

PERFECT CRYSTALS OF $U_q(G_2^{(1)})$

Shigenori Yamane

Department of Mathematical Science, Faculty of Engineering Science, Osaka University, Toyonaka, Osaka 560, Japan

yamane@sigmath.es.osaka-u.ac.jp

1 Introduction

The theory of crystal base was introduced by Kashiwara in \cite{K}. He proved the existence and uniqueness of this base for a quantized universal enveloping algebra $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is an arbitrary Kac-Moody Lie algebra with symmetrizable Cartan matrix. The theory of crystal base provides a powerful combinatorial tool for studying the structure of the integrable highest weight modules and the decomposition of tensor products of them.

Let $\mathfrak{g}$ be an arbitrary Kac-Moody Lie algebra with a symmetrizable Cartan matrix. Let $I$ be its index set. Denote the set of simple roots of $\mathfrak{g}$ by $\{\alpha_i \mid i \in I\}$. Let $P$ be the weight lattice of $\mathfrak{g}$. Take an integrable $U_q(\mathfrak{g})$-module $M$ and let $\hat{f}_i, \hat{e}_i$ be the Kashiwara operators (cf. \cite{K}) on $M$. Let $A$ be the subring of $\mathbb{Q}(q)$ consisting of $f \in \mathbb{Q}(q)$ that is regular at $q = 0$. A crystal lattice $L$ of an integrable $U_q(\mathfrak{g})$-module $M$ is free $A$-submodule of $M$ such that $M \cong \mathbb{Q}(q) \otimes_A L$, $L = \bigoplus_{\lambda \in P} L_\lambda$, where $L_\lambda = L \cap M_\lambda$ and $\hat{e}_i L \subset L$, $\hat{f}_i L \subset L$. A crystal base of the integrable $U_q(\mathfrak{g})$-module $M$ is a pair $(L, B)$ such that (i) $L$ is a crystal lattice of $M$, (ii) $B$ is a $Q$-base of $L/qL$, (iii) $B = \bigcup_{\lambda \in P} B_\lambda$, where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$, (iv) $\hat{e}_i B \subset B \cup \{0\}$, and (v) for $b, b' \in B$, $b' = \hat{f}_i b$ if and only if $b = \hat{e}_i b'$ for $i \in I$. We sometimes replace condition (ii) by: $B_{ps} = B' \cup (-B')$ where $B'$ is a $Q$-base of $L/qL$. In this case, we call $(L, B_{ps})$ a crystal pseudo base. The quotient $B_{ps}/\{\pm 1\}$ is called an associated crystal of $(L, B_{ps})$.

In \cite{K}, Kashiwara and Nakashima gave an explicit construction of crystal bases for all finite dimensional irreducible modules over $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is of type $A_n, B_n, C_n, D_n$, in terms of generalized Young tableaux. Kang and Misra gave a construction of crystal bases of $U_q(G_2)$-module by a method similar to Kashiwara and Nakashima’s in \cite{KM}.

The notion of perfect crystals was introduced in \cite{K} and \cite{KN} in order to compute the one-point functions of vertex model in 2 dimensional lattice statistical models. If a $U_q(\mathfrak{g})$-module $M$ has perfect crystal base, we have the notion of paths. Using paths, we can compute one-point functions of vertex models. We define classical part of the weight lattice by $P$. We have the notion of paths. Using paths, we can compute one-point functions of vertex models. We define classical part of the weight lattice by $P$. We have the notion of paths. Using paths, we can compute one-point functions of vertex models. We define classical part of the weight lattice by $P$. We have the notion of paths. Using paths, we can compute one-point functions of vertex models.

In \cite{KN}, they give examples of perfect crystals of arbitrary levels for algebras of the following types: $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$, $A_2^{(1)}$, $A_2^{(2)}$, $D_{n+1}^{(2)}$. Up to now we do not know simple criterion for perfectness of crystals. Further, perfect crystals with arbitrary level for $E_6^{(1)}$, $F_4^{(1)}$, $G_2^{(1)}$, $E_8^{(2)}$ are not known. In this paper, we give a series of perfect crystals of $U_q(G_2^{(1)})$, following method in \cite{KN}.

Let $U_q'(G_2^{(1)})$ be the subalgebra of $U_q(G_2^{(1)})$, generated by $e_i, f_i, q^h$ ($h \in P_\mathbb{Z} = \sum_{i \in \{0,1,2\}} \mathbb{Z}h_i$). We proceed in the following way.

1. As in \cite{KN}, for a given level $l$, our strategy is to employ the fusion procedure in order to construct finite dimensional irreducible module $V_l$ of $U_q'(G_2^{(1)})$ with a crystal pseudo base such that its associated crystal is perfect of level $l$. As a starting point, we take the following data. Let $V$ be the direct sum of the 14-dimensional module with highest weight $\Lambda_1$ and the trivial module over $U_q(G_2)$. The module $V$ has a crystal base which is characterized by a polarization on $V$. A polarization on $U_q(\mathfrak{g})$-module $M$
is the symmetric bilinear such that for $u, v \in M$ (i) $(q^b u, v) = (u, q^b v)$, (ii) $(e_i u, v) = (u, q_i^{-1} t_i^{-1} f_i v)$, (iii) $(f_i u, v) = (u, q_i^{-1} t_i e_i v)$ and (iv) positive definite. Let $B^1$ be the associated crystal of $U_q(G_2)$. We define the actions of $e_0$, $f_0$, and $q^{b_0}$ on $V$ to make it an irreducible module for $U_q'(G_2^{(1)})$. Then we show that there exists a polarization for $U_q'(G_2^{(1)})$. We see that $B^1$ is a perfect crystal of level 1 by direct calculation. Using the global base $\mathcal{B}$, we compute the R-matrix for $V$ explicitly. By the fusion construction, we obtain a certain finite-dimensional submodule $V_I$ of $V^\otimes l$. The existence of polarization on $V_I$ ensures that the submodule $V_l$ has a crystal pseudo base. The associated crystal of $V_l$ is isomorphic to a crystal of $U_q(G_2)$ and a crystal of $U_q(A_2)$ as crystals of $U_q(G_2)$ and $U_q(A_2)$ respectively. These isomorphisms show that the associated crystal is a crystal of $U_q'(G_2^{(1)})$. It is a perfect crystal of level $l$. This is shown in the section $\mathcal{B}$.

2. In section $\mathcal{B}$, we construct a perfect crystal $B^l$ of $U_q(G_2^{(1)})$ in a combinatorial way.

The Dynkin diagram of $G_2^{(1)}$ is $\begin{array}{c} 0 \bigoplus \begin{array}{ccc} & 1 \end{array} \bigoplus \begin{array}{ccc} & 2 \end{array} \end{array}$. Here, let $J_0 = \{1, 2\}$ and $J_2 = \{1, 0\}$ be the index set of $G_2$ and $A_2$, respectively. We define $i_t : J_t \to I$ by $i_t(j) = j (i = 0, 2)$. We define crystal $G^l$ to be

$$G^l = \bigoplus_{n=0}^{l} B^{G^2}(n \Lambda_1) \text{ (as crystals for } U_q(G_2)).$$

The affine crystal $b'$ for $U_q(G_2^{(1)})$ is constructed with $G^l$ and $\tilde{f}_0$. To define $\tilde{f}_0$ on $G^l$, we use crystals of $U_q(A_2)$. Here, we define $\mathcal{A}$ by

$$\mathcal{A}_i = \bigoplus_{i = 0}^{\lfloor \frac{l}{2} \rfloor} B^{A_2}(j_1 \Lambda_1 + j_0 \Lambda_0) \text{ (as crystals for } U_q(A_2)),$$

$$\mathcal{A} = \bigoplus_{i = 0}^{\lfloor \frac{l}{2} \rfloor} \mathcal{A}_i.$$ 

We define operators $E_A, F_A$ on $\mathcal{A}$ such that for $b, b' \in \mathcal{A}$, (C1) $E_A b = b'$, if and only if $F_A b' = b$, (C2) $E_A F_A b = \tilde{f}_0^A E_A b$, (C3) $\max \{m \mid F_A b = 0\} - \max \{m' \mid E_A b = 0\} = -2w_{A_1}(b) - w_{A_2}(b)$. Operators $E_A$ and $F_A$ are counterpart of $e_{i_1}^{G_2}, f_{i_1}^{G_2}$ on $G^l$. The crystal base $\mathcal{A}$ has an involution $C_A$ such that

$$C_A \left( (f_0^A)^r \left( \tilde{f}_0^A \right)^q \left( \tilde{f}_0^A \right)^{w_{i_1}^{(r)}} \tilde{b}_{(j,k)}^{(1)} \right) = \left( f_0^A \right)^{r+q-2p-r} \left( \tilde{f}_0^A \right)^{k+j-q} \left( \tilde{f}_0^A \right)^{k+q+p} \tilde{b}_{(j,k)}^{(1)},$$

where $0 \leq i \leq \lfloor \frac{l}{2} \rfloor$, $i \leq k, j \leq l-i$, $\tilde{b}_{(j,k)}^{(1)}$ is the highest weight element on $\mathcal{A}_i$ with weight $i \Lambda_1 + k \Lambda_0$, $0 \leq p \leq j$, $p \leq q \leq p + k$, $0 \leq r \leq j + q - 2p$. By calculation, we see $z \mathcal{A} = z G^l$. In view of this, we construct one-to-one map $\Phi : \mathcal{A} \to G^l$ such that for $b \in \mathcal{A}$ (E1) $e_i^{G_2} \Phi(b) = \Phi (e_i^G b)$, (E2) $\tilde{f}_0^2 \Phi(b) = \Phi (F_A b)$, (E3) $w_{A_1} \Phi(b) = w_{A_1}^G(b)$, (E4) $w_{A_2} \Phi(b) = -2w_{A_1}(b) - w_{A_2}(b)$, (E5) $e_0^A \Phi(b) = 0$ (resp. $f_0^A \Phi(b) = 0$) if and only if $\tilde{e}_0^G \Phi(b) = 0$ (resp. $\tilde{f}_0^G \Phi(b) = 0$), for $b \in \mathcal{A}$. Then we define $\tilde{f}_0$ on $G^l$ by $\Phi(f_0^A \Phi^{-1})$. By weight consideration, (C2), involution, etc., we see that the action of $\tilde{f}_0$ is unique. Therefore we have crystal base of $U_q(G_2^{(1)})$.

3. We show that perfectness are satisfied by direct calculation ($\mathcal{B}$, $\mathcal{A}$, $\mathcal{B}$, $\mathcal{A}$).

Graphs of the perfect crystal of level 1 and level 2 are given by Figure $\mathcal{B}$ and $\mathcal{A}$ respectively. In Figure $\mathcal{B}$ for simplicity, we omit arrows of $E_A(b)$ such that $e_0^A(b) \neq 0$. By definition, we have $B^2 = B^{G_2}(2 \Lambda_1) \oplus B^{G_2}(A_1) \oplus B^{G_2}(\phi)$ and $\mathcal{A}_0 = \oplus_{0 \leq j, 0 \leq k} B^{A_2}(j_1 \Lambda_1 + j_0 \Lambda_0), \mathcal{A}_1 = B^{A_2}(A_1 + \Lambda_0), \mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$. In this graph, the element $b \in B^2$ from which we draw arrows of $E_A(b)$ satisfies $\Phi(b) \in B^{G_2}(2 \Lambda_1), \Phi \left( \tilde{f}_0^A b \right) \notin B^{G_2}(A_1)$ and $\tilde{e}_0^G b = 0$. For $b \in \mathcal{A}$ such that $\Phi(b) \in B^{G_2}(A_1) \oplus B^{G_2}(\phi)$, the action of $E_A$ is same as the case of level 1. Using commutativity (C2), we obtain the completed graph of the perfect crystal of level 2. We obtain the perfect crystal $B^1$ of level 1 by restricting $B^2$ to $B^{G_2}(A_1) \oplus B^{G_2}(\phi)$. From these examples, we expect that we can obtain the perfect crystal $B^{l+1}$ by extending $B^l$.  

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To find perfect crystals of level $l$, we wrote a C program which judges whether a given representation has perfect crystal on a combinatorial level. In this program, we construct crystal bases $B$ using a Lakshmibai-Seshadri path $[1]$. A Lakshmibai-Seshadri path is a piecewise linear path $\pi : [0,1] \to \bigoplus Z\Lambda_i \otimes_Z \mathbb{R}$. Operators $e_\alpha$ and $f_\alpha$ on the path are modification by simple reflection $\sigma$ with respect to the simple root $\alpha$. When translated into the language of crystal base, a path $\pi$ is a base and operators $e_\alpha$ and $f_\alpha$ are Kashiwara operators $\tilde{e}_\alpha$ and $\tilde{f}_\alpha$. Let $\delta = \sum a_i \alpha_i$ be the null root. We define the operator $\tilde{f}_0 b (b \in B)$, by searching bases with weight $\text{wt}(b) + \sum_{i \neq 0} a_i \alpha_i$ and taking into account the conditions for perfectness. The conditions are $\langle c, \varphi(b) \rangle \geq l$, and the map $\varphi : B_l = \{ b \in B \mid \langle c, \varphi(b) \rangle = l \} \to (P^+_{cl})_l = \{ \lambda \in \bigoplus Z_{\geq 0} \Lambda_i \mid \langle c, \lambda \rangle = l \}$ are bijective ($c$ is the element of the center). If we find suitable action of $\tilde{f}_0$, $B$ is perfect crystal on a combinatorial level.

Using this program we find that $\bigoplus_{k=0}^l B(k \Lambda_1)$ is a candidate of perfect crystals of level $l$ also for $U_q(F^{(1)}_4)$, $U_q(E^{(1)}_6)$, etc, on a combinatorial level. We hope to report on these in a future publication.

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![Figure 1: Perfect crystal of level 1](image)

2 Results

Denote the index set of the simple roots of $G^{(1)}_2$ by $I = \{0, 1, 2\}$. We can take $J_0 = \{1, 2\}$ and $J_2 = \{1, 0\}$ as the index set of $G_2$ and $A_2$, respectively. We define $i_i : J_i \to I$ by $i_i(j) = j$ ($i = 0, 2$).

Proposition 2.1 For any integer $l \geq 1$, there exists the crystal $B^l$ for $G^{(1)}_2$ such that

$$i_0^*(B^l) \cong \bigoplus_{k=0}^l B^{G_2}(k \Lambda_1) \text{ (as crystals of } U_q(G_2)),$$
and
\[ t^*_i(B') \cong \bigoplus_{i=0}^l \bigoplus_{i+j_0 \leq l-i} B^{A_2}(j_1\Lambda_1 + j_0\Lambda_0) \text{ (as crystals of } U_q(A_2)). \]

**Theorem 2.2** The crystal \( B^l \) is a perfect crystal of level \( l \) for \( U_q(G_2^{(1)}) \).

### 3 Preliminaries

#### 3.1 Crystal base

We recall definitions of quantum enveloping algebras \( U_q(A_2), U_q(G_2), U_q(G_2^{(1)}) \) and crystal bases of \( U_q(A_2), U_q(G_2) \).

**3.1.1 Definition of \( U_q(\mathfrak{g}) \)**

Let \( \mathfrak{g} \) be a semi simple (resp. affine) Lie algebra generated by \( e_i, f_i \) (\( i \in \{1, \ldots, n\} \) (resp. \( \{0, 1, \ldots, n\} \)) and \( \mathfrak{h} \) the Cartan subalgebra over \( \mathbb{Q} \). Let \( \{\alpha_i \mid i \in I\} \subset \mathfrak{h}^* \) and \( \{h_i \mid i \in I\} \subset \mathfrak{h} \) denote the simple roots and simple coroots, respectively. We denote a non-degenerate invariant symmetric bilinear form on \( \mathfrak{h}^* \) by \( \langle, \rangle \). This bilinear form \( \langle, \rangle \) satisfies \( (\alpha_i, \alpha_i) \in \mathbb{Z}_{>0} \). Let \( P \) be the weight lattice and \( P^\ast \) be the dual lattice.

The quantized universal enveloping algebra \( U_q(\mathfrak{g}) \) is the \( \mathbb{Q}(q) \)-algebra generated by the symbols \( e_i, f_i \) (\( i \in I \)) and \( q^h \) (\( h \in P^\ast \)) with the following defining relations:

- \( q^0 = 1, q^h q^{h'} = q^{h+h'} \) for all \( h, h' \in P^\ast \),
- \( q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i, q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i \) for all \( i, h \in P^\ast \),
- \[ [e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}, \text{ where } q_i = q^{\alpha_i, \alpha_i} \text{ and } t_i = q^{\langle h_i, \alpha_i \rangle}, \]
- \[ \sum_{k=0}^b (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = 0 \text{ for } i \neq j \text{ and } b = 1 - \langle h_i, \alpha_j \rangle. \]
- \[ \sum_{k=0}^b (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0 \text{ for } i \neq j \text{ and } b = 1 - \langle h_i, \alpha_j \rangle. \]

Here we use the following notations: \( [m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}}, [k]_i = \prod_{m=1}^k [m]_i, \) and \( e_i^{(k)} = e_i^k/[k]_i!, f_i^{(k)} = f_i^k/[k]_i! \).

For \( k < 0 \), we put \( e_i^{(k)} = f_i^{(k)} = 0 \).

It is well-known that \( U_q(\mathfrak{g}) \) has a Hopf algebra structure with a comultiplication \( \Delta \) defined by

- \( \Delta(e_i) = e_i \otimes t_i^{-1} + 1 \otimes e_i \),
- \( \Delta(f_i) = f_i \otimes 1 + t_i \otimes f_i \),
- \( \Delta(q^h) = q^h \otimes q^h \).

for all \( i \in I, h \in P^\ast \). The tensor product of two \( U_q(\mathfrak{g}) \)-modules has a structure of \( U_q(\mathfrak{g}) \)-module by this comultiplication.
3.1.2 Definition of crystal bases[3]

For $U_q(\mathfrak{g})$-module $M$ and $\lambda \in P$, the $\lambda$-weight space of $M$ is defined by $M_\lambda = \{ u \in M \mid q^h u = q^{(h, \lambda)} u \text{ for all } h \in P^+ \}$. For $J \subset I$, let $U_q(\mathfrak{g}_J)$ be the subalgebra of $U_q(\mathfrak{g})$ generated by $e_i, f_i, t_i$ and $t_i^{-1}$ ($i \in J$). We say that $M$ is integrable if $M = \bigoplus_{\lambda \in P} M_\lambda$ and $M$ is a union of finite-dimensional $U_q(\mathfrak{g}_J)$-modules for each $i \in I$. By the representation theory of $U_q(\mathfrak{sl}_2)$ any element $u \in M_\lambda$ can be uniquely written as

$$u = \sum_{k \geq 0} f_i^{(k)} u_k,$$

where $u_k \in \ker e_i \cap M_{\lambda + k\alpha_i}$. We define the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $M$ as follows.

$$\tilde{e}_i u = \sum f_i^{(k-1)} u_k,$$

$$\tilde{f}_i u = \sum f_i^{(k+1)} u_k.$$

Let $A$ be the subring of $\mathbb{Q}(q)$ consisting of the rational functions regular at $q = 0$. A crystal lattice of integrable $U_q(\mathfrak{g})$-module $M$ is a free $A$-submodule of $M$ such that (1) $M \cong \mathbb{Q}(q) \otimes_A L$, (2) $L = \bigoplus_{\lambda \in P} L_\lambda$ where $L_\lambda = L \cap M_\lambda$, (3) $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$. A crystal base of the integrable $U_q(\mathfrak{g})$-module $M$ is a pair $(L, B)$ such that

(i) $L$ is a crystal lattice of $M$,

(ii) $B$ is a $\mathbb{Q}$-basis of $L/\mathbb{Q}L$,

(iii) $B = \bigcup_{\lambda \in P} B_\lambda$ where $B_\lambda = B \cap (L_\lambda/qL_\lambda)$,

(iv) $\tilde{e}_i B \subset B \cup \{ 0 \}$, $\tilde{f}_i B \subset B \cup \{ 0 \}$,

(v) for $b, b' \in B$, $b' \in \tilde{f}_i b$ if and only if $b = \tilde{e}_i b'$ for $i \in I$.

We sometimes replace (ii) by: $B_{ps} = B' \cup (-B')$ where $B'$ is a $\mathbb{Q}$-base of $L/\mathbb{Q}L$. We call $(L, B_{ps})$ a crystal pseudo-base and $B_{ps}/\{ \pm 1 \}$ the associated crystal of $(L, B_{ps})$.

For a dominant integrable weight $\lambda \in P_+ = \{ \lambda \in P \mid \langle h_i, \lambda \rangle \leq 0 \text{ for all } i \}$, let $V(\lambda)$ denote the irreducible integrable $U_q(\mathfrak{g})$-module with highest weight $\lambda$. Let $u_\lambda$ be the highest weight vector of $V(\lambda)$, and let $L(\lambda)$ be the smallest $A$-submodule of $V(\lambda)$ containing $u_\lambda$ and stable under $\tilde{f}_i$’s. Set $B(\lambda) = \{ b \in L(\lambda)/qL(\lambda) \mid b = \tilde{f}_i \cdots \tilde{f}_i u_\lambda \mod qL(\lambda) \} \setminus \{ 0 \}$. Then $(L(\lambda), B(\lambda))$ is crystal base of $V(\lambda)$. Crystal graph is an oriented colored (by 1) graph with $B(\lambda)$ as the set of vertices and colored arrow $b \overset{i}{\rightarrow} b'$ if and only if $b' = \tilde{f}_i b$ (hence $\tilde{e}_i b' = b$). Then graph completely describes the actions of $\tilde{e}_i$ and $\tilde{f}_i$ on $B(\lambda)$. For $b \in B(\lambda)$, we set

$$\varepsilon_i(b) = \max \{ k \geq 0 \mid \tilde{e}_i^k b \neq 0 \} ,$$

$$\varphi_i(b) = \max \{ k \geq 0 \mid \tilde{f}_i^k b \neq 0 \} .$$

Note that $B(\lambda) = \bigcup_{\mu \in P} B(\lambda)_\mu$ and for $b \in B(\lambda)_\mu$, we have $(h_i, \text{wt}(b)) = \varphi_i(b) - \varepsilon_i(b)$, where $\text{wt}(b) = \mu$ denotes the weight of $b$.

Proposition 3.1 Let $(L_j, B_j)$ be crystal bases of integrable $U_q(\mathfrak{g})$-modules $M_j$ ($j = 1, 2$). Set $L = L_1 \otimes_A L_2 \subset M_1 \otimes M_2$ and $B = \{ b_1 \otimes b_2 \mid b_j \in B_j (j = 1, 2) \} \subset L/\mathbb{Q}L$. Then $(L, B)$ is the crystal base of $M_1 \otimes M_2$ and the action of $\tilde{f}_i, \tilde{e}_i$ is given by

$$\tilde{f}_i (b_1 \otimes b_2) = \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases}$$

$$\tilde{e}_i (b_1 \otimes b_2) = \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2). \end{cases}$$
Corollary 3.2 For $b_j \in B_j$ ($j = 1, 2$), we have
\[
\varepsilon_i(b_1 \otimes b_2) = \max(\varepsilon_i(b_1) + \varepsilon_i(b_1) - \varphi_i(b_1)),
\]
\[
\varphi_i(b_1 \otimes b_2) = \max(\varphi_i(b_2) + \varphi_i(b_2) - \varepsilon_i(b_2)).
\]

Proposition 3.3 Let $(L_j, B_j)$ be crystal bases of integrable $U_q(g)$-modules $M_j$ ($j = 1, \ldots, N$), and let \{\(a_k\)\} be the sequence defined by $a_1 = 0$, $a_{k+1} = a_k + \varphi_i(b_k) - \varepsilon_i(b_{k+1})$. Then we have
\[
f_i(b_1 \otimes \cdots \otimes b_N) = b_1 \otimes \cdots \otimes b_{k-1} \otimes \tilde{f}_k \otimes b_{k+1} \otimes \cdots \otimes b_N,
\]
where $k = \max\{j \mid a_j = \min\{a_1, \ldots, a_N\}\}$,
\[
h_i(b_1 \otimes \cdots \otimes b_N) = b_1 \otimes \cdots \otimes b_{l-1} \otimes \tilde{e}_l b_l \otimes b_{l+1} \otimes \cdots \otimes b_N,
\]
where $l = \min\{j \mid a_j = \min\{a_1, \ldots, a_N\}\}$.

We regard integrable $U_q(g)$-module $M$ as a union of finite dimensional $U_q(g_{i})$-module for any $i \in I$. Let $u_0$ be the base of trivial representation of $U_q(\mathfrak{sl}_2)$-module, and let $u_+, u_-$ be a base of 2 dimensional $U_q(\mathfrak{sl}_2)$-module which satisfy
\[
f u_+ = u_-,
\]
where $f$ is one of generators of $U_q(\mathfrak{sl}_2)$.

In order to calculate the action of $\tilde{e}_i$ and $\tilde{f}_i$ on tensor products, we will use $u_+$, $u_-$ and $u_0$. For that purpose we prepare the following. Let $B$ be a crystal base. Let $\mathfrak{W}$ be the set of words generated by $u_+$, $u_-$, $u_0$.

Definition 3.4 For $i \in I$ we define a map
\[\Psi_i : B \to \mathfrak{W}\]
as follows:

1. for $b \in B$ such that $\tilde{e}_i b = \tilde{f}_i b = 0$, we define
\[\Psi_i(b) = u_0,
\]
2. if $b_k$ is an element of a maximally connected $i$-colored crystal subgraph $b_1 \overset{i}{\rightarrow} b_2 \overset{i}{\rightarrow} \cdots \overset{i}{\rightarrow} b_n$ where $n \in \mathbb{Z}_{\geq 2}$, we define
\[\Psi_i(b_k) = u_+^{k-1} u_-^{n-k}.
\]

We extend $\Psi_i$ to $b'_1 \otimes \cdots \otimes b'_N \in B^\otimes N$, $N \in \mathbb{Z}_{>0}$, by
\[\Psi_i(b'_1 \otimes \cdots \otimes b'_N) = \Psi_i(b'_1) \cdots \Psi_i(b'_N) = u_1 \cdots u_N,
\]
where $\Psi_i(b'_k) = u_k \in \mathfrak{W}$.

For $v = v_1 \cdots v_N \in \mathfrak{W}$, we define the length of the word $v$ by $l(v) = N'$. Let $\mathfrak{N}$ be the set of words generated by $\mathbb{Z}_{>0}$.

Definition 3.5 We define a map $\Psi_\mathfrak{m} : B^\otimes N \to \mathfrak{N}$ by
\[\Psi_\mathfrak{m}(b'_1 \otimes \cdots \otimes b'_N) = 1^{l(\Psi_i(b_1))} 2^{l(\Psi_i(b_2))} \cdots N^{l(\Psi_i(b_N))}
\]
For $r = r_1 \cdots r_{N''} \in \mathfrak{N}$, we also define the length of the word $r$ by $l(r) = N''$.

We put $\mathfrak{M}_j = \{v \in \mathfrak{W} \mid l(v) = j\}$, $\mathfrak{N}_j = \{r \in \mathfrak{N} \mid l(r) = j\}$, where $j \in \mathbb{Z}_{\geq 0}$.

Take words $v = v_1 \cdots v_M \in \mathfrak{W}_M$, $r = r_1 \cdots r_M \in \mathfrak{N}_M$, where $M \in \mathbb{Z}_{\geq 0}$. We put
\[s = \{s' \mid v s' = u_0, 1 \leq s' \leq M\},
\]
\[\{m_1, m_2, \ldots, m_s\} = \{s' \mid v s' = u_0, 1 \leq s' \leq M\},
\]
where $1 \leq m_1 < m_2 < \cdots < m_s \leq M$. 

