ON THE WAVE FUNCTIONS AND ENERGY SPECTRUM
FOR A SPIN 1 PARTICLE IN EXTERNAL COULOMB FIELD

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Quantum-mechanical system – spin 1 particle in external Coulomb field is studied on the base of the matrix Duffin – Kemmer – Petiau formalism with the use of the tetrad technique. Separation of the variables is performed with the help of Wigner functions \( D^{\pm 1}_{m, \sigma}(\phi, \theta, 0) \); \( \sigma = -1, 0, +1 \); \( j \) and \( m \) stand for quantum numbers determining the square and the third projection of the total angular momentum of the vector particle. With the help of parity operator, the radial 10-equation system is divided into two subsystem of 4 and 6 equations that correspond to parity \( P = (-1)^{j+1} \) and \( P = (-1)^j \) respectively. The system of 4 equation is reduced to a second order differential equation which coincides with that arising in the case of a scalar particle in Coulomb potential. It is shown that the 6-equation system reduces to two different second order differential equations for a ”main” function. One main equation reduces to a confluent Heun equation and provides us with energy spectrum. Another main equation is a more complex one, and any solutions for it are not constructed.

I. INTRODUCTION AND SEPARATION OF THE VARIABLES

Many years ago, a very peculiar behavior of a spin 1 particle in presence of the external Coulomb field was noticed by I.E. Tamm [1]. As far as we known the whole situation with this system stays much the same. In the present paper, we start examining the problem on the base of the matrix Duffin – Kemmer – Petiau formalism [2, 3, 4, 5] with the use of the tetrad generally covariant technique (for more details and references see in [16–18, 27]), it turns out to be more convenient than a common Proca tensor approach [6, 7]). Choosing a diagonal spherical tetrad according to

\[
\begin{align*}
    dS^2 &= dt^2 - dr^2 - r^2(d\theta^2 + \sin^2 \theta d\phi^2), \\
    e^0_{(0)} &= (1, 0, 0, 0), & e^0_{(3)} &= (0, 1, 0, 0), \\
    e^1_{(1)} &= (0, 0, 1, 0), & e^1_{(2)} &= (1, 0, 0, \frac{1}{r \sin \theta}).
\end{align*}
\]

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we reduce the main D-K-P stationary equation to the form
\[
\left[ \beta^0 (\epsilon + \frac{\alpha}{r}) + i \right. \\
\left. \beta^3 \partial_r + \frac{1}{r} (\beta^1 j^{31} + \beta^2 j^{32}) \right] + \frac{1}{r} \Sigma_{\theta, \phi} - M \right] \Phi(x) = 0 ,
\]
where \( \epsilon = E/(ch), \alpha = e^2/(ch), M = mc/\hbar \) and \( \Sigma_{\theta, \phi} \) stands for an angular operator (its form means that we have here a generalized Schrödinger – Pauli basis \([8, 9]\))
\[
\Sigma_{\theta, \phi} = i \beta^1 \partial_\theta + \beta^2 \frac{i \partial_\theta + i j^{12} \cos \theta}{\sin \theta} .
\]
Spherical waves with \((j, m)\) quantum numbers should be constructed within the following general substitution (we adhere notation developed in Red’kov \([16-22, 30]\); before similar techniques was applied by Dray \([12, 13]\), Krolikowski and Turski \([14]\), Turski \([15]\); many years ago such a tetrad basis was used by Schrödinger \([8]\) and Pauli \([9]\) when looking at the problem of single-valuedness of wave functions in quantum theory – then the case of spin \( S = 1/2 \) particle was specified)
\[
\Psi(x) = \{ \Phi_0(x), \Phi(x), \tilde{E}(x), \tilde{H}(x) \} ,
\]
\[
\Phi_0(x) = e^{-iEt/\hbar} \Phi_0(r) D_0 , \quad \Phi(x) = e^{-iEt/\hbar} \\
\tilde{E}(x) = e^{-iEt/\hbar} \begin{vmatrix} E_1(r) D_{-1} \\ E_2(r) D_0 \\ E_3(r) D_{+1} \end{vmatrix} , \quad \tilde{H}(x) = e^{-iEt/\hbar} \begin{vmatrix} H_1(r) D_{-1} \\ H_2(r) D_0 \\ H_3(r) D_{+1} \end{vmatrix} ;
\]
short notation for Wigner functions \([10]\) is used: \( D_\sigma = D^{j}_{-m, \sigma}(\phi, \theta, 0) , \quad \sigma = 0, +1, -1 \). In accordance with Pauli approach \([9]\) the quantum number \( j \) takes values 0,1,2, ... With the help of recurrent formulas \([11]\) (where \( \nu = \sqrt{j(j+1)} , \quad a = \sqrt{(j-1)(j+2)} \))
\[
\partial_\theta D_{-1} = \frac{1}{2} ( a D_{-2} - \nu D_0 ) , \quad \frac{m - \cos \theta}{\sin \theta} D_{-1} = \frac{1}{2} ( \nu D_{-1} + \nu D_{+1} ) , \quad \frac{m - \cos \theta}{\sin \theta} D_{0} = \frac{1}{2} ( \nu D_{-1} + \nu D_{+1} ) , \quad \frac{m - \cos \theta}{\sin \theta} D_{+1} = \frac{1}{2} ( a D_{-2} + \nu D_0 ) , \quad \frac{m + \cos \theta}{\sin \theta} D_{0} = \frac{1}{2} ( \nu D_{-1} + \nu D_{+1} ) , \quad \frac{m + \cos \theta}{\sin \theta} D_{+1} = \frac{1}{2} ( a D_{-2} + \nu D_0 ) , \quad \frac{m + \cos \theta}{\sin \theta} D_{+1} = \frac{1}{2} ( \nu D_{-1} + \nu D_{+1} ) ,
\]
after simple algebraic calculation we arrive at the radial equations (for clarity corresponding Proca tensor equations are written down as well)
where $\nu = \sqrt{j(j+1)/2}$ (note the factor $1/\sqrt{2}$).

Concurrently with $\vec{J}^2, J_3$ let us diagonalize an operator of spatial inversion $\hat{\Pi}$. After transition to spherical tetrad basis, and also to cyclic representation for D-K-P matrices $\beta^a$, for this discrete operator we get

$$\hat{\Pi} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & \Pi^3 & 0 & 0 \\ 0 & 0 & \Pi^3 & 0 \\ 0 & 0 & 0 & -\Pi^3 \end{vmatrix}, \quad \hat{P}, \quad \Pi_3 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{vmatrix}. \quad (7)$$

Eigen-value equation $\hat{\Pi}\Psi = P \Psi$ results in two different in parity states

$$P = (-1)^{j+1}, \quad \Phi_0 = 0, \quad \Phi_3 = -\Phi_1, \quad \Phi_2 = 0, \quad E_3 = -E_1, \quad E_2 = 0, \quad H_3 = H_1; \quad (8)$$

$$P = (-1)^j, \quad \Phi_3 = \Phi_1, \quad E_3 = +E_1, \quad H_3 = -H_1, \quad H_2 = 0. \quad (9)$$

