On the use of applying Lie-group symmetry analysis to turbulent channel flow with streamwise rotation

A comment on the article by Oberlack et al. (2006)

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Abstract

The study by Oberlack et al. (2006) consists of two main parts: a direct numerical simulation (DNS) of a turbulent plane channel flow with streamwise rotation and a preceding Lie-group symmetry analysis on the two-point correlation equation (TPC) to analytically predict the scaling of the mean velocity profiles for different rotation rates. We will only comment on the latter part, since the DNS result obtained in the former part has already been commented on by Recktenwald et al. (2009), stating that the observed mismatch between DNS and their performed experiment is possibly due to the prescription of periodic boundary conditions on a too small computational domain in the spanwise direction.

By revisiting the group analysis part in Oberlack et al. (2006), we will generate more natural scaling laws describing better the mean velocity profiles than the ones proposed. However, due to the statistical closure problem of turbulence, this improvement is illusive. As we will demonstrate, any arbitrary invariant scaling law for the mean velocity profiles can be generated consistent to any higher order in the velocity correlations. This problem of arbitrariness in invariant scaling persists even if we would formally consider the infinite statistical hierarchy of all multi-point correlation equations. The closure problem of turbulence simply cannot be circumvented by just employing the method of Lie-group symmetry analysis alone: as the statistical equations are unclosed, so are their symmetries! Hence, an \textit{a priori} prediction as how turbulence scales is thus not possible. Only \textit{a posteriori} by anticipating what to expect from numerical or experimental data the adequate invariant scaling law can be generated through an iterative trial-and-error process. Finally, apart from this issue, also several inconsistencies and incorrect statements to be found in Oberlack et al. (2006) will be pointed out.

\textbf{Keywords:} Symmetries, Lie Groups, Invariant Scaling Laws, Turbulence, Channel Flow, Rotation, Statistical Mechanics, Two-point Correlation Equation, Closure Problem of Turbulence;
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1. Lie-group symmetry analysis and the closure problem of turbulence

The object under investigation in Oberlack et al. (2006) for group analysis is the \textit{inviscid} ($\nu = 0$) two-point correlation equation (TPC) [Eq. (2.14)] to generate invariant scaling laws for large-scale quantities, such as mean velocities and Reynolds stresses. Small scale quantities, such as the dissipation, are not captured by this investigation. As said, “the basis for this assumption is the fact that, to leading order only, viscosity has no effect as $Re \to \infty$. Viscosity only affects the small scales of $O(\eta)$ where $\eta$ is the Kolmogorov length scale. Hence neglecting viscosity is only valid for $|r| > \eta$ [the large scales]” (Oberlack et al., 2006, p.388). Following this strategy, as

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also cited from Oberlack (2002), Oberlack & Guenther (2003) or Khujadze & Oberlack (2004), one may also take the one-point limit within this TPC [Eq. (2.14)] to obtain the inviscid ($\nu = 0$) Reynolds transport equations [Eqs.(2.1a)-(2.1c)], being valid then only “in regions sufficiently far from solid walls, [where] the viscous terms may be neglected to leading order, and the balance is dominated by the pressure and the turbulent stresses” (Oberlack et al., 2006, p. 385).\footnote{Note that by taking the one-point limit $r \to 0$ within the inviscid TPC [Eq. (2.14)], being itself only valid for the large scales $|r| > \eta$ where $\eta \neq 0$ is the Kolmogorov length scale, it is claimed that only an error of the order $O(Re^{-1/2})$ is made which becomes negligibly small if the Reynolds number is large enough; see e.g. the statement made in Khujadze & Oberlack (2004) [p. 399].}

The symmetry analysis on Eq. (2.14) was performed as a group classification in the way at looking which profiles for the mean velocities are admitted under a certain given set of symmetries for the higher-order moments including their coordinates. For the details to be discussed below, we will present the key results here again: If the mean velocity profiles $\bar{u}_1$ and $\bar{u}_3$ satisfy the constraint equations [Eqs. (2.18a)-(2.18b)]

$$\begin{cases}
[a_1x_2 + a_5] \frac{d\bar{u}_1(x_2)}{dx_2} - a_1\bar{u}_1(x_2) = c_1, \\
[a_1x_2 + a_5] \frac{d\bar{u}_3(x_2)}{dx_2} - a_1\bar{u}_3(x_2) = c_3,
\end{cases}$$

