Convergence of the Iterates in Mirror Descent Methods

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Abstract—We consider centralized and distributed mirror descent algorithms over a finite-dimensional Hilbert space, and prove that the problem variables converge to an optimizer of a possibly nonsmooth function when the step sizes are square summable but not summable. Prior literature has focused on the convergence of the function value to its optimum. However, applications from distributed optimization and learning in games require the convergence of the variables to an optimizer, which is generally not guaranteed without assuming strong convexity of the objective function. We provide numerical simulations comparing entropic mirror descent and standard subgradient methods for the robust regression problem.

I. INTRODUCTION

The method of Mirror Descent (MD), originally proposed by Nemirovski and Yudin [17], is a primal-dual method for solving constrained convex optimization problems. MD is fundamentally a subgradient projection (SGP) algorithm that allows one to exploit the geometry of an optimization problem through an appropriate choice of a strongly convex function [2]. This method not only generalizes the standard gradient descent (GD) method, but also achieves a better convergence rate. In addition, MD is applicable to optimization problems in Banach spaces where GD is not [4].

Of more recent interest, MD has been shown to be useful for efficiently solving large-scale optimization problems. In general, SGP algorithms are simple to implement, however they are typically slow to converge due to the fact that they are based in Euclidean spaces, and through a projection operator are inevitably tied to the geometry of these spaces. As a result, their convergence rate may be directly tied to the dimension d of the underlying Euclidean space in which the problem variables reside. Alternatively, MD can be adapted, or more specifically tailored to the geometry of the underlying problem space, potentially allowing for an improved rate of convergence; see [3] for an early example. Because of these notable potential benefits, MD has experienced significant recent attention for applications to large-scale optimization and machine learning problems in both the continuous and discrete time settings [11], [21], both the deterministic and stochastic scenarios [7], [15], [18], [20], and both the centralized and distributed contexts [14], [20]. MD has also been applied to a variety of practical problems, e.g., game-theoretic applications [24], and multi-agent distributed learning problems [8], [13], [16], [22], [23].

Hitherto, most of these prior studies have focused on studying the convergence rate of MD. In particular, if the step sizes are properly selected then MD can achieve a convergence rate of $O(1/k)$ or $O(1/\sqrt{k})$ for strongly convex or convex objective functions, respectively, [15], [17].

However, the convergence of the objective function value does not, in general, imply the convergence of the sequence of variables to an optimizer. To the best of the authors’ knowledge, there has not been any prior work establishing the convergence of these variables to an optimizer. Our motivation for pursuing a study of the convergence to an optimizer arises from potential applications in Distributed Lagrangian (DL) methods and Game Theory. Specifically, in the context of DL methods, we can apply distributed subgradient methods, or preferably distributed MD methods, to find the solution to the dual problem. In this setting, convergence to the dual optimizer is needed to complete the convergence analysis of DL methods [5], [6]. To motivate our study from a game theoretic viewpoint, note that the dynamics of certain natural learning strategies in routing games have been identified as the dynamics of centralized mirror descent in the strategy space of the players; see [12] for an example. In that context, convergence of the learning dynamics to the Nash equilibria (the minimizers of a convex potential function of the routing game) is critical; convergence to the optimal function value is not enough.

In this paper, our main contribution is thus a proof of convergence to an optimizer in the MD method, where the objective function is convex and not necessarily differentiable; we consider both the centralized and distributed settings.

II. CENTRALIZED MIRROR DESCENT

Let $(\mathbb{H}, \langle \cdot, \cdot \rangle)$ describe a finite-dimensional Hilbert space over reals, and $\mathbb{X}$ be a closed convex subset of $\mathbb{H}$. Consider a possibly nonsmooth convex and continuous function $f : \mathbb{X} \rightarrow \mathbb{R}$ that we seek to minimize via MD, starting from $x_0 \in \mathbb{X}$. We assume throughout that $f$ is finite-valued, and its effective domain contains $\mathbb{X}$. Further, we assume throughout that $f$ has at least one finite optimizer over $\mathbb{X}$.

