FRÉCHET DIFFERENTIABILITY IN FRÉCHET SPACES, AND DIFFERENTIAL EQUATIONS WITH UNBOUNDED VARIABLE DELAY

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Abstract. We introduce and discuss Fréchet differentiability for maps between Fréchet spaces. For delay differential equations \( x'(t) = f(x_t) \) we construct a continuous semiflow of continuously differentiable solution operators \( x_0 \mapsto x_t, \ t \geq 0 \), on submanifolds of the Fréchet space \( C^1((-\infty, 0], \mathbb{R}^n) \), and establish local invariant manifolds at stationary points by means of transversality and embedding properties. The results apply to examples with unbounded but locally bounded delay.

MSC 2010: 34K05, 37L05

Keywords: Fréchet space, Fréchet differentiability, delay differential equation, unbounded delay, semiflow, invariant manifolds

1. Introduction

Consider an autonomous delay differential equation

\[
x'(t) = f(x_t)
\]

with \( f: U \to \mathbb{R}^n \) defined on a set of maps \((-\infty, 0] \to \mathbb{R}^n\), and the segment, or history, \( x_t \) of the solution \( x \) at \( t \) defined by \( x_t(s) = x(t+s) \) for all \( s \leq 0 \). A solution on some interval \([t_0, t_e], t_0 < t_e \leq \infty\), is a map \( x: (-\infty, t_e) \to \mathbb{R}^n \) with \( x_t \in U \) for all \( t \in [t_0, t_e) \) so that the restriction of \( x \) to \([t_0, t_e)\) is differentiable and Eq. (1.1) holds on this interval. Solutions on the whole real line are defined accordingly. A toy example which can be written in the form (1.1) is the equation

\[
x'(t) = h(x(t-r)), \quad r = d(x(t))
\]

with functions \( h: \mathbb{R} \to \mathbb{R} \) and \( d: \mathbb{R} \to [0, \infty) \). Other examples arise from pantograph equations

\[
x'(t) = a x(\lambda t) + b x(t)
\]

with constants \( a \in \mathbb{C}, b \in \mathbb{R} \) and \( 0 < \lambda < 1 \), and from Volterra integro-differential equations

\[
x'(t) = \int_0^t k(t,s)h(x(s))ds
\]

with \( k: \mathbb{R}^{n \times n} \to \mathbb{R} \) and \( h: \mathbb{R}^n \to \mathbb{R}^n \) continuous. Eq. (1.3) is linear, and both equations (1.3) and (1.4) are non-autonomous. We shall come back to them in Section 9 below.

\[\text{Date: January 26, 2018.}\]
Building a theory of Eq. (1.1) which (a) covers examples with state-dependent delay like Eq. (1.2) and (b) results in solution operators \( x_0 \mapsto x_t, \ t \geq 0 \), which are continuously differentiable begins with the search for a suitable state space. For equations with bounded delay the basic steps of a solution theory were made in [18], starting from the observation that the domain of the functional on the right hand side of the differential equation must consist of maps which are continuously differentiable, and not merely continuous as in the by-now well established theory of retarded functional differential equations [5, 2]. Accordingly the functional \( f \) in Eq. (1.1) above should be defined on a subset \( U \) of the vector space \( C^1 = C^1((-\infty, 0], \mathbb{R}^n) \) of continuously differentiable maps \( (-\infty, 0] \to \mathbb{R}^n \). Linearization as in [18] suggests that in the new theory autonomous linear equations with constant delay, like for example,

\[
x'(t) = -\alpha x(t-1)
\]

will appear, which have solutions on \( \mathbb{R} \) with arbitrarily large exponential growth at \(-\infty\). In order not to lose such solutions we stay with the full space \( C^1 \) and work with the topology of locally uniform convergence of maps and their derivatives, which makes \( C^1 \) a Fréchet space.

In [20] we saw that under mild smoothness hypotheses on \( f \), which hold in examples with state-dependent delay, the set

\[
X_f = \{ \phi \in U : \phi'(0) = f(\phi) \}
\]

is a continuously differentiable submanifold of codimension \( n \) in \( C^1 \). Notice that \( X_f \) consists of the segments \( x_t, 0 \leq t < t_e \), of all continuously differentiable solutions \( x : (-\infty, t_e) \to \mathbb{R}^n \) on \( [0, t_e), 0 < t_e \leq \infty \), of Eq. (1.1). It is shown in [20] that these solutions constitute a continuous semiflow \( (t, x_0) \mapsto x_t \) on \( X_f \), with continuously differentiable solution operators \( x_0 \mapsto x_t, t \geq 0 \). Here continuous differentiability is understood in the sense of Michal [13] and Bastiani [1], which means for a continuous map \( f : V \supset U \to W, V \) and \( W \) topological vector spaces and \( U \subset V \) open, that all directional derivatives

\[
Df(u)v = \lim_{\theta \neq t \to 0} \frac{1}{t} (f(u + tv) - f(u))
\]

exist and that the map

\[
U \times V \ni (u, v) \mapsto Df(u)v \in W
\]

is continuous. Let us briefly speak of \( C^1_{MB} \)-smoothness.

It is convenient to call the set \( X_f \) the solution manifold associated with the map \( f \).

The mild hypotheses on \( f \) mentioned above are that \( f \) is \( C^1_{MB} \)-smooth and that

(e) each derivative \( Df(\phi) : C^1 \to \mathbb{R}^n, \phi \in U \), has a linear extension \( D_e f(\phi) : C \to \mathbb{R}^n \), with the map

\[
U \times C \ni (\phi, \chi) \mapsto D_e f(\phi) \chi \in \mathbb{R}^n
\]

being continuous.

Here \( C \) is the Fréchet space of continuous maps \( (-\infty, 0] \to \mathbb{R}^n \) with the topology of locally uniform convergence. Property (e) is closely related to the earlier notion of being almost Fréchet differentiable from [12], for maps on a Banach space of continuous functions.
An inspection of examples of differential equations with state-dependent delay for which the map \( f \) in Eq. (1.1) is \( C_{MB}^1 \)-smooth reveals that in these examples \( f \) is in fact better, namely, that it is continuously differentiable in the sense of the following definition.

**Definition 1.1.** A continuous map \( f : V \supset U \to W, \) \( V \) and \( W \) topological vector spaces and \( U \subset V \) open, is said to be \( C^1_F \)-smooth if all directional derivatives exist, if each map \( Df(u) : V \to W, \) \( u \in U, \) is linear and continuous, and if the map \( Df : U \ni u \mapsto Df(u) \in L_c(V,W) \) is continuous with respect to the topology \( \beta \) of uniform convergence on bounded sets, on the vector space \( L_c(V,W) \) of continuous linear maps \( V \to W. \)

The letter \( F \) in the symbol \( C^1_F \) stands for Fréchet because in case \( V \) and \( W \) are Banach spaces \( C^1_F \)-smoothness is equivalent to the familiar notion of continuous differentiability with Fréchet derivatives, see e.g. Proposition 3.4 below. In case \( V \) and \( W \) are Fréchet spaces \( C^1_F \)-smoothness is equivalent to \( C_{MB}^1 \)-smoothness combined with the continuity of the derivative with respect to the topology \( \beta \) on \( L_c(V,W) \), see e.g. Proposition 3.2 below.

It seems that \( C^1_F \)-smoothness of maps in Fréchet spaces which are not Banach spaces has not attracted much attention, compared to \( C_{MB}^1 \)-smoothness and further notions of smoothness [12]. For possible reasons, see [1] Chapter II, Section 3.

In any case, for the study of Eq. (1.1) the notion of \( C^1_F \)-smoothness is useful and yields, of course, slightly stronger results, compared to the theory based on \( C_{MB}^1 \)-smoothness in [20] [21]. The present report shows how to obtain solution manifolds, solution operators, and local invariant manifolds at stationary points, all of them \( C^1_F \)-smooth, and discusses Eqs. (1.2)-(1.4) as examples. We mention in this context that we do not touch upon higher order derivatives - in [11] it is shown that solution manifolds are in general not better than \( C^1_F \)-smooth. The same holds for infinite-dimensional local invariant manifolds in the solution manifold, whereas finite-dimensional local invariant manifolds may be \( k \) times continuously differentiable, \( k \in \mathbb{N} \), under appropriate hypotheses on the map \( f \) in Eq. (1.1) [9].

The present paper is divided into 3 parts. Part I with Sections 2-8 is about \( C^1_F \)-maps in general. Sections 2-4 collect what we need to know about \( C^1_F \)-maps, beginning in Section 2 with the topology \( \beta \) on \( L_c(V,W) \) for topological spaces \( V,W \). Section 3 discusses \( C^1_F \)-smoothness for maps between Fréchet spaces, and its relations to \( C_{MB}^1 \)-smoothness, and provides elements of calculus, including the chain rule for \( C^1_F \)-maps. In order to keep Section 3 short we make extensive use of [6], Part I on \( C_{MB}^1 \)-smoothness. Section 4 is about \( C^1_F \)-submanifolds of Fréchet spaces. Sections 5-7 contain a uniform contraction principle, an implicit function theorem, and simple transversality- and embedding results which yield \( C^1_F \)-submanifolds of finite dimension or finite codimension. All of these results are familiar in the Banach space case, and most of them are well-known also in the \( C_{MB}^1 \)-setting [3] [4] [20] [21].

In Section 8 examples illustrate the difference between \( C_{MB}^1 \)-smoothness and \( C^1_F \)-smoothness. None of them is related to a delay differential equation.

Part II with Sections 9-12 is about the construction of the semiflow on the solution manifold of Eq. (1.1) for \( C^1_F \)-maps \( f : C^1 \supset U \to \mathbb{R}^n \) which have property (e). In Section 9 these hypotheses are verified for maps \( f \) related to the examples (1.2)-(1.4). Notice that for the Volterra equation (1.4) the associated \( C^1_F \)-map \( f \) is defined on the big space \( C \supset C^1 \). - Proposition 9.5 guarantees that the set \( X_f \) is indeed a
bounded sets, compact sets are bounded. A set \( A \)
Recall that a subset \( B \)
all \( z \)
∈ \( C \)
For basic facts about topological vector spaces see [14]. The vector space of continuous linear maps on \( V \) is denoted by \( L_c(V,W) \).

Recall that a subset \( B \subset T \) of a topological vector space over the field \( \mathbb{K} \), \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \), is bounded if for every neighbourhood \( N \) of \( 0 \in T \) there exists a real \( r_N \geq 0 \) with \( B \subset rN \) for all reals \( r \geq r_N \). The points of convergent sequences form bounded sets, compact sets are bounded. A set \( A \subset T \) is balanced if \( zA \subset A \) for all \( z \in \mathbb{K} \) with \( |z| \leq 1 \). If \( A \) is balanced and \( |z| \geq 1 \) then \( A \subset zA \).

Part III with Sections 13-17 is based on [21]. We explain how to modify constructions in [21], in order to obtain local invariant manifolds at stationary points of the semiflow which are \( C^l \)-smooth, by means of the transversality- and embedding results from Section 7. An important ingredient is [20] Proposition 1.2] which says that a \( C^1_{MB} \)-map \( f : C^1 \supset U \rightarrow \mathbb{R}^n \) is of locally bounded delay, in the sense that

\[
(\text{ld}) \text{ for every } \phi \in U \text{ there are a neighbourhood } N(\phi) \subset U \text{ and } d > 0 \text{ such that for all } \chi, \psi \in N(\phi) \text{ with }
\]

\[
\chi(t) = \psi(t) \text{ for all } t \in [-d,0]
\]

we have \( f(\chi) = f(\psi) \).

It follows from property (ld) that solutions of Eq. (1.1) with segments close to a stationary point \( \phi \in X_f \subset U \) are given by solutions of an equation

\[
(1.5) \quad x'(t) = f_d(x_t)
\]

with a map \( f_d : U_d \rightarrow \mathbb{R}^n \) defined on an open neighbourhood \( U_d \) of the restriction \( \phi|_{[-d,0]} \), in the Banach space \( C_d^1 \) of continuously differentiable maps \( [-d,0] \rightarrow \mathbb{R}^n \), and with the segments \( x_t \) defined on \([-d,0]\).

The solutions of Eq. (1.5) generate a semiflow on the solution manifold \( X_{f_d} = \{ \psi \in U_d : \psi(0) = f_d(\psi) \} \) in the Banach space \( C_d^1 \). This will be exploited in the search for local invariant manifolds in the Fréchet space \( C^1 \). For example, in Section 15 a local stable manifold at a stationary point \( \phi \) of the semiflow on \( X_f \subset C^1 \) appears as a preimage of the restriction map \( C^1 \supset \phi \mapsto \phi|_{[-d,0]} \subset C_d^1 \) which is transversal to a local stable manifold at \( \phi|_{[-d,0]} \) in \( X_{f_d} \subset C_d^1 \). The latter was obtained in [7, Section 3.5].

In the earlier result in [21], on local invariant manifolds which are \( C^1_{MB} \)-smooth, it was necessary to add a technical hypothesis (d) which essentially requires that \( f_d \) is \( C^0 \)-smooth. Notice that in the present approach where \( f \) is \( C^1 \)-smooth and not only \( C^1_{MB} \)-smooth the hypothesis (d) is obsolete.

For other work on delay differential equations with states \( x_t \) in Fréchet spaces see [15],[16],[22].

**Notation, preliminaries.** The closure of a subset \( M \) of a topological space is denoted by \( \overline{M} \) and its interior is denoted by \( M^\circ \).

For basic facts about topological vector spaces see [14]. The vector space of continuous linear maps \( V \rightarrow W \) between topological vector spaces is denoted by \( L_c(V,W) \).
Continuous linear maps between topological vector spaces map bounded sets into bounded sets.

Products of topological vector spaces are always equipped with the product topology.

**Proposition 1.2.** ([22]) Suppose $T$ is a topological space, $W$ is a topological vector space, $M$ is a metric space with metric $d$, $g : T \times M \supset U \to W$ is continuous, $U \supset \{t\} \times K$, $K \subset M$ compact. Then $g$ is uniformly continuous on $\{t\} \times K$ in the following sense: For every neighborhood $N$ of 0 in $W$ there exist a neighborhood $T_N$ of $t$ in $T$ and $\epsilon > 0$ such that for all $t' \in T_N$, all $i \in T_N$, all $k \in K$, and all $m \in M$ with

$$d(m, k) < \epsilon \quad \text{and} \quad (t', k) \in U, \quad (i, m) \in U,$$

we have

$$g(t', k) - g(i, m) \in N.$$  

**Proof.** Choose a neighborhood $N'$ of 0 in $W$ with $N' + N' \subset N$. For every $k \in K$ there exist open neighborhoods $T(k)$ of $t$ in $T$ and $\delta(k) > 0$ such that for all $t' \in T(k)$ and all $m \in M$ with $d(m, k) < \delta(k)$ and $(t', m) \in U$ we have

$$g(t', m) - g(t, k) \in N',$$

due to continuity. The compact set $K$ is contained in a finite union of open neighbourhoods

$$\left\{ m \in M : d(m, k_j) < \frac{\delta(k_j)}{2} \right\}, \quad j = 1, \ldots, n,$$

with $k_1, \ldots, k_n$ in $K$. Set

$$\epsilon = \min \left\{ \frac{\delta(k_j)}{2} : j = 1, \ldots, n \right\} \quad \text{and} \quad T_N = \cap_{j=1}^{n} T(k_j).$$

Let $t' \in T_N$, $i \in T_N$, $k \in K$, and $m \in M$ with

$$d(m, k) < \epsilon, \quad (t', k) \in U, \quad (i, m) \in U$$

be given. For some $j \in \{1, \ldots, n\}$, $d(k, k_j) < \frac{\delta(k)}{2}$. By the triangle inequality,

$$d(m, k_j) < \frac{\delta(k)}{2}.$$

It follows that

$$g(t', k) - g(i, m) = (g(t', k) - g(t, k_j)) + (g(t, k_j) - g(i, m)) \in N' + N' \subset N.$$  

\[\square\]

A Fréchet space $F$ is a locally convex topological vector space which is complete and metrizable. The topology is given by a sequence of seminorms $| \cdot |_j$, $j \in \mathbb{N}$, which are separating in the sense that $|v|_j = 0$ for all $j \in \mathbb{N}$ implies $v = 0$. The sets

$$N_{j, k} = \left\{ v \in F : |v|_j < \frac{1}{k} \right\}, \quad j \in \mathbb{N} \quad \text{and} \quad k \in \mathbb{N},$$

form a neighbourhood base at the origin. If the sequence of seminorms is increasing then the sets

$$N_j = \left\{ v \in F : |v|_j < \frac{1}{j} \right\}, \quad j \in \mathbb{N},$$

form a neighbourhood base at the origin.

Products of Fréchet spaces, closed subspaces of Fréchet spaces, and Banach spaces are Fréchet spaces.
For a curve, a continuous map \( c : I \supset U \to F \), the tangent vector at \( t \in I \) is
\[
c'(t) = \lim_{0 \neq h \to 0} \frac{1}{h}(c(t + h) - c(t))
\]
provided the limit exists. As in [6, Part I] the curve is said to be continuously differentiable if it has tangent vectors everywhere and if the map
\[
c' : I \ni t \mapsto c'(t) \in F
\]
is continuous.

For a continuous map \( f : V \supset U \to F \), \( V \) and \( F \) Fréchet spaces and \( U \subset \mathbb{V} \) open, and for \( u \in U, v \in V \) the directional derivative is defined by
\[
Df(u)v = \lim_{0 \neq h \to 0} \frac{1}{h}(f(u + hv) - f(u))
\]
provided the limit exists. If for \( u \in U \) all directional derivatives \( Df(u)v \), \( v \in \mathbb{V} \) exists then the map \( Df(u) : V \ni v \mapsto Df(u)v \in F \) is called the derivative of \( f \) at \( u \).

For continuous maps \( f : U \to F \), \( V, W, F \) Fréchet spaces and \( U \subset V \times W \) open, partial derivatives are defined in the usual way. For example, \( D_{1}f(v, w) : V \to F \) is given by
\[
D_{1}f(v, w)\hat{v} = \lim_{0 \neq h \to 0} \frac{1}{h}(f(v + h\hat{v}, w) - f(v, w)).
\]

The tangent cone of a set \( M \subset F \), \( F \) a Fréchet space, at \( x \in M \) is the set \( T_{x}M \) of all tangent vectors \( v = c'(0) \) of continuously differentiable curves \( c : I \to F \) with \( I \) open, \( 0 \in I \), \( c(0) = x \), \( c(I) \subset M \).

We freely use facts about the Riemann integral for continuous maps \( [a, b] \to F \) into a Fréchet space and results from calculus based on continuous differentiability in the sense of Michal and Bastiani which can be found in [6, Sections I.1-I.4].

For maps \( \mathbb{R}^{k} \supset U \to \mathbb{R}^{n} \), \( U \subset \mathbb{R}^{k} \) open, \( C_{F}^{1} \)-smoothness and \( C_{MB}^{1} \)-smoothness are equivalent (see Propositions 3.2 and 3.3 below, for example), and we simply speak of continuously differentiable maps. Also for a curve \( c : I \to \mathbb{R}^{n} \) on an interval \( I \subset \mathbb{R} \) of positive length and not open, we only speak of continuous differentiability, with
\[
c'(t) = \lim_{0 \neq h \to 0} \frac{1}{h}(c(t + h) - c(t)) = Dc(t)1 \in \mathbb{R}^{n}
\]
at inner points and one-sided derivatives \( c'(t) \) at endpoints contained in \( I \).

