Double domination in lexicographic product graphs

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Abstract

In a graph $G$, a vertex dominates itself and its neighbours. A subset $S \subseteq V(G)$ is said to be a double dominating set of $G$ if $S$ dominates every vertex of $G$ at least twice. The minimum cardinality among all double dominating sets of $G$ is the double domination number. In this article, we obtain tight bounds and closed formulas for the double domination number of lexicographic product graphs $G \circ H$ in terms of invariants of the factor graphs $G$ and $H$.

Keywords: Double domination; total domination; total Roman $\{2\}$-domination; lexicographic product

1 Introduction

In a graph $G$, a vertex dominates itself and its neighbours. A subset $S \subseteq V(G)$ is said to be a dominating set of $G$ if $S$ dominates every vertex of $G$, while $S$ is said to be a double dominating set of $G$ if $S$ dominates every vertex of $G$ at least twice. A subset $S \subseteq V(G)$ is said to be a total dominating set of $G$ if every vertex $v \in V(G)$ is dominated by at least one vertex in $S \setminus \{v\}$. The minimum cardinality among all dominating sets of $G$ is the domination number, denoted by $\gamma(G)$. The double domination number and the total domination number of $G$ are defined by analogy, and are denoted by $\gamma_{\times 2}(G)$ and $\gamma_t(G)$, respectively. The domination number and the total domination number have been extensively studied. For instance, we cite the following books [19, 20, 21]. The double domination number, which has been less studied, was introduced in [18] by Harary and Haynes, and studied further in a number of works including [4, 10, 15, 17, 23].

Let $f : V(G) \to \{0, 1, 2\}$ be a function. For any $i \in \{0, 1, 2\}$ we define the subsets of vertices $V_i = \{v \in V(G) : f(v) = i\}$ and we identify $f$ with the three subsets of $V(G)$ induced by $f$. 
Thus, in order to emphasize the notation of these sets, we denote the function by \( f(V_0, V_1, V_2) \). Given a set \( X \subseteq V(G) \), we define \( f(X) = \sum_{v \in X} f(v) \), and the weight of \( f \) is defined to be \( \omega(f) = f(V(G)) = |V_1| + 2|V_2| \).

A function \( f(V_0, V_1, V_2) \) is a total Roman dominating function (TRDF) on a graph \( G \) if \( V_1 \cup V_2 \) is a total dominating set and \( N(v) \cap V_2 \neq \emptyset \) for every vertex \( v \in V_0 \), where \( N(v) \) denotes the open neighbourhood of \( v \). This concept was introduced by Liu and Chang [24]. For recent results on total Roman domination in graphs we cite [1, 2, 7, 9].

A function \( f(V_0, V_1, V_2) \) is a total Roman \( \{2\}\)-dominating function (TR2DF) if \( V_1 \cup V_2 \) is a total dominating set and \( f(N(v)) \geq 2 \) for every vertex \( v \in V_0 \). This concept was recently introduced in [6]. Notice that \( S \subseteq V(G) \) is a double dominating set of \( G \) if and only if there exists a TR2DF \( f(V_0, V_1, V_2) \) such that \( V_1 = S \) and \( V_2 = \emptyset \).

The total Roman domination number, denoted by \( \gamma_{tr}(G) \), is the minimum weight among all TRDFs on \( G \). By analogy, we define the total Roman \( \{2\}\)-domination number, which is denoted by \( \gamma_{r2}(G) \).

Notice that, by definition, \( \gamma_{r2}(G) \geq \gamma_{r2}(G) \). As an example of graph \( G \) for which \( \gamma_{r2}(G) > \gamma_{r2}(G) \) we consider a star graph \( K_{1,r} \) for \( r \geq 3 \). In this case, \( \gamma_{r2}(K_{1,r}) = r + 1 > 3 = \gamma_{r2}(K_{1,r}) \). We would point out that the problem of characterizing all graphs with \( \gamma_{r2}(G) = \gamma_{r2}(G) \) remains open. In this paper we show that the values of these two parameters coincide for any lexicographic product graph \( G \circ H \) in which graph \( G \) has no isolated vertices and graph \( H \) is not trivial. Furthermore, we obtain tight bounds and closed formulas for \( \gamma_{r2}(G \circ H) \) in terms of invariants of the factor graphs \( G \) and \( H \).

### 1.1 Additional concepts, notation and tools

All graphs considered in this paper are finite and undirected, without loops or multiple edges. As usual, the closed neighbourhood of a vertex \( v \in V(G) \) is denoted by \( N[v] = N(v) \cup \{v\} \). We say that a vertex \( v \in V(G) \) is a universal vertex of \( G \) if \( N[v] = V(G) \). By analogy with the notation used for vertices, for a set \( S \subseteq V(G) \), its open neighbourhood is the set \( N(S) = \bigcup_{v \in S} N(v) \), and its closed neighbourhood is the set \( N(S) = N(S) \cup S \). The subgraph induced by \( S \subseteq V(G) \) will be denoted by \( \langle S \rangle \), while the graph obtained from \( G \) by removing all the vertices in \( S \subseteq V(G) \) (and all the edges incident with a vertex in \( S \)) will be denoted by \( G - S \).

We will use the notation \( K_n, K_{1,n-1}, C_n, N_n, P_n \) and \( K_{n,n-r} \) for complete graphs, star graphs, cycle graphs, empty graphs, path graphs and complete bipartite graphs of order \( n \), respectively. A double star \( S_{n_1,n_2} \) is the graph obtained by joining the center of two stars \( K_{1,n_1} \) and \( K_{1,n_2} \) with an edge.

