Fourth-order superintegrable systems separating in polar coordinates. II. Standard potentials

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Abstract
Superintegrable Hamiltonian systems in a two-dimensional Euclidean space are considered. We present all real standard potentials that allow separation of variables in polar coordinates and admit an independent fourth-order integral of motion. The general form of the potentials satisfies a linear ODE. In the classical case, the standard potentials coincide with the Tremblay–Turbiner–Winternitz (TTW) or Post–Winternitz (PW) models. In the quantum case new superintegrable systems are obtained, in addition to the TTW and PW ones. Their classical limit is free motion.

Keywords: superintegrability, separation of variables, standard potentials, higher order integrals of motion

1. Introduction

The purpose of this paper is to obtain and classify all classical and quantum superintegrable Hamiltonians that allow separation of variables in polar coordinates and admit a fourth-order polynomial integral of motion \( Y \). We focus on standard potentials, namely those that satisfy (not trivially) a linear ODE. This linear ODE is nothing but the compatibility condition for the existence of the integral \( Y \). Superintegrable systems separating in polar coordinates, allowing fourth-order integrals of motion and involving exotic potentials were studied in an earlier article [12].
Roughly speaking, a Hamiltonian with $n$ degrees of freedom is called integrable if it allows $n$ independent well defined integrals of motion in involution. It is minimally superintegrable if it allows $n+1$ such integrals, and maximally superintegrable if it admits $2n-1$ integrals (where only subsets of $n$ integrals among them can be in involution).

The best known superintegrable systems are the harmonic oscillator with its $su(n+1)$ algebra of integrals, and the Kepler–Coulomb system with its $o(n+1)$ algebra (when restricted to fixed bound state energy values).

A more recent review article provides precise and detailed definitions, general settings and motivation for studying superintegrable systems \[41\]. A systematic search for superintegrable classical and quantum systems in $E_2$ and $E_3$ established an interesting connection between second-order superintegrability and multiseparability in the Schrödinger or Hamilton–Jacobi equation \[5, 6, 15, 16, 33, 35, 36\].

An extensive literature exists on second-order superintegrability in spaces of 2, 3 and $n$ dimensions, Riemannian and pseudo-Riemannian, real or complex \[26–30, 35\].

The systematic study of higher order integrability is more recent. Pioneering work is due to Drach \[10, 11\]. For more recent work see \[2, 7–9, 14, 21–24, 38–40, 45, 48–51, 53\].

The present article is a contribution to a series \[1, 12, 20, 21, 37, 40, 44, 47, 52\] devoted to superintegrable systems in $E_2$ with one integral of order $n \geq 3$ and one of order $n \leq 2$.

In this article we restrict ourselves to the space $E_2$. The Hamiltonian has the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y), \quad (1)$$

the mass $m = 1$, $V(x, y)$ is a scalar potential and in classical mechanics $p_x$ and $p_y$ are the momenta conjugate to the Cartesian coordinates $x$ and $y$, respectively. In quantum mechanics they are the corresponding operators $p_x = -i\hbar \frac{\partial}{\partial x}$, $p_y = -i\hbar \frac{\partial}{\partial y}$. In polar coordinates $(x, y) \equiv (r \cos \theta, r \sin \theta)$, the classical Hamiltonian reads

$$H = \frac{1}{2} \left( p_r^2 + \frac{p_\theta^2}{r^2} \right) + V(r, \theta), \quad (2)$$

where $p_r$ and $p_\theta$ are the associated canonical momenta. The corresponding quantum operator takes the form

$$H = -\frac{\hbar^2}{2} \left( \partial_r^2 + \frac{1}{r} \partial_r + \frac{1}{r^2} \partial_\theta^2 \right) + V(r, \theta). \quad (3)$$

In paper \[12\] we found all the exotic potentials, i.e. those which do not satisfy any linear differential equation. Here we solve the complementary problem, namely we find all the standard potentials which do satisfy a linear ODE. In all equations we keep the Planck constant $\hbar$ explicitly. Classical standard potentials will be obtained in the limit $\hbar \to 0$.

Before going into the question of standard superintegrable potentials with a fourth-order integral of motion, let us recall that two families of standard superintegrable potentials in $E_2$ are already known. Both allow separation of variables in polar coordinates and hence admit the second-order integral

$$X = p_\theta^2 + 2S(\theta), \quad p_\theta = L_\theta = -i\hbar \frac{\partial}{\partial \theta}. \quad (4)$$

They also admit a further polynomial integral of arbitrary order $N$. For even $N$, one of them is the Tremblay–Turbiner–Winternitz (TTW) potential.
where $k = mn$ and $m$ and $n$ are two integers (with no common divisors) \[50, 51\]. The other is the Post–Winternitz (PW) potential \[46\]

\[
V_{\text{PW}} = \frac{a}{r} + \frac{1}{r^2} \left[ \frac{\mu}{\cos^2 \left( \frac{k}{2} \theta \right)} + \frac{\nu}{\sin^2 \left( \frac{k}{2} \theta \right)} \right],
\]

where again $k = mn$ is rational.

In the TTW case (5) the lowest order of an additional integral $Z$ was shown to be \[17, 31, 32, 48\]

\[N = 2(m + n - 1),\]

both in classical and quantum mechanics.

The TTW and PW potentials are related by coupling constant metamorphosis \[4, 25, 34, 46\]. This transformation takes integrable and superintegrable systems into other systems that are also integrable or superintegrable, respectively. However, in general the order of the integrals of motion is not preserved, nor is their polynomial character. Special conditions \[34\] must be satisfied for polynomial integrals to be transformed into polynomials of the same order. It has been conjectured and verified for $k = 1$ and 2 that these conditions are satisfied for the TTW and PW systems \[46\].

Thus we may expect that in any study of superintegrable systems separating in polar coordinates in $E_2$ with an additional integral of order $N$ with $2 \leq N < \infty$ the TTW and PW systems will appear as ‘standard’ potentials.

A further comment is in order here and should be taken into account in any classification of superintegrable systems. If a certain potential $V(x, y)$ appears as superintegrable in $E_2$ with two integrals $X_1$ and $X_2$ that are polynomials of order $m$ and $n$, respectively, then the same potential will reappear at higher orders. Indeed $X_1X_2 + X_2X_1$ will be an integral of order $m + n$ for the same Hamiltonian. The commutator $[X_1, X_2]$ (or Poisson bracket $\{X_1, X_2\}_{\text{P. B.}}$) will be a polynomial integral of order $m + n - 1$. Such ‘trivially’ superintegrable systems should be weeded out of lists of new superintegrable systems.

For fourth order superintegrable potentials, in addition to the Hamiltonian $H$, we have two more conserved quantities. A second-order integral (4), associated with the separation of variables in polar coordinates for which the potential takes the form

\[
V(r, \theta) = R(r) + \frac{S(\theta)}{r^2},
\]

and a fourth-order one

\[
Y = A_1 \{L^1, p_x\} + A_2 \{L^2, p_x\} + A_3 \{L^3, p_x(p_x^2 + p_y^2)\} + A_4 \{L^4, p_x (p_y^2 + p_x^2)\} + B_1 (p_x^4 - p_y^4) + 2B_2 p_x p_y (p_x^2 + p_y^2) + B_3 \{L^2, p_x^2 - p_y^2\} + 2B_4 \{L_2^2, p_x^2\} + C_1 \{L^3, 3p_x^2 p_y - p_x^4\} + C_2 \{L^4, p_x^4 - 3p_y^2 p_x\} + D_1 (p_x^4 + p_y^4 - 6p_x^2 p_y^2) + 4D_2 p_x p_y (p_x^2 - p_y^2) + \{g_1(x, y), p_x^2\} + \{g_2(x, y), p_x p_y\} + \{g_3(x, y), p_y^2\} + g_4(x, y),
\]

where the $A_i, B_i, C_i, D_i$ are real constants. The bracket $\{\cdot, \cdot\}$ denotes an anticommutator and $R(r), S(\theta), g_{1,2,3,4}(x, y)$ are real functions such that

\[\left[H, Y\right] = \left[H, X\right] = 0.\]
Under rotations around the $z$-axis, each of the six pairs of parameters in (9)

$$(A_1, A_2), (A_3, A_4), (B_1, B_2), (B_3, B_4), (C_1, C_2), (D_1, D_2),$$
forms a doublet. In general, $Y$ may contain the $O(2)$ singlets but by linear combinations of the form $Y + u_1 X^2 + u_2 H^2 + u_3 X H$, where the $u_i$ are constants, we eliminated all such trivial terms. Under rotations through the angle $\theta$ the doublets $A_i, B_i, C_i, D_i$ rotate through $\theta, 2\theta, 3\theta, 4\theta$, respectively.