Correspondingly, eqs. (5) – (6) give 4 and 6 equations.
\( P = (-1)^{j+1}, \)
\[
+i(\epsilon + \frac{\alpha}{r}) E_1 + i\left(\frac{d}{dr} + \frac{1}{r}\right)H_1 + \frac{\nu}{r} H_2 = M \Phi_1 ,
\]
\[
-i(\epsilon + \frac{\alpha}{r}) \Phi_1 = ME_1 , \quad -i\left(\frac{d}{dr} + \frac{1}{r}\right)\Phi_1 = MH_1 , \quad 2i\frac{\nu}{r} \Phi_1 = MH_2 ,
\]

excluding \( E_1, H_1, H_2 \) we get a second order differential equation for \( \Phi_1 \)
\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + (\epsilon + \frac{\alpha}{r})^2 - M^2 - \frac{j(j+1)}{r^2} \right] \Phi_1 = 0 ,
\]
which coincides with that arising in the case of a scalar particle in Coulomb potential (it is the
known fact that was noted in Tamm’s first paper [1]. Its solution is well known and provides us
with the following energy spectrum (in usual units)
\[
E = \frac{mc^2}{\sqrt{1 + \alpha^2/N^2}} , \quad N = n + \frac{1}{2} + \sqrt{(j + 1)^2 - \alpha^2} .
\]

For states with parity \( P = (-1)^j \) we have the system

\[
P = (-1)^j ,
\]
\[
\left(\frac{d}{dr} + \frac{2}{r}\right)E_2 + 2\frac{\nu}{r} E_1 + M \Phi_0 = 0 ,
\]
\[
+i(\epsilon + \frac{\alpha}{r}) E_1 + i\left(\frac{d}{dr} + \frac{1}{r}\right)H_1 - M \Phi_1 = 0 ,
\]
\[
+i(\epsilon + \frac{\alpha}{r}) E_2 - 2i\frac{\nu}{r} H_1 - M \Phi_2 = 0 ,
\]
\[
-i(\epsilon + \frac{\alpha}{r}) \Phi_1 + \frac{\nu}{r} \Phi_0 - ME_1 = 0 ,
\]
\[
i(\epsilon + \frac{\alpha}{r}) \Phi_2 + \frac{d}{dr} \Phi_0 + ME_2 = 0 ,
\]
\[
i\left(\frac{d}{dr} + \frac{1}{r}\right)\Phi_1 + i\frac{\nu}{r} \Phi_2 + MH_1 = 0 .
\]

II. THE CASE OF MINIMAL VALUE \( j = 0 \)

The states with minimal value \( j = 0 \) can be treated straightforwardly (also see in [20]). In this
case, we should start with special substitution of the wave function
\[
\Phi_0(x) = e^{-iEt/h} \Phi_0(r) , \quad C(x) = e^{-iEt/h} C(r) , \quad C_0(x) = e^{-i\sigma} C_0(r) ,
\]
\[
\tilde{C}(x) = e^{-iEt/h} \begin{vmatrix} 0 \\ C_2(r) \end{vmatrix} , \quad \tilde{\Phi}_0(x) = e^{-i\sigma} \begin{vmatrix} 0 \\ \Phi_2(r) \end{vmatrix} ,
\]
\[
\tilde{E}(x) = e^{-iEt/h} \begin{vmatrix} 0 \\ E_2(r) \end{vmatrix} , \quad \tilde{H}(x) = e^{-i\sigma} \begin{vmatrix} 0 \\ H_2(r) \end{vmatrix} .
\]
The $\Sigma_{\theta,\varphi}$ acts on this function as a zero operator, and parity $P = (-1)^{0+1} = -1$. Now the radial system is (for eliminating imaginary unit $i$, we use slightly different variables: $\Phi_0 = \varphi_0$, $-i\Phi_1 = \varphi_1$, $-i\Phi_2 = \varphi_2$)

$$H_2 = 0, \quad -(\frac{d}{dr} + \frac{2}{r})E_2 = M\varphi_0,$$

$$(\epsilon + \frac{\alpha}{r})E_2 = M\varphi_2, \quad (\epsilon + \frac{\alpha}{r})\varphi_2 - \frac{d}{dr}\varphi_0 = ME_2. \quad (15)$$

From whence it follows a second order equation for $E_2$

$$\left[ \frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - \frac{2}{r^2} + (\epsilon + \frac{\alpha}{r})^2 - M^2 \right] E_2 = 0. \quad (16)$$

With substitution $E_2(r) = r^{-1}f(r)$, we get

$$\frac{d^2}{dr^2}f + (\epsilon^2 - M^2 + \frac{2\alpha\epsilon}{r} - \frac{2 - \alpha^2}{r^2})f = 0. \quad (17)$$

In dimensionless variables

$$x = r\epsilon = \frac{rE}{\hbar}, \quad \frac{M^2}{\epsilon^2} = \frac{m^2c^4}{E^2} = \lambda^2,$$

it reads

$$\frac{d^2}{dx^2}f + (1 - \lambda^2 + \frac{2\alpha}{x} - \frac{2 - \alpha^2}{x^2})f = 0. \quad (18)$$

With the substitution $f(x) = x^a e^{-bx}F(x)$, for $F$ we obtain

$$xF'' + (2a - 2bx)F' + \left[ \frac{a(a-1) + \alpha^2 - 2}{x} + (b^2 + 1 - \lambda^2)x + (2\alpha - 2ab) \right]F = 0.$$

Requiring

$$a = 1 \pm \sqrt{9 - 4\alpha^2}, \quad b = \pm \sqrt{\lambda^2 - 1} = \pm \sqrt{\frac{m^2c^4 - E^2}{E}}, \quad (19)$$

the choice of upper signs in the formulas provides us with good parameters for bound states, we get

$$xF'' + 2(a - bx)F' + 2(\alpha - ab)F = 0. \quad (20)$$

This equation is solved in confluent hypergeometric functions. Let us specify this solution in detail.

With the use of expansion for $F(x)$

$$F(x) = \sum_{k=0}^{\infty} C_k x^k, \quad F' = \sum_{k=1}^{\infty} kC_k x^{k-1}, \quad F'' = \sum_{k=2}^{\infty} k(k-1)C_k x^{k-2},$$
we get
\[\sum_{k=2}^{\infty} k(k - 1)C_k x^{k-1} + 2a \sum_{k=1}^{\infty} kC_k x^{k-1} - 2b \sum_{k=1}^{\infty} kC_k x^k + 2(\alpha - ab) \sum_{k=0}^{\infty} c_k x^k = 0,\]
from whence it follows
\[\sum_{k=1}^{\infty} kC_k x^{k-1} + 2\sum_{k=1}^{\infty} kC_k x^k + 2(\alpha - ab) \sum_{k=0}^{\infty} c_k x^k = 0.\]
Therefore, we arrive at recurrent formulas
\[\begin{align*}
C_1 &= - (\alpha - ab) C_0 = 0, \\
C_2 &= (1 + 2a) = 2 \left[ b - (\alpha - ab) \right] C_1 = 0, \quad n = 2, 3, 4, ..., \\
C_{n+1} &= (n + 1) (n + 2a) = 2 \left[ n b - (\alpha - ab) \right] C_n = 0. \quad (21)
\end{align*}\]
To get the polynomial we must require
\[C_{N+1} = 0 \implies \left[ n b - (\alpha - ab) \right] = 0.\]
This quantization rule gives
\[\frac{\alpha - ab}{b} = N, \quad a = \frac{1 + \sqrt{9 - 4\alpha^2}}{2}, \quad b = \frac{\sqrt{m^2c^4 - E^2}}{E},\]
so that
\[\frac{2\alpha \epsilon - (1 + \sqrt{9 - 4\alpha^2})\sqrt{m^2c^4 - E^2}}{2\sqrt{m^2c^4 - E^2}} = N;\]
its solution is
\[E = \frac{mc^2}{\sqrt{1 + \alpha^2/(\Gamma + N)^2}}, \quad 2\Gamma = 1 + \sqrt{9 - 4\alpha^2}. \quad (22)\]

