An Example of a Fractal Finitely Generated Solvable Group

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Abstract—A fractal group is a group with unbounded iterated identity. In this paper a finitely generated fractal solvable group is constructed.

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1. INTRODUCTION

In [2], A. Erschler developed the theory of iterated identities for groups. Recall the basic notions from [2]. Let $w = w(l_1, \ldots, l_n)$ be a group word on $n$ letters, $n \geq 1$. Consider its iterations defined as follows:

$$w(0)(l_1, \ldots, l_n) := w(l_1, \ldots, l_n), \quad w(i+1)(l_1, \ldots, l_n) = w(w(i)(l_1, \ldots, l_n), l_2, \ldots, l_n).$$

We say that a group $G$ satisfies an iterated identity $w$ if, for any $x_1, \ldots, x_n \in G$, there exists a number $j$ (which in general depends on the collection $x_1, \ldots, x_n$) such that $w(j)(x_1, \ldots, x_n) = 1$.

In [2] the iterated identities are called \textit{E-type} (or \textit{Engel type}) iterated identities, since the iterations of the word $[l_1 \ldots, l_2] := l_1^{-1}l_2^{-1}l_1l_2$ give the Engel brackets $[[\ldots [[l_1, l_2], l_2], \ldots], l_2]$. A group $G$ is said to be \textit{bounded} if, for any iterated identity $w(l_1, \ldots, l_n)$ of $G$, there is a number $j$ (which depends only on $G$) such that $w(j)(x_1, \ldots, x_n) = 1$ for all collections of elements $x_1, \ldots, x_n$ in $G$. A group $G$ is said to be \textit{fractal} if it satisfies some iterated identity and is not bounded. The simplest example of a fractal group is the quasicyclic group $\mathbb{Z}_{p^\infty} = \lim_{\rightarrow k} \mathbb{Z}/p^k$. It satisfies the iterated identity $w(l) = lp$, but is not bounded, since the orders of elements of $\mathbb{Z}_{p^\infty}$ are not bounded. In the same way, any finitely generated $p$-group of unbounded exponent, like the Golod–Shafarevich group [3] or the Grigorchuk group [4], gives an example of a finitely generated fractal group. Clearly, these examples of finitely generated fractal groups are not solvable.

It is shown in [2] that the finitely generated metabelian groups are bounded. It is natural to ask whether it is true that all finitely generated solvable groups are bounded as well.

The main result of this paper states that the property to be bounded cannot be extended to the class of finitely generated solvable groups of class 3. Let $p$ be a prime and $G$ be a group given by the following presentation:

$$G = \langle x, y, t \mid x^t = x^p, \quad y^{t^{-1}} = y^p, \quad [x, y] = 1, \quad [[x, y]^i, x] = 1, \quad i, j \in \mathbb{Z} \rangle.$$

Here we use the standard notation for group elements $a^b := b^{-1}ab$.

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The group $G$ is solvable of class 3. Let $w(l_1, l_2, l_3) = [l_1, [l_2, l_3]]^p$. Then $w$ is an unbounded iterated identity for $G$.

The iterations of the identity $w$ are

\[ [l_1, [l_2, l_3]]^p, \quad [l_1, [l_2, l_3], [l_2, l_3]]^p, \quad [[l_1, [l_2, l_3]]^p, [l_2, l_3]]^p, \quad [l_2, l_3]^p, \quad \ldots. \]

The group $G$ gives a required example of a finitely generated solvable fractal group.

Recall that there are a lot of properties that hold for the class of metabelian groups but do not hold for the class of solvable groups of class 3. For example, any quotient of a finitely presented metabelian group is finitely presented. However, this is not true for solvable groups of class 3 (see [1]). There exists a finitely presented solvable group of class 3 with unsolvable word problem [5]. The main result of the present paper gives another example of a property that distinguishes the metabelian groups from solvable groups of class 3.

