SLE curves and natural parametrization

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Abstract

Developing the theory of two-sided radial and chordal SLE, we prove that the natural parametrization on SLE_κ curves is well defined for all κ < 8. Our proof uses a two-interior-point local martingale.

Key words and Phrases: SLE, natural parametrization, Doob-Meyer decomposition, local martingale

AMS 2000 subject classification: 60D05, 60J60, 30C20, 28A80.

1 Introduction

1.1 Background and Motivation

Suppose that x_j, j = 1, 2, · · · are independent and identically distributed random vectors in Z^2 with probabilities

\[ \mathbb{P} \{ x_j = e \} = 1/4, \ |e| = 1. \]

It is well known that the scaled simple random walk in \( \mathbb{R}^2 \),

\[ B_{t}^{(n)} = n^{-1/2} \sum_{j=1}^{[nt]} x_j, \quad 0 \leq t \leq 1, \]

converges to a standard two-dimensional Brownian motion \( B_t, 0 \leq t \leq 1 \) as \( n \to \infty \). In the scaled walk, each step is traversed in the same amount of time. When passing to the scaling limit, this parameter \( t \) becomes the natural parametrization of Brownian motion. We can write the scaling factor as \( n^{-1/d} \), where \( d = 2 \) is the fractal dimension of the Brownian paths. The natural parametrization is a \( d \)-dimensional measure.

One variant of simple random walk is the loop erased random walk first appeared in [Law80]. Its definition is as follows. Consider any finite or recurrent connected graph \( G \),

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one vertex \( a \) and a set of vertices \( V \). Loop-erased random walk (LERW) from \( a \) to \( V \) is a random simple curve joining \( a \) to \( V \) obtained by erasing the loops in chronological order from a simple random walk started at \( a \) and stopped upon hitting \( V \). One can ask whether or not there is a corresponding result for LERW where the scaling factor is \( n^{-1/d} \) and \( d \) is the fractal dimension of the paths. Schramm [Sch00] introduced a process, now called the Schramm-Loewner evolution (\( SLE_\kappa \)), as a candidate for the scaling limit and gave a strong argument why \( SLE_2 \) should be the scaling limit of LERW. In order to use the Loewner equation, he used a capacity parametrization which is not the parametrization one would obtain by taking the limit above. In [LSW04], it was proved that the scaling limit of planar LERW \textit{in the capacity parametrization} is \( SLE_2 \). However, it is still open whether or not, one can take a limit as above. There are a number of other models that are known to converge to \( SLE_\kappa \) in the scaling limit using the capacity parametrization: critical site-percolation on the triangular lattice [Smi01], the level lines of the discrete Gaussian free field [SS09], the interfaces of the random cluster model associated with the Ising model [Smi07].

A start to taking limits as above is to define the natural parametrization for \( SLE_\kappa \). Possible definitions and constructions for a natural parametrization were suggested in [LS09]. As well as giving conjectures, one definition was proposed in terms of the Doob-Meyer decomposition of a path. We review this construction in Section 1.3. Although they conjectured that this definition is valid for all \( \kappa < 8 \), they were only able to establish the result for \( \kappa < \kappa_0 = 4(7 - \sqrt{33}) \). The technical problem came from difficult second moment estimates for the reverse Loewner flow.

In this paper, we prove that the definition in [LS09] is valid for \( \kappa < 8 \). Instead of using the reverse Loewner flow, we use a difficult estimate of Beffara [Bef08] on the forward Loewner flow to establish the necessary uniform integrability to apply the Doob-Meyer theorem. Beffara’s estimate was the key step in his proof of the Hausdorff dimension of \( SLE_\kappa \) curves. This estimate has recently been improved [LW10] and used to establish a multi-point Green’s function for \( SLE_\kappa \). We use this Green’s function to give an appropriate two-interior-point local martingale. By establishing a correlation inequality for this Green’s function, we are able to give a relatively simple proof of the existence of the natural parametrization.

1.2 Notation

In this subsection we set up the notation for \( SLE_\kappa \). For more background, see for example, [Wer04, GK04, Car05, Law05, Law09a].

Throughout this paper we let \( \kappa < 8 \) and \( a = 2/\kappa > 1/4 \), and we allow all constants to depend on \( \kappa \). We let \( d = 1 + \frac{\kappa}{8} = 1 + \frac{1}{4a} \) be the Hausdorff dimension of the paths. We parametrize the maps so that

\[
\partial_t g_t(z) = \frac{a}{g_t(z) - U_t},
\]

where \( U_t = -B_t \) is a standard Brownian motion. It can be shown [RS05, LSW04] that a.s. \( g_t^{-1} \) extends continuously to \( \mathbb{R} \) for every \( t \geq 0 \) and \( \gamma(t) := g_t^{-1}(U_t) \) is a continuous curve which is the \( SLE_\kappa \) path. The domain of definition of \( g_t \) is the unbounded connected component \( H_t \),
of $\mathbb{H} \setminus \gamma[0,t]$. We shall denote by $K_t$ the closure of the complement of $H_t$ in $\mathbb{H}$. If $z \in \overline{\mathbb{H}} \setminus \{0\}$, let

$$Z_t(z) = X_t(z) + iY_t(z) = g_t(z) + B_t.$$  

Then the Loewner equation can be written as

$$dZ_t(z) = \frac{a}{Z_t(z)} dt + dB_t,$$

$$dX_t(z) = \frac{aX_t(z)}{|Z_t(z)|^2} dt + dB_t, \quad \partial_t Y_t(z) = -\frac{aY_t(z)}{|Z_t(z)|^2}.$$  

The Loewner equation is valid up to the time

$$T_z = \sup\{t : Y_t(z) > 0\}.$$  

Let

$$\Upsilon_t(z) = \frac{Y_t(z)}{|g_t(z)|^2}, \quad \theta_t(z) = \arg Z_t(z), \quad S_t(z) = \sin \theta_t(z) = \frac{Y_t(z)}{|Z_t(z)|}.$$  

It is not difficult to see that $\Upsilon_t(z)$ is $1/2$ times the conformal radius of $H_t$ with respect to $z$, by which we mean that if $f : \mathbb{D} \to H_t$ is a conformal transformation with $f(0) = z$, then $|f'(0)| = 2 \Upsilon_t(z)$. Using the Schwarz lemma and the Koebe $(1/4)$-theorem we can see that

$$\Upsilon_t(z) \asymp_2 \text{dist}(z, \mathbb{R} \cup \gamma(0,t]),$$

where $\asymp_2$ means that both sides are bounded above by $2$ times the other side. Using the Loewner equation (1) we see that

$$\partial_t \Upsilon_t(z) = -\Upsilon_t(z) \frac{2aY_t(z)^2}{|Z_t(z)|^4}.$$  

(2)

In particular, $\Upsilon_t(z)$ decreases with $t$ and hence we can define

$$\Upsilon(z) = \lim_{t \to T_z-} \Upsilon_t(z)$$

which satisfies

$$\Upsilon(z) \asymp_2 \text{dist}[z, \mathbb{R} \cup \gamma(0,\infty)].$$  

Using Itô’s formula, we can see that

$$d\theta_t(z) = \frac{(1 - 2a) X_t(z) Y_t(z)}{|Z_t(z)|^4} dt - \frac{Y_t(z)}{|Z_t(z)|^2} dB_t.$$  

(3)

The Green’s function (for $SLE_\kappa$ from $0$ to $\infty$ in $\mathbb{H}$) is defined by

$$G(z) = y^{d-2} \sin \arg z^{1a-1} = y^{\frac{d}{2a} + 4a-2} |z|^{1-4a},$$

3
where $z = x + iy = |z|e^{i\theta}$. Itô’s formula shows that

$$M_t(z) = |g_t'(z)|^{2-d} G(z_t) = \Upsilon_t(z)^{d-2} S_t(z)^{4a-1}, \quad t < T_z,$$

is a local martingale satisfying

$$dM_t(z) = \frac{(1 - 4a) X_t(z)}{|Z_t(z)|^2} M_t(z) dB_t.$$

More generally, if $D$ is a simply connected domain, $z \in D$, and $w_1, w_2$ are distinct points in $\partial D$, we can define $\Upsilon_D(z), S_D(z; w_1, w_2)$ using the following scaling rules: if $f : D \to f(D)$ is a conformal transformation, then

$$\Upsilon_{f(D)}(f(z)) = |f'(z)| \Upsilon_D(z), \quad S_{f(D)}(f(z); f(w_1), f(w_2)) = S_D(z; w_1, w_2).$$

The Green’s function $G_D(z; w_1, w_2)$ is defined by

$$G_D(z; w_1, w_2) = \Upsilon_D(z)^{d-2} S_D(z; w_1, w_2)^{4a-1},$$

and satisfies the scaling rule

$$G_D(z; w_1, w_2) = |f'(z)|^{2-d} G_{f(D)}(f(z); f(w_1), f(w_2)).$$

Under this definition, the local martingale in (4) can be rewritten as

$$M_t(z) = G_{H_t}(z; \gamma(t), \infty).$$

The following easy lemma is useful for estimating $S_D(z; w_1, w_2)$.