In Part II the following Fréchet spaces are used: For \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_{0} \) and \( T \in \mathbb{R} \), \( C_{T}^{k} = C^{k}((-\infty, T], \mathbb{R}^{n}) \) denotes the Fréchet space of \( k \)-times continuously differentiable maps \( \phi : (-\infty, T] \to \mathbb{R}^{n} \) with the seminorms given by
\[
|\phi|_{k, T, j} = \sum_{\kappa=0}^{k} \max_{T-j \leq t \leq T} |\phi^{(\kappa)}(t)|, \quad j \in \mathbb{N},
\]
which define the topology of uniform convergence of maps and their derivatives on compact sets. Analogously we consider the space \( C_{\infty}^{k} = C_{T}^{k}(\mathbb{R}, \mathbb{R}^{n}) \), with
\[
|\phi|_{k, \infty, j} = \sum_{\kappa=0}^{k} \max_{j \leq t \leq j} |\phi^{(\kappa)}(t)|.
\]
In Part 3 we also need the Banach space 

\[ C^0, \cdot |_{k,j} = \cdot |_{k,0,j}, \text{ and also } C = C^0 = C^0_0, \]

\[ \cdot |_{j} = \cdot |_{0,j} = \cdot |_{0,0,j}. \]

In case \( T = \infty \) we abbreviate \( C^\infty = C^0_\infty, \cdot |_{\infty,j} = \cdot |_{0,\infty,j}. \)

The vector space \( C^\infty = \cap_{k=0}^\infty C^k \) will be used without a topology on it.

The differentiation map \( \partial_{s,T} : C^k_T \ni \phi \mapsto \phi' \in C^{k-1}_T \), \( k \in \mathbb{N} \) and \( T \in \mathbb{R} \) or \( T = \infty \), is linear and continuous. We abbreviate \( \partial_T = \partial_{1,T} \) and \( \partial = \partial_0 = \partial_{1,0}. \)

The following Banach spaces occur in Parts II and III: For \( n \in \mathbb{N} \) and \( k \in \mathbb{N}_0 \) and reals \( a < b \), \( C^k([a,b],\mathbb{R}^n) \) denotes the Banach space of \( k \)-times continuously differentiable maps \([a,b] \to \mathbb{R}^n\) with the norm given by

\[ |\phi|_{[a,b],k} = \sum_{k=0}^k \max_{a \leq t \leq b} |\phi^{(k)}(t)|. \]

We use various abbreviations, for \( S < T \) and \( d > 0 \) and \( k \in \mathbb{N}_0: \)

\[ C_{ST} = C^0([S,T],\mathbb{R}^n), \quad \cdot |_{ST} = \cdot |_{[S,T],0} \]

\[ C_{ST}^1 = C^1([S,T],\mathbb{R}^n), \quad \cdot |_{1,ST} = \cdot |_{[S,T],1} \]

\[ C^k_d = C^k([-d,0],\mathbb{R}^n), \quad \cdot |_{d,k} = \cdot |_{[-d],0,k} \]

It is easy to see that the linear restriction maps

\[ R_{d,k} : C^k \to C^k_d, \quad d > 0 \quad \text{and} \quad k \in \mathbb{N}_0, \]

and the linear prolongation maps

\[ P_{d,k} : C^k_d \to C^k, \quad d > 0 \quad \text{and} \quad k \in \mathbb{N}_0, \]

given by \( (P_{d,k}\phi)(s) = \phi(s) \) for \(-d \leq s \leq 0\) and

\[ (P_{d,k}\phi)(s) = \sum_{k=0}^k \frac{\phi^{(k)}(-d)}{k!}(s + d)^k \quad \text{for} \quad s < -d \]

are continuous, and for all \( d > 0 \) and \( k \in \mathbb{N}_0, \)

\[ R_{d,k} \circ P_{d,k} = \text{id}_{C^k_d}. \]

In Part II we also need the closed subspaces

\[ C_{0T,0} = \{ \phi \in C_{0T} : \phi(0) = 0 \} \quad \text{and} \quad C_{0T,0}^1 = \{ \phi \in C_{0T}^1 : \phi(0) = 0 = \phi'(0) \}. \]

In Part 3 we also need the Banach space \( B_a \), for \( a > 0 \) given, for all \( \phi \in C \) with

\[ \sup_{t \leq 0} |\phi(t)|e^{at} < \infty, \quad |\phi|_a = \sup_{t \leq 0} |\phi(t)|e^{at}, \]

and finally, the Banach space \( B^1_a \) of all \( \phi \in C^1 \) with

\[ \phi \in B_a, \quad \phi' \in B_a, \quad |\phi|_{a,1} = |\phi|_a + |\phi'|_a. \]

Solutions of equations

\[ x'(t) = g(x(t)), \quad \text{with} \quad g : C^1_d \supset U \to \mathbb{R}^n \quad \text{or} \quad g : B^1_a \supset U \to \mathbb{R}^n, \]

on some interval \( I \subset \mathbb{R} \) are defined as in case of Eq. (1.1): With \( J = [-d,0] \) or \( J = (-\infty,0] \), respectively, they are continuously differentiable maps \( x : J + I \to \mathbb{R}^n \) so that \( x_t \in U \) for all \( t \in I \) and the differential equation holds for all \( t \in I \). Observe that \( x_t \) may denote a map on \([-d,0]\) or on \((-\infty,0]\), depending on the context.

The following statement on “globally bounded delay” for continuous linear maps corresponds to a special case of [20, Proposition 1.2].
**Proposition 1.3.** [21] Proposition 1.2] For every continuous linear map $L : C \to B$, $B$ a Banach space, there exists $r > 0$ with $L\phi = 0$ for all $\phi \in C$ with $\phi(s) = 0$ on $[-r, 0]$.

For results on strongly continuous semigroups given by solutions of linear autonomous retarded functional differential equations

$$x'(t) = \Lambda x_t$$

with $\Lambda : C_d \to \mathbb{R}^n$ linear and continuous, see [2, 5].
2. Uniform convergence of continuous linear maps on bounded sets

Let $V, W$ be topological vector spaces over $\mathbb{R}$ or $\mathbb{C}$. On $L_c = L_c(V, W)$ the topology $\beta$ of uniform convergence on bounded sets is defined as follows. For a neighbourhood $N$ of 0 in $W$ and a bounded set $B \subset V$ the neighbourhood $U_{N,B}$ of 0 in $L_c$ is defined as

$$U_{N,B} = \{ A \in L_c : AB \subset N \}.$$  

Every finite intersection of such sets $U_{N_j,B_j}$, $j \in \{1, \ldots, J\}$, contains a set of the same kind, because we have

$$\cap_{j=1}^{J} U_{N_j,B_j} \supset \{ A \in L_c : A(\cup_{j=1}^{J} B_j) \subset \cap_{j=1}^{J} N_j \},$$

finite unions of bounded sets are bounded, and finite intersections of neighbourhoods of 0 are neighbourhoods of 0. Then the topology $\beta$ is the set of all $O \subset L_c$ which have the property that for each $A \in O$ there exist a neighbourhood $N$ of 0 in $W$ and a bounded set $B \subset V$ with $A + U_{N,B} \subset O$. It is the easy to show that indeed $\beta$ is a topology.

We call a map $A$ from a topological space $T$ into $L_c$ $\beta$-continuous at a point $t \in T$ if it is continuous at $t$ with respect to the topology $\beta$ on $L_c$.

Remark 2.1. (i) Convergence of a sequence in $L_c$ with respect to $\beta$ is equivalent to uniform convergence on every bounded subset $B \subset V$. (Proof: By definition convergence $A_j \rightarrow A$ with respect to $\beta$ is equivalent to convergence $A_j - A \rightarrow 0$ with respect to $\beta$. This means that for each neighbourhood $N$ of 0 in $W$ and for each bounded subset $B \subset V$ there exists $j_{N,B} \in \mathbb{N}$ so that for all integers $j \geq j_{N,B}$, $A_j - A \in U_{N,B}$. Or, for all integers $j \geq j_{N,B}$ and all $b \in B$, $(A_j - A)b \in N$. Now the assertion becomes obvious.)

(ii) If $V$ and $W$ are Banach spaces then $\beta$ is the norm topology on $L_c(V,W)$ given by $|A| = \sup_{\|v\| \leq 1} |Av|$.

(iii) In order to verify $\beta$-continuity of a map $A : T \rightarrow L_c$, $T$ a topological space, at some $t \in T$ one has to show that, given a bounded subset $B \subset V$ and a neighbourhood $N$ of 0 in $W$, there exists a neighbourhood $N_t$ of $t$ in $T$ such that for all $s \in N_t$ we have $(A(s) - A(t))(B) \subset N$.

In case $T$ has countable neighbourhood bases the map $A$ is $\beta$-continuous at $t \in T$ if and only if for any sequence $T \ni t_j \rightarrow t$ we have $A(t_j) \rightarrow A(t)$. For $A(t_j) \rightarrow A(t)$ we need that given a bounded subset $B \subset V$ and a neighbourhood $N$ of 0 in $W$, there exists $J \in \mathbb{N}$ with

$$(A(t_j) - A(t))(B) \subset N \quad \text{for all integers } j \geq J.$$  

In the sequel we shall use the previous statement frequently.

Proposition 2.2. Singletons $\{A\} \subset L_c$ are closed with respect to the topology $\beta$, and $L_c$ equipped with the topology $\beta$ is a topological vector space.

Proof. 1. (On singletons) Let $A \in L_c$ be given. We show $L_c \setminus \{A\} \in \beta$. Let $S \in L_c \setminus \{A\}$. For some $b \in V$, $Ab \neq Sb$. For some neighbourhood $N$ of 0 in $W$, $Ab \notin Sb + N$ [43, Theorem 1.12]. For all $S' \in U_{N,\{b\}}$ we have $S'b \in N$, hence $(S + S')b \in Sb + N$, and thereby, $S + S' \neq A$. Hence $S + U_{N,\{b\}} \subset L_c \setminus \{A\}$. 


2. (Continuity of addition) Assume $S, T$ in $L_c$ and let $U_{N,B}$ be given, $N$ a neighbourhood of 0 in $W$ and $B$ a bounded subset of $V$. We have to find neighbourhoods of $S$ and $T$ so that addition maps their Cartesian product into $S + T + U_{N,B}$. As $W$ is a topological vector space there are neighbourhoods $N_T, N_S$ of 0 in $W$ with $N_T + N_S \subset N$. For every $T' \in T + U_{N_T,B}$ and for every $S' \in S + U_{N_S,B}$ and for every $b \in B$ we get\((T' + S') - (T + S))b = T'b - Tb + S'b - Sb \in N_T + N_S \subset N,\) hence\((T' + S') - (T + S))B \subset N,\) or $T' + S' \in T + S + U_{N,B}$.

3. (Continuity of multiplication with scalars) In case of vector spaces over the field $\mathbb{C}$ let $c \in \mathbb{C}, T \in L_c$. Let a neighbourhood $N$ of 0 in $W$ and a bounded subset $B \subset V$ be given, and consider the neighbourhood $U_{N,B}$ of 0 in $L_c$. There is a neighbourhood $\hat{N}$ of 0 in $W$ with $\hat{N} + \hat{N} + \hat{N} \subset N$, see e. g. [14] proof of Theorem 1.10]. We may assume that $\hat{N}$ is balanced \([14] \text{Theorem 1.14}]\). As $TB$ is bounded there exists $r_N \geq 0$ such that for reals $r \geq r_N$, $TB \subset r\hat{N}$. We infer that for some $\epsilon > 0$, $(0, \epsilon)TB \subset \hat{N}$. As $\hat{N}$ is balanced we obtain that for all $z \in \mathbb{C}$ with $0 < |z| < \epsilon$,

\[
zTB = \frac{|z|}{|z|}TB = \frac{z}{|z|}z|TB \subset \frac{z}{|z|}\hat{N} \subset \hat{N}.
\]

With $0 \in \hat{N}$ we arrive at

\[
zTB \subset \hat{N} \quad \text{for all} \quad z \in \mathbb{C} \quad \text{with} \quad |z| < \epsilon.
\]

Because of continuity of multiplication $\mathbb{C} \times W \to W$ there are neighbourhoods $U_c$ of 0 in $\mathbb{C}$ and $N_c$ of 0 in $W$ with

\[
U_cN_c \subset \hat{N}, \quad cN_c \subset \hat{N}, \quad U_c \subset \{z \in \mathbb{C} : |z| < \epsilon\}.
\]

Consider $T' \in U_{N_c,B}$ and $c' \in U_c$. Observe

\[
(c + c')(T + T') = cT + (c'T + c'T'),
\]

For every $b \in B$ we get

\[
(c'T + c'T' + c'T)b = c'Tb + c'T'b + c'T'b \in c'TB + cN_c + U_cN_c \subset \hat{N} + \hat{N} + \hat{N} \subset N.
\]

Therefore, $c'T + c'T' + c'T' \in U_{N,B}$. It follows that

\[
(c + U_c)(T + U_{N_c,B}) \subset cT + U_{N,B},
\]

which yields the desired continuity at $(c, T)$. \hfill \Box

3. $C^1_{MB}$-maps in Fréchet spaces

In this section $V, V_1, V_2, F, F_1, F_2$ always denote Fréchet spaces. We begin with a few facts from [6] Part I] about $C^1_{MB}$-maps $f : V \supset U \to F$. Each derivative $Df(u) : V \to F$, $u \in U$, is linear and continuous. Differentiation of $C^1_{MB}$-maps is linear, and the chain rule holds. We have

\[
(3.1) \quad f(v) - f(u) = \int_0^1 Df(u + t(v - u))(v - u)dt \quad \text{for} \quad u + [0,1]v \subset U
\]

with the Riemann integral of continuous maps $[a, b] \to F$ from [6] Part I]. Linear continuous maps $T \in L_c(V, F)$ are $C^1_{MB}$-smooth with $DT(u) = T$ everywhere. If $f_1 : V \supset U \to F_1$ and $f_2 : V \supset U \to F_2$ are $C^1_{MB}$-smooth then also $f_1 \times f_2 : V \supset U \ni u \mapsto (f_1(u), f_2(u)) \in F_1 \times F_2$ is $C^1_{MB}$-smooth, with

\[
D(f_1 \times f_2)(u)v = (Df_1(u)v, Df_2(u)v).
\]
Proposition 3.1. (see [3] Part I). For continuous \( f : V_1 \times V_2 \supset U \to F \), \( U \) open, the following statements are equivalent.

(i) For every \( (u_1, u_2) \in U \), \( v_1 \in V_1 \), \( v_2 \in V_2 \) the partial derivatives \( D_k f(u_1, u_2)v_k \) exist and both maps
\[
U \times V_k \ni (u_1, u_2, v_k) \mapsto D_k f(u_1, u_2)v_k \in F, \quad k \in \{1, 2\},
\]
are continuous.

(ii) \( f \) is \( C^1_{MB} \)-smooth.

In this case,
\[
Df(u_1, u_2)(v_1, v_2) = D_1 f(u_1, u_2)v_1 + D_2 f(u_1, u_2)v_2
\]
for all \( (u_1, u_2) \in U \), \( v_1 \in V_1 \), \( v_2 \in V_2 \).

Proposition 3.2. Let \( U \subset V \) be open. A map \( f : U \to F \) is \( C^1_{L} \)-smooth if and only if it is \( C^1_{MB} \)-smooth with \( U \ni u \mapsto Df(u) \in L_c(V, F) \) \( \beta \)-continuous.

Proof. We only show that \( C^1_{L} \)-smoothness implies \( C^1_{MB} \)-smoothness. Assume \( f : V \ni u \mapsto f(u) \in F \) is \( C^1_{L} \)-smooth. Consider sequences \( U \ni u_j \to u \in U \) and \( v_j \to v \) in \( V \) and let a neighbourhood \( N \) of 0 in \( F \) be given. We have to show \( Df(u_j)v_j \to Df(u)v \) as \( j \to \infty \). The set \( B = \{v_j : j \in \mathbb{N}\} \) is bounded. For every \( j \in \mathbb{N} \),
\[
Df(u_j)v_j - Df(u)v = [Df(u_j) - Df(u)]v_j + Df(u)v_j - v.
\]
By continuity of \( Df(u) \) the last term tends to 0 as \( j \to \infty \). Now consider the points \( [Df(u_j) - Df(u)]v_j \in F, \ j \in \mathbb{N} \). By the \( \beta \)-continuity of \( Df : U \to L_c(V, F) \), \( Df(u_j) \in Df(u) + U_{N, B} \) for \( j \) sufficiently large. For these \( j \),
\[
[Df(u_j) - Df(u)]v_j \in [Df(u_j) - Df(u)]B \subset N.
\]
This yields \( [Df(u_j) - Df(u)]v_j \to 0 \) as \( j \to \infty \). \( \square \)

Proposition 3.3. In case \( E \) is a finite-dimensional normed space each \( C^1_{MB} \)-map \( f : E \supset U \to F \) is \( C^1_{L} \)-smooth.

Proof. Recall Remark 2.1 (iii). Let \( B \subset E \) be bounded and let \( N \) be a neighbourhood of 0 in \( F \). Because of \( \text{dim } E < \infty \) the closure \( \overline{B} \) is compact. The map \( U \times E \ni (y, x) \mapsto Df(y)x \in F \) is continuous. Apply Proposition 1.2 to the compact set \( \{u\} \times \overline{B} \subset U \times E \). This yields a neighbourhood \( N_u \) of \( u \) in \( U \) with
\[
(Df(v) - Df(u))b \in N \quad \text{for all} \quad v \in N_u, b \in \overline{B},
\]
hence \( Df(v) \in Df(u) + U_{N, B} \) for all \( v \in N_u \). \( \square \)

Proposition 3.4. For Banach spaces \( V \) and \( F \) and \( U \subset V \) open a map \( f : V \supset U \to F \) is \( C^1_{L} \)-smooth if and only if there exists a continuous map \( D_f : U \to L_c(V, F) \) such that for every \( u \in U \) and
\[
(f) \quad \text{for every } \epsilon > 0 \text{ there exists } \delta > 0 \text{ with}
\]
\[
|f(v) - f(u) - D_f(u)(v - u)| \leq \epsilon|v - u| \quad \text{for all} \quad v \in U \quad \text{with} \quad |v - u| < \delta.
\]
In this case, \( D_f(u)v \) is the directional derivative \( Df(u)v \), for every \( u \in U, v \in V \).

Proof. We only show that for a \( C^1_{L} \)-map \( f : V \supset U \to F \) and \( u \in U \) the map \( D_f(u) = Df(u) \in L_c(V, F) \) satisfies statement (F). Due to Proposition 3.2 we may
use the integral representation (3.1) for $C_{MB}^1$-maps. For $v$ in a convex neighbourhood $N \subset U$ of $u$ this yields
\[ |f(v) - f(u) - Df(u)(v-u)| = \left| \int_0^1 Df(u + s(v-u))[v-u] ds \right| \]
\[ = \left| \int_0^1 [Df(u + s(v-u)) - Df(u)][v-u] ds \right| \leq \int_0^1 |\ldots| ds \]
\[ \leq \max_{0 \leq s \leq 1} |Df(u + s(v-u)) - Df(u)||v-u|. \]

Now the continuity of $Df$ at $u$ completes the proof. □

Continuous linear maps $T : V \to F$ are $C_{MB}^1$-smooth because they are $C_{MB}^1$-maps with constant derivative, $DT(u) = T$ for all $u \in V$. Using Proposition 3.2 and continuity of addition and multiplication on $L_c(V,F)$ (with the topology $\beta$) one obtains from the properties of $C_{MB}^1$-maps that linear combinations of $C_{MB}^1$-maps are $C_{MB}^1$-maps. As $C_{MB}^1$-maps differentiation is linear, and the integral formula (3.1) holds. If $f_1 : V \supset U \to F_1$ and $f_2 : V \supset U \to F_2$ are $C_{MB}^1$-smooth then also $f_1 \times f_2 : V \supset U \ni u \mapsto (f_1(u), f_2(u)) \in F_1 \times F_2$ is $C_{MB}^1$-smooth, with
\[ D(f_1 \times f_2)(u)v = (Df_1(u)v, Df_2(u)v). \]

This follows easily from the analogous property for $C_{MB}^1$-maps, by means of the formula for the directional derivatives of $f_1 \times f_2$ and considering neighbourhoods of $0$ in $F_1 \times F_2$ which are products of neighbourhoods of $0$ in $F_j, j \in \{1,2\}$.

