Non-linear screening corrections of stellar nuclear reaction rates and their effects on solar neutrino fluxes

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Abstract

Non-linear electron-screening corrections of stellar nuclear fusion rates are calculated analytically in the framework of the Debye-Hückel model and compared with the respective ones of Salpeter's weak screening approximation. In typical solar conditions, the deviation from Salpeter's screening factor is less than one percent, while for hotter stars such corrections turn out to be of the order of one percent only over the limits of the Debye-Hückel model. Moreover, an investigation of the impact of such non-linear screening effects on the solar neutrino fluxes yields insignificant corrections for both the $pp$ and $CNO$ chain reactions.

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I. INTRODUCTION

Stellar nuclear reactions rates are influenced by the electron cloud that screens fusing nuclei from each other. In recent years, there has been a wide interest of the actual effect of such screening corrections on reaction rates. That interest stems mainly from the fact that any variation of the fusion reaction rates reflects on the solar neutrino fluxes. Therefore the investigation of the screening effect has produced a number of papers, most of which come to the conclusion that the solution of the neutrino problem cannot be found in the screening effect, though it has been argued that the discrepancy between theory and experiment might be reduced by means of a suitable screening correction [1]. Investigations pointing out the inability of screening corrections to reconcile theory and experiment are for example that of Ricci et al [2] and that of Gruzinov and Bahcall [3]. The former modifying Mitler’s model [4] showed that possible uncertainties due to screening are too small to solve the solar neutrino problem while the latter concluded that Salpeter’s weak screening formula is adequate for most solar fusion reactions, thus proving that screening can only modify solar reaction rates and neutrino fluxes by a few percent.

However, as there is an exhaustive effort for higher precision in the theoretical as well as experimental calculation of neutrino fluxes, even corrections of a few percent have become significant [5].

Most of the studies in that field depart from Salpeter’s weak screening formalism [1], and consider various corrections such as those arising from vacuum polarization [7], electron density distribution [3] etc. However, even authors who attempt to go beyond the linear regime make assumptions that lead to inaccuracies. For instance, a popular oversimplification is to consider that the electron density around the nucleus is equal to the average electron density in the plasma [4]. In a recent review of solar fusion cross sections [8], the absence of an analytical study of nonlinear screening effects was underlined, while at the same time it was suggested that electron degeneracy and non-linearities of the Debye-screening could produce corrections to Salpeter’s formula roughly of the order of a few percent for solar reaction. On the other hand, a numerical integration of the Poisson-Boltzmann equation for a mixture of electrons and ions [3], yielded non-linear corrections of the order of 1%.

The aim of the present paper is to study analytically non-linear (higher order) corrections for stellar nuclear reactions and determine their effects on the solar neutrino fluxes. The paper is organized as follows: In unit II the fundamentals of screened thermonuclear reaction are briefly reviewed. The non-linear screening formalism is derived in unit III and the effects of non-linearities on the most probable energy of interaction and the cross section factor are clarified. In unit IV the derived formula are implemented for various stellar reactions while in unit V the sensitivity of the solar neutrino fluxes to such non-linear corrections is investigated. Finally the conclusions of this paper are given in unit VI. An appendix at the end elucidates some misconceptions about the effects of electron screening on the astrophysical factor and the most probable energy of interaction.
II. FUNDAMENTALS OF ELECTRON SCREENING IN THERMONUCLEAR REACTIONS

In the stellar interior the nuclear fusion cross section \( \sigma (E) \) can be written as a product of a penetration factor \( P (E) \) times a nuclear factor \( \sigma_{\text{nuc}} (E) \). The penetration factor is actually the probability that two positively charged particles will tunnel through the Coulomb barrier that separates them:

\[
\sigma (E) = \sigma_{\text{nuc}} (E) P (E)
\]  

For two bare nuclei of mass numbers \( A_1 \) and \( A_2 \) respectively, that barrier is given by

\[
E_c = \frac{Z_1 Z_2 e^2}{R}
\]  

where the two nuclei can be considered sharp-edged spheres so that

\[
R = 1.4 \left( A_1^{1/3} + A_2^{1/3} \right) \text{ fm}
\]  

\( A_1, A_2 \) being the mass numbers of the fusing nuclei.

