Nonlinear Transformation for a Class of Gauged Schrödinger Equations with Complex Nonlinearities

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In the present contribution we consider a class of Schrödinger equations containing complex nonlinearities, describing systems with conserved norm $|\psi|^2$ and minimally coupled to an abelian gauge field. We introduce a nonlinear transformation which permits the linearization of the source term in the evolution equations for the gauge field, and transforms the nonlinear Schrödinger equations in another one with real nonlinearities. We show that this transformation can be performed either on the gauge field $A_\mu$ or, equivalently, on the matter field $\psi$. Since the transformation does not change the quantities $|\psi|^2$ and $F_{\mu\nu}$, it can be considered a generalization of the gauge transformation of third kind introduced some years ago by other authors.

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In a previous paper [1] we have introduced a nonlinear unitary transformation acting on a class of nonlinear Schrödinger equations (NLSEs) with complex nonlinearities in order to transform these equations in other NLSEs containing purely real nonlinearities. As a consequence, the continuity equation for the new matter field results to be linearized. This nonlinear transformation can be seen as a generalization of the Doebner-Goldin approach adopted to obtain the NLSE associated with unitary group representations [2, 3, 4, 5]. The present paper is a natural continuation of Ref. [1]. We consider a class of NLSEs minimally coupled with an abelian gauge field $A_\mu$. The $(n+1)$ model is described by the following Lagrangian density:

$$\mathcal{L} = \mathcal{L}_{\text{matter}} + \mathcal{L}_{\text{gauge}} ,$$

with

$$\mathcal{L}_{\text{matter}} = i\hbar \left[ \psi^* \frac{D_0 \psi}{\hbar} - \psi \frac{D_0 \psi^*}{\hbar} - \frac{\hbar^2}{2m} |D\psi|^2 - U(\psi, \psi^*, A) \right] ,$$

being $D_\mu = \partial_\mu + (i e/h c) A_\mu \equiv (D_0, D)$ the covariant derivative with $\partial_\mu \equiv (c^{-1} \partial_0, \nabla)$, $\mu = 0, 1, \cdots, n$.

The nonlinear potential $U(\psi, \psi^*, A)$ is assumed to be real. We use the notation $U([a])$ to indicate that $U$ is a function of the field $a$ and of its spatial covariant derivatives. After introducing the hydrodynamic fields $\rho(t, x)$ and $S(t, x)$ through:

$$\rho = \psi \psi^* ,$$

$$S = \frac{\hbar c}{2e} \log \left( \frac{\psi^*}{\psi} \right) ,$$

which represent the modulo and the phase of the field $\psi$ respectively:

$$\psi(t, x) = \rho^{1/2}(t, x) \exp \left[ \frac{i e}{\hbar c} S(t, x) \right] ,$$

we can write the nonlinear potential as $U = U(\rho, [S], A)$.

In the following, for simplicity, we assume for the Lagrangian $\mathcal{L}_{\text{gauge}}$ the standard form of the electromagnetic field:

$$\mathcal{L}_{\text{gauge}} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} ,$$

where the electromagnetic tensor is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. We obtain upper and lower indices by using the metric tensor $g^{\mu\nu} = \text{diag}(1, -1, \cdots, -1)$: $A^\mu = g^{\mu\nu} A_\nu$. Greek indices take the values $0, 1, 2, \cdots, n$ while the latin indices take the values $1, 2, \cdots, n$.

Starting from the action of the system $A = \int \mathcal{L} dt \, d^n x$, the evolution equations for the fields $a = \psi, \psi^*, A_\mu, \rho$ and $S$ can be obtained from the relation $\delta A/\delta a = 0$, where the functional derivative is defined as [8]:

$$\frac{\delta}{\delta a} \int F[a] dt \, d^n x = \sum_{k=0} (-1)^k \sum_I D_I \frac{\partial F[a]}{\partial (D_I a)} .$$

