Dissipative engineering of Gaussian entangled states in harmonic lattices with a single-site squeezed reservoir

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We discuss the dissipative engineering of quantum-correlated Gaussian states in bosonic lattice models, with particle-conserving quadratic Hamiltonians, which are coupled to a single localized squeezed reservoir. We show that it is possible to identify specific Hamiltonians which sustain in the stationary regime any Gaussian state that can be generated by applying a passive Gaussian transformation (which conserves the number of excitations) on a set of equally squeezed modes. The class of states that can be prepared includes non-trivial entangled states such as cluster states suitable for measurement-based quantum computation.

The harnessing of quantum many-body dynamics by engineered dissipation is interesting for applications in quantum technology \cite{1,2,3}. In these approaches the environment of many interacting quantum systems is designed in such a way that the interplay between controlled dissipation and interactions results in specific controlled system dynamics and in the robust preparation of non-trivial quantum global stationary states. Interestingly, it has been also shown that under certain conditions it is possible to engineer a single localized reservoir to have control over the global properties of the system \cite{4,5}.

In this work we are interested in strategies which make use of minimal resources, namely only one squeezed reservoir and a bosonic lattice with a passive (particle-conserving) quadratic Hamiltonian \cite{6,7,14}. It has been shown that these systems can be steered into peculiar entangled steady states, when the squeezed reservoir is coupled to single site of the lattice and the Hamiltonian is endowed with specific symmetries \cite{7,10}. Here we characterize the class of Gaussian pure states that can be achieved with this approach, and we show that it is composed of all the states that can be generated by applying any combination of particle-conserving quadratic operations (beam splitters and phase shifts) on a set of equally squeezed modes. We also identify the general properties of the Hamiltonians which enable the generation of these pure stationary states (showing, in particular, that they necessarily satisfy the chiral symmetry identified in Ref. \cite{10}), and, for each state, we discuss how to construct the specific Hamiltonian which sustain such state in the stationary regime. Interestingly, the class of states that can be obtained in this way includes Gaussian cluster states usable for universal quantum computation with continuous variables \cite{15}, and, as a prominent example, we study the performance of the present approach for the preparation of a cluster state in a square lattice.

In detail, we study the dissipative preparation of a zero-average pure Gaussian state of \( N \) bosonic modes \(|\Psi\rangle\), considering \( N + 1 \) bosonic modes (including an additional auxiliary mode). They are described by the annihilation operators \( b_j \) for \( j \in \{0, 1 \cdots N\} \), and we assume that only the auxiliary mode, that is the one with index \( j = 0 \), is coupled to a squeezed reservoir. In the ideal situation the auxiliary mode is the only open mode which is subject to dissipation in the squeezed reservoir. Additional dissipation acting on the other modes reduces the purity of the final state and will be addressed later on. We assume quadratic Hamiltonians \( H \) for the \( N + 1 \) modes, with only passive interaction terms, \( H = \hbar \sum_{j,k=0}^{N} f_{jk} b_j b_k \) (with \( f_{jk} = f_{kj}^* \)), which conserves the number of excitations, so that the existing quantum correlations in the steady state are a consequence of the correlations in the reservoirs only. The system is described by the master equation

\[
\dot{\rho} = -\frac{i}{\hbar} [H, \rho] + \mathcal{L} \rho, \tag{1}
\]

where the effect of the squeezed bath is given by the Lindblad term \( \mathcal{L} \rho = \kappa (\n \rho - \n \rho \rho - \rho \n \rho) \). This condition corresponds to a reservoir in a pure squeezed state; if \( |\bar{m}| < \sqrt{|n + 1|} \) the reservoir is not pure, and the states that we discuss here are modified, in a straightforward way, by a thermal component \( |\Psi\rangle \). The central result of this work is the following theorem.

