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Higher homotopy operations and André–Quillen cohomology

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Abstract

There are two main approaches to the problem of realizing a $Π$-algebra (a graded group $Λ$ equipped with an action of the primary homotopy operations) as the homotopy groups of a space $X$. Both involve trying to realize an algebraic free simplicial resolution $G_\bullet$ of $Λ$ by a simplicial space $W_\bullet$, and proceed by induction on the simplicial dimension. The first provides a sequence of André–Quillen cohomology classes in $H^{n+2}(Λ; Ω^n Λ)$ ($n ⩾ 1$) as obstructions to the existence of successive Postnikov sections for $W_\bullet$ (cf. Dwyer et al. (1995) [27]). The second gives a sequence of geometrically defined higher homotopy operations as the obstructions (cf. Blanc (1995) [8]); these were identified in Blanc et al. (2010) [16] with the obstruction theory of Dwyer et al. (1989) [25]. There are also (algebraic and geometric) obstructions for distinguishing between different realizations of $Λ$.

In this paper we

(a) provide an explicit construction of the cocycles representing the cohomology obstructions;
(b) provide a similar explicit construction of certain minimal values of the higher homotopy operations (which reduce to “long Toda brackets”); and
(c) show that these two constructions correspond under an evident map.

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0. Introduction

Secondary and higher order operations are often used in homotopy theory either as obstructions to resolving existence problems, or as computational tools. In the 1950’s, Adams used secondary cohomology operations in [1] to show the non-existence of elements of Hopf invariant one in the stable homotopy groups of spheres; at the same time, Toda employed his secondary compositions (Toda brackets) to calculate some of these homotopy groups in [48]. Higher homotopy and cohomology operations have since been applied in many areas, including $H$-spaces, rational homotopy, and stable homotopy theory (cf. [3,33,44,47]).

To make sense of the general notion of a higher order operation, note that many of the homotopy invariants of algebraic topology, such as homotopy or (co)homology groups, carry a further primary structure, definable in the homotopy category itself. For example, the homotopy groups of a pointed space $X$ have Whitehead products and composition operations, which together make $\pi_*X$ into a $\Pi$-algebra (see Section 1.1 below). Similarly, the mod $p$ cohomology of $X$ has the structure of an unstable algebra over the Steenrod algebra (cf. [43, §1.4]), and the stable homotopy groups of a commutative ring spectrum form a graded commutative ring.

The appropriate higher order operations (such as Massey products or Toda brackets) form a higher structure superimposed on the primary one: they are usually defined only when certain lower-order operations vanish. General higher homotopy operations were defined in [17,13] to be certain obstructions to rectifying homotopy-commutative diagrams $\tilde{X} : \Gamma \to \text{ho} \mathcal{M}$. Here $\mathcal{M}$ is a pointed simplicial model category and $\Gamma$ is a finite directed indexing category called a lattice (cf. Section 3.6). In [16], we show how this obstruction theory may be identified with that of Dwyer, Kan, and Smith (cf. [25]).

0.1. Realization problems

One natural question which arises in this context is whether a given abstract (primary) algebraic structure – such as a $\Pi$-algebra, an unstable algebra, or a graded ring – is in fact associated to some topological space or spectrum, and in how many ways. Such realization questions have a long history in algebraic topology (see [36,39,45] for the case of cohomology).

Here we consider the problem of realizing an abstract $\Pi$-algebra $\Lambda$ as the homotopy groups of a space $X$. To do so, we start with an (algebraic) free simplicial resolution $G_\bullet$ of $\Lambda$. This can always be realized by a “lax” simplicial space $\hat{W}_\bullet$, with each $\hat{W}_n$ homotopy equivalent to a wedge of spheres, where the simplicial identities hold only up to homotopy. If $\hat{W}_\bullet$ can be rectified to a strict simplicial space $W_\bullet$, then its geometric realization $X := \|W_\bullet\|$ has $\pi_*X \cong \Lambda$, as required.

There are two known approaches to solving this rectification problem (and thus the original realization problem):

I. The “geometric” approach proceeds by induction over the skeleta of $\hat{W}_\bullet$, yielding obstructions to the successive rectification problems in the form of higher homotopy operations (see [8,9]).
II. The “algebraic” approach of Dwyer, Kan, and Stover constructs inductively two sequences of André–Quillen cohomology obstructions: one sequence for the realization of $\Lambda$, and the other for two such realizations $X$ and $Y$ to be homotopy equivalent.

More explicitly, the second approach uses successive Postnikov approximations $W^{(n)}_{\bullet}$ to the putative simplicial space $W_{\bullet}$ to define two André–Quillen cohomology classes:

(a) An existence obstruction $\beta_n \in H^{n+2}(\Lambda; \Omega^n \Lambda)$, which vanishes if and only if $W^{(n)}_{\bullet}$ extends to an $(n + 1)$-Postnikov section $W^{(n+1)}_{\bullet}$.

(b) A difference obstruction $\delta_n \in H^{n+1}(\Lambda; \Omega^n \Lambda)$, for distinguishing between possible extensions $W^{(n+1)}_{\bullet}$.

See [27,14] for further details, with additional variants in [15].

A different version of this theory allows one to determine whether a graded commutative ring $R_\bullet$ is isomorphic to $\pi_* S$ for some commutative ring spectrum $S$ (see [30]). We note, however, that the main application of this theory (see [29]) relies on a large scale vanishing of relevant André–Quillen cohomology groups, from which the vanishing of the obstructions follow. The approach we describe here characterizes the obstructions directly, and more explicitly. We hope that this will open the door to addressing a broader range of realization questions using these techniques. For a simple example, see Section 8.

0.2. Main results

The aim of this paper is to make explicit the close connection between these two approaches, by showing that the higher homotopy operations correspond in a systematic way to the André–Quillen obstruction classes.

For this purpose, we first study André–Quillen cohomology for general universal algebras, showing how it can be calculated using a cochain complex, and providing an explicit description of the $k$-invariants of a simplicial algebra (see Proposition 2.4 and Corollary 2.14 below).

The $n$-th existence obstruction $\beta_n$ mentioned above is in fact the $k$-invariant for the simplicial $\Pi$-algebra $\pi_* W^{(n)}_{\bullet}$, so we can use this description to analyze $\beta_n$. We then explain how essentially the same inductive process for realizing $\Lambda$ (now using simply a truncated simplicial object $V_{\bullet}^{(n+1)}$, rather than a Postnikov section) has another obstruction theory in terms of higher homotopy operations, which are subsets $\langle \langle \Psi^{n+2}_0 \rangle \rangle$ of $[\bigvee_{\psi_0^{n+2}} \Sigma^n V_{n+2}, V_0]$. After defining a natural correspondence homomorphism $\tilde{\Phi}_n : [\bigvee_{\psi_0^{n+2}} \Sigma^n V_{n+2}, V_0] \to H^{n+2}(\Lambda; \Omega^n \Lambda)$ in Section 6.1, we construct certain natural minimal values in $\langle \langle \Psi^{n+2}_0 \rangle \rangle$ and prove:

**Theorem A.** The homomorphism $\tilde{\Phi}_n$ maps each minimal value of $\langle \langle \Psi^{n+2}_0 \rangle \rangle$ to the corresponding André–Quillen obstruction $\beta_n$ to realizing $\Lambda$, so if the minimal value vanishes, so does $\beta_n$. Conversely, if the cohomology obstruction $\beta_{n+1}$ associated to the next step vanishes, so does $\langle \langle \Psi^{n+2}_0 \rangle \rangle$.

[See Theorem 6.10 and Corollary 6.11].

Next, we also provide an explicit description of the cohomological difference obstructions $\delta_n$ for distinguishing between inequivalent $(n + 1)$-Postnikov sections $V_{\bullet}^{(a)}$ and $V_{\bullet}^{(b)}$ of resolutions...
of $\Lambda$. We use the same formalism employed in constructing $\langle\Psi_0^{n+2}\rangle$ to define a corresponding higher homotopy operation difference obstruction $\langle\tilde{d}_0^n, \tilde{d}_0^b\rangle$ in $[\Sigma^{n+1} V_{n+2}, V_0]$, with its own minimal values, and show:

**Theorem B.** The correspondence homomorphism maps a minimal value of $\langle\tilde{d}_0^n, \tilde{d}_0^b\rangle$ to the André–Quillen obstruction $\delta_n$.

[See Theorem 7.11].

As an application, we use this theory to show that for any connected graded Lie algebra $\Lambda$ over $\mathbb{Q}$, there is a branch of the obstruction theory for which all cycles representing the André–Quillen obstructions $\beta_n$ to realizing $\Lambda$ vanish (see Proposition 8.1 below). From this we recover the well-known fact, due to Quillen, that any simply-connected rational $\Pi$-algebra is realizable.

### 0.3. Notation and conventions

The category of topological spaces is denoted by $T$, and that of pointed connected spaces by $T_\ast$. For any category $C$, $sC := C^{\Delta^{op}}$ is the category of simplicial objects over $C$. We abbreviate $sSet$ to $S$ and $sSet_\ast$ to $S_\ast$; $S_\ast^{red}$ denotes the category of reduced simplicial sets. The constant simplicial object on an object $X \in C$ is written $c(X) \in sC$, and the $n$-truncation of $G_\ast \in sC$ (forgetting $G_i$ for $i > n$) is denoted by $\tau_n G_\ast$. The $n$-skeleton functor is left adjoint to the truncation functor $\tau_n$. However, we reserve the notation $sk_n : sC \to sC$ for the composite of the $n$-skeleton functor with $\tau_n$. The $n$-coskeleton functor $csk_n : sC \to sC$ is right adjoint to $sk_n$.

We denote by $\mathcal{G}p$ the category of groups, by $\mathcal{A}b\mathcal{G}p$ that of abelian groups, and by $\mathcal{S}pd$ that of groupoids. When $\Lambda$ is an abelian category, $\mathcal{C}h(\Lambda)$ denotes the category of (non-negatively graded) chain complexes over $\Lambda$.

If $\langle V, \otimes \rangle$ is a monoidal category, $V\text{-}\mathcal{C}at$ is the collection of all (not necessarily small) categories enriched over $V$ (see [19, §6.2]). For any set $\emptyset$, denote by $\emptyset\text{-}\mathcal{C}at$ the category of all small categories $\mathcal{D}$ with $\text{Obj} \mathcal{D} = \emptyset$. A $(V, \otimes)$-category is a category $\mathcal{D} \in \emptyset\text{-}\mathcal{C}at$ enriched over $V$, with mapping objects $\text{map}_{\mathcal{D}}(\emptyset, \emptyset) \in V$. The category of all small $(V, \otimes)$-categories will be denoted by $(V, \otimes)\text{-}\mathcal{C}at$. The main examples of $(V, \otimes)$ we have in mind are $(S, \times)$, $(S_\ast, \wedge)$, and $(\mathcal{S}pd, \otimes)$.

Note that because the Cartesian product on $S$ (and the smash product on $S_\ast$) are defined levelwise, we can think of an $(S, \emptyset)$- or $(S_\ast, \emptyset)$-category as a simplicial object over $\emptyset\text{-}\mathcal{C}at$—that is, a simplicial category with fixed object set $\emptyset$ in each dimension, and all face and degeneracy functors the identity on objects.

### 0.4. Organization

In Section 1 we provide some background on $\Pi_\Lambda$-algebras and the related resolution model categories used in this paper. In Section 2 we prove some basic facts about the cohomology of (graded) universal algebras. Section 3 discusses rectification of homotopy-commutative diagrams, and the higher homotopy operations which appear as the obstructions to such rectification. Section 4 analyzes the André–Quillen cohomology existence obstructions to realizing a $\Pi_\Lambda$-algebra $\Lambda$, and Section 5 defines the higher homotopy operation version of these obstructions. In Section 6 we describe certain minimal values of these higher homotopy operations, and prove Theorem 6.10. Section 7 shows how the difference obstructions (both cohomological and higher...
homotopy operation versions) may be treated analogously, yielding Theorem 7.11. Finally, Section 8 briefly discusses rational homotopy theory.

1. Model categories

This paper deals primarily with homotopy theory of pointed connected topological spaces, with the usual homotopy groups. However, some of the results hold more generally, so we now introduce axiomatic descriptions of some more general settings.

1.1. Definition. Let \( \mathcal{M} \) be a pointed simplicial model category (cf. [40, II, §1]), so that for every simplicial set \( K \) and \( X \in \mathcal{M} \) we have an object \( K \hat{\otimes} X \) in \( \mathcal{M} \). In particular, we call \( S^1 \hat{\otimes} X \) the half-suspension of \( X \), and by choosing a basepoint \( pt \) in \( S^1 \), we define the \( n \)-fold suspension \( \Sigma^n X \) to be \( (S^n \hat{\otimes} X)/(\{pt\} \hat{\otimes} X) \). Thus for \( X \in \mathcal{T}_n \), we have \( K \hat{\otimes} X = [K] \times X := ([K] \times X)/([K] \times \{\ast\}) \), so \( \Sigma^n X := [S^n] \wedge X \).

Now let \( \Lambda \) be a collection of homotopy cogroup objects in \( \mathcal{M} \) (sometimes called spherical objects). For any object \( Y \in \mathcal{M} \), its \( \Lambda \)-homotopy groups are \( \pi_*^\Lambda Y = (\pi_n^\Lambda Y)_{A \in \Lambda, n \in \mathbb{N}}, \) where \( \pi_n^\Lambda Y := [\Sigma^n A, Y]_{\mathcal{M}}, n = 0, 1, 2, \ldots \). A map \( f : Y \rightarrow Z \) in \( \mathcal{M} \) is called an \( \Lambda \)-equivalence if it induces an isomorphism in \( \pi_*^\Lambda \). We denote by \( \Pi_A \) the full subcategory of \( \text{ho} \mathcal{M} \) whose objects are finite coproducts of suspensions of elements of \( A \). A product-preserving functor \( \Lambda : \Pi_A^{\text{op}} \rightarrow \text{Set} \) is called a \( \Pi_A \)-algebra, and the category of such is denoted by \( \Pi_A^{\text{-Alg}} \). When we wish to emphasize the dependence on \( \mathcal{M} \), we call these \( \Pi_A^{\mathcal{M}} \)-algebras, and denote the category by \( \Pi_A^{\mathcal{M}} \text{-Alg} \). We write \( \Lambda(B) \) for the value of \( \Lambda \) at an object \( B \in \Pi_A \). We denote by \( \mathcal{M}_A \) the smallest full subcategory of \( \mathcal{M} \) containing \( \Lambda \) and closed under suspensions, arbitrary coproducts, and weak equivalences.

When \( \mathcal{M} = S^\text{red}_* \) (or \( \mathcal{T}_n \)) and \( A := \{S^1\} \) these are called simply \( \Pi \)-algebras, and the category is denoted by \( \Pi^{\text{-Alg}} \) (cf. [46]). Note the re-indexing \( \pi_n^\Lambda Y = \pi_{n+1} Y, \) for \( n \geq 0 \).

1.2. Example. The canonical example of a \( \Pi_A \)-algebra is a realizable one, denoted by \( \pi_*^\Lambda X \), and defined for fixed \( X \in \mathcal{M} \) by \( A \mapsto [A, X]_{\text{ho} \mathcal{M}} \) for all \( A \in \Pi_A \). This defines a functor \( \pi_*^\Lambda : \text{ho} \mathcal{M} \rightarrow \Pi_A^{\text{-Alg}} \). Thus when \( A = \{S^1\} \), so \( \Pi_A^{\text{-Alg}} = \Pi^{\text{-Alg}} \), a realizable \( \Pi \)-algebra consists of the sequence of groups \( \pi_* X \), equipped with the action of the primary homotopy operations on them (compositions, Whitehead products, and action of the fundamental group). This is called the homotopy \( \Pi \)-algebra of \( X \).

1.3. Definition. Let \( \Theta \) be an \( FP \)-sketch, in the sense of Ehresmann (cf. [28]) – that is, a small category with a distinguished collection \( \mathcal{P} \) of products. A \( \Theta \)-algebra is a functor \( \Lambda : \Theta \rightarrow \text{Set} \) which preserves the products in \( \mathcal{P} \). We think of a map \( \phi : \prod_{i=1}^n a_i \rightarrow \prod_{j=1}^m b_j \) in \( \Theta \) as representing an \( m \)-valued \( n \)-ary operation on \( \Theta \)-algebras, with gradings indexed by \( (a_i)^n_{i=1} \) and \( (b_j)^m_{j=1} \), respectively.

The category of \( \Theta \)-algebras is denoted by \( \Theta^{\text{-Alg}} \). If \( \text{Obj}(\Theta) \) is generated under the products in \( \mathcal{P} \) by a set \( \Theta \), there is a forgetful functor \( U : \Theta^{\text{-Alg}} \rightarrow \text{Set}^{\Theta} \) into the category of \( \Theta \)-graded sets, with left adjoint the free \( \Theta \)-algebra functor \( F : \text{Set}^{\Theta} \rightarrow \Theta^{\text{-Alg}} \).

1.4. Example. The main type of \( \Theta \)-algebras considered in this paper are those with \( \Theta = \Pi^{\text{op}}_A \), \( \Theta = \{\Sigma^n A : A \in \Lambda, n \in \mathbb{N}\} \), and \( \mathcal{P} \) the set of finite coproducts of objects in \( \Lambda \). Note that these are products in \( \Pi^{\text{op}}_A \).
In particular, let $\Pi_1$ denote the homotopy category of finite wedges of circles: that is, the full subcategory of $\text{ho} \mathcal{T}_n$ with object set $\{\bigvee_{i=1}^n S^1\}_{n=0}^\infty$. Then $\mathcal{G} := \Pi_1^{\text{op}}$, with $\mathcal{P} = \text{Obj}(\mathcal{G})$, is the theory representing groups — that is, $\mathcal{G}\text{-Alg}$ is naturally equivalent to $\mathcal{J} p$. Similarly, $\mathcal{G}^N$ represents $\mathbb{N}$-graded groups.

We define a $\mathcal{G}$-theory to be an FP-sketch $\Theta$ equipped with an embedding of sketches $\mathcal{G}^\mathcal{O} \hookrightarrow \Theta$, for $\mathcal{O}$ as in Section 1.3. In this case, any $\Theta$-algebra $X$ has a natural underlying $\mathcal{O}$-graded group structure. We do not require the operations of a $\mathcal{G}$-theory to be homomorphisms (that is, commute with the $\mathcal{G}$-structure).

1.5. Definition. As in [7] or [42, §1], we define a module over a $\Theta$-algebra $A$ to be an abelian group object $p: \mathcal{K} \to A$ in $\Theta\text{-Alg}/A$. Note that $0 \in \text{Hom}_{\Theta\text{-Alg}/A}(A, \mathcal{K})$ then provides a section for the structure map $p$. If for each $v \in \mathcal{O}$ we let $\overline{K}_v := \{x \in K(v): p(x) = 0\}$, then $\overline{K}_v$ has an abelian group structure induced from that of $\mathcal{K} \to A$, and for each $f: u \to v$ in $\Theta$ we have a homomorphism of abelian groups $f_*: \overline{K}_u \to \overline{K}_v$. Thus $\mathcal{K}$ is completely determined by the restricted module functor $\overline{K}: \Theta \to \text{Ab}\mathcal{O} p$. We denote the category of restricted modules by $\Lambda\text{-Mod}$, with $\mathcal{K} \mapsto \overline{K}$ defining an equivalence of categories $(\Theta\text{-Alg}/A)_{ab} \to \Lambda\text{-Mod}$ (cf. [12, Proposition 3.14] for more details).

1.6. Remark. The identification of the half-suspension $S^n \hat{\otimes} A$ with $\Sigma^n A \vee A$ for a homotopy cogroup object $A$ (cf. [6]) makes $[S^n \hat{\otimes} A, Y]$ into a module over $\pi^n_A Y$ in the sense of Section 1.5. This allows us to think of $\pi^n_A Y := [\Sigma^n A, Y]$ itself as a restricted $\pi^n_A Y$-module.

Moreover, for any $\Pi A$-algebra $A$, we may define an abelian $\Pi A$-algebra $\Omega A$ by setting $(\Omega A)(A) := A(\Sigma A)$. This has a natural structure of a restricted $A$-module (see [27, §9.4], and compare [5, §1.11]).

1.7. Resolution model categories

Let $\mathcal{M}$ be a pointed, cofibrantly generated, right proper simplicial model category, equipped with a collection $\mathcal{A}$ of homotopy cogroup objects. The category $s\mathcal{M}$ of simplicial objects over $\mathcal{M}$ has a resolution model category structure, in which a map $f: W_\bullet \to V_\bullet$ in $s\mathcal{M}$ is a weak equivalence if and only if the induced map of simplicial groups $f_\#: \pi^n_A W_\bullet \to \pi^n_A V_\bullet$ is a weak equivalence for each $A \in \mathcal{A}$ and $n \geqslant 0$. See [20] and [38] for further details.

