Exact asymptotics of the optimal $L_p$-error of asymmetric linear spline approximation†

Vladislav Babenko, Yuliya Babenko, Nataliya Parfinovych, Dmytro Skorokhodov

Abstract

In this paper we study the best asymmetric (sometimes also called penalized or sign-sensitive) approximation in the metrics of the space $L_p$, $1 \leq p \leq \infty$, of functions $f \in C^2([0,1]^2)$ with nonnegative Hessian by piecewise linear splines $s \in S(\Delta_N)$, generated by given triangulations $\Delta_N$ with $N$ elements. We find the exact asymptotic behavior of optimal (over triangulations $\Delta_N$ and splines $s \in S(\Delta_N)$) error of such approximation as $N \to \infty$.

Keywords: spline, asymmetric approximation, adaptive approximation, exact asymptotics, optimal error, anisotropic partitions, triangulations.

MSC: Primary 41A15; Secondary 41A25, 41A60.

§1. Introduction

The question of approximation of functions defined on a polytope by piecewise polynomial functions (splines), generated with the help of a mesh (partition of the domain), in various metrics is of a great importance in Approximation Theory and its applications (numerical solutions for PDE’s, surface simplification, image compression, terrain data processing etc.). Mainly, problems of approximation by interpolating splines have been considered. However, for various applications (as well as

†This research was partially conducted during visit of V. Babenko, N. Parfinovych and D. Skorokhodov to Kennesa State University (supported by QEP International Faculty Development grant and Simons Collaboration Grant AID 210363)
theoretical point of view) problems of best and best one-sided approximation are important.
Approximation Theory there exist two different tools to approximate functions: uniform methods that work rather well for all functions in a given class; and adaptive methods, or methods that take into account local variations (measured with the help of Hessian, curvature, their modifications, etc.) in the behavior of each given function. For methods which involve spline approximation, adaptivity affects the construction of domain partitions and geometry (both size and shape of elements, which can be highly anisotropic.

In this paper we will consider the problem of adaptive approximation of twice differentiable functions by linear splines (naturally the domain partitions in this case are triangulations). This has been studied extensively by many authors (see works [29, 15, 23, 5, 12, 6, 14, 28] and references therein), but many interesting (from both theoretical and applied point of view) questions remain open. In addition, note that the problem of surface approximation by linear splines is very natural. It is an important problem in Geometry on approximation (in various metrics) of smooth convex surfaces by various polytopes (inscribed, circumscribed, polytopes of best approximation etc.) After occasional results for bivariate functions, the book of L. Fejes Toth [20] was the first to provide a large number of problems, ideas, and results on polytopal approximation in dimensions two and three, concentrating specifically on extremal properties of regular polytopes. Many extensions have been made afterwards to higher dimensions, other metrics etc. (see [22, 10, 9, 21] and references therein).

In Approximation Theory there exists a tool to view both the problem of finding the best approximation without constraints and the problem of finding the best approximation with constraints “under one umbrella”. This can be viewed as the best approximation in the spaces with asymmetric norm, or so-called $(\alpha, \beta)$-approximation (see, for example, [1, 2, 25]), when positive and negative parts of the difference between function and the approximant are “weighted” differently. Such types of approximations are of a separate interest since they can be considered as the problems of approximation with non-strict constraints (see below for more precise statements), when constraints are allowed to be violated, but the penalty for their violation is introduced into the error measure. Within this framework we will consider the questions of best $(\alpha, \beta)$-approximation by linear splines. We believe that this approach could be interesting and useful for some questions in Geometry as well.

Note that the construction of the best (in a specified sense) triangulation for approximating an individual function, or construction of the best polytope for an individual convex body, is an extremely difficult problem, and therefore it is natural to consider asymptotically optimal sequences of partitions or asymptotically optimal sequences of triangulations (and splines defined on them).

One possible method to construct asymptotically optimal sequences of triangulations or partitions begins as follows. At the first step, we construct an intermediate approximation of the function (or convex body surface, respectively) by a piecewise quadratic function (surface). At the next
Optimal $L_p$-error of asymmetric linear spline approximation

approximate each quadratic piece by a piecewise linear function (spline) in the optimal way, generating a mesh of the domain. This in turn (at least in $\mathbb{R}^2$) requires solving the following optimization problem (we will give its statement for approximation of functions in $\mathbb{R}^d$).

Let a quadratic function $Q$ defined on $\mathbb{R}^d$ be given. Consider the best approximation ($L_p$, asymmetric, one-sided) of $Q$ by linear functions on simplex $T \subset \mathbb{R}^d$ of unit volume. The problem is to find a simplex $T^*$, for which the corresponding error is minimal.

This problem is important also in a number of questions of Geometry and Approximation Theory. In [7] we have proved the optimality of a regular simplex in the formulated problem for the $(\alpha, \beta)$-approximation in $L_p$-metric of function $Q(x) = \sum_{j=1}^d x_j^2$ by linear functions. Note that with help of linear transformations the solution of this problem allows us to obtain the solution of analogous optimization problems for an arbitrary positive definite quadratic form.

In this paper we will study the behavior of the optimal error of $(\alpha, \beta)$-approximation in $L_p$-metric of an arbitrary $C^2$ function with nonnegative Hessian. The main contributions of this paper are:

1. We present the construction of asymptotically optimal sequence of partitions and error estimates without assumption on the Hessian to be bounded away from zero. Remark that geometers able to remove restrictions of such type in some of their problems before (see [10]). How we use another technique to handle the problem.

2. We impose no restrictions on triangulations (many existing works require some type of “admissibility”).

3. We consider asymmetric approximation, which, as special cases, includes the cases of interpolating splines, splines of best approximation, splines of best one-sided approximation etc.

The paper is organized as follows. In Section 2, we begin by introducing major concepts and definitions, in particular related to asymmetric approximation and asymptotically optimal triangulations. In Subsection 2.3 we present the main questions that we will address in this paper, and state related geometric problems in Subsection 2.4. Subsection 2.5 contains statements of the main results. In Section 3 we introduce the major ideas for the proofs of the main results without much of technical details. In particular, we relate the problem of describing exact asymptotics of the optimal error to some geometric problems. Statements of the solutions to these problems are presented in Section 4, which also contains additional geometric observations needed later for the lower estimate of the error. Section 5 is dedicated to the construction of “good” triangulation for each fixed $N$ and the proofs of the upper estimate from above for the optimal error. The estimate from below is contained in Section 6.
§2. Notation, definitions, main questions, and results

Let the domain be \( D := [0, 1]^2 \subset \mathbb{R}^2 \). We use this region for simplicity; the approach presented in this paper can be applied to any bounded connected region which is a finite union of triangles. By \( C(D) \) we denote the space of functions continuous on \( D \). Let \( L_p := L_p(D), \ 0 < p \leq \infty \), be the space of measurable functions \( f : D \rightarrow \mathbb{R} \) such that \( \|f\|_p < \infty \) where

\[
\|f\|_p = \|f\|_{L_p(D)} := \begin{cases} 
\left( \int_D |f(x,y)|^p \, dx \, dy \right)^{1/p}, & \text{if } 0 < p < \infty, \\
\text{ess sup}\{ |f(x,y)| : (x,y) \in D\}, & \text{if } p = \infty.
\end{cases}
\]

For \( p > 1 \), \( \| \cdot \|_p \) is the standard norm in space \( L_p \). In addition, we use notation \( \| \cdot \|_p \) when \( p \) is not specified for statement of main results.

2.1. Asymmetric approximation

Let \( f \in L_p \) and let \( H \) be a subspace of \( L_p \). By \( E(f; H)_p \) we denote the best approximation of the function \( f \) by the subspace \( H \) in the \( L_p \)-metric, i.e.:

\[
E(f; H)_p := E(f; H)_{L_p(D)} = \inf\{ \|f - u\|_p : u \in H \}.
\]

In addition, by

\[
E^\pm(f; H)_p := E^\pm(f; H)_{L_p(D)} = \inf\{ \|f - u\|_p : \pm u(x,y) \leq \pm f(x,y), (x,y) \in D \text{ and } u \in H \}
\]

we denote the best one-sided approximation of the function \( f \) by the subspace \( H \) in the \( L_p \)-metric. In the case of “+” in the above definition we say that we have \textit{approximation from below}; in the case of “−” we say that we have \textit{approximation from above}.

For \( \alpha, \beta > 0 \) and \( f \in L_p \), \( 1 \leq p \leq \infty \), we define the \textit{asymmetric} \( (\alpha, \beta) \)-norm as follows

\[
\|f\|_{p;\alpha,\beta} = \|f\|_{L_p,\alpha,\beta(D)} = \|\alpha f^+ + \beta f^-\|_p,
\]

where \( g_{\pm}(x, y) = \max\{\pm g(x, y); 0\} \). Following the literature, we call \( \|f\|_{p;\alpha,\beta} \) the \textit{asymmetric norm}.

Note that it satisfies the norm axioms except for the fact that we only have \( \|\lambda f\|_{p;\alpha,\beta} = \lambda \|f\|_{p;\alpha,\beta} \) for \( \lambda > 0 \) (in particular, \( \|f\|_{p;\alpha,\beta} \neq \| - f\|_{p;\alpha,\beta} \) for \( \alpha \neq \beta \)). Asymmetric norms in connection with problems in Approximation Theory were considered in papers [26, 1, 16, 17] and books [27, 5].
Optimal $L_p$-error of asymmetric linear spline approximation

By $E(f; H)_{p;\alpha,\beta}$ we denote the best $(\alpha, \beta)$-approximation [1] of the function $f$ by the subspace $H$ in the $L_p$-metric, i.e.:

$$E(f; H)_{p;\alpha,\beta} := E(f; H)_{L_p;\alpha,\beta}(D) = \inf \{\|f - u\|_{p;\alpha,\beta} : u \in H\}.$$

Note that for $\alpha = \beta = 1$ we have $E(f; H)_{p;1,1} = E(f; H)_p$. V. Babenko proved in [1] that $H \subset L_p(D)$, $1 \leq p < \infty$, is locally compact, then for any $f \in L_p(D)$ the following limit relations are true (see also [25], Theorem 1.4.10):

$$\lim_{\beta \to +\infty} E(f; H)_{p;1,\beta} = E^+(f; H)_p \quad \text{and} \quad \lim_{\alpha \to +\infty} E(f; H)_{p;\alpha,1} = E^-(f; H)_p,$$

which are monotone in $\alpha$ and $\beta$. This allows us to include the problem of the best unconstrained approximation and the problem of the best one-sided approximation into the family of problems of the same type, and consider them from a general point of view (for more on this motivation, see [26]). In what follows we will allow the value $+\infty$ for $\alpha$ or $\beta$, in that case identifying $E(f; H)_{p;\alpha,\beta}$ with the corresponding one-sided approximation. Because of the relation

$$\|f - u\|_{p;1,\beta} = \|f - u\|_p^p + (\beta^p - 1) \|f - u\|_{p}^p, \quad \beta > 1,$$

the problem of the best $(1, \beta)$-approximation can be considered as the problem of the best approximation with non-strict constraint $f(x, y) \leq u(x, y), (x, y) \in D$. This constraint is allowed to be violated, but the penalty

$$(\beta^p - 1) \|f - u\|_{p}^p$$

for the violation is introduced into the error measure. In what follows we will allow the value $+\infty$ for $\alpha$ or $\beta$, in that case identifying $E(f; H)_{p;\alpha,\beta}$ with the corresponding one-sided approximation.

2.2. Optimal triangulations and asymptotically optimal sequences of triangulations

Let $N \in \mathbb{N}$. A collection $\triangle_N = \triangle_N(D) = \{T_i\}_{i=1}^N$ of $N$ triangles in the plane is called a triangulation of the set $D$ provided that

1) any pair of triangles from $\triangle_N$ intersect at most at a common vertex or along a common edge.

2) $D = \bigcup_{i=1}^N T_i$. 

