Entanglement quasidistributions for Bell-state measurements

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Abstract. Measurements in the quantum domain exceed classical notions. This concerns fundamental questions about the nature of the measurement process itself, as well as applications, such as their function as building blocks of quantum information processing protocols. In this paper, we explore the notion of entanglement for detection devices in theory and experiment. A method is devised that allows one to determine nonlocal quantum coherence of positive operator-valued measures via negative contributions in a joint distribution that fully describes the measurement apparatus under study. This approach is then applied to experimental data for detectors that ideally project onto Bell states. In particular, we describe the reconstruction of the aforementioned entanglement quasidistributions from raw data and compare the resulting negativities with the expected from theory. Therefore, our method provides a versatile toolbox for analysing measurements regarding their quantum-correlation features for quantum science and quantum technology.
1. Introduction

Quantum phenomena are understood today as novel resources for advanced quantum operations that constitute the foundation of modern quantum technologies. A variety of notions of nonclassicality, such as entanglement, are results of quantum superpositions of states. Such quantum interference phenomena—nowadays collectively referred to as quantum coherence—can provide the sought-after resources for quantum information processing [1–3]. While the notion of coherence has a longstanding tradition in quantum optics [4–8], only recently, broader concepts of quantum coherence have been recognized and extensively studied in the context of operational usefulness in quantum information theory. This encompasses entanglement of multipartite quantum states as the essential nonlocal component of quantum coherence [2, 9]. For instance, entanglement is the basis for steering [10], as well as generalized notions of conditional quantum correlations [11, 12].

Equally fundamental, yet less frequently addressed is the matter of the quantumness of measurements. Recently, however, this topic has gained considerable momentum, and multiple theoretical methods for the certification of quantum features of detectors were put forward [13–19]. Making the leap from state-based quantum coherence to quantifying the quantum performance of measurement devices is important for measurement-based quantum computation, providing an equivalent approach to state-based information processing [20, 21]. Beyond its relevance for such application, general observables—determined by so-called positive operator-valued measures (POVMs)—that project onto nonclassical states are essential for quantum protocols. For example, entangled Bell-state measurements (BSMs) are paramount in quantum teleportation and, by extension, in quantum repeaters for quantum communication via entanglement swapping; see, e.g., Ref. [22] for a recent experiment. In addition, pioneering experiments have reported on the quantumness of measurements [23–25]. For example, experiments have confirmed the noncommutativity of certain observables [26, 27] and proved the incompatibility of quantum measurements with classical statistical models on a quantitative basis. Furthermore, fundamental measurement-induced quantum coherence effects of sequential measurements have been investigated [28]. However, a generally applicable strategy for a theoretical and experimental certification of nonlocal coherence of measurements is still missing.

In the context of quantum optics, the close relation between entanglement and quantum coherence of multimode light is well known [29–31], and quantitative relations between single-mode nonclassicality and multimode entanglement have been established [32]. In this context, quasiprobabilities are arguably the most essential and widely applied tool for the characterization of quantum states of quantum fields; see Ref. [33] for a recent review. Nonclassical multimode radiation fields are identified through the failure of such quasiprobabilities to find a correspondence in classical probability theory, typically displayed through negativities. In general, the origin of such negativities, be it single-mode quantum effects or entanglement, cannot be distinguished; however,
notable exceptions exist [34]. Moreover, certain notions of quantum coherence are not detectable via quantum-optical quasiprobabilities. To mitigate this limitation, a construction of quasiprobabilities for general notions of quantum coherence of states has been formulated [35]. This includes the theory of entanglement quasiprobabilities, whose negativities are a necessary and sufficient criterion for the identification of entanglement, for either bi- and multipartite states [35]. Such entanglement quasiprobabilities even found applications in experiments to probe sources of entangled light [36].

The experience from quantum optics can serve as a guide to further advancing on detector characterization strategies to modern concepts. For example, the nonclassical properties of single-photon detectors have been studied in experiments via quantum-optical quasiprobabilities [37]. However, a similar methodology for detector entanglement has neither been established nor implemented to date. Moreover, very recently, the relation between quantum coherence and entanglement of measurements has been studied in theory [38], analogously to the connection of single-mode nonclassicality and entanglement for states. Despite this intriguing relationship, however, the approach does provide neither a practical nor intuitive tool for the quantitative assessment of detector entanglement akin to negativities in quasiprobabilities.

In this paper, we introduce and implement a methodology for the entanglement characterization of POVMs in terms of quasidistributions. This allows us to assess the entanglement of detection devices on the basis of negativities in those distributions, constituting a necessary and sufficient method to detect entanglement of measurements. Using data from detector tomography, we present in great detail the reconstruction of such quasidistributions for general two-qubit measurements in experiments. The resulting negativities of this treatment are then compared with the predictions for ideal BSMs to assess the quality of detector entanglement. By mixing POVM elements, we further show that non-entangled measurements are accompanied by nonnegative distributions. Thereby, we provide a practical toolbox for studying the quantum performance of detectors with respect to their entanglement features for fundamental studies in quantum science and applications in quantum technology.