Given two graphs \( G \) and \( H \), the lexicographic product of \( G \) and \( H \) is the graph \( G \circ H \) whose vertex set is \( V(G \circ H) = V(G) \times V(H) \) and \( (u,v)(x,y) \in E(G \circ H) \) if and only if \( ux \in E(G) \) or \( u = x \) and \( vy \in E(H) \). Notice that for any vertex \( u \in V(G) \) the subgraph of \( G \circ H \) induced by \( \{u\} \times V(H) \) is isomorphic to \( H \). For simplicity, we will denote this subgraph by \( H_u \). For basic properties of lexicographic product graphs we suggest the books [16, 22]. In particular, we cite the following works on domination theory of lexicographic product graphs: standard domination [25, 27, 31], Roman domination [28], total Roman domination [9], weak Roman domination [30], rainbow domination [29], \( k \)-rainbow independent domination [5], super domination [13], twin domination [26], power domination [14] and doubly connected domination [3].
For simplicity, for any \((u,v) \in V(G) \times V(H)\) and any TR2DF \(f\) on \(G \circ H\) we write \(N(u,v)\) and \(f(u,v)\) instead of \(N((u,v))\) and \(f((u,v))\), respectively.

For the remainder of the paper, definitions will be introduced whenever a concept is needed.

Now we present some tools that will be very useful throughout the work.

**Proposition 1.1.** [6] The following inequalities hold for any graph \(G\) with no isolated vertex.

(i) \(\gamma(G) \leq \gamma_{\{R2\}}(G) \leq 2\gamma(G)\).

(ii) \(\gamma_{\{R2\}}(G) \leq \gamma_{\times2}(G)\).

A double dominating set of cardinality \(\gamma_{\times2}(G)\) will be called a \(\gamma_{\times2}(G)\)-set. A similar agreement will be assumed when referring to optimal sets (and functions) associated to other parameters used in the article.

**Theorem 1.2.** If \(\gamma_{\times2}(G) = \gamma(G)\), then for any \(\gamma_{\times2}(G)\)-set \(D\) there exists an integer \(k \geq 1\) such that \(\langle D \rangle \cong \bigcup_{i=1}^{k} K_2\).

**Proof.** Let \(D\) be a \(\gamma_{\times2}(G)\)-set and suppose that \(\langle D \rangle\) has a component \(G'\) which is not isomorphic to \(K_2\). Let \(v \in V(G')\) be a vertex of minimum degree in \(G'\). Notice that the set \(D \setminus \{v\}\) is a total dominating set of \(G\). Hence, \(\gamma(G) \leq |D \setminus \{v\}| < |D| = \gamma_{\times2}(G)\), which is a contradiction. Therefore, the result follows.

**Theorem 1.3.** [6] The following statements are equivalent.

- \(\gamma_{\{R2\}}(G) = 2\gamma(G)\).
- \(\gamma_{\{R2\}}(G) = \gamma_{\times2}(G)\) and \(\gamma(G) = \gamma(G)\).

The following theorem merges two results obtained in [6] and [18].

**Theorem 1.4 ([6] and [18]).** The following statements are equivalent.

- \(\gamma_{\{R2\}}(G) = 2\).
- \(\gamma_{\times2}(G) = 2\).
- \(G\) has at least two universal vertices.

It is readily seen that if \(G'\) is a spanning subgraph of \(G\), then any \(\gamma_{\times2}(G')\)-set is a double dominating set of \(G\). Therefore, the following result is immediate.

**Theorem 1.5.** If \(G'\) is a spanning subgraph of \(G\) with no isolated vertex, then

\[
\gamma_{\times2}(G) \leq \gamma_{\times2}(G').
\]

In Proposition 4.7 we will show some cases of lexicographic product graphs for which the equality above holds.

**Remark 1.6.** For any integer \(n \geq 3\),
(ii) \( \gamma_{(R2)}(P_n) \) \( = \gamma_{(R2)}(C_n) \) \( = [6] \gamma_{(R2)}(P_n) \) \( = [4] \left\{ \begin{array}{ll} 2 \left\lfloor \frac{n}{3} \right\rfloor + 1, & \text{if } n \equiv 0 \pmod{3}, \\ 2 \left\lfloor \frac{n}{3} \right\rfloor, & \text{otherwise.} \end{array} \right. \)

The next theorem merges two results obtained in [28] and [31].

**Theorem 1.7** ([28] and [31]). For any graph \( G \) with no isolated vertex and any nontrivial graph \( H \),

\[
\gamma(G \circ H) = \left\{ \begin{array}{ll} \gamma(G), & \text{if } \gamma(H) = 1, \\ \gamma(G), & \text{if } \gamma(H) \geq 2. \end{array} \right.
\]

**Theorem 1.8.** [8] For any graph \( G \) with no isolated vertex and any nontrivial graph \( H \),

\[
\gamma(G \circ H) = \gamma(G).
\]

## 2 Main results on lexicographic product graphs

Our first result shows that the double domination number and the total Roman \( \{2\} \)-domination number coincide for lexicographic product graphs.

**Theorem 2.1.** For any graph \( G \) with no isolated vertex and any nontrivial graph \( H \),

\[
\gamma_{(R2)}(G \circ H) = \gamma_{(R2)}(G \circ H).
\]

**Proof.** Proposition 1.1 (ii) leads to \( \gamma_{(R2)}(G \circ H) \geq \gamma_{(R2)}(G \circ H) \). Let \( f(V_0, V_1, V_2) \) be a \( \gamma_{(R2)}(G \circ H) \)-function such that \( |V_2| \) is minimum. Suppose that \( \gamma_{(R2)}(G \circ H) > \gamma_{(R2)}(G \circ H) \). In such a case, \( V_2 \neq \emptyset \) and we can differentiate two cases for a fixed vertex \( (u, v) \in V_2 \).

Case 1. \( N(u, v) \cap (V_1 \cup V_2) \subseteq V(H_u) \). In this case, for any \( (u', v') \in N(u) \times V(H) \) we define the function \( g(V_0', V_1', V_2') \) where \( V_0' = V_0 \setminus \{(u', v')\} \), \( V_1' = V_1 \cup \{(u, v), (u', v')\} \) and \( V_2' = V_2 \setminus \{(u, v)\} \). Observe that \( V_1' \cup V_2' \) is a total dominating set of \( G \circ H \) and every vertex \( w \in V_0' \subseteq V_0 \) satisfies that \( g(N(w)) \geq 2 \). Hence, \( g \) is a \( \gamma_{(R2)}(G \circ H) \)-function and \( |V_2'| = |V_2| - 1 \), which is a contradiction.