**Proposition 3.5. (Chain rule).** If $f : V \supset U \to F$ and $g : F \supset W \to G$ are $C_{MB}^1$-maps, with $f(U) \subset W$, then also $g \circ f$ is a $C_{MB}^1$-map.

**Proof.** 1. The chain rule for $C_{MB}^1$-maps yields that $g \circ f$ is $C_{MB}^1$-smooth with $D(g \circ f)(u) = D(g(f(u))) \circ Df(u)$ for all $u \in U$. So it remains to prove that the map $U \ni u \mapsto Dg(f(u)) \circ Df(u) \in L_c(V,G)$ is $\beta$-continuous. As $V$ has countable neighbourhood bases it is enough to show that, given a sequence $U \ni u_j \to u \in U$, a bounded set $B \subset V$, and a neighbourhood $N$ of $0$ in $G$, we have
\[ |Dg(f(u_j)) \circ Df(u_j) - Dg(f(u)) \circ Df(u)|B \subset N \] for $j \in \mathbb{N}$ sufficiently large.

So let a sequence $U \ni u_j \to u \in U$, a bounded set $B \subset V$, and a neighbourhood $N$ of $0$ in $G$ be given.

2. There is a neighbourhood $N_1$ of $0$ in $G$ with $N_1 + N_1 + N_1 + N_1 \subset N_1$, see [14, proof of Theorem 1.10]. By linearity, for every $j \in \mathbb{N}$,
\[ Dg(f(u_j)) \circ Df(u_j) - Dg(f(u)) \circ Df(u) = |Dg(f(u_j)) - Dg(f(u))| \circ Df(u_j) \]
\[ + Dg(f(u)) \circ |Df(u_j) - Df(u)|. \]

3. Consider the last term. By continuity of $Dg(f(u))$ at $0 \in F$, there is a neighbourhood $N_2$ of $0$ in $F$ with $Dg(f(u))N_2 \subset N_1$. By $\beta$-continuity of $Df$ at $u \in U$, there is an integer $j$, such that for all integers $j \geq j_2$,
\[ |Df(u_j) - Df(u)|B \subset N_2. \]

Hence, for all integers $j \geq j_2$,
\[ Dg(f(u)) \circ |Df(u_j) - Df(u)|B \subset N_1. \]
4. $Df(u)B$ is bounded. Using $\beta$-continuity of $Dg$ at $f(u)$ and $\lim_{j \to \infty} f(u_j) = f(u)$ we find an integer $j_3 \geq j_2$ such that for all integers $j \geq j_3$ we have

$$[Dg(f(u_j)) - Dg(f(u))]Df(u)B \subset N_1.$$

5. Now we use the continuity of $W \times F \ni (w, h) \mapsto Dg(w)h \in G$ at $(f(u), 0)$. We find a neighbourhood $N_3$ of 0 in $F$ and an integer $j_4 \geq j_3$ such that for all integers $j \geq j_4$ we have $Dg(f(u_j))N_3 \subset N_1$ and $-Dg(f(u))N_3 \subset N_1$. This yields

$$[Dg(f(u_j)) - Dg(f(u))]N_3 \subset N_1 + N_1 \quad \text{for all integers} \quad j \geq j_4.$$

6. The $\beta$-continuity of $Df$ at $u \in U$ yields an integer $j_N \geq j_4$ such that for all integers $j \geq j_N$ we have

$$[Df(u_j) - Df(u)]B \subset N_3,$$

hence

$$Df(u_j)B \subset Df(u)B + N_3.$$

7. For integers $j \geq j_N$ we obtain

$$[Dg(f(u_j)) \circ Df(u_j) - Dg(f(u)) \circ Df(u)]B$$

$$= [Dg(f(u_j)) - Dg(f(u))] \circ Df(u_j) + Dg(f(u)) \circ [Df(u_j) - Df(u)]B \quad \text{(see part 2)}$$

$$\subset [Dg(f(u_j)) - Dg(f(u))]Df(u_j)B + [Dg(f(u)) \circ Df(u_j) - Df(u)]B$$

$$\subset [Dg(f(u_j)) - Dg(f(u))]Df(u_j)B + [Dg(f(u_j)) - Dg(f(u))]N_3 + N_1 \quad \text{(see parts 6 and 3)}$$

$$\subset [Dg(f(u_j)) - Dg(f(u))]Df(u_j)B + [Dg(f(u_j)) - Dg(f(u))]N_3 + N_1$$

$$\subset N_1 + (N_1 + N_1) + N_1 \quad \text{(see parts 4 and 5)}$$

$$\subset N.$$

\[\square\]

**Proposition 3.6.** For a continuous map $f : V_1 \times V_2 \supset U \to F$, $U$ open, the following statements are equivalent.

(i) For all $(u_1, u_2) \in U$ and all $v_k \in V_k$, $k \in \{1, 2\}$, $f$ has a partial derivative $D_k f(u_1, u_2)v_k \in F$, all maps

$$D_k f(u_1, u_2) : V_k \to F, \quad (u_1, u_2) \in U, \quad k \in \{1, 2\},$$

are linear and continuous, and the maps

$$U \ni (u_1, u_2) \mapsto D_k f(u_1, u_2) \in L_c(V_k, F), \quad k \in \{1, 2\},$$

are $\beta$-continuous.

(ii) $f$ is $C^1_{L^\infty}$-smooth.

In this case,

$$Df(u_1, u_2)(v_1, v_2) = D_1 f(u_1, u_2)v_1 + D_2 f(u_1, u_2)v_2$$

for all $(u_1, u_2) \in U$, $v_1 \in V_1$, $v_2 \in V_2$.

**Proof.** 1. Suppose (ii) holds. Then $f$ is $C^1_{L^\infty}$-smooth, and all statements in (i) up to the last one follow from Proposition 3.1 on partial derivatives. In order to deduce the last statement in (i) for $k = 1$ let a sequence $((u_{1j}, u_{2j}))_{j=1}^\infty$ in $U$ be given which converges to some $(u_1, u_2) \in U$. Let a neighbourhood $N$ of 0 in $F$ and a bounded set $B_1 \subset V_1$ be given, consider the neighbourhood $U_{N, B_1}$ of 0 in $L_c(V_1, F)$. As $V_1 \ni v \mapsto (v, 0) \in V_1 \times V_2$ is linear and continuous, $B_1 \times \{0\}$ is a bounded subset of $V_1 \times V_2$. As $f$ is $C^1_{L^\infty}$-smooth the map $Df$ is $\beta$-continuous, and
2. Suppose (i) holds.

2.1. Claim: Both maps $U \times V_k \ni (u_1, u_2, v_k) \mapsto D_k f(u_1, u_2) v_k \in F$, $k \in \{1, 2\}$, are continuous.

Proof for $k = 1$: Let a sequence $(u_{1j}, u_{2j}, v_{1j})^\infty_j$ in $U \times V_1$ be given which converges to some $(u_1, u_2, v_1) \in U \times V_1$. Then $v_{1j} \to v_1$ in $V_1$, and $B_1 = \{v_{1j} : j \in \mathbb{N}\} \cup \{v_1\}$ is a bounded subset of $V_1$. Let a neighbourhood $N$ of 0 in $F$ be given. By the $\beta$-continuity of $D_1 f$,

$$(D_1 f(u_{1j}, u_{2j}) - D_1 f(u_1, u_2)) B_1 \subset N \quad \text{for } j \text{ sufficiently large.}$$

For each $j \in \mathbb{N}$ we have

$$D_1 f(u_{1j}, u_{2j}) v_{1j} - D_1 f(u_1, u_2) v_1 = (D_1 f(u_{1j}, u_{2j}) - D_1 f(u_1, u_2)) v_{1j} + D_1 f(u_1, u_2)(v_{1j} - v_1).$$

Now it becomes obvious how to complete the proof, using the last equation, the statement right before it, and continuity of $D_1 f(u_1, u_2)$.

2.2. Proposition 3.1 on partial derivatives applies and yields that $f$ is $C^1_{\beta(B)}$-smooth, with

$$D f(u_1, u_2)(v_1, v_2) = D_1 f(u_1, u_2) v_1 + D_2 f(u_1, u_2) v_2$$

for all $(u_1, u_2) \in U$, $v_1 \in V_1$, $v_2 \in V_2$. According to Proposition 3.2 it remains to prove that the map $D f : U \to L_c(V_1 \times V_2, F)$ is $\beta$-continuous. The projections $pr_k$ of $V_1 \times V_2$ onto the factor $V_k$, for $k \in \{1, 2\}$, are linear and continuous. For every $(u_1, u_2) \in U$ we have

$$D f(u_1, u_2) = D_1 f(u_1, u_2) \circ pr_1 + D_2 f(u_1, u_2) \circ pr_2,$$

so it is sufficient to show that both maps

$$V_1 \times V_2 \ni (u_1, u_2) \mapsto D_k f(u_1, u_2) \circ pr_k \in L_c(V_1 \times V_2, F), \quad k \in \{1, 2\},$$

are $\beta$-continuous. We deduce this for $k = 1$. Let a sequence $(u_{1j}, u_{2j})^\infty_j$ in $U$ be given which converges to some $(u_1, u_2) \in U$, as well as a bounded subset $B \subset V_1 \times V_2$ and a neighbourhood $N$ of 0 in $F$. We need to show

$$(D_1 f(u_{1j}, u_{2j}) \circ pr_1 - D_1 f(u_1, u_2) \circ pr_1) B \subset N$$

for $j \in \mathbb{N}$ sufficiently large. $B_1 = pr_1 B$ is a bounded subset of $V_1$, and for every $j \in \mathbb{N}$ we have

$$(D_1 f(u_{1j}, u_{2j}) \circ pr_1 - D_1 f(u_1, u_2) \circ pr_1) B \subset (D_1 f(u_{1j}, u_{2j}) - D_1 f(u_1, u_2)) B_1.$$
4. $C^1_F$-submanifolds

$C^1_F$-submanifolds of a Fréchet space are defined in the same way as continuously differentiable submanifolds of a Banach space. Below we collect the simple facts which are used in Section 7 and in Parts II and III.

A $C^1_F$-diffeomorphism is an injective $C^1_F$-map from an open subset $U$ of a Fréchet space $F$ onto an open subset $W$ of a Fréchet space $V$ whose inverse defined on $W \subset V$ is a $C^1_F$-map.

Let $F = G \oplus H$ be a direct sum decomposition of a Fréchet space $F$ into closed subspaces. A subset $M \subset W$ of $F$ is a $C^1_F$-submanifold of $F$ (modelled over the Fréchet space $G$) if for every point $m \in M$ there are an open neighbourhood $U$ in $F$ and a $C^1_F$-diffeomorphism $K : U \to F$ onto $W = K(U)$ with

$$K(M \cap U) = W \cap G.$$  

The tangent cones of the $C^1_F$-submanifold $M$ are closed subspaces of $F$. For $K$ as before the map $(DK(m))^{-1}$ defines a topological isomorphism from $G$ onto $T_m M$, and $K^{-1}$ defines an injective map $P$ from the open neighbourhood $V \cap G$ of $K(m)$ in $G$ onto the open neighbourhood $U \cap M$ of $m$ in $M$.

Open subsets of $C^1_F$-submanifolds are $C^1_F$-submanifolds.

A $C^1_F$-map $h : M \to H$, $M$ a $C^1_F$-submanifold of $F$ and $H$ a Fréchet space, is defined by the property that for all local parametrizations $P$ as above the composition $f \circ P$ is a $C^1_F$-map.

For $h$ as before and $m \in M$ the derivative $T_m h : T_m M \to H$ is defined by $T_m h(t) = (h \circ c)'(0)$, for any continuously differentiable curve $c : I \to F$ with $c(0) = m$, $c(I) \subset M$, $c'(0) = t$. The map $T_m h$ is linear and continuous.

In case $h(M)$ is contained in a $C^1_F$-submanifold $M_H$ of $H$ and $z : M_H \to Z$ is $C^1_F$-smooth the chain rule holds, with $T_m h(T_m M) \subset T_{h(m)} M_H$ and $T_m(z \circ h) t = T_{h(m)} z T_m h(t)$.

The restriction of a $C^1_F$-map on an open subset of $F$ to a $C^1_F$-submanifold $M$ of $F$, with range in a Fréchet space $H$, is a $C^1_F$-map from $M$ into the target space.

5. Uniform contractions

The proof of Theorem 5.2 below employs twice the following basic uniform contraction principle.

**Proposition 5.1.** (See for example [2] Appendix VI, Proposition 1.2.) Let a Hausdorff space $T$, a complete metric space $M$, and a map $f : T \times M \to M$ be given. Assume that $f$ is a uniform contraction in the sense that there exists $k \in [0, 1)$ so that

$$d(f(t, x), f(t, y)) \leq k d(x, y)$$

for all $t \in T, x \in M, y \in M$, and $f(\cdot, x) : T \to M$ is continuous for each $x \in M$. Then the map $g : T \to M$ given by $g(t) = f(t, g(t))$ is continuous.
Theorem 5.2. Let a Fréchet space $T$, a Banach space $B$, open sets $V \subset F$ and $O_B \subset B$, and a $C^1_B$-map $A: V \times O_B \to B$ be given. Assume that for a closed set $M \subset O_B$ we have $A(V \times M) \subset M$, and $A$ is a uniform contraction in the sense that there exists $k \in [0, 1)$ so that 

$$|A(t, x) - A(t, y)| \leq k|x - y|$$

for all $t \in V, x, y \in O_B$. Then the map $g: V \to B$ given by $g(t) = A(t, g(t)) \in M$ is $C^1_B$-smooth.

Notice that the derivative $\Gamma = Dg(t)\dot{t}$ of the map $g$ satisfies the equation

$$\dot{t} = D_1A(t, g(t))\dot{t} + D_2A(t, g(t))\Gamma. \tag{5.1}$$

Proof of Theorem 5.2. 1. $A$ is continuous. So Proposition 5.1 applies to the restriction of $A$ to $V \times M$ and yields a continuous map $g: V \to B$ with $g(t) = A(t, g(t)) \in M$ for all $t \in V$. Choose $\kappa \in (k, 1)$. Each linear map $D_2A(t, x): B \to B, (t, x) \in V \times O_B$, is continuous. The contraction property yields

$$|D_2A(t, x)| = \sup_{|\hat{x}| \leq 1} |D_2A(t, x)\hat{x}| \leq \kappa \quad \text{for all} \quad (t, x) \in V \times O_B$$

since given $\epsilon = \kappa - k$ and $t \in V, x \in O_B$, and $\hat{x} \in B$ with $|\hat{x}| \leq 1$ there exists $\delta > 0$ such that for $h = \frac{\delta}{2}$,

$$x + h\hat{x} \in O_B \quad \text{and}$$

$$h^{-1}(A(t, x) - A(t, x + h\hat{x})) - D_2A(t, x)\hat{x} = |h^{-1}(A(t, x) - A(t, x + h\hat{x})) - DA(t, x)(0, \hat{x})| \leq \epsilon,$$

hence

$$|h||D_2A(t, x)\hat{x}| \leq \epsilon|h| + |A(t, x + h\hat{x}) - A(t, x)| \leq \epsilon|h| + k|h\hat{x}| \leq (\epsilon + k)|h| = \kappa|h|.$$

Divide by $|h| = h$.

2. It follows that each map $id_B - D_2A(t, x) \in L_c(B, B), t \in V$ and $x \in O_B$, is a topological isomorphism. As $A$ is $C^1_B$-smooth we get that the map

$$V \times O_B \ni (t, x) \mapsto D_2A(t, x) \in L_c(B, B)$$

is $\beta$-continuous, or equivalently, continuous with respect to the usual norm-topology on $L_c(B, B)$. As inversion is continuous we see that also the map

$$V \times O_B \ni (t, x) \mapsto (id_B - D_2A(t, x))^{-1} \in L_c(B, B)$$

is continuous.

3. For all $(t, x, \hat{t}) \in V \times O_B \times T$ and for all $\hat{x}, \hat{y} \in B$ we have

$$|DA(t, x)(\hat{t}, \hat{x}) - DA(t, x)(\hat{t}, \hat{y})| = |DA(t, x)(0, \hat{x} - \hat{y})| = |D_2A(t, x)(\hat{x} - \hat{y})| \leq \kappa|\hat{x} - \hat{y}|.$$

Hence Proposition 5.1 applies to the version

$$\Gamma = D_1A(t, x)\dot{t} + D_2A(t, x)\Gamma$$

of Eq. (5.1) with parameters $(t, x, \hat{t}) \in V \times O_B \times T$ and yields a continuous map $\gamma: V \times O_B \times T \to B$ with

$$\gamma(t, x, \hat{t}) = D_1A(t, x)\dot{t} + D_2A(t, x)\gamma(t, x, \hat{t}) \quad \text{for all} \quad (t, x, \hat{t}) \in V \times O_B \times T,$$
or equivalently,
\[ \gamma(t, x, \hat{t}) = (i_B - D_2A(t, x))^{-1}D_1A(t, x)\hat{t} \quad \text{for all } (t, x, \hat{t}) \in V \times O_B \times T. \]
This shows that each map \( \gamma(t, x, \cdot) \), \( (t, x) \in V \times O_B \), belongs to \( L_c(T, B) \).

Claim: The map
\[ \hat{\gamma} : V \times O_B \ni (t, x) \mapsto \gamma(t, x, \cdot) \in L_c(T, B) \]
is \( \beta \)-continuous.

Proof. Let a sequence \( (t_j, x_j) \) in \( V \times O_B \) converge to a point \( (t, x) \in V \times O_B \). Consider a neighbourhood \( N \) of 0 in \( B \) and a bounded set \( T_b \subset T \). We have to show that for \( j \in \mathbb{N} \) sufficiently large, \( (\hat{\gamma}(t_j, x_j) - \hat{\gamma}(t, x))T_b \subset N \). For all \( j \in \mathbb{N} \) and all \( t \in T_b \) we have
\[
|\hat{\gamma}(t_j, x_j) - \hat{\gamma}(t, x)| \hat{t} = |(i_B - D_2A(t_j, x_j))^{-1}D_1A(t_j, x_j) \\
- (i_B - D_2A(t, x))^{-1}D_1A(t, x)| \hat{t} \\
\leq |(i_B - D_2A(t_j, x_j))^{-1} - (i_B - D_2A(t, x))^{-1}| |D_1A(t_j, x_j)| | \hat{t} | \\
+ |(i_B - D_2A(t, x))^{-1}| ((D_1A(t_j, x_j) - D_1A(t, x))| \hat{t} |) \\
\leq |(i_B - D_2A(t_j, x_j))^{-1} - (i_B - D_2A(t, x))^{-1}| |(D_1A(t_j, x_j) - D_1A(t, x))| | \hat{t} |. \\
\]
Now it becomes obvious how to complete the proof, using
\[
|(i_B - D_2A(t_j, x_j))^{-1} - (i_B - D_2A(t, x))^{-1}| \to 0 \quad \text{as} \quad j \to \infty, \\
\]
boundedness of \( |D_1A(t, x)T_b| \), and \( \beta \)-continuity of the partial derivative
\[ D_1A : V \times O_B \to L_c(T, B) \]
due to Proposition 3.6.

\[ \lim_{\theta \to 0} \frac{1}{h}(g(t + h\hat{t}) - g(t)) = \hat{\gamma}(t, \hat{t}), \]
which means that the directional derivative \( Dg(t)\hat{t} \) exists and equals \( \hat{\gamma}(t, \hat{t}) \).