We have $[Y, X] = C \neq 0$ where $C$ is in general a 5th-order linear operator. We thus obtain a finitely generated polynomial algebra of integrals of motion. We are looking for fourth-order superintegrable systems, so at least one of the constants $A_i, B_i, C_i, D_i$ is different from zero. The operator $Y$ is the most general polynomial expression for a fourth-order Hermitian operator of the required form [12]. The commutator $[H, Y]$ contains derivatives of order up to three (see below).

The operator $Y$ in (9) is given in Cartesian coordinates for brevity. Putting

$$p_x = -i \hbar \left( \cos \theta \partial_r - \frac{\sin \theta}{r} \partial_\theta \right), \quad p_y = -i \hbar \left( \sin \theta \partial_r + \frac{\cos \theta}{r} \partial_\theta \right),$$
we obtain the corresponding expression in polar coordinates. Explicitly, the leading terms of the integral $Y$ are

$$Y = \hbar^4 \left( B_1 \cos 2\theta + B_2 \sin 2\theta + D_1 \cos 4\theta + D_2 \sin 4\theta \right) \partial_r^2 + \frac{1}{r} \left[ D_2 \sin 4\theta + D_1 \cos 4\theta - 2r (A_1 r^2 + A_3) \sin \theta - (B_2 + 2B_3 r^2) \sin 2\theta + 2r (A_3 r^2 + A_4) \cos \theta \right. \right.$$  

$$- \left. \left( B_2 + 2B_3 r^2 \right) \cos 2\theta - 2r \left( C_1 \cos 3\theta - C_2 \sin 3\theta \right) \right] \partial_\theta^2$$

$$- \frac{2}{r^2} \left( 3(D_1 \cos 4\theta + D_2 \sin 4\theta) - r^2 (B_1 \cos 2\theta + B_4 \sin 2\theta) \right)$$

$$+ r(A_3 \sin \theta - A_4 \cos \theta - 3(C_1 \cos 3\theta - C_2 \sin 3\theta)) \right] \partial_r^2 \partial_\theta^2$$

$$- \frac{2}{r} \left[ B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta) - r \left( C_1 \sin 3\theta + C_2 \cos 3\theta \right. \right.$$  

$$+ A_3 \cos \theta + A_4 \sin \theta \right) \left. \right] \partial_r \partial_\theta - \frac{2}{r^3} \left[ B_1 \sin 2\theta - B_2 \cos 2\theta \right.$$  

$$- 2(D_1 \sin 4\theta + D_2 \cos 4\theta) - r(A_3 \cos \theta + A_4 \sin \theta - 3(C_1 \sin 3\theta + C_2 \cos 3\theta \right)$$

$$+ 2r^2 (B_1 \sin 2\theta - B_2 \cos 2\theta) - r^3 \left( A_1 \cos \theta + A_3 \sin \theta \right) \right] \partial_r \partial_\theta^2 + \text{lower order terms}.$$

(11)

For convenience let us introduce the functions

$$G_1(r, \theta) = g_1 \cos^2 \theta + g_2 \sin^2 \theta + g_3 \cos \theta \sin \theta,$$

$$G_2(r, \theta) = \frac{g_1 \sin^2 \theta + g_2 \cos^2 \theta - g_3 \cos \theta \sin \theta}{r^2},$$
\[ G_3(r, \theta) = -\frac{g_1 \sin 2\theta - g_2 \sin 2\theta - g_3 \cos 2\theta}{r} , \]
\[ G_4(r, \theta) = g_4 . \]
(12)

Then, we can write
\[ \{g_1(x, y), p^2_r\} + \{g_2(x, y), p_r\} + \{g_3(x, y), p^2_\theta\} + g_4(x, y) = -\hbar^2 (\{G_1(r, \theta), \partial^2_r\} + \{G_3(r, \theta), \partial_r \partial_\theta\} + \{G_2(r, \theta), \partial^2_\theta\}) + G_4(r, \theta). \]
(13)

2. Determining equations for a fourth-order integral

2.1. Commutator \([H, Y]\]

Since \(Y\) is a fourth-order operator the commutator \([H, Y]\) would be a fifth-order one, i.e. we have
\[ [H, Y] = \sum_{k+l=0}^{N+1} Z_{k,l}(r, \theta) \frac{\partial^{k+l}}{\partial r^k \partial \theta^l}, \quad N = 4 \]
(14)

and we require \(Z_{k,l} = 0\) for all \(k\) and \(l\). The terms of order \(k + l = 5\) and \(k + l = 4\) vanish automatically. It was already shown that for arbitrary \(N\) the terms of order \(N + 1\) and \(N\) vanish [47]. Moreover, only terms with \(k + l\) of the opposite parity than \(N\) provide independent determining equations \((Z_{k,l} = 0)\). Those with the same parity provide equations that are differential consequences of the first ones.

Finally, we obtain
\[ [H, Y] = A_{rr} \frac{\partial^3}{\partial r^3} + A_{r\theta} \frac{\partial^3}{\partial r^2 \partial \theta} + A_{\theta\theta} \frac{\partial^3}{\partial r \partial \theta^2} + A_{\theta\theta\theta} \frac{\partial^3}{\partial \theta^3} + C_r \frac{\partial}{\partial r} + C_\theta \frac{\partial}{\partial \theta} = 0, \]
(15)

here the coefficients \(A_{rr}, A_{r\theta}, ...,\) are real functions of \(r\) and \(\theta\). The determining equations are obtained by setting all coefficients in (15) equal to zero. We thus obtain a total of 6 PDEs for the functions \(G_1, G_2, G_3, G_4, R\) and \(S\). Four equations are independent of \(\hbar\)
\[ G^{(1,0)}_1 = F_1 R' - \frac{2}{r^3} F_1 S + \frac{F_2}{r^2} S', \]
(16)
\[ \frac{1}{r^2} \left( G^{(0,1)}_2 + \frac{1}{r} G_1 \right) = F_3 R' - \frac{2}{r^3} F_3 S + \frac{F_4}{r^2} S'. \]
(17)
\[
\frac{1}{r^2} G^{(0,1)}_1 + G^{(1,0)}_3 = 3 F_3 R' - \frac{6 F_2}{r^3} S + \frac{F_5}{r^2} S'.
\]

\[
\frac{2}{r^2} G_1 + G^{(1,0)}_2 + \frac{1}{r^2} G^{(0,1)}_3 = F_3 R' - 2 \frac{F_5}{r^3} S + 3 \frac{F_3}{r^2} S'.
\]

