Von Neumann measurement of a spin-1 system

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Abstract

We derive exact formulas describing an indirect von Neumann measurement of a spin-1 system. The results hold for any interaction strength and for an arbitrary output variable $\hat{O}$. 

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I. INTRODUCTION

The simplest non-trivial Hilbert space is the two-dimensional one, which describes a spin 1/2 or a qubit. The measurement of a spin 1/2 as realized in the Stern-Gerlach experiment [1][2] epitomizes the ideal quantum measurement, even though a realistic description of the measurement involves some complications [3].

The next simplest system in quantum mechanics is provided by a three-dimensional Hilbert space, which can be realized, for instance, by a system with spin one. In the current jargon of quantum information, three-level systems are known as qutrits. They are known to provide higher-security quantum cryptography than qubits [4, 5]. Furthermore, it has been demonstrated that qutrits can be efficiently engineered and controlled, by using nonlinear optical techniques on biphotons [6, 7]. To the best of my knowledge, there is no study of the measurement of a spin-1, while a spin-1/2 has been treated quite extensively [3, 8–10]. In this manuscript, I am going to fill this gap, by studying a von Neumann measurement of a spin-1 system followed, possibly, by a postselection [11].

II. BACKGROUND

A. A useful property of a spin-one operator

In the following, we shall exploit the formula valid for a spin 1,

$$\exp(i\phi \hat{S}) = 1 + i \sin(\phi) \hat{S} - [1 - \cos(\phi)]\hat{S}^2,$$

which follows from

$$\hat{S}^3 = \hat{S}.$$  \hspace{1cm} (2)

We remark that this is the only property of the spin-one operator that we are going to exploit, so that the results presented here apply to any operator satisfying Eq. (2), not only operators on qutrits. In other words, the results of the present manuscript apply to any operator having eigenvalues in the set \{-1, 0, 1\}. Furthermore, the results can be trivially extended to any operator \(\hat{X}\) having three equally spaced eigenvalues \(x_1, x_2, x_3, x_2 - x_1 = x_3 - x_2 = \Delta x\), by making the shift and rescaling \(\hat{X} = \Delta x \hat{S} + x_2\).

In particular, an operator satisfying \(\hat{S}^2 = 1\), e.g. a Pauli matrix representing a spin-1/2, satisfies also Eq. (2), so that the following results apply to this case as well, after applying
the further restriction $\hat{S}^2 = 1$. As the exact solution of a measurement of a spin-1/2 is well known [8–10, 12–14], it will provide a reference check. Another example of particular relevance where Eq. (2) holds is that of two spin 1/2. Their total spin is a $4 \times 4$ matrix, giving a reducible representation of SU(2). The sector corresponding to the singlet is represented by the scalar 0, while the sector corresponding to the total spin 1 is represented by a $3 \times 3$ operator $S_3$, namely

$$S = \begin{pmatrix} 0 & 0 \\ 0 & S_3 \end{pmatrix}$$

with $0_3$ the null vector in three dimensions.

B. Description of the measurement

In a von Neumann measurement, before the interaction, the system and the detector are assumed to be uncorrelated, having a density matrix

$$\rho^- = \rho_i \otimes \rho_D;$$

the evolution operator of the system and the detector is taken to be

$$U = \exp(i\hat{Q}\hat{S}),$$

with $\hat{Q}$ an operator on the Hilbert space of the detector. The final entangled density matrix is thus

$$\rho^+ = \exp(i\hat{q}\hat{S}) (\rho_i \otimes \rho_D) \exp(-i\hat{q}\hat{S}).$$

We shall call the procedure a canonical measurement when the readout $\hat{P}$ has eigenstates $|j\rangle$ such that $\exp(i\hat{Q}\hat{S})$ translates one of them, say $|j_0\rangle$, into distinct eigenstates $|j_S\rangle$, with $S$ eigenvalues of the measured operator. Furthermore, we shall call the measurement ideal when the detector is prepared initially in the state $\rho_D = |j_0\rangle \langle j_0|$. In the present manuscript, however, we shall consider von Neumann measurements only in the sense that Eq. (5) is obeyed, and we shall not make the hypotheses of a canonical and ideal measurement, unless otherwise specified.
C. Postselection

The system may be postselected in a state $E_f$, represented by a positive operator not necessarily having trace one, by making a subsequent measurement. For instance, one could make a projective measurement of an observable $\hat{S}_f$, and analyze the output of the detector separately for each possible outcome $S_f$ [11]. In this case, the postselection states are the projectors $E_f = |S_f\rangle\langle S_f|$; or one could make a POV measurement of the system [15], then $E_f$ are not necessarily projectors; or, still, one could make a probabilistic postselection of the data [12].

