ASYMPTOTICS OF MOORE EXPONENT SETS

DANIELE BARTOLI\textsuperscript{1} AND YUE ZHOU\textsuperscript{2}\textsuperscript{†}

Abstract. Let $n$ be a positive integer and $I$ a $k$-subset of integers in $[0, n-1]$. Given a $k$-tuple $A = (\alpha_0, \cdots, \alpha_{k-1}) \in \mathbb{F}_q^k$, let $M_{A,I}$ denote the matrix $(\alpha_q^j)$ with $0 \leq i \leq k-1$ and $j \in I$. When $I = \{0, 1, \cdots, k-1\}$, $M_{A,I}$ is called a Moore matrix which was introduced by E. H. Moore in 1896. It is well known that the determinant of a Moore matrix equals 0 if and only if $\alpha_0, \cdots, \alpha_{k-1}$ are $\mathbb{F}_q$-linearly dependent. We call $I$ that satisfies this property a Moore exponent set. In fact, Moore exponent sets are equivalent to maximum rank-distance (MRD) code with maximum left and right idealisers over finite fields. It is already known that $I = \{0, 1, \cdots, k-1\}$ is not the unique Moore exponent set, for instance, (generalized) Delsarte-Gabidulin codes and the MRD codes recently discovered in [5] both give rise to new Moore exponent sets. By using algebraic geometry approach, we obtain an asymptotic classification result: for $q > 5$, if $I$ is not an arithmetic progression, then there exist an integer $N$ depending on $I$ such that $I$ is not a Moore exponent set provided that $n > N$.

1. Introduction

Let $q$ be a prime power and $n$ a positive integer. For a given $k$-tuple $A := (\alpha_0, \alpha_1, \cdots, \alpha_{k-1}) \in \mathbb{F}_q^k$, $k \leq n$, a square Moore matrix is defined as

$$M_A := \begin{pmatrix}
\alpha_0 & \alpha_0^q & \cdots & \alpha_0^{q^{k-1}} \\
\alpha_1 & \alpha_1^q & \cdots & \alpha_1^{q^{k-1}} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{k-1} & \alpha_{k-1}^q & \cdots & \alpha_{k-1}^{q^{k-1}}
\end{pmatrix},$$

which is a $q$-analog of the Vandermonde matrix introduced by Moore [14]. The determinant of $M_A$ can be expressed as

$$\det(M_A) = \prod_{c}(c_0\alpha_0 + c_1\alpha_1 + \cdots + c_{k-1}\alpha_{k-1}),$$

1\textsuperscript{Department of Mathematics and Computer Science, University of Perugia, 06123 Perugia, Italy}
2\textsuperscript{Department of Mathematics, National University of Defense Technology, 410073 Changsha, China}
\textsuperscript{†Corresponding Author}
E-mail addresses: daniele.bartoli@unipg.it, yue.zhou.ovgu@gmail.com.
Date: July 26, 2019.
2010 Mathematics Subject Classification. 15A15, 14G50, 51E22.
Key words and phrases. Moore matrix; Maximum rank-distance code; Finite geometry; Hasse-Weil bound.
where \( c = (c_0, c_1, \ldots, c_{k-1}) \) runs over all direction vectors in \( \mathbb{F}_q^k \), or equivalently we can say that \( c \) runs over \( \text{PG}(k-1, q) \). In other words,

\[
\det(M_A) = 0 \quad \text{if and only if} \quad \alpha_0, \ldots, \alpha_{k-1} \text{ are } \mathbb{F}_q\text{-linearly dependent.}
\]

We call \( \det(M_A) \) the Moore determinant.

We may replace the exponents of those elements in \( M_A \) in the following way: For \( I = \{i_0, i_1, \ldots, i_{k-1}\} \subseteq \mathbb{Z}_{\geq 0} \) and \( A = (\alpha_0, \alpha_1, \ldots, \alpha_{k-1}) \in \mathbb{F}_q^k \), define

\[
M_{A,I} := \begin{pmatrix}
\alpha_0^{q^{0i_0}} & \alpha_0^{q^{1i_1}} & \cdots & \alpha_0^{q^{(k-1)i_{k-1}}} \\
\alpha_1^{q^{0i_0}} & \alpha_1^{q^{1i_1}} & \cdots & \alpha_1^{q^{(k-1)i_{k-1}}} \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_{k-1}^{q^{0i_0}} & \alpha_{k-1}^{q^{1i_1}} & \cdots & \alpha_{k-1}^{q^{(k-1)i_{k-1}}}
\end{pmatrix},
\]

Besides \( I = \{0, 1, \ldots, k-1\} \), it is interesting to ask whether there exist other \( I \) sharing the same property \( \text{[1]} \). Namely we would like to investigate the following research question.

**Question 1.** Determine the value of \( q, n \) and \( I \) such that \( \det(M_{(\alpha_0, \ldots, \alpha_{k-1}), I}) = 0 \) if and only if \( \alpha_0, \ldots, \alpha_{k-1} \) are \( \mathbb{F}_q\text{-linearly dependent for all } k\text{-tuples } (\alpha_0, \ldots, \alpha_{k-1}) \in \mathbb{F}_q^k \).

For given \( q \) and \( n \), if \( I \) is such that the condition in Question \( \text{[1]} \) holds, then we say \( I \) is a Moore exponent set for \( q \) and \( n \).

Question \( \text{[1]} \) is strongly related to maximum rank-distance codes which are usually abbreviated to MRD codes. MRD codes have important applications in network coding and strong connections to semifield planes and linear sets in finite geometry; see \( \text{[9]} \) for a recent survey on them. It is already known that there are a huge number of inequivalent MRD codes consisting of \( m \times n \) matrices over finite fields with \( m - 1 < n \); see \( \text{[16]} \). However, there are only a few families of known MRD codes with \( m = n \). In this case, every MRD code over \( \mathbb{F}_q \) can be equivalently written as a set of \( q \)-polynomials. In particular, \( I = \{i_1, i_2, \ldots, i_{k-1}\} \) is a Moore exponent set for \( q \) and \( n \) if and only if the set of \( q \)-polynomials

\[
C = \left\{ a_0X^{q^{i_0}} + a_1X^{q^{i_1}} + \cdots + a_{k-1}X^{q^{(k-1)i_{k-1}}}: a_0, \ldots, a_{k-1} \in \mathbb{F}_q^n \right\}
\]

defines an MRD code in \( \mathbb{F}_q^{n \times n} \), i.e. each nonzero polynomial \( f \in C \) has at most \( q^k \) roots. The MRD code \( C \) associated with \( I \) has a special property: its right and left idealisers are both maximum; see \( \text{[10]} \) \( \text{[12]} \) for details of the right (left) idealisers of MRD codes. For more details on this special type of MRD codes, see \( \text{[2]} \). We refer to \( \text{[8]} \text{[6]} \text{[11]} \text{[13]} \text{[17]} \text{[18]} \text{[20]} \) for recent constructions of MRD codes and its link with finite geometries.

