Coding Theory and Uniform Distributions

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Abstract. In the present paper we introduce and study finite point subsets of a special kind, called optimum distributions, in the $n$-dimensional unit cube. Such distributions are closely related with known $(\delta, s, n)$-nets of low discrepancy. It turns out that optimum distributions have a rich combinatorial structure. Namely, we show that optimum distributions can be characterized completely as maximum distance separable codes with respect to a non-Hamming metric. Weight spectra of such codes can be evaluated precisely. We also consider linear codes and distributions and study their general properties including the duality with respect to a suitable inner product. The corresponding generalized MacWilliams identities for weight enumerators are briefly discussed. Broad classes of linear maximum distance separable codes and linear optimum distributions are explicitly constructed in the paper by the Hermite interpolations over finite fields.

1. Introduction

The present paper deals with a combinatorial structure of point sets uniformly distributed in the $n$-dimensional unit cube. We introduce point sets of a special kind, called optimum distributions, which are close related with known distributions, so-called $(\delta, s, n)$-nets, of low discrepancy. It turns out that optimum distributions have a rich interior combinatorial structure, namely, they can be characterized completely as maximum distance separable codes over finite fields with respect to a non-Hamming metric. Moreover, it is found that weight spectra of such codes can be evaluated precisely.

We also study linear codes and distributions. In this case new good codes and distributions can be obtained from given ones by duality with respect to a suitable inner product. The corresponding MacWilliams identities for weight enumerators in the non-Hamming metric are also briefly discussed in the paper. Notice that broad classes of linear maximum distance separable codes and linear optimum distributions can be explicitly given by interpolations over finite fields.

We emphasize that the specifying of a non-Hamming metric on the space of distributions is of crucial importance for our study. As soon as the metric is introduced, the basic concepts and methods of the modern coding theory may be applied to uniform distributions. In the present paper this approach is illustrated by several general results.

It is conceivable that the theory of uniform distributions may also work for error-correcting codes. Further studies of intimate linkages between coding theory and uniform distributions should be exceptionally intriguing.

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A non Hamming metric cited above has recently been proposed by Rosenbloom and Tsfasman [25] in coding theory. However, in the context of \((\delta, s, n)\)-nets a similar concept was implicitly introduced in the 1987 year paper [20] by Niederreiter. In fact, a respective non-Hamming weight of a linear code was defined in [20, Definitions 6.8 and 7.1] in terms of the code parity-check matrix.

In other contexts, the metric was independently rediscovered by several authors (see Martin and Stinson [18], and Skriganov [28]). It seems reasonable to say that subjects of such a kind are of great concern in many areas of combinatorial mathematics.

Since concepts and results from rather different fields are used in the present work, necessary comments and references will be given wherever possible. The reader interested in the fundamental aspects of coding theory is referred to the encyclopaedic book MacWilliams and Sloane [16] and references therein.

We recall basic facts in the theory of uniform distributions. For necessary details we refer to Beck and Chen [4], Kuipers and Niederreiter [12], and a recent book Matoušek [19].

Let \(D \subset U^n\) be a distribution of finitely many points in the \(n\)-dimensional unit cube \(U^n = [0, 1)^n\). The \(L_\infty\)-discrepancy \(\mathcal{L}[D]\) is defined by

\[
\mathcal{L}[D] = \sup_{Y \in U^n} |\#\{D \cap B(Y)\} - \#(D)\vol B(Y)|, \tag{1.1}
\]

where \(B(Y) = [0, y_1] \times \cdots \times [0, y_n), Y = (y_1, \ldots, y_n) \in U^n\), are rectangular boxes in \(U^n\), and \(\vol B(Y) = y_1 \cdots y_n\).

A main problem in the theory of uniform distributions is the construction of distributions with minimal discrepancies (1.1). It is known that for each \(N \geq 2\) there exist distributions \(D_N \subset U^n\) consisting of \(N\) points with

\[
\mathcal{L}[D_N] = O((\log N)^{n-1}), \tag{1.2}
\]

where the implied constant depends only on \(n\).

By a theorem of Roth for any distribution \(D_N\) of \(N \geq 2\) points one has the following lower bound

\[
\mathcal{L}[D_N] > c_n (\log N)^{\frac{1}{2}(n-1)}, \tag{1.3}
\]

and by a theorem of Schmidt in two dimensions the bound (1.3) admits the following improvements

\[
\mathcal{L}[D_N] > c \log N. \tag{1.4}
\]

In (1.3) and (1.4) \(c_n\) and \(c\) are positive absolute constants independent of \(N\).

Thus, in two dimensions bound (1.2) is best possible. In dimensions \(n > 2\) an exact minimal order of discrepancy (1.1) still remains inaccessible. By [4, Sections 1.1 and 9.2] this is a “Great open problem” in theory of uniform distributions. See also Baker [2] and Beck [3] where certain improvements of the lower bound (1.3) for discrepancy (1.1) are given in dimensions \(n > 2\).

One of the known approaches to the construction of distributions of low discrepancy can be outlined as follows. Fix a prime \(p\) and a prime power \(q = p^e, e \in \mathbb{N}\). We write \(\mathbb{N}\) for the set of natural numbers and \(\mathbb{N}_0\) for the set of all non-negative integers. Introduce elementary rectangular boxes \(\Delta^M_A \subset U^n\) by setting

\[
\Delta^M_A = \left[\frac{m_1}{q^{a_1}}, \frac{m_1 + 1}{q^{a_1}}\right) \times \cdots \times \left[\frac{m_n}{q^{a_n}}, \frac{m_n + 1}{q^{a_n}}\right), \tag{1.5}
\]
where $A = (a_1, \ldots, a_n)$, $M = (m_1, \ldots, m_n) \in \mathbb{N}_0^n$, and $m_j \in \{0, 1, \ldots, q^{a_j} - 1\}$, $1 \leq j \leq n$. We write $\mathbb{N}_0^n$ for the set of all vectors in $\mathbb{R}^n$ with non-negative integer coordinates. Notice that boxes (1.5) have volume
\[
\text{vol } \Delta^M_A = q^{a_1-1} - \cdots - a_n. \tag{1.6}
\]

Definition 1.1. Given integers $0 \leq \delta \leq s$. A subset $D \subset U^n$ consisting of $q^s$ points is called a $(\delta, s, n)$-net of deficiency $\delta$ in base $q$ if each elementary box $\Delta^M_A$ of volume $q^{\delta-s}$ contains exactly $q^\delta$ points of $D$.

Obviously $(\delta, s, n)$-nets fill out the unit cube $U^n$ very uniformly. It is easily shown (cf. [4, Section 3.2] and [20, Section 3]) that discrepancy $L[D]$ of a $(\delta, s, n)$-net $D$ satisfies the bound
\[
L[D] = O(q^\delta s^{n-1}) = O(q^\delta (\log N)^{n-1}), \tag{1.7}
\]
where $N = \# \{D\} = q^s$, and the implied constants in (1.7) depend only on $n$ and $q$. Hence, $(\delta, s, n)$-nets of bounded deficiency $\delta$ satisfy relation (1.2).

We mention the following simple corollary of Definition 1.1 (cf. [23, Lemma 9]) which shows that the base $p^e$ of a given net can be always reduced to $p$ at the cost of an increase in deficiency.

Lemma 1.1. Every $(\delta, s, n)$-net in base $p^e$ is a $(\delta', es, n)$-net of deficiency $\delta' = e\delta + (e-1)(n-1)$ in base $p$.

The first constructions of $(\delta, s, n)$-nets were given by Sobol [29, 30] and Faure [11]. Namely, in an arbitrary dimension $n$ for any $s \in \mathbb{N}$ $(\delta, s, n)$-nets of deficiency $\delta = O(n \log n)$ in base 2 were constructed in [29, 30], and $(0, s, n)$-nets of zeroth deficiency in arbitrary prime base $p \geq n - 1$ were given in [11].

A systematic study of $(\delta, s, n)$-nets (including nets in arbitrary integer bases $b \in \mathbb{N}$) was started by Niederreiter [20], see also a book [21] and a recent survey [22]. Among other things the following impressive result was recently obtained within the context of these studies. Niederreiter and Xing [23, 24] discovered that very good $(\delta, s, n)$-nets of deficiency $\delta = O(n)$ can be constructed in terms of algebraic curves over finite fields. Moreover, for a given base $q$ the bound $\delta = O(n)$ turns out to be best possible as $n \to \infty$.

It was observed in Adams and Shader [1], Clayman and Mullen [8], Edel and Bierbrauer [10], and Lawrence et al. [13] that properties of $(\delta, s, n)$-nets can be improved in some instances by error-correcting codes with large Hamming weights. The corresponding details and further references can also be found in the survey [24, Section 6].

It should be noted that $(\delta, s, n)$-nets can be used to construct new remarkable classes of uniform distributions. For example, probabilistic variations of $(\delta, s, n)$-nets were used in Chen [6] (see also [4, Section 3.4]) to determine exact minimal orders of the $L_W$-discrepancies for all $1 < W < \infty$.

Concerning alternative constructions of distributions of low discrepancy, we should mention the author’s paper [26] (see also [27]) where methods of the geometry of numbers had been used. In a sense the approach given in the present paper is based on similar geometric ideas adapted for finite fields.

Main concepts of the geometry of numbers (like sphere packings, homogeneous and inhomogeneous minima, etc.) arranged for finite fields can be found in Conway
and Sloane [9]. For general facts on geometry of vector spaces over finite fields we refer to Lang [14] and Lidl and Niederreiter [15].

In the present paper we study point sets of a special kind which are defined as follows. Introduce the following collection of elementary boxes (1.5)

$$
\mathcal{E}_s(q, n) = \{ \Delta^M_A : A = (a_1, \ldots, a_n), 0 \leq a_j \leq s, 1 \leq j \leq n \}. \quad (1.8)
$$

**Definition 1.2.** Given an integer $0 \leq k \leq n$. A subset $D \subset \mathbb{U}^n$ consisting of $q^k$ points is called an optimum $[ns, k]_s$-distribution in base $q$ if each elementary box $\Delta^M_A \in \mathcal{E}_s(q, n)$ of volume $q^{-k}$ contains exactly one point of $D$.

**Remark 1.1.** A meaning of Definition 1.2 is to settle on a relation between the number of points in a given distribution and the magnitude of denominators in the definition of elementary boxes (1.5).

It should be pointed out that in Definition 1.2 we may assume, without loss of generality, that $s \leq k \leq ns$. Indeed, for $0 \leq k < s$ each elementary box $\Delta^M_A \in \mathcal{E}_s(q, n)$ of volume $q^{-k}$ also belongs to a subcollection $\mathcal{E}_k(q, n)$, and the corresponding point set turns out to be an optimum $[nk, k]_k$-distribution. At the same time, for $s < k \leq ns$ such rescaling is impossible.

In the sequel an arbitrary subset $D \subset \mathbb{U}^n$ consisting of $q^k$ points is conveniently said to be an $[ns, k]_s$-distribution in base $q$ (with respect to a collection of elementary boxes (1.8)).

The following result is a simple corollary of Definition 1.2.

**Lemma 1.2.** Let $D \subset \mathbb{U}^n$ be an optimum $[ns, k]_s$-distribution in base $q$. Then,

(i) Each elementary box $\Delta^M_A \in \mathcal{E}_s(q, n)$ contains exactly $q^{k-a_1-\ldots-a_n}$ points of $D$, provided that $a_1 + \ldots + a_n \leq k$.

(ii) Each elementary box $\Delta^M_A \in \mathcal{E}_s(q, n)$ contains at most one point of $D$, provided that $a_1 + \ldots + a_n > k$.

**Proof.** (i) Choose integers $c_1, \ldots, c_n$ to satisfy the relations $c_1 + \ldots + c_n = k - a_1 - \ldots - a_n$, $0 \leq c_j \leq s$, $1 \leq j \leq n$. Let $b_j = a_j + c_j$, $1 \leq j \leq n$, then from (1.5) we obtain that

$$
\Delta^M_A = \left[ q^{c_1m_1} \frac{m_1}{q^{b_1}}, \frac{q^{c_1m_1} + q^{c_1}}{q^{b_1}} \right] \ldots \left[ q^{c_nm_n} \frac{m_n}{q^{b_n}}, \frac{q^{c_nm_n} + q^{c_n}}{q^{b_n}} \right].
$$

Hence, the box $\Delta^M_A$ is a union of $q^{c_1+\ldots+c_n}$ disjoint elementary boxes $\Delta^T_B \in \mathcal{E}_s(q, n)$ of volume $q^{-k}$, and (i) follows from Definition 1.2.

(ii) It suffices to notice that each elementary box $\Delta^M_A \in \mathcal{E}_s(q, n)$ with $a_1 + \ldots + a_n > k$ is contained in a bigger elementary box $\Delta^T_B \in \mathcal{E}_s(q, n)$ of volume $q^{-k}$, and (ii) follows from Definition 1.2. $\square$

Now a linkage between optimum distributions and nets can be described as follows.

**Proposition 1.1.** An optimum $[ns, k]_s$-distribution with $s \leq k \leq ns$ is a $(k - s, k, n)$-net in the same base $q$.

**Proof.** It suffices to compare Lemma 1.2 (i) with Definition 1.1 and take into account that any elementary box $\Delta^M_A$ of volume $q^{-s}$ belongs to the collection $\mathcal{E}_s(q, n)$. $\square$
From Proposition 1.1 and bound (1.7) we conclude that the discrepancy $L[D]$ of an optimum $[n,s,k]$-distribution $D$ with $s \leq k \leq ns$ satisfies the bound

$$L[D] = O(q^{k-s}s^{n-1}) = O(N^{n-s}(\log N)^{n-1}), \quad (1.9)$$

where $N = \#\{D\} = q^k$. Thus, optimum $[n,s,k]$-distributions coincide with $(0,s,n)$-nets of zeroth deficiency and they satisfy relation (1.2).

At first glance the foregoing implies that only optimum distributions with $k = s$ are of interest to the theory of uniform distributions. However, such is not the case, and distributions with $k > s$ can also be used to construct new ones which already satisfy relation (1.2). More precisely, from a given optimum $[gns, gs]$-distribution $D \subset U^{gn}$, $g \in \mathbb{N}$, we can obtain a new optimum $[gns, gs]$-distribution $\pi_{g,n}D \subset U^n$ by a mapping $\pi_{g,n} : U^{gn} \rightarrow U^n$ which is merely the known Peano’s bijection between points of the unit cubes $U^{gn}$ and $U^n$. Moreover, it turns out that the distributions $\pi_{g,n}D$ have additional intriguing properties. Concerning Peano’s mappings, the reader is referred to any text-book on elementary set theory.

The study of optimum $[n,s,k]$-distributions is a main goal of the present paper. Previously a part of our results was given in the author’s paper [28].

The paper is organized as follows. Section 2 contains general results on distributions and their $q$-ary codes. Our main concepts of a non-Hamming metric $\rho$ and maximum distance separable (or briefly MDS) codes in the metric $\rho$ are introduced in Sec. 2.2 and Sec. 2.3, respectively.

In Sec. 2.3 optimum distributions are characterized completely in terms of their $q$-ary codes. We show that a given distribution is an optimum distribution if and only if its code is an MDS code in the metric $\rho$ (see Theorem 2.1).

Notice that by definition MDS codes consists of the most widely spaced code words. For the Hamming metric $\kappa$ a general theory of MDS codes is given in [16, Chapter 11]. Metrics different from the Hamming one, say, the Lee metric, are known in coding theory and its applications (cf. [16]). Not all of them enable a rich theory of MDS codes. In Sec. 2.5 we shall mention a class of metrics, containing metrics $\kappa$ and $\rho$, which could be of interest from such standpoint (see Remark 2.1). Notice also that a nontrivial group of linear transformations preserving the metric $\rho$ can be given explicitly (see Remark 2.2).

It is an interesting question whether the non-Hamming metric $\rho$ can be practically applied to concrete communication systems. Some examples can be found in [25]. We shall also give examples of such a kind (see Remark 2.3) in order to illustrate main distinctions in applications of the metric $\rho$ to coding theory and uniform distributions.

Section 3 contains explicit formulas for weight spectra of optimum distributions and MDS codes in the metric $\rho$ (see Theorem 3.1). We emphasize that such formulas are fully inspired by coding theory where similar results are well known for MDS codes in the Hamming metric (cf. [16, Chapter 11, Theorem 6]).

Explicit formulas for weight spectra are proved in Section 3 for arbitrary (linear or non-linear) MDS codes and optimum distributions. Furthermore, such formulas imply certain necessary conditions for the existence of MDS codes and optimum distributions (see Proposition 3.1).

Section 4 contains results on linear codes and distributions. An additional group structure of these subjects allows the application of the Fourier analysis on Abelian groups to study linear codes and distributions. Here this approach is illustrated by several results of a general character.
We define the duality with respect to an inner product on the space of codes and distributions (see Definition 4.2), and show that the dual subjects to linear MDS \([ns, k]_s\)-codes and optimum \([ns, k]_s\)-distributions are linear MDS \([ns, ns - k]_s\)-codes, and correspondingly, optimum \([ns, ns - k]_s\)-distributions (see Theorem 4.1).

In particular, linear optimum \([ns, s]_s\)- and \([ns, (n - 1)s]_s\)-distributions are mutually dual. By the way, this confirms once more that the consideration of optimum \([ns, k]_s\)-distributions with \(k > s\) is of interest to us.

