The relativistic Aharonov–Bohm–Coulomb system with position-dependent mass

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Received 27 April 2019, revised 12 November 2019
Accepted for publication 28 November 2019
Published 7 January 2020

Abstract
In this work, we study the influence of the Aharonov–Bohm–Coulomb (ABC) system on the relativistic and non-relativistic quantum dynamics of a charged Dirac particle with position-dependent mass (PDM). In order to solve exactly our system, we use both the left-handed and right-handed projection operators. Next, we determine the Dirac spinor and the relativistic energy spectrum, where we observe that this spinor is written in terms of the generalized Laguerre polynomials and this spectrum depends explicitly on the quantum numbers \( n \) and \( m_l \), parameters \( Z \) and \( \Phi \) that characterize the ABC system, and on the parameter \( \kappa \) that characterizes the PDM. In particular, we analyse the behavior of the energies as a function of \( \kappa \) for different values of \( Z \). Finally, we consider the non-relativistic limit and we compared our problem with others works in the literature, where we verify that our results turn out to generalize some particular cases.

Keywords: dirac equation, Aharonov–Bohm–Coulomb system, position-dependent mass, relativistic bound states

1. Introduction

In the literature, there are a significant number of studies investigating the quantum dynamics of particles with a constant mass in the presence of the Aharonov–Bohm (AB) effect [1–5] and the 2D Coulomb potential [6–9]. In particular, due to its relevance, the Aharonov–Bohm–Coulomb (ABC) system described by a combination of the Coulomb and vector potentials provides an interesting physical system that has been receiving much attention over the years [10–13]. Moreover, the ABC system for relativistic spin-1/2 particles governed by the Dirac equation (DE) was studied in connection with the Feynman path integrals [14, 15], scattering...
processes [16], magnetic monopoles [17, 18], spontaneous creation of fermions pairs [19], Coulomb supercritical impurity [20], and so forth. Recently, the ABC system was applied in the study of graphene rings [21] and used for investigating the bound-state solutions of the Dirac oscillator (DO) [22].

On the other hand, physical systems with position-dependent mass (PDM) are of particular interest in theoretical and experimental physics. For instance, using the Schrödinger equation (SE) with PDM, we can investigate the electronic properties of semiconductors [23], quantum wells and quantum dots [24], 3He clusters [25], and quantum liquids [26]. In addition, the relativistic extension of this formalism has the advantage of eliminating the ordering ambiguity problem of the mass and momentum operators present in the kinetic term in the SE [27, 28]. Furthermore, using the DE with PDM, we can study problems involving scattering [29], Coulomb field [30, 31], cosmic strings [32], spin and pseudo-spin symmetry [33], solid state physics [34], PT-symmetry [35–37], infinite square well [38], and generalized uncertainty principle [39]. It is worth mentioning that, recently, the generalized DO with PDM was studied in the presence of an electric field via supersymmetric formalism [40].

In this work, we investigate the relativistic and non-relativistic quantum dynamics of the ABC system for a charged Dirac particle with PDM in (2+1)-dimensional Minkowski spacetime. To solve exactly our problem, we use the left-handed and right-handed projection operators [41]. In particular, these operators have practical use as an alternative method to determine the bound-state solutions of the 3D relativistic hydrogen atom. With respect to the PDM function, we assume the 2D version of a spherically symmetrical singular mass distribution presented in [30–33]. This case is particularly interesting because of its connection with the electronic properties of materials [42, 43] and high-energy physics [29]. Nevertheless, in these papers, this type of PDM is not taken into account in the context of the ABC system for either relativistic or non-relativistic regimes. Besides, planar (2D) solutions of the DE are not only interesting in themselves but also allow connection to the remarkable physics of 2D fermions in condensed matter systems, such as in graphene [44], electron gas [45], quantum Hall effect [46], and in quantum rings [47]. Recently, the 2D version of PDM found in [30–33] was applied in the study of the Klein–Gordon oscillator [48], scalar fields [49], Kaluza–Klein theory [50], and Lorentz symmetry breaking [51].