It is easy to see that \( I \) is a Moore exponent set if and only if \( I + s = \{i + s : i \in I\} \) is so, whence we may always assume that the smallest element in \( I \) is 0. Besides \( I = \{0, 1, \ldots, k-1\} \), there are other known examples of Moore exponent sets.

- \( I = \{0, 1, 3\} \) for \( n = 7 \) with odd \( q \);
- \( I = \{0, 1, 3\} \) for \( n = 8 \) with \( q \equiv 1 \pmod{3} \);
- \( I = \{0, d, \ldots, (k-1)d\} \) for any \( n \) satisfying \( \gcd(d, n) = 1 \)

The first two cases have been discovered recently in \( \text{[5]} \). The last case is equivalent to the so-called Delsarte-Gabidulin code (sometimes also called a Generalized Gabidulin code \( \text{[9]} \)).
It appears illusive to answer Question 1 by giving a complete list of Moore exponent sets. Instead, we would like to present an asymptotic answer in this paper which also implies an asymptotic classification of MRD codes with maximum left and right idealisers.

**Theorem 1.1.** Assume that $I$ is not an arithmetic progression and $q > 5$. Then there exist an integer $N$ depending only on $I$ such that $I$ is not a Moore exponent set for $q$ and $n$ provided that $n > N$.

In fact, for $q \leq 5$, we can get the same result for almost each $I$ which is not an arithmetic progression. The precise conditions on $I$ and $q$ are presented in the following theorem, from which one can directly derive Theorem 1.1. The main idea is to translate the determination of Moore exponent sets into an algebraic geometry problem.

**Theorem 1.2.** Assume that $I$ is not an arithmetic progression. Define $G_k : G_k(X_1, \ldots, X_k) = 0$ and $V_I : F_I(X_1, \ldots, X_k) = 0$, where

$$F_I(X_1, \ldots, X_k) = \det \begin{pmatrix} X_1^{q_i} & X_1^{q_1} & \cdots & X_1^{q^{k-1}} \\ X_2^{q_i} & X_2^{q_1} & \cdots & X_2^{q^{k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ X_k^{q_i} & X_k^{q_1} & \cdots & X_k^{q^{k-1}} \end{pmatrix},$$

and

$$G_k(X_1, \ldots, X_k) = \det \begin{pmatrix} X_1 & X_1^{q_1} & \cdots & X_1^{q^{k-1}} \\ X_2 & X_2^{q_1} & \cdots & X_2^{q^{k-1}} \\ \vdots & \vdots & \ddots & \vdots \\ X_k & X_k^{q_1} & \cdots & X_k^{q^{k-1}} \end{pmatrix}.$$  

Suppose that one of the following collections of conditions is satisfied.

(a) $i_2 - i_0 \neq 2(i_1 - i_0)$;
(b) $i_2 - i_0 = 2(i_1 - i_0)$, $k > 3$ and $q \geq 7$;
(c) $i_2 - i_0 = 2(i_1 - i_0)$, $k > 3$, $q = 4, 5$ and $i_1 - i_0 > 1$;
(d) $i_2 - i_0 = 2(i_1 - i_0)$, $k > 3$ and $q = 3$ with $i_1 - i_0 > 2$.

There exists an integer $N$ such that $V_I$ contains an $\mathbb{F}_{q^n}$-rational absolutely irreducible component and at least one $\mathbb{F}_{q^n}$-rational points not in $G_k$ provided that $n > N$.

The exact value of $N$ in Theorem 1.2 will be provided in Theorems 3.2 and 3.1.

The rest parts of this paper are organized as follows. In Section 2 we introduce some tools and results from algebraic geometry; in Section 3 we investigate the curve case of Theorem 1.2; finally in Section 4 we consider the general case of Theorem 1.2 and present a complete proof.

## 2. Preliminaries

To prove Theorem 1.2 we have to convert the original question into a problem of algebraic varieties over finite fields. In this section, we introduce some tools from algebraic geometry which will be used in the later parts.
The first one is a standard result on non-absolutely irreducible curves which can be found in \cite[Lemma 10]{7}.

**Lemma 2.1.** Let $F \in \mathbb{F}_q[X_1, \ldots, X_m]$ be a polynomial of degree $d$, irreducible over $\mathbb{F}_q$. Then there exists a natural number $s \mid d$ such that, over its splitting field, $F$ splits into $s$ absolutely irreducible polynomials, each of degree $d/s$.

**Lemma 2.2.** \cite[Lemma 2.1]{7} Let $H$ be a projective hypersurface and $X$ a projective variety of dimension $n-1$ in $\text{PG}(n, q)$. If $X \cap H$ has a reduced absolutely irreducible component defined over $\mathbb{F}_q$, then $X$ has a reduced absolutely irreducible component defined over $\mathbb{F}_q$.

Concerning the intersection number of two curves at a point, we need the following classical result which can be found in most of the textbooks on algebraic curves.

**Theorem 2.3** (Bézout’s Theorem). Let $A$ and $B$ be two projective plane curves over an algebraically closed field $\mathbb{K}$, having no component in common. Let $A$ and $B$ be the polynomials associated with $A$ and $B$ respectively. Then

$$\sum_P I(P, A \cap B) = (\deg A)(\deg B),$$

where the sum runs over all points in the projective plane $\text{PG}(2, \mathbb{K})$.

We also need the following results to estimate the intersection number, which is not difficult to prove (see Janwa, McGuire, and Wilson \cite[Proposition 2]{8}).

**Lemma 2.4.** Let $F$ be a polynomial in $\mathbb{F}_q[X, Y]$ and suppose that $F = AB$. Let $P = (u, v)$ be a point in the affine plane $\text{AG}(2, q)$ and write

$$F(X + u, Y + v) = F_m(X, Y) + F_{m+1}(X, Y) + \cdots,$$

where $F_i$ is zero or homogeneous of degree $i$ and $F_m \neq 0$. Let $L$ be a linear polynomial and suppose that $F_m = L^m$ and $L \mid F_{m+1}$. Then $I(P, A \cap B) = 0$, where $A$ and $B$ are the curves defined by $A$ and $B$ respectively.

The next result was proved in \cite[Lemma 4.3]{15} for $q$ even case. Actually it still holds when $q$ is odd and its proof is the same.

