Abstract

The objective of the present paper is to develop a minimax theory for the varying coefficient model in a non-asymptotic setting. We consider a high-dimensional sparse varying coefficient model where only few of the covariates are present and only some of those covariates are time dependent. Our analysis allows the time dependent covariates to have different degrees of smoothness and to be spatially inhomogeneous. We develop the minimax lower bounds for the quadratic risk and construct an adaptive estimator which attains those lower bounds within a constant (if all time-dependent covariates are spatially homogeneous) or logarithmic factor of the number of observations.

Keywords: varying coefficient model, sparse model, minimax optimality

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1 Introduction

One of the fundamental tasks in statistics is to characterize the relationship between a set of covariates and a response variable. In the present paper we study the varying coefficient model which is commonly used for describing time-varying covariate effects. It provides a more flexible approach than the classical linear regression model and is often used to analyze the data measured repeatedly over time.

Since its introduction by Cleveland, Grosse and Shyu [9] and Hastie and Tibshirani [14] many methods for estimation and inference in the varying coefficient model have been developed (see, e.g., [38] [15] [12] [19] for the kernel-local polynomial smoothing approach, [16] [18] [17] for the polynomial spline approach, [14] [15] [8] for the smoothing spline approach and [13] for a detailed discussion of the existing methods and possible applications). In the last five years, the varying coefficient model received a great deal of attention. For example, Wang et al. [36] proposed a new procedure based on a local rank estimator; Kai et al. [20] introduced a semi-parametric quantile regression procedure and studied an effective variable selection procedure. Lee et al. [22] extended the model to the case when the response is related to the covariates via a link function while Zhu et al. [41] studied the multivariate version of the model. Existing methods typically provide asymptotic evaluation of the precision of the estimation procedure under the assumption that the number of observations tends to infinity and is larger than the dimension of the problem.

Recently few authors consider still asymptotic but high-dimensional approach to the problem. Wei et al. [37] applied group Lasso for variable selection, while Lian [24] used extended
Bayesian information criterion. Fan et al. [11] applied nonparametric independence screening. Their results were extended by Lian and Ma [24] to include rank selection in addition to variable selection.

One important aspect that has not been well studied in the existing literature is the non-asymptotic approach to the estimation, prediction and variables selection in the varying coefficient model. Here, we refer to the situation where both the number of unknown parameters and the number of observations are large and the former may be of much higher dimension than latter. The only reference that we are aware of in this setting, is the recent paper by Klopp and Pensky [21]. Their method is based on some recent developments in the matrix estimation problem. Some interesting questions arise in this non-asymptotic setting. One of them is the fundamental question of the minimax optimal rates of convergence. The minimax risk characterizes the essential statistical difficulty of the problem. It also captures the interplay between different parameters in the model. To the best of our knowledge, our paper presents the first \textit{non-asymptotic minimax study} of the sparse heterogeneous varying coefficient model.

Modern technologies produce very high dimensional data sets and, hence, stimulate an enormous interest in variable selection and estimation under a sparse scenario. In such scenarios, penalization-based methods are particularly attractive. Significant progress has been made in understanding the statistical properties of these methods. For example, many authors have studied the variable selection, estimation and prediction properties of the LASSO in high-dimensional setting (see, e.g., [2], [4], [5], [34]). A related LASSO-type procedure is the group-LASSO, where the covariates are assumed to be clustered in groups (see, for example, [40, 1, 7, 26, 27, 25], and references therein).

In the present paper, we also consider the case when the solution is sparse, in particular, only few of the covariates are present and only some of them are time dependent. This setup is close to the one studied in a recent paper of Liang [23]. One important difference, however, is that in [23], the estimator is not adaptive to the smoothness of the time dependent covariates. In addition, Liang [23] assumes that all time dependent covariates have the same degree of smoothness and are spatially homogeneous. On the contrary, we consider a much more flexible and realistic scenario where the time dependent covariates possibly have different degrees of smoothness and may be spatially inhomogeneous.

In order to construct a minimax optimal estimator, we introduce the block Lasso which can be viewed as a version of the group LASSO. However, unlike in group LASSO, where the groups occur naturally, the blocks in block LASSO are driven by the need to reduce the variance as it is done, for example, in block thresholding. Note that our estimator does not require the knowledge which of the covariates are indeed present and which are time dependent. It adapts to sparsity, to heterogeneity of the time dependent covariates and to their possibly spatial inhomogeneous nature. In order to ensure the optimality, we derive minimax lower bounds for the risk and show that our estimator attains those bounds within a constant (if all time-dependent covariates are spatially homogeneous) or logarithmic factor of the number of observations. The analysis is carried out under the flexible assumption that the noise variables are sub-Gaussian. In addition, it does not require that the elements of the dictionary are uniformly bounded.

The rest of the paper is organized as follows. Section 1.1 provides formulation of the problem while Section 1.2 lays down a tensor approach to estimation. Section 2 introduces notations and assumptions on the model and provides a discussion of the assumptions. Section 3 describes the block thresholding LASSO estimator, evaluates the non-asymptotic lower and upper bounds for the risk, both in probability and in the mean squared risk sense, and ensures optimality of
the constructed estimator. Section 4 presents examples of estimation when assumptions of the paper are satisfied. Section 5 contains proofs of the statements formulated in the paper.

1.1 Formulation of the problem

Let \((W_i, t_i, Y_i), i = 1, \ldots, n\) be sampled independently from the varying coefficient model

\[ Y = W^T f(t) + \sigma \xi. \]  

(1.1)

Here, \(W \in \mathbb{R}^p\) are random vectors of predictors, \(f(\cdot) = (f_1(\cdot), \ldots, f_p(\cdot))^T\) is an unknown vector-valued function of regression coefficients and \(t \in [0, 1]\) is a random variable with the unknown density function \(g\). We assume that \(W\) and \(t\) are independent. The noise variable \(\xi\) is independent of \(W\) and \(t\) and is such that \(E(\xi) = 0\) and \(E(\xi^2) = 1\), \(\sigma > 0\) denotes the known noise level.

The goal is to estimate vector function \(f(\cdot)\) on the basis of observations \((W_i, t_i, Y_i), i = 1, \ldots, n\).

In order to estimate \(f\), following Klopp and Pensky (2013), we expand it over a basis \((\phi_l(\cdot))_l\) with \(\phi_0(t) = 1\). Expansion of the functions \(f_j(\cdot)\) over the basis, for any \(t \in [0, 1]\), yields

\[ f_j(t) = \sum_{l=0}^{L} a_{jl} \phi_l(t) + \rho_j(t) \quad \text{with} \quad \rho_j(t) = \sum_{l=L+1}^{\infty} a_{jl} \phi_l(t). \]  

(1.2)

If \(\phi(\cdot) = (\phi_0(\cdot), \ldots, \phi_L(\cdot))\) and \(A_0\) denotes a matrix of coefficients with elements \(A_0^{(l,j)} = a_{jl}\), then relation (1.2) can be re-written as \(f(t) = A_0^T \phi(t) + \rho(t)\), where \(\rho(t) = (\rho_1(t), \ldots, \rho_p(t))^T\). Combining formulae (1.1) and (1.2), we obtain the following model for observations \((W_i, t_i, Y_i), i = 1, \ldots, n\):

\[ Y_i = \text{Tr}(A_0^T \phi(t_i) W_i^T) + W_i^T \rho(t_i) + \sigma \xi_i, \quad i = 1, \ldots, n. \]  

(1.3)

Below, we reduce the problem of estimating vector function \(f\) to estimating matrix \(A_0\) of coefficients of \(f\).

1.2 Tensor approach to estimation

Denote \(a = \text{Vec}(A_0)\) and \(B_i = \text{Vec}(\phi(t_i) W_i^T)\). Note that \(B_i\) is the \(p(L+1)\)-dimensional vector with components \(\phi_l(t_i) W_i^{(j)}\), \(l = 0, \ldots, L, j = 1, \ldots, p\), where \(W_i^{(j)}\) is the \(j\)-th component of vector \(W_i\). Consider matrix \(B \in \mathbb{R}^{n \times p(L+1)}\) with rows \(B_i^T\), \(i = 1, \ldots, n\), vector \(\xi = (\xi_1, \ldots, \xi_n)^T\) and vector \(b\) with components \(b_i = W_i^T \rho(t_i), i = 1, \ldots, n\). Taking into account that

\[ \text{Tr}(A^T \phi(t_i) W_i^T) = B_i^T \text{Vec}(A) \]

we rewrite the varying coefficient model (1.3) in a matrix form

\[ Y = B a + b + \sigma \xi. \]  

(1.4)

In what follows, we denote

\[ \Omega_i = W_i W_i^T, \quad \Phi_i = \phi(t_i)(\phi(t_i))^T, \quad \Sigma_i = \Omega_i \otimes \Phi_i, \]  

(1.5)
where $\Omega_i \otimes \Phi_i$ is the Kronecker product of $\Omega_i$ and $\Phi_i$. Note that $\Omega_i$, $\Phi_i$ and $\Sigma_i$ are i.i.d. for $i = 1, \cdots, n$, and that $\Omega_i$ and $\Phi_i$ are independent for any $i_1$ and $i_2$. By simple calculations, we derive

$$a^T BB^T a = \sum_{i=1}^{n} (B_i^T a)^2 = \sum_{i=1}^{n} [\text{Tr}(A^T \phi(t_i) W_i^T)]^2$$

$$= \sum_{i=1}^{n} W_i^T A^T \phi(t_i) \phi^T(t_i) AW_i = \sum_{i=1}^{n} a^T (\Omega_i \otimes \Phi_i) a,$$

which implies

$$B^T B = \sum_{i=1}^{n} \Omega_i \otimes \Phi_i. \tag{1.6}$$

Let

$$\hat{\Sigma} = n^{-1} B^T B = n^{-1} \sum_{i=1}^{n} \Sigma_i. \tag{1.7}$$

Then, due to the i.i.d. structure of the observations, one has

$$\Sigma = \mathbb{E} \Sigma_1 = \Omega \otimes \Phi \quad \text{with} \quad \Omega = \mathbb{E}(W_1 W_1^T) \quad \text{and} \quad \Phi = \mathbb{E}(\phi(t_1) \phi^T(t_1)). \tag{1.8}$$

## 2 Assumptions and notations

### 2.1 Notations

In what follows, we use bold script for matrices and vectors, e.g., $A$ or $a$, and superscripts to denote elements of those matrices and vectors, e.g., $A^{(i,j)}$ or $a^{(j)}$. Below, we provide a brief summary of the notation used throughout this paper.

- For any vector $a \in \mathbb{R}^p$, denote the standard $l_1$ and $l_2$ vector norms by $\|a\|_1$ and $\|a\|_2$, respectively. For vectors $a, c \in \mathbb{R}^p$, denote their scalar product by $\langle a, c \rangle$.
- For any function $q(t)$, $t \in [0, 1]$, $\|q\|_2$ and $\langle \cdot, \cdot \rangle_2$ are, respectively, the norm and the scalar product in the space $L^2([0, 1])$. Also, $\|q\|_\infty = \sup_{t \in [0, 1]} |q(t)|$.
- For any vector function $q(t) = (q_1(t), \cdots, q_p(t))^T$, denote

$$\|q(t)\|_2 = \left[ \sum_{j=1}^{p} \|q_j\|_2^2 \right]^{1/2}.$$

- For any matrix $A$, denote its spectral and Frobenius norms by $\|A\|$ and $\|A\|_2$, respectively.
- Denote the $k \times k$ identity matrix by $I_k$.
- For any numbers, $a$ and $b$, denote $a \lor b = \max(a, b)$ and $a \land b = \min(a, b)$.
- In what follows, we use the symbol $C$ for a generic positive constant, which is independent of $n$, $p$, $s$ and $l$, and may take different values at different places.
• If \( r = (r_1, \cdots, r_p)^T \) and \( r'_j = r_j + 1/2 - 1/\nu_j \) for some \( 1 \leq \nu_j < \infty \), denote \( r^*_j = r_j \land r'_j \) and \( r^*_{\min} = \min_j r^*_j \).

