Abstract. We show that if $n \geq 1$, $\Omega \subset \mathbb{R}^{n+1}$ is a connected domain that is porous around a subset $E \subset \partial \Omega$ of finite and positive Hausdorff $\mathcal{H}^n$-measure, and the harmonic measure $\omega$ is absolutely continuous with respect to $\mathcal{H}^n$ on $E$, then $\omega|_E$ is concentrated on an $n$-rectifiable set.

1. Introduction

There is a strong connection between the rectifiability of the boundary of a domain in Euclidean space and the possible absolute continuity of harmonic measure with respect to Hausdorff measure. Recall that a set $E$ is $n$-rectifiable if it can be covered by a countable union of (possibly rotated) $n$-dimensional Lipschitz graphs up to a set of zero $n$-dimensional Hausdorff measure $\mathcal{H}^n$. The local F. and M. Riesz theorem of Bishop and Jones [BJ] says that, if $\Omega$ is a simply connected planar domain and $\Gamma$ is a curve of finite length, then $\omega \ll \mathcal{H}^1$ on $\partial \Omega \cap \Gamma$, where $\omega$ stands for harmonic measure. In the same paper, Bishop and Jones also provide an example of a domain $\Omega$ whose boundary is contained in a curve of finite length, but $\mathcal{H}^1(\partial \Omega) = 0 < \omega(\partial \Omega)$, thus showing that some sort of connectedness in the boundary is required.

A higher dimensional version of the theorem of Bishop and Jones does not hold, even when the analogous “connectivity” assumption holds for the boundary. In [Wu], Wu builds a topological sphere in $\mathbb{R}^3$ of finite surface measure bounding a domain whose harmonic measure charges a set of Hausdorff dimension 1 contained in $\mathbb{R}^2$. However, under some extra geometric assumptions, higher dimensional versions of the Bishop-Jones result do hold. For example, Wu shows in the same paper that if $\Omega \subset \mathbb{R}^{n+1}$ is a domain with interior corkscrews, meaning $\Omega \cap B(x, r)$ contains a ball of radius $r/C$ for every $x \in \partial \Omega$ and $r \in (0, \text{diam } \partial \Omega)$, then $\omega \ll \mathcal{H}^n$ on $\Gamma \cap \partial \Omega$ whenever $\Gamma$ is a bi-Lipschitz image of $\mathbb{R}^n$ (or in fact a somewhat more general surface).

Many results that establish absolute continuity follow a similar pattern to the results of Bishop, Jones, and Wu by considering portions of the boundary that are contained in nicer (and usually rectifiable) surfaces. For example, if $\Omega$ is a Lipschitz domain (meaning the boundary is a union of Lipschitz graphs), then Dahlberg shows in [Da] that $\omega \ll \mathcal{H}^n \ll \omega$ on $\partial \Omega$. The works of [Ba] and [DJ] also establish various degrees of mutual absolute continuity in nontangentially accessible domains when $\mathcal{H}^n|_{\partial \Omega}$ is Radon. Recall that a domain

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is nontangentially accessible (or NTA) \[\Omega\] if it has exterior corkscrews (meaning \((\Omega)^c\) has interior corkscrews) and it is uniform, meaning there is \(C > 0\) so that for every \(x, y \in \Omega\) there is a path \(\gamma \subset \Omega\) connecting \(x\) and \(y\) such that

(a) the length of \(\gamma\) is at most \(C|x - y|\) and

(b) for \(t \in \gamma\), \(\text{dist}(t, \partial\Omega) \geq \text{dist}(t, \{x, y\})/C\).

In \[\text{AZ}\], the first author shows that, for NTA domains \(\Omega \subset \mathbb{R}^{n+1}\), if \(\Gamma \subset \partial\Omega\) is an \(n\)-Ahlfors regular set (meaning \(H^n(B(x, r) \cap \Gamma) \sim r^n\) for any ball \(B(x, r)\) centered on \(\Gamma\) with \(r \in (0, \text{diam } \Gamma)\)), then \(\omega \ll H^n\) on \(\partial\Omega \cap \Gamma\) and \(\omega|_{\partial\Omega \cap \Gamma}\) is supported on an \(n\)-rectifiable set.

Without knowing that the portion of the boundary in question is contained in a nice enough surrogate set, things can go wrong. In \[\text{AMT}\], we constructed an NTA domain \(\Omega \subset \mathbb{R}^{n+1}\) with very flat boundary, with \(\partial\Omega\) containing a set \(E\) so that \(\omega(E) > 0 = H^n(E)\). Observe that, in this case, while \(\partial\Omega\) is still \(n\)-rectifiable, by the results of \[\text{AZ}\] described earlier, it follows that such a set \(E\) cannot intersect a Lipschitz graph (or any Ahlfors regular set) in a set of positive \(\omega\)-measure.

We think the result of \[\text{AMT}\] is quite surprising in light of the previous results involving rectifiability and harmonic measure, as one might think that the rectifiability of \(\partial\Omega\) should be enough to guarantee \(\omega \ll H^n\).

It is a natural question to ask then if the rectifiability of \(\omega\) is actually necessary for absolute continuity, that is, if the support of \(\omega\) can be exhausted up to a set of \(\omega\)-measure zero by \(n\)-dimensional Lipschitz graphs.\(^1\) Some results of this nature already exist. Recall that if \(\Omega\) is a simply connected planar domain, \(\phi : \mathbb{D} \to \Omega\) is a conformal map, and \(G \subset \mathbb{D}\) is the set of points where \(\phi\) has nonzero angular derivative, then there is \(S \subset \partial\Omega\) with \(H^1(S) = 0\) and \(\omega(S \cup \phi(G)) = 1\) (see Theorem VI.1.1 in \[\text{GM}\]). Thus, if \(E \subset \partial\Omega\) is such that \(0 < H^1(E) < \infty\) and \(\omega \ll H^1\) on \(E\), then \(\omega(E \cap S) = 0\), so \(\omega\) almost every point in \(E\) is in \(\phi(G)\). Since all points in \(\phi(G)\) are cone points (p. 208 of \[\text{GM}\]) and the set of cone points is a rectifiable set (Lemma 15.13 in \[\text{Ma}\]), \(\phi(G) \cap E\) is 1-rectifiable and thus \(\omega|_E\) is 1-rectifiable.