This paper is organized as follows. In section 2, we introduce the DE in polar coordinates for an electrically charged particle with PDM under the influence of the ABC system. Next, we apply the left-handed and right-handed projection operators in the DE and we obtain a second order differential equation. In this sense, in order to obtain this differential equation, we also use a similarity transformation given by a unitary operator. In section 3, we explicitly determine the Dirac spinor and the discrete energy spectrum for the bound-states of the particle. In section 4, we investigate the non-relativistic limit of our results. Finally, in section 5, we present our conclusions. In this work, we use the spacetime with signature (+−−).

2. The dirac equation with position-dependent mass in an Aharonov–Bohm–Coulomb system

The (2+1)-dimensional DE that governs the relativistic quantum dynamics of an electrically charged particle with PDM in the presence of an external electromagnetic field $A_{\mu}$ can be written as (in SI units) [52, 54]

$$[\gamma^\mu \Pi_\mu - m(r)c] \Psi(t, \mathbf{r}) = 0, \quad (\mu = 0, 1, 2),$$

(1)
where $\gamma^\mu = (\gamma^0,\gamma^1,\gamma^2)$ are $2 \times 2$ gamma matrices, $\Pi_\mu = p_\mu - qA_\mu$ is the kinetic momentum operator, $p_\mu = i\hbar \partial_\mu$ is the canonical momentum operator, $q < 0$ is the electric charge of the particle (electron) with a PDM given by $m(r)$, $\Psi(t, r)$ is the two-component Dirac spinor, and $A_\mu = (A_0/c, -A)$ with $A_0$ and $A$ being the two-dimensional scalar and vector potentials respectively.

Now, we introduce the following left-handed and right-handed projection operators [41]

$$P_L = \frac{1}{2}(1_{2 \times 2} - \gamma^5), \quad P_R = \frac{1}{2}(1_{2 \times 2} + \gamma^5),$$

which satisfy the properties $P_L^2 = P_L, P_R^2 = P_R, \{P_L, P_R\} = 0, P_L + P_R = 1_{2 \times 2}$ and $P_R \gamma^\mu = \gamma^\mu P_L$, where $\gamma^5 = \gamma_5 = \sigma^1$ is called chirality operator [52, 53]. Then, applying $P_L$ on equation (1) and defining the left-handed and right-handed spinors as $\Psi_L(t, r) = P_L \Psi(t, r)$ and $\Psi_R(t, r) = P_R \Psi(t, r)$, we get

$$\Psi_L(t, r) = \frac{1}{m(r)c} \gamma^\mu \Pi_\mu \Psi_R(t, r).$$

The above relation allows us to write the original Dirac spinor in the form

$$\Psi(t, r) = \frac{1}{m(r)c} [\gamma^\mu \Pi_\mu + m(r)c] \Psi_R(t, r),$$

where we used $\Psi(t, r) = \Psi_L(t, r) + \Psi_R(t, r)$.

Substituting the spinor (4) into equation (1), we obtain

$$[\gamma^\mu \Pi_\mu - m(r)c][\gamma^\nu \Pi_\nu + m(r)c] \Psi_R(t, r) = 0,$$

or in the differential form as

$$[\gamma^0 \Pi_0 - \gamma \cdot \Pi - m(r)c][\gamma^0 \Pi_0 - \gamma \cdot \Pi + m(r)c] \Psi_R(t, r) = 0,$$

where

$$\Pi_0 = \frac{1}{c} \left( i\hbar \frac{\partial}{\partial t} - qA_0 \right), \quad \Pi = (p - qA),$$

with $p = -i\hbar \nabla$ being the usual momentum operator.

Adopting now the polar coordinates system $(t, \rho, \theta)$ where the metric tensor is given by $g^{\mu\nu} = \text{diag}(1, -1, -\rho^2)$ [54], with $\rho = \sqrt{x^2 + y^2} > 0$ being the radial coordinate and $0 \leq \theta \leq 2\pi$ the angular coordinate, equation (5) becomes

