Experimental Eavesdropping Attack against Ekert’s Protocol based on Wigner’s Inequality

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We experimentally implemented an eavesdropping attack against the Ekert protocol for quantum key distribution based on the Wigner inequality. We demonstrate a serious lack of security of this protocol when the eavesdropper gains total control of the source. In addition we tested a modified Wigner inequality which should guarantee a secure quantum key distribution.

Quantum key distribution (QKD) provides a method for distributing a secret key for unconditional secret communications based on the “one time pad” because it guarantees that the presence of any eavesdropper compromising the security of the key is revealed. For a review on this topic see Ref. 1.

The first protocol for QKD has been proposed in 1984 by Bennett and Brassard 2, the worldwide famous BB84 protocol. In 1991 A. Ekert proposed a new QKD protocol whose security relies on the non-local behavior of quantum mechanics, i.e., on Bell’s inequalities 3.

Several groups around the world implemented and tested QKD systems based on variants of the BB84 protocol using either faint laser 4, 5, 6, 7, 8, 9 or entangled photons 10, 11, 12, 13, 14, while, to our knowledge, only recently two groups implemented the Ekert’s protocol 12, 13. In particular Naik et al. 13 implemented a variant of the Ekert’s protocol based on Clauser-Horne-Shimony-Holt (CHSH) inequality as proposed in Ekert’s paper 3, and Jennewein et al. 12 implemented the Ekert’s protocol based on the Wigner inequality.

In ref. 12 the Wigner inequality was first proposed to provide an easier and equally reliable eavesdropping test as the CHSH when the Ekert protocol is implemented. The necessary security proof of the Ekert protocol based on the Wigner inequality consists in verifying the violation of $W \geq 0$.

To obtain the Wigner inequality ($W \geq 0$) it is necessary to review the Wigner argument 14. Two main assumptions are stipulated in the proofs of the Wigner inequality: locality and realism. Locality means that Alice’s measurements do not influence Bob’s measurements, and vice versa. Realism means that, given any physical property, its value exists independently of its observation or measurement. The counterpart of the local-realistic theories is the non-locality behavior of quantum mechanics, a signature of quantum entanglement. In particular Wigner considered a quantum system prepared in the singlet state, and he obtained the violation of the inequality $W \geq 0$, i.e. $W = -0.125$. Furthermore, in the derivation of his inequality, Wigner assumed perfect anticorrelation in the measurement results. This assumption is obviously reasonable in the test of realism and locality of a physical theory (it reflects the classical counterpart of a quantum system prepared in the singlet state). Nevertheless, in terms of QKD this assumption corresponds to a lack of security.

In fact, when the eavesdropper, Eve, measures photons on either one or both of Alice and Bob channels, her presence should be revealed by a higher value of $W$ than the local-realistic theories limit $W = 0$, as it happens for the CHSH inequality 3. Unfortunately this is not the case. In fact, only when Eve adopts an intercept-resend strategy and detects one photon of the pair, the inequality becomes $W \geq 0.0625$, but, as we will show, this is not for eavesdropping on both channels, because in this case there is no limit 14.

In this letter we perform an experiment proving the weakness of the Wigner inequality as a security test for QKD, under the condition of Eve gaining total control of the source of photon pairs. Under this condition, she prepares each particle of the pair separately in a well-defined polarization direction, in other words she prepares the photon in Alice’s channel in the state $|φ_A⟩$, and the photon in Bob’s channel in the state $|φ_B⟩$, respectively

$$|φ_A⟩ = \cos φ_A |H_A⟩ + \sin φ_A |V_A⟩,$$

$$|φ_B⟩ = \cos φ_B |H_B⟩ + \sin φ_B |V_B⟩.$$  

Thus Eve has a perfect knowledge of the polarization of the photons sent and, even if the non-local behavior of the original quantum system (the singlet state) is completely removed, we prove that she can avoid disclosing herself.

We remind the reader that ref. 14 presented a modified version of the Wigner’s parameter $W$ which maintains the same limits, i.e. $W \geq 0$ for local realistic theories and $W = -0.125$ for the singlet state, but allows secure QKD, because $W$ contains the additional term accounting for the anticorrelation. In our experiment we
photons in the polarization bases. Two half-wave plates (HWPs) project the pair of photons by the couple of detectors $x_p$ and $x_B$ where $\phi_p = 98^\circ$ in agreement with the theory [16], ensuring a secure QKD.

$W$ is well above the limit for local-realistic theories in [16], respectively.

$\tilde{W} = 0.0025$ (light grey), $W(\tilde{W}) > 0.0625$ (white).