III. LORENTZ CONDITION IN PRESENCE OF COULOMB POTENTIAL

As known for a massive spin 1 particle there must exist a generalized Lorentz condition. Let us specify it for the problem under consideration. In the Proca tensor equations for the vector particle
\[\begin{align*}
D_\alpha \Psi_\beta - D_\beta \Psi_\alpha &= M \Psi_{\alpha \beta}, \\
D^\alpha \Psi_{\alpha \beta} &= M \Psi_\beta, \quad (23)
\end{align*}\]
where $D_\alpha = \nabla_\alpha + i (e/c) A_\alpha$, let us act by the operator $D_\alpha$ on the second equation in (23), we get

$$\left( \nabla_\alpha + i \frac{e}{c} A_\alpha \right) \Psi^\alpha = \frac{ie}{2cM} F_{\alpha\beta} \Psi^{\alpha\beta}.$$  (24)

This Lorentz condition should be translated to tetrad form. To this end, instead of $\Psi^\alpha$ and $\Psi^{\alpha\beta}$ one should introduced their tetrad components

$$\Psi^\alpha = e^{(\alpha)\alpha} \Psi_{(a)}, \quad \Psi^{\alpha\beta} = e^{(\alpha)\alpha} e^{(\beta)\beta} \Psi_{(a)(b)}.$$

Correspondingly, eq. (24) takes the form

$$(e^{(\alpha)\alpha} \Psi_a + e^{(\alpha)\alpha} \partial_\alpha ) \Psi_{(a)} + i \frac{e}{c} A^{(\alpha)} \Psi_{(a)} = i \frac{e}{2cM} F_{(a)(b)} \Psi_{(a)(b)}.$$  (25)

The Coulomb field $A_0 = e/r$, $F_{r0} = -e/r^2$ in the tetrad description looks

$$A^{(0)} = e^{(0)0} A_0 = \frac{e}{r}, \quad F^{(3)0} = e^{(3)0} e^{(0)0} F_{r0} = -\frac{e}{r^2}.  \quad (26)$$

Beside, after simple calculation we get

$$e^{(0)\alpha}_{i\alpha} = 0, \quad e^{(1)\alpha}_{i\alpha} = -\frac{\cos \theta}{r \sin \theta}, \quad e^{(2)\alpha}_{i\alpha} = 0, \quad e^{(3)\alpha}_{i\alpha} = -\frac{2}{r}.$$  

The components functions $\Psi_{(a)}$ and $\Psi_{(a)(b)}$ in (25) can be related with components of the D-K-P column as follows (transition between cyclic and Cartesian representations; $c \equiv 1/\sqrt{2}$)

| $\Psi_{(0)}$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\Phi_{(0)D_0}$ |
| $\Psi_{(1)}$ | $0$ | $-c$ | $0$ | $+c$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\Phi_{(1)D_{-1}}$ |
| $\Psi_{(2)}$ | $0$ | $-ic$ | $0$ | $-ic$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\Phi_{(2)D_0}$ |
| $\Psi_{(3)}$ | $0$ | $0$ | $1$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $\Phi_{(3)D_{+1}}$ |
| $\Psi_{(0)(1)}$ | $0$ | $0$ | $0$ | $0$ | $-c$ | $0$ | $+c$ | $0$ | $0$ | $E_{(1)D_{-1}}$ |
| $\Psi_{(0)(2)}$ | $0$ | $0$ | $0$ | $-ic$ | $0$ | $-ic$ | $0$ | $0$ | $0$ | $E_{(2)D_0}$ |
| $\Psi_{(0)(3)}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $0$ | $0$ | $0$ | $E_{(3)D_{+1}}$ |
| $\Psi_{(2)(3)}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-c$ | $0$ | $+c$ | $H_{(1)D_{-1}}$ |
| $\Psi_{(3)(1)}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $-ic$ | $0$ | $-ic$ | $H_{(2)D_0}$ |
| $\Psi_{(1)(2)}$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $0$ | $1$ | $0$ | $H_{(3)D_{+1}}$ |

We need only $\Phi_{(0)}$, $\Psi_{(1)}$, $\Psi_{(2)}$, $\Psi_{(3)}$, $\Psi_{(0)(3)}$

$$\Psi_{(1)} = e^{-iEt/h} \frac{1}{\sqrt{2}} \left( -\Phi_{(1)D_{-1}} + \Phi_{(3)D_{+1}} \right),$$

$$\Psi_{(2)} = e^{-iEt/h} \frac{i}{\sqrt{2}} \left( -\Phi_{(1)D_{-1}} + \Phi_{(3)D_{+1}} \right),$$

$$\Psi_{(0)} = e^{-iEt/h} \Phi_{(0)D_0}, \quad \Psi_{(3)} = e^{-iEt/h} \Phi_{(2)D_0},$$

$$\Psi_{(0)(3)} = e^{-iEt/h} E_{(2)D_0}.  \quad (27)$$
With the help of (27), eq. (25) gives

\[
\begin{align*}
\frac{1}{\sqrt{2}} r \Phi_1 (\partial_\theta D_{-1} - \frac{M - 1 \cos \theta}{\sin \theta} D_{-1}) - \\
\frac{1}{\sqrt{2}} r \Phi_3 (\partial_\theta D_{+1} + \frac{M + 1 \cos \theta}{\sin \theta} D_{+1}) + \\
+ D_0 (\frac{2}{r} \Phi_2 - i \epsilon \Phi_0 - \frac{d}{dr} \Phi_2) = i \frac{\alpha}{2Mr^2} E_2 D_0.
\end{align*}
\]

Now, by taking into account the recurrent formulas [11]

\[
\begin{align*}
\partial_\theta D_{-1} - \frac{M - 1 \cos \theta}{\sin \theta} D_{-1} &= -\sqrt{(j + 1)j} D_0, \\
\partial_\theta D_{+1} - \frac{M + 1 \cos \theta}{\sin \theta} D_{+1} &= -\sqrt{(j + 1)j} D_0,
\end{align*}
\]

we arrive at the Lorentz condition in radial form

\[
- i \epsilon \Phi_0 - \left( \frac{d}{dr} + \frac{2}{r} \right) \Phi_2 - \frac{\nu}{r} (\Phi_1 + \Phi_3) = i \frac{\alpha}{2Mr^2} E_2.
\]