Moreover, $s\mathcal{M}$ has its own simplicial structure $(s\mathcal{M}, \otimes)$ (cf. [40, II, §1]), and thus has a set of spherical objects: $(S^n \otimes c(\Sigma^i A))/([pt] \otimes c(\Sigma^i A))$ for $A \in \mathcal{A}$ and $i, n \in \mathbb{N}$. The natural homotopy groups of a simplicial object $X_\bullet \in s\mathcal{M}$ are defined by $\pi_n^{s, A} X_\bullet := [S^n \otimes c(\Sigma^i A))/([pt] \otimes c(\Sigma^i A))$, $X_\bullet|_{s\mathcal{M}}$ for $A \in \mathcal{A}$. Setting $\pi^n_A X_\bullet := \{\pi_n^{s, A} X_\bullet\}_{A \in \mathcal{A}, i \in \mathbb{N}}$, it may be shown that $\pi^n_A X_\bullet$ has a natural $\Pi A$-algebra structure (see [27, §5]).

To describe some basic constructions in $s\mathcal{M}$, recall that the $n$-th Moore chains object of a Reedy fibrant simplicial object $X_\bullet$ is defined:

$$C_n X_\bullet := \bigcap_{i=1}^n \text{Ker}[d_i: X_n \to X_{n-1}],$$

with differential $\partial_n^{X_\bullet} = \partial_n := (d_0)|_{C_n X_\bullet}: C_n X_\bullet \to C_{n-1} X_\bullet$. The $n$-th Moore cycles object is $Z_n X_\bullet := \text{Ker}(\partial_n^{X_\bullet})$. 


It turns out that under mild assumptions on $M$ the inclusion $i : C_{n+1}X_\bullet \hookrightarrow X_{n+1}$ induces an isomorphism

$$i_* : \pi_*^A C_{n+1}X_\bullet \to C_{n+1}X_\bullet$$

(see [46, Lemma 2.7] or [10, Prop. 2.7]), which fits into a commuting diagram of $\Pi_A$-algebras with exact rows:

$$\begin{array}{ccc}
\pi_*^A C_{n+1}X_\bullet & \xrightarrow{(\partial X_\bullet)^A} & \pi_*^A Z_n X \\
\downarrow i_* & \cong & \downarrow \hat{i}_* \\
C_{n+1}(\pi_* X_\bullet) & \xrightarrow{\hat{i}_*} & Z_n(\pi_* X_\bullet) \\
\end{array}
$$

This defines the Hurewicz map $h_n : \pi_*^A X_\bullet \to \pi_*^A X_\bullet$ in the spiral long exact sequence of $A$-graded groups:

$$\cdots \to \Omega \pi_*^A X_\bullet \xrightarrow{s_n} \pi_*^A X_\bullet \xrightarrow{h_n} \pi_*^A X_\bullet \xrightarrow{\partial_n} \cdots \to \pi_*^A X_\bullet$$

(cf. [27, 8.1]), where $s_n$ is induced by the connecting homomorphism in $\pi_*^A$ for the fibration sequence in $M$:

$$Z_n X_\bullet \xrightarrow{j_n} C_n X_\bullet \xrightarrow{d_0} Z_{n-1} X_\bullet.$$  (1.12)

1.13. Example. When $M = T_e$ and $A = \{S^1\}$, the resolution model category of simplicial spaces is the original $E^2$-model category of [26].

1.14. Remark. If $\Theta$ is a $G$-theory (Section 1.4), the monogenic free $\Theta$-algebras constitute a collection $A$ of (strict) cogroup objects in $M = \Theta-Alg$ (with the trivial model category structure). Since maps between free $\Theta$-algebras represent the operations in $\Theta$, in this case a $\Pi_A$-algebra may be identified with a $\Theta$-algebra, so there is a canonical equivalence of categories $\Pi_A^M-Alg \simeq M$. This applies in particular to $\Theta = \Pi_A^{op}$ itself, so that $\Pi_A^{\Pi_A-Alg} \simeq \Pi_A-Alg$.

In this case, the resolution model category structure on $s\Theta-Alg$ is Quillen’s model category for simplicial universal algebras (cf. [40, II, §4]), and $\pi_0^A G_\bullet$ is the graded group $\pi_0 G_\bullet$, equipped with a natural $\Theta$-algebra structure. Any $G_\bullet \in s\Theta-Alg$ for which each $G_n$ is free, and the degeneracy maps take generators to generators, is cofibrant.

1.15. $E^2$-model categories

If $M$ is a pointed model category as in Section 1.7 with a collection of spherical objects $A$ (Section 1.1), the resolution model category $sM$ is called an $E^2$-model category if it is equipped with:
(a) A functorial Postnikov tower of fibrations (in $sM$) for each $W_\bullet \in sM$:

$$
\ldots \rightarrow \mathbf{P}^n W_\bullet \xrightarrow{p(n)} \mathbf{P}^{n-1} W_\bullet \xrightarrow{p(n-1)} \ldots \rightarrow \mathbf{P}^0 W_\bullet.
$$

equipped with a weak equivalence $r : W_\bullet \rightarrow \mathbf{P}^\infty W_\bullet := \lim_n \mathbf{P}^n W_\bullet$, as well as fibrations $r(n) : \mathbf{P}^\infty W_\bullet \rightarrow \mathbf{P}^n W_\bullet$, such that $p(n)$ and $r(n)$ induce isomorphisms in $\pi^\natural_i$ for $i \leq n$, and $\pi^\natural_i \mathbf{P}^n W_\bullet = 0$ for $i > n$.

(b) For every $\Pi^M A$-algebra $\Lambda$, there is a functorial fibrant classifying object $B\Lambda \in sM$, unique up to homotopy, with $B\Lambda \simeq \mathbf{P}^0 B\Lambda$ and $\pi^\natural_0 B\Lambda \simeq \Lambda$.

(c) Given a $\Pi^M A$-algebra $\Lambda$ and a $\Lambda$-module $K$, for each $n \geq 1$ there is a functorial fibrant Eilenberg–Mac Lane object $E = E_\Lambda(K, n)$ in $sM/B\Lambda$, unique up to homotopy, equipped with a section for $(r(0) \circ r) : E \rightarrow \mathbf{P}^0 E \simeq B\Lambda$, such that $\pi^\natural_0 E \simeq K$ as a $\Lambda$-module, and $\pi^\natural_i E = 0$ for $1 \leq i \neq n$.

(d) For every $n \geq 0$, there is a functor that assigns to each $W_\bullet \in sM$ with $\pi^\natural_0 W_\bullet = \Lambda$ a homotopy pullback square:

$$
\begin{array}{ccc}
\mathbf{P}^{n+1} W_\bullet & \xrightarrow{p(n+1)} & \mathbf{P}^n W_\bullet \\
\downarrow \mathbf{PB} & & \downarrow k_n \\
B\Lambda & \rightarrow & E_\Lambda(\pi^\natural_{n+1} W_\bullet, n + 2)
\end{array}
$$

(1.16)

(in $sM$) with $k_n$ the $n$-th $k$-invariant for $W_\bullet$.

(e) A realization functor $J : sM \rightarrow M$, such that, for $\Lambda \in \Pi^M A$-Alg and cofibrant $X_\bullet \in sM$, if $\pi^A_n X_\bullet \simeq B\Lambda$ is a weak equivalence in $s\Pi^M A$-Alg (using the convention of Section 1.14 to define $B\Lambda \in s\Pi^M A$-Alg), there is an isomorphism:

$$
[A, JX_\bullet]_M \xrightarrow{\sim} \text{Hom}_{\Pi^M A\text{-Alg}}(\pi^A_n A, \Lambda),
$$

(1.17)

natural in $\Lambda$ and $A \in \mathcal{A}$.

1.18. Remark. In all the cases we are interested in, the coskeleton $\text{csk}_{n+1} W_\bullet$ (Section 0.3) provides the functorial Postnikov section $\mathbf{P}^n W_\bullet$ for Reedy fibrant simplicial objects $W_\bullet \in sM$ (cf. [35, §15]).

1.19. Examples. The two main $E^2$-model categories we have in mind are:

(1) $\mathcal{M} = T_s$ or $S_s$, with $\mathcal{A} = [S^1]$. In this case $J$ is the usual realization functor, and (1.17) follows from the collapse of the Bousfield–Friedlander spectral sequence (cf. [21, Theorem B.5]).

(2) $\mathcal{M} = \Theta$-Alg, the category of $\Theta$-algebras for some $\Theta$-theory $\Theta$, as in Section 1.14, and $\mathcal{A}$ the monogenic free $\Theta$-algebras. Here $JX_\bullet := \pi_0 X_\bullet$ (so $\pi^A_n X_\bullet \simeq B\Lambda$ is a weak equivalence in
\[ s\Pi^M_{\mathcal{A}}(\text{Alg}) \approx s\mathcal{M} \text{ as above if and only if } \varepsilon : X_{\bullet} \to \Lambda \text{ is a resolution}, \] and the isomorphism (1.17) is induced by \( \varepsilon \).

For further examples, see [15, §3].

\textbf{1.20. Definition.} Let \( \mathcal{M} \) be a pointed model category. A simplicial object \( G_{\bullet} \in s\mathcal{M} \) is called a CW object if:

1. For each \( n \geq 0 \) there is an object \( \bar{G}_n \in \mathcal{M} \) such that \( G_n = \bar{G}_n \sqcup L_n G_{\bullet} \). Here

\[
L_n G_{\bullet} := \bigsqcup_{0 \leq k \leq n} \bigsqcup_{0 \leq i_1 < \cdots < i_{n-k-1} \leq n-1} \bar{G}_k
\]  

(1.21)

is the \( n \)-th latching object of \( G_{\bullet} \), in which the copy of \( \bar{G}_k \) indexed by \( (i_1, \ldots, i_{n-k-1}) \) is in the image of \( s_{i_{n-k-1}} \cdots s_2 s_1 \).

2. There is an attaching map \( d_0^{G_n} : \bar{G}_n \to G_{n-1} \) with \( d_i \circ d_0^{G_n} = 0 \) for \( 0 \leq i \leq n-1 \), or equivalently, \( d_0^{G_n} \) factors through \( Z_{n-1} G_{\bullet} \subset G_{n-1} \).

3. The face maps of \( G_{\bullet} \) are determined by the simplicial identities and the requirement that

\[
(d_0)|_{\bar{G}_n} = d_0^{G_n} \text{ and } (d_i)|_{\bar{G}_n} = 0 \text{ for } 1 \leq i \leq n.
\]

The collection \( (\bar{G}_n)_{n=0}^{\infty} \) is called a CW basis for \( G_{\bullet} \).

When \( \mathcal{A} \) is a collection of homotopy cogroup objects in \( \mathcal{M} \), \( G_{\bullet} \to X \) is a cofibrant replacement in the resolution model category structure on \( s\mathcal{M} \) determined by \( \mathcal{A} \), and each \( \bar{G}_n \) in a CW basis for \( G_{\bullet} \) lies in \( \mathcal{M}_\mathcal{A} \) (Section 1.1), we call \( G_{\bullet} \) a CW resolution of \( X \).

\textbf{1.22. Remark.} The category of \( (n+2) \)-truncated CW objects \( V_{\bullet}^{(n+2)} \) in \( \mathcal{M} \) is equivalent to the category of pairs consisting of an \( (n+1) \)-truncated CW object \( V_{\bullet}^{(n+1)} \) and a map \( d_0^{V_{n+2}} : \overline{V}_{n+2} \to Z_{n+1} V_{\bullet}^{(n+1)} \): given such a pair \( (V_{\bullet}, d_0^{V_{n+2}}) \), we obtain an \( (n+2) \)-truncated CW object by setting \( V_{n+2} := \overline{V}_{n+2} \sqcup L_{n+2} V_{\bullet} \), with \( (d_0)|_{\overline{V}_{n+2}} := d_0^{V_{n+2}} \) and \( (d_i)|_{\overline{V}_{n+2}} = 0 \) for \( i > 0 \). The degeneracies are given by the obvious inclusions into \( L_{n+2} V_{\bullet} \), and the face maps on \( L_{n+2} V_{\bullet} \) are determined by the simplicial identities.

When \( \mathcal{M} := \Theta-\text{Alg} \), one can use this method inductively to construct a free CW-resolution of a \( \Theta \)-algebra \( \Lambda \) (with each \( \overline{V}_{n+2} \) free).

\textbf{2. Cohomology of \( \Theta \)-algebras}

In this section we recall the definition of André–Quillen cohomology for simplicial \( \Theta \)-algebras, provide a cochain description for their cohomology, and give an explicit construction of their \( k \)-invariants. Although most of the results are valid more generally, for simplicity we restrict attention to the case where the \( \Theta \)-algebras have a (possibly graded) underlying group structure.

\textbf{2.1. Definition.} Let \( \mathcal{A} \) be a collection of spherical objects in a pointed model category \( \mathcal{M} \), such that the resolution model category \( s\mathcal{M} \) is an \( E^2 \)-model category (Section 1.15). Assume given a \( \Pi^M_{\mathcal{A}} \)-algebra \( \Lambda \), a \( \Lambda \)-module \( \mathcal{K} \), and an object \( W_{\bullet} \in s\mathcal{M} \) equipped with a twisting map
\( \pi : \pi_0^A W_\bullet \to \Lambda \). We use \( \pi \), along with the natural map \( W_\bullet \to B_0^\pi A \), to think of \( W_\bullet \) as an object in \( s\mathcal{M}/B\Lambda \). Following [2,42], we define the \( n \)-th cohomology group of \( W_\bullet \) with coefficients in \( K \) to be

\[
H^n_\Lambda(W_\bullet; K) := \left[ W_\bullet, E_\Lambda(K, n) \right]_{s\mathcal{M}/B\Lambda} = \pi_0 \text{map}_{s\mathcal{M}/B\Lambda}(W_\bullet, E_\Lambda(K, n)),
\]

where the last mapping space is defined by the (homotopy) pullback:

\[
\begin{array}{ccc}
\text{map}_{s\mathcal{M}/B\Lambda}(W_\bullet, E_\Lambda(K, n)) & \to & \text{map}_{s\mathcal{M}}(W_\bullet, E_\Lambda(K, n)) \\
\downarrow & & \downarrow \\
\{ Bt \circ p^{(0)}_{W_\bullet} \} & \to & \text{map}_{s\mathcal{M}}(W_\bullet, B\Lambda).
\end{array}
\]

Typically, we have \( \Lambda = \pi_0^A W_\bullet \), with \( \iota \) an isomorphism; if in addition \( W_\bullet \simeq B\Lambda \), we denote \( H^n_\Lambda(W_\bullet; K) \) simply by \( H^n(\Lambda; K) \).

2.2. Definition. There is also a relative version, for a pair \((W_\bullet, Y_\bullet)\) — that is, a cofibration \( i : Y_\bullet \hookrightarrow W_\bullet \in s\mathcal{M} \) with \( \mathcal{K} \) a \( \Lambda \) module and \( t : \pi_0^A Y_\bullet \to \Lambda \) a twisting map as before. Let \( \mathcal{P}O(W_\bullet, Y_\bullet) \) denote the (homotopy) pushout in \( s\mathcal{M} \) of:

\[
Y_\bullet \xrightarrow{i} W_\bullet \xrightarrow{r} B\pi_0^A Y_\bullet \xrightarrow{j} \mathcal{P}O(W_\bullet, Y_\bullet).
\]

We define \( H^n_\Lambda(W_\bullet, Y_\bullet; \mathcal{K}) \) to be the group of homotopy classes of maps \( f : \mathcal{P}O(W_\bullet, Y_\bullet) \to (E_\Lambda(K, n), B\Lambda) \) in \( s\mathcal{M}/B\Lambda \) fitting into the commutative diagram:

\[
\begin{array}{ccc}
Y_\bullet & \xrightarrow{i} & W_\bullet \\
\downarrow r & & \downarrow j \\
B\pi_0^A Y_\bullet & \xrightarrow{j} & \mathcal{P}O(W_\bullet, Y_\bullet) \\
\downarrow & & \downarrow f \\
Bt & \xrightarrow{s} & E_\Lambda(K, n)
\end{array}
\]

(see [24, §2.1]).

2.3. Remark. Let \( \Theta \) be a \( \mathcal{G} \)-theory and \( \mathcal{M} = \Theta\text{-Alg} \), so \( \Pi_{\mathcal{A}}^{\mathcal{M}}\text{-Alg} \approx \mathcal{M} \) (cf. Section 1.14). The constant object \( c(\Lambda) \) is a fibrant model for \( B\Lambda \), so \( s\mathcal{M}/B\Lambda \cong \mathcal{M}/\Lambda \). In particular, this implies that \( s((\mathcal{M}/\Lambda)_{\text{ab}}) \) may be identified with the category \( (s\mathcal{M}/B\Lambda)_{\text{ab}} \) of abelian group objects in \( s\mathcal{M}/B\Lambda \), with abelianization functor \( \text{Ab}_\Lambda : (s\mathcal{M}/B\Lambda) \to (s\mathcal{M}/B\Lambda)_{\text{ab}} \) defined dimension-wise (cf. [12, §3.20]).
2.4. Proposition. Let $\Theta$ be a $\mathfrak{G}$-theory and $\mathcal{M} = \Theta -$Alg. For any $\Lambda \in \mathcal{M}$, $\Lambda$-module $\mathcal{K}$, cofibrant $W_\bullet \in s\mathcal{M}$ and twisting map $t$, we have chain complexes $\check{C}_* W_\bullet$ and $\check{E}_*'$ and maps

$$\text{Hom}_{s\mathcal{M}/BA}(W_\bullet, E_\Lambda(\mathcal{K}, n)) \xleftarrow{\xi} \text{Hom}_{\mathcal{M}/A}(\check{C}_* W_\bullet, \check{E}_*') \rightarrow \text{Hom}_{\mathcal{M}/A}(C_* \mathcal{A}b_\Lambda W_\bullet, \mathcal{K}),$$

natural in $W_\bullet$, inducing an isomorphism $H^n_{\Lambda}(W_\bullet; \mathcal{K}) \cong H^n\text{Hom}_{\mathcal{M}/\Lambda}(C_\ast A\mathcal{B} \Lambda W_\bullet, \mathcal{K})$ for each $n \geq 0$.

Proof. Step I. First, we show how $H^n_{\Lambda}(W_\bullet; \mathcal{K})$ may be described in terms of a mapping space of chain complexes:

Since $E := E_\Lambda(\mathcal{K}, n)$ can be chosen to be a strict abelian group object in $s\mathcal{M}/BA$ (see [15, §3.14]), we have a natural identification:

$$\text{map}_{s\mathcal{M}/BA}(W_\bullet, E) \cong \text{map}_{\mathcal{M}/BA}(A\mathcal{B} \Lambda W_\bullet, E).$$

Recall that the Moore chain functor $C_* : s(\mathcal{M}/\Lambda)_{ab} \rightarrow \text{Ch}((\mathcal{M}/\Lambda)_{ab})$ induces the Dold–Kan equivalence between simplicial objects and chain complexes over an abelian category (cf. [22, §1]). Composing with the equivalence $(\mathcal{M}/\Lambda)_{ab} \xrightarrow{\cong} \Lambda \text{-Mod}$ of Section 1.5 defines $C_* : s(\mathcal{M}/\Lambda)_{ab} \rightarrow \text{Ch}(\Lambda \text{-Mod})$.

Applying $C_* \text{to the simplicial } \Lambda$-module $A\mathcal{B} \Lambda W_\bullet$ yields $D_* := C_* A\mathcal{B} \Lambda W_\bullet$, with

$$\text{map}_{s\mathcal{M}/BA}(W_\bullet, E) \cong \text{map}_{\mathcal{M}/BA}(D_*, \check{C}_* E),$$

so applying $\pi_0$ yields the required cohomology group.

Step II. We now translate this into chain function complexes:

Both sides of (2.5) are simplicial abelian groups, so we can replace the right-hand side under the Dold–Kan equivalence with the usual mapping chain complex $F_* := \text{Hom}(D_*, \check{C}_* E)$, where $F_0 := \text{Hom}_{\mathcal{M}/\Lambda}(D_*, \check{C}_* E)$ and $F_1 := \text{Hom}_{\mathcal{M}/\Lambda}(D_*, P \check{C}_* E)$ for $P \check{C}_* E$ the path object on $\check{C}_* E$. Moreover, the differential $\partial_1 : \text{Hom}_j \rightarrow \text{Hom}_0$ is induced by the path fibration $p : P \check{C}_* E \rightarrow \check{C}_* E$.

