Homogenization of a two-phase problem with nonlinear dynamic Wentzell-interface condition for connected-disconnected porous media

M. Gahn

Abstract

We investigate a reaction-diffusion problem in a two-component porous medium with a nonlinear interface condition between the different components. One component is connected and the other one is disconnected. The ratio between the microscopic pore scale and the size of the whole domain is described by the small parameter $\varepsilon$. On the interface between the components we consider a dynamic Wentzell-boundary condition, where the normal fluxes from the bulk domains are given by a reaction-diffusion equation for the traces of the bulk-solutions, including nonlinear reaction-kinetics depending on the solutions on both sides of the interface. Using two-scale techniques, we pass to the limit $\varepsilon \to 0$ and derive macroscopic models, where we need homogenization results for surface diffusion. To cope with the nonlinear terms we derive strong two-scale convergence results.

Keywords: Homogenization; Two-scale convergence; Reaction-diffusion equations; Nonlinear interface conditions; Surface-diffusion.

MSC: 35K57; 35B27

1 Introduction

In this paper we derive homogenized models for nonlinear reaction-diffusion problems with dynamic Wentzell-boundary conditions in multi-component porous media. The domain consists of two components $\Omega_1^\varepsilon$ and $\Omega_2^\varepsilon$, where $\Omega_1^\varepsilon$ is connected, and $\Omega_2^\varepsilon$ is disconnected and consists of periodically distributed inclusions. The small scaling parameter $\varepsilon$ represents the ratio between the length of an inclusion and the size of the whole domain. At the interface $\Gamma_\varepsilon$ between the two components we assume a dynamic Wentzell-boundary condition, i.e., the normal flux at the surface is given by a reaction-diffusion equation on $\Gamma_\varepsilon$. More precisely, this boundary/interface condition describes processes like reactions, adsorption, desorption, and diffusion at the interface $\Gamma_\varepsilon$. Further it takes into account exchange of species between the different compartments, what can be modeled by nonlinear reaction-kinetics depending on the solutions on both sides of $\Gamma_\varepsilon$. The aim is the derivation of macroscopic models with homogenized diffusion coefficients for $\varepsilon \to 0$, the solution of which is an approximation of the...
microscopic solution. An additional focus of the paper is to provide general strong two-scale compactness results, which are based on \textit{a priori} estimates for the microscopic solution.

Reaction-diffusion processes play an important role in many applications, and our model is motivated by metabolic and regulatory processes in living cells. Here, an important example is the carbohydrate metabolism in plant cells, where biochemical species are diffusing and reacting within the (connected) cytoplasm and the (disconnected) organelles (like chloroplasts and mitochondria), and are exchanged between different cellular compartments. At the outer mitochondrial membrane takes place the process of metabolic channeling, where intermediates in metabolic pathways are passed directly from enzyme to enzyme without equilibrating in the bulk-solution phase of the cell [32]. This effect can be modeled by the dynamic Wentzell-boundary condition, see [15, Chapter 4] for more details about the modeling and the derivation of these boundary conditions, which can be derived by asymptotic analysis.

To pass to the limit $\epsilon \to 0$ in the variational equation for the microscopic problem we have to cope with several difficulties. The main challenges are the coupled bulk-surface diffusion in the perforated domains, as well as the treatment of the nonlinear terms, especially on the oscillating surface $\Gamma_\epsilon$. To overcome these problems we make use of the two-scale method in perforated domains and on oscillating surfaces, where we need two-scale compactness results for diffusion processes on surfaces. To pass to the limit in the nonlinear terms we need strong convergence results. Such results are quite standard for the connected domain, but the usual methods fail for the disconnected domain. Here we make use of the unfolding method, which gives us a characterization for the two-scale convergence via functions defined on fixed domains, and a Kolmogorov-Simon-type compactness result for Banach-valued function spaces. Additionally, due to the nonlinear structure of the problem and the weak assumptions on the data, we have to deal with low regularity for the time-derivative.

There exists a large amount of papers dealing with homogenization problems for parabolic equations in multi-component porous media. However, results for the connected-disconnected case for nonlinear problems, especially for nonlinear interface conditions, seem to be rare. In [18] and [19] systems of reaction-diffusion problems are considered with nonlinear interface conditions. In [18] surface concentration is included and an additional focus lies on the modeling part of the carbohydrate metabolism and the specific structure of the nonlinear reaction kinetics. In the present paper we extend those models to problems including an additional surface diffusion for the traces of the bulk-solutions in $\Omega^1_\epsilon$ and $\Omega^2_\epsilon$. The stationary case for different scalings with a continuous normal flux condition at the interface, given by a nonlinear monotone function depending on the jump of the solutions on both sides, is treated in [13]. There, the nonlinear terms in the disconnected domain only occurs for particular scalings and it is not straightforward to generalize those results to systems.

A double porosity model, where the diffusion inside the disconnected domain is of order $\epsilon^2$, is considered in [8, 29] for continuous transmission conditions at the interface for the solutions and the normal fluxes. In [8] a nonlinear diffusion coefficient is considered, and the convergence of the nonlinear term is obtained by using the Kirchhoff-transformation and comparing the microscopic and the macroscopic equation, where the last one was obtained by a formal asymptotic expansion. Nonlinear reaction-kinetics in the bulk domains and an additional ordinary differential equation on the interface is considered in [29], where the strong convergence is proved by showing that the unfolded sequence of the microscopic solution is a Cauchy-sequence. A
similar model with different kind of interface conditions is considered in [24], where the method of two-scale convergence is used and a variational principle to identify the limits of the nonlinear terms.

To pass to the limit in the diffusive terms on the interface $\Gamma_\epsilon$ arising from the Wentzell-boundary condition, compactness results for the surface gradient on an oscillating manifolds are needed. For such kind of problems in [4, 21] two-scale compactness results are derived for connected surfaces, where in [21] the method of unfolding is used. Compactness results for a coupled bulk-surface problem when the evolution of the trace of the bulk-solution on the surface $\Gamma_\epsilon$ is described by a diffusion equation, are treated in [5, 16]. In [5] continuity of the traces across the interface is assumed, where in [16] also jumps across the interface are allowed and also compactness results for the disconnected domain $\Omega_2^\epsilon$ are derived. In [5], the convergence results are applied to a linear problem with a dynamic Wentzell-interface condition.

A reaction-diffusion problem including dynamic Wentzell-boundary conditions and nonlinear reaction-rates in the bulk domain and on the surface is considered in [6] for a connected perforated domain.

In this paper we start with the microscopic model and establish existence and uniqueness of a weak solution. The appropriate function space for a weak solution is the space of Sobolev functions of first order with $H^1$-traces on the interface $\Gamma_\epsilon$, which we denote by $H(j,\epsilon)$ for $j = 1, 2$. To pass to the limit $\epsilon \to 0$ we make use of the method of two-scale convergence for domains and surfaces, see [2, 3, 26, 28]. For the treatment of the diffusive terms on the oscillating surface we use the methods developed in [16] for the spaces $H(j,\epsilon)$. Those two-scale compactness results are based on a priori estimates for the microscopic solution depending explicitly on $\epsilon$. However, to pass to the limit in the nonlinear terms, the usual (weak) two-scale convergence is not enough and we need strong two-scale convergence, what leads to difficulties especially in the disconnected domain $\Omega_2^\epsilon$. The strong convergence is obtained be applying the unfolding operator, see [11] for an overview of the unfolding method, to the microscopic solution and use a Kolmogorov-Simon-type compactness result for the unfolded sequence. We derive a general strong two-scale compactness result that is based only on a priori estimates and estimates for the difference between the solution and discrete shifts (with respect to the microscopic cells) of the solution. Since we only take into account linear shifts, which are not well defined for general surfaces, we use a Banach-valued Kolmogorov-Simon-compactness result, see [17]. Further, for our microscopic model we only obtain low regularity results for the time-derivative (which is a functional on $H(j,\epsilon)$), what leads to additional difficulties in the control of the time variable in the proof of the strong convergence.

The paper is organized as follows: In Section 2 we introduce the geometrical setting and the microscopic model. In Section 3 we show existence and uniqueness of a microscopic solution, and derive a priori estimates depending explicitly on $\epsilon$. In Section 4 we prove general strong two-scale compactness results for the connected and disconnected domain. In Section 5 we state the convergence results for the microscopic solution, formulate the macroscopic model, and show that the limit of the micro-solutions solves the macro-model. In the Appendix A we repeat the definition of the two-scale convergence and the unfolding operator and summarize some well known results from the literature.
2 The microscopic model

In this section we introduce the microscopic problem. We start with the definition of
the microscopic domains $\Omega_1^\epsilon$ and $\Omega_2^\epsilon$, as well as the interface $\Gamma_\epsilon$, and explain some
geometrical properties. Then we state the microscopic equation for given $\epsilon$ and give
the assumptions on the data.

2.1 The microscopic geometry

Let $\Omega \subset \mathbb{R}^n$ with Lipschitz-boundary and $\epsilon > 0$ a sequence with $\epsilon^{-1} \in \mathbb{N}$. We define
the unit cube $Y := (0,1)^n$ and $Y_2 \subset Y$ such that $\overline{Y_2} \subset Y$, so $Y_2$ strictly included in
$Y$. Further, we define $Y_1 := Y \setminus \overline{Y_2}$ and $\Gamma := \partial Y_2$, and we suppose that $\Gamma \in C^{1,1}$. We
assume that $Y_1$ is connected and for the sake of simplicity we also assume that $Y_2$
is connected. The general case of disconnected $Y_2$ is easily obtained by considering
the connected components of $Y_2$, see also Remark$^8$. Now, the microscopic domains
$\Omega_1^\epsilon$ and $\Omega_2^\epsilon$ are defined by scaled and shifted reference elements $Y_j$ for $j = 1,2$. Let
$K_\epsilon := \{ k \in \mathbb{Z}^n : \epsilon(k+Y) \subset \Omega \}$ and define

\[
\Omega_1^\epsilon := \bigcup_{k \in K_\epsilon} \epsilon(Y_2 + k), \quad \Omega_2^\epsilon := \Omega \setminus \overline{\Omega_2^\epsilon}, \quad \Gamma_\epsilon := \partial \Omega_2^\epsilon.
\]

Hence, $\Gamma_\epsilon$ denotes the oscillating interface between $\Omega_1^\epsilon$ and $\Omega_2^\epsilon$. Due to the assumptions
on $Y_1$ and $Y_2$ it holds that $\Omega_1^\epsilon$ is connected and $\Omega_2^\epsilon$ is disconnected, and $\Gamma_\epsilon \in C^{1,1}$ is
not touching the outer boundary $\partial \Omega$.

2.2 The microscopic model

We are looking for a solution $(u_1^\epsilon, u_2^\epsilon)$ with $u_j^\epsilon : (0,T) \times \Omega_j^\epsilon \to \mathbb{R}$ for $j = 1,2$, such that
it holds that

\[
\begin{align*}
\partial_t u_1^\epsilon &- \nabla \cdot (D_1^\epsilon \nabla u_1^\epsilon) = f_1^\epsilon(u_2^\epsilon) \quad &\text{in} \ (0,T) \times \Omega_1^\epsilon, \\
\epsilon \partial_t u_2^\epsilon &- \nabla_\epsilon \cdot \left( D_1^\epsilon \nabla_\epsilon u_2^\epsilon - b_1^\epsilon(u_1^\epsilon, u_2^\epsilon) \right) = -D_2^\epsilon \nabla u_2^\epsilon \cdot \nu - h_2^\epsilon(u_1^\epsilon, u_2^\epsilon) \quad &\text{on} \ (0,T) \times \Gamma_\epsilon, \\
-D_2^\epsilon \nabla u_2^\epsilon \cdot \nu &\quad = 0 \quad &\text{on} \ (0,T) \times \partial \Omega, \\
\quad u_1^\epsilon(0) &\quad = u_1^\epsilon, \quad \text{in} \ \Omega_1^\epsilon, \\
\quad u_2^\epsilon|_{\Gamma_\epsilon} (0) &\quad = u_2^\epsilon, \quad \text{on} \ \Gamma_\epsilon,
\end{align*}
\]

where $\nu$ denotes the outer unit normal (we neglect a subscript for the underlying
domain, since this should be clear from the context), and $u_2^\epsilon|_{\Gamma_\epsilon}$ denotes the trace of
$u_2^\epsilon$ on $\Gamma_\epsilon$. If it is clear from the context, we use the same notation for a
function and its trace, for example we just write $u_j^\epsilon$ for $u_j^\epsilon|_{\Gamma_\epsilon}$. The precise weak formulation of
the micro-model above is stated in Section$^3$, see$^3$, after introducing the necessary
function spaces.

