GEOMETRIC CLASSIFICATION OF TOTALLY STABLE STABILITY SPACES

YU QIU AND XIAOTING ZHANG

Dedicated to Alastair King on the occasion of his sixtieth birthday

Abstract. We construct a geometric model for the root category $D_\infty(Q)/[2]$ of any Dynkin diagram $Q$, which is an $h_Q$-gon $V_Q$ with cores, where $h_Q$ is the Coxeter number and $D_\infty(Q) = D^b(Q)$ is the bounded derived category associated to $Q$. As an application, we classify all spaces $\text{ToSt}D$ of totally stable stability conditions on triangulated categories $D$, where $D$ must be of the form $D_\infty(Q)$. More precisely, we prove that $\text{ToSt}D_\infty(Q)/C$ is isomorphic to the moduli spaces of stable $h_Q$-gons of type $Q$.

In particular, an $h_Q$-gon $V$ of type $D_n$ is a centrally symmetric doubly punctured $2(n-1)$-gon. $V$ is stable if it is convex and the punctures are inside the level-$(n-2)$ diagonal-gon. Another interesting case is $E_6$, where the (stable) $h_Q$-gon (dodecagon) can be realized as a pair of planar tiling pattern.

Key words: stability conditions, root system, Dynkin diagram, geometric model

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1. Introduction

The notion of stability conditions on a triangulated category $\mathcal{D}$ is introduced by Bridgeland [B1], whose motivation is $\Pi$-stability in string theory. They measure certain stability structure, as the name suggested, in physics as well as in mathematics. A stability condition $\sigma = (Z, \mathcal{P})$ on $\mathcal{D}$ consists of a group homomorphism $Z: K\mathcal{D} \to \mathbb{C}$, known as the central charge, and an $\mathbb{R}$-collection of abelian subcategories $\mathcal{P}(\phi)$, known as the slicing. The main theorem in [B1] states that all stability conditions on a triangulated category form a complex manifold. Moreover, one recent breakthrough in this area is the identification of stability spaces with the moduli spaces of quadratic differentials, cf. [BS, HKK]. Although there are already many studies, the known examples of global structures of stability spaces are still very limited.

1.1. Total stability. One key concept in the theory of stability conditions is stable object, i.e., simple object in some $\mathcal{P}(\phi), \phi \in \mathbb{R}$. It goes back to geometric invariant theory and King’s $\theta$-stability, which plays an important role in the study of Donaldson-Thomas invariant, as well as cluster theory, cf. [K]. When passing to the dynamical system side, i.e. identifying a stability condition with a quadratic differential on some Riemann/marked surface, stable objects correspond to non-broken geodesics (connecting zeroes) or saddles. An interesting question proposed in [Q1] is to find stability conditions that make all indecomposable objects in the triangulated category stable. Such stability conditions are called totally stable. The abelian version of this question is proposed by Reineke [R] when studying quantum dilogarithm identities of the module category $\text{mod} \ kQ$ of an ADE quiver $Q$.

Motivated by $q$-deformation of stability conditions in [IQ1, IQ2], Ikeda-Qiu introduce the notion of global dimension ($\mathbb{R}_{\geq 0}$-valued) function $\text{gldim}$ of a stability condition $\sigma$ to measure how stable a stability condition is. Qiu [Q2] shows that $\sigma$ is totally stable if and only if $\text{gldim} \sigma < 1$, which is very rare. In fact, such an existence of $\sigma$ implies that $\mathcal{D}$ must be the bounded derived category $\mathcal{D}_\infty(Q) = \mathcal{D}^b(Q)$ of a Dynkin diagram $Q$ (cf. [KOT, Q3]). In this paper, we classify all spaces $\text{ToSt}(Q)$ of totally stable stability conditions on the Dynkin category $\mathcal{D}_\infty(Q)$ by constructing their geometric models.

1.2. Geometric model for Dynkin categories. For type $A_n$, Qiu gives a geometric description of $\text{ToSt} \mathcal{D}_\infty(A_n)$, i.e. the moduli space of convex $(n+1)$-gon. It can be viewed as a variation/consequence of the fact that $\mathcal{D} = \mathcal{D}_\infty(Q)$ is the topological Fukaya category of $(n+1)$-gon as follows.

Let $S^\lambda$ be a graded marked surface, that is a topological surface $S$ with marked points on its boundary $\partial S$ and a grading $\lambda \in H_1(\mathbb{P}TS)$. Given an ‘$\infty$-angulation’ $A$ of $S^\lambda$, one obtains a graded gentle algebra $\Lambda_A$ and the topological Fukaya category $\mathcal{D}_\infty(S^\lambda)$ can be constructed as the bounded derived category $\mathcal{D}^b(\Lambda_A)$ of $\Lambda_A$. The arcs correspond to indecomposable objects in $\mathcal{D}^b(\Lambda_A)$, cf. [HKK]. We have $\mathcal{D}_\infty(S^\lambda) = \mathcal{D}_\infty(A_n)$ for a disk $S^\lambda$ with $(n+1)$ marked points. A geometric model for module category $\mathcal{H}$ of an $A_n$ quiver (with any orientation) is given in [BGMS] (and they construct a Tost with heart $\mathcal{H}$ as a consequence).
If $S^\lambda$ has punctures, which carry additional $\mathbb{Z}_2$-symmetry see [QZZ], cf. [FST, S, QZ], the story still works. Then $\Lambda_A$ is a graded skew-gentle algebra. In particular, $D_\infty(S^\lambda) = D_\infty(D_n)$ for a once-punctured disk $S$ with $n - 1$ marked points on $\partial S$.

In this paper, we introduce another model for $D_\infty(D_n)$, featuring double cover, which is similar, but different from the ones in [AP, AB]. More precisely, they take the double cover branching at punctures but we take the double cover of a once-punctured disk branching at a point other than the puncture. What we get is a doubly punctured disk with $2(n - 1)$ marked points on its boundary. As the other models mentioned above, we can also realized objects as arcs (without tagging). Moreover the two punctures in our model naturally correspond to the two ways of tagging in [FST, QZ, QZZ]. For instance, the tagged(-switching) rotation (introduced in [BQ]) corresponding to the Auslander-Reiten (AR) translation $\tau \in \text{Aut} D_\infty(Q)$ becomes the puncture-switching rotation in our setting. See Figure 1 that $\tau(B_-.V_3) = B_+.V_2$ in both cases, where the subscripts $+/-$ denote untagged/tagged respectively in the left picture.

Moreover, after straightening the model and making it centrally symmetric (then the geometric center coincide with the branching point), the (oriented) arcs corresponding to the objects naturally becomes their central charges. Furthermore, we can easily describe total stability using such a model, which we will mention in Section 1.3.

One step further, we manage to find similar geometric models (to describe objects, central charge and ToSt) for all exceptional type categories, i.e. $D_\infty(E_6,7,8)$ (see Remark 7.11). Such models shed lights on understanding the geometry of $\text{Stab} D_\infty(Q)$ and its connection to the root system/Kleinian singularities/Calabi-Yau categories.

1.3. Summary of notations and results. A (labelled) $h$-gon $V$ (on $\mathbb{C} = \mathbb{R}^2$) with vertices $V_0, V_1, \cdots, V_h = V_0$ and (oriented) edges $z_j = V_{j-1}V_j, \forall j \in \mathbb{Z}_h$. We write $VW$ for the vector (or oriented edges) $\overrightarrow{VW}$. It is convex if all other $V_i$ is on the left hand side of the edge $z_j = V_{j-1}V_j$.

For $1 \leq s \leq h/2$, the (oriented) length-$s$ diagonals of an $h$-gon are $V_jV_{j+s}$. For instance, length-1 diagonals are just edges. For a convex $h$-gon $V$, the level-$s$ diagonal-gon is the convex $h$-gon bounded by its length-$s$ diagonals (i.e. on the left hand side of).
Definition 1.1. An $h$-gon of type $Q$ is defined respectively as

- $A_n$: an $(n+1)$-gon.
- $B_n$: a centrally symmetric $2n$-gon with one puncture at its geometric center.
- $C_n$: a centrally symmetric $2n$-gon.
- $D_n$: a centrally symmetric doubly punctured $2(n-1)$-gon.
- $E_6$: a 12-gon satisfying (4 triangle relations and 3 square relations):
  \[
  \begin{align*}
  z_j + z_{j+4} + z_{j+8} &= 0, \\
  z_j - z_{j-3} + z_{j-6} - z_{j-9} &= 0,
  \end{align*}
  \]
  \[
  \forall j \in \mathbb{Z}_{12}.
  \tag{1.1}
  \]
  Note that the rank of the 7 relation equations is actually 6.
- $E_7$: a centrally symmetric 18-gon satisfying (3 hexagon relations):
  \[
  z_j + z_{j+1} + z_{j+6} + z_{j+7} + z_{j+12} + z_{j+13} = 0, \quad \forall j \in \mathbb{Z}_{18}.
  \tag{1.2}
  \]
  Note that after setting $z_{j+9} = -z_j$ (by the central symmetry), the rank of the 3 relation equations is actually 2.
- $E_8$: a centrally symmetric 30-gon satisfying (5 triangle relations and 3 pentagon relations):
  \[
  \begin{align*}
  z_j + z_{j+10} + z_{j+20} &= 0, \\
  z_j + z_{j+6} + z_{j+12} + z_{j+18} + z_{j+24} &= 0,
  \end{align*}
  \]
  \[
  \forall j \in \mathbb{Z}_{30}.
  \tag{1.3}
  \]
  Note that after setting $z_{j+15} = -z_j$ (by the central symmetry), the rank of the 8 relation equations is actually 7.
- $F_4$: a centrally symmetric 12-gon satisfying (1.1). Note that after setting $z_{j+6} = -z_j$ (by the central symmetry), the rank of the 7 relation equations in (1.1) further reduces to 4.
- $G_2$: a centrally symmetric 6-gon satisfying (2 triangle relations)
  \[
  z_j + z_{j+2} + z_{j+4} = 0, \quad \forall j \in \mathbb{Z}_6.
  \tag{1.4}
  \]
  Note that after setting $z_{j+3} = -z_j$ (by the central symmetry), the two relation equations are equivalent.

In particular, an $h$-gon of type $Q$ is an $hQ$-gon (satisfying extra conditions), where $h_Q$ is the Coxeter number associated to the Dynkin diagram $Q$.

For an exceptional case (or $F_4$ case), the relation (1.1), (1.2) or (1.3) induces a pair of $h_Q/2$-gons, called the ice and fire cores. See Figure 7 for the two cores in type $E_6$, the upper picture of Figure 9 for the ice core in type $E_7$ and Figure 10 for the ice core in type $E_8$. The fire core is the centrally symmetric mirror of the ice core in type $E_7/E_8$.

Definition 1.2. An $hQ$-gon of type $Q$ is stable if it is convex and moreover:

- $D_n$: the punctures are inside the level-$(n-2)$ diagonal-gon.
- $E_n$: the ice and fire cores are inside the level-$(n-3)$ diagonal-gon, for $n \in \{6, 7, 8\}$.
- $F_4$: same as $E_6$. Note that the pair of 6-gons are centrally symmetric to each other.

In particular, a stable $h$-gon means a convex $h$-gon without referring the types.
Two key observations are: (I) the central charges of objects in any \( \tau \)-orbit of the root category form an \( h_Q \)-gon (see Proposition 3.5) and (II) there are distinguished \( \tau \)-orbits, called the far-end \( \tau \)-orbits (cf. Lemma 2.5, as well as the middle-end/short-end \( \tau \)-orbits). We always fix a far-end \( \tau \)-orbit and call the corresponding \( h_Q \)-gon the far-end one.

