Generalized solutions of the Dirac equation, W bosons, and beta decay

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Abstract

We study the $7 \times 7$ Hagen-Hurley equations describing spin 1 particles. We split these equations, in the interacting case, into two Dirac equations with non-standard solutions. It is argued that these solutions describe decay of a virtual $W$ boson in beta decay.

1 Introduction

Recently, we have shown that in the free case covariant solutions of the $s = 0$ and $s = 1$ Duffin-Kemmer-Petiau (DKP) equations are generalized solutions of the Dirac equation [1]. These wavefunctions are non-standard since they involve higher-order spinors. We have demonstrated recently that in the $s = 0$ case the generalized solutions describe decay of a pion [2]. The aim of this work is to interpret spin 1 solutions, possibly in the context of weakly decaying particles.

There are several relativistic equations describing spin 1 particles, see [3, 4] for the reviews. The most common approach to study properties of spin 1 bosons is based on the $10 \times 10$ DKP equations (the DKP particles are bosons [5]). Several classes of potentials were used in DKP equations to investigate interactions of spin 1 particles [6–15]. However, we shall apply the $7 \times 7$ Hagen-Hurley equations [16–18] in spinor form [1, 19, 20]. Our motivation stems from the observation that these equations violate parity and thus should describe weakly interacting particles.

In the next Section we transform the Hagen-Hurley equations, in the interacting case, into two Dirac equations with non-standard solutions involving higher-order spinors, extending our earlier results described in [1]. These generalized solutions bear some analogy to generalized solutions of the Dirac equation argued to describe a lepton and three quarks [21]. In Section 3 we describe transition from non-standard solutions of two Dirac equations to the Dirac equation.
for a lepton and the Weyl equation for a neutrino. In the last Section we show that the transition is consistent with decay of a virtual $W$ boson in beta decay. In what follows we are using definitions and conventions of Ref. [22].

2 Generalized solutions of the Dirac equation in the interacting case

We have shown recently that, in the non-interacting case, solutions of the $s = 0$ and $s = 1$ DKP equations are generalized solutions of the Dirac equation [1]. In our derivation we have splitted the $10 \times 10$ DKP equations for $s = 1$ into two $7 \times 7$ Hagen-Hurley equations [16-18]. Let us note here that in the case of interaction with external fields such splitting is not possible since the identities (27) of Ref. [23], enabling the splitting, are not valid in the interacting case. Therefore, we shall base our theory on the $7 \times 7$ formulation, see Eqs. (18), (19) in [1] and Subsection 6 ii) in [19]. These equations violate parity $P$, where $P : x^0 \rightarrow x^0$, $x^i \rightarrow -x^i$ ($i = 1, 2, 3$), and thus one should expect a link with weak interactions.

We write one of these $7 \times 7$ equations (Eq. (19) of Ref. [1]), in the interacting case, in form:

$$\begin{align*}
\pi^A_B \zeta_A^B &= m \chi_B^D \\
\pi_A^D \chi_B^D &= -m \zeta_A^B
\end{align*}$$

(1)

and it is assumed that

$$\chi_B^D = \chi_D^B$$

(2)

what is the $s = 1$ constraint. In Eqs. (1) we have $\pi^{AB} = (\sigma^0 \pi^0 + \vec{\sigma} \cdot \vec{\pi})^{AB}$, $\pi^0 = p^0 - qA^0$, $\sigma^k$ ($k = 1, 2, 3$) are the Pauli matrices, and $\sigma^0$ is the $2 \times 2$ unit matrix. Let us note that equations (1), (2), which can be written in the $7 \times 7$ Hagen-Hurley form, were first proposed by Dirac [20].

Equations (1) in explicit form read:

$$\begin{align*}
- (\pi^0 + i \pi^2) \chi_{11} - (\pi^0 - \pi^3) \chi_{21} &= -m \zeta_{11} \\
(\pi^0 + \pi^3) \chi_{11} + (\pi^1 - i \pi^2) \chi_{21} &= -m \zeta_{21}
\end{align*}$$

(3a)

$$\begin{align*}
- (\pi^1 - i \pi^2) \chi_{12} - (\pi^0 - \pi^3) \chi_{22} &= -m \zeta_{12} \\
(\pi^0 + \pi^3) \chi_{12} + (\pi^1 + i \pi^2) \chi_{22} &= -m \zeta_{22}
\end{align*}$$

(3b)

where the condition $\chi_{BD} = \chi_{DB}$ is not imposed. We thus get two Dirac equations or, alternatively, a single Dirac equation with generalized solution

$$\begin{align*}
\pi^A_B \zeta_A^B &= m \chi_B^D \\
\pi_A^D \chi_B^D &= -m \zeta_A^B
\end{align*}$$
\[ E = \begin{pmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \\ \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{pmatrix} \]

\[
(\pi^0 \gamma^0 - \pi^1 \gamma^1 - \pi^2 \gamma^2 - \pi^3 \gamma^3) B = m B,
\]

generalizing Eq. (24) of Ref. [1].

