Abstract. In the paper, we give an explicit basis of the cyclotomic quiver Hecke algebra corresponding to a minuscule representation of finite type.

Introduction

The quiver Hecke algebras (also known as Khovanov-Lauda-Rouquier algebras) were introduced by Khovanov and Lauda [13, 14] and independently Rouquier [16] to categorify the half of a quantum group. For a dominant integral weight $\Lambda$, the quiver Hecke algebras have special quotient algebras $R^\Lambda$ which give a categorification of the irreducible highest weight module $V(\Lambda)$ [8]. In the viewpoint of categorification, these algebras have been the focus of many studies and various new features were discovered.

In the paper, we give an explicit basis of the cyclotomic quiver Hecke algebra corresponding to a minuscule representation in terms of crystals. A minuscule representation is an irreducible highest weight $U_q(\mathfrak{g})$-module $V(\Lambda_i)$ such that the Weyl group acts transitively on its weight space. The possible types for minuscule representations are $A_n$, $B_n$, $C_n$, $D_n$, $E_6$ and $E_7$, and the nontrivial highest weights of minuscule representations are marked by $\circ$ in (1.2). Let $V(\Lambda_i)$ be a minuscule representation. Since every weight space $V(\Lambda_i)_\xi$ is extremal, the corresponding cyclotomic quiver Hecke algebra $R^{\Lambda_i}(\xi)$ is simple. Thus, since every irreducible $R$-module is absolutely irreducible, we obtain an explicit basis of $R^{\Lambda_i}(\xi)$ by constructing a simple $R^{\Lambda_i}(\xi)$-module combinatorially in terms of the crystal $B(\Lambda_i)$. Our main tool is a homogeneous representation, which was introduced by Kleshchev and Ram [15]. In [15], they gave a combinatorial construction of homogeneous representations of simply-laced

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type and investigated a connection with fully commutative elements including dominant minuscule elements. The homogeneous representations of type $A_n$ and $D_n$ are classified and enumerated in [4, 5] by studying fully commutative elements.

For simply-laced $ADE$ type, essential parts for the main theorem come from the results of [15]. Let $b \in B(\Lambda_i)$, $\xi = \text{wt}(b)$ and $\mathcal{P}(b)$ the set of all paths from $b_{\Lambda_i}$ to $b$ on the crystal $B(\Lambda_i)$ (see (2.2)). We show that $\mathcal{P}(b)$ is in 1-1 correspondence to the set of reduced expressions of the minuscule element $w(b)$, which says that $\mathcal{P}(b)$ has a homogeneous $R_{\Lambda_i}(\xi)$-module structure by the result of [15] (see Theorem 2.8). For type $C_n$, since $V(\Lambda_n)\xi$ is 1-dimensional for all weights $\xi$, we easily obtain that the algebra $R_{\Lambda_1}(\xi)$ is trivial (see Proposition 2.10). The main novelty of this paper lies in the case of type $B_n$. Note that, since $B_n$ is not simply-laced, we cannot use the result of [15] directly. In type $B_n$, we show that there is a 1-1 correspondence between the set $\mathcal{P}_n$ of all strict partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \leq n$ and the Weyl group orbit $W \cdot \Lambda_1$. We then give a homogeneous $R_{\Lambda_1}(\xi)$-module structure on the set $\text{ST}(\lambda)$ of standard tableaux of shape $\lambda$ using combinatorics of shifted Young tableaux. Thus, we obtain a $k$-basis of $R_{\Lambda_1}(\xi)$ indexed by the set of all strict partitions of shape $\lambda$.

This paper is organized as follows: In Section 1, we review briefly minuscule representations of quantum groups of finite types. In Section 2, we review cyclotomic quiver Hecke algebras and show an explicit basis of $R_{\Lambda_1}(\xi)$ corresponding to minuscule representations for finite types $ADE$ and $C$. In Section 3, we construct simple $R_{\Lambda_1}$-module of type $B_n$ using combinatorics of shifted Young tableaux and show an explicit basis of $R_{\Lambda_1}(\xi)$.

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1. Minuscule representations

1.1. Quantum groups. Let $I = \{1, 2, \ldots, n\}$. A quintuple $(A, P, \Pi, P^\vee, \Pi^\vee)$ is called a Cartan datum if it consists of

(a) a generalized Cartan matrix $A = (a_{ij})_{i,j \in I}$,
(b) a free abelian group $P$, called the weight lattice,
(c) $\Pi = \{\alpha_i \mid i \in I\} \subset P$, called the set of simple roots,
(d) $P^\vee = \text{Hom}_\mathbb{Z}(P, \mathbb{Z})$, called the coweight lattice,
(e) $\Pi^\vee = \{h_i \in P^\vee \mid i \in I\}$, called the set of simple coroots

satisfying the following:
1.2. Minuscule representations. We write \( \Delta \) for the set of all roots associated with \( A \), and denote by \( W \) the Weyl group, which is the subgroup of \( \text{Aut}(P) \) generated by \( s_i(\lambda) := \lambda - \langle h_i, \lambda \rangle \alpha_i \) for \( i \in I \).

For \( \lambda \in P_+ \), an element \( w \in W \) is \( \lambda \)-minuscule if there exists a reduced expression \( w = s_{i_1} \cdots s_{i_\ell} \) such that

\[
\langle h_{i_k}, s_{i_{k+1}}s_{i_{k+2}} \cdots s_{i_\ell} \lambda \rangle = 1 \quad \text{for } k = 1, \ldots, \ell.
\]

A weight \( \lambda \in P \) is called minuscule if \( \langle \alpha_\vee, \lambda \rangle \in \{0, \pm 1\} \) for all \( \alpha \in \Delta \), where \( \alpha_\vee \) is the coroot corresponding to \( \alpha \). Irreducible highest weight module corresponding to the dominant minuscule weights are called minuscule representations. In the following Dynkin diagrams, the dominant minuscule weights \( \Lambda_i \) are marked by \( \circ \).

