \mathcal{P}[A] \). Suppose \( p: A \to \mathcal{P}[A] \) is a function. Define a logical function \( h: A \to y/n \) by \( h[a] := \text{not}(p[a])[a] \). Then \( h \) is not in the image of \( p \), so \( p \) is not surjective. \( \Box \)

### 5.3. Hessenberg’s theorem.

In the classical development this is a key fact about cardinality of sets. Here it is a key ingredient in showing ‘relaxed sets’ have the properties expected of sets. We go through the proof to check the use of quantification, and because the traditional proof is somewhat muddled by the identification of sets and elements.

**The canonical order.** Suppose \((A, \geq)\), is a linear order and \( A \supset B \). The canonical order on \( A \times B \) is a partially-symmetrized version of lexicographic order.

First define the maximum function \( \text{max}: A \times B \to A \) by \((a, b) \mapsto \max[a, b] \). This induces a pre-linear order on \( A \times B \), namely
\[ (a_2, b_2) \geq (a_1, b_1) := \max[a_2, b_2] > \max[a_1, b_1]. \]

The canonical order refines this to a linear order as follows: Fix \( c \in A \), then the elements of \( A \times B \) with \( \text{max} \) equal to \( c \) have the form \((\# < c, c)\) or \((c, \# \leq c)\). Each of these is given the order induced from \( A \), and pairs of the first form are defined to
be smaller than pairs of the second form. The canonical order is denoted by $\geq_{can}$.

Note that if $c \notin B$ then there are no elements of the first form in $A \times B$.

More explicitly, $(a_2, b_2) >_{can} (a_1, b_1)$ means:

$$(\max[a_2, b_2] > \max[a_1, b_1]) \text{ or } ((\max[a_2, b_2] = \max[a_1, b_1]) \& ((a_2 > a_1) \text{ or } (a_2 = a_1 \& b_2 > b_1))).$$

The following is straightforward:

**Lemma.** If $A, B$ are almost well-ordered, then the canonical order on $A \times B$ is an almost well-order.

The classifying function gives an order-preserving injection

$$\omega: (A \times B, \geq_{can}) \rightarrow (W, \geq)$$

with transitive image. The main result is a variation on a century-old theorem of Hessenberg (see [Jech], Th. 3.5) describing some of these images.

**Theorem.** If $c \in W$ is an infinite cardinal and $c \geq d$ is nonzero then

$$\omega[(\# < c) \times (\# < d), \geq_{can}] = (\# < c)$$

The maximal case $c = d$ is the most useful: it implies that for infinite $A$, $\text{card}[A \times A] = \text{card}[A]$.

To begin the proof, note the image in $W$ is transitive so is of the form $(\# < w)$ for some $w$. We want to show that $w = c$.

The first step is to note that, since $d > 0$, $(\# < c) \times (\# < d)$ contains a copy of $(\# < c)$. Its cardinality is therefore at least $c$. Since $c$ is a cardinal, this implies $c \leq w$.

Next we work out the consequences of strict inequality in this last: suppose $w > c$. Since the image is transitive, there is $(x, y) \in (\# < c) \times (\# < d)$ with $\omega(x, y) = c$. It follows that there is a bijection from $((\#_1, \#_2) \leq \text{can}(x, y))$ to $(\# < c)$, so these have cardinality $c$. Denote the maximum of $(x, y)$ by $m$, then $((x, y) >_{can} (\#_1, \#_2))$ is contained in $(\# < m) \times (\# < m)$, which is bijective to $(\# < \text{card}[m]) \times (\# < \text{card}[m])$. But $m < c$ and $c$ is a cardinal, so $\text{card}[m] < c$.

Putting these together we see that if $w > c$ then there is an infinite cardinal $n := \text{card}[m]$ with $n < c$ and $\text{card}[(\# < n) \times (\# < n)] \geq c$. This violates the ‘maximal case’ conclusion of the theorem for a cardinal smaller than $c$. We can therefore proceed by induction on $c$.

If the Proposition is false there is a least infinite cardinal for which it fails; call this $c$. The above shows that failure at $c$ implies there is an infinite cardinal $n < c$ so that $\text{card}[(\# < n) \times (\# < n)] \geq c$. But this means the Proposition fails at $n$, contradicting the minimality of $c$. Therefore the theorem must be true. □

**Products and unions.** For finite sets, the cardinality corresponds to the number of elements. Cardinality of a disjoint union is therefore the sum of the cardinalities, and cardinality of a product is the product. The preceding section implies that the situation is much simpler for infinite sets.

