Integral action for setpoint regulation control of a reaction-diffusion equation in the presence of a state delay

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Abstract

This paper is concerned with the regulation control of a one-dimensional reaction-diffusion equation in the presence of a state-delay in the reaction term. The objective is to achieve the PI regulation of the right Dirichlet trace with a command selected as the left Dirichlet trace. The control design strategy consists of the design of a PI controller on a finite dimensional truncated model obtained by spectral reduction. By an adequate selection of the number of modes of the original infinite-dimensional system, we show that the proposed control design procedure achieves both the exponential stabilization of the original infinite-dimensional system as well as the setpoint regulation of the right Dirichlet trace.

Key words: PI regulation; Reaction-diffusion equation; State-delay; Partial differential equation; Input-to-state stability.

1 Introduction

The proportional integral (PI) regulation control of infinite-dimensional systems, and in particular of partial differential equations (PDEs), has attracted much attention in the recent years. Early works dealt with bounded control operators [18] while the extension to unbounded control operators has been reported in [23]. The last decade has seen an intensification of the efforts in this research direction. PI boundary control of linear hyperbolic systems [2,6,10,24], as well as the extension to nonlinear transport equations [3,22] have been reported. Other types of PDEs have also been studied. This includes reaction-diffusion equations [13], wave equations used to model drilling systems [1,21], as well as semilinear wave equations [14]. The possible addition of an integral action to open-loop stable semigroups was investigated in [20].

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We study in this paper the boundary PI regulation control of a reaction-diffusion equation in the presence of a state-delay in the reaction term. Since delays are ubiquitous in practical applications, the topic of boundary stabilization of PDEs in the presence of delays, either in the control input [9,11,12,17,19] or in the state [7,8,15], has also attracted much attention in the recent years. However, it is worth noting that none of the aforementioned works embracing PI control design for PDEs was concerned with the possible presence of state-delays. This paper is a first step in that research direction. Specifically, the objective of this work is to extend the result reported in [15], which solely dealt with the boundary stabilization of a reaction-diffusion equation in the presence of a state-delay, to the PI regulation control of a Dirichlet trace. More precisely we consider the PDE:

\begin{align}
&y(t,x) = ay_{xx}(t,x) + by(t,x) + cy(t-h(t),x), \quad (1a) \\
&y(t,0) = u(t), \quad (1b) \\
&\cos(\theta)y(t,1) + \sin(\theta)y_x(t,1) = 0, \quad (1c) \\
&y(\tau,x) = \phi(\tau,x), \quad \tau \in [-h_{\text{M}},0] \quad (1d)
\end{align}

for \( t > 0 \) and \( x \in (0,1) \). Here \( a > 0, b, c \in \mathbb{R} \) with \( c \neq 0 \), \( h : \mathbb{R}_+ \to [h_{\text{m}},h_{\text{M}}] \) is continuous with \( 0 < h_{\text{m}} < h_{\text{M}} \), and \( \theta \in (0,\pi/2) \). The state at time \( t \) is \( y(t,\cdot) : [0,1] \to \mathbb{R} \). The control input is \( u(t) \in \mathbb{R} \) and applies to the left Dirichlet trace (1b). On the right-hand side of the do-
main, we consider the Robin boundary condition (1c). The initial condition is \(\phi : [-h_M, 0] \times (0, 1) \to \mathbb{R}\). The control objective is to design a PI controller in order to stabilize (1) while achieving the setpoint tracking of the right Dirichlet trace of the system. In particular, denoting by \(r : \mathbb{R}_+ \to \mathbb{R}\) a continuous reference signal, \(y(t, 1)\) must achieve the setpoint tracking of \(r(t)\).

The strategy for solving the above control design problem goes as follows. Inspired by [4], a finite dimensional-truncated model is obtained by spectral reduction. The order of the state-delayed truncated model is selected to ensure the stability of the residual infinite-dimensional dynamics. Then, inspired by [13] but with the challenge of a state-delayed term, the truncated model is augmented with an integral component to ensure the setpoint tracking of \(z(t)\). Finally, the feedback law is obtained by pole shifting. We assess the exponential stability of the closed-loop system, as well as the setpoint regulation control of the right Dirichlet trace. In the presence of an additive boundary perturbation in the control input, we show that the closed-loop system is exponentially input-to-state stable (ISS). This objective requires to work simultaneously with the original representation of the plant (for ISS purposes w.r.t. boundary disturbances) and an homogeneous version of the PDE (to analyze the system output) while handling the state-delay for both stability and setpoint regulation assessment.

The control design strategy is introduced in Section 2. The equilibrium conditions of the closed-loop system and the related dynamics of deviations are presented in Section 3. The stability analysis is reported in Section 4 while the reference tracking assessment is completed in Section 5. The robustness of the control strategy w.r.t. delay mismatches is studied in Section 6. Finally, numerical simulations are carried out in Section 7 while concluding remarks are formulated in Section 8.

## 2 Control design strategy

### 2.1 Spectral reduction and truncated model

Let \(\mathcal{H} = L^2(0, 1)\) with the inner product \(\langle f, g \rangle = \int_0^1 f g \, dx\). System (1) can be rewritten as

\[
\begin{align*}
\frac{dX}{dt}(t) &= AX(t) + cX(t - h(t)), \\
BX(t) &= u(t), \\
X(\tau) &= \Phi(\tau) = \phi(\tau, \cdot), \quad \tau \in [-h_M, 0]
\end{align*}
\]

