Limit theorems for Lévy walks in $d$ dimensions: rare and bulk fluctuations

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Abstract

We consider super-diffusive Lévy walks in $d \geq 2$ dimensions when the duration of a single step, i.e. a ballistic motion performed by a walker, is governed by a power-law tailed distribution of infinite variance and finite mean. We demonstrate that the probability density function (PDF) of the coordinate of the random walker has two different scaling limits at large times. One limit describes the bulk of the PDF. It is the $d$-dimensional generalization of the one-dimensional Lévy distribution and is the counterpart of the central limit theorem (CLT) for random walks with finite dispersion. In contrast with the one-dimensional Lévy distribution and the CLT this distribution does not have a universal shape. The PDF reflects anisotropy of the single-step statistics however large the time is. The other scaling limit, the so-called ‘infinite density’, describes the tail of the PDF which determines second (dispersion) and higher moments of the PDF. This limit repeats the angular structure of the PDF of velocity in one step. A typical realization of the walk consists of anomalous diffusive motion (described by anisotropic $d$-dimensional Lévy distribution) interspersed with long ballistic flights (described by infinite density). The long flights are rare but due to them the coordinate increases so much that their contribution determines the dispersion. We illustrate the concept by considering two types of Lévy walks, with isotropic and
anisotropic distributions of velocities. Furthermore, we show that for isotropic but otherwise arbitrary velocity distributions the $d$-dimensional process can be reduced to a one-dimensional Lévy walk. We briefly discuss the consequences of non-universality for the $d > 1$ dimensional fractional diffusion equation, in particular the non-uniqueness of the fractional Laplacian.

Keywords: random walk, central limit theorem, Lévy walks, infinite density, fractional diffusion equation, fractional Laplacian, large deviations

(Some figures may appear in colour only in the online journal)

1. Introduction

It is a universal consequence of microscopic chaos that the velocity $v(t)$ of a moving particle has a finite correlation time [1, 2]. The particle’s displacement $\int_0^t v(t')dt'$ on large time scales can be considered as an outcome of a sum of independent random ‘steps’. A single step here is the motion during the shortest time over which the velocity of the particle is correlated. Velocity correlations at different steps can be neglected, and the integral over the time interval $[0, t]$ can be replaced with the sum of integrals over disjoint steps. If variations of step durations and velocity fluctuations can be neglected as well, we arrive at the standard random walk consisting of steps that take the same fixed time. The distance covered during a single step is fixed, but the direction of the step is random. It can be, for example, a step along one of the basis vectors when the random walk is performed on a $d$-dimensional lattice or it can be a step in a completely random direction in $d$-dimensional space, as in the case of an isotropic walk. In many situations, though, the variation of the step’s duration and velocity cannot be disregarded. A famous example is Lévy walks (LWs) [3–27], which belong to a more general class of stochastic processes called continuous time random walks (CTRWs) [3, 4]. In CTRWs it is not only the direction of the displacement during one step that is random (as in the standard random walk) but also the length of the step and the time that it takes. This flexibility allows covering a large number of real-life situations including dynamics of an ordinary gas molecule when both the time between consecutive collisions and velocity of the molecule vary. In an ideal gas the probability of large (much larger than the mean free time) time between the collisions is negligibly small so the probability density function (PDF) of the step duration decays fast for large arguments. This is not the case for the so-called Lorentz billiard [2], in which the particle moves freely between collisions with scatterers arranged in a spatially periodic array. In the case without horizon, when infinitely long corridors between the scatterers are present, the particle can fly freely for a very long time if its velocity vector aligns close to the direction of the corridor. The distribution of times between consecutive collisions has a power-law tail and infinite variance [28]. The dynamics of a particle can be reproduced in great detail with a Lévy walk process [29, 30]. Lévy walks have been found in diverse real-life processes including the spreading of cold atoms in optical lattices, animal foraging, and diffusion of light in disordered glasses and hot atomic vapors (for more examples see a recent review [20] and references therein). In spite of these advances and new experimental findings, the theory of LWs remains mainly confined to the case of one-dimensional geometry.

In this work we study $d$-dimensional Lévy walks when the duration $\tau$ of single steps is characterized by a PDF $\psi(\tau)$ with power-law asymptotic form, $\psi(\tau) \propto \tau^{-1-\alpha}$. The most interesting and practically relevant is the so-called ‘sub-ballistic super-diffusive regime’ [20],
1 < \alpha < 2$, when the step time has finite mean but infinite variance\(^6\). We study the PDF \(P(x, t)\) of the walker’s coordinate \(x(t)\) at time \(t\) in this regime. We show that there are two ways of rescaling \(P(x, t)\) with powers of time that produce finite infinite time limits. The two limiting distributions describe the bulk and the tails of \(P(x, t)\). Both distributions are sensitive to the microscopic statistics of the velocity of walkers and are model specific.

We consider a random walk in \(\mathbb{R}^d\) with a PDF \(F(v)\) of the single step velocity that obeys \(F(-v) = F(v)\), see [31, 32]. Thus the process is unbiased and the average displacement is zero. In the case of the sum of a large number of independent and identically-distributed (i. i. d.) random variables with finite dispersion there is a well-known scaling,

\[
\lim_{t \to \infty} t^{d/2} P(x \sqrt{t}, t) = g(x),
\]

where \(g(x)\) is a Gaussian distribution [33]. A stronger, large deviation limit indicates that

\[
\lim_{t \to \infty} \frac{1}{t} \ln P(tx, t) = s(x),
\]

where the convex function \(s(x)\) is known as large deviations, or entropy, or rate, or Kramer’s function [34, 35]; see [36] for simple derivation. This limit describes the exponential decay of the probability of large deviations of finite-time value \(x(t)/t\) from its infinite time limit fixed by the law of large numbers, \(\lim_{t \to \infty} x(t)/t = 0\). It corresponds to the Boltzmann formula and indicates that the PDF of macroscopic thermodynamic variable (representable as the sum of large number of independent random variables) is exponential in the entropy, whose maximum’s location gives the average. The coefficients of the quadratic expansion near the maximum characterize thermodynamic fluctuations [35]. Thus in the case of finite dispersion there is one universal scaling with the limiting distribution (the special case of finite dispersion but power-law tail, \(\alpha > 2\), produces different limiting distribution [37]; we do not consider this case here).

There are two different scalings that were found in the case of one-dimensional superdiffusive LWs [17, 18]. One of them is a continuation of the central limit theorem (CLT) to the case of i. i. d. random variables with infinite dispersion, the so-called generalized central limit theorem (gCLT) [38]. The corresponding distribution is known as the celebrated Lévy distribution. We first generalize this scaling to the \(d\)-dimensional case and find significant difference from the one-dimensional case and ordinary \(d\)-dimensional random walks. Our result shows that for \(d\)-dimensional Lévy walks with power law tailed distribution of step duration \(\tau\) that has infinite moments starting from order \(\alpha\) between 1 and 2 there is a finite limit,

\[
\lim_{t \to \infty} t^{d/\alpha} P(\tau^{1/\alpha} x, t) = \int \exp \left[ i k \cdot x - \frac{A}{\langle \tau \rangle} \left| \cos \left( \frac{\pi \alpha}{2} \right) \frac{\langle |k \cdot v|^{\alpha} \rangle}{(2\pi)^d} \right| \right] \frac{dk}{(2\pi)^d},
\]

This holds for arbitrary statistics of a particle’s velocity that has finite moments and obeys \(F(v) = F(-v)\). We demonstrate that this result smoothly connects with the usual central limit theorem given by equation (1). Indeed equation (3) reproduces Gaussian distribution at \(\alpha = 2\). Thus our result includes the usual central limit theorem as a particular case and can be called generalized central limit theorem. We clarify in section 6 that equation (3) can be obtained from the distribution of sum of many independent identically distributed scalar random variables with power-law tailed distributions—see [8, 12], [38–41].

\(^6\)The regime with infinite mean step time is distinctive even in the one-dimensional case; see [17–19, 21].
The limiting distribution given by equation (3) has features that distinguish it from both the one-dimensional LW and \(d\)-dimensional random walks. In those cases there is universality: the form of the scale-invariant PDF does not depend on details of the single step statistics (for instance, ordinary two-dimensional random walks on triangular and square lattices are described by the same isotropic Gaussian PDF in the limit of large times [42]). In the case of \(d\)-dimensional LWs the anisotropy of the single step statistics is imprinted into the statistics of the displacement—no matter how large the observation time \(t\) is; see the discussion of the particular case of \(d = 2\) in [22].

In the language of field theory, ordinary random walks are renormalizable: in the long-time limit the information on macroscopic structure of the walks is reduced to finite number of constants [42]. Thus the Gaussian PDF of the sum of a large number of i. i. d. random variables is fully determined by the mean and the dispersion of those variables and their total number. The rest of the information on the statistics of these variables is irrelevant—when the Gaussian bulk of the PDF is addressed. In contrast, the entropy function of the large deviations theory is the Legendre transform of the logarithm of the characteristic function of the random variable in the sum. Thus the large deviations description is sensitive to the details of statistics of one step of the walk. We conclude that the Gaussian bulk of the PDF of the sum of large number of i. i. d. random variables with finite dispersion is determined by a finite number of constants characterizing the statistics of the single step, but the tail of the PDF is not.

In this work we show that the PDF of a \(d\)—dimensional LW is not universal and cannot be specified with a finite number of constants—already in the bulk—if the statistics of the velocity of single steps, given by the PDF \(F(v)\), is anisotropic. To stress this fact we call the corresponding distributions described by equation (3) ‘anisotropic Lévy distributions’. In contrast, if the velocity statistics is isotropic, for example, \(F(v)\) is given by the uniform distribution on the surface of the \(d\)-dimensional sphere of radius \(v_0 = \text{const}\) (see figure 1(a)), the description of the process can be reduced to the one-dimensional case and the bulk of the corresponding

**Figure 1.** Examples of Lévy walks in three dimensions. A walker has a constant (by absolute value) velocity \(v_0 = |v| = 1\). When performing uniform Lévy walk (a), the walker, after completing a ballistic flight, instantaneously selects a random time \(\tau\) for a new flight and randomly chooses a flight direction (it can be specified by a point on the surface of the unit sphere). The velocity PDF \(F(v)\) is described by the uniform distribution over the unit sphere’s surface. In the anisotropic XYZ Lévy walk (b), the walker is allowed to move only along one axis at a time. After the completion of a flight, the walker selects a random time \(\tau\) for a new flight, one out of six directions (with equal probability) and then moves along the chosen direction. The velocity PDF \(F(v)\) in this case is six delta-like distributions located at the points where the unit sphere is penetrated by the frame axes. The parameter \(\alpha\) is 3/2. In our simulations we used \(\psi(\tau) = (3/2)\tau^{-5/2}\) for \(\tau > 1\) otherwise it is zero.
PDF is fully determined by a finite number of constants which can be calculated from $F(v)$. Thus in the anisotropic case there is a dramatic difference between the bulk of the PDFs of a LW and the random walk with finite dispersion of the single step duration.

We demonstrate that besides the limiting distribution given by equation (3) there is another limiting distribution which is determined by

$$
\lim_{t \to \infty} P(v, t) = \frac{A}{v^d \Gamma(1 - \alpha)} |\Gamma(1 - \alpha)| (\tau) 
\times \int_{v' > v} F(v) v'^{d-1} dv'
\left[ \frac{v'^\alpha}{v^{1+\alpha}} - (\alpha - 1) \frac{v'^{\alpha-1}}{v^\alpha} \right],
$$

where $v = x/t$ is the effective velocity of the particle, $v = v\hat{v}$ and the limit exists because the tail of the PDF is determined by ballistic-type events. This distribution is called infinite density, where the word ‘infinite’ refers to the non-normalizable character of this function found previously in the one-dimensional case [17, 18, 43, 44]. This pointwise limit holds for $v \neq 0$ non-contradicting normalization of the PDF: in this limiting procedure the normalization is carried by the point $v = 0$. The existence of the other scaling limit is a unique property that has origins in the scale-invariance of the tail of distribution of $\tau$. This distribution describes the tail of the PDF of the particle’s coordinate. In this sense it is the counterpart of the large deviations result for ordinary random walks given by equation (2). There is, however, a significant difference: the large deviations function describes averages of high-order moments, but in the case of Lévy walks the infinite density already provides dispersion of the process. This can be seen on observing that, as we demonstrate, the distribution provided by equation (3) has power-law tail with divergent second moment. This is because for Lévy walks rare events when the walker performs extremely long ballistic flights have a substantial impact on the total displacement of the walker, even in the limit of long times. The probability of long ballistic steps is not negligibly small, and single ballistic steps could be discerned in a single trajectory of the walker for any time $t$. Such steps form the outer regions of the PDF.

Thus the two limit distributions describe the bulk and the tail, respectively, of the LW’s PDF. The bulk is formed by the accumulation of typical (most probable) steps. They are responsible for a diffusive motion (albeit an already anomalous one). In contrast, the PDF’s tails are formed by long ballistic flights, and are described by the infinite density. These flights are rare steps where the walker moves for a long (i.e. comparable to the total observation time $t$) time without changing its velocity. In the case of the Lorentz billiard this is the situation when the velocity vector of the particle aligns close to the direction of one of the ballistic corridors [29]. Though this happens relatively rarely, the distance covered by the walker during such a flight is so large that these flights give finite contribution to the probability that the walker displacement after time $t$ is of the order $v_0 t$. When the probability of long flights is, for example, exponentially small (as in the case of standard random walks) the contribution of the flights can be neglected. This is not, however, the case in LWs with power-law asymptotic $\psi(\tau)$, as we demonstrate in this paper.

The paper is organized as follows. In section 2 we introduce the basic definitions and the tool of the study—the Fourier–Laplace transform of the PDF of the walker’s coordinate. In the next section we provide a complete solution for the case of isotropic statistics of velocity of the walker. Central result of our work—the anisotropic CLT for the bulk of the PDF is derived in section 4. The next section describes universality of the tail of this non-universal bulk that helps finding low moments of the distance from the origin. Section 7 provides the other limiting theorem on infinite density that provides the tail of the distribution. The next section provides a detailed description of the moments of arbitrary order including anomaly in
growth due to anisotropy. In section 9 we provide the tail of the PDF and Conclusions résumé our work. A brief summary of some of our results in dimension 2 together with physical discussion on classification of models, e.g. chaotic motion in egg-crate potential, laser cooled atoms, etc, with respect to their underlying isotropy was recently published [22].

2. Fourier–Laplace transform of the PDF $P(x, t)$

In this section we specify the considered random process and introduce the main tool of the analysis on which all further results rest. This is the Fourier (in space)–Laplace (in time) transform of $P(x, t)$.