(28)

For states with parity \( P = (-1)^{j+1} \), this condition is satisfied identically. For states with parity \( P = (-1)^j \) it gives

\[
- i \epsilon \Phi_0 - \left( \frac{d}{dr} + \frac{2}{r} \right) \Phi_2 - \frac{2\nu}{r} \Phi_1 = i \frac{\alpha}{2Mr^2} E_2.
\]

(29)

From eq. (29) a very important relationship can be established. To this end, from eq. (29) let us exclude \( \Phi_2 \) with the help of the third equation in (13)

\[
i \epsilon M \Phi_0 + i(\epsilon + \frac{\alpha}{r})(\frac{d}{dr} + \frac{2}{r})E_2 - \frac{2i\nu}{r} (\frac{d}{dr} + \frac{1}{r})H_1 + \frac{2\nu M}{r} \Phi_1 = i \frac{\alpha}{2r^2} E_2.
\]

Let us transform the second and third terms with the help of 1-st and 2-nd equations in (13) – it results in

\[
E_2 = -2Mr \Phi_0.
\]

(30)

IV. THE MAIN FUNCTION OF THE FIRST TYPE, REDUCING THE PROBLEM TO THE CONFLUENT HEUN EQUATION

Let us examine eqs. (13), now with the use of the Lorentz condition (29) and its consequence (30). From the 1-st eq. in (13), we produce

\[
E_1 = \frac{Mr}{2\nu} (5 + 2r \frac{d}{dr})\Phi_0.
\]

(31)
With the help of (30), the fourth eq. in (13) gives
\[
\Phi_1 = \frac{-i}{\epsilon + \alpha/r} \left( \frac{\nu}{r} - \frac{5M^2}{2\nu} r - \frac{r^2M^2}{\nu} \frac{d}{dr} \right) \Phi_0. \quad (32)
\]

With the help of (30), the fifth eq. in (13) gives
\[
\Phi_2 = \frac{i}{\epsilon + \alpha/r} \left( \frac{d}{dr} - 2M^2 r \right) \Phi_0, \quad (33)
\]

The sixth eq. in (13) provides us with the representation for \( H_1 \) in term of \( \Phi_0 \) by means of a second order differential operator
\[
-MH_1 = \left[ \left( \frac{d}{dr} + \frac{1}{r} \right) \frac{1}{\epsilon + \alpha/r} \left( \frac{\nu}{r} - \frac{5M^2}{2\nu} r - \frac{r^2M^2}{\nu} \frac{d}{dr} \right) - \frac{1}{r \epsilon + \alpha/r} \left( \frac{d}{dr} - 2M^2 r \right) \right] \Phi_0. \quad (34)
\]

Take note that from the third eq. in (13) one can obtain another representation for \( H_1 \) that uses only the first order operator
\[
H_1 = -\frac{Mr}{2\nu} \frac{1}{\epsilon + \alpha/r} \left[ \frac{d}{dr} + 2r \left( (\epsilon + \frac{\alpha}{r})^2 - M^2 \right) \right] \Phi_0. \quad (35)
\]

These two representations for \( H_1 \) must be consistent with each other.

Now, turning to the Lorentz condition
\[
-i \epsilon \Phi_0 - \left( \frac{d}{dr} + \frac{2}{r} \right) \Phi_2 - \frac{2\nu}{r} \Phi_1 = i \frac{\alpha}{2Mr^2} E_2.
\]

one can readily derive a second order differential equation for \( \Phi_0 \):
\[
\frac{d^2\Phi_0}{dr^2} + \frac{1}{r} \left( 3 - \frac{\epsilon}{\epsilon + \alpha/r} \right) \frac{d\Phi_0}{dr} + \left( \epsilon^2 - \frac{\alpha^2}{r^2} - 3M^2 + 2M^2 \frac{\epsilon}{\epsilon + \alpha/r} - \frac{2\nu^2}{r^2} \right) \Phi_0 = 0. \quad (36)
\]

The function \( \Phi_0 \) will be termed as main function, 5 remaining ones \( \Phi_1, \Phi_2, E_1, H_1 \) are expressed through it
\[
E_2 = -2Mr \Phi_0, \quad 1
\]
\[
E_1 = \frac{Mr}{2\nu} \left( 5 + 2r \frac{d}{dr} \right) \Phi_0, \quad 2
\]
\[
\Phi_1 = \frac{-i}{\epsilon + \alpha/r} \left( \frac{\nu}{r} - \frac{5M^2}{2\nu} r - \frac{r^2M^2}{\nu} \frac{d}{dr} \right) \Phi_0, \quad 3
\]
\[
\Phi_2 = \frac{i}{\epsilon + \alpha/r} \left( \frac{d}{dr} - 2M^2 r \right) \Phi_0, \quad 4
\]
\[
H_1 = -\frac{Mr}{2\nu} \frac{1}{\epsilon + \alpha/r} \left[ \frac{d}{dr} + 2r \left( (\epsilon + \frac{\alpha}{r})^2 - M^2 \right) \right] \Phi_0. \quad 5
\]
Changing the variable

\[ x = -\frac{\epsilon}{\alpha} r < 0, \quad r = -\alpha \epsilon x; \]

eq (36) is reduced to the form

\[
\frac{d^2 \Phi_0}{dx^2} + \left( \frac{3}{x} - \frac{1}{x-1} \right) \frac{d\Phi_0}{dx} + \left( \alpha^2 - \Lambda^2 - \frac{\alpha^2 + 2\nu}{x} + \frac{2\Lambda^2}{x-1} \right) \Phi_0 = 0; \tag{38}
\]

where dimensionless parameters were used

\[ \Lambda^2 = \alpha^2 \lambda^2, \quad \lambda = \frac{mc}{E} > 1. \]

Let us consider behavior of the main function near the point \( x = 0 \):

\[
\frac{d^2 \Phi_0}{dx^2} + \frac{3}{x} \frac{d\Phi_0}{dx} - \frac{\alpha^2 + 2\nu}{x} \Phi_0 = 0, \quad \Phi_0 \sim \text{const} \ x^A, \quad A(A - 1) + 3A - \alpha^2 - 2\nu^2 = 0, \]

\[ A = -1 - \sqrt{1 + \alpha^2 + 2\nu^2}, \quad A = -1 + \sqrt{1 + \alpha^2 + 2\nu^2}; \tag{39}\]

to bound states there correspond positive values of \( A \). In the region near \( x = +\infty \), the main equation gives

\[
\frac{d^2 \Phi_0}{dx^2} + \frac{2}{x} \frac{d\Phi_0}{dx} + \left( \alpha^2 - \Lambda^2 \right) \Phi_0 = 0, \quad \Phi_0 = e^{+\sqrt{\alpha^2 - \Lambda^2} x} = e^{-\sqrt{m^2c^2-E^2}/hc}; \tag{40}\]

to bound states there correspond solutions vanishing at infinity.