\textbf{Theorem.} A zero-average pure Gaussian state which is factorized between the auxiliary mode \(|\Psi_0\rangle\) and the remaining \( N \) modes \(|\Psi\rangle\)

\[
|\Psi_{\text{tot}}\rangle = |\Psi_0\rangle |\Psi\rangle, \tag{2}
\]

and is generated from the vacuum \(|0\rangle\) by the unitary transformations \( U_0 \) and \( U \), such that \(|\Psi_0\rangle = U_0 |0\rangle \) and \(|\Psi\rangle = U |0\rangle \), is the unique steady state of Eq. (1) if and only if the following three propositions are true:

\textbf{I -} \( U_0 \) is the squeezing transformation \( U_{0} = e^{\frac{\pi}{2}(e^{\bar{n}b_0^*} - e^{-\bar{n}b_0})} \),

where the squeezing strength \( \zeta_0 \) and the squeezing phase \( \varphi_0 \) are determined by the squeezing of the reservoir according to the relations \( \tanh(\zeta_0) = \sqrt{|\bar{m}|/(|n + 1|)} \) and \( e^{i\varphi_0} = \bar{m} |\bar{m}| \);

\textbf{II -} \( U \) can be decomposed as \( U = U^{(p)} U^{(S)} \), where \( U^{(S)} \) is the product of \( N \) single-mode squeezing transformations with squeezing strength equal to that of the transformation \( U_0 \),

\[
U^{(p)} \text{ and } U^{(S)}
\]
i.e. \( U^{(S)} = U_1 \cdots U_N \), with \( U_j = e^{i \frac{1}{\hbar} (\phi_j b_j^+ - \phi_j b_j)} \), and \( U^{(p)} \) is a passive quadratic transformation (note that both \( U^{(S)} \) and \( U^{(p)} \) don’t operate on the auxiliary module); 

**III** - the passive quadratic Hamiltonian \( H \) for the \( N + 1 \) modes of Eq. (1) is given by \( H = U^{(p)} H^{(S)} U^{(p)\dagger} \), where \( H^{(S)} \) is any passive quadratic Hamiltonian for which the following propositions are true:

- \( H^{(S)} \) remains passive under the effect of the set of single-mode squeezing transformations for the \( N + 1 \) modes \( U_0 U^{(S)} \), i.e. \( U^{(S)\dagger} U_0^\dagger H^{(S)} U_0 U^{(S)} \) is passive; 
- all the normal modes of \( H^{(S)} \) have a finite overlap with the auxiliary mode (see [16]).

**Proof.** Part 1: If the propositions III are true then Eq. (2) is the only steady state. In the representation defined by the transformation \( U_0 U \), the transformed density matrix \( \tilde{\rho} = U_0^\dagger U^{\dagger} \rho U_0 U \), fulfills the master equation \( \dot{\tilde{\rho}} = -i [\tilde{H}, \tilde{\rho}] + \overline{\tilde{\rho}} \) where the dissipative term, \( \overline{\tilde{\rho}} = \kappa \mathcal{D}_{b_i^b} \tilde{\rho} \), describes pure dissipation in a vacuum reservoir, and the transformed Hamiltonian \( \tilde{H} = U_0^\dagger U^{(p)} H U_0 U \), can be written as \( \tilde{H} = U^{(S)\dagger} U_0^\dagger H^{(S)} U_0 U^{(S)} \). This shows that \( \tilde{H} \) is passive because of proposition III.a. The proposition III.b instead, entails that \( H^{(S)} \), and therefore also \( H \) and \( \tilde{H} \), have no dark modes [16], i.e. all the normal mode are coupled to the reservoir. Thus, the only steady state in the new representation is the vacuum, which is equal to Eq. (3) in the original representation.

Part 2: If Eq. (2) is the only steady state, then the propositions III are true. In the representation defined by the density matrix \( \tilde{\rho} \), the transformed steady state, \( \rho_{\text{tot}} = U_0^\dagger U^{\dagger} \rho U_0 U \), is the vacuum. This can be true only if the transformed Hamiltonian \( \tilde{H} \) is passive with no dark modes, and the dissipative term \( \overline{\tilde{\rho}} = U_0^\dagger \mathcal{L} \left( U_0^\dagger \tilde{\rho} U_0 \right) U_0 \) describes pure dissipation in a vacuum reservoir. For this to be true \( U_0 \) has to fulfill the proposition II.