To precisely define MD, consider a continuously differentiable $\mu$-strongly convex function on an open convex set $\mathbb{D}$ whose the closure contains $\mathbb{X}$. By that, we mean $\psi$ satisfies

$$\psi(y) \geq \psi(x) + \langle \nabla \psi(x), y - x \rangle + \frac{\mu}{2} \|y - x\|^2, \quad \forall x, y \in \mathbb{D}.$$  \hspace{1cm} (2)

Here, $\nabla \psi$ denotes the gradient of $\psi$ which is assumed to be diverged on the boundary of $\mathbb{D}$. In addition, $\|\cdot\|$ is the norm induced by the inner product. Define the Bregman divergence associated with $\psi$ for all $x$ and $y$ in $\mathbb{X}$ as

$$D_\psi(y, x) = \psi(y) - \psi(x) - \langle \nabla \psi(x), y - x \rangle. \hspace{1cm} (1)$$

Equipped with this notation, MD prescribes the following iterative dynamics, starting from some $x_0 \in \mathbb{X}$.

$$x_{k+1} = \arg\min_{z \in \mathbb{X}} \left\{ \langle \nabla f(x_k), z - x_k \rangle + \frac{1}{\alpha_k} D_\psi(z, x_k) \right\}. \hspace{1cm} (2)$$

$\alpha_k$ is the step size, and $D_\psi$ is the Bregman divergence. MD is also known as entropic mirror descent (EMD) when $\psi$ is the Kullback-Leibler divergence.

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One can only show the convergence of the sequence of these variables to the optimal set when this set is bounded.
Here, $\nabla f(x_k) \in \partial f(x_k)$ is an arbitrary subgradient of $f$ at $x_k$, the collection of which comprise the subdifferential set $\partial f(x_k)$, defined as

$$\partial f(x_k) := \{ g \in H \mid f(y) \geq f(x_k) + \langle g, y-x_k \rangle \text{ for all } y \in X \}.$$  

We note that MD enjoys an optimal $O\left(\frac{1}{\sqrt{k}}\right)$ convergence rate for nonsmooth functions [4], [17], i.e.,

$$f\left(\frac{1}{k} \sum_{t=1}^{k} x_t\right) - f^* \leq C\sqrt{\frac{1}{k}},$$

where $f^*$ is the optimal value of $f$ over $X$. Of interest to us in this work is the possible convergence of the problem variables themselves, i.e., whether $x_k$ converges to an optimizer for a suitable choice of step sizes $\alpha_k$. In the remainder of this section, we prove such a convergence result for centralized MD, and extend this to a distributed setting in the next section.

**Theorem 1:** Suppose

- $f$ is $L$-Lipschitz continuous over $X$, and
- $\{\alpha_k\}_{k=0}^{\infty}$ defines a nonincreasing sequence of positive step sizes that is square-summable, but not summable, i.e., $\sum_{k=0}^{\infty} \alpha_k = \infty$, $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$.

Then, $\lim_{k \to \infty} x_k$ optimizes $f$ over $X$ for $x_k$'s generated by MD in (2).

In proving the result, the following two properties of Bregman divergence will be useful. Their proofs are straightforward from its definition in (1).

$$D_f(x, y) = \langle \nabla f(x), y-x \rangle,$$

for arbitrary $x, y, z \in X$. We then deduce the result from (4).

**A. Proof of Theorem 1**

Our proof proceeds in two steps. We first show that consecutive iterates satisfy

$$D_f(z, x_{k+1}) - D_f(z, x_k) \leq \alpha_k \langle \nabla f(x_k), z-x_k \rangle + \frac{\alpha_k^2 L^2}{2\mu},$$

for each $z \in X$. We then deduce the result from (4).

