ON THE VARIATIONAL REGULARITY OF CAMERON–MARTIN PATHS

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Abstract. It is a well-known fact that finite $\rho$-variation of the covariance (in 2D sense) of a general Gaussian process implies finite $\rho$-variation of Cameron–Martin paths. In the special case of fractional Brownian motion (think: $2H = 1/\rho$), in the rougher than Brownian regime, a sharper result holds thanks to a Besov-type embedding; [Friz–Victoir, JFA 2006]. In the present note we give a general result which closes this gap. We comment on the importance of this result for various applications.

1. Introduction

In [FV06] the following embedding from fractional Sobolev (or Besov) spaces into $q$-variation spaces

\[(1.1) \quad \|h\|_{q-\text{var}([0,T])} := \left( \sup_{(t_i) \subseteq [0,T]} \sum_i |h_{t_{i+1}} - h_{t_i}|^q \right)^{1/q} \lesssim \left( \int_0^T \int_0^T \frac{|h_u - h_v|^p}{|v - u|^{p+q/p}} \right)^{1/p} du \; dv = |h|_{W_{p,q}^q}, \quad 1 \leq q < p \]

has been shown, based on localized Garcia–Rodemich–Rumsey estimates. Due to its Volterra structure, the Cameron–Martin space $\mathcal{H} = \mathcal{H}^H$ of fractional Brownian motion (fBM) $B^H$ (in the interesting regime $H \leq 1/2$) consists of indefinite, fractional integrals of $L^2([0,T])$-functions. Thereby, the above Besov regularity may be verified for suitable $p$ and from (1.1) one concludes

\[(1.2) \quad \mathcal{H}^H \hookrightarrow C^q_{\text{var}} \text{ for any } q > \frac{1}{H + \frac{1}{2}}.\]

This variational regularity is better than finite $1/H$-variation, which trivially follows from the embedding $\mathcal{H}^H \hookrightarrow C^H([0,T])$, the space of Hölder paths with exponent $H$.

The importance of this result is that it implies the so-called complementary Young regularity condition in the case of fBm for $H > \frac{1}{4}$. This condition means that a.e. sample path $X(\omega)$ (resp. every $h \in \mathcal{H}$) has $p$- (resp. $q$-) variation regularity such that Young’s condition $\frac{1}{p} + \frac{1}{q} > 1$ is satisfied. It follows, and

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this has been crucial in many recent applications, notably in the context of non-Markovian Hörmander theory (cf. [CF10, CHLT12] and also [Ina13, CLL13, FR13, BH07, HP11]), that perturbation in Cameron–Martin directions, such as
\[ dX \otimes dX \mapsto d(X + h) \otimes d(X + h), \]
can be done for all \( h \in \mathcal{H} \) at once on a set of full measure.

While the above Besov-variation embedding settles decisively the case of fractional Brownian motion\(^1\) it is too specific to apply in a general Gaussian context. In this case, under the assumption of finite \( \rho \)-variation of the covariance it has been shown in [FV10a] that
\[ \mathcal{H} \hookrightarrow C^{\rho-\text{var}}. \]
However, in view of (almost) finite \( 2\rho \)-variation of the sample paths, we are led to
\[ \frac{1}{2\rho} + \frac{1}{\rho} > 1 \iff \rho < \frac{3}{2}, \]
which does not cover the (general) rough regime \( \rho \in [\frac{3}{2}, 2) \). In particular, in case of fBM with \( H < \frac{1}{4} \) the previous Sobolev embedding is strictly better. Indeed, application of the general (Gaussian) result to fBM would only yield complementary Young regularity for \( H > \frac{1}{4} \). It is thus a repeated remark in the afore-mentioned literature that the main results hold in the general setting of \( \rho < \frac{3}{2} \) (which also covers fBM with \( H > \frac{1}{4} \)) and then also in the special case of fBM with \( H > \frac{1}{4} \). Our purpose here is to give a general result that bridges this gap\(^2\).

