Non-existence of positive weak solutions for some nonlinear \((p,q)\)-Laplacian systems

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Abstract

In this work we deal with the class of nonlinear \((p,q)\)-Laplacian system of the form

\[-\Delta_p u = \mu \rho_1(x) f(v) \quad \text{in } \Omega, \]
\[-\Delta_q v = \nu \rho_2(x) g(u) \quad \text{in } \Omega,
\]
\[u = v = 0 \quad \text{on } \partial \Omega.\]

where \(\Delta_p\) with \(p > 1\) denotes the \(p\)-Laplacian defined by \(\Delta_p u \equiv \text{div}[|\nabla u|^{p-2}\nabla u]\), \(\mu, \nu\) are positive parameters, \(\rho_1(x), \rho_2(x)\) are weight functions, \(f, g : [0, \infty) \to \mathbb{R}\) are continuous functions and \(\Omega \subset \mathbb{R}^N\) is a bounded domain with smooth boundary \(\partial \Omega\). Non-existence results of positive weak solutions are established under some certain conditions on \(f, g\) when \(\mu \nu\) is small.

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1 Introduction:

In this paper we first consider a non-existence result of positive weak solutions for the following nonlinear system

\[-\Delta_p u = \lambda a_1(x)v^{\alpha-1} - b_1(x)v^{\alpha-1} - c_1(x) \quad \text{in } \Omega,
\]
\[-\Delta_q v = \lambda a_2(x)u^{\beta-1} - b_2(x)u^{\beta-1} - c_2(x) \quad \text{in } \Omega,
\]
\[u = v = 0 \quad \text{on } \partial \Omega,\]

where \(\Delta_p\) with \(p > 1\) denotes the weighted \(p\)-Laplacian defined by \(\Delta_p u \equiv \text{div}[|\nabla u|^{p-2}\nabla u]\), \(a_i(x), b_i(x)\) and \(c_i(x), i = 1,2\) are weight functions, \(\alpha\) and \(\beta\) are positive constants and \(\Omega \subset \mathbb{R}^N\) is a bounded domain with smooth boundary \(\partial \Omega\).

We first show that if \(\lambda < \max(\lambda_p, \lambda_q)\), where \(\lambda_p, \lambda_q\) is the first eigenvalue of \(-\Delta_p, -\Delta_q\) respectively, then system (1.1) has no positive weak solutions.

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Next we consider the nonlinear system

\[
\begin{align*}
-\Delta_p u &= \mu \rho_1(x) f(v) \quad \text{in } \Omega, \\
-\Delta_q v &= \nu \rho_2(x) g(u) \quad \text{in } \Omega, \\
u = v &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

(1.2)

where \( \mu, \nu \) are positive parameters, \( \rho_1(x), \rho_2(x) \) are weight functions and \( \Omega \subset \mathbb{R}^N \) is a bounded domain with smooth boundary \( \partial \Omega \). Let \( f, g : [0, \infty) \to \mathbb{R} \) are continuous functions. Also, assume that there exist positive numbers \( K_i \) and \( M_i, i = 1, 2 \) such that

\[
f(v) \leq K_1 v^{p-1} - M_1, \quad \text{for all } v \geq 0
\]

(1.3)

and

\[
g(u) \leq K_2 u^{q-1} - M_2, \quad \text{for all } u \geq 0.
\]

(1.4)

We discuss a non-existence result for system (1.2) when \( \mu \nu \) is small.

Problems of the form (1.1) and (1.2) arise from many branches of pure mathematics as in the theory of quasiregular and quasiconformal mappings (see [13]) as well as from various problems in mathematical physics notably the flow of non-Newtonian fluids. The \( p \)-Laplacian also appears in the study of torsional creep (elastic for \( p = 2 \), plastic as \( p \to \infty \), (see [5]), glacial sliding (\( p \in (1; \frac{4}{3}] \), see [10] or flow through porous media (\( p = \frac{3}{2} \), see [11]). For existence and non-existence results of positive weak solutions for systems involving the weighted \( p \)-Laplacian, see ([2, 3, 6, 7, 8, 9, 12]).

This paper is organized as follows: In section 2, we introduce some technical results and notations, which are established in [4]. In section 3, we prove the non-existence of positive weak solutions for system (1.1) and (1.2).