If higher relative angular momenta are considered in the collisions then the height of the barrier is increased by

\[
E_l = \frac{l(l+1)\hbar^2}{2\mu r^2}
\]  

where \( \mu \) is the reduced mass.

In the stellar plasma the kinetic energy of the interacting particles is determined by a Maxwell-Boltzmann distribution of velocities corresponding to a thermal energy

\[
kT = 0.086 T_6 \text{ keV}
\]  

where \( T_6 \) the temperature in millions degrees Kelvin. In a semiclassical approach it can be shown that the two nuclei can only interact if

\[
l \leq 2 \cdot 10^{-3} R_{\text{fm}} \sqrt{AT_6}
\]  

where \( A \) is the reduced mass number. Note that all energies are assumed to be center of mass energies unless specified otherwise. By means of the above condition it is easy to show that in stellar fusion reactions between light nuclei \( s \)-wave interactions dominate. In such a case the interaction is adequately described by a single particle potential of the form:

\[
V (r) = V_N (r) + V_c (r)
\]  

where \( V_N (r) \) is the nuclear potential and \( V_c (r) \) is a potential which describes the Coulomb interaction and is not necessarily a pure Coulomb potential. Gamow first showed , in connection with the problem of \( \alpha \)-decay that for two bare nuclei of charge \( Z_1 \) and \( Z_2 \) moving
with relative velocity $u$ the probability to penetrate the Coulomb barrier is proportional to the factor $\exp(-2\pi n)$ where $n$ is the Sommerfeld parameter:

$$n(E) = Z_1 Z_2 e^2 \sqrt{\frac{\mu}{2\hbar^2}} E^{-1/2}$$  \hspace{1cm} (8)

The cross section of the nuclear fusion reaction is then given by:

$$\sigma_l (E) = \frac{S(E)}{E} T_l (E)$$  \hspace{1cm} (9)

where

$$T_l (E) = \exp \left[ -\frac{2\sqrt{2\mu}}{\hbar} \int_{R}^{r_c} \sqrt{V_c (r) + \frac{l(l+1)}{2\mu r^2}} - E \, dr \right]$$  \hspace{1cm} (10)

and $r_c$ is the classical turning point given by

$$V(r_c) + \frac{l(l+1)}{2\mu r_c^2} = E$$  \hspace{1cm} (11)

The lower limit $R$ of the integral is the radius of the nuclear forces given by (3) (assumed to be practically zero).

In the framework of the nearly perfect ionized gas, the presence of the electron cloud around the nuclei increases the reaction rates over their laboratory analogs. This screening effects has been studied by many authors and the most popular potential is the Debye-Hückel (D-H) screened Coulomb potential given by

$$V_c (r) = U_e e^{-u}$$  \hspace{1cm} (12)

where $u = r/r_D$ and $U_e = Z_1 Z_2 e^2 r_D^{-1}$ is the screening potential energy i.e. the Coulomb energy of the two colliding atoms at a distance equal to the D-H radius $r_D$ given by:

$$r_D^2 = \frac{4\pi e^2}{kT} \left( \sum_i Z_i^2 n_i + n_e \theta_e \right)$$  \hspace{1cm} (13)

where $n_i$ is the number density of ions with charge $Z_i$, $n_e$ is the average electron density and $\theta_e$ the electron degeneracy factor.

At this stage we need to define the limits of the Debye-Hückel model which are actually the limits of the results of this paper. This model assumes a nearly perfect gas at low density where the average Coulomb energy $\langle E_c \rangle$ between two adjacent nuclei is much smaller than the thermal kinetic energy of the plasma

$$\langle E_c \rangle \ll kT$$  \hspace{1cm} (14)

Therefore, in the calculations that follow we have to bear in mind that the derived non-linear corrections cannot be applied beyond the realm of validity of this condition.
III. NONLINEAR SCREENING FORMALISM

Let us assume a screening potential of the form \((12)\). The cross section for thermonuclear reactions can be written as:

\[
\sigma(E) = \frac{S(E)}{E} e^{-4n(E)I(x)}
\]  

(15)

where \(x = \frac{r_c}{r_D}\).