In the following, with $T_y \Gamma$ and $T_x \Gamma$, for $y \in \Gamma$ and $x \in \Gamma$, we denote the tangent
spaces of $\Gamma$ at $y$ and $\Gamma_x$ at $x$, respectively. The orthogonal projection $P_T(y) : \mathbb{R}^n \to T_y \Gamma$
for $y \in \Gamma$ is given by

\[
P_T(y) \xi = \xi - (\xi \cdot \nu(y)) \nu(y) \quad \text{for} \ \xi \in \mathbb{R}^n,
\]

where $\nu(y)$ denotes the outer unit normal at $y \in \Gamma$. Let us extend the unit normal
$Y$-periodically. Then, the orthogonal projection $P_T(x) : \mathbb{R}^n \to T_x \Gamma_\epsilon$ for $x \in \Gamma_\epsilon$ is
given by

\[
P_{\Gamma}(x)\xi = \xi - \left(\xi \cdot \nu \left(\frac{\xi}{\epsilon}\right)\right)\nu \left(\frac{x}{\epsilon}\right) \quad \text{for } \xi \in \mathbb{R}^n.
\]

**Assumptions on the data:**

In the following let \( j \in \{1, 2\} \).

(A1) For the bulk-diffusion we have \( D_j^i(x) := D_j^i \left(\frac{x}{\epsilon}\right) \) with \( D_j^i \in L_{\text{per}}^\infty (Y_j)^{n \times n} \) symmetric and coercive, i.e., there exists \( c_0 > 0 \) such that for almost every \( y \in Y_j \)

\[
D_j^i(y)\xi \cdot \xi \geq c_0|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n.
\]

(A2) For the surface-diffusion we suppose \( D_j^{\Gamma} (x) := D_j^i \left(\frac{x}{\epsilon}\right) \) with \( D_j^{\Gamma} \in L_{\text{per}}^\infty (\Gamma)^{n \times n} \) symmetric and \( D_j^{\Gamma}(y)|_{\Gamma_y} : T_y\Gamma \to T_y\Gamma \) for almost every \( y \in \Gamma \). Further, we assume that \( D_j^{\Gamma} \) is coercive, i.e., there exists \( c_0 > 0 \) such that for almost every \( y \in \Gamma \)

\[
D_j^{\Gamma}(y)\xi \cdot \xi \geq c_0|\xi|^2 \quad \text{for all } \xi \in T_y\Gamma.
\]

(A3) For the reaction-rates in the bulk domains we suppose that \( f_j^i(t, x, z) := f_j^i(t, \frac{x}{\epsilon}, z) \) with \( f_j^i \in L^\infty((0, T) \times Y_j \times \mathbb{R}) \) is \( Y \)-periodic with respect to the second variable and uniformly Lipschitz continuous with respect to the last variable, i.e., there exists \( C > 0 \) such that for all \( z, w \in \mathbb{R} \) and almost every \((t, y) \in (0, T) \times Y_j \) it holds that

\[
|f_j^i(t, y, z) - f_j^i(t, y, w)| \leq C|z - w|.
\]

(A4) For the reaction-rates on the surface we suppose that \( h_j^{\epsilon_i}(t, x, z_1, z_2) := h_j^i(t, \frac{x}{\epsilon}, z_1, z_2) \) with \( h_j^i \in L^\infty((0, T) \times \Gamma \times \mathbb{R}^2) \) is \( Y \)-periodic with respect to the second variable and uniformly Lipschitz continuous with respect to the last variable, i.e., there exists \( C > 0 \) such that for all \( z_1, z_2, w_1, w_2 \in \mathbb{R} \) and almost every \((t, y) \in (0, T) \times \Gamma \) it holds that

\[
|h_j^{\epsilon_i}(t, y, z_1, z_2) - h_j^i(t, y, w_1, w_2)| \leq C(|z_1 - w_1| + |z_2 - w_2|).
\]

(A5) For the initial conditions we assume \( u_{\epsilon, i} \in L^2(\Omega_j^i) \) and \( u_{\epsilon, i, \Gamma} \in L^2(\Gamma_{\epsilon}) \) and there exists \( u_{0,i} \in L^2(\Omega) \) and \( u_{0,i,\Gamma} \in L^2(\Gamma_{\epsilon}) \) such that

\[
\begin{align*}
    u_{\epsilon, i} & \to u_{0,i} \quad \text{in the two-scale sense,} \\
    u_{\epsilon, i, \Gamma} & \to u_{0,i,\Gamma} \quad \text{in the two-scale sense on } \Gamma_{\epsilon},
\end{align*}
\]

and it holds that

\[
\sqrt{\epsilon} u_{\epsilon, i, \Gamma} \in L^2(\Gamma_{\epsilon}) \leq C.
\]

Additionally we assume that the sequences \( u_{\epsilon, i}^2 \) and \( u_{\epsilon, i, \Gamma}^2 \) converge strongly in the two-scale sense.

For the definition of the two-scale convergence see Section [4]. We emphasize that due to the convergences in [A5] it holds that

\[
\|u_{\epsilon, i}^2\|_{L^2(\Omega_j^i)} + \sqrt{\epsilon} \|u_{\epsilon, i, \Gamma}^2\|_{L^2(\Gamma_{\epsilon})} \leq C.
\]
3 Existence of a weak solution and a priori estimates

The aim of this section is the investigation of the microscopic problem (1). We introduce appropriate function spaces and show existence and uniqueness of a microscopic solution. Further, we derive a priori estimates for the solution depending explicitly on $\epsilon$. These estimates form the basis for the derivation of the macroscopic problem (12) by using the compactness results from Section 4.

3.1 Function spaces

Due to the Laplace-Beltrami operator in the boundary condition in (1), it is not enough to consider the usual Sobolev space $H^1(\Omega^\epsilon_j)$ as a solution space, because we need more regular traces. This gives rise to deal with the following function spaces for $j = 1, 2$:

$$
\mathbb{H}_{j,\epsilon} := \left\{ \phi^\epsilon \in H^1(\Omega^\epsilon_j) : \phi^\epsilon|_{\Gamma_\epsilon} \in H^1(\Gamma_\epsilon) \right\},
$$

$$
\mathbb{H}_j := \left\{ \phi \in H^1(Y_j) : \phi|_\Gamma \in H^1(\Gamma) \right\},
$$

with the inner products

$$
(\phi^\epsilon, \psi^\epsilon)_{\mathbb{H}_{j,\epsilon}} := (\phi^\epsilon, \psi^\epsilon)_{H^1(\Omega^\epsilon_j)} + \epsilon (\phi^\epsilon, \psi^\epsilon)_{H^1(\Gamma_\epsilon)},
$$

$$
(\phi, \psi)_{\mathbb{H}_j} := (\phi, \psi)_{H^1(Y_j)} + \epsilon (\phi, \psi)_{H^1(\Gamma)},
$$

The associated norms are denoted by $\| \cdot \|_{\mathbb{H}_{j,\epsilon}}$ and $\| \cdot \|_{\mathbb{H}_j}$. Obviously, the spaces $\mathbb{H}_{j,\epsilon}$ and $\mathbb{H}_j$ are separable Hilbert spaces and we have the dense embeddings $C^\infty(\Omega^\epsilon_j) \subset \mathbb{H}_{j,\epsilon}$ and $C^\infty(Y_j) \subset \mathbb{H}_j$, see [16, Proposition 5]. We also define the space

$$
L_{j,\epsilon} := L^2(\Omega^\epsilon_j) \times L^2(\Gamma_\epsilon), \quad L_j := L^2(Y_j) \times L^2(\Gamma)
$$

with inner products

$$
(\phi^\epsilon, \psi^\epsilon)_{L_{j,\epsilon}} := (\phi^\epsilon, \psi^\epsilon)_{L^2(\Omega^\epsilon_j)} + \epsilon (\phi^\epsilon, \psi^\epsilon)_{L^2(\Gamma_\epsilon)},
$$

$$
(\phi, \psi)_{L_j} := (\phi, \psi)_{L^2(Y_j)} + \epsilon (\phi, \psi)_{L^2(\Gamma)},
$$

and again denote the associated norms by $\| \cdot \|_{L_{j,\epsilon}}$ and $\| \cdot \|_{L_j}$. Obviously, we have

$$
\mathbb{H}_{j,\epsilon} \ni \{ (u^\epsilon, v_\epsilon) \in H^1(\Omega^\epsilon_j) \times H^1(\Gamma_\epsilon) : u^\epsilon|_{\Gamma_\epsilon} = v_\epsilon \},
$$

and a similar result for $\mathbb{H}_j$. Therefore, we have the following Gelfand-triples:

$$
\mathbb{H}_{j,\epsilon} \hookrightarrow L_{j,\epsilon} \hookrightarrow \mathbb{H}'_{j,\epsilon}, \quad \mathbb{H}_j \hookrightarrow L_j \hookrightarrow \mathbb{H}_j.
$$

We will also make use for $\alpha \in \left( \frac{1}{2}, 1 \right]$ of the function space

$$
\mathbb{H}_j^\alpha := \{ \phi \in H^\alpha(Y_j) : \phi|_\Gamma \in H^\alpha(\Gamma) \}
$$

with inner product

$$
(\phi, \psi)_{\mathbb{H}_j^\alpha} := (\phi, \psi)_{H^\alpha(Y_j)} + \epsilon (\phi, \psi)_{H^\alpha(\Gamma)}.
$$

By definition we have $\mathbb{H}_j = \mathbb{H}_j^1$. Obviously, we have the compact embedding $\mathbb{H}_j \hookrightarrow \mathbb{H}_j^\alpha$ for $\alpha \in \left( \frac{1}{2}, 1 \right]$. 