\[
\begin{align*}
A_n & : 1 - 2 - \cdots - n - n - 1 \\
C_n & : 1 - 2 - \cdots - n - n - 1 \\
E_6 & : 1 - 2 - 3 - 4 - 5 - 6 \\
E_7 & : 1 - 2 - 3 - 4 - 5 - 6 - 7
\end{align*}
\]

We set \( D := \{A_n, B_n, C_n, D_n, E_6, E_7\} \) and, for \( X \in D \), denote by \( I_X \) the set of all indices marked by \( \circ \) in the above Dynkin diagram (1.2) of type \( X \).
For a dominant minuscule weight \( \Lambda_i \), the set of vertices of the crystal \( B(\Lambda_i) \) can be identified with the Weyl group orbit \( W \cdot \Lambda_i \). Set \( B(\Lambda_i) := W \cdot \Lambda_i \) and define

\[
\varepsilon_i(\mu) := \max\{0, -\langle h_i, \mu \rangle\}, \quad \varphi_i(\mu) := \max\{0, \langle h_i, \mu \rangle\},
\]

\[
\tilde{f}_i(\mu) := \begin{cases}
\mu - \alpha_i & \text{if } \langle h_i, \mu \rangle = 1, \\
0 & \text{otherwise},
\end{cases} \quad \tilde{e}_i(\mu) := \begin{cases}
\mu + \alpha_i & \text{if } \langle h_i, \mu \rangle = -1, \\
0 & \text{otherwise}.
\end{cases}
\]

for \( \mu \in B(\Lambda_i) \). Then one can show that \( B(\Lambda_i) \) is a crystal and it is isomorphic to the crystal \( B(\Lambda_i) \).

2. CYCLOTOMIC QUIVER HEcke ALGEBRAS AND MINUSCULE WEIGHTS

2.1. Quiver Hecke algebras. Let \( k \) be a field and \( (A, P, \Pi, P^\vee, \Pi^\vee) \) be a Cartan datum. Choose polynomials

\[
Q_{i,j}(u, v) = \delta_{i,j} \sum_{(p,q) \in Z_{\geq 0}^2} t_{i,j;p,q} u^p v^q \in k[u, v]
\]

with \( t_{i,j;p,q} \in k \), \( t_{i,j;p,q} = t_{j,i;q,p} \) and \( t_{i,j;-a_{ij},0} \in k^x \) such that \( Q_{i,j}(u, v) = Q_{j,i}(v, u) \) for \( i, j \in I \).

Let \( \mathfrak{S}_n = \langle s_1, \ldots, s_{n-1} \rangle \) be the symmetric group on \( n \) letters with the action of \( \mathfrak{S}_n \) on \( I^n \) by place permutation. For \( \beta \in Q_+ \) with \( \text{ht}(\beta) = n \), we set

\[
I^\beta = \{ \nu = (\nu_1, \ldots, \nu_n) \in I^n \mid \alpha_{\nu_1} + \cdots + \alpha_{\nu_n} = \beta \}.
\]

Definition 2.1. For \( \beta \in Q_+ \) with \( \text{ht}(\beta) = n \), the quiver Hecke algebra \( R(\beta) \) associated with \( \{Q_{i,j}\}_{i,j} \in I \) is the \( k \)-algebra generated by \( \{e(\nu)\}_{\nu \in I^\beta}, \{x_k\}_{1 \leq k \leq n}, \{\tau_m\}_{1 \leq m \leq n-1} \) satisfying the following defining relations:

\[
e(\nu)e(\nu') = \delta_{\nu,\nu'}e(\nu), \quad \sum_{\nu \in I^\beta} e(\nu) = 1,
\]

\[
x_kx_m = x_mx_k, \quad x_ke(\nu) = e(\nu)x_k,
\]

\[
\tau_m e(\nu) = e(s_m(\nu))\tau_m, \quad \tau_k \tau_m = \tau_m \tau_k \quad \text{if } |k - m| > 1,
\]

\[
\tau_k^2 e(\nu) = Q_{\nu_k,\nu_{k+1}}(x_k, x_{k+1})e(\nu),
\]

\[
(\tau_k x_m - x_{s_k(m)}\tau_k)e(\nu) = \begin{cases}
-e(\nu) & \text{if } m = k, \nu_k = \nu_{k+1}, \\
e(\nu) & \text{if } m = k + 1, \nu_k = \nu_{k+1}, \\
0 & \text{otherwise},
\end{cases}
\]
\[ (\tau_{k+1} - \tau_k)\tau_{k+1} - \tau_k\tau_{k+1} \tau_k) e(\nu) \]
\[ = \begin{cases} Q_{v_k,v_{k+1}}(x_k,x_{k+1}) - Q_{v_k,v_{k+1}}(x_{k+2},x_{k+1}) \frac{e(\nu)}{x_k - x_{k+2}} & \text{if } \nu_k = \nu_{k+2}, \\ 0 & \text{otherwise.} \end{cases} \]

Note that \( R(\rho) \) has the \( \mathbb{Z} \)-grading defined by
\[ \deg e(\nu) = 0, \quad \deg x_k e(\nu) = (\alpha_{\nu_k}, \alpha_{\nu_k}), \quad \deg \tau_i e(\nu) = -(\alpha_{\nu_i}, \alpha_{\nu_i}). \]

For \( n \in \mathbb{Z}_{\geq 0} \), we set \( R(n) := \bigoplus_{\beta \in \mathbb{Q}_+} [\beta] = n R(\beta) \). For \( w \in \mathcal{S}_n \), we fix a preferred reduced expression \( w = s_{i_1} \cdots s_{i_t} \) and define \( \tau_w := \tau_{s_{i_1}} \cdots \tau_{s_{i_t}} \). Note that \( \tau_w \) depends on the choice of reduced expression \( w \) unless \( w \) is fully commutative.