**Proof of (4).**

The optimality of $x_{k+1}$ in (2) implies

$$\langle \nabla f(x_k), z-x_k \rangle + \nabla D_f(x_{k+1}, x_k), z-x_{k+1} \rangle \geq 0. \quad (5)$$

Here, $\nabla D_f$ stands for the derivative of the Bregman divergence with respect to the first coordinate. The properties of the divergence in (3) yield

$$\langle \nabla D_f(x_{k+1}, x_k), z-x_k \rangle = \langle \nabla f(x_{k+1}) - \nabla f(x_k), z-x_k \rangle = D_f(z, x_{k+1}) - D_f(z, x_k) - D_f(x_k, x_{k+1}) \leq D_f(z, x_{k+1}) - D_f(z, x_k) - \frac{\mu}{2} \|x_{k+1} - x_k\|^2.$$

Substituting the above relation in (5), we get

$$D_f(z, x_{k+1}) - D_f(z, x_k) \leq \alpha_k \langle \nabla f(x_k), z-x_k \rangle + \frac{\alpha_k^2 L^2}{2\mu} \|x_{k+1} - x_k\|^2 + \frac{\mu}{2} \|x_{k+1} - x_k\|^2.$$

The above inequality and (6) together imply

$$D_f(z, x_{k+1}) - D_f(z, x_k) \leq \alpha_k^2 \|\nabla f(x_k)\|^2 + \frac{\mu}{2} \|x_{k+1} - x_k\|^2.$$

Lipschitz continuity of $f$ yields $\|\nabla f(x_k)\| \leq L$, from which we get (4).

**Deduction of Theorem 1 from (4).**

Let $X^*$ be the set of optimizers of $f$ over $X$, and $x^*$ be an arbitrary element in $X^*$. Then, the convexity of $f$ implies

$$\langle \nabla f(x_k), x^* - x_k \rangle \leq f^* - f(x_k).$$

Using the above relation in (4) with $z = x^*$ gives

$$D_f(x^*, x_{k+1}) - D_f(x^*, x_k) + \alpha_k [f(x_k) - f^*] \leq \frac{\alpha_k^2 L^2}{2\mu}.$$

Summing the above over $k$ from 0 to $K$, we get

$$D_f(x^*, x_{K+1}) - D_f(x^*, x_0) + \sum_{k=0}^{K} \alpha_k [f(x_k) - f^*] \leq \frac{L^2}{2\mu} \sum_{k=1}^{K} \alpha_k^2.$$

Taking $K \to \infty$, the right hand side remains bounded, owing to the square summability of the $\alpha_k$'s. Bregman divergence is always nonnegative, and so is each summand in the second term on the left hand side of the above inequality. Together, they imply that $\sum_{k=0}^{\infty} \alpha_k [f(x_k) - f^*] < \infty$ and $D_f(x_k, x^*)$ converges for each $x^* \in X^*$. The non-summability of the $\alpha_k$'s further yields

$$\lim_{k \to \infty} f(x_k) = f^*.$$

Convergence of $D_f(x^*, x_k)$ for each $x^* \in X^*$ implies the boundedness of the iterates $x_k$. Let $\{x_{k_{t}}\}_{t=0}^{\infty}$ be the bounded subsequence of $x_k$'s along which

$$\lim_{k \to \infty} f(x_k) = \lim_{t \to \infty} f(x_{k_{t}}) = f^*.$$  

(7)

This bounded sequence $\{x_{k_{t}}\}_{t=0}^{\infty}$ has a (strongly) convergent subsequence. Function evaluations over that subsequence tend to $f^*$. Continuity of $f$ implies that the subsequence converges to a point in $X^*$. Call this point $x^*$. Then, $D_f(x^*, x_{k_{t}})$ converges, and it converges to zero over said subsequence, implying

$$\lim_{k \to \infty} D_f(x^*, x_k) = 0.$$
Appealing to (3), we conclude \( \lim_{k \to \infty} x_k = x^* \). This completes the proof of Theorem 1.

**Remark 1:** This proof should be generalizable to \( \mathbb{H} \) being infinite dimensional, where one would consider weak convergence of \( x_k \) to an optimizer of \( f \) over \( \mathbb{X} \).

