Particle-Hole Symmetry and the $\nu = \frac{5}{2}$ Quantum Hall State

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We discuss the implications of approximate particle-hole symmetry in a half-filled Landau level in which a paired quantum Hall state forms. We note that the Pfaffian state is not particle-hole symmetric. Therefore, in the limit of vanishing Landau level mixing, in which particle-hole transformation is an exact symmetry, the Pfaffian spontaneously breaks this symmetry. There is a particle-hole conjugate state, which we call the anti-Pfaffian, which is degenerate with the Pfaffian in this limit. We observe that strong Landau level mixing should favor the Pfaffian, but it is an open problem which state is favored at filling fraction $\nu = 5/2$.

In this paper, we add a new wrinkle to this discussion. We note that the particle-hole conjugate of the Pfaffian is a new state which will be exactly degenerate in energy with the Pfaffian in the limit of vanishing Landau-level mixing. Landau level mixing is a symmetry-breaking perturbation which lifts the degeneracy between the two states. It is an open problem which state is favored at filling fraction $\nu = 5/2$.

When Landau level mixing is neglected, the Hamiltonian for electrons at filling fraction $\nu = N + \frac{1}{2}$ can be related to one for electrons at $\nu = N + 1 - \frac{1}{2}$ by an anti-unitary particle-hole transformation, $c_m^\dagger \rightarrow c_m$, $c_m \rightarrow c_m^\dagger$, with $m$ labeling orbitals within the Landau level. The Hamiltonian transforms according to $H_2 \rightarrow \tilde{H}_2 + \text{const.}$, where

$$H_2 = \sum_{klmn} V_{klmn} c_k^\dagger c_m^\dagger c_n c_l - \mu \sum_m c_m^\dagger c_m,$$

$$\tilde{H}_2 = \sum_{klmn} V_{klmn} c_k^\dagger c_m^\dagger c_n c_l + (\mu - 2\mu_{1/2}) \sum_m c_m^\dagger c_m. \tag{1}$$

Here, $V_{klmn}$ are the matrix elements of the Coulomb interaction between the corresponding orbital states, and $\mu_{1/2} = \sum_n V_{nnmm}$ (the sum is independent of $m$ in a translationally-invariant system). For the special case of $\frac{5}{2} = \frac{1}{2}$, for which $\mu = \mu_{1/2}$, this is a symmetry of the system.

It is widely believed, on the basis of numerical evidence, that the experimentally-observed plateau at $\nu = \frac{5}{2}$ is in the universality class of the Moore-Read Pfaffian state, by which it is meant that the lowest Landau level (of both spins) is filled, and the electrons in the first excited Landau level are fully spin-polarized and have a wavefunction which is in the same universality class as the one given by acting with Landau level raising operators on:

$$\Psi_{\text{PF}}(z_i) = \text{Pf} \left( \frac{1}{z_i - z_j} \right) \prod_{i < j} (z_i - z_j)^2 e^{-\sum \eta_i^2/4\ell_0^2}. \tag{2}$$

However, this state is not invariant under a particle-hole transformation. Let $\sigma_{xy}$ and $\kappa_{xy}$ denote the contributions to the electrical and thermal Hall conductivities (in units of $e^2/h$ and $\pi^2 k_B^2 T/3h$, respectively) of the fractional state in the lowest unfilled Landau level, in which the Pfaffian is assumed to form (which, in the case of $\nu = 5/2$, is the first excited Landau level). Under a particle-hole transformation of this Landau-level, $\sigma_{xy} \rightarrow 1 - \sigma_{xy}$ and $\kappa_{xy} \rightarrow 1 - \kappa_{xy}$. The Pfaffian state has $\kappa_{xy} = \frac{3}{2}$, as may be seen most easily from its edge theory, which has two modes, a chiral boson $\phi$ and a chiral Majorana fermion $\psi$, propagating in the same direction:

$$\mathcal{L}_\psi(\psi, \phi) = \psi(-i\partial_t + i\nu_x \partial_x)\psi + \frac{2}{4\pi} \partial_x \phi(i\partial_t + \nu_x \partial_x)\phi. \tag{3}$$

Here, $e^{i\phi}$ creates a charge $e/2$ semion with scaling dimension $1/4$, and $\nu_x, \nu_n$ denote the two edge velocities. A chiral boson contributes $\kappa_{xy} = 1$, while a chiral Majorana fermion carries $\kappa_{xy} = \frac{1}{2}$.