(1.1)

then the inviscid TPC system of equations [Eq. (2.14)]

$$\begin{align}
\frac{\partial R_{2j}}{\partial x_2} - \frac{\partial R_{ij}}{\partial r_i} &= 0, & \frac{\partial R_{ij}}{\partial r_j} &= 0, & \frac{\partial pu_i}{\partial r_j} &= 0, & \frac{\partial a_{2j}}{\partial r_i} - \frac{\partial a_{ij}}{\partial r_j} &= 0, \\
\frac{\partial T_{2j}^{(1)}}{\partial x_2} - \frac{\partial T_{ij}^{(1)}}{\partial r_i} &= 0, & \frac{\partial T_{ij}^{(2)}}{\partial x_2} - \frac{\partial T_{ij}^{(2)}}{\partial r_j} &= 0,
\end{align}$$

(1.2)

admits, when written in its infinitesimal generator form, the following four-parametric Lie-point symmetry group [Eqs. (2.16)-(2.17)]:

$$S_{1.1}^{(1)}: \xi_{x_1} = a_1x_1 + a_2, \quad \xi_{x_2} = a_1x_2 + a_4, \quad \xi_{x_3} = a_1x_3 + a_5,$$

$$\eta_{R_{ij}} = 2a_1R_{ij}, \quad \eta_{\rho u_i} = 3a_1\rho u_i, \quad \eta_{\rho p} = 3a_1\rho p, \quad \eta_{T_{ij}^{(1)}} = 2a_1T_{ij}^{(1)}, \quad \eta_{T_{ij}^{(2)}} = 2a_1T_{ij}^{(2)}$$

(1.3)

and vice versa. Note that in (1.2) we included all continuity conditions [Eq.2.5] consistently up to the highest (unclosed) order, where we denoted these moments collectively by $T_{ij}$, in particular as

$$T_{ij}^{(1)} = \frac{\partial R_{ij(k)}}{\partial r_k}, \quad T_{ij}^{(2)} = \frac{\partial R_{ij(2j)}}{\partial x_2} - \frac{\partial R_{ij(2j)}}{\partial r_j}.$$  

(1.4)

To simplify notation, we also suppressed the overall tilde-symbol used in Oberlack et al. (2006) to denote the re-scaling transformation [Eqs.(2.13a)-(2.13c)] relative to the rotation parameter $\Omega$. To ensure that in the following no ambiguity arises between the different notations, we will continually point out every time when this transformation back to the original variables is needed or performed.

\footnote{The notation $S_{1.1}^{(1)}$ clarifies that the symmetry $S_1$ (1.3) is connected to the constraint (1.1). In this respect it is also important to note that since the system (1.2) is unclosed, all admitted invariant transformations can only be regarded in the weak sense as equivalence transformations, and not as true symmetry transformations in the strong sense. For more details, we refer e.g. to Frewer et al. (2014a) and the references therein. In the following, however, we will continue to call them imprecisely as “symmetries”, like it was also done in Oberlack et al. (2006).}
Lie-group symmetry analysis in turbulence

The above result put forward in Oberlack et al. (2006), however, gives the misleading impression now that (1.3) is the only symmetry that can be connected to the mean velocity constraint (1.1). But by far this is not the case. Instead, when performing a systematic symmetry analysis assisted, e.g., by the Maple package DESOLV-II (Vu et al., 2012), one obtains an infinite-dimensional Lie-algebra involving arbitrary functions for the dependent variables. For example, the following symmetry

\[ S_2^{(1.1)} : \]
\[ \xi_1 = \alpha_1 r_1 + a_2, \quad \xi_2 = \alpha_1 r_2, \quad \xi_3 = \alpha_1 r_3 + a_4, \quad \xi_{x_2} = \alpha_1 x_2 + a_5, \]
\[ \eta_{R_1} = \alpha_2 R_{11} + f_{11}(x_2, r_2, r_3), \quad \eta_{R_{12}} = \alpha_2 R_{12} + f_{12}(x_2, r_3), \]
\[ \eta_{R_{23}} = \alpha_2 R_{23} + f_{23}(r_1), \quad \eta_{R_{231}} = \alpha_2 R_{231} + f_{31}(x_2, r_2), \quad \eta_{R_{232}} = \alpha_2 R_{232} + f_{32}(x_2, r_1), \]
\[ \eta_{R_{233}} = \alpha_2 R_{233} + f_{33}(x_2, r_1, r_2), \quad \eta_{\overline{p}u} = (\alpha_1 + \alpha_2)\overline{p}_u + h(x_2, r_2, r_3), \]
\[ \eta_{\overline{p}r} = (\alpha_1 + \alpha_2)\overline{p}_r + g(x_2, r_1, r_3), \quad \eta_{\overline{r}u} = (\alpha_1 + \alpha_2)\overline{r}_u + h(x_2, r_1, r_2), \]
\[ \eta_{\overline{r}r} = (\alpha_1 + \alpha_2)\overline{r}_r + h(x_2, r_1, r_2), \]
\[ \eta_{\overline{u}u} = \alpha_2 \overline{u}_u + h(x_2, r_3) + q_{11}(x_2, r_2, r_3). \]
is also compatible to the mean velocity constraint (1.1), where all $f_{ij}$, $g_i$, $h_i$, and $q_{ij}$ are arbitrary functions only restricted by the identity constraints [Eq. (2.6)]