Now, let us introduce substitution \( \Phi_0(x) = x^A e^{Bx} f(x) \), eq. (38) gives

\[
\frac{d^2 f}{dx^2} + \left[ 2B + \frac{2A + 3}{x} + \frac{1}{1 - x} \right] \frac{df}{dx} + \left[ B^2 + \alpha^2 - \Lambda^2 + \frac{2AB + A + 3B}{x} \right. \\
\left. + \frac{A(A - 1) + 3A - \alpha^2 - 2\nu^2}{x^2} + \frac{A + B - 2\Lambda^2}{1 - x} \right] f(x) = 0. \tag{41}\]

With restrictions on \( A \) and \( B \):

\[ A(A - 1) + 3A - \alpha^2 - 2\nu^2 = 0 \implies A = -1 + \sqrt{1 + 2\nu^2 + \alpha^2}; \]

\[ B^2 + \alpha^2 - \Lambda^2 = 0 \implies B = +\sqrt{\Lambda^2 - \alpha^2}, \tag{42}\]

eq (41) takes the form

\[
\frac{d^2 f}{dx^2} + \left[ 2B + \frac{2A + 3}{x} - \frac{1}{x - 1} \right] \frac{df}{dx} + \\
+ \left[ \frac{2AB + A + 3B}{x} + \frac{A + B - 2\Lambda^2}{1 - x} \right] f = 0. \tag{43}\]
It can be recognized as the confluent Heun’s equation
\[
f = f(a, b, c, d, h; z), \quad \frac{d^2 f}{dx^2} + \left(a + \frac{b + 1}{x} + \frac{c + 1}{x - 1}\right) \frac{df}{dx} - \frac{[-2d + a(-b - c - 2)]x + a(1 + b) + b(-1 - c) - c - 2h}{2x(x - 1)} f = 0 \quad (44)
\]
with parameters given by
\[
a = +2\sqrt{\Lambda^2 - \alpha^2}, \quad b = +2\sqrt{1 + 2\nu^2 + \alpha^2}, \\
c = -2, \quad d = 2\Lambda^2, \quad h = +2. \quad (45)
\]

The known condition for polynomial solutions is
\[
d = -a \left(n + \frac{b + c + 2}{2}\right), \quad (46)
\]
it gives the following quantization rule
\[
\frac{\Lambda^4}{\Lambda^2 - \alpha^2} = (n + \sqrt{1 + 2\nu^2 + \alpha^2})^2. \quad (47)
\]
Its physical solution is
\[
E^2 = m^2c^4 \frac{2\alpha^2}{N^2 - \sqrt{N^4 - 4\alpha^2 N^2}}. \quad (48)
\]
When \(N\) increases to infinity, we get
\[
N \to \infty, \quad E^2 = m^2c^4 \frac{2\alpha^2}{N^2 - N^2\sqrt{1 - 4\alpha^2 N^{-2}}} \approx m^2c^4 \frac{2\alpha^2}{2\alpha^2} = m^2c^4. \quad (49)
\]

To obtain a non-relativistic limit, one must impose special restriction
\[
N = n + \sqrt{1 + 2\nu^2 + \alpha^2} = n + \sqrt{1 + j(j + 1) + \alpha^2} \approx n + j \quad (50)
\]
which correlates with the known non-relativistic procedure
\[
m + \epsilon + \frac{\alpha}{r} \approx m + \epsilon.
\]

Taking into account eq. (50), one can we derive
\[
E^2 = m^2c^4 \frac{1}{2} (1 + \sqrt{1 - \frac{4\alpha^2}{N^2}}) \approx m^2c^4 (1 - \frac{\alpha^2}{N^2}) \quad N \approx n + j,
\]
that is
\[
E = mc^2 (1 - \frac{\alpha^2}{2N^2}) = mc^2 + E'. \quad (51)
\]

Thus, the non-relativistic energy levels are given by
\[
E' = -\frac{\alpha^2 mc^2}{2N^2} = -\frac{me^4}{\hbar^2 N^2}, \quad (52)
\]
which coincides with the known exact result (see below).
V. THE MAIN RADIAL FUNCTION OF THE SECOND TYPE, REDUCING THE PROBLEM TO ANOTHER SECOND ORDER DIFFERENTIAL EQUATION

From general considerations, we may expect two linearly independent solutions for 6-equation system for state with parity \( P = (-1)^j \). The above equation (36) provides us with only one class of these, what make us look for another class (may with with some different main function).

In this connection, let us turn back to eq. (35) multiplied by \(-M\) and compare it with eq. (34), from that it follows a second order differential equation \( \Phi_0 \), different from (36):

\[
\frac{d^2 \Phi_0}{dr^2} + \frac{1}{r} \left( 6 + \frac{\alpha}{r(\epsilon + \alpha/r)} \right) \frac{d\Phi_0}{dr} + \left[ \epsilon^2 - M^2 + \frac{2\epsilon^2\alpha}{er + \alpha} - \frac{\alpha\nu^2}{r^4M^2(er + \alpha)} - \frac{1}{2} \frac{\alpha(-15 + 4\nu^2 - 2\alpha^2)}{r^2(er + \alpha)} - \frac{\epsilon(-5 + 2\nu^2 - 3\alpha^2)}{r(er + \alpha)} \right] \Phi_0 = 0. \tag{53}
\]

In the variable \( x \), it reads

\[
\frac{d^2 \Phi_0}{dx^2} + \frac{1}{x} \left( 6 - \frac{x}{x - 1} \right) \frac{d\Phi_0}{dx} + \left[ (1 - \lambda^2)\alpha^2 - \frac{2\alpha^2}{x - 1} + \frac{\nu^2}{\alpha^2\lambda^2(x - 1)} + \frac{(-15 + 4\nu^2 - 2\alpha^2)}{2\alpha^2(x - 1)} - \frac{\epsilon(-5 + 2\nu^2 - 3\alpha^2)}{x(x - 1)} \right] \Phi_0 = 0. \tag{54}
\]

By means of the coordinate transformation \( y = x^{-1} \), eq. (54) becomes

\[
\frac{d^2 \Phi_0}{dy^2} + \frac{4y - 3}{y(1-y)} \frac{d\Phi_0}{dy} + \left[ (1 - \lambda^2)\alpha^2 - \frac{2\alpha^2}{y^4(1-y)} + \frac{\nu^2y}{\alpha^2\lambda^2(1-y)} - \frac{(15 - 4\nu^2 + 2\alpha^2)}{2y(1-y)} - \frac{(5 + 2\nu^2 - 3\alpha^2)}{y^2(1-y)} \right] \Phi_0 = 0. \tag{55}
\]

Both differential equations, (54) and (55), are very complex. We might expect that they can describe some third class of solutions, however any proof of this does not exist now.

VI. NON-RELATIVISTIC LIMIT, EXACT ENERGY SPECTRUM

In treating this point we will use results on non-relativistic limit for a vector particle according to [21, 22], general treatment of the problem of the wave equations for arbitrary spin particle see in [23], [24, 25]; also see in [26].