Case 2. \( N(u) \times V(H) \cap (V_1 \cup V_2) \neq \emptyset \). If \( f(u, v') > 0 \) for every vertex \( v' \in V(H) \), then the function \( g \), defined by \( g(u, v) = 1 \) and \( g(x, y) = f(x, y) \) whenever \( (x, y) \in V(G \circ H) \setminus \{(u, v)\} \), is a TR2DF on \( G \circ H \) and \( \omega(g) = \omega(f) - 1 \), which is a contradiction. Hence, there exists a vertex \( v' \in V(H) \) such that \( f(u, v') = 0 \). In this case, we define the function \( g(V_0', V_1', V_2') \) where \( V_0' = V_0 \setminus \{(u, v')\} \), \( V_1' = V_1 \cup \{(u, v), (u, v')\} \) and \( V_2' = V_2 \setminus \{(u, v)\} \). Notice that \( V_1' \cup V_2' \) is a total dominating set of \( G \circ H \) and every vertex \( w \in V_0' \subseteq V_0 \) satisfies that \( g(N(w)) \geq 2 \). Hence, \( g \) is a \( \gamma_{(R2)}(G \circ H) \)-function and \( |V_2'| = |V_2| - 1 \), which is a contradiction again.

According to the two cases above, we deduce that \( V_2 = \emptyset \). Therefore, \( V_1 \) is a \( \gamma_{(R2)}(G \circ H) \)-set and so \( \gamma_{(R2)}(G \circ H) = \gamma_{(R2)}(G \circ H) \). \( \square \)
From now on, the main goal is to obtain tight bounds or closed formulas for $\gamma_x(G \circ H)$ and express them in terms of invariants of $G$ and $H$.

A set $X \subseteq V(G)$ is called a 2-packing if $N[u] \cap N[v] = \emptyset$ for every pair of different vertices $u, v \in X$, [20]. The 2-packing number $\rho(G)$ is the maximum cardinality among all 2-packing sets of $G$. As usual, a 2-packing of cardinality $\rho(G)$ is called a $\rho(G)$-set.

**Theorem 2.2.** For any graph $G$ with no isolated vertex and any nontrivial graph $H$,

$$\max\{\gamma(G), 2\rho(G)\} \leq \gamma_x(G \circ H) \leq 2\gamma(G).$$

**Proof.** By Proposition 1.1 (i) and Theorem 1.8 we deduce that

$$\gamma(G) = \gamma(G \circ H) \leq \gamma_x(G \circ H) \leq 2\gamma(G) = 2\gamma(G).$$

Now, for any $\rho(G)$-set $X$ and any $\gamma_x(G \circ H)$-set $D$ we have that

$$\gamma_x(G \circ H) = |D| = \sum_{u \in V(G)} |D \cap V(H_u)| \geq \sum_{u \in X} \sum_{w \in N[u]} |D \cap V(H_w)| \geq 2|X| = 2\rho(G).$$

Therefore, the proof is complete. \qed

We would point out that the upper bound $\gamma_x(G \circ H) \leq \min\{2\gamma(G), \gamma(G)\gamma_x(H)\}$ was proposed in [12] for the particular case in which $G$ and $H$ are connected. Obviously, the connectivity is not needed, and the bound $\gamma_x(G \circ H) \leq \gamma(G)\gamma_x(H)$ also holds for any graph $G$ (even if $G$ is empty) and any graph $H$ with no isolated vertices.

In Theorem 2.4 we will show cases in which $\gamma_x(G \circ H) = 2\gamma(G)$, while in Theorem 2.8 (i) and (ii) we will show cases in which $\gamma_x(G \circ H) = 2\rho(G)$ or $\gamma_x(G \circ H) = \gamma(G)$.

**Corollary 2.3.** If $\gamma(G) = 1$, then for any nontrivial graph $H$,

$$2 \leq \gamma_x(G \circ H) \leq 4.$$

In Section 3 we characterize the graphs with $\gamma_x(G \circ H) \in \{2, 3\}$. Hence, by Corollary 2.3 the graphs with $\gamma_x(G \circ H) = 4$ will be automatically characterized whenever $\gamma(G) = 1$.

**Theorem 2.4.** If $G$ is a graph with no isolated vertex and $H$ is a nontrivial graph, then the following statements are equivalent.

(a) $\gamma_x(G \circ H) = 2\gamma(G)$.
(b) $\gamma_x(G \circ H) = \gamma_R(G \circ H)$ and $(\gamma(G) = \gamma(G) \text{ or } \gamma(H) \geq 2)$.

**Proof.** Assume that $\gamma_x(G \circ H) = 2\gamma(G)$. By Theorems 1.8 and 2.1 we deduce that

$$\gamma_{\{R\}}(G \circ H) = \gamma_x(G \circ H) = 2\gamma(G) = 2\gamma(G \circ H).$$

Hence, by Theorem 1.3 we have that $\gamma_x(G \circ H) = \gamma_R(G \circ H)$ and $\gamma(G \circ H) = \gamma(G \circ H) = \gamma(G)$. Notice that $\gamma(G \circ H) = \gamma(G)$ if and only if $\gamma(G) = \gamma(G) \text{ or } \gamma(H) \geq 2$, by Theorem 1.7. Therefore, (b) follows.
Conversely, assume that (b) holds. By Theorem 2.1 we have that
\[ \gamma_{(R2)}(G \circ H) = \gamma \times_2 (G \circ H) = \gamma_R(G \circ H). \] (1)
Now, if \( \gamma(G) = \gamma(H) \geq 2 \), by Theorems 1.7 and 1.8 we deduce that
\[ \gamma(G \circ H) = \gamma(G) = \gamma(G \circ H). \] (2)
Hence, Theorem 1.3 and equations (1) and (2) lead to \( \gamma \times_2 (G \circ H) = \gamma_{(R2)}(G \circ H) = 2\gamma(G \circ H) = 2\gamma(G) \), as required.

It was shown in [11] that for any connected graph \( G \) of order \( n \geq 3 \), \( \gamma(G) \leq \frac{2n}{3} \). Hence, Proposition 1.1 (i) and Theorem 2.1 lead to the following result.