**Lemma 1.1.** Suppose $D$ is a simply connected domain, $w_1, w_2 \in \partial D$, and $f : \mathbb{H} \to D$ is a conformal transformation with $f(0) = w_1, f(\infty) = w_2$. Let $\partial_+ = f([0, \infty]), \partial_- = f([-\infty, 0])$. If $z \in D$, let

$$q = q_D(z; w_1, w_2) = \min \{h_D(z, \partial_+), h_D(z, \partial_-)\},$$

where $h_D$ denotes harmonic measure. Then

$$2q \leq S_D(z; w_1, w_2) \leq \pi q.$$  

**Proof.** We first note that $q, h_D, S_D$ are conformal invariants, so it suffices to prove the result for $D = \mathbb{H}, w_1 = 0, w_2 = \infty$ and by symmetry we may assume that $\theta := \arg z \leq \pi/2$. By explicit calculation, we can see that $\theta = \pi q$, and hence (6) follows from the estimate

$$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1, \quad 0 < x \leq \frac{\pi}{2}.$$ 

\qed
Using Girsanov’s theorem, it can be shown that
\[
\lim_{\epsilon \to 0^+} \epsilon^d \mathbb{P}\{\Upsilon(z) \leq \epsilon\} = c_* G(z), \quad c_* = 2 \left[ \int_0^\pi \sin^4 a \, dx \right]^{-1}.
\]

A proof of this is given in [Law09a], but we include a self-contained proof in this paper (Proposition 2.3) which also estimates the error term. It follows that if \(D\) is a simply connected domain, \(z \in D\); \(\gamma\) is an SLE\(_{\kappa}\) curve connecting distinct boundary points \(w_1, w_2 \in \partial D\), and \(D_\infty\) denotes the component of \(D \setminus \gamma\) containing \(z\), then
\[
\mathbb{P}\{\Upsilon_{D_\infty}(z) \leq \epsilon\} \sim c_* \epsilon^{2-d} G_D(z; w_1, w_2), \quad \epsilon \to 0^+.
\]

### 1.3 Review of natural parametrization

We will briefly review the construction in [LS09]. The starting point is the following proposition.

**Proposition A.** ([LS09]) Suppose that there exists a parametrization for SLE\(_{\kappa}\) in \(\mathbb{H}\) satisfying the domain Markov property and the conformal invariance assumption. For a fixed Lebesgue measurable subset \(S \subset \mathbb{H}\), let \(\Theta_t(S)\) denote the process that gives the amount of time in this parametrization spent in \(S\) before time \(t\) (in the half-plane capacity parametrization), and suppose further that \(\Theta_t(S)\) is \(\mathcal{F}_t\) adapted for all such \(S\). If \(\mathbb{E}\Theta_\infty(D)\) is finite for all bounded domains \(D\), then it must be the case that (up to multiplicative constant)
\[
\mathbb{E}\Theta_\infty(D) = \int_D G(z) \, dA(z),
\]
where \(dA\) denotes integration with respect to area, and more generally,
\[
\mathbb{E}[\Theta_\infty(D) - \Theta_t(D) | \mathcal{F}_t] = \int_D M_t(z) \, dA(z).
\]

Let \(\mathcal{D}\) denote the set of bounded domains \(D \subset \mathbb{H}\) with \(\text{dist}(\mathbb{R}, D) > 0\). Write
\[
\mathcal{D} = \bigcup_{m=1}^\infty \mathcal{D}_m,
\]
where \(\mathcal{D}_m\) denotes the set of domains \(D\) with
\[
D \subset \{x + iy : |x| < m, 1/m < y < m\}.
\]
For any process \(\Theta_t(D)\) with finite expectations, by Proposition A., one has
\[
\Psi_t(D) = \mathbb{E}[\Theta_\infty(D) | \mathcal{F}_t] - \Theta_t(D),
\]
where \(\Psi_t(D) := \int_D M_t(z) \, dA(z)\), which is a supermartingale in \(t\) because \(M_t(z)\) is a non-negative local martingale. It is also not difficult to prove that \(\Psi_t(D)\) is in fact continuous.
as a function of $t$ by its definition. Assuming the conclusion of Proposition A, the first term on the right-hand side of (7) is a martingale and the map $t \mapsto \Theta_t(D)$ is increasing. Inspired by the continuous case of the standard Doob-Meyer theorem [DM82], any continuous supermartingale can be written uniquely as the sum of a continuous adapted decreasing process with initial value zero and a continuous local martingale, Lawler and Shieffeld in [LS09] introduce the following definition.

**Definition ([LS09])**

- If $D \in \mathcal{D}$, then the natural parametrization $\Theta_t(D)$ is the unique continuous, increasing process such that
  \[ \Psi_t(D) + \Theta_t(D) \]
  is a martingale (assuming such a process exists).
- If $\Theta_t(D)$ exists for each $D \in \mathcal{D}$, the natural parametrization in $\mathbb{H}$ is given by
  \[ \Theta_t = \lim_{m \to \infty} \Theta_t(D_m), \]
  where $D_m = \{x + iy : |x| < m, 1/m < y < m\}$.

The main result in that paper is the following.

**Theorem B. ([LS09])** If $\kappa < \kappa_0 := 4(7 - \sqrt{33})$, there is an adapted, increasing, continuous process $\Theta_t(D)$ with $\Theta_0(D) = 0$ such that

\[ \Psi_t(D) + \Theta_t(D) \]

is a martingale. Moreover, with probability one for all $t$

\[ \Theta_t(D) = \lim_{n \to \infty} \sum_{j \leq t2^n} \int_{\mathbb{H}} |\hat{f}_{j/2^n}(z)|^d \phi(z2^n/2) G(z) 1\{\hat{f}_{j/2^n}(z) \in D\} dA(z), \tag{8} \]

where $\phi(z)$ is defined by $\mathbb{E}[M_1(z)] = M_0(z)(1 - \phi(z))$, and $\hat{f}_s(z) = g_s^{-1}(z + U_s)$.

As for the proof, they start by discretizing time and finding an approximation for $\Theta_t(D)$. This time discretization is the first step in proving the Doob-Meyer decomposition for any supermartingale. The second step is to take the limit. For this purpose, they use the reverse-time flow for the Loewner equation to derive uniform second moment estimates for the approximations when $\kappa < \kappa_0$. This estimate is the most difficult one in all their derivation. Then they can take a limit both in $L^2$ and with probability one.
1.4 Multi-point Green’s function

As the main step in proving the Hausdorff dimension of the \(\text{SLE}\) curve, Beffara \[Bef08\] proved the following lemma.

**Lemma 1.2.** Suppose \(D\) is a bounded subdomain of \(\mathbb{H}\) with \(\operatorname{dist}(D,\mathbb{R}) > 0\). Then there exists \(c_D < \infty\) such that if \(z, w \in D\) and \(\epsilon > 0\),

\[
\mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon, \Upsilon_\infty(w) \leq \epsilon\} \leq c_D \epsilon^{2(2-d)} |z - w|^{d-2}.
\]

Recently, Lawler and Werness \[LW10\] extended Beffara’s argument to show that

\[
\mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon, \Upsilon_\infty(w) \leq \delta\} \leq c_D \epsilon^{2-d} \delta^{2-d} |z - w|^{d-2}. \tag{9}
\]

Building on this, they show that there is a multi-point Green’s function \(G(z, w)\) such that

\[
\lim_{\epsilon, \delta \to 0^+} \epsilon^{d-2} \delta^{d-2} \mathbb{P}\{\Upsilon_\infty(z) \leq \epsilon, \Upsilon_\infty(w) \leq \delta\} = c_*^2 G(z, w). \tag{10}
\]

Although a closed form of the function \(G(z, w)\) is not given, it is shown that

\[
G(z, w) = G(z) G(w) \left[ F(z, w) + F(w, z) \right] \tag{11}
\]

where

\[
F(z, w) = \frac{\mathbb{E}_z^* \left[ |g'_T(w)|^{2-d} G(Z_T(w)) \right]}{G(w)}.
\]

\(\mathbb{E}_z^*\) denotes expectation with respect to two-sided radial \(\text{SLE}_\kappa\) through \(z\) (see Section 2 for definitions) and \(T = T_z = \inf\{t : \gamma(t) = z\}\). Roughly speaking, \(G(z, w)\) represents the probability of going through \(z\) and \(w\), and \(G(z) G(w) F(z, w)\) represents the probability of going first through \(z\) and then through \(w\). Using (5), we can write

\[
G(w) F(z, w) = \mathbb{E}_z^* \left[ G_{D_T}(w; z, \infty) \right].
\]

From the definition, we can see that if \(r > 0\),

\[
G(z, w) = r^{2(2-d)} G(rz, rw), \quad F(z, w) = F(rz, rw).
\]

More generally, if \(D\) is a simply connected domain with boundary points \(z_1, z_2\), we can define \(F_D(z, w; z_1, z_2)\) by conformal invariance.

Let

\[
M_t(z, w) = |g'_t(z)|^{2-d} |g'_t(w)|^{2-d} G(Z_t(z), Z_t(w)) = \frac{M_t(z) M_t(w) G(Z_t(z), Z_t(w))}{G(Z_t(z)) G(Z_t(w))}. \tag{12}
\]

This is the so-called two-interior-point local martingale. A similar two-boundary-point local martingale appears in \[SZ10\]. Using (10), we can see if \(\epsilon > 0\) and \(T_\epsilon\) is the first time such
that \( \Upsilon_T(z) \geq \epsilon \) or \( \Upsilon_T(w) \geq \epsilon \), then \( \mathbb{E}[M_T(z, w)] = M_0(z, w) = G(z, w) \). Hence, by Fatou’s lemma, for every stopping time \( T \),

\[
G(z, w) \geq \mathbb{E}[M_T(z, w)].
\] (13)

For our main result we will need the following two estimates about \( G(z, w) \). The first follows immediately from (9) and (10); establishing the second is the main technical work in this paper.

**Lemma 1.3.**

- Suppose \( D \) is a bounded subdomain of \( \mathbb{H} \) with \( \text{dist}(D, \mathbb{R}) > 0 \). Then there exists \( c_D < \infty \) such that if \( z, w \in D \),

\[
G(z, w) \leq c_D |z - w|^{d-2}.
\]

- There exists \( c > 0 \) such that for all \( z, w \in \mathbb{H} \),

\[
G(z, w) \geq cG(z)G(w).
\] (14)

Note that (14) is equivalent to saying that there exists \( c \) such that

\[
F(z, w) + F(w, z) \geq c.
\]

We remark that

\[
\inf_{z,w} F(z, w) = 0.
\]

Indeed, one can check that

\[
\lim_{y \to \infty} F(iy, i/y) = 0.
\]

The basic idea is that if \( y \) is large then the chance that the SLE path goes through \( yi \) and then through \( i/y \) is much smaller than the probability of going through \( i/y \) and then through \( iy \).