First, let us consider the simpler system (56) for states with parity \( P = (-1)^{j+1} \):

\[
P = (-1)^{j+1}, \quad +i(\epsilon + \frac{\alpha}{r})E_1 + i\left( \frac{d}{dr} + \frac{1}{r} \right)H_1 + i\frac{\nu}{r}H_2 = M\Phi_1,
- \frac{d}{dr} + \frac{1}{r}\Phi_1 = ME_1, \quad -i\frac{d}{dr} + \frac{1}{r}\Phi_1 = MH_1, \quad 2i\frac{\nu}{r}\Phi_1 = MH_2. \tag{56}
\]
Here the \( H_1, H_2 \) represent non-dynamical variables, excluding them we obtain

\[
+i(\epsilon + \frac{\alpha}{r})E_1 + \frac{1}{M} \left( \frac{d}{dr} + \frac{1}{r} \right)^2 \Phi_1 - \frac{2\nu^2}{Mr^2} \Phi_1 = M\Phi_1, \quad -i(\epsilon + \frac{\alpha}{r})\Phi_1 = ME_1. \tag{57}
\]

Now we should make special substitution, introducing a big and small constituents (\( B_1(r) \) and \( M_1(r) \) respectively)

\[
\Phi_1 = B_1 + M_1, \quad iE_1 = B_1 - M_1; \tag{58}
\]

correspondingly eqs. \( 57 \) take the form

\[
(\epsilon + \frac{\alpha}{r})(B_1 - M_1) + \frac{1}{M} \left( \frac{d}{dr} + \frac{1}{r} \right)^2 (B_1 + M_1) - \frac{2\nu^2}{Mr^2} (B_1 + M_1) = M(B_1 + M_1),
\]

\[
(\epsilon + \frac{\alpha}{r})(B_1 + M_1) = M(B_1 - M_1). \tag{59}
\]

Summing and subtracting these two we get

\[
2(\epsilon + \frac{\alpha}{r})B_1 + \frac{1}{M} \left( \frac{\alpha}{r} \right)^2 (B_1 + M_1) - \frac{2\nu^2}{Mr^2} (B_1 + M_1) = 2MB_1,
\]

\[
-2(\epsilon + \frac{\alpha}{r})M_1 + \frac{1}{M} \left( \frac{\alpha}{r} \right)^2 (B_1 + M_1) - \frac{2\nu^2}{Mr^2} (B_1 + M_1) = 2MM_1. \tag{60}
\]

Now we should separate a rest energy by a formal change \( \epsilon \rightarrow \epsilon + M \); which results in

\[
2(\epsilon + \frac{\alpha}{r})B_1 + \frac{1}{M} \left( \frac{\alpha}{r} \right)^2 (B_1 + M_1) - \frac{2\nu^2}{Mr^2} (B_1 + M_1) = 0,
\]

\[
-2(\epsilon + \frac{\alpha}{r})M_1 + \frac{1}{M} \left( \frac{\alpha}{r} \right)^2 (B_1 + M_1) - \frac{2\nu^2}{Mr^2} (B_1 + M_1) = 4MM_1.
\]

Thus, we produce equation for a big \( B_1(r) \) and small \( M_1(r) \) components

\[
2(\epsilon + \frac{\alpha}{r})B_1 + \frac{1}{M} \left( \frac{\alpha}{r} \right)^2 B_1 - \frac{j(j+1)}{Mr^2} B_1 = 0, \tag{61}
\]

\[
\left( \frac{d}{dr} + \frac{1}{r} \right)^2 B_1 - \frac{j(j+1)}{r^2} B_1 = 4M^2M_1. \tag{62}
\]

Equation for the big component can be written as Schrödinger equation for a scalar particle

\[
\left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2M(\epsilon + \frac{\alpha}{r}) - \frac{j(j+1)}{r^2} \right] B_1 = 0. \tag{63}
\]

Corresponding non-relativistic 3-dimensional wave function for states with parity \( P = (-1)^{j+1} \) is

\[
P = (-1)^{j+1}, \quad \Psi = e^{-iEt/h} \begin{vmatrix}
+\Phi_1 + iE_1 & D_{-1} \\
0 & \Phi_1 + iE_1 \\
-(\Phi_1 + iE_1) & D_{+1}
\end{vmatrix}. \tag{64}
\]
Now let us consider radial equations for states with opposite parity \( P = (-1)^j \)

\[
-\left(\frac{d}{dr} + \frac{2}{r}\right)E_2 - 2\frac{\nu}{r}E_1 = M\Phi_0, \quad +i(\epsilon + \frac{\alpha}{r})E_1 + i\left(\frac{d}{dr} + \frac{1}{r}\right)H_1 = M\Phi_1,
\]

\[
+i(\epsilon + \frac{\alpha}{r})E_2 - 2i\frac{\nu}{r}H_1 = M\Phi_2, \quad -i(\epsilon + \frac{\alpha}{r})\Phi_1 + \frac{\nu}{r}\Phi_0 = ME_1,
\]

\[
-i(\epsilon + \frac{\alpha}{r})\Phi_2 - \frac{d}{dr}\Phi_0 = ME_2, \quad -i\left(\frac{d}{dr} + \frac{1}{r}\right)\Phi_1 - i\frac{\nu}{r}\Phi_2 = MH_1. \tag{65}
\]

Among four dynamical function \( \Phi_1, \Phi_2, E_1, E_2 \) separation of big and small constituents is performed as follows

\[
\Phi_1 = B_1 + M_1, \quad \Phi_2 = B_2 + M_2, \quad iE_1 = B_1 - M_1, \quad iE_2 = B_2 - M_2; \tag{66}
\]

the non-relativistic 3-dimensional wave function for states with parity \( P = (-1)^j \) is defined

\[
P = (-1)^j, \quad \Psi = e^{-iE_1/t} \begin{vmatrix}
(\Phi_1 + iE_1) D_{-1} \\
(\Phi_2 + iE_2) D_0 \\
(\Phi_1 + iE_1) D_{+1}
\end{vmatrix}. \tag{67}
\]

Excluding from (65) non-dynamical variables \( \Phi_0, H_1, \) we obtain the system (the rest energy is taken away as well: \( \epsilon \implies \epsilon + M \))

\[
i(\epsilon + M + \frac{\alpha}{r})E_1 + \frac{1}{M} \left(\frac{d}{dr} + \frac{1}{r}\right) \left[\frac{d}{dr} + \frac{1}{r}\right] \Phi_1 + \frac{\nu}{r}\Phi_2 = M\Phi_1,
\]

\[
i(\epsilon + M + \frac{\alpha}{r})E_2 - 2\frac{\nu}{Mr} \left[\frac{d}{dr} + \frac{1}{r}\right] \Phi_1 + \frac{\nu}{r}\Phi_2 = M\Phi_2,
\]

\[
-i(\epsilon + M + \frac{\alpha}{r})\Phi_1 + \frac{\nu}{r} \left[\frac{1}{d} + 2\frac{r}{d}\right] E_2 - 2\frac{\nu}{r} - E_1 = ME_1,
\]

\[
-i(\epsilon + M + \frac{\alpha}{r})\Phi_2 - \frac{1}{Mr} \left[\frac{1}{d} + \frac{2}{r}\right] E_2 - 2\frac{\nu}{r} - E_1 = ME_2. \tag{68}
\]