Denote by \( \text{Sth}(Q) \) the moduli space of stable \( h_Q \)-gons of type \( Q \), where two stable \( h_Q \)-gons are equivalent if and only if they are similar to each other.

**Theorem 1.** An \( h_Q \)-gon of type \( Q \) provides a geometric model for the root category \( D_\infty(Q)/[2] \), in the sense that it induces a central charge naturally. Moreover, we have the isomorphism (between complex manifolds)
\[
Z_h : \text{ToSt}(Q)/\mathbb{C} \cong \text{Sth}(Q),
\]
sending a Tost to the far-end stable \( h_Q \)-gon (of type \( Q \)).

A side product is that we have a simple description of the global dimension function on \( \text{ToSt}(Q) \) (Theorem 3.6).

**1.4. Connection to root systems.** In this subsection, we explain the connection between \( \text{ToSt}(Q) \), the root system \( \Lambda(Q) \) and the space \( \text{Stab} D_2(Q) \) of stability conditions on \( D_2(Q) \). Here, \( D_2(Q) \) is the finite dimensional derived category of the Calabi-Yau-2 Ginzburg dg algebra/derived preprojective algebra associated to \( Q \). It can also be constructed from coherent sheaves for the Kleinian singularity \( \mathbb{C}^2/G \), where \( G \) is the finite subgroup of \( \text{SL}_2(\mathbb{C}) \) corresponding to \( Q \) (i.e. the McKay correspondence).

Let \( g \) be the finite-dimensional complex simple Lie algebra associated to \( Q \), with Cartan subalgebra \( h \) and \( \Lambda(Q) \) the corresponding root system. Let \( h^{\text{reg}} \) be the complement of the root hyperplanes in \( h \):
\[
h^{\text{reg}} = \{ v \in h \mid v(\alpha) \neq 0, \forall \alpha \in \Lambda(Q) \}.
\]
The Weyl group \( W \), generated by reflections of the root hyperplanes, acts freely on \( h^{\text{reg}} \).

In [B2], Bridgeland shows
\[
\text{Stab} D_2(Q)/\text{Br} Q \cong h^{\text{reg}}/W Q.
\]
On the other hand, one expects that
\[
\text{Stab} D_\infty(Q) \cong h/W Q
\]
and it is confirmed in [HKK] for type \( A \). Clearly, there is a close relation between these two results. For instance, one can found a detailed case study in [BQS] for \( A_2 \) case. For all Dynkin case, there is also a conjectural description on the (almost) Frobenius structure on \( \text{Stab} D_2(Q) \), which has been proved in [IQ2] for type \( A \). More precisely, [IQ1] introduces the Calabi-Yau-\( X \) category to link the Calabi-Yau-\( \infty \) category and the Calabi-Yau-2 one. By the induction-reduction procedure there, each stability condition \( \sigma \) in \( \text{ToSt}(Q) \) induces a stability condition in \( \text{Stab} D_2(Q) \).

Another connection between \( \text{ToSt}(Q) \) and the root system is the Gepner point \( \sigma_G \), a stability condition with extra symmetry. In the Dynkin case, such a point is the (unique up to \( \mathbb{C} \)-action) solution to the equation (found by [KST])
\[
\tau(\sigma) = (-2/h_Q) \cdot \sigma,
\]
(1.5)
where \((-2/h_Q)\) is the \(C\)-action on the right hand side. The stability condition \(\sigma_G\) is also the most stable one, in the sense that it is the minimal point of the global dimension function \(\text{gldim}\), cf. [Q2, Thm. 4.7]. Interestingly, the central charge at the Gepner point for a Dynkin quiver \(Q\) is given by the projection of the root system on the Coxeter plane (Lemma 2.7). This was pointed out by Lutz Hille to Qy and Alastair King in Oberwolfach, Jan. 2020. We draw many pictures in Appendix A showing such projections (together with features of characterization of \(\text{ToSt}\), i.e. the stable \(h_Q\)-gon).

Here is the trailer:

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2. Preliminaries

2.1. Stability conditions. Following [B1], we recall the notion of stability conditions on a triangulated category. In this paper, \(D\) is a triangulated category with Grothendieck group \(K_D\) and assume that \(K_D \cong \mathbb{Z}^n\) for some \(n\). Denote by \(\text{Ind} D\) the set of (isomorphism classes of) indecomposable objects in \(D\).

Definition 2.1. A stability condition \(\sigma = (Z, P)\) on \(D\) consists of a group homomorphism \(Z: K D \to \mathbb{C}\), called the central charge, and a family of full additive subcategories \(P(\phi) \subset D\) for \(\phi \in \mathbb{R}\), called the slicing, satisfying the following conditions:

(a) if \(0 \neq E \in P(\phi)\), then \(Z(E) = m(E) \exp(i\pi\phi)\) for some \(m(E) \in \mathbb{R}_{>0}\),
(b) for all \(\phi \in \mathbb{R}\), \(P(\phi + 1) = P(\phi)[1]\),
(c) if \(\phi_1 > \phi_2\) and \(A_i \in P(\phi_i)\) \((i = 1, 2)\), then \(\text{Hom}(A_1, A_2) = 0\),
(d) for each object \(0 \neq E \in D\), there is a finite sequence of real numbers

\[\phi_1 > \phi_2 > \cdots > \phi_l\]
and a collection of exact triangles (known as the HN-filtration)

\[
0 = E_0 \rightarrow E_1 \rightarrow E_2 \rightarrow \cdots \rightarrow E_{l-1} \rightarrow E_l = E
\]

with \( A_i \in \mathcal{P}(\phi_i) \) for all \( i \).

(e) a technique condition, known as the support property, which holds automatically in our setting.

The categories \( \mathcal{P}(\phi) \) are then abelian. Their non-zero objects are called \textit{semistable of phase} \( \phi \) and simple objects \textit{stable of phase} \( \phi \). For a semistable object \( E \in \mathcal{P}(\phi) \), denote by \( \phi_\sigma(E) := \phi \) its phase.

There is a natural \( \mathbb{C} \)-action on the set \( \text{Stab} \mathcal{D} \) of all stability conditions on \( \mathcal{D} \), namely:

\[
s \cdot (Z, \mathcal{P}) = (Z \cdot e^{-i\pi s}, \mathcal{P}_{\text{Re}(s)}),
\]

where \( \mathcal{P}_x(\phi) = \mathcal{P}(\phi + x) \) and \( s \in \mathbb{C} \). There is also a natural action on \( \text{Stab} \mathcal{D} \) by the group of autoequivalences \( \text{Aut} \mathcal{D} \), namely:

\[
\Phi(Z, \mathcal{P}) = (Z \circ \Phi^{-1}, \Phi(\mathcal{P})),
\]

where \( \Phi \in \text{Aut} \mathcal{D} \).

The famous result in [B1] states that \( \text{Stab} \mathcal{D} \) is a complex manifold with dimension \( \text{rank} K \mathcal{D} \) and the local coordinate is provided by the central charge \( Z \).

### 2.2. Total (semi)stability via global dimension function.

Let \( \sigma = (Z, \mathcal{P}) \) in \( \text{Stab} \mathcal{D} \).

**Definition 2.2.** The global dimension of \( \sigma \) is defined as

\[
gldim \mathcal{P} = \sup \{ \phi_2 - \phi_1 \mid \text{Hom}(\mathcal{P}(\phi_1), \mathcal{P}(\phi_2)) \neq 0 \} \in \mathbb{R}_{\geq 0} \cup \{ +\infty \}, \tag{2.2}
\]

which is a continuous function and an invariant under both the \( \mathbb{C} \)-action and \( \text{Aut} \mathcal{D} \). The global dimension of \( \mathcal{D} \) is defined to be the inf of \( gldim \sigma \) for all \( \sigma \in \text{Stab} \mathcal{D} \).

Note that the notion generalizes the global dimension of an algebra/abelian category.

We recall the notion of total (semi)stability on triangulated categories, whose abelian version is due to Reineke [R], cf. [Q1, Conjecture 7.13] and comments there.

**Definition 2.3.** A stability condition \( \sigma \) is called \textit{totally (semi)stable}, if any indecomposable object in \( \mathcal{D} \) is (semi)stable w.r.t. \( \sigma \).
2.3. Dynkin diagrams and KOT-Q classification. A Dynkin quiver $Q$ is an oriented graph whose underlying graph $\overrightarrow{Q}$ is one of ADE (i.e. simply-laced) Dynkin diagram. Explicitly, $\overrightarrow{Q}$ is of the form $T_{p,q,r}$:

$$\begin{array}{cc}
\bullet & \cdots \\
\bigtriangleup & \\
\bullet & \cdots \\
\bullet & \circ \\
\end{array}$$

with $1 \leq p \leq q \leq r$ and

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

Then we have ($n = p + q + r - 2$)

- $Q$ is of type $A_n$ if $\overrightarrow{Q} = T_{1,1,n}$.
- $Q$ is of type $D_n$ if $\overrightarrow{Q} = T_{2,2,n-2}$ for $n \geq 4$.
- $Q$ is of type $E_n$ if $\overrightarrow{Q} = T_{2,3,n-3}$ for $n \in \{6, 7, 8\}$.

**Definition 2.4.** For a Dynkin quiver $Q$, we call its leaves (i.e. sinks and sources) the boundary vertices. Moreover, we call

- the leaf at the end of length $p$ branch of $Q$ the short-end vertex.
- the leaf at the end of length $q$ branch of $Q$ the middle-end vertex.
- the leaf at the end of length $r$ branch of $Q$ the far-end vertex.

Note that there is a choice involved for fixing the far-end vertex in type $A_n$, $D_4$ and $E_6$.

Denote by $\text{AR} D_\infty(Q)$ the AR quiver of $D_\infty(Q)$, which is isomorphic to $\mathbb{Z}Q$. Each vertex of $Q$ corresponding to a $\tau$-orbit (not necessarily different). The $\tau$-orbit that corresponds to an $xx$ vertex is call $xx$ $\tau$-orbit, for $xx$ being boundary/short-end/middle-end/far-end.

The following is well-known.

**Lemma 2.5.** For type $A_n$ and $E_6$, there are two choices of far-end $\tau$-orbits of $\text{AR} D_\infty(Q)$, which are shift [1] of each other. For type $D_4$, there are three choices of far-end $\tau$-orbit of $\text{AR} D_\infty(Q)$ (i.e. any boundary $\tau$-orbit). For other cases, there is a unique choice of the far-end $\tau$-orbit.

For the orbit categories of $D_\infty(Q)$ (e.g. the root category $D_\infty(Q)/[2]$ we are about to mention in particular), we keep the same notions. We will always choose a preferred far-end vertex/$\tau$-orbit in each case.