2. PROOF OF THEOREM 1

Step 1: solvability. First we prove that the group $G$ is solvable of class 3. Denote by $X$ the subgroup of $G$ generated by the elements $(x^{t_i}, i = 0, 1, \ldots)$ and by $Y$ the subgroup generated by the elements $(y^p, i = 0, 1, \ldots)$. The relations $x^t = x^p$ and $y^t = y^p$ imply that the subgroups $X$ and $Y$ are abelian and isomorphic to the subgroup of rationals $\mathbb{Z}[1/p] = \{m/p^k : m \in \mathbb{Z}, i \geq 0\}$. Denote by $H$ the normal closure $\langle x, y \rangle^G$. The subgroup $H$ is generated by the elements $x^{t_i}$ and $y^p$, $i = 0, 1, \ldots$.

The general element of $X$ is of the form $x^{mt_i}$ for some $i \geq 0$ and $m \in \mathbb{Z}$, and the general element of $Y$ is of the form $y^{mt_i}$ for some $i \geq 0$ and $m \in \mathbb{Z}$. For arbitrary $i, j, k \geq 0$ and $m, r, l \in \mathbb{Z}$, we will prove that the element

\[ [[x^{mt_i}, y^{rt_j}], x^{lt-k}] \]

is trivial in $G$. Using the standard commutator relations $[a, b^{-1}] = [a, b]^{-b^{-1}}$, $[ab, c] = [a, c][b, c]$, and $[a, bc] = [a, c][a, b]^c$, which hold in any group, one can reduce the proof to the case $l = m = 1$. Suppose that $k \geq i$. Then

\[ [[x^{t_i}, y^{rt_j}], x^{t-k}] = [[x^{t^i}, y^{rt_j+k}], x]^{t-k} = [[x^{t^i}, y^{rt+j}], x]^{t-k} = 1. \]

In the same way, if $i \geq k$, then

\[ [[x^{t-i}, y^{rt_j}], x^{t-k}] = [[x, y^{rt+j}], x^{t-k}]^{t-i} = [[x, y^{rt+j}], x^{p^j-k}]^{t-i} = 1. \]

That is, for all $a_1, a_2 \in X$ and $b \in Y$, we have $[[a_1, b], a_2] = [a_1^b, a_2] = 1$. Therefore, the subgroup generated by $X$ and $Y$ is a quotient of their wreath product $X \wr Y$. The wreath product of any pair of abelian groups is metabelian by construction. Therefore, the subgroup $H$ is metabelian, and $G$ is an extension of a cyclic group by a metabelian one and hence is solvable of class at most 3.

Step 2: iterated identity. Observe that the commutator subgroup $G'$ is generated by $[H, H]$ together with the elements $(x^{t^i})^{p^{i-1}}$ and $(y^{t^i})^{p^{i-1}}$, $i = 0, 1, 2, \ldots$. Since the subgroup $H$ is metabelian, the $j$th iteration of $w$ (for $j \geq 2$) can be rewritten on $G$ as

\[ \ldots [[[x_1, [x_2, x_3]]^p, [x_2, x_3]]^p]^{p^j}, [x_2, x_3]], \ldots, [x_2, x_3] \]

(here we use the fact that the element $[[x_1, [x_2, x_3]]^p, [x_2, x_3]]$ lies in $[H, H]$). Now we will prove that, for any element $u \in [H, H]$, there exists an $i$ such that $u^{p^i} = 1$. This will prove that $w$ is an iterated identity for $G$.
The subgroup \([H,H]\) is a normal closure of the elements \([x,y^i], i = 1, 2,\ldots\). Since the group \([H,H]\) is abelian, it is enough to show that \([x,y^i]^{p^i} = 1\). We will prove this by induction on \(i\). This is true for \(i = 0\), since \([x,y] = 1\) in \(G\). Suppose that
\[ [x,y^i]^{p^i} = 1 \]
for a given \(i\). Then
\[ 1 = [x,y^i]^{p^i}t = [x^t, y^{i+1}]^{p^i} = [x^p, y^{i+1}]^{p^i} = [x, y^{i+1}]^{p^i+1}. \]
Here we used the relation \([[[x,y^{i+1}]], x] = 1\), which holds in \(G\) by construction. We have proved that \(w\) is an iterated identity for \(G\).

