On the braiding of an Ann-category

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Abstract

A braided Ann-category $A$ is an Ann-category $A$ together with the braiding $c$ such that $(A, \otimes, a, c, (I, l, r))$ is a braided tensor category, and $c$ is compatible with the distributivity constraints. The paper shows the dependence of the left (or right) distributivity constraint on other axioms. Hence, the paper shows the relation to the concepts of distributivity category due to M. L. Laplaza and ring-like category due to A. Frohlich and C.T.C Wall.

The center construction of an almost strict Ann-category is an example of an unsymmetric braided Ann-category.

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Key words: Braided Ann-category, braided tensor category, distributivity constraint, ring-like category.

1 Introduction

The concept of a braided tensor category is introduced by André Joyal and Ross Street [2] which is a necessary extension of a symmetric tensor category, since the center of a tensor category is a braided tensor category but unsymmetric. The case of braided group categories was considered in the above work like a structure lift of the concept of group category [1].

In 1972, M.L. Laplaza introduced the concept of a distributivity category [4]. After that, in [1], A. Frohlich and C.T.C Wall introduced the concept of ring-like category, with the axiomatics which is asserted to be simpler than the one of M.L. Laplaza. These two concepts are categorization of the concept of commutative rings, as well as a generalization of the category of modules on a commutative ring $R$. In order to have descriptions on structures, and cohomological classify them, N. T. Quang has introduced the concept of Ann-categories
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[5], as a categorization of the concept of rings (unnecessarily commutative), with requirements of invertibility of objects and morphisms of the ground category, similar to of group categories (see [1]). With these requirements, we can prove that each congruence class of Ann-categories is completely defined by the invariants: the ring \( R \), the \( R \)-bimodule \( M \) and an element in the Mac Lane cohomology group \( H^3_{MacL}(R, M) \) (see [8]).

The concept of a braided Ann-category is a natural development of the concept of an Ann-category. The axioms of this concept presents natural relations between the constraints respect to \( \oplus, \otimes \). After that, we have proved the dependence of some axioms between the braiding and the distributivity constraints: thanks to the braiding \( c \), the distributivity constraint can be defined by only one side (right or left). Concurrently, the paper shows that each symmetric Ann-category (which the braiding is symmetric) satisfies the axiomatics due to M. L. Laplaza [4] and the axiomatics due to A. Frohlich and C. T. C. Wall [1]. Moreover, as a corollary, the paper shows a reduced system axiomatics of the one due to M. L. Laplaza, and prove that this axiomatics is equivalent to the one due to A. Frohlich and C. T. C. Wall.

In the last section, we present the center construction of an almost strict Ann-category, as an extension of the center construction of a tensor category due to Andr Joyal and Ross Street. This center construction is an example of the concept of a braided, but unsymmetric Ann-category.

In this paper, we sometimes denote \( XY \) instead of \( X \otimes Y \) of two objects \( X,Y \).

2 The definition of a braided Ann-category

Firstly, let us recall the definition of a braided tensor category according to [2].

A braiding for a tensor category \( \mathcal{V} \) consists of a natural collection of isomorphisms

\[
c = c_{A,B} : A \otimes B \xrightarrow{c} B \otimes A
\]

in \( \mathcal{V} \) such that the two diagrams (B1) and (B2) commute:

\[
\begin{align*}
& (A \otimes B) \otimes C \xrightarrow{c \otimes id} (B \otimes A) \otimes C \xrightarrow{a^{-1}} B \otimes (A \otimes C) \\
\downarrow a^{-1} & \downarrow id \otimes c

\end{align*}
\]

(B1)

\[
\begin{align*}
& A \otimes (B \otimes C) \xrightarrow{c} (B \otimes C) \otimes A \xrightarrow{a^{-1}} B \otimes (C \otimes A) \\
& A \otimes (B \otimes C) \xrightarrow{id \otimes c} A \otimes (C \otimes B) \xrightarrow{a} (A \otimes C) \otimes B \\
\downarrow a & \downarrow \varepsilon \otimes id

\end{align*}
\]

(B2)
If \(c\) is a braiding, so is \(c'\) given by \(c'_{A,B} = (c_{B,A})^{-1}\), since (B2) is just obtained from (B1) by replacing \(c\) with \(c'\). A symmetry is a braiding \(c\) which satisfies \(c' = c\).

A braided tensor category is a pair \((\mathcal{V}, c)\) consisting of a tensor category \(\mathcal{V}\) and a braiding \(c\).

**Definition 1.** [5, 7, 8] An Ann-category consists of:

i) A category \(\mathcal{A}\) together with two bifunctors \(\oplus, \otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}\).

ii) A fixed object \(0 \in \mathcal{A}\) together with naturality constraints \(a^+, c^+, g, d\) such that \((\mathcal{A}, \oplus, a^+, c^+, (0, g, d))\) is a symmetric categorical group.

iii) A fixed object \(1 \in \mathcal{A}\) together with naturality constraints \(a, l, r\) such that \((\mathcal{A}, \otimes, a, (1, l, r))\) is a monoidal \(\mathcal{A}\)-category.

iv) Natural isomorphisms \(\mathcal{L}, \mathcal{R}\)

\[
\mathcal{L}_{A,X,Y} : A \otimes (X \oplus Y) \to (A \otimes X) \oplus (A \otimes Y)
\]

\[
\mathcal{R}_{X,Y,A} : (X \oplus Y) \otimes A \to (X \otimes A) \oplus (Y \otimes A)
\]

such that the following conditions are satisfied:

(Ann-1) For each \(A \in \mathcal{A}\), the pairs \((L^A, L^A), (R^A, R^A)\) defined by relations:

\[
L^A = A \otimes - \quad R^A = - \otimes A
\]

\[
\mathcal{L}_{X,Y}^A = \mathcal{L}_{A,X,Y} \quad \mathcal{R}_{X,Y}^A = \mathcal{R}_{X,Y,A}
\]

are \(\oplus\)-functors which are compatible with \(a^+\) and \(c^+\).

