The Artificially Intelligent Deeply learned from Historical Battle data and Sun Tzu’s The Art of War Autonomous Forces on the March

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Abstract
Can autonomous forces be trained with artificially intelligence (AI) algorithms like deep learning from historical data and further be improved by the rules set by Sun Tzu’s Art of War for developing a state-of-the-art system of planning and analysis of military strategies? The application of Machine learning (ML) algorithms are general practice for extracting meaningful models from the data. ML could be unsupervised, supervised, or reinforcement learning. This article explores deep learning neural networks for extracting model parameters for generating autonomous forces. Although our main focus is on the world’s largest tank battle that is Battle of Kursk, we are also exploring other historical battle data of the world for validating the system. The battle of Kursk between the Soviets and Germans is known to be the biggest tank battle in history. This article explores the two-dimensional-simplex tank and artillery data from the Kursk database for analyzing a class of discrete time-homogeneous and heterogeneous Lanchester models. Under homogeneous form, the Soviet’s (or German’s) tank casualty is attributed to only the German’s (or Soviet’s) tank engagement. For heterogeneous form, the tank casualty is attributed to both tank and artillery engagements. For validating the models, different goodness-of-fit statistics are used for comparison.

KEYWORDS
Battle of Kursk, Homogeneous, Heterogeneous Lanchester Model, Goodness-of-fit, Kolmogorov-Smirnov, Sum-of-square residuals, Chi-square.

1. Introduction
During the past decades, many differential equation based models have gained significant importance for representing combat dynamics. These equations are widely used for modelling in warfare and representing the decrease in force levels over time commonly referred to as attrition process. Lanchester in 1914 proposed a set of differential equa-
tions, which quantify the importance of force concentration on the battlefield. Many authors have subsequently modified his original work to represent combat dynamics in modern warfare. In the recent time AI-enabled warfare involving 'killer Robots' has forced the designer to change the visualization of combat model development from traditional statistical methods to more practical adaptive and heuristic approaches. In reference we found that a fast large-scale theater model combining ground to ground battle attrition with air-to-ground strikes has been developed using such models. The features of the Lanchester equation that makes it suitable for analysis includes:

- **Applicability:** Lanchester models are widely used for historical battle analysis. Other than analysing human warfare Lanchester model have also been used for analysis of fights among social animals, market analysis.

- **Force Aggregation:** Lanchester models are found to be suitable for developing aggregated combat modelling using High Resolution Simulation Model. In reality, actual historical combat data is not easily available and common practice is to develop High Resolution simulation data with detailed design. Various literature have demonstrated that estimating attrition rates from high-resolution simulation and using Lanchester model for linking the various resolution of different simulation model.

- **Flexibility:** Lanchester models are flexible for both homogeneous as well as heterogeneous situations. Lanchester models are used for theoretically consistent force aggregation and dis-aggregation in two dimensions.

Regardless of credits of prior discovery, Lanchester’s equations are used worldwide for calculating attrition rates. We propose a general form of the heterogeneous Lanchester’s

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2
model as:

\[ \dot{X}_{i} = \sum_{i=1}^{F} a_i X_i^{p_i} Y_i^{q_i}, \forall i = 1, 2, \ldots F \]  

\[ \dot{Y}_{i} = \sum_{i=1}^{F} b_i X_i^{p_i} Y_i^{q_i}, \forall i = 1, 2, \ldots F \]  

where \( X_{i} \) denotes the strength of the \( i \)th type of Red forces at time \( t \) and \( Y_{i} \) denotes the strength of the \( i \)th type of Blue forces at time \( t \). \( \dot{X}_{i} \) and \( \dot{Y}_{i} \) are red and blue forces killed at time \( t \).

\( a_i \) represents attrition rate of \( i \)th type of Blue forces and \( b_i \) represents attrition rate of \( i \)th type of Red forces; 
\( \forall i = 1, 2, \ldots F \)

where \( F \) denotes the total number of forces. 
\( p_i \) the exponent parameter of the attacking force, 
\( q_i \) is the exponent parameter of the defending force.

Equations (1) and (2) involve unknown parameters \( a_i, b_i, p_i, \) and \( q_i \). A lot of work has been done to estimate these parameters using statistical estimation methods like Least Square, Maximum Likelihood, Bayes, Method of Moments etc. These estimates are most suitable for homogeneous situation. In heterogeneous situations these estimation procedures fail. In this article we propose AI based generalized reduced gradient (GRG) algorithm which is alternatively solved through Deep learning Neural Network. We are generally acquainted with two forms of these equations for homogeneous weapon engagement (when \( i \). Lanchester linear law in which \( p_i = q_i = 1 \) and force ratios remain equal if \( a_i X_i^{p_i} = b_i Y_i^{q_i} \). Lanchester’s linear law is interpreted as a model from a series of one-on-one duel between homogeneous forces and this law describes combat under ‘ancient conditions’. The equation is also considered a good model for area fire weapons, such as artillery. Lanchester square law in which , that is, force ratios remain equal if applied to modern warfare in which both sides are able to aim their fire or concentrate forces.

On integrating equation (1) and (2) we obtain the state equation:

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} a_i X_i \cdot X_j \cdot Y_i \cdot Y_j \]  

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} X_i^{0} \cdot X_j^{0} \cdot Y_i^{0} \cdot Y_j^{0} = 1 \]  

where \( X_0, X_0, Y_0, Y_0 \), represent the initial values of Blue and Red forces respectively. This equation says that the relationship between the power of the losses in any fixed
time period is equal to the inverse ratio of the attrition rate parameters. Equation (3) leads to the victory condition for Blue. Most forces have breakpoints at which they will cease fighting and either withdraw or surrender if:

\[
\sum_{i=1}^{X_N} \sum_{j=1}^{Y_N} a_i^j X_i Y_i j > 1
\]

Finally equations (4) may be solved in closed form as function of \( t \).

\[
X_p^i = \frac{1}{2} \left( (X_p^i Y_{q_i}^i - Y_{p_i}^i X_{q_i}^i) e^{\pm \sqrt{a_i b_i}} \right) + \frac{1}{2} \left( (X_p^i Y_{q_i}^i - Y_{p_i}^i X_{q_i}^i) e^{\mp \sqrt{a_i b_i}} \right)
\]

\[
Y_q^i = \frac{1}{2} \left( (Y_q^i X_{p_i}^i - X_{p_i}^i Y_{q_i}^i) e^{\pm \sqrt{a_i b_i}} \right) + \frac{1}{2} \left( (Y_q^i X_{p_i}^i - X_{p_i}^i Y_{q_i}^i) e^{\mp \sqrt{a_i b_i}} \right)
\]

There is another form of mixed combat model where attacker uses area fire \( (p_i = q_i = 1 \text{ i.e. linear form}) \) against a defender using aimed fire \( (p_i = 1, q_i = 0 \text{ i.e. square form}) \). This mixed form of combat model is known as ambush model proposed by Deitchman. Helmbold in 1965 studied the Iwo-Jima campaign between USA and Japan using one-sided homogeneous Lanchester model. Bracken in 1995 studied Ardennes campaign between Germany and USA. Clemens in 1997 and Lucas and Turkes in 2003 studied the Kursk campaign between Soviet and Germany. Willard has tested the capability of the Lanchester model for analyzing the historical battle data for the battles fought between the years of 1618-1905. Bracken (1995) used the database of the Ardennes campaign of World War II formulating four different models which are the variations of the basic Lanchester equations. The models developed in his study were homogeneous in nature in terms of tank, APC, artillery etc. He concluded that Lanchester linear model best fits the Ardennes campaign data in terms of minimizing the sum of squared residuals (SSR). This work validates the applicability of the Lanchester model for the historical Battle data. Fricker revised the Bracken’s models of the Ardennes campaign.
of World War II. He extended Bracken’s model by applying linear regression on the logarithmic transformed Lanchester equations and included the data from the entire campaign and air sortie data as well. Lastly, he concluded that neither of the Lanchester linear or square laws fit the data. A new form of Lanchester equations emerges with a physical interpretation.

Clemens fits the homogeneous version of Lanchester equations to the Battle of Kursk. He used two different techniques (i) Linear regression on logarithmic transformed equations (ii) a non-linear fit to the original equations using a numerical Newton-Raphson algorithm.

Hartley and Helmbold examined the validity of Lanchester’s square law using the one-sided data from the Inchon-Seoul Campaign. They have not found good fit using constant coefficient square law but better fit was found when the data was divided into a set of three separate battles. They concluded the Lanchester’s square law is not a proven attrition algorithm for warfare although they also commented that one-sided data is not sufficient to verify or validate Lanchester square law or any other attrition law. They have used linear regression, Akaikie Info criterion and Bozdogan’s consistency AIC(CAIC). Based on the regression analysis they have found the models with three regression parameters with intercept and without intercept was the best model with higher degree Coefficients of determination.

NR Johnson and Mackey analysed the Battle of Britain using the Lanchester model. This was a battle of an air combat between German and Britain.