**Step 3: fractality.** Now let us show that \(w\) is unbounded. Fix some \(i \geq 1\) and set
\[ x_1 = x^{t^{-i}}, \quad x_2 = t^{-1}, \quad x_3 = y^{-t^i}. \]
Then \([x_2, x_3] = (y^{t^i})^{p-1}\). For \(j \geq 1\), the \(j\)th iterated identity \(w\) applied to \(x_1, x_2, x_3\) gives an element
\[ [[[\ldots [[[x^{t^{-i}}, (y^{t^i})^{p-1}]^p, (y^{t^i})^{p-1}]^p, \ldots]], (y^{t^i})^{p-1}] = [x^{t^{-i}}(y^{t^i})^{p-1}]^{p^j}. \]
Here we used the standard notation for the Engel bracket \([a_i b] := [a, b]\) and \([a_{n+1} b] = [a_n b], b\].

Let us rewrite the generators of \(H\) as
\[ z_i := x^{t^{-i}}, \quad y_i := y^{t^i}, \quad i = 0, 1, \ldots. \]
The relations of \(H\) are
\[ [[z_i, y_j^s], z_k] = 1, \quad z_i = z_i^{p_{i+1}}, \quad y_i = y_i^{p_{i+1}}, \quad [z_0, y_0] = 1 \]
for all \(i, j, k = 0, 1, 2, \ldots\) and \(s \in \mathbb{Z}\). We examine the order of the element \([z_i, y_i^{p-1}]\) in \(H\). Take the quotient of \(H\) by the normal closure \(<z_0, y_0)^H\). The obtained group \(G := H/\langle z_0, y_0 \rangle^H\) is isomorphic to the wreath product of two \(p\)-quasicyclic groups:
\[ \Gamma \simeq \mathbb{Z}_{p}^\infty \rtimes \mathbb{Z}_{p}^\infty. \]
Denote by \(Z\) the subgroup generated by \(\{z_i\}_{i \geq 1}\) and by \(Y\) the subgroup generated by \(\{y_i\}_{i \geq 1}\). We have \(Z \simeq Y \simeq \mathbb{Z}_{p}^\infty\) and \(\Gamma = Z \rtimes Y\). We will use the multiplicative notation for the elements of \(Y\) and the additive notation for the elements of \(Z\). We will write \(u^n/p^i+z\) for the element \(y_i^n\) of \(Y\) and simply \(n/p^i+z\) for the element \(z_i^n\) of \(Z\). The elements of the wreath product \(Z \rtimes Y\) can be written as pairs \((\sum_{b \in \mathbb{Z}_{p}^\infty} a_b u^b, u^b_0), b_0 \in \mathbb{Z}_{p}^\infty\), with the product given as
\[ \left( \sum_{b \in \mathbb{Z}_{p}^\infty} a_b u^b, u^b_0 \right) \left( \sum_{b \in \mathbb{Z}_{p}^\infty} a'_b u^b, u'^b_0 \right) = \left( \sum_{b \in \mathbb{Z}_{p}^\infty} a_b u^b + \sum_{b \in \mathbb{Z}_{p}^\infty} a'_b u^{b+bo}, u^{b+bo}_0 \right). \]
In this notation, the element \([z_i, y_i^{p-1}]\) corresponds to the expression
\[ \left( \frac{1}{p^i+Z} \right) \left( \mu_{1/p^i}^{p-1} +Z, 1 \right)^j. \]
For \(j < i\), the order of this element is \(p^j\) (after opening the parentheses, there will be no term that could cancel \((-1)^j/p^i\)). This implies that for \(j < i\) the order of \([z_i, y_i^{p-1}]\) is \(p^i\). We conclude that
\[ [[[\ldots [[[x^{t^{-i}}, (y^{t^i})^{p-1}]^p, (y^{t^i})^{p-1}]^p, \ldots]], (y^{t^i})^{p-1}]^p, \ldots]], (y^{t^i})^{p-1}]^p, \ldots]] \neq 1 \]
for \(j < i\). That is, we need \(i\) iterations for the word \(w\) to become trivial for \(x_1 = x^{t^{-i}}, x_2 = t^{-1},\) and \(x_3 = y^{-t^i}\). The statement is proved.
3. FUNCTION-THEORETIC DESCRIPTION OF THE GROUP $G$