**Theorem 2.5.** For any connected graph \( G \) of order \( n \geq 3 \) and any graph \( H \),
\[ \gamma \times_2 (G \circ H) \leq 2 \left\lceil \frac{2n}{3} \right\rceil. \]

In order to show that the bound above is tight, we consider the case of rooted product graphs. Given a graph \( G \) and a graph \( H \) with root \( v \in V(H) \), the rooted product \( G \bullet_v H \) is defined as the graph obtained from \( G \) and \( H \) by taking one copy of \( G \) and \( |V(G)| \) copies of \( H \) and identifying the \( i^{th} \) vertex of \( G \) with vertex \( v \) in the \( i^{th} \) copy of \( H \) for every \( i \in \{1, \ldots, |V(G)|\} \). For instance, the graph \( P_3 \bullet_v P_3 \) where \( v \) is a leaf, is shown in Figure 1. Later, when we read Lemma 4.3, it will be easy to see that for \( n = |V(G \bullet_v P_3)| = 3|V(G)| \) we have that \( \gamma \times_2 ((G \bullet_v P_3) \circ H) = 4|V(G)| = 2\left\lceil \frac{2n}{3} \right\rceil \) whenever \( \gamma(H) \geq 3 \).

![Figure 1: The graph \( P_3 \bullet_v P_3 \)](image)

**Lemma 2.6.** For any graph \( G \) with no isolated vertex and any nontrivial graph \( H \), there exists a \( \gamma \times_2 (G \circ H) \)-set \( S \) such that \( |S \cap V(H_u)| \leq 2 \), for every \( u \in V(G) \).

*Proof.* Given a double dominating set \( S \) of \( G \circ H \), we define the set \( S_3 = \{ x \in V(G) : |S \cap V(H_x)| \geq 3 \} \). Let \( S \) be a \( \gamma \times_2 (G \circ H) \)-set such that \( |S_3| \) is minimum among all \( \gamma \times_2 (G \circ H) \)-sets. If \( |S_3| = 0 \), then we are done. Hence, we suppose that there exists \( u \in S_3 \) and let \( (u, v) \in S \). We assume that \( |S \cap V(H_u)| \) is minimum among all vertices in \( S_3 \). It is readily seen that if there exists \( u' \in N(u) \) such that \( |S \cap V(H_{u'})| \geq 2 \), then \( S' = S \setminus \{(u, v)\} \) is a double dominating set of \( G \circ H \), which is a contradiction. Hence, if \( u' \in N(u) \), then \( |S \cap V(H_{u'})| \leq 1 \), and in this case it is not difficult to check that for \( (u', v') \notin S \) the set \( S'' = (S \setminus \{(u, v)\}) \cup \{(u', v')\} \) is a \( \gamma \times_2 (G \circ H) \)-set such that \( |S''_3| \) is minimum among all \( \gamma \times_2 (G \circ H) \)-sets. If \( |S''_3| < |S_3| \), then we obtain a contradiction, otherwise \( u \in S''_3 \) and \( |S'' \cap V(H_u)| \) is minimum among all vertices in \( S''_3 \), so that we can successively repeat this process, until obtaining a contradiction. Therefore, the result follows. \qed
Theorem 2.7. Let $G$ be a graph with no isolated vertex and let $H$ be a nontrivial graph.

(i) If $\gamma(H) = 1$, then $\gamma \times 2(G \circ H) \leq \gamma \{R2\}(G)$.

(ii) If $H$ has at least two universal vertices, then $\gamma \times 2(G \circ H) \leq 2\gamma(G)$.

(iii) If $H$ has exactly one universal vertex, then $\gamma \times 2(G \circ H) = \gamma \{R2\}(G)$.

(iv) If $\gamma(H) \geq 2$, then $\gamma \times 2(G \circ H) \geq \gamma \{R2\}(G)$.

Proof. Let $f$ be a $\gamma \{R2\}(G)$-function and let $v$ be a universal vertex of $H$. Let $f'$ be the function defined by $f'(u,v) = f(u)$ for every $u \in V(G)$ and $f'(x,y) = 0$ whenever $x \in V(G)$ and $y \in V(H) \setminus \{v\}$. It is readily seen that $f'$ is a TR2DF on $G \circ H$. Hence, by Theorem 2.1 we conclude that $\gamma \times 2(G \circ H) = \gamma \{R2\}(G \circ H) \leq \omega(f') = \omega(f) = \gamma \{R2\}(G)$ and (i) follows.

Let $D$ be a $\gamma(G)$-set and let $y_1,y_2$ be two universal vertices of $H$. It is not difficult to see that $S = D \times \{y_1,y_2\}$ is a double dominating set of $G \circ H$. Therefore, $\gamma \times 2(G \circ H) \leq |S| = 2\gamma(G)$ and (ii) follows.

From now on, let $S$ be a $\gamma \times 2(G \circ H)$-set that satisfies Lemma 2.6 and assume that either $\gamma(H) \geq 2$ or $H$ has exactly one universal vertex. Let $g(V_0,V_1,V_2)$ be the function defined by $g(u) = |S \cap V(H_u)|$ for every $u \in V(G)$. We claim that $g$ is a TR2DF on $G$. It is clear that every vertex in $V_1$ has to be adjacent to some vertex in $V_1 \cup V_2$ and, if $\gamma(H) \geq 2$ or $H$ has exactly one universal vertex, then by Theorem 1.4 we have that $\gamma \times 2(H) \geq 3$, which implies that every vertex in $V_2$ has to be adjacent to some vertex in $V_1 \cup V_2$. Hence, $V_1 \cup V_2$ is a total dominating set of $G$. Now, if $x \in V_0$, then $S \cap V(H_x) = \emptyset$, and so $|N(V(H_x)) \cap S| \geq 2$. Thus, $g(N(x)) \geq 2$, which implies that $g$ is TR2DF on $G$ and so $\gamma \{R2\}(G) \leq \omega(g) = |S| = \gamma \times 2(G \circ H)$. Therefore, (iii) and (iv) follow.

