$p$-GROUPS FOR WHICH EACH OUTER $p$-AUTOMORPHISM CENTRALIZES ONLY $p$ ELEMENTS

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Abstract. An automorphism of a group is called outer if it is not an inner automorphism. Let $G$ be a finite $p$-group. Then for every outer $p$-automorphism $\phi$ of $G$ the subgroup $C_G(\phi) = \{ x \in G \mid x^{\phi} = x \}$ has order $p$ if and only if $G$ is of order at most $p^2$.

1. Introduction

An automorphism of a group is called outer if it is not an inner automorphism. Let $p$ be any prime number. An automorphism of a group is called $p$-automorphism if its order is a power of $p$. For any automorphism $\phi$ of a group $G$, $C_G(\phi)$ denotes the subgroup $\{ x \in G \mid x^{\phi} = x \}$. Berkovich and Janko proposed the following problem in [3, Problem 2008].

Problem 1.1. Study the $p$-groups $G$ such that for every outer $p$-automorphism $\phi$ of $G$ the subgroup $C_G(\phi)$ has order $p$.

Here we completely determine the structure of requested $p$-groups $G$ in Problem 1.1.

Theorem 1.1. Let $G$ be a finite $p$-group. For every outer $p$-automorphism $\phi$ of $G$ the subgroup $C_G(\phi)$ has order $p$ if and only if $G$ is of order at most $p^2$.

2. Preliminaries Results

We use the following results in the proof of Theorem 1.1.

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Remark 2.1. By a famous result of Gaschütz ([4]), if $G$ is a finite $p$-group of order greater than $p$, then $G$ admits an outer $p$-automorphism. Schmid ([8]) extended Gaschütz's result as follows: if $G$ is a finite nonabelian $p$-group, then $G$ admits an outer $p$-automorphism $\phi$ such that the center $Z(G)$ of $G$ is contained in $C_G(\phi)$. The reader may pay attention to [1] to see more recent results on the existence of noninner automorphism of order $p$ for finite nonabelian $p$-groups, a conjecture proposed by Y. Berkovich (see Problem 4.13 of [6]).

For any group $G$, we denote by $\text{Aut}'(G)$ the subgroup of all automorphisms of $G$ acting trivially on the factor group $G/\Phi(G)$, where $\Phi(G)$ denotes the Frattini subgroup of $G$, the intersection of all maximal subgroups of $G$. By a well-known result of Burnside, $\text{Aut}'(G)$ is a $p$-group whenever $G$ is a finite $p$-group. Note that the inner automorphism group $\text{Inn}(G)$ of $G$ is contained in $\text{Aut}'(G)$.

Remark 2.2 ([7, Theorem]). Let $G$ be a finite $p$-group which is neither elementary abelian nor extraspecial. Then $\text{Aut}'(G)$ properly contains $\text{Inn}(G)$.

Let $G$ be any group and $\phi$ is an automorphism of $G$. Let $N$ be a $\phi$-invariant subgroup of $G$; i.e., $N^\phi \subseteq N$. If $N$ is normal in $G$, the map defined on $G/N$ by $xN \mapsto x^\phi N$ for all $x \in G$ is an automorphism of $G/N$. We will denote the latter map by $\phi$.

Remark 2.3 ([5, Lemma 2.12]). Suppose that $\phi$ is an automorphism group of a finite group $G$ and $N$ is a normal $\phi$-invariant subgroup. Then $|C_{G/N}(\phi)| \leq |C_G(\phi)|$.

3. Proof of Theorem 1.1

Assume that for every outer $p$-automorphism $\phi$ of $G$ the subgroup $C_G(\phi)$ has order $p$.

Let $V$ be an elementary abelian group of order $p^d$ and $d > 2$. Suppose that $V = \langle v_1, \ldots, v_d \rangle$. Then the map defined by $v_1 \mapsto v_1 v_2$ and $v_i \mapsto v_i$ for all $i > 1$ can be extended to the automorphism $\phi$ of $V$ such that $|C_V(\phi)| = p^{d-1} > p$. The order of $\phi$ is $p$ and so it is an outer $p$-automorphism of $V$. Therefore, it follows that if $G$ is elementary abelian, the order of $G$ is at most $p^2$.

