Indeterministic Quantum Gravity and Cosmology

VI. Predynamical Geometry of Spacetime Manifold, Supplementary Conditions for Metric, and CPT

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Abstract

This paper is a continuation of the papers [1-5]. The introduction of a prior, i.e., predynamical global geometry of spacetime manifold is substantiated, and the geometry is specified. The manifold is an infinite four-cylinder, or tube in the five-dimensional Euclidean space, the orthogonal section of the cylinder being the unit three-sphere. Supplementary conditions for metric are introduced geometrically, coordinate-independently, as opposed to coordinate conditions. Parity and time-reversal transformations are extended to the manifold specified. $PT$ is a rotation through $\pi$ about an axis orthogonal to the cylinder axis. $CPT$ invariance is discussed.
One geometry cannot be more true than another; it can only be more *convenient*. Geometry is not true, it is advantageous.

Robert Pirsig

**Introduction**

In general relativity (GR), a spacetime manifold $M$ is a four-dimensional, real, Hausdorff, $C^\infty$, and paracompact manifold, which, however, possesses no prior, i.e., predynamical geometry. To emphasize the last issue, we quote [6]: ‘Mathematics was not sufficiently refined in 1917 to cleave apart the demands for “no prior geometry” and for a “geometric, coordinate-independent formulation of physics”. Einstein described both demands by a single phrase, “general covariance”. The “no-prior-geometry” demand actually fathered general relativity, but by doing so anonymously, disguised as “general covariance”, it also fathered half a century of confusion’.

Another characteristic feature of GR is locality, ‘This is surely Einstein’s concept that all physics takes place by “local action”’[6].

On the other hand, in the theory being developed in this series of papers, it is a prior global geometry of spacetime manifold that is of considerable importance. The first aim of this paper is to substantiate the introduction of this geometry and to specify the latter. (Since the role of the global geometric structure of the spacetime manifold is essential for the theory, we name it indeterministic quantum gravity and cosmology.)

A motivation for introducing the global structure is related to dynamical incompleteness of the Einstein equation and coordinate conditions. This issue was considered in [3,4]. The second aim of this paper is to continue the consideration.

The global structure allows for extending $P\,T$ transformation and $C\,P\,T$ invariance to curved spacetime, which is the third aim of this paper.

The main results are as follows. The spacetime manifold is an infinite four-cylinder, or tube in the five-dimensional Euclidean space, the orthogonal section of the cylinder being the unit three-sphere. Supplementary conditions for metric $g$ are introduced geometrically, coordinate-independently; $g = dt \otimes dt - h_t$ where $h_t$ is a time-dependent Riemannian metric on the sphere. $P\,T$ is equivalent to a rotation through $\pi$ about an axis orthogonal to the cylinder axis.

1 Motivation for predynamical global geometry

1.1 Dynamical incompleteness of the Einstein equation

A motivation for introducing a global predynamical structure of spacetime manifold in [3] was based on the dynamical incompleteness of the Einstein equation. Let us return to this issue. In the Einstein equation

$$G = T$$  

(1.1.1)

($G$ is the Einstein tensor and $T$ is the energy-momentum tensor), as it is well known, there are only six dynamical equations

$$G^{ij} = T^{ij}, \quad i, j = 1, 2, 3,$$

(1.1.2)
for six metric components $g_{ij}$, whereas four equations

$$G^{0\mu} = T^{0\mu}, \quad \mu = 0, 1, 2, 3,$$

are constraints on initial data, so that four components $g_{0\mu}$ are dynamically undetermined. The conventional way for circumventing the dynamical incompleteness consists in introducing gauge, or coordinate conditions [7,8]. The reasoning behind those conditions is as follows. Let $M$ be a spacetime manifold,

$$\mathcal{F}(M) = \bigcup_{p \in M} \mathcal{F}_p, \quad \mathcal{F} = (g, \mathcal{M})$$

where $\mathcal{M}$ is a set of matter fields, and

$$\mu : M \to M$$

be a differentiable transformation. The set $\mu_*\mathcal{F}$ is physically equivalent to $\mathcal{F}$. The transformation $\mu$ involves four gauge functions, which may be used to remove the indeterminacy of the $g_{0\mu}$’s. We stress that, as it is known, the gauge conditions cannot be formulated in covariant form [8].

