ENERGY DOMINANCE AND THE
HAWKING ELLIS VACUUM CONSERVATION THEOREM

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Abstract. At a time when uninhibited speculation about negative tension – and by implication negative mass density – world branes has become commonplace, it seems worthwhile to call attention to the risk involved in sacrificing traditional energy positivity postulates such as are required for the classical vacuum stability theorem of Hawking and Ellis. As well as recapitulating the technical content of this reassuring (when applicable) theorem, the present article provides a new, rather more economical proof.

1. Introduction.

Although overshadowed by other more recent contributions – such as the no-boundary recipe for creation of an entire universe – one of the most obvious subjects for reminiscence on the auspicious occasion of this 60th birthday celebration for Stephen Hawking is his central role in the foundation of classical black hole theory as a mathematical discipline in the late 1960’s and early 1970’s. It is remarkably fortunate that it has been possible, thirty years later, to assemble so many of the other protagonists in that memorable collective enterprise, including Roger Penrose, Werner Israel, Jim Hartle, Kip Thorne, Jim Bardeen, Charles Misner, Martin Rees, Gary Gibbons, and particularly George Ellis, the coauthor with Stephen of their landmark treatise “The Large Scale Structure of Space Time” (Cambridge 1973) \[1\], which remains unsuperseded as the definitive reference on this subject, having been published just at the time when, still under Stephen’s leadership, the emergence of black hole thermodynamics diverted the main thrust of progress in black hole theory from classical to quantum aspects.

A central problem in the classical theory to which Stephen made a particularly masterly and important contribution was the question of black hole equilibrium states, as described in my recent introductory historical overview \[2\]. Due to (local rather than global) limitations of both space and time I shall not attempt to take up the challenge of providing for this occasion a more general historical review of (dynamical as well as stationary) classical black hole the-
ory and of Stephen’s leading role therein. Before coming to the main point of this brief contribution I would just like to advertise the previously published personal reminiscences of some of those involved \cite{3,4} and to emphasize that a serious student of the subject could still not do better than to start by working through the relevant sections of Hawking and Ellis (1973) \cite{1}, as reprinted in its original form, which was crafted so well that, even after all these years, no new edition or replacement has been required. (The only small point on which I am conscious of the need for any caveat concerns the questionable assumptions in a heuristic energy extraction argument invoked \cite{1} – on page 328 – as a step towards the important conclusion that a stationary non-rotating black hole configuration should be strictly static. As briefly described in the cited overview \cite{2}, a mathematically sound basis for this conclusion has finally been provided \cite{5,6,7} by much more recent work under the leadership of Bob Wald.)

What I would like to do instead on this occasion is to draw attention to another quite distinct area (not specifically concerned with black holes) to which the treatise of Hawking and Ellis (1973) \cite{1} made a particularly significant contribution, namely the question of energy positivity conditions and their use in establishing the stability of the vacuum against spontaneous creation processes. This question has recently become rather topical in view of the current fashionability \cite{8} of higher dimensional theories involving what are euphemistically described as “negative tension branes”.

Generalising the familiar case of an ordinary membrane with a support surface having two space dimensions (as well as one dimension of time), the term “brane” has come to be used for the limit of a system confined within a neighbourhood of relatively small thickness about a supporting worldsheet surface of arbitrary dimension, as exemplified by the special cases of a “string” with only a single space dimension. Everyday architectural experience shows that negative tension can perfectly well be sustained by a supporting column of sufficient thickness relative to its length, but it can be shown quite generally \cite{9} that in the small thickness limit to which the term “brane” refers, negative tension will always be accompanied by instability against lateral – i.e. “wiggle” type – perturbations, an effect that can be dramatically demonstrated (as I found quite inadvertently!) by pushing too hard on an ordinary thin pointing rod (such as the one rashly provided to me by Gary Gibbons on this occasion).

However it is not this ordinary destabilising kind of negative tension that is involved in the higher dimensional brane world models of the kind considered by authors such as Gregory, Rubakov and Sibiryakov \cite{8}, but something far more exotic and dangerous. There are two reasons why the trouble with these models does not arise from ordinary wiggle instabilities. The first is that although a few authors have actually considered braneworlds having the full range \cite{10} of lateral degrees of freedom as in ordinary branes, the majority, starting with Randal and Sundrum \cite{11} have preferred to postulate that the relevant supporting worldsheets should be of bounding orbifold type, or subject to a reflection symmetry, which suppresses all the lateral degrees of freedom, thus
making it rather misleading to use the term “brane” at all in this context. The
other reason is that, unlike the non relativistic membranes and strings (or rods)
that are familiar in everyday life, whose tension (whatever its sign) has a mag-
nitude that is small (in relativistic units) compared with their mass densities,
the kinds of brane that occur in modern higher dimensional theories are typi-
cally of the Dirac type characterised by a tension that is approximately equal
to the corresponding surface mass density. Thus, in such a brane, negativity
of the tension has the alarming implication that the mass density itself should
also be negative. This avoids the risk of lateral instability (which arises only
when the signs are opposite) but at the expense of something that is far more
frightening, namely a flagrant violation of the principle that Hawking and Ellis
(1973) baptised as the “weak energy condition”, and thus a fortiori of the
usual “dominant energy” condition described below.

