SELF-DUAL CHERN–SIMONS SOLITONS
WITH NON-COMPACT GROUPS*

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ABSTRACT

It is shown how to couple non-relativistic matter with a Chern–Simons gauge field that belongs to a non-compact group. We treat in some details the $SL(2, \mathbb{R})$ and the Poincaré $ISO(2,1)$ groups. For suitable self-interactions, we are able to exhibit soliton solutions.
INTRODUCTION

Recently, there has been interest in the interaction of non-relativistic planar matter fields with Chern–Simons gauge fields. When self-interactions are suitably chosen, and the gauge group is compact, the system admits static solutions fulfilling a set of self-dual equations. The completeness of these static solutions has been discussed in Ref. [2]. A natural question addresses the generalization to arbitrary gauge groups. In this paper we present a general framework that encompasses models with compact as well as non-compact and non-semi-simple gauge groups. In Section I we generalize the notion of a Killing form that we need to define the system. In Section II we show that the reduction of the four-dimensional Yang–Mills self-dual equations leads to static solutions of our problem, provided the matter fields are taken in the adjoint representation. In Section III we treat as examples the semi-simple $SL(2, \mathbb{R})$ group and the non-semi-simple Poincaré group, $ISO(2, 1)$. In both cases, special Ansätze give explicit solutions to the static problem. Concluding remarks are given in Section IV, while an Appendix recalls some useful tools in Lie algebra theory.

I. GENERALIZATION TO NON-COMPACT LIE ALGEBRAS

Let us first recall the form taken by the Lagrangian density in the case of compact Lie algebras (i.e. Lie algebra of a compact Lie group):

$$\mathcal{L} = \frac{\kappa}{4} \epsilon^{\alpha\beta\gamma} \text{tr} A_\alpha \left( \partial_\beta A_\gamma + \frac{1}{3} [A_\beta, A_\gamma] \right) + i \bar{\psi} D_t \psi - \frac{1}{2} (D \bar{\psi})^* D \psi - \frac{g}{2} \sum_a (\bar{\psi} T_a \psi) (\bar{\psi} T_a \psi).$$

(1)

$\{T_a\}$ are the generators of the algebra and $\text{tr} T_a T_b \propto \delta_{ab}$ is its Killing form. As usual, $D_t = \partial_t + A_0$, $D = \partial - A$ are the covariant derivatives, and the matter field $\psi$ is an $n$-tuplet that transforms according to some finite-dimensional representation of the group. In the
generalization of this expression, we preserve the two main properties of \( \int d^3x \mathcal{L} \), namely its reality and its gauge invariance. In order to do so we replace the Killing form and the inner product in the representing vector space by suitable non-degenerate, Hermitian (for reality) and invariant (for gauge invariance) bilinear forms. Finite-dimensional representations of non-compact groups cannot be unitary; hence we do not expect these forms to be positive definite. Moreover, they do not exist in every representation (see Appendix).

Suppose that the adjoint representation possesses such a bilinear form denoted by \( \langle T_a, T_b \rangle_{\text{adj}} = \Omega_{ab} \). Suppose also that the matter fields belong to some representation with its own bilinear form \( \langle \quad , \quad \rangle \). If \( \Omega^{ab} \) is the inverse matrix of \( \Omega_{ab} \), the natural generalization of Eq. (1) is:

\[
\mathcal{L} = \frac{\kappa}{4} \epsilon^{\alpha\beta\gamma} \left\langle A_\alpha, \partial_\beta A_\gamma + \frac{1}{3} [A_\beta, A_\gamma] \right\rangle_{\text{adj}} + i \langle \psi, D_t \psi \rangle - \frac{1}{2} \langle D \psi, D \psi \rangle - g \sum_{a,b} \langle \psi, T_a \psi \rangle \Omega^{ab} \langle \psi, T_b \psi \rangle .
\]

(2)

Then most of the discussion of Ref. [1] can be followed in this more general case.

