Chimera and anticoordination states in learning dynamics

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In many real-life situations, individuals are dared to simultaneously achieve social objectives of acceptance or approval and strategic objectives of coordination. Since these two objectives may take place in different environments, a two-layer network is the simple and natural framework for the study of such kind of dynamical situations. In this paper we present a model in which the state of the agents corresponds to one of two possible strategies. They change their states by interaction with their neighbors in the network. Inside each layer the agents interact by a social pressure mechanism, while between the layers the agents interact via a coordination game. From an evolutionary approach, we focus on the asymptotic solutions for all-to-all interactions across and inside the layers and for any initial distribution of strategies. We find new asymptotic configurations which do not exist in a single isolated social network analysis. We report the emergence and existence of chimera states in which two different collective states coexist in the network. Namely, one layer reaches a state of full coordination while the other remains in a dynamical state of coexistence of strategies. In addition, the system may also reach a state of global anticoordination where a full coordination is reached inside each layer but with opposite strategies in each of the two network layers. We trace back the emergence of chimera states and global anticoordination states to the agents inertia against social pressure, referred here to as the level of skepticism, along with the degree of risk taken into account in a general coordination game.

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I. INTRODUCTION

The coexistence of coherent and incoherent states has received much attention as an intriguing manifestation of collective behaviour. This interesting behaviour was first observed by Kuramoto et al. [1] and then named it as chimera state. Although the literature about chimera states started with the study of interacting populations of oscillators in dynamical systems [3, 4, 5, 6, 7], it has been dizzily expanded to many fields in physics, chemistry, biology, etc. Also in social systems, situations of two interacting populations in which one exhibits a coherent or synchronized behaviour while the other is incoherent or desynchronized are commonly observed. This phenomena has also been addressed from the conceptual framework of chimera states.

Models based on individual interactions have been introduced for opinion formation and cultural dissemination [8] and [9] in order to analyze the emergence of localized coherent or chimeras states behavior in social contexts. The systems considered consist in two populations of social agents mutually coupled through global interaction fields that account for the state of the majority of the agents in each population. The internal interactions in each group have a condition that allow for non-interacting states. Two examples of such dynamics have been analyzed: (i) Axelrod’s model for social influence [10], and (ii) a discrete version of a bounded confidence model for opinion formation [11]. In both systems, there are localized coherent states for some parameter values, in which a group reaches a homogenous or consensus state, while the other group remains in a disordered or polarized state. In this paper we contribute further to the study of chimera states in social systems, searching for them in the context of models of social coordination and learning dynamics [12].

Coordination is an important issue in economics studies, being studied within a Game Theory approach in many theoretical and experimental studies, [13, 14, 15]. It has has become one of the most important challenges in modern societies: In spite of the fact that individuals are now more connected and handle more information due to technological progress, it seems that coordination to reach consensus is becoming an increasingly difficult goal. It is common to observe how some societies become polarized either by ideological or political opinions, or, the collective behaviour leads the population to states in which one part reaches a consensus while the other part behaves in an unstable manner, or it displays a dynamical coexistence of strategies. It can be argued that such diversity of outcomes are the result of a constant search for achieving simultaneously social and economic aims in modern societies. On one side, individuals, influenced by others, seek for social acceptance and recognition and on the other side they attempt to get higher gains. Whenever the social and the economic concerns take place in two different social networks or environments, the population unavoidably splits into two disjoint target groups. An example of different environments is the environments of family and friends versus the environments of work and business.

Here we present a simple model to illustrate such situation in which individuals of a population divided into two
groups are dared to coordinate in order to accomplish their social and economic goals. Our framework for studying such kind of population is a two-layer network. Each layer corresponds to a group of the population and interactions within the group aim to satisfy social concerns, while interactions between agents of different layers aim to satisfy economic concerns. In our system, the economic goal turns out to be reached when agents play the same strategy. Since there will be as many consensus as numbers of possible strategies a coordination problem arises. Also, a coordination problem to achieve social goals arises because there is a skepticism by individuals to be influenced by social pressure. A previous work of [16] shows that for an initial uniform distribution of two possible strategies the skepticism to follow the opposite strategy and the local connectivity are the driving forces to accomplish full coordination for this two-layer network. Here, we consider the role of different initial conditions leading to different asymptotic states of the dynamics and we determine the basin of attraction of these states. As one of our main results, we find two asymptotic states with nontrivial collective behaviour which can not be found in a single isolated network analysis. A first outcome is a social analogue of a chimera state with coexistence of coherent and incoherent states. In our model, one layer can reach a homogeneous state of full coordination while the other remains in a dynamical state of coexistence of strategies. The second non-trivial asymptotic state is the anticoordination or polarization state in which the system reaches coordination with a different strategy in each layer.

The paper is organized as follows. Section 2 introduces the general frame of our model. Based on a two-layer network, we describe the kind of interactions inside and between layers and the dynamical rule for individuals to update their strategies. Section 3 describes the possible asymptotic solutions of the collective dynamics reached by the system, as well as the basins of attraction to reach these solutions. In particular we describe the nontrivial chimera and anticoordination states and their basins of attraction. Section 4 shows bifurcation diagrams, in Sec. 4.1 for the case of a symmetric coordination game and in Sec. 4.2 for the case of an asymmetric coordination game. Finally, Section 5 summarizes our conclusions. In the Appendix, we present a mean-field theory for the time evolution of the system in the infinite size limit.