For a \( \mathbb{Z} \)-graded algebra \( A \), we denote by \( A\text{-Mod} \) the category of graded left \( A \)-modules, and write \( A\text{-proj} \) (resp. \( A\text{-gmod} \)) for the full subcategory of \( A\text{-Mod} \) consisting of finitely generated projective (resp. finite-dimensional) graded \( A \)-modules. We set \( R\text{-proj} := \bigoplus_{\beta \in \mathbb{Q}_+} R(\beta)\text{-proj} \) and \( R\text{-gmod} := \bigoplus_{\beta \in \mathbb{Q}_+} R(\beta)\text{-gmod} \).

For \( M \in R(\beta)\text{-Mod} \) and \( N \in R(\gamma)\text{-Mod} \), we define
\[ M \circ N = R(\beta + \gamma) e(\beta, \gamma) \otimes_{R(\beta) \otimes R(\gamma)} (M \otimes N), \]
where \( e(\beta, \gamma) = \sum_{\nu_1 \in I^\beta, \nu_2 \in I^\gamma} e(\nu_1 \ast \nu_2) \). Here \( \nu_1 \ast \nu_2 \) is the concatenation of \( \nu_1 \) and \( \nu_2 \). Then, the Grothendieck groups \( [R\text{-proj}] \) and \( [R\text{-gmod}] \) admit \( \mathbb{A} \)-algebra structures with the multiplication given by the functor \( \circ \).

**Theorem 2.2.** ([13, 14, 16]) There exist \( \mathbb{A} \)-algebra isomorphisms
\[ U^-_\mathbb{A}(g) \xrightarrow{\sim} [R\text{-proj}], \quad U^-_\mathbb{A}(g)^\vee \xrightarrow{\sim} [R\text{-gmod}]. \]

For \( \Lambda \in \mathbb{P}_+ \) and \( \beta \in \mathbb{Q}_+ \), let \( I^\Lambda_\beta \) be the two-sided ideal of \( R(\beta) \) generated by the elements \( \{ x^{(h_{\Lambda, \beta} - \Lambda)} e(\nu) \mid \nu \in I^\beta \} \). The **cyclotomic quiver Hecke algebra** \( R^\Lambda(\Lambda - \beta) \) is defined to be the quotient algebra
\[ R^\Lambda(\Lambda - \beta) := R(\beta)/I^\Lambda_\beta. \]

For \( m \in \mathbb{Z}_{\geq 0} \), we set \( R^\Lambda(m) := \bigoplus_{\beta \in \mathbb{Q}_+, |\beta| = m} R^\Lambda(\Lambda - \beta) \). When \( \Lambda \) is of type \( X \), we sometimes write \( R^X(m) \) and \( R^X(\mu) \) to emphasize the type \( X \).

For \( M \in R^\Lambda(\mu)\text{-Mod} \), we set \( \text{wt}(M) = \mu \) and define functors \( F_i^\Lambda \) and \( E_i^\Lambda \) by
\[ F_i^\Lambda M := R^\Lambda(\mu - \alpha_i) e(\alpha, \beta) \otimes_{R^\Lambda(\mu)} M \quad \text{and} \quad E_i^\Lambda M := e(\alpha, \beta - \alpha) M, \]
where \( \beta = \Lambda - \mu \). We denote by \( \text{Ind}_m^{m+1} \) and \( \text{Res}_m^{m+1} \) the induction and restriction functors via the embedding \( R^\Lambda(m) \hookrightarrow R^\Lambda(m+1) \). Note that \( \text{Ind}_m^{m+1} = \sum_{i \in I} F_i^\Lambda \) and \( \text{Res}_m^{m+1} = \sum_{i \in I} E_i^\Lambda \). The functors \( F_i^\Lambda \) and \( E_i^\Lambda \) give a \( U^\Lambda(g) \)-modules structures on \( [R^\Lambda\text{-proj}] \) and \( [R^\Lambda\text{-gmod}] \).
Theorem 2.3. ([8]) There exist $U_A(g)$-module isomorphisms
\[ [R^A_{\text{proj}}] \sim \rightarrow V_A(\Lambda), \quad [R^A_{\text{gmod}}] \sim \rightarrow V_A(\Lambda)^\vee. \]

If $\xi$ is an extremal weight of $V(\Lambda)$, then we know the representation type of $R^A(\xi)$.

Proposition 2.4. ([1, Corollary 4.7], [12, Lemma 1.11]) Let $\Lambda \in P_+$ and $\xi = w\Lambda$ for $w \in W$. Then the algebra $R^A(\xi)$ is simple.

Proposition 2.5. ([13, Corollary 3.19]) Every irreducible $R$-module is absolutely irreducible.

We now assume that $A$ is of simply laced type. Then the construction for homogeneous representations was given in [15]. We denote by $G_\beta$ the graph with the set of vertices $I^\beta$, and with $\nu, \mu \in I^\beta$ connected by an edge if and only if there exists $k$ such that $\mu = s_k\nu$ and $a_{\nu_k, \nu_{k+1}} = 0$. Let $C$ be a connected component of $G_\beta$. We say that $C$ is homogeneous if for each $\nu \in C$ the following condition holds:

\begin{equation}
\begin{aligned}
\text{if } \nu_r = \nu_s \text{ for some } r < s \text{ then there exist } t, u \text{ with } r < t < u < s \\
\text{such that } a_{\nu_r, \nu_t} = -1 \text{ and } a_{\nu_u, \nu_s} = -1.
\end{aligned}
\end{equation}

It was proved in [15, Theorem 3.4] that $C$ has a simple $R(\beta)$-module structure.

2.2. Minuscule representations of symmetric type $ADE$. Let $V(\Lambda_i)$ be a minuscule representation and $B(\Lambda_i)$ its crystal. Let $b \in B(\Lambda_i)$, $\xi = \text{wt}(b)$ and $\ell = \text{ht}(\Lambda_i - \xi)$. We set
\begin{equation}
P(b) = \{(i_1, \ldots, i_\ell) \in I^\ell \mid \bar{f}_{i_1} \cdots \bar{f}_{i_\ell} b_{\Lambda_i} = b\},
\end{equation}
where $b_{\Lambda_i}$ is the highest weight vector of $B(\Lambda_i)$. From a viewpoint of the crystal graph $B(\Lambda_i)$, an element of $P(b)$ can be thought as a sequence of arrows from $b_{\Lambda_i}$ to $b$ on the graph $B(\Lambda_i)$. In this sense, an element of $P(b)$ is called a path from $b_{\Lambda_i}$ to $b$ on the crystal $B(\Lambda_i)$.