for \(t \geq 0\) with \(A : D(A) \subset \mathcal{H} \to \mathcal{H}\) defined on \(D(A) = \{ f \in H^2(0, 1) : \cos(\theta)f(1) + \sin(\theta)f'(1) = 0 \}, \) with \(\theta \in (0, \pi/2),\) by \(Af = \alpha f'' + bf\) and the boundary operator \(B : D(B) \subset \mathcal{H} \to \mathbb{R}\) defined on \(D(B) = H^1(0, 1)\) by \(Bf = f(0).\) We define the disturbance free operator \(A_0 = A|_{D(A_0)}\) on \(D(A_0) = D(A) \cap \ker(B).\) It is well-known that \(A_0\) generates a \(C_0\)-semigroup. We introduce \(L \in L(\mathcal{H})\) defined for any \(u \in \mathcal{H}\) by \([Lu](x) = (1 - x)^2 u, \ x \in [0, 1]\). L has been selected such that its range satisfies \(R(L) \subset D(A)\) and \(BL = I_B.\) Hence, following the terminology of [5, Sec. 3.3], the pair \((A, B)\) defines a boundary control system with associated lifting operator \(L.\) We define \(A_\theta(t) \triangleq A + t \Phi(t)\) and \(A_{\leq 0}(t) \triangleq A_0 + t \Phi(t)\) on \(D(A_\theta) = D(A)\) and \(D(A_{\leq 0}) = D(A_0),\) respectively. From the Sturm-Liouville theory, it well known that the eigenvalues of \(A_{\leq 0}\) are simple and form a decreasing sequence \((\lambda_n)_{n \geq 0} \subset \mathbb{R}^n\) with \(\lambda_n \to -\infty\) when \(n \to +\infty.\) Moreover, one can select the associated eigenvectors such that \((e_n)_{n \geq 0}\) forms a Hilbert basis of \(\mathcal{H}.\) Using the terminology of [5, Def. 2.3.4], \(A_{\leq 0}\) is a Riesz spectral operator:

\[
D(A_{\leq 0}) = \{ f \in L^2(0, 1) : \sum_{n \geq 0} |\lambda_n|^2 \langle f, e_n \rangle^2 < \infty \}
\]

and \(\mathcal{A}_{\leq 0}f = \sum_{n \geq 0} \lambda_n \langle f, e_n \rangle e_n\) for all \(f \in D(A_{\leq 0}).\) Standard computations give \(\lambda_n = b + c - ar_n^2\) and \(e_n = 2\sqrt{2}b_n \sin(2r_n)\) with \(n \in \mathbb{N}\) where \(r_n > 0\) is the unique number \(r \in ((\pi, (n + 1)\pi)\) such that \(r \cot(r) = -\cos(\theta).\) This yields \(\lambda_n \sim -an^2\pi^2\) and \(e_n(1) = O(1)\) as \(n \to +\infty.\) Introducing \(x_n(t) = (X(t), c_n)\) the coefficients of projection of the system trajectory into the Hilbert basis, we have that \(X(t) = \sum_{n \geq 0} x_n(t)e_n\) and \(\|X(t)\|^2 = \sum_{n \geq 0} |x_n(t)|^2.\) Assuming that \(u\) is continuously differentiable and \(\phi\) is continuous, the mild solution \(X \in C^0([0, 1] ; \mathcal{H})\) of (2) is such that \(x_n\) is continuously differentiable and (see [15] for details)

\[
\dot{x}_n(t) = \lambda_n x_n(t) + c(x_n(t - h(t)) - x_n(t)) + (a_n + \lambda_n b_n)u(t)
\]

with

\[
a_n = \langle A, \mathbb{I}, e_n \rangle, \quad b_n = -\langle \mathbb{I}, e_n \rangle
\]

where \(\mathbb{I}\) denotes here the unit element of \(\mathbb{R}\). Note that due to the presence of the state-delay, there may exist delays for which certain modes \(x_n,\) hence the PDE, are unstable even if \(\lambda_n - c = b + c - ar_n^2 < 0\) and \(c < 0\) provided \(c\) is large enough [16, Sec. 3.3]. For a given integer \(N \geq 0\) selected such that \(\lambda_n < 0\) for all \(n \geq N + 1\) and which will be further constrained later, we define the following:

\[
\begin{align*}
Y(t) &= [x_0(t) \ldots x_N(t)]^T \in \mathbb{R}^{N+1}, \\
Y_\phi(\tau) &= [(\Phi(\tau), e_0) \ldots (\Phi(\tau), e_N)]^T \in \mathbb{R}^{N+1}, \\
A &= \text{diag}(\lambda_n)_{0 \leq n \leq N} \in \mathbb{R}^{(N+1) \times (N+1)}, \\
B &= (a_n + \lambda_n b_n)_{0 \leq n \leq N} \in \mathbb{R}^{N+1},
\end{align*}
\]

Then we obtain the truncated model:

\[
\begin{align*}
\dot{Y}(t) &= AY(t) + c(Y(t - h(t)) - Y(t)) + Bu(t) \quad (6a) \\
Y(\tau) &= Y_\phi(\tau), \quad \tau \in [-h_M, 0]\end{align*}
\]
2.2 Addition of an integral component

The objective is now to augment the truncated model (6) with an integral component to achieve the setpoint regulation control of the right Dirichlet trace \( z(t) = y(t, 1) \).

We first need to express the right Dirichlet trace \( y(t, 1) \) in function of the coefficients of projection \( x_n \).

**Lemma 1** Let \( \theta \in (0, \pi/2) \). For all \( f \in D(A_{c,0}) \) we have \( f(1) = \sum_{n \geq 0} \langle f, e_n \rangle e_n(1) \).

The proof of this Lemma, which is omitted, essentially relies on the Riesz-spectral property of \( A_{c,0} \). We cannot directly apply the above series expansion to the trajectory \( X \) of our system because, in general, \( X(t) \notin D(A_{c,0}) \). However, if we assume that \( X \in C^0(\mathbb{R}_+; D(A)) \cap C^1(\mathbb{R}_+; H) \) is a classical solution of (2), one has \( W(t) = X(t) - Lu(t) \in D(A_0) = D(A_{c,0}) \) with in particular \( y(t, 1) = \|X(t)(1)\| = \|W(t)(1)\| \). Hence, introducing \( w_n(t) = (W(t), e_n) = x_n(t) + b_n u(t) \), we obtain that \( y(t, 1) = \|X(t)(1)\| = \sum_{n \geq 0} w_n(t)e_n(1) \) for all \( t \geq 0 \). Since \( u \) is of class \( C^1 \) (we will actually need \( u \) of class \( C^2 \) to ensure the existence of classical solutions), we have that \( w_n \) is of class \( C^1 \) and, from (3),

\[
\dot{w}_n(t) = \lambda_n w_n(t) + c w_n(t - h(t)) - w_n(t) + a_n u(t) - cb_n (u(t - h(t)) - u(t)) + b_n u(t)
\]

for \( t \geq h_M \).