6
Definition 3.6 We define a map
\[ \text{Red}_0 : \bigcup_{j \in \mathbb{Z}_0} (\mathbb{M}_j \times \mathbb{N}_j) \rightarrow \bigcup_{j \in \mathbb{Z}_0} (\mathbb{M}_j \times \mathbb{N}_j). \]

The map \( \text{Red}_0 \) delete all \( v_m \), which equal to \( u_0 \) in \( v \) and \( r_m \), in \( r \). Namely we can express
\[ \text{Red}_0(v,r) = (v_1 \cdots v_{m_1} - v_{m_1+1} \cdots v_{m_s} - v_{m_s+1} \cdots v_M, r_1 \cdots r_{m_1} - r_{m_1+1} \cdots r_{m_s} - r_{m_s+1} \cdots r_M). \]

We define
\[ s_0 = \begin{cases} \min \{ m \mid v_m = u_+, v_{m+1} = u_- \} & \text{if there exists } m \text{ such that } v_m = u_+, v_{m+1} = u_-, \\ 0 & \text{otherwise}. \end{cases} \]

Definition 3.7 We define a map
\[ \text{Red}_+ : \bigcup_{j \in \mathbb{Z}_0} (\mathbb{M}_j \times \mathbb{N}_j) \rightarrow \bigcup_{j \in \mathbb{Z}_0} (\mathbb{M}_j \times \mathbb{N}_j), \]

for \( v = v_1 \cdots v_m \in \mathbb{M}_M, r = r_1 \cdots r_M \in \mathbb{N}_M \) by
\[ \text{Red}_+(v,r) = \begin{cases} (v_1 \cdots v_{s_0} - v_{s_0+1} \cdots v_M, r_1 \cdots r_{s_0} - r_{s_0+1} \cdots r_M) & \text{if } s_0 \geq 1, \\ (v,r) & \text{if } s_0 = 0. \end{cases} \]

Definition 3.8 We define a map
\[ \text{Red} : \bigcup_{j \in \mathbb{Z}_0} (\mathbb{M}_j \times \mathbb{N}_j) \rightarrow \bigcup_{j \in \mathbb{Z}_0} (\mathbb{M}_j \times \mathbb{N}_j) \]

by
\[ \text{Red} = \text{Red}_+ \text{Red}_0, \]

where \( n \in \mathbb{Z}_0 \) such that \( n = \min \{ n' \mid \text{Red}_+^{n'+1}(\text{Red}_0(v,r)) = \text{Red}_+^{n'}(\text{Red}_0(v,r)) \} \).

By Definition 3.8, we can denote
\[ \text{Red}((\Psi_i(b), \Psi_m(b))) = (v_1' v_2' \cdots v_M', r_1' r_2' \cdots r_M'). \]

where \( M' \in \mathbb{Z}_0 \). Let \( \text{Pr}_\mathbb{M}, \text{Pr}_\mathbb{N} \) be a projection of \( \bigcup_{j \in \mathbb{Z}_0} (\mathbb{M}_j \times \mathbb{N}_j) \) to \( \mathbb{M}, \mathbb{N} \) respectively. We define a map
\[ \text{Red}_i : B^\otimes N \rightarrow \mathbb{M}, \]

by
\[ \text{Red}_i = \text{Pr}_\mathbb{M} \cdot \text{Red}(\Psi_i, \Psi_m) \]

We define a map
\[ \text{Red}_m : B^\otimes N \rightarrow \mathbb{N}, \]

by
\[ \text{Red}_m = \text{Pr}_\mathbb{N} \cdot \text{Red}(\Psi_i, \Psi_m) \]

We set
\[ s_+ = \begin{cases} \min \{ s' \mid v_{s'} = u_+ \} & \text{if there exists } s' \text{ such that } v_{s'} = u_+, \\ 0 & \text{otherwise}, \end{cases} \]
\[ s_- = \begin{cases} \max \{ s' \mid v_{s'} = u_- \} & \text{if there exists } s' \text{ such that } v_{s'} = u_-, \\ 0 & \text{otherwise}. \end{cases} \]
Proposition 3.9 For $b = b_1 \otimes \cdots \otimes b_N \in B^\otimes N$, we have
\[
\begin{aligned}
\tilde{f}_i(b) &= \begin{cases} 
 b_1 \otimes \cdots \otimes \tilde{f}_i b_{r_+} \otimes \cdots \otimes b_N & \text{if } s_+ > 0, \\
 0 & \text{if } s_+ = 0,
\end{cases} \\
\tilde{e}_i(b) &= \begin{cases} 
 b_1 \otimes \cdots \otimes \tilde{e}_i b_{r_-} \otimes \cdots \otimes b_N & \text{if } s_- > 0, \\
 0 & \text{if } s_- = 0.
\end{cases}
\end{aligned}
\]

For $b = b_1 \otimes \cdots \otimes b_N \in B^\otimes N$, we denote $\Psi_i(b_k) = u_\varepsilon(b_k) u_\varphi(b_k)$. Then we see that Proposition 3.3 and Proposition 3.9 are equivalent.

Remark 3.10 For $b = b_1 \otimes \cdots \otimes b_N$, we denote $\text{Red}_i(b) = u_\varepsilon m_1 \varepsilon m_2$ $(m_1, m_2 \in \mathbb{Z}_{\geq 0})$. Then by Definition 3.4 and Proposition 3.9, we see
\[
\begin{align*}
\varepsilon_i(b) &= m_1, \\
\varphi_i(b) &= m_2.
\end{align*}
\]

Example 3.11 We assume $b = b_1 \otimes b_2 \otimes b_3$, $\Psi_i(b_1) = u_1 = u_-u_+, \Psi_i(b_2) = u_2 = u_0$ and $\Psi_i(b_3) = u_3 = u_-u_+$. We calculate $\tilde{f}_i b$ as follows:
\[
\begin{align*}
\Psi_i(b) &= \Psi_i(b_1)\Psi_i(b_2)\Psi_i(b_3) \\
&= u_1 u_2 u_3 \\
&= u_-u_+u_0u_-u_+, \\
\Psi_{\alpha_i}(b) &= 111133, \\
\text{Red}_0(\Psi_i(b), \Psi_{\alpha_i}(b)) &= (u_-u_+u_0u_-u_+, 111133), \\
\text{Red}(\Psi_i(b), \Psi_{\alpha_i}(b)) &= (u_-u_+u_+, 113), \\
\text{Red}_i(b) &= u_-u_+u_+.
\end{align*}
\]

Then we have $s_+ = 2$, and that $s_+$-th integer of the word 113 is 1. Therefore,
\[
\tilde{f}_i b = (\tilde{f}_i b_1) \otimes b_2 \otimes b_3.
\]

3.1.3 Crystal bases for $U_q(A_2)$-module[6]

Consider the finite dimensional simple Lie algebra of type $A_2$ with Cartan matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Let us denote by $\{\alpha_1, \alpha_2\}$ the set of simple roots and $\{h_1, h_2\}$ the set of simple coroots. Define $\Lambda_i \in \mathfrak{h}^*$ $(i = 1, 2)$ by $\Lambda_i(h_j) = \delta_{ij}$, $(j = 1, 2)$. We put $P = \mathbb{Z}\Lambda_1 \oplus \mathbb{Z}\Lambda_2$, $P^* = \mathbb{Z}h_1 \oplus \mathbb{Z}h_2$. Thus any dominant integral weight $\Lambda \in P_+$ is of the form $\Lambda = m\Lambda_1 + n\Lambda_2$, $(m, n \in \mathbb{Z}_+)$. We define a non-degenerate symmetric bilinear form $(,)$ on $\mathfrak{h}^*$ by $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2) = 1$, $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = -\frac{1}{2}$, $\langle h_i, \alpha_j \rangle$ is the $(i,j)$-th element of Cartan matrix and set $t_i = q^{h_i}$ for $i = 1, 2$. Then the corresponding quantized universal enveloping algebra $U_q(A_2)$ is the associative algebra over $\mathbb{Q}(q)$ generated by $e_i, f_i, t_i, t_i^{-1}$ $(i = 1, 2)$ satisfying the relations:
\[
\begin{align*}
t_i t_j &= t_j t_i, \\
t_i t_i^{-1} &= t_i^{-1} = t_i = 1, \\
t_i e_j t_i^{-1} &= q^{\langle h_i, \alpha_j \rangle} e_j, t_i f_j t_i^{-1} = q^{-\langle h_i, \alpha_j \rangle} f_j, \\
[e_i, f_j] &= \delta_{ij} \frac{t_i - t_i^{-1}}{q - q^{-1}}, \\
e_i^{(2)} e_j - e_i e_j e_i + e_j e_i^{(2)} &= f_i^{(2)} f_j - f_i f_j f_i + f_j f_i^{(2)} = 0 (i \neq j).
\end{align*}
\]
Let $V$ be the 3-dimensional $\mathbb{Q}(q)$-vector space with a basis $\{ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \}$. We define a $U_q(A_2)$-module structure on $V$ as follows:

\[
\begin{align*}
& e_1 1 = 0, \quad e_1 2 = 1, \quad e_1 3 = 0, \\
& e_2 1 = 0, \quad e_2 2 = 0, \quad e_2 3 = 2, \\
& f_1 1 = 2, \quad f_1 2 = 0, \quad f_1 3 = 0, \\
& f_2 1 = 0, \quad f_2 2 = 3, \quad f_2 3 = 0, \\
& t_1 1 = q 1, \quad t_1 2 = q^{-1} 2, \quad t_1 3 = 3, \\
& t_2 1 = 1, \quad t_2 2 = q 2, \quad t_2 3 = q^{-1} 3.
\end{align*}
\]

The module $V$ is isomorphic to the integrable highest weight $U_q(A_2)$-module $V(\Lambda_1)$ with highest weight $\Lambda_1$ and highest weight vector $[1]$

**Proposition 3.12** Put

\[
L(\Lambda_1) = A^1 \oplus A^2 \oplus A^3.
\]

\[
B(\Lambda_1) = \{ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} \}.
\]

Then $(L(\Lambda_1), B(\Lambda_1))$ is the crystal base of $V(\Lambda_1)$. The crystal graph $B(\Lambda_1)$ is given by:

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array} \xrightarrow{1} \begin{array}{c}
2 \\
3
\end{array} \xrightarrow{2} \begin{array}{c}
3
\end{array}.
\]

Define an ordering on the set $B(\Lambda_1)$ by

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array} < \begin{array}{c}
2 \\
3
\end{array} < \begin{array}{c}
3
\end{array}.
\]

We write $\begin{array}{c} a \\
b \end{array}$ for $a \otimes b$ ($a < b$). Similar to the $B(\Lambda_1)$-case, the crystal graph $B(\Lambda_2)$ is given by:

\[
\begin{array}{c}
1 \\
2 \\
3
\end{array} \xrightarrow{2} \begin{array}{c}
3
\end{array} \xrightarrow{3} \begin{array}{c}
3
\end{array}.
\]

**Proposition 3.13** The crystal bases of $B(m\Lambda_1 + n\Lambda_2)$ is given by the set of semi-standard Young tableaux of the same shape

\[
B(m\Lambda_1 + n\Lambda_2) = \left\{ \begin{array}{ccc}
& b_1 \cdots b_m & b_1 \\
& b_2 \cdots b_m & \vdots \\
& b_2 \cdots b_m & b_1
\end{array} \right| b_i^j \in \{1, 2, 3\}, b_i^j \leq b_i^{j-1}, b_i^j < b_i^{j+1} \}
\]

In particular, highest weight element $\hat{b} \in B(m\Lambda_1 + n\Lambda_2)$ is given by

\[
\hat{b} = \begin{array}{ccc}
1 & \cdots & 1 \\
1 & \cdots & 1 \\
2 & \cdots & 2
\end{array}
\]

By Definition 3.4 we have

\[
\Psi_i \left( \begin{array}{c} i \\
j
\end{array} \right) = \begin{cases}
 u_+ & \text{if } i = j, \\
 u_- & \text{if } i + 1 = j, \\
 u_0 & \text{otherwise}.
\end{cases}
\]

An element $b$ of $B(m\Lambda_1 + n\Lambda_2)$ can be expressed in the form

\[
\begin{array}{c}
\begin{array}{c}
b_1 \\
b_2 \\
b_3
\end{array} \otimes \begin{array}{c}
b_1 \cdots b_m \\
\begin{array}{c} b_1 \\
b_2 \\
b_3
\end{array} \otimes \begin{array}{c} b_1 \\
b_2 \\
b_3
\end{array}
\end{array} \otimes \cdots \otimes \begin{array}{c}
b_1 \cdots b_m \\
\begin{array}{c} b_1 \\
b_2 \\
b_3
\end{array} \otimes \begin{array}{c} b_1 \\
b_2 \\
b_3
\end{array}
\end{array} \in B(\Lambda_1)^{\otimes m+2n}.
\]

Then using Proposition 3.9, we have the action of $\hat{e}_i$ and $\hat{f}_i$. 

3.1.4 Crystal bases for $U_q(G_2)$-module[7]

Consider the finite dimensional simple Lie algebra of type $G_2$ with Cartan matrix $\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$.

Let $\{\alpha_1, \alpha_2\}$, $\{h_1, h_2\}$ be the set of simple root and simple coroot respectively. Let $\mathfrak{h}$, $\Lambda_i$ $(i = 1, 2)$, $P$ and $P^*$ be Cartan subalgebra, fundamental weight, weight lattice and dual lattice, respectively. We define a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ by $(\alpha_1, \alpha_1) = 3$, $(\alpha_2, \alpha_2) = 1$, $(\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = -\frac{3}{2}$, $(h_1, \alpha_i)$ is $(i, j)$-element of Cartan matrix and set $q_i = q^{(h_i, \alpha_i)}$, $t_i = q^{h_i}$ for $i = 1, 2$. Then the corresponding quantized universal enveloping algebra $U_q(G_2)$ is the associative algebra over $\mathbb{Q}(q)$ generated by $e_i, f_i, t_i, t_i^{-1}$ $(i = 1, 2)$ satisfying the relations:

$$t_i t_j = t_j t_i, \quad t_i t_i^{-1} t_i^{-1} = t_i = 1,$$

$$t_i e_j t_i^{-1} = q_i^{(h_i, \alpha_j)} e_j, \quad t_i f_j t_i^{-1} = q_i^{-(h_i, \alpha_j)} f_j,$$

$$[e_i, f_j] = \delta_{ij} t_i t_i^{-1} q_i - q_i^{-1},$$

$$e_1^{(2)} e_2 - e_1 e_2 e_1 + e_2 e_1^{(2)} = f_1^{(2)} f_2 - f_1 f_2 f_1 + f_2 f_1^{(2)} = 0,$$

$$\sum_{n=0}^{4} (-1)^n e_2^{(n)} e_1^{(4-n)} = \sum_{n=0}^{4} (-1)^n e_2^{(n)} e_1^{(4-n)} = 0.$$

Let $V$ be the 14-dimensional $\mathbb{Q}(q)$-vector space with a basis

$$\begin{bmatrix} 1 & 7 \\ 1 & 2 \end{bmatrix} \cup \begin{bmatrix} 1 & 6 \\ 2 & 5 \end{bmatrix}.$$
The other cases, $b \in B(\Lambda_1)$ is annihilated by $\dot{e}_i$ or $\dot{f}_i$.

The module $V$ is isomorphic to the integrable highest weight $U_q(G_2)$-module $V(\Lambda_1)$ with highest weight $\Lambda_1$ and highest weight vector $\mathbf{1}$.

**Proposition 3.14** Put

$$L(\Lambda_1) = \bigoplus_{i=1}^{6} \left( A[B_2] \oplus A[\bar{T}] \right) \oplus A[0_1] \oplus A[0_2].$$

$$B(\Lambda_1) = \left\{ [1], A[\bar{T}] \mid i = 1, 2, \ldots, 6 \right\} \cup \left\{ [0_1], [0_2] \right\}.$$

Then $(L(\Lambda_1), B(\Lambda_1))$ is the crystal base of $V(\Lambda_1)$. The crystal graph $B(\Lambda_1)$ is given by:

![Crystal Graph](image)

Define an ordering on the set $B(\Lambda_1)$ by

$$1 < 2 < 3 < 4 < 5 < 6 < 0_1 < \bar{T} < 7 < 8 < 9 < T.$$ 

We define the map $\sgn$, $x_i$, and $\pi_i$.

For $a \in \mathbb{R}$, we define $\sgn(a)$ by

$$\sgn(a) = \begin{cases} \frac{a}{|a|} & a \neq 0, \\ 0 & a = 0. \end{cases}$$

For $b = b_1 \cdots b_k$ we define $x_i(b)$, $\pi_i(b)$ by

$$x_i(b) = \mathbb{R} \left\{ b_k = i \right\} \mid k = 1, \ldots, n \} (i = 1, \ldots, 6, 0_1, 0_2),$$

$$\pi_i(b) = \mathbb{R} \left\{ b_k = \bar{T} \right\} \mid k = 1, \ldots, n \} (i = 1, \ldots, 6).$$

**Proposition 3.15** The crystal bases of $B(n\Lambda_1)$ is given by following set of restricted semi-standard Young tableaux:

$$\begin{cases} \begin{array}{l} b = b_1 \cdots b_k \\ = b_1 \otimes \cdots \otimes b_k \end{array} \mid \begin{array}{l} b_k \leq b_{k+1} (k = 1, \ldots, l - 1) \\ x_5(b) + x_4(b) + x_2(b) + x_1(b) \leq 1, \\ x_3(b) + x_4(b) + x_5(b) \leq 1, \\ x_5(b) + x_4(b) + x_3(b) \leq 1, \\ x_5(b) + \sgn(x_6(b)) + x_0_1(b) \leq 1, \\ x_0_1(b) + \sgn(x_6(b)) + x_5(b) \leq 1 \end{array} \end{cases}.$$

By Definition 3.4, we have

$$\begin{align*} 
\Psi_1(1) &= u_+, & \Psi_1(21) &= u_-, & \Psi_1(31) &= u_0, & \Psi_1(41) &= u_+, \\
\Psi_1(5) &= u_-, & \Psi_1(32) &= u_0^2, & \Psi_1(04) &= u_0, & \Psi_1(00) &= u_-u_+, \\
\Psi_1(6) &= u_0^2, & \Psi_1(61) &= u_+, & \Psi_1(10) &= u_-, & \Psi_1(7) &= u_0, \\
\Psi_1(2) &= u_+, & \Psi_1(51) &= u_-, 
\end{align*}$$
\[
\begin{align*}
\Psi_2(1) &= u_0, \quad \Psi_2(2) = u_+^3, \quad \Psi_2(3) = u_- u_+^2, \quad \Psi_2(4) = u_-^2 u_+,
\Psi_2(5) &= u_-^2, \quad \Psi_2(6) = u_-^3, \quad \Psi_2(7) = u_- u_+^2, \quad \Psi_2(8) = u_0.
\end{align*}
\]

(3.1.1)

### 3.1.5 Affine crystal

Here we recall the definition of affine crystal. Let \( g \) be an affine Lie algebra over \( \mathbb{Q} \) with an indecomposable generalized Cartan Matrix. Let \( c \in \sum_i \mathbb{Z}_{\geq 0} h_i \) be the canonical central element. Let \( \{\alpha_i \mid i \in I\} \subset h^* \) be the set of simple roots, and \( \{h_i \mid i \in I\} \subset h \) be the set of coroots. Let \( \delta \in \sum_i \mathbb{Z}_{\geq 0} h_i \) be the generator of null roots. Set \( h_{cl} = \bigoplus_{i \in I} \mathbb{Q} h_i \subset h \) and \( h^*_{cl} = (\bigoplus_{i \in I} \mathbb{Q} h_i)^* \). Let \( cl : h^* \to h^*_{cl} \) denote the canonical morphism. We have an exact sequence \( 0 \to Q \delta \to h^* \to h^*_{cl} \to 0 \). Fix \( i_0 \in I \) and take an integer \( d \) such that \( \delta - d\alpha_{i_0} \in \sum_{i \neq i_0} \mathbb{Z}\alpha_i \). For simplicity we write \( 0 \) for \( i_0 \). We define a map \( af : h^*_{cl} \to h^* \) satisfying: \( cl \circ af = id \) and \( cl \circ af(\alpha_i) = \alpha_i \) for \( i \neq 0 \). Let \( \Lambda_i \) be the element of \( h^*_{cl} \subset h^* \) such that \( \langle h_j, \Lambda_i \rangle = \delta_{ij} \). Hence we have \( \alpha_i = \sum_j \langle h_j, \alpha_i \rangle f(\Lambda_i) + \delta_{0i} d^{-1} \delta \). We define \( P = \sum_i \mathbb{Z} af(\Lambda_i) + \mathbb{Z} d^{-1} \delta \subset h^* \) and \( P_{cl} = cl(P) \subset h^*_{cl} \). An element of \( P \) is called an affine weight and an element in \( P_{cl} \) is called a classical weight. Let \( U_q'(g) \) be the quantized universal enveloping algebra associated with \( P_{cl} \). A \( P \)-weighted crystal is called an affine crystal and a \( P_{cl} \)-weighted crystal is called a classical crystal. Let \( \mathrm{Mod}^f(g, P_{cl}) \) be the category of \( U_q'(g) \)-module \( M \) satisfying the following conditions:

\[
M \text{ has the weight decomposition } M = \bigoplus_{\lambda \in P_{cl}} M_\lambda, \quad \text{and} \quad M \text{ is finite-dimensional over } \mathbb{Q}(q). \]

For a \( U_q'(g) \)-module \( M \) in \( \mathrm{Mod}^f(g, P_{cl}) \), we define the \( U_q(g) \)-module \( \mathrm{Aff}(M) \) by

\[
\mathrm{Aff}(M) = \bigoplus_{\lambda \in P} \mathrm{Aff}(M)_\lambda, \quad \mathrm{Aff}(M)_\lambda = M_{cl(\lambda)} \text{ for } \lambda \in P.
\]

The actions of \( e_i \) and \( f_i \) are defined by the commutative diagrams

\[
\begin{align*}
\mathrm{Aff}(M)_\lambda \xrightarrow{e_i} \mathrm{Aff}(M)_{\lambda + \alpha_i} & \quad \text{ and } \quad \mathrm{Aff}(M)_\lambda \xrightarrow{f_i} \mathrm{Aff}(M)_{\lambda - \alpha_i}, \\
M_{cl(\lambda)} \xrightarrow{e_i} M_{cl(\lambda + \alpha_i)} & \quad \text{ and } \quad M_{cl(\lambda)} \xrightarrow{f_i} M_{cl(\lambda - \alpha_i)}.
\end{align*}
\]

We define the \( U_q'(g) \)-linear automorphism \( T \) of \( \mathrm{Aff}(M) \) by

\[
\begin{align*}
\mathrm{Aff}(M)_\lambda \xrightarrow{T} \mathrm{Aff}(M)_{\lambda + \delta} & \quad \text{ and } \quad \mathrm{Aff}(M)_\lambda \xrightarrow{id} M_{cl(\lambda + \delta)}.
\end{align*}
\]

### 3.1.6 Quantized universal enveloping algebra \( U_q(G_2^{(1)}) \)-module

Consider the finite dimensional affine Lie algebra \( G_2^{(1)} \) with Cartan matrix

\[
\begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 \\
0 & -3 & 2
\end{pmatrix}.
\]

Let \( \{\alpha_0, \alpha_1, \alpha_2\} \), \( \{h_0, h_1, h_2\} \) be the set of simple root and simple coroot respectively. Let \( h \) and \( \Lambda_i (i = 0, 1, 2) \) be Cartan subalgebra, fundamental weight, respectively. We define a non-degenerate symmetric bilinear form \( (, ) \) on \( h^* \) by \( (\alpha_0, \alpha_0) = (\alpha_1, \alpha_1) = 3, (\alpha_2, \alpha_2) = 1 \), \( (\alpha_0, \alpha_1) = (\alpha_1, \alpha_0) = (\alpha_1, \alpha_2) = (\alpha_2, \alpha_1) = -\frac{3}{2} \). \( \langle h_i, \alpha_j \rangle \) is \( (i, j) \)-element of Cartan matrix and set \( q_i = q^{(\alpha_i, \alpha_i)}, t_i = q^{h_i} \) for \( i = 1, 2 \). Canonical central
element is \( c = h_0 + 2h_1 + h_2 \). Then the corresponding quantized universal enveloping algebra \( U_q(G_2^{(1)}) \) is the associative algebra over \( \mathbb{Q}(q) \) generated by \( e_i, f_i, t_i, t_i^{-1} \) \( (i = 0, 1, 2) \) satisfying the relations:

\[
trt = tjt, \quad tt_i^{-1} = t_i^{-1} = t_i = 1,
\]

\[
t_i e_j t_i^{-1} = q_i^{(h_i, \alpha_j)} e_j, t_i f_j t_i^{-1} = q_i^{-(h_i, \alpha_j)} f_j,
\]

\[
[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q_i - q_i^{-1}}
\]

\[
\sum_{n=0}^{l} (-1)^n e_i^n e_i^{(l-n)} = \sum_{n=0}^{l} (-1)^n e_i^n e_i^{(l-n)} = 0, \quad (i \neq j, l = 1 - (h_i, \alpha_j)).
\]

### 3.2 Perfect crystal [9]

Let \( B \) be a classical crystal. For \( b \in B \), we set \( \varepsilon(b) = \sum_i \varepsilon_i(b) \lambda_i \), and \( \varphi(b) = \sum_i \varphi_i(b) \lambda_i \). Note that \( \text{wt}(b) = \varphi(b) - \varepsilon(b) \). Set \( P_{\text{cl}}^+ = \{ \lambda \in P_{\text{cl}} | \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I \} \) and for \( l \in \mathbb{Z}_{\geq 0} \), let \( (P_{\text{cl}}^+)_l = \{ \lambda \in P_{\text{cl}}^+ | \langle c, \lambda \rangle = l \} \).

**Definition 3.16** (Definition 4.6.1 in [9])

For \( l \in \mathbb{Z}_{\geq 0} \), we say that \( B \) is a **perfect crystal of level** \( l \) if

1. \( B \otimes B \) is connected.
2. There exists \( \lambda_0 \in P_{\text{cl}} \) such that \( \text{wt}(B) = \lambda_0 + \sum_{i \neq 0} \mathbb{Z}_{\leq 0} \alpha_i \) and that \( \mathbb{B}(B_{\lambda_0}) = 1 \).
3. There is a finite dimensional \( U_q'(\mathfrak{g}) \)-module with a crystal pseudo-base \( (L, B_{ps}) \) such that \( B \cong B_{ps}/\pm 1 \).
4. For any \( b \in B \), we have \( \langle c, \varepsilon(b) \rangle \geq l \).
5. The maps \( \varepsilon, \varphi : B_l = \{ b \in B | \langle c, \varepsilon(b) \rangle = l \} \rightarrow (P_{\text{cl}}^+)_l \) are bijective.

The elements in \( B_l \) are called **minimal elements**.

**Definition 3.17** Let \( M \) be a \( U_q(\mathfrak{g}) \)-module. A symmetric bilinear form \( \langle , \rangle \) on \( M \) is called a prepolarization of \( M \) if it satisfies for \( u, v \in M \):

1. \( \langle q^h u, v \rangle = \langle u, q^h v \rangle \),
2. \( \langle e_i u, v \rangle = \langle u, q_i^{-1} t_i^{-1} f_i v \rangle \),
3. \( \langle f_i u, v \rangle = \langle u, q_i^{(-1)} t_i e_i v \rangle \).

A prepolarization is called a polarization if it is positive definite.

