Determining $g_A$ using non-perturbatively $O(a)$ improved Wilson fermions

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A completely non-perturbative estimate is given for $g_A$ using both quenched and unquenched $O(a)$ improved Wilson fermions. Particular attention is paid to the determination of the axial renormalisation constant, $Z_A$, using the Ward identity for the propagator. For the quenched case, we have results at three lattice spacings allowing a continuum extrapolation.

1. INTRODUCTION

The axial charge of the nucleon, $\Delta q$, is defined from the axial current by

$$\langle N(\vec{p},\vec{s})|A^\mu_R|N(\vec{p},\vec{s})\rangle = 2s^\mu \Delta q,$$

with $A^\mu_R = (\bar{q}\gamma^\mu\gamma_5 q)_R$, giving for the non-singlet charge

$$g_A \equiv g_A^{(3)} = \Delta u - \Delta d.$$

This determines the strength of $\beta$-decay and also occurs in the Bjorken sum rule where the lowest moment of the difference between the proton and neutron polarised structure function $g_1$ is proportional to the non-singlet charge. To find $g_A$ the programme is:

1. Compute $\langle N|\bar{u}\gamma^\mu\gamma_5 u - \bar{d}\gamma^\mu\gamma_5 d|N\rangle$ on the lattice for many quark masses ($\gg 3$) and many lattice spacings ($\gg 3$).

2. Determine non-perturbatively the renormalisation constant $Z_A$.

3. Chirally extrapolate $m_q$ (or $m^2_q$) → 0.

4. Continuum extrapolate $a^2$ → 0 (if using $O(a)$ improved fermions)

5. Do first for quenched (as much cheaper in CPU time) then repeat for unquenched fermions.

6. Compare with experiment, $g_A \approx 1.26$.

Our lattice simulation, point (1), is standard using ratios of nucleon three-point to two-point correlation functions and will not be discussed further here, except to note that because we are considering a non-singlet function, the difficult to compute quark line disconnected terms cancel. Here, in this talk, we shall mainly discuss point (3).

2. DETERMINING $Z_A$

On the lattice, for Wilson fermions, the chiral Ward Identity is only approximately true, due to discretisation effects. We have a finite renormalisation constant $Z_A(g)$ to determine by demanding that a Ward Identity is obeyed to $O(a^n)$ (with $n = 2$ for $O(a)$ improved fermions). The ALPHA collaboration [1] has found $Z_A$ for quenched fermions using the Schrödinger functional and PCAC; here, alternatively, we shall use the chiral Ward identity for the propagator. We shall first check our $Z_A$ results for quenched fermions before then determining $Z_A$ for unquenched fermions. The chiral Ward identity for the propagator, $S_R$, reads

$$\gamma_5 S_R^{-1} + S_R^{-1} \gamma_5 = 2 m_R \Lambda_R^P,$$

where $m_R$ is the quark mass, and $\Lambda_R^P$ the 1PI pseudoscalar vertex, obtained from the Green’s function $G_P$ by amputating the external propagators. On the lattice we take this
equation to be correct to $O(a^2)$,
\[
\text{Tr} S^{-1}_a = Z_A \bar{m} \text{Tr} \gamma_5 \Lambda_P^f,
\]
where $O(a)$ improved operators are denoted with a star. The bare quark mass $m_0$ has also been replaced by the quark mass obtained from PCAC, $\bar{m}$. These quark masses may be computed in the standard way using the appropriate axial vector matrix elements. The propagator and vertex can also be found on the lattice, in the Landau gauge, \[3, 4\]. A problem with using the propagator is that we now need $O(a)$ improvement off-shell. To try to ameliorate this problem we note that as dominant off-shell effects come at short distances, we shall subtract a contact term (ie delta function in position space), \[5\], from the propagator and Green’s function in addition to improving the appropriate operator. This procedure has been shown to work to first order in perturbation theory, \[6\], and we shall use the formulation given there. Note however that the pseudoscalar Green’s function (or $1PI$ vertex) is particularly simple: there is effectively only one additional $O(a)$ improvement term. However, initial attempts to satisfy the above equation fail because of the presence of large $O(a^2)$ terms mainly due to the Wilson term in the propagator, and so here we choose to re-write the propagator $S$ and vertex $\Lambda_P$ in an $O(a)$ equivalent form, \[6\], as solutions of
\[
S^{-1} = \left( 1 - am\psi \frac{\partial^2}{\partial^2} + \lambda \psi \frac{\partial^2}{\partial^2} S^{-1} \right) S^{-1} = \frac{1}{2} a \lambda \psi \frac{\partial^2}{\partial^2} S^{-1},
\]
and
\[
[1 - amdP] \Lambda_P^f = \Lambda_P^f + \frac{1}{2} a \lambda \psi \{S^{-1}_a, \Lambda_P^f\},
\]
\[
(\hat{P}_i = (1/a) \sin ap_i, \hat{\lambda}_i = (2/a) \sin ap_i/2)
\]
where $b_\psi = -1 + O(g^2)$ and $\lambda_\psi = 1 + O(g^2)$ are the improvement coefficient and coefficient of the contact term for the propagator and $d_P = b_\psi - \Lambda_P Z_m Z_P + b_A = -1 + O(g^2)$. This gives $S^{-1}_a = \hat{P} + m_+ + O(a^3 m^2 p^2)$ for the free field. The Wilson term has been suppressed, $O(a^2)$ terms being multiplied by an additional small $a^2 m^2$ term. As in this case $\text{Tr} \gamma_5 \Lambda_P^f \sim 1/(1 + (am)^2)$ it is little affected from its true value of 1.