The equations of motion read (\( \epsilon_{12} = 1 \)):

\[
F_{xy}^a = \partial_x A_y^a - \partial_y A_x^a + [A_x, A_y]^a = -\frac{i}{\kappa} \Omega^{ab} \langle \psi, T_b \psi \rangle ,
\]

(3a)

\[
F_{i0}^a = \frac{1}{2\kappa} \epsilon_{ij} \Omega^{ab} \left( \langle \psi, T_b D_j \psi \rangle - \langle D_j \psi, T_b \psi \rangle \right) ,
\]

(3b)

\[
i\partial_t \psi = -\frac{1}{2} D^2 \psi - i A_0 \psi + g \langle \psi, T_a \psi \rangle \Omega^{ab} T_b \psi .
\]

(3c)

Taking Eq. (3a) as a definition of \( A \), the last equation can also be derived from the Hamiltonian:

\[
H = \frac{1}{2} \int d^2r \left( \langle D \psi, D \psi \rangle + g \langle \psi, T_a \psi \rangle \Omega^{ab} \langle \psi, T_b \psi \rangle \right)
\]

\[
= \frac{1}{2} \int d^2r \left( D_\epsilon \psi, D_\epsilon \psi \right) + \left( g + \epsilon \frac{1}{\kappa} \right) \langle \psi, T_a \psi \rangle \Omega^{ab} \langle \psi, T_b \psi \rangle ,
\]

(4)
where the last equality involves the definition \( D_\epsilon \equiv D_x + i\epsilon D_y \ (\epsilon = \pm) \) and the discarding of a boundary term.

We can list the other conserved quantities generating symmetries in the system:

\[
\begin{align*}
P_i &= \int d^2 r T^{0i} & \text{momentum} , \\
J &= \int d^2 r \epsilon_{ij} r^i T^{0j} & \text{angular momentum} , \\
G^i &= t P^i - \int d^2 r r^i \langle \psi, \psi \rangle & \text{Galilean boost} , \\
D &= tH - \frac{1}{2} \int d^2 r r^i T^{0i} & \text{dilation} , \\
K &= -t^2 H + 2tD + \frac{1}{2} \int d^2 r r^2 \langle \psi, \psi \rangle & \text{conformal weight} .
\end{align*}
\]

In this system the momentum density \( T^{0i} \) corresponds to a current:

\[
T^{0i} = -\frac{i}{2} \left( \langle \psi, D_i \psi \rangle - \langle D_i \psi, \psi \rangle \right) .
\]

For static solutions, we deduce from the above that \( P^i, D, \) and especially \( H \) have to vanish.

With the special choice \( g = -\epsilon/\kappa \), the condition \( H = 0 \) is realized if \( \psi \) fulfills the first-order differential equation:

\[
D_\epsilon \psi = 0 .
\]

It is easy to see that the solutions of Eqs. (3a) and (7), together with:

\[
A_0^a = -\epsilon \frac{i}{2\kappa} \Omega^{ab} \langle \psi, T_b \psi \rangle ,
\]

are time-independent solutions of Eq. (3). But, unlike for the compact case, the converse is not true. Indeed, if \( \langle \ , \ \rangle \) is non-positive definite (as in the non-compact case) we cannot conclude that Eq. (7) is the only way to achieve \( H = 0 \) in Eq. (4).
II. REDUCTION OF THE YANG–MILLS SELF-DUAL EQUATION

It was already pointed out that static solutions of a Chern–Simons system are closely related to a reduction of self-dual equation expressed in four dimensions. This section provides a derivation of this fact for arbitrary Lie algebras. We consider either the $O(4)$ or $O(2,2)$ invariant metric in order to raise and lower indices. The self-dual Yang–Mills equation is $(\epsilon^{1234} = 1)$:

$$ F_{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} , $$

$$ F_{\alpha\beta} = \partial_\alpha W_\beta - \partial_\beta W_\alpha + [W_\alpha, W_\beta] , $$