**Lemma 3.18** (Lemma 3.4.4 in [9])

Let \( M \) be a \( U_q'(\mathfrak{g}) \)-module in \( \text{Mod}^f (\mathfrak{g}, P_{\text{cl}}) \). Assume that \( M \) has a crystal pseudo-base \( (L, B) \) such that

1. there exists \( \lambda \in P_{\text{cl}} \) such that \( \mathbb{B}(B/\{ \pm 1 \}) \lambda = 1 \),
2. \( B/\{ \pm 1 \} \) is connected.

Then \( M \) is an irreducible \( U_q'(\mathfrak{g}) \)-module.

Let us introduce the subalgebras \( A_Z \) and \( K_Z \) of \( \mathbb{Q}(q) \) as follows:

\[
A_Z = \{ f(q)/g(q) | f(q), g(q) \in \mathbb{Z}[q], g(0) = 1 \},
\]

\[
K_Z = A_Z[q^{-1}].
\]
Proposition 3.19 (Proposition 2.6.2 in [3])
Assume that \( g \) is finite dimensional and let \( M \) be a finite dimensional integrable \( U_q(\mathfrak{g}) \)-module of \( M \). Let \( (, \) be a prepolarization on \( M \), and \( M_{K_\mathbb{Z}} \) a \( U_q(\mathfrak{g}) \)-submodule of \( M \) such that \( (M_{K_\mathbb{Z}}, M_{K_\mathbb{Z}}) \subset K_\mathbb{Z} \). Let \( \lambda_1, \ldots, \lambda_m \in P^+ \) (\( \lambda_k = \sum a_{kj} \lambda_j, a_{kj} \in \mathbb{Z}_{\geq 0}, k = 1, \ldots, m \)), and we assume the following conditions.

(i) \( \dim M_{\lambda_k} \leq \sum_{j=1}^m \dim V(\lambda_j)_{\lambda_k} \) for \( k = 1, \ldots, m \).

(ii) There exists \( u_j \in (M_{K_\mathbb{Z}})_{\lambda_j} \) (\( j = 1, \ldots, m \)) such that \( (u_j, u_k) \in \delta_{jk} + qA \) and \( (e_i u_j, e_i u_j) \in q(q^{-2(1+\langle h_i, \lambda_j \rangle)} A)
\) for all \( i \in I \).

Set \( L = \{ U \in M \mid (u, u) \in A \} \) and set \( B = \{ b \in M_{K_\mathbb{Z}} \cap L/M_{K_\mathbb{Z}} \cap qL \mid (b, b)_0 = 1 \} \). Then we have

(i) \( (, )_0 \) is a polarization on \( M \),

(ii) \( M \cong \oplus V(\lambda_j) \),

(iii) \( (, )_0 \) is positive definite, and \( (L, B) \) is a crystal pseudobase of \( M \).

Lemma 3.20 (Lemma 4.1.2 [3]) A polarization \( (, )_0 \) for Kac-Moody Lie algebra with symmetrizable Cartan matrix satisfies following relations.

(i) Let \( b \) be a global base which satisfies \( e_i b = 0 \) and \( \langle h_i, \text{wt}(b) \rangle = 1 \). Then \( (b, b) = (f_i b, f_i b) \).

(ii) Let \( b \) be a global base which satisfies \( e_i b = 0 \) and \( \langle h_i, \text{wt}(b) \rangle = 2 \). Then \( (b, b) = (f_i^{(2)} b, f_i^{(2)} b) = q_i^{-1}[2]^{-1}(f_i b, f_i b) \).

(iii) Let \( b \) be a global base which satisfies \( e_i b = 0 \) and \( \langle h_i, \text{wt}(b) \rangle = 3 \). Then \( (b, b) = (f_i^{(3)} b, f_i^{(3)} b) = q_i^{-2}[3]^{-1}(f_i^{(2)} b, f_i^{(2)}) = q_i^{-2}[2]^{-1}(f_i b, f_i b) \).

The existance of polarization for Kac-Moody Lie algebra with symmetrizable Cartan matrix is proved in Proposition 3.4.4 [3].

Proposition 3.21 (Proposition 3.4.5 in [3])
Let \( m \) be a positive integer and assume the following conditions:

(i) \( \langle h_i, \lambda_0 + j\alpha_i \rangle \geq 0 \) for \( i \neq i_0 \) and \( 0 \leq j \leq m \),

(ii) \( \dim(V_{l_0 + k\alpha_i_0}) \leq \sum_{j=0}^m \dim(V(l_\lambda_0 + j\alpha_i)_{l_\lambda_0 + k\alpha_i} \) for \( 0 \leq k \leq m \), where \( V(\lambda) \) is an irreducible \( U_q(\mathfrak{g}_{l_\lambda_0}) \)-module with highest weight \( \lambda \),

(iii) There exists \( i_1 \in I \) such that \( \{ i \in I \mid \langle h_{i_0}, \alpha_i \rangle < 0 \} = \{ i_1 \} \),

(iv) \( -\langle h_{i_0}, \lambda_0 - \alpha_{i_1} \rangle \geq 0 \).

Then we have

\[ V_l \cong \oplus_{j=0}^m V(l_\lambda_0 + j\alpha_i) \text{ as a } U_q(\mathfrak{g}_{l_\lambda_0}) \text{-module}, \]

and \( V_l \) admits a crystal pseudobase as a \( U_q(\mathfrak{g}) \)-module.

4 Level-one representation and fusion construction

4.1 Construction of the polarization of \( V^1 \)

The space \( V^1 = V(\Lambda_1) \oplus V(0) \) is endowed with a \( U'_q(G_2^{(1)}) \)-module structure as follows.

\[ \begin{align*}
  f_0 [1] &= [2], & f_0 [2] &= [3], & f_0 [3] &= [4], & f_0 [4] &= [6],
\end{align*} \]
By Lemma 3.20, we have
\[ q^{h_0} = q^{-2k_1 - h_2}, \]
where \( \Box \) is the base of \( V(0) \).
We put
\[ B^1 = \{ i \ 00 \ 00 \ 00 \ 00 \ | i = 1, \ldots, 6 \}. \]

Let \((,)_1\) be the polarization on the \( U_q(G_2)\)-module \( V(\Lambda_1) \). We shall define a symmetric bilinear form
\((,)_1\) on \( V^1 \) by
\[ (\Box, u) = (u, \Box) = 0 \quad u \in V(\Lambda_1), \]
\[ (\Box, \Box) = q_0^{[2]}_0 \left( \begin{array}{ll} 1 & 1 \\ \end{array} \right)_1, \]
\[ (u, v) = (u, v)_1 \quad u, v \in V(\Lambda_1). \]

By Lemma 3.20, we have
\[ \left( a_1, a_1 \right)_1 = q_2^{-2[3]}_2 \left( \begin{array}{ll} 02 & 02 \\ \end{array} \right) = q_2^{-3[3]}_2^{-1} \left( \begin{array}{ll} 2 & 2 \\ \end{array} \right)_1 = q_1^{-1}[2]_1^{-1} \left( \begin{array}{ll} 0 & 0 \\ \end{array} \right)_1, \]
where \( a_1 = 1, 2, 6, \Box, \Box, \Box \), \( a_2 = 3, 4, 5, \Box, \Box, \Box \). It follows that \((,)_1\) is a polarization on the \( U_q(G_2^{(1)})\)-module \( V^1 \).

Therefore we see that \( B^1 \) is a crystal base and Kashiwara operators act as follows.
\[ \begin{align*}
\hat{f}_0 \begin{array}{ll} 5 \\ \end{array} & = \begin{array}{ll} 2 \\ \end{array}, \hat{f}_0 \begin{array}{ll} 7 \\ \end{array} = \begin{array}{ll} 3 \\ \end{array}, \hat{f}_0 \begin{array}{ll} 6 \\ \end{array} = \begin{array}{ll} 4 \\ \end{array}, \hat{f}_0 \begin{array}{ll} 2 \\ \end{array} = \begin{array}{ll} 6 \\ \end{array}, \hat{f}_0 \begin{array}{ll} 1 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \hat{f}_0 \begin{array}{ll} 0 \end{array} = \begin{array}{ll} 2 \end{array}, \\
\hat{f}_1 \begin{array}{ll} 1 \\ \end{array} & = \begin{array}{ll} 2 \\ \end{array}, \hat{f}_1 \begin{array}{ll} 4 \\ \end{array} = \begin{array}{ll} 5 \\ \end{array}, \hat{f}_1 \begin{array}{ll} 6 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \hat{f}_1 \begin{array}{ll} 1 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \hat{f}_1 \begin{array}{ll} 0 \end{array} = \begin{array}{ll} 0 \end{array}, \\
\hat{f}_2 \begin{array}{ll} 2 \\ \end{array} & = \begin{array}{ll} 3 \\ \end{array}, \hat{f}_2 \begin{array}{ll} 3 \\ \end{array} = \begin{array}{ll} 2 \end{array}, \hat{f}_2 \begin{array}{ll} 4 \\ \end{array} = \begin{array}{ll} 3 \end{array}, \hat{f}_2 \begin{array}{ll} 5 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \hat{f}_2 \begin{array}{ll} 0 \end{array} = \begin{array}{ll} 2 \end{array}, \hat{f}_2 \begin{array}{ll} 1 \end{array} = \begin{array}{ll} 0 \end{array}, \\
\hat{f}_2 \begin{array}{ll} 0 \end{array} & = \begin{array}{ll} 5 \\ \end{array}, \hat{f}_2 \begin{array}{ll} 0 \end{array} = \begin{array}{ll} 0 \end{array}, \hat{f}_2 \begin{array}{ll} 0 \end{array} = \begin{array}{ll} 0 \end{array}, \hat{f}_2 \begin{array}{ll} 0 \end{array} = \begin{array}{ll} 0 \end{array}, \hat{f}_2 \begin{array}{ll} 0 \end{array} = \begin{array}{ll} 0 \end{array}, \hat{f}_2 \begin{array}{ll} 0 \end{array} = \begin{array}{ll} 0 \end{array}, \\
\hat{e}_0 \begin{array}{ll} 6 \\ \end{array} & = \begin{array}{ll} 2 \end{array}, \hat{e}_0 \begin{array}{ll} 4 \\ \end{array} = \begin{array}{ll} 3 \end{array}, \hat{e}_0 \begin{array}{ll} 3 \\ \end{array} = \begin{array}{ll} 4 \end{array}, \hat{e}_0 \begin{array}{ll} 2 \\ \end{array} = \begin{array}{ll} 5 \end{array}, \hat{e}_0 \begin{array}{ll} 1 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \hat{e}_0 \begin{array}{ll} 0 \\ \end{array} = \begin{array}{ll} 2 \end{array}, \\
\hat{e}_1 \begin{array}{ll} 1 \\ \end{array} & = \begin{array}{ll} 2 \end{array}, \hat{e}_1 \begin{array}{ll} 4 \\ \end{array} = \begin{array}{ll} 5 \end{array}, \hat{e}_1 \begin{array}{ll} 6 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \hat{e}_1 \begin{array}{ll} 1 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \hat{e}_1 \begin{array}{ll} 0 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \hat{e}_1 \begin{array}{ll} 0 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \\
\hat{e}_2 \begin{array}{ll} 2 \\ \end{array} & = \begin{array}{ll} 3 \end{array}, \hat{e}_2 \begin{array}{ll} 3 \\ \end{array} = \begin{array}{ll} 2 \end{array}, \hat{e}_2 \begin{array}{ll} 4 \\ \end{array} = \begin{array}{ll} 3 \end{array}, \hat{e}_2 \begin{array}{ll} 5 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \hat{e}_2 \begin{array}{ll} 0 \\ \end{array} = \begin{array}{ll} 2 \end{array}, \hat{e}_2 \begin{array}{ll} 1 \\ \end{array} = \begin{array}{ll} 0 \end{array}, \\
\hat{e}_2 \begin{array}{ll} 4 \\ \end{array} & = \begin{array}{ll} 2 \end{array}, \hat{e}_2 \begin{array}{ll} 3 \\ \end{array} = \begin{array}{ll} 3 \end{array}, \hat{e}_2 \begin{array}{ll} 2 \\ \end{array} = \begin{array}{ll} 3 \end{array}, \hat{e}_2 \begin{array}{ll} 1 \\ \end{array} = \begin{array}{ll} 2 \end{array}. 
\end{align*} \]

By checking of conditions of Definition 5.16, we have following proposition.

**Proposition 4.1** \( B^1 \) is a perfect crystal of level 1.
4.2 Decomposition of the tensor product

The tensor product $V^1 \otimes V^1$ is decomposed in the following way:

\[
(V(\Lambda_1) \oplus V(0)) \otimes (V(\Lambda_1) \oplus V(0)) \\
\cong V(2\Lambda_1) \oplus V(3\Lambda_2) \oplus V(2\Lambda_2) \oplus V(\Lambda_1) \oplus V(0) \oplus V(0).
\] (4.2.1)

Lemma 4.2 Following vectors $u_{2\Lambda_1}$, $u_{3\Lambda_2}$, $u_{2\Lambda_2}$, $u_{\Lambda_1}$, $u_{\Lambda_2}$, $u_0$ and $u_0'$ are the highest weight vectors with the weights $2\Lambda_1$, $3\Lambda_2$, $2\Lambda_2$, $\Lambda_1$, $\Lambda_1$, $0$ and $0$ respectively.

\[
\begin{align*}
    u_{2\Lambda_1} & = 1 \otimes 1, \\
    u_{3\Lambda_2} & = 1 \otimes 2 - q^2 2 \otimes 1, \\
    u_{2\Lambda_2} & = 1 \otimes 5 - q^2 2 \otimes 4 + \frac{[2]}{[3]} q^2 3 \otimes 3 - q^2 4 \otimes 2 + q^1 5 \otimes 1 \\
    u_{\Lambda_1}^1 & = 1 \otimes 3, \\
    u_{\Lambda_1}^2 & = 1 \otimes 1, \\
    u_{\Lambda_2}^1 & = 1 \otimes 6 - [2] q^2 3 \otimes 6 + \frac{[2]}{[3]} q^2 3 \otimes 4 - \frac{[2]}{[3]} q^6 4 \otimes 3 \\
    + [2] q^6 6 \otimes 2 - q^2 0 \otimes 1, \\
    u_0^1 & = 1 \otimes 1, \\
    u_0^2 & = 1 \otimes 4 - q^2 2 \otimes 7 + \frac{1}{[3]} q^6 3 \otimes 7 - \frac{1}{[3]} q^6 4 \otimes 3 \\
    + \frac{1}{[3]} q^6 5 \otimes 5 + q^6 6 \otimes 6 - \frac{q^6}{[2][3]} 0 \otimes 0 - \frac{q^6}{[2]} 0 \otimes 0 \\
    + \frac{q^6}{[2][3]} 3 \otimes 3 + q^{12} 6 \otimes 6 - q^{12} 3 \otimes 4 + q^{12} 3 \otimes 3 \\
    - q^{15} 2 \otimes 2 + q^{15} 4 \otimes 4 - q^{15} 2 \otimes 1 - q^{15} 3 \otimes 3 - q^{15} 0 \otimes 3 - \frac{q^6}{[2]} 0 \otimes 0 - \frac{q^6}{[2]} 0 \otimes 0.
\end{align*}
\]

4.3 Calculation of the $R$-matrix

Let $V^1_x$ be $U_q^s(\mathfrak{g})$-module $Q[x, x^{-1}] \otimes V^1$ with the actions of $e_i$, $f_i$, and $t_i$ given by $x^\delta a e_i$, $x^{-\delta} f_i$, and $t_i$, respectively. The $R$-matrix for $V^1$ is an invertible $Q[x, x^{-1}] \otimes U_q(G_2)$-linear map

\[
R(x, y) : V_x \otimes V_y \rightarrow V_y \otimes V_x
\]
satisfying following properties.

1. $R(x, y) \in Q[q][x, y, y/x] \otimes \text{End}_{Q(q)}(V^1 \otimes V^1)$.
2. $(R(y, z))((1 \otimes R(x, z))((R(x, y) \otimes 1) = (1 \otimes R(x, y))(R(x, z) \otimes 1)(1 \otimes R(y, z))$.
3. $R(x, y)R(y, x) \in Q[q][x, y, y/x]$.

By the decomposition $[1.2]$, $U_q(G_2)$-linearity and $R = R(x, y)$, we have

\[
\begin{align*}
    R(u_{2\Lambda_1}) = a^{2\Lambda_1} u_{2\Lambda_1}, \\
    R(u_{3\Lambda_2}) = a^{3\Lambda_2} u_{3\Lambda_2}, \\
    R(u_{2\Lambda_2}) = a^{2\Lambda_2} u_{2\Lambda_2}, \\
    R(u_{\Lambda_1}^1) = \sum_{i=1}^3 a_{ij} u_{\Lambda_1}^1 (i = 1, 2, 3), \\
    R(u_0^1) = \sum_{i=1}^3 a_{ij} u_0^1 (i = 1, 2).
\end{align*}
\] (4.3.1)

Consider the highest weight vectors given in Lemma 4.2. Then we have in $V^1_x \otimes V^1_y$: 16
\[f_0^{(2)} f_1 f_2^{(3)} f_1 u_{\Lambda_1}^{1} = [2]_1 q^{-3} x^{-1} y^{-1} u_{2\Lambda_1},\]
\[f_0^{(2)} f_1 f_2^{(3)} f_1 u_{\Lambda_1}^{2} = [2]_1 q^{-3} x^{-1} y^{-1} u_{2\Lambda_1},\]
\[f_0^{(2)} f_1 f_2^{(3)} f_1 u_{\Lambda_1}^{3} = [2]_1 (x - q^6 y)(x + q^{12} y)q^{-6} x^{-2} y^{-2} u_{2\Lambda_1},\]
\[e_{12}^{(3)} (e_{12}^{(3)} e_{0 e}^{(2)} e_{2}^{(3)} e_{1 e} u_{\Lambda_1}^{1}) = [2]_1 y(x + y)u_{2\Lambda_1},\]
\[e_{12}^{(3)} (e_{12}^{(3)} e_{0 e}^{(2)} e_{2}^{(3)} e_{1 e} u_{\Lambda_1}^{2}) = [2]_1 q^{-6} x(x + y)u_{2\Lambda_1},\]
\[e_{12}^{(3)} (e_{12}^{(3)} e_{0 e}^{(2)} e_{2}^{(3)} e_{1 e} u_{\Lambda_1}^{3}) = [2]_2 q^6 (q^4 + 1)(x - q^6 y)u_{2\Lambda_1},\]
\[f_0 u_{\Lambda_1}^{1} = [2]_1 q^{-6} y^{-1} u_{2\Lambda_1},\]
\[f_0 u_{\Lambda_1}^{2} = [2]_1 x^{-1} u_{2\Lambda_1},\]
\[f_0 u_{\Lambda_1}^{3} = 0,\]
\[f_0^{(2)} u_{\Lambda_1}^{1} = [2]_1 q^{-3} x^{-1} y^{-1} u_{2\Lambda_1},\]
\[f_0^{(2)} u_{\Lambda_1}^{2} = q^{-12}(x^2 + q^{30} y^2)x^{-2} y^{-2} u_{2\Lambda_1},\]
\[f_0^{(2)} f_1 f_2^{(3)} f_1 f_2^{(3)} f_1 f_2 f_0 f_1 f_2^{(3)} f_1 f_0 u_{\Lambda_1}^{0} = [2]_2 q^{-6}(q^4 + q^2 + 1)q^{-8}(q^6 y^2 + x^2) x y u_{2\Lambda_1},\]
\[f_0^{(2)} f_1 f_2^{(3)} f_1 f_2^{(3)} u_{2\Lambda_1} = q^{-3}(x - q^6 y)(x + y)x^{-2} y^{-2} u_{2\Lambda_1},\]
\[f_0^{(2)} f_1 f_2^{(3)} f_1 f_2^{(3)} u_{2\Lambda_1} = q^{-8}(q^4 + q^2 + 1)(x - q^6 y)(x - q^{10} y)x^{-2} y^{-2} u_{2\Lambda_1}.\]

From these equations and the relation \([R, \Delta(X)] = 0\) \((X \in U_6(G_2))\), we obtain
\[a_{2\Lambda_1}(q^6 y, x, (q^4 + 1)[2]_2 q^4(x - q^6 y)) = (q^6 y, x, (q^4 + 1)[2]_2 q^4(y - q^6 x)) \cdot (a_{ij}^{\Lambda_1}),\]
\[a_{2\Lambda_1}(x, y, 0) = (y, x, 0) \cdot (a_{ij}^{\Lambda_1}),\]
\[a_{2\Lambda_1}([2]_2 q^6, (x^2 + q^{30} y^2)q^{-3} x^{-1} y^{-1}) = ([2]_2 q^6, (y^2 + q^{30} x^2)q^{-3} x^{-1} y^{-1}) \cdot (a_{ij}^{\Lambda_1}),\]
\[a_{2\Lambda_1}(s_1(x, y), s_2(x, y)) = (s_1(y, x), s_2(y, x)) \cdot (a_{ij}^{\Lambda_1}),\]
\[(x - q^6 y) a_{2\Lambda_1} = a_{3\Lambda_2}(y - q^6 x),\]
\[(x - q^6 y)(x - q^{10} y) a_{2\Lambda_1} = a_{2\Lambda_2}(y - q^6 x)(y - q^{10} x),\]
where
\[s_1(x, y) = [2]_1 (q^4 - a^2 + 1)(x^2 + q^6 y^2)\]
\[s_2(x, y) = (q^{22} + q^{18})y^2 + (q^6 + 1)(q^6 - q^{14} + q^{12} - 2q^{10} + q^6 + q^4 - q^2 + 1)xy + (q^4 + 1)x^2.\]

Let \(P_{2\Lambda_1}, P_{3\Lambda_2}, P_{2\Lambda_2}, P_{u_{\Lambda_1}}^{(i)} \) \((i = 1, 2, 3)\) and \(P_{u_{\Lambda_1}}^{(i)} \) \((i = 1, 2)\) be the projection from \(V^1 \otimes V^1\) to \(V(2\Lambda_1),\)
\(V(3\Lambda_2), V(2\Lambda_2), U_6(G_2)p_{u_{\Lambda_1}}^{(i)} \) \((i = 1, 2, 3)\) and \(U_6(G_2) u_{\Lambda_1}^{(i)} \) \((i = 1, 2)\) respectively.

**Proposition 4.3** Put \(z = xy^{-1}\). Since the \(R\)-matrix \(R(x, y)\) depends only on \(x/y\), we denote \(R(z) = R(x, y)\). Then we have
\[R(z)(u_{2\Lambda_1}) = (1 - q^{12} z)(1 - q^{10} z)(1 - q^6 z)(1 - q^8 z)u_{2\Lambda_1},\]
\[R(z)(u_{2\Lambda_2}) = (1 - q^{12} z)(1 - q^{10} z)(1 - q^8 z)(z - q^6)u_{3\Lambda_2},\]
\[R(z)(u_{2\Lambda_2}) = (1 - q^{12} z)(z - q^{10} z)(q^8(z - q^6)u_{2\Lambda_2}.\]
\[ R(u_{\lambda_i}) = \sum_{j=1}^{3} a_{ij}^{\Lambda_i} u_{\lambda_i}^j \quad (i = 1, 2, 3), \]
\[ R(u_0^i) = \sum_{j=1}^{2} a_{ij}^0 u_0^j \quad (i = 1, 2). \]

Here \((a_{ij}^{\Lambda_i})\) and \((a_{ij}^0)\) are given by
\begin{align*}
a_{11}^{\Lambda_1} &= (1 - q^{12}z)(q^6 - 1)(q^2 + 1)z(-q^{16} - q^{14} + q^{12} - q^{10} - q^6)z - (q^4 - q^2 + 1), \\
a_{12}^{\Lambda_1} &= (1 - q^{12}z)q^6(1 - z)(q^{12}z^2 + (q^{12} - q^6 - q^4 - q^2)z + 1), \\
a_{13}^{\Lambda_1} &= (1 - q^{12}z)q^6(q^6 - 1)z(z - 1), \\
a_{21}^{\Lambda_1} &= (1 - q^{12}z)q^6(1 - z)(q^{12}z^2 - (q^{10} + q^8 + q^6 - 1))z + 1), \\
a_{22}^{\Lambda_1} &= (1 - q^{12}z)(q^6 - 1)(q^2 + 1)z(-q^{16} + q^{14} - q^{12})z + (q^{10} + q^6 - q^4 + q^2 - 1), \\
a_{23}^{\Lambda_1} &= (1 - q^{12}z)q^6(q^6 - 1)(z - 1)z, \\
a_{31}^{\Lambda_1} &= (1 - q^{12}z)q^6(q^{12} - 1)(q^4 + 1)(q^2 + 1)z(1 - z)(z - q^6), \\
a_{32}^{\Lambda_1} &= (1 - q^{12}z)q^6(q^{12} - 1)(q^4 + 1)(q^2 + 1)(1 - z)(q^6 - z), \\
a_{33}^{\Lambda_1} &= (1 - q^{12}z)(z - q^6)(q^6z^2 + (q^{16} - q^{12} - q^{10} - q^8 - q^6 + 1)z + q^{12}), \\
a_{11}^0 &= q^{30}z^4 - q^{24}(q^4 + 1)(q^2 + 1)z^3 + (q^{36} - q^{30} + q^{22} + q^{20} + 2q^{18} + q^{16} + q^{14} - q^{6} - q^4 + 1)(q^2 + 1)z + q^6, \\
a_{12}^0 &= -q^3(q^2 - 1)(q^6 + 1)(1 - z)(1 + z)z, \\
a_{21}^0 &= -q^3q^{26}(1 - z)(1 + z)(q^{22} + q^{18}) + (q^{20} - q^{38} + q^{36} - q^{34} - q^{30} - q^{26} - q^{20} - q^{14} - q^{10} - q^6 + q^4 - q^2 + 1)z + (q^{22} + q^{18})z^2, \\
a_{22}^0 &= q^{30} - q^{24}(q^4 + 1)(q^2 + 1)z + (q^{36} - q^{30} + q^{22} + q^{20} + 2q^{18} + q^{16} + q^{14} - q^{6} + 1)z^2 - q^6(q^4 + 1)(q^2 + 1)z + 1. \end{align*}

### 4.4 Fusion construction

Let \(l\) be a positive integer and \(S_l\) the symmetric group of order \(l\). Let \(s_i\) be the simple reflection (the permutation of \(i\) and \(i + 1\)). Let \(l(w)\) be the length of \(w \in S_l\). Then for any \(w \in S_l\), we can define \(R_w(x_1, \ldots, x_l) : V_{x_1} \otimes \cdots \otimes V_{x_l} \rightarrow V_{x_{w(1)}} \otimes \cdots \otimes V_{x_{w(l)}}\) as follows:
\[ R_1(x_1, \ldots, x_l) = 1. \]
\[ R_{s_i}(x_1, \ldots, x_l) = \left( \otimes_{j < i} \text{id}_{V_{x_j}} \right) \otimes R(x_i, x_{i+1}) \otimes \left( \otimes_{j > i+1} \text{id}_{V_{x_j}} \right). \]

For \(w, w'\) with \(l(ww') = l(w) + l(w')\),
\[ R_{ww'}(x_1, \ldots, x_l) = R_{w'}(x_{w(1)}, \ldots, x_{w(l)}) \circ R_w(x_1, \ldots, x_l). \]

Fix \(r \in \mathbb{Z}_{>0}\). For each \(l \in \mathbb{Z}_{>0}\), we put
\[ R_l = R_{w_0} \left( q^{r(l-1)}, q^{r(l-3)}, \ldots, q^{-r(l-1)} \right); \]
\[ V_{q^{r(l-1)}} \otimes V_{q^{r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}} \rightarrow V_{q^{r(l-1)}} \otimes V_{q^{-r(l-3)}} \otimes \cdots \otimes V_{q^{-r(l-1)}}, \]
where \(w_0 \in S_l\) is the permutation given by \(i \mapsto l + 1 - i\). Then \(R_l\) is a \(U_q^r(\mathfrak{g})\)-linear homomorphism. We define
\[ V_l = \text{Im} R_l. \]

Hence taking \(\Lambda_1 = 2\lambda_0\) as \(\lambda_0\) in Definition 3.16. By Proposition 4.3, we have
\[ \varphi(z) = (1 - q^{12}z)(1 - q^{10}z)(1 - q^8z)(1 - q^6z). \]
Putting \( r = 3 \).

