Affine special Kähler structures in real dimension two

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We review properties of affine special Kähler structures focusing on singularities of such structures in the simplest case of real dimension two. We describe all possible isolated singularities and compute the monodromy of the flat symplectic connection, which is a part of a special Kähler structure, near a singularity. Beside numerous local examples, we construct continuous families of special Kähler structures with isolated singularities on the projective line.

1 Introduction

The notion of a special Kähler structure has its roots in physics [14, 9] and appears in supersymmetric field theories, \(\sigma\)-models, and supergravity. The following definition of an affine special Kähler structure is due to Freed [11].

**Definition 1.** Let \((M, g, I, \omega)\) be a Kähler manifold, where \(g\) is a Riemannian metric, \(I\) is a complex structure, and \(\omega(\cdot, \cdot) = g(I\cdot, \cdot)\) is the corresponding symplectic form. An **affine special Kähler structure** on \(M\) is a flat symplectic torsion-free connection \(\nabla\) on the tangent bundle \(TM\) such that

\[
(\nabla_X I)Y = (\nabla_Y I)X
\]

holds for all vector fields \(X\) and \(Y\).

**Remark 3.** In this contribution, we only consider affine special Kähler structures, which are simply abbreviated in what follows as special Kähler ones. These should not be confused with projective special Kähler structures, which are not considered in this article.

There are many reasons for a mathematician to care about special Kähler structures. Perhaps one of the most important is the so called c-map construction [5, 11, 23], which associates to a special Kähler structure on \(M\) a hyperKähler metric on the total space of \(T^*M\). Moreover, each cotangent space is a complex Lagrangian submanifold of \(T^*M\) with respect to a natural complex symplectic form on \(T^*M\), i.e., \(\pi: T^*M \to M\) is a holomorphic Lagrangian fibration. Conversely, if \(\pi: X \to M\) is an algebraic integrable system [11, Def. 3.1], then the base \(M\) carries a natural special Kähler structure.

Another reason to care about special Kähler structures is that some important moduli spaces are equipped with (or are closely related to) special Kähler structures. For example,
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moduli spaces of Calabi–Yau threefolds are bases of algebraic integrable systems \([10, 7]\) and thus carry special Kähler structures. The deformation space of a compact, complex Lagrangian submanifold of a complex Kähler symplectic manifold is special Kähler \([17]\). The Hitchin moduli space \([18]\) associated with a Riemann surface \(\Sigma\) is the total space of an algebraic integrable system, whose base is the space of quadratic differentials on \(\Sigma\). Hence, the space of quadratic differentials on \(\Sigma\) is equipped with a natural special Kähler structure.

A lot is known about properties of special Kähler structures (especially local ones) as well as about relations of special Kähler structures to other geometric structures. For example, an extrinsic characterisation of special Kähler structures has been obtained in \([7]\) (see also \([1, 8]\) as well as Section 2.2 below). It was shown in \([4]\) that a simply connected special Kähler manifold can be realised as a parabolic affine hypersphere. More recently, it has been realised \([2, 23]\) that the c-map construction combined with the quaternionic flip of \([15]\) is a useful tool in studies of quaternionic Kähler metrics (this is in turn related to projective special Kähler geometry).

The Riemannian geometry of the Hitchin moduli space is now being actively studied \([13, 25]\). The corresponding special Kähler structure plays a central role in the asymptotic description of the Riemannian metric near the ends of the moduli space.

Soon after special Kähler structures entered the mathematical scene, Lu \([22]\) proved that there are no complete special Kähler metrics besides flat ones. This motivates studying singular special Kähler metrics as the natural structure on bases of algebraic integrable systems with singular fibers.

