UNIVERSAL KZB EQUATIONS FOR ARBITRARY ROOT SYSTEMS

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Abstract. Generalising work of Calaque–Enriquez–Etingof [6], we construct a universal KZB connection $\nabla_{\text{ell}}$ for any finite (reduced, crystallographic) root system $\Phi$. $\nabla_{\text{ell}}$ is a flat connection on the regular locus of the elliptic configuration space associated to $\Phi$, with values in a graded Lie algebra $t_{\text{ell}}^\Phi$ with a presentation with relations in degrees 2, 3 and 4 which we determine explicitly. The connection $\nabla_{\text{ell}}$ also extends to a flat connection over the moduli space of pointed elliptic curves. We prove that its monodromy induces an isomorphism between the Malcev Lie algebra of the elliptic pure braid group $P_{\text{ell}}^\Phi$ corresponding to $\Phi$ and $t_{\text{ell}}^\Phi$, thus showing that $P_{\text{ell}}^\Phi$ is not 1–formal and extending a result of Bezrukavnikov valid in type $A$ [3]. We then study one concrete incarnation of our KZB connection, which is obtained by mapping $t_{\text{ell}}^\Phi$ to the rational Cherednik algebra $H_{\hbar,c}$ of the corresponding Weyl group $W$. Its monodromy gives rise to an isomorphism between appropriate completions of the double affine Hecke algebra of $W$ and $H_{\hbar,c}$.

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1. Introduction

The Knizhnik–Zamolodchikov–Bernard (KZB) connection was constructed by Felder–Wieczerkowski in [15]. It is a flat connection on the vector bundle of WZW conformal blocks on the moduli space $M_{1,n}$ of elliptic curves with $n$ marked points. Calaque–Enriquez–Etingof later constructed a universal KZB connection on $M_{1,n}$ [6], which has coefficients in an arbitrary vector bundle. The main goal of the current paper is to generalise the construction of Calaque–Enriquez–Etingof to the elliptic configuration space associated to an arbitrary root system.

1.1. The universal KZB connection.

1.1.1. Let $E$ be a Euclidean vector space, and $\Phi \subset E^*$ a finite, reduced, crystallographic root system. Let $Q \subset E^*$, $Q^\vee \subset E$ be the root and coroot lattices, and $P \subset E^*$, $P^\vee \subset E$ the corresponding weight and coweight lattices dual to $Q^\vee$ and $Q$ respectively. Denote the complexification of $E$ by $\mathfrak{h}$. 

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Let $\tau$ be a point in the upper half plane $\mathcal{H} = \{ z \in \mathbb{C} \mid \text{Im}(z) > 0 \}$, and set $\Lambda_{\tau} := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$. Consider the elliptic curve $\mathcal{E}_{\tau} := \mathbb{C}/\Lambda_{\tau}$ with modular parameter $\tau$. Let $T := \mathfrak{h}/(P^V + \tau P^V) \cong \text{Hom}_{\mathbb{Z}}(Q, \mathcal{E}_{\tau})$, which is non–canonically isomorphic to $\mathcal{E}_n^\phi$, where $n = \text{rank}(P^V)$. Any root $\alpha \in \Phi$ induces a map $\chi_{\alpha} : T \to \mathcal{E}_{\tau}$, with kernel $T_{\text{reg}}$. We refer to $T_{\text{reg}} = T \setminus \bigcup_{\alpha \in \Phi} T_{\alpha}$, as the elliptic configuration space associated to $\Phi$. If $E = \mathbb{R}^n$ with standard coordinates $\{ e_{ij} \}$, and $\Phi = \{ e_{ij} - e_{ij'} \}_{1 \leq i,j \leq n} \subset E^*$ is the root system of type $A_n - 1$, $T_{\text{reg}}$ is the configuration space of $n$ ordered points on the elliptic curve $\mathcal{E}_{\tau}$.

1.1.2. Let $\theta(z|\tau)$ be the Jacobi theta function, which is a holomorphic function $\mathbb{C} \times \mathcal{H} \to \mathbb{C}$, whose zero set is $\{ z \mid \theta(z|\tau) = 0 \} = \Lambda_{\tau}$, and such that its residue at $z = 0$ is $1$. (See Section §2.2.) Let $x$ be another complex variable, and set

$$k(z, x|\tau) := \frac{\theta(z + x|\tau)}{\theta(z|\tau)\theta(x|\tau)} - \frac{1}{x}.$$  

The function $k(z, x|\tau)$ has only simple poles at $z \in \Lambda_{\tau}$, and $k(z, x|\tau)$ is holomorphic in $x$. In other words, $k(z, x|\tau)$ belongs to $\text{Hol}(\mathbb{C} - \Lambda_{\tau})[x]$.  

1.1.3. Let $A$ be an algebra endowed with the following data: a set of elements $\{ t_{\alpha} \}_{\alpha \in \Phi}$, such that $t_{-\alpha} = t_{\alpha}$, and two linear maps $x : \mathfrak{h} \to A$, $y : \mathfrak{h} \to A$. We define an $A$–valued connection on $T_{\text{reg}}$. It takes the following form.

$$\nabla_{\text{KZB}} = d - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{x_{\alpha}}{2})|\tau)(t_{\alpha})d\alpha + \sum_{i=1}^{n} y(u^i)du_i,$$

where $\Phi_+ \subset \Phi$ is a chosen system of positive roots, $\{ u_i \}$, and $\{ u^i \}$ are dual basis of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively. When $\Phi$ is the root system of type $A_n$, the connection above coincides with the universal KZB connection introduced by Calaque–Enriquez–Etingof.

**Theorem A** (Theorem 2.5). The connection $\nabla_{\text{KZB}}$ is flat if and only if the following relations hold in $A$

1. For any rank 2 root subsystem $\Psi$ of $\Phi$, and $\alpha \in \Psi$,

$$[t_{\alpha}, \sum_{\beta \in \Psi} t_{\beta}] = 0.$$

2. For any $u, v \in \mathfrak{h}$

$$[x(u), x(v)] = 0 = [y(u), y(v)].$$

3. For any $u, v \in \mathfrak{h}$

$$[y(u), x(v)] = \sum_{v \in \Phi^+} \langle v, \gamma \rangle(u, \gamma) t_{\gamma}.$$

4. For any $\alpha \in \Phi$ and $u \in \mathfrak{h}$ such that $\alpha(u) = 0$

$$[t_{\alpha}, x(u)] = 0 = [t_{\alpha}, y(u)].$$

If, moreover, the Weyl group $W$ of $\Phi$ acts on $A$, then $\nabla_{\text{KZB}}$ is $W$–equivariant if and only

5. For any $\omega \in W$, $\alpha \in \Phi$, and $u, v \in \mathfrak{h}$,

$$w(t_{\alpha}) = t_{\omega \alpha}, \quad w(x(u)) = x(wu), \quad w(t_{\alpha}) = t_{\alpha} = y(wv)$$

Define $t_{\Phi}$ to be the Lie algebra generated by a set of elements $\{ t_{\alpha} \}_{\alpha \in \Phi}$, such that $t_{\alpha} = t_{-\alpha}$, and two linear maps $x : \mathfrak{h} \to t_{\Phi}$, $y : \mathfrak{h} \to t_{\Phi}$, satisfying the relations in Theorem A. The Weyl group $W$ acts on $t_{\Phi}$ according to relation 5 in Theorem A.

\[1\text{We assume further that } A \text{ is endowed with a topology such that the expressions } k(\alpha, \text{ad}(\frac{x_{\alpha}}{2})|\tau)(t_{\alpha}) \text{ converge. For example, } A \text{ could be complete with respect to a grading for which the elements } x_{\alpha} \text{ have positive degree.}\]
1.2. The Malcev Lie algebra of $P_{\text{ell}}^\Phi$. Let $P_{\text{ell}}^\Phi = \pi_1(T_{\text{reg}}, x_0)$ be the fundamental group of $T_{\text{reg}}$. We refer to $P_{\text{ell}}^\Phi$ as the \textit{elliptic pure braid group} corresponding to $\Phi$. The flatness of the universal KZB connection $\nabla_{\text{KZB},r}$ gives rise to the monodromy map

$$\mu : P_{\text{ell}}^\Phi \to \exp(\hat{t}_{\text{ell}}^\Phi),$$

where $\hat{t}_{\text{ell}}^\Phi$ is the (pro-nilpotent) completion of $t_{\text{ell}}^\Phi$ with respect to the grading given by $\deg(x(u)) = 1 = \deg(y(u))$, and $\deg(t_a) = 2$, and $\exp(t_{\text{ell}}^\Phi)$ is the corresponding pro-unipotent group.

Let $J \subseteq \mathbb{C}P_{\text{ell}}^\Phi$ be the augmentation ideal of the group ring $\mathbb{C}P_{\text{ell}}^\Phi$, and $\hat{\mathbb{C}}P_{\text{ell}}^\Phi$ the completion of $\mathbb{C}P_{\text{ell}}^\Phi$ with respect to $J$. Let $U(t_{\text{ell}}^\Phi)$ be the universal enveloping algebra of $t_{\text{ell}}^\Phi$, and $U(\hat{t}_{\text{ell}}^\Phi)$ its completion with respect to the grading on $t_{\text{ell}}^\Phi$. The monodromy map $\mu$ extends to the completions

**Theorem B** (Theorem 5.2).

1. The monodromy map extends to an isomorphism of Hopf algebras $\hat{\mu} : \hat{\mathbb{C}}P_{\text{ell}}^\Phi \to U(\hat{t}_{\text{ell}}^\Phi)$.
2. The restriction of $\hat{\mu}$ to primitive elements is an isomorphism of the Malcev Lie algebra of $P_{\text{ell}}^\Phi$ to the graded completion of $t_{\text{ell}}^\Phi$.

Let $\Gamma$ be an abstract group. Recall the Malcev Lie algebra $m_{\Gamma}$ of $\Gamma$ is the subspace of primitive elements in the completion of $\mathbb{C}\Gamma$ with respect to its augmentation ideal. By definition, $\Gamma$ is 1–formal if $m_{\Gamma}$ is isomorphic, as a filtered Lie algebra, to the graded completion of a quadratically presented Lie algebra (see, e.g. [9]).

Let $g \geq 1$, and $\Gamma_{g,n}$ be the fundamental group of the configuration space of $n$ ordered points on an oriented surface of genus $g$. Using the theory of minimal models, Bezrukavnikov gave an explicit presentation of the Malcev Lie algebra of $\Gamma_{g,n}$, and proved in particular that $\Gamma_{g,n}$ is 1–formal if and only if $g > 1$ or $g = 1$ and $n \leq 2$ [3]. Calaque–Enriquez–Etingof rederived Bezrukavnikov’s result in the case of genus $g = 1$ by using Chen’s iterated integrals associated to the universal KZB connection which they introduced [6]. A similar construction for surfaces of higher genus was carried by Enriquez [10]. A Tannakian interpretation of the construction of Calaque–Enriquez–Etingof was recently given by Enriquez–Etingof [11].

The proof of Theorem B is modelled on the approach of [6, 10].

**Theorem C** (Theorem 5.15). The Lie algebra $p_{\text{ell}}^\Phi$ is not quadratic. In particular, the pure elliptic braid group $P_{\text{ell}}^\Phi$ is not 1–formal.

1.3. Extension to the moduli space. Let $M_{1,n}$ be the moduli space of pointed elliptic curves associated to a root system $\Phi$. More explicitly, let $\mathfrak{h} \times \mathfrak{h}$ be the upper half plane. The semidirect product $(P^\vee \oplus P^\vee) \rtimes \text{SL}_2(\mathbb{Z})$ acts on $\mathfrak{h} \times \mathfrak{h}$. For $(n,m) \in (P^\vee \oplus P^\vee)$ and $(z, \tau) \in \mathfrak{h} \times \mathfrak{h}$, the action is given by translation: $(n,m) \star (z, \tau) := (z + n + \tau m, \tau)$. Let $\Phi(z, \tau) := (\frac{a}{c} \frac{b}{d}, \frac{\tau \tau + \delta}{c \tau + d})$. Let $\alpha : \mathfrak{h} \to \mathbb{C}$ be the map induced by the root $\alpha \in \Phi$. We define $\widetilde{H}_{\alpha,\tau} \subset \mathfrak{h} \times \mathfrak{h}$ to be

$$\widetilde{H}_{\alpha,\tau} = \{(z, \tau) \in \mathfrak{h} \times \mathfrak{h} | \alpha(z) \in \Lambda_\tau = \mathbb{Z} + \tau \mathbb{Z}\}.$$ 

The elliptic moduli space $M_{1,n}$ is defined to be the quotient of $\mathfrak{h} \times \mathfrak{h} \setminus \bigcup_{\alpha \in \Phi^+, \tau \in \mathfrak{h}} \widetilde{H}_{\alpha,\tau}$ by the action of $(P^\vee \oplus P^\vee) \rtimes \text{SL}_2(\mathbb{Z})$ action.

In the type $A$ case, $M_{1,n}$ is the moduli space of elliptic curves with $n$ marked points. We extend the KZB connection $\nabla_{\text{KZB},r}$ on $\mathcal{E}_r$ to a flat connection on $M_{1,n}$.

1.3.1. To begin with, we have the following derivation of $t_{\text{ell}}^\Phi$. Let $\mathfrak{d}$ be the Lie algebra with generators $\Delta_0, d, X, \text{and } \delta_{2m}(m \geq 1)$, and relations

$$[d, X] = 2X, \quad [d, \Delta_0] = -2\Delta_0, \quad [X, \Delta_0] = d,$$

$$[\delta_{2m}, X] = 0, \quad [d, \delta_{2m}] = 2m\delta_{2m}, \quad (ad \Delta_0)^{2m+1}(\delta_{2m}) = 0.$$
The Lie subalgebra generated by $\Delta_0, d, X$ is a copy of $\mathfrak{sl}_2$ in $\mathfrak{d}$. We can decompose $\mathfrak{d}$ as $\mathfrak{d} = \mathfrak{d}_+ \rtimes \mathfrak{sl}_2$, where $\mathfrak{d}_+$ is the subalgebra generated by $[\delta_{2m} | m \geq 1]$. In Proposition 6.1, we show $\mathfrak{d}$ acts on $t^\Phi_{\mathfrak{sl}_2}$ by derivation. (See Proposition 6.1 for the action of $\mathfrak{d}$ on $t^\Phi_{\mathfrak{sl}_2}$).

The Lie algebra $t^\Phi_{\mathfrak{sl}_2} \rtimes \mathfrak{d}$ is $\mathbb{Z}^2$–graded. The grading is given by
\[
\deg(\Delta_0) = (-1, 1), \quad \deg(d) = (0, 0), \quad \deg(X) = (1, -1), \quad \deg(\delta_{2m}) = (2m + 1, 1)
\]
and $\deg(x(u)) = (1, 0)$, $\deg(y(u)) = (0, 1)$, $\deg(t_a) = (1, 1)$.

Let $G_n := \exp(t^\Phi_{\mathfrak{sl}_2} \rtimes \mathfrak{d}_+) \rtimes SL_2(\mathbb{C})$ be the semiprodct of $\exp(t^\Phi_{\mathfrak{sl}_2} \rtimes \mathfrak{d}_+)$ and $SL_2(\mathbb{C})$, where the former is the completion of $t^\Phi_{\mathfrak{sl}_2} \rtimes \mathfrak{d}_+$ with respect to the grading above. Following [6], in Proposition 7.3, we construct a principal bundle $P_n$ on $M_{1,n}$ with structure group $G_n$, which is unique under certain conditions.

1.3.2. We extend the connection $\nabla_{KZB}$ to a connection $\nabla_{\mathfrak{sl}_2}$ on the principal bundle $P_n$. We now describe the extension $\nabla_{\mathfrak{sl}_2}$.

Let $g(z, x\tau) := k_\tau(z, x\tau)$ be the derivative of function $k(z, x\tau)$ with respect to variable $x$. Set $a_{2n} := e^{\frac{2\pi i}{n}} B_{2n} e^{\frac{2\pi i}{n}},$ where $B_m$ is the Bernoulli numbers and let $E_{2n+2}(\tau)$ be the Eisenstein series. Consider the following function on $\mathfrak{h} \times \mathbb{R}$
\[
\Delta := \Delta(\Delta_0, \tau) = -\frac{1}{2\pi i} \Delta_0 - \frac{1}{2\pi i} \sum_{n \geq 1} a_{2n} E_{2n+2}(\tau) \delta_{2n} + \frac{1}{2\pi i} \sum_{\beta \in \Phi^+} g(\beta, \text{ad} \frac{x_{\beta^\vee}}{2} | \tau)(t_\beta).
\]
This is a meromorphic function on $\mathfrak{h} \times \mathbb{R}$ valued in $(t^\Phi_{\mathfrak{sl}_2} \rtimes \mathfrak{d}_+) \rtimes \mathfrak{u}_+ \subset \text{Lie}(G_n)$, (where $\mathfrak{u}_+ = \mathbb{C} \Delta_0 \subset \mathfrak{sl}_2$). It has only poles along the hyperplanes $\bigcup_{\alpha \in \Phi^+, r \in \mathbb{R}} H_{\alpha, r}$.

**Theorem D** (Theorem 8.1). The following $t^\Phi_{\mathfrak{sl}_2} \rtimes \mathfrak{d}$-valued KZB connection on $M_{1,n}$ is flat.
\[
\nabla_{\mathfrak{sl}_2} = \nabla_{\mathfrak{sl}_2} - \Delta d\tau = d - \Delta d\tau - \sum_{\alpha \in \Phi} k(\alpha, \text{ad} \frac{x_{\alpha^\vee}}{2} | \tau)(t_\alpha) d\alpha + \sum_{i=1}^n y(u^i) du^i.
\]

1.4. Trigonometric degeneration.

1.4.1. Let $H = \mathfrak{h} / P^\vee \equiv \text{Hom}_{\mathbb{C}}(\mathbb{Q}, \mathbb{C}^\times)$ be the complex algebraic torus with Lie algebra $\mathfrak{h}$ and coordinate ring given by the group algebra $\mathbb{C}[Q]$. We denote the function corresponding to $\lambda \in Q$ by $e^{\lambda \mathfrak{h}}$, and set $H_{\text{reg}} = H \setminus \bigcup_{\alpha \in \Phi} \{ e^{\pi i \alpha} = 1 \}$. The first named author introduced a universal trigonometric connection $\nabla_{\text{trig}}$ on $H$, with logarithmic singularities on $H \setminus H_{\text{reg}}$ [22]. The connection is flat, $W$–equivariant, and takes values in a Lie algebra $t^\Phi_{\text{trig}}$ which is described as follows.

Let $A$ be an algebra endowed with a set of elements $\{t_\alpha\}_{\alpha \in \Phi}$ such that $t_{-\alpha} = t_\alpha$, and a linear map $X : \mathfrak{h} \to A$. The trigonometric connection $\nabla_{\text{trig}}$ is the $A$–valued connection on $H_{\text{reg}}$ given by
\[
\nabla_{\text{trig}} = d - \sum_{\alpha \in \Phi_+} \frac{2\pi i d\alpha}{e^{2\pi i \alpha} - 1} t_\alpha - du^i X(u^i).
\]

(1) For any rank 2 root subsystem $\Psi \subset \Phi$, and $\alpha \in \Psi$, $[t_{\alpha}, \sum_{\beta \in \Psi_{\alpha}} t_\beta] = 0$.

(2) For any $u, v \in \mathfrak{h}$, $[X(u), X(v)] = 0$.

(3) For any $\alpha \in \Phi_+, w \in W$ such that $w^{-1} \alpha$ is a simple root and $u \in \mathfrak{h}$, such that $\alpha(u) = 0$, $[t_\alpha, X_w(u)] = 0$,

where $X_w(u) = X(u) - \sum_{\beta \in \Phi_+, \gamma w \Phi_+} \beta(u)t_\beta$.

By definition, the Lie algebra $t^\Phi_{\text{trig}}$ is the Lie algebra presented by the above relations.
1.4.2. The connection $\nabla_{KZB,\tau}$ degenerates to a trigonometric connection as the imaginary part of the modular parameter $\tau$ tends to $\infty$.

**Theorem E** (Proposition 4.3). As $\text{Im}\,\tau \to +\infty$, the connection $\nabla_{KZB,\tau}$ degenerates to the following connection $\nabla^{\text{deg}}$ on $H_{\text{reg}}$

$$\nabla^{\text{deg}} = d - \sum_{\alpha \in \Phi^+} \frac{2\pi i t_\alpha}{e^{2\pi i t_\alpha} - 1} d\alpha + \sum_{i=1}^n \left( y(u^i) - \sum_{\alpha \in \Phi^+} (\alpha, u^i) \left( \frac{2\pi i e^{2\pi i t_\alpha}}{e^{2\pi i t_\alpha} - 1} - \frac{1}{\text{ad}(\frac{\alpha}{2})} \right) t_\alpha \right) du^i$$

By universality of $\Phi^{\text{reg}}$, the above degeneration gives rise to a map $\Phi^{\text{reg}} \to \Phi^{\text{reg}}$ given by

$$t_\alpha \mapsto t_\alpha \quad \text{and} \quad X(u) \mapsto -y(u) + \sum_{\alpha \in \Phi^+} (\alpha, u) \left( \frac{2\pi i e^{2\pi i t_\alpha}}{e^{2\pi i t_\alpha} - 1} - \frac{1}{\text{ad}(\frac{\alpha}{2})} \right) t_\alpha.$$

1.5. Rational Cherednik algebras and elliptic Dunkl operators. For the last part of this paper, we study one concrete incarnation of our KZB connection $\nabla_{KZB}$, which is obtained by mapping $\Phi^{\text{reg}} \to \Phi^{\text{reg}}$ to the rational Cherednik algebra of the Weyl group $W$. In special cases, this specialisation coincides with the elliptic Dunkl operators.

1.5.1. Let $H_{\hbar,c}$ be the rational Cherednik algebra of $W$ introduced in [12]. $H_{\hbar,c}$ is generated by the group algebra $CW$, together with a copy of $S$ and $S^*$, and depends on two sets of parameters (see [12], or Section 9.1 for the defining relations). Specifically, let $S \subset W$ be the set of reflections, $K$ the vector space of $W$-invariant functions $S \to \mathbb{C}$, and $\overline{K} = K \oplus \mathbb{C}$. $H_{\hbar,c}$ is an algebra over $\mathbb{C}[\overline{K}]$ which is $\mathbb{N}$-graded, provided the standard linear functions $\{c_{s_{\gamma}}\}_{s_{\gamma} \in S/W}$ and $\hbar$ on $K$ and $\mathbb{C}$ are given degree 2.

**Theorem F** (Prop. 9.3). For any $a, b \in \mathbb{C}$, there is a Lie algebra homomorphism $\Phi^{\text{reg}} \to H_{\hbar,c}$, defined as follows

$$x(v) \mapsto ax(v), \quad y(u) \mapsto bu, \quad \text{and} \quad t_\gamma \mapsto ab \left( \frac{\hbar}{\theta^2(\alpha|\alpha)} - \frac{2c_{s_{\gamma}}}{(\gamma|\gamma)} s_{\gamma} \right).$$

for $u, v \in \mathfrak{h}$ and $\gamma \in \Phi^+$, where $\pi : \mathfrak{h} \to \mathfrak{h}^*$ is the isomorphism induced by the non-degenerate bilinear form $(\cdot | \cdot)$ on $\mathfrak{h}$, $s_{\gamma}$ is the reflection corresponding to the root $\gamma$, and $\theta^2(\alpha|\alpha)$ is the dual Coxeter number of $\Phi$.

Set $a = b = 1$ for definiteness. Theorem F implies the following

**Theorem G** (Theorem 9.5). The universal KZB connection $\nabla_{KZB,\tau}$ specializes to the following elliptic KZ connection valued in the rational Cherednik algebra

$$\nabla_{H_{\hbar,c},\tau} = d + \sum_{\alpha \in \Phi^+} \frac{2c_{\alpha}}{(\alpha|\alpha)} k(\alpha, \text{ad}(\frac{\alpha}{2})(\gamma)) s_{\alpha} d\alpha - \sum_{\alpha \in \Phi^+} \frac{\hbar}{\theta^2(\alpha|\alpha)} \theta^2(\alpha|\alpha) d\alpha + \sum_{i=1}^n u^i d\mu_i.$$
The connection $\nabla_{H_{\ell, c}}$ is flat and $W$-equivariant. Its monodromy yields a homomorphism of $H_{\ell, g}$ onto the graded completion of $H_{\ell, c}$ which becomes an isomorphism after $H_{\ell, g}$ is also completed.

1.5.3. Relation with the affine Hecke algebra. We produce an algebra homomorphism from the degenerate affine Hecke algebra to the completion of the rational Cherednik algebra using the degeneration of Theorem E.

Cherednik in [8] constructed the affine KZ connection. It is a flat and $W$-equivariant connection on $H_{\ell, g}$ valued in the degenerate affine Hecke algebra $H'$. The degenerate affine Hecke algebra $H'$ is the associative algebra generated by $\mathbb{C} W$ and the symmetric algebra $S \mathfrak{h}$. Let $\{s_\alpha, x(u) \mid s_\alpha \in W, u \in \mathfrak{h}\}$ be the generators of $H'$. The affine KZ connection can be obtained by specializing the universal trigonometric KZ connection $\nabla_{\text{trig}}$. More precisely, we have a map $A_{\text{trig}} \to H'$, by $t_\alpha \mapsto k_\alpha s_\alpha$, and $X(u) \mapsto x(u)$, for $\alpha \in \Phi$, and $u \in \mathfrak{h}$. This gives the affine KZ connection

$$\nabla_{\text{AKZ}} = d - \sum_{\alpha \in \Phi^+} \frac{2\pi i t_\alpha}{e^{2\pi i \alpha} - 1} dk_\alpha s_\alpha - \sum_i x(u^i) du_i.$$ 

As $\text{Im} \tau \to \infty$, by Proposition 4.3 and §10, the elliptic KZ connection $\nabla_{H_{\ell, c}}$ degenerates to the following affine KZ connection.

$$\nabla = d - \sum_{\alpha \in \Phi^+} \left(\frac{2c_\alpha}{(\alpha | \alpha)} \left(\frac{2\pi i t_\alpha}{e^{2\pi i \alpha} - 1} + \frac{2\pi i \hbar}{(\alpha | \alpha)} \left(\frac{2\pi i e^{2\pi i \alpha} - 1}{e^{2\pi i \alpha} - 1} - \frac{1}{x_\alpha}\right)s_\alpha\right)\right) d\alpha + \sum_{i=1}^{n} y(u^i) du_i.$$ 

By the universality of the affine KZ connection $\nabla_{\text{AKZ}}$, we have

**Theorem H.** There is an algebra homomorphism from the degenerate affine Hecke algebra $H'$ to $\widehat{H}_{\ell, c}$ by

$$k_\alpha \mapsto -\frac{2c_\alpha}{(\alpha | \alpha)}, \quad w \mapsto w, \quad \text{for } w \in W,$$

$$x(u) \mapsto y(u) + \sum_{\alpha \in \Phi^+} \left(\alpha(u) \left(\frac{2\pi i t_\alpha}{e^{2\pi i \alpha} - 1} + \frac{2\pi i \hbar}{(\alpha | \alpha)} \left(\frac{2\pi i e^{2\pi i \alpha} - 1}{e^{2\pi i \alpha} - 1} - \frac{1}{x_\alpha}\right)s_\alpha\right)\right).$$

1.5.4. Relation with the elliptic Dunkl operator. In [5], Buchstab-Felder-Veselov defined elliptic Dunkl operators for Weyl groups. Etingof and Ma in [14] generalised these operators to an abelian variety $A$ with a finite group action, and defined an elliptic Cherednik algebra as a sheaf of algebras on $A$. They also constructed certain representations of the elliptic Cherednik algebra.

We show that those representations arise from the flat connection valued in the rational Cherednik algebra $H_{0, c}$, with parameter $\hbar = 0$.

**Theorem I (Proposition 11.1).** The flat connection $\nabla_{H_{0, c}}$ specialized on the vector bundle

$$b_{\mathbb{C}} \times_{(\rho^\vee \otimes \rho^\vee)} \mathbb{C} W$$

coincides with the elliptic Dunkl operator

$$\nabla = d - \sum_{w \in W} \sum_{\alpha \in \Phi^+} \frac{2c_\alpha}{(\alpha | \alpha)} \frac{\theta(\alpha + \alpha^\vee (w \rho))}{\theta(\alpha) \theta(\alpha^\vee (w \rho))} s_\alpha d\alpha$$

in [5] and [14].

1.6. The elliptic Casimir connection. In [24], we also give another concrete incarnation, the elliptic Casimir connection, of the universal connections $\nabla_{\text{KZB}, \tau}$ and $\nabla_{\text{KZB}}$. This is an elliptic analogue of the rational Casimir connection [21] and of the trigonometric Casimir connection [22]. This elliptic Casimir connection takes values in the deformed double current algebra $D_\hbar(\mathfrak{g})$, a deformation of the universal central extension of the double current algebra $g[u, v]$ recently introduced by Guay [16, 17]. The construction of a map from $t^\Phi_{\text{ell}}$ to $D_\hbar(\mathfrak{g})$ relies crucially on a double loop presentation of $D_\hbar(\mathfrak{g})$ in [16, 18].
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2. The universal connection for arbitrary root systems

2.1. The Lie algebra $t^\Phi_{\mathfrak{a}l}$. Let $E$ be a Euclidean vector space, and $\Phi \subset E^*$ a finite, reduced, crystallographic root system. Let $Q \subset E^*$ (resp. $Q^\vee \subset E$) be the root lattice generated by the roots $\{\alpha|\alpha \in \Phi\}$ (resp. coroots), and $P \subset E^*$, $P^\vee \subset E$ the corresponding weight and coweight lattices dual to $Q^\vee$ and $Q$ respectively. Denote the complexification of $E$ by $\mathfrak{h}$.