2. POVM entanglement

In this section, we establish the notion of entanglement of detection devices. The paper performs a different, unprecedented data analysis of the experiment reported in Ref. [39], implementing a detector tomography for BSMs, and develops the theory of entanglement quasidistributions for detectors, being based on entanglement quasiprobabilities for bi- and multipartite states [35]. An experimental reconstruction of entanglement quasiprobabilities for a Bell state was carried out [36]. Still, to date, no experimental entanglement characterization of POVMs has been carried out using the approach of entanglement quasiprobabilities. A key feature of our approach is that detector entanglement is intuitively displayed via negativities in joint distributions of
POVM elements. The underlying method for two qubits employs the two-qubit state representation known as standard form [40], being a correlation-diagonal representation in Pauli matrices, which is discussed later.

We formulate the formal aspects of POVM entanglement in Sec. 2.1. Theoretical expectations for BSMs are discussed in Sec. 2.2. The experiment under study is described in Sec. 2.3, and we conclude this section with an outline about the remainder of this work, Sec. 2.4.

2.1. Defining POVM entanglement

Let $\Pi_k$ be an element of a POVM, obeying $\forall k : \Pi_k \geq 0$ and $\sum_k \Pi_k = 1$. As it applies to the experiment under consideration, we restrict ourselves to finite-dimensional Hilbert spaces for sake of simplicity—especially, two qubits in the following sections. Similarly to the definition of separable states [41], we say that a POVM is separable if the decomposition

$$\Pi_k = \sum_{a,b} Q_k(a,b) |a\rangle \langle a| \otimes |b\rangle \langle b|,$$

in which $Q_k$ is a nonnegative joint distribution, holds true for all $k$. If this is not the case, we say the detection is entangled. Since local projectors form a generating set of the entire space of operators [42], the above decomposition is always possible when relaxing the nonnegativity constraint. Specifically, $\Pi_k$ is entangled if $Q_k \not\geq 0$, meaning there exist an entry $Q_k(a,b) < 0$ for at least one pair $(a,b)$. Please note that the wording quasidistributions—rather than quasiprobability—is appropriate since a POVM element $\Pi$ is not necessarily normalized, $\text{tr}(\Pi) \neq 1$, and quasiprobabilities require a unit normalization. In general, the representation via such quasidistributions is not unique [35]. The construction of optimal entanglement quasiprobabilities that ensure nonnegativity for the separable states was derived in Ref. [35]. This method straightforwardly extends to POVMs as $\Pi/\text{tr}(\Pi)$ describes a quantum state, being possible in finite dimensional spaces where $\text{tr}(\Pi) < \infty$ is always obeyed.

For the scenario of two qubits, and using local transformations, $T^{(A)} \otimes T^{(B)}$, two-qubit states can be put into the so-called standard form, $[T^{(A)} \otimes T^{(B)}] \rho [T^{(A)} \otimes T^{(B)}]^\dagger = \sum_w \rho_w \sigma_w^{\otimes 2}$, which is diagonal in the Pauli matrices $\sigma_w$, $w \in \{0, x, y, z\}$ [40]. This principle extends to general two-qubit POVM elements $\Pi$. How to obtain this standard form from data was derived in Ref. [36] and is the basis to obtain optimal quasidistributions via solutions of the so-called separability eigenvalue equations [35]. Thereby, optimal quasidistributions for entanglement can be computed that are negative if and only if the POVM element is entangled—extending beyond two qubits too [35].
Figure 1. Ideal entanglement quasidistributions $Q_k$ for $k \in \{0, x, y, z\}$ [Eq. (2)] for Bell-projection POVM elements [Eq. (3)]. Positive contributions (+1/3) display classical detector correlations while negativities (−1/6) are an unambiguous certification of POVM entanglement. Local projectors $|w\rangle\langle w|$ (left Bloch-sphere plots) are eigenstates of the Pauli matrices $\sigma_w$ for $w \in \{x, y, z\}$, allowing for a decomposition of the POVM elements according to Eq. (1).

2.2. Predictions for BSMs

Suppose the aforementioned standard form, $\Pi = \sum_{w \in \{0, x, y, z\}} \pi_w \sigma_w \otimes 2$, then the optimal quasidistribution in a compact form reads [35, 36]

$$Q\left(w^{(A)}_{\pm(A)}, w^{(B)}_{\pm(B)}\right) = \frac{q}{3} + |\pi_{w(A)}| \pm (A) \pm (B) \pi_{w(A)} \delta_{w(A), w(B)},$$

with the parameter $q = \pi_0 - |\pi_x| - |\pi_y| - |\pi_z|$ and the Kronecker symbol $\delta$. Furthermore, $w_\pm$ labels the eigenstates of Pauli operators, $\sigma_w |w_\pm\rangle = \pm |w_\pm\rangle$, and the superscripts determine the Alice’s (A) and Bob’s (B) subsystems, including signs of their eigenvalues. Please note that $q \geq 0$ and $q < 0$ respectively applies to separable and inseparable operators [35].

