Wave optics and image formation in gravitational lensing

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Abstract. We discuss image formation in gravitational lensing system using wave optics. Applying the Fresnel-Kirchhoff diffraction formula to waves scattered by a gravitational potential of a lens object, we demonstrate how images of source objects are obtained directly from waves without using a lens equation for gravitational lensing.

1. Introduction
Gravitational lensing is one of the prediction of Einstein’s general theory of relativity and many samples of images caused by gravitational lensing have been obtained observationally [1]. Light rays obey null geodesics in curved spacetime and they are deflected by the gravitational potential of lens objects. In weak gravitational field with thin lens approximation, a path of a light ray obeys the so called lens equation for gravitational lensing and many analysis concerning the gravitational lensing effect have been carried out based on this equation. Especially, we can obtain images of source objects by solving the lens equation using a ray tracing method. Since the path of light ray is derived as the high frequency limit of electromagnetic wave, wave effects of gravitational lensing become significant when the wavelength is not so much smaller than the size of lens objects and in such a situation, we must take into account of wave effects. For example, when we consider a gravitational wave which is scattered by gravitational lens objects, the wave effect gives significant impact on the amplification factor of intensity for waves [2, 3, 4]. Another example is the direct detection of black holes via their shadows [5, 6]. The apparent angular size of black hole shadows are so small that their detectability depends on the angular resolution of telescopes which is related to the diffraction limit of the image formation system.

Although interference and diffraction of waves by gravitational lensing have been discussed in connection with amplification of waves, little has been discussed about how images by gravitational lensing are obtained based on wave optics. For an electromagnetic wave, E. Herlt and H. Stephani [7] discussed the position of images by a spherical gravitational lens evaluating the Poynting flux of scattered wave at an observer. They claimed that there is a disagreement between wave optics and geometrical optics concerning the position of double images of a point source. But they have not presented a complete understanding of image formation. In this paper, we consider the image formation in gravitational lensing using wave optics and aim to understand how images by gravitational lensing are obtained in terms of waves. For this purpose, we adopt the diffraction theory of image formation in wave optics [8], which explains image formation in optical systems in terms of diffraction of waves.

2. Image formation based on wave optics
2.1. Image formation by a convex lens
To establish relation between the interference pattern of the wave and the images of the source in the gravitational lensing system, we first consider an image formation system composed of a single convex lens (Figure 1). We assume the distribution of the source field with the angular
frequency $\omega$ on the object plane $z = -a$ as $\Phi_0(x_0)$. Using the Fresnel-Kirchhoff diffraction formula, the amplitude of the wave at $x$ on the $z = b$ plane behind the lens is
\[
\Phi_2(x) \propto \int d^2x_0 \Phi_0(x_0) t_L(x_1) \exp \left[ i\omega \left( \frac{1}{2} \left( \frac{1}{a} + \frac{1}{b} - \frac{1}{f} \right) |x_1|^2 - \left( \frac{x_0}{a} + \frac{x}{b} \right) \cdot x_1 \right) \right] \times \exp \left[ i\omega \left( \frac{|x_0|^2}{a} + \frac{|x_1|^2}{b} \right) \right] \tag{1}
\]
where $x_1$ is the location of the light path on the lens plane, $t_L$ is the aperture function and $f$ represents the focal length of the lens. For a value of $b$ satisfying the lens equation for a convex thin lens, i.e. $1/a + 1/b = 1/f$, the wave amplitude becomes
\[
\Phi_2(x) \propto \int d^2x_0 \Phi_0(x_0) \left( \frac{2J_1(\omega D|x/b + x_0/a|)}{\omega D|x/b + x_0/a|} \right) \exp \left[ i\omega \frac{|x_0|^2}{2a} \right]. \tag{2}
\]
For $\omega D \to \infty$ limit, the Bessel function, i.e. $J_1$, in (2) becomes the delta function and we obtain the following wave amplitude on $z = b$:
\[
\Phi_2(x) \propto \int d^2x_0 \Phi_0(x_0) \times \delta^2 \left[ \frac{x_0}{a} + \frac{x}{b} \right] = \Phi_0 \left( -\frac{a}{b}x \right). \tag{3}
\]
Thus, a magnified image of the source field appears on the $z = b$ plane. This reproduces the result of image formation in geometric optics.