We consider an individual based model consisting in a population in which individuals interact in two different groups, A and B of sizes \( N_A \) and \( N_B \) respectively. We take \( N_A = N_B = N \). Using the frame of a two-layer network (see Figure 1), inside each layer the individuals interact with social objectives and between layers they interact with strategic or economic objectives, as described in the following.
A. Interactions between the layers: Coordination games

In Game Theory [13, 14, 15], the coordination game is a prototype model of strategic game with multiple Nash equilibria. Each player can choose a strategy, for example driving on the left (L) or on the right (R) side. In equilibrium none of the players want to change their choice of strategies, given the strategies of the other players. In this game players share the goal of coordinating on the same action, however, when they choose their strategies simultaneously, without knowing what the other players will choose, it becomes highly difficult for them to coordinate in the desirable way.

The coordination dynamics proceeds as follows. In each time step, each agent in a layer plays with each agent in the other layer a pairwise coordination game. Table I shows the pay-off normal form representation of a two-person, two-strategy coordination game: For example if one agent plays L and the other plays R, the pay off for the former is 0, and \(-b\) for the latter. We focus our analysis on two parametric settings, a pure or symmetric coordination game (SCG) in which \(s = 0\) and \(b = 0\) and a general or asymmetric coordination game (ACG) in which \(s = 1\) and \(b > 0\).

The profiles \((L, L)\) and \((R, R)\) are the two pure strategy Nash equilibria in both settings. A problem of equilibrium selection is present in both settings. In the symmetric coordination game both equilibria are equivalent. However, in the general coordination game the higher payoff is achieved when agents play the profile \((R, R)\), called the Pareto (payoff) dominant equilibrium. However, for \(b > 1\), the profile \((L, L)\) becomes the risk dominant equilibrium, in the sense of [17], since agents risk less by coordinating on \((L, L)\). Then, the socially efficient solution \((R, R)\) turns risky so that the possibility of coordinating in it decreases as \(b\) increases. The parameter \(b\) becomes a measure of risk for playing the strategy \(R\). For a complete review, see [18].

B. Interactions inside the layers: the effect of social pressure

Inside each layer, searching for social acceptance and approval, each agent observes the strategies being played by her partners. An agent may not feel at ease with her strategy when such strategy is not as popular as she wants in her social environment. The level of skepticism in the population is calibrated by a threshold \(T\) that determines the effect of the social pressure exerted on an individual. The criterion used by each player \(i\) is to measure how well she is doing by comparing the share of agents who are playing the opposite strategy to hers, denoted by \(d_i\), with the threshold \(T \in [0, 1]\). We may distinguish two types of populations. Herding population, for \(T < 0.5\), in which individuals are influenced by low levels of popularity of the opposite strategy, and, skeptical population, for \(T > 0.5\), where the social pressure has a weak effect on individuals: a feeling of disapproval only arises for high levels of popularity of the opposite strategy. Therefore, when \(d_i > T\), the social pressure is effective because player \(i\) generates a feeling of non-acceptance about the strategy she is currently playing and she is willing to revise it.

C. Inter-layer and intra-layer objectives: The degree of satisfaction

In the interactions across and inside the layers agents intend to satisfy two different objectives: social objectives of acceptance and approval, and strategic objectives of coordination. These objectives give rise to two sources of satisfaction: strategic satisfaction in terms of the payoff obtained in the coordination game and social satisfaction in
TABLE II: Degrees of dissatisfaction according to the fulfilment of social and strategical objectives.

| Term   | Expression                                      |
|--------|-------------------------------------------------|
| S      | $\pi_i = (1 + s)N$                              |
| P1     | $\pi_i > (1 + s)N$                              |
| P2     | $\pi_i < (1 + s)N$                              |
| U      | $\pi_i < (1 + s)N$                              |
| $d_i < T$ | $\pi_i = (1 + s)N$                              |
| $d_i > T$ | $\pi_i > (1 + s)N$                              |
| $d_i < T$ | $\pi_i < (1 + s)N$                              |
| $d_i > T$ | $\pi_i < (1 + s)N$                              |

terms of the effect of social pressure. Therefore, there are four degrees of satisfaction described in Table III where $\pi_i$ is the aggregate payoff that agent $i$ in a layer gets playing with all the other agents in the other layer.

The value of $s$ is derived from the parametric setting of the coordination game. When $s = 0$ the equality $\pi_i = N$ shows that the player $i$ coordinates with all the members of the other layer in the symmetric game. Then we say that agent $i$ is strategically satisfied. In the case of a general coordination game, $s = 1$, an agent is strategically satisfied when the coordination is on the socially efficient solution, i.e. the Pareto dominant strategy. When $d_i < T$ the share of agents inside the layer of player $i$ who play the same strategy as she does is high enough so the player $i$ feels socially satisfied with her current strategy. Then, the level of satisfaction of player $i$ can be: S (satisfied) when she is both socially ($d_i < T$) and strategically ($\pi_i = (1 + s)N$) satisfied, P1 or P2 (partially satisfied) when she is either socially ($d_i < T$) or strategically ($\pi_i > (1 + s)N$) satisfied and is U (unsatisfied) when she is both socially ($d_i > T$) and strategically ($\pi_i < (1 + s)N$) unsatisfied.

D. Learning dynamics

The learning dynamics in the system is described by the update rule used to change strategy: Following the learning dynamics of [16] and [19], at each elementary time step each player plays the coordination game with all the members in the other layer. Once the game is over and a payoff is assigned to each player, each agent observes and measures the popularity of her strategy in her own group. As a result, a level of satisfaction arises. Then, she might change her strategy impelled by her level of satisfaction. The process is repeated setting payoffs to zero. The synchronous update rule in which each player can change her current strategy according to her level of satisfaction is described as follows,

1. If her level of satisfaction is S, she remains with the same strategy.
2. If her level of satisfaction is P1 or P2, she imitates the strategy of her best performing agent inside the layer in case that such agent has received a larger payoff than the player herself, otherwise she remains with the same strategy.
3. If her level of satisfaction is U, she changes her current strategy.

