GROWTH OF HOMOTOPY GROUPS OF SPHERES
VIA THE GOODWILLIE TOWER

GUY BOYDE

Abstract. We bound the amount of 2-torsion in the homotopy groups of the Goodwillie approximations of a sphere in terms of the amount of 2-torsion in the stable homotopy groups of spheres. At the $2^k$-excisive approximation, this bound is obtained by ‘multiplying the stable answer by a polynomial of degree $k$’. The main tool is an EHP sequence due to Behrens which incorporates the Goodwillie Tower.

1. Introduction

Various authors [Sel82, BH83, Hen86, Iri87, Boy20] have used the classical EHP sequences of James and Toda [Jam57, Tod56] to prove exponential upper bounds for the volume of torsion in the unstable homotopy groups of spheres. The object of this paper is to show that the Goodwillie Tower realises such a bound as a limit of polynomial approximations, in analogy with the classical Taylor series $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

Let $p$ be prime. For an abelian group $A$, let $(p)A$ be the $p$-torsion subgroup: the subgroup consisting of elements of order a power of $p$. Let

$$\ell_p(A) := \log_p(\text{Card}((p)A)).$$

Write $\pi_*^S$ for the stable homotopy groups of spheres, and let

$$\ell^S_t := \max_{i \leq t} (\ell_p(\pi^S_i)),$$

that is, the maximum log-$p$-cardinality over the first $i$ stable homotopy groups.

The Goodwillie Tower of the identity, due to Goodwillie [Goo90, Goo92, Goo03] consists of the following data:

- For each $k \in \mathbb{Z}_{\geq 0}$, a functor $P_k$ from based spaces to based spaces, called the $k$-excisive approximation to the identity.

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• For each $k \in \mathbb{Z}_{\geq 0}$, natural transformations $P_{k+1}(X) \to P_k(X)$, and $X \to P_k(X)$.

Letting $D_k(X)$ be the homotopy fibre of $P_k(X) \to P_{k-1}(X)$, we obtain a functor $D_k$, which is called the $k$-th layer of the Goodwillie Tower. This data can be assembled into a commutative diagram as follows.

\[
\begin{array}{ccc}
  \vdots & & \\
  X & \longrightarrow & P_k(X) \\
  \downarrow & & \longleftrightarrow D_k(X) \\
  \downarrow & & \\
  P_{k-1}(X) & \longleftrightarrow & D_{k-1}(X) \\
  \downarrow & & \\
  \vdots & & \\
  \downarrow & & \\
  & & P_1(X).
\end{array}
\]

Henceforth, we will work 2-locally. Arone and Mahowald \cite{AM99} have shown that in the case of spheres, $D_k(S^n)$ is 2-locally contractible unless $k$ is a power of 2. It follows that $P_k(S^n)$ is 2-locally homotopy equivalent to $P_{2j}(S^n)$, where $2^j$ is the largest power of 2 which is at most $k$. It is therefore no loss to restrict attention to the layers $P_{2^k}(S^n)$ indexed by powers of 2.

Our main theorem is the following.

**Theorem 1.1.** For $k, t \geq 0$ and $n \geq 2$ we have

\[
\ell_2(\pi_{t+n}(P_{2^k}(S^n))) \leq \ell_2^S(t) \cdot f_k(t),
\]

where $f_k(t)$ is a polynomial in $t$ of degree $k$ with non-negative coefficients, and the coefficient of $t^k$ is equal to $\frac{1}{k!}$.

That is, the size of the 2-torsion in the unstable homotopy groups of the $2^k$-th stage of the Goodwillie Tower on a sphere exceeds the stable size by at most multiplication by a polynomial of degree $k$. The connectivity of $P_{2^k}(S^n)$ increases with $k$ (Lemma \cite{271}). This implies the following corollary.

**Corollary 1.2.** If $t \leq 2^k(n - 1) - 1$, then

\[
\ell_2(\pi_{t+n}(S^n)) \leq \ell_S^S(t) \cdot f_k(t),
\]

where $f_k(t)$ is as above. \qed
That is, our bound applies to the regular homotopy groups of spheres ‘below a line of gradient $2^k$’. Here is the approximate picture of which bound applies where:

![Diagram](image)

Theorem 1.1 is a statement ‘relative to the stable information’ encoded in $\ell^S_2(t)$. Isaksen, Wang and Xu have made the following conjecture about a closely related quantity.