• Denote

\[ t = (t_1, \cdots, t_n), \quad \mathbb{W} = (W_1, \cdots, W_n), \]

i.e., \( \mathbb{W} \) is the \( p \times n \) matrix with columns \( W_i, i = 1, \cdots, n \).

2.2 Assumptions

We impose the following assumptions on the varying coefficient model (1.3).

(A0). Only \( s \) out of \( p \) functions \( f_j \) are non-constant and depend on the time variable \( t \), \( s_0 \) functions are constant and independent of \( t \) and \( (p - s - s_0) \) functions are identically equal to zero. We denote by \( J \) the set of indices corresponding to the non-constant functions \( f_j \).

(A1). Functions \( (\phi_k(\cdot))_{k=0, \ldots, \infty} \) form an orthonormal basis of \( L_2([0,1]) \), and are such that \( \phi_0(t) = 1 \) and, for any \( t \in [0,1] \), any \( l \geq 0 \) and some \( C_\phi < \infty \)

\[ \sum_{k=0}^{l} \phi_k^2(t) \leq C_\phi^2(l+1). \]

(A2). The probability density function \( g(t) \) is bounded above and below \( 0 < g_1 \leq g(t) \leq g_2 < \infty \). Moreover, the eigenvalues of \( \mathbb{E}(\phi \phi^T) = \Phi \) are bounded from above and below

\[ 0 < \phi_{\min} = \lambda_{\min}(\Phi) \leq \lambda_{\max}(\Phi) = \phi_{\max} < \infty. \]

Here, \( \phi_{\min} \) and \( \phi_{\max} \) are absolute constants independent of \( L \).

(A3). Functions \( f_j(t) \) have efficient representations in basis \( \phi_i \), in particular, for any \( j = 1, \cdots, p \), one has

\[ \sum_{k=0}^{\infty} |a_{jk}|^{\nu_j}(k+1)^{\nu_j}r'_j \leq (C_a)^{\nu_j}, \quad r'_j = r_j + 1/2 - 1/\nu_j, \]

for some \( C_a > 0, 1 \leq \nu_j < \infty \) and \( r_j > \min(1/2, 1/\nu_j) \). In particular, if function \( f_j(t) \) is constant or vanishes, then \( r_j = \infty \). We denote vectors with elements \( \nu_j \) and \( r_j, j = 1, \cdots, p \), by \( \nu \) and \( r \), respectively, and the set of indices of finite elements \( r_j \) by \( J \):

\[ J = \{ j : r_j < \infty \}. \]

(A4). Variables \( \xi_i, i = 1, \cdots, n \), are i.i.d. sub-Gaussian such that

\[ \mathbb{E} \xi_i = 0, \quad \mathbb{E} \xi_i^2 = 1 \quad \text{and} \quad \| \xi_i \|_{\psi_2} \leq K. \]
where \( \| \cdot \|_{\psi^2} \) denotes the sub-Gaussian norm.

(A5) “Restricted Isometry in Expectation” condition. Let \( W_\Lambda, \Lambda \in \{1, \ldots, p\} \) be the sub-vector obtained by extracting the entries of \( W \) corresponding to indices in \( \Lambda \) and let \( \Omega_\Lambda = \mathbb{E} (W_\Lambda W_\Lambda^T) \). We assume that there exist two positive constants \( \omega_{\max}(\mathcal{R}) \) and \( \omega_{\min}(\mathcal{R}) \) such that for all subsets \( \Lambda \) with cardinality \( |\Lambda| \leq \mathcal{R} \) and all \( v \in \mathbb{R}^{|\Lambda|} \) one has

\[
\omega_{\min}(\mathcal{R}) \| v \|_2^2 \leq v^T \Omega_\Lambda v \leq \omega_{\max}(\mathcal{R}) \| v \|_2^2. \tag{2.5}
\]

Moreover, we suppose that \( \mathbb{E}(W^{(j)})^4 \leq V \) for any \( j = 1, \ldots, p \) and that, for any \( \mu \geq 1 \) and for all subsets \( \Lambda \) with \( |\Lambda| \leq \mathcal{R} \), there exist positive constants \( U_\mu \) and \( C_\mu \) and a set \( \mathcal{W}_\mu \) such that

\[
W_\Lambda \in \mathcal{W}_\mu \implies (\| W_\Lambda \|_2 \leq U_\mu) \cap \left( \max_{j \in \Lambda} |W^{(j)}_\Lambda| \leq C_\mu \right), \quad \mathbb{P}(W_\Lambda \in \mathcal{W}_\mu) \geq 1 - 2p^{-2\mu}. \tag{2.6}
\]

Here, \( U_\mu = U_\mu(\mathcal{R}), C_\mu = C_\mu(\mathcal{R}) \).

(A6). We assume that \((s + s_0)(1 + \log n) \leq p \) and that there exists a numerical constant \( C_\omega > 1 \) such that

\[
\log(n) \geq \frac{C_\omega \phi_{\max} \omega_{\max}((s + s_0) \log n)}{\phi_{\min} \omega_{\min}((s + s_0)(1 + \log n))}.
\]

We denote

\[
\omega^*_{\max} = \omega_{\max}((s + s_0)(1 + \log n)), \quad \omega^*_{\min} = \omega_{\min}((s + s_0)(1 + \log n)). \tag{2.7}
\]

2.3 Discussion of assumptions

- Assumptions (A0) corresponds to the case when \( s \) of the covariates \( f_j(t) \) are indeed functions of time, \( s_0 \) of them are time independent and \((p - s - s_0)\) are irrelevant.

- Assumption (A1) deals with the basis of \( L_2([0,1]) \). There are many types of orthonormal bases satisfying those conditions.

  a) Fourier basis. Choose \( \phi_0(t) = 1 \), \( \phi_k(t) = 2 \sin(2\pi kt) \) if \( k > 1 \) is odd, \( \phi_k(t) = 2 \cos(2\pi kt) \) if \( k > 1 \) is even. The basis functions are bounded and \( C_\phi = 2 \).

  b) Wavelet basis. Consider a periodic wavelet basis on \([0,1] \): \( \psi_{h,i}(t) = 2^{h/2} \psi(2^h t - i) \) with \( h = 0,1, \ldots, i = 0, \ldots, 2^h - 1 \). Set \( \phi_0(t) = 1 \) and \( \phi_j(t) = \psi_{h,i}(t) \) with \( j = 2^h + i + 1 \).

  If \( l = 2^l \), then the condition (2.2) is satisfied with \( C_\phi = \| \psi \|_\infty \). Observing that, for \( 2^l < l < 2^{l+1} \), we have \((l + 1) \geq (2^{l+1} + 1)/2\) one can take \( C_\phi = 2 \| \psi \|_\infty \).

- Assumption (A2) that \( \phi_{\min} \) and \( \phi_{\max} \) are absolute constants independent of \( L \) is guaranteed by the fact that the sampling density \( g \) is bounded above and below. For example, if \( g(t) = 1 \), one has \( \phi_{\min} = \phi_{\max} = 1 \).
Assumption (A3) describes sparsity of the vectors of coefficients of functions \( f_j(t) \) in basis \( \phi_t, j = 1, \cdots, p \) and its smoothness. For example, if \( \nu_j < 2 \), the vector of coefficients \( a_{jl} \) of \( f_j \) is sparse. In the case when basis \( \phi_t \) is formed by wavelets, condition (2.3) implies that \( f_j \) belongs to a Besov ball of radius \( C_a \). If we chose Fourier bases and \( \nu_j = 2 \), then \( f_j \) belongs to a Sobolev ball of smoothness \( r_j \) and radius \( C_a \). Note that Assumption (A3) allows each non-constant function \( f_j \) to have its own sparsity and smoothness patterns.

Assumption (A4) that \( \xi_i \) are sub-Gaussian random variables means that their distribution is dominated by the distribution of a centered Gaussian random variable. This is a convenient and reasonably wide class. Classical examples of sub-gaussian random variables are Gaussian, Bernoulli and all bounded random variables. Note that \( \mathbb{E} \xi_i^2 = 1 \) implies that \( K \leq 1 \).

Assumption (A5) is closely related to the Restricted Isometry (RI) conditions usually considered in the papers that employ LASSO technique or its versions, see, e.g., [2]. However, usually the RI condition is imposed on the matrix of scalar products of the elements of a deterministic dictionary while we deal with a random dictionary and require this condition to hold only for the expectation of this matrix.

Note that the upper bound in the condition (2.5) is automatically satisfied with \( \omega_{\text{max}} = \| \Omega \| \) where \( \| \Omega \| \) is the spectral norm of the matrix \( \mathbb{E}(WW^T) = \Omega \). If the smallest eigenvalue of \( \Omega \), \( \lambda_{\min}(\Omega) \), is non-zero, then the lower bound in (2.5) is satisfied with \( \omega_{\min} = \lambda_{\min}(\Omega) \). However, since the \( \lambda \)-restricted maximal eigenvalue \( \omega_{\text{max}}(\lambda) \) may be much smaller than the spectral norm of \( \Omega \) and \( \omega_{\min}(\lambda) \) may be much larger then \( \lambda_{\min}(\Omega) \), using those values will result in sharper bounds for the error. Note that in the case when \( W \) has i.i.d. zero-mean entries \( W^j \) with \( \mathbb{E} (W^j)^2 = \nu^2 \), we have \( \omega_{\max} = \omega_{\min} = \nu^2 \).

Assumption (A6) is usual in the literature on the high-dimensional linear regression model, see, e.g., [2]. For instance, if \( W \) has i.i.d. zero-mean entries \( W^j \) and \( g(t) = 1 \), this condition is satisfied for any \( 1 < C_\omega \leq \log(n) \).

3 Estimation strategy and non-asymptotic error bounds

3.1 Estimation strategy

Formulation (1.4) implies that the varying coefficient model can be reduced to the linear regression model and one can apply one of the multitude of penalized optimization techniques which have been developed for the linear regression. In what follows, we apply a block LASSO penalties for the coefficients in order to account for both the constant and the vanishing functions \( f_j \) and also to take advantage of the sparsity of the functional coefficients in the chosen basis.

In particular, for each function \( f_j, j = 1, \cdots, p \), we divide its coefficients into \( M + 1 \) different groups where group zero contains only coefficient \( a_{j0} \) for the constant function \( \phi_0(t) = 1 \) and \( M \) groups of size \( d \approx \log n \) where \( M = L/d \). We denote \( a_{j0} = a_{j0} \) and \( a_{jl} = (a_{j,l(\ell-1)+1}, \cdots, a_{j,d\ell})^T \) the sub-vector of coefficients of function \( f_j \) in block \( l, l = 1, \cdots, M \). Let \( K_l \) be the subset of indices associated with \( a_{jl} \). We impose block norm on matrix \( A \) as follows

\[
\| A \|_{\text{block}} = \sum_{j=1}^{p} \sum_{l=0}^{M} \| a_{jl} \|_2.
\]

(3.1)
Observe that $\|A\|_{\text{block}}$ indeed satisfies the definition of a norm and is a sum of absolute values of coefficients $a_{j0}$ of functions $f_j$ and $l_2$ norms for each of the block vectors of coefficients $a_{jl}$, $j = 1, \ldots, p$, $l = 1, \ldots, M$.

The penalty which we impose is related to both the ordinary and the group LASSO penalties which have been used by many authors. The difference, however, lies in the fact that the structure of the blocks is not motivated by naturally occurring groups (like, e.g., rows of the matrix $A$) but rather our desire to exploit sparsity of functional coefficients $a_{jl}$. In particular, we construct an estimator $\hat{A}$ of $A_0$ as a solution of the following convex optimization problem

$$\hat{A} = \arg \min_A \left\{ n^{-1} \sum_{i=1}^{n} \left[ Y_i - \text{Tr}(A^T \phi(t_i) W_i^T) \right]^2 + \delta \|A\|_{\text{block}} \right\},$$

(3.2)

where the value of $\delta$ will be defined later.