Although the update rule takes place inside the layers, individuals change their strategies by both social and strategical considerations. The imitation of the best performing individuals in her social environment aims to capture the individual behaviour observed in many complex real life situations.

II. RESULTS

A. Asymptotic solutions

Analytical equations for our model and their asymptotic solutions are discussed in the Appendix. For the general case of the asymmetric coordination game these solutions, described below, are the following:

Solution I: Coordination in strategy $L$: All agents in both layer play strategy $L$. It is linearly stable and exists for any $T \in [0, 1]$.

Solution II: Coordination in strategy $R$: All agents in both layer play strategy $R$. It is linearly stable and exists for any $T \in [0, 1]$.

Solutions III: Coexistence of strategies. These solution exist for any $T \neq 1$ and it occurs in two ways:

1. unstable fixed points and,
2. family of marginally stable periodic solutions.

Solutions III-a and III-b: Chimera solutions. This is an interesting case of coexistence of two distinct solutions, namely, solutions I and III. One of the layers goes to the absorbing state of coordination in strategy $L$, namely layer A for Solution III-a and layer B for solution III-b, while the other layer goes to a dynamical state of
coexistence of strategies, solution III type (2). We also find the case of coexistence of two solutions in which one layer coordinates in L and the other layer remains disordered with both strategies coexisting in the same proportion (Solution III type (1)). These solutions only appear when agents are playing the asymmetric coordination game. They exist for almost any $T < 0.5$ and almost any $b > 0$.

Although the strategies L and R are not equivalent in the asymmetric coordination game because the first is the socially undesired and the second the socially desired outcome, solutions III-a and III-b are equivalent in the sense that the layer reaching the absorbing state L is determined by the initial conditions of strategies in the two layers. We refer to these solutions as chimera states because of the coexistence of an ordered layer and a disordered layer. The disordered layer can be in a dynamical state (solution III (2)) or in a configuration in which the number of agents playing each strategy is equal and constant in time (solution III (1)).

Solutions IV Anticoordination states, layer A coordinates in strategy L while layer B in strategy R.

Solutions V Anticoordination states, layer A coordinates in strategy R while layer B in strategy L.

Solutions IV and V exist and they are linearly unstable and exist for any $T \in [0.5, 1]$.

We summarize in Figure 2 the domain of existence of the different asymptotic solutions for the general case of the asymmetric coordination game as a function of the threshold $T$ and the risk $b$ parameters. For this general case the parameters of the pay-off matrix take the values $s = 1$ and $b > 0$ for any $T \in [0, 1]$. Asymptotic solutions of the dynamics depend on initial conditions (see below Fig. 5), and in this sense we refer to $Q_1$ as those chimera states, solutions III-a and III-b, that are reached from initial conditions such that $x_{aa}^0 + x_{bb}^0 < 1$, where $x_{aa}^0$ and $x_{bb}^0$ are the initial conditions for $x_{aa}$ and $x_{bb}$ respectively. Likewise we refer to $Q_2$ as those chimera states, solutions III-a and III-b, that are reached from initial conditions such that $x_{aa}^0 + x_{bb}^0 > 1$. It turns out that solutions $Q_1$ exist for $b > 0$, zones A, C and D in Figure 2, while $Q_2$ only exist for $b > 1$, zones D and C. The range of values of $b$ and $T$ in which the system reaches solutions $Q_1$, $Q_2$, I, II and III corresponds to zones C and D in the Figure. The difference between zones C and D refers to the areas of the basins of attraction of each solution, as explained in Sect. 3.2. For the case of symmetric coordination game, the parameters take the values $s = 0$ and $b = 0$. The asymptotic solutions in this case are the same as those for the asymmetric coordination game except for Solutions III-a and III-b, the Chimera states. Chimera states could not be found for any level of skepticism in the case of the symmetric coordination game.

Given that the agent population is divided in two layers, A and B, we define $x_{aa}$ as the fraction of individuals playing strategy R in layer A, and $x_{bb}$ as the fraction of individuals playing strategy R in layer B. To describe the different asymptotic solutions we also introduce the order parameter $n_{AB}$ giving the density of inter-layer active links, i.e., the proportion of links connecting agents in different layers with opposite strategies. The order parameter $n_{AB}$

![FIG. 2: Domain of existence of the different asymptotic solutions in the Asymmetric Coordination Game (ACG) as a function of the threshold $T$ and the risk parameter $b$ as obtained from the analytic solution of the equations discussed in the Appendix. The two curves ($T = \frac{1}{3b}$ and $1 - T = \frac{1}{3b}$) and the two lines ($b = 1$, $T = 0.5$) divide the figure in eight zones from (A) to (H). In each of these zones the asymptotic solutions that exist are indicated. The red dots correspond to values of $T$ and $b$ used in Fig. 5.](image)
TABLE III: Asymptotic states and associated values of $x_{aa}, x_{bb}$ and $n_{AB}$. The last two columns indicate existence or non-existence of the state in the Symmetric Coordination Game (SCG) and in the Asymmetric Coordination Game (ACG). The parameter $b$ corresponds to the risk parameter of the ACG and the numbers $u, 1 - u$ and $v, 1 - v$ for $u, v \in (0, 1)$ describe the family of periodic solutions in layer A and layer B respectively. Chimera states in which the three variables $x_{aa}, x_{bb}$ and $n_{AB}$ take constant values, either 0 or 1/2, correspond to the case in which the disordered layer is in a solution III (1).