The following result is a direct consequence of Theorems 2.2 and 2.7. Recall that $\gamma \times 2(H) = 2$ if and only if $H$ has at least two universal vertices (see Theorem 1.4).

Theorem 2.8. Let $G$ be a graph with no isolated vertex and let $H$ be a nontrivial graph.

(i) If $\gamma(G) = \rho(G)$ and $\gamma \times 2(H) = 2$, then $\gamma \times 2(G \circ H) = 2\gamma(G)$.

(ii) If $\gamma \{R2\}(G) \in \{\gamma(G),2\rho(G)\}$ and $\gamma(H) = 1$, then $\gamma \times 2(G \circ H) = \gamma \{R2\}(G)$.

(iii) If $\gamma \{R2\}(G) = 2\gamma(G)$ and $\gamma(H) \geq 2$, then $\gamma \times 2(G \circ H) = \gamma \{R2\}(G)$.

It is well known that $\gamma(T) = \rho(T)$ for any tree $T$. Hence, the following corollary is a direct consequence of Theorem 2.8.

Corollary 2.9. For any tree $T$ and any graph $H$ with $\gamma \times 2(H) = 2$,

$$\gamma \times 2(T \circ H) = 2\gamma(T).$$

A double total dominating set of a graph $G$ is a set $S$ of vertices of $G$ such that every vertex in $V(G)$ is adjacent to at least two vertices in $S \{21\}$. The double total domination number of $G$, denoted by $\gamma_{2,1}(G)$, is the minimum cardinality among all double total dominating sets.
Theorem 2.10. [30] If \( G \) is a graph of minimum degree greater than or equal to two, then for any graph \( H \),
\[
\gamma_{2,t}(G \circ H) \leq \gamma_{2,t}(G).
\]

Theorem 2.11. Let \( G \) be a graph of minimum degree greater than or equal to two and order \( n \). The following statements hold.

(i) For any graph \( H \), \( \gamma_{\times 2}(G \circ H) \leq \gamma_{2,t}(G) \).

(ii) For any graph \( H \), \( \gamma_{\times 2}(G \circ H) \leq n \).

Proof. Since every double total dominating set is a double dominating set, we deduce that \( \gamma_{\times 2}(G \circ H) \leq \gamma_{2,t}(G \circ H) \). Hence, from Theorem 2.10 we deduce (i). Finally, since \( \gamma_{2,t}(G) \leq n \), from (i) we deduce (ii).

The following family \( \mathcal{H}_k \) of graphs was shown in [30]. A graph \( G \) belongs to \( \mathcal{H}_k \) if and only if it is constructed from a cycle \( C_k \) and \( k \) empty graphs \( N_{s_1}, \ldots, N_{s_k} \) of order \( s_1, \ldots, s_k \), respectively, and joining by an edge each vertex from \( N_{s_i} \) with the vertices \( v_i \) and \( v_{i+1} \) of \( C_k \). Here we are assuming that \( v_i \) is adjacent to \( v_{i+1} \) in \( C_k \), where the subscripts are taken modulo \( k \). Figure 2 shows a graph \( G \) belonging to \( \mathcal{H}_k \), where \( k = 4 \), \( s_1 = s_3 = 3 \) and \( s_2 = s_4 = 2 \).

Notice that \( \gamma_{\{R2\}}(G) = \gamma_{2,t}(G) \), for every \( G \in \mathcal{H}_k \). Hence, from Theorems 2.7 (iv) and 2.11 (i) we deduce that \( \gamma_{\times 2}(G \circ H) = \gamma_{2,t}(G) \) for any \( G \in \mathcal{H}_k \) and any graph \( H \) such that \( \gamma(H) \geq 2 \).

![Figure 2: The set of black-coloured vertices is a \( \gamma_{2,t}(G) \)-set.](image)

3 Small values of \( \gamma_{\times 2}(G \circ H) \)

First, we characterize the graphs with \( \gamma_{\times 2}(G \circ H) = 2 \).

Theorem 3.1. For any nontrivial graph \( G \) and any graph \( H \), the following statements are equivalent.
(i) $\gamma_{\ast 2}(G \circ H) = 2$.

(ii) $\gamma(G) = \gamma(H) = 1$ and ($\gamma_{\ast 2}(G) = 2$ or $\gamma_{\ast 2}(H) = 2$).

Proof. Notice that $G \circ H$ has at least two universal vertices if and only if $\gamma(G) = \gamma(H) = 1$, and also $G$ has at least two universal vertices or $H$ has at least two universal vertices. Hence, by Theorem 1.4 we conclude that (i) and (ii) are equivalent.

Next, we characterize the graphs that satisfying $\gamma_{\ast 2}(G \circ H) = 3$. Before we shall need the following definitions. For a set $S \subseteq V(G \circ H)$ we define the following subsets of $V(G)$.

$$
A_S = \{v \in V(G) : |S \cap V(H_v)| \geq 2\};
$$

$$
B_S = \{v \in V(G) : |S \cap V(H_v)| = 1\};
$$

$$
C_S = \{v \in V(G) : S \cap V(H_v) = \emptyset\}.
$$

**Theorem 3.2.** For any nontrivial graphs $G$ and $H$, $\gamma_{\ast 2}(G \circ H) = 3$ if and only if one of the following conditions is satisfied.

(i) $G \cong P_2$ and $\gamma(H) = 2$.

(ii) $G \not\cong P_2$ has at least two universal vertices and $\gamma(H) \geq 2$.

(iii) $G$ has exactly one universal vertex and either $\gamma(H) = 2$ or $H$ has exactly one universal vertex.

(iv) $G$ has exactly one universal vertex, $\gamma_{\ast 2}(G) = 3$ and $\gamma(H) \geq 3$.

(v) $\gamma(G) = 2$ and $\gamma_{\ast 2}(G) = 3$.

(vi) $\gamma(G) = 2$, $\gamma_{\ast 2}(G) = 3 < \gamma_{\ast 2}(G)$ and $\gamma(H) = 1$.