They are the same for classical or quantum systems, and two equations contain quantum corrections proportional to \(\hbar^2\):

\[
G_3 \left( \frac{1}{r^2} S' + 2 G_1 \left( R' - \frac{2}{r^3} S \right) - \frac{1}{2} G^{(1,0)}_4 \right)
+ \hbar^2 \left[ G^{(0,2)}_1 - \frac{3 G^{(0,1)}_3}{4r^2} + \frac{G^{(1,0)}_3}{4r^2} - \frac{3}{2} \frac{G^{(2,0)}_2}{r^2} + \frac{5 G^{(1,1)}_3}{4r} \right]
+ \frac{G^{(1,2)}_1}{2r^2} + \frac{1}{2} G^{(1,2)}_2 + \frac{2 G^{(2,0)}_1}{r} - \frac{1}{2} r G^{(2,0)}_2 + \frac{3}{4} G^{(2,1)}_3 + G^{(3,0)}_1 \right]
= \hbar^2 \left[ \left( \frac{6 F_6}{r^2} - \frac{24 F_1}{r^3} \right) S + \left( \frac{6 F_6}{r^2} - \frac{4 F_7}{r^3} + \frac{F_12}{r^2} \right) S' \right]
+ \left( \frac{F_8}{r^2} - \frac{2 F_5}{r^3} \right) S'' + F_3 \frac{r}{r^3} S''' + F_{11} R' + F_6 R'' + F_4 R'''.
\]

\[
G_3 \left( R' - \frac{2}{r^3} S \right) + 2 G_2 \frac{1}{r^2} S' - \frac{G^{(0,1)}_2}{2r^2}
+ \hbar^2 \left[ G^{(0,1)}_1 - \frac{G^{(0,2)}_3}{r^2} + \frac{5 G^{(0,2)}_3}{4r^4} + \frac{G^{(1,1)}_3}{4r^2} - \frac{G^{(1,0)}_3}{4r^2} + \frac{G^{(1,1)}_1}{r^3} \right]
+ \frac{3 G^{(1,2)}_1}{4r^2} + \frac{G^{(1,2)}_3}{2r^2} + \frac{G^{(2,1)}_1}{2r^2} + \frac{1}{2} G^{(2,1)}_2 + \frac{G^{(3,0)}_3}{4r} \right]
= \hbar^2 \left[ \left( \frac{6 F_7}{r^2} - \frac{24 F_2}{r^3} - \frac{2 F_{12}}{r^3} \right) S + \left( \frac{6 F_5}{r^2} - \frac{4 F_8}{r^3} + \frac{F_{13}}{r^2} \right) S' \right]
+ \left( \frac{F_9}{r^2} - \frac{6 F_3}{r^3} \right) S'' + \frac{F_4}{r^2} S''' + F_{12} R' + F_7 R'' + F_2 R''''.
\]

The functions \(F_1 \ldots F_{13}\) are completely determined by the constants \(A_i, B_i, C_i, D_i\) figuring in the leading part of the integral \(Y\). They are presented in the appendix.

The system of equations (16)–(21) is the main object of study of the present paper.

2.2. Linear compatibility condition

The compatibility equation for (16)–(19) is a linear equation
In the paper [12], a global factor for the functions is satisfied trivially. For instance (as are equations (16), ..., (19)).

We will henceforth define as those for which the compatibility condition vanish. Standard potentials must satisfy the linear PDE (22) nontrivially. Differentiating (22) with respect to the coefficients in the integral must be nonzero. It is the same in classical and quantum mechanics (as are equations (16), ..., (19)).

We will henceforth define exotic potentials as those for which the compatibility condition is satisfied trivially. For $N = 4$ this means all the coefficients in the integral $Y$ of (9) except $A_1, A_2, B_3$ and $B_4$ vanish. Standard potentials must satisfy the linear PDE (22) nontrivially. Hence at least one of the constants $A_1, B_1, C_1$ or $D_1$ in (22) must be nonzero.

Differentiating (22) with respect to $r$ we eliminate $S(\theta)$ from the equation and obtain 8 ODEs for $R(r)$ by setting the coefficients of $\cos k \theta$ and $\sin k \theta$ for $k = 1, 2, 3, 4$ equal to zero, for more details we refer the reader to the paper [12] where similar calculations were performed. In the paper [12], a global factor $\frac{1}{6r^2}$ is missing in the r.h.s. of equation (27). Including this
factor, the derivative of (27) with respect to $r$ gives exactly our equation (22). Summarizing, the above-mentioned 8 ODEs for $R(r)$ together with the determining equations imply that the only possible forms of $R(r)$ are

$$R(r) = 0, \quad R(r) = \frac{a}{r}, \quad R(r) = b r^2.$$  \quad (23)

Let us compare our result (23) with a related one due to Onofri and Pauri [43] (see also [18]). They proved that the only potentials that separate in polar coordinates and for which all bounded trajectories are closed are $R(r) = b r^2$ as in (23) plus a further family

$$R(r) = \frac{1}{r^2} \sqrt{a^2 r^2 + d},$$  \quad (24)

where $a$ and $d$ are constants. For $d = 0$, this reduces to the Kepler(Coulomb) potential. We on the other hand require the existence of a polynomial integral of motion (in this case a fourth order polynomial in the momenta, $N = 4$). We mention that no lower order polynomial integrals for the potential (24) with $d \neq 0$ exist [15, 52]. The case of higher order ($N > 4$) polynomial integrals for the radial part as in (24) will be analyzed in a forthcoming article.

The connection between maximal superintegrability, $(2n-1)$ independent integrals of motion for $n$ degrees of freedom, and periodic trajectories was established by Nekhoroshev in [42], namely in classical mechanics maximal superintegrability implies that all bounded trajectories are closed and the motion is periodic. We recall that Bertrand’s theorem states that the only two central potentials for which all bounded trajectories are closed are the Kepler(Coulomb) potential and the harmonic oscillator [3, 19]. Bertrand’s theorem assumes $S(\theta) = 0$, the result (24) does not. Let us now consider each of the cases listed in (23) separately.

### 3. Non-confining potential $V(r, \theta) = \frac{S(\theta)}{r^2}$

Here we address the case of a non-confining potential, namely $R(r) = 0$. It turns out that this case provides the most general form of the function $S(\theta)$. The same function $S(\theta)$ will reoccur for $R(r) \neq 0$.

The equations (16), (18) and (19) corresponding to the determining equations $A_{rrr} = A_{r\theta\theta} = A_{\theta\theta\theta} = 0$, respectively, take the following form

$$G_1^{(1.0)}(r, \theta) r^3 - \left( A_3 \cos \theta + A_4 \sin \theta + C_1 \cos 3 \theta + C_2 \cos 3 \theta \right)r$$

$$- \left( B_1 \sin 2 \theta - B_2 \cos 2 \theta + 2 \left( D_1 \sin 4 \theta - D_2 \cos 4 \theta \right) \right) S'$$

$$+ 4 \left( B_1 \cos 2 \theta + B_2 \sin 2 \theta + D_1 \cos 4 \theta + D_2 \sin 4 \theta \right) S = 0,$$  \quad (25)

$$G_3^{(1.0)}(r, \theta) r^4 + G_1^{(0.1)}(r, \theta) r^2 - 2 \left( \left( B_3 \cos 2 \theta + B_4 \sin 2 \theta \right) \right)^2$$

$$- \left( A_3 \sin \theta - A_4 \cos \theta - 3 \left( C_1 \cos 3 \theta - C_2 \sin 3 \theta \right) \right) r$$

$$- 3 \left( D_1 \cos 4 \theta + D_2 \sin 4 \theta \right) S'$$

$$+ 6 \left( A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3 \theta + C_2 \cos 3 \theta \right) r$$

$$- \left( B_1 \sin 2 \theta - B_2 \cos 2 \theta + 2 \left( D_1 \sin 4 \theta - D_2 \cos 4 \theta \right) \right) S = 0,$$  \quad (26)
The above equations can be integrated to find the functions $G_1(r, \theta)$, $G_3(r, \theta)$ and $G_2(r, \theta)$ in terms of $S(\theta)$. They are given by the following formulas:

\[
G_1(r, \theta) = \left( \frac{1}{2} \right) \left( B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta) \right) + \frac{1}{r} \left( A_1 \cos \theta + A_2 \sin \theta - C_1 \sin 3\theta + C_2 \cos 3\theta \right) S' + \frac{2}{r^2} \left( B_1 \cos 2\theta + B_2 \sin 2\theta + D_1 \cos 4\theta + D_2 \sin 4\theta \right) S + \gamma_1(\theta),
\]

