New Nonrelativistic Quantum Theory of Cold Dark Matter

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Abstract. Cold dark matter (DM) is conceived as a gas of massive particles that undergo collisions, interact gravitationally, and exchange quanta of energy. A new nonrelativistic quantum theory is presented for this model of DM, based on a recently discovered equation for a spinless, no charge, and free particle. This theory describes the quantum processes undergoing by the particles, specifies the required characteristic wavelength of the quanta of energy, gives constraints on the mass of DM particles, and predicts a detectable gravitational wave background associated with DM halos.

1. Introduction

The existence of dark matter (DM) is inferred from gravitational effects. While the Universe contains 4.9% of ordinary matter (OM), data from the Planck 2018 mission [1] suggests that 26.8% of DM constitutes the total mass-energy density.

Although different theories of DM have been proposed, the physical nature of the particles that make up DM remains unknown [2-8]. The most commonly accepted theory proposes a weakly interacting massive particle (WIMP). In the past several years, the search for WIMPs has intensified but thus far instruments have failed to detect them [8-10].

In this paper, a new nonrelativistic quantum theory of DM is presented, based on an equation describing a spinless, without charge, and free particle [11] that may be considered a candidate for DM. The existence of this equation is supported by the irreducible representations (irreps) of the extended Galilean group of the metric [12,13]. The irreps of the group also allow deriving the Schrödinger equation of quantum mechanics (QM) [14]. It was demonstrated that there is a special type of symmetry between these two equations and, since the Schrödinger equation describes the quantum structure of OM, the new equation may be used to represent DM [11].

The preliminary quantum model of DM reported in [11] is significantly extended here by taking into account collisions between massive DM particles and their gravitational interaction, as well as exchange of quanta of energy. The developed theory describes the quantum emission and absorption of the quanta of energy and...
the resulting equilibrium, and it also specifies the required characteristic wavelength of
the quanta of energy, gives constraints on the mass of dark matter particles, and predicts
a gravitational wave background for DM halos.

This paper is organized as follows: the basic equations are given in Section 2; a
model of dark matter is described in Section 3; a quantum theory of dark matter is
presented in Section 4; physical implications of the theory are discussed in Section 5;
and Conclusions are given in Section 6.

2. Basic equations of nonrelativistic quantum physics

In Galilean relativity, space and time are represented by different metrics that remain
invariant with respect to all transformations that form the Galilean group of the metric
\( G = [T(1) \otimes O(3)] \otimes [T(3) \otimes B(3)] \), where \( T(1), O(3), T(3), \) and \( B(3) \) are subgroups
of translations in time, rotations, translations in space, and boosts, respectively [12].
However, the Schrödinger equation is invariant with respect to the extended Galilean
group [15-17], whose structure is \( G_e = [O(3) \otimes_s B(3)] \otimes_s [T(3+1) \otimes U(1)] \), where \( T(3+1) \)
is an invariant Abelian subgroup of combined translations in space and time, and \( U(1) \)
is a one-parameter unitary subgroup [13].

Classification of the irreps of \( G_e \) by Bargmann [18] demonstrated that only the
scalar and spinor irreps are physical, but vectors and tensors are not because they
do not allow for elementary particle localizations. According to Wigner [19], a wave
function must transform like one of the irreps of the group because, only in this case,
all inertial observers identify the same physical object and agree on the description of
its physical properties [20].

It was shown [14] that the Wigner condition in the Galilean space and time can be
expressed mathematically by two eigenvalue equations [14,17], which were used [11] to
derive the following equations
\[
\left[ i \frac{\partial}{\partial t} + C_s \nabla^2 \right] \phi(t, \mathbf{x}) = 0 ,
\]
and
\[
\left[ \frac{\partial^2}{\partial t^2} - i C_w k \cdot \nabla \right] \phi(t, \mathbf{x}) = 0 .
\]
where \( \mathbf{x} = (x, y, z) \), with \( x, y \) and \( z \) being the Cartesian coordinates, \( C_s = \omega/k^2 \), and
\( C_w = \omega^2/k^2 \), with \( k^{2n} = (k \cdot k)^n \). In addition, \( \omega \) and \( k \) are labels of the irreps of \( G_e \),
which means that they can be any real numbers. As a result, there is an infinite number
of equations given by Eqs (1) and (2).