The conventional wisdom is that admissible theories must respect this
kind of energy positivity condition (at least on a classical macroscopically aver-
aged level) in order to avoid instability of the vacuum against a runaway process
of creation of positive and negative mass particles. It is perhaps conceivable that
the situation might be saved by specific restrictions forbidding the excitation of
the degrees of freedom that would be involved in such a runaway process, but in
the recent words of Ed Witten, it seems more “likely that physics with viola-
tion of the weak energy condition is unstable.” To be more specific, it is clear
that in the absence of such an energy condition it would no longer be possible
to invoke what Hawking and Ellis referred to simply as the “Conservation
Theorem”, a result that might be described more specifically as the Vacuum
Conservation Theorem, whose upshot (when applicable) is effectively that, at a
classical level, the vacuum must be stable against spontaneous matter creation
processes.

As well as recapitulating the technical content of this noteworthy vacuum
stability theorem, whose original proof sprawled over pages 92 to 94 of Hawking
and Ellis (1973), the main objective of this brief contribution is to offer a
modified derivation that is rather more concise.

2 The energy dominance condition

The Hawking Ellis vacuum conservation theorem – to the effect that one
cannot create something from nothing – applies to cases describable in terms of
classical fields characterised by a stress momentum energy density tensor subject
to a postulate that Hawking and Ellis (1973) referred to as the “dominant
energy” condition. Following the recapitulation of the contents of this postulate
immediately below in the present section, the formal statement and the proof
(in a new technically simpler version) of the theorem itself will be given in the
final section of this article.

In a spacetime characterised by a time orientable pseudo Riemannian
metric with components $g_{\mu\nu}$ and Lorentz type signature such that the condition
for a vector $u^\mu$ to be a timelike unit vector is

$$u^\mu u_\mu = -1,$$  \hspace{1cm} (1)

the meaning of the postulate that a (symmetric) stress momentum energy density tensor $T^{\mu\nu}$ satisfies the energy dominance condition in question is that for any future directed timelike unit vector $u^\mu$ the corresponding energy flux vector

$$E^\mu = -T^{\mu\nu}u_\nu,$$  \hspace{1cm} (2)

should be non-spacelike, i.e.

$$E^\mu E_\mu \leq 0,$$  \hspace{1cm} (3)

with future time orientation, i.e.

$$E^\mu u_\mu \leq 0.$$

(4)

This latter requirement (4) is evidently equivalent to the requirement that the corresponding energy density scalar be non negative, i.e.

$$T^{\mu\nu}u_\mu u_\nu \geq 0.$$  \hspace{1cm} (5)

It is to be observed that if $t^\mu$ is any vector that is also timelike,

$$t^\mu t_\mu < 0,$$  \hspace{1cm} (6)

then the condition (3) that the energy flux vector $E^\mu$ should be non-spacelike implies that its contraction with $t_\mu$ can vanish only if the energy flux vector itself vanishes, i.e.

$$E^\mu t_\mu = 0 \Rightarrow E^\mu = 0,$$  \hspace{1cm} (7)

and that if $t^\mu$ is past directed, i.e.

$$t_\mu u^\mu > 0,$$  \hspace{1cm} (8)

the contraction will in any case have the non-negativity property

$$E^\mu t_\mu \geq 0,$$  \hspace{1cm} (9)

which is equivalent to the condition that

$$T^{\mu\nu}u_\mu t_\nu \leq 0,$$  \hspace{1cm} (10)

for any pair of respectively future and past directed timelike vectors $u^\mu$ and $t^\mu$.

A further almost equally obvious consequence of the energy dominance condition is that $E^\mu$ cannot vanish for any particular unit vector $u^\mu$ unless it
vanishes for all such unit vectors, which will happen only in a vacuum where $T^{\mu\nu}$ vanishes altogether, i.e.

$$\mathcal{E}^\mu = 0 \quad \Rightarrow \quad T^{\mu\nu} = 0.$$  \hspace{1cm} (11)

It can thus be seen, by combining (7) and (11), that according to the energy dominance condition the possibility for the contraction $T^{\mu\nu} u_\mu t_\nu$ to vanish for any timelike vectors $t^\mu$ and $u^\mu$ is excluded everywhere except in a vacuum, i.e.

$$T^{\mu\nu} u_\mu t_\nu = 0 \quad \Rightarrow \quad T^{\mu\nu} = 0.$$  \hspace{1cm} (12)