$$A^- A^+ \Psi_R(t, \rho, \theta) = 0,$$

where the operators $A^\pm$ are defined in the form

$$A^\mp = \left[ \gamma^0 \left( i\hbar \frac{\partial}{\partial t} - \frac{qA_0}{c} \right) + \gamma^\rho \left( i\hbar \frac{\partial}{\partial \rho} - \frac{qA_\theta}{c} \right) + \gamma^\theta \left( \frac{i\hbar}{\rho} \frac{\partial}{\partial \theta} + qA_\theta \right) \right] m(r)c,$$

with

$$\gamma^0 = \gamma \cdot \hat{e}_\rho = \gamma^1 \cos \theta + \gamma^2 \sin \theta, \quad \gamma^\rho = \gamma \cdot \hat{e}_\theta = -\gamma^1 \sin \theta + \gamma^2 \cos \theta.$$ (10)

Here, we are assuming that the radial component of the vector potential is null ($A_\rho = 0$). Moreover, through a similarity transformation regarding the unitary operator $U(\theta) = e^{-i\theta \sigma_3/2}$, we can reduce the rotated matrices $\gamma^\rho$ and $\gamma^\theta$ to the fixed matrices $\gamma^1$ and $\gamma^2$ as follows [54]
On the other hand, since we are working in (2 + 1)-dimensional Minkowski spacetime, it is convenient to define the gamma matrices \( \gamma = (\gamma_1, \gamma_2) = (-\gamma_1, -\gamma_2) \) and \( \gamma^0 \) directly in terms of the Pauli matrices, i.e. \( \gamma_1 = \sigma_3 \sigma_1, \gamma_2 = \sigma_3 \sigma_2 \) and \( \gamma^0 = \sigma_3 \) \([52, 54]\). Therefore, using this information and the relations exhibited in (8), we rewrite equation (6) in the form

\[
B^\dagger B \psi_R(t, \rho, \theta) = 0,
\]

where

\[
B^\dagger = \left[ \sigma_3 \left( \frac{i\hbar}{c} \frac{\partial}{\partial t} - \frac{qA_0}{c} \right) + \sigma_2 \left( \frac{\hbar}{\rho} \frac{\partial}{\partial \rho} \right) + i\sigma_1 \left( \frac{i\hbar}{\rho} \frac{\partial}{\partial \theta} + qA_0 + \frac{\hbar \sigma_3}{2\rho} \right) \pm m(\rho)c \right],
\]

and

\[
\psi_R(t, \rho, \theta) = U^{-1}(\theta) \Psi_R(t, \rho, \theta).
\]

Let us now set the configuration of the scalar and vector potentials for our system. Explicitly, we choose the potential vector \( A \) associated with a constant magnetic flux in the form

\[
A = A_0 \hat{e}_\theta = \frac{\Phi}{2\pi \rho} \hat{e}_\theta, \quad (\rho > a),
\]

where \( \Phi = 2\pi a^2 B \) (\( \Phi > 0 \)) is the magnetic flux in the interior of a solenoid of radius \( a \). This potential gives rise to the AB effect experienced by a charged particle confined in the transverse plane outside the solenoid \([1]\). To fix the functional form of the scalar potential, let us assume a configuration analogous to the three-dimensional Coulomb potential, namely,

\[
A_0 = \frac{Ze}{4\pi \epsilon_0 \rho}, \quad (Ze > 0),
\]

so that \( Ze \) is an effective nuclear charge, with \( Z \) being the atomic number and \( e \) is the magnitude of the elementary electric charge. Here, it is worth to note that the Coulomb potential is genuinely logarithmic in two dimensions (see for instance \([56]\)). However, our choice for the scalar potential \((16)\) is not completely ad hoc, since considering the two-dimensional potential proportional to \( 1/\rho \) there exists many applications in problems associated with condensed matter physics, in particular involving graphene systems \([6–13]\).