The origin of both equations is the same, namely, they are derived from the
eigenvalue equations that guarantee that the wavefunction \( \phi(t, \mathbf{x}) \) transforms as one
of the irreps of \( G_e \). The equations reflect properties of the Galilean spatial and temporal
metrics, and they are the only second-order asymmetric differential equations allowed to
be constructed in the Galilean space and time. The equations complement each other
and they form a set of twin-equations, whose physical applications are significantly different as demonstrated in this paper.

Since the metrics for space and time in Galilean relativity are separated, it is required that the derived equations are asymmetric with respect to the space and time derivatives, which is the necessary condition to make the equations Galilean invariant. Because of its form, Eq. (1) is called a Schrödinger-like equation, while Eq. (2) is referred to as a new asymmetric equation [11].

Among an infinite number of Schrödinger-like equations, the Schrödinger equation of QM can be obtained by specifying the constant $C_s$, which can be done when the de Broglie relationship is used [11]. The result is

$$i \frac{\partial}{\partial t} + \frac{\hbar}{2m} \nabla^2 \phi(t, \mathbf{x}) = 0,$$

which is the Schrödinger equation for a free particle. By adding different potentials, the equation can be used to describe quantum states of OM in different physical settings [15].

There are also infinitely many new asymmetric equations. It is easy to verify that the de Broglie relationship cannot be used to evaluate the constant $C_w$. This means that $C_w$ does not depend on $\hbar$ but instead it requires a new constant of Nature, denoted as $\varepsilon_o$, which represents a quanta of energy. The resulting new asymmetric equation can be written as

$$\frac{\partial^2}{\partial t^2} - i \frac{\varepsilon_o}{2m} \mathbf{k} \cdot \nabla \phi(t, \mathbf{x}) = 0,$$

and it describes a free, spinless particle without charge. There are differences between this equation and the Schrödinger equation as Eq. (4) has the second-order derivative in time instead of the first-order, and the first-order derivative in space instead of the second-order; thus, there is a special kind of space and time symmetry between these two equations. In addition, the Schrödinger equation depends on the Planck constant that makes any quantized energy levels to depend on frequency, but the presence of quanta of energy $\varepsilon_o$ in the new asymmetric equation makes any quantized energy levels to be independent from frequency.

Among the main differences between the Schrödinger and the new asymmetric equations, the most important is that the latter cannot be made dependent on the Planck constant but instead it allows only for introducing the new constant $\varepsilon_o$. This prevents the new asymmetric equation from being applicable to any quantum description of OM. Because of this limitation, it was already suggested that the new asymmetric equation may describe correctly the quantum structure of DM [11], with $\varepsilon_o$ being the quanta of energy of DM. Since the new asymmetric equation does not depend on any potential term, it may describe only a free particle of DM. In the following, the equation is modified to account for the gravitational potential of DM, so the resulting new asymmetric equation can describe gravitationally interacting DM.
3. Dark matter model and governing equations

Let $S$ be a sphere of DM particles, or DM halo, and $m$ denotes the mass of each particle, which is charge and spin free. For the particles uniformly distributed inside the halo, the gravitational potential $V_h(R)$ is given by

$$V_h(R) = \frac{GM_h}{2R_h^3} \left(R^2 - 3R_h^2\right),$$

(5)

where $M_h$ and $R_h$ are the mass and radius of the halo, and $R$ is the spherical coordinate [21]. The acceleration of DM particles inside the halo resulting from this potential is

$$g_h(R) = \frac{dV_h(R)}{dR} = \frac{GM_h}{R_h^3} R,$$

(6)

which shows that $g_h(R)$ is a linear function of $R$ [21]. As a result, the derivative of $g_h(R)$ with respect to $R$ gives

$$\Omega_h^2 = \frac{dg_h(R)}{dR} = \frac{GM_h}{R_h^3} = \text{const},$$

(7)

which shows that $\Omega_h^2$ remains the same at every point inside the halo.