The focus of this article is on singularities of special Kähler metrics in the simplest case of real dimension two. All possible singularities of special Kähler metrics in two dimensions (under a mild assumption) were described in \([16]\). In this introductory article, after reviewing the basics of special Kähler geometry, we extend the results of \([16]\) by computing the monodromy of the flat symplectic connection near an isolated singularity, see Theorem 18. We also construct continuous families of special Kähler structures with isolated singularities on \(\mathbb{P}^1\), thus showing in particular that there is a non-trivial moduli space of singular special Kähler structures.

An interesting question, which is outside the scope of this article, is whether the result of the c-map construction applied to a singular special Kähler structure can be modified to yield a smooth hyperKähler metric. A proposal for such modification was given in \([12, 27]\). We leave this question for future research.

2 Special Kähler geometry in local coordinates

2.1 Local description in terms of special holomorphic coordinates

Locally, a special Kähler structure can be conveniently described in terms of special holomorphic coordinates. Following \([11]\), we say that a system of holomorphic coordinates \((z_1, \ldots, z_n)\) is special, if \(\nabla (\text{Re } dz_j) = 0\) for all \(j = 1, \ldots, n\). Two special coordinate systems \(\{z_j\}\) and \(\{w_k\}\) are said to be conjugate, if \(\{p_j := \text{Re } z_j, q_k := -\text{Re } w_k\}\) is a Darboux coordinate system, i.e.,

\[
\omega = \sum_j dp_j \wedge dq_j.
\]
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Such coordinate systems always exist in a neighbourhood of any point \([11]\). Moreover, for any \(j, k\) we have

\[
\frac{\partial w_j}{\partial z_k} = \frac{\partial w_k}{\partial z_j}
\]

and therefore there is a holomorphic function \(\mathfrak{F}\) such that

\[
w_k = \frac{\partial \mathfrak{F}}{\partial z_k}.
\]

This function \(\mathfrak{F}\), defined up to a constant, is called a holomorphic prepotential. The Kähler form can be expressed in terms of the holomorphic prepotential as follows:

\[
\omega = \frac{i}{2} \sum_{j,k} \text{Im} \left( \frac{\partial^2 \mathfrak{F}}{\partial z_j \partial z_k \partial z_j} \right) dz_j \wedge d\bar{z}_k.
\]  

(4)

One more useful object, which can be attached to a special Kähler structure, is the so called holomorphic cubic form, which is defined as follows. Consider the fiberwise projection \(\pi^{(1,0)}\) onto the \(T^{1,0}M \subset T_C M\) as a 1-form with values in \(T_C M\). Since this form vanishes on vectors of type \((0, 1)\), we can think of \(\pi^{(1,0)}\) as an element of \(\Omega^{1,0}(M; T_C M)\). Then, the holomorphic cubic form is

\[
\Xi := -\omega(\pi^{(1,0)}, \nabla \pi^{(1,0)}) \in H^0(M; \text{Sym}^3 T^* M).
\]

In terms of the holomorphic prepotential, the holomorphic cubic form can be expressed as follows:

\[
\Xi = \frac{1}{4} \sum_{j,k,l} \frac{\partial^3 \mathfrak{F}}{\partial z_j \partial z_k \partial z_l} dz_j \otimes dz_k \otimes dz_l.
\]

One can show that \(\Xi\) measures the difference between the flat connection \(\nabla\), which is part of the special Kähler structure, and the Levi–Civita connection \([11]\).

### 2.2 An extrinsic description

In \([11]\), the local description of special Kähler manifolds in terms of special holomorphic coordinates was reformulated as an extrinsic description of simply connected special Kähler manifolds:

Given an \(n\)-dimensional special Kähler manifold, then locally, special conjugate coordinate systems \(\{z_j\}\) and \(\{w_j\}\) define an immersion

\[
\phi = (z_1, \ldots, z_n, w_1, \ldots, w_n)
\]

into \(T^\ast \mathbb{C}^n = \mathbb{C}^{2n}\). Thinking of \((z_1, \ldots, z_n, w_1, \ldots, w_n)\) as a canonical coordinate system on \(T^\ast \mathbb{C}^n\), the standard complex symplectic form on \(T^\ast \mathbb{C}^n\) is

\[
\Omega = \sum_j dz_j \wedge dw_j.
\]