The quantity \(I(x)\) is given by:

\[
I(x) = e^{-x} \int_0^1 \frac{1}{u} e^{x(1-u)} - 1 du
\]  

(16)

where \(x = x(E)\) is the solution of the equation:

\[
x e^x = \frac{U_e}{E}
\]  

(17)

and \(I(0)\) is:

\[
I(0) = \int_0^1 \sqrt{\frac{1}{u} - 1} du = \frac{\pi}{2}
\]  

(18)

By introducing a multiplicative corrective term \(\xi(x)\) such that

\[
I(x) = I(0) \xi(x) = \frac{\pi}{2} \xi(x)
\]  

(19)

we obtain

\[
\xi(x) = e^{-x} \left( 1 + \frac{x}{2} + \frac{x^2}{16} + \frac{x^3}{48} - \frac{3x^4}{1024} + \frac{13x^5}{5120} - \frac{73x^6}{49152} + \ldots \right)
\]  

(20)

The screened reaction rate between nuclei \(i\) and \(j\) is:

\[
\nu_{ij}^{sc} = (1 + \delta_{ij})^{-1} n_i n_j \langle \sigma v \rangle^{sc}
\]  

(21)

where \(n_i, n_j\) are the number densities of nuclei \(i\) and \(j\) respectively, \(\delta_{ij}\) is the Kronecker delta, and the thermalized cross section per pair of particles is

\[
\langle \sigma v \rangle^{sc} = \sqrt{\frac{8}{\mu \pi}} (kT)^{-\frac{3}{2}} \int_0^\infty S(E) \exp \left[ -\frac{E}{kT} - 4n(E)I(x) \right] dE
\]  

(22)

Note that in the present paper the superscripts: \(nos, wes, sc, ss\), indicate respectively no-screening, weak-screening, non-linear screening and strong screening regimes.

Therefore

\[
\langle \sigma v \rangle^{sc} = \sqrt{\frac{8}{\mu \pi}} (kT)^{-\frac{3}{2}} f_0(E_0^{sc}) \int_0^\infty S(E) \exp \left[ -\frac{E}{kT} - 2\pi n(E) \right] dE
\]  

(23)
where the screening correction factor $f_0$ which is now evaluated at the most probable energy of the screened interaction is given by:

$$\ln f_0 = \pi n (E_0^{sc}) \left( x - \frac{x^2}{8} - \frac{x^3}{12} + \frac{35x^4}{512} - \frac{23x^5}{640} + \frac{449x^6}{24576} + O [x^7] \right)$$  \hspace{1cm} (24)$$

Following the method of the steepest descent \[9\], the most effective energy of interaction $E_0^{sc}$ between the two screened nuclei is:

$$\frac{d}{dE} \left( \frac{E}{kT} + 4n (E) I (x) \right)_{E=E_0^{sc}} = 0$$  \hspace{1cm} (25)$$

which yields (see Appendix I):

$$E_0^{sc} = \left( 2I (x) Z_1 Z_2 e^2 \sqrt{\frac{\mu}{2h^2 kT}} \right)^{\frac{2}{3}}$$  \hspace{1cm} (26)$$

where $x = x (E_0^{sc})$ is the solution of equation (17).

Hence

$$E_0^{sc} = \xi^{2/3} (x) E_0^{nos}$$  \hspace{1cm} (27)$$

On the other hand by using the formula for the most effective energy of interaction for two bare nuclei we obtain:

$$E_0^{sc} = 1.220 \cdot \left( Z_1^2 Z_2^2 AT_0^2 \right)^{1/3} \xi^{2/3} (x) \text{ keV}$$  \hspace{1cm} (28)$$

Eq.(17) now reads:

$$x e^{x} \xi^{2/3} (x) = \frac{U_e}{E_0^{nos}} = \frac{1180 (Z_1 Z_2)^{1/3}}{(AT_0^2)^{1/3} r_D (fm)}$$  \hspace{1cm} (29)$$

In the weak screening (wes) approximation, to first order in $x$, we obtain:

$$E_0^{wes} \simeq 1.220 \cdot \left( Z_1^2 Z_2^2 AT_0^2 \right)^{1/3} \left( 1 - \frac{x}{3} \right)$$  \hspace{1cm} (30)$$

which corrects the result of ref \[10\] according to which the energy should be shifted by $(1 - x)$, instead.