**Corollary 1.4.** If \( D \) is a bounded subdomain of \( \mathbb{H} \) with \( \text{dist}(D, \mathbb{R}) > 0 \), then there exists \( c_D < \infty \) such that if \( z, w \in D \), and \( T \) is a stopping time,

\[
\mathbb{E} [M_T(z) M_T(w)] \leq c_D |z - w|^{d-2}.
\] (15)

**Proof.** Using (12), (13), and Lemma 1.3 we get

\[
\mathbb{E} [M_T(z) M_T(w)] \leq c \mathbb{E} [M_T(z, w)] \leq c G(z, w) \leq c_D |z - w|^{d-2}.
\]
1.5 The main theorems

As in Lawler and Sheffield, we will prove that there is an adapted, increasing, continuous process \( \Theta_t(D) \) with \( \Theta_0(D) = 0 \) such that \( \Psi_t(D) + \Theta_t(D) \) is a martingale. The basic idea of the proof is the same. Here, we show how (15) yields the uniform integrability (class \( D \)) needed to establish the existence of the martingale.

**Theorem 1.** If \( 0 < \kappa < 8 \), there is an adapted, increasing, continuous process \( \Theta_t(D) \) with \( \Theta_0(D) = 0 \) such that

\[
\Psi_t(D) + \Theta_t(D)
\]

is a martingale. Moreover, let

\[
\Theta_{t,n}(D) = \sum_{j \leq t/2^n} \int_{H} \left| \hat{f}_{j-1}(z) \right|^d \phi(z2^n/2) G(z) 1\{ \hat{f}_{j-1}(z) \in D \} dA(z),
\]

then for any stopping time \( T \),

\[
\lim_{n \to \infty} \mathbb{E} \left[ |\Theta_{T,n}(D) - \Theta_T(D)| \right] = 0.
\]

**Remark** From Theorem B, we know that \( \mathbb{E} M_1(z) = M_0(z)(1 - \phi(z)) < M_0(z) \). So \( M_t(z) \) is a local martingale and a supermartingale which is not a proper martingale. This implies that \( \Psi_t(D) \) is not a proper martingale too. Since the theorem establishes that \( \Psi_t(D) + \Theta_t(D) \) is actually a martingale, \( \Theta_t(D) \) is non-trivial. In other words, it is not identically zero.

Before discussing how to prove Theorem 1, let us briefly review Doob-Meyer decomposition for supermartingales of class \( D \). First, we assume that \( \Psi_t \) is a supermartingale with respect to a filtration \( \mathcal{F}_t \), defined on the interval \([0, \infty)\). We also suppose that \( \mathcal{F}_t \) satisfies the usual conditions. The supermartingale \( \{\Psi_t, \mathcal{F}_t, t \geq 0\} \) is said to be of class \( D \) if the family \( \{\Psi_T : T \) is an almost surely finite stopping time\} is uniformly integrable. The following result is about Doob-Meyer decomposition for supermartingales of class \( D \). One can refer to Section 1.4 of [KS91] for more details.

**Theorem C.** [Doob-Meyer Decomposition] ([KS91]) Let \( \{\Psi_t, \mathcal{F}_t, t \geq 0\} \) be a continuous supermartingale of class \( D \). Then there exists a continuous predictable, non-decreasing process \( \{A_t, \mathcal{F}_t, t \geq 0\} \), such that \( A_0 = 0 \), \( A_\infty \) is integrable, and

\[
\Psi_t = M_t - A_t,
\]

where \( M_t := \mathbb{E}[A_\infty + \Psi_\infty | \mathcal{F}_t] \) is a martingale. This decomposition is unique up to indistinguishability, i.e. if \( \{M'_t, \mathcal{F}_t, t \geq 0\} \) and \( \{A'_t, \mathcal{F}_t, t \geq 0\} \) are a martingale and a predictable, non-decreasing process satisfying the above properties respectively, then

\[
\mathbb{P}\{M_t = M'_t, A_t = A'_t, \forall t \geq 0\} = 1.
\]
In order to prove Theorem 1, we need one result from Section 20 of Chapter VII in [DM82] about Doob-Meyer Decompositions of an increasing sequence of positive supermartingales.

**Theorem D.** ([DM82]). Let \( \{ \Psi_t^n, \mathcal{F}_t, t \geq 0 \} \) be an sequence of positive supermartingales, whose limit \( \{ \Psi_t, \mathcal{F}_t, t \geq 0 \} \) belongs to class \( \mathcal{D} \) and is regular. Let \( A_t^n \) and \( A_t \) denote the non-decreasing processes associated with \( \Psi_t^n \) and \( \Psi_t \), respectively. Then for any stopping time \( T \),

\[
\lim_{n \to \infty} E[|\Psi_T^n - \Psi_T|] = 0.
\]

**Proof of Theorem 1 given Lemma 1.3.** From Theorem C, in order to show the existence of \( \Theta_t(D) \) with the desired properties, we need to find a \( c(D) < \infty \) such that for every stopping time \( T \),

\[
E[\Psi_T^2(D)] \leq c(D).
\]

(16)

From (15), we have

\[
E[\Psi_T^2(D)] = \int_D \int_D E[M_T(z_1)M_T(z_2)] dA(z_1)dA(z_2)
\]

\[
\leq c_D \int_D \int_D |z_1 - z_2|^{d-2} dA(z_1)dA(z_2) \leq c(D).
\]

This gives the first result. Now we turn to the second result. For all \( t \) we set

\[
\Psi_t^n(D) = E[\Psi_{(i+2^{-n})}^2(D)|\mathcal{F}_i] \quad \text{if} \quad i2^{-n} \leq t < (i+1)2^{-n}.
\]

Hence \( \{ \Psi_t^n(D), \mathcal{F}_t, t \geq 0 \} \) is a positive supermartingale bounded above by \( \Psi_t(D) \), which increases to \( \Psi_t(D) \) as \( n \to \infty \). Let \( A_t^n \) be the non-decreasing process associated with \( \Psi_t^n(D) \). Then for \( i2^{-n} \leq t < (i+1)2^{-n} \),

\[
A_t^n = \sum_{0 \leq k < i} E[\Psi_{k2^{-n}}^2(D) - \Psi_{(k+1)2^{-n}}^2(D)|\mathcal{F}_{k2^{-n}}].
\]

Actually, by the change of variables and the scaling rule of \( \phi \), we have

\[
A_t^n = \Theta_{t,n}(D).
\]

By Theorem D we can complete the proof. \( \square \)

**Remark** Although we have used the existence of the multi-point Green’s function \( G(z, w) \) as established in [LW10], we could have proven our result here without its existence. In fact, an earlier draft of our paper derived the theorem from Beffara’s estimate replacing (14) with

\[
P \{ \Upsilon(z) \leq \epsilon, \Upsilon(w) \leq \delta \} \geq c P \{ \Upsilon(z) \leq \epsilon \} P \{ \Upsilon(w) \leq \epsilon \}.
\]

However, the argument is cleaner when written in terms of \( G(z, w) \) so we use it here.
The next result shows that our natural parametrization could be a kind of conformal Minkowski measure (although is not the same definition as the conformal Minkowski content defined in \cite{LS09}).

**Theorem 2.** Let $\gamma^\epsilon(0,t] = \{ z \in \mathbb{H} : \sup_{0 \leq s \leq t} M_s(z) \geq \epsilon^{2-d} \}$. If $0 < \kappa < 8$, then for any stopping time $T,$

$$\lim_{\epsilon \to 0} \mathbb{E} \left[ |\epsilon^{2-d} A(D \cap \gamma^\epsilon(0,T)) - \Theta_T(D)| \right] = 0.$$ 

**Proof of Theorem 2 given Lemma 1.3** Note that

$$\epsilon^{2-d} A(D \cap \gamma^\epsilon(0,T)) = \int_M M_{t \wedge \tilde{t}_\epsilon(z)}(z) I\{z \in \gamma^\epsilon(0,t]\} dA(z),$$

where $\tilde{t}_\epsilon(z) = \inf\{t \geq 0 : M_t(z) \geq \epsilon^{d-2}\}.$ The above integral appears in the identity

$$\int_D M_{t \wedge \tilde{t}_\epsilon(z)}(z) I\{z \notin \gamma^\epsilon(0,t]\} dA(z) = \int_D M_{t \wedge \tilde{t}_\epsilon(z)}(z) dA(z) - \int_D M_{t \wedge \tilde{t}_\epsilon(z)}(z) I\{z \in \gamma^\epsilon(0,t]\} dA(z).$$

This is actually the Doob-Meyer decomposition of the supermartingale $\int_D M_{t \wedge \tilde{t}_\epsilon(z)}(z) I\{z \notin \gamma^\epsilon(0,t]\} dA(z).$ By Theorem D and (16), we can complete the proof.

1.6 Outline of the paper

In Section 2 we will develop the theory of two-sided radial SLE in order to prove Proposition 3.1. Although this was discussed somewhat in \cite{Law09a}, our treatment here is self-contained. One main goal is the refined “one-point” estimate in Proposition 2.3. The first step of the proof goes back to \cite{RS05} in which the radial parametrization was used to establish a weaker form of the estimate. Here we use Girsanov’s theorem to reduce the question to the rate of convergence to equilibrium of a simple one-dimensional diffusion. The sharp estimate is used in \cite{LW10} to improve Beffara’s estimate and show existence of the multi-point Green’s function. When considering radial SLE\(_\kappa\) going through points $z = x + iy$ with $|x| \gg y$, it is useful to compare this to a process headed to $x$. We call the appropriate one-dimensional process “two-sided chordal SLE\(_\kappa\)” and give some of its properties.