(Ann-2) For all \(A, B, X, Y \in \mathcal{A}\), the following diagrams:

\[
\begin{array}{c}
(AB)(X \oplus Y) \xrightarrow{\alpha_{A,B,X \oplus Y}} A(B(X \oplus Y)) \xrightarrow{id_A \oplus \mathcal{L}^B} A(BX \oplus BY)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{L}^A \quad L^A
\end{array}
\]

(1)

\[
\begin{array}{c}
(AB)X \oplus (AB)Y \xrightarrow{\alpha_{A,B,X \oplus Y} \otimes \alpha_{A,B,Y}} A(BX) \oplus A(BY)
\end{array}
\]

\[
\begin{array}{c}
\mathcal{L}^A \quad L^A
\end{array}
\]

(2)

\[
\begin{array}{c}
X(AB) \oplus Y(AB) \xrightarrow{\alpha_{X,B,A} \oplus \alpha_{Y,B,A}} (X \otimes B)A \oplus (Y \otimes B)A
\end{array}
\]

(3)

\[
\mathcal{L}^A \quad L^A
\]

(4)
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commute, where \( v = v_{U,V,Z,T} : (U \oplus V) \oplus (Z \oplus T) \to (U \oplus Z) \oplus (V \oplus T) \) is the unique morphism built from \( a^+, c^+, \text{id} \) in the symmetric categorical group \((\mathcal{A}, \oplus)\).

(Ann-3) For the unity object \( 1 \in \mathcal{A} \) of the operation \( \oplus \), the following diagrams:

\[
\begin{array}{ccc}
1(X \oplus Y) & \overset{L^1}{\rightarrow} & 1X \oplus 1Y \\
\downarrow^{1X \oplus Y} & & \downarrow^{i_X \oplus i_Y} \\
X \oplus Y & \overset{i_X \oplus i_Y}{\rightarrow} & X \oplus Y
\end{array}
\]

\[
\begin{array}{ccc}
(X \oplus Y)1 & \overset{r^1}{\rightarrow} & X1 \oplus Y1 \\
\downarrow^{r_X \oplus r_Y} & & \downarrow^{r_X \oplus r_Y} \\
X \oplus Y & \overset{r_X \oplus r_Y}{\rightarrow} & X \oplus Y
\end{array}
\]

(5.1) (5.2)

commute.

**Definition 2.** A braided Ann-category \( \mathcal{A} \) is an Ann-category \( \mathcal{A} \) together with a braiding \( c \) such that \((\mathcal{A}, \otimes, a, c, (I, l, r))\) is a braided tensor category, and \( c \) makes the following diagram

\[
\begin{array}{ccc}
A.(X \oplus Y) & \overset{L_A \oplus Y}{\rightarrow} & A.X \oplus A.Y \\
\downarrow & & \downarrow \\
(X \oplus Y).A & \overset{R_A \oplus Y}{\rightarrow} & XA \oplus YA
\end{array}
\]

(6)

commutes and satisfies the condition: \( c_{0,0} = \text{id} \).

A braided Ann-category is called a symmetric Ann-category if the braiding \( c \) is a symmetric.

3 Some remarks on the axiomatics of a braided Ann-category

In the axiomatics of a braided Ann-category, we can see that the diagram (6) allows us to determine the right distributivity constraint \( \mathfrak{R} \) thanks to the left distributivity constraint and the braiding \( c \). So, we must consider the dependence or independence of the axioms which are related to the right distributivity constraint \( \mathfrak{R} \), as well as the dependence of some other conditions in our axiomatics.

Firstly, let us recall a result which has known.

**Proposition 3.1** (Proposition 2[7]). In the axiomatics of an Ann-category \( \mathcal{A} \), the compatibility of the functors \((L^A, \bar{L}^A), (R^A, \bar{R}^A)\) with the commutativity constraint \( c^+ \) can be deduced from the other axioms.

For the axiomatics of a braided Ann-category, we have the following result:

**Proposition 3.2.** In the axiomatics of a braided Ann-category \( \mathcal{A} \), the compatibility of the functor \((R^A, \bar{R}^A)\) [resp. \((L^A, \bar{L}^A)\)] with the associativity constraint \( a^+ \) can be deduced from the compatibility of the functor \((L^A, \bar{L}^A)\) [resp. \((R^A, \bar{R}^A)\)] with the constraint \( a^+ \) and the diagram (6).
Proof. To prove the compatibility of functor \((R^A, \tilde{R}^A)\) with the associativity constraint \(a^+\), let us consider the following diagram:

\[
\begin{array}{ccccccccc}
A((X \oplus Y) \oplus Z) & \xrightarrow{L} & A(X \oplus Y) \oplus AZ & \xrightarrow{\tilde{L} \oplus id} & (AX \oplus AY) \oplus AZ \\
(\l_\oplus a^+) & \xrightarrow{c} & (\l_\oplus c) & \xrightarrow{c \oplus c} & (c \oplus c) \oplus c \\
((X \oplus Y) \oplus Z) & \xrightarrow{\tilde{R}} & (X \oplus Y)A \oplus ZA & \xrightarrow{\tilde{R} \oplus id} & (XA \oplus YA) \oplus ZA \\
\xrightarrow{id \oplus a^+} & (I) & \xrightarrow{a^+ \oplus id_u} & (IV) & \xrightarrow{a^+} & (VII) \\
(X \oplus (Y \oplus Z)) & \xrightarrow{\tilde{L}} & AX \oplus A(Y \oplus Z) & \xrightarrow{id \oplus \tilde{L}} & AX \oplus (AY \oplus AZ) \\
\xrightarrow{c} & (V) & \xrightarrow{c \oplus c} & (VI) & \xrightarrow{c \oplus (c \oplus c)} \\
A(X \oplus (Y \oplus Z)) & \end{array}
\]

In that above diagram, the region (I) commutes thanks to the naturality of \(c\), the regions (II), (VIII), (IX) commute thanks to the diagram (6); the first component of the region (III) commutes thanks to the diagram (6), the second component one commutes thanks to the composition of morphisms, so the region (III) commutes; the first component of the region (VI) commutes thanks to the composition of morphisms, the second component one commutes thanks to the diagram (6), so the region (VI) commutes; the region (VII) commutes thanks to the naturality of the isomorphism \(a^+\), the perimeter commutes since \((L^A, \tilde{L}^A)\) is compatible with associativity constraint \(a^+\). Therefore, the region (IV) commutes, i.e., \((R^A, \tilde{R}^A)\) is compatible with \(a^+\).  