We consider a LW as an infinite sequence of flights (steps) of random duration $\tau_i$, where $i$ is the flight’s index in the chronologically ordered sequence. The process starts at time $t = 0$ at the point $x = 0$. The velocity of the walker during a flight is a random vector $v_i$ which remains constant during the flight. Upon the completion of the flight both the velocity $v_{i+1}$ and the duration $\tau_{i+1}$ of the next flight are randomly chosen, by using PDFs $F(v)$ and $\psi(\tau)$ respectively. The number of flights $N(t)$ performed during observation time $t$, is a random number constrained by $\tau = \sum_{i=1}^{N(t)} \tau_i + \tau_0$, where $\tau_0 = t - t_0$ is the so-called backward recurrence time [14]. The time $t$ coordinate of the particle is

$$x(t) = \sum_{i=1}^{N(t)} v_i \tau_i + v_{N(t)+1} \left( t - \sum_{i=1}^{N(t)} \tau_i \right).$$

The simplest model is the $d$-dimensional ‘Lévy plotter’, the product of $d$ independent one-dimensional walks along the basis vectors which span $\mathbb{R}^d$. The PDF of this process is the product of the corresponding one-dimensional PDFs [17, 18, 20]. This case demands no further calculation so we next consider non-trivial set-ups.

We consider two intuitive models, the uniform LW and anisotropic XYZ... LW. In the uniform model $F(v)$ is specified by the uniform distribution on the surface of $d$-dimensional unit sphere, so velocity has fixed magnitude 1; see figure 1(a). As we demonstrate in the next section, in many respects this model can be reduced to the one-dimensional case. In the anisotropic XYZ... model, the particle moves along one of the $d$ basis vectors at a time; see figure 1(b). The analysis we present below is valid for any PDF $F(v)$ obeying the symmetry $F(-v) = F(v)$. As an illustration, we consider a particular type of LWs in $\mathbb{R}^d$ with factorized velocity distribution $F(v) = F(|v|) \cdot F_\theta(v/|v|)$. In this product PDF the first multiplier controls the absolute value of the velocity (the simplest choice is $F(|v|) = \delta(|v| - v_0)$) while the second multiplier governs the direction statistics of steps. A PDF $F_\theta(v/|v|)$ is a subject of directional statistics [32] and can be specified with a probability distribution on the surface of the $(d-1)$ dimensional unit sphere in $\mathbb{R}^d$. For example, in $\mathbb{R}^3$ the continuous transition from the isotropic model to the XYZ LW can be realized with six von Mises–Fisher distributions [32] (centered at the points where the axes pierce the unit sphere) by tuning the concentration parameter of the distributions from zero to infinity. We demonstrate in the following sections that different statistics enter the PDF $P(x, t)$ through the moments $\langle k \cdot v \rangle^{2n}$. In particular, for the XYZ... model we have

$$\langle (k \cdot v)^{2n} \rangle = \frac{\sum_{i=1}^{d} k_i^{2n}}{d} (v_0)^{2n}.$$ 

In [22] we discuss physical models belonging to different classes of symmetry, e.g. the Lorentz gas with infinite horizon belongs to the XYZ... class.
The remaining PDF that defines the walk process is $\psi(\tau)$. Below we consider $\psi(\tau)$ that has the tail

$$\psi(\tau) \sim \frac{A}{\Gamma(-\alpha)} \tau^{-1-\alpha}, \quad 1 < \alpha < 2,$$

where $A > 0$ and $\Gamma(x)$ is the gamma function (observe that $\Gamma(-\alpha) > 0$ when $1 < \alpha < 2$). The factor $\Gamma(-\alpha)$ is introduced in order to make the Laplace transform $\psi(u)$ of $\psi(\tau)$

$$\psi(u) = \int_0^\infty \exp[-u\tau] \psi(\tau) d\tau,$$

have small $u$ behavior (that is determined by the tail of $\psi(\tau)$),

$$\psi(u) = 1 - \langle \tau \rangle u + Au^\alpha + \ldots,$$

where $\langle \tau \rangle = \int_0^\infty t \psi(t) dt$ is the average waiting time and dots stand for higher-order terms.

We use $k$ and $u$ to denote coordinates in Fourier and Laplace space respectively. By explicitly providing the argument of a function, we will distinguish between the normal or transformed space, for example $\psi(\tau) \rightarrow \psi(u)$ and $g(x) \rightarrow g(k)$.

The lower limit of $\tau$ for which equation (7) holds depends on the considered model. For instance the inverse gamma PDF,

$$\psi(\tau) = \pi \tau^{-2} \exp\left(-\frac{1}{2\tau}\right),$$

the tail described by equation (7) holds at $\tau \gg 1$ with $\alpha = 3/2$ and $A = 8/3$. The corresponding Laplace pair obeys

$$\psi(u) = [1 + 2\sqrt{u}] \exp[-2\sqrt{u}] \sim 1 - 2u + \frac{8u^{3/2}}{3},$$

which reproduces equation (9) where we use

$$\langle \tau \rangle = \int_0^\infty \frac{2\tau^{-3/2} d\tau}{\sqrt{\tau}} \exp\left[-\frac{1}{\tau}\right] = \frac{2\Gamma(1/2)}{\sqrt{\pi}} = 2.$$

We introduce the key instrument of our analysis, the Montroll–Weiss equation. It provides with the Laplace transform

$$P(k, u) = \int_0^\infty \exp[-u\tau] P(k, \tau) d\tau,$$

of the characteristic function of the position $x(t)$ of the random walker at time $t$,

$$P(k, t) = \langle \exp[i k \cdot x(t)] \rangle = \int \exp[i k \cdot x] P(x, t) dx,$$

in terms of averages over statistics of $v$ and $\tau$. We have

$$P(k, u) = \left\{ \frac{1 - \psi(u - ik \cdot v)}{u - ik \cdot v} \right\} \frac{1}{1 - \langle \psi(u - ik \cdot v) \rangle},$$

see appendix A. Here the angular brackets denote the averaging over the PDF $F(v)$. The technical problem is to invert this formula in the limit of large time.
3. Isotropic model

In this section we consider isotropic statistics of velocity where $F(v)$ depends on $|v|$ only. The magnitude of velocity is a random variable drawn from the PDF $S_{d-1}v^{d-1}F(v)$ where $S_{d-1} = 2\pi^{d/2}/\Gamma(d/2)$ is the area of unit sphere in $d$ dimensions ($2\pi$ in $d = 2$ and $4\pi$ in $d = 3$). We show that the PDF $P(x,t)$ of a LW in $\mathbb{R}^d$ can be derived from the one-dimensional distribution. Thus the well-developed theory of one-dimensional LWs can be used for describing $d$-dimensional isotropic LWs.

We start with the observation that for isotropic statistics of $v$, the average of an arbitrary function $h$ of $k \cdot v$ depends only on $|k|$; thus $\langle h(k \cdot v) \rangle$ can be obtained by taking $k = k\hat{x}$ (where $\hat{x}$ is the unit vector in the $x$-direction),

$$\langle h(k \cdot v) \rangle = \langle h(kv_x) \rangle = \int_{-\infty}^{\infty} h(kv_x)F(v_x)dv_x,$$

where $F(v_x)$ is the PDF of $x$-component of the velocity. It can be written in terms of the PDF $F(v) = F(v) = F_0(v/|v|)$ which obeys the normalization,

$$\int_{-\infty}^{\infty} F(v)dv = S_{d-1}\int_{0}^{\infty} v^{d-1}F(v)dv = 1.$$  \hspace{1cm} (17)

For $d > 2$ we have

$$F(v_x) = \frac{\int_{|v_x|}^{\infty} (v')^{d-1}F(v')dv'}{\int F(v')dv'} = \frac{\int_{|v_x|}^{\infty} (v')^{d-1}F(v')dv'\int_{0}^{\pi} \delta(v_x - v'\cos\theta)\sin^{d-2}\theta d\theta}{\int_{0}^{\infty} (v')^{d-1}F(v')dv'\int_{0}^{\pi} \sin^{d-2}\theta d\theta} \int_{|v_x|}^{\infty} (v')^{d-1}F(v')dv'\int_{0}^{\pi} \delta(v_x - v'\cos\theta)\sin^{d-2}\theta d\theta = \int_{|v_x|}^{\infty} (v')^{d-1}F(v')dv' \int_{-1}^{1} \delta(v_x - v'x)(1 - x^2)^{(d-3)/2}dx$$

$$= \frac{2\pi^{(d-1)/2}}{\Gamma[(d - 1)/2]} \int_{|v_x|}^{\infty} v^{d-2}\left(1 - \frac{v^2}{v_x^2}\right)^{(d-3)/2}F(v)dv,$$ \hspace{1cm} (18)

where we used $\int_{-1}^{1} (1 - x^2)^{(d-3)/2}dx = \sqrt{\pi} \Gamma[(d - 1)/2]/\Gamma(d/2)$ and equation (17). In the case of two dimensions $\theta$ varies between 0 and $\pi$ (not $2\pi$) but the calculation still holds. Thus equation (18) provides the distribution of $x$-component of velocity in arbitrary space dimension $d > 1$.

Equation (18) can be simplified further in the case of a uniform model with velocity $v_0$ where $F(v) = v_0^{-d}S_{d-1}^2\delta(v - v_0)$. Integration in equation (18) gives

$$F(v_x) = P_{S_{d/2-2},v_0}(v_x),$$

where $P_{S_{d/2-2},v_0}(v)$ is the (normalized) power semicircle PDF with range $v_0$ and shape parameter $d/2 - 2$ that vanishes when $|v| > v_0$ and for $|v| < v_0$ is given by $[45]$

$$P_{S_{d/2-2},v_0}(v) = \frac{\Gamma(d/2)}{\sqrt{\pi} v_0\Gamma[(d - 1)/2]} \left(1 - \frac{v^2}{v_0^2}\right)^{(d-3)/2}.$$ \hspace{1cm} (19)

The moments of this distribution read
\[ \langle |v_i|^\gamma \rangle = \sqrt{\gamma^{\gamma}} \frac{\Gamma(\gamma + 1/2) \Gamma(d/2)}{\sqrt{\pi} \Gamma[(d + \gamma)/2]} \]  
(20)

Note that the ratio of gamma functions can be rewritten as a product if \( d \) is an odd number.

Finally, from this PDF we can derive the PDF and the moments for arbitrary \( F(v) \). For the PDF we find from equations (18) and (19),
\[ F(v) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \int_{|v|}^\infty v^{d-1} P_{S_{d/2-2,v}}(v)F(v)dv, \]
which can also be seen directly from the definition. For the moments, interchanging the order of integrations, we obtain from equation (18) the identity
\[ \langle |v|^\gamma \rangle = \int_{|v|}^\infty v^{d-\gamma-1} F(v)dv. \]
(23)

For the Gaussian distribution \( F(v) = (2\pi \bar{v}_0^2)^{-d/2} \exp \left[-v^2/2\bar{v}_0^2\right] \) it gives
\[ \langle |v|^\gamma \rangle = \frac{2\gamma^{\gamma/2} \Gamma(\gamma + 1/2) \bar{v}_0^\gamma}{\sqrt{\pi}}, \]
(24)

which reproduces \( \bar{v}_0^2 = \bar{v}_0^2 \) when \( \gamma = 2 \). We observe that \( \langle |v|^\gamma \rangle = S_{d-1} \int_0^\infty v^{d + \gamma - 1} F(v)dv \) so that equation (23) implies the identity
\[ \langle |v|^\gamma \rangle = \frac{\Gamma[(\gamma + 1)/2] \Gamma(d/2)}{\Gamma[(d + \gamma)/2] \sqrt{\pi}} \langle |v|^\gamma \rangle. \]
(25)

This formula is a consequence of the isotropy of the process and for any random vector \( x \) whose PDF depends on \( |x| \) only we have
\[ \langle |x|^\gamma \rangle = \frac{\Gamma[(\gamma + 1)/2] \Gamma(d/2)}{\Gamma[(d + \gamma)/2] \sqrt{\pi}} \langle |x|^\gamma \rangle, \]
(26)

where \( x_s, s \in \{1, 2, ..., d\}, \) is one of the Cartesian coordinates of \( x \).

3.1. Bulk statistics: \( d \)-dimensional Lévy distributions

We use equation (16) to rewrite equation (15) in the one-dimensional form,
\[ P(k, u) = \left\{ \begin{array}{ll} 1 - \psi(u - ikv) & \text{for } u - ikv > 0 \\ \frac{1}{u} & \text{for } u - ikv < 0 \end{array} \right\}, \]
(27)

where the averaging is taken over the distribution of \( v \), given by equations (18) and (21). Thus we can directly use the results for \( P(k, t) \) in the one-dimensional case. The difference from the one-dimensional case is in how the real space PDF is reproduced from \( P(k, t) \); here the formula for the inverse Fourier transform of radially symmetric function in \( d \) dimensions has to be used. We find from [17] that the bulk of the PDF is described by
\[ P_{\text{cen}}(k, t) \sim \exp \left[ -K_d |k|^\alpha \right], \quad \text{(28)} \]

\[ K_\alpha = \frac{A}{\langle \tau \rangle} \left( \langle |v|^{\alpha} \rangle \right)^{1/2} \left( \frac{\pi \alpha}{2} \right), \quad \text{(29)} \]

where \( \langle |v|^{\alpha} \rangle \) is given by equation (23) with \( \gamma = \alpha \). Here the subscript in \( P_{\text{cen}} \) was introduced in [17]. It stands for the centre (or bulk) part of the PDF. Briefly, to obtain this result we expand \( P(k, u) \) using the scaling assumption that \( k^\alpha \) is of the order \( u \) when both are small. Later, we will derive these results as a special case of the more general non-isotropic model (see equation (55) below).

We can write \( K_\alpha \) in terms of \( \langle |v|^{\alpha} \rangle \) using equation (25),

\[ K_\alpha = \frac{A\Gamma[(\alpha + 1)/2] \Gamma(d/2) \langle |v|^{\alpha} \rangle^{1/2}}{\langle \tau \rangle^{(d + \alpha)/2} \sqrt{\pi}} \left( \frac{\pi \alpha}{2} \right). \quad \text{(30)} \]

This coefficient reduces to the diffusion coefficient of one-dimensional walk found in [17] setting \( d = 1 \). In the case where \( v \) is a conserved constant \( v_0 \) (modelling conservation of energy), we find

\[ K_\alpha = \frac{A\Gamma[(\alpha + 1)/2] \Gamma(d/2) v_0^{\alpha/2}}{\langle \tau \rangle^{(d + \alpha)/2} \sqrt{\pi}} \left( \frac{\pi \alpha}{2} \right). \quad \text{(31)} \]

Using the inverse Fourier transform we find for the PDF’s bulk,

\[ P_{\text{cen}}(x, t) \sim \frac{1}{(K_\alpha)^{d/\alpha}} L_d \left( \frac{x}{(K_\alpha)^{d/\alpha}} \right), \quad \text{(32)} \]

\[ L_d(x) = \int \exp[ik \cdot x - k^\alpha] \frac{dk}{(2\pi)^d}. \quad \text{(33)} \]

These formulas provide generalized CLT for \( d \)-dimensional isotropic LWs. Below, these will be generalized to the case of arbitrary (not necessarily isotropic) statistics of velocity. We provide the form of \( L_d(x) \) in the cases of physical interest \( d = 2 \) and \( d = 3 \). In the two-dimensional case we have,

\[ L_2(x) = \int_0^\infty k J_0(kx) \exp \left[ -k^\alpha \right] \frac{dk}{2\pi}. \quad \text{(34)} \]

which can be called two-dimensional isotropic Lévy density. Here \( J_0(x) \) is the Bessel function of the first kind. In three dimensions we find (\( L_3(x) \) is normalized \( \int L_3(x) \, dx = 1 \)),

\[ L_3(x) = -\frac{1}{x} (\partial_x) \int_0^\infty \frac{dk}{2\pi^2} \cos(kx) \exp[-k^\alpha] = \frac{L(x)}{2\pi x}, \quad \text{(35)} \]

where \( L(x) = L_1(x) \) is the standard Lévy distribution,

\[ L(x) = -\int_{-\infty}^\infty \exp(ikx) \frac{dk}{2\pi}. \quad \text{(36)} \]

Thus in three dimensions the isotropic Lévy density can be obtained from the one-dimensional one by differentiation (this is an improved version of the old result of [12] valid for arbitrary velocity distribution). This is true for any odd-dimensional case. It can be demonstrated that in
the case of even dimension $L_{2n}(x)$ can be obtained from $L(x)$ using derivative operator of half integer order, see appendix E and see [21].