The reduced density matrix of the detector, for a given postselection, is

$$\rho_{D|f} = \frac{\text{Tr}_{sys}[(E_f \otimes 1)\rho^+]}{\text{Tr}_{sys,D}[(E_f \otimes 1)\rho^+]}$$

with $\text{Tr}$ the trace, and $\text{Tr}_{sys}$ the partial trace on the Hilbert space of the system. The normalization factor $\text{Tr}_{sys,D}[(E_f \otimes 1)\rho^+]$ is the probability of successful postselection $P_f$.

III. RESULTS

A. General formula

Usually, the output to be observed in the detector is $\hat{P}$, the variable conjugated to $\hat{Q}$. This implicitly requires that the detector has an infinite-dimensional Hilbert space, so that one can define canonically conjugated position and momentum operators. We shall not make this assumption and let, instead, the Hilbert space of the detector to be arbitrary.

Let us start by computing the probability of postselection. After substitution of Eq. (1) into Eq. (6), and expressing the trace over the detector Hilbert space in terms of position eigenstate, $\text{Tr}_D[\ldots] = \int dq\langle q|\ldots|q\rangle$, we have

$$P_f = \omega \left\{ 1 - 2\hat{s}A'' + \hat{s}^2B'' - 2\hat{t}C'' + 2\hat{s}\hat{t}D'' + \hat{t}^2E_w \right\},$$

where we defined

$$\hat{s} = \sin \hat{Q},$$

$$\hat{t} = 1 - \cos \hat{Q},$$
the overline indicating average with respect to $\rho_D$, we introduced the fidelity

$$\omega = \text{Tr}_{\text{sys}}[E_f \rho_i],$$

i.e. the overlap between preparation and the postselection, and we defined the weak values

$$A_w = \omega^{-1} \text{Tr}_{\text{sys}}[E_f \hat{S} \rho_i],$$

$$B_w = \omega^{-1} \text{Tr}_{\text{sys}}[E_f \hat{S} \rho_i \hat{S}],$$

$$C_w = \omega^{-1} \text{Tr}_{\text{sys}}[E_f \hat{S}^2 \rho_i],$$

$$D_w = \omega^{-1} \text{Tr}_{\text{sys}}[E_f \hat{S} \rho_i \hat{S}^2],$$

$$E_w = \omega^{-1} \text{Tr}_{\text{sys}}[E_f \hat{S}^2 \rho_i \hat{S}].$$

Notice that $B_w$ and $E_w$ are real, while $A_w$, $C_w$, and $D_w$ are complex. For brevity, we are indicating with a single prime the real part of a complex number, and with a double prime its imaginary part, $A_w = A_w' + iA_w''$, etc. The quantities defined in Eqs. (11) are called weak values just in analogy with the quantity defined in Ref. [16], but we are not assuming anything here about the strength of the interaction.

Without loss of generality, $\hat{O} = 0$, i.e. the average output before the interaction vanishes, which means that the detector is unbiased. Otherwise, if $\overline{\hat{O}} \neq 0$, one should substitute in the following $\delta \hat{O} = \hat{O} - \overline{\hat{O}}$ for $\hat{O}$. The average output is then

$$\langle O \rangle_f = \text{Tr}_D[\hat{O} \rho_D|f] = \frac{\omega}{P_f} \left\{ i \overline{\{\overline{O}, \hat{s}\}} A_w' - \{\overline{O}, \hat{s}\} A_w'' - \overline{\{\overline{O}, \hat{t}\}} C_w' - i \overline{\{\overline{O}, \hat{t}\}} C_w'' + \hat{s} \overline{\hat{s}} B_w + i \hat{t} \overline{\hat{s}} D_w' - i \overline{\hat{t}} \hat{s} D_w'' + i \overline{\hat{t}} E_w \right\}. \quad (12)$$

