To the nonlinear quantum mechanics

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Abstract

The Schrödinger equation with the nonlinear term $-b(\ln |\Psi|^2)\Psi$ is derived by the natural generalization of the hydrodynamical model of quantum mechanics. The nonlinear term appears to be logically necessary because it enables explanation of the classical limit of the wave function, the collaps of the wave function and solves the Schrödinger cat paradox.
1 Introduction

Many authors have suggested that the quantum mechanics based on linear Schrödinger equation is only an approximation of some more nonlinear theory with the nonlinear Schrödinger equation. The motivation for considering the nonlinear equations is to get some more nonstandard solution in order to get the better understanding of the synergism of wave and particle.

The ambitious program to create nonlinear wave mechanics was elaborated by de Broglie [1] and his group. Bialynicki-Birula and Mycielski [2] considered the generalized Schrödinger equation with the additional term $F(|\Psi|^2)\Psi$ where $F$ is some arbitrary function which they later specified to $-b(\ln |\Psi|^2)$, $b > 0$. The nonlinear term was selected by assuming the factorization of the wave function for the composed system.

The most attractive feature of the logarithmic nonlinearity is the existence of the lower energy bound and validity of Planck’s relation $E = \hbar \omega$. At the same time the Born interpretation of the wave function cannot be changed. In this theory the estimation of $b$ was given by the relation $b < 4 \times 10^{-10} eV$ following from the agreement between theory and the observed $2S - 2P$ Lamb shift in hydrogen. This implies an upper bound to the electron soliton spatial width of 10 $\mu$m.

Shimony [3] proposed an experiment which is based on idea that a phase shift occurs when an absorber is moved from one point to another along the path of one of the coherent split beams in a neutron interferometer. In case of the logarithmic nonlinearity Shull et al. [4] performed the experiment with a two-crystal interferometer. They searched for a phase shift when an attenuator was moved along the neutron propagation direction in one arm of the interferometer. A sheet of Cd, 0.086 mm thick, was used for the absorber. They obtained the upper bound on $b$ of $3.4 \times 10^{-13} eV$ which is more than three orders of magnitude smaller than the bound estimated by Bialynicky-Birula and Mycielski [2].

The best upper limit on $b$ has been reported by Gähler, Klein and Zeilinger [5] who has searched for variations in the free space propagation of neutrons. 20 $\AA$ neutrons were diffracted from an abrupt highly absorbing knife edge at the object position. By comparing the experimental results with the solution to the ordinary Schrödinger equation they were able to get the limit $b < 3 \times 10^{-15} eV$, which corresponds to an electron soliton width of 3 mm. The similar results was obtained by the same group from diffraction a 100 $\mu$m boron wire.

To our knowledge the Mössbauer effect was not used to determine the constant $b$ although this effect allows to measure energy losses smaller than $10^{-15} eV$. Similarly the Josephson effect has been not applied for the determination of the constant $b$.

We see that the constant $b$ is very small, nevertheless we cannot it neglect a priori, because we do not know its role in the future physics. The corresponding analogon is the Planck constant which is also very small, however, it plays the fundamental role in physics.

The goal of this article is to give the new derivation of the logarithmic nonlinearity, to find the solution of the nonlinear Schrödinger equation of the one-dimensional case and to show that in the mass limit $m \to \infty$ we get exactly the delta-function behavior of the probability of finding the particle at point $x$. It means that there exists the classical motion of a particle with sufficient big mass. The nonlinearity of the Schrödinger equation also solves the collapses of the wave function and the Schrödinger cat paradox. We will start from
the hydrodynamical formulation of quantum mechanics. The mathematical generalization of the Euler hydrodynamical equations leads automatically to the logarithmic term with $b > 0$.

2 The derivation of the nonlinear Schrödinger equation

We respect here the so called Dirac heuristical principle [6] according to which it is useful to postulate some mathematical requirement in order to get the true information about nature. While the mathematical assumption is intuitive, the consequences have the physical interpretation or in other words they are physically meaningful. In derivation of the logarithmic nonlinearity we use just the Dirac method.

