On $n$-maximal subgroups of finite groups

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Abstract

We describe finite soluble groups in which every $n$-maximal subgroup is $F$-subnormal for some saturated formation $F$.

1 Introduction

Throughout this paper, all groups are finite and $G$ always denotes a finite group. We use $\mathcal{U}$, $\mathcal{N}$ and $\mathcal{N}^r$ to denote the class of all supersoluble groups, the class of all nilpotent groups and the class of soluble groups of nilpotent length at most $r$ ($r \geq 1$). The symbol $\mathbb{P}$ denotes the set of all primes, $\pi(G)$ denotes the set of prime divisors of the order of $G$. If $p$ is a prime, then we use $\mathcal{S}_p$ to denote the class of all $p$-groups.

Let $\mathcal{F}$ be a class of groups. If $1 \in \mathcal{F}$, then we write $G^\mathcal{F}$ to denote the intersection of all normal subgroups $N$ of $G$ with $G/N \in \mathcal{F}$. The class $\mathcal{F}$ is said to be a formation if either $\mathcal{F} = \emptyset$ or $1 \in \mathcal{F}$ and every homomorphic image of $G/G^\mathcal{F}$ belongs to $\mathcal{F}$ for any group $G$. The formation $\mathcal{F}$ is said to be: saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$ for any group $G$; hereditary if $H \in \mathcal{F}$ whenever $G \in \mathcal{F}$ and $H$ is a subgroup of $G$. A group $G$ is called $\mathcal{F}$-critical provided $G$ does not belong to $\mathcal{F}$ but every proper subgroup of $G$ belongs to $\mathcal{F}$.

For any formation function $f : \mathbb{P} \to \{\text{group formation}\}$, the symbol $LF(f)$ denotes the collection of all groups $G$ such that either $G = 1$ or $G \neq 1$ and $G/C_G(H/K) \in f(p)$ for every chief factor $H/K$ of $G$ and every $p \in \pi(H/K)$. It is well-known that for any non-empty saturated formation $\mathcal{F}$, there

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is a unique formation function \( F \) such that \( \mathcal{F} = LF(F) \) and \( F(p) = \mathcal{S}_pF(p) \subseteq \mathcal{F} \) for all primes \( p \), where \( \mathcal{S}_pF(p) \) is the set of all groups \( G \) such that \( G^{F(p)} \in \mathcal{S}_p \) (see Proposition 3.8 in [1, Chapter IV]). The formation function \( F \) is called the \textit{canonical local satellite} of \( \mathcal{F} \). A chief factor \( H/K \) of \( G \) is called \( \mathcal{F} \)-central in \( G \) provided \( G/C_{G}(H/K) \in F(p) \) for all primes \( p \) dividing \( |H/K| \), otherwise it is called \( \mathcal{F} \)-eccentric.

Fix some ordering \( \phi \) of \( \mathbb{P} \). The record \( p \phi q \) means that \( p \) precedes \( q \) in \( \phi \) and \( p \neq q \). Recall that a group \( G \) of order \( p_{1}^{\alpha_{1}}p_{2}^{\alpha_{2}}\ldots p_{n}^{\alpha_{n}} \) is called \( \phi \)-dispersive whenever \( p_{1}\phi p_{2}\phi \ldots \phi p_{n} \) and for every \( i \) there is a normal subgroup of \( G \) of order \( p_{1}^{\alpha_{1}}p_{2}^{\alpha_{2}}\ldots p_{i}^{\alpha_{i}} \). Furthermore, if \( \phi \) is such that \( p \phi q \) always implies \( p > q \), then every \( \phi \)-dispersive group is called \( \textit{Ore dispersive} \).

By definition, every formation is \( 0 \)-multiply saturated and for \( n \geq 1 \) a formation \( \mathcal{F} \) is called \( n \)-multiply saturated if \( \mathcal{F} = LF(f) \), where every non-empty value of the function \( f \) is an \((n-1)\)-multiply saturated formation (see [2] and [3]). In fact, almost saturated formations met in mathematical practice are \( n \)-multiply saturated for every natural \( n \). For example, the formations of all soluble groups, all nilpotent groups, all \( p \)-soluble groups, all \( p \)-nilpotent groups, all \( p \)-closed groups, all \( p \)-decomposable groups, all Ore dispersive groups, all metanilpotent groups are \( n \)-multiply saturated for all \( n \geq 1 \). Nevertheless, the formations of all supersoluble groups and all \( p \)-supersoluble groups are saturated, but they are not \( 2 \)-multiply saturated formations.

Recall that a subgroup \( H \) of \( G \) is called a \( 2 \)-maximal (second maximal) subgroup of \( G \) whenever \( H \) is a maximal subgroup of some maximal subgroup \( M \) of \( G \). Similarly we can define \( 3 \)-maximal subgroups, and so on.

The interesting and substantial direction in finite group theory consists in studying the relations between the structure of the group and its \( n \)-maximal subgroups. One of the earliest publications in this direction is the article of Huppert [4] who established the supersolubility of a group \( G \) whose all second maximal subgroups are normal. In the same article Huppert proved that if all \( 3 \)-maximal subgroups of \( G \) are normal in \( G \), then the commutator subgroup \( G' \) of \( G \) is nilpotent and the chief rank of \( G \) is at most 2. These two results were developed by many authors. Among the recent results on \( n \)-maximal subgroups we can mention [5], where the solubility of groups is established in which all \( 2 \)-maximal subgroups enjoy the cover-avoidance property, and [6, 7, 8], where new characterizations of supersoluble groups in terms of \( 2 \)-maximal subgroups were obtained. The classification of non-nilpotent groups whose all \( 2 \)-maximal subgroups are \( TI \)-subgroups appeared in [9]. Description was obtained in [10] of groups whose every \( 3 \)-maximal subgroup permutes with all maximal subgroups. The non-nilpotent groups are described in [11] in which every two \( 3 \)-maximal subgroups are permutable. The groups are described in [12] whose all \( 3 \)-maximal subgroups are \( S \)-quasinormal, that is, permute with all Sylow subgroups. Subsequently this result was strengthened in [13] to provide a description of the groups whose all \( 3 \)-maximal subgroups are subnormal.

Despite of all these and many other known results about \( n \)-maximal subgroups, the fundamental work of Mann [14] still retains its value. It studied the structure of groups whose \( n \)-maximal
subgroups are subnormal. Mann proved that if all $n$-maximal subgroups of a soluble group $G$ are subnormal and $|\pi(G)| \geq n+1$, then $G$ is nilpotent; but if $|\pi(G)| \geq n-1$, then $G$ is $\phi$-dispersive for some ordering $\phi$ of $P$. Finally, in the case $|\pi(G)| = n$ Mann described $G$ completely.

Let $\mathcal{F}$ be a non-empty formation. Recall that a subgroup $H$ of a group $G$ is said to be $\mathcal{F}$-subnormal in $G$ if either $H = G$ or there exists a chain of subgroups $H = H_0 < H_1 < \ldots < H_n = G$ such that $H_{i-1}$ is a maximal subgroup of $H_i$ and $H_i/(H_{i-1})H_i \in \mathcal{F}$, for $i = 1, \ldots, n$.

The main goal of this article is to prove the following formation analogs of Mann’s theorems.

**Theorem A.** Let $\mathcal{F}$ be an $r$-multiply saturated formation such that $N \subseteq \mathcal{F} \subseteq N^{r+1}$ for some $r \geq 0$. If every $n$-maximal subgroup of a soluble group $G$ is $\mathcal{F}$-subnormal in $G$ and $|\pi(G)| \geq n+r+1$, then $G \in \mathcal{F}$.