$$
R_{ij}(x, r) = R_{ji}(x + r, -r), \quad u_{ij}(x, r) = \overline{u_{ij}}(x + r, -r),$$

$$
T_{ij}^{(1)}(x, r) = T_{ij}^{(2)}(x, r) \bigg|_{x=x+r; r=-r},
$$

(1.6)
to be satisfied when generating any invariant functions for $R_{ij}$, $\overline{u_{ij}}$, $\overline{p_{ij}}$, $T_{ij}^{(1)}$, or $T_{ij}^{(2)}$ from this symmetry. Note that (1.5) is not the most general symmetry which the inviscid TPC system (1.2) under the mean velocity constraint (1.1) can admit. It is only a particular subgroup of a more general one not shown here. The particular choice (1.5) should only give an idea as how such a symmetry involving arbitrary functions can look like. By specifying $\alpha_1 = a_1$ and $\alpha_2 = 2a_1$, and by putting all arbitrary functions in (1.5) to zero, this symmetry reduces to (1.3), that is, $S_1^{(1,1)} \subset S_2^{(1,1)}$ is a subgroup of the symmetry group (1.5).

It is also important to note that although the symmetry (1.5) is consistent from the outset only up to second order in the moments, this result can be made consistent to any order. Because, when augmenting the inviscid TPC system (1.2) by transport equations for the next higher-order moments, one way of ensuring the stability for the second-moment generators of (1.5) is simply to enforce them as a constraint in the symmetry-finding algorithm for the next higher order; similar as to the procedure for the mean velocities, where (1.1) acted as the lower order constraint for the symmetries of the next higher-order quantities $R_{ij}$, $\overline{u_{ij}}$, and $\overline{p_{ij}}$ in (1.2).

Due to an infinite hierarchy of equations, this procedure is realizable at any order, since at each order there always will be enough (unclosed) higher-order moments which can compensate for the given constraints at lower order. And this strategy is independent of whether one augments the inviscid TPC system (1.2) by higher-order transport equations within the two-point correlations directly, or indirectly by first going over to the equations for the three-point correlations and to then take the two-point limit at the end of the performed symmetry analysis.

Although our new symmetry $S_2^{(1,1)}$ (1.5) is more general than the symmetry $S_1^{(1,1)}$ (1.3) proposed in Oberlack et al. (2006), there is no reason to rejoice. The problem is that we are faced with complete arbitrariness in constructing invariant scaling laws for the second order moments from (1.5). Due to the arbitrary and thus unknown functions $f_{ij}$, $g_i$ and $h_i$, any arbitrary scaling law or scaling dependency, in particular in the inhomogeneous direction $x_3$, can be generated now by also showing full compatibility to the velocity constraint (1.1). And in knowing that the new symmetry (1.5) basically only forms a particular subgroup of a functionally much wider group makes the situation in the problem of invariant scaling even more worse.

As if that were not enough, the problem of arbitrariness in invariant scaling not only extends in the direction of higher orders, but also in the opposite direction to lower orders, namely directly down to the initial constraints we pose. In other words, the constraint (1.1) for the mean velocities itself is not unique. Any other constraint can be posed. For example, instead of posing linear functions for both the mean stream- and spanwise velocity according to (1.1), we will now pose a linear function only for the streamwise velocity $\bar{u}_1$, while for the more complex spanwise velocity $\bar{u}_3$ we will pose a quadratic profile which, of course, as a profile with at most three parameters, will match the DNS data better than only a linear profile with at most two parameters. Hence, if we pose instead of (1.1), for example, the following (closed form) constraint for the mean velocities

$$
\bar{u}_1(x_2) = c_{11}x_2 + c_{12}, \quad \bar{u}_3(x_2) = c_{31}x_2^2 + c_{32}x_2 + c_{33},
$$

(1.7)