Remind that our objective is to achieve the setpoint regulation control of the system output \( z(t) = y(t, 1) \). In order to introduce in a comprehensive manner the proposed integral component \( \zeta(t) \in \mathbb{R} \) that will be used to augment the truncated model (6), consider first the classical integral component given by \( \chi(t) = y(t, 1) - \zeta(t) = \sum_{n \geq 0} w_n(t)e_n(1) - r(t) \) for \( t \geq 0 \). Here \( r(t) \in \mathbb{R} \) stands for a reference signal. Recall that the second equality holds only when considering classical solutions for (2). As the above series expansion involves all the modes of the system, and in particular the coefficients of projection \( w_n(t) \) for \( n \geq N + 1 \), the integral component \( \chi \) cannot be directly included into the dynamics of the truncated model (6). To solve this issue, we introduce the following preliminary change of variable \( \zeta_0(t) = \chi(t) + \sum_{n \geq N + 1} \frac{\epsilon_n(1)}{\lambda_n} (b_n u(t) - w_n(t)) \). Note that the convergence of the series follow from \( \lambda_n \sim -an^2 \pi^2 \) and \( \epsilon_n(1) = O(1) \) as \( n \to +\infty \). Based on (7) we obtain, for \( t \geq h_M \), \( \zeta_0(t) = \sum_{n=0}^N \frac{\epsilon_n(1)}{\lambda_n} (w_n(t - h(t)) - w_n(t)) + c \sum_{n \geq N + 1} \frac{\epsilon_n(1)}{\lambda_n} b_n (u(t - h(t)) - u(t)) \) where

\[
\alpha = \sum_{n=0}^N b_n e_n(1) - \sum_{n \geq N + 1} \frac{\epsilon_n}{\lambda_n} e_n(1).
\]

We now note that the two last terms of the above identity describing the \( \zeta_0 \)-dynamics have a null contribution at equilibrium. This observation motivates the introduction of the below \( \zeta \)-dynamics. Assuming that the delay \( h \) is known (robustness w.r.t. delay mismatches will be discussed later in Section 6), we mimic the structure of the dynamics of the truncated model (6) by defining for \( t \geq 0 \) the integral component \( \zeta(t) \in \mathbb{R} \) as follows:

\[
\dot{\zeta}(t) = \sum_{n=0}^N x_n(t)e_n(1) + c(\zeta(t-h(t)) - \zeta(t)) \tag{9a} + au(t) - r(t),
\]

\[
\zeta(\tau) = \zeta_0(\tau), \quad \tau \in [-h_M, 0] \tag{9b}
\]

**Remark 1** The \( \zeta \)-dynamics achieves the same equilibrium condition as the \( \zeta_0 \)-dynamics. As we will show later in Section 3, the integral component (9) ensures that the equilibrium condition (\( X_e, \zeta_e \)) of the forthcoming closed-loop system, associated with some constant reference signal \( r(t) = r_e \), achieves the desired reference tracking for the right Dirichlet trace, i.e., \( X_e(1) = r_e \).

**Remark 2** Even if (9) has been motivated and derived by considering classical solutions of (2), the dynamics (9) actually makes sense for any mild solutions of (2).

Since (9) only involves the \( N + 1 \) first modes of the system, we can now augment the dynamics of the truncated model (6) with the \( \zeta \)-dynamics as follows:

\[
\dot{Y}_a(t) = A_a Y_a(t) + c Y_a(t-h(t)) - Y_a(t) + B_a u(t) + \Gamma(t), \tag{10a}
\]

\[
Y_a(\tau) = Y_{\Phi,a}(\tau), \quad \tau \in [-h_M, 0] \tag{10b}
\]

where \( C = [\epsilon_0(1) \ldots \epsilon_N(1)] \in \mathbb{R}^{1 \times (N+1)} \),

\[
Y_a(t) = \begin{bmatrix} Y(t) \\ \zeta(t) \end{bmatrix}, \quad Y_{\Phi,a}(\tau) = \begin{bmatrix} Y_{\Phi}(\tau) \\ \zeta_0(\tau) \end{bmatrix}, \tag{11a}
\]

\[
A_a = \begin{bmatrix} A \\ C \end{bmatrix}, \quad B_a = \begin{bmatrix} B \\ \alpha \end{bmatrix}, \quad \Gamma(t) = \begin{bmatrix} 0 \\ -r(t) \end{bmatrix}. \tag{11b}
\]

2.3 Proposed control strategy

The proposed control strategy consists of a stabilizing state feedback of the truncated model (10). Such a procedure is allowed by the following lemma.

**Lemma 2** \((A_a, B_a)\) satisfies the Kalman condition.

**Proof.** We define the matrix \( T = \begin{bmatrix} A & B \\ C & \alpha \end{bmatrix} \in \mathbb{R}^{(N+2) \times (N+2)} \).