### III. Distributed Mirror Descent

In this section, we consider a distributed variant of MD. More precisely, we consider a collection of \( N \) agents who collectively seek to minimize \( f(x) := \sum_{i=1}^{N} f_i(x) \) over \( \mathbb{X} \). Agent \( i \) only knows the convex but possibly non-smooth function \( f_i \), and thus, the agents must solve the problem cooperatively. The agents are allowed to exchange their iterates only with their neighbors in an undirected graph \( \mathcal{G} \). Starting from \( x^{i_0}_1, \ldots, x^{i_0}_N \), each agent communicates with its neighbors in \( \mathcal{G} \) and updates its iterates \( x^i_k \) at time \( k \) as follows.

\[
v^i_k = \sum_{j=1}^{N} A_{ij} x^j_k,
\]

\[
x^i_{k+1} = \arg\min_{x \in \mathbb{X}} \left\{ \langle \nabla f(x^i_k), z - v^i_k \rangle + \frac{1}{\alpha_k} D_{\psi}(z, v^i_k) \right\}.
\]

(Math 8)

Matrix \( A \) thus encodes the communication graph \( \mathcal{G} \), i.e., \( A_{ij} \neq 0 \) if and only if agent \( j \) can communicate to agent \( i \) its current iterate, denoted by an edge between \( i \) and \( j \) in \( \mathcal{G} \). Rates for convergence of the function value in the above distributed mirror descent (DMD) algorithm have been reported in [14, Theorem 2]. We prove that DMD drives \( x^i_k, \ldots, x^N_k \) to a common optimizer \( x^* \) of \( f \) over \( \mathbb{X} \).

**Theorem 2:** Suppose

- \( f^i \) is \( L \)-Lipschitz continuous over \( \mathbb{X} \),
- \( \{\alpha_k\}_{k=0}^{\infty} \) defines a nonincreasing sequence of positive step sizes that is square-summable, but not summable, i.e., \( \sum_{k=0}^{\infty} \alpha_k = \infty \), \( \sum_{k=0}^{\infty} \alpha_k^2 < \infty \),
- \( y \mapsto D_{\psi}(x, y) \) is convex,
- \( A \) is doubly stochastic, irreducible, and aperiodic.

Then, \( \lim_{k \to \infty} x^i_k \) is identical across \( i = 1, \ldots, N \), and the limit optimizes \( \sum_{i=1}^{N} f^i \) over \( \mathbb{X} \) for \( x^* \)’s generated by DMD in (8).

The first two assumptions are identical to the centralized counterpart in Section II. The third one is special to the distributed setting, and is crucial to the proof of the result. We remark that Bregman divergence \( D_{\psi} \) is always strictly convex in its first argument. Our result requires convexity in the second argument. A sufficient condition is derived in [1], that requires \( \psi \) to be thrice continuously differentiable and satisfy \( H_\psi(x) \geq 0 \) and \( H_\psi(x) + \nabla H_\psi(x)(x-y) \geq 0 \) for all \( x \) and \( y \) in \( \mathbb{X} \), where \( H_\psi \) stands for the Hessian of \( \psi \).

The last assumption defines a requirement on the information flow. Stated in terms of graph \( \mathcal{G} \) that defines the connectivity among agents, it is sufficient to have \( \mathcal{G} \) being connected with at least one node with a self-loop.

**A. Proof of Theorem 2**

We first appeal to the optimality of \( x^i_{k+1} \) in (8) to conclude

\[
\langle \alpha_k \nabla f^i(v^i_k), \nabla_{\psi} D_{\psi}(x^i_{k+1}, v^i_k), z - x^i_{k+1} \rangle \geq 0
\]

for every \( z \in \mathbb{X} \). The properties of Bregman divergence in (3) yield

\[
\langle \nabla D_{\psi}(x^i_{k+1}, v^i_k), z - x^i_{k+1} \rangle = \langle \nabla \psi(x^i_{k+1}) - \nabla \psi(v^i_k), z - x^i_{k+1} \rangle
\]

\[
= D_{\psi}(z, v^i_k) - D_{\psi}(z, x^i_{k+1}) - D_{\psi}(v^i_k, x^i_{k+1})
\]

(10)

Substituting the above equality in (9), and summing over \( i = 1, \ldots, N \), we get an inequality of the form

\[
\sum_{i=1}^{N} S^i_k(z) + \sum_{i=1}^{N} T^i_k(z) \geq 0,
\]

where

\[
S^i_k(z) := \langle \alpha_k \nabla f^i(v^i_k), z - x^i_{k+1} \rangle,
\]

\[
T^i_k(z) := D_{\psi}(z, v^i_k) - D_{\psi}(z, x^i_{k+1}) - D_{\psi}(x^i_{k+1}, v^i_k)
\]

for each \( z \in \mathbb{X} \).