With respect to the configuration of the variable mass, we consider the following PDM

\[
m(\rho) = m_0 + \frac{\kappa}{\rho}, \quad (\kappa > 0),
\]

where \( m_0 \) is the rest mass of the particle and \( \kappa \) is a real parameter. In particular, the PDM \((17)\) is the two-dimensional version of a spherically symmetrical singular mass distribution worked in \([30, 31, 33]\) with connection to the study of a charged particle at the high-energy regime. In fact, since the PDM \((17)\) diverges at the origin (\( \rho = 0 \)), the parameter \( \kappa \) can be interpreted as an effective renormalization scale in quantum field theory to eliminate the ultraviolet divergences \([30]\). Thereby, we can write \( \kappa = m_0 \mu \lambda^2 \), where \( \lambda \) is the Compton wavelength of the particle and \( \mu \) is a real scale parameter with dimensions of inverse length \([30]\), or even as \( \kappa = m_0 \alpha \), with \( 0 < \alpha < r_0 \) and \( r_0 \) being the classical electron radius \([31]\).

Therefore, using the configurations \((15)–(17)\), we transform equation \((12)\) as follows

\[
\left[ \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \left( \frac{\Gamma^2}{\hbar^2} - \Gamma^2 \frac{1}{4} + \frac{\Delta}{\rho} - \Sigma \right) \right] \psi_R(t, \rho, \theta) = 0,
\]

where \( \theta = \arctan(\rho_0/\rho), \quad \rho_0 = \sqrt{a^2 - \rho^2} \).
where we define the following operators
\[
\Gamma^2 = \left( L_z + \frac{\Phi}{\Phi_0} \right)^2 - (Z\alpha)^2 + (ck)^2, \quad \Gamma = \left( L_z + \frac{\Phi}{\Phi_0} \right) \sigma_3 - iZ\alpha \sigma_1 - c\kappa \sigma_2.
\] (19)

\[
\Delta = \frac{2iZ\alpha}{c} \frac{\partial}{\partial \rho} - \frac{2m_0c^2\kappa}{\hbar^2}, \quad \Sigma = \frac{1}{c^2} \frac{\partial^2}{\partial \rho^2} + \frac{m_0^2c^2}{\hbar^2},
\] (20)

with \( \alpha = e^2/4\pi \epsilon_0 \hbar \approx 1/137 \) being the Sommerfeld fine structure constant, \( \Phi_0 = \hbar/e \) is the magnetic flux quantum and \( L_z = -i\hbar \partial/\partial \theta \) is the projection of the orbital angular momentum operator \( \mathbf{L} \) on the \( z \)-axis.

Now, writing the two-component Dirac spinor in the form \([52, 54]\)
\[
\psi_R(t, \rho, \theta) = e^{i\frac{(ml\theta - Et)}{\hbar}} \sqrt{\frac{2}{\pi \rho}} \left( \phi^+(\rho) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right), \quad (m_l = \pm 1/2, \pm 3/2, \ldots).
\] (21)

Equation (18) becomes compacted in the following differential equation
\[
\left[ \frac{d^2}{d\rho^2} - \frac{\gamma(\gamma - s)}{\rho^2} + \rho_0 \rho + \frac{E^2 - m_0^2c^4}{\hbar^2 c^2} \right] \phi'(\rho) = 0, \quad (s = \pm 1),
\] (22)

where
\[
\gamma = \sqrt{\left( m_l + \frac{\Phi}{\Phi_0} \right)^2 - (Z\alpha)^2 + \left( \frac{ck}{\hbar} \right)^2} > 0, \quad \rho_0 = \left( \frac{2Z\alpha E}{\hbar c} - \frac{2m_0^2c^2\kappa}{\hbar^2} \right).
\] (23)

with \( \phi'(\rho) \) being real radial functions, \( E \) is the relativistic total energy of the particle and \( m_l = \pm 1/2, \pm 3/2, \ldots \) is the orbital magnetic quantum number.

### 3. Bound-state solutions and energy spectrum

In order to solve equation (22), we will introduce now a new dimensionless variable given by
\[
z = 2\eta \rho, \quad \eta = \sqrt{m_0^2c^4 - E^2/\hbar c},
\]
with the condition \( m_0^2c^4 > E^2 \) that ensures the existence of relativistic bound-states (for the case \( m_0^2c^4 < E^2 \) the spectrum is continuous associated with scattering states) [14, 52]. Thus, performing a variable changing in equation (23), we obtain
\[
\left[ \frac{d^2}{dz^2} - \frac{\gamma(\gamma - s)}{z^2} + \frac{z_0}{z} - \frac{1}{4} \right] \phi'(z) = 0,
\] (24)

where
\[
z_0 = \frac{\rho_0}{2\eta}.
\] (25)