So let \( t \in V \) and \( \hat{t} \in T \) be given. Choose a convex neighbourhood \( N_B \subset O_B \) of \( g(t) \). There exists \( \delta > 0 \) such that for \( -\delta \leq h \leq \delta \),
\[
t + h\hat{t} \in V \quad \text{and} \quad g(t + h\hat{t}) \in N_B. \\
\]
Notice that for all \( h \in [-\delta, \delta] \) and for all \( \theta \in [0, 1], \)
\[ g(t) + \theta(g(t + h\hat{t}) - g(t)) \in N_B. \]

With the abbreviation
\[ \xi = \xi(t, \hat{t}) = \hat{\gamma}(t, g(t), \hat{t}) = D_1A(t, g(t))\hat{t} + D_2A(t, g(t))\gamma(t, g(t), \hat{t}) \]
\[ = D_1A(t, g(t))\hat{t} + D_2A(t, g(t))\xi \]
one finds that
\[ h^{-1}(g(t + h\hat{t}) - g(t)) - \xi = h^{-1}(A(t + h\hat{t}, g(t + h\hat{t})) - A(t, g(t)) - \xi, \quad \text{with} \quad 0 < |h| < \delta, \]
\[ h^{-1}(A(t + h\hat{t}, g(t + h\hat{t})) - A(t, g(t)) - \xi, \quad \text{with} \quad 0 < |h| < \delta, \]

\[ 17 \]
The first term in the last expression converges to 0 as 0 ≠ h → 0. The map 
\[-\delta, \delta] \times [0, 1] \ni (h, \theta) \mapsto \{D_2 A(t + h \hat{t}, g(t) + \theta |g(t + h \hat{t}) - g(t)|) - D_2 A(t, g(t))\} |x| \xi \in B
is uniformly continuous with value 0 on \{0\} \times [0, 1]. This implies that for 0 ≠ h → 0 the last integrand converges to 0 uniformly with respect to \theta \in [0, 1]. Therefore the last integral tends to 0 as 0 ≠ h → 0. □
6. AN IMPLICIT FUNCTION THEOREM

From Theorem 5.2 one obtains the following Implicit Function Theorem in the usual way, paying attention to $C^1_{Fr}$-smoothness.

**Theorem 6.1.** Let a Fréchet space $T$, Banach spaces $B$ and $E$, an open set $U \subset T \times B$, a $C^1_{Fr}$-map $f : U \to E$, and a zero $(t_0,x_0) \in U$ of $f$ be given. Assume that $D_2f(t_0,x_0) : B \to E$ is bijective. Then there are open neighbourhoods $V$ of $t_0$ in $T$ and $W$ of $x_0$ in $B$ with $V \times W \subset U$ and a $C^1_{Fr}$-map $g : V \to W$ with $g(t_0) = x_0$ and

$$\{(t,x) \in V \times W : f(t,x) = 0\} = \{(t,x) \in V \times W : x = g(t)\}.$$  

**Proof.** 1. (A fixed point problem) Choose an open neighbourhood $N_{T,1}$ of $t_0$ and a convex open neighbourhood $N_B$ of $x_0$ in $B$ with $N_{T,1} \times N_B \subset U$. The equation

$$f(t,x) = f(t,x_0) + D_2f(t_0,x_0)[x - x_0] + R(t,x)$$

defines a $C^1_{Fr}$-map $R : N_{T,1} \times N_B \to E$, with $R(t,x_0) = 0$ for all $t \in N_{T,1}$,

$$D_2R(t,x) = D_2f(t,x) - D_2f(t_0,x_0)$$

for all $t \in N_{T,1}$ and $x \in N_B$, and in particular, $D_2R(t_0,x_0) = 0$. The map

$$N_{T,1} \times N_B \ni (t,x) \mapsto D_2R(t,x) \in L_c(B,E)$$

is $\beta$-continuous. In order to solve the equation $0 = f(t,x)$, $(t,x) \in N_{T,1} \times N_B$, for $x$ as a function of $t$, observe that this equation is equivalent to

$$0 = f(t,x_0) + D_2f(t_0,x_0)[x - x_0] + R(t,x),$$

or,

$$x = x_0 + (D_2f(t_0,x_0))^{-1}[-f(t,x_0) - R(t,x)]$$

$$= x_0 - (D_2f(t_0,x_0))^{-1}f(t,x_0) - (D_2f(t_0,x_0))^{-1}R(t,x).$$

The last expression defines a map

$$A : N_{T,1} \times N_B \to B$$

with $A(t_0,x_0) = x_0$, and for $(t,x) \in N_{T,1} \times N_B$,

$$0 = f(t,x)$$

if and only if $x = A(t,x)$.

The map $A$ is $C^1_{Fr}$-smooth since the linear map $D_2f(t_0,x_0))^{-1} : E \to B$ is continuous, due to the open mapping theorem.

2. (Contraction) For all $t \in N_{T,1}$ and for all $x, \hat{x}$ in $N_B$,

$$|A(t,\hat{x}) - A(t,x)| = |(D_2f(t_0,x_0))^{-1}R(t,\hat{x}) + (D_2f(t_0,x_0))^{-1}R(t,x)|$$

$$\leq |(D_2f(t_0,x_0))^{-1}| \int_0^1 |D_2R(t,x + s(\hat{x} - x))[\hat{x} - x]| ds.$$  

Let

$$\epsilon = \frac{1}{2|(D_2f(t_0,x_0))^{-1}|}.$$  

There are an open neighbourhood $N_{T,2} \subset N_{T,1}$ of $t_0$ and $\delta > 0$ such that for all $t \in N_{T,2}$ and all $x \in B$ with $|x - x_0| \leq \delta$,

$$x \in N_B \quad \text{and} \quad |D_2R(t,x)| = |D_2R(t,x) - D_2R(t_0,x_0)| < \epsilon.$$
For all $x \neq \hat{x}$ in $B$ with $|x - x_0| \leq \delta$ and $|\hat{x} - x_0| \leq \delta$ and for all $t \in N_{T,2}$ and $s \in [0,1]$ it follows that $|x + s(\hat{x} - x) - x_0| \leq \delta$, hence
\[
\left\lvert D_2R(t, x + s(\hat{x} - x)) \frac{1}{|\hat{x} - x|}[\hat{x} - x] \right\rvert < \epsilon,
\]
and thereby
\[
|A(t, \hat{x}) - A(t, x)| \leq \epsilon|\hat{x} - x|(|D_2f(t_0, x_0)|^{-1}) = \frac{1}{2}|\hat{x} - x|.
\]

3. (Invariance) By continuity there is an open neighbourhood $N_{T,3} \subset N_{T,2}$ of $t_0$ such that
\[
|A(t, x_0) - A(t_0, x_0)| < \frac{\delta}{4} \quad \text{for all} \quad t \in N_{T,3}.
\]

For all $t \in N_{T,3}$ and $x \in B$ with $|x - x_0| \leq \delta$ this yields
\[
|A(t, x) - x_0| = |A(t, x) - A(t_0, x_0)| \leq |A(t, x) - A(t, x_0)| + |A(t, x_0) - A(t_0, x_0)| < \frac{1}{2}|x - x_0| + \frac{\delta}{4} \leq \frac{\delta}{2} + \frac{\delta}{4} = \frac{3\delta}{4}.
\]

4. Set $V = N_{T,3}$, $O_B = \{x \in B : |x - x_0| < \delta\}$, and
\[
M = \left\{ x \in B : |x - x_0| \leq \frac{3\delta}{4} \right\},
\]
and apply Theorem 5.2 to the restriction of $A$ to the set $V \times O_B$. This yields a $C_1^p$-map $g : V \to B$ with $g(t) = A(t, g(t)) \in O_B$ for all $t \in V$. Using Part 3 we get
\[
|g(t) - x_0| < \frac{3\delta}{4}
\]
for all $t \in V$. Set
\[
W = \left\{ x \in B : |x - x_0| < \frac{3\delta}{4} \right\}.
\]

Then $g(V) \subset W$. From $g(t) = A(t, g(t))$ for all $t \in V$ we obtain $0 = f(t, g(t))$ for these $t$. Conversely, if $0 = f(t, x)$ for $(t, x) \in V \times W \subset V \times M$, then $x = A(t, x)$, hence $x = g(t)$. In particular, $x_0 = g(t_0)$. $\square$

7. Submanifolds by Transversality and Embedding

**Proposition 7.1.** Let a $C_1^p$-map $g : F \supset U \to G$ and a $C_1^p$-submanifold $M \subset G$ of finite codimension $m$ be given. Assume that $g$ and $M$ are transversal at a point $x \in g^{-1}(M)$ in the sense that
\[
G = Dg(x)F + T_{g(x)}M.
\]
Then there is an open neighbourhood $V$ of $x$ in $U$ so that $V \cap g^{-1}(M)$ is a $C_1^p$-submanifold of codimension $m$ in $F$, and $T_x(g^{-1}(M) \cap V) = Dg(x)^{-1}T_{g(x)}M$.

In case $\dim G = m$, $M = \{g(x)\}$, and $Dg(x)$ surjective the assertion holds with $T_{g(x)}M = \{0\}$

**Proof** for $M \neq \{g(x)\}$.

1. There are an open neighbourhood $N_G$ of $\gamma = g(x)$ in $G$ and a $C_1^p$-diffeomorphism $K : N_G \to G$ onto an open set $U_G \subset G$ such that $K(\gamma) = 0$, $K(N_G \cap M) = U_G \cap T_\gamma M$. We may assume $DK(\gamma) = id$ since otherwise we can replace $K$ with $DK(\gamma)^{-1} \circ K$. Then $DK(\gamma) = id$ maps $T_\gamma M$ onto itself.
2. By transversality and codim $M = m$ we find a subspace $Q \subset Dg(x)F$ of dimension $m$ which complements $T_\gamma M$ in $G,$

$$G = T_\gamma M \oplus Q.$$  

The projection $P : G \to Q$ along $T_\gamma M$ onto $Q$ is linear and continuous (see [14, Theorem 5.16]), and $PD\kappa \gamma(x) = PDg(x)$ is surjective. The preimage $U_F = g^{-1}(N_G)$ is open, with $x \in U_F \subset U.$ For $z \in U_F$ we have

$$z \in g^{-1}(M) \cap U_F \iff g(z) \in M \cap N_G \iff PK(g(z)) = 0.$$  

For the $C^1_F$-map $h = P \circ K \circ (g|_{U_F})$ we infer $g^{-1}(M) \cap U_F = h^{-1}(0).$ The derivative $Dh(x) : F \to Q$ is surjective. It follows that there is a subspace $R$ of $F$ with $\dim R = \dim Q = m$

$$F = Dh(x)^{-1}(0) \oplus R.$$  

The restriction $Dh(x)|_R$ is an isomorphism.

3. The $C^1_F$-map

$$H : \{(z, r) \in Dh(x)^{-1}(0) \times R : x + z + r \in U_F\} \ni (z, r) \mapsto h(x + z + r) \in Q$$  

satisfies $H(0, 0) = 0.$ Because of $D_2H(0, 0) = Dh(x) \hat{r}$ for all $\hat{r} \in R$ and $\dim R = \dim Q$ the map $D_2H(0, 0)$ is an isomorphism. Theorem 6.1 yields convex open neighbourhoods $V_H$ of 0 in $Dh(x)^{-1}(0)$ and $V_R$ of 0 in $R,$ with $x + V_H + V_R \subset U_F,$ and a $C^1_F$-map $w : V_H \to V_R$ with $w(0) = 0$ and

$$\{(v, r) \in V_H \times V_R : r = w(z)\}.$$  

For every $y \in x + V_H + V_R,$ $y = x + z + r$ with $z \in V_H$ and $r \in V_R,$ we have

$$y \in g^{-1}(M) \cap U_F \iff h(y) = 0 \iff h(x + z + r) = 0 \iff H(z, r) = 0 \iff r = w(z).$$  

Hence $g^{-1}(M) \cap (x + V_H + V_R) = \{x + z + w(z) : z \in V_H\},$ which implies that $g^{-1}(M) \cap (x + V_H + V_R)$ is a $C^1_F$-submanifold of $F,$ with codimension equal to $\dim R = \dim Q = m.$ Set $V = x + V_H + V_R.$

4. (On tangent spaces) From $g^{-1}(M) \cap U_F = h^{-1}(0)$ and $h(x) = 0$ we get $h(g^{-1}(M) \cap V) = \{0\},$ hence $Dh(x)|_V(H^{-1}(M) \cap V) = \{0\},$ or

$$T_x(g^{-1}(M) \cap V) \subset Dh(x)^{-1}(0).$$  

As both spaces have the same codimension $m$ they are equal. For every $v \in F$ we have

$$v \in Dh(x)^{-1}(0) \iff Dh(x)v = 0 \iff PDg(x)v = 0 \iff Dg(x)v \in P^{-1}(0) = T_{g(x)}M \iff v \in Dg(x)^{-1}T_xM.$$  

Using this we obtain

$$T_x(g^{-1}(M) \cap V) = Dh(x)^{-1}(0) = Dg(x)^{-1}T_xM.$$  

\[\square\]

**Proposition 7.2.** Suppose $W$ is an open subset of a finite-dimensional normed space $V,$ $b \in W,$ $F$ is a Fréchet space, $j : V \supset W \to F$ is a $C^1_F$, map, and $Dj(b)$ is injective. Then there is an open neighbourhood $N$ of $j(b)$ in $F$ such that $N \cap j(W)$ is a $C^1_F$-submanifold of $F,$ with $T_{j(b)}(N \cap j(W)) = Dj(b)V$ (hence $\dim (N \cap j(W)) = \dim V$).
Proposition 8.1. The map $f$ is $C_{MB}^1$-smooth, with

$$Df(\xi)\eta = (f_j'(x_j)y_j)^\infty$$

for $\xi = (x_j)^\infty$, $\eta = (y_j)^\infty$. 

Proof. 1. The topology induced by $F$ on the finite-dimensional subspace $Y = Dj(b)V$ of $F$ is given by a norm [14 Section 1.19], and $Y$ has a closed complementary space $Z \subset F$, see [14 Lemma 4.21]. The projection $P : F \to F$ along $Z$ onto $Y$ is linear and continuous ([14 Theorem 5.16]). The map $P \circ j$ is $C_{F}^1$-smooth and defines a $C_{F}^1$-map $W \to Y$. Its derivative at $b$ is an isomorphism $V \to Y$ (use $Py = y$ on $Y$ and the injectivity of $Dj(b)$). The Inverse Mapping Theorem (for maps between finite-dimensional normed spaces) yields a $C_{F}^1$-map $g : Y \cap U \to V$, $U$ open in $F$ and $P(j(b)) \in Y \cap U$, such that $g(P(j(b))) = b$, and an open neighbourhood $W_1 \subset W$ of $b$ in $V$ such that $g(Y \cap U) = W_1$, $(P \circ j)(W_1) = Y \cap U$, $(g \circ (P \circ j))(v) = v$ on $W_1$, and $((P \circ j) \circ g)(y) = y$ on $Y \cap U$. It follows that the map $h : Y \cap U \to Z$ given by

$$h(y) = ((id_P - P) \circ j \circ g)(y)$$

is $C_{F}^1$-smooth.

2. Proof of $j(W_1) = \{y + h(y) : y \in Y \cap U\}$: (a) For $y \in Y \cap U$,

$$y + h(y) = y + ((id_P - P) \circ j \circ g)(y) = ((P \circ j) \circ g)(y) + (j \circ g)(y) - ((P \circ j) \circ g)(y) = j(g(y)) \in j(W_1).$$

(b) For $x \in j(W_1)$ there exists $y \in Y \cap U$ with

$$x = j(g(y)) = ((P \circ j) \circ g)(y) + j(g(y)) - (P \circ j)(g(y)) = y + ((id_P - P) \circ j \circ g)(y) = y + h(y).$$

The graph representation of $j(W_1)$ now yields that it is a $C_{F}^1$-submanifold of $F$. □

8. $C_{MB}^1$-Maps which are not $C_{F}^1$-Smooth

Let $N$ denote the Banach space of sequences $\xi = (x_j)^\infty$ in $\mathbb{R}$ with limit 0, with $|\xi| = \max_{j \in \mathbb{N}} |x_j|$. For $j \in \mathbb{N}$ choose a continuously differentiable function $f_j : \mathbb{R} \to \mathbb{R}$ with $f_j(0) = 0$ and $f_j'(u) = ju$ on $[-1/j, 1/j]$ and

$$\sup_{u \in \mathbb{R}} |f_j'(u)| \leq 2$$

for all $j \in \mathbb{N}$.

Then the sequence $(f_j')^\infty$ is not equicontinuous.

For every $\xi = (x_j)^\infty \in N$ and $\eta = (y_j)^\infty \in N$ we have

$$|f_j(x_j)| \leq 2|x_j| \quad \text{and} \quad |f_j(x_j) - f_j(y_j)| \leq 2|x_j - y_j|$$

for all $j \in \mathbb{N}$, and we obtain a Lipschitz continuous map

$$f : N \ni \xi \mapsto (f_j(x_j))_{j \in \mathbb{N}} \in N.$$

Notice that for $\xi$ and $\eta$ in $N$ we also have $(f_j'(x_j)y_j)^\infty \in N$.

Proposition 8.1. The map $f$ is $C_{MB}^1$-smooth, with

$$Df(\xi)\eta = (f_j'(x_j)y_j)^\infty$$

for $\xi = (x_j)^\infty$, $\eta = (y_j)^\infty$. 

Proof. 1. (Directional derivatives) For $\xi = (x_j)_1^\infty \in N$ and $\eta = (y_j)_1^\infty \in N$ set $A(\xi, \eta) = (f'_j(x_j)y_j)_1^\infty \ (N)$. For every real $h \neq 0$ we have

$$|h^{-1}(f(\xi + h\eta) - f(\xi)) - A(\xi, \eta)| = \sup_{j \in N} |h^{-1}(f_j(x_j + h y_j) - f_j(x_j)) - f'_j(x_j)y_j|$$

$$= \sup_{j \in N} \left| \int_0^1 (f'_j(x_j + \theta h y_j) - f'_j(x_j)y_j) d\theta \right|$$

$$\leq \sup_{j \in N} \max_{0 \leq \theta \leq 1} |(f'_j(x_j + \theta h y_j) - f'_j(x_j)y_j)|.$$ 

Let $\epsilon > 0$. There exists $j(\epsilon) \in \mathbb{N}$ with

$$|y_j| \leq \frac{\epsilon}{8} \quad \text{for all integers} \quad j > j(\epsilon).$$

For each $j \in \mathbb{N}$ with $j \leq j(\epsilon)$ the continuity of $f'_j$ yields $h_j > 0$ such that for all $h \in (-h_j, h_j)$ and for all $\theta \in [0, 1]$ we have

$$|f'_j(x_j + \theta h y_j) - f'_j(x_j)| < \frac{\epsilon}{2(|\eta| + 1)}.$$ 

For reals $h$ with $|h| < \min\{h_j : j \in \mathbb{N}, 1 \leq j \leq j(\epsilon)\}$ we obtain

$$|h^{-1}(f(\xi + h\eta) - f(\xi)) - A(\xi, \eta)|$$

$$\leq \max_{j \in N} \max_{0 \leq \theta \leq 1} |(f'_j(x_j + \theta h y_j) - f'_j(x_j)y_j)|$$

$$\leq \sum_{j=1}^{j(\epsilon)} \max_{0 \leq \theta \leq 1} |(f'_j(x_j + \theta h y_j) - f'_j(x_j)y_j)|$$

$$+ \sup_{j \in N,j > j(\epsilon)} \max_{0 \leq \theta \leq 1} |(f'_j(x_j + \theta h y_j) - f'_j(x_j)y_j)|$$

$$\leq \frac{\epsilon|\eta|}{2(|\eta| + 1)} + (2 + 2) \frac{\epsilon}{8} < \epsilon.$$ 

We have shown that

$$Df(\xi) \eta = \lim_{0 \neq h \to 0} h^{-1}(f(\xi + h\eta) - f(\xi))$$

exists and equals $A(\xi, \eta)$. 

