Classical solutions and higher regularity for nonlinear fractional diffusion equations

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Abstract

We study the regularity properties of the solutions to the nonlinear equation with fractional diffusion

$$\partial_t u + (-\Delta)^{\sigma/2} \varphi(u) = 0,$$

posed for $x \in \mathbb{R}^N$, $t > 0$, with $0 < \sigma < 2$, $N \geq 1$. If the nonlinearity satisfies some not very restrictive conditions: $\varphi \in C^{1,\gamma}(\mathbb{R})$, $1 + \gamma > \sigma$, and $\varphi'(u) > 0$ for every $u \in \mathbb{R}$, we prove that bounded weak solutions are classical solutions for all positive times. We also explore sufficient conditions on the nonlinearity to obtain higher regularity for the solutions, even $C^\infty$ regularity. Degenerate and singular cases, including the power nonlinearity $\varphi(u) = |u|^{m-1}u$, $m > 0$, are also considered, and the existence of classical solutions in the power case is proved.

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1 Introduction

This paper is devoted to establish the regularity of bounded weak solutions for the nonlinear parabolic equation involving fractional diffusion

\begin{equation}
\partial_t u + (-\Delta)^{\sigma/2}\varphi(u) = 0 \quad \text{in } Q = \mathbb{R}^N \times (0, \infty).
\end{equation}

Here \((-\Delta)^{\sigma/2} = \mathcal{F}^{-1}(|\cdot|^\sigma \mathcal{F})\), where \(\mathcal{F}\) denotes Fourier transform, is the usual fractional Laplacian with \(0 < \sigma < 2\) and \(N \geq 1\). The constitutive function \(\varphi\) is assumed to be at least continuous and nondecreasing. Further conditions will be introduced as needed.

The existence of a unique weak solution to the Cauchy problem for equation (1.1) has been fully investigated in \([16, 17]\) for the case where \(\varphi\) is a positive power. The solution in that case is in fact bounded for positive times even if the initial data are not, provided they are in a suitable integrability space. Such theory can be easily extended to the case of more general functions \(\varphi\); see Section 8 at the end of the paper for some details.

When \(\varphi(u) = u\) the equation is the so-called fractional heat equation, that has been studied in a number of papers, mainly coming from probability. Explicit representation with a kernel allows to show in this case that solutions are \(C^\infty\) smooth and bounded for every \(t > 0, x \in \mathbb{R}^N\), under the assumption that the initial data are integrable. In the nonlinear case such a representation is not available. Nevertheless, we will still be able to obtain that bounded weak solutions are smooth if the equation is “uniformly parabolic”, \(0 < c \leq \varphi'(u) \leq C < \infty\).

Classical solutions. Our first result establishes that if the nonlinearity is smooth enough, compared to the order of the equation, \(\max\{1, \sigma\}\), then bounded weak solutions are classical solutions.

**Theorem 1.1** Let \(u\) be a bounded weak solution to (1.1), and assume \(\varphi \in C^{1,\gamma}(\mathbb{R})\), \(0 < \gamma < 1\), and \(\varphi'(s) > 0\) for every \(s \in \mathbb{R}\). If \(1 + \gamma > \sigma\), then \(\partial_t u\) and \((-\Delta)^{\sigma/2}\varphi(u)\) are Hölder continuous functions and (1.1) is satisfied everywhere.

The precise regularity of the solution is determined by the regularity of the nonlinearity \(\varphi\); see Section 5 for the details. Notice that the condition \(\varphi' > 0\) together with the boundedness of \(u\) implies that the equation is uniformly parabolic.

The idea of the proof is as follows: thanks to the results of Athanasopoulos and Caffarelli [2], we already know that bounded weak solutions are \(C^\alpha\) regular for some \(\alpha \in (0, 1)\). In order to improve this regularity we write the equation (1.1) as a fractional linear heat equation with a source term. This term is in principle not very smooth, but it has some good properties. To be precise, given \((x_0, t_0) \in Q\), we have

\begin{equation}
\partial_t u + (-\Delta)^{\sigma/2}u = (-\Delta)^{\sigma/2}f,
\end{equation}

where

\[f(x, t) := u(x, t) - \frac{\varphi(u(x, t))}{\varphi'(u(x_0, t_0))}.\]
after the time rescaling $t \to t/\varphi'(u(x_0, t_0))$. It turns out, as we will prove in Sections 4 and 5, that solutions to the linear equation (1.2) have the same regularity as $f$.

Next, using the nonlinearity we observe that $f$ in the actual right-hand side is more regular than $u$ near $(x_0, t_0)$; see formula (4.1). We are thus in a situation that is somewhat similar to the one considered by Caffarelli and Vasseur in [7], where they deal with an equation, motivated by the study of geostrophic equations, of the form

$$\partial_t u + (-\Delta)^{1/2} u = \text{div}(vu),$$

where $v$ is a divergence free vector. Comparing with (1.2), we see two differences: in their case $\sigma = 1$, and the source term is local. These two differences will significantly complicate our analysis.

In order to obtain the above-mentioned regularity for the solutions $u$ to (1.2), we will use the fact that they are given by the representation formula

$$u(x, t) = \int_{\mathbb{R}^N} P_\sigma(x - \bar{x}, t) u(\bar{x}, 0) d\bar{x} + \int_0^t \int_{\mathbb{R}^N} (-\Delta)^{\sigma/2} P_\sigma(x - \bar{x}, t - \bar{t}) f(\bar{x}, \bar{t}) d\bar{x} d\bar{t}, \quad (x, t) \in Q,$$

where $P_\sigma$ is the kernel of the $\sigma$-fractional linear heat equation; see Section 3 for a proof of this fact, that falls into the linear theory. Therefore, we are led to study the singular kernel $A_\sigma(x, t) := (-\Delta)^{\sigma/2} P_\sigma(x, t)$. Unfortunately, $P_\sigma$, and hence $A_\sigma$, is only explicit when $\sigma = 1$. However, using the self-similar structure of $P_\sigma$, we will be able to obtain the required estimates and cancellation properties for $A_\sigma$; see Section 2.

SINGULAR AND DEGENERATE EQUATIONS. The hypotheses made in Theorem 1.1 excludes all the powers $\varphi(u) = |u|^{m-1}u$ for $m > 0$, $m \neq 1$, since they are degenerate ($m > 1$) or singular ($m < 1$) at the level $u = 0$. Nevertheless, a close look at our proof shows that we may in fact get a “local” result, Theorem 7.1. Therefore, we get for these nonlinearities (and also for more general ones) a regularity result in the positivity (negativity) set of the solution that implies that bounded weak solutions with a sign are classical; see Section 7.

HIGHER REGULARITY. If $\varphi$ is $C^\infty$ we prove that solutions are $C^\infty$. The result will be a consequence of the regularity already provided by Theorem 1.1 plus a result for linear equations with variable coefficients, Theorem 6.1, which has an independent interest. The case $\sigma < 1$ is a little bit more involved since we first have to raise the regularity in space exponent from $\sigma$ to 1. See more in Section 6, where higher regularity results depending on the smoothness of $\varphi$ are given.

**Theorem 1.2** Let $u$ be a bounded weak solution to equation (1.1). If $\varphi \in C^\infty(\mathbb{R})$, $\varphi' > 0$ in $\mathbb{R}$, then $u \in C^\infty(Q)$. 

As a direct precedent of the present work, let us mention the paper [18], where we consider the nonlinearity $\varphi(u) = \log(1 + u)$ in the case $\sigma = 1 = N$, and prove that
solutions with initial data in some \( L \log L \) space become instantaneously bounded and \( C^\infty \). Notice that in this case \( \varphi'(u) = 1/(1+u) \), and hence the equation is uniformly parabolic.

We expect some of these ideas to have a wider applicability. We point out several possible extensions, together with some comments and applications of equation (1.1) in Section 9.

Let us remark that Kiselev et al. give a proof of \( C^\infty \) regularity of a class of periodic solutions of geostrophic equations in 2D with \( C^\infty \) data [13]. Their methods are completely different to the ones used in the present paper.

2 Kernel properties

In this section we consider two issues for the kernel \( A_\sigma = (-\Delta)^{\sigma/2} P_\sigma \) which play an important role in the study of regularity, namely some estimates and a cancelation property. Before doing this, it will be convenient to introduce certain Hölder space adapted to equation (1.1), together with appropriate notations. For simplicity, we will omit the subscript \( \sigma \) in what follows when no confusion arises. It will also be convenient to use the notation \( Y = (x, t) \in \mathbb{R}^{N+1} \).

The kernel \( P \) has as Fourier transform \( \hat{P}(\xi, t) = e^{-|\xi|^{\sigma} t} \). Therefore, it has the self-similar form

\[
(2.1) \quad P(x, t) = t^{-N/\sigma} \Phi(z), \quad z = xt^{-1/\sigma} \in \mathbb{R}^N, \quad t > 0.
\]

Moreover, the profile \( \Phi \) is a \( C^\infty \) positive, radially decreasing function \( \Phi(z) = \bar{\Phi}(|z|) \), satisfying \( \bar{\Phi}(s) \sim s^{-N-\sigma} \) for \( s \to \infty \), cf. [4]. We will exploit all these properties fruitfully in what follows.

**The \( \sigma \)-distance and the associated Hölder space.** The self-similar structure of \( P \) motivates the use of the \( \sigma \)-parabolic “distance” \( |Y_1 - Y_2|_\sigma \), where

\[
|Y|_\sigma := \left( |x|^2 + |t|^{2/\sigma} \right)^{1/2} = t^{1/\sigma}(|z|^2 + 1)^{1/2}.
\]

This is not really a distance unless \( \sigma \geq 1 \), since the triangle inequality does not hold if \( \sigma < 1 \). However it is a quasimetric, with relaxed triangle inequality

\[
(2.2) \quad |Y_1 - Y_3|_\sigma \leq 2^{(1-\sigma)/\sigma} \left( |Y_1 - Y_2|_\sigma + |Y_2 - Y_3|_\sigma \right).
\]

This will be enough for our purposes.

Observe the relation between the standard Euclidean distance and this \( \sigma \)-parabolic distance:

\[
(2.3) \quad |Y| \leq c|Y|_\sigma^{\nu} \text{ for every } |Y| \leq 1, \quad \nu := \min\{1, \sigma\}.
\]
The \( \sigma \)-parabolic ball is defined as \( B_R := \{ Y \in \mathbb{R}^{N+1} : |Y|_\sigma < R \} \). Performing the change of variables

\[
s = |x| \, |t|^{-1/\sigma}, \quad r = (|x|^2 + |t|^{2/\sigma})^{1/2},
\]

we get for all \( \delta > -N - \sigma \),

\[
\int_{B_R} |Y|^\delta dY = 2\sigma N \omega_N \int_0^R r^{\delta + \sigma + 1} dr \int_0^\infty \frac{s^{N-1}}{(1 + s^2)^{N+\delta}} ds = cR^{\delta + \sigma}.
\]

In particular, the volume of the ball \( B_R \) is proportional to \( R^{N+\sigma} \). In the same way, \( \int_{B_R} |Y|^{-\delta} dY = cR^{-\delta + N + \sigma} \) for every \( \delta > N + \sigma \).