Analyzing the asymptotic behavior of equation (24) for \( z \to 0 \) and \( z \to \infty \), we obtain
\[
\phi'(z) = C' z^{(1-s)/2} e^{-z/2} R'(z),
\] (26)

where \( C' \) are normalization constants and \( R'(z) \) are unknown functions to be determined.
In this way, substituting (26) into equation (24), we have
\[ z \frac{d^2 R(z)}{dz^2} + (b^e - z) \frac{dR(z)}{dz} - a^e R(z) = 0, \] (27)
where
\[ b^e = (2 \gamma + 1 - s), \quad a^e = \left( \frac{b^e}{2} - z_0 \right). \] (28)

It is not difficult to note that equation (27) has the form of a generalized Laguerre equation, whose solution are the generalized Laguerre polynomials \( R^e(z) = L_N^{(2\gamma+1-2)}(z) \) [55]. Besides that, for \( \phi^e(z) \) to be a normalized solution, we must impose that the parameter \( a^e \) be equal to a non-positive integer number \(-N\) \((N = 0, 1, 2, \ldots)\). Therefore, using this condition and the relations (25) and (10), we obtain the following energy spectrum of the Dirac particle with PDM under the influence of the ABC system
\[ E_{n,m_l} = m_0 c^2 \left[ \frac{Z_{\text{OCR}}}{\hbar n} + \sqrt{1 + \left( \frac{Z_{\text{OCR}}}{\hbar n} \right)^2 - \frac{1}{n}} \left( \frac{Z\alpha}{\hbar} \right)^2 + \left( \frac{c \kappa}{\hbar} \right)^2 \right], \] (29)
where
\[ \bar{n} = \left( n - |m_l| - \frac{1}{2} \right) + \sqrt{\left( m_l + \frac{\Phi}{\Phi_0} \right)^2 - (Z\alpha)^2 + \left( \frac{c \kappa}{\hbar} \right)^2} + (Z\alpha)^2, \] (30)
with \( n = N + |m_l| + 1/2 = 1, 2, 3, \ldots \) being the main quantum number.

We see that the spectrum (29) is quantized in terms on the quantum numbers \( n \) and \( m_l \) and explicitly depends on the parameters \( Z \) and \( \Phi \) that characterize the ABC system, and also the parameter \( \kappa \) which characterizes the PDM. Nevertheless, the negative sign in (29) was excluded because for a positively charged solenoid \((Z\alpha > 0)\), the negative energy states \((E < 0)\) do not satisfy the relation: \( z_0 = N + b^e / 2 > 0 \) (the quantization condition). In this sense, only particles can interact with the ABC system, rather than antiparticles ones. Besides, we see that the condition \((Z\alpha)^2 > (m_l + \Phi/\Phi_0)^2 + \kappa^2\) is not acceptable since this results in imaginary energies and the Dirac Hamiltonian becomes anti-Hermitian (collapse of the vacuum) [14, 19, 52]. In addition, the maximum energy for the particle is given by \( E_0 = m_0 c^2 \) and it corresponds to the continuous (classical) limit of the spectrum (29). This limit is achieved when the magnetic flux is either intense \((\Phi \rightarrow \infty)\) or when one considers a very large quantum numbers \((n = m_l \rightarrow \infty)\). Therefore, the energies increase as a function of \( n, m_l \) and \( \Phi \).