**Proposition.** Suppose $A, B$ are $Q^1$ domains, and at least one is infinite. Then

1. $\text{card}[A \times B] = \max[\text{card}[A], \text{card}[B]]$, and
2. if $A, B \subseteq D$ then $\text{card}[A \cup B] = \max[\text{card}[A], \text{card}[B]]$. 


Proof: suppose \( \text{card}[A] \geq \text{card}[B] \), so \( \max[\text{card}[A], \text{card}[B]] = \text{card}[A] \). Then
\[
\text{card}[A] \leq \text{card}[A \times B] \leq \text{card}[A \times A] = \text{card}[A]
\]
The last step being the Proposition above. This gives (1).

For (2), \( \text{card}[A] \leq \text{card}[A \cup B] \leq \text{card}[A \times B] = \text{card}[A] \), by (1). \( \square \)

5.4. The Cantor Beth function. Cantor introduced several functions related to cardinals. Only one is needed here.

Definition. ‘Beth’ (\( \beth \)) is the second character in the Hebrew alphabet, and was used by Cantor to denote the “iterated powerset function”. \( \beth \) is briefly mentioned in [Jech] §5, p. 55. This, and the associated rank function, play major roles in the construction of the ZFC set theory in §6.

Proposition. There is a unique maximal function \( \beth : D \to \mathbb{W} \) satisfying:

1. \( D \) is a transitive subdomain of \( \mathbb{W} \);
2. if \( a = 0 \) then \( \beth[a] = \text{card}[N] \);
3. if \( a > 0 \) is not a limit and \( \beth[a - 1] \) is defined and \( a = Q^2 \), then \( a \in D \) and \( \beth[a] = \text{card} [P(\beth[a - 1])] \); and
4. if \( a \) is a limit of \( D \) then \( a \in D \) and \( \beth[a] = \sup \beth[# < a] \).

‘sup’ in (3) is the ‘supremum’, which is a convenient shorthand for the minimum of elements greater or equal to the image: \( \sup[h] := \min[\# \geq (\forall a | h[a] = \text{yes})] \).

It is straightforward to formulate a recursive condition on partially defined functions so that conditions (1)–(4) correspond to fully recursive. Existence of a maximal such function therefore follows from recursion. \( \square \)

We can be more precise about the domain of \( \beth \). If \( Q^1 = Q^\infty \) then \( D = \mathbb{W} \) and the image is cofinal in \( \mathbb{W} \). If \( Q^1 \neq Q^\infty \) then, as in §4.7, define \( q_\infty := \text{card}[Q^\infty \mathbb{W}] \). The Union hypothesis implies \( q_\infty \) is a regular cardinal, so \( \beth[q_\infty] = q_\infty \). \( D \) is a finite extension: \( D = (\# < q_\infty + \tau - 1) \), where \( \tau \) is the quantification threshold of §4.7.

Limits and strong limits. Recall that \( a \in \mathbb{W} \) is a limit if \( (# < a) \) does not have a maximal element. It is a strong limit if \( b < a \) implies \( \text{card}[P[# < b] < a] \).

Proposition. (1) \( b \leq \beth[b] \);

2. \( a \in \mathbb{W} \) is a strong limit if and only if \( a = \beth[x] \) for either \( x = 0 \) or \( x \) a limit; and

3. if \( b \) is a limit then the image \( \beth[# < b] \) is cofinal in \( (# < \beth[b]) \).

Proof: (1) is standard, and easily proved by considering the least element that fails.

For (2), suppose \( a \) is a strong limit. \( \beth \) is increasing, so \( \beth^{-1}[#] < a \) is transitive and therefore of the form \( (\# < x) \). Since \( b \notin (\beth^{-1}[#] < a) \), \( \beth[b] \geq a \).

We show \( b \) is a limit. If not then \( (# < b) \) has a maximal element, \( b - 1 \).
\( \beth[b - 1] < a \), so by definition of strong limit \( \text{card}[P[# < \beth[b - 1]]] < a \), but the left side is the definition of \( \beth[b] \), so this contradicts the choice of \( b \). We conclude \( (# < b) \) does not have a maximal element, so \( b \) is a limit.

