Exact Solutions for an MHD Generalized Burgers fluid: Stokes’ Second Problem

Masood Khan, Rabia Malik and Asia Anjum
Department of Mathematics, Quaid-i-Azam University, Islamabad 44000, Pakistan

Abstract: This paper offers the exact analytical solutions for the magnetohydrodynamic (MHD) flow of an incompressible generalized Burgers fluid corresponding to the second problem of Stokes in the presence of the transverse magnetic field. Modified Darcy’s law has been taken into account. The expression for the velocity field and associated tangential stress, presented as a sum of the steady-state and transient solutions, are obtained by means of the integral transforms. Moreover, several figures are plotted to investigate the effects of various emerging parameters on the velocity field. The obtained results show that the magnitude of the velocity and boundary layer thickness significantly reduce in the presence of magnetic field.

Keywords: Generalized Burgers fluid; MHD flow; Porous medium; Exact solutions.

1 Introduction

Considerable progress has been made in studying flows of non-Newtonian fluids throughout the last few decades. Due to their viscoelastic nature, non-Newtonian fluids, such as oils, paints, ketchup, liquid polymers, asphalt and for forth exhibit some remarkable phenomena. Amplifying interest of many researchers has shown that these flows are imperative in industry, manufacturing of food and paper, polymer processing and technology. Dissimilar to the Newtonian fluid, the flows of non-Newtonian fluids cannot be explained by a single constitutive model. Therefore, models have been recognized due to the rheological properties of non-Newtonian fluids. Amongst them rate type fluid model [1] has received great devotion. These fluids exhibit the relaxation and retardation phenomena. The simplest subclasses of rate type fluids are the Maxwell and Oldroyd-B fluids. It is not an easy task to obtain the analytical solutions for such fluids. In spite of several challenges, many researchers have established the analytical solutions regarding these fluids [2 − 10]. However, these fluids do not predict the rheological properties of some fluids like cheese in food products and asphalt in geomechanics. In 1935, one-dimensional rate-type model known as the Burgers model [11] was put in a thermodynamic framework, which was later extended to the frame-indifferent three-dimensional form by Krishnan and Rajagopal [12]. This model has been utilized to describe the motion of the earth mental. This model is also the preferred model to explain the response of asphalt and asphalt concrete [13]. In addition, Burgers model is sometimes used to model other geological structures, such as Olivine rocks [14] and the propagation of seismic waves in the interior of the earth [15]. Here, we mention some studies [16 − 20] related to the Burgers fluid. Moreover, MHD flow involving such fluids has promising applications on the development of energy generation, astrophysics and geophysics fluid dynamics. Recently the theory of MHD has received great attention [21, 22] and references therein. The effects of transverse magnetic field in the porous space over the unsteady non-Newtonian fluids are analyzed by several researchers [23 − 25].

1 Corresponding author: Electronic mail: mkhan@qau.edu.pk; mkhan_21@yahoo.com (M. Khan)
Motivated by the above mentioned studies, this work presents an MHD flow of a generalized Burgers fluid through a porous space due to the oscillation of an infinite rigid plate. The transverse magnetic field and modified Darcy’s law and their influence on the flow are considered. The solutions are presented as a sum of steady-state and transient solutions. The effects of some physical parameters are discussed through graphical illustrations.

2 Governing Equations

The Cauchy stress tensor $\mathbf{T}$ in a generalized Burgers fluid is given by [16–20]

$$
\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \lambda_1 \frac{\partial \mathbf{S}}{\partial t} + \lambda_2 \frac{\partial^2 \mathbf{S}}{\partial t^2} = \mu \left[ \mathbf{A}_1 + \lambda_3 \frac{\partial \mathbf{A}_1}{\partial t} + \lambda_4 \frac{\partial^2 \mathbf{A}_1}{\partial t^2} \right],
$$

where $\mathbf{S}$ is the extra-stress tensor, $p$ the pressure, $\mathbf{I}$ the identity tensor, $\mu$ the dynamic viscosity, $\mathbf{A}_1 = \mathbf{L} + \mathbf{L}^T$ the first Rivlin-Ericksen tensor with $\mathbf{L}$ as the velocity gradient, $\lambda_1$ and $\lambda_3$ ($\leq \lambda_1$) are relaxation and retardation times, respectively, $\lambda_2$ and $\lambda_4$ are material parameters of generalized Burgers fluid and $\delta/\delta t$ denotes the upper convected time derivative defined by

$$
\frac{\delta \mathbf{S}}{\delta t} = \frac{d\mathbf{S}}{dt} - \mathbf{L}\mathbf{S} - \mathbf{S}\mathbf{L}^T, \quad \frac{\delta^2 \mathbf{S}}{\delta t^2} = \frac{\delta}{\delta t} \left( \frac{\delta \mathbf{S}}{\delta t} \right),
$$

in which $d/dt$ is the material time derivative.