Moreover, in the weak screening approximation, the integral $I (x)$ can be written:

$$I^{wes} (x) \simeq \frac{\pi}{2} \left( 1 - \frac{x}{2} \right)$$  \hspace{1cm} (31)$$

and therefore one obtains Salpeter’s weak screening correction \[10\]:

$$f_0^{wes} \simeq e^{\pi n x}$$  \hspace{1cm} (32)$$
Note that the assumption made in ref [10] about the classical turning point being equal for both the screened and unscreened cases is not necessary [11]. In fact the actual approximation is

\[ r_{cs} \approx r_{c}^{nos} (1 - x) \] (33)

Regarding the rate of thermonuclear reactions, it has been shown that in the stellar interior where the temperature is \( T \) degrees Kelvin and the density is \( \rho \) in g/cm\(^3\) the nuclear reaction rate is:

\[ r_{ij} = \frac{2.62 \cdot 10^{29}}{(1 + \delta_{ij})} \rho^{2} X_{i} X_{j} \int_{0}^{\infty} S_{eff} \tau^{2} e^{-\tau} \ cm^{-3} \ sec^{-1} \] (34)

where

\[ \tau = \frac{3E_{0}^{nos}}{kT} \] (35)

and \( S_{eff} \) is expressed in keV-barns. The quantities \( X_{i,j}, Z_{i,j}, A_{i,j} \) are the mass fraction, charge and mass number respectively of the nucleus \( i, j \).

To first order in \( \tau^{-1} \) [12] :

\[ S_{eff} (E_{0}) \approx S(0) \left[ 1 + \frac{5}{12\tau} + S^{-1}(0) \left( \frac{dS}{dE} \right)_{E=0} \left( E_{0} + \frac{35}{36} kT \right) \right] \] (36)

Although in Eq.(34) the screening enhancement factor \( f_{0} \) is now evaluated at \( E_{0}^{sc} \) the effective astrophysical factor \( S_{eff} \) is always evaluated at \( E_{0}^{nos} \) as the screening effects have already been worked out of the integral so the correction assumed in ref. [11] is not applicable (see Appendix II).

**IV. RESULTS FOR VARIOUS STELLAR FUSION REACTIONS**

In the solar region of maximum energy production \( (R/R_{\odot} = 0.09) \), the temperature and the density are respectively \( T_{6} = 13.5 \) and \( \rho = 93.3 \) g/cm\(^3\). Using recently calculated isotopic abundances of the solar interior [13] the average internuclear distance given by

\[ a = \frac{1}{(4\pi \rho N_{0})^{1/3}} = 51000 \rho^{-1/3} \ \text{fm} \] (37)

is found to be \( a \approx 11244 \ \text{fm} \) and the Debye-Huckel radius is \( r_{D} = 25719 \ \text{fm} \), where the weak electron degeneracy has been taken into account \( (\theta_{e} \approx 0.92) \). Moreover, the thermal kinetic energy is \( kT = 1.161 \) keV and is always higher than the average Coulomb energy between two adjacent ions according to condition [14]. Therefore the potential of Eq.(12) can be used in order to study the screening corrections to solar fusion reactions. The results are depicted in Table 1 that follows:
Table I. The electron screening corrections for various solar fusion reactions.