Theorem 1 puts forward the notion of mixed \((1, \rho)\)-variation under which this improved embedding holds true. In order to illustrate how to work with mixed variation, we formulate a sufficient condition for finite mixed \((1, \rho)\)-variation for processes with stationary increments (Theorem 2). Let us remark that finite mixed \((1, \rho)\) implies finite \( \rho \)-variation (in 2D sense), an essentially sharp condition for the existence of the stochastic area (resp. iterated integrals and then rough paths) for zero-mean, Gaussian processes with independent components, each having covariance of finite \( \rho \)-variation.

In summary, the practical value of our results is to replace two standard (but hard-to-check) conditions in (Gaussian) rough path theory (namely: finite \( \rho \)-variation of the covariance and complementary Young regularity) by one simpler condition: mixed \((1, \rho)\)-condition. Let us conclude the introduction with a non-complete list of recent works where the these conditions are crucial: support theorem for Gaussian rough paths (cf. [FV10b, Ch. 19] and the references therein); Malliavin calculus for rough differential equations (RDE) driven by Gaussian rough paths and non-Markovian Hörmander theory (cf. [CF10, CHLT12] and the references therein); Laplace method for RDE [Ina13]; every use of Borell’s inequality in the (Gaussian) rough path context, notably non-linear Fernique theory (cf. [FO10]) and the integrability estimates for RDE driven by Gaussian rough paths (cf. [CLL13, FR13]), [FR13] also discusses applications related to rough SPDE; strong rates and multilevel Monte Carlo for RDE driven by Gaussian rough paths (forthcoming work by Bayer et al.).

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\(^1\)Complementary Young regularity always holds since \( H + (H + \frac{1}{2}) > 1 \iff H > \frac{1}{4} \).

\(^2\)We even slightly improve \((1.2)\) by allowing equality \( q = \frac{1}{H + \frac{1}{2}} \).
2. Results

Let \( X : [0, T] \to \mathbb{R} \) be a real valued, centered, continuous Gaussian process with covariance

\[
R_X(s, t) = \mathbb{E}X_sX_t.
\]

We will denote the associated Cameron-Martin space by \( \mathcal{H} \). It is well-known that \( \mathcal{H} \subset C([0, T], \mathbb{R}) \) and each \( h \in \mathcal{H} \) is of the form \( h_t = \mathbb{E}X_t \) with \( Z \) being an element of the \( L^2 \)-closure of \( \text{span}\{X_t|t \in [0, T]\} \), a Gaussian random variable. If \( h_t = \mathbb{E}X_t \), \( h'_t = \mathbb{E}Z'X_t \), \( h, h' \in \mathcal{H} \).

For any function \( h : [0, T] \to \mathbb{R} \) we define \( h_{s,t} := h_t - h_s \) for all \( s, t \in [0, T] \). We recall the definition of mixed right \((\gamma, \rho)\)-variation given in [Tox02]: For \( \gamma, \rho \geq 1 \)

\[
V_{\gamma,\rho}(R_X; [s, t] \times [u, v]) := \sup_{(t_i) \in \mathcal{D}(s, t)} \left( \sum_{i} \left( \sum_{\substack{j \in \mathcal{D}(u, v) \cap \mathcal{D}(t_i) \cap \mathcal{D}(s, t)}} \left| R_X \left( \frac{t_{i+1} - t_i}{2}, \frac{t_{i+1} - t_i}{2} \right) \right| \right)^{\gamma} \right)^{\frac{1}{\gamma}},
\]

where \( \mathcal{D}(s, t) \) denotes the set of all dissections of \([s, t]\) and

\[
R_X \left( \frac{t_{i+1} - t_i}{2}, \frac{t_{i+1} - t_i}{2} \right) = \mathbb{E}X_{t_i, t_{i+1}}X_{t'_j, t'_{j+1}}.
\]

The notion of the 2D \( \rho \)-variation is recovered as \( V_{\rho} = V_{\rho, \rho} \). Recall that \( V_{\rho} \)-regularity plays a key role in Gaussian rough path theory [FV10b, FV10a, FH13] and in particular yields a stochastic integration theory for large classes of multidimensional Gaussian processes. Clearly, \( V_{\gamma,\rho}(R; A) \leq V_{\gamma,\rho}(R; A) \leq V_{\gamma,\rho}(R; A) \) for all rectangles \( A \subseteq [0, T]^2 \). As the main result of this paper we present the following embedding theorem for the Cameron-Martin space.