From now on, we suppose that $X \in \{A_n, D_n, E_6, E_7\}$ and $i \in I_X$.

Lemma 2.6. For $(i_1, \ldots, i_\ell), (j_1, \ldots, j_\ell) \in P(b)$, we have
\[ s_{i_1} \cdots s_{i_\ell} = s_{j_1} \cdots s_{j_\ell}. \]

Proof. Let $(i_1, \ldots, i_\ell) \in P(b)$ and set $w := s_{i_1} \cdots s_{i_\ell}$. Since $B(\Lambda_i)$ is minuscule, $s_i b' = \bar{f}_i b'$ for any $b' \in B(\Lambda_i)$ with $\varphi_i(b') > 0$. Thus, the reduced expression $w = s_{i_1} \cdots s_{i_\ell}$ satisfies (1.1), i.e. $w$ is $\Lambda_i$-minuscule. It was shown in [17, Proposition 2.1] that every reduced expression of $w$ satisfies (1.1). We take $j \in I$ with $j \neq i$.

We suppose that $\ell(ws_j) < \ell(w)$. Then one can write a reduced expression of $w = s_{t_1} \cdots s_{t_{\ell-1}} s_j$ for some $t_1, \ldots, t_\ell \in I$. Since $\langle h_j, \Lambda_i \rangle = 0$, $w = s_{t_1} \cdots s_{t_{\ell-1}} s_j$ is not $\Lambda_i$-minuscule which is a contradiction. Thus, we have
\begin{equation}
\ell(ws_j) > \ell(w) \text{ for all } j \in I \text{ with } j \neq i.
\end{equation}
We now consider an element \((j_1, \ldots, j_\ell) \in P(b)\). Set \(u := s_{j_1} \cdots s_{j_\ell}\). By the same argument as above,

\[ \ell(us_j) > \ell(u) \text{ for all } j \in I \text{ with } j \neq i \]  

(2.4)

Considering the realization \(B(\Lambda_i)\) given in Section 1.2, we have \(w \cdot \Lambda_i = u \cdot \Lambda_i = \text{wt}(b)\), which tells us that

\[ u^{-1} w \cdot \Lambda_i = \Lambda_i. \]

Hence \(u^{-1} w\) is contained in the parabolic subgroup of \(W\) generated by \(s_j\) for \(j \in I \setminus \{i\}\).

Suppose that \(u^{-1} w \neq \text{id}\). Then there exists \(j \in I \setminus \{i\}\) such that \(\ell(u^{-1} ws_j) = \ell(u^{-1} w) - 1\).

It follows from (2.4) that

\[ \ell(ws_j) = \ell(u(u^{-1} ws_j)) = \ell(u) + \ell((u^{-1} w)s_j) \]

\[ = \ell(u) + \ell(u^{-1} w) - 1 = \ell(u(u^{-1} w)) - 1 = \ell(w) - 1, \]

which is a contradiction to (2.3). Therefore, we conclude that \(u^{-1} w = \text{id}\). \(\square\)

We define \(w(b) := s_{i_1} \cdots s_{i_\ell} \in W\), where \((i_1, \ldots, i_\ell)\) is an element of \(P(b)\). Note that it is well-defined by Lemma 2.6.

**Lemma 2.7.** For \(b \in B(\Lambda_i)\), the set \(P(b)\) is in 1-1 correspondence to the set of all reduced expressions of \(w(b)\) via the map \((i_1, \ldots, i_\ell) \mapsto s_{i_1} \cdots s_{i_\ell}\).

**Proof.** Let \(E(b)\) be the set of all reduced expressions of \(w(b)\). We consider the map

\[ F : P(b) \rightarrow E(b), \quad (i_1, \ldots, i_\ell) \mapsto s_{i_1} \cdots s_{i_\ell}. \]

Thanks to Lemma 2.6, the map \(F\) is well-defined.

Recall the crystal realization \(B(\Lambda_i)\) given in Section 1.2. Note that, for \(\mu \in B(\Lambda_i)\), \(\tilde{f}_i(\mu) = s_i \mu\) if \(\langle h_i, \mu \rangle = 1\). Since every reduced expression of \(w(b)\) satisfies (1.1) by [17, Proposition 2.1], the map

\[ G : E(b) \rightarrow P(b), \quad s_{i_1} \cdots s_{i_\ell} \mapsto (i_1, \ldots, i_\ell), \]

is well-defined. Then, it is clear that \(F \circ G = \text{id}\) and \(G \circ F = \text{id}\). \(\square\)

Since \(w(b)\) is fully commutative [17, Proposition 2.1], Lemma 2.7 says that for \(\nu, \mu \in P(b)\), there exists \(w_{\nu, \mu} \in S_\ell\) such that \(\nu = w_{\nu, \mu} \mu\). We set

\[ c_{\nu, \mu} := e(\nu) \tau_{w_{\nu, \mu}} e(\mu). \]

(2.5)

Note that \(c_{\nu, \mu}\) does not depend on the choice of the reduced expressions of \(w_{\nu, \mu}\).
We now set
\[ S(b) := \bigoplus_{\nu \in \mathcal{P}(b)} k \mu(\nu) \]
and define an \( R(\beta) \)-module structure on \( S(b) \) by
\[
e(\mu) p(\nu) = \delta_{\nu, \mu} p(\nu), \quad x_i e(\mu) p(\nu) = 0, \quad \tau_j e(\mu) p(\nu) = \begin{cases} p(s_j \nu), & \text{if } \nu = \mu \text{ and } s_j \nu \in \mathcal{P}(b), \\ 0 & \text{otherwise,} \end{cases}
\]
for \( \nu \in \mathcal{P}(b) \) and admissible \( i, j, \mu \).