The basic equations governing the unsteady flow of an incompressible fluid are

$$
\text{div} \mathbf{V} = 0, \quad \rho \frac{d\mathbf{V}}{dt} = \text{div} \mathbf{T} - \sigma B_0^2 \mathbf{V} + \mathbf{R},
$$

where $\mathbf{V}$ is the velocity, $\rho$ the density of the fluid, $\sigma$ the electrical conductivity of the fluid, $B_0$ the magnitude of applied magnetic field and $\mathbf{R}$ denotes the Darcy’s resistance.

For the problem under consideration we shall assume the velocity and stress fields of the form

$$
\mathbf{V} = \mathbf{V}(y, t) = u(y, t)\mathbf{i}, \quad \mathbf{S} = \mathbf{S}(y, t),
$$

where $\mathbf{i}$ is the unit vector along the $x$–coordinate direction. The velocity field (5) automatically satisfies the continuity equation (3).

Substitution of Eq. (5) in Eq. (1) and having in mind the initial conditions

$$
\mathbf{S}(y, 0) = \frac{\partial \mathbf{S}(y, 0)}{\partial t} = \mathbf{0},
$$

yields $S_{yy} = S_{yz} = S_{zz} = S_{xz} = 0$ and

$$
\left( 1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2} \right) S_{xy} = \mu \left( 1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2} \right) \frac{\partial u}{\partial y},
$$

where $S_{xy}$ is the tangential stress.

In view of the reference [22], we have the following relation of $\mathbf{R}$ for a generalized Burgers fluid

$$
\left( 1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2} \right) \mathbf{R} = -\frac{\mu\sigma}{k} \left( 1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2} \right) \mathbf{V}(y, t),
$$
where \( \varphi \) is the porosity and \( k \) the permeability of the porous medium.

By substituting Eq. (5) in the balance of linear momentum (4), having in mind Eqs. (7) and (8) and assuming that there is no pressure gradient in the flow direction, one finds the following governing equation

\[
\begin{align*}
\left( 1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2} \right) \frac{\partial u(y,t)}{\partial t} & = \nu \left( 1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2} \right) \frac{\partial^2 u(y,t)}{\partial y^2} \\
- \frac{\sigma B_0^2}{\rho} \left( 1 + \lambda_1 \frac{\partial}{\partial t} + \lambda_2 \frac{\partial^2}{\partial t^2} \right) u(y,t) - \frac{\nu \varphi}{k} \left( 1 + \lambda_3 \frac{\partial}{\partial t} + \lambda_4 \frac{\partial^2}{\partial t^2} \right) u(y,t),
\end{align*}
\]

where \( \nu = \mu/\rho \) is the kinematic viscosity of the fluid.

3 Statement of the Problem and its Solution

Let we consider an incompressible and electrically conducting generalized Burgers fluid. The fluid occupies the porous space above the flat plate perpendicular to the \( y \)-axis and permeated by an applied magnetic field \( B_0 \) normal to the flow. For \( t > 0 \) the plate starts to oscillate in its own plane with velocity \( U H(t) \cos(\omega t) \) or \( U H(t) \sin(\omega t) \) with \( H(\cdot) \) as the Heaviside unit step function and \( U \) the amplitude of the velocity of the plate. Due to the shear, the fluid above the plate is gradually moved. The governing equation of the problem is (9) and the associated initial and boundary conditions are

\[
\begin{align*}
u \frac{\partial^2 u(y,0)}{\partial y^2} & = 0; \quad y > 0, \\
\frac{\partial u(0,t)}{\partial y} & = \frac{\partial^2 u(0,t)}{\partial y^2} = 0; \quad t > 0, \\
u \frac{\partial u(0,t)}{\partial y} & = H(t) \cos(\omega t) \quad \text{or} \quad u(0,t) = H(t) \sin(\omega t); \quad t > 0,
\end{align*}
\]

and

\[
u \frac{\partial u(y,t)}{\partial y} \to 0 \quad \text{as} \quad y \to \infty \quad \text{and} \quad t > 0.
\]

In the non-dimensional form we can write the above problem as

\[
\begin{align*}
\left( 1 + \frac{\partial}{\partial t} + \beta \frac{\partial^2}{\partial t^2} \right) \frac{\partial u(y,t)}{\partial t} & = \nu \left( 1 + \alpha \frac{\partial}{\partial t} + \gamma \frac{\partial^2}{\partial t^2} \right) \frac{\partial^2 u(y,t)}{\partial y^2} - M^2 \left( 1 + \frac{\partial}{\partial t} + \beta \frac{\partial^2}{\partial t^2} \right) u(y,t) \\
- \frac{1}{K} \left( 1 + \alpha \frac{\partial}{\partial t} + \gamma \frac{\partial^2}{\partial t^2} \right) u(y,t),
\end{align*}
\]

where \( \begin{align*}
u \frac{\partial^2 u(y,0)}{\partial y^2} & = 0; \quad y > 0, \\
\frac{\partial u(0,t)}{\partial y} & = \frac{\partial^2 u(0,t)}{\partial y^2} = 0; \quad t > 0, \\
u \frac{\partial u(0,t)}{\partial y} & = H(t) \cos(\omega t) \quad \text{or} \quad u(0,t) = H(t) \sin(\omega t); \quad t > 0,
\end{align*}\)