3 Decay of spin 1 bosons

We note that solutions of two Dirac equations are non-standard since they involve higher-order spinors rather than spinors \( \xi_A, \eta_B \). To interpret Eqs. (3), we put:

\[
\chi_{\alpha B}(x) = \eta_B(x) \alpha_{\beta}(x) \quad (5a)
\]

\[
\zeta_{\alpha B}(x) = \xi_A(x) \alpha_{\beta}(x) \quad (5b)
\]

where \( \alpha_{\beta}(x) \) is the Weyl spinor while \( \eta_B(x), \xi_A(x) \) are the Dirac spinors. Note that now \( \chi_{12} \neq \chi_{21} \) and, accordingly, the spin is not determined – more exactly, the spin is in the 0±1 space. It means that we consider virtual (off-shell) bosons. This substitution is in the spirit of the method of fusion of de Broglie [24,25] (similar ansatz was used in the \( s = 0 \) case [2]). After the substitution of (5) into Eqs. (4) we obtain two equations:

\[
\begin{align*}
- (\pi^1 + i \pi^2) \eta_1 \alpha_{\dot{A}} - (\pi^0 - \pi^3) \eta_2 \alpha_{\dot{A}} &= -m \xi_1 \alpha_{\dot{A}} \\
(\pi^0 + \pi^3) \eta_1 \alpha_{\dot{A}} + (\pi^1 - i \pi^2) \eta_2 \alpha_{\dot{A}} &= -m \xi_2 \alpha_{\dot{A}} \\
- (\pi^1 - i \pi^2) \xi_1 \alpha_{\dot{A}} - (\pi^0 - \pi^3) \xi_2 \alpha_{\dot{A}} &= m \eta_1 \alpha_{\dot{A}} \\
(\pi^0 + \pi^3) \xi_1 \alpha_{\dot{A}} + (\pi^1 + i \pi^2) \xi_2 \alpha_{\dot{A}} &= m \eta_2 \alpha_{\dot{A}}
\end{align*}
\]

where \( \dot{A} = 1, 2 \), and, after substituting solution of the Weyl equation

\[ p^{\dot{A} \dot{B}} \alpha_{\dot{B}} = 0, \]

\( \alpha_{\dot{A}}(x) = \hat{\alpha}_{\dot{A}} e^{ik\cdot x}, k^\mu k_\mu = 0 \), we get a single Dirac – equation for spinors \( \xi_A(x), \eta_B(x) \):

\[
\begin{align*}
- (\hat{\pi}^1 + i \hat{\pi}^2) \eta_1 - (\hat{\pi}^0 - \hat{\pi}^3) \eta_2 &= -m \xi_1 \\
(\hat{\pi}^0 + \hat{\pi}^3) \eta_1 + (\hat{\pi}^1 - i \hat{\pi}^2) \eta_2 &= -m \xi_2 \\
- (\hat{\pi}^1 - i \hat{\pi}^2) \xi_1 - (\hat{\pi}^0 - \hat{\pi}^3) \xi_2 &= m \eta_1 \\
(\hat{\pi}^0 + \hat{\pi}^3) \xi_1 + (\hat{\pi}^1 + i \hat{\pi}^2) \xi_2 &= m \eta_2
\end{align*}
\]

where \( \hat{\pi}^\mu \equiv \pi^\mu + k^\mu \), since components \( \alpha_1(x), \alpha_2(x) \) cancel out.

Equations (7), (8) describe two spin \( \frac{1}{2} \) particles, whose spins can couple to \( s = 0 \) or \( s = 1 \), i.e. \( \frac{1}{2} \otimes \frac{1}{2} = 0 \oplus 1 \).
4 Conclusions

Results obtained in Sections 2, 3 cast new light on the Hagen-Hurley equations as well as on weak decays of spin 1 bosons. We have shown that transition from equation (1), describing a spin $s = 1$ particle, to equations (7), (8), via substitution (5) – which means that now $s \in 0 \oplus 1$, corresponds to decay of this particle into a Weyl antineutrino, cf. Eq. (7), and a Dirac lepton, cf. Eq. (8). Indeed, it should be a weak decay since Eq. (1) violates parity. The spin of this particle becomes undetermined in the process of decay, more exactly it belongs to the $0 \oplus 1$ space – this suggests that this is a virtual particle. Therefore, the products, a lepton and an antineutrino, should have total spin 0 or 1 and there should be a third particle to secure spin conservation.

The above description fits a (three-body) beta decay with formation of a virtual $W^-$ boson, decaying into a lepton and antineutrino. This is most conveniently explained in the case of a mixed beta decay [26]:

$$n (\uparrow) \to \begin{cases} p (\downarrow) + [e (\uparrow) \bar{\nu}_e (\uparrow)] & \text{Gamow-Teller transition} \\ p (\uparrow) + [e (\uparrow) \bar{\nu}_e (\downarrow)] & \text{Fermi transition} \end{cases}$$ (9)

where products of the $W^-$ boson decay (see [27]) are shown in square brackets and $\uparrow$ denotes spin $\frac{1}{2}$ – this seems to correspond well to the proposed transition from Eq. (1) to Eqs. (7), (8). Since spin of the products of decay of the virtual $W^-$ boson belongs to the $0 \oplus 1$ space, their spin can be $s = 0$ or $s = 1$. Moreover, in the case of the Gamow-Teller transition there must be a spin-flip in the decaying nucleon. Let us add here, that in the reaction (9) some neutrons (82%) decay according to the Gamow-Teller mechanism while some (18%) undergo the Fermi transition [26]. This mixed mechanism is explained by decoupled spins of the just born products – indeed, the condition $\chi_{\dot{1}\dot{2}} = \chi_{\dot{2}\dot{1}}$ for the spinor $\chi_{\dot{A}\dot{B}}$, due to the substitution (5a), does not hold and spin of the products is in the $0 \oplus 1$ space.

It is now obvious that another set of $7 \times 7$ equations, involving spinor $\eta_{\dot{A}\dot{B}}$ rather than $\chi_{\dot{A}\dot{B}}$, see Eq. (18) of Ref. [1], describes a $\beta^+$ decay with intermediate $W^+$ boson. Let us note finally, that kinematics of the neutrino appears in the Dirac equation for the electron with arbitrary neutrino four-momentum, suggesting a continuous distribution of neutrino energy.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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