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3.2 Existence and uniqueness of a weak solution

A weak solution of Problem (1) is defined in the following way: The tuple \((u_1^j, u_2^j)\) is a weak solution of (1) if for \(j = 1, 2\)
\[
u_j^1 \in L^2((0, T), H_{\text{div}}^s) \cap H^1((0, T), H_{\text{div}}^s),
\]
and \(u_j^1\) fulfill the initial condition \(u_j^1(0) = u_{j,0}\) and \(u_j^1|_{\Gamma_0} = 0\), and for all \(\phi_j^0 \in H_{\text{div}}^s\) it holds everywhere in \((0, T)\)
\[
\langle \partial_t u_j^1, \phi_j^0 \rangle_{H_{\text{div}}^s, H_{\text{div}}^s} + \langle D_j^1 \nabla u_j^1, \nabla \phi_j^0 \rangle_{H_j^s, H_j^s} + \epsilon \left( \langle D_j^1 \nabla u_j^1, \nabla \phi_j^0 \rangle_{H_j^s, H_j^s} + \epsilon \langle D_j^1 \nabla u_j^1, \nabla \phi_j^0 \rangle_{\Gamma_0} \right),
\]
(3)
Here \(\langle \cdot, \cdot \rangle_U\) stands for the inner product on \(L^2(U)\), for a suitable set \(U \subset \mathbb{R}^n\), and for a Banach space \(X\) and its dual \(X'\) we write \(\langle \cdot, \cdot \rangle_{X', X}\) for the duality pairing between \(X'\) and \(X\). The scaling factor \(\epsilon\) for the time-derivative in (1) is included in the duality pairing \(\langle \cdot, \cdot \rangle_{H_{\text{div}}^s, H_{\text{div}}^s}\). In fact, if additionally it holds that \(\partial_t u_j^1 \in L^2((0, T), H^1(\Omega_j^1)')\) and \(\partial_t u_j^1|_{\Gamma_0} \in L^2((0, T), H^1(\Gamma_0))\) with respect to the Gelfand-triples \(H^1(\Omega_j^1) \rightarrow L^2(\Omega_j^1) \rightarrow H^1(\Gamma_0)\), we get for all \(\phi_j^0 \in H_j^s\)
\[
\langle \partial_t u_j^1, \phi_j^0 \rangle_{H_{\text{div}}^s, H_{\text{div}}^s} = \langle \partial_t u_j^1, \phi_j^0 \rangle_{\mathcal{H}(\Omega_j^1), H^1(\Omega_j^1)} + \epsilon \langle \partial_t u_j^1|_{\Gamma_0}, \phi_j^0|_{\Gamma_0} \rangle_{H^1(\Gamma_0), H^1(\Gamma_0)}.
\]

Proposition 1. There exists a unique weak solution \(u_\epsilon = (u_1^\epsilon, u_2^\epsilon)\) of the microscopic problem (1).

Proof. This is an easy consequence of the Galerkin-method and the Leray-Schauder principle, where we have to use similar estimates as in Proposition 2 below. The uniqueness follows from standard energy estimates. For more details see [15].

3.3 A priori estimates

We derive a priori estimates for the microscopic solution depending explicitly on \(\epsilon\). These estimates are necessary for the application of the two-scale compactness results from Section 3 to derive the macroscopic model. In a first step, we give estimates in the spaces \(L^2((0, T), H_{\text{div}}^s)\) and \(H^1((0, T), H_{\text{div}}^s)\). Such kind of estimates are also needed to establish the existence of a weak solution via the Galerkin-method. In a second step, we derive estimates for the difference of shifted microscopic solution with respect to the macroscopic variable. These estimates are necessary for strong two-scale compactness results in the disconnected domain.

The following trace inequality for perforated domains will be used frequently throughout the paper and follows easily by a standard decomposition argument and the trace inequality on the reference element \(Y_j\), see also [21] Theorem II.4.1 and Exercise II.4.1: For every \(\theta > 0\) there exists a \(C(\theta) > 0\) such that for every \(v_\epsilon \in H^1(\Omega_j^1)\) it holds that
\[
\|v_\epsilon\|_{L^2(\Gamma_\epsilon)} \leq \frac{C(\theta)}{\sqrt{\epsilon}} \|v_\epsilon\|_{L^2(\Omega_j^1)} + \theta \sqrt{\epsilon} \|\nabla v_\epsilon\|_{L^2(\Omega_j^1)}.
\]
(4)

Proposition 2. The weak solution \(u_\epsilon = (u_1^\epsilon, u_2^\epsilon)\) of the microscopic problem (1) fulfills the following a priori estimate
\[
\|\partial_t u_1^\epsilon\|_{L^2((0, T), H_{\text{div}}^s)} + \|u_2^\epsilon\|_{L^2((0, T), H_{\text{div}}^s)} \leq C,
\]
for a constant \(C > 0\) independent of \(\epsilon\).
Proof. We choose $u_j^\varepsilon$ as a test-function in (3) (for $j = 1, 2$) to obtain with the Assumptions ([A3]) and ([A4]) on $f_j$ and $h^j$

$$\frac{1}{2} \frac{d}{dt} \left\| u_j^\varepsilon \right\|_{L^2_k}, + (D^j f_j \nabla u_j^\varepsilon, \nabla u_j^\varepsilon)_{\Omega^j} + \epsilon (D^j \nabla \nabla u_j^\varepsilon, u_j^\varepsilon)_{\Gamma^j} = (f_j^\varepsilon (u_j^\varepsilon), u_j^\varepsilon)_{\Omega^j} + \epsilon (h_j^\varepsilon (u_j^\varepsilon), u_j^\varepsilon)_{\Gamma^j} \leq C \left( 1 + \left\| u_j^\varepsilon \right\|_{L^2(\Omega^j)}^2 + \epsilon \left\| u_j^\varepsilon \right\|_{L^2(\Gamma^j)}^2 + \epsilon \left\| \nabla u_j^\varepsilon \right\|_{L^2(\Gamma^j)}^2 \right)$$

Using the coercivity of $D^j f_j$ and $D^j f_j$ from the Assumptions ([A1]) and ([A2]) we obtain for $j = 1, 2$

$$\frac{d}{dt} \left\| u_j^\varepsilon \right\|_{L^2_k}, + \epsilon \left\| \nabla u_j^\varepsilon \right\|_{L^2(\Omega^j)}^2 \leq C \left( 1 + \left\| u_j^\varepsilon \right\|_{L^2(\Omega^j)}^2 + \epsilon \left\| u_j^\varepsilon \right\|_{L^2(\Gamma^j)}^2 \right).$$

Summing over $j = 1, 2$, integrating with respect to time, the Assumption ([A5]) and the Gronwall-inequality implies the boundedness of $\left\| u_j^\varepsilon \right\|_{L^2((0,T),H^1_\Omega)}$ uniformly with respect to $\varepsilon$.

It remains to check the bound for the time-derivative $\partial_t u_j^\varepsilon$. As a test-function in (3) we choose $\phi_j^\varepsilon \in H_{\text{div}}$ with $\left\| \phi_j^\varepsilon \right\|_{H_{\text{div}}} \leq 1$ to obtain (using the boundedness of the diffusion tensors and again the growth condition for $h^j$ and $f_j$):

$$\langle \partial_t u_j^\varepsilon, \phi_j^\varepsilon \rangle_{H^1_{\text{div}}} \leq C \left( \left\| u_j^\varepsilon \right\|_{H^1_{\text{div}}} + \left\| u_j^\varepsilon \right\|_{H^2_{\text{div}}} \right) \left\| \phi_j^\varepsilon \right\|_{H^1_{\text{div}}} \leq C \left( \left\| u_j^\varepsilon \right\|_{H^1_{\text{div}}} + \left\| u_j^\varepsilon \right\|_{H^2_{\text{div}}} \right).$$

Squaring, integrating with respect to time, and the boundedness of $u_j^\varepsilon$ for $j = 1, 2$ already obtained above implies the desired result.

Next, we derive estimates for the difference of the shifted functions. First of all, we introduce some additional notations. For $h > 0$ let us define

$$\Omega_h := \{ x \in \Omega : \text{dist}(x, \partial \Omega) > h \},$$

$$K^\varepsilon_h := \{ k \in \mathbb{Z}^n : \varepsilon (Y + k) \in \Omega_h \},$$

$$\Omega^2_{h, \varepsilon} := \bigcup_{k \in K^\varepsilon_h} \varepsilon (Y + k),$$

and the related perforated domains and the related surface

$$\Omega^1_{h, \varepsilon} := \Omega_{h, \varepsilon} \setminus \Omega^2_{h, \varepsilon}, \quad \Omega_{h, \varepsilon} := \Omega^1_{h, \varepsilon} \setminus \Omega^2_{h, \varepsilon}.$$ 

For $l \in \mathbb{Z}^n$ with $|l| < h$ and $G_{e, h} \in \{ \Omega_{h, \varepsilon} \cap \Omega_{h, \varepsilon}^1 \},$ we define for an arbitrary function $v_\varepsilon : G_{e, h} \to \mathbb{R}$ the shifted function

$$v_l^\varepsilon(x) := v_\varepsilon(x + le),$$

and the difference between the shifted function and the function itself

$$\delta_l v_\varepsilon(x) := \delta v_\varepsilon(x) := v_l^\varepsilon(x) - v_\varepsilon(x) = v_\varepsilon(x + le) - v_\varepsilon(x).$$

Here, in the writing $\delta v_\varepsilon$ we neglect the dependence on $l$ if it is clear from the context. Further, we define $\mathbb{H}_{j, \varepsilon}^l$ in the same way as $\mathbb{H}_{j, \varepsilon}$ in (2) by replacing $\Omega^l_{1, \varepsilon}$ and $\Gamma^l_{1, \varepsilon}$ with $\Omega^l_{1, \varepsilon}$ and $\Gamma^l_{1, \varepsilon}$. In the same way we define $L_{j, \varepsilon}^l$. Further, for any function $\phi_{e, h} \in \mathbb{H}_{2, \varepsilon}^l$ we write $\overline{\phi}_{e, h}$ for the zero extension to $\Omega^l_{2, \varepsilon}$. Especially it holds that $\overline{\phi}_{e, h} \in \mathbb{H}_{2, \varepsilon}^l$, since $\Omega^l_{2, \varepsilon}$ is disconnected.
Proposition 3. Let $0 < h \ll 1$, then for all $l \in \mathbb{Z}^n$ with $|\ell| < h$, it holds that
\[
\|\delta u^2_l\|_{L^\infty((0,T),L^2,\Omega^2_{s,x})} + \|D\delta u^2_l\|_{L^2((0,T),L^2,\Omega^2_{s,x})} \\
\leq C \left(\|\delta u^1_l\|_{L^\infty((0,T),L^2,\Omega^1_{s,x})} + \|\delta u^2_l\|_{L^2(\Gamma_{s,x})} + \|\nabla u^2_l\|_{L^2(\Omega^2_{s,x})} + \epsilon, \right)
\]
for a constant $C > 0$ independent of $h$, $\epsilon$, and $l$.

Proof. Let $0 < h \ll 1$ and $l \in \mathbb{Z}^n$ with $|\ell| < h$, and we shortly write $u^{2,l}_l := (u^2_l|_{\Omega^2_{s,x}})^t$, i.e., the shifts with respect to $\ell$ of the restriction $u^2_l|_{\Omega^2_{s,x}}$ (we neglect the index $h$).

