Betti numbers and the curvature operator of the second kind

Jan Nienhaus\textsuperscript{1} \quad Peter Petersen\textsuperscript{2} \quad Matthias Wink\textsuperscript{1}

\textsuperscript{1}Mathematisches Institut, Universität Münster, Einsteinstraße, Münster, Germany
\textsuperscript{2}Department of Mathematics, University of California Los Angeles, Los Angeles, California, USA

Correspondence
Matthias Wink, Mathematisches Institut, Universität Münster, Einsteinstraße 62, 48149 Münster, Germany.
Email: mwink@uni-muenster.de

Funding information
Deutsche Forschungsgemeinschaft, Grant/Award Number: EXC 2044–390685587; Alexander von Humboldt Foundation

Abstract
We show that compact, $n$-dimensional Riemannian manifolds with $\frac{n+2}{2}$-nonnegative curvature operators of the second kind are either rational homology spheres or flat. More generally, we obtain vanishing of the $p$th Betti number provided that the curvature operator of the second kind is $C(p,n)$-positive. Our curvature conditions become weaker as $p$ increases. For $p = \frac{n}{2}$, we have $C(p,n) = \frac{3n}{2} \cdot \frac{n+2}{n+4}$, and for $5 \leq p \leq \frac{n}{2}$, we exhibit a $C(p,n)$-positive algebraic curvature operator of the second kind with negative Ricci curvatures.

MSC 2020
53C25 (primary), 53C15, 53C20 (secondary)

INTRODUCTION

It is an important topic in geometry to understand how geometric assumptions restrict the topology of the underlying Riemannian manifold. For example, D. Meyer [21] showed that manifolds with positive curvature operators are rational homology spheres. Gallot–Meyer [13] proved the corresponding rigidity theorem. That is, manifolds with nonnegative curvature operators are either reducible, locally symmetric or their universal cover has the cohomology of a sphere or a complex projective space.

With Ricci flow techniques, these results were improved to diffeomorphism classifications. In particular, due to the work of Hamilton [14, 15], Chen [11] and Böhm-Wilking [9], manifolds with 2-positive curvature operators are diffeomorphic to space forms. The corresponding rigidity result was obtained by Ni–Wu [25]. Generalizations of these results in the context of isotropic curvatures were proven by Brendle–Schoen [7, 8] and Brendle [5].
Moreover, the second and third authors proved vanishing and rigidity theorems for $p$-forms based on the corresponding assumption on the sum of the lowest $(n - p)$ eigenvalues of the curvature operator in [27].

In addition to the curvature operator

$$\mathfrak{R} : \Lambda^2TM \to \Lambda^2TM, \quad (\mathfrak{R}(\omega))_{ij} = \sum_{k,l} R_{ijkl} \omega_{kl},$$

the curvature tensor of a Riemannian manifold also induces a self-adjoint operator on the space of symmetric $(0,2)$-tensors

$$\overline{R} : S^2(TM) \to S^2(TM), \quad (\overline{R}(h))_{ij} = \sum_{k,l} R_{ijkl} h_{kl}.$$

The curvature operator of the second kind is the induced map on the space of trace-free symmetric $(0,2)$-tensors:

$$\mathcal{R} : S^2_0(TM) \to S^2_0(TM), \quad \mathcal{R} = \text{pr}_{S^2_0(TM)} \circ \overline{R}_{|S^2_0(TM)}.$$

It was already studied by Bourguignon–Karcher in [3]. In contrast to $\overline{R}$, the curvature operator of the second kind $\mathcal{R}$ satisfies the natural geometric condition that $\mathcal{R} \geq \kappa$ implies that all sectional curvatures are bounded from below by $\kappa$.

Ogiue–Tachibana [26] proved that similarly to D. Meyer’s result, compact manifolds with positive curvature operators of the second kind are rational homology spheres. Both proofs rely on the Bochner technique.

Nishikawa [23] conjectured that compact manifolds with positive curvature operators of the second kind are diffeomorphic to spherical space forms. In [10], Cao–Gursky–Tran proved Nishikawa’s conjecture. In fact, they proved Nishikawa’s conjecture for manifolds with 2-positive curvature operators of the second kind. Subsequently, X. Li [20] relaxed the assumption to 3-positive curvature operator of the second kind. The proofs are based on the observation that these manifolds satisfy the PIC1 condition and thus Brendle’s [5] convergence result for the Ricci flow applies.

In addition, the rigidity part of Nishikawa’s conjecture [23] asserts that a manifold with non-negative curvature operator of the second kind is diffeomorphic to a locally symmetric space. In [20], X. Li proved that a Riemannian manifold of dimension $n \geq 4$ with 3-nonnegative curvature operator of the second kind is either diffeomorphic to a spherical space form, flat, or $n \geq 5$ and the universal cover is isometric to a compact irreducible symmetric space.

The first main theorem of the paper rules out the third option, even under a weaker assumption on the eigenvalues of the curvature operator of the second kind.

**Definition.** A self-adjoint operator $\mathcal{R}$ with eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N$ is called $k$-nonnegative for some $k \geq 1$ if $\lambda_1 + \cdots + \lambda_{[k]} + (k - [k]) \lambda_{[k]+1} \geq 0$.

Note that $\mathcal{R}$ is $k$-nonnegative if it is $\lfloor k \rfloor$-nonnegative. We say $\mathcal{R}$ is nonnegative if it is 1-nonnegative.

**Theorem A.** Let $(M, g)$ be a compact, $n$-dimensional Riemannian manifold. If the curvature operator of the second kind is $\frac{n+2}{2}$-nonnegative, then $(M, g)$ is either flat or a rational homology sphere.
Compact symmetric spaces which are real cohomology spheres were classified by Wolf in [29]. Apart from spheres, $SU(3)/SO(3)$ is the only simply connected example. However, according to Example 4.5, the curvature operator of the second kind of $SU(3)/SO(3)$ is 9-positive but not 8-nonnegative. Thus, combining Theorem A with X. Li’s result explained above [20], we obtain the following improvement on Nishikawa’s conjecture.

**Corollary.** Let $n \geq 4$ and let $(M, g)$ be a compact, $n$-dimensional Riemannian manifold. If the curvature operator of the second kind is 3-nonnegative, then $(M, g)$ is either flat or diffeomorphic to a spherical space form.

In dimension $n = 3$, X. Li proved the above result in [20].

The proof of Theorem A is an application of a Bochner formula for the curvature operator of the second kind. In the case of $\frac{n+2}{2}$-nonnegative curvature operator, we are able to obtain control on all Betti numbers. This is also the case for Einstein manifolds:

**Theorem B.** Let $(M, g)$ be a compact, $n$-dimensional Einstein manifold. Let $N = \frac{3n}{2} \frac{n+2}{n+4}$.

(a) If the curvature operator of the second kind is $N$-positive, then $M$ is a rational homology sphere.
(b) If the curvature operator of the second kind is $N'$-nonnegative for some $N' < N$, then $(M, g)$ is either flat or a rational homology sphere.
(c) If the curvature operator of the second kind is $N$-nonnegative, then all harmonic forms are parallel.

**Remark.** By the theory of Diophantine equations, $\frac{3n}{2} \frac{n+2}{n+4}$ is only an integer if $n = 0, 2, 8$. Therefore, unless $n = 2, 8$, if the curvature operator of the second kind is $\lfloor N \rfloor$-nonnegative, part (b) applies.

Furthermore, the curvature condition in part (b) implies that either $R$ is $N$-positive or 1-nonnegative, cf. Theorem 3.6(d).

Theorem B amplifies the work of Cao–Gursky–Tran [10], who proved that Einstein manifolds with 4-nonnegative curvature operators of the second kind are locally symmetric, and have constant sectional curvature in the case of 4-positivity. This is a consequence of their observation that manifolds with 4-nonnegative curvature operators of the second kind have nonnegative isotropic curvature, and Brendle’s theorem [6] on Einstein manifolds with nonnegative isotropic curvature. Much earlier, Kashiwada [16] proved the theorem for manifolds with positive curvature operators of the second kind.

For a general Riemannian manifold, we obtain the following vanishing and rigidity results for the $p$th Betti number. Note that due to Poincaré duality, we may assume $p \leq \frac{n}{2}$. Set

$$C_p = C_p(n) = \frac{3}{2} \frac{n(n+2)p(n-p)}{n^2p - np^2 - 2np + 2n^2 + 2n - 4p}.$$

**Theorem C.** Let $(M, g)$ be a compact, $n$-dimensional Riemannian manifold and let $p \leq \frac{n}{2}$.

(a) If the curvature operator of the second kind is $C_p$-positive, then the $p$th Betti number $b_p(M, \mathbb{R})$ vanishes.
(b) If the curvature operator of the second kind is $C'$-nonnegative for some $C' < C_p$, then $b_p(M, \mathbb{R})$ vanishes or $(M, g)$ is flat.
(c) If the curvature operator of the second kind is $C_p$-nonnegative, then all harmonic $p$-forms are parallel.

Note that $C_p$ increases as $p \leq \frac{n}{2}$ increases. In particular, the curvature conditions become weaker as $p$-increases. Therefore, unless $(M, g)$ is flat, if the curvature operator of the second kind is $C_p$-nonnegative, then all harmonic $k$-forms vanish for $p < k < n - p$. Furthermore, we obtain the weakest curvature condition for $p = \frac{n}{2}$. In this case, we have $C_{\frac{n}{2}} = \frac{3n}{2} \frac{n+2}{n+4}$ as in the Einstein case in Theorem B.

The effect that curvature conditions become weaker as $p$ increases also occurs for holomorphic $p$-forms on a compact Kähler manifold according to a result of Bochner [4]. However, in the case of manifolds with generic holonomy, this is a new phenomenon.

Due to a result of X. Li [20], $n$-nonnegativity of $\mathcal{R}$ implies $\text{Ric} \geq \frac{\text{scal}}{n(n+1)} \geq 0$. However, $\mathcal{R}$ being $(n + 1)$-nonnegative does not imply nonnegativity of Ricci curvature, according to Example 4.4.

Notice that for any fixed $p$, we have

$$C_p(n) \sim \frac{3n}{2} \frac{p}{p + 2}$$

for large $n$. In particular, asymptotically for large $n$, our curvature conditions do not imply lower Ricci curvature bounds for $p \geq 5$. Specifically, for $p = 5$, we have $C_5(n) \geq \frac{15n}{14} \geq n + 1$ if $n \geq 14$. In contrast, the results for the (standard) curvature operator in [27] imply lower Ricci curvature bounds for any $p$.