Note that with the tensor approach which we used in Section 1.2, optimization problem (3.2) can be re-written in terms of vector $\alpha = \text{Vec}(A)$ as

$$\hat{a} = \arg \min_\alpha \left\{ n^{-1} \|Y - B\alpha\|^2 + \delta \|\alpha\|_{\text{block}} \right\},$$

(3.3)

where $\|\alpha\|_{\text{block}} = \|A\|_{\text{block}}$ is defined by the right-hand side (3.1) with vectors $a_{jl}$ being sub-vectors of vector $\alpha$. Subsequently, we construct an estimator $\hat{f}(t) = (\hat{f}_1(t), \ldots, \hat{f}_p(t))^T$ of the vector function $f(t)$ using

$$\hat{f}_j(t) = \sum_{k=0}^{L} \hat{a}_{jk} \phi_k(t), \quad j = 1, \ldots, p. \quad (3.4)$$

In what follows, we derive the upper bounds for the risk of the estimator $\hat{a}$ (or $\hat{A}$) and suggest a value of parameter $\delta$ which allows to attain those bounds. However, in order to obtain a benchmark of how well the procedure is performing, we determine the lower bounds for the risk of any estimator $\hat{A}$ under assumptions (A0)–(A4).

**Remark 1** Assumption that $K = L/d$ is an integer is not essential. Indeed, we can replace the number of groups $K$ by the largest integer below or equal to $L/d$ and then adjust group sizes to be $d$ or $d + 1$ where $d = \lfloor \log n \rfloor$, the largest integer not exceeding $\log n$.

### 3.2 Lower bounds for the risk

In this section we will obtain the lower bounds on the estimation risk. We consider a class $\mathcal{F} = \mathcal{F}_{s_0, s, \nu, \tau}(C_0)$ of vector functions $f(t)$ such that $s$ of their components are non-constant with coefficients satisfying condition (2.3) in (A3), $s_0$ of the components are constant and $(p - s - s_0)$ components are identically equal to zero. We construct the lower bound for the minimax quadratic risk of any estimator $\hat{f}$ of the vector function $f \in \mathcal{F}_{s_0, s, \nu, \tau}(C_0)$. Let $\omega^*_\max$ be given by formula (2.7). Denote $r_{\max} = \max\{r_j : j \in \mathcal{J}\}$ and

$$n_{\text{low}} = \frac{2 \sigma^2 \kappa}{C_0^2 \omega^*_\max \phi_{\max}} \max \left\{ 1, \left( \frac{6}{s} \right)^{2r_{\max} + 1} \right\}, \quad (3.5)$$
\[
\Delta_{lower}(s_0, s, n, r) = \max \left\{ \frac{\kappa \sigma^2 s_0}{4n \omega_{max}^* \phi_{max}}, \frac{1}{8} \sum_{j \in j} C_n^{2r_j+1} \left( \frac{\sigma^2 \kappa}{n \omega_{max}^* \phi_{max}} \right)^{2r_j+1} \right\}. \tag{3.6}
\]

Then, the following statement holds.

**Theorem 1** Let \( s \geq 1 \) and \( s_0 \geq 3 \). Consider observations \( Y_i \) in model (1.3) with \( W_i \), \( i = 1, \ldots, n \) and \( t \) satisfying assumptions (A5) and (A2), respectively. Assume that, conditionally on \( W_i \) and \( t_i \), variables \( \xi_i \) are Gaussian \( N(0,1) \) and that \( n \geq n_{low} \). Then, for any \( \kappa < 1/8 \) and any estimator \( \hat{f} \) of \( f \), one has

\[
\inf \sup_{f \in F} \mathbb{P} \left( \| \hat{f} - f \|_2^2 \geq \Delta_{lower}(s_0, s, n, r) \right) \geq \frac{\sqrt{2}}{1 + \sqrt{2}} \left( 1 - 2\kappa - \sqrt{\frac{2\kappa}{\log 2}} \right). \tag{3.7}
\]

Note that condition \( s_0 \geq 3 \) is not essential since, for \( s_0 < 3 \), the first term in (3.7) is of parametric order. Condition \( n \geq n_{low} \) is a purely technical condition which is satisfied for the collection of \( n \)'s for which upper bounds are derived. Observe also that inequality (3.7) immediately implies that

\[
\inf \sup_{f \in F} \mathbb{E} \| \hat{f} - f \|_2^2 \geq \Delta_{lower}(s_0, s, n, r) \left[ \frac{\sqrt{2}}{1 + \sqrt{2}} \left( 1 - 2\kappa - \sqrt{\frac{2\kappa}{\log 2}} \right) \right]. \tag{3.8}
\]

### 3.3 Adaptive estimation and upper bounds for the risk

In this section we derive an upper bound for the risk of the estimator (3.2). For this purpose, first, we shall show that, with high probability, the ratio between the restricted eigenvalues of \( \check{\Sigma}_n \) is immediately implies that

\[
\inf \sup_{f \in F} \mathbb{E} \| \hat{f} - f \|_2^2 \geq \Delta_{lower}(s_0, s, n, r) \left[ \frac{\sqrt{2}}{1 + \sqrt{2}} \left( 1 - 2\kappa - \sqrt{\frac{2\kappa}{\log 2}} \right) \right]. \tag{3.8}
\]

**Lemma 1** Let \( n \geq N(\mathbb{R}) \) and \( \mu \) in (2.6) be large enough, so that

\[
p^\mu \geq \max \left\{ \frac{\sqrt{2V n}}{8 \mu \sqrt{\mathbb{R} U^2 (\mathbb{R}) \log (p + L)}}, \frac{2n}{2} \right\}, \tag{3.10}
\]

where \( V \) is defined in Assumption (A5). Then, for any \( \Lambda \in \{1, \ldots, p\} \)

\[
\inf_{\Lambda: |\Lambda| \leq \mathbb{N}} \mathbb{P} \left( \| \check{\Sigma}_\Lambda - \Sigma_\Lambda \| < \phi_{\min} \omega_{\min}(\mathbb{R}) \right) \cap \mathcal{W}_{\mu}^{\otimes n}) \geq 1 - 2p^{-\mu}, \tag{3.11}
\]

where \( \mathcal{W}_{\mu} \) is the set of points in \( \mathbb{R}^p \) such that condition (2.6) holds and \( \mathcal{W}_{\mu}^{\otimes n} \) is the direct product of \( n \) sets \( \mathcal{W}_{\mu} \).

Moreover, on the set \( \mathcal{W}_{\mu}^{\otimes n} \), with probability at least \( 1 - 2p^{-\mu} \), one has simultaneously

\[
\inf_{\Lambda: |\Lambda| \leq \mathbb{N}} \lambda_{\min}(\check{\Sigma}_\Lambda) \geq (1 - h)\phi_{\min} \omega_{\min}(\mathbb{R}), \quad \sup_{\Lambda: |\Lambda| \leq \mathbb{N}} \lambda_{\max}(\check{\Sigma}_\Lambda) \leq (1 + h)\phi_{\max} \omega_{\max}(\mathbb{R}). \tag{3.12}
\]
Lemma ensures that the restricted lowest eigenvalue of the regression matrix is within a constant factor of the respective eigenvalue of matrix . Since may be large, this is not guaranteed by a large value of (as it happens in the asymptotic setup) and leads to additional conditions on the relationship between parameters , , and .

By applying a combination of LASSO and group LASSO arguments, we obtain the following theorem that gives an upper bound for the quadratic risk of the estimator . We set 

\[
U_\mu = U_\mu(s + s_0).
\]

Define

\[
N = \max \left\{ \frac{64 \mu (s + s_0) C_2^2 U_\mu^2 (L + 1) \phi_{\max} \omega_{\max}^* \log(p + L)}{h^2 \phi_{\min}^2 (\omega_{\min}^*)^2}, \frac{U_\mu^2 C_0^2 (L + 1) \mu \log p}{g_2 \omega_{\max}(s)}, 3 C_2^2 g_2 s \omega_{\max}(s) \right\}
\]

and

\[
\hat{\delta} = 2 \left( \sigma C_{\omega} K \sqrt{\mu} + 1 \right) \sqrt{\frac{(1 + h) \phi_{\max} \omega_{\max}(1) \log p}{n}}.
\]

**Theorem 2** Let \( \min k(r_k \wedge r'_k) \geq 2, \ L + 1 \geq n^{1/2} \) and \( n \geq N \). Let \( \mu \) in (2.6) be large enough, so that

\[
p^{\mu} \geq \max \left\{ \frac{\sqrt{2} V n}{8 \mu \sqrt{s + s_0} U_\mu^2 \log(p + L)}, \frac{2L}{\log n}, 2n \right\}.
\]

If \( \hat{a} \) is an estimator of \( a \) obtained as a solution of optimization problem (3.3) with \( \delta = \hat{\delta} \), and the vector function \( \hat{f} \) is recovered using (3.4), then, one has

\[
P \left( \| \hat{f} - f \|_2^2 \leq \Delta(s_0, s, n, r) \right) \geq 1 - 8 p^{-\mu}
\]

where

\[
\Delta(s_0, s, n, r) = \frac{C_2^2 s}{n^2} + \frac{C_B (1 + h) \omega_{\max}^* \phi_{\max}}{(1 - h) \omega_{\min}^* \phi_{\min}} \left( \frac{C_\omega \sigma^2 K^2 \mu + 1}{n(1 - h) \omega_{\min}^* \phi_{\min}} \frac{(s + s_0) \log p}{\log n} \right)^{2r_j} \left( \frac{\nu_j (2r_j + 1)}{g_2 r_j} \right)^{2r_j}.
\]

Note that construction (3.3) of the estimator \( \hat{a} \) does not involve knowledge of unknown parameters \( r \) and \( \nu \) or matrix \( \Sigma \); therefore, estimator \( \hat{a} \) is fully adaptive. Moreover, conclusions of Theorem 2 are derived without any asymptotic assumptions on \( n, p \) and \( L \).

In order to assess the optimality of estimator \( \hat{a} \), we consider the case of the Gaussian noise, i.e. \( K = 1 \). Observe that, under Assumption (A2), the values of \( \phi_{\min} \) and \( \phi_{\max} \) are independent of \( n \) and \( p \), so that the only quantities in (3.16) which are not bounded from above and below by an absolute constant are \( \sigma, \omega_{\min}^*, \omega_{\max}^*, s \), and \( s_0 \). Hence, \( \Delta(s_0, s, n, r) \leq C \Delta_{\text{upper}}(s_0, s, n, r) \) with

\[
\Delta_{\text{upper}}(s_0, s, n, r) = \frac{\omega_{\max}^*}{\omega_{\min}^*} \left[ \frac{\sigma^2 s_0 \log p}{n \omega_{\min}^*} + \sum_{j \in J} C_2^2, \phi_{\min} \right] \left( \frac{\sigma^2}{n \omega_{\min}^*} \right)^{2r_j} \left( \log p \right)^{2r_j} \leq 0.
\]
Inequality (3.17) implies that, for any values of the parameters, the ratio between the upper and the lower bound for the risk (3.6) is bounded by $C \log(p) \omega^*_{\max}^2 / \omega^*_{\min}^2$. Note that $\omega^*_{\max} / \omega^*_{\min}$ is the condition number of matrix $\Omega A | A | = (s + s_0)(1 + \log n)$. Hence, if matrix $\Omega A$ is well conditioned, so that $\omega^*_{\max} / \omega^*_{\min}$ is bounded by a constant, the estimator $\hat{f}$ attains optimal convergence rates up to a $\log p$ factor.

Consider the case when all functions $f_j(t)$ are spatially homogeneous, i.e., $\min_j \nu_j \geq 2$ and $n$ is large enough, i.e. there exists a positive $\beta$ such that $n^\beta \geq p$. Then, the estimator $\hat{f}$ attains optimal convergence rates up to a constant factor, if $s / s_0$ is bounded or $n \omega^*_{\min} / \sigma^2$ is relatively large. In particular, if all functions in assumption (A3) belong to the same space, then the following corollary is valid.