In the same way we define $u^{1,l}_l$. Let $\phi_{s,x} \in \mathbb{H}^2_{s,x}$. Then, for $x \in \Omega^2_{s,x} \setminus (\Omega^2_{s,x} + \ell)$ it holds that $x - \ell \notin \Omega^2_{s,x}$ and therefore $\overline{\phi}^{l,x}_l(x) = 0$ and similar from $x \in \Gamma_{s,x} \setminus (\Gamma_{s,x} + \ell)$ it follows $\overline{\phi}^{l,x}_l(x) = 0$. This implies for all $\psi \in C^\infty_0(0,T)$
\[
\int_0^T \left(\int_{\Omega^2_{s,x}} u^{2,l}_l(t,x) \phi_{s,x}(x) dx + \epsilon \int_{\Gamma_{s,x}} u^{2,l}_l(t,x) \phi_{s,x}(x) d\sigma \right) \psi(t) dt \\
= \int_0^T \left(\int_{\Omega^2_{s,x}} u^{2,l}_l(t,x) \phi^{l,x}_l(x) dx + \epsilon \int_{\Gamma_{s,x}} u^{2,l}_l(t,x) \phi^{l,x}_l(x) d\sigma \right) \psi(t) dt \\
= \int_0^T \left(\int_{\Omega^2_{s,x}} u^{2,l}_l(t,x) \phi^{l,x}_l(x) dx + \epsilon \int_{\Gamma_{s,x}} u^{2,l}_l(t,x) \phi^{l,x}_l(x) d\sigma \right) \psi(t) dt \\
= \int_0^T \left(\int_{\Omega^2_{s,x}} u^{2,l}_l(t,x) \phi^{l,x}_l(x) dx + \epsilon \int_{\Gamma_{s,x}} u^{2,l}_l(t,x) \phi^{l,x}_l(x) d\sigma \right) \psi(t) dt
\]
Hence, we have $\partial_t u^{2,l}_l \in L^2((0,T),\mathbb{H}^1_{2,s,x})$ with
\[
\partial_t u^{2,l}_l, \phi_{s,x} \in \mathbb{H}^2_{2,s,x}
\]
almost everywhere in $(0,T)$. Using $\phi^{l,x}_l \in \mathbb{H}^2_{2,s,x}$ as a test-function in (3), we obtain using the periodicity of $D^2$, $D^2_t$, $f^2$, and $h^2$, by an elemental calculation
\[
\partial_t u^{2,l}_l, \phi_{s,x} \in \mathbb{H}^2_{2,s,x}
\]
Subtracting the above equation for $l = 0$ and arbitrary $l \in \mathbb{Z}^n$ with $|\ell| < h$ we obtain
\[
\partial_t \delta u^2_l, \phi_{s,x} \in \mathbb{H}^2_{2,s,x}
\]
Choosing $\phi_{s,x} := \delta u^2_l$ (more precisely we take the restriction of $u^2_l$ to $\Omega^2_{s,x}$) we obtain with the coercivity of $D^2$ and $D^2_t$, as well as the Lipschitz continuity of $f^2$ and $h^2$
\[
\frac{1}{2} \frac{d}{dt} \|\delta u^2_l\|^2_{L^2,\Omega^2_{s,x}} + c_0 \|\nabla \delta u^2_l\|^2_{L^2,\Omega^2_{s,x}} \leq C \left(\|\delta u^2_l\|^2_{L^2,\Omega^2_{s,x}} + \epsilon \sum_{j=1}^2 \|\delta u^2_l\|^2_{L^2,\Gamma_{s,x}} \right)
\]
\[
\leq \epsilon \left(\sum_{j=1}^2 \|\delta u^2_l\|^2_{L^2,\Omega^2_{s,x}} + \epsilon \|\nabla \delta u^2_l\|^2_{L^2,\Omega^2_{s,x}} \right),
\]
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for arbitrary \( \theta > 0 \), where we used the trace-inequality (4). Choosing \( \theta \) small enough the gradient term for \( j = 2 \) can be absorbed from the left-hand side. Integrating with respect to time, using the \textit{a priori} estimates from Proposition 2 for the gradients of \( u_j^\varepsilon \), as well as the Gronwall-inequality, we obtain the desired result. \( \square \)

4 Two-scale compactness results

In this section we prove general strong two-scale compactness results for functions in the space \( L^2((0, T), H^p_j,_{\text{loc}}) \cap H^4((0, T), H^p_j,_{\text{loc}}) \) for \( j \in \{1, 2\} \) based on suitable \textit{a priori} estimates. These estimates are fulfilled by the microscopic solution \( u_\varepsilon = (u_j^1, u_j^2) \) which fulfills Proposition 2 and Proposition 3, but are not restricted to them. The connected and disconnected case are completely different and are therefore treated separately. These strong compactness results are enough to pass to the limit in the nonlinear terms in the microscopic equation (3), in fact we have:

Lemma 1. Let \( p \in (1, \infty) \).

(i) For \( j \in \{1, 2\} \) let \( (u_j^\varepsilon) \subset L^p((0, T) \times \Omega^j) \) be a sequence converging strongly in the two-scale sense to \( u_j^0 \in L^p((0, T) \times \Omega \times Y_j) \). Further \( f : [0, T] \times Y_j \times \mathbb{R} \to \mathbb{R} \) is continuous, \( Y \)-periodic with respect to the second variable, and fulfills the growth condition

\[
|f(t, y, z)| \leq C(1 + |z|) \quad \text{for all } (t, y, z) \in [0, T] \times Y_j \times \mathbb{R}.
\]

Then it holds up to a subsequence

\[
f\left(t, \frac{\cdot}{\varepsilon}, u_j^\varepsilon\right) \to f\left(t, \cdot, u_j^0\right) \quad \text{in the two-scale sense in } L^p.
\]

(ii) Let \( (u_\varepsilon) \subset L^p((0, T) \times \Gamma,_{\text{loc}}) \) be a sequence converging strongly in the two-scale sense on \( \Gamma,_{\text{loc}} \) to \( u_0 \in L^p((0, T) \times \Omega \times \Gamma) \). Further \( h : [0, T] \times \Gamma \times \mathbb{R} \to \mathbb{R} \) is continuous, \( Y \)-periodic with respect to the second variable, and fulfills the growth condition

\[
|h(t, y, z)| \leq C(1 + |z|) \quad \text{for all } (t, y, z) \in [0, T] \times \Gamma \times \mathbb{R}.
\]

Then it holds up to a subsequence

\[
h\left(t, \frac{\cdot}{\varepsilon}, u_\varepsilon\right) \to h\left(t, \cdot, u_0\right) \quad \text{in the two-scale sense on } \Gamma,_{\text{loc}} \text{ in } L^p.
\]

We emphasize that for functions \( f \) and \( h \) uniformly Lipschitz-continuous with respect to the last variable, the growth conditions are fulfilled. For such Lipschitz-continuous functions we also easily obtain the strong two-scale convergence of the whole sequence.

Proof. We only prove (ii). The other statement follows the same way. Due to Lemma 1 the sequence \( T_{\varepsilon}u_\varepsilon \) converges in \( L^p((0, T) \times \Omega \times \Gamma) \) to \( u_0 \). Hence, up to a subsequence, \( T_{\varepsilon}u_\varepsilon \to u_0 \) almost everywhere in \( (0, T) \times \Omega \times \Gamma \). Further we have

\[
T_{\varepsilon}\left(h\left(t, \frac{\cdot}{\varepsilon}, u_\varepsilon\right)\right) = h\left(t, y, T_{\varepsilon}u_\varepsilon\right) \to h\left(t, \cdot, u_0\right) \quad \text{a.e. in } (0, T) \times \Omega \times \Gamma.
\]
The growth condition on \( h \) implies \( \mathcal{T}_e \left( h \left( t, \frac{x}{\epsilon}, u_e \right) \right) \) bounded in \( L^p((0,T) \times \Omega \times \Gamma) \). Egorov’s theorem (see also [23, Theorem 13.44]) implies

\[
\mathcal{T}_e \left( h \left( t, \frac{x}{\epsilon}, u_e \right) \right) \rightharpoonup h(\cdot, \cdot, u_0) \quad \text{weakly in} \quad L^p((0,T) \times \Omega \times \Gamma).
\]

As a direct consequence we obtain by density:

**Lemma 2.** Let \( f \) be as in Lemma [7]. Further, let \( \{v_\epsilon\} \) be a bounded sequence in \( L^2((0,T) \times \Omega_1^\epsilon) \) which converges strongly in the \( L^p \)-two-scale sense to \( v_0 \in L^2((0,T) \times \Omega \times Y) \) for \( p \in [1,2) \). Then it holds that

\[
f \left( t, \frac{x}{\epsilon}, v_\epsilon(x) \right) \rightharpoonup f(\cdot, \cdot, v_0) \quad \text{in the two-scale sense in} \quad L^2.
\]

A similar result holds on the oscillating surface \( \Gamma_\epsilon \).

### 4.1 The connected domain \( \Omega_1^\epsilon \)

Here we give a strong compactness result for a sequence in the connected domain \( \Omega_1^\epsilon \) under suitable \textit{a priori} estimates. The case of a connected perforated domain can be treated more easily than a disconnected domain, because we can extend a bounded sequence in \( H^1(\Omega_\epsilon) \) to a bounded sequence in \( H^1(\Omega) \), due to [11,12]. Hence, we can work in fixed Bochner spaces (not depending on \( \epsilon \)) and use standard methods from functional analysis. For this we need control for the time-variable, which can be obtained from the uniform bound of the time-derivative \( \partial_t u_\epsilon^i \). However, since \( \partial_t u_\epsilon^i \) is pointwise only an element in the space \( H^1(\Omega_\epsilon) \), it is not clear if the time-derivative of the extension of \( u_\epsilon^i \) exists and if it is bounded uniformly with respect to \( \epsilon \) (Unfortunately, this circumstance is often overseen in the existing literature). The following Lemma gives us an estimate for the difference of the shifts with respect to time for functions with generalized time-derivative. It is just an easy generalization of [10, Lemma 9].

**Lemma 3.** Let \( V \) and \( H \) be Hilbert spaces and we assume that \((V,H,V')\) is a Gelfand-triple. Let \( v \in L^2((0,T),V) \cap H^1((0,T),V') \). Then, for every \( \phi \in V \) and almost every \( t \in (0,T), s \in (-T,T) \), such that \( t + s \in (0,T) \), we have

\[
\left| \left( v(t+s) - v(t), \phi \right)_H \right| \leq \sqrt{|s|} \left\| \partial_t v \right\|_{L^2((t,t+s),V')} \left\| \phi \right\|_V.
\]

Especially, it holds that

\[
\left\| v(t+s) - v(t) \right\|_H^2 \leq \sqrt{|s|} \left\| \partial_t v \right\|_{L^2((t,t+s),V')} \left\| \phi \right\|_V.
\]

**Proof.** The proof follows the same lines as the proof of [10, Lemma 9], if we replace the Gelfand-triple \((H^1(\Omega_\epsilon),L^2(\Omega_\epsilon),H^1(\Omega_\epsilon)')\) by the Gelfand-triple \((V,H,V')\).

In the following, for \( v_\epsilon \in H^1(\Omega_\epsilon^\epsilon) \) we denote by \( \tilde{v}_\epsilon \in H^1(\Omega) \) the extension from [11,12] with

\[
\left\| \tilde{v}_\epsilon \right\|_{L^2(\Omega)} \leq C \left\| v_\epsilon \right\|_{L^2(\Omega_\epsilon^\epsilon)}, \quad \left\| \nabla \tilde{v}_\epsilon \right\|_{L^2(\Omega)} \leq C \left\| \nabla v_\epsilon \right\|_{L^2(\Omega_\epsilon^\epsilon)},
\]

with a constant \( C > 0 \) independent of \( \epsilon \).
Proposition 4. Let \((v_r) \in L^2((0,T), \mathbb{H}_{1,r}) \cap H^1((0,T), \mathbb{H}'_{1,r})\) be a sequence with
\[
\|\partial_t v_r\|_{L^2((0,T), \mathbb{H}'_{1,r})} + \|v_r\|_{L^2((0,T), \mathbb{H}_{1,r})} \leq C. \tag{6}
\]
There exists \(v_0 \in L^2((0,T), H^1(\Omega))\) such that for all \(\beta \in \left(\frac{1}{2}, 1\right)\) up to a subsequence it holds that
\[
\tilde{v}_r \to v_0 \quad \text{in} \quad L^2((0,T), H^\beta(\Omega)).
\]
Further, it holds that (up to a subsequence)
\[
\mathcal{T}_\epsilon v_r \to v_0 \quad \text{in} \quad L^2((0,T) \times \Omega, \mathbb{H}_{1}).
\]

Proof. Since \(\tilde{v}_r\) is bounded in \(L^2((0,T), H^1(\Omega))\) there exists \(v_0 \in L^2((0,T), H^1(\Omega))\), such that up to a subsequence \(v_r\) converges weakly to \(v_0\) in \(L^2((0,T), H^1(\Omega))\). Lemma \(\mathcal{3}\) and inequality \(\mathcal{4}\) imply for \(0 < h < 0\)
\[
\int_0^{T-h} \|v_r(t+h) - v_r\|_{L^2_\epsilon(\Omega)}^2 \, dt \leq C \sqrt{T} \|\partial_t v_r\|_{L^2((0,T), \mathbb{H}'_{1,r})} \int_0^{T-h} \|v_r(t+h) - v_r\|_{\mathbb{H}_{1,r}} \, dt \leq C \sqrt{T}.
\]

Now, from the properties of the extension \(\tilde{v}_r\) we obtain
\[
\int_0^{T-h} \|	ilde{v}_r(t+h) - \tilde{v}_r\|_{L^2(\Omega)}^2 \, dt \leq C \int_0^{T-h} \|v_r(t+h) - v_r\|_{L^2(\Omega)}^2 \, dt \leq C \sqrt{T}.
\]

Since \(H^1(\Omega) \to H^\beta(\Omega)\) is compact for \(\beta \in \left(\frac{1}{2}, 1\right)\) we can apply \(\mathcal{5}\) Theorem 1 to \((\tilde{v}_r)\) as a sequence in \(L^2((0,T), H^\beta(\Omega))\) and obtain the strong convergence of \(\tilde{v}_r\) to \(v_0\) in \(L^2((0,T), H^\beta(\Omega))\).