From (11), the Hautus test shows that the pair
Let \(0 < h_m < h_M\), \(h \in C^0(\mathbb{R}_+)\) with \(h_m \leq h(t) \leq h_M\), \(\Phi \in C^1([-h_M,0];\mathcal{H})\), \(\mathcal{G}_0 \in C^1([-h_M,0]), \) \(p \in C^0(\mathbb{R}_+)\), and \(r \in C^0(\mathbb{R}_+).\) Then there exists a unique mild solution \(X_{\zeta} = (X,\zeta) \in C^0(\mathbb{R}_+;\mathcal{H})\) of (2) and (9) with control input (12). Moreover we have \(u,\zeta \in C^1(\mathbb{R}_+).\)
introducing for $n \geq 0$ the quantities $w_{n,e} = x_{n,e} + b_{n}u_{e}$, we have for $n \geq N + 1$ that $w_{n,e} = -\sum_{\kappa=0}^{N} a_{n}\kappa$ showing that $(w_{n,e})_{n\geq0} \in \ell^{2}(\mathbb{N})$ and $(\lambda_{n}w_{n,e})_{n\geq0} \in \ell^{2}(\mathbb{N})$. This allows the introduction of $W_{e} = \sum_{n\geq0} w_{n,e}e_{n} \in D(A_{0}) = D(A_{-0})$. Moreover, from the definition of $b_{n}$ given by (4), we have $X_{e} = W_{e} + Lu_{e} \in D(A_{e})$ hence $B_{X} = u_{e}$. Furthermore, since $a_{n}w_{n,e} + a_{n}u_{e} = 0$ for all $n \geq 0$, we have from the definition of $a_{n}$ given by (4) that $A_{-0}W_{e} + A_{e}Lu_{e} = 0$ hence $A_{e}X_{e} = 0$. Using now Lemma 1, (8), and the above relations between $w_{n,e}$ and $w_{n,e}$, we obtain from $0 = \sum_{n=0}^{N} x_{n,e}e_{n}(1) + \alpha u_{e} - r_{e}$ that $W_{e}(1) = r_{e}$. Since $X_{e} \in D(A_{e}) \in H^{1}(0,1)$, we infer that $X_{e}(1) = W_{e}(1) + [Lu_{e}](1) = r_{e}$, which provides the desired reference tracking.

3.2 Dynamics of deviations

Let $r_{e}, p_{e} \in \mathbb{R}$ be arbitrary and consider the different equilibrium quantities defined above. We can introduce the dynamics of deviations of the system trajectory with respect to the considered equilibrium condition. These deviations are denoted by the symbol “$\Delta$”. For instance, $\Delta X(t) = X(t) - X_{e}$.

The main result of this section is stated as follows.

Theorem 1 Let $0 < h_{m} < h_{M}$ be arbitrarily given. Let $N \geq 0$ be such that $\lambda_{N+1} < -2\sqrt{3}|c|$ and consider the matrices $A_{e}$ and $B_{e}$ defined by (11). Let $K \in \mathbb{R}^{1 \times (N+2)}$ be such that $A_{K} = A_{a} + B_{a}K$ is Hurwitz with simple eigenvalues $\mu_{1}, \ldots, \mu_{N} \in \mathbb{C}$ satisfying $\text{Re} \mu_{n} < -3|c|$ for all $1 \leq n \leq N$. Then there exist constants $\kappa, C_{0}, C_{1} > 0$ such that, for any $x_{0} \in C^{0}([-h_{M}, 0]; \mathbb{R}^{N})$, any $h \in C^{0}(\mathbb{R}_{+})$ with $h_{m} < h(t) \leq h_{M}$, and any $q \in C^{0}(\mathbb{R}_{+}; \mathbb{R})$, the trajectory of

\[ \dot{x}(t) = Ax(t) + c\{x(t - h(t)) - x(t)\} + q(t), \]

\[ x(\tau) = x_{0}(\tau), \quad \tau \in [-h_{M}, 0] \]

satisfies, for all $t \geq 0$,

\[ \|x(t)\| \leq C_{0}e^{-\kappa t} \sup_{\tau \in [-h_{M}, 0]} \|x_{0}(\tau)\| + C_{1} \sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)}\|q(\tau)\|. \]  

(16)

From the assumptions of Thm. 1, Lemma 4 applies to the truncated model (14a) with initial condition (14d).

4.2 Residual infinite-dimensional dynamics

We now need to investigate the selection of the integer $N \geq 0$ such that the residual dynamics composed of (14b) and (14e) is exponentially stable.

Lemma 5 Let $0 < h_{m} < h_{M}$ and $\sigma, C_{2}, C_{3} > 0$ be arbitrarily given. Let $N \geq 0$ be such that $\lambda_{N+1} < -2\sqrt{5}|c|$. Then, there exist constants $\kappa \in (0, \sigma)$ and $C_{4}, C_{5} > 0$ such that, for all $h \in C^{0}(\mathbb{R}_{+})$ with $h_{m} \leq h(t) \leq h_{M}$,
\( \Phi \in C^0([-h_M,0]; \mathcal{H}), \zeta_0 \in C^0([-h_M,0]), p \in C^1(\mathbb{R}_+), r \in C^0(\mathbb{R}_+), \text{ and } u \in C^1(\mathbb{R}_+) \) such that

\[
|\Delta u(t)| \leq C_2e^{-\sigma t} \sup_{\tau \in [-h_M,0]} (||\Delta \Phi(\tau)|| + |\Delta \zeta_0(\tau)|) \\
+ C_3 \sup_{\tau \in [0,t]} e^{-\sigma(t-\tau)} (|\Delta p(\tau)| + |\Delta r(\tau)|) \\
(17)
\]

for all \( t \geq 0 \), the mild solution \( X_\zeta = (X, \zeta) \in C^0(\mathbb{R}_+; \mathcal{H}_\zeta) \) of (2) and (9) satisfies for all \( t \geq 0 \)

\[
\sum_{n \geq N+1} |\Delta x_n(t)|^2 \\
\leq C_4 e^{-2\kappa t} \sup_{\tau \in [-h_M,0]} (||\Delta \Phi(\tau)|| + |\Delta \zeta_0(\tau)|)^2 \\
+ C_5 \sup_{\tau \in [0,t]} e^{-2\kappa(t-\tau)} (|\Delta p(\tau)| + |\Delta r(\tau)|)^2. \\
(18)
\]

**Remark 5** The design constraint \( \lambda_{N+1} < -2\sqrt{\nu}|c| \) is the same as in [15, Lem. 10]. However, the proof reported therein does not apply in the presence of the boundary perturbation \( p \). Indeed, following the lines of [15, Lem. 10], one gets an estimate similar to (18) but with the occurrence of the extra term \( |\Delta \psi(\tau)| \) in the term evaluating the contribution of \( \Delta p \) and \( \Delta r \). We refine here the stability analysis in order to obtain the claimed estimate (18) involving only \( \Delta p \), and not \( \Delta \psi \).