and

\[
u \frac{\partial u(y,t)}{\partial y} \to 0 \quad \text{as} \quad y \to \infty \quad \text{and} \quad t > 0,
\]

where

\[
u^* = \frac{u}{U}, \quad t^* = \frac{t}{\lambda_1}, \quad y^* = \frac{y}{\sqrt{\nu \lambda_1}}, \quad \alpha = \frac{\lambda_3}{\lambda_1}, \quad \beta = \frac{\lambda_2}{\lambda_1}, \quad \gamma = \frac{\lambda_4}{\lambda_1}, \quad M^2 = \frac{\sigma B_0^2 \lambda_1}{\rho}, \quad \frac{1}{K} = \frac{\lambda_1 \nu \varphi}{k},
\]

with asterisks have been omitted for simplicity.
3.1 Calculation of the Velocity Field

In order to obtain the exact solution describing the flow for small and large times, we shall use the Fourier sine and Laplace transforms [26]. Thus, multiplying both sides of Eq. (13) by \( \sin(y\xi) \) integrating the result with respect to \( y \) from 0 to infinity, and taking into account the initial and boundary conditions (14) – (16), we find that

\[
\beta \frac{\partial^3 u_s(\xi, t)}{\partial t^3} + \left( 1 + \xi^2 \gamma + \beta M^2 + \frac{\gamma}{K} \right) \frac{\partial^2 u_s(\xi, t)}{\partial t^2} \\
+ \left( 1 + \alpha \xi^2 + M^2 + \frac{\alpha}{K} \right) \frac{\partial u_s(\xi, t)}{\partial t} + \left( \xi^2 + M^2 + \frac{1}{K} \right) u_s(\xi, t) \\
= \left( 1 + \alpha \frac{\partial}{\partial t} + \gamma \frac{\partial^2}{\partial t^2} \right) \xi H(t) \cos(wt); \quad \xi, t > 0,
\]

respectively,

\[
\beta \frac{\partial^3 u_s(\xi, t)}{\partial t^3} + \left( 1 + \xi^2 \gamma + \beta M^2 + \frac{\gamma}{K} \right) \frac{\partial^2 u_s(\xi, t)}{\partial t^2} \\
+ \left( 1 + \alpha \xi^2 + M^2 + \frac{\alpha}{K} \right) \frac{\partial u_s(\xi, t)}{\partial t} + \left( \xi^2 + M^2 + \frac{1}{K} \right) u_s(\xi, t) \\
= \left( 1 + \alpha \frac{\partial}{\partial t} + \gamma \frac{\partial^2}{\partial t^2} \right) \xi H(t) \sin(wt); \quad \xi, t > 0,
\]

where the Fourier sine transform has to satisfy the initial conditions

\[
u_s(\xi, 0) = \frac{\partial u_s(\xi, 0)}{\partial t} = \frac{\partial^2 u_s(\xi, 0)}{\partial t^2} = 0; \quad \xi > 0.
\]

Now taking the Laplace transform of Eqs. (18) and (19) subject to the initial conditions (20), we obtain

\[
\bar{U}_s(\xi, q) = \frac{\xi}{\beta} \frac{\xi}{q(q^2 + w^2)} \left[ (1 + \alpha q + \gamma q^2) q^2 \\
+ \left( \frac{1 + \xi^2 \gamma + a_m}{\beta} \right) q^2 \\
+ \left( \frac{1 + \alpha \xi^2 + c_m}{\beta} \right) q + \left( \frac{\xi^2 + b_m}{\beta} \right) \right],
\]

respectively,

\[
\bar{U}_s(\xi, q) = \frac{\xi}{\beta} \frac{w(1 + \alpha q + \gamma q^2) q}{q^2 + w^2} \left[ q^2 + \left( \frac{1 + \xi^2 \gamma + a_m}{\beta} \right) q^2 \\
+ \left( \frac{1 + \alpha \xi^2 + c_m}{\beta} \right) q + \left( \frac{\xi^2 + b_m}{\beta} \right) \right],
\]

where \( \bar{U}_s(\xi, q) \) is the Laplace transform of \( u_s(\xi, t) \), \( q \) the transform parameter and

\[
a_m = \beta M^2 + \frac{\gamma}{K}, \quad b_m = M^2 + \frac{1}{K}, \quad c_m = M^2 + \frac{\alpha}{K}.
\]

Rewriting Eqs. (21) and (22) in simpler form as:

\[
\bar{U}_s(\xi, q) = \frac{\xi G(\xi, q)}{\beta q},
\]

where
Equations (24) and (25) can also be written as

\[
G(\xi, q) = \frac{(1 + \alpha q + \gamma q^2) q^2}{(q^2 + w^2)(q - q_1)(q - q_2)(q - q_3)},
\]
respectively
\[
G(\xi, q) = \frac{w (1 + \alpha q + \gamma q^2) q}{(q^2 + w^2)(q - q_1)(q - q_2)(q - q_3)}.
\]