Wiper, Pettit and Young applied Bayesian computational techniques to fit the Ardennes Campaign data. They studied stochastic form of Lanchester model and enquired whether there is role of any attacking and defending army on the number of casualties of the battle. They compared their results with the results of the Bracken and Fricker and results were found to be different. They concluded that logarithmic and linear-logarithmic forms fits more appropriately as compared to the linear form found by Bracken. They also concluded that the Bayesian approach is more appropriate to make inferences for battles in progress as it uses the prior information from experts or previous battles. They have applied the Gibbs sampling approach along with Monte Carlo simulation for deriving the distribution patterns of the parameters involved.

Turkes extended the previous work for the validation of Lanchester models with real data. He stated that historical data for validation of attrition model is poor. Mostly, the data contained starting sizes and casualties only for one side. He applied various derivatives of Lanchester equations for fitting model on the Kursk Database. The results found in his study were different with earlier studies on the Ardennes campaign. He found that wide variety of models fit the data as well. He has shown none of the basic Lanchester models fit the data, bringing into question their use in combat modelling. Lucas and Turkes used a new approach to find the optimal parameters for fitting

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Lanchester models on the data of Battles of Kursk and Ardennes. They have gained an understanding of how well various parameter combinations explain the battles. They have found that variety of models fits the data. They concluded that none of the basic laws (i.e. square, linear and logarithmic) fit the data correctly and raises the question of utility of basic Lanchester model for combat modelling. They also suggested finding new ways to model the aggregated attrition process to provide a good-fitting Lanchester model.

US Army’s Colonel Trevor N. Dupuy discussed about different weapon system in his book about the Evolution of Weapons and Warfare (1990) which had evolved from 2000 BCE onwards till the Cold War and their tactical impact on combat. Despite its Western bias, the book is good for detailed description of the military hardware which modern Europe produced. Eminent author K Roy described a global history of warfare from slings to drones also includes discussion on insurgency, civil war, sieges, skirmishes, ambushes and raids.

The main aim of this paper is to fit Lanchester Model based on Kursk data. For that we require to estimate attrition rates and exponent parameters. There are several approaches to estimate the parameters. We shall consider two common and rational procedures namely, Least Square Estimation (LSE) and Maximum Likelihood Estimation (MLE). These two estimation procedures will be applied through AI techniques like Deep Learning. The authors have demonstrated Deep Reinforcement Learning (DRL) AI techniques for solving complex problem, our autonomous forces are configured with this technique. DRL is an iterative optimization process. Our autonomous force higher commanders are established on the logic of Deep Deterministic Policy Gradient Agent (DDPGAgent). These DDPGAgents are represented by actor-critic relationship where Actor is \( \pi(S, \theta) \), for observations \( S \) with parameter \( \theta \) and critic \( Q(\phi, S, A) \) with parameter, observation and action are \( \phi, S, A \) respectively. The parameter \( \theta \) is periodically updated for optimal value of long term reward function that produces the target Actor \( \phi(S, \theta_t) \) and target critic \( Q(\phi_t, S, A) \). In the present article the observations are historical battle data, actions are mathematical functions formulated from 'Art of the War', the parameters \( \theta \) estimated are attrition rate coefficients, exponents parameters which are being estimated through Deep Learning Neural Network, the critic parameters \( \phi_t \) are estimated from the same data set with strategic inputs and rules from the 'Art of the War'. The long term reward function is designed from the GOF measures as defined in subsequent sections.

In the next section we have discussed in detail the mathematical formulations of homogeneous and heterogeneous situations. We have seen in Bracken and Fricker.

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Clemens, Turkes, and Lucas that LSE method have been applied for evaluating the parameters for fitting the homogeneous Lanchester equations to the historical battle data. The MLE method has not been explored particularly for fitting the historical battle data till date. Also only one measure i.e. Sum-of-squared-residuals (SSR) has been explored for measuring the Goodness-of-fit (GOF). The main objective of this study is to assess the performance of the MLE approach for fitting homogeneous as well as heterogeneous Lanchester equations to the Battle of Kursk. Various measures of GOF viz. Kolmogorov-Smirnov, Chi-square and $R^2$ have been computed for comparing the fits and to test how well the model fits the observed data. Applying the various GOF measures considering the artillery strength and casualties of Soviet and German sides from the Kursk battle data of World War-II validates the performance of MLE technique. Section 2 presents in brief the overview of the battle of Kursk. Section 3 describes the mathematical formulation of likelihood estimation in case of both homogeneous as well as heterogeneous situations. Section 4 describes the Tank and Artillery data of Battle of Kursk and discusses the methodology for implementing the proposed as well as other approaches. Also, this section contains a performance appraisal of the MLE using various GOF measures. Section 5 analyses the results after observing various tables and figures and discusses how well the MLE fits the data. Section 6 summarizes the important aspects of the paper.

2. HISTORY OF THE BATTLE OF KURSK

After suffering a terrible defeat at Stalingrad in the winter of 1943, the Germans desperately wanted to regain the initiative. In the spring of 1943, the Eastern front was conquered by a salient, 200 km wide and 150 km deep, centred on the city of Kursk. The Germans planned in a classic pincer operation named Operation Citadel, to eliminate the salient and destroy the Soviet forces in it. On 2 July 1943, Hitler declared, "This attack is of decisive importance and it must succeed, and it must do so rapidly and convincingly. It must secure for us the initiative.... The victory of Kursk must be a blazing torch to the world." The Germans started the Battle of Kursk on July 4, 1943 on the southern half of the Kursk salient, but this was merely to gain better artillery observation points. The battle began in earnest early in the morning of July 5, when Soviets conducted an artillery barrage before the Wehrmacht attacked. The Germans countered with their own planned barrages shortly thereafter and seized the initiative on both fronts. Soviet General Rokossovsky redeployed his reserves on the night of July 5 in order to attack the following day. The divisions of the 17th and 18th Guards Rifle Corps, with support from the 3rd, 9th, 16th, and 19th Tank Corps, were beginning offensive operations at 5:30 a.m. on July 6 in support of 13th Army. July 6 were considered as worst single day of...
CITADEL for German tank losses. On 7th July the German attack with armour forces in the northern and northeast side and captured the village Lutschki and continued advancing towards the village of Tetsreino against the very strong Soviet infantry and armor battle. By evening July 7 the Germans were able to capture the village Tetrevino. On 8th July the German attacked with armour forces and barely captured the Teploye village. The Soviet counterattacked and recaptured the lost Teploye village. The Soviet defended very well on that day although they lost 315 tanks on that day in comparison to 108 tank loss of the Germans. The German forces wanted to develop a sharp wedge towards Kursk via Oboyan village. The German forces attacked with more than 500 tanks. The Soviet forces defended the Oboyan with sophisticated artillery guns. Despite the strong defense the Germans were able to foothold over the Pena River. During the period of July 7-9 both the sides had suffered largest number of tank losses. Due to this reason the Germans planned to attack from less resistance Prokorovka side. After changing the direction of the attack the German reached and seized the village of Novoselovka. The Soviets understood the German’s plan and they started using their reserve units. But despite of that the Germans were able to break through the Soviet defenses by evening of the July 10. The German intention at this point was to cross the river Psel to the extent of as many as troops and vehicles are possible. The Germans were able to seize a bridge. The engagement was at this point between artillery and tank. Both the forces were preparing for the battle of Prokhorovka. The Battle of Prokhorovka was the decisive phase of the Battle of Kursk. The Soviets started with artillery defense and later it turns out to be totally tank against tank meeting engagement. The German tanks had to face minefield as well as well defended Soviet anti-tank weapons. The resulting titanic battle was a tactical draw. The Germans lost 98 tanks against 414 Soviet tank losses. Hitler called off the battle. The losses from the fighting over July 12 and 13 were extensive on both sides. In KUTUZOV there was a heavy armour engagement between the two forces. The German forces destroyed 117 Soviet tanks. The Soviets also damaged 57 German tanks. The battle on Prokorovka still continued on 14th July. The German planned an offensive operation named operation Roland. It started on July 14. The aim was to destroy the Soviet armor reservoirs. The German armor units fought with the artillery forces that were defending the armor reservoirs of the Soviet in the southern part of the Prokorovka. Several tactical positions were captured by the German forces. On this day the Germans were capable of performing minor offensive operations and they were launching attacks to form the Gostishchevo-Liski pocket. Hitler redirected it back to Isyum on July 16. According to the various war analysts it is being considered that because of Hitler’s decision, von Manstein lost the availability of a powerful mobile formation that could have been very useful in the battle. During this time most of the engagements were between German infantry and Soviet tanks. Most of the damage was suffered by the Soviet because they were not equipped with modern antitank weapons that can deal effectively with the Soviet armour. The Soviets launched their counteroffensive along the Mius River on July 17. The Southwestern Front, commanded by Colonel General Tolbukhin, attacked the heavily fortified Mius River line defenses. The Soviet counter attack was known as operation RUMANTSYEV.