2. (Continuity of $N \times N \ni (\xi, \eta) \mapsto Df(\xi) \eta \in N$). Let $\xi_0 = (x_{o_j})_1^\infty \in N$ and $\eta_0 = (y_{o_j})_1^\infty \in N$ be given. For all $\xi = (x_j)_1^\infty \in N$ and $\eta = (y_j)_1^\infty \in N$ we have

$$|Df(\xi) \eta - Df(\xi_0) \eta_0| \leq |(Df(\xi) - Df(\xi_0)) \eta| + |Df(\xi_0)(\eta - \eta_0)|,$$

and $|Df(\xi_0)(\eta - \eta_0)| = \sup_{j \in \mathbb{N}} |f'_j(x_{o_j})(y_j - y_{o_j})| \leq 2|\eta - \eta_0|$ while

$$|(Df(\xi) - Df(\xi_0)) \eta| = \sup_{j \in \mathbb{N}} |(f'_j(x_j) - f'_j(x_{o_j}))y_j|$$

$$\leq \sup_{j \in \mathbb{N}} |(f'_j(x_j) - f'_j(x_{o_j}))(y_j - y_{o_j})| + \sup_{j \in \mathbb{N}} |(f'_j(x_j) - f'_j(x_{o_j}))y_{o_j}|$$

$$\leq (2 + 2)|\eta - \eta_0| + \sup_{j \in \mathbb{N}} |(f'_j(x_j) - f'_j(x_{o_j}))y_{o_j}|.$$ 

From the preceding estimates it is obvious how to complete the proof provided we have

$$\sup_{j \in \mathbb{N}} |(f'_j(x_j) - f'_j(x_{o_j}))y_{o_j}| \to 0 \quad \text{as} \quad \xi \to \xi_0.$$
In order to prove this let $\epsilon > 0$ be given. There exists $j(\epsilon) \in \mathbb{N}$ such that for all integers $j > j(\epsilon)$ we have $4|y_{0j}| < \frac{\epsilon}{2}$. For each $j \in \mathbb{N}$ with $1 \leq j \leq j(\epsilon)$ the continuity of $f'_j$ yields $\delta_j > 0$ with
\[
|f'_j(x) - f'_j(x_{0j})| < \frac{\epsilon}{2(|y_{0j}| + 1)} \quad \text{for all } x \in \mathbb{R} \text{ with } |x - x_{0j}| < \delta_j.
\]
For every $\xi = (x_j)_{j=1}^\infty \in N$ with $|\xi - \xi_0| < \min_{j=1,\ldots,j(\epsilon)} \delta_j$ we get
\[
\max_{j=1,\ldots,j(\epsilon)} |(f'_j(x_j) - f'_j(x_{0j}))y_{0j}| \leq \frac{\epsilon|y_{0j}|}{2(|y_{0j}| + 1)} < \frac{\epsilon}{2}.
\]
It follows that for such $\xi$,
\[
\sup_{j \in \mathbb{N}} |(f'_j(x_j) - f'_j(x_{0j}))y_{0j}| \leq \max_{j=1,\ldots,j(\epsilon)} |(f'_j(x_j) - f'_j(x_{0j}))y_{0j}|
\]
\[
\quad + \sup_{j \in \mathbb{N}, j > j(\epsilon)} |(f'_j(x_j) - f'_j(x_{0j}))y_{0j}|
\]
\[
< \frac{\epsilon}{2} + (2 + 2) \sup_{j \in \mathbb{N}, j > j(\epsilon)} |y_{0j}| \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

\[\square\]

**Proposition 8.2.** There is a sequence $(\xi_k)_{k=1}^\infty$ in $N$ with $\lim_{k \to \infty} \xi_k = 0 \in N$ such that $(Df(\xi_k))_{k=1}^\infty$ does not converge to $Df(0)$ in the $\beta$-topology.

**Proof.** Recall $f'_j(1/j) = 1$ for all $j \in \mathbb{N}$. For $k \in \mathbb{N}$ consider $\eta_k = (\delta_{kj})_{j=1}^\infty \in N$ and $\xi_k = \frac{1}{k}\eta_k \in N$. We have $|\eta_k| = 1$ for all $k \in \mathbb{N}$, and $\xi_k \to 0$ in $N$ as $k \to \infty$. With $Df(0) = 0$,
\[
|Df(\xi_k) - Df(0)| = |Df(\xi_k)| \geq |Df(\xi_k)\eta_k| = |f'_k(1/k)| = 1
\]
for all $k \in \mathbb{N}$.

So, $f$ is not $C^1_{F^}\$-smooth.

Next, consider the Banach space
\[
l^1 = \left\{ (\xi_k)_{k=1}^\infty \in N : \sum_{k=1}^\infty |\xi_k| < \infty \right\} \text{ with } |(\xi_k)_{k=1}^\infty| = \sum_{k=1}^\infty |\xi_k|.
\]

Obviously,
\[
\sum_{k=1}^\infty |f_k(\xi_k) - f_k(\eta_k)| \leq 2 \sum_{k=1}^\infty |\xi_k - \eta_k| \quad \text{for all } (\xi_k)_{k=1}^\infty \in l^1, \quad (\eta_k)_{k=1}^\infty \in l^1,
\]
and the map $f$ from Proposition 8.1 defines a map $\hat{f} : l^1 \to l^1$ which is Lipschitz continuous.

**Proposition 8.3.** $\hat{f}$ is $C^1_{MB}$-smooth with
\[
D\hat{f}(\xi)\eta = (f'_j(\xi_j)\eta_j)_{j=1}^\infty \quad \text{for } \xi = (\xi_j)_{j=1}^\infty \in l^1, \quad \eta = (\eta_j)_{j=1}^\infty \in l^1.
\]

Before giving the proof (which is similar to the proof of Proposition 8.1) consider the composition $s \circ \hat{f} : l^1 \to \mathbb{R}$ with the continuous linear map
\[
s : l^1 \ni (\xi_j)_{j=1}^\infty \mapsto \sum_{j=1}^\infty \xi_j \in \mathbb{R}.
\]
The composition is \(C^1_{MB}\)-smooth but not \(C^2_{B}\)-smooth because we have \(D(s \circ \hat{f})(0) = sD\hat{f}(0) = 0\) and, for sequences \(l^1 \ni \xi_k \to 0 \in l^1\) and \(\eta_k \in l^1\) with \(|\eta_k| = 1\) as in the proof of Proposition 8.2,

\[
|D(s \circ \hat{f})(\xi_k) - D(s \circ \hat{f})(0)| = |sD\hat{f}(\xi_k)| \geq |sD\hat{f}(\xi_k)\eta_k| = |f'_k(1/k)| = 1
\]

for all \(k \in \mathbb{N}\), which excludes \(\beta\)-continuity of \(D(s \circ \hat{f})\).

**Proof** of Proposition 8.3. 1. (Directional derivatives) For \(\xi = (x_j)_1^\infty \in l^1\) and \(\eta = (y_j)_1^\infty \in l^1\) set \(A(\xi, \eta) = (f'_j(x_j)y_j)_1^\infty \in l^1\). For every real \(h \neq 0\) we have

\[
|h^{-1}(\hat{f}(\xi + h\eta) - \hat{f}(\xi)) - A(\xi, \eta)| = \sum_{1}^{\infty} |h^{-1}(f_j(x_j + hy_j) - f_j(x_j)) - f'_j(x_j)y_j| = \sum_{1}^{\infty} \left| \int_{0}^{1} (f'_j(x_j + \theta hy_j)y_j - f'_j(x_j)y_j) d\theta \right| \leq \sum_{1}^{\infty} \max_{0 \leq \theta \leq 1} |(f'_j(x_j + \theta hy_j) - f'_j(x_j))y_j|.
\]

Let \(\epsilon > 0\). There exists \(j(\epsilon) \in \mathbb{N}\) with

\[
\sum_{j(\epsilon)}^{\infty} 4|y_j| \leq \frac{\epsilon}{2}.
\]

For each \(j \in \mathbb{N}\) with \(j \leq j(\epsilon)\) we obtain from the continuity of \(f'_j\) that there exists \(h_j > 0\) such that for all \(u \in (-h_j, h_j)\) and for all \(\theta \in [0, 1]\) we have

\[
|f'_j(x_j + \theta hy_j) - f'_j(x_j)||y_j| |2 j(\epsilon) < \epsilon.
\]

For reals \(h\) with \(|h| < \min_{j=1,\ldots,j(\epsilon)} h_j\) we obtain

\[
|h^{-1}(\hat{f}(\xi + h\eta) - \hat{f}(\xi)) - A(\xi, \eta)| \leq \sum_{1}^{\infty} \max_{0 \leq \theta \leq 1} |(f'_j(x_j + \theta hy_j) - f'_j(x_j))y_j| 
\leq \sum_{1}^{j(\epsilon)} \max_{0 \leq \theta \leq 1} |(f'_j(x_j + \theta hy_j) - f'_j(x_j))y_j| + \sum_{j(\epsilon)+1}^{\infty} \max_{0 \leq \theta \leq 1} |(f'_j(x_j + \theta hy_j) - f'_j(x_j))y_j|
\leq \frac{\epsilon}{2} + \sum_{j(\epsilon)+1}^{\infty} (2 + 2|y_j|) < \epsilon.
\]

We have shown that

\[
D\hat{f}(\xi)\eta = \lim_{0 \neq h \to 0} h^{-1}(\hat{f}(\xi + h\eta) - \hat{f}(\xi))
\]

exists and equals \(A(\xi, \eta)\).

2. (Continuity of \(l^1 \times l^1 \ni (\xi, \eta) \mapsto D\hat{f}(\xi)\eta \in l^1\)) Let \(\xi_0 = (x_{0_j})_1^\infty \in l^1\) and \(\eta_0 = (y_{0_j})_1^\infty \in l^1\) be given. For all \(\xi = (x_j)_1^\infty \in l^1\) and \(\eta = (y_j)_1^\infty \in l^1\) we have

\[
|D\hat{f}(\xi)\eta - D\hat{f}(\xi_0)\eta_0| \leq |(D\hat{f}(\xi) - D\hat{f}(\xi_0))\eta| + |D\hat{f}(\xi_0)(\eta - \eta_0)|,
\]

and

\[
|D\hat{f}(\xi_0)(\eta - \eta_0)| = \sum_{1}^{\infty} |f'_j(x_{0_j})(y_j - y_{0_j})| \leq 2|\eta - \eta_0| \leq 2|\eta - \eta_0|\]

while

\[
|(D\hat{f}(\xi) - D\hat{f}(\xi_0))\eta| = \sum_{1}^{\infty} |(f'_j(x_j) - f'_j(x_{0_j}))y_j|.
\]
\[
\sum_{j=1}^{\infty} |(f_j'(x_j) - f_j'(x_{0j}))(y_j - y_{0j})| + \sum_{j=1}^{\infty} |(f_j'(x_j) - f_j'(x_{0j})))y_{0j}|
\]

\[
\leq (2 + 2)|\eta - \eta_0| + \sum_{j=1}^{\infty} |(f_j'(x_j) - f_j'(x_{0j}))y_{0j}|
\]

From the preceding estimates it is obvious how to complete the proof provided we have

\[
\sum_{j=1}^{\infty} |(f_j'(x_j) - f_j'(x_{0j}))y_{0j}| \to 0 \quad \text{as} \quad \xi \to \xi_0.
\]

In order to prove this let \( \epsilon > 0 \) be given. There exists \( j(\epsilon) \in \mathbb{N} \) such that for all \( \xi = (x_j)_{j=1}^{\infty} \in l^1 \) we have

\[
\sum_{j=1}^{j(\epsilon)+1} |(f_j'(x_j) - f_j'(x_{0j}))y_{0j}| \leq \sum_{j=1}^{j(\epsilon)+1} (2 + 2)|y_{0j}| < \frac{\epsilon}{2}.
\]

For each \( j \in \mathbb{N} \) with \( 1 \leq j \leq j(\epsilon) \) there exists \( \delta_j > 0 \) with

\[
|f_j'(x_{0j} + z) - f_j'(x_{0j})| |y_{0j}| 2 j(\epsilon) < \epsilon \quad \text{for all} \quad z \in (-\delta_j, \delta_j).
\]

Let \( \delta = \min_{j \in \mathbb{N}: 1 \leq j \leq j(\epsilon)} \delta_j \). For every \( \xi = (x_j)_{j=1}^{\infty} \in l^1 \) with \( |\xi - \xi_0| < \delta \) we get

\[
|x_j - x_{0j}| < \delta_j \quad \text{for all} \quad j \in \{1, \ldots, j(\epsilon)\},
\]

which yields

\[
\frac{\sum_{j=1}^{j(\epsilon)} |(f_j'(x_j) - f_j'(x_{0j}))y_{0j}|}{2 j(\epsilon)} = \frac{\epsilon}{2}.
\]

It follows that

\[
\sum_{j=1}^{\infty} |(f_j'(x_j) - f_j'(x_{0j}))y_{0j}| = \sum_{j=1}^{j(\epsilon)} |(f_j'(x_j) - f_j'(x_{0j}))y_{0j}| + \sum_{j=1}^{\infty} |(f_j'(x_j) - f_j'(x_{0j}))y_{0j}|
\]

\[
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \quad \Box.
\]

Now we use the \( C^1_{MB} \)-map \( f : N \to N \) from Proposition 8.1 in order to construct \( C^1_{MB} \)-maps \( C \to N \) and \( C^1 \to N \) which are not \( C^1_k \)-smooth, for the spaces \( C \) and \( C^1 \) with \( n = 1 \). Let \( C_* \) denote the closed hyperplane given by \( \phi(0) = 0 \). Then

\[
C = C_* \oplus \mathbb{R} \eta
\]

where \( \eta(t) = 1 \) for all \( t \leq 0 \). Let \( P_* : C \to C \) denote the projection onto \( C_* \) along \( \mathbb{R} \eta \). Choose a strictly increasing sequence of points \( t_j < 0, j \in \mathbb{N} \), with limit 0. Choose continuous functions \( e_j : (-\infty, 0] \to [0, 1] \subset \mathbb{R}, j \in \mathbb{N} \), with \( e_j(t_j) = 1 \) for all \( j \in \mathbb{N} \), and with mutually disjoint supports in \( (-\infty, 0) \). For every \( x = (x_j)_{j=1}^{\infty} \in N \) the equation

\[
Jx(t) = \sum_{j=1}^{\infty} x_j e_j(t)
\]

defines a continuous function \( (-\infty, 0] \to \mathbb{R} \) with \( Jx(0) = 0 \), and

\[
\max_{t \leq 0} |Jx(t)| = \max_{j \in \mathbb{N}} |x_j| = |x|.
\]

The map \( J : N \to C_* \) is injective, linear, and continuous.
(1) A $C^1_{MB}$-map $C \rightarrow N$ which is not $C^1_F$-smooth. Consider the linear evaluation map
$$E : C_* \ni \phi \mapsto (\phi(t_j))_{j=1}^\infty \in N.$$ which is continuous. The map $g = f \circ E \circ P_*$ from $C$ into $N$ is $C^1_{MB}$-smooth, due to the chain rule for $C^1_{MB}$-maps. We show that it is not $C^1_F$-smooth: Otherwise the composition $g \circ J$ from $N$ into $N$ is $C^1_F$-smooth (due to the chain rule for $C^1_F$-maps). For each $x \in N$ we have
$$\begin{align*}
(g \circ J)(x) &= f(E(P_* Jx)) = f(E(Jx)) \quad \text{(since } Jx \in C_{I*}) \\
&= f(x)
\end{align*}$$ which yields a contradiction to the fact that $f$ is not $C^1_F$-smooth.

(2) A $C^1_{MB}$-map $C^1 \rightarrow N$ which is not $C^1_F$-smooth. Let $C^1_* \subset C^1$ denote the closed hyperplane given by $\phi'(0) = 0$. We have $C^1 = C^1_* \oplus \mathbb{R} \iota$ with $\iota(t) = t$ for all $t \leq 0$. Let $P^1_* : C^1 \rightarrow C^1$ denote the projection onto $C^1_*$ along $\mathbb{R} \iota$. The equation
$$\text{int}(\phi)(t) = -\int_t^0 \phi(s)ds$$ defines a continuous linear map $\text{int} : C \rightarrow C^1$ with $\text{int}(C_*) \subset C^1_*$. Observe that $\partial(C^1_*) \subset C_*$, and $\partial(\text{int} \phi) = \phi$ for all $\phi \in C$. Consider $g = f \circ E \circ \partial \circ P^1_*$ from $C^1$ into $N$, which is $C^1_{MB}$-smooth due to the chain rule for $C^1_{MB}$-maps. We show that $g$ is not $C^1_F$-smooth: Otherwise the composition $g \circ \text{int} \circ J$ from $N$ into $N$ is $C^1_F$-smooth as well, due to the chain rule for $C^1_F$-maps. For each $x \in N$ we have
$$\begin{align*}
(g \circ \text{int} \circ J)(x) &= f(E(\partial(P^1_*(\text{int}(Jx)))))) \\
&= f(E(\partial(\text{int}(Jx)))) \quad \text{(since } Jx \in C_* \text{ and } \text{int}(Jx) \in C^1_*) \\
&= f(E(Jx)) = f(x)
\end{align*}$$ which yields a contradiction to the fact that $f$ is not $C^1_F$-smooth.

In the same way one finds examples of maps from Banach spaces $C^{ST}$ and $C^{1}_{ST}$ into $N$ which are $C^1_{MB}$-smooth but not $C^1_F$-smooth.

**Remark 8.4.** The examples above with their infinite-dimensional target spaces are not related to the delay differential equation (1.1). See [23] for the construction of maps $C \rightarrow \mathbb{R}^n$ and $C^1 \rightarrow \mathbb{R}^n$ which are $C^1_{MB}$-smooth but not $C^1_F$-smooth.


9. Examples, and the Solution Manifold

We begin with the toy example (1.2),

\[ x'(t) = h(x(t - d(x(t)))) \]

with continuously differentiable functions \( h : \mathbb{R} \to \mathbb{R} \) and \( d : \mathbb{R} \to [0, \infty) \subset \mathbb{R} \). For continuously differentiable functions \( (-\infty, t_e), 0 < t_e \leq \infty \), which satisfy Eq. (1.2) for \( 0 \leq t < t_e \) this delay differential equation has the form (1.1) for \( U = C^1 \) with \( n = 1 \) and \( f = f_{h,d} \) given by

\[ f_{h,d}(\phi) = h(\phi(-d(\phi(0)))) \]

In order to see that \( f_{h,d} \) is a composition of \( C^1 \)-maps all defined on open sets of Fréchet spaces it is convenient to introduce the odd prolongation maps \( P_{odd} : C \to C_\infty \) (with \( n = 1 \)) and \( P_{odd,1} : C^1 \to C^1_\infty \) (with \( n = 1 \)) which are defined by the relations

\[ (P_{odd}\phi)(t) = \phi(t) \quad \text{for} \quad t \leq 0, \quad (P_{odd}\phi)(t) = -\phi(-t) + 2\phi(0) \quad \text{for} \quad t > 0, \]

and \( P_{odd,1}\phi = P_{odd}\phi \) for \( \phi \in C^1 \). Both maps are linear and continuous. With the evaluation map

\[ ev_{\infty,1} : C^1_\infty \times \mathbb{R} \to \mathbb{R}, \quad ev_{\infty,1}(\phi, t) = \phi(t), \]

we have

\[ f_{h,d}(\phi) = h \circ ev_{\infty,1}(P_{odd,1}\phi, -d(\phi(0))) \]

for all \( \phi \in C^1 \). We also need the evaluation map \( ev_{\infty} : C_\infty \times \mathbb{R} \to \mathbb{R} \) given by \( ev_{\infty}(\phi, t) = \phi(t) \).