**Theorem 1.** Assume that the covariance \( R_X \) has finite mixed \((1, \rho)\)-variation in 2D sense. Then there is a continuous embedding

\[
\mathcal{H} \hookrightarrow C^q\text{-var} \quad \text{with} \quad q = \frac{1}{2\rho} + \frac{3}{2} < 2.
\]

More precisely,

\[
\|h\|_{q\text{-var};[s, t]} \leq \|h\|_{\mathcal{H}} \sqrt{V_{1,\rho}(R_X; [s, t]^2)}, \quad \forall [s, t] \subseteq [0, T].
\]

**Remark 2.** (1) As explained in the introduction, this embedding is crucial in applications of rough paths to Malliavin calculus. In the particular example of fBM with \( H \leq \frac{1}{2} \) we see that Cameron-Martin paths enjoy finite \( q = \frac{1}{H+\frac{3}{2}} \)-variational regularity. The case \( q > \frac{1}{H+\frac{3}{2}} \) was previously obtained in [FV06].

(2) Note that for \( \rho > 1 \) we have \( q < \rho \) in Theorem 1 which improves [FV10a, Proposition 17] where it has been shown

\[
\|h\|_{\rho\text{-var};[s, t]} \leq \|h\|_{\mathcal{H}} \sqrt{V_{\rho}(R_X; [s, t]^2)}.
\]

(3) Strictly speaking, for the sole purpose of Theorem 1 it would have been enough to consider identical dissections \((t_i) \equiv (t'_i)\) in the definition of mixed variation \( V_{\gamma,\rho} \) in 2.1. The criteria in Theorem 6 below would then allow for a mildly simplified proof. On the other hand, this criteria derived in Theorem 6 below are also sufficient (and interesting) for finite \( \rho \)-variation.
$V_\rho = V_{\rho, \rho}$ as needed in Gaussian rough path theory \cite{FV10}: hence the additional generality of different vertical and horizontal dissections.

Remark 3. Let $X: [0, T] \to \mathbb{R}^d$ be a multidimensional centered Gaussian process. Then every path $h$ in the associated Cameron-Martin space $\mathcal{H}$ is of the form $h_t = EZX_t$ with $Z$ being an element of the $L^2$-closure of span$\{X_t^i \mid t \in I, i = 1, \ldots, d\}$ and $\|h\|_{\mathcal{H}} = \|Z\|_{L^2}$. The $q$-variation of $h$ is finite if and only if the $q$-variation of every $h^i = EZX^i$ is finite and we obtain the bound

$$\|h\|_{q-\text{var};[s,t]} \leq C \max_{i=1,\ldots,d} \sqrt{V_{1,\rho}(R_{X^i}; [s,t]^2)}$$

where $C$ is a constant depending only on the dimension $d$.

Theorem 4. Consider a centered, continuous Gaussian process $X$ with covariance of finite $\rho$-variation, $\rho < 2$. Then $X$ has vanishing compensated quadratic variation, i.e.

$$\sum_{[s,t] \in D^n} (X^2_{s,t} - EX^2_{s,t}) \to 0 \text{ as } n \to \infty$$

in probability and in $L^q(\Omega)$ for all $q \in [1, \infty)$, for every sequence of dissections $(D^n)$ whose mesh-size tends to zero.