| Asymptotic state | $n_{AB}$ | $x_{aa}$ | $x_{bb}$ | solution | SCG ACG |
|------------------|----------|----------|----------|-----------|---------|
| Coordination     | 0        | 0        | 0        | 1         | ✓ ✓     |
|                  | 1        | 1        | 1        | II        | ✓ ✓     |
| Anticorrelation  | 1        | 0        | 1        | IV        | ✓ ✓     |
|                  | 1        | 0        | 0        | V         | ✓ ✓     |
| Coexistence of strategies | $1 \over 2$ | $1 \over 2$ | $1 \over 2$ | III   | ✓ ✓ |
|                  | $1+\sqrt{\frac{3}{2}}$ | $1+\sqrt{\frac{3}{2}}$ | $1+\sqrt{\frac{3}{2}}$ | ✓ ✓ |
|                  | $u + v - 2uw$ | $u - l u$ | $v - l v$ | ✓ ✓ |
| Chimera states   | $v, 1-v$ | 0        | $v, 1-v$ | ✓ ✓ |
|                  | $1 \over 2$ | $1 \over 2$ | $1 \over 2$ | ✓ ✓ |
|                  | $u, 1-u$ | $u, 1-u$ | 0        | III-b    | ✓ ✓ |
|                  | $1 \over 2$ | $1 \over 2$ | 0        | ✓ ✓ |

The different solutions described in terms of the asymptotic values of $n_{AB}, x_{aa}$ and $x_{bb}$ are shown in Table III. These solutions follow from the mean-field analysis described in the Appendix. We next describe these solutions as obtained from numerical simulations. In these simulations, we have fixed the system size $N_A + N_B = 2000$ where $N_A = N_B = 1000$.

Figure [3] shows the time evolution of $x_{aa}$ and $x_{bb}$, as obtained from the simulation of our individual based model, for different initial conditions that lead to the asymptotic solutions I, II, III, IV and V. Figures 3A to 3G correspond to the symmetric coordination game (SCG), while chimeras states appearing in the asymmetric coordination game (ACG) are shown in Figures 3H to 3J. In Figures 3A and 3B the order parameter $n_{AB} = 0$, indicates that the system goes to an absorbing state in which the agents in both layers play the same strategy. Figure 3A shows that after a short transient time the fractions $x_{aa} = x_{bb} = 0.0$ and the density of inter-layer active links is $n_{AB} = 0$ corresponding to solution I, while for Figure 3B after a short transient time the fractions $x_{aa} = x_{bb} = 1$ and $n_{AB} = 0$ corresponding to solution II.

In Figures 3C (solution IV) and 3D (solution V), the value $n_{AB} = 1$ indicates that in both cases the system goes to an anticoordination absorbing state, i.e. agents in each layer are playing opposite strategies. This state of anticoordination can emerge only in skeptical populations where $T > 0.5$. It is interesting to notice that the layer with an initial higher proportion of $R$ is the layer that ends playing $L$. A complete analysis of the anticoordination solutions for a skeptical two-group population can be found in [19].

It is important to note that there exist absorbing states other than solutions I, II, IV and V. They correspond to an unstable fixed point $x_{aa} = x_{bb} = r$ of the dynamics for $0 < r < 1$. These solutions correspond to the classification (1) of Solution III, namely, $r = 1/2$ in the SCG for all $T \in (0, 1)$ and in the ACG for $T < 1/2$, or the fixed point $r = \frac{1+\sqrt{3}}{3}$ in ACG when $T > 1/2$.

Figures 3E to 3G display the temporal evolution of the system for marginally stable periodic solutions (solutions III (2)) in the case of the symmetric coordination game. The asymptotic dynamical configurations show phase (Figure 3F) and anti-phase (Figure 3E) oscillations of strategies between the two layers. Note that $0 < n_{AB} < 1$ remains constant during these oscillations.

The Chimera solutions are illustrated in Figure 3H to 3J for the ACG with parameter values $T = 0.25, b = 0.5$. Figure 3H corresponds to a solution III-a, in which all agents play strategy $L$ in layer $A$, i.e. $x_{aa} = 0$ while a configuration of dynamical coexistence of strategies takes place in layer $B$, i.e. $0 < x_{bb} < 1$. Note that the fraction of agents that choose to play strategy $L$ or $R$ in layer $B$ changes over time, so that we also observe an oscillation of $n_{AB}$. Figure 3I corresponds to a solution III-b. In this case the behaviour is similar to the one in Figure 3H, but now the $L$ coordinating absorbing state occurs in layer $B$, while the dynamical coexistence of strategies appears in layer $A$. We also notice that possibly similar solutions with $x_{aa} \neq 0$ constant, and $x_{bb}$ oscillating are not found in our model due to the nonequivalence of the $L$ and $R$ strategies in the ACG. In Figure 3J we illustrate the particular case of a
T = 0 and the initial and the final states are the same when \( x = 0 \).