**Theorem B.** Let $\mathcal{F} = LF(F)$ be a saturated formation such that $N \subseteq \mathcal{F} \subseteq \mathcal{U}$, where $F$ is the canonical local satellite of $\mathcal{F}$. Let $G$ be a soluble group with $|\pi(G)| \geq n + 1$. Then all $n$-maximal subgroups of $G$ are $\mathcal{F}$-subnormal in $G$ if and only if $G$ is a group of one of the following types:

I. $G \in \mathcal{F}$.

II. $G = A \rtimes B$, where $A = G^\mathcal{F}$ and $B$ are Hall subgroups of $G$, while $G$ is Ore dispersive and satisfies the following:

1. $A$ is either of the form $N_1 \times \ldots \times N_t$, where each $N_i$ is a minimal normal subgroup of $G$, which is a Sylow subgroup of $G$, for $i = 1, \ldots, t$, or a Sylow $p$-subgroup of $G$ of exponent $p$ for some prime $p$ and the commutator subgroup, the Frattini subgroup, and the center of $A$ coincide, while $A/\Phi(A)$ is an $\mathcal{F}$-eccentric chief factor of $G$;

2. every $n$-maximal subgroup of $G$ belongs to $\mathcal{F}$ and induces on the Sylow $p$-subgroup of $A$ an automorphism group which is contained in $F(p)$ for every prime divisor $p$ of $|A|$.

In the proof of Theorem B we often use Theorem A and the following useful fact.

**Theorem C.** Let $\mathcal{F}$ be a hereditary saturated formation such that every $\mathcal{F}$-critical group is soluble and it has a normal Sylow $p$-subgroup $G_p \neq 1$ for some prime $p$. Then every 2-maximal subgroup of $G$ is $\mathcal{F}$-subnormal in $G$ if and only if either $G \in \mathcal{F}$ or $G$ is an $\mathcal{F}$-critical group and $G^\mathcal{F}$ is a minimal normal subgroup of $G$.

**Theorem D.** Let $\mathcal{F}$ be a saturated formation such that $N \subseteq \mathcal{F} \subseteq \mathcal{U}$. If every $n$-maximal subgroup of a soluble group $G$ is $\mathcal{F}$-subnormal in $G$ and $|\pi(G)| \geq n$, then $G$ is $\phi$-dispersive for some ordering $\phi$ of $P$.

All unexplained notation and terminology are standard. The reader is referred to [11] or [15] if necessary.
2 Preliminary Results

Let $\mathcal{F}$ be a non-empty formation. Recall that a maximal subgroup $H$ of $G$ is said to be $\mathcal{F}$-normal in $G$ if $G/H \in \mathcal{F}$, otherwise it is said to be $\mathcal{F}$-abnormal in $G$.

We use the following results.

**Lemma 2.1.** Let $\mathcal{F}$ be a formation and $H$ an $\mathcal{F}$-subnormal subgroup of $G$.

1. If $\mathcal{F}$ is hereditary and $K \leq G$, then $H \cap K$ is an $\mathcal{F}$-subnormal subgroup in $K$ [15, Lemma 6.1.7(2)].

2. If $N$ is a normal subgroup in $G$, then $HN/N$ is an $\mathcal{F}$-subnormal subgroup in $G/N$ [15, Lemma 6.1.6(3)].

3. If $K$ is a subgroup of $G$ such that $K$ is $\mathcal{F}$-subnormal in $H$, then $K$ is $\mathcal{F}$-subnormal in $G$ [15, Lemma 6.1.6(1)].

4. If $\mathcal{F}$ is hereditary and $K$ is a subgroup of $G$ such that $G^F \leq K$, then $K$ is $\mathcal{F}$-subnormal in $G$ [15, Lemma 6.1.7(1)].

The following lemma is evident.

**Lemma 2.2.** Let $\mathcal{F}$ be a hereditary formation. If $G \in \mathcal{F}$, then every subgroup of $G$ is $\mathcal{F}$-subnormal in $G$.

**Lemma 2.3.** Let $\mathcal{F}$ be a hereditary saturated formation. If every $n$-maximal subgroup of $G$ is $\mathcal{F}$-subnormal in $G$, then every $(n-1)$-maximal subgroup of $G$ belongs to $\mathcal{F}$ and every $(n+1)$-maximal subgroup of $G$ is $\mathcal{F}$-subnormal in $G$.

**Proof.** We first show that every $(n-1)$-maximal subgroup of $G$ belongs to $\mathcal{F}$. Let $H$ be an $(n-1)$-maximal subgroup of $G$ and $K$ a maximal subgroup of $H$. Then $K$ is an $n$-maximal subgroup of $G$ and so, by hypothesis, $K$ is $\mathcal{F}$-subnormal in $G$. Hence $K$ is $\mathcal{F}$-subnormal in $H$ by Lemma 2.1(1). Thus all maximal subgroups of $H$ are $\mathcal{F}$-normal in $H$. Therefore $H \in \mathcal{F}$ since $\mathcal{F}$ is saturated.

Now, let $E$ be an $(n+1)$-maximal subgroup of $G$, and let $E_1$ and $E_2$ be an $n$-maximal and an $(n-1)$-maximal subgroup of $G$, respectively, such that $E \leq E_1 \leq E_2$. Then, by the above, $E_2 \in \mathcal{F}$, so $E_1 \in \mathcal{F}$. Hence $E$ is $\mathcal{F}$-subnormal in $E_1$ by Lemma 2.2. By hypothesis, $E_1$ is $\mathcal{F}$-subnormal in $G$. Therefore $E$ is $\mathcal{F}$-subnormal in $G$. The lemma is proved.

**Lemma 2.4** (See [16] Chapter VI, Theorem 24.2]). Let $\mathcal{F}$ be a saturated formation and $G$ a soluble group. If $G^F \neq 1$ and every $\mathcal{F}$-abnormal maximal subgroup of $G$ belongs to $\mathcal{F}$, then the following hold:

1. $G^F$ is a $p$-group for some prime $p$;

2. $G^F/\Phi(G^F)$ is an $\mathcal{F}$-eccentric chief factor of $G$;

3. if $G^F$ is a non-abelian group, then the center, commutator subgroup, and Frattini subgroup of $G$ coincide and are of exponent $p$;
(4) if $G^F$ is abelian, then $G^F$ is elementary;

(5) if $p > 2$, then $G^F$ is of exponent $p$; for $p = 2$ the exponent of $G^F$ is at most 4;

(6) every pair of $\mathcal{F}$-abnormal maximal subgroups of $G$ are conjugate in $G$.

**Lemma 2.5** (See [16, Chapter VI, Theorem 24.5]). Let $\mathcal{F}$ be a saturated formation. Let $G$ be an $\mathcal{F}$-critical group and $G$ has a normal Sylow $p$-subgroup $G_p \neq 1$ for some prime $p$. Then:

1. $G_p = G^F$;
2. $F(G) = G_p\Phi(G)$;
3. $G_{p'} \cap C_G(G_p/\Phi(G_p)) = \Phi(G) \cap G_{p'}$, where $G_{p'}$ is some complement of $G_p$ in $G$.

**Lemma 2.6** (See [16, Chapter VI, Theorems 26.3 and 26.5]). Let $G$ be an $U$-critical group. Then:

1. $G$ is soluble and $|\pi(G)| \leq 3$;
2. if $G$ is not a Schmidt group, then $G$ is Ore dispersive;
3. $G^U$ is the unique normal Sylow subgroup of $G$;
4. if $S$ is a complement of $G^U$ in $G$, then $S/S \cap \Phi(G)$ is either a primary cyclic group or a Miller-Moreno group.

Recall that the product of all normal subgroups of a group $G$ whose $G$-chief factors are $\mathcal{F}$-central in $G$ is called $\mathcal{F}$-hypercentre of $G$ and denoted by $Z_\mathcal{F}(G)$ [11 p. 389].

**Lemma 2.7** (See [17, Lemma 2.14]). Let $\mathcal{F}$ be a saturated formation and $F$ the canonical local satellite of $\mathcal{F}$. Let $E$ be a normal $p$-subgroup of a group $G$. Then $E \leq Z_\mathcal{F}(G)$ if and only if $G/C_G(E) \in F(p)$.

The product $\mathcal{MH}$ of the formations $\mathcal{M}$ and $\mathcal{H}$ is the class of all groups $G$ such that $G^\mathcal{H} \in \mathcal{M}$.

**Lemma 2.8** (See [3, Corollary 7.14]). The product of any two $n$-multiply saturated formations is an $n$-multiply saturated formation.

We shall also need the following evident lemma.