The immersion \(\phi\) is holomorphic and Lagrangian \((\phi^\ast \Omega = 0)\).
Furthermore, consider the real structure $\tau: \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ and $\gamma = \mathbf{i}(\cdot, \tau\cdot)$. Then the special Kähler metric is given by $g = \text{Re}(\phi^*\gamma)$ and the Kähler form is $\omega = \phi^*\alpha$, where $\alpha = 2 \sum_j dp_j \wedge dq_j$.

In particular, any simply connected special Kähler manifold $(M, I, \omega, g, \nabla)$ of dimension $n$ admits such a holomorphic, non-degenerate (i.e., $\phi^*\gamma$ is non-degenerate) Lagrangian immersion [1, Thm. 4(iii)].

The converse also holds: Let $\phi: M \to T^*\mathbb{C}^n = \mathbb{C}^{2n}$ be a non-degenerate holomorphic Lagrangian embedding of an $n$-dimensional complex manifold $(M, I)$ and let $g = \text{Re}(\phi^*\gamma)$ be a Kähler metric. Then one can prove that $\text{Re}(\phi)$ is also an immersion and obtain global coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$, which induce a flat torsion-free connection on $M$. With this connection, $(M, g = \text{Re}(\phi^*\gamma), I, \omega = \phi^*\alpha, \nabla)$ is special Kähler (cf. [1, Thm. 3]).

Example 5. The basic example in this context is given by a closed holomorphic 1-form $\vartheta = \sum_j F_j dz_j$ on $M = \mathbb{C}^n$ with invertible real matrix $\text{Im}(\frac{F_j}{\tau z_k})$. Then, the image of $M \overset{\vartheta}{\to} T^*\mathbb{C}^n = \mathbb{C}^{2n}$ is a special Kähler manifold. Locally, $\vartheta = d\mathfrak{F}$ is the differential of a holomorphic function $\mathfrak{F}$, the holomorphic prepotential.

2.3 Local description in terms of solutions of the Kazdan–Warner equation

In the simplest case of complex dimension one, the following alternative local description was obtained in [16]. The main advantage of this description is that it does not rely on the existence of a special holomorphic coordinate. This is particularly important in the case of special Kähler structures with singularities, since near the singularities there may be no special holomorphic coordinates which extend over the singularities, and the traditional approach becomes less helpful.

Let $\Omega \subset \mathbb{C}$ be any domain, which is viewed as being equipped with a holomorphic coordinate $z = x + yi$ and the flat Euclidean metric $|dz|^2 = dx^2 + dy^2$. The coordinate $z$ does not need to be special.

Write a special Kähler metric $g$ on $\Omega$ in the form

$$g = e^{-u} |dz|^2.$$ 

Using the global trivialisation of $T\Omega$ provided by the real coordinates $(x, y)$ the connection $\nabla$ is described by its connection 1-form $\omega_{\nabla} \in \Omega^1(\Omega; \text{gl}(2, \mathbb{R}))$. A computation shows that $\nabla$ is torsion-free and satisfies (2) if and only if $\omega_{\nabla}$ can be written in the form

$$\omega_{\nabla} = \begin{pmatrix} \omega_{11} & -\ast \omega_{11} \\ \ast \omega_{22} & \omega_{22} \end{pmatrix}. \tag{6}$$
Here \( * \) denotes the Hodge star operator with respect to the flat metric. Moreover, \( \nabla \) preserves the symplectic form \( \omega = 2e^{-u} d\sigma \wedge dy \) if and only if \( \text{tr} \omega \nabla = \omega_{11} + \omega_{22} = -du \). Thus, \( \nabla \) is parameterised by a single 1-form, say \( \omega_{11} \).