The Hölder space \( C^\alpha_\sigma(Q) \), \( \alpha \in (0, \nu) \), will consist of functions \( u \) defined in \( Q \) such that for some constant \( c > 0 \)

\[
|u(Y_1) - u(Y_2)| \leq c|Y_1 - Y_2|_\sigma^\alpha \quad \text{for every } Y_1, Y_2 \in Q.
\]

**The estimates.** Using formula (2.1) we deduce that \( A = (-\Delta)^{\sigma/2} P \) has the self-similar expression

\[
A(x,t) = t^{-1-\frac{\sigma}{2}} \tilde{\Psi}(z),
\]

where \( z = xt^{-1/\sigma} \), \( \Psi(z) = (-\Delta)^{\sigma/2} \Phi(z) \). This is the basis for the estimates.

**Proposition 2.1** For every \( Y \in Q \) the kernel \( A \) satisfies

\[
|A(Y)| \leq \frac{c}{|Y|_{\sigma}^{N+\sigma}}, \quad |\partial_r A(Y)| \leq \frac{c}{|Y|_{\sigma}^{N+2\sigma}}, \quad |\nabla_x A(Y)| \leq \frac{c}{|Y|_{\sigma}^{N+\sigma+1}}.
\]

**Proof.** We observe that \( \tilde{\Phi}(\xi) = e^{-|\xi|^\sigma} \), hence \( \tilde{\Psi}(\xi) = |\xi|^\sigma e^{-|\xi|^\sigma} \). Using the expression of the inverse Fourier transform of a radial function, putting \( \Psi(z) = \tilde{\Psi}(|z|) \),

\[
\tilde{\Psi}(s) = c_N s^{1-\frac{N}{2}} \int_0^\infty e^{-r^\sigma} r^{\frac{N}{2}+\sigma} J_{N/2}(rs) dr,
\]

we get the decay \( |\tilde{\Psi}(s)| = O(s^{-N-\sigma}) \) for \( s \) large [12] Lemma 1]. Since \( \tilde{\Psi} \) is bounded, we have

\[
|\tilde{\Psi}(|z|)| = |(-\Delta)^{\sigma/2} \Phi(z)| \leq c(1 + |z|^2)^{-\frac{N+\sigma}{2}},
\]

which implies

\[
|A(Y)| \leq \frac{c}{t^{1+\frac{\sigma}{2}} (1 + |xt^{-1/\sigma}|^2)^{\frac{N+\sigma}{2}}} = \frac{c}{|Y|_{\sigma}^{N+\sigma}}.
\]

The estimate for the time derivative is a consequence of

\[
\partial_t A(Y) = -(-\Delta)^\sigma P(x,t) = -t^{-N/\sigma-2}(-\Delta)^\sigma \Phi(z),
\]
which follows immediately from the equation satisfied by \(P\), and (2.6). Indeed,

\[
|\partial_t A(Y)| \leq \frac{c}{t^{2+\frac{N}{\sigma}}(1 + |xt^{-1/\sigma}|^2)^{\frac{N+2\sigma}{2}}} = \frac{c}{|Y|^\frac{N}{\sigma}+2\sigma}.
\]

In order to estimate the spatial derivative \(\nabla_x A(Y)\), we consider the equation relating the profiles \(\Phi\) and \(\Psi\),

\[
\sigma(-\Delta)^\sigma \Phi(z) - (N+\sigma)\Psi(z) - z \cdot \nabla \Psi(z) = 0,
\]

which follows from (2.7). It implies that

\[
|\nabla \Psi(z)| \leq \frac{c}{|z|}(|\Psi(z)| + |(-\Delta)^\sigma \Phi(z)|).
\]

Since \(\nabla \Psi\) is bounded, we deduce the estimate \(|\tilde{\Psi}'(s)| \leq c(1 + s^2)^{-\frac{N+\sigma+1}{\sigma}}\). Finally

\[
|\nabla_x A(Y)| = t^{-\frac{N+1}{\sigma}} |\tilde{\Psi}'(s)| \leq \frac{c}{t^{1+\frac{N+1}{\sigma}}(1 + |xt^{-1/\sigma}|^2)^{\frac{N+2\sigma}{2}}} = \frac{c}{|Y|^\frac{N}{\sigma}+\sigma+1}.
\]

□

Let us point out that further derivatives may be estimated in a similar way.

**Cancelation.** We now show that the function \(A\) has zero integral in the sense of principal value adapted to the self-similar variables: we take out a small \(\sigma\)-ball and integrate, and then pass to the limit.

**Proposition 2.2** For every \(R > \epsilon > 0\),

\[
(2.8) \quad \int_{B_R^+ - B_\epsilon} A(x, t) \, dx \, dt = \int_{B_R^- - B_\epsilon} A(x, t) \, dx \, dt = 0
\]

where \(B_R^+ = B_R \cap \{t > 0\}\), \(B_R^- = B_R \cap \{t < 0\}\).

**Proof.** From the equation for the profile \(\Phi\),

\[
\sigma(-\Delta)^{\sigma/2}\Phi(z) - N\Phi(z) - z \cdot \nabla \Phi(z) = 0,
\]

we get an alternative expression for the profile of \(A\),

\[
\tilde{\Psi}(s) = \frac{1}{\sigma}(N\tilde{\Phi}(s) + s\tilde{\Phi}'(s)) = \frac{s^{1-N}}{\sigma}(s^N\tilde{\Phi}(s))'.
\]

Hence, using the change of variables (2.4) and the behaviour of \(\tilde{\Phi}\) at infinity, we get

\[
\int_{B_R^+ - B_\epsilon} A(x, t) \, dx \, dt = N \omega_N \sigma \int_\epsilon^R \int_0^\infty \frac{s^{N-1} \frac{\tilde{\Psi}(s)}{r}}{r} \, ds \, dr = N \omega_N \log(R/\epsilon)(s^N\tilde{\Phi}(s)) \big|_0^\infty = 0.
\]

□
3 The linear problem

As we have said in the Introduction, the solution $u$ to equation (1.1) will be analyzed by writing it as a solution of a linear problem with a particular right hand side. This leads to the representation of $u$ by means of a variation of constants formula. We give a proof of this independent fact and then proceed to establish the regularity of this linear problem.

3.1 A representation formula

Let us consider the Cauchy problem associated to the fractional linear heat equation with a source term,

$$
\begin{aligned}
&\frac{\partial u}{\partial t} + (-\Delta)^{\sigma/2} u = (-\Delta)^{\sigma/2} f, \\
&u(x,0) = u_0(x),
\end{aligned}
$$

We assume that $f \in L^\infty_{\text{loc}}((0,\infty) : L^1(\mathbb{R}^N, \rho \, dx))$ with $\rho(x) = (1+|x|)^{-(N+\sigma)}$. We define a very weak solution to problem (3.1) as a function $u \in C([0,\infty) : L^1(\mathbb{R}^N, \rho \, dx))$, such that

$$
\int_Q u(x,t) \partial_t \zeta(x,t) \, dx \, dt = \int_Q (u - f)(x,t)(-\Delta)^{\sigma/2} \zeta(x,t) \, dx \, dt
$$

for all $\zeta \in C^\infty_0(Q)$, and $u(x,0) = u_0$ almost everywhere.

**Theorem 3.1** If $f \in L^\infty_{\text{loc}}([0,\infty) : L^1(\mathbb{R}^N, \rho \, dx)) \cap C^\alpha_\sigma(Q)$, $0 < \alpha \leq \min\{1,\sigma\}$, and $u_0 \in L^p(\mathbb{R}^N)$ for some $1 \leq p \leq \infty$, there is a unique very weak solution of problem (3.1), which is given by representation using Duhamel’s formula:

$$
\begin{aligned}
u(x,t) &= \int_{\mathbb{R}^N} P(x - \overline{x},t)u_0(\overline{x}) \, d\overline{x} \\
&\quad + \int_0^t \int_{\mathbb{R}^N} (-\Delta)^{\sigma/2} P(x - \overline{x},t - \overline{t}) f(\overline{x},\overline{t}) \, d\overline{x} d\overline{t}, \quad (x,t) \in Q.
\end{aligned}
$$

**Proof.** **Step 1. Uniqueness.** We may assume $u_0 = f = 0$ and then apply the results of [3] where a wider class of data and solutions is treated.

**Step 2. u is well defined.** We only have to take care of the last term in (3.2), which can be written as

$$
\int_Q A(Y - \overline{Y}) \chi_{\{\overline{Y} \leq \rho\}} f(\overline{Y}) \, d\overline{Y},
$$

$Y \in Q \subset \mathbb{R}^{N+1}$. In order to prove that this integral is well defined we decompose $Q = B^-_r \cup (Q \setminus B^-_r)$, with $r > 0$ small, where $B^-_r = \{\overline{Y} = (\overline{x},\overline{t}) : |Y - \overline{Y}|_{\sigma} < r, \overline{t} \leq t\}$. 


The cancellation property (2.8) allows us to estimate the inner integral,

\[
\left| \int_{B^{-r}_r} A(Y - \overline{Y}) \chi_{\{\overline{t} < t\}} f(Y) \, d\overline{Y} \right| = \left| \int_{B^{-r}_r} A(Y - \overline{Y}) \chi_{\{\overline{t} < t\}} \left( f(Y) - f(\overline{Y}) \right) \, d\overline{Y} \right| \\
\leq \int_{B^{-r}_r} |A(Y - \overline{Y})||f(Y) - f(\overline{Y})| \, d\overline{Y} \\
\leq c \int_{B^{-r}_r} \frac{d\overline{Y}}{|Y - \overline{Y}|^{N+\sigma-\alpha}} \leq c.
\]

The outer integral is bounded by using estimate (2.5).

Step 3. u IS A VERY WEAK SOLUTION. In order to justify the representation formula we proceed by approximation. Let \( t > 0 \) and take \( f \in C^\infty_c(Q) \) with \( f(x, \overline{t}) = 0 \) for \( \overline{t} \geq t - r, r \) small, thus avoiding the singularity. In that case the integral in the ball \( B^{-r}_r \) vanishes identically. Since moreover \( u \) given by (3.2) is a bounded classical solution, hence a very weak solution, the assertion holds.