**Lemma 2.9.** If $G = AB$, then $G = AB^x$ for all $x \in G$.

Let $\mathcal{F}$ be a class of groups and $t$ a natural number with $t \geq 2$. Recall that $\mathcal{F}$ is called $\Sigma_t$-closed if $\mathcal{F}$ contains all such groups $G$ that $G$ has subgroups $H_1, \ldots, H_t$ whose indices are pairwise coprime and $H_i \in \mathcal{F}$, for $i = 1, \ldots, t$.

**Lemma 2.10** (See [16, Chapter I, Lemma 4.11]). Every formation of nilpotent groups is $\Sigma_3$-closed.

If $\mathcal{F} = LF(f)$ and $f(p) \subseteq \mathcal{F}$ for all primes $p$, then $f$ is called an integrated local satellite of $\mathcal{F}$. Let $\mathcal{X}$ be a set of groups. The symbol $l_n\text{form}\mathcal{X}$ denotes the intersection of all $n$-multiply saturated formations $\mathcal{F}$ such that $\mathcal{X} \subseteq \mathcal{F}$. In view of [15 Remak 3.1.7], $l_n\text{form}\mathcal{X}$ is an $n$-multiply saturated
Lemma 2.11 (See [8] Theorem 8.3). Let $\mathcal{F}$ be an $n$-multiply saturated formation. Then $\mathcal{F}$ has an integrated local satellite $f$ such that $f(p) = l_{n-1}\text{form}(G/O_{p',p}(G)|G \in \mathcal{F})$ for all primes $p$.

Lemma 2.12 (See [18] Section 1.4). Every $r$-multiply saturated formation contained in $\mathcal{N}^{r+1}$ is hereditary.

Lemma 2.13 (See [16] p. 35). For any ordering $\phi$ of $\mathbb{P}$ the class of all $\phi$-dispersive groups is a saturated formation.

Lemma 2.14 (See [17] Corollary 1.6). Let $\mathcal{F}$ be a saturated formation containing all nilpotent groups and $E$ a normal subgroup of $G$. If $E/E \cap \Phi(G) \in \mathcal{F}$, then $E \in \mathcal{F}$.

Lemma 2.15 (See [16] Theorem 15.10). Let $\mathcal{F}$ be a saturated formation and $G$ a group such that $G^{3}$ is nilpotent. Let $H$ and $M$ be subgroups of $G$, $H \in \mathcal{F}$, $H \leq M$ and $HF(G) = G$. If $H$ is $\mathcal{F}$-subnormal in $M$, then $M \in \mathcal{F}$.

3 Proof of Theorem A

First we give two propositions which may be independently interesting since they generalize some known results.

Proposition 3.1. Suppose that $G = A_{1}A_{2} = A_{2}A_{3} = A_{1}A_{3}$, where $A_{1}$, $A_{2}$ and $A_{3}$ are soluble subgroups of $G$. If the indices $|G : N_{G}(A_{1})|$, $|G : N_{G}(A_{2})|$, $|G : N_{G}(A_{3})|$ are pairwise coprime, then $G$ is soluble.

Corollary 3.2. Suppose that $G = A_{1}A_{2} = A_{2}A_{3} = A_{1}A_{3}$, where $A_{1}$, $A_{2}$ and $A_{3}$ are soluble subgroups of $G$. If the indices $|G : N_{G}(A_{1})|$, $|G : N_{G}(A_{2})|$, $|G : N_{G}(A_{3})|$ are pairwise coprime, then $G$ is soluble.

Corollary 3.3 (H. Wielandt). If $G$ has three soluble subgroups $A_{1}$, $A_{2}$ and $A_{3}$ whose indices $|G : A_{1}|$, $|G : A_{2}|$, $|G : A_{3}|$ are pairwise coprime, then $G$ is itself soluble.

Proposition 3.4. Let $\mathcal{M}$ be an $r$-multiply saturated formation and $N \subseteq \mathcal{M} \subseteq \mathcal{N}^{r+1}$ for some $r \geq 0$. Then, for any prime $p$, both formations $\mathcal{M}$ and $\exists_{p}\mathcal{M}$ are $\Sigma_{r+3}$-closed.

Proof. Let $M$ be the canonical local satellite of $\mathcal{M}$. Let $\mathcal{F}$ be one of the formations $\mathcal{M}$ or $\exists_{p}\mathcal{M}$. Let $G$ be any group such that for some subgroups $H_{1}, \ldots, H_{r+3}$ of $G$ whose indices $|G : H_{1}|, \ldots, |G : H_{r+3}|$ are pairwise coprime we have $H_{1}, \ldots, H_{r+3} \in \mathcal{F}$. We shall prove $G \in \mathcal{F}$. Suppose that this is false and let $G$ be a counterexample with $r + |G|$ minimal. Let $N$ be a minimal normal subgroup of $G$.

(1) $N = G^{3}$ is the only minimal normal subgroup of $G$ and $N \leq O_{q}(G)$ for some prime $q$. Hence if $\mathcal{F} = \exists_{p}\mathcal{M}$, then $q \neq p$.

It is clear that the hypothesis holds for $G/N$, so $G/N \in \mathcal{F}$ by the choice of $G$. Hence $N = G^{3}$ since $G \notin \mathcal{F}$. Moreover, $N$ is a $q$-group for some prime $q$ since $G$ is soluble by Proposition 3.1.
Finally, if \( \mathcal{F} = \mathcal{S}_p \mathcal{M} \) and \( p = q \), then

\[
G \in \mathcal{S}_p(\mathcal{S}_p \mathcal{M}) = \mathcal{S}_p \mathcal{F} = \mathcal{F},
\]
a contradiction. Hence we have (1).

Since the indices \( |G : H_1|, \ldots, |G : H_{r+3}| \) are pairwise coprime, in view of (1) we may assume without loss of generality that \( N \leq H_i \) for all \( i = 2, \ldots, r + 3 \).

(2) \( C_G(N) = N \).

First we show that \( N \not\subseteq \Phi(G) \). Suppose that \( N \leq \Phi(G) \). If \( r > 0 \), then \( \mathcal{F} \) is saturated by Lemma 2.8, so \( G \in \mathcal{F} \). This contradiction shows that \( r = 0 \) and so \( \mathcal{F} = \mathcal{S}_p \mathcal{M} \) by Lemma 2.10 and the choice of \( G \). Hence \( q \neq p \) by (1). Let \( O/N = O_p(G/N) \) and \( P \) be a Sylow \( p \)-subgroup of \( O \). Then \( G = ON_G(P) = NPN_G(P) = NN_G(P) = N_G(P) \) by the Frattini Argument since \( N \leq \Phi(G) \). Hence in view of (1), \( O_p(G/N) = 1 \) and so \( G/N \in \mathcal{M} \) since \( G/N \in \mathcal{F} = \mathcal{S}_p \mathcal{M} \). But then \( G \) is a \( p' \)-group. Hence \( H_1, H_2, H_3 \in \mathcal{M} \). Thus \( G \in \mathcal{M} \subseteq \mathcal{F} \) by Lemma 2.10. This contradiction shows that \( N \not\subseteq \Phi(G) \). But then \( C_G(N) = N \) by (1) and \( \mathcal{M} \), Theorem 15.2).

(3) \( r > 0 \).

Suppose that \( r = 0 \). Then \( \mathcal{F} = \mathcal{S}_p \mathcal{M} \), where \( \mathcal{M} \) is a formation of nilpotent groups. Since \( N \leq H_2 \in \mathcal{F} \) and, by (2), \( C_G(N) = N \), \( O_p(H_2) = 1 \). Hence \( H_2 \) is a \( p' \)-group. Similarly, \( H_3 \) is a \( p' \)-group. Hence \( G = H_1H_2 \) is a \( p' \)-group. But then \( H_1 \in \mathcal{M} \), so \( G \in \mathcal{F} \) by Lemma 2.10. This contradiction shows that we have (3).

(4) \( H_i/N \in M(q) \) for all \( i = 2, \ldots, r + 3 \).

Let \( i \in \{2, \ldots, r+3\} \). Then \( H_i \in \mathcal{M} \). Indeed, if \( \mathcal{F} = \mathcal{S}_p \mathcal{M} \), then \( q \neq p \) by (1). On the other hand, in view of (2), \( C_G(N) = N \). Hence \( O_p(H_i) = 1 \), which implies that \( H_i \in \mathcal{M} \). But then, by (2) and Lemma 2.7, \( H_i/N = H_i/C_{H_i}(N) \in M(q) \).