**Lemma 2.5.** Let $F$ be a polynomial in $\mathbb{F}_q[X, Y]$ and suppose that $F = AB$. Let $P = (u, v)$ be a point in the affine plane $\text{AG}(2, q)$ and write

$$F(X + u, Y + v) = F_m(X, Y) + F_{m+1}(X, Y) + \cdots,$$

where $F_i$ is zero or homogeneous of degree $i$ and $F_m \neq 0$. Let $L$ be a linear polynomial and suppose that $F_m = L^m$, $L \mid F_{m+1}$, $L^2 \mid F_{m+1}$. Then $I(P, A \cap B) = 0$ or $m$, where $A$ and $B$ are the curves defined by $A$ and $B$ respectively.

An algebraic hypersurface defined over a field $\mathbb{K}$ is **absolutely irreducible** if the associated polynomial is irreducible over every algebraic extension of $\mathbb{K}$. An absolutely irreducible $\mathbb{K}$-rational component of a hypersurface $\mathcal{V}$, defined by the polynomial $F$, is simply an absolutely irreducible hypersurface such that the associated polynomial has coefficients in $\mathbb{K}$ and it is a factor of $F$.

**Theorem 2.6** (Hasse-Weil Theorem). For an absolutely irreducible curve $C$ in $\text{PG}(2, q)$, then

$$|\#C(\mathbb{F}_q) - q - 1| \leq (d - 1)(d - 2)\sqrt{q},$$

where $d$ is the degree of the defining polynomial for $C$. 
We also need two results concerning the number of rational points on an absolutely irreducible hypersurface.

**Theorem 2.7.** [21, Theorem 2] Let $G$ be an absolutely irreducible hypersurface of degree $f$ over $\mathbb{F}_q$, and $H$ a hypersurface of degree $e$ over $\mathbb{F}_q$ not divisible by $G$. Then provided that
\[ q > \frac{1}{4} \left( \alpha + \sqrt{\alpha^2 + 4\beta} \right)^2 \]
where $\alpha = (f - 1)(f - 2)$ and $\beta = 5f^{13/3} + f(f + e - 1)$, there is a nonsingular point of $G$ that is not a point of $H$.

**Theorem 2.8.** [21, Theorem 3] Let $F$ be an absolutely irreducible hypersurface of degree $f$ over $\mathbb{F}_q$. Then provided that
\[ q > \frac{3f^4 - 4f^3 + 5f^2}{2} \]
there is a nonsingular point of $F$.

**Lemma 2.9.** Let $S$ be a hypersurface containing $O = (0,0,\ldots,0)$ of the affine equation $F(X_1, \ldots, X_n) = 0$, where
\[ F(X_1, \ldots, X_n) = F_d(X_1, \ldots, X_n) + F_{d+1}(X_1, \ldots, X_n) + \cdots, \]
with $F_d$ the homogeneous part of the smallest degree $d$ of $F(X_1, \ldots, X_n)$. Let $P$ be an $\mathbb{F}_q$-rational simple point of the variety
\[ F_d(X_1, X_2, \ldots, X_{n-1}, X_n) = 0. \]
Then there exists an $\mathbb{F}_q$-rational plane $\pi$ through the line $\ell$ joining $O$ and $P$ such that $\pi \cap S$ has $\ell$ as non-repeated tangent $\mathbb{F}_q$-rational line at the origin.

**Proof.** Without loss of generality we can suppose that $P = (0,0,\ldots,0,1)$. This means that
\[ F_d(X_1, \ldots, X_n) = X_n^{d-1} \left( \sum_{i=1}^{n-1} \alpha_i X_i \right) + \cdots, \]
with at least one of the $\alpha_i$’s different from 0. Hence there exists at least one ($n-2$)-tuple $(\lambda_2, \lambda_3, \ldots, \lambda_{n-1}) \in \mathbb{F}_q^{n-2}$ such that the line $m$ given by $\{(t, \lambda_2 t, \ldots, \lambda_{n-1} t, 1) : t \in \mathbb{F}_q\}$ intersects the variety $F_d(X_1, \ldots, X_{n-1}, 1) = 0$ with multiplicity 1 at $P$. This means that
\[ A = \alpha_1 + \sum_{i=2}^{n-1} \alpha_i \lambda_i \neq 0. \]

Let $\pi$ be the plane generated by $m$ and $O$. Then $\pi$ is the set of points
\[ \{(t, \lambda_2 t, \ldots, \lambda_{n-1} t, u) : t, u \in \mathbb{F}_q\}. \]
The intersection between $\pi$ and $S$ is given by
\[ F(X, \lambda_2 X, \ldots, \lambda_{n-1} X, Y) = F_d(X, \lambda_2 X, \ldots, \lambda_{n-1} X, Y) + \cdots = AY^{d-1} X + \cdots \]
This shows that $X \parallel F(X, \lambda_2 X, \ldots, \lambda_{n-1} X, Y)$, which means that the line $X = 0$ in the plane $\pi$ is a non-repeated tangent line at the origin for the $\mathbb{F}_q$-rational curve $\pi \cap S$. \[ \square \]
The next result can be simply proved by counting argument. It tells us the number of $\mathbb{F}_{q^n}$-rational points in $G_m$.

**Lemma 2.10.** Let $m \leq n$ be two positive integers. The total number of points $(x_1, x_2, \cdots, x_m) \in PG(m-1, q^n)$ such that $x_i$'s are linearly dependent equals $q^{n(m-1)} - (q^n - q)(q^n - q^2) \cdots (q^n - q^{m-1}) + q^{n(m-1) - 1} \cdot \frac{m!}{m! - 1}$. 

3. CURVES

Let $i, j$ be positive integers such that $j > i$ and consider $I = \{0, i, j\}$. Let $G_3$ and $V_I$ be the curves of the affine equations $G_3(X, Y, T) = 0$ and $F_I(X, Y, T)/G_3(X, Y, T) = 0$, where $F_I$ and $G_3$ are as in (2) and (3). Note that $G_3$ coincides with the set of points in $PG(2, \mathbb{F}_q)$ lying on the union of all lines defined over $\mathbb{F}_q$.