We find from equation (35) that $L_{3}(x) \propto |x|^{-\alpha-3}$ at large argument, where we use the well-known behavior $L(x) \sim |x|^{-\alpha-1}$, see e.g. [14]. (Similarly it will be demonstrated below that $L_{d}(x) \sim |x|^{-\alpha-d}$.) This tail must fail at larger arguments because it would give divergent dispersion $(x^2(t)) \propto \int x^2L_{d}(x)dx = \infty$, which is wrong provided that the moments of $F(v)$ are finite, an assumption we use throughout this paper. The PDF must necessarily decay fast at $x > v_{t}$ where $v_{t}$ is the typical value of velocity. Thus the Lévy density does not provide a valid description of the tail of the PDF that determines dispersion. This necessitates the study of the tail of the distribution performed below.

3.2. Infinite density

The description of the tail of the PDF is performed using the reduction to the one-dimensional case where the problem was solved in [17]. This solution is based on asymptotic resummation of the series for the characteristic function.

We observe that for isotropic statistics of velocity $P(k,t)$ obeys

$$P(k, t) = \langle \exp [-ik \cdot x] \rangle = 1 + \sum_{n=1}^{\infty} \frac{(-1)^{n} k^{2n} (\lambda_{1}^{n})^{2n}}{(2n)!}, \quad (37)$$

where we used the facts that odd moments of $k \cdot x$ vanish and that isotropy implies that $\langle (k \cdot x)^{2m}\rangle$ is independent of direction of $k$, so it can be obtained setting $k = k\hat{k}$, giving $\langle (k \cdot x)^{2m}\rangle = k^{2n}(\lambda_{1}^{n})^{2n}$, see with similar consideration for velocity.

Though $\langle x_{1}^{2n}\rangle$ cannot be found completely at all times, it was discovered in [17] that this can be done asymptotically in the limit of large times. The proper adaptation of the result tells that using small $k$ and $u$ expansion of the quasi-one-dimensional Montroll–Weiss equation (15) when keeping the ratio $k/u$ fixed we find,

$$P(k, t) \sim 1 + \frac{A}{\langle \tau \rangle} \sum_{n=1}^{\infty} \frac{\Gamma(2n - \alpha)(-1)^{n} t^{2n+1-\alpha} \langle v_{y}^{2n}\rangle}{(2n-1)! \Gamma(1-\alpha) \Gamma(2n + 2 - \alpha)}. \quad (38)$$

This result was obtained in the limit of long times asymptotically—that is, the $n$–th term in the series is valid provided time is large. How large this time is depends on $n$: the higher $n$ is, the larger times are needed for the validity of the asymptotic form. Thus at arbitrarily large but finite $t$ the terms of the series fail starting from some large but finite $n$. Thus, in contrast, to equation (86) the resummation of the series given by equation (38) does not have to lead to $P(x, t)$ because there is no time for which all terms in the series of equation (38) are valid.

It is the finding of [17] that resummation of the series in equation (38) still produces a function that has physical meaning. That function, called the infinite density, gives a valid description of the tail of $P(x, t)$ but fails in the bulk. This is because the bulk corresponds to small $x$ and large $k \propto 1/x$. For very small $x$ very large $k$ are relevant, implying that terms with very large $n$ become relevant in the sum. However these terms are not valid at finite $t$, leaving the small $x$ inaccessible to the sum in equation (38).

Since statistics of both $x$ and $v$ are isotropic then,

$$\frac{\langle x_{1}^{2n}\rangle}{\langle x^{2n}\rangle} = \frac{\langle v_{y}^{2n}\rangle}{\langle v^{2n}\rangle}, \quad (39)$$
see equation (26). We find, comparing the series in equations (38)–(86),
\[
\langle x^{2n}(t) \rangle = \frac{2nA\langle v^{2n} \rangle}{\langle \tau \rangle |\Gamma(1-\alpha)|(2n+1-\alpha)(2n-\alpha)} t^{2n+1-\alpha}.
\] (40)

Remarkably this is independent of dimension and thus coincides with the one-dimensional case (isotropy implies that the geometry disappears from \(x^{2n} \) because of equation (39)).

The use of these moments for formal reconstruction of the long-time limit of the PDF
\[
\delta(x) = \langle \delta(|x(t)\rangle - x) \rangle
\]
of \(|x(t)\rangle \) through \(P_\alpha(x,t) \) defined by
\[
P_\alpha(x,t) = \frac{\delta x^n t^{n-1}}{\langle \tau \rangle |\Gamma(1-\alpha)|} \int_{|v|/t}^{\infty} v^{d-1}F(v)dv \times \left[ \frac{|v|^\alpha}{|v|^\alpha - (\alpha - 1)|x(t)|^{\alpha - 1}} \right],
\] (42)

which holds for \(x \neq 0\). The function \(P_\alpha(x,t) \) clearly describes the long-time behavior of the moments of \(|x(t)\rangle \) via
\[
\langle x^{2n}(t) \rangle \sim \int_0^\infty P_\alpha(x,t)x^{2n}dx, \ n \geqslant 1.
\] (43)

This function, however, does not describe the normalization (obtained as \(n = 0\)) since \(P_\alpha(x,t) \sim x^{-(1+\alpha)} \) for \(x \to 0\). Hence \(P_\alpha(x,t) \) is not normalizable, for which reason it is called infinite density. However it does describe the integer order moments \(\int_0^\infty P(x,t)x^{2n}dx \) where \(P(x,t) \) is the PDF of the distance to the origin \(|x(t)\rangle \). Thus \(P(x,t) \sim P_\alpha(x,t) \) is true for integrals with integer powers. It can be seen that this function describes the tail of the PDF \(P(x,t) \) at \(x \sim t \) (that is, at large times \(P(x,t) \sim P_\alpha(x,t) \) holds for large \(x \propto t) \) while the bulk corresponds to \(x \sim t^{1/\alpha} \). This fits the moments of integer order being determined by the tail of \(P(x,t) \), as clarified in the coming sections. Below we derive the infinite density in \(d \) dimensions in different form, proving the existence of finite long-time limit \(\lim_{t \to \infty} t^{d-1+\alpha} P(x,t) \) for arbitrary (anisotropic) statistics of velocity. The descriptions of the bulk and the tail of the PDF together with the moments provide a complete description of the \(d \)-dimensional walk with isotropic statistics.

In figure 2 we present simulations of the uniform model in dimension 3. Excellent agreement is found between numerical results and theory, both for the bulk distribution described by the Lévy density and the tail of the density corresponding to the infinite density. These finite time simulations demonstrate that while the infinite density is describing rare events it still can be sampled with a finite number of particles.

4. CLT for anisotropic Lévy walks

Numerical simulations presented in figure 3, for the XYZ and the uniform models clearly demonstrate considerable variation between the models and non-universal features of the global shape of the density. Thus we now turn to investigate in detail non-isotropic Lévy walks. We start the analysis of anisotropic LWs with the derivation of the generalized CLT describing the PDF \(P(x,t) \) of the walker. We consider in this section random walks whose single step duration’s PDF \(\psi(u) \) obeys equation (9) at small \(u \), but we let the range of \(\alpha \) considered include
$\alpha = 2$. That is, we consider $\psi(u)$ in equation (9) with $1 < \alpha \leq 2$. In the case of $\alpha = 2$, though, $\psi(\tau)$ does not obey equation (7). This is because equation (9) with $\alpha = 2$ describes the case of finite dispersion of $\tau$ given by $\langle \tau^2 \rangle = \psi''(u = 0) = 2A$. In contrast, for $\alpha = 2$ equation (7) gives a $\tau^{-3}$ tail, for which the dispersion is infinite. Thus $\psi(\tau)$ obeying equation (9) with $\alpha = 2$ decays faster than $\tau^{-3}$.

We demonstrate that for $\psi(u)$ obeying equation (9) with $1 < \alpha \leq 2$ there is finite limit

$$\lim_{t \to \infty} t^{\alpha/2} P(t^{1/\alpha} x, t) = \int \exp \left[ i k \cdot x - \frac{A}{\langle \tau \rangle} \cos \left( \frac{\pi \alpha}{2} \left| \langle k \cdot \psi \rangle \right| \right) \right] \frac{dk}{(2\pi)^d}.$$

(44)
which holds for arbitrary statistics of \( \nu \) obeying \( F(\nu) = F(-\nu) \). It is proper to call this result a generalized CLT because for \( \alpha = 2 \) it reproduces the central limit theorem,

\[
\lim_{t \to \infty} t^{1/2} P(x/\sqrt{t}, t) = \int \exp \left[ i k \cdot x - \frac{\Gamma_{j\ell} k_j k_\ell}{2} \right] \frac{dk}{(2\pi)^d},
\]

where the RHS defines \( g(x) \) in equation (1), the covariance matrix \( \Gamma \) is defined by

\[
\Gamma_{j\ell} = \frac{\langle \tau^2 \rangle (\nu, \nu)}{\tau},
\]

and we have used \( A = \langle \tau^2 \rangle /2 \). In equation (45) we use Einstein’s summation rule over the repeated indices. We observe that the units of \( k \) in equation (45) are \( l/t^{1/2} \), the units of \( x \) are \( l/t^{1/2} \) and those of \( \Gamma \) are \( l^2/t \), so that the argument of the exponent is dimensionless.

The form of the covariance matrix could be seen considering the second moment of displacement \( \tau = \sum_{k} x_k^2 \),

\[
\tau = \frac{\langle N(t) \nu^2 \tau^2 \rangle}{\tau},
\]

where we have used the implication of the law of large numbers that \( \lim_{t \to \infty} \langle N(t) \nu^2 \rangle = 1/\tau \).

In the Gaussian case the details of statistics of individual steps of the walk become irrelevant in the long-time limit: they get summarized in \( (d(d+1)/2) \) independent coefficients of the covariance matrix \( \Gamma \). The second moments of velocity \( \nu^2 \) determine the long-time statistics of the displacement uniquely. In contrast, in \( \alpha < 2 \) case the details of the walk influence the displacement’s PDF however large time is via \( \langle |k \cdot \nu|^{\alpha} \rangle \). This is a non-trivial function of the direction of \( k \), that depends on which directions of motion are more probable in one step. This is a function of a continuous variable rather than \( \Gamma \), that depends on a finite number of discrete indices. Here we assume that isotropy is broken—in the isotropic case the degree of universality of \( \alpha < 2 \) and \( \alpha = 2 \) is the same: \( \langle |k \cdot \nu|^{\alpha} \rangle \) uniquely determines the long-time behavior given by equation (30). Correspondingly infinite variability of shapes of \( P(x, t) \), is possible in contrast with fixed Gaussian shape in \( \alpha = 2 \) case.

We start the derivation of equation (44). The calculation below holds for \( 1 < \alpha \leq 2 \). We use

\[
t^{d/\alpha} P(t^{1/\alpha} x, t) = t^{d/\alpha} \int \frac{dk}{(2\pi)^d} \exp \left[ i k \cdot x \right] P(k, t)
\]

\[
= \int \frac{dk}{(2\pi)^d} \exp \left[ i k \cdot x \right] P(t^{-1/\alpha} k, t).
\]

Thus we have to prove the existence of the limit,

\[
\lim_{t \to \infty} P(t^{-1/\alpha} k, t) = \int \frac{du}{2\pi i} \exp[u] \lim_{t \to \infty} \frac{1}{t} P\left( \frac{k}{t^{1/\alpha}}, \frac{u}{t} \right).
\]

We use that \( \langle k \cdot \nu \rangle = 0 \),

\[
\left\{ \frac{1 - \psi(ut^{-1} - i k \cdot \nu t^{-1/\alpha})}{ut^{-1} - i k \cdot \nu t^{-1/\alpha}} \right\} = \langle \tau \rangle + o(t),
\]

\[
1 - \left\{ \psi \left( \frac{u}{t} - i \frac{k \cdot \nu}{t^{1/\alpha}} \right) \right\} = \frac{\langle \tau \rangle u}{t} - A \left( \frac{u}{t} - i \frac{k \cdot \nu}{t^{1/\alpha}} \right)^\alpha
\]

\[
+ o(t) = \frac{\langle \tau \rangle u + A \cos(\pi \alpha/2) \langle |k \cdot \nu|^\alpha \rangle}{t^{1/\alpha}}.
\]
where we used that $F(v) = F(-v)$ implies [17],
\[
\langle (-ik \cdot v)\rangle = \left\langle |k \cdot v|^\alpha \exp \left[ -\frac{i\pi \alpha}{2} \text{sign}(k \cdot v) \right] \right\rangle = \left\langle |k \cdot v|^\alpha \right\rangle \cos \left( \frac{\pi \alpha}{2} \right).
\]
(52)

This formula holds in the $\alpha = 2$ case as well. We find, using equations (49)–(51) in the Montroll–Weiss equation, that
\[
\lim_{t \to \infty} \frac{1}{t^{1/\alpha}} P \left( \frac{k}{t^{1/\alpha}} , \frac{u}{t} \right) = \frac{1}{u + \hat{A} |\cos(\pi\alpha/2)| \left\langle |k \cdot v|^\alpha \right\rangle},
\]
where $\hat{A} = A/\tau$, see one-dimensional case in [17]. We conclude that
\[
\lim_{t \to \infty} P(t^{-1/\alpha}k, t) = \int \frac{du}{2\pi i} \frac{\exp[u]}{u + \hat{A} |\cos(\pi\alpha/2)| \left\langle |k \cdot v|^\alpha \right\rangle},
\]
(53)
see equation (48). We find, performing the integration,
\[
\lim_{t \to \infty} P(t^{-1/\alpha}k, t) = \exp \left[ -\hat{A} \left\langle \frac{\pi \alpha}{2} \right\rangle \left\langle |k \cdot v|^\alpha \right\rangle \right],
\]
(54)
completing the proof of equation (44). We find asymptotically at large times that
\[
P(k, t) \sim \exp \left[ -\frac{tA}{\left\langle \tau \right\rangle} \cos \left( \frac{\pi \alpha}{2} \right) \left\langle |k \cdot v|^\alpha \right\rangle \right].
\]
(55)
This is one of our main results as it provides the generalized CLT for non-isotropic LWs. For the isotropic case equation (55) reduces to equations (29).