In Eq. (7), the second sum is over the multi-index $I \equiv (i_1, i_2, \cdots, i_n)$, and $0 \leq i_p \leq k, \sum_i i_p = k$, $D_I \equiv \partial^{i_1}/(\partial x^{i_1} \cdots \partial x^{i_k})$. When $a = \psi^*$ we obtain the following gauged NLSE:

$$i c \hbar D_0 \psi = -\frac{\hbar^2}{2m} D^2 \psi + W \psi + i W \psi ,$$

where the real $W$ and imaginary $W$ parts of the nonlinearity in Eq. (8) are given by:

$$W(\rho, [S], A) = \frac{\delta}{\delta \rho} \int U(\rho, [S], A) dt \, d^n x ,$$

$$W(\rho, [S], A) = \frac{\hbar c}{2e \rho} \frac{\delta}{\delta S} \int U(\rho, [S], A) dt \, d^n x .$$
Differently, for \( a = S \) we obtain:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot J = \frac{c}{e} \int \frac{\partial}{\partial S} U([\rho], [S], A) \, dt \, d^nx = 0 \tag{11}
\]

which can be rewrite in:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot J = \frac{c}{e} \int \frac{\partial}{\partial S} U([\rho], [S], A) \, dt \, d^nx \tag{12}
\]

where the current is given by

\[
J_i = \frac{c}{m} (\partial_i S + A_i) \rho + \frac{c}{e} \frac{\delta}{\delta (\partial_i S)} \int U \, dt \, d^nx \tag{13}
\]

If the nonlinear potential \( U \) depends on \( S \) only trough its derivatives, the r.h.s of Eq. (12) vanish and its became a continuity equation of the field \( \rho \):

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \tag{14}
\]

which implies the conservation of the total charge:

\[
Q = e \int \rho \, d^nx \tag{15}
\]

Following the Noether theorem, Eq. (14) implies the existence of a symmetry for the system. The action \( A \) is invariant over global transformation of the \( U(1) \) group: \( \psi \to \psi' = \psi \exp(i \epsilon) \). We remark that this symmetry exists only if the nonlinear potential \( U \) depend on \( S \) only through its derivatives.

Analogously the evolution equation \( \delta A / \delta A_\nu = 0 \) of the field \( A_\nu \) becomes:

\[
\partial^\mu F_{\mu\nu} = \frac{e}{c} J_\nu \tag{16}
\]

where the covariant current \( J_\nu \) is given by: \( J_\nu \equiv (c \rho, -J) \).

Now we will introduce a nonlinear and nonlocal transformation whose effect is to eliminate the imaginary part of the nonlinearity in the NLSE so that the particle current associated to the new transformed Schrödinger equation assumes the standard bilinear structure. As it will be shown, this can be performed in two different ways: or by a unitary transformation acting on the field \( \psi \) or by a gauge transformation acting on the field \( A_\mu \).

We start by considering the transformation acting on the field \( \psi \):

\[
\psi(t, x) \to \phi(t, x) = U([\rho], [S], A) \psi(t, x) \tag{17}
\]

which allows to eliminate the imaginary part \( W \) of the nonlinearity in the evolution equation (8), and to standardize the expression of the current:

\[
J([\rho], [S], A) \to J(\rho, \nabla s, A) = \frac{e}{mc} (\nabla s - A) \rho \tag{18}
\]

The functional \( U \), with \( U^* = U^{-1} \), can be written as:

\[
U([\rho], [S], A) = \exp \left[ \frac{i c}{\hbar} \sigma ([\rho], [S], A) \right] \tag{19}
\]

where the generator \( \sigma \) of the transformation is defined through the relation:

\[
\partial_i \sigma ([\rho], [S], A) = \frac{mc^2}{\hbar e \rho} \frac{\delta}{\delta (\partial_i S)} \int U \, dt \, d^nx \tag{20}
\]

Eq. (20) imposes a condition on the form of the nonlinear potential \( U \) which can be obtained using the relation:

\[
\partial_i \sigma = \partial_i \sigma:
\]

\[
\left\{ \frac{\partial_i \left[ \frac{1}{\rho} \frac{\delta}{\delta (\partial_j S)} \right]}{\partial_j \left[ \frac{1}{\rho} \frac{\delta}{\delta (\partial_i S)} \right]} \right\} \int U \, dt \, d^nx = 0 \tag{21}
\]

for all \( i, j = 1, \cdots, n \). We remark that the condition (21) select the potential \( U([\rho], [S], A) \) and the nonlinear systems where we can perform the transformation (17).

From Eqs. (17) and (19) it is easy to obtain the phase \( s \) of the new field \( \phi \):

\[
s = S + \sigma([\rho], [S], A) \tag{22}
\]

while the modulo of \( \phi \) is equal to the modulo of \( \psi \), because of the unitarity of the transformation. If Eq. (22) is invertible, we can express the old phase \( S \) as a functional of \( \rho, s \) and \( A \): \( S = S([\rho], [s], A) \).