\[
G_2(r, \theta) = \frac{1}{r} \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right) S' + \frac{1}{r^2} \left[ \frac{3}{2} \left( A_3 \cos \theta - A_4 \cos \theta \right) - 3 \left( C_1 \cos 3\theta - C_2 \sin 3\theta \right) \right] S' + \frac{1}{r^3} \left[ - \frac{10}{3} \left( B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta) \right) S \right] + \frac{1}{r^4} \left[ B_1 \cos 2\theta + B_2 \sin 2\theta + 4(D_1 \cos 4\theta + D_2 \sin 4\theta) \right] S' + \frac{1}{r^5} \left[ B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta) \right] S'' + \frac{1}{r^6} \gamma_3(\theta) + \gamma_3(\theta),
\]
\[ G_2(r, \theta) = -\frac{3}{r} (A_1 \cos \theta + A_2 \sin \theta) S' + \frac{1}{r^2} \left[ (2 (B_3 \cos 2\theta + B_4 \sin 2\theta)) S + 5 (B_1 \sin 2\theta - B_4 \cos 2\theta) S' - (B_3 \cos 2\theta + B_4 \sin 2\theta) S'' \right] \\
+ \frac{1}{r^2} \left[ -\frac{7}{3} \left( (A_3 \sin \theta - A_4 \cos \theta) - 3 (C_1 \cos 3\theta - C_2 \sin 3\theta) \right) S \\
- \frac{1}{6} \left( (A_3 \cos \theta + A_4 \sin \theta) - 47 (C_1 \sin 3\theta + C_2 \cos 3\theta) \right) S' \\
+ \frac{2}{3} \left( (A_3 \sin \theta - A_4 \cos \theta) - 3 (C_1 \cos 3\theta - C_2 \sin 3\theta) \right) S'' \\
- \frac{1}{6} \left( (A_3 \cos \theta + A_4 \sin \theta) + (C_1 \sin 3\theta + C_2 \cos 3\theta) \right) S''' \right] \\
+ \frac{1}{r^2} \left[ -\frac{2}{3} \left( (B_1 \cos 2\theta + B_2 \sin 2\theta) + 13 (D_1 \cos 4\theta + D_2 \sin 4\theta) \right) S \\
- \frac{1}{3} \left( (B_1 \sin 2\theta - B_2 \cos 2\theta) + 20 (D_1 \sin 4\theta - D_2 \cos 4\theta) \right) S' \\
+ \frac{1}{3} \left( (B_1 \cos 2\theta + B_2 \sin 2\theta) + 4 (D_1 \cos 4\theta + D_2 \sin 4\theta) \right) S'' \\
+ \frac{1}{24} \left( (B_1 \sin 2\theta - B_2 \cos 2\theta) + 2 (D_1 \sin 4\theta - D_2 \cos 4\theta) \right) S''' \right] \\
+ \frac{1}{r} \gamma_1(\theta) + \frac{1}{r^2} \gamma_1'(\theta) + \frac{1}{2r^2} \gamma_1''(\theta) + \gamma_2(\theta) . \tag{30} \]

where \( \gamma_1(\theta) \), \( \gamma_2(\theta) \) and \( \gamma_3(\theta) \) are functions of \( \theta \) still to be determined.

The determining equation \( A_{\theta \theta \theta} = 0 \) (17) has not been used yet. We now substitute the expressions (28)–(30) into (17) in order to obtain further information on functions of \( \theta \), namely \( S(\theta) \) and \( \gamma_k(\theta) \), \( k = 1, 2, 3 \). The variable \( r \) still figures in the obtained result but only explicitly as \( r^k \), \( k = 0, 1, 2, 3, 4 \) (no unknown functions of \( r \) remain). Thus, the coefficient of \( r^k \) in \( A_{\theta \theta \theta} = 0 \) (17) must vanish for each \( k \) and we obtain a system of five equations for the four remaining functions of \( \theta \). They are

\[ 0 = \gamma_2', \tag{31} \]

\[ 0 = 4 \left( A_1 \cos \theta + A_2 \sin \theta \right) S + 14 \left( A_1 \sin \theta - A_2 \cos \theta \right) S' \\
- 6 \left( A_1 \cos \theta + A_2 \sin \theta \right) S'' + 2 \gamma_3' + 2 \gamma_3, \tag{32} \]

\[ 0 = -16 \left( B_3 \sin 2\theta - B_4 \cos 2\theta \right) S + 28 \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right) S' \\
+ 14 \left( B_3 \sin 2\theta - B_4 \cos 2\theta \right) S'' - 2 \left( B_3 \cos 2\theta + B_4 \sin 2\theta \right) S^{(3)} \\
+ 4 \gamma_1' + \gamma_3^{(3)}, \tag{33} \]
Equations (32) and (33) must be used to determine $J$. In both cases it is still necessary to solve the determining equations. Let us now discuss these equations.

1. The equation (31) determines $\gamma_2 = \text{constant}$.

2. Equations (32) and (33) must be used to determine $\gamma_1(\theta)$ and $\gamma_3(\theta)$ in terms of the function $S(\theta)$ figuring in the potential.

3. Of the 12 ‘doublet coefficients’ figuring in the leading part of the integral $Y$, four figure only in (34), namely $B_1, B_2, D_1, D_2$ and four only in (35), namely $A_3, A_4, C_1, C_2$. The remaining ones ($A_1, A_2$) and ($B_3, B_4$) appear only in (32) and (33), respectively. By definition, the linear compatibility condition (22) must be satisfied trivially for exotic potentials at $R(r) = 0$. Hence, for these potentials the only possible constants figuring in $Y$ are ($A_1, A_2$) and ($B_3, B_4$). This case was analyzed conclusively and in detail in our previous article [12].

Let us now turn to the problem of finding all standard superintegrable potentials. To do this we proceed as follows:

- **I.** Assume that at least one of the constants $B_1, B_2, D_1, D_2$ is not vanishing and solve (34) for $S(\theta)$. The general solution for $S(\theta)$ will depend on these constants and on additional integration constants.

- **II.** We proceed similarly with equation (35), i.e. solve it assuming that at least one of the $A_3, A_4, C_1, C_2$ is non-zero. In both cases it is still necessary to solve the determining equations

$$C_r = 0, \quad C_\theta = 0,$$

with $C_r$ and $C_\theta$ figuring in (15), (see (20) and (21)). The determining equation $C_r = 0$ defines $G_4(r, \theta), C_\theta = 0$ will provide further information and make it possible to determine $S(\theta)$ completely and hence also the integrals $X$ and $Y$.

- **III.** It is also possible for both of the above sets of constants to contain nonzero constants. The corresponding potentials will then allow more than one fourth-order integral. However, at most 3 integrals ($H, X$ and $Y$) can be polynomially independent. In any case, the function $S_{III}(\theta)$ obtained in this case will be a special case of both $S_1$ and $S_{II}$ and we shall not pursue this issue further.
3.1. General form of the angular part $S(\theta)$

**Case I.** $(B_1, B_2, D_1, D_2) \neq (0, 0, 0, 0)$

We concentrate in equation (34). We first define $T_I(\theta)$ by putting

$$T'_I(\theta) \equiv S_I(\theta). \quad (37)$$

The general solution of the fifth-order linear ODE for $T_I$ obtained from (34) is

$$T_I(\theta) = \frac{1}{M_I}(s_1 + s_2 \sin 2\theta + s_3 \cos 2\theta + s_4 \sin 4\theta + s_5 \cos 4\theta), \quad (38)$$

where the $s_i$ are integration constants and

$$M_I = B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta).$$

Using (37) we obtain $S_I(\theta)$ as

$$S_I(\theta) = -\frac{1}{M^2_I} \left( 2s_1[B_1 \cos 2\theta + B_2 \sin 2\theta + 4(D_1 \cos 4\theta + D_2 \sin 4\theta)] 
+ 2s_2[B_1 + D_1 \cos 6\theta + D_2 \sin 6\theta + 3(D_1 \cos 2\theta + D_2 \sin 2\theta)] 
+ 2s_3[B_2 - D_1 \cos 6\theta + D_1 \sin 6\theta + 3(D_2 \cos 2\theta - D_1 \sin 2\theta)] 
+ s_4[8D_1 - B_1 \cos 6\theta - B_2 \sin 6\theta + 3(B_1 \cos 2\theta - B_2 \sin 2\theta)] 
+ s_5[8D_2 + B_2 \cos 6\theta - B_1 \sin 6\theta + 3(B_2 \cos 2\theta + B_1 \sin 2\theta)] \right). \quad (39)$$

There seem to be too many integration constants in (39), since (34) is a fourth-order ODE for $S(\theta)$. Indeed the five solutions corresponding to $s_1, \ldots, s_5$, are not independent and one can be eliminated in terms of the others. We shall not go into this here since constraints on the constants $s_i$ will be imposed by the last still unsolved determining equation, namely $C_\theta = 0$ and these will always leave less than five free constants.