$W_\alpha$ being the gauge potentials with value in the Lie algebra. The reduction to two dimensions is achieved by imposing translation invariance with respect to $x^3$ and $x^4$. Take $\kappa$ positive and write:

$$ x = x^1 \quad y = x^2 \quad \Psi = \sqrt{\frac{\kappa}{2}} (W_3 - iW_4) $$

$$ A_x = W_1 \quad A_y = W_2 \quad \bar{\Psi} = -\sqrt{\frac{\kappa}{2}} (W_3 + iW_4) . $$

In these variables and with the definitions $\partial_{\pm} = \partial_x \pm i\partial_y, A_{\pm} = A_x \pm iA_y$ and $D_\epsilon$ as in Eq. (4), Eq. (9) reduces to:

$$ \partial_- A_+ - \partial_+ A_- + [A_- , A_+] = \frac{2}{\kappa} [\bar{\Psi} , \Psi] $$

$$ D_\epsilon \Psi = 0 , $$

where in the last equation $\epsilon$ is correlated with the metric: $\epsilon = +1$ for $O(4)$, and $\epsilon = -1$ for $O(2,2)$.

Introducing $A_+ = A_+ + \sqrt{2/\kappa} \Psi, A_- = \sqrt{2/\kappa} \bar{\Psi}$ we see that if (and only if) $\epsilon = -1$, Eqs. (11) are equivalent to a zero curvature condition:

$$ \partial_- A_+ - \partial_+ A_- + [A_- , A_+] = 0 . $$
For compact groups, Dunne\textsuperscript{2} has found explicitly all the solutions of this last equation. However, it is not clear that his construction works for non-compact groups. Namely, Eq. (12) is solved in a matrix representation and the solution is a matrix which does not necessarily belong to the algebra we are interested in. For example, we are not able to exhibit a solution by this method in the \textit{ISO}(2,1) case.

We observe that the reduced self-dual equations (11) are the same as Eqs. (3a) and (7) provided we take the matter fields in the adjoint representation, \textit{i.e.}:

\begin{equation}
\Psi = \sum_a \psi^a T_a, \quad \bar{\Psi} = - \sum_a (\psi^a)^* T_a.
\end{equation}

Henceforth we shall work with this representation. Moreover, in the compact case, only the choice \(\epsilon = -1\) leads to regular solutions. In that case, Eq. (12) is relevant and one can follow the general discussion of Ref. [1] involving chiral currents and give explicit solutions. But in the non-compact case, different signs conspire to ensure the existence of regular solutions only in the opposite case, \(\epsilon = +1\), where Eq. (12) is no longer valid. In our following illustrations we shall consider only this case.

\section*{III. SOLITON SOLUTIONS IN THE ADJOINT REPRESENTATION}

We take now two examples of non-compact groups: the semi-simple \textit{SL}(2, \mathbb{R}) and the non-semi-simple \textit{ISO}(2,1). The matter field is in the adjoint representation and we shall present different \textit{Ansätze} to solve the self-dual equations with \(\epsilon = 1\).
III.1 The $SL(2, \mathbb{R})$ Case

As discussed in the Appendix, the adjoint representation carries a Killing form $\Omega = \text{diag}(1,-1,-1)$. In order to get simple differential equations from Eq. (11), we try a solution with the gauge field $A$ in a maximal commutative subalgebra. Since $SL(2, \mathbb{R})$ is semi-simple it does have a decomposition of Cartan type, but due to the non-compactness, different decompositions are not equivalent. Indeed, we can make two choices $A \propto J_0$ and $A \propto J_2$ (or $J_1$).