**Corollary 1** Let conditions of Theorem 2 hold with $r_j = r$ and $\nu_j = \nu$, $j = 1, \ldots, p$, and matrix $\Omega$ be well conditioned, i.e. $\omega^*_{\max} / \omega^*_{\min}$ is bounded by some absolute constant independent of $n$, $p$ and $\sigma$. Then,

$$\Delta_{\text{upper}}(s_0, s, n, r) = \begin{cases} C \log p, & \text{if } \sigma^2 (s / s_0)^{2r+1} \geq n \omega^*_{\min}, \\ C(\log p)^{2r+1}, & \text{if } \sigma^2 (s / s_0)^{2r+1} < n \omega^*_{\min}, 1 \leq \nu < 2, \\ C, & \text{if } \sigma^2 (s / s_0)^{2r+1} < n \omega^*_{\min}, \nu \geq 2. \end{cases}$$

\[(3.18)\]

3.4 Adaptive estimation with respect to the mean squared risk

Theorem 2 derives upper bounds for the risk with high probability. Suppose that an upper bound on the norms of functions $f_j$ is available due to physical or other considerations:

$$\max_{1 \leq j \leq p} \|f_j\|_2^2 \leq C_f^2.$$  

Then, $\|a\|_2^2 \leq p C_f^2$ and $\hat{a}$ given by (3.3) can be replaced by the solution of the convex problem

$$\hat{a} = \arg \min_a \left\{ n^{-1} \|Y - B a\|_2^2 + \delta \|a\|_{\text{block}} \text{ s.t. } \|a\|_2^2 \leq p C_f^2 \right\}$$

with $\delta = \hat{\delta}$ where $\hat{\delta}$ is defined in (3.14), and estimators $\hat{f}_j$ of $f_j$, $j = 1, \ldots, p$, are constructed using formula (3.4). Choose $\mu$ in (2.6) large enough, so that

$$16 n C_f^2 \leq p^{\mu - 1}.$$  

Then, the following statement is valid.

**Theorem 3** Under the assumptions of the Theorem 2 and for any $\mu$ satisfying condition (3.21), one has

$$E \|\hat{f} - f\|_2^2 \leq C \Delta_{\text{upper}}(s_0, s, n, r)$$

\[(3.22)\]

where $C$ is an absolute constant independent of $n$, $p$ and $\sigma$. 

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4 Examples and discussion

4.1 Examples

In this section we provide several examples when assumptions of the paper are satisfied. For simplicity, we assume that \( g(t) = 1 \), so that \( \phi_{\text{min}} = \phi_{\text{max}} = 1 \).

**Example 1 Normally distributed dictionary** Let \( W_i, i = 1, \ldots, n \), be i.i.d. standard Gaussian vectors \( N(0, I_p) \). Then, \( \Omega = I_p \), so that \( \omega_{\text{min}} = \omega_{\text{max}} = 1 \). Moreover, \( W^{(j)} \) are independent standard Gaussian variables and \( (W_i)_{\Lambda}^T(W_i)_{\Lambda} \) are independent chi-squared variables with \( |\Lambda| = \aleph \) degrees of freedom. Using inequality (see, e.g., [3], page 67)

\[
P\left( \chi^2_{\aleph} \leq \aleph + 2\sqrt{\aleph x} + 2x \right) \geq 1 - e^{-\mu_1^2}, \quad x > 0,
\]

for any \( \mu_1 \geq 0 \), derive

\[
P\left( (W_1)^T_{\Lambda}(W_1)_{\Lambda} \leq (\sqrt{\aleph} + \sqrt{2\mu_1})^2 \right) \geq 1 - \exp(-\mu_1^2).
\]

Choose any \( \mu > 0 \) and set \( \mu_1^2 = 2\mu \log(p) \). Then, using a standard bound on the maximum of \( p \) Gaussian variables one obtains that Assumption (A5) holds with

\[
C_{\mu} = \sqrt{2\log p}, \quad U_{\mu}^2 = (\sqrt{\aleph} + 2\sqrt{\mu \log p})^2.
\]

**Example 2 Symmetric Bernoulli dictionary** Let \( W^{(j)}_i, i = 1, \ldots, n, j = 1, \ldots, p \), be independent symmetric Bernoulli variables

\[
P(W^{(j)}_i = 1) = P(W^{(j)}_i = -1) = 1/2.
\]

Then, \( \Omega = I_p, \omega_{\text{min}} = \omega_{\text{max}} = 1 \) and, for any \( \mu \),

\[
C_{\mu} = 1, \quad U_{\mu}^2(\aleph) = \aleph.
\]

In both cases, \( N \) in (3.13) is of the form \( N = C(s + s_0)^2(L + 1) \log(p) \). Under conditions of Theorem 2 the upper bounds for the risk are of the form

\[
\Delta_{\text{upper}}(s_0, s, n, r) = C \left[ \frac{\sigma^2 s_0 \log n}{n} + \sum_{j \in J} \left( \frac{\sigma^2}{n} \right)^{2r_j+1} C_{a}^{2/(2r_j+1)} (\log n)^{\frac{2r_j}{r_j(2r_j+1)+(2r_j+1)}} (\log p)^{\frac{2r_j}{2r_j+1}} \right]
\]

where \( C \) is a numerical constant, so that it follows from Corollary that the block LASSO estimator is minimax optimal up to, at most logarithmic factor of \( p \).

The two examples above illustrate the situation when estimator (3.4) attains nearly optimal convergence rates when \( p > n \). This, however, is not always possible. Note that our analysis of the performance of the estimator (3.4) relies on the fact that the eigenvalues of any sub-matrix \( \hat{\Sigma}_\Lambda \) are close to those of matrix \( \Sigma_\Lambda \) (Lemma 1). The latter requires \( n \geq N \) where \( N \) depends on the nature of vectors \( W_i \). The next example shows that sometimes \( n < p \) prevents Lemma 1 from being valid.

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Example 3 Orthonormal dictionary Let $W_i, i = 1, \ldots, n$, be uniformly distributed on a set of canonical vectors $e_k, k = 1, \ldots, p$. Then, $\Omega = I_p/p$, so that $\omega_{\text{min}} = \omega_{\text{max}} = 1/p$. Moreover, $\|W_1\|_2^2 = 1$ and $|W(j)| \leq 1$. Therefore, for any $\mu > 0$, $C_\mu = 1$, $U_\mu^2(\mathcal{N}) = 1$.

In the case of the orthonormal dictionary, $N$ in (3.13) is of the form $N = C(s + s_0)(L + 1)p \log(p)$. Under conditions of Theorem 2, the upper bound for the risk is of the form

$$\Delta_{\text{upper}}(s_0, s, n, r) = C\left[\frac{\sigma^2ps_0\log n}{n} + \sum_{j \in \mathcal{S}} \left(\frac{\sigma^2}{n}\right)^{2r_j/r_j} C_a^{2/(2r_j+1)}(\log n)^{\frac{2r_j}{r_j(2r_j+1)}}(\log p)^{\frac{2r_j}{2r_j+1}}\right],$$

so, $n \geq N$ implies $n > C\sigma^2p(s_0 + s)\log n$ which also guarantees that the risk of the estimator is small. This, indeed, coincides with one’s intuition since one would need to sample more than $p$ vectors in order to ensure that each component of the vector has been sampled at least once.

4.2 Discussion

In the present paper, we provided a non-asymptotic minimax study of the sparse high-dimensional varying coefficient model. To the best of our knowledge, this has never been accomplished before. An important feature of our analysis is its flexibility: it distinguishes between vanishing, constant and time-varying covariates and, in addition, it allows the latter to be heterogeneous (i.e., to have different degrees of smoothness) and spatially inhomogeneous. In this sense, our setup is more flexible than the one usually used in the context of additive or compound functional models (see, e.g., [10] or [30]).

The adaptive estimator is obtained using block LASSO approach which can be viewed as a version of group LASSO where groups do not occur naturally but are rather driven by the need to reduce the variance, as it is done, for example, in block thresholding. Since we used tensor approach for derivation of the estimator, we believe that the results of the paper can be generalized to the case of the multivariate varying coefficient model studied in [41].

An important feature of our estimator is that it is fully adaptive. Indeed, application of the proposed block LASSO technique does not require the knowledge of the number of the non-zero components of $f$. It only depends on the highest diagonal element of matrix $\Omega$ which can be estimated with high precision even when $n$ is quite small due to Lemma 1.

Note that, even when $p$ is larger than $n$, the vector function $f$ is completely identifiable due to Assumption (A5), as long as the number of non-zero components of $f$ does not exceed $N$ in condition (2.5) and the number of observations $n$ is large enough. The examples of the paper deal with the dictionaries such that (2.5) holds for all $\mathcal{N} = 1, \ldots, p$. The latter ensures identifiability of $f$ provided $n \geq N$ where $N$ is specified for each type of the random dictionary and depends on the sparsity level of $f$. On the other hand, large values of $p$ ensure great flexibility of the choice of $f$, so one can hope to represent the data using only few components of it.

Finally, we want to comment on the situation when the requirement $n \geq N$ is not met due to lack of sparsity or insufficient number of observations. In this case, $f$ is not identifiable and one cannot guarantee that $\hat{f}$ is close to the true function $f$. However, this kind of situations occur in all types of high-dimensional problems.
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5 Proofs

5.1 Proofs of the lower bounds for the risk

In order to prove Theorem 1, we consider a set of test vector functions $f_\omega(t) = (f_1, \omega, \cdots, f_p, \omega)^T$ indexed by binary sequences $\omega$ with components

$$f_k, \omega(t) = \omega_{k0}u_k + \sum_{l=1}^{2l_{0k}-1} \omega_{kl}v_k\phi_l(t), \quad (5.1)$$

where $\omega_{kl} \in \{0, 1\}$ for $l = l_{0k}, l_{0k}+1, \cdots, 2l_{0k} - 1$, $k = 1, \cdots, p$. Let $K_0$ and $K_1$, respectively, be the sets of indices such that $K_0 \cap K_1 = \emptyset$ and $u_k = u$ if $k \in K_1$ and $u_k = 0$ otherwise, $v_k = v$ if $k \in K_0$ and $v_k = 0$ otherwise, so that $u_kv_k = 0$.

By simple calculations, it is easy to verify that condition (5.2) is satisfied if we set

$$u \leq C_\alpha, \quad v = C_\alpha(2l_{0k})^{-(r_k+1/2)}, \quad (5.3)$$

where the constancy of $v$ implies that $l_{0k}$ in (5.3) are different for different values of $k$.

Consider two binary sequences $\omega$ and $\tilde{\omega}$ and the corresponding test functions $f(t) = f_\omega(t)$ and $\tilde{f}(t) = f_{\tilde{\omega}}(t)$ indexed by those sequences. Then, the total squared distance in $L_2([0, 1])$ between $f_\omega(t)$ and $f_{\tilde{\omega}}(t)$ is equal to

$$D^2 = u^2 \sum_{k \in K_1} |\omega_{k0} - \tilde{\omega}_{k0}| + v^2 \sum_{k \in K_0} \sum_{l=l_{0k}}^{2l_{0k}-1} |\omega_{kl} - \tilde{\omega}_{kl}|, \quad (5.4)$$

Let $P_f$ and $P_{\tilde{f}}$ be probability measures corresponding to test functions $f$ and $\tilde{f}$, respectively. We shall consider two cases. In Case 1, the first $s_0$ functions are constant and the rest of the functions are equal to identical zero. In Case 2, the first $s$ functions are time-dependent and the rest of the functions are equal to identical zero. In both cases, $f(t)$ and $\tilde{f}(t)$ contain at most $s + s_0$ non-zero coordinates. Using that conditionally on $W_i$ and $t_i$, the variables $\xi_i$ are Gaussian $N(0, 1)$, we obtain that the Kullback-Leibler divergence $K(P_f, P_{\tilde{f}})$ between $P_f$ and $P_{\tilde{f}}$ satisfies

$$K(P_f, P_{\tilde{f}}) = (2\sigma^2)^{-1} \sum_{i=1}^{n} \left[ Q_i(f) - Q_i(\tilde{f}) \right]^2 = (2\sigma^2)^{-1} n \sum_{i=1}^{n} \left[ Q_i(f) - Q_i(\tilde{f}) \right]^2$$

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where
\[ Q_i(f) = W^T_i f(t_i) \]
and, due to conditions (A2) and (A5),
\[
\mathbb{E} \left[ Q_i(f) - Q_j(f) \right]^2 = \mathbb{E} \left( (f - \tilde{f})^T(t_1) W_1 W_i^T(f - \tilde{f})(t_1) \right) = \mathbb{E} \left( (f - \tilde{f})^T(t_1) \Omega (f - \tilde{f})(t_1) \right) \\
\leq \omega^*_\max \mathbb{E} \left( \| (f - \tilde{f})(t_1) \|_2^2 \right) \leq \omega^*_\max \phi^*_\max D^2,
\]
where \( \omega^*_\max \) and \( D^2 \) are defined in (2.7) and (5.4), respectively.