We provide upper bounds on each of the summations separately. In the sequel, we use the notation

\[
x_k = \frac{1}{N} \sum_{i=1}^{N} x^i_k.
\]

(12)

- **An upper bound for \( \sum_{i=1}^{N} S^i_k(z) \).**

\[
\sum_{i=1}^{N} S^i_k(z) = \sum_{i=1}^{N} \langle \alpha_k \nabla f^i(v^i_k), z - v^i_k \rangle
\]

\[
+ \sum_{i=1}^{N} \langle \alpha_k \nabla f^i(v^i_k), v^i_k - x^i_{k+1} \rangle.
\]

(13)

We bound each summand in both the summations above. Using the convexity of \( f^i \), we get

\[
\langle \alpha_k \nabla f^i(v^i_k), z - v^i_k \rangle \leq f^i(z) - f^i(v^i_k)
\]

\[
= f^i(z) - f^i(x_k) + f^i(x_k) - f^i(v^i_k)
\]

\[
\leq f^i(z) - f^i(x_k) + \langle \nabla f^i(v^i_k), x_k - v^i_k \rangle
\]

\[
\leq f^i(z) - f^i(x_k) + L \|x_k - v^i_k\|.
\]

The last line follows from the Cauchy-Schwarz inequality and that \( f^i \) is \( L \)-Lipschitz. Further, the Cauchy-Schwarz and arithmetic-geometric mean inequalities yield

\[
\langle \alpha_k \nabla f^i(v^i_k), v^i_k - x^i_{k+1} \rangle \leq \frac{\alpha_k^2}{2\mu} \|\nabla f^i(v^i_k)\|^2 + \frac{\mu}{2} \|v^i_k - x^i_{k+1}\|^2
\]

\[
\leq \frac{\alpha_k^2 L^2}{2\mu} + \frac{\mu}{2} \|v^i_k - x^i_{k+1}\|^2.
\]

(14)

The last line is a consequence of \( f^i \) being \( L \)-Lipschitz. Combining (13) and (14) then allows us to deduce

\[
\sum_{i=1}^{N} S^i_k(z) \leq \alpha_k \|f(z) - f(x_k)\| + \alpha_k L \sum_{i=1}^{N} \|x_k - v^i_k\|
\]

\[
+ \alpha_k^2 \frac{NL^2}{2\mu} + \frac{\mu}{2} \sum_{i=1}^{N} \|v^i_k - x^i_{k+1}\|^2.
\]

(15)
We mimic the style of arguments in the proof of Theorem 1 to complete the derivation, and provide an upper bound on the double summation on the right hand side of the above inequality.

- **An upper bound for** \( \sum_{i=1}^{N} T^i_k(z) \).

To derive this bound, we fix an orthonormal basis for \( H \) with \( \dim H = d \). Let \( x^i_k \in \mathbb{R}^d \) denote the coordinates of \( x^i_k \) in that basis. The coordinates for the centroid \( \bar{x}_k \) are given by \( \bar{x}_k \). The inner product in \( H \) becomes the usual dot product among the corresponding coordinates. The norm becomes the usual Euclidean 2-norm in the coordinates. Define

\[
X^\top_k := (x^1_k \ldots x^N_k) \in \mathbb{R}^{d \times N}.
\]

Let \( I \in \mathbb{R}^N \) denote a vector of all ones and \( I \in \mathbb{R}^{N \times N} \) be the identity matrix. Also, define

\[
P := I - \frac{1}{N}I \otimes I^\top
\]

for convenience. Equipped with this notation, we then have

\[
\sum_{i=1}^{N} \| x_k - x_k^i \|^2 = \sum_{i=1}^{N} \| \bar{x}_k - x^i_k \|^2 \\
\leq \sqrt{N} \| PX_k \|_F \|
\]

(19)

Here, \( \| \cdot \|_F \) and \( \| \cdot \| \) denote the Frobenius and the 2-norm of matrices, respectively. In what follows, we bound \( \sum_{k=0}^{K} \alpha_k \| PX_k \|_F^2 \) from above.