It turns out that linear \((\delta, s, n)\)-nets can also be characterized completely in terms of weights of their dual subjects (see Theorem 4.2). Such results on \((\delta, s, n)\)-nets can be treated as a metric interpretation of those given earlier in [20].

Furthermore, in Sec. 4.4 we shall take a quick look at generalized MacWilliams relations for weight enumerators in the metric \(\rho\). However, an extended discussion of this matter is beyond the scope of the paper.

Notice that an alternative approach to linear \((\delta, s, n)\)-nets, including MacWilliams type theorems, has recently been developed by Martin and Stinson [18] in terms of association schemes.

In Sec. 4.5 we evaluate the non-Hamming weight of a linear code in terms of its parity-check matrix (see Proposition 4.2). This simple result confirms in passing that the corresponding definitions given in [20] and [25] are equivalent for linear codes. Certainly, the definition based on the metric \(\rho\) is preferable from a geometric standpoint.

Section 5 contains explicit examples of linear MDS codes in the metric \(\rho\) and linear optimum distributions constructed by the Hermite interpolations over finite fields. A general theory of the Hermite interpolation problem over the field of complex numbers \(\mathbb{C}\) can be found in Berezin and Zhidkov [5, Chapter 2, Section 11]. It should be pointed out that in the problem over finite fields we replace the usual formal derivatives by hyperderivatives introduced by Hasse and Teichmüller (see [15, Section 6.4] and references therein). Such an approach turns out to be very convenient for calculations in fields of finite characteristic.

Constructions of MDS codes in the Hamming metric by interpolations are also known in coding theory (see, for example, Mandelbaum [17]). From this standpoint our constructions of MDS codes in the metric \(\rho\) can be regarded as a suitable generalization of the known Reed–Solomon codes, or more generally, the redundant residue codes (see [16, Chapter 10]). Notice that interpolations with usual formal derivatives were also used in [25].

It was mentioned above that new algebraic-geometric constructions of very good \((\delta, s, n)\)-nets had been given by Niederreiter and Xing [23, 24]. In the spirit of our discussion of parallels between coding theory and uniform distributions these constructions should be interpreted as the known Goppa codes adapted for the non-Hamming metric \(\rho\). Concerning the Goppa codes and related topics we refer to [16, Chapter 12], Stichtenoth [31], and Tsfasman and Vladut [32]. Notice also that algebraic-geometric codes for the metric \(\rho\) are briefly discussed in [25]. Unfortunately, in the present paper we have to leave aside these extremely interesting questions.

Section 6 contains more specific results. Here we study the behavior of distributions under Peano’s bijection \(\pi_{g,n} : U^{gn} \to U^n\). It is worth noting that the reconstruction of codes and distributions by the mapping \(\pi_{g,n}\) is also inspired by coding theory, where reconstructions of a similar kind are known, say, for the so-called concatenated codes (cf. [16, Chapter 10]).
We show that for a given optimum \([gns, gk]_s\)-distribution \(D \subset U^{gn}\), \(g \in \mathbb{N}\), the image \(\pi_{g,n} D \subset U^n\) is an optimum \([ngs, gk]_{g^s}\)-distributions. It turns out that the distribution \(\pi_{g,n} D\) also has a large weight in the usual Hamming metric \(\kappa\) (see Propositions 6.1 and 6.3). We give explicit examples of codes and distributions with large weights simultaneously in both metrics \(\rho\) and \(\kappa\) (see Theorems 6.1 and 6.2). These topics are close related with a recent joint work of Chen and the author [7]. Conceivably, the treatment of codes and distributions simultaneously in several different (or even in all possible) metrics should be of much interest to clarify the structure of these subjects.

The author hopes that the foregoing discussion could show the benefits of interactions between coding theory and the theory of uniform distributions.

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2. Distributions and Codes

2.1 Preliminaries. Consider the \(q\)-ary representation of a point \(x \in [0, 1)\),

\[
x = \sum_{i \geq 1} \eta_i(x) q^{-i}, \quad \eta_i(x) \in \{0, 1, \ldots, q - 1\}.
\]  

(2.1)

Representation (2.1) is uniquely defined if we agree that the series (2.1) is finite for rational \(q\)-ary points \(x = \frac{m}{q^n}, a \in \mathbb{N}, m \in \{0, 1, \ldots, q^a - 1\}\). We write \(\mathbb{Q}(q^s)\), \(s \in \mathbb{N}\), for the set of all points \(x = \frac{m}{q^n} \in [0, 1)\) with \(a \leq s\), and \(\mathbb{Q}^n(q^s)\) for the set of all points \(X = (x_1, \ldots, x_n)^T \in U^n\) with coordinates \(x_j \in \mathbb{Q}(q^s), 1 \leq j \leq s\). Here \((\cdot)^T\) denotes the transpose of a matrix \((\cdot)\).

For each \(x \in [0, 1)\) we define a projection \(\tau_s x\) onto \(\mathbb{Q}(q^s)\) by

\[
\tau_s x = \sum_{i=1}^{s} \eta_i(x) q^{-i}.
\]  

(2.2)

Obviously, we have

\[
x - \tau_s x = \sum_{i \geq s+1} \eta_i(x) q^{-i} \in [0, q^{-s}).
\]  

(2.3)

Similarly, for each point \(X = (x_1, \ldots, x_n)^T \in U^n\) we define a projection \(\tau_s X\) onto \(\mathbb{Q}^n(q^s)\) by

\[
\tau_s X = (\tau_s x_1, \ldots, \tau_s x_n)^T.
\]  

(2.4)

**Lemma 2.1.** Let \(D \subset U^n\) be an optimum \([ns, k]_s\)-distribution, then its projection \(\tau_s D \subset \mathbb{Q}^n(q^s)\) is also an optimum \([ns, k]_s\)-distribution.

**Proof.** From relations (2.2), (2.3), and (2.4) we conclude that a point \(X\) falls into an elementary box \(\Delta_A^M \in \mathcal{E}_s(q,n)\) if and only if its projection \(\tau_s X\) does, and Lemma 2.1 follows at once from Definition 2.1. □

Thus, without loss of generality, we may consider in the sequel only optimum \([ns, k]_s\)-distribution which are subsets in \(\mathbb{Q}^n(q^s)\).
Representation (2.1) is convenient to write in the form

\[ x = \sum_{i=1}^{s} \xi_i(x)q^{i-s-1}, \tag{2.5} \]

where \( \xi_i(x) = \eta_{s+1-i}(x) \).

Coefficients \( \xi_i(x) \) in (2.5) can be identified with elements of a finite field. Let \( \mathbb{F}_q \) be a finite field with \( q = p^e \) elements. Fix a basis in \( \mathbb{F}_q \) over \( \mathbb{F}_p \), and represent elements \( \mu = (\mu^{(1)}, \ldots, \mu^{(e)}) \in \mathbb{F}_q \) as \( e \)-term sets of elements \( \mu^{(i)} \in \mathbb{F}_p \). In so doing, elements of the field \( \mathbb{F}_p \) are thought of as reduced residues \( \{0, 1, \ldots, p-1\} \) modulo \( p \) (see [15] for details). With these conventions we have the following bijection between elements \( \mu \in \mathbb{F}_q \) and integers \( m \in \{0, 1, \ldots, q-1\} \):

\[ \mathbb{F}_q \ni \mu = (\mu^{(1)}, \ldots, \mu^{(e)}) \iff m = \sum_{i=1}^{e} \mu^{(i)}p^{i-1} \in \{0, 1, \ldots, q-1\}. \tag{2.6} \]

By the use of bijection (2.6), coefficients \( \xi_i(x) \) in (2.5) may be regarded as elements of the field \( \mathbb{F}_q \), and therefore, a point \( x \in \mathbb{Q}(q^s) \) can be identified with the following row matrix,

\[ \omega(x) = (\omega(x_1), \ldots, \omega(x_s)) = \mathbf{1} \]

We write \( \text{Mat}_{n,s}(k) \) for the linear space of all matrices with \( n \) rows and \( s \) columns over a field \( k \).

Using (2.7), we identify each point \( X \in \mathbb{Q}^n(q^s) \) with the following matrix, the code word of \( X \),

\[ \Omega(X) = (\omega(x_1), \ldots, \omega(x_n))^T \in \text{Mat}_{n,s}(\mathbb{F}_q). \tag{2.8} \]

For a given distribution \( D \subset \mathbb{Q}^n(q^s) \) we define its code \( \mathcal{C}(D) \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) as the following array of the corresponding code words (2.8):

\[ \mathcal{C}(D) = \{ \Omega(X), X \in D \}. \tag{2.9} \]

Obviously, each subset \( \mathcal{C} \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) is code (2.8) for a distribution \( D = D(\mathcal{C}) \subset \mathbb{Q}^n(q^s) \).

By relations (2.5), (2.7), and (2.8) the structure of the vector space \( \text{Mat}_{n,s}(\mathbb{F}_q) \) can be transferred onto the set of all rational \( q \)-ary points \( \mathbb{Q}^n(q^s) \). Namely, for any two points \( X = (x_1, \ldots, x_n)^T \) and \( Y = (y_1, \ldots, y_n)^T \in \mathbb{Q}^n(q^s) \) and two elements \( \alpha \) and \( \beta \in \mathbb{F}_q \) we define a linear combination \( \alpha X \oplus \beta Y = (\alpha x_1 \oplus \beta y_1, \ldots, \alpha x_n \oplus \beta y_n)^T \in \mathbb{Q}^n(q^s) \) by setting

\[ \xi_i(\alpha x_j \oplus \beta y_j) = \alpha \xi_i(x_j) + \beta \xi_i(y_j), \quad 1 \leq i \leq s, \quad 1 \leq j \leq n, \tag{2.10} \]

where \( \xi_i(x) \in \mathbb{F}_q \) are coefficients in representation (2.5).

With respect to arithmetic operations (2.10) the set \( \mathbb{Q}^n(q^s) \) is a vector space of dimension \( ns \) over \( \mathbb{F}_q \).

We need to describe special affine subspaces in \( \mathbb{Q}^n(q^s) \) associated with elementary boxes (1.5).
Lemma 2.2. For each elementary box $\Delta^M_A \in \mathcal{E}_s(q,n)$ the intersection $V^M_A = \mathbb{Q}^n(q^*) \cap \Delta^M_A$ is an affine subspace in $\mathbb{Q}^n(q^*)$ of dimension $ns - a_1 - \ldots - a_n$ parallel to the subspace $V^0_A = \mathbb{Q}^n(q^*) \cap \Delta^0_A$, that is, $V^M_A = V^0_A \oplus Y^M_A$, where a vector $Y^M_A \in \mathbb{Q}^n(q^*)$ is defined uniquely up to translations along $V^0_A$; in particular one may take $Y^M_A = \left(\frac{m_1}{q^1}, \ldots, \frac{m_n}{q^s}\right)^T$.

More precisely, the affine subspace $V^M_A$ consists of points $X = (x_1, \ldots, x_n)^T \in \mathbb{Q}^n(q^*)$ which satisfy the following system of $a_1 + \ldots + a_n$ linear independent equations

\[
\xi_i(x_j) = \xi_i \left( \frac{m_j}{q^a_j} \right), \quad s \geq i \geq s + 1 - a_j, \quad 1 \leq j \leq n, \tag{2.11}
\]

where $\xi_i(.)$ are coefficients in q-ary representation (2.5).

Proof. From representation (2.1) we easily conclude that a point $x \in \mathbb{Q}(q^*)$ falls into an interval $\left[ \frac{m}{q^a}, \frac{m+1}{q^a} \right]$, $a \in \mathbb{N}_0$, $0 \leq a \leq s$, $m \in \{0,1,\ldots,q^a-1\}$, if and only if $\eta_i(x) = \eta_i \left( \frac{m}{q^a} \right)$ for $1 \leq i \leq a$. Writing these equations in terms of coefficients $\xi_i(.)$ in (2.5), and using definition (1.8), we obtain (2.11). It is evident that equations (2.11) imply all other statements of Lemma 2.2. □

2.2. Definition of a non-Hamming Metric. We introduce the following non-Hamming weight $\rho$ on the space $\operatorname{Mat}_{n,s}(\mathbb{F}_q)$ (cf. [25]). At first, let $n = 1$ and $\omega = (\xi_1, \ldots, \xi_s) \in \operatorname{Mat}_{1,s}(\mathbb{F}_q)$. Then we put $\rho(0) = 0$, and

\[
\rho(\omega) = \rho_q(\omega) = \max \{ j : \xi_j \neq 0 \} \tag{2.12}
\]

for $\omega \neq 0$.

Notice that in the sequel the base $q$ is usually assumed to be fixed, and we drop it from our notation. Only in Sec. 6.4 we shall need to indicate the dependence on $q$ in order to treat the variation of $q$ from $p^e$ to $p$.

It is obvious that $\rho(\omega) > 0$ for $\omega \neq 0$, $\rho(\alpha \omega) = \rho(\omega)$ for $\alpha \in \mathbb{F}_q^* = \mathbb{F}_q \setminus \{0\}$, and

\[
\rho(\omega + \omega') \leq \max \{ \rho(\omega), \rho(\omega') \} \leq \rho(\omega) + \rho(\omega')
\]

for all $\omega, \omega' \in \operatorname{Mat}_{1,s}(\mathbb{F}_q)$. Moreover, we have trivially that

\[
\rho(\omega) \leq s \tag{2.13}
\]

for all $\omega \in \operatorname{Mat}_{1,s}(\mathbb{F}_q)$.

Let $\Omega = (\omega(1), \ldots, \omega(n))^T \in \operatorname{Mat}_{n,s}(\mathbb{F}_q)$, $\omega(j) \in \operatorname{Mat}_{1,s}(\mathbb{F}_q)$, $1 \leq j \leq n$. Then we put

\[
\rho(\Omega) = \rho_q(\Omega) = \sum_{j=1}^n \rho(\omega(j)). \tag{2.14}
\]

For example, let $n = 2$, $s = 3$, and $\Omega = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then $\rho(\Omega) = \rho((110)) + \rho((001)) = 2 + 3 = 5$.

Obviously, $\rho(\Omega) > 0$ for $\Omega \neq 0$, $\rho(\alpha \Omega) = \rho(\Omega)$ for $\alpha \in \mathbb{F}_q^*$, and

\[
\rho(\Omega + \Omega') \leq \rho(\Omega) + \rho(\Omega')
\]
for all $\Omega, \Omega' \in \text{Mat}_{n,s}(\mathbb{F}_q)$. Thus, $\rho(\Omega - \Omega')$ is a metric (or a distance) on the space $\text{Mat}_{n,s}(\mathbb{F}_q)$.

Recall that by definition the Hamming weight $\kappa(\Omega) = \kappa_q(\Omega)$, $\Omega \in \text{Mat}_{n,s}(\mathbb{F}_q)$ is equal to the number of non-zero entries in the matrix $\Omega$ (cf. [16]). It is easily seen that the weights $\kappa$ and $\rho$ are related by

$$\kappa(\Omega) \leq \rho(\Omega) \leq s\kappa(\Omega), \quad (2.15)$$

and these inequalities can not be improved on the whole space $\text{Mat}_{n,s}(\mathbb{F}_q)$. In this sense the metric $\rho$ is stronger than $\kappa$ for large $s$.

### 2.3. MDS Codes in the Metric $\rho$

Given an arbitrary metric $\lambda$ on the space $\text{Mat}_{n,s}(\mathbb{F}_q)$. Using relations (2.5), (2.7), and (2.8), the metric $\lambda$ can be transferred onto the set of all rational $q$-ary points $\mathbb{Q}^n(q^s)$ by setting

$$\lambda(X) = \lambda(\langle X \rangle) \quad (2.16)$$

for the $\lambda$-weight of a point $X \in \mathbb{Q}^n(q^s)$, and

$$\lambda(X \ominus X') = \lambda(\langle X \rangle - \langle X' \rangle) \quad (2.17)$$

for the $\lambda$-distance between two points $X, X' \in \mathbb{Q}^n(q^s)$. In (2.17) the symbol $\ominus$ denotes the subtraction with respect to the addition $\oplus$, which was defined in (2.10).

For each code $C \subset \text{Mat}_{n,s}(\mathbb{F}_q)$, which is assumed to contain at least two elements, its weight $\lambda(C)$ is defined by

$$\lambda(C) = \min\{\lambda(\Omega - \Omega') : \Omega, \Omega' \in C, \Omega \neq \Omega'\} \quad (2.18)$$

Similarly, for each distribution $D \subset \mathbb{Q}^n(q^s)$, which contains at least two points, its weight $\lambda(D)$ is defined by

$$\lambda(D) = \min\{\lambda(X \ominus X') : X, X' \in D, X \neq X'\}. \quad (2.19)$$

It is obvious that $\lambda(D) = \lambda(C \langle D \rangle)$ and $\lambda(C) = \lambda(D \langle C \rangle)$.

In the sequel $\lambda$ is equal to either $\kappa$ or $\rho$. So that, we write $\kappa(C)$ and $\rho(C)$, and correspondingly $\kappa(D)$ and $\rho(D)$, for the Hamming and non-Hamming weights of codes $C \subset \text{Mat}_{n,s}(\mathbb{F}_q)$ and distributions $D \subset \mathbb{Q}^n(q^s)$.