Now, in order to demonstrate the validity of the other propositions, we note that it is always possible to decompose a unitary transformation \( U \), which generates a zero-average pure Gaussian state, in a form similar to the one defined in the proposition II, where \( U^{(S)} \) is a set of single-mode squeezing transformations which can be, in general, of different strength, and \( U^{(p)} \) is a multi-mode passive transformation. This can be seen by using the Bloch-Messiah decomposition [16]. Thus, Eq. (2) can be always written in the form \( \rho_{\text{tot}} = U_0^\dagger U^{(p)} H^{(S)} U^{(p)\dagger} U_0 \). In the representation defined by the transformed density matrix \( \tilde{\rho}^{(S)} = U^{(p)\dagger} \rho U^{(p)} \), which fulfill the equation \( \dot{\tilde{\rho}}^{(S)} = -i \hbar [H^{(S)}, \tilde{\rho}^{(S)}] \) + \( \overline{\tilde{\rho}}^{(S)} \), the Hamiltonian \( H^{(S)} = U^{(p)\dagger} H U^{(p)} \) is passive (because \( U^{(p)} \) and \( H \) are passive), and remains passive under the effect of \( U_0 \) (in fact \( U^{(S)\dagger} U_0^\dagger H^{(S)} U_0 U^{(S)} = \tilde{H} \) which, as we have seen, has to be passive), and therefore the proposition II.a is true. Moreover, \( \tilde{H} \) has no dark modes (because we are assuming that the system has a single steady state), and thus the proposition II.b is true as well [16]. Finally, this also means that all the modes are connected (even if not directly) by the interactions terms of \( H^{(S)} \), and this together with the following lemma guarantees that the strength of all the squeezing transformations which constitute \( U^{(S)} \) are equal. In particular they have to be equal to the squeezing strength of the auxiliary mode \( \tilde{z}_0 \), which is fixed by the squeezing strengths of the reservoir, so also the proposition II is true.

Let us now introduce the following lemma which describes the precise structure of the Hamiltonian \( H^{(S)} \).

**Lemma.** Given a passive quadratic Hamiltonian, \( H^{(S)} = \hbar \sum_{j,k=0}^N J^{(S)}_{jk} b_j^\dagger b_k \), with \( J^{(S)}_{jk} = J^{(S)}_{kj} \) for \( j < k \), and \( z_j \) is the vacuum. This can be true only if the transformed Hamiltonian \( \tilde{H} = U_N^\dagger \cdots U_0^\dagger H^{(S)} U_0 \cdots U_N \), with \( U_j = e^{i \frac{1}{\hbar} (\varphi_j b_j^\dagger - \varphi_j b_j)} \), is passive, if and only if (i) \( J^{(S)}_{jk} = 0 \) for all \( j \) with \( z_j \neq 0 \), (ii) \( \varphi_j - \varphi_k + \pi / 2 \) for \( j < k \) (with \( n \in \mathbb{Z} \)), and \( z_j = z_k \) for all \( j \neq k \) with \( J^{(S)}_{jk} \neq 0 \). Moreover, if \( \tilde{H} \) is passive then \( \tilde{H} = H^{(S)} \). (The proof of this lemma is straightforward and is reported in [16].)

It is, now, important to point out that, for any given state \( |\Psi\rangle \) which fulfills the proposition II each quadratic Hamiltonian \( H^{(S)} \) which fulfills the propositions III.a and III.b (and the lemma) can be used to construct a (different) Hamiltonian \( H \) (see the proposition II) of model (I) which sustain the given state in the stationary regime. Thus the same steady state can be obtained with many different Hamiltonians. The specific form of \( H \) can determine how fast (and therefore how efficiently, when additional noise sources affect the system dynamics) the system approaches the steady state. We also note that both \( H^{(S)} \) and \( H \) satisfy the chiral symmetry identified in Ref. [10] (see [16]). This implies that the chiral symmetry of \( H \), is also a necessary condition (not only a sufficient one, as suggested in Ref. [10]) for the existence of the pure steady state (2) of Eq. (1).