The particle-hole conjugate of the Pfaffian, which we will call the anti-Pfaffian ($\overline{\text{PF}}$), with wavefunction of the form:

$$\Psi_{\text{PF}}(z_i) = \int \prod_{\alpha} d^2 \eta_\alpha \prod_{i < j} (z_i - z_j) e^{-\sum \eta_i^2/4\ell_0^2} \times$$

$$\prod_{\alpha} (z_i - \eta_\alpha) \prod_{\beta < \gamma} (\eta_\beta - \eta_\gamma) e^{-\sum \eta_\alpha^2/4\ell_0^2} \Psi_{\text{PF}}(\overline{\alpha}) \tag{4}$$
must have $\kappa_{xy} = -\frac{1}{2}$, $\sigma_{xy} = \frac{1}{2}$. Therefore, the anti-Pfaffian has counter-propagating edge modes, which will have direct experimental significance, as we discuss later.

Landau level mixing breaks particle-hole symmetry. If we treat it perturbatively (although it is not particularly small in experiments) then, in addition to renormalizing the Coulomb repulsion, which does not break particle-hole symmetry, it also generates three-body interactions, which do break the symmetry. In fact, the Pfaffian (2) is the exact ground state of the simplest repulsive non-vanishing three-body interaction. The anti-Pfaffian (4) is, therefore, the exact ground state of the particle-hole conjugate of this interaction, which is an attractive three-body interaction together with a repulsive two-body interaction with the same coefficient. When two-body Coulomb interactions and weaker three-body interactions are present, it is unclear which phase occurs. (Although we expect that the specific wavefunction (2) is lower in energy than (4) if the three-body interaction is repulsive, this does not tell us which phase the actual ground state of the Hamiltonian is in).

Fermion Chern-Simons theory, at the mean-field level, which includes non-perturbative mixing between all Landau levels, implies that the Pfaffian is the ground state [4]. However, it is an open question which state is favored by the moderate Landau level mixing which actually occurs in experiments, where the strength of Landau level mixing, $(e^2/\epsilon\ell_0)/\hbar\omega_c \sim 1$. On the sphere, which does not have translational symmetry, a particle-hole transformation is not a symmetry of (1); the Pfaffian and anti-Pfaffian states of $N$ particles occur at magnetic fluxes of $N_\Phi = 2N - 3$ and $N_\Phi = 2N + 1$, respectively. This may explain why the anti-Pfaffian has not been identified in numerical studies of finite systems on the sphere [3]. A particle-hole transformation is a symmetry of (1) on the torus. However, in a finite system, we expect mixing between the Pfaffian and the anti-Pfaffian so that the symmetric combination is the ground state, consistent with numerics (3).

Edge Excitations of the anti-Pfaffian

Under a particle-hole transformation, the edge between the anti-Pfaffian and the vacuum ($\nu = 0$) is mapped onto the edge between the Pfaffian and a $\nu = 1$ Hall liquid. (At $\nu = 5/2$, there are also the edges of the lowest Landau level of both spins, but they play no role in our discussion, so we drop them for simplicity.) Therefore, we can deduce the former from the properties of the latter, in which the edge of the Pfaffian (3) is coupled to a counter-propagating free chiral Dirac fermion (or its bosonized equivalent):

$$\mathcal{L} = \frac{1}{4\pi} \partial_x \phi_1 ( -i \partial_t + v_1 \partial_x ) \phi_1 + \mathcal{L}_\text{pert}(\psi_1, \phi_2)$$

$$+ \frac{1}{4\pi} 2 v_{12} \partial_x \phi_1 \partial_x \phi_2 + \xi(x) \psi_1 e^{i(\phi_1 - 2\phi_2)} + \text{h.c.} \quad (5)$$

We have included a density-density interaction between the $\nu = 1$ and $\nu = 1/2$ edge modes, and also an inter-mode electron tunneling term. With an assumed inter-mode momentum mismatch and in the presence of impurities, the electron tunneling amplitude $\xi(x)$ can be taken as a random (complex) function with zero mean and short-ranged correlations, $\xi^a(\mathbf{x})\xi^c(\mathbf{x}^c) = W \delta(\mathbf{x} - \mathbf{x}^c)$. For large $v_{12}$ the tunneling term is relevant, and can then be conveniently analyzed by introducing a charge/neutral decomposition, $\phi_0 = \phi_1 - \phi_2$ and $\phi_0 = \phi_1 + 2\phi_2$, and then fermionizing the neutral chiral boson: $e^{i\phi_0} \equiv \psi_2 + i\psi_3$. The Lagrangian then takes the form $\mathcal{L} = \mathcal{L}_\text{sym} + \mathcal{L}_\text{pert}$, where:

$$\mathcal{L}_\text{sym} = \frac{2}{4\pi} \partial_x \phi_1 ( -i \partial_t + v_1 \partial_x ) \phi_1 + \psi_1 ( -i \partial_t + iv_0 \partial_x ) \psi_1$$

$$+ \delta v_1 i \partial_x \psi_1 + i \psi_2 \psi_3 \partial_x \phi_0 \quad (6)$$

$$\mathcal{L}_\text{pert} = 2i \psi_1 (\xi_1 \psi_3 + \xi_2 \psi_2) + \delta v_1 i \partial_x \psi_1 + i \psi_2 \psi_3 \partial_x \phi_0 \quad (7)$$

with $\xi_1$, $\xi_2$ the real and imaginary parts of $\xi(x)$. The three Majorana fermions $\psi_a$, $a = 1, 2, 3$ form an SU(2)$_2$ triplet in the absence of the symmetry-breaking terms (7).

The first terms in (7) can be eliminated from the action by an SU(2) rotation, $\psi = O \tilde{\psi}$, with $O(x) = P \exp \left( -i \int_{-\infty}^{x} dx' (\xi_1(x')T_2 - \xi_2(x')T_3) / \nu_0 \right)$, where $P$ denotes a path ordering of the integral, and $\nu_0$, $a = 1, 2, 3$ are the SU(2) generators in the spin-1 representation. In the transformed variables, the second and third terms in (7) will have spatially dependent random coefficients. Integrating out $\xi_1$, $\xi_2$ (using the replica method, for instance), results in terms which have scaling dimension $(-1)$ and are, therefore, perturbatively irrelevant. Hence, we obtain (6) as the action for the edge between the Pfaffian state and a $\nu = 1$ Hall droplet. The edge theory between the anti-Pfaffian state and the vacuum is obtained by flipping the directions of all the modes in (6). There are three counter-propagating neutral Majorana fermion edge modes, which yields the expected value for the thermal Hall conductivity, $\kappa_{xy} = -1/2$.

In the anti-Pfaffian edge theory, the minimal dimension electron operator is $e^{i\phi_1} = (\psi_2 - iv_0) e^{i2\phi_2}$. There are 6 primary fields which are local with respect to this electron operator: 1, $e^{i\phi_1}$, $\psi_a$, $\sigma_a e^{i\phi_1}$, $\phi_1^{\pm} e^{i\phi_1/2}$, $\phi_1^{\pm} e^{i\phi_1/2}$. The spin-1/2 primary fields of SU(2)$_2$, denoted $\phi_1^{\pm}$, can be written in terms of the Ising spin and disorder fields $\sigma_a$ and $\mu_a$ of the three Majorana fermions: $\phi_1^{\pm} = \sigma_1 \sigma_2 \sigma_3 + i \mu_1 \mu_2 \sigma_3 + \mu_1 \mu_2 \sigma_3 + i \sigma_1 \sigma_2 \sigma_3 + i \sigma_1 \sigma_2 \sigma_3 - i \sigma_1 \mu_2 \sigma_3 + \mu_1 \sigma_2 \sigma_3$. The fields $\phi_1^{\pm}$ thus act to switch between periodic and anti-periodic boundary conditions on all three Majorana fermions. Note that $\phi_1^{\pm}$ is a dimension 3/16 operator, unlike $\sigma$, which has dimension 1/16. The difference in scaling dimension has consequences for quasiparticle tunneling as we discuss below.

Topological Properties of the anti-Pfaffian

The 6 primary fields of the conformal field theory for the anti-Pfaffian correspond to its 6-fold degeneracy on the torus. A 2d (bulk) effective field theory for the anti-Pfaffian state which encodes this degeneracy as well as the other bulk topological properties, can be deduced from consistency with the edge theory or, alternatively, in the following way. We begin by bosonizing the action for electrons at $\nu = 1/2$ employing a Chern-Simons
gauge field $c_\mu$:

$$L = \Phi^*(i\partial_0 + c_0 + A_0) \Phi + \frac{1}{2m}(i\partial_i + c_i + A_i) \Phi^2 + V(\Phi) + \frac{1}{4\pi} c_\mu c_\nu \partial_\mu c_\lambda,$$  