Relations (2.15) imply the similar inequality for weights

$$\kappa(C) \leq \rho(C) \leq s\kappa(C), \quad \kappa(D) \leq \rho(D) \leq s\kappa(D). \quad (2.20)$$

Notice also that if a code or a distribution coincides with the whole space, then

$$\rho(\text{Mat}_{n,s}(\mathbb{F}_q)) = \rho(\mathbb{Q}^n(q^s)) = 1. \quad (2.21)$$

These relations follow at once from definitions (2.12), (2.14), and (2.18), (2.19).

Given an integer $0 \leq k \leq ns$. An arbitrary code $C \subset \text{Mat}_{n,s}(\mathbb{F}_q)$ consisting of $q^k$ elements is conveniently said to be an $[ns,k]_{q^s}$-code.

The following basic result was given in [25, Theorem 1]. For the reader convenience we also reproduce its proof.
**Proposition 2.1.** For every code \( C \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) its cardinality \(#\{C\}\) and weight \( \rho(C) \) are related by
\[
#\{C\} \leq q^{ns-\rho(C)+1}. \tag{2.22}
\]
In particular, every \([ns,k]_s\)-code satisfies the bound
\[
\rho(C) \leq ns - k + 1. \tag{2.23}
\]

**Proof.** In a matrix \( \Omega = (\xi_{ij}) \in \text{Mat}_{n,s}(\mathbb{F}_q), 1 \leq j \leq n, 1 \leq i \leq s \), we enumerate entries \( \xi_{ij} \) lexicographically, and mark first \( \rho(C) - 1 \) positions. Now, from definition (2.18) of the weight \( \rho(C) \) we conclude that for two distinct code words \( \Omega_1, \Omega_2 \in C \) their entries can not coincide in all other \( ns - \rho(C) + 1 \) positions, because otherwise \( \rho(\Omega_1 - \Omega_2) \leq \rho(C) - 1 \). Therefore, relation (2.22) holds and bound (2.23) follows. \( \square \)

**Proposition 2.1** leads immediately to the following.

**Definition 2.1.** An \([ns,k]_s\)-code \( C \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) is called a maximum distance separable (or briefly MDS) \([ns,k]_s\)-code in the metric \( \rho \) if its weight \( \rho(C) = ns - k + 1 \).

### 2.4. Optimum Distributions and MDS Codes in the Metric \( \rho \).

Our first result on optimum distributions can be stated as follows.

**Theorem 2.1.** Let \( D \subset \mathbb{Q}^n(q^s) \) be a distribution and \( C(D) \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) be its code (2.9). Then the following two statements are equivalent:

(i) \( D \) is an optimum \([ns,k]_s\)-distribution,

(ii) \( C(D) \) is an MDS \([ns,k]_s\)-code in the metric \( \rho \).

For the proof of Theorem 2.1 we need the following.

**Lemma 2.3.** (i) A given point \( X \in \mathbb{Q}^n(q^s) \) falls into an elementary box \( \Delta_A^{0} \in \mathcal{E}_s(q,n) \) of volume \( q^{-k} \) if and only if the weight \( \rho(X) \leq ns - k \).

(ii) Two given points \( X, X' \in \mathbb{Q}^n(q^s) \) fall simultaneously into an elementary box \( \Delta_A^{M} \in \mathcal{E}_s(q,n) \) of volume \( q^{-k} \) if and only if the distance \( \rho(X \ominus X') \leq ns - k \).

**Proof.** First of all, we notice that definition (2.11) and representation (2.5) imply the inequality
\[
x = \sum_{i=1}^{\rho(x)} \xi_i(x)q^{i-s-1} \leq (q-1) \sum_{i=1}^{\rho(x)} q^{i-s-1} < q^{\rho(x)-s} \tag{2.24}
\]
for all \( x \in \mathbb{Q}(q^s) \) and the inequality
\[
x = \sum_{i=1}^{\rho(x)} \xi_i(x)q^{i-s-1} \geq q^{\rho(x)-s-1} \tag{2.25}
\]
for \( x > 0, x \in \mathbb{Q}(q^s) \).

(i) Let \( \rho(X) \leq ns - k \). Writing inequality (2.24) for all coordinates of the point \( X = (x_1, \ldots, x_n)^T \in \mathbb{Q}^n(q^s) \) and using definition (1.5), we conclude that
\(X \in \Delta_{A(X)}^0\) with \(A(X) = (a_1(x_1), \ldots, a_n(x_n))\) where \(a_j(x_j) = s - \rho(x_j), 1 \leq j \leq n\). Therefore, we have (cf. (2.14), (2.17))

\[
a_1(x_1) + \ldots + a_n(x_n) = ns - \rho(X) \geq k,
\]

and \(0 \leq a_j(x_j) \leq s, 1 \leq j \leq n\). Hence (cf. (1.6), (1.8)), there exists an elementary box \(\Delta_A^0 \in \mathcal{E}_s(q, n)\) of volume \(q^{-k}\) such that \(\Delta_A^0 \supseteq \Delta_{A(X)}^0 \ni X\).

Let \(\rho(X) > ns - k\). Writing inequality (2.25) for all non-zero coordinates of the point \(X = (x_1, \ldots, x_n)^T \in \mathbb{Q}^n(q^s)\) and using definitions (1.5) and (1.8), we observe the following: if the point \(X\) falls into an elementary box \(\Delta_A^0 \in \mathcal{E}_s(q, n)\) with \(A = (a_1, \ldots, a_n)\), then \(0 \leq a_j \leq s - \rho(x_j)\) for \(x_j > 0\) and \(0 \leq a_j \leq s\) for \(x_j = 0, 1 \leq j \leq n\). Since \(\rho(0) = 0\), we have

\[
a_1 + \ldots + a_n \leq ns - \rho(x_1) - \ldots - \rho(x_n) = ns - \rho(X) < k.
\]

Hence (cf. (1.6)), the elementary box \(\Delta_A^0 \in \mathcal{E}_s(q, n)\) has volume strictly bigger than \(q^{-k}\). The proof of the statement (i) is complete.

(ii) Since the distance \(\rho(X \ominus X')\) is invariant under translations \(X \rightarrow X + Y, X' \rightarrow X' + Y, Y \in \mathbb{Q}^n(q^s)\), the statement (ii) follows at once from the statement (i) with the help of Lemma 2.2. \(\square\)

**Proof of Theorem 2.1.** (i) Let \(D\) be an optimum \([ns,k]_s\)-distribution. Then, by Definition 1.2, each elementary box \(\Delta_A^M \in \mathcal{E}_s(q, n)\) of volume \(q^{-k}\) contains exactly one point of \(D\), and by Lemma 2.3 (ii), the distance \(\rho(X \ominus X') > ns - k\) for any pair of distinct points \(X, X' \in D\). Therefore, by Definition 2.1, \(\mathcal{C}(D)\) is an MDS \([ns,k]_s\)-code in the metric \(\rho\).

(ii) Let \(\mathcal{C}(D)\) be an MDS \([ns,k]_s\)-code in the metric \(\rho\). Then, by Definition 2.1, the distance \(\rho(X \ominus X') > ns - k\) for any pair of distinct points \(X, X' \in D\), and by Lemma 2.3 (ii), each elementary box \(\Delta_A^M \in \mathcal{E}_s(q, n)\) of volume \(q^{-k}\) contains at most one point of \(D\). Now we observe that for a given \(A = (a_1, \ldots, a_n) \in \mathbb{N}_0^n\) with \(a_1 + \ldots + a_n = k\) and \(0 \leq a_j \leq s, 1 \leq j \leq n\), the number of the corresponding elementary boxes \(\Delta_A^M \in \mathcal{E}_s(q, n)\) with \(M = (m_1, \ldots, m_n) \in \mathbb{N}_0^n\), \(m_j \in \{0,1,\ldots,q^{a_j}-1\}, 1 \leq j \leq n\), is equal to \(q^k\), that coincides with the cardinality of \(D\). Therefore, each of these elementary boxes \(\Delta_A^M \in \mathcal{E}_s(q, n)\) of volume \(q^{-k}\) contains exactly one point of \(D\). Thus, \(D\) is an optimum \([ns,k]_s\)-distribution. \(\square\)

**2.5. Additional Remarks.** We complete the present section with few remarks.

**Remark 2.1.** In the special of \(s = 1\) the metrics \(\rho\) and \(\varkappa\) coincide. Hence, the results and methods of the present paper contain those for the Hamming metric \(\varkappa\) as a specific case. In particular, for \(s = 1\) Definition 2.1 coincides with the known definition of MDS codes in the Hamming metric (cf. [16, Chapter 11]).

It should be pointed out that both metrics \(\varkappa\) and \(\rho\) belong to a broad class of metrics of the following kind. Let \(E\) be a vectors space of finitely many dimensions over an arbitrary field. Fix a collection \(\mathcal{M}\) of subspaces \(V \subset E\) such that \(E \in \mathcal{M}, \{0\} \in \mathcal{M},\) and for any two subspaces \(V_1, V_2 \in \mathcal{M}\) their vector sum \(V_1 + V_2 \in \mathcal{M}\). Now we put

\[
\sigma(\mathcal{M}; X) = \min\{\dim V : X \in V \in \mathcal{M}\}, \quad X \in E.
\]  

(2.26)

It is obvious that \(\sigma(\mathcal{M}; \cdot)\) is a metric on \(E\).
Certainly, the metric (2.26) may be trivial for a bad choice of the collection \( M \). For example, \( \sigma(M; X) = 1 \) for all \( X \in E \setminus \{0\} \) if \( M \) contains each one dimensional subspace of \( E \).

However, specific choices of \( M \) lead to very interesting metrics. For example, \( \sigma(M; X) = 1 \) for all \( X \in E \setminus \{0\} \) if \( M \) contains each one dimensional subspace of \( E \).

A rich theory of MDS codes in metrics (2.26) can be developed for appropriate choices of collections \( M \). Furthermore, elements of such MDS codes turn out to be uniformly distributed in a set of affine subspaces in \( E \) parallel to subspaces \( V \in M \). Curiously enough, the indicated combinatorial constructions can be given over arbitrary fields, including the usual fields of real and complex numbers.

Remark 2.2. It should be of interest to determine transformations preserving the metric \( \rho \). Obviously, the symmetric group \( S_n \) of all permutations of rows of a matrix \( \Omega \in \text{Mat}_{n,s}(\mathbb{F}_q) \) preserves the weight \( \rho(\Omega) \) (and the weight \( \kappa(\Omega) \) as well). Examples of another kind can be given as follows.

Let \( T_s \) denote a group of all nonsingular lower triangular \( s \times s \) matrices over \( \mathbb{F}_q \) (with arbitrary nonzero diagonal elements). Then, it is readily seen that \( \rho(\omega v) = \rho(\omega) \) for \( \omega = (\xi_1, \ldots, \xi_s) \in \text{Mat}_{1,s}(\mathbb{F}_q) \) and \( v \in T_s \). Therefore, the product

\[
H = S_n \times T_s \times \ldots \times T_s = S_n \times T_s^n
\]

(2.27)
forms a group of linear transformations preserving the weight \( \rho \).

We shall return to the group (2.27) once again in Sec. 4.4 below in the context of MacWilliams-type theorems for the metric \( \rho \).

Remark 2.3. It is instructive to compare the metric \( \rho \) adoption in coding theory and the theory of uniform distributions. It is well known that the major field of application for coding theory is related with the transmission of information (cf. [16]). Whether the metric \( \rho \) can be used in this area? Very schematically the problem can be outlined as follows.

Suppose that we wish to transmit a sequence \( \Omega = (\eta_1, \eta_2, \ldots) \) of digits \( \eta_j \in \mathbb{F}_q \) across a noisy channel. If the channel noise generates equiprobable errors in all possible positions, the concept of the Hamming metric is quite adequate to the problem. However, the Hamming metric turns out to be very crude if possible errors form patterns of a specific shape.

Consider, for example, a channel where errors are generated by a periodic spike-wise perturbation of period \( s \). We split the sequence of digits \( \Omega \) of length \( ns \) to \( n \) blocks of length \( s \): \( \Omega = (\omega_1, \ldots, \omega_n) \), \( \omega_j \in \text{Mat}_{1,s}(\mathbb{F}_q) \), \( 1 \leq j \leq n \). In each block \( \omega_j = (\xi_1^{(j)}, \ldots, \xi_s^{(j)}) \) the errors have a tendency to occur at the left end, just after a spike pulse, whereas the probability of errors at the right end, like \((0, \ldots, 0, 1)\), is negligible.

Now, it is obvious that the non-Hamming metric defined in (2.14) depicts the actual structure of the indicated problem much more adequately. Moreover, a code \( C \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) with a large weight \( \rho(C) \) is capable of detecting and correcting all the channel errors \( \Omega' \in \text{Mat}_{n,s}(\mathbb{F}_q) \) with \( 2\rho(\Omega') < \rho(C) \), even though its Hamming weight \( \kappa(C) \) is small (see (2.20)).

We refer to [25] for another possible application of the metric \( \rho \) to the transmission of information. Presumably, efficient decoding algorithms can be also given for MDS codes in the metric \( \rho \) (cf. [16, Sections 9.6 and 10.10]). Certainly, the study
of such algorithms may be of interest only if the metric $\rho$ is really practicable for realistic systems of communication.

The foregoing implies in passing that our basic parameters $n$ and $s$ play distinct roles in applications of the metric $\rho$ to coding theory and the theory of uniform distributions. In the former case the parameter $s$ is given by constructional features of the channel, while the parameter $n$ is free and can be chosen arbitrary large if we wish to transmit long messages.

Exactly the converse situation arises from problems related with uniform distributions. Here $s \to \infty$ while $n$ is fixed, because we are interested in more and more uniformly distributed point sets in the unit cube of a given dimension $n$.

A comprehensive study of uniform bounds for main code characteristics at arbitrary values of the parameters $n$ and $s$ should be of great importance.

3. Weight Spectra

3.1. Spheres and Balls in the Metric $\rho$.

Definition 3.1. Given a code $C \subset \text{Mat}_{n,s}(\mathbb{F}_q)$. The following set of $ns$ nonnegative integers

\[ w_r(\Omega') = w_r(C; \Omega') = \#\{\Omega \in C : \rho(\Omega - \Omega') = r\}, \quad r \in \mathbb{N}_0, \quad 0 \leq r \leq ns, \quad (3.1) \]

is called the weight spectrum of the code $C$ relative to an element $\Omega' \in C$. Similarly, for a distribution $D \subset \mathbb{Q}^n(q^s)$ the set of nonnegative integers

\[ w_r(X') = w_r(D; X') = \#\{X \in D : \rho(X \ominus X') = r\}, \quad r \in \mathbb{N}_0, \quad 0 \leq r \leq ns, \quad (3.2) \]

is called the weight spectrum of the distribution $D$ relative to a point $X' \in D$.

It is obvious that weight spectra of distributions and the corresponding codes coincide: $w_r(D; X') = w_r(C(D); \Omega(X'))$.

A remarkable thing is that weight spectra of all MDS codes in the metric $\rho$ as well as optimum distributions can be evaluated precisely. For MDS codes in the Hamming metric such results are well known in coding theory (see [16, Chapter 11]).

Clearly, weights (3.1) and (3.2) are equal to the number of elements $\Omega \in C$ or points $X \in D$ lying on spheres in the metric $\rho$. Hence, we need to consider these geometric subjects closer. Here we consider only the space $\mathbb{Q}^n(q^s)$. Certainly, the corresponding definitions and results for the space $\text{Mat}_{n,s}(\mathbb{F}_q)$ can easily be given as well. Let

\[ \mathcal{S}(r) = \{X \in \mathbb{Q}^n(q^s) : \rho(X) = r\}, \quad 0 \leq r \leq ns, \quad (3.3) \]

be a sphere of radius $r \in \mathbb{N}_0$ in the metric $\rho$ on the space $\mathbb{Q}^n(q^s)$.

Split the unit interval into union of the following $s + 1$ subintervals

\[ [0, 1) = \bigcup_{0 \leq b \leq s} g_b, \quad (3.4) \]

where $g_0 = [0, q^{-s})$, $g_1 = [q^{-s}, q^{-s+1})$, \ldots, $g_s = [q^{-1}, 1]$. Then, the unit cube splits into a union of the following $(s + 1)^n$ disjoint boxes

\[ U^n = \bigcup_{B} G_B, \quad G_B = \prod_{j=1}^{n} g_{b_j}, \quad (3.5) \]
where \( B = (b_1, \ldots, b_n) \in \mathbb{N}_0^n, 0 \leq b_j \leq s, 1 \leq j \leq n \). The boxes \( G_B \) are said to be fragments.

**Lemma 3.1.** Each sphere \((3.3)\) is a disjoint union of the following fragments

\[
\mathcal{G}(r) = \bigcup_{b_1 + \ldots + b_n = r} G_B. \tag{3.6}
\]

**Proof.** Comparing \( q \)-ary representation (2.5) with the definitions of the \( \rho \)-weight (2.11) and subintervals \( g_b \) in (3.4), we observe the following: a point \( x \in \mathbb{Q}(q^s) \) belongs to a subinterval \( g_b \) if and only if its weight \( \rho(x) = b \). Thus, a point \( X = (x_1, \ldots, x_n)^T \in \mathbb{Q}^n(q^s) \) falls onto a sphere \( \mathcal{G}(r) \) if and only if \( X \) belongs to one of fragments \( G_B \) with \( b_1 + \ldots + b_n = r \).