Furthermore, by a direct computation one obtains that the flatness of \( \nabla \) implies that \( \eta := e^{-u} \omega_{11} \) is closed. Hence, \( \nabla \) is in fact parameterized by a single closed 1-form \( \eta \); Moreover, \( \nabla \) is flat if and only if
\[
\eta = \frac{1}{2} d(h + e^{-u}).
\] A computation shows that (7) is equivalent to
\[
\Delta h = 0, \quad \Delta u = |dh + 2\psi|^2 e^{2u}.
\] (8)

Hence, we obtain the following proposition, which is a slight generalisation of [16, Cor. 2.3].

**Proposition 9.** For any solution \((h, u, \psi) \in C^\infty(\Omega) \times C^\infty(\Omega) \times \Omega^1(\Omega)\) of
\[
\Delta h = 0, \quad \Delta \psi = 0, \quad \Delta u = |dh + 2\psi|^2 e^{2u},
\] (10)the metric \( g = e^{-u} |dz|^2 \) is special Kähler. Moreover, the connection 1-form of the flat connection \( \nabla \) is given by (6) with
\[
2\omega_{11} = e^u (dh + \psi) - du, \quad 2\omega_{22} = -e^u (dh + \psi) - du.
\] (11)

Conversely, if any de Rham cohomology class in \( H^1(\Omega; \mathbb{R}) \) can be represented by a harmonic 1-form, then any special Kähler structure on \( \Omega \) yields a solution of Eq. (10).

In particular, for the punctured disc \( B^*_1 := B_1 \setminus \{0\} \) the first de Rham cohomology group is generated by the following harmonic 1-form:
\[
\varphi = \frac{y \, dx - x \, dy}{x^2 + y^2}.
\]

Hence, we have the following result.

**Corollary 12 ([16, Cor.2.3]).** Any triple \((h, u, a) \in C^\infty(B^*_1) \times C^\infty(B^*_1) \times \mathbb{R}\) satisfying
\[
\Delta h = 0, \quad \Delta u = |dh + a\varphi|^2 e^{2u}
\] (13)Determines a special Kähler structure on \( B^*_1 \). Conversely, any special Kähler structure on \( B^*_1 \) determines a solution of (13).

**Remark 14.** The last equation of (10) is the celebrated Kazdan–Warner equation [19].
A straightforward computation shows that in the setting of Corollary 12 the holomorphic cubic form is given by

\[ \Xi = \Xi_0 \, dz^3 = \frac{1}{2} \left( \frac{a}{2z} - \frac{\partial h}{\partial z} \right) \, dz^3. \]

Let \( N \in \mathbb{Z} \) denote the order of \( \Xi \) at the origin. This means the following: If \( N > 0 \), then the origin is a zero of \( \Xi_0 \) of multiplicity \( N \); If \( N < 0 \), then the origin is the pole of \( \Xi_0 \) of order \( |-N| \); lastly, if \( N = 0 \), \( \Xi_0 \) is holomorphic at the origin and does not vanish there. In particular, by saying that \( N \) is an integer, we exclude essential singularities as well as the case \( \Xi \equiv 0 \), which corresponds to the flat special Kähler metric with \( \nabla \) being the Levi–Civita connection \([11]\).

The description of special Kähler metrics given in Corollary 12 can be used to prove the following result.

**Theorem 15** ([16, Thm. 1.1]). Let \( g = w \, |dz|^2 \) be a special Kähler metric on \( B^*_1 \). Assume that \( \Xi \) is holomorphic on the punctured disc and the order of \( \Xi \) at the origin is \( N \in \mathbb{Z} \). Then,

\[ w = -|z|^{N+1} \log |z|(C + o(1)) \quad \text{or} \quad w = |z|^\beta (C + o(1)) \]  \hspace{1cm} (16)

as \( z \to 0 \), where \( C > 0 \) and \( \beta < N + 1 \).