The consistency to second order is simply due to the fact that the symmetry (1.5) is only being admitted by a second order system (1.2) with unclosed third-order moments $R_{i(jk)}$ and $R_{(ijk)}$, or, respectively, expressed as $T_{ij}^{(3)}$ and $T_{ij}^{(2)}$ via the divergence relation (1.4). For a higher order consistency, additional transport equations for the unclosed higher order moments need to be considered. Hereby it does not matter whether the equations are formulated for the $R$- or for the $T$-quantities, because if the transformation rule for one of these quantities is known then the transformation rule for the other quantity can be straightforwardly reconstructed via (1.4) by just transforming along the coordinates $(x, r)$ of the underlying symmetry.
then the inviscid TPC system \((1.2)\) being consistent up to second order, will admit, for example, the following symmetry group\(^1\)

\[ S_3^{(1.7)} : \xi_{r_1} = b_1, \quad \xi_{r_2} = 0, \quad \xi_{r_3} = b_3, \quad \xi_{\tau_2} = 0, \]

\[ \eta_{R_{11}} = \beta R_{11} + \theta_{11}(x_2, r_2, r_3), \quad \eta_{R_{12}} = \beta R_{12} + \theta_{12}(x_2, r_3), \quad \eta_{R_{13}} = \beta R_{13} + \theta_{13}(x_2, r_2), \]

\[ \eta_{R_{21}} = \beta R_{21} + \theta_{21}(r_3, \tau_3), \quad \eta_{R_{22}} = \beta R_{22} + \theta_{22}(r_1, r_3), \quad \eta_{R_{23}} = \beta R_{23} + \theta_{23}(r_1), \]

\[ \eta_{R_{31}} = \beta R_{31} + \theta_{31}(x_2, r_2), \quad \eta_{R_{32}} = \beta R_{32} + \theta_{32}(x_2, r_1), \quad \eta_{R_{33}} = \beta R_{33} + \theta_{33}(x_2, r_1, r_2), \]

being again only a particular subgroup of a much wider and more general group. Also here, as in symmetry \(S_2^{(1.3)}\) \((1.5)\), we face again the same kind of arbitrariness in invariant scaling for the second-order moments due to the appearance of the unknown and thus arbitrary functions \(\theta_{ij}\), \(\phi_i\) and \(\psi_i\), which again are only restricted by the identity constraints \((1.6)\). To briefly illustrate the action of these constraints, let us consider for example the invariant of the diagonal component \(R_{11}\) under the simplified condition that \(b_3 = 0\), and \(\beta = 0\), for which it then takes the invariant form

\[ R_{11}(x_2, r) = \theta_{11}(x_2, r_2, r_3) \cdot \frac{r_1}{b_1} + \Lambda_{11}(x_2, r_2, r_3), \quad (1.9) \]

where \(\Lambda_{11}\) is an arbitrary integration function. Indeed, a quick check shows that \((1.9)\) stays invariant under the transformation

\[ T_1 : \quad x_2^* = x_2, \quad r_1^* = r_1 + b_1, \quad r_2^* = r_2, \quad r_3^* = r_3, \quad R_{11}^* = R_{11} + \theta_{11}(x_2, r_2, r_3), \quad (1.10) \]

induced by the symmetry generators \(\xi_{x_2} = \xi_{r_2} = \xi_{r_3} = 0, \xi_{r_1} = b_1\) and \(\eta_{R_{11}} = \theta_{11}(x_2, r_2, r_3)\) of \((1.8)\). Now, the only way for \((1.9)\) to satisfy the constraint \(R_{11}(x_2, r) = R_{11}(x_2 + r_2, -r)\) is to restrict the arbitrary functions \(\theta_{11}\) and \(\Lambda_{11}\) to the following adapted but still invariant form\(^1\)

\[ R_{11}(x_2, r) = \widehat{\theta}_{11}(2x_2 + r_2, r_2, r_3) \cdot \frac{r_1}{b_1} + \widehat{\Lambda}_{11}(2x_2 + r_2, r_2, r_3), \quad (1.11) \]

where \(\widehat{\theta}_{11}\) and \(\widehat{\Lambda}_{11}\) have to satisfy the following conditions in their second and third argument:

\[ \widehat{\theta}_{11}(\cdot, -r_2, -r_3) = -\widehat{\theta}_{11}(\cdot, r_2, r_3), \quad \widehat{\Lambda}_{11}(\cdot, -r_2, -r_3) = \widehat{\Lambda}_{11}(\cdot, r_2, r_3). \quad (1.12) \]