| Fusion reaction | $f_{0}^{wes}$ | $f_{0}^{sc}$ | $\Delta f_{a}$ | $\Delta f_{n}$ | $E_{0}^{\text{nos}}$ (keV) | $E_{0}^{\text{sc}}$ (keV) | $E_{c}$ (keV) | $U_{e}$ (keV) |
|-----------------|----------------|-------------|---------------|---------------|-----------------|----------------|-------------|-------------|
| $H^{1} (p, e^{+} \nu_{e}) H^{2}$ | 1.049 | 1.048 | 0.030 | 0.5 | 5.624 | 5.471 | 514 | 0.056 |
| $He^{3} (2He^{4}, \gamma) Be^{7}$ | 1.212 | 1.210 | 0.129 | 1.7 | 20.860 | 20.786 | 1358 | 0.223 |
| $He^{3} (2He^{3}, 2p) He^{4}$ | 1.212 | 1.210 | 0.135 | – | 19.952 | 19.877 | 1426 | 0.224 |
| $Be^{7} (p, \gamma) B^{8}$ | 1.212 | 1.210 | 0.161 | 1.5 | 16.671 | 16.596 | 1412 | 0.224 |
| $N^{14} (p, \gamma) O^{15}$ | 1.400 | 1.396 | 0.332 | 0.8 | 24.735 | 24.606 | 2111 | 0.391 |
| $C^{12} (p, \gamma) N^{13}$ | 1.335 | 1.331 | 0.272 | – | 22.238 | 22.127 | 1876 | 0.348 |

Note that the statement [8] that non-linearities of the Debye screening might produce a correction to Salpeter’s formula of the order of a few percent can be now upgraded. According to the above results, as far as the Debye-Hückel mean field potential is concerned, the actual deviation is less than 0.5% for all solar fusion reactions.

A very interesting fact is that such non-linear corrections are largely due to the presence of Eq. (29) which supersedes its linear oversimplified counterpart of the weak screening regime:

$$x = \frac{U_{e}}{E_{0}^{\text{nos}}}$$

(38)

That means that for most stellar conditions instead using Eq. (24) we can safely neglect higher order terms in order to use:

$$\ln f_{0} = \pi n \left( E_{0}^{\text{sc}} \right) x \left( E_{0}^{\text{sc}} \right)$$

(39)

where $x \left( E_{0}^{\text{sc}} \right)$ is always the solution of Eq. (29).

Moreover, the usual assumption [2] that the screening enhancement factors are independent of the isotope is not necessarily valid when nonlinear corrections are considered. This is also clear from Eq. (29) where $x$ depends on the reduced mass number. However, as can be readily seen from table I, for typical solar conditions, isotopic dependence is indeed practically negligible. On the other hand the Gamow peak energy, which is usually considered screening independent, in the non-linear regime shows a maximum variation of the order of 0.5%.

For purposes of illustration nonlinear screening corrections have also been calculated for the cases considered by Salpeter [4]. Following the notation of table I we have obtained the following results:

a) In a red dwarf (main sequence star cooler than the sun), with typical central conditions $\rho = 100, T_{6} = 8, \theta_{e} = 0.8$, the screening corrections for the $pp$ reaction are:
we obtain is strong, though not strong enough to assume absolute validity. By applying formula (45)
where \( \mu \) corrections are:

\[
\begin{array}{ccccccc}
\rho & \Delta f (\%) & E_{0}^{\text{nos}} (keV) & E_{0}^{\text{sc}} (keV) & E_{c} (keV) & U_{e} (keV) & kT (keV) \\
1.116 & 1.114 & 0.132 & 3.873 & 3.848 & 514 & 0.076 & 0.688 \\
\end{array}
\] (40)

b) Salpeter also considers the reaction \( N^{14} (p, \gamma) O^{15} \) for the interior of the hotter main sequence stars for a combination: \( \rho = 122, T_{0} = 11.6, \theta_{e} = 0.85 \). By taking into account non-linear corrections the results are:

\[
\begin{array}{ccccccc}
\rho & \Delta f (\%) & E_{0}^{\text{nos}} (keV) & E_{0}^{\text{sc}} (keV) & E_{c} (keV) & U_{e} (keV) & kT (keV) \\
1.648 & 1.636 & 0.685 & 22.356 & 22.193 & 2111 & 0.499 & 0.997 \\
\end{array}
\] (41)