Proposition 3.3. In a braided Ann-category, the commutativity of the diagrams (2) and (3) can be deduced from the other axioms.

Proof. To prove that the diagram (3) commutes, we consider the following dia-
In the above diagram, the regions (I) and (V) commute thanks to the diagram (6), the regions (II) and (IV) commute thanks to the naturality of $\mathcal{L}$, the regions (III) and (V) commute thanks to the diagram (1), the region (VI) commutes thanks to the naturality of $c$, the regions (VIII) and (IX) commute thanks to the diagram (B2). Therefore, the perimeter commutes, i.e., the diagram (3) commutes.

To prove that the region (2) commutes, we consider the following diagram:
In the above diagram, the regions (I) and (VII) commute since \((\mathcal{A}, \otimes)\) is a braided tensor category; the regions (II), (IV) and (VIII) commute thanks to the diagram (6); the region (III) commutes thanks to Proposition 3.2; the region (VI) commutes thanks to the naturality of \(\mathcal{L}\); the perimeter commutes thanks to the diagram (1). Therefore, the diagram (VI) commutes, i.e., the diagram
(2) commutes.

**Proposition 3.4.** In the braided Ann-category $\mathcal{A}$, the commutativity of the diagram (5.2) can be deduced from the diagrams (5.1), (6) and the compatibility of $c$ with the unicity constraint $(I, l, r)$.

**Proof.** Consider the diagram:

![Diagram](image)

In the above diagram, the regions (I) and (IV) commute thanks to the compatibility of $c$ with the unicity constraint $(I, l, r)$; the region (II) commutes thanks to the diagram (5.1); the perimeter commutes thanks to the diagram (6). Therefore, the region (III) commutes, i.e., the diagram (5.2) commutes.

**Remark 1.** According to Proposition 3.1 - 3.4, in the axiomatics of a braided Ann-category, we can omit the diagrams (2), (3), (5.2) and the compatibility of the functors $(R^\mathcal{A}, \tilde{R}^\mathcal{A})$ with the constraints $a^+, c^+$ of the operator $\oplus$.

## 4 The object O

In an Ann-category, the object O has important properties which is used to define the $\Pi_0$-structure bimodule on $\Pi_1 = Aut(0)$, where $\Pi_0$ is the ring of congruence classes of objects.

**Proposition 4.1** (Proposition 1 [7]). In the Ann-category $\mathcal{A}$, there exist uniquely the isomorphisms:

$$\hat{L}^\mathcal{A} : A \otimes 0 \to A, \quad \hat{R}^\mathcal{A} : 0 \otimes A \to A$$

such that $(L^\mathcal{A}, \hat{L}^\mathcal{A}, \tilde{L}^\mathcal{A})$, $(R^\mathcal{A}, \hat{R}^\mathcal{A}, \tilde{R}^\mathcal{A})$ are the functors which are compatible with the unicity constraints of the operator $\oplus$ (also called $U$-functors).

A part from some properties of the object 0 in an Ann-category (see [9]), we have the following proposition.

**Proposition 4.2.** In a braided Ann-category, the following diagram

$$X.0 \xrightarrow{c} 0.X \xleftarrow{\hat{L} \hat{R}} 0$$

(7)
commutes.

Proof. Firstly, we can easily prove the following remark:

Let \((F, \tilde{F}), (G, \tilde{G}) : C \to C'\) be \(\oplus\)-functors which are compatible with the unitivity constraints and \(\tilde{F} : F(0) \to 0', \tilde{G} : G(0) \to 0'\) are isomorphisms, respectively. If \(\alpha : F \to G\) is an \(\oplus\)-morphism such that \(\alpha_0\) is an isomorphism, then the following diagram commutes.

\[
\begin{array}{ccc}
F0 & \xrightarrow{\alpha_0} & G0 \\
\downarrow & & \downarrow \\
\tilde{F} & \xleftarrow{\hat{F}} & \tilde{G}
\end{array}
\]

We now apply the above remark to the two functors \(F : A \otimes -\), \(G : - \otimes A\).

The morphism \(\alpha : F \to G\) is defined by

\[
\alpha_X = c_{A,X} : A \otimes X \to X \otimes A
\]

is an \(\oplus\)-morphism thanks to the diagram (6). Since \(\alpha_0 = c_{A,0}\) is an isomorphism, according to the above remark the diagram (7) commutes. \(\square\)

5 The relation between a symmetric Ann-category and a distributivity category due to Miguel L. Laplaza

Following, we will establish the relation between a symmetric Ann-category and a distributivity category due to [4], and deduce the coherence in a symmetric Ann-category [4].

