\( \mathcal{W} \)-strings from \( N = 2 \) Hamiltonian reduction and classification of \( N = 2 \) super \( \mathcal{W} \)-algebras

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Abstract

We present an algebraic approach to string theory, using a Hamiltonian reduction of \( N = 2 \) WZW models. An embedding of \( sl(1|2) \) in a Lie superalgebra determines a nilpotent subalgebra. Chirally gauging this subalgebra in the corresponding WZW action leads to an extension of the \( N = 2 \) superconformal algebra. We classify all the embeddings of \( sl(1|2) \) into Lie superalgebras: this provides an exhaustive classification and characterization of all extended \( N = 2 \) superconformal algebras. Then, twisting these algebras, we obtain the BRST structure of a string theory. We characterize and classify all the string theories which can be obtained in this way.

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The work we present is based on a collaboration with A. Sevrin and P. Sorba [1]. We will present a classification of $N = 2$ super $\mathcal{W}$-algebras obtained from Hamiltonian Reduction (HR) of superalgebras and apply it to the classification of $\mathcal{W}$-string. Thus, our purpose is twofold: first, show you how $\mathcal{W}$-strings are related to $N = 2$ super $\mathcal{W}$-algebras through BRST approach, and second to classify all the $N = 2$ super $\mathcal{W}$-algebras obtained from HR of superalgebras. In the first part, we will present you an algebraic definition for $\mathcal{W}$-string (i.e. $\mathcal{W}$-gravity) [3, 4, 5, 6] which is self-contained, while the second part will provide the $N = 2$ multiplets contents of any $N = 2$ super $\mathcal{W}$-algebras.

Our presentation will naturally follow these 2 ideas ($\mathcal{W}$-gravity and $\mathfrak{sl}(2|1)$ HR). To simplify, we will mainly treat an example, the bosonic string, and connect it with a (twisted) $N = 2$ superconformal algebra (SCA2). Then, we will present a classification of $\mathfrak{sl}(1|2)$ embeddings: although quite technical, this part is interesting since it directly provides a way to compute multiplet contents of $N = 2$ super $\mathcal{W}$-algebras.

1 Bosonic string and twisted $N = 2$ SC algebra

The action of the bosonic string is constituted with 3 terms: a Liouville term which corresponds to the dilaton of the gravity sector; a ghost sector which comes from the change of variable in the metric $g^{ab} = e^{\phi} \eta^{ab}$ (with $\phi$ Liouville field and $\eta^{ab}$ background metric); and then a matter sector on which we do not make any assumption.

The energy-momentum tensor is then $T = T_L + T_{bc} + T_m$ with

\[
T_L = \frac{1}{2} \partial \phi \partial \phi + \sqrt{\frac{25 - c_m}{12}} \partial^2 \phi \tag{1.1}
\]

\[
T_{bc} = -2b \partial c - (\partial b) c \tag{1.2}
\]

where $c_m$ is the central charge of the matter-sector, $c_{bc} = -26$ is the central charge of the ghost part, while the background charge for the Liouville field has been adjusted to get $c_L = 26 - c_m$ (so that $c_{\text{tot}} = 0$). The BRS charge for the string obeys to

\[
Q_{\text{BRS}}(b) = T \quad \text{with} \quad Q_{\text{BRS}}^2 = 0 \tag{1.3}
\]

In the BV formalism, its action is computed using a BRS current $J_{\text{BRS}}(z)$, with

\[
Q_{\text{BRS}}[F(z)] = \oint_z \frac{dx}{2\pi i} J_{\text{BRS}}(x) F(z) \tag{1.4}
\]

The usual choice for $J_{\text{BRS}}$ is $J_{\text{BRS}} = c \left( T_L + T_M + \frac{1}{2} T_{bc} \right)$, which satisfies (1.3). However, a $J_{\text{BRS}}(z)$ is defined up to total derivative terms. Then, choosing $J_{\text{BRS}}(z) = c(T_L + T_M + \frac{1}{2} T_{bc}) + \alpha \partial(c \partial \phi) + \beta \partial^2 c \tag{1.5}$
with
\[ \alpha = -\frac{\sqrt{3}}{6} \left( \sqrt{1 - c_m} + \sqrt{25 - c_m} \right) \quad \text{and} \quad \beta = -\frac{1}{12} \left( 7 - c_m + \sqrt{(1 - c_m)(25 - c_m)} \right) \] (1.6)

one can verify that \( J_{\text{BRS}}(z)J_{\text{BRS}}(w) = 0 \), while \( Q_{\text{BRS}}(b) = T \) is translated into
\[ J_{\text{BRS}}(z)b(w) = \frac{c_2}{(z - w)^3} + \frac{U(w)}{(z - w)^2} + \frac{T}{z - w} \quad \text{with} \quad c_2 = -\frac{1}{2} \left( 7 - c_m + \sqrt{(1 - c_m)(25 - c_m)} \right) \] (1.7)