(5) \( G/N \in M(q) \).

By Lemma 2.11 and [11], Chapter IV, Proposition 3.8], \( M(q) = S_q \mathcal{M}_0 \), where \( \mathcal{M}_0 = l_{r-1} \)-form \( (G/O_{q',q}(G))/G \in \mathcal{M} \). Since \( \mathcal{M} \subseteq N^{r+1} \), \( G/O_{q',q}(G) \in N^r \), so \( \mathcal{M}_0 \subseteq N^r \) since \( \mathcal{M}_0 \) is an \( (r-1) \)-multiply saturated formation. Therefore, the minimality of \( r + |G| \) and Claim (4) imply that \( G/N \in M(q) \).

Final contradiction. Since \( N \) is a \( q \)-group by (1), from (5) it follows that \( G \in \mathcal{S}_q \mathcal{M}(q) = M(q) \subseteq \mathcal{M} \subseteq \mathcal{S}_p \mathcal{M} \). This contradiction completes the proof of the proposition.

**Corollary 3.5** (See [20], Satz 1.3]. Every saturated formation contained in \( N^2 \) is \( \Sigma_4 \)-closed.

**Corollary 3.6**. The class of all soluble groups of nilpotent length at most \( r \) (\( r \geq 2 \)) is \( \Sigma_{r+2} \)-closed.

**Proof.** It is clear that \( N^r \) is hereditary formation. Moreover, in view of Lemma 2.8, \( N^r \) is an \( (r-1) \)-multiply saturated formation. So \( N^r \) is \( \Sigma_{r+2} \)-closed by Proposition 3.4.

**Proof of Theorem A.** Suppose that the theorem is false and consider some counterexample
G of minimal order. Take a maximal subgroup M of G. Then by hypothesis all \((n - 1)\)-maximal subgroups of M are \(\mathcal{F}\)-subnormal in G, and so they are \(\mathcal{F}\)-subnormal in M by Lemmas 2.1(1) and 2.12. The solubility of G implies that either \(|\pi(M)| = |\pi(G)|\) or \(|\pi(M)| = |\pi(G)| - 1\), so \(M \in \mathcal{F}\) by the choice of G. Hence G is an \(\mathcal{F}\)-critical group.

Since G is soluble, G has a maximal subgroup T with \(|G : T| = p^a\) for any prime p dividing \(|G|\). On the other hand, \(\mathcal{F}\) is \(\Sigma_{r+3}\)-closed by Proposition 3.4. Hence \(|\pi(G)| \leq r + 2\). Moreover, by hypothesis, \(|\pi(G)| \geq n + r + 1\). Therefore \(n = 1\). Thus all maximal subgroups of G are \(\mathcal{F}\)-normal, so \(G/\Phi(G) \in \mathcal{F}\). But \(\mathcal{F}\) is a saturated formation and hence \(G \in \mathcal{F}\). This contradiction completes the proof of the result.

**Corollary 3.7** (See [11, Theorem 6]). If each \(n\)-maximal subgroup of a soluble group G is subnormal, and if \(|\pi(G)| \geq n + 1\), then G is nilpotent.

**Corollary 3.8** (See [21, Theorem A]). If every \(n\)-maximal subgroup of a soluble group G is \(\mathcal{U}\)-subnormal in G and \(|\pi(G)| \geq n + 2\), then G is supersoluble.

**Corollary 3.9.** Let \(\mathcal{F}\) be the class of all groups G with \(G' \leq F(G)\). If every \(n\)-maximal subgroup of a soluble group G is \(\mathcal{F}\)-subnormal in G and \(|\pi(G)| \geq n + 2\), then G \(\in \mathcal{F}\).

**Corollary 3.10.** If every \(n\)-maximal subgroup of a soluble group G is \(\mathcal{N}\)-subnormal in G \((r \geq 1)\) and \(|\pi(G)| \geq n + r\), then G \(\in \mathcal{N}\).

## 4 Proofs of Theorems B, C, and D

**Proof of Theorem C.** First suppose that every 2-maximal subgroup of G is \(\mathcal{F}\)-subnormal in G. Assume that \(G \notin \mathcal{F}\). We shall show that G is an \(\mathcal{F}\)-critical group and \(G^{\mathcal{F}}\) is a minimal normal subgroup of G. Let M be a maximal subgroup of G and T a maximal subgroup of M. By hypothesis, T is \(\mathcal{F}\)-subnormal in G. Therefore T is \(\mathcal{F}\)-normal in M by Lemma 2.1(1), so \(M/T_M \in \mathcal{F}\). Since T is arbitrary and \(\mathcal{F}\) is saturated, \(M \in \mathcal{F}\). Consequently, all maximal subgroups of G belong to \(\mathcal{F}\). Hence G is an \(\mathcal{F}\)-critical group. Then by hypothesis, G is soluble and it has a normal Sylow p-subgroup \(G_p \neq 1\) for some prime p. Thus \(G_p = G^{\mathcal{F}}\) by Lemma 2.5. On the other hand, by Lemma 2.4, \(G_p/\Phi(G_p)\) is a chief factor of G.

Let M be an \(\mathcal{F}\)-abnormal maximal subgroup of G. Then \(G_p \notin M\), so \(G = G_pM\) and \(M = (G_p \cap M)G_{p'} = \Phi(G_p)G_{p'}\), where \(G_{p'}\) is a Hall \(p'\)-subgroup of G. Assume that \(\Phi(G_p) \neq 1\). It is clear that \(\Phi(G_p) \notin \Phi(M)\). Let T be a maximal subgroup of M such that \(\Phi(G_p) \notin T\). Then \(M = \Phi(G_p)T\). Since T is \(\mathcal{F}\)-subnormal in G, there is a maximal subgroup L of G such that \(T \leq L\) and \(G/L_G \in \mathcal{F}\). Then \(G_p \leq LG\), so \(G = G_pM = G_p\Phi(G_p)T = G_pT \leq L\), a contradiction. Hence \(\Phi(G_p) = 1\). Therefore \(G_p = G^{\mathcal{F}}\) is a minimal normal subgroup of G by Lemma 2.4.

Now suppose that G is an \(\mathcal{F}\)-critical group and \(G^{\mathcal{F}}\) is a minimal normal subgroup of G. Let T be a 2-maximal subgroup of G and M a maximal subgroup of G such that T is a maximal subgroup
of \( M \). Since \( M \in \mathcal{F} \), \( T \) is \( \mathcal{F} \)-subnormal in \( M \) by Lemma 2.2. Therefore, if \( M \) is \( \mathcal{F} \)-normal in \( G \), then \( T \) is \( \mathcal{F} \)-subnormal in \( G \) by Lemma 2.1(3). Assume that \( M \) is \( \mathcal{F} \)-abnormal in \( G \). Then \( G^\mathcal{F} \not\leq M \). Therefore, since \( G^\mathcal{F} \) is a minimal normal subgroup of \( G \) by hypothesis, \( G = G^\mathcal{F} \rtimes M \) and \( G^\mathcal{F}T \) is a maximal \( \mathcal{F} \)-normal subgroup of \( G \). Moreover, since \( G \) is an \( \mathcal{F} \)-critical group, \( G^\mathcal{F}T \in \mathcal{F} \) and hence \( T \) is \( \mathcal{F} \)-subnormal in \( G^\mathcal{F}T \) by Lemma 2.2. Hence, \( T \) is \( \mathcal{F} \)-subnormal in \( G \). The theorem is proved.

From Theorem C and Lemma 2.6 we get

**Corollary 4.1** (See [21, Theorem 3.1]). Every 2-maximal subgroup of \( G \) is \( \mathcal{U} \)-subnormal in \( G \) if and only if \( G \) is an \( \mathcal{U} \)-critical group and \( G^{\mathcal{U}l} \) is a minimal normal subgroup of \( G \).

**Proof of Theorem B.** First suppose that all \( n \)-maximal subgroups of \( G \) are \( \mathcal{F} \)-subnormal in \( G \). We shall show, in this case, that either \( G \in \mathcal{F} \) or \( G \) is a group of the type II. Assume that this is false and consider a counterexample \( G \) for which \( |G| + n \) is minimal. Therefore \( A = G^\mathcal{F} \not= 1 \). Then:

(a) The hypothesis holds for every maximal subgroup of \( G \).