As an example, consider a BSM, represented by the set $\{\Pi_0, \Pi_x, \Pi_y, \Pi_z\}$. Each element $\Pi_w$ is a projector $\Pi_w = |\psi_w\rangle\langle \psi_w|$ for a Bell state $|\psi_w\rangle$ that is already in standard form. That is, we have

$$\Pi_0 = \frac{\sigma_0^{02} - \sigma_x^{02} - \sigma_y^{02} - \sigma_z^{02}}{4}, \quad \Pi_x = \frac{\sigma_0^{02} - \sigma_x^{02} + \sigma_y^{02} + \sigma_z^{02}}{4},$$
$$\Pi_y = \frac{\sigma_0^{02} + \sigma_x^{02} - \sigma_y^{02} + \sigma_z^{02}}{4}, \quad \Pi_z = \frac{\sigma_0^{02} + \sigma_x^{02} + \sigma_y^{02} - \sigma_z^{02}}{4}$$

(3)
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Figure 2. Sketch of the detection scheme that implements a BSM. A comprehensive characterization of our realization can be found in Ref. [39]. Because of the final detection in $D$-$A$ basis, the POVM elements are labeled as $\Pi_{AA}$, $\Pi_{AD}$, $\Pi_{DA}$, and $\Pi_{DD}$ throughout this work, indicating between which detectors coincidences have been recorded.

for the Bell states $|\psi_0\rangle = (|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)/\sqrt{2}$, $|\psi_x\rangle = (|0\rangle \otimes |0\rangle - |1\rangle \otimes |1\rangle)/\sqrt{2}$, $|\psi_y\rangle = (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)/\sqrt{2}$, and $|\psi_z\rangle = (|0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle)/\sqrt{2}$, respectively, using the computational basis $\{|0\rangle, |1\rangle\}$. Applying Eq. (2), the resulting ideal quasidistributions for BSMs are depicted in Fig. 1. The negativities in those quasidistributions tell us that projective measurements onto Bell states are, in fact, entangled. Those entangled POVM elements of ideal Bell projectors will be compared with our experimental reconstruction later in this paper.

2.3. Experimental setup

Consider the entangling detector, Fig. 2, which is based on the use of a photonic control-sign gate (C-SIGN) for polarisation qubits [39, 43–45]. Two photons from degenerate spontaneous down-conversion arrive at a partially polarizing beam splitter (PPBS), whose transmittivities are $T_H = 1$ for the horizontal ($H$) component and $T_V = 1/3$ for the vertical ($V$) component; therefore, quantum interference can only occur for vertical components. Two extra beam splitters of this kind, rotated by $90^\circ$, are inserted on the two output ports in order to balance polarization-dependent loss [46]. The gate works in post-selection, accepting only events leading to a coincidence between the two outputs.

The action of the gate leads the entangled states $\sqrt{2}^{-1/2}(|z_+ x_+\rangle \pm |z_- x_-\rangle)$ to $|x_\pm x_\pm\rangle$, and, similarly, $\sqrt{2}^{-1/2}(|z_+ x_-\rangle \pm |z_- x_+\rangle)$ to $|x_\pm x_-\rangle$. Considering the polarization encoding in Table 1, this implies that the four states in the Bell basis can be discriminated after the gate by a separable measurement in the diagonal basis. In combination, we expect POVM elements to correspond to projectors on Bell states [39]. The main factors causing a departure from the ideal can be identified in the actual value of $T_H$ and $T_V$, imperfect visibility, and local phase shifts. These not only cause mixtures but also an unbalance of the expected probabilities. A complete characterization of the
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Table 1. The table shows the computational bases (first column) as chosen for Alice (second column) and Bob (third column) in terms of polarization states. Also, the relation to eigenstates \( |w_{\pm}\rangle \) of Pauli matrices \( \sigma_w \) are provided. (Please note that irrelevant global phases are not included.)

| computational | Alice  | Bob  |
|---------------|--------|------|
| \( |0\rangle = |z_+\rangle \) | \( |H\rangle \) | \( |D\rangle \) |
| \( |1\rangle = |z_-\rangle \) | \( |V\rangle \) | \( |A\rangle \) |
| \( \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |x_+\rangle \) | \( |D\rangle \) | \( |H\rangle \) |
| \( \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |x_-\rangle \) | \( |A\rangle \) | \( |V\rangle \) |
| \( \frac{|0\rangle + i|1\rangle}{\sqrt{2}} = |y_+\rangle \) | \( |L\rangle \) | \( |R\rangle \) |
| \( \frac{|0\rangle - i|1\rangle}{\sqrt{2}} = |y_-\rangle \) | \( |R\rangle \) | \( |L\rangle \) |

BSM experiment was presented in Ref. [39]; we use the same set of data for our analysis of quasidistributions here.