In the work \[\text{HMU}\], Hofmann, Martell and Uriarte-Tuero show that if \(\Omega \subset \mathbb{R}^{n+1}\) is a uniform domain, \(\partial\Omega\) is Ahlfors regular, and harmonic measure satisfies the weak-A\(_\infty\) condition, then \(\partial\Omega\) is uniformly rectifiable. The weak-A\(_\infty\) condition is a stronger assumption than \(\omega\) being absolutely continuous, but \(\partial\Omega\) being uniformly rectifiable is a stronger conclusion than just being rectifiable. We omit the definitions of these terms and refer the reader to these references.

Our main result is the following.

**Theorem 1.1.** Let \(n \geq 1\) and \(\Omega \subset \mathbb{R}^{n+1}\) be a proper domain of \(\mathbb{R}^{n+1}\) and let \(\omega\) be the harmonic measure in \(\Omega\). Suppose that there exists \(E \subset \partial\Omega\) with \(0 < H^n(E) < \infty\) and that \(\partial\Omega\) is porous in \(E\), i.e. there is \(r_0 > 0\) so that every ball \(B\) centered at \(E\) of radius at most \(r_0\) contains another ball \(B' \subset \mathbb{R}^{n+1} \setminus \partial\Omega\) with \(r(B) \sim r(B')\), with the implicit constant depending only on \(E\). If \(\omega|_E\) is absolutely continuous with respect to \(H^n|_E\), then \(\omega|_E\) is \(n\)-rectifiable, in the sense that \(\omega\)-almost all of \(E\) can be covered by a countable union of \(n\)-dimensional (possibly rotated) Lipschitz graphs.

\(^1\)We stress that when we speak of a measure \(\omega\) being rectifiable, we mean that it may be covered up to a set of \(\omega\)-measure zero by \(n\)-dimensional Lipschitz graphs. This is a stronger criterion than rectifiability of measures as defined by Federer in \[\text{Fed}\], who defines this as being covered up to a set of \(\omega\)-measure zero by Lipschitz images of subsets of \(\mathbb{R}^n\).
We list a few observations about this result:

1. Theorem 1.1 is local: we don’t assume $H^n|_{\partial \Omega}$ is a Radon measure, only on the subset $E$.

2. We don’t assume any strong connectedness property like uniformity, or a uniform exterior or interior corkscrew property, which the higher dimensional results mentioned earlier all rely upon. Aside from basic connectivity in Theorem 1.1, we only need a large ball in the complement of $\partial \Omega$ in each ball centered on $E \subset \partial \Omega$ with no requirement whether that ball is in $\Omega$ or its complement.

3. Examples of domains with porous boundaries are uniform domains, John domains, interior or exterior corkscrew domains, and the complement of an $n$-Ahlfors regular set.

4. The theorem establishes rectifiability of the measure $\omega|_E$ and not of the set $E$: the set $E$ may very well contain a purely $n$-unrectifiable subset, but that subset must have $\omega$-measure zero.

5. As far as we know, in the case $n = 1$, the theorem is also new.

The following is an easy consequence of our main result.

**Corollary 1.2.** Suppose that $n \geq 1$, $\Omega \subset \mathbb{R}^{n+1}$ is a connected domain, and $E \subset \partial \Omega$ is a set such that $0 < H^n(E) < \infty$, $\partial \Omega$ is porous in $E$, and $H^n \ll \omega$ on $E$. Then $E$ is $n$-rectifiable.

Indeed, by standard arguments, there is $E' \subset E$ such that $H^n(E \setminus E') = 0$ and $\omega \ll H^n \ll \omega$ on $E'$. By Theorem 1.1 $\omega|_{E'}$ is $n$-rectifiable, but since $H^n \ll \omega$ on $E$, we also have that $E'$ is $n$-rectifiable, and thus $E$ is $n$-rectifiable.

We also mention that from Theorem 1.1 in combination with the results of [Az] we obtain the next corollary.

**Corollary 1.3.** Let $\Omega \subset \mathbb{R}^{n+1}$ be an NTA domain, $n \geq 1$, and let $E \subset \partial \Omega$ be such that $0 < H^n(E) < \infty$. Then $\omega|_E \ll H^n|_E$ if and only if $E$ may be covered by countably many $n$-dimensional Lipschitz graphs up to a set of $\omega$-measure zero.

The forward direction is just a consequence of Theorem 1.1 and the reverse direction follows from the result from [Az] as described earlier since $n$-dimensional Lipschitz graphs are $n$-Ahlfors regular.

During the preparation of this manuscript, we received a preprint by Hofmann and Martell [HM] that shows that the result from [HMu] described above holds not only for uniform domains, but also for domains which are complements of Ahlfors regular sets, again under the assumption that harmonic measure is weak-$A_{\infty}$. We thank Steve Hofmann for making his joint work available to us. We remark that our method of proof of Theorem 1.1 is completely independent of the techniques in [HM] and previous works such as [HMu]. We also mention that after having written a first version of the present paper, José María Martell informed us that in a joint work with Akman, Badger and Hofmann in preparation, they have obtained some result in the spirit of Corollary 1.2 under some stronger assumptions (in particular, assuming $\partial \Omega$ to be Ahlfors regular).
2. Some notation

We will write \( a \lesssim b \) if there is \( C > 0 \) so that \( a \leq Cb \) and \( a \lesssim_t b \) if the constant \( C \) depends on the parameter \( t \). We write \( a \sim b \) to mean \( a \lesssim b \lesssim a \) and define \( a \sim_t b \) similarly.

For sets \( A, B \subset \mathbb{R}^{n+1} \), we let

\[
\text{dist}(A, B) = \inf \{|x - y| : x \in A, y \in B\}, \quad \text{dist}(x, A) = \text{dist}(\{x\}, A),
\]

We denote the open ball of radius \( r \) centered at \( x \) by \( B(x, r) \). For a ball \( B = B(x, r) \) and \( \delta > 0 \) we write \( r(B) \) for its radius and \( \delta B = B(x, \delta r) \). We let \( \Upsilon(B) \) to be the \( \varepsilon \)-neighborhood of a set \( A \subset \mathbb{R}^{n+1} \). For \( A \subset \mathbb{R}^{n+1} \) and \( 0 < \delta \leq \infty \), we set

\[
\mathcal{H}_\delta^n(A) = \inf \left\{ \sum_i \text{diam}(A_i)^n : A_i \subset \mathbb{R}^{n+1}, \text{diam}(A_i) \leq \delta, A \subset \bigcup_i A_i \right\}.
\]

Define the \( n \)-dimensional Hausdorff measure as

\[
\mathcal{H}^n(A) = \lim_{\delta \downarrow 0} \mathcal{H}_\delta^n(A)
\]

and the \( n \)-dimensional Hausdorff content as \( \mathcal{H}^n_\infty(A) \). See Chapter 4 of [Ma] for more details.