Now we prove the convergence of \(\mathcal{T}_\epsilon v_r\). It holds that
\[
\|\mathcal{T}_\epsilon v_r - v_0\|_{L^2((0,T) \times \Omega, \mathbb{H}_{1})} = \|\mathcal{T}_\epsilon v_r - v_0\|_{L^2((0,T) \times \Omega, H^1(\Omega_1))} + \|\mathcal{T}_\epsilon v_r - v_0\|_{L^2((0,T) \times \Omega, H^1(\Gamma))}.
\]

We only prove the convergence to zero for the second term, since the first one can be treated in a similar way. We obtain from the properties of the unfolding operator from Lemma \(\mathcal{6}\) the trace inequality, and the inequality \(\mathcal{7}\)
\[
\|\mathcal{T}_\epsilon v_r - v_0\|_{L^2((0,T) \times \Omega, H^1(\Omega_1))} \leq C \left[\|\mathcal{T}_\epsilon v_r - v_0\|_{L^2((0,T) \times \Omega, H^1(\Omega))} + C \|\nabla_{\epsilon,y}\mathcal{T}_\epsilon v_r\|_{L^2((0,T) \times \Omega, \mathbb{H}_{1})}\right]
\leq C \left[\|v_r - v_0\|_{L^2((0,T) \times \Omega, H^1(\Omega))} + \|\mathcal{T}_\epsilon v_r - v_0\|_{L^2((0,T) \times \Omega, H^1(\Omega))} + C \|\nabla_{\epsilon,y}\mathcal{T}_\epsilon v_r\|_{L^2((0,T) \times \Omega, \mathbb{H}_{1})}\right]
\leq C \left[\|v_r - v_0\|_{L^2((0,T) \times \Omega, H^1(\Omega))} + \|\mathcal{T}_\epsilon v_r - v_0\|_{L^2((0,T) \times \Omega, H^1(\Omega))} + \epsilon\right] + C \|\nabla_{\epsilon,y}\mathcal{T}_\epsilon v_r\|_{L^2((0,T) \times \Omega, \mathbb{H}_{1})} + \epsilon.
\]

The first term converges to zero for \(\epsilon \to 0\), due to the strong convergence of \(\tilde{v}_r\) to \(v_0\), and the second term because of \(\mathcal{11}\) Prop. 4.4. This gives the desired result. \(\Box\)

Remark 1.

(i) The extension of a bounded sequence \(v_r \in L^2((0,T), H^1(\Omega_1))\) to \(\tilde{v}_r \in L^2((0,T), H^1(\Omega))\) preserving the boundedness is only possible for the connected domain \(\Omega_1\). This implies in a simple way the strong convergence of \(\tilde{v}_r\), if there is also a control for the time-derivative. For the disconnected domain this method fails and therefore we use a Kolmogorov-Simon-compactness result for the unfolded sequence, where we need an additional condition to control the shifts with respect to the macroscopic variable, see Theorem \(\mathcal{7}\).
(ii) If we replace in the assumptions of Proposition the space $H^1(\Omega_2')$, we obtain the well known result that a bounded sequence in $L^2((0,T), H^1(\Omega_2')) \cap H^1((0,T), H^1(\Omega_2'))$ has an extension converging strongly in $L^2((0,T) \times \Omega)$, see [22], and also in $L^2((0,T), H^2(\Omega))$ for $\beta \in (\frac{1}{2}, 1)$. The proof above for the strong convergence of $v_\varepsilon$ can be easily adapted to this situation. However, we emphasize that in our situation we cannot guarantee for the solution $(u^1_\varepsilon, u^2_\varepsilon)$ that $\partial_t u^1_\varepsilon \in L^2((0,T), H^1(\Omega_2'))$.

4.2 The disconnected domain $\Omega_2'\varepsilon$

In this section we give a strong two-scale compactness result for the disconnected domain $\Omega_2'$ of Kolmogorov-Simon-type, i.e., it is based on a priori estimates for the difference of discrete shifts, see condition \((\text{iii})\) in Theorem 4. As already mentioned above it is in general not possible to find an extension for a function in $H^1(\Omega_2')$ to the whole domain $\Omega$ which preserves the a priori estimates. Hence, the method from Section \([4,4]\) for the connected domain fails. Therefore, we consider the unfolded sequence in the Bochner space $L^p(\Omega, L^2((0,T) \times \Omega_2'))$ with $\beta \in (\frac{1}{2}, 1)$ and $p \in (1,2)$, and apply the Kolmogorov-Simon-compactness result from \([17]\), which gives an extension of \([30, \text{Theorem 1}]\) to higher-dimensional domains of definition. Here, a crucial point is the estimate for the shifts. An important reason to work here with general Bochner spaces, i.e., Banach-valued functions spaces, is that we are dealing with manifolds and therefore linear shifts with respect to the space variable are not well defined.

In the following Lemma we estimate the shifts of the unfolded sequence with respect to the macroscopic variable by the shifts of the function itself, see also Section \([5,3]\) for the notations.

**Lemma 4.** Let $v_\varepsilon \in L^2((0,T) \times \Omega_2')$ for $j \in \{1,2\}$ and $w_\varepsilon \in L^2((0,T) \times \Gamma_j)$. Then, for $0 < \varepsilon \ll 1$, $|z| < h$, and $\varepsilon$ small enough it holds that

\[
\left\| T_{\varepsilon} v_\varepsilon(t, x + z, y) - T_{\varepsilon} v_\varepsilon \right\|_{L^2(0,T) \times \Omega_{2h \times Y_j}} \leq \sum_{k \in \{0,1\}^n} \left\| \delta v_\varepsilon \right\|_{L^2(0,T) \times \Omega_{2h \times Y_j}},
\]

\[
\left\| T_{\varepsilon} w_\varepsilon(t, x + z, y) - T_{\varepsilon} w_\varepsilon \right\|_{L^2(0,T) \times \Omega_{2h \times Y_j}} \leq \sum_{k \in \{0,1\}^n} \left\| \delta w_\varepsilon \right\|_{L^2(0,T) \times \Gamma_j},
\]

with $l = l(\varepsilon, z, k) = k + \lceil \frac{z}{h} \rceil$.

**Proof.** The proof for a thin layer can be found in \([27, \text{p. 709}]\) and can be easily extended to our setting. See also \([19]\) for more details.

**Theorem 1.** Let $v_\varepsilon \in L^2((0,T), \mathbb{H}_{2,\varepsilon}) \cap H^1((0,T), \mathbb{H}_{2,\varepsilon}')$ with:

(i) It holds the estimate

\[
\left\| v_\varepsilon \right\|_{L^2((0,T), \mathbb{H}_{2,\varepsilon})} + \left\| \partial_t v_\varepsilon \right\|_{L^2((0,T), \mathbb{H}_{2,\varepsilon}')} \leq C.
\]

(ii) For $0 < \varepsilon \ll 1$ and $l \in \mathbb{Z}^n$ with $|\varepsilon| < h$ it holds that

\[
\left\| \delta v_\varepsilon \right\|_{L^2((0,T), L^2(\Omega_{2h}))} \to 0.
\]
Then, there exists \( v_0 \in L^2((0,T), L^2(\Omega)) \), such that for \( \beta \in \left( \frac{1}{p}, 1 \right) \) and \( p \in [1, 2) \) it holds up to a subsequence that
\[
\mathcal{T}_t v_e \rightharpoonup v_0 \quad \text{in } L^p(\Omega, L^2((0,T), H^2_\alpha)).
\]

Especially, \( v_e \) and \( v_e|_{\Gamma_e} \) converge strongly in the two-scale sense to \( v_0 \) (with respect to \( L^p \)).

**Proof.** Our aim is to apply [17, Corollary 2.5] to \( (\mathcal{T}_t v_e) \) as a sequence in \( L^p(\Omega, L^2((0,T), H^2_\alpha)) \) for \( p \in [1,2) \) and \( \beta \in \left( \frac{1}{2}, 1 \right) \). Hence, we have to check the following three conditions:

(K1) For every measurable set \( A \subset \Omega \) the sequence \( \{\int_A v_t \, dx\} \) is relatively compact in \( L^2((0,T), H^2_\alpha) \),

(K2) for all \( 0 < h \ll 1 \) and \( |z| < h \) it holds that
\[
\sup_{\epsilon} |v_t(\cdot + z) - v_t|_{L^p(\Omega_h,B)} \to 0 \quad \text{for } z \to 0,
\]

(K3) for \( h > 0 \) it holds that \( \sup_{\epsilon} \int_{\Omega \setminus \Omega_h} |v_t(x)|^p \, dx \to 0 \) for \( h \to 0 \).

We start with the condition (K1). Let \( A \subset \Omega \) be measurable and we define \( V_\epsilon(t,y) := \int_A \mathcal{T}_t v_e(t,x,y) \, dx \). To show the relative compactness of \( (V_\epsilon) \), we use again [30, Theorem 1] as in the proof of Proposition 4. First of all, due to our assumptions on \( v_e \) and the properties of the unfolding operator, for \( t_1, t_2 \in (0,T) \) it holds that
\[
\left\| \int_{t_1}^{t_2} V_\epsilon \, dt \right\|_{H^2_\alpha} \leq \left\| \mathcal{T}_t v_e \right\|_{L^2((0,T) \times \Omega, H^2_\alpha)} \leq C.
\]

Due to the compact embedding \( H^2_\alpha \hookrightarrow H^1_\alpha \) we obtain that \( \int_{t_1}^{t_2} V_\epsilon \, dt \) is relatively compact in \( H^1_\alpha \). Further, for \( 0 < s \ll 1 \) we obtain with the estimates for \( v_e \) and the trace inequality [1]
\[
\left| V_\epsilon(t+s,y) - V_\epsilon \right|_{L^2((0,T-s), H^2_\alpha)} \leq \left| \mathcal{T}_t v_e(t+s,x,y) - \mathcal{T}_t v_e \right|_{L^2((0,T-s) \times \Omega, H^2_\alpha)}
\]
\[
\leq C \left| v_t(t+s,x) - v_t \right|_{L^2((0,T-s) \times \Omega, H^2_\alpha)} + C \left\| \nabla v_t(t+s,x) - \nabla v_t \right\|_{L^2((0,T-s) \times \Omega, H^2_\alpha)}
\leq C \left( s^{\frac{3}{2}} + \epsilon \right),
\]

where for the last inequality we used Lemma 3 to estimate the first term in the line before by using the same arguments as for the inequality [1] in the proof of Proposition 4. Hence, [30, Theorem 1] implies that \( (V_\epsilon) \) is relatively compact in \( L^2((0,T), H^1_\alpha) \), i.e., condition (K1).