3. ESTIMATION

Let $S$ denote the time between two consecutive casualties for a side, its probability density function is denoted by $f_S(S)$. Let $(m_k^i, n_k^i)$ represents the $i^{th}$ type force strengths (e.g. tank, artillery etc.) of blue and red forces of a battle for the $k^{th}$ time instance.
respectively. Let us also denote (for $k = 1, 2 \ldots K$) the time (a random variable) at which $K^{th}$ casualty occurs on $T_k$ (with realization $t_k$). Let the Blue and Red casualties $X_{t_{i}}$ and $Y_{t_{i}}$ in a combat are r.v. whose densities are defined by $f_{sx}(s|a_{i},p_{i},q_{i})$ and $f_{sy}(s|b_{i},p_{i},q_{i})$ respectively where forms of the densities are known except the unknown parameters ($a_{i}, b_{i}, p_{i}, q_{i}$). It is assumed that the of a random sample ($x_{t_{i}^{'},y_{t_{i}^{'}}}^{'}$) from $f_{S}(S)$ can be observed. On the basis of the observed sample values ($x_{t_{i}^{'},y_{t_{i}^{'}}}^{'}$) it is desired to estimate the value of the unknown parameters ($a_{i}, b_{i}, p_{i}, q_{i}$). We further assume that the times between casualties are exponentially distributed, then the pdf of casualty for the $K$ sides associated to the equation (1) and (2) can be represented as in the equations (7) and (8):

$$f_{S_{x_{t_{i}^{'}}}^{'}}(s) = \prod_{i=1}^{F}(a_{i}x_{t_{i}^{'}}^{p_{i}}y_{t_{i}^{'}}^{q_{i}}).exp(-\sum_{i=1}^{F}a_{i}x_{t_{i}^{'}}^{p_{i}}y_{t_{i}^{'}}^{q_{i}})s$$

$$-\infty < a_{i}, p_{i}, q_{i} < \infty, \forall t_{i}^{'} = 1, 2, \ldots, F$$ \hspace{1cm} (7)

$$f_{S_{y_{t_{i}^{'}}}^{'}}(s) = \prod_{i=1}^{F}(b_{i}x_{t_{i}^{'}}^{p_{i}}y_{t_{i}^{'}}^{q_{i}}).exp(-\sum_{i=1}^{F}b_{i}x_{t_{i}^{'}}^{p_{i}}y_{t_{i}^{'}}^{q_{i}})s$$

$$-\infty < b_{i}, p_{i}, q_{i} < \infty, \forall t_{i}^{'} = 1, 2, \ldots, F$$ \hspace{1cm} (8)

The likelihood equation of $n$ pairs of random variable ($x_{t_{i}^{'}, y_{t_{i}^{'}}}^{'}$) is defined as the joint density of the $n$ pairs of random variables, which is considered to be a function of ($a_{i}, b_{i}, p_{i}, q_{i}$). In particular, if ($x_{t_{i}^{'}, y_{t_{i}^{'}}}^{'}$) are independently and identically distributed random sample from the density $f_{S}(S)$, then the likelihood function is:

$$f(x_{t_{i}^{'}}|a_{i}, p_{i}, q_{i}).f(y_{t_{i}^{'}}|b_{i}, p_{i}, q_{i}) \ldots f(x_{t_{i}^{'}}|a_{i}, p_{i}, q_{i}).f(y_{t_{i}^{'}}|b_{i}, p_{i}, q_{i})$$ \hspace{1cm} (9)

Then, the joint pdf will be:

$$L(a_{i}, b_{i}, p_{i}, q_{i}) = \prod_{i=1}^{F} \prod_{i=1}^{F}(a_{i}x_{t_{i}^{'}}^{p_{i}}y_{t_{i}^{'}}^{q_{i}})^{x_{t_{i}^{'}}}(b_{i}y_{t_{i}^{'}}^{q_{i}}.x_{t_{i}^{'}}^{p_{i}})^{y_{t_{i}^{'}}}.exp(-\sum_{i=1}^{F}a_{i}x_{t_{i}^{'}}^{p_{i}}y_{t_{i}^{'}}^{q_{i}}) + b_{i}y_{t_{i}^{'}}^{q_{i}}.x_{t_{i}^{'}}^{p_{i}})s)$$ \hspace{1cm} (10)

To construct the likelihood function from the available data set, it is generally observed that casualty figures are generally available at daily interval. Let $L(a_{i}, b_{i}, p_{i}, q_{i})$ be the likelihood function for the random variables ($x_{t_{i}^{'}, y_{t_{i}^{'}}}^{'}$). If $\hat{a}_{i}, \hat{b}_{i}, \hat{p}_{i}, \hat{q}_{i}$ are the values of $a_{i}, b_{i}, p_{i}, q_{i}$ which maximizes $L(a_{i}, b_{i}, p_{i}, q_{i})$ , then $\hat{a}_{i}, \hat{b}_{i}, \hat{p}_{i}, \hat{q}_{i}$ are the maximum-likelihood estimates of $a_{i}, b_{i}, p_{i}, q_{i}$. Now, instead of maximizing the likelihood function we will maximize its logarithmic form since both the maximum values occur at the same
point and logarithmic form is easily imputable. Thus, on taking log of equation (8), we have

$$\ln L = \left( \prod_{i=1}^{F} \prod_{t=1}^{N} \dot{x}_i^t \ln(a_i \cdot x_i^{p_i} y_i^{q_i}) \right) +$$

$$\dot{y}_i^t \ln(b_i \cdot x_i^{q_i} y_i^{p_i}) - (a_i \cdot x_i^{p_i} y_i^{q_i} + b_i \cdot y_i^{p_i} x_i^{q_i})s$$  \hspace{1cm} (11)

Differentiating the Log-likelihood function (9) partially with respect to $a_i$ and $b_i$ and equating it to zero, we have:

$$\frac{d\ln L}{da_i} = \sum_{t=1}^{N} \frac{\dot{x}_i^t}{a_i} - \sum_{i=1}^{F} \sum_{t=1}^{N} \frac{\dot{y}_i^t}{y_i^{q_i}} x_i^{q_i} s = 0$$  \hspace{1cm} (12)

and

$$\frac{d\ln L}{db_i} = \sum_{t=1}^{N} \frac{\dot{y}_i^t}{b_i} - \sum_{i=1}^{F} \sum_{t=1}^{N} \frac{\dot{x}_i^t}{x_i^{p_i}} y_i^{p_i} s = 0$$  \hspace{1cm} (13)

This gives

$$\sum_{t=1}^{N} \dot{x}_i^t a_i = \sum_{i=1}^{F} \sum_{t=1}^{N} x_i^{p_i} y_i^{q_i} s$$  \hspace{1cm} (14)

and

$$\sum_{t=1}^{N} \dot{y}_i^t b_i = \sum_{i=1}^{F} \sum_{t=1}^{N} y_i^{p_i} x_i^{q_i} s$$  \hspace{1cm} (15)

Thus, the maximum likelihood estimates are:

$$\hat{a}_i = \frac{\sum_{t=1}^{N} \dot{x}_i^t}{\sum_{i=1}^{F} \sum_{t=1}^{N} x_i^{p_i} y_i^{q_i} s}$$  \hspace{1cm} (16)

$$\hat{b}_i = \frac{\sum_{t=1}^{N} \dot{y}_i^t}{\sum_{i=1}^{F} \sum_{t=1}^{N} y_i^{p_i} x_i^{q_i} s}$$  \hspace{1cm} (17)

4. OVERVIEW OF KURSK DATABASE

The Kursk Data Base (KDB) is developed by Dupuy Institute (DPI) and is reformatted into a computerized database in 1998. KDB is documented in the KOSAVE (Kursk Operation Simulation and Validation Exercise)\textsuperscript{39}. The KDB contains daily on hand and

\textsuperscript{39} W.J. Bauman and Army concepts analysis agency bethesda md, Kursk Operation Simulation and Validation Exercise - Phase II (KOSAVE II), AD-a360 311 (Army concepts analysis agency bethesda md, 1998), https://books.google.co.in/books?id=pVSqnQEACAAJ
Table 1. Daily Soviet’s and Germans on hand and losses data for tanks and artillery from Kursk Battle.