A stable DM halo requires hydrostatic equilibrium, which means that

$$\frac{dp_h(R)}{dR} = -\rho_h(R) g_h(R),$$

(8)

where $p_h(R)$ and $\rho_h(R)$ the pressure and density of DM inside the halo. By writing the above equation in the form

$$g_h(R) = -\frac{1}{\rho_h} \frac{dp_h(R)}{dR},$$

(9)

the acceleration $g(R)$ becomes a pressure-gradient force per unit mass [22].

The particles of the spherical DM halo are allowed to collide, interact gravitationally, and also exchange energy by quantum processes. To account for gravitational interaction between a pair of DM particles, a local spherical coordinate system $(r, \theta, \phi)$ centered at one of these particles is considered, with $r$ representing the distance between the particles. In this model, the only spatial variable considered is $r$, which means that there are neither changes with respect to $\theta$ nor $\phi$.

The gravitational potential $V(r)$ is the work that is needed to bring the DM particle from infinity to its location, and is given as $V(r) = -Gm/r$. By taking the negative gradient of $V(r)$, it yields the acceleration per unit mass of the particle, $a(r) = Gm/r^2$ (see Eq. (6)). Taking the derivative with respect to $r$ one more time, the result is

$$\Omega_g^2(r) = \frac{da(r)}{dr} = -\frac{2Gm}{r^3}.$$

(10)

which can be included into Eq. (4) to account for gravitational interaction between a pair of DM particles. Then, Eq. (4) becomes

$$\left[\partial_t^2 - \frac{\varepsilon_o}{2m}(\mathbf{k} \cdot \hat{r}) \frac{\partial}{\partial r} - \Omega_g^2(r)\right] \phi(t, r) = 0,$$

(11)
with \( r = r \hat{r} \). Since \( \Omega_h \) is constant anywhere inside the DM halo (see Eq. [7]), the above new asymmetric equation with the \( \Omega^2_g(r) \) term remains valid for any pair of particles located at any point inside the DM halo. This is an important result as it shows that the developed model of gravitationally interacting pair of DM particles is valid for the entire DM halo. The model also allows for the particles of the pair to change their kinetic energy by colliding with other particles in the DM halo.

According to [1-8,23-26], DM is not a source of any form of electromagnetic, weak or strong interactions, but it is known to interact gravitationally with OM. However, in the model of DM considered in this paper, OM is not included, but only gravitationally interacting DM particles are taken into account.

4. Quantum theory of dark matter

The quantum theory of DM is formulated by considering a pair of DM particles, whose evolution in time and space is described by Eq. (11). By separation of variables, \( \phi(t,r) = \chi(t)\eta(r) \), the following equation is obtained

\[
\frac{1}{\chi} \frac{d^2\chi}{dt^2} = i \frac{\varepsilon_o}{2m} (k \cdot \hat{r}) \frac{1}{\eta} \frac{d\eta}{dr} + \frac{2Gm}{r^3} = -\mu ,
\]

where \( \mu \) is a separation constant to be determined. For a time-independent model of DM, the equation to be solved is

\[
\frac{d\eta}{\eta} = -4i\frac{Gm^2}{\varepsilon_o} \frac{dr}{(k \cdot \hat{r})r^3} - 2i\frac{m\mu}{\varepsilon_o} \frac{dr}{(k \cdot \hat{r})} ,
\]

and the resulting solution is

\[
\eta(r) = \eta_o \exp \left( \frac{2i(\varepsilon_g - 2\varepsilon_k)}{\varepsilon_o (k \cdot \hat{r})} \right) ,
\]

where \( \eta_o \) is an integration constant, \( \varepsilon_g = Gm^2/r \) is the gravitational potential energy of DM particles with mass \( m \), and \( \varepsilon_k = mv^2/2 \) is the kinetic energy of the particles with their thermal velocity \( v(r) \). The particle’s thermal velocity is determined by the temperature of DM and rate of collisions between the particles in the DM halo.