\( U(w) \) is the ghost number current (up to derivative):
\[ U(w) = -bc - \partial \phi \] (1.8)

Moreover, computing all the OPE's between \( T, J_{\text{BRS}}, b \) and \( U \), one realizes that they form a closed algebra which is nothing but the twisted SCA2. Identifying \( J_{\text{BRS}} \) with \( G_+ \), \( b \) with \( G_- \), and \( T_{N=2} = T - \frac{1}{2} \partial U \), we get the SCA2 with central charge \( c_2 \).

If the matter sector is a reduction of an \( sl(2) \) WZW model \( c_m = 1 - 6\frac{q}{p} \), we get \( c_2 = 3(1 - 2\frac{p}{q}) \), and for \( (p, q) = (1, k + 2) \) we recover the \( N=2 \) unitary minimal models \( c_2 = 3 \frac{k}{k+2} \).

Since we get the SCA2, it is natural to ask whether this approach can be connected with another way of obtaining the SCA2, namely the HR of \( sl(2|1) \). It is well-known that the reduction of the WZW model based on \( sl(1|2) \) gives the SCA2, but the point is to see whether one can get the above realization of this algebra.

## 2 Hamiltonian reductions of \( sl(2|1) \)

### 2.1 The usual Hamiltonian reduction

We start with a WZW model based on \( sl(2|1) \). We recall that the \( sl(2|1) \) algebra is formed by a (bosonic) \( sl(2) \oplus U(1) \) algebra, together with four fermions gathered into two doublets (under \( sl(2) \)). The WZ action \( S_-(g) \) is invariant under (semi-local) \( sl(2|1) \) transformations, and the associated currents \( J = g^{-1} \partial g \), which are chiral on-shell (\( \bar{\partial} J = 0 \)), form an affine \( sl(2|1) \) algebra.

Using the \( sl(2) \)-Cartan generator \( e_0 \), one grades the currents
\[ J = J^a(z)t_a = J^a(z)t_a + J^{\dot{a}}(z)t_{\dot{a}} \quad \text{with} \quad [e_0, t_a] = g_a t_a, \quad g_a \in \mathbb{R} \quad \text{and} \quad g_a < 0, \quad g_{\dot{a}} \geq 0 \] (2.1)

and impose the constraints
\[ J(z)|_{e_0} = e_- + \tau(z) \quad \text{with} \quad e_- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau(z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_2(z) \\ \tau_1(z) & 0 & 0 \end{pmatrix} \] (2.2)
These constraints generate gauge transformation (with group $sl(2|1)_{>0}$) and the action:

$$S = S_-(g) + \int d^2 x \text{str}\{A(J - e_+ - \tau) + [e_+, \tau]\bar{\partial}\tau\} \quad \text{with} \quad A \in sl(2|1)_{>0}$$  \hspace{1cm} (2.3)

where $A(x)$ are the gauge fields, and play the role of Lagrange multipliers.

A good choice for a gauge fixing is $A = 0$, and at the quantum level, this provides Fateev-Popov (FP) ghosts $(\beta, \gamma)$ that belong to $sl(2|1)_{<}$. The gauge fixed action reads:

$$S_{g.f.} = S_- + \int d^2 x \text{str}\{(\bar{\partial}\beta)\gamma + [e_+, \tau]\bar{\partial}\tau\}$$  \hspace{1cm} (2.4)

with gauge transformations

$$\delta A = \bar{\partial}\eta + [\eta, A] \quad \eta \in sl(1|2)_{>0}$$

$$\delta g = \eta g \quad \delta \tau = -\Pi_{\frac{1}{2}} \eta$$  \hspace{1cm} (2.5)

where $\Pi_{\frac{1}{2}}$ denotes the projector onto $G_{\frac{1}{2}}$. The BRS current associated to this gauge invariance reads:

$$j_{\text{BRS}} = \text{str}[\gamma(J - e_+ - \tau + \frac{1}{2}J_{gh})], \quad J_{gh} = \beta\gamma \gamma \quad \text{and} \quad s_{\text{BRS}}[F(z)] = \oint_{z} \frac{dx}{2\pi i} j_{\text{BRS}}(x)F(z)$$  \hspace{1cm} (2.6)

and the gauge invariant quantities are in the cohomology of $s_{\text{BRS}}$, namely $H_0(\Omega, s_{\text{BRS}}) = \text{Ker}(s_{\text{BRS}})/\text{Im}(s_{\text{BRS}})$, with $\Omega$ the enveloping algebra generated by $J|_{>0}$, $\beta$ and $\gamma$.