For any subset $\Psi \subset \Phi$, and subring $R \subset \mathbb{R}$ of real numbers, let $\langle \Psi \rangle_R \subset \mathfrak{h}^*$ be the $R$–span of $\Psi$. By definition, a root subsystem of $\Phi$ is a subset $\Psi \subset \Phi$ such that $\langle \Psi \rangle \cap \Phi = \Psi$. If $\Psi \subset \Phi$ is a root subsystem, we set $\Psi_+ = \Psi \cap \Phi_+$.  

Definition 2.1. Let $t^\Phi_{\mathfrak{a}l}$ be the Lie algebra generated by a set of elements $\{t_\alpha|\alpha \in \Phi\}$, such that $t_\alpha = t_{-\alpha}$, and two linear maps $x: \mathfrak{h} \rightarrow t^\Phi_{\mathfrak{a}l}$, $y: \mathfrak{h} \rightarrow t^\Phi_{\mathfrak{a}l}$. Those generators satisfy the following relations

(tt): For any root subsystem $\Psi$ of $\Phi$, we have $[t_\alpha, \sum_{\beta \in \Psi} t_\beta] = 0$.

(xx): $[x(u), x(v)] = 0$, for any $u, v \in \mathfrak{h}$;

(yy): $[y(u), y(v)] = 0$, for any $u, v \in \mathfrak{h}$;

(xy): $[y(u), x(v)] = \sum_{\gamma \in \Phi^*} \langle v, \gamma \rangle \langle u, \gamma \rangle t_\gamma$.

(tx): $[t_\alpha, x(u)] = 0$, if $\langle \alpha, u \rangle = 0$.

(ty): $[t_\alpha, y(u)] = 0$, if $\langle \alpha, u \rangle = 0$.

The Lie algebra $t^\Phi_{\mathfrak{a}l}$ is bigraded, with grading $\deg(x(u)) = (1, 0), \deg(y(v)) = (0, 1)$, and $\deg(t_\alpha) = (1, 1)$, for any $u, v \in \mathfrak{h}$ and $\alpha \in \Phi$.

Remark 2.2. Unlike the rational and the trigonometric cases, there is no embedding of the Lie algebra $t^\Phi_{\mathfrak{a}l}$ of rank 1 into the Lie algebra $t^\Phi_{\mathfrak{a}l}$ of higher rank, such that, $x(u) \mapsto x(u), y(v) \mapsto y(v)$, and $t_\alpha \mapsto t_\alpha$. This fails since the defining relation

$$[y(u), x(v)] = \sum_{\gamma \in \Phi^*} \langle v, \gamma \rangle \langle u, \gamma \rangle t_\gamma$$

is not preserved under the map.

2.2. Theta functions. In this subsection, we recall some basic facts about theta functions.

Let $\Lambda_\tau := \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ and $\mathfrak{H}$ be the upper half plane, i.e. $\mathfrak{H} := \{z \in \mathbb{C} | \text{Im}(z) > 0\}$. The following properties of $\theta(z|\tau)$ uniquely characterize the theta function $\theta(z|\tau)$ (see also [6])

1. $\theta(z|\tau)$ is a holomorphic function $\mathbb{C} \times \mathfrak{H} \rightarrow \mathbb{C}$, such that $|z \mid \theta(z|\tau) = 0\} = \Lambda_\tau$.

2. $\frac{d\theta}{dz}(0|\tau) = 1$.

3. $\theta(z + 1|\tau) = \theta(z|\tau)$, and $\theta(z + \tau|\tau) = -e^{-\pi i \tau} e^{-2\pi i z} \theta(z|\tau)$.

4. $\theta(z + 1|\tau) = \theta(z|\tau)$, while $\theta(-z|\tau) = -e^{-\pi i \tau} \theta(z|\tau)$.

5. Let $q := e^{2\pi i \tau}$ and $\eta(\tau) := q^{1/24} \prod_{n \geq 1} (1 - q^n)$. If we set $\theta(z|\tau) := \eta(\tau)^3 \theta(z|\tau)$, then $\theta(z|\tau)$ satisfies the differential equation

$$\frac{\partial \theta(z, \tau)}{\partial \tau} = \frac{1}{4\pi i} \frac{\partial^2 \theta(z, \tau)}{\partial z^2}.$$

We have the following product formula of $\theta(z|\tau)$. Let $u = e^{2\pi i z}$, we have

$$\theta(z|\tau) = u^z \prod_{s > 0} (1 - q^s u) \prod_{s \geq 0} (1 - q^s u^{-1}) \frac{1}{2\pi i} \prod_{s > 0} (1 - q^s)^{-2}. \quad (2)$$
Remark 2.3. The theta function $\theta(z|\tau)$ is normalized such that $\frac{\partial}{\partial z}(0|\tau) = 1$. One could take the Jacobi theta function $\theta_1(z, q) = 2 \sum_{n=0}^{\infty} q^{\frac{n^2}{2}} \sin(nz)$ in [25, Page 463-465], then we have

$$\theta(z|\tau) := \frac{\theta_1(\pi z, q)}{2q^{\frac{1}{2}} \prod_{n=1}^{\infty}(1 - q^n)^3}.$$ 

2.3. Principal bundles on elliptic configuration space. Consider the elliptic curve $E := \mathbb{C}/\Lambda$ with modular parameter $\tau \notin \mathbb{R}$. Let $T := \mathfrak{h}/(P^\vee \oplus \tau P^\vee)$ be the adjoint torus, which non-canonically isomorphic to $E^*$, for $n = \text{rank}(P^\vee)$. For any root $\alpha \in \mathfrak{Q} \subset \hat{\mathfrak{h}}^*$, the linear map $\alpha : \mathfrak{h} = P^\vee \otimes \mathbb{C} \rightarrow \mathbb{C}$ induces a natural map

$$\chi_\alpha : T = \mathfrak{h}/(P^\vee \oplus \tau P^\vee) \rightarrow E.$$ 

Denote kernel of $\chi_\alpha$ by $T_\alpha$, which is a divisor of $T$. Denote $T_{\text{reg}} = T \setminus \bigcup_{\alpha \in \Phi} T_\alpha$, which we will refer as the elliptic configuration space. In the type A case, when $\Phi = \{e_i - e_j \mid 1 \leq i \neq j \leq n\}$, $T_{\text{reg}}$ is the configuration space of $n$-points on elliptic curve $E$. 

The Lie algebra $t^0_{\text{cat}}$ is positively bi-graded. Let $\widehat{t^0_{\text{cat}}}$ be the degree completion of $t^0_{\text{cat}}$ and $\exp(\widehat{t^0_{\text{cat}}})$ be the corresponding group of $\widehat{t^0_{\text{cat}}}$. We now describe a principal bundle $P_{T,n}$ on the elliptic configuration space $T_{\text{reg}}$ with structure group $\exp(\widehat{t^0_{\text{cat}}})$. 

The lattice $\Lambda_\tau \otimes P^\vee$ acts on $\mathfrak{h} = \mathbb{C} \otimes P^\vee$ by translations, whose quotient is $T$. For any $g \in \exp(\widehat{t^0_{\text{cat}}})$, and the standard basis $\{\lambda_i^\vee\}_{1 \leq i \leq \ell}$ of $P^\vee$, we define an action of $\Lambda_\tau \otimes P^\vee$ on the group $\exp(\widehat{t^0_{\text{cat}}})$ by

$$\lambda_i^\vee(g) = g \text{ and } \tau \lambda_i^\vee(g) = e^{-2\pi i x_i^\vee} g.$$ 

We can then form the twisted product $\tilde{P} := \mathfrak{h} \times_{\Lambda_\tau \otimes P^\vee} \exp(\widehat{t^0_{\text{cat}}})$. It is a principal bundle on $T$ with structure group $\exp(\widehat{t^0_{\text{cat}}})$. Denote by $P_{T,n}$ the restriction of the bundle $\tilde{P}$ over $T_{\text{reg}} \subset T$.

Equivalently, let $\pi : \mathfrak{h} \rightarrow \mathfrak{h}/(P^\vee \oplus \tau P^\vee)$ be the natural projection. For an open subset $U \subset T_{\text{reg}}$, the sections of $P_{T,n}$ on $U$ are

$$\{f : \pi^{-1}(U) \rightarrow \exp(\widehat{t^0_{\text{cat}}}) \mid f(z + \lambda_i^\vee) = f(z), f(z + \tau \lambda_i^\vee) = e^{-2\pi i x_i^\vee} f(z)\}.$$ 

2.4. The universal KZB connection. In this subsection, we construct the universal KZB connection for root system $\Phi$. As in [6], we set

$$k(z, x|\tau) := \frac{\theta(z + x|\tau)}{\theta(z|\tau)} \frac{1}{x^\vee}.$$ 

(3)

When $\tau$ is fixed, the element $k(z, x|\tau)$ belongs to $\text{Hol}(\mathbb{C} - \Lambda_\tau)[[x]]$. For $x(u) = x_u \in t^0_{\text{cat}}$, substituting $x = \text{ad} x_u$ in (3), we get a linear map $t^0_{\text{cat}} \to (t^0_{\text{cat}} \otimes \text{Hol}(\mathbb{C} - \Lambda_\tau))^\lambda$, where $(-)^\lambda$ is taking the completion.

We consider the $t^0_{\text{cat}}$-valued connection on $T_{\text{reg}}$

$$\nabla_{\text{KZB,}\tau} = d - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{\alpha^\vee}{2}) - (t_\alpha)\alpha + \sum_{i=1}^{n} y(u_i) du_i,$$ 

(4)

where $\Phi^+ \subset \Phi$ is a chosen system of positive roots, $\{u_i\}$, and $\{u_i^*\}$ are dual basis of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively.

Note that the form (4) is independent of the choice of $\Phi^+$. It follows from the equality $k(z, x|\tau) = -k(-z, -x|\tau)$, which is a direct consequence of the fact that theta function $\theta(z|\tau)$ is an odd function.

We now show that the KZB connection $\nabla_{\text{KZB,}\tau}$ (4) is a connection on the principal bundle $P_{T,n}$. In order to do this, we rewrite $\nabla_{\text{KZB,}\tau}$ (4) as the form

$$\nabla_{\text{KZB}} = d - \sum_{i=1}^{n} K_i d\lambda_i^\vee = d - \sum_{i=1}^{n} \left( \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) k(\alpha, \text{ad}(\frac{\alpha^\vee}{2} - (t_\alpha)\alpha - y(\alpha)) \right) d\lambda_i^\vee.$$ 


**Proposition 2.4.** For any \(1 \leq i \leq n\), the function \(K_i\) satisfies the conditions
\[
K_i(z + \lambda^\prime_j) = K_i(z) \quad \text{and} \quad K_i(z + \tau \lambda^\prime_j) = e^{-2\pi i x(\lambda^\prime_j)/2} K_i(z).
\]
As a consequence, the KZB connection \(\nabla_{KZB,\tau}\) (4) is a connection on the bundle \(P_{\tau,\eta}\).

**Proof.** Using the properties of the theta function \(\theta(z|\tau)\), we have, for any integer \(m \in \mathbb{Z}\),
\[
k(z + m, x|\tau) = k(z, x|\tau)
\]
\[
k(z + \tau m, x|\tau) = e^{-2\pi i m x} k(z, x|\tau) + \frac{e^{-2\pi i m x} - 1}{x}.
\]
Therefore, it is obvious that \(K_i(\alpha + \lambda^\prime_j) = K_i(\alpha)\), for any \(\alpha \in \Phi, \lambda^\prime_j \in P^\vee\). We now check the second equality.
It follows from the following computation.
\[
K_i(\alpha + \tau \lambda^\prime_j) = \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) k(\alpha + \tau(\alpha, \lambda^\prime_j)) \text{ad}(\frac{x(\alpha^\prime)}{2}) (t_\alpha) - y(\alpha_i)
\]
\[
= \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) \left( \exp \left( -2\pi i(\alpha, \lambda^\prime_j) \frac{x(\alpha^\prime)}{2} \right) k(\alpha, \text{ad}(\frac{x(\alpha^\prime)}{2}) (t_\alpha) + \frac{\exp \left( -2\pi i(\alpha, \lambda^\prime_j) \frac{x(\alpha^\prime)}{2} \right) - 1}{x(\alpha^\prime)/2} (t_\alpha) - y(\alpha_i) \right)
\]
by the relation \([-x(\alpha^\prime)/2 + x(\lambda^\prime_j), t_\alpha] = 0.
\]
\[
= \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) \exp \left( -2\pi i x(\lambda^\prime_j) \right) k(\alpha, \text{ad}(\frac{x(\alpha^\prime)}{2}) (t_\alpha) + \frac{\exp \left( -2\pi i x(\lambda^\prime_j) \right) - 1}{x(\lambda^\prime_j)} \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i)(\alpha, \lambda^\prime_j) (t_\alpha) - y(\alpha_i)
\]
\[
= \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) \exp \left( -2\pi i x(\lambda^\prime_j) \right) k(\alpha, \text{ad}(\frac{x(\alpha^\prime)}{2}) (t_\alpha) - \exp \left( -2\pi i x(\lambda^\prime_j) \right) y(\alpha_i))
\]
by the relation \([y(\alpha_i), x(\lambda^\prime_j)] = \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i)(\alpha, \lambda^\prime_j) t_\alpha.
\]
\[
= \sum_{\alpha \in \Phi^+} (\alpha, \alpha_i) \exp \left( -2\pi i x(\lambda^\prime_j) \right) k(\alpha, \text{ad}(\frac{x(\alpha^\prime)}{2}) (t_\alpha) - \exp \left( 2\pi i x(\lambda^\prime_j) \right) y(\alpha_i))
\]
\[
= \exp(-2\pi i x(\lambda^\prime_j)) K_i(\alpha).
\]
This completes the proof.  

Let \(W\) be the Weyl group generated by reflections \(\{s_\alpha \mid \alpha \in \Phi\}\). Assume \(W\) acts on the algebra \(t^\Phi_{\text{int}}\).

**Theorem 2.5.**

1. The connection \(\nabla_{KZB,\tau}\) (4) is flat if and only if the defining relations \((tt), (xx), (yy), (yx), (tx), (ty)\) of \(t^\Phi_{\text{int}}\) hold.

2. The connection \(\nabla_{KZB,\tau}\) is \(W\)-equivariant if and only if the action of \(W\) on \(t^\Phi_{\text{int}}\) is given by
\[
s_\alpha(t_\beta) = t_{s_\alpha(\beta)}, \quad s_\alpha(x(\mu)) = x(s_\alpha \mu), \quad s_\alpha(y(\nu)) = y(s_\alpha \nu).
\]

**Proof.** The necessity of claim (1) follows from Corollary 5.3. We postpone the proof of sufficiency of part (1) to the next section. We now show part (2) of the claim. The connection \(\nabla_{KZB,\tau}\) is \(W\)-equivariant if and
only if \( s_i^* \nabla_{KZB, \tau} = \nabla_{KZB, \tau} \), for simple reflection \( s_i \in W \). Note that \( s_i \) permutes the set \( \Phi^+ \setminus \{ \alpha_i \} \), and we have the equality \( k(-\alpha, -\text{ad} \frac{\alpha_i}{2}) = k(\alpha, \text{ad} \frac{\alpha_i}{2}) \). Therefore,

\[
s_i^* \nabla_{KZB, \tau} = d - \sum_{\alpha \in \Phi^+} k(s_i \alpha, \text{ad} (\frac{s_i \alpha}{2})) \tau(s_i t_0) d(s_i \alpha) + \sum_{j=1}^{n-1} (s_j y(u^j)) d(s_j u_j)
\]

\[
= d - \sum_{\beta \in \Phi^+} k(\beta, \text{ad} (\frac{s_i \alpha}{2})) \tau(s_i t_0) d\beta + \sum_{j=1}^{n-1} (s_j y(u^j)) d(s_j u_j)
\]

\[
= d - \sum_{\beta \in \Phi^+} k(\beta, \text{ad} (\frac{s_i \alpha}{2})) \tau(t_0) d\beta + \sum_{j=1}^{n-1} (y((s_i u^j))) d(s_j u_j) = \nabla_{KZB, \tau}.
\]

This completes the proof of part (2).

\[\square\]

**Example 2.6.** In the type A case, the root system is \( \Phi = \{ \alpha_i = \epsilon_i - \epsilon_j \mid 1 \leq i \neq j \leq n \} \), where \( \{\epsilon_1, \ldots, \epsilon_n\} \) is a standard orthonormal basis of \( \mathbb{C}^n \). The connection \( \nabla_{KZB, \tau} \) (4) can be rewritten as \( \nabla_{KZB} := d - \sum_{i=1}^{n} K_i(z|\tau) dz_i \), where

\[
K_i(z|\tau) = -y_i + \sum_{j \neq i} K_{ij}(z_{ij}|\tau) := -y_i + \sum_{j \neq i} k(z_{ij}, \text{ad} x_i|\tau)(t_{ij}).
\]

The above follows from the relation

\[
(\text{ad} x_i)^k(t_{ij}) = (-\text{ad} x_j)^k(t_{ij}) = (\text{ad} \frac{x_i - x_j}{2})^k(t_{ij}),
\]

for any \( k > 0 \), since \([x_i + x_j, t_{ij}] = 0\). This form of KZB connection (5) is constructed in [6].

### 3. Flatness of the universal KZB connection

In this section, we prove Theorem 2.5 (1). Set

\[
A := \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad} (\frac{x^\alpha}{2})) |\tau(t_0) d\alpha - \sum_{j=1}^{n} y(u^j) du_j.
\]

Since \( dA = 0 \), the connection \( \nabla_{KZB, \tau} \) (4) is flat if and only if the curvature \( \Omega := A \wedge A \) is zero. We write the curvature as \( \Omega = \Omega_1 + \Omega_2 + \Omega_3 \), where

\[
\Omega_1 = \frac{1}{2} \sum_{\alpha \neq \beta} \left[ k(\alpha, \text{ad} (\frac{x^\alpha}{2})) |\tau(t_0), k(\beta, \text{ad} (\frac{x^\beta}{2})) |\tau(t_0) \right] d\alpha \wedge d\beta.
\]

\[
\Omega_2 = \sum_{\alpha, j} \left[ k(\alpha, \text{ad} (\frac{x^\alpha}{2})) |\tau(t_0), y(u^j) \right] d\alpha \wedge du_j.
\]

\[
\Omega_3 = \frac{1}{2} \sum_{i \neq j} \left[ y(u^i), y(u^j) \right] du_i \wedge du_j.
\]

The defining relation \([y(u^i), y(u^j)] = 0\) of \( t_0^\Phi \) implies that \( \Omega_3 = 0 \). In the rest of this section, we show that \( \Omega_1 = \Omega_2 \). This gives the flatness of the KZB connection \( \nabla_{KZB, \tau} \), which finishes the proof of Theorem 2.5 (1).
Lemma 3.1. The following identities follow from a direct calculation. We have

$$\omega(u^\gamma, v) = \frac{-2(v, v)}{(u, u)(v, v) - (u, v)^2} u + \frac{2(u, v)}{(u, u)(v, v) - (u, v)^2} v,$$

where $u^\gamma := \frac{2v}{(u, u)}$. The vector $\omega(u^\gamma, v)$ in $\mathfrak{h}$ can be characterized by the following property.

- The vector $\omega(u^\gamma, v)$ is a linear combination of vectors $u$ and $v$.
- One has $(u^\gamma + \omega(u^\gamma, v)) \perp u$ and $\omega(u^\gamma, v) \perp v$.

The following identities follow from a direct calculation.

**Proposition 3.2.** Modulo the relations $(tx), (xx)$ of $\Phi^\prime$, we have, for any $\alpha, \beta \in \Phi^+$

$$[k(\alpha, \text{ad}(\frac{X^\gamma}{2})(\tau))(t_\alpha), k(\beta, \text{ad}(\frac{X^\gamma}{2})(\tau))(t_\beta)] = k(\alpha, \text{ad}(\frac{X_{\omega(\alpha^\gamma, \beta)}}{2})(\tau)k(\beta, \text{ad}(\frac{X_{\omega(\beta^\gamma, \alpha)}}{2})(\tau)[t_\alpha, t_\beta].$$

**Proof.** If $\alpha = \beta$, it is clear that both sides are the same as zero. Therefore, the identity holds.

We now assume $\alpha \neq \beta$. Note that the function $k(z, x)$ is a formal power series in $x$, with coefficient in $\mathbb{C}[z]$. To show the above identity, it suffices to show the identity with $k(z, x)$ replaced by $x^n$, for any $n \in \mathbb{N}$. We have

$$[(\text{ad } x_{\alpha^\gamma})^n(t_\alpha), (\text{ad } x_{\beta^\gamma})^m(t_\beta)] = \left(-\text{ad } x_{\omega(\alpha^\gamma, \beta)} - \text{ad } x_{\omega(\beta^\gamma, \alpha)}\right)^m(t_\beta)\left(-\text{ad } x_{\omega(\alpha^\gamma, \beta)} - \text{ad } x_{\omega(\beta^\gamma, \alpha)}\right)^n(t_\alpha).$$

The first equality follows from the relation $[x_{\alpha^\gamma}, t_\alpha] = -[x_{\omega(\alpha^\gamma, \beta)}, t_\alpha]$, since $\alpha^\gamma + \omega(\alpha^\gamma, \beta)$ is perpendicular to $\alpha$. The second equality follows from $[x_{\omega(\alpha^\gamma, \beta)}, \alpha] = [x_{\omega(\beta^\gamma, \alpha)}, \alpha] = 0$. This completes the proof. \[Q.E.D.\]

**Corollary 3.3.** The curvature term $\Omega_1$ (6) is equal to

$$\Omega_1 = \frac{1}{2} \sum_{\alpha \neq \beta} k(\alpha, \text{ad}(\frac{X_{\omega(\alpha^\gamma, \beta)}}{2})(\tau)k(\beta, \text{ad}(\frac{X_{\omega(\beta^\gamma, \alpha)}}{2})(\tau)[t_\alpha, t_\beta]d\alpha \wedge d\beta. \quad (9)$$

**Proposition 3.4.** Suppose $u, \omega, \alpha, \beta$ modulo the relations $(ty), (yx), (tx), (xx)$ of $\Phi^\prime$, the following identity holds for any $\alpha \in \Phi^+, u \in \mathfrak{h}^*$.

$$[\gamma(u), k(\alpha, \text{ad}(\frac{X^\gamma}{2})(\tau))(t_\alpha)] = \sum_{\gamma \in \Phi^+} (\alpha^\gamma, \gamma)(u, \gamma) \frac{k(\alpha, \text{ad}(\frac{X^\gamma}{2})(\tau) - k(\alpha, \text{ad}(\frac{X_{\omega(\alpha^\gamma, \gamma)}}{2})(\tau))}{\text{ad } x_{\alpha^\gamma} + \text{ad } x_{\omega(\alpha^\gamma, \gamma)}}[t_\gamma, t_\alpha].$$
Proof. As before, note that the function $k(z, x)$ is a formal power series in $x$, with coefficient in $\mathbb{C}[z]$. To show the above identity, it suffices to show the identity with $k(z, x)$ replaced by $x^n$, for any $n \in \mathbb{N}$. We have

$$[y(u), (\text{ad } x_\omega^\nu)^n(t_\alpha)]$$

$$= \sum_{s=0}^{n-1} (\text{ad } x_\omega^\nu)^s([y(u), x_\omega^\nu])(\text{ad } x_\omega^\nu)^{n-1-s}(t_\alpha)$$

$$= \sum_{\gamma \in \Phi^+} \sum_{s=0}^{n-1} (\alpha^\nu, \gamma)(\alpha, \gamma) \sum_{s=0}^{n-1} (\text{ad } x_\omega^\nu)^s(\text{ad } t_\gamma)(\text{ad } x_\omega^\nu)^{n-1-s}(t_\alpha) \quad \text{by the relation (}\gamma x)$$

$$= \sum_{\gamma \in \Phi^+} \sum_{s=0}^{n-1} (\alpha^\nu, \gamma)(\alpha, \gamma) (\text{ad } x_\omega^\nu)^s(\text{ad } t_\gamma)(\text{ad } x_\omega^\nu)^{n-1-s}(t_\alpha) \quad \text{by } [x_\omega^\nu, t_\alpha] = -[x_\omega^\nu(t_\alpha), t_\alpha]$$

$$= \sum_{\gamma \in \Phi^+} \sum_{s=0}^{n-1} (\alpha^\nu, \gamma)(\alpha, \gamma) f(\text{ad } x_\omega^\nu, t_\gamma, (\text{ad } x_\omega^\nu, t_\alpha)), \quad \text{where } f(u, v) = \frac{u^n - v^n}{u - v}.$$ 

Therefore, the assertion follows. \[ \square \]

Corollary 3.5. The curvature term $\Omega_2$ (7) is equal to

$$\Omega_2 = \sum_{\alpha \neq \gamma \in \Phi^+} \frac{(\alpha^\nu, \gamma)k(\alpha, \text{ad}(\frac{x_\omega^\nu}{2})|\tau) - k(\alpha, \text{ad}(\frac{x_\omega^\nu}{2})|\tau)}{\text{ad}x_\alpha^\nu + \text{ad}x_{\omega(\alpha^\nu, \gamma)}|\tau}[t_\alpha, t_\gamma]d\alpha \wedge d\gamma.$$ 

Proof. By definition,

$$\Omega_2 = \sum_{\alpha} \sum_i \left[k(\alpha, \text{ad}(\frac{x_\omega^\nu}{2})|\tau) t_\alpha, y(u_i) \right] d\alpha \wedge du_i.$$ 

Choose the basis $\{u_i\} \subset \mathfrak{b}^*$ to be such that $u_1 = \alpha$ and $u_i \perp \alpha$ for $i \geq 2$. Then since $d\alpha \wedge du_1 = 0$, by Proposition 3.4, the above is equal to

$$- \sum_{\alpha \neq \beta \neq \gamma \in \Phi^+} \sum_{\gamma \in \Phi^+} (u_1, \gamma)(\alpha^\nu, \gamma) \frac{k(\alpha, \text{ad}(\frac{x_\omega^\nu}{2})|\tau) - k(\alpha, \text{ad}(\frac{x_\omega^\nu}{2})|\tau)}{\text{ad}x_\alpha^\nu + \text{ad}x_{\omega(\alpha^\nu, \gamma)}|\tau} [t_\gamma, t_\alpha] d\alpha \wedge du_1$$

$$= \sum_{\alpha \neq \beta \neq \gamma \in \Phi^+} \sum_{\gamma \in \Phi^+} (u_1, \gamma)(\alpha^\nu, \gamma) \frac{k(\alpha, \text{ad}(\frac{x_\omega^\nu}{2})|\tau) - k(\alpha, \text{ad}(\frac{x_\omega^\nu}{2})|\tau)}{\text{ad}x_\alpha^\nu + \text{ad}x_{\omega(\alpha^\nu, \beta)}|\tau} [t_\alpha, t_\beta] d\alpha \wedge d\beta,$$ 

which is equal to the claimed result since $\gamma = (\gamma, u_i)u_i$. \[ \square \]

3.2. In this subsection, we use the expressions of $\Omega_1$ and $\Omega_2$ from previous subsection to compute the difference $\Omega_1 - \Omega_2$. By Corollary 3.3 and Corollary 3.5, we write

$$\Omega_1 - \Omega_2$$

$$= \sum_{\alpha \neq \beta} \left(\frac{1}{2}k(\alpha, \text{ad}(\frac{x_\omega(\alpha^\nu, \beta)}{2})|\tau) k(\beta, \text{ad}(\frac{x_\omega(\beta^\nu, \alpha)}{2})|\tau) - (\alpha^\nu, \beta) \frac{k(\alpha, \text{ad}(\frac{x_\omega^\nu}{2})|\tau) - k(\alpha, \text{ad}(\frac{x_\omega^\nu}{2})|\tau)}{\text{ad}x_\alpha^\nu + \text{ad}x_{\omega(\alpha^\nu, \beta)}|\tau} [t_\alpha, t_\beta] d\alpha \wedge d\beta.$$
Let \( k(\alpha, \beta)[t_\alpha, t_\beta] \) be the difference of the coefficient of \( d\alpha \wedge d\beta \) and the coefficient of \( d\beta \wedge d\alpha \) in the formula of \( \Omega_1 - \Omega_2 \), that is

\[
k(\alpha, \beta) = k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) k(\beta, \text{ad}(\frac{x_{\omega(\beta', \alpha)}}{-2}) | \tau) - (\alpha^\vee, \beta) \frac{k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) - k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\alpha^\vee} + dx_{\omega(\alpha, \beta)}}
\]

\[
- (\beta^\vee, \alpha) \frac{k(\beta, \text{ad}(\frac{x_{\omega(\beta', \alpha)}}{-2}) | \tau) - k(\beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\beta^\vee} + dx_{\omega(\beta', \alpha)}}
\]

**Proposition 3.6.** With notations as above, for any \( \alpha, \beta \in \Phi \), we have

\[
k(\alpha, \beta) d\alpha \wedge d\beta + k(\alpha, \alpha + \beta) d(\alpha + \beta) \wedge d\alpha + k(\beta, \alpha + \beta) d\beta \wedge d(\alpha + \beta) = 0.
\]

**Proof.** It is clear that the statement is equivalent to \(-k(\alpha, \beta) + k(\alpha, \alpha + \beta) + k(\beta, \alpha + \beta) = 0\). We now check this equality. By definition, we have

\[
-k(\alpha, \beta) + k(\alpha, \alpha + \beta) + k(\beta, \alpha + \beta)
\]

\[
e - k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) k(\beta, \text{ad}(\frac{x_{\omega(\beta', \alpha)}}{-2}) | \tau) + k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) k(\beta, \alpha + \beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)
\]

\[
+ k(\beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) k(\alpha + \beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)
\]

\[
+ (\alpha^\vee, \beta) \frac{k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) - k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\alpha^\vee} + dx_{\omega(\alpha, \beta)}} + (\beta^\vee, \alpha) \frac{k(\beta, \text{ad}(\frac{x_{\omega(\beta', \alpha)}}{-2}) | \tau) - k(\beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\beta^\vee} + dx_{\omega(\beta', \alpha)}}
\]

\[
- (\alpha^\vee, \alpha + \beta) \frac{k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) - k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\alpha^\vee} + dx_{\omega(\alpha, \alpha + \beta)}} - ((\alpha + \beta)^\vee, \alpha) \frac{k(\alpha + \beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) - k(\alpha + \beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\alpha + \beta} + dx_{\omega(\alpha, \beta)}}
\]

\[
- (\beta^\vee, \alpha + \beta) \frac{k(\beta, \text{ad}(\frac{x_{\omega(\beta', \alpha)}}{-2}) | \tau) - k(\beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\beta^\vee} + dx_{\omega(\beta', \alpha)}} - ((\alpha + \beta)^\vee, \beta) \frac{k(\alpha + \beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) - k(\alpha + \beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\alpha + \beta} + dx_{\omega(\alpha + \beta, \beta)}}
\]