Remark 5. Let us discuss briefly a consequence of Theorem 3 when $\rho = 1$. If one assumes in addition the existence of a continuous mean quadratic variation, in the sense that for some sequence of dissections $(D^n)$ whose mesh tends to zero,

$$(2.2) \quad \varphi(t) := \lim_{n \to \infty} \sum_{[u,v] \in D^n, u \leq t} \mathbb{E}X^2_{u,v}$$

exists and depends continuously on $t$, then it already follows that the pathwise quadratic variation of $X$ in the sense of Föllmer exists, along a suitable subsequence of $(D^n)$, and indeed is given by $t \mapsto \varphi(t)$. Typically, (2.2) is easy to check. For instance, if $\mathbb{E}X^2_{u,u+h} = f(u)h + o(h)$ for some continuous $f$ then $\varphi$ is precisely the indefinite integral of $f$. The interest here is the possibility of a pathwise Itô calculus (cf. \cite{Fol81}). In particular, Itô integrals of gradient 1-forms are defined in a pathwise manner through the Itô-formula. Remark that the Föllmer theory and in particular the integration of gradient 1-forms - extends without problem to the multidimensional setting via polarization. On the other hand, the integration of general 1-forms is impossible in the (pathwise) Föllmer setting and the starting point of integration against rough paths.

In the following we give a sufficient criterion for finite mixed $(1, \rho)$-variation of stochastic processes. Let $X$ be a centered, continuous stochastic process with \textit{stationary increments} in the sense that

$$\mathbb{E}X^2_{t,t+u} = \mathbb{E}X^2_{0,u}, \quad \forall t, u \geq 0.$$

We set

$$\sigma^2(u) := \mathbb{E}X^2_{0,u} = \mathbb{E}(X_u - X_0)^2.$$

\footnote{In fact, if the $\rho$-variation of the covariance is Hölder dominated in the sense of \cite{FV10} (a sufficient condition for this is given in Theorem 3 below) and $(D^n)$ is dyadic, then a Borel–Cantelli argument may be used to see that there is no need to pass to a subsequence.}
Theorem 6. Let $X$ be a centered, continuous stochastic process on $[0, T]$ with stationary increments. Assume that $\sigma^2$ is concave and that $|\sigma^2(\cdot)| \leq C_\sigma |\cdot|^\rho$ for some $\rho \geq 1$. Then there is an $h > 0$ such that the covariance of $X$ is of finite mixed $(1, \rho)$-variation on every square $[s, t]^2 \subseteq [0, T]^2$ with $|t - s| \leq h$. More precisely,

$$V_{1, \rho}(R; [s, t]^2) \leq 5C_\sigma |t - s|,$$

for all $[s, t]^2 \subseteq [0, T]^2$, $|t - s| \leq h$.

Remark that the "stationary increment" assumption is by no means crucial, and it is not hard to envision generalizations. In essence, one needs to assume that the covariance of non-overlapping increments of the process has a sign. Note also that $V_{1, \rho}$ regularity implies finite $\rho$-variation in the 2D sense, the key condition for existence of Gaussian rough paths, as put forward in [FV10a, FV10b, FV11]. We have

Corollary 7. Every Gaussian process $X$ with independent components, each of which is assumed to meet the assumptions put forward in the above theorem satisfies complementary Young regularity and if $\rho < 2$ may be lifted to a Gaussian rough path.

Fractional Brownian motion with $H \in (1/4, 1/2]$ is clearly covered with $\rho = 1/(2H)$. But even the Brownian regularity regime ($\rho = 1$) contains new and interesting examples: for instance, the "spatial" Gaussian rough path constructed in Hairer’s recent work on rough SPDE [Hai11] falls in the present framework, as do various fractional SPDE (to be discussed in forthcoming work).

3. Proofs

3.1. Proof of Theorem 6

Proof. Let $h = \mathbb{E} ZX \in \mathcal{H}$. Fix a dissection $D = (t_j) \subset [s, t]$, write $h_j \equiv h_{t_j, t_{j+1}}, X_j = X_{t_j, t_{j+1}}$ and also $\|h\|^q = \sum_j |h_j|^q$. Let $q'$ and $\rho'$ be the conjugate exponents of $q$ and $\rho$. An easy calculation shows that $\rho' = q' / 2$. By duality,

$$\|h\|_q = \sup_{\beta: \|\beta\|_q' \leq 1} \sum_j \beta_j h_j = \sup_{\beta: \|\beta\|_q' \leq 1} \mathbb{E} \left( Z \sum_j \beta_j X_j \right),$$