A distributivity category \(\mathcal{A}\) consists of:

i) Two bifunctors \(\otimes, \oplus : A \times A \to A\);

ii) Two fixed objects \(O\) and \(1\) of \(\mathcal{A}\), called null and unit objects;

iii) Natural isomorphisms:

\[
\begin{align*}
    a_{A,B,C} : A \otimes (B \otimes C) & \to (A \otimes B) \otimes C, \\
    a_{A,B,C}^+ : A \oplus (B \oplus C) & \to (A \oplus B) \oplus C, \\
    l_A : 1 \otimes A & \to A, \\
    r_A : A \otimes 1 & \to A, \\
    g_A : O \oplus A & \to A, \\
    d_A : A \oplus O & \to A \\
    \hat{L}^A : A \otimes O & \to O, \\
    \hat{R}^A : O \otimes A & \to O
\end{align*}
\]

and natural isomorphisms:

\[
\begin{align*}
    \mathcal{L}_{A,B,C} : A \otimes (B \oplus C) & \to (A \otimes B) \oplus (A \otimes C), \\
    \mathcal{R}_{A,B,C} : (A \oplus B) \otimes C & \to (A \otimes C) \oplus (B \otimes C)
\end{align*}
\]
for all \(A, B, C\) of \(\mathcal{A}\).

These natural isomorphisms satisfy the coherence conditions for \(\{a, c, l, r\}\) and \(\{a^+, c^+, g, d\}\); the functors \((\hat{L}^A, \hat{L}^A), (\hat{R}^A, \hat{R}^A)\) (similar as in the concept of an Ann-category) are compatible with the associativity, commutativity constraints of the operator \(\oplus\), and satisfy the commutative diagrams (1), (2), (3), (4), (5.1), (5.2), (6) and 13 following diagrams:

\[
\hat{L}^O = \hat{R}^O : O \otimes O \to O \quad (L1)
\]
\[
O(A \oplus B) \xrightarrow{g} OA \oplus OB \quad (L2)
\]
\[
d_O(\hat{L}^A \oplus \hat{L}^B) \hat{R}_{A,B,O} = \hat{L}^A \oplus B : (A \oplus B)O \to O \quad (L3)
\]
\[
r_1 = g_O : 1 \otimes O \to O \quad (L4)
\]
\[
l_1 = d_O : O \otimes 1 \to O \quad (L5)
\]
\[
\hat{L}^A = \hat{R}^A, e_{A,O} : A \otimes O \to O \quad (L6)
\]
\[
O(AB) \xrightarrow{g_{O,A,B}} (OA)B \quad (L7)
\]
\[
A(AB) \xrightarrow{a_{A,O,B}} (AO)B \quad (L8)
\]
\[
\hat{L}^AB \otimes a_{A,O,B} = \hat{L}^A \otimes (id_A \otimes \hat{L}^B) : A(BO) \to O \quad (L9)
\]
\[
A(O \oplus B) \xrightarrow{g_{A,O,B}} AO \oplus AB \quad (L10)
\]
\[
g_{BA}(\hat{L}^A \oplus id_{BA}) \hat{R}_{O,B,A} = g_B \otimes id_A : (O \oplus B)A \to BA \quad (L11)
\]
\[
d_{AB}(id_{AB} \oplus \hat{L}^A) \hat{R}_{A,B,O} = id_A \otimes d^B : A(B \oplus O) \to AB \quad (L12)
\]
\[
d_{AB}(id_{AB} \oplus \hat{R}^B) \hat{R}_{A,O,B} = d_A \otimes id_B : (A \oplus O)B \to AB \quad (L13)
\]
**Proposition 5.1.** Each symmetric Ann-category is a distributivity category.

**Proof.** Let $\mathcal{A}$ be a symmetric Ann-category. From the definitions of a symmetric Ann-category and a distributivity category, we just must prove that $\mathcal{A}$ satisfies the conditions (L1)-(L13).

According to Proposition 3.4 [9], the diagrams (L2), (L3) commute.

According to Proposition 3.5 [9], we have the equations (L4), (L5).

According to Proposition 4.2, the diagram (L6) commutes.

According to Proposition 3.3 [9], the diagrams (L7), (L8), (L9) commute.

According to Proposition 3.1 [9], the diagrams (L10), (L11), (L12), (L13) commute.

From the condition $c_{0,0} = id$ and Proposition 4.2, we obtain $\hat{L}^{o} = \hat{R}^{o}$, i.e., the condition (L1) is satisfied. So, each symmetric Ann-category is a distributivity category.

In the above proof, only proof of the condition (L6) is related to the groupoid property of the ground category, so we have

**Corollary 5.2.** In the axiomatics of a distributivity category, the axioms (L1)-(L5), (L7)-(L13) are dependent.

**Corollary 5.3.** Let $\mathcal{A}$ be a category. $\mathcal{A}$ is a symmetric Ann-category iff $\mathcal{A}$ satisfies the two following conditions:

(i) $\mathcal{A}$ is a distributivity category;

(ii) Objects of category $(\mathcal{A}, \oplus)$ are all invertible and the ground category of $(\mathcal{A}, \oplus)$ is a groupoid.

**Proof.** The necessary condition can be deduced from the definition of a symmetric Ann-category and Proposition 5.1.

The sufficient condition is obvious.

We remark that the commutation of the diagrams in a symmetric Ann-category $\mathcal{A}$ is independent of the invertibility of objects and morphisms in the category $(\mathcal{A}, \oplus)$. Therefore, from the coherence theorem in a distributivity category [4], we have:

**Corollary 5.4.** (Coherence theorem) In a symmetric Ann-category, each morphism built from $a^{+}, c^{+}, g, d, a, c, l, r, \mathcal{L}, \mathcal{R}$ is just dependent on the source and the target.

The coherence theorem for a braided Ann-category is still an open problem.