(8)

where $A_\mu$ is the electromagnetic field. At the mean-field level, this is now a system of bosons at $\nu = 1$. We next use boson-vortex duality\cite{17} to transform to a system of vortices, also at $\nu_{eff} = 1$, minimally coupled to a dual gauge field $b_\mu$. Integrating out $c_\mu$ induces a Chern-Simons term for the $b_\mu$:

$$L = \Phi^*(i\partial_0 + b_0) \Phi_v + \frac{1}{2m_v}(i\partial_i + b_i) \Phi_v^2 + U(\Phi_v) - \frac{1}{4\pi} b_\mu c_\nu \partial_\nu b_\lambda - \frac{1}{2\pi} b_\mu c_\nu \partial_\nu A_\mu.$$  

(9)

We now shift $b_\mu \rightarrow b_\mu - A_\mu$ and refermionize $\Phi_v$ since $b_\mu$ attatches one flux quantum to it:

$$L = \bar{\chi}(i\partial_0 - A_0 - \mu) \chi + \frac{1}{2m_0}(i\partial_i - A_i) \chi^2$$

$$+ \frac{1}{4\pi} A_\mu c_\nu \partial_\nu A_\lambda + \cdots$$  

(10)

The fermion field $\chi^\dagger$ creates holes which carry electrical charge opposite to that of the electron. Therefore, when we attach flux to these fermions, pair them, and condense the pairs\cite{4,15}, we will obtain a negative contribution to the Hall conductivity. Furthermore, if the pairing is due to gauge field fluctuations, as in\cite{4}, the pairs will have a $p - ip$ pairing symmetry. The resulting effective action can be written:

$$L = \bar{\eta} i \partial_0 \eta - m_\eta \eta - \frac{2}{4\pi} a_\mu c_\nu \partial_\nu a_\lambda - \frac{1}{4\pi} a_\mu c_\nu \partial_\nu a_\lambda - \frac{1}{2\pi} A_\mu c_\nu \partial_\nu a_\lambda + \frac{1}{2\pi} A_\mu c_\nu \partial_\nu a_\lambda.$$  

(11)

The negative sign for the mass of the 2d Majorana fermion $\eta$ is a consequence of the reversed pairing symmetry. The last term in\cite{10} has been rewritten by introducing an auxiliary gauge field $\bar{a}_\mu$.

Finally, we introduce $a_\mu^r = a_\mu - \bar{a}_\mu$ and $a_\mu^a = 2a_\mu - \bar{a}_\mu$ to obtain the desired $2 + 1d$ effective action for the anti-Pfaffian:

$$L = \bar{\eta} i \partial_0 \eta - m_\eta \eta - \frac{2}{4\pi} a_\mu^r c_\nu \partial_\nu a_\lambda^r + \frac{1}{4\pi} a_\mu^r c_\nu \partial_\nu a_\lambda^r$$

$$- \frac{1}{2\pi} A_\mu c_\nu \partial_\nu a_\mu^a.$$  

(12)

The action\cite{12} corresponds closely to the edge effective action\cite{6,16}. The most salient topological feature is the reversed chirality of the neutral fermion sector. As a result, the braid matrices are complex conjugated as compared to the Pfaffian state\cite{5,16}: $T_{ij} = e^{i\gamma_3} e^{i\pi \gamma_3}$.

**Toy Model** We next describe a simple lattice model of spinless fermions which has similar physics to the neutral sectors of the Pfaffian and anti-Pfaffian phases. The fermions hop only between near neighbor sites of a square lattice. At half-filling the model is invariant under the anti-unitary symmetry $c_i \rightarrow (-1)^i c_i^\dagger$, $c_{j-i} \rightarrow (-1)^j c_i^\dagger$. With interactions present we suppose that the fermions can develop the following order parameters: $\Delta_{ij}$, which is a $\Delta(p) = \sin px + i \sin py$ superconducting order parameter which spontaneously breaks $U(1)$; $\varphi$, which spontaneously breaks particle-hole symmetry by enabling next-nearest-neighbor hopping; and $\theta$, which spontaneously breaks $\pi/2$ rotational symmetry. The mean-field Hamiltonian with these order parameters is:

$$H = \sum_{(i,j)} \left(-tc_i c_j + \Delta_{ij} c_i^\dagger c_j^\dagger + h.c.\right) - \mu \sum_i c_i^\dagger c_i - \sum_{(i,j)} \varphi c_i^\dagger c_j + h.c. + \sum_i \theta \left(c_i^\dagger c_{i+a\hat{x}} - c_i^\dagger c_{i+a\hat{y}}\right) + h.c.$$  

When $\mu = \varphi = 0$, this Hamiltonian is particle-hole symmetric. When $\varphi = \theta = \mu = 0$, there are gapless excitations at two "nodes", with momenta $k = (\pi, 0), (0, \pi)$. Such gapless excitations would not be present for an off-lattice $p + ip$ superconductor. The low-energy excitations in the vicinity of each node is a (two-component) Majorana fermion, which can be combined into a single Dirac fermion $\psi$. Non-zero $\varphi$ and $\theta$ are, respectively, Dirac and Majorana mass terms for $\psi$. The former breaks particle-hole symmetry, but not the O(2) which rotates one Majorana fermion into the other (a $\pi/2$ lattice rotation is $\pi$ rotation within this O(2)); the latter does not break particle-hole symmetry but breaks O(2).