**Theorem 3.1.** Assume that $\gcd(i, j) = 1$ and $j > 2$. The curve $V_I$ is absolutely irreducible and the set of singular points of $V_I$ is either $PG(2, q^{j-1})$ or $PG(2, q^{j-1}) \setminus PG(2, q)$, in which the latter case happens if and only if $i = 1$. Moreover

$$V_I \cap G_3 = \begin{cases} (PG(2, q^k) \setminus PG(2, q)) \cap G_3, & i = 1; \\ PG(2, q^j) \cap G_3, & \text{otherwise}. \end{cases}$$

By Lemma 2.10 we have

$$\#(PG(2, q^k) \cap G_3) = (q^k - q + 1) \frac{q^3 - 1}{q - 1}.$$ 

Hence

$$\#(V_I \cap G_3) = \begin{cases} (q^{j-1} - q) \frac{q^3 - 1}{q - 1}, & i = 1; \\ (q^{j-i} - q + 1) \frac{q^3 - 1}{q - 1}, & \text{otherwise}. \end{cases}$$

By the Hasse-Weil theorem (see Theorem 2.6), the number of $\mathbb{F}_{q^n}$-rational points of $V_I$ satisfies

$$\#V_I(\mathbb{F}_{q^n}) \geq q^n + 1 - (\ell - 1)(\ell - 2)\sqrt{q^n}$$

$$\geq q^n + 1 - q^{2j+n/2} - 2q^{j+i+n/2} - q^{2i+n/2},$$

where $\ell = q^j + q^i - q^2 - q$ is the degree of $\frac{F_I}{G_3}$.

When $\gcd(i, j) = 1$, we can derive that

$$\#V_I(\mathbb{F}_{q^n}) > \#(V_I \cap G_3)$$

provided $n > 4j + 2$.

When $\gcd(i, j) = d$ and $j \neq 2i$, $V_I$ has two components

$$\frac{F_I(X, Y, T)}{G_3(X, Y, T)} = \frac{F_I(X, Y, T)}{H_d(X, Y, T)} \frac{H_d(X, Y, T)}{G_3(X, Y, T)},$$

where

$$H_d(X, Y, T) = \begin{vmatrix} X & Xq^d & Xq^{2d} \\ Y & Yq^d & Yq^{2d} \\ T & Tq^d & Tq^{2d} \end{vmatrix}.$$ 

Suppose that $i = i'd$ and $j = j'd$. Let $C'$ and $H$ be the curves defined by $\frac{F_I(X, Y, T)}{H_d(X, Y, T)} = 0$ and $H_d(X, Y, T) = 0$, respectively. By Theorem 3.1 $C'$ is irreducible. Here we are just considering Theorem 3.1 on $q' = q^d$ with exponents $i'$ and $j'$. It is obvious that $G_3$ is a component of $H$. 
Theorem 4.1. The degree of $\frac{E(X,Y,T)}{n_2(X,Y,T)}$ is $\ell' = q^i + q^j - q^{2d} - q^i$. By the Hasse-Weil bound, we have

\[
\#\mathcal{C}'(F_q^n) \geq q^n + 1 - (\ell' - 1)(\ell' - 2)\sqrt{q^n} \\
\geq q^n + 1 - q^{2i+n/2} - 2q^{i+n/2} - q^{2i+n/2},
\]

which is the same as the lower bound of $\#V_I(F_q^n)$ obtained in (4).

Therefore, one of the following two conditions implies that $\#V_I(F_q^n) \geq \#\mathcal{C}'(F_q^n) > \#(V_I \cap G_3)$.

- $\#\mathcal{C}'(F_q^n) > \#(\mathcal{C}' \cap \mathcal{H})$ which holds if $n > 4j + 2$;
- $G_3(F_q^n) \subseteq \mathcal{H}(F_q^n)$ which holds if $\gcd(n,d) > 1$.

Therefore we have proved the following result.

Theorem 3.2. Let $i, j$ be two positive integer such that $j > i$ and $j \neq 2i$. For integer $n$ satisfying $n > 4j + 2$ or $\gcd(n,i,j) > 1$ and any prime power $q$, $\{0,i,j\}$ is not a Moore exponent set.

Remark 3.3. The lower bound on $n$ in Theorem 3.2 holds for all prime power $q$. When $q$ or the gap between $j$ and $i$ is large enough, one may get a better lower bound $n > 4j$.

4. General Case

In this section, we investigate the general case of Theorem 1.2 and we prove the following.

Theorem 4.1. Suppose that $k > 3$ and $I = \{0,i_1,i_2,\ldots,i_{k-1}\}$ is not an arithmetic progression. Assume that one of the following collections of conditions hold.

(a) $i_2 \neq 2i_1$;
(b) $i_2 = 2i_1$ and $q \geq 7$;
(c) $i_2 = 2i_1$, $q = 4, 5$ and $i_1 > 1$;
(d) $i_2 = 2i_1$, $q = 3$ with $i_1 > 2$.

For $n > \frac{13}{4}i_{k-1} + 2$, $V_I$ contains a simple $F_{q^n}$-rational point which is not contained in $G_k$ (see (2) and (3)), whence $I$ is not a Moore exponent set.

Depending on whether $i_2 = 2i_1$, we separate the proof of the existence of an $F_{q^n}$-rational absolutely irreducible component of $V_I$ into two parts.

Theorem 4.2. Let $I = \{0,i_1,\ldots,i_k\}$ be a set of positive integers satisfying $i_1 < \cdots < i_k$. Let $F_I(X_1,\ldots,X_k,1)$ and $G_{k+1}(X_1,\ldots,X_k,1)$ be as in (2) and (3).

Suppose that $i_2 \neq 2i_1$ and that $n > 4i_{k-1} + 2$. Then the affine hypersurface $V_I$ of the affine equation $\frac{F_I(X_1,X_2,\ldots,X_k,1)}{G_{k+1}(X_1,X_2,\ldots,X_k,1)} = 0$ contains an $F_{q^n}$-rational absolutely irreducible component.

Proof. We prove the existence of an $F_{q^n}$-rational absolutely irreducible component by induction on $k$.

First let us consider the case $k = 3$. Let $d_1 = q^{i_2} + q^{i_1} + 1$ and $d_2 = q^2 + q + 1$. The homogeneous parts of the smallest degrees of $F_I$ and $G_4$ are

\[
\Phi_{d_1}(X_1, X_2, X_3) = \begin{vmatrix}
X_1 & X_1^{q^{i_1}} & X_1^{q^{i_2}} \\
x_2 & X_2^{q^{i_1}} & X_2^{q^{i_2}} \\
x_3 & X_3^{q^{i_1}} & X_3^{q^{i_2}}
\end{vmatrix}
\quad \text{and} \quad
\Gamma_{d_2}(X,Y,Z) = \begin{vmatrix}
X_1 & X_1^{q} & X_1^{q^{2}} \\
x_2 & X_2^{q} & X_2^{q^{2}} \\
x_3 & X_3^{q} & X_3^{q^{2}}
\end{vmatrix}.
\]
respectively.