Next, \( \beth[# < b] \) is bounded by \( a \), so \( \sup[\beth[# < b]] \) is defined, is \( \leq a \), and is the definition of \( \beth[b] \). But the choice of \( b \) requires \( \beth[b] \geq a \), so \( \beth[b] = a \), as required.

For the other direction of ‘if and only if’, suppose \( b \) is a limit. We want to show that \( a = \beth[b] := \sup[\beth[# < b]] \) is a strong limit. Suppose \( x < a \). Then there is \( y < b \) with \( x \leq \beth[y] \). Thus \( \text{card}[P[# < x]] \leq \text{card}[P[# < \beth[y]]] \). Next, since \( b \) is a
limit, \( y + 1 < b \). But the definition of \( \beth \) gives \( \text{card}[P(# < \beth[y])] = \beth[y + 1] \). Since \( \beth[y + 1] < \sup[\beth[# < b]] = a \), we get \( \text{card}[P(# < x)] < a \). This verifies the definition of strong limit.

The cofinality conclusion follows from the definition of ‘sup’. \( \square \)

A corollary of (3) in the Proposition is that the cofinalities are the same: \( cf[\beth[b]] = cf[b] \). Therefore the strong limit \( \beth[b] \) is regular if and only if \( b = \beth[b] \), and \( b \) is regular.

**Beth rank.** Rank plays a central role in the construction in Section 6. This rank follows the \( \beth \) function closely, and is a bit different from the rank function defined in [Jech] §6.2: it starts with \( \text{rank} = 0 \leftrightarrow \text{‘finite’} \), rather than \( \text{rank} = 0 \leftrightarrow \text{‘empty’} \), and otherwise differs by +1.

The (Beth) \( \text{rank} \) is the function \( \mathbb{W} \rightarrow \mathbb{W} \) given by

\[
\text{rank}[x] := \min[(\# \mid x < \beth[\#])],
\]

provided \( (x < \beth[\#]) \) is nonempty. It can be empty only if the Quantification hypothesis is false. In this case there is a maximal element \( \mu \) in the domain of \( \beth \), and we define the rank of \( x \geq \beth[\mu] \) to be \( \mu + 1 \).

Note that if \( a \) is a limit then there are no elements of rank \( a \). The reason is that \( \beth[a] = \sup[\beth[# < a]] \), so if \( x < \beth[a] \) then there is some \( b < a \) with \( x < \beth[b] \). These hidden values come into play in ranks of logical functions, defined next.

If \( h : \mathbb{W} \rightarrow \mathbb{W} \) has bounded \( (\Leftrightarrow Q^1) \) support then define

\[
\text{rank}[h] := \sup[\text{rank}((\# \mid h[\#])].
\]

**Examples and special cases.** If \( h[\#] = (\# = a) \), the function that detects \( a \), then the ranks of \( a \) and \( h \) are the same. More generally, \( h[a] \Rightarrow (\text{rank}[a] \leq \text{rank}[h]) \).

If \( h \) is cofinal in \( g \) then \( \text{rank}[h] = \text{rank}[g] \).

Finally, consider the logical function \( (\# < a) \). \( \text{rank}[\# < a] = \text{rank}[a] \) unless \( a = \beth[b] \) for some \( b \), in which case \( \text{rank}[\# < a] = b \) and \( \text{rank}[a] = b + 1 \).

### 6. The universal well-founded pairing

This section provides an interface between the present theory and traditional axiomatic set theory. The result is roughly that the universal well-founded pairing is a ZFC set theory, and a domain is a relaxed set (ie. \( Q^\infty \)) if and only if there is a bijection with a set in this ZFC theory.

#### 6.1. Main result.

**Definitions.** Most of these are standard, but included for precision. Suppose \( (A, \epsilon) \) is a logical pairing, and \( A \supseteq B \) is a subdomain.