In addition to vanishing and rigidity results, our methods also yield estimation results in the presence of lower Ricci curvature bounds. This is a consequence of the techniques developed by Gallot [12] and P. Li [19]. In particular, Gallot proved estimation theorems for the Betti numbers of manifolds with upper diameter bounds and lower bounds on the (standard) curvature operator. The curvature assumption was weakened in [27] to a lower bound on the average of the lowest $(n - p)$-eigenvalues of the curvature operator.

For the curvature operator of the second kind, Lemma 3.14 yields a lower bound on the Ricci curvature provided that the average of the lowest $n$ eigenvalues of the curvature operator of the second kind is bounded from below. Therefore, the techniques of Gallot and Li yield the following:

**Theorem D.** Let $n \geq 3$, $D > 0$ and $\kappa \leq 0$. Let $(M, g)$ be a compact, $n$-dimensional Riemannian manifold. Let $\lambda_1 \leq \ldots \leq \lambda_{\frac{n+2}{2}}$ be the eigenvalues of the curvature operator of the second kind of $(M, g)$. There is $C(n, D\kappa^2) > 0$ such that if $\text{diam}(M) < D$ and

$$\begin{align*}
\lambda_1 + \ldots + \lambda_{\frac{n+2}{2}} &+ \frac{1}{2} \cdot \lambda_{\frac{n+2}{2}} \geq \frac{n+2}{2} \cdot \kappa, \text{ if } n \text{ odd}, \\
\lambda_1 + \ldots + \lambda_{\frac{n+2}{2}} &\geq \frac{n+2}{2} \cdot \kappa, \text{ if } n \text{ even},
\end{align*}$$

then

$$b_p(M) \leq \binom{n}{p} \exp \left( C(n, \kappa D^2) \cdot \sqrt{-\kappa D^2 p(n-p)} \right).$$

In particular, there is $\varepsilon(n) > 0$ such that $\kappa D^2 > -\varepsilon$ implies $b_p(M) \leq \binom{n}{p}$. 
In the Einstein case, a lower bound on the scalar curvature implies a lower bound on the Ricci curvature. Thus, we obtain the bound on the Betti numbers \( b_p(M) \) in Theorem D for all \( p \) provided the Riemannian manifold \((M, g)\) is Einstein, \( \text{diam}(M) < D \) and \( \lambda_1 + \cdots + \lambda_{\lfloor N \rfloor} + (N - \lfloor N \rfloor) \cdot \lambda_{\lfloor N \rfloor + 1} \geq N \kappa \), where \( N = \frac{3n}{2} \frac{n+2}{n+4} \).

In order to obtain an estimation analog of Theorem C, we impose an explicit lower Ricci curvature bound:

**Corollary.** Let \( n \geq 3, D > 0 \) and \( \kappa \leq 0 \). Let \((M, g)\) be a compact, \( n \)-dimensional Riemannian manifold. Let \( \lambda_1 \leq \cdots \leq \lambda_{\frac{1}{2}(n+2)(n-1)} \) denote the eigenvalues of the curvature operator of the second kind of \((M, g)\). There is \( C(n, D\kappa^2) > 0 \) such that if \( \text{diam}(M) < D \), \( \text{Ric} \geq (n-1)\kappa \) and \( \lambda_1 + \cdots + \lambda_{\lfloor C_p \rfloor} + (C_p - \lfloor C_p \rfloor) \lambda_{\lfloor C_p \rfloor + 1} \geq C_p \kappa \),

then

\[
b_p(M) \leq \left( \frac{n^p}{p^p} \right) \exp \left( C(n, \kappa D^2) \cdot \sqrt{-\kappa D^2 p(n-p)} \right).
\]

In particular, there is \( \varepsilon(n) > 0 \) such that \( \kappa D^2 > -\varepsilon \) implies \( b_p(M) \leq \left( \frac{n}{p} \right) \).

The proofs of the main theorems are based on the Bochner technique. By Hodge theory, every de Rham cohomology class is represented by a harmonic form. If \( \omega \) is a harmonic \( p \)-form, then it satisfies the Bochner formula

\[
\Delta \frac{1}{2} |\omega|^2 = |\nabla \omega|^2 + g(\text{Ric}_L(\omega), \omega).
\]

We establish that the curvature term of the Lichnerowicz Laplacian satisfies the equation

\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) = \sum_{\alpha=1}^{N} \lambda_\alpha |S_\alpha \omega|^2 + \frac{p(n-2p)}{n} \sum_{j, k} \sum_{l_1, \ldots, l_p} R_{jk} \omega_{l_1 \ldots l_p} \omega_{k l_1 \ldots l_p} + \frac{p^2}{n^2} \text{scal} |\omega|^2,
\]

where \( \{ S_\alpha \} \) is an orthonormal eigenbasis of the curvature operator of the second kind with corresponding eigenvalues \( \{ \lambda_\alpha \} \), and \( N = \dim S_0^2(TM) = \frac{1}{2}(n-1)(n+2) \).

We are able to control the first term by understanding the interaction of trace-free, symmetric tensors on forms, adapting ideas of [27]. The key point is that all weights \( |S_\alpha \omega|^2 \) are bounded by \( \frac{p(n-p)}{n} |\omega|^2 \) while the total weight \( \sum_\alpha |S_\alpha \omega|^2 = \frac{p(n-p) n+2}{n^2} |\omega|^2 \) is large in comparison. In particular, \( \sum_\alpha \lambda_\alpha |S_\alpha \omega|^2 \geq 0 \) if the curvature operator of the second kind is \( \frac{n+2}{2} \)-nonnegative. Moreover, \( \frac{n+2}{2} \)-nonnegativity also implies nonnegative Ricci curvature and hence \( g(\text{Ric}_L(\omega), \omega) \geq 0 \).

It follows that every harmonic form on a manifold with \( \frac{n+2}{2} \)-nonnegative curvature operator of the second kind is parallel and satisfies \( 0 \geq \text{scal} |\omega|^2 \). This implies \( \omega \) vanishes unless \( M \) is flat. We remark that this final conclusion is also possible with the formula obtained by Ogiue–Tachibana [26], cf. Remark 2.6, provided the curvature operator of the second kind is positive. However, this argument has not been pointed out before.

For the general case, we also incorporate the Ricci and scalar curvature terms in a single estimate on the eigenvalues of the curvature operator of the second kind. This places different weights on the eigenvalues. The main technical tool, the weight principle 3.6, is a refinement of the ideas.
above and allows us obtain eigenvalue estimates for sums with different weights. In particular, the weight principle 3.6 extends [27, Lemma 2.1] to an abstract setting.

The curvature operator of the second kind also naturally occurs in the context of deformations of Einstein structures, cf. Berger–Ebin [1], Besse [2] or Koiso [17, 18], as well as in Bochner–Weitzenböck formulas for symmetric tensors, cf. Mikeš–Rovenski–Stepanov [22] or Shandra–Stepanov–Mikeš [28].

Restrictions on the restricted holonomy groups of not necessarily complete manifolds which satisfy nonnegativity or nonpositivity conditions on the eigenvalues of the curvature operator of the second kind are studied by the authors and W. Wylie in [24].

**Structure**

Section 1 collects some preliminary results and sets up notation. In Section 2, we provide a brief introduction to the Bochner technique and in particular establish a Bochner formula for the curvature operator of the second kind in Proposition 2.1. In Section 3, we prove the key technical tool, the weight principle 3.6. As an application we obtain estimates on the curvature terms in the Bochner formula. For example, Proposition 3.13 provides a simple, preliminary estimate that provides asymptotically the same result as Theorem C, cf. Example 4.2. Theorem C itself relies on the refined estimate in Proposition 3.15. The proofs of the main theorems are given in Section 4. Section 4 also contains the example of an \((n + 1)\)-positive algebraic curvature operator of the second kind with negative Ricci curvatures, and discusses the curvature of the rational homology sphere \(SU(3)/SO(3)\).

1 | **PRELIMINARIES**

Let \((V, g)\) be an \(n\)-dimensional Euclidean vector space, and let \(R\) be an algebraic \((0,4)\)-curvature tensor on \(V\). Let \(S^2(V)\) denote the space of symmetric \((0,2)\)-tensors on \(V\). The subspace of trace-free symmetric \((0,2)\)-tensors is denoted by \(S^2_0(V)\). Recall that

\[
S^2(V) = S^2_0(V) \oplus \mathbb{R} g
\]

is the decomposition of \(S^2(V)\) into \(O(n)\)-invariant, irreducible subspaces.

For an algebraic curvature tensor \(R\), set

\[
\overline{R} : S^2(V) \rightarrow S^2(V),
\]

\[
h \mapsto \sum_{k,l=1}^{n} R_{kl} h_{kl},
\]

where the components are with respect to an orthonormal basis \(e_1, \ldots, e_n\) for \(V\).

Note that \(\overline{R}\) is self-adjoint and

\[
\overline{R}(g) = \sum_{k,l=1}^{n} R_{kl} \delta_{kl} = -\text{Ric}.
\]
Furthermore, the operator $\overline{R}$ leaves the subspace $S^2_0(V)$ invariant if and only if $R$ is Einstein. Define

$$\hat{R} : S^2(V) \to S^2_0(V),$$

$$\hat{R} = \text{pr}_{S^2_0(V)} \circ R = \overline{R} + g(\text{Ric}, \cdot) \frac{g}{n}.$$ 

The induced operator $\mathcal{R} = \hat{R}|_{S^2_0(V)} : S^2_0(V) \to S^2_0(V)$ is called curvature operator of the second kind. Note that $\mathcal{R}$ is again self-adjoint.

**Example 1.1.** If $R$ is the curvature tensor of the round sphere, then $\overline{R}(h) = h - \text{tr}(h) \text{id}$. Indeed, note that $R_{ijkl} = \delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}$ and thus

$$\left(\overline{R}(h)\right)_{ij} = \sum_{k,l} R_{iklj}h_{kl} = \sum_{k,l} \left(\delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl}\right)h_{kl} = h_{jl} - \delta_{ij} \sum_{k=1}^{n} h_{kk} = h_{jl} - \delta_{ij} \text{tr}(h).$$

In particular, $\hat{R}|_{S^2_0(V)} = \text{id}_{S^2_0(V)}$.

**Proposition 1.2.**

$$\text{tr}(R) = \frac{\text{scal}}{2} \text{ and } \text{tr}(R) = \frac{n + 2}{2n} \text{ scal}.$$  

**Proof.** If $e_1, \ldots, e_n$ is an orthonormal basis for $V$, then

$$g(R(e^i \otimes e^i), e^i \otimes e^i) = 0,$$

$$g\left(\overline{R}\left(\frac{1}{\sqrt{2}}(e^i \otimes e^j + e^j \otimes e^i)\right), \frac{1}{\sqrt{2}}(e^i \otimes e^j + e^j \otimes e^i)\right) = R_{iij}. $$

This implies $\text{tr}(R) = \sum_{i<j} R_{iij} = \frac{\text{scal}}{2}$. Furthermore, note that $\text{tr}(R) = \text{tr}(\overline{R}) - g(\overline{R}(\frac{g}{\sqrt{n}}), \frac{g}{\sqrt{n}}) = \frac{\text{scal}}{2} + \frac{\text{scal}}{n} = \frac{n + 2}{2n} \text{ scal}.\Box$

Via the metric, we identify symmetric (0,2)-tensors with self-adjoint endomorphisms of $V$.