**Conjecture 1.3 ([IWX20]).** There exists a nonzero constant $C$ such that

$$\lim_{t \to \infty} \frac{\sum_{i=1}^{t} \ell_2(\pi_i^S)}{t^2} = C.$$ 

Since

$$\ell^S_2(t) := \max_{i \leq t} (\ell_2(\pi_i^S)) \leq \sum_{i=1}^{t} \ell_2(\pi_i^S),$$

truth of the conjecture would imply the existence of constants $a$ and $b$ such that $\ell^S_2(t) \leq at^2 + b$, and hence by Theorem 1.1 that the 2-torsion in the homotopy groups of $P_{2k}(S^n)$ satisfies the absolute bound

$$\ell_2(\pi_{t+n}(P_{2k}(S^n))) \leq (at^2 + b)f_k(t).$$

In particular, $\ell_2(\pi_{t+n}(P_{2k}(S^n)))$ would be bounded above by a polynomial of degree $k + 2$ in $t$.

In the unpublished [AK95], Arone and Kankaanrinta give an analogy between the Goodwillie Tower and the Taylor Series of the logarithm function, inverse to an analogy between stable homotopy and $e^{x-1}$. On this view one should think of the Goodwillie Tower as an infinite product, rather than an infinite sum, analogous to the following equation, which is obtained by exponentiating the Taylor Series of $\ln(1+(x-1))$.

$$e^{x-1} \cdot e^{\frac{(x-1)^2}{2}} \cdot e^{\frac{(x-1)^3}{3}} \cdots = x.$$
It is interesting to consider Theorem 1.1 from this point of view.

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2. Preliminaries

We require the following lemma on connectivity of \(P_{2k}(S^n)\), which is noted by Johnson \[Joh95\]. It follows immediately from the proof of Theorem 1.13 in \[Goo03\], using the fact that the identity functor is 1-analytic.

**Lemma 2.1.** Let \(n \geq 2\). The map \(S^n \to P_{2k}(S^n)\) induces an isomorphism on \(\pi_i\) for \(i \leq (2^k + 1)(n - 1)\). \(\square\)

It will be convenient to translate between two pairs of variables, \((t, j)\), and \((i, n)\). The variables \(i\) and \(n\) refer straightforwardly to the homotopy groups \(\pi_i(P_{2k}(S^n))\). We define \(t\) and \(j\) via

\[
\begin{align*}
  t &= i - n, \\
  j &= t + 2 - n = i + 2 - 2n,
\end{align*}
\]

and hence obtain, for translating back,

\[
\begin{align*}
  n &= t + 2 - j, \\
  i &= t + n = 2t + 2 - j.
\end{align*}
\]

The condition \(n \geq 1\) becomes \(j \leq t + 1\). We will use this in the proof. The variable \(t\) is the stem. The variable \(j\) describes the position on the stem, and is determined by insisting that \(j = 0\) on the lowest-dimensional sphere \(S^n\) for which the \(t\)-th stem is stable, and that decreasing \(n\) by 1 increases \(j\) by 1.

3. EHP Sequences

The classical 2-primary EHP sequence is due to James.

**Theorem 3.1.** \[Jam57\] For \(n \geq 1\), there is a fibre sequence

\[
\ldots \to P \to S^n \xrightarrow{E} \Omega S^{n+1} \xrightarrow{H} \Omega S^{2n+1} \xrightarrow{P} \ldots
\]

\(\square\)

Behrens \[Beh12\] gives the following refinement, which incorporates the Goodwillie Tower.
Theorem 3.2. [Beh12, Corollary 2.1.4] For $k \geq 0$ and $n \geq 1$, there is a fibre sequence

$$\ldots \overset{P}{\to} P_{2k+1}(S^n) \overset{E}{\to} \Omega P_{2k+1}(S^{n+1}) \overset{H}{\to} \Omega P_{2k}(S^{2n+1}) \overset{P}{\to} \ldots \quad \Box$$

The proof of Theorem 1.1 is essentially an induction over the long exact sequence on homotopy groups derived from this fibration. We take the point of view of the following corollary of Theorem 3.2.

Corollary 3.3. Let $k \geq 0$ and $n \geq 1$. If $i = 2n - 1$ then

$$\ell_2(\pi_i(P_{2k+1}(S^n))) \leq \ell_2(\pi_{i+1}(P_{2k+1}(S^{n+1}))) + 1,$$

and otherwise we have

$$\ell_2(\pi_i(P_{2k+1}(S^n))) \leq \ell_2(\pi_{i+1}(P_{2k+1}(S^{n+1}))) + \ell_2(\pi_{i+2}(P_{2k}(S^{2n+1}))).$$

The following lemma will be used to prove Corollary 3.3.

Lemma 3.4. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence of abelian groups, with $A$ torsion, and let $p$ be a prime. Then $\ell_p(B) \leq \ell_p(A) + \ell_p(C)$.