| Days | Soviet's Tank On Hand | Soviet's Tank Losses | German Tank On Hand | German Tank Losses | Soviet's Arty On Hand | Soviet's Arty Losses | German Arty On Hand | German Arty Losses |
|------|-----------------------|----------------------|---------------------|--------------------|-----------------------|----------------------|---------------------|---------------------|
| 1    | 2396                  | 105                  | 198                 | 705                | 24                    | 1166                 | 24                  |
| 2    | 2967                  | 117                  | 749                 | 676                | 30                    | 1161                 | 5                   |
| 3    | 2084                  | 259                  | 673                 | 121                | 661                   | 1154                 | 7                   |
| 4    | 17546                 | 315                  | 596                 | 108                | 648                   | 1213                 | 13                  |
| 5    | 1495                  | 289                  | 490                 | 139                | 640                   | 1210                 | 6                   |
| 6    | 1406                  | 377                  | 545                 | 36                 | 629                   | 1199                 | 12                  |
| 7    | 1351                  | 135                  | 563                 | 63                 | 628                   | 1206                 | 15                  |
| 8    | 977                   | 414                  | 600                 | 98                 | 613                   | 1194                 | 12                  |
| 9    | 978                   | 117                  | 495                 | 57                 | 606                   | 1187                 | 7                   |
| 10   | 907                   | 118                  | 480                 | 46                 | 603                   | 1184                 | 5                   |
| 11   | 883                   | 96                   | 426                 | 79                 | 601                   | 1185                 | 4                   |
| 12   | 985                   | 27                   | 495                 | 23                 | 600                   | 1179                 | 4                   |
| 13   | 978                   | 42                   | 557                 | 7                  | 602                   | 1182                 | 2                   |
| 14   | 948                   | 105                  | 986                 | 198                | 705                   | 1166                 | 24                  |

Figure 1. Comparison of daily number of tank losses of the Battle of Kursk of WW II.

losses for the four categories viz. manpower, tanks, APC and artillery for the Soviets and Germans for each of the 15 days of battle. Evidences of multiple force interaction in Kursk Battle shows multiple forces were fighting in the war. Therefore, developing heterogeneous model on this data is justified. In the present study, we have considered only the tank and artillery data for developing heterogeneous Lanchester model. Table 1 shows the tank and artillery weapons on hand and losses during the 14 days of battle. Figure 1 shows a comparison between the Soviet and German’s tank losses during the 14 days of battle.

This paper fits the generalized form of Heterogeneous Lanchester equations to the Battle of Kursk data using the method of Maximum Likelihood estimation and compares the performance of MLE with the techniques studied earlier such as the Sum of squared residuals (SSR), Linear regression and Newton-Raphson iteration. Different authors applied different methodologies for fitting Lanchester equations to the different battle data. The methodologies of Bracken, Fricker, and Clemen are applied to the Tank data of Battle of Kursk and results are shown in Table 3.
5. GOODNESS-OF-FIT STATISTICS

First, we applied the technique of Least Square for estimating the parameters of the heterogeneous Lanchester model. The GRG algorithm is applied for maximizing the MLE and for minimizing the LSE. For implementing the Least Square approach, the Sum of Squared Residuals (SSR) is minimized. The expression of SSR for the equation (1) and (2) is given as:

\[
SSR = \sum_{t=1}^{14} (\dot{x}_{t} - \sum_{i=1}^{2} a_{i} x_{t}^{p_{i}} y_{t}^{q_{i}})^{2} + \sum_{t=1}^{14} (\dot{y}_{t} - \sum_{i=1}^{2} b_{i} y_{t}^{p_{i}} x_{t}^{q_{i}})^{2}.
\]

(18)

For implementing this expression from table 1 we have taken zero as initial values for all the unknown parameters. Then we start running the GRG algorithm iterative. The GRG algorithm is available with the Microsoft Office Excel (2007) Solver and MATLAB. The GRG solver uses iterative numerical method. The derivatives (and Gradients) play a crucial role in GRG. We have run the program for 1000 iterations for getting the stabilized values of these parameters. Once, we have the parameters we compute the estimated casualties. With the difference between the estimated and observed casualties, we computed the Sum of Squared Residuals. Similarly, we applied the GRG algorithm for optimizing the objective function as given in equation (9). We check the graphs of estimated and observed casualties for both the LS and MLE based approaches and found that if we divide the data set into several subsets then we can improve the fit. As we increase the number of divisions, the fit turns out to be better. The estimated casualty converges to the observed casualty. We have considered tank and artillery data for mixing the forces therefor \(a_{1}\) (or \(b_{1}\)) represents effectiveness of Soviet (or Germans) tanks against Germans (or Soviets) tanks and \(a_{2}\) (or \(b_{2}\)) represents effectiveness of Soviets (or Germans) Artillery against Germans (or Soviets) tanks. The variation of attrition rates throughout battle tells us how the different player in the battle performs. Whether they are acting defensively or offensively.

The basic idea of using GRG algorithm is to quickly find optimal parameters that maximize the log-likelihood. The objective is to find the parameters that maximize the log-likelihood or in other words provide the best fit. Given the values in Table 1, we investigate what values of the parameters best fit the data. Although we derived the estimates for \(a\) and \(b\) using the MLE approach in equations (8) and (9), they are not applied directly. Log Likelihood is calculated using the equation (7) considering 0.5 as the initial value of the parameters. Then, we optimized the entire duration of the battle of the likelihood function using the GRG algorithm. The model obtained after estimation of parameters is:

\[
\dot{x}_{1} = (1.46)x_{1}^{1.129}y_{1}^{0.404} + (.906)x_{1}^{1.38}y_{2}^{1.36}
\]
\[ \hat{y}_1 = (0.704) y_1^{1.29} x_1^{1.404} + (0.953) y_1^{1.138} x_2^{1.136} \]  

As the data for the first day is extremely low, we drop it since it will pose a problem in the computation of the likelihood and SSR function. Also, the extremely low casualty levels on the first day represent large outliers; thus, including the data of the first day affects the outcome to a great extent. Thus, the first day was dropped in fitting the data to the models. This approach is also justified by the historical account of the battle of Kursk, because the fight did not begin until July 5, the second day of the battle. Thus, dropping the data for the first day and dividing the remaining 14 days data into five phases, the total number of optimal parameters with each day as single phase is 102. This is a much better fit than any of the homogeneous model because both the residual as well as the likelihood are optimized. Log-likelihood is calculated using equation (7) and is maximized separately for each of the five phases. Let \( t \) denote the days, then the division is made as \((t_2 - t_3), (t_4 - t_6), (t_7 - t_8), (t_9 - t_11), \) and \((t_{12} - t_{15}).\) Fitting the model over multiple phases results in a better overall fit because there are additional parameters to explain the variation in casualties. The model has been improved from partitioning the battle into 14 phases. Each day of the battle is treated as mini-battle. For the purpose of comparing models, \( R^2 \) value is calculated along with the Sum of squared residuals (SSR). \( R^2 \) value is calculated as:

\[
R^2 = 1 - \frac{SSR}{SST} = 1 - \frac{\sum_{t=1}^{15} (\hat{x}_i - \hat{\hat{x}}_i)^2 + \sum_{t=1}^{15} (\hat{y}_i - \hat{\hat{y}}_i)^2}{\sum_{t=1}^{15} (x_i - \hat{x}_i)^2 + \sum_{t=1}^{15} (y_i - \hat{y}_i)^2}
\]

A larger \( R^2 \) value indicates better fit. Also, Goodness-of-fit measures namely; Kolmogorov-Smirnov statistic\(^{43}\) and Chi-square (\( \chi^2 \))\(^{44}\) have been calculated for the accuracy assessment of the MLE to that of the conventional approaches. Kolmogorov-Smirnov statistic is a measure of Goodness-of-fit, that is, the statistic tells us how well the model fits the observed data. The Kolmogorov-Smirnov (KS) statistic is based on the largest vertical difference between the theoretical and empirical (data) increasing distribution function.

\[
KS = \max_{1 \leq t \leq 30} |F(e_{t*}) - \frac{t^*}{30} - \frac{1}{30} - F(\hat{e}_{t*})|
\]

where \( F(\hat{e}_{t*}) \) is the cumulative distribution function of the estimated error between the observed losses and the estimated losses for both sides. Chi-Square (\( \chi^2 \)) is another measure of Goodness-of-fit. Chi-Square is given as:

\[
\chi^2 = \sum_{t=1}^{14} \frac{(\hat{x}_i - \hat{x}_{i*})^2}{\hat{x}_{i*}} + \sum_{t=1}^{14} \frac{(\hat{y}_i - \hat{y}_{i*})^2}{\hat{y}_{i*}}
\]

where \( \hat{x}_{i*} \) and \( \hat{x}_{i*} \) are the observed and expected casualties respectively.

This article also explores other historical battles and estimated goodness-of-fits statistics to analyse the military strategies for modeling the Lethal Behavior\(^{45}\) of autonomous forces. These models are given in table 6. The directional field plots or D-field plots are a visual representation of the system of differential equations. These are shown in

43. D’Agostino, *Goodness-of-Fit-Techniques*
44. D’Agostino
45. Arkin, *Governing Lethal Behavior in Autonomous Robots*
These graphs give an insight about the data and show the divergent and convergent properties of the historical battles. Generally, we see that for developing combat models of aggregated forces, different elements of the forces are aggregated together in terms of Combat Potential or Lethal Behavior and then differential equations are used for representing their decay over time frame. These are pseudo-aggregated models, for the purpose of modeling heterogeneous forces, are blended together and total aggregated losses are estimated and then again dis-aggregated for allocating these losses in different individual forces. For doing so, a significant amount of information is being lost. In the present approach, these information losses are being managed by ignoring the aggregation and dis-aggregation. Here, we directly estimate the relative attrition coefficients of one element against another one. The limitations of the current approach are that it does not consider other influencing factors of combat. If we follow the Dupuy’s QJMA concept along with the Lethal Behavior of autonomous forces for quantification and to define the Combat Potential and Lethal Behavior (CPLB) of an autonomous force $i$ with $n_i$ autonomous fighting elements as:

$$CP_i = n_i, OL_i, m_i, v_i, l_i, t_i, e_i, mo_i, po_i, pW_{att/Def}$$ (23)

where,

$n_i$: number of autonomous elements in the $i^{th}$ force

$OL_i$: Operational Lethality Index of the $i^{th}$ element

$m_i$: mobility factor

$v_i$: vulnerability of the force

$l_i$: leadership

$t_i$: training

$e_i$: ethical

$mo_i$: morale

$po_i$: political

$pW_{att/Def}$: weather effect on attacker or defender.