3. The vacuum conservation theorem

Exploiting the existence (demonstrated in Section 6.4 of their book [1]) of a globally well behaved time coordinate, $\tau$ – i.e. a field with everywhere strictly timelike gradient $\tau_{,\mu}$ – and assuming the validity of the ordinary local energy momentum conservation condition

$$T^{\mu\nu;\nu} = 0,$$  \hspace{1cm} (13)

(using a semi colon for Riemannian covariant differentiation) Hawking and Ellis showed, in Section 4.3 of their book [1], how the energy dominance postulate that has just been described can be used to derive a vacuum conservation theorem whose purport is as follows: if the boundary of a compact causally well behaved space-time volume, $V$ say, consists just of an “initial” (but not necessarily spacelike) vacuum hypersurface $\Sigma^{(0)}$ say – i.e. a hypersurface where $T^{\mu\nu}$ vanishes – together with a future boundary hypersurface, $\Sigma^{(1)}$ say, that is spacelike, then the entire space time volume $V$ will be characterised by the vacuum property, $T^{\mu\nu} = 0$.

This result is obtainable as an immediate corollary of a lemma to the effect that the vacuum property will hold on the future boundary $\Sigma^{(1)}$ (and thus on the entire boundary): it evidently suffices to apply this lemma to the intersection of $V$ with the past of a timelike hypersurface (given by a fixed global time $\tau$) through any point under consideration.

It is evident from (10) and (12) that to establish the required lemma, it will be sufficient to demonstrate the non-positivity, and hence the vanishing, of an integral of the (generically positive) form

$$\mathcal{I} = \int_{\Sigma^{(1)}} \mathcal{E}^\mu t_\mu d\Sigma,$$  \hspace{1cm} (14)

for some pair of respectively future and past directed timelike vector fields $u^\mu$ and $t^\mu$ on the “final” hypersurface $\Sigma^{(1)}$ in question.

In order to do this, let us consider the case for which $u^\mu$ is taken to be the unit future (i.e. outward) directed normal to the hypersurface. The corresponding normal surface element will then be expressible as $d\Sigma_\mu = -u_\mu d\Sigma$, so
that we shall obtain
\[ \mathcal{I} = \int_{\Sigma(t)} T^{\mu \nu} t_{\nu} d\Sigma_{\mu} \geq 0. \] (15)

If \( t^\mu \) is taken to be proportional to any one of the globally well behaved past directed timelike unit vector fields that can always be constructed (see Section 2.6 of Hawking and Ellis [1]) in any time orientable space time manifold, one can use Green’s theorem to convert the surface integral (15) to the form
\[ \mathcal{I} = \int_{\mathcal{V}} (T^{\mu \nu} t_{\nu})_{;\mu} d\mathcal{V}, \] (16)
as a consequence of the postulate that the vacuum condition should already be satisfied on the remaining “initial” part of the boundary of the relevant space time volume \( \mathcal{V} \).

More specifically (relying on the the causal good behaviour postulate) let \( \tau \) be the globally well defined time coordinate field invoked above. Then – as pointed out by Hawking and Ellis [1] in their Section 4.3 – the energy dominance condition ensures that in any compact space time region \( \mathcal{V} \) there will be some finite positive constant, \( C > 0 \), such that
\[ |T^{\mu \nu} \tau_{;\mu\nu}| \leq C T^{\mu \nu} \tau_{;\mu} \tau_{;\nu}. \] (17)
If we now choose the timelike vector in (16) to be the gradient of a new exponentially related time coordinate \( t \) according to a specification of the form
\[ t_{\mu} = t_{;\mu}, \quad C(t - t_{\infty}) = -e^{-C \tau}, \] (18)
for some constant \( t_{\infty} \), then it can be seen from (13) that the expression (16) will reduce to the form
\[ \mathcal{I} = \int_{\mathcal{V}} T^{\mu \nu} t_{\mu \nu} d\mathcal{V}, \] (19)
with
\[ t_{;\mu\nu} = e^{C \tau} (\tau_{;\mu\nu} - C \tau_{;\mu} \tau_{;\nu}). \] (20)
It then follows from (17) that we shall have
\[ T^{\mu \nu} t_{;\mu\nu} \leq 0, \] (21)
and hence
\[ \mathcal{I} \leq 0, \] (22)
which is compatible with the non-negative nature of the integral (13) only if it vanishes.

This completes the proof of the lemma (and hence of the theorem) since, as noted above, the conclusion that the integral will vanish,
\[ \mathcal{I} = 0, \] (23)
implies Q.E.D, namely that, by (10) and (12), the vacuum condition

$$T^{μν} = 0,$$

(24)

will indeed have to be satisfied everywhere on the “final” hypersurface Σ (and hence throughout V).

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