Moreover, for any \( N \in \mathbb{Z} \) and \( \beta \in \mathbb{R} \) such that \( \beta < N + 1 \), there is an affine special Kähler metric satisfying (16). (In particular, for any \( N \in \mathbb{Z} \) there is an affine special Kähler metric satisfying \( w = -|z|^{N+1} \log |z|(C + o(1)).)\)

**Remark 17.** In [16], the first formula of (16) appears in the form \( w = -|z|^{N+1} \log |z| e^{O(1)} \), which follows from McOwen’s analysis of solutions of the Kazdan–Warner equation \([26]\). The asymptotics as stated in Theorem 15 can be obtained from \([24, \text{Prop. 3.1}]\), which in fact provides even more refined asymptotics near the origin.

By analysing the asymptotic behaviour of solutions of the Kazdan–Warner equation with singular coefficients it is possible to compute the monodromy of the flat symplectic connection. Namely, we have the following result, whose proof will appear elsewhere.

**Theorem 18.** Let \( g = w \, |dz|^2 \) be a special Kähler metric on \( B^*_1 \) such that

\[ w = |z|^\beta (C + o(1)) \quad \text{or} \quad w = -|z|^{N+1} \log |z|(C + o(1)), \]

where \( \beta < N + 1 \) (in the second case, we put by definition \( \beta = N + 1 \)). Let \( \text{Hol}(\nabla) \) denote the monodromy of \( \nabla \) along a loop that goes once around the origin. Then, the following holds:

- If \( \beta \notin \mathbb{Z} \), \( \text{Hol}(\nabla) \) is conjugate to \( \left( \begin{array}{cc} \cos \pi \beta & - \sin \pi \beta \\ \sin \pi \beta & \cos \pi \beta \end{array} \right) \);

- If \( \beta \in 2\mathbb{Z} \), \( \text{Hol}(\nabla) \) is trivial or conjugate to \( \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \);

- If \( \beta \in 2\mathbb{Z} + 1 \), \( \text{Hol}(\nabla) \) is \(-\text{id}\) or conjugate to \( \left( \begin{array}{cc} -1 & 1 \\ 0 & -1 \end{array} \right) \). \hspace{1cm} \Box

**Corollary 19.** \( \text{Hol}(\nabla) \) is conjugate to a matrix lying in \( \text{Sp}(2, \mathbb{Z}) \) if and only if \( \beta \in \frac{1}{2} \mathbb{Z} \cup \frac{1}{4} \mathbb{Z} \).

**Proof.** Since \( \text{Hol}(\nabla) \in \text{Sp}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R}) \), the characteristic polynomial of \( \text{Hol}(\nabla) \) has integer coefficients if and only if \( \text{tr} \, \text{Hol}(\nabla) \in \mathbb{Z} \). This implies that \( \text{Hol}(\nabla) \) is conjugate to a matrix lying in \( \text{Sp}(2, \mathbb{Z}) \) if and only if \( \cos \pi \beta \in \{0, \pm \frac{1}{2}, \pm 1\} \). \hspace{1cm} \Box
2.4 A link between two local descriptions

Our next goal is to obtain a link between the two descriptions of special Kähler structures in terms of solutions of (13) and in terms of special holomorphic coordinates. Notice that special holomorphic coordinates always exist in a neighbourhood of a point, where the special Kähler structure is regular. However, in a neighbourhood of a singular point there may be no special holomorphic coordinates. More precisely, we have the following.

Proposition 20. Let $\Omega$ be a disc or a punctured disc. A special Kähler structure on $\Omega$ admits a special holomorphic coordinate on $\Omega$ if and only if the triple $(h, u, a)$ appearing in Corollary 12 is given by $(h, u, a) = (h, -\log(-h), 0)$ for some negative harmonic function $h$ on $\Omega$.