Let $S$ be an extraspecial $p$-group of order $p^3$. Assume that $p > 2$. Suppose first that the exponent of $S$ is $p$. Then $S$ has a presentation as follows:

$$\langle x, y \mid x^p = y^p = 1, [x, y]^p = [x, y]^m = [x, y]^x \rangle.$$  

Now the map defined by $x \mapsto xy$ and $y \mapsto y$ determines the noninner automorphism $\alpha$ of order $p$ such that $\langle y, [x, y] \rangle \leq C_S(\alpha)$. To see the former claim, one may use von Dyck's Theorem, as the $x^\alpha$ and $y^\alpha$ satisfy the same relations as $x$ and $y$ do, $\alpha$ can be extended to an endomorphism of $S$. Since
$S = \langle x^\alpha, y^\alpha \rangle$, $\alpha$ is an epimorphism and since $S$ is finite, $\alpha$ is an automorphism of $S$.

Now suppose that $S$ is of exponent $p^2$. Then $S$ has a presentation as follows:

$$\langle x, y \mid x^{p^2} = y^p = 1, x^y = x^{1+p} \rangle.$$  

The map defined by $x \mapsto xy$ and $y \mapsto y$ determines the noninner automorphism $\beta$ of order $p$ such that $\langle y, [x, y] \rangle \leq C_S(\beta)$. Showing that $\beta$ is an automorphism of $S$ is similar to that of $\alpha$, one may use the presentation of $S$ and observe that $x^\beta$ and $y^\beta$ satisfy corresponding relations as $x$ and $y$ do respectively.

Now assume that $S = Q_8$ the quaternion group of order 8 or $S = D_8$ the dihedral group of order 8. We know that $Q_8$ and $D_8$ have the following presentations:

$Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, x^y = x^{-1} \rangle$, $D_8 = \langle x, y \mid x^4 = y^2 = 1, x^y = x^{-1} \rangle$.

The map defined on $S$ by $x \mapsto x$ and $y \mapsto xy$ can be extended to the noninner automorphism $\alpha$ of order 4 and $\langle x \rangle \leq C_S(\alpha)$.

The following way to obtain such an automorphism $\alpha$ for $S \in \{D_8, Q_8\}$ is suggested by the referee. Let $D$ be the semidihedral group of order 16 and $C = \langle c \rangle$ the cyclic subgroup of index 2 in $D$. Both types of $S$ are subgroups of $D$ (see e.g., Theorem 1.2 of [2]). Let $\bar{c}$ be the conjugation of $D$ by $c$. Then the fixed points of the restriction of $\bar{c}$ to $S$ constitute the intersection $C \cap S$ of order 4. Clearly, the restriction is a noninner automorphism of $S$.

It follows that if $G$ is an extraspecial $p$-group, then $|G| > p^3$. Thus $G$ is a central product of an extraspecial group $A$ of order $p^3$ and another extraspecial group $B$. By previous paragraph, $A$ has an outer $p$-automorphism $\theta$ leaving $Z(A)$ elementwise fixed. Now it is not hard to see that the map $\theta$ on $G$ defined by $ab \mapsto a^\theta b$ for all $a \in A$ and $b \in B$ is an outer $p$-automorphism fixing both $Z(A)$ and $B$ elementwise. This contradicts the assumption, since $|Z(A)B| > p$.

Now assume that $G$ is neither elementary abelian nor extraspecial. By Remark 2.2, there exists some $\phi \in \text{Aut}^G(G) \setminus \text{Inn}(G)$ so that $|C_G(\phi)| = p$ by hypothesis. It follows from Remark 2.3 that $|C_G/\Phi(G)(\overline{\phi})| \leq p$. Thus $|G/\Phi(G)| = |C_G/\Phi(G)(\overline{\phi})| \leq p$. This means that $G$ is a cyclic $p$-group. If $G = \langle a \rangle$ and $|a| = p^n > p^2$, then $\phi : a \mapsto a^{p^{n-1}+1}$ is an automorphism of order $p$. Now $|a^p| \leq C_G(\phi)$, a contradiction. Thus $|G| = p^2$. The converse obviously holds. This completes the proof.

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