The $OLI$ of an autonomous mobile weapon is defined as

$$OLI_{Mobile} = ((fmr) + p)RFAmC$$

$$OLI_{Non-Mobile} = (r^*tRAR_l m_a g_m b_m l_t A_m)/d$$

where

46. Arkin, Governing Lethal Behavior in Autonomous Robots

47. Arkin, Governing Lethal Behavior in Autonomous Robots; T. N. Dupuy, “Application of the Quantified Judgment Method of Analysis of Historical Combat to Current Force Assessments,” in Military Strategy and Tactics: Computer Modeling of Land War Problems, ed. Reiner K. Huber, Lynn F. Jones, and Egil Reine (Boston, MA: Springer US, 1975), 133–151. ISBN: 978-1-4757-0958-2, https://doi.org/10.1007/978-1-4757-0958-2_18
$f =$ firepower  
$m =$ mobility  
$r =$ radius  
$p =$ punishment  
$R =$ Rapidity  
$F =$ Fire control effect  
$A_s =$ Ammunition Supply  
$A_m =$ Aircraft Mount  
$C =$ Ceiling  
$r^* =$ Rate of fire  
$t =$ Number of targets  
$i =$ incapacitating  
$R =$ range  
$A =$ Accuracy  
$R_l =$ Reliability  
$m_s =$ self-propelled mobility  
$g =$ guidance  
$c_m =$ charges(multiple)  
$b_m =$ barrel(multiple)  
$h_t =$ Wheel Track  
$d =$ dispersion

let us assume an autonomous Armour Troop with 3 Tanks of Type 1, [hence Troop OLI is =Troop OLI(Tank$_1$)=3.OLI$_{Tank_1}$ = 3.$(f_1 m_1 r_1) + p_1) R_1 F_1 A_s, A_m, C_1$ is fighting with another autonomous troop with Tank$_2$.  
Troop OLI$_{Tank_2}$=3.$(f_2 m_2 r_2) + p_2) R_2 F_2 A_s, A_m, C_2$

Combat Potentials(CP) of these sides are:

$$\begin{align*}
CP_1 &= n_1.OLI_1.m_1.v_1.l_1.t_1, m_0, p_{o1}, W_{att} \\
CP_2 &= n_2.OLI_2.m_2.v_2.l_2.t_2, m_0, p_{o2}, W_{def}
\end{align*}$$

Differentiating the above equations over time $t$ we get,

$$\begin{align*}
\frac{dCP_1}{dt} &= n_1.OLI_1.m_1.v_1.l_1.t_1, m_0, p_{o1}, \frac{dW_{att}}{dt} \\
\frac{dCP_2}{dt} &= n_2.OLI_2.m_2.v_2.l_2.t_2, m_0, p_{o2}, \frac{dW_{def}}{dt}
\end{align*}$$
dividing above equations we have
\[
\frac{dCP_1}{dCP_2} = c \frac{n_1}{n_2} \frac{dW_{att}}{dW_{def}} \quad (28)
\]

where \( c = \frac{OLI_1.m_1.v_1.l_1.t_1.mo_1.p_1}{OLI_2.m_2.v_2.l_2.t_2.mo_2.p_2} \), if defender is at advantageous positions then \( \frac{dCP_1}{dCP_2} < 1 \),

\[\Rightarrow c \frac{n_1}{n_2} \frac{dW_{att}}{dW_{def}} < 1 \quad (29)\]

If weather effect is same for both the attacker and defender then \( \frac{dW_{att}}{dW_{def}} = 1 \) it implies that
\[
\frac{m_2.v_2.l_2.t_2.mo_2.p_2}{m_1.v_1.l_1.t_1.mo_1.p_1} > \frac{OLI_1.n_1}{OLI_2.n_2} \quad (30)
\]

5.1. **Theorem-1**

Consider a combat between autonomous \( FORCE_1 (n_1|OLI_1, m_1, v_1, l_1, t_1, mo_1, p_1) \) and \( FORCE_2 (n_2|OLI_2, m_2, v_2, l_2, t_2, mo_2, p_2) \) where \( m_i, v_i, l_i, t_i, mo_i, p_i \in \mathbb{R} \)

\[
CP_1 = -a.CP_2 \quad \text{and} \quad CP_2 = -b.CP_1 \quad \text{(if direct fire mode)} \quad (31)
\]

\[
CP_2 = -a.CP_1 \quad \text{and} \quad CP_2 = -b.CP_2 \quad \text{(if indirect fire mode)} \quad (32)
\]

where \( a, b \in \mathbb{R} \) and for \( t = 0, CP_1 = CP_{10} \) and \( CP_2 = CP_{20} \) at \( t = 0 \) are real function, then:

(i) if direct and indirect fire both are in cycle then adding above two equations we have
\[
CP_1 = -\frac{a}{2}(CP_1 + CP_2) \quad (33)
\]

and
\[
CP_2 = -\frac{b}{2}(CP_1 + CP_2) \quad (34)
\]

If effectiveness of force is different in direct fire mode and indirect fire mode then
\[
CP_1 = -a_1.CP_1 - a_2.CP_2 \quad (35)
\]

and
\[
CP_2 = -b_1.CP_1 - b_2.CP_2 \quad (36)
\]
(ii) Numerical strength increases to attacker whereas numerical strength reduces to defender thus if side 1 is attacker and side 2 is defender a tactical parameter is being introduced i.e.

$$\dot{C}P_{1_{\text{Attacker}}} = -a(d).CP_2$$  \hspace{1cm} (37)$$

and

$$\dot{C}P_{2_{\text{Defender}}} = -b(1/d).CP_1$$  \hspace{1cm} (38)$$

$$\frac{CP_1}{CP_2} = \frac{-adCP_2}{-bCP_2/d} = \frac{a}{b} \cdot \frac{CP_2}{CP_1}.(d^2).$$

Therefore force ratio gets squared time increased. The ethical reasoning of the autonomous forces are divided into two processes as we have seen in the Ethical Governor in MissionLab. These are Evidential Reasoning and Constraint Application. For constrain application we follow the GOF mathematics as described above and for Evidential Reasoning we adopted the Sun Tzu’s Art of War. Now let us consider Side 1 is attacker therefore,

$$CP_1 = n_1.(d).OLI_1,m_1.v_1.t_1,m_0.p_0_1.W_{att}$$

$$CP_2 = n_2.(1/d).OLI_2,m_2.v_2.t_2,m_0.p_0_2.W_{def}$$  \hspace{1cm} (39)$$

If the defender does not have the exact location of battle he will spread across a large area which will reduce the force concentration. Where as attacker is going to hit a particular point with much higher force concentration, therefore combat density at hit point is:

$$\int_0^t \int_0^{l_1} CP_1.dt.dl_1 \text{ and } \int_0^t \int_0^{l_2} CP_2.dt.dl_2$$  \hspace{1cm} (40)$$

where \(l_2 \geq l_1\).

Let us consider the defender is deployed and divided over 4 sub-units in 4 different positions, front, rear, right and left. Hence, \(CP_2 = CP_2^{\text{Front}} + CP_2^{\text{Rear}} + CP_2^{\text{Right}} + CP_2^{\text{Left}} = \cup_{i=1}^4 CP_2^i\). When time(\(t\)) and location(\(l\)) are unknown

$$\int_0^t \int_0^{l_1} CP_1.dt.dl = 0 \Rightarrow \int_0^t \int_0^{l_2} \cup_{i=0}^4 CP_2^i.dl.dl = 0,$$  \hspace{1cm} (41)$$

Succor or Support

$$P(t \in T, l \in L, \cup_{i=0}^4 \int_0^l \int_0^t CP_2^i.dl.dl < 0)$$  \hspace{1cm} (42)$$

So even \(n_2 > n_1\) but the succor or support during an attack denoted by \(P(t \in T, l \in L, \cup_{i=0}^4 \int_0^l \int_0^t CP_2^i.dl.dl < 0)\) so the plan is (time,location)(\(t_i,p_i\)) is the

48. Arkin, *Governing Lethal Behavior in Autonomous Robots*
governing factor, the likelihood of the plan is

\[ L(t_i \in T, x_i, y_i \in X, Y) = \int_0^t \int_0^{x \in X, y \in Y} CP_{x_i, y_i} \, dx \, dy \, < 0 \]  

(43)

Soldier exceeds in number that shall matter nothing in the matter of victory, consider a situation if \( n_1 > n_2 \) and

\[ S_1^p = P(t \in T, l \in L, \cup_{i=0}^n \int_t^l \int_0^{CP_{1, i}} \, dt \, dn_1) \]  

(44)

\[ S_2^p = P(t \in T, l \in L, \cup_{i=0}^n \int_t^l \int_0^{CP_{2, i}} \, dt \, dn_2) \]  

(45)

\( S_1^p < S_2^p \) then \( S_1^p \) will not contribute in victory \( S_2^p \) will contribute in victory.