Let us first try the following Ansatz:

$$\Psi = u^0 J_0 + u^+ \frac{1}{\sqrt{2}} (J_1 + iJ_2) + u^- \frac{1}{\sqrt{2}} (J_1 - iJ_2) \ ,$$

$$A_+ = \omega J_0 \ .$$

If $u^+$ is non-zero, the self-dual equations (with $\epsilon = 1$) become:

$$u^0 = 0 \ , \quad \partial_+ (u^+ u^-) = 0 \ , \quad \omega = i\partial_+ \ln u^+ \ ,$$ (15a)

$$\nabla^2 \ln |u^+|^2 = -\frac{2}{\kappa} \left(|u^+|^2 - |u^-|^2\right) \ .$$ (15b)

We know$^1$ that regular solutions are found only if $u^- = 0$. The last equation is then the Liouville equation for the norm of $u^+$. Its phase is fixed (up to a gauge transformation) by requiring regularity for $\omega$. The radially symmetric solutions are:

$$u^+ = 2\sqrt{\kappa} N \frac{1}{r} \frac{1}{(r/r_0)^N + (r_0/r)^N} e^{i(1-N)\theta} \ ,$$ (16a)

$$\omega = -4iN \frac{1}{r} \frac{(r/r_0)^N}{(r/r_0)^N + (r_0/r)^N} e^{-i\theta} \ .$$ (16b)

This solution carries an angular momentum $J = -\kappa 2N$ and a conformal weight $K = -\pi \kappa r_0^2 \csc(\pi/N)$ (note the opposite sign with respect to the compact $SU(2)$ case). With
the other choice for $\varepsilon$ we would have found the opposite sign in the Liouville equation leading
to no regular solution.

Consider now the other possibility for the Cartan subalgebra:

$$
\Psi = u^0 J_0 + u^+ \frac{1}{\sqrt{2}} (J_1 + J_0) + u^- \frac{1}{\sqrt{2}} (J_1 - J_0) ,
$$

\[ A_+ = \omega J_2 . \tag{17} \]

The self-dual equations become:

$$
\begin{align*}
    u^+ u^- &= C(x^-) , \\
    \omega &= \partial_+ \ln u^+ ,
\end{align*} \tag{18a,b}
$$

with an arbitrary complex function $C(x^-)$. The combination $\phi = 2 \arg u^+ + \arg C$ obeys the
sine-Gordon equation in Euclidean space:

$$
\nabla^2 \phi = -\frac{2}{\kappa} |C| \sin \phi . \tag{19}
$$

To solve it explicitly we take $C$ constant and we find “multi-kink” solutions, regular
everywhere.\textsuperscript{3} However, they do not lead to a function $\omega$ decreasing at infinity, unless $C = 0$.

In that case the solution is:

$$
\phi = \text{constant} , \quad u^+ = |u^+| e^{i\phi} , \quad \omega = \partial_+ \ln |u^+| , \tag{20}
$$

which is gauge equivalent to the trivial solution $u^+ = \omega = 0$ and thus gives no new soliton
solution.

Equation (16) gives regular radially symmetric solutions. Like in the $SU(2)$ case\textsuperscript{2} we
expect that all of them are obtained through the \textit{Ansatz} (14).
III.2 The \textit{ISO}(2,1) Case

The Poincaré group is an example of a non-compact and non-semi-simple Lie group.

The six generators and the bilinear form \( \Omega \) of the adjoint representation are described in the Appendix. In this non-semi-simple algebra the Cartan decomposition has no meaning, but it is still useful to take the gauge field in a maximal Abelian subalgebra. Again we have two choices. Let us make the first \textit{Ansatz}:

\[
\Psi = u^0 J_0 + u^+ \frac{1}{\sqrt{2}} (J_1 + i J_2) + u^- \frac{1}{\sqrt{2}} (J_1 - i J_2) \\
+ v^0 P_0 + v^+ \frac{1}{\sqrt{2}} (P_1 + i P_2) + v^- \frac{1}{\sqrt{2}} (P_1 - i P_2) ,
\]

\[ A_+ = \omega J_0 + e^0 P_0 . \tag{21} \]