For the same reaction in the center of Sirius where \( \rho = 80, T_{0} = 20, \theta_{e} \simeq 1 \), the corrections are:

\[
\begin{array}{ccccccc}
\rho & \Delta f (\%) & E_{0}^{\text{nos}} (keV) & E_{0}^{\text{sc}} (keV) & E_{c} (keV) & U_{e} (keV) & kT (keV) \\
1.209 & 1.207 & 0.120 & 32.145 & 32.037 & 2111 & 0.327 & 1.720 \\
\end{array}
\] (42)

c) In the more luminous main sequence stars where Hydrogen has been exhausted, the first stage of the two-stage reaction \( 3He^{4} \rightarrow C^{12} + \gamma \) is the temporary formation of \( Be^{8} \), through the reaction \( He^{4} + He^{4} \rightarrow B^{8} \). The screening corrections for such a collision between two alpha particles, for indicative conditions considered by Salpeter are:

i) \( \rho = 10^{4}, T_{0} = 150, \theta_{e} = 0.87 \)

\[
\begin{array}{ccccccc}
\rho & \Delta f (\%) & E_{0}^{\text{nos}} (keV) & E_{0}^{\text{sc}} (keV) & E_{c} (keV) & U_{e} (keV) & kT (keV) \\
1.049 & 1.048 & 0.017 & 109.346 & 109.141 & 1295 & 0.619 & 12.9 \\
\end{array}
\] (43)

Despite the high density, obviously condition (14) still holds here, therefore Salpeter’s weak screening approximation is still reliable.

ii) \( \rho = 10^{6}, T_{0} = 150, \theta_{e} \simeq 0 \)

\[
\begin{array}{ccccccc}
\rho & \Delta f (\%) & E_{0}^{\text{nos}} (keV) & E_{0}^{\text{sc}} (keV) & E_{c} (keV) & U_{e} (keV) & kT (keV) \\
1.491 & 1.474 & 1.136 & 109.346 & 107.680 & 1295 & 5.540 & 12.9 \\
\end{array}
\] (44)

In the above case (ii), the electron screening effect is strong enough for the weak screening approximation to become inaccurate and therefore nonlinear corrections become important. Unfortunately, even nonlinear corrections cannot redeem the inaccuracies of the Debye-Hückel potential which is only applicable to fusion reactions where condition (14) holds. On the other hand the strong screening formula [3]

\[ f_{o}^{ss} = \exp \left[ 0.205 \left( \frac{\rho}{\mu_{e}} \right)^{1/3} T_{6}^{-1} \left( Z_{1} + Z_{2} \right)^{5/3} \left( Z_{1}^{1/3} - Z_{2}^{5/3} \right) \right] \] (45)

where \( \mu_{e} \) is the mean molecular weight per electron, can be cautiously used here as screening is strong, though not strong enough to assume absolute validity. By applying formula (45) we obtain \( f_{o}^{ss} = 1.498 \) which is reasonably close to its linear and non-linear counterparts.
V. EFFECTS ON SOLAR NEUTRINO FLUXES

It has been suggested [14] that the electron screening effect may be important to the solar neutrino production as it enhances the reaction rates in the interior of the sun. More precisely, if we consider the Gamow peak screening-independent then any variations of screening reflects on the cross section \( S(E) \) which is in fact multiplied by this screening factor so that:

\[
S^{sc}(E) = f_0 S^{nos}(E)
\]  
(46)

In fact, it has been shown that the solar neutrino fluxes \( \Phi \) can be given as a function of the screening factors of the \( pp \) and \( CNO \) chains [2]. In that work the screening factor \( f \) was assumed to be isotope independent. Although in typical solar conditions this assumption was shown to be valid (see table I) when nonlinear screening effects are considered the derived formula should be modified as follows:

\[
\Phi_{Be^7(e^-,\nu_e)}^{sc}(Be^7) = \Phi_{Be^7}^{nos}(f_{p+p})^{10/8} \frac{f_{He^3+He^4}}{\sqrt{f_{He^3+He^3}}} \]  
(47)

\[
\Phi_{B^8(e^+,\nu_e)}^{sc}(B^8^*) = \Phi_{B^8}^{nos}(f_{p+p})^{-23.6/8} (f_{p+p}^{Be}) \frac{f_{He^3+He^4}}{\sqrt{f_{He^3+He^3}}} \]  
(48)