2.4. Preliminary summary and outline

Analogous to the notion of inseparability of states [41], the notion of inseparable POVMs was established in this section, which naturally extends to more than two parties. Since even nonlocal operators can be expanded via products of local operators [42], we argued that entangled POVM elements may be expressed in this manner, however, requiring negative expansion coefficients which are not needed for separable detectors. This defines the concept of entanglement quasidistributions of POVMs, and negativities in such distributions are a necessary and sufficient criterion for entanglement of detection devices. This approach also unifies quasiprobabilities for entangled states [35] with quasidistributions for detector entanglement. As examples with specific relevance for the continuation of this work, we considered BSMs in two-qubit systems.

In the remainder of this paper, the specific steps from raw data to entanglement quasidistributions for a BSM are laid out, further applying to arbitrary two-qubit measurements. Specifically, the data processing for the detector tomography is described in Sec. 3. Further processing then yields the sought-after entanglement quasidistributions, Sec. 4. A concluding discussion is given in Sec. 5, which includes a comparison with our theoretical predictions. For instance, the quality of the BSM is assessed by analysing the maximal negativities, the negativities with the highest statistical significance, as well as the total negativities. Also, a comparison with separable detectors is carried out, where non-entangled POVMs can be straightforwardly mimicked by mixing data to eliminate quantum correlations. To our knowledge, this leads to the first description of POVMs via quasiprobabilities with negativities in joint distributions as unique signatures of its entanglement.
Figure 3. Raw data in form of total number of measured coincidences from both detectors (top-left plot) and the resulting relative coincidences for each POVM element $\Pi_k$. Axes label the input product states for horizontal (H), vertical (V), diagonal (D), antidiagonal (A), right-circular (R) and left-circular (L) polarization for realizing the detector tomography. Note that bars for relative coincidences are filled to the value $1/4$ to easily distinguish between above- and below-average count rates—i.e., the deviation from uniformity of total counts distributed among the four individual POVM elements.

3. Data processing I: Detector tomography

In this section, we start with presenting the measured data in Sec. 3.1, suitable local computational bases are established in Sec. 3.2, the POVM is reconstructed in Sec. 3.3, and corrections for unphysical features of POVM elements are discussed in Sec. 3.4.

3.1. Raw data

Data for each POVM element are recorded for product states $|a\rangle \otimes |b\rangle$ with known polarization to implement a detector tomography. For each element, the coincidence counts $E_k(a,b)$ for $k \in \{AA, AD, DA, DD\}$ can be summed to obtain the total counts for each probe state, $E(a,b) = \sum_k E_k(a,b)$. Thereby, relative frequencies

$$p_k(a,b) = \frac{E_k(a,b)}{E(a,b)} = \text{tr} \left( \Pi_k |a\rangle \langle a| \otimes |b\rangle \langle b| \right),$$

(4)

can be defined, yielding the probabilities for the following reconstruction. Those total counts and relative coincidences are shown in Fig. 3, representing our raw data. The probe states for both subsystems comprise $|H\rangle$, $|V\rangle$, $|D\rangle = (|H\rangle + |V\rangle)/\sqrt{2}$, $|A\rangle = (|H\rangle - |V\rangle)/\sqrt{2}$, $|R\rangle = (|H\rangle - i|V\rangle)/\sqrt{2}$, and $|L\rangle = (|H\rangle + i|V\rangle)/\sqrt{2}$, being the common mutually unbiased bases of a polarization qubit.
### 3.2. Local computational bases

As outlined in Ref. [39], it is convenient to consider well-chosen local bases. That is, Alice uses a horizontal-vertical basis, and Bob employs a diagonal-antidiagonal one. This choice has no effect on the entanglement but changes representations of Pauli matrices that are used to determine correlations and that are formulated in terms of the computational basis \{0, 1\}. \(\sigma_z = |0\rangle \langle 0| - |1\rangle \langle 1|\), \(\sigma_x = |0\rangle \langle 1| + |1\rangle \langle 0|\), and \(\sigma_y = i|1\rangle \langle 0| - i|0\rangle \langle 1|\). In Table 1, the bases choices for the measured data are provided. With that, local Pauli matrices can be straightforwardly obtained,

\[
\sigma_w = |w_+\rangle \langle w_+| - |w_\mp\rangle \langle w_\mp|, \quad \text{for} \ w \in \{x, y, z\}, \tag{5}
\]

which is relevant for determining the correlations \(\text{tr}(\Pi_k \sigma_w^{(A)} \otimes \sigma_w^{(B)})\) for the POVM elements \(\Pi_k\) under study. For completeness, the \(2 \times 2\) identity can be expressed symmetrically as

\[
\sigma_0 = \frac{1}{3} \sigma_0 + \frac{1}{3} \sigma_0 + \frac{1}{3} \sigma_0 = \frac{1}{3} \sum_{w \in \{x, y, z\}} (|w_+\rangle \langle w_+| + |w_\mp\rangle \langle w_\mp|). \tag{6}
\]