Let $\mathcal{P}_1$ be the set of bivariate linear polynomials $p(x, y) = ax + by + c$, with $a, b, c \in \mathbb{R}$. For a triangulation $\triangle_N$, define the class of linear splines $S(\triangle_N)$ as follows

$$S(\triangle_N) := \{ f \in C(D) : \forall i = 1, ..., N \ \exists p_i \in \mathcal{P}_1 \text{ such that } f|_{T_i} = p_i|_{T_i} \}.$$ 

Now let the function $f \in C^2(D)$ and the number $N$ of triangles be fixed. Set

$$R_N(f, L_{p;\alpha,\beta}) := \inf_{\triangle_N} E(f; S(\triangle_N))_{p;\alpha,\beta} = \inf_{\triangle_N} \inf_{s \in S(\triangle_N)} \| f - s \|_{p;\alpha,\beta}.$$ 

This quantity we will call the optimal $L_p$-error of piecewise linear $(\alpha, \beta)$-approximation of the function $f$ on triangulations with $N$ elements. A triangulation $\triangle_0^N$ and the corresponding spline $s_0^N \in S(\triangle_0^N)$ are called $L_{p;\alpha,\beta}$-optimal for the given function $f$ if

$$\| f - s_0^N \|_{p;\alpha,\beta} = R_N(f, L_{p;\alpha,\beta}).$$ 

Note that the quantity $R_N(f, L_{p;1,1})$ coincides with the error of the best $L_p$-approximation of the function $f$ by linear splines from $S(\triangle_N)$. In addition, in view of (2.1) in the case $\alpha = 1$ and $\beta = 1$ and $\alpha \rightarrow \infty$) the quantity $R_N(f, L_{p;\alpha,\beta})$ tends to the error of the best $L_p$-approximation of the function $f$ from above (below) by splines from $S(\triangle_N)$. Remark that the latest statement will immediately follow from (2.1). However, the proof of it is rather simple and we omit it here.

### 2.3. Main questions

In this paper we will work with functions $f \in C^2(D)$, where $D$ for simplicity is taken to be a square $[0, 1]^2 \subset \mathbb{R}^2$. It is well known that for most of such functions (i.e. for functions with Hessian not identically equal to zero) the order of the optimal error $R_N(f, L_{p;\alpha,\beta})$ is $\frac{1}{N}$ as $N \rightarrow \infty$. Our goal is to study sharp asymptotic behavior of the quantity $R_N(f, L_{p;\alpha,\beta})$ as $N \rightarrow \infty$. We end, we will prove the existence of the limit of $N \cdot R_N(f, L_{p;\alpha,\beta})$ as $N \rightarrow \infty$, and will find its value. In turn, our analysis will allow one to obtain information about construction of asymptotically optimal sequences of triangulations.

We will study this problem in the case when the given function has nonnegative Hessian. In the case when the Hessian is strictly positive there exist a lot of results (see works [29, 15, 5, 12, 6, and references therein]). One of the most studied questions is the problem of approximation by interpolating splines. Note that in the case when Hessian is strictly positive interpolating splines obviously coincide with splines of best (one-sided) approximation from above. However, even in the case of strictly positive Hessian the questions of finding sharp asymptotics of the optimal
the cases of approximating by splines of best approximation, best approximation from below, and, in general, best asymmetric approximation remain open. Finding solutions to these questions is one of the two main goals of the present paper. Besides that, in all these questions we will remove the restriction of Hessian being bounded away from zero to allow algorithms to be applicable for wider range of surfaces. In [10] Böröczky addressed this nontrivial question in the related case of approximating smooth convex bodies by some inscribed polytopes.

As for the little investigated case of negative Hessian, the only known (at least to us) explicit result is [5].

### 2.4. Related geometric problems

The essential role in further results is played by the solution of the following extremal problems on the function $q(x,y) := x^2 + y^2$.

**Problem 1.** For $\alpha, \beta > 0$ and $1 \leq p \leq \infty$, find

$$C_{p;\alpha,\beta} := \inf_T \frac{E(q; P_1)_{L^p(T)}}{|T|^{1+1/p}},$$

where the infimum is taken over all triangles $T$ in $\mathbb{R}^2$ and $|T|$ stands for the area of triangle $T$.

Solution to this problem allows solving a similar problem for arbitrary positive definite quadratic form (see Section 4.1 for details).

Note that Problem 1 is a generalization of the following problems.

**Problem 2.** For $1 \leq p \leq \infty$, find

$$C_p := \inf_T \frac{E(q; P_1)_{L^p(T)}}{|T|^{1+1/p}} \quad \text{and} \quad C_p^\pm := \inf_T \frac{E^\pm(q; P_1)_{L^p(T)}}{|T|^{1+1/p}}.$$

The constant $C_p^-$ coincides with the best $L_p$-error of interpolation of $q$ by linear functions on triangles of unit area.

To the best of our knowledge the progress on the problem of computing the constant $C_p^-$ (Problem 3 above) can be outlined as follows:

1. $p = \infty$ (D’Azevedo and Simpson [15], 1989);
2. $p = 1$ (Böröczky, Ludwig [9], 1999);
3. $p = 2$ (Pottmann et al [30], 2000);
4. $p \in \mathbb{N}$ (Chen [12], 2007);
5. $p \in (1, \infty)$ (V. Babenko, Yu. Babenko, and Skorokhodov [6], and independently Chen 2008) for any dimension $d$.

The most general constant $C_{p;\alpha,\beta}$ for any dimension $d \in \mathbb{N}$ was found by the authors in [7] that the infimum in the definition of constant $C_{p;\alpha,\beta}$ is achieved only on regular simplices.

**Remark 1.** Note that the inf in all the constants $C_{p;\alpha,\beta}$, $C_p$, and $C^\pm_p$, that are solutions to Problems 1, 2 and 3 posed in the previous section, are achieved on regular triangles. This $C_{p;\alpha,\beta}$ was proved in greater generality (for any dimension $d$) in [7].

### 2.5. Main results

The following theorem is the main result of this paper.

**Theorem 2.1.** Let $f \in C^2(D)$ be such that $H(f; x, y) \geq 0$ for all $(x, y) \in D$. Then for all $1 \leq p \leq \infty$,

$$
\lim_{N \to \infty} N \cdot R_N(f, L_{p;\alpha,\beta}) = 2^{-1} C_{p;\alpha,\beta} \cdot \|H\|_{p+1}^{1/(p+1)},
$$

where $C_{p;\alpha,\beta}$ was defined in (2.2).

For regular (no constraints) best and best one-sided $L_p$-approximations, we obtain the following corollaries.

**Corollary 2.2.** Let $f \in C^2(D)$, $H(f; x, y) \geq 0$ for all $(x, y) \in D$. Then for all $1 \leq p \leq \infty$,

$$
\lim_{N \to \infty} N \cdot R_N(f, L_p) = 2^{-1} C_p \cdot \|H\|_{p+1}^{1/(p+1)},
$$

$$
\lim_{N \to \infty} N \cdot \inf_{\triangle N} E^\pm(f; S(\triangle_N))_p = 2^{-1} C^\pm_p \cdot \|H\|_{p+1}^{1/(p+1)},
$$

where $C_p$ and $C^\pm_p$ were defined in (2.3).

**Remark 2.** Corollary 2.2 generalizes Theorem 2 in paper [6] (case of interpolating splines).

**Remark 3.** Assertion of Theorem 2.1 remains true if we replace the space $S(\Delta)$ of complete piecewise linear splines on triangulation $\Delta$ by wider space $\overline{S}(\Delta)$ of arbitrary piecewise linear elements from triangulation $\Delta$ splines which are not necessarily continuous.
§3. Ideas used in the proof of the main result

In order to make the reading of the rest of the paper easier we would like to devote this section to introduce the main ideas without technical details.

Let $f \in C^2(D)$ be such that $H(f; x, y) \geq 0$ for all $(x, y) \in D$. To prove Theorem 2.1 we will prove that for any $\varepsilon > 0$

$$\limsup_{N \to \infty} N \cdot R_N(f, L_{p;\alpha,\beta}; D) \leq 2^{-1} C_{p;\alpha,\beta} \cdot \|H\|_{\frac{p}{p+1}} (1 + \varepsilon),$$

and

$$\liminf_{N \to \infty} N \cdot R_N(f, L_{p;\alpha,\beta}; D) \geq 2^{-1} C_{p;\alpha,\beta} \cdot \|H\|_{\frac{p}{p+1}} (1 - \varepsilon).$$

Further, we will refer to the proof of inequality (3.1) as “estimate from above”, and to inequality (3.2) as “estimate from below”.

Note that the following ideas in the case of strictly positive Hessian and for the case of approximation from above (interpolation) have already been introduced in [5] and [6].

3.1. Main ideas used in the proof of estimate from above

To prove inequality (3.1), for every $\varepsilon > 0$, we will present a particular sequence of triangulations $\{\Delta_N(\varepsilon)\}_{N=1}^\infty$ and corresponding sequence of splines $\{s_N(\varepsilon)\}_{N=1}^\infty$, $s_N(\varepsilon) \in S(\Delta_N(\varepsilon))$, such that

$$\limsup_{N \to \infty} N \cdot \|f - s_N(\varepsilon)\|_{L_{p;\alpha,\beta}(D)} \leq 2^{-1} C_{p} \cdot \|H\|_{\frac{p}{p+1}} \cdot (1 + \varepsilon).$$

We begin by following the ideas developed in [5, 6]. For each $N$ we will construct a triangulation based on the idea of intermediate approximation of functions from the class $C^2(D)$ by piecewise quadratic functions.

First, for $\delta > 0$, define the modulus of continuity of $g \in C(D)$ by

$$\omega(g, \delta) := \sup \{ |g(x') - g(x'')| : (x', y') \in D, |x' - x''| \leq \delta, |y' - y''| \leq \delta \},$$

and for $(x, y) \in D$, consider

$$\lambda_{\min}(x, y) := (f_{xx}(x, y) + f_{yy}(x, y)) / 4 - \sqrt{(f_{xx}(x, y) + f_{yy}(x, y))^2 / 16 - H(f; x, y)/4}.$$
We begin by splitting the domain $D$ into small in comparison with $N$ ($o(N)$ as $N \to \infty$) $m_N^2$ of subdomains ($m_N^2 \to \infty$ as $N \to \infty$). In the case $D = [0, 1]^2$ we will take these subdomains to be squares (for simplicity). Denote them by $D_i^N$, $i = 1, \ldots, m_N^2$. On each $D_i^N$ we will consider intermediate approximation of $f$ by $f_{N,i}$ that is second degree Taylor polynomial of $f$ constructed at some point (we take center) inside of $D_i^N$. Observe that the order of this approximation is higher than the order of the optimal error $R_N(f, L_p(\alpha, \beta))$ and therefore will not affect the main term asymptotics of $R_N(f, L_p(\alpha, \beta))$. In addition, note that to approximate quadratic function $f_{N,i}$ by splines is the same as to approximate its quadratic part (denote it by $Q_{N,i}$) by linear splines on every square. This switch to an intermediate approximation is one of the major ideas in obtaining asymptotically optimal sequences of triangulations and exact asymptotics of the optimal error $R_N(f, L_p(\alpha, \beta))$.

The coefficients of quadratic form $Q_{N,i}$, which obviously depend on $f$, determine the geometry of optimal mesh element (triangle) on the particular subdomain $D_i^N$ as follows. We begin by finding the eigenvalues $\lambda_{\text{min}}^{N,i}$ and $\lambda_{\text{max}}^{N,i}$ of the quadratic form $Q_{N,i}$ for each $i = 1, \ldots, m_N^2$. Depending on the magnitude of the eigenvalues on each subdomain, we will split all the squares $D_i^N$ into four groups.

1. The first group contains squares with $\lambda_{\text{max}}^{N,i} \geq \lambda_{\text{min}}^{N,i} > \varepsilon$.

In this case, the intermediate approximation $Q_{N;i}$ of $f$ on $D_i^N$ is an elliptic paraboloid. We find the optimal triangle (its shape and orientation in the plane) by solving a local optimization problem of minimizing $\ell_p$-error of approximation of $Q_{N;i}$ by linear functions. For the description of the solution see Sections 4.1 and 1.2. We then choose the size of the triangles so that the amount of $\ell_p$-error of approximation on triangles from the first group is minimized. Note that the error on squares from the first group will be a small contribution to the global error and main focus of our attention.

Let $T_i^N$ be an optimal triangle which solves the minimization problem on $D_i^N$. We use $T_i^N$ together with its reflections and translations, to provide a triangulation of each $D_i^N$ with not too small ($o(m_{N}^{-2}N)$ as $N \to \infty$) number of other triangles (created by refining shapes along boundary of $D_i^N$).

2. The second group contains squares with $\omega \left( \lambda_{\text{min}}, m_N^{-1} \right) < \lambda_{\text{min}}^{N,i} \leq \varepsilon$.

There will be very few squares from this group and its contribution to the global error will be significant. The squares from this group we subdivide into equal right isosceles triangles, the amount of $o(m_{N}^{-2}N)$.

3. The third group contains squares with $\lambda_{\text{min}}^{N,i} \leq \omega \left( \lambda_{\text{min}}, m_N^{-1} \right)$ and $\lambda_{\text{max}}^{N,i} \geq \varepsilon$.