Equations (24) and (25) can also be written as

\[
G(\xi, q) = \frac{q_1^2(1 + \alpha q_1 + \gamma q_1^2)}{(q_1 - q_2)(q_1 - q_3)(q_1^2 + w^2)} \cdot \frac{1}{(q - q_1)} - \frac{q_2^2(1 + \alpha q_2 + \gamma q_2^2)}{(q_1 - q_2)(q_2 - q_3)(q_2^2 + w^2)} \cdot \frac{1}{(q - q_2)} + \frac{q_3^2(1 + \alpha q_3 + \gamma q_3^2)}{(q_1 - q_3)(q_2 - q_3)(q_3^2 + w^2)} \cdot \frac{1}{(q - q_3)} + \frac{\psi_1}{\eta} \cdot \frac{w}{(q^2 + w^2)} + \frac{\psi_2}{\eta} \cdot \frac{q}{(q^2 + w^2)},
\]
respectively,
\[
G(\xi, q) = \frac{w q_1(1 + \alpha q_1 + \gamma q_1^2)}{(q_1 - q_2)(q_1 - q_3)(q_1^2 + w^2)} \cdot \frac{1}{(q - q_1)} - \frac{w q_2(1 + \alpha q_2 + \gamma q_2^2)}{(q_1 - q_2)(q_2 - q_3)(q_2^2 + w^2)} \cdot \frac{1}{(q - q_2)} + \frac{w q_3(1 + \alpha q_3 + \gamma q_3^2)}{(q_1 - q_3)(q_2 - q_3)(q_3^2 + w^2)} \cdot \frac{1}{(q - q_3)} + \frac{\psi_1}{\eta} \cdot \frac{w}{(q^2 + w^2)} - \frac{\psi_2}{\eta} \cdot \frac{q}{(q^2 + w^2)},
\]

where

\[
q_j = s_j - \frac{1 + \xi^2 \gamma + a_m}{3 \beta^2}, \quad j = 1, 2, 3;
\]
\[
s_1 = 3 \sqrt{-\frac{p_1}{2} + \sqrt{\frac{p_1^3}{4} + \frac{p_1^5}{27}}} + 3 \sqrt{-\frac{p_1}{2} - \sqrt{\frac{p_1^3}{4} + \frac{p_1^5}{27}}},
\]
\[
s_2 = h \sqrt{3 \sqrt{-\frac{p_1}{2} + \sqrt{\frac{p_1^3}{4} + \frac{p_1^5}{27}}} + h^2 \sqrt{-\frac{p_1}{2} - \sqrt{\frac{p_1^3}{4} + \frac{p_1^5}{27}}},
\]
\[
s_3 = h^3 \sqrt{-\frac{p_1}{2} + \sqrt{\frac{p_1^3}{4} + \frac{p_1^5}{27}}} + h^3 \sqrt{-\frac{p_1}{2} - \sqrt{\frac{p_1^3}{4} + \frac{p_1^5}{27}}},
\]
\[
p_1 = \frac{\xi^2 + b_m}{\beta} - (1 + \xi^2 \gamma + a_m) \left(1 + \alpha \xi^2 + c_m\right) + \frac{2}{27 \beta^3} (1 + \xi^2 \gamma + a_m)^3,
\]
\[
p_2 = \frac{1 + \alpha \xi^2 + c_m}{\beta} - \frac{(1 + \xi^2 \gamma + a_m)^2}{3 \beta^2}, \quad h = -\frac{1 + i \sqrt{3}}{2},
\]
\[
\psi_1 = q_1 q_2 + q_1 q_3 + q_2 q_3 - w^2 + q_1 w^2 \alpha - q_2 w^2 \alpha - q_3 w^2 \alpha - q_1 q_2 w^2 \gamma - q_1 q_3 w^2 \gamma - q_2 q_3 w^2 \gamma + w^4 \gamma,
\]
\[
\psi_2 = -q_1 q_2 q_3 + q_1 w^2 + q_2 w^2 + q_3 w^2 + q_1 q_2 w^2 \alpha + q_1 q_3 w^2 \alpha + q_2 q_3 w^2 \alpha - w^4 \alpha + q_1 q_2 q_3 w^2 \gamma - q_1 w^4 \gamma - q_2 w^4 \gamma - q_3 w^4 \gamma,
\]
\[
\psi_3 = (q_1^2 + w^2)(q_2^2 + w^2)(q_3^2 + w^2).
\]