If enemy is stronger in number we can prevent him from fighting scheme is to know the plans and likelihood of success. The likelihood of success of a plan depends on the Combat Potential (at location \( l \) at time \( t \). therefore the likelihood of success of a plan is the ratio of combat densities of two sides. Therefore,

\[ L(x \in win|n_1, n_2, t, l) = \frac{\int t \int_t^l CP_{1dt} \, dl}{\int t \int_t^l CP_{2dt} \, dl} \]  

(46)

Principle of activity i.e. the functional properties of CP as a function of \((l,t)\)

\[ CP_1(l, t) = n_1(l, t)(d).OLI_1.m_1(l, t).v_1(l, t)l_1.t_1.m_01.po_1(l, t)W_{att}(l, t) \]  

\[ CP_2(l, t) = n_2(l, t)(1/d).OLI_2.m_2(l, t).v_2(l, t)l_2.t_2.m_02.po_2(l, t)W_{def}(l, t) \]  

(47)

Taking logarithms on both sides we have

\[ \log CP_1(l, t) = \log(n_1(l, t)) + \log(d) + \log(OLI_1) + \log(m_1(l, t)) \]  

\[ + \log(v_1(l, t)) + \log(l_1) + \log(t_1) + \log(m_01) + \log(po_1(l, t)) + \log(W_{att}(l, t)) \]  

(48)

\[ \log CP_2(l, t) = \log(n_2(l, t)) + \log(1/d) + \log(OLI_2) + \log(m_2(l, t)) \]  

\[ + \log(v_2(l, t)) + \log(2_1) + \log(t_2) + \log(m_02) + \log(po_2(l, t)) + \log(W_{def}(l, t)) \]  

(49)

Collect samples from various locations \( CP(x^1, y^1, t), CP(x^2, y^2, t), \ldots, CP(x^n, y^n, t^n) \in F_0 \). Carefully compare the opposing Army and Conceal Tactical Dispositions: that means we have to camouflage the center location of the CP’s and the distribution pattern or distribution dispersion.

5.2. Theorem-2

Victory depends not only on the \( n, m, l, m_0, p \ldots \) other factors which is important is the force concentration of CP at time \( t \) and location \( l \). If comparatively it is more
than the enemy then only victory can be produced, it is not simply the number
\( n_i, v_i, OLI_i, l_i, t_i, mo_i, p_i, W_{att/def} \) it is combined effect over location and times.

5.3. **Theorem-3**

Consider system

\[
CP_{1A} = n_1.(d).OLI_1.m_1.v_1.l_1.t_1.mo_1.po_1.W_{att}
\]

\[
CP_{2B} = n_2.(1/d).OLI_2.m_2.v_2.l_2.t_2.mo_2.po_2.W_{def}
\]  

(50)

with \( n_i, OLI_i, m_i, v_i, l_i, t_i, mo_i, po_i \neq 0 \). Let \( W_{att} \) be the function of \( l \) and \( t \) and \( W_{att} = \sin(tx) \) with \( \alpha, \beta \neq 0 \), the CP value is maximum at location \( l \) and decays exponentially with parameter estimated \( \hat{CP} \)

\[
B(l, t) = \frac{1}{\sqrt{(2.\pi).\sigma_1.\sigma_2|\sum|}}\exp(-\frac{1}{2}((x - \mu)\sum(x - \mu')')) = CP
\]  

(51)

if \( \sigma_1 = \sigma_2 = \sigma \)

then

\[
\frac{1}{\sqrt{(2.\pi).\sigma|\sum|}}\exp(-\frac{1}{2}((x - \mu)\sum(x - \mu')')) = CP
\]  

(52)

The density function is the tactics and its total area is the overall strategies. Do not repeat the tactics these tactics are governed by the plan is dependent on \( l \) and \( t \). So a particular plan is just the realization during these period. So the \( \hat{l} \) and \( \hat{t} \) should not be repeated \( l \) and \( t \) are \( \in \mathbb{L}, \mathbb{R}^2 \). Military tactics are like Unto water. So strike on weak, \( Foe \equiv Ground, Water \equiv Soldier \). As water has no constant shape war has no constant shape. Heaven born leader modify his tactics in relation to his desire. Earth, metal and planet are not important what is important is the seasonal changes of water a they one season gives way to the another season. this article uses the concept of Art of War\(^{49}\) for establishing the above mathematical co notations.

5.4. **Theorem-4**

Consider the battle

\[
CP_1 = af(CP_1) + bg(CP_2), CP_2 = af(CP_2) + bg(CP_1)
\]  

(53)

being \( a, b, c, d \in \mathbb{R} \) and \( f \) and \( g \) are smooth real function such that \( f(0) = g(0) = 0 \) then:

(i) if \( abcd \leq 0 \) battle system (23) has no limit cycle.

(ii) Assume that \( f \) and \( g \) are analytically

\[
f(CP_1) = (CP_1)^{2k-1} + O(CP_1)^2 and CP_2 = (CP_2)^{2k-1} + O(CP_2)^2
\]  

(54)

\(^{49}\) Tzu and Yet, *The art of war*
for some positive integer number \( k \) and \( l \) where \( k \neq l \). Then there exits \( abcd \) such that system (23) has at least one limit cycle surrounded the origin which whenever exists is hyperbolic.

(iii) there exists \( f \) and \( g \) such that for same values of \( abcd \) system (23) has more than one limit cycle surrounding the origin. More over same values using \( g(CP_2) \equiv CP_2 \) that is for system (23).

5.5. Theorem-5
Consider the battle

\[
\dot{CP}_1 = Af(CP_1) + Bg(CP_2), \dot{CP}_2 = Cf(CP_2) + Dg(CP_1)
\]  

(55)

being \( ABCD \in \mathbb{R} \) and iid random variables with \( N(0,1) \) Gaussian Distribution and where \( f \) and \( g \) are smooth real function such that \( f(0) = g(0) = 0 \) then the probability that it does not have periodic orbits is greater than or equalto \( \frac{1}{2} \). Equivalently, the probability of having some limit cycles is smaller than or equal to \( \frac{1}{2} \), then:

(i) if \( abcd \leq 0 \) battle system (23) has no limit cycle.

(ii) Assume that \( f \) and \( g \) are analytically

\[
f(CP_1) = (CP_1)^{2k-1} + O((CP_1)^2l) \text{ and } \dot{CP}_2 = (CP_2)^{2k-1} + O((CP_2)^2l)
\]  

(56)

for some positive integer number \( k \) and \( l \) where \( k \neq l \). Then there exits \( abcd \) such that system (23) has at least one limit cycle surrounded the origin which whenever exists is hyperbolic.

(iii) there exists \( f \) and \( g \) such that for same values of \( abcd \) system (23) has more than one limit cycle surrounding the origin. More over same values using \( g(CP_2) \equiv CP_2 \) that is for system (23).

Proof. the system does not have periodic orbits \( P(ABCD < 0) \) \( ABCD \) and \( -ABCD \)

have same distribution \( A, -A, P(ABCD < 0) = P(-ABCD < 0) = P(ABCD > 0) \)

Since \( P(ABCD = 0) = 0P(ABCD > 0) = P(-ABCD < 0) = \frac{1}{2} \), thus the probability of the system pairing atleast one limit cycle is \( \leq \frac{1}{2} \) \( \square \)

5.6. Theorem-6
Consider the random combat system

\[
\dot{CP}_1 = Af(CP_1) + BCP_2, \dot{CP}_2 = Cf(CP_2) + DCP_1
\]  

(57)

where \( f(CP_1) = \alpha CP_1^k + \sum_{k<\ell<m} f_i(CP_i)^k + \beta(CP_1)^m \), with \( \alpha \beta \neq 0, k \leq 0 \) odd integers, \( m \geq 1 \) and \( ABCD \) iid \( N(0,1) \) random variables. Assume also that \( x=0 \) is the unique real root of \( f(CP_1) = 0 \). Then: \( k > 1 \), the probability of having an odd number of limit cycles is \( 1/8 \), and the probability of not having limit cycles or having an even number of there is \( 7/8 \).