We characterize different statistics of velocity with structure function $s(\hat{k})$ that depends on the unit vector $\hat{k} = k/|k|,$
\[
s(\hat{k}) = \frac{\Gamma[(d + \alpha)/2]\sqrt{\pi}}{\Gamma[(\alpha + 1)/2] \Gamma(d/2)} \left\langle |\hat{k} \cdot v|^\alpha \right\rangle / \left\langle |v|^\alpha \right\rangle.
\]
(56)
This is defined so that for isotropic statistics $s(\hat{k}) = 1$ (in that case we have $\left\langle |\hat{k} \cdot v|^\alpha \right\rangle = \left\langle |v|^\alpha \right\rangle$ where $\left\langle |v|^\alpha \right\rangle$ is determined by equation (25)). We have, with this definition,
\[
\lim_{t \to \infty} t^{d/\alpha} P(t^{1/\alpha}x, t) = \int \exp[ik \cdot x - K_\alpha k^\alpha s(\hat{k})] \frac{dk}{(2\pi)^d},
\]
(57)
where we use $K_\alpha$ defined in equation (30) (in the anisotropic case $K_\alpha$ does not have a direct interpretation of the diffusion coefficient so this is to be taken as a mathematical definition). The PDF of the displacement obeys
\[
P(x, t) \sim \frac{1}{(K_\alpha t)^{d/\alpha}} \hat{L}_d \left( \frac{x}{(K_\alpha t)^{1/\alpha}} \right).
\]
(58)
\[
\hat{L}_d(x) = \int \exp[ik \cdot x - k^\alpha s(\hat{k})] \frac{dk}{(2\pi)^d}
\]
(59)
For the isotropic model there is no modulation and \( \hat{L}_d(x) \) is the universal function \( L_d(x) \) introduced previously, see equation (33). Thus different isotropic statistics produces the same long-time PDF in the bulk, differing only by the value of \( K_i \). For the XYZ... model \( \langle |k \cdot v|^\alpha \rangle = v_0^\alpha \sum_{\alpha} |k_i|^\alpha \delta d \) we find that the distribution factorizes in the product of one-dimensional distributions. Thus in this case the bulk of the PDF coincides with that of independent walks along different axes. In these cases the functional shape is universal.

The chief feature introduced by the passage from one dimension to the higher-dimensional case is that quite arbitrary angular structure of the distribution becomes possible. The structure function \( s(\hat{k}) \) describes positive angular modulation in \( k \)–space that correspondingly changes the functional form in real space—see equation (59). This function does not seem to obey strong constraints that would strongly limit the possible forms of \( \hat{L}_d(x) \). We stress this fact, calling distribution \( \hat{L}_d(x) \) ‘anisotropic \( d \)-dimensional Lévy distribution’ in contrast with \( d \)-dimensional isotropic Lévy distributions introduced previously [12] in the context of Lévy flights that have universal shape.

Considering marginal distributions of components of \( x \), one reduces the problem to one dimension, restoring universality of the distribution. Integrating equation (58),

\[
P(x_i, t) = \int P(x, t) \prod_{k=1}^{d} dx_k \sim \frac{1}{(K_d)^{1/\alpha}} L\left( \frac{x_i}{(K_d)^{1/\alpha}} \right),
\]

where \( L(x) \) is defined in equation (36) and \( K_i \) is ‘diffusion coefficient in \( i \)th direction’,

\[
K_i = \frac{\alpha}{\tau} \cos \left( \frac{\pi \alpha}{2} \right) \hat{|x|}^\alpha.
\]

Thus marginal PDFs are given by the standard one-dimensional symmetric Lévy stable law. In contrast the PDF \( \hat{P}(x, t) = \int \delta(\hat{|x(t)|} - x) P(x, t) dx \) of the distance \( x(t) = |x(t)| \) from the origin is not universal. We find integrating equation (58) that

\[
P(x, t) \sim \frac{1}{t^{1/\alpha}} P_0\left( \frac{x}{t^{1/\alpha}} \right)
\]

\[
P_0(x) = \int x^{d/2} J_{d/2-1}(kx) dk (2\pi)^{d/2} k^{d/2-1} \exp \left[ -K_0 s(\hat{k}) \right],
\]

(see the next section for further details of notation choices) where we used

\[
\int \exp \left[ |k \cdot x| \right] \delta(\hat{|x(t)|} - x) dx = \frac{2\pi x}{k^{d/2-1}} J_{d/2-1}(kx),
\]

where \( J_\nu(\cdot) \) is the Bessel function of the first kind of order \( \nu \). We observe that integration over angles implied by the definition of the PDF of the distance from the origin does not bring the universal form of the PDF (which would then coincide with the PDF for isotropic statistics).

The structure function is present in equation (63) and can produce quite different forms of \( P(x, t) \). We have using Taylor series for the Bessel function in equation (63),

\[
P_0(x) = x^{d-1} \sum_{n=0}^{\infty} \frac{(-1)^n c_n}{n! \Gamma(d/2 + n) 2^{d/2-1}} \left( \frac{x}{2} \right)^{2n},
\]

\[
c_n = \int \frac{dk}{(2\pi)^{d/2} k^{2n}} \exp \left[ -K_0 s(\hat{k}) \right],
\]

\[\text{I Fouxon et al \ J. Phys. A: Math. Theor. 50 (2017) 154002}\]
where $x^{d-1}$ factor describes the contribution of the surface of the sphere of radius $x$. In the isotropic case we have

$$c_n = \int \frac{dk}{(2\pi)^d} k^{2n} \exp \left[-K_x k^2\right] = \frac{S_{d-1} \Gamma[(2n+d)/\alpha]}{(2\pi)^d \alpha K_{\alpha}^{2n+d}/\alpha}.$$  

We find that in the isotropic case $P_0(x) = P_0(x)/[x^{d-1}S_{d-1}]$ using the Taylor series for $P_{cen}(x,t)$ defined in equation (33),

$$L_d(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma[(2n+d)/\alpha]}{n! \Gamma(n+d/2)2^{2n+d-1}n!} x^{2n}.$$  

The factor of $\Gamma(n+d/2)$ can be simplified in cases of odd and even dimensions using $\Gamma(n+1) = n!$ and $\Gamma(n+1/2) = 2^{-n} \sqrt{\pi}(2n-1)!!$. In the case of $d = 1$ the series reproduces the one-dimensional formula [14].

As mentioned, for arbitrary statistics of velocity there seems to be no constraint on the Taylor coefficients $c_n$ that would uniquely determine the functional form of $P_0(x)$. However, despite that the small $x$ expansion of $P_0(x)$ is not universal, the large $x$ behavior of $P_0(x)$ is universal. This will be demonstrated in the next section.

5. Universal tail of anisotropic Lévy distribution and low-order moments

In this section we demonstrate that the PDF of the distance of the walker from the walk’s origin has power-law tail with universal (independent of statistics of velocity) exponent. This has the implication that moments of distance from the origin with order smaller than $\alpha$ are determined by the bulk of the PDF but moments of higher order are determined by the PDF’s tail (whose description is different from the Lévy distribution and is provided further in the text). Thus dispersion is determined by the tail of the PDF independently of the statistics of velocity.

We observe from equation (63) that

$$P_0(x) - \delta(x) = x^{d/2} \int \frac{dk}{(2\pi)^{d/2}} k^{1-d/2} \delta_{d/2-1}(kx) \times \left( \frac{A}{\langle \tau \rangle} \cos \left( \frac{\pi \alpha}{2} \right) \left| \langle k \cdot v \rangle \right| - 1 \right),$$

where we used the Fourier transform representation of the $\delta$-function and restored the definitions of the constants. By using the large argument asymptotic expansion of the Bessel function and rescaling the integration variable by $x$, we find

$$P_0(x) \sim \frac{\sqrt{\alpha}}{x^{1+\alpha} \sqrt{\pi} \langle \tau \rangle} \int \frac{dk}{(2\pi)^{d/2}} k^{1-d/2} \cos(k - \pi(d-1)/4) \times \left( \frac{A \langle k \cdot v \rangle}{x^{\alpha} \langle \tau \rangle} \cos \left( \frac{\pi \alpha}{2} \right) \right) \times \left| \langle k \cdot v \rangle \right| - 1 \right) \exp \left[-ek\right].$$  

where infinitesimal $\epsilon$ in the last term is the convergence factor introduced for convergence of the large $x$ expansion found by expanding the exponent in brackets. The leading order term is

$$P_0(x) \sim -\frac{A \sqrt{\alpha}}{x^{1+\alpha} \sqrt{\pi} \langle \tau \rangle} \left| \cos \left( \frac{\pi \alpha}{2} \right) \right| \int \frac{dk}{(2\pi)^{d/2}} k^{(1-d)/2} \cos(k - \pi(d-1)/4) \exp \left[-ek\right].$$  

(67)
The angular integral gives
\[ \int dk \langle |k \cdot v|^\alpha \rangle = \langle v^\alpha \rangle \int dk \langle |\hat{k} \cdot \hat{v}|^\alpha \rangle, \quad (68) \]
where we have interchanged the orders of averaging and integration and used that the last integral is independent of \( \hat{v} \). However, since it is independent of \( \hat{v} \) then it can be obtained taking \( \hat{v} \) in the \( x \)-direction which gives,
\[ \int dk \langle |k \cdot v|^\alpha \rangle = \langle v^\alpha \rangle \int dk \langle |k|^\alpha \rangle. \quad (69) \]
We find using this in equation (67),
\[ P_0(x) \sim -\frac{\sqrt{2} \Gamma[(d + \alpha)/2]K_\alpha}{x^{1+\alpha}(2\pi)^{d/2} \sqrt{\pi}} \int k^{1-d/2}dk \times |k|^\alpha \cos \left( k - \frac{\pi(d - 1)}{4} \right) \exp [-\epsilon k], \quad (70) \]
where we use \( K_\alpha \) defined in equation (30). It is seen readily from equation (66) that the condition of applicability of large \( x \) expansion is
\[ x \gg \alpha xK_1. \]
The \( x \)-axis defining \( k_x \) in the integrand is an arbitrary direction in space. We observe that \( |k|^\alpha \) averaged over the angles can be found using in equation (26) the isotropic statistics with \( x \) switched by \( k \):
\[ \langle |k|^\alpha \rangle_{\text{angle}} = \frac{\Gamma[(\alpha + 1)/2] \Gamma(d/2)}{\Gamma[(d + \alpha)/2] \sqrt{\pi}} |k|^\alpha, \quad (71) \]
where we set \( \langle |k|^\alpha \rangle = |k|^\alpha \). We find
\[ P_0(x) \sim -\frac{\sqrt{2} K_\alpha S_{d-1}}{x^{1+\alpha}(2\pi)^{d/2} \sqrt{\pi}} \int_0^\infty k^{\alpha+(d-1)/2}dk \times \cos \left( k - \frac{\pi(d - 1)}{4} \right) \exp [-\epsilon k], \]
where we performed the integral over angles. We write,
\[ P_0(x) \sim -\frac{\sqrt{2} K_\alpha S_{d-1}}{x^{1+\alpha}(2\pi)^{d/2} \sqrt{\pi}} \Re \left[ \exp \left( \frac{i\pi(d - 1)}{4} \right) \times \int_0^\infty \exp [-i(k - \epsilon)k] k^{\alpha+(d-1)/2}dk \right]. \quad (72) \]
By using integration variable \( t = (\epsilon + i)x \) we find
\[ P_0(x) \sim \frac{\sqrt{2} K_\alpha S_{d-1} \Gamma[(\alpha + 1)/2]}{x^{1+\alpha}(2\pi)^{d/2} \sqrt{\pi}} \Re \left[ \exp \left( \frac{i\pi(d - 1)}{4} \right) (\epsilon + i)^{-\alpha-(d+1)/2} \right]. \quad (73) \]
We conclude that at \( x \gg K_\alpha^{1/\alpha} \),
\[ P_0(x) \sim \frac{\sqrt{2} K_\alpha S_{d-1} \Gamma[(\alpha + 1)/2]}{x^{1+\alpha}(2\pi)^{d/2} \sqrt{\pi}} \sin(\pi\alpha/2). \quad (74) \]
For the probability density function we find on restoring physical units that, at \( x \gg (K_\alpha t)^{1/\alpha} \),
\[ P(x, t) \sim \frac{\sqrt{2} K_\alpha S_{d-1} \Gamma[(\alpha + 1)/2]}{x^{1+\alpha}(2\pi)^{d/2} \sqrt{\pi}} \sin(\pi\alpha/2)t, \quad (75) \]
where we used equation (62). Another form is obtained using the value of $K_α$ given by equation (30),

$$P(x,t) \sim \frac{\Gamma[(\alpha + 1)/2][\Gamma(\alpha + (d + 1)/2)\alphaA(|v|^α)]}{2^{d-1}[(1 - \alpha)/\Gamma(1 - \alpha)]\Gamma(\alpha)(d + \alpha)/2[\Gamma(\tau)x^{1+\alpha}]}.$$  

(76)

where we used $\Gamma(\alpha)\Gamma(1 - \alpha) = \pi\sin(\pi\alpha)$. In three-dimensional and one-dimensional cases we find

$$P(x,t) \sim \frac{\alphaA(|v|^α)}{\Gamma(1 - \alpha)[\tau]x^{1+\alpha}}, \; d = 1, 3.$$  

(77)

In one dimension this is a known result [14, 17, 18]. The coincidence of tails in one- and three-dimensional cases does not have clear origin. In other dimensions including the physically relevant two-dimensional case the tail is different and dimension-dependent.

We conclude that despite the PDF of $x$ being non-universal, its tail obeys the universal power-law with decay exponent $\alpha + 1$. Furthermore though the velocity statistics can be non-isotropic the tail is uniquely determined by $\langle v^α \rangle$, so effective isotropization occurs. This may appear to contradict the non-isotropy of both the bulk and the tail of the PDF demonstrated in previous and coming sections. In fact isotropization happens only in the leading order term of large $x$ series of $P(x,t)$. It is readily seen from equation (66) that the next order term involves the angular integral $\int d\hat{k}\left(\hat{k} \cdot v^α\right)^2$ that depends on non-isotropy of the velocity statistics (the orders of integration and averaging can no longer be interchanged because of non-linearity). Thus the large $x$ series of $P(x,t)$ is isotropic in leading order only.