A particularly simple Hamiltonian \( H^{(S)} \) that fulfills the propositions III and the lemma is the Hamiltonian for a linear chain with open boundary conditions (for which the normal modes have always a finite overlap with the end modes)

\[
H^{(S)} = i \hbar \sum_{j=0}^N J^{(S)}_{j,j-1} \left( e^{i \theta_j} b_j^+ b_{j-1} - e^{-i \theta_j} b_{j-1}^\dagger b_j \right),
\]

where \( \theta_j = (\varphi_j - \varphi_{j-1}) / 2 \), with \( \varphi_j \), the squeezing phases introduced in the proposition II. This means that Eq. (3) can be used to construct the Hamiltonian \( H \) corresponding to any state that fulfills the proposition II. Specific examples of multi-mode entangled states that can be prepared with this strategy have been discussed in Ref. [17][12].

It is interesting to note that the class of states that can be prepared with our approach is wide and it includes also cluster states which are the main resource of measurement-based quantum computation [15]. In particular all the cluster states that have been proposed and prepared by manipulating one or two squeezed light beams with a complex interferometer [13].
can be also generated following our approach. The difference between these results and the present approach is that, while in these works the state is prepared in traveling wave beams of light, our results show how to generate similar states, in a robust way, as stationary states of a dissipative dynamics. This approach is, hence, attractive in situations in which the quantum modes are localized, as for example in a solid-state or atomic device.

Dissipative generation of a cluster state. Let us now investigate the potentiality of our result to design a model which sustain in the stationary regime a cluster state in a square lattice which constitute a universal resource for quantum computation. To be specific, we consider a cluster state of $N = 25$ modes with a $N \times N$ real symmetric adjacency matrix $A$ (with non-zero entries equal to one) which represents the square lattice. This state can be generated by the multi-mode squeezing transformation \[ U_z = e^{-i \frac{1}{2} \sum_{j<k} (Z_{jk} b_j^* b_k + Z_{kj} b_k^* b_j)}, \]
where the $N \times N$ matrix of interaction coefficients is given by $Z = -i (A - i I) (A + i I)^{-1}$. What characterizes this as cluster state is the fact that the covariance matrix of the $N$ operators $x_j = p_j - \sum_{k=1}^N A_{jk} q_k$ [with $q_j = b_j + b_j^*$ and $p_j = -i (b_j - b_j^*)$], called nullifiers, approaches the null matrix in the limit of infinite squeezing, $z \to \infty$. The transformation $U_z$ can be decomposed, similarly to the definition in the proposition 4 of the theorem, as $U_z = U_z^{(p)} U_z^{(S)}$, with $U_z^{(S)}$ given by the product of $N$ single-mode squeezing transformations (where $\varphi_j = 0$ for all $j$), and with $U_z^{(p)}$ which fulfills the relation $U_z^{(S)} b_j U_z^{(p)} = \sum_{k=1}^N (-i Z)^{1/2}_{jk} b_k$ [16]. The fact that $U_z^{(S)}$ describes the equal squeezing of all the modes implies, according to our theorem, that $U_z |0\rangle$ is the steady state of Eq. (1) when

\[ H = U_z^{(p)} H^{(S)} U_z^{(p)^*}, \]

where $H^{(S)}$ is the Hamiltonian for the linear chain. In Fig. 1 and 2 we show the results for the preparation of this cluster state. We have studied how the present approach performs in non-ideal situations which include additional noise sources, with dissipation rate $\gamma$, and random deviations from the optimal system Hamiltonian defined in Eq. (4).

In particular, in Fig. 1 and 2 we characterize the steady state $\rho_{st}$ of

\[ \dot{\rho}' = -i [H, \rho'] + \mathcal{L} \rho' + \sum_{j=0}^N \mathcal{D}_{\varphi_j} \rho', \]

in terms of its fidelity with respect to the steady state $\rho_{st}$ achievable with $\gamma = 0$ [black solid line, panel (a)], and in terms of the variance of the nullifiers over $\rho_{st}$, relative to the variance over the vacuum [dark gray lines, panels (b)]. We observe that significant reduction of the variance (squeezing) of the nullifiers (which indicates that the state is close to the cluster state) is observed when $\gamma (N + 1) \ll \kappa$, namely when the total added dissipation is much weaker than the dissipation in the squeezed reservoir. The thin lines in panel (a) describe how the model is sensitive to deviation form the ideal Hamiltonian.
system is significantly more stable with respect to the latter. In any case, even when the fidelity is very low, the nullifiers always exhibit significant squeezing [panel (c)].