1. As with well-orders, ‘transitive’ means \( b \in B \) and \( \epsilon[a, b] = \text{yes} \) implies \( a \in B \).
2. An element \( c \in A - B \) is minimal if \( \epsilon[a, c] \Rightarrow a \in B \).
3. A pairing \( (A, \epsilon) \) is well-founded if \( A \) is \( Q^\infty \), and if \( B \subset A \) is transitive with \( A - B \) nonempty, then there is a minimal element in \( A - B \).
4. For the “almost” version we need the **transitive closure** of \( B \subset A \). This is the smallest transitive subdomain containing \( \epsilon[b, \#] \) for all \( b \in B \). More explicitly, it consists of \( a \) such that there is a finite sequence \( c_0, c_1, \ldots, c_n \), \( n \geq 1 \), with \( c_0 \in B \), \( c_n = a \), and \( \epsilon[c_{i+1}, c_i] = \text{yes} \). We denote this by \( \text{tcl}[B] \).
5. Now we define a pairing \( (A, \epsilon) \) to be almost well-founded if the restriction to the transitive closure of any point, \( \text{tcl}[a], \epsilon \), is well-founded.
We note that $Q^\infty$ is not necessary for the definition in (2) to make sense. However we have a clean result only for $Q^\infty$, and putting it in the definition means we can avoid cluttering theorem statements with it.

**Main result.** Some of the terms are defined after the statement.

**Theorem. Existence:** There is a pairing $\in: Q^\infty W \times Q^\infty W \to \gamma/n$ so that the adjoint of the opposite gives a bijection $\in^{\text{opadj}}: Q^\infty W \to bP[Q^\infty W]$, such that

1. the restriction $\mathbb{N} \to bP[\mathbb{N}]$ is the canonical order isomorphism; and
2. if $x \notin \mathbb{N}$ then $\text{rank}[\in^{\text{opadj}}[x]] = \text{rank}[x] - 1$.

**Universality:** Suppose $(A, \epsilon)$ is an almost well-founded pairing, and the adjoint of the opposite $\epsilon^{\text{opadj}}: A \to P[A]$ is injective. Then there is a unique injective function $\omega: A \to Q^\infty W$ with $\epsilon$-transitive image, giving a morphism of pairings from $\epsilon$ to the restriction of $\in$ to $\text{im}[\omega]$.

**Set theory:** $(Q^\infty W, \in)$ satisfies the ZFC axioms.

Clarifications: the adjoint of the opposite is the function $a \mapsto [\#a]$. $\mathbb{N}$ is the natural numbers. The opposite adjoint of `$\epsilon$` takes elements of rank 1 to the cofinal functions on $\mathbb{N}$. Technically, all functions on $\mathbb{N}$ have rank 0, but the bounded ones are accounted for by Existence (1). In Existence (2), recall that ranks of elements cannot be limits, so $\text{rank}[x] - 1$ is defined.

It is not used here, but two such pairings `$\epsilon$` differ by a bijection $Q^\infty W \to Q^\infty W$ that is the identity on $\mathbb{N}$ and preserves the rank function.

The universality property is a version of Mostowski collapsing ([Jech], §6.15). The injective-adjoint condition is traditionally called “extensional”, and corresponds to the set-theory axiom that sets are determined by their elements.

**Proof of Existence.** We will show that for every $a$ there is a bijection, from elements of rank $a + 1$ to logical functions of rank $a$. The rest of the proof is short: According to the axiom of choice we can make a simultaneous choice of bijections for all $a$. These fit together to give a bijection $Q^\infty W \to bP[Q^\infty W]$. This is the opposite adjoint of a pairing with properties (1) and (2).

Now we show that there are bijections as claimed. Suppose $a \in W$. There is a bijection from elements of rank $a + 1$ to functions of rank $a$ if these collections have the same cardinality.

The elements of rank $a + 1$ are $(\exists[a] \leq \# < \#[a + 1])$. This is the complement of $(\# < \#[a + 1])$ in $(\# < \#[a + 1])$. Since $\#[a + 1] = \text{card}[P[\# < \#[a]]]$, we are removing a set of smaller cardinality. According to §5.3, this does not change cardinality. The cardinality of the collection of elements is therefore $\text{card}[P[\# < \#[a]]]$.

There are two cases for functions. Suppose $a$ is not a limit. Then functions of rank $a$ are the complement of $P[\# < \#[a - 1]]$ in $P[\# < \#[a]]$. Again the smaller subdomain has smaller cardinality, so removing it does not change cardinality. Cardinality of the functions is therefore $\text{card}[P[\# < \#[a]]]$, same as the elements.