2 Technical Results

Let us introduce the Sobolev space \( W^{1,p} (\Omega), 1 < p < \infty \), defined as the completion of \( C^\infty (\Omega) \) with respect to the norm (see [4])

\[
\| u \|_{W^{1,p} (\Omega)} = \left[ \int_\Omega |u|^p + \int_\Omega |\nabla u|^p \right]^\frac{1}{p} < \infty.
\]

(2.1)

Since we are dealing with the Dirichlet problem, we define the space \( W^{1,p}_0 (\Omega) \) as the closure of \( C_0^\infty (\Omega) \) in \( W^{1,p} (\Omega) \) with respect to the norm

\[
\| u \|_{W^{1,p}_0 (\Omega)} = \left[ \int_\Omega |\nabla u|^p \right]^\frac{1}{p} < \infty,
\]

(2.2)

which is equivalent to the norm given by (2.1). Both spaces \( W^{1,p} (\Omega) \) and \( W^{1,p}_0 (\Omega) \) are well defined reflexive Banach Spaces.

Now, we introduce some technical results concerning the eigenvalue problem

\[
\begin{align*}
-\Delta_p u &= \lambda a(x)|u|^{p-2}u \quad \text{in } \Omega, \\
u = 0 &= \quad \text{on } \partial \Omega.
\end{align*}
\]

(2.3)
We will say $\lambda \in R$ is an eigenvalue of (2.3) if there exists $u \in W^{1,p}_0(\Omega)$, $u \neq 0$, such that
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \phi \, dx = \lambda \int_{\Omega} a(x) u^{p-2} u \phi \, dx, \tag{2.4}
\]
holds for $\phi \in W^{1,p}_0(\Omega)$. Then $u$ is called an eigenfunction corresponding to the eigenvalue $\lambda$.

**Lemma 1** There exists the first eigenvalue $\lambda_p > 0$ and precisely one corresponding eigenfunction $\phi_p \geq 0$ a.e. in $\Omega$ of the eigenvalue problem (2.3). Moreover, it is characterized by
\[
\lambda_p = \frac{\int_{\Omega} |\nabla \phi_p|^p}{\int_{\Omega} a(x)|\phi_p|^p} = \inf_{u \in W^{1,p}_0(\Omega)} \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} a(x)|u|^p} \leq \frac{\int_{\Omega} |\nabla u|^p}{\int_{\Omega} a(x)|u|^p} = \lambda.
\]

**Definition 1** A pair of non-negative functions $(u, v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$ are called a weak solution of (1.2) if they satisfy
\[
\int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla \zeta \, dx = \mu \int_{\Omega} p_1(x) f(v) \zeta \, dx,
\]
\[
\int_{\Omega} |\nabla v|^{q-2} \nabla v \nabla \eta \, dx = \nu \int_{\Omega} p_2(x) g(u) \eta \, dx,
\]
for all test functions $\zeta \in W^{1,p}_0(P, \Omega), \eta \in W^{1,q}_0(\Omega)$.

### 3 Non-existence Results

In this section we state our main results. Throught this section, we assume $q$ be such that $\frac{1}{p} + \frac{1}{q} = 1$.

**Theorem 2** For $\lambda \leq \lambda^*$, system (1.1) has no positive weak solution.

**Proof.** Assume that there exist a positive solution $(u, v) \in W^{1,p}_0(\Omega) \times W^{1,q}_0(\Omega)$ of (1.1). Multiplying the first equation of (1.1) by $u$, we have
\[
\int_{\Omega} |\nabla u|^p \, dx = \int_{\Omega} |\lambda a_1(x) v^{p-1} - b_1(x) v^{\alpha-1} - c_1(x)| u \, dx \tag{3.1}
\]
\[
< \int_{\Omega} |\lambda a_1(x) v^{p-1} - c_1(x)| u \, dx.
\]
But, from the characterization of the first eigenvalue, we have
\[
\lambda_p \int_{\Omega} a(x)|u|^p \leq \int_{\Omega} |\nabla u|^p. \tag{3.2}
\]
Combining (1.1) and (3.2), we have
\[
\lambda_p \int_{\Omega} a_1(x) u^p < \int_{\Omega} \lambda a_1(x) v^{p-1} u \, dx - \int_{\Omega} c_1(x) u \, dx. \tag{3.3}
\]
Similarly, from the second equation of (1.1), we obtain
\[
\lambda_q \int_{\Omega} a_2(x) v^q < \int_{\Omega} \lambda a_2(x) v^{q-1} v \, dx - \int_{\Omega} c_2(x) v \, dx. \tag{3.4}
\]
Adding (3.3) and (3.4), we get

$$\lambda_p \int_\Omega a_1(x)u^p + \lambda_q \int_\Omega a_2(x)v^q < \int_\Omega \lambda a_1(x)v^p - u dx + \int_\Omega \lambda a_2(x)v^q - u dx - \int_\Omega c_1(x)u dx - \int_\Omega c_2(x)v dx$$