**Proposition 9.1.** The map \( ev_{\infty} \) is continuous and the map \( ev_{\infty,1} \) is \( C^1 \)-smooth with

\[ Dev_{\infty,1}(\phi, t)(\chi, s) = D_1ev_{\infty,1}(\phi, s)\chi + D_2ev_{\infty,1}(\phi, s)t = \chi(t) + s\phi'(t) \]

**Proof.** Arguing as in the proof of [20, Proposition 2.1] one shows that \( ev_{\infty} \) is continuous and that \( ev_{\infty,1} \) is \( C^1_{MB} \)-smooth, and that the directional derivatives satisfy the equations in the proposition. It remains to prove that the map \( C^1_\infty \times \mathbb{R} \ni (\phi, t) \mapsto D ev_{\infty,1}(\phi, t) \in L_c(C^1_\infty \times \mathbb{R}, \mathbb{R}) \) is \( \beta \)-continuous. As \( C^1_\infty \times \mathbb{R} \) has countable neighbourhood bases it is enough to show that, given a sequence \( C^1_\infty \times \mathbb{R} \ni (\phi_k, t_k) \to (\phi, t) \in C^1_\infty \times \mathbb{R} \) for \( k \to \infty \), a neighbourhood \( N \) of 0 in \( \mathbb{R} \) and a bounded subset \( B \subset C^1_\infty \times \mathbb{R} \), we have

\[ (D ev_{\infty,1}(\phi_k, t_k) - D ev_{\infty,1}(\phi, t))B \subset N \quad \text{for} \quad k \quad \text{sufficiently large}. \]

In order to prove this, choose \( j \in \mathbb{N} \) with \( |t| < j \) and \( |t_k| < j \) for all \( k \in \mathbb{N} \). By [14, Theorem 1.37], \( c_j = \sup_{(\chi, s) \in B} |(\chi|_{1,\infty,j} + |s|) \subset \mathbb{R} \). For every \( k \in \mathbb{N} \) and \( (\chi, s) \in B \),

\[
\begin{align*}
|D ev_{\infty,1}(\phi_k, t_k) - D ev_{\infty,1}(\phi, t))(\chi, s)| &= |\chi(t_k) - \chi(t) + s(\phi_k'(t_k) - \phi'(t))| \\
&\leq \max_{-j \leq u \leq j} |\chi'(u)||t_k - t| + c_j(\max_{-j \leq u \leq j} |\phi_k'(u) - \phi'(u)| + |\phi'(t_k) - \phi'(t)|)) \\
&\leq c_j(|t_k - t|) + |\phi_k - \phi|_{\infty,1,j} + |\phi'(t_k) - \phi'(t)|),
\end{align*}
\]

and it becomes obvious how to complete the proof. \( \square \)
The map $ev_1(\cdot, 0) : C^1 \ni \phi \mapsto \phi(0) \in \mathbb{R}$ is linear and continuous, and the evaluation $ev : C \times (-\infty, 0] \ni (\phi, t) \mapsto \phi(t) \in \mathbb{R}$ is continuous, see \cite{20} Proposition 2.1.

The next result says that $f_{h, d}$ satisfies the hypotheses for the results on semiflows and local invariant manifolds in the subsequent sections.

**Corollary 9.2.** For $d : \mathbb{R} \to [0, \infty) \subset \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ continuously differentiable the map $f_{h, d}$ is $C^1$-smooth and has property (e).

**Proof.** The functions $d$ and $h$ are $C^1$-smooth. The map $ev_1(\cdot, 0)$ is linear and continuous, hence $C^1$-smooth. It follows that $P_{odd,1} \times (-d \circ ev_1(\cdot, 0)) : C^1 \to C^1 \times \mathbb{R}$ is $C^1$-smooth, by the chain rule (Proposition 3.5) and by $C^1$-smoothness of maps into product spaces. Now use that $ev_{\infty,1}$ is $C^1$-smooth, due to Proposition 9.1, and apply the chain rule to the composition

$$f_{h, d} = h \circ ev_{\infty,1} \circ (P_{odd,1} \times (-d \circ ev_1(\cdot, 0))).$$

It follows that $f_{h, d}$ is $C^1$-smooth with

$$Df_{h, d}(\phi) \chi = h'(\phi(0))\chi + \phi'(0)\chi.$$

For each $\phi \in C^1$ this term on the right hand side of this equation defines a linear continuation $Df_{h, d}(\phi) : C \to \mathbb{R}$ of $Df_{h, d}(\phi)$. Using that the evaluation $ev$ and differentiation $C^1 \to C$ are continuous one finds that the map

$$C^1 \times C \ni (\phi, \chi) \mapsto Df_{h, d}(\phi) \chi \in \mathbb{R}$$

is continuous.

The pantograph equation (1.3), namely,

$$x'(t) = ax(\lambda t) + bx(t)$$

with constants $a \in \mathbb{C}$, $b \in \mathbb{R}$ and $0 < \lambda < 1$, was extensively studied in \cite{8}. For real parameters $a, b$ and arguments $t > 0$ this is a nonautonomous linear equation with unbounded delay $\tau(t) = (1 - \lambda)t > 0$ since $\lambda t = t - \tau(t)$. Define $F : \mathbb{R} \times C^1 \to \mathbb{R}$ by

$$F(t, \phi) = a(P_{odd,1}(\phi)(-\tau(t))) + b\phi(0),$$

or,

$$F = a \circ ev_{\infty,1} \circ ((P_{odd,1} \circ pr_2) \times (-\tau \circ pr_1)) + b \circ ev_{\infty,1}(\cdot, 0) \circ P_{odd,1} \circ pr_2,$$

with the projections $pr_j$ onto the first and second component, respectively. The map $F$ is $C^1$-smooth, and every continuously differentiable function $x : (-\infty, t_e) \to \mathbb{R}$, $0 < t_e \leq \infty$, which satisfies the pantograph equation for $0 \leq t < t_e$ also solves the nonautonomous equation

$$x'(t) = F(t, x_t)$$

for $0 \leq t < t_e$. The role of the odd prolongation map in the definition of $F$ is to allow arguments $(t, \phi)$ with $t < 0$, for which $-\tau(t) > 0$. The solutions of Eq. (9.1) can be obtained from the autonomous equation (1.1) with $n = 2$ and $f : C^1 \to \mathbb{R}^2$ given by

$$f_1(\phi_1, \phi_2) = 1, \quad f_2(\phi_1, \phi_2) = F(\phi_1(0), \phi_2)$$

in the familiar way: If the continuously differentiable map $x : (\infty, t_e) \to \mathbb{R}$ satisfies Eq. (9.1) for $t_0 \leq t < t_e \leq \infty$ then $(s, z) : (-\infty, t_e - t_0) \to \mathbb{R}^2$ given by

$$s(t) = t + t_0 \quad \text{and} \quad z(t) = x(t + t_0)$$
satisfies the system
\[
\begin{align*}
s'(t) &= 1 = f_1(s_t, z_t) \\
z'(t) &= F(s(t), z_t) = f_2(s_t, z_t)
\end{align*}
\]
for \(0 \leq t < t_e\) and \(s(0) = t_0\). The map \(f\) is \(C^1\)-smooth and has the extension property (e).

For the Volterra integro-differential equation (1.4),
\[
x'(t) = \int_0^t k(t, s)h(x(s))ds
\]
with \(k : \mathbb{R}^{n \times n} \to \mathbb{R}^n\) and \(h : \mathbb{R}^n \to \mathbb{R}^n\) continuously differentiable the scenario is simpler than in both cases above where delays are discrete. In [22] it is shown that every continuous function \((-\infty, t_e) \to \mathbb{R}^n\), \(0 < t_e \leq \infty\), which for \(0 < t < t_e\) is differentiable and satisfies Eq. (1.4), also satisfies an equation of the form (9.1) for \(0 < t < t_e\), with the \(C^1_T\)-map \(F = F_{k,h}\) in Eq. (9.1) defined on the space \(\mathbb{R} \times C\). The associated autonomous equation of the form (1.1) is given by the \(C^1_T\)-map \(f_{k,h} : C((\infty, 0], \mathbb{R}^{n+1}) \to \mathbb{R}^{n+1}\) with
\[
f_{k,h} = f_1 \times \hat{f}, \quad f_1(\psi) = 1, \quad \psi = (\psi_1, \phi), \quad \hat{f}(\psi) = F(\psi_1(0), \phi).
\]
It follows that the restriction of \(f_{k,h}\) to \(C^1((\infty, 0], \mathbb{R}^{n+1})\) is \(C^1_T\)-smooth and has property (e), which means that the hypotheses for the theory of Eq. (1.1) in the following sections, with a semiflow on the solution manifold in \(C^1((\infty, 0], \mathbb{R}^{n+1})\), are satisfied. However, in the present case we also get a nice semiflow without recourse to this theory. A result in [22] for Eq. (1.1) with a map \(f : C \supset U \to \mathbb{R}^n\) which is \(C^1_T\)-smooth establishes a continuous semiflow on \(U\), with all solution operators \(C^1_T\)-smooth. In the present case, with \(f = f_{k,h}\), the semiflow yields a process of solution operators for the nonautonomous equation (9.1), all of them defined on open subsets of \(C((\infty, 0], \mathbb{R}^n)\) and \(C^1_T\)-smooth. The process incorporates all solutions of the Volterra integro-differential equation.

The first ingredient of the present, more general theory of Eq. (1.1) is the solution manifold
\[
X_f = \{\phi \in U : \phi'(0) = f(\phi)\}.
\]

**Proposition 9.3.** For a \(C^1_T\)-map \(f : C \supset U \to \mathbb{R}^n\) with property (e) and \(X_f \neq \emptyset\) the set is a \(C^1_T\)-submanifold of codimension \(n\) in the space \(C^1\), with tangent spaces
\[
T_\phi X_f = \{\chi \in C^1 : \chi'(0) = Df(\phi)\chi\} \quad \text{for all} \quad \phi \in X_f.
\]

**Proof.** \(X_f\) is the preimage of \(0 \in \mathbb{R}^n\) under the map \(C^1_T\)-map \(g : C^1 \supset \emptyset \ni \phi \mapsto \phi'(0) - f(\phi) \in \mathbb{R}^n\). [23] Proposition 2.2 applies as \(f\) also is \(C^1_{MB}\)-smooth. It follows that all derivatives \(Dg(u), u \in U\), are surjective. Apply Proposition 7.1 to \(g\) and \(M = \{0\}\).

\[\square\]

10. Evaluation maps

For the construction of solutions of Eq. (1.1) we need a few facts about evaluation maps. The segment evaluation maps
\[
E_T : C_T \times (-\infty, T] \ni (\phi, t) \mapsto \phi_t \in C,
\]
\[
E_T^1 : C^1_T \times (-\infty, T] \ni (\phi, t) \mapsto \phi_t \in C^1
\]
and

\[ E_T^{10} : C_T^1 \times (-\infty, T] \ni (\phi, t) \mapsto \phi_t \in C \]

for \( T \in \mathbb{R} \) and their analogues \( E_\infty^{1}, E_{10}^{1} \) for \( T = \infty \) are all linear in the first argument.

**Proposition 10.1.** Let \( T \leq \infty \).

(i) The maps \( E_T \) and \( E_T^{10} \) are continuous.

(ii) For every \( \phi \in C_T^1 \), the curve \( \Phi : (-\infty, T) \ni t \mapsto \phi_t \in C \) is continuously differentiable, with \( \Phi'(t) = E_T(\partial_T \phi, t) \).

(iii) The map \( E_T^{10} : C_T^1 \times (-\infty, T) \) is \( C^1 \)-smooth, with

\[
D_1 E_T^{10}(\phi, t) \hat{\phi} = E_T^{10}(\hat{\phi}, t) = \hat{\phi}_t \quad \text{and} \quad D_2 E_T^{10}(\phi, t) s = s E_T(\partial_T \phi, t) = s(\phi')_t.
\]

**Proof.** 1. For assertions (i) and (ii), and for the fact that the map \( E_T^{10} : C_T^1 \times (-\infty, T) \to C_T^1 \) is \( C^1 \)-smooth, see the proof of [20 Proposition 3]. It remains to show that \( D E_T^{10} \) is \( \beta \)-continuous. Let a sequence \((\phi_j, t_j)^\infty \) in \( C_T^1 \times (-\infty, T) \) be given which converges to some \((\phi, t) \in C_T^1 \times (-\infty, T) \). Let a bounded subset \( B \subset C_T^1 \times \mathbb{R} \) and a neighbourhood \( V \) of 0 in \( C \) be given. We may assume

\[ V = \{ \chi \in C : |\chi| < \frac{1}{T} \} \quad \text{for some integer } \ l > 0 \]

and have to show that for \( j \) sufficiently large,

\[ (D E_T^{10}(\phi_j, t_j) - D E_T^{10}(\phi, t)) B \subset V. \]

For every \((\hat{\phi}, \hat{t}) \in C_T^1 \times \mathbb{R} \) and for all \( j \in \mathbb{N} \),

\[ (D E_T^{10}(\phi_j, t_j) - D E_T^{10}(\phi, t))(\hat{\phi}, \hat{t}) = \hat{\phi}_{t_j} - \hat{\phi}_t + \hat{t}[(\phi_j')_{t_j} - (\phi')_t]. \]

2. As the projections from \( C_T^1 \times \mathbb{R} \) onto \( C_T^1 \) and onto \( \mathbb{R} \) are continuous and linear they map the bounded set \( B \) into bounded sets, and we obtain that for some real \( r > 0 \) and for all \( k \in \mathbb{N} \),

\[ \{ \hat{t} \in \mathbb{R} : \text{For some } \hat{\phi} \in C_T^1, (\hat{\phi}, \hat{t}) \in B \} \subset [-r, r] \]

and

\[ \sigma_{1,T,k} = \sup\{(|\hat{\phi}|_{1,T,k} \in \mathbb{R} : \text{For some } \hat{t} \in \mathbb{R}, (\hat{\phi}, \hat{t}) \in B \} < \infty. \]

3. Choose \( k \in \mathbb{N} \) so large that for all \( s \in [-l, 0] \) and for all \( j \in \mathbb{N} \),

\[ T - k < s + t_j < T \quad \text{and} \quad T - k < s + t < T. \]

Consider \((\hat{\phi}, \hat{t}) \in B \). For each \( j \in \mathbb{N} \) we have

\[
|\hat{\phi}_{t_j} - \hat{\phi}_t| = \max_{-l \leq s \leq 0} |\hat{\phi}(t_j + s) - \hat{\phi}(t + s)| \\
\leq \max_{-T-k \leq u \leq T} |(\phi'_j)(u)||t_j - t| \leq \sigma_{1,T,k}|t_j - t|
\]

and

\[
|\hat{t}[(\phi_j')_{t_j} - (\phi')_t]| \leq r[|((\phi_j')_{t_j} - (\phi')_t)| + |(\phi_j')_{t_j} - (\phi')_t|] \\
\leq r \max_{-l \leq s \leq 0} |(\phi_j')(s + t_j) - (\phi')(s + t_j)| + \max_{-l \leq s \leq 0} |(\phi')_j(s + t_j) - (\phi')(s + t)| \\
\leq r \max_{-T-k \leq u \leq T} |(\phi'_j)(u) - (\phi')(u)| + \max_{-l \leq s \leq 0} |(\phi')(s + t_j) - (\phi')(s + t)|. \]
Altogether, for every \( j \in \mathbb{N} \) and for all \((\hat{\phi}, \hat{t}) \in B\),

\[
| (DE^{10}_T(\phi_j, t_j) - DE^{10}_T(\phi, t))(\hat{\phi}, \hat{t}) | \\
\leq \sigma_{1,T,k}|t_j - t| + r|\phi_j - \phi|_{1,T,k} + \max_{-\infty \leq s \leq 0} |(\phi')(s + t_j) - (\phi')(s + t)|.
\]

Using \( t_j \to t \) and \( |\phi_j - \phi|_{1,T,k} \to 0 \) as \( j \to \infty \) and the uniform continuity of \( \phi' \) on \([T - k, T]\) one finds \( J \in \mathbb{N} \) such that for all integers \( j \geq J \) and for all \((\hat{\phi}, \hat{t}) \in B\),

\[
| (DE^{10}_T(\phi_j, t_j) - DE^{10}_T(\phi, t))(\hat{\phi}, \hat{t}) | < \frac{1}{7}
\]

It follows that \((DE^{10}_T(\phi_j, t_j) - DE^{10}_T(\phi, t))B \subset V\) for all integers \( j \geq J\). \( \square \)

11. The fixed point problem, and a substitution operator

In the sequel we always assume that \( U \subset C^1 \) is open and that \( f : U \to \mathbb{R}^n \) is \( C^1 \)-smooth and has the property (e).

Following \[20\] we rewrite the initial value problem

(11.1)

\[
x'(t) = f(x_t) \quad \text{for} \quad t \geq 0, \quad x_0 = \phi \in X_f
\]

as a fixed point equation: Suppose \( x : (-\infty, T] \to \mathbb{R}^n, T > 0 \), is a solution of Eq. (1.1) on \([0, T]\) with \( x_0 = \phi \). Extend \( \phi \) by \( \phi(t) = \phi(0) + t\phi'(0) \) to a continuously differentiable function \( \hat{\phi} : (-\infty, T] \to \mathbb{R}^n \). Then \( y = x - \hat{\phi} \) satisfies \( y(t) = 0 \) for \( t \leq 0 \), the curve \((\infty, T] \ni s \mapsto x_s \in C^1 \) is continuous (use \( x_s = E^1_T(x, s) \) and apply Proposition 10.1 (i)), as well as the curves \((\infty, T] \ni s \mapsto y_s \in C^1 \) and \((\infty, T] \ni s \mapsto \hat{\phi}_s \in C^1 \). For \( 0 \leq t \leq T \) we get

\[
y(t) = x(t) - \hat{\phi}(t) = x(0) + \int_0^t f(x_s)ds - \phi(0) - t\phi'(0)
\]

\[
= \int_0^t f(y_s + \hat{\phi}_s)ds - t f(\phi)
\]

\[
= \int_0^t f(y_s + \hat{\phi}_s) - f(\phi)ds.
\]

holds Obviously, \( y(0) = 0 = y'(0) \). So \( \eta = y|_{[0,T]} \in C^1_{0,T,0} \) satisfies the fixed point equation

(11.2)

\[
\eta(t) = \int_0^t (f(\eta_s + \hat{\phi}_s) - f(\phi))ds, \quad 0 \leq t \leq T,
\]

where \( \eta \in C^1_T \) is the prolongation of \( \eta \) given by \( \eta(t) = 0 \) for all \( t < 0 \). In order to find a solution of the initial value problem (11.1) one solves the fixed point equation (11.2) by means of a parametrized contraction on a subset of the Banach space \( C^1_{0,T,0} \) with the parameter \( \phi \in U \) in the Fréchet space \( C^1 \). For \( \phi \in X_f \) the associated fixed point \( \eta = \eta_\phi \) yields a solution \( x = \eta + \hat{\phi} \) of the initial value problem (11.1).

The application of a suitable contraction mapping theorem, namely, Theorem 5.2, requires some preparation. We begin with the substitution operator

\[
F_T : \text{dom}_T \to C^1_{0,T}
\]

which for \( 0 < T < \infty \) is given by

\[
\text{dom}_T = \{ \phi \in C^1_T : \text{For } 0 \leq s \leq T, \phi_s \in U \}
\]

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Proof: Consider a seminorm due to lack of smoothness of the map $F_T$ is contained in the neighbourhood $U$.

Proposition 11.1. The map $F_T$, $0 < T < \infty$, is $C^1_T$-smooth.

Proof. 1. Let $\phi \in \text{dom}_T \subset C^1_T$, $\epsilon > 0$, and a bounded set $B \subset C^1_T$ be given. Using the norm on $C_{0T}$ we have to find a neighbourhood $N$ of $\phi$ in $C^1_T$ so that for every $\psi \in N$ and for all $\hat{\phi} \in B$,

$$\max_{0 \leq s \leq T} |((DF_T(\psi) - DF_T(\phi))\hat{\phi})(s)| < \epsilon.$$ 

Define $B_T = \{E^1_T(\hat{\phi}, s) \in C^1 : 0 \leq s \leq T, \hat{\phi} \in B\}$

Claim: $B_T \subset C^1$ is bounded.