In order to derive the lower bounds for the risk, we use Theorem 2.5 of Tsybakov (2009) which implies that, if a set \( \Theta \) of cardinality \( M + 1 \) contains sequences \( \omega_0, \cdots, \omega_M \) with \( M \geq 2 \) such that, for any \( j = 1, \cdots, M \), one has \( \| f_{\omega_0} - f_{\omega_j} \|_2 \geq D > 0 \), \( P_{\omega_j} \ll P_{\omega_0} \) and \( \mathcal{K}(P_{\omega_j}, P_{\omega_0}) \leq \kappa \log M \) with \( 0 \leq \kappa \leq 1/8 \), then
\[
\inf_{\omega} \sup_{f_{\omega}, \omega \in \Theta} \mathbb{P} \left( \| f_{\omega} - f_0 \|_2 \geq D/2 \right) \geq \frac{\sqrt{M}}{1 + \sqrt{M}} \left( 1 - 2\kappa - \sqrt{\frac{2\kappa}{\log M}} \right). \tag{5.5}
\]
Now, we consider two separate cases.

\textbf{Case 1.} Let the first \( s_0 \) functions be constant and the rest of the functions be equal to identical zero. Then, \( v = 0 \) and \( K_1 = \{1, \cdots, s_0\} \). Use the Varshamov-Gilbert Lemma (Lemma 2.9 of [33]) to choose a set \( \Theta \) of \( \omega \) with \( \operatorname{card}(\Theta) \geq 2^{s_0/8} \) and \( D^2 \geq u^2 s_0/8 \). Inequality \( \mathcal{K}(P_{f_j}, P_{f_0}) \leq (2\sigma^2)^{-1} n \omega^*_\max \phi^*_\max u^2 s_0/8 \leq \kappa \log (\operatorname{card}(\Theta)) \) holds if \( u^2 = 2\sigma^2 \kappa/(n \omega^*_\max \phi^*_\max) \). Then, \( D^2 = (s_0 \sigma^2 \kappa)/(4n \omega^*_\max \phi^*_\max) \) and \( u \leq C_a \) provided \( n \geq 2\sigma^2 \kappa/(C_a^2 \omega^*_\max \phi^*_\max) \).

\textbf{Case 2.} Let the first \( s \) functions be time-dependent and the rest of the functions be equal to identical zero. Then \( u = 0 \), \( v \) is given by formula (5.3), \( K_0 = \{1, \cdots, s\} \). Let \( r_k, k \in K_0 \), coincide with the values of finite components of vector \( r \). Denote
\[
\mathcal{L} = \sum_{k=1}^{s} l_{0k}.
\]
Use Varshamov-Gilbert Lemma to choose a set \( \Theta \) of \( \omega \) with \( \operatorname{card}(\Theta) \geq 2^{\mathcal{L}/8} \) and \( D^2 \geq v^2 \mathcal{L}/8 \). Inequality \( \mathcal{K}(P_{f_j}, P_{f_0}) \leq \kappa \mathcal{L}/8 \) holds if
\[
v^2 \leq (\sigma^2 \kappa)/(4n \omega^*_\max \phi^*_\max),
\]
which, together with (5.3) and (5.4) imply that
\[
l_{0k} = \left[ \frac{1}{2} \left( \frac{C_a^2 n \omega^*_\max \phi^*_\max}{\sigma^2 \kappa} \right)^{1/2k+1} \right] + 1, \quad D^2 \geq \frac{C_a^2}{16} \sum_{k=1}^{s} \left( \frac{4 C_a^2 n \omega^*_\max \phi^*_\max}{\sigma^2 \kappa} \right)^{-\frac{2k}{2k+1}}
\]
where \( \lfloor x \rfloor \) denotes the integer part of \( x \). Condition \( \mathcal{L} \geq 3 \) is satisfied for any \( s \geq 1 \) provided
\[ n \geq n_{\text{low}}. \]
5.2 Proofs of the upper bounds for the risk

Proof of Theorem 2

For any $\alpha \in \mathbb{R}^{p(L+1)}$, one has

$$n^{-1}\|B\hat{a} - Y\|^2_2 + \delta\|\hat{a}\|_{\text{block}} \leq n^{-1}\|B\alpha - Y\|^2_2 + \delta\|\alpha\|_{\text{block}}.$$  

Consider a set $F_1 \subseteq W_\mu^n$ such that (5.12) holds for any $t$ and $W \in F_1$, and a set $F_2 \subseteq W_\mu^n$ such that, (5.33) hold. Let $F = F_1 \cap F_2$. Lemma 2 implies that on the event $F$

$$\frac{2}{n} \left| \langle \hat{a} - \alpha, B^T b \rangle \right| \leq \hat{\delta}\|\hat{a} - \alpha\|_{\text{block}}.$$  

Consider a set $\Xi$ of values of the vector $\xi$ such that

$$2\sigma n^{-1} \left| \langle \hat{a} - \alpha, B^T \xi \rangle \right| \leq \hat{\delta}\|\hat{a} - \alpha\|_{\text{block}} \quad \text{for} \quad \xi \in \Xi. \quad (5.6)$$

Using (1.4), for $\xi \in \Xi$, any $t$ and $W \in F$, one obtains

$$\frac{\|B(\hat{a} - a_0)\|^2_2}{n} \leq \frac{\|B(\alpha - a_0)\|^2_2}{n} - 2\hat{\delta}\|\hat{a}\|_{\text{block}} + 2\hat{\delta}\|\alpha\|_{\text{block}} + \hat{\delta}\|\hat{a} - \alpha\|_{\text{block}}. \quad (5.7)$$

Since $\alpha$ is an arbitrary vector, setting $\alpha = a$ in (5.7) yields

$$\hat{\delta}\|\hat{a}\|_{\text{block}} - \hat{\delta}\|a\|_{\text{block}} \leq 0.5\hat{\delta}\|\hat{a} - a\|_{\text{block}}.$$  

Let the set $J_0$ contain the indices of nonzero blocks of $a_0$:

$$J_0 = \{(j, l) : \|a_{jl}\|_2 \neq 0\}. \quad (5.8)$$

Then, the last inequality implies

$$\sum_{(i,j) \in J_0^C} \| (a - \hat{a})_{ij} \| \leq 3 \sum_{(i,j) \in J_0} \| (a - \hat{a})_{ij} \|. \quad (5.9)$$

From Lemma 1 it follows that

$$\lambda_{\min}(\hat{\Sigma}) \geq (1 - h)\phi_{\min} \omega_{\min}^* \quad \text{and} \quad \lambda_{\max}(\hat{\Sigma}) \leq (1 + h)\phi_{\max} \omega_{\max}^*$$

where $\omega_{\min}$ and $\omega_{\max}$ are defined in (2.7). For $1 \leq j \leq p$ consider sets

$$G_{00} = \{j : 1 \leq j \leq p, \alpha_{j0} = 0\}, \quad G_{01} = \{j : 1 \leq j \leq p, \alpha_{j0} \neq 0\},$$

$$G_{j0} = \{l : 1 \leq l \leq M, \|\alpha_{jl}\|_2 = 0\}, \quad G_{j1} = \{l : 1 \leq l \leq M, \|\alpha_{jl}\|_2 \neq 0\}.$$  

We choose $\alpha_{j0} = a_{j0}$ if $a_{j0} \neq 0$ and $\alpha_{j0} = 0$ otherwise. Let sets $G_{j1}$ be so that $l \in G_{j1}$ if

$$\|a_{jl}\|_2 > \varepsilon = \frac{8^2(\sigma^2 C_\omega K^2 \mu + 1) \log p}{n\lambda_{\min}(\hat{\Sigma})}.$$  

We set $\alpha_{jl} = a_{jl}$ if $j \in J$ and $l \in G_{j1}$ and $\alpha_{jl} = 0$ otherwise where $J$ is the set of indices corresponding to non-constant functions $f_j$.
With $\delta = 2 \hat{\delta}$, inequality (5.9) and Lemma 4 guarantee that

$$\frac{\|B(\bar{a} - a)\|^2}{n} \geq C_B \lambda_{\min}(\hat{\Sigma}) \|\bar{a} - a\|^2.$$  (5.10)

On the other hand, Lemma 11 and the definition of $\alpha$ imply that

$$\frac{\|B(\alpha - a)\|^2}{n} \leq \lambda_{\max}(\hat{\Sigma}) \|\alpha - a\|^2.$$  (5.11)

Then, using (5.10) and (5.11), we rewrite inequality (5.7) as

$$C_B \lambda_{\min}(\hat{\Sigma}) \|\bar{a} - a_0\|^2 \leq \lambda_{\max}(\hat{\Sigma}) \|\alpha - a_0\|^2$$  (5.12)

Using (5.33), we obtain

$$\|\bar{a} - a_0\|^2 \leq 2 \hat{\delta} \sum_{j \in J} \|\bar{a}_j - a_j\|^2 + \sum_{j \in J} \|\bar{a}_j - a_j\|^2.$$  (5.13)

and similar inequality applies to the first sum in (5.12). By subtracting $0.5 C_B \lambda_{\min}(\hat{\Sigma}) \|\bar{a} - a\|^2$ from both sides of (5.12) and plugging in the values of $\delta$ and $\alpha$, derive:

$$\|\bar{a} - a\|^2 \leq C_B \lambda_{\min}(\hat{\Sigma}) \left[ \sum_{j=1}^p \|a_j\|^2 + \frac{8^2 (\sigma^2 C_\omega K^2 \mu + 1) (s_0 + s) \log p}{n \lambda_{\min}(\hat{\Sigma})} \right]$$  (5.14)

Observe that for any $1 \leq j \leq p$ one has

$$\sum_{l \in G_{j_0}} \|a_{jl}\|^2 + \frac{8^2 (\sigma^2 C_\omega K^2 \mu + 1) \log p}{n \lambda_{\min}(\hat{\Sigma})} \leq \sum_{l=1}^M \min \left( \|a_{jl}\|^2, \frac{8^2 (\sigma^2 C_\omega K^2 \mu + 1) \log p}{n \lambda_{\min}(\hat{\Sigma})} \right).$$

Application of inequality (5.31) with $\varepsilon = \frac{8^2 (\sigma^2 C_\omega K^2 \mu + 1) \log p}{n \lambda_{\min}(\hat{\Sigma}) \log n}$ (see Lemma 4) yields

$$\|\bar{a} - a\|^2 \leq C_B \lambda_{\max}(\hat{\Sigma}) \left[ \frac{(C_\omega \sigma^2 K^2 \mu + 1) (s_0 + s) \log p}{n \lambda_{\min}(\hat{\Sigma})} \right]$$  (5.15)

Denote $r^*_j = r_j \wedge r^*_j$ and $r^*_{\min} = \min_j r^*_j$. Choose $L + 1 \geq n^{1/2}$ and note that $r^*_{\min} \geq 2$. Using (5.33), we obtain

$$\|\bar{f} - f\|^2 \leq \|\bar{a} - a\|^2 + C_\alpha^2 s (L + 1)^{-2r^*_\min},$$

which implies

$$\|\bar{f} - f\|^2 \leq \|\bar{a} - a\|^2 + \frac{C_\alpha^2 s}{n^2}.$$
Now, (5.14) together with Lemmas 1, 2 and 4 imply
\[ \mathbb{P} \left( \| \hat{f} - f \|_2^2 \leq \Delta(s_0, s, n, r) \right) \geq 1 - 8p^{-\mu}. \]

**Proof of Theorem 3.** Let sets \( \tilde{F} \) be such that (5.14) holds. Then, \( P(\tilde{F}) \geq 1 - 8p^{-\mu} \) and (5.14) yields
\[ \mathbb{E} \| \hat{a} - a \|_2^2 \leq \mathbb{E} \left[ \| \hat{a} - a \|_2^2 \mathbb{I}(\tilde{F}^C) \right] + \mathbb{E} \left[ \| \hat{a} - a \|_2^2 \mathbb{I}(\tilde{F}) \right] \leq C \Delta_{\text{upper}}(s_0, s, n, r), \]
due to (3.21).