### 3.3. Correlation matrix and POVM reconstruction

Our goal is now to decompose the elements \(\Pi_k\) in terms of Pauli matrices, \(\Pi_k = \sum_{w^{(A)},w^{(B)} \in \{0,x,y,z\}} \pi_{w^{(A)},w^{(B)}|k} \sigma_{w^{(A)}} \otimes \sigma_{w^{(B)}}\), also defining a Pauli-correlation matrix \(C_{k} = [\pi_{w^{(A)},w^{(B)}|k}]_{w^{(A)},w^{(B)} \in \{0,x,y,z\}}\). This further yields the computational basis expansion of \(\Pi_k\) from the informationally complete set of measurements. For a measured probe state, e.g., \(|a\rangle \otimes |b\rangle = |w^{(A)}_\pm\rangle \otimes |w^{(B)}_\pm\rangle\), we can use Eq. (4) and the bases in Table 1 to obtain the desired coefficients from the data in Fig. 3. To this end, we can define the matrix of relative coincidences, \(P_k = [p_k(a,b)]_{a,b \in \{H,V,D,A,R,L\}}\) and sampling matrices

\[
S^{(A)} = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
1 & -1 & 0 & 0 & 0 & 0
\end{bmatrix} \quad \text{and} \quad S^{(B)} = \begin{bmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\
1 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & -1 \\
0 & 0 & 1 & -1 & 0 & 0
\end{bmatrix}. \tag{7}
\]

Together, those matrices deliver the sought-after expansion coefficients via

\[
C_{k} = \frac{1}{4} S^{(A)} P_k S^{(B)^T} = \left[\text{tr} \left( \Pi_k \frac{1}{2} \sigma_{w^{(A)}} \otimes \frac{1}{2} \sigma_{w^{(B)}} \right) \right]_{w^{(A)},w^{(B)} \in \{0,x,y,z\}} \tag{8}
\]

using orthogonality in the form \(\text{tr}(\sigma_w \sigma_{w'}) = 2 \delta_{w,w'}\). \(\text{The coefficients of the sampling matrices in Eq. (7) describe the relations in Eqs. (5) and (6) as well as Table 1.}\)

From the computed coefficients, i.e., elements of \(C_k\), one also obtains the basis expansion of \(\Pi_k\) as the expansion of the Pauli matrices is known. In Fig. 4, this
Figure 4. Reconstructed $\Pi_k$ in terms of real (left column) and imaginary (right column) part in the computational basis, $|k^{(A)}\rangle\langle l^{(A)}| \otimes |k^{(B)}\rangle\langle l^{(B)}|$ for $k^{(A)}, l^{(A)}, k^{(B)}, l^{(B)} \in \{0, 1\}$. Imaginary contributions are comparably small. Real parts show relations to measurements in terms of Bell-state projectors. No correction for imperfections have been carried out to determine the shown decomposition.

bipartite basis expansion as obtained from the data is depicted. We find that, from top to bottom, the POVM elements resemble projective measurements for the Bell states $|\psi_0\rangle$, $|\psi_x\rangle$, $|\psi_z\rangle$, and $|\psi_y\rangle$. The fidelities with those projectors were reported previously [39], using a different reconstruction approach, and are all above 90%.

3.4. Noise addition for indefiniteness

Within the numerical precision ($10^{-9}$), the reconstructed POVM elements satisfy $\sum_k \Pi_k = 1$. However, the positive semidefiniteness, $\Pi_k \geq 0$, is slightly violated, constituting a common issue in tomographic reconstruction schemes. This indefiniteness
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can be easily accounted for without falsely increasing POVM entanglement properties; this is discussed in the following.

For the aforementioned correction, we consider a uniform white-noise addition,

$$\Pi_k \mapsto (1 - p)\Pi_k + p\frac{1}{4}$$

(9)

for all four POVM elements and $0 \leq p \leq 1$. Since the two-qubit identity $\mathbb{1} = \sigma_0 \otimes \sigma_2$ is a product—hence, uncorrelated—operator, the above mixing operation with the separable $\mathbb{1}$ cannot increase inseparability. Moreover, $\sum_k \Pi_k = \mathbb{1}$ is also not influenced by this mapping. In the following, the mixing probability $p$ is chosen such that $\Pi_k \geq 0$ is simultaneously satisfied for all $k$. Importantly, this procedure makes sure that negativities we observe in quasidistributions are a result of entanglement and not a result from slightly unphysical POVM reconstructions.