A Dynkin specie $Q^\iota$ is the $\iota$-orbit of a Dynkin quiver $Q$, where $\iota$ is an admissible automorphism of $Q$. All possible cases are listed as follows (we omit the orientations but they should be compatible with $\iota$):
$B_n$: $Q$ is of type $D_{n+1}$ and $Q^\iota$ is of type $B_n$ while $\iota$ exchanges the black bullets in the same column.

\[
\begin{array}{c}
\bullet \\
\circ \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \\
\bullet \\
\end{array} \quad \iota \quad \begin{array}{c}
\circ \\
\bullet \longrightarrow \circ \longrightarrow \cdots \longrightarrow \circ \\
1 \quad 1 \quad 1 \quad 2
\end{array}
\]

$C_n$: $Q$ is of type $A_{2n-1}$ and $Q^\iota$ is of type $C_n$ while $\iota$ exchanges the black bullets in the same column.

\[
\begin{array}{c}
\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \\
\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \\
2 \quad 2 \quad 1
\end{array} \quad \iota \quad \begin{array}{c}
\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet \\
2 \quad 2 \quad 1
\end{array}
\]

$F_4$: $Q$ is of type $E_6$ and $Q^\iota$ is of type $F_4$ while $\iota$ exchanges the black bullets in the same column.

\[
\begin{array}{c}
\bullet \longrightarrow \bullet \\
\bullet \longrightarrow \bullet \\
1 \quad 1 \quad 2 \quad 2
\end{array} \quad \iota \quad \begin{array}{c}
\bullet \longrightarrow \bullet \longrightarrow \bullet \\
1 \quad 1 \quad 2 \quad 2
\end{array}
\]

$G_2$: $Q$ is of type $D_4$ and $Q^\iota$ is of type $G_2$ while $\iota$ rotates the black bullets in the same column.

\[
\begin{array}{c}
\bullet \longrightarrow \bullet \\
\bullet \longrightarrow \bullet \\
1 \quad 3
\end{array} \quad \iota \quad \begin{array}{c}
\bullet \longrightarrow \\
1 \quad 3
\end{array}
\]

The label below any black bullet in $Q^\iota$ is its weight.

Denote by $D_\infty(Q) = \mathcal{D}^b(kQ)$ the bounded derived category of the path algebra $kQ$ for a Dynkin quiver $Q$. For a Dynkin specie $Q^\iota$, denote by $D_\infty(Q^\iota)$ the $\iota$-stable category $D_\infty(Q^\iota)^\iota$. In the specie case, the boundary/far-end vertices (of $Q^\iota$) and $\tau$-orbits (of $\text{AR} \ D_\infty(Q^\iota)$) are induced from $Q$. Lemma 2.5 implies that there is a unique far-end $\tau$-orbit in any specie case.

By abuse of notation, when we mention a Dynkin diagram $Q$, we mean either it is a Dynkin quiver $Q$ or a Dynkin specie $Q^\iota$. Denote by $\text{ToSt} \mathcal{D}$ the subspace of $\text{Stab} \mathcal{D}$ consisting of all totally stable stability conditions on a triangulated category $\mathcal{D}$. The classification theorem below is a combination of the following two results:

- [Q2, Prop. 3.5]: $\sigma$ is totally stable if and only if $\text{gldim} \sigma < 1$.
- [Q3, Thm. 3.2]: $\mathcal{D}$ admits a stability condition $\sigma$ with $\text{gldim} \sigma < 1$ if and only if $\mathcal{D}$ equals $D_\infty(Q)$ for a Dynkin diagram $Q$.

Note that the second result is essentially due to Kikuta-Ouchi-Takahashi, where they impose a mild condition that excludes the non simply-laced case.

**Theorem 2.6.** $\text{ToSt} \mathcal{D}$ is non-empty if and only if $\mathcal{D}$ equals $\mathcal{D}_\infty(Q)$ for a Dynkin diagram $Q$. In such a case, we write $\text{ToSt}(Q)$ for $\text{ToSt} \mathcal{D}_\infty(Q)$. 
2.4. **Root system.** Let $\mathfrak{g}$ be the complex simple Lie algebra of type $Q$ and $\mathfrak{h}$ its Cartan subalgebra. Denote by $\Lambda(Q)$ the associated root system and $h_Q$ the Coxeter number.

The Coxeter element $w$ is the product of all simple reflections. Although it depends on the order of the product, all such elements are conjugate to each other. Thus, up to symmetry, there is essentially one Coxeter element. After fix the Coxeter element $w$, there is a unique plane $P_w$, known as the Coxeter plane, on which $w$ acts by rotation by $2\pi/h_Q$.

For instance, the projection of the root system on the Coxeter plane $P_w$ of type $D_5$, $E_6$, $E_7$ and $E_8$ are shown in Figures 13, 14, 15 and 16 respectively.

2.5. **Root categories.** Note that the dimension function $\dim$ depends on the orientation of $Q$ but the root category does not. Recall the famous Gabriel’s theorem that the root category categorifies the root system, in the sense that there is a bijection
\[
\dim : \text{Ind} \mathcal{D}_\infty(Q)/[2] \rightarrow \Lambda(Q).
\]
(2.4)

Set $\tau = \tau^{-1}$. It is well-known that $\tau^{h_Q} = [2]$ and $S = [1] \circ \tau$ is the Serre functor. The following observation is due to L. Hille.

**Lemma 2.7.** Under the bijection in (2.4), the projection of the root system on the Coxeter plane gives the central charge of the Gepner point $\sigma_G$ of $\mathcal{D}_\infty(Q)$.

**Proof.** As the Coxeter element $w$ acts by rotation by $2\pi/h_Q$, the roots with the same length spread evenly in $P_w$. This matches the Gepner equation (1.5) and implies the lemma. See the top pictures in Figures 14 and 15. \(\square\)

3. **Prototype of ToSt**

In the proofs, we only consider the quiver/simply-laced/ADE case for simplicity. But all statements work for the specie/BCFG cases too (via folding).

3.1. **The $h_Q$-gons induced by $\tau$-orbits.**

**Lemma 3.1.** $\text{id} - \tau$ (or $\text{id} + S$) is a non-singular linear map on $\mathbb{R}^n = K \mathcal{D}_\infty(Q) \otimes \mathbb{R}$.

**Proof.** Taking the basis $\{\dim P_i \mid i \in Q_0\}$ of $\mathbb{R}^n$ for $P_i$ being the projectives of $Q$. As $\tau(P_i) = I_i[-1]$ for $I_i$ being the injectives, the matrix presentation $C(Q)$ of $\tau$, as a linear transformation on $K \mathcal{D}_\infty(Q)$ is the so-called Coxeter transformation of $Q$. It is well-known that the eigenvalues of the Coxeter transformation is not 1 for Dynkin type (cf. [L]). Thus the lemma follows. \(\square\)

For any $M$ in $\text{Ind} \mathcal{D}_\infty(Q)/[2]$, define
\[
g(M) : = \sum_{j=1}^{h_Q} [\tau^j M],
\]
to be the sum of all objects in the $\tau$-orbit of $M$ in $KD_\infty(Q)$.

**Proposition 3.2.** The function $g \equiv 0$.

**Proof.** Note that $\tau$ is a group automorphism of $KD_\infty(Q)$. Then $\tau^{h_Q} = [-2] = \text{id}$ implies

$$(\text{id} - \tau)(\text{id} + \tau + \cdots \tau^{h_Q - 1}) = 0.$$ 

Then Lemma 3.1 implies the proposition.

An alternative categorical proof is as follows, which fits better in our content. We claim that any $g(P_i), i \in Q_0$ are proportional. Then

$$\sum_{i \in Q_0} g(P_i) = \sum_{M \in \text{Ind} D_\infty(Q)/[2]} [M] = \sum_{M \in \text{Ind} h_Q} \left([M] + [M[1]]\right) = 0$$

implies the proposition. For the claim, let $i$ be a leaf of $Q$ (e.g. the boundary vertex, cf. Definition 2.4) and $j$ be its neighbour vertex. Summing all the mesh relations in $KD_\infty(Q)$ at the $\tau$-orbit of $P_i$, which are triangle relations, we have

$$2g(P_i) = g(P_j).$$

If $j$ is not the tri-valent vertex in (2.3), let $k$ be its neighbour vertex other than $i$. Summing all the mesh relation in $KD_\infty(Q)$ at the $\tau$-orbit of $P_i$, which are square relations, we have

$$2g(P_j) = g(P_i) + g(P_k)$$

and hence $g(P_k) = 3g(P_i)$. Do all such calculations, we see the claim holds. \qed

Let $Z$ be any central charge and denote by $\tau = \tau^{-1}$. Then we have the following fact:

**Corollary 3.3.** For any $M \in \text{Ind} D_\infty(Q)$, the vectors $z_1 = Z(\tau M), z_2 = Z(\tau^2 M), \ldots, z_{h_Q} = Z(\tau^{h_Q} M)$ form an $h_Q$-gon.

**Definition 3.4.** The $h_Q$-gon (w.r.t. to any given central charge $Z$) induced by a boundary/?-end $\tau$-orbit (of $\text{AR} D_\infty(Q)/[2]$) is called the boundary/?-end $h_Q$-gon, for ? being short/middle/far.

Once we choose our favorite far-end vertex/\tau-orbit, we will have the corresponding far-end $h_Q$-gon. It will not make much difference in type $A_n$ and $E_6$. But as we will explain in Section 4.4, the choice matters in type $D_4$.

### 3.2. Convexity as stability for $h_Q$-gon and global dimension function.

**Proposition 3.5.** If $\sigma = (Z, P) \in \text{ToSt}(Q)$, then, for any $M \in \text{Ind} D_\infty(Q)$, the vectors $Z(\tau M), Z(\tau^2 M), \ldots, Z(\tau^{h_Q} M)$ form a convex $h_Q$-gon.

**Proof.** In the AR quiver $D_\infty(Q)$, there are paths from $\tau^i M$ to $\tau^{i+1} M$. As every indecomposable object is stable, we have

$$\phi_\sigma(M) < \phi_\sigma(\tau M) < \cdots < \phi_\sigma(\tau^{h_Q - 1} M) < \phi_\sigma(\tau^{h_Q} M) = \phi_\sigma(M[2]) = \phi_\sigma(M) + 2.$$

Let $z_i = Z(\tau^i M), \forall i \in Z_{h_Q}$. Then arg $z_0 = \phi_\sigma(M) \cdot \pi$. So we have

$$\text{arg } z_0 < \text{arg } z_1 < \cdots < \text{arg } z_{h_Q - 1} < \text{arg } z_{h_Q} + 2\pi = \text{arg } z_0 + 2\pi.$$
where arg takes values in \([\phi_\sigma(M) \cdot \pi, \phi_\sigma(M) \cdot \pi + 2\pi]\). It implies the \(h_Q\)-gon with edges \(z_i\) is indeed convex.

Up to translation, each \(\tau\)-orbit induces one convex \(h_Q\)-gon for a chosen \(\sigma \in \text{ToSt}(Q)\). We call all such \(h_Q\)-gons the \(\sigma\)-induced convex \(h_Q\)-gons.

A direct consequence is the following characterization of \(\text{gldim}\) on \(\text{ToSt}(Q)\), which generalizes [Q2, Prop. 3.6] for type \(A_n\) to all Dynkin cases.

**Theorem 3.6.** If \(\sigma \in \text{ToSt}(Q)\), then \(\text{gldim} \sigma\) equals \(\pi \cdot \max\{\phi(\tau(E[1])) - \phi(E) | E \in \text{Ind} \text{D}_\infty(Q)\}\).