**5.3 Proof of Lemma 1.**

In order to simplify the notations, we set \( \omega_{\text{max}}(\aleph) = \omega_{\text{max}} \) and \( U_{\mu} = U_{\mu}(\aleph) \). Let \( W_{\mu} \) be the set of points described in condition (2.6) of Assumption (A5). Denote the direct product of \( n \) sets \( W_{\mu} \) by \( W_{\mu}^{\otimes n} \). Then,
\[ \mathbb{P}(W_{\mu}^{\otimes n}) \geq 1 - 2np^{-2\mu}. \]

Consider random matrices
\[ Z_i = (\Sigma_i)\Lambda - \Sigma = (\Omega_i)\Lambda \otimes \Phi_i - \Omega \Lambda \otimes \Phi, \quad \zeta_i = (\Sigma_i)\Lambda \mathbb{I} - \mathbb{E}((\Sigma_i)\Lambda \mathbb{I}(W_{\mu})). \]

Then, \( \zeta_i \) are i.i.d. with \( \mathbb{E}\zeta_i = 0 \). We apply the matrix version of Bernstein’s inequality, given in Tropp [32]:

**Proposition 1 (Theorem 1.6, Tropp (2011))** Let \( \zeta_1, \ldots, \zeta_n \) be independent random matrices in \( \mathbb{R}^{m_1 \times m_2} \) such that \( \mathbb{E}(\zeta_i) = 0 \). Define
\[ \sigma_{\zeta} = \max \left\{ \left\| \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left( \zeta_i \zeta_i^T \right) \right\|^{1/2}, \left\| \frac{1}{n} \sum_{i=1}^n \mathbb{E} \left( \zeta_i^T \zeta_i \right) \right\|^{1/2} \right\}. \]

and suppose that \( \| \zeta_i \| \leq T \) for some \( T > 0 \). Then, for all \( t > 0 \), with probability at least \( 1 - e^{-t} \) one has
\[ \left\| \frac{1}{n} \sum_{i=1}^n \zeta_i \right\| \leq 2 \max \left\{ \sigma_{\zeta} \sqrt{\frac{t + \log(d)}{n}}, T \frac{t + \log(d)}{n} \right\}, \quad (5.15) \]
where \( d = m_1 + m_2 \).
In order to find $\sigma_\xi$, note that

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(\zeta_i^T \zeta_i^T) \right\| = \left\| \mathbb{E}(\zeta_1^T \zeta_1^T) \right\| \leq \left\| \mathbb{E}((\Sigma_1)\Lambda((\Sigma_1)^T \Lambda)^T(W_\mu)) \right\| + \left\| \mathbb{E}((\Sigma_1)\Lambda(W_\mu)) \right\| \left\| \mathbb{E}((\Sigma_1)^T \Lambda(W_\mu)) \right\|
\]

\[
= \left\| \mathbb{E}(((\Omega_1)\Lambda \otimes \Phi_i)((\Omega_1)\Lambda \otimes \Phi_i)^T(W_\mu)) \right\| + \left\| \mathbb{E}(((\Omega_1)\Lambda \otimes \Phi_i)^T(W_\mu)) \right\|^2
\]

\[
= \left\| \mathbb{E}(((\Omega_1)\Lambda \otimes \Phi_i)((\Omega_1)\Lambda \otimes \Phi_i)^T(W_\mu)) \right\| + \left\| \mathbb{E}(((\Omega_1)\Lambda(W_\mu)) \right\| \left\| \mathbb{E}((\Omega_1)\Lambda(W_\mu)) \right\| \left\| \mathbb{E}(\Phi_i) \right\|^2
\]

\[
\leq \left\| \mathbb{E}(\Phi_i \Phi_i^T) \right\| \left\| \mathbb{E}(((\Omega_1)\Lambda((\Omega_1)^T \Lambda)^T(W_\mu)) \right\| + \left\| \mathbb{E}(((\Omega_1)\Lambda(W_\mu)) \right\| \left\| \mathbb{E}(\Phi_i) \right\|^2.
\]

and, similarly,

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(\zeta_i^T \zeta_i^T) \right\| \leq \left\| \mathbb{E}(\Phi_i \Phi_i^T) \right\| \left\| \mathbb{E}(((\Omega_1)\Lambda((\Omega_1)^T \Lambda)^T(W_\mu)) \right\| + \left\| \mathbb{E}((\Omega_1)\Lambda(W_\mu)) \right\| \left\| \mathbb{E}(\Phi_i) \right\|^2.
\]

Here,

\[
\| \mathbb{E}(\Phi_i \Phi_i^T) \| \leq \| C_\Phi^2 (L + 1) \Phi \| = C_\Phi^2 (L + 1) \phi_{\max}
\]

and

\[
\| \mathbb{E}(((\Omega_1)\Lambda((\Omega_1)^T \Lambda)^T(W_\mu)) \| = \| \mathbb{E}([W_\Lambda W_\Lambda^T W_\Lambda W_\Lambda^T \Phi_i(W_\mu)]) \|
\]

\[
\leq U_\mu^2 \| \mathbb{E}(W_\Lambda W_\Lambda^T) \| = U_\mu^2 \| \Omega_\Lambda \| = U_\mu^2 \omega_{\max},
\]

so that

\[
\sigma_\xi^2 \leq 2 C_\Phi^2 U_\mu^2 (L + 1) \phi_{\max} \omega_{\max}.
\]

Now, observe that, since matrix $\mathbb{E}((\Sigma_i)\Lambda(W_\mu))$ is non-negative definite and matrices $\Phi_i$ and $(\Omega_i)\Lambda$ have rank one for any $i$, one has

\[
T = \sup \| \zeta_i \| \leq 2 \sup \| (\Sigma_1)\Lambda(W_\mu) \| = 2 \sup \| (\Omega_1)\Lambda(W_\mu) \| \| \Phi_i \| \leq 2 C_\Phi^2 U_\mu^2 (L + 1).
\]

Apply Bernstein inequality \(^{(5.15)}\) with $\sigma_\xi^2$ and $T$ given by formulae \(^{(5.16)}\) and \(^{(5.17)}\), respectively. Then, we obtain for any $t > 0$, with probability at least $1 - e^{-t}$

\[
\left\| \frac{1}{n} \sum_{i=1}^{n} \zeta_i \right\| \leq 4 \max \left\{ \frac{C_\phi U_\mu \sqrt{(L + 1)(t + \log(LN) \phi_{\max}) \omega_{\max}}}{\sqrt{n}}, \frac{C_\phi^2 U_\mu^2 (L + 1)(t + \log(LN))}{n} \right\}.
\]

In order to apply inequality \(^{(5.18)}\) to $Z_i$, observe that $Z_i - \zeta_i = (\Sigma_i)\Lambda(W_\mu^c) - \mathbb{E}((\Sigma_i)\Lambda(W_\mu^c))$ and

\[
\| \mathbb{E}((\Sigma_i)\Lambda(W_\mu^c)) \|^2 = \| \mathbb{E}((\Sigma_i)\Lambda(W_\mu^c) \otimes \Phi_i) \|^2 \leq \| \mathbb{E}((\Sigma_i)\Lambda(W_\mu^c)) \|^2 \| \Phi_i \|^2
\]

\[
\leq \left( \| \mathbb{E}((\Sigma_i)\Lambda(W_\mu^c)) \right\|^2 \| \Phi_i \|^2 \leq 2 C_\phi^4 V \Lambda p^{-2\mu} (L + 1)^2,
\]

due to the fact that $\mathbb{E}(W^{(j)})^4 \leq V$ for any $j = 1, \ldots, p$. 

Now, we use the union bound over all \( \Lambda \) such that \(|\Lambda| \leq N\), the inequality \( \frac{p}{N} \leq \left(\frac{\log(p)}{N}\right)^8 \) and choose \( t = 2\mu N \log\left(\frac{p}{N}\right)^8 \). Combining (5.18) and (5.19), for all \( \Lambda \) such that \(|\Lambda| \leq N\), we derive

\[
\begin{align*}
&\inf_{\Lambda: |\Lambda| \leq N} \mathbb{P}\left(\left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i \right\| < z \right\} \cap \{ \mathcal{W} \in \mathcal{W}_\mu^{\otimes n} \} \right) \\
&\geq \inf_{\Lambda: |\Lambda| \leq N} \mathbb{P}\left(\left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i \right\| < z - \|\mathbb{E}(\mathbf{\Sigma}_1)\| \right\} \cap \{ \mathcal{W} \in \mathcal{W}_\mu^{\otimes n} \} \right) \\
&\geq 1 - \sup_{\Lambda: |\Lambda| \leq N} \mathbb{P}\left(\left\{ \left\| \frac{1}{n} \sum_{i=1}^{n} \mathbf{z}_i \right\| \geq z - C_\phi^2 (L+1) p^{-\mu} \sqrt{2N} \right\} \cap \{ \mathcal{W} \in \mathcal{W}_\mu^{\otimes n} \} \right) \geq 1 - (ep)^{-\mu} - 2np^{-2\mu}
\end{align*}
\]

for any \( z \) such that

\[
z \geq 8 \max \left\{ \frac{C_\phi U_\mu \sqrt{\mu N (L+1) \log(p+L) \phi_{\max} \omega_{\max}}}{\sqrt{n}}, \frac{\mu N C_\phi^2 U_\mu^2 (L+1) \log(p+L)}{n} \right\} + C_\phi^2 (L+1) p^{-\mu} \sqrt{2N}.
\]

(5.20)

Note that, under condition (3.10), one has

\[
C_\phi^2 (L+1) p^{-\mu} \sqrt{2N} \leq 8 \mu N C_\phi^2 U_\mu^2 (L+1) \log(p+L) n^{-1}.
\]

It is easy to check that, whenever \( n \geq N(\lambda) \) where \( N(\lambda) \) is defined in (3.9), condition (5.20) is satisfied with

\[
z = \frac{8 C_\phi U_\mu \sqrt{\mu N (L+1) \phi_{\max} \omega_{\max} \log(p+L)}}{\sqrt{n}} + \frac{9 \mu N C_\phi^2 U_\mu^2 (L+1) \log(p+L)}{n} \leq h \omega_{\min} \phi_{\min},
\]

which, together with condition \( p^\mu \geq 2n \), implies that

\[
\inf_{\Lambda: |\Lambda| \leq \Lambda} \mathbb{P}\left( \|\tilde{\mathbf{\Sigma}}_\Lambda - \mathbf{\Sigma}_\Lambda\| \leq h \omega_{\min} \phi_{\min} \right) \geq 1 - p^{-\mu} - p^{-\mu}.
\]

(5.21)

In order to complete the proof, observe that \( \lambda_{\min} (\tilde{\mathbf{\Sigma}}_\Lambda) \geq \lambda_{\min} (\mathbf{\Sigma}_\Lambda) - \|\tilde{\mathbf{\Sigma}}_\Lambda - \mathbf{\Sigma}_\Lambda\| \) and \( \lambda_{\max} (\tilde{\mathbf{\Sigma}}_\Lambda) \leq \lambda_{\max} (\mathbf{\Sigma}_\Lambda) + \|\tilde{\mathbf{\Sigma}}_\Lambda - \mathbf{\Sigma}_\Lambda\| \).

### 5.4 Proofs of supplementary lemmas

**Lemma 2** Let vector \( \mathbf{a} \in \mathbb{R}^{p(L+1)} \) be partitioned into subgroups the way it has been done for vector \( \mathbf{a} = \text{Vec} (\mathbf{A}) \). Let \( L+1 \geq n^{1/2} \) and \( n \geq N \) where \( N \) is defined in (3.15). Suppose that Assumptions (A5), (A4) are valid and \( \mu \) in (2.6) is large enough, so that condition (3.15) holds. Then,

\[
\text{(i)} \quad \mathbb{P}\left( \frac{2\sigma |(\mathbf{\alpha}, \mathbf{B}^T \mathbf{\xi})|}{n} \leq \delta \|\mathbf{\alpha}\|_{\text{block}} \right) \geq 1 - 5p^{-\mu};
\]

(5.22)
\[(ii) \quad \mathbb{P} \left( \frac{2 |\langle \alpha, B^T b \rangle|}{n} \leq \hat{\delta} \|\alpha\|_{\text{block}} \right) \geq 1 - 4p^{-\mu}, \quad (5.23) \]

where \(\hat{\delta}\) is defined in \((5.14)\).