**Definition 1.3.** Let $V$ be a finite-dimensional Euclidean vector space. Let $\mathcal{T}^{(0,k)}(V)$ denote the vector space of (0, $k$)-tensors on $V$. For $S \in S^2(V)$ and $T \in \mathcal{T}^{(0,k)}(V)$, set

$$(ST)(X_1, \ldots, X_k) = \sum_{i=1}^{k} T(X_1, \ldots, SX_i, \ldots, X_k)$$

and define $T^S \in \mathcal{T}^{(0,k)}(V) \otimes S^2(V)$ via

$$g\left(T^S(X_1, \ldots, X_k), S\right) = (ST)(X_1, \ldots, X_k).$$
In particular, if \( \{ S_\alpha \} \) is an orthonormal basis for \( S^2(V) \), then

\[
T^{S^2} = \sum_{S_\alpha} S_\alpha T \otimes S_\alpha.
\]

Similarly, we define

\[
T^{S^2_0} = \sum_{S_\alpha} S_\alpha T \otimes S_\alpha,
\]

where \( \{ S_\alpha \} \) is an orthonormal basis for \( S^2_0(V) \).

**Remark 1.4.**

(a) We have \( T^{S^2} = T^{S^2_0} + \frac{1}{\sqrt{n}} (gT) \otimes \frac{1}{\sqrt{n}} g \). The observation

\[
gT = \sum_{i=1}^{k} T(\ldots, \text{id}, \ldots) = kT
\]

thus implies the important relation

\[
T^{S^2} = T^{S^2_0} + \frac{k}{n} T \otimes g.
\]

(b) If \( \omega \in \bigwedge^p V^* \) is a \( p \)-form and \( S \in S^2(V) \), then \( S\omega \) is again a \( p \)-form. Indeed,

\[
(S\omega)(X, X, X_3, \ldots, X_p) = \omega(SX, X, X_3, \ldots, X_p) + \omega(X, SX, X_3, \ldots, X_p) + 0 = 0.
\]

(c) We use the standard norm on \( \bigotimes^k V^* \).

In particular, if \( \omega \) is a \( p \)-form, then

\[
|\omega|^2 = \sum_{i_1, \ldots, i_p} (\omega_{i_1 \ldots i_p})^2 = p! \sum_{i_1 < \ldots < i_p} (\omega_{i_1 \ldots i_p})^2.
\]

**Example 1.5.** Set \( e^i \circ e^j = \frac{1}{2}(e^i \otimes e^j + e^j \otimes e^i) \) and note that

\[
||e^i \circ e^j||^2 = \begin{cases} 1 & \text{if } i = j, \\ \frac{1}{2} & \text{if } i \neq j. \end{cases}
\]

Thus,

\[
T^{S^2} = \sum_{i \neq j}(e^i \circ e^j)T \otimes \frac{e^i \circ e^j}{||e^i \circ e^j||^2}
\]

\[
= \sum_i (e^i \circ e^j)T \otimes e^i \circ e^j + \sum_{i \neq j}(e^i \circ e^j)T \otimes e^i \circ e^j
\]

\[
= \sum_{i, j}(e^i \circ e^j)T \otimes e^i \circ e^j.
\]
Furthermore, note that
\[
\overline{\mathbf{R}}(e^i \tilde{\circ} e^j) = \frac{1}{2} \sum_{k,l} R_{kli} (\delta_{ki} \delta_{lj} + \delta_{kj} \delta_{li}) = \frac{1}{2} (R_{ij} + R_{ji})
\]
and thus
\[
g\left(\overline{\mathbf{R}}(e^i \tilde{\circ} e^j), e^k \tilde{\circ} e^l\right) = \frac{1}{2} (R_{kijl} + R_{kjl}).
\]

**Definition 1.6.** For an algebraic curvature tensor \( R \) and a \((0,k)\)-tensor \( T \), set
\[
\overline{R}(T^{S^2}) = \sum_{\alpha} S_\alpha T \otimes \overline{R}(S_\alpha).
\]
In particular, for all \( S \in S^2(V) \) and \( X_1, \ldots, X_k \in V \), we have
\[
g\left(\overline{R}(T^{S^2})(X_1, \ldots, X_k), S\right) = (\overline{R}(S)T)(X_1, \ldots, X_k).
\]
Similarly we define \( \overline{R}(T^{S^2}_0) \) and \( R(T^{S^2}_0) \).

**Example 1.7.** If \( \{S_\alpha\} \) is an orthonormal eigenbasis for \( R \) with corresponding eigenvalues \( \{\lambda_\alpha\} \), then
\[
g\left(R(T^{S^2}_0), T^{S^2}_0\right) = \sum_{\alpha,\beta} g(S_\alpha T, S_\beta T) \ g(R(S_\alpha), S_\beta) = \sum_{\alpha,\beta} \lambda_\alpha g(S_\alpha T, S_\beta T) = \sum_{\alpha} \lambda_\alpha |S_\alpha T|^2
\]
and thus in particular
\[
|T^{S^2}_0|^2 = \sum_{\alpha} |S_\alpha T|^2.
\]

**Proposition 1.8.** If \( T \) is a \((0,k)\)-tensor, then
\[
g\left(\overline{R}(T^{S^2}_0), T^{S^2}_0\right) = g\left(R(T^{S^2}_0), T^{S^2}_0\right).
\]

**Proof.** Let \( \{S_\alpha\} \) denote an orthonormal basis for \( S^2_0 \). Recall that \( \overline{R} = \mathcal{R} + \frac{\mathcal{R}(\text{Ric}, \cdot)}{n} \). Since any \( S_\alpha \) is trace-free and hence orthogonal to \( g \), we have
\[
g(\overline{R}(S_\alpha), S_\beta) = g((\text{pr}_{S^2_0} \circ \overline{R})(S_\alpha), S_\beta) = g(\mathcal{R}(S_\alpha), S_\beta).
\]
Thus, we obtain
\[
g\left( \bar{R} \left( T^{S_0^2} \right), T^{S_0^2} \right) = \sum_{\alpha, \beta} g(S_{\alpha} T, S_{\beta} T) g(\bar{R}(S_{\alpha}), S_{\beta})
\]
\[
= \sum_{\alpha, \beta} g(S_{\alpha} T, S_{\beta} T) g(R(S_{\alpha}), S_{\beta}) = g\left( R \left( T^{S_0^2} \right), T^{S_0^2} \right). \tag*{□}
\]

## 2 A BOCHNER FORMULA FOR THE CURVATURE OPERATOR OF THE SECOND KIND

The Bochner technique relies on the observation that on \( p \)-forms
\[
\Delta_{\text{Hodge}} = \nabla^* \nabla + \text{Ric}_L,
\]
where
\[
\text{Ric}_L(\omega)(X_1, \ldots, X_p) = \sum_{i=1}^{p} \sum_{j=1}^{n} (R(X_i, e_j) \omega)(X_1, \ldots, e_j, \ldots, X_p).
\]
In particular, cf. [2], the curvature term on \( p \)-forms is given by
\[
g(\text{Ric}_L(\omega), \omega) = p \sum_{l_2, \ldots, l_p} \sum_{i,j} R_{ij} \omega_{i l_2 \ldots l_p} \omega_{j l_2 \ldots l_p}
\]
\[
- \frac{p(p-1)}{2} \sum_{l_3, \ldots, l_p} \sum_{i,j,k,l} R_{ijkl} \omega_{ij l_3 \ldots l_p} \omega_{k l l_3 \ldots l_p}.
\]
If \( \omega \) is a harmonic \( p \)-form, \( \Delta_{\text{Hodge}} \omega = 0 \), then
\[
\Delta \frac{1}{2} |\omega|^2 = |\nabla \omega|^2 - g(\nabla^* \nabla \omega, \omega) = |\nabla \omega|^2 + g(\text{Ric}_L(\omega), \omega).
\]
In particular, if \( M \) is compact and \( g(\text{Ric}_L(\omega), \omega) \geq 0 \), then \( \omega \) is parallel.

Estimation results on the dimension of the kernel of the Hodge Laplacian follow if there are constants \( \kappa \leq 0 \) and \( C > 0 \) such that \( g(\text{Ric}_L(\omega), \omega) \geq \kappa C |\omega|^2 \) and \( \text{Ric} \geq (n-1)\kappa \), cf. [27, Theorem 1.9].

The connection to the curvature operator of the second kind is given by the following observation.

**Proposition 2.1.** Let \( R \) be an algebraic curvature tensor and let \( \omega \) be a \( p \)-form. With respect to an orthonormal basis, the curvature term in the Bochner formula satisfies
\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) = g\left( R \left( \omega_{s_0^2} \right), \omega_{s_0^2} \right) + \frac{p(n-2p)}{n} \sum_{j,k} \sum_{l_2, \ldots, l_p} R_{jk \omega_{j l_2 \ldots l_p}} \omega_{k l l_2 \ldots l_p} + \frac{p^2}{n^2} \text{scal} |\omega|^2.
\]
If the orthonormal basis diagonalizes the Ricci tensor, then furthermore

\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) = g\left( \mathcal{R}\left(\omega^{S_0^2}\right), \omega^{S_0^2}\right) + \frac{n - 2p}{n} \sum_{l=1}^{n} \left( \sum_{i \in \mathcal{I}} R_{ii} \right) \omega_i^2 + \frac{p^2}{n^2} \text{scal} |\omega|^2.
\]

In particular, if R is Einstein, \( \text{Ric} = \frac{\text{scal}}{n} g \), then

\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) = g\left( \mathcal{R}\left(\omega^{S_0^2}\right), \omega^{S_0^2}\right) + \frac{p(n-p)}{n^2} \text{scal} |\omega|^2.
\]

**Proof.** Recall that due to Proposition 1.8, we may consider the term \( g(\bar{R}(\omega^{S_0^2}), \omega^{S_0^2}) \). Since \( \omega^{S_0^2} = \omega^{S_2} - \frac{p}{n} \omega \otimes g \) and \( \bar{R}(g) = -\text{Ric} \), it follows that

\[
g\left( \bar{R}\left(\omega^{S_0^2}\right), \omega^{S_0^2}\right) = g\left( \bar{R}\left(\omega^{S_2}\right), \omega^{S_2}\right) + \frac{2p}{n} g\left( \omega^{S_2}, \omega \otimes \text{Ric} \right) - \frac{p^2}{n^2} \text{scal} |\omega|^2.
\]

