AN ANALYTICAL SOLUTION OF THE WEIGHTED FERMAT-TORRICELLI PROBLEM ON THE UNIT SPHERE

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Abstract. We obtain an analytical solution for the weighted Fermat-Torricelli problem for an equilateral geodesic triangle \( \triangle A_1A_2A_3 \) which is composed by three equal geodesic arcs (sides) of length \( \frac{\pi}{2} \) for given three positive unequal weights that correspond to the three vertices on a unit sphere. This analytical solution is a generalization of Cockayne’s solution given in [4] for three equal weights. Furthermore, by applying the geometric plasticity principle and the spherical cosine law, we derive a necessary condition for the weighted Fermat-Torricelli point in the form of three transcendental equations with respect to the length of the geodesic arcs \( A_1A_1', A_2A_2' \) and \( A_3A_3' \), to locate the weighted Fermat-Torricelli point \( A_0 \) at the interior of a geodesic triangle \( \triangle A_1' A_2' A_3' \) on a unit sphere with sides less than \( \frac{\pi}{2} \).

1. Introduction

Let \( \triangle A_1A_2A_3 \) be a geodesic triangle and \( A_0 \) a point on a unit sphere. We denote by \( a_{ij} \) the length of the geodesic arc \( A_iA_j \), which is part of a great circle of unit radius and \( \alpha_{ijk} \) the angle between the geodesic arcs \( A_iA_k \) and \( A_kA_j \) for \( i, j, k = 0, 1, 2, 3, i \neq j \neq k \).

The weighted Fermat problem on the unit sphere refers to the following problem:

Problem 1. Consider a positive constant weight \( w_i \) that correspond to the vertex \( A_i \), for \( i = 1, 2, 3 \). Find a point \( A_0 \) (weighted Fermat point) for which the sum

\[
\sum_{i=1}^{3} w_i a_{0i} \tag{1.1}
\]

is minimized.

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The existence and uniqueness for the weighted Fermat point on a convex surface has been studied in [10], [6], [2], [11] (see also in [7], Chapter II, pp. 208).

Concerning some studies that focus on the geometric properties of the weighted Fermat point on the two dimensional sphere and on a convex surface we refer to the studies of [4], [5], [9], [3], [7], [8] and [13].

The following results (Proposition 1, 2) characterize the solutions of the weighted Fermat problem on a $C^2$ surface and they have been proved in [11], proposition 6, page 53 and proposition 7, page 55:

**Proposition 1** (Floating Case). [11, Proposition 6, p. 53], [2] If $\vec{U}_{A_iA_j}$ is the unit tangent vector of the geodesic arc $A_iA_j$ at $A_i$ and $D$ is the domain of a $C^2$ surface $M$ bounded by $\triangle A_1A_2A_3$, for $i, j = 1, 2, 3$ then the following (I), (II), (III) conditions are equivalent: (I) All the following inequalities are satisfied simultaneously:

$$\left\|w_2\vec{U}_{A_1A_2} + w_3\vec{U}_{A_1A_3}\right\| > w_1,$$  \hspace{1cm} (1.2)

$$\left\|w_1\vec{U}_{A_2A_1} + w_3\vec{U}_{A_2A_3}\right\| > w_2,$$  \hspace{1cm} (1.3)

$$\left\|w_1\vec{U}_{A_3A_1} + w_2\vec{U}_{A_3A_2}\right\| > w_3.$$  \hspace{1cm} (1.4)

(II) The point $A_0$ is an interior point of $\triangle A_1A_2A_3$ (weighted Fermat-Torricelli point) and does not belong to the geodesic arcs $A_1A_2$, $A_2A_3$ and $A_1A_3$.

(III) $\vec{U}_{A_0A_1} + \vec{U}_{A_0A_2} + \vec{U}_{A_0A_3} = \vec{0}$.

**Proposition 2** (Absorbed Case). [11, Proposition 7, p. 55], [2] The following (I), (II) conditions are equivalent.

(I) One of the following inequalities is satisfied:

$$\left\|w_2\vec{U}_{A_1A_2} + w_3\vec{U}_{A_1A_3}\right\| \leq w_1,$$  \hspace{1cm} (1.5)

or

$$\left\|w_1\vec{U}_{A_2A_1} + w_3\vec{U}_{A_2A_3}\right\| \leq w_2,$$  \hspace{1cm} (1.6)

or

$$\left\|w_1\vec{U}_{A_3A_1} + w_2\vec{U}_{A_3A_2}\right\| \leq w_3.$$  \hspace{1cm} (1.7)

(II) The point $A_0$ (weighted Fermat-Cavalieri point) is attained at $A_1$ or $A_2$ or $A_3$, respectively.
We note that there is no analytical solution with respect to the weighted Fermat-Torricelli problem on the unit sphere, except of Cockayne’s solution given in [4] for an equilateral geodesic triangle having sides with length $\frac{\pi}{2}$ for three equal weights.