Next, for any \( f \in C^\infty(Q) \) and compactly supported in space (in a uniform way), we use approximation with functions \( f_n \) as before by modifying \( f \) in the time interval \( t - r_n \leq \overline{t} \leq t \). Using the fact that the fractional Laplacian can be applied to \( f_n \) instead of \( P \), it is easy to see that we can pass to the limit \( u \) of the solutions \( u_n \), which is still a bounded classical solution. Moreover, the formula as it is written holds for functions \( f \) of this class by integrating by parts and the integrability estimate from Step 2.

Finally, for general \( f \) as in the statement, we use approximation of \( f \) in a compact set by functions \( f_n \in C^\infty(Q) \) compactly supported in space. Passing to the limit in the very weak formulation, which is again justified thanks to Step 2, we obtain that \( u = \lim u_n \) is a very weak solution. \( \square \)

### 3.2 Regularity of the linear problem

The first term in the right-hand side of the representation formula (3.2) is regular. Hence, \( u \) has the same regularity as

\[
(3.3) \quad g(Y) = \int_{\mathbb{R}^{N+1}_+} A(Y - \overline{Y}) \chi_{\{\overline{t} < t\}} f(\overline{Y}) \, d\overline{Y}.
\]

We start by proving that \( g \) has the same \( \sigma \)-Hölder regularity as \( f \).

**Lemma 3.1** Let \( f \in C_\sigma^\alpha(Q) \cap L^\infty(Q) \) for some \( 0 < \alpha < \nu \), and \( g \) given by (3.3). Then \( g \in C_\sigma^\alpha(Q) \cap L^\infty(Q) \).

**Proof.** Let \( Y_1 = (x_1, t_1), Y_2 = (x_2, t_2) \in Q \) be two points with \( |Y_1 - Y_2|_\sigma = h > 0 \) small. We have to estimate the difference

\[
(3.4) \quad g(Y_1) - g(Y_2) = \int_{\mathbb{R}^{N+1}_+} \left( A(Y_1 - \overline{Y}) \chi_{\{\overline{t} < t_1\}} - A(Y_2 - \overline{Y}) \chi_{\{\overline{t} < t_2\}} \right) f(\overline{Y}) \, d\overline{Y}
\]
and see if we get that it is $O(h^\alpha)$. We decompose $Q$ into four regions, depending on the sizes of $|\tau - x_1|$ and $|\bar{t} - t_1|$, see Figure I.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{integration_regions.png}
\caption{Integration regions.}
\end{figure}

(i) The small “ball” $B_{\varrho h}(Y_1)$, where $\varrho > 1$ is a constant to be fixed later. We take $h$ small enough ($\varrho h < \min\{t_1, 1\}$) so that, on the one hand $B_{\varrho h} \subset Q$, and, on the other hand, we can use the relation (2.3). The difficulty in this region is the non-integrable singularity of $A(Y)$ at $Y = 0$. Integrability will be gained thanks to the regularity of $f$. We first have, repeating the computations in Step 2 of the proof of Theorem 3.1

\[
\left| \int_{B_{\varrho h}(Y_1)} A(Y_1 - \bar{Y}) \chi_{(\bar{t} < t_1)} f(\bar{Y}) d\bar{Y} \right| \leq ch^\alpha.
\]

To estimate the second term in (3.4), we consider the ball $B_{\varrho h}(Y_2)$. To be sure that the distance from $\partial B_{\varrho h}(Y_1)$ to $B_{\varrho h}(Y_2)$ is positive, we take $\varrho = \max\{4, 2^{2/\sigma}\}$; see (2.2). Using again the cancelation property (2.8), we get

\[
\int_{B_{\varrho h}(Y_1)} A(Y_2 - \bar{Y}) \chi_{(\bar{t} < t_1)} f(\bar{Y}) d\bar{Y} =
\]

\[
\int_{B_{\varrho h}(Y_1)} A(Y_2 - \bar{Y}) \chi_{(\bar{t} < t_2)} f(\bar{Y}) d\bar{Y} + f(Y_2) \int_{B_{\varrho h}(Y_1) - B_{\varrho h}(Y_2)} A(Y_2 - \bar{Y}) \chi_{(\bar{t} < t_2)} d\bar{Y}.
\]

The first integral $I_1$ satisfies, as before, $|I_1| \leq c h^\alpha$. As to $I_2$, since we are far from the singularity of $A$,

\[
|I_2| \leq c h^\alpha \int_{B_{\varrho h}(Y_1) - B_{\varrho h}(Y_2)} \frac{d\bar{Y}}{h^{N+\sigma}} \leq c h^\alpha.
\]
(ii) The narrow strip $S_h = \{Y \in B_{\varrho h}(Y_1), |\bar{t} - t_1| < h^\sigma\}$. In this region we have $|\bar{Y} - Y_1|_\sigma \leq \varrho_1|\bar{Y} - Y_2|_\sigma$ and $|\bar{t} - x_1| > \varrho_2 h$, for some positive constants $\varrho_1, \varrho_2$ depending only on $\sigma$. Therefore,

$$
\left| \int_{S_h} \left( A(Y_1 - \bar{Y}) \chi_{t < t_1} - A(Y_2 - \bar{Y}) \chi_{t < t_2} \right) f(Y) \, d\bar{Y} \right|
\leq \int_{S_h} |A(Y_1 - \bar{Y})| + |A(Y_2 - \bar{Y})| |f(Y)| \, d\bar{Y} \leq \int_{S_h} \frac{d\bar{Y}}{|Y - Y_1|_{\sigma}^{N+\sigma-\alpha}} \leq c \int_{t_1 - h^\sigma}^{t_1 + h^\sigma} \int_{|\bar{t} - x_1| > \varrho_2 h} \frac{d\bar{x}d\bar{t}}{(|\bar{t} - x_1|^{N+\sigma-\alpha}} \leq ch^\alpha.
$$

Notice that $\alpha < \sigma$, so that the last integral is convergent.

(iii) The complement of the ball $B_{\varrho h}(Y_1)$ for large times, $T_h = \{Y \in B_{\varrho h}(Y_1), \bar{t} > t_1 + h^\sigma\}$. The integral in this region is 0, since here we have

$$A(Y_1 - \bar{Y}) \chi_{t < t_1} = A(Y_2 - \bar{Y}) \chi_{t < t_2} = 0.$$

(iv) The complement of the ball $B_{\varrho h}(Y_1)$ for small times, $D_h = \{Y \in B_{\varrho h}(Y_1), \bar{t} < t_1 - h^\sigma\}$. The required estimate is obtained here using the fact that

$$A(Y_1 - \bar{Y}) \chi_{t < t_1} - A(Y_2 - \bar{Y}) \chi_{t < t_2} = A(Y_1 - \bar{Y}) - A(Y_2 - \bar{Y}).$$

Thus we are integrating a difference of $A$’s, so there will be some cancelation. Indeed, by the Mean Value Theorem,

$$|A(Y_1 - \bar{Y}) - A(Y_2 - \bar{Y})| \leq |Y_1 - Y_2| \max\{|\nabla_x A(\theta)|, |\partial_\theta A(\theta)|\}$$



where $\theta = s(Y_1 - \bar{Y}) + (1-s)(Y_2 - \bar{Y})$ for some $s \in (0,1)$.

Since $|Y_1 - Y_2| \leq |Y_1 - Y_2|_\sigma^\nu = h^\nu$, and in $D_h$ it holds $|Y_1 - \bar{Y}|_\sigma \leq 2|\theta|_\sigma$, Proposition 2.1 yields

$$|A(Y_1 - \bar{Y}) - A(Y_2 - \bar{Y})| \leq \frac{ch^\nu}{|\theta|_{\sigma}^{N+\sigma+\nu}} \leq \frac{ch^\nu}{|Y - Y_1|_{\sigma}^{N+\sigma+\nu}}.$$

Therefore we conclude that

$$\left| \int_{D_h} \left( A(Y_1 - \bar{Y}) \chi_{t < t_1} - A(Y_2 - \bar{Y}) \chi_{t < t_2} \right) f(Y) \, d\bar{Y} \right|
\leq ch^\nu \int_{D_h} \frac{d\bar{Y}}{|Y - Y_1|_{\sigma}^{N+\sigma+\nu-\alpha}} \leq ch^\nu \int_{|\bar{t} > \varrho h} \frac{d\bar{Y}}{|\bar{t} - x_1|_{\sigma}^{N+\sigma+\nu-\alpha}} = ch^\alpha,$$

assuming $\alpha < \nu$ so that the last integral is convergent.

$\square$

Remark. Notice that if some derivative (even a fractional one) of $f$ belongs to $C^\alpha_\sigma(Q) \cap L^\infty(Q)$, then a computation analogous to that in the above lemma shows that the convolution of this derivative against the kernel $A$ also belongs to $C^\alpha_\sigma(Q) \cap L^\infty(Q)$. We conclude that $g$ has the same regularity as $f$.

As a corollary of Lemma 3.1, we obtain a maximal regularity result for the linear equation with a standard right hand side that has independent interest.
Corollary 3.1 Let \( f \in C_\sigma^\alpha(Q) \cap L^\infty(Q) \), \( 0 < \alpha < \min\{1, \sigma\} \). If \( w \) is a very weak solution to

\[
\partial_t w + (-\Delta)^{\sigma/2} w = f,
\]

then \( \partial_t w, (-\Delta)^{\sigma/2} w \in C_\sigma^\alpha(Q) \cap L^\infty(Q) \).

Proof. The function \( u = (-\Delta)^{\sigma/2} w \) solves (3.1) in the distributional sense. Hence \( u \in C_\sigma^\alpha(Q) \cap L^\infty(Q) \). The result now follows noticing that \( \partial_t w = f - u \).

4 Improving \( \sigma \)-Hölder regularity

We now return to the nonlinear equation (1.1). For bounded weak energy solutions the equation is neither degenerate nor singular. Hence, the results from [2] guarantee that they are \( C_\sigma^\alpha \) for some \( \alpha \in (0, \nu) \). The aim of this section is to improve this regularity showing that the solutions belong to \( C_\sigma^\alpha \) for all \( \alpha < \nu \). Further regularity, showing that the solution is classical, will be obtained in Section 5.