Let \( M \) be a maximal subgroup of \( G \). Then by hypothesis, all \((n - 1)\)-maximal subgroups of \( M \) are \( \mathcal{F} \)-subnormal in \( G \), and so they are \( \mathcal{F} \)-subnormal in \( M \) by Lemmas 2.1(1) and 2.12. Moreover, the solubility of \( G \) implies that either \( |\pi(M)| = |\pi(G)| \) or \( |\pi(M)| = |\pi(G)| - 1 \).

(b) If \( M \) is a maximal subgroup of \( G \) and \( |\pi(M)| = |\pi(G)| \), then \( M \in \mathcal{F} \).

In view of hypothesis and Lemmas 2.1(1) and 2.12, all \((n - 1)\)-maximal subgroups of \( M \) are \( \mathcal{F} \)-subnormal in \( M \). Since \( |\pi(M)| = |\pi(G)| \geq n + 1 = n - 1 + 2 \), \( M \in \mathcal{F} \) by Theorem A.

(c) If \( W \) is a Hall \( q' \)-subgroup of \( G \) for some \( q \in \pi(G) \), then either \( W \in \mathcal{F} \) or \( W \) is a group of the type II.

If \( W \) is not a maximal subgroup of \( G \), then there is a maximal subgroup \( V \) of \( G \) such that \( W \leq V \) and \( |\pi(V)| = |\pi(G)| \). By (b), \( V \in \mathcal{F} \). Hence \( W \in \mathcal{F} \) by Lemma 2.12. Suppose that \( W \) is a maximal subgroup of \( G \). Then by (a), the hypothesis holds for \( W \), so either \( W \in \mathcal{F} \) or \( W \) is a group of the type II by the choice of \( G \).

(d) The hypothesis holds for \( G/N \), where \( N \) is a minimal normal subgroup of \( G \).

If \( N \) is not a Sylow subgroup of \( G \), then \( |\pi(G/N)| = |\pi(G)| \). Moreover, if \( H/N \) is an \( n \)-maximal subgroup of \( G/N \), then \( H \) is an \( n \)-maximal subgroup of \( G \). Therefore \( H \) is \( \mathcal{F} \)-subnormal in \( G \). Consequently, \( H/N \) is \( \mathcal{F} \)-subnormal in \( G/N \) by Lemma 2.1(2). But if \( G/N \) has no \( n \)-maximal subgroups, then by the solubility of \( G \), the identity subgroup of \( G/N \) is \( \mathcal{F} \)-subnormal in \( G/N \) and it is the unique \( i \)-maximal subgroup of \( G/N \) for some \( i < n \) with \( i < |\pi(G/N)| \). Finally, consider the case that \( N \) is a Sylow \( p' \)-subgroup of \( G \). Let \( E \) be a Hall \( p' \)-subgroup of \( G \). It is clear that \( |\pi(E)| = |\pi(G)| - 1 \) and \( E \) is a maximal subgroup of \( G \).

Let \( H/N \) be an \((n - 1)\)-maximal subgroup of \( G/N \). Then \( H \) is an \((n - 1)\)-maximal subgroup of \( G \) and \( H = H \cap NE = N(H \cap E) \). There is a chain of subgroups \( H = H_0 < H_1 < \ldots < H_{n-1} = G \) of \( G \), where \( H_{i-1} \) is a maximal subgroup of \( H_i \) \((i = 1, \ldots, n - 1) \). Then \( H_{i-1} \cap E \) is a maximal
subgroup of $H_i \cap E$, for $i = 1, \ldots, n - 1$. Indeed, suppose that for some $i$ there is a subgroup $K$ of $H_i \cap E$ such that $H_{i-1} \cap E \leq K \leq H_i \cap E$. Then $(H_{i-1} \cap E)N \leq KN \leq (H_i \cap E)N$, so $H_{i-1} = H_{i-1} \cap EN \leq KN \leq H_i \cap EN = H_i$. Whence either $KN = H_{i-1}$ or $KN = H_i$. If $KN = H_{i-1}$, then $H_{i-1} \cap E = KN \cap E = K(\cap N \cap E) = K$. In the second case we have $H_{i-1} \cap E = KN \cap E = K(\cap N \cap E) = K$. Therefore $H_{i-1} \cap E$ is a maximal subgroup of $H_i \cap E$, so $H \cap E$ is an $(n-1)$-maximal subgroup of $E$. Since $E$ is a maximal subgroup of $G$, $H \cap E$ is an $n$-maximal subgroup of $G$. Hence $H \cap E$ is $\mathcal{F}$-subnormal in $G$ by hypothesis. Therefore $H/N = (H \cap E)N/N$ is $\mathcal{F}$-subnormal in $G/N$ by Lemma 2.1(2).

(e) $|\pi(G)| > 2$.

If $|\pi(G)| = 2$, then $n = 1$ and so all maximal subgroups of $G$ are $\mathcal{F}$-normal by hypothesis. Hence $G \in \mathcal{F}$ since $\mathcal{F}$ is a saturated formation, a contradiction.

(f) $G$ is an Ore dispersive group.

Suppose that this is false. Take a minimal normal subgroup $N$ of $G$. Then by (d), the hypothesis holds for $G/N$, so either $G/N \in \mathcal{F}$ or $G/N$ is a group of the type II. Thus, in view of $\mathcal{F} \subseteq \mathcal{U}$ and the choice of $G$, $G/N$ is an Ore dispersive group. By Lemma 2.13, the class of all Ore dispersive groups is a saturated formation. Therefore $N$ is the unique minimal normal subgroup of $G$ and $N \not\subseteq \Phi(G)$. Hence $\Phi(G) = 1$ and there is a maximal subgroup $L$ of $G$ such that $G = N \times L$ and $L_G = 1$. Thus $C_G(N) = N$ by [I, A, Theorem 15.2].

Since $G$ is soluble, $G$ has a normal maximal subgroup $M$ with $|G : M| = p$ for some prime $p$ and either $|\pi(M)| = |\pi(G)|$ or $|\pi(M)| = |\pi(G)| - 1$. By (a), the hypothesis holds for $M$. Therefore, in view of $\mathcal{F} \subseteq \mathcal{U}$ and the choice of $G$, $M$ is an Ore dispersive group. Denote by $q$ the greatest number in $\pi(M)$. Take a Sylow $q$-subgroup $M_q$ of $M$. Since $M_q$ is a characteristic subgroup of $M$, $M_q$ is normal in $G$. Consider the case $|\pi(M)| = |\pi(G)|$ first. Then $q$ is the greatest prime divisor of the order of $G$ and $M_q \neq 1$. Hence $G/M_q$ is an Ore dispersive group, and by the maximality of $q$, so is $G$. Suppose now that $|\pi(M)| = |\pi(G)| - 1$. If $q > p$, then, as above, we conclude that $G$ is an Ore dispersive group as well. Let $p > q$. Then $p$ is the greatest prime divisor of $|G|$. Since $M_q \neq 1$, it follows that $N \leq M_q$, so $N$ is a $q$-group. In addition, since $|\pi(G)| > 2$ by (e), there is a prime divisor $r$ of the order of $G$ such that $q \neq r \neq p$. Take a Hall $r'$-subgroup $W$ of $G$. Then $PN \leq W$ for some Sylow $p$-subgroup $P$ of $G$. Moreover, by (c), $W$ is an Ore dispersive group. Hence $P$ is normal in $W$, and so $P \leq C_G(N) = N$. The resulting contradiction shows that $G$ is an Ore dispersive group.

(g) $A$ is a nilpotent group.

