Featuring Causal Order in Teleportation With Two Quantum Teleportation Channels

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Abstract. Causal order can improve the quantum information transmission in teleportation due to a noisy entangled resource being present in it by using additionally an appropriate measurement state on the control state ruling the causal order. In this work, we get analytically the fidelity for the entire process under an arbitrary initialization of such control state and performing an optimal measurement on it, thus obtaining a perfect teleportation. We also analyse other values characterizing the imperfect entangled state where a perfect teleportation cannot be reached. Notably, we determine that the best fidelity does not depend on the preparation of the control state but instead on the imperfect initial entangled resource.

1. Introduction

Quantum communication is always looking for improvements. Recently, it has been shown that a couple of depolarizing channels becomes transparent as a result of interference under indefinite causal order, thus reaching enhancement with the assistance of causal order activation [1]. After this, indefinite causal order has been widely studied. It is the case for the so called quantum switch [2].

Causal order enhancements in communication have had approaches both theoretical and experimental. In the case of two quantum channels, it has been shown that even though no information can be transmitted through depolarizing channels by classical means due to the noise, it could be possible to transmit information by combining two depolarizing channels in a superposition of causal orders [1, 3]. Thus, it has been extended to more than two channels, showing that the enhancement of transmission for the three channel scenario [4] improves the amount of information being transmitted. Alternatively, for the quantum teleportation algorithm where very noisy singlets introduce serious imperfections in the process, it has been shown the possibility of still transmitting perfectly an arbitrary information state by applying superposition of causal order of two such channels [5]. Such work states that for an egalitarian superposition on the causal order, certain final measurements on it let to reach a perfect teleportation for the noisiest entangled resource using a two teleportation channels assisted by an indefinite causal order scheme.

This work shows that a perfect quantum teleportation can be still reached using causal order superposition with an arbitrary control state by stating a proper selection of the post-measurement state on it, thus extending the outcomes obtained in [5]. Section 2 presents the details of the teleportation circuit in causal order for two channels. Section 3 shows the main findings for optimal measurements on the control state through their corresponding fidelity,
analyzing the conditions for which an optimal teleportation can be obtained. The final section states the conclusions.

2. Formalism to set quantum teleportation under an indefinite causal order scheme with two channels

The aim of this work is the teleportation of the state $|\psi\rangle = \cos \theta |0\rangle + \sin \theta e^{i\phi} |1\rangle$. In [5] the proposed imperfect entangled state assessing the teleportation is stated as $|\chi\rangle = \sum_{i=0}^{3} \sqrt{p_i} |\beta_i\rangle$; where $|\beta_i\rangle$ are the Bell states $|\beta_0\rangle = |\beta_{00}\rangle$, $|\beta_1\rangle = |\beta_{01}\rangle$, $|\beta_2\rangle = |\beta_{11}\rangle$ and $|\beta_3\rangle = |\beta_{10}\rangle$, with:

$$|\beta_{ij}\rangle = \frac{1}{\sqrt{2}} (|0\rangle + (-1)^i |1\rangle) \oplus 1.$$

(1)

The perfect teleportation process could be achieved with $p_0 = 1$, ($p_1 = p_2 = p_3 = 0$). If the Bell state $|\beta_0\rangle$ is pretended to be used as the successful entanglement resource, the output of the teleportation channel is given by $\Lambda[|\rho\rangle] = \sum_{i=0}^{3} p_i \tilde{\sigma}_i |\rho\rangle |\tilde{\sigma}_i\rangle$, with $\tilde{\sigma}_i = \sigma_i$ if $i = 0, 1, 3$ and $\tilde{\sigma}_2 = i\sigma_2$, where $|\rho\rangle = |\psi\rangle \langle \psi|$. Thus, for a single teleportation channel, the corresponding Kraus operators are $K_i = \sqrt{p_i} \sigma_i$. Recently, [5] presents a version of two teleportation channels in superposition of causal orders controlled by the state:

$$|\rho_c\rangle = \sum_{k, k' = 0}^{1} \sqrt{q_{k}q_{k'}} |k\rangle \langle k'|.$$

(2)

Figure 1 shows the deployment of two teleportation channels a) - b) in a definite causal order as function of the control state, $T_1$ first and then $T_2$ if $|\psi_c\rangle = |0\rangle$ or $T_2$ first and then $T_1$ if $|\psi_c\rangle = |1\rangle$; and c) in an indefinite causal order if $|\psi_c\rangle$ is a superposition of the two last control states with probabilities $q_0$ and $q_1 = 1 - q_0$ respectively as in (2).

In such process, a final measurement on the control state shows that when the outcome is $|\psi_m\rangle = |+\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$, a perfect teleportation process is obtained for the worst case given by $p_1 = p_2 = p_3 = p = \frac{1}{3}$. In this work, instead, we consider the general state for the measurement as (which could exhibit different probabilities of success $P$):

$$|\psi_m\rangle = \cos \frac{\theta}{2} |0\rangle + \sin \frac{\theta}{2} e^{i\phi} |1\rangle.$$

(3)

Following [5], we construct the Kraus operators for two consecutive teleportation channels in causal order, but instead of apply a measurement on the control state based on the states

![Figure 1](image_url)
we apply a general measurement based on the state (3) and their correspondent orthogonal state $|\psi_m^\perp\rangle = \sin \theta^2 |0\rangle - \cos \theta^2 e^{-i\phi} |1\rangle$. For the first state, we get the unnormalized output:

$$\Lambda^2_{\text{un}}[\rho] = \sum_{i,j=0}^{3} p_ip_j \left( \left( \frac{1}{2} + (q_0 - \frac{1}{2}) \cos \theta \right) \sigma_i \sigma_j \rho \sigma_i \sigma_j + \sqrt{q_0q_1} \sin \theta \cos \phi \sigma_i \sigma_j \rho \sigma_i \sigma_j \right).$$