**Proof.** By AR duality, we have \(\text{Hom}(E, F) = D\text{Hom}(F, \tau(E[1]))\). As any indecomposable is stable, we have

\[
\text{gldim} \sigma = \max\{\phi(\tau(E[1])) - \phi(E) | E \in \text{Ind} \text{D}_\infty(Q)\}
\]

by Hammock property. Thus the statement follows from that \(\pi \cdot (\phi(\tau(E[1])) - \phi(E))\) is indeed an interior angle of the \(h_Q\)-gons induced by the \(\tau\)-orbit containing \(E\) (with respect to \(\sigma\)). It follows \(\Box\)

3.3. **ToSt of type \(A_n\).** Although the orientation of the quiver/species does not matter when considering \(\text{ToSt}(Q)\), we will choose our favorite one for convenience (e.g. to describe objects).

**Sep-up 3.7.** Take the \(A_n\) quiver with straight orientation

\[
1 \longleftrightarrow 2 \longleftrightarrow \cdots \longleftrightarrow n
\]

(3.1)

We declare vertex 1 as the fixed far-end vertex and label the indecomposable objects in \(\text{D}_\infty(A_n)\) by

\[
P^j_i := \tau^j P_i, \quad j \in \mathbb{Z}, 1 \leq i \leq n,
\]

(3.2)

where each \(P_i\) is the projective corresponding to vertex \(i\). The AR quiver of \(\text{D}_\infty(A_n)/[2]\) is illustrated as in Figure 2, where \(j(j + i)\) denotes \(P^j_i\) and \((j + i)j\) denotes \(P^j_i[1]\) for \(1 \leq i \leq n\) and \(0 \leq j \leq n - i\).

Recall that \(\text{Sth}(A_n)\) denotes the moduli space of stable \((n + 1)\)-gons on \(\mathbb{R}^2 = \mathbb{C}\).

**Proposition 3.8.** [Q2, Prop. 3.6] There is a natural isomorphism

\[
Z_h: \text{ToSt}(A_n)/\mathbb{C} \rightarrow \text{Sth}(A_n)
\]

sending a Tost \(\sigma\) to the far-end \((n + 1)\)-gon.

More precisely, we have \(z_j = Z(P^j_i - 1)\) for \(j \in \mathbb{Z}_{n+1}\) for the edges of the far-end \((n + 1)\)-gon and

\[
Z(P^j_i) = V_j V_{j+i}, \quad 1 \leq i \leq n, 0 \leq j \leq n - i.
\]

A study of totaly stability for the module category \(\mathcal{H}\) of type \(A\) is in [Ki], which can be deduced from the description above after fixing the heart of a ToSt to be \(\mathcal{H}\).
Figure 2. AR quiver of $\mathcal{D}_\infty(A_n)$

Figure 3. The 6-gon of type $A_5$

4. ToSt of type $D_n$

**Sep-up 4.1.** Set $m = n - 1$ and then $h_Q = 2m$ for $Q = D_n$. We choose an orientation as follows:

1 $\leftarrow$ 2 $\leftarrow$ \ldots $\leftarrow$ $n - 2$ $\leftarrow$ $m$

Moreover, we fix 1 as our favorite far-end vertex for $n = 4$ and it is the unique far-end vertex for $n \geq 5$. Label the indecomposable objects in the $\mathcal{D}_\infty(D_n)$ still by (3.2). Thus the $i^{th}$-$\tau$-orbit consisting of objects $\{P^j_i\}_{j \in \mathbb{Z}}$ is the one corresponding to vertex $i$.

As $\tau^{2m} = [2]$, the indecomposables in the root category are $\{P^j_i \mid 1 \leq i \leq n, j \in \mathbb{Z}_{2m} \}$. For instance, see the AR quiver $\text{AR} \mathcal{D}_\infty(D_5)/[2]$ in the low picture of Figure 4, where
$j(j + i)$ denotes $P^j_i$, $j \varrho(j)$ denotes $P^j_m$ and $j \varrho(j + 1)$ denotes $P^j_n$. Here

$$\varrho(x) = \text{sign}(-1)^x : \mathbb{Z} \to \{\pm\}$$

is the parity function.

Figure 4. The 8-gon of type $D_5$ and AR quiver of $D_{\infty}(D_5)$.

### 4.1. Centrally symmetric doubly punctured $h$-gons.

**Definition 4.2.** A centrally symmetric $h$-gon $V$ is an $h$-gon (cf. notations in Section 1.3) such that $h$ is even and

$$z_{j + h/2} = -z_j. \quad (4.1)$$

Let $O$ be its geometric center. It is doubly punctured if there is a pair of punctures $B_{\pm}$ such that $O$ is the middle point of them.
Given a central charge $Z : K D_{\infty}(D_n) \to \mathbb{C}$, we construct a centrally symmetric doubly punctured $2m$-gon $(V_Z, B_{\pm})$ as follows:

$$
\begin{cases}
V_j = -\left( \sum_{t=j}^{j+m-1} Z(P_t) \right)/2, \\
B_+ = (Z(P_m) - Z(P_n))/2 \\
B_- = (Z(P_n) - Z(P_m))/2.
\end{cases}
$$

(4.2)

So $z_j = V_j - V_j = Z(P_{j+1})$ noticing $Z(P_{j+m-1}) = -Z(P_{j-1})$. More precisely,

- $V = V_Z$ is the far-end $2m$-gon, which is indeed centrally symmetric since
  $$
  \tau^m = [1]
  $$
  holds on the far-end $\tau$-orbit $\{P_t | t \in \mathbb{Z}_{2m}\}$ that implies (4.1).
- Moreover, the geometric center of $V$ is at origin as
  $$
  \begin{cases}
  V_j = -V_{j+m}, & j \in \mathbb{Z}_{2m} \\
  B_+ = -B_-.
  \end{cases}
  $$

(4.3)

**Definition 4.3.** An $h$-gon of type $D_n$ is a centrally symmetric doubly punctured $2m$-gon. It is stable (of type $D_n$) if it is convex and the punctures are inside the level-$(n-2)$ diagonal-gon. Denote by $\text{Sth}(D_n)$ the moduli space of stable $2m$-gon of type $D_n$ up to similarity, which has complex dimension $m = n - 1$.

Next we show that total stability implies the stability of $2m$-gon of type $D_n$.

**Proposition 4.4.** If $Z$ is the central charge of some $\sigma \in \text{ToSt}(D_n)$, then the centrally symmetric doubly punctured $2m$-gon $(V, B_{\pm})$ defined above is a stable $2m$-gon of type $D_n$. We call it the far-end stable $2m$-gon with respect to $\sigma$.

**Proof.** We need to show that the total stability of $\sigma$ implies the stability of the corresponding far-end $h$-gon $V$ together with punctures $B_{\pm}$.

Firstly, the convexity $V$ follows from Proposition 3.5.

Secondly, we will check that $B_{\pm}$ is inside the level-$(m-1)$ diagonal-gon. Namely, for any $j \in \mathbb{Z}_{2m}$, it suffices to show that $B_{\pm}$ is on the left hand side of $V_{j+1}V_{j+m}$:
Note that we have AR triangles \((i \in \{m, n\})\)

\[ P_i \to P_{i+1} \to P_{i-1} \to P_i[1] \]

with central charges

\[ V_j B_{\pm} = \begin{cases} V_j, & i = m \text{ and } \pm = \varrho(j) \text{ if } i = m \text{ and } \pm = \varrho(j+1) \text{ if } i = n \end{cases} \]

Here \(\pm = \varrho(j)\) if \(i = m\) and \(\pm = \varrho(j+1)\) if \(i = n\) to be more precise. Since all \(P_i\) are stable objects, their phases are increasing. Therefore we have

\[ \arg B_{\pm} V_j < \arg V_{j+1} V_j < \arg B_{\pm} V_{j+m} + \pi, \]

where \(\arg\) takes values in \([\phi(\varrho(P_i)) \cdot \pi, \phi(\varrho(P_i)) \cdot \pi + 2\pi]\). This completes the proof. \(\Box\)

4.2. Geometric model for root category of type \(D_n\). Suppose that we have a \(2m\)-gon \((V, B_{\pm})\) of type \(D_n\) (with vertices \(V_j\) and punctures \(B_{\pm}\)). Up to translation, we may assume its geometric center is at origin, i.e. \((4.3)\) holds. Then we have

\[ \begin{align*}
V_j V_k &= V_{k+m} V_j, \\
V_j B_+ &= B_- V_{j+m}.
\end{align*} \]

(4.4)

**Theorem 4.5.** A \(2m\)-gon of type \(D_n\) is a geometric model for the root category \(\mathcal{D}_\infty(D_n)/[2]\) in the sense that by setting

\[ \begin{align*}
Z(P_j^i) &= V_j V_{j+i}, & j & \in \mathbb{Z}_{2m}, 1 \leq i \leq m - 1, \\
Z(P_j^m) &= V_j B_{\varrho(j)}, & j & \in \mathbb{Z}_{2m}, \\
Z(P_j^n) &= V_j B_{\varrho(j+1)}, & j & \in \mathbb{Z}_{2m},
\end{align*} \]

(4.5)

we obtain a central charge \(Z : K \mathcal{D}_\infty(D_n) \to \mathbb{C}\).

**Proof.** To show that \(Z\) is a group homomorphism, we only need to check all the mesh relation in the root category still holds after applying \(Z\). To start with, notice that

\[ Z(P_j^i) + Z(P_j^i) = V_j V_{j+m} \]

(4.6) by \((4.4)\). The rest of the proof is just a direct checking:

1°. At the \(\tau\)-orbit of \(i = 1\) the mesh relation has the form

\[ [P_j^i] + [P_j^{i+1}] = [P_j^i] \]

and indeed we have

\[ V_j V_{j+1} + V_{j+1} V_{j+2} = V_j V_{j+2}. \]

2°. At the \(\tau\)-orbit of \(1 < i < n - 2\) the mesh relation has the form

\[ [P_i^j] + [P_i^{j+1}] = [P_i^{j+1}] + [P_i^j] \]

and indeed we have

\[ V_j V_{j+i} + V_{j+i} V_{j+i+1} = V_{j+i} V_{j+i} + V_j V_{j+i+1}. \]
At the $\tau$-orbit of $i = n - 2$ the mesh relation has the form
\[ [P_{n-2}^j] + [P_{n-3}^{j+1}] = [P_{n-1}^j] + [P_{n-1}^j] \]
and indeed we have, using (4.6),
\[ V_j V_{j+n-2} + V_{j+1} V_{j+n-1} = V_{j+1} V_{j+n-2} + V_j V_{j+n-1}. \]

At the $\tau$-orbit of $i \in \{m, n\}$, the mesh relation has the form
\[ [P_i^j] + [P_i^{j+1}] = [P_{n-2}^j] \]
and indeed we have, using (4.4),
\[ V_j B_{\pm} + V_{j+1} B_{\mp} = B_{\pm} V_j V_{j+m} + V_{j+1} B_{\mp} = V_{j+1} V_{j+m}. \]

Note that here the sign $\pm$ depends on $i \in \{m, n\}$ and the parity function $\rho$.

Note that all these calculations can be easily checked in Figure 4 for case $n = 5$. □

Example 4.6 (Type $Q = D_5$). In Figure 4, we have

- The objects drawn in blue/violet/green circle, in the $\tau$-orbit of $ARD_\infty(Q)$, correspond to length-1/2/3 diagonals drawn in blue/violet/green respectively as in the upper 8-gons.
- The objects drawn in orange circle, in the upper/lower $\tau$-orbit of $ARD_\infty(Q)$, correspond to solid/dashed orange line segments respectively as in the right upper 8-gon.

4.3. From stable $h$-gon to ToSt of type $D_n$. Given a stable $2m$-gon $V$ of type $D_n$, we can construct a ToSt $\sigma = (Z, P)$ with $Z$ defined as in Theorem 4.5.