Since (2.5) induces an identification: $\pi_0 \text{map}_{s\mathcal{M}/BA}(W_\bullet, E) \cong H_0\text{Hom}(\check{D}_*, \check{C}_* E)$, we have a right-exact sequence:

$$\text{Hom}_{\mathcal{M}/\Lambda}(\check{D}_*, P \check{C}_* E) \xrightarrow{p_*} \text{Hom}_{\mathcal{M}/\Lambda}(\check{D}_*, \check{C}_* E) \rightarrow H^n_{\Lambda}(W_\bullet; \mathcal{K}) \rightarrow 0.$$  (2.6)

Here $p : P \check{C}_* E \rightarrow \check{C}_* E$ has the obvious minimal model:

$$\begin{array}{cccccc}
\check{P}_* & = & \ldots & 0 & \rightarrow & \check{K} \\
\downarrow q & & & & & \downarrow = \\
\check{E}_* & = & \ldots & 0 & \rightarrow & 0 \ldots \\
\dim & n + 1 & n & n - 1 & n - 2
\end{array}$$

since $H_i \check{C}_* E \cong H_i \check{C}_* E \cong \pi_i E$, which is $\check{K}$ for $i = n$ and 0 otherwise (note that $\check{A} = 0$!).
This means that we can choose a cofibrant model $E'_* \rightarrow C_*$ fitting into a span – that is, a commuting diagram:

$$
\begin{array}{ccc}
P C_* E & \xrightarrow{p'} & P E'_* \\
\downarrow p & & \downarrow p' \\
C_* E & \xrightarrow{\zeta} & E'_* \xrightarrow{\eta} E_*.
\end{array}
$$

The horizontal weak equivalences $\zeta, \eta, P \zeta, \text{and} P \eta$ induce a span of quasi-isomorphisms from (2.6) to

$$
\text{Hom}_{\mathcal{C}(\Lambda\text{-Mod})}(D_*, P_*) \xrightarrow{q_*} \text{Hom}_{\mathcal{C}(\Lambda\text{-Mod})}(D_*, E_*) \xrightarrow{\sim} H^n_A(W_*; K) \rightarrow 0,
$$

which are both the identity on $H^n_A(W_*; K)$.

Step III. Finally, we produce a commuting diagram:

$$
\begin{array}{ccc}
\text{Hom}_{\mathcal{C}(\Lambda\text{-Mod})}(D_*, P_*) & \xrightarrow{q_*} & \text{Hom}_{\mathcal{C}(\Lambda\text{-Mod})}(D_*, E_*) \\
\delta^{n-1} = (\partial n)^* & \text{and} & \beta
\end{array}
\xrightarrow{n-1} Z^n D^*
$$

where the dual cochain complex $D^*$ is defined by applying $\text{Hom}_{\mathcal{C}(\Lambda\text{-Mod})}(-, K)$ dimension-wise to $D_*$, and $Z^n D^*$ are its $n$-cocycles.

To describe $\alpha$, note that a chain map $f$ in $\text{Hom}_{\mathcal{C}(\Lambda\text{-Mod})}(D_*, P_*)$ is given by a commuting diagram:

$$
\begin{array}{ccc}
\cdots & \xrightarrow{\partial n} & \cdots \\
\downarrow \psi \circ \partial n & & \downarrow \psi \\
0 & \xrightarrow{=} & 0
\end{array}
$$

so that $\alpha(f) := \psi$.

Similarly, a chain map $g$ in $\text{Hom}_{\mathcal{C}(\Lambda\text{-Mod})}(D_*, E_*)$ is given by a commuting diagram:

$$
\begin{array}{ccc}
\cdots & \xrightarrow{\partial n} & \cdots \\
\downarrow \phi & & \downarrow \\
0 & \xrightarrow{=} & 0
\end{array}
$$

with $\beta(g) := \phi$; indeed, $\beta(g) \in Z^n D^*$ since $\phi \circ \partial n+1 = 0$. 

Since (2.10) clearly commutes, and the cokernel of the bottom map is $H^n_A D^*$, we conclude from (2.9) and the identification $\text{Hom}_{\mathcal{M}/\Lambda}(-, \mathcal{K}) = \text{Hom}_{(\mathcal{M}/\Lambda)_{ab}}(-, \mathcal{K})$ that:

$$H^n_A(W_s; \mathcal{K}) = H^n(\text{Hom}_{\mathcal{A}-\text{Mod}}(\overline{C}_s A W_s, \overline{\mathcal{K}})) \cong H^n(\text{Hom}_{\mathcal{M}/\Lambda}(C_s A W_s, \mathcal{K})), $$

again using the equivalence of categories $\mathcal{A}-\text{Mod} \approx (\mathcal{M}/\Lambda)_{ab}$ of Section 1.5, and the fact that $\overline{\mathcal{K}}$ is an abelian group object in $\mathcal{M}/\Lambda$.  \qed

2.11. Lemma. Under the assumptions of Proposition 2.4, if $W_s$ has a CW-basis $(\overline{W}_n)_{n=0}^\infty$ (Section 1.20), then for each $n > 0$ the natural map $\overline{W}_n \rightarrow C_n A^0 A W_s$ induces an isomorphism

$$\text{Hom}_{\mathcal{A}}(C_n A^0 A W_s, \mathcal{K}) \rightarrow \text{Hom}_{\mathcal{M}}(\overline{W}_n, \overline{\mathcal{K}}).$$

Proof. Let $s_I(n) : W_0 \rightarrow W_n$ be the unique iterated degeneracy map, making $W_0$ a coproduct summand in $L_n W_s \cong L_n W_s \sqcup W_0$, where by (1.21), $L_n' W_s$ is a coproduct of the images under various iterated degeneracies of (copies of) the basis objects $(\overline{W}_n)_{n=1}^{\infty}$.

Since the structure map $W_n \rightarrow \Lambda$ for the augmentation $W_s \rightarrow c(\Lambda)$ factors through the retract $W_n \rightarrow W_0$ for $s_I(n)$, in fact $W_n \rightarrow \Lambda$ is a coproduct in $\mathcal{M}/\Lambda$ of $0 : \overline{W}_n \sqcup L_n W_s \rightarrow \Lambda$ and $\varepsilon : W_0 \rightarrow \Lambda$, where the first summand further splits as a coproduct of objects of the form $0 : \overline{W}_j \rightarrow \Lambda$.

Because the abelianization functor $A^0 A : \mathcal{M}/\Lambda \rightarrow (\mathcal{M}/\Lambda)_{ab}$ is a left adjoint, it commutes with coproduts, and when applied to $0 : \Lambda \rightarrow \Lambda$ yields $0 : A^0 A \Lambda \rightarrow A^0 A \Lambda$, where $A^0 A : \mathcal{M} \rightarrow \mathcal{M}_{ab}$ is the usual abelianization of $\Theta$-algebras.

$$A^0 A W_n = A^0 A(\overline{W}_n \sqcup L_n W_s) \sqcup A^0 A W_0 \cong (A^0 A \overline{W}_n \sqcup A^0 A L_n' (W_s)) \sqcup A^0 A W_0$$

$$\cong (A^0 A \overline{W}_n \sqcup L_n' (A^0 A W_s)) \sqcup A^0 A W_0 \cong A^0 A \overline{W}_n \sqcup A^0 A L_n (A^0 A W_s)$$

where the last coproduct is actually the direct sum

$$A^0 A \overline{W}_n \oplus L_n (A^0 A W_s) \quad (2.12)$$

in the abelian category $(\mathcal{M}/\Lambda)_{ab}$, and all the abelianizations are applied dimensionwise (cf. Section 2.3).

This implies that

$$C_n A^0 A W_s \cong N_n (A^0 A W_s) = A^0 A \overline{W}_n = A^0 A \overline{W}_n, \quad (2.13)$$

since over the abelian category $(\mathcal{M}/\Lambda)_{ab}$ the Moore chains $C_s$ can be identified with the normalized chains $N_s : s(\mathcal{M}/\Lambda)_{ab} \rightarrow \text{Ch}((\mathcal{M}/\Lambda)_{ab})$, where by definition, $N_n A_s = A_n / L_n A_s$ for $A_s \in s(\mathcal{M}/\Lambda)_{ab}$ (cf. (2.12) and [22, (1.12)]).

The abelianization $A^0 A \overline{W}_n$ of 0 : $\overline{W}_n \rightarrow \Lambda$, is just 0 : $A^0 A \overline{W}_n \rightarrow \Lambda$, where $A^0 A \overline{W}_n$ has a trivial restricted $\Lambda$-module structure (cf. Section 1.5). Therefore maps into $(p : \mathcal{K} \rightarrow \Lambda)$ (over $\Lambda$) factor through $\text{Ker} p \equiv \overline{\mathcal{K}}$. Applying $\text{Hom}_A (-, \mathcal{K})$ to (2.13) thus yields:
\[
\text{Hom}_A(C_n Ab A W_\bullet, K) \cong \text{Hom}_A(\text{Ab}_A W_n, K) \cong \text{Hom}_{A-\text{Mod}}(\text{Ab} W_n, \overline{K}) \\
\cong \text{Hom}_{A_{\text{ab}}}(\text{Ab} W_n, \overline{K}) \cong \text{Hom}_{A}(W_n, \overline{K}),
\]
as required. \(\square\)

Combining Lemma 2.11 and Proposition 2.4, we have the following analogue of the cellular cohomology of a CW-complex (in \(T\)):

2.14. Corollary. For \(W_\bullet\) as above, every cohomology class in \(H^n_A(W_\bullet; K)\) is determined by a map of \(\Theta\)-algebras \(\phi : \overline{W}_n \to \overline{K}\).

Notice the map \(\phi : \overline{W}_n \to \overline{K}\) represents zero in \(H^n_A(W_\bullet; K)\) if and only if there is a commuting diagram in \(\mathcal{M}\):

\[
\begin{array}{c}
\overline{W}_n \\
\downarrow \phi \\
\overline{K}
\end{array} \xymatrix{ & W_{n-1} \\
& \overline{K} \ar[u]^i \\
\overline{K} \ar[r]^(0.6){\rho} & \Lambda 
\end{array}
\]

2.15. Description of \(k\)-invariants

Let \(K_\bullet \in s\mathcal{M}\) for \(\mathcal{M} = \Theta-\text{Alg}\) as above, with \(\mathbb{P}^n K_\bullet\) its \(n\)-th Postnikov section. Recall from (1.16) that the functorial \(n\)-th \(k\) invariant \(k_n \in H^{n+2}(\mathbb{P}^n K_\bullet; \pi_{n+1} K_\bullet)\) fits into a homotopy pullback square:

\[
\begin{array}{c}
\mathbb{P}^{n+1} K_\bullet \\
\downarrow \text{PB} \\
B A
\end{array} \xymatrix{ & \mathbb{P}^n K_\bullet \\
& E_A(\pi_{n+1} K_\bullet, n + 2) \\
B A \ar[r] & E_A(\pi_{n+1} K_\bullet, n + 2) 
\end{array}
\]

for \(A = \pi_0 K_\bullet\). This is constructed as in [14, §6] by first taking the homotopy pushout \(Z\) of the upper left corner of (2.16), and then noting that \(\mathbb{P}^{n+2} Z \simeq E_A(\pi_{n+1} K_\bullet, n + 2)\).

We can represent \(k_n\) for \(K_\bullet\) by the map in \(\mathcal{M}/A\):

\[
b : (\mathbb{P}^n K_\bullet)_{n+2} \to \pi_{n+1} K_\bullet
\]

which sends any \((n + 2)\)-simplex \(\sigma \in (\text{csk}_{n+1} K_\bullet)_{n+2}\) to the class in \(\pi_{n+1} K_\bullet\) represented by the matching collection of \((n + 1)\)-faces

\[
(d_{0}^{n+2}\sigma, \ldots, d_{n+2}^{n+2}\sigma) \subseteq (\text{csk}_{n+1} K_\bullet)_{n+1} = K_{n+1}.
\]

Note that this collection need no longer have an \((n + 2)\)-dimensional fill-in in \(K_\bullet\), but it does represent a map from the boundary of an \((n + 2)\)-simplex into \(K_\bullet\), and so an element of \(\pi_{n+1} K_\bullet\).
In forming the homotopy pushout, we actually replace the map \( p^{(n+1)} \) which is the identity up through dimension \( n + 1 \), by a cofibration, by adding a \( k \)-simplex to \( \mathbf{P}^n \mathcal{K}_n \) for each non-unique filler of a matching collection of \( (n + 1) \)-faces in \( \mathbf{P}^{n+1} \mathcal{K}_n \) for \( k \geq n + 2 \). However, any matching collection in \( \mathbf{P}^{n+1} \mathcal{K}_n \) with a filler represents zero in \( \pi_{n+1} \mathcal{K}_n \) so this does not affect \( [b] \), while the above description remains valid otherwise.

Since \( \pi_{n+1} \mathcal{K}_n \) is an abelian group object in \( \mathcal{M}/\mathcal{A} \) for \( \Lambda = \pi_0 \mathcal{K}_n \), the map \( b \) factors through the abelianization \( \mathcal{A} \mathcal{b}_n(\mathbf{P}^n \mathcal{K}_n)_{n+2} \), so we can apply Proposition 2.4 to produce a cohomology class.

3. Rectification of diagrams and higher homotopy operations

Higher homotopy or cohomology operations have been studied extensively since they were first discovered over fifty years ago, but there is still no completely satisfactory theory that adequately covers all known examples and explains their properties. In the approach we take here, based on that of [17] (as modified in [13]), they appear as “geometric” obstructions to realizing homotopy-commutative diagrams.

3.1. The rectification problem

Let \( \mathcal{M} \) be a model category, and \( \mathcal{D} \) a small category. We start with a functor \( \tilde{X} : \mathcal{D} \to \text{ho} \mathcal{M} \), which we would like to rectify – that is, lift to a functor \( X : \mathcal{D} \to \mathcal{M} \).

By definition, we can choose a function \( X_{\text{arr}} : \text{Arr} \mathcal{D} \to \text{Arr} \mathcal{M} \) which assigns to each arrow \( \phi : a \to b \) in \( \mathcal{D} \) a map \( X_{\text{arr}}(\phi) : \tilde{X}(a) \to \tilde{X}(b) \) representing \( \tilde{X}(\phi) \). Moreover, for each two composable arrows \( a \xrightarrow{\phi} b \xrightarrow{\psi} c \) in \( \mathcal{D} \) we can choose a homotopy \( H(\psi, \phi) : X_{\text{arr}}(\psi \circ \phi) \sim X_{\text{arr}}(\psi) \circ X_{\text{arr}}(\phi) \). The idea is that if we can choose these homotopies compatibly, in an appropriate sense, then the diagram \( \tilde{X} \) can be rectified; and that higher homotopy operations arise as the obstructions to making such a choice.

To make this precise, we need suitable function complexes for \( \mathcal{M} \), in which to house the higher homotopies: we work here with the more familiar simplicial enrichment of \( \mathcal{M} \) (although the cubical version is more economical) – more precisely, with the pointed version, enriched in \( S_* \).

In fact, we only need the mapping spaces between the objects of \( \mathcal{M} \) which are in the image of \( \tilde{X} \), so let \( \mathcal{M}_0 \) denote the \((S_*, \mathcal{O})\)-category with object set \( \mathcal{O} := \text{Obj} \mathcal{D} \) and map \( \pi_0(\mathcal{M}) \) of \( \mathcal{M}_0 \) may be thought of as a subcategory of the original \( \text{ho} \mathcal{M} \), and we can replace \( \tilde{X} \) by a functor \( \tilde{X} : \mathcal{D} \to \mathcal{M}_0 \). If we think of \( \mathcal{D} \) as the constant \((S_*, \mathcal{O})\)-category \( c(\mathcal{D}) \), rectifying \( \tilde{X} \) is equivalent to lifting \( \tilde{X} \) to an \((S_*, \mathcal{O})\)-functor \( \tilde{X} : c(\mathcal{D}) \to \mathcal{M}_0 \).

In [23, §1], Dwyer and Kan define a simplicial model category structure on \((S, \mathcal{O})\)-\textit{Cat}, also valid for \((S_*, \mathcal{O})\)-\textit{Cat} (cf. [37, Prop. 1.1.8]), in which the fibrations and weak equivalences are defined objectwise (that is, on each mapping space \( S_*(a, b) \)). Thus the above lifting problem can be stated in a homotopically meaningful way if we use a cofibrant replacement for \( c(\mathcal{D}) \), and require \( \mathcal{M}_0 \) to be fibrant (which just means that each mapping space of \( \mathcal{M}_0 \) is a Kan complex).

The cofibrant objects in \((S_*, \mathcal{O})\)-\textit{Cat} are not easy to describe, in general. However, a canonical cofibrant replacement for \( c(\mathcal{D}) \) is given by the simplicial category \( F_i \mathcal{D} \) obtained by iterating the comonad \( F_\mathcal{O} : \mathcal{O}\text{-Cat} \to \mathcal{O}\text{-Cat} \), where the free category functor \( F : \mathcal{D} \mathcal{S}_* \to \mathcal{C} \text{at} \) is left adjoint to the forgetful functor \( U : \mathcal{C} \text{at} \to \mathcal{D} \mathcal{S}_* \) to the category of directed pointed graphs (that is, directed graphs in which the set of arrows from one vertex to another is pointed).
A lift of $\tilde{X} : D \to \pi_0 \mathcal{M}_\emptyset$ to an $(S_\emptyset, \emptyset)$-functor $\hat{X} : F_\emptyset D \to \mathcal{M}_\emptyset$ is called an $\infty$-homotopy commuting version of the original $\tilde{X}$, and by [18, Theorem IV.4.37] or [25, Theorem 2.4], its existence is equivalent (in a homotopy-invariant sense) to solving the original rectification problem, in the case where $\mathcal{M} = T_\emptyset$ or $S_\emptyset$.

3.2. The inductive process of rectification

Following [25], we wish to construct such an $\infty$-homotopy commuting lift $\hat{X}$ of $\tilde{X}$ by induction over the Postnikov sections (applied to each simplicial mapping space of the target category $\mathcal{M}$). For our purposes we need the relative version, in which $\mathcal{C}$ is a subcategory of the given indexing category $\mathcal{D}$, and $\tilde{X}|\mathcal{C}$ has already been rectified.

Let $\emptyset = \text{Obj} \mathcal{C}$ and $\emptyset_+ := \text{Obj} \mathcal{D}$, and note that the inclusion $i : \mathcal{C} \hookrightarrow \mathcal{D}$ induces a cofibration $F_\emptyset i : F_\emptyset \mathcal{C} \hookrightarrow F_\emptyset \mathcal{D}$ in $(S_\emptyset, \emptyset_+)$-$\text{Cat}$ (where we think of an $(S_\emptyset, \emptyset)$-category as an $(S_\emptyset, \emptyset_+)$-category by extending trivially). Thus the following pushout in $(S_\emptyset, \emptyset_+)$-$\text{Cat}$:

$$
\begin{array}{c}
F_\emptyset \mathcal{C} \\
\downarrow r
\end{array}
\begin{array}{c}
\xrightarrow{F_\emptyset i}
F_\emptyset \mathcal{D}
\end{array}
\begin{array}{c}
\downarrow s
\end{array}
\begin{array}{c}
c(\mathcal{C}) \\
\downarrow c(i)
\end{array}
\begin{array}{c}
\xrightarrow{F_\emptyset j}
F_\emptyset (\mathcal{D}, \mathcal{C})
\end{array}
$$

is in fact a homotopy pushout (and the vertical maps are weak equivalences).

We begin the induction with an $(S_\emptyset, \emptyset)$-functor $X : c(\mathcal{C}) \to \mathcal{M}_\emptyset$ (a rectification of $\tilde{X}|\mathcal{C}$), and a compatible $(S_\emptyset, \emptyset_+)$-functor $X_0 : F_\emptyset \mathcal{D} \to \mathbf{P}^0 \mathcal{M}_\emptyset$ lifting $\tilde{X}$, which together induce $\hat{X}^0 : F_\emptyset (\mathcal{D}, \mathcal{C}) \to \mathbf{P}^0 \mathcal{M}_\emptyset$.

Now assume by induction on $n > 0$ that we can lift $\hat{X}^n$ to $\hat{X}^{n-1} : F_\emptyset (\mathcal{D}, \mathcal{C}) \to \mathbf{P}^{n-1} \mathcal{M}_\emptyset$, making the following diagram commute:

$$
\begin{array}{c}
F_\emptyset \mathcal{C} \\
\downarrow r
\end{array}
\begin{array}{c}
\xrightarrow{F_\emptyset i}
F_\emptyset \mathcal{D}
\end{array}
\begin{array}{c}
\downarrow s
\end{array}
\begin{array}{c}
c(\mathcal{C}) \\
\downarrow X
\end{array}
\begin{array}{c}
\xrightarrow{F_\emptyset j}
F_\emptyset (\mathcal{D}, \mathcal{C})
\end{array}
\begin{array}{c}
\xrightarrow{\hat{X}^{n-1}}
\mathcal{M}_\emptyset
\end{array}
\begin{array}{c}
\xrightarrow{j}
\mathcal{M}_\emptyset
\end{array}
\begin{array}{c}
\xrightarrow{q^n}
\mathbf{P}^{n-1} \mathcal{M}_\emptyset
\end{array}
$$

and our goal is to identify the obstruction to lifting $\hat{X}^{n-1}$ to $\hat{X}^n : F_\emptyset (\mathcal{D}, \mathcal{C}) \to \mathbf{P}^n \mathcal{M}_\emptyset$.