On the other hand for the \( CNO \) cycle which is governed by the slowest reaction \( N^{14}(p,\gamma)O^{15} \) the enhancement on the neutrino fluxes are [2]:

\[
\Phi_{N,O}^{sc} = \Phi_{N,O}^{nos}(f_{p+p})^{-22/8} (f_{p+14N}) \]  
(49)

Conservation of luminosity yields the \( pp \) neutrino fluxes by means of

\[
\Phi_{pp}^{sc} + \Phi_{Be}^{sc} + \Phi_{N}^{sc} + \Phi_{O}^{sc} = \Phi_{pp}^{nos} + \Phi_{Be}^{nos} + \Phi_{N}^{nos} + \Phi_{O}^{nos}
\]  
(50)

whereas the \( pep \) neutrino fluxes can be obtained by the observation that in any solar model:

\[
\frac{\Phi_{pep}}{\Phi_{pp}} = \frac{\Phi_{pep}^{nos}}{\Phi_{pp}^{nos}}
\]  
(51)

Instead of solving algebraic equations, an easier way of obtaining corrections to \( pp \) and \( pep \) neutrino fluxes is by using the proportionality formula [15]:

\[
\Phi_{pp} \sim S_{pp}^{0.14} S_{He^3+He^4}^{0.03} S_{He^3+He^3}^{-0.06}
\]  
(52)

which readily yields:

\[
\frac{\Phi_{pp}^{sc}}{\Phi_{pp}^{nos}} = f_{p+p}^{0.14} f_{He^3+He^3}^{0.03} f_{He^3+He^4}^{-0.06}
\]  
(53)

By using the above formulae the corrections to the solar neutrino fluxes both in the weak screening and non-linear screening regimes are shown in table II.
Table II. The nonlinear screening corrections of solar neutrino fluxes

| Neutrino source | $\Phi^{\text{wes}} / \Phi^{\text{nos}}$ | $\Phi^{\text{sc}} / \Phi^{\text{nos}}$ |
|-----------------|-------------------------------|-------------------------------|
| $H^1 (p, e^+ \nu_e) H^2$ | 1.000                         | 1.000                         |
| $H^1 (p e^-, \nu_e) H^2$ | 1.001                         | 1.001                         |
| $Be^7 (e^-, \nu_e) Li^7$ | 1.037                         | 1.037                         |
| $B^8 (e^+, \nu_e) B^{8*}$ | 1.158                         | 1.159                         |
| $N^{13} (e^+ \nu) C^{13}$ | 1.227                         | 1.227                         |
| $O^{15} (e^+, \nu_e) N^{15}$ | 1.227                         | 1.227                         |

Admittedly, it has been argued that the weak-screening factors are more appropriate for the $pp$ reaction rate where condition (14) is fully satisfied, whereas for reactions with $Z_1Z_2 > 4$ Mitler’s formula should be used, instead. In our approach (Table II) we have used the weak-screening formula for all solar reactions following the recent suggestion of ref. [3]. Note that in ref. [3], where Mitler’s screening factors were used, the ratio $\Phi^{\text{sc}} / \Phi^{\text{nos}}$ for the $pp$ neutrino was found to be 0.995 while for the $N^{13}$ neutrino it was 1.13. This discrepancy is unimportant to the results of this paper according to which there is a negligible contribution of the investigated non-linear screening effects to solar neutrino fluxes well below the experimental errors of any currently imaginable neutrino detector.

VI. CONCLUSIONS

In this paper we have studied the non-linear effects of electron screening on stellar nuclear fusion rates calculating the respective corrections analytically. The formalism employed has been based on the Debye-Hückel model. In typical solar conditions such non-linear effects are shown to be negligible proving Salpeter’s linear approach to be sufficient for the study of solar nuclear reactions. Regarding the solar neutrino problem it was also shown that, non-linear screening leads to a negligible correction of the solar neutrino fluxes of the $pp$ and $CNO$ chains. Moreover, non-linear corrections are shown to be of some importance only in the intermediate screening regime, where the average Coulomb energy begins to challenge the average thermal kinetic energy and the Debye-Hückel model begins to break down.