Now suppose $a$ is a limit. The functions on $\# < \#[a]$ of rank $a$ are the cofinal ones. These are the complement of the bounded functions: $\text{cfP}[\# < \#[a]] = P[\# < \#[a]] - bP[\# < \#[a]]$. We want to show that removing the bounded functions does not change the cardinality. For this it is sufficient to show that the cardinality of the cofinal functions is at least as large as that of the bounded ones. This is so because there is an injection $bP[\# < \#[a]] \to \text{cfP}[\# < \#[a]]$ defined by $h \mapsto \text{not}[h]$. 

The conclusion is that the cardinality of the functions is \( \text{card}[P[#] \subset B[a]] \), again the same as the elements.

This completes the existence part of the Theorem.

6.2. Well-founded recursion, and universality. See Jech, [Jech] p. 66.

Suppose \((A, \lambda)\) is an almost well-founded pairing. As in §4.2, a recursion condition is a partially-defined function \( R : \text{tfn}[A, B] \times A \to B \), where ‘tfn’ denotes partially-defined functions with \( \lambda \)-transitive domains.

Again as in §4.2, a partially-defined function \( f : A \to B \) is \( R \)-recursive if:

1. \( \text{dom}[f] \) is \( \lambda \)-transitive; and
2. for every \( c \in \text{dom}[f] \), \( f[c] = R[(f \uparrow (\text{tcl}[c] - c), c) \).

Then, exactly as in §4.2, there is a unique maximal \( R \)-recursive partially-defined function \( A \to B \).

For the proof of universality of \( \varepsilon \), we suppose \((A, \lambda)\) is an almost well-founded pairing and define a recursion condition \( R : \text{tfn}[A, W] \times A \to Q^\infty W \) by:

1. \((f, a) \in \text{dom}[R]\) if the image of \( f \) is transitive and \((\lambda[#], a) \subset \text{dom}[f] \);
2. In this case \( R[f, a] := (\varepsilon^{\text{opadj}})^{-1}[(\lambda[f^{-1}[#]), a)] \).

Comment. We unwind \((2)\). Given \((f, a)\), define a logical function on \( W \) by \( # \mapsto \lambda[f^{-1}[#], a] \). When this is used, \( f \) is injective and the inverse function can be taken literally. To avoid building this into the definition we define \( \lambda[a, C] \), where \( C \) is a subdomain of \( A \) (here \( f^{-1}[#] \)), by: \( \exists c \in C \mid \lambda[a, c] = \text{yes} \). The above definition then makes sense if the \( f^{-1}[#] \) are subset rather than just points. In any case it is easy to see that this logical function is bounded.

Recall that the opposite adjoint \( \varepsilon^{\text{opadj}} : Q^\infty W \to bP[Q^\infty W] \) is a bijection. Therefore there is \( b \in Q^\infty W \) so that \( \varepsilon[#] = \lambda[f^{-1}[#], a] \). \( R[f, a] \) is defined to be this element \( b \).

Note that undoing adjoints gives \( \lambda[#] = \varepsilon[f[#], R[f, a]]\). If \( f[a] = R[f, a] \) then this becomes \( \lambda[#] = \varepsilon[f[#], a] \), which is part of the condition that \( f \) is a morphism of pairings. Note also that \( \varepsilon[#] = R[f, a] \subset \text{im}[f \uparrow (\text{tcl}[a] - a)] \), so the image is still transitive after extending \( f \) by \( f[a] = R[f, a] \).

We return to the proof. By recursion, there is a unique maximal \( R \)-recursive function \( f : A \to Q^\infty W \). The maximality criterion for recursion implies the maximal has domain \( A \). As noted above, the recursion condition implies that \( f \) is a morphism of pairings \((A, \lambda) \to (Q^\infty W, \varepsilon)\).

In general, morphisms of pairings need not be injective. However it is straightforward to see that if the opposite adjoint \( \Lambda^{\text{opadj}} : A \to P[A] \) is injective then \( f \) is injective. This completes the proof of the uniqueness part of the theorem.

6.3. Reformulation of the ZFC axioms. A traditional set theory is a Universe \( U \) of potential elements and a binary logical operator ‘\( \varepsilon \)’. We use the prefix form: define \( \varepsilon : U \times U \to y/n \) by \( \varepsilon(a, b) := a \varepsilon b \).

A set in the theory is a logical function on \( U \) of the form \(# \mapsto \varepsilon[#], a\), for some \( a \in U \). Assume these sets are relaxed sets, by restricting the theory if necessary, see Note (2) below.