Applying the Young inequality on the right hand side of the above equation, we have

$$\lambda_p \int_\Omega a_1(x)u^p + \lambda_q \int_\Omega a_2(x)v^q < \int_\Omega \lambda a_1(x)v^p - u dx + \int_\Omega \lambda a_2(x)v^q - u dx$$

Now, we discuss the following two cases:

Case I, if $u \leq v$ for all $x$, then (3.5) becomes

$$\lambda_p \int_\Omega a_1(x)u^p + \lambda_q \int_\Omega a_2(x)v^q < \int_\Omega \lambda a_1(x)v^p - u dx + \int_\Omega \lambda a_2(x)v^q - u dx.$$

Hence,

$$(\lambda_p - \lambda) \int_\Omega a_1(x)u^p + (\lambda_q - \lambda) \int_\Omega a_2(x)v^q < 0$$

which is a contradiction if $\lambda \leq \max(\lambda_p, \lambda_q) = \lambda^\ast$.

Case II, if $u \geq v$ for all $x$, then (3.5) becomes

$$\lambda_p \int_\Omega a_1(x)u^p + \lambda_q \int_\Omega a_2(x)v^q < \int_\Omega \lambda a_1(x)v^p - u dx + \int_\Omega \lambda a_2(x)v^q - u dx.$$

Hence,

$$(\lambda_p - \lambda) \int_\Omega a_1(x)u^p + (\lambda_q - \lambda) \int_\Omega a_2(x)v^q < 0$$

which is a contradiction if $\lambda \leq \max(\lambda_p, \lambda_q) = \lambda^\ast$. The proof complete.

Now we consider the main result for system (2.2):

**Theorem 3** Let (1.3) and (1.4) hold. Then system (1.2) has no positive weak solution if $\mu \nu \leq \frac{\lambda^2}{K_1K_2}$. 

**Proof.** Suppose $u > 0$ and $v > 0$ be such that $(u, v)$ is a solution of (2.2). We prove our theorem by arriving at a contradiction. Multiplying the first equation in (2.2) by a positive eigenfunction say $\phi_p$ corresponding to $\lambda_p$, we obtain

$$- \int \Delta_p u \phi_p dx = \mu \int_\Omega \rho_1(x)f(v)\phi_p dx,$$

and hence using (2.1) and (1.3), we have

$$\lambda_p \int_\Omega \rho_1(x)u^{p-1}\phi_p dx \leq \mu \int_\Omega \rho_1(x)[K_1v^{q-1} - M_1]\phi_p dx.$$
Similarly using the second equation in (2.2) and (1.4) we obtain
\[
\lambda_q \int_\Omega \rho_2(x) v^{q-1} \phi_q \, dx \leq \nu \int_\Omega \rho_2(x) [K_2 u^{p-1} - M_2] \phi_q \, dx.
\] (3.7)

From (3.7), we have
\[
v^{q-1} \leq \frac{\nu}{\lambda_q} [K_2 u^{p-1} - M_2]
\] (3.8)

Combining (3.6) and (3.8) we obtain
\[
[\lambda_p - \mu \nu \frac{K_1 K_2}{\lambda_q}] \int_\Omega \rho_1(x) u^{p-1} \phi_p \leq -\mu \int_\Omega \rho_1(x) \left[ \frac{\nu K_1 M_2}{\lambda_q} + M_1 \right] \phi_p < 0.
\]

Hence system (1.2) has no positive weak solution if \( \mu \nu \leq \frac{\lambda_p \lambda_q}{K_1 K_2} \).

Remark 4 If \( f, g \) be such that
\[
f(v) \geq K_1 v^{q-1} + M_1, \quad \text{for all} \quad v \geq 0,
\] (3.9)
and
\[
g(u) \geq K_2 u^{p-1} + M_2, \quad \text{for all} \quad u \geq 0,
\] (3.10)
then we have the following theorem:

**Theorem 5** Let (3.9) and (3.10) hold. Then system (1.2) has no positive weak solution if \( \mu \nu \geq \frac{\lambda_p \lambda_q}{K_1 K_2} \).

**Proof.** The proof proceeds in the same way as for Theorem 6.

Remark 6 When \( p = q \), \( m_1(x) = m_2(x) = m, m = a, b, c \) is constant , and \( \alpha = \beta \), we have some results for (1.1) in [1].

Remark 7 When \( p = q \) and \( \rho_1(x) = \rho_2(x) = 1 \), we have some results for (1.2) in [1].

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