Now, from the equation $C_r = 0$ (20) we obtain the function $G_4(r, \theta)$

$$48r^4 G_4(r, \theta) = 4 \left[ 32SS' (B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta)) 
+ 48S^2 (B_1 \cos 2\theta + B_2 \sin 2\theta + D_1 \cos 4\theta + D_2 \sin 4\theta) 
+ S' \left[ S'' (B_2 \cos 2\theta - B_1 \sin 2\theta - 2(D_1 \sin 4\theta - D_2 \cos 4\theta)) 
- 6S' (B_1 \cos 2\theta + B_2 \sin 2\theta + 4(D_1 \cos 4\theta + D_2 \sin 4\theta)) \right] \right]
+ h^2 \left[ 3072S (D_1 \cos 4\theta + D_2 \sin 4\theta) 
+ 64S' (B_1 \sin 2\theta - B_2 \cos 2\theta + 62(D_1 \sin 4\theta - D_2 \cos 4\theta)) 
- 96S'' (B_1 \cos 2\theta + B_2 \sin 2\theta + 20(D_2 \sin 4\theta + D_1 \cos 4\theta)) 
+ S^{(3)} (52(B_2 \cos 2\theta - B_1 \sin 2\theta) + 440(D_2 \cos 4\theta - D_1 \sin 4\theta)) 
+ 12S^{(4)} (B_1 \cos 2\theta + B_2 \sin 2\theta + 4(D_2 \sin 4\theta + D_1 \cos 4\theta)) 
+ S^{(5)} (B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta)) \right]. \quad (40)$$
The determining equation $C_\theta = 0$ reads

\[
4 \left( 256 S^2 (B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta)) \right.
- 240 S S' (B_2 \sin 2\theta + B_1 \cos 2\theta + 4(D_1 \cos 4\theta + D_2 \sin 4\theta))
+ 60 (S')^2 (B_2 \cos 2\theta - B_1 \sin 2\theta - 8(D_1 \sin 4\theta - D_2 \cos 4\theta))
+ 40 S S'' (B_2 \cos 2\theta - B_1 \sin 2\theta - 2(D_1 \sin 4\theta - D_2 \cos 4\theta))
+ 30 S' S''' (B_1 \cos 2\theta + B_2 \sin 2\theta + 4(D_1 \cos 4\theta + D_2 \sin 4\theta))
\]

\[
+ (S'')^2 (B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta))
+ 3 S^{(3)} S' (B_1 \sin 2\theta - B_2 \cos 2\theta + 2(D_1 \sin 4\theta - D_2 \cos 4\theta)) \right)
\]
where \( q_{\ell}, \ell = 1, 2, 3, 4 \), are the four roots of the quartic equation
\[
\hbar^8 [D^4 q^4 + 4 B_2 D_1 D^3 q^3 + (B^2_2 (D^4_1 + 6 D^2_1 D^2_1) - 16 D^2_1 (D^2_1 + D^2_2)) q^2
- 2 B_2 D_1 (8 D^2_1 - B^2_2) D_2 (D^2_1 + 2 D^2_2) q + (B^2_2 - 8 D^2_1) \cdot \cdot \cdot ] = 0.
\] (43)

In (42), we assume that the discriminant \( \Gamma \) of the quartic equation (43)
\[
\Gamma = -256 \hbar^8 D^4_1 D^2_2 (B^2_2 - 8 D^2_1)^2 [B^2_1 (60 D^2_1 - 48 D^2_2) + 768 B^2_2 (D^2_1 + D^2_2)^2 + B^2_2 - 4096 (D^2_1 + D^2_2)^3],
\] (44)
is not equal to zero, \( \Gamma \neq 0 \). Such discriminant is zero if and only if at least two roots of (43) are equal. If the discriminant is negative there are two real roots and two complex conjugate roots. If it is positive the roots are either all real or all non-real. From a physical point of view the real solutions are admitted. In general, by substituting (42) into (39) we obtain an angular component \( S_1(\theta) \) proportional to \( h^2 \) with no classical analog, it cannot be transformed or reduced to that of the TTW model.

In particular, the discriminant (44) vanishes at \( h = 0 \). The highest order terms in (41) are proportional to \( h^2 \), therefore the limit \( h \to 0 \) is singular and the solutions (42) are no longer valid at \( h = 0 \). Let us analyze the zeros of the discriminant (44).

3.1.1. Case \( h = 0 \). At \( h = 0 \), non-trivial solutions exist for \( B_2 = 0 \) only. The corresponding coefficients take the values
\[
s_1 = s_1, \quad s_2 = 0, \quad s_3 = 0, \quad s_4 = s_4, \quad s_5 = s_5,
\]
which yields the potential
\[
S_1(\theta) = \frac{4 (D_1 \cos 4 \theta + D_2 \sin 4 \theta) s_1 + 4 (D_3 s_3 + D_2 s_4)}{(D_1^2 - D_2^2) \cos 8 \theta + 2 D_1 D_2 \sin 8 \theta - (D_1^2 + D_2^2)}. \] (45)

As mentioned in the Introduction, the angular component of the TTW potential \([50, 51]\) is
\[
S_{TTW}(\theta) = \frac{\alpha k^2}{\cos^2 (k\theta)} + \frac{\beta k^2}{\sin^2 (k\theta)}
= \frac{4 k^2 (\alpha - \beta) \cos 2 k \theta - 4 k^2 (\alpha + \beta)}{\cos 4 k \theta - 1}. \] (46)

In (46), it is easy to check that for \( \theta \to \theta + \frac{\pi}{4} \arctan(-D_2/D_1) \), \( \alpha = -\frac{\sqrt{D_1^2 + D_3 s_3 + D_2 s_2}}{8(D_1^2 + D_2^2)} \), \( \beta = -\frac{D_1 s_1 + \sqrt{D_1^2 D_2 s_1 + D_2 (D_3 s_3 + \sqrt{D_1^2 + D_2^2})}}{8(D_1^2 + D_2^2)^{3/2}} \) and \( k = 2 \) we recover (45). Therefore, the potential (45) corresponds to a rotated TTW model (with no radial component \( R(r) = 0 \)) which is a superintegrable system in both the classical and quantum cases.

3.1.2. Case \( D_1 = 0 \). For \( D_1 = 0 \), the discriminant (44) vanishes. The corresponding coefficients vanish, \( s_1 = s_2 = s_3 = s_4 = s_5 = 0 \), which gives the trivial solution \( S_1(\theta) = 0 \).

3.1.3. Case \( D_2 = 0 \). For \( D_2 = 0 \), the discriminant (44) vanishes again. The corresponding coefficients are given by
\[
s_1 = s_1, \quad s_2 = 0, \quad s_3 = \frac{B^2_2 s_4 - 8 D^2_2 (s_3 + s_4)}{2 B_2 D_1}, \quad s_4 = s_4, \quad s_5 = 0,
\]
thus

\[ S(\theta) = -\frac{2(B_2 s_4 + 2D_1 s_1 \sin 2\theta + 2D_1 s_4 \sin 2\theta)}{B_2 D_1 (1 + \cos 4\theta)}. \]

This solution corresponds to the angular component of the TTW model (46) with \( k = 1 \).