We illustrate the conclusions of this section with the two-dimensional case of the $XYZ$... model, called the non-symmetric $x - y$ model. In this model, diffusion coefficients in $x$ and $y$ directions differ, so diffusion is non-symmetric. The velocity vector is in $x$ or $-x$ direction with probability $\lambda/2$ and in $y$ or $-y$ directions with probability $(1 - \lambda)/2$. The magnitude of velocity $v_0$ is fixed. Thus $\langle \hat{k} \cdot v^α \rangle = v_0^α (\lambda k_x^α + (1 - \lambda)k_y^α)$. We find from equation (44) that,

$$P(x,t) \sim \frac{1}{(K_xK_y^2)^{1/2}}L\left(\frac{x}{(K_x)^{1/2}}\right)L\left(\frac{y}{(K_y)^{1/2}}\right),$$

$$K_x = \frac{A\lambda}{\tau}\cos\left(\frac{\pi\alpha}{2}\right), \quad K_y = \frac{A\lambda}{\tau}\cos\left(\frac{\pi\alpha}{2}\right).$$

This distribution is non-symmetric, where the asymmetry results from different scaling factors of one-dimensional distributions in $x$ and $y$ directions. The PDF $P(x,t)$ of the distance $x$ from the origin is (in the previous equation $x$ is $x$ component of $x$, below $x = |x|$ with no ambiguity),

$$P(x,t) \sim \frac{x}{(K_xK_y^2)^{1/2}}\int_0^{2\pi}L\left(\frac{x\cos\phi}{(K_x)^{1/2}}\right)\times L\left(\frac{x\sin\phi}{(K_y)^{1/2}}\right)d\phi.$$  

(78)

This PDF depends on the degree of asymmetry via $\lambda$ included in $K_x, K_y$. In the limits of $\lambda \to 0$ or $\lambda \to 1$ it gives LW in the direction of the corresponding axis. When $\lambda = 1/2$ we find the distribution of $x - y$ model. In contrast, the large $x$ asymptotic form of $P(x,t)$ depends on $\langle |v|^α \rangle = v_0^α$ only and thus is independent of $\lambda$, see equation (76). The simplest way of verifying this seems to be using integral representation of $L(x)$ and repeating the steps of the previous derivation in the general case.
The universal power-law tail obtained above has non-trivial implications for the moments of distance from the origin. The moment of $q$-th order $\langle x^q(t) \rangle$ diverges at large $x$ for $q > \alpha$. Consequently the second moment is determined by the tail of the PDF independently of the statistics of velocity. The moments of order $0 < q < \alpha$ are determined by the bulk of the PDF. We presume that the moments are determined either by the bulk or by the tail of the PDF, which is usually the case and is confirmed below. Then demanding self-consistency we find the described conclusions. We find from equations (62),

$$\langle |x|^q(t) \rangle \sim c_q t^{\alpha/q}, \quad q < \alpha$$  \hspace{1cm} (79)

where the constants $c_q$ are given by

$$c_q = \int_0^\infty x^q P_0(x) dx,$$  \hspace{1cm} (80)

where $P_0(x)$ is given by equation (63). For $q > \alpha$ the integral diverges and the formula fails, demanding finding the tail of the PDF. This is obtained in section 7. Before we study the tail of the distribution we provide consequences of our consideration for sums of random variables.

6. Statistics of sum of power-law tailed variables in $d$ dimensions

In this section we describe the correspondence between our results and distribution of sum of independent identically distributed random vectors whose PDF has power-law tail with infinite dispersion but finite mean, see [12, 40]. We observe that at large times the bulk distribution of $x(t)$ is identical with that of

$$x_0(t) = \sum_{i=1}^{N(t)} v_i \tau_i.$$  \hspace{1cm} (81)

Indeed, comparing this definition with equation (5) we see that equating distributions in the bulk is equivalent to neglecting the displacement at large times due to the last step of the walk. This point is obvious for ordinary random walks but not for LWs: it breaks down for ballistic LW with infinite average duration of the step [19]. In our case the proof of the distributions’ equality is done using the Montroll–Weiss equation. It is demonstrated in the appendix A that the Montroll–Weiss equation for the variable $x_0(t)$ is very similar to that for $x(t)$. The prefactor that is different obeys the small $u$ behavior

$$\frac{1 - \psi(u^{-1})}{u^{-1}} = \langle \tau \rangle + o(t),$$  \hspace{1cm} (82)

which is identical with that in equation (48). Thus the long-time limits for the bulk of the PDFs of $x(t)$ and $x_0(t)$ coincide.

We obtain characteristic function of

$$Y_N = \frac{\sum_{i=1}^{N} v_i \tau_i}{N^{-\alpha}}$$  \hspace{1cm} (83)

in the $N \to \infty$ limit and then demonstrate that it is equivalent to our previous results. We have

$$P_N(k) = \langle \exp[i k \cdot Y_N] \rangle = \left\{ \exp \left( \frac{i k \cdot \nu \tau}{\lambda^{\alpha/\alpha}} \right) \right\}^N,$$  \hspace{1cm} (84)

where we used independence of the summands in $Y_N$. Performing averaging over $\tau$,
\[
\tilde{P}(k, u) = \exp \left(-A \cos \left(\frac{\pi \alpha}{2} \left| |k| \cdot v \right|\right) \right),
\]
where we used equation (52). This result reproduces equation (44) for distribution of \(x_0(t)\) using that with probability one, \(x_0(t) = \sum_{i=1}^{N(t)} x_i \). We have from the Montroll–Weiss equation (15) that the Fourier-Laplace transform \(\tilde{\psi}_P(u, t)\) of \(\tilde{P}(k, u, t)\) obeys
\[
\tilde{\psi}_P(u, t) = \tilde{\psi}_P(u, t) - \tilde{\psi}_P(u, t) + \frac{1}{1 - \tilde{\psi}_P(u, t)} - \tilde{\psi}_P(u, t) - 1.
\]

We use
\[
\tilde{P}(w, t) = \int \frac{dk'}{(2\pi)^d} \frac{du'}{2\pi i} \exp [i |k'| \cdot w + u' t] \tilde{P}(k', u')
\]
\[
= t^{-d-1} \int \frac{dk}{(2\pi)^d} \frac{du}{2\pi i} \exp [i k \cdot w] \tilde{P} \left( \frac{k}{t}, \frac{u}{t} \right)
\]
where \(i \) stands for ‘infinite’. We have

\[
\tilde{P}(k, u) = \lim_{t \to \infty} t^{d-2} \tilde{P} \left( \frac{k}{t}, \frac{u}{t} \right)
\]
\[
\lim_{t \to \infty} \frac{P(w, t)}{t^{1-d-\alpha}} = \int \frac{dk}{(2\pi)^d} \frac{du}{2\pi i} \exp[i k \cdot w + u] P(k, u).
\]

We use the large \(t\) asymptotic forms \((k \cdot v) = 0\) for \(\psi\) which is the Laplace transform of \(\psi(\tau)\),

\[
\left\{ 1 - \frac{\psi(ut^{-1} - i k \cdot v t^{-1})}{ut^{-1} - i k \cdot v t^{-1}} \right\} \sim \left( \frac{u}{t} - \frac{i k \cdot v}{t} \right)^{\alpha - 1},
\]

and

\[
\left[ 1 - \left( \frac{u}{t} - \frac{i k \cdot v}{t} \right) \right]^{-1} \sim \frac{t}{\langle \tau \rangle u},
\]

\[
+ \left( \frac{u}{t} \right)^{\alpha - 2} \frac{A}{\langle \tau \rangle^2} \left( 1 - \frac{i k \cdot v}{u} \right)^\alpha,
\]

holding when the rest of the arguments are held fixed. We have neglected higher order terms.

The use of these identities in equation (89) gives,

\[
\frac{u^{2-\alpha} P(k, u)}{A(\tau)} = \left( 1 - \frac{i k \cdot v}{u} \right)^\alpha - \left( 1 - \frac{i k \cdot v}{u} \right)^{\alpha - 1}.
\]

We find, using equation (92) and \(P(x, t) = P(x, t)\) for \(x = 0\),

\[
\lim_{t \to \infty} \frac{P(w, t)}{t^{1-d-\alpha}} = \frac{A}{\langle \tau \rangle} \int \frac{dk}{(2\pi)^d} \frac{du}{2\pi i} \exp[i k \cdot w + u]
\left[ \left( 1 - \frac{i k \cdot v}{u} \right)^\alpha - \left( 1 - \frac{i k \cdot v}{u} \right)^{\alpha - 1} \right], \quad w \neq 0.
\]

It can readily be seen that in one dimension equation (95) reproduces the infinite density obtained in [17]. We can use the method of calculation proposed in that work for finding the integral in equation (95). We use series

\[
(1 - x)^\alpha - (1 - x)^{\alpha - 1} = \sum_{n=1}^{\infty} \frac{(-\alpha)_n x^n}{\alpha(n - 1)!},
\]

where \((a)_n = \Gamma(a + n)/\Gamma(a) = a(a + 1) \ldots (a + n - 1)\) is the Pochhammer symbol. We find

\[
\lim_{t \to \infty} \frac{P(w, t)}{t^{1-d-\alpha}} = \frac{A}{\langle \tau \rangle} \int \frac{dk}{(2\pi)^d} \frac{du}{2\pi i} \exp[i k \cdot w + u]
\frac{(-\alpha)_2 \left( (i k \cdot v)^2 \right)}{\alpha(2n - 1)!} = \frac{A}{\langle \tau \rangle} \int \frac{dk}{(2\pi)^d} \left( \frac{i k \cdot w}{\Gamma(1 - \alpha)} \right),
\]

where we have defined

\[
G_\alpha(y) = \sum_{n=1}^{\infty} \frac{(-1)^n y^{2n}}{(2n - 1)! (2n - \alpha)(2n + 1 - \alpha)},
\]

using \((-\alpha)_2 \Gamma(2n + 2 - \alpha) = (2n - \alpha)(2n + 1 - \alpha)/\Gamma(-\alpha)\). The function \(G_\alpha(y)\) was introduced in [17], where it was demonstrated that
\[ G(y) = \alpha B_\alpha(y) - (\alpha - 1)B_{\alpha-1}(y), \tag{99} \]

\[ B_\alpha(y) = \int_0^1 \frac{\cos(\omega y) - 1}{\omega^{\frac{1}{\alpha} + 1}} d\omega. \tag{100} \]

Thus to find the inverse Fourier transform (97) we consider,

\[ \tilde{B}_\alpha (w, v) = \int \frac{dk}{(2\pi)^d} \exp [ik \cdot w] \tilde{B}_\alpha (k \cdot v) \]

\[ = \int \frac{dk}{(2\pi)^d} \exp [ik \cdot w] \int_0^1 \frac{\cos(\omega k \cdot v) - 1}{\omega^{\frac{1}{\alpha} + 1}} d\omega. \tag{101} \]

We observe that

\[ \int \frac{dk}{(2\pi)^d} \exp [ik \cdot w] \cos(\omega k \cdot v) = \frac{\delta(w + \omega v) + \delta(w - \omega v)}{2}. \]

Thus we find

\[ \tilde{B}_\alpha (w, v) = \int_0^1 \frac{\delta(\omega + \omega v) + \delta(\omega - \omega v) - 2\delta(x)}{2\omega^{\frac{1}{\alpha} + 1}} d\omega, \tag{102} \]

where \( v = v \hat{v} \). We find using the angular \( \delta \)-function \( \delta_\alpha (\hat{u}) \) obeying \( \delta(w - w') = w^{1-\alpha} \delta(w - w') \delta_\alpha (\hat{u} - \hat{v'}) \) that, for \( w \neq 0 \),

\[ \tilde{B}_\alpha (w, v) = v^{\alpha} \int_0^1 \delta(w - \omega') \frac{\delta_\alpha (\hat{w} + \hat{v}) + \delta_\alpha (\hat{w} - \hat{v})}{2\omega^{\frac{1}{\alpha} + 1}w^{d-1}} d\omega', \tag{103} \]

where \( \omega' = \omega v \). This gives that

\[ \tilde{B}_\alpha (w, v) = \frac{\delta_\alpha (\hat{w} + \hat{v}) + \delta_\alpha (\hat{w} - \hat{v})}{2v^{\alpha}w^{d-1}}, \quad |w| < |v|, \tag{104} \]

\[ \tilde{B}_\alpha (w, v) = 0, \quad |w| > |v|. \tag{105} \]

We find, using equations (97) and (99), that (we switch \( w \) with dimensions of velocity by \( v \) in the final formula) for \( v \neq 0 \),

\[ I(v) = \lim_{t \to \infty} \frac{P(v, t)}{t^{d-\alpha}} = \frac{A}{v^{d-1} \Gamma(1 - \alpha)(\tau)} \]

\[ \times \int_{v' > v} F(v' \hat{v}) v'^{d-1} dv' \left[ \alpha - (\alpha - 1) v^{\alpha - 1} \right], \tag{106} \]

where \( I(v) \) is the infinite density. We separated the dependence in the velocity’s PDF \( F(v) \) on the magnitude \( v \) and direction \( \hat{v} \) and used \( F(v) = F(-v) \). This is the chief result of our work, which provides the infinite density as the statement on the existence of long-time scaling limit of the PDF similar to the central limit theorem or the limit introduced in the previous section. The point \( v = 0 \) describes the region of not too large \( |x| \) where Lévy distribution describes \( P(x, t) \) well. This region that determines the normalization shrinks in the considered large times’ scaling limit to the point \( x = 0 \).

Infinite density inherits anisotropy of \( F(v) \): all angular harmonics present in the expansion of \( F(v) \) in spherical harmonics will be present in the expansion of \( I(v) \), see equation (106) (disregarding degenerate cases when the integral that provides the corresponding coefficient
vanishes). Thus in contrast with the case of isotropic statistics described by equation (42) the angular structure is non-isotropic when the velocity statistics is not. Furthermore, though anisotropy of the statistics of the single step of the walk influences the distribution in the bulk, described by the anisotropic Lévy distribution, for the tail this influence is more immediate.

The infinite density limit (106) implies that at large times,

$$P(x, t) \sim \frac{1}{t^{d+\alpha-1}} I_{\tau}(\frac{x}{t})$$  \quad x \neq 0.  \quad (107)$$

This is complementary to the other limit implied by equation (57),

$$P_L(x, t) \sim \frac{1}{(K(t))^{d+\alpha}} \tilde{I}_{\tau}(\frac{x}{(K(t))^{\frac{1}{\alpha}}}).  \quad (108)$$

In fact, both limits characterize different asymptotic regions of the PDF. It is clear from the structure of the scaling limits that Lévy distribution describes the bulk of the PDF, whose scale grows proportionally to $t^{1/\alpha}$, and infinite density describes the tail with $|x| \propto t$. In the next section we demonstrate that the integer order moments are determined by the infinite density tail of the PDF.

8. Integer order moments and dispersion at large times

In this section we calculate the long-time limit of the moments of integer order from the Montroll–Weiss equation. The basic result that we derive is that

$$\langle x(t)x(t) \rangle = \frac{2At^{3-\alpha}}{|\Gamma(1-\alpha)(2-\alpha)(3-\alpha)(\tau)\rangle(\nu k)_i}.  \quad (109)$$

The property $F(v) = F(-v)$ implies that $(\nu k)_i = 0$ for $i \neq k$ so we can write

$$\langle x(t)x(t) \rangle = \frac{2At^{3-\alpha}}{|\Gamma(1-\alpha)(2-\alpha)(3-\alpha)(\tau)\rangle(\nu^2)\delta_{ik}}.  \quad (110)$$

The trace of this equation gives

$$\langle x^2(t) \rangle = \frac{2A(\nu^2)}{|\Gamma(1-\alpha)(2-\alpha)(3-\alpha)(\tau)\rangle}t^{3-\alpha}.  \quad (111)$$

This is a universal formula that holds in arbitrary dimension for arbitrary (possibly strongly anisotropic) statistics of velocity. In the case of isotropic statistics this reduces to the previously derived equation (40). In one dimension this reproduces the formula of [17].