Given a signed Radon measure \( \nu \) in \( \mathbb{R}^{n+1} \) we consider the \( n \)-dimensional Riesz transform

\[
\mathcal{R}\nu(x) = \int \frac{x - y}{|x - y|^{n+1}} d\nu(y),
\]

whenever the integral makes sense. For \( \varepsilon > 0 \), its \( \varepsilon \)-truncated version is given by

\[
\mathcal{R}_\varepsilon \nu(x) = \int_{|x - y| > \varepsilon} \frac{x - y}{|x - y|^{n+1}} d\nu(y).
\]

For \( \delta \geq 0 \) we set

\[
\mathcal{R}_s, \delta \nu(x) = \sup_{\varepsilon > \delta} |\mathcal{R}_\varepsilon \nu(x)|.
\]

We also consider the maximal operator

\[
M_\delta^n \nu(x) = \sup_{r > \delta} \frac{|\nu|(B(x, r))}{r^n},
\]

In the case \( \delta = 0 \) we write \( \mathcal{R}_s \nu(x) := \mathcal{R}_{s,0} \nu(x) \) and \( M^n \nu(x) := M_0^n \nu(x) \).

3. The strategy

We fix a point \( p \in \Omega \) far from the boundary to be specified later. To prove that \( \omega^p|_E \) is rectifiable we will show that any subset of positive harmonic measure of \( E \) contains another subset \( G \) of positive harmonic measure such that \( \mathcal{R}_{s, \omega^p|_E} \) is bounded in \( L^2(\omega^p|_G_0) \).

Then from the results of Nazarov, Tolsa and Volberg in [NToV1] and [NToV2], it follows that \( \omega^p|_{G_0} \) is \( n \)-rectifiable. This suffices to prove the full \( n \)-rectifiability of \( \omega^p|_E \).

One of the difficulties of Theorem 1.1 is due to the fact that the non-Ahlfors regularity of \( \partial \Omega \) makes it difficult to apply some usual tools from potential of theory, such as the ones developed by Aikawa in [Ai1] and [Ai2]. In our proof we solve this issue by applying some stopping time arguments involving the harmonic measure and a suitable Frostman measure.
The connection between harmonic measure and the Riesz transform is already used, at least implicitly, in the work of Hofmann, Martell and Uriarte-Tuero [HMU], and more explicitly in the paper by Hofmann, Martell and Mayboroda [HMM]. Indeed, in [HMU], in order to prove the uniform rectifiability of $\partial \Omega$, the authors rely on the study of a square function related to the double gradient of the single layer potential and the application of an appropriate rectifiability criterion due to David and Semmes [DS]. Note that the gradient of the single layer potential coincides with the Riesz transform away from the boundary.

We think that the Riesz transform is a much more flexible tool than the square function used in [HMU]. Indeed, to work with the Riesz transform with minimal regularity assumptions one can apply the techniques developed in the last so many years in the area of the so-called non-homogeneous Calderón-Zygmund theory. However, it is not clear to us if the aforementioned square function behaves reasonably well without strong assumptions such as the $n$-Ahlfors regularity of $\partial \Omega$.

4. Harmonic and Frostman Measures

We start by reviewing a result of Bourgain from [Bo].

**Lemma 4.1.** There is $\delta_0 > 0$ depending only on $n \geq 1$ so that the following holds for $\delta \in (0, \delta_0)$. Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a domain, $\xi \in \partial \Omega$, $r > 0$, $B = B(\xi, r)$, and set $\rho := \mathcal{H}^s_{\infty}(\partial \Omega \cap \delta B)/(\delta r)^s$ for some $s > n - 1$. Then

$$\omega^u_{\Omega}(B) \gtrsim n \rho \quad \text{for all } x \in \delta B.$$

**Proof.** We only prove the case $n \geq 2$, the $n = 1$ case is similar, although one uses $-\log |\cdot|$ instead of $|\cdot|^{1-n}$ to define Green’s function.

Without loss of generality, we assume $\xi = 0$ and $r = 1$. Let $\mu$ be a Frostman measure supported in $\delta B \cap \partial \Omega$ so that

- $\mu(B(x, r)) \leq r^s$ for all $x \in \mathbb{R}^{n+1}$ and $r > 0$,
- $\rho \delta^s \geq \mu(\delta B \cap \partial \Omega) \geq c \rho \delta^s$ where $c = c(n) > 0$.

Define a function

$$u(x) = \int \frac{1}{|x - y|^{n-1}} d\mu(y),$$

which is harmonic out of $\text{supp} \mu$ and satisfies the following properties:

(i) For $x \in \delta B$,

$$u(x) \geq 2^{1-n} \delta^{1-n} \mu(\delta B) \geq c 2^{1-n} \delta^{s-n+1} \rho.$$

(ii) For $x \in \delta B$,

$$u(x) \leq \sum_{j=0}^{\infty} \int_{\delta^{2-j} \leq |x-y| < \delta^{2-j+1}} \frac{1}{|x-y|^{n-1}} d\mu(x) \leq \sum_{j=0}^{\infty} (2^{-j} \delta^s)(2^{-j} \delta)^{1-n} \sim \delta^{s-n+1}.$$

(iii) For $x \in B^c$,

$$u(x) = \int \frac{1}{|x - y|^{n-1}} d\mu(x) \leq 2^{n-1} \mu(\delta B) \leq 2^{n-1} \rho \delta^s.$$

(iv) Thus, by the maximum principle, we have that $u(x) \leq \delta^{s-n+1}$ for all $x \in \mathbb{R}^{n+1}$.