The idea is to use that the solution \( u \) to the nonlinear equation (1.1) is a solution to the linear equation

\[
\partial_t u + (-\Delta)^{\sigma/2} u = (-\Delta)^{\sigma/2} f, \quad f(Y) = u(Y) - \frac{\varphi(u(Y))}{\varphi'(u(Y_0))}.
\]

Since \( u \in C_\sigma^\alpha(Q) \), \( \varphi \) is uniformly parabolic, and \( \varphi' \in C^\gamma(\mathbb{R}) \), applying the Mean Value Theorem we get

\[
|f(Y_1) - f(Y_2)| = \left| 1 - \frac{\varphi'(\theta)}{\varphi'(u(Y_0))} \right| |u(Y_1) - u(Y_2)|
\]

\[
\leq \frac{|u(Y_0) - \theta|^\gamma}{\varphi'(u(Y_0))} |Y_1 - Y_2|^\alpha_{\sigma}
\]

\[
\leq c \max\{|u(Y_1) - u(Y_0)|^\gamma, |u(Y_2) - u(Y_0)|^\gamma\} |Y_1 - Y_2|^\alpha_{\sigma}
\]

\[
\leq c \max\{|Y_1 - Y_0|^\alpha_{\sigma}, |Y_2 - Y_0|^\alpha_{\sigma}\} |Y_1 - Y_2|^\alpha_{\sigma},
\]

where \( \theta \) is some value between \( u(Y_1) \) and \( u(Y_2) \). This gives not only that \( f \) has the same regularity as \( u \), namely \( f \in C_\sigma^\alpha(Q) \), but a bit more that will be enough to improve the \( \sigma \)-Hölder regularity of \( u \) by a constant factor.

Lemma 4.1 Let \( f \in L^\infty(Q) \) and let \( g \) be the function defined in (3.3). Assume that there exist \( c > 0 \), \( \delta_0 > 0 \) and \( \epsilon > 0 \), \( \alpha + \epsilon < \nu \), such that

\[
|f(Y) - f(Y_0)| \leq c|Y - Y_0|^\alpha + c^+\epsilon,
\]

\[
|f(Y_1) - f(Y_2)| \leq c\delta^\epsilon |Y_1 - Y_2|^\alpha_{\sigma},
\]

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for all $0 < \delta < \delta_0$, $Y, Y_1, Y_2 \in B_0(Y_0)$. Then,

$$|g(Y) - g(Y_0)| \leq c'|Y - Y_0|^\alpha + \epsilon,$$

for all $Y \in B_0(Y_0)$, where $c'$ depends on $c$.

Proof. The fact that $f$ is $C^{\alpha+\epsilon}_\sigma$ at $Y_0$ (with $\alpha + \epsilon < \nu$) implies that all the estimates used to prove Lemma 3.1 work and yield terms which are $O(h^{\alpha+\epsilon})$, except that for the integral $I_1$ in (3.5). To estimate this term take $\rho h < \delta_0$ and observe that (4.3) gives

$$|I_1| = \left| \int_{B_{\rho h}(Y_0)} A(Y - \nabla) \chi_{\{t < t_0\}} \left( f(Y) - f(Y) \right) dY \right| \leq c h^\epsilon \int_{B_{\rho h}(Y_0)} |Y - Y|^\alpha \sigma dY \leq c h^{\alpha+\epsilon}.$$

Applying this lemma a finite number of times we obtain the desired regularity.

**Theorem 4.1** Let $u$ be a bounded weak solution to equation (1.1). Assume $\varphi \in C^{1,\gamma}(\mathbb{R})$ and $\varphi'(s) > 0$ for every $s \in \mathbb{R}$. Then $u$ belongs to $C^\alpha_\sigma(Q)$ for all $\alpha \in (0, \nu)$.

We must remark that the restriction $\alpha + \epsilon < \nu$ in Lemma 4.1 is only needed to make the outer integrals convergent; the estimate in the ball $B_{\rho h}(Y_0)$ is true for any $\alpha \in (0, \nu)$, $\epsilon \in (0, 1]$. This observation turns to be of great importance in obtaining further regularity in the next section.

## 5 Classical solutions

Our next aim is to go beyond the $C^\alpha_\sigma$ threshold of regularity. We encounter here an additional difficulty, stemming from the nonlocal character of the fractional Laplacian operator, which is not present in the work [7], namely that $A(Y - \nabla) \chi_{\{t < t_0\}} \neq (-\Delta)\sigma/2 \chi_{\{t < t_0\}}$. For that reason we must treat the second order estimates in the time and space variables separately. We begin by improving regularity in space, to obtain $u(\cdot, t) \in C^{\alpha}(\mathbb{R}^N)$ uniformly in $t$ for some $\alpha > \nu$ depending on the regularity of the nonlinearity $\varphi$. We then use equation (1.1) to get Lipschitz regularity in time, which is later improved to get $u(x, \cdot) \in C^{\nu(1+\gamma)/\sigma} \chi_{\{t < t_0\}}(\mathbb{R}_+)$ uniformly in $x$. The last step is to reach the desired smoothness in space, $u(\cdot, t) \in C^{\nu(1+\gamma)}(\mathbb{R}^N)$ uniformly in $t$.

**Notation.** By $u \in C^{\alpha}$ with $\alpha \in [1, 2)$ we mean $u \in C^{1,\alpha-1}$ if $\alpha \in (1, 2)$, and $u \in C^{0,1}$ if $\alpha = 1$.

**Lemma 5.1** Let $f \in L^\infty(Q)$ satisfy (1.2) and (1.3) with $0 < \alpha < \nu$, $0 < \epsilon < 1$, and let $g$ be the function defined in (3.3). Then, for every $e \in \mathbb{R}^N$, $|e| = 1$,

$$|g(x_0 + he, t_0) - 2g(x_0, t_0) + g(x_0 - he, t_0)| \leq c h^{\alpha+\epsilon} \quad \text{for every } h > 0 \text{ small.}$$
Proof. Put $Y = Y_0 + (he, 0)$ and let $Y^* = 2Y_0 - Y$ be its symmetric point with respect to $Y_0$. We have to estimate the second difference

$$g(Y) - 2g(Y_0) + g(Y^*) = \int_{\mathbb{R}_+^{N+1}} A(Y, Y_0, \overline{Y}) f(\overline{Y}) d\overline{Y},$$

where

$$A(Y, Y_0, \overline{Y}) = A(Y - \overline{Y}) \chi_{\{\tau < t\}} - 2A(Y_0 - \overline{Y}) \chi_{\{\tau < t_0\}} + A(Y^* - \overline{Y}) \chi_{\{\tau < t^*\}}$$

$$= \left( A(Y - \overline{Y}) - 2A(Y_0 - \overline{Y}) + A(Y^* - \overline{Y}) \right) \chi_{\{\tau < t_0\}}.$$ 

As in the proof of Lemma 3.1 we consider separately the contributions to the integral in several regions, though here we only need to consider the ball $B_{\phi h}(Y_0)$ and its complement $B_{\phi h}(Y_0)$. The contribution of the integral in $B_{\phi h}(Y_0)$ is decomposed as the sum $J_1 - 2J_2 + J_3$, where

$$J_1 = \int_{B_{\phi h}(Y_0)} A(Y - \overline{Y}) \chi_{\{\tau < t_0\}} f(\overline{Y}) d\overline{Y},$$

$$J_2 = \int_{B_{\phi h}(Y_0)} A(Y - \overline{Y}) \chi_{\{\tau < t_0\}} f(\overline{Y}) d\overline{Y},$$

$$J_3 = \int_{B_{\phi h}(Y_0)} A(Y^* - \overline{Y}) \chi_{\{\tau < t_0\}} f(\overline{Y}) d\overline{Y}.$$ 

Thus $|\int_{B_{\phi h}(Y_0)} A f| \leq |J_1| + 2J_2 + |J_3|$. The integrals $J_1$ and $J_2$ were already estimated in the course of the proof of Lemma 3.1 modified as in Lemma 4.1 (see the comment after that lemma), thus obtaining $O(h^{\alpha + \epsilon})$. Since $Y^* - Y_0 = Y_0 - Y$, the integral $J_3$ is estimated just in the same way.

To estimate the contribution in the complement of the ball we use Taylor’s formula. We have, by Proposition 2.1

$$|A(Y, Y_0, \overline{Y})| = |A(Y - \overline{Y}) - 2A(Y_0 - \overline{Y}) + A(2Y_0 - Y - \overline{Y})|$$

$$\leq ch^2 |D^2 A(\theta)| \leq \frac{ch^2}{|Y_0 - \overline{Y}|^{N+\sigma+2}},$$

where $\theta$ is as before some intermediate point. This gives

$$\int_{B_{\phi h}(Y_0)} |A(Y, Y_0, \overline{Y})| |f(\overline{Y})| d\overline{Y} \leq ch^2 \int_{B_{\phi h}(Y_0)} \frac{d\overline{Y}}{|Y_0 - \overline{Y}|^{N+\sigma+2-\alpha-\epsilon}} = ch^{\alpha + \epsilon}.$$ 

We have used that $\alpha + \epsilon < 2$, and so the integral converges. This completes the desired estimate.

\[\square\]

**Lemma 5.2** Under the hypotheses of Theorem 1.1 bounded weak solutions $u$ to equation (1.1) satisfy $u(\cdot, t) \in C^{\alpha(1+\gamma)}(\mathbb{R}^N)$ for every $\alpha \in (0, \nu)$ uniformly in $t \geq \tau > 0$ for every $\tau > 0$. 

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Proof. For each given $Y_0 \in Q$ we define a function $g$ and, as in the proof of Lemma 4.1, we deduce estimate (5.1) with $\epsilon = 1 + \gamma$ at that point, which is translated into the same estimate for the solution $u$. Since the constants do not depend on the particular point chosen, we get that $u$ satisfies
\[
|u(x + he, t) - 2u(x, t) + u(x - he, t)| \leq c h^{\alpha(1 + \gamma)},
\]
with constant uniform in $Q$. We can thus prove that $(-\Delta)^{\delta/2}u$ is bounded in $\mathbb{R}^N$ for every $t > \tau > 0$ and every $\delta \in (0, \alpha(1 + \gamma))$. Indeed,
\[
|(-\Delta)^{\delta/2}u(x, t)| = \left| c \delta \int_{\mathbb{R}^{N+1}} \frac{u(x + z, t) - 2u(x, t) + u(x - z, t)}{|z|^{N+\delta}} dz \right| \\
\leq c \int_{\{|z| < \tau\}} \frac{|z|^{\alpha(1 + \gamma)}}{|z|^{N+\delta}} dz + c \int_{\{|z| > \tau\}} \frac{dz}{|z|^{N+\delta}} \leq c.
\]
The result now follows from [20, Proposition 2.9].

Lemma 5.3 Under the hypotheses of Theorem 1.1, $u(x, \cdot) \in C^{\nu(1+\gamma)/\sigma}(\mathbb{R}_+)$ uniformly in $x \in \mathbb{R}^N$.