The only thing we need to prove Theorem 1 is the estimate (14). We do not know the optimal value for $c$, but we can guarantee that our proof does not give the optimal value. In Section 3.1 we spend a relatively long time deriving an estimate that states roughly that there is a positive probability that a two-sided radial path stays with a small distance of an “L”-shape. The proof uses two well-known ideas:

- If $f : [0,1] \to \mathbb{R}$ is a continuous function with $f(0) = 0$ and $\epsilon > 0$, and $U_t$ is a standard Brownian motion, then the probability that $\|f - U\|_\infty < \epsilon$ is positive, where $\| \cdot \|_\infty$ denotes the supremum norm.
• If the driving functions (using the Loewner equation) of two curves are close in the supremum norm, then the curves are close in the Hausdorff metric. This follows in a straightforward manner from the Loewner equation. (It is possible that the curves are not close in the supremum norm on curves, but this is not important for us).

It takes a little time to write out the details because our driving functions are those for two-sided radial and we need some uniformity in the estimates. Readers who are convinced that this can be done can skip the proof of Proposition 3.1. Proposition 3.2 is a simple deterministic estimate which uses Proposition 3.1 to establish (14) for \( z,w \) far apart.

It remains to prove (14) for \( z,w \) close and this is the goal of Section 3.2. Indeed, this does not seem that it should be difficult, since the events of getting close to \( z \) and getting close to \( w \) should be very positively correlated! We give the argument in two cases: when \( |z-w| \) is much smaller than \( \text{Im}(z) \) and when \( z,w \) are close to each other and also close to the boundary.

## 2 Two-sided radial SLE

One can see that with probability one \( M_{T_z}(z) = 0 \); indeed, with probability one \( \Upsilon(z) > 0 \) and \( S_{T_z}(z) = 0 \). This local martingale blows up on the event of probability zero that \( z \in \gamma(0,\infty) \). To be more precise, let

\[
\tau_\epsilon(z) = \inf\{t : \Upsilon_t(z) \leq \epsilon\}.
\]

Then \( M_{t \wedge \tau_\epsilon}(z) \) is a martingale. For convenience, we will fix \( z \) and write \( M_t, X_t, Y_t, \tau_\epsilon, \ldots \) for \( M_t(z), X_t(z), Y_t(z), \tau_\epsilon(z), \ldots \).

Two-sided radial SLE\(_\kappa\) through \( z \) is the process obtained from the Girsanov transformation by weighting by the local martingale \( M_t \). We should really call this two-sided radial SLE\(_\kappa\) from 0 to \( \infty \) going through \( z \) stopped when it reaches \( z \), but we will just say two-sided radial SLE\(_\kappa\) through \( z \). This could also be called SLE\(_\kappa\) conditioned to go through \( z \) (and stopped at \( T_z \)) although this is a conditioning on an event of probability zero. Using (4) and the Girsanov theorem, we see that for each \( \epsilon \)

\[
\frac{1 - 4a}{|Z_t|^2} dt + dW_t, \quad t \leq \tau_\epsilon,
\]

where \( W_t \) is a standard Brownian motion in the weighted measure \( \mathbb{P}^* \) which can be defined by saying that if \( V \) is an event depending only on \( \{B_t : 0 \leq t \leq \tau_\epsilon\} \),

\[
\mathbb{P}^*(V) = \mathbb{P}^*_z(V) = G(z)^{-1} \mathbb{E}[M_{\tau_\epsilon} 1_V].
\]

We write \( \mathbb{E}^* \) for expectations with respect to \( \mathbb{P}^* \). There is an implicit \( z \) dependence in \( \mathbb{P}^* \) and \( \mathbb{E}^* \); when we need to make this explicit, we write \( \mathbb{P}^*_z, \mathbb{E}^*_z \). It is not hard to show (see Section
that for every $\epsilon > 0$, $\mathbb{P}^*\{\tau_\epsilon < \infty\} = 1$. Since the SDE in (17) has no $\epsilon$ dependence, we can let $0 \leq t < T_\epsilon$. Note that

$$dX_t = \frac{(1 - 3a)}{|Z_t|^2} X_t \, dt + dW_t,$$

$$-U_t = B_t = X_t - a \int_0^t \frac{X_s}{|Z_s|^2} \, ds.$$

### 2.1 Radial parametrization

When studying the behavior of an $SLE_\kappa$ curve near an interior point $z$, it is useful to reparametrize the curve so that $\log \Upsilon_t(z)$ decays linearly. This is the parametrization generally used for radial $SLE$ and for this reason we call it the *radial parametrization*. We will study the radial parametrization in this subsection. We denote this time change as a function $\sigma(t)$ and we write $\hat{X}_t = X_{\sigma(t)}$, $\hat{Y}_t = Y_{\sigma(t)}$, etc.

We define $\sigma$ by asserting that

$$\hat{Y}_t = Y_{\sigma(t)} = e^{-2at}.$$

Using (2), we see that

$$-2a\hat{Y}_t = \partial_t [\hat{Y}_t] = -\hat{Y}_t \frac{2aY_t^2}{|Z_t|^4} [\partial_t \sigma(t)],$$

and hence

$$\partial_t \sigma(t) = \frac{|\hat{Z}_t|^4}{Y_t^2}.$$

From (3), we see that $\hat{\theta}_t := \theta_{\sigma(t)}$ satisfies

$$d\hat{\theta}_t = 2a \cot \hat{\theta}_t \, dt + d\hat{W}_t, \quad \hat{W}_t \text{ is a standard Brownian motion with respect to the weighted measure } \mathbb{P}^*.$$

Since $a > 1/4$, the process (18) stays in $(0, \pi)$ for all times. This implies that for all $\epsilon > 0$,

$$\mathbb{P}^*\{\tau_\epsilon < \infty\} = 1. \quad (19)$$

### 2.1.1 The corresponding SDE

The analysis of two-sided radial $SLE$ relies on detailed properties of the equation (18) which fortunately are not very difficult to obtain. In this subsection we focus on this SDE and set $r = 2a > 1/2$. Constants in this section may depend on $r$. We consider the equation

$$dX_t = r \cot X_t \, dt + dB_t, \quad X_0 \in (0, \pi). \quad (20)$$
In studying this equation, it is useful to note that this is the equation that one gets if one starts with a Browian motion $X_t$ and then weights paths locally by $\sin^r X_t$. To be more precise, if $X_t$ is a standard Brownian motion, then

$$M_t = [\sin X_t]^r \exp \left\{ -\frac{r(r-1)}{2} \int_0^t \frac{ds}{\sin^2 X_s} + \frac{1}{2} r^2 t \right\}$$

is a local martingale satisfying

$$dM_t = r \cot X_t M_t dX_t.$$ 

If we weight by the local martingale, then $X_t$ satisfies (20) where $B_t$ is a Brownian motion in the weighted measure. In fact, since the weighted process stays in $(0, \pi)$, one can see that $M_t$ is actually a martingale.

The equation (20) is closely related to the Bessel equation

$$dX_t = \frac{r}{X_t} dt + dB_t. \quad (21)$$

This equation is obtained by starting with a Brownian motion $X_t$ and weighting locally by $X_t^r$, i.e., by the martingale

$$N_t = X_t^r \exp \left\{ -\frac{r(r-1)}{2} \int_0^t \frac{ds}{X_s^2} \right\},$$

which satisfies

$$dN_t = r \frac{1}{X_t} N_t dX_t.$$ 

The next lemma gives a precise statement about the relationship.

**Lemma 2.1.** There exists $c < \infty$ such that the following is true. Suppose $\mu_1$ is the probability measure on paths given by $X_t$, $0 \leq t \leq 1 \wedge T$, where $X_t$ satisfies (20) with $X_0 = x$ and

$$T = \inf\{t : X_t \geq \pi/2\}.$$

Let $\mu_2$ be the analogous probability measure using the equation (21). Then

$$\frac{1}{c} \leq \frac{d\mu_1}{d\mu_2} \leq c. \quad (22)$$

**Proof.** From the explicit forms of $M_t, N_t$, one can see there exists $c$ such that

$$c^{-1} N_t \leq M_t \leq c N_t, \quad 0 \leq t \leq T \wedge 1.$$
By comparison with the Bessel process we see that the process satisfying (20) never leaves $(0, \pi)$. The invariant density for the process is

$$h(x) = C_{2r} \sin^{2r} x, \quad C_{2r}^{-1} = \int_0^\pi \sin^{2r} y \, dy.$$ 

(This can be derived in a number of ways. It essentially follows because the invariant density for Brownian motion weighted by a function $F$ is proportional to $F^2$.) In particular,

$$\int_0^\pi h(x) [\sin x]^{1-2r} \, dx = 2C_{2r}. \quad (23)$$

If $t \geq 0, x \in (0, \pi)$ we define

$$\psi(t, x) = \mathbb{E}^x[(\sin X_t)^{1-2r}],$$

where $X_t$ satisfies (20). Note that $\psi(t, x) = \psi(t, \pi - x)$. Let

$$\tilde{\psi}(t, x) = \mathbb{E}^x[X_t^{1-2r}],$$

where $X_t$ satisfies (21). Using the previous lemma, we can see there exist $c_1, c_2$ such that

$$c_1 \psi(t, x) \leq \tilde{\psi}(t, x) \leq c_2 \psi(t, x), \quad 0 < x \leq \pi/2, \quad 0 \leq t \leq 1. \quad (24)$$

Let

$$v(t, x) = [x \vee \sqrt{t} \wedge 1]^{1-4a}.$$ 

The next lemma collects the facts about the SDE that we will need.