Set

$$v(x) = \frac{u(x) - \sup_{\partial B} u}{\sup u}.$$

Then
(a) \( v \) is harmonic in \((\delta B \cap \partial \Omega)^c\),
(b) \( v \leq 1 \),
(c) \( v \leq 0 \) on \( B^c \),
(d) for \( x \in \delta B \) and \( \delta \) small enough,

\[
v(x) \gtrsim \frac{c_\delta^{s+1-n} \rho - 2^{n-1} \rho \delta^s}{c_\delta^{s-n+1}} \gtrsim \rho.
\]

Let \( \phi \) be any continuous compactly supported function equal to 1 on \( B \). Then \( \int \phi d\omega_\Omega^x \) is at least any subharmonic function \( f \) with \( \limsup_{x \in \Omega \to \xi} f(x) \leq \phi(\xi) \). The function \( v \) satisfies this, and so we have \( \int \phi d\omega_\Omega^x \geq v(x) \). Taking the infimum over all such \( \phi \), we get that \( \omega_\Omega^x(B) \geq v(x) \), and the lemma follows.

The proof of the next lemma is fairly standard but we include it for the sake of completeness.

**Lemma 4.2.** Let \( \Omega \subsetneq \mathbb{R}^{n+1} \) be an open Greenian domain, \( n \geq 1 \), \( \xi \in \partial \Omega \), \( r > 0 \) and \( B := B(\xi, r) \). Suppose that there exists a point \( x_B \in \Omega \) so that the ball \( B_0 := B(x_B, r/C) \) satisfies \( 4B_0 \subset \Omega \cap B \) for some \( C > 1 \). Then, for \( n \geq 2 \),

\[
\omega_\Omega^x(B) \gtrsim \omega_\Omega^{x_B}(B)r^{n-1}G_\Omega(x, x_B) \quad \text{for all } x \in \Omega \setminus B_0.
\]

In the case \( n = 1 \),

\[
\omega_\Omega^x(B) \gtrsim \omega_\Omega^{x_B}(B)\left|G_\Omega(x, x_B) - G_\Omega(x, z)\right| \quad \text{for all } x \in \Omega \setminus B_0 \text{ and } z \in \frac{1}{2}B_0.
\]

Note that the class of domains considered in Theorem 4.1 are Greenian. Indeed, all open subsets of \( \mathbb{R}^{n+1} \) are Greenian for \( n \geq 2 \) (Theorem 3.2.10 [Hel]), and in the plane, if \( \mathcal{H}^1(\partial \Omega) > 0 \), then \( \partial \Omega \) is nonpolar (p. 207 Theorem 11.14 of [HKM]) and domains with nonpolar boundaries are Greenian by Myrberg’s Theorem (see Theorem 5.3.8 on p. 133 of [AG]). For the definitions of Greenian and polar sets, see [Hel].

**Proof.** First we consider the case \( n \geq 2 \). Without loss of generality, we assume that \( \omega_\Omega^{x_B}(B) > 0 \) since otherwise (4.1) is trivial. We define a new domain \( \Omega' := \Omega \setminus B_0 \subset \Omega \). From the definition of the Green function we have

\[
G_\Omega(x, x_B) \lesssim r(B)^{1-n} \quad \text{for } x \in \partial B_0.
\]

Since the set of Wiener irregular boundary points is polar (Corollary 4.5.5 [Hel]), it holds that \( G_\Omega(x, x_B) = 0 \) for all \( x \in \partial \Omega' \cap \partial \Omega \) apart from a polar set. Moreover, for \( x \in \partial B_0 \) we have from (4.3) that

\[
G_\Omega(x, x_B) \leq c_0 \frac{1}{|x - x_B|^{n-1}} \leq c_1 \omega_\Omega^{x_B}(B),
\]

for some purely dimensional constant \( c_1 > 0 \), where the fact that \( \omega^{x_B}(B)/\omega^{x_B}(B) \sim 1 \) follows from the standard interior Harnack’s inequality for \( 2B_0 \).

Define now \( u(x) = c_1 r^{1-n} \omega^{x_B}(B)/\omega^{x_B}(B) - G_\Omega(x, x_B) \) for all \( x \in \Omega' \cup \partial \Omega' \), which is harmonic in \( \Omega' \). Using that \( G_\Omega(x, x_B) \lesssim |x - x_B|^{1-n} \lesssim r^{1-n} \) for any \( x \in \Omega' \), we obtain that \( u \geq -c_2 r^{1-n} \) in \( \Omega' \). Therefore, by [Hel] Theorem 4.2.21, in view of the fact that \( u \) is harmonic and bounded below in \( \Omega' \), \( u \geq 0 \) on \( \partial \Omega' \) except for a polar set, and \( \liminf_{|x| \to \infty} u(x) \geq 0 \), we conclude (4.1).
Now we deal with the case $n = 1$. Again we assume that $\omega^{1,2}(B) > 0$ and we take $\Omega' = \Omega \setminus B_0 \subset \Omega$, as above. From the definition of the Green function, for $x \in \partial B_0$ and $z \in \frac{1}{2} B_0$ we have

$$
(4.4) \quad |G_{\Omega}(x, x_B) - G_{\Omega}(x, z)| = \left| \log \frac{|x - z|}{|x - x_B|} - \int \log \frac{|\xi - z|}{|\xi - x_B|} \, d\omega^{1,2}(\xi) \right| \lesssim 1,
$$

since

$$
\frac{|x - z|}{|x - x_B|} \approx \frac{|\xi - z|}{|\xi - x_B|} \approx 1 \quad \text{for } x \in \partial B_0, \xi \in \partial \Omega, \text{ and } z \in \frac{1}{2} B_0.
$$

Arguing as in the case $n \geq 2$, we deduce that

$$
|G_{\Omega}(x, x_B) - G_{\Omega}(x, z)| \leq c'_0 \leq c'_1 \frac{\omega^{1,2}(B)}{\omega^{1,2}(B^{1,2})},
$$

for some absolute constant $c'_1 > 0$, where the fact that $\omega^{1,2}(B)/\omega^{1,2}(B) \sim 1$ follows from the standard interior Harnack’s inequality for $2B_0$.

For $x \in \Omega' \cup \partial \Omega'$ and a fixed $z \in \partial \frac{1}{2} B_0$, consider the function

$$
u(x) = c'_1 \frac{\omega^{1,2}(B)}{\omega^{1,2}(B^{1,2})} - |G_{\Omega}(x, x_B) - G_{\Omega}(x, z)|.
$$

This is superharmonic in $\Omega'$ and uniformly bounded. Therefore, since $\nu \geq 0$ on $\partial \Omega'$ except for a polar set and $\lim \inf_{|x| \to \infty} \nu(x) \geq 0$, we obtain (4.2).

From now on, $\Omega$ and $E$ will be as in Theorem 1.1. Also, fix a point $p \in \Omega$ and consider the harmonic measure $\omega^p$ of $\Omega$ with pole at $p$. The reader may think that $p$ is point deep inside $\Omega$.