(4)

3. Fidelity of quantum teleportation under an indefinite causal order scheme

We use the fidelity $F = \text{Tr} (\Lambda^2[\rho]|\rho\rangle)$ to assess the entire process (a comparative measure between the input and the teleported state) together with the probability of successful measurement $P = \text{Tr} (\Lambda^2[\rho])$:

$$F_{\text{un}} = \sum_{i,j=0}^{3} p_ip_j \left( \left( \frac{1}{2} + (q_0 - \frac{1}{2}) \cos \theta \right) \text{Tr}(\rho \sigma_i \sigma_j \rho \sigma_i \sigma_j) + \sqrt{q_0q_1} \sin \theta \cos \phi \text{Tr}(\rho \sigma_i \sigma_j \rho \sigma_i \sigma_j) \right),$$

(5)

$$P = \sum_{i,j=0}^{3} p_ip_j \left( \left( \frac{1}{2} + (q_0 - \frac{1}{2}) \cos \theta \right) \text{Tr}(\sigma_i \sigma_j \rho \sigma_i \sigma_j) + \sqrt{q_0q_1} \sin \theta \cos \phi \text{Tr}(\sigma_i \sigma_j \rho \sigma_i \sigma_j) \right),$$

(6)

then, dividing (5) by (6), we get the normalized fidelity:

$$F = \frac{\sum_{i,j=0}^{3} p_ip_j \left( \left( \frac{1}{2} + (q_0 - \frac{1}{2}) \cos \theta \right) \text{Tr}(\rho \sigma_i \sigma_j \rho \sigma_i \sigma_j) + \sqrt{q_0q_1} \sin \theta \cos \phi \text{Tr}(\rho \sigma_i \sigma_j \rho \sigma_i \sigma_j) \right)}{\sum_{i,j=0}^{3} p_ip_j \left( \left( \frac{1}{2} + (q_0 - \frac{1}{2}) \cos \theta \right) \text{Tr}(\sigma_i \sigma_j \rho \sigma_i \sigma_j) + \sqrt{q_0q_1} \sin \theta \cos \phi \text{Tr}(\sigma_i \sigma_j \rho \sigma_i \sigma_j) \right)}. $$

(7)

**Figure 2.** In the contour plots a) - e) for $F$, it is indicated the values for $\theta$ and $\phi$ such that $F = 1$ is reached ($q_0 = 0.1, 0.3, 0.5, 0.7, 0.9$ and $P = 0.12, 0.28, 0.33, 0.28, 0.12$ respectively). Color bar shows the values of fidelity. Plot f) exhibits the relation between $\theta$ and $q_0$ under the election of the best control measurement ($\phi = 0$ always).
We can still simplify last formula using the Pauli matrices and the trace operation properties: \( \text{Tr}(\sigma_i \sigma_j \rho \sigma_j \sigma_i) = 1 \) and \( \text{Tr}(\sigma_i \sigma_j \rho \sigma_i \sigma_j) = \delta_{ij} + (1 - \delta_{ij})(1 - 2 \text{sgn}(ij)) \), with \( \text{sgn}(x) \) the sign function. Now, our task is to demonstrate for the worst case \( p_1 = p_2 = p_3 = p = 1/3 \) and \( p_0 = 0 \), that it is possible to reach a perfect teleportation \( (F = 1) \) by choosing adequately the measurement state \( \phi \). We solve numerically the optimization problem by fixing \( p \) and \( q \) state. Contour plots a) - e) correspond to \( p = 1/3 \) and \( q = 1/2 \) (reported on the color-scale besides). In fact, surprisingly the outcome is \( \sqrt{2} \) value from 0 -red- to \( \pi \) -blue- (for \( p = 3 - \sqrt{3}/6 \)) to 1 -blue- (for \( p = 0, 1/3 \)).

**Figure 3.** Outcomes for the fidelity \( F \) for other values of \( p \) different from \( 1/3 \). a) Shows the contour plot for \( F \) for \( p = 1/3 \) and \( q_0 = 1/2 \); b)-c) depicts the dependence of \( F \) and \( \theta \) from \( p \) and \( q_0 \). There, color reports the \( \theta \) values and the \( F \) values respectively (see the text).

### 4. Conclusions

We have analyzed the features of two teleportation channels in superposition of causal order for the \( p_1 = p_2 = p_3 = p, p_0 = 1 - 3p \) case for the deformed entangled resource required in each teleportation process. For the specific case with \( p = 1/3 \), we have shown that \( F = 1 \) can be reached not only with the state \( |+\rangle \) for \( q_0 = 1/2 \), but as far as we choose the measurement state given by (3) in the range for \( \phi = 0 \) and \( \theta \in [0, \pi] \), a perfect teleportation can be done as function of an arbitrary \( q_0 \). Nevertheless, the probability of success is maximum for the \( q_0 = 1/2 \) case with the optimal measurement \( |\psi_m\rangle = |+\rangle \).

If we analyze other values for \( p \neq 1/3 \), we can not reach \( F = 1 \) no matter the measurement state chosen. Nevertheless, we can figure out the best choice for the measurement state in...
order to reach the optimal fidelity, noting it does not depend on the value for $q_0$, so that the maximum for $\mathcal{F}$ is fixed once selected $p$ and it can be reached with the correct values for $\theta$ and $\phi$ in the measurement basis. Future work should to analyze similar outcomes for $\mathcal{F}$ by managing independently the $p_0, p_1, p_2, p_3, q_0$ values together with $\theta$ and $\phi$ in order to search the maximum $\mathcal{F}$ for each case and the improvement of $\mathcal{P}$.

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