Proof. First, note that if an element of $B$ has order a power of $p$, then its image in $C$ also has order a power of $p$. Second, note that any element of $(p)B$ which is hit by an element of $A$ must actually be hit by an element of $(p)A$. To see this, suppose that some nonzero $y \in (p)B$ is the image of $x \in A$. Since $A$ is torsion, $x$ must have finite order. The order of $y$ is $p^b$, for some $b \in \mathbb{N}$, so the order of $x$ is $up^a$, for $a \geq b$ and $u$ coprime to $p$. Then $ux$ lies in $(p)A$, and $f(ux) = uy$. Multiplication by $u$ is an isomorphism on $p$-torsion subgroups, so there is a unique element $z$ of $(p)A$ with $uz = ux$, and we must have $f(z) = u^{-1}f(ux) = y$, as desired.

From these two observations we obtain an exact sequence

$$(p)A \to (p)B \to (p)C.$$ 

It follows that $\text{Card}((p)B) \leq \text{Card}((p)A)\text{Card}((p)C)$, and the result follows by taking logarithms. \Box

Proof of Corollary 3.3. The point is that the behaviour of Behrens’ EHP sequence is sufficiently well governed by the classical one that, morally, one need only establish the result in that setting.

Consider the following portion of the long exact sequence on homotopy groups induced by the fibration of Theorem 3.2.

$$\ldots \to \pi_{i+2}(P_{2k}(S^{2n+1})) \overset{P}{\to} \pi_i(P_{2k+1}(S^n)) \overset{E}{\to} \pi_{i+1}(P_{2k+1}(S^{n+1})) \to \ldots$$

By [AM99, Proposition 3.1], since $S^{2n+1}$ is an odd sphere, $D_{2k}(S^{2n+1})$ (the homotopy fibre of $P_{2k}(S^{2n+1}) \to P_{2k-1}(S^{2n+1}))$ is rationally contractible for $2^k > 1$, i.e. for $k > 0$. This means that $\pi_{i+2}(P_{2k}(S^{2n+1}))$
contains a class of infinite order if and only if $\pi_{i+2}(S^{2n+1})$ does, which happens if and only if $i + 2 = 2n + 1$ [Ser53]. By Lemma 3.4 the result then follows for $i \neq 2n - 1$.

We therefore restrict attention to the case $i = 2n - 1$. Consider the following commutative diagram, where the rows are the relevant portions of the long exact sequences on homotopy groups obtained from the EHP sequences, and the vertical maps are the natural transformations of the Goodwillie Tower.

$$
\begin{array}{ccc}
\pi_{2n+1}(S^{n+1}) & \xrightarrow{H_*} & \pi_{2n+1}(S^{2n+1}) \\
\downarrow & & \downarrow \\
\pi_{2n+1}(P_{2k+1}(S^{n+1})) & \xrightarrow{H_*} & \pi_{2n+1}(P_{2k}(S^{2n+1})) \\
& & \downarrow \\
& & \pi_{2n-1}(P_{2k+1}(S^n))
\end{array}
$$

By Lemma 2.1, the middle vertical map is an isomorphism, and therefore $\pi_{2n+1}(P_{2k+1}(S^{n+1})) \cong \mathbb{Z}$. We are interested in the contribution made by this copy of $\mathbb{Z}$ to the $2$-torsion in $\pi_{2n-1}(P_{2k+1}(S^n))$.

If $n$ is even, then we have seen that $\pi_{2n+1}(P_{2k+1}(S^{2n+1}))$ does not contain a class of infinite order, so $H_*$ is trivial. By exactness, $P_*$ is an injection. This means that no contribution is made to the torsion. Precisely, continuing the sequence to the right, the torsion subgroup of $\pi_{2n-1}(P_{2k+1}(S^n))$ maps injectively into that of $\pi_{2n}(P_{2k+1}(S^{n+1}))$, so $\ell_2(\pi_{i}(P_{2k+1}(S^n))) \leq \ell_2(\pi_{i+1}(P_{2k+1}(S^{n+1})))$, which implies the first case of the result.

If $n$ is odd, then the image of $H_*$ in the top row certainly contains twice the generator of $\pi_{2n+1}(S^{2n+1})$. By commutativity, this must also be true in the bottom row, so, again extending to the right, we obtain an exact sequence

$$A \to \pi_{2n-1}(P_{2k+1}(S^n)) \to \pi_{2n}(P_{2k+1}(S^{n+1})),
$$

with $A$ equal to either $\mathbb{Z}/2$ or $0$. Applying Lemma 3.4 to this sequence then gives the first case of the result, as required.