Proof: Consider a seminorm $| \cdot |_{1,j}$, $j \in \mathbb{N}$. Choose an integer $k \geq j + T$. The seminorm $| \cdot |_{1,T,k}$ is bounded on $B$. For every $\hat{\phi} \in B$ and for all $s \in [0,T]$ we see from

$$|E^1_T(\hat{\phi}, s)|_{1,j} = \max_{-j \leq u \leq 0} |\hat{\phi}(s + u)| + \max_{-j \leq u \leq 0} |\hat{\phi}'(s + u)|$$

$$\leq \max_{-j \leq w \leq 0} |\hat{\phi}(w)| + \max_{-j \leq w \leq 0} |\hat{\phi}'(w)| \leq |\hat{\phi}|_{1,T,k}$$

that $| \cdot |_{1,j}$ is bounded on $B_T$.

2. For every $\psi \in \text{dom}_T$, $\hat{\phi} \in B$, $s \in [0,T]$ we have

$$((DF_T(\psi) - DF_T(\phi))\hat{\phi})(s) = (D\epsilon f(E^1_T(\psi, s)) - D\epsilon f(E^1_T(\phi, s))E^{10}_T(\hat{\phi}, s)$$

(see [20] Proposition 3.2))

$$= (D f(E^1_T(\psi, s)) - D f(E^1_T(\phi, s))E^{1}_T(\hat{\phi}, s)$$

(with $E^{10}_T(\hat{\phi}, s) \in C^1$),

where $E^1_T(\hat{\phi}, s)$ is in the bounded set $B_T$. As $f$ is $C^1_T$-smooth the composition

$$C^1_T \times \mathbb{R} \supset \text{dom}_T \times [0,T] \ni (\psi, s) \mapsto D f(E^1_T(\psi, s)) \in \mathcal{L}_c(C^1, \mathbb{R}^n)$$

is $\beta$-continuous, hence uniformly $\beta$-continuous on the compact set $\{ \phi \} \times [0,T]$ (see Proposition 1.2). It follows that there is a neighbourhood $N$ of $\phi$ in $C^1_T$ such that for every $\psi \in N$ and for all $s \in [0,T]$ the difference

$$D f(E^1_T(\psi, s)) - D f(E^1_T(\phi, s))$$

is contained in the neighbourhood $U_{U_T(0), B_T}$ of 0 in $\mathcal{L}_c(C^1, \mathbb{R}^n)$, with $U_T(0) = \{ x \in \mathbb{R}^n : |x| < \epsilon \}$. Finally, we obtain for each $\psi \in N$, $s \in [0,T]$, $\hat{\phi} \in B$,

$$|((DF_T(\psi) - DF_T(\phi))\hat{\phi})(s)| = |(D f(E^1_T(\psi, s)) - D f(E^1_T(\phi, s))E^{1}_T(\hat{\phi}, s)| < \epsilon.$$
The prolongation maps

\[ P_T : C^1 \to C^1_T, \quad 0 < T \leq \infty, \]
given by

\[ P_T \phi(t) = \phi(t) \quad \text{for} \quad t \leq 0, \quad P_T \phi(t) = \phi(0) + t\phi'(0) \quad \text{for} \quad 0 < t \leq T, \]
given by

\[ P_{ST} : C^1_{0S} \to C^1_{0T}, \quad 0 < S < T < \infty, \]
given by

\[ P_{ST} \phi(t) = \phi(t) \quad \text{for} \quad 0 \leq t \leq S, \quad P_{ST} \phi(t) = \phi(S) + (t-S)\phi'(S) \quad \text{for} \quad S < t \leq T, \]
given by

\[ Z_T : C^1_{0T,0} \to C_T, \quad 0 < T < \infty \]
given by

\[ Z_T \phi(t) = \phi(t) \quad \text{for} \quad 0 \leq t \leq T, \quad Z_T(\phi)(t) = 0 \quad \text{for} \quad t < 0, \]
and the integration operators

\[ I_T : C^1_{0T,0} \to C^1_{0T,0}, \quad 0 < T < \infty, \quad \text{given by} \quad I_T \phi(t) = \int_0^t \phi(s)ds \]
are all linear and continuous. We have \( Z_TC^1_{0T,0} \subset C^1_T, \) and the induced map \( C^1_{0T,0} \xrightarrow{Z_T} C^1_T \) is continuous, too. For \( P_{ST}, \) \( 0 < S < T, \)

\[ P_{ST}C^1_{0S,0} \subset C^1_{0T,0}, \]
and

\[ |P_{ST}\phi|_{1,0T} \leq (2 + T)|\phi|_{1,0S} \quad \text{for all} \quad \phi \in C^1_{0S} \]
because of the estimate

\[
|P_{ST}\phi|_{1,0T} = \max_{0 \leq t \leq T} |P_{ST}\phi(t)| + \max_{0 \leq t \leq T} |(P_{ST}\phi)'(t)| \\
\leq \max_{0 \leq t \leq S} |\phi(t)| + |\phi(S)| + |\phi'(S)|T + \max_{0 \leq t \leq S} |\phi'(t)|.
\]

It follows that for every \( T > 0 \) the set

\[ D_T = \{(\phi, \eta) \in U \times C^1_{0T,0} : P_T \phi + Z_T \eta \in \text{dom}_T \} \]
is open. Let \( pr_1 \) and \( pr_2 \) denote the projections from \( C^1 \times C^1_{0T,0} \) onto the first and second factor, respectively. Define \( \tau : \mathbb{R}^n \to C^1_{0T} \) by \( \tau(\xi)(t) = \xi. \) Both projections and \( \tau \) are continuous linear maps. Using Proposition 11.1, the chain rule, and linearity of differentiation we infer that the map

\[ G_T : C^1 \times C^1_{0T,0} \ni U \times C^1_{0T,0} \ni D_T \to C^1_{0T,0} \in C^1_{0T} \]
given by

\[ G_T(\phi, \eta) = F_T(P_T pr_1(\phi, \eta) + Z_T pr_2(\phi, \eta)) - \tau \circ f \circ pr_1(\phi, \eta) \]
(notice that \( G_T(\phi, \eta)(0) = f((P_T \phi + Z_T \eta)_0) - f(\phi) = f(\phi + 0) - f(\phi) = 0 \)
is \( C^1_T \)-smooth. For the derivatives we obtain the following result.
Corollary 11.2. Let $0 < T < \infty$. For $(\phi, \eta) \in D_T$ and $\hat{\phi} \in C^1$, $\hat{\eta} \in C^1_{0,T,0}$,

$$DG_T(\phi, \eta)(\hat{\phi}, \hat{\eta}) = D_T(P_T \phi + Z_T \eta)(P_T \hat{\phi} + Z_T \hat{\eta}) - \tau(D_T \phi) \hat{\phi},$$

and for $0 \leq t \leq T$,

$$DG_T(\phi, \eta)(\hat{\phi}, \hat{\eta})(t) = (D_{e,f}(E_T^1(P_T \phi + Z_T \eta, t)E_T^{0}(P_T \hat{\phi} + Z_T \hat{\eta}, t)
- \tau(D_T \phi) \hat{\phi})(t)
= D_{e,f}((P_T \hat{\phi})_t + (Z_T \hat{\eta})_t)((P_T \hat{\phi})_t + (Z_T \hat{\eta})_t)
- \tau(D_T \phi) \hat{\phi}.$$

The map $A_T = I_T \circ G_T$ is $C^1_\rho$-smooth. We now restate [20, Proposition 3.4], which prepares the proof that $A_T$ with $T > 0$ sufficiently small defines a uniform contraction on a small ball in $C^1_{0,T,0}$.

Proposition 11.3. Let $\phi \in U$ be given. There exist $T = T_\phi > 0$, a neighbourhood $V = V_\phi$ of $\phi$ in $U$, $\epsilon = \epsilon_\phi > 0$, and $j = j_\phi \in \mathbb{N}$ such that for all $S \in (0, T)$, $\chi \in V$, $\eta$ and $\tilde{\eta}$ in $C^1_{0,S,0}$ with $|\eta|_{1,0S} < \epsilon$ and $|\tilde{\eta}|_{1,0S} < \epsilon$, $w \in [0, S]$, and $\theta \in [0, 1]$,

$$(P_S \chi)_w + (Z_S \eta)_w + \theta[(Z_S \tilde{\eta})_w - (Z_S \eta)_w] \in U$$

and

$$|D_{e,f}((P_S \chi)_w + (Z_S \eta)_w + \theta[(Z_S \tilde{\eta})_w - (Z_S \eta)_w])[(Z_S \tilde{\eta})_w - (Z_S \eta)_w]| \leq 2j |\tilde{\eta} - \eta|_{0S}.$$

Proof. See the proof of [20, Proposition 3.4] \qed

Let $\phi \in U$, and let $T = T_\phi > 0$, a convex neighbourhood $V = V_\phi$ of $\phi$ in $U$, $\epsilon = \epsilon_\phi > 0$, and $j = j_\phi \in \mathbb{N}$ be given as in Proposition 11.3.

Then Propositions 4.1, 4.2, 4.3 from [20] hold, with verbatim the same proofs. We restate these propositions as follows.

Proposition 11.4. For every $S \in (0, T)$, $\chi \in V$, $\eta$ and $\tilde{\eta}$ in $C^1_{0,S,0}$ with $|\eta|_{1,0S} < \epsilon$ and $|\tilde{\eta}|_{1,0S} < \epsilon$,

$$(\chi, \eta) \in D_S, (\chi, \tilde{\eta}) \in D_S, \text{ and } |A_S(\chi, \tilde{\eta}) - A_S(\chi, \eta)|_{1,0S} \leq 2j(S + 1)|\tilde{\eta} - \eta|_{1,0S}.$$

Proposition 11.5. $\lim_{S \searrow 0} A_S(\phi, 0) = 0$.

Proposition 11.6. There exist $S_\phi \in (0, T_\phi)$ and an open neighbourhood $W_\phi$ of $\phi$ in $V_\phi$ such that for all $\chi \in W_\phi$, for all $S \in (0, S_\phi)$, and all $\eta \in C^1_{0,S,0}$ and $\tilde{\eta} \in C^1_{0,S,0}$ with $|\eta|_{1,0S} \leq \frac{\epsilon_\phi}{2}$ and $|\tilde{\eta}|_{1,0S} \leq \frac{\epsilon_\phi}{2}$,

$$(\chi, \eta) \in D_S, (\chi, \tilde{\eta}) \in D_S,$$

$$|A_S(\chi, \eta)|_{1,0S} \leq \frac{\epsilon_\phi}{2} \text{ and } |A_S(\chi, \tilde{\eta}) - A_S(\chi, \eta)|_{1,0S} \leq \frac{1}{2} |\tilde{\eta} - \eta|_{1,0S}.$$

For each $S \in (0, S_\phi)$ now the uniform contraction result Theorem 5.2 applies to the map

$$W_\phi \times \{ \eta \in C^1_{0,S,0} : |\eta|_{1,0S} < \epsilon_\phi \} \ni (\chi, \eta) \mapsto A_S(\chi, \eta) \in C^1_{0,S,0},$$

with $M = M_\phi = \{ \eta \in C^1_{0,S,0} : |\eta|_{1,0S} \leq \frac{\epsilon_\phi}{2} \}$, and yields a $C^1_\rho$-map

$$W_\phi \ni \chi \mapsto \eta_\chi \in C^1_{0,S,0}$$
given by \( \eta \in M_\phi \) and \( A_S(\chi, \eta) = \eta \chi \). As the maps \( P_S \) and \( C^1_{0,S,0} \) \( \odot \) \( C^1_{S} \) are linear and continuous it follows from linearity of differentiation and by means of the chain rule that also the map

\[
\Sigma_\phi : W_\phi \ni \chi \mapsto P_S \chi + Z_S \eta \chi \in C^1_S
\]

is \( C^1_F \)-smooth. An application of the chain rule to the compositions of this map with the continuous linear maps \( E^1_S(\cdot, t) : C^1_S \to C^1, 0 \leq t \leq S \), yields that all maps

\[
W_\phi \ni \chi \mapsto E^1_S(\Sigma_\phi(\chi), t) \in C^1, \quad 0 \leq t \leq S,
\]

are \( C^1_F \)-smooth. As \( E^1_S \) is continuous we obtain that the composition

\[
[0, S] \times W_\phi \ni (t, \chi) \mapsto E^1_S(\Sigma_\phi(\chi), t) \in C^1
\]

is continuous.

Proposition 5.3 and in its proof the words continuously differentiable are replaced by the expression \( C^1_F \)-smooth. Thus \( \Sigma_\phi \) is continuous, with each \( C^1_F \)-submanifold \( X_f \) into \( C^1 \).

12. THE SEMIFLOW ON THE SOLUTION MANIFOLD

The uniqueness results [20] Propositions 4.5 and 5.1 remain valid, with the same proofs. As in [20] Section 5 we find maximal solutions \( x^\phi : ( -\infty, t_\phi ) \to \mathbb{R}^n \), \( 0 < t_\phi \leq \infty \), of the initial value problems

\[
x'(t) = f(x_t) \quad \text{for} \quad t > 0, \quad x_0 = \phi \in X_f,
\]

which are solutions on \([0, t_\phi]\) and have the property that any other solution on some interval with left endpoint 0, of the same initial value problem, is a restriction of \( x^\phi \). The relations

\[
\Omega_f = \{ (t, \phi) \in [0, \infty) \times X_f : t < t_\phi \}, \quad \Sigma_f(t, \phi) = x^\phi_t
\]

define a semiflow \( \Sigma_f : \Omega_f \to X_f \) on \( X_f \), compare [20] Proposition 5.2. In [20] Proposition 5.3 and in its proof the words continuously differentiable can everywhere be replaced by the expression \( C^1_F \)-smooth. Thus \( \Sigma_f \) is continuous, with each domain

\[
\Omega_{f,t} = \{ \phi \in X_f : (t, \phi) \in \Omega_f \}, \quad t \geq 0,
\]

an open subset of \( X_f \) and the time-\( t \)-map

\[
\Sigma_{f,t} : \Omega_{f,t} \to X_f, \quad \Sigma_{f,t}(\phi) = \Sigma_f(t, \phi), \quad t \geq 0,
\]

\( C^1_F \)-smooth in case \( \Omega_{f,t} \neq \emptyset \).
[20] Proposition 5.5] and its proof remain valid. In the proof of [20] Proposition 6.1] the words continuously differentiable can everywhere be replaced by the expression $C^1$-smooth. This yields

$$D_2 \Sigma f(t, \phi) \chi = D \Sigma f, t(\phi) \chi = v^{\phi, \chi}_t$$

with the unique maximal continuously differentiable solution $v = v^{\phi, \chi}$ of the initial value problem

$$v'(t) = Df(\Sigma f(t, \phi))v_t \quad \text{for} \quad t > 0, \quad v_0 = \chi \in T_{\phi}X_f.$$
Part III

13. ON LOCALLY BOUNDED DELAY, THE EXTENSION PROPERTY, AND PROLONGATION AND RESTRICTION

Assume as in Part II that \( f : C^1 \supset U \to \mathbb{R}^n \) is \( C^1_p \)-smooth and that \( f \) has property (e). It is convenient from here on to abbreviate \( X = X_f, \Omega = \Omega_f, \text{ and } \Sigma = \Sigma_f. \)

Let a stationary point \( \bar{\phi} \in X \) of \( \Sigma \) be given, \( \Sigma(t, \bar{\phi}) = \bar{\phi} \) for all \( t \geq 0. \) Then \( \bar{\phi} \) is constant. (Proof of this: The solution \( x \) of Eq. (1.1) on \([0, \infty)\) with \( x_0 = \phi \) satisfies \( x(t) = x_1(0) = \Sigma(t, \phi)(0) = \phi(0) \) for all \( t \geq 0. \) For all \( s < 0 \) we have \( x(s) = \phi(s) = \Sigma(-s, \phi)(s) = x(\bar{x}_s(s) = x(0) = \phi(0).) \)

Choose an open neighbourhood \( N \) of \( \bar{\phi} \) in \( U \) and \( d > 0 \) according to property (lbd). We restate [21] Proposition 2.1 as follows.

**Proposition 13.1.** For every \( \phi \in N \) we have

\[
Df(\phi)\psi = 0 \quad \text{for all } \psi \in C^1 \quad \text{with } \psi(0) = 0 \quad \text{on } [-d, 0],
\]

and

\[
D_{e}f(\phi)\chi = 0 \quad \text{for all } \chi \in C \quad \text{with } \chi(0) = 0 \quad \text{on } [-d, 0].
\]

Set \( \bar{\phi}_d = R_{d,1}\bar{\phi} = \bar{\phi}|_{[-d,0]} \). As \( \bar{\phi} \) is constant we have

\[
P_{d,1}\bar{\phi}_d = \bar{\phi} \in N,
\]

and it follows that there exist neighbourhoods \( U_d \) of \( \bar{\phi}_d \) in \( C^1 \) with \( P_{d,1}U_d \subset N. \)

Due to the chain rule the map

\[
f_d : C^1 \supset U_d \to \mathbb{R}^n, \quad f_d(\phi) = f(P_{d,1}\phi),
\]

is \( C^1 \)-smooth, with

\[
Df_d(\phi)\chi = Df(P_{d,1}\phi)P_{d,1}\chi.
\]

According to [21] Proposition 2.2 \( f_d \) has property (e). Results from [18] apply and show that the equation

\[
x'(t) = f_d(x_t)
\]

(with segments \( x_t : [-d, 0] \ni s \mapsto x(t + s) \in \mathbb{R}^n \)) defines a continuous semiflow \( \Sigma_d : \Omega_d \to X_d \) on the submanifold

\[
X_d = \{ \phi \in U_d : \phi'(0) = f_d(\phi) \}, \quad \text{codim } X_d = n,
\]

of the Banach space \( C^1. \) In the terminology of the present paper, the manifold \( X_d \) and all solution operators \( \Sigma_d(t, \cdot), t \geq 0, \) with non-empty domain are \( C^1 \)-smooth.

The proofs of [21] Propositions 2.3-2.5 remain valid without change. We restate the result as follows.

**Proposition 13.2.** (i) \( X_d = R_{d,1}(X \cap N \cap R_{d,1}^{-1}(U_d)) \)

(ii) For every \( \phi \in X \cap N \cap R_{d,1}^{-1}(U_d), \)

\[
T_{R_{d,1}\phi}X_d = R_{d,1}T_{\phi}X.
\]

(iii) For \((t, \phi) \in \Omega \) with \( \Sigma([0, t] \times \{ \phi \}) \subset N \cap R_{d,1}^{-1}(U_d), \)

\[
(t, R_{d,1}\phi) \in \Omega_d \quad \text{and } \quad \Sigma_d(t, R_{d,1}\phi) = R_{d,1}\Sigma(t, \phi).
\]
(iv) If \((t, \chi) \in \Omega_d\) and if \(x : (-\infty, t] \to \mathbb{R}^n\) given by \(x(s) = x^\chi(s)\) on \([-d, t]\) and by \(x(s) = (P_{d,1})x\) for \(s < -d\) satisfies \(\{x_s : 0 \leq s \leq t\} \subset N\) then
\[
(t, P_{d,1}) \in \Omega\quad \text{and} \quad R_{d,1} \Sigma(t, P_{d,1}) = \Sigma_d(t, \chi).
\]

Proposition 13.2 (iii) shows that \(\bar{\phi}_d\) is a stationary point of the semiflow \(\Sigma_d\).

For \(t \geq 0\) consider the operators \(T_t = D_2 \Sigma(t, \bar{\phi}_d)\) on \(T_d\) and \(T_{d,t} = D_2 \Sigma_d(t, \bar{\phi}_d)\) on \(T_d\). The proof of [21, Corollary 2.6] remains valid. We state the result as follows.