Let \( d_1 = d_2 - d_1 \) and let \( \Psi_{d_1}(X_1, X_2, X_3) \) be the homogeneous part of the smallest degree of the polynomial \( H(X_1, X_2, X_3) = \frac{F_3(X_1, X_2, X_3)}{G_3(X_1, X_2, X_3)} \). Then \( \Phi_{d_1}(X_1, X_2, X_3) = \Gamma_{d_2}(X_1, X_2, X_3) \Psi_{d_3}(X_1, X_2, X_3) \), that is the tangent cone at \( O = (0, 0, 0) \) of \( \mathcal{V}_l \) is given by

\[
\Psi_{d_3}(X_1, X_2, X_3) = \begin{vmatrix}
X_1 X_1^{q_1} X_1^{d_2} \\
X_2 X_2^{q_1} X_2^{d_2} \\
X_3 X_3^{q_1} X_3^{d_2}
\end{vmatrix} = \begin{vmatrix}
X_1 X_1^{q_1} X_1^{d_2} \\
X_2 X_2^{q_1} X_2^{d_2} \\
X_3 X_3^{q_1} X_3^{d_2}
\end{vmatrix},
\]

where \( d = \gcd(i_1, i_2) \) and we denote the first component by \( C(X_1, X_2, X_3) \).

By Theorem 4.1, the curve defined by \( C(X_1, X_2, X_3) = 0 \) is absolutely irreducible and the set of its singular points is either \( PG(2, q^{i_2 - i_1}) \) or \( PG(2, q^{i_2 - i_1}) \setminus PG(2, q^d) \), in which the latter case happens if and only if \( i_1 = d \). By the Hasse-Weil theorem, its \( F_{q^n} \)-rational simple points are at least

\[
q^n + 1 - (\ell - 1)(\ell - 2) - q^{2(i_2 - i_1)} - q^{i_2 - i_1} - 1,
\]

where \( \ell = q^{i_2} + q^{i_1} - q^{2d} - q^d \).

When \( n > 4i_2 + 2 \), which always holds provided that \( n > 4i_{k-1} + 2 \), it is straightforward to check that there is at least on simple point on the curve defined by \( C(X_1, X_2, X_3) = 0 \).

Hence, by the existence of a simple \( F_{q^n} \)-rational point \( P \) in the curve \( C(X_1, X_2, X_3) \), one can show, by Lemma 2.3, that there exists an \( F_{q^n} \)-rational plane \( \pi \) through the origin and \( P \) such that \( \pi \cap \mathcal{V}_l \) contains an absolutely irreducible component defined over \( F_{q^n} \). Such a component cannot be repeated (remember that the number of singular points is finite) and then, by Lemma 2.2, there is an \( F_{q^n} \)-rational component \( \mathcal{V}_l' \) in \( \mathcal{V}_l \) which is absolutely irreducible.

Suppose now that for each \( I = \{0, i_1, \cdots, i_j\} \), \( j < s - 1 \), the hypersurface \( \mathcal{V}_l \) contains an \( F_{q^n} \)-rational absolutely irreducible component. Let us prove the case \( I = \{0, i_1, \cdots, i_s\} \). The tangent cone at the origin of \( \mathcal{V}_l \) is \( \mathcal{V}_l' \), where \( I' = \{0, i_1, \cdots, i_{s-1}\} \), which by induction contains an absolutely irreducible \( F_{q^n} \)-rational component \( \mathcal{W}_l' \).

Note that the degree of \( \mathcal{V}_l' \) is \( d = q^{i_{s-1}} + q^{i_{s-2}} + \cdots + q^{i_1} - (q^{s-1} + q^{s-2} + \cdots + q) \). By assumption \( n > 4i_{k-1} + 2 \), which implies \( q^n > \frac{3}{2}d^4 > \frac{3d^4 - 4d^4 + 5d^2}{2} \).

By Theorem 2.8, component \( \mathcal{W}_l' \), which has degree at most \( d \), has a simple \( F_{q^n} \)-rational point. This shows, by Lemma 2.3, that there exists an \( F_{q^n} \)-rational plane \( \pi \) through the origin and \( P \) such that \( \pi \cap \mathcal{V}_l \) contains an absolutely irreducible component defined over \( F_{q^n} \). Using Lemma 2.2 inductively, it is readily seen that there is a reduced \( F_{q^n} \)-rational component in \( \mathcal{V}_l \) which is absolutely irreducible, since the set of singular points of \( S_s \) is finite. \( \square \)
Next let us turn to the case \( i_2 = 2i_1 \). We define

\[
\begin{align*}
L(U, Z_1, \ldots, Z_{k-3}) &= \begin{vmatrix}
U & Uq^{i_2} & \cdots & Uq^{i_{k-1}} \\
Z_1 & Z_1q^{i_2} & \cdots & Z_1q^{i_{k-1}} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{k-3} & Z_{k-3}q^{i_2} & \cdots & Z_{k-3}q^{i_{k-1}} \\
1 & 1 & \cdots & 1
\end{vmatrix}, \\
M(U, Z_1, \ldots, Z_{k-3}) &= \begin{vmatrix}
U & Uq^{i_2-i_1} & \cdots & Uq^{i_{k-1}-i_1} \\
Z_1 & Z_1q^{i_2-i_1} & \cdots & Z_1q^{i_{k-1}-i_1} \\
\vdots & \vdots & \ddots & \vdots \\
Z_{k-3} & Z_{k-3}q^{i_2-i_1} & \cdots & Z_{k-3}q^{i_{k-1}-i_1} \\
1 & 1 & \cdots & 1
\end{vmatrix}, \\
N(Z_1, \ldots, Z_{k-3}) &= \begin{vmatrix}
Z_1^{i_2-i_1} & \cdots & Z_1^{i_{k-1}-i_1} \\
\vdots & \ddots & \vdots \\
Z_{k-3}^{i_2-i_1} & \cdots & Z_{k-3}^{i_{k-1}-i_1} \\
1 & \cdots & 1
\end{vmatrix}, \\
R(Z_2, \ldots, Z_{k-3}) &= \begin{vmatrix}
Z_2^{i_3-i_1} & \cdots & Z_1^{i_{k-1}-i_1} \\
\vdots & \ddots & \vdots \\
Z_{k-3}^{i_3-i_1} & \cdots & Z_{k-3}^{i_{k-1}-i_1} \\
1 & \cdots & 1
\end{vmatrix}.
\end{align*}
\]

**Proposition 4.3.** Suppose \( k > 3 \) and \( i_2 = 2i_1 \). If \( M(U, Z_1, \ldots, Z_{k-3}) \) divides \( L(U, Z_1, \ldots, Z_{k-3}) \) then \( \gamma_1 \mid i_j \) for each \( j = 1, \ldots, k - 1 \).