Proof. Observe first, that for any negative harmonic function $h$ the triple $(h, u, a) = (h, -\log(-h), 0)$ is a solution of (13). Moreover, in this case by (11) we have $\omega_{11} = 0$, which implies that $\nabla d\bar{x} = 0$. In other words, $\bar{z}$ is a special holomorphic coordinate.

Assume now that $\bar{z}$ is a special holomorphic coordinate. Then $\nabla \text{Re } d\bar{z} = 0$ implies $\omega_{11} = 0$, which yields $dh + a\varphi - e^{-u} du = 0$. Since $\varphi$ is not exact, we must have $a = 0$. This yields $h = -e^{-u}$, i.e., $h$ is a negative harmonic function. □

For the special Kähler structure determined by a single negative harmonic function as in the above proposition, we compute

$$g = -h|dz|^2, \quad \Xi = -\frac{1}{2} \frac{\partial h}{\partial \bar{z}} dz \bar{z}^3, \quad \omega_\nabla = \frac{1}{h} \begin{pmatrix} 0 & 0 \\ 0 & dh \end{pmatrix}.$$

Furthermore, by (4) the holomorphic prepotential satisfies

$$\text{Im} \frac{\partial^2 \mathfrak{F}}{\partial z \partial \bar{z}} = -2h.$$

If $\Omega$ is a disc, this equality determines $\mathfrak{F}$ up to a polynomial of degree 2, cf. [11, Prop. 1.38(c)].

If $\Omega = B_1^+$, by [3, Thm. 3.9] there is $A \geq 0$ such that $h = A \log |z| + h_0$, where $h_0$ is a smooth harmonic function on $B_1$. Hence, we have the following result.

Corollary 21. If a special Kähler structure on $B_1^+$ admits a special holomorphic coordinate in a neighbourhood of the origin, there are some constants $A \geq 0$ and $B$ such that

$$g = (-A \log |z| + B + o(1)) \ |dz|^2 \quad \text{as } z \to 0,$$

Moreover, $\text{ord}_0 \Xi \geq -1$. □

In particular, a special Kähler structure on $B_1^+$ such that $\text{ord}_0 \Xi \leq -2$ does not admit a special holomorphic coordinate in a neighbourhood of the origin.

2.5 A relation with metrics of constant negative curvature

Recall that if $g$ and $\tilde{g} = e^{2u} g$ are two metrics on a two-manifold, then their curvatures $K$ and $\tilde{K}$ are related by $\Delta u = K - \tilde{K} e^{2u}$. In particular, if $g$ is flat, then $\Delta u = -\tilde{K} e^{2u}$. Hence, with the help of (7) we conclude: If $g = e^{-u} |dz|^2$ is special Kähler, then $\tilde{g} = e^{2u} |dz|^2$ has a non-positive curvature.
On the other hand, if $\tilde{g} = e^{2u}|dz|^2$ is a metric of constant negative curvature on some domain $\Omega \subset \mathbb{C}$, then $u$ solves $\Delta u = Ke^{2u}$ with $K > 0$. Hence, the triple $(h, u, \psi)$ with $h(x, y) = \sqrt{K} x$ and $\psi = 0$ solves (10) and therefore the metric $g = e^{-u}|dz|^2$ is special Kähler. Summarising, we obtain the following result.

**Proposition 22** ([16, Prop. 3.1]). If $\tilde{g} = w|dz|^2$ is a metric of constant negative curvature $-K$ on a domain $\Omega$, then $g = \frac{1}{\sqrt{w}}|dz|^2$ is a special Kähler metric on the same domain $\Omega$. Moreover, the associated holomorphic cubic form is given by

$$\Xi = -\sqrt{K}i dz^3.$$



### 2.6 Examples

**Example 23.** In the setting of Proposition 20, choose $h = A \log |z| + B$, where $A \geq 0$ and $B$ are some constants. We require also $B < 0$ so that $h$ is negative on $B_1^*$. For the corresponding special Kähler structure we have

$$g = -(A \log |z| + B) |dz|^2, \quad \Xi = \frac{Ai}{4z} dz^3, \quad \omega_{\nabla} = \frac{A}{h} \begin{pmatrix} 0 & 0 \\ *d \log |z| & d \log |z| \end{pmatrix}.$$