Construction 4.7. Let us construct a slicing $P$ as follows.

Step 1: Assign a real number $\phi(M)$ for each object $M$ in the far-end $\tau$-orbit of $ARD_\infty(D_n)$:

- Let
  \[ \phi(P_i^j[k]) = \text{arg} Z(P_i^j)/\pi + k, \]
  for any $k \in \mathbb{Z}$, where arg takes values in $[0, 2\pi]$.
- For $1 \leq j \leq m - 1$ and $k \in \mathbb{Z}$, let
  \[ \phi(P_i^j[k]) = \text{arg} Z(P_i^j)/\pi + k, \]
  where arg takes values in $[\text{arg} Z(P_i), Z(P_i) + 2\pi)$.

So we have the monotonicity and periodicity:
\[
\begin{cases}
\phi(P_i^j) < \phi(P_i^{j+1}), \\
\phi(P_i^{j+m}) = \phi(P_i^j[1]) = \phi(P_i^j) + 1,
\end{cases} \quad \forall j \in \mathbb{Z}. \tag{4.7}
\]

Step 2: For any $2 \leq i \leq n - 2 < m$ and $j \in \mathbb{Z}$, we have
\[ Z(P_i^j) = V_j V_{j+i} = \sum_{t=j}^{j+i-1} V_t V_{t+1} = Z(P_i^j) + \cdots Z(P_i^{j+i-1}). \]
Note that by Step 1, the convexity of $V$ gives
$$\phi(P^j_1) < \phi(P^j_1+1) < \cdots < \phi(P^j_{1+i-1}) < \phi(P^j_{1+m}) = \phi(P^j_1[1]) = \phi(P^j_1) + 1$$
Then let
$$\phi(P^j_i[k]) = \arg Z(P^j_i)/\pi + k, \hspace{1cm} (4.8)$$
where $\arg$ takes values in $[\phi(P^j_1) \cdot \pi, \phi(P^j_1) \cdot \pi + 2\pi]$. It is straightforward to check that the monotonicity and periodicity of (4.7) are inherited:
$$\phi(P^j_i) < \phi(P^j_{i+1}) < \phi(P^j_{i+m}) = \phi(P^j_i[1]) = \phi(P^j_i) + 1, \hspace{1cm} \forall j \in \mathbb{Z}.$$ Note that, locally, the additive subcategory of $\text{AR}\mathcal{D}_\infty(D_n)$ generated by
$$\begin{cases} P^j_1, \ldots, P^j_{1+i-2}, P^j_{1+i-1} \\ P^j_2, \ldots, P^j_{2+i-2} \\ \vdots \\ P^j_i \end{cases} \hspace{1cm} (4.9)$$
is isomorphic to $\text{mod} kA_i$ for an $A_i$ quiver with straight orientation. Inductively, the convexity implies that
$$\begin{cases} \phi(P^s_t) < \phi(P^s_{t+1}) < \phi(P^s_{t+i-1}) & 1 < t \leq i, j \leq s < j + i - t \\ \phi(P^s_t) < \phi(P^s_{t+i-1}) < \phi(P^s_{t+i}) & 1 \leq t \leq i, j \leq s < j + i - t. \end{cases} \hspace{1cm} (4.10)$$

**Step 3:** For $i \in \{m,n\}$ and $j \in \mathbb{Z}$, let
$$\phi(P^j_i) = \arg Z(P^j_i)/\pi, \hspace{1cm} (4.11)$$
where $\arg$ takes value in $[\phi(P^j_{n-2}) \cdot \pi, \phi(P^j_{n-2}) \cdot \pi + 2\pi]$. Note that
$$\phi(P^j_{n-2} + m) = \phi(P^j_{n-2}[1]) = \phi(P^j_{n-2}) + 1,$$ and
$$\begin{cases} \phi(P^j_m + m) = \phi(P^j_m[1]) = \phi(P^j_m) + 1, \\ \phi(P^j_{n-1} + m) = \phi(P^j_{n-1}[1]) = \phi(P^j_{n-1}) + 1, \end{cases}$$
if $m$ is even and
$$\begin{cases} \phi(P^j_m + m) = \phi(P^j_m[1]) = \phi(P^j_m) + 1, \\ \phi(P^j_{n-1} + m) = \phi(P^j_{n-1}[1]) = \phi(P^j_{n-1}) + 1, \end{cases}$$
if $m$ is odd. This completes the assigning $\phi$.

**Step 4:** Define
$$\mathcal{P}(\varphi) = \text{Add} \bigoplus_{P \in \text{Ind } \mathcal{D}_\infty(D_n) \atop \phi(P) = \varphi} P \hspace{1cm} (4.12)$$

**Proposition 4.8.** $\sigma = (Z, \mathcal{P})$ defined as above is a ToSt on $\mathcal{D}_\infty(D_n)$.

**Proof.** By construction, we already have...
• all indecomposable objects are in some $\mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$ and
• $\mathcal{P}$ is compatible with the central charge $Z$ as well as the shift $[1]$.

Thus, what is left to show is $\phi(M) < \phi(L)$ whenever there is a non-zero map from $M$ to $L$ for any indecomposable objects $M, L$. This amounts to check that, for any arrow $M \to L$ in $\text{AR} \mathcal{D}_\infty(D_n)$, we have $\phi(M) < \phi(L)$. There are three cases:

• If the arrow is between the far-end $\tau$-orbit (corresponding to vertex 1) and the double-trivalent $\tau$-orbit (corresponding to vertex $n - 2$), then it is a type $A_n$ issue, cf. (4.9), and (4.10) gives the inequality.
• If the arrow is from the double-trivalent $\tau$-orbit to the boundary $\tau$-orbits (corresponding to vertex $m$ or $n$), then (4.11) implies the inequality.
• If the arrow is from the boundary $\tau$-orbits (corresponding to vertex $m$ or $n$) to the double-trivalent $\tau$-orbit, then we need to examine the triangle

Then, from the stability condition of $(\mathbf{V}, B_\pm)$, we know that $B_\pm$ is bounded by solid lines in the above picture. By (4.8) and (4.11), we deduce that both $\phi(P_{i}^{j})$, $i \in \{m, n\}$, and $\phi(P_{n-2}^{j+1})$ are in $[\phi(P_{n-2}^{j}), \phi(P_{n-2}^{j}) + 2)$. Thus we have

$$\phi(P_{n-2}^{j}) = \frac{\arg V_j V_j + V_{j+m-1}}{\pi}$$

$$< \frac{\arg V_j B_\pm}{\pi} = \phi(P_{i}^{j})$$

$$< \frac{\arg V_j + V_{j+m}}{\pi} = \phi(P_{n-2}^{j+1})$$

where $\pm$ in the second row depends on $i \in \{m, n\}$ and the parity function $\rho$. Note that $	ext{arg}$ takes values in $[\phi(P_{n-2}^{j}) \cdot \pi, \phi(P_{n-2}^{j}) \cdot \pi + 2\pi)$. 

In all, the first case uses convexity of $V$ and the last two cases uses the extra condition of stability of $(V, B_{\pm})$. □

**Remark 4.9.** Alternatively, one can define $\mathcal{P}$ as follows:

- For any $1 \leq i \leq n$, let
  \[ \phi(P_i) = \arg Z(P_i)/\pi, \]
  where arg takes value in $[\arg Z(P_1), Z(P_1) + 2\pi)$.

- For fixed $1 \leq i \leq n$, let
  \[ \phi(P^j_i) = \arg Z(P^j_i)/\pi \]
  for $1 \leq j \leq m$, where arg takes values in $[\arg Z(P_i), Z(P_i) + 2\pi)$.

- Define $\mathcal{P}$ as in (4.12).

But the simplicity of this construction will trade off more complexity of the proof above.

**Theorem 4.10.** There is a natural isomorphism

\[ Z_h: \text{ToSt}(D_n)/\mathbb{C} \rightarrow \text{Sth}(D_n), \]

sending a ToSt $\sigma$ to the far-end stable $2m$-gon.

**Proof.** It follows from combining Propositions 4.4 and 4.8, noticing that the $\mathbb{C}$-action on ToSt$(D_n)$ and the similarity equivalence on the moduli space Sth$(D_n)$ neutralize each other. □

---

**Figure 5.** Three far-end 6-gons of type $D_4$ for a given central charge
4.4. Example: $D_4$ with three far-end stable 6-gons. As mentioned above couple of times, there are three choices of far-end vertices/$\tau$-orbits/6-gons. The isomorphism in Theorem 4.10 depends on such a choice. Here, we give an example, as shown in Figure 5, of how the three (far-end) stable 6-gons look like (3 hexagons on the left), with respect to a fixed $\sigma \in \text{ToSt}(D_4)$ (whose central charge is on the right).

5. $\text{ToSt}$ of the exceptional type $E_6$

5.1. The $h$-gon of type $E_6$.

Definition 5.1. An $h$-gon $V$ of type $E_6$ is a 12-gon satisfying (1.1).

A direct calculation shows that:

Lemma 5.2. The set of 7 equations (1.1) has rank 6. So the space of 12-gons of type $E_6$ has complex dimension $12 - 6 = 6$.

By (1.1), we have the following hexagon relations.

$$z_j + z_{j+1} + z_{j+2} + z_{j+6} + z_{j+7} + z_{j+8} = 0, \quad \forall j \in \mathbb{Z}_{12}. \quad (5.1)$$

Construction 5.3. Using the triangle relations in (1.1), we can draw triangles

$$T_j: = V_{j-1}V_jW_{j-1} \quad \forall j \in \mathbb{Z}_{12}$$

with edges

$$V_{j-1}V_j = z_j, \quad V_jW_{j-1} = z_{j+4}, \quad W_{j-1}V_{j-1} = z_{j+8}.$$

Note that we have 4 sets of 3 congruent triangles

$$\{T_{j+4k} \mid k = 0, 1, 2\} \quad \text{for } j \in \mathbb{Z}_4,$$

drawn in orange/blue/green/violet respectively. Moreover, the square relations in (1.1) correspond to the squares

$$S_j: = V_{j-1}V_jW_jW_{j-2} \quad \forall j \in \mathbb{Z}_{12}$$

with edges

$$V_{j-1}V_j = z_j, \quad V_jW_j = -z_{j-3}, \quad W_jW_{j-2} = z_{j-6}, \quad W_{j-2}V_{j-1} = -z_{j-9}.$$

Note that we have 3 sets of 4 congruent squares

$$\{S_{j+3k} \mid k = 0, 1, 2, 3\} \quad \text{for } j \in \mathbb{Z}_3,$$

see the squares in Figure 7.

Thus it is better to have two copies of $V$ (called ice and fire, cf. Figure 7):

$V_{\text{ice}}$: the one with 6 triangles $\{T_{2j} \mid j \in \mathbb{Z}_6\}$ and 6 squares $\{S_{2j+1} \mid j \in \mathbb{Z}_6\}$. Then we have a hexagon ice core $V_{\text{ice}}^\oplus$ of $V$ with vertices $W_{2j+1}$.

$V_{\text{fire}}$: the one with 6 triangles $\{T_{2j+1} \mid j \in \mathbb{Z}_6\}$ and 6 squares $\{S_{2j} \mid j \in \mathbb{Z}_6\}$. Then we have a hexagon fire core $V_{\text{fire}}^\oplus$ of $V$ with vertices $W_{2j}$.