For a given integer vector \( B = (b_1, \ldots, b_n) \in \mathbb{N}_0^n \) we denote by \( \kappa(B) \) the number of nonzeroth coordinates \( b_j, 1 \leq j \leq n \), thus, \( \kappa(B) \) is equal to the Hamming weight of the vector \( B \). Consider a subset \( S_l(r) \subset \mathcal{S}(r) \) consisting of fragments \( G_B \subset \mathcal{S}(r) \) with \( \kappa(B) = l \),

\[
S_l(r) = \bigcup_{b_1 + \ldots + b_n = r, \kappa(B) = l} G_B. \tag{3.7}
\]

From (3.6) and (3.7) we obtain that

\[
\mathcal{S}(r) = \bigcup_{0 \leq l \leq n} S_l(r). \tag{3.8}
\]

Certainly, some of subsets \( S_l(r) \) in (3.8) may be empty.

We introduce nonnegative integers \( \sigma_s(l, r), r \in \mathbb{N}_0, l \in \mathbb{N}_0, 0 \leq l \leq n \), by setting

\[
\sigma_s(l, r) = \# \{ A = (a_1, \ldots, a_l) \in \mathbb{N}^l : a_1 + \ldots + a_l = r, 0 < a_j \leq s, 1 \leq j \leq l \} \tag{3.9}
\]

for \( l \geq 1 \), for \( l = 0 \) we put \( \sigma_s(0, 0) = 1 \) and \( \sigma_s(0, r) = 0 \) for \( r \geq 1 \). In particular, for \( s = 1 \) we have

\[
\sigma_1(l, r) = \delta_{l,r}, \tag{3.10}
\]

where \( \delta_{l,r} \) is the Kronecker symbol.

Moreover, from definition (3.9), we can easily obtain the following asymptotics

\[
\sigma_s(l, st + c) = \sqrt{t} v(l, t) s^{l-1} + O(s^{l-2}) \tag{3.11}
\]

as \( s \to \infty \). In (3.11) \( 1 \leq t \leq l - 1 \), \( c \in \mathbb{N} \) are integers, and \( v(l, t) \) is the \((l - 1)\)-dimensional volume of a section of the \( l \)-dimensional unit cube \( U^l \) by the hyperplane \( x_1 + \ldots + x_l = t \). It is evident that \( v(l, t) > 0 \).

We write \( \binom{n}{i} \) for the usual binomial coefficients, moreover, \( \binom{n}{i} = 0 \) for \( i > n \).

**Lemma 3.2.** For each subset \( \mathcal{S}_l(r) \) the number of fragments \( G_B \) in union (3.7) is equal to \( \binom{n}{l} \sigma_s(l, r) \).
Proof. Let \( l = 0 \). Then, there is only one fragment \( G_B = G_0 = [0, q^{-s})^n \) with \( \kappa(B) = 0 \). It is obvious that the fragment \( G_0 \) is present in union (3.7) only for \( t = 0 \). This proves Lemma 3.2 for \( l = 0 \).

Let \( l \geq 1 \). Consider one of \( \binom{n}{l} \) possible choices of \( l \) indices \( J = \{j_1, \ldots, j_l\} \subset \{1, \ldots, n\} \). It is obvious (cf. Lemma 3.1) that the numbers of fragments \( G_B \subset S(l, r) \) with \( 0 < b_j \leq s \) for \( j \in J \) and \( b_j = 0 \) for \( j \notin J \) is equal to \( \sigma_s(l, r) \). This proves Lemma 3.2 for \( l \geq 1 \). □

Using Lemma 3.2 and representations (3.7) and (3.8), we can easily evaluate the number of points \( X \in Q^n(q^s) \) lying on a sphere \( S(r) \). First of all, we notice that

\[
\#\{Q^n(q^s) \cap G_B\} = (q - 1)^{\kappa(B)} q^{b_1 + \ldots + b_n - \kappa(B)}.
\]

Therefore, using Lemma 3.2 and (3.7), (3.8), we obtain the relation

\[
\#\{Q^n(q^s) \cap S(r)\} = \sum_{l=0}^{n} \binom{n}{l} \sigma_s(l, r)(q - 1)^l q^{-l}.
\]

(3.12)

Notice that the terms with \( l > r \) are absent in (3.12), because \( \sigma_s(l, r) = 0 \) if \( l > r \) (cf. (3.9)). Let

\[
\mathcal{B}(t) = \{X \in Q^n(q^s) : \rho(X) \leq t\}
\]

(3.13)

be a ball of radius \( t \in \mathbb{N}_0 \) in the metric \( \rho \). It is obvious that

\[
\mathcal{B}(t) = \bigcup_{0 \leq r \leq t} S(r).
\]

(3.14)

Let \( D \subset Q^n(q^s) \) be a distribution of \( N \) points. Assume that its weight \( \rho(D) \geq 2t + 1 \), \( t \in \mathbb{N}_0 \) (see (2.20)). Then the balls \( \{\mathcal{B}(t) \oplus X, X \in D\} \) are pairwise disjoint. This observation together with relations (3.12), (3.13), and (3.14) implies the following ball packing bound (cf. [25, Theorem 3])

\[
N \sum_{r=0}^{t} \sum_{l=0}^{n} \binom{n}{l} \sigma_s(l, t)(q - 1)^l q^{-l} \leq q^{ns}.
\]

(3.15)

In view of (3.10) relation (3.15) for \( s = 1 \) coincides with the well known Hamming bound for usual codes (cf. [9, 15, 16]). In a similar spirit known results of coding theory (like the Gilbert–Varshamov and Plotkin bounds) can be also given with respect to the metric \( \rho \) (cf. [25, Theorems 2 and 4]). In connection with this, we should mention a very interesting combinatorial problem on the description of perfect distributions \( D \subset Q^n(q^s) \) (and the corresponding perfect codes \( C(D) \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) in metric \( \rho \) which provide exact close-packings (without gaps and overlappings) of the space \( Q^n(q^s) \) by balls \( \{\mathcal{B}(t) \oplus X, X \in D\} \) or spheres \( \{S(r) \oplus X \in D\} \).
where \( A^* = (a_1^*, \ldots, a_n^*) \) with \( a_j^* = s - a_j, 0 \leq a_j \leq s, 1 \leq j \leq n \).

**Proof.** Our arguments here are quite similar to those in the proof of Lemma 2.3. Let \( X \in \mathfrak{B}(t) \). Writing inequality (2.24) for all coordinates of the point \( X \), we conclude that \( X \in \Delta_{A(X)}^0 \) with \( A(X) = (a_1(x_1), \ldots, a_n(x_n)) \) where \( a_j(x_j) = s - \rho(x_j), 1 \leq j \leq n \), and \( \rho(x_1) + \ldots + \rho(x_n) = \rho(X) \leq t \). Hence, there exists an elementary box \( \Delta_{A}^0 \) in the union (3.16) such that \( \Delta_{A}^0 \supseteq \Delta_{A(X)}^0 \supseteq X \).

Let \( X \not\in \mathfrak{B}(t) \), that is, \( \rho(X) > t \). Writing inequalities (2.13) and (2.25) correspondingly for zeroth and nonzeroth coordinates of the point \( X \), we observe the following: if the point \( X \) falls into an elementary box \( \Delta_{A^*}^0 \in \mathfrak{E}_s(q, n) \) with \( A^* = (a_1^*, \ldots, a_n^*) \) and \( a_j^* = s - a_j, 0 \leq a_j \leq s, 1 \leq j \leq n \), then \( \rho(x_j) \leq a_j \leq s, 1 \leq j \leq n \). Therefore, we have \( a_1 + \ldots + a_n \geq \rho(X) > t \), and hence, the elementary box \( \Delta_{A^*}^0 \) does not belong to union (3.16). \( \square \)

### 3.2. Weight Spectra of MDS Codes and Optimum Distributions.

Our main result on weight spectra can be stated as follows.

**Theorem 3.1.** Let \( D \subset \mathbb{Q}^n(q^s) \) be an optimum \([ns, k,s]-distribution \) and \( C = C(D) \) the corresponding MDS \([ns, k,s]-code \) in the metric \( \rho \). Then weights (3.1) and (3.2) are independent of elements \( \Omega \in C \) and points \( X' \in D \), and \( w_r = w_r(\Omega') = w_r(X') \) are given by

\[
 w_0 = 1, \quad w_r = 0
\]

for \( 0 \leq r < \rho(C) = ns - k + 1 \), and

\[
 w_r = \sum_{l=1}^{\rho(C)} \sigma_s(l, r) \sum_{t=0}^{r-\rho(C)} (-1)^t \left( \begin{array}{c} l \\ t \end{array} \right) q^{r-\rho(C)+1-t-1}
\]

\[
 = (q - 1) \sum_{l=1}^{\rho(C)} \sigma_s(l, r) \sum_{t=0}^{r-\rho(C)} (-1)^t \left( \begin{array}{c} l-1 \\ t \end{array} \right) q^{r-\rho(C)-t}
\]

for \( \rho(C) \leq t \leq ns \).

We wish to discuss some corollaries of Theorem 3.1. First of all, we notice that in view of (3.10) relations (3.18) for \( s = 1 \) imply

\[
 w_r = \sum_{t=0}^{r-\rho(C)} (-1)^t \left( \begin{array}{c} l \\ t \end{array} \right) q^{r-\rho(C)+1-t-1}
\]

\[
 = (q - 1) \sum_{t=0}^{r-\rho(C)} (-1)^t \left( \begin{array}{c} l-1 \\ t \end{array} \right) q^{r-\rho(C)-t}.
\]
These relations coincide with known identities (cf. [16, Chapter 11, Theorem 6]) for weight spectra of MDS codes in the Hamming metric (see Remark 2.1).

Now we consider the special case of $k = s$. Introduce the following positive integers

$$\tilde{\sigma}_s(n, r) = \# \{ A = (a_1, \ldots, a_n) \in \mathbb{N}_0^n : a_1 + \ldots + a_n = r, 0 \leq a_j \leq s, 1 \leq j \leq n \}. \quad (3.19)$$

Comparing the definitions (3.9) and (3.19), we immediately obtain the relation

$$\tilde{\sigma}_s(n, r) = \sum_{l=1}^{n} \binom{n}{l} \sigma_s(l, r), \quad r \geq 1. \quad (3.20)$$

Theorem 3.1 implies the following result.

**Theorem 3.2.** Let $D$ be an optimum $[ns, s]_s$-distribution or, that is the same (cf. Proposition 1.1), a $(0, s, n)$-net. Then weights (3.2) are given by

$$w_0 = 1, \quad w_r = 0 \quad (3.21)$$

for $0 \leq r < \rho(D) = (n - 1)s + 1$, and

$$w_r = \tilde{\sigma}_s(n, r) \sum_{t=0}^{r - \rho(D)} (-1)^t \binom{n}{t} (q^{r - \rho(D) + 1} - 1)$$

$$= \tilde{\sigma}_s(n, r)(q - 1) \sum_{t=0}^{r - \rho(D)} (1 - 1)^t \binom{n - 1}{t} q^{\rho(D) - t} \quad (3.22)$$

for $\rho(D) \leq r \leq ns$. In particular,

$$w_r = \tilde{\sigma}_s(n, r)(q - 1)^n q^{\rho(D) - n + 1} \quad (3.23)$$

for $\rho(D) + n - 1 \leq r \leq ns$.

Proof. Evidently, relations (3.17) imply (3.21). Let $\rho(D) = (n - 1)s + 1 \leq r \leq ns$, then $\sigma_s(l, r) = 0$ for $1 \leq l \leq n - 1$ and $\sigma_s(n, r) = \tilde{\sigma}_s(n, r)$ (see (3.9), (3.19), and (3.20)). Hence, relations (3.18) imply (3.22). Finally, relations (3.23) follow at once from (3.22), because, for $\rho(D) + n - 1 \leq r \leq ns$ the summation in (3.22) is taken over all indices $0 \leq t \leq n - 1$. \[\square\]

By definition, weights (3.1) and (3.2) are nonnegative. Together with Theorems 3.1 and 3.2 this leads to a series of necessary conditions for the existence of optimum distributions and MDS codes. From relation (3.17) we find the first nontrivial terms in the weight spectrum

$$w_{\rho(D)} = \tilde{\sigma}_s(n, \rho(D))(q - 1), \quad (3.24)$$

and

$$w_{\rho(D) + 1} = (q - 1) \sum_{l=1}^{n} \binom{n}{l} (q + 1 - l) \sigma_s(l, \rho(D) + 1). \quad (3.25)$$
Notice that relation (3.20) has been used in (3.24). Moreover, writing relation (3.25), we assume that \( \rho(D) + 1 \leq ns \); this means that \( s > 1 \) if \( k = s \).

For \( k = s \) relation (3.25) takes the form

\[
 w_{\rho(D)+1} = (q - 1)\sigma_s(n, \rho(D) + 1)(q - n + 1),
\]

(3.26)

because \( \sigma_s(l, (n - 1)s + 2) = 0 \) for \( l < n \) (see (3.9)).

Since the weight (3.26) is nonnegative, we obtain the following necessary condition for the existence of optimum \([ns, s]_s\)-distributions:

\[
 q \geq n - 1.
\]

(3.27)

For \((0, n, s)\)-nets condition (3.27) is well known (cf. [4, Lemma 3.28] and [20, Corollary 5.11]). The above arguments can be considered as an interpretation of condition (3.27) in terms of weight spectra.

From relation (3.25) we can also derive the following general result.

**Proposition 3.1.** Suppose that \( k = st \) for given \( 1 \leq t \leq n - 1 \), and for an infinite sequence of \( s \in \mathbb{N} \) there exist optimum \([ns, k]_s\)-distributions (or, that is the same, MDS \([ns, k]_s\)-codes in the metric \( \rho \)). Then condition (3.27) holds.

**Proof.** Substituting asymptotics (3.11) to relation (3.25), we find that

\[
(q - 1)(q + 1 - n)\sqrt{n}v(n, n - t)s^{n-1} + O(s^{n-2}) \geq 0, \quad s \to \infty,
\]

with the constant \( v(n, n - t) > 0 \), and inequality (3.27) follows. \( \square \)

**Remark 3.1.** For given \( 1 \leq k \leq ns, q \geq n - 1 \) and arbitrary \( s \in \mathbb{N} \) optimum \([ns, k]_s\)-distributions will be explicitly constructed in Section 5 (see Theorem 5.2 below). Hence, condition (3.27) is actually necessary and sufficient for the existence of infinite sequences of optimum distributions and MDS codes in the metric \( \rho \).

### 3.3. Proof of Theorem 3.1.

First of all, we notice that Theorem 3.1 may be proved only for optimum distributions, because a given MDS \([ns, k]_s\)-code \( C \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) can be identified with the corresponding optimum \([ns, k]_s\)-distribution \( D = D(C) \subset \mathbb{Q}^n(q^s) \) by Theorem 2.1.

Next, from Definition 2.1 and Theorem 2.1 we conclude that if \( D \subset \mathbb{Q}^n(q^s) \) is an optimum \([ns, k]_s\)-distribution, then so is any distribution \( D \oplus X' \) shifted by an arbitrary vector \( X' \in \mathbb{Q}^n(q^s) \). Thus, without loss of generality, in proving of Theorem 3.1 we may assume that \( 0 \in D \) and consider weight spectrum (3.2) only relative to the origin \( X' = 0 \). In so doing, we put \( w_r = w_r(0) \).

Notice also that relations (3.17) hold trivially, in view of Definition 2.1 and Theorem 2.1. Thus, we have to consider only weights \( w_r \) with \( r \geq \rho(D) = ns - k + 1 \).