Proof. We first show that $|(-\Delta)^{\sigma/2}\varphi(u)|$ is bounded in $Q$. For that purpose we estimate the second differences in $x$ of $\varphi(u)$ in terms of second differences in $x$ of $u$ and use the previous result. If $Z = (he, 0), e \in \mathbb{R}^N, |e| = 1$,
\[
|\varphi(u(Y_0 + Z)) - 2\varphi(u(Y_0)) + \varphi(u(Y_0 - Z))| \\
\leq |\varphi(u(Y_0 + Z)) - 2\varphi(u(Y_0)) + \varphi(2u(Y_0) - u(Y_0 + Z))| \\
+ |\varphi(u(Y_0 - Z)) - \varphi(2u(Y_0) - u(Y_0 + Z))| \\
\leq |\varphi|_{C^{1+\gamma}} |u(Y_0 + Z) - u(Y_0)|^{1+\gamma} \\
+ \|\varphi'(u)||_{L^\infty(Q)} |u(Y_0 + Z) - 2u(Y_0) + u(Y_0 - Z)| \leq c h^{\alpha(1+\gamma)}
\]
for every $\alpha < \nu$. Since $\nu(1 + \gamma) > \sigma$ we get, analogously to how we obtained (5.2), that $|(-\Delta)^{\sigma/2}\varphi(u)| \leq c$ in $Q$. Now, using the equation we get that $|\partial_t u| \leq c$ in $Q$, that is, $u$ is Lipschitz continuous in time, uniformly in space. This means $u \in C^\sigma_{r}(Q)$. With this information we now try to repeat the above calculations of Lemma 5.3 with $x_0$ fixed and varying $t$. To this end we consider the point $Y = Y_0 + (0, h)$, $h > 0$ (for simplicity), and we replace $h$ by $h^{1/\sigma}$ in the regions of integration, see the proof of Theorem 3.1.

First, the integral in the ball $B_{gh^{1/\sigma}}(Y_0)$ is estimated as in Lemma 4.1, taking note of (1.1), which holds with $\alpha = \nu$. Thus $|\int_{B_{gh^{1/\sigma}}(Y_0)} A f| = O(h^{\nu(1+\gamma)/\sigma})$.

Now consider the region $D_{h^{1/\sigma}} = \{ \overline{Y} \in \overline{B}_{gh^{1/\sigma}}(Y_0), \overline{t} < t_0 - h \}$. The idea here is that the characteristic functions take all the value one. Thus, by using Taylor’s expansion,
since only $t$ varies, we have
\[
|\mathcal{A}(Y, Y_0, \overline{Y})| = |A(Y - \overline{Y}) - 2A(Y_0 - \overline{Y}) + A(2Y_0 - Y - \overline{Y})| \\
\leq ch^2 |\partial_t^2 A(\theta)| \leq \frac{ch^2}{|Y_0 - \overline{Y}|^{N+3\sigma}},
\]
where $\theta$ is some intermediate point. This gives
\[
\int_{D_{h^{1/\sigma}}} |\mathcal{A}(Y, Y_0, \overline{Y})| |f(\overline{Y})| d\overline{Y} \leq ch^2 \int_{D_{h^{1/\sigma}}} \frac{|Y_0 - \overline{Y}|^{\nu(1+\gamma)}}{|Y_0 - \overline{Y}|^{N+3\sigma}} d\overline{Y} \leq ch^{\nu(1+\gamma)/\sigma}.
\]

We now turn our attention to the difficult part, the small slice $S_{h^{1/\sigma}} = \{\overline{Y} \in \overline{B}_{gh^{1/\sigma}}(Y_0), \ |\overline{t} - t_0| < h \}$, where we have to look more carefully at the possible cancellations. We have
\[
\int_{S_{h^{1/\sigma}}} \mathcal{A}(Y, Y_0, \overline{Y}) f(\overline{Y}) d\overline{Y} \\
= \int_{S_{h^{1/\sigma}}} (A(Y - \overline{Y}) \chi_{\{\overline{t}<t_0\}} - 2A(Y_0 - \overline{Y}) \chi_{\{\overline{t}<t_0\}}) f(\overline{Y}) d\overline{Y} \\
+ \int_{S_{h^{1/\sigma}}} (A(Y_0 - \overline{Y}) \chi_{\{\overline{t}>t_0\}} - A(Y_0 - \overline{Y}) \chi_{\{\overline{t}<t_0\}}) f(\overline{Y}) d\overline{Y}.
\]

First, by the Mean Value Theorem applied to $A$ in the time variable, together with the regularity $C^\nu_\sigma$ of $u$ and Lemma 3.1, we have
\[
|J_1| \leq \int_{t_0-h}^{t_0+h} \int_{\{|\overline{t} - x_0| > gh^{1/\sigma}\}} \frac{ch|Y - Y_0|^{\nu(1+\gamma)}}{|Y - Y_0|^{N+2\sigma}} d\overline{t} dx \\
\leq \frac{d}{\{\{|\overline{t} - x_0| > gh^{1/\sigma}\}} \frac{|Y - Y_0|^{N+2\sigma-\nu(1+\gamma)}}{N+2\sigma-\nu(1+\gamma)} = ch^{\nu(1+\gamma)/\sigma}.
\]

As to the second integral $J_2$, performing the change of variables $\overline{Y} \to Z_1 = \overline{Y}' \equiv (\overline{t}, 2t_0 - \overline{t})$, symmetric in time, in the second term (and writing again $\overline{Y}$ instead of $Z_1$), we have
\[
J_2 = \int_{S_{h^{1/\sigma}}} A(Y_0 - \overline{Y}) \chi_{\{\overline{t}>t_0\}} f(\overline{Y}) d\overline{Y} - \int_{S_{h^{1/\sigma}}} A(y - Y_0) \chi_{\{\overline{t}>t_0\}} f(\overline{Y}') d\overline{Y} \\
= \int_{S_{h^{1/\sigma}}} A(Y_0 - \overline{Y}) \chi_{\{\overline{t}>t_0\}} (f(\overline{Y}) - f(\overline{Y}')) d\overline{Y}.
\]
Now we observe that
\[ |f(Y) - f(Y^*)| \leq c |u(Y) - u(Y_0)|^{\gamma} |u(Y) - u(Y^*)| \leq ch |Y - Y_0|^{\mu}, \]
see (4.1). Thus
\[ |J_2| \leq ch \int_{t_0}^{t_0+h} \int_{\{|x-x_0|>\epsilon h^{1/\sigma}\}} |Y - Y_0|^{\mu} d\sigma d\tilde{t} \leq ch^{1+\nu \gamma / \sigma}. \]
We conclude by noting that \( 1 + \nu \gamma / \sigma \geq \nu (1 + \gamma) / \sigma \). □

**Lemma 5.4** Under the hypotheses of Theorem 1.1, \( u(\cdot, t) \in C^{\nu(1+\gamma) / \sigma} (\mathbb{R}^N) \) uniformly in \( t \).

**Proof.** Once we know that \( u(x, \cdot) \in C^{(1+\gamma)/\sigma}(0, \infty) \) uniformly in \( x \in \mathbb{R}^N \), we can repeat the calculations in the proof of Lemma 5.2 with \( \alpha \) replaced by \( \nu \). □

Using the worst case we can write the joint regularity in the form
\[
\begin{cases} 
C^{(1+\gamma)/\sigma}(Q) & \text{if } \sigma \geq 1, \\
C^\sigma(1+\gamma)(Q) & \text{if } \sigma \leq 1.
\end{cases}
\]
with both variables playing the same role. We also have that the solution is classical since it has continuous derivatives in the sense required in the equation.

**Corollary 5.1** Under the hypotheses of Theorem 1.1, the function \( z := \partial_t u = -(-\Delta)^{\sigma/2} \varphi'(u) \) satisfies \( z \in C^{\nu(1+\gamma)-\sigma, (\nu(1+\gamma)-\sigma)/\sigma} (x, t) \).

**Proof.** We point out that both sides of the equation are bounded functions and equal almost everywhere. We also know that \( \partial_t u \) is Hölder continuous as a function of \( t \) for a.e. \( x \), and the Hölder continuity is locally uniform. On the other hand, we easily conclude that \( (-\Delta)^{\sigma/2} \varphi'(u) \) is Hölder continuous as a function of \( x \) for a.e. \( t \), and this happens again locally uniformly. Hölder continuity everywhere in both variables follows. □

Let us recall that under our assumptions \( \sigma < \nu (1 + \gamma) \), so that we are getting Hölder regularity for \( \partial_t u \) in all cases.

### 6 Higher regularity

We have already proved that solutions of (1.1) are differentiable in time. However, in view of Lemma 5.4 at this stage they are only known to be differentiable in space if \( \sigma(1 + \gamma) > 1 \), where \( \gamma \) is the Hölder exponent of \( \varphi' \). This assumption can we weakened.
Proposition 6.1 Under the assumptions of Theorem 6.1, if \( \sigma < 1 \) and \( \gamma + \sigma > 1 \), then \( u \in C^{1,\alpha}(Q) \) for some \( \alpha \in (0,1) \).

Proof. We only need to study the case \( \sigma(1+\gamma) \leq 1 \), in which case necessarily \( \sigma < 1 \).

We consider the function \( z = \partial_t u \), which belongs to \( C^\sigma_\alpha(Q) \) for all \( \alpha < \sigma \). Let \( Y_0 = (x_0, t_0) \in Q \) be fixed and denote \( a(Y) = \varphi'(u(Y)) \), \( z_0 = z(Y_0) \), \( a_0 = a(Y_0) \). Then \( z \) is a distributional solution to the inhomogeneous fractional heat equation

\[
\partial_t z + a_0(-\Delta)^{\sigma/2} z = (-\Delta)^{\sigma/2} F_1 + (-\Delta)^{\sigma/2} F_2,
\]

where

\[
F_1 = -(a-a_0)(z-z_0), \quad F_2 = -z_0a.
\]

We decompose \( z \) as \( z_1 + z_2 \), where \( z_i \) is a solution to

\[
\partial_t z_i + a_0(-\Delta)^{\sigma/2} z_i = (-\Delta)^{\sigma/2} F_i, \quad i = 1, 2.
\]

The term \( z_2 \) inherits the regularity of \( F_2 \), that is, the regularity of \( a(Y) \). As to \( z_1 \), we use the fact that the function \( F_1 = (a-a_0)(v-v_0) \) satisfies conditions \( (4.2) \) and \( (4.3) \), which implies, thanks to Lemma 4.1, that \( z_1 \) is more smooth than \( a \), hence more smooth than \( z_2 \). Therefore, we concentrate on the ‘bad’ term, \( z_2 \).