**Theorem 2.8.** Let \( X \in \{ A_n, D_n, E_6, E_7 \} \) and \( i \in l_X \). Let \( b \in B(\Lambda_i) \) and write \( \xi = \text{wt}(b) \).

1. \( S(b) \) is a homogeneous simple \( R^{\Lambda_i}(\xi) \)-module.
2. \( R^{\Lambda_i}(\xi) \) has a \( k \)-basis

\[ C(b) := \{ c_{\nu, \mu} \mid \nu, \mu \in \mathcal{P}(b) \}, \]

where \( c_{\nu, \mu} \) is given in (2.5).

**Proof.** (1) By Lemma 2.7, the set \( \mathcal{P}(b) \) can be identify with the set of all reduced expression of \( w(b) \). Then it follows from [17, Proposition 2.5] (c.f. [15, Proposition 3.8]) that \( \mathcal{P}(b) \) satisfies the condition (2.1). Thus, \( \mathcal{P}(b) \) has a homogeneous simple \( R \)-module structure which is same as the above actions on \( S(b) \). Moreover, since \( \nu_\ell = i \) for any \( (\nu_1, \ldots, \nu_\ell) \in \mathcal{P}(b) \) and \( x_\ell \) acts as zero, \( S(b) \) is an \( R^{\Lambda_i}(\xi) \)-module.

(2) Since every element of \( B(\Lambda_i) \) is extremal, \( R^{\Lambda_i}(\xi) \) is simple by Proposition 2.4. Thus, by Proposition 2.5, \( R^{\Lambda_i}(\xi) \) is isomorphic to the endomorphism ring of \( S(b) \), which implies that \( C(b) \) is a basis of \( R^{\Lambda_i}(\xi) \).

**Remark 2.9.** For type \( A_n \), the simple module \( S(b) \) can be realized as the set of standard tableaux of a usual Young diagram (see [3, Section 5]). Thus, the algebra \( R^{\Lambda_i}(m) \) has a \( k \)-basis indexed by all pairs of standard tableaux. In Section 3, we shall give a \( B_n \)-type analogue using shifted Young tableaux.

### 2.3. Minuscule representations of type \( C_n \)

In this subsection, we assume that \( X = C_n \) and consider the dominant minuscule weight \( \Lambda_n \). Then the crystal \( B(\Lambda_n) \) is given as follows:

\[
\begin{array}{cccccccc}
\hline
\hline
n & \cdots & 3 & \hline
\hline
\end{array}
\]

For \( b \in B(\Lambda_n) \), if we write \( b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} b_{\Lambda_n} \), then we set
\[ \nu_b := (i_1, \ldots, i_\ell). \]

We set \( S(b) := k \mu(\nu_b) \) and define an \( R \)-module structure by
\[
e(\mu) p(\nu_b) = \delta_{\nu_b, \mu} p(\nu_b), \quad x_i e(\mu) p(\nu_b) = \tau_j e(\mu) p(\nu_b) = 0,
\]
for admissible $i, j, \mu$. Then one can prove that $S(b)$ is a 1-dimensional $R^{A_n}$-module. Proposition 2.4 and Proposition 2.5 imply the following immediately.

**Proposition 2.10.** Let $X = C_n$, $b \in B(\Lambda_n)$, and write $\xi = \text{wt}(b)$. Then $R^{A_n}(\xi)$ is isomorphic to $k$.

### 3. Simple $R^{A_1}$-modules of type $B_n$

**3.1. Shifted Young tableaux.** For a strict partition $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0)$ with $N = \sum_{i=1}^{\ell} \lambda_i$, we write $\lambda \vdash N$ and set $|\lambda| := N$. We identify strict partitions with shifted Young diagrams and depict shifted Young diagrams using English convention. We write $b = (i, j) \in \lambda$ if there exists a box in the $i$th row and the $j$th column. By an addable (resp. a removable) box $b \in \lambda$, we mean a box which can be added to (resp. removed from) $\lambda$ to obtain a valid shifted Young diagram $\lambda \nearrow b$ (resp. $\lambda \searrow b$). For $n \in \mathbb{Z}_{>0}$, let $\mathcal{P}_n$ be the set of all strict partitions $(\lambda_1, \lambda_2, \ldots)$ such that $\lambda_1 \leq n$. It is easy to check that $|\mathcal{P}_0| = 1$ and $|\mathcal{P}_n| = 2|\mathcal{P}_{n-1}|$ for $n \geq 1$, which say that

$$|\mathcal{P}_n| = 2^n \quad \text{for } n \geq 1. \tag{3.1}$$

A standard tableau $T$ of shape $\lambda \vdash n$ is a filling of boxes of $\lambda$ with $\{1, 2, \ldots, n\}$ such that (i) each entry is used exactly once, (ii) the entries in rows and columns increase from left to right and top to bottom, respectively. Let $\text{ST}(\lambda)$ be the set of all standard tableaux of shape $\lambda$. For example, the following is a standard tableau of shape $\lambda = (7, 4)$.

\[
\begin{array}{cccccc}
1 & 2 & 4 & 5 & 7 & 8 & 11 \\
3 & 6 & 9 & 10 \\
\end{array}
\]

Let $\lambda \vdash n$ be a shifted Young diagram. We define the residue of $(i, j) \in \lambda$ as follows:

$$\text{res}(i, j) := j - i + 1. \tag{3.2}$$

For example, the residues for $\lambda = (9, 6, 3, 1)$ are as follows. In this example, $\text{res}(2, 6) = 5$.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & 2 & 3 & 4 & 5 & 6 \\
1 & 2 & 3 \\
1 \\
\end{array}
\]

For $T \in \text{ST}(\lambda)$, we define the residue sequence of $T$ by

$$\text{res}(T) = (\text{res}(i_1, j_1), \ldots, \text{res}(i_2, j_2), \text{res}(i_1, j_1)) \in I^n,$$
where \((i_k, j_k) \in \lambda\) is the box filled with \(k\) in \(T\).