In this paper, we find an analytical solution of the weighted Fermat-Torricelli problem for an equilateral geodesic triangle on a unit sphere which is composed by three equal geodesic arcs of length $\frac{\pi}{2}$, by using as variables the two angles of longitude and latitude from the spherical coordinates and by applying the spherical sine law in some specific geodesic triangles (Theorem 1). The geometric plasticity principle which has been proved in [13] yields a class of geodesic triangles such that the corresponding weighted Fermat-Torricelli point remains the same. By applying the geometric plasticity principle, we find a class of geodesic triangles by using the cosine law on the unit sphere, such that the weighted Fermat-Torricelli point is the same with the weighted Fermat-Torricelli point which corresponds to the equilateral geodesic triangle. Finally, by applying the geometric plasticity principle and the spherical cosine law, we derive a necessary condition for the weighted Fermat-Torricelli point in the form of three transcendental equations with respect to some specific three length of geodesic arcs to locate the weighted Fermat-Torricelli point $A_0$ at the interior of a geodesic triangle $\triangle A_1'A_2'A_3'$ on a unit sphere with sides less than $\frac{\pi}{2}$ (Proposition 1).

2. Analytical solution of the weighted Fermat-Torricelli problem on the unit sphere

Let $\triangle A_1A_2A_3$ be a geodesic triangle on the unit sphere $S : x^2 + y^2 + z^2 = 1$, such that $a_{12} = a_{23} = a_{31} = \frac{\pi}{2}$ and $A_1 = (1, 0, 0)$, $A_2 = (0, 1, 0)$, $A_3 = (0, 0, 1)$.

Lemma 1. [10] Theorem 1, [12] If $A_0$ is the weighted Fermat-Torricelli point of $\triangle A_1A_2A_3$, then each angle $\alpha_{i0j}$ is expressed as a function of $w_1$, $w_2$ and $w_3$:

$$
\alpha_{i0j} = \arccos \left( \frac{w_k^2 - w_i^2 - w_j^2}{2w_iw_j} \right)
$$

(2.1)

for $i, j, k = 1, 2, 3$, and $k \neq i \neq j$.

We start by expressing the position of the weighted Fermat-Torricelli point $A_0 = (x, y, z)$ in terms of the spherical coordinates $(\omega, \varphi)$:

$$
A_0 = (\cos \omega \cos \varphi, \cos \omega \sin \varphi, \sin \omega).
$$

Theorem 1. The analytical solution of the weighted Fermat-Torricelli problem of $\triangle A_1A_2A_3$ on the unit sphere is given by the following two relations:
\[ \varphi = \arccos \left( \sqrt{\frac{w_1^2 + w_3^2 - w_2^2}{2w_3^2}} \right) \] (2.2)

and

\[ \omega = \arccos \left( \frac{w_1^2 + w_2^2 - w_3^2}{2w_1w_2 \sin \left( \arccos \left( \frac{w_1^2 - w_2^2 - w_3^2}{2w_1w_2} \right) \right) \sin \left( \arccos \left( \frac{w_2^2 - w_1^2 - w_3^2}{2w_1w_3} \right) \right) \right) \],

which yield the exact location of the weighted Fermat-Torricelli point A_0.

**Proof.** The location of \( A_0 = (\cos \omega \cos \varphi, \cos \omega \sin \varphi, \sin \omega) \). is determined by \( \omega \) and \( \varphi \).

We proceed by calculating \( \omega \) and \( \varphi \) with respect to the given positive weights \( w_1, w_2 \) and \( w_3 \).

By applying the sine law in \( \triangle A_1A_0A_3, \triangle A_1A_0A_2 \) and \( \triangle A_2A_0A_3 \), we get, respectively:
\[
\frac{1}{\sin \alpha_{103}} = \frac{\sin a_{03}}{\sin \alpha_{013}} = \frac{\sin a_{01}}{\sin \alpha_{130}}, \tag{2.4}
\]
\[
\frac{1}{\sin \alpha_{102}} = \frac{\sin a_{02}}{\sin \left(\frac{\pi}{2} - \alpha_{013}\right)} = \frac{\sin a_{01}}{\sin \alpha_{120}} \tag{2.5}
\]
and
\[
\frac{1}{\sin \alpha_{203}} = \frac{\sin a_{03}}{\sin \left(\frac{\pi}{2} - \alpha_{120}\right)} = \frac{\sin a_{02}}{\sin \left(\frac{\pi}{2} - \alpha_{130}\right)} \tag{2.6}
\]