Propositions 2.3 and 2.4 below imply

\[
g\left( \bar{R}\left(\omega^{S_0^2}\right), \omega^{S_0^2}\right) = -\frac{3}{2} \frac{p(p-1)}{2} \sum_{i,j,k,l} \omega_{ijl3...lp} \omega_{ki3...lp} R_{ijkl} + \left( \frac{p}{2} + \frac{2p^2}{n} \right) \sum_{i,j,l} R_{ij} \omega_{ijl3...lp} \omega_{ji3...lp} - \frac{p^2}{n^2} \text{scal} |\omega|^2
\]

\[
= -\frac{3}{2} \frac{p(p-1)}{2} \sum_{i,j,k,l} \omega_{ijl3...lp} \omega_{ki3...lp} R_{ijkl} + \frac{p(n+4p)}{2n} \sum_{i,j,l} R_{ij} \omega_{ijl3...lp} \omega_{ji3...lp} - \frac{p^2}{n^2} \text{scal} |\omega|^2.
\]

Thus,

\[
g(\mathcal{R}(\omega^{S_0^2}), \omega^{S_0^2}) + \frac{p(n-2p)}{n} \sum_{j,k,l} R_{jk} \omega_{jkl3...lp} \omega_{ki3...lp} + \frac{p^2}{n^2} \text{scal} |\omega|^2 =
\]

\[
= \frac{3p}{2} \sum_{l_2,...,l_p} \sum_{l_{i,j}} R_{ij} \omega_{i2...lp} \omega_{j2...lp} - \frac{3}{2} \frac{p(p-1)}{2} \sum_{i_3,...,i_p} \sum_{l_1,...,l_p} R_{ijkl} \omega_{ijkl3...lp} \omega_{kl3...lp}
\]

\[
= \frac{3}{2} g(\text{Ric}_L(\omega), \omega).
\]

**Proposition 2.2.** For an orthonormal basis \( e_1, ..., e_n \), set \( e^i \circ e^j = \frac{1}{2} (e^i \otimes e^j + e^j \otimes e^i) \). Then, every \( p \)-form \( \omega \) satisfies

\[
g((e^i \circ e^j) \omega, \omega) = p \sum_{i_2,...,i_p} \omega_{i_2...i_p} \omega_{j_2...i_p}
\]
and
\[
g((e^i \circ e^j)\omega, (e^k \circ e^l)\omega) = \frac{p}{4} \sum_{i_2, \ldots, i_p} \left( \delta_{jk} \omega_{i_2 \ldots i_p}, \omega_{i_2 \ldots i_p} + \delta_{jl} \omega_{i_2 \ldots i_p}, \omega_{k_2 \ldots i_p} + \delta_{ik} \omega_{j_2 \ldots i_p}, \omega_{l_2 \ldots i_p} + \delta_{il} \omega_{j_2 \ldots i_p}, \omega_{k_2 \ldots i_p} \right)
+ \frac{p(p-1)}{2} \sum_{i_3, \ldots, i_p} \left( \omega_{i_3 \ldots i_p}, \omega_{j_3 \ldots i_p} + \omega_{i_3 \ldots i_p}, \omega_{j_3 \ldots i_p} \right).
\]

**Proof.** If \(\Omega\) is another \(p\)-form, then
\[
g((e^i \circ e^j)\omega, \Omega) = \sum_{i_1, \ldots, i_p} ((e^i \circ e^j)\omega)_{i_1 \ldots i_p} \Omega_{i_1 \ldots i_p}
= \frac{1}{2} \sum_{i_1, \ldots, i_p} \sum_{k=1}^p \delta_{ik} \omega_{i_1 \ldots j \ldots i_p} \Omega_{i_1 \ldots i_p} + \frac{1}{2} \sum_{i_1, \ldots, i_p} \sum_{k=1}^p \delta_{ji} \omega_{i_1 \ldots i_p} \Omega_{i_1 \ldots j \ldots i_p}
= \frac{1}{2} \sum_{k=1}^p \sum_{i_1, \ldots, i_{k-1}, i_{k+1}, \ldots, i_p} \left( \omega_{i_1 \ldots j \ldots i_p} \Omega_{i_1 \ldots i_p} + \omega_{i_1 \ldots i_p} \Omega_{i_1 \ldots j \ldots i_p} \right)
= \frac{p}{2} \sum_{i_2, \ldots, i_p} \left( \omega_{i_2 \ldots i_p}, \Omega_{j_2 \ldots i_p} + \omega_{j_2 \ldots i_p}, \Omega_{i_2 \ldots i_p} \right).
\]

This implies the first claim. Moreover, note that
\[
((e^k \circ e^l)\omega)_{i_2 \ldots i_p} = \frac{1}{2} \left( \delta_{ik} \omega_{i_2 \ldots i_p} + \delta_{il} \omega_{k_2 \ldots i_p} \right) + \frac{1}{2} \sum_{\alpha=2}^p \left( \delta_{k \alpha} \omega_{i_2 \ldots i_p} \Omega_{i_2 \ldots i_p} + \delta_{l \alpha} \omega_{i_2 \ldots i_p} \Omega_{i_2 \ldots i_p} \right).
\]

Thus,
\[
g((e^i \circ e^j)\omega, (e^k \circ e^l)\omega) = \frac{p}{4} \sum_{i_2, \ldots, i_p} \left( \delta_{jk} \omega_{i_2 \ldots i_p}, \omega_{i_2 \ldots i_p} + \delta_{jl} \omega_{i_2 \ldots i_p}, \omega_{k_2 \ldots i_p} \right)
+ \frac{p}{4} \sum_{i_2, \ldots, i_p} \sum_{\alpha=2}^p \left( \delta_{k \alpha} \omega_{i_2 \ldots i_p}, \omega_{j_2 \ldots i_p} + \delta_{l \alpha} \omega_{i_2 \ldots i_p}, \omega_{j_2 \ldots i_p} \right)
+ \text{both sums with } i \text{ and } j \text{ reversed}
= \frac{p}{4} \sum_{i_2, \ldots, i_p} \left( \delta_{jk} \omega_{i_2 \ldots i_p}, \omega_{i_2 \ldots i_p} + \delta_{jl} \omega_{i_2 \ldots i_p}, \omega_{k_2 \ldots i_p} \right)
+ \delta_{ik} \omega_{j_2 \ldots i_p}, \omega_{i_2 \ldots i_p} + \delta_{il} \omega_{j_2 \ldots i_p}, \omega_{k_2 \ldots i_p} \right)
+ \frac{p}{2} \sum_{\alpha=2}^p \sum_{i_2, \ldots, i_p} \left( \omega_{i_2 \ldots k \ldots i_p}, \omega_{j_2 \ldots i_p} + \omega_{i_2 \ldots i_p}, \omega_{j_2 \ldots k \ldots i_p} \right)
\[
\frac{p}{4} \sum_{i_2, \ldots, i_p} \left( \delta_{jk} \omega_{i_1i_2\ldots i_{p-1}} \omega_{i_2\ldots i_{p}} + \delta_{jl} \omega_{i_1i_2\ldots i_{p-1}} \omega_{ki_2\ldots i_{p}} + \delta_{ik} \omega_{j_1i_2\ldots i_{p-1}} \omega_{i_2\ldots i_{p}} + \delta_{il} \omega_{j_1i_2\ldots i_{p-1}} \omega_{ki_2\ldots i_{p}} \right) + \frac{p(p-1)}{2} \sum_{i_3, \ldots, i_{p}} \left( \omega_{i ki_3\ldots i_{p}} \omega_{jli_3\ldots i_{p}} + \omega_{i li_3\ldots i_{p}} \omega_{jki_3\ldots i_{p}} \right). \]

**Proposition 2.3.** Every \( p \)-form \( \omega \) satisfies

\[
g\left( \omega^{S^2}, \omega \otimes \text{Ric} \right) = p \sum_{i,j} \sum_{i_2, \ldots, i_{p}} R_{ij} \omega_{i_2\ldots i_{p}} \omega_{j\ldots i_{p}}.
\]

**Proof.** Due to Example 1.5, we have

\[
g\left( \omega^{S^2}, \omega \otimes \text{Ric} \right) = \sum_{i,j} g\left( (e^i \circ e^j) \omega, e^i \otimes e^j, \omega \otimes \text{Ric} \right) = \sum_{i,j} g\left( (e^i \circ e^j) \omega, \omega \right) R_{ij}
\]

and thus Proposition 2.2 implies the claim. \( \square \)

**Proposition 2.4.** If \( \omega \) is a \( p \)-form, then

\[
g\left( \overline{R} \left( \omega^{S^2} \right), \omega^{S^2} \right) = \frac{p}{2} \sum_{i,j} \sum_{i_2, \ldots, i_{p}} R_{ij} \omega_{i_2\ldots i_{p}} \omega_{j\ldots i_{p}} - \frac{3}{2} \frac{p(p-1)}{2} \sum_{i,j,k,l} \sum_{i_3, \ldots, i_{p}} R_{ijkl} \omega_{i_1i_2\ldots i_{p}} \omega_{kli_3\ldots i_{p}}
\]

and thus

\[
\frac{3}{2} g(\text{Ric}_L (\omega), \omega) = g\left( \overline{R} \left( \omega^{S^2} \right), \omega^{S^2} \right) + p \sum_{i,j} \sum_{i_2, \ldots, i_{p}} R_{ij} \omega_{i_2\ldots i_{p}} \omega_{j\ldots i_{p}}.
\]