We observe that directly repeating the steps of one-dimensional calculation in [17] with $k \cdot v$ instead of $kv$ we find the asymptotic expansion,

$$P(k, t) \sim 1 + \frac{A}{(\tau)} \sum_{n=1}^{\infty} \frac{\Gamma(2n-\alpha)(-1)^n t^{2n+1-\alpha} \langle (k \cdot v)^{2n} \rangle}{(2n-1)!|\Gamma(1-\alpha)|\Gamma(2n+2-\alpha)},$$

which is the direct continuation of equation (39) for arbitrary statistics of velocity. For the XYZ... model we have $\langle (k \cdot v)^{2n} \rangle = \nu_0^{2n} [\sum_{i=1}^{d} k_i^{2n}] / d$ so that $P(k, t)$ has the structure of $P(k, t) = \sum_{i=1}^{d} P_i(k_i, t)$ where $P_i(k_i, t)$ is the corresponding one-dimensional distribution. Thus the inverse Fourier transform is a sum of products on one-dimensional distribution $P_i(x_i, t)$ times $\delta$-functions of the remaining coordinates. For instance, in three dimensions we have
\[ P(x, t) \sim P_1(x, t)\delta(y)\delta(z) + \delta(x)P_1(y, t)\delta(z) + \delta(z)\delta(y)P_1(z, t), \]  
(112)

where \( P_1(x, t) \) is the one-dimensional distribution studied in [17]. Correspondingly, the temporal growth of the moments is as in one dimension.

Comparing the asymptotic form of \( P(k, t) \) with the characteristic function, see equation (86),

\[ P(k, t) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n \langle (k \cdot x)^{2n} \rangle}{(2n)!} , \]
(113)

we read off the moments. We find

\[ \left( \prod_{i=1}^{d} x_i^{\alpha}(t) \right) \sim \frac{2nA \left( \prod_{i=1}^{d} x_i^{\alpha} \right) t^{2n+1-\alpha}}{\Gamma(1-\alpha)(2n-\alpha)(2n+1-\alpha)(\tau)} , \]
(114)

where \( \sum_{i=1}^{d} t_i = 2n \). The \( n = 1 \) term gives equation (110). On the level of dispersion there is no qualitative difference between isotropic and anisotropic statistics of velocity. The formula (111) for \( (x^2(t)) \) is dimension-independent. This is also true for \( 2n \)-th moment of the magnitude of the distance from the walk’s origin \( (x^{2n}(t)) \),

\[ \langle x^{2n}(t) \rangle \sim \frac{2nA (v^2)^{2n+1-\alpha}}{\Gamma(1-\alpha)(2n-\alpha)(2n+1-\alpha)(\tau)} , \]

as can be seen by opening the brackets in \( (x^{2n}(t)) = \left( \left( \sum_{i=1}^{d} x_i^{2}(t) \right)^n \right) \) and using equation (114).

This reproduces equation (40) in the isotropic case and reproduces the result of [17, 18] for \( (x^{2n}(t)) \) in one dimension. Similarly the growth of one of the components is universal:

\[ \langle x_i^{2n}(t) \rangle \sim \frac{2nA (v_i^{2n}t^{2n+1-\alpha})}{\Gamma(1-\alpha)(2n-\alpha)(2n+1-\alpha)(\tau)} . \]
(115)

It is readily confirmed using equation (114) and the construction of the infinite density in the previous section that the identity

\[ \left( \prod_{i=1}^{d} x_i^{\alpha}(t) \right) \sim \frac{1}{t^{d+\alpha-1}} \int \prod_{i=1}^{d} x_i^{\alpha} \left( \frac{x}{\tau} \right) dx , \]

holds. The calculation can be done by direct transfer of the calculation in one dimension [17] using \( k \cdot v \) instead of \( k v \) in the derivations. Thus integer order moments are determined by the infinite density. We saw previously that the moments of order higher than \( \alpha \) are described by the tail of the PDF. This confirms that the tail of the PDF is indeed described by the infinite density (unless the tail cannot be reconstructed from moments of integer order, which is not the case here).

Previous results on the moments coincided with those in one dimension [17]. The difference from the one-dimensional case holds when cross correlations of different components of \( \mathbf{x}(t) \) are considered. We consider the difference in the case of fourth-order moments. We have

\[ \langle x_i^2(t)x_j^2(t) \rangle \sim \frac{4A (v_i^2v_j^2)}{\Gamma(1-\alpha)(4-\alpha)(5-\alpha)(\tau)} t^{5-\alpha} . \]
(116)

In the XYZ \ldots model \( (v_i^2v_j^2) = 0 \) for \( i \neq k \) so that this formula gives zero. This does not indicate that the positive quantity \( x_i^2(t)x_j^2(t) \) has zero average but rather that higher order corrections
for the asymptotic calculation at large times are needed. This can be done by direct study of the Montroll–Weiss equation.

9. Tail of the PDF of anisotropic LW and fractional order moments

We observe from equation (106) that angular structure of the infinite density, and thus of the PDF’s tail, directly reflects the angular structure of the velocity PDF. This is a consequence of the tail being formed by rare long ballistic flights. We find for the angular average

\[
I_0(\nu) = \int I(\nu \hat{v}) d\hat{v} = \frac{A}{v^{d-1}|\Gamma(1-\alpha)|} \int_{v' > v} F(v') dv' \times \left[ \frac{\nu'^\alpha}{v^{1+\alpha}} - (\alpha - 1) \frac{\nu'^\alpha - 1}{v^\alpha} \right].
\]

(117)

We conclude from equation (107) that the probability density function of the distance \(x\) from the walk’s origin obeys

\[
P(x, t) = x^{d-1} \int P(x, t, \hat{x}) d\hat{x} \sim x^{d-1} \frac{1}{r^{d+\alpha-1}} I_0 \left( \frac{x}{t} \right) = \frac{A}{|\Gamma(1-\alpha)|} \int_{v' > x/t} F(v') dv' \left[ \frac{\alpha \nu'^\alpha}{x^{1+\alpha}} - (\alpha - 1) \frac{\nu'^\alpha - 1}{x^\alpha} \right].
\]

(118)

This has universal asymptotic form at small \(x\),

\[
P(x, t) \sim \frac{\alpha A(\nu^\alpha)}{|\Gamma(1-\alpha)|} \langle \tau \rangle x^{d+\alpha}, \quad \frac{\nu^\alpha}{x} \ll 1,
\]

(119)

where \(\nu^\alpha\) is the characteristic value of the velocity. This is independent of dimension and details of statistics of velocity. This coincides with the tail of the Lévy distribution in one and three-dimensional cases, see equation (77). This was observed in one dimension in [17]. However in other dimensions there is a multiplicative factor difference between the two asymptotic forms. For \(I_0(x)\) we have

\[
I_0(x) \sim \frac{\alpha A(\nu^\alpha)}{|\Gamma(1-\alpha)|} \langle \tau \rangle x^{d+\alpha}.
\]

(120)

It can be demonstrated that the small \(x\) form of the infinite density and the large \(x\) form of the Lévy distribution have to be of the same order. We observe that \(P_L(x, t)\) in equation (108) obeys at large \(t\) and \(v \neq 0\)

\[
\frac{P_L (\nu v, t)}{t^{1-d-\alpha}} \propto \frac{1}{t^{1-d-\alpha + d\alpha}} \hat{L}_d \left( t^{1-1/\alpha} \frac{v}{(K_n)^{1/\alpha}} \right) \propto \text{const},
\]

where we use that at large arguments \(I_0(x) \propto x^{-d-\alpha}\). Thus \(P_L(x, t)\) would give finite constant contribution in the scaling limit described by the infinite density. This contribution comes from the tail of the Lévy distribution. Then the consideration performed in the one-dimensional case in [17] holds in higher-dimensional cases, demonstrating that the small \(x\) form of the infinite density and the large \(x\) form of the Lévy distribution have to be of the same order.

Based on the infinite density tail we can readily derive the moments of order higher than \(\alpha\). We have
\[
\langle |x(t)|^q \rangle \sim \int \frac{x^q}{t^{d+\alpha-1}} \left( \frac{x}{t} \right) \, dx = \tilde{c}_q t^{\alpha+1-q},
\]

\[
\tilde{c}_q = \int |x|^{\alpha+1-d} I_0(x) \, dx.
\] (121)

This formula holds for \( q > \alpha \) where the formula’s breakdown at smaller \( q \) is signalled by the small \( x \) divergence of the integral in \( \tilde{c}_q \) see equation (120).

The scaling exponents of the moments depend on the moment’s order linearly for both \( q \leq \alpha \) and \( q > \alpha \), albeit with different linear dependencies [48]. This bilinear behavior of the moments of the distance from the origin was observed in the one-dimensional case in [17] and continues to hold in the higher-dimensional case.

We find the infinite density in the uniform model with fixed velocity \( v_0 \). In this case the PDF of \( |v| \) is \( \delta (v - v_0) \) so that

\[
I(x) = \frac{A v_0^{\alpha-1}(\alpha - 1)}{|\Gamma(1 - \alpha)| \pi^{\alpha}} \left[ \frac{\alpha v_0}{x(\alpha - 1)} - 1 \right], \quad x < v_0.
\] (122)

If \( x > v_0 \) then \( I(x) = 0 \). Since \( x(t)/t \propto t^{1-\alpha/\alpha} \) is a decaying function of time then at large times the infinite density describes a non-trivial region of the probability density function. We find that if \( x(t) < x < v_0 t \) then

\[
P_d(x, t) \sim \frac{A \Gamma(d/2)v_0^{\alpha-1}(\alpha - 1)}{2 \pi^{d/2}x^{d+\alpha-1} |\Gamma(1 - \alpha)| \tau} \left[ \frac{\alpha v_0 t}{x(\alpha - 1)} - 1 \right].
\] (123)

The probability density function at \( x > v_0 t \) vanishes since a particle moving at constant speed \( v_0 \) cannot pass distances longer than \( v_0 t \) in time \( t \).

10. Fractional diffusion equation and confined motion

Our results show that the fractional space diffusion equation in dimension larger than unity should be used with care. Using the Lévy walk approach it becomes clear that there is no unique Laplacian for the fractional diffusion equation, which is the result of our discovery that the bulk density is described by the \( d \)-dimensional Lévy law, which in turn depends on the distribution of velocity. Thus in general one cannot reduce the problem to a simple fractional diffusion equation with a single diffusion constant, or a finite set of \( d \) constants. In other words general transport coefficients like diffusivity do not exist. Each velocity distribution will yield a different fractional Laplacian operator namely the fractional diffusion equation in \( k \) space reads

\[
\frac{\partial}{\partial t} P(k, t) = -\tau^{\alpha-1} \langle |k \cdot v|^{\alpha} \rangle P(k, t),
\] (124)

which is easily obtained from equation (55) and \( \tau^{\alpha-1} \equiv A |\cos(\pi \alpha/2)| \langle \tau \rangle \). The Weyl–Reitz [13] operator gives \(-|k|^{\alpha} P(k, t)\) on the right hand side of equation (124), which corresponds to the isotropic case, while a non-isotropic case was mathematically treated in [49]. Hence when applying fractional space equations to real physical systems a case-by-case analysis is needed, while the widely explored one dimensional counterpart is unique (the velocity distribution enters through a single moment \( \langle |v|^{\alpha} \rangle \) as is the normal Gaussian case \( \alpha = 2 \) in dimension \( d \)). It is left for future mathematical work to explore how to define boundary conditions for the fractional equation (124).
The anisotropy of the $d$ dimensional Lévy walk is not limited to unbounded free diffusion. Lévy walks in confined systems, and in the presence of binding potential fields exhibit signatures of non-universality in the stationary state and intricate correlations and structure of the phase space [50–53]. This was investigated for Langevin equations with Lévy noise [50, 51] where the increase of dimension beyond one yields new non-trivial behaviors. Stationary states in phase space, in the presence of a confining field, were recently investigated based on a microscopical theory of Sisyphus laser cooling [52, 53], a system described by Lévy statistics [24].

11. Conclusions

We have obtained the PDF $P(x, t)$ for $d$-dimensional Lévy walks. We derived two complimentary limiting distributions describing the PDF at long times.

One of these scalings has origin similar to the central limit theorem, providing the $d$-dimensional counterpart of one-dimensional Lévy distribution that describes the bulk of the PDF. There is, however, significant difference from the one-dimensional case. In one dimension different statistics of velocity result in the same universal (up to rescaling of the density) shape of the bulk of the PDF $P(x, t)$. This is no longer true in higher dimensions: different velocity statistics result in different shapes of the PDF bulks. Only in the case of isotropic statistics does a certain universality hold. This is in sharp contrast with the normal diffusion, where the $d$-dimensional Gaussian profiles are universal attractors.

We have demonstrated that despite the non-universality of the PDF in the bulk region, the tail of PDF of the distance follows the universal power-law with exponent $\alpha - 1$. The prefactor of the law depends only on angle-averaged statistics of velocity via $\langle |v|^\alpha \rangle$ (in contrast, the bulk of the PDF depends on probabilities of moving in different directions). This provides a direct generalization of the one-dimensional result. Thus dependence on anisotropy of velocity statistics and non-universality of the shape of the PDF in the bulk disappear at large distances, where universality is restored. The higher order corrections are anisotropic and non-universal though.

This universality of the tail’s exponent guarantees that, independently of details of anisotropy of the statistics, the moments of order smaller than $\alpha$, where $1 < \alpha < 2$, are determined by the PDF’s bulk. However, higher order moments, including dispersion, are due to the PDF’s tail.

The existence of the complimentary scaling, unfamiliar in the field until very recently, is the consequence of the scaling of the tail of the PDF $\psi(\tau)$ of the single step duration $\tau$. This limit describes the tail of $P(x, t)$ formed by ballistic motions with very large duration $\tau$. It describes $x$ that scale proportionally with $t$ (ballistic scaling). In contrast the displacements described by the bulk of the PDF are formed by accumulation of lots of typical diffusive increments. Thus the bulk describes $x$ that have (anomalous) diffusive scaling proportional to $t^{1/\alpha}$ (we remark that $t^{1/\alpha} \ll t$ at large $t$ because of $\alpha > 1$).

The function that describes the PDF’s tail is called the ‘infinite density’ because it is not normalizable. This is caused by divergence of normalization at small arguments. It is important to understand that ‘infinite density’ is not a probability density function (which must be normalized). The infinite density is a point-like limit of the rescaled PDF and does not have to be normalizable. This is because for different spatial arguments the pointwise limit holds at different times. No matter how large time is, the infinite density never converges to the PDF uniformly in space. For any large but finite time the bulk of the PDF that determines the normalization holds around the origin $x = 0$, and can be described by the corresponding
anisotropic Lévy distribution. The complete picture of the PDF is simpler in three and one-dimensional cases. There the small argument form of the infinite density tail continuously transforms in the universal tail of the PDF’s bulk. This makes it plausible that an intermediate region between the bulk and the tail is described by these asymptotic forms. Both the bulk anisotropic Lévy distribution and its tail shrink to zero when rescaled with ballistic scaling $t$ of the infinite density scaling limit. In the two-dimensional case the intermediate region between the bulk and the tail demands separate study.