### 6 The relation between a symmetric Ann-category and a ring-like category

Frohlich and C. T. C. Wall have presented the concept of a ring-like category as a generalization of the category of modules on a commutative ring $R$ (see [1]).
According to [1], a ring-like category consists of:
i) Two monoidal structures, given by the two functors: $\oplus, \otimes$;
ii) The distributivity isomorphisms:
\[ \mathcal{L} : A \otimes (B \oplus C) \rightarrow A \otimes B \oplus A \otimes C \]
such that the constraints $a^+, c^+, a, c, \mathcal{L}$ satisfy the coherence conditions, and the pairs $(L^A, \bar{L}^A)$ are the compatible with associativity, commutativity constraints of the operator $\oplus$, satisfying the diagram (1) and the following diagram (8):

![Diagram 8](image)

**Proposition 6.1.** Each symmetric Ann-category is a ring-like category.

**Proof.** According to the definition of a symmetric Ann-category and the definition of a ring-like category, we just prove that in a symmetric Ann-category, the diagram (8) commutes. Indeed, we consider the following diagram:

![Diagram 9](image)

In the above diagram, the region (I) commutes thanks to the diagram (6), the region (II) commutes thanks to the diagram (4); according to the diagram...
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(6), each component of the region (III) commutes, so does the region (III). Then, the perimeter commutes, i.e., the diagram (8) commutes. \[ \square \]

**Proposition 6.2.** Let \( \mathcal{A} \) be a category. Then, \( \mathcal{A} \) is a symmetric Ann-category iff it satisfies the following conditions:

i) \( \mathcal{A} \) is a ring-like category;

ii) All objects of the category \( (\mathcal{A}, \oplus) \) are invertible and the ground of the \( (\mathcal{A}, \oplus) \) is a groupoid.

**Proof.** The necessary condition can be implied from the definition of a symmetric Ann-category and Proposition 6.1.

The sufficient condition can be implied from the definition of a ring-like category and the following Proposition 6.3.

In the section 2, we have proved: In a symmetric Ann-category, the diagrams (2), (3), (4) and (5.2) can be omitted. Now, we prove that we can omit the right distributivity constraint.

**Proposition 6.3.** Let \( \mathcal{A} \) be a category and two bi-functors \( \oplus, \otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A} \). Then, \( \mathcal{A} \) is a symmetric Ann-category iff the following conditions are satisfied:

i) There is an object \( O \in \mathcal{A} \) and naturality isomorphisms \( a^+, c^+, g, d \) such that \( (\mathcal{A}, \oplus, a^+, c^+, (O, g, d)) \) is a symmetric categorical group.

ii) There is an object \( 1 \in \mathcal{A} \) together with naturality constraints \( a, c, l, r \) such that \( (\mathcal{A}, \otimes, a, c, (1, l, r)) \) is a symmetric monoidal category.

iii) There is a naturality isomorphism \( \Sigma : \mathcal{A} \otimes (X \oplus Y) \rightarrow (\mathcal{A} \otimes X) \oplus (\mathcal{A} \otimes Y) \) such that \( L^A = A \otimes - \), \( \bar{L}^A_{X,Y} = \Sigma_{A,X,Y} \) are \( \oplus \)-functors which are compatible with \( a^+ \) and \( c^+ \), and the diagrams (1), (8), (5.1) commute.

**Proof.** The necessary condition is obviously.

The sufficient condition: Let \( \mathcal{A} \) be a category satisfying all conditions in the above proposition. The right distributivity constraint in \( \mathcal{A} \) is defined by

\[ \mathcal{R} : (X \oplus Y) \otimes A \rightarrow X \otimes A \oplus Y \otimes A \]

thanks to the commutative diagram (6). Then, according to Propositions 3.2 – 3.4, to prove that \( \mathcal{A} \) is a symmetric Ann-category, it remains to prove that the diagram (4) commutes.

We consider the diagram (9). In this diagram, the regions (I) and (III) commute thanks to the definition of the distributivity constraint \( \mathcal{R} \), the perimeter commutes thanks to the diagram (8). Hence, the region (II) of the diagram (9) commutes, i.e., the diagram (4) commutes. The proposition has been proved. \[ \square \]

A. Frohlich and C. T. C Wall commented that the axiomatics of a distributivity category due to M. L. Laplaza [4] is complicated (see [1]). So, they presented the definition of a ring-like category in order to present a reduced version. However, in [1], authors have not show the relation between these two definitions.

As a corollary of the Propositions 5.3, 6.2, we have the following result:
Proposition 6.4. The two concepts of: a distributivity category and a ring-like category are equivalent.

Proof. Let $\mathcal{A}$ be a distributivity category. From the definition of a distributivity category and a ring-like category, it remains to show that in a ring-like category, the diagram (8) commutes.

Consider the diagram (9). Since, in a distributivity category, the diagrams (4) and (6) commute, so from the proof of Proposition 6.1, we imply that the perimeter of the diagram (9) commutes, i.e., the diagram (8) commutes.

Inversely, let $\mathcal{A}$ be a ring-like category. In $\mathcal{A}$, we put

$$\mathfrak{R} : (X \oplus Y)A \to XA \oplus YA.$$ 

It is a naturality constraint which is defined by the commutative diagram (6). Then, we prove similarly as Propositions 3.1, 3.2, 3.3, in a ring-like category, we deduced that the diagrams (2), (3) commute and we deduced that the functors $(R^A, \tilde{R}^A)$ are compatible with the associativity, commutativity of the operator of the $\oplus$.

Finally, it remains that the diagram (4) commutes. We consider the diagram (9). Since, in a ring-like category, the diagrams (6) and (8) commute, according to the proof of Proposition 6.1, we obtain the region (II) of the diagram (9) commutes, i.e., the diagram (4) commutes. So, each ring-like category is a distributivity category.

\[ \square \]

7 The center of an almost strict Ann-category

Andrè Joyal and Ross Street [2] have built a braided tensor $L_\mathcal{V}$ of a strict tensor category $(\mathcal{V}, \otimes)$. C. Kassel [3] has presented one for an arbitrary tensor category.