**Corollary 13.3.**

(i) For \((t, \phi) \in \Omega\) as in Proposition 13.2 (iii) and for all \(\chi \in T_d\),
\[
R_{d,1} \chi \in T_{R_{d,1} \phi} X_d\quad \text{and} \quad R_{d,1} D_2 \Sigma(t, \phi) \chi = D_2 \Sigma_d(t, R_{d,1} \phi) R_{d,1} \chi.
\]

(ii) For all \(\chi \in T_d\) and for all \(t \geq 0\),
\[
R_{d,1} \chi \in T_{\bar{\phi}_d} X_d\quad \text{and} \quad R_{d,1} T_t \chi = T_{d,t} R_{d,1} \chi.
\]

From [7, Sections 3.5 and 4.1-4.3] and from [10] we get local stable, center, and unstable manifolds of \(\Sigma_d\) at \(\bar{\phi}_d \in X_d \subset C^1_d\), all of them \(C^1_P\)-smooth.

## 14. Decomposition of the Tangent Space

Let \(Y = T_d X\). In this section we recall from [21, Section 3] the definitions of the linear stable, center, and unstable spaces of the operators \(T_t : Y \to Y, \ t \geq 0\).

The linear stable space in \(Y\) is defined by
\[
Y_s = R_{d,1}^{-1} Y_{d,s}
\]
with the linear stable space \(Y_{d,s}\) of the strongly continuous semigroup \((T_{d,t})_{t \geq 0}\) on the tangent space \(Y_d = T_{\bar{\phi}_d} X_d \subset C^1_d\). We have \(Y_{d,s} = Y_d \cap C_{d,s}\) with the linear stable space \(C_{d,s}\) of the strongly continuous semigroup of solution operators \(T_{d,e,t} : C_d \to C_d, \ t \geq 0\), which is defined by the equation
\begin{equation}
(14.1) \quad v'(t) = D_e f_d(\bar{\phi}_d) v_t.
\end{equation}

Let \(C_{d,c}\) and \(C_{d,u}\) denote the finite-dimensional linear center and unstable spaces of the semigroup on \(C_d\). Each \(\chi \in C_{d,c} \oplus C_{d,u}\) uniquely defines an analytic solution \(v = v^\chi\) on \(\mathbb{R}\) of Eq. (14.1). The injective map
\[
I : C_{d,c} \oplus C_{d,u} \ni \chi \mapsto \chi|_{(-\infty,0]} \in C^1
\]
is linear, and continuous (as its domain is finite-dimensional). The center and unstable spaces in \(Y\) are defined as
\[
Y_c = IC_{d,c}\quad \text{and} \quad Y_u = IC_{d,u},
\]
respectively. They are finite-dimensional and the maps \(T_t, \ t \geq 0\), act as isomorphims on each of them. The stable space \(Y_s\) is closed and positively invariant under each map \(T_t, \ t \geq 0\), and we have the decomposition
\[
Y = Y_s \oplus Y_c \oplus Y_u.
\]

Finally, observe
\[
Y_u \subset B^1_a
\]
since each \(v^\chi, \chi \in C_{d,u}\), and its derivative both have limit 0 at \(-\infty\).
15. The local stable manifold

We begin with the local stable manifold $W^s_d \subset X_d$ of the semiflow $\Sigma_d$ at the stationary point $\bar{\phi}_d \in X_d \subset C^1_d$ as it was obtained in [7]. It is easy to see that $W^s_d$ is a continuously differentiable submanifold of the Banach space $C^1_d$ which is locally positively invariant under $S_d$, with tangent space

$$T_{\bar{\phi}_d}W^s_d = Y_{d,s}$$

at $\bar{\phi}_d$, and that it has the following properties (I) and (II), for some $\beta > 0$ chosen with

$$\Re z < -\beta < 0$$

for all $z$ with $\Re z < 0$ in the spectrum of the generator of the semigroup on $C_d$, and for some $\gamma > \beta$.

(I) There are an open neighbourhood $\hat{W}^s_d$ of $\bar{\phi}_d$ in $W^s_d$ such that $[0, \infty) \times \hat{W}^s_d \subset \Omega_d$ and $\Sigma_d([0, \infty) \times \hat{W}^s_d) \subset W^s_d$, and a constant $\tilde{c} > 0$ such that for all $\psi \in \hat{W}^s_d$ and all $t \geq 0$,

$$|\Sigma_d(t, \psi) - \bar{\phi}_d|_{d,1} \leq \tilde{c} e^{-\gamma t}|\psi - \bar{\phi}_d|_{d,1}.$$

(II) There exists a constant $\tilde{c} > 0$ such that each $\psi \in X_d$ with $[0, \infty) \times \{\psi\} \subset \Omega_d$ and

$$e^{\beta t}|\Sigma_d(t, \psi) - \bar{\phi}_d|_{d,1} \leq \tilde{c}$$

for all $t \geq 0$ belongs to $W^s_d$.

The codimension of $W^s_d$ in $C^1_d$ is equal to

$$n + \dim Y_{d,c} + \dim Y_{d,u} = n + \dim C_{d,c} + \dim C_{d,u}.$$

As the continuous linear map $R_{d,1} : C^1 \to C^1_d$ is surjective we can apply Proposition 7.1 and obtain an open neighbourhood $V$ of $\bar{\phi}$ in $N \subset U \subset C^1$ so that

$$W^s = W^s(\bar{\phi}) = V \cap R_{d,1}^{-1}(W^s_d)$$

is a $C^1$-submanifold of $C^1$ with codimension $n + \dim C_{d,c} + \dim C_{d,u}$ and tangent space

$$T_{\bar{\phi}}W^s = R_{d,1}^{-1}(T_{\bar{\phi}_d}W^s_d) = R_{d,1}^{-1}(Y_{d,s}) = Y_s.$$  

The next proposition shows that $W^s$ is the desired local stable manifold of $\Sigma$ at $\bar{\phi}$.

**Proposition 15.1.**

(i) $W^s \subset X$, and $W^s$ is locally positively invariant.

(ii) There are an open neighbourhood $\hat{V}$ of $\bar{\phi}$ in $V$ with $[0, \infty) \times (\hat{V} \cap W^s) \subset \Omega$ and a constant $\tilde{c} > 0$ such that for all $\phi \in \hat{V} \cap W^s$ the solution $x : \mathbb{R} \to \mathbb{R}^n$ on $[0, \infty)$ of Eq. (1.1) with $x_0 = \phi$ satisfies

$$|x(t) - \bar{\phi}(0)| + |x'(t)| \leq \tilde{c} e^{-\gamma t}|R_{d,1}\phi - \bar{\phi}_d|_{d,1} \quad \text{for all} \quad t \geq 0.$$

(iii) There are an open neighbourhood $\hat{V}$ of $\bar{\phi}$ in $V$ and a constant $\tilde{c} > 0$ such that for every solution $x : \mathbb{R} \to \mathbb{R}^n$ on $[0, \infty)$ of Eq. (1.1) with $x_0 \in \hat{V} \cap X$ and

$$|x(t) - \bar{\phi}(0)| + |x'(t)| \leq \tilde{c} e^{-\beta t} \quad \text{for all} \quad t \geq 0$$

we have $x_0 \in W^s$.

Proposition 15.1 is proved exactly as [24] Propositions 4.1, 4.2], using the properties of $W^s_d$ stated above.
16. The Local Unstable Manifold

In this section all segments $x_t$ are defined on $(-\infty, 0]$. Fix some $a > 0$ and consider the Banach spaces $B_a \subset C$ and $B^1_a \subset C^1$ introduced in Section 1. It is easy to see that the linear inclusion maps

$$j_0 : B_a \to C \quad \text{and} \quad j_1 : B^1_a \to C^1$$

are continuous, as well as the restriction and prolongation maps

$$R_{a,d,1} : B^1_a \ni \phi \mapsto R_{d,1}\phi \in C^1_d \quad \text{and} \quad P_{a,d,1} : C^1_d \ni \chi \mapsto P_{d,1}\chi \in B^1_a.$$ 

The set $U_a = j_1^{-1}(N) \cap R_{a,d,1}(U_d) \subset B^1_a$ is open and contains $\bar{\phi}$, and the $C^1_p$-map

$$f_a : U_a \to \mathbb{R}^n, \quad f_a(\phi) = f(j_1(\phi)),$$

satisfies $f_a(\bar{\phi}) = 0$. Notice that every solution of the equation

$$(16.1) \quad x'(t) = f_a(x)$$

on some interval also is a solution of Eq. (1.1) on this interval. The proof of [21, Proposition 5.1] remains valid. Therefore we have

$$(16.2) \quad f_a(\phi) = f_a(\psi) \quad \text{for all } \phi \in U_a, \ \psi \in U_a \quad \text{with} \ \phi(s) = \psi(s) \ \text{on} \ [-d, 0],$$

each derivative $Df_a(\phi) : B^1_a \to \mathbb{R}^n$, $\phi \in U_a$, has a linear extension $D_c f_a(\phi) : B_a \to \mathbb{R}^n$, and the map

$$U_a \times B_a \ni (\phi, \chi) \mapsto D_c f_a(\phi)\chi \in \mathbb{R}^n$$

is continuous. Now results from [19] show that $X_a = \{ \phi \in U_a : \phi'(0) = f_a(\phi) \}$ is a $C^1_p$-submanifold of $B^1_a$; that the solutions of Eq. (16.1) define a continuous semiflow $\Sigma_a : \Omega_a \to X_a$ on $X_a$, and that there is a local unstable manifold $W^u_a \subset X_a$ at the stationary point $\bar{\phi} \in W^u_a$. $W^u_a$ is a $C^1_p$-submanifold of $B^1_a$ consisting of data $\phi \in X_a$ which are solutions of Eq. (16.1) on $(-\infty, 0]$ with $\phi_s \to \bar{\phi}$ as $s \to -\infty$, and

$$T_{\bar{\phi}} W^u_a = Y_a.$$ 

(In order to verify the last equation observe that in [13] the tangent space of $W^u_a$ at $\bar{\phi}$ is obtained as the vector space of all maps $\chi : (-\infty, 0] \to \mathbb{R}^n$ with $\chi_0 = \chi \in C_{d,u}$ which for some $t > 0$ and for all integers $j < 0$ satisfy

$$\dot{\chi}_{jt} = \Lambda^{-j}\chi$$

where $\Lambda : C_{d,u} \to C_{d,u}$ is the isomorphism whose inverse is given by $T_{d,e,t}$. The maps in the vector space $Y_a = IC_{d,u}$ share the said property. The dimension of both vector spaces equals $\dim(C_{d,u})$)

Moreover, there exist $\beta > \gamma > 0$ and $c_u > 0$ so that

$$\text{(I)} \quad |\phi_s - \bar{\phi}|_{a,1} \leq c_u e^{\beta s} |\phi - \bar{\phi}|_{a,1} \quad \text{for all } \phi \in W^u_a \quad \text{and} \quad s \leq 0,$$

and

$$\text{(II)} \quad \text{for every solution } \psi \in B^1_a \text{ of Eq. (16.1) on } (-\infty, 0] \text{ with }$$

$$\sup_{s \leq 0} |\psi_s - \bar{\psi}|_{a,1} e^{-\gamma s} < \infty$$

there exists $s_\psi \leq 0$ with $\psi_s \in W^u_a$ for all $s \leq s_\psi$.

From a manifold chart at $\bar{\phi}$ we obtain $\epsilon > 0$ and a $C^1_p$-map

$$w^u_a : Y_a(\epsilon) \to B^1_a, \quad Y_a(\epsilon) = \{ \phi \in Y_a : |\phi|_{a,1} < \epsilon \},$$

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with $w_a^n(0) = \tilde{\phi}$, $w_a^n(Y_u(\epsilon))$ an open subset of $W^u_a$, and $Dw_a^n(0)\eta = \eta$ for all $\eta \in Y_u$. Proposition 7.2 applies to the $C^1_{\mathcal{F}}$-map $j_1 \circ w_a^n$. So we may assume that

$$W^u = W^u(\tilde{\phi}) = j_1w_a^n(Y_u(\epsilon))$$

is a $C^1_{\mathcal{F}}$-submanifold of the Fréchet space $C^1$ with

$$T_\phi W^u = j_1 Dw_a^n(0)Y_u = Y_u.$$  

The proof of [21 Proposition 5.2] remains valid in the present setting. We state the result about the properties of the local unstable manifold $W^u$ as follows.

**Proposition 16.1.** (i) Every $\phi \in W^u$ is a solution of Eq. (1.1) on $(-\infty, 0]$, with $\phi_s \to \tilde{\phi}$ as $s \to -\infty$, and for all $s \leq 0$,

$$|\phi(s) - \tilde{\phi}(0)| \leq c_a e^{3s}|\phi - \tilde{\phi}|_{\alpha,1} \quad \text{and} \quad |\phi'(s)| \leq c_a e^{3s}|\phi - \tilde{\phi}|_{\alpha,1}.$$  

(ii) For every $\psi \in X$ which is a solution of Eq. (1.1) on $(-\infty, 0]$ with

$$\sup_{s \leq 0} e^{-\gamma s}|\psi(s) - \tilde{\phi}(0)| < \infty \quad \text{and} \quad \sup_{s \leq 0} e^{-\gamma s}|\psi'(s)| < \infty$$

there exists $s(\psi) \leq 0$ with $\psi_s \in W^u$ for all $s \leq s(\psi)$.

17. LOCAL CENTER MANIFOLDS

In this section we assume

$$\{0\} \neq Y_c$$

which is equivalent to

$$\{0\} \neq C_{d,c}.$$  

In the sequel we recall the steps which in [21 Section 6] led to a local center manifold at $\tilde{\phi}$ which is $C^1_{MB}$-smooth, and point out the observation which yields $C^1_{\mathcal{F}}$-smoothness.

The approach from [21 Section 6] first follows constructions from the proof of [10 Theorem 2.1] which were done for the case $\tilde{\phi}_d = 0$. Therefore we introduce $V_d = U_d - \tilde{\phi}_d$ and the $C^1_{\mathcal{F}}$-map $g_d : V_d \to \mathbb{R}^n$. Then $g_d(0) = 0$ and $Dg_d(0) = Df_d(\tilde{\phi}_d)$.

There is a decomposition

$$C^1_d = C^1_{d,s} \oplus C_{d,c} \oplus C_{d,u}, \quad C^1_{d,s} = C^1_d \cap C_{d,s},$$

into closed subspaces which defines a projection $P_{d,c} : C^1_d \to C^1_d$ onto $C_{d,c}$, and there is a norm $\|\cdot\|_{d,1}$ on $C^1_d$ which is equivalent to $\|\cdot\|_{d,1}$ and whose restriction to $C_{d,c} \setminus \{0\}$ is $C^\infty$-smooth.

Next there exists $\Delta > 0$ with

$$N_\Delta = \{\phi \in C^1_d : \|\phi\|_{d,1} < \Delta\}$$

contained in $V_d$ so that the restricted remainder map

$$N_\Delta \ni \phi \mapsto g_d(\phi) - Dg_d(0)\phi \in \mathbb{R}^n$$

has a global continuation

$$r_{d,\Delta} : C^1_d \to \mathbb{R}^n$$

with Lipschitz constant

$$\lambda = \sup_{\phi \neq \psi} \frac{\|r_{d,\Delta}(\phi) - r_{d,\Delta}(\psi)\|_{d,1}}{\|\phi - \psi\|_{d,1}} < 1.$$
The desired local center manifold at \( \tilde{\phi} \in X \) will be given, up to translation, by segments \((-\infty, 0] \to \mathbb{R}^n\) of solutions on \( \mathbb{R} \) of the equation

\[
(17.1) \quad x'(t) = Dg_d(0)x_t + r_d,\Delta(x_t) \quad \text{(with segments in } C^1_d) 
\]

which do not grow too much at \( \pm \infty \).

For \( \eta > 0 \) let \( C^1_{d, \eta} \) denote the Banach space of all continuous maps \( u : \mathbb{R} \to C^1_d \) with

\[
\sup_{t \in \mathbb{R}} e^{-\eta|t|}|u(t)|_{d, 1} < \infty
\]

and the norm given by the preceding supremum. There exists \( \eta_1 > 0 \) so that for every \( \phi \in C^1_{d, c} \) there is a unique continuously differentiable map

\[
x^{[\phi]} : \mathbb{R} \to \mathbb{R}^n
\]

which satisfies Eq. (17.1) for all \( t \in \mathbb{R} \) and \( P^1_{d,e}x^{[\phi]}_0 = \phi \) and has the continuous map \( \mathbb{R} \ni t \mapsto x^{[\phi]}_t \in C^1_d \) contained in the space \( C^1_{d, \eta_1} \). Observe that we have

\[
x^{[\phi]}(0) = 0 \quad \text{for all } t \in \mathbb{R}.
\]

Incidentally, from here on the proof in [21, Section 6] deviates from the approach in [10].

Now consider the map

\[
J : C^1_{d, c} \ni \phi \mapsto \tilde{\phi} + x^{[\phi]}|_{(-\infty, 0]} \in C^1.
\]

Observe that the proof of [21 Corollary 6.2] shows that the map \( J \) is in fact \( C^1_F \)-smooth, not only \( C^1_{MB} \)-smooth, and

\[
DJ(0)\phi = I\phi \quad \text{for all } \phi \in C^1_{d, c}.
\]

As \( C^1_{d, c} \) is finite-dimensional and as \( I \) is injective Proposition 7.2 yields an open neighbourhood \( N_{d, c} \) of \( 0 \) in \( C^1_{d, c} \) so that the image

\[
W^c = J(N_{d, c})
\]

is a \( C^1_F \)-submanifold of the Fréchet space \( C^1 \), with

\[
T_{\tilde{\phi}}W^c = IC^1_{d, c} = Y_c.
\]

By continuity of \( J \) and \( J(0) = \tilde{\phi} \) we may assume \( J(N_{d, c}) \subset N \subset U \). By continuity of the map

\[
C^1_{d, c} \ni \phi \mapsto R_d,1(J(\phi) - \tilde{\phi}) \in C^1_d
\]

at \( 0 \in C^1_{d, c} \) we also may assume that for all \( \phi \in N_{d, c} \) we have

\[
\|x^{[\phi]}_0\|_{d, 1} < \Delta \quad \text{for all } \phi \in N_{d, c}
\]

or, \( x^{[\phi]}_0 \in N_\Delta \) for all \( \phi \in N_{d, c} \), with segments \( x^{[\phi]}_0 \) defined on \([-d, 0]\).

We take \( W^c \), with tangent space \( Y_c \) at the stationary point \( \tilde{\phi} \in X \), as the desired local center manifold of the semiflow \( \Sigma \) and verify that it has the appropriate properties. Following the proof of [21 Proposition 6.3] we get

\[
W^c \subset X.
\]

Next, choose an open neighbourhood \( U_* \) of \( \tilde{\phi} \) in \( N \subset U \) so small that

\[
R_{d, 1}U_* \subset U_d \cap (N_\Delta + \tilde{\phi}_d)
\]
and for all $\psi \in U_*$,
\[ p_{d,c}^1 R_{d,1}(\psi - \bar{\phi}) \in N_{d,c}. \]

Then the proofs of [21, Proposition 6.4, Proposition 6.5] remain valid. We state the result as follows.

**Proposition 17.1.** (i) (Local positive invariance) For every $(t, \psi) \in \Omega$ with $\psi \in W^c \subset X$ and $\Sigma([0, t] \times \{\psi\}) \subset U_*$ we have $\Sigma([0, t] \times \{\psi\}) \subset W^c$.

(ii) For every solution $y : \mathbb{R} \to \mathbb{R}^n$ of Eq. (1.1) on $\mathbb{R}$ with $y_t \in U_*$ for all $t \in \mathbb{R}$ we have $y_t \in W^c$ for all $t \in \mathbb{R}$.

Observe that the proofs of both parts of Proposition 17.1 make use of [21, Lemma 7.1] on uniqueness for an initial value problem with data in $C^1$.

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