Collecting the coordinates of \( v^i_k \) in \( V_k \in \mathbb{R}^{N \times d} \) similarly to \( X_k \), we have

\[
\| PX_{k+1} \|_2 = \| P (AX_k + X_{k+1} - V_k) \|_2 \\
\leq \| APX_k \|_2 + \| P (X_{k+1} - V_k) \|_2
\]

(20)

since \( A \) commutes with \( P \). We bound each term on the right hand side above. For the first term, notice that

\[
PX_k = \left( I - \frac{1}{N}I \otimes I^\top \right) X_k \perp I.
\]

Since \( A \) is doubly stochastic (i.e., \( A I = I \)), the Perron-Frobenius theorem [9, Theorem 8.4.4] and the Courant-Fischer theorem [9, Theorem 4.2.11] together yield

\[
\| APX_k \|_2 \leq \sigma_2(A) \| PX_k \|_2,
\]

(21)

where \( \sigma_2(A) \) is the second largest singular value of \( A \). Further, \( A \) is irreducible, and aperiodic, implying \( \sigma_2(A) < 1 \). To bound the second term on the right hand side of (20), we use that matrix norms are submultiplicative, and hence we have

\[
\| P (X_{k+1} - V_k) \|_2 \leq \| P \|_2 \| X_{k+1} - V_k \|_2 \\
= \sum_{i=1}^{N} \| v_k^i - x_{k+1}^i \|,
\]

(22)

because \( \| P \|_2 = 1 \). To bound each term in the above summation, we utilize (9) and (10) with \( z = v_k^i \) to obtain

\[
\langle \alpha_k \nabla f^i(v_k), v_k^i - x_{k+1}^i \rangle \\
\geq \langle \nabla \psi(v_k) - \nabla \psi(x_{k+1}), v_k^i - x_{k+1}^i \rangle.
\]

(23)
Since $f^i$ is $L$-Lipschitz, and $\psi$ is $\mu$-strongly convex, we have the following two inequalities
\[
(\alpha_k \nabla f^i(v^i_k), v^i_k - x^i_{k+1}) \leq \alpha_k L \|v^i_k - x^i_{k+1}\|
\]
\[
(\nabla \psi(x_k) - \nabla \psi(x^i_{k+1}), v^i_k - x^i_{k+1}) \geq \mu \|v^i_k - x^i_{k+1}\|^2.
\]
that together with (23) gives
\[
\|v^i_k - x^i_{k+1}\| \leq \frac{\alpha_k L}{\mu}.
\]
Summing the above over $i = 1, \ldots, N$, we obtain an upper bound on the right hand side of (22). Utilizing that bound and (21) in (20), we get
\[
\|PX_{k+1}\|_2 \leq \sigma_2(A) \|PX_k\|_2 + \alpha_k \frac{NL}{\mu}. \tag{24}
\]
For convenience we suppress the dependency of $\sigma_2$ on $A$ in the sequel. Iterating the above inequality gives
\[
\|PX_k\|_2 \leq \sigma_2^k \|PX_0\|_2 + \frac{NL}{\mu} \sum_{\ell=0}^{k-1} \alpha_\ell \sigma_2^{k-\ell-1}
\]
for each $k \geq 1$, which further yields
\[
\sum_{k=0}^{K} \alpha_k \|PX_k\|_2 \leq \alpha_0 \|PX_0\|_2 + \sum_{k=0}^{K} \alpha_k \sigma_2^k \|PX_0\|_2 + \frac{NL}{\mu} \sum_{k=0}^{K} \alpha_k \sum_{\ell=0}^{k-1} \sigma_2^{k-\ell-1}. \tag{25}
\]
Now, $\sigma_2 < 1$ and the $\alpha_k$’s are nonincreasing. Thus, we have
\[
\alpha_0 + \sum_{k=1}^{K} \alpha_k \sigma_2^k \leq \alpha_0 \sum_{k=0}^{\infty} \sigma_2^k = \alpha_0(1-\sigma_2)^{-1},
\]
and using this in (25) gives the following required bound.
\[
\sum_{k=0}^{K} \sum_{i=1}^{N} \alpha_k \|x_k - x^i_k\| 
\leq N^\prime \sum_{k=0}^{K} \alpha_k \|PX_k\|_2 
\leq \frac{\alpha_0 N^\prime}{1-\sigma_2} \|PX_0\|_2 + \frac{NN^\prime L}{\mu} \sum_{k=0}^{K} \alpha_k \sum_{\ell=0}^{k-1} \sigma_2^{k-\ell-1} 
\leq \frac{\alpha_0 N^\prime}{1-\sigma_2} \|PX_0\|_2 + \frac{NN^\prime L}{\mu} \sum_{k=0}^{K} \alpha_k \sum_{\ell=0}^{k-1} \sigma_2^{k-\ell-1} 
\leq \frac{\alpha_0 N^\prime}{1-\sigma_2} \|PX_0\|_2 + \frac{N^\prime N L^2}{\mu(1-\sigma_2)} \sum_{k=0}^{K} \alpha_k \sum_{\ell=0}^{k-1} \sigma_2^\ell. \tag{26}
\]