Let us suppose that $\varphi$ orders. Then $\mu$ must be adjusted to maintain half-filling. If $\varphi > 0$, we need $\mu > 0$, and the electron Fermi surface (when $\Delta = 0$) is closed and the hole Fermi surface is open. The system is adiabatically connect to electrons in the continuum, which essentially form a continuum $p + ip$ superconductor. In particular, it supports a gapless chiral Majorana fermion edge mode\cite{13} with $\kappa_{xy} = \frac{1}{2}$. The long-distance form of the pair wavefunction is $g(r) \sim 1/z$. Hence, we identify this phase with the neutral sector of the Pfaffian. For $\varphi < 0$ the masses of the two nodal Majorana fermions change sign, giving $\kappa_{xy} = -\frac{1}{2}$. In this case $\mu < 0$, and the holes form a closed Fermi surface rather than the electrons. The spin $p_x + i \sin p_y$ pairs of electrons are, in this phase, better interpreted as $\sin p_x - i \sin p_y$ pairs of holes. This is consistent with our effective field theory of the anti-Pfaffian\cite{12}, which has $p - ip$ pairs of composite fermions obtained by attaching flux to holes. Moreover, $g(r) \sim 1/(z - (1)^2/z - (1)^3/z)$. We identify this phase with the neutral sector of the anti-Pfaffian.

At the transition between these two phases, $\varphi = 0$, and there are gapless excitations described by a single massless Dirac fermion:

$$L = \bar{\psi} i \partial_0 \psi - g(\psi \psi)^2,$$  

(13)
a 2d analogue of the Gross-Neveu model. For \( g < g_c \), the \( \mathbb{Z}_2 \) symmetry \( \psi(t, x, y) \rightarrow \gamma^1 \psi(t, -x, y) \), \( \psi(t, x, y) \rightarrow \psi(t, -x, y) \gamma^1 \) is unbroken (\( \gamma^1 \) is a purely imaginary \( \gamma \) matrix), and the nodal fermions are gapless. However, for \( g > g_c \), particle-hole symmetry can spontaneously break \( \varphi \propto \langle \overline{\psi} \psi \rangle \neq 0 \), and by varying explicit symmetry-breaking terms, such as \( \mu \) or a second-neighbor hopping \( \nu' \), the system can be driven through a first-order transition between the two phases. At \( g = g_c \), the system is critical. Exponents are known in the large-\( N \) limit, where \( N \) is the number of flavors of fermions, e.g., \( \nu = 1 + 8/(3\pi^2 N) \)[13]. As the critical point is approached from within the symmetry-broken phase, the velocity of the gapless chiral Majorana fermion edge mode vanishes. The small value of the neutral Majorana fermion edge mode in numerical studies of the Pfaffian state may indicate proximity to such a critical point[19]. Note that with \( \theta \neq 0 \), there is also an intermediate phase with \( \kappa_{xy} = 0 \) which breaks rotational symmetry (perhaps slightly reminiscent of the ne-8
al predictions made by Levin[6], Laughlin quasiparticle also gives \( \kappa_{xy} = 3/2 \) and \( \kappa_{xy} = 1/2 \), respectively. Electrical transport measurements through a point contact will also differ in the two phases. As described in Ref.[21], weak tunneling of the charge \( e/4 \) non-Abelian quasiparticles between the edges of a Pfaffian Hall bar leads to \( R_{xx} \sim T^{-3/2} \). For the anti-Pfaffian one obtains \( R_{xx} \sim T^{-1} \), different due to the extra edge modes present. (See Ref.[22] for experiments in this direction.) Interestingly, weak inter-edge tunneling of the charge \( e/2 \) Laughlin quasiparticle also gives \( R_{xx} \sim T^{-1} \), in both the Pfaffian and anti-Pfaffian states. The existence of counter-propagating neutral modes in the anti-Pfaffian state might also be detectable, and would have implications for interferometry experiments[22][10][11][12] if it proves to be real-4
ized at \( \nu = 5/2 \).

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