and so by Cauchy–Schwarz

$$\|h\|_q^2 \leq \mathbb{E} \left( \sum_j \beta_j \beta_k ZX_j X_k \right).$$

Set $R_{j,k} = \mathbb{E} X_j X_k$. Then, using the symmetry of $R$ and Hölder’s inequality,

$$\sum_{j,k} \beta_j \beta_k R_{j,k} \leq \frac{1}{2} \sum_{j,k} \beta_j^2 |R_{j,k}| + \frac{1}{2} \sum_{j,k} \beta_k^2 |R_{j,k}|$$

$$= \sum_j \beta_j^2 \sum_k |R_{k,j}|$$

$$\leq \|\beta\|_{q'}^2 \left( \sum_j \left( \sum_k |R_{i,k}| \right)^\rho \right)^{1/\rho}$$

$$\leq V_{1, \rho}(R; [s, t]^2)$$

when $\|\beta\|_{2\rho'} = \|\beta\|_{q'} \leq 1$ which shows the claim. □
Lemma 8. Assume that \( \sigma^2 \) is concave and \( \sigma^2 \) restricted to \([0, h]\) is non-decreasing, for some \( h > 0 \). Then,

\[
0 \leq \mathbb{E}X_{s,t}X_{u,v} \leq \mathbb{E}X_{u,v}^2 = \sigma^2 (v - u), \quad \forall [u, v] \subseteq [s, t] \subseteq [0, h].
\]

Proof. Note \( X_{s,t}X_{u,v} = (a + b + c)b \) where \( a = X_{s,u}, b = X_{u,v}, c = X_{v,t} \). Applying the algebraic identity

\[
2(a + b + c)b = (a + b)^2 - a^2 + (c + b)^2 - c^2
\]

and taking expectations yields

\[
2\mathbb{E}X_{s,t}X_{u,v} = (\sigma^2(v - s) - \sigma^2(u - s)) + (\sigma^2(v - s) - \sigma^2(u - s)) \geq 0 + 0,
\]

where we used that \( \sigma^2 \) is non-decreasing. We thus see

\[
0 \leq \mathbb{E}X_{s,t}X_{u,v}.
\]

From [MIR06] Lemma 7.2.7 we know that due to concavity of \( \sigma^2 \), non-overlapping increments are non-positively correlated, in the sense that

\[
\mathbb{E}X_{s,t}X_{u,v} = R_X \left( \frac{s}{u}, \frac{t}{v} \right) \leq 0, \quad \forall 0 \leq s \leq t \leq u \leq v \leq h.
\]

Using this and \((a + b + c)b = b^2 + ab + cb\), yields

\[
\mathbb{E}X_{s,t}X_{u,v} = \mathbb{E}X_{u,v}^2 + \mathbb{E}X_{s,u}X_{u,v} + \mathbb{E}X_{v,t}X_{u,v} \leq \mathbb{E}X_{u,v}^2.
\]

This concludes the proof.
Lemma 9. Let \( \sigma^2 : [0, T] \to \mathbb{R}_+ \) be a continuous, concave function with \( \sigma^2(0) = 0 \) for some \( T > 0 \). Then there is an \( h \in (0, T] \) such that \( \sigma^2 \) is non-decreasing on \( [0, h] \).

Proof. Since \( \sigma^2 : [0, T] \to \mathbb{R}_+ \) is a continuous, concave function, each local maximum is a global maximum. Let \( h := \inf \{ t \in [0, T] \mid \sigma^2(t) = \max_{r \in [0, T]} \sigma^2(r) \} \). Without loss of generality we may assume \( \sigma^2 \neq 0 \). Thus, \( h > 0 \) and \( \sigma^2 \) is non-decreasing on \( [0, h] \), since otherwise \( \sigma^2 \) would have a local maximum in \( [0, h) \), thus attaining \( \max_{r \in [0, T]} \sigma^2(r) \) in \( [0, h) \) in contradiction to the definition of \( h \). \( \square \)