The regularity of \( F_2 \), that is, the regularity of \( \varphi'(u) \), coincides with the minimum between the regularities of \( \varphi' \) and \( u \). Therefore, \( F_2(x,\cdot) \) belongs to \( C^\gamma(\mathbb{R}) \) uniformly in \( x \). As for spatial regularity, at this stage we know that \( F_2(\cdot,t) \) is \( C^\alpha(\mathbb{R}^N) \) uniformly in \( t \) for all \( \alpha < \min\{\sigma(1+\gamma),\gamma\} \).

If \( \sigma(1+\gamma) \geq \gamma \), we get \( z_2(\cdot,t) \in C^\gamma(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) uniformly in \( t \), and so is \( z = \partial_t u \).

Using the equation we conclude that \( u(\cdot,t) \in C^{\gamma+\sigma}(\mathbb{R}^N) \) uniformly in time. Since we have assumed that \( \gamma + \sigma > 1 \), this means that \( u \) is differentiable also in \( x \).

If \( \sigma(1+\gamma) < \gamma \), we get \( z_2(\cdot,t) \in C^{\sigma(1+\gamma)}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) uniformly in \( t \). Since \( w = (-\Delta)^{\sigma/2} z_2 \) satisfies

\[
\partial_t w + a_0(-\Delta)^{\sigma/2} w = (-\Delta)^{\sigma/2} (-\Delta)^{\sigma/2} F_2,
\]

the regularity of \( w \) is given by the regularity of \( (-\Delta)^{\sigma/2} F_2 \). This implies that \( z_2(\cdot,t) \in C^\alpha(\mathbb{R}^N) \) uniformly in \( t \) for all \( \alpha < \min\{\sigma(2+\gamma),\gamma\} \). Repeating this argument as many times as needed, we finally obtain that \( z(\cdot,t) \in C^\alpha(\mathbb{R}^N) \) for all \( \alpha < \gamma \) uniformly in \( t \).

Using the equation, we obtain that \( u \in C^{\alpha,1+\gamma}(Q) \) for all \( \alpha < \sigma + \gamma \). \( \square \)

A similar argument allows to prove a regularity result for linear equations with variable coefficients that has an independent interest.

Theorem 6.1 Let \( u \) be a bounded very weak solution to \( \partial_t u + (-\Delta)^{\sigma/2} (au + b) = 0 \), where the coefficients satisfy \( a, b \in C^{1,\alpha}(Q) \cap L^\infty(Q) \), \( a(x,t) \geq \delta > 0 \). If \( u \in C^\alpha(Q) \) then \( \partial_t u, \partial_x u \in C^\alpha(Q \cap \{t > \tau\}) \cap L^\infty(Q \cap \{t > \tau\}) \), \( i = 1, \ldots, N \), for every \( \tau > 0 \).
The proof of $C^{1,\alpha}$ regularity is done by considering the linear equations satisfied by the derivatives. Boundedness for the derivatives then immediately follows, since $u \in C^{1,\alpha}(Q \cap \{ t > \tau \}) \cap L^{\infty}(Q \cap \{ t > \tau \})$.

This linear result is used now to obtain further regularity for the nonlinear problem, which covers in particular Theorem 1.2.

**Theorem 6.2** If in addition to the hypotheses of Theorem 1.1, $\varphi \in C^{k,\gamma}({\mathbb R})$ for some $k \geq 2$ and $0 < \gamma < 1$, then $u \in C^{k,\alpha}(Q)$ for some $\alpha \in (0,1)$.

**Proof.** We proceed by induction in the derivation order. Since $\varphi \in C^{1,\theta}({\mathbb R})$ for all $\theta \in (0,1)$, Proposition 6.1 yields $\partial_t u, \partial_{x_i} u \in C^{\alpha}(Q) \cap L^{\infty}(Q)$ for some $\alpha \in (0,1)$.

Assume that the result is true for derivatives of order $j \leq k - 1$. Let

$$v_{\beta} = \partial_t^{\beta_0} \partial_{x_1}^{\beta_1} \cdots \partial_{x_N}^{\beta_N} u, \quad \sum_{i=0}^{N} \beta_i = j.$$

It is easily checked that $v_{\beta}$ satisfies an equation of the form

$$\partial_t v_{\beta} + (-\Delta)^{\sigma/2}(\varphi(u)v_{\beta} + b_{\beta}) = 0,$$

where $b_{\beta}$ is a polynomial in $v_{\beta}$, $\beta_i' \leq \beta_i$, $i = 0, \ldots, N$, $\sum_{i=0}^{N} \beta_i' \leq j-1$, with coefficients involving the derivatives $\varphi^{(l)}(u)$, $0 < l \leq j$. By hypothesis, $b_{\beta} \in C^{1,\alpha}(Q) \cap L^{\infty}(Q)$ for some $\alpha \in (0,1)$. Since $u$ is bounded, $a = \varphi'(u) \geq \delta > 0$. Hence we may apply Theorem 6.1 to conclude the result. \hfill $\Box$

## 7 Nonlinear degenerate and singular equations

A careful inspection of the proof of Theorem 1.1 shows that the result has a local nature, and this will be exploited here to treat more general equations.

**Theorem 7.1** Let $u$ be a bounded weak solution of equation (1.1) such that $u \in C_\sigma^\alpha(\Omega)$ for some $\alpha \in (0,1)$ and some subdomain $\Omega \subset Q$. Let $\varphi \in C^{1,\gamma}(J)$, $J = (a,b)$, where $\varphi'(s) \geq c > 0$ for every $s \in J$. Under these assumptions we conclude that $\partial_t u$ and $(-\Delta)^{\sigma/2}\varphi(u)$ are H"older continuous functions in $\mathcal{O} = \Omega \cap \{(x,t) : u(x,t) \in J\}$. Hence $u$ is a classical solution of (1.1) in that set.

**Proof.** We have to revise the proofs of all the results in Subsection 3.2 and Sections 4 and 5. For instance, in the proof of Lemma 3.1, we have to replace the assumption $f \in C_\sigma^\alpha(Q)$ by $f \in C_\sigma^\alpha(\Omega)$ to conclude that $g$ belongs to the same space, and this is true since $f$ is also bounded. The same holds for Lemma 4.1.

As to Lemma 5.2, we observe that the estimate of the second order differences holds uniformly in every compact $K \subset \Omega$. Now take a smooth cut-off function $\phi$ with
support contained in $\Omega$, with $\phi \equiv 1$ in a subset $\Omega' \subset \Omega$. Observe that the second difference of $\psi = g\phi$ satisfies
\[
\psi(Y_0 + Z) - 2\psi(Y_0) + \psi(Y_0 - Z) = \left( g(Y_0 + Z) - 2g(Y_0) + g(Y_0 - Z) \right)\phi(Y_0) \\
+ \left( \phi(Y_0 + Z) - 2\phi(Y_0) + \phi(Y_0 - Z) \right)g(Y_0) \\
+ \left( \phi(Y_0) - \phi(Y_0 - Z) \right) \left( g(Y_0 + Z) - g(Y_0 - Z) \right) = O(h^a(1+\gamma)),
\]
uniformly in $Q$, $Z = (he,0)$, $e \in \mathbb{R}^N$, $|e| = 1$, $0 < \alpha < \nu$. Then $(-\Delta)^{\delta/2}\psi$ is bounded for every $\delta < \alpha(1 + \gamma)$, which implies that $\psi$ is $C^a(1+\gamma)$, and thus $g \in C^a_x(1+\gamma)(\Omega')$, for every $0 < \alpha < \nu$. The rest proceeds in the same way. \hfill \Box

In order to apply this result we need to make sure that the solution is $C^a$ in some set $\Omega \subset Q$. This has been proved under certain conditions in [2]: for some $A$, $B \in \mathbb{R}$, $A < B$, there exists a constant $C = C(A,B) > 0$ such that
\[
(7.1) \quad \sup_{s \in [s_1,s_2]} \varphi'(s) \leq C \frac{\varphi(s_2) - \varphi(s_1)}{s_2 - s_1}, \quad A \leq s_1 < s_2 \leq B.
\]
Indeed, in this case every bounded weak solution $u$ to the equation in (1.1) satisfying
\[ A \leq \text{ess inf}_\Omega u \leq \text{ess sup}_\Omega u \leq B \]
belongs to $C^\varepsilon(\Omega)$ for some $\varepsilon = \varepsilon(C)$, and thus $u \in C^a(\Omega)$, $\alpha = \nu \varepsilon$.

**Application to the fractional porous medium equation.** When $\varphi(u) = |u|^{m-1}u$, $m \geq 1$, hypothesis (7.1) is satisfied with a constant $C = m$ which does not depend on $A, B$. Therefore, bounded weak solutions are uniformly Hölder continuous in $\mathbb{R}^N \times (\tau, \infty)$, $\tau > 0$. However, the equation degenerates when $u = 0$. Hence the application of Theorem 7.1 only yields the regularity stated there in the set \{u $\neq$ 0\}.

In the fast diffusion case $m < 1$ hypothesis (7.1) only holds if $A > 0$ or $B < 0$. Thus, $C^\alpha$ regularity is only guaranteed in the positivity set (or negativity set) of a solution. Nevertheless, the application of our result leads to the same conclusion as in the case $m > 1$ in the set \{u $\neq$ 0\}.

On the other hand, in our paper [17] we prove, for all $m > 0$, that when the initial value is nonnegative the solution is strictly positive everywhere for positive times. We obtain that the solution belongs to $C^\alpha$ in this case, and the application of the results of the present paper then imply that the solution is classical. The positivity property holds for all $m > 0$, which is in sharp contrast with the nonlinear theory with the standard Laplacian and $m > 1$, where the existence of free boundaries is well-known [22].

8 **Theory of existence and basic properties**

As a complement to the previous regularity theory, we devote this section to provide a survey of the main facts of the existence and uniqueness theory for the Cauchy
problem for equation (1.1),

\[
\begin{cases}
\partial_t u + (-\Delta)^{\sigma/2} \varphi(u) = 0 & \text{in } Q, \\
u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^N.
\end{cases}
\] (CP)

Such a theory has been developed in great detail in the paper [17] for the case where \( \varphi \) is a power function. As in the case of the standard (local) porous medium equation, many of the basic features of the theory can be extended to more general nonlinearities \( \varphi \), as long as they are continuous and nondecreasing, cf. [9]. Therefore, we will outline here how such extension can be done in the fractional case \( \sigma \in (0, 2) \), with special attention to the points where the arguments differ.