The Green function of $\Omega$ will be denoted just by $G(\cdot, \cdot)$.

Let $g \in L^1(\omega^p)$ be such that

$$\omega^p|_E = g \mathcal{H}^n|_{\partial \Omega}.$$

Given $M > 0$, let

$$E_M = \{x \in \partial \Omega : M^{-1} \leq g(x) \leq M\}.$$

Take $M$ big enough so that $\omega^p(E_M) \geq \omega^p(E)/2$, say. Consider an arbitrary compact set $F_M \subset E_M$ with $\omega^p(F_M) > 0$. We will show that there exists $G_0 \subset F_M$ with $\omega^p(G_0) > 0$ which is $n$-rectifiable. Clearly, this suffices to prove that $\omega^p|_{E_M}$ is $n$-rectifiable, and letting $M \to \infty$ we get the full $n$-rectifiability of $\omega^p|_E$.

Let $\mu$ be an $n$-dimensional Frostman measure for $F_M$. That is, $\mu$ is a non-zero Radon measure supported on $F_M$ such that

$$\mu(B(x, r)) \leq C r^n \quad \text{for all } x \in \mathbb{R}^{n+1}.
$$

Further, by renormalizing $\mu$, we can assume that $\|\mu\| = 1$. Of course the constant $C$ above will depend on $\mathcal{H}^n(0)$, and the same may happen for all the constants $C$ to appear, but this will not bother us. Notice that $\mu \ll \mathcal{H}^n|_{F_M} \ll \omega^p$. In fact, for any set $H \subset F_M$,

$$
(4.5) \quad \mu(H) \leq C \mathcal{H}^n(H) \leq C \mathcal{H}^n(H) \leq C M \omega^p(H).
$$
5. The Dyadic Lattice of David and Mattila

Now we will consider the dyadic lattice of cubes with small boundaries of David-Mattila associated with $\omega^p$. This lattice has been constructed in [DM, Theorem 3.2] (with $\omega^p$ replaced by a general Radon measure). Its properties are summarized in the next lemma.

**Lemma 5.1** (David, Mattila). Consider two constants $C_0 > 1$ and $A_0 > 5000 C_0$ and denote $W = \text{supp} \omega^p$. Then there exists a sequence of partitions of $W$ into Borel subsets $Q, Q \in D_k$, with the following properties:

- For each integer $k \geq 0$, $W$ is the disjoint union of the “cubes” $Q, Q \in D_k$, and if $k \leq l$, $Q \in D_k$, and $R \in D_k$, then either $Q \cap R = \emptyset$ or else $Q \subset R$.

- The general position of the cubes $Q$ can be described as follows. For each $k \geq 0$ and each cube $Q \in D_k$, there is a ball $B(Q) = B(z_Q, r(Q))$ such that

$$ z_Q \in W, \quad A_0^{-k} \leq r(Q) \leq C_0 A_0^{-k}, $$

$$ W \cap B(Q) \subset Q \subset W \cap \{28 r(Q)\}, $$

and the balls $5 B(Q), Q \in D_k$, are disjoint.

- The cubes $Q \in D_k$ have small boundaries. That is, for each $Q \in D_k$ and each integer $l \geq 0$, set

$$ N_l^{\text{ext}}(Q) = \{x \in W \setminus Q : \text{dist}(x, Q) < A_0^{-k-l}\}, $$

$$ N_l^{\text{int}}(Q) = \{x \in Q : \text{dist}(x, W \setminus Q) < A_0^{-k-l}\}, $$

and

$$ N_l(Q) = N_l^{\text{ext}}(Q) \cup N_l^{\text{int}}(Q). $$

Then

$$ \omega^p(N_l(Q)) \leq (C^{-1} C_0^{-3d-1} A_0)^{-l} \omega^p(90 B(Q)). $$

(5.1)

- Denote by $D_k^{\text{db}}$ the family of cubes $Q \in D_k$ for which

$$ \omega^p(100 B(Q)) \leq C_0 \omega^p(B(Q)). $$

(5.2)

We have that $r(Q) = A_0^{-k}$ when $Q \in D_k \setminus D_k^{\text{db}}$ and

$$ \omega^p(100 B(Q)) \leq C_0^{-1} \omega^p(100^{l+1} B(Q)) \quad \text{for all } l \geq 1 \text{ such that } 100^l \leq C_0 \text{ and } Q \in D_k \setminus D_k^{\text{db}}. $$

(5.3)

We use the notation $D = \bigcup_{k \geq 0} D_k$. Observe that the families $D_k$ are only defined for $k \geq 0$. So the diameter of the cubes from $D$ are uniformly bounded from above. We set $\ell(Q) = 56 C_0 A_0^{-k}$ and we call it the side length of $Q$. Notice that

$$ \frac{1}{28} C_0^{-1} \ell(Q) \leq \text{diam}(Q) \leq \ell(Q). $$

Observe that $r(Q) \sim \text{diam}(Q) \sim \ell(Q)$. Also we call $z_Q$ the center of $Q$, and the cube $Q' \in D_{k-1}$ such that $Q' \supset Q$ the parent of $Q$. We set $B_Q = 28 B(Q) = B(z_Q, 28 r(Q))$, so that

$$ W \cap \frac{1}{28} B_Q \subset Q \subset B_Q. $$
We assume $A_0$ big enough so that the constant $C^{-1}C_0^{-3d-1}A_0$ in (5.1) satisfies

$$C^{-1}C_0^{-3d-1}A_0 > A_0^{1/2} > 10.$$  

Then we deduce that, for all $0 < \lambda \leq 1$, 

$$\omega^p\left(\{x \in Q : \text{dist}(x, W \setminus Q) \leq \lambda \ell(Q)\}\right) + \omega^p\left(\{x \in 3.5B_Q : \text{dist}(x, Q) \leq \lambda \ell(Q)\}\right) \leq c\lambda^{1/2}\omega^p(3.5B_Q).$$  

(5.4)

We denote $D_{db} = \bigcup_{k \geq 0} D_k$. Note that, in particular, from (5.2) it follows that

$$\omega^p(3B_Q) \leq \omega^p(100B(Q)) \leq C_0\omega^p(Q)$$  

if $Q \in D_{db}$.  

(5.5)

For this reason we will call the cubes from $D_{db}$ doubling.