B. Canonical continuous von Neumann measurement

In the following, we shall consider the case when $\hat{Q}$ has a continuous unbounded spectrum, so that it can be assimilated, say, to a position operator. The readout is taken to be its conjugated variable, $\hat{P}$. For brevity, we shall overload the bar symbol with the following meaning: when applied to a function of $Q$ and $P$, with no hats, it represents quasi-averages, i.e. averages with respect to the initial Wigner function of the probe, namely

$$\overline{f}(P, Q) = \int dP dQ W_D(P, Q) f(P, Q), \quad (13)$$
with the Wigner function defined as
\begin{equation}
W_D(P, Q) = \int \frac{dq}{2\pi} e^{iqP} \langle Q - \frac{q}{2} | \rho_D | Q + \frac{q}{2} \rangle = \int \frac{dp}{2\pi} e^{iQp} \langle P + \frac{p}{2} | \rho_D | P - \frac{p}{2} \rangle.
\end{equation}

We note that when \( f \) is a function only of \( P \) or \( Q \), then the quasi-averages are ordinary averages, \( f(Q) = \text{Tr}[f(\hat{Q}) \rho_D] \) and \( f(P) = \text{Tr}[f(\hat{P}) \rho_D] \). Then, the average output is
\begin{equation}
\langle P \rangle_f = \frac{\omega}{P} \left\{ \cos QA' - 2P \sin QA'' - 2P(1 - \cos Q)C'w + P \sin^2 QB_w 
+ \cos Q(1 - \cos Q)D'_w + 2P \sin Q(1 - \cos Q)D''_w + (1 - \cos Q)^2 E_w \right\}. \tag{15}
\end{equation}

C. Canonical discrete von Neumann measurement

Here, we shall not assume that \( \hat{Q} \) has a continuous spectrum. Let \( d = 2J + 1 \geq 3 \) the dimension of the Hilbert space of the detector. We shall assume that the readout \( \hat{P} \) has eigenstates \( |j\rangle, j \in \mathcal{I} = \{-J, -J+1, \ldots, J\} \) and eigenvalues \( P = j/\sqrt{d} \), such that \( \exp[ik\hat{Q}] \) translates periodically them into each other, for any integer \( k \). Namely,
\begin{equation}
\exp[ik\hat{Q}] |j\rangle = (-1)^{(d-1)r_{j+k}} |j \oplus k\rangle \tag{16}
\end{equation}
with \( \oplus \) modular addition, i.e., the result of the ordinary sum \( j + k \) is reduced to the interval \( \mathcal{I} \) by adding or subtracting an appropriate multiple of \( d \), \( r_{j+k}d \). As discussed in Ref. [17], the two operators \( \hat{Q} \) and \( \hat{P} \) having this property can be considered a generalization of canonically conjugated operators in finite-dimensional Hilbert spaces. Actually, here we are abounding in requiring that Eq. (16) holds for all integer \( k \). It would be sufficient, e.g., that \( \exp[iS\hat{Q}] = | -J \oplus S\rangle \) for \( S \in \{-1, 0, +1\} \).

We call this case the canonical discrete von Neumann measurement because, if the initial state of the detector is \( \rho_D = | -J \rangle \langle -J | \) and the system is in an eigenstate of \( \hat{S} \), \( \rho_i = |S\rangle \langle S | \), then the final state of the detector is one of the three orthogonal states \( | -J \oplus S\rangle \), so that the von Neumann measurement criterion is satisfied [18]. However, as in the rest of this manuscript, we shall not make the further hypothesis that \( \rho_D \) is an eigenprojector of \( \hat{P} \).

It follows that, in this case, the average readout is given by Eq. (15), as in the continuous case, but with the discrete Wigner function defined as
\begin{equation}
W_D(P, Q) = \text{Re}[\langle \tilde{k} | j \rangle \langle j | \rho_D | \tilde{k} \rangle], \tag{17}
\end{equation}

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with $|j\rangle$ the eigenstate of $\hat{P}$ corresponding to the eigenvalue $P = j/\sqrt{d}$ and $|\tilde{k}\rangle$ the eigenstate of $\hat{Q}$ corresponding to the eigenvalue $Q = 2\pi k/\sqrt{d}$,

$$|\tilde{k}\rangle = \frac{1}{\sqrt{d}} \sum_{j \in I} \exp[-2\pi ijk/d]|j\rangle.$$  \hspace{1cm} (18)

Notice that the definition (17) ensures that the marginal probability obtained by either summing over $j$ or over $k$ is positive-definite

$$\sum_{Q} W_D(P, Q) = \sum_{k} \text{Re}\{\langle \tilde{k}|j\rangle \langle j|\rho_D|\tilde{k}\rangle\} = \langle j|\rho_D|j\rangle,$$  \hspace{1cm} (19)

$$\sum_{P} W_D(P, Q) = \sum_{j} \text{Re}\{\langle \tilde{k}|j\rangle \langle j|\rho_D|\tilde{k}\rangle\} = \langle \tilde{k}|\rho_D|\tilde{k}\rangle.$$  \hspace{1cm} (20)

This property holds for any two bases $|j\rangle$, $|\tilde{j}\rangle$, not just for the canonically conjugated bases specifically considered here. For a review of discrete Wigner functions, see Ref. [19].