![Image of gravitational lens system](image)

**Figure 1.** The left panel: one lens image formation system. The right panel: configuration of a gravitational lens with a convex lens system. $r_L$ is the distance from the observer to the lens object and $r_S$ is the distance from the observer to the source. $r_{LS} = r_S - r_L$.

### 2.2. Image formation in gravitational lens system

As we have observed that a convex lens can be a device for image formation in wave optics, we combine it with a gravitational lensing system and obtain images by gravitational lensing. We consider a configuration of the gravitational lens system shown in the right panel in Figure 1 and examine how the images of the source object appear using wave optics. As source object, we assume a point source of wave with intensity $a_0$. For a monochromatic wave with angular frequency $\omega$, the wave function obeys
\[
(\nabla^2 + \omega^2)\Phi = 4\omega^2 U(r)\Phi \tag{4}
\]
where $U$ is the gravitational potential of the lens object. Introducing the two dimensional gravitational potential $\psi(\xi) = 2 \int_{-\infty}^{\infty} dz U(\xi, z)$, the wave amplitude on the image plane becomes
\[
\Phi_1(y, d) = \frac{a_0}{r_S} \int_{|d'| \leq d_0} d^2d' F(y + d') \times \exp \left[ iw \left( \frac{r_{LS}}{2r_{LS}} \left( \frac{1}{r_S} + \frac{1}{z^2} - \frac{1}{f} \right) d^2 - \left( y + \frac{r_{LS}}{r_L} d \right) \cdot d' + \frac{1}{2} \left( \frac{r_{LS}}{r_L} y^2 + \frac{r_{LS}}{r_L} z^2 d^2 \right) \right) \right] \tag{5}
\]
where $U$ is the gravitational potential of the lens object.
where \( F \) is the amplification factor by a gravitational lensing given by
\[
F(y, d) = \frac{w}{2\pi i} \int d^2 x \exp \left[ iw \left( \frac{1}{2}(x - y - d)^2 - \psi(x) \right) \right]
\] (6)
and the following dimensionless variables are introduced
\[
x = \frac{\xi}{\xi_0}, \quad y = \frac{r_L}{r_S} \eta, \quad d = \left( 1 - \frac{r_L}{r_S} \right) \frac{\Delta}{\xi_0}, \quad w = \frac{r_S \xi_0^2}{r_S \xi_0^2 + 1} \omega, \quad \psi = \frac{r_L r_{LS}}{r_S \xi_0^2} \tilde{\psi}
\]
and we choose \( \xi_0 \) as \( \xi_0 = r_L \theta_E, \theta_E = \sqrt{4M r_{LS} / (r_L r_S)} \). \( M \) is the mass of the gravitational source, \( \xi_0 \) and \( \theta_E \) represent the Einstein radius and the Einstein angle, respectively.