3.2. Irreducible representations. For \(\lambda \in \mathfrak{P}_n\), we set

\[
\text{wt}(\lambda) := \Lambda_1 - \sum_{b \in \lambda} \alpha_{\text{res}(b)} \in \mathcal{P}.
\]

Recall the crystal realization \(\mathbb{B}(\Lambda_1)\) given in Section 1.2.

**Lemma 3.1.** The map \(\lambda \mapsto \text{wt}(\lambda)\) gives 1-1 correspondence between \(\mathfrak{P}_n\) and \(\mathbb{B}(\Lambda_1)\).

**Proof.** Let \(f : \mathfrak{P}_n \to \mathcal{P}\) be the map defined by

\[
f(\lambda) = \text{wt}(\lambda).
\]

Let \(\lambda \in \mathfrak{P}_n\). We first show that \(f(\lambda) \in \mathcal{W} \cdot \Lambda_1\) by induction argument on \(|\lambda|\). Since \(\text{wt}(\emptyset) = \Lambda_1 \in \mathcal{W} \cdot \Lambda_1\), we assume that \(|\lambda| > 0\). Thus we can write \(\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_\ell > 0) \in \mathfrak{P}_n\), and set

\[
\beta_k := \sum_{t=1}^{k} \alpha_t \quad \text{for } k = 1, \ldots, n.
\]

Then we have \(\text{wt}(\lambda) = \Lambda_1 - \sum_{k=1}^{\ell} \beta_{\lambda_k}\) and

\[
\langle h_i, \beta_{\lambda_k} \rangle = \begin{cases} 
2 & \text{if } i = \lambda_j = 1, \\
1 & \text{if } i = \lambda_j > 1, \\
-1 & \text{if } i = \lambda_j + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \(b := (\ell, \lambda_\ell) \in \lambda\) and \(\mu := \lambda \nearrow b\). By the induction hypothesis, we have \(\text{wt}(\mu) \in \mathcal{W} \cdot \Lambda_1\). Setting \(i := \text{res}(b)\), we obtain

\[
\langle h_i, \text{wt}(\mu) \rangle = \langle h_i, \Lambda_1 \rangle - \sum_{k=1}^{\ell} \langle h_i, \beta_{\lambda_k} \rangle + \langle h_i, \alpha_i \rangle = 1,
\]

which tells us that

\[
\text{wt}(\lambda) = \text{wt}(\mu) - \alpha_i = s_i(\text{wt}(\mu)) \in \mathcal{W} \cdot \Lambda_1.
\]

Thus, we have that \(\text{Im}(\mathfrak{P}_n) \subset \mathcal{W} \cdot \Lambda_1\).

We now show that \(f\) is injective. Let \(\lambda = (\lambda_1, \lambda_2, \ldots), \mu = (\mu_1, \mu_2, \ldots) \in \mathfrak{P}_n\), and assume that \(\lambda \neq \mu\), i.e., there exists \(k\) such that \(\lambda_i = \mu_i\) for \(i < k\) and \(\lambda_k \neq \mu_k\). Then by the definition, we have

\[
\max\{\lambda_k, \mu_k\} \in \text{supp}(\beta_{\lambda_k} - \beta_{\mu_k}) \quad \text{and} \quad \max\{\lambda_k, \mu_k\} \notin \text{supp}(\beta_{\lambda_t} - \beta_{\mu_t}) \text{ for } t > k,
\]
where \( \text{supp}(\beta) := \{ i \mid b_i \neq 0 \} \) for \( \beta = \sum_{i \in I} b_i \alpha_i \in \mathbb{Q} \). This tells us that
\[
\text{wt}(\lambda) - \text{wt}(\mu) = \beta_\lambda k - \beta_\mu k + \sum_{t > k} (\beta_\lambda - \beta_\mu) \neq 0.
\]

Hence \( f \) is injective.

Moreover, if we denote by \( P \) the subgroup of \( W \) generated by \( s_2, \ldots, s_n \), then by (3.1), we have
\[
|P_n| = 2^n = \frac{2^n n!}{n!} = \left| \mathcal{W} \cdot \Lambda_1 \right|.
\]
Therefore, the map \( f \) gives a bijection between \( P_n \) and \( \mathcal{W} \cdot \Lambda_1 \).

**Remark 3.2.** One can describe the \( U_q(B_n) \)-crystal structure on \( \mathfrak{P}_n \) directly via the 1-1 correspondence given in Lemma 3.1. In this description, the crystal operators \( \tilde{f}_i \) and \( \tilde{e}_i \) are defined as follows: for \( i \in I \) and \( \lambda \in \mathfrak{P}_n \),
\[
\tilde{f}_i \lambda = \begin{cases} 
\lambda \uparrow b & \text{if } b \text{ is an addable box of } \lambda \text{ with residue } i, \\
0 & \text{otherwise},
\end{cases}
\]
\[
\tilde{e}_i \lambda = \begin{cases} 
\lambda \downarrow b & \text{if } b \text{ is a removable box of } \lambda \text{ with residue } i, \\
0 & \text{otherwise}.
\end{cases}
\]
Thus, for \( b \in B(\Lambda_1) \), we can identify \( \text{ST}(\lambda_b) \) with the set \( \mathcal{P}(b) \) of all paths from \( b_{\Lambda_1} \) to \( b \) on the crystal \( B(\Lambda_1) \). This crystal realization also can be obtained from the \( B_\infty \)-Fock space given in [7] by restricting to type \( B_n \).