3.1.4. Case \( B_2^2 - 8D_2^2 = 0 \). For \( B_2^2 - 8D_2^2 = 0 \), the discriminant (44) vanishes as well. For simplicity, we put \( D_1 = D_2 = 1 \) thus \( B_2 = \sqrt{8} \). In this case, the coefficients

\[ s_1 = 0, \quad s_2 = -2\sqrt{2}h^2, \quad s_3 = 2\sqrt{2}h^2, \quad s_4 = -4h^2, \quad s_5 = 0, \]

lead to

\[ S(\theta) = 2h^2 \frac{\sqrt{2} \cos 6\theta - \sqrt{2} \sin 6\theta + 2}{(\cos 4\theta - \sin 4\theta + \sqrt{2} \cos 2\theta)^2}, \]

which is a pure quantum potential. It can not be reduced to that of the TTW model either. A similar analysis can be done for the last factor of the discriminant (44). This may lead to new quantum potentials vanishing in the classical limit.

**Case II:** \((A_3, A_4, C_1, C_2) \neq (0, 0, 0, 0)\).

Now, we proceed to the case II. First, the solution to the second linear equation (35) is

\[ T_{\text{II}}(\theta) = \frac{1}{M_{\text{II}}}(s_1 + s_2 \sin \theta + s_3 \cos \theta + s_4 \sin 3\theta + s_5 \cos 3\theta), \quad (47) \]

where

\[ M_{\text{II}} = A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3\theta + C_2 \cos 3\theta. \]

Consequently

\[
S_{\text{II}}(\theta) = \frac{1}{M_{\text{II}}} \left( 2s_1[A_3 \sin \theta - A_4 \cos \theta + 3(C_2 \sin 3\theta - C_1 \cos 3\theta)] \\
+ s_2[A_3 - C_2 \cos 4\theta - C_1 \sin 4\theta + 2(C_2 \cos 2\theta - C_1 \sin 2\theta)] \\
- s_3[A_4 + C_1 \cos 4\theta - C_2 \sin 4\theta + 2(C_1 \cos 2\theta - C_2 \sin 2\theta)] \\
+ s_4[3C_2 + A_3 \cos 4\theta + A_4 \sin 4\theta + 2(A_3 \cos 2\theta - A_4 \sin 2\theta)] \\
- s_5[3C_1 - A_4 \cos 4\theta + A_3 \sin 4\theta + 2(A_4 \cos 2\theta + A_3 \sin 2\theta)] \right). \quad (48)
\]

Again the five solutions corresponding to the 5 constants \( s_i \), are not linearly independent.

As mentioned above, the determining equation \( C_r = 0 \) (20) defines the function \( G_4(r, \theta) \).

The last equation \( C_\theta = 0 \) (21) constrains the coefficients \( s_i \).

From the determining equation \( C_r = 0 \) (20) we obtain the function \( G_4(r, \theta) \)
\[48 \pi^3 G_2(r, \theta) = 16 S' \left[ 3 S' \left( A_4 \cos \theta - A_3 \sin \theta - 3(C_2 \sin 3\theta - C_1 \cos 3\theta) \right) \right. \]
\[ \quad + \left. (S'' - 14 S) \left( A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right]\]
\[+ \hbar^2 \left[ 16 S \left( 2(A_4 \cos \theta - A_3 \sin \theta) + 15(C_2 \sin 3\theta - C_1 \cos 3\theta) \right) \right. \]
\[ \quad + \left. 4 S' \left( 7(A_3 \cos \theta + A_4 \sin \theta) - 127 (C_1 \sin 3\theta + C_2 \cos 3\theta) \right) \right. \]
\[ + \left. 4 S'' \left( A_3 \sin \theta - A_4 \cos \theta + 93(C_1 \cos 3\theta - C_2 \sin 3\theta) \right) \right. \]
\[ + \left. 5 S^{(3)} \left( 9(A_3 \cos \theta + A_4 \sin \theta) + 121 (C_1 \sin 3\theta + C_2 \cos 3\theta) \right) \right. \]
\[ + \left. 6 S^{(4)} \left( A_3 \sin \theta - A_4 \cos \theta + 3(C_2 \sin 3\theta - C_1 \cos 3\theta) \right) \right. \]
\[ \quad - \left. 5 S^{(5)} \left( A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right], \tag{49} \]

and correspondingly \( C_\theta = 0 \) (21) takes the form
\[4 \left[ -36 S^2 \left( A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right. \]
\[ + \left. 15 S^2 \left( A_3 \cos \theta + A_4 \sin \theta + 9(C_1 \sin 3\theta + C_2 \cos 3\theta) \right) \right. \]
\[ + \left. 20 S S' \left( A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right. \]
\[ + \left. (S'')^2 \left( A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right. \]
\[ + \left. 3 S' \left( 20 S \left( A_4 \cos \theta - A_3 \sin \theta + 3(C_1 \cos 3\theta - C_2 \sin 3\theta) \right) \right. \right. \]
\[ + \left. 5 S' \left( A_3 \sin \theta - A_4 \cos \theta + 3(C_2 \sin 3\theta - C_1 \cos 3\theta) \right) \right. \]
\[ - \left. S^{(3)} \left( A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right] \right]\]
\[+ \hbar^2 \left[ 96 S \left( A_3 \cos \theta + A_4 \sin \theta + 39 \left( C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right) \right. \]
\[ + \left. 24 S' \left( A_3 \sin \theta - A_4 \cos \theta + 303 \left( C_2 \sin 3\theta - C_1 \cos 3\theta \right) \right) \right. \]
\[ + \left. 8 S' \left( 21 \left( A_3 \cos \theta + A_4 \sin \theta \right) - 719 \left( C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right) \right. \]
\[ + \left. 30 S^{(3)} \left( 3 \left( A_3 \sin \theta - A_4 \cos \theta \right) + 79 \left( C_1 \cos 3\theta - C_2 \sin 3\theta \right) \right) \right. \]
\[ + \left. 8 S^{(4)} \left( 3 \left( A_3 \cos \theta + A_4 \sin \theta \right) + 67 \left( C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right) \right. \]
\[ + \left. 21 S^{(5)} \left( A_3 \sin \theta - A_4 \cos \theta + 3 \left( C_2 \sin 3\theta - C_1 \cos 3\theta \right) \right) \right. \]
\[ - \left. 3 S^{(6)} \left( A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right] \right] = 0. \tag{50} \]

Again, by an appropriate rotation we can always set \( A_3 = 0 \). Then, we substitute the expression (48) into the remaining determining equation (50). As a result we obtain algebraic
equations for the coefficients \( s_i \) \((i = 1, 2, 3, 4, 5)\) by setting the coefficients of \( \cos k\theta \) and \( \sin k\theta \) for \( k = 1, 2, 3, \ldots, 16 \) equal to zero. In general, for \( h \neq 0 \) we obtain the following three solutions

\[
\begin{align*}
    s_1 &= 0, \\
    s_2 &= s_2, \\
    s_3^{(\ell)} &= q_\ell h^2, \\
    s_4^{(\ell)} &= h^2 \frac{(2C_1^2 + C_1A_4 + 3C_2^2) q_\ell + C_1C_2s_2}{A_4C_2} \left( C_1 + A_4 \right) q_\ell^2 - q_\ell^3, \\
    s_5^{(\ell)} &= h^2 \frac{\left( (4C_1^2 + C_1A_4 + 6C_2^2 - A_2^2) q_\ell + (2C_1 + A_4) C_2s_2 \right) - (2C_1 + 3A_4) q_\ell^2 - 2q_\ell^3}{A_4 \left( 2C_1 + A_4 \right)},
\end{align*}
\]

(51)

where \( q_\ell, \ell = 1, 2, 3, \) is the solution to the cubic equation

\[
h^6 \left[ q^3 + A_4 q^2 - q \left( 3C_1^2 + 2A_4C_1 + 3C_2^2 \right) + \left( 2C_1 + A_4 \right)^2 C_1^2 + 2C_2^2 \right] = 0.
\]

(52)

Again, substituting (51) into (48) we obtain the corresponding angular component \( S_{II}(\theta) \) of the potential. Notice that there exist potentials \( S_{II}(\theta) \) proportional to \( h^2 \) that have no classical analog.