From Eq. (8) and taking into account Eqs. (17), (19) and (20), it is easy to obtain the following NLSE for the new field \( \phi \):

\[
i \hbar D_0 \phi = -\frac{\hbar^2}{2m} \nabla^2 \phi + \widetilde{W}([\rho], [s], A) \phi \tag{23}
\]

where the real nonlinearity \( \widetilde{W} \) is given by:

\[
\widetilde{W}([\rho], [s], A) = W + \frac{e^2}{2mc^2} (\nabla \sigma)^2 - \frac{e}{c} \frac{\mathcal{J} \cdot \nabla \sigma}{\rho} - \frac{e}{c} \frac{\partial \sigma}{\partial t} \tag{24}
\]

with \( W \equiv W([\rho], [S([\rho], [s], A)] \).

As a consequence of Eq. (23), the continuity equation becomes:

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0 \tag{25}
\]

Finally, the evolution equation (16) for the gauge field \( A_\mu \) becomes:

\[
\partial^\mu F_{\mu\nu} = \frac{e}{c} J_\nu \tag{26}
\]

where now also the source \( J_\nu \equiv (c \rho, -\mathcal{J}) \) results linearized.
We consider now the transformation acting on the gauge field $A_{\mu}$:

$$A \rightarrow \chi = A - \nabla \sigma .$$  \hspace{1cm} (27)

It is well known that the gauge field is the fundamental quantity of the theory, while the physical informations are carried out by the tensor field $F_{\mu\nu}$ [9]. We require that the transformation leaves unchanged this quantity:

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu \chi_\nu - \partial_\nu \chi_\mu .$$  \hspace{1cm} (28)

When $\mu$, $\nu$ are spatial indices, Eq. (28) is satisfied if Eq. (21) still holds. When $\mu$ or $\nu$ is equal to zero, Eq. (28) implies the following transformation for $A_0$:

$$A_0 \rightarrow \chi_0 = A_0 + \frac{1}{c} \frac{\partial \sigma}{\partial t} ,$$  \hspace{1cm} (29)

where $\sigma$ is given by Eq. (20). Then, from Eq. (8), and taking into account Eqs. (27) and (29) we obtain:

$$i \hbar \partial_0 \psi = -\frac{\hbar^2}{2m} D^2 \psi + \nabla \left( [\rho, [S], \chi] \right) \psi ,$$  \hspace{1cm} (30)

which has the same form of Eq. (23) but $\nabla_\mu = \partial_\mu + \left( i c/\hbar c \right) \chi_\mu$ and the real nonlinearity is:

$$\nabla \left( [\rho, [S], \chi] \right) = W + \frac{\hbar^2}{2m c^2} (\nabla \sigma)^2 - \frac{\hbar^2}{c} \frac{\mathcal{I} \cdot \nabla \sigma}{\rho} - \frac{\hbar^2}{c} \frac{\partial \sigma}{\partial t} ,$$  \hspace{1cm} (31)

with $W \equiv W([\rho], [S], \chi + \nabla \sigma]$ , ) and the new linearized current

$$\mathcal{I} = \frac{e}{m c} (\nabla S - \chi) \rho ,$$  \hspace{1cm} (32)

obeys the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathcal{I} = 0 .$$  \hspace{1cm} (33)

On the same foot, the evolution equation (16) for the gauge field becomes:

$$\partial^\mu F_{\mu\nu} = \frac{e}{c} \mathcal{I}_\nu ,$$  \hspace{1cm} (34)

where $\mathcal{I}_\nu \equiv (c \rho, -\mathcal{I})$.

In order to show how the method above described can be used, we consider the canonical subclass of the Doebner and Goldin equations [4, 5]:

$$i \hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + m \nu \nabla \left( \frac{J_0}{\rho} \right) \psi$$
$$-2 \alpha \frac{\hbar^2}{m} \left( \frac{\Delta \rho}{\rho} - \frac{\hbar^2}{2m} \left( \frac{\nabla \rho}{\rho} \right)^2 \right) \psi + \frac{\hbar^2}{2\nu} \frac{\Delta \rho}{\rho} \psi ,$$  \hspace{1cm} (35)

where $J_0 = (-i \hbar/2m) (\psi^* \nabla \psi - \psi \nabla \psi^*)$ is the standard quantum-mechanical current, $\nu$ is a diffusion coefficient, and $\alpha$ a dimensionless coupling constant. Eq. (35) is obtainable starting from the nonlinear potential:

$$U_{DG}([\rho], [S]) = \frac{\nu}{2} (\rho \Delta S - \nabla \rho \cdot \nabla S) + \frac{\hbar^2}{m} \frac{(\nabla \rho)^2}{\rho} .$$  \hspace{1cm} (36)