The calculation for this kind of cohomology is known [7], and the result here is just the SCA2. Computing a representant of each cohomological class gives a realization of the SCA2 in terms of the unconstrained generators of the affine $sl(2|1)$ algebra. Of particular importance is the restriction of this realization to the $(0,0)$-grade part: this map is an algebra homomorphism and allows a free field realization of the SCA2. It is called the quantum Miura-map. Here it gives

$$T_{N=2} = \partial\varphi\partial\bar{\varphi} - \sqrt{\kappa + 1}\partial^2(\varphi + \bar{\varphi}) - \frac{1}{2}(\chi\partial\bar{\chi} - \partial\chi\bar{\chi})$$

$$G_+ = -\chi\partial\varphi + \sqrt{\kappa + 1}\partial\chi$$

$$G_- = -\bar{\chi}\partial\bar{\varphi} + \sqrt{\kappa + 1}\partial\bar{\chi}$$

$$\chi = \tau_1\sqrt{\kappa}$$

$$\frac{\partial\varphi}{\sqrt{\kappa + 1}}(\hat{E}^0 + \hat{U}^0)$$

$$\bar{\chi} = \tau_2\sqrt{\kappa}$$

$$\frac{\partial\bar{\varphi}}{\sqrt{\kappa + 1}}(\hat{E}^0 - \hat{U}^0)$$  \hspace{1cm} (2.7)

Clearly, the realization we obtain has nothing to do with the one obtained from the basonic string, since $G_+$ and $G_-$ have a symmetric construction here, while we had before $G_+ = J_{\text{BRS}}$ and $G_- = b$. In particular, remark that $G_-$ is not composite, a fact difficult to tackle in the usual HR. Thus, if one has to compare the bosonic string with the HR of $sl(1|2)$, we must take another (hence a Non-Standard) HR.
2.2 Non Standard HR

To give an intuitive insight to the HR we are looking for, we come back to the realization of the SCA2 we have obtained from the bosonic string. The point is to remark that one of the supersymmetry charge is a simple (i.e., non composite) field. From the point of view of HR, it is a non-trivial information, because in the usual HR, only the spin 1 fields can be simple. Thus, to get the spin 3/2 field \( G^- \) simple, we have to impose it by hand. For such a purpose, we take another grading of the \( sl(2|1) \) algebra, namely \( h = e_0 + 2u_0 \). This grading is non standard in the sense that first it is associated to a \( sl(2) \oplus gl(1) \) decomposition (see however [8] for a general approach), and second it does not satisfy the "non-degeneracy" condition:

\[
\mathcal{K} = \text{Ker}(ade_+) \cap G_{<0} \neq \{0\}
\]  

This implies that there are some highest weights which have negative grades and should be constrained. To avoid this, we must introduce new auxiliary fields \( \Psi \) that belong to \( \text{Ker}(ade_+) \cap G_{<0} \), and which restore the freedom of the highest weight through the new form of the constraints:

\[
J|_{<0} = e_- + \Psi \text{ with } \Psi(z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \psi(z) & 0 \end{pmatrix}
\]  

(2.9)

Note that \( \tau \) and \( \Psi \) are both auxiliary fields, but not of same origin. \( \tau \) is just linked to the half-integral grading and the fact that we need first class constraints, it appeared already at the level of standard HR. On the contrary, \( \Psi \) is completely new and due to violation of the non-degeneracy condition.

At the level of action, we have to introduce a partner \( \bar{\psi} \) to \( \psi \) for the action

\[
S = S_-(g) + \int d^2 x \text{ str} \{A(J - e_- - \Psi) + \Psi \bar{\partial} \bar{\Psi}\}
\]  

(2.10)

to be invariant under:

\[
\begin{align*}
\delta A &= \bar{\partial} \eta + [\eta, A] \\
\delta g &= g \eta \\
\delta \Psi &= 0
\end{align*}
\]  

\[
\delta \tau = -\Pi_2 \eta
\]

(2.11)

\[
\delta \bar{\Psi} = \Pi_\mathcal{K} \eta \text{ with } \bar{\Psi} \in \mathcal{K} = \text{Ker}(ade_-) \cap G_{>0}
\]

Note that the emergence of \( \bar{\Psi} \) is not surprising if one thinks at the bosonic string: if \( \Psi \) plays the rôle of the \( b \) field, that we must find the \( c \) field somewhere in the game.