The discriminant \( \Omega \) of the cubic equation (52) is given by

\[
\Omega = -4 h^4 C_2 \left[ A_4^4 + 8 A_4^3 C_1 + 18 A_4^2 (C_1^2 + C_2^2) - 27 (C_1^2 + C_2^2)^2 \right].
\]

(53)

The discriminant (53) is zero if and only if at least two roots are equal. It is positive if the roots are all distinct real numbers, and negative if there exist one real and two complex conjugate roots. The solutions (51) are valid for \( \Omega \neq 0 \) only.

For \( h = 0 \), the corresponding coefficients take the values

\[
\begin{align*}
    s_1 &= s_1, \\
    s_2 &= 0, \\
    s_3 &= 0, \\
    s_4 &= s_4, \\
    s_5 &= s_5,
\end{align*}
\]

which yields the potential

\[
S_{II}(\theta) = \frac{6 (s_5 + s_1 \cos 3\theta)}{C_1 (1 - \cos 3\theta)},
\]

(54)

\((A_4 = C_2 = 0)\) it corresponds to that of the TTW potential with \( k = \frac{3}{2} \). Similarly we can show that for \( C_2 = 0 \), thus \( \Omega = 0 \), the TTW (angular) potentials with \( k = \frac{1}{2} \) and \( k = 1 \) occur.

4. Confining potentials

4.1. Deformed Coulomb potential

Here we address the case of the deformed Coulomb potential

\[
V(r, \theta) = \frac{a}{r} + \frac{S(\theta)}{r^2},
\]

(55)
where \( a \neq 0 \) is a real constant.

Similarly to the non-confining case \( R(r) = 0 \), from (16), (18) and (19) corresponding to the determining equations \( \mathcal{A}_{rr} = \mathcal{A}_{r\theta} = \mathcal{A}_{\theta \theta} = 0 \), respectively, we determine the functions \( G_1(r, \theta), G_2(r, \theta), \) and \( G_3(r, \theta) \) up to arbitrary additive functions \( \gamma_1(\theta), \gamma_3(\theta) \) and \( \gamma_2(\theta) \), respectively.

Then, we substitute such functions \( G_{1,2,3} \) into the determining equation \( \mathcal{A}_{\theta \theta \theta} = 0 \) (17). By doing so, such equation yields again a set of five linear ODE

- 1. Two fourth-order ODE for the angular component \( S(\theta) \). The first ODE contains the doublets \( (B_1, B_2), (D_1, D_2) \), while the second one depends on the doublets \( (A_3, A_4), (C_1, C_2) \) and the parameter \( a \) figuring in the Coulomb term.
- 2. Three linear ODE that define the functions \( \gamma_1(\theta), \gamma_2(\theta) \) and \( \gamma_3(\theta) \), respectively. The equation for \( \gamma_1 \) depends on the doublet \( (B_3, B_4) \) and the parameter \( a \). For the function \( \gamma_3 \) the doublet \( (A_1, A_2) \) and the parameter \( a \) are involved and the equation for \( \gamma_2 \) contains only the parameter \( a \).

For \( a \neq 0 \) the most general form for the angular component \( S(\theta) \) is given by the case II (48) for which at least one of the constants \( C_1, C_2, A_3, A_4 \) is not vanishing. Therefore, the function \( S(\theta) \) is not new with respect to the non-confining case \( R(r) = 0 \) and it was already described in detail in the section 3.

The corresponding functions \( G_{1,2,3} \) are given by

\[
\begin{align*}
G_1(r, \theta) &= G_1^{(0)}(r, \theta), \\
G_2(r, \theta) &= G_2^{(0)}(r, \theta) + 5a \left[ \frac{A_4 \cos \theta - A_3 \sin \theta + 3(C_1 \cos 3\theta - C_2 \sin 3\theta)}{2r^2} \right], \\
G_3(r, \theta) &= G_3^{(0)}(r, \theta) + 3a \left[ \frac{A_1 \cos \theta + A_4 \sin \theta + C_2 \cos 3\theta + C_4 \sin 3\theta}{r} \right].
\end{align*}
\]

(56)

where the \( G_i^{(0)}(r, \theta), i = 1, 2, 3, \) coincide with those of the non-confining potential given by (28)–(30), respectively, putting \( B_1 = B_2 = D_1 = D_2 = 0 \) and \( \gamma_1 = \gamma_2 = \gamma_3 = 0 \).

From the determining equation \( C_1 = 0 \) given in (20) we calculate the corresponding function \( G_4(r, \theta) \)

\[
G_4(r, \theta) = G_4^{(0)}(r, \theta) + a \left[ \frac{1}{r^2} \left( \frac{1}{4} \hbar^2 (7(A_4 \cos \theta - A_3 \sin \theta) + 99(C_2 \sin 3\theta - C_1 \cos 3\theta)) \right) \\
+ 5S'(\theta) \left( A_4 \sin \theta + A_3 \cos \theta + C_1 \sin 3\theta + C_2 \cos 3\theta \right) \right],
\]

(57)

here \( G_4^{(0)}(r, \theta) \) is given in (49). The angular component \( S(\theta) \) is given by (48) with the coefficients \( s_i \) (51) presented in the case II.

### 4.2. Deformed harmonic oscillator potential

Here we address the case of the confining potential

\[
V(r, \theta) = br^2 + \frac{S(\theta)}{r^2},
\]

(58)
with \( b \neq 0 \) a real constant.

Following the same strategy as for the previous case, we obtain the following functions

\[
G_1(r, \theta) = G_1(0)(r, \theta) + 2br^2(B_1 \cos 2\theta + B_2 \sin 2\theta + D_1 \cos 4\theta + D_2 \sin 4\theta),
\]

\[
G_2(r, \theta) = G_2(0)(r, \theta),
\]

\[
G_3(r, \theta) = G_3(0)(r, \theta) + 2br(B_2 \cos 2\theta - B_1 \sin 2\theta + 2(D_2 \cos 4\theta - D_1 \sin 4\theta)),
\]

\[
G_4(r, \theta) = G_4(0)(r, \theta),
\]

(59)

where the \( G_i(0)(r, \theta), i = 1, 2, 3, \) coincide with those of the non-confining potential \( R(r) = 0, \) given in (28)–(30), respectively, putting \( A_3 = A_4 = C_1 = C_2 = 0 \) and \( \gamma_1 = \gamma_2 = \gamma_3 = 0. \) The function \( G_4(0)(r, \theta) \) is given in (40).

For \( b \neq 0, \) the angular component \( S(\theta) \) is given by (39) with the coefficients \( s_i \) (42) presented in the case I for which at least one of the constants \( B_1, B_2, D_1, D_2 \) is not vanishing.