The behavior of the ground-state energy \((n = 1, m_l = 1/2)\) is shown in figure 1 as a function of the parameter \( \kappa \) for three different values of the atomic number \( Z \), where for simplicity we take \( \hbar = c = m_0 = a = 1 \) and \( \Phi = \Phi_0 \). We note that in all cases, the curves tend to a maximum value, i.e. the rest energy, and then decrease rapidly as \( \kappa \) increases, going to zero when \( k \rightarrow \infty \). From figure 2, we plot the energies of ground state \((n = 1)\), first \((n = 2)\) and second \((n = 3)\) excited states as a function of the atomic number \( Z \), with \( m_l = 1/2 \) and \( \kappa = 0.5 \). We note that in all cases, the curves decrease when \( Z \) increase, however, they saturate at a critical value from which the energy becomes imaginary. This result corroborates with the condition previously obtained, where \((Z\alpha)^2 > (m_l + \Phi/\Phi_0)^2 + \kappa^2\) entails an instability of the energy levels. Besides, the energies of the excited states are greater than that of the ground state as it should be.
Now, let us take a general comparison of expression (29) with the corresponding results of the other works. Initially, we observe that in the absence of the AB effect ($\Phi \to 0$), the energy spectrum becomes similar to that obtained in [30, 31] for a Dirac particle with PDM in the presence of the 3D Coulomb potential. Furthermore, in the limit of the constant mass ($\kappa \to 0$), the usual spectrum of the relativistic ABC system is recovered [14, 19]. It is also interesting to note that even in the absence of the ABC system ($Z = \Phi = 0$), the free Dirac particle with PDM still has discrete energy spectrum given by:

$$E_{N,m_l} = m_0 c^2 \sqrt{1 - \left(\frac{c \kappa}{\hbar}\right)^2 \left(N^2 + \frac{m_l^2}{\kappa^2} + \left(\frac{c \kappa}{\hbar}\right)^2\right)} - 2.$$

In this case, we can interpret the PDM as a type of scalar coupling in the DE, where we have the Lorentz-scalar potential $V_s(\rho) = \frac{\kappa}{\rho}$ [30]. In fact, according to [52], this result appears because of a spin-1/2 particle with PDM is equivalent to a DE with scalar coupling.

Hereafter, we will focus on obtaining the analytical expression for the Dirac spinor. Substituting the variable $z = 2\eta \rho$ in the radial function (26), the spinor (21) becomes

$$\psi_R(t, \rho, \theta) = e^{i(m \theta - E t/\hbar)} \left( D^+ \rho^{\gamma-1/2} e^{-\eta \rho} L_N^{\gamma-1}(2\eta \rho) - D^- \rho^{\gamma+1/2} e^{-\eta \rho} L_N^{\gamma+1}(2\eta \rho) \right),$$

where

$$D^t = \frac{C'(2\eta)^{\gamma+1-s/2}}{\sqrt{2\pi}}, \quad (s = \pm 1).$$

Now, substituting the spinor (14) into spinor (4) and using the operators (13) together with the configurations (15) and (16), we obtain

$$\Psi = \frac{1}{m(\rho)c} U \left[ \sigma_3 \left( \frac{ih}{c} \frac{\partial}{\partial t} + \frac{Z \alpha h}{\rho} \right) + \sigma_2 \left( \frac{h}{\rho} \frac{\partial}{\partial \rho} \right) + i\sigma_1 \left( \frac{ih}{\rho} \frac{\partial}{\partial \rho} + \frac{h}{\rho} \Phi + \frac{h \sigma_3}{2 \rho} \right) + m(\rho)c \right] \psi_R.$$

Thus, substituting the spinor (31) in (33), the original Dirac spinor is written in the form

$$\Psi_{N,m_l}(t, \rho, \theta) = e^{i(m \theta - E t/\hbar)} \left( F^+(\rho) + iG^- (\rho) \right) \left( F^- (\rho) - iG^+ (\rho) \right),$$

Figure 1. The behavior of the ground-state energy as a function of $\kappa$ for different values of $Z$ (we take $\hbar = c = m_0 = \alpha = 1$ and $\Phi = \Phi_0$ in equation (29)).
where

\[
F_s(\rho) \equiv D_s \rho^{\gamma - s/2} e^{-\eta \rho} L_N^{2\gamma - s}(2\eta \rho) \left[ \frac{s}{m(\rho)c} \left( \frac{E}{\epsilon} - \frac{Z\alpha\hbar}{\rho} \right) + 1 \right],
\]

(35)

\[
G_s(\rho) \equiv \frac{\hbar D_s}{m(\rho)c} \rho^{\gamma - s/2} e^{-\eta \rho} \left[ L_N^{2\gamma - s}(2\eta \rho) \left( \eta + \frac{(s m_l - \frac{1}{2} - \gamma + s \frac{\Phi}{\Phi_0})}{\rho} \right) + L_{N-1}^{2\gamma + 1 - s}(2\eta \rho) \right],
\]

(36)

with \( m_j = m_l \pm 1/2 \) \([54]\).