We see that \( \varphi(q^{2kr}) \) does not vanish for any \( k > 0 \).

By Proposition 4.3, we see

\[
R(q^{2r})(u_{2\Lambda_1}) = (1 - q^{18})(1 - q^{16})(1 - q^{14})(1 - q^{12}),
\]

\[
R(q^{2r})(u_{1\Lambda_1}) = \sum_{j=1}^{3} a_{1j}^\Lambda_1 u_{1j}^\Lambda_1,
\]

\[
R(q^{2r})(u_0^\Lambda_1) = \sum_{j=1}^{2} a_{0j}^\Lambda_1 u_{0j}^\Lambda_1,
\]

\[
R(q^{2r})(u_{2\Lambda_2}) = 0,
\]

\[
R(q^{2r})(u_{1\Lambda_2}) = 0,
\]

\[
R(q^{2r})(u_{2\Lambda_1} - u_{2\Lambda_2}) = 0,
\]

\[
R(q^{2r})(u_{2\Lambda_1}^3) = 0.
\]

Then \( N = \ker R(q^{2r}) \) contains \( u_{3\Lambda_2}, u_{2\Lambda_2}, u_{1\Lambda_1} - u_{2\Lambda_1}, u_{3\Lambda_1} \) and \( q^3(q^{12} - q^6 + 1)u_0^\Lambda_1 - (q^6 + 1)u_0^\Lambda_1 \). Therefore by (4.2.1), we have

\[
N \cong U_q(G_2)u_{3\Lambda_2} \oplus U_q(G_2)u_{2\Lambda_2} \oplus U_q(G_2)u_{1\Lambda_1}^3
\]

\[
\oplus U_q(G_2)(u_{1\Lambda_1} - u_{2\Lambda_1}) \oplus U_q(G_2)(q^3(q^{12} - q^6 + 1)u_0^\Lambda_1 - (q^6 + 1)u_0^\Lambda_1).
\]

Then we have

\[
\dim(V^\otimes 2/N) = \dim \left( \bigoplus_{j=0}^{2} V(j\Lambda_1) \right)
\]

\[
= z \left\{ \begin{array}{c}
\begin{array}{c} \square \otimes a \\
\end{array}
\end{array} \right| a, b \in B(2\Lambda_1) \right\}
\]

\[
+ z \left\{ \begin{array}{c}
\begin{array}{c} a \otimes \square \\
\end{array}
\end{array} \right| a \in B(\Lambda_1) \right\}
\]

where \( a, b \in \{1, \ldots, 6, 0_1, 0_2, \bar{6}, \ldots, 1\} \).

Hence,

\[
\dim \left( V^{\otimes l} / \sum_{i} V^{\otimes i} \otimes N \otimes V^{\otimes (l-2-i)} \right)
\]

\[
= z \left\{ \begin{array}{c}
\begin{array}{c} b_0 \otimes \cdots \otimes b_i \otimes \square \otimes \cdots \otimes \square \\
\end{array}
\end{array} \right| \begin{array}{c}
\begin{array}{c} b_i, b_n \in B(2\Lambda_1), \ 0 \leq i < n \leq l \\
\end{array}
\end{array} \right\}
\]

\[
= \dim \left( \bigoplus_{j=0}^{l} V(j\Lambda_1) \right).
\]

By (3.3.10) of \( \square \), \( V_l \) is the quotient of \( V^{\otimes l} / \sum_{i=0}^{l-2} V^{\otimes i} \otimes N \otimes V^{\otimes (l-2-i)} \).

Therefore we have

\[
\dim(V_l) \leq \sum_{j=0}^{l} \dim V(j(\Lambda_1 - 2\Lambda_0))_\lambda.
\]

We set \( i_0 = 0 \), then \( \langle h_1, \lambda_0 \rangle = 1, \langle h_2, \lambda_0 \rangle = 0, \{ i_1 \} = \{1\}, \langle h_0, \lambda_0 \rangle = -2 \). Therefore applying Proposition 3.21, we obtain the following results.

**Proposition 4.4**

1. \( V_l \) has a crystal pseudobase.
2. \( V_l \cong \bigoplus_{j=0}^{l} (V(j(\Lambda_1 - 2\Lambda_0))) \) as a \( U_q(G_2) \)-module.
5 Perfectness of the crystals

The algebra $U_q(G_2^{(1)})$ is $\mathbb{Q}(q)$-algebra generated by $\{e_i, f_i, t_i, t_i^{-1} \ (i \in I = \{0, 1, 2\})\}$. Let $U_q((G_2^{(1)})_J)$ be a $\mathbb{Q}(q)$-algebra generated by $\{e_i, f_i, t_i, t_i^{-1} \ (i \in J \subset I)\}$.

The algebras $U_q((G_2^{(1)})_{\{1, 2\}})$ and $U_q((G_2^{(1)})_{\{0, 1\}})$ are isomorphic to $U_q(G_2)$ and $U_q(A_2)$ respectively. Let $J_0 = \{1, 2\}$ and $J_2 = \{1, 0\}$ be the index set of $A_2$ and $G_2$, respectively. We define $i_1 : J_1 \to I$ by $i_1(j) = j$ $(i = 0, 2)$. In order to show that the crystal $B^l$ is perfect, we have to show the following conditions.

1. There exists a crystal base of $U_q(G_2)$ isomorphic to $i_1^*(B^l)$.
   There exists a crystal base of $U_q(A_2)$ isomorphic to $i_2^*(B^l)$. Crystal base $B^l$ is connected.

2. For any $b \in B^l$, $(c, \varphi(b)) \geq l$.

3. The maps $\varepsilon$ and $\varphi : B_l = \{b \in B^l \mid \langle c, \varepsilon(b) \rangle = l\} \to (P_\lambda)_l = \{\lambda \in \sum \mathbb{Z}_{\geq 0}A_i \mid \langle c, \lambda \rangle = 1\}$ are bijective.

4. Crystal graph of $B^l \otimes B^l$ is connected.

At first we construct crystal $B^l$ which satisfies condition (1) (5.2, 5.3). Then we prove conditions (2)(3) in 5.3 and (4) in 5.3.

5.1 Preparation

Let $B^A_2(\lambda)$ (resp. $B^G_2(\lambda)$) be a crystal base of $U_q(A_2)$ (resp. $U_q(G_2)$) with the highest weight $\lambda$. We write $[b_1 \cdots b_k]$ instead of Young tableau $b_1 \cdots b_k$. We introduce following notations:

$$[g]^k = \underbrace{[g] \cdots [g]}_{k \text{ times}} \ (a = 1, \ldots, 6, 0_1, 0_2, 0, \ldots, 1),$$

$$[c]^k = j_6 [6]^k = \begin{cases} 6^{\frac{k}{2}} \begin{bmatrix} 6 \end{bmatrix}^{\frac{k}{2}}, & k \equiv 0 \pmod{2}, \\ 6^{\frac{k}{2}} \begin{bmatrix} 6 \end{bmatrix}^{\frac{k}{2}}, & k \equiv 1 \pmod{2}. \end{cases}$$

$$[\mathcal{W}]^k = \tilde{j}_2 [2]^k = \begin{cases} 2^{\frac{k}{2}} \begin{bmatrix} 6 \end{bmatrix}^{\frac{k}{2}} & k \equiv 0 \pmod{3}, \\ 2^{\frac{k}{2}} \begin{bmatrix} 6 \end{bmatrix}^{\frac{k}{2}} & k \equiv 1 \pmod{3}, \\ 2^{\frac{k}{2}+1} \begin{bmatrix} 4 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix}^{\frac{k}{2}} & k \equiv 2 \pmod{3}. \end{cases}$$

$$[\mathcal{W}]^k = \hat{c_6} [2]^k = \begin{cases} 6^{\frac{k}{2}} \begin{bmatrix} 2 \end{bmatrix}^{\frac{k}{2}} & k \equiv 0 \pmod{3}, \\ 6^{\frac{k}{2}} \begin{bmatrix} 2 \end{bmatrix}^{\frac{k}{2}} & k \equiv 1 \pmod{3}, \\ 6^{\frac{k}{2}} \begin{bmatrix} 2 \end{bmatrix}^{\frac{k}{2}} + 1 & k \equiv 2 \pmod{3}. \end{cases}$$

Then we easily have

$$[c]^k = [6] [c]^{k-2} [5], \quad (5.1.1)$$

$$[\mathcal{W}]^k = [2]^2 [\mathcal{W}]^{k-3} [6], \quad (5.1.2)$$

$$[\mathcal{W}]^k = [6] [\mathcal{W}]^{k-3} [2]^2, \quad (5.1.3)$$
\[ \tilde{d}[c]^k = [c]^{k-1}[6], \quad (5.1.4) \]
\[ \tilde{e}_2[W]^k = [2][W]^{k-1}, \quad (5.1.5) \]
\[ \tilde{e}_2[W]^k = [6][W]^{k-2}[2]. \quad (5.1.6) \]

**Proposition 5.1** For \([c]^k, [W]^k, [W]^k\) \((k \in \mathbb{Z}_{\geq 0})\), We have

\[ \text{Red}_2([c]^k) = u_0, \quad (5.1.7) \]
\[ \text{Red}_1([W]^k) = u_0, \quad (5.1.8) \]
\[ \text{Red}_1([W]^k) = u_0. \quad (5.1.9) \]

**Proof** We prove (5.1.7). By Definition 3.4 and (3.1.1) we have

\[ \Psi_2([c]^{2k}) = \Psi_2([6]^k[\pi]^k) = u_3^k u_3^k, \]
\[ \text{Red}_2([c]^{2k}) = u_0, \]
\[ \Psi_2([c]^{2k+1}) = \Psi_2([6]^k[\pi]^k) = u_3^k u_0 u_3^k, \]
\[ \text{Red}_2([c]^{2k+1}) = u_0. \]

In a similar way, we can prove (5.1.8), (5.1.9). \(\square\)

We use the following notations for \(b \in B_{A^2}(m\Lambda_1 + n\Lambda_2)\),

\[ b = \begin{bmatrix} b_1^{m+n} & \cdots & b_1^{m+1} & \cdots & b_1^1 \b_2^{m+n} & \cdots & b_2^{m+1} \end{bmatrix} \in B_{A^2}(m\Lambda_1 + n\Lambda_2) \]
\[ = \left[ [b_1^{m+n}, b_1^{m+n}] \cdot [b_1^{m+1}, b_2^{m+1}] [b_1^{m}], \ldots, [b_1^1] \right]. \quad (5.1.10) \]

The highest weight element is given by

\[ \begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 \ 
2 & \cdots & 2 \end{bmatrix} = \left[ [1, 2]^m \cdot [1]^m \right]. \]

**Proposition 5.2** Fix \(l \in \mathbb{Z}_{>0}\). Take \(i, j, p\) such that \(0 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor, i \leq j \leq l - i, 0 \leq p \leq j\). Let \(\bar{b} \in B_{A^2}(i+j)\Lambda_1 + l\Lambda_2\) be the highest weight element. Let \(\underline{b} \in B_{A^2}(i+j)\Lambda_1 + l\Lambda_2\) be the lowest weight element. Then we have

\[ \varepsilon_2 \left( \tilde{f}_1^{l+i} \tilde{f}_2^{l+j} \bar{b} \right) = 0, \quad (5.1.11) \]
\[ \varphi_2 \left( \tilde{f}_1^{l+i} \tilde{f}_2^{l+j} \bar{b} \right) = j, \quad (5.1.12) \]
\[ \tilde{f}_2 \tilde{f}_1^{l+i} \tilde{f}_2^{l+j} \bar{b} = \tilde{f}_1^{l+i-p} \tilde{f}_2^{l+p} \bar{b}, \quad (5.1.13) \]
\[ \tilde{f}_1^{l+i-p} \tilde{f}_2^{l+p} \bar{b} = \varepsilon_2 \varepsilon_1 \tilde{f}_1^{l-i} \bar{b}. \quad (5.1.14) \]
Proof By Proposition 3.13, we have
\[
\overline{b} = [\{1, 2\}^{j} [1]^{i+j}],
\]
\[
\underline{b} = [\{2, 3\}^{j} [3]^{i+j}].
\]
We prove (5.1.11), (5.1.12). Using Proposition 3.9, we have
\[
\Psi_2(\overline{b}) = u_0^{i+j}(u_0u_+)^j,
\]
\[
\text{Red}_2(\overline{b}) = u_+^j.
\]
Then we have
\[
\tilde{f}_2^j \overline{b} = [\{1, 3\}^{j} [1]^{i+j}].
\]
Similarly, we have
\[
\tilde{f}_2^{i+j} \tilde{f}_2^j \overline{b} = [\{1, 3\}^{i+j} [2]^{i+j}].
\]
Using Proposition 3.9, we have
\[
\Psi_2 \left( [\{1, 3\}^i [2, 3]^{l-i} [2]^{i+j}] \right) = u_0^{i+j}(u_+u_-)^{l-i}(u_0u_-)^j,
\]
\[
\text{Red}_2 \left( [\{1, 3\}^i [2, 3]^{l-i} [2]^{i+j}] \right) = u_+^j.
\]
Therefore, we have
\[
\varepsilon_2 \left( [\{1, 3\}^i [2, 3]^{l-i} [2]^{i+j}] \right) = 0,
\]
\[
\varphi_2 \left( [\{1, 3\}^i [2, 3]^{l-i} [2]^{i+j}] \right) = j.
\]
Relations (5.1.13), (5.1.14) can be proved by using
\[
\tilde{f}_1^{i+j-p} \tilde{f}_2^{i+p} \tilde{f}_1^j \overline{b} = [\{1, 3\}^{i+j-p} [3]^{i+j}].
\]
\[\square\]

Proposition 5.3 Let \( \overline{b} \) be the highest weight element of the crystal base \( B^{A_2}(k\Lambda_1 + j\Lambda_2) \) of \( U_q(A_2) \). Then for \( 0 \leq q < p \leq j \) we have,
\[
\tilde{f}_1 \tilde{f}_2 \left( \tilde{f}_1^q \tilde{f}_2^p \overline{b} \right) = \tilde{f}_2 \tilde{f}_1 \left( \tilde{f}_1^q \tilde{f}_2^p \overline{b} \right).
\]

Proof By (5.1.10), we have
\[
\overline{b} = [\{1, 2\}^{j} [1]^{k}].
\]
Using Proposition 3.9, we have
\[
\Psi_2(\overline{b}) = u_0^k(u_0u_+)^j,
\]
\[
\text{Red}_2(\overline{b}) = u_+^j.
\]
Therefore we have
\[
\tilde{f}_2^p \overline{b} = [\{1, 3\}^{j-p} [1, 3]^{p} [1]^{k}].
\]
Using Proposition 3.9 again, we have
\[
\Psi_1 \left( \tilde{f}_2^p \overline{b} \right) = u_+^p(u_+u_-)^p(u_+u_-)^{j-p}.
\]
Thus we have
\[
\tilde{f}_1^q \tilde{f}_2^p = \begin{cases} 
[1,2]^{j-p} [1,3]^p [1]^{k-q} [2]^q & (k > q), \\
[1,2]^{j-p} [1,3]^{p-q+k} [2,3]^{q-k} [2]^k & (k \leq q).
\end{cases}
\]

In a similar way we have,
\[
\tilde{f}_2 \tilde{f}_1 \left( \tilde{f}_1^q \tilde{f}_2^p \right) = \tilde{f}_1 \tilde{f}_2 \left( \tilde{f}_1^q \tilde{f}_2^p \right) = \begin{cases} 
[1,2]^{j-p} [1,3]^p [1]^{k-q-1} [2]^q [3] & (k > q), \\
[1,2]^{j-p} [1,3]^{p-q+k-1} [2,3]^{q-k+1} [2]^{k-1} [3] & (k < q).
\end{cases}
\]

In a similar way, we have the following Proposition.

**Proposition 5.4** Let \( \vec{b} \) be the highest weight element of \( B^{A_2}(k \Lambda_1 + j \Lambda_2) \). For \( 0 \leq p, j, 0 \leq s \leq k \), we have
\[
\tilde{e}_1 \tilde{f}_2 \tilde{f}_1^{p+s+1} \tilde{f}_2^k \tilde{b} = \tilde{f}_2 \tilde{f}_1^{p+s} \tilde{f}_2^k \tilde{b} \neq \tilde{f}_2^{s+1} \tilde{f}_1^{p+s} \tilde{f}_2^k \tilde{b}, \tag{5.1.15}
\]
\[
\tilde{e}_1 \tilde{f}_2^{s+1} \tilde{f}_1^{p+s+1} \tilde{f}_2^k \tilde{b} = \tilde{f}_2^{s+2} \tilde{f}_1^{p+s} \tilde{f}_2^k \tilde{b} \neq \tilde{f}_2^{s+1} \tilde{f}_1^{p+s} \tilde{f}_2^k \tilde{b}. \tag{5.1.16}
\]

**Proposition 5.5** Let \( \vec{b} \) be the highest weight element of crystal base \( B^{A_2}(k \Lambda_1 + j \Lambda_2) \). For an element \( \tilde{f}_1^q \tilde{f}_2^p \tilde{b} \) (\( p \leq q \leq p + k \)), we have
\[
\varphi_2 \left( \tilde{f}_1^q \tilde{f}_2^p \tilde{b} \right) = j + q - 2p. \tag{5.1.17}
\]

**Proof** Using Proposition 5.9 we have,
\[
\text{Red}_2 \left( \tilde{f}_1^q \tilde{f}_2^p \tilde{b} \right) = u_+^{j+p-2q}.
\]

**Proposition 5.6** For \( b \in B^{G_2}(l \Lambda) \) such that \( \tilde{f}_2 b \neq 0 \), there exists unique \( k' \in \{0, \ldots, \varphi_2(b) - 1\} \) such that
\[
\varphi_1 (\tilde{f}_2^{(k+1)}(b)) = \begin{cases} 
\varphi_1 (\tilde{f}_2^k(b)) & k \leq k', \\
\varphi_1 (\tilde{f}_2^k(b)) + 1 & k > k'.
\end{cases}
\]

where \( k = 0, \ldots, \varphi_2(b) - 1 \).

**Proof** Put \( b = b_1 \otimes \cdots \otimes b_l \), and \( r_1 r_2 \cdots r_{\varphi_1(b)} + \varphi_1(b) = \text{Red}_n(b) \). Let \( k_1 \) be \( r_{\varepsilon_1(b)}(b) \) if \( \varepsilon_1(b) \neq 0 \), 0 if \( \varepsilon_1(b) = 0 \). By Proposition 3.3, there exists integers \( k_2 \in \mathbb{Z}_{\geq 0} \) such that \( \tilde{f}_2 b = b_1 \otimes \cdots \otimes \tilde{f}_2 b_{k_2} \otimes \cdots \otimes b_l \).

First we consider \( \Psi_1(\tilde{f}_2^k b_{k_2}) \), using \( u_+, u_- \) in Definition 3.4
\[
\text{if } b_{k_2} = [2], [5], [6], [4], \text{ then } \Psi_1(b_{k_2}) = u_n^+, \Psi_1(\tilde{f}_2^k b_{k_2}) = u_n^{n-1} (n = 1, 2). \tag{5.1.18}
\]
\[
\text{if } b_{k_2} = [3], [4], [0], [3], \text{ then } \Psi_1(b_{k_2}) = u_n^+, \Psi_1(\tilde{f}_2^k b_{k_2}) = u_n^{n+1} (n = 0, 1). \tag{5.1.19}
\]

We consider \( \Psi_1(\tilde{f}_2 b) \). By Remark 3.10, we have \( \text{Red}_1(b) = u_0^{\varepsilon_1(b)} u_+^{\varepsilon_2(b)} \).

The case of \( 1 \leq k_2 \leq k_1 \). If (5.1.18), we see that
\[
\text{Red}_1(\tilde{f}_2 b) = u_0^{\varepsilon_1(b)-1} u_+^{\varepsilon_2(b)}. \tag{5.1.20}
\]

If (5.1.19), we see that the increased \( u_+ \) is paired with a neighboring \( u_- \) and reduced. Then we have (5.1.20).
The case of \(k_1 + 1 \leq k_2 \leq l\).

If \((5.1.18)_i\), the decreased \(u_-\) is paired with \(u_+\) in \(\Psi_1(b)\). Then we see

\[
\text{Red}_1(\tilde{f}_2b) = u_-^{\varepsilon_i(b)}u_+^{\varphi_i(b)+1}.
\] (5.1.21)

If \((5.1.19)_i\), we see \((5.1.20)_i\). Therefore, we have

\[
\tilde{f}_2b' = b'_1 \otimes \cdots \otimes b'_l.
\]

If \(\tilde{f}_2b' \neq 0\), we have

\[
\tilde{f}_2b' = b'_1 \otimes \cdots \tilde{f}_2b_k \otimes \cdots \otimes b'_l,
\]

where \(k' \in \mathbb{Z}_{\geq 0}\). Here the action of \(\tilde{f}_2\) is given by changing the leftmost \(u_+\) to \(u_-\) in \(\text{Red}_2(b)\) and \(\text{Red}_2(b') = u_-^{\varepsilon_i(b')}u_+^{\varphi_i(b')}\). Then we see that \(k' \geq k_2\).

Thus we have the proposition.

### 5.2 Construction of crystal base \(B^l\) of \(U_q(G_2^{(1)})\)

#### 5.2.1 Structure of \(B^l\)

As we noted earlier, \(U_q((G_2^{(1)})_{\{1,2\}})\) is isomorphic to \(U_q(G_2)\) and \(U_q((G_2^{(1)})_{\{1,0\}})\) is isomorphic to \(U_q(A_2)\). Since the structure of \(U_q(A_2)\) is simpler than that of \(U_q(G_2)\), we define the actions of \(\tilde{f}_0\) on \(U_q(G_2)\) by exploiting operators on \(U_q(A_2)\). In view of isomorphism between \(U_q((G_2^{(1)})_{\{1,0\}})\) and \(U_q(A_2)\), we write \([1,0]\) as index set of roots of \(U_q(A_2)\). The affine crystal \(B^l\) for \(U_q(G_2^{(1)})\) is constructed on \(G^l\) with \(\tilde{f}_0^G\).

**Definition 5.7** Fix \(l \in \mathbb{Z}_{>0}\). We define crystal base \(G^l\) of \(U_q(G_2)\) and crystal base \(A\) of \(U_q(A_2)\) as follows.

\[
G^l = \bigoplus_{n=0}^l B^{G_2}(nA_1),
\]

\[
A_i = \bigoplus_{i \leq j, j_0 \leq l-i} B^{A_2}, (j_1A_1 + j_0A_0)
\]

\[
A = \bigoplus_{i=0}^{24} A_i.
\]

We denote \(\tilde{f}_i, \tilde{e}_i, \varphi_i, \varepsilon_i,\) wt, wt\(_i\) \((i = 1, 0)\) on \(A\) by \(\tilde{f}_i^A, \tilde{e}_i^A, \varphi_i^A, \varepsilon_i^A,\) wt\(_i^A\) respectively. In a similar way, we denote \(\tilde{f}_i, \tilde{e}_i, \varphi_i, \varepsilon_i,\) wt, wt\(_i\) \((i = 1, 2)\) on \(G\) by \(\tilde{f}_i^G, \tilde{e}_i^G, \varphi_i^G, \varepsilon_i^G,\) wt\(_i^G\) respectively.

**Proposition 5.8**

\[
\mathcal{Z}G^l = \mathcal{Z}A
\]

**Proof** The proposition amounts to show

\[
\sum_{n=0}^l \mathcal{Z}B^{G_2}(nA_1) = \sum_{i=0}^{24} \sum_{i \leq j, j_2 \leq l-i} \mathcal{Z}B^{A_2}(j_1A_1 + j_2A_2).
\] (5.2.1)

Let \(\Delta^+\) be the set of positive roots, put \(\delta = \frac{1}{2} \sum_{\beta \in \Delta^+} \beta\), and let \(d(\Lambda)\) be the dimension of finite dimensional irreducible representation with highest weight \(\Lambda\). Then we have

\[
d(\Lambda) = \prod_{\beta \in \Delta^+} \frac{(\Lambda + \delta, \beta)}{(\delta, \beta)}
\]

by \([3]\). Therefore, we have

\[
\mathcal{Z}B^{G_2}(nA_1) = (n+1) \left(\frac{1}{2}n+1\right) \left(\frac{2}{3}n+1\right) \left(\frac{3}{4}n+1\right) \left(\frac{4}{5}n+1\right),
\]

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It is possible that there exists crystal bases with same highest weight in \( \Lambda_1 \). By calculation, we see that

\[
\sharp B^{A_2}(n_1\Lambda_1 + n_2\Lambda_2) = \frac{1}{2}(n_1 + 1)(n_2 + 1)(n_1 + n_2 + 2).
\]

By calculation, we see that

\[
\sharp B^{G_2}(l\Lambda_1) = \sum_{i=0}^{\left\lfloor \frac{l}{2} \right\rfloor} \left( \sum_{k=i}^{l-i-1} \sharp B^{A_2}(k\Lambda_1 + (l-i)\Lambda_2) + \sum_{j=i}^{l-i} \sharp B^{A_2}((l-i)\Lambda_1 + j\Lambda_0) \right). \tag{5.2.2}
\]

Thus, we have (5.2.1).

By Proposition 5.8, we can construct one-to-one correspondence between \( G \) and \( A \) as sets.

Let \( \overline{r}_{j_1,j_0}^{l,i} \in A_i \) be the heighest weight element with weight \( j_1\Lambda_1 + j_0\Lambda_0 \). Let \( \overline{r}_{-j_1,-j_0}^{l,i} \in A_i \) be the lowest weight element with weight \(-j_1\Lambda_1 - j_0\Lambda_0 \). Elements \( \overline{r}_{j_1,j_0}^{l,i} \) and \( \overline{r}_{-j_1,-j_0}^{l,i} \) satisfy

\[
\overline{r}_{j_1,j_0}^{l,i} = (\varepsilon_0^A)^{-j_0}(\varepsilon_1^A)^{j_1+j_0}(\varepsilon_0^A)^{j_1}\overline{r}_{-j_1,-j_0}^{l,i}.
\]

For a Lie algebra of type \( A_2 \), it is known that all elements \( b \) in the crystal base \( B^{A_2}(j_1\Lambda_1 + j_0\Lambda_0) \) are uniquely expressed as

\[
b = (\tilde{f}_0^A)^r(\tilde{f}_1^A)^q(\tilde{f}_0^A)^p\overline{r}_{j_1,j_0}^{l,i} \quad (0 \leq p \leq j_0, p \leq q \leq p + j_1, 0 \leq r \leq j_0 + q - 2p), \tag{5.2.3}
\]

It is possible that there exists crystal bases with same highest weight in \( A_i \) and \( A_i' \) (\( i \neq i' \)). To distinguish them, we define crystal base \( B_{j_1,j_0}^{l,i} \subset A_i \) by

\[
B_{j_1,j_0}^{l,i} = \left\{ (\tilde{f}_0^A)^r(\tilde{f}_1^A)^q(\tilde{f}_0^A)^p\overline{r}_{j_1,j_0}^{l,i} \mid 0 \leq p \leq j_0, p \leq q \leq p + j_1, 0 \leq r \leq j_0 + q - 2p \right\}. \tag{5.2.4}
\]

By calculation, we have the following proposition.