(ii) when \( k = 1 \) and \( \beta > 0 \), the probability of having an odd number of limit cycles is \( P^+(\alpha) \leq 1/2 \) and the probability of not having limit cycles or the probability of having an even number of them is \( 1 - P^+(\alpha) \). Here \( P^+: \mathbb{R} \rightarrow (0,1/2) \) is a decreasing further that satisfies \( \lim_{\alpha \rightarrow \infty} P^+(\alpha) = 1/2, P^+(0) = 1/8, \lim_{\alpha \rightarrow \infty} P^+(\alpha) = 0 \) given by: \( P^+(\alpha) = \)
\[ \frac{1}{4\pi^2} \iint_{S} \exp \left( -\frac{(a^2+b^2+c^2+d^2)}{2} \right) \, d\alpha \, d\beta \] 

where \( T(\alpha) = (a, b, c, d) : ad - bc > 0; a(a + d) \leq s \) when \( k = 1 \) and \( \beta \leq 0 \), then some results as in item (ii) hold but changing \( P^+ \) by \( P^- \) where \( P^+(\alpha) = P^+(-\alpha) \) In all the cases, each limit cycle is counted with its multiplicity.

6. DISCUSSIONS

Figures 2 and 3 show the graphs of Soviet and German Tank losses along with the losses estimated through maximum likelihood approach. In this model a single set of parameters are estimated for representing the entire 14 days of the battle. Figures 4 and 5 show the performance of the same model when entire data set is divided into 5 phases. From these figures, it is apparent that fitting the models with division into 5 phases resulted in a much better fit. Figures 6 and 7 show the further improvement in the data set by dividing it into 14 phases where each day is considered as a mini battle. Further, the total losses are divided into two components: Losses due to tank and Losses due to Artillery. The overall SSR and likelihood values are functions of \( p_i \)'s and \( q_i \)'s. Figures 8 and 9 shows the 3D surfaces and contour plots of SSR as a function of \( p_1, q_1 \) and \( p_2, q_2 \) respectively. From these figures, we can see that the minimum SSR zone is represented by contours of \( 1.5E + 5 \) and \( 2.5E + 5 \). Using a grid search in this zone, the best or optimal fit is obtained at \( p_1 = .129, q_1 = .404, p_2 = .138, q_2 = .136 \) with SSR \( 1.19E + 5 \). The \( a_1, b_1, a_2, b_2 \) values corresponding to the optimal fit are \( 1.14, 0.70, 0.90, 0.95 \) respectively. Figures 10 and 11 shows the surface and contour plots of likelihood as a function of \( p_1, q_1 \) and \( p_2, q_2 \) respectively. From these figures, we can see that the zone of maximum likelihood is represented by contours of \( 6.0E + 3 \) and \( 5.0E + 3 \) with MLE \( 5.11E + 3 \). Using a grid search in this zone, the best or optimal fit is obtained at \( p_1 = .21, q_1 = .28, p_2 = .02, q_2 = .04 \). The \( a_1, b_1, a_2, b_2 \) values corresponding to the optimal fit are \( 0.99, 0.88, 0.89, 0.96 \) respectively. Table 2 shows the results of Bracken, Fricker, Clemens and MLE approaches applied on the tank versus tank and artillery data under heterogeneous situation. This table shows the KS statistic for MLE (with 14 divisions) is 0.08674, which is less than any other estimation methods implying that the method of MLE fits better as compared to the other methods. Also, \( R^2 \) is a measure of goodness of fit. Larger values of \( R^2 \) implies a good fit to the data. The \( R^2 \) value of MLE (with 14 divisions) is 1. For comparing the efficiency of the different approaches, the root mean square error (RMSE) criteria is used. The RMSE of MLE with 5 divisions is 88.13 and the RMSE of MLE with 14 divisions is 0.005, which is found to be the minimum. The RMSE of Clemen’s Newton-Raphson Iteration model is 116.19, which is found to be the maximum. Therefore, efficiency (E) is measured with respect to the RMSE of the MLE with 14 divisions. Thus, the E for MLE is maximum i.e. equal to 1 and E of Clemen’s model is minimum i.e. equal to 4.30E – 06. If the comparison is made among Bracken’s, Fricker’s and Clemens approaches, we can say that the Bracken approach is better. However, in all the cases the MLE outperforms other approaches. Based on all the GOF measures, it can be concluded that MLE provides better fits. In the present research we just demonstrate that if it is possible for mixing two forces it is also applicable for more than two forces. The number of parameters to be estimated increases fourth folded for mixing one additional force. With the estimated parameters, we computed the casualty due to tank component and

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50. Bracken, “Lanchester models of the ardennes campaign”
51. Fricker, “Attrition models of the Ardennes campaign”
52. Clemens, “The application of Lanchester Models to the Battle of Kursk, unpublished manuscript”
casualty due to artillery component (See Table 3). When the 14 days Battle data is considered without any division, $a$ and $b$ parameters are significantly small and $a_1 > b_1$ which implies German tanks were more effective than Soviet tanks. Similarly, when we compare $a_2$ against $b_2, b_2 > a_2$ which implies Soviet artillery were more effective than German artillery. Table 4 shows maximised log-likelihood values with divisions into 14 phases where each day is treated as a mini-battle. Table 3 shows the optimal parameters of heterogeneous Lanchester model with an $R^2$ of 1, RMSE of 0.0005, chi-square of 1.9E-5, SSR of 3.3E-6 and MLE of 13202. The parameters are obtained from maximum likelihood estimation of heterogeneous Lanchester model of tank and Artillery data (table 1) from Kursk Database with each day as single phase. The GRG algorithm is applied for maximizing the likelihood function given in equation (9). Also, the parameter estimates $a_i, b_i, p_i, q_i$ are given corresponding to the maximised log-likelihood values with divisions. From this table we can see that the patterns of the parameters for each day of the battle are same for both the sides. In addition the tank component parameters are seen to be playing major role in the entire duration of the battle. Out of 14 days, 10 days the tank component parameters came out to be the maximum. That’s why the result justified the Battle of Kursk and was correctly termed as the largest tank battle in the history.

7. CONCLUSION

Although mathematical formulations are well established for heterogeneous Lanchester model, very few studies have been done to model actual battle scenario. We have developed heterogeneous Lanchester model for Kursk Battle from World War II using tank and artillery data. All the previous studies on Kursk Battle were done to capture the homogeneous weapon system (Tank against Tank or Artillery against Artillery). The working principles of this model were only applicable for homogeneous situation. So extending those models in heterogeneous situation both theoretically and practically were main focus of this paper. We have formulated the likelihood expression under heterogeneous situation and applied to fit model under heterogeneous Lanchester model for Kursk database. We have estimated the MLE of the different parameters that are proved to be statistically more accurate. The unfamiliarity to deal with the heterogeneous situation by the previous approaches motivated us to venture the minute details of the Kursk Battle. The estimates are cross-validated to control the problem of the over fitting. Also, these estimates possess the optimal properties of consistency, sufficiency and efficiency. So compared to the previous work, the present paper opens up the opportunity for exploring the complicated structure of Kursk Battle of World War II.

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Table 2. Comparison of different estimation methods with respect to common Goodness-of-fit measures such as sum of square residuals (SSR), Kolmogorov-Smirnov (KS) Statistic, Chi-Square and R-square (R2), root mean square error (RMSE) and efficiency (E) from homogeneous tank against tank data of the Kursk battle.

| Sl. No. | Approaches                        | SSR/log-Likelihood | KS    | $\chi^2$ | $R^2$ | RMSE | E    |
|--------|-----------------------------------|--------------------|-------|----------|-------|------|------|
| 1      | Bracken\(^a\) Model I            | 1.19E+5            | 0.1567| 2964     | 0.48  | 92.19| 0.9559|
| 2      | Fricker\(^b\) Model I            | 1.29E+5            | 0.1065| 3082     | 0.44  | 95.99| 0.9181|
| 3      | Clement\(^c\) Linear Regression I| 1.88E+5            | 0.1063| 3854     | 0.22  | 115.88| 0.7605|
|        | Clement\(^c\) Newton-Raphson Iteration | 1.89E+5        | 0.1123| 3520     | 0.22  | 116.19| 0.7584|
| 4      | Without-division                  | 13203              | 0.1053| 2580     | 0.71  | 89.72| 0.9822|
| 5      | With-divisions (4 phases)         | 13313              | 0.0909| 2670     | 0.82  | 88.13| 1     |

\(^a\) Jerome Bracken, “Lanchester models of the ardennes campaign,” Naval Research Logistics (NRL) 42, no. 4 (June 1995): 559–577, https://doi.org/10.1002/1520-6750(199506), https://ideas.repec.org/a/wly/navres/v42y1995i4p559-577.html.
\(^b\) Ronald D. Fricker, “Attrition models of the Ardennes campaign,” Naval Research Logistics (NRL) 45, no. 1 (February 1998): 1–22, https://doi.org/10.1002/(SICI)1520-6750(1, https://ideas.repec.org/a/wly/navres/v45y1998i1p1-22.html.
\(^c\) Clemens, “The application of Lanchester Models to the Battle of Kursk, unpublished manuscript” (Yale University, New Haven, CT, 5 May 1997).
\(^d\) Sumanta Das, Pankaj Sati, and Rajiv Kumar Gupta, “Aggregate Combat Modelling Using High-resolution Simulation: The ’Meeting Engagement’ Scenario as a Case Study,” Journal of Battlefield Technology 10 (2007): 37, https://api.semanticscholar.org/CorpusID:112782649.
Table 3. Maximization of likelihood estimation with divisions from homogeneous tank against tank data of Kursk battle.