If we now turn into the one-point limit \((r \to 0)\), where the two-point correlation \(R_{11}\) reduces to the diagonal Reynolds stress component \(\tau_{11}\) (up to an error \(O(Re^{-1/2})\); see first footnote on p. 2) and where we assume that the free functions \(\theta_{11}\) and \(\Lambda_{11}\) behave smoothly in this limit, then we will obtain, according to \((1.11)\), the following fully arbitrary result

\[ \lim_{r \to 0} R_{11}(x_2, r) = \widehat{\Lambda}_{11}(2x_2) =: \tau_{11}(x_2), \quad (1.13) \]

which gives no information at all as how the Reynolds stress tensor \(\tau_{11}\) should scale if we assume a linear and a quadratic mean velocity profile for \(\bar{u}_1\) and \(\bar{u}_3\), respectively, according to the given constraint \((1.7)\). Any function \(\Lambda_{11}\) within any region of \(x_2\) can thus be chosen such that the numerical or experimental results are matched satisfactorily.

\(^1\)The infinitesimal generators for the unclosed higher-order moments were not listed in \((1.8)\) since they are similar to those listed in \((1.5)\), except for an extra quadratic term in some components. Using the same strategy as just discussed before, also symmetry \((1.8)\) can be made consistent beyond the second order by just keeping the result for the generators up to second order fixed in posing them as a constraint for all higher orders.

\(^1\)The structure of \((1.11)\) is based on the fact that \(2x_2 + r_2\) is an invariant under the transformation \(x_2 \to x_2 + r_2\) and \(r_2 \to -r_2\).
Even if we relax the condition of zero to a non-zero scaling \( \beta \neq 0 \), the problem of arbitrariness remains. Instead of (1.9), the invariant function now takes the form
\[
R_{11}(x_2, r) = -\frac{1}{\beta} \cdot \theta_{11}(x_2, r_2, r_3) + e^{\beta x_1/b_1} \cdot \Gamma_{11}(x_2, r_2, r_3),
\] (1.14)
which indeed stays invariant under the 1-parametric (\( \epsilon \)) group transformation
\[
T_2 : x_2^* = x_2, \quad r_1^* = r_1 + b_1 \epsilon, \quad r_2^* = r_2, \quad r_3^* = r_3, \quad R_{11}^* = e^{\beta \epsilon} R_{11} + \frac{e^{\beta \epsilon} - 1}{\beta} \cdot \theta_{11}(x_2, r_2, r_3),
\] (1.15)
induced by the generators \( \xi_{x_2} = \xi_{r_2} = \xi_{r_3} = 0, \xi_{r_1} = b_1 \) and \( \eta_{R_{11}} = \beta R_{11} + \theta_{11}(x_2, r_2, r_3) \) of symmetry (1.8). Again, for (1.14) to satisfy the constraint \( R_{11}(x_2, r) = R_{11}(x_2 + r_2, -r) \), the arbitrary integration function \( \Gamma_{11} \) has to be turned down to zero, while \( \theta_{11} \) needs to be restricted to \( \hat{\theta}_{11} \) has to be a symmetric function now in its second and third argument:
\[
\hat{\theta}_{11}(\cdot, -r_2, -r_3) = \hat{\theta}_{11}(\cdot, r_2, r_3).
\] (1.17)
Hence, as before in (1.13), the one-point limit of (1.16) leads again to a fully arbitrary result for the invariant Reynolds stress
\[
\lim_{r_3 \to 0} R_{11}(x_2, r) = -\frac{1}{\beta} \cdot \hat{\theta}_{11}(2x_2) =: \tau_{11}(x_2).
\] (1.18)

Coming back to the initial ansatz (1.7), it is clear that for the mean streamwise velocity field \( \bar{u}_1 \) we could have also chosen a different functional dependency than the linear one proposed in Oberlack et al. (2006). Obviously, scaling the corresponding DNS results [Fig. 3, p. 393] by eye it is reasonable to assume a linear scaling law for \( \bar{u}_1 \) in the range \( x_2 \sim 0.2-0.6 \) for \( Ro > 2.5 \). But a systematic group analysis does not uniquely predict this behavior. It can also be a weak (non-linear) power law, which, for example, could serve as an alternative constraint for \( \bar{u}_1 \) in (1.7). Analytically with group theory alone, it is not possible to tell how the mean velocity profile \( \bar{u}_1 \) really scales. In particular, it is not clear at all how the scaling behavior will change with ever increasing rotation rates. Maybe the “linear scaling” weakens and gets less obvious by eye.