According to Madelung [7], Bohm and Vigier [8], Wilhelm [9], Rosen [10] and others, the original Schrödinger equation can be transformed into the hydrodynamical system of equations by using the so called Madelung ansatz:

$$\Psi = \sqrt{n} e^{iS}$$,

(1)

where $n$ is interpreted as the density of particles and $S$ is the classical action for $\hbar \to 0$. The mass density is defined by relation $\rho = nm$ where $m$ is mass of a particle.

It is well known that after insertion of the relation (1) into the original Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi,$$

(2)

where $V$ is the potential energy, we get, after separating the real and imaginary parts, the following system of equations:

$$\frac{\partial S}{\partial t} - \frac{1}{2m} (\nabla S)^2 + V = \frac{\hbar^2}{2m} \Delta \sqrt{n}$$

(3)

$$\frac{\partial n}{\partial t} + \text{div}(n \mathbf{v}) = 0$$

(4)

with

$$\mathbf{v} = \frac{\nabla S}{m}.$$  

(5)

Equation (3) is the Hamilton-Jacobi equation with the additional term

$$V_q = \frac{\hbar^2}{2m} \Delta \sqrt{n},$$

(6)

which is called the quantum Bohm potential and equation (4) is the continuity equation.

After application of operator $\nabla$ on eq. (3), it can be cast into the Euler hydrodynamical equation of the form:

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{m} \nabla (V + V_q).$$

(7)

It is evident that this equation is from the hydrodynamical point of view incomplete as a consequence of the missing term $-\rho^{-1} \nabla p$ where $p$ is hydrodynamical pressure. We use
here this fact just as the crucial point for derivation of the nonlinear Schrödinger equation. We complete the equation (7) by adding the pressure term and in such a way we get the total Euler equation in the form:

\[
m \left( \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla (V + V_q) - \nabla F, \tag{8}
\]

where

\[
\nabla F = \frac{1}{n} \nabla p. \tag{9}
\]

The equation (8) can be obtained by the Madelung procedure from the following extended Schrödinger equation

\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi + F \Psi \tag{10}
\]

on the assumption that it is possible to determine \( F \) in term of the wave function. From the vector analysis follows that the necessary condition of the existence of \( F \) as the solution of the equation (9) is \( \text{rot} \text{grad} f = 0 \), or,

\[
\text{rot}(n^{-1} \nabla p) = 0, \tag{11}
\]

which enables to take the linear solution in the form

\[
p = -bn = -b |\Psi|^2, \tag{12}
\]

where \( b \) is some arbitrary constant. We do not consider the more general solution of eq. (11). Then, from eq. (8) i.e. \( \text{grad} F = \mathbf{a} \) we have:

\[
F = \int a_i dx_i = -b \int \frac{1}{n} dn = -b \ln |\Psi|^2, \tag{13}
\]

where we have omitted the additive constant which plays no substantial role in the Schrödinger equation.

Now, we can write the generalized Schrödinger equation which correspond to the complete Euler equation (8) in the following form:

\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi - b(\ln |\Psi|^2) \Psi. \tag{14}
\]

Let us approach the solving the equation (14).

### 3 The soliton-wave solution of the nonlinear Schrödinger equation

Let be \( c, \text{Im} c = 0, v, k, \omega \) some parameters and let us insert function

\[
\Psi(x, t) = cG(x - vt)e^{ikx - i\omega t} \tag{15}
\]
into the one-dimensional equation (14) with $V = 0$. Putting the imaginary part of the new equation to zero, we get

$$v = \frac{\hbar k}{m}$$

and for function $G$ we get the following nonlinear equation (symbol ' denotes derivation with respect to $\xi = x - vt$):

$$G'' + AG + B(\ln G)G = 0,$$

where

$$A = \frac{2m}{\hbar} \omega - k^2 + \frac{2m}{\hbar^2} b \ln c^2$$

$$B = \frac{4mb}{\hbar^2}.$$  

After multiplication of eq. (17) by $G'$ we get:

$$\frac{1}{2} [G'^2]' + \frac{A}{2} [G^2]' + B \left[ \frac{G^2}{2} \ln G - \frac{G^2}{4} \right]' = 0,$$

or, after integration

$$G'^2 = -AG^2 - BG^2 \ln G + \frac{B}{2} G^2 + \text{const.}.$$  

If we choose the solution in such a way that $G(\infty) = 0$ and $G'(\infty) = 0$, we get $\text{const.} = 0$ and after elementary operations we get the following differential equation to be solved:

$$\frac{dG}{G\sqrt{a - B \ln G}} = d\xi,$$

where

$$a = \frac{B}{2} - A.$$  

Equation (22) can be solved by the elementary integration and the result is

$$G = e^{\frac{a}{2} \xi} e^{-\frac{B}{4}(\xi+d)^2},$$

where $d$ is some constant.