According to [3], the center $L_\mathcal{V}$ of the tensor category $\mathcal{V}$ is a category whose objects are pairs $(A, u)$, where $A \in Ob(\mathcal{V})$ and

$$u_X : A \otimes X \to X \otimes A$$

is a naturality transformation satisfying following commutative diagrams:

\begin{align*}
A \otimes I & \xrightarrow{u_I} I \otimes A \\
\downarrow \phi & \downarrow \phi \\
A & \xrightarrow{i} A \\
\end{align*}

\begin{align*}
2
(A \otimes X) \otimes Y & \xrightarrow{u_{A,X,Y}} A \otimes (X \otimes Y) \\
\downarrow u_{X,Y} & \downarrow u_{X,Y} \\
(X \otimes Y) \otimes A & \xrightarrow{\alpha_{X,Y}} (X \otimes Y) \otimes A \\
\end{align*}

\begin{align*}
(X \otimes A) \otimes Y & \xrightarrow{u_{X,A,Y}} X \otimes (A \otimes Y) \\
\downarrow id_X \otimes \alpha_{Y} & \downarrow id_X \otimes \alpha_{Y} \\
X \otimes (Y \otimes A) & \xrightarrow{\alpha_{X,Y}} X \otimes (Y \otimes A) \\
\end{align*}
A morphism $f : (A, u) \to (B, m)$ in $L_V$ is a morphism $f : A \to B$ satisfies the condition

$$m_X \circ (f \otimes id) = (id \otimes f) \circ u_Y$$  \hspace{1cm} (10)$$

for all $X \in V$.

The tensor product of two objects in $L_V$ is defined:

$$(A, u) \times (B, m) = (A \otimes B, u \times m)$$

in which $u \times m$ is a morphism defined by the following commutative diagram:

$$
\begin{array}{ccc}
(A \otimes B) \otimes X & \xrightarrow{a} & A \otimes (B \otimes X) & \xrightarrow{id \otimes m_X} & A \otimes (X \otimes B) \\
(u \times m)_X & \downarrow & & \downarrow & \\
X \otimes (A \otimes B) & \xrightarrow{a} & (X \otimes A) \otimes B & \xrightarrow{u_X \otimes id} & (A \otimes X) \otimes B
\end{array}
$$

Then $L_V$ is a braided tensor category with the braiding defined by

$$c_{(A, u), (B, m)} = u_B : (A, u) \times (B, m) \to (B, m) \times (A, u).$$

Now, we will build the tensor of an Ann-category $A$ and the main result of this section is Theorem 7.3. Firstly, let us recall that each Ann-category is Ann-equivalent to the reduced one of the type $(R, M)$. Moreover, each Ann-category of the type $(R, M)$ is equivalent to an almost strict Ann-category on it (see [6]), and in this category, the family of morphisms $i_X : X \oplus X' \to O$ is identity (an Ann-category is called almost strict if all its natural constraints, except for the commutativity constraint of the operation $\oplus$ and the left distributivity constraint, are identities). So, in this section, we always assume that $A$ is an almost strict Ann-category and the morphisms $i_X : X \oplus X' \to O$ are identity.

**Definition 3.** The center of an Ann-category $A$, denoted by $C_A$, is a category in which objects are pairs $(A, u)$, with $A \in Ob(A)$ and

$$u_X : A \otimes X \to X \otimes A$$

is a natural transformation satisfying the three conditions

$$(C1) : \hspace{1cm} u_I = id$$

$$(C2) : \hspace{1cm} u_{X \otimes Y} = (id_X \otimes u_Y) \circ (u_X \otimes id_Y)$$

$$(C3) : \hspace{1cm} u_{X \otimes Y} = (u_X \otimes u_Y) \circ \tilde{L}_{A, X, Y}.$$

The morphism $f : (A, u) \to (B, m)$ of $C_A$ is a morphism $f : A \to B$ of $A$ satisfying the condition (10).

**Proposition 7.1.** The center of an almost strict Ann-category $A$ is a symmetric categorical group with the sum of two objects are defined by

$$(A, u) + (B, m) = (A \oplus B, u + m)$$
On the braiding of an Ann-category

in which:

\[(u + m)_X = \tilde{L}^{-1}_{X,A,B} \circ (u_X \oplus m_X)\]

and the sum of two morphisms of \(C_A\) is the sum of morphisms in \(A\).

Proof. Firstly, we prove that for two objects \((A,u), (B,m)\) of \(C_A\) then \((A \oplus B, u + m)\) defined above is an object of \(C_A\). Indeed, since \(u_1 = \text{id}, m_1 = \text{id}, \tilde{L}_{1,A,B} = \text{id}\), we have \((u + m)_1 = \text{id}. On the other hand, we have

\[
(u + m)_{XY} = \tilde{L}^{-1}_{XY,A,B} \circ (u_{XY} \oplus m_{XY})
\]

\[
= \tilde{L}^{-1}_{X,Y,A,B} \circ (id_X \otimes \tilde{L}_{Y,A,B})^{-1} \circ \tilde{L}^{-1}_{X,Y,A,YB} \circ (id_X \otimes u_Y \oplus \text{id}_X \otimes m_Y)
\]

\[
= (id_X \otimes \tilde{L}_{Y,A,B})^{-1} \circ (id_X \otimes (u_Y \oplus m_Y)) \circ \tilde{L}^{-1}_{X,Y,A,B} \circ (u_X \otimes \text{id}_Y \oplus m_X \otimes \text{id}_Y)
\]

(\text{thanks to the definition of an Ann-category})

\[
= (id_X \otimes \tilde{L}_{Y,A,B})^{-1} \circ (id_X \otimes (u_Y \oplus m_Y)) \circ \tilde{L}^{-1}_{X,Y,A,B} \circ (u_X \otimes \text{id}_Y \oplus m_X \otimes \text{id}_Y)
\]

(\text{thanks to the naturality of} \(\tilde{L}\))

\[
= (id_X \otimes (u + m)_Y) \circ \tilde{L}^{-1}_{X,Y,A,B} \circ (u_X \otimes \text{id}_Y \oplus m_X \otimes \text{id}_Y)
\]

\[
= (id_X \otimes (u + m)_Y) \circ \tilde{L}^{-1}_{X,Y,A,B} \circ (u_X \otimes \text{id}_Y \oplus m_X \otimes \text{id}_Y)
\]

(\text{thanks to the definition of} \(u + m\))

So, \(u + m\) satisfies the condition \((C2)\).