For (K2) we fix \( 0 < h \ll 1 \) and choose \( |z| < h \). Lemma 4 with \( l = k + \left[ \frac{z}{h} \right] \) (see the definition of the difference \( \delta \) in (3)), the conditions (i) and (ii) as well as the trace inequality [13] imply
\[
\left\| \mathcal{T}_t v_e(t,x+z,y) - \mathcal{T}_t v_e \right\|_{L^2(\Omega_{2h}, L^2((0,T), H^2_\alpha))}
\leq C \sum_{k \in \{0,1\}^n} \left( \left| \delta v_t \right|_{L^2((0,T) \times \Omega^2_{\alpha,k})} + \epsilon \left\| \nabla v_t \right\|_{L^2((0,T) \times \Omega^2_{\alpha,k})} + 2 \left\| \delta \nabla v_t \right\|_{L^2((0,T) \times \Gamma_{\alpha,k})} \right)
\leq C \left( \sum_{j \in \{0,1\}^n} \left| \delta v_t \right|_{L^2((0,T) \times \Omega^2_{\alpha,k})} + \epsilon \right) \xrightarrow{\epsilon \to 0} 0.
\]
We show that this implies the uniform convergence in (K2) with respect to $\epsilon$, see also [27] p.710-711 or [14] p.1476-1477. Let $0 < \rho$. Due to our previous results there exist $0 < \epsilon_0, \delta_0$, such that for all $\epsilon \leq \epsilon_0$ and $|z| \leq \delta_0$ it holds that
\[
\|T_\epsilon v_\epsilon(t,x+z,y) - T_\epsilon v_\epsilon\|_{L^2(\Omega_{z\to}(0,T),H^2_z)} \leq \rho.
\] (8)

Since $\epsilon^{-1} \in \mathbb{N}$, there are only finitely many elements $\epsilon_i$ with $i = 1, \ldots, N$, such that $\epsilon_0 < \epsilon_i$. For every $\epsilon_i$ there exists a $0 \leq \delta_i$, such that (8) is valid for $\epsilon = \epsilon_i$ and all $|z| \leq \delta_i$. Choosing $\delta := \max_{i \leq i_0} \sqrt{\delta_i}$, inequality (8) holds uniformly with respect to $\epsilon$ for all $|z| \leq \delta$. This implies (K2). For the last condition (K3) we use the Hölder-inequality to obtain for $\rho \in [1,2]$ and $0 < h \ll 1$
\[
\|T_\epsilon v_\epsilon\|_{L^p(\Omega,\Omega_{z\to}(0,T),H^2_z)} \leq \|\nabla v_\epsilon\|_{L^p(\Omega,\Omega_{z\to}(0,T),H^2_z)} \leq \|\nabla v_\epsilon\|_{L^p(\Omega,\Omega_{z\to}(0,T),H^2_z)} \leq C h^{p-1} \frac{h}{h} \to 0,
\]
where we used again estimate (i) Now, [14] Corollary 2.5) implies the strong convergence of $T_\epsilon v_\epsilon$ up to a subsequence in $L^p(\Omega,\Omega_{z\to}(0,T),H^2_z)$ to a limit function $v_0$. Lemma 7 implies the strong two-scale convergence of $v_\epsilon$ to the same limit. The fact $v_0 \in L^2((0,T),L^2(\Omega))$ follows from standard two-scale compactness results, see [2], based on the estimate (i)

Remark 2. Theorem 7 and its proof remain valid if we replace $\mathbb{H}_{2,\epsilon}$ and $\mathbb{H}_3$ by $H^1(\Omega_{z2})$ and $H^p(\Omega_{z2})$. 

5 Derivation of the macroscopic model

The aim of this section is the derivation of the macroscopic model [12] from Theorem 2 for $\epsilon \to 0$. Therefore we make use of compactness results from Section 4 and the a priori estimates from Section 3 In the following Proposition we collect the convergence results for the microscopic solution $u_\epsilon = (u_\epsilon^1, u_\epsilon^2)$:

Proposition 5. Let $u_\epsilon = (u_\epsilon^1, u_\epsilon^2)$ be the microscopic solution of the problem (1). There exist
\[
u_0^1 \in L^2((0,T),H^1(\Omega)), \quad u_\epsilon^1 \in L^2((0,T),H_1(\mathbb{R}), \quad u_\epsilon^2 \in L^2((0,T) \times \Omega),
\]
such that up to a subsequence it holds for $\rho \in [1,2]$
\[
\begin{align*}
&u_\epsilon \to u_0^1 \\
&\nabla u_\epsilon \to \nabla u_0^1 + \nabla y u_0^1 \\
&u_\epsilon^1|_{\Gamma} \to u_0^1 \\
&\nabla \nabla u_\epsilon \to \nabla \nabla u_0^1 + \nabla \nabla y u_0^1 \\
&u_\epsilon^2 \to u_0^2 \\
&\nabla u_\epsilon \to 0 \\
&u_\epsilon^2|_{\Gamma} \to u_0^2 \\
&\nabla \nabla u_\epsilon \to 0
\end{align*}
\]
strongly in the two-scale sense, strongly in the two-scale sense, strongly in the two-scale sense on $\Gamma$, strongly in the two-scale sense on $\Gamma_c$, strongly in the two-scale sense in $L^p$, strongly in the two-scale sense in $L^p$, strongly in the two-scale sense in $L^p$ on $\Gamma_c$, strongly in the two-scale sense on $\Gamma_c$, strongly in the two-scale sense on $\Gamma_c$, strongly in the two-scale sense on $\Gamma_c$. 

Proof. The convergence results (9a) - (9d) follow immediately from Proposition 1, Lemma 5 and the a priori estimates in Proposition 2.
For (10) we first notice that due to Lemma 5 there exists \( u_0^1 \in L^2((0, T) \times \Omega) \) and \( u_1^2 \in L^2((0, T) \times \Omega, H_2/R) \), such that up to a subsequence

\[
\begin{align*}
    u_0^2 &\to u_0^2 & \text{in the two-scale sense,} \\
    \nabla u_0^2 &\to \nabla_y u_1^2 & \text{in the two-scale sense,} \\
    u_1^2|_{\Gamma_\varepsilon} &\to u_0^2 & \text{in the two-scale sense on } \Gamma_\varepsilon, \\
    \nabla \Gamma, u_1^2|_{\Gamma_\varepsilon} &\to \nabla \Gamma u_1^2 & \text{in the two-scale sense on } \Gamma_\varepsilon.
\end{align*}
\]

For the strong two-scale convergence of \( u_0^2 \) and \( u_1^2|_{\Gamma_\varepsilon} \), we make use of Theorem 1 where we have to check the conditions (1) and (2). The first one is just the a priori estimate from Proposition 2. For (2) we use Proposition 3 to obtain for fixed \( 0 < h \ll 1 \) and \( l \in \mathbb{Z}^n \) with \( |l| < h \)

\[
\|\delta u_1^2\|_{L^2((0,T),L^2(\Omega^2))} \leq C \left( \|\delta u_1^1\|_{L^2((0,T),L^2(\Omega^2))} + \|\delta(u_1^1, u_1^2, u^2, \Gamma_\varepsilon, z)\|_{L^2(0,T \times \Omega \times \Gamma)} + \varepsilon \right).
\]

For the first term on the right-hand side we have

\[
\|\delta u_1^2\|_{L^2((0,T),L^2(\Omega^2))} \leq \|\mathcal{T}_l u_1^2(x + h, y) - \mathcal{T}_l u_1^1\|_{L^2((0,T) \times \Omega \times \Gamma)}.
\]

The right-hand side converges to zero, due to the strong two-scale convergence of \( u_1^2 \), i.e., the strong convergence of \( \mathcal{T}_l u_1^1 \) in \( L^2((0, T) \times \Omega \times \Omega_1) \), and the standard Kolmogorov-compactness theorem. For the \( L^2 \)-norm of \( \delta u_1^2 \), the second term we argue in a similar way, where we can use the strong two-scale convergence in the Assumption (A5). For the norm of \( \delta u_1^2 \), we can use the Kolmogorov-Simon compactness result from Corollary 2.5, applied to the strong convergent sequence \( (\mathcal{T}_l u_1^2, \mathcal{T}_l u_1^2, \mathcal{T}_l u_1^2) \) in \( L^2((0, T) \times \Omega, H_2) \).

To prove (10) and (11) we have to show that \( u_1^2 \) is constant with respect to \( y \). Therefore, we choose \( \phi_\epsilon(t, x) := \phi(t, x, \frac{x}{\epsilon}) \) with \( \phi \in C_0^\infty((0, T) \times \Omega \times \Omega_2) \) (periodically extended in the last variable) as a test-function in (3) for \( j = 2 \) and integrate with respect to time to obtain

\[
\begin{align*}
    \int_0^T (\partial_t u_2^2, \phi_\epsilon)_{\Omega_2} \, dt &+ \int_0^T \int_{\Omega_2} \partial_t^2 \nabla \phi_{\epsilon} \cdot \left( \epsilon \nabla_x \phi \left( t, x, \frac{x}{\epsilon} \right) + \nabla_y \phi \left( t, x, \frac{x}{\epsilon} \right) \right) \, dx \, dt \\
    &+ \epsilon \int_0^T \int_{\Gamma_\varepsilon} \partial_t^2 \nabla_y u_2^2 \cdot \left( \epsilon P_t \nabla_x \phi \left( t, x, \frac{x}{\epsilon} \right) + P_t \nabla_y \phi \left( t, x, \frac{x}{\epsilon} \right) \right) \, d\sigma \, dt \\
    &+ \epsilon \int_0^T \int_{\Omega_2} \partial_t \left( u_2^2 \phi \left( t, x, \frac{x}{\epsilon} \right) \right) \, dx \, dt + \epsilon \int_0^T \int_{\Omega_2} h_2^2(u_0^2, u_2^2) \phi \left( t, x, \frac{x}{\epsilon} \right) \, dx \, dt.
\end{align*}
\]

(10)

For the first term on the left-hand side including the time-derivative we get by integration by parts in time

\[
\begin{align*}
    \int_0^T (\partial_t u_2^2, \phi_\epsilon)_{\Omega_2} \, dt &= -\epsilon \int_0^T \int_{\Omega_2} \partial_t \phi \left( t, x, \frac{x}{\epsilon} \right) u_2^2 \, dx \, dt \\
    &- \epsilon \int_0^T \int_{\Gamma_\varepsilon} \partial_t \phi \left( t, x, \frac{x}{\epsilon} \right) u_2^2 \, d\sigma \, dt.
\end{align*}
\]

The right-hand side is of order \( \varepsilon \), due to the estimates in Proposition 2. Hence, all terms in (10) except the terms including the \( \nabla_y \) are of order \( \varepsilon \) (again because of
Proposition 2 and we obtain for $\epsilon \to 0$

$$
\int_0^T \int_{\Omega} D^2(y) \nabla_y u_1^2(t, x, y) \cdot \nabla_y \phi(t, x, y) dy dx dt
+ \int_0^T \int_{Y} D^2(y) \nabla_{\Gamma,y} u_1^2(t, x, y) \cdot \nabla_{\Gamma,y} \phi(t, x, y) d\sigma_y dx dt = 0.
$$

Due to the density of $C^\infty(\overline{Y_2})$ in $H_2$, see Lemma 2.1, the equation above holds for all $\phi \in L^2((0, T) \times \Omega, H_2)$. This implies $u_1^2 = 0$. The Proposition is proved.