With the above conventions we turn to the proof of relations (3.18). From Lemma 3.1 we obtain the relation

\[
 w_r = \#\{D \cap \mathcal{G}(r)\} = \sum_{b_1+\ldots+b_n=r} \#\{D \cap G_B\}.
\]

(3.28)

Consider a fragment \( G_B, B = (b_1, \ldots, b_n) \), in (3.28) with \( \kappa(B) = l \geq 1 \). Notice that a fragment \( G_B \) with \( \kappa(B) = 0 \) is absent in (3.28) since \( r > 0 \). Suppose that
\[ b_j \neq 0 \quad (0 < b_j \leq s) \] for \( j \in J \) and \( b_j = 0 \) for \( j \notin J \), where \( J = \{j_1, \ldots, j_l\} \subseteq \{1, \ldots, n\} \) is a subset of \( l \) indices. Using (3.4), (3.5), we conclude that the fragment \( G_B \) has the following representations in terms of elementary boxes (1.5)

\[
G_B = \Delta_{A_0}^0 \setminus \bigcup_{i \in J} \Delta_{A_i}^0,
\]  

(3.29)

where \( A_0 = (a_1^{(0)}, \ldots, a_n^{(0)}) \) with

\[ a_j = s - b_j, \quad 1 \leq j \leq n, \]  

(3.30)

and \( A_i = (a_1^{(i)}, \ldots, a_n^{(i)}) \) with

\[
a_j^{(i)} = \begin{cases} 
  s & \text{if } j \notin J, \\
  s - b_j & \text{if } j \in J \text{ and } j \neq i, \\
  s - b_i + 1 & \text{if } j \in J \text{ and } j = i.
\end{cases}
\]  

(3.31)

Obviously, we have the equality

\[ a_1^{(0)} + \ldots + a_n^{(0)} = ns - b_1 - \ldots - b_n = ns - r. \]  

(3.32)

Let \( I = \{i_1, \ldots, i_t\} \subseteq J = \{j_1, \ldots, j_l\} \) be a subset of \( t \) indices. Then,

\[
\Delta_{A_{i_1}}^0 \cap \ldots \cap \Delta_{A_{i_t}}^0 = \Delta_{A_I}^0,
\]  

(3.33)

where \( A_I = (a_1^{(I)}, \ldots, a_n^{(I)}) \) with

\[
a_j^{(I)} = \max\{a_j^{(i_1)}, \ldots, a_j^{(i_t)}\} = \begin{cases} 
  s - b_j & \text{if } j \notin I, \\
  s - b_j + 1 & \text{if } j \in I.
\end{cases}
\]  

(3.34)

Notice that relations (3.31) have been used in (3.34). Obviously, we have the equality

\[ a_1^{(I)} + \ldots + a_n^{(I)} = ns - r + l. \]  

(3.35)

Applying the principle of inclusion and exclusion to the summand \( \#\{D \cap G_B\} \) in (3.28), and using relations (3.29), (3.32), we obtain the formula

\[
\#\{D \cap G_B\} = \{D \cap \Delta_{A_0}^0\} - \sum_{t=1}^{l} \sum_{\substack{I \subseteq J, \\
\#\{I\} = t}} (-1)^t \#\{D \cap \Delta_{A_I}^0\},
\]  

(3.36)

where the inner sum is taken over all \( \binom{l}{t} \) subsets \( I = \{i_1, \ldots, i_t\} \subseteq J = \{j_1, \ldots, j_l\} \) of \( t \) indices.

From relations (3.30) and (3.34) we conclude that all elementary boxes \( \Delta_{A_0}^0 \) and \( \Delta_{A_I}^0 \) in (3.36) belong to collection (1.8). Now we observe the following. Let an elementary box \( \Delta_{A}^0 \in \mathcal{E}_s(q, n) \), then

\[
\#\{D \cap \Delta_A^0\} = \begin{cases} 
  q^{s-a_1-\ldots-a_n} & \text{if } a_1 + \ldots + a_n < k, \\
  1 & \text{if } a_1 + \ldots + a_n \geq k.
\end{cases}
\]  

(3.37)
The first equality in (3.37) follows from Lemma 1.2 (i), and the second follows from Lemma 1.2 (ii) and the above assumption that \( 0 \in D \).

Substituting (3.37) to (3.36), and using (3.32) and (3.35), we find that

\[
\#\{D \cap G_B\} = \sum_{t=0}^{r-ns-k-1} (-1)^t \binom{l}{t} q^{r-ns-k-t} + \sum_{t=r-ns-k}^l (-1)^t \binom{l}{t}
\]

\[
= \sum_{t=0}^{r-\rho(D)} (-1)^t \binom{l}{t} (q^{r-\rho(D)+1-t} - 1)
\]

\[
= (q-1) \sum_{t=0}^{r-\rho(D)} (-1)^t \binom{l-1}{t} q^{r-\rho(D)-t}.
\]

(3.38)

Notice that the well known identities for binomial coefficients:

\[
\sum_{t=0}^l (-1)^t \binom{l}{t} = 0 \quad \text{and} \quad \binom{l}{t} = \binom{l-1}{t} + \binom{l-1}{t-1}
\]

have been used in calculation (3.38).

Relation (3.28) can be written in the form

\[
w_r = \sum_{l=1}^n \sum_{B: \kappa(B) = l} \#\{D \cap G_B\},
\]

(3.39)

where the inner sum is taken over all fragments \( G_B \subset \Theta(r) \) with \( \kappa(B) = l \). The number of such fragments is equal to \( \binom{l}{t} \sigma_s(l, r) \) by Lemma 3.2.

Substituting (3.38) to (3.39), we obtain the relation in question (3.28).

The proof of Theorem 3.1 is complete. □

4. Linear Codes and Distributions

4.1. Duality for Codes and Distributions.

Definition 4.1. A subset \( C \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) is called a linear code if \( C \) is a subspace in \( \text{Mat}_{n,s}(\mathbb{F}_q) \). Similarly, a subset \( D \subset \mathbb{Q}^n(q^s) \) is called a linear distribution if \( D \) is a subspace in the vector space \( \mathbb{Q}^n(q^s) \) over \( \mathbb{F}_q \) with respect to the arithmetic operations (2.10).

For linear codes and distributions the definitions of their weights (2.17) and (2.18) take the form

\[
\lambda(C) = \min\{\lambda(\Omega) : \Omega \in C \setminus \{0\}\}, \quad \lambda(D) = \min\{\lambda(X) : X \in D \setminus \{0\}\},
\]

(4.1)

where \( \lambda \) is one of the metrics \( \kappa \) or \( \rho \).

Notice that relations (4.1) are quite similar to the known definition of the homogeneous minimum of a lattice in Euclidean spaces (cf. [9] and [16]).
We introduce the following inner product $\langle \cdot , \cdot \rangle$ on the space $\text{Mat}_{n,s}(\mathbb{F}_q)$. At first, let $n = 1$ and $\omega_1 = (\xi_1', \ldots , \xi_s')$, $\omega_2 = (\xi_1'', \ldots , \xi_s'') \in \text{Mat}_{1,s}(\mathbb{F}_q)$, then we put

$$\langle \omega_1, \omega_2 \rangle = \langle \omega_2, \omega_1 \rangle = \sum_{i=1}^{n} \xi_i' \xi_{i+1}'' = 1.$$  \hspace{1cm} (4.2)

Now, let $\Omega_i = (\omega_i^{(1)}, \ldots, \omega_i^{(n)})^T \in \text{Mat}_{n,s}(\mathbb{F}_q)$, $i = 1, 2$, where $\omega_i^{(j)} \in \text{Mat}_{1,s}(\mathbb{F}_q)$, $1 \leq j \leq n$. Then we put

$$\langle \Omega_1, \Omega_2 \rangle = \langle \Omega_2, \Omega_1 \rangle = \sum_{j=1}^{n} \langle \omega_1^{(j)}, \omega_2^{(j)} \rangle.$$  \hspace{1cm} (4.3)

Certainly, instead of (4.2) we could write the inner product in a common form $\langle \omega_1, \omega_2 \rangle = \xi_1' \xi_1'' + \ldots + \xi_s' \xi_s''$. In doing so, $\langle \omega_1, \omega_2 \rangle = \langle \omega_1, \Phi \omega_2 \rangle$ with a nonsingular $s$ by $s$ matrix $\Phi_0$ over $\mathbb{F}_q$. Recall that an arbitrary nonsingular bilinear form $\Phi(\omega_1, \omega_2) = \langle \omega_1, \Phi \omega_2 \rangle$, det $\Phi \neq 0$, could be used as an appropriate inner product on the space $\text{Mat}_{1,n}(\mathbb{F}_q) \simeq \mathbb{F}_q^n$ (cf. [14, Chapters 13, 14]). Definition (4.2) is preferable here, because otherwise, the matrix $\Phi_0$ would be involved in all subsequent considerations.

Using relation (2.8), we can transfer inner product (4.3) onto the vector space of rational $q$-ary points $\mathbb{Q}^n(q^s)$ by setting

$$\langle X_1, X_2 \rangle = \langle \Omega(X_1), \Omega(X_2) \rangle, \quad X_1, X_2 \in \mathbb{Q}^n(q^s).$$  \hspace{1cm} (4.4)

**Definition 4.2.** Given a linear code $C \subset \text{Mat}_{n,s}(\mathbb{F}_q)$, the dual code $C^\perp \subset \text{Mat}_{n,s}(\mathbb{F}_q)$ is defined by

$$C^\perp = \{ \Omega_2 \in \text{Mat}_{n,s}(\mathbb{F}_q) : \langle \Omega_1, \Omega_2 \rangle = 0 \quad \text{for all} \quad \Omega_1 \in C \}.$$  \hspace{1cm} (4.5)

Similarly, for a given linear distribution $D \subset \mathbb{Q}^n(q^s)$ the dual distribution $D^\perp \subset \mathbb{Q}^n(q^s)$ is defined by

$$D^\perp = \{ X_2 \in \mathbb{Q}^n(q^s) : \langle X_1, X_2 \rangle = 0 \quad \text{for all} \quad X_1 \in D \}.$$  \hspace{1cm} (4.6)

The following simple corollaries of Definition 4.2 should be mentioned. Dual codes (4.5) and distributions (4.6) are also linear subspaces. Moreover,

$$C^\perp = C, \quad D^\perp = D,$$  \hspace{1cm} (4.7)

and (cf. (2.9))

$$C \langle D^\perp \rangle = (C \langle D \rangle)^\perp, \quad D \langle C^\perp \rangle = (D \langle C \rangle)^\perp.$$  \hspace{1cm} (4.8)

Thus, in view of (4.7), linear codes $C, C^\perp \subset \text{Mat}_{n,s}(\mathbb{F}_q)$ and distributions $D, D^\perp \subset \mathbb{Q}^n(q^s)$ are pairs of mutually dual subspaces.

Let $C \subset \text{Mat}_{n,s}(\mathbb{F}_q)$ and $D \subset \mathbb{Q}^n(q^s)$ be subspaces of dimension $k$, then $C^\perp$ and $D^\perp$ are subspaces of dimension $k^\perp = ns - k$. This implies in passing the following relations for their cardinalities

$$\# \{ C \} = \# \{ D \} = q^k, \quad \# \{ C^\perp \} = \# \{ D^\perp \} = q^{k^\perp},$$  \hspace{1cm} (4.9)
and
\[ \#\{C\} \#\{C^\perp\} = \#\{D\} \#\{D^\perp\} = q^{ns}. \] (4.10)

Furthermore, let \( C \) and \( D \) be linear an \([ns, k]_s\)-code and, correspondingly, an \([ns, k]_s\)-distribution, then \( C^\perp \) and \( D^\perp \) are linear an \([ns, ns - k]_s\)-code and, correspondingly, \([ns, ns - k]_s\)-distribution.

Our results on metric properties of mutually dual codes and distributions can be stated as follows.

**Theorem 4.1.** (i) Let \( C \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) be a linear MDS \([ns, k]_s\)-code in the metric \( \rho \). Then \( C^\perp \) is a linear MDS \([ns, ns - k]_s\)-code in the metric \( \rho \).

(ii) Let \( D \subset \mathbb{Q}^n(q^s) \) be a linear optimum \([ns, k]_s\)-distribution. Then \( D^\perp \) is a linear optimum \([ns, ns - k]_s\)-distribution.

It is notable that linear \((\delta, n, s)\)-nets (cf. Definition 1.1) can also be characterized completely in terms of their dual distributions (cf. [20, Theorems 6.10 and 6.14]).

**Theorem 4.2.** Let \( D, D^\perp \subset \mathbb{Q}^n(q^s) \) be mutually dual linear distributions of dimensions \( k = s \) and \( k^\perp = (n - 1)s \), respectively. Then the following two statements are equivalent:

(i) \( D \) is a \((\delta, n, s)\)-net of deficiency \( \delta \),

(ii) \( D^\perp \) has a weight \( \rho(D^\perp) \geq s + 1 - \delta \).

It should be noted that the statements of Theorem 4.1 for \( k = s \) and Theorem 4.2 for \( \delta = 0 \) actually coincide (cf. Proposition 1.1 and Theorem 2.1).

For the proof of Theorems 4.1 and 4.2 we need to recall some known facts about the Fourier transform on vector spaces over finite fields. For details we refer to [15] and [16]. Notice also that the necessary facts are given here in the form adapted to the vector spaces \( \mathbb{Q}^n(q^s) \).

### 4.2. Fourier Transform on the Space of Distributions

Introduce a function \( \Psi(X_1, X_2), X_1, X_2 \in \mathbb{Q}^n(q^s) \), by setting

\[ \Psi(X_1, X_2) = \exp \left( 2\pi \sqrt{-1} \frac{1}{p} \text{Tr} \langle X_1, X_2 \rangle \right), \] (4.11)

where we write \( \text{Tr} \xi = \xi + \xi^p + \xi^{p^2} + \ldots + \xi^{p^{n-1}} \) for the trace of an element \( \xi \in \mathbb{F}_q \), \( q = p^e \), over \( \mathbb{F}_p \).

It is obvious that function (4.11) satisfies the relation

\[ \Psi(X, X_1 \oplus X_2) = \Psi(X, X_1) \Psi(X, X_2). \] (4.12)

Hence, for a given \( X \) the function \( \Psi(X, \cdot) \) is an additive character on the vector space \( \mathbb{Q}^n(q^s) \). One can show (cf. [15, Section 5.1]) that each character of the additive group of \( \mathbb{Q}^n(q^s) \) coincides with \( \Psi(X, \cdot) \) for a suitable \( X \).

The Fourier transform \( \hat{f} \) of a complex-valued function \( f : \mathbb{Q}^n(q^s) \to \mathbb{C} \) is defined by

\[ \hat{f}(Y) = \sum_{X \in \mathbb{Q}^n(q^s)} \Psi(Y, X) f(X). \] (4.13)

The following results are well known in the theory of Abelian groups (see, for example, [15, Chapters 5, 9] and [16, Chapter 5]).
Lemma 4.1. Let $D$, $D^\perp \subset \mathbb{Q}^n(q^*)$ be mutually dual linear distributions. Then,

(i) One has the relation

\[ \sum_{X \in D} \Psi(Y, X) = \begin{cases} \# \{D\} & \text{if } Y \in D^\perp, \\ 0 & \text{if } Y \not\in D^\perp. \end{cases} \] (4.14)

(ii) One has the Poisson summation formula

\[ \sum_{X \in D} f(X) = q^{-ns} \# \{D\} \sum_{Y \in D^\perp} \tilde{f}(Y) \] (4.15)

for an arbitrary function $f$.

Consider affine subspaces $V_A^M = \mathbb{Q}^n(q^*) \cap \Delta_A^M$, $\Delta_A^M \in \mathcal{C}(q, n)$ as distributions in $\mathbb{Q}^n(q^*)$ (see Lemma 2.2). We write $\chi^M_A(\cdot)$ for the indicator function of $V_A^M$, that is,

\[ \chi^M_A(X) = \begin{cases} 1 & \text{if } X \in V_A^M, \\ 0 & \text{if } X \not\in V_A^M. \end{cases} \] (4.16)

We wish to find out distributions $(V_A^0)^\perp$ dual to subspaces $V_A^0$ and evaluate the Fourier transform $\tilde{\chi}^M_A(\cdot)$ of $\chi^M_A(\cdot)$.

Notice that in view of (4.12), (4.13), and (4.16) we have the relation

\[ \tilde{\chi}^M_A(Y) = \Psi(Y, Y_A^M) \chi^0_A(Y). \] (4.17)

Lemma 4.2. For a given $A = (a_1, \ldots, a_n) \in \mathbb{N}_0^n$, $0 \leq a_j \leq s$, define the vector $A^* = (a_1^*, \ldots, a_n^*) \in \mathbb{N}_0^n$, $0 \leq a_j^* \leq s$, $1 \leq j \leq n$, by $a_j^* = s - a_j$, $1 \leq j \leq n$. Then one has the relations

\[ (V_A^0)^\perp = V_A^{0*}, \] (4.18)

and

\[ \tilde{\chi}^0_A(Y) = q^{ns-a_1-\ldots-a_n} \chi_A^{0*}(Y). \] (4.19)

Proof. From Lemma 2.2 we derive that the subspace $V_A^0$ consists of points $X = (x_1, \ldots, x_n)^T \in \mathbb{Q}^n(q^*)$ with coordinates

\[ x_j = \sum_{i=1}^{s-a_j} \xi_j(x_j) q^{i-s-1}, \quad 1 \leq j \leq n, \]

where $\xi_j(x_j)$ are arbitrary elements of the field $\mathbb{F}_q$. Using definitions (4.2), (4.3), (4.4), and (4.6), we conclude that the dual subspace $(V_A^0)^\perp$ consists of points $Y = (y_1, \ldots, y_n)^T \in \mathbb{Q}^n(q^*)$ which satisfy the equations

\[ \xi_{s+1-i}(y_j) = 0 \quad \text{for } 1 \leq i \leq s - a_j, \quad 1 \leq j \leq n, \]
or in an equivalent form

$$\xi_i(y_j) = 0 \quad \text{for} \quad s \geq i \geq s + 1 - a^*_j, \quad 1 \leq j \leq n.$$ 

By Lemma 2.2 this implies relation (4.18).