It should be noted here that our spinor (34) incorporates the positive and negative values of the quantum number \( m_j \) (or \( m_l \)), which does not happen for the case of the spinor worked in \([54]\).

4. Non-relativistic limit

Finally, we analyze the non-relativistic limit of our results. According to the literature \([52]\), this limit is obtained considering the following conditions: \( E \cong \epsilon + m_0 c^2 \), where \( \epsilon \ll m_0 c^2 \), \( (Z\alpha) \ll 1 \) or \( Z \ll 137 \) (weak Coulomb field) and \( \kappa \ll 1 \) or \( (c\kappa/\hbar) \ll 1 \) (small effective mass). Thus, applying these conditions in equation (22), we obtain the following 2D SE for a spinless particle with PDM under the influence of the ABC system

\[
\left[ \frac{\hbar^2}{2m_0} \left( \frac{d^2}{d\rho^2} + \frac{1}{\rho} \frac{d}{d\rho} - \frac{(m + \frac{\Phi}{\Phi_0})^2}{\rho^2} \right) + \frac{1}{\rho} \left( c^2 \kappa - \frac{Ze^2}{4\pi\epsilon_0} \right) \right] f^+(\rho) = \epsilon f^+(\rho),
\]

(37)

where \( m \equiv m_l - 1/2 = 0, \pm 1, \pm 2, \ldots \), \( f^+(\rho) = \phi^+(\rho)/\sqrt{\rho} \) is the wave function and we use the parameter \( s \) with the positive sign \( (s = +1) \), which represents the state of the particle. It is worth mentioning that although the constant \( c^2 \) appears explicitly in equation (37), such constant is canceled due to the existence of parameter \( \kappa \). Comparing equation (37) with the

\[
F^+(\rho) = m \rho^{\gamma - s/2} e^{-\eta \rho} L_N^{2\gamma - s}(2\eta \rho) \left[ \frac{s}{m(\rho)c} \left( \frac{E}{\epsilon} - \frac{Z\alpha\hbar}{\rho} \right) + 1 \right],
\]

(38)

\[
G^+(\rho) = \frac{\hbar D^+}{m(\rho)c} \rho^{\gamma - s/2} e^{-\eta \rho} \left[ L_N^{2\gamma - s}(2\eta \rho) \left( \eta + \frac{(s m_l - \frac{1}{2} - \gamma + s \frac{\Phi}{\Phi_0})}{\rho} \right) + L_{N-1}^{2\gamma + 1 - s}(2\eta \rho) \right],
\]

(39)

It should be noted here that our spinor (34) incorporates the positive and negative values of the quantum number \( m_j \) (or \( m_l \)), which does not happen for the case of the spinor worked in \([54]\).
literature, we see that in the limit of the constant mass ($\kappa \to 0$), we obtain the SE for a particle under the influence of the ABC system [10]. Besides, in the constant mass limit and in the absence of the AB effect ($\Phi \to 0$), we have the SE for the 2D non-relativistic hydrogen atom ($Z = 1$) [7]. Now, in the absence of the ABC system ($\Phi = Z = 0$), we obtain the SE for a free particle with PDM.

To analyze the non-relativistic limit of the energy spectrum (29), we can apply a Taylor series expansion for a two-variable function. Defining $x = (Z\alpha)^2$ and $y = (e\kappa/h)^2$ in expression (29), we can write the non-relativistic energy spectrum as follows

$$\varepsilon(x, y) = m_0c^2 f(x, y) - m_0c^2,$$

where

$$f(x, y) = \left[ \frac{\sqrt{xy}}{n(x, y)} + \sqrt{1 + \left( \frac{\sqrt{xy}}{n(x, y)} \right)^2} \right],$$

with $n(x, y)$ being

$$n(x, y) = \left( n - |m| - \frac{1}{2} + \sqrt{\left( m + \frac{\Phi}{\Phi_0} \right)^2 - x + y} \right)^2 + x.$$

Expanding $f(x, y)$ into a first-order Taylor series, we obtain

$$f(x, y) \approx 1 - \frac{1}{2} \frac{(x + y)}{(n - \frac{1}{2} + Q)^2},$$

with $Q = -|m| + |m + \Phi/\Phi_0|$ a quantity called ‘quantum defect’ or Rydberg correction [10].