**Proposition 5.9** For \( B_{j_1,j_0}^{l,i} \) \( (i < k, j \leq l - i) \), we have

\[
B_{j_1,j_0}^{l,i} \cup \left\{ (\tilde{f}_1^A)^q(\tilde{f}_0^A)^p\overline{r}_{j_1,j_0}^{l,i} \mid i < k, j \leq l - i, 0 \leq p, 0 \leq q \leq p + k \right\} \cup \left\{ (\varepsilon_0^A)^q(\varepsilon_1^A)^p\overline{r}_{-j_1,-j_0}^{l,i} \mid 0 \leq p, 0 \leq q \leq p + k \right\}
\]

\[
\cong B^{A_2}((k - 1)\Lambda_1 + (j - 1)\Lambda_0). \tag{5.2.5}
\]

For \( l \in \mathbb{Z}_{>0} \), we define

\[
A_{l}^{(i)} = \left( \bigcup_{i \leq k \leq l - i} B_{k,i}^{i} \right) \bigcup \left( \bigcup_{i \leq j \leq l - q} B_{i,j}^{i} \right) \bigcup \left( \bigcup_{i \leq k \leq l - i} (\tilde{f}_1^A)^q(\tilde{f}_0^A)^p\overline{r}_{j_1,j_0}^{l,i} \right) \bigcup \left( \bigcup_{i \leq k, j \leq l - i, 0 \leq p, 0 \leq q \leq p + k} (\varepsilon_0^A)^q(\varepsilon_1^A)^p\overline{r}_{-j_1,-j_0}^{l,i} \right).
\]

\[
\mathcal{A}^l = \bigcup_{0 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor} A_{l}^{(i)}.
\]

**Proposition 5.10** For \( l \in \mathbb{Z}_{>0} \), we have

\[
\sharp B^{G_2}(l\Lambda_1) = \sharp \mathcal{A}^l.
\]
Proof By Proposition 5.9, we have

\[ \#B^{A_2} ((l - i)A_1 + (l - k)A_0) = \#B^{l_{(k,i)}} + \sum_{m=1}^{l-i-k} \left\{ \left( \bar{f}_0^A \right)^q \left( \bar{f}_0^A \right)^p \bar{u}^l_{(k+m,i+m)} \right\} 0 \leq p \leq i + m \]

\[ + \sum_{m=1}^{l-i-k} \left\{ \left( \tilde{e}_1^A \right)^q \left( \tilde{e}_0^A \right)^p \bar{u}^l_{(k-m,i-k-m)} \right\} 0 \leq q \leq p + k + m \]

\[ + \sum_{m=1}^{l-i-j} \left\{ \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \bar{u}^l_{(i+m,j+m)} \right\} 0 \leq p \leq j + m \]

\[ + \sum_{m=1}^{l-i-j} \left\{ \left( \tilde{e}_1^A \right)^q \left( \tilde{e}_0^A \right)^p \bar{u}^l_{(j-m,i-m)} \right\} 0 \leq q \leq p + j + m \].

By (5.2.2), we have

\[ \sum_{j=i+1}^{l-i} \left( \#B^{A_2} ((l - i)A_1 + (l - k)A_0) \right) = \#B^{G_{26}} (lA_1). \]

By Proposition 5.9 and Proposition 5.10, we have the following lemma.

**Lemma 5.11** For \( l \in \mathbb{Z}_{\geq 0} \) we have

\[ G^l \setminus A^{(l)} \cong G^{l-1}. \]

as crystal base of \( U_q(A_2) \).

### 5.2.2 Operators \( E_A \) and \( F_A \) on \( A \)

We are going to define operators \( E_A \) and \( F_A \) on \( A \) which satisfy properties

(C1) for \( b, b' \in A, E_A b = b' \), if and only if \( F_A b' = b \),

(C2) for \( b \in A, E_A \tilde{f}_0^A (b) = \tilde{f}_0^A E_A (b) \),

(C3) for \( b \in A, \max\{ m \mid F^{\lambda^d}_{\lambda^d}(b) \neq 0 \} - \max\{ m' \mid E^{\lambda^d}_{\lambda^d}(b) \neq 0 \} = -2w^i d_1(b) - w^i d_0(b) \).

Operators \( E_A \) and \( F_A \) are counterpart of \( \tilde{e}_0^A \) and \( \tilde{f}_0^A \), respectively. In view of Lemma 5.2.3 and (C2) we define \( E_A \) and \( F_A \) inductively. If \( l = 0, A \) is trivial. If \( l > 0, \) we use following relation for induction

\[ E_A \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \bar{u}^l_{(k,j)} \right) = \tilde{e}_0^A \left( E_A \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \bar{u}^l_{(k-1,j-1)} \right) \right). \] (5.2.6)

Put \( A^{(l)}_+ = \{ b \mid \tilde{e}_0^A (b) = 0 \} \). Using Proposition 3.3 we see that \( A^{(l)}_+ \) is given by

\[ A^{(l)}_+ = \left\{ \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \bar{u}^l_{(k,j)} \mid 0 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor, l \leq i \leq l - i, 0 \leq p \leq j, p \leq q \leq p + k \right\}. \]

By (C1) and (C2), it is sufficient to define the operator \( E_A \) for \( b \in A^{(l)}_+ \).

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Definition 5.12 We define $E_A$ by

$$E_A \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tilde{T}_{l,i}^{k,j} \right) = \begin{cases} 
\left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tilde{b}_{(k-1,j,i)}^{l,i}, & \text{(a1)} \\
\left( \tilde{f}_1^A \right)^{q+1} \left( \tilde{f}_0^A \right)^{p+1} \tilde{b}_{(k,j+1,i)}^{l,i}, & \text{(a2)} \\
\left( \tilde{f}_1^A \right)^{q+1} \left( \tilde{f}_0^A \right)^p \tilde{b}_{(k+1,j+1,l,i)}^{l,i+1}, & \text{(a3)} \\
\left( \tilde{f}_1^A \right)^{q+1} \left( \tilde{f}_0^A \right)^p \tilde{b}_{(k+1,j,l,i)}^{l,i+1}, & \text{(a4)} \\
\left( \tilde{f}_1^A \right)^{q+1} \left( \tilde{f}_0^A \right)^p \tilde{b}_{(k+1,j,l,i)}^{l,i+1}, & \text{(a5)} \\
0, & \text{(a6)} \\
\tilde{e}_0^A \left( E_A \left( \left( \tilde{f}_1^A \right)^{q-1} \left( \tilde{f}_0^A \right)^p \tilde{b}_{(k-1,j-1,l,i)}^{l-1,i} \right) \right). & \text{(5.2.6)}
\end{cases}$$

Conditions for (a1)-(a6) and (5.2.6) are as follows:

1. If $i < k \leq l - i$, $i < j \leq l - i$,
   
   (i) if $0 \leq q \leq j - 1 - \left[ \frac{i-k}{3} \right]$, $p = \min(q,j)$, then the action of $E_A$ is given by (a1),
   
   (ii) if $j < l - i$, $j - \left[ \frac{i-k}{3} \right] \leq q \leq j + k$, $p = \min(q,j)$, then the action of $E_A$ is given by (a2),
   
   (iii) if $j = l - i$, $j - \left[ \frac{i-k}{3} \right] \leq q \leq j + k$, $p = \min(q,j)$, then the action of $E_A$ is given by (a6),

2. If $k = i$, $i + 2 \leq j \leq l - i$,
   
   (i) if $0 \leq p \leq j - 1 - \left[ \frac{i-k}{3} \right]$, $p \leq q \leq p + i$ then the action of $E_A$ is given by (a3),
   
   (ii) if $j \leq k$, $j - \left[ \frac{i-k}{3} \right] \leq p \leq j$, $p \leq q \leq p + i$ then the action of $E_A$ is given by (a2),

3. If $k = j$, $j = i + 1$,
   
   (i) if $0 \leq p \leq i$, $p \leq q \leq p + i$ then the action of $E_A$ is given by (a4),
   
   (ii) if $p = i + 1$, $p \leq q \leq p + i$ then the action of $E_A$ is given by (a2),

4. If $k = i$, $j = i$,
   
   (i) if $0 \leq p \leq i - 1$, $p \leq q \leq p + i$ then the action of $E_A$ is given by (a5),
   
   (ii) if $p = i$, $i \leq q \leq 2i$ then the action of $E_A$ is given by (a2).

5. If $i + 1 \leq k \leq l - i$, $j = i$,
   
   (i) if $0 \leq p \leq i$, $p \leq q \leq p - 1 - \left[ \frac{i-k}{3} \right]$ then the action of $E_A$ is given by (a1),
   
   (ii) if $p = i$, $p - \left[ \frac{i-k}{3} \right] \leq q \leq p + k$ then the action of $E_A$ is given by (a2),
   
   (iii) if $0 \leq p \leq i - 1$, $p - \left[ \frac{i-k}{3} \right] \leq q \leq p + k$ then the action of $E_A$ is given by (a5).
By (C1) and definition of $E_A$, we see the action of $F_A$ on $A^{(l)}_+$ is given by

$$F_A\left(\left(\tilde{f}^A_1\right)^q \left(\tilde{f}^A_0\right)^p \tilde{b}_{(k,j)}\right) = \begin{cases} 
\tilde{f}^A_1 \tilde{f}^A_0 \tilde{b}_{(k+1,j)}, 
\tilde{f}^A_1 \tilde{f}^A_0 \tilde{b}_{(k,j-1)}, 
\tilde{f}^A_1 \tilde{f}^A_0 \tilde{b}_{(k-1,j+1)}, 
\tilde{f}^A_1 \tilde{f}^A_0 \tilde{b}_{(k-1,j+1)}, 
\tilde{f}^A_1 \tilde{f}^A_0 \tilde{b}_{(k-1,j+1)}, 
0, 
\tilde{e}^A_0 \left(F_A\left(\left(\tilde{f}^A_1\right)^{q-1} \left(\tilde{f}^A_0\right)^p \tilde{b}_{(k-1,j-1)}\right)\right). 
\end{cases} \tag{5.2.6}'
$$

As in the case of $E_A$, we use induction \([5.2.6]'\). Conditions for \((a1)'-(a6)'\) are as follows:

1. If $i < k \leq l - i$, $i < j \leq l - i$,
   (i) if $k < l - i$, $0 \leq q \leq j - 1 - \left\lfloor \frac{i - k - 1}{3} \right\rfloor$, $p = \min(q, j)$, then the action of $F_A$ is given by \((a1)'\),
   (ii) if $k = l - i$, $0 \leq q \leq j - 1 - \left\lfloor \frac{i - k - 1}{3} \right\rfloor$, $p = \min(q, j)$, then the action of $F_A$ is given by \((a6)'\),
   (iii) if $j - \left\lfloor \frac{i - k - 1}{3} \right\rfloor \leq q \leq j + k$, $p = \min(q, j)$, then the action of $F_A$ is given by \((a2)'\),
   (iv) if $0 \leq p \leq j - 1$, $p + 1 \leq q \leq p + k$, then we use induction \([5.2.8]'\).
2. If $k = i$, $i + 1 \leq j \leq l - i$,
   (i) if $0 \leq p \leq j - 1 - \left\lfloor \frac{i - k - 1}{3} \right\rfloor$, $q = p$ then the action of $F_A$ is given by \((a1)'\),
   (ii) if $0 \leq p \leq j - 1 - \left\lfloor \frac{i - k - 1}{3} \right\rfloor$, $p + 1 \leq q \leq p + k + 1$ then the action of $F_A$ is given by \((a3)'\),
   (iii) if $j - \left\lfloor \frac{i - k - 1}{3} \right\rfloor \leq p \leq j$, $p \leq q \leq p + k$, then the action of $F_A$ is given by \((a2)'\).
3. If $k = i$, $j = i$,
   (i) if $0 \leq p \leq i$, $p = q$ then the action of $F_A$ is given by \((a2)'\),
   (ii) if $0 \leq p \leq i$, $p + 1 \leq q \leq p + i$ then the action of $F_A$ is given by \((a3)'\).
4. If $k = i + 1$, $j = i$,
   (i) if $0 \leq p \leq i$, $p = q$ then the action of $F_A$ is given by \((a2)'\),
   (ii) if $0 \leq p \leq i$, $p + 1 \leq q \leq p + i + 1$ then the action of $F_A$ is given by \((a4)'\).
5. If $i + 2 \leq k \leq l - i$, $j = i$,
   (i) if $0 \leq p \leq i$, $p \leq q \leq p - 1 - \left\lfloor \frac{i - k - 1}{3} \right\rfloor$ then the action of $F_A$ is given by \((a1)'\),
   (ii) if $0 \leq p \leq i$, $p - \left\lfloor \frac{i - k - 1}{3} \right\rfloor \leq q \leq p + k$ then the action of $F_A$ is given by \((a5)'\).

For $b \in A$, we put

$$\varepsilon_A(b) = \max\{m \mid F^m_A b \neq 0\},$$

$$\varphi_A(b) = \max\{m' \mid E^{m'}_A b \neq 0\}.$$

We verify that $E_A$ is well-defined. By the definition of $F_A$, we see that for $b, b' \in A^{(l)}_+$,

if $E_A(b) = E_A(b') \neq 0$, then $b = b'$. \tag{5.2.7}
In order to verify (C2), we prove that $E_A(f^A_0(b)) \neq 0$ if and only if $f^A_0(E_A(b)) \neq 0$. For this, it is sufficient to prove that for $b \in A^{(l)}_+$,

$$\varphi_0(b) = \varphi_0(E_A(b)).$$

(5.2.8)

Using Proposition 5.3, we have following formula in the case of (a1) of Definition 5.12

$$\varphi_0\left(f^q_1 f^p_0 b_{(k,j)} \right) = j + q - 2p,$$

and

$$\varphi_0\left(E_A f^q_1 f^p_0 b_{(k,j)} \right) = \varphi_0\left(f^q_1 f^p_0 b_{(k-1,j)} \right) = j + q - 2p.$$

In a similar way we can prove other cases. Therefore we see that for $b \in A$ such that $E_A(b) \neq 0$, $f^A_0 b \neq 0$ we have $E_A(f^A_0 b) \neq 0$. Similarly we can show that for $b \in A$ such that $E_A(b) = 0$ or $f^A_0 b = 0$ we have $E_A(f^A_0 b) = 0$.

We verify (C3). By calculation, we see

$$\text{wt}^A\left(f^q_1 f^p_0 b_{(k,j)} \right) = (k + p - 2q)A_1 + (j - 2p + q)A_0.$$  

(5.2.9)

If the action of $E_A$ is given by (a1), we have

$$\text{wt}^A\left(E_A f^q_1 f^p_0 b_{(k,j)} \right) = \text{wt}^A\left(f^q_1 f^p_0 b_{(k-1,j)} \right) = (k - 1 + p - 2q)A_1 + (j - 2p + q)A_0.$$

Therefore, we have

$$-2\text{wt}^A(E_A b) - \text{wt}^A(E_A b) = -2\text{wt}^A(b) - 2\text{wt}^A(b) + 2.$$  

(5.2.10)

Similarly we can show (5.2.10) when the action of $E_A$ is given by (a2) - (a5).

We prove (C3) for $b = \left(f^A_1 \right)^q \left(f^A_0 \right)^p b_{(i-j)} \in A^{(l)}_+$ where $i \leq j \leq l - i$, $0 \leq p \leq j$, $p \leq q \leq p + l - i$. By Definition 5.12, $b$ satisfies $F_A(b) = 0$ and

$$\varepsilon_A(b) = 2l - 2i + j - 3q.$$  

(5.2.11)

1. If $0 \leq p = q \leq \left[\frac{i-j}{2}\right],$

The action of $E_A$ on $f^p_1 f^p_0 b_{(i-j)}$ ($i < k \leq l - i$) is given by (a1). Then we have

$$E_A^{l-2i}(b) = f^p_1 f^p_0 b^{(i-j)}_{(i,j)}.$$  

If $\left[\frac{i-j}{2}\right] > 0$, the action of $E_A$ on $f^p_1 f^p_0 b^{(i-j)}_{(i-j)}$ is given by (a3). We put $x = \frac{i+j}{2}$ and $\hat{x} = \frac{i+j+1}{2}$. Then we have

$$E_A^{l-3i+|x|}(b) = \begin{cases}
(f^p_1 f^p_0 b^{(i-j)}_{(x-x)}) & j - i \text{ is odd}, \\
(f^p_1 f^p_0 b^{(i-j)}_{(x-x)}) & j - i \text{ is even}.
\end{cases}$$

If $i - j$ is odd, the action of $E_A$ on $f^p_1 f^p_0 b^{(i-j)}_{(i-j-1)}$ is given by (a4). Then we have

$$E_A^{l-3i+|x|}(b) = f^p_1 f^p_0 b^{(i-j)}_{(i-j-1)}.$$  

If $j - p - [\hat{x} - i] > 0$, then the action of $E_A$ on $f^p_1 f^p_0 b^{(i-j)}_{(x-x)}$, $f^p_1 f^p_0 b^{(i-j-1)}_{(x-x)}$ are given by (a5). Then we have

$$E_A^{l-2i+|x|}(b) = f^p_1 f^p_0 b^{(i-j)}_{(i-j-i-p,p)}.$$  

Moreover, the action of $E_A$ on $f^p_1 f^p_0 b^{(i-j)}_{(i-j-i-p,p)}$ is given by (a2). Then we have

$$E_A^{l-2i+|x|}(b) = f^p_1 f^p_0 b^{(i-j-i-j+p)}_{(i-j-i-p,l-p)}.$$  

(5.2.12)

Finally, the action of $E_A$ on $f^p_1 f^p_0 b^{(i-j)}_{(i-j-i-p,l-p)}$ is given by (a6). Then we have

$$E_A^{l-2i+|x|}(b) = 0.$$
2. If \( \left\lfloor \frac{i+1}{3} \right\rfloor < p = q < \frac{2i+1}{3} \), we can calculate in a similar way:

\[
E_{A}^{-2i}(b) = f_{1}^{p} f_{0}^{p(i,j)}, \quad (a1)
\]

\[
E_{A}^{i+2j-3p}(b) = f_{1}^{p} f_{0}^{p(2j+3p-j-i+3p)}, \quad (a3)
\]

\[
E_{A}^{2i-2j-3p}(b) = f_{1}^{p} f_{0}^{p(2j+2i-3p-l_2-j+2i+3p)}, \quad (a2)
\]

\[
E_{A}^{2i-2j-3p+1}(b) = 0. \quad (a6)
\]

3. If \( \frac{2i+1}{3} < p = q < j \) or \( p < q < y \), we have

\[
E_{A}^{i+2j-3q}(b) = f_{1}^{p} f_{0}^{p(3q-j-i)}, \quad (a1)
\]

\[
E_{A}^{2i-2j-3q}(b) = f_{1}^{p} f_{0}^{p(2j+3q-4p, l-p)}, \quad (a2)
\]

\[
E_{A}^{2i-3q+1}(b) = 0. \quad (a6)
\]

4. If \( p < j = i, p < q < y + j \), we have

\[
E_{A}^{i+2j-3(p-q)}(b) = f_{1}^{p} f_{0}^{p(i+3(p-q), i)}, \quad (a1)
\]

\[
E_{A}^{i+q+2p}(b) = f_{1}^{p} f_{0}^{p(2i+3q-4p, l-p)}, \quad (a5)
\]

\[
E_{A}^{2i-3q}(b) = f_{1}^{p} f_{0}^{p(2i+3q-4p, l-p)}, \quad (a2)
\]

\[
E_{A}^{2i-3q+1}(b) = 0. \quad (a6)
\]

5. Other cases are reduced to \( G^{l-1} \), using (5.2.4).

Then for \( b = (f_{A}^{l})^{q} (f_{0}^{l})^{p} b_{(i,j)}^{l,i} \in A^{(l)} \) where \( i \leq j \leq l-i, \ 0 \leq p \leq j, \ p \leq q \leq p + l - i \), we have

\[-2\text{wt}^{A}_{1}(b) - \text{wt}^{A}_{0}(b) = -(2l - 2i + j - 3q).\]

By (5.2.11), we see

\[\varphi_{A}(b) - \varepsilon_{A}(b) = -(2l - 2i + j - 3q).\]

Therefore by (5.2.10), we verify inductively that \( E_{A} \) satisfies (C3).

5.2.3 An involution on \( A \)

Let \( b_{(i,j)}^{l,i} \) be the lowest weight element in \( A_{i,j} \) with weight \( j_{1}A_{1} + j_{0}A_{0} \). We define a map \( C_{A} : A \rightarrow A \) by:

\[
C_{A} \left( \left( f_{0}^{A} \right)^{q} \left( f_{1}^{A} \right)^{p} b_{(i,j)}^{l,i} \right) = \left( \tilde{e}_{0}^{A} \right)^{q} \left( \tilde{e}_{1}^{A} \right)^{p} \left( \tilde{b}_{(i,j)}^{l,i} \right)
\]

\[
= \left( f_{0}^{A} \right)^{j+q+2p} \left( f_{1}^{A} \right)^{k+l+q} \left( f_{0}^{A} \right)^{k-q+p} \left( f_{1}^{A} \right)^{k+q-p} \left( \tilde{b}_{(i,j)}^{l,i} \right). \quad (5.2.16)
\]

where \( 0 \leq p \leq j, \ p \leq q \leq p + k, \ 0 \leq r \leq j + q - 2p \). It is easy to see that \( C_{A} \) is an involution,

\[
C_{A} \left( C_{A}(b) \right) = b \ (b \in A).\]

Proposition 5.13 For \( b \in A \), we have

\[
C_{A} \left( E_{A}b \right) = F_{A}(C_{A}(b)). \quad (5.2.17)
\]
Proof It is sufficient to prove (5.2.17) for \( b \in \mathcal{A} \) such that \( \varepsilon_0^{\mathcal{A}}(b) = 0 \). We assume that for \( b = \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tilde{b}_{l,i}^{(i,j)} \in \mathcal{A} \), \( k, j, p, q \) satisfy \( i + 1 \leq k \leq l - i, \, i + 1 \leq j \leq l - i, \) \( 0 \leq q \leq j - 1 - \frac{k - l}{3} \), \( p = \min(q, j) \). Since this is the condition (1)(i) in Definition 5.12 the action of \( E_\mathcal{A} \) is given by (a1). Then we have

\[
C_\mathcal{A}(E_\mathcal{A} b) = \left( \tilde{f}_0^A \right)^{i+j+q} \left( \tilde{f}_1^A \right)^{k+j+q-1} \left( \tilde{f}_0^A \right)^{k+q+p-1} \tilde{b}_{l,i}^{(i,k-1)}.
\]

We consider the action of \( F_\mathcal{A} \) on \( C_\mathcal{A}(b) \). By applying the involution, we have

\[
C_\mathcal{A}(b) = \left( \tilde{f}_0^A \right)^{i+j-2p} \left( \tilde{f}_1^A \right)^{k+j-q} \left( \tilde{f}_0^A \right)^{k+q+p} \tilde{b}_{l,i}^{(i,k)}.
\]

Thus the action of \( F_\mathcal{A} \) on \( C_\mathcal{A}(b) \) is given by \((a2)\)' Therefore we have

\[
F_\mathcal{A}(C_\mathcal{A}(b)) = \left( \tilde{f}_0^A \right)^{i+j+q} \left( \tilde{f}_1^A \right)^{k+j+q-1} \left( \tilde{f}_0^A \right)^{k+q+p} \tilde{b}_{l,i}^{(i,k-1)}.
\]

In a similar way, we have

if the action of \( E_\mathcal{A} b \) is given by \((a1)\), then the action of \( F_\mathcal{A}(C_\mathcal{A}(b)) \) is given by \((a2)\)',

if the action of \( E_\mathcal{A} b \) is given by \((a2)\), then the action of \( F_\mathcal{A}(C_\mathcal{A}(b)) \) is given by \((a1)\)',

if the action of \( E_\mathcal{A} b \) is given by \((a3)\), then the action of \( F_\mathcal{A}(C_\mathcal{A}(b)) \) is given by \((a5)\)',

if the action of \( E_\mathcal{A} b \) is given by \((a4)\), then the action of \( F_\mathcal{A}(C_\mathcal{A}(b)) \) is given by \((a4)\)',

if the action of \( E_\mathcal{A} b \) is given by \((a5)\), then the action of \( F_\mathcal{A}(C_\mathcal{A}(b)) \) is given by \((a3)\)',

if the action of \( E_\mathcal{A} b \) is given by \((a6)\), then the action of \( F_\mathcal{A}(C_\mathcal{A}(b)) \) is given by \((a6)\)'.

Thus we have \( C_\mathcal{A}(E_\mathcal{A}(b)) = F_\mathcal{A}(C_\mathcal{A}(b)) \).

For \( a \in \mathbb{Z} \), we define \( a_+ \) by

\[
a_+ = \begin{cases} a & (a > 0), \\ 0 & (a \leq 0). \end{cases}
\]

We set \( B_C, B_W, B_U, B_R \subset \mathcal{A}^{(l)} \) by

\[
B_C = \left\{ \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tilde{b}_{l,i}^{(i,j)} \mid 0 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor, \, i \leq j \leq l - i, \, 0 \leq q \leq p \leq j \right\},
\]

\[
B_W = \left\{ \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tilde{b}_{l,i}^{(i,j)} \mid 0 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor, \, j = i, \, 0 \leq p \leq j, \, p < q \leq y + j \right\},
\]

\[
B_U = \left\{ \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tilde{b}_{l,i}^{(i,j)} \mid 0 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor, \, i \leq j \leq l - i, \, p = j, \, j < q \leq y + 2j - i \right\},
\]

\[
B_R = \left\{ \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tilde{b}_{l,i}^{(i,j)} \mid 0 \leq i \leq \frac{l}{2}, \, i < j \leq l - i, \, 0 \leq q < j, \, p < q \leq y + j - (i - p)_+ \right\},
\]

where \( y = y \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tilde{b}_{l,i}^{(i,j)} \right) = \frac{l - 3i - 1}{3} \).