By taking the orthogonal projection of \(A_0\) with respect to the \(xy\) plane and by using the Euclidean sine law, we express \(a_{01}, a_{02}\) and \(a_{03}\) as functions of \(\omega\) and \(\varphi\):

\[
\cos a_{03} = \sin \omega, \tag{2.7}
\]

\[
\cos a_{01} = \cos \omega \cos \varphi \tag{2.8}
\]

and

\[
\cos a_{02} = \cos \omega \sin \varphi. \tag{2.9}
\]

By replacing (2.7), (2.8) and (2.9) in (2.4), (2.5) and (2.6) we obtain:

\[
\frac{1}{\sin \alpha_{103}} = \frac{\cos \omega}{\sin \alpha_{013}} = \frac{\sqrt{1 - \cos^2 \omega \cos^2 \varphi}}{\sin \alpha_{130}}, \tag{2.10}
\]

\[
\frac{1}{\sin \alpha_{102}} = \frac{\sqrt{1 - \cos^2 \omega \sin^2 \varphi}}{\cos \alpha_{013}} = \frac{\sqrt{1 - \cos^2 \omega \cos^2 \varphi}}{\sin \alpha_{120}} \tag{2.11}
\]

and

\[
\frac{1}{\sin \alpha_{203}} = \frac{\cos \omega}{\cos \alpha_{120}} = \frac{\sqrt{1 - \cos^2 \omega \sin^2 \varphi}}{\cos \alpha_{130}}. \tag{2.12}
\]

From (2.10) we get:

\[
\sin \alpha_{013} = \sin \alpha_{103} \cos \omega \tag{2.13}
\]

and

\[
\sin \alpha_{130} = \sin \alpha_{103} \sqrt{1 - \cos^2 \omega \cos^2 \varphi}. \tag{2.14}
\]

From (2.11) we get:

\[
\cos \alpha_{013} = \sin \alpha_{102} \sqrt{1 - \cos^2 \omega \sin^2 \varphi} \tag{2.15}
\]
and
\[ \sin \alpha_{120} = \sin \alpha_{102} \sqrt{1 - \cos^2 \omega \cos^2 \varphi}. \]  (2.16)

From (2.12) we get:
\[ \cos \alpha_{120} = \sin \alpha_{203} \cos \omega \]  (2.17)

and
\[ \cos \alpha_{130} = \sin \alpha_{203} \sqrt{1 - \cos^2 \omega \sin^2 \varphi}. \]  (2.18)

By squaring both parts of (2.13) and (2.15) and by adding the two derived equations we obtain:
\[ \sin^2 \alpha_{103} \cos^2 \omega + \sin^2 \alpha_{102} (1 - \cos^2 \omega \sin^2 \varphi) = 1. \]  (2.19)

By squaring both parts of (2.14) and (2.18) and by adding the two derived equations we obtain:
\[ \sin^2 \alpha_{203} \cos^2 \omega + \sin^2 \alpha_{102} (1 - \cos^2 \omega \cos^2 \varphi) = 1. \]  (2.20)

By subtracting (2.20) from (2.19) and taking into account (2.1) from lemma 1 we derive:
\[ \cos 2\varphi = \frac{B_1^2 - B_3^2}{B_3^2} \]  (2.21)

which yields (2.2).

By adding (2.20) from (2.19) and taking into account the trigonometric identity
\[ \sin^2 \alpha_{102} = \sin^2 \alpha_{203} \cos^2 \alpha_{103} + \cos^2 \alpha_{203} \sin^2 \alpha_{103} + 2 \sin \alpha_{203} \cos \alpha_{203} \sin \alpha_{103} \cos \alpha_{103} \]
and (2.11) from lemma 1 we derive (2.3).

\[ \text{Corollary 1. If } w_1 = w_2 = w_3, \text{ then } \omega = \frac{\pi}{4} \text{ and } \varphi = \arccos \sqrt{\frac{2}{3}} \text{ and } A_0 = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right). \]

Proof. By replacing \( w_1 = w_2 = w_3 \) in (2.2) and (2.3) we derive that \( \omega = \frac{\pi}{4} \) and \( \varphi = \arccos \sqrt{\frac{2}{3}} \) and we deduce the position of the Fermat-Torricelli point \( A_0 = \left( \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right). \)
Proposition 3 (Geometric plasticity Principle). [13] Theorem 3, Proposition 8] Suppose that the weighted floating case of the weighted Fermat point $A_0$ point with respect to $A_1A_2A_3A_4$ is valid:

$$\|w_Q\vec{U}_{RQ} + w_S\vec{U}_{RS} + w_P\vec{U}_{RP}\| > w_R,$$

for each $R, Q, S, P \in \{A_1, A_2, A_3, A_4\}$. If $A_0$ is connected with every vertex $R$ for $R \in \{A_1, A_2, A_3, A_4\}$ and we select a point $R'$ with non-negative weight $w_R$ which lies on the shortest arc $RA_0$ and the quadrilateral $A_1'A_2'A_3'A_4'$ is constructed such that:

$$\|w_Q\vec{U}_{R'Q} + w_S\vec{U}_{R'S} + w_P\vec{U}_{R'P}\| > w_{R'},$$

for $R', Q', S', P' \in \{A_1', A_2', A_3', A_4'\}$. Then the weighted Fermat-Torricelli point $A_0'$ is identical with $A_0$.