**Proof.** In order to prove (i), note that
\[
n^{-1/2} |\langle \alpha, B^T \xi \rangle| \leq n^{-1/2} \max_{1 \leq j \leq p} |\langle B^T \xi \rangle^{(j,0)}| \sum_{j=1}^{p} |\alpha_j| \quad (5.24)
\]

\[+ \quad n^{-1/2} \max_{1 \leq j \leq p} \left[ \sum_{k \in K_i} \left| \langle B^T \xi \rangle^{(j,k)} \right|^2 \right]^{1/2} \sum_{j=1}^{p} \sum_{l=1}^{M} \sqrt{\sum_{k \in K_i} \alpha_{jk}^2}. \]

Fix vectors \(W_1, \cdots, W_n\) and \(t\). Using Hoeffding-type inequality for sub-gaussian random variables (see, e.g., Proposition 5.2 in \([35]\)) we obtain that, for any \(\mu \geq 1\),
\[
\mathbb{P} \left( n^{-1/2} \max_{1 \leq j \leq p} |\langle B^T \xi \rangle^{(j,0)}| \leq \sqrt{CK^2n_{\max}(\tilde{\Sigma}_\Lambda) \log(p)} \right) \geq 1 - e^{1-2\mu} \quad (5.25)
\]

where \(C > 0\) is an absolute constant and \(|\Lambda| \leq 1\).

For the second maximum in \((5.24)\), we use a corollary of the Hanson-Wright inequality for sub-gaussian random vectors (see Theorem 2.1 in \([29]\)), which states that, for any matrix \(Q \in \mathbb{R}^{n \times d}\) and any \(\mu \geq 1\), one has
\[
\mathbb{P} \left( \|Q\|_2 \leq C K \lambda_{\max}(Q) \sqrt{\mu \log(p)} \right) \geq 1 - 2p^{-3\mu}.
\]

We apply this inequality with matrix \(Q = n^{-1/2} B_{j,t}\), where \(B_{j,t}\) is a \(n \times d\) sub-matrix of matrix \(B\). Note that \(\lambda_{\max}(Q) \leq \sqrt{\lambda_{\max}(\tilde{\Sigma}_\Lambda)}\) where \(|\Lambda| \leq 1\). Using \(\|Q\|_2 \leq d \lambda_{\max}(\tilde{\Sigma}_\Lambda)\) and \(d \approx \log n\), obtain for any \(\mu \geq 1\)
\[
\mathbb{P} \left( n^{-1/2} \max_{1 \leq j \leq p} \sum_{k \in K_i} \left[ \langle B^T \xi \rangle^{(j,k)} \right] \leq \left( 1 + CK \sqrt{\mu} \right) \sqrt{\lambda_{\max}(\tilde{\Sigma}_\Lambda) \log(p)} \right) \geq 1 - \frac{2pL}{p^{3\mu} \log(n)} \quad (5.26)
\]

where we used \(M = L / \log(n)\).

Now, consider a set \(F_1 \subseteq W_{\mu}^{\otimes n}\) such that \((3.12)\) holds for any \(t\) and any \(W \in F_1\). By Lemma\(\square\) obtain \(\mathbb{P}(F_1) \geq 1 - 2p^{-\mu}\). Then,
\[
n^{-1} |\langle \alpha, B^T \xi \rangle| \leq CK \sqrt{\frac{\mu(1 + h) \phi_{\max} \omega_{\max}(1) \log p}{n}}
\]

with probability at least \(1 - 5p^{-\mu}\), due to condition \((3.15)\).

Inequality (ii) is proved in a similar manner. Indeed, one has
\[
n^{-1/2} |\langle \alpha, B^T b \rangle| \leq n^{-1/2} \max_{1 \leq j \leq p} |\langle B^T b \rangle^{(j,0)}| \sum_{j=1}^{p} |\alpha_j| \quad (5.27)
\]

\[+ \quad n^{-1/2} \max_{1 \leq j \leq p} \left[ \sum_{k \in K_i} \left| \langle B^T b \rangle^{(j,k)} \right|^2 \right]^{1/2} \sum_{j=1}^{p} \sum_{l=1}^{M} \sqrt{\sum_{k \in K_i} \alpha_{jk}^2}. \]
Consider a set $F_1 \subseteq \mathcal{W}^{\otimes n}_{\mu}$ such that (5.12) holds for any $t$ and any $W \in F_1$. By Lemma 4, we obtain that $\mathbb{P}(\mathcal{F}_1) \geq 1 - 2^p r^{-\mu}$. Then, on the event $F_1$, the following inequalities are valid

$$n^{-1/2} \max_{1 \leq j \leq p} \left| (B^T b)^{j,0} \right| \leq \sqrt{1 + h} \phi \omega (1) \|b\|_2,$$

$$n^{-1/2} \max_{1 \leq j \leq p} \sqrt{\sum_{k \in K_i} \left[ (B^T b)^{j,k} \right]^2} \leq \sqrt{1 + h} \phi \omega (1) \|b\|_2.$$  \hspace{1cm} (5.28)

Now consider a set $F_2 \subseteq \mathcal{W}^{\otimes n}_{\mu}$ such that (5.36) holds. Let $F = F_1 \cap F_2$. Lemmas 4 and 5 imply that on the event $F$

$$n^{-1} \max_{1 \leq j \leq p} \left| (B^T b)^{j,0} \right| \leq C_a (L + 1)^{-2} \sqrt{3} g_2 s (1 + h) \phi \omega (s) \omega (1)$$

and

$$n^{-1} \max_{1 \leq j \leq p} \sqrt{\sum_{k \in K_i} \left[ (B^T b)^{j,k} \right]^2} \leq C_a (L + 1)^{-2} \sqrt{3} g_2 s (1 + h) \phi \omega (s) \omega (1).$$

Now, the statement of the Lemma is valid due to assumption $(L + 1)^2 \geq n$.

**Lemma 3** Let $S \in \{1, \ldots, p\}$ be a subset of indices with $|S| \leq (s + s_0)$ and $J = S \times \{0, \ldots, L\} \subset \{1, \ldots, p\} \times \{0, \ldots, L\}$. We define $A$ to be the following set of vectors

$$A = \left\{v \in \mathbb{R}^{(L+1)p} : \sum_{(i,j) \in J^C} \|v_{ij}\| \leq 3 \sum_{(i,j) \in J} \|v_{ij}\| \right\}.$$  \hspace{1cm}

Suppose that the assumptions of Lemma 4 and Assumption (A6) hold. Then, there exists a numerical constant $C_B$ such that

$$\min_S \min_{v \in A} \frac{\|Bv\|}{\|v\|} \geq C_B (1 - h) \phi \omega^*$$

where $\omega^*$ is defined in (2.7).

**Proof.** The proof follows the lines of the proof of Lemma 4.1 in [2].

**Lemma 4** Let function $f(t) = \sum_{k=1}^{\infty} a_k \phi_k(t)$ satisfy condition (A4), i.e.

$$\sum_{k=0}^{\infty} \|a_k\|^{\nu} (k + 1)^{-r'} \leq C_a^{\nu}, \quad r' = r + 1/2 - 1/\nu,$$  \hspace{1cm} (5.30)

for some $C_a > 0$, $1 \leq \nu < \infty$ and $r > \min(1/2, 1/\nu)$. Let $a_k$ be blocks of coefficients $a_k$ of length $d$, so that $a_k = (a_{(i-1)d+1}, \ldots, a_{id})$. Then, for any $\varepsilon > 0$ one has

$$\sum_{l=0}^{\infty} \min_{\varepsilon d} (\|a_k\|_2^2, \varepsilon) \leq C_a^{2d} \varepsilon \cdot \varepsilon d \cdot d^{(2-\nu)/2}.$$  \hspace{1cm} (5.31)

Moreover, if $r' \geq 2$ and basis $\{\phi_k\}$ satisfies assumption (A1), then

$$\|f - f_J\|_{\infty} \leq C_a C_{\phi} (J + 1)^{-(r^*-1/2)}$$

with $f_J(t) = \sum_{k=1}^{J} a_k \phi_k(t).$  \hspace{1cm} (5.32)

Here, $r^* = \min(r, r')$, $(x)_+ = x$ if $x > 0$ and zero otherwise.
Proof. First, let us show that for $J \geq 1$

$$\sum_{k=J+1}^{\infty} a_k^2 \leq C_a^2 (J + 1)^{-2r^*}. \quad (5.33)$$

Indeed, if $\nu \geq 2$, the Cauchy inequality yields

$$\sum_{k=J+1}^{\infty} a_k^2 \leq \left( \sum_{k=J+1}^{\infty} |a_k|^\nu (k^r)^{\nu/2} \right)^{2/\nu} \left( \sum_{k=J+1}^{\infty} \frac{2r^*}{\nu - 2} \right)^{1-2/\nu} \leq C_a^2 \left( \frac{\nu - 2}{2r^*} \right)^{\nu/2 - 1} (J + 1)^{-2r}.$$

Since $(\nu - 2)/(2r^*) < 1$ for $\nu \geq 2$, inequality (5.33) holds. If $1 \leq \nu < 2$, then

$$\sum_{k=J+1}^{\infty} a_k^2 \leq \left( \max_{k \geq J+1} |a_k| \right)^{2-\nu} (J + 1)^{-r^*} \left( \sum_{k=J+1}^{\infty} |a_k|^\nu (k^r)^{\nu/2} \right) \leq C_a^2 (J + 1)^{-2r^*},$$

so that (5.33) is valid.

Now, using (5.33), we prove (5.31). Again, we consider cases $\nu \geq 2$ and $1 \leq \nu < 2$, separately. If $\nu \geq 2$, then partitioning the sum into the portion for $l \leq J$ and $l > J$ (which corresponds to $k > J d$), we derive

$$\sum_{l=1}^{\infty} \min(\|a_l\|_2^2, \varepsilon d) \leq J \varepsilon d + C_a^2 (J d)^{-2r^*}.$$

Minimizing the last expression with respect to $J$, we obtain (5.31) without the log-factor. If $1 \leq \nu < 2$, then

$$\sum_{l=1}^{\infty} \min(\|a_l\|_2^2, \varepsilon d) \leq J \varepsilon d + (d \varepsilon)^{1-\nu/2} \sum_{l=J+1}^{\infty} \|a_l\|_2^\nu.$$

Since for $1 \leq \nu < 2$

$$\|a_l\|_2^2 = \sum_{k=(l-1)d+1}^{ld} a_k^2 \leq \left( \sum_{k=(l-1)d+1}^{ld} |a_k|^{\nu} \right)^{2/\nu},$$

one has

$$\sum_{l=J+1}^{\infty} \|a_l\|_2^\nu \leq \sum_{k=Jd+1}^{\infty} |a_k|^{\nu} \leq C_a^\nu (Jd + 1)^{-r^* \nu}$$

and

$$\sum_{l=1}^{\infty} \min(\|a_l\|_2^2, \varepsilon d) \leq J \varepsilon d + C_a^\nu (d \varepsilon)^{1-\nu/2} (Jd)^{-r^* \nu}.$$

Minimization of the last expression with respect to $J$ yields (5.31).
In order to prove (5.32), observe that for any \( J > 1 \) and \( r^* > 3/2 \), one has

\[
\|f - f_J\|_\infty \leq \sup_{t \in [0,1]} \sum_{l=1}^{\infty} \left[ \sum_{k=Jl}^{J(l+1)-1} a_k^2 \right]^{1/2} \sum_{k=Jl}^{J(l+1)-1} \phi_k^2(t) \leq C_a C_\phi \sum_{l=1}^{\infty} \sqrt{J(1 + Jl)}(1^{-r^*})
\]

which completes the proof.