We use the following identities in Lemma 3.1

\[
\frac{\alpha^\vee + \omega(\alpha^\vee, \beta)}{(\alpha^\vee, \beta)} = \frac{\alpha^\vee + \omega(\alpha^\vee, \alpha + \beta)}{(\alpha^\vee, \alpha + \beta)}, \quad \frac{\beta^\vee + \omega(\beta^\vee, \alpha)}{(\beta^\vee, \alpha)} = \frac{\beta^\vee + \omega(\beta^\vee, \alpha + \beta)}{(\beta^\vee, \alpha + \beta)},
\]

\[
\frac{(\alpha + \beta)^\vee + \omega(\alpha + \beta)^\vee, \alpha}{((\alpha + \beta)^\vee, \alpha)} = - \frac{(\alpha + \beta)^\vee + \omega(\alpha + \beta)^\vee, \beta}{((\alpha + \beta)^\vee, \beta)}.
\]

and the fact that \( x : \mathfrak{h} \to \mathfrak{h}^* \) is a linear function. We could rewrite the terms (12), and (13) using the identities. For example, we have

\[
(a^\vee, \alpha + \beta) \frac{k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) - k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\alpha^\vee} + dx_{\omega(\alpha, \alpha + \beta)}} = (a^\vee, \beta) \frac{k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) - k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\alpha^\vee} + dx_{\omega(\alpha, \beta)}}.
\]

After cancellations with (11), we have

\[
(11) + (12) + (13)
\]

\[
= (\alpha^\vee, \beta) \frac{k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) - k(\alpha, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\alpha^\vee} + dx_{\omega(\alpha, \beta)}} + (\beta^\vee, \alpha) \frac{k(\beta, \text{ad}(\frac{x_{\omega(\beta', \alpha)}}{-2}) | \tau) - k(\beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\beta^\vee} + dx_{\omega(\beta', \alpha)}}
\]

\[
- ((\alpha + \beta)^\vee, \alpha) \frac{k(\alpha + \beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau) - k(\alpha + \beta, \text{ad}(\frac{x_{\omega(\alpha, \beta)}}{-2}) | \tau)}{dx_{\alpha + \beta} + dx_{\omega(\alpha + \beta, \beta)}}.
\]
For simplicity, we make a change of variable. Set $u := \frac{\omega(\alpha^, \beta)}{2}$ and $v := \frac{\omega(\beta^, \alpha + \beta)}{2}$. Using again the identities from Lemma 3.1, we get

$$u = \frac{\omega(\alpha^, \beta)}{2} = \frac{\beta^ + \omega(\beta^, \alpha)}{(\beta^, \alpha)} - 2 = \frac{\omega((\alpha + \beta)^, \beta)}{2},$$

$$v = \frac{\omega(\beta^, \alpha + \beta)}{2} = -\frac{\omega(\alpha^, \alpha + \beta)}{(\alpha + \beta, \alpha)} = -\frac{\omega((\alpha + \beta)^, \alpha)}{2},$$

$$u + v = \frac{\omega(\beta^, \alpha)}{2} = \frac{\alpha^ + \omega(\alpha^, \beta)}{(\alpha^, \beta)} = \frac{\omega((\alpha + \beta)^, \alpha)}{2}.$$

Plugging $u$, $v$ and $u + v$ into $-k(\alpha, \beta) + k(\alpha, \alpha + \beta) + k(\beta, \alpha + \beta)$, we have

$$- k(\alpha, \beta) + k(\alpha, \alpha + \beta) + k(\beta, \alpha + \beta)$$

$$= - k(\alpha, \omega(\beta^, \alpha)) k(\beta(\alpha + \beta, \alpha + v) + k(\beta, \alpha + \beta, \alpha + v) + k(\alpha, \alpha + \beta, \alpha + v) - k(\alpha, \alpha + \beta, \alpha + v) - v.$$
3.3.2. Case $A_2$. Let $\epsilon_1, \epsilon_2, \epsilon_3$ be the standard orthonormal basis of $\mathbb{R}^3$. Let $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2 - \epsilon_3$ be the simple roots for $A_2$. Then, we have $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2\}$. By definition,
\[
\begin{align*}
\Omega_1 - \Omega_2 &= k(\alpha_1, \alpha_2)[t_{\alpha_1}, t_{\alpha_2}]d\alpha_1 \wedge d\alpha_2 + k(\alpha_1, \alpha_1 + \alpha_2)[t_{\alpha_1}, t_{\alpha_1+\alpha_2}]d\alpha_1 \wedge d(\alpha_1 + \alpha_2) \\
&\quad + k(\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_2}, t_{\alpha_1+\alpha_2}]d\alpha_2 \wedge d(\alpha_1 + \alpha_2).
\end{align*}
\]
By Proposition 3.6, we have
\[
k(\alpha_1, \alpha_1 + \alpha_2)d\alpha_1 \wedge d(\alpha_1 + \alpha_2) = k(\alpha_1, \alpha_2)d\alpha_1 \wedge d\alpha_2 + k(\alpha_2, \alpha_1 + \alpha_2)d\alpha_2 \wedge d(\alpha_1 + \alpha_2).
\]
Plugging the above expression into (14), we have
\[
\Omega_1 - \Omega_2 = k(\alpha_1, \alpha_2)[t_{\alpha_1}, t_{\alpha_2} + t_{\alpha_1+\alpha_2}]d\alpha_1 \wedge d\alpha_2 + k(\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_2}, t_{\alpha_1+\alpha_2}]d\alpha_2 \wedge d(\alpha_1 + \alpha_2).
\]
Using (tt)-relations $[t_{\alpha_1}, t_{\alpha_2} + t_{\alpha_1+\alpha_2}] = [t_{\alpha_1 + t_{\alpha_2}}, t_{\alpha_1+\alpha_2}] = 0$, we conclude $\Omega_1 - \Omega_2 = 0$ in the case of $A_2$.

The above procedure can be represented as the graph
\[
\begin{array}{c}
(a_1, \alpha_1 + \alpha_2) \\
\end{array} \quad \begin{array}{c}
(a_1, \alpha_2) - - - - - - - - - (a_2, \alpha_1 + \alpha_2)
\end{array}
\]

3.3.3. Case $B_2$. Let $\epsilon_1, \epsilon_2$ be the standard orthonormal basis of $\mathbb{R}^2$. Let $\alpha_1 = \epsilon_1 - \epsilon_2, \alpha_2 = \epsilon_2$ be the simple roots for $B_2$. Then, we have $\Phi^+ = \{\alpha_1, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_2\}$. By definition,
\[
\begin{align*}
\Omega_1 - \Omega_2 &= k(\alpha_1, \alpha_2)[t_{\alpha_1}, t_{\alpha_2}]d\alpha_1 \wedge d\alpha_2 + k(\alpha_1, \alpha_1 + \alpha_2)[t_{\alpha_1}, t_{\alpha_1+\alpha_2}]d\alpha_1 \wedge d(\alpha_1 + \alpha_2) \\
&\quad + k(\alpha_1, \alpha_1 + 2\alpha_2)[t_{\alpha_1}, t_{\alpha_1+2\alpha_2}]d\alpha_1 \wedge d(\alpha_1 + 2\alpha_2) \\
&\quad + k(\alpha_2, \alpha_1 + 2\alpha_2)[t_{\alpha_2}, t_{\alpha_1+2\alpha_2}, t_{\alpha_1+2\alpha_2}]d\alpha_2 \wedge d(\alpha_1 + 2\alpha_2) \\
&\quad + k(\alpha_2, \alpha_1 + 2\alpha_2)[t_{\alpha_2}, t_{\alpha_1+2\alpha_2}, t_{\alpha_1+2\alpha_2}]d\alpha_2 \wedge d(\alpha_1 + 2\alpha_2) \\
&\quad + k(\alpha_1 + \alpha_2, \alpha_2)[t_{\alpha_1+\alpha_2}, t_{\alpha_1+2\alpha_2} + t_{\alpha_1+2\alpha_2}]d(\alpha_1 + \alpha_2) \wedge d(\alpha_1 + 2\alpha_2) \\
&\quad + k(\alpha_1 + \alpha_2, \alpha_2)[t_{\alpha_1+\alpha_2}, t_{\alpha_1+2\alpha_2} + t_{\alpha_1+2\alpha_2}]d(\alpha_1 + \alpha_2) \wedge d(\alpha_1 + 2\alpha_2).
\end{align*}
\]
We use Proposition 3.6 and split $k(\alpha, \beta)d\alpha \wedge d\beta$ according to the following graph
\[
\begin{array}{c}
(a_1, \alpha_1 + \alpha_2) \\
\end{array} \quad \begin{array}{c}
(a_1 + \alpha_2, \alpha_1 + \alpha_2) \\
\end{array} \quad \begin{array}{c}
(a_1 + \alpha_2, \alpha_1 + \alpha_2) \\
\end{array}
\]
\[
(a_1, \alpha_1 + \alpha_2) \quad (a_1 + \alpha_2, \alpha_1 + \alpha_2) \quad (a_1 + \alpha_2, \alpha_1 + \alpha_2)
\]
\[
\begin{array}{c}
(a_1, \alpha_1 + \alpha_2) \\
\end{array} \quad \begin{array}{c}
(a_1 + \alpha_2, \alpha_1 + \alpha_2) \\
\end{array} \quad \begin{array}{c}
(a_1 + \alpha_2, \alpha_1 + \alpha_2) \\
\end{array}
\]
\[
(a_1, \alpha_1 + \alpha_2) \quad (a_1 + \alpha_2, \alpha_1 + \alpha_2) \quad (a_1 + \alpha_2, \alpha_1 + \alpha_2)
\]
After simplification, we have
\[
\begin{align*}
\Omega_1 - \Omega_2 &= k(\alpha_1, \alpha_2)[t_{\alpha_1}, t_{\alpha_2} + t_{\alpha_1+\alpha_2}]d\alpha_1 \wedge d\alpha_2 + k(\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_1 + t_{\alpha_2}}, t_{\alpha_1+\alpha_2}, t_{\alpha_1+\alpha_2}]d\alpha_2 \wedge d(\alpha_1 + \alpha_2) \\
&\quad + k(\alpha_2, \alpha_1 + 2\alpha_2)[t_{\alpha_2}, t_{\alpha_1+2\alpha_2}, t_{\alpha_1+2\alpha_2}]d\alpha_2 \wedge d(\alpha_1 + 2\alpha_2).
\end{align*}
\]
Using the (tt) relations $[t_{\alpha}, \sum_{\beta \in \Psi} t_{\beta}]$ for $B_2$ root system, each summand of the above formula equals to zero. Therefore, we conclude $\Omega_1 - \Omega_2 = 0$ in the case of $B_2$.

3.3.4. Case $G_2$. Let $\epsilon_1, \epsilon_2, \epsilon_3$ be the standard orthonormal basis of $\mathbb{R}^3$. The simple roots of $G_2$ root system are $\alpha_1 = 2\epsilon_2 - \epsilon_1 - \epsilon_3, \alpha_2 = \epsilon_1 - \epsilon_2$ and we have $\Phi^+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2, 2\alpha_1 + 3\alpha_2\}$. We
use Proposition 3.6 and split $k(\alpha, \beta) d\alpha \wedge d\beta$ according to the following graph

\[
\begin{align*}
& (\alpha_1 + \alpha_2, 2\alpha_1 + 3\alpha_2) \\
& (2\alpha_1 + 3\alpha_2, \alpha_1 + 2\alpha_2) \\
& (\alpha_1 + 3\alpha_2, \alpha_2) \\
& (\alpha_1 + 2\alpha_2, \alpha_1 + 3\alpha_2)
\end{align*}
\]

After simplification, we have

\[
\begin{align*}
\Omega_1 - \Omega_2 &= k(\alpha_1, \alpha_2)[t_{\alpha_1}, t_{\alpha_2} + t_{\alpha_1+\alpha_2}] d\alpha_1 \wedge d\alpha_2 + k(\alpha_2, \alpha_1 + \alpha_2)[t_{\alpha_1}, t_{\alpha_2}, t_{\alpha_1+\alpha_2}] d\alpha_2 \wedge d(\alpha_1 + \alpha_2) \\
&+ k(\alpha_1 + 2\alpha_2, \alpha_2)[t_{\alpha_1+2\alpha_2}, t_{\alpha_2} + t_{\alpha_1+\alpha_2} + t_{\alpha_1+3\alpha_2}] - [t_{\alpha_1+\alpha_2}, t_{2\alpha_1+3\alpha_2}] d(\alpha_1 + 2\alpha_2) \wedge d\alpha_2 \\
&+ k(2\alpha_1 + 3\alpha_2, \alpha_1 + 2\alpha_2)[t_{2\alpha_1+3\alpha_2}, t_{\alpha_1+2\alpha_2} + t_{\alpha_1+\alpha_2}] d(2\alpha_1 + 3\alpha_2) \wedge d(\alpha_1 + 2\alpha_2) \\
&+ k(\alpha_1 + 3\alpha_2, \alpha_2)[t_{\alpha_1+3\alpha_2}, t_{\alpha_2} + t_{\alpha_1+2\alpha_2}] d(\alpha_1 + 3\alpha_2) \wedge d\alpha_2
\end{align*}
\]

Using the (tt) relations $[t_{\alpha}, \sum_{\beta \in \Psi^+} f_{\beta}]$ for $G_2$ root system, each summand of the above formula equals to zero, where the second line follows from the equality

\[
[t_{\alpha_1+2\alpha_2}, t_{\alpha_2} + t_{\alpha_1+\alpha_2} + t_{\alpha_1+3\alpha_2}] - [t_{\alpha_1+\alpha_2}, t_{2\alpha_1+3\alpha_2}] - [t_{\alpha_1+\alpha_2}, t_{\alpha_1+2\alpha_2}, t_{2\alpha_1+3\alpha_2}].
\]

Therefore, we conclude $\Omega_1 - \Omega_2 = 0$ in the case of $G_2$.

4. Degeneration of the elliptic connection

In this section, we show that as the imaginary part of $\tau$ tends $\infty$, the connection $\nabla_{KZB, \tau}$ (4) degenerates to a trigonometric connection of the form considered in [22]. This gives a map from the trigonometric Lie algebra $A_{\text{trig}}$ to the completion of $V_{\Phi}^\Phi$.

4.1. The trigonometric connection. In [22], Toledano Laredo introduced the trigonometric connection, which we recall it here. Let $H = \text{Hom}_\mathbb{C}(Q, \mathbb{C}^*)$ be the complex algebraic torus with Lie algebra $\mathfrak{h}$ and coordinate ring given by the group algebra $\mathbb{C}[Q]$. We denote the function corresponding to $\lambda \in Q$ by $e^\lambda \in \mathbb{C}[H]$, and set

\[
H_{\text{reg}} = H \setminus \bigcup_{\alpha \in \Phi} \{ e^\alpha = 1 \}
\]

Let $A_{\text{trig}}$ be an algebra endowed with the following data:

- a set of elements $\{t_\alpha\}_{\alpha \in \Phi} \subset A_{\text{trig}}$ such that $t_{-\alpha} = t_\alpha$,
- a linear map $X : h \to A_{\text{trig}}$.

Consider the $A_{\text{trig}}$-valued connection on $H_{\text{reg}}$ given by

\[
\nabla_{\text{trig}} = d - \sum_{\alpha \in \Phi^+} \frac{d\alpha}{e^\alpha - 1} t_\alpha - du_i X(u^i),
\]

where $\Phi^+ \subset \Phi$ is a chosen system of positive roots, $\{u_i\}$ and $\{u^i\}$ are dual bases of $\mathfrak{h}^*$ and $\mathfrak{h}$ respectively, the differentials $du_i$ are regarded as translation invariant one forms on $H$ and the summation over $i$ is implicit.

**Theorem 4.1 ([22]).** The connection (16) is flat, if and only if the following relations hold

(tt): For any rank 2 root subsystem $\Psi \subset \Phi$, and $\alpha \in \Psi$, $[t_\alpha, \sum_{\beta \in \Psi^+} f_{\beta}] = 0$.

(XX): For any $u, v \in \mathfrak{h}$, $[X(u), X(v)] = 0$. 

\((tX)\): For any \(\alpha \in \Phi_+, w \in W\) such that \(w^{-1}\alpha\) is a simple root and \(u \in \mathfrak{h}\), such that \(\alpha(u) = 0\),
\[
\left[t_w, X_w(u)\right] = 0,
\]
where \(X_w(u) = X(u) - \sum_{\beta \in \Phi_+ \cap w\Phi_+} \beta(u)t_\beta\).

Assume now that the algebra \(A_{\text{trig}}\) is acted upon by the Weyl group \(W\) of \(\Phi\).

**Proposition 4.2** ([22]). The connection \(\nabla_{\text{trig}}\) is \(W\)–equivariant if and only if for any \(\alpha \in \Phi\), simple reflection \(s_\alpha \in W\) and \(x \in \mathfrak{h}\),
\[
\begin{align*}
    s_\alpha(t_u) &= t_{s_\alpha(u)}, \\
    s_\alpha(X(x)) - X(s_\alpha(x)) &= (\alpha_i, x)t_{\alpha_i}.
\end{align*}
\]

### 4.2. The degeneration of elliptic configuration space.

In this subsection, we show as \(\Im \tau \to \infty\). The elliptic configuration space \(T_{\text{reg}}\) degenerates to \(H_{\text{reg}}\). We start with analyzing the elliptic curve \(E_\tau\) as \(\Im \tau \to \infty\). Let \(\phi(z)\) be the Weierstrass function with respect to the lattice \(\mathbb{Z} + \tau\mathbb{Z}\),
\[
\phi(z) = \frac{1}{\tau^2} + \sum_{m,n} \left( \frac{1}{(z - m\tau + n)^2} - \frac{1}{(m\tau + n)^2} \right).
\]
Differentiating \(\phi(z)\) term by term, we get \(\phi'(z)\). For \(q = e^{2\pi i\tau}\), the points \((\phi(z), \phi'(z))\) lie on the curve \(E\) defined by the equation
\[
y^2 = 4x^3 - g_2x - g_3,
\]
where
\[
g_2 = \frac{(2\pi i)^4}{12} \left( 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{(1 - q^n)} \right),
\]
\[
g_3 = \frac{(2\pi i)^6}{63} \left( -1 + 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{(1 - q^n)} \right).
\]
The cubic polynomial \(4x^3 - g_2x - g_3\) has a discriminant given by \(\Delta = g_2^3 - 27g_3^2\).

The map
\[
E_\tau = \mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z}) \to E \subset \mathbb{P}(C), \quad z \mapsto (1, \phi(z), \phi'(z))
\]
is an isomorphism of complex Lie groups. As \(\Im \tau \to +\infty\), we have \(q \to 0\). The elliptic curve \(E\) degenerates to \(\tilde{E}\) whose defining equation is
\[
y^2 = 4x^3 - \frac{(2\pi i)^4}{12} x + \frac{(2\pi i)^6}{63} = -\frac{4}{27} (2\pi^2 - 3x)(3x + \pi^2)^2,
\]
with discriminant \(\Delta = 0\). Therefore, \(\tilde{E}\) has a singular point at \(x_0 = (x, y) = (\frac{\pi^2}{3}, 0)\).

Removing the singular point \(x_0\) of \(\tilde{E}_\tau\), topologically, the open subset \(\tilde{E}_\tau \setminus x_0\) is homeomorphic to the complex torus \(\mathbb{C}^*\). Thus, we have a continues map on topological spaces \(\phi : \mathbb{C}^* \otimes P^V \to \tilde{E}_\tau \otimes P^V\).

The pullback the bundle \(\mathcal{P}_{\tau, \nu}\) under the map \(\phi\) is a trivial principal bundle on \(\mathbb{C}^* \otimes P^V\) with structure group \(\exp(t_{\text{trig}}\mathfrak{h})\). The section of this trivial bundle can be described as
\[
(f(z) : \pi^{-1}(U) \to \exp(t_{\text{trig}}\mathfrak{h}) | f(z + 1) = f(z)),
\]
where \(U \subset \mathbb{C}^* \otimes P^V\) and \(\pi = \exp : \mathbb{C} \to \mathbb{C}^*\) be the natural exponential map.
4.3. The degeneration of KZB connection. We describe the degeneration of the connection $\nabla_{KZB,\tau}$ (4) as $\Im \tau \to +\infty$ in this subsection. As $\Im \tau \to +\infty$, using the product formula (2) of theta function, the theta function $\theta(z|\tau)$ tends to
\[
\theta(z|\tau) \to \frac{u^\frac{1}{2}(1-u^{-1})}{2\pi i} = \frac{e^{\pi i z} - e^{-\pi i z}}{2\pi i}.
\]
(19)
Thus, by (19), as $q \to 0$, we have $k(\alpha, \text{ad}(\frac{X_\tau}{2})|\tau)$ tends to
\[
k(\alpha, \text{ad}(\frac{X_\tau}{2})|\tau) \to 2\pi i \left( \frac{e^{\pi i (\alpha + \text{ad}(\frac{X_\tau}{2}))} - e^{-\pi i (\alpha + \text{ad}(\frac{X_\tau}{2}))}}{e^{\pi i \text{ad}(\frac{X_\tau}{2})} - e^{-\pi i \text{ad}(\frac{X_\tau}{2})}} \right) - \frac{1}{\text{ad}(\frac{X_\tau}{2})}
\]
To simplify, we have
\[
=2\pi i \left( \frac{1}{e^{2\pi i \alpha} - 1} + \frac{e^{2\pi i \text{ad}(\frac{X_\tau}{2})}}{e^{2\pi i \text{ad}(\frac{X_\tau}{2})} - 1} \right) - \frac{1}{\text{ad}(\frac{X_\tau}{2})}
\]
Therefore, as $\Im \tau \to +\infty$, the connection $\nabla_{KZB}$ (4) degenerates to a flat connection $\nabla^{\text{deg}}$ on the space
\[
(C^* \otimes P^\vee) \setminus \left( \cup_{\alpha \in \Phi^+} (e^{2\pi i \alpha} - 1) \right).
\]
The flat connection $\nabla^{\text{deg}}$ is defined on the trivial principal bundle with fibers isomorphic to $\exp(\widehat{t}_{\text{ell}})$.  

**Proposition 4.3.** The flat connection $\nabla^{\text{deg}}$ on $H_{\text{reg}}$ takes the following form
\[
\nabla^{\text{deg}} = d - \sum_{\alpha \in \Phi^+} \left( \frac{2\pi i}{e^{2\pi i \alpha} - 1} + \frac{2\pi i e^{2\pi i \text{ad}(\frac{X_\tau}{2})}}{e^{2\pi i \text{ad}(\frac{X_\tau}{2})} - 1} - \frac{1}{\text{ad}(\frac{X_\tau}{2})} \right) t_{\alpha} d\alpha + \sum_{i=1}^n y(u^i) du^i
\]
\[
= d - \sum_{\alpha \in \Phi^+} \frac{2\pi it_{\alpha}}{e^{2\pi i \alpha} - 1} d\alpha + \sum_{i=1}^n \left( y(u^i) - \sum_{\alpha \in \Phi^+} (\alpha, u^i) \left( \frac{2\pi i e^{2\pi i \text{ad}(\frac{X_\tau}{2})}}{e^{2\pi i \text{ad}(\frac{X_\tau}{2})} - 1} - \frac{1}{\text{ad}(\frac{X_\tau}{2})} \right) t_{\alpha} \right) du^i.
\]

We modify the trigonometric connection (16) slightly by changing of variables. On the torus $(C^* \otimes P^\vee) \setminus \cup_{\alpha \in \Phi^+} (e^{2\pi i \alpha} - 1)$, we consider an $A_{\text{trig}}$-valued flat trigonometric connection
\[
\nabla_{\text{trig}} = d - \sum_{\alpha \in \Phi^+} \frac{2\pi it_{\alpha}}{e^{2\pi i \alpha} - 1} - du^i X(u^i).
\]
(20)
By universality of the trigonometric Lie algebra $A_{\text{trig}}$, Proposition 4.3 gives rise to a map $A_{\text{trig}} \to \widehat{t}_{\text{ell}}$. The map is given by
\[
t_{\alpha} \mapsto t_{\alpha}, \quad X(u) \mapsto -y(u) + \sum_{\alpha \in \Phi^+} (\alpha, u) \left( \frac{2\pi i e^{2\pi i \text{ad}(\frac{X_\tau}{2})}}{e^{2\pi i \text{ad}(\frac{X_\tau}{2})} - 1} - \frac{1}{\text{ad}(\frac{X_\tau}{2})} \right) t_{\alpha}.
\]
Note that this map does not preserve the gradings of $A_{\text{trig}}$ and $\widehat{t}_{\text{ell}}$, but only the corresponding descending filtrations.

5. The Malcev Lie algebra of the pure elliptic braid group

The following definition can be found in [9, Definition 2.4].

**Definition 5.1.** A finitely presented group $\Gamma$ is $1$–formal if its Malcev Lie algebra $m_\Gamma$ is isomorphic to the completion of its holonomy Lie algebra as filtered Lie algebras. Equivalently, $\Gamma$ is $1$–formal if, and only if $m_\Gamma$ is isomorphic, as a filtered Lie algebra, to the graded completion of a quadratic Lie algebra.
5.1. Let $P_{\text{ell}}^\Phi$ be the pure elliptic braid group $P_{\text{ell}}^\Phi := \pi_1(T_{\text{reg}}, x_0)$. Then, we have the short exact sequence of groups (see [1, Example 2.20])

$$1 \to P_{\text{ell}}^\Phi \to B_{\text{ell}} \to W \to 1,$$

where $B_{\text{ell}} := \pi_1^{\text{ orb}}(T_{\text{reg}}/W, x_0)$ is the elliptic braid group and $W$ is the Weyl group. The flatness of the universal KZB connection $\nabla_{\text{KZB},r}$ (4) gives rise to the monodromy map

$$\mu : P_{\text{ell}}^\Phi \to \exp(\hat{t}_{\text{ell}}),$$

where $\hat{t}_{\text{ell}}$ is the completion of $t_{\text{ell}}^\Phi$ with respect to the grading $\deg(x(u)) = \deg(y(u)) = 1$ and $\deg(t_\alpha) = 2$.

Let $J \subseteq P_{\text{ell}}^\Phi$ be the augmentation ideal of group ring $\mathbb{C}P_{\text{ell}}^\Phi$, and let $\hat{C}P_{\text{ell}}^\Phi$ be the completion with respect to the augmentation ideal $J$. We denote by $U((t_{\text{ell}}^\Phi))$ the universal enveloping algebra of the Lie algebra $t_{\text{ell}}^\Phi$. Then, the monodromy map $\mu$ induces the following map on the completions

$$\hat{\mu} : \hat{C}P_{\text{ell}}^\Phi \to U((t_{\text{ell}}^\Phi)).$$

**Theorem 5.2.** The induced map $\hat{\mu} : \hat{C}P_{\text{ell}}^\Phi \to U((t_{\text{ell}}^\Phi))$ is an isomorphism of Hopf algebras.

Taking the primitive elements of the Hopf algebras $\hat{C}P_{\text{ell}}^\Phi$ and $U((t_{\text{ell}}^\Phi))$, we get an isomorphism of the Malcev Lie algebra of $P_{\text{ell}}^\Phi$ and the Lie algebra $t_{\text{ell}}^\Phi$ defined in Definition 2.1.

**Corollary 5.3.** The defining relations of the Lie algebra $t_{\text{ell}}^\Phi$ are necessary to make connection of the form (4) flat.

**Proof.** Assume $\nabla$ is a flat connection defined on a principal bundle with structure group $\exp(A)$ with the form (4). The monodromy of $\nabla$ induces a map $\hat{C}P_{\text{ell}}^\Phi \to \hat{U}(A)$. By Theorem 5.2, $\hat{C}P_{\text{ell}}^\Phi \cong \hat{U}((t_{\text{ell}}^\Phi))$. Therefore, we have a well-defined map $t_{\text{ell}}^\Phi \to \hat{A}$. This implies the claim. $\square$

The rest of the section is devoted to prove Theorem 5.2. Note that both $\hat{C}P_{\text{ell}}^\Phi$ and $\hat{t}_{\text{ell}}^\Phi$ are $\mathbb{N}$–filtered and $\hat{\mu}$ preserves the grading. It suffices to show the associated graded

$$\text{gr}(\hat{\mu}) : \text{gr}(\hat{C}P_{\text{ell}}^\Phi) \to \text{gr}(\hat{U}((t_{\text{ell}}^\Phi))) = \hat{U}(t_{\text{ell}}^\Phi)$$

is an isomorphism.

We first describe the generators of the pure elliptic braid group $P_{\text{ell}}^\Phi$. The following proposition can be found in [4, Proposition A.1].

**Proposition 5.4.** Let $i$ be the injection of an irreducible divisor $D$ in a smooth connected complex variety $Y$ and base point $x_0 \in Y - D$. Then, the kernel of the morphism

$$\pi_1(i) : \pi_1(Y - D, x_0) \to \pi_1(Y, x_0)$$

is generated by all the generators of the monodromy around $D$.

Using induction on the number of irreducible divisors, we have the following.