5. Conclusions

We considered superintegrable systems in a two-dimensional Euclidean space. Classical and quantum fourth-order superintegrable standard potentials separating in polar coordinates were derived. We can summarize the main results via the following theorems

**Theorem 1.** In classical mechanics, the standard superintegrable confining systems, \( ab \neq 0, \) correspond to the TTW potential

\[
V_{\text{TTW}}(r, \theta) = br^2 + \frac{1}{r^2} \left[ \frac{\alpha}{\cos^2 2\theta} + \frac{\beta}{\sin^2 2\theta} \right],
\]

(60)

(\( \alpha, \beta \) real constants), and the PW potential

\[
V_{\text{PW}}(r, \theta) = ar + \frac{1}{r^2} \left[ \frac{\mu}{\cos^2 \frac{3}{2}\theta} + \frac{\nu}{\sin^2 \frac{3}{2}\theta} \right],
\]

(61)

(\( \mu, \nu \) real constants). The corresponding leading terms of the integral \( Y \) in (9) are proportional to

\[
(p_x^4 + p_y^4 - 6p_x^2 p_y^2),
\]

and

\[
\{Lz, 3p_x^2 p_y - p_y^3 \},
\]

respectively. These terms are independent of \( a \) and \( b. \)

**Theorem 2.** In quantum mechanics, the new confining superintegrable systems correspond to

\[
V(r, \theta) = br^2 + \frac{1}{r^2} S_i(\theta),
\]

and
where $S_I(\theta)$ is given by (39) with (42), and $S_{II}(\theta)$ takes the form (48) with (51). In general, the functions $S_I$ and $S_{II}$ are proportional to $\hbar^2$ and cannot be reduced to a TTW or PW potential. The corresponding leading terms of the integral $Y$ in (9) are

$$2B_2 p_x p_y (p_x^2 + p_y^2) + D_1 (p_x^4 + p_y^4 - 6 p_x^2 p_y^2) + 4 D_2 p_x p_y (p_x^3 - p_y^3),$$

$(B_2, D_1, D_2) \neq (0, 0, 0)$ and

$$A_4 \{L_z, p_y (p_x^2 + p_y^2)\} + C_1 \{L_z, 3 p_x^2 p_y - p_y^3\} + C_2 \{L_z, p_x^3 - 3 p_x^2 p_y\},$$

$(A_4, C_1, C_2) \neq (0, 0, 0)$ respectively.

We emphasize that these are pure quantum potentials. As particular cases, both potentials $V_{TTW}$ and $V_{PW}$ appear in the quantum case as well.

Work has been presented on a general and unified study of $N$th-order exotic and standard potentials separating in polar coordinates [13]. Within this study [13] the TTW and PW models correspond to standard classical potentials. For some cases we plan to present the polynomial algebra generated by the integrals of motion and to use it to calculate the energy spectrum and the wave functions in the quantum case.

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**Appendix**

Explicitly, the functions $F_1, \ldots, F_{13}$ figuring in (16)–(21) are given by
\[ F_1 = 2 \left( B_1 \cos 2 \theta + B_2 \sin 2 \theta + D_1 \cos 4 \theta + D_2 \sin 4 \theta \right), \]
\[ F_2 = \frac{1}{r} \left( B_2 \cos 2 \theta - B_1 \sin 2 \theta - 2 (D_1 \sin 4 \theta - D_2 \cos 4 \theta) \right) \\
\quad + A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3 \theta + C_2 \cos 3 \theta, \]
\[ F_3 = \frac{1}{r^2} \left( B_2 \cos 2 \theta - B_1 \sin 2 \theta + 2(D_1 \sin 4 \theta - D_2 \cos 4 \theta) \right) \\
\quad + \frac{1}{r^2} \left( A_4 \sin \theta + A_3 \cos \theta - 3(C_2 \cos 3 \theta + C_1 \sin 3 \theta) \right) \\
\quad - \frac{2}{r} \left( B_3 \sin 2 \theta - B_4 \cos 2 \theta \right) + A_2 \sin \theta + A_1 \cos \theta, \]
\[ F_4 = \frac{2}{r^2} \left( D_1 \cos 4 \theta + D_2 \sin 4 \theta - B_1 \cos 2 \theta - B_2 \sin 2 \theta \right) \\
\quad + \frac{4}{r^2} \left( A_4 \cos \theta - A_3 \sin \theta - C_1 \cos 3 \theta + C_2 \sin 3 \theta \right) \\
\quad - \frac{4}{r^2} \left( B_3 \cos 2 \theta - B_4 \sin 2 \theta \right) + \frac{4}{r} \left( A_2 \cos \theta - A_1 \sin \theta \right), \]
\[ F_5 = \frac{6}{r^2} \left( D_1 \cos 4 \theta + D_2 \sin 4 \theta \right) + \frac{2}{r} \left( A_4 \cos \theta - A_3 \sin \theta \right) \\
\quad + 3(C_1 \cos 3 \theta - C_2 \sin 3 \theta) + 2(B_3 \cos 2 \theta + B_4 \sin 2 \theta), \]
\[ F_6 = \frac{3}{r^2} \left( A_4 \cos \theta - A_3 \sin \theta + 3(C_1 \cos 3 \theta - C_2 \sin 3 \theta) \right) - \frac{9}{r} \left( D_1 \cos 4 \theta + D_2 \sin 4 \theta \right), \]
\[ F_7 = \frac{12}{r^2} \left( D_1 \sin 4 \theta - D_2 \cos 4 \theta \right) + \frac{1}{2r} \left( A_4 \sin \theta + A_3 \cos \theta \right) \\
\quad - 15 \left( C_1 \sin 3 \theta + C_2 \cos 3 \theta \right) - 2 \left( B_3 \sin 2 \theta - B_4 \cos 2 \theta \right), \]
\[ F_8 = \frac{3}{r^2} \left( B_1 \cos 2 \theta + B_2 \sin 2 \theta - 5(D_1 \cos 4 \theta + D_2 \sin 4 \theta) \right) \\
\quad + \frac{3}{2r^2} \left( A_4 \cos \theta - A_3 \sin \theta - 9(C_1 \cos 3 \theta - C_2 \sin 3 \theta) \right) \\
\quad - \frac{15}{2r} \left( B_3 \cos 2 \theta - B_4 \sin 2 \theta \right) + \frac{3}{2} \left( A_2 \cos \theta - A_1 \sin \theta \right), \]
\[ F_9 = \frac{9}{r^2} \left( B_1 \sin 2 \theta - B_2 \cos 2 \theta - 2(D_1 \sin 4 \theta - D_2 \cos 4 \theta) \right) \\
\quad + \frac{15}{2r^2} \left( 3(C_1 \sin 3 \theta + C_2 \cos 3 \theta) - A_3 \cos \theta - A_4 \sin \theta \right) \\
\quad + \frac{12}{r^2} \left( B_3 \sin 2 \theta - B_4 \cos 2 \theta \right) - \frac{9}{2r} \left( A_1 \cos \theta + A_2 \sin \theta \right), \]
\[ F_{10} = \frac{3}{r} \left( (B_2 \cos 2 \theta - B_1 \sin 2 \theta) + 2(D_2 \cos 4 \theta - D_1 \sin 4 \theta) \right) \\
\quad + 3 \left( A_3 \cos \theta + A_4 \sin \theta + C_1 \sin 3 \theta + C_2 \cos 3 \theta \right), \]
\[ F_{11} = \frac{3}{r^2} \left( B_1 \cos 2 \theta + B_2 \sin 2 \theta - 5(D_1 \cos 4 \theta + D_2 \sin 4 \theta) \right) - 4(B_3 \cos 2 \theta + B_4 \sin 2 \theta) \\
\quad + \frac{1}{r} \left( A_4 \cos \theta - A_3 \sin \theta - 9(C_1 \cos 3 \theta - C_2 \sin 3 \theta) \right), \]
\[ F_{12} = \frac{2}{r^3} (B_1 \sin 2\theta - B_2 \cos 2\theta - 14 (D_1 \sin 4\theta - D_2 \cos 4\theta)) \]
\[ + \frac{3}{2r^2} (11 (C_1 \sin 3\theta + C_2 \cos 3\theta) - A_3 \cos \theta - A_4 \sin \theta) \]
\[ + \frac{6}{r} (B_3 \sin 2\theta - B_4 \cos 2\theta - \frac{3}{2} (A_1 \cos \theta + A_2 \sin \theta), \]
\[ F_{13} = \frac{4}{r^4} (2 (B_1 \cos 2\theta + B_2 \sin 2\theta) - 11 (D_1 \cos 4\theta + D_2 \sin 4\theta)) \]
\[ - \frac{2}{r^3} (A_4 \cos \theta - A_3 \sin \theta - 17 (C_1 \cos 3\theta - C_2 \sin 3\theta)) \]
\[ + \frac{12}{r^2} (B_3 \cos 2\theta + B_4 \sin 2\theta) - \frac{3}{2} (A_2 \cos \theta - A_1 \sin \theta). \]

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