FIG. 3: Time evolution of \( x_{aa} \) (circles), \( x_{bb} \) (squares) and the density of active inter-layer links \( n_{AB} \) (dashed lines). Panels A to D show the time evolution to coordination (solutions I and II) and anti-coordination states (solutions IV and V) for symmetric game \( (s = 0, b = 0) \) with \( T = 0.75 \). Panels E, F and G show the phase and anti-phase oscillations (solutions III (2)) of the fraction of strategies \( x_{aa} \) and \( x_{bb} \) for symmetric game \( (s = 0, b = 0) \) with \( T = 0.25 \). Panels H, I and J show the temporal evolution of \( x_{aa} \) and \( x_{bb} \) and the active inter-layer links \( n_{AB} \), for the asymmetric coordination game \( (s = 1, b = 0.5) \) with \( T = 0.25 \). Panels H and I show the chimera states, solution III-a and III-b respectively and Panel J shows the case when the initial and the final states are the same when \( x_{aa} = 0 \) and \( x_{bb} = 0.5 \).

FIG. 4: Plot in color scale of the fraction of active links between layer A and B in the asymptotic solution of the dynamics as a function of the initial density of \( x_{aa} \) and \( x_{bb} \) for the Symmetric Coordination Game \( (s = 0, b = 0) \). The color scale defines the values of the fraction of actives links, \( n_{AB} = 1 \) black color and \( n_{AB} = 0 \) white color. Asymptotic solutions are as indicated. Panels A to H correspond to different values of \( T \): A) \( T = 0 \), B) \( T = 0.25 \), C) \( T = 0.5 \), D) \( T = 0.75 \), E) \( T = 0.85 \) and F) \( T = 1 \). System size, \( N_A + N_B = 1000 + 1000 = 2000 \).
inter-layer active links. Both figures show how the basins of attraction in terms of the initial conditions are determined. Herding behaviour while the other is incoherent or desynchronized. Likewise we have two populations of interacting agents described by means of bifurcations diagrams obtained in terms of the control parameters. The color code defines the solutions in terms of the fraction of actives links, $n_{AB} = 1$ black color and $n_{AB} = 0$ white color. Asymptotic solutions are as indicated. Panels A to H correspond to the values of the threshold $T$ and parameter $b$ indicated by red dots in Fig. 2. A) $T = 0.25, b = 0.50$, B) $T = 0.46, b = 0.50$, C) $T = 0.25, b = 0.30$, D) $T = 0.25, b = 3.00$, E) $T = 0.55, b = 3.00$, F) $T = 0.80, b = 3.00$, G) $T = 0.55, b = 0.50$, H) $T = 0.80, b = 0.50$. System size, $N_A + N_B = 1000 + 1000 = 2000$.

chimera state in which one layer coordinates in $L$, in this case layer B, while the other layer remains in a solution III (1) in which agents in that layer start and continue playing for all times both strategies with equal proportions.

The particular collective behaviors described by solutions III-a and III-b, are the social analogue of a chimera state arising in two interacting populations of oscillators observed in dynamical systems. In general a chimera state describes a situation where two populations that interact with each other, one exhibits a coherent or synchronized behaviour while the other is incoherent or desynchronized. Likewise we have two populations of interacting agents such that one reaches an absorbing state, while the other remains in a dynamically disordered state. Our chimera states only arise in our system when the population is herding ($T < 0.5$) and play an asymmetric coordination game. This means that beside the initial distribution of strategies, a herding behaviour is the mechanism that allows to reach chimera states when the two equilibria of the coordination game are not equivalent in terms of payoff.

B. Basins of attraction: The global picture

Depending on the initial conditions for $x_{aa}$ and $x_{ab}$, the system reaches different asymptotic solutions characterized by their value of the order parameter $n_{AB}$. Extensive numerical simulations of our individual based model are summarized in Figure 4 and Figure 5 that show the basins of attraction of the different asymptotic solutions in terms of the initial densities of $x_{aa}$ and $x_{ab}$, and for different values of the threshold parameter $T$ for the SCG and for different values of $T$ and $b$ for the ACG, respectively. The color code defines the solutions in terms of the fraction of inter-layer active links. Both figures show how the basins of attraction in terms of the initial conditions are determined by the value of the threshold parameter $T$ in the case of the SCG and the threshold $T$ along with the parameter $b$ in the case of the ACG.

III. BIFURCATION DIAGRAMS

In this section we consider possible transitions among the different solutions discussed before. These transitions are described by means of bifurcations diagrams obtained in terms of the control parameters $T$ and $b$. 
the population is extremely skeptical of global coordination, Solutions I and II. Strategies, Solution III for almost all initial conditions, to states of anticoordination, Solutions IV and V, and states approach, indicating domains of existence of different asymptotic solutions in the $b$ gradient.

**FIG. 6**: Bifurcation diagrams for the average of $n_{AB}$ as function of $T$ for fixed initial conditions for the SCG, panels (A-C), and for the ACG, panels (D-G). Panel A: $x_{aa} = 0.40$ and $x_{bb} = 0.10$ (circles), $x_{aa} = 0.40$ and $x_{bb} = 0.30$ (squares). Panel B: $x_{aa} = 0.40$ and $x_{bb} = 0.10$ (circles), $x_{aa} = 0.10$ and $x_{bb} = 0.90$ (squares). Panel C: $x_{aa} = 0.25$ and $x_{bb} = 0.51$ (circles), $x_{aa} = 0.37$ and $x_{bb} = 0.52$ (squares). Panel D: $x_{aa} = 0.30$ and $x_{bb} = 0.45$ with $T = 0.75$. Panel E: $x_{aa} = 0.10$ and $x_{bb} = 0.48$ with $T = 0.15$. Panel F: $x_{aa} = 0.25$ and $x_{bb} = 0.1$ with $T = 0.7$. Panel G: $x_{aa} = 0.45$ and $x_{bb} = 0.65$ with $T = 0.75$.