Applying the inverse Laplace transform to Eq. (23), one obtains
respectively,

\[ u_s(\xi, t) = \frac{\xi}{\beta} \left[ \frac{\chi_1 e^{q_1 t}}{(q_1 - q_2)(q_1 - q_3)} - \frac{\chi_2 e^{q_2 t}}{(q_1 - q_2)(q_2 - q_3)} + \frac{\chi_3 e^{q_3 t}}{(q_1 - q_3)(q_2 - q_3)} + \frac{\psi_1}{\eta} \sin(\omega t) + \frac{\psi_2}{\eta} \cos(\omega t) \right], \quad (28) \]

\[ u_s(\xi, t) = \frac{\xi}{\beta} \left[ \frac{\chi_1^* e^{q_1 t}}{(q_1 - q_2)(q_1 - q_3)} - \frac{\chi_2^* e^{q_2 t}}{(q_1 - q_2)(q_2 - q_3)} + \frac{\chi_3^* e^{q_3 t}}{(q_1 - q_3)(q_2 - q_3)} + \frac{\psi_1}{\eta} \sin(\omega t) - \frac{\psi_2}{\eta} \cos(\omega t) \right], \quad (29) \]

where

\[ \chi_i = \frac{q_i(1 + \alpha q_i + \gamma q_i^2)}{(q_i^2 + w^2)}, \quad \chi_i^* = \frac{w(1 + \alpha q_i + \gamma q_i^2)}{(q_i^2 + w^2)}, \quad i = 1, 2, 3. \]

Inverting Eqs. (28) and (29) by means of the Fourier’s sine formulae [26], we can write the starting solutions as

\[ u(y, t) = H(t) \frac{1}{\beta} \int_0^\infty \left( \frac{\chi_1 e^{q_1 t}}{(q_1 - q_2)(q_1 - q_3)} - \frac{\chi_2 e^{q_2 t}}{(q_1 - q_2)(q_2 - q_3)} + \frac{\chi_3 e^{q_3 t}}{(q_1 - q_3)(q_2 - q_3)} \right) \xi \sin(\xi y) d\xi \]

\[ + \frac{\psi_1}{\beta X} H(t) \sin(\omega t) \int_0^\infty \frac{\xi \sin(\xi y)}{(\xi^2 + Y^2)^2 + Z^2} d\xi + H(t) \cos(\omega t) \int_0^\infty \frac{(\xi^2 + Y^2) \sin(\xi y)}{(\xi^2 + Y^2)^2 + Z^2} d\xi, \]

\[ (30) \]

respectively,

\[ u(y, t) = \frac{H(t)}{\beta} \int_0^\infty \left( \frac{\chi_1^* e^{q_1 t}}{(q_1 - q_2)(q_1 - q_3)} - \frac{\chi_2^* e^{q_2 t}}{(q_1 - q_2)(q_2 - q_3)} + \frac{\chi_3^* e^{q_3 t}}{(q_1 - q_3)(q_2 - q_3)} \right) \xi \sin(\xi y) d\xi \]

\[ + H(t) \sin(\omega t) \int_0^\infty \frac{(\xi^2 + Y^2) \sin(\xi y)}{(\xi^2 + Y^2)^2 + Z^2} d\xi - \frac{\psi_1}{\beta X} H(t) \cos(\omega t) \int_0^\infty \frac{\xi \sin(\xi y)}{(\xi^2 + Y^2)^2 + Z^2} d\xi, \]

\[ (31) \]

with

\[ X = \frac{1 + (\alpha^2 - 2\gamma) w^2 + \gamma^2 w^4}{\beta^2 w^4}, \]

\[ Y^2 = \frac{[\gamma (1 + a_m) - \alpha \beta] w^4 + [\alpha (1 + c_m) - (1 + a_m + \gamma b_m)] w^2 + b_m}{1 + (\alpha^2 - 2\gamma) w^2 + \gamma^2 w^4}, \]

\[ Z^2 = \frac{w^2 [1 + c_m - b_m \alpha - c_m w^2 \gamma + w^2 (\alpha + a_m \alpha - \beta - \gamma + w^2 \beta \gamma)]^2}{1 + w^2 \{\alpha^2 + \gamma (-2 + w^2 \gamma)\}^2}. \]

As we know the following relations

\[ \int_0^\infty \frac{\xi \sin(\xi y)}{(\xi^2 + Y^2)^2 + Z^2} d\xi = \frac{\pi}{2Z} e^{-Ay} \sin(By), \quad \int_0^\infty \frac{(\xi^2 + Y^2) \sin(\xi y)}{(\xi^2 + Y^2)^2 + Z^2} d\xi = \frac{\pi}{2} e^{-Ay} \cos(By), \]

\[ (32) \]

where

\[ 2A^2 = \sqrt{Y^4 + Z^2} + Y^2, \quad 2B^2 = \sqrt{Y^4 + Z^2} - Y^2. \]
Using the above relations into Eqs. (30) and (31) we get the following simplified expressions for the starting solutions

\[
\begin{align*}
u(y, t) &= \frac{H(t)}{\beta} \int_0^\infty \left( \frac{\chi_1 e^{\eta t}}{(q_1 - q_2)(q_1 - q_3)} - \frac{\chi_2 e^{\eta t}}{(q_1 - q_2)(q_2 - q_3)} + \frac{\chi_3 e^{\eta t}}{(q_1 - q_3)(q_2 - q_3)} \right) \xi \sin(\xi d) d\xi \\
&+ \frac{\pi}{2} H(t) e^{-Ay} \cos(\omega t - By),
\end{align*}
\]

respectively,

\[
\begin{align*}
u(y, t) &= \frac{H(t)}{\beta} \int_0^\infty \left( \frac{\chi_1 e^{\eta t}}{(q_1 - q_2)(q_1 - q_3)} - \frac{\chi_2 e^{\eta t}}{(q_1 - q_2)(q_2 - q_3)} + \frac{\chi_3 e^{\eta t}}{(q_1 - q_3)(q_2 - q_3)} \right) \xi \sin(\xi d) d\xi \\
&+ \frac{\pi}{2} H(t) e^{-Ay} \sin(\omega t - By).
\end{align*}
\]