| Sl. No. | Likelihood | a   | b   | p   | q   |
|--------|------------|-----|-----|-----|-----|
| Phase 1 | 2621.35    | 0.9031 | 1.0887 | 0.4793 | 0.3079 |
| Phase 2 | 6169.72    | 1.1327 | 0.7884 | 0.1418 | 0.4720 |
| Phase 3 | 2354.21    | 1.0300 | 0.8816 | 0.3889 | 0.4838 |
| Phase 4 | 2168.54    | 1.0221 | 0.8939 | 0.2793 | 0.3483 |

Table 4. The Parameters of Heterogeneous Lanchester Model. The parameters are obtained from maximum likelihood estimation from heterogeneous tank against tank and artillery data of Kursk battle with each day as single phase.

| phase | Likelihood | a_1 | a_2 | b_1 | b_2 | p_1 | p_2 | q_1 | q_2 |
|-------|------------|-----|-----|-----|-----|-----|-----|-----|-----|
| Phase I | 1232.743 | 0.929 | 0.456 | 0.935 | 1.395 | 0.011 | 0.467 | 5E-11 | 0.273 |
| Phase II | 1559.505 | 0.795 | 0.670 | 1.119 | 1.188 | 0.539 | 0.039 | 0.182 | 0.000 |
| Phase III | 1639.509 | 1.062 | 1.021 | 0.808 | 0.887 | 0.015 | 0.371 | 0.285 | 0.377 |
| Phase IV | 1894.741 | 0.989 | 1.143 | 0.870 | 0.753 | 0.000 | 0.352 | 0.113 | 0.418 |
| Phase V | 1895.489 | 1.140 | 0.882 | 0.815 | 0.979 | 0.215 | 0.024 | 0.574 | 0.000 |
| Phase VI | 229.8775 | 1.501 | 0.897 | 0.534 | 0.967 | 0.000 | 0.056 | 0.660 | 0.945 |
| Phase VII | 2422.045 | 1.290 | 1.332 | 0.568 | 0.516 | 0.020 | 0.388 | 0.597 | 0.416 |
| Phase IX | 613.6283 | 1.196 | 0.889 | 0.714 | 0.967 | 0.184 | 0.000 | 0.498 | 0.039 |
| Phase X | 575.0583 | 1.270 | 0.906 | 0.612 | 0.960 | 0.140 | 0.072 | 0.535 | 0.106 |
| Phase XI | 608.3638 | 1.017 | 0.904 | 0.888 | 0.972 | 0.300 | 0.138 | 0.390 | 0.176 |
| Phase XII | 111.104 | 0.996 | 0.291 | 0.585 | 0.968 | 0.210 | 0.022 | 0.252 | 0.048 |
| Phase XIII | 121.6655 | 1.369 | 0.907 | 0.266 | 0.936 | 0.002 | 0.000 | 0.492 | 0.000 |
| Phase XIV | 297.3759 | 1.590 | 0.911 | 0.119 | 0.945 | 0.007 | 0.005 | 0.576 | 0.01 |

Figure 2. Fitted Losses plotted versus real losses for the (a) German’s Tanks Losses (b) Soviet’s Tanks Losses without any division of the Battle of Kursk of WW II.

Figure 3. Fitted tank Losses plotted versus real tank losses for the (a) Soviets and (b) German’s with division over multiple phases of the Battle of Kursk of WW II. The multiple phases are arranged as division 1(with day 1, 2), division 2 (with day 3-7), division 3 (with day 8), division 4 (with day 9-14).
Table 5. Fitted Tank Losses and residual sum of square using Heterogeneous Lanchester model. The tank and arty components of the fitted models are obtained through maximum likelihood estimation method from heterogeneous tank against tank and artillery data of Kursk battle.

| Phase | Days | SLossFit | STank Comp. | SArty Comp. | SResidual | GLossFit | GTank Comp. | GArty Comp. | GResidual |
|-------|------|----------|-------------|-------------|-----------|-----------|-------------|-------------|-----------|
| phase 1 | 105.00 | 1.01 | 103.99 | 1.66E-08 | 198.00 | 1.02 | 196.98 | 3.72E-08 |
| phase 2 | 117.00 | 116.12 | 0.88 | 1.82E-08 | 248.00 | 246.47 | 1.53 | 1.34E-07 |
| phase 3 | 259.00 | 248.66 | 9.09E-08 | 121.00 | 5.81 | 115.19 | 3.33E-08 |
| phase 4 | 315.00 | 312.71 | 1.1E-07 | 108.00 | 1.79 | 106.21 | 3.9E-08 |
| phase 5 | 289.00 | 287.95 | 1.04 | 1.03E-07 | 139.00 | 137.86 | 1.14 | 7.21E-10 |
| phase 6 | 157.00 | 155.19 | 1.81 | 5.37E-07 | 36.00 | 34.20 | 1.81 | 3.48E-07 |
| phase 7 | 135.00 | 134.04 | 0.96 | 8.63E-08 | 63.00 | 61.96 | 1.04 | 2.05E-08 |
| phase 8 | 411.00 | 366.23 | 1.4E-07 | 98.00 | 15.18 | 82.82 | 4.39E-08 |
| phase 9 | 117.00 | 115.91 | 1.09 | 2.76E-08 | 57.00 | 55.83 | 1.17 | 1.32E-08 |
| phase 10 | 118.00 | 114.91 | 3.09 | 2.68E-08 | 46.00 | 43.08 | 2.92 | 1.06E-08 |
| phase 11 | 96.00 | 88.11 | 7.89 | 4.33E-08 | 79.00 | 72.21 | 6.79 | 7.96E-10 |
| phase 12 | 27.00 | 25.55 | 1.45 | 2.02E-08 | 23.00 | 21.50 | 1.50 | 9.59E-09 |
| phase 13 | 42.00 | 41.09 | 0.91 | 2.43E-07 | 7.00 | 6.06 | 0.94 | 5.48E-09 |
| phase 14 | 85.00 | 83.94 | 1.06 | 2.64E-07 | 6.00 | 4.91 | 1.09 | 9.07E-07 |
| Sl. No. | Battle Name | Equations | Parameters | Parameters |
|--------|-------------|-----------|------------|------------|
| 1      | Iwo Jima    | \( \dot{S} = -a \cdot S \cdot p \cdot G \) | \( a = 0.0577 \), \( b = 0.0106 \) | |
| 2      | Battle of Atlantic | \( \dot{M} = 2 \cdot (S/E) \), \( \dot{S} = -1.6 \times 10^{-6} E + 8 \times 10^{-4} S \) | | |
| 3      | Battle of Trafalgar | \( \dot{A} = -bB \), \( \dot{B} = -aA \) | | |
| 4      | Battle of Kursk | \( \dot{S} = -a \cdot S \cdot p \cdot G \), \( \dot{G} = -b \cdot S \cdot q \cdot G \) | \( p = 5.87, q = 0.61, a = 4.9 \times 10^{-3}, b = 3.52 \times 10^{-3} \) | |
| 5      | Battle of Ardennes | \( \dot{S} = -a \cdot S \cdot p \cdot G \), \( \dot{G} = -b \cdot S \cdot q \cdot G \) | \( p = 0.91, q = -0.61, a = 4.9 \times 10^{-3}, b = 3.52 \times 10^{-3} \) | |
| 6      | Battle of Britain | \( \dot{B} = -a \cdot B \cdot p \cdot G \) | \( a = -5.4 \times 10^{-3}, p = 1.2, q = 0.9 \) | |

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Figure 4. Fitted (a) Soviet’s tank losses due to German’s and (b) German’s tank losses due to Soviet’s tank and artillery using heterogeneous Lanchester model of the Battle of Kursk of WW II. Total losses are divided into two components, losses due to tank and losses due to arty.

Figure 5. Contour plot of (a) Log-likelihood and (b) SSR values for the tank data of Soviet and German sides of the Battle of Kursk of WW II. The p and q values are varied between -3 (-6) and 3 (6). The parameters are estimated using the MLE and the Least Square approaches.

Figure 6. 3D plot of (a) Log-likelihood and (b) SSR values for the tank data of the Battle of Kursk of WW II., p and q values are varied between -3(-6) to 3(6), a and b values depend on p and q. The parameters are estimated using the MLE approach.
Considerable insight into the Combat system described through equations as referred in the table can be gained by examining the behaviour of the differential equations through the use of plots known as Directional field plots or D-field plots. The graph gives an idea whether the system is going to stable at a point or diverges without actually knowing the solutions. These graphs show the convergent and divergent properties of the battles.
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