The measured quantities in our experiment are $W$ and $\tilde{W}$, respectively

$$W = p_{0,0}^{A,B}(+A,+B) + p_{0,0}^{A,B}(+A,+B)$$

$$\tilde{W} = p_{0,0}^{A,B}(+A,+B) + p_{0,0}^{A,B}(+A,+B)$$

where $p_{0,0}^{A,B}(x_A,y_B)$ are the probabilities of detecting the pair of photons by the couple of detectors $x_A, y_B$ ($x_A = +A, -A$ and $y_B = +B, -B$) when in the detection apparatuses two half-wave plates (HWPs) project photons in the polarization bases

$$|s_{\alpha_z} \rangle = \cos \alpha_z |H_z \rangle + \sin \alpha_z |V_z \rangle,$$

$$|s^\perp_{\alpha_z} \rangle = \sin \alpha_z |H_z \rangle - \cos \alpha_z |V_z \rangle,$$

with $z = A, B$. In formula

$$p_{A,A,B}(+A,+B) = |\langle \phi_A|s_{\alpha_A} \rangle \langle \phi_B|s_{\alpha_A} \rangle|^2$$

$$p_{A,A,B}(+A,-B) = |\langle \phi_A|s_{\alpha_A} \rangle \langle \phi_B|s^\perp_{\alpha_A} \rangle|^2$$

$$p_{A,A,B}(-A,+B) = |\langle \phi_A|s^\perp_{\alpha_A} \rangle \langle \phi_B|s_{\alpha_A} \rangle|^2$$

$$p_{A,A,B}(-A,-B) = |\langle \phi_A|s^\perp_{\alpha_A} \rangle \langle \phi_B|s^\perp_{\alpha_A} \rangle|^2.$$ (3)

In Fig. 1 (a) and (b) we present the calculated contour plots of $W$ and $\tilde{W}$ versus the polarization directions $\phi_A$ and $\phi_B$ of the photons of the pair sent by Eve. Highest values of $W$ and $\tilde{W}$ ($max(W) \simeq max(\tilde{W}) \simeq 0.9557$) corresponds to the center of white regions of Figs. 1. Darker regions correspond to lower range of values for $W$ and $\tilde{W}$.

The values $min(W) \simeq -0.2121$ are in the middle of black regions of Fig. 1 (a) along "Fig. 3 (a)" line, while $min(\tilde{W}) \simeq 0.0443$ are almost in the middle of dark grey regions of Fig. 1 (b). The straight lines for $\phi_B = 0^\circ, 62^\circ, 98^\circ$ represent sections of the plots where the theoretical predictions are compared with the experimental results of Figs. 3 (a), (b) and (c).

In Fig. 2 we depict the experimental scheme considering the situation in which Eve has total control of the source. In this scheme, the source under Eve’s control replaces the source of entangled photon pairs of a typical QKD scheme [10, 11, 12, 13, 14]. Eve’s source is obtained by a 1 mm length LiIO3 nonlinear crystal (NLC2) pumped by ultrashort pulses (150 fs) at 415 nm generated from a second harmonic (obtained from NLC1). In Fig. 2 we depict the experimental scheme considered the situation in which Eve has total control of the source.
of a ultrashort mode-locked Ti-Sapphire with a repetition rate of 76 MHz pumped by a 532 nm green laser. The NLC2 realizes a non-collinear type I phase matching and Eve selects two quantum correlated optical channels along which the twin photons at 830 nm (emitted at 3.4°) are sent towards Alice and Bob’s detection apparatuses [17, 18]. The down-converted photons of a pair have the same polarization state (ordinary waves) and Eve can modify deterministically the polarization state of the photon by means of a half-wave-plate (HWP) in each channel, in other words Eve sends photon pairs to Alice and Bob with polarization state |ϕ_A⟩ and |ϕ_B⟩, respectively.

Alice and Bob’s detection apparatuses are identical and are composed of an open air-fiber coupler to collect the down-converted light by single-mode optical fibers. The detection of photons in the proper polarization basis is guaranteed by a HWP before the fiber coupler and a fiber-integrated polarizing beam splitter (PBS). Photons at the two output ports of the PBS are sent to fiber coupled photon counters (Perkin-Elmer SPCM-AQR-14) [19]. Interference filters peaked at 830 nm with 11 nm bandwidth are placed in front of the fiber couplers to reduce straylight.

Coincident counts between any of Alice’s detectors (+A, −A) and any of Bob’s detectors (+B, −B) are obtained from an Elsag prototype of four-channel coincident circuit [20, 21]. Single-counts and coincidences are counted by a National Instruments sixteen channels counter plug-in PC card.