An important property used in deriving Eq. (15) for the discrete case is that, even though the canonical commutation relation $[\hat{Q}, \hat{P}] = i$ cannot be obeyed for finite $d$, however, the commutation relations $[\hat{P}, \exp(\pm i\hat{Q})] = \pm \exp(\pm i\hat{Q})$ still hold, so that, e.g., $[\hat{P}, \sin(\hat{Q})] = -i \cos(\hat{Q})$, as if $\hat{P} = -i \partial/\partial \hat{Q}$, formally.

### D. Spin 1/2 or $\hat{S}^2 = 1$

We remark that, for a spin 1/2, or, more generally, for an operator satisfying $\hat{S}^2 = 1$, the following identities hold: $C_w = 1$, $D_w = A_w$, $E_w = 1$. Then, as expected, Eqs. (12) and (8) reduce to the expressions for a spin 1/2, as reported for instance in Refs. [12, 20].

### E. Preparation or postselection commuting with $\hat{S}$

It may happen that either $[E_f, \hat{S}] = 0$ or $[\rho_i, \hat{S}] = 0$. Two important instances are when no postselection is made, $E_f \propto \mathbb{1}$, and when the initial state is the completely unpolarized state $\rho_i = \mathbb{1}/3$. Other important cases are when the system is either prepared or postselected in an eigenstate of $\hat{S}$. When this happens, all the weak values defined in Eqs. (11) are real and furthermore $C_w = E_w = B_w$, $D_w = A_w$. 

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F. Preparation or postselection commuting with $\hat{S}^2$

It may happen that either $[E_f, \hat{S}^2] = 0$ or $[\rho_i, \hat{S}^2] = 0$, but $[E_f, \hat{S}] \neq 0$ and $[\rho_i, \hat{S}] \neq 0$. While this case is less interesting than the former one, we shall treat it for completeness. The following relations hold among the weak values: $C_w$ is real, $D_w = A_w$, and $E_w = B_w$.

G. Preparation and postselection in pure states

Since in this case $E_f \propto |f\rangle \langle f|$ and $\rho_i = |i\rangle \langle i|$, $B_w = |A_w|^2$, $E_w = |C_w|^2$, and $D_w = A_w C_w^*$. There are thus only four independent real parameters.

IV. WEAK MEASUREMENT LIMIT

In this section, we shall compare the weak limit of our main result Eqs (8) and (12) with the formulas for a weak measurement of any operator $\hat{S}$, which were given by Jozsa [21] for the linear regime, and by me [12] in a more general case including orthogonal preparation and postselection. As expected, the formulas coincide.

In order to keep track of the perturbative expansion, it is better to introduce a coupling constant so that the time-evolution of Eq. (5) reads

$$U = \exp(i \lambda \hat{Q} \hat{S}).$$

A. Conditions of validity for the weak measurement

A measurement is called weak when $\lambda$ is sufficiently small. A sloppy way to characterize the strength of the measurement consists in saying that in the limit $\lambda \to \infty$ the measurement is strong, and in the limit $\lambda \to 0$ it is weak. However, $\lambda$ is a dimensionful constant (its dimension being the inverse of the dimension of $Q$, if we consider $S$ dimensionless), and we all learnt that dimensionful quantities are to be considered large or small always in comparison with another homogeneous quantity. Thus, the question to ask is: $\lambda$ is small compared to what? In the seminal paper of Aharonov et al., which considered a canonical von Neumann measurement, it was assumed that $\lambda \ll \sigma_P$, $\sigma_P^2$ being the initial variance in the canonical readout variable, i.e. the initial uncertainty over the pointer variable. However,
as pointed out in Ref. [9], the coherence of the detector, relative to the readout basis, is an essential requisite for the weak measurement to show its stranger features. Indeed, this was quantified better in Ref. [10], where the initial state of the detector was considered to be a mixed Gaussian state

\[
\langle P | \rho_D | P' \rangle = \exp\left[ -(P + P')^2 / 8\sigma_P^2 - \sigma_Q^2 (P - P')^2 / 2 - i\tilde{Q} (P - P') \right], \tag{22}
\]