As shown in [DM, Lemma 5.28], every cube $R \in D$ can be covered $\omega^p$-a.e. by a family of doubling cubes:

**Lemma 5.2.** Let $R \in D$. Suppose that the constants $A_0$ and $C_0$ in Lemma 5.1 are chosen suitably. Then there exists a family of doubling cubes $\{Q_i\}_{i \in I} \subset D_{db}$, with $Q_i \subset R$ for all $i$, such that their union covers $\omega^p$-almost all $R$.

The following result is proved in [DM, Lemma 5.31].

**Lemma 5.3.** Let $R \in D$ and let $Q \subset R$ be a cube such that all the intermediate cubes $S$, $Q \subset S \subset R$ are non-doubling (i.e. belong to $D \setminus D_{db}$). Then

$$\omega^p(100B(Q)) \leq A_0^{-10n(J(Q)-J(R)-1)}\omega^p(100B(R)).$$  

(5.6)

Given a ball $B \subset \mathbb{R}^{n+1}$, we consider its $n$-dimensional density:

$$\Theta_\omega(B) = \frac{\omega^p(B)}{r(B)^n}.$$  

From the preceding lemma we deduce:

**Lemma 5.4.** Let $Q, R \in D$ be as in Lemma 5.3. Then

$$\Theta_\omega(100B(Q)) \leq C_0 A_0^{-9n(J(Q)-J(R)-1)}\Theta_\omega(100B(R))$$

and

$$\sum_{S \in D : Q \subset S \subset R} \Theta_\omega(100B(S)) \leq c \Theta_\omega(100B(R)),$$

with $c$ depending on $C_0$ and $A_0$.

For the easy proof, see [To3, Lemma 4.4], for example.

From now on we will assume that $C_0$ and $A_0$ are some big fixed constants so that the results stated in the lemmas of this section hold. Further, we will choose the pole $p \in \Omega$ of the harmonic measure $\omega^p$ so that $\text{dist}(p, \partial \Omega) \geq 10C_0$. The existence of such point $p$ can be assumed by dilating $\Omega$ by a suitable factor if necessary.
6. The Bad Cubes

Now we need to define a family of bad cubes. We say that $Q \in \mathcal{D}$ is bad and we write $Q \in \text{Bad}$, if $Q \in \mathcal{D}$ is a maximal cube satisfying one of the conditions below:

(a) $\mu(Q) \leq \tau \omega^p(Q)$, where $\tau > 0$ is a small parameter to be fixed below, or

(b) $\omega^p(3B_Q) \geq A r(B_Q)^n$, where $A$ is some big constant to be fixed below.

The existence of maximal cubes is guaranteed by the fact that all the cubes from $\mathcal{D}$ have side length uniformly bounded from above (since $\mathcal{D}_k$ is defined only for $k \geq 0$). If the condition (a) holds, we write $Q \in \text{LM}$ (little measure $\mu$) and in the case (b), $Q \in \text{HD}$ (high density).

On the other hand, if a cube $Q \in \mathcal{D}$ is not contained in any cube from $\text{Bad}$, we say that $Q$ is good and we write $Q \in \text{Good}$.

Notice that

$$\sum_{Q \in \text{LM}} \mu(Q) \leq \tau \sum_{Q \in \text{LM}} \omega^p(Q) \leq \tau \|\omega\| = \tau \mu(F_M).$$

Therefore, taking into account that $\tau \leq 1/2$ and that $\omega^p|_{F_M} = g(x) \mathcal{H}^n|_{F_M}$ with $g(x) \geq M$, we have by (4.5)

$$\frac{1}{2} \omega^p(F_M) \leq \frac{1}{2} = \frac{1}{2} \mu(F_M) \leq \mu\left(F_M \setminus \bigcup_{Q \in \text{LM}} Q\right) \leq C \mathcal{H}^n\left(F_M \setminus \bigcup_{Q \in \text{LM}} Q\right) \leq C M \omega^p\left(F_M \setminus \bigcup_{Q \in \text{LM}} Q\right).$$

On the other hand, since $\Theta^{n,*}(x, \omega^p) := \limsup_{r \to 0} \frac{\omega^p(B(x,r))}{(2r)^n} < \infty$ for $\omega^p$-a.e. $x \in \mathbb{R}^{n+1}$, it is also clear that for $A$ big enough

$$\omega^p\left(\bigcup_{Q \in \text{HD}} Q\right) \ll \omega^p(F_M).$$

From the above estimates it follows that

$$(6.1) \quad \omega^p\left(F_M \setminus \bigcup_{Q \in \text{Bad}} Q\right) > 0$$

if $\tau$ and $A$ have been chosen appropriately.

For technical reasons we have now to introduce a variant of the family $\mathcal{D}^{db}$ of doubling cubes defined in Section 5. Given some constant $T \geq C_0$ (where $C_0$ is the constant in Lemma 5.1) to be fixed below, we say that $Q \in \tilde{\mathcal{D}}^{db}$ if

$$\omega^p(100B(Q)) \leq T \omega^p(Q).$$

We also set $\tilde{\mathcal{D}}^{db}_k = \tilde{\mathcal{D}}^{db} \cap \mathcal{D}_k$ for $k \geq 0$. From 5.5 and the fact that $T \geq C_0$, it is clear that $\mathcal{D}^{db} \subset \tilde{\mathcal{D}}^{db}$.

**Lemma 6.1.** If the constant $T$ is chosen big enough, then

$$\omega^p\left(F_M \cap \bigcup_{Q \in \tilde{\mathcal{D}}^{db}_0} Q \setminus \bigcup_{Q \in \text{Bad}} Q\right) > 0.$$

Notice that above $\tilde{\mathcal{D}}^{db}_0$ stands for the family of cubes from the zero level of $\tilde{\mathcal{D}}^{db}$.
Proof. By the preceding discussion we already know that

\[ \omega^p \left( F_M \setminus \bigcup_{Q \in \text{Bad}} Q \right) > 0. \]

If \( Q \not\in \tilde{D}^{db} \), then \( \omega^p(Q) \leq T^{-1} \omega^p(100B(Q)) \). Hence by the finite overlap of the balls \( 100B(Q) \) associated with cubes from \( D_0 \) we get

\[ \omega^p \left( \bigcup_{Q \in D_0 \setminus \tilde{D}^{db}} Q \right) \leq \frac{1}{T} \sum_{Q \in D_0} \omega^p(100B(Q)) \leq \frac{C}{T} \| \omega^p \| = \frac{C}{T}. \]