We argue against that approach to the incompleteness of dynamics as follows. There are two alternative approaches: Spacetime manifold is a set of events $E$ [9,10,6]; spacetime manifold is an abstract manifold $M$ [7].

In the first case, we have $\mathcal{F}(E)$, so that $\mu_*\mathcal{F}$ is not physically equivalent to $\mathcal{F}$ since at least $g_e, e \in E$, may be measured [7] and $(\mu_*g)_e \neq g_e$. Let us try to remedy the situation. Let

$$\nu : E \to M$$

be a diffeomorphism, where $M$ is an abstract manifold, then

$$\nu_*\mathcal{F}(E) = \mathcal{F}^M(M).$$

Now we introduce $\mu$ (1.1.3) and obtain

$$\mathcal{F}^M = \mu_*\mathcal{F}^M,$$

which is physically equivalent to $\mathcal{F}^M$. In $\mathcal{F}^M$ the indeterminacy of the $g_{0\mu}$’s may be removed, so that $\mathcal{F}^M$ may be considered to be known. But we cannot find

$$\mathcal{F} = \nu^*\mu_*\mathcal{F}^M$$

since $\mu$ is not known.

In the second case, we invoke the geometric principle, i.e., the demand for a geometric, coordinate-independent formulation of physics [6]. Since the principle cannot be realized locally in a covariant form, it should be realized globally—-in the form of predynamical global geometry and related supplementary (not coordinate!) conditions for metric.

At the conclusion of this subsection, we note that any coordinate conditions imply a single global coordinate system, which, in general, does not exist.
1.2 Cauchy problem vicious circle

An initial-value problem in dynamics is the Cauchy problem. Metric is obtained from a solution to this problem. The Cauchy problem for a given spacetime manifold demands a given family of Cauchy surfaces. But a Cauchy surface is determined by metric. Thus we come to a vicious circle: Metric implies the Cauchy problem, the Cauchy problem implies a Cauchy surface, the Cauchy surface implies metric,

\[ \text{metric} \rightarrow \text{Cauchy surface} \leftarrow \text{Cauchy problem} \]

where arrow stands for implies.

To break the vicious Cauchy circle, it is necessary to introduce a family of Cauchy surfaces, which may be done on the basis of a predynamical global geometry.

1.3 Quantum jumps

To construct an indeterministic theory, it is necessary to incorporate quantum jumps into dynamics. The jumps imply the existence of a preferred time, i.e., cosmic time, which, in its turn, implies a specific global structure of spacetime manifold.

2 Maximally symmetric cylindrical manifold

2.1 Spacetime as a set

To be a mathematical object, spacetime should first of all be defined as a set. We invoke Georg Cantor [11,12]: ‘I call a manifold (a totality, a set) of elements which belong to some conceptual sphere well-defined, if on the basis of its definition and as a consequence of the logical principle of excluded middle it must be seen as internally determined both whether some object belonging to the same conceptual sphere belongs to the imagined manifold as an object or not, as well as whether two objects belonging to the set are equal to one another or not, despite formal differences in the way they are given’. In the case of spacetime manifold \( M \), a straightforward definition may be based on the Whitney embedding theorem: The problem of the identification of points of \( M \) is naturally solved by considering \( M \) as a subset of \( R^p \), since points of \( R^p \) are identified. Here \( R^p \) is the Euclidean (or arithmetical) space with

\[ 4 \leq p \leq 9 \ (= 2 \cdot 4 + 1). \]  

(2.1.1)

In fact, there is no other natural way for solving the problem.