The starting solutions (33) and (34) are presented as a sum of the steady-state and transient solutions. The steady-state solutions are

\[
u(y, t) = \frac{\pi}{2} H(t) e^{-Ay} \cos(\omega t - By),
\]

respectively,

\[
u(y, t) = \frac{\pi}{2} H(t) e^{-Ay} \sin(\omega t - By).
\]

### 3.2 Calculation of the Shear Stress

To determine the shear stress, we use relation (7) in the non-dimensional form

\[
\left( 1 + \frac{\partial}{\partial t} + \beta \frac{\partial^2}{\partial t^2} \right) \tau = \left( 1 + \alpha \frac{\partial}{\partial t} + \gamma \frac{\partial^2}{\partial t^2} \right) \frac{\partial U}{\partial y}.
\]

Applying the Laplace transform to Eq. (37) one obtains

\[
\bar{\tau}(y, q) = \frac{1 + \alpha q + \gamma q^2}{\beta(q - q_4)(q - q_5)} \frac{\partial \bar{U}(y, q)}{\partial y},
\]

where

\[
q_4 = -1 + \frac{\sqrt{1 - 4\beta}}{2\beta}, \quad q_5 = -1 - \frac{\sqrt{1 - 4\beta}}{2\beta},
\]

and \(\bar{\tau}(y, q)\) is the Laplace transform of \(\tau(y, t)\) and the image function \(\bar{U}(y, q) = L[u(y, t)]\) has been obtained through Eqs. (21) and (22) by applying the inverse Fourier sine transform and gets

\[
\bar{U}(y, q) = \frac{2}{\pi \beta} \int_0^{\infty} \frac{(1 + \alpha q + \gamma q^2) q^2}{q(q^2 + w^2)(q - q_1)(q - q_2)(q - q_3)} \xi \sin(\xi d) d\xi,
\]

respectively

\[
\bar{U}(y, q) = \frac{2}{\pi \beta} \int_0^{\infty} \frac{(1 + \alpha q + \gamma q^2) q w}{q(q^2 + w^2)(q - q_1)(q - q_2)(q - q_3)} \xi \sin(\xi d) d\xi.
\]

Substituting Eqs. (39) and (40) in Eq. (38), we reach at the following expressions

\[
\tau(y, q) = \frac{2}{\pi \beta^2} \int_0^{\infty} \frac{(1 + \alpha q + \gamma q^2) q^2}{q(q^2 + w^2)(q - q_1)(q - q_2)(q - q_3)(q - q_4)(q - q_5)} \xi^2 \cos(\xi d) d\xi,
\]
respectively

\[ \tau(y, q) = \frac{2}{\pi \beta^2} \int_0^\infty \frac{(1 + \alpha q + \gamma q^2) w}{q(q^2 + w^2)(q - q_1)(q - q_2)(q - q_3)(q - q_4)(q - q_5)} \xi^2 \cos(\xi y) d\xi, \quad (42) \]

or in simpler form we can write as

\[ \bar{\tau}(y, q) = \frac{2}{\pi \beta^2} \int_0^\infty \frac{F(\xi, q)}{q} \xi^2 \cos(\xi y) d\xi, \quad (43) \]

where

\[ F(\xi, q) = \frac{(1 + \alpha q + \gamma q^2) q^2}{(q^2 + w^2)(q - q_1)(q - q_2)(q - q_3)(q - q_4)(q - q_5)}, \quad (44) \]

respectively

\[ F(\xi, q) = \frac{(1 + \alpha q + \gamma q^2) w}{(q^2 + w^2)(q - q_1)(q - q_2)(q - q_3)(q - q_4)(q - q_5)}. \quad (45) \]

Inverting Eq. (43) by means of the Laplace transform, following the same way as for the velocity field, we find the following expressions for the tangential stress

\[ \tau(y, t) = \frac{2H(t)}{\pi \beta^2} \int_0^\infty \left( \frac{\zeta_1 e^{\eta t}}{(q_1 - q_2)(q_1 - q_3)} + \frac{\zeta_2 e^{\eta t}}{(q_2 - q_1)(q_2 - q_3)} + \frac{\zeta_3 e^{\eta t}}{(q_1 - q_3)(q_2 - q_3)} \right) \xi^2 \cos(\xi y) d\xi \]

\[ - \frac{H(t)}{\beta^2} e^{-By} \left[ [A \sin(Ay) - B \cos(Ay)] [\psi_4 \cos(wt) + \psi_5 \sin(wt)] 
- [A \cos(Ay) - B \sin(Ay)] [\psi_6 \cos(wt) + \psi_7 \sin(wt)] \right], \quad (46) \]