Remark 5.4. An interesting feature is that both ice and fire configurations induce planar tiling patterns, see Figure 17 in Appendix A. In other words, a 12-gon of type $E_6$ is equivalent to ‘A Tiling of Ice of Fire’ as shown there.
5.2. Categorical set up.

Figure 6. The 12-gon of type $E_6$ and AR quiver of $D_\infty(E_6)$.
**Sep-up 5.5.** The lower picture in Figure 6 is part of AR $\mathcal{D}_\infty(E_6)$, which is in fact the AR quiver of $\text{Ind mod } kE_6$ for the $E_6$ quiver with alternative orientation:

\[
\begin{array}{cccc}
3 \\
\downarrow \\
1 & \leftrightarrow & 2 & \rightarrow & 4 & \leftrightarrow & 5 & \rightarrow & 6
\end{array}
\] (5.2)

We still label the indecomposable objects in the $\mathcal{D}_\infty(E_6)$ by (3.2).

Fix 6 to be the far-end vertex so that 1 is the middle-end vertex. Note that 3 is the short-end vertex. The labelling of the AR quiver $\text{AR mod } kE_6$ in Figure 6, is given by:

- **1st-$\tau$-orbit:** $j+6/j+5$ denotes $C_j := P_{1}^{j-1}$ (drawn in green). This is the mid-end $\tau$-orbit.
- **2nd-$\tau$-orbit:** $j+7/j+5$ denotes $P_{2}^{j-1}$.
- **3rd-$\tau$-orbit:** Let $M_j := P_{3}^{j-1}$ (drawn in yellow). This is the short-end $\tau$-orbit.
- **4th-$\tau$-orbit:** $j-2/j+1$ denotes $P_{4}^{j-1}$.
- **5th-$\tau$-orbit:** $j-1/j+1$ denotes $P_{5}^{j-1}$.
- **6th-$\tau$-orbit:** $j-1/j$ denotes $B_j := P_{6}^{j-1}$ (drawn in blue). This is the far-end $\tau$-orbit.

Given any central charge $Z: \mathcal{D}_\infty(E_6) \to \mathbb{C}$. Let $V_Z$ be the far-end 12-gon of $\mathcal{D}_\infty(E_6)$ with edges $z_j = Z(B_j)$ for $1 \leq j \leq 12$.

**Lemma 5.6.** The set $\{ [B_j] \mid 1 \leq j \leq 6 \}$ is a basis of $\mathcal{K} \mathcal{D}_\infty(E_6)$. Moreover, $V_Z$ is a 12-gon of type $E_6$ and its ice/fire core is formed by (the odd/even) half of

\[ \{ w_j := Z(C_j) = -z_{j+6} \mid 1 \leq j \leq 12 \} \]

**Proof.** The first statement can be checked directly.

\[ \text{Figure 7. Configurations Ice and Fire} \]
Now we will show that all $z_j$ satisfy (1.1). On one hand, we have triangles in $D_\infty(E_6)$:

$$B_j \to C_{j+2} \to B_{j+4} \to B_j[1],$$  

(5.3)

which implies that

$$[B_j] + [B_{j+4}] = [C_{j+2}].$$

Due to $C_{j+2} = B_{j+8}[1]$, we have

$$[B_j] + [B_{j+4}] + [B_{j+8}] = 0.$$ 

Thus the triangle relations in (1.1) holds. On the other hand, we have triangles in $D_\infty(E_6)$:

$$\begin{cases}
B_j \to M_{j+1} \to C_{j+3} \to B_j[1], \\
C_j \to M_{j+1} \to B_{j+3} \to C_j[1].
\end{cases}$$  

(5.4)

Then we have

$$[B_j] + [C_{j+3}] = [M_{j+1}] = [C_j] + [B_{j+3}].$$

Noticing that $B_j = C_{j+6}[1]$ and $B_{j+3} = B_{j-9}$, the above equation becomes

$$[B_j] - [B_{j-3}] = -[B_{j-6}] + [B_{j-9}],$$

again using. Thus the square relations in (1.1) holds.

In particular, we see that the edges of the ice/fire core are also $w_j$’s. □

![Figure 8. The triangles/squares for $E_6$](image_url)
5.3. Geometric model for the root category of type $E_6$. In this subsection, we will construct a central charge from a 12-gon $\mathbf{V}$ of type $E_6$. Recall that $\mathbf{V}$ has edges $z_j = V_{j-1}V_j$, $j \in \mathbb{Z}_{12}$, as shown in the upper picture of Figure 6. Moreover, we have the associated ice/fire configuration/core according to Construction 5.3.

Theorem 5.7. A 12-gon $\mathbf{V}$ of type $E_6$ is a geometric model for the root category $\mathcal{D}_\infty(E_6)/[2]$ in the sense that we have a central charge defined as follows:

- central charges of objects in the 1st-$\tau$-orbits are given by edges of the ice and fire cores of $\mathbf{V}$, i.e., $Z(C_j) = Z(P_1^{j-1}) = W_{j-2}W_j$.
- central charges of objects in the 2nd-$\tau$-orbits are given by $Z(P_2^{j-1}) = V_{j-2}W_j$.
- central charges of objects in the 3rd-$\tau$-orbits are given by $Z(M_j) = Z(P_3^{j-1}) = V_{j-2}W_{j-1}$.
- central charges of objects in the $i$th-$\tau$-orbits are given by length-$\left(7-i\right)$ diagonals of $\mathbf{V}$, for $i = 4, 5, 6$. In particular, $Z(B_j) = Z(P_6^{j-1}) = V_{j-1}V_j$.

Remark 5.8. In fact, there are several ways to realize the indecomposable objects in the $\tau$-orbits. For instance, for any object $M \in \text{Ind mod} kE_6$ that is not in the 3rd-$\tau$-orbit, it admits a labeling $j/k$ as in Set-up 5.5, which is also the labelling corresponding to its image under the central charge, namely $Z(M) = V_jV_k$ and $Z(M[1]) = V_kV_j$;

for each $M_j$ in the 3rd-$\tau$-orbits, it can be realized in the following ways:

$$W_{j+3}V_{j+4} = V_{j-2}W_{j-1} = W_jV_{j+2} = V_{j+8}W_{j+6}. \quad (5.5)$$

See the thicken black line segments in Figure 8 for $j = 2$.

In type $E_6$, the far-end and middle-end $\tau$-orbits are shift of each other which enable us to realize the objects in many ways. However, in types $E_7$ and $E_8$, this is not the case. Therefore, we write the statement in the above theorem of the form that can be generalized to types $E_7$ and $E_8$.

Proof of Theorem 5.7. The proof follows the same way as in the proof of Theorem 4.5. One has to show that the central charge $Z$ preserves the mesh relations of AR $\mathcal{D}_\infty(E_6)/[2]$. Here we only point out the reason that $Z(M_j)$ can be realized as in (5.5) is due to the triangles in (5.4) and Construction 5.3. The rest finite calculations are left to the readers. \hfill \Box

5.4. Stability of 12-gon for $E_6$. Suppose that $\mathbf{V}$ is a convex 12-gon of type $E_6$. We have the following observations.

- Its ice/fire core is also convex. This follows from the fact that the edges of ice/fire core are exactly half of the edges of the 12-gon induced by the middle-end $\tau$-orbit (which is just $-\mathbf{V}$).
- Any length-3 diagonal $V_jV_{j+3}$ is a long diagonal of a fat hexagon

$$\mathbf{H}_j = V_{j-1}V_jV_{j+1}V_{j+2}W_{j+1}W_{j-1}$$

that corresponds to (5.1). Note that $\mathbf{H}_j$ and $\mathbf{H}_{j+6}$ are related by a translation $V_{j+2}V_{j+5}$. Hence they inherit the convexity of $\mathbf{V}$.
• See the left picture in Figure 7, the configuration ice contains 6 fat hexagons \( \{H_{2j}\} \) and the corresponding length-3 diagonals (blue dashed lines in the left picture) bound another hexagon, we call the ice-boundary, which contains the ice core automatically due to the convexity of the fat hexagons.

• Dually, see the right picture in Figure 7, the configuration fire contains 6 fat hexagons \( \{H_{2j+1}\} \) and the corresponding length-3 diagonals (red dashed lines in the right picture) bound another hexagon, we call the fire-boundary, which also contains the fire core automatically.

**Definition 5.9.** A 12-gon \( \mathbf{V} \) of type \( E_6 \) is stable if it is convex and both its ice and fire cores are inside the level-3 diagonal-gon.

Denote by \( \text{Sth}(E_6) \) the moduli space of stable 12-gon of type \( E_6 \) up to similarity. By Lemma 5.2, the complex dimension of \( \text{Sth}(E_6) \) is \( 6 - 1 = 5 \).

Using the above observations, one can simplify the stability condition of 12-gons a bit as in the lemma below.

**Lemma 5.10.** A convex 12-gon \( \mathbf{V} \) of type \( E_6 \) is stable if and only if its ice/fire core is inside its fire/ice boundary respectively.

**Proof.** As we know, the convexity of \( \mathbf{V} \) implies the convexity of the fat hexagons. Then, the ice/fire core is automatically inside ice/fire boundary respectively. Moreover, the level-3 diagonal-gon is the intersection of the ice and fire boundaries. Thus, the stability condition of \( \mathbf{V} \) can be simplified to the condition stated in the lemma, provided the convexity of \( \mathbf{V} \).

Now we proceed to show that total stability of \( \sigma \) deduces stability of its far-end 12-gon.

**Proposition 5.11.** If \( \sigma = (Z, P) \in \text{ToSt}(E_6) \), then its far-end 12-gon is a stable 12-gon of type \( E_6 \).

**Proof.** By Proposition 3.5, the far-end 12-gon \( \mathbf{V} \) is convex. Since the equation (1.1) holds due to Lemma 5.6, we can follow Construction 5.3 to obtain its ice/fire core with vertices \( W_j, j \in \mathbb{Z}_{12} \) via triangles (cf. Figure 6). Then we only need to show that each \( W_j \) are bounded by length-3-diagonals of \( \mathbf{V} \). By Lemma 5.10, it suffices to check that \( W_{2j+1} \) is bounded by the fire boundary and \( W_{2j} \) is bounded by the ice boundary. By the observation of the fat hexagons, it is equivalent to prove that each \( W_j \) is on the left side of the diagonal \( V_{j-1}V_{j+2} \).

Consider the triangle \( V_{j-1}W_jV_{j+2} \) with edges

\[
V_{j-1}W_j = Z(P_3^j), \quad W_jV_{j+2} = Z(P_3^{j-1}), \quad V_{j-1}V_{j+2} = Z(P_4^j).
\]

These edges correspond to the central charges of the terms in the following AR triangle:

\[
P_3^{j-1} \rightarrow P_4^j \rightarrow P_3^j \rightarrow P_3^{j-1}[1]. \quad (5.6)
\]

See the yellow triangle (and its mirror in lighter yellow) in Figure 8 for \( j = 1 \). And the total stability implies

\[
\phi_\sigma(P_3^{j-1}) < \phi_\sigma(P_4^j) < \phi_\sigma(P_3^j) < \phi_\sigma(P_3^{j-1}) + 1
\]
By taking arg in $[\pi \cdot \phi_\sigma(P_j^{j-1}), \pi \cdot \phi_\sigma(P_j^{j-1}) + 2\pi)$, we conclude that each $W_j$ is indeed on the left side of $Z(P_j^j) = V_{j-1}^j V_{j+2}$.