Now, apart from the adjoinction of these new auxiliary fields, the calculation is the same as in the standard HR: we choose the gauge fixing \( A = 0 \) and add the corresponding FP ghosts \( (\beta, \gamma) \) to get

\[
S_{g.f.} = S_-(g) + \int d^2 x \text{ str} \{(\bar{\partial} \beta) \gamma + \Psi \bar{\partial} \bar{\Psi}\}
\]  

(2.12)

\[
j_{BRS} = \text{ str}[\gamma(J - e_- - \Psi + \frac{1}{2}J_{gh})]
\]  

(2.13)
Computing the cohomology of $s_{BRS}$, we get the SCA2, and the Miura map gives a free field realization:

$$
T_{N=2} = \partial \varphi \partial \bar{\varphi} - \frac{\kappa}{2\sqrt{\kappa + 1}} \partial^2 (\varphi + \bar{\varphi}) - \frac{1}{2\sqrt{\kappa + 1}} \partial^2 (\varphi - \bar{\varphi}) - \frac{\kappa}{2} (3\psi \partial \bar{\psi} - \bar{\psi} \partial \psi)
$$

$$
U = -\sqrt{\kappa + 1} \partial (\varphi - \bar{\varphi}) - \kappa \psi \bar{\psi}
$$

$$
G_+ = \kappa \bar{\psi} \partial \varphi \partial \bar{\psi} - \frac{\kappa(2\kappa + 1)}{2} \partial^2 \bar{\psi} - \frac{\kappa^2}{\sqrt{\kappa + 1}} \bar{\psi} \partial \bar{\varphi} + \kappa \sqrt{\kappa + 1} \partial \bar{\psi} \partial (\varphi - \bar{\varphi})
$$

$$
G_- = \psi
$$

using the definitions introduced in the usual HR for the fields $\varphi$ and $\bar{\varphi}$.

Identifying $\psi$ with the ghost $b$, $\kappa \bar{\psi}$ with the ghost $c$ and $\sqrt{2} \partial (\varphi - \bar{\varphi})$ with the Liouville field $\partial \varphi_L$, we find exactly the bosonic-string realization given in section 1, with the matter sector in a WZW model based on $sl(2)$, where the matter field is $\varphi_m = i\sqrt{2}/16 \partial (\varphi + \bar{\varphi})$ (Note the correspondence $k_{sl(2)} \rightarrow k_{sl(2|1)} + 1$ in the levels). In other words, we have found an algebraic approach to the bosonic string, through a non-standard HR of $sl(2|1)$ WZW model.

We can summarize this result with the following picture:

Gravity in the conformal gauge
with matter in $sl(2)$ WZW model
"BRS algebra" $(J_{BRS}, T, U, b)$

Gravity in the conformal gauge
with matter in $sl(2)$ WZW model
"BRS algebra" $(J_{BRS}, T, U, b)$

\[
\uparrow
\]

WZW model based on $sl(1|2)$

\[
\rightarrow
\]

SC($N=2$) algebra
$c_{N=2} = 3(1 - 2(\kappa + 2))$

"twisted" HR
$h = e_0 + 2u_0$

Miura map

where we have called "twisted" the non-standard HR because of the presence of the $U(1)$ factor. The double arrow symbolizes the direction we will take in this paper, namely defining the gravity through an HR of $sl(1|2)$.

The generalization of this picture to general $\mathcal{W}$-gravity is quite obvious. One has just to replace the Liouville sector by a Toda sector and take the matter sector in a model with the same $\mathcal{W}$ symmetry. Then the associated "BRS algebra" will be a (twisted) $N = 2$ super $\mathcal{W}$-algebra. The problem of finding the BRS current (which is in general not known) can now be replaced by the (algebraic) HR of a given Lie superalgebra w.r.t. some $sl(1|2)$ embedding.

We are thus naturally led to study the classification of $sl(2|1)$ embedding into superalgebras. This group theoretical approach will provide a classification of $N = 2$ super $\mathcal{W}$-algebra
that we obtain through HR. The advantage of this approach is that we will obtain automatically the BRS operator in terms of the fields of the theory, together with the multiplet contents. The disadvantage of this approach is that you need to read now a group theory.