Let \( \Delta\varepsilon = \varepsilon_g - 2\varepsilon_k \), and \( \eta_r(r) \) be the real part of the solution of \( \eta(r) \) given by

\[
\eta_r(r) = \eta_o \cos \left( \frac{2\Delta\varepsilon}{\varepsilon_o (k \cdot \hat{r})} \right) .
\]

The following quantum processes are allowed: 

(i) \( \Delta\varepsilon = 0 \) requires \( \varepsilon_g = 2\varepsilon_k \), which means that

\[
\cos \left( \frac{2\Delta\varepsilon}{\varepsilon_o (k \cdot \hat{r})} \right) = 1 ,
\]

and that a pair of DM particles reaches its dynamical equilibrium, which is established between the gravitational potential and kinetic energies of the particles.

In the equilibrium, the velocities of both particles of the pair are the same and given by \( v_e = \sqrt{Gm/r_e} \), where the subscript ‘e’ stands for equilibrium. Thus, the time
$t_e$ required by a particle with its thermal velocity $v_e$ to travel the distance $r_e$ is $t_e = r_e/v_e$. This allows finding the separation constant to be $\mu = 1/t_e^2$ or $\mu = Gm/r_e^3$, whose value is fixed. Particles whose $v(r) \neq v_e$ travel in time $t_e$ the distance $r = v(r)t_e$.

(ii) $\Delta \varepsilon > 0$ corresponds to emission of the quanta $\varepsilon_o$ by the pair $(2\Delta \varepsilon)$. However, the condition for the emission process for one DM particle of the pair is

$$\frac{\Delta \varepsilon}{\varepsilon_o(k \cdot r)} = n\pi,$$

with $n = 1, 2, 3, \ldots$, as it guarantees that after this process takes place the pair reaches its dynamical equilibrium.

(iii) $\Delta \varepsilon < 0$ represents absorption of the quanta $\varepsilon_o$ by the pair, and the condition for this process to occur for one DM particle of the pair is

$$\frac{\Delta \varepsilon}{\varepsilon_o(k \cdot r)} = -n\pi,$$

because after this process the pair returns to its dynamical equilibrium.

In general, the quantization rules for one DM particle of the pair can be written as

$$\Delta \varepsilon = \pm n\pi\varepsilon_o(k \cdot r),$$

where $n = 0, 1, 2, 3, \ldots$, with $n = 0$ representing the dynamical equilibrium of the pair.

A new result, when compared to QM, is the presence of the term $(k \cdot r)$, which shows that the emission and absorption processes depend on the direction between the vectors $k$ and $r$. The latter connects the particles of the pair but the former is the label of the irreps of $G_e$ and it represents the inverse of the characteristic wavelength $\lambda_o$ associated with the quanta $\varepsilon_o$. The wavelength is given by

$$\lambda_o = \frac{Gm^2}{\varepsilon_o},$$

with $k = (1/\lambda_o)\hat{k}$, and plays the same role as the Compton wavelength $\lambda_c = h/mc$ in QM. However, while $\lambda_c$ depends on the Planck constant $h$ and the speed of light $c$, the characteristic wavelength for DM particles depends on the gravitational constant $G$ and on the new constant $\varepsilon_o$, implying that $\varepsilon_o$ may become important in quantization of gravity, instead of the Planck constant.

The presented theory of quantum DM is nonrelativistic and based on a scalar wavefunction, as required by the irreps of $G_e$; neither vector nor tensor wavefunctions can be used because they do not allow for localization of elementary particles [18]. Another nonrelativistic quantum theory based on the Schrödinger equation was proposed [27], and the theory postulated the existence of extremely light bosonic particles; however, more detailed studies [28,29] showed that the theory failed as it required a DM particle of different mass for different types of galaxies.

Other theories of DM based on scalar [30,31] or other wavefunctions [32] are relativistic, developed within the framework of quantum field theory and typically they do not take gravitational interaction of DM into account. Attempts to modify Einstein’s
General Relativity (GR) have been also made [33,34], but none of them is widely accepted because of the well-known predictive power of GR and its solid observational verification. Thus, the main difference between the previous theories of DM and the one developed in this paper is that the latter is nonrelativistic and based on the new asymmetric equation that accounts for gravitationally interacting DM.