**Proof.** Let \( N \geq 0 \) be such that \( \lambda_{N+1} < -2\sqrt{\nu}|c| \). We define \( \eta = -\lambda_{N+1}/2 > \sqrt{\nu}|c| \geq 0 \), which is such that \( \lambda_n \leq \lambda_{N+1} = -2\eta < 0 \) for all \( n \geq N + 1 \). Note that, in this proof, we always consider integers \( n \geq N + 1 \). Let \( \kappa \in (0, \min(\eta, \sigma)) \) be arbitrarily given and to be specified later. We introduce, for \( t \geq 0 \), the estimated \( \Delta v_n(t) = \Delta x_n(t) - \Delta x_n(t - h(t)) \), yielding

\[
\Delta x_n(t) = \lambda_n \Delta x_n(t) - c \Delta v_n(t) + (a_n + \lambda_n b_n) \Delta u(t) \\
(19)
\]

for all \( t \geq 0 \). We also consider the series

\[
S_x(t) = \sum_{n \geq N+1} |\Delta x_n(t)|^2, \\
t \geq -h_M; \\
S_v(t) = \sum_{n \geq N+1} |\Delta v_n(t)|^2, \\
t \geq 0
\]

which are finite because \( S_x(t) \leq ||\Delta X(t)||^2 \) and \( S_v(t) \leq 2S_x(t) + 2S_x(t - h(t)) \). Finally, we introduce for any \( t_1 < t_2 \) and any real-valued and continuous function \( \psi \) the notation \( I(\psi, t_1, t_2) = \int_{t_1}^{t_2} e^{-\eta|\psi(\tau)|} \psi(\tau) \, d\tau \). We have

\[
I(\psi, t_1, t_2) \leq \frac{1 - e^{-2\eta|\psi(t_2 - t_1)|}}{2(\eta - \kappa)} \sup_{\tau \in [t_1, t_2]} e^{-2\kappa|\psi(\tau)|} \\
\text{and } I(\psi, t_1, t_2)^2 \leq \frac{1 - e^{-2\eta|\psi(t_2 - t_1)|}}{2\eta} I(\psi^2, t_1, t_2).
\]

By integrating (19), we obtain for \( t \geq h_M \)

\[
\Delta v_n(t) = \left(e^{\lambda_n h(t)} - 1\right) \Delta x_n(t - h(t)) + \int_{t-h(t)}^{t} e^{\lambda_n (t-\tau)} \{ -c \Delta v_n(\tau) + (a_n + \lambda_n b_n) \Delta u(\tau) \} \, d\tau
\]

hence, using \( \lambda_n \leq -2\eta \),

\[
|\Delta v_n(t)| \leq |\Delta x_n(t - h(t))| + |cI(\Delta v_n, t - h(t), t) + |a_n|^2 I(\Delta u, t - h(t), t) \\
+ |b_n|^2 \left|\lambda_n \int_{t-h(t)}^{t} e^{\lambda_n (t-\tau)} \Delta u(\tau) \, d\tau \right|.
\]

Since \( \kappa < \eta \) we have

\[
\left|\lambda_n \int_{t-h(t)}^{t} e^{\lambda_n (t-\tau)} \Delta u(\tau) \, d\tau \right| \leq \frac{2\eta}{2\eta - \kappa} \sup_{\tau \in [-h(t), t]} e^{-\kappa|\tau|} |\Delta u(\tau)|
\]

because \( \lambda_n \leq -2\eta < -\eta < -\kappa < 0 \). Combining the two latter estimates and using Young’s inequality we obtain

\[
|\Delta v_n(t)|^2 \leq 4|\Delta x_n(t - h(t))|^2 + 2\gamma_2 |c|^2 I(\Delta v_n^2, t - h(t), t) \\
+ \gamma_1 |a_n|^2 I(\Delta u^2, t - h(t), t) \\
+ \frac{16\eta^2}{(2\eta - \kappa)^2} |b_n|^2 \sup_{\tau \in [-h(t), t]} e^{-2\kappa|\tau|} |\Delta u(\tau)|^2
\]

for all \( t \geq h_M \) where \( \gamma_1 = \frac{2}{n}(1 - e^{-2\eta h_M}) \). Summing for \( n \geq N + 1 \), we obtain for \( t \geq h_M \)

\[
S_v(t) \leq 4S_x(t - h(t)) + 2\gamma_2 |c|^2 \sup_{\tau \in [-h(t), t]} e^{-2\kappa|\tau|} S_v(\tau) \\
+ \gamma_3(\kappa) \sup_{\tau \in [-h(t), t]} e^{-2\kappa|\tau|} \|\Delta u(\tau)\|^2
\]

where \( a = A_{\kappa}L_{11}, b = -L_{11}, \gamma_2(\kappa) = \frac{1}{\eta(\eta - \kappa)} (1 - e^{-2\eta h_M}) (1 - e^{-2(\eta - \kappa) h_M}) \) and \( \gamma_3(\kappa) = \gamma_2(\kappa) \|a\|^2 + \frac{16\eta^2}{(2\eta - \kappa)^2} \|b\|^2 \). This implies that, for all \( t \geq h_M \),

\[
\sup_{\tau \in [h_M, t]} e^{2\kappa\tau} S_v(\tau) \leq 4e^{2\kappa h_M} \sup_{\tau \in [0, t-h_M]} e^{2\kappa\tau} S_v(\tau) \\
+ \gamma_3(\kappa) \sup_{\tau \in [0, t]} e^{2\kappa\tau} \|\Delta u(\tau)\|^2.
\]