**Proof.** Suppose, by way of contraction, that not all the \( i_j \) are divisible by \( i_1 \).

Clearly, if \( M(U, Z_1, \ldots, Z_{k-3}) \) divides \( L(U, Z_1, \ldots, Z_{k-3}) \), then in particular \( M(U, V, z_2, \ldots, z_{k-3}) \) also divides \( L(U, V, z_2, \ldots, z_{k-3}) \) for all \( (z_2, \ldots, z_{k-3}) \). Choose \( z_2, \ldots, z_{k-3} \in \mathbb{F}_q \) such that none of the lower \( (k-3) \times (k-3) \) minors in the determinant \( M(U, V, z_2, \ldots, z_{k-3}) \) vanishes.

The tangent cone at the origin of the curve \( D \) defined by the affine equation \( M(U, V, z_2, \ldots, z_{k-3}) = 0 \) is \( R(z_2, \ldots, z_{k-3})(UVq^{i_1} - VUq^{i_1}) \). Now the origin is an ordinary \( q^{i_1} + 1 \)-fold singular point of \( D \) (since the polynomial \( UVq^{i_1} - VUq^{i_1} \) factorizes in non-repeated linear factors over \( \mathbb{F}_q \)). Therefore there are exactly \( q^{i_1} + 1 \) branches centered at the origin and they correspond to the elements of \( \mathbb{F}_q^{i_1} \cup \{ \infty \} \).

Given \( \lambda \in \mathbb{F}_q^{i_1} \cup \{ \infty \} \), let \( \gamma_\lambda \) denote the corresponding branch of the curve \( D \) centered at the origin. Since \( i_1 \) does not divide all the \( i_j \) the line \( V = \lambda U \) is not a component of \( M(U, V, z_2, \ldots, z_{k-3}) = 0 \) and the branch \( \gamma_\lambda \) is of the type \( (t, \lambda t + \mu t^\alpha + \cdots) \) for some nonzero \( \mu \in \mathbb{F}_q \) and \( \alpha > 1 \). So \( \gamma_\lambda \) belongs to the curve
Let 
\begin{align*}
L(U, V, z_2, \ldots, z_{k-3}) &= 0 \\
L(t, \lambda t + \mu t^\alpha + \cdots, z_2, \ldots, z_{k-3}) =
\begin{bmatrix}
  t & t^q & \cdots & t^{q^{k-1}} \\
  \lambda t + \mu t^\alpha + \cdots & \lambda t^{q^{2i_1}} + \mu t^{2i_1} t^\alpha q^{2i_1} + \cdots & \cdots & \cdots \\
  \vdots & \vdots & \cdots & \vdots \\
  z_{k-3} & z_{k-3} & \cdots & z_{k-3} \\
  1 & 1 & \cdots & 1
\end{bmatrix},
\end{align*}

we must have that the above power series in \( t \) vanishes. We may subtract the second row by the first row times \( \lambda \). By checking the term of the smallest degree, we must have \( \alpha + q^{2i_1} = \alpha q^{2i_1} + 1 \) which yields \( \alpha = 1 \) or \( q^{2i_1} = 1 \), a contradiction. \( \square \)

**Theorem 4.4.** Let \( k \) be an integer larger than 3 and
\[
F(X, Y) = \begin{bmatrix}
  X & Xq & Xq^2 & \cdots & Xq^{k-1} \\
  Y & Yq & Yq^2 & \cdots & Yq^{k-1} \\
  z_1 & z_1 q & z_1 q^2 & \cdots & z_1 q^{k-1} \\
  \vdots & \vdots & \vdots & \cdots & \vdots \\
  z_{k-3} & z_{k-3} q & z_{k-3} q^2 & \cdots & z_{k-3} q^{k-1} \\
  1 & 1 & 1 & \cdots & 1
\end{bmatrix},
\]

and
\[
G(X, Y) = \begin{bmatrix}
  X & X^q & X^q^2 & \cdots & X^q^{k-1} \\
  Y & Y^q & Y^q^2 & \cdots & Y^q^{k-1} \\
  z_1 & z_1^q & z_1^q^2 & \cdots & z_1^q^{k-1} \\
  \vdots & \vdots & \vdots & \cdots & \vdots \\
  z_{k-3} & z_{k-3}^q & z_{k-3}^q^2 & \cdots & z_{k-3}^q^{k-1} \\
  1 & 1 & 1 & \cdots & 1
\end{bmatrix},
\]

Assume that there exists \( (z_1, \ldots, z_{k-3}) \in \mathbb{F}_{q^n}^{k-3} \) such that \( N(z_1, \ldots, z_{k-3}) \neq 0 \) and \( M(U, z_1, \ldots, z_{k-3}) \) does not divide \( L(U, z_1, \ldots, z_{k-3}) \). If one of the following conditions is satisfied,

- \( q \geq 7 \),
- \( q = 4, 5 \) and \( i_1 > 1 \),
- \( q = 3 \) and \( i_1 > 2 \),

then the curve \( C \) of the affine equation \( \frac{F(X, Y)}{G(X, Y)} = 0 \) contains an \( \mathbb{F}_q \)-rational absolutely irreducible component.

**Proof.** We want to study the intersection multiplicity of two putative components \( A \) and \( B \) of \( C \) at its singular points.

By direct computation, affine singular points of \( F(X, Y) = 0 \) satisfy
\[
\frac{\partial F(X, Y)}{\partial X} = (M(Y, z_1, \ldots, z_{k-3}))^{q^{i_1}}, \\
\frac{\partial F(X, Y)}{\partial Y} = (M(X, z_1, \ldots, z_{k-3}))^{q^{i_1}}, \\
F(X, Y) = 0.
\]

Consider now a singular point \((\alpha, \beta)\) of \(C\). Then
\[
M(\alpha, z_1, \ldots, z_{k-3}) = M(\beta, z_1, \ldots, z_{k-3}) = 0.
\]

Expanding \(F(X + \alpha, Y + \beta)\), one can see that the terms of the smallest degree appearing in it are

\[
L(\alpha, z_1, \ldots, z_{k-3}) Y^{q^{i_1}} - L(\beta, z_1, \ldots, z_{k-3}) X^{q^{i_1}} + (N(z_1, \ldots, z_{k-3}))^{q^{i_1}} (X Y^{q^{i_1}} - Y X^{q^{i_1}}) + \cdots,
\]

where \(L\) is as in (9).