In this case, the intermediate approximation $Q_{N;i}$ on $D_i^N$ is close to a parabolic cylinder.
squares $D_i^N$ from this group will be divided into $\varepsilon \cdot o \left( m^{-2}_N N \right)$ of right triangles with the longer side positioned in the direction of eigenvector corresponding to the smallest eigenvalue.

4. The fourth group contains squares with $\lambda_{\text{min}}^{N,i} \leq \omega \left( \lambda_{\text{min}}, m^{-1}_N \right)$ and $\lambda_{\text{max}}^{N,i} \leq \varepsilon$.

The intermediate approximation is almost a plane. The squares from this group we subdivide into equal right isosceles triangles in the amount of $\varepsilon \cdot o \left( m^{-2}_N N \right)$.

“Gluing” (without adding new vertices) triangulations of all $D_i^N$, we obtain the desired triangulation $\Delta_N(\varepsilon)$ of $D$.

Having the triangulation $\Delta_N(\varepsilon)$ of the whole domain $D$, we will then define the spline $s_N(\varepsilon; x, y)$ from $S(\Delta_N(\varepsilon))$ which approximate $f$ sufficiently well. First of all, on the union of all triangles (denoted $U_N$), each of which is contained in the interior of the corresponding square $D_i^N$ from the first group, we set $s_N(\varepsilon; x, y) := \tilde{s}_N(\varepsilon; x, y)$, where $\tilde{s}_N$ is the spline of the best $L_{p;\alpha,\beta}$-approximation (on $U$) of respective $Q_{N,i}$. On triangles that are contained in $D \setminus \text{int}(U_N)$, we let $s_N(\varepsilon; x, y)$ to interpolate the function $f$ at the vertices of $\Delta_N(\varepsilon)$ which are located in $D \setminus U_N$, and interpolate $\tilde{s}_N$ at points on the boundary of $U_N$. Note that the spline defined in such a way is continuous on the whole domain $D$.

The constructed sequence of triangulations $\{\Delta_N(\varepsilon)\}_{N=1}^{\infty}$ and corresponding sequence of splines $\{s_N(\varepsilon)\}_{N=1}^{\infty}$ will allow us to prove the estimate from above (3.3).

3.2. Main ideas used in the estimate from below

To prove estimate (3.2) we will show that for every (sufficiently small) number $\varepsilon > 0$ and every sequence $\{\Delta_N\}_{N=1}^{\infty}$ of triangulations with property

$$
\liminf_{N \to \infty} N \cdot \inf_{s \in S(\Delta_N)} \|f - s\|_{L_{p;\alpha,\beta}(D)} < \infty,
$$

the following inequality holds true

$$
\liminf_{N \to \infty} N \cdot \inf_{s \in S(\Delta_N)} \|f - s\|_{L_{p;\alpha,\beta}(D)} \geq 2^{-1} C_p \|\sqrt{H}\|_{p \frac{1}{p+1}} (1 - \varepsilon).
$$

The general idea of the proof of inequality (3.5) is to classify the triangles from triangulation into two categories: “good” triangles and “bad” triangles. First we show that the errors $\inf_{P \in P_1} P \|L_{p;\alpha,\beta}(T)\|$ for every “bad” triangle $T$ can be neglected. According to this observation, we can study the errors $\inf_{P \in P_1} \|f - P\|_{L_{p;\alpha,\beta}(T)}$ only for “good” triangles $T \in \Delta_N$. For such triangles $T$ we use the intermediate approximation and substitute the function $f$ by its second degree Taylor polynomial $f_N;T$ constructed at an arbitrary point inside $T$. Then we use results of Sections 4.1 and 4.2 together with the Jensen inequality to obtain the desired inequality (3.5).
Next, we clarify the classification of triangles $T \in \triangle_N$ into “good” and “bad”. To this end, triangles of each triangulation $\triangle_N$, $N \in \mathbb{N}$, we divide into five groups according to the following:

Assume that $T \in \triangle_N$. Then:

1. $T \in A_1^N$ iff $H(f; x, y) < 2\varepsilon$ at every point $(x, y) \in T$;

2. $T \in A_2^N$ iff $T \not\in A_1^N$, $H(f; x, y) \geq \varepsilon$ at every point $(x, y) \in T$ and $\|f - f_N; T\|_{L_p}$ significantly lower than $\inf_{P \in P_1} \|f - P\|_{L_{p,\alpha,\beta}(T)}$;

3. $T \in A_3^N$ iff $T \not\in A_1^N$, $H(f; x, y) \geq \varepsilon$ at every point $(x, y) \in T$ and diam $T$ is very large;

4. $T \in A_4^N$ iff there exist two points $(x', y') \in T$ and $(x'', y'') \in T$ such that $H(f; x', y') \geq 2\varepsilon$.

In Lemmas 6.1 and 4.6 we will show that the overall area of triangles belonging to sets $A_3^N$, $A_5^N$ is less than $\varepsilon$. Therefore, this fact and the definition of the group $A_1^N$, allow us to classify triangles $T \in A_1^N \cup A_3^N \cup A_4^N \cup A_5^N$ as “bad” and the triangles $T \in A_2^N$ as “good” respectively.

§4. Construction of an optimal mesh element

4.1. Optimality of a regular triangle for $E(x^2 + y^2; P_1)_{L_p;\alpha,\beta}$

Here we would like to state the solution of Problem 1 posed in Section 2.4. For the proof of this result we refer the reader to [7].

**Theorem 4.1.** Let $q(x, y) = x^2 + y^2$. Then for every $\alpha, \beta > 0$ and $1 \leq p \leq \infty$,

$$C_{p;\alpha,\beta} = E(q; P_1)_{L_p;\alpha,\beta}(T_0),$$

where $C_{p;\alpha,\beta}$ was defined in (2.2) and $T_0$ is equilateral triangle of unit area.

This result can be proved in a significantly simpler and more elegant way (comparing to the result for any dimension $d$ in [7]) using the idea of symmetry and averaging, which was also used in [7].

**Remark 4.** Let $P$ be the polynomial of the best approximation of $q$ on equilateral triangles. Using arguments about symmetry and rotational invariance we can conclude that the difference $E(q; P)_{L_p;\alpha,\beta}(T_0)$ attains the same values at three vertices of $T_0$.

In certain cases, the constant $C_{p;\alpha,\beta}$ can be found explicitly. For instance,

$$C_{\infty;\alpha,\beta} = 4 \cdot 3^{-3/2} \alpha \beta (\alpha + \beta)^{-1},$$
and in the case $3^{3/2} \pi^{-1} \alpha \leq \alpha + \beta$,

$$C_{1,\alpha,\beta} = 3^{3/2} \alpha - 2^{-1} \pi^{-1} \alpha^2 (\alpha + \beta)^{-1}.$$

### 4.2. Geometry of optimal triangle for $E(Ax^2 + By^2 + 2Cxy; P_1)_{L_p,\alpha,\beta}$

In previous section we found that the optimal triangle for approximation of the form $q(x, y) = x^2$ is the equilateral triangle. Let us consider general positive definite quadratic form $Q(x, y) = Ax^2 + By^2 + 2Cxy$ (i.e. $AB > C^2$) and find a unit area triangle $T$ that delivers the infimum in the problem

$$E(Q; P_1)_{L_p,\alpha,\beta} \rightarrow \text{inf}_T.$$

Let $\lambda_1, \lambda_2$ be the eigenvalues of the matrix $S = \begin{pmatrix} A & C \\ C & B \end{pmatrix}$, and by $U$ we denote $2 \times 2$ matrix composed from eigenvectors of $S$ having unit length. Then the linear mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} = U \begin{pmatrix} \lambda_1^{-1/2} & 0 \\ 0 & \lambda_2^{-1/2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

transforms quadratic form $Q(x, y)$ into the form $q(u, v)$. Hence, optimal triangle $T$ for problem (4.2) is obtained from the equilateral triangle by applying the inverse transformation to (4.2). Therefore, we established the following fact.

**Corollary 4.2.** Let $Q(x, y) = Ax^2 + By^2 + 2Cxy$ be a positive definite quadratic form, i.e. $AB - C^2 > 0$. Then for every $\alpha, \beta > 0$ and $1 \leq p \leq \infty$,

$$\inf_T \frac{E(Q; P_1)_{L_p,\alpha,\beta}(T)}{|T|^{1+1/p}} = C_{p,\alpha,\beta} \sqrt{AB - C^2} = E(q; P_1)_{L_p,\alpha,\beta(T_0)} \sqrt{AB - C^2},$$

where, as defined above, $T_0$ is a regular triangle of unit area.

**Remark 8.** Let triangle $T$ deliver the infimum in problem (4.1) for positively definite quadratic form $Q$. If $P$ is the linear polynomial of the best approximation of $Q$ on $T$ then the difference $Q$ attains equal values at three vertices of $T$.

In addition to Corollary 4.2 we need the following two lemmas.

**Lemma 4.3.** Let us consider the collection of quadratic forms $Ax^2 + By^2 + 2Cxy$ which satisfy conditions $0 < A \leq A^+, 0 < B \leq B^+$ and $H = AB - C^2 \geq K$, where $A^+, B^+, K$ are some positive numbers. Then for any such form $\lambda_{\text{min}} \geq \sqrt{K} > 0.$
Proof trivially follows from the fact that the function \( g(u, v) = u - \sqrt{u^2 - v} \) is decreasing in \( u \) and is increasing in \( v \).

**Lemma 4.4.** For the collection of quadratic forms satisfying the assumptions of Lemma 4.3, the diameter of the optimal triangle to the square root of the area of this triangle is bounded by the constant independent of \( A^+, B^+ \), and \( K \).

This statement follows from Lemma 4.3.

### 4.3. Additional geometric observations

In paper [6] the following lemma was proved.

**Lemma 4.5.** Let \( f \in C^2(D); H(f; x, y) \geq K > 0 \) for all \((x, y) \in D\). If \( \bar{n} \) is an arbitrary unit vector in the plane, then

\[
\left| \frac{\partial^2 f}{\partial \bar{n}^2} \right| \geq \frac{K}{2} \min \left\{ \frac{1}{\| f_{xx} \|_{\infty}}, \frac{1}{\| f_{yy} \|_{\infty}} \right\}.
\]

The following geometric lemma, together with Lemma 4.5, plays crucial role in the proof of the lower estimate in Theorem 2.1.

**Lemma 4.6.** Let \( T \) be an arbitrary triangle in the plane with \( \text{diam} T \leq \sqrt{2} \). Let also \( O \) be an arbitrary point inside \( T \), \( \delta > 0 \) be a fixed number and \( K_\delta := \min \{ 1; \delta^2/2 \} \). Then there exists triangle \( T' \) that lies completely in the intersection of \( T \) and the disk \( B \) centered at \( O \) and having the radius \( \delta \) such that

\[
|T'| \geq K_\delta^2 \cdot |T| \quad \text{and} \quad \text{diam} T' \geq K_\delta \cdot \text{diam} T.
\]

The proof of this result requires only elementary geometric observations and is omitted here.

### §5. Error of asymmetric approximation of \( C^2 \) functions by linear splines: estimate from above

Recall that we consider functions \( f : D \to \mathbb{R} \) with Hessian that is nonnegative on \( D \). For definiteness, we assume that function is convex. In this section we will show that for every such function \( 1 \leq p < \infty \)

\[
\limsup_{N \to \infty} N \cdot R_N(f, L_{p;\alpha,\beta}) \leq 2^{-1} C_{p;\alpha,\beta} \left\| \sqrt{H} \right\|_{\frac{p}{p+1}}.
\]
Remark that letting $p \to \infty$ we also could prove lower estimate (5.1) in case $p = \infty$. To prove it in Section 5.2, for every $\varepsilon > 0$ we will construct a suitable family of triangulations $\{\Delta^\varepsilon_N\}_{N=1}^\infty$ and a family of corresponding piecewise-linear splines $\{s^\varepsilon_N\}_{N=1}^\infty$, $s^\varepsilon_N \in S(\Delta^\varepsilon_N)$. Following that, in Section 5.2 we will show that

$$
\lim_{\varepsilon \to 0} \left( \limsup_{N \to \infty} N \cdot \|f - s^\varepsilon_N\|_{p;\alpha,\beta} \right) \leq 2^{-1} C_{p;\alpha,\beta} \left\| \sqrt{H} \right\|_{p+\varepsilon}^{-1}.
$$

The latter inequality implies the desired estimate (5.1).

In what follows, we fix the number $1 \leq p < \infty$.