Therefore, replacing (41) in (38), we explicitly obtain the energy spectrum of a particle with PDM under the influence of the ABC system

$$\varepsilon_{n,m} = -\frac{Z^2\varepsilon + \nu}{(n - \frac{1}{2} + Q)^2},$$

where

$$\varepsilon = \frac{1}{2}m_0c^2\alpha^2 \approx 13.6 \text{ eV}, \quad \nu = \frac{1}{2}m_0\left( \frac{e^2\kappa}{\hbar} \right)^2 \approx 11.9 \text{ eV}.$$  

As we can see from the energy spectrum (42), the term $\nu$ due to PDM has the function of increasing the absolute values of the energies. It arises as a consequence of the Taylor series expansion that the non-relativistic spectrum is always negative. However, this does not mean that we have an antiparticle in the energy spectrum. In fact, we have bound states ($E < 0$), where the expression (42) provides the energies required for confinement of a particle around the charged solenoid [52]. Moreover, we see that for $m_l > 0$, we have $Q = \Phi/\Phi_0$, i.e. the spectrum does not depend on the quantum number $m_l$. On the other hand, for $m_l < 0$, we have $Q \neq \Phi/\Phi_0$, i.e. the energy spectrum depends on the quantum number $m_l$. The limit of the continuous (classical) case $\varepsilon \to 0$ is obtained by assuming a constant mass ($\kappa \to 0$) with the absence of the Coulomb field ($Z = 0$), or when we have an intense magnetic flux ($\Phi \to \infty$) or for a very large quantum numbers ($n = m_l \to \infty$). Now, comparing the spectrum (42) with the literature, we verify that, in the limit of the constant mass, we obtain the usual spectrum of a spinless particle under the influence of the ABC system [10]. Besides, in the absence of the
AB effect ($\Phi \to 0$) with $\kappa \to 0$, we get the energy spectrum of a particle interacting with the Coulomb field, or, the Bohr formula for the 2D non-relativistic hydrogen atom [7]. Thus, this shows that our non-relativistic spectrum generalizes some particular cases of the literature.

5. Conclusion

In this paper, we investigate the bound-state solutions of the relativistic and non-relativistic ABC system for a charged Dirac particle with PDM. We apply the left-handed and right-handed projection operators in the DE which is written in polar coordinates for the sake of obtaining a second-order differential equation. After analyzing the asymptotic behavior of this differential equation for both $z \to 0$ and $z \to \infty$, we obtain a generalized Laguerre equation. From this result, we determine the energy spectrum and the Dirac spinor for the bound-states of the particle. In addition, we also observe that the relativistic energy spectrum depends explicitly on the parameters $Z$ and $\Phi$ that characterize the ABC system and on the parameter $\kappa$ that characterizes the PDM system. In particular, we verify that the energies grow in values as a function of $n, m_l$ and $\Phi$, and in the limit of the constant mass ($\kappa \to 0$), the energy spectrum of the relativistic ABC system is recovered. It is also important to mention that even in the absence of the ABC system ($\Phi = Z = 0$), the free Dirac particle with PDM still has a discrete energy spectrum. This occurs because the PDM system can be interpreted as a type of scalar coupling into DE. We finish this work with the non-relativistic limit of our results. We observe in this limit that the parameter $\kappa$ has the function of increasing the absolute values of the energies. Besides, these energies grow quadratically with the increase of $Z$. In particular, in the limit of the constant mass, we obtain the energy spectrum of the non-relativistic ABC system.

Acknowledgments

The authors would like to thank the Fundação Cearense de apoio ao Desenvolvimento Científico e Tecnológico (FUNCAP), the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), and the Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) for financial support. R V M and C A S A thank CNPq Grant N° 307556/2018-2 and 308638/2015-8 for supporting this project.

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