5.5. From stable 12-gon to ToSt for $E_6$. Finally, we would like to construct a slicing $P$ from a stable 12-gon $V$, which is compatible with the central charge defined in Theorem 5.7 and the shift $[1]$. Such a construction is basically taking arg in appropriate length $2\pi$ intervals for central charges of each indecomposables. It follows exactly the same line of work as in Section 4.3. Thus, we have a similar statement in type $E_6$.

**Theorem 5.12.** There is a natural isomorphism

$$Z_h: \text{ToSt}(E_6)/\mathbb{C} \rightarrow \text{Sth}(E_6),$$

sending a ToSt $\sigma$ to the far-end 12-gon.

6. ToSt of the exceptional type $E_7$

6.1. The h-gon of type $E_7$.

**Definition 6.1.** An $h$-gon $V$ of type $E_7$ is a centrally symmetric 18-gon satisfying (1.2).

A direct calculation shows that:

**Lemma 6.2.** After setting $z_{j+9} = -z_j$ for $j \in \mathbb{Z}_{18}$, the set of 3 equations (1.2) has rank 2. So the space of 18-gons of type $E_7$ has complex dimension $9 - 2 = 7$.

**Construction 6.3.** Using the hexagon relations in (1.2), we can draw hexagons

$$(\text{L}_{2j}) = V_{2j-1}^1 V_{2j+1}^j W_{2j+1}^j U_{2j}^j W_{2j-1}^j$$

with edges

$$V_{2j-1}^j V_{2j}^j = z_{2j}, \quad V_{2j}^j V_{2j+1}^j = z_{2j+1}, \quad V_{2j+1}^j W_{2j+1}^j = z_{2j+6},$$

$$W_{2j+1}^j U_{2j}^j = z_{2j+7}, \quad U_{2j}^j W_{2j-1}^j = z_{2j+12}, \quad W_{2j-1}^j V_{2j-1}^j = z_{2j+13}$$

as shown in the upper picture of Figure 9. Note that we have 3 sets of 3 congruent hexagons

$$\{\text{L}_{2j+6k} \mid k = 0, 1, 2\} \quad \text{for } j \in \mathbb{Z}_3,$$

drawn in blue/green/orange respectively.

Denoting

$$z_{j+5/2} = z_j + z_{j+5}, \quad j \in \mathbb{Z}_{18}, \quad (6.1)$$

each hexagon decomposes into 3 triangles corresponding to the above equation and 1 triangle corresponding to the relation

$$z_{j+1/2} + z_{j+7/2} + z_{j+13/2} = 0.$$

So there is a 9-gon $V_{\text{ice}}$, called the ice core of $V$, with vertices $\{W_{2j+1} \mid j \in \mathbb{Z}_9\}$ and edges $W_{2j-1}^j W_{2j+1}^j = z_{(j+1)/2}.$
Remark 6.4. We take nine $L_{2j}$ with even indices to obtain $V_{\text{ice}}$. One can also take the other nine $L_{2j+1}$ with odd indices to obtain another 9-gon $V_{\text{fire}}$, called the fire core of $V$, with vertices $\{W_{2j} \mid j \in \mathbb{Z}_9\}$ and edges $W_{2j}W_{2j+2} = z_{(4j+3)/2}$. However, since $V$ is centrally symmetric, these two cores are also centrally symmetric to each other.

\[ z_j + z_{j+1} + z_{j+6} + z_{j+7} + z_{j+12} + z_{j+13} = 0, \quad \forall j \in \mathbb{Z}_9 \]

Figure 9. The (regular) 18-gon of type $E_7$ and AR-quiver of $D_\infty(E_7)$.
6.2. Categorical set up.

Separation 6.5. The lower picture in Figure 9 is part of $\mathcal{D}_\infty(E_7)$, which is in fact the AR quiver of $\text{Ind mod } kE_7$ for the $E_7$ quiver with alternative orientation:

$$
\begin{array}{ccccccc}
3 & \downarrow \\
1 & \leftarrow & 2 & \rightarrow & 4 & \leftarrow & 5 & \rightarrow & 6 & \leftarrow & 7
\end{array}
$$

(6.2)

We still label the indecomposable objects in the $\mathcal{D}_\infty(E_7)$ by (3.2). So 7,1 and 3 are the far-end, middle-end and short-end vertices respectively. For simplicity, we only label the indecomposable objects in the boundary $\tau$-orbits:

1st-$\tau$-orbit: Denote $C_j := P_1^{j-1}$ (drawn in green). This is the mid-end $\tau$-orbit.

3rd-$\tau$-orbit: Denote $M_j := P_3^{j-1}$ (drawn in yellow). This is the short-end $\tau$-orbit.

7th-$\tau$-orbit: Denote $B_j := P_7^{j-1}$ (drawn in blue). This is the far-end $\tau$-orbit.

Given any central charge $Z: K\mathcal{D}_\infty(E_7) \to \mathbb{C}$. Let $V_Z$ be the far-end 18-gon of $\mathcal{D}_\infty(E_7)$ with edges $z_j = Z(B_j)$ for $1 \leq j \leq 18$.

Lemma 6.6. The set $\{[B_j] | 1 \leq j \leq 9\}$ is a basis of $K\mathcal{D}_\infty(E_7)$. Moreover, $V_Z$ is a 18-gon of type $E_7$ and its ice/fire core is formed by (the odd/even) half of

$$\{w_{j-1/2}: = Z(C_j) | 1 \leq j \leq 18\}.$$

Proof. The first statement can be checked directly.

Now we will show that all $z_j$ satisfy (1.2) Note that we have triangles in $\mathcal{D}_\infty(E_7)$:

$$B_j \to C_{j+3} \to B_{j+5} \to B_j[1],$$

(6.3)

which implies that

$$[C_{j+3}] = [B_j] + [B_{j+5}].$$

Applying the central charge, we obtain

$$Z(C_{j+3}) = z_j + z_{j+5},$$

noting (6.1), and thus $w_{j+5/2} = z_{j+5/2}$. Moreover, we also have triangles in $\mathcal{D}_\infty(E_7)$:

$$\begin{cases}
B_j \to M_{j+2} \to C_{j+4} \to B_j[1], \\
C_{j+1} \to M_{j+2} \to B_{j+4} \to C_{j+1}[1].
\end{cases}$$

(6.4)

Then we have

$$[B_j] + [C_{j+4}] = [M_{j+2}] = [C_{j+1}] + [B_{j+4}],$$

and thus

$$z_j + w_{j+7/2} = w_{j+1/2} + z_{j+3}.$$ 

Substitute (6.1) to kill $w_{j+7/2}$, we obtain

$$z_j + (z_{j+1} + z_{j+6}) = (z_{j+3} + z_{j-2}) + z_{j+4}.$$ 

Noticing that $z_{k+9} = -z_k$, the above equation becomes (1.2).

Thanks to (6.1), the last statement is clear. \qed
6.3. *Geometric model for the root category of type $E_7$. Now we describe a geometric model in $E_7$ case.*

**Theorem 6.7.** An 18-gon $V$ of type $E_7$ is a geometric model for the root category $\mathcal{D}_\infty(E_7)/\mathbb{Z}$ in the sense that we have a central charge defined as follows:

- central charges of objects in the 1st-τ-orbits are given by edges of the ice and fire cores of $V$, i.e., $Z(C_j) = Z(P_1^{j-1}) = W_{j-2}W_j$.
- central charges of objects in the 2nd-τ-orbits are given by $Z(P_2^{j-1}) = V_{j-3}W_j$.
- central charges of objects in the 3rd-τ-orbits are given by $Z(M_j) = Z(P_3^{j-1}) = V_{j-3}W_{j-1}$.
- central charges of objects in the $i$th-τ-orbits are given by length-(8−$i$) diagonals of $V$, for $i = 4, 5, 6, 7$. In particular, $Z(B_j) = Z(P_7^{j-1}) = V_{j-1}V_j$.

For instance, the (edges of the) yellow triangle in the top picture of Figure 9 corresponds to the (central charges of) triangle $M_1 \rightarrow X \rightarrow M_4 \rightarrow M_3[1]$ (6.5) in $\mathcal{D}_\infty(E_7)$. Such a type of triangles also appears in type $E_6$, see (5.6) in the proof of Proposition 5.11 and the corresponding yellow triangle (for $j = 1$) in Figure 8.

6.4. *Stability of 18-gon for $E_7$.*

**Definition 6.8.** An $h$-gon $V$ of type $E_7$ is *stable* if it is convex and its ice/fire core is inside the level-4 diagonal-gon.

Note that the convexity of $V$ will be inherited by its ice and fire cores as in the $E_6$ case. Denote by $\text{Sth}(E_7)$ the moduli space of stable 18-gon of type $E_7$ up to similarity. By Lemma 6.2, the complex dimension of $\text{Sth}(E_7)$ is $7 - 1 = 6$.

**Proposition 6.9.** If $\sigma = (Z, P) \in \text{ToSt}(E_7)$, then its far-end 18-gon is a stable 18-gon of type $E_7$.

**Proof.** To prove the statement, one needs to use the AR triangle (6.5) similar as in type $E_6$, whose central charges of its terms form the yellow triangle in Figure 9. Then the rest argument follows the same way as in Proposition 5.11. □

**Theorem 6.10.** There is a natural isomorphism

$$Z_h: \text{ToSt}(E_7)/\mathbb{C} \rightarrow \text{Sth}(E_7).$$

sending a ToSt $\sigma$ to the far-end 18-gon.

7. **ToSt of the exceptional type $E_8$**

7.1. *The h-gon of type $E_8$.*

**Definition 7.1.** An $h$-gon $V$ of type $E_8$ is a centrally symmetric 30-gon satisfying (1.3).

A direct calculation shows that:
Lemma 7.2. After setting $z_{j+15} = -z_j$ for $j \in \mathbb{Z}_{30}$, the set of 8 equations (1.3) has rank 7. So the space of 30-gons of type $E_8$ has complex dimension $15 - 7 = 8$.

Construction 7.3. Using the triangle/pentagon relations in (1.3), we can draw pentagons

$P_{2j+1} : = V_{2j}V_{2j+1}W_{2j+1}U_{2j+1}W_{2j-1} \quad \forall j \in \mathbb{Z}_{15}$

with edges

\[
\begin{align*}
V_{2j}V_{2j+1} &= z_{2j+1}, \\
V_{2j+1}W_{2j+1} &= z_{2j+7}, \\
W_{2j+1}U_{2j+1} &= z_{2j+13}, \\
U_{2j+1}W_{2j-1} &= z_{2j+19}, \\
W_{2j-1}V_{2j} &= z_{2j+25}
\end{align*}
\]

and triangles

$T_{2j} : = V_{2j-1}V_{2j}W_{2j-1} \quad \forall j \in \mathbb{Z}_{15}$

Figure 10. The (regular) 30-gon of type $E_8$
with edges
\[ V_{2j-1}V_{2j} = z_{2j}, \quad V_{2j}W_{2j-1} = z_{2j+10}, \quad W_{2j-1}V_{2j-1} = z_{2j+20}, \]
as shown in Figure 10. Note that we have 3 sets of 5 congruent pentagons
\[ \{ P_{2j+1+6k} \mid k = 0, 1, 2, 3, 4 \}, \quad \text{for } j \in \mathbb{Z}_3 \]
drawn in orange/green/blue respectively and 5 sets of 3 congruent triangles
\[ \{ T_{2j+10k} \mid k = 0, 1, 2 \}, \quad \text{for } j \in \mathbb{Z}_5 \]
drawn in violet with different opacity respectively.