**Proof.** Notice that

\[
\sum_{i,j,k,l} \sum_{i_3, \ldots, i_{p}} \omega_{i_1i_2\ldots i_{p}} \omega_{jli_3\ldots i_{p}} R_{ijkl} = \frac{1}{2} \sum_{i,j,k,l} \sum_{i_3, \ldots, i_{p}} \omega_{jili_3\ldots i_{p}} \omega_{ki_3\ldots i_{p}} R_{ijkl},
\]

since \( R_{ijkl} = -(R_{kijl} + R_{ikjl}) = R_{ilkj} + R_{ikjl} \) implies

\[
\sum_{i,j,k,l} \sum_{i_3, \ldots, i_{p}} \omega_{i_1i_2\ldots i_{p}} \omega_{jli_3\ldots i_{p}} R_{ijkl} = \sum_{i,j,k,l} \sum_{i_3, \ldots, i_{p}} \left( \omega_{i_1i_2\ldots i_{p}} \omega_{jli_3\ldots i_{p}} R_{ilkj} + \omega_{i_1i_2\ldots i_{p}} \omega_{jli_3\ldots i_{p}} R_{ikjl} \right)
\]

\[
= - \sum_{i,j,k,l} \sum_{i_3, \ldots, i_{p}} \omega_{i_1i_2\ldots i_{p}} \omega_{jli_3\ldots i_{p}} R_{ijkl}
\]

\[
+ \sum_{i,j,k,l} \sum_{i_3, \ldots, i_{p}} \omega_{jli_3\ldots i_{p}} \omega_{ki_3\ldots i_{p}} R_{ijkl}.
\]
Due to Example 1.5 and Proposition 2.2, we thus obtain

\[
g\left(R\left(\omega^{S^2}\right),\omega^{S^2}\right) = \sum_{i,j,k,l} g\left((e^i \circ e^j) \omega \otimes R(e^i \circ e^j), (e^k \circ e^l) \omega \otimes R(e^k \circ e^l)\right)
\]

\[
= \frac{1}{4} \sum_{i,j,k,l} g\left((e^i \circ e^j) \omega, (e^k \circ e^l) \omega\right)(R_{ki,jl} + R_{kj,il} + R_{li,jk} + R_{lj,ik})
\]

\[
= \frac{1}{2} \sum_{i,j,k,l} g\left((e^i \circ e^j) \omega, (e^k \circ e^l) \omega\right)(R_{ki,jl} + R_{kj,il})
\]

\[
= \frac{p}{8} \sum_{i,j,k,l} \sum_{1 \leq l_2, \ldots, l_p \leq n} \Delta \left(\omega_{l_2 \ldots, l_p} \omega_{l_2 \ldots, l_p} + \delta_{ik, \omega_{l_2 \ldots, l_p}} + \delta_{jl, \omega_{l_2 \ldots, l_p}} \right)(R_{ki,jl} + R_{kj,il})
\]

\[
= \frac{p(p-1)}{4} \sum_{i,j,k,l} \sum_{1 \leq l_2, \ldots, l_p \leq n} \Delta \left(\omega_{l_2 \ldots, l_p} \omega_{l_2 \ldots, l_p} \right)(R_{ki,jl} + R_{kj,il})
\]

\[
= \frac{p}{8} \sum_{i,j} \sum_{1 \leq l_2, \ldots, l_p \leq n} R_{ij} \omega_{l_2 \ldots, l_p} \omega_{l_2 \ldots, l_p} - \frac{p(p-1)}{2} \sum_{i,j,k,l} \sum_{1 \leq l_2, \ldots, l_p \leq n} \omega_{l_2 \ldots, l_p} \omega_{l_2 \ldots, l_p} R_{ijkl}
\]

where we used the initial observation for the last equality.

**Remark 2.5.** Every \((0,p)\)-tensor \(T\) satisfies

\[
\frac{3}{2} g(Ric_L(T), T) = g\left(R\left(T^{S^2}_0\right), T^{S^2}_0\right) + \frac{p(n-2p)}{n} \sum_{j,k} R_{jk} T_{ji_2 \ldots, l_p} T_{ki_2 \ldots, l_p} + \frac{p^2}{n^2} \text{scal} |T|^2
\]

\[
+ \sum_{1 \leq r \neq s \leq p} \sum_{l \in I^n} \sum_{i,j,k,l} T_{ij}^{r,s} T_{kl}^{r,s} (R_{ijkj} + R_{kjil}),
\]
where

\[ T^{rs} = \{(i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_{s-1}, i_{s+1}, \ldots, i_p) \in \{1, \ldots, n\}^{p-2}\}, \]

\[ T'_{ij} = T_{i_1 \ldots i_{r-1} i_{r+1} \ldots i_{s-1} j_{s+1} \ldots i_p}. \]

Note that the last term vanishes for \( p \)-forms. For symmetric \((0,2)\)-tensors, the last term reads

\[ 4 \sum_{i,j,k,l} T_{ij} T_{kl} R_{klij} = 8 \sum_{i<j} \sum_{k<l} T_{ij} T_{kl} (R_{klij} + R_{lijk}). \]

**Remark 2.6.** In order to recover the Bochner formula of Ogiue–Tachibana [26] from Proposition 2.1, set

\[ e^i \odot e^j = e^i \otimes e^j + e^j \otimes e^i - \frac{2}{n} \delta_{ij} \sum_{k=1}^{n} e^k \otimes e^k. \]

Note that the \( e^i \odot e^j \) are trace-free but not orthogonal. For every \( p \)-form \( \omega \), we have

\[ ((e^i \odot e^j) \omega)_{i_1 \ldots i_p} = \sum_{k=1}^{p} \left( \delta_{ik} \omega_{i_1 \ldots j \ldots i_p} + \delta_{ji} \omega_{i_1 \ldots i \ldots i_k} \right) - \frac{2}{n} \delta_{ij} \omega_{i_1 \ldots i_p}. \]

In [26], Ogiue–Tachibana observed that

\[ \frac{3}{2} g(\text{Ric}_L(\omega), \omega) = \frac{1}{4} \sum_{i,j,k,l} \sum_{i_1 \ldots i_p} ((e^i \odot e^j) \omega)_{i_1 \ldots i_p} ((e^j \odot e^k) \omega)_{i_1 \ldots i_p} R_{ijkl} \]

\[ + \frac{p(n - 2p)}{n} \sum_{j,k} \sum_{i_2 \ldots i_p} R_{jk} \omega_{j_2 \ldots i_p} \omega_{k_2 \ldots i_p} + \frac{p^2}{n^2} \text{scal} \, |\omega|^2. \]

In fact, it is straightforward to check that for any \((0, k)\)-tensor

\[ T^{S_0^2} = T^{S_0^2} - \frac{k}{n} T \otimes g = \frac{1}{2} \sum_{i,j} (e^i \odot e^j) T \otimes (e^i \bar{\circ} e^j) = \frac{1}{4} \sum_{i,j} (e^i \odot e^j) T \otimes (e^i \odot e^j). \]

With \( g(R(e^i \bar{\circ} e^j), e^k \bar{\circ} e^l) = \frac{1}{2} (R_{kijl} + R_{kjl}) \), we thus directly obtain

\[ g\left( \bar{R} \left( T^{S_0^2}, T^{S_0^2} \right), \right) = \frac{1}{4} \sum_{i,j,k,l} \sum_{i_1 \ldots i_p} g((e^i \odot e^j) T, (e^k \odot e^l) T) g\left( \bar{R} \left( e^i \bar{\circ} e^j \right), e^k \bar{\circ} e^l \right) \]

\[ = \frac{1}{4} \sum_{i,j,k,l} \sum_{i_1 \ldots i_p} ((e^i \odot e^j) T)_{i_1 \ldots i_p} ((e^l \odot e^k) T)_{i_1 \ldots i_p} R_{ijkl} \]

and together with Proposition 2.1, we recover the formula of Ogiue–Tachibana.
3 | THE WEIGHT PRINCIPLE

Let $R$ be an operator with eigenvalues $\lambda_i \in \mathbb{R}$. In this section, we introduce a calculus to estimate finite weighted sums $\sum_i \omega_i \lambda_i$ with weights $\omega_i \geq 0$. The main result is the weight principle 3.6. As an application, we estimate the curvature term in the Bochner formula for the curvature operator of the second kind.

**Definition 3.1.** Let $\omega_i \geq 0$ with $\Omega = \max_i \omega_i$ and set $S = \sum_i \omega_i$. We call $S$ the total weight and $\Omega$ the highest weight.

We will use the notation

$$[R, \Omega, S]$$

to denote any finite weighted sum $\sum_i \omega_i \lambda_i$ in terms of the eigenvalues $\lambda_i$ of the operator $R$ with highest weight $\Omega$ and total weight $S$. In particular, if $F(R)$ is a (geometric) quantity depending on $R$ and $R = R(R)$ is an operator, then we will write

$$F(R) \geq [R, \Omega, S]$$

provided $F(R)$ is bounded from below by a weighted sum in terms of the eigenvalues of $R$ with highest weight $\Omega$ and total weight $S$.

We write

$$[R, \Omega, S] \geq [R, \bar{\Omega}, \bar{S}]$$

provided for every sum $\sum_i \omega_i \lambda_i$ with $\sum_i \omega_i = S$ and $\max_i \omega_i = \Omega$, there is a sum $\sum_i \bar{\omega}_i \lambda_i$ with $\sum_i \bar{\omega}_i = \bar{S}$ and $\max_i \bar{\omega}_i = \bar{\Omega}$ such that

$$\sum_i \omega_i \lambda_i \geq \sum_i \bar{\omega}_i \lambda_i.$$

Similarly, if $c \in \mathbb{R}$, we write

$$[R, \Omega, S] \geq c$$

provided every sum $\sum_i \omega_i \lambda_i$ with $\sum_i \omega_i = S$ and $\max_i \omega_i = \Omega$ satisfies

$$\sum_i \omega_i \lambda_i \geq c.$$

**Example 3.2.** If $R$ denotes the curvature operator of the second kind of an $n$-dimensional Riemannian manifold, then

$$\text{scal} \geq \frac{2n}{n+2} \left[ R, 1, \frac{(n-1)(n+2)}{2} \right],$$

since $\text{scal} = \frac{2n}{n+2} \text{tr}(R)$ and $\dim S^2_0(TM) = \frac{(n-1)(n+2)}{2}$. 
**Lemma 3.3.** Let \([R, \Omega, S], [R, \tilde{\Omega}, \tilde{S}]\) denote weighted sums of eigenvalues of \(R\) with highest weights \(\Omega, \tilde{\Omega}\) and total weights \(S, \tilde{S}\), respectively.

(a) If \(c > 0\), then

\[ [R, c\Omega, cS] = c \cdot [R, \Omega, S]. \]

(b) If \(\Omega \leq \tilde{\Omega}\), then

\[ [R, \Omega, S] \geq [R, \tilde{\Omega}, S]. \]

(c)

\[ [R, \Omega, S] + [R, \tilde{\Omega}, \tilde{S}] \geq [R, \Omega + \tilde{\Omega}, S + \tilde{S}]. \]

**Proof.** Part (a) is immediate. For part (b), note that any sum \([R, \Omega, S] = \sum_i \omega_i \lambda_i\) is bounded from below by the corresponding sum with decreasing weights \(\omega_j \geq \omega_{j+1}\) and increasing \(\lambda_j \leq \lambda_{j+1}\). Increasing the highest weight in the rearranged sum while keeping the total weight fixed decreases the total sum. For part (c), note that the highest weight is bounded by \(\Omega + \tilde{\Omega}\) and its total weight is \(S + \tilde{S}\). Thus, the claim follows from (b).