### 5.1. Additional notations

This subsection contains several simple, yet important observations, and notations that will be used throughout this section.

**Lemma 5.1.** Let $f \in C^2(D)$. If $P_2 = P_2(f; x, y; x_0, y_0)$ denotes the second degree Taylor polynomial of $f$ at a point $(x_0, y_0)$ inside the square $D_h \subset D$ with side length equal to $h$, then we have the following estimate:

$$
\|f - P_2\|_{L^\infty(D_h)} \leq 2h^2 \omega_2(f, h).
$$

This lemma is obvious, and we omit the proof here.

Now we consider functions

$$
\lambda_{\min}(x, y) := (f_{xx} + f_{yy})/4 - \sqrt{(f_{xx} + f_{yy})^2/16 - (f_{xx}f_{yy} - f^2_{xy})}/4,
$$

$$
\lambda_{\max}(x, y) := (f_{xx} + f_{yy})/4 + \sqrt{(f_{xx} + f_{yy})^2/16 - (f_{xx}f_{yy} - f^2_{xy})}/4.
$$

For every $\varepsilon > 0$, we define the following set

$$
G_{0;2\varepsilon} := \{(x, y) \in D : 0 < \lambda_{\min}(x, y) < \varepsilon\}.
$$

Note that $\mu(G_{0;2\varepsilon}) \to 0$ as $\varepsilon \to 0$.

In addition, for a fixed $\varepsilon \in (0, 1)$ and every $N \in \mathbb{N}$, we define

$$
m_N = m_N(\varepsilon) := \min \{m > 0 : 2m^{-2} \max \{\alpha, \beta\} \omega_2(f, m^{-1}) \leq \varepsilon N^{-1}\}.
$$

Observe that $m_N \to \infty$ as $N \to \infty$. In addition, note that

$$
m_N^2 N^{-1} \to \infty, \quad N \to \infty,
$$

$$
m_N^2 N^{-1} \to \infty, \quad N \to \infty,
$$
i.e. $m_N = o\left(\sqrt{N}\right)$ as $N \to \infty$ (see, for instance, Section 4 in [5]). In what follows we will also assume that $N$ is large enough so that inequality $\omega(\lambda_{\text{min}}, m_{N}^{-1}) \leq \varepsilon$ holds true.

Now, let us subdivide the square $D$ into squares of size $m_{N}^{-1} \times m_{N}^{-1}$ with the sides parallel to the sides of $D$. By $D_{i}^{N} := D_{i}^{N}(\varepsilon)$, $i = 1, \ldots, m_{N}^{2}$, we denote the resulting squares enumerated in arbitrary order.

For $i \in \{1, \ldots, m_{N}^{2}\}$, the center of the square $D_{i}^{N}$ we denote by $(x_{i}^{N}, y_{i}^{N})$. Let

$$ A_{i}^{N} := f_{xx}(x_{i}^{N}, y_{i}^{N})/2, \quad B_{i}^{N} := f_{yy}(x_{i}^{N}, y_{i}^{N})/2, \quad C_{i}^{N} := f_{xy}(x_{i}^{N}, y_{i}^{N})/2, $$

$$ \lambda_{\text{min};i}^{N} := (A_{i}^{N} + B_{i}^{N})/2 - \sqrt{(A_{i}^{N} + B_{i}^{N})^2/4 - (A_{i}^{N}B_{i}^{N} - (C_{i}^{N})^2)/4}, $$

$$ \lambda_{\text{max};i}^{N} := (A_{i}^{N} + B_{i}^{N})/2 + \sqrt{(A_{i}^{N} + B_{i}^{N})^2/4 - (A_{i}^{N}B_{i}^{N} - (C_{i}^{N})^2)/4}. $$

Note that

$$ H(f; x_{i}^{N}, y_{i}^{N}) = 4\left(A_{i}^{N}B_{i}^{N} - (C_{i}^{N})^2\right) = 4\lambda_{\text{min};i}^{N}\lambda_{\text{max};i}^{N}. $$

Finally, let $f_{N;i}$ be the second degree Taylor polynomial of $f$ constructed at the point $(x_{i}^{N}, y_{i}^{N})$. Let $H(f; x_{i}^{N}, y_{i}^{N})$ denote its quadratic part by

$$ Q_{N,i} = Q_{N,i}(x, y) := A_{i}^{N}x^2 + 2C_{i}^{N}xy + B_{i}^{N}y^2, \quad (x, y) \in \mathbb{R}^2. $$

### 5.2. Construction of the family of “good” triangulations and the family of corresponding splines

In what follows, let the fixed number $\varepsilon > 0$ satisfy the restriction $23\varepsilon + 2\mu(G_{0;2\varepsilon}) < 1$. Our aim in this subsection is to construct “nearly” optimal triangulation $\Delta_{N;i} = \Delta_{N;i}(\varepsilon)$ of the square $D$. Then we “glue” these triangulations into one triangulation $\Delta_{N}^{\varepsilon}$ of the square $D$. Once the construction of triangulation $\Delta_{N}^{\varepsilon}$ is complete, we will define the spline $s_{N}^{\varepsilon} \in S_{N}$ which is “nearly” optimal.

We will split all the set of indices $i = 1, \ldots, m_{N}^{2}$ into the following four groups:

$$ I_{1}(\varepsilon; N) := \{ i \in \{1, \ldots, m_{N}^{2}\} : \lambda_{\text{min};i}^{N} \geq \varepsilon \}, $$

$$ I_{2}(\varepsilon; N) := \{ i \in \{1, \ldots, m_{N}^{2}\} : \omega(\lambda_{\text{min}}, m_{N}^{-1}) < \lambda_{\text{min};i}^{N} < \varepsilon \}, $$

$$ I_{3}(\varepsilon; N) := \{ i \in \{1, \ldots, m_{N}^{2}\} : \lambda_{\text{min};i}^{N} \leq \omega(\lambda_{\text{min}}, m_{N}^{-1}), \lambda_{\text{max};i}^{N} \geq \varepsilon^2 \}, $$

$$ I_{4}(\varepsilon; N) := \{ i \in \{1, \ldots, m_{N}^{2}\} : \lambda_{\text{min};i}^{N} < \omega(\lambda_{\text{min}}, m_{N}^{-1}), \lambda_{\text{max};i}^{N} < \varepsilon^2 \}. $$
Evidently, the sets $I_1(\varepsilon; N)$, $I_2(\varepsilon; N)$, $I_3(\varepsilon; N)$ and $I_4(\varepsilon; N)$ are pairwise non-intersecting for every $\varepsilon$.

Let us describe how we will construct triangulations of each $D^N_i$ depending on which out of above groups index $i$ is in.

**a)** For this $i \in I_1(\varepsilon; N)$, we set $n^N_i(\varepsilon) := \left[ \frac{N(1 - 23\varepsilon - 2\mu(G_{0;2\varepsilon})) H^{\frac{p}{2(p+1)}}(f; x^N_i, y^N_i)}{\sum_{j \in I_1(\varepsilon; N)} H^{\frac{p}{2(p+1)}}(f; x^N_j, y^N_j)} \right]^{-1}$

Here $[a]$ stands for the integer part of a real number $a$. This quantity $n^N_i(\varepsilon)$ is the number of triangles in the triangulation of $D^N_i$ (before refining shapes along the boundary) and is obtained by minimizing the global error under the condition $\sum_{i \in I_1(\varepsilon; N)} n^N_i(\varepsilon) \approx N$.

Since $n^N_i(\varepsilon) > (1 - 23\varepsilon - 2\mu(G_{0;2\varepsilon}))(4\varepsilon^2)^{\frac{p}{2(p+1)}} \|H\|_{\infty}^{\frac{p}{2(p+1)}} \cdot m^{-2} N$, and due to relation (5.3), we see that $n^N_i$ tends to infinity as $N$ gets large.

In order to formalize further constructions, for an arbitrary triangle $T$ in the plane, by $\text{Til}(T)$ denote the tiling of the plane, generated by $T$ in the following way: we take a triangle $\tilde{T}$ symmetric to $T$ with respect to the midpoint of one of its sides, and then we tile $\mathbb{R}^2$ with the shifts $T \cup \tilde{T}$.

Let us describe the algorithm for construction of triangulation $\triangle_{N;i}$:

1. Let $T^N_i$ be the triangle that delivers infimum in (4.1).
2. By $T^N_i$ we denote a re-scaling of $F(T)$ such that $|T^N_i| = m^{-2} (n^N_i)^{-1}$.
3. With the help of the triangle $T^N_i$ we generate the tiling $\text{Til}(T^N_i)$ of the plane.
4. Every triangle from $\text{Til}(T^N_i)$ that lies completely inside the square $D^N_i$ we include into triangulation $\triangle_{N;i}$.
5. For every triangle $T \in \text{Til}(T^N_i)$ that has common points with the boundary of $D^N_i$, we consider the intersection $T \cap D^N_i$. Evidently, it is a polygon with at most seven vertices. We then split this polygon into at most five triangles without adding new vertices and include them into triangulation $\triangle_{N;i}$.

Let us estimate the number of triangles in $\triangle_{N;i}$. Since the quadratic form $Q_{N;i}$ satisfies conditions of Lemma 1.4 we derive that there exists a constant $c_1 = c_1(\varepsilon)$, independent of $N$, that

$$\text{diam } T^N_i \leq c_1 m^{-1} (n^N_i)^{-1/2}, \quad N \to \infty.$$
Consequently, the number of triangles $T \in \text{Til} \left( T_i^N \right)$ that have nonempty intersection with the boundary of $D_i^N$ is $O \left( \sqrt{n_i^N} \right)$ as $N \to \infty$. Therefore, the total number of triangles in $\triangle_{N;i}$ is

$$n_i^N + O \left( \sqrt{n_i^N} \right) = n_i^N + o \left( m_i^{-2} N \right), \quad N \to \infty.$$

b) Let $i \in I_2(\varepsilon;N)$. Since $m_i^{-2} N \to \infty$ with $N$, for each $N$ there exists an integer, denote $r_1 = r_1(N)$, such that

$$2^{-1} m_i^{-2} N \leq r_1^2 \leq m_i^{-2} N.$$

Let us subdivide the square $D_i^N$ into squares of the size $m_i^{-1} r_1^{-1} \times m_i^{-1} r_1^{-1}$ whose sides are parallel to the sides of $D_i^N$. Then inside each small square we draw one of its diagonals. Thus, we obtain the triangulation $\triangle_{N;i}$ of $D_i^N$ consisting of $2r_1^2$ equal isosceles right triangles. In addition, for an arbitrary such triangle and denote it by $T_i^N$.

c) Let $i \in I_3(\varepsilon;N)$. Denote by $\xi_i$ and $\overline{\eta}_i$ the eigenvectors of the quadratic form $Q_{N;i}$ corresponds to the eigenvalues $\lambda_{\text{min};i}^N$ and $\lambda_{\text{max};i}^N$. For this $\varepsilon > 0$, and for each $N$, there exists an integer, denote by $r_2$, such that

$$2^{-1} \varepsilon m_i^{-2} N \leq r_2^2 \leq \varepsilon m_i^{-2} N.$$

Let $\Pi_i^N$ be the rectangle of the size $m_i^{-1} \times m_i^{-1} r_2^{-1}$ whose sides are parallel to vectors $\overline{\xi}_i$, respectively. We will draw inside this rectangle one of its diagonals and denote any of two common triangles by $T_i^N$. Let $T$ be an arbitrary triangle from $\text{Til} \left( T_i^N \right)$. If $T$ lies completely in the interior of $D_i^N$ then we include it into triangulation $\triangle_{N;i}$. Otherwise, we split every polygon, which is intersection of $D_i^N \cap T$, into at most five triangles without adding new vertices and include them into triangulation $\triangle_{N;i}$. Note that the number of triangles in $\triangle_{N;i}$ does not exceed $10r_2^2$.

d) Finally, let $i \in I_4(\varepsilon;N)$. Similarly to the case b), we subdivide the square $D_i^N$ into squares of the size $m_i^{-1} r_2^{-1} \times m_i^{-1} r_2^{-1}$ whose sides are parallel to the sides of $D_i^N$. Then inside each small square we draw one of its diagonals. Thus, we obtained the triangulation $\triangle_{N;i}$ of $D_i^N$ consisting of $2r_2$ isosceles right triangles (denoted by $T_i^N$).