Let us recall the concept of weak solution to the Cauchy problem (CP): a function \( u \in C([0, \infty) : L^1(\mathbb{R}^N)) \) such that (i) \( \varphi(u) \in L^2_{\text{loc}}((0, \infty) : \dot{H}^{\sigma/2}(\mathbb{R}^N)) \); (ii) identity

\[
\int_0^\infty \int_{\mathbb{R}^N} u \partial_t \zeta \, dx \, dt - \int_0^\infty \int_{\mathbb{R}^N} (-\Delta)^{\sigma/4} \varphi(u) (-\Delta)^{\sigma/4} \zeta \, dx \, dt = 0
\]

holds for every \( \zeta \in C_0^\infty(Q) \); and (iii) \( u(\cdot, 0) = u_0 \) almost everywhere. The (homogeneous) fractional Sobolev space \( \dot{H}^{\sigma/2}(\mathbb{R}^N) \) is the space of locally integrable functions \( \zeta \) such that \( (-\Delta)^{\sigma/4} \zeta \in L^2(\mathbb{R}^N) \). We point that this is a convenient choice among other possible notions of weak solution, and it can be described as a weak \( L^1 \)-energy solution to be specific.

### 8.1 Solutions with bounded initial data

We will start by considering the theory for initial data

\[ u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N). \]

Existence and uniqueness are proved by using the definition of the fractional Laplace operator based in the extension technique developed by Caffarelli and Silvestre [6], which is a generalization of the well-known Dirichlet to Neumann operator corresponding to \( \sigma = 1 \). Thus, if \( g = g(x) \) is a smooth bounded function defined in \( \mathbb{R}^N \), its \( \sigma \)-harmonic extension to the upper half-space, \( v = E(g) \), is the unique smooth bounded solution \( v = v(x, y) \) to

\[
\begin{cases}
\nabla \cdot (y^{1-\sigma} \nabla v) = 0 & \text{in } \mathbb{R}^{N+1}_+, \\
v(\cdot, 0) = g & \text{in } \mathbb{R}^N.
\end{cases}
\] (8.1)

Then it turns out, see [6], that

\[
\frac{\partial v}{\partial y^\sigma} = -\mu_\sigma \lim_{y \to 0^+} y^{1-\sigma} \frac{\partial v}{\partial y} = (-\Delta)^{\sigma/2} g(x),
\] (8.2)

where \( \mu_\sigma = 2^{\sigma-1} \Gamma(\sigma/2)/\Gamma(1-\sigma/2) \). In (8.1) the operator \( \nabla \) acts in all \( (x, y) \) variables, while in (8.2) \( (-\Delta)^{\sigma/2} \) acts only on the \( x = (x_1, \cdots, x_N) \) variables.
Using this approach, problem (CP) can be written in an equivalent local form. If $u$ is a solution, then $w = E(\varphi(u))$ solves
\[
\begin{cases}
\nabla \cdot (y^{1-\sigma} \nabla w) = 0, & (x, y) \in \mathbb{R}^{N+1}_+, t > 0, \\
\frac{\partial w}{\partial y^\sigma} - \frac{\partial \beta(w)}{\partial t} = 0, & x \in \mathbb{R}^N, y = 0, t > 0, \\
\beta(w) = u_0, & x \in \mathbb{R}^N, y = 0, t = 0,
\end{cases}
\tag{8.3}
\]
where $\beta = \varphi^{-1}$. Conversely, if we obtain a solution $w$ to (8.3), then $u = \beta(w)|_{y=0}$ is a solution to (CP).

We use the concept of weak solution for problem (8.3) obtained by multiplying by a test function $\zeta$,
\[
\int_0^\infty \int_{\mathbb{R}^N} \beta(w) \frac{\partial \zeta}{\partial t} dx dt - \mu_\sigma \int_0^\infty \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} \langle \nabla w, \nabla \zeta \rangle dx dy dt = 0.
\]

We then introduce the energy space $X^\sigma(\mathbb{R}^{N+1}_+)$, the completion of $C^\infty_c(\mathbb{R}^{N+1}_+)$ with the norm
\[
\|v\|_{X^\sigma} = \left( \mu_\sigma \int_{\mathbb{R}^{N+1}_+} y^{1-\sigma} |\nabla v|^2 dx dy \right)^{1/2}.
\]

In order to solve the evolution problem, which is our concern, we use the Nonlinear Semigroup Generation Theorem due to Crandall-Liggett [8]. We are thus reduced to deal with the related elliptic problem
\[
\begin{cases}
\nabla \cdot (y^{1-\sigma} \nabla w) = 0, & x \in \mathbb{R}, y > 0, \\
-\frac{\partial w}{\partial y^\sigma} + \beta(w) = g, & x \in \mathbb{R}, y = 0,
\end{cases}
\tag{8.4}
\]
with $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$. As in the case treated in [17], in order to get a solution by variational techniques, it is convenient to replace the half space $\mathbb{R}^{N+1}_+$ by a half ball $B^+_R = \{ |x|^2 + y^2 < R^2, x \in \mathbb{R}^N, y > 0 \}$. We impose zero Dirichlet data on the “new part” of the boundary. Therefore we are led to study the problem
\[
\begin{cases}
\nabla \cdot (y^{1-\sigma} \nabla w) = 0 & \text{in } B^+_R, \\
\partial w/\partial y^\sigma + \beta(w) = g & \text{on } \partial B^+_R \cap \{ y > 0 \}, \\
\partial w/\partial y^\sigma + \beta(w) = g & \text{on } D_R := \{ |x| < R, y = 0 \},
\end{cases}
\tag{8.5}
\]
with $g \in L^\infty(D_R)$ given. Minimizing the functional
\[
J(w) = \frac{\mu_\sigma}{2} \int_{B^+_R} y^{1-\sigma} |\nabla w|^2 + \int_{D_R} B(w) - \int_{D_R} wg,
\]
$B' = \beta$, in the admissible set $\mathcal{A} = \{ w \in H^1(B^+_R; y^{1-\sigma}) : 0 \leq \beta(w) \leq \|g\|_{L^\infty} \}$, we obtain a unique solution $w = w_R$ to problem (8.5). Moreover, if $g_1$ and $g_2$ are two admissible data, then the corresponding weak solutions satisfy the $L^1$-contraction property
\[
\int_{D_R} (\beta(w_1(x, 0)) - \beta(w_2(x, 0)))_+ dx \leq \int_{\mathbb{R}} (g_1(x) - g_2(x))_+ dx.
\]
The passage to the limit $R \to \infty$ uses the monotonicity in $R$ of the approximate solutions $w_R$. We obtain a function $w_\infty = \lim_{R \to \infty} w_R$ which is a weak solution to problem (8.3). The above contractivity property also holds in the limit. Moreover, $\|\beta(w_\infty(\cdot,0))\|_{L^\infty(\mathbb{R})} \leq \|g\|_{L^\infty(\mathbb{R})}$, and $w_\infty \geq 0$, since $g \geq 0$.

Now, using the Crandall-Liggett Theorem we obtain the existence of a unique mild solution $\overline{w}$ to the evolution problem (8.3). To prove that $\overline{w}$ is moreover a weak solution to problem (8.3), one needs to show that it lies in the right energy space. This is done using the same technique as in [15], which yields the energy estimate
\[
\mu_\sigma \int_0^T \int_{\mathbb{R}^N} y^{1-\sigma} |\nabla w(x,y,t)|^2 \, dx \, dy \, dt \leq \int_{\mathbb{R}^N} B(u_0(x)) \, dx \quad \text{for every } T > 0.
\]
Hence the function $u = \beta(\overline{w}(\cdot,0))$ is a weak solution to problem (CP). In addition, $\|\beta(\overline{w}(\cdot,0))\|_{L^\infty(\mathbb{R} \times (0,\infty))} \leq \|u_0\|_{L^\infty(\mathbb{R})}$, and $\overline{w} \geq 0$. Recalling the isometry between $\dot{H}^{\sigma/2}(\mathbb{R}^N)$ and $X^{\sigma}(\mathbb{R}^{N+1}_+)$, we obtain
\[
\int_0^T \int_{\mathbb{R}^N} |(-\Delta)^{\sigma/4} \varphi(u)(x,t)|^2 \, dx \, dt \leq \int_{\mathbb{R}^N} B(u_0(x)) \, dx \quad \text{for every } T > 0.
\]

The Semigroup Theory also guarantees that the constructed solutions satisfy the $L^1$-contraction property $\|u(\cdot,t) - \tilde{u}(\cdot,t)\|_1 \leq \|u_0 - \tilde{u}_0\|_1$.

Uniqueness follows by the standard argument due to Oleinik et al. [15], using here the test function
\[
\zeta(x,t) = \begin{cases} 
\int_t^T (\varphi(u) - \varphi(\tilde{u}))(x,s) \, ds, & 0 \leq t \leq T, \\
0, & t \geq T,
\end{cases}
\]
in the weak formulation for the difference of two solutions $u$ and $\tilde{u}$.

Summarizing, we have proved the following existence and uniqueness result.

**Theorem 8.1** Let $\varphi \in C(\mathbb{R})$ be nondecreasing. Given $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ there exists a unique bounded weak $L^1$-energy solution to problem (CP).

### 8.2 Solutions with unbounded data. Boundedness and decay

If the (nondecreasing) nonlinearity $\varphi$ satisfies $\varphi'(u) \geq C |u|^{m-1}$ for some $m \in \mathbb{R}$ and $|u| \geq C$, then weak solutions with initial data in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, where $p \geq 1$ satisfies $p > p(m) = (1 - m)N/\sigma$, become immediately bounded, hence, thanks to our results, classical.

The idea is to take as test function in the weak formulation $\zeta = (|u| - 1)^{p-1} \text{sign}(u)$. Though $u$ is not differentiable in time a.e. for a general $\varphi$, this is not needed for the proof, since a regularization procedure, using some Steklov averages, allows to bypass this difficulty; see for example the classical paper [11] for the case of local operators. Hence, we only have to check that $\zeta \in L^2_{\text{loc}}((0,\infty) : H^{\sigma/2}(\mathbb{R}^N))$ for every $p \geq 2$. This will follow from the following result applied to $v = \varphi(u)$. 