*Proof of the result by combining all upper bounds.*

Utilizing (26) in (18) yields
\[
\sum_{i=1}^{N} [D_\psi(x^*, x_{K+1}^i) - D_\psi(x^*, x_0)] + \sum_{k=0}^{K} \alpha_k [f(x_k) - f(x^*)] 
\leq \frac{\alpha_0 N^\prime L}{1-\sigma_2} \|PX_0\|_2 + \frac{NN^\prime L^2}{\mu(1-\sigma_2)} \sum_{k=0}^{K} \alpha_k^2 + \frac{NL^2}{2\mu} \sum_{k=0}^{K} \alpha_k^2. \tag{27}
\]

Driving $K \uparrow \infty$, the right hand sides of (26) and (27) converge as the sequence of $\alpha_k$’s are square summable. Further, the $\alpha_k$’s are non-square summable, and hence we conclude
\[
1) \quad \lim_{K \rightarrow \infty} \inf_{\epsilon=1}^{N} \sum_{i=1}^{N} \|X_K - x_K^i\| = 0,
\]
\[
2) \quad \lim_{K \rightarrow \infty} f(S_K) = f^*,
\]
\[
3) \quad \sum_{i=1}^{N} D_\psi(x^*, x_{K+1}^i) \text{ converges}.
\]
Convergence of the Bregman divergences for each $x^* \in \mathbb{X}^*$ implies the boundedness of the iterates $x_K^i$ for each $i = 1, \ldots, N$. Recall that $X_K$ denotes the collective iterate for all agents at time $K$. Consider the bounded subsequence of $X_K$, denoted $X_K^i$, along which
\[
\lim_{\ell \rightarrow \infty} \sum_{i=1}^{N} \|X_{K_{\ell}} - x_{K_{\ell}}^i\| = \lim_{K \rightarrow \infty} \inf_{i=1}^{N} \sum_{i=1}^{N} \|X_K - x_K^i\| = 0,
\]
\[
\lim_{\ell \rightarrow \infty} f(S_{K_{\ell}}) = \lim_{K \rightarrow \infty} f(S_K) = f^*.
\]

This bounded sequence $(X_{K_{\ell}})_{\ell=0}^{\infty}$ has a (strongly) convergent subsequence. Over that subsequence, the agents’ iterates converge to the centroid, and the function evaluations over the centroid tend to $f^*$. Continuity of $f$ implies that each agent’s iterate over that subsequence converges to the same point in $\mathbb{X}^*$. Call this point $x^*$. Since $D_\psi(x^*, x_{K_{\ell}}^i)$ converges, it converges to zero over that subsequence, implying $\lim_{K \rightarrow \infty} D_\psi(x^*, x_{K_{\ell}}^i) = 0$ for each $i$. Appealing to (3), we infer $\lim_{K \rightarrow \infty} x_{K_{\ell}}^i = x^*$, concluding the proof.