Proof of Theorem 6. Let \( [s, t]^2 \subseteq [0, T]^2 \) with \( |t - s| \leq h \), where \( h \) is as in Lemma 9. For any \( t_i < t_{i+1} \in [s, t] \) we have

\[
\sum \left| \mathbb{E}X_{t_i, t_{i+1}} X_{t'_j, t'_{j+1}} \right| \leq \left| \mathbb{E}X_{t_i, t_{i+1}} X_{|1-\text{var};s,t|} \right|
\]

(3.2)

(I): Recall that by [MR06, Lemma 7.2.7] non-overlapping increments are non-positively correlated (cf. (3.1)). Therefore, for any dissection \( t'_j \) of \([s, t_i]\) we have

\[
\sum \left| \mathbb{E}X_{t_i, t_{i+1}} X_{t'_j, t'_{j+1}} \right| = \left| \mathbb{E}X_{t_i, t_{i+1}} X_{s,t_i} \right|
\]

By Lemma 8 we have

\[
\left| \mathbb{E}X_{t_i, t_{i+1}} X_{s,t_i} \right| \leq \left| \mathbb{E}X_{t_i, t_{i+1}} X_{s,t_{i+1}} \right| + \left| \mathbb{E}X_{t_i, t_{i+1}} X_{t_i,t_{i+1}} \right| \leq 2\sigma^2(t_{i+1} - t_i).
\]

Hence,

\[
\sum \left| \mathbb{E}X_{t_i, t_{i+1}} X_{t'_j, t'_{j+1}} \right| \leq 2\sigma^2(t_{i+1} - t_i) \leq 2C_\sigma |t_{i+1} - t_i|^\frac{1}{\rho},
\]

which yields

(I) = \( \left| \mathbb{E}X_{t_i, t_{i+1}} X_{|1-\text{var};s,t_i|} \right| \leq 2C_\sigma |t_{i+1} - t_i|^\frac{1}{\rho} \).

(II): Using Lemma 8 for any dissection \( t'_j \) of \([t_i, t_{i+1}]\) we have

\[
\sum \left| \mathbb{E}X_{t_i, t_{i+1}} X_{t'_j, t'_{j+1}} \right| = \left| \mathbb{E}X_{t_i, t_{i+1}} \right| = \sigma^2(t_{i+1} - t_i) \leq C_\sigma |t_{i+1} - t_i|^\frac{1}{\rho}.
\]

Hence,

(II) = \( \left| \mathbb{E}X_{t_i, t_{i+1}} X_{|1-\text{var};t_i,t_{i+1}|} \right| = \sup_{D \subseteq [t_i, t_{i+1}]} \sum_{t'_j \in D'} \left| \mathbb{E}X_{t_i, t_{i+1}} X_{t'_j, t'_{j+1}} \right| \leq C_\sigma |t_{i+1} - t_i|^\frac{1}{\rho}.
\)

(III): As for (I).

From (3.2) we conclude
Thus, \[ V_{1,\rho}(R_X; [s, t]^{2}) = \sup_{(t_i), (t'_j) \in D([s, t])} \left( \sum_{t_i} \sum_{t'_j} \left| \mathbb{E} X_{t_i, t_{i+1}} X_{t'_j, t'_{j+1}} \right| \right)^{\rho} \leq 5^\rho C_{\sigma} |t - s|. \]

\[ \square \]

**Proof of Corollary 7.** Let \([s, t] \subseteq [0, T]\). Choosing some dissection \(s = t_0 \leq t_1 \leq \ldots \leq t_N = t\) with mesh-size \(\max_{i=1, \ldots, N} |t_i - t_{i-1}| \leq h\), where \(h\) is as in Theorem 6, we note \[ \|h\|_{q-\text{var}; [s, t]} \leq C \sum_{i=1}^{N} \|h\|_{q-\text{var}; [t_{i-1}, t_i]}, \] for some constant \(C = C(N, q)\). We conclude the proof by applying Theorem 6 and Theorem 1 on each interval \([t_{i-1}, t_i]\). \[ \square \]

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