- If one between \(\alpha\) and \(\beta\) does not satisfy \(L(U, z_1, \ldots, z_{k-3}) = 0\) then \(P\) has multiplicity \(q^{i_1}\) and the intersection multiplicity of the two components at \(P\) is either 0 or \(q^{i_1}\), by Lemmas 2.4 and 2.5.
- If \(L(\alpha, z_1, \ldots, z_{k-3}) = L(\beta, z_1, \ldots, z_{k-3}) = 0\) then the intersection multiplicity of the two components at \(P\) is at most \((q^{i_1} + 1)^2/4\). Since by assumption \(M(U, z_1, \ldots, z_{k-3})\) does not divide \(L(U, z_1, \ldots, z_{k-3})\), the number of points of the second type is at most \(q^{2(\deg M)-1} = q^{2(i_k-1-i_1-1)}\).
- Consider now an ideal point \(P = (\alpha, \beta, 0)\). Such a point is equivalent (up to a change of variables) to an affine singular point of the the curve \(C\). So we can suppose that the intersection multiplicity of the two components \(A\) and \(B\) at \(P\) is at most \((q^{i_1} + 1)^2/4\). The total number of ideal points is at most \((q^{i_1} + 1)^2/4\), since the term of the highest degree in \(F(X, Y)\) is

\[
\left(\frac{XY^{q^{i_1}-i_k-2} - Xq^{i_1}Y^{q^{i_1}-i_k-2}}{Y}ight)^{q^{i_k-2}}.
\]

The largest possible value for the sum of the multiplicities of intersection of two components \(A\) and \(B\) of \(C\) is

\[
\tau = \left(\frac{q^{2(i_k-1-i_1}) - q^{2(i_1-i_1-1)}}{q^{2(i_k-1-i_1-1)}}\right) q^{i_1} + q^{2(i_k-1-i_k-2)} (q^{i_1} + 1)^2/4
\]

(9)

\[
\text{Affine Type I}
\]

\[
\left(\frac{q^{i_k-1-i_k-2} + 1}{q^{i_1} + 1}\right) (q^{i_1} + 1)^2/4.
\]

\[
\text{Infinity}
\]

\[
\tau \leq q^{2i_k-1} \left(\frac{1}{q^{i_1}} \left(1 - \frac{1}{q^2}\right) + \frac{(q^{i_1} + 1)^2}{4q^{2i_1+2}} + \frac{q^{i_1} + 4}{4q^{2i_1+2-1-i_1}}\right) + \frac{(q^{i_1} + 1)^2}{4}
\]

(10)

\[
\tau \leq q^{2i_k-1} \left(\frac{1}{q^{i_1}} \left(1 - \frac{1}{q^2}\right) + \frac{(q^{i_1} + 1)^2}{4q^{2i_1+2}} + \frac{1}{4q^{2i-5}} + \frac{1}{4q^{2k-4}}\right) + \frac{(q^{i_1} + 1)^2}{4}.
\]
because
\[
\frac{(q^{i_1-1} - i_k - 2 + 1)(q^{i_1} + 1)^2}{4} = \frac{q^{i_1-1} - i_k - 2 + i_i}{4} \left( \frac{q^{i_1} + 2 + \frac{1}{q^{i_1}}}{q^{i_1}} \right) + \frac{(q^{i_1} + 1)^2}{4} \\
\leq \frac{(q^{i_1} + 4)q^{2i_k-1}}{4q^{i_1+1}q^{i_k-2}} + \frac{(q^{i_1} + 1)^2}{4}.
\]

Assume that
\[
F(X, Y)/G(X, Y) = W_1(X, Y)W_2(X, Y) \cdots W_r(X, Y)
\]
is the decomposition over \(\mathbb{F}_{q^n}\) with \(\deg W_\ell = d_\ell\) and
\[
d = \sum_{k=1}^r d_\ell = \sum_{j=1}^{\ell-1}(q^{i_j} - q^j).
\]

Since we have already shown that for any two components \(A\) and \(B\) their total intersection number has \((9)\) as upper bound, \(W_{\ell_1}\) and \(W_{\ell_2}\) must be relatively prime for any distinct \(\ell_1\) and \(\ell_2\).

By Lemma 2.1 there exist natural numbers \(s_\ell\) such that \(W_\ell\) splits into \(s_\ell\) absolutely irreducible factors over \(\mathbb{F}_{q^n}\), each of degree \(d_\ell/s_\ell\). Assume, by way of contradiction, that \(C\) has no absolutely irreducible component over \(\mathbb{F}_{q^n}\), i.e. \(s_\ell > 1\) for \(\ell = 1, 2, \ldots, r\). Define two polynomials \(A(X, Y, T)\) and \(B(X, Y, T)\) by
\[
A(X, Y, T) = \prod_{\ell=1}^r \prod_{j=1}^{[s_\ell/2]} Z_{\ell,j}(X, Y, T), \quad B(X, Y, T) = \prod_{\ell=1}^r \prod_{j=[s_\ell/2]+1}^{s_\ell} Z_{\ell,j}(X, Y, T),
\]
where \(Z_{\ell,1}(X, Y, T), \ldots, Z_{\ell,s_\ell}(X, Y, T)\) are the absolutely irreducible components of \(W_\ell(X, Y, T)\). Let \(\alpha\) and \(\alpha + \beta\) be the degrees of \(A(X, Y, T)\) and \(B(X, Y, T)\) respectively. Then
\[
2\alpha + \beta = d, \quad \beta \leq \alpha, \quad \beta \leq \frac{d}{3}.
\]

Let \(A\) and \(B\) be the curves defined by \(A(X, Y, T)\) and \(B(X, Y, T)\), respectively. It is clear that
\[
(\deg A)(\deg B) = (\alpha + \beta)\alpha = \frac{d^2 - \beta^2}{4} \geq \frac{2}{9}d^2.
\]

By looking at the value of \(d\), we have
\[
\frac{2}{9}d^2 \geq \frac{2}{9} \left( (q^{i_k-1} - q^k - 1)^2 + (q^{i_1} - q)^2 \right)
\]
(11)
\[
\geq \frac{2}{9} \left( 1 - \frac{1}{q^{i_k-1} - (k-1)} \right)^2 q^{2i_k-1} + \frac{2}{9} (q^{i_1} - q)^2.
\]

By comparing the bounds on \(\tau\) and \(\frac{2}{9}d^2\) by (10) and (11), we see that \(\tau < (\deg A)(\deg B)\) if one of the assumptions on \(q\) and \(i_1\) holds, a contradiction to Bézout’s theorem; see Theorem 2.3 \(\square\)

Suppose that \(I = \{0, i_1, i_2, \ldots, i_k-1\}\) is not an arithmetic progression. Then, by the proof of Proposition 4.3 there does exist \((z_1, \ldots, z_{k-3}) \in \mathbb{F}_{q^n}^k\) such that \(N(z_1, \ldots, z_{k-3}) \neq 0\) and \(M(U, z_1, \ldots, z_{k-3})\) does not divide \(L(U, z_1, \ldots, z_{k-3})\). Thus, by Theorem 4.4 there exists an \(\mathbb{F}_{q^n}\)-rational absolutely irreducible component in \(C\). By using Lemma 2.2 recursively, we obtain the following result.
Corollary 4.5. Suppose that $i_2 = 2i_1$ and $I = \{0, i_1, i_2, \ldots, i_k-1\}$ is not an arithmetic progression. If one of the following conditions satisfies,

- $q \geq 7$,
- $q = 4, 5$ and $i_1 > 1$,
- $q = 3$ and $i_1 > 2$,

then the variety $V_I$ contains an $\mathbb{F}_{q^n}$-rational absolutely irreducible component.