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**Conclusion**

As a result of this section we can conclude that a Lie-group symmetry analysis on the unclosed TPC equation (1.2) cannot analytically predict its scaling behavior \textit{a priori}. For that, modelling procedures and exogenous information from numerical simulations or physical experiments are needed to get further insights. Ultimately this just reflects the closure problem of turbulence, which, as we have clearly demonstrated in this section, cannot be solved or bypassed by the method of Lie-groups alone, as misleadingly and continually claimed by Oberlack et al. also again in their latest contribution Oberlack & Rosteck (2016).

Neither by augmenting the unclosed TPC equation (1.2) with transport equations for its higher-order moments, nor by including the three-point correlation equations, the problem of arbitrary scaling cannot be circumvented. Every systematic Lie-group symmetry analysis will always lead to a sufficient number of free functions such that for every turbulent flow quantity any kind of invariant scaling law can be generated, which, within a trial and error procedure, can be always chosen such as to fit any numerical or experimental data adequately.

\(^1\)The restriction from (1.14) to (1.16) did not change the considered invariance property, that is, function (1.16) is still invariant under the considered transformation (1.15).
Also when considering the strategy as initially proposed in Oberlack & Rosteck (2010) and then as later applied in Oberlack & Zieleniewicz (2013), Avsarkisov et al. (2014), Wachawczyk et al. (2014) and Oberlack et al. (2015), namely to formally consider the infinite hierarchy of multi-point equations, the problem of arbitrariness in invariant scaling remains, independent of whether the pure fluctuating $R$- or the instantaneous $H$-approach in Oberlack & Rosteck (2010) is used. The key problem here is that the infinite multi-point system is a forward recursive hierarchy (Frewe, 2015 $a,b$), in which the (unclosed) higher-order $n$-point correlations can only be obtained by the next higher $(n+1)$-point correlation equation, but which by itself is again unclosed, rendering thus the infinite hierarchy to an unclosed system admitting arbitrary symmetries, because as with each higher order new arbitrary functions in the symmetry finding process will appear. Particularly in the instantaneous $H$-approach the closure problem can be experienced directly when performing a symmetry analysis on the infinite hierarchy of multi-point equations. Because, since the hierarchy in this representation is linear, it naturally admits the symmetry of linear superposition, thus giving raise to a symmetry which is unclosed per se: Any solution solving the underlying (unclosed) system of multi-point equations up to a certain order can be added or superposed to an already given symmetry to obtain a new symmetry. Incrementally one can therefore now improve the symmetry for any turbulent flow quantity such as to obtain an invariant scaling law that will adequately fit the data. Hence, the systematic Lie-group symmetry approach degenerates down to a non-predictive incremental trial-and-error method. For more details on this issue, see e.g. the instructive example in Khujadze & Frewe (2016).

Hence, in this sense Lie-group theory offers no answer, nor does it give any prediction a priori in how turbulence should scale. As already said, this failure simply reflects the classical closure problem of turbulence, which, also with the powerful and appealing Lie-group symmetry method, cannot be solved or bypassed analytically. However, using this method to nevertheless systematically generate such invariant scaling laws would be the same as guessing it, and if one knows what to expect a posteriori then, of course, one can manually arrange everything backwards and pretend that theory is predicting these results. But such an approach has nothing to do with science (Frewe et al., 2014 $b$).

2. List of inconsistencies and incorrect statements in Oberlack et al. (2006)

This section will reveal all inconsistent and incorrect information that can be found in Sec. 2.3 [pp. 388-392] of Oberlack et al. (2006). They will be listed and discussed in the order as they appear in the text.

$(1)$: It is claimed that “for physical reasons the translation invariance of $r_{ij}$ is not meaningful”, with the argument that “since $R_{ij}$ reaches its finite maximum at $|\vec{r}| = 0$ and tends to zero for $|\vec{r}| \to \infty$, a shift in the correlation space cannot be a new solution” [p. 390]. If one would strictly follow this argument, then consequently it also has to apply to the translation invariance of $x_2$. Because, since $\tilde{R}_{ij}$ has one of its minima always at $\tilde{x}_2 = \pm 1$ due to the no-slip condition at the walls, a shift in wall-normal direction thus cannot be a new solution, too. Hence, according to Oberlack et al. (2006) also the translation invariance of $\tilde{x}_2$ should not be physically meaningful, and thus in the same way as the translation parameters $a_2$-$a_4$ for the correlations lengths $\tilde{r}_i$ were put to zero, so has the translation parameter $a_5$ in the wall-normal $\tilde{x}_2$-direction be put to zero, if one would strictly follow the reasoning in Oberlack et al. (2006). However, when putting $a_5 = 0$ has a significant negative effect on the results obtained for the similarity variables $\eta_i$ [Eq. (2.23a)]. Because, in or near the channel center plane $\tilde{x}_2 = 0$ we would have infinitely large values for all three independent similarity variables $\eta_i$, which again would yield the unphysical result of zero correlations in that region, since by definition $\tilde{R}_{ij}$ tends to zero for $|\eta| \to \infty$.