So there is a 15-gon \( \mathbf{V}_{\text{ice}} \), called the ice core of \( \mathbf{V} \) with vertices \( \{ W_{2j-1} \mid j \in \mathbb{Z}_{15} \} \) and edges \( w_{2j+1} = W_{2j-1}W_{2j+1}. \)

Remark 7.4. We take fifteen \( P_{2j+1} \) with odd indices and fifteen \( T_{2j} \) with even indices to obtain \( \mathbf{V}_{\text{ice}} \). One can also take the other fifteen \( P_{2j+1} \) with odd indices and fifteen \( T_{2j} \) with even indices to obtain another 15-gon \( \mathbf{V}_{\text{fire}} \), called the fire core of \( \mathbf{V} \), with vertices \( \{ W_{2j} \mid j \in \mathbb{Z}_{15} \} \) and edges \( w_{2j} = W_{2j-2}W_{2j}. \) However, since \( \mathbf{V} \) is centrally symmetric, these two cores are also centrally symmetric to each other.

7.2. Categorical set up.

Sep-up 7.5. Figure 11 is part of \( AR\mathcal{D}_\infty(E_8) \), which is in fact the AR quiver of \( \text{Ind mod } kE_8 \) for the \( E_8 \) quiver with alternative orientation:

\[
\begin{array}{cccccccc}
& & & & & & & \\
& & & & & 3 & & \\
& & & & 1 & 2 & 4 & \\
& & & 5 & & 6 & & 7 & 8
\end{array}
\]

We still label the indecomposable objects in the \( \mathcal{D}_\infty(E_8) \) by (3.2). So 8, 1 and 3 are the far-end, middle-end and short-end vertices respectively. For simplicity, we only label the indecomposable objects in the boundary \( \tau \)-orbits:

- **1st-\( \tau \)-orbit**: Denote \( C_j : = P_j^{1-1} \) (drawn in green). This is the mid-end \( \tau \)-orbit.
- **3rd-\( \tau \)-orbit**: Denote \( M_j : = P_j^{3-1} \) (drawn in yellow). This is the short-end \( \tau \)-orbit.
- **8th-\( \tau \)-orbit**: Denote \( B_j : = P_j^{8-1} \) (drawn in blue). This is the far-end \( \tau \)-orbit.

Given any central charge \( Z : K\mathcal{D}_\infty(E_8) \to \mathbb{C} \). Let \( \mathbf{V}_Z \) be the far-end 30-gon of \( \mathcal{D}_\infty(E_8) \) with edges \( z_j = Z(B_j) \) for \( 1 \leq j \leq 30. \)

Lemma 7.6. The set \( \{ [B_j] \mid 1 \leq j \leq 15 \} \) is a basis of \( K\mathcal{D}_\infty(E_8) \). Moreover, \( \mathbf{V}_Z \) is a 30-gon of type \( E_8 \) and its ice/fire core is formed by (the odd/even) half of \( \{ w_j : = Z(C_j) \mid 1 \leq j \leq 30 \}. \)

Proof. The first statement can be checked directly.

Now we will show that all \( z_j \) satisfy (1.3). Note that we have triangles in \( \mathcal{D}_\infty(E_8) \):

\[ B_j \to C_{j+3} \to B_{j+6} \to B_j[1]. \]
which implies that

\[ w_{j+3} = z_j + z_{j+6}. \]  \hspace{1cm} (7.2)

Moreover, we have four families of triangles in \( D_\infty(E_8) \) that form a family of octahedral diagrams

\[
\begin{align*}
B_j & \rightarrow B_j \\
C_{j+1} & \rightarrow M_{j+2} \rightarrow B_{j+5} \rightarrow C_{j+1}[1] \\
C_{j+1} & \rightarrow C_{j+4} \rightarrow B_{j+10} \rightarrow C_{j+1}[1] \\
B_j[1] & \rightarrow B_j[1]
\end{align*}
\]
Then we obtain
\[
\begin{align*}
&z_j - z_{j+5} + z_{j+10} = 0, \\
&w_{j+1} - w_{j+4} + z_{j+10} = 0,
\end{align*}
\forall j \in \mathbb{Z}_{30}
\tag{7.3}
\]
corresponding to the third column/row respectively. Noticing \(-z_{j+5} = z_{j+20}\), the first equation of (7.3) becomes the triangle relation in (1.3). Substitute (7.2) to kill \(w_{\gamma}\)'s in the second equation of (7.3), we have
\[
z_j - 2z_{j+4} + z_{j+1} - z_{j+7} + z_{j+10} = 0, \quad \forall j \in \mathbb{Z}_{30}.
\]
Noticing \(z_k = -z_{k+15}\), the above equation becomes the pentagon relation in (1.3). □

7.3. Geometric model for the root category of type \(E_8\). Now we describe a geometric model in type \(E_8\).

**Theorem 7.7.** A 30-gon \(V\) of type \(E_8\) is a geometric model for the root category \(D_\infty(E_8)/[2]\) in the sense that we have a central charge defined as follows:

- central charges of objects in the 1st-\(\tau\)-orbits are given by edges of the ice and fire cores of \(V\), i.e., \(Z(C_j) = Z(P_j^{-1}) = W_{j-2}W_j\).
- central charges of objects in the 2nd-\(\tau\)-orbits are given by \(Z(P_j^2) = V_{j-4}W_j\).
- central charges of objects in the 3rd-\(\tau\)-orbits are given by \(Z(M_j) = Z(P_j^3) = V_{j-3}W_{j-1}\).
- central charges of objects in the \(i^{\text{th}}\)-\(\tau\)-orbit are given by length-(9 - \(i\)) diagonals of \(V\), for \(i = 4, 5, 6, 7, 8\). In particular, \(Z(B_j) = Z(P_j^8) = V_{j-1}V_j\).

For instance, the (edges of the) yellow triangle in the top picture of Figure 10 corresponds to the (central charges of) triangle \(M_3 \rightarrow X \rightarrow M_4 \rightarrow \) in \(D_\infty(E_8)\).

7.4. Stability of 30-gon for \(E_8\).

**Definition 7.8.** A 30-gon \(V\) of type \(E_8\) is **stable** if it is convex and its ice/fire core is inside the level-5 diagonal-gon.

Denote by \(\text{St}(E_8)\) the moduli space of stable 30-gon of type \(E_8\) up to similarity. By Lemma 7.2, the complex dimension of \(\text{St}(E_8)\) is \(8 - 1 = 7\).

As above in type \(E_7\), we have the following proposition and theorem for \(E_8\).

**Proposition 7.9.** If \(\sigma = (Z, P) \in \text{ToSt}(E_8)\), then its far-end 30-gon is a stable 30-gon of type \(E_8\).

**Theorem 7.10.** There is a natural isomorphism
\[
Z_h: \text{ToSt}(E_8)/\mathbb{C} \rightarrow \text{St}(E_8).
\]
sending a ToSt \(\sigma\) to the far-end 30-gon.
7.5. **Summary of exceptional cases.**

**Remark 7.11.** A geometric model $V$ for the exceptional case $Q = E_n$, where $n \in \{6, 7, 8\}$, consists of the following data:

- $V$ is an $h_Q$-gon whose edges are given by some central charge of the indecomposable objects in the far-end $\tau$-orbit of $\mathcal{D}_\infty(Q)$.
- Its ice/fire core is an $h_Q/2$-gon whose edges are given by the (same) central charge of half of the indecomposable objects in the middle-end $\tau$-orbit of $\mathcal{D}_\infty(Q)$.

Such an $h_Q$-gon is stable if and only if both the ice and fire cores are inside the length-$(n - 3)$ diagonal-gon of $V$. Moreover, the usual types can also be thought as a degeneration of the model above, in the sense that:

- for $Q = D_n$, the ice/fire core shrinks to the puncture $B_\pm$.
- for $Q = A_n$, the ice/fire core vanishes.

**Remark 7.12.** Another interesting numerical observation is that the triangle/square relations in type $E_6$ correspond to $x^4$ and $y^3$, and triangle/pentagon relations in type $E_6$ correspond to $x^5$ and $y^3$, respectively, in the corresponding simple (surface) singularity

$$f(x, y, z) = \begin{cases} 
  x^{n+1} + y^2 + z^2, & Q = A_n \ (n \geq 1); \\
  x^{n-1} + xy^2 + z^2, & Q = D_n \ (n \geq 4); \\
  x^4 + y^3 + z^2, & Q = E_6; \\
  x^5 + y^3 + z^2, & Q = E_7; \\
  x^5 + y^3 + z^2, & Q = E_8. 
\end{cases}$$

These equations are used to defined $\mathcal{D}_\infty(Q)$ via (graded) matrix factorizations in [KST]. The correspondence in type $E_7$ is not as nice as the other two. Nevertheless, such numerical coincidence makes the story very convincing without further calculation. It is also noticed by Alastair King.

8. **Non simply-laced types via folding**

A Dynkin specie $Q^\iota$ is obtained by folding the corresponding Dynkin quiver $Q$, see Section 2.3. The derived category/stability conditions for $Q^\iota$ are the $\iota$-stable part of the one for $Q$ (cf. [CQ] for more details), Thus, the stable $h_Q$-gons for $Q^\iota$ are just $\iota$-stable $h$-gons, i.e. satisfying extra symmetry. More precisely, we have the following:

- **$B_n$:** It is obtained by folding the corresponding quiver of type $D_{n+1}$. Thus the extra symmetry condition on the corresponding doubly punctured $2n$-gon is the two punctures coincide (and thus at the geometric center of the $2n$-gon).
- **$C_n$:** It is obtained by folding the corresponding quiver of type $A_{2n-1}$. Thus the extra symmetry condition on the corresponding $2n$-gon is central symmetry.
- **$F_4$:** It is obtained by folding the corresponding quiver of type $E_6$. Thus the extra symmetry condition on the corresponding 12-gon is central symmetry.
- **$G_2$:** It is obtained by folding the corresponding quiver of type $D_4$. Thus the extra condition is: concision of the punctures, central symmetry and (1.4).
Figure 12. Deforming stable $h$-gon and central charge of $D_4$ to $B_3$ to $G_2$

So Theorem 1 also holds in these cases.

Example 8.1. In Figure 12, we show that how to deform the central charge and associated far-end stable $h_Q$-gon of type $D_4$ into the ones of type $B_3$ and of type $G_2$, respectively.

From $D_4$ to $B_3$, it requires the (orange/violet) central charges of objects in the two boundary (besides the chosen far-end) $\tau$-orbits coincides correspondingly. Equivalently, the two punctures coincide as mentioned above.

From $D_4$ to $G_2$, it requires the (orange/violet/blue) central charges of objects in all boundary $\tau$-orbits coincides correspondingly. Equivalently, the long diagonals of the $h_Q = 6$-gon intersect at its geometric center that divide it into six congruent triangles.

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Appendix A. Tikz Art Gallery by Qy

In this appendix, we collection figures of stable $h$-gons at Gepner points that interact with the corresponding projection of the root systems in the Coxeter plans, as well as a tilting of Ice and Fire of type $E_6$.

Figure 13. The (Coxeter projection of the) root system of type $D_5$ and the stable $h$-gon
Figure 14. The central charges, root system and stable $h$-gons of type $E_6$
Figure 15. The central charges, root system and stable $h$-gons of type $E_7$
Figure 16. The root system and (two) stable $h$-gons of type $E_8$
Figure 17. A Tiling of Ice and Fire