### A. Symmetric Coordination Game

We have shown in the previous section that the solution obtained for the SCG, and for a fixed initial condition, depends on the value of the threshold parameter $T$, so that by varying $T$ we find transitions among those solutions. Examples of these transitions are shown in Figure 6A to 6C. These are bifurcation diagrams that give the average of the fractions of inter-layer active links $n_{AB}$ or $1 - n_{AB}$ as a function of the threshold $T$. These bifurcation diagrams describe transitions that occur, for threshold values of $T$, between solutions III to I, III to V and I to V respectively for different fixed values of the initial conditions. Each panel shows two examples. We also find subsequent transitions among three solutions. For instance, the bifurcation diagram Figure 7A for the average of $n_{AB}$, illustrates a first transition between solution III and solution I, followed by a second transition between I and V as $T$ increases. These results show the effect of the skepticism on the collective behavior in a two-layer network. Tuning the level of skepticism from the limit value $T = 0$, where the population is extremely herding to the limit value $T = 1$ where the population is extremely skeptical, Figure 4 indicates that the system goes from a state of complete coexistence of strategies, Solution III for almost all initial conditions, to states of anticoordination, Solutions IV and V, and states of global coordination, Solutions I and II.

### B. Asymmetric Coordination Game

As discussed before, Figure 2 shows for the ACG case a phase diagram, obtained from a mean field theoretical approach, indicating domains of existence of different asymptotic solutions in the $b - T$ parameter space. In comparison with the SCG case, the additional parameter $b$ allows for new transitions that occur for a threshold value of $b$ and fixed $T$, including transition to chimera states. Examples of these transitions are shown in Figures 3D to 3G. Figs 3D shows a transition between a state of anticoordination IV and a state of full coordination I and Fig 3G shows a transition between a state of full coordination II and a state of anticoordination V, while Figs 3E and 3F show transitions between a state of coordination II or dynamical coexistence III and a chimera state III-a. On the other hand, Figure 7B shows an example of subsequent transitions as $T$ increases for fixed $b = 3$, with a first transition between a state of dynamical coexistence III and a chimera state III-a, followed by a transition between III-a and a state of full coordination II and a final transition between II and a state of anticoordination IV.

A different form of bifurcation diagrams can be obtained by considering the area of the basin of attraction of a given solution in the parameter space of the initial conditions $x_{aa}$ and $x_{bb}$. Results for this area are indicated in
FIG. 7: Panel (A): Bifurcation diagram among solutions III, I and V of the average of $n_{AB}$ as function of $T$ in the SCG for a fixed initial condition $x_{aa} = 0.20$ and $x_{bb} = 0.20$. Panel (B): Bifurcation diagram among solutions III, III-a, II and IV of the average of $n_{AB}$ as function of $T$ for a fixed initial condition $x_{aa} = 0.78$ and $x_{bb} = 0.60$ where the risk parameter $b = 3$ in the ACG.

Table IV: Areas of the basin of attraction of chimera states in the parameter space of initial conditions of $x_{aa}$ and $x_{bb}$ according to the zones described in Figure 2.

| zones | ranges | $Q_1$ | $Q_2$ |
|-------|--------|------|------|
| A     | $T < \frac{1+b}{1+b} < 0.5$ | $2 \left( \frac{1+b}{1+b} - T \right) T$ | 0 |
| B     | $\frac{1+b}{1+b} < T < 0.5$ | 0 | 0 |
| C     | $1 - \frac{1+b}{1+b} < T < 0.5$ | $2(1-2T)T$ | $2(1-2T)T$ |
| D     | $T < 1 - \frac{1+b}{1+b} < 0.5$ | $2 \left( \frac{1+b}{1+b} - T \right) T$ | $2 \left( \frac{2+T}{2+T} - 1 \right) T$ |

Table [IV] for the chimera states $Q_1$ and $Q_2$ and the different zones of the $T$-$b$ parameter space of Fig. 2. We recall that $Q_1$ are those solutions III-a and III-b reached from initial conditions such that $x_{aa}^0 + x_{bb}^0 < 1$, while $Q_2$ are those obtained when $x_{aa}^0 + x_{bb}^0 > 1$. The areas of the basins of attractions $A_{Q_1}$, $A_{Q_2}$ for $Q_1$ and $Q_2$ respectively are plotted versus $T$ and $b$ in Figure [IV]. Using $A_{Q_1}$, $A_{Q_2}$ as order parameters, these figures show bifurcation diagrams for the transition from existence $A_{Q} \neq 0$ to non-existence $A_{Q} = 0$ of a chimera state.

Figures [IV][A] and [IV][B] show a threshold value $T = 0.5$, so that chimera states exist for $T < 0.5$ in agreement with the phase diagram of Fig. 2. They also show that as $T$ increases the areas of the basin of attraction of chimera states first increase until a certain value of $T$ and then they decrease to become zero for $T = 0.5$. In addition Figures [IV][C] and [IV][D] identify a threshold value of $b$ for the existence of chimera states. For $Q_2$ chimeras this is fixed at $b = 1$ independently of $T$, while for $Q_1$ it depends on $T$, with $Q_1$ chimera states existing for all values of $b$ and $T$ small enough. For both $Q_1$ and $Q_2$ we also identify a characteristic $T$-dependent value of $b$ beyond which the area of the basin of attraction remains constant.