**Lemma 3.4.** If \(\lambda_1 \leq \ldots \leq \lambda_N\) denote the eigenvalues of \(R\), then for \(m \in \mathbb{N}\)

\[ [R, \Omega, S] \geq (S - m\Omega)\lambda_{m+1} + \Omega \sum_{i=1}^{m} \lambda_i. \]

**Proof.** If \(\omega_i\) denote the corresponding weights with \(\Omega = \max \omega_i\) and \(S = \sum \omega_i\), then

\[ [R, \Omega, S] = \sum_{i=1}^{N} \omega_i \lambda_i \geq \sum_{i=1}^{m} \omega_i \lambda_i + \sum_{i=m+1}^{N} \omega_i \lambda_{m+1} = S\lambda_{m+1} + \sum_{i=1}^{m} \omega_i (\lambda_i - \lambda_{m+1}) \]

\[ \geq S\lambda_{m+1} + \Omega \sum_{i=1}^{m} (\lambda_i - \lambda_{m+1}) = (S - m\Omega)\lambda_{m+1} + \Omega \sum_{i=1}^{m} \lambda_i. \]

Recall that by definition \(R\) is \(k\)-nonnegative for some \(k \geq 1\) provided its eigenvalues \(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N\) satisfy \(\lambda_1 + \ldots + \lambda_{[k]} + (k - [k])\lambda_{[k]+1} \geq 0\).

**Proposition 3.5.** Let \(\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_N\) denote the eigenvalues of \(R\).

(a) \(R\) is \(k\)-nonnegative if and only if \([R, 1, k] \geq 0\).

(b) Let \(c \in \mathbb{R}\). Then, \(\lambda_1 + \ldots + \lambda_{[k]} + (k - [k])\lambda_{[k]+1} \geq c\) if and only if \([R, 1, k] \geq c\).

**Proof.**

(a) By definition we have \(\lambda_1 + \ldots + \lambda_{[k]} + (k - [k])\lambda_{[k]+1} \geq [R, 1, k]\). On the other hand, by Lemma 3.4, any sum in \([R, 1, k]\) is bounded from below by \(\lambda_1 + \ldots + \lambda_{[k]} + (k - [k])\lambda_{[k]+1}\).

(b) Follows as in (a).
Theorem 3.6 (Weight principle). Let $\mathcal{R}$ be an operator on a finite-dimensional vector space with real eigenvalues. Then,

(a) $[\mathcal{R}, \Omega, S] > 0$ if and only if $[\mathcal{R}, 1, \frac{S}{\Omega}] > 0$ if and only if $\mathcal{R}$ is $\frac{S}{\Omega}$-positive.

(b) $[\mathcal{R}, \Omega, S] \geq 0$ if and only if $[\mathcal{R}, 1, \frac{S}{\Omega}] \geq 0$ if and only if $\mathcal{R}$ is $\frac{S}{\Omega}$-nonnegative.

(c) Let $\kappa \in \mathbb{R}$. $[\mathcal{R}, \Omega, S] \geq \kappa S$ if and only if $[\mathcal{R}, 1, \frac{S}{\Omega}] \geq \kappa \frac{S}{\Omega}$.

(d) Let $k' < k$. If $\mathcal{R}$ is $k'$-nonnegative, then either $\mathcal{R}$ is $k$-positive or $1$-nonnegative.

Proof. Parts (a)–(c) are an immediate consequence of Lemma 3.3 and Proposition 3.5. For part (d) observe that if $\mathcal{R}$ is not $k$-positive, then $\lambda_{[k'+1]} = 0$. Thus, $k'$-nonnegativity implies $\lambda_1 = \cdots = \lambda_{[k'+1]} = 0$ and in particular $\lambda_i \geq 0$ for all $i$.

Lemma 3.7. Let $\omega$ be a $p$-form and $S \in S^2_0(V)$. Then,

(a) $|\omega_S^2|^2 = \frac{p(n-p)(n+2)}{n} |\omega|^2$.

(b) $|S\omega|^2 \leq \frac{p(n-p)}{n} |S|^2 |\omega|^2 = \frac{2}{n+2} |S|^2 |\omega_S^2|^2$.

Proof.

(a) We apply the formula for $g(\mathcal{R}(\omega_S^2), \omega_S^2)$ in the proof of Proposition 2.1 to the curvature tensor $R_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}$ of the round sphere. Hence, we have $\mathcal{R} = \text{id}$ on $S^2_0(V)$ and

$$|\omega_S^2|^2 = \frac{p}{2n} (-3(p-1)n + (n+4p)(n-1) - 2p(n-1)) |\omega|^2 = \frac{p}{2n} (n+2)(n-p) |\omega|^2.$$

(b) For $S \in S^2_0(V)$, there is an orthonormal basis $e_1, \ldots, e_n$ for $V$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $S(e_i) = \mu_i e_i$ for $i = 1, \ldots, n$. It follows that

$$(S\omega)_{i_1 \ldots i_p} = \left( \sum_{i \in [i_1, \ldots, i_p]} \mu_i \right) \omega_{i_1 \ldots i_p},$$

and

$$|S\omega|^2 = \sum_{I = [i_1, \ldots, i_p]} \left( \sum_{i \in I} \mu_i \right)^2 |\omega_{i_1 \ldots i_p}|^2.$$
yields Lagrange multipliers $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\left(\sum_{j \in \{j_1, \ldots, j_p\}} \mu_j \right)^2 - \alpha_1 \omega_{j_1 \ldots j_p} = 0,$$

$$2 \sum_{I=(i_1, \ldots, i_p)} \left(\sum_{i \in I} \mu_i \right) \chi_I(j) \left(\omega_{i_1 \ldots i_p} \right)^2 - 2\alpha_2 \mu_j - \alpha_3 = 0$$

for all $j, j_1, \ldots, j_p = 1, \ldots, n$, where for $I = (i_1, \ldots, i_p)$,

$$\chi_I(i) = \begin{cases} 1 & i \in I, \\ 0 & i \notin I \end{cases}$$

is the characteristic function.

In particular, if $\omega_{j_1 \ldots j_p} \neq 0$, then $\left(\sum_{j \in \{j_1, \ldots, j_p\}} \mu_j \right)^2 = \alpha_1$ is constant and thus $|S\omega|^2 = \alpha_1^2$. Therefore, it suffices to show that

$$\left(\sum_{i=1}^{p} \mu_i \right)^2 \leq \frac{p(n-p)}{n}$$

provided that

$$\sum_{i=1}^{n} \mu_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{n} \mu_i = 0.$$

This again yields Lagrange multipliers $\beta_1, \beta_2 \in \mathbb{R}$ such that

$$1 - 2\beta_1 \mu_j - \beta_2 = 0 \quad \text{for } 1 \leq j \leq p,$$

$$-2\beta_1 \mu_j - \beta_2 = 0 \quad \text{for } p+1 \leq j \leq n.$$

This implies $\mu_1 = \cdots = \mu_p$ and $\mu_{p+1} = \cdots = \mu_n$. Solving

$$p\mu_1^2 + (n-p)\mu_n^2 = 1,$$

$$p\mu_1 + (n-p)\mu_n = 0$$

yields $\mu_1^2 = \frac{n-p}{pn}$ and $\mu_n^2 = \frac{p}{(n-p)n}$ and thus

$$\left(\sum_{i=1}^{p} \mu_i \right)^2 = p^2 \mu_1^2 = \frac{p(n-p)}{n}$$

as claimed. \qed
Example 3.8. The estimate in Lemma 3.7(ii) is sharp for \( \omega = e^1 \wedge \ldots \wedge e^p \) and \( S \in S^2_0(V) \) given by \( S(e_i) = \mu_i e_i \) with

\[
\mu_1 = \ldots = \mu_p = \sqrt{\frac{n-p}{np}},
\]
\[
\mu_{p+1} = \ldots = \mu_n = -\sqrt{\frac{p}{(n-p)n}}.
\]

Corollary 3.9. If the curvature operator of the second kind is \( \frac{n+2}{2} \)-nonnegative, then

\[
g(\mathcal{R}(\omega S^2_0), \omega S^2_0) \geq 0.
\]

Proof. Recall that

\[
g(\mathcal{R}(\omega S^2_0), \omega S^2_0) = \sum_{\alpha} \lambda_{\alpha} |S_{\alpha} \omega|^2,
\]

where \( \{S_{\alpha}\} \) is an orthonormal eigenbasis for \( \mathcal{R} \) with corresponding eigenvalues \( \{\lambda_{\alpha}\} \). In particular, the total weight is \( |\omega S^2_0|^2 \) and highest weight is bounded by \( \frac{p(n-p)}{n} |\omega|^2 \) due to Lemma 3.7. Thus,

\[
g(\mathcal{R}(\omega S^2_0), \omega S^2_0) \geq \left[ \mathcal{R}, \frac{p(n-p)}{n}, \frac{p(n-p)(n+2)}{2} \right] \cdot |\omega|^2
\]

and the weight principle 3.6 implies the claim. \( \square \)

X. Li [20] observed that \( \text{Ric} \geq \text{scal} \geq \frac{n}{n(n+1)} > 0 \) provided the curvature operator of the second kind is \( n \)-nonnegative. An application of the Bochner technique hence yields Theorem A.

Proof of Theorem A. By passing to the orientation double cover if needed, we may assume that \((M, g)\) is oriented. Thus, we may assume \( p \leq \frac{n}{2} \) due to Poincaré duality. Proposition 2.1 and Corollary 3.9 hence show that \( g(\text{Ric}_L(\omega), \omega) \geq 0 \), and thus all harmonic forms are parallel.

If \( \omega \) is a parallel \( p \)-form for \( 1 \leq p \leq \frac{n}{2} \) and there is \( q \in M \) with \( \text{scal}_q > 0 \), then Proposition 2.1 implies \( \text{scal} |\omega|^2 = 0 \) at \( q \). In particular, \( \omega \) vanishes at \( q \) and consequently \( \omega = 0 \).

Otherwise, \((M, g)\) is scalar flat and hence \( \mathcal{R} = 0 \). In particular, \((M, g)\) is flat. \( \square \)

Proposition 3.10. The trace-free, symmetric (0,2)-tensors

\[
\phi_{ij} = \frac{1}{\sqrt{2}} \left( e^i \otimes e^j + e^j \otimes e^i \right), \quad 1 \leq i < j \leq n,
\]
\[
\psi_k = -\frac{1}{\sqrt{(n-k+1)(n-k)}} \left( -k e^k \otimes e^k + \sum_{l=k+1}^n e^l \otimes e^l \right), \quad k = 1, \ldots, n-1,
\]
form an orthonormal basis for \( S^2_0(V) \). Moreover, \( g(\mathcal{R}(\phi_{ij}), \phi_{ij}) = R_{ijij} \) and in particular

\[
\sum_{j=1, j \neq i}^{n} g(\mathcal{R}(\phi_{ij}), \phi_{ij}) = R_{ii},
\]

\[
\sum_{k=1}^{p} g(\mathcal{R}(\psi_k), \psi_k) = \frac{2}{n-p} \left( \sum_{k=1}^{p} R_{kk} - \sum_{1 \leq k < l \leq p} R_{kkll} \right) - \frac{p}{(n-p)n} \text{scal}.
\]

**Proof.** This is a straightforward computation. \( \square \)

**Lemma 3.11.** For an algebraic curvature tensor \( \mathcal{R} \), let \( \mathcal{R} \) denote the corresponding curvature operator of the second kind.