Next, we verify that the morphism \(u + m\) satisfies the condition \((C3)\).

\[
(u + m)_{X \otimes Y} = \tilde{L}^{-1}_{X \otimes Y,A,B} \circ (u_{X \otimes Y} \oplus m_{X \otimes Y})
\]

\[
= \tilde{L}^{-1}_{X \otimes Y,A,B} \circ [(u_X \otimes m_X) \oplus (u_Y \otimes m_Y)](\tilde{L}_{A,X,Y} \circ \tilde{L}_{B,X,Y})
\]

(\text{since} \(u, m\) satisfy the condition \((C3)\))

\[
= \tilde{L}^{-1}_{X \otimes Y,A,B} \circ \psi_{X,A,YA,XB,YB} \circ [(u_X \otimes m_X) \oplus (u_Y \otimes m_Y)] \circ \psi_{AX,AY,BX,BY}
\]

\[
\circ \tilde{L}_{A,X,Y} \circ \tilde{L}_{B,X,Y}
\]

(\text{thanks to the naturality of the isomorphism} \(\psi\))

\[
= (\tilde{L}_{X,Y,A,B})^{-1} \circ ((u_X \otimes m_X) \oplus (u_Y \otimes m_Y)) \circ \tilde{L}_{A \oplus B,X,Y}
\]

(\text{thanks to the definition of an Ann-category})

\[
= ((u + m)_X \oplus (u + m)_Y) \circ \tilde{L}_{A \oplus B,X,Y}
\]

(\text{thanks to the definition of} \(u + m\))

So \(u + m\) satisfies the condition \((C3)\), i.e., \((A \oplus B, u + m)\) is an object of \(C_A\).

Now, we assume that \(f : (A,u) \rightarrow (B,m), g : (A',u') \rightarrow (B',m')\) are morphisms of \(C_A\). From the definition of the sum of two objects, the sum of morphisms in \(C_A\), and the naturality of the isomorphism \(\mathcal{L}\), and \(R = \text{id}\), we can
verify that \( f + g = f \oplus g \) is a morphism of the category \( C_A \). Furthermore, we can verify that

\[
id : ((A, u) + (B, m)) + (C, w) \to (A, u) + ((B, m) + (C, w))
\]

is the associativity constraint of the category \( C_A \), \( ((O, \theta_X = \hat{L}_X^{-1}), \text{id}, \text{id}) \) is the unitivity constraint of \( C_A \), and

\[
e^+_{A,B} : (A, u) + (B, m) \to (B, m) + (A, u)
\]

is the commutativity constraint of \( C_A \). Finally, we prove that each object of \( C_A \) is invertible.

Let \((A, u)\) be an object of the category \( C_A \). According to the Ann-category \( A \), there exists \( A' \in \text{Ob}(A) \) such that \( A \oplus A' = O \). We define a natural transformation \( u'_X : \hat{A}' X \to X A' \) as follows:

\[
u_X + u'_X = \hat{L}_{X,A,A'} \circ \theta_X
\]

Then, we have

\[
\theta_X = \theta_X \circ (u_X + u'_X) = \hat{L}_{X,A,A'} \circ (u_X + u'_X)
\]

We can easily verify that \( u' \) satisfies the conditions (C1) and (C2). Now, we will verify that the morphism \( u' \) satisfies the condition (C3).

\[
u_{X \otimes Y} + u'_{X \otimes Y} = \hat{L}_{X \otimes Y,A,A'} \circ \theta_{X \otimes Y}
\]

(thanks to the definition of \( u' \))

\[
\hat{L}_{X \otimes Y,A,A'} = \hat{L}_{X,A,A' \otimes Y} \circ \theta_{X \otimes Y}
\]

(according to the definition of an Ann-category)

\[
\hat{L}_{X \otimes Y,A,A'} = (\hat{L}_{X,A} \otimes \hat{L}_{A',Y}) \circ (\theta_X \otimes \theta_Y)
\]

(since \( A \oplus A' = O \))

\[
\hat{L}_{X \otimes Y,A,A'} = (u_X \oplus u'_X) \oplus (u_Y \oplus u'_Y)
\]

(thanks to the definition of \( u' \) and \( u + u' = \theta \))

\[
((u_X \oplus u_Y) \oplus (u'_X \oplus u'_Y)) \circ v_{AX,A',AY,A'}
\]

(thanks to the naturality of \( v \))

\[
((u_X \oplus u_Y) \oplus (u'_X \oplus u'_Y)) \circ (\hat{L}_{A,X,Y} \oplus \hat{L}_{A',X,Y})
\]

(thanks to the definition of an Ann-category).

Together with the equation \( u_{X \otimes Y} = (u_X \oplus u_Y) \circ \hat{L}_{A,X,Y} \), we have:

\[
u'_{X \otimes Y} = (u'_X \oplus u'_Y) \circ \hat{L}_{A',X,Y}
\]

So \((A', u')\) is an object of the category \( C_A \) and it is the invertible object of \((A, u)\) respect to the operator \(+\).

\textbf{Proposition 7.2.} The \( C_A \) is a braided tensor category where the tensor product of two objects is defined by

\[
(A, u) \times (B, m) = (A \otimes B, u \times m)
\]
in which \( u \times m \) is the morphism given by the diagram (11), and the tensor product of two morphisms in \( C_A \) is indeed the tensor product in \( \mathcal{A} \).