5. Physical implications

In the developed quantum theory of cold DM, massive particles are confined to a sphere (DM halo), and they form pairs bounded gravitationally. The particles within the pairs may collide with other DM particles, changing the kinetic energy of the colliding particles. A pair reaches its equilibrium when the gravitational potential energy equals the particle’s kinetic energy. If the gravitational energy exceeds the kinetic energy, a pair emits the quanta of energy $\varepsilon_o$; in the opposite case, a pair absorbs the same quanta. This shows that $\varepsilon_o$ plays the same role for DM as the Planck constant does for OM.

However, the main difference between $\hbar$ and $\varepsilon_o$ is that while the former is present in all quantum theories of OM that deal with electromagnetic, weak and strong interactions, the latter appears in the quantum theory of DM that deals with gravitational interaction, which is treated here classically. From a physical point of view, the situation resembles what is known in QM, where the electron described by the Schrödinger equation with the Coulomb potential requires photons to change orbits in the atom; however, nonrelativistic QM does not deal with photons, which are introduced by relativistic QM or quantum electrodynamics.

Similarly, in the presented quantum theory of DM, the emission and absorption processes take place when the quanta $\varepsilon_o$ are exchanged. Since the gravitational interaction is the only one considered in the theory, it is suggested that the quanta $\varepsilon_o$ are called here dark gravitons, to emphasize that they are associated with DM.

Dark gravitons may be abundant in DM halos. Such a large number of $\varepsilon_o$ in the sea of dark gravitos may be responsible for generating a gravitational wave background that would be specific for DM halos and, therefore, observable by the gravitational wave detectors as clearly distinct from the other proposed forms of stochastic gravitational wave background [26].

The concept of massive gravitons has gained interest among researchers (e.g., [35-37]) because it can be used to account for DM, and depending on the graviton’s mass, it may also be used to explain dark energy [36]. Having massive gravitons implies that gravitational waves do not propagate with the speed of light but their characteristic speed is lower. The LIGO observations set up the limit on possible mass of gravitons as being smaller than $\approx 10^{-23} \text{ eV/c}^2$ [38]. Dark gravitons introduced in this paper are distinct from either gravitons or masssive gravitons as the quantization procedure to obtain dark gravitons is significantly different than the quantum field procedure that gives gravitons or massive gravitons.

Finally, it is interesting to consider the Compton wavelength $\lambda_c$ to be of the same
order as the characteristic wavelength $\lambda_o$ of the quanta of $\varepsilon_o$ given by Eq. (20). More specifically, let $\lambda_c = \lambda_o$, which gives

$$m = \sqrt[3]{\frac{\hbar \varepsilon_o}{Gc}}. \quad (21)$$

This shows that the mass of DM particles is fully determined by all four constants of Nature, with $\varepsilon_o$ being currently unknown. However, if Eq. (21) is valid, then $\varepsilon_o$ would be known when the mass of DM particles could be experimentally established. The four constants give the following characteristic length $l_c = \sqrt[3]{G\hbar^2/(\varepsilon_o c^2)}$.

6. Conclusions

A spherical DM halo filled with massive DM particles is considered. The particles are allowed to form gravitationally bounded pairs, whose dynamical stability can be impaired by particle collisions. By using a recently discovered asymmetric equation for spinless, no charge, and free particles [11], a quantum theory of cold DM is developed, showing that the dynamically unstable pairs can regain their stability by emitting or absorbing the quanta of energy $\varepsilon_o$. The quantum rules for these processes are presented and the characteristic wavelength corresponding to $\varepsilon_o$ is obtained. Since the rules are independent from frequency, $\varepsilon_o$ is both a new constant of Nature as well as the quanta called dark gravitons associated exclusively with DM. The abundance of $\varepsilon_o$ in DM halos forms the dark gravitons sea, which generate a gravitational wave background (GWB) that is specific for these halos and, therefore, potentially observable by the gravitational wave detectors, as clearly distinct from other proposed forms of stochastic GWB [26].