Taking into account (66), transform (68) into

\[
(\epsilon + M + \frac{\alpha}{r})(B_1 - M_1) + \frac{1}{M} \left(\frac{d}{dr} + \frac{1}{r}\right)^2 (B_1 + M_1) + \frac{\nu}{M} \left(\frac{d}{dr} + \frac{1}{r}\right)^2 (B_2 + M_2) = M(B_1 + M_1),
\]

\[
(\epsilon + M + \frac{\alpha}{r})(B_2 - M_2) - 2\frac{\nu}{Mr} \left(\frac{d}{dr} + \frac{1}{r}\right)(B_1 + M_1) - 2\frac{\nu^2}{Mr^2} (B_2 + M_2) = M(B_2 + M_2),
\]

\[
(\epsilon + M + \frac{\alpha}{r})(B_1 + M_1) - \frac{\nu}{Mr} \left(\frac{d}{dr} + \frac{2}{r}\right)(B_2 - M_2) - 2\frac{\nu^2}{Mr^2} (B_1 - M_1) = M(B_1 - M_1),
\]

\[
(\epsilon + M + \frac{\alpha}{r})(B_2 + M_2) + \frac{1}{Mr} \left(\frac{d}{dr} + \frac{1}{r}\right)^2 (B_1 - M_1) + \frac{2\nu}{r^2} (B_2 - M_2) + 2\frac{\nu}{Mr} \frac{d}{dr} \frac{1}{r} (B_1 - M_1) = M(B_2 - M_2).
\]
From whence, we get

\[(\epsilon + \frac{\alpha}{r})(B_1 - M_1) + \frac{1}{M}(\frac{d}{dr} + \frac{1}{r})^2(B_1 + M_1) + \frac{\nu}{M}(\frac{d}{dr} + \frac{1}{r}) \frac{1}{r}(B_2 + M_2) = +2M \ M_1,\]

\[(\epsilon + \frac{\alpha}{r})(B_1 + M_1) - \frac{\nu}{Mr}(\frac{d}{dr} + \frac{2}{r})(B_2 - M_2) -
\frac{2\nu^2}{Mr^2}(B_1 - M_1) = -2M \ M_1,\]

\[(\epsilon + \frac{\alpha}{r})(B_2 - M_2) - \frac{2\nu}{Mr}(\frac{d}{dr} + \frac{1}{r})(B_1 + M_1) -
\frac{2\nu^2}{Mr^2}(B_2 + M_2) = +2M \ M_2,\]

\[(\epsilon + \frac{\alpha}{r})(B_2 + M_2) + \frac{1}{M} \frac{d}{dr} \frac{d}{dr} + \frac{2}{r}(B_2 - M_2) +
\frac{2\nu}{Mr} \frac{d}{dr} \frac{1}{r}(B_1 - M_1) = -2M \ M_2.\]

Now summing and subtracting equation within the first couple, and doing the same within second
couple, we arrive at

\[(\epsilon + \frac{\alpha}{r})(B_1 - M_1) + \frac{1}{M}(\frac{d}{dr} + \frac{1}{r})^2(B_1 + M_1) + \frac{\nu}{M}(\frac{d}{dr} + \frac{1}{r}) \frac{1}{r}(B_2 + M_2) +
+(\epsilon + \frac{\alpha}{r})(B_1 + M_1) - \frac{\nu}{Mr}(\frac{d}{dr} + \frac{2}{r})(B_2 - M_2) - \frac{2\nu^2}{Mr^2}(B_1 - M_1) = 0 ,\]

\[(\epsilon + \frac{\alpha}{r})(B_1 - M_1) + \frac{1}{M}(\frac{d}{dr} + \frac{1}{r})^2(B_1 + M_1) + \frac{\nu}{M}(\frac{d}{dr} + \frac{1}{r}) \frac{1}{r}(B_2 + M_2) -
-(\epsilon + \frac{\alpha}{r})(B_1 + M_1) + \frac{\nu}{Mr}(\frac{d}{dr} + \frac{2}{r})(B_2 - M_2) +
\frac{2\nu^2}{Mr^2}(B_1 - M_1) = +4M \ M_1 ,\]

\[(\epsilon + \frac{\alpha}{r})(B_2 - M_2) - \frac{2\nu}{Mr}(\frac{d}{dr} + \frac{1}{r})(B_1 + M_1) - \frac{2\nu^2}{Mr^2}(B_2 + M_2) +
+(\epsilon + \frac{\alpha}{r})(B_2 + M_2) + \frac{1}{M} \frac{d}{dr} \frac{d}{dr} + \frac{2}{r}(B_2 - M_2) + \frac{2\nu}{Mr} \frac{d}{dr} \frac{1}{r}(B_1 - M_1) = 0 ,\]

\[(\epsilon + \frac{\alpha}{r})(B_2 - M_2) - \frac{2\nu}{Mr}(\frac{d}{dr} + \frac{1}{r})(B_1 + M_1) - \frac{2\nu^2}{Mr^2}(B_2 + M_2) -
-(\epsilon + \frac{\alpha}{r})(B_2 + M_2) - \frac{1}{M} \frac{d}{dr} \frac{d}{dr} + \frac{2}{r}(B_2 - M_2) -
\frac{2\nu}{Mr} \frac{d}{dr} \frac{1}{r}(B_1 - M_1) = +4M \ M_2 .\]

Now, using the same method as in [21, 22] (consider $B_1, B_2$ as big, and $M_1, M_2$ as small), we
arrive at two equations for big components, and two equations defining small components through
The right-hand part can be brought to a diagonal form

$$\begin{align*}
1 + \frac{d}{dr} \left( \frac{1}{r} \right)^2 &+ \frac{\nu}{M} \left( \frac{d}{dr} + \frac{1}{r} \right)^2 B_1 + \frac{\nu}{M} \left( \frac{d}{dr} + \frac{2}{r} \right)^2 B_2 + \frac{2\nu^2}{Mr^2} B_1 = +4M M_1, \\
-2\nu \left( \frac{d}{dr} + \frac{1}{r} \right) B_1 - \frac{2\nu^2}{Mr^2} B_2 - \frac{1}{M} \frac{d}{dr} \left( \frac{d}{dr} + \frac{2}{r} \right) B_2 - 2\nu \frac{d}{Mr} \frac{1}{r} B_1 = +4M M_2 , \\
2(\epsilon + \frac{a}{r}) B_2 - 2\nu \left( \frac{d}{dr} + \frac{1}{r} \right) B_1 - \frac{2\nu^2}{Mr^2} B_2 + \frac{1}{M} \frac{d}{dr} \left( \frac{d}{dr} + \frac{2}{r} \right) B_2 + 2\nu \frac{d}{Mr} \frac{1}{r} B_1 = 0 , \\
2(\epsilon + \frac{a}{r}) B_1 + \frac{1}{M} \left( \frac{d}{dr} + \frac{1}{r} \right)^2 B_1 + \nu \frac{d}{Mr} \left( \frac{d}{dr} + \frac{2}{r} \right) B_2 - \nu \frac{d}{Mr} \left( \frac{d}{dr} + \frac{2}{r} \right) B_2 - \frac{2\nu^2}{Mr^2} B_1 = 0 .
\end{align*}$$