In view of Definition 5.12, the conditions \( \varepsilon_0^{\mathcal{A}}(b) = 0 \), \( \varphi_\mathcal{A}(b) = 0 \) on \( b = \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tilde{b}_{l,i}^{(i,j)} \) \( 0 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor, \, i \leq j \leq l - i, \, 0 \leq p \leq j \) are rephrased as

\[
\begin{align*}
0 & \leq q \leq p - \frac{t + i + j}{3} + (j - i) \quad (p \leq i, \, l - i - j = 0 \mod 3), \\
0 & \leq q \leq p - 1 - \frac{t + i + j}{3} + (j - i) \quad (p \leq i, \, l - i - j \neq 0 \mod 3), \\
0 & \leq q \leq j - \frac{t + i + j}{3} \quad (p > i, \, l - i - j = 0 \mod 3), \\
0 & \leq q \leq j - 1 - \frac{t + i + j}{3} \quad (p > i, \, l - i - j \neq 0 \mod 3).
\end{align*}
\]
Summarizing, we have

\[ 0 \leq q \leq \left\lfloor \frac{l - i - j}{3} \right\rfloor + j - (i - p)_+ = y + j - (i - p)_+. \]

Thus we have

\[ \{ b \in A | e^A_0(b) = 0, \varphi_A(b) = 0 \} = \left\{ \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tau_{l,i} b_{l-i,j} \right\} \right| 0 \leq i \leq \left\lfloor \frac{l}{2} \right\rfloor, \quad i \leq j \leq l - i, \quad 0 \leq p \leq j, \quad p \leq q \leq y + j - (i - p)_+ \right\}. \]

Hence, we see easily that

\[ \{ b \in A | e^A_0(b) = 0, \varphi_A(b) = 0 \} \subset B_C \cup B_W \cup B_U \cup B_R. \]

By Proposition 5.3, we have

\[ \{ \left( \tilde{f}_0^A \right)^r \left( \tilde{f}_1^A \right)^p \left( \tilde{f}_0^A \right)^p \tau_{l,i} b_{l-i,j} \} \right| 0 \leq p \leq j, \quad 0 \leq r \leq j - p \} = B_C. \quad (5.2.18) \]

**Proposition 5.14** For \( b \in B_W \cup B_U \), we have

\[ C_A \left( E^A_{\varphi_A(b)} \right) \in B_C. \]

**Proof** By calculation for \( b = \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^p \tau_{l,i} b_{l-i,j} \in B_W \) (0 \( p < i, \quad p < q \leq y + i \)) we have

\[ C_A \left( E^A_{\varphi_A(b)} \right) = \left( \tilde{f}_1^A \right)^{i+2q-4p} \left( \tilde{f}_0^A \right)^{2i+3q-4p} b_{l-p,2i+3q-4p}. \]

Since \( p < q, \quad p < i \), we have

\[ 2i + 3q - 4p - (i + 2q - 2p) = (i - p) + (q - p) > 0. \]

Then we have

\[ C_A \left( E^A_{\varphi_A(b)} \right) \in B_C. \]

In a similar way, for \( b \in B_U \) we have

\[ C_A \left( E^A_{\varphi_A(b)} \right) \in B_C. \]

Similar to Proposition 5.14, we have following proposition:

**Proposition 5.15** For \( b \in B_R \), we have

\[ C_A \left( E^A_{\varphi_A(b)} \right) \in B_R. \]

**5.2.4 Definition of \( \Phi \) on \( A \) and \( \tilde{f}_0^G \) on \( G^l \)**

In this section, we define one-to-one map \( \Phi : A \to G^l \) and Kashiwara operator \( \tilde{f}_0^G \) on \( G^l \) exploiting operators on \( A \).

Since roots \( \alpha_0 \) and \( \alpha_2 \) of algebra \( U_q(C_2^{(1)}) \) are orthogonal, the Kashiwara operator \( \tilde{f}_0^G \) commutes with \( \tilde{f}_2^G \).

\[ \tilde{f}_0^G \tilde{f}_2^G(b) = \tilde{f}_2^G \tilde{f}_0^G(b), \quad \tilde{e}_0^G \tilde{e}_2^G(b) = \tilde{e}_2^G \tilde{e}_0^G(b), \quad b \in G^l. \]

In view of Proposition 5.8, we are going to construct the one-to-one map

\[ \Phi : A \to G^l, \]

with the following properties;
We verify the properties (E1)–(E5).

By (E3) and (E4), we see that $\Phi$ is unique:

(5.2.20)

$\Phi \left( (\tilde{e}^A_0)^p k_{l,-l} \right) = \left[ [2]^{l-p} [0]^p \right] (0 \leq p \leq l)$.

We verify (E1). Since $\tilde{e}^A_1 = 0$, we verify $\tilde{e}^A_0 [2] = 0$. Using Proposition 3.3, we have

$\Psi_1 ([2]) = u_{+,1}$.

Then we have

$\tilde{e}^0 [2] = 0$

In a similar way, we verify (E2). We consider $\left( \tilde{f}^A_0 \right)^p k_{l,-l}$. We see that the element $b \in A$ which satisfies the following formula is unique:

$\tilde{e}^0 (b) = (l + k) \Lambda_1 - 3l \Lambda_2$.

Thus, we can define

$\Phi \left( (\tilde{f}^A_0)^p k_{l,-l} \right) = \left[ [6]^{p} [2]^{l-p} \right] (0 \leq p \leq l)$.

We define

$\Phi (\tilde{f}^A_0) = \tilde{f}^A_0$

for $b \in A$.

We call an element $b \in A$ an $A$-terminal if $F_A(b) = 0$. An $E_A$-string is a sequence of elements in $A$, $b$, $E_A(b)$, $E_A^2(b)$, ..., $E_A^n(b)$, where $b$ is an $A$-terminal and $E_A^{n+1}(b) = 0$ ($n \in \mathbb{Z}_{\geq 0}$). We first define $\Phi$ for $A$-terminal elements using (E3)(E4) mainly. Next we define the the action of $\Phi$ for elements of an $E_A$-string using (E2). We verify (E1) completely in §5.3.

Definition 5.16 We define

(5.2.19)

$\Phi \left( (\tilde{f}^A_0)^p k_{l,-l} \right) = \left[ [6]^{p} [2]^{l-p} \right] (0 \leq p \leq l)$.

We verify the properties (E1)–(E5).

By the definition of $k_{l,-l}$,

$\Phi \left( (\tilde{e}^A_0)^p k_{l,-l} \right) = \left[ [2]^{l-p} [0]^p \right] (0 \leq p \leq l)$.

We verify (E1). Since $\tilde{e}^A_1 = 0$, we verify $\tilde{e}^A_0 [2] = 0$. Using Proposition 3.3, we have

$\Psi_1 ([2]) = u_{+,1}$.

Then we have

$\tilde{e}^0 [2] = 0$

In a similar way, we verify (E2). We consider $\left( \tilde{f}^A_0 \right)^p k_{l,-l}$. We see that the element $b \in G$ which satisfies the following formula is unique:

$\tilde{e}^0 (b) = (l + k) \Lambda_1 - 3l \Lambda_2$.

Thus, we can define

$\Phi \left( (\tilde{f}^A_0)^p k_{l,-l} \right) = \left[ [6]^{p} [2]^{l-p} \right] (0 \leq p \leq l)$.
Similarly, by the calculation of weight, we can define uniquely

\[ \Phi \left( \tilde{e}_0^A \right)^p \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 6^{l-p} \end{bmatrix} \right] \quad (0 \leq p \leq l). \]

By (E5) and (5.2.22), we define the action of \( \tilde{e}_0^G \) and \( \tilde{f}_0^G \) for \( \begin{bmatrix} 6 \end{bmatrix}^p \begin{bmatrix} 2 \end{bmatrix}^{l-p} \) \((0 \leq p \leq l)\) by

\[ \tilde{f}_0^G \begin{bmatrix} 6 \end{bmatrix}^p \begin{bmatrix} 2 \end{bmatrix}^{l-p} = \begin{cases} \begin{bmatrix} 6 \end{bmatrix}^{p+1} \begin{bmatrix} 2 \end{bmatrix}^{l-p-1} & 0 \leq p \leq l - 1 \\ 0 & p = l \end{cases} \quad (5.2.23) \]

\[ \tilde{e}_0^G \begin{bmatrix} 6 \end{bmatrix}^p \begin{bmatrix} 2 \end{bmatrix}^{l-p} = \begin{cases} \begin{bmatrix} 6 \end{bmatrix}^{p-1} \begin{bmatrix} 2 \end{bmatrix}^{l-p+1} & 1 \leq p \leq l \\ 0 & p = 0 \end{cases} \quad (5.2.24) \]

In view of Definition 5.12, 5.16 and (E2)', following definition is led.

**Definition 5.17** We define

\[ \Phi \left( \left( \begin{bmatrix} 6 \end{bmatrix} \right)^p \begin{bmatrix} 2 \end{bmatrix}^{l-p} \right) = \begin{bmatrix} 6 \end{bmatrix}^{l-k} \begin{bmatrix} 6 \end{bmatrix}^p \begin{bmatrix} 2 \end{bmatrix}^{l-p} \] \quad (5.2.25)

where \( 0 \leq k, p \leq l \).

We verify the properties (E1)--(E5). The property (E2) is obvious. We verify (E1)(E3). We see

\[ \varepsilon_1^A \left( \left( \begin{bmatrix} 6 \end{bmatrix} \right)^p \begin{bmatrix} 2 \end{bmatrix}^{l-p} \right) = 0, \]

\[ \varphi_1^A \left( \left( \begin{bmatrix} 6 \end{bmatrix} \right)^p \begin{bmatrix} 2 \end{bmatrix}^{l-p} \right) = p + k. \]

On the other hand,

\[ \left( \begin{bmatrix} 6 \end{bmatrix} \right)^{l-k} \begin{bmatrix} 6 \end{bmatrix}^{p-(l-k)/3} \begin{bmatrix} 2 \end{bmatrix}^{l-p} \]

\[ = \begin{cases} \begin{bmatrix} 2 \end{bmatrix}^{(l-k)/3} \begin{bmatrix} 6 \end{bmatrix}^{p-(l-k)/3} \begin{bmatrix} 2 \end{bmatrix}^{l-p} & 0 \leq l - k \leq 3p, \ l - k \equiv 0 \pmod{3} \\ \begin{bmatrix} 2 \end{bmatrix}^{(l-k)/3} \begin{bmatrix} 6 \end{bmatrix}^{p-1-(l-k)/3} \begin{bmatrix} 2 \end{bmatrix}^{l-p} & 0 < l - k < 3p, \ l - k \equiv 1 \pmod{3} \\ \begin{bmatrix} 2 \end{bmatrix}^{(l-k)/3} \begin{bmatrix} 6 \end{bmatrix}^{p-1-(l-k)/3} \begin{bmatrix} 2 \end{bmatrix}^{l-p} & 0 < l - k < 3p, \ l - k \equiv 2 \pmod{3} \\ \begin{bmatrix} 2 \end{bmatrix}^{p-k-3p} \begin{bmatrix} 2 \end{bmatrix}^{k+2p} & l - k > 3p \end{cases} \]

Using Proposition 3.9, we have

\[ \varepsilon_1 \left( \left( \begin{bmatrix} 6 \end{bmatrix} \right)^{l-k} \begin{bmatrix} 6 \end{bmatrix}^{p} \begin{bmatrix} 2 \end{bmatrix}^{l-p} \right) = 0, \quad (5.2.26) \]

\[ \varphi_1 \left( \left( \begin{bmatrix} 6 \end{bmatrix} \right)^{l-k} \begin{bmatrix} 6 \end{bmatrix}^{p} \begin{bmatrix} 2 \end{bmatrix}^{l-p} \right) = p + k. \quad (5.2.27) \]

By calculation, we can verify (E4). We verify (E5). By (5.2.24) and (D1), we must have

\[ \tilde{e}_0^G \left( \left( \begin{bmatrix} 6 \end{bmatrix} \right)^{l-k} \begin{bmatrix} 6 \end{bmatrix}^{p} \begin{bmatrix} 2 \end{bmatrix}^{l-p} \right) = 0 \quad (5.2.28) \]

By Definition 5.17 and (E1)', following definition is led.
Definition 5.18 We define
\[ \Phi \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^t \tilde{b}_{l(k,l)}^0 \right) = \Phi \left( \left( \tilde{f}_1^A \right)^t \tilde{b}_{l(k,l)}^0 \right) (0 \leq p \leq l+j). \]

We can verify (E1), (E2), (E3), (E4). We see easily
\[ \tilde{f}_0^A \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^t \tilde{b}_{l(k,l)}^0 \right) = 0 \ (0 \leq q \leq l). \]

By (5.1.11), we see
\[ \tilde{c}_0^A \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^t \tilde{b}_{l(k,l)}^0 \right) = 0 \ (l \leq q \leq l+j). \]

In order to satisfy (E4), we define
\[ \tilde{f}_0^A \left( \Phi \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^t \tilde{b}_{l(k,l)}^0 \right) \right) = 0 \ (0 \leq q \leq l), \]
\[ \tilde{c}_0^A \left( \Phi \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^t \tilde{b}_{l(k,l)}^0 \right) \right) = 0 \ (l \leq q \leq l+j). \]

For \( b \in B_{(k,j)} \) we define \( y(b) \in \mathbb{Z}_{\geq 0} \) by
\[ y(b) = \left\lfloor \frac{l-i-j}{3} \right\rfloor. \]

We often write \( y \) instead of \( y(b) \) for simplicity.

Definition 5.19 For \( b_{l(i-j)} \), we define
\[ \Phi \left( b_{l(i-j)} \right) = \begin{cases} \left[ \begin{array}{ccc} 6 & \varepsilon & 2^y \varepsilon \end{array} \right] \left[ \begin{array}{cc} 2^y & \varepsilon \\
\varepsilon & 2^y \varepsilon \end{array} \right] j-i & \text{if } l-i-j \equiv 0 \ (\bmod 3) \\
\left[ \begin{array}{ccc} 6 & \varepsilon & 4 \varepsilon \end{array} \right] \left[ \begin{array}{cc} 2^y & \varepsilon \\
\varepsilon & 2^y \varepsilon \end{array} \right] j-i & \text{if } l-i-j \equiv 1 \ (\bmod 3) \\
\left[ \begin{array}{ccc} 6 & \varepsilon & 3 \varepsilon \end{array} \right] \left[ \begin{array}{cc} 2^y & \varepsilon \\
\varepsilon & 2^y \varepsilon \end{array} \right] j-i & \text{if } l-i-j \equiv 2 \ (\bmod 3). \end{cases} \]

We verify the properties (E1)–(E5). We verify (E1) and (E3). We see easily
\[ \tilde{c}_1^A \left( b_{l(i-j)} \right) = 0, \tilde{f}_1^A \left( b_{l(i-j)} \right) = l-i. \]

By proposition 3.9, we have
\[ \Psi_1 \left( \left[ \begin{array}{ccc} 6 & \varepsilon & 2^y \varepsilon \end{array} \right] \left[ \begin{array}{cc} 2^y & \varepsilon \\
\varepsilon & 2^y \varepsilon \end{array} \right] \right) = u_{+}^{y+j}u_{-}^{y+i}u_{+}^{2y}, \]
\[ \text{Red}_1 \left( \left[ \begin{array}{ccc} 6 & \varepsilon & 2^y \varepsilon \end{array} \right] \left[ \begin{array}{cc} 2^y & \varepsilon \\
\varepsilon & 2^y \varepsilon \end{array} \right] \right) = u_{+}^{l-i}. \]

Then we have
\[ \tilde{c}_1^A \left( \left[ \begin{array}{ccc} 6 & \varepsilon & 2^y \varepsilon \end{array} \right] \left[ \begin{array}{cc} 2^y & \varepsilon \\
\varepsilon & 2^y \varepsilon \end{array} \right] \right) = 0, \]
\[ \tilde{f}_1^A \left( \left[ \begin{array}{ccc} 6 & \varepsilon & 2^y \varepsilon \end{array} \right] \left[ \begin{array}{cc} 2^y & \varepsilon \\
\varepsilon & 2^y \varepsilon \end{array} \right] \right) = l-i. \]
We set
\[ e_i^p \left( \begin{bmatrix} 6 & y+1 \end{bmatrix} \begin{bmatrix} y+i-1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix} \right) = 0, \]
\[ f_i^p \left( \begin{bmatrix} 6 & y+1 \end{bmatrix} \begin{bmatrix} y+i-1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix} \right) = l - i, \]
\[ e_i^1 \left( \begin{bmatrix} 5 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \right) = 0, \]
\[ f_i^1 \left( \begin{bmatrix} 5 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \right) = l - i, \]
\[ e_i^1 \left( \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} y+1 \end{bmatrix} \begin{bmatrix} y+i-1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix} \right) = 0, \]
\[ e_i^1 \left( \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} y+1 \end{bmatrix} \begin{bmatrix} y+i-1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix} \right) = l - i, \]

By calculation, we can verify (E2) and (E4). We verify (E5). We put \( b = \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} y+i \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix} \). By calculation and Lemma 5.17, we have
\[ (e_i^p)^{2j-2i+j} b = \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y(b) \end{bmatrix} \begin{bmatrix} y(b)+i \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y(b)+j \end{bmatrix}, \]
\[ (e_i^1)^{l+j} (e_i^p)^{2j-2i+j} b = \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y(b) \end{bmatrix} \begin{bmatrix} 2y(b)+i \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y(b)+j \end{bmatrix}, \]
\[ (e_i^1)^{l-i-j} (e_i^1)^{l+j} (e_i^p)^{2j-2i+j} b = \begin{bmatrix} 6 \end{bmatrix}. \]

By Definition 5.18, we must have
\[ \Phi \left( (f_i^1)^{l+j} \left( f_0^1 \right)^{l-i} b_{l-i+j,i} \right) = \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j+i \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix}. \]
\[ e_i^p \left[ \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+i \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix} \right] = 0. \]

By (D1), we must have
\[ e_i^p \left[ \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} y+i \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix} \right] = 0. \]

Similarly, we can prove other cases. □

**Definition 5.20** We set \( b = b_{l-i,j} \). We define the action of \( \Phi \) as follows:
If \( 0 \leq p \leq i, \)
\[ \Phi \left( (f_0^1)^{l-p+i} b \right) = \begin{cases} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} y+i-p \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix} & (l - i - j \equiv 0 \pmod{3}), \\ \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} y+i+1 \end{bmatrix} \begin{bmatrix} y+i-p-1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix} & (y + i > p, l - i - j \equiv 1 \pmod{3}), \\ \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} & (y + i = p, l - i - j \equiv 1 \pmod{3}), \\ \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} y+i-p \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \begin{bmatrix} y+j \end{bmatrix} & (l - i - j \equiv 2 \pmod{3}), \end{cases} \]
If \( i < p \leq j, \)
\[ \Phi \left( (f_0^1)^{l-p+i} b \right) = \begin{cases} \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} p-i+y \end{bmatrix} \begin{bmatrix} y+j-p+i \end{bmatrix} & (l - i - j \equiv 0 \pmod{3}), \\ \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} p-i+y+1 \end{bmatrix} \begin{bmatrix} y-i-1 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j-p+i \end{bmatrix} & (y > 0, l - i - j \equiv 1 \pmod{3}), \\ \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} p-i \end{bmatrix} \begin{bmatrix} 7 \end{bmatrix} \begin{bmatrix} j-p+i \end{bmatrix} & (y = 0, l - i - j \equiv 1 \pmod{3}), \\ \begin{bmatrix} 1 \end{bmatrix} \begin{bmatrix} 6 \end{bmatrix} \begin{bmatrix} p-i+y+1 \end{bmatrix} \begin{bmatrix} 3 \end{bmatrix} \begin{bmatrix} 2 \end{bmatrix} \begin{bmatrix} y+j-p+i \end{bmatrix} & (l - i - j \equiv 2 \pmod{3}). \end{cases} \]
By calculation, we can verify (E1)–(E4). In order to satisfy (E5), we define \( \tilde{e}_G^0 \) on \( \Phi \left( \left( \tilde{f}_1^A \right)^p \left( \tilde{f}_0^A \right)^p \tilde{b}_{(l-i,j)} \right) \) (0 ≤ p ≤ j) and \( \tilde{f}_G^0 \) on \( \Phi \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^q \tilde{b}_{(l-i,j)} \right) \) (0 ≤ q ≤ j).

Using Proposition 5.14 and (E2), it is enough to define the action of \( \Phi \) on \( B_C \cup B_R \).

By (E1)', we are led to the following definition.

**Definition 5.21** We define

\[
\Phi \left( \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^q \tilde{b}_{(l-i,j)} \right) = \left( \tilde{f}_1^A \right)^q \left( \tilde{f}_0^A \right)^q \Phi \left( \left( \tilde{f}_1^A \right)^p \left( \tilde{f}_0^A \right)^p \tilde{b}_{(l-i,j)} \right),
\]

where (0 ≤ p ≤ j, 0 ≤ q ≤ y + j − (i − p)).

Thus we have defined \( \Phi(b) \) where \( b \in B_C \cup B_R \).

**Definition 5.22** For \( b = [b_1] \cdots [b_n] \in B^{G_2}(nA_1) \) (0 ≤ n ≤ l), we define the involution on \( G^l \) by

\[
C_G(b) = [b_n] \cdots [b_1],
\]

where if \( b_i = m \) (m = 1, . . . , 6) then \( [b_i] = [m] \), if \( b_i = 0_1, 0_2 \), then \( [b_i] = [b_i] \).

**Remark 5.23** Let \( \bar{b} \) (resp. \( b \)) be the highest (resp. lowest) weight element of \( B^{G_2}(lA_1) \). By Proposition 3.9, 7.14 and 7.15, we have

\[
\bar{b} = [1],
\]

and

\[
b = [1].
\]

**Proposition 5.24** We define \( \Phi \) for \( b \in B_C \cup B_R \)

\[
\Phi(E^{m_A}_b) = \left( \tilde{e}_G^0 \right)^m \Phi(b) \quad (0 \leq m \leq \varepsilon_A(b)).
\]

We define \( \Phi \) for \( b \in B_W \cup B_U \),

\[
\Phi(E^{m_A}_b) = C_G(\Phi(C_A(E^{m_A}_b))) \quad (0 \leq m \leq \varepsilon_A(b)).
\]

For \( b \in B_R \), we define

\[
\Phi \left( E^{m_A}_a \left( \bar{f}_0^A \right)^{\varepsilon_{(b)}}(b) \right) = C_G \left( \Phi \left( F_{A_2}^{\varepsilon_A(b)-m}(b) \right) \right).
\]

**Remark 5.25** By Definition 5.22, we see that the relation (5.2.30) satisfies (E2). By (5.2.30) and Proposition 5.14, we see that \( C_A(E^{m_A}_b) \) is already defined. By (5.2.31), we see \( C_A(C_A(E^{m_A}_b)) = b \) (0 ≤ m ≤ \( \varepsilon_A(b) \)).

We have defined \( \Phi(b) \) where \( b \in A^{(l)} \).
Proposition 5.26 For \( b \in A^{(l)} \),
\[
C_G (\Phi (b)) = \Phi (C_A (b)).
\]

Proof We put \( b = (\tilde{f}_1^{A})^q (\tilde{f}_0^{A})^j \tilde{b}_{(l-i,j)} \). By \((5.2.12), (5.2.13)\) and \((5.2.14)\),
\[
E_{A}^{\varepsilon A}(b) (\tilde{c}_0^{A})^{j-q} b = \begin{cases} 
(\tilde{f}_1^{A})^{l+j-2q} (\tilde{f}_0^{A})^{l-q} \tilde{b}_{(j+i-q, l-q)} & \text{if } 0 \leq q \leq \left[ \frac{i+j}{2} \right], \\
(\tilde{f}_1^{A})^{l+j-2q} (\tilde{f}_0^{A})^{l-i-j+q} \tilde{b}_{(l+i+2j-3q, l-2i+2j-3q)} & \text{if } \left[ \frac{i+j}{2} \right] < q \leq \frac{2j+i}{3}, \\
(\tilde{f}_1^{A})^{l-i-j+q} (\tilde{f}_0^{A})^{l-i-j+q} \tilde{b}_{(l-2j+3q, l-i)} & \text{if } \frac{2j+i}{3} < q \leq j.
\end{cases}
\]

If \( 0 \leq q \leq \left[ \frac{i+j}{2} \right] \),
\[
\Phi \left( (\tilde{f}_0^{A})^j \tilde{b}_{(l-i,j)} \right) = \left( [1] [6]^{j+i} [3]^y [7]^y^{j+i} \right),
\]
\[
\Phi \left( (\tilde{f}_1^{A})^q (\tilde{f}_0^{A})^j \tilde{b}_{(l-i,j)} \right) = \begin{cases} 
[1] [6]^{j+i+y} [3]^y [7]^y^{j+i} [7]^y & (i > q), \\
[1] [6]^{j+q+y} [3]^y [7]^y^{j+i} [7]^y & (i \leq q),
\end{cases}
\]
\[
\Phi \left( E_{A}^{\varepsilon A}(b) (\tilde{c}_0^{A})^{q} (\tilde{f}_0^{A})^j \tilde{b}_{(l-i,j)} \right) = \begin{cases} 
[1] [2]^{j+i} [3]^y [7]^y^{j+i} [7]^y & (i > q), \\
[1] [2]^{j+q} [3]^y [7]^y^{j+i} [7]^y & (i \leq q),
\end{cases}
\]
\[
C_G \left( \Phi \left( E_{A}^{\varepsilon A}(b) (\tilde{c}_0^{A})^{q} (\tilde{f}_0^{A})^j \tilde{b}_{(l-i,j)} \right) \right) = \begin{cases} 
[1] [6]^{i} [3]^y [7]^y^{j+i} [7]^y & (i > q), \\
[1] [6]^{i} [3]^y [7]^y^{j+i} [7]^y & (i \leq q).
\end{cases}
\]

On the other hand,
\[
\Phi \left( (\tilde{f}_0^{A})^i \tilde{b}_{(l-q, i+j-q)} \right) = \begin{cases} 
[1] [6]^{i-q+y} [3]^y [7]^y^{j+i} & (i > q), \\
[1] [6]^{i+q-i} [3]^y [7]^y^{j+i-q} & (i \leq q),
\end{cases}
\]
\[
\Phi \left( C_A \left( E_{A}^{\varepsilon A}(b) \right) \right) = \Phi \left( (\tilde{f}_1^{A})^i (\tilde{f}_0^{A})^j \tilde{b}_{(l-q, i+j-q)} \right) = \begin{cases} 
[1] [6]^{i-q+y} [3]^y [7]^y^{j+i} [7]^y & (i > q), \\
[1] [6]^{i} [3]^y [7]^y^{j+i-q} [7]^y & (i \leq q).
\end{cases}
\]

By \((5.2.31)\), we have
\[
C_G (\Phi (E_{A}^{m}(b)) = \Phi (C_A (E_{A}^{m}(b))) \quad (0 \leq m \leq \varepsilon_0 (b))
\]

In a similar way, we can calculate other cases. \(\square\)

Definition 5.27 For \( b \in G^{j} \), we define \( \tilde{f}_0^{G}(b) \) and \( \varepsilon_0^{G}(b) \) by
\[
\tilde{f}_0^{G}(b) = \Phi f_0^A \Phi^{-1}(b),
\]
\[
\varepsilon_0^{G}(b) = \Phi^{-1} f_0^A \Phi(b).
\]
Therefore, it is sufficient to verify that $\Phi(b)$ and $\varphi_0^G(b)$ by

$$
\varepsilon_0^G(b) = \max \left\{ n \left| (\varepsilon_0^G)^n b \neq 0 \right. \right\}, \\
\varphi_0^G(b) = \max \left\{ n \left| (\varphi_0^G)^n b \neq 0 \right. \right\}.
$$

By (C2), it is obvious that $\tilde{f}_0^G = \Phi f_0^A \Phi^{-1}$, $\tilde{e}_0^G = \Phi e_0^A \Phi^{-1}$ satisfy (D1).

**Remark 5.28** By Definition 5.27, for $b \in A$ we have

$$
\varepsilon_0^G(\Phi(b)) = \varepsilon_0^A(b),
$$

$$
\varphi_0^G(\Phi(b)) = \varphi_0^A(b).
$$

**Proposition 5.29** The action of $\tilde{f}_0^G$ is unique.

**Proof** Let $\Phi, \Phi'$ be one-to-one maps $A \to G$ which satisfy conditions (E1)(E2)(E3) in §5.2.4. Here, by (E2) we have $\Phi F_A = \tilde{f}_0^G \Phi$, $\Phi' F_A = \tilde{f}_0^G \Phi'$. We see that if for $b \in A$

$$
\Phi(f_0^A b) \neq \Phi'(f_0^A b), \quad \text{for some } b \in A
$$

we have

$$
\Phi \left( \tilde{f}_0^G F_A \tilde{\varphi}_0^A(b) \right) \neq \Phi' \left( \tilde{f}_0^G F_A \tilde{\varphi}_0^A(b) \right).
$$

Therefore it is sufficient to verify that $\Phi(b)$ is unique, for $b \in A$ such that $F_A(b) = 0$. By (E1), $\Phi \tilde{f}_1^A = \tilde{f}_1^G \Phi$, then it is enough to verify that

$$
\Phi \left( \tilde{f}_0^A \right)^p \ell_{i,j} \left( \tilde{f}_0^A \right) \left( 0 \leq i \leq \left[ \frac{l}{2} \right], 0 \leq j \leq l - i, 0 \leq p \leq j \right)
$$

is unique. This is obvious by Proposition 5.20.