Always with \( \epsilon = 1 \), the self-dual equation implies for \( u^+ , u^- \) and \( \omega \) the same equations (15) as in the previous example. Regular solutions were obtained only with \( u^- = 0 \). With this condition the other equations are:

\[
\begin{align*}
    u^0 &= v^0 = 0 \tag{22a} \\
    \nabla^2 \Re \left[ \frac{v^+}{u^+} \right] &= -\frac{2}{\kappa} |u^+|^2 \Re \left[ \frac{v^+}{u^+} \right] , \tag{22b} \\
    e^0 &= i \partial_+ \left( \frac{v^+}{u^+} \right) , \tag{22c} \\
    \partial_+ (u^+ v^-) &= 0 . \tag{22d}
\end{align*}
\]

At first sight it seems that there is not enough constraints, as Eq. (22b) determines the real part of \( (v^+/u^+) \) but \( e^0 \) in Eq. (22c) depends also on its imaginary part. Nevertheless, by a gauge transformation we can always shift \( e^0 \) by the total derivative of a regular and real quantity and set the imaginary part of \( (v^+/u^+) \) to what we want. We have used the same kind of reasoning to determine the phase of \( u^+ \).
We recognize Eq. (22b) as the deformation of the Liouville equation (15b) (with $u^- = 0$). Namely, if $|u^+|^2 = \ln \left(1 + |\phi|^2\right)$ is the general solution involving some analytical function $\phi(x^+)$, we find the solutions of Eq. (22b) by making an arbitrary deformation $\phi(x^+) \rightarrow \phi(x^+) (1 + \epsilon \psi (x^+))$:

$$|u^+|^2 \Re \left(\frac{v^+}{u^+}\right) = \kappa \nabla^2 \left(\frac{|\phi|^2}{1 + |\phi|^2} (\psi + \psi^*)\right).$$

In the “radially symmetric” case — with $\phi(x^+) \propto (x^+)^{-N}$ and $\psi(x^+) \propto (x^+)^M$ — Eq. (23) reads:

$$\Re \left(\frac{v^+}{u^+}\right) = \frac{1}{N} \frac{r^M}{(r/r_0)^N + (r_0/r)^N} \times \left[(M - N) \left(\frac{r}{r_0}\right)^N + (M + N) \left(\frac{r_0}{r}\right)^N\right] (a_M \cos M\theta + b_M \sin M\theta).$$

The gauge freedom we have allows a convenient choice of its imaginary part:

$$v^+ = 2\sqrt{\kappa} C_{N,M} \frac{r^{M-1}}{\left[(r/r_0)^N + (r_0/r)^N\right]^2} \times \left[(M - N) \left(\frac{r}{r_0}\right)^N + (M + N) \left(\frac{r_0}{r}\right)^N\right] e^{i(1-N-M)\theta},$$

where we have used the expression (16a) for $u^+$. Equation (22c) then gives:

$$e^0 = 4iN C_{N,M} \frac{r^{M-1}}{\left[(r/r_0)^N + (r_0/r)^N\right]^2} e^{i(1-M)\theta}.$$

In order to avoid singularities at $r = 0$ and $r = \infty$ we have to restrict the integer values of $M$ to $1 - N \leq M \leq 1 + N$.

Finally, Eq. (22d) is trivially solved by $v^- = f(x^-)/u^+$. As “radially symmetric” choice we take $f(\bar{z}) = \bar{z}^\alpha$ and the regular solution is:

$$v^- = C_N \left[\left(\frac{r}{r_0}\right)^N + \left(\frac{r_0}{r}\right)^N\right] \left[C_1 \left(\frac{r}{r_0}\right)^L e^{-iL\theta} + C_2 \left(\frac{r_0}{r}\right)^L e^{iL\theta}\right] e^{iN\theta},$$
with an integer \( L \geq N \). Equations (16), (25), (26) and (27) together with \( u^0 = v^0 = u^- = 0 \) give a soliton solution to our self-dual problem.