The Ricci tensor satisfies \( \text{Ric} \geq [\mathcal{R}, 1, (n-1)] \) and for \( p \geq 2 \), we have

\[
\sum_{i=1}^{p} R_{ii} \geq [\mathcal{R}, 2, p(n-1)]
\]

with respect to any orthonormal basis \( e_1, \ldots, e_n \) for \( V \).

**Proof.** Proposition 3.10 implies

\[
\sum_{i=1}^{p} R_{ii} = 2 \sum_{1 \leq i < j \leq p} g(\mathcal{R}(\phi_{ij}), \phi_{ij}) + \sum_{1 \leq i \leq p} \sum_{p+1 \leq j \leq n} g(\mathcal{R}(\phi_{ij}), \phi_{ij}).
\]

Since the \{\phi_{ij}\} are orthonormal, we obtain

\[
\sum_{i=1}^{p} R_{ii} \geq [\mathcal{R}, 2, 2 \cdot \frac{p(p-1)}{2} + p(n-p)] = [\mathcal{R}, 2, p(n-1)]. \square
\]

**Remark 3.12.** The same technique also yields that for \( p \geq 2 \), the Ricci tensor is \( p \)-nonnegative provided that any sum of \( \frac{p(n-1)}{2} \) sectional curvatures \( R_{ijij} \) is nonnegative. Indeed, note that

\[
\sum_{i=1}^{p} R_{ii} = \sum_{i=1}^{p} \sum_{j=1}^{n} R_{ijij} = 2 \sum_{1 \leq i < j \leq p} R_{ijij} + \sum_{1 \leq i \leq p} \sum_{p+1 \leq j \leq n} R_{ijij} = [R_{ijij}, 2, p(n-1)]
\]

and the weight principle 3.6 applies.

**Proposition 3.13.** If \( p \leq \frac{n}{2} \) and \( \omega \) is a \( p \)-form, then

\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) \geq \left[ \mathcal{R}, \frac{1}{n(n+2)}(n^2p - np^2 - 2np + 2n^2 + 4n - 8p), \frac{3}{2}p(n-p) \right] \cdot |\omega|^2.
\]
Proof. Due to Propositions 1.2 and 2.1 and Lemmas 3.3, 3.7, and 3.11, we have with respect to an orthonormal basis that diagonalizes the Ricci tensor

\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) = g\left( R\left( \omega^{S_0}, \omega^{S_0} \right) \right) + \frac{n - 2p}{n} \sum_{I_l = (i_1, \ldots, i_p)} \left( \sum_{i \in I} R_{ii} \right) \omega^2_I + \frac{p^2}{n^2} \text{scal} |\omega|^2 \\
\geq \left[ R, \frac{p(n - p)}{n} \right] \frac{p(n - p)}{n} \cdot \frac{2n}{2} \cdot |\omega|^2 + \frac{n - 2p}{n} \left[ R, 2, p(n - 1) \right] \cdot |\omega|^2 \\
+ \frac{p^2}{n^2} \cdot \frac{2n}{n + 2} \left[ R, 1, \frac{(n + 2)(n - 1)}{2} \right] \cdot |\omega|^2 \\
\geq \left[ R, \frac{1}{n} \left( p(n - p) + 2(n - 2p) + \frac{2p^2}{n + 2} \right) \right], \\
\frac{p}{n} \left( \frac{(n - p)}{2} + (n - 2p)(n - 1) + p(n - 1) \right) \cdot |\omega|^2.
\]

Note that the condition \( p \leq \frac{n}{2} \) ensures that the \((n - 2p)\) factor in the second term is nonnegative. □

By considering both \( \{\phi_{ij}\} \) and \( \{\psi_k\} \) from Proposition 3.10, we can refine Lemma 3.11.

Lemma 3.14. For an algebraic curvature tensor \( R \), let \( \mathcal{R} \) denote the curvature operator of the second kind.

With respect to any orthonormal basis, the Ricci tensor satisfies

\[
R_{11} \geq \frac{n - 1}{n + 1} [\mathcal{R}, 1, n] + \frac{1}{n(n + 1)} \text{scal}
\]

and for \( p \geq 2 \), we have

\[
\sum_{i=1}^{p} R_{ii} \geq \frac{n - p + 1}{n - p + 2} \left[ \mathcal{R}, 2, p(n - 1) \right] + \frac{p}{n(n - p + 2)} \text{scal}.
\]

Proof. Proposition 3.10 implies

\[
\sum_{1 \leq k < l \leq p} g(\mathcal{R}(\phi_{kl}), \phi_{kl}) + \sum_{1 \leq k \leq p \leq l \leq n} g(\mathcal{R}(\phi_{kl}), \phi_{kl}) + \sum_{k=1}^{p} g(\mathcal{R}(\psi_k), \psi_k) = \\
= \frac{n - p + 2}{n - p} \left( \sum_{k=1}^{p} R_{kk} - \sum_{1 \leq k < l \leq p} R_{kll} \right) - \frac{p}{(n - p)n} \text{scal}.
\]

Since \( \{\phi_{kl}\} \cup \{\psi_k\} \) is orthonormal, the sum is nonnegative provided that \( \mathcal{R} \) is \( \frac{6}{2}(2n - p + 1) \)-nonnegative.
Moreover, it follows that

\[
\sum_{i=1}^{p} R_{ii} = \frac{2(n-p+1)}{n-p+2} \sum_{1 \leq k \leq l \leq p} R_{kkl} + \frac{n-p}{n-p+2} \left( \sum_{1 \leq k \leq p} \sum_{p+1 \leq l \leq n} R_{kkl} + \sum_{k=1}^{p} g(\tilde{R}(\psi_k), \psi_k) \right)
\]

\[
+ \frac{p}{(n-p+2)n} \text{scal}.
\]

In particular, with regard to eigenvalues of \(R\), the terms in the first line have highest weight \(\frac{2(n-p+1)}{n-p+2}\) and total weight \(\frac{(n-p+1)p(n-1)}{n-p+2}\).

\[\square\]

**Proposition 3.15.** If \(p \leq \frac{n}{2}\) and \(\omega\) is a \(p\)-form, then

\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) \geq \left[ \mathcal{R}, \frac{1}{n(n+2)} (n^2p - np^2 - 2np + 2n^2 + 2n - 4p), \frac{3}{2} p(n-p) \right] \cdot |\omega|^2.
\]

**Proof.** The proof is analogous to the proof of Proposition 3.13. Instead of Lemma 3.11, one uses Lemma 3.14.  \(\square\)

**Remark 3.16.** A 1-form \(\omega\) in fact satisfies the slightly improved estimate

\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) \geq \left[ \mathcal{R}, \frac{2n-1}{n+2}, \frac{3(n-1)}{2} \right] \cdot |\omega|^2.
\]

**Proposition 3.17.** Let \(p \leq \frac{n}{2}\) and let \(\omega\) be a \(p\)-form. If \(\text{Ric} = \frac{\text{scal}}{n} g\), then

\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) \geq \frac{p(n-p)}{n} \left[ \mathcal{R}, \frac{n+4}{n+2}, \frac{3n}{2} \right] \cdot |\omega|^2.
\]

**Proof.** According to Proposition 2.1, we have

\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) = g\left( \mathcal{R} \left( \omega^\otimes_0, \omega^\otimes_0 \right) + \frac{p(n-p)}{n^2} \text{scal} |\omega|^2. \right.
\]

Thus, the weight principle 3.6, Lemma 3.7 and Proposition 1.2 yield

\[
\frac{3}{2} g(\text{Ric}_L(\omega), \omega) \geq \frac{p(n-p)}{n} \left[ \mathcal{R}, 1, \frac{n+2}{2} \right] \cdot |\omega|^2
\]

\[
+ \frac{p(n-p)}{n^2} \frac{2n}{n+2} \left[ \mathcal{R}, 1, \frac{(n-1)(n+2)}{2} \right] \cdot |\omega|^2
\]

\[
\geq \frac{p(n-p)}{n} \left[ \mathcal{R}, \frac{n+4}{n+2}, \frac{3n}{2} \right] \cdot |\omega|^2. \quad \square
\]

4 | PROOFS OF THE MAIN THEOREMS

In this section, we prove Theorems B–D. The proof of Theorem A was given after Corollary 3.9.

We conclude the section with an example of an algebraic, \((n+1)\)-positive curvature operator of
the second kind with negative Ricci curvatures, and the example of the rational homology sphere $SU(3)/SO(3)$.

**Proof of Theorem B.** Recall that $N = \frac{3n}{2} \frac{n+2}{n+4}$.

(c) The fact that all forms are parallel if the curvature operator of the second kind is $N$-nonnegative is a direct consequence of Proposition 3.17, the weight principle 3.6 and the Bochner technique as outlined at the beginning of Section 2.

(a) To obtain vanishing of the Betti numbers, let $\omega$ be a harmonic $p$-form and suppose that the curvature operator of the second kind is $N$-positive. Since $\omega$ is parallel, Proposition 3.17 yields

$$0 = \frac{3}{2} g(\text{Ric}_L(\omega), \omega) \geq \frac{p(n-p)}{n} \left[ R, \frac{n+4}{n+2}, \frac{3n}{2} \right] |\omega|^2.$$  

The weight principle 3.6 implies $\left[ R, \frac{n+4}{n+2}, \frac{3n}{2} \right] > 0$, and hence $\omega$ vanishes.

(b) If $R$ is $N'$-nonnegative for some $N' < N$, then by the weight principle 3.6(d), either $R$ is $N$-positive and $M$ is a rational homology sphere by part (a), or $R$ is 1-nonnegative and $(M, g)$ is either flat or a rational homology sphere by Theorem A.  \[\Box\]

**Proof of Theorem C.** The proof is analogous to the proof of Theorem B. Instead of Proposition 3.17, one uses 3.15.

**Proof of Theorem D.** The weight principle 3.6 and the estimate $\text{Ric} \geq \left[ R, 1, (n-1) \right]$ in Proposition 3.11 immediately imply $\text{Ric} \geq (n-1)\kappa$ provided that the average of the lowest $(n-1)$ eigenvalues of the curvature operator of the second kind is bounded from below by $\kappa$. The methods of Gallot and P. Li imply Theorem D, cf. [27, Theorem 1.9]. \[\Box\]

**Remark 4.1.** Note that Proposition 3.14 in fact provides a lower bound on Ricci curvature if the average of the lowest $n$ eigenvalues of $R$ is bounded from below by $\kappa$.