3.5. The Dwyer–Kan–Smith obstruction theory

Although $(S_\emptyset, \emptyset)$-$\text{Cat}$ (the category of simplicially enriched categories with fixed object set $\emptyset$) is not quite a resolution model category as defined here, Dwyer and Kan have shown that it has a notion of $(S_\emptyset, \emptyset)$-cohomology, represented by Eilenberg–Mac Lane objects $\mathcal{E}_\mathcal{D}(\mathcal{K}, n)$ in
(\mathcal{S}_n, \mathcal{O})\text{-}Cat, defined much as in Section 2.1 (if we replace \(\Pi_{\mathcal{A}}\)-algebras by \((\mathcal{S} pd, \mathcal{O})\)-categories as the target of \(\pi_0^A\) throughout), including a relative version \(H^*_{SO}(\mathcal{D}, \mathcal{C}; \kappa)\).

We can thus use \(\hat{\mathcal{E}}^n\) to pull back the \((n - 1)\)-st \(k\)-invariant \(k_{n-1} : \mathcal{P}^n_1 \mathcal{M}_{\mathcal{O}+} \to \mathcal{E}_\mathcal{D}(\pi_n \mathcal{M}_{\mathcal{O}^+}, n + 1)\) for \(\mathcal{M}_{\mathcal{O}^+}\), as in (1.16), to a map \(h_{n-1} := k_{n-1} \circ \hat{\mathcal{E}}^{n-1} : F_\mathcal{S}(\mathcal{D}, \mathcal{C}) \to \mathcal{E}_\mathcal{D}(\pi_n \mathcal{M}_{\mathcal{O}^+}, n + 1)\), and deduce that the map \(\hat{\mathcal{E}}^{n-1}\) lifts to \(\hat{\mathcal{E}}^n\) if and only if \([h_{n-1}]\) vanishes in \(H^{n+1}_{SO}(\mathcal{D}, \mathcal{C}; \pi_n \mathcal{M}_{\mathcal{O}^+})\) (see [25, Proposition 4.8]).

Our goal is to replace these \((\mathcal{S}_n, \mathcal{O})\)-cohomology obstructions by geometrically defined higher homotopy operations, which we can then identify in cases of interest with the André–Quillen cohomology obstructions of [14]. For this purpose, we must restrict attention to a more limited class of indexing categories \(\mathcal{D}\), defined as follows:

**3.6. Definition.** A finite non-unital category \(\Gamma\) will be called a *lattice* if it has no self-maps and is equipped with a *(weakly) initial* object \(v_{\text{init}}\) and a *(weakly) final* object \(v_{\text{fin}}\), such that there is a unique \(\phi_{\text{max}} : v_{\text{init}} \to v_{\text{fin}}\). We say that \(\Gamma\) is *pointed* if it is enriched in pointed sets. In this case necessarily \(\phi_{\text{max}} = \ast\).

A composable sequence of \(n\) arrows in \(\Gamma\) will be called an *\(n\)-chain*. For any finite category \(\Gamma\), the maximal occurring \(n\) is its *length*. When \(\Gamma\) is a lattice, this is necessarily for a chain from \(v_{\text{init}}\) to \(v_{\text{fin}}\), factorizing \(\phi_{\text{max}}\).

**3.7. Example.** The simplest pointed lattice of interest to us is the *Toda lattice* of length 3:

\[
v_{\text{init}} \xrightarrow{f} u \xrightarrow{g} w \xrightarrow{h} v_{\text{fin}}.
\]

**3.8. Higher homotopy operations**

Now let \(\mathcal{D} = \Gamma^+\) be a (pointed) lattice, with objects \(\mathcal{O}_+ := \text{Obj} \Gamma^+\), and \(\mathcal{C} = \Gamma\) a subcategory of \(\Gamma^+\) with object set \(\mathcal{O} \subseteq \mathcal{O}_+\). We assume given a (pointed) diagram \(\tilde{X} : \Gamma^+ \to \text{ho}\mathcal{M}\) for an \(\mathcal{S}_n\)-category \(\mathcal{M}\), which we wish to rectify as above. In the cases of interest to us \(\mathcal{O}_+ \setminus \mathcal{O}\) will consist of a single object (either \(v_{\text{init}}\) or \(v_{\text{fin}}\)), and the maps in \(\Gamma^+\) which are not in \(\Gamma\) will be (essentially) only zero maps.

In the approach of [17,13], we try to extend an \((\mathcal{S}_n, \mathcal{O})\)-functor \(X : c(\mathcal{C}) \to \mathcal{M}_\mathcal{O}\) to \(\tilde{X} : F_\mathcal{S}(\Gamma^+, \Gamma) \to \mathcal{M}_{\mathcal{O}^+}\) (see Section 3.1), by induction over the skeleta of \(F_\mathcal{S}(\Gamma^+, \Gamma)\). When \(\mathcal{M}_{\mathcal{O}^+}\) is fibrant, this is essentially the same as the induction in Section 3.2, since the \(k\)-coskeleton functor is a \((k - 1)\)-Postnikov section for a fibrant \((\mathcal{S}_n, \mathcal{O})\)-category. The obstruction to extending to the \((k - 1)\)-stage thus lies in a set of relative homotopy classes of \((\mathcal{S}_n, \mathcal{O}_+)\)-functors from \((\text{sk}_{k+1} F_\mathcal{S}(\Gamma^+, \Gamma), \text{sk}_k F_\mathcal{S}(\Gamma^+, \Gamma))\) to \(\mathcal{M}_{\mathcal{O}^+}\), which are in general hard to describe. However, if \(n + 1\) is the length of \(\Gamma^+\), then \(F_\mathcal{S}(\Gamma^+, \Gamma)\), and thus \(F_\mathcal{S}(\Gamma^+, \Gamma)\), is \(n\)-dimensional, and in this case the last obstruction is adjoint to a wedge of maps \(\Sigma^{n-1} \tilde{X}(v_{\text{init}}) \to \tilde{X}(v_{\text{fin}})\) in \(\mathcal{M}\) (see [16, Proposition 3.21]). We then define the associated \(n\)-th order higher homotopy operation associated to \(\tilde{X}\) to be the collection of elements:

\[
\langle \langle \tilde{X} \rangle \rangle \subseteq \left[ \Sigma^{n-1} \tilde{X}(v_{\text{init}}), \tilde{X}(v_{\text{fin}}) \right]_{\text{ho}\mathcal{M}}.
\]
obtained in this way from all possible extensions of $X$ to $\sk_{n-1} F_{\Lambda} (\Gamma_{+}, \Gamma)$. Each such element is called a value of $\langle \langle X \rangle \rangle$.

In the cases of interest to us here, $\mathcal{M}$ is equipped with a collection $\mathcal{A}$ of homotopy cogroup objects and $\widetilde{X} (v_{\text{init}})$ is in $\mathcal{M}_{\mathcal{A}}$ (Section 1.1) so $\langle \langle X \rangle \rangle$ takes values in the $\mathcal{A}$-homotopy groups of $\widetilde{X} (v_{\text{init}})$.

In [16, Theorem 4.14] the set $\langle \langle X \rangle \rangle$ was shown to be equivalent (for $\mathcal{D} = \Gamma_{+}$ a lattice) to the set of $(\mathcal{S}, \emptyset)$-cohomology classes appearing as the final obstructions to rectification in [25, Theorem 2.4] – with a particular cohomology class associated to each value of $\langle \langle X \rangle \rangle$ in such a way that they vanish simultaneously.

Note that the obstruction theory of [17,13] is actually defined for $\mathcal{M}$ enriched in (pointed) cubical sets (rather than in $\mathcal{S}$), using Boardman and Vogt’s $W$-construction instead of $F_{\Lambda} \mathcal{D}$ (which is in fact a canonical triangulation thereof).

4. The André–Quillen cohomology obstructions

In this section, we study the André–Quillen cohomological existence obstructions to realizing an abstract $\Pi_{\mathcal{A}}$-algebra $\Lambda$, with a view to comparing them to the higher homotopy operation obstructions described in the next section.

4.1. The general setting

Given an $E^{2}$-model category $\mathcal{M}$ (Section 1.1.5) with a collection of spherical objects $\mathcal{A}$, and an abstract $\Pi_{\mathcal{A}}$-algebra $\Lambda$, one can try to realize it by finding an object $X \in \mathcal{M}$ with $\pi_{\mathcal{A}} \Gamma X \cong \Lambda$.

For this purpose, we try to construct a cofibrant object $V_{\ast}$ in $s \mathcal{M}$ realizing a free simplicial resolution $G_{\ast} \rightarrow \Lambda$ in $s \Pi_{\mathcal{A}} \mathcal{A}$-Alg, (i.e. $\pi_{\mathcal{A}} \Gamma V_{\ast} = G_{\ast}$) and (1.17) then implies that one can choose $X = J V_{\ast}$. On the other hand, we know simplicial resolutions exist in both $s \Pi_{\mathcal{A}} \mathcal{A}$-Alg and $s \mathcal{M}$, and it follows from [10, Proposition 3.13] that if $\Lambda$ is realizable, any choice of $G_{\ast}$ must also be realizable in the sense that there is a CW object $V_{\ast} \in s \mathcal{M}$ with $\pi_{\mathcal{A}} \Gamma V_{\ast} \cong G_{\ast}$. Thus, the realization problem for $\Lambda$ is reduced to one of realizing CW objects in $s \Pi_{\mathcal{A}} \mathcal{A}$-Alg.

Therefore, assume given a free simplicial $\Pi_{\mathcal{A}} \mathcal{A}$-algebra resolution $G_{\ast}$ of a $\Pi_{\mathcal{A}} \mathcal{A}$-algebra $\Lambda$ with CW-basis $\{ \varpi_{n} \}_{n=0}^{\infty}$ and $(n+2)$-attaching map $d_{0}^{\varpi_{n+2}} : \varpi_{n+2} \rightarrow \mathbb{Z}_{n+1} G_{\ast} \subset C_{n+1} G_{\ast}$. In our inductive approach, we also assume given an $(n+1)$-truncated realization $V_{\ast}^{(n+1)}$ of $G_{\ast}$: that is, the $(n+1)$-trunctations $\tau_{n+1} \pi_{\mathcal{A}} \Gamma V_{\ast}^{(n+1)}$ and $\varpi_{n+1} G_{\ast}$ are isomorphic. We also choose a map $d_{0}^{\varpi_{n+2}} : \varpi_{n+2} \rightarrow C_{n+1} V_{\ast}^{(n+1)}$ realizing the attaching map $d_{0}^{\varpi_{n+2}} : \varpi_{n+2} \rightarrow \mathbb{Z}_{n+1} G_{\ast}$.

We may assume $V_{\ast}^{(n+1)}$ is Reedy fibrant (as a truncated simplicial object). If we apply the $(n-1)$-Postnikov section functor to $V_{\ast}^{(n+1)}$, we obtain a (full) simplicial object $W_{\ast}^{(n)} := \text{csk}_{n} V_{\ast}^{(n+1)}$ (cf. Section 1.18).

To start the induction, note that each basis object $\varpi_{n}$, and thus each $G_{n}$, is free, so if we choose any realization for $d_{0}^{\varpi_{1}} : \varpi_{1} \rightarrow G_{0}$, we obtain a strict 1-truncated realization $V_{\ast}^{(1)}$ of $G_{\ast}$.

We can further choose a realization $d_{0}^{\varpi_{2}} : \varpi_{2} \rightarrow C_{1} V_{\ast}^{(1)}$ of $d_{0}^{\varpi_{2}} : \varpi_{2} \rightarrow Z_{1} G_{\ast} \subset C_{1} G_{\ast}$, again by (1.9). Finally, we can use [15, Corollary 4.2] to guarantee that $d_{0}^{\varpi_{1}} \circ d_{0}^{\varpi_{2}} = 0$ on the nose, thus extending $V_{\ast}^{(1)}$ to $V_{\ast}^{(2)}$, so there are no obstructions for $n = 0$.
At the $n$-th stage, we deduce from (1.11) that:

$$
\pi_k^{\check W(n)} \cong \begin{cases} 
\Omega^k \Lambda & \text{for } 0 \leq k < n, \\
0 & \text{otherwise}
\end{cases} \quad (4.2)
$$

as well as:

$$
\pi_k \pi^A_{\check W(n)} \cong \begin{cases} 
\Lambda & \text{for } k = 0, \\
\Omega^n \Lambda & \text{for } k = n + 1, \\
0 & \text{otherwise}
\end{cases}
$$

(4.3)

Any simplicial object over $\mathcal{M}$ satisfying (4.2) and (4.3) is called an $(n - 1)$-semi-Postnikov section for $\Lambda$ (these were called potential $(n - 1)$-stages in [14, §9.1]).

4.4. The cohomology existence obstruction

There are several variant descriptions of the André–Quillen cohomological existence obstructions, given by [9, Theorem 4.15], [14, §9], and [15, Theorem 6.4(b)], respectively. We recall the third version:

Under the setting of Section 4.1, $W(n)$ need not extend to a full resolution $W_*$, with $\pi_0 \pi^A_* W_* \cong \Lambda$, and $\pi_i \pi^A_* W_* = 0$ for $i > 0$. If it does so extend, then the structure map $r(n - 1)$: $W_* \rightarrow P^{n-1} W = W(n)$ induces a map of simplicial $\Pi_A$-algebras $r(n - 1): B \Lambda \rightarrow \pi^A_* W(n)$ over $\Lambda$, which serves as a zero section, and implies that $\pi_*^A W(n) \cong E_A(\Omega^n \Lambda, n + 1)$. In other words, if $W(n)$ extends to a full resolution, then the simplicial $\Pi_A$-algebra $\pi_*^A W(n)$ has trivial $k$-invariants. Thus, we distinguish those $(n - 1)$-semi-Postnikov sections where the $n$-th $k$-invariant for $\pi_*^A W(n)$ vanishes by calling them $(n - 1)$-quasi-Postnikov sections.

It turns out that if $W(n)$ is an $(n - 1)$-semi-Postnikov section, the $n$-th $k$-invariant for the simplicial $\Pi_A$-algebra $\pi_*^A W(n)$, which we denote by:

$$
\beta_n \in H^{n+2}(\Lambda, \Omega^n \Lambda),
$$

(4.5)

is precisely the obstruction to extending $W(n)$ to an $n$-semi-Postnikov section $W(n+1)$ with $\text{csk}_n W(n+1) \cong W(n)$ (see [14, §9]). In other words, an $(n - 1)$-semi-Postnikov section extends one step further toward a full resolution if and only if it is an $(n - 1)$-quasi-Postnikov section. Here we use (4.3) to identify $P^d \pi_*^A W(n)$ with $B \Lambda$ (a simplicial $\Pi_A$-algebra classifying object for $\Lambda$).

Of course, there may be many ways to choose the extension $W(n+1)$ once it is known to exist; these are classified by difference obstructions (see Section 7 below).

4.6. Representing the $k$-invariant explicitly

Suppose that the map $\bar{d}_{0}^{\check V_{n+2}}: V_{n+2} \rightarrow C_{n+1} V^{(n+1)}$ actually lands in the $(n + 1)$-cycle object $Z_{n+1} V^{(n+1)}$. This would yield an extension of $V^{(n+1)}$ to an $(n + 2)$-truncated realization of $G_*$ (cf. Section 1.22). Since $d_i \bar{d}_{0}^{\check V_{n+2}} = 0$ for $i \geq 1$ by (1.8), the composite map $d_0 \bar{d}_{0}^{\check V_{n+2}}$ is truly the only obstruction to extending the realization. Moreover, since at the $\Pi_A$-algebra level $\bar{d}_{0}^{\check V_{n+2}}$
lands in the cycles, rather than just the chains, from (1.9) we know that \( [d \bar{d}_0 V_{n+2}] = 0 \). Thus the real question is whether by varying \( d \bar{d}_0 V_{n+2} : \bar{V}_{n+2} \to C_{n+1} V_{(n+1)} \) within its homotopy class (determined by \( d \bar{G}_{n+2} \)), we can force the null homotopic composite \( d \bar{d}_0 V_{n+2} \) to be strictly zero. This sets the stage for constructing a higher homotopy operation in Section 5. First, we show how the cohomology class \( \beta_n \) may be described in terms of \( d \bar{d}_0 V_{n+2} \):

### 4.7. Proposition

Under the assumptions of Section 4.1 for \( n \geq 1 \), the obstruction class \( \beta_n \) of (4.5) is represented in the sense of Corollary 2.14 by the map \( \bar{G}_{n+2} \to \pi^n_0 V_{(n+1)} \) induced by \( d_0 V_{n+1} \circ d \bar{d}_0 V_{n+2} : \bar{V}_{n+2} \to Z_n V_{(n+1)} \).

**Proof.** Let \( W^{(n)} := \text{csk}_n V_{(n+1)} \) be an \( (n - 1) \)-semi-Postnikov section for \( \Lambda \) as above, and let \( \mathcal{K}_{\bullet} := \pi^n_\bullet W^{(n)} \) be the corresponding simplicial \( \Pi_{\Lambda} \)-algebra. We begin by constructing a weak equivalence \( f : \mathcal{G}_{\bullet} \to \mathcal{P}^n \mathcal{K}_{\bullet} = \text{csk}_{n+1} \mathcal{K}_{\bullet} \) as follows:

Since \( \mathcal{P}^n \mathcal{K}_{\bullet} \) is \( (n+1) \)-coskeletal, it suffices to define \( f \) on \( \text{sk}_n G_{\bullet} \). On the other hand, since \( W^{(n)} := \text{csk}_n V_{(n+1)} \) and \( \tau_{n+1} \pi^n_\bullet V_{(n+1)} = \tau_{n+1} G_{\bullet} \), we see that \( \text{sk}_n \mathcal{P}^n \mathcal{K}_{\bullet} \) is isomorphic to \( \text{sk}_n G_{\bullet} \), and thus for simplicity we assume that \( f \) is the identity through simplicial dimension \( n \).

Since we want \( f_{n+1} \) to be a map of \( \Pi_{\Lambda} \)-algebras, and \( G_{n+1} \) is free, by the Yoneda Lemma it is enough to say where \( f_{n+1} \) takes the tautological \( (n+1) \)-simplex \( \iota_{n+1} \in G_{n+1} V_{(n+1)} \) corresponding to \( \text{Id} \in [V_{n+1}, V_{n+1}] \cong G_{n+1} V_{(n+1)} \). In order to describe:

\[
\mathcal{K}_{n+1} V_{(n+1)} = (\pi^n_\bullet \text{csk}_n V_{(n+1)})_{n+1} V_{(n+1)},
\]

we need a map \( V_{n+1} \to (\text{csk}_n V_{(n+1)})_{n+1} = M_{n+1} V_{\bullet} \) (the matching object for \( V_{\bullet} \)): that is, a strict matching collection of \( n + 2 \) maps \( V_{n+1} \to V_n \). This is provided by:

\[
(d_{n+1} V_{n+1}, d_1 V_{n+1}, \ldots, d_{n+1} V_{n+1}).
\]

(4.8)

Since \( f \) so defined obviously commutes with the face and degeneracy maps, this defines a map of simplicial \( \Pi_{\Lambda} \)-algebras \( f : \mathcal{G}_{\bullet} \to \mathcal{P}^n \pi^n_\bullet W^{(n)} \). Moreover, \( f \) is necessarily a weak equivalence, since the only non-trivial homotopy group on either side is in dimension 0.