Table [IV] identifies the existence of chimera states for fixed values of $b$ and $T$ small enough. More generally and on a qualitative basis, it follows from Figs. 4 and 5 that, for any fixed $b$, the basins of attraction of solution III and chimeras states disappear as $T$ increases, so that and in the limiting case of $T = 1$, only Solutions I, II, IV and V can be reached by the system. Another interesting limiting case is the one of the risk parameter $b \to \infty$, where it is extremely risky to play strategy $R$. It can be expected that in this limit solution I becomes preponderant. Indeed, we show in Fig. 9, as compared with Fig. 5, that the basin of attraction of solution I increases, solution II disappears and solution III and chimeras remain for fixed values of $T < 0.5$. As both $b$ and $T$ increase, solutions II, III, IV and V disappear and the basin of attraction of solution I increases. In the limit, $T = 1$ and $b \to \infty$, solution I becomes the main solution in the system for almost every initial condition.
FIG. 8: Sizes of the basins of attraction of chimera states categorized by $Q_1$ and $Q_2$, denoted by $A_{Q_1}$ and $A_{Q_2}$, as function of $T$ for different values of $b$ (Panels A and B) and as function of $b$ for different values of $T$ (Panels C and D).

FIG. 9: Plot in color scale of the fraction of active links between layer A and B in the asymptotic solution of the dynamics as a function of the initial density of $x_{aa}$ and $x_{bb}$ for the Asymmetric Coordination Game. The color scale defines the values of the fraction of active links, $n_{AB} = 1$ black color and $n_{AB} = 0$ white color. Asymptotic solutions are as indicated. Panel A: $b = 100$, $T = 0.25$, and panel B: $b = 100$, $T = 0.50$. 
We have considered a model of evolutionary game of a population divided into two groups where individuals are searching to fulfil their social and strategic objectives. The framework for this situation has been a multilayer network of two layers. Interactions within each layer aim to fulfil social objectives associated with learning dynamics, while interactions across layer consist in a coordination game, therefore involving strategic objectives. Our analysis, based on a mean-field theoretical approach and corroborated by numerical simulations of the model, reveals the existence of collective behaviors commonly observed on nature but impossible to find on a single isolated network analysis. In our multilayer framework we find states different of those of global coordination or dynamical states of coexistence of the strategies. Namely, in the multilayer coordination challenge, anticoordination and chimera states solution emerge. In the former the dynamics of the system polarizes the population, with each layer coordinating in a different strategy. This can also happen in the asymmetric coordination game where the two strategies correspond to different Nash equilibria: the socially efficient or Pareto dominant, and the risk dominant equilibrium. In the chimera states one layer coordinates in the risk dominant equilibrium, while the second remains disordered, that is with coexistence of the two strategies. This coexistence can be time independent or in the form of periodic solutions.

In connection with the standard notion of chimera states in two populations of dynamical oscillators having global or long range interactions \[2, 20–22\], we note that in our evolutionary game theory framework we also have the basic ingredients of two non-linear dynamical systems which are globally coupled. In chimera states of coupled oscillators, one population is in a coherent state and coexists with the other population in an incoherent state. In our social analogue of the chimera state we have interpreted the coordination states in one layer as a coherent or ordered state, while we identify the incoherent state with the layer that exhibits coexistence of the two strategies. In most cases this coexistence is of dynamical nature, being the disordered layer in an active dynamical state of oscillation between the two possible strategies. Our model only incorporates two possible individual states of the agents, but we envisage that in social models including more individual states or strategies, such as those in reference \([8, 9]\), the disordered layer coordinates in the risk dominant equilibrium, while the second remains disordered, that is with coexistence of the two strategies. This coexistence can be time independent or in the form of periodic solutions.

In the context of coordination in social systems, our contribution brings a more realistic insight about the consequences of a collective behavior that makes a distinction between social and strategic objectives. This collective behavior may lead herding societies to chimera states and skeptical societies to polarized states of anticoordination.

Appendix

Mean-field approach: symmetric coordination game

We present here the mean-field equations for the time evolution of a system divided in two groups. Our analysis reveals a highly nontrivial behavior of the learning and coordinating process depending on the initial conditions of the system.

Let \(x_{aa}\) be the fraction of individuals playing \(R\) in layer \(A\) and \(x_{bb}\) be the fraction of individuals playing \(R\) in layer \(B\). In the limit of infinite population size \((N_A = N_B \to \infty)\), and according to the levels of satisfaction described in Section 2 for the symmetric coordination case, the time derivative \(\dot{x}_{aa}\) is given by:

\[
\dot{x}_{aa} = -x_{aa}\Theta(1-x_{aa}-T)\Theta^*(1/2-x_{bb})H(x_{bb} = 1) \\
-x_{aa}[1-\Theta(1-x_{aa}-T)]H(x_{bb} < 1)\Theta^*(1/2-x_{bb})\Theta^*(1-x_{aa}) \\
-x_{aa}\Theta(1-x_{aa}-T)H(x_{bb} < 1) \\
+(1-x_{aa})\Theta(x_{aa}-T)\Theta^*(x_{bb}-1/2)H(x_{bb} = 0) \\
+(1-x_{aa})(1-\Theta(x_{aa}-T)]H(x_{bb} > 0)\Theta^*(x_{bb} - 1/2)\Theta^*(x_{aa}) \\
+(1-x_{aa})\Theta(x_{aa}-T)H(x_{bb} > 0)
\]
Figure 4 shows results of numerical simulations of our individual based model for different parameter values and different initial conditions of the different solutions I, II, III, IV and V.