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Appendix A. Montroll–Weiss equation from sum over trajectories

In this section we derive the Montroll–Weiss (MW) equation by using the Laplace transform of the characteristic function of the coordinate of random walker. Though this equation is not solvable in the general case, it does help in evaluating the long-time asymptotic form of the PDF $P(x,t)$. The MW-equation is well-known in one-dimensional setups, see e.g. [20] and references therein. Here we provide alternative derivation in terms of sums over all possible trajectories. This can help in the study of multi-time statistics of the walk, fluctuations, and other quantities.

Trajectories during the period $[0,t]$ can be characterized by the integer number $N \geq 0$ of time renewals that occurred during this time interval. Since events with different $N$ form a complete set of non-overlapping events, we can write averages as sums of contributions of events with different $N$. This is realized by using the identity

$$1 = \sum_{N=1}^{\infty} \int_0^t \int_{-t}^t \int_{-t}^t d\tau d\tau' d\tau'' \delta \left( \sum_{i=1}^N \tau_i + t'' - t'' - \tau - \tau' \right)$$

which holds for arbitrary infinite sequences of positive numbers $\tau_i$ and $t$. Only one term on the RHS is non-zero and is one. This is the term of $N$ renewals for which $\tau_i > t_i$ or the last term of no renewals if $\tau_i > t$. The identity can be used to average an arbitrary function of $\tau_i$ as the sum of contributions of events with different $N$. The simplest is the probability $P_N(t)$ of having $N$ renewals before the time $t$,

$$P_N(t) = \left( \int_0^t d\tau \int_{-t}^t d\tau' \int_{-t}^t d\tau'' \delta \left( \sum_{i=1}^N \tau_i - t'' - \tau - \tau' \right) \delta \left( \sum_{i=1}^N \tau_i - t'' - \tau - \tau' \right) \right)$$

$$= \int_0^t p_N(t')d\tau' \int_{-t}^t \psi(t')d\tau'$$

$$= \int_{-t}^t \psi(t')d\tau'$$

where $p_N(t)$ is the PDF of $\sum_{i=1}^N \tau_i$. The Laplace transform of $p_N(t)$ obeys
\[ p_N(u) = \int_0^\infty \exp[-u t] p_N(t) dt = \left( \exp \left( -u \sum_{i=1}^N \tau_i \right) \right) = \psi^N(u), \quad (A.3) \]

where \( \psi(u) \) is the Laplace transform of \( \psi(\tau) \). The Laplace transform of convolution in equation \( (A.2) \) gives

\[ P_N(u) = \frac{\psi^N(u)[1 - \psi(u)]}{u}, \quad (A.4) \]

where we used \( \int_0^\infty \psi(\tau) d\tau = 1 \). This formula includes the \( N = 0 \) case, implying that the Laplace transform of the generating function

\[ p(s, u) = \frac{[1 - \psi(u)]}{u[1 - s\psi(u)]}, \quad (A.5) \]

where we use that \( \psi(s) \sim 1 \), with equality only at \( u = 0 \). The normalization condition \( \sum_{N=0}^\infty P_N(u) = 1 \) implies \( \sum_{N=0}^\infty P_N(u) = 1/u \) is obeyed. The Laplace transform of the average number of renewals \( \langle N(t) \rangle = \sum NP_N(t) \) is

\[ \langle N(u) \rangle = \sum_{N=1}^\infty NP_N(u) = \sum_{N=1}^\infty \frac{\psi^N(u)}{u[1 - \psi(u)]} = \frac{\psi(u)}{u[1 - \psi(u)]}, \quad (A.6) \]

which can also be obtained from \( \langle N(u) \rangle = \nabla p(s = 1, u) \). We compare the behavior of \( \langle N(t) \rangle \) in cases with finite and infinite dispersion of \( \tau \). If dispersion is finite then the small \( u \) behavior of \( \psi(u) \) is described by \( \psi(u) \sim 1 - \langle \tau \rangle u + \langle \tau^2 \rangle u^2/2 \). This gives

\[ \langle N(u) \rangle \sim \frac{1}{\langle \tau \rangle u^2} + \frac{\sigma^2 - \langle \tau \rangle^2}{2\langle \tau \rangle^2} + o(u), \quad (A.7) \]

where \( \sigma^2 = \langle \tau^2 \rangle - \langle \tau \rangle^2 \). We find the long-time behavior,

\[ \langle N(t) \rangle \sim \frac{t}{\langle \tau \rangle} + \frac{\sigma^2 - \langle \tau \rangle^2}{2\langle \tau \rangle^2}, \quad (A.8) \]

see e.g. [46]. Thus the leading order correction to the law of large numbers is constant. In the case of divergent \( \langle \tau^2 \rangle \) the correction grows with time. Using the small \( u \) behavior \( \psi(u) \sim 1 - \langle \tau \rangle u + Au^\alpha \) we have

\[ \langle N(u) \rangle = \frac{u^\alpha}{\langle \tau \rangle^2 u^2[(\tau) - Au^\alpha]} - \frac{1}{\langle \tau \rangle u^2} = \frac{Au^\alpha - 3}{\langle \tau \rangle^2}. \quad (A.9) \]

This yields the long-time asymptotic behavior,

\[ \langle N(t) \rangle \sim \frac{t}{\langle \tau \rangle} + \frac{A t^{2-\alpha}}{\langle \tau \rangle^3 \Gamma(3 - \alpha)}, \quad (A.10) \]

This indicates that convergence of \( \langle N(t) \rangle \) to its long-time probability one limit \( 1/\langle \tau \rangle \) is slower than in the case of finite \( \langle \tau^2 \rangle \) because the average of \( N(t)/t - 1/\langle \tau \rangle \) decays as \( t^{1-\alpha} \) which is slower than \( 1/t \) law of finite dispersion.

Similarly for \( \langle N^2(u) \rangle \) we find,

\[ \langle N^2(u) \rangle - \langle N(u) \rangle = \nabla^2 p(s = 1, u) = \frac{2\psi^2(u)}{u[1 - \psi(u)]^2}, \quad (A.11) \]
In the case of finite $\langle \tau^2 \rangle$ this has small $u$ behavior,

$$\langle N^2(u) \rangle - \langle N(u) \rangle \sim \frac{2(1 - 2\langle \tau \rangle u)}{\langle \tau^3 \rangle u^2 [1 - \langle \tau^2 \rangle u \langle \tau \rangle]}$$

$$\sim \frac{2}{\langle \tau^3 \rangle u^2} - \frac{4}{u^2 \langle \tau \rangle} + \frac{2\langle \tau^2 \rangle}{u^2 \langle \tau \rangle^3}. \quad (A.12)$$

This gives the long-time behavior,

$$\langle N^2(t) \rangle - \langle N(t) \rangle \sim \frac{t^2}{\langle \tau \rangle^2} - \frac{4t}{\langle \tau \rangle} + \frac{2t\langle \tau^2 \rangle}{\langle \tau \rangle^3}. \quad (A.13)$$

Using the previous result for $\langle N(t) \rangle$ we find for variance,

$$\langle N^2(t) \rangle - \langle N(t) \rangle^2 \sim \frac{\sigma^2 t}{\langle \tau \rangle^3}. \quad (A.14)$$

Thus in the case of finite dispersion of $\tau$ variance of $N(t)$ equals the mean up to multiplicative constant.

In contrast, in the case of infinite $\langle \tau^2 \rangle$ strong violation of Poissonicity holds. Using $\psi(u) \sim 1 - \langle \tau \rangle u + Au^\alpha$ we have

$$\langle N^2(u) \rangle - \langle N(u) \rangle \sim \frac{2(1 - 2\langle \tau \rangle u)}{\langle \tau^3 \rangle u^2 [1 - 2Au^\alpha - I\langle \tau \rangle]}$$

$$\sim \frac{2}{\langle \tau^3 \rangle u^2} + \frac{4Au^{\alpha - 4}}{\langle \tau \rangle^3}, \quad (A.15)$$

which gives the long-time behavior,

$$\langle N^2(t) \rangle - \langle N(t) \rangle \sim \frac{t^2}{\langle \tau \rangle^2} + \frac{4At^{3 - \alpha}}{\langle \tau \rangle^3 \Gamma(4 - \alpha)}. \quad (A.16)$$

We find, for variance,

$$\langle N^2(t) \rangle - \langle N(t) \rangle^2 \sim \frac{2(\alpha - 1)At^{3 - \alpha}}{\langle \tau \rangle^3 \Gamma(4 - \alpha)}. \quad (A.17)$$

where $\langle N(t) \rangle$ is a small correction to the RHS. We have super-Poissonian behavior,

$$\frac{\langle N^2(t) \rangle - \langle N(t) \rangle^2}{\langle N(t) \rangle} \sim \frac{2(\alpha - 1)At^{3 - \alpha}}{\langle \tau \rangle^3 \Gamma(4 - \alpha)}. \quad (A.18)$$

The PDF that is of interest for us here is that of the particle’s displacement $x(t)$ in time $t$. If $N$ renewals occurred in time $t$ then the displacement $x_N(t)$ obeys,

$$x_N(t) = \sum_{j=1}^N v_j t + v_{N+1} \left( t - \sum_{j=1}^N t \right), \quad x_0 = v_1 t. \quad (A.19)$$

Inserting 1 in $P(x, t) = \langle 1 \times \delta(x(t) - x) \rangle$ in the form given by equation (A.1) we find for the PDF of the particle’s position that
\[
\begin{align*}
P(x, t) = & \langle \delta(x(t) - x) \rangle = \int_0^\infty \langle \delta(v_{j}t - x)\delta(\tau_{j} - t') \rangle dt' \\
& + \sum_{N=1}^{\infty} \left\{ \int_0^t dt' \int_0^\infty dt'' \delta(\sum_{i=1}^{N} \tau_{i} - t') \delta(\tau_{N+1} - t'') \right\} \\
& \delta \left( \sum_{i=1}^{N} v_{i}\tau_{i} + v_{N+1}(t - t') - x \right).
\end{align*}
\]

The characteristic function \( P(k, t) = \langle \exp[ik \cdot x(t)] \rangle \) obeys
\[
P(k, t) = \int_0^\infty \langle \exp[i(k \cdot v_{j}t)]\delta(\tau_{j} - t') \rangle dt' + \sum_{N=1}^{\infty} \left\{ \int_0^t dt' \int_0^\infty dt'' \delta(\sum_{i=1}^{N} \tau_{i} - t') \psi(t'') \right\} \\
\exp \left[ \sum_{i=1}^{N} ik \cdot v_{i}\tau_{i} + ik \cdot v_{N+1}(t - t') \right].
\]

The Laplace transform over \( t \) gives the Montroll–Weiss equation in \( d \) dimensions,
\[
P(k, u) = \left\{ \frac{1 - \psi(u - ik \cdot v)}{u - ik \cdot v} \right\} + \sum_{N=1}^{\infty} \left\{ \frac{1 - \psi(u - ik \cdot v)}{u - ik \cdot v} \right\} \langle \psi(u - ik \cdot v) \rangle^N
\]
\[
= \left\{ \frac{1 - \psi(u - ik \cdot v)}{u - ik \cdot v} \right\} 1 - \langle \psi(u - ik \cdot v) \rangle.
\]

The derivations above hold irrespectively of the form of the PDFs of flight times and velocities.

Statistics of random walks given by equations (A.19) and (A.23) are different though as we demonstrate in the main text are identical in the bulk.

**Appendix B. Small-argument expansion for logarithm of radially symmetric \( L^d \) distribution**

Here we consider the small argument behavior of \( L_d(x) \) defined by equation (33). Since \( L_d(x) \) is positive function with maximum at \( x = 0 \) it can be advantageous having Taylor expansion for \( \ln L_d(x) \) rather than \( L_d(x) \) itself (which is provided by equation (65)). We perform asymptotic study of \( L_d(r) \) in equation (33). The neighbourhood of the maximum of \( L_d(r) \) that holds at \( r = 0 \) can be described writing
\[ \ln L_d(r) = \ln \left[ \langle \exp[ik \cdot r] \rangle_k \int \exp[-k^\alpha] \frac{dk}{(2\pi)^d} \right], \quad (B.1) \]

where we defined

\[ \langle \exp[ik \cdot r] \rangle_k = \frac{\int \exp[ik \cdot r - k^\alpha]dk}{\int \exp[-k^\alpha]dk}, \quad (B.2) \]

which can be considered as an average over the statistics of \( k \) defined by the probability density function \( P(k) \).

\[ \langle \exp[ik \cdot r] \rangle_k = \int \exp[ik \cdot r]P(k)dk, \quad (B.3) \]

\[ P(k) = \frac{\exp[-k^\alpha]}{\int \exp[-k^\alpha]dk} = \frac{\alpha \Gamma(d/2)}{2\pi^{d/2} \Gamma(d/\alpha)} \exp[-k^\alpha]. \quad (B.4) \]

The writing of the integral as average over a statistical distribution is useful because we can use the cumulant expansion theorem for writing \( \langle \exp[ik \cdot r] \rangle_k \) as series in \( r \). We find

\[ L_d(r) = \frac{2^{1-d}\Gamma(d/\alpha)}{\pi^{d/2}\alpha \Gamma(\alpha)} \exp \left[ \sum_{n=1}^{\infty} \frac{(-1)^n \langle (k \cdot r)^n \rangle_{\alpha,k}}{(2n)!} \right], \quad (B.5) \]

where \( c \) stands for cumulant (see below) and odd-order moments vanish because \( P(k) = P(-k) \).

The quadratic cumulant is the dispersion,

\[ \langle (k \cdot r)^2 \rangle_{\alpha,k} = r^2 \langle k^2 \rangle = \frac{r^2 \Gamma[(d + 2)/\alpha]}{d \Gamma(d/\alpha)}. \quad (B.6) \]

The next non-vanishing term in the series is the quartic cumulant,

\[ \langle (k \cdot r)^4 \rangle_{\alpha,k} = \langle (k \cdot r)^4 \rangle - 3 \langle (k \cdot r)^2 \rangle^2 = \frac{3r^4}{d \Gamma(d/\alpha)} \left( \frac{\Gamma[(d + 4)/\alpha]}{d + 2} - \frac{\Gamma[(d + 2)/\alpha]}{d \Gamma(d/\alpha)} \right), \quad (B.7) \]

where we used isotropy,

\[ \langle (k \cdot r)^4 \rangle = r r_p r_p \left[ \delta_{i\alpha} \delta_{p\alpha} + \delta_{i\beta} \delta_{p\beta} + \delta_{i\gamma} \delta_{p\gamma} \right] = \frac{\langle k^4 \rangle}{d(d + 2)} = \frac{3r^4 \langle k^4 \rangle}{d(d + 2)}. \quad (B.8) \]

The series is useful for studying the vicinity of the maximum when \( r \ll 1 \). Higher \( r \) demand higher order cumulants whose form is quite cumbersome.

The cumulant expansion described above corresponds to resummation of Taylor series for \( L_d(r) \) for \( \ln L_d(r) \). Both series are useful when \( r \ll 1 \) is considered.