If the eigenvalues $\lambda_1 \leq \cdots \leq \lambda_N$ of the curvature operator of the second kind satisfy $\lambda_1 + \cdots + \lambda_m \geq m\kappa$, then $\lambda_j \geq \kappa$ for $j > m$ and thus

$$\text{scal} = \frac{2n}{n+2} \text{tr}(R) \geq \frac{2n}{n+2} \kappa \cdot \dim(S^2_0(TM)) = n(n-1)\kappa.$$  

In particular, if in addition $(M, g)$ is Einstein, then $\text{Ric} = \frac{\text{scal}}{n} g \geq (n-1)\kappa g$ and we obtain the estimation theorem corresponding to Theorem B from the weight principle 3.6 and the work of Gallot and P. Li as before.

The proof of the estimation theorem corresponding to Theorem C is analogous, provided a lower bound on the Ricci curvature is assumed explicitly, cf. Examples 4.2 and 4.4.

**Example 4.2.** Let $S = \frac{3}{2} p(n-p)$. Let

$$\Omega_{\text{pre}} = \frac{1}{n(n+2)} \left( n^2 p - np^2 - 2np + 2n^2 + 4n - 8p \right)$$
be the highest weight obtained with the preliminary estimate in Proposition 3.13 and let

$$\Omega = \frac{1}{n(n+2)}(n^2p - np^2 - 2np + 2n^2 + 2n - 4p)$$

be the highest weight in Proposition 3.15.

The difference of the highest weights is $\Omega^{\text{pre}} - \Omega = \frac{2(n-2p)}{n(n+2)}$.

Furthermore, the quotient $\frac{S}{\Omega}$ is increasing in $p$.

Note that we require $p \leq \frac{n}{2}$ in the estimates above due to the $(n - 2p)$ factor of the Ricci curvature term. Thus, we get the weakest curvature condition for $p = \frac{n}{2}$.

For $p = 2$, we obtain

$$\frac{S}{\Omega} = \frac{3n}{4} \frac{n^2 - 4}{n^2 - \frac{3}{2}n - 2}.$$

For $p = 4$ and $n \geq 4$, we have

$$\frac{S}{\Omega} = n \frac{n^2 - 2n - 8}{n^2 - \frac{11}{3}n - \frac{8}{3}} > n.$$

For $p = 5$ and $n \geq 5$, we have

$$\frac{S}{\Omega} = \frac{15n}{14} \frac{n^2 - 3n - 10}{n^2 - \frac{33}{7}n - \frac{20}{7}} > \frac{15n}{14}.$$

For $p = \frac{n}{2}$, we have

$$\frac{S}{\Omega} = \frac{3n}{2} \frac{n + 2}{n + 4}$$

as in the Einstein case. Furthermore, note that for any fixed $p$, we have

$$\lim_{n \to \infty} \frac{S}{n \cdot \Omega} = \frac{3p}{2(p + 2)}.$$

Recall that X. Li [20] proved the lower Ricci curvature bound $\text{Ric} \geq \frac{\text{scal}}{n(n+1)} \geq 0$, provided the curvature operator of the second kind is $n$-nonnegative. In contrast, Example 4.4 below exhibits an $(n+1)$-positive curvature operator of the second kind with negative Ricci curvatures.

In particular, for $p = 5, \ldots, \frac{n}{2}$, our curvature conditions do not imply nonnegative Ricci curvature, while we are still able to control the Betti numbers. For example, as a special case of Theorem C, we have the following:

**Corollary 4.3.** Let $(M, g)$ be a compact $n$-dimensional Riemannian manifold. Let $n \geq 14$ and $5 \leq p \leq n - 5$.

If the curvature operator of the second kind is $(n+1)$-nonnegative, then all harmonic $p$-forms are parallel.

If in addition $\text{scal} > 0$ at a point in $M$, then the $p$th Betti number $b_p(M, \mathbb{R})$ vanishes.
Example 4.4. In [20], X. Li observed that the curvature operator of the second kind of $S^1 \times S^{n-1}$ has the eigenvalues $-\frac{n-2}{n}$ with multiplicity one, 0 with multiplicity $n - 1$ and 1 with multiplicity $\frac{(n-2)(n+1)}{2}$. In particular, the curvature operator of the second kind is $(n + 1)$-positive, but not $n$-nonnegative. For small $\kappa < 0$, we obtain an algebraic, $(n + 1)$-positive curvature operator of the second kind with Ricci curvature $R_{11} < 0$ by adding the curvature tensor $\frac{\kappa}{2} g \otimes g$ of constant sectional curvature $\kappa$ to the curvature tensor of $S^1 \times S^{n-1}$.

Example 4.5. Consider the irreducible symmetric space $M = SU(3) / SO(3)$. As Wolf [29] observed, $M$ is a rational homology sphere. However, note that $H_2(M, \mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$.

The curvature operator $\mathcal{R} : \bigwedge^2 TM \to \bigwedge^2 TM$ has a 7-dimensional kernel and the nonzero eigenvalue $5/2$ with multiplicity 3. In particular, the Ricci tensor satisfies $\text{Ric} = 3g$.

The curvature operator of the second kind $\mathcal{R} : S^2_0(TM) \to S^2_0(TM)$ has eigenvalues $-3/2$ with multiplicity 5 and 2 with multiplicity 9. In particular, it is 9-positive but not 8-nonnegative.

ACKNOWLEDGEMENTS
JN acknowledges support by the Alexander von Humboldt Foundation through Gustav Holzegel’s Alexander von Humboldt Professorship endowed by the Federal Ministry of Education and Research. JN and MW are funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany’s Excellence Strategy EXC 2044–390685587, Mathematics Münster: Dynamics–Geometry–Structure.

Open access funding enabled and organized by Projekt DEAL.

JOURNAL INFORMATION
The Journal of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.

REFERENCES
1. M. Berger and D. Ebin, Some decompositions of the space of symmetric tensors on a Riemannian manifold, J. Differential Geometry 3 (1969), 379–392.
2. A. L. Besse, Einstein manifolds, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge, Band 10, Springer, Berlin, Heidelberg, 1987.
3. J.-P. Bourguignon and H. Karcher, Curvature operators: pinching estimates and geometric examples, Ann. Sci. École Norm. Sup. (4) 11 (1978), no. 1, 71–92.
4. S. Bochner, Vector fields and Ricci curvature, Bull. Amer. Math. Soc. 52 (1946), 776–797.
5. S. Brendle, A general convergence result for the Ricci flow, Duke Math. J. 145 (2008), 585–601.
6. S. Brendle, Einstein manifolds with nonnegative isotropic curvature are locally symmetric, Duke Math. J. 151 (2010), no. 1, 1–21.
7. S. Brendle and R. M. Schoen, Classification of manifolds with weakly $1/4$-pinched curvatures, Acta Math. 200 (2008), no. 1, 1–13.
8. S. Brendle and R. Schoen, Manifolds with $1/4$-pinched curvature are space forms, J. Amer. Math. Soc. 22 (2009), no. 1, 287–307.
9. C. Böhm and B. Wilking, Manifolds with positive curvature operators are space forms, Ann. of Math. (2) 167 (2008), 1079–1097.
10. X. Cao, M. J. Gursky, and H. Tran, Curvature of the second kind and a conjecture of Nishikawa, Math. Helv. 98 (2023), no. 1, 195–216.
11. H. Chen, *Pointwise \( \frac{1}{4} \)-pinched 4-manifolds*, Ann. Global Anal. Geom. 9 (1991), no. 2, 161–176.
12. S. Gallot, *Estimées de Sobolev quantitatives sur les variétés riemanniennes et applications*, C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 6, 375–377.
13. S. Gallot and D. Meyer, *Opérateur de courbure et laplacien des formes différentielles d’une variété riemannienne*, J. Math. Pures Appl. (9) 54 (1975), no. 3, 259–284.
14. R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), 255–306.
15. R. S. Hamilton, *Four-manifolds with positive curvature operator*, J. Differential Geom. 24 (1986), 153–179.
16. T. Kashiwada, *On the curvature operator of the second kind*, Natur. Sci. Rep. Ochanomizu Univ. 44 (1993), no. 2, 69–73.
17. N. Koiso, *A decomposition of the space \( \mathcal{M} \) of Riemannian metrics on a manifold*, Osaka Math. J. 16 (1979), no. 2, 423–429.
18. N. Koiso, *On the second derivative of the total scalar curvature*, Osaka Math. J. 16 (1979), no. 2, 413–421.
19. P. Li, *On the Sobolev constant and the \( p \)-spectrum of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. (4) 13 (1980), no. 4, 451–468.
20. X. Li, *Manifolds with nonnegative curvature operator of the second kind*, Commun. Contemp. Math. (2023), 2350003. https://doi.org/10.1142/S0219199723500037
21. D. Meyer, *Sur les variétés riemanniennes à opérateur de courbure positif*, C. R. Acad. Sci. Paris Sér. A-B 272 (1971), A482–A485.
22. J. Mikeš, V. Rovenski, and S. E. Stepanov, *An example of Lichnerowicz-type Laplacian*, Ann. Global Anal. Geom. 58 (2020), no. 1, 19–34.
23. S. Nishikawa, *On deformation of Riemannian metrics and manifolds with positive curvature operator*, Curvature and topology of Riemannian manifolds (Katata, 1985), Lecture Notes in Math., vol. 1201, Springer, Berlin, 1986, pp. 202–211.
24. J. Nienhaus, P. Petersen, M. Wink, and W. Wylie, *Holonomy restrictions from the curvature operator of the second kind*, Differential Geom. Appl. 88 (2023), no. 9, 102010.
25. L. Ni and B. Wu, *Complete manifolds with nonnegative curvature operator*, Proc. Amer. Math. Soc. 135 (2007), no. 9, 3021–3028.
26. K. Ogiue and S.-i. Tachibana, *Les variétés riemanniennes dont l’opérateur de courbure restreint est positif sont des sphères d’homologie réelle*, C. R. Acad. Sci. Paris Sér. A-B 289 (1979), no. 1, A29–A30.
27. P. Petersen and M. Wink, *New curvature conditions for the Bochner technique*, Invent. Math. 224 (2021), no. 1, 33–54.
28. I. G. Shandra, S. E. Stepanov, and J. Mikeš, *On higher-order Codazzi tensors on complete Riemannian manifolds*, Ann. Global Anal. Geom. 56 (2019), no. 3, 429–442.
29. J. Wolf, *Symmetric spaces which are real cohomology spheres*, J. Differential Geometry 3 (1969), 59–68.