Relation (4.19) follows at once from (4.18). It suffices to write relation (4.14) for $D = V^0_A$ and take into account that $\#\{V^0_A\} = q^{a_s-a_1-\ldots-a_n}$ (cf. relation (4.9) and Lemma 2.2). \( \Box \)

**Lemma 4.3.** Let $D, D^\perp \subset \mathbb{Q}^n(q^s)$ be mutually dual linear distributions. Then for each elementary box $\Delta^M_A \in \mathcal{E}_s(q,n)$ one has the relation

$$\#\{D \cap \Delta^M_A\} = q^{-a_1-\ldots-a_n} \#\{D\} \sum_{Y \in D^\perp} \Psi(Y, Y^M_A) \chi^0_A(Y). \quad (4.20)$$

In particular,

$$\#\{D \cap \Delta^0_A\} = q^{-a_1-\ldots-a_n} \#\{D\} \#\{D^\perp \cap \Delta^0_A\}. \quad (4.21)$$

**Proof.** It suffices to substitute (4.17) and (4.19) to the Poisson summation formula (4.15) and use the relation

$$\#\{D \cap \Delta^M_A\} = \sum_{X \in D} \chi^M_A(X). \quad \Box$$

**4.3. Proof of Theorems 4.1 and 4.2.** Theorems 4.1 and 4.2 are direct corollaries of the following general result.

**Proposition 4.1.** Let $D, D^\perp \subset \mathbb{Q}^n(q^s)$ be mutually dual linear distributions of dimensions $d$ and $d^\perp = ns - d$, respectively. Then, for a given integer $0 \leq \delta \leq d$ the following two statements are equivalent:

(i) Each elementary box $\Delta^M_A \in \mathcal{E}_s(q,n)$ of volume $q^{-d+\delta}$ contains exactly $q^\delta$ points of $D$.

(ii) $D^\perp$ has a weight $\rho(D^\perp) \geq ns - d^\perp - \delta + 1 = d - \delta + 1$.

By relation (4.8) the statement (i) and (ii) of Theorem 4.1 are equivalent. At the same time Theorem 4.1 (ii) follows at once from Theorem 2.1 and Proposition 4.1 for $d = k$ and $\delta = 0$. Similarly, Theorem 4.2 follows from Definition 1.1 and Proposition 4.1 for $d = s$ and $\delta$ equal to the deficiency of the corresponding net.

The proof of Theorems 4.1 and 4.2 is complete. \( \Box \)

**Proof of Proposition 4.1.** (i) Let the statement (i) be held. Then, writing equality (4.21) for an elementary box $\Delta^0_A \in \mathcal{E}_s(q,n)$ of volume $q^{-d+\delta}$, and using relations (1.6) and (4.9), we conclude that $\#\{D^\perp \cap \Delta^0_A\} = 1$ for an arbitrary elementary box $\Delta^0_A \in \mathcal{E}_s(q,n)$ with $a^*_1 + \ldots + a^*_n = ns - d + \delta = d^\perp + \delta$. Hence each indicated elementary boxes $\Delta^0_A$ contains a single point $Y = 0$ belonging to $D^\perp$. By Lemma 3.3 this means that the ball $\mathcal{B}(t) \subset \mathbb{Q}^n(q^s)$ of radius $t = ns - d^\perp - \delta = d - \delta$ does not contain points of $D^\perp$ other than the origin $Y = 0$. Therefore $\rho(D^\perp) \geq ns - d^\perp - \delta + 1 = d - \delta + 1$ (cf. definition (4.1)).
(ii) Let the statement (ii) be held. Then, by Lemma 3.3 each elementary box $\Delta_{A^*}^0 \in \mathcal{E}_s(q, n)$ with $a_1^* + \ldots + a_n^* = d + \delta = ns - d + \delta$ contains a single point $Y = 0$ belonging to $D^\perp$. Writing equality (4.20) for the indicated boxes $\Delta_{A^*}^0$, we find that $\# \{D \cap \Delta_{A^*}^M \} = q^\delta$ for each elementary box $\Delta_{A^*}^M \in \mathcal{E}_s(q, n)$ of volume $q^{-d + \delta}$. □

4.4. Enumerators of Mutually Dual Distributions. Relations (4.21) can be expressed in terms of the corresponding generating functions. For a given distribution $D \subset \mathbb{Q}^n(q^s)$ we define its box enumerator $\varphi(D; z_1, \ldots, z_n)$, $(z_1, \ldots, z_n) \in \mathbb{C}^n$, by

$$\varphi(D; z_1, \ldots, z_n) = \sum_{0\leq a_j \leq s, \atop 1 \leq j \leq n} \# \{D \cap \Delta_{A^*}^0 \} z_1^{a_1} \ldots z_n^{a_n}$$

$$= \sum_{X \in D} \sum_{0\leq a_j \leq s, \atop 1 \leq j \leq n} \chi_{A^*}^0(X) z_1^{a_1} \ldots z_n^{a_n}. \quad (4.22)$$

Expression (4.22) is a polynomial of degree $s$ in each complex variable $z_j$.

Substituting relation (4.21) to (4.22), we immediately obtain the following.

**Theorem 4.3.** Let $D$ and $D^\perp \subset \mathbb{Q}^n(q^s)$ be linear mutually dual distributions. Then, one has the relation

$$\varphi(D; z_1, \ldots, z_n) = \# \{D\} \left(\frac{z_1 \ldots z_n}{q^n}\right)^s \varphi \left(\frac{D^\perp; q}{z_1}, \ldots, \frac{q}{z_n}\right). \quad (4.23)$$

For a given linear distribution $D \subset \mathbb{Q}^n(q^s)$ we can also introduce its weight enumerator $W(D; z)$, $z \in \mathbb{C}$, by

$$W(D; z) = \sum_{r=0}^{ns} w_r(D) z^r = \sum_{r=0}^{ns} \# \{D \cap \mathcal{S}(r)\} z^r, \quad (4.24)$$

where $w_0(D) = 1$, $w_1(D), \ldots, w_{ns}(D)$ is the weight spectrum of $D$ in the metric $\rho$ (cf (4.1)), and $\mathcal{S}(r)$ is sphere (3.3).

The Hamming metric weight enumerators of mutually dual codes are related by the well known MacWilliams identity (see [16, Chapter 5, Theorem 13]). Is there an extension of such identities to the metric $\rho$?

Close inspection of this question shows that the problem is intimately related with the action of group (2.27) preserving the metric $\rho$. One can check that the group $H$ is transitive on each sphere $\mathcal{S}(r)$ only in the two special cases of $s = 1$ or $n = 1$. For arbitrary $n$ and $s$ spheres $\mathcal{S}(r)$ split into finitely many $H$-orbits. It turns out that weight enumerators associated with such $H$-orbits satisfy remarkable relations for mutually dual codes and distributions. Such relations should be treated as a proper extension of the MacWilliams identities to the metric $\rho$. Unfortunately, within the limits of the present paper we can not discuss these questions in detail. Here we consider only the special case of $n = 1$. The author hopes to return to this matter elsewhere.

**Theorem 4.4.** (i) For every linear distribution $D \subset \mathbb{Q}(q^s)$ its box and weight enumerators $\varphi(D; z)$ and $W(D; z)$ are related by

$$W(D; z) = z^s(1 - z) \varphi \left(D; \frac{1}{z}\right) + \# \{D\} z^{s+1}. \quad (4.25)$$
(ii) For linear mutually dual distributions $D$ and $D^\perp \subset \mathbb{Q}(q^s)$ one has the identity

$$(qz - 1)\mathcal{W}(D; z) + 1 - z = \#\{D\} z^{s+1} \left(q(1-z)\mathcal{W}\left(D^\perp; \frac{1}{qz}\right) + qz - 1\right),$$

(4.26)

or more symmetrically,

$$v(D; z) = \#\{D\} q z^{s+2} v\left(D^\perp; \frac{1}{qz}\right),$$

(4.27)

where $v(D; z) = (qz - 1)\mathcal{W}(D; z) + 1 - z$.

Proof. By Lemma 3.1 every sphere $\mathcal{S}(r) \subset \mathbb{Q}(q^s)$ consists of a single fragment $g_r$. Precisely, the spheres have the following structure: $\mathcal{S}(0) = [0, q^{-s}) = \Delta^0_{q-s}$, and

$$\mathcal{S}(r) = [q^{-s+r-1}, q^{-s+r}) = \Delta^0_{q-s-r} \setminus \Delta^0_{q-s-r+1}, 1 \leq r \leq s,$$

where $\Delta^0_a = [0, q^{-a}), 0 \leq a \leq s$, are elementary boxes (1.5) for $n = 1$. Notice also that $\#\{D \cap \Delta^0_a\} = \#\{D\}$, and $\#\{D \cap \Delta^0_{q-s}\} = 1$, since $D$ is a linear distribution.

Substituting the foregoing relations to (4.24), we find that

$$\mathcal{W}(D, z) = 1 + \sum_{r=1}^{s} (\#\{D \cap \Delta^0_{q-s-r}\} - \#\{D \cap \Delta^0_{q-s-r+1}\}) z^r$$

$$= 1 + \sum_{a=0}^{s} \#\{D \cap \Delta^0_a\} z^{s-a} - \sum_{a=0}^{s} \#\{D \cap \Delta^0_a\} z^{s+1-a}$$

$$- \#\{D \cap \Delta^0_{q-s}\} + \#\{D \cap \Delta^0_{q-s+1}\} z^{s+1}$$

$$= \varphi\left(D; \frac{1}{z}\right) z^s - \varphi\left(D; \frac{1}{z}\right) z^{s+1} + \#\{D\} z^{s+1}.$$

This proves relation (4.25).

(ii) Identities (4.26) and (4.27) follow from relations (4.23) and (4.25) by a direct calculation. □

4.5. Parity-check Matrices and $\rho$-Weights. Linear $[ns, k]$-codes can be treated as null subspaces of linear surjections $\text{Mat}_{n, s}(\mathbb{F}_q) \to \mathbb{F}_q^k$. For a given code $C$ such a mapping, written in a fixed basis, is said to be a parity-check matrix of $C$.

More precisely, let $\mathcal{H}$ be an $ns$ by $k$ matrix over $\mathbb{F}_q$ of rank $k \geq 1$. Write $\mathcal{H} = (\mathcal{H}_1, \ldots, \mathcal{H}_n)$, where $\mathcal{H}_j$, $1 \leq j \leq n$, are $s$ by $k$ matrices. Then, a linear $[ns, k]$-code $C \subset \text{Mat}_{n, s}(\mathbb{F}_q)$ with the parity-check matrix $\mathcal{H}$ is defined by the following $k$ linear independent equations

$$\mathcal{H}_1 \omega^T + \ldots + \mathcal{H}_n \omega^T_n = 0,$$

(4.28)

where $\Omega = (\omega_1, \ldots, \omega_n) \in C$ and $\omega_j \in \text{Mat}_{1, s}(\mathbb{F}_q), 1 \leq j \leq n$.

We wish to evaluate the $\rho$-weight of a linear code in terms of its parity-check matrix. For the Hamming weight the corresponding result is well known (cf. [16, Chapter 1, Theorem 10]).
Introduce the following characteristic of a matrix $H = (H_1, \ldots, H_n)$. For given integers $0 \leq d_j \leq s$, $1 \leq j \leq n$, we form a set $Q(d_1, \ldots, d_n)$ of $d_1 + \ldots + d_n$ vectors in $\mathbb{F}_q^k$ selecting first $d_1$ columns from the matrix $H_1$, first $d_2$ columns from the matrix $H_2$, etc. Now we put

$$\rho^\sharp(H) = \min \sum_{j=1}^n d_j,$$

where the minimum is extended over all integers $(d_1, \ldots, d_n) \neq (0, \ldots, 0)$ with linear dependent sets $Q(d_1, \ldots, d_n)$.

Characteristic (4.29) was originally introduced in [20, Definitions 6.8 and 7.1] for linear $(\delta, s, n)$-nets. It turns out that quantity (4.29) coincides with the $\rho$-weight of the corresponding code (4.28).

**Proposition 4.2.** Let $C \subset \text{Mat}_{n,s}(\mathbb{F}_q)$ be a linear code with a parity-check matrix $H$. Then

$$\rho(C) = \rho^\sharp(H).$$

**Proof.** For each $\Omega \in C \setminus \{0\}$ columns of the matrices $H_j$, $1 \leq j \leq n$, are related by (4.28). Comparing definitions (2.12), (2.14) with (4.28), we conclude the following: for given integers $0 \leq d_j \leq s$, $1 \leq j \leq n$, $(d_1, \ldots, d_n) \neq (0, \ldots, 0)$, there exists a code word $\Omega = (\omega_1, \ldots, \omega_n) \in C \setminus \{0\}$ with $\rho(\omega_j) = d_j$, $1 \leq j \leq n$, if and only if the set $Q(d_1, \ldots, d_n)$ is linear dependent. Since $\rho(\Omega) = d_1 + \ldots + d_n$, equality (4.30) follows at once from (4.29) and definition (4.1) for the weight $\rho(C)$. □

5. Constructions of Linear MDS Codes and Optimum Distributions

5.1 Hermite Interpolations over Finite Fields. Let $\mathbb{F}_q[z]$, $q = p^e$, denote the ring of polynomials

$$f(z) = \sum_{i=0}^{t-1} f_i z^i$$

with coefficients $f_i \in \mathbb{F}_q$, $t = \deg f + 1$. For a given polynomial (5.1) its $j^{th}$ hyperderivative $\partial^j f \in \mathbb{F}_q[z]$ is defined by

$$\partial^j f(z) = \sum_{i=0}^{t-1} \binom{i}{j} f_i z^{i-j},$$

(cf. [15, Section 6.4]). From here on we shall write $\binom{i}{j}$ for binomial coefficients modulo $p$, moreover, as it usually is, $\binom{i}{j} = 0$ for $j > i$.

In particular, definition (5.2) implies

$$\partial^j(z - \beta)^i = \binom{i}{j} (z - \beta)^{i-j}$$

for each $\beta \in \mathbb{F}_q$. Using (5.2) and (5.3), it is easy to check the following expansion

$$f(z) = \sum_{j=0}^{t-1} \partial^j f(\beta)(z - \beta)^j,$$
where \( \partial^j f(\beta) \) denotes the value of \( \partial^j f(z) \) at \( \beta \in \mathbb{F}_q \).

It should be noted that for \( j < p \) hyperderivative (5.2) and the usual \( j^{th} \) formal derivative \( f^{(j)}(z) \) are related by

\[
\partial^j f(z) = \frac{1}{j!} f^{(j)}(z). \tag{5.5}
\]

Certainly, in the ring of polynomials over a field of infinite characteristic relation (5.5) holds for all \( j \in \mathbb{N}_0 \).

We can formally add a point \( \infty \) to \( \mathbb{F}_q \). In so doing, all arithmetic operations (except \( 0 \cdot \infty \)) may be defined in the obvious way on the set \( \mathbb{F}_q \cup \{\infty\} \) of \( q + 1 \) elements. For a polynomial (5.1) of degree \( t - 1 \) we put by definition

\[
\partial^j f(\infty) = \begin{cases} f_{t-1-j} & \text{if } 0 \leq j \leq t-1, \\ 0 & \text{if } j > t-1. \end{cases} \tag{5.6}
\]

Obviously, \( \partial^j f(\infty) = \partial^j \bar{f}(0) \), where \( \bar{f}(z) = z^{t-1} f \left( \frac{1}{z} \right) = f_{t-1} + f_{t-2} z + \ldots + f_0 z^{t-1} \) is a polynomial reciprocal to \( f(z) \).

Let \( \mathbb{M}^t \subset \mathbb{F}_q[z] \) denote the set of all polynomials (5.1) of degree less than \( t, t \in \mathbb{N}_0 \), then \( \mathbb{M}^t \) is a vector space of dimension \( t \) over \( \mathbb{F}_q \). Consider the following **Hermite interpolation problem**: Find a polynomial \( f \in \mathbb{M}^t \) which satisfies the equations

\[
\partial^j f(\beta_i) = a_i^{(j)}, \quad j = 0, \ldots, t_i - 1, \quad 1 \leq i \leq l, \tag{5.7}
\]

where \( l \) distinct elements \( \beta_1, \ldots, \beta_l \in \mathbb{F}_q \cup \{\infty\} \) (interpolation nodes) are fixed, and integers \( t_i \in \mathbb{N} \) and coefficients \( a_i^{(j)} \in \mathbb{F}_q \) are given. Certainly, we assume here that \( q \geq l - 1 \).

**Proposition 5.1.** (i) The Hermite interpolation problem (5.7) has a unique solution \( f \in \mathbb{M}^t \), provided that \( t_1 + \ldots + t_l = t \).

(ii) The homogeneous Hermite interpolation problem (5.7) with all coefficients \( a_i^{(j)} = 0 \) has a unique solution \( f = 0 \) in the space \( \mathbb{M}^t \), provided that \( t_1 + \ldots + t_l \geq t \).

**Proof.** (i) At first, we suppose that all interpolation nodes \( \beta_i \in \mathbb{F}_q, 1 \leq i \leq l \). Consider the polynomials

\[
r_i(z) = \sum_{j=0}^{t_i-1} a_i^{(j)} (z - \beta_i)^j, \quad 1 \leq i \leq l. \tag{5.8}
\]

From relations (5.3), (5.4), and (5.8) we conclude that equations (5.7) are equivalent to the following congruences in the ring \( \mathbb{F}_q[z] \)

\[
f(z) = r_i(z) \mod (z - \beta_i)^{t_i}, \quad 1 \leq i \leq l. \tag{5.9}
\]

In (5.9) the polynomials \( (z - \beta_i)^{t_i}, 1 \leq i \leq l \), are pairwise coprime. Therefore, congruences (5.9) have a unique solution \( f \in \mathbb{M}^t, t = t_1 + \ldots + t_l \) by the Chinese remainder theorem in the ring \( \mathbb{F}_q[z] \) (see [16, Section 10.9, Theorem 5]).