In the indeterministic cosmology being developed in this series of papers, the universe should be spatially finite—otherwise the cosmic energy determinacy principle [1] would make no sense. Assuming, in addition, that the universe is spatially closed we have

\[ 5 \leq p \leq 9. \]  

(2.1.2)

We adopt the simplest possibility, \( p = 5 \). Thus

\[ M \subset R^5. \]  

(2.1.3)
2.2 Trivial bundle. Cylindrical manifold

The existence of cosmic time implies a fibration of spacetime manifold:

\[ \pi : M \overset{S}{\longrightarrow} T, \quad (2.2.1) \]

so that \( M \) is a bundle space, a base space \( T \) is cosmic time, a standard fibre \( S \) is cosmic space, and \( \pi \) is the projection. Cosmic time is an open interval in the real axis,

\[ T = (a, b) \subset \mathbb{R}. \quad (2.2.2) \]

The simplest way of forming a fibre bundle (2.2.1) is to take the product

\[ M = T \times S, \quad (2.2.3) \]

i.e., the trivial bundle. Thus \( M \) is a four-dimensional cylinder, or tube in \( \mathbb{R}^5 \).

2.3 Maximal symmetry. Spherical space manifold

From simplicity desideratum, we impose the maximal symmetry conditions on the manifold \( M \) (2.2.3): In the Euclidean metric of \( \mathbb{R}^5 \)

\[ T \subset \mathbb{R} \perp \mathbb{R}^4 \supset S \quad (2.3.1) \]

and

\[ S = S^3 \quad (2.3.2) \]

where \( S^3 \) is the unit 3-sphere, so that \( M \) is a right 4-cylinder whose orthogonal section is \( S^3 \).

Note that \( M \) corresponds to the manifold in the Robertson-Walker spacetime of positive spatial curvature.

3 Supplementary conditions for metric

3.1 Tangent space

By eq. (2.2.3) the tangent space at a point \( p \in M \) is

\[ M_p = T_p \oplus S_p. \quad (3.1.1) \]

In the metric induced on \( M \) by the Euclidean metric of \( \mathbb{R}^5 \), we have

\[ T_p \perp S_p \text{ in the Euclidean metric.} \quad (3.1.2) \]
3.2 Orthogonality condition

As a key condition on the metric $g$, we adopt the following: $g$ should respect the orthogonality condition (3.1.2),

$$T_p \perp S_p \quad \text{in the metric } g.$$  \hspace{1cm} (3.2.1)

This condition is introduced from symmetry, i.e., once again, simplicity desideratum. It should be particularly emphasized that the condition is formulated geometrically, coordinate-independently.

Eq. (3.2.1) implies

$$g = g_T + g_S, \quad g_T = g_{00}, \quad g_S = g_{ij}. \quad \text{(3.2.2)}$$

Using a standard scaling for time and taking into account the Lorentzian character of the metric, we obtain

$$g = dt \otimes dt - h_t, \quad \text{(3.2.3)}$$

where $h_t$ is a time-dependent Riemannian metric on $S^3$. In the coordinate form, eq. (3.2.3) reads

$$ds^2 = dt^2 - h_{ij} dx^i dx^j, \quad dx^0 = dt. \quad \text{(3.2.4)}$$

Thus the dynamics of spacetime is described by six metric components $g_{ij} = -h_{ij}$, which corresponds to the six dynamical equations (1.1.2).

4 CPT and the eternal universe

The aim of this section is to extend parity and time-reversal transformation to a curved space-time with the manifold $M$ given by eqs. (2.2.3), (2.2.2), (2.3.2),

$$M = (a, b) \times S^3, \quad \text{(4.0.1)}$$

so as to obtain the possibility of $CPT$ invariance.

4.1 Parity

We begin with introducing parity $P$ for the unit sphere. For the sake of clarity of presentation, we consider 1- and 3-sphere in $R^2$ and $R^4$ respectively.