The affine crystal $B^l$ which is constructed with $G$ and $\tilde{f}_0^G$ satisfies Proposition 2.1.

### 5.3 Proof of commutativity of $\Phi \tilde{f}_1^A = \tilde{f}_1^G \Phi$

In this section we prove (E1) in §5.2.4, namely for $b \in A$

$$
\begin{array}{ccc}
\quad & \Phi(b) & \\
\downarrow & & \downarrow \\
\tilde{f}_1^A b & \quad & \tilde{f}_1^G (\Phi(b)).
\end{array}
$$

In the proof of commutativity we use $\tilde{\varepsilon}_1^A$, $\tilde{\varepsilon}_1^G$ instead of $\tilde{f}_1^A$, $\tilde{f}_1^G$ respectively. Consider an $E_A$-string $b$, $E_A(b)$, $\ldots$, $E_A \tilde{\varphi}_A(b)$, where $b \in A$ is $A$-terminal. For any $b \in A$, we can denote

$$
b = E_A^n b',
$$

where $n' \in \mathbb{Z}_{\geq 0}$, $b' \in A$ is an $A$-terminal element, namely $F_A(b') = 0$. Similarly, we can denote

$$
\tilde{e}_1^A b = E_A^n b'',
$$

where $n'' \in \mathbb{Z}_{\geq 0}$, $b'' \in A$ is an $A$-terminal element. In order to show (E1), we will verify

$$
\tilde{e}_1^G \left( \Phi \left( E_A^n b' \right) \right) = \Phi \left( E_A^n b'' \right). \quad (5.3.1)
$$
Let us define $A_+, A_-$ by

$$A_+ = \bigoplus_{i=0}^{n} \bigoplus_{1 \leq k \leq l-i} B_{(k,j)}^i,$$

$$A_- = \bigoplus_{i=0}^{n} \bigoplus_{1 \leq j < k \leq l-i} B_{(k,j)}^i.$$  

For $b' \in A_+$ such that $E_A(b') \in A_-$, we see that the action of $E_A$ is given by (a3) or (a4) or (a5). We take $b = (\tilde{f}_i^A)^q (\tilde{f}_0^A)^p \tilde{b}_{l(i,j)}^{i,j} \in B_C$ ($0 \leq q \leq p \leq j$), which is an $A$-terminal element of an $E_A$-string.

By consulting the proof of (C3) in §5.2.4, we see that for $b$ the action of $E_A$ has following relations, using sequence depending $b$, $0 \leq n_1 \leq n_2 \leq \cdots \leq n_6 = \varepsilon_A(b)$.

the action of $E_A$ on $E_A^n b$ ($0 \leq n' < n_1$) is given by (a1), if $n_1 \neq 0$,
the action of $E_A$ on $E_A^n b$ ($n_1 \leq n' < n_2$) is given by (a3), if $n_1 < n_2$,
the action of $E_A$ on $E_A^n b$ ($n_2 \leq n' < n_3$) is given by (a4), if $n_2 < n_3$,
the action of $E_A$ on $E_A^n b$ ($n_3 \leq n' < n_4$) is given by (a5), if $n_3 < n_4$,
the action of $E_A$ on $E_A^n b$ ($n_4 \leq n' < n_5$) is given by (a2), if $n_4 < n_5$,
the action of $E_A$ on $E_A^n b$ is given by (a6).

Then we see that the number $\tilde{n}(b) \in \mathbb{Z}_{\geq 0}$ which satisfy following condition is at most one:

$$E_A^{\tilde{n}(b)} b \in A_+, \quad E_A^{\tilde{n}(b)+1} b \in A_-.$$  

If there does not exist such $\tilde{n}(b)$, we put

$$\tilde{n}(b) = \varepsilon_A(b) = 2l - 2i + j - 3q.$$

By Proposition 5.13 and induction 5.2.6 it is sufficient to prove 5.3.1 for $b' = E_A^n b$ ($0 \leq n \leq \tilde{n}(b), b \in B_C$).

5.3.1 Actions of $\tilde{e}_1^A$

Proposition 5.30 For $(\tilde{f}_i^A)^q (\tilde{f}_0^A)^p \tilde{b}_{l(i,j)}^{i,j} \in B_C$ ($0 \leq q \leq p \leq j$), we have following relations.

Put $b_1 = (\tilde{f}_i^A)^q (\tilde{f}_0^A)^p \tilde{b}_{l(i,j)}^{i,j} \in B_C$ ($3q \geq i + 2j - \left[\frac{i - 1}{2}\right], p + 2q > i + 2j$). We have

$$\tilde{e}_1^A(E_A^n b_1) = E_A^n \left((\tilde{f}_i^A)^q (\tilde{f}_0^A)^p \tilde{b}_{l(i,j)}^{i,j}\right),$$  

(5.3.2)

$$\tilde{e}_1^A\left(E_A^{l-i-j+3(j-q)+1} b_1\right) = E_A^{l-i-j+3(j-q)+1} \left((\tilde{f}_i^A)^q (\tilde{f}_0^A)^p \tilde{b}_{l(i,j+1)}^{i,j+1}\right),$$  

(5.3.3)

$$\tilde{e}_1^A\left(E_A^{l-i-j+3(j-q)+2} b_1\right) = E_A^{l-i-j+3(j-q)+2} \left((\tilde{f}_i^A)^q (\tilde{f}_0^A)^p \tilde{b}_{l(i,j+2)}^{i,j+2}\right),$$  

(5.3.4)

$$\tilde{e}_1^A\left(E_A^{l-i-j+3(j-q)+n'} b_1\right) = E_A^{l-i-j+3(j-q)+n'} \left((\tilde{f}_i^A)^q (\tilde{f}_0^A)^p \tilde{b}_{l(i,j+3)}^{i,j+3}\right),$$  

(5.3.5)

where $0 \leq n \leq l - j - i + 3(j - q)$, $0 \leq n' \leq 3(y - 1), y = \left[\frac{l-i-j}{3}\right]$.

Put $b_2 = (\tilde{f}_i^A)^q (\tilde{f}_0^A)^p \tilde{b}_{l(i,j)}^{i,j} \in B_C$ ($p - q \leq \left[\frac{i - 1}{2}\right], p + 2q \leq i + 2j$). We have

$$\tilde{e}_1^A(E_A^n b_2) = E_A^n \left((\tilde{f}_i^A)^q (\tilde{f}_0^A)^p \tilde{b}_{l(i,j)}^{i,j}\right),$$  

(5.3.6)

$$\tilde{e}_1^A\left(E_A^{2l-[j-i)/2]-3-n'} b_2\right) = E_A^{2l-[j-i)/2]-3-n'} \left((\tilde{f}_i^A)^q (\tilde{f}_0^A)^p \tilde{b}_{l(i,j+3)}^{i,j+3}\right),$$  

(5.3.7)
where $0 \leq n \leq l - 2i + p - q, 0 \leq n' \leq \left\lfloor \frac{i+j}{2} \right\rfloor$.

Put $b_3 = \left(\tilde{f}_1^A\right)^q \left(\tilde{f}_0^A\right)^p \tilde{b}_{(i-j)}^l$ ($3q < i + 2j - \left\lfloor \frac{i+j}{2} \right\rfloor, \left\lfloor \frac{i+j}{2} \right\rfloor < p - q$), then we have

$$
\tilde{c}_1^A(E_A^p b_3) = E_A^p \left(\left(\tilde{f}_1^A\right)^q \left(\tilde{f}_0^A\right)^p \tilde{b}_{(i-j)}^l\right),
$$

where $0 \leq n \leq l - 2i + \left\lfloor \frac{j+i}{2} \right\rfloor$.

**Proof** We consider the action of $\tilde{c}_1^A$ for $\left(\tilde{f}_1^A\right)^q \left(\tilde{f}_0^A\right)^p \tilde{b}_{(k,j)}^l$ ($0 \leq p \leq j, 0 \leq q \leq p + k, 0 \leq r \leq j + q - 2p$). By Proposition 5.4, if $q = p$ we have

$$
\tilde{c}_1^A \left(\tilde{f}_1^A\right)^q \left(\tilde{f}_0^A\right)^p \tilde{b}_{(k,j)}^l = \left(\tilde{f}_1^A\right)^r \left(\tilde{f}_0^A\right)^{p-1} \tilde{b}_{(k,j)}^l
$$

where $0 \leq r \leq j - p$. If $q > p$,

$$
\tilde{c}_1^A \left(\tilde{f}_1^A\right)^q \left(\tilde{f}_0^A\right)^p \tilde{b}_{(k,j)}^l \neq \left(\tilde{f}_1^A\right)^r \left(\tilde{f}_0^A\right)^{p-1} \tilde{b}_{(k,j)}^l
$$

where $q - p \leq r \leq j + q - 2p$.

We consider the case of $q = p$ in part I, and the case of $q > p$ in part II.

**Part I** We consider $\left(\tilde{f}_1^A\right)^q \left(\tilde{f}_0^A\right)^p \tilde{b}_{(k,j)}^l$ ($0 \leq p \leq j, 0 \leq r \leq j - p$).

We define $b, b' \in B_{(k,j)}^l$ by

$$
b = \left(\tilde{f}_1^A\right)^p \left(\tilde{f}_0^A\right)^p \tilde{b}_{(k,j)}^l,
$$

$$
b' = \left\{ \begin{array}{ll}
\left(\tilde{f}_1^A\right)^{p-1} \left(\tilde{f}_0^A\right)^{p-1} \tilde{b}_{(k,j)}^l & (p > 0) \\
0 & (p = 0) 
\end{array} \right.
$$

Then by Proposition 5.13 we have

$$
\tilde{c}_1^A \left(\tilde{f}_0^A\right)^r b = \left(\tilde{f}_0^A\right)^{r+1} b' ,
$$

(Figure 2) We define $m_b$ by

$$
m_b = j - k - 3(j - p) = -2j - k + 3p,
$$

We consider an $E_A^*$-string. We can express $b$ using an $A$-terminal element. By Definition 5.12, we have

$$
b = E_A^{l-i-k+(m_b)+} \left(\left(\tilde{f}_1^A\right)^{p-(m_b)+} \left(\tilde{f}_0^A\right)^p \tilde{b}_{(l-i,j-(m_b)+)}^l\right)
$$

(5.3.9)

If $m_b < 0$ the action of $E_A^*$ is given by (a1) or (a3) of Definition 5.12, and if $m_b \geq 0$ the action is given by (a2).

**The case of** (a1): $E_A^* b \in B_{(k-1,j)}^l$.

This case is $b = E_A^{n'} b_1$ ($0 \leq n' < \min\{l - 2i, l - i + 2j - 3q\}$), $b = E_A^{n'} b_2$ ($0 \leq n' < l - 2i$), $b = E_A^{n'} b_3$ ($0 \leq n' < l - 2i$).

If $p = 0$, we have

$$
\tilde{c}_1^A E_A^* b = 0.
$$

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If $p > 0$, since $m_b < 0$, we have $m_{b'} < 0$. Then the actions of $E_A$ for $b'$ is given by (a1). Then we have

\[ E_A b = \left( \tilde{f}_0^A \right)^p \left( \tilde{f}_1^A \right)^p b_{(k-1,j)}^{l,i}, \]
\[ E_A b' = \left( \tilde{f}_0^A \right)^{p-1} \left( \tilde{f}_1^A \right)^{p-1} b_{(k-1,j)}^{l,i}, \]

(Figure 3). Then we obtain

\[ \tilde{e}_1^A E_A \left( \left( \tilde{f}_0^A \right)^r b \right) = E_A \left( \left( \tilde{f}_0^A \right)^{r+1} b' \right). \] (5.3.10)

We have the case of $(0 \leq n \leq \min\{l-2i, l-i+2j-3q\})$ of (5.3.2), the case of $(0 \leq n \leq l-2i)$ of (5.3.6), and the case of $(0 \leq n \leq l-2i)$ of (5.3.8).

**The case of (a2): $E_A b \in B_{(k,j+1)}.$**

This case is $b = E_A b_1 (l-i+2j-3q \leq n' < 2l-2i+j-3q, l-i+2j-3q < l-2i)$. We denote

\[ E_A b = \left( \tilde{f}_1^A \right)^{p+1} \left( \tilde{f}_0^A \right)^{p+1} b_{(k,j+1)}^{l,i}. \]

if $m_b = 0, 1, 2$, since $m_{b'} < 0 \leq m_b$, then the action of $E_A$ for $b'$ is given by (a1). Then we have

\[ E_A b' = \left( \tilde{f}_1^A \right)^{p-1} \left( \tilde{f}_0^A \right)^{p-1} b_{(k-1,j)}^{l,i}. \]

Therefore, we have

\[ \tilde{e}_1^A E_A \left( \left( \tilde{f}_0^A \right)^r b \right) \neq E_A \left( \left( \tilde{f}_0^A \right)^{r+1} b' \right). \]
By (5.3.3), we have
\[ \hat{c}^A E_A \left( (\tilde{f}_0^A)^r \right) b = E_A^{l-i-k} \left( (\tilde{f}_0^A)^{r+1} (\tilde{f}_1^A)^p (\tilde{f}_0^A)^{p-m_i} b_{(l-i,j+1)}^{l} \right). \]

If \( m_b \geq 3 \), then the action of \( E_A \) for \( b' \) is given by (a2), since \( 0 \leq m_{b'} < m_b \). Thus we have
\[ E_A b' = (\tilde{f}_1^A)^p (\tilde{f}_0^A)^{p-m_i} b_{(k,j+1)}^{l}. \]

(Figure 4). Therefore, we have
\[ \hat{c}^A E_A \left( (\tilde{f}_0^A)^r \right) b = E_A \left( (\tilde{f}_0^A)^{r+1} b' \right) = E_A^{l-i-k+m_b-2} \left( (\tilde{f}_0^A)^{r+1} (\tilde{f}_1^A)^{p-m_b+2} (\tilde{f}_0^A)^{p-m_i} b_{(l-i,j-m_b+3)}^{l} \right), \]

(5.3.11)

Therefore we have (3.3.3), (3.3.4), (5.3.3), if \( l-i+2j-3q < l-2i \).

The case of (a3): \( E_A b \in B_{(i+1,j-1)}^{l}. \)

This case is \( b = E_A^{l-2i}b_1, b = E_A^{l-2i}b_2, b = E_A^{l-2i}b_3. \)

We denote
\[ E_A b = (\tilde{f}_1^A)^p (\tilde{f}_0^A)^{p-l+i} b_{(k+1,j-1)}^{l}. \]

Since \( m_{b'} < m_b < 0 \) the actions of \( E_A \) for \( b' \) is given by (a3). Then we have
\[ E_A b' = (\tilde{f}_1^A)^p (\tilde{f}_0^A)^{p-l+i} b_{(k+1,j-1)}^{l}. \]

Here by Proposition 5.4 we have
\[ \hat{c}_1^A (\tilde{f}_0^A)^r (\tilde{f}_1^A)^{p+l_i} (\tilde{f}_0^A)^{p-l_i} b_{(k,j)}^{l} \neq (\tilde{f}_0^A)^r (\tilde{f}_1^A)^{p-l_i} b_{(k,j)}^{l} \neq (\tilde{f}_0^A)^r (\tilde{f}_1^A)^{p-l_i} b_{(k,j)}^{l} (r > 0), \]

(Figure 5). Then we have
\[ \hat{c}_1^A (\tilde{f}_0^A)^r E_A b = (\tilde{f}_0^A)^{r+1} E_A b' (r > 0), \]

(5.3.12)

\[ \hat{c}_1^A E_A b = E_A^{l-2i+1} \left( (\tilde{f}_1^A)^p (\tilde{f}_0^A)^{p-l_i} b_{(i-1,j-1)}^{l} \right). \]

(5.3.13)

Therefore we have the case of \( n = l - 2i + 1 \) of (5.3.2), (5.3.6), (5.3.8).

Part II. We consider \( \hat{c}_1^A E_A (\tilde{f}_0^A)^r (\tilde{f}_1^A)^{p+s} (\tilde{f}_0^A)^{p-l_i} b_{(k,j)}^{l} (i \leq k, j \leq l - i, k - i \geq s > 0) \)

If \( i < k, j \leq l - i \), we use induction (5.2.6). Otherwise, using involution, we have
\[ C_{A} (\tilde{b}_{j,i}^{l}) = (\tilde{f}_0^A)^i (\tilde{f}_1^A)^{i+j} (\tilde{f}_0^A)^j b_{(i,j)}^{l}. \]

Then it is sufficient to consider following element:
\[ \hat{c}_1^A E_A (\tilde{f}_0^A)^r (\tilde{f}_1^A)^{p+s} (\tilde{f}_0^A)^{p-l_i} b_{(i,j)}^{l} (i < j \leq l - i, 0 < s \leq k - i). \]
Figure 4: Part I, the case of (a2), the location of $E_A b$ in the crystal graph of $B_{(k,j+1)}$.

Figure 5: Part I, the case of (a3), the location of $E_A b$ and $E_A b'$ in the crystal graph $B_{(k+1,j-1)}^{i+1}$.

We denote $b, b', b'' \in B_{(k,j)}$ by

$$b = (\tilde{f}_1^A)^{p+s} (\tilde{f}_0^A)^{p-1} b_{(k,j)}^{l,i},$$

$$b' = (\tilde{f}_1^A)^{p+s+1} (\tilde{f}_0^A)^{p-1} b_{(k,j)}^{l,i},$$

$$b'' = e_{\tilde{A}}^A b.$$ 

Then we have

$$e_{\tilde{A}}^A (\tilde{f}_1^A)^r b = (\tilde{f}_0^A)^r b'' \quad (r < s),$$

$$e_{\tilde{A}}^A (\tilde{f}_0^A)^r b = (\tilde{f}_0^A)^{r+1} b' \quad (r \geq s),$$

(Figure 6). We rewrite $m_b$

$$m_b = (j + s) - (k - s) + s - 3((j + s) - p) = -2j - k + 3p.$$ 

Then we denote $b$ using the $A$-terminal element in an $E_A$-string.

$$b = \begin{cases} 
E_A^{s+1-2(k-s+1)} (\tilde{f}_1^A)^p (\tilde{f}_0^A)^p b_{(l-(k-s+1),j+s)}^{l-k-s+1} & \text{if } m_b < 0, \\
E_A^{s+1-2(k-s+1)+m_b} (\tilde{f}_1^A)^{p-m_b} (\tilde{f}_0^A)^{p-m_b} b_{(l-(k-s+1),j+s-m_b)}^{l-k-s+1} & \text{if } m_b \geq 0.
\end{cases} \quad (5.3.14)$$

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The case of (a2): $E_A b \in B^{i+1}_{i+1,j+1}$. 

This case is $b = E_A^l b_1 \ (l - 2i < l - i + 2j - 3q \leq n' < 2l - 2i + j - 3q)$. We denote

$$E_A b = (\tilde{f}_1^A)^{p+s+1}(\tilde{f}_0^A)^{p+1}\bar{b}_{(i,j+1)}.$$ 

if $r < s$, since $m_b = m_{b''}$, then the action of $E_A$ for $b'$ is given by (a2). Then we have

$$E_A b'' = (\tilde{f}_1^A)^{p+s}(\tilde{f}_0^A)^{p+1}\bar{b}_{(i,j+1)}.$$ 

Therefore we have,

$$\hat{e}_1^A E_A \left((\tilde{f}_0^A)^r b \right) = E_A \left((\tilde{f}_0^A)^r b'' \right).$$ 

If $r \geq s$, $m_b = 0, 1, 2$, since $m_{b''} < 0$, then the action of $E_A$ for $b'$ is given by (a3). Then we have

$$E_A b' = (\tilde{f}_1^A)^{p+s}(\tilde{f}_0^A)^{p-1}\bar{b}_{(i+1,j-1)},$$ 

therefore we have

$$\hat{e}_1^A E_A \left((\tilde{f}_0^A)^r b \right) = E_A \left((\tilde{f}_0^A)^r b' \right)$$ 

$$= E_A^{s+2l-i-2}\left((\tilde{f}_0^A)^{r+1}\left((\tilde{f}_1^A)^p\bar{b}_{(i+1,j-1)}\right)\right).$$ 

(5.3.15)

If $r \leq s$, $m_b \geq 3$, then the actions of $E_A$ for $b'$ is given by (a2), since $m_{b''} \geq 0$. Then we have,

$$E_A b' = (\tilde{f}_1^A)^{p+s}(\tilde{f}_0^A)^{p}\bar{b}_{(i,j+1)},$$

(Figure 7) Thus we have

$$\hat{e}_1^A E_A \left((\tilde{f}_0^A)^r b \right) = E_A \left((\tilde{f}_0^A)^r b' \right)$$

$$= E_A^{s+2l-i-2+m-2}\left((\tilde{f}_0^A)^{r+1}\left((\tilde{f}_1^A)^p\bar{b}_{(i+1,j-1)}\right)\right).$$ 

(5.3.16)

Therefore we have the case of (5.3.3), (5.3.4), (5.3.5), if $l - 2i \leq l - i + 2j - 3q$.

The case of (a3): $E_A b \in B^{i+1}_{i+1,j-1}$. 

This case is $b = E_A^l b_1 \ (l - 2i < n' < l - i + 2j - 3q), b = E_A^l b_2 \ (l - 2i < n' < \lceil \frac{l-i}{2} \rceil), b = E_A^l b_3 \ (l - 2i < n' < \lceil \frac{l-i}{2} \rceil).$ (5.3.8).

Since $m_{b'} < m_{b''} = m_b < 0$, the action of $E_A$ for $b', b''$ is given by (a3). Then we have

$$E_A b = (\tilde{f}_1^A)^{p+s+1}(\tilde{f}_0^A)^{p}\bar{b}_{(i+1,j-1)},$$

$$E_A b' = (\tilde{f}_1^A)^{p+s}(\tilde{f}_0^A)^{p-1}\bar{b}_{(i+1,j-1)},$$

$$E_A b'' = (\tilde{f}_1^A)^{p+s}(\tilde{f}_0^A)^{p-1}\bar{b}_{(i+1,j-1)},$$

(Figure 8) With Proposition 5.4, we have

$$\hat{e}_1^A E_A \left((\tilde{f}_0^A)^r b \right) = E_A \left((\tilde{f}_0^A)^r b'' \right) \quad (r < s + 1),$$

$$\hat{e}_1^A E_A \left((\tilde{f}_0^A)^r b \right) = E_A \left((\tilde{f}_0^A)^{r+1} b' \right) \quad (r \geq s + 1).$$ 

(5.3.17)
Figure 6: Part II, the location of $b$, $b'$ and $b''$ in the crystal graph $B_{i(j)}^i$.

Figure 7: Part II, the case of (a2), the location of $E_Ab$ and $E_Ab''$ in the crystal graph of $B_{i(j+1)}^i$.

Therefore we have the case of $(l - 2i < n \leq l - i + 2j - 3q)$ of (5.3.3), the case of $(l - 2i < n \leq \left\lceil \frac{i}{2} \right\rceil)$ of (5.3.6) and (5.3.7).

By (5.3.11) – (5.3.17), we have Proposition 5.30.

5.3.2 Actions of $\tilde{e}_G^G$

Proposition 5.31 For $b \in G$, $n \in \mathbb{Z}_{>0}$ if $\tilde{f}_1 (\tilde{e}_2^G)^n b = (\tilde{e}_2^G)^n \tilde{f}_1 b \neq 0$, we have

$$\tilde{f}_1 (\tilde{e}_2^G)^k b = (\tilde{e}_2^G)^k \tilde{f}_1 b \ (0 \leq k \leq n).$$

Proof We prove the proposition by the induction on $k$. We assume

$$\tilde{f}_1 (\tilde{e}_2)^{k'} b = (\tilde{e}_2)^{k'} \tilde{f}_1 b \ (0 \leq k' < k),$$

then we prove

$$\tilde{f}_1 \tilde{e}_2 (\tilde{e}_2)^{k-1} b = \tilde{e}_2 \tilde{f}_1 (\tilde{e}_2)^{k-1} b. \quad (5.3.18)$$

We write $\tilde{e}_2^{k-1} b = b_1 \otimes \cdots \otimes b_l$. We have

$$\tilde{f}_1 (\tilde{e}_2)^{k-1} b = b_1 \otimes \cdots \otimes \tilde{f}_1 b_{k_1} \otimes \cdots \otimes b_l, \quad (5.3.19)$$

$$\tilde{e}_2 (\tilde{e}_2)^{k-1} b = b_1 \otimes \cdots \otimes \tilde{e}_1 b_{k_2} \otimes \cdots \otimes b_l, \quad (5.3.20)$$

where $1 \leq k_1, k_2 \leq l$.

The case of $k_1 > k_2$. Similar to Proposition 5.6, by Proposition 5.3 we see that the action of $\tilde{f}_1$ increase the number of $u_+ \text{ of } \text{Red}_2(b)$. Then the operator $\tilde{f}_1$ does not influence the action of $\tilde{e}_2$. In a similar way, we see that the operator $\tilde{e}_2$ does not influence the action of $\tilde{f}_1$. Therefore we have (5.3.18).
Figure 8: Part II, the case of (a3), the location of $E_A b$, $E_A b'$ and $E_A b''$ in the crystal graph of $B_{i+1}^{(i,j+1)}$.

The case of $k_1 \leq k_2$. We assume

$$\tilde{f}_1\tilde{e}_2 (\tilde{e}_2)^{k-1} b \neq \tilde{e}_2\tilde{f}_1 (\tilde{e}_2)^{k-1} b.$$  \hfill (5.3.21)

By assumption, at most one of following relations are satisfied:

$$\tilde{f}_1\tilde{e}_2 (\tilde{e}_2)^{k-1} b = \begin{cases} b_1 \otimes \ldots \otimes \tilde{f}_1\tilde{e}_2 b_{k_1} \otimes \ldots \otimes b_l, \\ b_1 \otimes \ldots \otimes \tilde{e}_2 b_{k_1} \otimes \ldots \otimes \tilde{f}_1 b_{k_1} \otimes \ldots \otimes b_l, \end{cases}$$  \hfill (5.3.22)

$$\tilde{e}_2\tilde{f}_1 (\tilde{e}_2)^{k-1} b = \begin{cases} b_1 \otimes \ldots \otimes \tilde{e}_2 b_{k_1} \otimes \ldots \otimes b_l, \\ b_1 \otimes \ldots \otimes \tilde{e}_2 b_{k_1} \otimes \ldots \otimes \tilde{f}_1 b_{k_1} \otimes \ldots \otimes b_l, \end{cases}$$  \hfill (5.3.23)

Since $\tilde{f}_1u_+ = u_-$ and $\Psi_1(b) = u_+^{z_1(b)}u_+^{\varphi_1(b)}$, we have $k_1 < k_1'$. In a similar way we have $k_2' < k_2$. If (5.3.22) is satisfied, we see that $k_1$-th element of $\tilde{e}_2^k\tilde{f}_1\tilde{e}_2^k b$ and $\tilde{f}_1\tilde{e}_2^k\tilde{e}_2^k b$ ($1 \leq k' \leq n - k$) are different. Then we have

$$\tilde{e}_2^k\tilde{f}_1 b \neq \tilde{f}_1\tilde{e}_2^k b.$$  \hfill (5.3.24)

This is contradiction. In a similar way, if (5.3.23) is satisfied, we have contradiction. Thus we have (5.3.18).