We note that the overlaps between normal modes and auxiliary mode [see panel (e)] determine the rates at which each normal mode is coupled to the squeezed reservoir, so, in the ideal case, they determine how fast each normal mode approach the steady state. The optimal situation is the one in which all the overlaps are equal and are as large as possible so that all the normal modes are optimally coupled to the reservoir. This is described by Fig. 2 which shows that in this case the system is significantly more resistant to deviation from the ideal configuration. We also note that the overlaps are the same for both $H^{(S)}$ and $H$ (because $U^{(p)}$ does not operate on the auxiliary mode [16]). And this means that the time to reach the steady state is entirely determined by the dynamics of the linear chain [Eq. (3)].

In conclusion, we have shown that, by squeezing the local environment of a single site of an harmonic lattice, it is possible to steer the whole system toward any pure Gaussian state that can be generated by a passive multi-mode transformation which operates on a batch of many equally squeezed modes. In particular, given one of these states, we have shown how to determine a passive quadratic Hamiltonian which sustain it in the stationary regime (and which necessarily fulfills the chiral symmetry identified in Ref. 10). This Hamiltonian is not unique, and we have shown, by studying the generation of a cluster state in a square lattice, that the efficiency for the preparation of the chosen state, in non-ideal situations, depends critically on the specific ideal Hamiltonian that one consider. Understanding which Hamiltonian corresponds to a model which is more resistant to imperfections, is a question which deserve further investigation. Another interesting related question regards the possibility to extend this approach to spin systems [6].

We finally note, that this approach can be particularly valuable for implementations of quantum information devices, with circuit QED systems which have been used to realize various lattice models [36,37]. The squeezed reservoir, instead, can be realized using a squeezed field with sufficiently large bandwidth [8,13], or can be engineered with bichromatic drives [7].

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[1] B. Kraus, H. P. Büchler, S. Diehl, A. Kantian, A. Micheli, and P. Zoller, “Preparation of entangled states by quantum Markov processes,” Phys. Rev. A 78, 042307 (2008).
[2] S. Diehl, A. Micheli, A. Kantian, B. Kraus, H. P. Büchler, and P. Zoller, “Quantum States and Phases in Driven Open Quantum Systems with Cold Atoms,” Nat. Phys. 4, 878–883 (2008).
[3] Frank Verstraete, Michael M. Wolf, and Ignacio Cirac, “Quantum computation and quantum-state engineering driven by dissipation,” Nat. Phys. 5, 633–636 (2009).
[4] G. Barontini, R. Labouvie, F. Stubenrauch, A. Vogler, V. Guarrera, and H. Ott, “Controlling the Dynamics of an Open Many-Body Quantum System with Localized Dissipation,” Phys. Rev. Lett. 110, 033002 (2013).
[5] F. Tonelli, R. Fazio, S. Diehl, and J. Marino, “Orthogonality catastrophe in dissipative quantum many body systems,” Phys. Rev. Lett. 122, 040604 (2019).
[6] S. Zippilli, M. Paternostro, G. Adesso, and F. Illuminati, “Entanglement Replication in Driven Dissipative Many-Body systems,” Phys. Rev. Lett. 110, 040503 (2013).
[7] Stefano Zippilli, Jie Li, and David Vitali, “Steady-state nested entanglement structures in harmonic chains with single-site squeezing manipulation,” Phys. Rev. A 92, 032319 (2015).
[8] Muhammad Asjad, Stefano Zippilli, and David Vitali, “Mechanical Einstein-Podolsky-Rosen entanglement with a finite-bandwidth squeezed reservoir,” Phys. Rev. A 93, 062307 (2016).
[9] Shan Ma, Matthew J. Woolley, Ian R. Petersen, and Naoki Yamamoto, “Pure Gaussian states from quantum harmonic oscillator chains with a single local dissipative process,” J. Phys. A: Math. Theor. 50, 135301 (2017).
[10] Yariv Yanay and Aashish A. Clerk, “Reservoir engineering of bosonic lattices using chiral symmetry and localized dissipation,” Phys. Rev. A 98, 043615 (2018).
[11] Yariv Yanay and Aashish A. Clerk, “Reservoir engineering with localized dissipation: Dynamics and prethermalization,” Phys. Rev. Research 2, 023177 (2020).
[12] Yariv Yanay, “An algorithm for tailoring a quadratic lattice with a local squeezed reservoir to stabilize generic chiral states with non-local entanglement,” arXiv:2002.05224 (2020).