In fact, it is possible to consider a more general Ansatz with the gauge field in a larger subalgebra than the maximal Abelian one:

\[
\Psi = u^+ \frac{1}{\sqrt{2}}(J_1 + iJ_2) + v^0 P_0 + v^+ \frac{1}{\sqrt{2}}(P_1 + iP_2) + v^- \frac{1}{\sqrt{2}}(P_1 - iP_2) ,
\]

\[
A_+ = \omega J_0 + e^0 P_0 + e^+ \frac{1}{\sqrt{2}}(P_1 + iP_2) + e^- \frac{1}{\sqrt{2}}(P_1 - iP_2) .
\]

First of all we remark that since the commutators of \( J_0, P_0 \) with \( P_1, P_2 \) only produce \( P_1, P_2 \) terms, the gauge choice previously made for \( \omega, e^0 \) can still be achieved. Moreover, a gauge transformation parallel to \( P_1, P_2 \) transforms \( e^+, e^- \) like (\( \Lambda \) is a regular complex function):

\[
e^+ \rightarrow e^+ + \partial_+ \Lambda + i \omega \Lambda ,
\]

\[
e^- \rightarrow e^- + \partial_+ \Lambda^* - i \omega \Lambda^* ,
\]

while leaving \( \omega, e^0 \) unchanged. Thus in a suitable gauge we can also take \( e^+ = 0 \).

For \( u^+, v^+, v^-, \omega, e^0 \) the equations are similar to the previous ones. The two remaining equations for the two unknown functions \( v^0, e^- \) look rather simple:

\[
\partial_+ \left( \partial_- v^0 - v^0 \partial_- \ln |u^+|^2 \right) = 0 , \quad e^- = (u^+)^{-1} \partial_+ v^0 .
\]

The first one is integrated with the help of two arbitrary functions:

\[
v^0 = C_1 (x^+) |u^+|^2 + \int_{x^-}^{x^+} dy^- C_2 (y^-) |u^+|^2 (x^+, y^-) .
\]

As an explicit example we choose \( C_2 = 0 \), a constant \( C_1 \) and the “radially symmetric” case:

\[
v^0 = 4C_1 \kappa N^2 \frac{1}{r^2} \frac{1}{\left[ (r/r_0)^N + (r_0/r)^N \right]^2} ,
\]

\[
e^- = 2iC_1 \sqrt{\kappa} N \frac{1}{r^2} \frac{1}{\left[ (r/r_0)^N + (r_0/r)^N \right]^2} \left[ (1 + N) \left( \frac{r}{r_0} \right)^N + (1 - N) \left( \frac{r_0}{r} \right)^N \right] e^{iN \theta} .
\]
If we choose the gauge field in another direction in the algebra (e.g. $A \propto J_2$) we would find the same trivial solution as in the $SL(2, \mathbb{R})$ example. The set of equations (16), (25), (26), (27), (32) gives a large class of soliton solutions in the $ISO(2,1)$ case. However, the conserved quantities (5) give nothing interesting on these solutions. Namely the non-trivial ones are given here by:

$$ J = -\int d^2r \mid u^+ \mid^2 \Re \left( \frac{v^+}{u^+} \right), $$

$$ G^i = \int d^2r r^i \mid u^+ \mid^2 \Re \left( \frac{v^+}{u^+} \right), $$

$$ K = -\frac{1}{2} \int d^2r r^2 \mid u^+ \mid^2 \Re \left( \frac{v^+}{u^+} \right). $$

But due to the angular dependence of $\Re (v^+/u^+)$ [cf. Eq. (24)], $J = K = 0$ and $G^i$ is non-vanishing only for $M = 1$, where $e^0$ is radially symmetric.

**IV. CONCLUSION**

We have shown how to couple non-relativistic matter to non-compact Chern–Simons theory. This is not always possible since the matter field must be in a representation that carries an invariant bilinear form. In that case, static equations are nicely related to the reduction of four-dimensional Yang–Mills self-dual equations that leads to non-trivial solutions. The presence of these solitons can be useful to understand Euclidean gravity in two dimensions as a reduction of a Chern–Simons system in three dimensions. Although the matter is taken as non-relativistic, this study can also give some insight into the question of coupling matter, in a gauge invariant form, to $2 + 1$ gravity seen as a Chern–Simons theory.