If we choose the location of the image plane \( z_2 \) to satisfy the following “lens equation” for a convex lens \( 1/r_S + 1/z_2 = 1/f \), then the wave amplitude on the image plane becomes
\[
\Phi_f(y, d) = \frac{a_0}{r_S} \int_{|d'| \leq d_0} d^2 d' F(y + d') \exp \left[ -iw \left( y + \frac{r_L r_{LS}}{r_S f} d' \right) \cdot d' \right] \times e^{i(w/2)\sigma(y,d)}
\] (7)
where \( g = (r_L r_{LS}/r_L) y^2 + (r_L r_{LS}/r_L z_2) d^2 \). Thus the wave amplitude on the image plane is the Fourier transform of the amplification factor \( F \) which gives the interference fringe pattern. Under the geometrical optics limit \( w \gg 1 \), \( x \) integral in the amplification factor (6) can be approximated by the WKB form \( F(y + d') \approx A e^{i\omega [\frac{1}{2}(x_\ast - y - d')^2 - \psi(x_\ast)]} \), where \( x_\ast(y, d') \) is the solution of the lens equation
\[
0 = x - y - d' - \nabla_x \psi(x) \approx x - y - \nabla_x \psi(x).
\] (8)
We have assumed that the aperture of the convex lens is sufficiently smaller than the size of the gravitational lensing system and \( |d'| \leq d_0 \ll 1 \) holds. Then, the wave amplitude on the image plane is
\[
\Phi_f(y, d) \propto A e^{i\omega [\frac{1}{2}(x_\ast - y)^2 - \psi(x_\ast)]} e^{i\omega g(y,d)/2} \times 2\pi d_0^2 \frac{J_1(w|x_\ast + \beta d|d_0)}{w|x_\ast + \beta d|d_0}, \quad \beta = \frac{r_L r_{LS}}{r_S f}
\] (9)
For \( w d_0 \to \infty \) limit (large lens aperture limit or high frequency limit), we obtain \( \Phi_f(y, d) \propto \delta^2 \left[ x_\ast(y) + \frac{r_L r_{LS}}{r_S f} d \right] \) and the image of the point source appears at the location on the image plane determined by the lens equation (8). This reproduces the same result of image formation in the geometrical optics (ray tracing) in terms of the wave optics.

As an example of image formation in a gravitational lensing system using wave optics, we present the wave optical images of a point source by the gravitational lensing of a point mass (Figure 2). These images are obtained by Fourier transformation of the amplification factor \( F \) (equation (7)). This procedure corresponds to image formation by a convex lens. We can observe the wave effect in these images.

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Figure 2. Wave optical images of a point source by the gravitational lensing of a point mass. Parameters are \( w = 40, d_0 = 0.5 \) (aperture of a convex lens), \( y = 0, 0.5, 1, 1.5 \).
In each one of these images, we observe a concentric interference pattern which is caused by the finite size of the lens aperture and this is not an intrinsic feature of the gravitational lensing system. We can also observe radial non-concentric patterns. These are caused by interference between double images and represent the intrinsic feature of the gravitational lensing system. For \( y = 0 \) case which corresponds to the Einstein ring in the geometrical optics limit, we can observe a bright spot at the center of the ring, which is the result of constructive interference and does not appear in geometric optics. For sufficiently large values of \( wd_0 \), the wave amplitude at the observer coincides with the result obtained by geometric optics.

3. Summary

We have investigated image formation in gravitational lensing system based on wave optics. Instead of using a ray tracing method, we obtained images directly from wave functions at the observer. For this purpose, we introduced a “telescope” with a single convex thin lens, which acts as a Fourier transformer for the interference pattern formed at an observer. The analysis in this article relates the wave amplitude and images by the gravitational lensing directly. In the geometric optics limit of waves, images by lens systems are obtained by a lens equation which determines paths of each light rays. As light rays are trajectories of massless test particles (photons), expressing images in terms of waves is like expressing particles in terms of waves.

As an application and extension of analysis presented in this paper, we plan to investigate gravitational lensing by a black hole and obtain wave optical images of black holes. This is related to the topic of observation of black hole shadows \([5, 6]\). As the apparent angular sizes of black hole shadows are so small, the diffraction effect on images is crucial to resolve black hole shadows in observation using radio interferometer. For SgrA*, which is the black hole candidate at Galactic center, the apparent angular size of its shadow is estimated to be \( \sim 30\mu \) arc seconds and this value is the largest among black hole candidates. For a sub-mm VLBI with a baseline length \( D \), using the resolving power of images \( \theta_0 = \lambda/D \) determined by the diffraction limit, the condition to resolve the shadow requires \( D > 1000 \text{ km} \), and this shows the possibility to detect the black hole shadow of SgrA* using the present day technology of VLBI telescope. Thus, analysis of black hole shadows based on wave optics is important to evaluate detectability of shadows and determination of black hole parameters via imaging of black holes.

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