Two last equation provides us with non-relativistic radial equations – they can be written as

$$\begin{align*}
r^2 \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2M(\epsilon + \frac{\alpha}{r}) - \frac{2\nu^2}{r^2} \right] B_2 &= 2B_2 + 4\nu B_1 , \\
r^2 \left[ \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2M(\epsilon + \frac{\alpha}{r}) - \frac{2\nu^2}{r^2} \right] B_1 &= 2\nu B_2 .
\end{align*}$$

(69)

It is convenient to presents eqs. (69) in a matrix form

$$\begin{align*}
\frac{1}{2} r^2 \Delta & \begin{vmatrix}
B_1 \\
B_2
\end{vmatrix} = \begin{vmatrix}
0 & \nu \\
2\nu & 1
\end{vmatrix} \begin{vmatrix}
B_1 \\
B_2
\end{vmatrix} .
\end{align*}$$

(70)

The right-hand part can be brought to a diagonal form

$$\begin{align*}
\begin{vmatrix}
f_1 \\
f_2
\end{vmatrix} = \begin{vmatrix}
a & c \\
d & b
\end{vmatrix} \begin{vmatrix}
B_1 \\
B_2
\end{vmatrix} , \\
r^2 \Delta \begin{vmatrix}
f_1 \\
f_2
\end{vmatrix} = \begin{vmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{vmatrix} \begin{vmatrix}
f_1 \\
f_2
\end{vmatrix} .
\end{align*}$$

(71)

The problem is to solve two systems

$$\begin{align*}
\begin{vmatrix}
a & c \\
d & b
\end{vmatrix} \begin{vmatrix}
0 & \nu \\
2\nu & 1
\end{vmatrix} = \begin{vmatrix}
\lambda_1 & 0 \\
0 & \lambda_2
\end{vmatrix} \begin{vmatrix}
a & c \\
d & b
\end{vmatrix} ,
\end{align*}$$

from whence it follows

$$\begin{align*}
\begin{cases}
\lambda_1 a - 2\nu c = 0 \\
-\nu a + (\lambda_1 - 1) c = 0 ,
\end{cases} & \lambda_1 = \frac{1 + \sqrt{1 + 4j(j + 1)}}{2} = j + 1 , \quad c = \frac{\lambda_1}{2\nu} a ; \\
\begin{cases}
\lambda_2 d - 2\nu b = 0 \\
-\nu d + (\lambda_2 - 1) b = 0 ,
\end{cases} & \lambda_2 = \frac{1 - \sqrt{1 + 4j(j + 1)}}{2} = -j , \quad b = \frac{\lambda_2}{2\nu} d .
\end{align*}$$
The transformation matrix we need is given by
\[
\begin{vmatrix}
  f_1 & a \lambda_1 a/2\nu \\
  f_2 & d \lambda_2 d/2\nu
\end{vmatrix} = \begin{vmatrix} B_1 \\ B_2 \end{vmatrix}.
\] (72)

Thus, the system (70) is led to the diagonal form
\[
\begin{align*}
\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2M(\epsilon + \frac{\alpha}{r}) - \frac{2\nu^2}{r^2} - \frac{2\lambda_1}{r^2}\right] f_1 &= 0, \\
\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2M(\epsilon + \frac{\alpha}{r}) - \frac{2\nu^2}{r^2} - \frac{2\lambda_2}{r^2}\right] f_2 &= 0.
\end{align*}
\] (73)

Note simple relations
\[
\frac{2\nu^2}{r^2} + \frac{2\lambda_1}{r^2} = \frac{j(j + 1) + 2(j + 1)}{r^2} = \frac{(j + 1)(j + 2)}{r^2},
\]
\[
\frac{2\nu^2}{r^2} + \frac{2\lambda_2}{r^2} = \frac{j(j + 1) - 2j}{r^2} = \frac{(j - 1)j}{r^2}.
\]

Thus, we have two problem of one the same type (below to \( f_1 \) and \( f_2 \) correspond \( \nu = j - 1 \) and \( \nu = j + 1 \) respectively)
\[
\left[\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2M(\epsilon + \frac{\alpha}{r}) - \frac{\nu(\nu + 1)}{r^2}\right] f = 0.
\] (74)

Changing the variable \( x = 2\sqrt{-2\epsilon M} r \)
\[
\left[\frac{d^2}{dx^2} + \frac{2}{x} \frac{d}{dx} - \frac{1}{4} - \frac{\alpha M}{x \sqrt{-2\epsilon M}} - \frac{\nu(\nu + 1)}{x^2}\right] f(x) = 0,
\] (75)
and introducing the substitution \( f(x) = x^a e^{-bx} F(x) \), eq. (75) is brought to
\[
\frac{d^2 F}{dx^2} + (2a + 2 - 2bx) \frac{dF}{dx} + \left[\frac{a(a + 1) - \nu(\nu + 1)}{x} - 2b - 2ab + \frac{\alpha M}{\sqrt{-2\epsilon M}} + (b^2 - \frac{1}{4})x\right] F = 0.
\] (76)

When \( b = +1/2, \ a = +\nu \), eq. (76) is simplified
\[
\frac{d^2 F}{dx^2} + (2\nu + 2 - x) \frac{dF}{dx} - \left[1 + \nu - \frac{\alpha M}{\sqrt{-2\epsilon M}}\right] F = 0,
\] (77)
what is the confluent hypergeometric equation for \( F(A, C; x) \) with parameters given by
\[
A = 1 + \nu - \frac{\alpha M}{\sqrt{-2\epsilon M}}, \quad C = 2\nu + 2.
\]
The quantization condition is \( A = -n \), which gives
\[
1 + \nu - \frac{\alpha M}{\sqrt{-2\epsilon M}} = -n \quad \Rightarrow \quad \epsilon = -\frac{\alpha^2 M}{2(1 + \nu + n)^2} = -\frac{me^4}{2\hbar^2(1 + \nu + n)^2}.
\] (78)

remembering that to linearly independent solutions with parity \( P = (-1)^j \) correspond \( \nu = j - 1 \) and \( \nu = j + 1 \).
VII. CONCLUSIONS

Quantum-mechanical system – spin 1 particle in external Coulomb field is studied on the base of the matrix Duffin – Kemmer – Petiau formalism with the use of the tetrad technique. With the help of parity operator, the radial 10-equation system is divided into two subsystem of 4 and 6 equations that correspond to parity $P = (-1)^{j+1}$ and $P = (-1)^{j}$ respectively. The system of 4 equation is reduced to a second order differential equation which coincides with that arising in the case of a scalar particle in Coulomb potential. It is shown that the 6-equation system reduces to two different differential equations for a "main" function. One main equation reduces to to a confluent Heun equation and provides us with energy spectrum. Another main equation is a more complex one, and any solutions for it are not constructed. In radial equations, transition to non-relativistic case is performed. In this limit, three types of linearly independent solutions have been constructed in terms of hypergeometric functions.

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