It is easy to verify that the current $j$ associated to Eq. (35) is given by:

$$j = \frac{\nabla S}{m} \rho - \nu \nabla \rho ,$$  \hspace{1cm} (37)

and obeys to the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot j = 0 ,$$  \hspace{1cm} (38)

which is the well-known Fokker-Planck equation. In the case of interacting charged particles, Eq. (35) must be replaced by the following NLSE:

$$i \hbar D_0 \psi = -\frac{\hbar^2}{2m} D^2 \psi + m \nu \nabla \left( \frac{J_0}{\rho} \right) \psi$$
$$-2 \alpha \frac{\hbar^2}{m} \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 \right] \psi + \frac{\hbar^2}{2\nu} \frac{\Delta \rho}{\rho} \psi ,$$  \hspace{1cm} (39)

whose associated nonlinear potential is given by:

$$U([\rho], [S], A) = \frac{\nu e}{2c} \rho \nabla \left( \nabla S - \nabla A \right) - \nabla \rho \cdot \left( \nabla S - \nabla A \right)$$
$$\quad + \frac{\hbar^2}{m} \frac{(\nabla \rho)^2}{\rho} .$$  \hspace{1cm} (40)

We note that the current $J_0$ is given by:

$$J_0 = -\frac{ie}{2m} \left[ \psi^* \left( \nabla - \frac{ie}{\hbar c} A \right) \psi - \psi \left( \nabla + \frac{ie}{\hbar c} A \right) \psi^* \right] .$$  \hspace{1cm} (41)

The current $J$ of the system (39) assumes the form $J = J_0 - \nu \nabla \rho$ or explicitly:

$$J = \frac{e}{m c} \rho \left( \nabla S - A \right) - \nu \nabla \rho .$$  \hspace{1cm} (42)

Performing the unitary transformation $\psi \rightarrow \phi = U \psi$; $A_\mu \rightarrow A_\mu$ defined through Eqs. (19)-(20), we obtain

$$U = \exp \left( -i \frac{m}{\hbar} \nu \log \rho \right) .$$  \hspace{1cm} (43)

It is easy to verify that Eq. (39) is transformed as:

$$i \hbar D_0 \phi = -\frac{\hbar^2}{2m} D^2 \phi$$
$$\quad + \left( m \nu^2 - 2 \alpha \frac{\hbar^2}{m} \right) \left[ \frac{\Delta \rho}{\rho} - \frac{1}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 \right] \phi .$$  \hspace{1cm} (44)

We note that the transformation (43) is the same previously introduced by Doebner and Goldin [3, 4]. The
problem concerning the elimination of the imaginary part in the nonlinearity of Doebner-Goldin equation has been considered previously in Ref. [6].

Equation (44) can be obtained considering the gauge transformation $\psi \rightarrow \psi'$; $A_\mu \rightarrow \chi_\mu$ with:

$$\chi = A + \frac{m c}{e} \nu \frac{\nabla \rho}{\rho},$$

$$\chi_0 = A_0 + \frac{\nu}{c \rho} \nabla \left[ (\nabla S - \chi) \rho \right].$$

We conclude the discussion on the gauged Doebner and Goldin equation, by observing that Eq. (44) can be written in the form:

$$i c k \cdot D_0 \phi = -\frac{\hbar^2}{2m} D^2 \phi,$$

where $\hbar^2 = \hbar^2 (1 + 8 \alpha) - 4 m^2 \nu^2$. The procedure used to transform Eq. (44) into Eq. (47) is the same used in the case $A_\mu = 0$, described in Ref. [10].

In conclusion we have shown here two different ways to reduce the complex nonlinearity of a NLSE into a real one. In order to obtain this, we can perform a nonlinear unitary transformation on the matter field of Eq. (17) or, alternatively, by performing a transformation on the gauge field (27). As a working example of the method here discussed we have considered the gauged Doebner and Goldin equation, describing a system of interacting charged particles.

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