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Remark. If moreover \( f \) is convex then, noting that \( \| E(v) \|_{\infty} \leq \| v \|_{\infty} \), we deduce the estimate
\[
\| f(v) \|_{H^{\sigma/2}(\mathbb{R}^N)} \leq \| f'(v) \|_{\infty} \| v \|_{H^{\sigma/2}(\mathbb{R}^N)}.
\]

If we use the above test function and apply the generalized Stroock-Varopoulos inequality, proved in [17, Lemma 5.2], we obtain
\[
\int_{\mathbb{R}^N} (|u| - 1)^p_+ (x, t) \, dx = - \int_{\mathbb{R}^N} (|u| - 1)_{+}^{p} (x, \tau) \, dx - c \int_{\tau}^{\tilde{t}^2} \int_{\mathbb{R}^N} |(-\Delta)^{\sigma/4} G(u)|^2 \, dx \, dt,
\]
where
\[
|G'(u)|^2 = \varphi'(u)(|u| - 1)_+^{p-2} \geq c(|u| - 1)_+^{m+p-3}.
\]
Now using the Hardy-Littlewood-Sobolev inequality [12], [21] if \( N > \sigma \), we get
\[
\int_{\mathbb{R}^N} (|u| - 1)^p_+ (x, \tau) \, dx \geq \frac{c}{\int_{\tau}^{\tilde{t}^2}} \left( \int_{\mathbb{R}^N} (|u| - 1)^{(m+p-1)}_+ \, dx \right)^{\frac{N-\sigma}{N}} \, dt
\]
for every \( p \geq 2 \). If in addition \( p > (1-m)N/\sigma \), this inequality is enough to apply a standard Moser’s iteration technique to obtain an \( L^p-L^\infty \) smoothing effect. We can then weaken the restriction \( p \geq 2 \) to \( p \geq 1 \) using interpolation; see [17]. Take note that in the case \( N = 1 \leq \sigma < 2 \) we must replace the Hardy-Littlewood-Sobolev inequality by a Nash-Gagliardo-Nirenberg inequality; see [17, Lemma 5.3]. We omit further details.

Let us state precisely the smoothing result thus obtained for future reference.

Theorem 8.2 Let \( \varphi \in C^1(\mathbb{R}) \) be such that \( \varphi'(u) \geq C |u|^{m-1} \) for some \( m \in \mathbb{R} \) and \( |u| \geq C \). If \( u_0 \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N) \), where \( p \geq 1 \) satisfies \( p > (1-m)N/\sigma \), then there exists a unique weak \( L^1 \)-energy solution to the Cauchy problem (CP) which is bounded in \( \mathbb{R}^N \times (\tau, \infty) \) for all \( \tau > 0 \). This solution moreover satisfies
\[
\sup_{x \in \mathbb{R}^N} |u(x, t)| \leq \max \{ C, C_1 \, t^{-\gamma_p} \| u_0 \|_{\delta_p} \}
\]
with \( \gamma_p = N/(N(m-1) + \sigma p) \) and \( \delta_p = \sigma p \gamma_p/N \), the constant \( C_1 \) depending on \( m, p, \sigma, C, \) and \( N \).
Remark. If the function $\varphi$ satisfies the condition $\varphi'(u) \geq C|u|^{m-1}$ for every $u \in \mathbb{R}$ and some fixed $m > 0$, a classical scaling argument allows to obtain a decay estimate for every $t > 0$; see for instance [23]. Indeed, the function $v(x, t) = \lambda^{\gamma_p} u(\lambda^{p\gamma_p/N} x, \lambda t)$ is a solution to the equation $\partial_t v + (-\Delta)^{\sigma/2} \tilde{\varphi}(v) = 0$ with $\tilde{\varphi}(s) = \lambda^{m\gamma_p} \varphi(\lambda^{-\gamma_p} s)$, which satisfies the same condition on the derivative. Thus, applying (8.6) at $t = 1$ and putting $\lambda = t$ we get

$$\|u(\cdot, t)\|_{\infty} = t^{-\gamma_p} \|v(\cdot, 1)\|_{\infty} \leq C_1 t^{-\gamma_p} \|u_0\|_{L^p}$$

for every $t > 0$.

Existence for data which are unbounded is proved by approximation; see [17] for the details in the case where the nonlinearity is a pure power. As for uniqueness, continuity in $L^1$ guarantees that two solutions with the same initial data do not differ more than $\varepsilon$ in $L^1$ norm for some small enough time. Since for positive times solutions are assumed to be bounded, we may use the $L^1$ contraction property to prove that the distance between the two solutions stays smaller than $\varepsilon$ for any later time. Since $\varepsilon$ is arbitrary, uniqueness follows.

9 Extensions and comments

Some applications. Equation (1.1) appears in the study of hydrodynamic limits of interacting particle systems with long range dynamics. Thus, in [11], Jara and co-authors study the non-equilibrium functional central limit theorem for the position of a tagged particle in a mean-zero one-dimensional zero-range process. The asymptotic behavior of the particle is described by a stochastic differential equation governed by the solution of (1.1).

In several space dimensions, equations like (1.1) occur in boundary heat control, as already mentioned by Athanasopoulos and Caffarelli [2], where they refer to the model formulated in the book by Duvaut and Lions [10], and use the extension technique of Caffarelli and Silvestre.

For a more thorough discussion on applications see [5].

Regularity for unbounded solutions. In our proofs we are requiring the solutions to be bounded in order to make the integrals on unbounded sets convergent. However, this requirement may be not needed to this aim. It may be enough that the solutions belong to $C([0, T] : L^1(\mathbb{R}^N, \rho \, dx))$. It would be interesting to explore this possibility, since this may be helpful in the study of higher regularity.

Higher regularity for the fractional porous medium equation. The main difficulty to obtain further regularity in this case is that, since the equation is not uniformly parabolic at infinity (it is not true that $0 \leq c \leq \varphi'(u) \leq C < \infty$), we do not know the derivatives to be bounded. Hence, we cannot apply Theorem 6.1 directly. However, as mentioned in the previous paragraph, this might be circumvented by substituting the boundedness requirement by some less restrictive condition. The
precise quantitative statements of the positivity property obtained in [5] might be helpful to this aim.

**The fractional porous medium equation with sign changes.** Our results only give that the equation is satisfied in a classical sense where the solution is different from 0. It remains to determine what is the optimal regularity for changing sign solutions. A first step would be to study whether solutions are strong, i.e., whether $\partial_t u$ (and hence $(-\Delta)^{\sigma/2} u$) are functions, and not only distributions.

**The very fast fractional porous medium equation.** The nonlinearities $\varphi(u) = (1+u)^{m-1}/m$, $m \neq 0$, are uniformly parabolic if we restrict ourselves to nonnegative solutions. Moreover, they fall within the hypotheses of Theorem 8.2 if we modify the nonlinearity suitably for $u < 0$, which does not matter if we only consider nonnegative solutions. Therefore, we obtain existence of $C^\infty$ solutions for all nonnegative initial data in $L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ with $p$ large enough. If $\sigma > 1 - m$ and $N = 1$ we can even take $p = 1$.

The nonlinearity $\varphi(u) = \log(1+u)$ is also uniformly parabolic if we restrict to nonnegative solutions. In addition, after a suitable modification for $u < 0$, it satisfies the hypotheses of Theorem 8.2 with $m = 0$. Thus, if $N = 1$ and $\sigma = 1$ we are in the critical case where we need a bit more than integrability to have existence. In [18] we proved that it is enough for $u_0$ to belong to some $L \log L$ space. The solution is then guaranteed to be $C^\infty$.

The singular nonlinearities $\varphi(u) = u^m/m$, $m < 0$, and $\varphi(u) = \log u$ (with $u > 0$) cannot be treated in the same way, and require new ideas.

**The fractional Stefan problem.** For the Stefan nonlinearity $\varphi(u) = (u-1)_+$, hypothesis (7.1) holds if $A > 1$. Hence bounded weak solutions are $C^\alpha$ in the set where $u > 1$ and our main result proves that they are $C^{1,\gamma}$, hence classical, in that set for all $\gamma \in (0,1)$. Let us mention that $u$ is known to be continuous everywhere, though not $C^\alpha$. It would be interesting to determine what is the optimal regularity for this problem.

**Problems in fluid mechanics.** We now explore an interesting and enlightening connection, in the case $N = \sigma = 1$, between equation (1.1) and Morlet’s family of 1-dimensional nonlocal transport equations with viscosity [14],

\begin{equation}
\partial_t v - \delta \partial_y(H(v)v) - (1 - \delta)H(v)\partial_y v = -(-\Delta)^{1/2} v, \quad 0 \leq \delta \leq 1.
\end{equation}

For a nonnegative solution $u$ to equation (1.1), we consider the change of variables $(x,t,u) \mapsto (y,\tau,v)$ given by the Bäcklund type transform

\[ y = \int_0^x (1 + u(s,t)) \, ds - c(t), \quad \tau = t, \quad v(y,\tau) = \varphi(u(x,t)), \]

with $c'(t) = H(\varphi(u))(0,t)$. We denote $(y,\tau) = J(x,t)$. Notice that the Jacobian of the transformation $J$ is $\frac{\partial(u,\tau)}{\partial(x,t)} = 1 + u \neq 0$, since $u \geq 0$. Then we may write the
inverse

\[ x = \int_0^y \frac{d\theta}{1 + \varphi^{-1}(v(\theta, \tau))} - \varphi(\tau), \]

with \( \varphi'(\tau) = -H(\varphi(u))(0, t)/(1 + u(0, t)) \).

We recall that, if the operators are acting on smooth enough functions, then the half-Laplacian \((-\Delta)^{1/2}\) can be written in terms of the Hilbert transform

\[ Hf(x) = \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{f(y)}{x - y} dy. \]

as \((-\Delta)^{1/2} = H\partial_x = \partial_x H\). Therefore, we have

\[ \partial_x y = 1 + u, \quad \partial_t y = -H(\varphi(u)) = -\tilde{H}(v), \]

where \( \tilde{H}(v) = H(v \circ J) \circ J^{-1} \) is the conjugate of the Hilbert transform \( H \) by the transformation \( J \). Specifically,

\[
\tilde{H}(v(y, \tau)) = H(\varphi(u(x, t))) = \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{\varphi(u(x', t))}{x - x'} dx' = \frac{1}{\pi} \text{P.V.} \int_{\mathbb{R}} \frac{v(y', \tau)}{(1 + \varphi^{-1}(v(y', \tau)))} \int_{y'}^{y} \frac{dy'}{1 + \varphi^{-1}(v(y', \tau))} dy'.
\]

If \( \varphi(u) = \frac{(1 + u)^m - 1}{m}, \ m \in [-1, 0) \), then equation \( \Box \) becomes

\[ \partial_t v - \tilde{H}(v) \partial_y v = -(1 + mv)\partial_y \tilde{H}(v). \]

If instead of \( \tilde{H} \) we had the standard Hilbert transform \( H \), and we take \( m = -\delta \), we would have an equation in Morlet’s family \( \Box \). The connection also works for the case \( m = 0 \), if we take \( \varphi(u) = \log(1 + u) \); see \( \Box \).

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