respectively

\[ \tau(y, t) = \frac{2H(t)}{\pi \beta^2} \int_0^\infty \left( \frac{\zeta_1^* e^{\eta t}}{(q_1 - q_2)(q_1 - q_3)} + \frac{\zeta_2^* e^{\eta t}}{(q_2 - q_1)(q_2 - q_3)} + \frac{\zeta_3^* e^{\eta t}}{(q_1 - q_3)(q_2 - q_3)} \right) \xi^2 \cos(\xi y) d\xi 
+ \frac{H(t)}{\beta^2} e^{-By} \left[ [A \sin(Ay) - B \cos(Ay)] [\psi_4^* \cos(wt) - \psi_5 \sin(wt)] 
- [A \cos(Ay) - B \sin(Ay)] [\psi_6^* \cos(wt) - \psi_7 \sin(wt)] \right], \quad (47) \]

where we have used notations

\[ \zeta_i = \frac{q_i(1 + \alpha q_i + \gamma q_i^2)^2}{(q_i^2 + w^2)(q_i - q_1)(q_i - q_5)}, \quad \zeta_i^* = \frac{w(1 + \alpha q_i + \gamma q_i^2)^2}{(q_i^2 + w^2)(q_i - q_4)(q_i - q_5)}, \quad i = 1, 2, 3, \]
ψ_4 = \frac{\beta^2 (1 + w^2 \alpha - w^2 \beta - w^2 \gamma + w^4 \beta \gamma)}{1 + w^2 (1 + \beta (-2 + w^2 \beta))}, \quad ψ_5 = \frac{w \beta^2 (1 - \alpha + w^2 \alpha \beta - w^2 \gamma)}{1 + w^2 (1 + \beta (-2 + w^2 \beta))},

ψ_6 = \frac{\beta^2}{[1 + w^2 (1 + \beta (-2 + w^2 \beta))] [1 + w^2 (\alpha^2 + \gamma (-2 + w^2 \gamma))]} \left[ w^8 \left( -2 \alpha \beta^2 \gamma + 2 \beta \gamma^2 + a_m \beta \gamma^2 \right) + b_m \right.

+ w^2 (-2 - a_m - c_m + 2 \alpha + 2 c_m \alpha - b_m ((-2 + \alpha) \alpha + \beta + 2 \gamma))

\times \left( -2 \alpha \beta^2 - a_m \beta^2 + 2 (1 + a_m) \alpha \gamma - 4 \beta \gamma - 2 a_m \beta \gamma - 2 \gamma^2 \right)

+ 2 (2 + a_m + c_m) \alpha \gamma - 2 a_m \alpha \gamma + b_m (\alpha^2 \beta - 2 \alpha \gamma + \gamma (2 \beta + \gamma))

\left. \right] \left[ (w^2 (1 + a_m - \alpha (1 + c_m - w^2 \beta)) - (1 + a_m) w^4 \gamma + b_m (-1 + w^2 \gamma)) \right]

\times (1 + w^2 (\alpha^2 + \gamma (-2 + w^2 \gamma))) (1 + w^2 (\alpha - \beta - \gamma) + w^4 \gamma)

\frac{1}{(1 + \alpha (-1 + w^2 \beta) - w^2 \gamma)} \right],

ψ_7 = \frac{w \beta^2}{[1 + w^2 (1 + \beta (-2 + w^2 \beta))] [1 + w^2 (\alpha^2 + \gamma (-2 + w^2 \gamma))]} \left[ (1 + w^2 (\alpha - \beta)) (1 - 2 b_m \alpha + w^2 (2 + 2 a_m - \alpha) \alpha + (-1 + w^2 \alpha^2) (-c_m + w^2 \beta)) \right.

- w^2 \gamma \left( \frac{2 + 2 c_m + (1 + a_m) w^2 - b_m (1 + \alpha (2 + w^2 (\alpha - 2 \beta)))}{+ w^2 ((1 + a_m) \alpha (2 + w^2 \alpha) - 2 (2 + c_m + (1 + a_m) w^2 \alpha) \beta + 2 w^2 \beta^3)} \right)

\times \left[ \frac{1 - 2 b_m + w^2 (2 + 2 a_m - \alpha - 2 \beta) + w^4 \beta (\alpha + \beta)}{- c_m (-1 + w^2 (\alpha + \beta))} \right]

- w^6 (-b_m + (1 + a_m) w^2) \gamma^3

4 Graphical Results and Discussion

To study the significant physical effects of the obtained results, the impact of the material parameters on the fluid motion is highlighted by graphical illustration of the velocity profiles for the flow due to the sinusoidal oscillations of the infinite plate. To illustrate the difference, we depicted the velocity profile for both cosine and sine oscillations of the boundary. The numerical results are plotted for different values of time t, magnetic parameter M, permeability parameter K, and the rheological parameters \( \beta \) and \( \gamma \) of Burgers and generalized Burgers fluids.