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APPENDIX

I. The most effective energy of interaction.

The method of the steepest decent yields the maximum of the integrand of Eq.(22):

$$
\frac{d}{dE} \left( \frac{E}{kT} + 4n(E) I(x) \right)_{E=E^{sc}} = 0
$$

In fact the quantity $x$ is energy dependent $x = x(E)$ by means of Eq.(17), therefore upon differentiation we obtain:
\[ \frac{1}{kT} + 2\pi \left[ \left( \frac{dn (E)}{dE} \right)_{E=E_0^{sc}} \xi \left( x_0^{sc} \right) + n \left( E_0^{sc} \right) \left( \frac{dx (E)}{dE} \right)_{E=E_0^{sc}} \left( \frac{d\xi (x)}{dx} \right)_{x=x_0^{sc}} \right] = 0 \] (55)

where \( n (E) \) is given by (8) and \( x_0^{sc} \) is the solution of Eq. (17). If we assume that throughout the integral which appears in the thermalized cross section in Eq. (22) there is only a negligible variation of \( \xi (x) \) then the above integral yields Eq. (26). This assumption is better than assuming that \( \xi (x) = 1 \), which practically yields \( E_{0^{nos}} = E_0^{sc} \), as the latter disregards \textit{a priori} all the corrections of the screened interaction to the effective energy of interaction.

### II. Screening independence of \( S_{eff}^{sc} \)

The basic formula for the thermalized cross section of the non-resonant screened thermonuclear reaction is:

\[ \left\langle \sigma v \right\rangle^{sc} = \frac{8}{\mu \pi} (kT)^{-\frac{3}{2}} \int_0^\infty S (E) \exp \left[ -\frac{E}{kT} - 4n (E) I (x) \right] dE \] (56)

We have to work out the screening factor \( f_0 \) first otherwise the introduction of an \( S_{eff}^{sc} \) for the screened case replaces the integral above by an average (corrected) expression as follows [12]:

\[ \int_0^\infty S (E) \exp \left[ -\frac{E}{kT} - 4n (E) I (x) \right] dE = 2E_0^{sc} \left( \frac{\pi \tau}{2} \right)^{1/2} e^{-\tau} S_{eff} (E_0^{sc}) \] (57)

Once the integral has been replaced the screening effect cannot be parametrized in the usual way by means of the screening factor \( f_0 \).

Therefore, after we work out the screening enhancement factor \( f_0 \) we obtain:

\[ \left\langle \sigma v \right\rangle^{sc} = \frac{8}{\mu \pi} (kT)^{-\frac{3}{2}} f_0 (E_0^{sc}) \int_0^\infty S (E) \exp \left[ -\frac{E}{kT} - 2\pi n (E) \right] dE \] (58)

where it should be noted that the evaluation of \( f_0 \) is performed at the most probable energy of the screened interaction. The remaining integral is actually unaware of the screening effects as the former is calculated by the method of the steepest decent around its maximum value, which is the (unscreened) most effective energy of interaction \( E_{0^{nos}} \).

Therefore, the energy \( E_0 \) appearing in the formula:

\[ S_{eff} = S (E_0) \left\{ 1 + \tau^{-1} \left[ \frac{5}{12} + \frac{5}{2} \frac{S' (E_0) E_0}{S (E_0)} + \frac{S'' (E_0) E_0^2}{S (E_0)} \right] \right\} + O \left[ \tau^{-2} \right] \] (59)

can only be the quantity \( E_{0^{nos}} = 1.220 \cdot (Z_1^2 Z_2^2 A T_0^2)^{1/3} \), which corresponds to the no-screening regime. As a result, no screening correction can be incorporated into the frequently used formula:

\[ S_{eff} \simeq S (0) \left\{ 1 + \frac{5}{12\tau} + \frac{S' (0) E_0}{S (0)} \left( E_0 + \frac{35}{36} kT \right) + \frac{S'' (0) E_0^2}{S (0)} \right\} \] (60)

because \( E_0 \) cannot be replaced by \( E_0^{sc} \). Note that even if that was the case one should have to modify \( \tau = \tau (E_{0^{nos}}) \) as well.
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