$^1$To note is that in particular the later studies Oberlack & Rosteck (2010), Oberlack & Zieleniewicz (2013), Wachawczyk et al. (2014) and Avsarkisov et al. (2014) also suffer from the additional problem that new unphysical symmetries are generated, which in turn violate the classical principle of cause and effect. For more details, we refer to our other comments and reviews, Frewe et al. (2014a, 2015a, 2016a,$b$); Frewe & Khujadze (2016a); Khujadze & Frewe (2016), and to our reactions in Frewe (2015$c$); Frewe et al. (2015$b$, 2016$c$); Frewe & Khujadze (2016$b$).
Figure 1: DNS result for the mean streamwise velocity field at $Re_\tau = 180$ for different rotation numbers $Ro = 0, 2.5, 6.5$ and 10 (from top to bottom). The numerical simulation was repeated with the same code (Lundbladh et al., 1992) for the same parameters, domain sizes and resolution as chosen in Oberlack et al. (2006). The left plot exactly corresponds to Fig. 3 [p. 393] in Oberlack et al. (2006) and displays the velocity profile normalized on $u_\tau$, which for all of the considered rotation rates takes the same unchanged value $u_\tau \sim 0.011$ at $Re_\tau = 180$. The right plot displays the dimensionalized or non-normalized velocity profile including the best fit of the invariant linear scaling law $\bar{u}_1 = m \cdot x_2 + b$ according to (2.1). Already with a normal sense of proportion one can see that the proposed linear scaling cannot meet the conditions as claimed in Oberlack et al. (2006) for the range $|x_2| = 0.2-0.6$: The slope $m$ does not proportionally scale with the rotation rate $Ro \sim \Omega$, but rather stays invariant at a value $|m| = 0.042$, while the constant $b$ contrarily exhibits a strong dependency on $\Omega$: Increasing the rotation rate from $Ro = 6.5$ to 10, the constant already decreases by more than 5%, from $b = 0.192$ down to 0.181.

(2): Since Eqs. (2.21a)-(2.21b) result from Eq. (2.14) which by itself does not show any explicit dependence on the rotation parameter $\Omega$, it is correct that all group and integration constants appearing in Eqs. (2.21a)-(2.21b) do not depend on $\Omega$. The dependence on the rotation rate only enters when transforming this result back to its original variables according to Eqs. (2.13a)-(2.13c). Along with the claim that “the function $\gamma$ behaves as $\gamma \sim 1/\Omega$” [p. 390], this transformation then yields the invariant result [Eqs. (2.22a)-(2.22b)]

$$\bar{u}_1 \sim C_1 \Omega x_2 + C_1 a_5/a_1 - c_1/a_1, \quad \bar{u}_3 \sim C_3 \Omega x_2 + C_3 a_5/a_1 - c_3/a_1, \quad (2.1)$$

with the conclusion that “only the slope of the linear scaling laws depends on the rotation rate” [p. 390]. But such a parametric scaling is inconsistent to the DNS results shown in Fig. 3 in Oberlack et al. (2006). Because, when fitting a linear law $\bar{u}_1 = m \cdot x_2 + b$ for the streamwise velocity field to the DNS data, we obtain, as can be seen from Figure 1, a contrary result to (2.1): (i) For the lowest chosen rotation rate $Ro = 2.5$ no convincing linear scaling over a longer range can be detected. (ii) When increasing the rotation rate from $Ro = 6.5$ to 10, the slope $m$ of the linear law does not proportionally increase along as prescribed by (2.1); instead it rather remains constant. (iii) The claim that “the two additive constants appearing in the scaling laws (2.22a) and (2.22b) do not depend on $\Omega$” [p. 390] cannot be confirmed; the DNS results in Figure 1 clearly show the opposite, namely a fairly strong dependence of $b = B_1$ on the rotation rate $\Omega$.