Integrating now (19) on \([0, t]\) for \( t \geq 0 \), using again \( \lambda_n \leq -2\eta \), and proceeding as in the previous paragraph, we infer that, for all \( t \geq 0 \),

\[
S_x(t) \leq 4e^{-2\kappa t} S_x(0) + \gamma_4(\kappa) |c|^2 \sup_{\tau \in [0, t]} e^{-2\kappa(\tau - t)} S_v(\tau) \\
+ \gamma_5(\kappa) \sup_{\tau \in [0, t]} e^{-2\kappa|\tau|} \|\Delta u(\tau)\|^2
\]

where \( \gamma_4(\kappa) = \frac{1}{2\eta(\eta - \kappa)} \) and \( \gamma_5(\kappa) = \gamma_4(\kappa) \|a\|^2 + \frac{16\eta^2}{(2\eta - \kappa)^2} \|b\|^2 \). Combining (20-21) and noting that \( S_x(0) \leq \|\Delta \Phi(0)\|^2 \), we obtain for \( t \geq h_M \)

\[
\sup_{\tau \in [h_M, t]} e^{2\kappa\tau} S_v(\tau) \leq 16e^{2\kappa h_M} \|\Delta \Phi(0)\|^2
\]
\[ + \xi(k) \sup_{\tau \in [0,t]} e^{2\kappa \tau} S_{\nu}(\tau) + \gamma_6(k) \sup_{\tau \in [0,t]} e^{2\kappa \tau} |\Delta u(\tau)|^2 \]

with \( \gamma_6(k) = \gamma_3(k) + 4e^{2h_M} \gamma_5(k) \) and

\[ \xi(k) = \gamma_2(k)|c|^2 + 4e^{2h_M} \gamma_4(k)|c|^2 \]

\[ = \frac{|c|^2}{\eta(\eta - \kappa)} \left( 4e^{2h_M} + (1 - e^{-2\eta h_M})(1 - e^{-2(\eta - \kappa) h_M}) \right). \]

Recalling from the design constraint \( \lambda_M + 1 < -2\sqrt{5}|c| \)

that \( \eta > 2\sqrt{5}|c| \), we have \( 5|c|^2/\eta^2 < 1 \). Hence, a

continuity argument at \( k = 0 \) shows the existence of \( \kappa \in (0, \min(\eta, \eta)) \) such that \( 0 \leq \xi(k) < 1 \). We fix such a

\( \kappa \in (0, \min(\eta, \eta)) \) for the rest of the proof. Since all the

considered supremums are finite, we deduce from the latter estimate that, for all \( t \geq h_M, \)

\[ \sup_{\tau \in [h_M,t]} e^{2\kappa \tau} S_{\nu}(\tau) \leq \frac{16e^{2h_M}}{1 - \xi} ||\Delta \Phi(0)||^2 \tag{22} \]

\[ + \frac{\xi}{1 - \xi} \sup_{\tau \in [0,h_M]} e^{2\kappa \tau} S_{\nu}(\tau) + \frac{\gamma_6}{1 - \xi} \sup_{\tau \in [0,t]} e^{2\kappa \tau} |\Delta u(\tau)|^2 \]

where we dropped the dependency of \( \gamma_6, \xi \) on the parameter \( \kappa \) which is now fixed. To conclude the proof, we need to estimate the term \( \sup_{\tau \in [0,t]} e^{2\kappa \tau} S_{\nu}(\tau) \) for \( t \in [0, h_M] \)

From the definition of \( S_{\nu} \), we have, for any \( t \in [0, h_M], \)

\[ \sup_{\tau \in [0,t]} e^{2\kappa \tau} S_{\nu}(\tau) \leq 4e^{2h_M} \sup_{\tau \in [h_M,t]} S_{\nu}(\tau). \]

From (14b) and recalling that \( n \geq N + 1 \) with \( \lambda_n < -2\eta < -2\sqrt{5}|c| \), we have \( \lambda_n - c \leq \lambda_n + |c| < -(2\sqrt{5} - 1)|c| \leq 0 \)

\[ |\Delta x_n(t)| \leq |\Delta x_n(0)| + |c| \sqrt{h_M} \int_0^t |\Delta x_n(\tau - h)|^2 | \ dx \]

\[ + (|a_n| h_M + |b_n| h_M) \sup_{\tau \in [0,t]} |\Delta u(\tau)| \]

for all \( t \in [0, h_M] \). Using Young’s inequality and summing for \( n \geq N + 1 \), we obtain

\[ S_{\nu}(t) \leq 3S_{\nu}(0) + 3|c|^2 h_M^2 \sup_{\tau \in [h_M,t-h_n]} S_{\nu}(\tau) \]

\[ + 6(||a||^2 h_M^2 + ||b||^2 e^{2|c|h_M}) \sup_{\tau \in [0,t]} |\Delta u(\tau)|^2 \]

for all \( t \in [0, h_M]. \) This implies, for all \( t \in [0, h_M], \)

\[ \sup_{\tau \in [0,t]} S_{\nu}(\tau) \leq 3(1 + |c|^2 h_M^2) \sup_{\tau \in [h_M,0]} ||\Delta \Phi(\tau)||^2 \]

\[ + 3|c|^2 h_M^2 \sup_{\tau \in [0,\max(1-h_M,0)]} S_{\nu}(\tau) \]

\[ + 6(||a||^2 h_M^2 + ||b||^2 e^{2|c|h_M}) \sup_{\tau \in [0,t]} |\Delta u(\tau)|^2. \]

By a simple induction argument (since \( h_M > 0 \)), we obtain the existence of a constant \( \gamma_7 \) such that, for all \( t \in [0, h_M], \)

\[ \sup_{\tau \in [0,t]} e^{2\kappa \tau} S_{\nu}(\tau) \leq \gamma_7 \sup_{\tau \in [-h_M,0]} ||\Delta \Phi(\tau)||^2 + \gamma_7 \sup_{\tau \in [0,t]} |\Delta u(\tau)|^2. \]

We deduce (see beginning of this paragraph) the existence of a constant \( \gamma_8 > 0 \) such that, for all \( t \in [0, h_M], \)

\[ \sup_{\tau \in [0,t]} e^{2\kappa \tau} S_{\nu}(\tau) \leq \gamma_8 \sup_{\tau \in [-h_M,0]} ||\Delta \Phi(\tau)||^2 + \gamma_8 \sup_{\tau \in [0,t]} e^{2\kappa \tau} |\Delta u(\tau)|^2. \]