3 Classification of $sl(2|1)$ embedding into superalgebra

Let us immediately start with the remark that in the chain

\[
\begin{array}{ccc}
sl(2) & \rightarrow & osp(1|2) \\
\downarrow & & \downarrow \\
\text{conformal alg.} & \rightarrow & \text{superconformal alg.} \\
\end{array}
\rightarrow \text{SCA2}
\]

(3.1)

$sl(2|1)$ is the first (super)algebra that possesses 2 Dynkin diagrams:

\[
\begin{array}{ccc}
\circ & \rightarrow & \bullet \\
\end{array}
\rightarrow \begin{cases}
\otimes \otimes \\
\otimes \\end{cases}
\]

(3.2)

In other words, contrarily to $sl(2)$ and $osp(1|2), sl(2|1)$ has two non equivalent simple root systems. This may be the reason why it is the $N = 2$ supersymmetries that plays a special rôle. Let also remark that $sl(1|2)$ has rank 2, while the others have rank 1, so that the modification of the grading is possible inside the algebra for the first time.

3.1 The $sl(2|1)$ superalgebra

We have already seen that the $sl(2|1)$ superalgebra contains an $sl(2) \oplus U(1)$ bosonic part and two doublets (fermionic). This superalgebra is isomorphic to the $osp(2|2)$ superalgebra (which contains a $so(2) + sp(2)$ bosonic algebra), but they differ at the level of supergroup. In the following, we will call $(e_\pm, e_0)$ the $sl(2)$ algebra and $u_0$ the $U(1)$, the fermions being $f_\pm$ and $\bar{f}_\pm$. If one decomposes $sl(1|2)$ w.r.t. its bosonic subalgebra, we get:

\[
\begin{array}{ccc}
sl(2) & \rightarrow & osp(1|2) \\
\downarrow & & \downarrow \\
\text{D}_1(0) & \rightarrow & \begin{cases}
\otimes \otimes \\
\otimes \\end{cases}
\end{array}
\]

(3.3)

which corresponds to the structure of a $(0,1)$ representation of $sl(2|1)$ (see below).
3.2 Classification of $sl(2|1)$ embedding

The classification of $sl(2|1)$ embedding follows the classification of $sl(2)$ embeddings in algebra. We first have to define what is the principal embedding. As for $sl(2)$, it is related to the simple roots of the (super)algebra, but here these roots must be all fermionic and satisfies other properties that imposes constraints on the superalgebra. In fact, only (sums of) $sl(n \pm 1|n)$ superalgebras possess a principal $sl(2|1)$. It is defined through its positive fermionic roots:

$$f_+ = \sum_{i=1}^n f_i \quad \text{and} \quad \bar{f}_+ = \sum_{i=1}^n \bar{f}_i \quad \text{with} \quad \{f_i, f_j\} = 0 = \{ar{f}_i, \bar{f}_j\}$$

(3.4)

where $(f_i, \bar{f}_j)$ are the simple roots of $sl(n \pm 1|n)$. Now that we have defined the principal $sl(2|1)$ embedding (in $\oplus i sl(n_i \pm 1|n_i)$), the process to classify the $sl(2|1)$ in a given superalgebra $G$ is quite simple.

We first select in $G$ all the possible regular $sl(n \pm 1|n)$ subalgebra and then, any $sl(2|1)$ in $G$ will be conjugate to the principal $sl(2|1)$ in these $sl(n \pm 1|n)$ regular subalgebra.

We have introduced the notion of regular subalgebra: $H$ is regular in $G$ if the set of root generators of $H$ is a subset of root generators of $G$. Thus, the principal embedding is not a regular embedding (except when $G$ is $sl(1|2)$ itself). The classification of regular sub(super)algebras of $G$ is done using its Dynkin diagramm(s) (DD): one first extend the DD of $G$ to the affine DD, and then removes nodes: all the DDs obtained in this way will lead to regular sub(super)algebras of $G$ (see [9] for details).

Thus, starting with a $G$ superalgebra the following process classify all the $sl(2|1)$ embeddings

$$G \quad \longrightarrow \quad H = \oplus i sl(n_i \pm 1|n_i) \quad \longrightarrow \quad sl(1|2) \quad \text{principal}$$

Dynkin Diag.-
like process regular in $G$ $f_+ = \sum_{i=1}^n f_i$ \quad $\bar{f}_+ = \sum_{i=1}^n \bar{f}_i$ \quad in $H$

(3.5)

Small exception: in $osp(m|m)$, one has to treat the regular $osp(2|2)$ separately from the regular $sl(2|1)$. Tables corresponding to the classification of all $sl(1|2)$ embeddings in Lie superalgebras with $\text{rank}(G) \leq 4$ can be found in [1].