Now, let one of interpolation nodes, say \( \beta_l \), coincides with \( \infty \). Then, using (5.6), we put

\[
f(z) = g(z) + r_l(z), \tag{5.10}
\]
where
\[ r_l(z) = \sum_{j=t-t_l}^{t-1} a_i^{(j)} z^j, \]
and \( g(z) \) is a polynomial of degree less than \( t - t_l \).

Substituting (5.10) to first \( l - 1 \) equations (5.7), we find that the polynomial \( g \in \mathbb{M}^{t-t_l} \) is a solution of the following Hermite interpolation problem with \( l - 1 \) nodes \( \beta_1, \ldots, \beta_{l-1} \in \mathbb{F}_q \):
\[
\partial^j g(\beta_i) = b_i^{(j)}, \quad j = 0, \ldots, t_i - 1, \quad 1 \leq i \leq l - 1,
\]
where \( b_i^{(j)} = a_i^{(j)} - \partial^j r(\beta_i) \), and \( t_1 + \ldots + t_{l-1} = t - t_l \). This problem has a unique solution by the above arguments.

(ii) This statement follows at once from (i) because we may choose integers \( t_i' \in \mathbb{N}, t_i' \leq t_i \), to satisfy the equality \( t_1' + \ldots + t_{l'}' = t \). \( \square \)

**Remark 5.1.** The Hermite interpolations may be treated as a confluent Lagrange interpolation problem (cf. [5, Chapter 2, Section 11]). This leads to another proof of Proposition 5.1. Substituting relations (5.2) for a polynomial \( f \in \mathbb{M}^t \) to (5.7), we obtain a system of \( t_1 + \ldots + t_l \) linear equations for \( t \) unknown coefficients \( f_i \). If \( t_1 + \ldots + t_l = t \), then the system has a unique solution, since the corresponding \( t \) by \( t \) matrix turns out to be a confluent Vandermonde matrix, and Proposition 1.1 follows. However, the above arguments based on the Chinese remainder theorem seem to be the most natural even in the case of the Hermite interpolations over the field of complex numbers.

**5.1. Explicit Constructions.** Henceforward we assume that \( q \geq n - 1 \). Fix \( n \) distinct elements \( \beta_1, \ldots, \beta_n \in \mathbb{F}_q \cup \{\infty\} \), and define a linear mapping \( \Gamma_{n,s} : \mathbb{F}_q[z] \to \text{Mat}_{n,s}(\mathbb{F}_q) \) by setting
\[
\Gamma_{n,s} : \mathbb{F}_q[z] \ni f \to \Gamma_{n,s} f = (\partial^{s-1} f(\beta_i)) \in \text{Mat}_{n,s}(\mathbb{F}_q), \tag{5.11}
\]
where \( 0 \leq j \leq s - 1 \) and \( 1 \leq i \leq n \).

Thus, for each polynomial \( f \in \mathbb{F}_q[z] \) the matrix
\[
\Gamma_{n,s} f = (\omega_f^{(1)}, \ldots, \omega_f^{(n)})^T \in \text{Mat}_{n,s}(\mathbb{F}_q)
\]
consists of \( n \) rows of the form
\[
\omega_f^{(i)} = (\partial^{s-1} f(\beta_i), \ldots, \partial f(\beta_i), f(\beta_i)) \in \text{Mat}_{1,s}(\mathbb{F}_q), \quad 1 \leq i \leq n. \tag{5.12}
\]

**Lemma 5.1.** For \( t \leq ns \) the image \( \Gamma_{n,s} \mathbb{M}^t \subset \text{Mat}_{n,s}(\mathbb{F}_q) \) of the vector space \( \mathbb{M}^t \subset \mathbb{F}_q[z] \) under mapping (5.11) is a subspace of dimension \( t \).

**Proof.** It suffices to prove that \( \Gamma_{n,s} \mathbb{M}^{ns} = \text{Mat}_{n,s}(\mathbb{F}_q) \), since \( \mathbb{M}^t \subset \mathbb{M}^{ns} \) for \( t \leq ns \) and the cardinalities of \( \mathbb{M}^{ns} \) and \( \text{Mat}_{n,s}(\mathbb{F}_q) \) coincide. Let \( f \in \mathbb{M}^{ns} \) and \( \Gamma_{n,s} f = 0 \), then from (5.11) we conclude that \( f \) is a solution of a homogeneous Hermite interpolation problem (5.7) with \( l = n, t = ns, t_1 = \ldots = t_n = s \). Therefore \( f = 0 \) by Proposition 5.1. \( \square \)
Theorem 5.1. For each integer \(1 \leq k \leq ns\) the subspace \(\Gamma_{n,s} M^{k} \subset \text{Mat}_{n,s}(\mathbb{F}_{q})\) is an MDS \([ns,k]_{s}\)-code in the metric \(\rho\).

Proof. By Definition 2.1 and relations (4.1) it suffices to show that \(\rho(\Gamma_{n,s} f) \geq ns - k + 1\) for all \(f \in M^{ks} \setminus \{0\}\). Suppose on the contrary that there is a non-zero polynomial \(f \in M^{ks}\) such that \(\rho(\Gamma_{n,s} f) \leq ns - k\). By definitions (2.14), and (5.11), (5.12) this yields

\[
\rho(\Gamma_{n,s} f) = \rho(\omega^{(1)}_f) + \ldots + \rho(\omega^{(n)}_f) \leq ns - k. \tag{5.13}
\]

From (5.13) we conclude that the strict inequality

\[
\rho(\omega^{(i)}_f) < s \tag{5.14}
\]

holds at least for one of indices \(i\). Without loss of generality, we may assume that inequality (5.14) holds for \(i = 1, \ldots, l\) with \(l \geq 1\) and \(\rho(\omega^{(i)}_f) = s\) for \(i = l+1, \ldots, n\).

Using (5.11), (5.12), and (5.14), we find that the polynomial \(f \in M^{k}\) is a solution of the following homogeneous Hermite interpolation problem (5.7)

\[
\partial^j f(\beta_i) = 0, \quad j = 0, \ldots, t_i - 1, \quad 1 \leq i \leq l,
\]

where \(t_i = s - \rho(\omega^{(i)}_f) \in \mathbb{N}\).

It is easy to check that \(t_1 + \ldots + t_l = ns - \rho(\Gamma_{n,s} f)\). Therefore, \(t_1 + \ldots + t_l \geq k\) and \(f = 0\) by Proposition 1.1, a contradiction. \(\square\)

Theorem 5.1 gives broad classes of linear MDS \([ns,k]_{s}\)-codes in the metric \(\rho\) with arbitrary values of the parameters \(n, k, s\). By Theorem 2.1 such MDS codes correspond to linear optimum \([ns,k]_{s}\)-distributions, which can be explicitly described as follows. Introduce a linear mapping \(\gamma_{n,s} : \mathbb{F}_{q}[z] \to \mathbb{Q}^n(q^s)\) by setting

\[
\gamma_{n,s} : \mathbb{F}_{q}[z] \ni f \to \gamma_{n,s} f = X = (x_1, \ldots, x_n)^T \in \mathbb{Q}^n(q^s), \tag{5.15}
\]

where coordinates \(x_i\) of the point \(X\) are given by

\[
x_i = \sum_{j=1}^{s} \partial^{j-1} f(\beta_i) q^{-j}. \tag{5.16}
\]

Using (5.1), (5.2), and (5.6), we can write the coefficients \(\partial^{j-1} f(\beta_i)\) in the form

\[
\partial^{j-1} f(\beta_i) = \sum_{l=0}^{t_i-1} \binom{j - 1}{l} \beta_i^{l+1-j} f_l \tag{5.17}
\]

for \(\beta_i \in \mathbb{F}_{q}\), and

\[
\partial^{j-1} f(\beta_i) = f_{t_i-j} \tag{5.18}
\]

for \(\beta_i = \infty\).

Theorem 5.2. For each integer \(1 \leq k \leq ns\) the subspace \(\gamma_{n,s} M^{k} \subset \mathbb{Q}^n(q^s)\) is an optimum \([ns,k]_{s}\)-distribution.
Proof. It suffices to compare relations (5.15), (5.16) with (5.11), (5.12), and use Theorems 5.1 and 2.1. □

In the special case of \( k = s \) an optimum \([n_s, s]_s\)-distribution \( \gamma_{n_s, s} M^s \) (defined by the relations from (5.15) to (5.18)) coincides with a \((0, s, n)\)-net of zeroth deficiency (cf. Proposition 1.1). Nets of such a kind were given previously in [11] and [20].

6. Reconstructions of Codes and Distributions

6.1. Peano’s Bijection. As usual, for a matrix \( \Omega \in \text{Mat}_{g,s}(\mathbb{F}_q) \) we write

\( \Omega = (\omega_1, \ldots, \omega_g)^T \), where \( \omega_j = (\xi^{(j)}_1, \ldots, \xi^{(j)}_s) \in \text{Mat}_{1,s}(\mathbb{F}_q) \). Introduce a mapping \( \pi_g \) from \( \text{Mat}_{g,s}(\mathbb{F}_q) \) to \( \text{Mat}_{1,gs}(\mathbb{F}_q) \) by

\[
\pi_g : \text{Mat}_{g,s}(\mathbb{F}_q) \ni \Omega \mapsto \pi_g \Omega = \left( \xi_1^{(1)}, \ldots, \xi_1^{(g)}, \ldots, \xi_s^{(1)}, \ldots, \xi_s^{(g)} \right) \in \text{Mat}_{1,gs}(\mathbb{F}_q).
\] (6.1)

It is obvious that mapping (6.1) is an isomorphism of vector spaces \( \text{Mat}_{g,s}(\mathbb{F}_q) \) and \( \text{Mat}_{1,gs}(\mathbb{F}_q) \).

By relations (2.4), (2.5), and (2.6) mapping (6.1) can be considered as an isomorphism of vector spaces \( \mathbb{Q}^q(q^s) \) and \( \mathbb{Q}(q^{gs}) \). In this case \( \pi_g \) coincides with the well known Peano’s mapping (restricted on \( \mathbb{Q}^q(q^s) \)) which gives a bijection between points of the unit cube \( U^g \) and the unit segment \([0, 1]\).

More generally, we introduce an isomorphism \( \pi_{g,n} \) of vector spaces \( \text{Mat}_{gn,s}(\mathbb{F}_q) \) and \( \text{Mat}_{n,gs}(\mathbb{F}_q) \) as follows. For a matrix \( \Omega \in \text{Mat}_{gn,s}(\mathbb{F}_q) \) we write \( \Omega = (\Omega_1, \ldots, \Omega_n)^T \), where \( \Omega_j \in \text{Mat}_{g,s}(\mathbb{F}_q), 1 \leq j \leq n \). Now we put

\[
\pi_{g,n} : \text{Mat}_{gn,s}(\mathbb{F}_q) \ni \Omega \mapsto \pi_{g,n} \Omega = (\pi_g \Omega_1, \ldots, \pi_g \Omega_n)^T \in \text{Mat}_{n,gs}(\mathbb{F}_q).
\] (6.2)

In other words, we split \( gn \) rows of a matrix \( \Omega \in \text{Mat}_{gn,s}(\mathbb{F}_q) \) into \( n \) subsets \( \Omega_1, \ldots, \Omega_n \), each of which consists of successive \( g \) rows; and subsequently, we map by \( \pi_g \) first \( g \) rows of \( \Omega_1 \) to the first row of the matrix \( \pi_{g,n} \Omega \in \text{Mat}_{n,gs}(\mathbb{F}_q) \), second \( g \) rows of \( \Omega_2 \) to the second row of \( \pi_{g,n} \Omega \), and so on, last \( g \) rows \( \Omega_n \) are mapped to the \( n^{th} \) row of \( \pi_{g,n} \Omega \).

Using relations (2.8) and (6.2), we define an isomorphism

\[
\pi_{g,n} : \mathbb{Q}^{gn}(q^s) \ni X \mapsto \pi_{g,n} X \in \mathbb{Q}^n(q^{gs})
\] (6.3)

of vector spaces \( \mathbb{Q}^{gn}(q^s) \) and \( \mathbb{Q}^n(q^{gs}) \) by setting

\[
\Omega(\pi_{g,n} X) = \pi_{g,n} \Omega(X).
\] (6.4)

It is obvious that mapping (6.3), (6.4) is a restriction on \( \mathbb{Q}^{gn}(q^s) \) and \( \mathbb{Q}^n(q^{gs}) \) of the well known Peano’s bijection between points of the unit cubes \( U^{gn} \) and \( U^n \).

The foregoing implies the following.

Lemma 6.1. (i) Let \( C \subset \text{Mat}_{gn,s}(\mathbb{F}_q) \) be a \([gns, gk]_s\)-code and \( D \subset \mathbb{Q}^{gn}(q^s) \) be a \([gns, gk]\_s\)-distribution, then \( \pi_{g,n} C \subset \text{Mat}_{n,gs}(\mathbb{F}_q) \) is a \([ngs, gk]_{gs}\)-code and \( \pi_{g,n} D \subset \mathbb{Q}^n(q^{gs}) \) is a \([ngs, gk]_{gs}\)-distribution.
(ii) If a code $C \subset \text{Mat}_{n,s}(\mathbb{F}_q)$ and a distribution $D \subset Q^{gn}(q^s)$ are linear, then the code $\pi_{g,n} \subset \text{Mat}_{n,gs}(\mathbb{F}_q)$ and the distribution $\pi_{g,n}D \subset Q^n(q^{gs})$ are also linear.

Now our concern is with the behavior of the Hamming and non-Hamming weights $\kappa$ and $\rho$ under mappings (6.2) and (6.3).

**Lemma 6.2.** (i) For arbitrary matrices $\Omega \in \text{Mat}_{n,s}(\mathbb{F}_q)$ and points $X \in Q^{gn}(q^s)$ one has the relations

$$\kappa(\pi_{g,n}\Omega) = \kappa(\Omega), \quad \kappa(\pi_{g,n}X) = \kappa(X),$$

and

$$\rho(\pi_{g,n}\Omega) \geq \rho(\Omega), \quad \rho(\pi_{g,n}X) \geq \rho(X).$$

(ii) For arbitrary codes $C \subset \text{Mat}_{n,s}(\mathbb{F}_q)$ and distributions $D \subset Q^{gn}(q^s)$ one has the relations

$$\kappa(\pi_{g,n}C) = \kappa(C), \quad \kappa(\pi_{g,n}D) = \kappa(D)$$

and

$$\rho(\pi_{g,n}C) \geq \rho(C), \quad \rho(\pi_{g,n}D) \geq \rho(D).$$

**Proof.** (i) It suffices to prove (6.5) and (6.6) only for matrices $\Omega \in \text{Mat}_{n,s}(\mathbb{F}_q)$.

To prove (6.5) we simply notice that the matrix $\pi_{g,n}\Omega$ is obtained by a reordering (see (6.1), (6.2)) of elements of $\Omega$. Therefore, the matrices $\pi_{g,n}\Omega$ and $\Omega$ have the same number of non-zero entries.

To prove (6.6) we first consider the behavior of the $\rho$-weight under mapping (6.1). We wish to prove that for an arbitrary matrix $\Omega \in \text{Mat}_{n,s}(\mathbb{F}_q)$ one has the inequality

$$\rho(\pi_{g,n}\Omega) \geq \rho(\Omega).$$

Since (6.9) holds trivially for $\Omega = 0$, we suppose that $\Omega \neq 0$. Let $\Omega = (\omega_1, \ldots, \omega_g)^T$ where $\omega_j \in \text{Mat}_{1,s}(\mathbb{F}_q)$, $1 \leq j \leq g$. Introduce an integer $l$, $1 \leq l \leq g$, by setting

$$l = l(\Omega) = \max\{j : \rho(\omega_j) \neq 0\}.$$  

Comparing definitions (2.12) and (6.1), we conclude that

$$\rho(\pi_{g,n}\Omega) = \rho(\omega_l) + (l - 1)s.$$  

Moreover, definition (2.14) implies

$$\rho(\Omega) = \sum_{j=1}^{l} \rho(\omega_j).$$  

Using relations (2.13), (6.11), and (6.12), we find that

$$\rho(\Omega) = \rho(\omega_l) + \sum_{j=1}^{l-1} \rho(\omega_j) \leq \rho(\omega_l) + (l - 1)s = \rho(\pi_{g,n}\Omega).$$

The proof of inequality (6.9) is complete.
Now, let an arbitrary matrix \( \Omega \in \text{Mat}_{gn,s}(\mathbb{F}_q) \) be given. Then \( \Omega = (\Omega_1, \ldots, \Omega_n)^T \), where \( \Omega_j \in \text{Mat}_{g,s}(\mathbb{F}_q) \), \( 1 \leq j \leq n \). Comparing definitions (2.14) and (6.2), we conclude that

\[
\rho(\Omega) = \sum_{j=1}^{n} \rho(\Omega_j), \quad \rho(\pi_{g,n}\Omega) = \sum_{j=1}^{n} \rho(\pi_g\Omega_j).
\] (6.13)

Substituting inequality (6.9) to (6.13), we obtain inequality (6.6).