Combining this latter estimate with (22), we infer the existence of a constant \( \gamma_9 > 0 \) such that, for all \( t \geq 0, \)

\[ \sup_{\tau \in [0,t]} e^{2\kappa \tau} S_{\nu}(\tau) \leq \gamma_9 \sup_{\tau \in [-h_M,0]} ||\Delta \Phi(\tau)||^2 \tag{23} \]

\[ + \gamma_9 \sup_{\tau \in [0,t]} e^{2\kappa \tau} |\Delta u(\tau)|^2. \]

Substituting this estimate into (21), we obtain the existence of a constant \( \gamma_10 > 0 \) such that, for all \( t \geq 0, \)

\[ S_{\nu}(t) \leq \gamma_10 e^{-2\kappa t} \sup_{\tau \in [-h_M,0]} ||\Delta \Phi(\tau)||^2 + \gamma_10 \sup_{\tau \in [0,t]} e^{-2\kappa(t-\tau)} |\Delta u(\tau)|^2. \]

The claimed estimate (18) now directly follows from the assumption that \( \Phi \) satisfies (17) and the fact that \( 0 < \kappa < \sigma. \) \( \square \)

4.3 Completion of the proof of Theorem 1

By applying first the result of Subsection 4.1 and then the result of Subsection 4.2, the claimed estimate (15) follows from \( |\Delta \zeta(t)| \leq ||\Delta Y_{\nu}(t)||, ||\Delta \chi(t)|| \leq ||\Delta Y_{\nu}(t)|| + \sqrt{\sum_{n \geq N+1} |\Delta x_n(t)|^2}, \) and (14c). This completes the proof of Theorem 1.

5 Setpoint regulation assessment

We now address the setpoint regulation of the closed-loop system for classical solutions.

Theorem 2 Under the assumptions of Theorem 1, and for the same constant \( \kappa > 0, \) there exist constants \( \overline{c}_2, \overline{c}_3 > 0 \) such that, for all \( h \in C^1([h_M,0]) \) with \( h_n \leq h(t) \leq h_M \) and so that \( t \to t - h(t) \) crosses \( 0 \) a finite number of times, \( \Phi \in C^1([-h_M,0]; H) \) with \( \Phi(0) \in D(\Delta), \zeta_0 \in C^1([-h_M,0]), p \in C^2(R^+), \)

and \( r \in C^1(R^+), \) all such that the compatibility condition (13) holds, we have, for all \( t \geq 0, \)

\[ ||X(t)||(1) - r(t) \leq \overline{c}_2 e^{-\kappa t} \left\{ \sup_{\tau \in [-h_M,0]} (||\Delta \Phi(\tau)|| + |\Delta \zeta_0(\tau)|) + ||A_r \Phi(0)|| \right\} \]

\[ + \overline{c}_3 \sup_{\tau \in [0,t]} e^{-(\kappa(t-\tau))} (|\Delta p(\tau)| + |\Delta \psi(\tau)| + |\Delta r(\tau)|). \]

Corollary 3 In the context of Theorem 2, assume that \( r(t) \to r_c, p(t) \to p_c, \) and \( p(t) \to 0 \) as \( t \to +\infty \). Then \( |X(t)(1)| \to r_c \) as \( t \to +\infty \) with exponential vanishing of the contribution of the initial conditions.

Proof of Theorem 2. Recalling that, for classical solutions, \( W(t) = X(t) - Lu(t) \in D(A_{c,0}), \) and since
\( W_e = X_e - Lu_e \in D(A_{e,0}) \) with \( X_e(1) = W_e(1) = r_e \), we have \( ||X(t) - (1 - r(t)) \leq ||W(t) - (1 - r)|| + ||X(t) + \Delta W|| \leq ||X(t)|| + \Delta X(t) \). To obtain (24), we first need to investigate the term \( ||\Delta W(t)|| \). To do so, since \( \lambda_n \sim -an^2\pi^2 \) as \( n \to +\infty \), let \( \delta > 0 \) and an integer \( M \geq N \) such that \( \lambda_n \leq -2(2\delta + \delta) < 0 \) and \( |\lambda_n| \leq |\lambda_n| \) for all \( n \geq M + 1 \). Then we have \( \Delta W(t) = \sum_{n \geq 0} \Delta w_n(t)c_n(1) \leq \sqrt{\sum_{n \geq 0} |c_n(1)|^2 ||\Delta W(t)|| + \sum_{n \geq M+1} |c_n(1)|^2 \sqrt{\sum_{n \geq M+1} |\lambda_n||\Delta w_n||^2} \) where it can be seen from \( \lambda_n \sim -an^2\pi^2 \) and \( c_n(1) = O(1) \) as \( n \to +\infty \).

The two latter inequalities imply the existence of a constant \( \gamma_{11} > 0 \) such that

\[
\frac{1}{\gamma_{11}} \sum_{n \geq M+1} |\lambda_n||\Delta w_n(t)||^2 \leq \tag{25}
\]
\[
e -2\kappa e^{-2\kappa t} \sum_{n \geq M+1} |\lambda_n||\Delta w_n(0)||^2 + e^{-2\kappa t} \sup_{\tau \in [-\delta,0]} \|\Delta \Phi(\tau)||^2 + \sup_{\tau \in [0,t]} e^{-2\kappa (t-\tau)}|\Delta u(\tau)||^2 \]

for all \( t \geq 0 \). Since \( \Delta W(0) \in D(A_{e,0}) \) and \( |\lambda_n| \leq |\lambda_n| \) for all \( n \geq M + 1 \), we note that \( \sum_{n \geq M+1} |\lambda_n||\Delta w_n(0)||^2 \leq ||A_{e,0}\Delta W(0)||^2 \leq ||A_{e,0}X(0)||^2 + ||L||||u(0)||^2 \) with \( \Delta X(0) = \Delta \Phi(0) \) and \( ||u(0)|| \leq \|K\|\|\Delta Y(u(0))\| + ||\Delta p(0)|| \leq \|K\|\|\Delta \Phi(0)|| + ||\Delta p(0)|| \). To conclude the proof, it is sufficient to study the two last terms of (25). The estimation of the term involving \( \Delta u \) immediately follows from (15). Hence, only the term involving \( \Delta v \) needs to be investigated. From (14a) and (14c), we have, for all \( t \geq 0 \), \( |\Delta u(t)|| \leq ||K|||\Delta v(t)|| + ||\Delta \phi(0)|| \) with \( ||\Delta v(t)|| \leq ||A_k - cI||||\Delta s(0)|| + ||c||||\Delta s \in h(t)|| + ||B_n||||\Delta p(0)|| + ||\Delta \phi(0)|| \). The claimed conclusion now follows from \( ||\Delta Y(u(t))|| \leq ||\Delta X(t)|| + ||\Delta \zeta(t)|| \) for \( \tau \geq -\delta \).