**Corollary 5.5.** Suppose $X = Y - \bigcup_{i=1}^n D_i$, where $D_i$ are irreducible divisors of $Y$. Then, the fundamental group $\pi_1(X, x_0)$ is generated by all the generators of the monodromy around the divisors $D_i$ and generators of $\pi_1(Y, x_0)$.

As a consequence, the generators of $P_{\text{ell}}^\Phi$ can be chosen as $T_\alpha, X_1, \ldots, X_n, Y_1, \ldots, Y_n$, where $T_\alpha$ is the path around the divisor $H_\alpha$, for $\alpha \in \Phi^+$ and $\{X_i, Y_i\}$ are the standard generators of the torus $E \otimes P^\vee \cong E^\vee$. The presentation of $\hat{C}P_{\text{ell}}^\Phi$ can be described as follows. Suppose $P_{\text{ell}}^\Phi$ is presented by generators $g_1, \ldots, g_n$ and relations $R_i(g_1, \ldots, g_n)$, $i = 1, \ldots, p$. Then $\hat{C}P_{\text{ell}}^\Phi$ is the quotient of the free Lie algebra generated by
\(\gamma_1, \ldots, \gamma_n\) by the ideal generated by \(\log(R_i(e^{\gamma}), \ldots, e^{\gamma}))\), \(i = 1, \ldots, p\). We will use notation and denote the generators of \(\hat{\mathbb{C}P}^\Phi_{\text{all}}\) by \(T_\alpha, X_1, \ldots, X_n, Y_1, \ldots, Y_n\) for \(\alpha \in \Phi^+\).

We construct a map

\[ p : U(\mathbb{C}P^\Phi_{\text{all}}) \to \text{gr}(\hat{\mathbb{C}P}^\Phi_{\text{all}}), \]

by sending \(t_\alpha \mapsto T_\alpha, \lambda(\gamma) \mapsto X_i, \) and \(\gamma(m) \mapsto Y_i.\) By definition, it is clear that \(p\) is surjective. We now check the following in the next two subsections.

1. \(p\) is a homomorphism.
2. \(\text{gr}(\hat{\mu}) \circ p\) is an isomorphism.

This implies that \(\text{gr}(\hat{\mu})\) is an isomorphism, which in turn proves Theorem 5.2.

5.2. \(p\) is a homomorphism.

5.2.1. The map \(p\) respects the \((tt)\) relations. We follow the approach in [22] to show that \(p\) preserves the \((tt)\) relations.

Let \(T := \text{Hom}_\mathbb{Z}(Q, E) = P^\vee \otimes_\mathbb{Z} E\) be the torus. Denote \(\ker(\chi_\alpha)\) by \(T_\alpha \subseteq T\).

**Lemma 5.6.** There exist a component \((T_\alpha \cap T_\beta)_X\), such that \((T_\alpha \cap T_\beta)_X\) contained in a subtorus \(T_Y\) if and only if \(\gamma \in \mathbb{Z}\alpha + \mathbb{Z}\beta\).

**Proof.** Without loss of generality, we assume \(\alpha\) is a simple root. Note \(\mathbb{Z}\alpha + \mathbb{Z}\beta\) may not be a primitive sublattice in \(Q\). Pick a vector \(e_2\) such that \(\alpha, e_2\) generate the primitive \(\mathbb{Z}\alpha + \mathbb{Z}\beta\) of \(\mathbb{Z}\alpha + \mathbb{Z}\beta\). Write \(\beta = a\alpha + be_2\). Extend the set \([\alpha, e_2]\) to a basis \(\{e_1 := \alpha, e_2, e_3, \ldots, e_n\}\) of \(Q\). Choose the corresponding dual basis \(\{f_1, f_2, f_3, \ldots, f_n\}\) of \(P^\vee\). Recall that the coweight lattice \(P^\vee\) is dual to root lattice \(Q\).

Identify \(T = P^\vee \otimes_\mathbb{Z} E\) with \((E)^n\) using the basis \(\{f_1, f_2, f_3, \ldots, f_n\}\) of \(P^\vee\). Then, the maps \(\chi_\alpha, \chi_\beta\) are giving by

\[ \chi_\alpha : T = (E)^n \to E, \sum_{i=1}^n z_i f_i \mapsto z_1. \]

\[ \chi_\beta : T = (E)^n \to E, \sum_{i=1}^n z_i f_i \mapsto az_1 + bz_2. \]

Thus, \(T_\alpha \cong \{0\} \times (E)^{n-1} \subseteq (E)^n,\) and \(T_\alpha \cap T_\beta = \{(0, z_2, \ldots, z_n) \subseteq (E)^n \mid b z_2 = 0\}.\) It is clear that the number of connected components of \((T_\alpha \cap T_\beta)_X\) is \(b^2\). Taking the component \((T_\alpha \cap T_\beta)_X\) to be \((0, \frac{1}{b}; z_3, \ldots, z_n) \subseteq (E)^n\).

Write \(\gamma = \sum_{i=1}^n m_i e_i,\) then, the following holds.

\[ (T_\alpha \cap T_\beta)_X \subseteq T_Y \iff \frac{1}{b} m_2 + \sum_{i=3}^n m_i z_i \in \mathbb{Z} + \tau \mathbb{Z}, \forall \mathbb{Z} z_i \in \mathbb{C}, i = 3, \ldots, n. \]

\[ \iff b \mid m_2, \quad \text{and} \quad m_3 = \cdots = m_n = 0. \]

\[ \iff \gamma \in \mathbb{Z}\alpha + \mathbb{Z}\beta. \]

This completes the proof. \(\square\)

**Proposition 5.7.** The \((tt)\)-relation \([t_\alpha, \sum_{\beta \in \Psi} t_\beta] = 0\) holds in \(\text{gr}(\hat{\mathbb{C}P}^\Phi_{\text{all}}).\)

**Proof.** Fix two roots \(\alpha, \beta \in \Phi^+\), let \((T_\alpha \cap T_\beta)_X\) be the component as in Lemma 5.6. Choose a point \(x_0 \in (T_\alpha \cap T_\beta)_X\), such that \(x_0 \notin T_Y\), for any \(T_Y\) satisfying \((T_\alpha \cap T_\beta)_X \subseteq T_Y.\)
Take an open disc neighborhood $D$ of $x_0$, such that $D \cap T_\gamma = \emptyset$, for those $T_\gamma$, satisfying $(T_\alpha \cap T_\beta)_\gamma \notin T_\gamma$. By Lemma 5.6, we have

$$D \cap (\bigcup_{\gamma \in \Phi^+} T_\gamma) = \bigcup_{\gamma \in \mathbb{Z}_a + \mathbb{Z}_b} (D \cap T_\gamma).$$

[23, Lemma 1.45] implies the $(tt)$ relation in $\text{gr}(\widehat{\mathbb{C}P}^{\Phi}_\text{et})$. □

5.2.2. The map $p$ respects the $(yx)$ relation.

**Proposition 5.8.** The map $p$ sends the $(yx)$ relation $[y(u), x(v)] = \sum_{\gamma \neq \Phi^+} \langle v, \gamma \rangle (u, \gamma) t_\gamma$ to zero in $\text{gr}(\widehat{\mathbb{C}P}^{\Phi}_\text{et})$.

**Proof.** Note that the loop $[\alpha]_{\gamma}$ for some coefficient $C_{i, j, \gamma} \in \mathbb{Q}$. We now determine the coefficients $C_{i, j, \gamma}$. Consider the map

$$\chi_\alpha : (P^\vee \otimes \mathbb{E}) \setminus \ker \chi_\alpha \to \mathbb{E} \setminus \{0\}.$$

Denote by $\varphi_\alpha$ the composition

$$\varphi_\alpha : T_{\text{reg}} := (P^\vee \otimes \mathbb{E}) \setminus (\cup_{\gamma \neq \Phi^+} \ker \chi_\alpha) \hookrightarrow (P^\vee \otimes \mathbb{E}) \setminus \ker \chi_\alpha \to \mathbb{E} \setminus \{0\};$$

Let $x(\lambda^\vee_j) = (t + \frac{1}{s} + \frac{1}{r}) \lambda^\vee_j$, where $0 \leq t \leq 1$, and $y(\lambda^\vee_j) = (s \tau + \frac{1}{s} + \frac{1}{r}) \lambda^\vee_j$, where $0 \leq s \leq 1$ be two loops in $T_{\text{reg}}$. Using [2, Corollary 5.1], one has the following relation in the fundamental group of the punctured torus $\mathbb{E} \setminus \{0\}$:

$$\left[ \frac{1}{(\lambda^\vee_j, \alpha)} \varphi_\alpha(y(\lambda^\vee_j)), \frac{1}{(\lambda^\vee_j, \alpha)} \varphi_\alpha(x(\lambda^\vee_j)) \right] = t_\alpha,$$

where $t_\alpha$ is the loop around the puncture $[0]$. The above relation can be rewritten as $[\varphi_\alpha(y(u)), \varphi_\alpha(x(v))] = (u, \alpha)(v, \alpha)t_\alpha$. This determines the coefficient $C_{i, j, \gamma} = (\lambda^\vee_j, \gamma)(\lambda^\vee_j, \gamma)$. This completes the proof. □

5.2.3. The map $p$ respects the $(xx)$ (yy) and $(tx)$ (ty) relations. We show the relations $[x(u), x(v)] = 0$, $[y(u), y(v)] = 0$, for any $u, v \in \mathbb{B}$ and the relations $[t_\alpha, x(u)] = 0$, $[t_\gamma, y(u)] = 0$ for $(\alpha, u) = 0$ in the pure elliptic braid group $P^{\Phi}_{\text{et}}$. There is an embedding of groups $P^{\Phi}_{\text{et}} \hookrightarrow B_{\text{et}}$. It is suffices to show the relations in the elliptic braid group $B_{\text{et}}$.

We first recall some results from [7] about the topological interpretation of the double affine braid group $B_{\text{et}}$.

Let $\mathbb{U} = \{z \in \mathbb{C}^n \mid (\alpha, z) \notin \mathbb{Z} + i\mathbb{Z}, \text{ for any root } \alpha \in \Phi\}$, $\bar{\mathbb{U}} = \mathbb{C}^n \times \mathbb{U}$.

Let $\widetilde{\mathcal{W}}$ be the 2-extended Weyl group $\mathcal{W} := \mathcal{W} \ltimes (P^\vee \oplus iP^\vee)$. For any element $\tilde{w} = (w, a, b) \in \widetilde{\mathcal{W}}$, the action of $\tilde{w}$ on $\tilde{z} = (z, z^0) \in \bar{\mathbb{U}}$ is given by

$$\tilde{w}(\tilde{z}) = ((-1)^{l'(w)}, w(z) + a + ib),$$

where $l'(w)$ is the modified length of $w$.

We fix a base point $\tilde{z}^0 = (z^0_+, z^0_-)$ such that the real and imaginary parts of $(z^0_j, z^0_j, 1 \leq j \leq n)$ are positive and sufficiently small. Note that the action of $\widetilde{\mathcal{W}}$ on $\bar{\mathbb{U}}$ is not free.

**Definition 5.9.** [7, Definition 2.3] The double affine braid group $\widetilde{B}_{\text{et}}$ is formed by the paths $\gamma \subset \bar{\mathbb{U}}$ joining $\tilde{z}^0$ with points from $\{\tilde{w}(\tilde{z}^0), \tilde{w} \in \widetilde{\mathcal{W}}\}$ modulo the homotopy and the action of $\widetilde{\mathcal{W}}$. 
We introduce the elements $t_j := t_{a_j}$, $x_j, y_j$ for $1 \leq j \leq n$ and $c \in \widetilde{B}_{el}$ by the following paths, for $1 \leq \psi \leq 1$,

$$t_j(\psi) = (z^0_{\psi}) \exp(t_j^0 \pi \psi), \quad x_j(\psi) = (z^0_{\psi}, z^0 + \psi b_j), \quad y_j(\psi) = (z^0_{\psi}, z^0 + i \psi b_j),$$

$$c(\psi) = (z^0_{\psi}) \exp(-2\pi i \psi), \quad z^0.$$

**Theorem 5.10.** [7, Theorem 2.4] The group $\widetilde{B}_{el}$ is generated by $\{t_i, 1 \leq i \leq n\}$, the two sets $\{x_i\}, \{y_i\}$ of pairwise commutative elements and central $c$ with the following relations

1. $t_j t_k \cdots t_j t_k \cdots = t_j t_k \cdots $, $m_{ij}$ factors on each side.
2. $t_i x_i = x_i^{-1} t_i^{-1}$ and $t_i y_i = y_i^{-1} t_i^{-1}$.
3. $p_i x_i p_i^{-1} = x_i^{-1}$, $p_r x_r p_r^{-1} = x_r^{-1}$.

where $c_i = c^{(i)}$, $p_r := y_r c^{-1}$ for $w = \sigma_r^{-1}$.

As discussed in [7], the map $T_i \mapsto t_i, X_b \mapsto x_b c^{p_i, b}$ and $Y_b \mapsto y_b c^{p_i, b}$, identifies the double affine Hecke group (see [7] for the definition) with a subgroup of $B_{el}$. In particular, we have the relations

$$T_i X_b = X_b T_i, \quad T_i Y_b = Y_b T_i, \quad \text{if } (b, \alpha_i) = 0.$$

**Corollary 5.11.** The orbifold fundamental group $B_{el}$ of the space $T_{reg}/W$ is isomorphic to the double affine braid group $\widetilde{B}_{el}$ modulo the central element $c$.

Therefore, the $(xx)$ $(yy)$ and $(tx)$ $(ty)$ relations hold in $P_{\widetilde{B}_{el}}^\Phi$.

5.3. $\text{gr}(\hat{\mu}) \circ p$ is an isomorphism.

**Lemma 5.12.** The map $\text{gr}(\hat{\mu}) : \text{gr}(\widetilde{P}_{\widetilde{B}_{el}}^\Phi) \rightarrow U(t_{\Phi_{\widetilde{B}_{el}}})$ is given by $X_{\phi} \mapsto -y(u'), Y_{\phi} \mapsto 2\pi i x(u') - \tau y(u')$, and $T_\alpha \mapsto 2\pi i t_\alpha$.

**Proof.** Recall the KZB connection (4) is

$$\nabla_{\text{KZB}} = d - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad} \frac{X_{\phi}}{2})(t_\alpha) d\alpha + \sum_{i=1}^n y(u^i) d\mu,$$

where $k(\alpha, \text{ad} \frac{X_{\phi}}{2})(t_\alpha) = \frac{1}{\alpha} t_\alpha + \text{terms of degree } \geq 3$. Let $F_{\phi}(u)$ be the horizontal section of $\nabla_{\text{KZB}}$. Then, log $F_{\phi}(u) = -\sum_{i=1}^n (u_i - u_i^0) y(u^i) + \text{terms of degree } \geq 2$.

**Corollary 5.13.** For some open sets $U$ and $V$, we have

$$F^U_{\phi}(u + \delta_i) = F^U_{\phi}(u) \mu_{\phi}(x(u^i)), \quad F^V_{\phi}(u + \tau \delta_i) = e^{-2\pi i x(u^i)} F^V_{\phi}(u) \mu_{\phi}(y(u^i)),$$

where $\mu$ is the monodromy representation.

**Proof.** This follows directly from Proposition 2.4. \hfill $\square$

By Corollary 5.13, we get

$$\log \mu_{\phi}(x(u^i)) = -y(u^i) + \text{terms of degree } \geq 2,$$

$$\log \mu_{\phi}(y(u^i)) = 2\pi i x(u^i) - \tau y(u^i) + \text{terms of degree } \geq 2.$$
Therefore, \( \text{gr}(\hat{\mu})(x(u')) = -y(u') \) and \( \text{gr}(\hat{\mu})(y(u')) = 2\pi i(x(u') - \tau y(u')). \) Note that
\[
\hat{\mu}(T_\alpha) = \int_{T_\alpha} \frac{1}{t_\alpha + \text{terms of degree } \geq 3} d\alpha = 2\pi i t_\alpha + \text{terms of degree } \geq 3.
\]
So we have \( \text{gr}(\hat{\mu})(T_\alpha) = 2\pi i t_\alpha. \) The argument above shows the following. \( \square \)

**Corollary 5.14.** The composition \( \text{gr}(\hat{\mu}) \circ p : U(t^\Phi_{\text{ell}}) \to U(t^\Phi_{\text{ell}}) \) is given by \( x(u') \mapsto -y(u'), \) \( y(u') \mapsto 2\pi i x(u') - \tau y(u') \), and \( t_\alpha \mapsto 2\pi i t_\alpha. \) In particular, \( \text{gr}(\hat{\mu}) \circ p \) is an isomorphism.

5.4 In this section, we prove the following.

**Theorem 5.15.**

1. The Lie algebra \( t^\Phi_{\text{ell}} \) is not quadratic.
2. The elliptic pure braid group is not 1–formal.

**Proof.** By Theorem 5.2, (2) is a direct consequence of (1). Set \( \text{deg}(x(u)) = \text{deg}(y(u)) = 1 \), and \( \text{deg}(t_\gamma) = 2 \), for \( \gamma \in \Phi^+. \) Denote by \( (t^\Phi_{\text{ell}})^{(2)} \) the vector space of degree 2 elements in \( t^\Phi_{\text{ell}}. \) If \( t^\Phi_{\text{ell}} \) is quadratic, which we henceforth assume, it is generated by \( x(u), y(v) \) and the commutators \( [x(u), y(v)] \) span \( (t^\Phi_{\text{ell}})^{(2)}. \) Since \( [x(u), y(v)] = [x(v), y(u)] \) by relation (yx) of Definition 2.1, \( (t^\Phi_{\text{ell}})^{(2)} \) is therefore of dimension at most \( n(n + 1)/2 \), where \( n = \dim \mathfrak{h} \).

By Proposition 9.3, there is an algebra homomorphism \( \rho \) from \( t^\Phi_{\text{ell}} \) to the rational Cherednik algebra \( H_{h,c}, \) which maps the element \( t_\gamma \) to the reflection \( s_\gamma \) in the group algebra \( \mathbb{C}W, \) which is a subalgebra of \( H_{h,c}. \) Since the set \( \{s_\gamma \mid \gamma \in \Phi^+\} \) is linearly independent in \( \mathbb{C}W, \) it follows that \( \{t_\gamma \mid \gamma \in \Phi^+\} \) is linearly independent in \( (t^\Phi_{\text{ell}})^{(2)}. \) For any root system \( \Phi \) other than type \( A, \) we have \( [\Phi^+] > \frac{n(n+1)}{2} \), which is a contradiction. If the root system \( \Phi \) is of type \( A, \) it is shown in [3] that \( t^\Phi_{\text{ell}} \) is not quadratic, as the degree 3 relations \( (tx), (ty) \) in Definition 2.1 can not be obtained from the degree 2 relations. \( \square \)

6. Derivations of the Lie algebra \( t^\Phi_{\text{ell}} \)

Let \( \mathfrak{d} \) be the Lie algebra defined in [6] with generators \( \Delta_0, d, X, \) and \( \delta_{2m} (m \geq 1), \) and relations
\[
[d, X] = 2X, \quad [d, \Delta_0] = -2\Delta_0, \quad [X, \Delta_0] = d,
\]
\[
[\delta_{2m}, X] = 0, \quad [d, \delta_{2m}] = 2m\delta_{2m}, \quad (\text{ad} \Delta_0)^{2m+1}(\delta_{2m}) = 0.
\]
We define a Lie algebra morphism \( \mathfrak{d} \to \text{Der}(t^\Phi_{\text{ell}}), \) denoted by \( \xi \mapsto \hat{\xi}. \) The image \( \hat{\xi} \) of \( \xi \) acts on \( t^\Phi_{\text{ell}} \) by the following formulas.
\[
\hat{\xi}(x(u)) = x(u), \quad \hat{\xi}(y(u)) = -y(u), \quad \hat{\xi}(t_\alpha) = 0,
\]
\[
\hat{\Delta}_0(x(u)) = y(u), \quad \hat{\Delta}_0(y(u)) = 0, \quad \hat{\Delta}_0(t_\alpha) = 0,
\]
\[
\hat{\delta}_{2m}(x(u)) = 0, \quad \hat{\delta}_{2m}(y(u)) = 0, \quad \hat{\delta}_{2m}(t_\alpha) = [t_\alpha, (\text{ad} x(\alpha)^{-1})^{2m}(t_\alpha)],
\]
and
\[
\hat{\delta}_{2m}(y(u)) = \frac{1}{2} \sum_{\alpha, \beta \in \Phi^+} \alpha(u) \sum_{p+q=2m-1} \left[ (\text{ad}x(\alpha)^{-1})^p(t_\alpha), (\text{ad}x(\alpha)^{-1})^q(t_\alpha) \right].
\]

**Proposition 6.1.** The above map \( \mathfrak{d} \to \text{Der}(t^\Phi_{\text{ell}}) \) is a Lie algebra homomorphism.

For any generator \( \xi \) of \( \mathfrak{d}, \) we check \( \hat{\xi} \) is a derivation of \( t^\Phi_{\text{ell}}. \) By definition, it is clear that \( \hat{d}, \hat{X} \) and \( \hat{\Delta}_0 \) are derivations of \( t^\Phi_{\text{ell}}. \) It is also clear that \( \hat{\delta}_{2m} \) respects the relations \( [x(u), x(v)] = 0 \) and \( [t_\alpha, x(u)] = 0, \) if \( (\alpha, u) = 0. \) It remains to show that \( \hat{\delta}_{2m} \) preserves the following relations of \( t^\Phi_{\text{ell}}. \)
Example 6.2. Let $\Psi$ be a root system of the root system $\Phi$. In this subsection, we introduced the set $S_{\Psi}^{(\alpha)}$, for a rank 2 root subsystem $\Psi$, and $\alpha \in \Phi$ a root. This section is to study the (tt) relations in detail.

Let $\Phi$ be a root system, for any root $\alpha \in \Phi$, we define a set $S_{\Psi}^{(\alpha)}$ by

$$S_{\Psi}^{(\alpha)} = \{ \beta \in \Phi | \langle \alpha, \beta \rangle \in \Psi \},$$

where $\Psi$ is a rank 2 subsystem of $\Phi$.

Example 6.3. Let $\Phi$ be the root system of $B_2$, and $\alpha_1$ be the simple long root, $\alpha_2$ be the simple short root. There are two root subsystems. One is $\Psi_1 = \{ \pm \alpha_1, \pm(\alpha_1 + 2\alpha_2) \}$ consisting of all long roots, and the other is $\Phi$ itself. The set of all short roots is not a root system.

Therefore, we have the following set

$$S_{\Psi_1}^{(\alpha_1)} = \{ \pm(\alpha_1 + 2\alpha_2) \}, \quad S_{\Phi}^{(\alpha_1)} = \{ \pm(\alpha_1 + \alpha_2), \pm \alpha_2 \},$$

$$S_{\Psi_1}^{(\alpha_2)} = \emptyset, \quad S_{\Phi}^{(\alpha_2)} = \{ \pm \alpha_1, \pm(\alpha_1 + \alpha_2), \pm(\alpha_1 + 2\alpha_2) \}.$$

Lemma 6.4. For $\Psi_1 \neq \Psi_2$, we have $S_{\Psi_1}^{(\alpha)} \cap S_{\Psi_2}^{(\alpha)} = \emptyset$. Furthermore, we have the decomposition of disjoint sets

$$\Phi = (\sqcup_{\Psi \subseteq \Phi} S_{\Psi}^{(\alpha)}) \sqcup \{ \pm \alpha \}. \quad (21)$$

Proof. For any $\beta \in \Phi$, such that $\beta \neq \alpha$, consider $\Psi := \langle \alpha, \beta \rangle \cap \Phi$, then $\beta \in S_{\Psi}^{(\alpha)}$. Thus, $\beta$ lies in the right hand side of (21). It is clear that the right hand side of (21) is a disjoint union.

Corollary 6.5.

(1) For any rank 2 root subsystem $\Psi \subset \Phi$, we have the (tt) relation $[t_\alpha, \sum_{\beta \in S_{\Psi}^{(\alpha)}} t_\beta] = 0$.

(2) For any $\beta \in S_{\Psi}^{(\alpha)}$, we have $[t_\beta, \sum_{\gamma \in S_{\psi}^{(\alpha)}} t_\gamma + t_\alpha] = 0$.

Proof. Let $\Psi \subset \Phi$ be a rank 2 root subsystem containing $\alpha$, we have $S_{\Psi}^{(\alpha)} = (\Psi \setminus \{ \pm \alpha \}) \cup \cup_{\Psi \subseteq \Psi \subseteq \Phi} S_{\Psi}^{(\alpha)}$. The conclusion follows from the (tt) relations $[t_\beta, \sum_{\gamma \in S_{\psi}^{(\alpha)}} t_\gamma] = 0$, and $[t_\beta, \sum_{\gamma \in S_{\psi}^{(\alpha)}} t_\gamma] = 0$. □

Lemma 6.6. For any $u_1, u_2 \in S_{\Psi}^{(\beta)}$, we have

$$\omega(u_1^\vee, \beta) = \pm \omega(u_2^\vee, \beta).$$
Proof. For \( u_i \in S_\Psi^{(b)} \), \( i = 1, 2 \). Since \( \langle u_i, \beta \rangle = \Psi \), the transition matrix between the two integral basis \( \{ u_1, \beta \} \) and \( \{ u_2, \beta \} \) is of the form \[
abla = \begin{pmatrix} 1 & y \\ 0 & x \end{pmatrix} \] thus \( x = \pm 1 \). Write \( u_1 = \pm u_2 + y\beta \). Then, \( \omega(u_1, \beta) = \omega((\pm u_2 + y\beta), \beta) = \pm \omega(u_2, \beta) \). So the assertion follows.

\[ \square \]

6.2. The map \( \tilde{\delta}_{2m} \) preserves relation (yx). We show

\[ [\tilde{\delta}_{2m}(y(u)), x(v)] + [y(u), \tilde{\delta}_{2m}(x(v))] = \sum_{\gamma \in \Phi^+, \delta} (u, \gamma)(v, \gamma)\tilde{\delta}_{2m}(t_\gamma) \]  

in this subsection.

By definition of \( \tilde{\delta}_{2m} \), the left hand side of (22) is

\[ [\tilde{\delta}_{2m}(y(u)), x(v)] + [y(u), \tilde{\delta}_{2m}(x(v))] = \left[ \frac{1}{2} \sum_{\gamma \in \Phi^+} \gamma(u) \sum_{p+q=2m-1} [(\text{ad} \frac{x_\gamma}{2})^p(t_\gamma), (\text{ad} -\frac{x_\gamma}{2})^q(t_\gamma)], x(v) \right]. \]

By linearity of \( x(v) \) in \( v \), we have \( x(v) = \frac{1}{2}(v, \gamma)x_\gamma + x(v') \), where \( (\gamma, v') = 0 \). Using the (tx)-relation \( [t_\gamma, x(v')] = 0 \), we have

\[ ([\text{ad} \frac{x_\gamma}{2})^p(t_\gamma), (\text{ad} -\frac{x_\gamma}{2})^q(t_\gamma)], x(v') \]  

Taking into account this simplification, we have

\[ [\tilde{\delta}_{2m}(y(u)), x(v)] + [y(u), \tilde{\delta}_{2m}(x(v))] \]

\[ = \frac{1}{2} \sum_{\gamma \in \Phi^+} \gamma(u) \sum_{p+q=2m-1} [(\text{ad} \frac{x_\gamma}{2})^p(t_\gamma), (\text{ad} -\frac{x_\gamma}{2})^q(t_\gamma)] \]

\[ = \frac{1}{2} \sum_{\gamma \in \Phi^+} (v, \gamma)\gamma(u) [\sum_{p+q=2m-1} ([(\text{ad} \frac{x_\gamma}{2})^p(t_\gamma), (\text{ad} -\frac{x_\gamma}{2})^q(t_\gamma)] - [(\text{ad} \frac{x_\gamma}{2})^p(t_\gamma), (\text{ad} -\frac{x_\gamma}{2})^q(t_\gamma)] - [(\text{ad} \frac{x_\gamma}{2})^{p+1}(t_\gamma), (\text{ad} -\frac{x_\gamma}{2})^q(t_\gamma)])]

\[ = \sum_{\gamma \in \Phi^+} (v, \gamma)\gamma(u) [\text{ad} \frac{x_\gamma}{2})^2(t_\gamma)] = \sum_{\gamma \in \Phi^+} (v, \gamma)\gamma(u) \tilde{\delta}_{2m}(t_\gamma). \]

This completes the proof that \( \tilde{\delta}_{2m} \) preserves relation (yx).

6.3. The map \( \tilde{\delta}_{2m} \) preserves relation (tt). We show in this subsection that

\[ [\tilde{\delta}_{2m}(t_\alpha), \sum_{\beta \in \Psi^+} t_\beta] + [t_\alpha, \sum_{\beta \in \Psi^+} \tilde{\delta}_{2m}(t_\beta)] = 0. \]  

Lemma 6.7. Let \( \Psi \) be a rank 2 root system, and suppose we have the relation \( [t_\alpha, \sum_{\beta \in \Psi^+} t_\beta] = 0. \) Then,

\[ \tilde{\delta}_{2m}(t_\beta) = \sum_{\gamma \in \Psi^+} \text{ad} \frac{x_\gamma}{2})^{2m}(t_\gamma). \]

Proof. To show the claim, it suffices to show \( \sum_{\gamma \in \Psi^+} \text{ad} \frac{x_\gamma}{2})^{2m}(t_\gamma) = 0. \)

Using the (tx) relation, we have

\[ \sum_{\gamma \in \Psi^+} \text{ad} \frac{x_\gamma}{2})^{2m}(t_\gamma) = \sum_{\gamma \in \Psi^+} \text{ad} \frac{x_\gamma}{2})^{2m}(t_\gamma) = \sum_{\gamma \in \Psi^+} \text{ad} \frac{x_\gamma}{2})^{2m}(t_\gamma), \]

where \( \text{ad} \frac{x_\gamma}{2}) \) is an element of the root system.
where the last equality follows from the decomposition in Lemma 6.4. By Lemma 6.6, the term \((\text{ad} \frac{x_{(y',a)}}{2})^{2m}\) is independent of \(y \in S_{y'}^{(b)}\). Using the relation \([t_{\beta}, x_{(y',a)}] = 0\), we have

\[
\sum_{y \in S_{y'}^{(b)} \cap \Psi^+} [t_{\beta}, (\text{ad} \frac{x_{(y',a)}}{2})^{2m}(t_{\gamma})] = (\text{ad} \frac{x_{(y',a)}}{2})^{2m}[t_{\beta}, \sum_{y \in S_{y'}^{(b)} \cap \Psi^+} t_{\gamma}] = 0.
\]

The last equality follows from the (tt) relation in Corollary 6.5. This completes the proof. 