Important to note here is that for Figure 1 we have rerun the DNS with the same code (Lundbladh et al., 1992) for the same parameters, domain sizes and resolution as chosen in Oberlack et al. (2006), with the only aim to check the internal consistency of its part on group-analysis. We did not check the consistency of the DNS results themselves as presented in Oberlack et al. (2006), which would be a study in itself, in particular as these results were criticized by Recktenwald et al. (2009) as not being reliable due to the prescription of periodic boundary conditions on a too small computational domain in the spanwise direction.
The result Eq. (2.23b) for the two-point velocity correlation $\tilde{R}_{ij}$ is not consistent with the underlying one-point momentum equation equation Eq. (2.1a) in the limit of large Reynolds numbers. Because, when taking the one-point limit of $\tilde{R}_{ij}$ [Eq. (2.23b)]

$$\lim_{r \to 0} \tilde{R}_{ij}(\bar{x}_2, \bar{r}) = \lim_{r \to 0} F_{ij}(\eta) \cdot (\bar{x}_2 + a_5/a_1)^2$$

$$= \tilde{\gamma}_{ij}(\bar{x}_2) + \mathcal{O}(Re^{-1/2}) = F_{ij}(0) \cdot (\bar{x}_2 + a_5/a_1)^2, \quad (2.2)$$

we obtain a quadratic law for all Reynolds stresses (up to an error $\mathcal{O}(Re^{-1/2})$, however which becomes negligibly small in the limit of large Reynolds numbers; see first footnote on p. 2):

$$\tau_{ij}(x_2) = \Omega^2 \cdot C_{ij} \cdot \left( x_2 + \gamma(\Omega) \cdot a_5/a_1 \right)^2, \quad (2.3)$$

where $C_{ij} = F_{ij}(0)$ is the arbitrary integration constant. However, such a law does not solve the inviscid ($\nu = 0$) momentum equation Eq. (2.1a)

$$0 = K - \frac{d\tau_{ij}}{dx_2}, \quad (2.4)$$

when taking the key assumption of Oberlack et al. (2006) that (2.3) is valid “in regions sufficiently far from solid walls, [where] the viscous terms may be neglected to leading order” [p. 385]. The quantity $K$ denotes the constant mean streamwise pressure gradient $K \sim -\partial p/\partial x_1$ that drives the flow. As also can be clearly seen from the DNS result in Fig. 6 [p. 395], the shear stress $\tau_{12} = \bar{u}_1\bar{u}_2$ follows a linear law away from the walls consistent with equation (2.4), and not according to the quadratic law (2.3) as incorrectly proposed in Oberlack et al. (2006).

The result Eq. (2.26) [p. 391] is incorrect. A substantial factor is missing, which, when included, invalidates the conclusion in Oberlack et al. (2006) that “relation (2.26) gives raise to a new symmetry transformation” [p. 391]. Also the claim that the validity of this symmetry transformation [Eq. (2.27)] “can be verified by substituting (2.27) into (2.14) after the similarity coordinate (2.25) and the linear profiles (2.21a) and (2.21b) have been employed” [p. 391], cannot be confirmed, neither with the incorrect relation as given by Eq. (2.26), nor with the correct relation (2.6), which will be derived now: Using the invariant result of $\tilde{R}_{ij}$ [Eq. (2.23b)], the identity $\tilde{R}_{ij}(\bar{x}_2, \bar{r}) = \tilde{R}_{ij}(\bar{x}_2 + \bar{r}, -\bar{r})$ [Eq. (2.6)] and the abbreviation $\tilde{x}_2 = \bar{x}_2 + a_5/a_1$ [Eq. (2.24)], then the correct derivation of relation Eq. (2.26) reads

$$\tilde{R}_{ij}(\bar{x}_2, \bar{r}) = F_{ij}(\eta) \cdot \left( \frac{\bar{x}_2 + a_5}{a_1} \right)^2$$

$$= F_{ij} \left( \frac{\bar{r}}{\bar{x}_2} \right) \cdot \bar{x}_2^2 = F_{ji} \left( \frac{-\bar{r}}{\bar{x}_2 + \bar{r}_2} \right) \cdot \left( \bar{x}_2' + \bar{r}_2 \right)^2$$

$$= F_{ji} \left( \frac{-\bar{r}}{\bar{x}_2' + \left(1 + \bar{r}_2/\bar{x}_2' \right)} \right) \cdot \bar{x}_2'^2 \cdot \left(1 + \bar{r}_2/\bar{x}_2' \right)^2$$

$$= F_{ji} \left( -\frac{\eta}{1 + \eta_2} \right) \cdot \bar{x}_2'^2 \cdot (1 + \eta_2)^2, \quad (2.5)$$

which then, since $\tilde{R}_{ij}(\bar{x}_2, \bar{r}) = F_{ij}(\eta) \cdot \bar{x}_2'^2$ [Eq. (2.23b)], finally leads to the different result

$$F_{ij}(\eta) = F_{ji} \left( -\frac{\eta}{1 + \eta_2} \right) \cdot (1 + \eta_2)^2. \quad (2.6)$$

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