Figure 1 compares the profiles of velocity for different values of time for cosine and sine oscillations of the boundary, respectively. It is noted that the two oscillations have similar amplitudes and a phase shift that persevere for all times. Here we can observe that as the bottom plate is set into motion the velocity near the bottom plate is developing and fluctuating around zero with the same frequency as the plate. Additionally, the fluid oscillation has maximum amplitude adjacent the bottom plate and reduces far away from
the plate and approaches to zero.

Figure 2 presents the profiles of the velocity for different values of the magnetic parameter \( M \) for both cosine and sine oscillations of the boundary respectively. Since, magnetic field is applied in the transverse direction and it is a force which resists the flow. Therefore, with the increase in the values of the magnetic parameter \( M \), the amplitude of the oscillation tends to decrease. Also, a comparison shows that the amplitude of oscillation is larger for hydrodynamic case \( (M = 0) \) when compare with hydromagnetic case \( (M \neq 0) \) case. The effects of the permeability parameter \( K \) are depicted in figure 3. It has quite opposite effect on the velocity profile to that of the magnetic parameter \( M \). It is clearly seen that with the increase of \( K \) the velocity profile increases.

Figure 4 shows the effect of the material parameter \( \beta \) on the velocity profile for both cosine and sine oscillations of the boundary, respectively. We can clearly see from the figure that the velocity decreases slightly by increasing the parameter \( \beta \). Further, the effect of the rheological parameter \( \gamma \) of the generalized Burgers fluid is shown in figure 5 for both cosine and sine oscillations of the boundary, respectively. It is noted that an increase in the rheological parameter \( \gamma \) of generalized Burgers fluid yield an effect opposite to that of the parameter \( \beta \). From these figures, it is noticed that the profile of the velocity for the sine oscillation are more sensitive compared to the cosine oscillations of the boundary.

Figure 1 : Profiles of velocity \( u(y,t) \) given by Eqs. (33) and (34) for different values of time \( t \) for cosine and sine oscillations of the boundary, respectively.
Figure 2: Profiles of velocity $u(y,t)$ given by Eqs. (33) and (34) for different values of $M$ for cosine and sine oscillations of the boundary, respectively.

Figure 3: Profiles of velocity $u(y,t)$ given by Eqs. (33) and (34) for different values of $K$ for cosine and sine oscillations of the boundary, respectively.
Figure 4: Profiles of velocity $u(y,t)$ given by Eqs. (33) and (34) for different values of $\beta$ for cosine and sine oscillations of the boundary, respectively.

Figure 5: Profiles of velocity $u(y,t)$ given by Eqs. (33) and (34) for different values of $\gamma$ for cosine and sine oscillations of the boundary, respectively.

5 Brief Summary

In this article the problem of an MHD flow through porous medium involving generalized Burgers fluid has been discussed for cosine and sine oscillations of the boundary. Modified Darcy’s law for a generalized Burgers fluid is used in the modelling of the governing equations. The results for the flow are constructed by means of the Fourier sine and Laplace transforms. These solutions, presented as a sum of the steady-state and transient parts, explain the motion of the fluid for some times after its initiation. After that time, when the transients disappear, the motion of the fluid is described by the steady-state, which is independent of initial conditions. Based on the solutions derived, we have analyzed the influence of the various parameters on the velocity profile of the fluid.
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$u(y, t)$

(a)

$\alpha = 0.1$

$\alpha = 1.0$

$\beta = 1, \gamma = 1, M = 3,$
$K = 0.01, w = 1$
\( u(y,t) \)

\( \alpha = 0.1 \), \( \alpha = 1.0 \)

\( \beta = 1, \gamma = 1, M=3, K = 0.01, w=1 \)

- \( t = 0 \)
- \( t = \pi \)
- \( t = 3\pi/2 \)
- \( t = \pi/2 \)
\[ t = \frac{\pi}{12}, \alpha = 1, \beta = 1, \gamma = 1, K=0.1, w=1 \]
\( t = \pi/12, \alpha = 1, \beta = 1, \gamma = 1, K=0.1, w=1 \)
$u(y,t)$

t = $\pi/12$, $\alpha = 1$, $\beta = 1$, $\gamma = 1$, $M=3$, $w=1$

- $K = 0.01$
- $K = 0.05$
- $K = 0.1$
- $K = 0.5$
\[ u(y,t) \]

\[ K = 0.01, K = 0.05, K = 0.1, K = 0.5 \]

\[ t = \pi/12, \alpha = 1, \beta = 1, \gamma = 1, M=3, w=1 \]
\( t = \pi/12, \alpha = 1, \gamma = 1, M=3, K=0.1, w=1 \)
(b) $t = \pi/12, \alpha = 1, \gamma = 1, M=3, K=0.1, w=1$
t = \pi/12, \alpha = 1, \beta = 1, M=3, K=0.1, w=1
$y$ vs $u(y,t)$

(b)

$t = \pi/12$, $\alpha = 1$, $\beta = 1$, $M=3$, $K=0.1$, $w=1$