The corresponding soliton-wave function is evidently in the one-dimensional free particle case of the form:

$$\Psi(x, t) = ce^{\frac{a}{2} \xi} e^{-\frac{B}{4}(x-\omega t+d)^2} e^{ikx-i\omega t}.$$  

4 Normalization and the classical limit

It is not necessary to change the standard probability interpretation of the wave function. It means that the normalization condition in our one-dimensional case is

$$\int_{-\infty}^{\infty} \Psi^* \Psi \, dx = 1.$$  

5
Using the Gauss integral
\[ \int_0^\infty e^{-\lambda^2 x^2} dx = \frac{\sqrt{\pi}}{2\lambda}, \]  
(27)
we get with \( \lambda = \left( \frac{B}{2} \right)^{\frac{1}{2}} \)
\[ c^2 e^{\frac{2\pi}{\lambda}} = \left( \frac{B}{2\pi} \right)^{\frac{1}{2}} \]  
(28)
and the density probability \( \Psi^* \Psi = \delta_m(\xi) \) is of the form (with \( d = 0 \)):
\[ \delta_m(\xi) = \sqrt{\frac{m\alpha}{\pi}} e^{-\alpha m\xi^2} \; ; \; \alpha = \frac{2b}{\hbar^2}. \]  
(29)
It may be easy to see that \( \delta_m(\xi) \) is the delta-generating function and for \( m \to \infty \) is just the Dirac \( \delta \)-function.
It means that the motion of a particle with sufficiently big mass \( m \) is strongly localized and in other words it means that the motion of this particle is the classical one. Such behaviour of a particle cannot be obtained in the standard quantum mechanics because the plane wave
\[ e^{i k x - i \omega t} \]  
(30)
corresponds to the free particle with no possibility of localization for \( m \to \infty \).
Let us still remark that coefficient \( c^2 \) is real and positive number because it is a result of the solution of equation (28) which can be transformed into equation \( (x = c^2) \)
\[ x^{1-r} = \text{constant}. \]  
(31)

5 The collapse of the wave function

The nonlinear quantum mechanics explains by the easy way the collapse or the reduction of the wave function. Let us suppose that an electron is impinging on the screen. It means in other words that it is captured by a system of atoms in the screen. However, the system of a great amount of particles has mass \( M \) which is enormous in comparison with the mass of electron. It means that the electron together with the system of particles have mass \( M + m \) which is practically sufficiently big in order to get the soliton-wave solution of the form which describes the strong localization of the system consisting of electron and surrounding particles. The classical analogon of this situation is obviously the ballistic pendulum. Such localization is not possible in the standard quantum mechanics of the known textbooks. Therefore the nonlinear quantum mechanics of the above type with the logarithmic nonlinearity solves one of the old problems of the standard quantum mechanics.

6 The principle of superposition and the Schrödinger cat

If \( \varphi_1 \) and \( \varphi_2 \) are two different solution of the nonlinear Schrödinger equation then the linear combination \( \varphi = a\varphi_1 + b\varphi_2 \) where \( a \) and \( b \) are the arbitrary constants is not the solution of the same equation because of its nonlinearity. In other words the original
principle of superposition of the standard quantum mechanics is broken. The consequence of the breaking of the principle of superposition is the resolution of the Schrödinger cat paradox.

In the cat paradox [11], Schrödinger imagined the arrangement in which a cat is confined to a box which contains a lethal device that may be either triggered or left passive according to whether radioactive nucleus decays or fails do decay. If the radioactive decay takes place, which he assumed would happen half the time, a hammer would strike a vial of cyanide and cat would be dispatched. It means that the quantum signal is amplified in order to be possible to move by some equipment the hammer.