Suppose that this is false. Let $N$ be a minimal normal subgroup of $G$. Then by (d), $(G/N)^{\mathcal{F}} = G^{\mathcal{F}} N/N \simeq G^{\mathcal{F}} / G^{\mathcal{F}} \cap N$ is a nilpotent group. It is known that the class of all nilpotent groups is a saturated formation. Hence in the case when $G$ has a minimal normal subgroup $R \neq N$ we have $G^{\mathcal{F}} / (G^{\mathcal{F}} \cap N) \cap (G^{\mathcal{F}} \cap R) \simeq G^{\mathcal{F}}$ is nilpotent. Thus $N$ is the unique minimal normal subgroup of $G$ and $N \leq G^{\mathcal{F}}$. If $N \leq \Phi(G)$, then $G^{\mathcal{F}} / G^{\mathcal{F}} \cap \Phi(G) \simeq (G^{\mathcal{F}} / N) / ((G^{\mathcal{F}} \cap \Phi(G)) / N)$ is nilpotent, so $G^{\mathcal{F}}$ is nilpotent by Lemma 2.14. Therefore $N \not\subseteq \Phi(G)$. Hence $\Phi(G) = 1$ and there is a maximal subgroup
of $G$ such that $G = N \times L$ and $L_G = 1$. Thus $C_G(N) = N$ by [1], A, Theorem 15.2 and $N \neq A$.

Case 1: $|\pi(G)| = 3$. By hypothesis, either all maximal subgroups of $G$ or all its 2-maximal subgroups are $\mathcal{F}$-subnormal in $G$. In the first case we infer that $G \in \mathcal{F}$, which contradicts the choice of $G$. Hence all 2-maximal subgroups of $G$ are $\mathcal{F}$-subnormal. Since $\mathcal{F} \subseteq \mathcal{U}$, in view of Lemma 2.6, every $\mathcal{F}$-critical group has a normal Sylow subgroup. Whence Theorem C implies that $G$ is an $\mathcal{F}$-critical group and $A = G^\mathcal{F}$ is a minimal normal subgroup of $G$. Therefore $A = N$, a contradiction.

Case 2: $|\pi(G)| \geq 4$. Assume that $N$ is a $p$-group, and take a Sylow subgroup $P$ of $G$ such that $N \leq P$. Observe that if $N \neq P$, then $L \in \mathcal{F}$ by (b), and so $A = N$, a contradiction. Hence $N = P$.

Case 2.1: $|\pi(G)| = 4$.

(1) All 3-maximal subgroups of $G$ are $\mathcal{F}$-subnormal in $G$ and $L$ is an $\mathcal{F}$-critical group.

Since $G \notin \mathcal{F}$ and $|\pi(G)| = 4$, either all 2-maximal subgroups of $G$ or all its 3-maximal subgroups are $\mathcal{F}$-subnormal in $G$. In the first case $G$ is an $\mathcal{F}$-critical group and $A = G^\mathcal{F}$ is a minimal normal subgroup of $G$ by Theorem C. Hence $A = N$, a contradiction. Therefore all 3-maximal subgroups of $G$ are $\mathcal{F}$-subnormal in $G$. Thus all second maximal subgroups of $G$ belong to $\mathcal{F}$ by Lemma 2.3. Consequently, either $L \in \mathcal{F}$ or $L$ is an $\mathcal{F}$-critical group. But in the first case $N = A$, a contradiction. Therefore $L$ is an $\mathcal{F}$-critical group.

(2) $L = Q \times (R \times T)$, where $Q, R, T$ are Sylow subgroups of $G$, $Q = L^\mathcal{F}$ is a minimal normal subgroup of $L$, and $G^\mathcal{F} = PQ$.

Since $N = P$ is a Sylow $p$-subgroup of $G$ and $|\pi(G)| = 4$, $|\pi(L)| = 3$. Hence in view of (f), $L = Q \times (R \times T)$, where $Q, R, T$ are Sylow subgroups of $G$. Moreover, $Q = L^\mathcal{F}$ by Lemma 2.5 and $Q$ is a minimal normal subgroup of $L$ by Theorem C since every 2-maximal subgroup of $L$ is $\mathcal{F}$-subnormal in $L$ by (1) and Lemmas 2.1(1) and 2.12. Finally, since $G/N \notin \mathcal{F}$ and $G/PQ \simeq L/Q \in \mathcal{F}$, we have $G^\mathcal{F} = PQ$.

(3) $V = PQR$ is not supersoluble. Hence $V \notin \mathcal{F}$.

Assume that $V$ is a supersoluble group. Since $F(V)$ is a characteristic subgroup of $V$ and $V$ is a normal subgroup of $G$, $F(V)$ is normal in $G$. Hence every Sylow subgroup of $F(V)$ is normal in $G$. But $N$ is the unique minimal normal subgroup of $G$. Therefore $F(V) = N = P$. Thus $V/P \simeq QR$ is an abelian group. Hence $R$ is normal in $L$ and so $R \leq F(L)$. In view of Lemma 2.5, $F(L) = Q\Phi(L)$. Whence $R \leq \Phi(L)$. This contradiction shows that $V$ is not supersoluble. Thus $V \notin \mathcal{F}$ since $\mathcal{F} \subseteq \mathcal{U}$ by hypothesis.

(4) $V$ is a maximal subgroup of $G$. Hence $|T| = t$ is a prime.

If $V$ is not a maximal subgroup of $G$, then there is a maximal subgroup $U$ of $G$ such that $V \leq U$ and $|\pi(U)| = |\pi(G)|$. Hence $U \in \mathcal{F}$ by (b), so $V \in \mathcal{F}$ by Lemma 2.12, a contradiction. Therefore $V$ is a normal maximal subgroup of $G$. Whence $|T|$ is a prime.

(5) $|Q| = q$ is a prime and $R = \langle x \rangle$ is a cyclic group.
Since $V$ is a maximal subgroup of $G$ by (4), all 2-maximal subgroups of $V$ are $\mathcal{F}$-subnormal in $V$ by (1) and Lemmas 2.1(1) and 2.12. Hence, in view of (3), $V$ is an $\mathcal{F}$-critical group by Theorem C. Therefore, in fact, $V$ is an $\mathcal{U}$-critical group by (3) since $\mathcal{F} \subseteq \mathcal{U}$. Hence $QR$ is supersoluble. Since $V$ is normal in $G$ and $\Phi(G) = 1$, $\Phi(V) = 1$. Therefore $QR$ is a Schmidt group by Lemma 2.6. Hence $R$ is cyclic and $Q$ is a minimal normal subgroup of $QR$ by Lemma 2.4. Whence $|Q|$ is a prime.

$(6)$ $|R| = r$ is a prime and $C_{\ell}(Q) = Q$.

By (4) and (5), $L$ is a supersoluble group. Suppose that $|R| = r^b$ is not a prime and let $M$ be a maximal subgroup of $L$ such that $|L : M| = r$. Let $W = PM$. Then $\pi(W) = \pi(G)$, so $W \in \mathcal{F}$ by (b) and hence $W$ is supersoluble. Since $C_{\ell}(N) = N$, $F(W) = P$. Hence $W/P \simeq M$ is abelian. It is clear that $Q \leq M$, so $M \leq C_{\ell}(Q)$. Hence $T \leq F(L)$. On the other hand, $F(L) = Q\Phi(L)$ by Lemma 2.5. Therefore $T \not\leq F(L)$. This contradiction shows that $|R| = r$ and so $C_{\ell}(Q) = Q$ by Lemma 2.5.

$(7)$ $1 \neq C_{\ell}(x) \cap PQ = P_1 \leq P$.

Suppose that $C_{\ell}(x) \cap PQ = 1$. Then by the Thompson’s theorem [22, Theorem 10.5.4], $PQ$ is a nilpotent group, so $Q \leq C_{\ell}(P) = P$, a contradiction. Thus $C_{\ell}(x) \cap PQ \neq 1$. Suppose that $q$ divides $|C_{\ell}(x) \cap PQ|$. Then, by (5), for some $a \in P$ we have $Q^a \leq C_{\ell}(x) \cap PQ$, so $\langle Q^a, RT \rangle \leq N_{\ell}(R)$. Hence if $E$ is a Hall $p'$-subgroup of $N_{\ell}(R)$, then $E \simeq L$. Therefore $L$ has a normal $r$-subgroup, so $C_{\ell}(Q) \neq Q$, a contradiction. Thus $C_{\ell}(x) \cap PQ = P_1 \leq P$.