Finally we can prove our main result in this section.

Proof of Theorem 4.1. By Theorem 4.2 and Corollary 4.5, there exists an $\mathbb{F}_{q^n}$-rational absolutely irreducible component $V'_I$ of $V_I$.

Let us consider $V'_I$ and $G_k$ using Theorem 2.7. It is clear that the degree of $G_k$ is $e = \sum_{j=1}^{k-1} q^j$ and the degree $V'_I$ is $f \leq \sum_{j=1}^{k-1} q^j - e$. Hence

$$\alpha + \sqrt{\alpha^2 + 4\beta} \leq f^2 - 3f + 2 + \sqrt{f^4 - 6f^3 + 13f^2 - 12f + 4 + 20f^{13/3} + 4f^2 + 4\beta} - 4f$$

$$\leq f^2 - 3f + 2 + \sqrt{21f^{13/6}}$$

$$\leq \sqrt{22f^{13/6}}.$$

When integer $n > \frac{13}{3}i_k - 1 + 2$, we have

$$q^n > \frac{22}{4} f^{13/3} > \frac{1}{4} \left( \alpha + \sqrt{\alpha^2 + 4\beta} \right)^2.$$

By Theorem 2.7, there exists a nonsingular $\mathbb{F}_{q^n}$-rational point of $V'_I$ that is not a point of $G_k$.

Remark 4.6. In this paper, we have proved an asymptotic classification result for Moore exponent sets for $q > 5$. It appears that the same result could be also true for $q = 2, 3, 4$ and 5. To prove this result, one could try to get a better estimation of the upper bound for the sum of the multiplicities of intersection of two putative components in Theorem 4.4.

Acknowledgment

The work of Daniele Bartoli was supported by the Italian National Group for Algebraic and Geometric Structures and their Applications (GNSAGA - INdAM). Yue Zhou is partially supported by the National Natural Science Foundation of China (No. 11771451). This work is partially done during the visit of the second author to the University of Perugia. He would like to thank Daniele Bartoli and Massimo Giulietti for their hospitality.

References

[1] Y. Aubry, G. McGuire, and F. Rodier. A few more functions that are not apn infinitely often, finite fields theory and applications. In Ninth International conference Finite Fields and Applications, McGuire et al. editors, Contemporary Math. No. 518, AMS, Providence (RI), pages 23–31, 2010.
[2] H. Borges. On multi-frobenius non-classical plane curves. Archiv der Mathematik, 93(6):541–553, Nov 2009.
[3] B. Csajbók, G. Marino, and O. Polverino. Classes and equivalence of linear sets in PG(1, $q^n$). Journal of Combinatorial Theory, Series A, 157:402 – 426, 2018.
[4] B. Csajbók, G. Marino, O. Polverino, and C. Zanella. A new family of MRD-codes. Linear Algebra and its Applications, 548:203–220, July 2018.
[5] B. Csajbók, G. Marino, O. Polverino, and Y. Zhou. Maximum rank-distance codes with maximum left and right idealisers. [arXiv:1807.08774 [math]], 2018.

[6] B. Csajbók, G. Marino, and F. Zullo. New maximum scattered linear sets of the projective line. Finite Fields and Their Applications, 54:133 – 150, 2018.

[7] F. Hernando and G. McGuire. Proof of a conjecture on the sequence of exceptional numbers, classifying cyclic codes and APN functions. Journal of Algebra, 343(1):78–92, Oct. 2011.

[8] H. Janwa, G. Mcguire, and R. Wilson. Double-error-correcting cyclic codes and absolutely irreducible polynomials over GF(2). Journal of Algebra, 178(2):665–676, Dec. 1995.

[9] A. Kshevetskiy and E. Gabidulin. The new construction of rank codes. In International Symposium on Information Theory, 2005. ISIT 2005. Proceedings, pages 2105–2108, Sept. 2005.

[10] D. Liebhold and G. Nebe. Automorphism groups of Gabidulin-like codes. Archiv der Mathematik, 107(4):355–366, Oct. 2016.

[11] G. Lunardon. MRD-codes and linear sets. Journal of Combinatorial Theory, Series A, 149:1–20, July 2017.

[12] G. Lunardon, R. Trombetti, and Y. Zhou. On kernels and nuclei of rank metric codes. Journal of Algebraic Combinatorics, 46(2):313–340, Sep 2017.

[13] G. Lunardon, R. Trombetti, and Y. Zhou. Generalized twisted gabidulin codes. Journal of Combinatorial Theory, Series A, 159:79 – 106, 2018.

[14] E. H. Moore. A two-fold generalization of Fermat’s theorem. Bull. Amer. Math. Soc., 2(7):189–199, 04 1896.

[15] K.-U. Schmidt and Y. Zhou. Planar functions over fields of characteristic two. Journal of Algebraic Combinatorics, 40(2):503–526, Sept. 2014.

[16] K.-U. Schmidt and Y. Zhou. On the number of inequivalent Gabidulin codes. Designs, Codes and Cryptography, 86(9):1973–1982, 2018.

[17] J. Sheekey. A new family of linear maximum rank distance codes. Advances in Mathematics of Communications, 10(3):475–488, 2016.

[18] J. Sheekey. New semifields and new MRD codes from skew polynomial rings. [arXiv:1806.00251 [math]], June 2018.

[19] J. Sheekey. MRD Codes: Constructions and Connections. [arXiv:1904.05813 [math]], April 2019.

[20] R. Trombetti and Y. Zhou. A new family of MRD codes in $F_q^{2n\times 2n}$ with right and middle nuclei $F_{q^n}$. IEEE Transactions on Information Theory, 65(2):1054–1062, Feb 2019.

[21] J. Zahid. Nonsingular points on hypersurfaces over $F_q$. Journal of Mathematical Sciences, 171(6):731–735, Dec 2010.