with \( \sigma_Q^2 \) the initial variance of the write-in variable \( \hat{Q} \). Reference [10] showed that the relevant criterion for the weak measurement is that

\[
2\lambda \sigma_J = \lambda / \delta P \ll 1, \tag{23}
\]

where \( \delta P \) was defined as the coherence scale relative to the \( |P\rangle \) basis, i.e. the scale over which the offdiagonal elements \( \rho_D(P + p/2, P - p/2) \) vanish with respect to \( \rho_D(P, P) \) for increasing \( |p| \) and fixed \( P \). Here and in the following, we assume that the range of the eigenvalues of \( \hat{S} \) is \( O(1) \). If this were not the case, one could always redefine \( \lambda \) and \( \hat{S} \) appropriately. Since, by the Kennard uncertainty relation [22], \( \sigma_J \geq 1 / 2\sigma_Q = \delta P \), Eq. (23) implies that \( \lambda \ll \sigma_J \), but the vice versa may not be true. However, for a pure Gaussian state, \( \sigma_J = 1 / 2\sigma_Q = \delta P \). Since this case was the one mostly considered in the literature, following Ref. [16], the two scales \( \delta P \) and \( \sigma_J \) were not discriminated from each other, so that the correct condition for the weakness of the measurement, Eq. (23), was found only long after the concept of weak measurement had been established [23]. Furthermore, it is also required that \( \lambda \tilde{Q} \ll 1 \), but if this condition is not obeyed, one can gauge out \( \tilde{Q} \) [12]. In Refs. [24, 25], the importance of the coherent behavior of the detector was stressed as well. There, however, the initial state of the detector was assumed to be that of an ideal von Neumann measurement, and the coherence was created by the very act of measuring the spin of several electrons sequentially. A more precise condition of validity for the perturbative expansion, including Eq. (23) as a necessary condition, has been recently provided in Ref. [26],

\[
(2\lambda)^n \max\{|S|\}^n |\tilde{Q}|^n \leq \delta^n, \forall n \in \mathbb{N} \tag{24}
\]

with \( \delta \) a small positive number.
B. Result

By expanding the various terms in Eqs. (8) and (12) up to second order in $\lambda$

$$P_f \simeq \omega \left\{ 1 - 2\lambda \bar{Q} A''_w + \lambda^2 \bar{Q}^2 (B_w - C'_w) \right\},$$
\(25\)

and

$$\langle O \rangle_f \simeq \frac{\omega}{P_f} \left\{ \lambda \left( i[\hat{O}, \hat{Q}] A'_w - \{\hat{O}, \hat{Q}\} A''_w \right) + \frac{1}{2} \lambda^2 \left( -\{\hat{O}, \hat{Q}^2\} C'_w - i[\hat{O}, \hat{Q}^2] C''_w + 2\tilde{Q}\tilde{Q}\tilde{O} B_w \right) \right\},$$
\(26\)
in agreement with the result of Ref. [12]. In particular, for $\hat{O} = \hat{Q}$ and $\hat{O} = \hat{P}$, the results of Ref. [27] are recovered. Furthermore, as noted in Refs. [12] and [28], one can effectively neglect the $C_w$ terms, leading to the simplified interpolation formulas

$$P_f \simeq \omega \left\{ 1 - 2\lambda \bar{Q} A''_w + \lambda^2 \bar{Q}^2 B_w \right\},$$
\(27\)

and

$$\langle O \rangle_f \simeq \frac{\omega}{P_f} \left\{ \lambda \left( i[\hat{O}, \hat{Q}] A'_w - \{\hat{O}, \hat{Q}\} A''_w \right) + \lambda^2 \bar{Q}\tilde{Q}\bar{O} B_w \right\}.$$  
\(28\)

These formulas, originally derived and justified in Ref. [12], were also independently rediscovered by Kofman et al. [20].

Finally, let us assume that it is admissible to make a Taylor expansion of $\omega/P_f$, which is the case when $|A_w|$ and $B_w$ are not too large, i.e. when the preparation and the postselection have not too small an overlap $\omega = \text{Tr}(E_f \rho_i)$. Then, we recover the formula due to Josza [21],

$$\langle O \rangle_f \simeq \lambda \left( i[\hat{O}, \hat{Q}] A'_w - \{\hat{O}, \hat{Q} - \tilde{Q}\} A''_w \right).$$
\(29\)

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