Combined with the description of the \( n \)-th \( k \)-invariant in Section 2.15, we see that the obstruction \( \beta_n \in H^{n+2}(\Lambda, \Omega^n \Lambda) \) of (4.5) is represented by the map of simplicial \( \Pi_{\Lambda} \)-algebras \( k_n \circ f : \mathcal{G}_{\bullet} \to \overline{E}_n(\pi_{n+1} \mathcal{K}_{\bullet}, n+2) \). This in turn is determined by the cocycle (\( \Pi_{\Lambda} \)-algebra map) \( b \circ C_{n+2}(f) : C_{n+2} G_{\bullet} \to \pi_{n+2} \mathcal{K}_{\bullet} \), where \( b \) is the cocycle of (2.17), by the naturality in Proposition 2.4. By Lemma 2.11, it suffices to say where \( b \circ C_{n+2} \) sends the tautological \( (n+2) \)-simplex \( \iota_{n+2} \in C_{n+2} G_{\bullet} \{ \bar{V}_{n+2} \} \subset G_{n+2} \{ \bar{V}_{n+2} \} \), corresponding to the inclusion \( \bar{G}_{n+2} \subset G_{n+2} \). Since the target is \( (n+1) \)-coskeletal, the \( (n+2) \)-simplex \( f_{n+2}(\iota_{n+2}) \) in \( (\text{csk}_{n+1} K_{\bullet})_{n+2} \{ \bar{V}_{n+2} \} \) is given by the matching collection of \( n + 3 \) elements in \( K_{n+1} \{ \bar{V}_{n+2} \} \):

\[
(f(d_{n+2} G_{n+2} \circ \iota_{n+2}), f(d_{n+2} G_{n+2} \circ \iota_{n+2}), \ldots, f(d_{n+2} G_{n+2} \circ \iota_{n+2})).
\]

This is simply \( f(d_{n+2} G_{n+2}, 0, 0, \ldots, 0) \), since \( \iota_{n+2} : \bar{G}_{n+2} \to G_{n+2} \) lands in \( C_{n+2} G_{\bullet} \), so \( d_{n+2} G_{n+2} \circ \iota_{n+2} = 0 \) for \( i > 0 \).
Each element \( f(\alpha) \in C_{n+1}K_\bullet(V_{n+2}) \) is represented by a map \( \tilde{\alpha} : V_{n+2} \to \text{csk}_n V_\bullet \) induced by (4.8) so it is the homotopy class of the collection of strictly matching maps \( V_{n+2} \to V_n \):

\[
\left( d_{V_n+1}^{\alpha}, d_1^{V_{n+1}} \tilde{\alpha}, d_2^{V_{n+1}} \tilde{\alpha}, \ldots, d_{n+1}^{V_{n+1}} \tilde{\alpha} \right).
\]

The class \( \alpha = \overline{d_0}^{n+2} \) is obtained by precomposing the tautological class \( \iota_{n+1} \in G_{n+1}\{V_{n+1}\} \) with the homotopy class of \( \overline{d_0}^{V_{n+2}} : V_{n+2} \to V_{n+1} \). Thus by (4.8):

\[
f(\alpha) = f(\iota_{n+1} \circ [\overline{d_0}^{V_{n+2}}])
= f(\iota_{n+1}) \circ [\overline{d_0}^{V_{n+2}}]
= [(d_{V_n+1}^{\alpha}, d_1^{V_{n+1} \circ \overline{d_0}^{V_{n+2}}}, \ldots, d_{n+1}^{V_{n+1} \circ \overline{d_0}^{V_{n+2}}})].
\]

Since we assumed that \( \overline{d_0}^{n+2} : V_{n+2} \to V_{n+1} \) lands in \( C_{n+1}V_\bullet^{(n+1)} \), this collection is equal to \([d_{V_n+1}^{\alpha} \circ \overline{d_0}^{V_{n+2}}, 0, 0, \ldots, 0]\).

Since \( W_\bullet^{(n)} := \text{csk}_n V_\bullet^{(n+1)} \), we find that

\[
C_{n+1}W_\bullet \cong \mathbb{Z}_n V_\bullet
\]

(see [11, Fact 3.3]). We have thus described a representing map \( V_{n+2} \to \mathbb{Z}_n V_\bullet^{(n+1)} \), as required.

4.10. Ladder diagrams

In order to identify the target of the map representing \( \beta_n \) in Proposition 4.7, we must analyze the isomorphisms \( s_i \) in the spiral long exact sequence (1.11). For this, we need the following technical tool:

Given \( X \in \mathcal{M}_A \) and \( Y_\bullet \in \mathcal{S} \mathcal{M} \) with \( \gamma_m : X \to Z_m Y_\bullet \) a morphism in \( \mathcal{M} \), let \( \text{Co}(X) \) denote the cone on \( X \), and \( i : X \to \text{Co}(X) \), \( j_m : Z_m Y_\bullet \to C_m Y_\bullet \) the inclusions. If \( g_m = j_m \gamma_m \) is null homotopic, then a choice of null homotopy \( H_m \) yields a commutative diagram:

\[
\begin{array}{ccc}
X & \xrightarrow{\gamma_m} & \text{Co}(X) \\
\downarrow & & \downarrow \gamma_m \\
Z_m Y_\bullet & \xrightarrow{j_m} & C_m Y_\bullet
\end{array}
\]

\[
\begin{array}{ccc}
\Sigma X & \xrightarrow{H_m} & \Sigma X \\
\downarrow & & \downarrow \\
Z_m Y_\bullet & \xrightarrow{d_0 j_m} & Z_{m-1} Y_\bullet
\end{array}
\]

The reason is \( d_0 j_m = 0 \) by the definition of \( Z_m Y_\bullet \), hence it follows that \( d_0 g_m = d_0 j_m \gamma_m = 0 \), so \( d_0 H_m \) descends to a map \( \text{Co}(X)/X \to Z_{m-1} Y_\bullet \) and identifying \( \text{Co}(X)/X \) with \( \Sigma X \) produces a map \( \gamma_{m-1} : \Sigma X \to Z_{m-1} Y_\bullet \).
4.11. Definition. A ladder diagram from $\gamma_m$ to $\gamma_k$ is a commutative diagram of the form:

\[
\begin{array}{cccccccc}
X & \rightarrow & \text{Co}(X) & \rightarrow & \Sigma X & \rightarrow & \cdots & \rightarrow & \Sigma^{m-k-1}X & \rightarrow & \text{Co}(\Sigma^{m-k-1}X) & \rightarrow & \Sigma^{m-k}X \\
\downarrow \gamma_m & & \downarrow \gamma_m & & \downarrow \gamma_{m-1} & & \cdots & & \downarrow \gamma_{k+1} & & \downarrow H_{k+1} & & \downarrow \gamma_k \\
Z_m Y_\bullet & \rightarrow & C_m Y_\bullet & \rightarrow & Z_{m-1} Y_\bullet & \rightarrow & \cdots & \rightarrow & Z_{k+1} Y_\bullet & \rightarrow & C_{k+1} Y_\bullet & \rightarrow & Z_k Y_\bullet.
\end{array}
\]

Consider any ladder diagram ending in:

\[
\begin{array}{cccccccc}
W & \rightarrow & \text{Co}(W) & \rightarrow & \Sigma W & \rightarrow & \cdots & \rightarrow & \Sigma^{n-1}W & \rightarrow & \text{Co}(\Sigma^{n-1}W) & \rightarrow & Z_n Y_\bullet, \\
\downarrow \gamma_n & & \downarrow \gamma_n & & \downarrow \gamma_{n-1} & & \cdots & & \downarrow \gamma_{n-1} & & \downarrow \gamma_{n-1} & & \downarrow \gamma_{n-1} \\
Z_n Y_\bullet & \rightarrow & C_n Y_\bullet & \rightarrow & Z_{n-1} Y_\bullet & \rightarrow & \cdots & \rightarrow & Z_0 Y_\bullet & \rightarrow & C_0 Y_\bullet & \rightarrow & Z_0.
\end{array}
\]

(4.12)

In order to continue building it to the right, we would need to have chosen $H_n$ carefully to induce $\gamma_n - 1$ such that $g_{n-1} = j_{n-1} \circ \gamma_{n-1}$ is null homotopic. For this purpose, we can sometimes use the following:

4.13. Lemma. If $W \in \mathcal{M}_A$ and $\pi_{n-1} \pi^*_A Y_\bullet \{\Sigma W\} = 0$, then the null homotopy $H_n$ in (4.12) can be chosen so the map $g_{n-1}$ it induces is null homotopic.

Proof. Suppose $H_n$ is chosen so that the resulting $[g_{n-1}] \neq 0$. Then by assumption the map $d_0 : C_n \pi^*_A Y_\bullet \{\Sigma X\} \to Z_{n-1} \pi^*_A Y_\bullet \{\Sigma X\}$ is surjective. By (1.9), there is then a map $\alpha_n : \Sigma X \to C_n Y_\bullet$ with $d_0[\alpha_n] = -[\gamma_{n-1}]$, and we choose $H'_n = H_n \star \alpha_n$ (using the notation of [44, §2] for the coaction of $\{\Sigma X, Z\}$ on a null homotopy $H : \text{Co}(X) \to Z$). This new null homotopy of $\gamma_n$, induces a map $\gamma'_n : \Sigma X \to Z_{n-1} Y_\bullet$ as above, which in turn yields $g'_{n-1}$. In fact,

\[
[g'_{n-1}] = (j_{m-1})_*[d_0(H_n * \alpha_n)] = (j_{n-1})_*[\gamma_{n-1}] + (j_{n-1})_*(-[\gamma_{n-1}]) = [g_{n-1}] - [g_{n-1}] = 0,
\]

as required. $\square$

We now exploit the usual proof of exactness for the homotopy sequence of a fibration at the fiber position to show:

4.14. Proposition. Under the assumptions of Section 4.1, given $\gamma_n = d_0 d_0^{-1} \tilde{V}_{n+2} : \tilde{V}_{n+2} \to Z_n V_\bullet^{(n+1)}$ there exists a ladder diagram from $\gamma_n$ to $\gamma_0 : \Sigma^n \tilde{V}_{n+2} \to Z_0 V_\bullet^{(n+1)} = V_0$, with adjoint whose homotopy class $[\tilde{\gamma}_0]$ in $\Omega^n \pi_0 \pi^*_A V_\bullet^{(n+1)} \{\tilde{V}_{n+2}\} \cong \Omega^n A(\tilde{V}_{n+2})$ represents the obstruction class $\beta_n$.

Proof. First, since $V_\bullet^{(n+1)}$ is a quasi-Postnikov section, it follows from (4.2) and (4.3) that the homomorphisms $s_i$ in the spiral long exact sequence (1.11) are isomorphisms for $0 \leq i \leq n$. We
assume by downward induction on \(i\) (starting with \(i = n\) from Proposition 4.7) that \(\gamma_i\) represents \(\hat{\beta}_n\) in \(\pi_i^s V^{(n+1)} \cdot (\Sigma^{n-i} \nabla_{n+2}) \cong \Omega^{n-i} \pi_i^s V^{(n+1)} \cdot \nabla_{n+2}\).

By definition, \(s_i\) is induced by the connecting homomorphism for the fibration sequence (1.12) (see Section 1.7). Thus, a preimage of \(\gamma_i : \Sigma^{n-i} \nabla_{n+2} \to Z_i \cdot V^{(n+1)}\) under \(s_i\) is obtained by choosing a null homotopy \(H_i : \text{Co}t(\Sigma^{n-i} \nabla_{n+2}) \to C_i \cdot V^{(n+1)}\) for \(j_i \circ \gamma_i\), noting that \(d_0 \circ H_i |_{\Sigma^{n-i} \nabla_{n+2}} = 0\) so \(H_i\) induces a map \(\gamma_{i-1} : \Sigma^{n-i+1} \nabla_{n+2} \to Z_{i-1} \cdot V^{(n+1)}\) with \(s_i[\gamma_{i-1}] = [\gamma_i]\). Evidently, this is just extending a ladder diagram from \(\gamma_n\) to \(\gamma_i\) and thereby producing a ladder diagram from \(\gamma_n\) to \(\gamma_i\), so is possible by (4.3) verifying the vanishing condition required in order to apply Lemma 4.13. □

5. Higher homotopy operations as existence obstructions

In Section 4.6 we saw that vanishing of the \(n\)-th “algebraic” obstruction to realizing the \(\Pi_A\)-algebra \(A\) determines whether we can choose a representative \(\tilde{d}_0^{G_{n+2}}\) for the attaching map \(d_0^{G_{n+2}}\) so that the composite \(d_0 \circ \tilde{d}_0^{V_{n+2}} : \nabla_{n+2} \to Z_n \cdot V^{(n+1)}\) is zero (and not just null homotopic).

The same question may also be addressed using the inductive rectification process of Section 3.2, since we can view this as trying to find a strict representative for a (pointed) diagram in the homotopy category. We need the relative version of Section 3.5 for this, since we want to ensure that \(d_0^{V_{n+1}}\) remains strictly zero for \(i > 0\), and want to leave the \((n + 1)\)-truncation \(V^{(n+1)}\) untouched.

Because the Dwyer–Kan–Smith (\(S_n, \emptyset\))-cohomology obstructions are difficult to compute, we prefer the more “geometric” approach of Section 3.8; but for this the indexing category must be a lattice. This is not true of (truncated) simplicial objects, because of the degeneracy maps. However, in our case we can work with a smaller indexing category, which is a lattice, by using the CW structure to avoid dealing with the degeneracies. We can further reduce the complexity by careful initial choices (see Section 6 below).

5.1. Remark. By [10, Theorem 3.16] any chosen CW-resolution \(G\) of a realizable \(\Pi_A\)-algebra \(A := \pi_A^s X\) (cf. Section 1.20) can be realized by a CW-resolution \(V\) of \(X\) in \(sM\), with \(\pi_A^s V \cong \overline{G}\) and \((\tilde{d}_0^V)_\# = \tilde{d}_0^G\), and thus \(\pi_A^s V \cong G\).

We observe also that if we want to realize a CW-resolution \(G \to A\) in \(s\Pi_A\cdot\text{Alg} – \text{or equivalently, to rectify the corresponding simplicial object up-to-homotopy in } s(hoM),\) obtained by choosing some \(\nabla_n \in M_A\) with \(\pi_A^s V \cong \overline{G}\) for each \(n \geq 0\), with \(\tilde{d}_0^V\) uniquely determined up to homotopy by \(\tilde{d}_0^G\) – it suffices to inductively rectify the corresponding restricted simplicial object (in which we forget the degeneracies), since the degeneracies can then be reconstructed from (1.21).

5.2. Definition. Write \(\tilde{A} \subseteq A^{\text{op}}\) for the indexing category for restricted simplicial objects, with \(\text{Obj}(\tilde{A}) := \{n\}^{\infty}_{n=0}\) and maps \(d_0, \ldots, d_n : n \to n - 1\) satisfying the simplicial identities \(d_i d_j = d_{j-1} d_i\) for \(i < j\). (This is the opposite of the usual indexing category, for restricted cosimplicial objects.)

Similarly, augmented restricted simplicial objects are represented by \(\tilde{A}_+\), with an additional object \(-1\) and \(e = d_0 : 0 \to -1\). The truncated categories \(\tilde{A}^n\) and \(\tilde{A}^n_+\) are the full subcategories with objects \(n, n - 1, \ldots, 0\) and \(n, n - 1, \ldots, 0, -1\), respectively. The latter is a lattice of length \(n + 1\), with (weakly) terminal object \(\nu_{\text{fin}} := -1\).
Finally, let $\tilde{\Delta}^{n+2}$ be the category with object set $\{0, 1, \ldots, n+2\}$, all the maps of $\tilde{\Delta}^{n+1}$, and one additional map $\tilde{d}_0 : n+2 \to n+1$ such that $d_i^{n+1} \circ \tilde{d}_0 = 0$ for $0 \leq i \leq n+1$.

Thus $V_{(n+1)}$, together with $\tilde{V}_{n+2}$ and its face maps, define a functor $\hat{V}_{(n+2)} : \tilde{\Delta}^{n+2} \to \text{ho} \mathcal{M}$. Moreover, we have maps $d_i^{V_k} : V_k \to V_{k-1}$ and $d_0^{\tilde{V}_{n+2}} : \tilde{V}_{n+2} \to V_{n+1}$ which, together with $\hat{V}_{(n+1)}$, “almost” define a functor $\hat{V}_{(n+2)} : \tilde{\Delta}^{n+2} \to \mathcal{M}$ lifting $\hat{V}_{(n+2)}$, in that all (pointed) identities of $\tilde{\Delta}^{n+2}$ hold strictly, except:

$$d_0 \circ d_0^{\tilde{V}_{n+2}} \sim 0$$

(5.3)

(the composite need not be strictly zero).

Rectifying $\hat{V}_{(n+2)}$ (thus turning $\hat{V}_{(n+2)}$ into a strict functor) is equivalent to producing a full $(n+2)$-truncated simplicial object $V_{(n+2)}$ realizing $\tau_{n+2}G_\bullet$, by Remark 1.22.

5.4. Using the Dwyer–Kan–Smith approach

In order to apply the inductive procedure of Section 3.2, let $\mathcal{C} := \tilde{\Delta}^{n+1}$ and $\mathcal{D} = \tilde{\Delta}^{n+2}$, so $\mathcal{O} = \{0, 1, \ldots, n+1\}$ and $\mathcal{O}_+ = \mathcal{O} \cup \{n+2\}$. The $(n+1)$-truncated simplicial object $V_{(n+1)}$ provides the strictly commuting diagram $\tilde{\Delta}^{n+1} \to \mathcal{M}$, and thus the constant $(S_\bullet, \mathcal{O})$ map $X : c(\tilde{\Delta}^{n+1}) \to \mathcal{M}_\mathcal{O}$.

The lifting diagram of $(S_\bullet, \mathcal{O}_+)$-categories representing (3.4) in our case is:

$$\begin{array}{cccc}
F_\mathcal{C} \tilde{\Delta}^{n+1} & \xrightarrow{F_i} & F_\mathcal{C} \tilde{\Delta}^{n+2} \\
\xrightarrow{\sim} & \xrightarrow{\sim} \\
c(\tilde{\Delta}^{n+1}) & \xrightarrow{i_\bullet} & Q & \xrightarrow{\tilde{\mathcal{E}}^\ell} \mathbf{P}^\ell \mathcal{M}_\mathcal{O}_+ \\
\xrightarrow{X} & \xrightarrow{\tilde{\mathcal{E}}^{\ell-1}} & & \xrightarrow{k_{\ell-1}} E_{\Delta^{n+2}}(\pi_{\ell} \mathcal{M}_\mathcal{O}_+, \ell + 1) \\
\mathcal{M}_\mathcal{O} & j & \mathcal{M}_\mathcal{O}_+ & d^n \mathcal{M}_\mathcal{O}_+ \\
\end{array}$$

(5.5)

for each $1 \leq \ell \leq n$, where $Q$ is the pushout $F_\mathcal{C}(\tilde{\Delta}^{n+2}, \tilde{\Delta}^{n+1})$ in $(S_\bullet, \mathcal{O}_+)\text{-}\mathcal{Cat}$. The initial choice of $\tilde{X}^0 : \mathcal{Q} \to \mathbf{P}^0 \mathcal{M}_\mathcal{O}_+$ is actually equivalent to the “lax functor” $\hat{V}_{(n+2)}$, together with choices of null homotopies $H_i : d_i \circ d_0^{\tilde{V}_{n+2}} \sim 0$ ($0 \leq i \leq n+1$) in (5.3).

5.6. The geometry of $Q$

To analyze the $(S_\bullet, \mathcal{O}_+)$-maps $\tilde{X}^k : \mathcal{Q} \to \mathbf{P}^k \mathcal{M}_\mathcal{O}_+$, or their adjoints $\text{sk}_{k+1} \mathcal{Q} \to \mathcal{M}_\mathcal{O}_+$, we need an explicit description of the simplices of each simplicial mapping space map$(\mathbf{m}, \mathbf{j})$. We denote the $i$-th “internal” face map of each such simplicial set by $d_i$, to avoid confusion with the “categorical” face maps $d_i$ (morphisms of $\tilde{\Delta}^{n+1}$).

Recall that the free simplicial resolution $F_\mathcal{K}$ of a category $\mathcal{K}$ is constructed by iterating the free category monad $FU$ (cf. Section 3.1). We denote a $k$-simplex $\sigma^k$ of $F_\mathcal{K}$ by a $(k+1)$-fold
parentheses of a sequence of maps from $\mathcal{K}$, which we think of as representing a $k$-th order homotopy between the vertices of $\sigma$. The $i$-th face $\partial_i \sigma^k$ is obtained by omitting the $(i + 1)$-st level of parentheses, and the degeneracy $s_j \sigma^k$ is obtained by iterating the $(j + 1)$-st level of parentheses ($0 \leq i, j \leq k$). The categorical composition is denoted by concatenation. The augmentation $\varepsilon : F_i \mathcal{K} \to \mathcal{K}$ is defined by dropping all parentheses (and composing in $\mathcal{K}$).

Note that the mapping space $\text{map}_\mathcal{Q}(\mathbf{m}, \mathbf{j})$ is discrete unless $\mathbf{m} = \mathbf{n} + 2$, since otherwise $F_i i$ (at the top of (5.5)), and consequently $i_n$, too, are isomorphisms. Consequently each pair of parentheses not surrounding a map out of $\mathbf{n} + 2$ represents a homotopy in a discrete space, so will be omitted. Thus we only need to consider nested parentheses having $\bar{d}_0$ (the only new 0-vertex in $\mathcal{Q} \setminus F_i \bar{\Delta}^{n+1}$) in the innermost parentheses.

Therefore, we further abbreviate the standard notation by omitting all right parentheses, and replacing left parentheses by vertical bars. Moreover, every string representing a simplex $\sigma$ of $\text{map}_\mathcal{Q}(\mathbf{n} + 2, \mathbf{j})$ has $\bar{d}_0$ as its last entry, so we can omit it. Thus, $|d_i| |d_3| |s_0(d_i|d_3|))$ in the standard notation, while $d_i |d_3| = s_0(d_i |d_3|)$ is decomposable (that is, represents a composition with a degeneracy of a zero simplex in some discrete case) because there is no vertical bar on the extreme left.