Proof. Let \((A, u), (B, m)\) be objects of \( C_A \). According to A. Joyal and R. Street [2], \( u \times m \) satisfies the two conditions (C1) and (C2). On the other hand, \( u \times m \) satisfies the condition (C3), since:

\[
(u \times m)_{X \otimes Y} = (u_{X \otimes Y} \otimes id) \circ (id_A \otimes m_{X \otimes Y})
\]

\[
= ((u_X \otimes u_Y) \otimes id_B) \circ (\tilde{L}_{A, X, Y} \otimes id_B) \circ (id_A \otimes m_{X \otimes Y})
\]

(since \( u \) satisfies the condition (C3))

\[
= ((u_X \otimes u_Y) \otimes id_B) \circ (\tilde{L}_{A, X, Y} \otimes id_B) \circ (id_A \otimes (m_X \otimes m_Y)) \circ (id_A \otimes \tilde{L}_{B, X, Y})
\]

(since \( m \) satisfies the condition (C3))

\[
= (u_X \otimes id_B \otimes u_Y \otimes id_B) \circ (\tilde{L}_{A, X, Y} \otimes id_B) \circ (id_A \otimes (m_X \otimes m_Y)) \circ (id_A \otimes \tilde{L}_{B, X, Y})
\]

(thanks to the definition of the Ann-category \( \mathcal{A} \) and \( \tilde{R} = id \))

\[
= (u_X \otimes id_B \otimes u_Y \otimes id_B) \circ (\tilde{L}_{A, X, Y} \otimes id_B) \circ (id_A \otimes (m_X \otimes m_Y)) \circ (id_A \otimes \tilde{L}_{B, X, Y})
\]

(according to the definition of an Ann-category)

\[
= ((u \otimes id_B) \circ (u_Y \otimes id_B)) \circ (id_A \otimes m_A \otimes id_A \otimes m_Y) \circ (id_A \otimes \tilde{L}_{B, X, Y})
\]

\[
\circ (id_A \otimes \tilde{L}_{B, X, Y})
\]

(since \( \tilde{L} \) is natural)

\[
= ((u \times m)_{X} \oplus (u \times m)_{Y}) \circ (id_A \otimes \tilde{L}_{B, X, Y})
\]

(thanks to the definition of \( u + m \))

\[
= ((u \times m)_{X} \oplus (u \times m)_{Y}) \circ (id_A \otimes \tilde{L}_{B, X, Y})
\]

So \((A \otimes B, u \times m)\) is an object of the category \( C_A \).

Assume that \( f : (A, u) \rightarrow (B, m) \), \( g : (A', u') \rightarrow (B', m') \) are two morphisms of \( C_A \). According to [2], the morphism

\[
f \times g = f \otimes g : (A, u) \times (A', u') \rightarrow (B, m) \times (B', m')
\]

satisfies the condition (10), i.e., \( f \times g \) is a morphism of \( C_A \). According to [2], \( C_A \) has an associativity constraint as follows:

\[
id : ((A, u) \times (B, m)) \times (c, w) \rightarrow (A, u) \times ((B, m) \times (C, w)).
\]

We can easily verify that \((I, id)\) is an object of \( C_A \) and it together with the identity constraints \( l = id, r = id \) is the unicity constraint respect to the operator \( \times \) of \( C_A \). Finally, according to [2], \( C_A \) is a braided tensor category with the braiding \( c \) given by:

\[
c_{(A, u), (B, m)} = u_B : (A, u) \times (B, m) \rightarrow (B, m) \times (A, u).
\]

\( \Box \)

Theorem 7.3. \( C_A \) is a braided Ann-category.
Proof. According to Proposition 7.1, \((C_A, +)\) is a symmetric categorical group. According to Proposition 7.2, \((C_A, \times)\) is a braided tensor category. Moreover, we can verify that \(C_A\) has a distributivity constraints:

\[
\hat{L}_{(A,u),(B,m),(C,w)} = \hat{L}_{A,B,C}, \quad \hat{R}_{(A,u),(B,m),(C,w)} = \text{id}
\]

Since \(A\) is an Ann-category, so is \(C_A\). On the other hand, each object \((A,u)\) of \(C_A\), natural isomorphism \(u\) satisfies the condition \((C3)\), so the constraints \(\hat{L}, \hat{R} = \text{id}, c\) of the category \(C_A\) satisfy the diagram (6). Since \(id_0 = \text{id}, c_{(0, id),(0, id)} = \text{id}\). So \(C_A\) is a braided Ann-category.

When \(A\) is an almost strict Ann-category of the type \((R, M)\), we can describe more detail the center of \(A\).

Let \(I\) be an almost strict Ann-category of the type \((R, M)\). Then, the center \(C_I\) of \(I\) is a category whose objects are pairs \((a, u)\), with \(a \in R\),

\[
u(x) : ax \to xa
\]

is a natural transformation satisfying the conditions \((C1)\), \((C2)\), \((C3)\).

Firstly, since \(u(x) : ax \to xa\) is a morphism of a category of the type \((R, M)\), so we have

\[
a x = xa \quad \forall x \in R
\]

So \(a \in Z(R)\).

Next, since \(u\) satisfies the condition \((C1)\), we have \(u(1) = 0\). The morphism \(u\) satisfies the condition \((C2)\), we have:

\[
u(xy) = xu(y) + yu(x).
\]

Finally, \(u\) satisfies the condition \((C3)\), i.e.,

\[
\lambda(a, x, y) = u(x) - u(x + y) + u(y)
\]

So, each object of \(I\) is a pair \((a, u)\), in which \(a \in Z(R)\) and

\[
u : R \to M
\]

is a function satisfying the three above conditions.

Let \(f : (a, u) \to (b, m)\) be a morphism of the category \(C_I\). Since \(f : a \to b\) is a morphism of \(I\), we have \(a = b\). The morphism \(f\) satisfies the condition \((10)\), i.e.,

\[
m(x) + (a, f) \otimes (0, x) = (0, x) \otimes (a, f) + u(x) \iff m(x) + xf = xf + u(x)
\]

So \(u = m\), i.e., morphisms of \(C_I\) are all edomorphisms.
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