Here we present an alternative function-theoretic description of the group $G$ in Theorem 1. The construction of this section is due to S. Gorchinskiy; the author thanks him for drawing the author’s attention to this interpretation of the group $G$.

Given a set $\Sigma$ and an abelian group $\Lambda$, denote by $\text{Fun}^f(\Sigma, \Lambda)$ the abelian group of all functions from $\Sigma$ to $\Lambda$ with finite support, with the group structure defined pointwise by the group structure on $\Lambda$.

As above, put $X = Y = \mathbb{Z}[1/p]$. We will use the subgroups $p^i\mathbb{Z}$, $i \in \mathbb{Z}$, in $X$ and $Y$. Define

$$F = \text{Fun}^f(Y, X), \quad F_i = \text{Fun}^f(Y, p^{-i}\mathbb{Z}), \quad i \in \mathbb{Z}.$$ 

In particular, we have $F = \bigcup_{i \in \mathbb{Z}} F_i$ and there is an equality $F_i = pF_{i+1}$ of subgroups in $F$. Since the group $Y$ acts on itself by translations, $Y$ acts naturally on the groups $F$ and $F_i$, $i \in \mathbb{Z}$.

Let $A_i$, $i \in \mathbb{Z}$, be the subgroup of $F_i$ generated by the elements of type $u(f) - f$ with $f \in F_i$ and $u \in p^i\mathbb{Z}$. Put

$$A = \sum_{i \in \mathbb{Z}} A_i, \quad E = F/A, \quad E_i = \text{Im}(F_i \to E).$$

Given a function $f \in F$, by $\overline{f}$ denote its class in $E$. It is easy to see that $A$ is invariant under the action of $Y$, and so the action of $Y$ on $F$ defines the action of $Y$ on $E$ and $E_i$, $i \in \mathbb{Z}$.

One can show that for each $i \in \mathbb{Z}$ there is an exact sequence of $Y$-modules

$$0 \to E_{i-1} \to E_i \xrightarrow{\lambda_i} \text{Fun}^f(Y/p^i\mathbb{Z}, \mathbb{Z}/p\mathbb{Z}) \to 0.$$ 

Here, given $f \in F_i$, to define $\lambda_i(f)$ we first take the direct image of $f$ with respect to the homomorphism $Y \to Y/p^i\mathbb{Z}$ and then take the values of the function modulo $p^{-i+1}\mathbb{Z}$, thus obtaining values in the quotient $p^{-i}\mathbb{Z}/p^{-i+1}\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z}$. To prove the exactness of the sequence above, one shows that there is an embedding $F_{i-1} \cap A_i \subset A_{i-1}$ in $F_{i-1}$, whence $E_i = F_{i}/(\sum_{j \leq i} A_j)$.