Axioms. Given all this, we translate the ZFC axioms described in [Jech] Ch. 1

1. (Well-founded, or Regular) \( \varepsilon \) is almost well-founded.
2. (Extensionality) \( \varepsilon[#], a] = \varepsilon[#], b \) implies \( a = b \) (see note 3);
(3) (Union) \( \exists (\cup a) \in U \) such that \( \#, \epsilon[\cup a] = (\epsilon \ast \epsilon)[\#, a] \). \( \epsilon \ast \epsilon \) is the composition of pairings, defined by \( (\epsilon \ast \epsilon)[\#, a] := (\exists b \mid \epsilon[\#, b] \land \epsilon[b, a]) \).

(4) (Powerset) \( \exists P[a] \in U \) such that \( \epsilon[b, P[a]] = (\epsilon[\#, b] \subset \epsilon[\#, a]) \);

(5) (Infinity) There is an \( \epsilon \) with \( \epsilon[\#, \epsilon] \) infinite;

(6) (Choice) There is a partially-defined function \( ch : U \to U \) with domain \( (a \in U \mid \epsilon[\#, a] \neq U) \) satisfying \( \epsilon[ch[a], a] = \text{no} \) (see note 4);

(7) (Separation) If \( P \) is a logical function on \( U \) given by a first-order formula in the set operations, and \( A \) then the intersection of a set with \( P \) is also a set (see note 5);

(8) (Replacement) Suppose \( A \) is a set and \( f : A \to U \) is a ZFC function (see note 6). Then the image of \( f \) is a set.

Notes.

(1) ‘Well-founded’ is usually put near the end of axiom lists. The universality theorem indicates that it is a key ingredient, so we put it first.

(2) External quantification requires that there be a logical function, defined on all possible functions \( A \to y/n \), that detects the empty function. The Internal quantification of ZFC requires only that \( \emptyset \) be detectable among functions of the form \( \epsilon[\#, b] \). In principle there could be many fewer of these, so, again in principle, there might be theories with sets that support internal quantification but not external. This seems unlikely, but if it happens then we discard such sets.

(3) The domain \( U \) may not support quantification, so we may not be able to identify functions \( \epsilon[\#, a] \) among all possible logical functions on \( U \). But we can identify them within functions of the form \( \epsilon[\#, x] \) as follows. \( \epsilon[a, \#] = \epsilon[b, \#] \) means: \( (\forall x \in \epsilon[a, \#])(\epsilon[b, x] = \text{yes}) \), and similarly \( (\forall x \in \epsilon[b, \#])(\epsilon[a, x] = \text{yes}) \).

(4) In this formulation the choice function provides an element not in the given set, rather than (as more usual) one in the set. This is the form used to show \( U \) has an almost well-order, which implies any other form of Choice one might want. Note that if \( U \) does not support quantification then \( \epsilon[\#, a] \neq U \) does not make good sense. In this case omit this condition: it is redundant anyway because \( \epsilon[\#, a] \) is assumed to be a set, and therefore cannot be all of \( U \).

(5) The objective of the “Separation” axiom is to ensure there are enough sets to transact the business of set theory.

(6) The intent of “Replacement” is that functions should take sets to sets. Beware, however, that “ZFC function” means “graph is a set”, so there will generally be functions in the naïve sense that are not functions in the theory (see below). In relaxed set theory all subdomains of a set are sets, so the “functions” of the theory match the naïve version.

Verifying the axioms. We verify that \( (\mathbb{W}, \epsilon) \) satisfies the translated axioms.

(1) ‘Well-founded’ follows from the fact that the pairing reduces rank, and rank takes values in an almost well-ordered domain.

(2) The domains \( \epsilon[\#, a] \) are relaxed sets because they are bounded subdomains of \( Q^{\sim}\mathbb{W} \).

(3) ‘Extension’ is equivalent to injectivity of the adjoint of \( \epsilon \), and this is a design requirement in the construction.
(4) ‘Infinity’ is a primitive axiom in the system used here.
(5) Similarly, ‘Choice’ is a primitive axiom.
(6) For Separation, note that in the universal theory the intersection of a set with any logical function is again a set, so this is certainly true of the ones coming from first-order logic.
(7) We interpret ‘Replacement’ in a strong way, namely that any function \( Q^\infty \mathcal{W} \rightarrow Q^\infty \mathcal{W} \) should take bounded logical functions to bounded logical functions. If QH holds, this is included in §4.6. If QH does not hold, it is the Union Hypothesis.