We have the following representation of $u_1^1$:

**Corollary 1.** *Almost everywhere in $(0, T) \times \Omega \times Y_1$ it holds that*

$$
u(t, x, y) = \sum_{i=1}^n \partial_{x_i} u_1^0(t, x) w_i^1(y),
$$

*where $w_i^1 \in H_1/\mathbb{R}$ with $Y$-periodic boundary conditions is the unique weak solution of the following cell problem $(i = 1, \ldots, n)$

$$
-\nabla_y \cdot \left( D(\nabla_y w_i^1 + e_i) \right) = 0
$$

in $Y_1$,

$$
-\nabla_y w_i^1 + e_i \cdot \nu = -\nabla_{\Gamma,y} \cdot \left( D^1(\nabla_{\Gamma,y} w_i^1 + \nabla_{\Gamma,y} y_i) \right)
$$

on $\Gamma$,

$w_i^1$ is $Y$-periodic and $\int_Y w_i^1 d\sigma = 0$.

**Proof.** The procedure is quite standard, see e.g., [2], but for the sake of completeness we give the main steps. We choose $\phi_\epsilon(t, x) = \epsilon \phi(t, x, \frac{y}{\epsilon})$ with $\phi \in C^\infty(\overline{(0, T) \times \Omega, Y_1})$ as a test-function in (3) and integrate with respect to time to obtain (10) if we replace $j = 2$ by $j = 1$. From Proposition 3 we get for $\epsilon \to 0$

$$
0 = \int_0^T \int_{\Omega} \int_{Y_1} D^1(y) \left[ \partial_x u_1^0(t, x) + \nabla_y u_1^1(t, x, y) \right] \cdot \nabla_y \phi(t, x, y) dy dx dt
+ \int_0^T \int_{\Omega} \int_{Y} D^1(y) \left[ P_1(y) \nabla_x u_1^0(t, x) + \nabla_{\Gamma,y} u_1^1(t, x, y) \right] \cdot \nabla_{\Gamma,y} \phi(t, x, y) d\sigma_y dx dt.
$$

Due to the Lax-Milgram Lemma this problem has a unique solution $u_1^1$ and due to its linearity we easily obtain the representation (11).

Now, we are able to formulate the macroscopic model. We show that $u_0 = (u_0^1, u_0^0)$ from Proposition 3 is the unique weak solution (the definition of a weak solution is given below) of the macro-model

$$
(\Omega + |\Gamma|) \partial_t u_0^1 - \nabla \cdot (\bar{D} \nabla u_0^1) = \int_{Y_1} f_1(t, y, u_0^1) dy + \int_{\Omega} h_1(t, y, u_1^0, u_0^2) d\sigma_y
$$

in $(0, T) \times \Omega$,

$$
(\Omega + |\Gamma|) \partial_t u_0^2 = \int_{Y_2} f_2(t, y, u_0^2) dy + \int_{\Omega} h_2(t, y, u_1^0, u_0^2) d\sigma_y
$$

in $(0, T) \times \Omega$,

$$
-\bar{D} \nabla u_0^1 \cdot \nu = 0
$$

on $(0, T) \times \partial\Omega$.

$$
u_0^0(0) = \frac{|\Omega| u_0^1 + |\Gamma| u_0^2}{|\Omega| + |\Gamma|}
$$

in $\Omega$,

$$
(12)
$$
where the homogenized diffusion coefficient $\hat{D}^1 \in \mathbb{R}^{n \times n}$ is defined by $(i, l = 1, \ldots, n)$

$$(\hat{D}^1)_{il} := \int_{Y_1} D^1(\nabla_y w^i_1 + e_i) \cdot (\nabla_y w^l_1 + e_l) dy$$

$$+ \int_{\Gamma} D^1_l(\nabla_{\gamma, y} w^i_1 + \nabla_{\gamma, y} y_l) \cdot (\nabla_{\gamma, y} w^l_1 + \nabla_{\gamma, y} y_l) d\sigma,$$

and $w^i_1 \in H_1 / \mathbb{R}$ (see Section 5.1 for the definition of this space) for $i = 1, \ldots, n$ are the solutions of the cell problems

$$-\nabla_y \cdot (D^1(\nabla_y w^i_1 + e_i)) = 0 \quad \text{in } Y_1,$$

$$\quad -D^1(\nabla_y w^i_1 + e_i) \cdot \nu = -\nabla_{\gamma, y} \cdot (D^1_l(\nabla_{\gamma, y} w^i_1 + \nabla_{\gamma, y} y_l)) \quad \text{on } \Gamma,$$

(13)

$$w^i_1 \text{ is } Y \text{-periodic and } \int_{\Gamma} w^i_1 d\sigma = 0.$$

We say that $u_0 = (u^1_0, u^2_0)$ is a weak solution of the macroscopic model, if

$$u^1_0 \in L^2((0, T), H^1(\Omega)) \cap H^1((0, T), H^1(\Omega)),$$

$$u^2_0 \in L^2((0, T) \times \Omega) \cap H^1((0, T), L^2(\Omega)),$$

the equation for $\partial_t u^i_0$ in (12) is valid in $L^2((0, T) \times \Omega)$, and for all $\phi \in H^1(\Omega)$ it holds almost everywhere in $(0, T)$

$$\left( |Y_1| + |\Gamma| \right) \langle \partial_t u^i_0, \phi \rangle_{H^1(\Omega), H^1(\Omega)} + \int_{\Omega} \hat{D}^1 \nabla u^i_0 \cdot \nabla \phi dx$$

$$= \int_{\Omega} \int_{Y_1} f^i_1(y, u^1_0) \phi dy dx + \int_{\Gamma} \int_{\Gamma} h^1_1(y, u^1_0, u^2_0) \phi d\sigma,$$

$$\text{together with the initial conditions from (12).}$$

**Theorem 2.** The limit function $u_0 = (u^1_0, u^2_0)$ from Proposition 5 is the unique solution of the macroscopic problem (12).

**Proof.** We illustrate the procedure for $j = 1$ (the case $j = 2$ follows by similar arguments, where the diffusion terms vanishes in the limit). As a test-function in (13) for $j = 1$ we choose $\phi \in C^\infty_0((0, T) \times \Omega)$ and integrate with respect to time. By integration by parts in time we obtain

$$-\int_0^T \int_{\Omega_1} u^i_1 \partial_t \phi dx dt - \epsilon \int_0^T \int_{\Gamma_1} u^i_1 \partial_t \phi d\sigma dt$$

$$+ \int_0^T \int_{\Omega_1} D^1 \nabla u^i_1 \cdot \nabla \phi dx dt + \epsilon \int_0^T \int_{\Gamma_1} D^1_1 \nabla_{\gamma, y} u^i_1 \cdot \nabla_{\gamma, y} \phi d\sigma dt$$

$$= \int_0^T \int_{\Omega_1} f^i_1(u^1_0) \phi dx dt + \epsilon \int_0^T \int_{\Gamma_1} h^1_1(u^1_0, u^2_0) \phi d\sigma dt$$

$$+ \int_{\Omega_1} u^i_1 \phi dx + \epsilon \int_{\Gamma_1} u^i_1 \phi d\sigma.$$

Using the convergence results from Proposition 5, Corollary 4 and Lemma 1, as well as the Assumption (A5) on the initial conditions, we obtain for $\epsilon \to 0$

$$-\left( |Y_1| + |\Gamma| \right) \int_0^T \int_{\Omega_1} u^i_0 \partial_t \phi dx dt + \int_0^T \int_{\Omega_1} \hat{D}^1 \nabla u^i_0 \cdot \nabla \phi dx dt$$

$$= \int_0^T \int_{\Omega_1} Y_1 f^i_1(u^1_0) \phi dy dx dt + \epsilon \int_0^T \int_{\Gamma_1} h^1_1(u^1_0, u^2_0) \phi d\sigma dx dt$$

$$+ \int_{\Omega} |Y_1| u^1_0 \phi(0) dx + \int_{\Gamma} |\Gamma| u^1_0 \phi(0) dx.$$
Choosing \( \phi \) with compact support in \((0, T)\) we get \( \partial_t u^1_0 \in L^2((0, T), H^1(\Omega')) \) (see also Remark 3) with \( u^1_0(0) = \frac{\|u^0_1\|_{\Omega'} + |\Gamma|}{\|\nabla u^0_1\|} \), and by density we obtain that \( u^1_0 \) is a weak solution of the macroscopic equation for \( j = 1 \) in [12]. Uniqueness follows by standard energy estimates.

Remark 3.

(i) The regularity of the time-derivative is also directly obtained from the a priori estimates in Proposition 3. In fact, define for \( 0 < h \ll 1 \) and \( v : (0, T) \to X \) for a Banach space \( X \) the difference quotient for \( t \in (0, T-h) \)

\[
\partial_t^h v(t) := \frac{v(t+h) - v(t)}{h}.
\]

Then for all \( \phi \in C^\infty_0((0,T), C^\infty(\Omega)) \) it holds, due to Proposition 3 and the a priori estimates for the time-derivative in Proposition 2,

\[
\langle \partial_t^h u^0_1, \phi \rangle_{L^2((0,T-h), H^1(\Omega')), L^2((0,T-h), H^1(\Omega))} = \int_0^{T-h} \int_{\Omega} \partial_t^h u^1_0 \phi dx dt
\]

\[
= \lim_{\epsilon \to 0} \frac{1}{|Y_1| + |\Gamma|} \left( \int_0^{T-h} \int_{\Omega} \partial_t^h u^1_0 \phi dx + \epsilon \int_{\Gamma} \partial_t^h u^1_0 \phi d\sigma dt \right)
\]

\[
\leq \lim_{\epsilon \to 0} \frac{1}{|Y_1| + |\Gamma|} \int_0^T \langle \partial_t^h u^1_0, \phi \rangle_{H^1_B(\Sigma_1, \Sigma_1, \Omega)} dt.
\]

\[
\leq \lim_{\epsilon \to 0} \frac{1}{|Y_1| + |\Gamma|} \left| \langle \partial_t^h u^1_0 \rangle_{L^2((0,T), H^1(\Omega'))} \right| \| \phi \|_{L^2((0,T), H^1(\Omega'))}
\]

\[
\leq C \lim_{\epsilon \to 0} \| \partial_t u^1_0 \|_{L^2((0,T), H^1(\Omega'))} \| \phi \|_{L^2((0,T), H^1(\Omega'))}
\]

where at the end we used that \( P^1_\Gamma \) is an orthogonal projection. By density and the reflexivity of \( L^2((0,T-h), H^1(\Omega)) \) we obtain the boundedness

\[
\| \partial_t^h u^1_0 \|_{L^2((0,T-h), H^1(\Omega'))} \leq C,
\]

for a constant \( C \) independent of \( h \). This implies \( \partial_t u^1_0 \in L^2((0, T), H^1(\Omega')) \). A similar argument implies \( \partial_t u^2_0 \in L^2((0, T), H^1(\Omega')) \). However, the limit equation for \( u^0_0 \) even improves the regularity of \( \partial_t u^2_0 \).

(ii) We can also consider the case of a connected-connected porous medium (for \( n \geq 3 \) and a domain \( \Omega \) which can be decomposed in microscopic cells, for example a rectangle with integer side length, and an additional boundary condition on \( \partial \Gamma_1 \) is needed). In this case both macroscopic solutions are described by a reaction-diffusion equation as for \( u^1_0 \) in Theorem 4. The derivation of the macroscopic model for the connected-connected case even gets simpler, because we only need the a priori estimates from Proposition 3 and the convergence results for the connected domain in Section 4.2. The estimates for the shifts in Proposition 3 are no longer necessary.