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APPENDIX

In this Appendix we discuss the existence of non-degenerate, Hermitian and invariant bilinear forms in a finite-dimensional representation of a Lie algebra. If $u, v$ belong to a representing vector space and if the generators of the Lie algebra act on them by $u \rightarrow Tu$, we are looking for a non-degenerate bilinear form $\langle \ , \ \rangle$ such that:

\[
\langle u, v \rangle = \langle v, u \rangle^* \ , \quad \langle Tu, v \rangle + \langle u, Tv \rangle = 0 .
\] (A.1)

In matrix notations we write $\langle u, v \rangle = (u^m)^* \Omega_{mn} v^n$ with $(\Omega_{mn})$ invertible and:

\[
\Omega^\dagger = \Omega \ , \quad T^\dagger = -\Omega T \Omega^{-1} .
\] (A.2)

If the algebra is semi-simple, the adjoint representation carries such a form: the Killing form. But for non-semi-simple or for other representations this is not always true.

Let us consider the following examples:

A) Compact, semi-simple Lie algebra like $SU(n)$. All irreducible representations are unitary, thus in all representations $\Omega \propto I$.

B) Non-compact, semi-simple Lie algebra. Our prototype is $SL(2, \mathbb{R})$:

\[
[J_a, J_b] = \epsilon_{abc} J_c ,
\] (A.3)

with $a, b, c = 0, 1, 2$, $\epsilon_{012} = 1$, $\epsilon_a^c = \eta^{c}{}^c{}' \epsilon_{ab}{}'$, $\eta^{ab} = \text{diag} (1, -1, -1)$. In the three-dimensional adjoint representation we have the Killing form $\Omega = \text{diag} (1, -1, -1)$. In the two-dimensional fundamental representation:

\[
J_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad J_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} ,
\] (A.4)
we have $\Omega = 2iJ_0$.

C) Non-compact, non-semi-simple Lie algebra. Here we consider the Poincaré algebra $ISO(2,1)$ $(a, b, c = 0, 1, 2)$:

$$[J_a, J_b] = \epsilon_{abc} J_c, \quad [J_a, P_b] = \epsilon_{abc} P_c, \quad [P_a, P_b] = 0.$$ \hfill (A.5)

In the six-dimensional adjoint representation, it turns out that there is still a bilinear form with the good properties (which of course is not the Killing form):

$$\langle J_a, J_b \rangle_{\text{adj}} = c_1 \eta_{ab}, \quad \langle J_a, P_b \rangle_{\text{adj}} = c_2 \eta_{ab}, \quad (c_2 \neq 0),$$ \hfill (A.6)

with $\eta_{ab}$ being the diagonal matrix $\text{diag}(1, -1, -1)$. But this is not true in all representations. For example, in the four-dimensional fundamental one [with $\tilde{J}_a$ given by (A.4)]:

$$J_a = \begin{pmatrix} \tilde{J}_a & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$P_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$ \hfill (A.7)

we cannot find an invertible matrix $\Omega$ with the properties (A.2). On the other hand, there is another four-dimensional representation given in terms of $4 \times 4$ gamma matrices $\Gamma^A$ $(A = 0, 1, 2, 3)$ of the four-dimensional Minkowskian space ($\Gamma_A = \eta_{AB} \Gamma^B$, $\Gamma^5 = i\Gamma^0\Gamma^1\Gamma^2\Gamma^3$):

$$J_a = -\frac{1}{4} \epsilon_{abc} \Gamma^b \Gamma^c,$$ \hfill (A.8)

$$P_a = i\beta \Gamma_a (1 + \Gamma^5), \quad (\beta \in \mathbb{R}),$$

which carries the bilinear form $\Omega = \Gamma_0$. 

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