Now any non-degenerate $k$-simplex $\sigma$ of $\mathcal{Q}$ ($k \geq 1$) is necessarily in $\text{map}_\mathcal{Q}(\mathbf{n} + 2, \mathbf{j})$ for some $j \leq n - k + 1$. It can thus be written uniquely as:

$$\sigma = d_{I_0}|d_{I_1}|d_{I_2}| \cdots |d_{I_k}|d_{I_{k+1}}$$

with a total of $k + 1$ vertical bars. Either or both of $I_0$ and $I_{k+1}$ can be empty.

Any $k$-simplex in $\text{map}_\mathcal{Q}(\mathbf{n} + 2, \mathbf{n} - k + 1)$ (the maximal dimension here) will be called atomic. In particular, the basic atomic $k$-simplex is:

$$\tau_k := |d_0| |d_0| \cdots |d_0|.$$  (5.7)

5.8. Definition. For fixed $n \geq 1$, a $k$-flag is a sequence $\varphi = (i_1, i_2, \ldots , i_k)$ of $|\varphi| := k$ integers with $0 \leq i_1 < i_2 < \cdots < i_k \leq n + 1$. The collection $\Psi^{n+2}_{n-k+1}$ of $k$-flags is in one-to-one correspondence with the set $\text{Hom}_{\bar{\Delta}^{n+2}}(\mathbf{n} + 2, \mathbf{n} - k + 1)$, where $\varphi$ represents the map $d_\varphi = d_{i_1} d_{i_2} \ldots d_{i_k} \bar{d}_0 : \mathbf{n} + 2 \to \mathbf{n} - k + 1$ in $\bar{\Delta}^{n+2}$ (cf. Section 5.2). It thus determines a $k$-dimensional simplicial set $\mathcal{K}_\varphi$ – namely, the (pointed) component of $(d_\varphi)$ in $\mathcal{Q}(\mathbf{n} + 2, \mathbf{n} - k + 1)$, which we call the flag complex for $\varphi$. We may describe $\mathcal{K}_\varphi$ explicitly as follows:

A $j$-simplex $\sigma$ of $\mathcal{K}_\varphi$ is determined by an expression of the form

$$d_{\ell'_1} \cdots d_{\ell'_0} |d_{m'_0} |d_{\ell'_1} |d_{m'_1} | \cdots |d_{\ell'_{j+1}} |d_{m'_{j+1}} ,$$

with $\ell'_1 < \ell'_2 < \cdots < \ell'_m$, and $j + 1$ vertical bars, where the composite (obtained by omitting all bars) is $d_\varphi$, and we allow bars at either end of the sequence, as well as repeated bars. The face map $\partial_j$ removes the $(i + 1)$-st vertical bar (rewriting the resulting sequence in standard form, if necessary), and the $i$-th degeneracy repeats the $(i + 1)$-st bar, as in $\mathcal{Q}$.

5.9. Example. For $\varphi = (2 < 4 < 5)$, we have a 3-simplex $|d_1| |d_3| |d_2|$ in $\mathcal{K}_\varphi$, since $d_3 d_3 d_2 = d_2 d_4 d_5$. The 2-simplices include $|d_2 d_4| |d_5|$, $d_2 |d_4| |d_5|$ and $|d_2 d_4| |d_4|$. Moreover, $\partial_1 [|d_3| |d_3| |d_2|] = |d_3 d_4| |d_2|$ and $\partial_0 [|d_2 d_4| |d_3|] = |d_2 d_4| |d_3|$. 


5.10. Definition. For any flag $\varphi$ as above, the boundary of the flag complex $K_\varphi$, denoted by $\partial K_\varphi$, is the $(k-1)$-dimensional subsimplicial set spanned by $\partial_i \sigma$, for $\sigma$ a $k$-simplex of $K_\varphi$ ($0 \leq i \leq k$). The subcomplex of $\partial K_\varphi$ spanned by the zero-faces $\partial_0 \sigma$ is called the base complex, written $\partial_0 K_\varphi$; and the subcomplex spanned by the other faces ($1 \leq i \leq k$) is written $\tilde{\partial} K_\varphi$, so $\partial K_\varphi \cong \partial_0 K_\varphi \cup \tilde{\partial} K_\varphi$.

The vertex $|d_0|d_1|d_2| \ldots |d_k$ of $K_\varphi$, lying in $\tilde{\partial} K_\varphi$, will be called its cone point, and denoted by $c_\varphi$ (see Lemma 5.12 below).

5.11. Examples. (1) Fig. 1 shows the 2-dimensional flag complex $K_\varphi$ for $\varphi = (0 < 1)$. It contains the basic atomic 2-simplex $\tau_2$ as the left 2-simplex (cf. (5.7)).

The base complex $\partial_0 K_\varphi$ consists of the top two 1-simplices $\partial_0 \tau_2 = d_0|d_0|$ and $d_0|d_1|$. Note that $\partial_0 K_\varphi$ is a triangulated dual 2-permutahedron on the set $\{0, 1\}$, where $d_0d_1$ corresponds to $(0, 1)$ and $d_0d_0$ corresponds to $(1, 0)$ since the simplicial identities are involved in permuting face maps.

(2) Fig. 2 shows the atomic 3-simplex $\sigma = |d_0|d_1|d_0|$ of map $Q(n + 2, n - 2)$: more precisely, we have cut open its boundary $\partial \sigma$, so that it can be depicted in the plane, with identifications of the outer edges indicated by dotted arrows. The cone point $c_\varphi$ corresponds to the outer vertices...
Fig. 3. $\partial_0 K_\phi \subset \text{map}_Q(n+2, n-2)$ for $\phi = (0 < 1 < 2)$.

$|d_0 d_1 d_2|$ (all identified) and the central facet $\partial_0 \sigma$ is part of $\partial_0 K_\phi$, while the bottom 2-simplex in the figure $\partial_2 \sigma$ forms part of the (upper) boundary $\tilde{\partial} K_\phi$.

(3) Fig. 3 depicts the base complex $\partial_0 K_\phi$ (a triangulated dual 3-permutohedron) for the first flag of length 3 $\phi = (0 < 1 < 2)$ in map$_Q(n+2, n-2)$.

**5.12. Lemma.** For any flag $\phi$ with $|\phi| = k$, the base complex $\partial_0 K_\phi$ is isomorphic to a triangulated dual $k$-permutohedron (a $(k-1)$-dimensional convex polytope), and $K_\phi$ is the combinatorial cone on its zero-face $\partial_0 K_\phi$, with cone point $c_\phi$.

**Proof.** The $(k-1)$-simplices of the base complex $\partial_0 K_\phi$ may be listed by decomposing $d_\phi$ in all possible atomic (mostly non-standard) forms, so as a composite of $k$ face maps with vertical bars inserted to the right of each map (but not at the left end). If we let the standard representation (in ascending order) $d_1 | d_2 | \ldots | d_k$ correspond to the identity permutation, we have a faithful, effective, and transitive action of the symmetric group $\Sigma_k$ on the $(k-1)$-simplices of $\partial_0 K_\phi$, in which any adjacent transposition $(j, j+1)$ takes $d_1 | \ldots | d_{j-1} | d_j | d_{j+1} | \ldots | d_k$ to $d_1 | \ldots | d_{j-1} | d_{j+1} | \ldots | d_{j+1} | d_{j} | \ldots | d_k$, by applying the simplicial identity $d_j \circ d_{j+1} = d_{j+1} \circ d_j$. This shows that the base complex $\partial_0 K_\phi$ is indeed the dual of the triangulated $k$-permutohedron.

To see that $K_\phi = \text{Co}(\partial_0 K_\phi)$, note that if we compose all the factors in any representation $|d_{i_1}| \ldots |d_{i_k}$ as above, we obtain the same map $d_\phi$, which implies that the initial vertex of each $k$-simplex $\sigma$ of $K_\phi$ is precisely the cone point $c_\phi$, explicitly $|d_1 \ldots d_k|$. □
5.13. Remark. The terminal vertex of each simplex of $\partial_0 K_\varphi$ is $d_{i_1} \ldots d_{i_k}$, so the base complex is again the cone on its own boundary.

5.14. Corollary. For any flag $\varphi$ with $|\varphi| = k$, the boundary $\partial K_\varphi$ of the flag complex is a combinatorial $(k - 1)$-sphere.

Proof. The permutohedron $P^k$ on $k$ elements is a $(k - 1)$-polytope—that is, a convex $(k - 1)$-dimensional polyhedron in $\mathbb{R}^{k-1}$ (cf. [32]) so its dual $\partial_0 K_\varphi$ is also a $((k - 1))$-polytope (see [31, §3.4]). Thus the cone $K_\varphi = \text{Co}(\partial_0 K_\varphi)$ is a $k$-polytope, and its boundary $\partial K_\varphi$ is combinatorially equivalent to a $(k - 1)$-sphere. \hfill \square

We can now describe the decomposition of the mapping spaces in $Q$:

5.15. Lemma. The space $\text{map}_Q(m, j)$ is empty if $m < j$, discrete for $m \neq n + 2$, and for $0 \leq k \leq n + 1$,

$$\text{map}_Q(n + 2, n - k + 1) \cong \bigvee_{\varphi \in Q_{n+2}^{n-k+1}} K_\varphi,$$

where the vertex $|d_{i_1}d_{i_2} \ldots d_{i_k}$ is chosen as basepoint in each flag complex.

Proof. In the pushout (5.5) defining $Q$, for entries with $m \neq n + 2$, the top map is an isomorphism, hence the bottom map is as well, so $\text{map}_Q(m, j)$ is discrete.

If $m > j$, by construction, the pointed mapping space $\text{map}_Q(m, j)$ decomposes into a wedge of natural summands corresponding to distinct maps in $\hat{\Delta}^{n+1}$—that is, iterated face maps $m \to j$. However, these summands are just the flag complexes, since the composite obtained by omitting the bars from any decomposition of $\varphi$ always equals $d_{i_1}d_{i_2} \ldots d_{i_k}$. \hfill \square

5.16. Remark. Note that the basic atomic $k$-simplex $\tau_k = |d_0|d_0| \ldots |d_0|$ (cf. Section 5.6) is a top dimensional simplex of $K_{0<1<2< \ldots < k-1}$ in each case, while $\partial_0 \tau_k$ decomposes (in the simplicial enrichment of $Q$) as $\tau_{k-1}$ followed by (the degeneracy of) the 0-simplex $d_0 : j + 1 \to j$. We will see later, that with appropriate initial choices, we can send any top dimensional simplex other than $\tau_k$ to zero, and one only really needs to extend over this single simplex, imposing this one relation, at each stage of the induction.

Given a flag $\varphi = \{0 \leq i_1 < i_2 < \cdots < i_k \leq n + 1\}$, write $\varphi^{ji} := \{0 \leq i_1 < i_2 < \cdots < i_{j-1} < i_{j+1} < \cdots < i_k \leq n + 1\}$. For $j < \ell$, write $\varphi^{ji,\ell} := \{0 \leq i_1 < \cdots < i_{j-1} < i_{j+1} \cdots i_{\ell-1} < i_{\ell+1} < \cdots < i_k \leq n + 1\}$. Let $d_{i_j-j+1} \circ K_{\varphi, j}$ denote the subcomplex of $\partial_0 K_\varphi$ spanned by simplices of the form $d_{i_j-j+1}\sigma$, where $\sigma$ represents a simplex of $K_{\varphi, j}$.

5.17. Lemma. Each component $K_\varphi$ of the mapping space $\text{map}_Q(n + 2, n - k + 1)$ has $\partial_0 K_\varphi \cong \bigcup_j d_{i_j-j+1} \circ K_{\varphi, j}$, where each $(k - 2)$-simplex not in the boundary is shared by only two $(k - 1)$-simplices, and if $j < \ell$:

$$d_{i_j-j+1} \circ K_{\varphi, j} \cap d_{i_{\ell-1+1}} \circ K_{\varphi, \ell} = d_{i_j-j+1}d_{i_{\ell-1+1}} \circ K_{\varphi, j, \ell}.$$
Proof. This is a straightforward calculation, based on the simplicial identity
\[ d_1d_2\ldots d_k = d_{i_j-(j-1)}d_{i_{j-1}}\ldots d_{i_{j+1}}\ldots d_k, \]
since the face map \( d_{ij} \) is moved forward past \( j - 1 \) others, whose indices are always lower by assumption. Permutations naturally break up according to those which move a fixed term to the front, with each such piece a copy of a permutation group on a set with one less element. The same applies a second time to get the intersection statement. □

5.18. The inductive procedure

Let \( Q[n + 2, j] \) denote the full subsimplicial category of \( Q \) which only contains the objects \( n + 2, n + 1, \ldots, j \). Note that any simplicial functor \( \hat{X}^k : Q[n + 2, n - k + 1] \to M_{O_+} \) (compatible with \( X : \Delta^{n+1} \to M(\hat{\phi}) \)) extends uniquely to a simplicial functor \( \partial_0 Q[n + 2, n - k] \to M_{O_+} \) (where the face is only taken in the last mapping space) by Lemma 5.17 (for each \( \hat{\phi} \)) and extends by zero to \( \partial K_{\phi} \) (the rest of \( \partial K_{\phi} \)). Together with \( \hat{X}^k \), this yields a simplicial functor \( \hat{X}^k : \partial Q[n + 2, n - k] \to M_{O_+} \) (where again the boundary is only taken in the last mapping space).

For any flag with \( |\phi| = k + 1 \), note that \( \hat{X}^k|_{\partial K_{\phi}} \) sends \( \partial K_{\phi} \) to the zero map. Since the target in \( M_{O_+} \) is assumed to be a Kan complex, we can instead consider the induced map \( f_{\phi} \) from the quotient \( \partial K_{\phi}/\partial K_{\phi} \cong \partial_0 K_{\phi}/\partial_0 K_{\phi} \), which is a \( k \)-sphere by Lemma 5.12. Moreover, the adjoint map \( \tilde{f}_{\phi} : \Sigma^k \nabla_{n+2} \to V_{n-k} \) is null homotopic precisely when \( f_{\phi} \) is such, or equivalently, when \( \hat{X}^k|_{\partial K_{\phi}} \) has a filler to all of \( K_{\phi} \).

5.19. Proposition. A simplicial functor \( \hat{X}^k : Q[n + 2, n - k + 1] \to M_{O_+} \) (compatible with a fibrant \( X : \Delta^{n+1} \to M(\hat{\phi}) \)) extends to a simplicial functor \( \hat{X}^{k+1} : Q[n + 2, n - k] \to M_{O_+} \) if and only if for each flag of length \( k + 1 \) the induced map \( f_{\phi} \) is null homotopic.

Proof. If there is an extension, the fact that the full \( K_{\phi} \) serves as a cone on its boundary \( k \)-sphere means the extension serves as a null homotopy of the restriction to \( \partial K_{\phi} \), thereby implying that \( f_{\phi} \) is also null homotopic.

Conversely, if \( f_{\phi} \) is null homotopic, given the choice of a null homotopy \( H \) for \( \hat{X}^k|_{\partial K_{\phi}} \) and an \( i \)-simplex \( \sigma \in K_{\phi} \setminus \partial K_{\phi} \) (with \( i = k \) or \( k + 1 \)), \( H \) determines an \( i \)-simplex \( \hat{X}^{k+1}(\sigma) \in \text{map}_{M}(\hat{X}^k(n + 2), \hat{X}^k(n - k)) \) since the target is a Kan complex. Note that any such \( i \)-simplex \( \sigma \) is indecomposable, so \( \hat{X}^{k+1} \) so defined (and extending \( \hat{X}^k \)) is indeed a simplicial functor. □

5.20. Definition. Given a flag \( \phi = (0 \leq i_1 < \cdots < i_{k+1} \leq n + 1) \), with corresponding map \( d_{\phi} = d_{i_1} \ldots d_{i_{k+1}}d_0 : n + 2 \to n - k \), by Corollary 5.14 the boundary \( \partial K_{\phi} \) of the flag complex is a simplicial \( k \)-sphere. Therefore, the adjoint of \( f_{\phi} = \hat{X}^k|_{\partial K_{\phi}} : \partial K_{\phi} \to \text{map}_{M}(\nabla_{n+2}, V_{n-k}) \) may be thought of as a map \( \tilde{f}_{\phi} : \Sigma^k \nabla_{n+2} \to V_{n-k} \) (after identifying \( \partial K_{\phi} \) with the cone point \( c_{\phi} \)). We define the \((k + 1)\)-st order higher homotopy operation obstruction to realizing \( \Lambda \) to be the subset:

\[ \{\psi_{n-k}+2\} \subseteq \left[ \bigvee_{\psi \in \psi_{n-k}+2} \Sigma^k \nabla_{n+2}, V_{n-k} \right]. \]
consisting of all homotopy classes:

\[
\langle f_{n-k}^{n+2} \rangle := \left[ \bigvee_{\varphi \in \Psi_{n-k}^{n+2}} \tilde{f}_{\varphi} \right] \in \left[ \bigvee_{\varphi \in \Psi_{n-k}^{n+2}} \Sigma^k V_{n+2}, V_{n-k} \right]
\]

obtained by varying the inductively defined choice of \( \hat{X}^k \). Each such class (5.21) is thus a value, in the sense of Section 3.8, of the \((k+1)\)-st order higher homotopy operation \( \langle\langle \Psi_{n-k}^{n+2} \rangle\rangle \).

5.22. Theorem. Under the assumptions of Section 5.4, the homotopy class \( \langle f_{n-k}^{n+2} \rangle \) of (5.21) vanishes if and only if the restriction of the lifting \( \hat{X}^k \) in diagram (5.5) to \( Q[n+2, n-k] \) exists.

Proof. This follows by induction from Proposition 5.19. □

5.23. Corollary. The last \((n\text{-th order})\) higher homotopy operation \( \langle\langle \Psi_0^{n+2} \rangle\rangle \) is the \((\text{final})\) obstruction to extending the \((n+1)\)-truncated simplicial object \( V_{\pi_0^{n+1}} \) to a rectification of \( \tilde{V}_{\pi_0^{n+2}} \), and thus to a realization of \( \tau_{n+2} G_\cdot \).

Proof. The induction of Section 5.18 (and Proposition 5.19) is different \textit{prima facie} from that of Section 3.2, since we enlarge the indexing categories \( \hat{\Delta}_{n+2}^{n+2} \) at each stage. However, this is no longer true at the last stage, when \( k = n \), so our last obstruction is for the full extension to \( M_{O_+} \), as in Section 3.2. □

6. Minimal higher homotopy operations

We would like to relate the higher homotopy operation obstructions of Section 5 to the cohomological obstructions of Section 4. Evidently, these two obstructions do not take values in the same groups, so they can not be identified \textit{per se}. In order to compare them, we define a homomorphism between the target groups, as follows:

6.1. The correspondence homomorphism

By adjointness, there is a natural isomorphism \( \Sigma^n V_{n+2}, X \cong [\tilde{V}_{n+2}, \Omega^n X] \) and by \( \tilde{V}_{n+2} \in M_{A} \), we have a natural isomorphism \( [\tilde{V}_{n+2}, Y] \cong \text{Hom}_{\text{Alg}}(\pi_*^A \tilde{V}_{n+2}, \pi_*^A Y) \). However, \( \Omega^n \pi_*^A X := \pi_*^A \Omega^n X \) so the combination gives a natural isomorphism

\[
[\Sigma^n V_{n+2}, V_0] \cong \text{Hom}_{\text{Alg}}(\pi_*^A \tilde{V}_{n+2}, \pi_*^A V_0).
\]

Next, post-composition with the looped augmentation map \( \epsilon : G_0 = \pi_*^A V_0 \to A \) induces a (surjective) homomorphism

\[
\text{Hom}_{\text{Alg}}(\tilde{G}_{n+2}, \Omega^n A) \to \text{Hom}_{\text{Alg}}(\tilde{G}_{n+2}, \Omega^n A).
\]

If we identify \( \tilde{G}_{n+2} \) with \( \pi_*^A \tilde{V}_{n+2} \), we get a homomorphism

\[
[\Sigma^n V_{n+2}, V_0] \to \text{Hom}_{\text{Alg}}(\tilde{G}_{n+2}, \Omega^n A).
\]
Finally, by Corollary 2.14 there is a homomorphism
\[ \text{Hom}_{\Pi_{\mathcal{A}} - \text{Alg}}(G_{n+2}, \Omega^n A) \to H^{n+2}(A; \Omega^n A) \]
as well. Combining these maps and identifications yields:
\[ \Phi : \left( \Sigma^n \overline{V}_{n+2}, V_0 \right) \to H^{n+2}(A; \Omega^n A). \tag{6.2} \]

6.3. Definition. In the setting of Section 4.1, the \( n \)-th correspondence homomorphism is the map
\[ \tilde{\Phi}_n : \bigoplus_{\phi \in \Psi_0^{n+2}} \left( \Sigma^n \overline{V}_{n+2}, V_0 \right) \to H^{n+2}(A; \Omega^n A) \]
obtained by adding up the homomorphisms \( \Phi \) of (6.2), whose target is an abelian group.