1-sphere:

$$x^2 + y^2 = 1, \quad y = \sin \varphi, \quad x = \cos \varphi, \quad -\pi \leq \varphi \leq \pi;$$

$$P : \varphi \rightarrow -\varphi,$$

$$y \rightarrow -y, \ x \rightarrow x; \quad \text{(4.1.1)}$$

3-sphere:
\[ x^2 + y^2 + z^2 + u^2 = 1, \]
\[ u = \sin \chi \]
\[ z = \cos \chi \cdot \sin \vartheta \]
\[ y = \cos \chi \cdot \cos \vartheta \cdot \sin \varphi \]
\[ x = \cos \chi \cdot \cos \vartheta \cdot \cos \varphi, \]
\[-\pi/2 \leq \chi \leq \pi/2, \ -\pi/2 \leq \vartheta \leq \pi/2, \ -\pi \leq \varphi \leq \pi; \]
\[ P : \chi \rightarrow -\chi, \ \vartheta \rightarrow -\vartheta, \ \varphi \rightarrow -\varphi, \]
\[ u \rightarrow -u, \ z \rightarrow -z, \ y \rightarrow -y, \ x \rightarrow x. \]  
\[ (4.1.2) \]

Thus for the \((2n+1)\)-sphere, \(n = 0, 1, \ldots\), the parity transformation \(P\) is given by the reflections in \(2n+1\) hyperplanes which intersect along a diameter and are mutually orthogonal, or, to put this another way, by the inversion with respect to the diameter. In eqs.(4.1.1), (4.1.2), the diameter is along the \(x\) axis.

For the \((2n+1)\)-sphere in \(R^{2n+3}\), \(P\) is equivalent to a rotation through \(\pi\) about the axis \(x\), i.e., through \(\pi\) in \(n + 1\) 2-planes orthogonal to the axis \(x\) and to each other.

### 4.2 Time reversal and the eternal universe

To introduce time-reversal transformation,
\[ T : t \rightarrow -t, \]  
(4.2.1)

we should adopt in eqs.(2.2.2),(4.0.1)
\[ (a, b) = R, \]  
(4.2.2)

so that \(M\) becomes an infinite cylinder,
\[ M = R \times S^3. \]  
(4.2.3)

For the infinite cylindrical manifold, time reversal is the reflection in a hyperplane orthogonal to the cylinder axis.

The infinite time interval (4.2.2) implies the universe to be eternal. One of the aspects of such a universe (namely, related to the oscillating model) was discussed in [1].

### 4.3 PT

For \(2m\)-cylinder in \(R^{2m+1}\) (in our case \(m = 2\)), \(PT\) is equivalent to the rotation through \(\pi\) about an axis (in our case the \(x\) axis) orthogonal to the cylinder axis, so that we introduce the notation
\[ R = PT \]  
(4.3.1)
4.4 CPT

Now it is possible to consider CPT invariance conditions. The dynamics of indeterministic cosmology is described by equations of motion between quantum jumps [2] and relations for the jumps [5]. The equations of motion are (II.4.1,4.2):

\[ G_S = (\Psi, T_S \Psi), \quad \text{(4.4.1)} \]
\[ H \Psi = \varepsilon \Psi. \quad \text{(4.4.2)} \]

The jump relations reduce, in the final analysis, to the between-jump dynamics and standard probabilities

\[ w = |(\Psi_2, \Psi_1)|^2. \quad \text{(4.4.3)} \]

We have

\[ G = G(g, g', g''), \quad T = T(g, g'), \quad H = H[g, g'] \quad \text{(4.4.4)} \]

where prime denotes derivatives with respect to \( x^\mu, \mu = 0, 1, 2, 3 \).

CPT transformation is

\[ g \rightarrow Rg, \quad \Psi \rightarrow V \Psi \quad \text{(4.4.5)} \]

where

\[ V = U_C U_P V_T, \quad \text{(4.4.6)} \]

the \( U \)'s are unitary and the \( V \)'s antiunitary. Transformed eqs.(4.4.1)-(4.4.3) are

\[ G_S(Rg, Rg', Rg'') = (V \Psi, T_S(Rg, Rg')V \Psi), \quad \text{(4.4.7)} \]
\[ H[Rg, Rg']V \Psi = \varepsilon V \Psi, \quad \text{(4.4.8)} \]
\[ w = |(V \Psi_2, V \Psi_1)|^2. \quad \text{(4.4.9)} \]

Since

\[ (V \Psi_2, V \Psi_1) = (\Psi_1, \Psi_2), \quad \text{(4.4.10)} \]

eq.(4.4.9) reduces to eq.(4.4.3).