We prove from (5.3.2) to (5.3.8) exchanging $\tilde{e}_1^A$ with $\tilde{e}_1^G$. For example, we denote following equations exchanging $\tilde{e}_1^A$ with $\tilde{e}_1^G$ from (5.3.2) to (5.3.5). We put $b_1 = (\tilde{f}_1^A)^{q} (\tilde{f}_0^A)^{p} b_{(l-i,j)} \in B_C$ ($3q \geq i + 2j - \lfloor \frac{i - j}{2} \rfloor$, $p + 2q > i + 2j$).

We are going to show

$$\tilde{e}_1^G (\Phi (E_A^1 b_1)) = \Phi \left( E_A^1 \left( (\tilde{f}_1^A)^{q-1} (\tilde{f}_0^A)^{p} b_{(l-i,j)} \right) \right),$$  \hfill (5.3.25)

$$\tilde{e}_1^G \left( \Phi \left( E_A^{j-i-j+3(j-q)+1} b_1 \right) \right) = \Phi \left( E_A^{j-i-j+3(j-q)+1} (\tilde{f}_1^A)^{q} (\tilde{f}_0^A)^{p+1} b_{(l-i,j+1)} \right).$$  \hfill (5.3.26)
\[
\tilde{c}_1^G \left( \Phi \left( E_{\mathcal{A}}^{l-i-j+3(j-q)+2} b_1 \right) \right) = \Phi \left( E_{\mathcal{A}}^{l-i-j+3(j-q)} \left( \left( f_1^A \right)^{q+1} \left( f_0^A \right)^{p+2} \bar{b}_{l,i}^{(i-j+2)} \right) \right), \quad (5.3.27)
\]
\[
\tilde{c}_1^G \left( \Phi \left( E_{\mathcal{A}}^{l-i-j+3(j-q)+n'} b_1 \right) \right) = \Phi \left( E_{\mathcal{A}}^{l-i-j+3(j-q)+n'} \left( \left( f_1^A \right)^{q+2} \left( f_0^A \right)^{p+3} \bar{b}_{l,i}^{(i-j+3)} \right) \right), \quad (5.3.28)
\]

where \(0 < n < l - j - i + 3(j - q), 0 < n' < 3(y - 1)\).

Here we prove the case of \(l - i - j \equiv 0 \pmod{3}\), we can prove other cases similarly.

By Proposition 5.24, we have
\[
\Phi(b_1) = \left[ \begin{array}{c} 1 \\ 1 \\ 6 \\ 6 \\ 6 \\ y^{q+p-j-i} \\ 2 \\ y^{-1} \\ 1 \\ 1 \end{array} \right]^{i+p}.
\]

**The case of (5.3.25), \(n = 0\).** By (E1), we have
\[
\tilde{c}_1^G \left( \Phi(b_1) \right) = \Phi \left( \left( f_1^A \right)^{q+1} \left( f_0^A \right)^{p+2} \bar{b}_{l,i}^{(i-j)} \right).
\]

**The case of (5.3.26), \(n = l - i - j + 3(j - q)\).** Put \(b_4 = E_{\mathcal{A}}^{l-j-i+3(j-q)} b_1\). We see
\[
\Phi(b_4) = \left[ \begin{array}{c} 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ y^{q+p-j-i} \\ 2 \\ y^{-1} \\ 1 \\ 1 \end{array} \right]^{j+i-p}.
\]

Using Definition 3.4,
\[
\Psi_1(b_4) = u_{l-j-i} y^{q+p-j-i} y^{q+p-j-i} u_{l-j-i} u_{l-j-i},
\]
\[
\text{Red}_1(b_4) = u_{l-j-i} y^{q+p-j-i} u_{l-j-i} u_{l-j-i}.
\]

By Proposition 3.9, we have
\[
\tilde{c}_1^G \left( \Phi(b_4) \right) = \left[ \begin{array}{c} 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ y^{q+p-j-i} \\ 2 \\ y^{-1} \\ 1 \\ 1 \end{array} \right]^{j+i-p}.
\]

Put \(b_4 = E_{\mathcal{A}}^n \left( \left( f_1^A \right)^{q+1} \left( f_0^A \right)^{p+2} \bar{b}_{l,i}^{(i-j)} \right)\). We see
\[
\Phi(b_4) = \left[ \begin{array}{c} 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ y^{q+p-j-i} \\ 2 \\ y^{-1} \\ 1 \\ 1 \end{array} \right]^{j+i-p}.
\]

Thus we have
\[
\tilde{c}_1^G \left( \Phi \left( E_{\mathcal{A}}^n b_1 \right) \right) = \Phi \left( E_{\mathcal{A}}^n \left( \left( f_1^A \right)^{q+1} \left( f_0^A \right)^{p+2} \bar{b}_{l,i}^{(i-j)} \right) \right).
\]

**The case of (5.3.26).** We set \(b_5 = E_{\mathcal{A}}^{l-j-i+3(j-q)+1} b_1\).
\[
\Phi(b_5) = \left[ \begin{array}{c} 1 \\ 1 \\ 2 \\ 2 \\ 2 \\ y^{q+p-j-i} \\ 2 \\ y^{-1} \\ 1 \\ 1 \end{array} \right]^{j+i-p}.
\]

We consider \(\tilde{c}_1^G(\Phi(b_5))\) with \(u_+\) and \(u_-\).
\[
\Psi_1(\Phi(b_5)) = u_{l-j-i} u_{p+1} y^{q+p-j-i} y^{q+p-j-i} y^{q+p-j-i} u_{l-j-i} u_{l-j-i} u_{l-j-i} u_{l-j-i},
\]
\[
\text{Red}_1(\Phi(b_5)) = u_{l-j-i} y^{q+p-j-i} y^{q+p-j-i} u_{l-j-i} u_{l-j-i}.
\]
Then we have
\[ \hat{e}_1^G (\Phi(b_0)) = [[1][2] j-p [2] y \hat{e}_1^G [2] y \varepsilon \hat{e}_1^G [3] [2] y-1 [7] j+i-p ] = [[1][2] j-p [2] y \varepsilon \hat{e}_1^G [3] [2] y+q-p-j-i [7] [2] y-1 [7] j+i-p ]. \]

We put \( b_0' = E_A^{-l-i-j+3(j-q)} \left( (\hat{f}_l^A)^q (\hat{f}_0^A)^{p+1} \overline{b}_{l,i,j+1} \right) \). Then we have
\[ \Phi(b_0') = [[1][2] j-q [2] y \varepsilon \hat{e}_1^G [3] [2] y+q-j-i-1 [3] [2] y-1 [7] j+i-p ]. \]

Thus we have
\[ \hat{e}_1^G (\Phi(E_A^{-l-i-j+3(j-q)+1} b_1)) = E_A^{-l-i-j+3(j-q)} \left( (\hat{f}_l^A)^q (\hat{f}_0^A)^{p+1} \overline{b}_{l,i,j+1} \right). \]

Similarly, in case of (5.3.27), \( n' = 0 \) and \( n' = 2(l - i - j) + 3(j - q) \) of (5.3.28), we have
\[ \hat{e}_1^G (\Phi(E_A^{-l-i-j+3(j-q)+2} b_1)) = \Phi \left( E_A^{-l-i-j+3(j-q)} \left( (\hat{f}_l^A)^{q+1} (\hat{f}_0^A)^{p+2} \overline{b}_{l,i,j+2} \right) \right), \]
\[ \hat{e}_1^G (\Phi(E_A^{-l-i-j+3(j-p)+3} b_1)) = \Phi \left( E_A^{-l-i-j+3(j-p)} \left( (\hat{f}_l^A)^{q+2} (\hat{f}_0^A)^{p+3} \overline{b}_{l,i,j+3} \right) \right), \]
\[ \hat{e}_1^G (\Phi(E_A^{-2(l-i-j)+3(j-p)} b_1)) = \Phi \left( E_A^{-2(l-i-j)+3(j-q)-1} \left( (\hat{f}_l^A)^{q+2} (\hat{f}_0^A)^{p+3} \overline{b}_{l,i,j+3} \right) \right). \]

By Proposition 5.31, we have \( \hat{e}_1^G \Phi = \Phi \hat{e}_1^A \) for \( E_A^n b_1 \) \( 0 \leq n \leq \varepsilon_A(b_1) = 2(l - i - j) + 3(j - p) \).

Similarly, we can prove \( \hat{e}_1^G \Phi = \Phi \hat{e}_1^A \) for any \( b \in A \).

### 5.4 Selection of minimal

#### 5.4.1 Selection of minimal elements in \( B^{G_2}(l\Lambda_1) \)

In this section, we only consider crystal \( B^l \), so we denote \( \hat{e}_1, \hat{e}_2 \) instead of \( \hat{e}_1^G, \hat{e}_2^G \) respectively for simplicity.

By definition 3.10, minimal elements are \( b \in B^l \) such that \( \langle c, \varphi(b) \rangle = l \). At first we search for the element \( b \in B(l\Lambda_1) \) such that \( \langle c, \varphi_1(b) \Lambda_1 + \varphi_2(b) \Lambda_2 \rangle = l \). Next we verify that there does not exist \( b \in B(l\Lambda_1) \) such that \( \langle c, \varphi_1(b) \Lambda_1 + \varphi_2(b) \Lambda_2 \rangle < l \). Then we verify that for \( b \in B^l \) such that \( \langle c, \varphi_1(b) \Lambda_1 + \varphi_2(b) \Lambda_2 \rangle = l \), we have \( \varphi_0(b) = 0 \)

**Proposition 5.32** For \( b \in B^{G_2}(l\Lambda_1) \) such that \( \varepsilon_2(b) = 0 \), we have
\[ \min \{ 2 \varphi_1(\tilde{f}_2^{(k)}(b)) + \varphi_2(\tilde{f}_2^{(k)}(b)) \mid k = 0, \ldots, \varphi_2(b) \} \geq l - \left[ \frac{1}{2} w_{t_0}(b) \right], \]

**Proof** We consider
\[ \min \{ 2 \varphi_1(b') + \varphi_2(b') \mid b' = \tilde{f}_2^{(k)}(b), k = 0, \ldots, \varphi_2(b) \}. \]

(5.4.1)

By Proposition 5.6, we have
\[ 2 \varphi_1(\tilde{f}_2^{(k+1)}(b)) + \varphi_2(\tilde{f}_2^{(k+1)}(b)) - (2 \varphi_1(\tilde{f}_2^{(k)}(b)) + \varphi_2(\tilde{f}_2^{(k)}(b))) = \begin{cases} 1 & (k < k') \\ -1 & (k \leq k') \end{cases}. \]
We set \( b' = f_{\hat{\varphi}_2(b)} \). Then (5.4.1) is

\[
2\varphi_1(b) + \varphi_2(b) - \frac{\varphi_2(b) - (2\varphi_1(b') - (2\varphi_1(b) + \varphi_2(b)))}{2} = \varphi_1(b) + \varphi_1(b').
\]

Then we prove

\[
\varphi_1(b) + \varphi_1(b') - l + \left[ \frac{1}{2} \text{wt}_0(b) \right] \geq 0.
\]

We recall that \( x_i(b) = \{ k = 1, \ldots, n \} \) \( i = 1, \ldots, 6, 0_1, 0_2 \), \( \Psi_i(b) = \{ k = 1, \ldots, n \} \) \( i = 1, \ldots, 6 \). We consider \( b \) such that \( \varepsilon_2(b) = 0 \). Such an element \( b \) satisfy following conditions:

1. \( 0 \leq x_1(b) + \Psi_1(b) \leq l \). This is because \( \bar{f}_2 \left( \begin{array}{c} 0_2 \end{array} \right) = 0, \bar{c}_2 \left( \begin{array}{c} 0_2 \end{array} \right) = 0, \bar{f}_2 \left( \begin{array}{c} 0_2 \end{array} \right) = 0, \bar{c}_2 \left( \begin{array}{c} 0_2 \end{array} \right) = 0. 

2. \( x_{0_2}(b) = 0 \) or \( 1 \). Since \( \bar{f}_2 \left( \begin{array}{c} 0_2 \end{array} \right) = 0, \bar{c}_2 \left( \begin{array}{c} 0_2 \end{array} \right) = 0 \) and by Proposition 3.15 \( x_{0_2}(b) < 2 \).

3. \( \Psi_2(b) = \Psi_3(b) = \Psi_4(b) = \Psi_5(b) = \Psi_{0_2}(b) = 0 \). Because if \( \Psi_2(b) > 0 \) or \( \Psi_3(b) > 0 \) or \( \Psi_4(b) > 0 \) or \( \Psi_5(b) > 0 \) or \( \Psi_{0_2}(b) > 0 \) then \( \varepsilon_2(b) > 0 \).

4. \( x_3(b) + x_4(b) + x_5(b) = 0 \) or \( 1 \). By Proposition 3.13

5. \( x_6(b) \leq \Psi_6(b) \). In particular, if \( x_3(b) = 1 \) or \( x_4(b) = 1 \) then \( x_6(b) < \Psi_6(b) \) and if \( x_5(b) \) then \( x_6(b) = 0 \).

6. \( \sum_{i=1}^{6} (x_i(b) + \Psi_i(b)) + x_{0_1}(b) + x_{0_2}(b) = l \).

We start with the case of \( x_5(b) = 0 \).

By calculation, we have \( x_1(b') = x_1(b), x_6(b') = x_2(b) + x_3(b) + x_4(b) + x_6(b), x_{0_2}(b') = x_{0_2}(b), \Psi_6(b') = x_6(b), \Psi_3(b') = x_3(b), \Psi_4(b') = \Psi_4(b), \Psi_5(b') = \Psi_5(b) - x_{0_2}(b) - x_6(b) - x_4(b) - x_3(b), \Psi_1(b') = \Psi_1(b) \). We put \( \delta = \text{wt}_0(b) \) mod 2. Thus we have

\[
-wt_0(b) = 2x_1(b) + x_3(b) + x_4(b) + x_6(b) - \Psi_6(b) - 2\Psi_1(b),
\]

\[
l = x_1(b) + x_2(b) + x_3(b) + x_4(b) + x_6(b) + x_{0_2}(b) + \Psi_6(b) + \Psi_1(b),
\]

\[
-\frac{1}{2} \text{wt}_0(b) = 2x_1(b) + \frac{3}{2} x_2(b) + \frac{1}{2} x_3(b) + \frac{1}{2} x_4(b) + \frac{3}{2} x_6(b) + \frac{1}{2} \Psi_6(b) + \frac{1}{2} \delta.
\]

\[
\varphi_1(b) = x_1(b) + (x_{0_2}(b) + 2x_6(b) + x_4(b) - x_2(b))_+,
\]

\[
\varphi_1(b') = x_1(b') + x_{0_2}(b') + 2x_6(b') + (\Psi_4(b') - \Psi_6(b') - x_{0_2}(b'))_+,
\]

\[
x_1(b) + 2x_2(b) + 2x_3(b) + 2x_4(b) + 2x_6(b) + x_{0_2}(b) + \Psi_6(b) - x_{0_2}(b) - 3x_6(b) - 2x_4(b) - x_3(b))_+.
\]

\[
\varphi_1(b) + \varphi_1(b') - l + \left[ \frac{1}{2} \text{wt}_0(b) \right] = \frac{1}{2} \left[ x_{0_2}(b) + 2x_6(b) + x_4(b) - x_2(b) \right] + \frac{1}{2} \frac{1}{2} \Psi_6(b) - x_{0_2}(b) - 3x_6(b) - 2x_4(b) - x_3(b) - \frac{\delta}{2} \geq 0 \quad \text{integer greater than } -\frac{1}{2}.
\]

The case of \( x_5(b) = 1 \). By Proposition 3.13 \( x_3(b) = x_4(b) = x_6(b) = x_{0_2}(b) = 0 \). Similar to the case of \( x_5(b) = 0 \), we have

\[
\varphi_1(b) + \varphi_1(b') - l + \left[ \frac{1}{2} \text{wt}_0(b) \right] = \frac{1}{2} x_2(b) + \frac{1}{2} \Psi_6(b) - \frac{\delta}{2} \geq 0 \quad \text{integer greater than } -\frac{1}{2}
\]

\( \Box \)

A minimal element is an element \( b \in B' \) such that \( \langle c, \varphi(b) \rangle = l \). By \( \langle c, \varphi(b) \rangle = \varphi_0(b) + 2 \varphi_1(b) + \varphi_2(b) \) and Proposition 5.32 if \( \text{wt}_0(b) > 0 \) we have \( \langle c, \varphi(b) \rangle > l \). Therefore for a minimal element \( b \) we must have \( \text{wt}_0(b) = 0 \).
Lemma 5.33 we have
\[ \min \{ \langle c, \varphi(b) \rangle \mid b \in B(n\Lambda_1) \} = l. \]

**Proof** By (5.4.3) and (5.4.3), the elements \( b \) such that \( \varepsilon_2(b) = 0, \varphi_1(b) + \varphi_1(f_2^{2\varepsilon_2(b)}(b)) = l \) satisfy the following conditions.

If \( x_5(b) = 0, \)
\[ x_2(b) = 2x_6(b) + x_0(b) + x_4(b), \]
\[ \Psi_2(b) = 3x_6(b) + x_0(b) + 2x_4(b) + x_3(b). \]

If \( x_5(b) = 1, \)
\[ x_2(b) = 0, \]
\[ x_6(b) = 0. \]

Then we can express \( b \) such that \( \varepsilon_2(b) = 0, \varphi_1(b) + \varphi_1(f_2^{2\varepsilon_2(b)}(b)) = l \) as:
\[
\begin{cases}
[1]^{m} [2]^{n} [3]^{n} [4]^{n} [5]^{m} \quad (l = 2m + 3n), \\
[1]^{m} [2]^{n} [3]^{n} [4]^{n} [5]^{n+1} [6]^{m} \quad (n > 0, l = 2m + 3n + 1), \\
[1]^{m} [3]^{n} [5]^{m} \quad (l = 2m + 1), \\
[1]^{m} [2]^{n} [3]^{n} [4]^{n} [5]^{n+1} [6]^{m} \quad (l = 2m + 3n + 2),
\end{cases}
\]

where \( m, n \in \mathbb{Z}_{\geq 0} \). We put
\[
\tilde{b}(m) = f_2^{3n}b = \begin{cases}
[1]^{m} [2]^{n} [3]^{n} [4]^{n} [5]^{m} \quad (l = 2m + 3n), \\
[1]^{m} [2]^{n} [3]^{n} [4]^{n} [5]^{n+1} [6]^{m} \quad (n > 0, l = 2m + 3n + 1), \\
[1]^{m} [0]^{m} [1]^{m} \quad (l = 2m + 1), \\
[1]^{m} [2]^{n} [3]^{n} [4]^{n} [5]^{m} \quad (l = 2m + 3n + 2),
\end{cases}
\]

where \( m \in \mathbb{Z}_{\geq 0} \) such \( l - 2m \geq 0 \). We prove \( \langle c, 2\varepsilon_1(\tilde{b}(m))\Lambda_1 + \varphi_2(\tilde{b}(m))\Lambda_2 \rangle = l \) \((k = 1, 2, 3)\). We prove the case of \( l = 2m + 3n \). By Proposition 3.3
\[
\Psi_1(\tilde{b}(m)) = u_-u_+^nu_-^n, \quad \text{Red}_1(\tilde{b}(m)) = u_-^n,
\]
we have
\[ \varphi_1(\tilde{b}(m)) = m. \]

By Proposition 3.3
\[
\Psi_2(\tilde{b}(m)) = u_0u_-^nu_0^2u_+^n, \quad \text{Red}_2(\tilde{b}(m)) = u_-^n,
\]
we have
\[ \varphi_2(\tilde{b}(m)) = 3n. \]
Therefore, 
\[ \langle c, 2\varphi_1(\tilde{b}(m))\Lambda_1 + \varphi_2(\tilde{b}(m))\Lambda_2 \rangle = 2m + 3n. \]

In similar way, we can prove other cases.

Let \( B^0 \cong \{ b \mid b \in B(l\Lambda_1), \langle c, \varphi_1(b)\Lambda_1 + \varphi_2(b) \rangle = l \}, \) \( (P_{cl})_0^+ \cong \{ \lambda \in \mathbb{Z}\Lambda_1 + \mathbb{Z}\Lambda_2 \mid \langle c, \lambda \rangle = l \}. \) Then we see following proposition easily.

**Proposition 5.34** The maps \( \varepsilon, \varphi : B^0 \to (P_{cl})_0^+ \) are bijective.

### 5.4.2 Existence of minimal elements on \( B^{G_2}(l\Lambda_1) \)

We consider \( \left( \tilde{f}_1 \right)^i \left( \tilde{f}_0 \right)^i \tilde{b}_{(i,i)} \). By Definition 5.12 we have
\[
\left( \tilde{f}_1 \right)^i \left( \tilde{f}_0 \right)^i \tilde{b}_{(i,i)} = (\tilde{e}_2)^{i-2i} \left( \tilde{f}_1 \right)^i \left( \tilde{f}_0 \right)^i \tilde{b}_{(i-i,i)}.
\]

For \( b \in B^{l}_{k\Lambda_1,j\Lambda_0} \), we define \( y = y(b) = \left[ \frac{l-i-1}{3} \right] \).

By Proposition 5.2 and Lemma 5.2,
\[
\Phi \left( \left( \tilde{f}_1 \right)^i \left( \tilde{f}_0 \right)^i \tilde{b}_{(i-i,i)} \right) =
\begin{cases}
\left[ [1] [0] [y] [\tilde{z}]^y [\tilde{t}]^y [\tilde{t}]' \right] & (l - 2i \equiv 0 \pmod{3}), \\
\left[ [1] [y] [\tilde{t}]' \right] & (l - 2i \equiv 1 \pmod{3}, y = 0), \\
\left[ [1] [0] [y] [0] [y]^{-1} [\tilde{t}] [\tilde{z}]^y [\tilde{t}]' \right] & (l - 2i \equiv 1 \pmod{3}, y > 0), \\
\left[ [1] [0] [0] [y] [\tilde{z}]^y [\tilde{t}]^y [\tilde{t}]' \right] & (l - 2i \equiv 2 \pmod{3}),
\end{cases}
\]

we have
\[
\Phi \left( (\tilde{e}_2)^{i-2i} \left( \tilde{f}_1 \right)^i \left( \tilde{f}_0 \right)^i \tilde{b}_{(i-i,i)} \right) =
\begin{cases}
\left[ [1] [2] [y] [\tilde{z}]^y [\tilde{t}]' \right] & (l - 2i \equiv 0 \pmod{3}), \\
\left[ [1] [0] [\tilde{t}]' \right] & (l - 2i \equiv 1 \pmod{3}, y = 0), \\
\left[ [1] [2] [1] [y]^{-1} [\tilde{t}] [\tilde{z}]^y [\tilde{t}]' \right] & (l - 2i \equiv 1 \pmod{3}, y > 0), \\
\left[ [1] [2] [y] [3] [\tilde{z}]^y [\tilde{t}]^y [\tilde{t}]' \right] & (l - 2i \equiv 2 \pmod{3}),
\end{cases}
\]

Therefore we have proved existence of minimal elements. By 5.1.12 we see
\[
\varphi_0 \left( \left( \tilde{f}_1 \right)^i \left( \tilde{f}_0 \right)^i \tilde{b}_{(i,i)} \right) = 0.
\]

Thus we have
\[
\langle c, \varphi(\tilde{b}(m)) \rangle = l,
\]

where \( m \in \mathbb{Z}_{\geq 0} \).
5.4.3 Selection of minimals on $B^l$

By (5.2.7), for $n < l$, we have

$$
\varphi_0 \big|_{B^{n+1}} (b) = \varphi_0 \big|_{B^n} (b) + 1.
$$

Then for $b \in B^n$ such that $\langle c, \varphi(b) \rangle = n$, we have $\langle c, \varphi(b) \rangle = n + 1$ on $B^{n+1}$. If $l = 1$, by calculation we see

$$
\min \left\{ \langle c, \varphi(b) \rangle \mid b \in B^1 \right\} = 1.
$$

Then we see

$$
\min \left\{ \langle c, \varphi(b) \rangle \mid b \in B^l \right\} = l,
$$

inductively. Therefore a minimal element for level $l - 1$ is also a minimal element for level $l$. Thus it is obvious that $\varepsilon$ and $\varphi$ are bijective.

**Example 5.35** Following elements are minimal ($0 \leq l \leq 7$).

- if $l \geq 0$ \( \phi \),
- if $l \geq 1$ \( \mathbb{J}_1 \),
- if $l \geq 2$ \( \left[ [1] [3] \right] \),
- if $l \geq 3$ \( \left[ [1] [1] [1] \right] \),
- if $l \geq 4$ \( \left[ [1] [1] [1] [1] [1] \right] \),
- if $l \geq 5$ \( \left[ [1] [1] [1] [1] [1] \right] \),
- if $l \geq 6$ \( \left[ [1] [1] [1] [1] [1] \right] \),
- if $l \geq 7$ \( \left[ [1] [1] [1] [1] [1] [1] \right] \).

5.5 Connectedness of $B^l \otimes B^l$

We will show connectedness of $B^l \otimes B^l$, by showing that any element of $B^l \otimes B^l$ is connected with $\phi \otimes \phi$. We consider decomposition of tensor products $B^{G_2}(m_1 \Lambda_1) \otimes B^{G_2}(m_2 \Lambda_1)$. Each connected component has a lowest weight element. Then, we prove the lowest elements connected with $\phi \otimes \phi$.

For $b_1 \in B^{G_2}(m_1 \Lambda_1)$, $b_2 \in B^{G_2}(m_2 \Lambda_1)$, there exist a sequence $(i_1, \ldots, i_k)$ $(i_n \in \{1, 2\}, n = 1, \ldots, k)$ such that $\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} b_1 \otimes b_2$ is a lowest weight element $b'_1 \otimes \left[ [\Gamma]^{m_2} \right] \in B^{G_2}(m_1 \Lambda_1) \otimes B^{G_2}(m_2 \Lambda_2)$. We can express

$$
\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} b_1 \otimes b_2 = b'_1 \otimes \left[ [\Gamma]^{m_2} \right].
$$

Then there exists $b''_1 \in B^{G_2}(m'_1 \Lambda_1)$ such that

$$
\tilde{f}_0^{(\varphi_0(b'_1)-m_2)+m_2} b'_1 \otimes \left[ [\Gamma]^{m_2} \right] = b''_1 \otimes \phi.
$$

There exist another sequence $(i'_1, \ldots, i'_{k'})$ $(i'_n \in \{1, 2\}, n = 1, \ldots, k')$ such that $\tilde{f}_0^{m'_1} \tilde{f}_{i'_1} \cdots \tilde{f}_{i'_{k'}} b''_1$ is lowest weight element $\left[ [\Gamma]^{m'_1} \right]$. Therefore we can express as

$$
\tilde{f}_0^{m'_1} \tilde{f}_{i'_1} \cdots \tilde{f}_{i'_{k'}} b''_1 \otimes \phi = \tilde{f}_0^{m'_1} \left[ [\Gamma]^{m'_1} \right] \otimes \phi = \phi \otimes \phi.
$$
Then we have 

\[ b_1 \otimes b_2 \text{ is connected with } \phi \otimes \phi. \]

By §5.4, §5.5 we have Theorem 2.2.
Perfect crystal of level 2

Figure 9: Perfect crystal of level 2
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