We now convert Lemma 3.3 into $(t, j)$-coordinates. Define

$$\ell_{2,k}(t, j) := \ell_2(\pi_{2t+2-j}(P_{2k}(S^{t+2-j}))).$$

Changing variables in Lemma 3.3 gives

**Corollary 3.5.** Let $k \geq 0$ and $j \leq t + 1$. If $j = 1$ then

$$\ell_{2,k+1}(t, j) \leq \ell_{2,k+1}(t, j - 1) + 1,$$

and otherwise we have

$$\ell_{2,k+1}(t, j) \leq \ell_{2,k+1}(t, j - 1) + \ell_{2,k}(j - 1, 3j - 2t - 4).$$
4. Proof of Theorem 1.1

Proof. We proceed by induction on $k$. Since the 1-excisive approximation $P_1 = P_{2^0}$ is just the stable homotopy functor, we may take $f_0$ to be the constant polynomial equal to 1. This establishes the case $k = 0$.

We now assume that the result has been proven for some $k \geq 0$ (that is, that there exists a polynomial $f_k(t)$ of degree $k$, with leading coefficient equal to $\frac{1}{k!}$, such that $\ell_{2,k}(t, j) \leq f_k(t) \cdot \ell^S_{2}(t)$) and attempt to show it for $k + 1$. When $t \leq 0$ (i.e. on non-positive stems), Lemma 2.1 implies that $\pi_{t+n}(P_{2k+1}(S^n))$ contains no 2-torsion, so the result is automatic.

We must still prove the result on positive stems. We begin by applying Corollary 3.5 inductively along the stem, relating $\ell_{2,k+1}(t, j)$ to $\ell_{2,k+1}(t, 0)$. Since, by construction, $j = 0$ lies in the stable range, $\ell_{2,k+1}(t, 0) = \ell^S_{2}(t)$. The first case of the corollary applies only when $j = 1$, and the rest of the time the second case applies. We therefore obtain

$$\ell_{2,k+1}(t, j) \leq \ell^S_{2}(t) + 1 + \sum_{\theta=2}^{j} \ell_{2,k}(\theta - 1, 3\theta - 2t - 4)$$

$$\leq \ell^S_{2}(t) + 1 + \sum_{\theta=2}^{j} f_k(\theta - 1) \cdot \ell^S_{2}(\theta - 1)$$

$$= \ell^S_{2}(t) + 1 + \sum_{\tau=1}^{j-1} f_k(\tau) \cdot \ell^S_{2}(\tau)$$

$$\leq \ell^S_{2}(t) + 1 + \sum_{\tau=1}^{j-1} f_k(\tau) \cdot \ell^S_{2}(t),$$

applying the inductive hypothesis and the fact that $\tau \leq j - 1 \leq t$ (recall that $j \leq t + 1$ is equivalent under our change of coordinates to $n \geq 1$, where $n$ is the dimension of the sphere).

Since $f_k$ has nonnegative coefficients, it is nondecreasing on $\tau \geq 0$, so we obtain

$$\frac{\ell_{2,k+1}(t, j) - 1}{\ell^S_{2}(t)} - 1 \leq \sum_{\tau=1}^{j-1} f_k(\tau) \leq \int_{1}^{j} f_k(\tau)d\tau,$$
which by inductive hypothesis is of the form
\[
\int_1^j \frac{1}{k!} \tau^k + \alpha_{k-1} \tau^{k-1} + \cdots + \alpha_1 \tau + \alpha_0 d\tau
\]
\[
= \left[ \frac{1}{(k+1)!} \tau^{k+1} + \frac{\alpha_{k-1}}{k} \tau^k + \cdots + \frac{\alpha_1}{2} \tau^2 + \alpha_0 \right]_1^j
\]
\[
= \frac{1}{(k+1)!} j^{k+1} + \text{terms of lower order}.
\]

Now, for \( t \geq 1 \) we have \( \ell^S_2(t) \geq 1 \) (since \( \pi^S_1 \cong \mathbb{Z}/2 \) [Fre38]), so
\[
\frac{\ell_{2,k+1}(t, j)}{\ell^S_2(t)} - 2 \leq \frac{\ell_{2,k+1}(t, j) - 1}{\ell^S_2(t)} - 1.
\]
Since \( j \leq t + 1 \), this implies that
\[
\frac{\ell_{2,k+1}(t, j)}{\ell^S_2(t)} \leq \frac{1}{(k+1)!} t^{k+1} + \text{terms of lower order}
\]
All of the coefficients of the new polynomial are positive, so this completes the inductive step, and hence the proof. \( \square \)

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