**Appendix C. Dispersion and fourth order moments at all times**

Here we find formulas for dispersion and fourth order moments of the particle’s position valid at all times. These are obtained using differentiation of the Montroll–Weiss equation over \( k \) and setting \( k = 0 \). This direct procedure brings formulas that hold at arbitrary times in contrast with the asymptotic study in the main text that holds at large times. Thus we find more detailed information on the temporal growth of the moments. This includes corrections to
the long-time behavior described in the main text that can in some cases become dominant because of the vanishing of the leading order term. This direct calculation becomes cumbersome for higher order moments, so we perform calculation of dispersion and fourth-order moments only. Thus we describe the difference in temporal behavior of \(\langle x_i^2(t) x_k^2(t) \rangle\) for \(i \neq k\) in isotropic and XYZ... models.

The displacement’s dispersion is found from

\[
\langle x_i^2 \rangle = \frac{\partial^2}{\partial k_i^2} \left[ \frac{1 - \psi(u - i k \cdot v)}{u - i k \cdot v} \right],
\]

where the RHS is taken at \(k = 0\). We find using that odd moments of \(v_i\) vanish that

\[
\langle x_i^2 \rangle = \frac{\langle v_i^2 \rangle}{1 - \psi(u)} \left( \frac{1 - \psi(u)}{u} \right)^n + \frac{\langle v_i^2 \rangle \psi''(u)}{u[1 - \psi(u)]} = \frac{2\langle v_i^2 \rangle}{u^3} + \frac{2\langle v_i^2 \rangle \psi'(u)}{u^2[1 - \psi(u)]}.
\]

(C.1)

We find summing over \(i\) that,

\[
\langle x^2(t) \rangle = 2\langle v^2 \rangle \int_{-\infty}^{\infty} \frac{\exp[iu\tau]}{2\pi i} \frac{u\psi'(u) - \psi(u) + 1}{u^2[1 - \psi(u)]}.
\]

(C.2)

which can be used for finding detailed temporal dependence of the displacement for given \(\psi(\tau)\). This formula is identical for all statistics of velocity. The leading order term in the limit of large times can be obtained using the small \(u\)-expansion \(\psi(u) \sim 1 - \langle \tau \rangle u + Au^\alpha\). We have

\[
u_i^2(u) - \psi(u) + 1 \sim A(\alpha - 1)u^\alpha,
\]

\[
u_i^2(u) - \psi(u) + 1 \sim A\alpha u^\alpha.
\]

(C.3)

Further,

\[
\frac{\psi'(u)}{\psi(u) - 1} = \frac{-\langle \tau \rangle + A\alpha u^{\alpha - 1}}{-\langle \tau \rangle u + Au^\alpha} = \frac{1}{u[1 - Au^{\alpha - 1}\langle \tau \rangle]}
\]

\[
-\frac{A\alpha u^{\alpha - 2}}{\langle \tau \rangle} \approx \frac{1}{u} - \frac{A(\alpha - 1)u^{\alpha - 2}}{\langle \tau \rangle}.
\]

We conclude that the leading order term at small \(u\) is

\[
\frac{\langle x_i^2(u) \rangle}{\langle v_i^2 \rangle} = \frac{\langle r^2(u) \rangle}{\langle v^2 \rangle} \sim \frac{2A(\alpha - 1)u^{\alpha - 4}}{\langle \tau \rangle}.
\]

Thus we find in the \(t \to \infty\) limit,

\[
\langle x^2(t) \rangle \sim \frac{2A\langle v^2 \rangle t^{3-\alpha}}{|\Gamma(1 - \alpha)(2 - \alpha)(3 - \alpha)(\tau)|}.
\]

This reproduces the result of the main text directly from equation (C.2). A similar conclusion is reached for

\[
\langle x_i^4 \rangle = \frac{\partial^4}{\partial k_i^4} \left[ \frac{1 - \psi(u - i k \cdot v)}{u - i k \cdot v} \right] \frac{1}{1 - \psi(u - i k \cdot v)}.
\]
where the RHS is taken at $k = 0$. We find, using that odd moments of $v_i$ vanish, that

$$
\langle x_i^4 \rangle = \frac{\langle v_i^4 \rangle}{1 - \psi(u)} \left( \frac{1 - \psi(u)}{u} \right)^{(4)} + \frac{6(v_i^2)^2 \psi''(u)}{[1 - \psi(u)]^2} \\
\times \left( \frac{1 - \psi(u)}{u} \right)^\eta + \frac{\langle v_i^4 \rangle \psi''(u)}{u[1 - \psi(u)]} + \frac{6(v_i^2)^2 \psi''(u)}{u[1 - \psi(u)]^2}.
$$

(C.4)

We observe that at small $u$ we have

$$
1 - \psi(u) = \langle \tau \rangle - Au^{\alpha - 1}, \quad 1 - \psi(u) = \langle \tau \rangle u - Au^\alpha,
$$

so that

$$
\langle x_i^4(u) \rangle \sim \frac{4A \langle v_i^4 \rangle (-\alpha) u^{\alpha - 6}}{\langle \tau \rangle^\alpha}.
$$

(C.6)

where the neglected terms involving $\langle v_i^2 \rangle^2$ are proportional to $u^{2\alpha - 7}$ and can be neglected at small $u$ because of $\alpha > 1$. This reproduces the result of the main text,

$$
\langle x_i^4(t) \rangle \sim \frac{4A \langle v_i^4 \rangle}{[\Gamma(1 - \alpha)\Gamma(4 - \alpha)\Gamma(5 - \alpha)]^2} t^{5 - \alpha}.
$$

(C.7)

We consider the cross-correlation $\langle x_i x_k \rangle$. We have

$$
\langle x_i^2(t)x_k^2(t) \rangle = \frac{\partial^4}{\partial k_0^2 \partial k_0^2} \left[ \frac{1 - \psi(u - ik \cdot v)}{u - ik \cdot v} \right] \frac{1}{1 - \langle \psi(u - ik \cdot v) \rangle},
$$

where the RHS is taken at $k = 0$. We find, using that odd moments of $v_i, v_k$ vanish, that

$$
\langle x_i^2(t)x_k^2(t) \rangle = \frac{\langle v_i^2 v_k^2 \rangle}{1 - \psi(u)} \left( \frac{1 - \psi(u)}{u} \right)^{(4)} + \frac{2(v_i^2)(v_k^2) \psi''(u)}{[1 - \psi(u)]^2} \\
\times \left( \frac{1 - \psi(u)}{u} \right)^\eta + \frac{\langle v_i^2 v_k^2 \rangle \psi''(u)}{u[1 - \psi(u)]} + \frac{2(v_i^2)(v_k^2) \psi''(u)}{u[1 - \psi(u)]^2}.
$$

This can be used for detailed study of temporal behavior of $\langle x_i^2(t)x_k^2(t) \rangle$. We use this formula in finding the leading order long-time behavior in XYZ ... model. There we have $\langle v_i^2 v_k^2 \rangle = 0$ for $i \neq k, \langle v_i^2 \rangle = v_i^2/d$ which gives

$$
\langle x_i^2(t)x_k^2(t) \rangle = \frac{2v_i^2 v_k^2 \psi''(u)}{[1 - \psi(u)]^2 d^2} \left[ \left( \frac{1 - \psi(u)}{u} \right)^\eta + \frac{\psi''(u)}{u} \right].
$$

The leading order term when $u$ is small is

$$
\langle x_i^2(t)x_k^2(t) \rangle = \frac{4v_i^2 \psi''(u)}{\langle \tau \rangle^2 d^2}.
$$

(C.8)

Performing inverse Laplace transform we find

$$
\langle x_i^2(t)x_k^2(t) \rangle = \frac{4v_i^2 \psi''(u)}{\Gamma(7 - 2\alpha)\langle \tau \rangle^2 d^2} t^{6 - 2\alpha}.
$$

(C.9)
The probability distribution is characterized by constant time-independent ratio

$$\frac{\langle x_i^2(t) x_j^2(t) \rangle}{\langle x_i^2(t) \rangle \langle x_j^2(t) \rangle} = \frac{4\alpha \Gamma^2(2 - \alpha)}{\Gamma(7 - 2\alpha)},$$

(C.10)

indicating that interdependence of the displacement’s components becomes constant at large times. In contrast using \( \langle v_i^2 \rangle = v_i^2 d \) (this can be found from \( v_i^2 = \left\langle \sum_{k} v_k^2 \right\rangle = d (v_k^2) \) where we use that cross-correlations of velocity vanish) we find that the ratio

$$\frac{\langle x_i^2(t) x_j^2(t) \rangle}{\langle x_i^2(t) \rangle + \langle x_j^2(t) \rangle} = \frac{\alpha(\alpha - 1)(4 - \alpha)(5 - \alpha)\Gamma(2 - \alpha)}{(2d)(\tau)\Gamma(7 - 2\alpha) r_{\alpha - 1}}$$

decreases with time indefinitely, characterizing growing anisotropy of the distribution.

**Appendix D. Infinite density is the generating function of the moments**

We demonstrate that moments of integer order are described by the infinite density. We observe that equations (97) and (98) give for the Fourier transform of \( I(v) \) that

$$I(k) = \frac{A}{|\Gamma(1 - \alpha)| \langle \tau \rangle} \sum_{n=1}^{\infty} \frac{(2n)(-1)^n \langle (k \cdot v)^{2n} \rangle}{(2n)! (2n - \alpha)(2n + 1 - \alpha)}.$$

Comparing this with equation (114) we conclude that in the limit of large times,

$$\frac{\langle x_{i_1}(t) x_{i_2}(t) \cdots x_{i_n}(t) \rangle}{t^{2n + 1 - \alpha}} \sim \int v_{i_1} v_{i_2} \cdots v_{i_n} I(v) dv.$$

This implies that

$$\langle x_{i_1}(t) x_{i_2}(t) \cdots x_{i_n}(t) \rangle \sim \int \frac{x_{i_1} x_{i_2} \cdots x_{i_n}}{t^{d + 1 - \alpha}} \left( \frac{x}{t} \right) dx,$$

that is, \( P(x, t) \) provided by equation (107) describes the long-time limit of the moments. In the one-dimensional case this result was derived in [17].

**Appendix E. Fractional derivative form of Fourier transform and isotropic Lévy distributions**

In the recent work [21] numerical study of two-dimensional Lévy distributions was performed observing that the distribution can be written as fractional derivative of the distribution in one dimension and using the Matlab code for the fractional derivative. Here we observe that this consideration has universal applicability. We demonstrate that \( d \)-dimensional Fourier transform of arbitrary radially symmetric function can be written as the fractional derivative of order \((d - 1)/2\) of the one-dimensional Fourier transform of that function. We clarify that this fact deserves to be known because it gives a simple way of studying \( d \)-dimensional transforms based on simpler one-dimensional transforms. Series expansions are obtained immediately using fractional derivatives of powers if term-by-term differentiation of series for one-dimensional Fourier transforms is valid. Similarly the asymptotic form of the transform at large argument can be found by differentiation of the simpler one-dimensional form. Numerically \( d \)-dimensional transforms are obtained by applying fractional derivative code to well-developed one-dimensional Fourier transform code.
We do the calculations for inverse Fourier transform with formulas for direct transform implied. We consider the $d$-dimensional Fourier transform of radially symmetric function $f_d(x)$,
\[ f(k) = \int \exp[-i k \cdot x] f_d(x) dx, \tag{E.1} \]
where $|x'| = x$. The formula for the inverse Fourier transform of radially symmetric function whose Fourier transform is $f(k)$ where $k = |k|$ is
\[ f_d(x) = \frac{x^{1-d/2}}{(2\pi)^{d/2}} \int_0^\infty J_{d/2-1}(kx) k^{d/2} f(k) dk. \tag{E.2} \]
We introduce $\tilde{f}_d(x) = \tilde{f}_d(x^2)$ where
\[ \tilde{f}_d(x) = \frac{x^{1/2-d/4}}{(2\pi)^{d/4}} \int_0^\infty J_{d/2-1}(k\sqrt{x}) k^{d/4} f(k) dk, \tag{E.3} \]
where $d$ is the dimension of space. We find the dependence of $f_d$ on $d$ when $f(k)$ is fixed function. We observe that operator of fractional derivative of order 1/2 whose action on arbitrary well-behaved function $h$ obeys
\[ D^{1/2} h = -\frac{d}{dx} \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{h(x') dx'}{\sqrt{x' - x}}, \tag{E.4} \]
is dimension raising,
\[ \frac{1}{\sqrt{\pi}} D^{1/2} \tilde{f}_d = \tilde{f}_{d+1}. \tag{E.5} \]
We use that the identity
\[ \frac{d}{dx} [x^{-\nu} J_\nu(x)] = -x^{-\nu} J_{\nu+1}(x) \tag{E.6} \]
implies
\[ \frac{d}{dx} [x^{-\nu/2} J_\nu(k\sqrt{x})] = -x^{-\nu/2} J_{\nu+1}(k\sqrt{x}) \tag{E.7} \]
where we use that
\[ \int_0^\infty \frac{h(x') dx'}{\sqrt{x' - x}} = \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{h(x') dx'}{\sqrt{x' - x}}, \tag{E.8} \]
We observe that $D^{1/2}$ can be written in terms of the right-side fractional integral $I^{1/2}$ defined as \cite{47}
\[ I^{1/2}[h] = \frac{1}{\sqrt{\pi}} \int_x^\infty \frac{h(x') dx'}{\sqrt{x' - x}}. \tag{E.8} \]
We have
\[ D^{1/2} [h] = -I^{1/2} [h'], \tag{E.9} \]
where we use that
\[ \frac{d}{dx} \int_x^\infty \frac{h(x') dx'}{\sqrt{x' - x}} = 2 \frac{d}{dx} \int_x^\infty h(x') dx' \frac{d}{dx'} \sqrt{x' - x} \]
\[ = -2 \frac{d}{dx} \int_x^\infty h'(x') dx' \sqrt{x' - x} = \int_x^\infty h'(x') dx' \sqrt{x' - x}, \tag{E.10} \]
We find using fractional integral from [54] that
\[
\frac{1}{\sqrt{\pi}}D^{1/2} \left[ x^{-\nu/2} J_\nu(k \sqrt{x}) \right] = \frac{k}{2\pi} \int_x^\infty \frac{x^{t-(\nu+1)/2} dx'}{\sqrt{x' - x}}
\]
\[
\times J_{\nu + 1/2}(k \sqrt{x}) = \frac{k}{2\pi} \sqrt{x}^{-(\nu+1)/2} J_{\nu + 1/2}(k \sqrt{x}).
\] (E.11)

Thus we find equation (E.5) by acting on equation (E.3) with \(\frac{D^{1/2}}{x}\) and using identity (E.11). 

Further applying \(d - 1\) times \(\frac{D^{1/2}}{x}\) on \(f_1\) and using equation (E.5) we obtain that
\[
\frac{1}{\pi^{(d-1)/2}} \left[ D^{1/2} f_1^{d-1} \right]_t = f_d.
\] (E.12)

If \(D^{1/2}\) were the ordinary derivative then the above would imply
\[
\frac{1}{\pi^{(d-1)/2}} D^{d-1} f_1 = f_d,
\] (E.13)

where the fractional derivative of order \(\alpha\) is
\[
D^\alpha f = \frac{(-1)^n}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_x^\infty \frac{f(x') dx'}{(x' - x)^{n - \alpha - 1}}.
\] (E.14)

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