Let $-\lambda_{\text{max. neg.}}$ be the minimal eigenvalue of all POVM elements. (We set $\lambda_{\text{max. neg.}} = 0$ if all elements are already positive semidefinite.) To further enhance numerical stability, we can add a small extra margin, $\lambda_{\text{max. neg.}} \mapsto \lambda_{\text{max. neg.}} + 10^{-5}$, implying positive definiteness, $\Pi_k > 0$. In our case, we get $\lambda_{\text{max. neg.}} \approx 0.05$ in this manner, which is comparably small considering maximal positive eigenvalues that are close to unity. Finally, the map in Eq. (9) results in a proper (i.e., physical) POVM for $p = \lambda_{\text{max. neg.}} / (\lambda_{\text{max. neg.}} + 1/4)$. The thereby obtained POVM is used for further entanglement characterization, despite resulting in reduced quantum correlations because of the extra uncorrelated noise.

4. Data processing II: Quasidistribution reconstruction

In this section, the reconstruction of entanglement quasidistributions is carried out, which is based on the results of the previous section. The transformation of correlation matrices to the standard form is presented in Sec. 4.1, and the thereby implied transformation of local bases states is given in Sec. 4.2. The propagation of uncertainties via a common Monte Carlo approach is explained for completeness in Sec. 4.3.

4.1. Transformation to standard form

As developed in Ref. [36], the numerical transformation of the correlation matrices $C_k$, containing the expansion coefficients of $\Pi_k$ in Pauli-operator expansion, is a two-step process. The first transformation removes local elements such that coefficients for $\sigma_w \otimes \sigma_0$ and $\sigma_0 \otimes \sigma_w$ vanish for all $w \in \{x, y, z\}$. The second step is a rotation for concluding the diagonalization, meaning that coefficients for $\sigma_w \otimes \sigma_{w'}$ become zero for $w \neq w'$. In Fig. 5, the two steps are depicted, exemplified for $\Pi_k$ with $k = AA$. One can see how $C_k$ is successively becoming more diagonal. The first step, (i) $\mapsto$ (ii), acts like a Lorentz boost operation on the Pauli-expansion [36, 40]. In terms of the operators
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Figure 5. Transformation to standard form of the correlation matrix $C_{AA} = [\pi_{w,w'}| AA]_{w,w' \in \{0,x,y,z\}}$ that describes the POVM element $\Pi_{AA}$ as an example. The initial matrix (i) is transformed such that $\pi_{w,0| AA} = 0$ holds true for (ii), thus removing local correlations that are given via an identity in one subsystem. Then the fully diagonal form in (iii) is obtained by local rotations, resulting in coefficients $\pi_{w,w'| AA} = 0$ for $w \neq w'$.

themselves, this describes a local invertible operation,

$$\Pi \mapsto \left[ L^{(A)} \otimes L^{(B)} \right] \Pi \left[ L^{(A)} \otimes L^{(B)} \right]^\dagger,$$

where the inverse for both $L^{(A)}$ and $L^{(B)}$ exist but is generally not unitary. The second transformation, (ii) $\mapsto$ (iii), is a SO(3) rotation in the Pauli representation. It is worth mentioning that the rotations are chosen such that the ordering of magnitudes and signs along the diagonal is preserved, which helps to minimize rotations on the Bloch sphere and preserves directionality to some extent when compared with our initial local basis choice. The obtained rotations act as a local unitary on the operators,

$$\Pi \mapsto \left[ U^{(A)} \otimes U^{(B)} \right] \Pi \left[ U^{(A)} \otimes U^{(B)} \right]^\dagger.$$

Eventually, we obtain the sought-after standard form to which we can apply the quasidistribution as expressed in Eq. (2), likewise

$$\left[ U^{(A)} L^{(A)} \otimes U^{(B)} L^{(B)} \right] \Pi \left[ U^{(A)} L^{(A)} \otimes U^{(B)} L^{(B)} \right]^\dagger = \sum_{w \in \{0,x,y,z\}} \pi_w \sigma_w \otimes 2 \sum_{k,l=1}^6 Q(a_k, b_l) |a_k\rangle \langle a_k| \otimes |b_l\rangle \langle b_l|,$$

where we relabel the states $[x_+, x_-, y_+, y_-, z_+, z_-]$ as $[a_1, \ldots, a_6]$ and $[b_1, \ldots, b_6]$ for convenience, and $Q$ is the solution in standard form, Eq. (2). The numerical specifics for determining the boost-like and rotation operations can be found in the supplemental material of Ref. [36].
4.2. Local basis transformations