This completes the ZFC axioms, and the proof of the Theorem. □

6.4. Application: ZFC sets bijective with non-sets. Suppose \((U, \epsilon)\) is a ZFC set theory in the sense of [Jech] or [Manin].

Proposition. Either \((U, \epsilon)\) is a truncation of the universal theory, or there is a ZFC set \(A \subset U\) and a (naïve) injective function \(A \rightarrow U\) whose image is not a ZFC set.

The replacement axiom of ZFC requires images of ZFC functions to be ZFC sets, so the injection of the proposition cannot be a ZFC function. It is an injective function in the naïve sense that it is a one-to-one way to assign an element of \(U\) to each element of \(A\). In contrast, ZFC functions are those whose graphs are ZFC sets. The injection of the Proposition evidently fails this.

Proof. According to the Theorem there is a (unique) injection \(U \rightarrow \mathcal{W}\) with transitive image, such that \(\epsilon\) is the restriction of \(\in\) to \(U \times U\). If the image is not all of \(\mathcal{W}\) then the complement is non-empty, and therefore has minimal elements \(M\). Minimal means elements in \(M\) are elements of \(U\): \(\# \in M \subset U\). Since \(M \notin U\), this is a relaxed set that is not a ZFC set.

\(\mathcal{W}\) has, by construction, an almost well-order. The standard identification of \(U\) with the ordinal numbers of the theory gives an almost well-order on \(U\). If the subdomain \((\# \in M) \subset U\) is cofinal in \(U\) for every minimal \(M\), then \(U\) is a truncation of \(\mathcal{W}\). Specifically, \((U, \geq)\) is a regular strong-limit cardinal and \(\epsilon\) is the restriction of \(\in\) to the elements below this. If there is a minimal \(M\) that is not cofinal then there is a ZFC set \(K\) with greater (relaxed) cardinality. The induced well-orders give an injection \(M \rightarrow K\) with transitive image. This means there is \(m \in K\) so that the image is \((\# \in K) \& (\# < m)\) This formula shows that the image is a ZFC subset of \(K\), as desired. □

7. The quantification gap

The picture so far is that logical domains that support infinite-order quantification give a complete and convenient set theory. To recap, \(Q^\infty \mathcal{W}\) encodes set theory, in the sense that a domain is a set if and only if it is bijective to a bounded subdomain of this. The usual paradoxes show that \(Q^\infty \mathcal{W}\) itself is not a set, but is the smallest object that is not a set. The Union hypothesis gives enough information for the trans-set needs of category theory; see [Quinn].

\(\mathcal{W}\) encodes \(Q^1\) domains in the same sense. The region between \(Q^1\) and \(Q^\infty\) is the “quantification gap”. In more detail, there is a quantification threshold \(\tau \geq 1\) so that \(Q^\infty = Q^\tau \neq Q^{\tau-1}\) (§4.7).
The Quantification Hypothesis (QH) asserts that $Q^\infty = Q^1$. In this case $\tau = 1$, there is no gap, and the object just beyond set theory is $\mathcal{W}$.

If $\tau > 1$ then the gap is nontrivial, that is, there are domains that support finite-order quantification but not infinite-order. $Q^\infty \mathcal{W}$ is the smallest of these, and is $Q^{\tau-1}$ but not $Q^\tau$. In this case the cardinality of $Q^\infty$ is the maximal strong-limit cardinal, and the Union hypothesis asserts that it is a regular cardinal.

The Hypotheses assumed so far do not seem to constrain $\tau$ or, if $\tau > 1$, the objects in the gap. So far, experimentation has failed to yield any good ideas about this.

REFERENCES

[Jech] Thomas J. Jech: *Set theory – 3rd Millennium ed, rev. and expanded*. Springer monographs in mathematics (2002) ISBN 3-540-44085-2

[Manin] Yu. I. Manin: *A Course in Mathematical Logic for Mathematicians, second edition*. Springer Graduate Texts in Mathematics 53 (2010)

[Quinn] Frank Quinn *Object descriptors: usage and foundations*

[Quinn old] Frank Quinn *A construction of set theory* http://arxiv.org/abs/2110.01489