(iii) The results can be easily extended to systems, see [15] for more details.
6 Discussion

By the methods of two-scale convergence and the unfolding operator we derived a macroscopic model for a reaction-diffusion equation in a connected-disconnected porous medium with a nonlinear dynamic Wentzell-interface condition across the interface. The crucial point was to pass to the limit in the nonlinear terms, especially on the interface. Therefore, we established strong two-scale compactness results just depending on a priori estimates for the sequence of solutions. For the proof we used the unfolding operator and a Banach-valued Kolmogorov-Simon-compactness argument, which was necessarily for the disconnected domain. In fact, while the solutions in the connected domain $\Omega^1_\varepsilon$ can be extended to the whole domain $\Omega$ preserving the a priori estimates, this is not possible anymore for the disconnected domain.

We emphasize that the strong compactness result in Theorem 1 is not restricted to our specific problem, but on the a priori estimates and the estimates for the shifts for the sequence. Therefore it can be easily applied to other problems. Especially, the results above can be extended to systems in an obvious way.

The time-derivative in the Wentzell-boundary condition on the interface $\Gamma_\varepsilon$ regularizes the problem and leads to a simple variational structure with respect to the Gelfand-triple $(\mathbb{H}_{j,\varepsilon}, L_{j,\varepsilon}, H_{j,\varepsilon}')$, see (3). Hence, the problem seems to be more complex regarding stationary interface conditions (neglecting the time-derivative). On the other hand, neglecting the diffusion term on the surface leads to an ordinary differential equation on the surface. Hence, we loose spatial regularity on the surface and therefore we have to replace the space $\mathbb{H}_{j,\varepsilon}$ by the function space $\langle u_\varepsilon, v_\varepsilon \rangle \in H_0^1(\Omega_j) \times L^2(\Gamma_\varepsilon) : u_\varepsilon|_{\Gamma_\varepsilon} = v_\varepsilon$ with norm $\sqrt{\|u_\varepsilon\|_{H_0^1(\Omega_j)}^2 + \varepsilon \|v_\varepsilon\|_{L^2(\Gamma_\varepsilon)}^2}$. For this choice it could be expected that the methods in the paper can be adapted to the case without surface diffusion. Nevertheless, both cases should be considered in more detail and are part of my ongoing work.

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A Two-scale convergence and unfolding operator

We repeat the definition of the two-scale convergence and the unfolding operator and summarize some well known properties and compactness results.

A.1 Two-scale convergence

In the following, unless stated otherwise, we assume that $p \in (1, \infty)$ and $p'$ is the dual exponent of $p$. We start with the definition of the two-scale convergence, see [2, 28].

Definition 1. We say the sequence $u_\varepsilon \in L^p((0, T) \times \Omega)$ converges in the two-scale sense (in $L^p$) to a limit function $u_0 \in L^p((0, T) \times Y)$, if for all $\phi \in L^{p'}((0, T) \times \Omega, C^0_{p\varepsilon}(Y))$
Then it holds that
\[
\lim_{\epsilon \to 0} \int_0^T \int_\Omega u_\epsilon(t,x) \phi \left( t,x,\frac{x}{\epsilon} \right) \, dx \, dt = \int_0^T \int_\Omega \int_\Gamma u_0(t,x,y) \phi(t,x,y) \, dy \, dx \, dt.
\]

We say the sequence converges strongly in the two-scale sense in \( L^p \), if it holds that
\[
\lim_{\epsilon \to 0} \left\| u_\epsilon \right\|_{L^p((0,T) \times \Omega)} = \left\| u_0 \right\|_{L^p((0,T) \times \Omega \times \Gamma)}.
\]

**Remark 4.**

(i) For sequences in \( L^p((0,T) \times \Omega_\epsilon^j) \) on the perforated domain we also use the designation "two-scale convergence". The definition is also valid for such functions by extension by zero (or with the extension operator from [11]), and considering suitable test-functions.

(ii) The two-scale convergence introduced above should actually be referred to as "weak two-scale convergence". However, in accordance with the definition in [2], we neglect the word "weak" and only use "strong" to highlight the "strong two-scale convergence".

(iii) For the "two-scale convergence in \( L^2 \)" we just write "two-scale convergence".

Next, we give the definition of the two-scale convergence on oscillating surfaces, see [3, 26].

**Definition 2.** We say the sequence \( u_\epsilon \in L^p((0,T) \times \Gamma_\epsilon) \) converges in the two-scale sense (in \( L^p \)) to a limit function \( u_0 \in L^p((0,T) \times \Omega \times \Gamma) \), if for all \( \phi \in C^0([0,T) \times \overline{\Omega}, C^0_{\text{per}}(\Gamma)) \) it holds that
\[
\lim_{\epsilon \to 0} \int_0^T \int_{\Gamma_\epsilon} u_\epsilon(t,x) \phi \left( t,x,\frac{x}{\epsilon} \right) \, d\sigma \, dt = \int_0^T \int_\Omega \int_\Gamma u_0(t,x,y) \phi(t,x,y) \, d\sigma \, dy \, dx \, dt.
\]

We say the sequence converges strongly in the two-scale sense, if it holds that
\[
\lim_{\epsilon \to 0} \epsilon^2 \left\| u_\epsilon \right\|_{L^p((0,T) \times \Gamma_\epsilon)} = \left\| u_0 \right\|_{L^p((0,T) \times \Omega \times \Gamma)}.
\]

In accordance with Remark 3, we proceed analogously for the two-scale convergence on \( \Gamma_\epsilon \), and neglect the word "weak" and the addition "\( L^2 \)".

To pass to the limit \( \epsilon \to 0 \) in the diffusion terms in the bulk domain \( \Omega_\epsilon^j \) and the surface \( \Gamma_\epsilon \) in the microscopic equation [3] we need compactness results for the spaces \( H^{1,\epsilon}_j \). In the following Lemma we summarize some weak two-scale compactness results for such functions, which can be found in [10]:

**Lemma 5.** For \( j \in \{1,2\} \) let \( u_\epsilon^j \in L^2((0,T), H^{1,\epsilon}_j) \) be a sequence with
\[
\left\| u_\epsilon^j \right\|_{L^2((0,T), H^{1,\epsilon}_j)} \leq C.
\]

Then it holds:

(i) For \( j = 1 \) there exist \( u_0^1 \in L^2((0,T), H^1(\Omega)) \) and a \( Y \)-periodic function \( u_1^1 \in L^2((0,T) \times \Omega, H^1_1(\R)) \), such that up to a subsequence
\[
\begin{align*}
\nabla u_\epsilon^1 \to & \ u_0^1 \quad \text{in the two-scale sense}, \\
\nabla_x u_\epsilon^1 \to & \ \nabla_x u_0^1 + \nabla_y u_1^1 \quad \text{in the two-scale sense}, \\
\n\nabla u_\epsilon^1 |_{\Gamma_\epsilon} \to & \ \nabla u_0^1 \quad \text{in the two-scale sense on} \ \Gamma_\epsilon, \\
\n\nabla u_\epsilon^1 |_{\Gamma_\epsilon} \to & \ \nabla u_0^1 + P T \nabla u_1^1 \quad \text{in the two-scale sense on} \ \Gamma_\epsilon.
\end{align*}
\]
Lemma 6. It holds that \( \text{and also } [7, 8, 9, 10, 31] \). In the following we consider the tuple \((G_\varepsilon, G) \in \{ (\Omega, Y), (\Omega_1^1, Y_1), (\Omega_2^2, Y_2), (\Gamma_\varepsilon, \Gamma) \} \) and we define

\[
\bar{G}_\varepsilon := \text{int} \bigcup_{k \in K_\varepsilon} \epsilon (G + k), \quad \Lambda_\varepsilon := \Omega \times \bar{G}_\varepsilon.
\]

Then, for \( p \in (1, \infty) \) we define the unfolding operator

\[
\mathcal{T}_\varepsilon : L^p((0, T) \times G_\varepsilon) \to L^p((0, T) \times \Omega \times G),
\]

with

\[
\mathcal{T}_\varepsilon(\phi) \cdot (t, x, y) := \begin{cases} 
\phi_\varepsilon \left(t, \epsilon \left[ \frac{x}{\varepsilon} \right] + \varepsilon y \right) & \text{for } x \in \bar{G}_\varepsilon, \\
0 & \text{for } x \in \Lambda_\varepsilon.
\end{cases}
\]

We emphasize that we use the same notation for the unfolding operator for the different choices of the tuple \((G_\varepsilon, G)\). It should be clear from the context in which sense it has to be understood. Further, we mention that unfolding operator commutes with the trace operator in the following sense: For \( \phi_\varepsilon \in L^p((0, T), W^{1,p}(\Omega_j)) \) for \( j \in \{1, 2\} \) it holds that

\[
\mathcal{T}_\varepsilon(\phi_\varepsilon |_{\Gamma_\varepsilon}) = (\mathcal{T}_\varepsilon(\phi_\varepsilon)) |_{\Gamma_\varepsilon}.
\]

**Lemma 6.**

(a) For \( (G_\varepsilon, G) \in \{ (\Omega, Y), (\Omega_1^1, Y_1), (\Omega_2^2, Y_2) \} \) we have:

(i) For \( \phi_\varepsilon \in L^p((0, T) \times G_\varepsilon) \) it holds that

\[
\| \mathcal{T}_\varepsilon \phi_\varepsilon \|_{L^p((0, T) \times \Omega \times G)} = \| \phi_\varepsilon \|_{L^p((0, T) \times \bar{G}_\varepsilon)}.
\]

(ii) For \( \phi_\varepsilon \in L^p((0, T), W^{1,p}(G_\varepsilon)) \) it holds that

\[
\nabla_y \mathcal{T}_\varepsilon \phi_\varepsilon = \varepsilon \mathcal{T}_\varepsilon \nabla_x \phi_\varepsilon.
\]

(b) For the unfolding operator on the surface we have:

(i) For \( \phi_\varepsilon \in L^p((0, T) \times \Gamma_\varepsilon) \) it holds that

\[
\| \mathcal{T}_\varepsilon \phi_\varepsilon \|_{L^p((0, T) \times \Omega \times \Gamma)} = \| \phi_\varepsilon \|_{L^p((0, T) \times \Gamma_\varepsilon)}.
\]
(ii) For \( \phi_\epsilon \in L^p((0,T), W^{1,p}(\Gamma_\epsilon)) \) it holds that

\[
\nabla_{\Gamma, y} T_\epsilon \phi_\epsilon = \epsilon T_\epsilon \nabla_{\Gamma, y} \phi_\epsilon.
\]

Proof. For (a) and (b)(i) see [11]. A proof for (b)(ii) can be found in [22].

In the following Lemma we give an equivalent relation between the unfolding operator and the two-scale convergence. For a proof see for example [8, 9, 11].

Lemma 7. Let \( p \in (1, \infty) \).

(a) For \( (G, G) \in \{(\Omega, Y), (\Omega^1, Y_1), (\Omega^2, Y_2)\} \) and a sequence \( u_\epsilon \in L^p((0,T) \times G_\epsilon) \), the following statements are equivalent:

- (a) \( u_\epsilon \rightharpoonup u_0 \) weakly/strongly in the two-scale sense in \( L^p \),
- (b) \( T_\epsilon u_\epsilon \rightharpoonup u_0 \) weakly/strongly in \( L^p((0,T) \times \Omega \times G) \).

(b) For a sequence \( u_\epsilon \in L^p((0,T) \times \Gamma_\epsilon) \) with \( \epsilon \| u_\epsilon \|_{L^p((0,T) \times \Gamma_\epsilon)} \leq C \), the following statements are equivalent:

- (a) \( u_\epsilon \rightharpoonup u_0 \) weakly/strongly in the two-scale sense on \( \Gamma_\epsilon \) in \( L^p \),
- (b) \( T_\epsilon u_\epsilon \rightharpoonup u_0 \) weakly/strongly in \( L^p((0,T) \times \Omega \times \Gamma) \).

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