The correspondence homomorphism is hard to evaluate, in general. However, there is a special class of values of the higher homotopy operation \( \langle \psi_0^{n+2} \rangle \) for which this evaluation is possible:

6.4. Definition. A value \( \langle f \rangle \in \left\{ \bigvee_{\phi \in \Psi_j^{n+2}} \Sigma^{n-j} \overline{V}_{n+2}, V_j \right\} \) of the higher homotopy operation \( \langle \psi_j^{n+2} \rangle \) defined in Section 5.20 is called minimal if it is represented by a map \( f = \bigvee_{\phi \in \Psi_j^{n+2}} \tilde{f}_\phi \), as in (5.21), for which \( \tilde{f}_\phi \) is a constant map (that is, takes values in degenerate 0-simplices) for all but the particular flag \( \phi_j := (0 < 1 < 2 \cdots < n - j) \) in \( \psi_j^{n+2} \) (corresponding to \( d_0d_0 \cdots \tilde{d}_0 \)), and the map \( f_{\psi_j} : \partial K_{\psi_j} \to \text{map}_{\mathcal{M}}(\overline{V}_{n+2}, V_j) \) is constant on all but the basic atomic \( k \)-simplex \( \tau_k = |d_0| \cdots |d_0| \) of \( K_{\psi_j} \) (Section 5.6) for each \( 1 \leq k \leq n - j \).

6.5. Remark. More generally, we could replace \( \phi_j \) by another map \( \phi' \in \Psi_j^{n+2} \), and simply require that \( \tilde{f}_{\phi'} \) be constant on all but one \( k \)-simplex \( \sigma_k \) of \( K_{\phi'} \) for each \( 1 \leq k \leq n - j \). Note that for minimal cases, \( \langle f \rangle \in \left\{ \bigvee_{\phi \in \Psi_j^{n+2}} \Sigma^{n-j} \overline{V}_{n+2}, V_j \right\} \) is completely determined by the homotopy class of \( \tilde{f}_{\phi_j} \) corresponding to a single map \( \Sigma^{n-j} \overline{V}_{n+2} \to V_j \).

6.6. Definition. A ladder diagram from \( \gamma_n = d_0d_0^n : \overline{V}_{n+2} \to \mathcal{Z}_n \mathbf{V}_{n+1} \) to \( \gamma_j : \Sigma^{n-j} \overline{V}_{n+2} \to \mathcal{Z}_j \mathbf{V}_{n+1} \) (cf. Section 4.10) is said to be equivalent to a ladder diagram from the same \( \gamma_n \) to \( \gamma'_j : \Sigma^{n-j} \overline{V}_{n+2} \to \mathcal{Z}_j \mathbf{V}_{n+1} \) if \( \gamma_j \sim \gamma'_j \).

6.7. Proposition. There is a bijection between equivalence classes of ladder diagrams from \( \gamma_n = d_0d_0^n : \overline{V}_{n+2} \to \mathcal{Z}_n \mathbf{V}_{n+1} \) to \( \gamma_j : \Sigma^{n-j} \overline{V}_{n+2} \to \mathcal{Z}_j \mathbf{V}_{n+1} \) and minimal values of the higher homotopy operation \( \langle f_{\gamma_j} \rangle \in \langle \psi_j^{n+2} \rangle \). Moreover, if \( \gamma_j \sim 0 \) then \( \langle f_{\gamma_j} \rangle \) vanishes (and conversely for the appropriate minimal value).

Proof. Given such a ladder diagram, we inductively define suitable simplicial functors \( \tilde{X}^k : \text{sk}_k \mathcal{Q}[n + 2, n - k] \to \mathcal{M}_{\mathcal{O}_+} \) extending the given \( X : \tilde{\Delta}^{n+1} \to \mathcal{M}_{\mathcal{O}} \), in the notation of (5.5). To do so, we only need to specify \( \tilde{X}^k \) on \( \tau_k \), with all other simplices of \( \mathcal{Q}[n + 2, n - k] \setminus F \tilde{\Delta}^{n+1} \)
sent to zero. The compatibility condition reduces to \( \partial_0 \hat{X}^k(\tau_k) = d_0^{V_{n-k}+2} \hat{X}^{k-1}(\tau_{k-1}) \) and \( \partial_i \hat{X}^k(\tau_k) = 0 \) for \( i > 0 \).

Given the map \( g_i : W \to C_i Y \) in a ladder diagram, let \( g'_i := j'_i \circ g_i \), where \( j'_i : C_i Y \to Y_i \) is the inclusion.

To begin, we define \( \hat{X}^0 : sk_0 Q[n+2, n] \to M_{O^+} \) by mapping \( \partial_0 \tau_1 = d_0 \) to \( g'_n \in M_{O^+}(\bar{V}_{n+2}, V_n) \). Recall that \( (\Delta[1] \otimes \bar{V}_{n+2})/(\Lambda_0^1 \otimes \bar{V}_{n+2}) \) provides a model for the cone \( Co(\bar{V}_{n+2}) \), where \( \Lambda_0^1 \subseteq \partial \Delta[n] \) is the horn omitting the \( k \)-th face. Hence, the map in the ladder diagram \( H_n : Co(\bar{V}_{n+2}) \to C_n V \subseteq V_n \) defines a map \( \bar{H}_n : \Delta[1] \otimes \bar{V}_{n+2} \to V_n \) whose restriction to \( \Lambda_0^1 \otimes \bar{V}_{n+2} \) is zero, and whose restriction to \( \partial \Delta[1] \otimes \bar{V}_{n+2} \) is \( g'_n \). The adjoint \( \bar{H}_n : \Delta[1] \to M_{O^+}(\bar{V}_{n+2}, V_n) \) of \( \bar{H}_n \) restricts to the zero map on the horn \( \Lambda_0^1 \). We can therefore extend \( \hat{X}^0 \) to a simplicial functor \( sk_1 Q[n+2, n] \to M_{O^+} \) by sending \( \tau_2 \) to \( \bar{H}_n \in M_{O^+}(\bar{V}_{n+2}, V_n) \), with \( d_1 \bar{H}_n = 0 \) and \( d_0 \bar{H}_n = g'_n \) by construction. Since \( d_0^{V_n} g'_n = 0 \), \( H_n \) induces a map

\[
\begin{array}{l}
g'_{n-1} : \Sigma \bar{V}_{n+2} \cong (\Delta[1] \otimes \bar{V}_{n+2})/(\partial \Delta[1] \otimes \bar{V}_{n+2}) \to C_{n-1} V \subseteq V_{n-1},
\end{array}
\]

and we define \( \hat{X}^1 : sk_1 Q[n+2, n-1] \to M_{O^+} \) extending the previous choices by sending \( \partial_0 \tau_2 \) to \( g'_{n-1} \in M_{O^+}(\bar{V}_{n+2}, V_{n-1}) \).

At the \( k \)-th stage, assume we have defined \( \hat{X}^k : sk_k Q[n+2, n-k] \to M_{O^+} \) sending \( \partial_0 \tau_{k+1} \) to \( g'_{n-k} \in M_{O^+}(\bar{V}_{n+2}, V_{n-k}) \). Note that \( (\Delta[k+1] \otimes \bar{V}_{n+2})/(\Lambda_0^{k+1} \otimes \bar{V}_{n+2}) \) is a model for \( Co(\Sigma^k \bar{V}_{n+2}) \), so \( H_{n-k} \) defines a map \( \bar{H}_{n-k} : \Delta[k+1] \to M_{O^+}(\bar{V}_{n+2}, V_{n-k}) \) whose restriction to \( \Lambda_0^{k+1} \) is zero. Viewed as a \( (k+1) \)-simplex in the mapping space, this means that \( d_i \bar{H}_{n-k} = 0 \) for \( i > 0 \), while \( d_0 \bar{H}_{n-k} = g'_{n-k} \). We may therefore extend \( \hat{X}^k : sk_k Q[n+2, n-k] \to M_{O^+} \) to \( \bar{H}_{n-k} \) by mapping \( \tau_{k+1} \) to \( \bar{H}_{n-k} \). Since \( d_0^{V_{n-k}} g'_{n-k} = 0 \), \( H_{n-k} \) induces a map

\[
\begin{array}{l}
g'_{n-k-1} : \Sigma^{n-k} \bar{V}_{n+2} \cong (\Delta[k+1] \otimes \bar{V}_{n+2})/(\partial \Delta[k+1] \otimes \bar{V}_{n+2}) \to C_{n-k-1} V \subseteq V_{n-k-1}.
\end{array}
\]

We define \( \hat{X}^{k+1} : sk_{k+1} Q[n+2, n-k-1] \to M_{O^+} \) extending the previous choices by sending \( \partial_0 \tau_{k+2} \) to \( g'_{n-k-1} \in M_{O^+}(\bar{V}_{n+2}, V_{n-k-1}) \).

Conversely, given \( \hat{X}^j : sk_j Q[n+2, n-j] \to M_{O^+} \) representing a minimal value of \( [\bar{V}_{\phi \in \psi_{n+2}^j} \Sigma^{n-j} \bar{V}_{n+2}, V_j] \), we define the corresponding ladder diagram by setting \( \bar{H}_n = \hat{X}^j(\tau_{n-m+1}) \), and \( g'_m = \hat{X}^j(\partial_0 \tau_{n-m+1}) \). Note that the adjoint \( H_n \) factors through \( C_m V \subseteq V_m \), since \( \partial_i \tau_{n-m+1} = 0 \) for \( i > 0 \), while the adjoint \( g'_m \) factors through \( Z_m V \subseteq V_m \) since \( \partial_i \partial_0 \tau_{n-m+1} = 0 \) for all \( i \).

Combining Proposition 6.7 with Proposition 4.14 yields:

**6.8. Corollary.** Under the assumptions of Section 4.1, the last higher operation \( \langle\langle \Psi_0^{n+2} \rangle\rangle \) has a minimal value. As a consequence, the operation is non-empty (well defined) and vanishes if any minimal value vanishes.
6.9. **Remark.** Note that minimal values of the higher operation are values of long Toda brackets of the form:

\[ V_{n+2} \xrightarrow{d_0} C_{n+1} V \xrightarrow{d_0^{n+1}} C_n V \xrightarrow{d_0^n} C_{n-1} V \xrightarrow{d_0^{n-1}} \cdots \xrightarrow{d_0^1} C_1 V \xrightarrow{d_0^0} V_0, \]

with linear indexing category, in which all but the first composite is strictly 0.

6.10. **Theorem.** In the situation of Section 4.1, the correspondence homomorphism \( \tilde{\Phi}_n \) maps a minimal value of \( \langle\langle \Psi_0^{n+2} \rangle\rangle \) to the André–Quillen obstruction \( \beta_n \) to realizing \( \Lambda \).

**Proof.** This follows from Propositions 4.14 and 6.7 with \( j = 0 \).

6.11. **Corollary.** The vanishing of any minimal value of \( \langle\langle \Psi_0^{n+2} \rangle\rangle \) implies the vanishing of \( \beta_n \). Conversely, given an \((n + 1)\)-semi-Postnikov section \( W_{\bullet}^{(n+1)} \) for \( \Lambda \), as in Section 4.4, there is an \((n + 1)\)-truncated simplicial object \( V_{\bullet}^{(n+1)} \) realizing \( \tau_{n+1}G_{\bullet} \), and if the cohomology obstruction \( \beta_{n+1} \) associated to \( W_{\bullet}^{(n+1)} \) vanishes, so does \( \langle\langle \Psi_0^{n+2} \rangle\rangle \) for \( V_{\bullet}^{(n+1)} \).

Note the (necessary) shift in indexing of the two series of obstructions, because of the different things they measure: the \( k \)-invariant \( \beta_n \) is the obstruction (4.5) to obtaining an \( n \)-semi-Postnikov section for \( \Lambda \), extending a given \((n - 1)\)-semi-Postnikov section, while the higher homotopy operation \( \langle\langle \Psi_0^{n+2} \rangle\rangle \) is the obstruction (Corollary 5.23) to constructing \( V_{\bullet}^{(n+2)} \) realizing the \((n + 2)\)-truncation of a given resolution \( G_{\bullet} \) for \( \Lambda \).

7. **Difference obstructions**

The next question arising in the inductive procedure for realizing a \( \Pi_{\mathcal{A}} \)-algebra \( \Lambda \) described in Section 4.1 is that of distinguishing between different extensions of a given \((n - 1)\)-semi-Postnikov section \( W_{\bullet}^{(n)} \) to an \( n \)-semi-Postnikov section \( W_{\bullet}^{(n+1)} \).

Recall that if \( \Lambda \) is realizable, then any \( X \in \mathcal{M} \) with \( \pi^\mathcal{A}_* X \cong \Lambda \) has a free resolution \( W_{\bullet} \xrightarrow{c(X)} E^2_{\lambda} \mathcal{M} \) (by [46]), with \( J W_{\bullet} \mathcal{A} \)-equivalent to \( X \) (Section 1.15), and \( \pi^\mathcal{A}_* W_{\bullet} \) a free \( \Pi_{\mathcal{A}} \)-algebra resolution of \( \Lambda \). Since \( J \) preserves weak equivalences, classifying realizations (in \( \mathcal{M} \)) of free simplicial resolutions of \( \Lambda \) subsumes (and refines) the classification of all realizations of \( \Lambda \) (in \( \mathcal{M} \)) up to \( \mathcal{A} \)-equivalence. However, every free simplicial \( \Pi_{\mathcal{A}} \)-algebra \( G_{\bullet} \) has a CW basis, and every CW resolution \( G_{\bullet} \xrightarrow{\sim} \Lambda \) can be realized as a resolution \( W_{\bullet} \xrightarrow{\sim} X \) in \( \mathcal{M} \). Thus we can in fact apply the inductive procedure described in Section 4.1, starting with a specific CW basis for \( G_{\bullet} \).

Once more, we have two methods of constructing the difference obstructions for inductively distinguishing between realizations of such a CW resolution \( G_{\bullet} \xrightarrow{\sim} \Lambda \): in terms of André–Quillen cohomology classes, and in terms of higher homotopy operations.
7.1. Cohomology difference obstructions

Again, there are a number of equivalent descriptions of the difference obstructions in cohomology, and we give one based on [14], but stated in terms of a fixed simplicial $\Pi_A$-algebra resolution $G_* \to A$ with given CW basis $(G_i)_{i=0}^\infty$.

Assume as in Section 4.1 that we have chosen a realization $V_{n+1}^{(n+1)}$ for $\tau_{n+1}G_*$, which has two different extensions $V_{n+1}^{(a)}$ and $V_{n+1}^{(b)}$, both realizing $\tau_{n+2}G_*$. Since we are not concerned now with the existence problem, we may assume that $V_{n+1}^{(a)}$ and $V_{n+1}^{(b)}$ are $(n+2)$-coskeletal objects (that is, $P_{n+1}$-simplicial objects) in $sM$, with $W_* := \text{csk}_{n+1} V_{n+1}^{(a)} = \text{csk}_{n+1} V_{n+1}^{(b)}$ as their common $n$-th Postnikov section. In particular, they are both $n$-quasi-Postnikov sections.

The question is whether $V_{n+1}^{(a)}$ and $V_{n+1}^{(b)}$ are weakly equivalent (relative to $W_*$), that is, whether there is a map $\varphi$ fitting into a commuting diagram of vertical fibration sequences in $sM/B\Lambda$, with horizontal weak equivalences:

$$
\begin{array}{ccc}
\mathbb{E}(\Omega^{n+1}A, n+1) & \xrightarrow{\varphi_*} & \mathbb{E}(\Omega^{n+1}A, n+1) \\
\downarrow & & \downarrow \\
V_{n+1}^{(a)} & \xrightarrow{\varphi} & V_{n+1}^{(b)} \\
\downarrow p_{n+1}^{(a)} & & \downarrow p_{n+1}^{(b)} \\
W_* & = & W_* \\
\downarrow k_n^{(a)} & & \downarrow k_n^{(b)} \\
\mathbb{E}(\Omega^{n+1}A, n+2) & \xrightarrow{W\varphi_*} & \mathbb{E}(\Omega^{n+1}A, n+2)
\end{array}
$$

(7.2)

where $p_{n+1}^{(t)} : V_{n+1}^{(t)} \to W_*$ are the structure maps in the Postnikov towers, and $k_n^{(t)} : W_* = \mathbb{P}^n V_{n+1}^{(t)} \to \mathbb{E}(\Omega^{n+1}A, n+2)$ are the (functorial) $k$-invariants for $V_{n+1}^{(t)}$ ($t = a, b$).

By [14, Prop. 8.7] (or [15, Prop. 5.3]), for any $\Lambda$-module $K$ there is a natural isomorphism

$$
\left[ W_*, E^s_{A}(K, n) \right]_{sM/BA} \to \left[ \pi_*^A W_*, E_{\Lambda}(K, n) \right]_{s\Pi_A-\text{Alg}/A} \cong H^n(\pi_*^A W_*/A; K)
$$

for $n \geqslant 2$.

Therefore, the $k$-invariants $k_n^{(t)}$ are determined by the induced maps of simplicial $\Pi_A$-algebras $(k_n^{(t)})_* : \pi_*^A W_* \to E_{\Lambda}(\Omega^{n+1}A, n+2)$ (where both source and target are Eilenberg–Mac Lane objects in $s\Pi_A-\text{Alg}$, by Section 4.4). Since the vertical fibration sequences of (7.2) induce long exact sequences in $\pi_*^A$ by [15, Lemma 5.11], we see that $(k_n^{(t)})_*$ is an isomorphism. Thus the choice of the $k$-invariants is determined solely by the Eilenberg–Mac Lane structure on $\pi_*^A W_*$, which is determined in turn by the choice of sections $s^{(a)}, s^{(b)} : \mathbb{P}^i \pi_*^A W_* \simeq B\Lambda \to \pi_*^A W_*$ (where $\mathbb{P}^i$ is the $i$-th Postnikov section in $s\Pi_A-\text{Alg}$).
Thus the cohomology difference obstruction for $V_\bullet^{(a)}$ and $V_\bullet^{(b)}$ (relative to $P^n V_\bullet^{(a)} = P^n V_\bullet^{(b)} = W_\bullet$) is defined to be

$$\delta_n := [s^{(a)}] - [s^{(b)}] \in H^{n+2}(\Lambda, \Omega^{n+1}\Lambda).$$

(7.3)

See [14, §8 and Proposition 9.12] and [15, Theorem 5.7(c)].

7.4. Proposition. Under the assumptions of Section 4.1 for $n \geq 1$, with the two attaching maps $\tilde{d}_0^t: \overline{V}_{n+2} \to Z_{n+1} V_\bullet^{(t)} = Z_{n+1} W_\bullet$ ($t = a, b$), the obstruction class $\delta_n$ is represented in the sense of Corollary 2.14 by the map $\tilde{\delta}_n: \pi^{\natural}_n V_\bullet^{(t)} \to H^{n+2}(\Lambda, \Omega^{n+1}\Lambda)$ induced by $\delta := \tilde{d}_0^a \cdot (\tilde{d}_0^b)^{-1}: \overline{V}_{n+2} \to Z_{n+1} W_\bullet$.

7.5. Remark. Note that even though $W_\bullet = P^n V_\bullet^{(t)}$ is only $(n + 1)$-coskeletal, it is an $n$-quasi-Postnikov section for $\Lambda$, and in either extension we have $Z_{n+1} V_\bullet^{(t)} = Z_{n+1} W_\bullet$. Thus we have a canonical identification $\pi^{\natural}_n V_\bullet^{(a)} \cong \pi^{\natural}_n V_\bullet^{(b)}$, induced by the identity on $Z_{n+1}$ (cf. (1.10)). Thus the difference $\tilde{d}_0^a \cdot (\tilde{d}_0^b)^{-1}$ makes sense; it is defined using the homotopy cogroup structure on $\overline{V}_{n+2}$.