**Lemma 2.2.** There exists $c < \infty$ such that the following is true for all $x \in (0, \pi/2]$.

- If $t \geq 1$,
  $$|\psi(t, x) - 2C_{2r}| \leq ce^{-(r+\frac{1}{2})t}. \quad (25)$$

- If $t \geq 0$,
  $$c^{-1} v(t, x) \leq \psi(t, x) \leq c v(t, x). \quad (26)$$

- For every $\epsilon > 0$, there is a $\delta > 0$ such that if $t > 0, 0 < x < \pi$, and $I \subset \mathbb{R}$ with
  $$\mathbb{P}^x\{X_t \in I\} \geq \epsilon,$$
  then
  $$\mathbb{E}^x[(\sin X_t)^{1-2r}; X_t \in I] \geq \delta v(t, x). \quad (27)$$

**Proof.** The fact that

$$\lim_{t \to \infty} \psi(t, x) = 2C_{2r}$$
follows from (23). To get the error estimate, one needs the next eigenvalue. We will pull it out of the hat. Suppose $x_0 \leq \pi/2$ and let $T$ be the first time that the process reaches $\pi/2$. A simple application of Itô’s formula shows that

$$M_t = \cos(X_t) e^{(r+\frac{1}{2})t},$$

is a martingale. Using this one can show that

$$\mathbb{P}^{x_0} \{ T \geq t \} \leq c e^{-(r+\frac{1}{2})t},$$

with a constant independent of the starting point and then a coupling argument can be used to show that if we start two processes, one at $x_0$ and the other in the invariant density, then the probability that they do not couple by time $t$ is $O(e^{-(r+\frac{1}{2})t})$.

Note that (26) for $t \geq 1$ follows immediately from (25), so it suffices to prove it for $t \leq 1$. A coupling argument shows that if $x \leq x_1$, then

$$\mathbb{P}^x \{ X_t \leq y \} \geq \mathbb{P}^{x_1} \{ X_t \leq y \} \leq c(y/x)^{2r+1}.$$

By integrating, we see for all $t$,

$$\mathbb{E}^x[(\sin X_t)^{1-2r}] \leq c x^{1-2r} + \mathbb{E}^x[(\sin X_t)^{1-2r} ; X_t \leq x] \leq c x^{1-2r}.$$

This gives the upper bound for $t \leq x^2$; for the lower bound for these $t$, we need only note that there is a positive probability that the process starting at $x$ stays within distance $x/2$ of $x$ up to time $x^2$.

For the remainder we consider $x^2 \leq t \leq 1$. The estimates are more easily done for the Bessel process using scaling. Since $t \leq 1$, it suffices by (24) to prove that

$$\tilde{\psi}(t,x) \asymp t^{\frac{1}{2}-r}, \quad x^2 \leq t \leq 1, \quad 0 < x \leq \pi/2,$$

where the implicit constants in the $\asymp$ notation are uniform over $t,x$. Suppose $X_t$ satisfies (21) with $X_0 = x$. Then $X_t$ has the same distribution as $x Y_{t/x^2}$, where $Y_t$ satisfies (21) with $Y_0 = 1$. We consider $Y_t$ for $1 \leq t \leq x^{-2}$.

To study this equation it is convenient to consider $Z_t = e^{-t/2} Y_{t}$. Note that

$$dZ_t = \left[ -e^{-t/2} Y_{t} e^{-t/2} \frac{r e^{t}}{Y_e} \right] dt + e^{-t/2} dB_{e^{t/2}} = \left[ -Z_t + \frac{r}{Z_t} \right] dt + d\tilde{B}_t,$$

where $\tilde{B}_t$ is a standard Brownian motion. This is a positive recurrent SDE with invariant density proportional to

$$x^{2r} e^{-x^2}. $$

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We consider this with the initial condition $Z_{-\infty} = \infty$. (One can show this makes sense by writing the equation for $R_t = 1/Z_t$ and showing that this can be well defined with $R_{-\infty} = 0$.) By time 0, the process is within a constant multiple of the invariant density. All we will need from this are the following easily derived facts.

\[
\mathbb{E} \left[ Z_t^{1-2r} \right] \leq c, \quad t \geq 0.
\]

Secondly, for every $\epsilon > 0$, there is a $\delta > 0$ such that if $t \geq 0$ and

\[
P\{Z_t \geq K\} \geq \epsilon,
\]

then

\[
\mathbb{E} \left[ Z_t^{1-2r}; Z_t \geq K \right] \geq \delta.
\]

\[\square\]

### 2.1.2 The one-point estimate

The results of the previous subsection will be used with $r = 2a > 1/2$. We use the radial parametrization to prove the next proposition.

**Proposition 2.3.** If $z = x + iy$ and $\theta = \arg z$, then for $\epsilon \leq y$,

\[
P\{\tau_\epsilon < \infty\} = \epsilon^{2-d} G(z) \psi(t, \theta),
\]

where $t$ satisfies $\epsilon = ye^{-2ta}$, i.e.,

\[
P\{\tau_\epsilon < \infty\} = \epsilon^{2-d} G(z) \psi \left( \frac{\log(y/\epsilon)}{2a}, \theta \right).
\]

It follows that for $1/2 \leq \epsilon \leq 1$,

\[
P\{\tau_{\epsilon y} < \infty\} \geq \theta^{4a-1} \left[ \theta \vee \sqrt{1-\epsilon} \right]^{1-4a} = \min \{1, (\theta/\sqrt{1-\epsilon})^{4a-1} \}.
\]

For $\epsilon \leq 1/2$,

\[
P\{\tau_{\epsilon y} < \infty\} \geq \epsilon^{2-d} \theta^{4a-1}.
\]

**Proof.** Let $\tau = \tau_\epsilon$. Since $M_{t \land \tau}$ is a martingale, we have

\[
G(z) = \mathbb{E} [M_{t \land \tau}] = \mathbb{E} [M_t; t < \tau] + \mathbb{E} [M_\tau; \tau \leq t].
\]

We can let $t \to \infty$, and from (19) and the dominated convergence theorem, we can deduce that

\[
G(z) = \mathbb{E} [M_\tau; \tau < \infty].
\]

Also,

\[
P\{\tau < \infty\} = \epsilon^{2-d} \mathbb{E} \left[ \Upsilon_{\tau}^{-d-2}; \tau < \infty \right] = \epsilon^{2-d} \mathbb{E} [M_\tau S_{\tau}^{-4a}; \tau < \infty] = \epsilon^{2-d} G(z) \mathbb{E}^* \left[ S_{\tau}^{-4a} \right].
\]

From Girsanov, we see that

\[
\mathbb{E}^* \left[ S_{\tau}^{-4a} \right] = \psi(t, \theta).
\]

\[\square\]
The following is a corollary of this and Lemma 2.2.

**Proposition 2.4.** For $\epsilon \leq y$,

$$
P\{\tau_\epsilon < \infty\} = c_\epsilon G(z) \epsilon^{2-d} \left[1 + O\left(\frac{\epsilon}{y}^{1+\frac{d}{2}}\right)\right]. \tag{28}
$$

In particular, there exists $c < \infty$ such that for all $z$ and all $\epsilon$,

$$
P\{\tau_\epsilon < \infty\} \leq c [\epsilon/\text{Im}(z)]^{2-d}. \tag{29}
$$

It follows that for all $z, \epsilon$,

$$
P\{\tau_\epsilon < \infty\} \geq c G(z) \epsilon^{2-d}. \tag{30}
$$

### 2.2 Two-sided chordal $SLE_\kappa$

When $z = x + iy$ with $|x| \gg y$, then the radial parametrization is not so useful, because the path can get close to $z$ without significantly decreasing the radial parametrization. Fortunately, one can study such processes by studying another process which we call *two-sided chordal $SLE_\kappa$ through $x$*. It is $SLE_\kappa$ weighted locally by $X_t^{1-4a}$ (as compared to $|Z_t|^{1-4a}$ for two-sided radial).

Let us fix $z = x + iy$ and for convenience we assume $x > 0$. We write $\tilde{X}_t = Z_t(x) = g_t(x) - U_t$ which satisfies

$$
d\tilde{X}_t = \frac{a}{\tilde{X}_t} \ dt + dB_t.
$$

Itô’s formula shows that

$$
d\tilde{X}_t^{1-4a} = X_t^{1-4a} \left[\frac{a(4a - 1)}{X_t^2} \ dt + \frac{1 - 4a}{X_t} dB_t\right],
$$

and hence

$$
N_t = \tilde{X}_t^{1-4a} \exp\left\{-a(4a - 1) \int_0^t \frac{ds}{X_s^2}\right\} = \tilde{X}_t^{1-4a} g'_t(x)^{4a-1},
$$

is a local martingale satisfying

$$
dN_t = \frac{1 - 4a}{\tilde{X}_t} M_t \ dB_t.
$$

Hence,

$$
 dB_t = \frac{1 - 4a}{\tilde{X}_t} \ dt + d\tilde{W}_t,
$$

where $\tilde{W}$ is a Brownian motion in the weighted measure. In particular,

$$
 d\tilde{X}_t = \frac{1 - 3a}{\tilde{X}_t} \ dt + d\tilde{W}_t.
$$
From this we see that the terminal time $T_x$ is finite with probability one in the weighted measure. Proposition 2.6 shows that $\gamma(T_x) = x$. (The proof is similar to the proof that SLE$\kappa$ hits points for $\kappa \geq 8$.) We precede this with a standard lemma about Bessel process whose proof we omit.