Final contradiction for Case 2.1. Let $D = \langle P_1, RT \rangle$. Then $D \leq N_{\ell}(R)$. If $q$ divides $|D|$, then, as above, we have $C_{\ell}(Q) \neq Q$. Thus $D \cap Q^a = 1$ for all $a \in P$. Moreover, if $P \leq D$, then $PR = P \times R$ and $R \leq C_{\ell}(P) = P$. Therefore $P \not\leq D$ and $D$ is not a maximal subgroup of $G$. Hence $D$ is a $k$-maximal subgroup of $G$ for some $k \geq 2$. Then there is a 3-maximal subgroup $S$ of $G$ such that $RT \leq S \leq D$. By hypothesis, $S$ is $\mathcal{F}$-subnormal in $G$. Hence at least one of the maximal subgroups $L$ or $PRT$ is $\mathcal{F}$-normal in $G$, contrary to (2).

Case 2.2: $|\pi(G)| > 4$. If $\pi(L) = \{p_1, \ldots, p_t\}$, then $t > 3$. Let $E_i$ be a Hall $p_i'$-subgroup of $L$ and $X_i = P_Ei$. We shall show that $E_i \in \mathcal{F}$ for all $i = 1, \ldots, t$. By (c), either $X_i \in \mathcal{F}$ or $X_i$ is a group of the type II, for $i = 1, \ldots, t$. In the former case we have $E_i \simeq X_i/P \in \mathcal{F}$. Assume that $X_i$ be a group of the type II. Then $X_i^\mathcal{F}$ is nilpotent, so $X_i^\mathcal{F} \leq F(X_i)$. But since $P$ is normal in $X_i$ and $C_{\ell}(P) = P$, $F(X_i) = P$. Hence $X_i^\mathcal{F} = P$, so $E_i \in \mathcal{F}$. Since $t > 3$, Proposition 3.4 implies that then $L \in \mathcal{F}$. Therefore $A = N$, a contradiction. Hence we have (g).

(h) $A$ is a Hall subgroup of $G$.

Suppose that this is false. Since $G$ is Ore dispersive by (f), for the greatest prime divisor $p$ of $|G|$ the Sylow $p$-subgroup $P$ is normal in $G$. Assume that $P$ is not a minimal normal subgroup of $G$. Then there is a maximal subgroup $M$ of $G$ such that $G = PM$ and $P \cap M \neq 1$. Since $|\pi(M)| = |\pi(G)|$, $M \in \mathcal{F}$ by (b). Hence $G/P \simeq M/M \cap P \in \mathcal{F}$, so $A = C_{\mathcal{F}} \leq P$. Suppose that $\Phi(P) \neq 1$. Let $N$ be a minimal normal subgroup of $G$ such that $N \leq \Phi(P)$. By (d), the hypothesis holds for $G/N$, so either $G/N \in \mathcal{F}$ or $G/N$ is a group of the type II by the choice of $G$. If $G/N \in \mathcal{F}$,
then $A = N \leq \Phi(P)$. Since $P$ is normal in $G$, $\Phi(P) \leq \Phi(G)$. Thus $A \leq \Phi(G)$ and so $G \in \mathcal{F}$, a contradiction. Hence $G/N$ is a group of the type II. Therefore $AN/N = G^\mathcal{F}N/N = (G/N)^\mathcal{F}$ is a Hall subgroup of $G/N$. Consequently, $AN = P$. Hence $A\Phi(P) = P$, so $A = P$, a contradiction. Thus $\Phi(P) = 1$. By Maschke’s theorem, $P = N_1 \times \ldots \times N_k$ is the direct product of some minimal normal subgroups of $G$. If $N_1 \neq P$, then $G/N_1 \in \mathcal{F}$ and $G/N_2 \in \mathcal{F}$ by Theorem A. Consequently, so is $G$. This contradiction shows that $P$ is a minimal normal subgroup of $G$.

By (d), the hypothesis holds for $G/P$, so either $G/P \in \mathcal{F}$ or $G/P$ is a group of the type II by the choice of $G$. If $G/P \in \mathcal{F}$, then $A = P$, a contradiction. Hence $G/P$ is a group of the type II. Therefore $AP/P = G^\mathcal{F}P/P = (G/P)^\mathcal{F}$ is a Hall subgroup of $G/P$. If $P \leq A$, then $A = P \rtimes A_{p'}$, where $A_{p'}$ is a Hall $p'$-subgroup of $A$. But since $A_{p'} \simeq A/P$ and $AP/PA$ is a Hall subgroup of $G/P$, $A$ is a Hall subgroup of $G$. Therefore $P \cap A = 1$, so $A$ is a Hall subgroup of $G$ since $AP/P \simeq A/A \cap P \simeq A$.

(i) $A$ is either of the form $N_1 \times \ldots \times N_t$, where each $N_i$ is a minimal normal subgroup of $G$, which is a Sylow subgroup of $G$, for $i = 1, \ldots, t$, or a Sylow $p$-subgroup of $G$ of exponent $p$ for some prime $p$ and the commutator subgroup, the Frattini subgroup, and the center of $A$ coincide, while $A/\Phi(A)$ is an $\mathcal{F}$-eccentric chief factor of $G$.

Suppose that $A$ is not a minimal normal subgroup of $G$. Take a Sylow $p$-subgroup $P$ of $A$, where $p$ divides $|A|$. Claims (g) and (h) imply that $P$ is a normal Sylow subgroup of $G$. Let $N$ be a minimal normal subgroup of $G$ with $N \leq P$. First suppose that $N \leq \Phi(G)$, and take a maximal subgroup $M$ of $G$ with $P \nsubseteq M$. Then $M \in \mathcal{F}$ by (b). Therefore $G/P \cong M/M \cap P \in \mathcal{F}$. In this case $A = P$. Moreover, if $S$ is a maximal subgroup of $G$ such that $P \nsubseteq S$, then $S \in \mathcal{F}$. Observe also that for every maximal subgroup $X$ of $G$ with $P \leq X$ we have $X$ is $\mathcal{F}$-subnormal in $G$. Thus, by Lemma 2.4, $A = G^\mathcal{F}$ satisfies condition II(1).

Suppose that for every minimal normal subgroup $R$ of $G$ such that $R \leq A$ we have $R \nsubseteq \Phi(G)$. Then there is a maximal subgroup $L$ of $G$ such that $G = N \rtimes L$. If $N \neq P$, then $L \in \mathcal{F}$ by (b). Therefore $A = N$, a contradiction. Consequently, all Sylow subgroups of $A$ are minimal normal subgroups of $G$. Therefore $A = N_1 \times \ldots \times N_t$, where $N_i$ is a minimal normal subgroup of $G$, for $i = 1, \ldots, t$.

(j) Every $n$-maximal subgroup of $G$ belongs to $\mathcal{F}$ and induces on the Sylow $p$-subgroup of $A$ the automorphism group which is contained in $F(p)$ for every prime divisor $p$ of $|A|$.

Let $H$ be any $n$-maximal subgroup of $G$. Suppose that $H$ is a maximal subgroup of $V$, where $V$ is an $(n - 1)$-maximal subgroup of $G$. Since $V \in \mathcal{F}$ by Lemmas 2.3 and 2.12, $H \in \mathcal{F}$.

Let $E = AH$. Since $A$ is normal in $E$ and $A$ is nilpotent by (g), $A \leq F(E)$. Whence $E = F(E)H$. Since $H$ is $\mathcal{F}$-subnormal in $G$, $H$ is $\mathcal{F}$-subnormal in $E$ by Lemmas 2.1(1) and 2.12. Moreover, $H \in \mathcal{F}$. Therefore $E \in \mathcal{F}$ by Lemma 2.15. Let $P$ be a Sylow $p$-subgroup of $A$ and $K/L$ a chief factor of $E$ such that $1 \leq L < K \leq P$. Since $E \in \mathcal{F}$, $E/C_E(K/L) \leq F(p)$. Hence $P \leq Z_\mathcal{F}(E)$, so $E/C_E(P) \in F(p)$ by Lemma 2.7. Then $H/C_H(P) = H/C_E(P) \cap H \cong HC_E(P)/C_E(P) \in F(p)$. 