Let us estimate the overall number $\tilde{N}$ of triangles in above-constructed triangulations $\triangle_{N;i}$ to the choice of numbers $r_1$ and $r_2$ we obtain

$$\tilde{N} \leq \sum_{i \in I_1(\varepsilon;N)} \left( n_i^N + o \left( m_i^{-2} N \right) \right) + \sum_{i \in I_2(\varepsilon;N)} 2r_1^2 + \sum_{i \in I_3(\varepsilon;N)} 10r_2^2 + \sum_{i \in I_4(\varepsilon;N)} 2r_2^2 \leq (1 - 23\varepsilon - 2\mu (G_0;2\varepsilon))N + o(N) + 2\mu (G_0;2\varepsilon) N + 12m_i^2 N r_2^2 \leq (1 - 23\varepsilon) N + o(N) + 12\varepsilon N = (1 - 11\varepsilon) N + o(N) \quad \text{as} \quad N \to \infty.$$
Optimal $L_p$-error of asymmetric linear spline approximation

Now we "glue" triangulations $\Delta_{N;i}$ according to the following rule:

1. We include into triangulation $\Delta_{N;i}$ every triangle $T \in \Delta_{N;i}$, $i = 1, \ldots, m_N^2$, which does intersect with the boundary of the square $D_N^i$.

2. For every $i = 1, \ldots, m_N^2$, denote by $W_i^N$ the set of the vertices of triangulation $\Delta_{N;i}$ which are on the boundary of $D_N^i$. For arbitrary $i, j = 1, \ldots, m_N^2$, $i \neq j$, we set $S_{i,j} = D_N^i \cap D_N^j$.

3. We subdivide every triangle $T \in \Delta_{N;i}$ that has non-empty intersection with $S_{i,j}$ by joining the vertices of $T$ with the points from $W_N^j \cap T$. Finally, we include all obtained triangles into triangulation $\Delta_{N;i}$.

Let us estimate the number $\hat{N}$ of triangles in $\Delta_{N;i}$. By $\#A$ we denote the number of points in a finite set $A$. Then

$$\hat{N} \leq \tilde{N} + \sum_{i=1}^{m_N^2} \#(W_i^N).$$

Note that $\#(W_i^N) \leq 10r_N^2$ for every $i \in I_3(\varepsilon; N)$ and $\#(W_i^N) = o(m_N^{-2}N)$ otherwise. Hence,

$$\hat{N} \leq (1 - 11\varepsilon)N + o(N) + m_N^2 \cdot (10r_N^2 + o(m_N^{-2}N)) \leq (1 - \varepsilon)N + o(N).$$

Therefore, $\hat{N} \leq N$ for all $N$ large enough.

Now we are ready to construct the "nearly" optimal spline $s_N^\varepsilon$ on the triangulation $\Delta_{N;i}^\varepsilon$. For every $i \in I_1(\varepsilon; N)$ and $T \in \Delta_{N;i}^\varepsilon \cap \text{int} D_N^i$, let $s_N^\varepsilon$ be the sum of two linear polynomials: $f_{N,i} - Q_{N,i}$, polynomial of the best approximation of $f_{N,i}$ on triangle $T$. Then due to Remark 4 we see that $s_N^\varepsilon$ is continuous on the union of interior triangles from $\Delta_{N;i}^\varepsilon \cap \text{int} D_N^i$.

Finally, let $s_N^\varepsilon$ interpolate the function $f$ at the remaining vertices of $\Delta_{N;i}$, i.e. at the vertices located in the interior of squares $D_N^i$ with $i \in I_2(\varepsilon; N) \cup I_3(\varepsilon; N) \cup I_4(\varepsilon; N)$ as well as at the vertices located along the boundaries of all $D_N^i$'s. This would automatically "glue" the spline $s_N^\varepsilon$.

Therefore, for every sufficiently small $\varepsilon > 0$, there exists $N(\varepsilon) \in \mathbb{N}$ such that for each $N > N(\varepsilon)$ we have constructed the triangulation $\Delta_{N;i}^\varepsilon$ with at most $N$ triangles and corresponding continuous piecewise linear spline $s_N^\varepsilon \in S(\Delta_{N;i}^\varepsilon)$.

5.3. The proof of estimate from above

In this subsection we will prove that

$$\lim_{\varepsilon \to 0} \left( \liminf_{N \to \infty} N \cdot \|f - s_N^\varepsilon\|_{p;\alpha,\beta} \right) \leq 2^{-1} C_{p;\alpha,\beta} \frac{\sqrt{H}}{\frac{p}{p+1}}.$$
Due to the triangle inequality, we have

$$\Omega_i^N := \{(x, y) \mid \exists T \in \tilde{\Delta}_i^N \text{ such that } (x, y) \in T\}.$$

Let us find the upper estimates for the quantity $\|f - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(D_i^N)}$ separately in two situations: 1) $i \in \{1, \ldots, m_2^N\} \setminus I_3(\varepsilon; N)$ and 2) $i \in I_3(\varepsilon; N)$.

1) As a first step let $i$ be an arbitrary index from the set $\{1, \ldots, m_2^N\} \setminus I_3(\varepsilon; N)$. Note that

$$\|f - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(D_i^N)}^p = \|f - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(\Omega_i^N)}^p + \sum_{T \in \Delta_i^N \setminus \tilde{\Delta}_i^N} \|f - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(T)}^p.$$

Due to the triangle inequality, we have

$$\|f - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(\Omega_i^N)} \leq \|f - f_{N;i}\|_{L_{p;\alpha,\beta}(\Omega_i^N)} + \|f_{N;i} - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(\Omega_i^N)}.$$

We estimate the first term in the right-hand side of (6.7) with the help of Lemma 5.1

$$\|f - f_{N;i}\|_{L_{p;\alpha,\beta}(\Omega_i^N)} \leq \max\{\alpha; \beta\} \|f - f_{N;i}\|_{L_{\infty}(D_i^N)} |D_i^N|^{1/p} \leq \max\{\alpha; \beta\} m_N^{-2-2/p} \omega_2(f, m_N^{-1}) \leq \varepsilon m_N^{-2/p} N.$$

As for the second term in the right-hand side of (6.7), we observe that

$$\|f_{N;i} - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(\Omega_i^N)}^p = \sum_{T \in \tilde{\Delta}_i^N} \|f_{N;i} - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(T)}^p = \left(\#\tilde{\Delta}_i^N\right) \|f_{N;i} - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(T_i^N)}^p.$$

In order to obtain upper estimates for $\|f_{N;i} - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(T_i^N)}$ we should consider three cases: a) $i \in I_1(\varepsilon; N)$, b) $i \in I_2(\varepsilon; N)$, and c) $i \in I_4(\varepsilon; N)$.

a) First, we assume that $i \in I_1(\varepsilon; N)$. Due to the algorithm for construction of triangles $\tilde{\Delta}_i^N(\varepsilon)$ described in the previous subsection and due to Corollary 4.2 we have that $\#\tilde{\Delta}_i^N \leq \varepsilon^2 m_N^{2+2/p} (n_i^N)^{1+1/p}$. Therefore

$$\|f_{N;i} - s_N^\varepsilon\|_{L_{p;\alpha,\beta}(T_i^N)} = 2^{-1} C_{p;\alpha,\beta} H^{1/2}(f; x_i^N, y_i^N) |T_i^N|^{1+1/p} = C_{p;\alpha,\beta} H^{1/2}(f; x_i^N, y_i^N) 2 m_N^{2+2/p} (n_i^N)^{1+1/p}.$$
From here and from (5.9), we obtain
\[ \|f_{N;i} - s_N^\varepsilon\|_{L_p;\alpha,\beta(\Omega_i^N)}^p \leq 2^{-p} C^{P}_{p;\alpha,\beta} \cdot H^{P/2} \left( f; x_i^N, y_i^N \right) \cdot m_N^{-(p+1)} \left( n_i^N \right)^{-p}. \]  

b) Next, we assume that \( i \in I_2(\varepsilon; N) \). Then \( \# \tilde{\Delta}_i^N \leq 2r_i^2 \). For every triangle \( T \) and continuous on \( T \) function \( g \), by \( s_{g,T} \) we denote the linear function interpolating \( g \) at the vertices of \( T \). We have
\[ \left| \frac{\partial^2 f_{N;i}}{\partial \pi^2} \right| \leq \lambda_{\text{max};i}^N \] for every unit vector \( \pi \), after change of variables, we have (see [31, 24])
\[ \|f_{N;i} - s_N^\varepsilon\|_{L_p;\alpha,\beta(T_i^N)} \leq \frac{\max\{\alpha; \beta\} \lambda_{\text{max};i}^N}{2} \left( \int_{T_i^N} \left| q(x, y) - s_{q,T_i^N}(x, y) \right|^p \, dx \, dy \right)^{1/p} \leq 2^{-1} k_1 \left( m^{2r_i^2}_N \right)^{-1-1/p} \leq k_1 \left( m^{2r_i^2}_N \right)^{-1-1/p}. \]

Here we recall that \( q(x, y) = x^2 + y^2 \), and denote by \( k_1, k_2, \ldots \) constants that are independent of \( \varepsilon \) and \( \alpha \). Therefore,
\[ \|f_{N;i} - s_N^\varepsilon\|_{L_p;\alpha,\beta(\Omega_i^N)}^p \leq 2k_1^p m_N^{-2} N^p. \]  
c) Finally, let \( i \in I_4(\varepsilon; N) \). Then like to the previous case we obtain that \( \# \tilde{\Delta}_i^N \leq 2r_i^2 \) and
\[ \|f_{N;i} - s_N^\varepsilon\|_{L_p;\alpha,\beta(\Omega_i^N)}^p \leq 2k_1^p \varepsilon^p m_N^{-2} N^p. \]

The analysis of three above cases is complete. Next we will estimate the deviation of spline \( s_N^\varepsilon \) from the function \( f \) on arbitrary triangle \( T \in \Delta_i^N \setminus \tilde{\Delta}_i^N \). For every such triangle,
\[ \|f - s_N^\varepsilon\|_{L_p;\alpha,\beta(T)} \leq \max\{\alpha; \beta\} \left( \|f - s_{f,T}\|_{L_p(T)} + \|s_{f,T} - s_N^\varepsilon\|_{L_p(T)} \right). \]

By the triangle inequality we obtain that
\[ \|f - s_{f,T}\|_{L_p(T)} \leq \|f - f_{N;i} - s_{f,N;i,T}\|_{L_p(T)} + \|f - f_{N;i}\|_{L_p(T)} + \|s_{f,T} - s_{f,N;i,T}\|_{L_p(T)} \].

In view of Lemma 5.1 we have that
\[ \|f - f_{N;i}\|_{L_p(T)} \leq \|f - f_{N;i}\|_{L_\infty(T)} |T|^{1/p} \leq m_N^{-2} \omega_2 \left( f, m_N^{-1} \right) |T|^{1/p} = O \left( N^{-1-1/p} \right), \]
\[ \|s_{f,T} - s_{f,N;i,T}\|_{L_p(T)} \leq \|f - f_{N;i}\|_{L_\infty(T)} |T|^{1/p} = O \left( N^{-1-1/p} \right), \]
\[
\|s_f,T - s_N^\varepsilon\|_{L_p(T)} \leq \|f - f_{N;i}\|_{L_\infty(T)}|T|^{1/p} = O\left(N^{-1-1/p}\right),
\]
as \(N \to \infty\). In addition, by Corollary 4.2
\[
\|f_{N;i} - s_{f_{N;i},T}\|_{L_p(T)} \leq \|f_{N;i} - s_{f_{N;i},T_i^N}\|_{L_p(T_i^N)} \leq 2^{-1} \lambda_{\max;i}^{N} \|q - s_{q,T_i^N}\|_{L_p(T_i^N)}.
\]
Due to inequality (5.4) in the case \(i \in I_1(\varepsilon; N)\) we have
\[
\|q - s_{q,T_i^N}\|_{L_p(T_i^N)} \leq O\left(N^{-1-1/p}\right).
\]
In the case \(i \in I_2(\varepsilon; N) \cup I_4(\varepsilon; N)\) we have
\[
\|q - s_{q,T_i^N}\|_{L_p(T_i^N)} = k_2 m_N^{-2-2/p} f_1^{-2-2/p} = O\left(N^{-1-1/p}\right).
\]
Therefore,
\[
\|f - s_N^\varepsilon\|_{L_p(\alpha, \beta)(T)} = O\left(N^{-1-1/p}\right), \quad N \to \infty.
\]
Let us remind that we are considering the case when \(i \in \{1, \ldots, m_N^2\} \setminus I_3(\varepsilon; N)\). Due to the algorithm for construction of triangulation \(\Delta_N(\varepsilon)\), we conclude that the number of triangles \(\Delta_i^N\) is \(o(m_N^{-2}N)\) as \(N \to \infty\). This implies that
\[
\sum_{T \in \Delta_i^N} \|f - s_N^\varepsilon\|_{L_p(\alpha, \beta)(T)} = o\left(m_N^{-2}N^{-p}\right).
\]
2) Now let us consider \(i \in I_3(\varepsilon; N)\). It can be easily seen that
\[
\|f - s_N^\varepsilon\|_{L_p(\alpha, \beta)(D_i^N)} \leq (\max\{\alpha, \beta\})^p \sum_{T \in \Delta_i^N} \|f - s_N^\varepsilon\|_{L_p(T)}.
\]
Let \(T\) be an arbitrary triangle in \(\Delta_i^N\). Then
\[
\|f - s_N^\varepsilon\|_{L_p(T)} \leq \|f - f_{N;i}\|_{L_p(T)} + \|f_{N;i} - s_{f_{N;i},T}\|_{L_p(T)} + \|s_N^\varepsilon - s_{f_{N;i},T}\|_{L_p(T)}.
\]
With the help of Lemma 5.1 we obtain the following upper estimates
\[
\|f - f_{N;i}\|_{L_p(T)} \leq \|f - f_{N;i}\|_{L_\infty(T)}|T|^{1/p} \leq \varepsilon N^{-1} \cdot (2m_N^2 r_2)^{-1/p},
\]
Let us estimate each of four terms in (5.17) independently.