For \( b \in B(\Lambda_1) \), we denote by \( \lambda_b \) the strict partition corresponding to \( b \) under the bijection given in Lemma 3.1. We set
\[
\mathcal{S}(b) := \bigoplus_{T \in \text{ST}(\lambda_b)} kT,
\]
and define an \( R(\text{wt}(b)) \)-module structure on \( \mathcal{S}(b) \) by
\[
\begin{align*}
e(\nu)T &= \delta_{\nu, \text{res}(T)} T, \quad x_i e(\nu)T = 0, \quad \tau_j e(\nu)T = \begin{cases} 
s_jT & \text{if } \nu = \mu \text{ and } s_jT \in \text{ST}(\lambda_b), \\
0 & \text{otherwise},
\end{cases}
\end{align*}
\]
for \( T \in \text{ST}(\lambda_b) \) and admissible \( i, j, \nu \). Here, \( s_jT \) is the tableau obtained from \( T \) by exchanging the entries \( j \) and \( j + 1 \). For \( k \in \mathbb{Z}_{\geq 0} \), we simply write \( S(k) \) for the simple module \( \mathcal{S}(b) \) where \( b \) is the element of \( B(\Lambda_1) \) such that \( \lambda_b = (k) \). For \( S, T \in \text{ST}(\lambda_b) \), let \( w_{T,S} \) be an element of \( \mathfrak{G}_{|\lambda|} \) such that \( T = w_{T,S}S \), and set
\[
e_{T,S} := e(\text{res}(T))\tau_{w_{T,S}}e(\text{res}(S)).
\]

**Theorem 3.3.** Let \( X = B_n, b \in B(\Lambda_1) \), and write \( \xi = \text{wt}(b) \).
(1) \( S(b) \) is a homogeneous simple \( R^A_1(\xi) \)-module.

(2) If we write \( \lambda_b = (\lambda_1, \ldots, \lambda_\ell) \), then

\[
S(b) \simeq \text{hd}(S(\lambda_\ell) \circ \cdots \circ S(\lambda_2) \circ S(\lambda_1)),
\]

where \( \text{hd}(M) \) denotes by the head of a module \( M \).

(3) Let \( m = \text{ht}(A_1 - \xi) \). Then

\[
\text{Ind}_{m-1}^m S(b) \simeq \bigoplus_{i \in \bar{I}, \bar{\delta}_i \neq 0} S(\bar{\delta}_i b), \quad \text{Res}_{m-1}^m S(b) \simeq \bigoplus_{i \in \bar{I}, \bar{\delta}_i \neq 0} S(\bar{\delta}_i b),
\]

where \( \text{Ind}_{m-1}^m \) and \( \text{Res}_{m-1}^m \) are the induction and restriction functors given in Section 2.1.

(4) \( R^A_1(\xi) \) has a \( k \)-basis

\[
\mathcal{C}(b) := \{ c_{T,S} \mid T, S \in \text{ST}(\lambda_b) \},
\]

where \( c_{T,S} \) is given in (3.3).

**Proof.** (1) Since any choices of the polynomial \( Q_{i,j}(u, v) \) yield isomorphic algebras when it is of finite type (c.f. [2, Lemma 3.2]), without loss of generality, we may choose the following \( Q_{i,j}(u, v) \): for \( i < j \),

\[
Q_{i,j}(u, v) = \begin{cases} 
   u^2 - v & \text{if } i = 1, j = 2, \\
   u - v & \text{if } j = i + 1 > 1, \\
   1 & \text{otherwise}.
\end{cases}
\]

Let \( T \in \text{ST}(\lambda_b), N := |\lambda_b| \) and write \( \text{res}(T) = (i_1, \ldots, i_N) \). Considering the residue pattern (3.2) and combinatorics of standard tableaux, we have

(i) if \( i_k \neq i_{k+1} \) for any \( k = 1, \ldots, N - 1 \),

(ii) \( s_k T \) is standard if and only if \( a_{i_k, i_{k+1}} = 0 \),

(iii) if \( i_k = i_{k+2} \), then \( (i_k, i_{k+1}, i_{k+2}) = (1, 2, 1) \).

We shall only check that the defining relations for \( (\tau_k x_m - x_{s_k(m)} \tau_k), \tau_k^2 \) and \( \tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_k \) since it is straightforward to check other relations.

By (i), it suffices to show that \( (\tau_k x_m - x_{s_k(m)} \tau_k)e(\nu) = 0 \), which always holds by the definition of \( S(b) \).

If \( a_{i_k, i_{k+1}} \neq 0 \), then \( \deg Q_{i_k, i_{k+1}}(u, v) > 0 \) and \( s_k T \notin \text{ST}(\lambda_b) \) by (ii). Thus we have

\[
\tau_k^2 e(\nu) T = 0 = Q_{i_k, i_{k+1}}(x_k, x_{k+1}) e(\nu) T.
\]

If \( a_{i_k, i_{k+1}} = 0 \), then \( Q_{i_k, i_{k+1}}(u, v) = 1 \) and

\[
\tau_k^2 e(\nu) T = \delta_{\nu, \text{res}(T)} T = Q_{i_k, i_{k+1}}(x_k, x_{k+1}) e(\nu) T
\]

by (ii). Thus, the defining relation for \( \tau_k^2 e(\nu) \) holds.
We now consider the last case. If \( i_k \neq i_{k+2} \), then it is easy to check that \( \tau_{k+1} \tau_k \tau_{k+1} e(\nu) T = \tau_k \tau_{k+1} \tau_{k} e(\nu) T \). Suppose that \( i_k = i_{k+2} \). Then (iii) says that \( (i_k, i_{k+1}, i_{k+2}) = (1, 2, 1) \). Thus, by the definition of \( S(b) \), we have

\[
(\tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_{k}) e(\nu) T = 0.
\]

On the other hand, since \( (i_k, i_{k+1}, i_{k+2}) = (1, 2, 1) \), we obtain

\[
\frac{Q_{1,2}(x_k, x_{k+1}) - Q_{1,2}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} = x_k + x_{k+2},
\]

which implies that

\[
\frac{Q_{1,2}(x_k, x_{k+1}) - Q_{1,2}(x_{k+2}, x_{k+1})}{x_k - x_{k+2}} e(\nu) T = 0.
\]

Therefore, the defining relations for \( \tau_{k+1} \tau_k \tau_{k+1} - \tau_k \tau_{k+1} \tau_{k} \) holds, which completes the proof.