[13] S. Zippilli and F. Illuminati, “Non-Markovian dynamics and steady-state entanglement of cavity arrays in finite-bandwidth squeezed reservoirs,” Phys. Rev. A 89, 033803 (2014).

[14] Shan Ma and Matthew J. Woolley, “Entangled pure steady states in harmonic chains with a two-mode squeezed reservoir,” J. Phys. A: Math. Theor. 52, 325301 (2019).

[15] Mile Gu, Christian Weedbrook, Nicolas C. Menicucci, Timothy C. Ralph, and Peter van Loock, “Quantum computing with continuous-variable clusters,” Phys. Rev. A 79, 062318 (2009).

[16] See Supplemental Material for additional details on the Gaussian steady states which can be prepared with the present proposal and on the Hamiltonians which enable the preparation of these states.

[17] Gaetana Spedalieri, Christian Weedbrook, and Stefano Pirandola, “A limit formula for the quantum fidelity,” J. Phys. A: Math. Theor. 46, 025304 (2012).

[18] Nicolas C. Menicucci, Steven T. Flammia, Hussain Zaidi, and Olivier Pfister, “Ultracompact generation of continuous-variable cluster states,” Phys. Rev. A 76, 010302(R) (2007).

[19] Nicolas C. Menicucci, Steven T. Flammia, and Olivier Pfister, “One-Way Quantum Computing in the Optical Frequency Comb,” Phys. Rev. Lett. 101, 130501 (2008).

[20] Steven T. Flammia, Nicolas C. Menicucci, and Olivier Pfister, “The optical frequency comb as a one-way quantum computer,” J. Phys. B: At. Mol. Opt. Phys. 42, 114009 (2009).

[21] Nicolas C. Menicucci, Xian Ma, and Timothy C. Ralph, “Arbitrarily Large Continuous-Variable Cluster States from a Single Quantum Nondemolition Gate,” Phys. Rev. Lett. 104, 250503 (2010).

[22] Nicolas C. Menicucci, “Temporal-mode continuous-variable cluster states using linear optics,” Phys. Rev. A 83, 062314 (2011).

[23] Moran Chen, Nicolas C. Menicucci, and Olivier Pfister, “Experimental Realization of Multipartite Entanglement of 60 Modes of a Quantum Optical Frequency Comb,” Phys. Rev. Lett. 112, 120505 (2014).

[24] Shota Yokoyama, Ryuji Ukiy, Seiji C. Armstrong, Chanond Sornphiphathpong, Toshiyuki Kaji, Shigenari Suzuki, Jun-ichi Yoshikawa, Hidehiro Yonezawa, Nicolas C. Menicucci, and Akira Furusawa, “Ultra-large-scale continuous-variable cluster states multiplexed in the time domain,” Nat Photon 7, 982–986 (2013).

[25] Rafael N. Alexander, Pei Wang, Niranjani Sridhar, Moran Chen, Olivier Pfister, and Nicolas C. Menicucci, “One-way quantum computing with arbitrarily large time-frequency-continuous-variable cluster states from a single optical parametric oscillator,” Phys. Rev. A 94, 032327 (2016).

[26] Y. Cai, J. Roslund, G. Ferrini, F. Arzani, X. Xu, C. Fabre, and N. Treps, “Multimode entanglement in reconfigurable graph states using optical frequency combs,” Nat. Commun. 8, 15645 (2017).

[27] Rafael N. Alexander, Shota Yokoyama, Akira Furusawa, and Nicolas C. Menicucci, “Universal quantum computation with temporal-mode bilayer square lattices,” Phys. Rev. A 97, 032302 (2018).