The terms \( p_{α_A,α_B}(x_A, y_B) \) are estimated in terms of the number of coincident counts:

\[
p_{α_A,α_B}(x_A, y_B) = \frac{N_{α_A,α_B}(x_A, y_B)}{[N_{α_A,α_B}(+A, +B) + N_{α_A,α_B}(+A, −B) + N_{α_A,α_B}(−A, +B) + N_{α_A,α_B}(−A, −B)]}
\]

where \( N_{α_A,α_B}(x_A, y_B) \) is the number of coincidences measured by the couple of detectors \( x_A, y_B (x, y = +, −) \) when Alice and Bob’s detection apparatuses project photons in the polarization bases at Eqs. [4].

In Fig. 3 (a) we present our main result: photons sent by Eve in a definite polarization state violate the limit of local-realistic theories. Experimental data for \( W \) (circles) and \( \tilde{W} \) (squares) are presented versus \( ϕ_A \), with \( ϕ_B \) fixed approximately at 62° and show a good agreement with theoretical predictions (lines).

As expected from the theory [17], not only does \( W \) violate the limit of local-realistic theories \((W = 0)\), but also some data points pass the quantum limit \((W = −0.125)\); while \( \tilde{W} \) is well above the limit of local-realistic theories. The theoretical curves are calculated with \( ϕ_A = 62° \), and the discrepancy between theory and experiment can be explained by noting the difficulties in the proper angular positioning of the four HWPs and in the noise introduced by real optical devices, e.g., fibers, PBSs, detectors dark counts and straylight.

In Fig. 3 (b) we present the experimental data and the theoretical curve obtained with \( ϕ_B = 98° \), corresponding to a position close to the minima of \( W \) as predicted by the theory. Fig. 3 (b) shows a good agreement between
experimental data (circles) and theoretical predictions of $\tilde{W}$ and the minimum of experimental values, 0.0685, is slightly higher than the theoretical predictions of 0.0466.

Furthermore, Fig. 3 (b) shows also the experimental data for $W$ (small squares) together with the associated theoretical curve, and we observe that also in this case a violation of the local-realistic theories limit occurs.

According to Eq.s (1) and (2), we note that $\tilde{W}$ differs from $W$ only because of the term $p_{A}^{x}p_{B}^{y}(\phi_{A}=-\phi_{B})$, thus if $p_{A}^{x}p_{B}^{y}(\phi_{A}=-\phi_{B})=0$ then $\tilde{W}=W$, and this occurs when $\phi_{A}=0^\circ$, $180^\circ$ or $\phi_{B}=0^\circ$, $180^\circ$.

In Fig. 3 (c) we consider the situation when $\phi_{B}=0^\circ$ and we observe that the experimental data for $\tilde{W}$ (small circles) are almost superimposed to the $W$ ones (squares) in good agreement with the theoretical prediction, $\tilde{W}=W$ (line).

Some further analysis of $\tilde{W}$ must be considered for the practical implementation of the Ekert protocol based on Wigner’s inequality. According to [12] we highlight that the Ekert’s protocol based on modified Wigner’s inequality still guarantees a simplification with respect to the one based on the CHSH inequality, because Alice and Bob randomly choose between two rather than three bases. Though the necessity of an experimental evaluation of the term $p_{A}^{x}p_{B}^{y}(\phi_{A}=-\phi_{B})$ forces Alice and Bob to sacrifice part of the key for the sake of security, we note that in any practical implementation of QKD protocols, Alice and Bob distill from the noisy sifted key a nearly noise-free corrected key by means of error correction procedures subjected to the constraint of knowing the quantum bit error rate (QBER). Also, the QBER is estimated at the cost of losing part of the key. Thus, we suggest using the same sacrificed part of the key to estimate both $W$ and QBER.

To perform a proper comparison of the performances of Ekert protocols based on Wigner’s inequality versus the one CHSH-based [3], it is necessary to consider situations where the same number of analyzer settings are employed. In particular, we consider the modified protocol based on Wigner inequality proposed in [17] where Alice and Bob measure randomly using three analyzer settings (as in the case of CHSH). This protocol is more efficient than the protocol based on CHSH. In particular, for CHSH only 2/9 of the qubits exchanged are devoted to the key generation [3], while here we can improve till 1/3 depending on the security needs. Furthermore in this protocol none of the qubits exchanged are discarded while in the case of CHSH 1/3 of the qubits are discarded [10].

In conclusion, this paper highlights the insecurity of Ekert’s protocol based on the Wigner inequality. We performed an experiment simulating the total control of photons in Alice and Bob channels by an eavesdropper, proving that the QKD Ekert protocol based on Wigner’s inequality presents a serious lack of security. In addition, we proved that a modified version of the Wigner security parameter guarantees secure QKD.

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