We use Lemma 6.7 to show the equality (23). By Lemma 6.7, we have

\[
[t_{\beta}, \sum_{\beta \in \Psi^+} t_{\beta}] + [t_{\alpha}, \sum_{\beta \in \Psi^+} \delta_{2m}(t_{\beta})] = -[t_{\alpha}, \sum_{\beta \in \Psi^+} [t_{\beta}, (\text{ad} \frac{x_{y'}}{2})^{2m}(t_{\gamma})]] + [t_{\alpha}, \sum_{\beta \in \Psi^+} \sum_{y \in \Psi^+} (\text{ad} \frac{x_{y'}}{2})^{2m}(t_{\gamma})] \]

\[
= [t_{\alpha}, \sum_{\beta \in \Psi^+} \sum_{y \in \Psi^+} (\text{ad} \frac{x_{(y',a)}}{2})^{2m}(t_{\gamma})] = \sum_{\Psi' \subseteq \Psi} [t_{\alpha}, \sum_{\beta \in \Psi^+} (\text{ad} \frac{\omega(y',a)}{2})^{2m}(t_{\gamma})],
\]

where the last equality follows from the decomposition in Lemma 6.4. By Lemma 6.6, the term \((\text{ad} \frac{x_{(y',a)}}{2})^{2m}\) is independent of \(y \in S_{y'}^{(b)}\). Using the relation \([t_{\beta}, x_{(y',a)}] = 0\), we have

\[
[t_{\beta}, \sum_{y \in S_{y'}^{(b)} \cap \Psi^+} (\text{ad} \frac{\omega(y',a)}{2})^{2m}(t_{\gamma})] = (\text{ad} \frac{\omega(y',a)}{2})^{2m}[t_{\beta}, \sum_{y \in S_{y'}^{(b)} \cap \Psi^+} t_{\gamma}] = 0.
\]

The last equality follows from the (tt) relation in Corollary 6.5. This implies the equality (23).

6.4. The map \(\delta_{2m}\) preserves relation (ty). We show in this subsection

\[
[y(u), \delta_{2m}(t_{\alpha})] + [\delta_{2m}(y(u)), t_{\alpha}] = 0, \quad \text{if} \ (\alpha, u) = 0.
\]

(24)

We first calculate the term \([\delta_{2m}(y(u)), t_{\alpha}]\).

**Lemma 6.8.** We write \(\{\Psi \subseteq \Phi\}\) for the subset of rank 2 root subsystems of \(\Phi\). We have

\[
[\delta_{2m}(y(u)), t_{\alpha}] = \frac{1}{2} \sum_{\Psi \subseteq \Phi} \sum_{(y, \beta) \in S_{y'}^{(b)} \cap \Psi^+ \cap (y \neq \beta)} y(u) t_{\alpha}, \quad (\text{ad} \frac{x_{(y',a)}}{2})^{2m} = (\text{ad} \frac{x_{(y',a)}}{2})^{2m} - \epsilon(\beta, \gamma) \frac{x_{(\gamma',a)}}{2} - \phi_{(\beta, \gamma)}(t_{\beta}, t_{\gamma}),
\]

(25)

where \(\epsilon(\beta, \gamma) = \pm 1\). It is determined by the equality \(\omega(y', a) = \epsilon(\beta, \gamma) \omega(\beta', a)\).

**Proof.** For fixed \(p, q\), such that \(p + q = 2m - 1\). Using the relations \([x_{y'}, t_{\gamma}] = [x_{(y',a)}, t_{\gamma}] = 0\) and \(x_{(y',a)}, t_{\alpha} = 0\), we have

\[
\begin{align*}
&\left(\text{ad} \frac{x_{y'}}{2}\right)^2(t_{\gamma}) \quad (\text{ad} \frac{x_{y'}}{2})^2(t_{\gamma})] \\
&= \left[\left(\text{ad} \frac{x_{y'}}{2}\right)^2(t_{\gamma}), \left(\text{ad} \frac{x_{y'}}{2}\right)^2(t_{\gamma})\right] \\
&= \left[\left(\text{ad} \frac{x_{(y',a)}}{2}\right)^2(t_{\gamma}), \left(\text{ad} \frac{x_{y'}}{2}\right)^2(t_{\gamma})\right] \\
&= \left[\left(\text{ad} \frac{x_{(y',a)}}{2}\right)^2(t_{\gamma}), \left(\text{ad} \frac{x_{(y',a)}}{2}\right)^2(t_{\gamma})\right] \\
&= \left[\left(\text{ad} \frac{x_{(y',a)}}{2}\right)^2(t_{\gamma}), \left(\text{ad} \frac{x_{(y',a)}}{2}\right)^2(t_{\gamma})\right].
\end{align*}
\]

(26)
Taking summation over all pairs \( (p, q) \) with \( p + q = 2m - 1 \), the two summands in (26) could be combined. Therefore, we have

\[
\sum_{p+q=2m-1} \left[ (\text{ad} \frac{X_{\gamma'}^\nu}{2})^p(t_\gamma), (\text{ad} - \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right] = 2 \sum_{p+q=2m-1} \left[ (\text{ad} \frac{X_{\alpha\beta}(\gamma', \alpha)}{2})^p[t_\alpha, t_\gamma], (\text{ad} \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right].
\]

By (26), together with the (tt) relation \([t_\alpha, t_\gamma] = -\sum_{\beta \in S(\nu)^{\text{even}} \setminus \Phi^+} [t_\alpha, t_\beta] \) and the fact \( \Phi \setminus \{\pm \alpha\} = \sqcup_{\psi < \Phi} S^{\nu}(\psi) \), we have

\[
\sum_{\gamma \in \Phi^+} \gamma(u) \sum_{p+q=2m-1} \left[ (\text{ad} \frac{X_{\gamma'}^\nu}{2})^p(t_\gamma), (\text{ad} - \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right] = 2 \sum_{\gamma \in \Phi^+} \gamma(u) \sum_{p+q=2m-1} \left[ (\text{ad} \frac{X_{\alpha\beta}(\gamma', \alpha)}{2})^p[t_\alpha, t_\gamma], (\text{ad} \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right] = -2 \sum_{\Psi \subset \Phi^+ \setminus \{\pm \alpha\}} \left\{ \sum_{\gamma \in S^{\nu}(\psi)} \sum_{\beta \in \Psi \cap \Phi^+} \gamma(u) \sum_{p+q=2m-1} \gamma(u) \sum_{p+q=2m-1} \left[ [t_\alpha, (\text{ad} \frac{X_{\alpha\beta}(\gamma', \alpha)}{2})^p[t_\beta], (\text{ad} \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right] (27)
\]

For a pair \((\gamma, \beta) \in S^{\nu}(\psi)\), as shown in Lemma 6.6, \( \omega(\gamma', \alpha) = \epsilon(\beta, \gamma) \omega(\beta', \alpha) \), where \( \epsilon(\beta, \gamma) = \pm 1 \). Therefore, we simplify (27) further.

\[
\left[ [t_\alpha, (\text{ad} \frac{X_{\alpha\beta}(\gamma', \alpha)}{2})^p[t_\beta], (\text{ad} \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right] = \epsilon(\beta, \gamma)^p [t_\alpha, (\text{ad} \frac{X_{\alpha\beta}(\gamma', \alpha)}{2})^p[t_\beta], (\text{ad} \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right] = (-\epsilon(\beta, \gamma))^p [t_\alpha, (\text{ad} \frac{X_{\alpha\beta}(\gamma', \alpha)}{2})^p[t_\beta], (\text{ad} \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right]. (28)
\]

We plug (28) into (27) and then split (27) into two equal parts. We use the trick that switching the pair \((\beta, \gamma)\), and then the pair of two indices \((p, q)\). The identity \( \gamma(u) = \epsilon(\beta, \gamma) \beta(u) \) is useful in the simplification, which follows from the fact \((\alpha, u) = 0\). For \( p + q = 2m - 1 \), we have \((-\epsilon(\beta, \gamma))^p = (-\epsilon(\beta, \gamma))^p \). More precisely, we have

\[
2[\delta_{2m}(\gamma(u)), t_\alpha] = \sum_{\gamma \in \Phi^+} \gamma(u) \sum_{p+q=2m-1} \left[ (\text{ad} \frac{X_{\gamma'}^\nu}{2})^p(t_\gamma), (\text{ad} - \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right] = -2 \sum_{\Psi \subset \Phi^+ \setminus \{\pm \alpha\}} \left\{ \sum_{\gamma \in S^{\nu}(\psi)} \sum_{\beta \in \Psi \cap \Phi^+} \gamma(u) \sum_{p+q=2m-1} \gamma(u) \sum_{p+q=2m-1} \left[ [t_\alpha, (\text{ad} \frac{X_{\alpha\beta}(\gamma', \alpha)}{2})^p[t_\beta], (\text{ad} \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right] = -\sum_{\Psi \subset \Phi^+ \setminus \{\pm \alpha\}} \left\{ \sum_{\gamma \in S^{\nu}(\psi)} \sum_{\beta \in \Psi \cap \Phi^+} \beta(u) \sum_{p+q=2m-1} \left[ [t_\alpha, (\text{ad} \frac{X_{\alpha\beta}(\gamma', \alpha)}{2})^p[t_\beta], (\text{ad} \frac{X_{\gamma'}^\nu}{2})^q(t_\gamma) \right]
\]
\[
\begin{align*}
&= - \sum_{\Psi^* \subseteq \Phi^*} \sum_{\{(\gamma, \beta) \in S_\Psi^* \cap \Phi^* | \gamma \neq \beta\}} \gamma(u) \sum_{p+q=2m-1} (-\epsilon(\beta, \gamma))^p [t_\alpha, (\text{ad} \frac{X_{y^\gamma}}{2})^p t_\beta, (\text{ad} \frac{X_{y^\gamma}}{2})^q(t_\gamma)] \\
&- \sum_{\Psi^* \subseteq \Phi^*} \sum_{\{(\beta, \gamma) \in S_\Psi^* \cap \Phi^* | \gamma \neq \beta\}} \gamma(u) \sum_{p+q=2m-1} (-\epsilon(\beta, \gamma))^p [\text{ad} \frac{X_{y^\beta}}{2})^p(t_\beta, [t_\alpha, (\text{ad} \frac{X_{y^\gamma}}{2})^q(t_\gamma)] \\
&= - \sum_{\Psi^* \subseteq \Phi^*} \sum_{\{(\beta, \gamma) \in S_\Psi^* \cap \Phi^* | \gamma \neq \beta\}} \gamma(u) \sum_{p+q=2m-1} (-\epsilon(\beta, \gamma))^p [t_\gamma, \{(\text{ad} \frac{X_{y^\beta}}{2})^p t_\beta, (\text{ad} \frac{X_{y^\gamma}}{2})^q(t_\gamma)] \\
&= \sum_{\Psi^* \subseteq \Phi^*} \sum_{\{(\gamma, \beta) \in S_\Psi^* \cap \Phi^* | \gamma \neq \beta\}} \gamma(u) \sum_{p+q=2m-1} (-\epsilon(\beta, \gamma))^p [t_\alpha, (\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^{2m} - (\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^{2m} - \epsilon(\beta, \gamma) \text{ad} \frac{X_{a\alpha^{\gamma}}}{2} - \text{ad} \frac{X_{a\alpha^{\gamma}}}{2}](t_\beta, t_\gamma). \\
\end{align*}
\]

This completes the proof. \(\square\)

We then calculate the term \([y(u), \tilde{\delta}_{2m}(t_\alpha)]\). Using the relation \([y(u), t_\alpha] = 0\), we have

\[
[y(u), \tilde{\delta}_{2m}(t_\alpha)] = [t_\alpha, [y(u), (\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^{2m}(t_\alpha)] \\
\]

By \([y(u), t_\alpha] = 0\).

\[
= \left[\frac{1}{2} \sum_{\gamma \in \Phi^* | \gamma \neq \alpha} (\alpha^{\gamma}, \gamma)(u, \gamma) \sum_{s=0}^{2m} (\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^s(\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^{2m-1-s}[t_\gamma, t_\alpha] \right] \\
\]

By Jacobi identity and the relation \((yx)\).

\[
= \left[\sum_{\gamma \in \Phi^* | \gamma \neq \alpha} \gamma(u) \frac{(\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^{2m} - (\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^{2m}}{\text{ad} \frac{X_{a\alpha^{\gamma}}}{2}}[t_\gamma, t_\alpha] \right] \\
\]

By the fact that \(\frac{\alpha^{\gamma} + \omega(\alpha^{\gamma}, \gamma)}{\omega(\alpha^{\gamma}, \gamma)} = \frac{\omega(\alpha^{\gamma}, \gamma)}{2}\).

Using the decomposition in Lemma 6.4, and the (tt) relation in Corollary 6.5, the above is the same as

\[
[t_\alpha, \sum_{\Psi \subseteq \Phi} \sum_{\Psi^* \subseteq \Phi^*} \gamma(u) \frac{(\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^{2m} - (\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^{2m}}{\text{ad} \frac{X_{a\alpha^{\gamma}}}{2}}[t_\gamma, t_\alpha] = t_\alpha, \sum_{\Psi \subseteq \Phi} \sum_{\Psi^* \subseteq \Phi^*} \gamma(u) \frac{(\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^{2m} - (\text{ad} \frac{X_{a\alpha^{\gamma}}}{2})^{2m}}{\text{ad} \frac{X_{a\alpha^{\gamma}}}{2}}[t_\beta, t_\gamma]. \\
\]

(29)
Switching $\beta$ and $\gamma$, we have

$$[y(u), \delta_{2m}(t_\alpha)] = \left[ t_\alpha, \sum_{\Psi \subset \Phi} \sum_{\gamma, \beta \in S^{(\alpha)}_\Psi \cap \Phi^+} \gamma(u) \frac{\langle \delta \rangle^{2m} - \langle \delta \rangle^{2m}}{2} \left[ t_\beta, t_\gamma \right] \right]$$

$$= \frac{1}{2} \left[ t_\alpha, \sum_{\Psi \subset \Phi} \sum_{\gamma, \beta \in S^{(\alpha)}_\Psi \cap \Phi^+} \gamma(u) \frac{\langle \delta \rangle^{2m} - \langle \delta \rangle^{2m}}{2} \left[ t_\beta, t_\gamma \right] \right]$$

$$- \frac{1}{2} \left[ t_\alpha, \sum_{\Psi \subset \Phi} \sum_{\gamma, \beta \in S^{(\alpha)}_\Psi \cap \Phi^+} \beta(u) \frac{\langle \delta \rangle^{2m} - \langle \delta \rangle^{2m}}{2} \left[ t_\beta, t_\gamma \right] \right]$$

$$= \frac{1}{2} \left[ t_\alpha, \sum_{\Psi \subset \Phi} \sum_{\gamma, \beta \in S^{(\alpha)}_\Psi \cap \Phi^+} \gamma(u) \frac{\langle \delta \rangle^{2m} - \langle \delta \rangle^{2m}}{2} \left[ t_\beta, t_\gamma \right] \right]. \quad (30)$$

For fixed $\alpha$ and $\Psi$. The set $S^{(\alpha)}_\Psi$ is given in Examples 6.2 and 6.3. The relation (24) follows from direct computation by comparing (25) and (30).

More precisely, we assume the pair $\beta, \gamma$ satisfy $\beta = \epsilon(\beta, \gamma)\gamma + \alpha$, then, we have

$$\omega(\beta, \gamma) = \omega(\alpha^\vee, \gamma), \quad \omega(\gamma, \beta) = -\epsilon(\beta, \gamma)\omega(\alpha^\vee, \beta), \quad \epsilon(\beta, \gamma)\omega(\beta, \gamma) + \omega(\gamma, \beta) = \omega(\gamma, \alpha).$$

Therefore, the relation (24) holds under this assumption.

The assumption $\beta = \epsilon(\beta, \gamma)\gamma + \alpha$ does not hold only when $\alpha$ is the short root of $G_2$. For the exceptional case, we have $\beta = \epsilon(\beta, \gamma)\gamma + 3\alpha$. We modify the equality of (29) to use a more refined (tt) relation for the root system $\Phi_{G_2}$, see Example 6.3. Then, the corresponding term of $[y(u), \delta_{2m}(t_\alpha)]$ in (30) will be modified to

$$\frac{1}{2} \left[ t_\alpha, \gamma(u) \frac{\langle \delta \rangle^{2m} - \langle \delta \rangle^{2m}}{2} \left[ t_\beta, t_\gamma \right] \right].$$

The rest of the proof is exactly the same as before. This concludes the relation (24).

6.5. The map $\delta_{2m}$ preserves relation (yy). We show in this subsection

$$[y(u), \delta_{2m}(y(v))] + [\delta_{2m}(y(u)), y(v)] = 0, \text{ for any } u, v \in \mathfrak{h}. \quad (32)$$

By definition of $\delta_{2m}$, we have

$$[y(u), \delta_{2m}(y(v))] + [\delta_{2m}(y(u)), y(v)] = [y(u), \frac{1}{2} \sum_{\gamma \in \Phi^+} \gamma(v) \sum_{p+q=2m-1} \{ \langle \delta \rangle^{p+q} \rangle^p(t_\gamma), \langle \delta \rangle^{p+q} \rangle^q(t_\gamma) \}]$$

$$- [y(v), \frac{1}{2} \sum_{\gamma \in \Phi^+} \gamma(u) \sum_{p+q=2m-1} \{ \langle \delta \rangle^{p+q} \rangle^p(t_\gamma), \langle \delta \rangle^{p+q} \rangle^q(t_\gamma) \}]$$

$$= \sum_{\gamma \in \Phi^+} \left[ y_\gamma, \frac{1}{2} \sum_{p+q=2m-1} \{ \langle \delta \rangle^{p+q} \rangle^p(t_\gamma), \langle \delta \rangle^{p+q} \rangle^q(t_\gamma) \} \right]$$

$$= \sum_{\gamma \in \Phi^+} \sum_{p+q=2m-1} \left[ \gamma_\gamma, \langle \delta \rangle^{p+q} \rangle^p(t_\gamma), \langle \delta \rangle^{p+q} \rangle^q(t_\gamma) \right], \quad (33)$$

where $y_\gamma = y(v)y(u) - y(u)y(v)$ and therefore $[y_\gamma, t_\gamma] = 0.$
We use the relation \((yx, \frac{x y}{2}) = (yx, \frac{x y}{2})^{p-1}([y x, \frac{x y}{2}](\text{ad}(\frac{x y}{2})^{p-1}([y x, \frac{x y}{2}]))

= \sum_{s=0}^{p-1} \sum_{\beta \in \Phi^+} (\beta, \eta, \frac{x y}{2})(\text{ad}(\frac{x y}{2})^{s}(\text{ad}(\frac{x y}{2})^{p-1-s}([y x, \frac{x y}{2}])

= \sum_{s=0}^{p-1} \sum_{\beta \in \Phi^+} \text{det}(\beta, \gamma, u, v)(\frac{x y}{2})(\text{ad}(\frac{x y}{2})^{s}(\text{ad}(\frac{x y}{2})^{p-1-s}([y x, \frac{x y}{2}])

= \sum_{\beta \in \Phi^+} \text{det}(\beta, \gamma, u, v)(\frac{x y}{2}) - (\text{ad}(\frac{x y}{2})^{p-1-s}([y x, \frac{x y}{2}])

= \sum_{\beta \in \Phi^+} \text{det}(\beta, \gamma, u, v)(\frac{x y}{2}) - (\text{ad}(\frac{x y}{2})^{p-1}([y x, \frac{x y}{2}])

where \(\text{det}(\beta, \gamma, u, v) = \beta(u)\gamma(v) - \beta(v)\gamma(u)\). The last equality of (34) follows from the linearity of \(x(u)\), and the equality \(\gamma(x y, \frac{x y}{2}) = \frac{\omega(y, \frac{x y}{2})}{(y, \frac{x y}{2})}\).

We now use the decomposition \(\Phi = \cup_{\psi \in \Phi^+} S_{\psi}^{(y)} \cup \{\pm y\}\), and the (tt relation in Corollary 6.5. By fixing \(\gamma \in \Phi^+\), we have

\[
(34) = \sum_{\psi \in \Phi^+} \sum_{\beta \in S_{\psi}^{(y)} \gamma \Phi^+} \text{det}(\beta, \gamma, u, v)(\text{ad}(\frac{x y}{2})^{p} - (\text{ad}(\frac{x y}{2})^{p})^{\beta}([y x, \frac{x y}{2}])

= - \sum_{\psi \in \Phi^+} \sum_{\beta, \omega \in S_{\psi}^{(y)} \gamma \Phi^+} \text{det}(\beta, \gamma, u, v)(\text{ad}(\frac{x y}{2})^{p} - (\text{ad}(\frac{x y}{2})^{p})^{\beta}([y x, \frac{x y}{2}])

= - \frac{1}{2} \sum_{\psi \in \Phi^+} \sum_{\beta, \omega \in S_{\psi}^{(y)} \gamma \Phi^+} \text{det}(\beta, \gamma, u, v)(\text{ad}(\frac{x y}{2})^{p} - (\text{ad}(\frac{x y}{2})^{p})^{\beta}([y x, \frac{x y}{2}])

+ \frac{1}{2} \sum_{\psi \in \Phi^+} \sum_{\beta, \omega \in S_{\psi}^{(y)} \gamma \Phi^+} \text{det}(\alpha, \gamma, u, v)(\text{ad}(\frac{x y}{2})^{p} - (\text{ad}(\frac{x y}{2})^{p})^{\alpha}([y x, \frac{x y}{2}])

= - \frac{1}{2} \sum_{\psi \in \Phi^+} \sum_{\beta, \omega \in S_{\psi}^{(y)} \gamma \Phi^+} \text{det}(\beta, \gamma, u, v)(\text{ad}(\frac{x y}{2})^{p} - (\text{ad}(\frac{x y}{2})^{p})^{\beta}([y x, \frac{x y}{2}])

The last equality follow from the fact that for \(\alpha, \beta \in S_{\psi}^{(y)}\),

\[
\text{ad}(\frac{x y}{2})^{p} / \text{det}(\beta, \gamma, u, v) = \text{ad}(\frac{x y}{2})^{p} / \text{det}(\alpha, \gamma, u, v).

For \(\alpha, \beta \in S_{\psi}^{(y)}\), we have \(\alpha, \beta, \gamma\) are colinear. Therefore, there exist three integers \(A, B, C \in \mathbb{Z}\), such that \(A\alpha + B\beta + C\gamma = 0\). From Examples 6.2 and 6.3, we know for \(\alpha, \beta \in S_{\psi}^{(y)}\), we have either \(\pm(\alpha \pm \beta) = \gamma\) or
\[ \pm (\alpha \pm \beta) = 3\gamma. \] We first deal with the case \( \pm (\alpha \pm \beta) = \gamma. \) Without loss of generality, we assume first that \( C = 1, \) and \( A = \pm 1, B = \pm 1. \) Then we have the equalities
\[
\omega(\gamma', \alpha) = -\frac{1}{B}\omega(\beta', \alpha), \quad \omega(\gamma', \beta) = -\frac{1}{A}\omega(\alpha', \beta),
\]
\[
\omega(\gamma', \alpha) - \omega(\gamma', \beta) = -\frac{1}{B}\omega(\beta', \gamma).
\]
Using the above equalities, we compute the equation (34) further.

\[
(34) = \frac{1}{2} \sum_{\Phi \subset \Phi} \sum_{\beta, \alpha \in S^{(\gamma')}_{\psi}} \frac{\det(\beta, \gamma)_{\nu, \nu}}{B} \sum_{s+k=p-1} \left[ \left( \frac{1}{B} \text{ad}_{\frac{\gamma'}{-2}} \text{ad}_{\frac{\alpha}{-2}} \text{ad}_{\frac{\beta}{-2}} \right)^{k} \right] [t_{\beta}, t_{\alpha}]
\]
\[
= \frac{1}{2} \sum_{\Phi \subset \Phi} \sum_{\beta, \alpha \in S^{(\gamma')}_{\psi}} \frac{\det(\beta, \gamma)_{\nu, \nu}}{B} \sum_{s+k=p-1} \left[ \left( \frac{1}{B} \text{ad}_{\frac{\gamma'}{-2}} \text{ad}_{\frac{\alpha}{-2}} \text{ad}_{\frac{\beta}{-2}} \right)^{k} \right] [t_{\beta}, t_{\alpha}]
\]
\[
= \frac{1}{2} \sum_{\Phi \subset \Phi} \sum_{\beta, \alpha \in S^{(\gamma')}_{\psi}} \frac{\det(\beta, \gamma)_{\nu, \nu}}{B} \sum_{s+k=p-1} \left[ \left( \frac{1}{B} \text{ad}_{\frac{\gamma'}{-2}} \text{ad}_{\frac{\alpha}{-2}} \text{ad}_{\frac{\beta}{-2}} \right)^{k} \right] [t_{\beta}, t_{\alpha}].
\] (35)

Plugging the formula (35) into the equation (33), we have
\[
[\tilde{\delta}_{2m}(y(u)), y(v)] + [y(u), \tilde{\delta}_{2m}(y(v))]
\]
\[
= \frac{1}{2} \sum_{s+k=q=2m-2} \sum_{\gamma \in \Phi^{+}} \sum_{\beta, \alpha \in S^{(\gamma')}_{\psi}} \frac{\det(\beta, \gamma)_{\nu, \nu}}{B} \sum_{s+k=p-1} \left[ \left( \frac{1}{B} \text{ad}_{\frac{\gamma'}{-2}} \text{ad}_{\frac{\alpha}{-2}} \text{ad}_{\frac{\beta}{-2}} \right)^{k} \right] [t_{\beta}, t_{\alpha}]
\]
\[
= \frac{1}{2} \sum_{s+k=q=2m-2} \sum_{\beta, \alpha \in S^{(\gamma')}_{\psi}} \frac{\det(\beta, \gamma)_{\nu, \nu}}{B^{q+k-1}} \sum_{s+k=p-1} \left[ \left( \frac{1}{B} \text{ad}_{\frac{\gamma'}{-2}} \text{ad}_{\frac{\alpha}{-2}} \text{ad}_{\frac{\beta}{-2}} \right)^{k} \right] [t_{\beta}, t_{\alpha}].
\] (36)

Exactly the same proof as before, using \( AB\alpha + By + \beta = 0, \) we have
\[
[\tilde{\delta}_{2m}(y(u)), y(v)] + [y(u), \tilde{\delta}_{2m}(y(v))]
\]
\[
= \frac{1}{2} \sum_{s+k=q=2m-2} \sum_{\beta, \alpha \in S^{(\gamma')}_{\psi}} \frac{\det(\beta, \gamma)_{\nu, \nu}}{B^{q+k-1}} \sum_{s+k=p-1} \left[ \left( \frac{1}{B} \text{ad}_{\frac{\gamma'}{-2}} \text{ad}_{\frac{\alpha}{-2}} \text{ad}_{\frac{\beta}{-2}} \right)^{k} \right] [t_{\beta}, t_{\alpha}].
\] (37)

and using \( Ay + AB\beta + \alpha = 0, \) one get
\[
[\tilde{\delta}_{2m}(y(u)), y(v)] + [y(u), \tilde{\delta}_{2m}(y(v))]
\]
\[
= \frac{1}{2} \sum_{s+k=q=2m-2} \sum_{\beta, \alpha \in S^{(\gamma')}_{\psi}} \frac{\det(\beta, \gamma)_{\nu, \nu}}{A^{1+s+q}B^{q+k-1}} \sum_{s+k=p-1} \left[ \left( \frac{1}{B} \text{ad}_{\frac{\gamma'}{-2}} \text{ad}_{\frac{\alpha}{-2}} \text{ad}_{\frac{\beta}{-2}} \right)^{k} \right] [t_{\beta}, t_{\alpha}].
\] (38)

It is obvious that \( \det(\beta, \gamma)_{\nu, \nu} = -\det(\gamma, \beta)_{\nu, \nu} = -A \det(\beta, \alpha)_{\nu, \nu}. \) Therefore, taking into account that \( s + k + q = 2m - 2, \) the coefficients satisfy the equality
\[
\frac{\det(\beta, \gamma)_{\nu, \nu}}{B^{q+k-1}A^{k}} = -\frac{\det(\gamma, \beta)_{\nu, \nu}}{B^{q+k-1}A^{k}} = -\frac{\det(\beta, \alpha)_{\nu, \nu}}{A^{1+s+q}B^{q+k-1}}.
\]

Taking the summation of (36), (37) and (38), we have
\[
3\left( [\tilde{\delta}_{2m}(y(u)), y(v)] + [y(u), \tilde{\delta}_{2m}(y(v))] \right) = 0
\]
by the Jacobi identity.
When $\alpha \alpha + B\beta + 3\gamma = 0$, which is the case $G_2$. We modify the above proof using the refined (tt) relations. Then the equation (36) is modified to

$$[\tilde{\delta}_{2m}(y(u)), y(v)] + [y(u), \tilde{\delta}_{2m}(y(v))]$$

$$= \frac{1}{2} \sum_{x+k+q=2m-2} \sum_{y \in \Phi^+ \subseteq \Phi} \sum_{\xi \in \Phi_{\beta,\alpha} \subseteq \Phi^+} \sum_{\gamma \in \Phi} BC \det(\beta, \gamma)_{\alpha,\beta} \left[ \left( \frac{1}{B} \text{ad} \frac{X_{2\gamma}}{2} \right)^{x}(t_{\beta}), \left( \frac{1}{A} \text{ad} \frac{X_{\alpha+\gamma}}{2} \right)^{y}(t_{\alpha}), \left( \frac{1}{A} \text{ad} \frac{X_{\alpha+\gamma}}{2} \right)^{y}(t_{\beta}) \right].$$

The rest of the proof is similar.