3.3 $sl(2|1)$ representations

First of all, let us come back the $sl(2|1)$ superalgebra in itself: we have seen that its decomposition w.r.t. its $sl(2) \oplus U(1)$ subalgebra, takes the form (3.3). This representation is the $(0,1)$ representation of $sl(2|1)$. It is a "usual" (typical) representation of $sl(2|1)$ and $(0,1)$ indicates the eigenvalues $(b,j)$ with respect to $(u_0,e_0)$. 
More generally a typical \((b,j)\) representation takes the form:

\[
\begin{align*}
\text{(b,j) representation, } b \neq \pm j: & \quad D_j(b) \\
\dim(b,j) = 8j: & \quad D_{j-1/2}(b + \frac{1}{2}) \quad \searrow \quad \downarrow \quad D_{j-1/2}(b - \frac{1}{2}) \\
& \quad \swarrow \\
& \quad D_{j-1}(b)
\end{align*}
\]

(3.6)

The typical representations are just usual representations, as one encounters in bosonic Lie algebras. If one considers the principal \(osp(1|2)\) contained in \(sl(1|2)\), the \((b,j)\) representation decomposes as \((b,j) = R_j \oplus R_{j-1/2}\), where \(R_j\) is the \((4j + 1)\)-dimensional representation of \(osp(1|2)\) (see e.g. [10, 11] for \(osp(1|2)\) embeddings and representations). The point is that sometimes, there are representations that possess a null vector, and thus are not irreducible. These “atypical” representations, once we have quotiented by the null vector, take the form:

\[
\begin{align*}
\text{typical representations, } b \neq \pm j: & \quad D_j(j) \quad \swarrow \\
& \quad \downarrow D_{j-1/2}(j + \frac{1}{2}) \\
& \quad \swarrow \quad \downarrow \quad D_{j-1/2}(-j + \frac{1}{2}) \\
\text{atypical representations, } b = \pm j: & \quad D_{j-1}(j) \quad \swarrow \\
& \quad \downarrow \quad \swarrow \quad D_{j-1}(j)
\end{align*}
\]

(3.7)

These representation have dimension \(4j+1\), and under \(osp(1|2)\), they decompose as \(\pm (\pm j,j)\). The point is that sometimes, there are representations that possess a null vector, and thus are not irreducible.

Note that for each \(sl(1|2)\) embeddings, we are able to give the decomposition of the fundamental representation of \(G\) once \(H = \oplus_i sl(n_i \pm 1|n_i)\) is given: if \(G = sl(m|n)\), each \(sl(n_i + 1|n_i)\) or \(sl(n_i|n_i + 1)\) will provide a \((\pm \frac{m}{2}, \frac{n}{2})\) representation (in the fundamental representation of \(G\)), while if \(G = osp(m|n)\), any \(sl(n_i + 1|n_i)\) or \(sl(n_i|n_i + 1)\) will lead to a sum \((\frac{m}{2}, \frac{n}{2}) \oplus (-\frac{m}{2}, \frac{n}{2})\). The special case of a regular \(osp(2|2)\) embedding in \(osp(m|n)\) will correspond to a \((0, \frac{1}{2})\) representation in the fundamental. Then, the determination of the decomposition of the fundamental of \(G\) completely fixes the decomposition of the adjoint representation of \(G\), thanks to algebraic rules given in [1].
4  sl(2|1) Hamiltonian Reduction of superalgebras

4.1  Usual HR

For each \( sl(2|1) \), we take as grading operator \( e_0 \in sl(2) \), and perform the HR. The action

\[
S = S_- (g) + \int d^2 x \, \text{str} \{ A(J - e_+ - \tau) + [e_+, \tau] \bar{\partial} \tau \} \quad (\tau \in \Pi_{\frac{1}{2}} G) \tag{4.1}
\]

is invariant under gauge transformation, and the gauge fixing \( A = 0 \) leads to

\[
S = S_- (g) + \int d^2 x \, \text{str} \{ \beta \bar{\partial} \gamma + [e_+, \tau] \bar{\partial} \tau \} \tag{4.2}
\]

The cohomology of the BRS operator \( j_{\text{BRS}} = \text{str}[(J - e_\tau) \gamma + \beta \gamma \gamma] \) will provide an \( N = 2 \) super \( \mathcal{W} \)-algebra.