Lemma 2. The geometric plasticity principle holds for a geodesic triangle on the unit sphere.

Proof. By replacing $w_4 = 0$ in Proposition 3, we deduce the geometric plasticity principle of a geodesic triangle $\triangle A_1A_2A_3$ on the unit sphere. □

Let $\triangle A_1'A_2'A_3'$ be a geodesic triangle on the unit sphere, such that $A_i'$ belongs to the geodesic arc $A_0A_i$, for $i = 1, 2, 3$, where $A_0$ is the weighted Fermat-Torricelli point of $\triangle A_1A_2A_3$ (Fig. 1).

We assume that $a_{12}', a_{23}', a_{31}' \leq \frac{\pi}{2}$, in order to locate the geodesic triangle $\triangle A_1'A_2'A_3'$ at the interior of $\triangle A_1A_2A_3$.

Furthermore, we assume that the same weight $w_i$ that corresponds to the vertex $A_i$ corresponds to the vertex $A_i'$, for $i = 1, 2, 3$, such that the inequalities of the weighted floating case hold (Proposition 1).

We denote by $A_0'$ the corresponding weighted Fermat-Torricelli point of $\triangle A_1'A_2'A_3'$.

We denote by $a$ the length of the geodesic arc $A_1A_1'$, by $b$ the length of the geodesic arc $A_2A_2'$ and by $c$ the length of the geodesic arc $A_3A_3'$.

Proposition 4. The following system of three equations with respect to $a, b$ and $c$ provide a necessary condition to locate the weighted Fermat-Torricelli point $A_0 \equiv A_0'$ at the interior of a geodesic triangle $\triangle A_1'A_2'A_3'$ on a unit sphere with sides less than $\frac{\pi}{2}$:

$$\cos a_{12}' = \cos(a_{01} - a) \cos(a_{02} - b) + \sin(a_{01} - a) \sin(a_{02} - b) \frac{w_2^2 - w_1^2 - w_2^2}{2w_1w_2},$$

(2.22)
\[
\cos a'_{23} = \cos(a_{02} - b) \cos(a_{03} - c) + \sin(a_{02} - b) \sin(a_{03} - c) \frac{w_1^2 - w_2^2 - w_3^2}{2w_2w_3},
\]

and

\[
\cos a'_{13} = \cos(a_{01} - a) \cos(a_{03} - c) + \sin(a_{01} - a) \sin(a_{03} - c) \frac{w_2^2 - w_1^2 - w_3^2}{2w_1w_3}.
\]

**Proof.** From lemma 2, the geometric plasticity holds on the units sphere. Therefore, \( A_0 = A'_0 \). By applying the cosine law in \( \triangle A'_1A_0A'_2 \), \( \triangle A'_2A_0A'_3 \) and \( \triangle A'_1A_0A'_3 \), we obtain (2.22), (2.23) and (2.24), respectively. The equations (2.22), (2.23) and (2.24) yield a system of three equations with respect to \( a, b \) and \( c \), because \( a_{01}, a_{02} \) and \( a_{03} \) could be expressed explicitly as functions of \( w_1, w_2 \) and \( w_3 \) taking into consideration the exact location of \( \triangle A_1A_2A_3 \) which has been given in Theorem 1.

\[\square\]

**Remark 1.** By replacing the Weirstrass transformations \( \sin a = \frac{2t_a}{1+t_a^2}, \cos a = \frac{1-t_a^2}{1+t_a^2}, \sin b = \frac{2t_b}{1+t_b^2}, \cos b = \frac{1-t_b^2}{1+t_b^2}, \sin c = \frac{2t_c}{1+t_c^2}, \cos c = \frac{1-t_c^2}{1+t_c^2} \), in (2.22), (2.23) and (2.24) we get a system of three rational equations with respect to \( t_a, t_b \) and \( t_c \). By solving the first derived equation of second degree with respect to \( t_a \), we may obtain two solutions \( t_{a1} = f_1(t_b) \) and \( t_{a2} = f_2(t_b) \). Similarly, by solving the second derived equation of second degree with respect to \( t_c \), we may obtain two solutions \( t_{c1} = f_1(t_b) \) and \( t_{c2} = f_2(t_b) \).

By replacing these pairs of solutions with respect to \( (t_a(t_b), t_c(t_b)) \) in the third derived equation we obtain a rational equation which depend only on \( t_b \).

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