Moreover, $z$ is a special holomorphic coordinate. The dual “coordinate” $w$ is given by

$$w = 2(B - A)iz + 2Ai \log z.$$

Of course, $w$ is not a coordinate in any neighbourhood of the origin if $A \neq 0$, but choosing a suitable branch of the logarithm the above expression defines a dual coordinate in a neighbourhood of any point in $B_1^*$.

Similarly, on a suitable branch of the logarithm (or by going to the universal covering of $B_1^*$), the extrinsic description of this metric is given by the holomorphic 1-form

$$\vartheta = w \, dz = 2((B - A)iz + Ai \log z) \, dz,$$

or the corresponding prepotential

$$\mathcal{F} = i(B - A)z^2 + iA \left(z^2 \log(z) - \frac{z^2}{2}\right)$$

with $\frac{\partial}{\partial z} = w$, as in Example 5.

This special Kähler structure is related to the Ooguri–Vafa metric [28], see [6].

The monodromy of $\nabla$ along the circle of radius 1 centered at the origin can be computed explicitly and equals

$$\text{Hol}(\nabla) = \begin{pmatrix} 1 - \frac{4\pi}{1} \\ 0 \end{pmatrix}.$$

**Example 24.** Apply Proposition 22 to the Poincaré metric on the punctured disc $\tilde{g} = |z|^{-2}(\log |z|)^{-2} |dz|^2$ to obtain that the metric

$$g = -|z| \log |z| \, dz^2$$

is special Kähler.
Example 25 (Special Kähler metrics via meromorphic functions). If $\Omega$ is simply connected, the conformal factor of any metric of constant negative Gaussian curvature $K$ can be written \[ w = 4 \frac{|f'(z)|^2}{(1 + K |f(z)|^2)^2}, \] where $f$ is a meromorphic function on $\Omega$ with at most simple poles such that $f'(z) \neq 0$ on $\Omega$. Hence, by Proposition 22 the metric
\[ g = \frac{1}{2} \frac{1 + K |f(z)|^2}{|f'(z)|} |dz|^2 \]
is special Kähler for any meromorphic $f$ as above.

For example, put $K = -1$ and $f(z) = z^n$, where $n \geq 1$. Then we obtain that
\[ g = \frac{1}{2n} |z|^{1-n} \left( 1 - |z|^{2n} \right) |dz|^2 \]
is a special Kähler metric on $B^n_1$.

Example 26. By a classical result of Picard [29], for any given $n \geq 3$ pairwise distinct points $(z_1, \ldots, z_n)$ in $\mathbb{C}$ and any $n$ real numbers $(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_j < 1$ and $\sum \alpha_j > 2$, there exists a metric of constant negative curvature $\tilde{g}$ on $\mathbb{C} \setminus \{z_1, \ldots, z_n\}$ satisfying $\tilde{g} = |z - z_j|^{-2\alpha_j}(c + o(1))|dz|^2$ near $z_j$. Hence, the corresponding special Kähler metric $g$ has a conical singularity near $z_j$,
\[ g = |z - z_j|^\alpha_j(c + o(1))|dz|^2. \]

Explicit examples of constant negative curvature — hence special Kähler — metrics on the three times punctured complex plane can be found in [20] and references therein.

3 Some global aspects of special Kähler geometry on $\mathbb{P}^1$

Even though the methods of Section 2.3 are mainly local, some global conclusions can be also derived. The main objective for this section is to show that by allowing singular special Kähler metrics we have a lot of examples on a compact manifold and even a non-trivial moduli space.