7. A Principal Bundle on the Moduli Space $M_{1,n}$

Let $e, f, h$ be the standard basis of $sl_2$. There is a Lie algebra morphism $\mathfrak{d} \rightarrow sl_2$ defined by $\delta_{2m} \mapsto 0, d \mapsto h, X \mapsto e, \Delta_0 \mapsto f$. Let $\mathfrak{d} \subset \mathfrak{d}$ be the kernel of this homomorphism. Since the morphism has a section, which is given by $e \mapsto X, f \mapsto \Delta_0$ and $h \mapsto d$, we have a semidirect decomposition $\mathfrak{d} = \mathfrak{d}_+ \rtimes sl_2$. As a consequence, we have the decomposition

$$t_{\mathfrak{d}}^\Phi \rtimes \mathfrak{d} = (t_{\mathfrak{d}}^\Phi \rtimes \mathfrak{d}_+) \rtimes sl_2.$$  

**Lemma 7.1.** ([6, Lemma 8]) The Lie algebra $t_{\mathfrak{d}}^\Phi \rtimes \mathfrak{d}_+$ is positively graded.

**Proof.** Define a $\mathbb{Z}^2$-grading of $\mathfrak{d}$ and $t_{\mathfrak{d}}^\Phi$ by

$$\deg(\Delta_0) = (-1, 1), \quad \deg(d) = (0, 0), \quad \deg(X) = (1, -1), \quad \deg(\delta_{2m}) = (2m + 1, 1)$$

and $\deg(x(u)) = (1, 0)$, $\deg(y(u)) = (0, 1)$, $\deg(t_{\alpha}) = (1, 1)$.

It is straightforward to check this Lie algebra is positively graded. \qed

We form the following semidirect product

$$G_n := \exp(t_{\mathfrak{d}}^\Phi \rtimes \mathfrak{d}_+) \rtimes \text{SL}_2(\mathbb{C}),$$

where $t_{\mathfrak{d}}^\Phi \rtimes \mathfrak{d}_+$ is the completion of $t_{\mathfrak{d}}^\Phi \rtimes \mathfrak{d}_+$ with respect to the grading in Lemma 7.1.

Let $P^\vee$ be the coweight lattice, the semidirect product $(P^\vee \oplus P^\vee) \rtimes \text{SL}_2(\mathbb{Z})$ acts on $\mathfrak{h} \times \mathfrak{h}$. For $(n, \tau m) \in (P^\vee \oplus P^\vee)$ and $(z, \tau) \in \mathfrak{h} \times \mathfrak{h}$, the action is given by translation. $(n, \tau m) \ast (z, \tau) := (z + n + \tau m, \tau)$. For $(a/b, c/d) \in \text{SL}_2(\mathbb{Z})$, the action is given by $(a/b, c/d) \ast (z, \tau) := (az + \tau b + d, \tau c + d)$.

Let $\alpha(-) : \mathfrak{h} \rightarrow \mathbb{C}$ be the map induced by the root $\alpha \in \mathfrak{h}^\vee$. We define $\widetilde{H}_{\alpha,\tau} \subset \mathfrak{h} \times \mathfrak{h}$ to be

$$\widetilde{H}_{\alpha,\tau} = \{(z, \tau) \in \mathfrak{h} \times \mathfrak{h} | \alpha(z) \in \Lambda_\tau = \mathbb{Z} + \tau \mathbb{Z}\}.$$  

**Lemma 7.2.** The group $(P^\vee \oplus P^\vee) \rtimes \text{SL}_2(\mathbb{Z})$ preserves the hyperplane complement $\mathfrak{h} \times \mathfrak{h} \setminus \bigcup_{\alpha \in \Phi^+, \tau \in \mathbb{Z}} \widetilde{H}_{\alpha,\tau}$.

**Proof.** It is clear that $\alpha(n + \tau m) \in \Lambda_\tau$, for any $\alpha \in \Phi$. Therefore, for $(z, \tau) \in \mathfrak{h} \times \mathfrak{h}$ such that $\alpha(z) \notin \Lambda_\tau$, we have $\alpha(z + n + \tau m) \notin \Lambda_\tau$.

Let $(z, \tau) \in \mathfrak{h} \times \mathfrak{h}$ be the element such that $\alpha(z) \notin \Lambda_\tau$ for any $\alpha \in \Phi$. Suppose there exists some $\beta \in \Phi$ and $n, m \in \mathbb{Z}$, such that $\beta(z) = n + m(\tau + d)$. It is equivalent to $\beta(z) = n(\tau + d) + m(\tau + b) \in \Lambda_\tau$, which is contradicting with the choice of $(z, \tau)$. This completes the proof. \qed

We define the elliptic moduli space $M_{1,n}$ to be the quotient of $\mathfrak{h} \times \mathfrak{h} \setminus \bigcup_{\alpha \in \Phi^+, \tau \in \mathbb{Z}} \widetilde{H}_{\alpha,\tau}$ by the group $(P^\vee \oplus P^\vee) \rtimes \text{SL}_2(\mathbb{Z})$ action. Let $\pi : \mathfrak{h} \times \mathfrak{h} \setminus \bigcup_{\alpha \in \Phi^+, \tau \in \mathbb{Z}} \widetilde{H}_{\alpha,\tau} \rightarrow M_{1,n}$ be the natural projection. We define a principal $G_n$-bundle $P_n$ on the elliptic moduli space $M_{1,n}$ in this section.

For $u \in \mathbb{C}^*$, $u^d := \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \subset G_n$ and for $v \in \mathbb{C}$, $e^{iv} := \begin{pmatrix} 1 & iv \\ 0 & 1 \end{pmatrix} \in \text{SL}_2(\mathbb{C}) \subset G_n$. 
Proposition 7.3. There exists a unique principal $G_n$-bundle $P_n$ over $M_{1,n}$, such that a section of $U \subset M_{1,n}$ is a function $f : \pi^{-1}(U) \to G_n$, with the properties that
\[
f(z + \lambda_i^\vee | \tau) = f(z | \tau), \quad f(z + \tau \lambda_i^\vee | \tau) = e^{-2 \pi i x_i} f(z | \tau), \quad f(z | \tau + 1) = f(z | \tau),
\]
\[
f(\frac{z}{\tau} - \frac{1}{\tau}) = \tau^d \exp(\frac{2 \pi i}{\tau} \sum_i z_i x_i^\vee + X)f(z | \tau).
\]

Proof. In [6, Proposition 10], Calaque–Enriquez–Etingof constructed a principal bundle $\tilde{P}_n$ on $h \times \mathcal{S}/((\mathbb{Z})^2 \rtimes \text{SL}_2(\mathbb{Z}))$ with the structure group $G_n$. The vector bundle $P_n$ can be obtained by restricting this bundle $\tilde{P}_n$ on $M_{1,n} \subset h \times \mathcal{S}/((\mathbb{Z})^2 \rtimes \text{SL}_2(\mathbb{Z}))$. \hfill $\square$

8. Flat connection on the elliptic moduli space

In this section, we construct the universal flat connection on the moduli space $M_{1,n}$ of (pointed) elliptic curves associated to a root system $\Phi$. This connection is an extension of the universal KZB connection $\nabla_{KZB}$. Recall that in Section §2 (3), we have the function $k(z, x | \tau) = \frac{\theta(z + x | \tau)}{\theta(z | \tau)} - \frac{1}{x} \in \text{Hol}(\mathbb{C} - \Lambda_\tau)[[x]]$. Let
\[
g(z, x | \tau) := k(z, x | \tau) = \frac{\theta(z + x | \tau)}{\theta(z | \tau)} \left( \frac{\theta'(z + x | \tau)}{\theta(z + x | \tau)} - \frac{\theta'(z | \tau)}{\theta(z | \tau)} \right) + \frac{1}{x^2}
\]
be the derivative of function $k(z, x | \tau)$ with respect to variable $x$. We have $g(z, x | \tau) \in \text{Hol}(\mathbb{C} - \Lambda_\tau)[[x]]$.

For a power series $\psi(x) = \sum_{n \geq 1} b_{2n} x^{2n} \in \mathbb{C}[[x]]$ with positive even degrees, we consider the following two elements in $t_0^{\Phi} \ltimes \mathfrak{d}$
\[
\delta_\psi := \sum_{n \geq 1} b_{2n} \delta_{2n}, \quad \Delta_\psi := \Delta_0 + \delta_\psi = \Delta_0 + \sum_{n \geq 1} b_{2n} \delta_{2n}.
\]
As in [6], we define the power series $\varphi(x)$ to be
\[
\varphi(x) = g(0, 0 | \tau) - g(0, x | \tau) = -\frac{1}{x^2} - \left( \frac{\theta'}{\theta} \right)'(x | \tau) + \left( \frac{1}{x^2} + \left( \frac{\theta'}{\theta} \right)'(x | \tau) \right)_{|x=0} \in \mathbb{C}[[x]],
\]
which has positive even degrees. We set $a_{2n} := -\frac{(2n+1)! B_{2n}}{(2n+2)!}$, where $B_n$ are the Bernoulli numbers determined by the expansion $\frac{t}{e^t - 1} = \sum_{n \geq 0} \frac{B_n}{n!} t^n$. Then, the power series $\varphi(x)$ has the expansion $\varphi(x) = \sum_{n \geq 1} a_{2n} E_{2n+2}(\tau) x^{2n}$ for some coefficients $E_{2n+2}(\tau)$ only depend on $\tau$. By our convention, we have the following two elements in $t_0^{\Phi} \ltimes \mathfrak{d}$
\[
\delta_\varphi := \sum_{n \geq 1} a_{2n} E_{2n+2}(\tau) \delta_{2n}, \quad \Delta_\varphi := \Delta_0 + \delta_\varphi = \Delta_0 + \sum_{n \geq 1} a_{2n} E_{2n+2}(\tau) \delta_{2n}.
\]
Consider the following function on $h \times \mathcal{S}$
\[
\Delta := \Delta(a, \tau) = -\frac{1}{2\pi i} \Delta_\varphi + \frac{1}{2\pi i} \sum_{\beta \in \Phi^+} g(\beta, \text{ad} \frac{x_{\beta}^{\vee}}{2} | \tau)(t_\beta)
\]
\[
= -\frac{1}{2\pi i} \Delta_0 - \frac{1}{2\pi i} \sum_{n \geq 1} a_{2n} E_{2n+2}(\tau) \delta_{2n} + \frac{1}{2\pi i} \sum_{\beta \in \Phi^+} g(\beta, \text{ad} \frac{x_{\beta}^{\vee}}{2} | \tau)(t_\beta).
\]
This is a meromorphic function on $\mathbb{C}^n \times \mathcal{S}$ valued in $(t_0^{\Phi} \ltimes \mathfrak{d}) \rtimes \mathfrak{n}_+ \subset \text{Lie}(G_n)$, where $\mathfrak{n}_+ = \mathfrak{c}_0 \subset \mathfrak{s}l_2$. It has only poles along the hyperplanes $\bigcup_{\alpha \in \Phi^+, \tau \in \mathcal{S}} \mathfrak{H}_n, \tau$.

Theorem 8.1. The following $t_0^{\Phi} \ltimes \mathfrak{d}$-valued KZB connection on $M_{1,n}$ is flat.
\[
\nabla_{KZB} = \nabla_{KZB, \tau} - \Delta d\tau = d - \Delta d\tau - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad} \frac{x_{\alpha}^{\vee}}{2} | \tau)(t_\alpha) d\alpha + \sum_{i=1}^n \gamma(u') du_i.
\]
Note we have already shown the flatness of $\nabla_{KZB, \tau}$ in Theorem 2.5. We prove Theorem 8.1 in the rest of this section. Set $A := \Delta d\tau + \sum_{\alpha \in \Phi^+}^n k(\alpha, \text{ad } \frac{x_{\alpha}}{\tau}(t_\alpha) d\tau - \sum_{i=1}^n y(\alpha^i d\alpha_i$ so that $\nabla_{KZB} = d - A$. By definition of flatness, it suffices to show the curvature $dA + A \wedge A$ is zero.

8.1. Proof of $dA = 0$. In this subsection, we show the differential $dA$ vanishes. We have

$$dA = \frac{\partial A}{\partial \tau} d\tau + \sum_{i=1}^n \frac{\partial A}{\partial \alpha_i} d\alpha_i$$

$$= \sum_{\alpha \in \Phi^+} \frac{\partial}{\partial \tau}(k(\alpha, \text{ad } \frac{x_{\alpha}}{2}(t_\alpha)) d\tau \wedge d\alpha + \sum_{\alpha \in \Phi^+} \frac{\partial}{\partial \alpha_i} \Delta(\alpha, \tau) d\alpha_i \wedge d\tau$$

$$= \sum_{\alpha \in \Phi^+} \frac{\partial}{\partial \tau}(k(\alpha, \text{ad } \frac{x_{\alpha}}{2}(t_\alpha)) d\tau - \frac{1}{2\pi} \sum_{\beta \in \Phi^+} \frac{\partial}{\partial \alpha_i} g(\beta, \text{ad } \frac{x_{\alpha}}{2}(t_\alpha)) d\tau \wedge d\alpha_i$$

which is 0 by the differential equation $(\partial, k)(z, x|\tau) = \frac{1}{2\pi}(\partial, g)(z, x|\tau)$. See [6, Page 190] for the proof of this equation.

8.2. Simplification of $A \wedge A$. In this subsection, we simplify the term $A \wedge A$. By definition, we have,

$$A \wedge A = \sum_{\alpha \in \Phi^+}^n \sum_{\beta \in \Phi^+} [\Delta(\alpha, \tau), k(\alpha, \text{ad } \frac{x_{\alpha}}{2}(t_\alpha))] d\tau \wedge d\alpha - \sum_{\alpha \in \Phi^+}^n \sum_{\beta \in \Phi^+} [\Delta(\alpha, \tau), y(\alpha^i \wedge a)] d\tau \wedge d\alpha_i$$

$$= -\frac{1}{2\pi} \left( \sum_{\alpha \in \Phi^+}^n \sum_{\beta \in \Phi^+} [\Delta(\alpha, \tau), k(\alpha, \text{ad } \frac{x_{\alpha}}{2}(t_\alpha))] d\tau \wedge d\alpha - \sum_{\alpha \in \Phi^+}^n \sum_{\beta \in \Phi^+} [\Delta(\alpha, \tau), y(\alpha^i \wedge a)] d\tau \wedge d\alpha_i \right)$$

$$= -\frac{1}{2\pi} \left( \sum_{\alpha \in \Phi^+}^n \sum_{\beta \in \Phi^+} [\Delta(\alpha, \tau), y(\alpha^i \wedge a)] d\tau \wedge d\alpha_i + \sum_{\alpha \in \Phi^+}^n \sum_{\beta \in \Phi^+} [\Delta(\alpha, \tau), y(\alpha^i \wedge a)] d\tau \wedge d\alpha_i \right)$$

We now simplify each summand of (40), (41). In the following lemma, we rewrite the second term of (40).

**Lemma 8.2.** Modulo the relations $(tx), (sx)$ of $t_{\alpha}^{\Phi}$, the following identity holds for any $\alpha, \beta \in \Phi^+$

$$[g(\beta, \text{ad } \frac{x_{\beta}}{2}(t_{\beta}), k(\alpha, \text{ad } \frac{x_{\beta}}{2}(t_{\beta})) = g(\beta, (\text{ad } \frac{x_{\beta}}{2}(t_{\beta})) \alpha)(\text{ad } \frac{x_{\beta}}{2}(t_{\beta}))) t_{\beta}, t_{\alpha}].$$

**Proof.** When $\alpha = \beta$, it is clear that both sides of the identity are zero, hence they are equal. We now assume $\alpha \neq \beta$. The desired identity follows from the relations

$$[x_{\beta}, x_{\text{ad}(\alpha^i \wedge a), t_{\beta}}] = 0, \quad [x_{\text{ad}(\alpha^i \wedge a), t_{\alpha}}] = 0,$$

and $[x_{\alpha^i \wedge a}, x_{\text{ad}(\alpha^i \wedge a), t_{\beta}}] = 0, \quad [x_{\text{ad}(\alpha^i \wedge a), t_{\beta}}] = 0.$

$\square$

In the following lemma, we rewrite the second term of (41).
Lemma 8.3. Modulo the relations (tx), (ty),(yx), (xx) of $t_{v_0}$, the following identity holds.

$$
\sum_{i=1}^{n} \left[ \gamma(u^i), g(\beta, \text{ad} \frac{x_{\beta'}}{2}[\tau](t_\beta))d\tau \wedge du_i - \sum_{\beta \in \Phi^+} \left[ \gamma(\beta'), g(\beta, \text{ad} \frac{x_{\beta'}}{2}[\tau](t_\beta))d\tau \wedge d\beta \right] \right.
$$

$$
= \sum_{\gamma \neq \beta \in \Phi^+} \left[ \gamma(\beta'), g(\beta, \text{ad} \frac{x_{\beta'}}{2}[\tau]) - g(\beta, (\text{ad} \frac{x_{\beta'}}{2}[\tau])\right] \left[ t_\gamma, t_\beta \right] d\tau \wedge d\gamma
$$

$$
- \sum_{\gamma \neq \beta \in \Phi^+} \left[ \gamma(\beta'), g(\beta, \text{ad} \frac{x_{\beta'}}{2}[\tau]) - g(\beta, (\text{ad} \frac{x_{\beta'}}{2}[\tau])\right] \left[ t_\gamma, t_\beta \right] d\tau \wedge d\beta.
$$

Proof. If $\beta(u') = 0$, by Proposition 3.4, we have

$$
\left[ \gamma(u^i), g(\beta, \text{ad} \frac{x_{\beta'}}{2}[\tau](t_\beta)) \right] = \sum_{\gamma \in \Phi^+} \gamma(\beta') \gamma(u^i) \frac{g(\beta, \text{ad} \frac{x_{\beta'}}{2}[\tau]) - g(\beta, (\text{ad} \frac{x_{\beta'}}{2}[\tau])}{ad x_{\beta'} + ad x_{\omega(\beta', \gamma)}} \left[ t_\gamma, t_\beta \right].
$$

For a fixed root $\beta$, extend $\{\beta\}$ to a basis $\{u_1, \ldots, u_n\}$ of $b^*_+$, with $u_i = \beta$. Let $\{u^1, \ldots, u^n\}$ be the corresponding dual basis. We then have $u^1 = \frac{\gamma'}{2}$, and $\beta(u') = 0$, for $i \neq 1$. For such choice of $\{u_i\}$, we have

$$
\sum_{i=1}^{n} \left[ \gamma(u^i), g(\beta, \text{ad} \frac{x_{\beta'}}{2}[\tau](t_\beta))d\tau \wedge du_i - \left[ \gamma(\beta'), g(\beta, \text{ad} \frac{x_{\beta'}}{2}[\tau](t_\beta))d\tau \wedge d\beta \right] \right.
$$

$$
= \sum_{i=1}^{n} \sum_{\gamma \in \Phi^+} \gamma(\beta') \gamma(u^i) \frac{g(\beta, \text{ad} \frac{x_{\beta'}}{2}[\tau]) - g(\beta, (\text{ad} \frac{x_{\beta'}}{2}[\tau])}{ad x_{\beta'} + ad x_{\omega(\beta', \gamma)}} \left[ t_\gamma, t_\beta \right] d\tau \wedge du_i
$$

$$
- \sum_{\gamma \neq \beta \in \Phi^+} \gamma(\beta') \gamma(\beta') \frac{g(\beta, \text{ad} \frac{x_{\beta'}}{2}[\tau]) - g(\beta, (\text{ad} \frac{x_{\beta'}}{2}[\tau])}{ad x_{\beta'} + ad x_{\omega(\beta', \gamma)}} \left[ t_\gamma, t_\beta \right] d\tau \wedge d\beta.
$$

The last equality follow from the identity $\frac{x_{\beta'} + x_{\omega(\beta', \gamma)}}{\gamma(\beta')} = \frac{x_{\omega(\beta', \gamma)}}{2}$ in Lemma 3.1. Thus, the conclusion follows.

\[\square\]

In the following lemma, we rewrite the first term of (41).
Lemma 8.4. Modulo the relation \((\delta y)\) of \(t_0^\Phi \cong \mathfrak{d}\), we have

\[
\sum_{i=1}^{n} [\Delta_x, y(u')] du_i = \sum_{a \in \Phi^+} \sum_{s} \left[ f_s(\text{ad} \frac{x_\alpha y}{2})(t_{a}), \ g_s(\text{ad} \frac{x_\alpha y}{2})(t_{a}) \right] d\tau \land d\alpha,
\]

where the functions \(f_s\) and \(g_s\) are determined by the equality \(\frac{1}{2} \frac{\varphi(u) - \varphi(v)}{u - v} = \sum s f_s(u) g_s(v)\).

Proof. Recall that by definition, we have \([\Delta_0, y(u')] = 0\), and the action of \(\delta_{2m}\) on \(y(u')\) is given by

\[
[\delta_{2m}, y(u')] = \frac{1}{2} \sum_{\alpha \in \Phi^+} \sum_{p+q=2m-1} \left[ (\text{ad} \frac{x_\alpha y}{2})(t_{\alpha}), (\text{ad} \frac{x_\alpha y}{2})(t_{\alpha}) \right].
\]

Let \(\varphi(x) = \sum_{n \geq 1} b_{2n} x^{2n}\), then,

\[
\frac{1}{2} \frac{\varphi(u) - \varphi(v)}{u - v} = \frac{1}{2} \sum_{n \geq 1} b_{2n} \sum_{p+q=2n-1} u^p v^q = \sum_s f_s(u) g_s(v).
\]

Therefore,

\[
[\Delta_x, y(u')] = \delta_{\varphi}(y(u')) = \sum_{n \geq 1} b_{2n} \delta_{2n}(y(u'))
= \frac{1}{2} \sum_{n \geq 1} b_{2n} \sum_{\alpha \in \Phi^+} \alpha(u') \sum_{p+q=2n-1} \left[ (\text{ad} \frac{x_\alpha y}{2})(t_{\alpha}), (\text{ad} \frac{x_\alpha y}{2})(t_{\alpha}) \right]
= \sum_{\alpha \in \Phi^+} \sum_s f_s(\text{ad} \frac{x_\alpha y}{2})(t_{\alpha}), g_s(\text{ad} \frac{x_\alpha y}{2})(t_{\alpha})].
\]

This implies the conclusion. \(\square\)

We now compute the first term of (40). First, we have the Lemma.

Lemma 8.5. Modulo the relations \((\delta x)\) and \((\delta t)\) of \(t_0^\Phi \cong \mathfrak{d}\), we have

\[
[\delta_x, k(\alpha, \text{ad} \frac{x_\alpha y}{2})(t_{a})] = \sum_{s} \left[ f_s(\text{ad} \frac{x_\alpha y}{2})(t_{a}), m_s(\text{ad} \frac{x_\alpha y}{2})(t_{a}) \right],
\]

where \(k(\alpha, u + v)\varphi(v) = \sum_s f_s(u) m_s^2(v)\).

Proof. Let \(\varphi(x) = \sum_{n \geq 1} b_{2n} x^{2n}\), and \(k(x) = \sum_{m \geq 0} a_m x^m\) be the expansions of \(\varphi(x)\) and \(k(x)\), then we have

\[
k(\alpha, u + v)\varphi(v) = \sum_{m \geq 0} a_m \sum_{p+q=m} \binom{m}{p} \sum_{n \geq 1} b_{2n} u^p v^{2n+q} = \sum_s f_s(u) m_s^2(v).
\]

Therefore,

\[
[\delta_x, k(\alpha, \text{ad} \frac{x_\alpha y}{2})(t_{a})] = \sum_{n \geq 1} b_{2n} k(\alpha, \text{ad} \frac{x_\alpha y}{2})(t_{a})
= \sum_{m \geq 0} a_m \sum_{n \geq 1} b_{2n} (\text{ad} \frac{x_\alpha y}{2})^m (t_{a}), (\text{ad} \frac{x_\alpha y}{2})^{2n+q}(t_{a})
= \sum_s f_s(\text{ad} \frac{x_\alpha y}{2})(t_{a}), m_s^2(\text{ad} \frac{x_\alpha y}{2})(t_{a})].
\]

This completes the proof. \(\square\)
We need the following Lemma for the first term of (40).

**Lemma 8.6.** *Modulo the relations* \([\Delta_0, x], [\Delta_0, y], [\Delta_0, l], \) *and* \((yx)\) *of* \(t_i^0 \gg 0\), *we have*

\[
[\Delta_0, k(\alpha, \text{ad } \frac{x_{\alpha}^\vee}{2}[\tau](t_\alpha))] = \left[\frac{y_{\alpha}^\vee}{2}, g(\alpha, \text{ad } \frac{x_{\alpha}^\vee}{2}[\tau](t_\alpha))\right] - \sum_s [h_s(\text{ad } \frac{x_{\alpha}^\vee}{2}(t_\alpha), k_s(\text{ad } \frac{x_{\alpha}^\vee}{2}(t_\alpha)] - \sum_{y \neq \alpha} k(\alpha, \text{ad } \frac{x_{\alpha}^\vee}{2}) - k(\alpha, \text{ad } \frac{x_{\alpha}^\vee}{-2}) - (\text{ad } \frac{x_{\alpha}^\vee}{2} + \text{ad } \frac{x_{\alpha}^\vee}{-2})\) \(g(\alpha, \text{ad } \frac{x_{\alpha}^\vee}{2})\]

*where the functions* \(h_s, k_s\) *are determined by the equality*

\[
\frac{(\kappa_y^\vee(\alpha - s) - \kappa_y^\vee(\alpha + s))}{(\kappa_y^\vee(\alpha - s) - \kappa_y^\vee(\alpha + s))} = \frac{1}{2} = - \sum_s h_s(u)k_s(v).
\]

**Proof.** Using the relations \([\Delta_0, t_\alpha] = 0\) \(\) and \([\Delta_0, x_{\alpha}^\vee] = y_{\alpha}^\vee\), *we have*

\[
[\Delta_0, (\text{ad } x_{\alpha}^\vee)^s(t_\alpha)] = \sum_{s=0}^{n-1} (\text{ad } x_{\alpha}^\vee)^s(\text{ad } x_{\alpha}^\vee)^s(t_\alpha) = \sum_{s=0}^{n-1} (\text{ad } x_{\alpha}^\vee)^s(\text{ad } x_{\alpha}^\vee)^s(t_\alpha).
\]

*Using the Jacobi identity, we have the following equality, for* \(0 \leq i \leq n - 2\).*

*Taking the summation over* \(\{i \mid 0 \leq i \leq s - 1\}, *we have*

\[
\sum_{i=0}^{s-1} (\text{ad } x_{\alpha}^\vee)^i(\text{ad } [x_{\alpha}^\vee, y_{\alpha}^\vee])(\text{ad } x_{\alpha}^\vee)^{n-2-i}(t_\alpha)
\]

\[
= \sum_{i=0}^{s-1} (\text{ad } x_{\alpha}^\vee)^i(\text{ad } y_{\alpha}^\vee)(\text{ad } x_{\alpha}^\vee)^{n-2-i}(t_\alpha) - \sum_{i=0}^{s-1} (\text{ad } x_{\alpha}^\vee)^i(\text{ad } y_{\alpha}^\vee)(\text{ad } x_{\alpha}^\vee)^{n-2-i}(t_\alpha)
\]

\[
= (\text{ad } x_{\alpha}^\vee)^i(\text{ad } y_{\alpha}^\vee)(\text{ad } x_{\alpha}^\vee)^{n-2-i}(t_\alpha) - (\text{ad } x_{\alpha}^\vee)^i(\text{ad } y_{\alpha}^\vee)(\text{ad } x_{\alpha}^\vee)^{n-2-i}(t_\alpha).
\]

*Therefore, taking the summation over* \(\{s \mid 0 \leq s \leq n - 1\}, *we have*

\[
[\Delta_0, (\text{ad } x_{\alpha}^\vee)^s(t_\alpha)] = \sum_{s=0}^{n-1} (\text{ad } x_{\alpha}^\vee)^s(\text{ad } y_{\alpha}^\vee)(\text{ad } x_{\alpha}^\vee)^{n-1-s}(t_\alpha)
\]

\[
=n \text{ad } y_{\alpha}^\vee(\text{ad } x_{\alpha}^\vee)^{n-1}(t_\alpha) + \sum_{s=1}^{n-1} \sum_{i=0}^{s-1} (\text{ad } x_{\alpha}^\vee)^i(\text{ad } [x_{\alpha}^\vee, y_{\alpha}^\vee])(\text{ad } x_{\alpha}^\vee)^{n-2-i}(t_\alpha)
\]

\[
=n \text{ad } y_{\alpha}^\vee(\text{ad } x_{\alpha}^\vee)^{n-1}(t_\alpha) - \sum_{s=1}^{n-1} \sum_{i=0}^{s-1} (\text{ad } x_{\alpha}^\vee)^i(\text{ad } y_{\alpha}^\vee)(\text{ad } x_{\alpha}^\vee)^{n-2-i}(t_\alpha)
\]

\[
=n \text{ad } y_{\alpha}^\vee(\text{ad } x_{\alpha}^\vee)^{n-1}(t_\alpha) - \sum_{s=1}^{n-1} \sum_{i=0}^{s-1} (\text{ad } x_{\alpha}^\vee)^i(\text{ad } y_{\alpha}^\vee)(\text{ad } x_{\alpha}^\vee)^{n-2-i}(t_\alpha)
\]

\[
- \sum_{s=1}^{n-1} \sum_{i=0}^{s-1} (\text{ad } x_{\alpha}^\vee)^i(\text{ad } y_{\alpha}^\vee)(\text{ad } x_{\alpha}^\vee)^{n-2-i}(t_\alpha)
\]