**Lemma 2.5.** Suppose $r < 1/2$ and $X_t$ satisfies the Bessel equation

$$dX_t = \frac{r}{X_t} dt + dB_t, \quad X_0 = 1.$$

Let $T = \inf\{t : X_t = 0\}$. Then with probability one,

$$T < \infty, \quad \int_0^T \frac{dt}{X_t} < \infty, \quad \int_0^T \frac{dt}{X_t^2} = \infty.$$

Moreover, for every $\delta > 0$,

$$\mathbb{P}\left\{ 0 \leq X_t \leq 1 + \delta, 0 \leq t \leq \delta; T \leq \delta; \int_0^T \frac{dt}{X_t} \leq \delta \right\} > 0. \quad (31)$$

If $B_0 = 0$, then

$$-B_t = X_0 - X_t + \int_0^t \frac{r}{X_s} ds = 1 - X_t + \int_0^t \frac{r}{X_s} ds$$

and hence on the event described in (31)

$$-\delta \leq -B_t \leq 1 + \delta, \quad 0 \leq t \leq T.$$

**Proposition 2.6.** Suppose $0 < x < x_1$ and let $R_t = Z_t(x_1) = g_t(x_1) - U_t$. Then with probability one, if $\gamma$ is two-sided chordal to $x$ with terminal time $T = T_x$,

$$R_T > 0.$$

**Proof.** Since $1 - 3a < \frac{1}{2}$, Lemma 2.5 implies that with probability one $T < \infty$ and

$$\int_0^T \frac{dt}{X_t} < \infty, \quad (32)$$

$$\int_0^T \frac{dt}{X_t^2} = \infty. \quad (33)$$

Let $q(x, y)$ denote the probability that $R_T > 0$ given $\tilde{X}_0 = x, R_0 = y$. By Bessel scaling, $q(x, y) = q(1, y/x)$. We claim that $q(1, r) \to 1$ as $r \to \infty$. Indeed, we have

$$\log R_T = \log[R_T - \tilde{X}_T] = \log(y - x) + \int_0^T \partial_t[\log(R_t - \tilde{X}_t)] dt = \log(y - x) - \int_0^T \frac{a dt}{X_t R_t}.$$
From this and (32), we can deduce this fact. From this and the strong Markov property, it follows that on the event

$$\sup_{0 \leq t < T} \frac{R_t - \tilde{X}_t}{X_t} = \infty,$$

we must have $R_T > 0$.

If

$$L_t = \log \frac{R_t - \tilde{X}_t}{\tilde{X}_t},$$

then

$$dL_t = \left[ -a \frac{\tilde{X}_t}{R_t} - \frac{1}{2} - 3a \right] dt - \frac{1}{X_t} d\tilde{W}_t.$$

Under a suitable time change $\hat{L}_t = L_{\sigma(t)}$, this can be written as

$$d\hat{L}_t = \left[ -a \frac{\hat{X}_t}{\hat{R}_t} - \left( \frac{1}{2} - 3a \right) \right] dt + d\hat{B}_t,$$

for a standard Brownian motion $\hat{B}_t$. By (33), in this time change it takes infinite time for $\hat{X}_t$ to reach zero, i.e., $\sigma(\infty) = T$. Since $\hat{X}_t \leq \hat{R}_t$, the drift term is bounded below by

$$-a - \left( \frac{1}{2} - 3a \right) > 0,$$

and hence $\hat{L}_t \to \infty$ as $t \to \infty$. In particular, (34) holds.

If $0 < x < x_1 < x_2$, then the random variable

$$\Delta = \Delta(x, x_1, x_2) = \max_{0 \leq t \leq T_x} \frac{Z_t(x_2)}{Z_t(x_1)}$$

is well defined and satisfies $1 < \Delta < \infty$. Moreover, Bessel scaling implies that the distribution of $\Delta(x, x_1, x_2)$ is the same as that of $\Delta(rx, rx_1, rx_2)$. The next lemma gives uniform bounds on the distribution in terms of $(x_2 - x)/(x_1 - x)$.

**Lemma 2.7.** For every $\rho > 0$ there exists $C < \infty$ such that the following is true. Suppose $x < x_1 < x_2$ with $x_1 - x \geq \rho(x_2 - x)$. Then with probability at least $1 - \rho$, $\Delta(x, x_1, x_2) \leq C$.

**Proof.** We fix $\rho$ and allow constants to depend on $\rho$. By scaling and monotonicity, we may assume $x = 1$ and $x_1 - x = \rho(x_2 - x)$. Hence we write $x_1 = 1 + s\rho$, $x_2 = 1 + s$ where $s > 0$ (our estimates must be uniform over $s$). If $s$ is very large, then since $T_x < \infty$, with probability one, one can find a $K$ such that with probability at least $1 - \rho$ for all $\bar{x} \geq K\rho$,

$$\frac{Z_0(\bar{x})}{2} \leq Z_t(\bar{x}) \leq 2Z_0(\bar{x}), \quad 0 \leq t \leq T_x,$$
and hence with probability at least $1 - \rho$, $\Delta(1, 1 + s\rho, 1 + s) \leq 4$. For $1/2 \leq s \leq K$, we can bound

$$\Delta(1, 1 + s\rho, 1 + s) \leq \Delta \left(1, 1 + \frac{\rho}{2}, 1 + K\right).$$

For the remainder we assume that $s \leq 1/2$. Let

$$\xi = \inf\{t : |\gamma(t) - 1| = 2s\}.$$

By distortion estimates, we see that

$$g_t'(x) \asymp g_t'(x_1) \asymp g_t'(x_2), \quad 0 \leq t \leq \xi$$

with the implicit constants uniform over $s$. Since $g_t'(x')$ is an increasing function of $x'$, this implies

$$Z_t(x_1) \asymp Z_t(x_2), \quad 0 \leq t \leq \xi,$$

which gives uniform estimates (with probability one) on

$$\max_{0 \leq t \leq \xi} \frac{Z_t(x_2)}{Z_t(x_1)}$$

Also, using the 1/4-theorem with distortion estimates,

$$Z_\xi(x) \asymp Z_\xi(x_1) - Z_\xi(x) \asymp Z_\xi(x_2) - Z_\xi(x),$$

with again the estimates uniform over $s$. The conditional distribution of

$$\max_{\xi \leq t \leq T} \frac{Z_t(x_2)}{Z_t(x_1)}$$

given $\mathcal{F}_\xi$ is the distribution of $\Delta(Z_\xi(x), Z_\xi(x_1), Z_\xi(x_2))$. Hence, we reduce this to the case where $x, x_1, x_2$ are comparable which we have already stated and handled.

**Lemma 2.8.** For every $\rho > 0$, there is a $u > 0$ such that the following holds. Suppose $x > 0$ and $\gamma$ is two-sided chordal $SLE_\kappa$ to $x$. Let $0 < \epsilon \leq x$ and

$$\xi = \inf\{t : |\gamma(t) - x| = \epsilon\}.$$

Define $\psi$ by

$$\gamma(\xi) = x + \epsilon e^{i\psi}.$$

Then with probability at least $1 - \rho$, $\psi \geq u$.

**Proof.** As $\psi \to 0$, the extremal distance between $(x, x + \epsilon/2)$ and $(x + 2\epsilon, \infty)$ in $H_\xi$ tends to $\infty$. By conformal invariance of extremal distance, if $\psi$ is very close to zero, then we can see that $Z_\xi(x + 2\epsilon) - Z_\xi(x) \gg Z_\xi(x + \frac{\epsilon}{2}) - Z_\xi(x)$. The previous lemma bounds the probability that this happens.
2.2.1 Comparison with two-sided radial

In this section, we fix \( x > 0 \) and \( \varepsilon < x/2 \). If \( z \in \mathbb{H} \), we write \( Z_t = Z_t(z) = X_t + iY_t \), \( \tilde{X}_t = Z_t(x) \) as above. Recall that two-sided radial to \( z \) is the process obtained by weighting with respect to the local martingale

\[
M_t = |Z_t|^{1-4a} (Y_t/y)^{\frac{1}{4a}-1} (Y_t/y)^{4a-1}.
\]

Here we have included a constant \( y^{2-4a-\frac{1}{4a}} \) to the usual local martingale, but this does not affect the weighting. Under this choice of \( M_t \), \( M_0 = |z|^{1-4a} \). Two-sided chordal is obtained by weighting by the local martingale

\[
N_t = |\tilde{X}_t|^{1-4a} g_t'(x)^{4a-1}.
\]

Let

\[
\sigma = \inf \{ t : |\gamma(t) - x| = 2\varepsilon \}.
\]

Distortion estimates and the 1/4-theorem imply that for \( 0 \leq t \leq \sigma \) and \( |z - x| \leq \varepsilon \),

\[
g_t'(x) \asymp |g_t'(z)|, \quad \Upsilon_t \asymp \Upsilon_0 = y, \quad Y_t \asymp y |g_t'(z)|, \quad \tilde{X}_t \asymp |Z_t|
\]

and hence

\[
M_t \asymp N_t, \quad 0 \leq t \leq \sigma,
\]

where the implicit constants are uniform over \( z \). We have just proved the following.

**Proposition 2.9.** There exists \( c < \infty \), such that for every \( z \), if \( \mu \) denotes the measure on paths

\[
\gamma(t), \quad 0 \leq t \leq \sigma,
\]

given by two-sided radial \( \text{SLE}_\kappa \) to \( z = x + iy \) and \( \nu \) denotes the analogous measure on paths using two-sided chordal to \( x \), then

\[
c^{-1} \leq \frac{d\mu}{d\nu} \leq c.
\]

The following is a corollary of this proposition and Lemma 2.8.

**Lemma 2.10.** For every \( \rho > 0 \), there is a \( u > 0 \) such that the following is true. Suppose \( x > 0 \) and \( \delta < x \). Suppose \( z \in \mathbb{H} \) with \( |z - x| \leq \delta/2 \). Let \( \gamma \) be a two-sided radial \( \text{SLE}_\kappa \) path through \( z \), and let

\[
\sigma = \inf \{ t : |\gamma(t) - x| = \delta \}.
\]

Then,

\[
\mathbb{P}^*_z \{ S_{\xi} \geq u \} \geq 1 - \rho.
\]
3 Proof of (14)

3.1 Probability of \( L \)-shapes

If \( z = x + iy \), let \( L_z \) denote the “\( L \)”-shape

\[
\begin{align*}
L_z &= [0, x] \cup [x, x + iy], \quad x \geq 0, \\
L_z &= [x, 0] \cup [x, x + iy], \quad x \leq 0,
\end{align*}
\]

and if \( \rho > 0 \),

\[
\begin{align*}
L_{z, \rho} = \{ z' \in \mathbb{H} : \text{dist}(z', L_z) \leq \rho |z| \}.
\end{align*}
\]

The goal of this section is to prove the following proposition.