(2) We set \( e := e(\nu_{\lambda_i} \ast \ldots \ast \nu_{\lambda_2} \ast \nu_{\lambda_1}) \) where \( \nu_k = (k, \ldots, 2, 1) \) for \( k \in \mathbb{Z}_{\geq 0} \). Considering the combinatorics of shifted Young tableaux, it is easy to see that \( e S(b) = k T_0 \), where \( T_0 \) is the initial tableau, which is filled with \( \{1, \ldots, N\} \) from left to right and top to bottom in order. Thus, there exists a surjective homomorphism

\[
S(\lambda_{\ell}) \otimes \cdots \otimes S(\lambda_2) \otimes S(\lambda_1) \twoheadrightarrow e S(b),
\]

which yields the surjective homomorphism

\[
S(\lambda_{\ell}) \circ \cdots \circ S(\lambda_2) \circ S(\lambda_1) \twoheadrightarrow S(b).
\]

On the other hand, by the shuffle lemma [13, Lemma 2.20], we have

\[
\dim e(S(\lambda_{\ell}) \circ \cdots \circ S(\lambda_2) \circ S(\lambda_1)) = 1.
\]

For any quotient \( Q \) of \( S(\lambda_{\ell}) \circ \cdots \circ S(\lambda_2) \circ S(\lambda_1) \), the natural surjection

\[
S(\lambda_{\ell}) \circ \cdots \circ S(\lambda_2) \circ S(\lambda_1) \twoheadrightarrow Q
\]

tells us that \( \dim(eQ) = 1 \). Thus, we conclude that \( S(\lambda_{\ell}) \circ \cdots \circ S(\lambda_2) \circ S(\lambda_1) \) has a unique simple head. Hence, (3.4) implies the desired result.

(3) Note that every element of \( B(\Lambda_1) \) is extremal. By Proposition 2.4, \( R^{\Lambda_1}(\zeta) \) is simple for any weight \( \zeta \) with \( B(\Lambda_1) \zeta \neq 0 \), which says that \( R^{\Lambda_1}(t) \) is semisimple for any \( t > 0 \). Since \( S(b) \) has a basis indexed by \( S T(\lambda_b) \), we have \( F_i^{\Lambda_1} S(b) \simeq S(\bar{e}_i b) \) by considering their characters. Here we set \( S(0) := 0 \). Thus, we have \( \text{Res}_{m-1}^{m} S(b) = \bigoplus_{i \in I, \bar{e}_i b \neq 0} S(\bar{e}_i b) \). The isomorphism for Ind_{m+1} follows from the fact that \( F_i^{\Lambda} \) and \( E_i^{\Lambda} \) form an adjoint pair [11, Theorem 3.5].

(4) By Proposition 2.4 and Proposition 2.5, \( R^{\Lambda_i}(\xi) \) is isomorphic to the endomorphism ring of \( S(b) \) and \( C(b) \) is a basis. \( \square \)
Corollary 3.4. For \( n \in \mathbb{Z}_{\geq 2} \) and \( m \in \mathbb{Z}_{\geq 0} \), there exist algebra isomorphisms
\[
R_{\Lambda_1}^\Lambda(n) \simeq R_{\Lambda_1}^\Lambda(m) \simeq R_{\Lambda_2}^\Lambda(m).
\]
In particular, \( \dim R_{\Lambda_1}^\Lambda(n) = \dim R_{\Lambda_1}^\Lambda(m) = \dim R_{\Lambda_2}^\Lambda(m) = \sum \lambda \vdash m | \text{ST}(\lambda)|^2 \)

Proof. It is clear that \( R_{\Lambda_1}^\Lambda(m) \simeq R_{\Lambda_2}^\Lambda(m) \) by the symmetry of the Dynkin diagram. We shall define a simple \( R_{\Lambda_1}^\Lambda(n) \)-module using shifted Young tableaux. Let \( \lambda \vdash m \) be a strict partition. We define
\[
\text{res}_D(i, j) := \begin{cases} 1 & \text{if } i = j \text{ and } i \text{ is odd}, \\ 2 & \text{if } i = j \text{ and } i \text{ is even}, \\ j - i + 2 & \text{otherwise}, \end{cases}
\]
and \( \text{wt}_D(\lambda) := \Lambda_1 - \sum_{b \in \lambda} a_{\text{res}_D(b)}. \) For \( T \in \text{ST}(\lambda) \), we set
\[
\text{res}_D(T) = (\text{res}_D(i_m, j_m), \ldots, \text{res}_D(i_1, j_1)),
\]
where \((i_k, j_k) \in \lambda\) is the box filled with \( k \) in \( T \). For example, the residue are given as follows:

\[
\begin{array}{cccccccccc}
1 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
2 & 3 & 4 & 5 & 6 & 7 \\
1 & 3 & 4 \\
2 \\
\end{array}
\]

Then one can show that there exists \( b \in B(\Lambda_1) \) with \( \text{wt}(b) = \text{wt}_D(\lambda) \). We set
\[
S(b) := \bigoplus_{T \in \text{ST}(\lambda)} kT,
\]
and define an \( R_{\Lambda_1}^\Lambda \)-module structure on \( S(b) \) by
\[
e(\nu)T = \delta_{\nu, \text{res}_D(T)}T, \quad x_i e(\nu)T = 0, \quad \tau_j e(\nu)T = \begin{cases} s_j T & \text{if } \nu = \mu \text{ and } s_j T \in \text{ST}(\lambda), \\ 0 & \text{otherwise}, \end{cases}
\]
for \( T \in \text{ST}(\lambda) \) and admissible \( i, j, \nu \). Then it is straightforward to check that \( S(b) \) is a homogeneous \( R_{\Lambda_1}^\Lambda \)-module and it is isomorphic to \( S(b) \) given in Section 2.2.

Therefore, for each \( \lambda \vdash m \), \( R_{\Lambda_1}^\Lambda(\text{wt}(\lambda)) \) and \( R_{\Lambda_1}^\Lambda(\text{wt}_D(\lambda)) \) are simple and their simple modules have the same dimension. Thus we conclude that \( R_{\Lambda_1}^\Lambda(\text{wt}(\lambda)) \simeq R_{\Lambda_1}^\Lambda(\text{wt}_D(\lambda)) \), which implies the desired result. \( \square \)
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