Proof. Notice that with the above premises, $G$ does not have isolated vertices. Let $S$ be a $\gamma_{\ast 2}(G \circ H)$-set that satisfies Lemma 2.6 and assume that $|S| = 3$. By Theorems 1.8 and 1.2 we have that $3 = \gamma_{\ast 2}(G \circ H) > \gamma(G \circ H) = \gamma(G) \geq 2$, which implies that $\gamma(G) = 2$ and so $\gamma(G) \in \{1, 2\}$. We differentiate two cases.

Case 1. $\gamma(G) = 1$. In this case, Theorem 3.1 leads to $\gamma_{\ast 2}(H) \geq 3$. Now, we consider the following subcases.

Subcase 1.1. $G \cong P_2$. Notice that Theorem 3.1 leads to $\gamma(H) \geq 2$. Suppose that $\gamma(H) \geq 3$ and let $V(G) = \{u, w\}$. Observe that $S \cap V(H_u) \neq \emptyset$ and $S \cap V(H_w) \neq \emptyset$. Without loss of generality, let $S \cap V(H_u) = \{(u, v_1), (u, v_2)\}$ and $|S \cap V(H_v)| = 1$. Since $\gamma(H) \geq 3$, we have that $\{v_1, v_2\}$ is not a dominating set of $H$, which implies that no vertex in $\{u\} \times (V(H) \setminus (N(v_1) \cup N(v_2)))$ has two neighbours in $S$, which is a contradiction. Hence $\gamma(H) = 2$. Therefore, (i) follows.

Subcase 1.2. $G \not\cong P_2$ has at least two universal vertices. In this case, $\gamma_{\ast 2}(G) = 2$ and by Theorem 3.1 we deduce that $\gamma(H) \geq 2$. Thus, (ii) follows.

Subcase 1.3. $G$ has exactly one universal vertex. If $\gamma(H) \leq 2$, then by Theorem 3.1 we deduce that either $\gamma(H) = 2$ or $H$ has exactly one universal vertex, so that (iii) follows. Assume
leads to

leads to

Proposition 3.3. Let $H$ be a nontrivial graph. For any integer $n \geq 3$, the following statements hold.

(i) $\gamma_2(K_n \circ H) = \begin{cases} 2 & \text{if } \gamma(H) = 1, \\ 3 & \text{otherwise.} \end{cases}$

(ii) $\gamma_2(K_{1,n-1} \circ H) = \begin{cases} 2 & \text{if } \gamma_2(H) = 2, \\ 3 & \text{if } \gamma_2(H) \geq 2 \text{ and } \gamma(H) \leq 2, \\ 4 & \text{otherwise.} \end{cases}$
We now consider the cases in which $G$ is a double star graph or a complete bipartite graph. The following result is a direct consequence of Theorems 2.2, 3.1 and 3.2.

**Proposition 3.4.** Let $H$ be a graph. For any integers $n_2 \geq n_1 \geq 2$, the following statements hold.

(i) $\gamma_2(S_{n_1,n_2} \circ H) = 4$.

(ii) $\gamma_2(K_{n_1,n_2} \circ H) = \begin{cases} 3 & \text{if } n_1 = 2 \text{ and } \gamma(H) = 1; \\ 4 & \text{otherwise} \end{cases}$. 

4 All cases where $G \cong P_n$ or $G \cong C_n$

**4.1 Cases where $\gamma(H) = 1$**

**Proposition 4.1.** Let $n \geq 3$ be an integer and let $H$ be a nontrivial graph. If $\gamma(H) = 1$, then

$$\gamma_2(P_n \circ H) = \begin{cases} \left\lceil \frac{2n}{3} \right\rceil + 1, & \text{if } \gamma_2(H) \geq 3 \text{ and } n \equiv 0 \pmod{3}, \\ 2 \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise}. \end{cases}$$

**Proof.** If $\gamma_2(H) = 2$, then by Corollary 2.9 we deduce that $\gamma_2(P_n \circ H) = 2\gamma(P_n)$. Now, if $\gamma_2(H) \geq 3$, then $H$ has exactly one universal vertex and by Theorem 2.7 (iii) we deduce that $\gamma_2(G \circ H) = \gamma_2(P_n)$. \qed

From now on we assume that $V(C_n) = \{u_1, \ldots, u_n\}$, where the subscripts are taken modulo $n$ and consecutive vertices are adjacent.

**Proposition 4.2.** Let $n \geq 3$ be an integer and let $H$ be a graph. If $\gamma(H) = 1$, then

$$\gamma_2(C_n \circ H) = \left\lceil \frac{2n}{3} \right\rceil.$$

**Proof.** If $H$ is a trivial graph, then we are done, by Remark 1.6. From now on we assume that $H$ has at least two vertices. If $\gamma(H) = 1$, then by combining Theorem 2.7 (i) and Remark 1.6 (ii), we deduce that $\gamma_2(C_n \circ H) \leq \left\lceil \frac{2n}{3} \right\rceil$.

Now, let $S$ be a $\gamma_2(C_n \circ H)$-set. Notice that for any $i \in \{1, \ldots, n\}$ we have that

$$\left| S \cap \left( \bigcup_{j=0}^{2} V(H_{u_{i+j}}) \right) \right| \geq 2.$$

Hence,

$$3\gamma_2(C_n \circ H) = 3|S| = \sum_{i=1}^{n} \left| S \cap \left( \bigcup_{j=0}^{2} V(H_{u_{i+j}}) \right) \right| \geq 2n.$$

Therefore, $\gamma_2(C_n \circ H) \geq \left\lceil \frac{2n}{3} \right\rceil$, and the result follows. \qed
4.2 Cases where $\gamma(H) = 2$

To begin this subsection we need to state the following four lemmas.

**Lemma 4.3.** Let $G$ be a nontrivial connected graph and let $H$ be a graph. The following statements hold for every $\gamma_{\times 2}(G \odot H)$-set $S$ that satisfies Lemma 2.6.

(i) If $\gamma(H) \geq 2$ and $x \in B_S \cup C_S$, then $\sum_{u \in N(x)} |S \cap V(H_u)| \geq 2$.