(ii) The proposition (ii) follows at once from (i) and definitions (2.18), (2.19) of weights \( \kappa \) and \( \rho \). □

**Proposition 6.1.** Let \( C \subset \text{Mat}_{gn,s}(\mathbb{F}_q) \) be an MDS \( [gns, gk]_s \)-code in the metric \( \rho \) and \( D \subset Q^{gn}(q^s) \) be an optimum \( [gns, gk]_s \)-distribution. Suppose that \( k = st, 1 \leq t \leq n - 1 \). Then,

\[
\rho(\pi_{g,n}C) = (n-t)gs + 1, \quad \rho(\pi_{g,n}D) = (n-t)gs + 1.
\] (6.14)

Hence, \( \pi_{g,n}C \subset \text{Mat}_{n,gs}(\mathbb{F}_q) \) is an MDS \( [ngs, gk]_gs \)-code in the metric \( \rho \) and \( \pi_{g,n}D \subset Q^n(g^gs) \) is an optimum \( [ngs, gk]_gs \)-distribution.

Furthermore, the corresponding Hamming weights satisfy the following inequalities

\[
\kappa(\pi_{g,n}C) \geq (n-t)g + 1, \quad \kappa(\pi_{g,n}D) \geq (n-t)g + 1.
\] (6.15)

**Proof.** It suffices to prove relations (6.14) and (6.15) only for codes.

To prove (6.14) we notice that \( \rho(\pi_{g,n}C) \geq (n-t)gs + 1 \) by Definition 2.1 and inequality (6.8). It remains to refer to Proposition 2.1.

To prove (6.15) we notice that

\[
\kappa(C) \geq s^{-1} \rho(C) = (n-t)g + s^{-1}
\]

by inequality (2.15) and Definition 2.1. Therefore \( \kappa(C) \geq (n-t)g + 1 \), since the weight \( \kappa(C) \) is an integer. It remains to refer to equality (6.7). □

### 6.2. Duality and Peano’s Bijection

We wish to consider the behavior of inner products (4.2) and (4.3) under mappings (6.1) and (6.2). We introduce an isomorphism \( J_g \) of the space \( \text{Mat}_{g,s}(\mathbb{F}_q) \) by setting

\[
J_g : \text{Mat}_{g,s}(\mathbb{F}_q) \ni \Omega = (\omega_1, \ldots, \omega_g)^T 
\rightarrow J_g\Omega = (\omega_g, \ldots, \omega_1)^T \in \text{Mat}_{g,s}(\mathbb{F}_q),
\] (6.16)

where \( \omega_j \in \text{Mat}_{1,s}(\mathbb{F}_q) \), \( 1 \leq j \leq g \). Thus, \( J_g \) is merely a permutation of rows in inverse order.

By a direct calculation from definitions (4.2), (4.3) and (6.1), (6.16) we derive the identity

\[
\langle \pi_g\Omega_1, \pi_g\Omega_2 \rangle = \langle J_g\Omega_1, \Omega_2 \rangle = \langle \Omega_1, J_g\Omega_2 \rangle
\] (6.17)

for all \( \Omega_1, \Omega_2 \in \text{Mat}_{g,s}(\mathbb{F}_q) \).

Now we introduce an isomorphism \( J_{g,n} \) of the space \( \text{Mat}_{gn,s}(\mathbb{F}_q) \) by setting

\[
J_{g,n} : \text{Mat}_{gn,s}(\mathbb{F}_q) \ni \Omega = (\Omega_1, \ldots, \Omega_n)^T 
\rightarrow J_{g,n}\Omega
\]
\[(J_g \Omega_1, \ldots, J_g \Omega_n)^T \in \text{Mat}_{gn,s}(F_q), \quad (6.18)\]

where \( \Omega_j \in \text{Mat}_{g,s}(F_q) \).

Using definitions (4.2), (4.3), (6.1), (6.2), and relation (6.17), we obtain the identity

\[ \langle \pi_{g,n} \Omega_1, \pi_{g,n} \Omega_2 \rangle = \langle J_{g,n} \Omega_1, \Omega_2 \rangle = \langle \Omega_1, J_{g,n} \Omega_2 \rangle \quad (6.19) \]

for all \( \Omega_1, \Omega_2 \in \text{Mat}_{gn,s}(F_q) \).

**Proposition 6.2.** Given a linear code \( C \subset \text{Mat}_{gn,s}(F_q) \) and its image \( \pi_{g,n} C \subset \text{Mat}_{n,gs}(F_q) \) under mapping (6.2). Then the dual codes \( C^\perp \) and \( (\pi_{g,n} C)^\perp \) are related by

\[ (\pi_{g,n} C)^\perp = \pi_{g,n} (J_{g,n} C^\perp). \quad (6.20) \]

**Proof.** Since mapping (6.2) is a bijection, we can write \( (\pi_{g,n} C)^\perp = \pi_{g,n} C_1 \) with a linear code \( C_1 \subset \text{Mat}_{n,gs}(F_q) \). Using Definition 4.1 and relation (6.19), we conclude that \( J_{g,n} C_1 = C^\perp \). Therefore, \( C_1 = J_{g,n} C^\perp \), because \( J_{g,n}^2 \) is the identity, and relation (6.20) follows. \( \Box \)

For linear codes and distributions Proposition 6.1 can be supplemented with the following result.

**Proposition 6.3.** Let \( C \subset \text{Mat}_{gn,s}(F_q) \) be a linear MDS \( [gns, gk]_s \)-code in the metric \( \rho \) and \( D \subset \mathbb{Q}^{gn}(q^s) \) be a linear optimum \( [gns, gk]_s \)-distribution. Suppose that \( k = st, 1 \leq t \leq n - 1 \). Then, the Hamming and non-Hamming weights of the dual subjects \( (\pi_{g,n} C)^\perp \) and \( (\pi_{g,n} D)^\perp \) satisfy the following relations

\[ \rho((\pi_{g,n} C)^\perp) = kg + 1, \quad \kappa((\pi_{g,n} D)^\perp) = kg + 1 \]

and

\[ \kappa((\pi_{g,n} C)^\perp) \geq tg + 1, \quad \kappa((\pi_{g,n} D)^\perp) \geq tg + 1. \]

**Proof.** It suffices to prove Proposition 6.3 only for codes. First of all, we notice that

\[ \rho(J_{g,n} C_1) = \rho(C_1) \quad \kappa(J_{g,n} C_1) = \kappa(C_1) \quad (6.21) \]

for an arbitrary code \( C \subset \text{Mat}_{gn,s}(F_q) \), because the mapping \( J_{g,n} \) is nothing but a permutation of rows of the corresponding code words, that preserves both weights \( \rho \) and \( \kappa \) (cf. Remark 2.2).

Now let \( C \subset \text{Mat}_{gn,s}(F_q) \) be a linear MDS \( [gns, gk]_s \)-code in the metric \( \rho \). Then, by Theorem 4.1 (i), \( C^\perp \subset \text{Mat}_{gn,s}(F_q) \) is an MDS \( [gn, g(n-k), s] \)-code in the metric \( \rho \). Therefore, relations (6.7), (6.15) (where \( k \) is replaced by \( ns - k \)), (6.20), and (6.21) imply

\[ \rho((\pi_{g,n} C)^\perp) = \rho(\pi_{g,n} (J_{g,n} C^\perp)) = \rho(J_{g,n} C^\perp) = \rho(C^\perp) = kg + 1, \]

and similarly,

\[ \kappa((\pi_{g,n} C)^\perp) = \kappa(\pi_{g,n} (J_{g,n} C^\perp)) = \kappa(J_{g,n} C^\perp) = \kappa(C^\perp) \geq tg + 1. \]
6.3. Codes and Distributions with large $\rho$- and $\kappa$-Weights. Now we can give explicit constructions of codes and distributions with large weights simultaneously in both metrics $\rho$ and $\kappa$.

Given arbitrary integers $g \in \mathbb{N}$ and $k = st$, $1 \leq t \leq n - 1$. Define a linear $[gns, gk]_s$-code $C(g) \subset \text{Mat}_{gn,s}(\mathbb{F}_q)$ and a linear $[gns, gk]_s$-distribution $D(g) \subset \mathbb{Q}^{gn}(q^s)$ by
\[
C(g) = \Gamma_{gn,s}M^{gk}, \quad D(g) = \gamma_{gn,s}M^{gk},
\] (6.22)
where the mappings $\Gamma_{gn,s}$ and $\gamma_{gn,s}$ are given in (5.11). Certainly, we assume here that
\[
q = p^e \geq gn - 1.
\] (6.23)

Next, we introduce an $[ngs, gk]_s$-code $\pi_{g,n}C(g) \subset \text{Mat}_{ngs}(\mathbb{F}_q)$ and an $[ngs, gk]_s$ distribution $\pi_{g,n}D(g) \subset \mathbb{Q}^{ngs}(q^{gs})$, where $\pi_{g,n}$ are Peano’s bijections (6.2) and (6.3). We consider also the dual $[ngs, ngs - k]_s$-code $(\pi_{g,n}C(g))^\perp \subset \text{Mat}_{ngs}(\mathbb{F}_q)$ and $[ngs, ngs - k]_s$-distribution $(\pi_{g,n}D(g))^\perp \subset \mathbb{Q}^{ngs}(q^{gs})$.

It is worthwhile writing coordinates of points $X = (x_1, \ldots, x_n) \in D(g)$ explicitly. Comparing relations (5.15), (5.16) with (6.1), (6.2) and (6.22), we find that
\[
x_j = \sum_{l=1}^{g} q^{-(l-1)s} \sum_{i=1}^{s} q^{i-1} f(\beta_{j,l}) q^{-i},
\] (6.24)
where $f \in M^{gk}$, and $\beta_{j,l}, 1 \leq j \leq n, 1 \leq l \leq g$, are $ng$ distinct elements in $\mathbb{F}_q \cup \{\infty\}$. Notice that such elements $\beta_{j,l}$ exist by (6.23).

**Theorem 6.1.** With the above notation one has the following relations for the corresponding Hamming and non-Hamming weights
\[
\rho(\pi_{g,n}C(g)) = (ns - k)g + 1, \quad \rho(\pi_{g,n}D(g)) = (ns - k)g + 1,
\]
\[
\kappa(\pi_{g,n}C(g)) \geq (n - t)g + 1, \quad \kappa(\pi_{g,n}D(g)) \geq (n - t)g + 1,
\] (6.25)
and
\[
\rho((\pi_{g,n}C(g))^\perp) = kg + 1, \quad \rho((\pi_{g,n}D(g))^\perp) = kg + 1,
\]
\[
\kappa((\pi_{g,n}C(g))^\perp) \geq tg + 1, \quad \kappa((\pi_{g,n}D(g))^\perp) \geq tg + 1,
\] (6.26)

**Proof.** As usual, we prove Theorem 6.1 only for codes. By Theorem 5.1 the code $C(g) \subset \text{Mat}_{gn,s}(\mathbb{F}_q)$ given in (6.22) is a linear MDS $[gns, gk]_s$-code in the metric $\rho$. This implies relations (6.25) by Proposition 6.1 and relations (6.26) by Proposition 6.3. \( \Box \)

6.4. Variations of the Base $q$. Every distribution in the space $\mathbb{Q}^{n}(q^s), q = p^e$, can be treated as a distribution in the space $\mathbb{Q}^{n}(p^{es})$ (cf. Lemma 1.1). We wish to find out what happens to metric properties of a given distribution $D \subset \mathbb{Q}^{n}(q^s)$ when the base $q$ varies from $p^e$ to $p$.

In what follows we write $\kappa_q$, $\rho_q$ for the corresponding Hamming and non-Hamming weights on the space $\mathbb{Q}^{n}(q^s), q = p^e$, and $\kappa_p$, $\rho_p$ for those on the space $\mathbb{Q}^{n}(p^{es})$ (cf. Sec. 2.2).
Let \( x \in \mathbb{Q}(q^s) \). Consider representation (2.5) in bases \( q \) and \( p \),

\[
x = \sum_{i=1}^{s} \xi_i(x)q^{i-s-1} = \sum_{j=1}^{es} \theta_j(x)p^{j-es-1},
\]

(6.27)

where \( \xi_i(x) = \xi_i = \mu_i^{(1)} + \mu_i^{(2)}p + \ldots + \mu_i^{(e)}p^{e-1} \), and \( \mu_i^{(e)} = \mu_i^{(e)}(x) \in \{0,1,\ldots,p-1\} \) (cf. (2.6)). The coefficients \( \theta_j = \theta_j(x) \) in (6.27) are determined by the following rule for the corresponding code words:

\[
\text{Mat}_{1,s}(\mathbb{F}_q) \ni \omega_q(x) = (\xi_1, \ldots, \xi_s)
\]

\[
= (\mu_1^{(1)}, \ldots, \mu_1^{(e)}, \mu_2^{(1)}, \ldots, \mu_2^{(e)}, \ldots, \mu_s^{(1)}, \ldots, \mu_s^{(e)})
\]

\[
= (\theta_1, \ldots, \theta_{es}) = \omega_p(x) \in \text{Mat}_{1,es}(\mathbb{F}_p).
\]

(6.28)

Since \( \rho_q(0) = \rho_p(0) = 0 \), we may consider only \( x > 0 \). With the above notation definition (2.12) takes the form

\[
\rho_q(x) = \max\{i : \xi_i \neq 0\}, \quad \rho_p(x) = \max\{j : \theta_j \neq 0\}.
\]

(6.29)

Suppose that \( \rho_q(x) = r, r \in \mathbb{N} \), then using (6.28) and (6.29), we conclude that \( \theta_j = 0 \) for all \( j > er \) and there is an element \( \theta_j \neq 0 \) with \( e(r-1) + 1 \leq j \leq er \). Therefore,

\[
e(\rho_q(x) - 1) + 1 \leq \rho_p(x) \leq e\rho_q(x).
\]

(6.30)

Combining inequality (6.30) with definition (2.14), we find the following relation

\[
e(\rho_q(X) - 1) + 1 - (e-1)(n-1) \leq \rho_p(X) \leq e\rho_q(X)
\]

(6.31)

where \( X \in \mathbb{Q}^n(q^s) \) is an arbitrary point. Certainly, the left side of (6.30) and (6.31) is informative only for sufficiently large weights \( \rho_q \).

Similarly, using rule (6.28), we obtain the following inequality for the Hamming weights

\[
\kappa_q(X) \leq \kappa_p(X) \leq e\kappa_q(X),
\]

(6.32)

where \( X \in \mathbb{Q}^n(q^s) \) is an arbitrary point. Notice that bounds of such a kind are well known in coding theory (cf. [16, Section 7.7]).

The foregoing immediately implies the following result.

**Proposition 6.4.** Given a distribution \( D \subset \mathbb{Q}^n(q^s) \), \( q = p^e \). Then, its Hamming and non-Hamming weights in bases \( q \) and \( p \) are related by

\[
e(\rho_q(D) - 1) + 1 - (e-1)(n-1) \leq \rho_p(D) \leq e\rho_q(D)
\]

(6.33)

and

\[
\kappa_q(D) \leq \kappa_p(D) \leq e\kappa_q(D).
\]

(6.34)

In particular, if \( D \subset \mathbb{Q}^n(q^s) \) is an optimum \([ns,k]_s\)-distribution in base \( q \) then its weight \( \rho_p(D) \) in base \( p \) satisfies the inequality

\[
\rho_p(D) \geq (ns-k)e + 1 - (e-1)(n-1).
\]

(6.35)
Proof. Inequalities (6.33), (6.34) follow from (6.31), (6.32), and definition (2.18). If \( D \subset \mathbb{Q}^n(q^*) \) is an optimum \([ns,k]_s\)-distribution, then \( \rho_q(D) = ns - k + 1 \) by Theorem 2.1, and inequality (6.35) follows from (6.33). □

We return to the \([ns,k]_s\)-distribution \( \pi_{g,n}D^{(g)} \subset \mathbb{Q}^n(q^{gs}) \) and the dual \([ns,ns-k]_s\)-distribution \( (\pi_{g,n}D^{(g)})^\perp \subset \mathbb{Q}^n(q^{gs}) \) which were introduced in Sec. 6.3. The bounds for their Hamming and non-Hamming weights \( \varkappa = \varkappa_q \) and \( \rho = \rho_q \) in base \( q \) are given in Theorem 6.1. Now we wish to estimate the corresponding weights \( \varkappa_p \) and \( \rho_p \) in base \( p \).

**Theorem 6.2.** With the notation of Sec. 6.3 one has the following relations for the corresponding Hamming and non-Hamming weights in base \( p \)

\[
\rho_p(\pi_{g,n}D^{(g)}) \geq (ns - k)eg + 1 - (e - 1)(n - 1),
\]

\[
\varkappa_p(\pi_{g,n}D^{(g)}) \geq (n - t)g + 1,
\]  

and

\[
\rho_p((\pi_{g,n}D^{(g)})^\perp) \geq k eg + 1 - (e - 1)(n - 1),
\]

\[
\varkappa_p((\pi_{g,n}D^{(g)})^\perp) \geq t eg + 1,
\]

where integers \( e, g \in \mathbb{N} \) and the prime \( p \) are related by (6.23).

Proof. It suffices to compare Theorem 6.1 and Proposition 6.4; in so doing inequalities (6.36) and (6.37) follow from inequalities (6.25) and (6.26), respectively. □

Thus, we have constructed explicitly the distributions \( \pi_{g,n}D^{(g)} \) and \((\pi_{g,n}D^{(g)})^\perp\) with large weights simultaneously in metrics \( \rho_p \) and \( \varkappa_p \). As mentioned in the Introduction distributions with such metric properties are of crucial importance for further applications (see [7]).

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