Proof. Note that $G_\bullet$ is a cofibrant model for $B\Lambda$, while the sections $s^{(t)}$ evidently factor through $P^{n+1} \pi^A_\bullet V_\bullet^{(t)} \simeq P^{n+1} G_\bullet$ ($t = a, b$), and are thus induced in $\pi^A_\bullet$ by the Postnikov structure maps $p_{n+1}^{(t)}: V_\bullet^{(t)} \to W_\bullet$ in $s.M$. The cohomology class $\delta_n$ is thus represented by the difference $(p_{n+1}^{(a)})# - (p_{n+1}^{(b)})#$ (now taking values in abelian groups) mapping $P^{n+1} G_\bullet$ to $\pi^A_\bullet W_\bullet$. Since by assumption the maps $p_{n+1}^{(t)}$ are the identity through simplicial dimension $n + 1$, and their source is $(n + 2)$-coskeletal, the map $\tilde{\delta}$ is determined by what each map $p_{n+1}^{(t)}$ does to $(n + 2)$-simplices. As in the proof of Proposition 4.7, $(p_{n+1}^{(t)})#$ is determined by its value on the tautological class $\iota_{n+2} \in G_{n+2}(\overline{V}_{n+2})$, which maps to the class represented by the matching collection $\tilde{d}_0^{\overline{V}_{n+2}}(0, \ldots, 0) \in \text{Hom}_M(\overline{V}_{n+2}, M_{n+2} V_\bullet^{(t)})$, corresponding to $\tilde{d}_0^{\overline{V}_{n+2}}: \overline{V}_{n+2} \to Z_{n+1} V_\bullet^{(t)} = Z_{n+1} W_\bullet$. □

7.6. The representing cocycles

The cohomology class $\delta_n$ evidently vanishes if the cocycle representing it does – that is, if the attaching maps $\tilde{d}_0^t: \overline{V}_{n+2} \to Z_{n+1} W_\bullet$ for $V_\bullet^{(t)}$ ($t = a, b$) are homotopic – though the contrary need not hold. Note that if we post-compose the maps $\tilde{d}_0^t$ with the inclusion $j_{n+1}: Z_{n+1} W_\bullet \hookrightarrow C_{n+1} W_\bullet$, the resulting maps $j_{n+1} \circ \tilde{d}_0^t: \overline{V}_{n+2} \to C_{n+1} W_\bullet$ both represent $\tilde{d}_0^{\overline{V}_{n+2}}: \overline{G}_{n+2} \to C_{n+1} G_\bullet$, by (1.9), so that we again face a situation similar to that of Section 4.10, where we want to lift a null homotopy for $j_{n+1} \circ \tilde{\delta}: \overline{V}_{n+2} \to C_{n+1} W_\bullet$ to a null homotopy for $\tilde{\delta}$:

$$\begin{array}{ccc}
\overline{V}_{n+2} & \xrightarrow{i} & \text{Co}(\overline{V}_{n+2}) \\
\downarrow \tilde{\delta} & & \downarrow H' \\
Z_{n+1} W_\bullet & \xrightarrow{j_{n+1}} & C_{n+1} W_\bullet.
\end{array}$$

If we can do so, $V_\bullet^{(a)}$ and $V_\bullet^{(b)}$ are weakly equivalent (relative to $W_\bullet$).
We can therefore compare the André–Quillen difference obstruction of [14] and [15] (described above) with the construction of [9], as follows:

In the notation of [9, §4], for each \( t = a, b \) the attaching map \( \overline{d}^t_0 : V_{n+2} \to \mathbb{Z}_{n+1} V_{(t)} = \mathbb{Z}_{n+1} W_\bullet \) is determined by a map \( \lambda : \overline{G}_{n+2} \cong \pi_*^A V_{n+2} \to \pi_*^A \mathbb{Z}_{n+1} V_{(t)} \), fitting into the commuting diagram:

\[
\begin{array}{ccc}
\pi_*^A V_{n+2} & \xrightarrow{\lambda} & \pi_*^A \mathbb{Z}_{n+1} V_{(t)} \\
\overline{G}_{n+2} & \xrightarrow{\overline{d}^t_0} & \mathbb{Z}_{n+1} G_\bullet \cong \mathbb{Z}_{n+1}(\pi_*^A V_{(t)}) .
\end{array}
\]

(7.7)

The map \((j_{n+1})^\#\) is surjective by the commutative diagram before [11, §4.12], which also shows that there is a short exact sequence:

\[
0 \to \Omega \pi^2_n V_{(t)} \hookrightarrow \pi_*^A \mathbb{Z}_{n+1} V_{(t)} \xrightarrow{(j_{n+1})^\#} \mathbb{Z}_{n+1}(\pi_*^A V_{(t)}) \to 0.
\]

Thus the choices for the lift \( \lambda \), and thus the difference \( \delta \) between \( \overline{d}^a_0 \) and \( \overline{d}^b_0 \), are in fact parametrized by \( \Omega^{n+1} A \).

We have thus shown that the cohomology obstructions of [9, Theorem 4.18], described there in terms of the lift \( \lambda \), may be identified with \( \delta_n \) of (7.3).

7.8. The difference higher homotopy operation

The problem of lifting the null homotopy in (7.7) can be stated in terms of a ladder diagram as in Section 4.10, namely:

\[
\begin{array}{ccc}
V_{n+2} & \xrightarrow{\delta} & \text{Co}(V_{n+2}) \xrightarrow{H_m} \Sigma V_{n+2} \xrightarrow{\gamma_{m-1}} \cdots \\
\mathbb{Z}_{n+1} W_\bullet & \xrightarrow{j_m} & C_{n+1} W_\bullet \xrightarrow{d_0} \mathbb{Z}_n W_\bullet \xrightarrow{\gamma_n} \cdots \\
\end{array}
\]

This in turn can be recast as a special instance of rectifying a suitable homotopy commutative diagram (compare Proposition 6.7).

However, to avoid describing this diagram explicitly, we instead define a certain \((n + 2)\)-truncated simplicial object \( Y_\bullet \) and an attaching map \( \overline{d}^{n+3}_0 : \overline{V}_{n+3} \to \mathbb{Z}_{n+2} Y_\bullet \) which allows us to use the definitions of Section 5 verbatim, in the setting of Section 7.1:

We begin with the \((n + 1)\)-truncation \( \tau_{n+1} Y_\bullet := \tau_{n+1} W_\bullet \), and set \( Y_{n+2} := \mathbb{Z}_{n+1} W_\bullet \) with \( d^{n+2}_0 \) the inclusion \( i_{n+1} : \mathbb{Z}_{n+1} Y_\bullet \hookrightarrow Y_{n+1} \) and \( d^{n+2}_i = 0 \) for \( i > 0 \). This indeed constitutes an \((n + 2)\)-truncated simplicial object, since \( d^{n+1}_i d^{n+2}_j = 0 \) for any \( 0 \leq i, j \leq n + 2 \). Now set \( \overline{V}_{n+3} := \overline{V}_{n+2} \) and let \( \overline{d}^{n+3}_0 = \overline{\delta} \), which lands in \( \mathbb{Z}_{n+1} V_{\bullet}^{(n+1)} = Y_{n+2} = C_{n+2} Y_\bullet \) by construction. However, since \( \mathbb{Z}_{n+2} Y_\bullet = 0 \) (since \( d^{n+2}_0 \) is monic) it follows that \( \overline{\delta} = 0 \) if and only if it factors through
\[ Z_{n+2}Y_\bullet \). Note that \( d_0^{n+2} \circ \bar{d}_0^{n+3} = i_{n+1} \circ \bar{\delta} \) is null homotopic, since \( i_{n+1} \circ \bar{d}_0^a \) and \( i_{n+1} \circ \bar{d}_0^b \) both realize \( \bar{d}_0^{G_{n+2}} : \bar{\mathcal{S}}_{n+2} \to C_{n+1}G_\bullet \), by (1.9).

Thus the \((n + 2)\)-truncated simplicial object \( \mathcal{T}_{n+2}Y_\bullet \), together with \( \bar{\mathcal{Y}}_{n+3} \) and \( \bar{d}_0^{T_{n+3}} \), satisfies the assumptions of Section 4.1, and we may therefore apply the constructions of Section 5 to make the following:

7.9. Definition. In the setting of Section 7.1, the higher homotopy operation difference obstruction associated to \( \bar{d}_0^a \) and \( \bar{d}_0^b \) is defined to be the subset of \( [\Sigma^{n+1}\bar{\mathcal{Y}}_{n+3}, Y_\bullet] = [\Sigma^{n+1}\bar{\mathcal{Y}}_{n+2}, V_\bullet] \) associated as in Section 5.20 to \( Y_\bullet \). We denote this subset by \( \langle\langle \bar{d}_0^a, \bar{d}_0^b \rangle\rangle \).

Note that the source is suspended once further than in the existence case. Thus, the correspondence homomorphism yields a map \( \bar{\mathcal{S}}_{n+2} \to \Omega^{n+1}\Lambda \), which represents a cohomology class in \( H^{n+2}(\Lambda, \Omega^{n+1}\Lambda) \), with coefficient module determined by the target in Proposition 2.4.

7.10. Proposition. In the setting of Section 7.1, the difference obstruction \( \langle\langle \bar{d}_0^a, \bar{d}_0^b \rangle\rangle \) always has a minimal value, so it is well defined (and non-empty). It vanishes if and only if \( V_\bullet^{(a)} \simeq V_\bullet^{(b)} \).

Proof. Corollary 6.8 applies here, too, so \( \langle\langle \bar{d}_0^a, \bar{d}_0^b \rangle\rangle \) has a minimal value. Combining Proposition 6.7 with Proposition 4.14 for \( Y_\bullet \), we see that vanishing of some value implies that \( \bar{\delta} \) can be chosen to factor through \( Z_{n+2}Y_\bullet = 0 \). Consequently \( [\bar{\delta}] = 0 \), so \( \bar{d}_0^a \sim \bar{d}_0^b \). Hence \( V_\bullet^{(a)} \simeq V_\bullet^{(b)} \).

The proof of Theorem 6.10 transfers word for word to our setting to show:

7.11. Theorem. The correspondence homomorphism \( \bar{\Phi}_n(Y_\bullet) \) maps a minimal value of \( \langle\langle \bar{d}_0^a, \bar{d}_0^b \rangle\rangle \) to the André–Quillen difference obstruction between the two realizations of \( \Lambda \).

7.12. Remark. The rectification problem of Section 7.8 is somewhat unsatisfactory, in that the truncated simplicial object \( Y_\bullet \) which we are trying to rectify does not consist solely of “wedges of spheres” (objects in \( \mathcal{M}_A \)) in each simplicial dimension, so that (unlike the existence obstructions of Section 5.20) the higher homotopy operation of Section 7.9 cannot be described in terms of \( \Pi_\Lambda = \Lambda \).

However, \( s\mathcal{M} \) is a pointed simplicial category in the sense of [40, II, §1], and for any \( A \in \mathcal{M} \) we have an object \( S^n \wedge A := (S^n \otimes A)/(\{\text{pt}\} \otimes A) \) in \( s\mathcal{M} \) (having \( A \) in dimension \( n \) and \( * \) below), with a natural bijection between maps \( s\mathcal{M}(S^{n+1} \wedge A, W_\bullet) \) and maps \( \mathcal{M}(A, Z_{n+1}W_\bullet) \). Therefore, \( \bar{\delta} \sim * \) in Proposition 7.4 – or equivalently, \( \bar{d}_0^a \sim \bar{d}_0^b : \bar{\mathcal{Y}}_{n+2} \to Z_{n+1}W_\bullet \) – if and only if the diagram:

\[
\begin{array}{ccc}
S^{n+1} \wedge \bar{\mathcal{Y}}_{n+2} & \xrightarrow{\bar{d}_0^a} & W_\bullet \\
\bar{d}_0^b & \sim & \end{array}
\]

(7.13)

homotopy commutes (i.e., the corresponding maps \( \bar{d}_0^a \) and \( \bar{d}_0^b \) are homotopic in \( s\mathcal{M} \)). Thus the vanishing of the higher homotopy operation \( \langle\langle \bar{d}_0^a, \bar{d}_0^b \rangle\rangle \) is equivalent to rectifying the diagram (7.13), whose entries are all in \( \mathcal{M}_A \).
7.14. The geometric correspondence homomorphism

In the setting of Section 7.1, assume that we have changed the two maps \( \tilde{d}^a_0 \) and \( \tilde{d}^b_0 \) in (7.13) into cofibrations, so that we have cofibration sequences:

\[
S^{n+1} \wedge \mathbb{V}_{n+2} \xrightarrow{\tilde{d}^{a}_0} W_\bullet \rightarrow \text{sk}_{n+2} V^{(t)}_\bullet \quad (t = a, b)
\]

in \( sM \) (cf. [40, I, §2]).

If \( \tilde{d}^a_0 \sim \tilde{d}^b_0 \), we have the solid homotopy-commutative diagram with vertical weak equivalences:

\[
\begin{array}{ccc}
S^{n+1} \wedge \mathbb{V}_{n+2} & \xrightarrow{\tilde{d}^{a}_0} & W_\bullet \\
\downarrow & = & \downarrow \\
S^{n+1} \wedge \mathbb{V}_{n+2} & \xrightarrow{\tilde{d}^{b}_0} & W_\bullet
\end{array}
\]

\[
\xrightarrow{\sim} \phi
\]

(7.15)

inducing the dotted weak equivalence \( \phi \).

In general, since homotopy colimits preserve weak equivalences and commute with each other, applying hocolim over \( \Delta^{op} \) to the top row of (7.15) yields a horizontal homotopy cofibration sequence (in \( M \)):

\[
\begin{array}{ccc}
\Sigma^{n+1} \mathbb{V}_{n+2} & \xrightarrow{\text{hocolim} \tilde{d}^{a}_0} & \text{hocolim} W_\bullet \\
\downarrow & = & \downarrow \\
\Sigma^{n+1} \mathbb{V}_{n+2} & \xrightarrow{\text{hocolim} \tilde{d}^{b}_0} & \text{hocolim} W_\bullet
\end{array}
\]

\[
\xrightarrow{\sim} \phi
\]

because hocolim(\( S^{n+1} \wedge \mathbb{V}_{n+2} \)) \( \simeq \Sigma^{n+1} \mathbb{V}_{n+2} \) by construction.

Note that \( g \) exists (making the diagram commute up to homotopy) if and only if the class \( \Theta^{(b)}_{n+2} := \text{hocolim} \text{hocolim} i^b \tilde{d}^b_0 \) is zero. A sufficient (but not necessary) condition for this to happen is that the left square in (7.15) commutes up to homotopy (in which case \( g := \text{hocolim} \phi \) is a weak equivalence).

Therefore, we can think of \( \Theta^{(b)}_{n+2} \) in \( [\Sigma^{n+1} \mathbb{V}_{n+2}, \text{hocolim} \text{sk}_{n+2} V^{(b)}_\bullet] \) as the obstruction to extending the weak equivalence \( \text{sk}_{n+1} V^{(a)}_\bullet \rightarrow \text{sk}_{n+1} W_\bullet \rightarrow \text{sk}_{n+1} V^{(b)}_\bullet \) to a weak equivalence \( \text{sk}_{n+2} V^{(a)}_\bullet \rightarrow \text{sk}_{n+2} V^{(b)}_\bullet \).

Now assume that we have two full simplicial realizations \( V^{(a)}_\bullet \) and \( V^{(b)}_\bullet \) of \( G_\bullet \to \Lambda \) in \( sM \), and that the two associated realizations \( X^{(t)} := J V^{(t)}_\bullet \) of \( \Lambda \) in \( M \) (cf. Section 1.15) are given by hocolim \( V^{(t)}_\bullet \). In this case, we can think of the filtrations \( \mathcal{T}_n X^{(t)} := \text{hocolim} \text{sk}_n V^{(t)}_\bullet \) \( (n \geq 0) \) as successive approximations to the objects \( X^{(t)} \), where each inclusion \( \text{sk}_n V^{(t)}_\bullet \hookrightarrow V^{(t)}_\bullet \) induces a map \( j^{(t)}_n : \mathcal{T}_n X^{(t)} \to X^{(t)} \). Moreover, the resulting sequence of obstructions are just traditional “higher Toda brackets” (cf. [49]) appearing in trying to extend the inclusion of hocolim \( \text{sk}_0 V^{(a)}_\bullet = W_0 \) into hocolim \( \text{sk}_0 V^{(b)}_\bullet \). There is also an “inverse” obstruction \( \Theta^{(a)}_{n+2} \) in \( [\Sigma^{n+1} \mathbb{V}_{n+2}, \text{hocolim} \text{sk}_{n+2} V^{(a)}_\bullet] \). Note that one vanishes if and only if the other does.
Considering $\Theta^{(b)}_{n+2}$ as an element in $(\Omega^{n+1}^{A}F_{n+2}X^{(b)})[V_{n+2}]$, and applying $(j_{n}^{(t)})_{\#}$ to $\Theta^{(b)}_{n+2}$ yields a class in:

$$\left(\Omega^{n+1}^{A}X^{(b)}\right)[V_{n+2}] = \left(\Omega^{n+1}A\right)[V_{n+2}] \cong \text{Hom}_{\Pi_{A}^{A}-\text{Alg}}(G_{n+2}, \Omega^{n+1}A).$$

The procedure defined above should therefore be thought of as a “geometric version” of the correspondence homomorphism $\widetilde{\Phi}_{n}(Y_{*})$ of Section 6.3 applied to a minimal value of $(\langle d_{0}^{n}, d_{0}^{1}\rangle)$.

To see this, we specialize to the case where $\mathcal{M} = S_{e}$ and $A = \{S^{1}\}$. In this case the simplicial sets $F_{n}X^{(t)} := \text{hocolim} \text{sk}_{n}V_{*}^{(t)}$ are indeed successive approximations to $X^{(t)}$, since from (4.3) and the Bousfield–Friedlander spectral sequence (cf. [21, Theorem B.5]), we see that $\pi_{*}F_{n}X^{(t)}$ agrees with $A$ through dimension $n$ (and in fact much more is true: see [5, §10]).

Moreover, we can use the description of the $E_{n+1}$-term of the spectral sequence for $\text{sk}_{n+1}V_{*}^{(b)}$ in [4, §3] (combined with [27, §§4]) to see that the minimal value for the higher homotopy operation $\langle d_{0}^{n}, d_{0}^{1}\rangle$ is defined precisely when the differential $d_{n+1}^{0}$ vanishes on the element in $E_{n+2,0}$ represented by $\delta \in (\pi_{*}C_{n+1})[V_{n+2}] = E_{n+2,0}$. In this case $[\delta]$ lifts to $C_{0}W_{*} = W_{0}$, and post-composing with the structure map $W_{0} \to \text{hocolim} \text{sk}_{n+1}V_{*}^{(b)}$ yields (one value for) $\Theta^{(b)}_{n+2}$ as constructed above.

8. An application to rational homotopy

Let $\mathcal{M} = \mathcal{L}_{Q}$ be the category of reduced differential graded Lie algebras (DGLs) over $Q$ – a model for the rational homotopy theory of simply-connected pointed spaces (cf. [41, §4]). If we let our collection $A$ of homotopy cogroup objects consist of the rational DGL 2-sphere $S_{Q}^{2}$ in $\mathcal{L}_{Q}$, we see that a $\Pi_{A}$-algebra is just a connected graded Lie algebra over $Q$. Note the shift in dimension, due to the fact that we use $\pi_{*}(\Omega X; Q)$ as the homotopy $\Pi_{A}$-algebra of $X_{Q}$, so we have Samelson products, which respect the grading of $\pi_{*}(\Omega X; Q)$, rather than Whitehead products.

By Hilton’s theorem (cf. [34, Theorem A]), if $W$ is a wedge of rational spheres of dimension $\geq 2$, $\Gamma := \pi_{*}(\Omega W; Q)$ is a free Lie algebra, so $\Gamma$ is (intrinsically) coformal – that is, $\Gamma$, equipped with zero differentials, is itself a minimal DGL model for $W_{Q}$. Moreover, no higher $A$-homotopy operations exist in $\pi_{*}(\Omega W; Q)$, because no non-trivial rational homotopies exist for maps between (fibrant and cofibrant) DGLs with zero differential.

Thus any free simplicial $\Pi_{A}$-algebra resolution $G_{*} \to A$ of a rational $\Pi$-algebra $A$ is canonically realizable by a (strict) simplicial DGL $W_{*} \in s\mathcal{L}_{Q}$, whose geometric realization $\|W_{*}\|$ is the coformal realization of $A$ (unique up to weak equivalence in $\mathcal{M}$).

Since this $W_{*}$ is coformal in each simplicial dimension, all (higher) homotopies used to define all values of the existence obstruction $(\langle \Psi_{0}^{n+2}\rangle)$ based on the $(n+1)$-truncation of $W_{*}$, for any $n \geq 1$, necessarily vanish. In particular, this will hold for any of the minimal values (which exist by Corollary 6.8). In light of Theorem 6.10, this implies:

8.1. Proposition. If $\mathcal{M} = \mathcal{L}_{Q}$, $A := \{S_{Q}^{2}\}$, and $A \in \Pi_{A}-\text{Alg}$, all the André–Quillen obstructions $b_{n}$ to realizing $A$ vanish (for one branch of the inductive procedure in Section 3.2).

Of course, by Corollaries 5.23 and 6.11, this implies in turn Quillen’s corollary to [41, Theorem 1], stating that any simply-connected rational $\Pi$-algebra $A$ is realizable.
8.2. Remark. Note that if we replace $W_\bullet$ by a weakly equivalent Reedy fibrant simplicial object $W'_\bullet$, the differentials in each DGL $W_i$ need no longer vanish; so we cannot apply the above argument (for the vanishing of the higher homotopy operations) to calculating the cocycle representing the cohomology obstruction directly.

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