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References
[1] N. Aghanim, et al., Astron. Astrophys. 641 (2020) A6 (67 pages)
[2] M.J. Rees, Dark Matter - Introduction, Astro-Physics 361 (2003) 2427
[3] K. Freeman, and G. McNamara, In Search of Dark Matter, Springer, Praxis, Chichester, 2006
[4] L. Papantonopoulos, L. (Editor), The Invisible Universe: Dark Matter and Dark Energy, Lecture Notes in Physics 720, Springer, Berlin – Heidelberg, 2007
[5] J.A. Frieman, M.B. Turner, and D. Huterer, Ann. Rev. Astr. Astrophys. 46 (2008) 385
[6] K. Sugita, Y. Okamoto, M. Sekine, Int. J. Theor. Phys. 47 (2008) 2875
[7] R.H. Sanders, The Dark Matter Problem: A Historical Perspective, Cambridge Uni. Press, Cambridge, 2010
[8] T.M. Undagoita, and L. Rauch, [arXiv:1509.08767v1 [physics.ins-det] 26 Sep 2015]
[9] S. Giagu, Front. Phys. 7 (2019) 75
[10] Y.J. Ko, and H. K. Park, arXiv:2105.11109v3 [hep-ph] 4 June 2021
[11] Z.E. Musielak, Int. J. Mod. Phys. A, 28 (2021) 2150042 (12pp)
[12] J.-M. Levy-Leblond, Comm. Math. Phys. 6 (1967) 286
[13] J.-M. Levy-Leblond, J. Math. Phys. 12 (1969) 64
[14] Z.E. Musielak and J.L. Fry, Ann. Phys. 324 (2009) 296
[15] E. Merzbacher, Quantum Mechanics, Wiley & Sons, Inc., New York, 1998
[16] A.B. van Oosten, Apeiron, 13 (2006) 449
[17] Z.E. Musielak and J.L. Fry, Int. J. Theor. Phys. 48 (2009) 1194
[18] V. Bargmann, Ann. Math. 59 (1954) 1
[19] E.P. Wigner, Ann. Math. 40, 149 (1939)
[20] Y.S. Kim and M.E. Noz, Theory and Applications of the Poincaré Group, Reidel, Dordrecht, 1986
[21] W. Lowrie, A Student’s Guide to Geophysics Equations, Cambridge Uni. Press, Cambridge, 2011
[22] J.A. Knauss, and N. Garfield, Introduction to Physical Oceanography, Waveland Press, Long Grove, 2017
[23] S.W. Randall, M. Markevitch, D. Clowe, A.H. Gonzalez, and M. Bradac, Astrophys. J., 679 (2008) 1173
[24] F. Kahlhoefer, K. Schmidt-Hoberg, M.T. Frandsen, and S. Sarkar, MNRAS 437 (2014) 2865
[25] M. Ajello, et al., Astrophys. J. 833 (2019) 33 (12pp)
[26] J.D. Romano, and N.J. Cornish, Living Rev Relativ. 20 (2017) (1): 2
[27] Sin, S.-J. 1994, Phys. Rev. D 50, 3650
[28] Spivey, S.C., Musielak, Z.E. and Fry, J.L., 2013, MNRAS 428, 712
[29] Spivey, S.C., Musielak, Z.E. and Fry, J.L., 2015, MNRAS 448, 1574
[30] Böhmer, C.G., and Harko, T., 2007, J. Cosm. Astropart. Phys. 06, 025
[31] G. Arcadi, O. Lebedev, S. Pokorski and T. Toma, J. High Energy Phys. 8 (2019) 050
[32] P. Adshead, and K.D. Lozanov, Phys. Rev. D, 103 (2021) 103501
[33] R. Kh. Karimov, R.N. Izmailov and K.K. Nandi, J. Mod. Phys. D 30 (2021) 024001
[34] G. Nash, Gen. Relativ. Gravit. 51 (2019) 53
[35] R.A. El-Nabulsi, Int. J. Theor. Phys. 51 (2012) 1230015
[36] K. Aoki and S. Mukohyama, Phys. Rev. D 94 (2016) 024001
[37] H. Cai, G. Cacciapaglia and S.J. Lee, Phys. Rev. Lett. 128 (2022) 081806
[38] B.P. Abbott et al., Phys. Rev. Lett. 118 (2017) 221101