(ii) If $\gamma(H) = 2$ and $x \in A_S$, then $\sum_{u \in N(x)} |S \cap V(H_u)| \geq 1$.

(iii) If $\gamma(H) \geq 3$ and $x \in V(G)$, then $\sum_{u \in N(x)} |S \cap V(H_u)| \geq 2$.

**Proof.** First, we suppose that $\gamma(H) = 2$. If there exists either a vertex $x \in B_S \cup C_S$ such that $\sum_{u \in N(x)} |S \cap V(H_u)| \leq 1$ or a vertex $x \in A_S$ such that $\sum_{u \in N(x)} |S \cap V(H_u)| = 0$, then there exists a vertex in $V(H_x) \setminus S$ which does not have two neighbours in $S$. Therefore, (ii) follows, and (i) follows for $\gamma(H) = 2$. Now, let $x \in V(G)$. Since $S$ satisfies Lemma 2.6, if $\gamma(H) \geq 3$, then there exists a vertex in $V(H_x) \setminus S$ which does not have neighbours in $S \cap V(H_x)$, which implies that $\sum_{u \in N(x)} |S \cap V(H_u)| \geq 2$ and so (i) and (iii) follows. Therefore, the proof is complete.

![Figure 3](image_url)

**Lemma 4.4.** For any integer $n \geq 3$ and any graph $H$ with $\gamma(H) = 2$,

$$
\gamma_{\times 2}(P_n \odot H) \leq \begin{cases} 
n - \left\lfloor \frac{n}{7} \right\rfloor + 1 & \text{if } n \equiv 1, 2 \pmod{7}, 
n - \left\lfloor \frac{n}{7} \right\rfloor & \text{otherwise}. 
\end{cases}
$$

\[12\]
Proof. In Figure 3 we show how to construct a double dominating set $S$ of $P_n \circ H$ for $n \in \{2, \ldots, 8\}$. In this scheme, the circles represent the copies of $H$ in $P_n \circ H$, two dots in a circle represent two vertices belonging to $S$, which form a dominating set of the corresponding copy of $H$, while a single dot in a circle represents one vertex belonging to $S$.

We now proceed to describe the construction of $S$ for any $n = 7q + r$, where $q \geq 1$ and $0 \leq r \leq 6$. We partition $V(P_n) = \{u_1, \ldots, u_n\}$ into $q$ sets of cardinality 7 and for $r \geq 1$ one additional set of cardinality $r$, in such a way that the subgraph induced by all these sets are paths. For any $r \neq 1$, the restriction of $S$ to each of these $q$ paths of length 7 corresponds to the scheme associated with $P_7 \circ H$ in Figure 3, while for the path of length $r$ (if any) we take the scheme associated with $P_r \circ H$. The case $r = 1$ and $q \geq 2$ is slightly different, as for the first $q - 1$ paths of length 7 we take the scheme associated with $P_7 \circ H$ and for the path associated with the last 8 vertices of $P_n$ we take the scheme associated with $P_8 \circ H$.

Notice that, for $n \equiv 1, 2 \pmod{7}$, we have that $\gamma_{\times 2}(P_n \circ H) \leq |S| = 6q + r + 1 = n - \left\lfloor \frac{n}{7} \right\rfloor + 1$, while for $n \not\equiv 1, 2 \pmod{7}$ we have $\gamma_{\times 2}(P_n \circ H) \leq |S| = 6q + r - n - \left\lfloor \frac{n}{7} \right\rfloor$. Therefore, the result follows.

Lemma 4.5. Let $P_r = w_1, \ldots, w_7$ be a subgraph of $C_n$. Let $H$ be a graph such that $\gamma(H) = 2$ and $W = \{w_1, \ldots, w_7\} \times V(H)$. If $S$ is a double dominating set of $C_n \circ H$ which satisfies Lemma 2.6, then

$$|S \cap W| \geq 6.$$ 

Proof. By Lemma 4.3 (i) and (ii) we have that $|S \cap (\{w_1, w_2, w_3\} \times V(H))| \geq 2$ and $|S \cap (\{w_4, w_5, w_6, w_7\} \times V(H))| \geq 3$. If $|S \cap (\{w_1, w_2, w_3\} \times V(H))| \geq 3$, then we are done. Hence, we assume that $|S \cap (\{w_1, w_2, w_3\} \times V(H))| = 2$. In this case, and by applying again Lemma 4.3 (i) and (ii) we deduce that $|S \cap (\{w_4, w_5, w_6, w_7\} \times V(H))| \geq 4$, which implies that $|S \cap W| \geq 6$, as desired. Therefore, the proof is complete.

Lemma 4.6. For any integer $n \geq 3$ and any graph $H$ with $\gamma(H) = 2$,

$$\gamma_{\times 2}(C_n \circ H) \geq \begin{cases} n - \left\lfloor \frac{n}{4} \right\rfloor + 1 & \text{if } n \equiv 1, 2 \pmod{7}, \\ n - \left\lfloor \frac{n}{4} \right\rfloor & \text{otherwise.} \end{cases}$$

Proof. It is easy to check that $\gamma_{\times 2}(C_n \circ H) = n$ for every $n \in \{3, 4, 5, 6\}$. Now, let $n = 7q + r$, with $0 \leq r \leq 6$ and $q \geq 1$. Let $S$ be a $\gamma_{\times 2}(C_n \circ H)$-set that satisfies Lemma 2.6.

If $r = 0$, then by Lemma 4.5 we have that $|S| \geq 6q = n - \left\lfloor \frac{n}{7} \right\rfloor$. From now on we assume that $r \geq 1$. By Theorem 1.5 and Lemma 4.4 we deduce that $\gamma_{\times 2}(C_n \circ H) \leq \gamma_{\times 2}(P_n \circ H) < n$, which implies that $A_S \neq \emptyset$, otherwise there exists $u \in V(C_n)$ such that $N(u) \cap C_S \neq \emptyset$ and so $|N(u) \cap B_S| \leq 1$, which is a contradiction. Let $x \in A_S$ and, without loss of generality, we can label the vertices of $C_n$ in such a way that $x = u_1$, and $u_2 \in A_S \cup B_S$ whenever $r \geq 2$. We partition $V(C_n)$ into $X = \{u_1, \ldots, u_r\}$ and $Y = \{u_{r+1}, \ldots, u_n\}$. Notice that Lemma 4.5 leads to $|S \cap (Y \times V(H))| \geq 6q$.