*where the last equality follows from the* \((yx)\) *relation* \([x_{\alpha}^\vee, y_{\alpha}^\vee] = - \sum_{\gamma \neq \alpha} (\alpha, \gamma) \Phi_y\). *We now use the following identity which can be shown by induction*

\[
\sum_{s=1}^{n-1} \sum_{i=0}^{s-1} a'^i b'^{n-2-i} = \frac{a'^n - b'^n - nb'^{n-1}(a - b)}{(a - b)^2}, \text{ for } n \geq 2.
\]
The argument similar to the proof of Lemma 8.5 shows
\[
[\Delta_0, k(\alpha, \text{ad} \frac{x_{\alpha'}}{2} \tau(t_\alpha))] = \left[\frac{Y_\alpha}{2}, g(\alpha, \text{ad} \frac{x_{\alpha'}}{2} \tau(t_\alpha))\right] - \sum_s \left[h_s(\text{ad} \frac{x_{\alpha'}}{2})(t_\alpha), k_s(\text{ad} \frac{x_{\alpha'}}{2})(t_\alpha)\right] - \sum_{\{\gamma \in \Phi^+, \gamma \neq \alpha\}} (\alpha', \gamma)^2 k(\alpha, \text{ad} \frac{x_{\alpha'}}{2}) - \frac{k(\alpha, \text{ad} \frac{x_{\alpha'}}{2}) - \frac{(\alpha', \gamma)^2}{(\alpha', \gamma)^2}}{(\text{ad} x_{\alpha'})^2 + (\text{ad} x_{\alpha'})^2}[t_\gamma, t_\alpha].
\]

The conclusion now follows from the equality \( \frac{x_{\alpha'} + x_{\alpha'(\gamma')}}{(\alpha', \gamma')^2} = \frac{x_{\alpha'(\gamma')}}{2}. \)

Plugging the formulas in Lemmas 8.2, 8.3, 8.4, 8.5, and 8.6 into the formula (40) + (41) of \( A \wedge A \), we get
\[
-2\pi i A \wedge A = \sum_{\alpha \in \Phi^+} \sum_s \left(F_s^\gamma(\text{ad} \frac{x_{\alpha}}{2}), G_s^\gamma(\text{ad} \frac{x_{\alpha}}{2})\right) d\tau \wedge d\alpha + \sum_{\{\alpha, \beta \in \Phi^+, \gamma \neq \alpha\}} H(\alpha, \gamma)[t_\gamma, t_\alpha] d\tau \wedge d\alpha,
\]
where \( \sum_s F_s(u)^\gamma G_s(u)(v) = -L(\alpha, u, v) \), and
\[
L(z, u, v) = \frac{1}{2} \left(\frac{\varphi(u) - \varphi(v)}{u + v} + \frac{1}{2} k(z, u + v)(\varphi(u) - \varphi(v)) + \frac{1}{2} (g(z, u) k(z, v) - k(z, u) g(z, v)) - \frac{1}{2} \left(\frac{k(z, u + v) - k(z, u) - \text{ad} g(\gamma, \text{ad} \frac{x_{\alpha(\gamma')}}{2})}{u^2} - \frac{k(z, u + v) - k(z, v) - \text{ad} g(\gamma, \text{ad} \frac{x_{\alpha(\gamma')}}{2})}{v^2}\right)\right).
\]

As shown in [6], we have \( L(z, u, v) = 0 \). Therefore, the first summand of (42) is zero. In (42), we have
\[
H(\alpha, \gamma) = -\frac{k(\alpha, \text{ad} \frac{x_{\alpha'(\gamma')}}{2}) - k(\alpha, \text{ad} \frac{x_{\alpha'(\gamma')}}{2})}{(\text{ad} \frac{x_{\alpha'(\gamma')}}{2})^2} + \frac{\gamma(\alpha') g(\alpha, \text{ad} \frac{x_{\alpha'(\gamma')}}{2})}{2 \text{ad} \frac{x_{\alpha'(\gamma')}}{2}} + \frac{g(\gamma, \text{ad} \frac{x_{\alpha'(\gamma')}}{2}) - g(\gamma, \text{ad} \frac{x_{\alpha'(\gamma')}}{2})}{\text{ad} \frac{x_{\alpha'(\gamma')}}{2}} - \frac{g(\gamma, \text{ad} \frac{x_{\alpha'(\gamma')}}{2})}{\text{ad} \frac{x_{\alpha'(\gamma')}}{2}} k(\alpha, \text{ad} \frac{x_{\alpha'(\gamma')}}{2}).
\]

To show the vanishing of (42), we first need the following identity of the coefficients \( H(\alpha, \gamma) \).

**Proposition 8.7.** The following identity holds
\[
H(\alpha, \gamma) - H(\alpha, \alpha + \gamma) - H(\gamma + \alpha, \gamma) + H(\gamma + \alpha, \alpha) \equiv 0.
\]

**Proof.** We have the equalities \( x_{\omega(\gamma')\alpha} = x_{\omega(\alpha + \gamma)\alpha} \), and \( x_{\omega(\alpha')\gamma} = x_{\omega((\alpha + \gamma)\gamma'\gamma)}. \) Plugging them into the definition of \( H(\alpha, \gamma) \), we can simplify as follows.
\[
H(\alpha, \gamma) - H(\alpha, \alpha + \gamma) - H(\gamma + \alpha, \gamma) + H(\gamma + \alpha, \alpha) = -\frac{k(\alpha, \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2}) - k(\alpha, \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2})}{(\text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2})^2} - \frac{g(\alpha, \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2})}{2 \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2}} + \frac{g(\alpha + \gamma, \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2}) - k(\alpha + \gamma, \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2})}{(\text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2})^2} + \frac{g(\gamma, \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2})}{2 \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2}} + \frac{g(\gamma, \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2})}{\text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2}} - \frac{g(\gamma, \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2})}{2 \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2}} k(\alpha, \text{ad} \frac{x_{\alpha'(\alpha + \gamma)}}{2}).
\]

We choose \( z = \alpha, z' = \alpha + \gamma \), then \( z' - z = \gamma \). Choose \( v = \frac{x_{\alpha'(\alpha + \gamma)}}{2} = \frac{x_{\omega(\alpha + \gamma)\alpha}}{2} \), and \( u = \frac{x_{\alpha'(\gamma')}}{2} = \frac{x_{\omega(\alpha + \gamma)\gamma'}}{2} \). Note that \( k(z, x) = -k(z, x) \) and \( g(z, x) = \text{ad} g(\gamma, \text{ad} \frac{x_{\alpha'(\gamma')}}{2}) \). The above expression is...
of the following form
\[
H(z, z', u, v) = \frac{k(z, u + v) - k(z, u) - vg(z, u)}{u^2} - \frac{k(z', u + v) - k(z', v) - ug(z', v)}{u^2}
+ \frac{g(z' - z, -u) - g(z' - z, v)}{u + v} - g(-z' - v)k(-z, -u) + g(-z, -u)k(-z', -v)
- g(z' - z, v)k(z, u + v) + g(z' - z, -u)k(z', u + v).
\]
It is shown in [6] that \(H(z, z', u, v) = 0\). This implies the conclusion.

To show the vanishing of the second term of (42), by Lemma 3.8, it suffices to show the vanishing of of (42) for rank two root system \(\Phi\). We show (42) is zero case by case. The mail tool is the identity in Proposition 8.7.

8.3. Case \(A_1 \times A_1\). It is obvious that (42) is zero since the (tt) relation \([t_{a_1}, t_{a_2}] = 0\).

8.4. Case \(A_2\). According to Proposition 8.7, we have the equality
\[
H(a_1, a_2) = H(a_1, a_1 + a_2) - H(a_1 + a_2, a_1) - H(a_1, a_2, a_2).
\]
We use the graph \(H(a_1, a_2) \rightarrow \{ H(a_1, a_1 + a_2), H(a_1 + a_2, a_1), H(a_1 + a_2, a_2) \}\) to represent we split the term \(H(a_1, a_2)\) according to Proposition 8.7. We plug it into \(\sum_{\alpha \in \Phi^*} (\sum_{\gamma \neq 0} H(\alpha, \gamma)[t_{a_1}, t_{a_2}]) dt \wedge d\alpha\). By computation, the coefficient of \(dt \wedge da_1\) is
\[
H(a_1, a_1 + a_2)[t_{a_1}, t_{a_1 + a_2} + t_{a_2}] + H(a_1 + a_2, a_1)[t_{a_1 + a_2}, t_{a_2}] + H(a_1 + a_2, a_2)[t_{a_1 + a_2} + t_{a_1}, t_{a_2}].
\]
It vanishes because of the (tt) relations.

We now use the graph \(H(a_2, a_1) \rightarrow \{ H(a_2, a_1 + a_2), H(a_1 + a_2, a_2), H(a_1 + a_2, a_1) \}\). In this case, the coefficient of \(dt \wedge da_2\) becomes
\[
H(a_2, a_1 + a_2)[t_{a_2}, t_{a_1 + a_2} + t_{a_1}] + H(a_1 + a_2, a_1)[t_{a_1 + a_2} + t_{a_2}, t_{a_1}] + H(a_1 + a_2, a_2)[t_{a_1 + a_2} + t_{a_1}, t_{a_2}].
\]
This also vanishes because of the (tt) relations. Therefore, (42) is zero in the case of \(A_2\).

8.5. Case \(B_2\). According to Proposition 8.7, we rewrite \(H(\alpha, \gamma)\) following the graphs
\[
H(a_1, a_2) \rightarrow \{ H(a_1, a_1 + a_2), H(a_1 + a_2, a_1), H(a_1 + a_2, a_2) \}\)
and \(H(a_1 + 2a_2, a_2) \rightarrow \{ H(a_1 + 2a_2, a_1 + a_2), H(a_1 + 2a_2, a_1 + 2a_2), H(a_1 + a_2, a_2) \}\)
We plug them into \(\sum_{\alpha \in \Phi^*} (\sum_{\gamma \neq 0} H(\alpha, \gamma)[t_{a_1}, t_{a_2}]) dt \wedge d\alpha\). By computation, the coefficient of \(dt \wedge da_1\) is
\[
H(a_1, a_1 + a_2)[t_{a_1}, t_{a_1 + a_2} + t_{a_2}] + H(a_1 + a_2, a_1)[t_{a_1 + a_2} + t_{a_2}, t_{a_1}] + H(a_1 + a_2, a_2)[t_{a_1 + a_2} + t_{a_1}, t_{a_2}]
+ H(a_1 + a_2, a_2)[t_{a_1 + a_2} + t_{a_1} + t_{a_1 + 2a_2}, t_{a_2}] + H(a_1 + 2a_2, a_1 + a_2)[t_{a_1 + 2a_2}, t_{a_1 + a_2} + t_{a_2}].
\]
It vanishes because of the (tt) relations of root system \(B_2\).

Similarly, we rewrite \(H(\alpha, \gamma)\) following the graphs, using the identity in Proposition 8.7.
\[
H(a_2, a_1) \rightarrow \{ H(a_2, a_1 + a_2), H(a_1 + a_2, a_2), H(a_1 + 2a_2, a_2) \}
and \(H(a_1 + a_2, a_1) \rightarrow \{ H(a_1 + a_2, a_2), H(a_1 + 2a_2, a_1 + a_2), H(a_2, a_1 + a_2) \}\)
and $H(a_1 + a_2, a_1 + 2a_2) \rightarrow \begin{cases} H(a_1 + 2a_2, a_1 + a_2) \\ H(a_1 + 2a_2, a_2) \\ H(a_1 + a_2, a_2) \end{cases}$. By computation, the coefficient of $d\tau \wedge da_2$ is

\[
H(a_2, a_1 + a_2)[t_{a_2}, t_{a_1 + a_2} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_1 + a_2)[t_{a_1 + 2a_2}, t_{a_1 + a_2} + t_{a_1} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_1)[t_{a_1 + a_2} + t_{a_2}, t_{a_1}] + H(a_1 + a_2, a_2)[t_{a_1 + 2a_2} + t_{a_1 + a_2} + t_{a_1}, t_{a_2}].
\]

It vanishes because of the (tt) relations of root system $B_2$. This shows (42) is zero for $B_2$.

8.6. **Case $G_2$**. According to Proposition 8.7, we rewrite $H(a, \gamma)$ following the graphs in order.

\[
H(a_1, a_1 + 3a_2) \rightarrow \begin{cases} H(a_1, 2a_1 + 3a_2) \\ H(a_1 + 2a_2, a_1) \\ H(2a_1 + 3a_2, a_1 + 3a_2) \end{cases} \quad H(a_1, a_2) \rightarrow \begin{cases} H(a_1, a_1 + a_2) \\ H(a_1 + a_2, a_1) \\ H(a_1 + a_2, a_2) \end{cases}
\]

\[
H(a_1 + 3a_2, a_1) \rightarrow \begin{cases} H(a_1 + 3a_2, 2a_1 + 3a_2) \\ H(2a_1 + 3a_2, a_1 + 3a_2) \\ H(2a_1 + 3a_2, a_1 + 3a_2) \end{cases} \quad H(a_1 + 3a_2, a_2) \rightarrow \begin{cases} H(a_1 + 3a_2, a_1 + 2a_2) \\ H(a_1 + 2a_2, a_2) \\ H(a_1 + 2a_2, a_1 + 3a_2) \end{cases}
\]

\[
H(a_1 + a_2, 2a_1 + 3a_2) \rightarrow \begin{cases} H(a_1 + a_2, a_1 + a_2) \\ H(2a_1 + 3a_2, a_1 + 2a_2) \\ H(2a_1 + 3a_2, a_1 + a_2) \end{cases} \quad H(a_1 + 3a_2, a_1) \rightarrow \begin{cases} H(a_1 + 2a_2, a_1 + a_2) \\ H(2a_1 + 3a_2, a_1 + 3a_2) \\ H(a_1 + 2a_2, a_1 + 3a_2) \end{cases}
\]

We plug them into $\sum_{a \in \Phi^+}(\sum_{\gamma \neq a} H(a, \gamma)[t_{a_1}, t_{\gamma}])d\tau \wedge da_1$. By computation, the coefficient of $d\tau \wedge da_1$ is

\[
H(a_1, a_1 + a_2)[t_{a_1}, t_{a_1+a_2} + t_{a_1} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_1)[t_{a_1+a_2}, t_{a_1+2a_2} + t_{2a_1+3a_2} + t_{a_1} + t_{a_2}] + H(a_1 + a_2, a_1)[t_{a_1+2a_2} + t_{a_1+a_2} + t_{a_1} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_2)[t_{a_1+2a_2}, t_{a_1+a_2} + t_{a_1} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_1)[t_{a_1+a_2} + t_{a_1+a_2} + t_{2a_1+3a_2} + t_{a_1} + t_{a_2}] + H(a_1 + 2a_2, a_2)[t_{a_1+a_2} + t_{2a_1+3a_2} + t_{a_1} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_1)[t_{a_1+2a_2} + t_{2a_1+3a_2} + t_{a_1} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_2)[t_{a_1+2a_2} + t_{2a_1+3a_2} + t_{a_1} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_1)[t_{a_1+a_2} + t_{2a_1+3a_2} + t_{a_1} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_2)[t_{a_1+a_2} + t_{2a_1+3a_2} + t_{a_1} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_1)[t_{a_1+a_2} + t_{2a_1+3a_2} + t_{a_1} + t_{a_1 + 2a_2}] + H(a_1 + 2a_2, a_2)[t_{a_1+a_2} + t_{2a_1+3a_2} + t_{a_1} + t_{a_1 + 2a_2}]
\]

The coefficient vanishes because of the (tt) relations of root system $G_2$.

Similarly, we rewrite $H(a, \gamma)$ according to Proposition 8.7 by following the graphs in order.

\[
3H(a_1 + 3a_2, a_1) \rightarrow \begin{cases} 3H(a_1 + 3a_2, 2a_1 + 3a_2) \\ 3H(a_1 + 3a_2, a_1) \\ 3H(a_1 + 3a_2, a_1 + 3a_2) \end{cases} \quad H(a_2, a_1 + a_2) \rightarrow \begin{cases} H(a_2, a_1 + a_2) \\ H(a_1 + a_2, a_2) \\ H(a_1 + a_2, a_1) \end{cases}
\]

\[
H(2a_1 + 3a_2, a_1 + a_2) \rightarrow \begin{cases} H(2a_1 + 3a_2, a_1 + 2a_2) \\ H(a_1 + a_2, a_1 + 2a_2) \\ H(a_1 + a_2, 2a_1 + 3a_2) \end{cases} \quad 2H(2a_1 + 3a_2, a_1 + a_2) \rightarrow \begin{cases} 2H(2a_1 + 3a_2, a_1 + 2a_2) \\ 2H(a_1 + 2a_2, a_1 + 2a_2) \\ 2H(a_1 + 2a_2, 2a_1 + 3a_2) \end{cases}
\]
Similar computation shows the coefficient of \( dt \wedge da_2 \) is
\[
H(a_1 + 2a_2, a_2) \rightarrow \begin{cases} 
H(a_1 + 2a_2, a_1 + a_2) \\
H(a_1 + a_2, a_2) \\
H(a_1 + a_2, a_1 + 2a_2) \\
H(a_1 + 2a_2, a_1 + 2a_2)
\end{cases} \quad H(a_1 + 2a_2, a_2) \rightarrow \begin{cases} 
H(a_1 + 2a_2, a_1 + a_2) \\
H(a_2, a_1 + a_2) \\
H(a_2, a_1 + 2a_2)
\end{cases}
\]
\[
H(a_1 + 3a_2, a_2) \rightarrow \begin{cases} 
H(a_1 + 3a_2, a_1 + a_2) \\
H(a_1 + 2a_2, a_2) \\
H(a_1 + 2a_2, a_1 + 3a_2) \\
H(a_1 + 2a_2, a_1 + a_2)
\end{cases} \quad H(a_1 + 2a_2, a_2) \rightarrow \begin{cases} 
H(a_1 + a_2, a_1 + a_2) \\
H(a_1 + 2a_2, a_1 + a_2) \\
H(a_1 + 2a_2, a_1 + 2a_2)
\end{cases}
\]
\[
H(a_2, a_1 + 3a_2) \rightarrow \begin{cases} 
H(a_2, a_1 + 2a_2) \\
H(a_1 + 2a_2, a_1 + 3a_2) \\
H(a_1 + 2a_2, a_1 + a_2)
\end{cases} \quad H(a_1 + a_2, a_1 + a_2) \rightarrow \begin{cases} 
H(a_1 + 2a_2, a_1 + a_2) \\
H(a_1 + 2a_2, a_1 + 2a_2)
\end{cases}
\]

This coefficient vanishes because of the \((tt)\) relations of root system \(G_2\). Thus, \((42)\) is zero in the case of \(G_2\).

9. The elliptic connection valued in rational Cherednik algebras

In this section, we specialise the coefficients of the connections \(\nabla_{KZB,T} \) \((4)\) and \(\nabla_{KZB} \) \((39)\) to the rational Cherednik algebra of a Weyl group \(W\). This specialisation is the elliptic analogue of the Coxeter \(KZ\) connection valued in the group algebra \(\mathbb{C}W\), and Cherednik’s affine \(KZ\) connection valued in the degenerate affine Hecke algebra of \(W\).

9.1. The rational Cherednik algebra. In this subsection, we review some basic facts about the rational Cherednik algebras. For details, see \([12], [13]\).

Let \(W\) be a Weyl group and \(\mathfrak{h}\) be its complexified reflection representation. Let \(S \subseteq W\) be the set of reflections and, for any \(s \in S\), fix \(\alpha_s \in \mathfrak{h}^*\), such that \(s(\alpha_s) = -\alpha_s\). Let \(K\) be the vector space of \(W\)-invariant functions \(S \rightarrow \mathbb{C}\), and \(\overline{K} = K \oplus \mathbb{C}\). We denote the standard linear functions on \(\overline{K}\) by \(\{c_s\}_{s \in S/W}\) and \(h\).

**Definition 9.1.** The rational Cherednik algebra \(H_{h,c}\) is the quotient of the algebra \(\mathbb{C}W \ltimes T(\mathfrak{h} \oplus \mathfrak{h}^*)[\overline{K}]\) by the ideal generated by the relations
\[
[x, x'] = 0, \quad [y, y'] = 0, \quad [y, x] = h(y, x) - \sum_{s \in S} c_s(\alpha_s, y)\langle \alpha_s^\vee, x \rangle s,
\]
where \(x, x' \in \mathfrak{h}^*, y, y' \in \mathfrak{h}\). The algebra is \(\mathbb{N} \times \mathbb{N}\)-graded by \(\deg(x) = (1, 0), x \in \mathfrak{h}\), \(\deg(y) = (0, 1), y \in \mathfrak{h}^*\), \(\deg(w) = 0, w \in W\), \(\deg(h) = (1, 1)\) and \(\deg(c_s) = (1, 1)\).
Let \( \{y_i \mid 1 \leq i \leq n\} \) be the basis of \( \mathfrak{h} \), and \( \{x_i \mid 1 \leq i \leq n\} \) be the corresponding dual basis of \( \mathfrak{h}^* \). We have the following elements of the rational Cherednik algebras.

\[
\mathbf{h} := \sum_{i=1}^{n} x_i y_i + \frac{1}{2} \dim \mathfrak{h} - \sum_{s \in S} c_s s = \sum_{i=1}^{n} \frac{x_i y_i + y_i x_i}{2}, \quad \mathbf{E} := -\frac{1}{2} \sum_{i=1}^{n} x_i^2, \quad \text{and} \quad \mathbf{F} := \frac{1}{2} \sum_{i=1}^{n} y_i^2.
\]

The following properties of \( \mathbf{h} \), \( \mathbf{E} \) and \( \mathbf{F} \) can be found in [13, Prop. 3.18, 3.19].

**Proposition 9.2.** The following holds.

(i) For any \( x \in \mathfrak{h}^* \), \( y \in \mathfrak{h} \), \([\mathbf{h}, x] = hx, [\mathbf{h}, y] = -hy\).

(ii) The elements \( \frac{1}{\mathbf{h}} \mathbf{h}, \mathbf{E}, \frac{1}{\mathbf{h}} \mathbf{F} \) form an \( \mathfrak{sl}_2 \)-triple.

9.2. **Specialising the KZB connection \( \nabla_{KZB, r} \) to the rational Cherednik algebra.** In this subsection, we specialize the KZB connection \( \nabla_{KZB, r} \) to the rational Cherednik algebra \( H_{h,c} \) by constructing a homomorphism from the Lie algebra \( t_{\text{cl}}^{\Phi} \) to \( H_{h,c} \).

Let \( \tilde{a} \) be the highest root of Lie algebra \( \mathfrak{g} \). Write \( \tilde{a}^\vee = \sum_{i=1}^{n} g_i \tilde{a}_i^\vee \), and let \( h^\vee = 1 + \sum_{i=1}^{n} g_i \) be the dual Coxeter number of \( \mathfrak{g} \).

**Proposition 9.3.** For any \( a, b \in \mathbb{C} \), there is a bigraded Lie algebra homomorphism \( \xi_{a,b} : t_{\text{cl}}^{\Phi} \rightarrow H_{h,c} \), defined as follows

\[
x(v) \mapsto a\pi(v), \quad y(u) \mapsto bu, \quad t_\gamma \mapsto ab \left( \frac{h}{h^\vee} - \frac{2c_\gamma}{(\gamma|\gamma)s_\gamma} \right),
\]

for \( a, b \in \mathbb{C} \) and \( \gamma \in \Phi^+ \), where \( \pi : \mathfrak{h} \rightarrow \mathfrak{h}^* \) is the isomorphism induced by the non-degenerate bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{h} \), and \( s_\gamma \) is the reflection corresponding to the root \( \gamma \).

For simplicity and without lost of generality we assume \( a = b = 1 \). To prove the Proposition, we need to show the map \( \xi_{a,b} \) respects all defining relations of \( t_{\text{cl}}^{\Phi} \). The only non-trivial relation to check is the \( (yx) \) relation

\[
[y(u), x(v)] = \sum_{\gamma \in \Phi^+} \langle v, \gamma \rangle \langle u, \gamma \rangle t_\gamma.
\]

In the remainder of this subsection, we check \( \xi \) preserves the \( (yx) \) relation of \( t_{\text{cl}}^{\Phi} \).

**Lemma 9.4.** The following equality holds

\[
\langle \cdot, \cdot \rangle = 1 \langle \cdot, \cdot \rangle \sum_{\gamma \in \Phi^+} \langle \cdot, \gamma \rangle \langle \gamma, \cdot \rangle.
\]

**Proof.** Both \( \langle \cdot, \cdot \rangle : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{C} \), and \( \sum_{\gamma \in \Phi^+} \langle \cdot, \gamma \rangle \langle \gamma, \cdot \rangle : \mathfrak{h} \times \mathfrak{h} \rightarrow \mathbb{C} \) are two symmetric bilinear forms on \( \mathfrak{h} \). They are both positive definite, and invariant under the Weyl group \( W \). Note that there is only one such bilinear form up to constants. Therefore, there exists some constant \( k \in \mathbb{Q} \), such that

\[
\langle \cdot, \cdot \rangle = k \sum_{\gamma \in \Phi^+} \langle \cdot, \gamma \rangle \langle \gamma, \cdot \rangle.
\]

It remains to show the constant \( k \) is the same as \( \frac{1}{h^\vee} \). It follows from [19, Lemma1.2] that \( \sum_{\gamma \in \Phi^+} \langle \tilde{a}^\vee, \gamma \rangle \langle \tilde{a}^\vee, \gamma \rangle = 2h^\vee \). On the other hand, \( (\tilde{a}^\vee | \tilde{a}^\vee) = 2 \). This gives \( k = \frac{1}{h^\vee} \). Note that in the case of type \( A_n \), we have \( k = \frac{1}{n+1} \), which coincides with the constant in [6]. \( \square \)
Proof of Proposition 9.3. We now use Lemma 9.4 to prove Proposition 9.3. Under the map $\hat{\xi}_{a,b}$, we have
\[
[y(u), x(v)] \mapsto [u, \pi(v)] = \hbar \langle u, \pi(v) \rangle - \sum_{x \in S} c_x(\alpha_x, u)(\alpha_x^\vee, \pi(v)) s_x = \hbar (u|v) - \sum_{y \in \Phi^+} c_y(\gamma, u)(\gamma^\vee, \pi(v)) s_y.
\]
\[
\sum_{y \in \Phi^+} \langle y, \gamma \rangle (u, y) \tau_y \mapsto \sum_{y \in \Phi^+} \langle y, \gamma \rangle (u, y) \left( \frac{\hbar}{\hbar^\vee} - \frac{2c_y}{(\gamma^\vee) s_y} \right) = \hbar (u|v) - \sum_{y \in \Phi^+} c_y(\pi(v), \gamma^\vee) (u, y) s_y.
\]
Therefore, $\hat{\xi}_{a,b}(y(u), x(v)) = \hat{\xi}_{a,b}(\sum_{y \in \Phi^+} \langle y, \gamma \rangle (u, y) \tau_y)$. This completes the proof.

Theorem 9.5. The universal KZB connection specializes to the following elliptic KZ connection valued in the rational Cherednik algebra
\[
\nabla_{\mathbb{H}^c} = d + \sum_{\alpha \in \Phi^+} \frac{2c_\alpha}{(\alpha|\alpha)} k(\alpha, \text{ad}(\frac{\alpha^\vee}{2})|\tau) s_\alpha d\alpha - \sum_{\alpha \in \Phi^+} \frac{\hbar}{\hbar^\vee} \frac{\theta'(\alpha|\tau)}{\theta(\alpha|\tau)} d\alpha + \sum_{i=1}^n u^i du_i.
\]
The elliptic connection is flat and $W$-equivariant.

Proof. By Proposition 9.3, the universal KZB connection specializes to the following connection
\[
\nabla_{\mathbb{H}^c} = d - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{\alpha^\vee}{2})|\tau) \left( \frac{\hbar}{\hbar^\vee} - \frac{2c_\alpha}{(\alpha|\alpha)} s_\alpha \right) d\alpha + \sum_{i=1}^n u^i du_i.
\]
We now simplify the above expression. Note that $k(z, 0|\tau) = \frac{\theta'(z|\tau)}{\theta(\tau)}$. Therefore,
\[
k(\alpha, \text{ad}(\frac{\alpha^\vee}{2})|\tau) \left( \frac{\hbar}{\hbar^\vee} - \frac{2c_\alpha}{(\alpha|\alpha)} s_\alpha \right) = k(\alpha, \text{ad}(\frac{\alpha^\vee}{2})|\tau) \left( \frac{\hbar}{\hbar^\vee} - \frac{2c_\alpha}{(\alpha|\alpha)} s_\alpha \right) \frac{\hbar}{\hbar^\vee} \frac{\theta'(\alpha|\tau)}{\theta(\alpha|\tau)} - k(\alpha, \text{ad}(\frac{\alpha^\vee}{2})|\tau) \left( \frac{2c_\alpha}{(\alpha|\alpha)} s_\alpha \right).
\]
This completes the proof.

9.3. The monodromy of $\nabla_{\mathbb{H}^c}$. The connection $\nabla_{\mathbb{H}^c}$ is flat and $W$-equivariant. Its monodromy yields a one parameter family of monodromy representations of the elliptic braid group $B_{\#} = \pi_1^{orb}(T_{reg}/W)$. The double affine Hecke algebra $\mathbb{H}_{\#}$ is the quotient of the group algebra $\mathbb{C}[B_{\#}]$ by the quadratic relations
\[
(S_i - qt_i)(S_i + qt_i^{-1}) = 0,
\]
where $q = e^{\pi i \frac{\hbar}{\hbar^\vee}}$, and $t_i = e^{-\pi i \frac{\hbar^\vee}{(\alpha^\vee)|\alpha^\vee}}$.

Proposition 9.6. The monodromy of the elliptic connection $\nabla_{\mathbb{H}^c}$ factors through the double affine Hecke algebra $\mathbb{H}_{\#}$.