\textbf{Remark 6} In the context of Theorem 2 dealing with classical solutions, the stability result stated by Theorem 1 can be strengthened as follows. First, it can be shown similarly to [19, Eq. 42] that \( \|f\|^2 = -\cot(\theta)|f(1)|^2 + \frac{b+c}{a}||f||^2 - \frac{1}{a} \sum_{n \geq 0} \lambda_n|\langle f, e_n \rangle|^2 \) for any \( f \in D(A_{e,0}) \). Considering classical solutions, we can apply this identity to \( \Delta W(t) \in D(A_{e,0}) \) where we note that estimates of \( ||\Delta W(t)|| \) and \( ||\Delta W(t)\|| \) are provided by Theorem 1 and Theorem 2, respectively, while the series \( \sum_{n \geq 0} |\lambda_n||\Delta w_n(t)||^2 \) has been evaluated in the proof of Theorem 2. Since \( \Delta X(t) = \Delta W(t) + Lu(t) \in D(A_e) \subset H^2(0,1) \) with \( \|\Delta u(t)|| \leq \frac{1}{\gamma_{11}} \|\Delta w_n(t)||^2 \), we infer that \( ||\Delta X(t)|| \), and hence \( \|\Delta X(t)|| \), is upper bounded by a term similar (i.e., with different constants \( \gamma_{11} \)) to the right-hand side of (24). If we further make the assumptions of Corollary 3, we obtain that \( X(t) \) converges in \( H^1(0,1) \) norm and hence, by the continuous embedding \( H^1(0,1) \subset C^0([0,1]) \), in \( L^\infty \) norm to \( X_e \) when \( t \to +\infty \).

\section{Robustness with respect to delay mismatches}

In the previous sections, we have assumed the perfect knowledge of the state-delay \( h \). This was used to build the dynamics of the integral component \( \zeta \) given by (9). In this section, we discuss the robustness of the proposed control strategy with respect to delay mismatches. Assume that we dispose of an estimate \( \hat{h} \) of the actual delay \( h \) such that \( |\hat{h} - h| \leq \delta \) for some constant \( \delta > 0 \). In this case, we replace the integral component \( \zeta \), originally
\textbf{7 Simulation results}

We set \(a = 0.2, b = 2, c = 1, \) and \(\theta = \pi/3\). The first eigenvalues of \(A_{c,0}\) are approximately given by \(\lambda_0 \approx 2.301, \lambda_1 \approx -1.668 > -2\sqrt{5}c, \) and \(\lambda_2 \approx -9.567 < -2\sqrt{5}c\). Hence we set \(N = 1\). The feedback gain \(K \in \mathbb{R}^{1 \times 3}\) is computed such that \(AK = A_0 + B_0K\) is Hurwitz with simple eigenvalues \(\mu_1 = -4, \mu_2 = -5, \mu_3 = -6\), selected such that \(\mu_i < -3|c|\). The initial conditions of the plant and the integral component are set as \(\phi(t, \tau) = 10\cos(3\pi\tau)x(1 - x)^2\) and \(\zeta_i(\tau) = \cos(3\pi\tau)\zeta_0\) where \(\zeta_0 \in \mathbb{R}\) is selected such that (13) holds. The numerical scheme consists of the modal approximation of the reaction–diffusion equation using its first 40 modes.

The behavior of the closed-loop system composed of (2), (9), and (12) is illustrated for the time varying delay \(h(t) = 1 + \frac{1}{2}\sin(5\pi t + \pi/4)\) and the boundary perturbation \(p(t)\) as shown in Fig. 1(d). The results are depicted in Fig. 1. During the 10 first seconds we observe that the control law achieves the stabilization of the closed-loop system: both the state and the regulated output converge to zero in spite of a constant perturbation \(p(t) = 1\). Then, in order to evaluate the setpoint tracking capabilities of the system output (see Thm. 2), the reference
signal is set as $r(t) = 5$ for $t > 20$ s after an oscillatory transient. In conformity with the tracking estimate (24), we observe that the control strategy ensures the setpoint tracking of the reference signal $r(t)$ by the right Dirichlet trace $y(t, 1)$. Around $t = 30$ s, the boundary perturbation $p(t)$ increases to reach (approximately) the value of 25 and then decreases to converge to the value of 6. It is seen that the impact of this perturbation on both the state trajectory and the regulated output are successfully eliminated due to the presence of the integral component.

Finally, Fig. 2 illustrates the impact of delay mismatches on the closed-loop system performance. Here we set $\hat{h} = 1$ while considering increasing values for the actual delay $h \in \{1, 2, 3, 4\}$. The boundary perturbation is set as $p = 1$. As expected, we observe a smooth degradation of the performances of the resulting closed-loop system.

8 Conclusion

This paper has investigated the boundary PI regulation control of a reaction-diffusion equation in the presence of a state-delay in the reaction term. Our modal-based approach ensures the stability of the resulting closed-loop system as well as the setpoint regulation of the right Dirichlet trace. Future research directions may be concerned with extensions to the PI regulation control of either linear wave equations or semilinear heat equations in the presence of a state-delay.

Fig. 1. Time evolution of the closed-loop system

Fig. 2. Impact of delay mismatches

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