Conversely to the previous relation, we can express the POVM element as through the inverse transformation. That is, we have

$$\Pi = \sum_{k,l=1}^{6} Q(\tilde{a}_k, \tilde{b}_l)|\tilde{a}_k\rangle \langle \tilde{a}_k| \otimes |\tilde{b}_l\rangle \langle \tilde{b}_l|$$

$$= \left[ U^{(A)} L^{(A)} \otimes U^{(B)} L^{(B)} \right]^{-1} \left[ \sum_{k,l=1}^{6} Q(a_k, b_l)|a_k\rangle \langle a_k| \otimes |b_l\rangle \langle b_l| \right] \left[ U^{(A)} L^{(A)} \otimes U^{(B)} L^{(B)} \right]^{-\dagger}.$$  

(13)
In this formula, we use normalized states and a correspondingly renormalized distribution, given as follows:

\[
\langle \hat{a}_k | \hat{a}_k \rangle = \frac{L^{(A)}-1 U^{(A)} | a_k \rangle \langle a_k | U^{(A)} L^{(A)\dagger}}{\langle a_k | U^{(A)} L^{(A)\dagger} L^{(A)-1} U^{(A)\dagger} | a_k \rangle},
\]

\[
\langle \hat{b}_l | \hat{b}_l \rangle = \frac{L^{(B)}-1 U^{(B)} | b_l \rangle \langle b_l | U^{(B)} L^{(B)\dagger}}{\langle b_l | U^{(B)} L^{(B)\dagger} L^{(B)-1} U^{(B)\dagger} | b_l \rangle},
\]

\[
Q(\hat{a}_k, \hat{b}_l) = Q(a_k, b_l) \langle a_k | U^{(A)} L^{(A)\dagger} L^{(A)-1} U^{(A)\dagger} a_k \rangle \langle b_l | U^{(B)} L^{(B)\dagger} L^{(B)-1} U^{(B)\dagger} | b_l \rangle.
\]

Similarly to the previous two-step description, we depicted the resulting transformation of both local states from standard form over rotations \([U^{(A)} \otimes U^{(B)}]^{-1}\) to the final boost transformations \([L^{(A)} \otimes L^{(B)}]^{-1}\) in Fig. 6.

We can also confirm that the POVM element expressed by this quasidistribution and the corresponding local states does, within the numerical precision, exactly describe the previously reconstructed POVM element in Fig. 4. This demonstrates the successful representation of an entangled POVM via quasidistributions.

4.3. Error propagation

The methods described so far have been applied to estimate mean values. To determine uncertainties, a random sample for a Monte Carlo error propagation is prepared. Each sample element undergoes the aforementioned processing, allowing one to estimate the resulting fluctuations. The ensuing error estimates for our quasidistributions are depicted in Fig. 7.

To implement the error propagation, a sample of 10,000 relative coincidence matrices \([P_k(a, b)]_{k \in \{AA, AD, DA, DD\}}\) is generated for each probe-state setting \((a, b)\). This sample is distributed with a mean that corresponds to previously determined relative frequencies, \(\mu_k = P_k(a, b) = E_k(a, b)/E(a, b) [\text{Eq. (4)}]\). Fluctuations are implemented through the covariance matrix of the counting statistics, \(\Sigma_{k,k'} = \left[\delta_{k,k'} P_k(a, b) - P_k(a, b) P_k(a, b')/|E(a, b) - 1|\right]\), to describe the standard deviation as well as cross-correlations in the data. These uncertainties are multiplied by 1.05 to provide an extra 5% error margin as a safeguard to counter common issues, such as undersampling. The sample elements generated in this manner are further chosen to be normalized and nonnegative as they resemble probabilities, \(P_k(a, b) \geq 0\) and \(\sum_k P_k(a, b) = 1\).

As mentioned before, each sample element is treated with the reconstruction approaches established in Secs. 3 and 4. The standard deviation, for example, of the resulting sample of quasidistributions then provides the error margin, as depicted in Fig. 7. This concludes the full reconstruction from detector-tomography raw data to quasidistributions.

5. Discussion and conclusion

After comprehensively presenting the data processing approach, our conclusions from this reconstruction is presented in this section, together with the implications pertaining...
Figure 7. Reconstructed quasidistributions (left column) for all POVM elements, including a one-standard-deviation error margin (black bars). The right column shows the corresponding local states for the decomposition according to Eq. (1). In comparison with the ideal cases, Fig. 1, one can observe here the same general structure of POVM elements. In theory, the maximal negativity is $-\frac{1}{6} \approx -0.17$. Here, we find the highest negativities as $-0.20 \pm 0.06$ (please mind the error margin), $-0.14 \pm 0.01$, $-0.13 \pm 0.02$, and $-0.14 \pm 0.02$ for $AA$, $AD$, $DA$, and $DD$, respectively.
to the entangled nature of the realized BSM. A brief discussion is provided in Sec. 5.1. An additional comparison with separable POVMs is done in Sec. 5.2. Eventually, we summarize the findings of the paper in Sec. 5.3.