The monodromy of $\nabla_{\mathbb{H}^c}$ gives an algebra homomorphism
\[
\mathbb{H}_{\#} \to \mathbb{H}_{\#}^{\hat{\cdot}}
\]
where $\mathbb{H}_{\#}^{\hat{\cdot}}$ is the completion of $\mathbb{H}_{\#}$ with respect to the $\mathbb{N}$-grading on $\mathbb{H}_{\#}$. Furthermore, it induces an isomorphism between the completion $\mathbb{H}_{\#}$ and $\mathbb{H}_{\#}^{\hat{\cdot}}$. Indeed, $\mathbb{H}_{\#}$ and $\mathbb{H}_{\#}^{\hat{\cdot}}$ are both flat deformations of $\mathbb{C}[[\hbar]] \otimes \mathbb{C}[[\hbar^\vee]] \otimes CW$. Thus, the algebra homomorphism $\mathbb{H}_{\#} \to \mathbb{H}_{\#}^{\hat{\cdot}}$ is an isomorphism.
9.4. Specialisation of $\nabla_{\mathrm{KZB}}$ to the rational Cherednik algebra. In this subsection, we specialize the KZB connection $\nabla_{\mathrm{KZB}}$ to the rational Cherednik algebra $H_{h,c}$ by constructing a homomorphism from the Lie algebra $t^\Phi_{el} \cong \mathfrak{d}$ to $H_{h,c}$.

In the formula (43) of $E$, $F$ and $H$, we assume furthermore that $\{y_i\}$ is an orthonormal basis, and $\{x_i\}$ is the corresponding dual basis.

Proposition 9.7. The homomorphism $\xi_{a,b} : t^\Phi_{el} \rightarrow H_{h,c}$ can be extended to the homomorphism $\tilde{\xi}_{a,b} : U(t^\Phi_{el} \cong \mathfrak{d}) \rightarrow W \rightarrow H_{h,c}$ by the following formulas

$$w \mapsto w, \quad d \mapsto \frac{h}{h}, \quad X \mapsto ab\frac{F}{h}, \quad \Delta_0 \mapsto ba\frac{F}{h},$$

$$\delta_{2m} \mapsto -\frac{2a^{2m-1}b^{-1}}{h} \sum_{\alpha \in \mathfrak{d}^*} \frac{c_{\alpha}^2}{(\alpha, \alpha)} (x_\alpha y_\alpha)^{2m}.$$

Proof. For simplicity and without loss of generality, we assume that $a = b = 1$. We first show the above homomorphism preserves the relations of $\mathfrak{d}$, see Section 96 for the relations. From Proposition 9.2, we know the triple $\frac{h}{h}, \frac{E}{h}, \frac{F}{h}$ form an $\mathfrak{sl}_2$-triple. Thus, the homomorphism $\tilde{\xi}$ preserves the relations of the triple $d, X, \Delta_0$.

It is obvious that $\tilde{\xi}$ preserves the relation $[\delta_{2m}, X] = 0$, since the images of $\delta_{2m}$ and $X$ under $\tilde{\xi}$ lie in $\mathbb{C}[h]$.

The fact that $(h, x) = hx$ implies $\tilde{\xi}$ preserves the relation $[d, \delta_{2m}] = 2m\delta_{2m}$. Now we check $\tilde{\xi}$ preserves the relation $(\text{ad} \Delta_0)^{2m+1}(\delta_{2m}) = 0$. We have the following relation

$$[F, x_j] = h y_j, \quad \text{for any } 1 \leq j \leq n.$$

Since

$$2[F, x_j] = \sum_{i=1}^n y_i^2 x_j = \sum_{i=1}^n \left(y_i [y_i, x_j] + [y_i, x_j] y_i\right)$$

$$= 2hy_j - \sum_i \sum_s c_s(y_i, \alpha_s)(x_j, \alpha_s')(y_i s + sy_i)$$

$$= 2hy_j - \sum_i \sum_s c_s(x_i, \alpha_s)(x_j, \alpha_s')(y_i s + sy_i) \quad \text{by } (y_i, \alpha_s) = (x_i, \alpha_s)$$

$$= 2hy_j - \sum_{\alpha \in \mathfrak{d}^*} \sum_i \langle x_i, \alpha\rangle (\sum_i \langle x_i, \alpha\rangle (y_i s + sy_i))$$

$$= 2hy_j - \sum_{\alpha \in \mathfrak{d}^*} c_s(x_j, \alpha')(s_\alpha x_\alpha + s_\alpha s_\alpha) = 2hy_j. \quad \text{by } s_\alpha \alpha = -\alpha s_\alpha$$

Using above relation, we obtain

$$(\text{ad} F)^{2m+1} x_{\alpha'}^{2m} = \text{ad} F)^{2m}[F, x_{\alpha'}^{2m}] = h(\text{ad} F)^{2m}(x_{\alpha'}^{2m-1} y_{\alpha'} + y_{\alpha'} x_{\alpha'}^{2m-1})$$

$$= h(\text{ad} F)^{2m} x_{\alpha'}^{2m-1} y_{\alpha'} + h y_{\alpha'} (\text{ad} F)^{2m} x_{\alpha'}^{2m-1} = 0.$$  

The last equality follows from the fact $(\text{ad} F)^{2m-1} x_{\alpha'}^{2m-1} \in \mathbb{C}[y_{\alpha'}]$, and $[F, y_{\alpha'}] = 0$, for any $n$. Therefore, $(\text{ad} F)^{2m} x_{\alpha'}^{2m-1} = 0$. This shows $\tilde{\xi}$ preserves the relation $(\text{ad} \Delta_0)^{2m+1}(\delta_{2m}) = 0$. Therefore, $\tilde{\xi}$ preserves relations of $\mathfrak{d}$.

It is clear that the map $\xi : t^\Phi_{el} \rightarrow H_{h,c}$ is compatible with the Weyl group $W$ action. The Weyl group $W$ acts on $\mathfrak{d}$ trivially. We now check that for any $\eta \in \mathfrak{d}$, we have $w(\eta) = 0$. First, we have $h = w(h)$. Indeed, for any $w \in W$, $w(x_i)$ are a basis of $\mathfrak{d}^*$, and $w(y_i)$ are the corresponding dual basis of $\mathfrak{d}$. Then, we have $\sum_{i=1}^n (x_i y_i + y_i x_i) = \sum_{i=1}^n (w(x_i)w(y_i) + w(y_i)w(x_i))$. Thus, $h = w(h)$. We now show that $w(F) = F$, for any
\( w \in W \). This follows from the computation
\[
2 \sum_{i=1}^{n} y_i^2 = \frac{n}{2} (w(y))^2 = \frac{n}{2} (y_i)^2.
\]
The last equality follows from the fact that \( \{ y_i \mid 1 \leq i \leq n \} \) is an orthonormal basis. Similarly, \( w(E) = E \), for any \( w \in W \). We now show \( w(\xi(\delta_{2m})) = \xi(\delta_{2m}) \). This follows from the fact that \( w \) permutes the root system \( \Phi \) and preserves the killing form.

Next, we show that the map \( \tilde{\xi} \) preserves the relations between \( ts \) and \( b \), see Section 6 for the relations. By Proposition 9.2, it is clear that \( \tilde{\xi} \) preserves the relations \( \tilde{d}(x(u)) = x(u) \) and \( \tilde{d}(y(u)) = -y(u) \). The map \( \tilde{\xi} \) preserves the relation \( \tilde{d}(t_a) = 0 \) follows from the fact \( w(h) = h \), for any \( w \in W \). Indeed, \( [h, w] = (h - w(h))w = 0 \).

We check \( \tilde{\xi}_{a,b} \) preserves the relation \( \tilde{\delta}_{2m}(t_a) = [t_a, (ad x(\alpha))^{2m}(t_a)] \). We have already shown that \( [\tilde{\xi}(\delta_{2m}), w] = 0 \), for any \( w \in W \). It suffices to show \( [s_a, (ad x(\alpha))^{2m}(s_a)] = 0 \). Using the fact \( \frac{x(\alpha)}{2}, s_a = x(\alpha^\vee)^{2m} s_a \), we have
\[
[s_a, (ad x(\alpha))^{2m}(s_a)] = [s_a, x(\alpha^\vee)^{2m} s_a] = s_a x(\alpha^\vee)^{2m} s_a - x(\alpha^\vee)^{2m} = 0.
\]

Finally, we show that \( \tilde{\xi}_{a,b} \) preserves relation
\[
\tilde{\delta}_{2m}(y(u)) = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha(u) \sum_{p+q=2m-1} [(ad x(\alpha^\vee))^{p}(t_a), (ad x(\alpha^\vee))^{q}(t_a)].
\]

We first compute
\[
\sum_{p+q=2m-1} [(ad x(\alpha^\vee))^{p}(t_a), (ad x(\alpha^\vee))^{q}(s_a)]
\]
\[
= \sum_{p+q=2m-1} (-1)^q [x(\alpha^\vee)^{p} s_a, x(\alpha^\vee)^{q} s_a]
\]
\[
= \sum_{p+q=2m-1} (-1)^q (-1)^p x(\alpha^\vee)^{2m-1} = 4m x(\alpha^\vee)^{2m-1}.
\]

Therefore, under the map \( \tilde{\xi} \), the right hand side of (44) maps to
\[
\frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha(u) 4m \left( \frac{2 c_{\alpha}}{(\alpha, \alpha)} \right)^2 x(\alpha^\vee)^{2m-1} = 4m \sum_{\alpha \in \Phi^+} \frac{c_{\alpha}^2}{(\alpha, \alpha)} \alpha^\vee(u) x_{\alpha^\vee}^{2m-1}.
\]

We now compute the image of the left hand side of (44). We have
\[
[\delta_{2m}, y(u)] = -\frac{2}{\hbar} \sum_{\alpha \in \Phi^+} \frac{c_{\alpha}^2}{(\alpha, \alpha)} [(x_{\alpha^\vee})^{2m}, y(u)]
\]
\[
= -\frac{2}{\hbar} \sum_{\alpha \in \Phi^+} \frac{c_{\alpha}^2}{(\alpha, \alpha)} \left( \sum_{p+q=2m-1} x_{\alpha^\vee}^p(y(u)) x_{\alpha^\vee}^q \right)
\]
\[
= -\frac{2}{\hbar} \sum_{\alpha \in \Phi^+} \frac{c_{\alpha}^2}{(\alpha, \alpha)} \left( \sum_{p+q=2m-1} x_{\alpha^\vee}^p \left( -\hbar(u, \alpha^\vee) + \sum_{\gamma \in \Phi^+} c_{\gamma}(\gamma^\vee, u)(\gamma, \alpha^\vee) s_{\gamma} \right) x_{\alpha^\vee}^q \right)
\]
\[
= 4m \sum_{\alpha \in \Phi^+} \frac{c_{\alpha}^2}{(\alpha, \alpha)} \alpha^\vee(u) x_{\alpha^\vee}^{2m-1}.
\]
The last equality follows from the following calculation.

$$
\sum_{\alpha \in \Phi^+} \frac{c^2_\alpha}{(\alpha, \alpha)} \sum_{p+q=2m-1} \langle \gamma, \alpha^\vee \rangle \left( x^p_\alpha, s, x^q_\alpha \right) 
$$

$$
\frac{1}{2} \sum_{\alpha \in \Phi^+} \frac{c^2_\alpha}{(\alpha, \alpha)} \sum_{p+q=2m-1} \langle \gamma, \alpha^\vee \rangle x^p_\alpha (x_{s_\gamma(\alpha^\vee)})^q s_y + \frac{1}{2} \sum_{\alpha \in \Phi^+} \frac{c^2_\alpha}{(\alpha, \alpha)} \sum_{p+q=2m-1} \langle \gamma, s_y(\alpha^\vee) \rangle (x_{s_\gamma(\alpha^\vee)})^q (x_{\alpha^\vee})^p s_y
$$

$$
= \frac{1}{2} \sum_{\alpha \in \Phi^+} \frac{c^2_\alpha}{(\alpha, \alpha)} \sum_{p+q=2m-1} \langle \gamma, \alpha^\vee \rangle (x_{s_\gamma(\alpha^\vee)})^q (x_{\alpha^\vee})^p s_y = 0,
$$

since $$\langle \gamma, \alpha^\vee + s_y(\alpha^\vee) \rangle = 0$$. Therefore, $$\tilde{\xi}_{a,b}$$ preserves the relation (44). This completes the proof. □

10. The degenerate affine Hecke algebra and the rational Cherednik algebra

In this section, we construct a map from the degenerate affine Hecke algebra $$\mathcal{H}$$ to the completion of the rational Cherednik algebra $$H_{\mathfrak{h}, \mathfrak{c}}$$ by degenerating the connection in Section §9. The completion is with respect to the grading of the rational Cherednik algebra $$H_{\mathfrak{h}, \mathfrak{c}}$$

$$\deg(x) = \deg(y) = 1, \deg(w) = 0, \deg(h) = \deg(c) = 2,$$

for $$x \in \mathfrak{h}^*, y \in \mathfrak{h}$$ and $$w \in W$$.

**Definition 10.1.** The degenerate affine Hecke algebra $$\mathcal{H}$$ is the associative algebra generated by $$\mathbb{C}W$$ and the symmetric algebra $$\mathbb{S}\mathfrak{h}$$, subject to the relations,

$$s_ix_i - x_is_i = k_i(u, \alpha_i),$$

for any simple reflection $$s_i \in W$$ and linear generator $$x_i, u \in \mathfrak{h}$$, and $$k_i \in \mathbb{C}$$.

Cherednik in [8] constructed the affine KZ connection. It is a flat and $$W$$-equivariant connection on $$H_{\text{reg}}$$ valued in the degenerate affine Hecke algebra $$\mathcal{H}$$. The affine KZ connection can be obtained by specializing the trigonometric KZ connection in Section §4. More precisely, we have a map $$A_{\text{trig}} \rightarrow \mathcal{H}$$, by $$t_{\alpha} \mapsto k_\alpha s_\alpha$$, and $$X(u) \mapsto x(u)$$, for $$\alpha \in \Phi$$, and $$u \in \mathfrak{h}$$. This gives the affine KZ connection

$$\nabla_{\text{AKZ}} = d - \sum_{\alpha \in \Phi^+} \frac{2\pi i \alpha}{e^{2\pi i \alpha} - 1} d\alpha k_\alpha s_\alpha - \sum_i x(u^i) du_i.$$

As $$\text{Im} \tau \rightarrow \infty$$, by Proposition 4.3, the elliptic KZ connection valued in the rational Cherednik algebra degenerates to the following trigonometric connection.

$$\nabla = d - \sum_{\alpha \in \Phi^+} \frac{2\pi i \alpha}{e^{2\pi i \alpha} - 1} \left( \frac{\hbar}{h^\vee} - \frac{2\alpha_\alpha}{(\alpha, \alpha)} s_\alpha \right) - \sum_{\alpha \in \Phi^+} \frac{2\pi i e^{2\pi i \text{ad}(\frac{\alpha}{2})}}{e^{2\pi i \text{ad}(\frac{\alpha}{2})} - 1} - \frac{1}{\text{ad}(\frac{\alpha}{2})} \left( \frac{\hbar}{h^\vee} - \frac{2\alpha_\alpha}{(\alpha, \alpha)} s_\alpha \right) \alpha + \sum_{i=1}^n y(u^i) du_i$$

$$= d - \sum_{\alpha \in \Phi^+} \frac{2\pi i \alpha}{(\alpha, \alpha)} e^{2\pi i \alpha} - 1 - \frac{1}{\text{ad}(\frac{\alpha}{2})} \left( \frac{\hbar}{h^\vee} - \frac{2\alpha_\alpha}{(\alpha, \alpha)} s_\alpha \right) \alpha + \sum_{i=1}^n y(u^i) du_i.$$

Note that the constant term of $$\frac{2\pi ie^{2\pi i \text{ad}(\frac{\alpha}{2})}}{e^{2\pi i \text{ad}(\frac{\alpha}{2})} - 1} - \frac{1}{\text{ad}(\frac{\alpha}{2})}$$ is $$\pi i$$.

By the universality of the affine KZ connection $$\nabla_{\text{AKZ}}$$, we have an algebra homomorphism $$\mathcal{H} \rightarrow H_{\mathfrak{h}, \mathfrak{c}}$$ by

$$k_\alpha \mapsto -\frac{2\alpha_\alpha}{(\alpha, \alpha)} w \mapsto w, \text{ for } w \in W,$$

$$x(u) \mapsto -y(u) + \sum_{\alpha \in \Phi^+} \alpha(u) \left( \frac{e^{2\pi i \alpha}}{e^{2\pi i \alpha} - 1} + \frac{1}{\text{ad}(\frac{\alpha}{2})} \left( \frac{2\pi i e^{2\pi i \alpha}}{e^{2\pi i \alpha} - 1} - \frac{1}{\alpha^\vee} \right) s_\alpha \right).$$
11. The elliptic Dunkl operators

In [5], Buchstaber-Felder-Veselov defined elliptic Dunkl operators for an arbitrary Weyl groups. Etingof and Ma in [14] generalised the elliptic Dunkl operators to an abelian variety with a finite group action. Using the elliptic Dunkl operators, Etingof-Ma defined the elliptic Cherednik algebras as a sheaf of algebras on the abelian variety. They also constructed certain representations of the elliptic Cherednik algebra.

In this section, we show that those representations constructed in [14] of the elliptic Cherednik algebra arise from the flat connections valued in the rational Cherednik algebra $H_{0,c}$.

Let $T = P^∨ \otimes E_r$ be the abelian variety and $\mathcal{F}$ be the sheaf on $T$ considered in [14, Section 5]. The sheaf $\mathcal{F}$ can be identified with the vector bundle $h_\mathbb{C} \times_{P^∨ \otimes (\mathbb{Z}+\mathbb{Z})} \mathbb{C}W$ on $T$, whose fiber is the regular representation of $W$. We construct an action of the rational Cherednik algebra $H_{0,c}$ on the fiber $\mathbb{C}W$. Such action induces a flat connection on the vector bundle $h_\mathbb{C} \times_{P^∨ \otimes (\mathbb{Z}+\mathbb{Z})} \mathbb{C}W$, see Section §9.

**Proposition 11.1.** The flat connection of Section §9 specialized on the vector bundle $h_\mathbb{C} \times_{P^∨ \otimes (\mathbb{Z}+\mathbb{Z})} \mathbb{C}W$ coincides with the elliptic Dunkl operator

$$\nabla = d - \sum_{w \in W} \sum_{\alpha \in \Phi^+} \frac{2c_{\alpha}}{(\alpha(\alpha) \theta(\alpha)\theta(\alpha^∨(wp)))} s_\alpha d\alpha$$

in [5] and [14].

11.1. The vector bundle on $T$. In this subsection, we recall the construction of elliptic Dunkl operators in [14]. Let $T = P^∨ \otimes E_r$ be the abelian variety. In [14], the sheaf $\mathcal{F}$ on $T$ is defined as follows. Choose $\rho \in h^*$, such that $\rho$ is not fixed by any $w \in W$. Consider the trivial rank 1 bundle $h \times \mathbb{C}$ on $h$. We define the $P^∨ \otimes \tau P^∨$ action on $h \times \mathbb{C}$ by

$$\lambda_i^\vee : (z, \xi) \mapsto (z + \lambda_i^\vee, \xi), \quad \tau^\vee \lambda_i^\vee : (z, \xi) \mapsto (z + \tau \lambda_i^\vee, \exp(-2\pi i \rho(\lambda_i^\vee))\xi).$$

Here, we assume $\text{Im}(\tau) = 1$. Denote by $L_w$ the line bundle on $T$ which is the quotient of $h \times \mathbb{C}$ by the group $P^∨ \otimes \tau P^∨$ action. We use the notation as in [14] that $\mathcal{L}_w := L_w$, for $w \in W$. The sheaf $\mathcal{F} = \oplus_{w \in W} L_w$ is defined by taking direct sum of $L_w$ for all $w \in W$. Labelling each fiber of $L_w$ by $C_w$, we identify the vector bundle $\mathcal{F}$ with the vector bundle $h_\mathbb{C} \times_{P^∨ \otimes (\mathbb{Z}+\mathbb{Z})} \mathbb{C}W$ with rank $|W|$. Under this identification, the action of $P^∨ \otimes \tau P^∨$ on the fiber $\mathbb{C}W$ is given by

$$\lambda_i^\vee : (z, \xi) \mapsto (z + \lambda_i^\vee, \xi), \quad \text{and} \quad \tau^\vee \lambda_i^\vee : (z, \xi) \mapsto (z + \tau \lambda_i^\vee, \exp(-2\pi i \rho(\lambda_i^\vee))\xi),$$

where $\xi = (\xi_w)$, for $\xi_w \in C_w$.

11.2. The action of the rational Cherednik algebra $H_{0,c}$. In this subsection, we define the action of rational Cherednik algebra $H_{0,c}$ on the fiber $\mathbb{C}W$, such that the element $\lambda_i^\vee \in h$ acts by multiplication of $wp(\lambda_i^\vee)$.

We first recall some facts of the rational Cherednik algebra $H_{0,c}$, see [12, 13] for details. By the Sataka isomorphism, the center $Z_{0,c}$ of $H_{0,c}$ is isomorphic to the spherical subalgebra $\mathbb{C}H_{0,c}$, where

$$e = \frac{1}{|W|} \sum_{w \in W} w \in \mathbb{C}W$$

is the idempotent element. Any irreducible representation of $H_{0,c}$ as a representation of $W$ is isomorphic to the regular representation $\mathbb{C}W$. In particular, the dimension is $|W|$. The moduli space of irreducible representations of $H_{0,c}$ is the Calogero-Moser space $\text{Spec}(Z_{0,c})$.

We construct an action of $H_{0,c}$ on $\mathbb{C}W$. This following action is a generalization of type $A$ case in [13, Section 9.6]. As before, we choose $\rho \in h$ such that $\rho$ is not fixed by any $w \in W$. Let $E = E_{\rho,\mu}$ be the space of the complex valued functions on the $W$ orbit of $(\rho, \mu) \in h \times h^*$. As a representation of the Weyl group $W$, $E_{\rho,\mu}$ is isomorphic to the regular representation of $W$. 

We define the action of $H_{0,c}$ on $E_{\rho,\mu}$ as follows. For $x \in \mathfrak{h}^*$, and $y \in \mathfrak{h}$,

$$x \cdot F(a, b) = (x, a)F(a, b),$$

$$y \cdot F(a, b) = (y, b)F(a, b) + \sum_{a \in \Phi^+} c_a \frac{\alpha(y)}{\alpha(a)} (1 - s_a)F(a, b).$$

We now check that it is a well-defined action. We check the action respects the relation $[y, y'] = 0$. We have

$$yy'F(a, b) = (y, b)(y', b)F(a, b) + \sum_{a \in \Phi^+} c_a \frac{\alpha(y')}{\alpha(a)} (1 - s_a)(y', b)F(a, b)$$

$$+ \sum_{\beta, a \in \Phi^*} \beta(y) \sum_{\beta, a \in \Phi^*} \frac{\beta(y)}{\beta(a)} (1 - s_\beta) \frac{\alpha(y)}{\alpha(a)} (1 - s_a)F(a, b)$$

$$= (y, b)(y', b)F(a, b) + \sum_{a \in \Phi^+} c_a \frac{\alpha(y)}{\alpha(a)} ((y', b)F(a, b) - (y', b)F(s_\alpha a, s_\alpha b))$$

$$+ \sum_{\beta, a \in \Phi^*} \beta(y) \frac{\alpha(y')}{\alpha(a)} (1 - s_\beta)(1 - s_a)F(a, b).$$

Therefore, the action of the commutator $[y, y']$ is given by

$$[y, y']F(a, b) = - \sum_{a \in \Phi^+} c_a \frac{\alpha(y)}{\alpha(a)} ((y', b)F(s_\alpha a, s_\alpha b)) - \sum_{a \in \Phi^+} c_a \frac{\alpha(y')}{\alpha(a)} ((y, b)F(s_\alpha a, s_\alpha b))$$

$$+ \sum_{a \in \Phi^+} c_a \frac{\alpha(y)}{\alpha(a)} ((y, b)F(s_\alpha a, s_\alpha b)) + \sum_{a \in \Phi^+} c_a \frac{\alpha(y')}{\alpha(a)} ((y, b)F(s_\alpha a, s_\alpha b))$$

$$= - \sum_{a \in \Phi^+} c_a \frac{\alpha(y)}{\alpha(a)} \alpha(y')(b, \alpha^\vee)F(s_\alpha a, s_\alpha b) + \sum_{a \in \Phi^+} c_a \frac{\alpha(y')}{\alpha(a)} \alpha(y)(b, \alpha^\vee)F(s_\alpha a, s_\alpha b)$$

$$= 0.$$

We then check the action respects the relation $[y, x] = - \sum_{\alpha \in \Phi^+} c_\alpha \alpha(y)\alpha^\vee(x) s_\alpha$. We have

$$[x, y]F(a, b) = \sum_{a \in \Phi^+} c_a \frac{\alpha(y)}{\alpha(a)} ((x, a)F(a, b) - (x, s_\alpha a)F(s_\alpha a, s_\alpha b)) - \sum_{a \in \Phi^+} c_a \frac{\alpha(y)}{\alpha(a)} ((x, a)F(a, b) - (x, a)F(s_\alpha a, s_\alpha b))$$

$$= \sum_{a \in \Phi^+} c_a \frac{\alpha(y)}{\alpha(a)} \alpha^\vee(x) F(s_\alpha a, s_\alpha b) = \sum_{\alpha \in \Phi^+} c_\alpha \alpha(y)\alpha^\vee(x) s_\alpha F(a, b).$$

It is straightforward to show that the action respects to other defining relations of $H_{0,c}$.

Therefore, we have an irreducible representation $\mathbb{C}W$ of the rational Cherednik algebra $H_{0,c}$. It has the following properties. As a representation of the Weyl group $W$, it is isomorphic to the regular representation of $W$. And the element $\lambda^\vee_{\gamma} \in \mathfrak{h}$ acts by multiplication of $w_\rho(\lambda^\vee_{\gamma})$. 
11.3. The comparison. The KZB connection valued in \( H_{\hbar,c} \) in Section §9 can be simplified as follows, when \( \hbar = 0 \).

\[
\nabla = d - \sum_{\alpha \in \Phi^+} k(\alpha, \text{ad}(\frac{\alpha}{2})\tau)_{t_\alpha} d\alpha + \sum_{i=1}^{n} y(u^i) du_i
\]

\[
= d - \sum_{\alpha \in \Phi^+} \frac{\theta(\alpha + \text{ad}(\frac{\alpha^\vee}{2}))_{t_\alpha} d\alpha}{\theta(\alpha)(\text{ad}(\frac{\alpha^\vee}{2}))} + \sum_{i=1}^{n} \left( \sum_{\alpha \in \Phi^+} \frac{\alpha(u^i)}{\text{ad}(\frac{\alpha^\vee}{2})} t_\alpha + y(u^i) \right) du_i
\]

\[
= d - \sum_{\alpha \in \Phi^+} \frac{\theta(\alpha + \text{ad}(\frac{\alpha^\vee}{2}))_{t_\alpha} d\alpha}{\theta(\alpha)(\text{ad}(\frac{\alpha^\vee}{2}))}.
\]  
(45)

The equality (45) follows from the following computations. For any \( \nu \in \mathfrak{h} \), we apply \([x(\nu), -]\) to the term \( \sum_{\alpha \in \Phi^+} \frac{\alpha(u^\nu)}{\text{ad}(\frac{\alpha^\vee}{2})} t_\alpha + y(u^\nu) \). Using the relation of \( H_{\hbar,c} \), we have \([x(\nu), y(u^\nu)] = -\sum_{\alpha \in \Phi^+} (\alpha, \nu)(\alpha, u^\nu)t_\alpha \). On the other hand, we have

\[
[x(\nu), \sum_{\alpha \in \Phi^+} \frac{\alpha(u^\nu)}{\text{ad}(\frac{\alpha^\vee}{2})} t_\alpha] = \sum_{\alpha \in \Phi^+} \frac{(\alpha, u^\nu)}{\text{ad}(\frac{\alpha^\vee}{2})} [x(\nu), t_\alpha] = \sum_{\alpha \in \Phi^+} \frac{(\alpha, u^\nu)(\alpha, \nu)}{\text{ad}(\frac{\alpha^\vee}{2})} t_\alpha = \sum_{\alpha \in \Phi^+} (\alpha, u^\nu)(\alpha, \nu),
\]

where the second equality is obtained by decomposing the vector \( \nu \) as \( \nu = \nu_{\parallel} + \nu_{\perp} \), where \( \nu_{\parallel} = (\alpha, \nu)\frac{\alpha^\vee}{2} \), and we have the relation \([x(\nu_{\parallel}), t_\alpha] = 0 \), since \((\nu_{\parallel}, \alpha) = 0 \). This implies (45).

We specialize the connection (45) to the vector bundle \( \mathfrak{h} \times_{P_{\nu}\otimes \Lambda_\nu} \mathbb{C}W \). Recall that the action of \( \Lambda_\nu^\nu \in \mathfrak{h} \) is given by multiplication of \( w_p(\Lambda_\nu^\nu) \). This gives the \( W \)-equivariant flat connection on the bundle \( \mathfrak{h} \times_{P_{\nu}\otimes \Lambda_\nu} \mathbb{C}W \).

It takes the following form

\[
\nabla = d - \sum_{w \in W} \sum_{\alpha \in \Phi^+} \frac{2c_\alpha}{(\alpha)(\text{ad}(\alpha^\vee(w_p)))} s_\alpha d\alpha.
\]

This is exactly the elliptic Dunkl operator in [5] and [14]. The flat connection defines a representation of \( W \times D_{T_{\text{reg}}} \) on the bundle \( \mathcal{F}|_{T_{\text{reg}}} \), where \( D_{T_{\text{reg}}} \) is ring of differential operators on \( T_{\text{reg}} \). This restricts to the action of the elliptic Cherednik algebra defined in [14].

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