**IV. NUMERICAL EXPERIMENTS**

Theorems 1 and 2 guarantee the convergence of the iterates to the optimizer, but do not provide convergence rates with non-summable but square summable step-sizes. Given the lack of rates, we empirically illustrate that mirror descent is both in centralized and distributed settings — often outperforms vanilla subgradient methods on simple examples with our step sizes. Our simulations are different from many prior works, e.g., [15], where they choose $\alpha_k = \frac{\alpha}{\sqrt{k+1}}, \alpha > 0$ to guarantee the fastest convergence of the function value.

Consider the following robust linear regression problem over a simplex.

\[
\text{minimize } \|Gx - h\|_1, \text{ subject to } 1^\top x = 1, x \geq 0. \tag{28}
\]

Robust regression fits a linear model to the data $G \in \mathbb{R}^{N \times d}, h \in \mathbb{R}^N$. It differs from ordinary least squares in that the objective function penalizes the entry-wise absolute deviation from the linear fit rather than the squared residue, and is known to be robust to outliers [10]. Consider two different Bregman divergences on the $d$-dimensional simplex $\mathbb{X}$ defined by the Euclidean distance $\psi_1(x) := \frac{1}{2} \|x\|_2^2$, and negative entropy $\psi_2(x) := \sum_{j=1}^{d} x^j \log x^j$. Centralized mirror descent with $D_{\psi_1}$ amounts to a projected subgradient algorithm where each iteration is a subgradient step followed by a projection on $\mathbb{X}$. With $D_{\psi_2}$, the updates define an exponentiated gradient method, also known as the entropic mirror descent algorithm (cf. [2], [19]). Its updates are given by
\[
x_{k+1}^i := x_k^i \exp \left(-\alpha_k \|\nabla f(x_k^i)\|\right) \sum_{\ell=1}^{\alpha_k} x_k^\ell \exp \left(-\alpha_k \|\nabla f(x_k^\ell)\|\right),
\]
where the objective in (28) is $f(x)$, and
\[
\nabla f(x) = \sum_{j=1}^{N} \text{sgn} \left( \langle g_j^i \rangle^T x - h_i \right) g_j^i.
\]

Here, \(\text{sgn}(\cdot)\) denotes the sign of the argument, and \(g_j^i\) is the \(i\)-th row of \(G\). Negative entropy being a ‘natural’ function over simplex, entropic mirror descent enjoys faster convergence than projected subgradient descent, as shown in Figure 1a using step sizes \(\alpha_k = \frac{1}{M+k+1}\).

Next, consider the case where each node \(i = 1, \ldots, N\) in a graph only knows \(g_i^i\) and \(h_i\), and they together seek to minimize \(\sum_{i=1}^{N} \| g_i^i \|^2 \frac{1}{2} x - h_i \|\). Figures 1b and 1c show how the distributed variant of entropic mirror descent outperforms that of projected subgradient method with steps-sizes \(\alpha_k = 1/(M+k+1)\). We choose \(A\) as the transition probabilities of a Markov chain in the Metropolis-Hastings algorithm for the respective graphs. Centralized algorithms converge faster than distributed algorithms; however, the denser the graph, the faster the convergence is of the distributed algorithms.

![Fig. 1: Diagram showing the convergence behavior of projected subgradient method (---) and entropic mirror descent (---) for (28) with \(N = 100\) and \(d = 10\). All entries of \(G\) and \(h\) are chosen uniformly at random from \([0,1]\). The algorithm was initialized at a random point in \(\mathbb{R}\). Plot (a) shows the dynamics of the centralized algorithms. Plots (b) and (c) show dynamics of the distributed variants over the respective networks with 939 and 2678 edges, respectively.](image)

**V. CONCLUSION**

In this paper, we proved guaranteed convergence of the iterates (i.e., the problem variables) to an optimizer in MD on a finite dimensional Hilbert space, using a specific choice of step size in both centralized and distributed settings. The convergence holds even when minimizing possibly non-smooth and non-strongly convex functions. This convergent behavior generalizes a similar property of subgradient methods. Extension to the case with additive noise with bounded support, and to infinite dimensional Hilbert and Banach spaces remain interesting directions for future research.

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