The \( N = 2 \) multiplets of this \( \mathcal{W} \)-algebra are known very easily from the previous group analysis. In fact, the \( N = 2 \) multiplets of the \( \mathcal{W} \)-algebra are in one-to-one correspondence with the \( sl(1|2) \) representations that enter in the decomposition of the adjoint of \( G \). A \((b,j)\) representation will correspond to a \((q,s)\) multiplet of conformal spin \( s = j + 1 \) and \( U(1) \)-hypercharge \( q = b \). To each typical \( sl(2|1) \) representation will correspond a full \( N = 2 \) superfield in \( \mathcal{W} \); to each atypical \( sl(2|1) \) representation will correspond a chiral/antichiral superfield in \( \mathcal{W} \).

Thus, we have most of the structure of the \( \mathcal{W} \)-algebra (see e.g. tables for \( \text{rank}(G) \leq 4 \)).

We have seen that some products of irreducible representations are not fully reducible. It is the case for the \((0, \frac{1}{2})\) representation. This implies that if one reduces an \( osp(m|2n) \) with respect to \( N osp(2|2) \) with \( N > 1 \), the adjoint of \( osp(m|2n) \) is not the sum of \( sl(2|1) \) irreducible representations (this has been explicitly checked for \( osp(4|2) \) in [1]). We do not know what it means for the corresponding \( \mathcal{W} \)-algebra, a complete example of HR has still to be done.

4.2  Unusual Hamiltonian Reduction of superalgebras

We come back to our physical motivations, namely the algebraic study of \( \mathcal{W} \)-gravity from the point of view of \( sl(2|1) \) HR.

we recall that we have seen that the gravity structure can be described using a special realization of SCA2, and that this realization can be obtained through the Miura map in the HR of a WZW model based on \( sl(2|1) \). The HR we performed was unusual in the sense tha the grading was not an \( sl(2) \) Cartan generator, but was ”twisted” by a \( U(1) \) factor. This ”twist” was introduced to make one of the h.w. of negative grade, so that we were ”allowed” to introduce the ghost field that was lacking in the usual HR.

It is the same point of view that we will take for a general HR of superalgebra. For each \((b,j)\) representation we want to introduce one ghost that will be the ”BRS partner” of the field associated to the h.w. of \((b,j)\).
Now, for a general grading $h = e_0 + 2Nu_0$, and a $(b,j)$ representation, the grades of the h.w. are $N2b, j - \frac{1}{2} + 2Nb \pm N, j - 1 + 2bN$. Then, if $b \neq 0$, we cannot be sure that there will be one and only one ”ghost” by $sl(2|1)$ multiplet.

For that reason, we consider only the reductions that leads to $b = 0$ irreducible representations. They are of the form

$$sl(1|2) \subset_{pal} p \cdot sl(2j + 1|2j) \oplus q \cdot sl(2j|2j + 1) \subset_{reg} sl(p(2j + 1) + q(2j)|p(2j) + q(2j + 1))$$

$$osp(2|2) \subset_{reg} osp(m|2n)$$

where we have used the notations $\subset_{pal}$ for a principal embedding and $\subset_{reg}$ for a regular one.

In these cases, the gradation we take is $h = e_0 + 2j_{\text{max}}u_0$, where $j_{\text{max}}$ is the highest value we obtain in the decomposition of the superalgebra.

This gradation is such that one and only one h.w. by multiplet has negative grade. Thus, for each multiplet, you will get a $W$ generator ($sl(2|1)$ h.w.) with its ”$b$-partner” ($sl(2|1)$ h.w. with $h < 0$).

Let us remark that in particular, one can consider:

$$sl(1|2) \subset_{pal} sl(2n \pm 1|2n) \text{ and } osp(2|2) \subset_{reg} osp(m|2n)$$

(4.3)

as subcases of our approach: these two cases were studied in [4] and [5] respectively.

Looking at the action, we start with a gauged WZW model and impose as constraint:

$$J|_{<0} = e_- + \tau + \Psi + [\bar{\Psi}, \Psi] \text{ with } \Psi \in \mathcal{K} = Ker(ade_+) \cap \mathcal{G}_{<0}$$

(4.4)

and see whether the action is invariant under the gauge transformation. It happens not and in fact the constraints are not first class.