**Lemma 5** Let \( r_j^* = r_j \land r_j' \), \( r^* = \min_j r_j^* \geq 2 \) in assumption (A3). Let \( \mu \) in (5.4) be large enough, so that

\[
p^\mu \geq 2n
\]

and \( n \) be such that

\[
n \geq \frac{U_\mu^2 C_\phi^2 (L + 1) \mu \log p}{g_2 \omega_{\max}(s)}
\]

where \( U_\mu = U_\mu(s + s_0) \). Then, one has

\[
\mathbb{P} \left\{ n^{-1} \|b\|_2^2 \leq 3g_2 \omega_{\max}(s) s C_\phi^2 (L + 1)^{-4} \cap \{W \in W_{\mu}^{\otimes n}\} \right\} \geq 1 - 2p^{-\mu}.
\]

Here \( b \) is the vector with components \( b_i = W_i^T \rho(t_i) \), \( i = 1, \ldots, n \), where \( \rho(t) = (\rho_1(t), \ldots, \rho_p(t))^T \) and \( \rho_i(t) \) are defined in (5.2).

**Proof.** In order to estimate \( \|b\|_2^2 \) we apply Bernstein inequality to the centered random variables \( \beta_i = b_i^2 \|W_{\mu}\| = \mathbb{E} b_i^2 \|W_{\mu}\| - \mathbb{E} (b_i^2 \|W_{\mu}\|) \). We start with establishing an upper bound for \( \mathbb{E} b_i^2 \):

\[
\mathbb{E} b_i^2 = \mathbb{E} (n^{-1} \|b\|_2^2) = \mathbb{E} (\rho(t) \Omega_i \rho(t)) \leq \omega_{\max}(s) \mathbb{E} \|\rho(t)\|_2^2.
\]

Using (5.33) and the orthonormality of the basis, we derive

\[
\mathbb{E} \|\rho(t)\|_2^2 \leq 2g_2 C_\phi^2 (L + 1)^{-2r^*}.
\]

Plugging (5.38) into (5.37), we obtain

\[
\mathbb{E} b_i^2 \leq 2g_2 \omega_{\max}(s) s C_\phi^2 (L + 1)^{-2r^*}.
\]

Now, we use the upper bound (5.32) to establish an upper bound for the variance \( \sigma_b^2 \) of \( b_i^2 \):

\[
\sigma_b^2 \leq \mathbb{E} (b_i^2 \|W_{\mu}\|) \leq U_{\mu}^2 \mathbb{E} \|\rho(t)\|_2^4 \leq 2g_2 U_{\mu}^4 s^2 C_\phi^4 (L + 1)^{-4r^* + 2}.
\]

It is also easy to see that, for any \( i \) and \( W \in W_{\mu}^{\otimes n} \), by (5.32), one has

\[
U_b = \max_t (b_i^2) \leq \|W_{\mu}\| \max_t \|\rho(t)\|_2^2 \leq U_\mu^2 C_\phi^2 (L + 1)^{-(2r^* - 1)} s.
\]

With \( \sigma_b^2 \) and \( U_b \) given by (5.39) and (5.40), respectively, and \( t = \mu \log(p) \), one obtains, that with probability at least \( 1 - p^{-\mu} \)

\[
\left| \frac{1}{n} \sum_{i=1}^{n} \beta_i \right| \leq 2 \max \left\{ \sigma_b \sqrt{\frac{\mu \log p}{n}}, U_b \frac{\mu \log p}{n} \right\}.
\]
Since \( b_i^2 - \beta_i = b_i^2 \mathbb{P}(W_i^\mu) + \mathbb{E}(b_i^2 \mathbb{P}(W_i)) \), we derive

\[
\mathbb{P} \left( \left\{ n^{-1} \| b \|^2 < z \right\} \cap \left\{ \mathbb{W} \in \mathcal{W}_\mu^{\otimes n} \right\} \right) \geq \mathbb{P} \left( \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} \beta_i \right| < z - \mathbb{E} \left( n^{-1} \| b \|^2 \mathbb{P}(W_i) \right) \right\} \cap \left\{ \mathbb{W} \in \mathcal{W}_\mu^{\otimes n} \right\} \right)
\]

\[
\geq \mathbb{P} \left( \left\{ \left| \frac{1}{n} \sum_{i=1}^{n} \beta_i \right| < z - g_2 \omega_{\max}(s) C_a^2(L + 1)^{-2r^*} \right\} \cap \left\{ \mathbb{W} \in \mathcal{W}_\mu^{\otimes n} \right\} \right) \geq 1 - p^{-\mu} - 2n p^{-2\mu} \geq 1 - 2p^{-\mu}
\]

for any \( z \) such that

\[
z \geq 2 \max \left\{ \sigma_b \sqrt{\frac{\mu \log p}{n}}, U_b \frac{\mu \log p}{n} \right\} + g_2 \omega_{\max}(s) s C_a^2(L + 1)^{-2r^*}.
\]

For \( n \), satisfying condition (5.35), one can choose

\[
z = 3 g_2 \omega_{\max}(s) s C_a^2(L + 1)^{-2r^*}
\]

which together with \( r^* \geq 2 \) implies the statement of the Lemma.

References

[1] Bach, F. (2008) Consistency of the group lasso and multiple kernel learning. J. Mach. Learn. Res., 9, 1179 - 1225.

[2] Bickel, P.J., Ritov, Y. and Tsybakov, A. (2009) Simultaneous analysis of Lasso and Dantzig selector. Ann. Statist., 37(4), 1705 - 1732.

[3] Birgè, L., Massart, P. (2007) Minimal Penalties for Gaussian Model Selection. Probab. Theory Related Fields, 138, 33-73

[4] Bunea, F., Tsybakov, A. and Wegkamp, M. (2007) Aggregation for Gaussian regression. Ann. Statist., 35(4), 1674 - 1697.

[5] Bunea, F., Tsybakov, A. and Wegkamp, M. (2007) Sparsity oracle inequalities for the Lasso. Electron. J. Stat., 1, 169 - 194.

[6] Bühlmann, P. and van de Geer, S. (2011) Statistics for High-Dimensional Data: Methods, Theory and Applications. Springer.

[7] Chesneau, C. and Hebiri, M. (2008) Some theoretical results on the grouped variables Lasso. Math. Methods Statist., 17(4), 317 - 326.

[8] Chiang, C.-T., Rice, J. A. and Wu, C. O. (2001). Smoothing spline estimation for varying coefficient models with repeatedly measured dependent variables. J. Amer. Statist. Assoc., 96, 605-619.

[9] Cleveland, W.S., Grosse, E. and Shyu, W.M. (1991) Local regression models. Statistical Models in S (Chambers, J.M. and Hastie, T.J., eds), 309-376. Wadsworth and Books, Pacific Grove.
[10] Dalalyan, A., Ingster, Y., Tsybakov, A.B. (2013) Statistical inference in compound functional models. *Probab. Theory Rel. Fields*, to appear.

[11] Fan, J., Ma, Y., and Dai, W. (2013) Nonparametric Independence Screening in Sparse Ultra-High Dimensional Varying Coefficient Models. [arXiv:1303.0458v1](http://arxiv.org/abs/1303.0458v1)

[12] Fan, J. and Zhang, W. (1999). Statistical estimation in varying coefficient models. *Ann. Statist.*, **27**, 1491-1518.

[13] Fan, J., and Zhang, W. (2008) Statistical methods with varying coefficient models. *Statistics and Its Interface*, **1**, 179-195.

[14] Hastie, T.J. and Tibshirani, R.J. (1993) Varying-coefficient models. *J. Roy. Statist. Soc. B* (Chambers, J.M. and Hastie, T.J., eds), **55**, 757-796.

[15] Hoover, D. R., Rice, J. A., Wu, C. O. and Yang, L.-P. (1998). Non-parametric smoothing estimates of time-varying coefficient models with longitudinal data. *Biometrika*, **85**, 809-822.

[16] Huang, J. Z., Wu, C. O. and Zhou, L. (2002). Varying-coefficient models and basis function approximations for the analysis of repeated measurements. *Biometrika*, **89**, 111-128.

[17] Huang, J. Z. and Shen, H. (2004). Functional coefficient regression models for nonlinear time series: A polynomial spline approach. *Scandinavian Journal of Statistics*, **31**, 515-534.

[18] Huang, J. Z., Wu, C. O. and Zhou, L. (2004). Polynomial spline estimation and inference for varying coefficient models with longitudinal data. *Statistica Sinica*, **14**, 763-788.

[19] Kauermann, G. and Tutz, G. (1999). On model diagnostics using varying coefficient models. *Biometrika*, **86**, 119-128.

[20] Kai, B., Li, R., and Zou, H. (2011) New efficient estimation and variable selection methods for semiparametric varying-coefficient partially linear models. *Ann. Stat.*, **39**, 305-332.

[21] Klopp, O., Pensky, M. (2013) Non-asymptotic approach to varying coefficient model. *Electronic Journal of Statistics*, **7**, 454-479.

[22] Lee, Y.K., Mammen, E., and Park, B.U. (2012) Flexible generalized varying coefficient regression models. *Ann. Stat.*, **40**, 1906-1933.

[23] Lian, H. (2012) Variable selection for high-dimensional generalized varying-coefficient models. *Statistica Sinica*, **22**, 1563-1588.

[24] Lian, H., and Ma, S. (2013) Reduced-rank Regression in Sparse Multivariate Varying-Coefficient Models with High-dimensional Covariates. [arXiv:1309.6058v1](http://arxiv.org/abs/1309.6058v1)

[25] Lounici, K., Pontil, M., Tsybakov, A. and van de Geer, S. (2010) Oracle inequalities and optimal inference under group sparsity. *Annals of Statistics*, **39**(4), 2164-2204.

[26] Meier, L., van de Geer, S. and Bühlmann, P. (2008) The group Lasso for logistic regression. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, **70**(1), 53 - 71.
[27] Meier, L., van de Geer, S. and Bühlmann, P. (2009) High-dimensional additive modeling. *Ann. Statist.*, 37(6B), 3779 - 3821.

[28] Mallat, S. (2009) *A Wavelet Tour of Signal Processing*, Third Ed., Elsevier, New York

[29] Rudelson, M., and Vershynin, R. (2013). Hanson-Wright inequality and sub-gaussian concentration. [arXiv:1306.2872](https://arxiv.org/abs/1306.2872)

[30] Raskutti, G., Wainwright, M.J., Yu, B. (2012) Minimax-optimal rates for sparse additive models over kernel classes via convex programming. *Journ. Machine Learning Research*, 13, 389-427.

[31] Senturk, D. and Mueller, H. G. (2010) Functional varying coefficient models for longitudinal data. *J. Amer. Statist. Assoc.*, 105, 1256-1264.

[32] Tropp, J.A. (2011) User-friendly tail bounds for sums of random matrices. *Found. Comput. Math.*, 11(4).

[33] Tsybakov, A. (2009) *Introduction to Nonparametric Estimation*, Springer Series in Statistics.

[34] van de Geer, S. (2008) High-dimensional generalized linear models and the Lasso. *Ann. Statist.*, 36, 614 - 645.

[35] Vershynin, R. (2012) *Introduction to the non-asymptotic analysis of random matrices*. In *Compressed Sensing, Theory and Applications*, ed. Y. Eldar and G. Kutyniok, Chapter 5. Cambridge University Press.

[36] Wang, L., Kai, B., and Li, R. (2009) Local Rank Inference for Varying Coefficient Models. *J. Amer. Statist. Assoc.*, 104, 1631-1645.

[37] Wei, F., Huang, J. and Li, G. (2011) Variable Selection and Estimation in High-Dimensional Varying-Coefficient Models. *Statistica Sinica.*, 21, 1515-1540.

[38] Wu, C. O., Chiang, C. T. and Hoover, D. R. (1998). Asymptotic confidence regions for kernel smoothing of a varying-coefficient model with longitudinal data. *J. Amer. Statist. Assoc.*, 93, 1388-1402.

[39] Yang, L., Park, B.U., Xue, L. and Hardle, W. (2006) Estimation and Testing for Varying Coefficients in Additive Models With Marginal Integration. *J. Amer. Statist. Assoc.*, 101, 1212-1227

[40] Yuan, M. and Lin, Y. (2006) Model selection and estimation in regression with grouped variables. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 68(1), 49 - 67.

[41] Zhu, H., Li, R., and Kong, L. (2012) Multivariate varying coefficient model for functional responses. *Ann. Stat.*, 40, 2634-2666