We shall first determine $Z_A$ for quenched fermions. We have generated data sets at $\beta = 6.0, 6.2, 6.4$, each data set with 3 (or more) quark masses. To determine $Z_A$, we attempt to minimise $R(p) \equiv \text{Tr} S^{-1}_a / \bar{m} \text{Tr} \gamma_5 \Lambda_P^f = \text{const}$ for many momenta (a three parameter fit). A typical result is given in Fig. 1. The gradient gives an estimate for $Z_A$. Deviations are seen for the higher momenta. This is more clearly seen in Fig. 2 which tells us how far we can go, a good fit being obtained for $(ap)^2 \lesssim 2.5$, say. Once the quark propagator has been obtained, we may then use it to find $Z_{\psi}^{\text{MOM}}$ and $m_q^{\text{MOM}}$, \[7\]. We shall not be concerned with this here, \[7\], but just concentrate on $Z_A$. In Fig. 3 we compare our result here with the previously known result, \[7\]. Reasonably good agreement is found. (In any case, as different methods have different $O(a^2)$ corrections complete agreement does not have to be found.)

Buoyed up by this we now turn to unquenched $n_f = 2$ fermions where we have analysed three UKQCD data sets, \[7\], $(\beta, \kappa_{sea}) = (5.29, 0.1340), (5.26, 0.1345)$ and $(5.20, 0.1350)$. These are matched at an approximately constant $r_0$ value, corresponding to $a \sim 0.105$fm.

**Figure 1.** \(\text{Tr} S^{-1}_a\) plotted against \(\bar{m} \text{Tr} \gamma_5 \Lambda_P^f\) for \(\beta = 6.20\) for 4 quark masses. Heavier masses lie on the RHS of the picture. Smaller momenta are on the RHS of each data set.
Figure 2. $\text{Tr}S^{-1}$ plotted with $Z_A \bar{m} \text{Tr} \gamma_5 A^a_x$ for $\beta = 6.20$ for the 4 quark masses (the heavier quark masses are higher).

| $(\beta, \kappa_{\text{sea}})$ | $Z_A |_{c_A = -0.02}$ | $Z_A |_{c_A = -0.08}$ |
|-----------------|-----------------|-----------------|
| (5.29, 0.1340) | 0.65(2)         | 0.73(2)         |
| (5.26, 0.1345) | 0.68(3)         | 0.75(2)         |
| (5.20, 0.1350) | 0.64(1)         | 0.71(1)         |

Table 1
Preliminary values of $Z_A$ obtained for unquenched fermions.

(Using $r_0 = 0.5$fm). A typical result is shown in Fig. 3. The preliminary values obtained are given in Table 1. Unfortunately for constructing $\bar{m}$, the axial current improvement coefficient $c_A$ is unknown; we shall use here two values, the first $c_A \sim -0.02$ corresponding to a tadpole improved result, while the second $c_A \sim -0.08$ is the non-perturbative quenched value at $\beta = 6.0$. While this does not change the fit, i.e. the product $Z_A \bar{m}$, the 10% change in $\bar{m}$ directly produces a 10% change in $Z_A$.

As a further check we have first computed $f_\pi$. After performing the chiral extrapolation, for the three data points given in Table 1 we find the results given in Fig. 4. Although a satisfactory result is obtained, one should not read too much into this at present, as apart from the uncertainty in $Z_A$ only three quark masses are used in the chiral extrapolation.

Figure 3. $Z_A$ versus $g^2$ (filled circles). Also shown is the ALPHA non-perturbative determination, together with the first order perturbation theory result.

3. RESULTS FOR $g_A$

We are now ready to present our results for $g_A$. Again after the (linear) chiral extrapolation we plot our results as a function of $a^2$. The current results are shown in Fig. 5. All results, at present, lie too low in comparison with the experimental number. Even allowing for the uncertainty in the determination

A typical chiral extrapolation for $\beta = 6.0$ quenched fermions is shown in Fig. 1 of [11]. (Note that we take the unknown improvement coefficient $b_A = 0$ here. We have checked that in the chiral limit, as expected, this plays no role in the numerical value.)
4. CONCLUSIONS

A completely non-perturbative evaluation of a nucleon matrix element needs

- large physical boxes
- good determination of $Z$
- chiral extrapolation (delicate)
- continuum extrapolation (delicate)

How far have we got? We have concentrated on $g_A$ here and tried to develop a way of determining $Z_A$ using the propagator and pseudoscalar vertex. It is apparent though that we are only at the beginning, not only for the unquenched results, but even for the quenched results many more quark mass and $a$ values must be used to allow for reasonably reliable chiral and continuum extrapolations. We finally note that the successful determination of $g_A$ is a real test of QCD as it is a relatively simple (nucleon) matrix element, well measured in experiment.

ACKNOWLEDGEMENTS

The numerical computations were performed on the Quadrics machines at NIC (Zeuthen) as well as the Cray T3Es at ZIB (Berlin), EPCC (Edinburgh) and NIC (Jülich). We wish to thank all these institutions for their support. UKQCD acknowledges PPARC grants GR/L22744 and PPA/G/S/1998/000777.

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