Now, if $r \in \{1, 2\}$, then $|S \cap (X \times V(H))| \geq r + 1$, which implies that $|S| \geq r + 1 + 6q = n - \left\lfloor \frac{n}{7} \right\rfloor + 1$. Analogously, if $r = 3$, then $|S \cap (X \times V(H))| \geq r$ and so $|S| \geq r + 6q = n - \left\lfloor \frac{n}{7} \right\rfloor$.

Finally, if $r \in \{4, 5, 6\}$, then by Lemma 4.3 (i) and (ii) we deduce that $|S \cap (X \times V(H))| \geq r$, which implies that $|S| \geq r + 6q = n - \left\lfloor \frac{n}{7} \right\rfloor$.

The following result is a direct consequence of Theorem 1.5 and Lemmas 4.4 and 4.6.
Proposition 4.7. For any integer n ≥ 3 and any graph H with γ(H) = 2,
\[ γ_Î©(C_n ∘ H) = γ_Î©(P_n ∘ H) = \left\{ \begin{array}{ll} n - \lfloor \frac{n}{2} \rfloor + 1 & \text{if } n \equiv 1, 2 \pmod{7}, \\ n - \lfloor \frac{n}{2} \rfloor & \text{otherwise}. \end{array} \right. \]

4.3 Cases where γ(H) ≥ 3
To begin this subsection we need to recall the following well-known result.

Remark 4.8. [21] For any integer n ≥ 3,
\[ γ(P_n) = γ(C_n) = \left\{ \begin{array}{ll} \frac{n}{2} & \text{if } n \equiv 0 \pmod{4}, \\ \frac{n+1}{2} & \text{if } n \equiv 1, 3 \pmod{4}, \\ \frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{4}. \end{array} \right. \]

Lemma 4.9. Let P_n = u_1 u_2 ... u_n be a path of order n ≥ 6, where consecutive vertices are adjacent, and let H be a graph. If γ(H) ≥ 3, then there exists a γ₂(P_n ∘ H)-set S such that u_n, u_{n−3} ∈ C_S and u_{n−1}, u_{n−2} ∈ A_S.

Proof. Let S be a γ₂(P_n ∘ H)-set that satisfies Lemma 2.6 such that |A_S| is maximum. First, we observe that u_{n−1} ∈ A_S by Lemma 4.3. Now, by applying again Lemma 4.3, we have that |S ∩ V(H_{u_n})| + |S ∩ V(H_{u_{n−2}})| ≥ 2. Hence, without loss of generality we can assume that u_{n−2} ∈ A_S and u_n ∈ C_S as |A_S| is maximum. If u_{n−3} ∈ C_S, then we are done. On the other hand, if u_{n−3} ∉ C_S, then as every vertex of V(H_{u_{n−3}}) has two neighbours in S ∩ V(H_{u_{n−2}}), we can redefine S by replacing the vertices in S ∩ V(H_{u_{n−2}}) with vertices in V(H_{u_{n−4}}) ∪ V(H_{u_{n−5}}) and obtaining a new γ₂(P_n ∘ H)-set S satisfying that u_{n−3} ∈ C_S, as desired. Therefore, the result follows.

Proposition 4.10. Let n ≥ 3 be an integer and let H be a graph. If γ(H) ≥ 3, then
\[ γ_Î©(P_n ∘ H) = 2γ(P_n) = \left\{ \begin{array}{ll} n & \text{if } n \equiv 0 \pmod{4}, \\ n + 1 & \text{if } n \equiv 1, 3 \pmod{4}, \\ n + 2 & \text{if } n \equiv 2 \pmod{4}. \end{array} \right. \]

Proof. Since Proposition 1.1 leads to γ₂(P_n ∘ H) ≤ 2γ(P_n), we only need to prove that γ₂(P_n ∘ H) ≥ 2γ(P_n). We proceed by induction on n. By Propositions 3.3 and 3.4 we obtain that γ₂(P_n ∘ H) = 2γ(P_n) for n = 3, 4. By Lemma 4.3 it is easy to see that γ₂(P_5 ∘ H) = 2γ(P_5). This establishes the base case. Now, we assume that n ≥ 6 and that γ₂(P_k ∘ H) ≥ 2γ(P_k) for k < n. Let S be a γ₂(P_n ∘ H)-set that satisfies Lemma 4.9. Let D = V(P_n ∘ H) \ (∪_{i=0}^{k−1} V(H_{u_{i−1}})). Notice that S ∩ D is a double dominating set of (P_n ∘ H) − D ≅ P_{n−4} ∘ H. Hence, by applying the induction hypothesis,
\[ γ_Î©(P_n ∘ H) ≥ γ_Î©(P_{n−4} ∘ H) + 4 ≥ 2γ(P_{n−4}) + 4 ≥ 2γ(P_n), \]
as desired. To conclude the proof we apply Remark 4.8.
Proposition 4.11. Let \( n \geq 3 \) be an integer and let \( H \) be a graph. If \( \gamma(H) \geq 3 \), then
\[
\gamma_{\times 2}(C_n \circ H) = n.
\]

Proof. From Theorem 2.11 we know that \( \gamma_{\times 2}(C_n \circ H) \leq n \). We only need to prove that \( \gamma_{\times 2}(C_n \circ H) \geq n \). Let \( S \) be a \( \gamma_{\times 2}(G \circ H) \)-set that satisfies Lemma 2.6. Since \( \gamma(H) \geq 3 \), by Lemma 4.3 (iii) we deduce that
\[
2\gamma_{\times 2}(C_n \circ H) = 2|S| = \sum_{x \in V(C_n)} \sum_{u \in N(x)} |S \cap V(H_u)| \geq 2n.
\]
Therefore, the result follows. \( \square \)

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