Thus for \( T \) big enough we derive

\[ \omega^p \left( \bigcup_{Q \in D_0 \setminus \tilde{D}^{db}} Q \right) \leq \frac{1}{2} \omega^p \left( F_M \setminus \bigcup_{Q \in \text{Bad}} Q \right), \]

and then the lemma follows. \( \square \)

Notice that for the points \( x \in F_M \setminus \bigcup_{Q \in \text{Bad}} Q \), from the condition (b) in the definition of bad cubes, it follows that

\[ \omega^p(B(x, r)) \lesssim A r^n \quad \text{for all } 0 < r \leq 1. \]

Trivially, the same estimate holds for \( r \geq 1 \), since \( \| \omega^p \| = 1 \). So we have

(6.2) \[ M^n \omega^p(x) \lesssim A \quad \text{for } \omega^p\text{-a.e. } x \in F_M \setminus \bigcup_{Q \in \text{Bad}} Q. \]

7. **The Key Lemma about the Riesz Transform of \( \omega^p \) on the Good Cubes**

**Lemma 7.1** (Key lemma). Let \( Q \in \text{Good} \) be contained in some cube from the family \( \tilde{D}^{db}_0 \), and \( x \in B_Q \). Then we have

(7.1) \[ |R_{r(B_Q)} \omega^p(x)| \leq C(A, M, T, \tau, d_p), \]

where, to shorten notation, we wrote \( d_p = \text{dist}(p, \partial \Omega) \).

**Proof.** To prove the lemma, clearly we may assume that \( r(B_Q) \ll \text{dist}(p, \partial \Omega) \) and that \( r(P) < r_0 \) for any \( P \in \text{Good} \), where \( r_0 \) is as in the statement of Theorem 1.1. First we will prove \( (7.1) \) for \( Q \in \tilde{D}^{db} \cap \text{Good} \). In this case, by definition we have

\[ \mu(Q) > \tau \omega^p(Q) \quad \text{and} \quad \omega^p(3B_Q) \leq T \omega^p(Q). \]

We consider a ball \( B \) centered on \( Q \cap \text{supp } \mu \) with \( \delta^{-1}B \subset 2B_Q \) (where \( \delta \) is the constant in Lemma 4.1) such that \( \mu(B) \gtrsim \mu(Q) \) and \( r(B) \sim r(B_Q) \). Also, appealing to the porosity condition of \( \partial \Omega \) in \( E \) and the fact that \( \text{supp } \mu \subset E \), we may take another ball \( B_0 \) such that \( B_0 \subset B \setminus \partial \Omega \) with

\[ r(B_0) \sim r(B) \sim r(B_Q). \]

Here (as well as in the rest of the lemma) all implicit constants may depend on \( \delta \).

Denote by \( \mathcal{E}(x) \) the fundamental solution of the Laplacian in \( \mathbb{R}^{n+1} \), so that the Green function \( G(\cdot, \cdot) \) of \( \Omega \) equals

(7.2) \[ G(x, p) = \mathcal{E}(x - p) - \int \mathcal{E}(x - y) d\omega^p(y). \]
Notice that the kernel of the Riesz transform is
\begin{equation}
K(x) = c_n \nabla E(x),
\end{equation}
for a suitable absolute constant \(c_n\). For \(x \in \mathbb{R}^{n+1} \setminus \overline{\Omega}\), since \(K(x - \cdot)\) is harmonic in \(\Omega\), we have
\begin{equation}
\mathcal{R}\omega^p(x) = \int K(x - y) \, d\omega^p(y) = K(x - p).
\end{equation}

For \(x \in \Omega\), by (7.3) and (7.2) we have
\begin{equation}
\mathcal{R}\omega^p(x) = c_n \nabla \int E(x - y) \, d\omega^p(y) = c_n \nabla x (E(x - p) - G(x, p))
\end{equation}
\begin{equation}
= K(x - p) - c_n \nabla x G(x, p).
\end{equation}

So if \(B_0 \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}\), then (7.4) holds for all \(x \in B_0\), while if \(B_0 \subset \Omega\), then every \(x \in B_0\) satisfies (7.5). We claim that, in any case, for the center \(z_{B_0}\) of \(B_0\) we have
\begin{equation}
|\mathcal{R}\omega^p(z_{B_0})| \lesssim 1.
\end{equation}
This is clear if \(B_0 \subset \mathbb{R}^{n+1} \setminus \overline{\Omega}\), since in this case
\begin{equation}
|\mathcal{R}\omega^p(z_{B_0})| = |K(p - z_{B_0})| \sim \text{dist}(p, \partial\Omega)^{-n}.
\end{equation}

Suppose now that \(B_0 \subset \Omega\). From (7.5) we infer that for all \(x \in B_0\) we have
\begin{equation}
|\mathcal{R}\omega^p(x)|^2 \lesssim 1 + |\nabla x G(x, p)|^2.
\end{equation}
Averaging this with respect to the Lebesgue measure \(m\) on \(\frac{1}{2}B_0\) and applying Caccioppoli’s inequality,
\begin{equation}
\int_{\frac{1}{2}B_0} |\mathcal{R}\omega^p|^2 \, dm \lesssim 1 + \int_{\frac{1}{2}B_0} |\nabla x G(x, p)|^2 \, dm(x)
\end{equation}
\begin{equation}
\lesssim 1 + \int_{\frac{1}{2}B_0} \frac{|G(x, p) - G(z_{B_0}, p)|^2}{r(B_0)^2} \, dm(x).
\end{equation}