In fact, in the case $\mathcal{G} = sl(m|n)$, the constraint, to be first class, need $\Psi$ to be not in $Ker(e_+) \cap \mathcal{G}_{<0}$, but more precisely in $\mathcal{K}_0$, a vector space isomorphic to $Ker(e_+) \cap \mathcal{G}_{<0}$ and which is defined as follows:

1) Take $\mathcal{K} = Ker(e_+) \cap \mathcal{G}_{<0}$
2) Define $e_+ \in sl(2) \subset_{pal} p \cdot sl(2j + 1) \oplus q \cdot sl(2j + 1)$
3) For each element $\Psi$ of $\mathcal{K}$, select its component $\Pi_{\mathcal{K}}\Psi$ that is in $\mathcal{K}' = Ker(e_+)$
4) $\mathcal{K}_0$ is defined by

$$\mathcal{K}_0 = \{ \Pi_{\mathcal{K}}\Psi, \text{ with } \Psi \in \mathcal{K} \}$$

(4.5)

In the case of $osp(2|2)$, $\mathcal{K}_0$ will simply be $\mathcal{K} = Ker(e_+) \cap \mathcal{G}_{<0}$.

With this definition for $\mathcal{K}_0$, the following action is invariant:

$$S = S_-(g) + \int d^2x \text{ str} \{ A(J - e_- - \Psi) + [e_+, \tau] \phi + \Psi \phi^* + A\Psi + \Psi^* \phi - A[\bar{\Psi}, \Psi] \}$$

(4.6)

with the gauge transformations

$$\delta A = \bar{\phi} \eta + [\eta, A] \quad \delta \Psi = \Pi_{\mathcal{K}_0}(\eta + [\eta, \bar{\Psi}]) \quad \delta \tau = -\Pi_{\mathcal{K}_0} \Pi_{fm(ade_+)} \eta$$

$$\delta g = g \eta \quad \delta \bar{\Psi} = \Pi_{\mathcal{K}_0}(\eta + [\eta, \Psi])$$

(4.7)

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4.3 Identification of the matter sector for the string theory

The decomposition of the $N = 2$ $\mathcal{W}$-algebra into $N = 2$ multiplets allows us to deduce the WZW model that realizes the matter sector. In fact, for each $N = 2$ multiplet, we get as $W$ generator the field corresponding to the highest weight of the $sl(2|1)$-representation. This means that, at the level of the string theory, we will get an $sl(2)$-representation. Thus from the decomposition $\oplus_i (b_i, j_i)$ results a decomposition $\oplus_i D_{j_i}$, and it is an easy task to recognize the reduction that provides such a representation. The results are:

| Starting Gauged WZW model corresponding to $\mathcal{W}(\mathcal{G},\mathcal{H})$ with | Resulting matter sector Gauged WZW model with $\mathcal{W}(\mathcal{G}',\mathcal{H}')$ for |
|---|---|
| $\mathcal{G} = sl[p(2j + 1) + q2j|p2j + q(2j + 1)]$ | $\mathcal{G}' = sl[p(2j + 1)|q(2j + 1)]$ |
| $\mathcal{H} = p sl(2j + 1|2j) \oplus q sl(2j|2j + 1)$ | $\mathcal{H}' = p sl(2j + 1) \oplus q sl(2j + 1)$ |
| $\mathcal{G} = osp(m|2n)$ | $\mathcal{G}' = osp(m - 2|2n)$ |
| $\mathcal{H} = osp(2|2)$ | $\mathcal{H}' = sp(2n)$ |

One remarks that, in the first case, we get bosonic matter sector as soon as $q = 0$ (we have loosely noted $sl(m|0) = sl(m)$), as it has been already constructed for the classical $\mathcal{W}_m$-gravity [4]. In the second case, we recover the construction of $N$-extended superstring from $osp(2|2N)$ [4].

5 Conclusion

We have shown how the non-standard HR associated to $sl(2|1)$ embeddings can be related to $\mathcal{W}$-gravity. This approach provide an algebraic tool for the calculation of BRS cohomology of $\mathcal{W}$-string.

Then, for such a purpose, we have classified all the $sl(2|1)$ embeddings in superalgebras. We have given general formulae that allow the calculation of the multiplet content for the corresponding $N = 2$ super $\mathcal{W}$-algebra, a result which is in itself interesting.

Finally, we have introduced a non-standard HR (in the case $b = 0$) and exhibited invariant actions.

The generalizations of this approach are numerous: first one has to find a correct action in the case $b \neq 0$, together with a good interpretation of the non-standard HR; second, one can do an $N = 1$ superfield approach of non-standard HR, as it has already been done for usual HR associated to $osp(1|2)$ embeddings [10, 11, 13]: in that case, we should find super $\mathcal{W}$-gravity (in $N = 1$ formalism) and relate it to $osp(3|2)$ embeddings.

Finally, a special treatment as to be done for the cases which are not fully reducible: what happen at the level of the $N = 2$ super $\mathcal{W}$-algebra?

Works are in progress on the two first points. For the last one, we hope to do the simplest example, namely $osp(4|2)/2osp(2|2)$.
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