**Proposition 3.1.** For every \( \rho > 0 \), there is a \( u > 0 \) such that for all \( z = x + iy \in \mathbb{H} \),

\[
\mathbb{P}_z^{*} \{ \gamma : [0, T_z] \subset L_{z, \rho} \} \geq u.
\]

The important thing is to show that we can choose \( u \) uniformly over \( z \). Before discussing the proof of this proposition, let us show how it can be used to prove (14) for \( z \) and \( w \) sufficiently spread apart. We use the following deterministic estimate.

**Proposition 3.2.** For every \( 0 < \rho \leq 1/4 \) there exists \( c < \infty \) such that the following holds. Suppose \( z \in \mathbb{H} \), \( w \in \mathbb{H} \setminus L_{z, 2\rho} \), and suppose \( \gamma : [0, T] \to \mathbb{H} \) is a curve with \( \gamma(0) = 0 \), \( \gamma(T) = z \) and \( \gamma[0, T] \subset L_{z, \rho} \). Let \( g = g_T \) be the corresponding conformal transformation and let \( Z = g(w) - g(z) \). Then

\[
G(Z) \geq c G(w),
\]

\[
c^{-1} \leq |g'(w)| \leq c.
\]

**Proof.** By scaling it suffices to consider \( |z| = 1 \) which we assume. The estimate is easy for \( |w| \geq 2 \) so we assume \( w \in \mathbb{H} \setminus L_{z, 2\rho} \) with \( |w| \leq 2 \). We fix \( \rho \) and allow (implicit or explicit) constants to depend on \( \rho \). Let \( J = \{ w' : \text{Im}(w') \geq 3 \} \). Using conformal invariance, one can see that \( \text{Im}[g(w)] \) is comparable to the probability that a Brownian motion starting at \( w \) reaches \( J \) without hitting \( \mathbb{R} \cup \gamma[0, T] \) and this is not smaller than the the probability of reaching \( J \) before hitting \( \mathbb{R} \cup L_{z, \rho} \). It is not difficult to see that this is bounded below by a constant times \( \text{Im}(w) \) and hence

\[
c \text{Im}[w] \leq \text{Im}[g(w)] \leq \text{Im}[w].
\]

Using (6), we also see that

\[
S_{H_T}(w) \simeq \text{Im}[g(w)] \asymp \text{Im}[w] \simeq S_0(w)
\]

(remember that the implicit constants depend on \( \rho \)). Also, \( \text{dist}(w, \mathbb{R} \cup L_{z, \rho}) \asymp \text{Im}(w) \), and hence by the Koebe-1/4 theorem,

\[
|g'(w)| \asymp 1.
\]

Therefore, \( G(Z) \asymp G(w) \). \( \square \)
Corollary 3.3. For every $\rho > 0$, there exists $c > 0$ such that if $z \in \mathbb{H}$ and $w \in \mathbb{H} \setminus L_{z,\rho}$, then $F(z, w) \geq c$.

Proof. By Propositions 3.1 and 3.2, there exists $u = u(\rho) > 0$ such that

\[ \mathbb{P}^*_z \{ G_{H_T}(w; z, \infty) \geq uG(w) \} \geq u. \]

Therefore,

\[ \mathbb{E}^*_z [G_{H_T}(w; z, \infty)] \geq u^2 G(w). \]

The proof of Proposition 3.1 combines a probabilistic estimate for the driving function and a deterministic estimate using the Loewner equation. Assume $z = x + iy, x \geq 0, y > 0$. Note that one obtains the hull $[x, x + iy]$ by solving (1) with $U_t \equiv x, 0 \leq t \leq y^2/(2a)$. For every $\delta > 0$, let $E_{z, \delta}$ denote the event

\[ E_{z, \delta} = \left\{ T_z \leq \frac{y^2}{2a} + \delta; -\delta \leq U_t \leq x + \delta, 0 \leq t \leq \delta; |U_t - x| \leq \delta, \delta \leq t \leq T_z \right\}. \]

Lemma 3.4. For every $\rho > 0$, there exists $\delta > 0$ such that the following holds. Suppose $z = x + iy$ with $0 \leq x \leq 1, 0 < y \leq 1$. Then on the event $E_{z, \delta}$,

\[ \gamma[0, T_z] \subset L_{z, \rho}. \]

Proof. This is a straightforward deterministic estimate using (1). One first shows that if $\delta$ is sufficiently small, then $|g_{\delta}(w) - w| \leq \rho/100$ for $\text{dist}(w, [0, x]) \geq \rho$. For $\delta \leq t \leq T_z$, we compare $g_t$ to the corresponding function $\tilde{g}_t$ obtained with $\tilde{U}_t \equiv x$. These estimates are standard (see, e.g., [Law05, Proposition 4.47]), and we omit the details.

Therefore, Proposition 3.1 reduces to the following probabilistic estimate on the driving function.

Lemma 3.5. For every $\delta > 0$, there exists $u > 0$ such that if $z = x + iy$ with $0 \leq x \leq 1, 0 < y \leq 1$, then

\[ \mathbb{P}^*_z [E_{z, \delta}] > u. \]

We will use the next lemma in the proof of Lemma 3.5. The lemma may seem to follow immediately from $T_z < \infty$, but it is important to establish uniformity in $z$.

Lemma 3.6. For every $\epsilon > 0$, there exists $r < \infty$ such that if $z = x + iy \in \mathbb{H}$,

\[ \mathbb{P}^*_z \{ T_z \leq r|z|^2; |U_t| \leq r|z|, 0 \leq t \leq T_z \} \geq 1 - \epsilon. \] (35)
Proof. By scaling and symmetry, it suffices to prove the result for \(|z| = 1, x \geq 0\). For fixed \(z\), the result is immediate from the fact that \(\mathbb{P}_z^*\{T_z < \infty\} = 1\). It is not difficult to extend this as follows: for every \(u > 0\) and every \(\epsilon > 0\), there exists \(r\) such that if \(|z| = 1\) and \(S_0(z) \geq u\), then
\[
\mathbb{P}_z^*\{T_z \leq r ; |U_t| \leq r, 0 \leq t \leq T_z\} \geq 1 - \epsilon.\tag{36}
\]
For \(y < 1/100\), let \(\mathbb{P}_1^*\) denote probabilities for two-sided chordal SLE\(_\kappa\) to 1. Again, since \(\mathbb{P}_1^*\{T_1 < \infty\} = 1\), it is easy to see that for every \(r\), there exists \(\epsilon > 0\) such that
\[
\mathbb{P}_1^*\{T_1 \leq r ; |U_t| \leq r, 0 \leq t \leq T_1\} \geq 1 - \epsilon.\tag{37}
\]
Suppose \(z = x + iy \in \mathbb{H}\) with \(|z - 1| = \delta/2 < 1/4\). Let
\[
\sigma = \inf\{t : |\gamma(t) - 1| = \delta\},
\]
and define \(\xi\) by \(\gamma(\sigma) = 1 + \delta e^{i\xi}\). From (37) and Lemma 2.9, we see that there exists \(c_1\) such that
\[
\mathbb{P}_z^*\{\sigma \leq r ; |U_t| \leq r, 0 \leq t \leq \sigma\} \geq 1 - c_1\epsilon.
\]
Using Lemma 2.10, we see that there exists \(u > 0\), such that
\[
\mathbb{P}_z^*\{\sigma \leq r ; |U_t| \leq r, 0 \leq t \leq \sigma ; S_\sigma(z) \geq u\} \geq 1 - 2c_1\epsilon.
\]
Using the form of the Poisson kernel in the upper half plane, we can see that \(|Z_\sigma(z)| \leq c_2\delta\).

Proof of Lemma 3.5. By scaling and symmetry, we may assume that \(z = x + iy\) with \(|z| = 1\) and \(x \geq 0, y > 0\). We will show that there exists \(c < \infty\) such that for each \(\rho\), there exists \(q(\rho) > 0\) such that for all \(z\),
\[
\mathbb{P}_z^*\left[E_{z,c \sqrt{\rho}}\right] \geq q(\rho).
\]
If suffices to consider \(0 < \rho \leq 1/1000\). We will consider two cases: \(y \leq 10\rho\) and \(y > 10\rho\). In this proof, constants \(c_1, c_2, \ldots\) are independent of \(\rho\), but constants \(\delta, q_1, q_2, \ldots\) may depend on \(\rho\).

First assume \(y \leq 10\rho \leq 1/100\), and hence \(3/4 < x \leq 1\). Let
\[
\eta = \eta_{\rho,z} = \inf\{t : \text{Re}[\gamma(t)] = x - 4\rho\}.
\]
For every \(\delta > 0\), consider the event \(V_\delta = V_{\delta,\rho,z}\) given by
\[
V_\delta = \{\eta \leq \delta; -\delta \leq U_t \leq 1 + \delta, \quad 0 \leq t \leq \eta\}.
\]
Using the deterministic estimate, Lemma 3.6, we can see that by choosing $\delta$ sufficiently small, then on the event $V_\delta$, $\text{Im}[\gamma(\eta)] \leq \rho$. By choosing $\delta$ smaller if necessary, we assume $\delta < \rho$.

Using Lemma 2.5, we can see that $\mathbb{P}[V_\delta] \geq q_1 > 0$. There is a curve of length at most $11\rho$ in $H_\eta$ connecting $z$ and $\gamma(\eta)$. Hence, using the Beurling estimate, there exists $c_1$ such that

$$|Z_\eta(z)| \leq c_1 \sqrt{\rho}.$$  

(Actually, we can get an estimate of $O(\rho)$, but the estimate above suffices for our purposes.)