Define automorphisms of the groups $Y$ and $F$, which we denote by the same letter by abuse of notation:

$$t: Y \xrightarrow{\sim} Y, \quad u \mapsto p^{-1}u, \quad t: F \xrightarrow{\sim} F, \quad f \mapsto (v \mapsto pf(pv)).$$

One checks directly that the action of $Y$ on $F$ is $t$-equivariant, that is, we have $t(u(f)) = t(u)(t(f))$ for all $u \in Y$ and $f \in F$. In addition, $t$ induces isomorphisms

$$t: F_{i+1} \xrightarrow{\sim} F_i, \quad t: A_{i+1} \xrightarrow{\sim} A_i, \quad i \in \mathbb{Z}.$$ 

Thus, $A$ is invariant under $t$, and so the action of $t$ on $F$ defines the action of $t$ on $E$ and induces isomorphisms

$$t: E_{i+1} \xrightarrow{\sim} E_i, \quad i \in \mathbb{Z}.$$ 

Using the action of $Y$ on $E$, we form the semidirect product $H := E \rtimes Y$. So, there is an exact sequence

$$1 \to E \to H \to Y \to 1.$$ 

It follows that $t$ acts on $H$. Then we form the semidirect product $G = H \rtimes \mathbb{Z}$. In particular, there is an exact sequence

$$1 \to H \to G \to \mathbb{Z} \to 1$$ 

and the group $G$ has a normal filtration with abelian adjoint quotients $\mathbb{Z}$, $Y$, and $E$. 

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There is an embedding $X \subset E$ that sends an element $r \in X$ to the class of the delta function at $0 \in Y$ with the value $r$. It is easy to see that the group $G$ is generated by the elements $x = 1 \in X = \mathbb{Z}[1/p]$, $y = 1 \in Y = \mathbb{Z}[1/p]$, and $t$. Moreover, one can show that $G$ is isomorphic to the group in Theorem 1.

Let us show within this interpretation that $G$ is fractal. Note that for any $u \in Y$ we have

$$[u, t^{-1}] = ut^{-1}(-u)t = u - t^{-1}(u) = (1 - p)u \in Y.$$ 

Hence, for all $u \in Y$, $f \in E$, and $i \geq 0$, the following equalities hold in $E$:

$$w^{(i)}(\bar{f}, u, t^{-1}) = \sum_{j=0}^{i} (-1)^{j} \binom{j}{i} (j(1 - p)u)(p^{j} \bar{f}),$$

where we consider $1 - (1 - p)u$ as an element of the group algebra $\mathbb{Z}[Y]$.

Now suppose that $u \in \mathbb{Z} \setminus p\mathbb{Z} \subset Y$. Then the numbers $j(1 - p)u$, $0 \leq j \leq i$, have pairwise different residues in $\mathbb{Z}/p^{i+1}\mathbb{Z}$, forming a subset $M \subset Y/p^{i+1}\mathbb{Z}$ of $i + 1$ elements. Let $f \in F_{2i} \setminus F_{2i-1}$ be the delta function at $0 \in Y$ with the value $p^{-2i}$. Then $\lambda(p^{i}f) \in \text{Fun}^{1}(Y/p^{i+1}\mathbb{Z}, \mathbb{Z}/p\mathbb{Z})$ is the delta function $\delta_{0}$ at $0 \in Y/p^{i+1}\mathbb{Z}$ with the value 1. Therefore, $\lambda(w^{(i)}(\bar{f}, u, t^{-1}))$ is a linear combination of delta functions at elements of $M \subset Y/p^{i+1}\mathbb{Z}$. Moreover, in this linear combination, the coefficient of $\delta_{0}$ equals 1. In particular, this function is nonzero and we obtain $w^{(i)}(\bar{f}, u, t^{-1}) \neq 0$.

Note that, for $u$ and $f$ as above, we have $w^{(2)}(\bar{f}, u, t^{-1}) = 0$, because $p^{2i}f \in E_{0}$ is invariant under the action of the subgroup $\mathbb{Z} \subset Y$ and, in particular, is invariant under $[u, t^{-1}] \in \mathbb{Z}$ (one shows similarly in general that $w$ is an iterated identity for $G$).

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