3.1 A constraint from the Gauss–Bonnet formula

Let $g$ be a special Kähler metric on the complex projective line $\mathbb{P}^1$ with singularities at $\{z_1, \ldots, z_k\}$. Assume that at each $z_j$ the metric $g$ has a conical singularity of order $\beta_j/2 > -1$, i.e.,
\[ g = |z|^{\beta_j}(C_j + o(1))|dz|^2, \]
where $C_j$ is positive. A restriction on the cone angles of special Kähler metrics as above can be obtained from the Gauss–Bonnet formula [30, Prop. 1], which in this case reads
\[ \frac{1}{2\pi} \int_{\mathbb{P}^1} K = \chi(\mathbb{P}^1) + \frac{1}{2} \sum_{j=1}^k \beta_j. \]
Here, $K$ is the curvature of $g$ and $\chi$ is the Euler characteristic. Since $K \geq 0$, compare [11, Rem. 1.35], we obtain
\[ \sum_{j=1}^{k} \beta_j \geq -2\chi(\mathbb{P}^1) = -4. \]  
\[ (27) \]

### 3.2 Families of special Kähler metrics on $\mathbb{P}^1$

Just like in Example 26, for any $k \geq 3$ points $z_1, \ldots, z_k$ on $\mathbb{P}^1$ and any $\alpha_1, \ldots, \alpha_k$ such that
\[ \alpha_j < 1 \quad \text{and} \quad \sum_{j=1}^{k} \alpha_j > 2, \]
there is a unique metric $\tilde{g}$ of constant negative curvature on $\mathbb{P}^1$ with conical singularity at $z_j$ of order $-\alpha_j$. Think of $\mathbb{P}^1$ as $\mathbb{C} \cup \{\infty\}$, where each $z_j$ is distinct from $\infty$. If $z$ is a holomorphic coordinate on $\mathbb{C}$, we can write
\[ \tilde{g} = w(z, \bar{z})|dz|^2 \]
with
\[ w(z, \bar{z}) = |z|^{-4}\left(c + o(1)\right) \quad \text{as} \quad z \to \infty. \]

Applying Proposition 22 we obtain a special Kähler metric $g$ on $\mathbb{C}$ with conical singularity of order $\alpha_j/2$ at $z_j$ for each $j = 1, k$. Moreover,
\[ g = |z|^2\left(c + o(1)\right)|dz|^2, \quad \text{as} \quad z \to \infty. \]
\[ (28) \]

In other words, $g$ can be thought of as a special Kähler metric on $\mathbb{P}^1$ with conical singularities at $z_1, \ldots, z_k$ and $z_{k+1} = \infty$ of order $\alpha_1/2, \ldots, \alpha_k/2$, and $-3$, respectively. Summarising, we obtain the following result.

**Proposition 29.** For any $k \geq 3$ points on $\mathbb{P}^1$ and any $\alpha_1, \ldots, \alpha_k$ such that
\[ \alpha_j < 1 \quad \text{and} \quad \sum_{j=1}^{k} \alpha_j < 2, \]
there is a special Kähler metric on $\mathbb{P}^1$ such that
\[ g = |z - z_j|^\alpha_j\left(c_j + o(1)\right)|dz|^2 \]
for all $j = 1, k$. Moreover, near $\infty$, this metric satisfies Eq. (28), which corresponds to
\[ g = |\zeta|^{-6}\left(c + o(1)\right)|d\zeta|^2 \]
in a local coordinate $\zeta$ near $\infty$. \[ \Box \]

Applying a Möbius transformation, we can move $(z_1, z_2, z_3)$ into any given triple of points. Hence, Proposition 29 yields a family of special Kähler metrics with singularities at $k + 1 \geq 4$ points parameterised by $k + 2(k - 3) = 3k - 6$ real parameters.

**Remark 30.** Restriction (27) does not apply to the special Kähler metrics constructed in Proposition 29, since such metrics always have singularities of order $-3$. 

10
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