Hence, taking into account (5.2), we obtain
\[
\| f_{N;i} - s_{f_{N;i},T} \|_{L_p(T)}^p \leq \int_0^1 \int_0^1 \left| \bar{q}(x, y) - s_{\bar{q}, T_i}(x, y) \right|^p dy dx \leq \int_0^1 \int_0^1 \left[ \omega \left( \frac{\lambda_{\min, i} m^{-1}_N}{m^2 N r_2} \right) u^2 + \frac{\lambda_{\max} \|u\|_{L^\infty}}{m^2 N r_2} v^2 - s_{\bar{q}, T_i}(x, y) \right]^p dx \leq \frac{\omega^p \left( \frac{\lambda_{\min, i} m^{-1}_N}{m^2 N r_2} \right)}{m^{2(p+1)} N^{2(p+2)}} \cdot \int_0^1 \int_0^1 |u^2 - v|^p dv.
\]

Hence, taking into account (5.2), we obtain
\[
\| f_{N;i} - s_{f_{N;i},T_i} \|_{L_p(T)}^p \leq \frac{k_3}{(m^2 N r_2)^{1/p}} \left( \frac{m^{-2}_N \omega \left( \frac{\lambda_{\min, i} m^{-1}_N}{m^2 N r_2} \right)}{\epsilon N r_2^2} \right) \leq \frac{k_3 (\epsilon + o(1))}{N \cdot (m^2 N r_2)^{1/p}}
\]
as \( N \to \infty \).

Therefore, we arrive at the following estimate in the case when \( i \in I_3(\epsilon; N) \)
\[
\| f - s_N \|_{L_p(T)} \leq \frac{2 \epsilon}{N^{2/p}} + \frac{1}{N^{1/p}} + \frac{k_3 (\epsilon + o(1))}{N \cdot (m^2 N r_2)^{1/p}} \leq \frac{(k_3 + 2) (\epsilon + o(1))}{N (m^2 N r_2)^{1/p}}.
\]

Now, we are ready to prove inequality (5.5). Indeed,
\[
\| f - s_N \|_{L_p(\alpha, \beta)}^p = \sum_{i=1}^{N} \| f - s_N \|_{L_p(\alpha, \beta)(D_i^N)}^p = \sum_{j=1}^{q} \sum_{i \in I_j(\epsilon; N)} \| f - s_N \|_{L_p(\alpha, \beta)(D_i^N)}^p.
\]

Let us estimate each of four terms in (5.17) independently.

1) Combining inequalities (5.6), (5.7), (5.8), (5.10) and (5.13) we see that for every \( i \in I_1(\epsilon; N) \)
\[
\| f - s_{N} \|_{L_p(\alpha, \beta)(D_i^N)}^p \leq \left[ \frac{C_{p, \alpha, \beta}}{2} \cdot \frac{H^{1/2} \left( f; x_i^N, y_i^N \right)}{m^2 N r_{i}^{p+1}} + \frac{\epsilon}{N m^{p/2}} \right]^p \approx \frac{C_{p, \alpha, \beta}}{2} \cdot \frac{H^{p/2} \left( f; x_i^N, y_i^N \right)}{m^2 N r_{i}^{p+1} (n_i^N)^p} + \frac{k_4 \epsilon}{N \cdot m^2 N} + o \left( \frac{1}{N m^2 N} \right), \quad N \to \infty.
\]
From the latter inequality, definition of numbers $n_i^N$, and the Riemann integrability of $\sqrt{H(f;x,y)}$ we obtain

$$
\sum_{i \in I_1(\varepsilon;N)} \| f - s_N^\varepsilon \|^p_{L^p(\alpha,\beta)(D_i^N)} \leq \frac{C_{p;\alpha,\beta}^p}{2^p m_N^{2(p+1)}} \sum_{i \in I_1(\varepsilon;N)} \frac{H^{p/2}(f;x_i^N,y_i^N)}{(n_i^N)^p} + k_4 \varepsilon + o(1)
$$

as $N \to \infty$.

2) Let $i \in I_2(\varepsilon;N)$. In view of inequalities (5.6), (5.7), (5.8), (5.11), and (5.13) we obtain

$$
\| f - s_N^\varepsilon \|^p_{L^p(\alpha,\beta)(D_i^N)} \leq \left( \left( 2^{1/p} k_1 + \varepsilon \right)^p + o(1) \right) m_i^{-2} N^{-p}, \quad N \to \infty.
$$

From this we derive that

$$
\sum_{i \in I_2(\varepsilon;N)} \| f - s_N^\varepsilon \|^p_{L^p(\alpha,\beta)(D_i^N)} \leq \mu(G_{0;2\varepsilon}) \left( \left( 2^{1/p} k_1 + \varepsilon \right)^p + o(1) \right) N^{-p}, \quad N \to \infty.
$$

3) Combining inequalities (5.14) and (5.15), we obtain that

$$
\sum_{i \in I_3(\varepsilon;N)} \| f - s_N^\varepsilon \|^p_{L^p(\alpha,\beta)(D_i^N)} \leq \left( \max\{\alpha,\beta\} \right)^p \sum_{i \in I_3(\varepsilon;N)} \sum_{T \in \Delta_i^N} \| f - s_N^\varepsilon \|^p_{L^p(T)} \leq \left( \max\{\alpha,\beta\} \right)^p \sum_{i \in I_3(\varepsilon;N)} \sum_{T \in \Delta_i^N} \frac{(k_3 + 2)^p (\varepsilon + o(1))^p}{N^p m_i^2 r_i^2}
$$

$$
= \left( \max\{\alpha,\beta\} \right)^p \cdot 10 r_i^2 m_i^2 \frac{(k_3 + 2)^p (\varepsilon + o(1))^p}{N^p m_i^2 r_i^2} = 10 \frac{\left( \max\{\alpha,\beta\} \right) (k_3 + 2) \varepsilon^p + o(1)}{N^p} \text{ as } N \to \infty.
$$

4) Finally, in view of inequalities (5.6), (5.7), (5.8), (5.12), and (5.13) we conclude that

$$
\sum_{i \in I_4(\varepsilon;N)} \| f - s_N^\varepsilon \|^p_{L^p(\alpha,\beta)(D_i^N)} \leq ((2k_1^p + 1)\varepsilon^p + o(1)) N^{-p}, \quad N \to \infty.
$$
Now, we combine estimates (5.17), (5.18), (5.19), (5.20), and (5.21). As a result we obtain
\[
\|f - s^\epsilon_N\|_{p;\alpha,\beta}^p \leq \frac{C^p_{p;\alpha,\beta}}{2^p (1 - 23\varepsilon - 2\mu (G_{0;2\varepsilon}))^p N^p} \left\| \sqrt{H} \right\|_{\frac{p}{p+1}}^p N^p + \frac{k_4\varepsilon + \mu (G_{0;2\varepsilon}) \cdot (2k_1 + \varepsilon)^p}{N^p} + 
10 (\max\{\alpha; \beta\})^p (k_3 + 2)^p \varepsilon^p + (2k_1 + 1)^p \varepsilon^p + o(1), \quad N \to \infty.
\]

Therefore,
\[
\limsup_{N \to \infty} N^p \cdot \|f - s^\epsilon_N\|_{p;\alpha,\beta}^p \leq \frac{C^p_{p;\alpha,\beta}}{2^p (1 - 23\varepsilon - 2\mu (G_{0;2\varepsilon}))^p} \left\| \sqrt{H} \right\|_{\frac{p}{p+1}} + k_4\varepsilon + \mu (G_{0;2\varepsilon}) \cdot (2k_1 + \varepsilon)^p + 
10 (\max\{\alpha; \beta\})^p (k_3 + 2)^p \varepsilon^p + (2k_1 + 1)^p \varepsilon^p.
\]

The latter upper estimate implies inequality (5.17).

§6. Error of asymmetric approximation of $C^2$ functions by linear splines: lower estimate

To prove the lower estimate of the optimal error, we need the following lemma. We omit the proof here as the lemma itself is rather evident.

**Lemma 6.1.** Let $T$ be an arbitrary triangle. Then for every function $f \in C^2(T)$, $H(f; x, y) \geq K$ on $T$, it follows that there exists a constant $\Upsilon_f > 0$ (independent of $T$) such that
\[
E(f, P_1)_{L^p;\alpha,\beta(T)} \geq K\Upsilon_f (\text{diam} T)^{2/|T|^{1/p}}.
\]

Let the number $1 \leq p < \infty$ be fixed. In this section we develop ideas of the paper [6] to prove for every function $f : D \to \mathbb{R}$ with nonnegative Hessian the following inequality holds true
\[
\liminf_{N \to \infty} N : R_N(f, L^p;\alpha,\beta) \geq 2^{-1} C^p_{p;\alpha,\beta} \left\| \sqrt{H} \right\|_{\frac{p}{p+1}}.
\]

For every $\varepsilon > 0$, we define the sets $A_\varepsilon$ and $F_\varepsilon$ in the following way
\[
A_\varepsilon := \{(x, y) \in D : H(f; x, y) < \varepsilon\},
\]
\[
F_\varepsilon := D \setminus A_\varepsilon = \{(x, y) \in D : H(f; x, y) \geq \varepsilon\}.
\]
For an arbitrary triangle $T$ in the plane, denote by $\text{diam} T$ and $|T|$ the length of the longest side and the area of $T$, respectively. In addition, let $U_T$ be an arbitrary point inside $T$.

Let $N \in \mathbb{N}$ and let $\triangle = \{T_i\}_{i=1}^N$ be an arbitrary triangulation of the square $D$ consisting of $N$ triangles. We need to distinguish (in triangulation $\triangle$) several types of triangles: normal, extra-long, and the triangles where the Hessian of function $f$ is relatively small. To this end for $N \in \mathbb{N}$, we set $I_N := \{1, \ldots, N\}$, and define the following five subsets of $I_N$:

- $M_1(\triangle; \varepsilon) := \{i \in I_N : T_i \subset A_{2\varepsilon}\}$;
- $M_2(\triangle; \varepsilon) := \left\{ i \in I_N : T_i \subset F_{\varepsilon}, \frac{(\text{diam} T_i)^2 \omega_2(f, \text{diam} T_i)}{\sqrt{H(f; U_T)|T_i|}} \leq \frac{\varepsilon C_{p,\alpha,\beta}}{4 \max\{\alpha; \beta\}} \right\}$;
- $M_3(\triangle; \varepsilon) := \left\{ i \in I_N : T_i \subset F_{\varepsilon}, \frac{(\text{diam} T_i)^2 \omega_2(f, \text{diam} T_i)}{\sqrt{H(f; U_T)|T_i|}} > \frac{\varepsilon C_{p,\alpha,\beta}}{4 \max\{\alpha; \beta\}}, \text{diam} T_i \leq \varepsilon \right\}$;
- $M_4(\triangle; \varepsilon) := \left\{ i \in I_N : T_i \subset F_{\varepsilon}, \text{diam} T_i > \varepsilon N^{-1/4} \right\}$;
- $M_5(\triangle; \varepsilon) := \left\{ i \in I_N : T_i \cap A_{\varepsilon} \neq \emptyset, T_i \cap F_{2\varepsilon} \neq \emptyset \right\}$.