In the mathematical form, to the microstate $\varphi_a$ corresponds the macrostate $\varphi_A$ where the microstate is the decay of the nucleus and the macrostate is the striking hammer a vial cyanide. If to the nondecay of the nucleus corresponds the state $\varphi_b$ then according to Schrödinger the corresponding macrostate is $\varphi_B$. In the standard quantum mechanics it is possible the superposition $\varphi_a + \varphi_b$. To this state corresponds the superposition $\varphi_A + \varphi_B$, which we can interpreted as the cat is partly dead and partly alive, which is impossible because the vial of cyanide cannot be partly broken and partly unbroken.

The resolution of this contradiction in the nonlinear quantum mechanics with the logarithmic nonlinearity is that the superposition of the states is not solution of the nonlinear Schrödinger equation and therefore the vial is or broken or not. The answer is unambiguous.

7 Discussion

We have seen that the introduction of the logarithmic nonlinearity in the Schrödinger equation was logically supported by the fact that the nonlinear Schrödinger equation gives results which are physically meaningful. We have obtained the correct mass limit of the wave function and explained the collapse of it. The Schrödinger cat paradox was also explained.

The further strong point of the nonlinear Schrödinger equation (14) is the result (16) which is equivalent to the famous de Broglie relation

$$\lambda = \frac{h}{p}$$

(32)

because of $\lambda = 2\pi/k = 2\pi(h/mv) = 2\pi(h/2\pi)(1/p)$ and it means that de Broglie relation is involved in this form of the nonlinear quantum mechanics.

The nonlinear equation (14) has also the normalized plane-wave solution

$$\Psi(x, t) = \frac{1}{\sqrt{2\pi}} e^{ikx-i\omega t}.$$  

(33)

After insertion of eq. (33) into eq. (14), we get the following dispersion relation:

$$\hbar \omega = \hbar^2 k^2 \frac{1}{2m} + b \ln(2\pi),$$

(34)

from which the relations follows:

$$\hbar \omega = b \ln(2\pi); \quad k = 0$$

(35)
\[ k = \pm i \sqrt{\frac{2m}{\hbar^2} \ln(2\pi)}; \quad \omega = 0. \] (36)

It is no easy to give the physical interpretation of eqs. (35) and (36) and so we cannot say that the plane-solution of the nonlinear Schrödinger equation is physically meaningful. Only the soliton-wave solution of the nonlinear Schrödinger equation can be taken as relevant. Only this solution is suitable for the physical verification. The possible new tests of the nonlinear quantum mechanics are discussed in the author article [12].

The generalization to the motion of particle in the electromagnetic field with potentials \( \varphi(x, t) \) and \( \mathbf{A}(x, t) \) can be performed by the standard transformation

\[ \frac{\hbar}{i} \nabla \rightarrow \frac{\hbar}{i} \nabla - \left( \frac{e}{c} \right) \mathbf{A}(x, t) \] (37)

and adding the scalar potential energy \( e\varphi(x, t) \) in the Schrödinger equation for the free particles. According to [2] the solution of the equation in this case can be taken in the form

\[ \Psi(x, t) = e^{iS}G(x - u(t)), \] (38)

where function \( G \) is necessary to determine. In the similar form the problem was yet solved [13].

Kamesberger and Zeilinger [14] have given the numerical solution of the original Schrödinger equation and this equation with the nonlinear term \(-b(\ln|\Psi|^2)\Psi\) in order to visualize the spreading of the diffractive waves. When comparing the evolution patterns of the nonlinear case with the linear one, one notices that the maxima are more pronounced in the nonlinear solution. It can be understood as a mechanism compressing the wave maxima spatially. In the quantitative comparison of the both cases this enhacement of the maxima and minima can be seen very clearly.

Although we have given reasons for the introducing of the nonlinear Schrödinger equation it is obvious that only the crucial experiments can establish the physical and not only logical necessity of such equation. In case that the nonlinear Schrödinger equation will be confirmed by experiment, then it can be expected that it will influence other parts of theoretical physics.

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