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Now suppose that either \( G \in \mathcal{F} \) or \( G \) is a group of type II. If \( G \in \mathcal{F} \), then every subgroup of \( G \) is \( \mathcal{F} \)-subnormal in \( G \) by Lemma 2.2. Let \( \mathcal{F} \) be a group of type II. Take an \( n \)-maximal subgroup \( H \) of \( G \). Put \( E = G^\mathcal{F}H \). Let \( P \) be a Sylow \( p \)-subgroup of \( G^\mathcal{F} \) and \( K/L \) a chief factor of \( E \) such that \( 1 \leq L < K \leq P \). By hypothesis, \( H/C_H(P) \in F(p) \), so \( H/C_H(K/L) \simeq (H/C_H(P))/(C_H(K/L)/C_H(P)) \in F(p) \). Since \( G^\mathcal{F} \) is normal in \( E \) and \( G^\mathcal{F} \) is nilpotent, \( G^\mathcal{F} \leq F(E) \leq C_E(K/L) \). Hence

\[
E/C_E(K/L) = E/C_E(K/L) \cap E = E/C_E(K/L) \cap G^\mathcal{F}H = E/G^\mathcal{F}(C_E(K/L) \cap H) = \]

\[
= G^\mathcal{F}H/G^\mathcal{F}C_H(K/L) \simeq H/G^\mathcal{F}C_H(K/L) \cap H = H/C_H(K/L)(G^\mathcal{F} \cap H) = H/C_H(K/L),
\]

so \( E/C_E(K/L) \in F(p) \) since \( F(p) \) is hereditary by Lemma 2.12 and \([15, \text{Proposition 3.1.40}] \). Then \( P \leq Z_\mathcal{F}(E) \), whence \( G^\mathcal{F} \leq Z_\mathcal{F}(E) \). Thus \( E/Z_\mathcal{F}(G) \in \mathcal{F} \). Hence \( E \in \mathcal{F} \), so \( H \) is an \( \mathcal{F} \)-subnormal subgroup of \( G^\mathcal{F} = E \). Since \( G^\mathcal{F} \leq G^\mathcal{F}H \), \( G^\mathcal{F}H \) is \( \mathcal{F} \)-subnormal in \( G \) by Lemma 2.1(4). Consequently, in view of Lemma 2.1(3), \( H \) is \( \mathcal{F} \)-subnormal in \( G \). The theorem is proved.

**Corollary 4.2** (See [21 Theorem B]). Given a soluble group \( G \) with \( |\pi(G)| \geq n+1 \), all \( n \)-maximal subgroups of \( G \) are \( \mathcal{U} \)-subnormal in \( G \) if and only if \( G \) is a group of one of the following types:

I. \( G \) is supersoluble.

II. \( G = A \rtimes B \), where \( A = G^\mathcal{U} \) and \( B \) are Hall subgroups of \( G \), while \( G \) is Ore dispersive and satisfies the following:

(1) \( \mathcal{F} \) is either of the form \( N_1 \times \ldots \times N_t \), where each \( N_i \) is a minimal normal subgroup of \( G \), which is a Sylow subgroup of \( G \), for \( i = 1, \ldots, t \), or a Sylow \( p \)-subgroup of \( G \) of exponent \( p \) for some prime \( p \) and the commutator subgroup, the Frattini subgroup, and the center of \( A \) coincide, every chief factor of \( G \) below \( \Phi(G) \) is cyclic, while \( A/\Phi(A) \) is a noncyclic chief factor of \( G \);

(2) for every prime divisor \( p \) of the order of \( A \) every \( n \)-maximal subgroup \( H \) of \( G \) is supersoluble and induces on the Sylow \( p \)-subgroup of \( A \) an automorphism group which is an extension of some \( p \)-group by abelian group of exponent dividing \( p-1 \).

**Proof of Theorem D.** Assume that this is false and consider a counterexample \( G \) for which \( |G| + n \) is minimal.

(a) \( G \) has a unique minimal normal subgroup \( N \) such that \( C_G(N) = N \) and \( N \) is not a Sylow subgroup of \( G \).

Let \( N \) be a minimal normal subgroup of \( G \). Then the hypothesis holds for \( G/N \) (see Claim (d) in the proof of Theorem B). Consequently, \( G/N \) is \( \phi \)-dispersive for some ordering \( \phi \) of \( \mathbb{P} \) by the choice of \( G \). Therefore \( N \) is not a Sylow subgroup of \( G \). Moreover, by Lemma 2.13, \( N \not\subseteq \Phi(G) \). Therefore \( G \) has a maximal subgroup \( M \) such that \( G = N \rtimes M \). By Lemmas 2.1(1) and 2.12 all \((n-1)\)-maximal subgroups of \( M \) are \( \mathcal{F} \)-subnormal in \( M \). Moreover, \( |\pi(M)| = |\pi(G)| \). Therefore Theorem B implies that \( G/N \simeq M \) is an Ore dispersive group. Hence in the case when \( G \) has a minimal normal subgroup \( R \neq N \) we have \( G/N \cap R \simeq G \) is an Ore dispersive group. Thus \( N \) is the unique minimal normal subgroup of \( G \), and so \( C_G(N) = N \) by [1, A, Theorem 15.2].
(b) If $W$ is a Hall $q'$-subgroup of $G$ for some $q \in \pi(G)$, then $W$ is $\phi$-dispersive for some ordering $\phi$ of $\mathbb{P}$.

If $W$ is not a maximal subgroup of $G$, then there is a maximal subgroup $V$ of $G$ such that $W \leq V$ and $|\pi(W)| = |\pi(G)|$. By hypothesis, every $(n - 1)$-maximal subgroup of $V$ is $\mathcal{F}$-subnormal in $G$, so it is $\mathcal{F}$-subnormal in $V$ by Lemmas 2.1(1) and 2.12. Then, in view of Theorem B, $V$ is Ore dispersive. Hence $W$ is Ore dispersive. Suppose that $W$ is a maximal subgroup of $G$. Then $|\pi(W)| = |\pi(G)| - 1$ and every $(n - 1)$-maximal subgroup of $W$ is $\mathcal{F}$-subnormal in $W$ in view of hypothesis and Lemmas 2.1(1) and 2.12. Therefore $W$ is $\phi$-dispersive for some ordering $\phi$ of $\mathbb{P}$ by the choice of $G$.

(c) $|\pi(G)| > 2$.

Suppose that $|\pi(G)| = 2$. Then by hypothesis, either all maximal subgroups of $G$ or all its 2-maximal subgroups are $\mathcal{F}$-subnormal in $G$. Therefore every maximal subgroup of $G$ belongs to $\mathcal{F}$ in view of Lemmas 2.3 and 2.12. Consequently, either $G \in \mathcal{F}$ or $G$ is an $\mathcal{F}$-critical group. Since $\mathcal{F} \subseteq \mathcal{U}$, $G$ is either a supersoluble group or an $\mathcal{U}$-critical group. Therefore, in view of Lemma 2.6, $G$ is $\phi$-dispersive for some ordering $\phi$ of $\mathbb{P}$, a contradiction.

Final contradiction. Suppose that $N$ is a $p$-group, and take a prime divisor $q$ of $|G|$ such that $q \neq p$. Take a Hall $q'$-subgroup $E$ of $G$. Then $N \leq E$. By (b), $E$ is $\phi$-dispersive for some ordering $\phi$ of $\mathbb{P}$. Consequently, some Sylow subgroup $R$ of $E$ is normal in $E$. Furthermore, if $N \nleq R$, then $R \leq C_G(N) = N$. Hence $R$ is a Sylow $p$-subgroup of $E$. It is clear also that $R$ is a Sylow $p$-subgroup of $G$ and $(|G : N_G(R)|, r) = 1$ for every prime $r \neq q$. Since $|\pi(G)| > 2$ by (c), $R$ is normal in $G$. Hence $G$ is $\phi$-dispersive for some ordering $\phi$ of $\mathbb{P}$, a contradiction. The theorem is proved.

Corollary 4.3 (See [21, Theorem C]). If every $n$-maximal subgroup of a soluble group $G$ is $\mathcal{U}$-subnormal in $G$ and $|\pi(G)| \geq n$, then $G$ is $\phi$-dispersive for some ordering $\phi$ of the set of all primes.

Finally, note that there are examples which show that the restrictions on $|\pi(G)|$ in Theorems A, B, and D cannot be weakened.

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