Since \( G \) is quadratic in differentiation operators,

\[ G_{S_p}(Rg, Rg', Rg'') = G_{S_{RP}}(g, g', g'') \quad \text{(4.4.11)} \]

holds. We have

\[ (V \Psi, T_S(Rg, Rg')V \Psi) = (V \Psi, V \Psi^{-1} T_S(Rg, Rg')V \Psi). \quad \text{(4.4.12)} \]

Let

\[ V^{-1} T_{S_p}(Rg, Rg')V = T_{S_{RP}}(g, g') \quad \text{(4.4.13)} \]

be fulfilled, then

\[ (V \Psi, T_S(Rg, Rg')V \Psi) = (V \Psi, V T_{S_{RP}}(g, g') \Psi) = (\Psi, T_{S_{RP}}(g, g') \Psi)^* = (\Psi, V T_{S_{RP}}(g, g') \Psi), \quad \text{(4.4.14)} \]

so that eq.(4.4.7) reduces to eq.(4.4.1).
We have from (4.4.8)
\[ V^{-1}H[Rg, Rg']V\Psi = \varepsilon\Psi. \quad (4.4.15) \]

Let
\[ V^{-1}H[Rg, Rg']V = H[g, g'] \quad (4.4.16) \]

be fulfilled, then eq.(4.4.8) reduces to eq.(4.4.2).

Thus the \textit{CPT} invariance conditions are eqs.(4.4.13), (4.4.16).

5 Quantum jumps and cosmology

In conclusion, note the following. In the framework of special relativity (SR), i.e., without taking into account gravity, it is impossible to construct a dynamics of quantum jumps: There is no preferred time in the structure of Minkowskian spacetime. Both gravity and quantum jumps are not consistent with SR, which links them. Both cosmology and quantum jumps demand a preferred time, which links them. A realization of that link is at the heart of indeterministic cosmology.

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References

[1] Vladimir S. Mashkevich, \textit{Indeterministic Quantum Gravity} (gr-qc/9409010, 1994).
[2] Vladimir S. Mashkevich, \textit{Indeterministic Quantum Gravity II. Refinements and Developments} (gr-qc/9505034, 1995).
[3] Vladimir S. Mashkevich, \textit{Indeterministic Quantum Gravity III. Gravidynamics versus Geometrodynamics: Revision of the Einstein Equation} (gr-qc/9603022, 1996).
[4] Vladimir S. Mashkevich, \textit{Indeterministic Quantum Gravity IV. The Cosmic-length Universe and the Problem of the Missing Dark Matter} (gr-qc/9609035, 1996).
[5] Vladimir S. Mashkevich, \textit{Indeterministic Quantum Gravity V. Dynamics and Arrow of Time} (gr-qc/9609046, 1996).
[6] Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, Gravitation (W.H. Freeman and Company, San Francisco, 1973).
[7] S.W. Hawking, G.F.R. Ellis, The Large Scale Structure of Space-Time (Cambridge University Press, 1973).
[8] Steven Weinberg, Gravitation and Cosmology (John Wiley and Sons, Inc., New York etc., 1972).
[9] Albert Einstein, The Meaning of Relativity (Princeton, 1953).

[10] J.L. Synge, Relativity: the General Theory (North-Holland Publishing Company, Amsterdam, 1960); Talking About Relativity (North-Holland Publishing Company, Amsterdam, London, 1970).

[11] G. Cantor, Mathematishe Annalen, 20, 113 (1882).

[12] Michael Hallet, Cantorian set theory and limitation of size (Clarendon Press, Oxford, 1984).