Table V summarizes the results for $\dot{x}_{aa}$ in group 1.

| $\dot{x}_{aa}$ | $x_{aa}$ | $\dot{x}_{bb}$ | $x_{bb}$ |
|----------------|----------|----------------|---------|
| 0              | 0        | 0              | [0, 1]  |
| $(0,1-T)$      | $-x_{aa}$ | $[0,1/2]$      | $(1/2,1)$ |
| $(1-T,T)$      | $-x_{aa}$ | $[0,1/2]$      | $(1/2,1)$ |
| $(T,1)$        | $-x_{aa}$ | 0              | $(0,1/2)$ |
| $(T,1)$        | $-x_{aa}$ | 0              | $(0,1/2)$ |
| $1$            | 0        | 0              | $[0,1]$  |

Table V: Results of $\dot{x}_{aa}$ for the different values of $x_{aa}$ and $x_{bb}$ in the SCG. $T > 1/2$ (left) and $T < 1/2$ (right).

where $\Theta(z) = 1$ if $z \geq 0$ and $\Theta^*(z) = 1$ if $z > 0$, otherwise $\Theta(z) = 0$, and $H(z) = 1$ if $z$ is true, otherwise is 0. Since the inequalities $1/2 > x_{bb}$ and $x_{bb} = 1$ cannot hold simultaneously, and neither $x_{bb} > 1/2$ and $x_{bb} = 0$, the previous equation can be shortened as:

$$
\dot{x}_{aa} = -x_{aa}[1 - \Theta(1 - x_{aa} - T)]H(x_{bb} < 1)\Theta^*(1/2 - x_{bb})\Theta^*(1 - x_{aa}) - x_{aa}\Theta(1 - x_{aa} - T)H(x_{bb} < 1) + (1 - x_{aa})[1 - \Theta(x_{aa} - T)]H(x_{bb} > 0)\Theta^*(x_{bb} - 1/2)\Theta^*(x_{aa}) + (1 - x_{aa})\Theta(x_{aa} - T)H(x_{bb} > 0)
$$

As mentioned before, the equation is derived considering the different levels of satisfaction of agents playing $R$ or $L$ in group $A$. Next, in order to simplify the analysis we consider two cases depending on whether $T$ is bigger than $1/2$. Table V summarizes the results for $\dot{x}_{aa}$ for the different values of $x_{aa}$ and $x_{bb}$ when we distinguish the two cases, $T > 1/2$ and $T < 1/2$. Carrying out the same analysis for $\dot{x}_{bb}$ we can easily check that the solutions are as follows:

- Time independent coordination or anticoordination solutions:
  1. $x_{aa} = x_{bb} = 0$. Solution I,
  2. $x_{aa} = x_{bb} = 1$. Solution II,
  3. $x_{aa} = 0$, $x_{bb} = 1$. Solution IV,
  4. $x_{aa} = 1$, $x_{bb} = 0$. Solution V.

- Time independent coexistence solution: $x_{aa} = x_{bb} = 1/2$. Solution III(1).

- A family of periodic solutions of coexistence: $x_{aa}$ oscillates between $s$ and $1 - s$ and $x_{bb}$ between $r$ and $1 - r$ for some $0 < s, r < 1$. Solutions III(2).

Figure 4 shows results of numerical simulations of our individual based model for different parameter values and different initial conditions $x_{aa,0}$ and $x_{bb,0}$ for $x_{aa}$ and $x_{bb}$ respectively. These results identify the basins of attraction of the different solutions I, II, III, IV and V.

**Mean-field approach: asymmetric coordination game**

By the same arguments than for the symmetric coordination case, and in the limit of infinite population size ($N_A = N_B \to \infty$), the time derivative $\dot{x}_{aa}$ is given by:

$$
\dot{x}_{aa} = -x_{aa}[1 - \Theta(1 - x_{aa} - T)]H(x_{bb} < 1)\Theta^*(1 + \frac{1}{2} - x_{bb})\Theta^*(1 - x_{aa}) - x_{aa}\Theta(1 - x_{aa} - T)H(x_{bb} < 1) + (1 - x_{aa})[1 - \Theta(x_{aa} - T)]H(x_{bb} > 0)\Theta^*(x_{bb} - 1 + \frac{1}{2})\Theta^*(x_{aa}) + (1 - x_{aa})\Theta(x_{aa} - T)H(x_{bb} > 0)
$$
The term $\frac{1+b}{3+b}$ arises when the partially unsatisfied player with strategy $R$ will change to strategy $L$ if the players with the strategy $L$ gain a higher payoff than her. This happens when the fraction of players playing $R$ in the other layer is not larger than $\frac{1+b}{3+b}$. Table VI summarizes the results for $\dot{x}_{aa}$ in the two cases, $T > 1/2$ and $T < 1/2$. As in the symmetric coordination case, the equations display the time independent coordination or anticorordination solutions: $x_{aa} = x_{bb} = 0$ and $x_{aa} = x_{bb} = 1$ and $x_{aa}$ and $x_{bb}$ equal to 0 or 1, the time independent coexistence solution $x_{aa} = x_{bb} = 1/2$ for $T < 1/2$ and the family of periodic solutions of coexistence with $0 < x_{aa} < 1$ and $0 < x_{bb} < 1$. Additionally, we find three new types of solutions:

1. For $T > 1/2$: a time independent coexistence solution: $x_{aa} = x_{bb} = \frac{1+b}{3+b}$. Solution III(1).

2. For $T < 1/2$: Two chimera solutions with one dynamically disordered layer:
   - $x_{aa} = 0$, and a periodic solution for $x_{bb}$. Solution III-a.
   - $x_{bb} = 0$ and a periodic solution for $x_{aa}$. Solution III-b.

3. For $T < 1/2$: Two chimera solutions with one time independent disordered layer:
   - $x_{aa} = 0$, $x_{bb} = 0.5$ or $x_{aa} = 0.5$, $x_{bb} = 0$. Solutions III-a or III-b.

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