Using Lemma 3.6, we can say there exists $c_2$ such that

$$\mathbb{P}\left\{ T - \eta \leq c_2 \sqrt{\rho}, \sup_{\eta \leq t \leq T} |U_t - U_\eta| \leq c_2 \sqrt{\rho} \mid V_\delta \right\} \geq \frac{1}{2}.$$  

Therefore, with probability at least $q_1/2$,

$$T \leq (c_2 + 1) \sqrt{\rho}, \quad -(1 + c_2) \sqrt{\rho} \leq U_t \leq 1 + (1 + c_2) \sqrt{\rho}.$$  

We now assume $y \geq 10\rho$. Let

$$\eta = \eta_{\rho,z} = \inf\{t : \text{Im}[\gamma(t)] = y - 4\rho\}.$$  

Let $W_t$ denote a standard Brownian motion and consider the event $E = E_{\delta,x}$ that

$$|W_t + (tx/\delta)| < \delta, \quad 0 \leq t \leq \delta,$$

$$|W_t - x| < \delta, \quad \delta \leq t \leq 1/a.$$  

Using standard estimates for Brownian motion (including the Cameron-Martin formula), it is standard to show that for every $\delta > 0$ there exists $u_1 > 0$ such that for all $0 \leq x \leq 1$, $\mathbb{P}(E) \geq u_1$. If we let $U_t = W_t$, then by choosing $\delta$ sufficiently small, we see that $\mathbb{P}[E] \geq u_1$.

We claim that there exists $c_3 > 0$ such that on the event $E$,

$$S_\eta(z) \geq c_3.$$  

To show this, we consider the path $\gamma[0, \eta]$. Let $\gamma^+$ be the part of the path mapped to $[U_\eta, \infty)$ under $g_\eta$ and let $\gamma^-$ be the part mapped to $(-\infty, U_\eta)$. Using the fact that $\gamma[0, \eta] \subset L_\rho$; $\text{Im}[\gamma(\eta)] = y - 4\rho$ and $\text{Im}[\gamma(t)] < y - 4\rho, t < \eta$, we can see geometrically that there is a positive probability $u_2$ such that a Brownian motion starting at $z$ exists $\eta$ at $\gamma^+$ with probability at least $c_2'$ and at $\gamma^-$ with probability at least $c_2'$. This combined with (5) gives the lower bound on $S_\eta(z)$. Since $\Upsilon_t(z)$ decreases with $t$, we get a lower bound on $M_\eta(z)$. Therefore, there exists $q_2 > 0$ such that

$$\mathbb{P}_z^{*} \{ \gamma[0, \eta] \subset L_{z,\rho} \} \geq q_2.$$  

(The reader may note that we have used the trivial bound $\Upsilon_\eta(z) \leq 1$. In fact, $\Upsilon_\eta(z) \asymp \rho$, so we can improve the last estimate but we do not need to. For $\rho$ small, it is much more likely for two-sided radial SLE to follow the $L$-shape to $z$ then for usual SLE.)

On the event $E$, there is a curve connecting $\gamma(\eta)$ to $z$ in $H_\eta$ of length $O(\rho)$. Using the Beurling estimate, we can see that there exists $c_4$ such that $|Z_\eta(z)| \leq c_4 \sqrt{\rho}$. The proof proceeds as in the previous case.  

\[ \square \]
3.2 Remainder of proof

To finish the proof, we need to consider \( z, w \) that are close. In this case we will take a stopping time \( \sigma \) such that \( z, w \) are not so close in the domain \( H_\sigma \). The next lemma is easy, but it is useful to state it.

**Lemma 3.7.** There exists \( c > 0 \) such that the following is true. Suppose \( z, w \in \mathbb{H}, u \geq 0 \) and \( \sigma \) is a stopping time for SLE\(_\kappa\) such that

\[
|\gamma(t) - z| \geq 3|z - w|, \quad 0 \leq t \leq \sigma,
\]

and such that

\[
P_z^* \{ F_{H_\sigma}(z, w; \gamma(\sigma), \infty) \geq u \} \geq u. \tag{38}
\]

Then

\[ F(z, w) \geq cu^3. \]

**Remark** Implicit in the assumptions is \( |z| \geq 3|z - w| \). We do not assume that the disk of radius \( |z - w| \) about \( z \) is contained in \( \mathbb{H} \). In particular, \( \text{Im}(z), \text{Im}(w) \) can be small and very different.

**Proof.** Let \( r = |z - w| \) and let \( B \) denote the open disk of radius \( 2r \) centered at \( z \). By assumption \( 0 \not\in B \). It is not necessarily the case that \( B \subset \mathbb{H} \); however, for \( t \leq \sigma \), the conformal map \( g_\sigma \) can be extended to \( B \) by Schwarz reflection.

Let \( E = E_u \) denote the event that \( F_{H_\sigma}(z, w; \gamma(\sigma), \infty) \geq u \). Using the Beurling estimate, we can see that \( M_{t\wedge\sigma}(z), M_{t\wedge\sigma}(w) \) are bounded martingales and hence

\[ G(z) = M_0(z) = \mathbb{E}[M_\sigma(z)], \quad G(w) = M_0(w) = \mathbb{E}[M_\sigma(w)]. \]

Also, by definition of \( \mathbb{E}^*_z \),

\[
\mathbb{E}^*_z [M_\sigma(z) 1_E] = \frac{\mathbb{E}[M_\sigma(z)^2 1_E]}{M_0(z)} = \frac{\mathbb{E}[M_\sigma(z)^2 1_E]}{\mathbb{E}[M_\sigma(z)]} \geq u \frac{\mathbb{E}[M_\sigma(z)^2 1_E]}{\mathbb{E}[M_\sigma(z) 1_E]}
\]

\[ \geq u \mathbb{E}[M_\sigma(z) 1_E] \geq u^2 G(z). \]

The first inequality uses \[ \text{(38)} \]. Using the distortion theorem on \( B \), we can see that \( |g_\sigma'(z)| \asymp |g_\sigma'(w)| \). We also claim that

\[ \frac{S_\sigma(z)}{S_\sigma(w)} \asymp \frac{\text{Im}(z)}{\text{Im}(w)}. \tag{39} \]

To see this, consider the first time that a Brownian motion starting at \( z, w \) reaches \( \mathbb{R} \cup \partial B \). If \( p(z), p(w) \) denotes the probabilities that the process hits \( \partial B \cap \mathbb{H} \) before leaving \( \mathbb{H} \), then standard estimates (gambler’s ruin estimate) show that \( p(z)/p(w) \asymp \text{Im}(z)/\text{Im}(w) \). Also, the conditional distributions given that one hits \( \partial B \) are mutually absolutely continuous (here we use either a boundary Harnack principle or the explicit form of the Poisson kernel in a half disk). Given this and \[ \text{(39)} \), we can conclude \[ \text{(39)} \].

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Therefore, using (5), we see that
\[ \frac{M_\sigma(z)}{M_\sigma(w)} = \frac{G_{H_\sigma}(z; \gamma(\sigma), \infty)}{G_{H_\sigma}(w; \gamma(\sigma), \infty)} \approx \frac{G(z)}{G(w)}, \]
and hence,
\[ \mathbb{E}_z^* [M_\sigma(w) 1_E] \geq c u^2 G(w). \]
Also,
\[ \mathbb{E}_z^* [G_{H_{T_z}}(w; z, \infty) 1_E | F_\sigma] = 1_E u G_{H_\sigma}(w; \gamma(\sigma), \infty) = 1_E u M_\sigma(w). \]
Taking expectations, we get
\[ \mathbb{E}_z^* [G_{H_{T_z}}(w; z, \infty)] \geq \mathbb{E}_z^* [G_{H_{T_z}}(w; z, \infty) 1_E] \geq c u^3 G(w), \]
which implies \( F(z, w) \geq c u^3. \)

**Proof of (14).** By symmetry and scaling, it suffices to consider \( 1 = |z| \leq |w| \) with \( \text{Re}(z) \geq 0. \)
If \( |w| > 1.01, \) then \( w \notin L_{z,1/100} \) and hence Corollary 3.3 implies that \( F(z, w) \geq c > 0. \)

Similarly, if \( \text{Im}(z) > \text{Im}(w) + (1/100), \) then \( z \notin L_{w,1/100}, \) and Corollary 3.3 implies that \( F(w, z) \geq c. \) Using similar facts about real parts and interchanging \( z, w, \) it suffices to consider \( z = x + iy, w = \tilde{x} + i\tilde{y} \) with \( 1 \leq |z|, |w| \leq 1.01, x \geq 0 \) and
\[ y \leq \tilde{y} \leq y + \frac{1}{100}, |x - \tilde{x}| \leq \frac{1}{100}. \]
We now split into two cases: \( \tilde{y} \geq 1/10 \) and \( \tilde{y} < 1/10. \)

For \( \tilde{y} \geq 1/10, \) let \( \tau = \inf \{ t : |t| = 10|z - w| \}, T = T_z. \) Using Lemma 2.2, we see that there exists \( u > 0 \) such that
\[ \mathbb{P}_z^* \{ S_\tau(z) \geq 1/4 \} \geq u. \]
Using distortion estimates and Corollary 3.3, we can see that there exists \( c > 0 \) such that on the event that \( S_\tau(z) \geq 1/4, \)
\[ F_{H_\sigma}(z, w; \gamma(\tau), \infty) \geq c. \]
We can now apply Lemma 3.7.

If \( \tilde{y} < 1/10 \) and \( |z - w| \leq \tilde{y}/20, \) we can do similarly as above, interchanging the roles of \( z, w. \)

For the remainder, we assume that \( \tilde{y} < 1/10 \) and \( |z - w| \geq \tilde{y}/20. \) Note that \( x, \tilde{x} > 9/10. \) Let
\[ r = \max \{ \tilde{y}, |z - w| \} < 1/10. \]
Let \( \tau = \inf \{ t : |\gamma(t) - \tilde{x}| = 4r \}. \) If we write
\[ \gamma(\tau) = \tilde{x} + 4r e^{i\xi}, \]
then by Lemma 2.10 there exists \( u \) such that
\[ \mathbb{P}_z^* \{ \xi > u \} \geq u. \]
On this event, distortion estimates and Corollary 3.3 imply that \( F_{H_\sigma}(z, w; \gamma(\tau), \infty) \geq c \) for some \( c \) (depending on \( \xi). \) We can now apply Lemma 3.7. 

\[ \square \]
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