According to the given definition, every index $i \in I_N$ belongs to at least one (possibly more) sets $M_j(\triangle; \varepsilon), j = 1, \ldots, 5$.

Note that the set $M_3(\triangle; \varepsilon)$ consists of narrow triangles while the sets $M_4(\triangle; \varepsilon)$ and $M_5(\triangle; \varepsilon)$ consists of extra long triangles. In the next three propositions we will show that the overall number of these “bad” triangles in “nearly” optimal triangulation $\triangle$ is relatively small.

**Lemma 6.2.** Let $\varepsilon > 0$ and let $\{\triangle_N\}_{N=1}^\infty$ be the sequence of triangulations $\triangle_N = \{T_i^N\}_{i=1}^N$ such that

$$\liminf_{N \to \infty} \left( N \cdot \inf_{s \in S(\triangle_N)} \|f - s\|_{p,\alpha,\beta} \right) < \infty$$

then for all $N$ large enough

$$\sum_{i \in M_3(\triangle_N; \varepsilon)} |T_i^N| < \varepsilon.$$

**Proof.** Let number $\varepsilon > 0$ be fixed. Assume to the contrary that there exists a subsequence $\{\triangle_{N_k}\}$ of positive integers, such that $N_k \to \infty$ as $k \to \infty$, and

$$\sum_{i \in M_3(\triangle_{N_k}; \varepsilon)} |T_i^{N_k}| \geq \varepsilon.$$
Optimal $L_p$-error of asymmetric linear spline approximation

Without loss of generality we let $N_k = k$ for every $k \in \mathbb{N}$. Applying Lemma 6.1 and the definition of the set $M_3(\Delta_N; \varepsilon)$, for every $N \in \mathbb{N}$ and $i \in M_3(\Delta_N; \varepsilon)$, we obtain

$$E(f, \mathcal{P}_1)_{L_p, \alpha, \beta}(T_i^N) \geq \varepsilon \mathcal{Y}_f \left( \text{diam } T_i^N \right)^2 |T_i^N|^{1/p} > \varepsilon \mathcal{Y}_f \cdot \frac{\varepsilon C_p, \alpha, \beta}{4 \max \{\alpha; \beta\}} \cdot \frac{H^{1/2} \left( f; U_{T_i^N} \right) |T_i^N|^{1+1/p}}{\omega_2 \left( f, \text{diam } T_i^N \right)} =: \frac{c_2 H^{1/2} \left( f; U_{T_i^N} \right) |T_i^N|^{1+1/p}}{\omega_2 \left( f, \text{diam } T_i^N \right)}.$$  

Here $c_2 = c_2(\varepsilon)$ is the constant independent of $N$. Then

$$\mathcal{F}(\Delta_N) := \inf_{s \in S(\Delta_N)} \| f - s \|^p_{L_p, \alpha, \beta} \geq \sum_{i=1}^N E_p \left( f, \mathcal{P}_1 \right)_{L_p, \alpha, \beta}(T_i^N) \geq \sum_{i \in M_3(\Delta_N; \varepsilon)} E_p \left( f, \mathcal{P}_1 \right)_{L_p, \alpha, \beta}(T_i^N) \geq c_2^p \sum_{i \in M_3(\Delta_N; \varepsilon)} \frac{H^{p/2} \left( f; U_{T_i^N} \right) |T_i^N|^{1+1/p}}{\omega_2 \left( f, \text{diam } T_i^N \right)} \geq \frac{c_2^p \varepsilon^{p/2}}{\omega_2 \left( f, \varepsilon N^{-1/4} \right)} \sum_{i \in M_3(\Delta_N; \varepsilon)} |T_i^N|^{p+1} \geq \frac{c_2^p \varepsilon^{1+3p/2}}{N^p \omega_2 \left( f, \varepsilon N^{-1/4} \right)} |\# M_3(\Delta_N; \varepsilon)|^{-p} \geq \frac{c_2^p \varepsilon^{1+3p/2}}{N^p \omega_2 \left( f, \varepsilon N^{-1/4} \right)} |\# M_3(\Delta_N; \varepsilon)|^{-p} \geq \frac{c_2^p \varepsilon^{1+3p/2}}{N^p \omega_2 \left( f, \varepsilon N^{-1/4} \right)}.$$  

Using convexity of the function $t^{p+1}$ and assumption (6.4), we have

$$\mathcal{F}(\Delta_N) \geq \frac{c_2^p \varepsilon^{p/2}}{\omega_2 \left( f, \varepsilon N^{-1/4} \right)} \sum_{i \in M_3(\Delta_N; \varepsilon)} |T_i^N|^{p+1} \geq \frac{c_2^p \varepsilon^{1+3p/2}}{N^p \omega_2 \left( f, \varepsilon N^{-1/4} \right)} \geq \frac{c_2^p \varepsilon^{1+3p/2}}{N^p \omega_2 \left( f, \varepsilon N^{-1/4} \right)} \geq \frac{c_2^p \varepsilon^{1+3p/2}}{N^p \omega_2 \left( f, \varepsilon N^{-1/4} \right)}.$$  

Therefore,

$$\liminf_{N \to \infty} N \cdot \inf_{s \in S(\Delta_N)} \| f - s \|_{p, \alpha, \beta} \geq c_2^p \varepsilon^{3/2+1/p} \cdot \omega_2^{-1} \left( f, \varepsilon N^{-1/4} \right) = +\infty.$$  

The latter contradicts the assumption (6.2). The lemma is proved.

**Lemma 6.3.** Let $\varepsilon > 0$ and let $\{\Delta_N\}_{N=1}^\infty$ be the sequence of triangulations $\Delta_N = \{T_i^N\}_{i=1}^N$. If (6.2) holds true then for all $N$ large enough inequality (6.3) also holds true.

**Proof.** Let number $\varepsilon > 0$ be fixed. Assume to the contrary that there exists a subsequence $\{N_k\}_{k=1}^\infty$ of positive integers, such that $N_k \to \infty$ as $k \to \infty$ and inequality (6.3) holds true. Without loss...
Lemma 6.4. Let \( \varepsilon > 0 \) and let \( \{\Delta_N\}_{N=1}^{\infty} \) be the sequence of triangulations \( \Delta_N = \{T_i^N\}_{i=1}^{N} \) such that \( \sup_{N} |\Delta_N| = \mathcal{O}(N) \). Assume to the contrary that there exists a subsequence \( \{N_k\}_{k=1}^{\infty} \) of positive integers, such that \( N_k \to \infty \) as \( k \to \infty \) and inequality (6.4) holds true. Without loss of generality we let \( N_k \) be fixed and let \( T \in \mathbb{N} \), we have

\[
\mathcal{F}(\Delta_N) := \inf_{s \in \mathbb{S}(\Delta_N)} \|f - s\|_{L^p(\mu)}^p \geq \sum_{i=1}^{N} E^p(f, \mathcal{P}_1)_{L^p(\mu, \beta)}(T_i^N) \geq \sum_{i \in M_5(\Delta_N; \varepsilon)} \varepsilon E^p(f, \mathcal{P}_1)_{L^p(\mu, \beta)}(T_i^N) \geq \varepsilon^p \mathcal{Y}_f K_{\delta_0} \sum_{i \in M_5(\Delta_N; \varepsilon)} (\text{diam } T_i^N)^{2p} |T_i^N| \geq \varepsilon^p \mathcal{Y}_f K_{\delta_0}^{p+2} (2\delta_0)^{2p}.
\]

Therefore,

\[
\liminf_{N \to \infty} N \cdot \inf_{s \in \mathbb{S}(\Delta_N)} \|f - s\|_{L^p(\mu)}^p \geq \varepsilon^{3+1/p} \mathcal{Y}_f \lim_{N \to \infty} \sqrt{N} = +\infty.
\]

The latter contradicts to assumption (6.2) of lemma. The lemma is proved.
Optimal $L_p$-error of asymmetric linear spline approximation

Therefore,
\[
\liminf_{N \to \infty} N \cdot \inf_{s \in S(\Delta_N)} \| f - s \|_{p;\alpha,\beta} = +\infty
\]
which contradicts to assumption (6.2) of lemma. The lemma is proved.

Now we have all facts needed to prove the lower estimate (6.1). To that end we need to show for every sequence \( \{\Delta N\}_{N=1}^{\infty} \) of triangulations \( \Delta N = \{T_i^N\}_{i=1}^N \) of \( D \),
\[
\liminf_{N \to \infty} N \cdot \inf_{s \in S(\Delta_N)} \| f - s \|_{p;\alpha,\beta} \geq 2^{-1} C_{p;\alpha,\beta} \| \sqrt{H} \|_{p+1}.
\]
Without loss of generality we consider only those sequences \( \{\Delta N\}_{N=1}^{\infty} \) for which
\[
\liminf_{N \to \infty} N \cdot \inf_{s \in S(\Delta_N)} \| f - s \|_{p;\alpha,\beta} < \infty.
\]
For every \( i \in M_2(\Delta_N;\varepsilon) \) we substitute the function \( f \) on the triangle \( T_i^N \) by its Taylor polynomial of second order \( f_{N,i} \) constructed at the point \( U_{T_i^N} \). In view of Lemma 5.1 we have
\[
\| f - f_{N,i} \|_{L_{p;\alpha,\beta}(T_i^N)} \leq \max\{\alpha; \beta\} \| f - f_{N,i} \|_{L_{\infty}(T_i^N)} |T_i^N|^{1/p} \\
\leq 2 \max\{\alpha; \beta\} (\text{diam } T_i^N)^2 \omega_2(f, \text{diam } T_i^N) |T_i^N|^{1/p}.
\]
By the definition of the set \( M_2(\Delta_N;\varepsilon) \),
\[
\| f - f_{N,i} \|_{L_{p;\alpha,\beta}(T_i^N)} \leq 2^{-1} \varepsilon C_{p;\alpha,\beta} \cdot H^{1/2} \left( f; U_{T_i^N} \right) |T_i^N|^{1+1/p}.
\]
Applying the triangle inequality and Corollary 12 we obtain
\[
E(f, \mathcal{P}_1)_{L_{p;\alpha,\beta}(T_i^N)} \geq E(f_{N,i}, \mathcal{P}_1)_{L_{p;\alpha,\beta}(T_i^N)} - \| f - f_{N,i} \|_{L_{p;\alpha,\beta}(T_i^N)} \\
\geq 2^{-1} (1 - \varepsilon) C_{p;\alpha,\beta} \cdot H^{1/2} \left( f; U_{T_i^N} \right) |T_i^N|^{1+1/p}.
\]
Now,
\[
\mathcal{F}(\Delta_N) := \inf_{s \in S(\Delta_N)} \| f - s \|_{p;\alpha,\beta} \geq \sum_{i=1}^N E^p(f, \mathcal{P}_1)_{L_{p;\alpha,\beta}(T_i^N)} \geq \sum_{i \in M_2(\Delta_N;\varepsilon)} E^p(f, \mathcal{P}_1)_{L_{p;\alpha,\beta}(T_i^N)} \\
\geq 2^{-p} (1 - \varepsilon) C_{p;\alpha,\beta} \sum_{i \in M_2(\Delta_N;\varepsilon)} H^{p/2} \left( f; U_{T_i^N} \right) |T_i^N|^{p+1}.
\]
Application of the Jensen inequality for the function $t^{p+1}$ implies that

\[
F(\Delta_N) \geq \frac{(1 - \varepsilon)^p C_{p,\alpha,\beta}^p}{2^p (\#M_2(\Delta_N; \varepsilon))^p} \left( \sum_{i \in M_2(\Delta_N; \varepsilon)} H^{\frac{p}{2(p+1)}} \left( f; U_{T_i^N} \right) |T_i^N| \right)^{p+1} \\
\geq \frac{(1 - \varepsilon)^p C_{p,\alpha,\beta}^p}{2^p N^p} \left( \sum_{i \in M_2(\Delta_N; \varepsilon)} H^{\frac{p}{2(p+1)}} \left( f; U_{T_i^N} \right) |T_i^N| \right)^{p+1}.
\]