5.1. Results

The entangled POVM in Fig. 7 are locally described via Eq. (1), however, requiring the depicted negativities in the joint distribution to capture the detector entanglement of the experimentally implemented device. Furthermore, these results structurally relate quite well to the ideal POVM elements that one expects for unperturbed BSMs (Fig. 1). For instance, the nonlocal (negative) and local (positive) contributions are found in the same pattern that one can see in the theory plots.

In terms of statistical significance, we find that the most significant negativities are 15 standard deviations for \( \Pi_{AA} \), 17 standard deviations for \( \Pi_{AD} \), 8 standard deviations for \( \Pi_{DA} \), and 16 standard deviations for \( \Pi_{DD} \) below the classical threshold of zero. Therefore, our results show a highly significant POVM entanglement of the implemented detection scheme. (Please note that those highest significances not necessarily coincide with the ones that exhibit the highest absolute negativity reported in Fig. 7). Furthermore, and using the same POVM element order, the cumulative negativities, i.e., the sums over all negative entries of \( Q \), are \(-0.77 \pm 0.08, -0.65 \pm 0.03, -0.65 \pm 0.05, \) and \(-0.72 \pm 0.03 \). For comparison, the perfect case yields minus one for all Bell-type POVM elements (Fig. 1 with six negative contributions with the value \(-1/6 \)), not being drastically larger than we find for our data.

We emphasize that, except for accounting for unphysical eigenvalues of POVM elements by mixing with separable noise, no corrections for imperfections, such as deconvolutions of attenuations and postprocessing for other sources of noise, have been carried out. Still, a highly significant verification of detector entanglement that is essential for quantum information processing was confirmed with our methodology. Furthermore, the framework provides an intuitive (visual) signature of entanglement of detectors and yields a unified foundation with entanglement of states by virtue of analogous entanglement quasiprobability methods [36]. Also, since such analogue methods for states extend to multipartite and qudit entanglement [35], the POVM framework discussed here is similarly extendable to high-dimensional scenarios.

5.2. Comparison with separable POVMs

For completeness, we may also show that separable POVMs truly lead to nonnegative—i.e., classical—distributions. To this end, it is worth mentioning the known fact that the uniform mixture of a Bell state with any of the other Bell states results in a separable operator. Thus, we might mix our data accordingly to probe if this indeed produces non-entangled POVMs.

For example, we can combine POVM elements which are identical in one of the indices to produce new POVMs, such as \( \{ \Pi_{A*} = \Pi_{AA} + \Pi_{AD}, \Pi_{D*} = \Pi_{DA} + \Pi_{DD} \} \) and
Figure 8. Quasidistributions for POVMs that combine two Bell-like states and are therefore expected to be separable. Our reconstruction correctly reveals this feature.

\( \{ \Pi_{aA} = \Pi_{AA} + \Pi_{AD}, \Pi_{dD} = \Pi_{AD} + \Pi_{DD} \} \). In terms of data, this means adding counts \( E_k(a, b) \) accordingly and follow the same reconstruction approach as carried out for the BSM. The results of this treatment can be found in Fig. 8. Indeed, the quasidistributions for both sets of POVMs appear to be nonnegative as one expects for separable POVMs. In terms of our device, this means that it is vital to record the outcomes of both detectors to be able to harness the detector entanglement in quantum protocols as the loss of that information results in a local model of the measurement device.

5.3. Summary

In summary, we introduced a framework to assess entanglement of POVMs. Based on our definition of a separable POVM in terms of nonnegative mixtures of local (i.e., product) projectors, entangled POVMs cannot exhibit such a local representation. Rather, nonlocal coherence contributes to the recorded measurement outcome from entangled POVMs, even when separable states are measured. As an intuitive approach, it was shown that these nonlocal measurement features can be represented in terms of pseudomixtures. Meaning a local-like representation (preserving product projectors) is possible when allowing for mixing ratios that include negative contributions, while being strictly nonnegative for classically correlated detection schemes. The negativity of the thereby defined joint quasidistribution constitutes a necessary and sufficient criterion for POVM entanglement.

Specifically, we studied BSMs because of their fundamental and applied importance.
We provided the detailed step-by-step reconstruction from raw detector-tomography data to fully reconstructed quasidistributions, including error estimates. With high statistical significance, it was then certified that the experimental detection scheme under study does include negativities in the quasidistributions of each POVM element. This means that the operation of this detector cannot be explained in terms of local coherence effects alone for any of the possible measurement outcome, identifying entanglement as a essential quantum resource for function of this device. As a counterexample, we also showed that our data processing method correctly leads to nonnegative joint distributions for separable POVMs, thus admitting a local measurement model.

Therefore, our highly sensitive and comparably easily accessible diagnostic tool offers a novel approach to characterizing quantum detectors regarding their quantum-correlation properties. This not only renders it possible to decide the fundamental question whether a local description of a specific quantum measurement is possible but also provides a practical means to assess the nonlocal performance of detection devices for quantum-technological applications.

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