For \(x \in \frac{1}{2}B_0\), in the case \(n = 1\), by Lemma 4.2 we have
\begin{equation}
|G(x, p) - G(z_{B_0}, p)| \lesssim \frac{\omega^p(\delta^{-1}B)}{r(\delta^{-1}B)^{n-1}} \frac{1}{\omega^{z_{B_0}}(\delta^{-1}B)}
\end{equation}
The same estimate holds for \(n \geq 2\) using that
\begin{equation}
|G(x, p) - G(z_{B_0}, p)| \lesssim |G(x, p)| + G(z_{B_0}, p) \lesssim G(z_{B_0}, p),
\end{equation}
by Harnack’s inequality, and then plugging Lemma 4.2 again. Also, from Bourgain’s Lemma 4.1 and 4.5 we get
\begin{equation}
\omega^{z_{B_0}}(\delta^{-1}B) \geq \frac{\mu(B)}{r(B)^n}.
\end{equation}
Therefore,
\begin{equation}
|G(x, p) - G(z_{B_0}, p)| \lesssim \frac{\omega^p(\delta^{-1}B)}{r(\delta^{-1}B)^{n-1}} \frac{r(B)^n}{\mu(B)} \lesssim \frac{\omega^p(2B_Q) \, r(B)}{\mu(Q)}.
\end{equation}
From the fact that \(Q\) is doubling (from \(D^{db}\) and good), we deduce that \(\omega^p(2B_Q) \lesssim \omega^p(Q) \lesssim \tau^{-1} \mu(Q)\), and so
\begin{equation}
|G(x, p) - G(z_{B_0}, p)| \leq C(\tau) \, r(B) \quad \text{for all } x \in \frac{1}{2}B_0.
\end{equation}
Thus, by the harmonicity of $R \omega^p$ in $B_0$, Hölder’s inequality, (7.8), and the last estimate, we get
\[ |R \omega^p(z_{B_0})| = \left| \int_{\frac{1}{2}B_0} R \omega^p dm \right| \lesssim \int_{\frac{1}{2}B_0} |R \omega^p|^2 dm \lesssim 1 + \int_{\frac{1}{2}B_0} \frac{|G(x, p) - G(z_{B_0}, p)|^2}{r(B_0)^2} dm(x) \lesssim 1 \]
with the implicit constant depending on $\tau$ and other parameters of the construction, and so (7.6) holds in this case too.

From standard Calderón-Zygmund estimates and the fact that
\[ |R_r(B_0)\omega^p(z_{B_0})| = |R_{r(B_0)}\omega^p| \lesssim 1, \]
we derive that, for all $y \in B_Q$,
\[ |R_r(B_Q)\omega^p(y)| \lesssim |R_{r(B_0)}\omega^p(z_{B_0})| + M'_{\ell(Q)}\omega^p(z_{B_Q}) \lesssim A 1, \]
where $z_{B_Q}$ is the center of $B_Q$. In the last estimate we took into account that $Q$ and hence all its ancestors are good and thus $Q \not\in \text{HD}$. Hence the lemma holds when $Q \in \tilde{D}^{db} \cap \text{Good}$.

Consider now the case $Q \in \text{Good} \setminus \tilde{D}^{db}$. Let $Q' \supset Q$ be the cube from $\tilde{D}^{db}$ with minimal side length. The existence of $Q'$ is guarantied by the assumption in the lemma regarding the existence of some cube from $D_0^{db}$ containing $Q$. For all $y \in B_Q$ then we have
\[ |R_{r(B_Q)}\omega^p(y)| \leq |R_{r(B_0)}\omega^p(y)| + C \sum_{P \in D : Q \subset P \subset Q'} \Theta_\omega(2B_P). \]
The first term on the right hand side is bounded by some constant depending on $A, M, \tau, \ldots$. To bound the last sum we can apply Lemma 5.4 (because the cubes that are not from $\tilde{D}^{db}$ do not belong to $D^{db}$ either) and then we get
\[ \sum_{P \in D : Q \subset P \subset Q'} \Theta_\omega(2B_P) \lesssim C \Theta_\omega(4B'_{Q'}). \]
Finally, since $Q' \not\in \text{HD}$, we have $\Theta_\omega(4B'_{Q'}) \lesssim C A$. So (7.1) also holds in this case. $\square$

From the lemma above we deduce the following corollary.

**Lemma 7.2.** For $Q \in \text{Good}$ and $x \in B_Q$, we have
\[ (7.9) \quad R_{r(B_Q)}\omega^p(x) \leq C(A, M, \tau, d_p), \]
where, to shorten notation, we wrote $d_p = \text{dist}(p, \partial \Omega)$.

### 8. The end of the proof of Theorem 1.1

Set
\[ G = F_M \cap \bigcup_{Q \in \tilde{D}^{db}} Q \setminus \bigcup_{Q \in \text{Bad}} Q. \]
and recall that, by Lemma 6.1
\[ \omega^p(G) > 0. \]
As shown in (6.2), we have
\begin{equation}
M^\alpha \omega^p(x) \lesssim A \quad \text{for } \omega^p\text{-a.e. } x \in G.
\end{equation}
On the other hand, from Lemma 7.2 is also clear that
\begin{equation}
R_* \omega^p(x) \leq C(A, M, \tau, d_p) \quad \text{for } \omega^p\text{-a.e. } x \in G.
\end{equation}

Now we will apply the following result.

**Theorem 8.1.** Let \( \sigma \) be a Radon measure with compact support on \( \mathbb{R}^d \) and consider a \( \sigma \)-measurable set \( G \) with \( \sigma(G) > 0 \) such that
\[
G \subset \{ x \in \mathbb{R}^d : M^\alpha \sigma(x) < \infty \text{ and } R_* \sigma(x) < \infty \}.
\]
Then there exists a Borel subset \( G_0 \subset G \) with \( \sigma(G_0) > 0 \) such that \( \sup_{x \in G_0} M^\alpha \sigma|_{G_0}(x) < \infty \) and \( R_\sigma|_{G_0} \) is bounded in \( L^2(\sigma|_{G_0}) \).

This result follows from the deep non-homogeneous Tb theorem of Nazarov, Treil and Volberg in [NTrV] (see also [Vol]) in combination with the methods in [To1]. For the detailed proof in the case of the Cauchy transform, see [To2, Theorem 8.13]. The same arguments with very minor modifications work for the Riesz transform.

From (8.1), (8.2) and Theorem 8.1 applied to \( \sigma = \omega^p \) in case that \( \partial \Omega \) is compact, we infer that there exists a subset \( G_0 \subset G \) such that the operator \( R_\omega^p|_{G_0} \) is bounded in \( L^2(\omega^p|_{G_0}) \).

If \( \partial \Omega \) is non-compact, then we consider a ball \( B(0, R) \) such that \( \omega^p(G \cap B(0, R)) > 0 \) and we set \( \sigma = \chi_{B(0,2R)} \omega^p \). Since
\[
R_*(\chi_{B(0,2R)} \omega^p)(x) \leq \frac{\omega^p(B(0,2R)^c)}{R} < \infty \quad \text{for all } x \in B(0, R),
\]
from (8.2) we infer that \( R_\omega^p(x) < \infty \) for all \( x \in G \cap B(0, R) \), and so we can argue as above.

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