Let us subdivide each triangle $T_i^N$, $i \in I_N \setminus M_2(\Delta_N; \varepsilon)$, into $n_i^N$ smaller triangles $T_{i,j}^N$, $j = 1, \ldots, n_i^N$, enumerated in arbitrary order, such that $\text{diam} \ T_{i,j}^N \to 0$ as $N \to \infty$ for all $j = 1, \ldots, n_i^N$. For $i \in M_2(\Delta_N; \varepsilon)$, set $n_i^N := 1$ and $T_{i,1}^N := T_i^N$. Observe that $\bigcup_{i=1}^N n_i^N = D$, and for every appropriate $i$ and $j$, it follows that $\text{diam} \ T_{i,j}^N \to 0$ as $N \to \infty$. Then

\[
\sum_{i \in M_1(\Delta_N; \varepsilon)} \sum_{j=1}^{n_i^N} H^{\frac{p}{2(p+1)}} \left( f; U_{T_{i,j}^N} \right) |T_{i,j}^N| \leq (2\varepsilon)^p \sum_{i \in M_1(\Delta_N; \varepsilon)} |T_i^N| \leq (2\varepsilon)^p \cdot 1 = (2\varepsilon)^p.
\]

In addition, by Lemmas 6.2, 6.3 and 6.4 for all $N$ large enough we have for $r = 3, 4, 5,$

\[
\sum_{i \in M_r(\Delta_N; \varepsilon)} \sum_{j=1}^{n_i^N} H^{\frac{p}{2(p+1)}} \left( f; U_{T_{i,j}^N} \right) |T_{i,j}^N| \leq \|H\|_{\frac{p}{2(p+1)}} \sum_{i \in M_r(\Delta_N; \varepsilon)} |T_i^N| \leq \varepsilon \|H\|_{\frac{p}{2(p+1)}}.
\]

Therefore,

\[
F(\Delta_N) \geq \frac{(1 - \varepsilon)^p C_{p,\alpha,\beta}^p}{2^p N^p} \left( \sum_{i=1}^N \sum_{j=1}^{n_i^N} H^{\frac{p}{2(p+1)}} \left( f; U_{T_{i,j}^N} \right) |T_{i,j}^N| - 3\varepsilon \|H\|_{\frac{p}{2(p+1)}} - (2\varepsilon)^p \right) \\
= \frac{C_{p,\alpha,\beta}^p}{2^p N^p} \left( \int_D H^{\frac{p}{2(p+1)}}(f; x, y) \, dx \, dy + o(1) - 3\varepsilon \|H\|_{\frac{p}{2(p+1)}} - (2\varepsilon)^p \right)^{p+1},
\]

as $N \to \infty$. Hence,

\[
\liminf_{N \to \infty} N^p \inf_{s \in S(\Delta_N)} \|f - s\|_{p,\alpha,\beta}^p \geq 2^{-p} C_{p,\alpha,\beta}^p \left( \int_D H^{\frac{p}{2(p+1)}}(f; x, y) \, dx \, dy - 3\varepsilon \|H\|_{\frac{p}{2(p+1)}} - (2\varepsilon)^p \right)^{p+1},
\]

Since $\varepsilon$ is arbitrary, we obtain the desired inequality (6.1).
§7. Asymptotically optimal sequences of triangulations

We now choose the asymptotically optimal sequence of triangulations \( \{\triangle^*_{N}\}_{N=1}^{\infty} \) and the corresponding piecewise linear splines \( \{s^*_{N}\}_{N=1}^{\infty} \), that will give the answer to the second question addressed in this paper.

Let \( \{\varepsilon_k\}_{k=1}^{\infty} \) be a decreasing sequence of positive numbers which tends to zero as \( k \to \infty \). For every \( \varepsilon > 0 \), we have constructed sequences of triangulations \( \{\triangle^\varepsilon_{N}\}_{N=1}^{\infty} \) of the set and corresponding piecewise linear splines \( \{s^\varepsilon_{N}\}_{N=1}^{\infty} \) for every \( N \) large enough. By \( N(\varepsilon) \) let us denote the minimal number \( N \) for which the triangulation \( \triangle^\varepsilon_{N} \) and \( s^\varepsilon_{N} \) were constructed. Without loss of generality we may assume that the sequence of numbers \( \{N(\varepsilon_k)\}_{k=1}^{\infty} \) is strictly increasing. Then, we define the set

\[
\triangle^*_{N} := \triangle^\varepsilon_{N}, \quad s^*_{N} := s^\varepsilon_{N}, \quad \text{if} \quad N(\varepsilon_k) < N \leq N(\varepsilon_{k+1}), \quad k \in \mathbb{N},
\]

where triangulations \( \triangle^\varepsilon_{N} \) and splines \( s^\varepsilon_{N} \) were defined in Section 5.2. For \( 1 \leq N \leq N(\varepsilon_1) \), we take \( \triangle^*_{N} \) to be an arbitrary triangulation of \( D \), and \( s^*_{N} \in S(\triangle^*_{N}) \) to be an arbitrary spline.

The above constructed sequence will be asymptotically optimal. Indeed, because of (5.22), for \( 1 \leq p < \infty \) and for every \( N(\varepsilon_k) < N \leq N(\varepsilon_{k+1}) \), we have

\[
R_N(f, L_{p;\alpha,\beta}) \leq \|f - s^*_N\|_{p;\alpha,\beta} \leq \frac{(1 + k_2 \varepsilon_k)}{2N} \left( \int_D H_{\frac{p}{2}(p+1)}(f; x, y) \, dxdy \right)^{1+1/p}.
\]

On the other hand, for every \( N \) large enough,

\[
R_N(f, L_{p;\alpha,\beta}) \geq \frac{C_{p;\alpha,\beta}}{2(1 + k_3 \varepsilon_k) N} \left( \int_D H_{\frac{p}{2}(p+1)}(f; x, y) \, dx \, dy \right)^{1+1/p}.
\]

Combining the last two inequalities and letting \( k \to \infty \), we obtain the desired

\[
\lim_{N \to \infty} N \cdot R_N(f, L_{p;\alpha,\beta}) = \lim_{N \to \infty} N \|f - s^*_N\|_{p;\alpha,\beta}.
\]

References

[1] Babenko V. F. (1982)
Non-symmetric approximations in spaces of summable functions, Ukrain. Mat. Zh., 34 (1982), pp. 409–416; English transl. Ukrainian Math. J., 34, pp. 323–336.

[2] Babenko V. F. (1983)
Asymmetric extremal problems in approximation theory, Dokl. USSR 269(3), pp. 521–524 [Russian]
[3] Babenko V. F. (1984)
Duality theorems for some problems in approximation theory, Contemp. questions of complex analysis, Kiev, In-t math. AN USSR, pp. 3–13. [in Russian]

[4] Babenko V. F. (1987)
Approximations, widths and optimal quadrature formulae for classes of periodic functions rearrangement invariant sets of derivatives, Anal. Math. 13, pp. 15–28.

[5] Babenko V., Babenko Y., Ligun A., Shumeiko A. (2006)
On asymptotical behavior of the optimal linear spline interpolation error of $C^2$ functions, J. Approx., 12(1), pp. 71–101.

[6] Babenko V., Babenko Y., Skorokhodov D. (2008)
Exact asymptotics of the optimal $L_{p,\Omega}$-error of linear spline interpolation, East J. Approx., pp. 285–237.

[7] Babenko V., Babenko Y., Parfinovych N., Skorokhodov D. (2009)
On one extremal property of a regular simplex, Comm. Anal. Geom. 17, no. 4, pp. 685–699.

[8] Babenko Y. (2006)
On the asymptotic behavior of the optimal error of spline interpolation of multivariate functions, PhD thesis.

[9] Böröczky K., Ludwig M. (1999)
Approximation of Convex Bodies and a Momentum Lemma for Power Diagrams, Monatshefte für Mathematik, 127(2), pp. 101–110.

[10] Böröczky K. (2000)
Approximation of general smooth convex bodies, Adv. in Math., 153 pp. 325–341.

[11] Brezin M. (1992)
A solution-based triangular and tetrahedral mesh quality indicator, SIAM Journal on Scientific Computing 19, pp. 979–997.

[12] Chen L., Sun P., Xu J. (2007)
Optimal anisotropic meshes for minimizing interpolation errors in $L_p$-norm, Math. Comp. pp. 179–204.
Optimal $L_p$-error of asymmetric linear spline approximation

[13] Chen L. (2008)
On minimizing the linear interpolation error of convex quadratic functions and the optimal simplex, East J. Approx. 10(3), pp. 271–284.

[14] Cohen A., Mirebeau J.-M. (2009)
Adaptive and anisotropic piecewise polynomial approximation, chapter 4 in Multiscale, Nonlinear and Adaptive Approximation, Springer.

[15] D’Azevedo E. F., Simpson R. B. (1989)
On optimal interpolation triangle incidences (1989), SIAM J. Sci. Statist. Comput. 10(6), 1063–1075.

[16] Dolzhenko E. P., Sevast’yanov E. A. (1998)
Approximations with a sign-sensitive weight: existence and uniqueness theorems, Izv. RAS, Math., 62:6, pp. 59–102; English transl.: Izv. Math., 62:2, pp. 1127–1168.

[17] Dolzhenko E. P., Sevast’yanov E. A. (1999)
Approximations with a sign-sensitive weight. Stability, applications to the theory of snakes Hausdorff approximations, Izv. RAS, Ser. Math., 63:3, pp. 77–118; English transl.: Izv. Math., 63:3, pp. 495–534.

[18] Dyn N., Levin D., Rippa S. (1990)
Data dependent triangulations for piecewise linear interpolation, IMA J. Numer. Anal., 10, 1, pp. 137–154.

[19] Dyn N., Levin D., Rippa S. (1992)
Boundary correction for piecewise linear interpolation defined over data-dependent triangulations, Journal of Computational and Applied Mathematics, 39, pp. 179–192.

[20] Fejes Toth L. (1972)
Lagerungen in der Ebene, auf der Kugel und im Raum, 2nd ed. Berlin: Springer.

[21] Goodman J., O’Rourke J., (eds.) (2004)
Handbook of Discrete and Computational Geometry, CRC Press.

[22] Gruber P. (1988)
Volume approximation of convex bodies by inscribed polytopes, Math. Ann., 281, pp. 229–247.
[23] Huang, W.; Sun, W. (2003)
Variational mesh adaptation. II. Error estimates and monitor functions. J. Comput. Phys. 196 (2004), no. 2, pp. 619–648.

[24] Kilizhekov Yu. A. (1996)
Approximation error for linear polynomial interpolation on $n$-simplices, Math. Notes, 60:4, 378–382.

[25] Korneichuk N. P. (1987)
Exact constants in approximation theory, Nauka, Moscow; translated from Russian by K. Wordsworth, Encyclopedia of Mathematics and its Applications, 38. Cambridge University Press, Cambridge, 1991.

[26] Krein M. G. (1962)
The L-Problem in an abstract linear normed space, in: Some Questions in the Theory of Linear Operators, N. I. Akhiezer and M. G. Krein (eds.), Am. Math. Soc., Providence, pp. 175–204.

[27] Krein M.G., Nudel’man A. A. (1977)
The Markov Moment Problem and Extremal Problems, Translations of Mathematical Monographs, V. 50; 417 pp.

[28] Mirebeau J.-M. (2010)
Optimally adapted finite elements meshes, Constructive Approximation, Vol 32, N. 2, pp. 365–383.

[29] Nadler E. (1986)
Piecewise linear best $L_2$ approximation on triangles, in: Chui, C.K., Schumaker, L.L. and J.D. (eds.), Approximation Theory V, Academic Press, pp. 499–502.

[30] Pottmann H., Krasauskas R., Hamann B, Joy K., Seibold W. (2000)
On piecewise linear approximation of quadratic functions, J. Geom. Graph. 4(1), pp. 39–53.

[31] Subbotin Yu. N. (1989)
The dependence of estimates of a multidimensional piecewise-polynomial approximation on geometric characteristics of a triangulation, A work collection of the All-Union school on approximation theory (Dushanbe, August 1986), Trudy Mat. Inst. Steklov., 189, Nauka, Moscow, pp. 136–137.
Optimal $L_p$-error of asymmetric linear spline approximation

Vladislav Babenko,
Department of Mathematical Analysis and Theory of Functions
Dnepropetrovsk National University
pr. Gagarina, 72,
Dnepropetrovsk, UKRAINE, 49050
babenko.vladislav@gmail.com

Yuliya Babenko,
Department of Mathematics and Statistics
Kennesaw State University
1000 Chastain Road, #1601
Atlanta, GA, USA 30144-5591
ybabenko@kennesaw.edu

Nataliya Parfinovych,
Department of Mathematical Analysis and Theory of Functions
Dnepropetrovsk National University
pr. Gagarina, 72,
Dnepropetrovsk, UKRAINE, 49050
nparfinovich@yandex.ru

Dmytro Skorokhodov,
Department of Mathematical Analysis and Theory of Functions
Dnepropetrovsk National University
pr. Gagarina, 72,
Dnepropetrovsk, UKRAINE, 49050
dmitriy.skorokhodov@gmail.com