[28] Daiqin Su, Krishna Kumar Sabapathy, Casey R. Myers, Haoyu Qi, Christian Weedbrook, and Kamil Bradler, “Implementing quantum algorithms on temporal photonic cluster states,” Phys. Rev. A 98, 032316 (2018).

[29] Mikkel V. Larsen, Xueyi Guo, Casper R. Breum, Jonas S. Neergaard-Nielsen, and Ulrik L. Andersen, “Deterministic generation of a two-dimensional cluster state,” Science 366, 369–372 (2019).

[30] Bo-Han Wu, Rafael N. Alexander, Shuai Liu, and Zheshen Zhang, “Quantum computing with multidimensional continuous-variable cluster states in a scalable photonic platform,” Phys. Rev. Research 2, 023138 (2020).

[31] Warit Asavanant, Yu Shiozawa, Shota Yokoyama, Baramee Charoensombutamon, Hiroki Emura, Rafael N. Alexander, Shuntaro Takeda, Jun-ichi Yoshikawa, Nicolas C. Menicucci, Hidehiro Yonezawa, and Akira Furusawa, “Generation of time-domain-multiplexed two-dimensional cluster state,” Science 366, 373–376 (2019).

[32] Warit Asavanant, Baramee Charoensombutamon, Shota Yokoyama, Takeru Ebihara, Tomohiro Nakamura, Rafael N. Alexander, Mamoru Endo, Jun-ichi Yoshikawa, Nicolas C. Menicucci, Hidehiro Yonezawa, and Akira Furusawa, “One-hundred step measurement-based quantum computation multiplexed in the time domain with 25 MHz clock frequency,” arXiv:2006.11537 (2020).

[33] Tomoki Ozawa, Hannah M. Price, Alberto Amo, Nathan Goldman, Mohammad Hafezi, Ling Lu, Mikael C. Rechtsman, David Schuster, Jonathan Simon, Oded Zilberberg, and Iacopo Carusotto, “Topological photonics,” Reviews of Modern Physics 91, 015006 (2019).

[34] Michał Tomza, Krzysztof Jachymski, Ren GERRITSSMA, Antonio Negretti, Tommaso Calarco, Zbigniew Idziaszek, and Paul S. Julienne, “Cold hybrid ion-atom systems,” Reviews of Modern Physics 91, 035001 (2019).

[35] Stefano Zippilli and David Vitali, “Any Gaussian cluster state can be generated by a multi-mode squeezing transformation,” arXiv: 2007.12772 (2020).

[36] Matthias Fitzpatrick, Neereja M. Sundaresan, Andy C. Y. Li, Jens Koch, and Andrew A. Houck, “Observation of a Dissipative Phase Transition in a One-Dimensional Circuit QED Lattice,” Phys. Rev. X 7, 011016 (2017).

[37] Ruichao Ma, Brendan Saxberg, Clai Owens, Nelson Leung, Yao Lu, Jonathan Simon, and David I. Schuster, “A dissipatively stabilized Mott insulator of photons,” Nature 566, 51–57 (2019).
Supplemental material for “Dissipative engineering of Gaussian entangled states in harmonic lattices with a single-site squeezed reservoir”

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UNIQUENESS OF THE STEADY STATE

Here we show that the proposition $\{III.b\}$ of the main text guarantees that the system has no dark modes. We consider the linear system of equations for the average annihilation operators $\langle \hat{b}_j \rangle = -i (\| \hat{b}_j, H_{\text{SS}} \| ) - \kappa \langle \hat{b}_j \rangle$ which can be written in matrix form as $\langle \hat{b}_j \rangle = - \sum_{k=0}^{\infty} R_{jk} \langle \hat{b}_k \rangle$, with $R = i 2 \mathcal{J}(S) + \Gamma$, where $\mathcal{J}(S)$ is the hermitian matrix of coefficients corresponding to the Hamiltonian part and $\Gamma$ is the matrix for the dissipative part which has a single non-zero entry

$$\Gamma = \begin{pmatrix} \kappa & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \ddots & \ddots \end{pmatrix}.$$  \hfill (S.1)

First we note that when we say that our model has no dark modes, we mean that the matrix $R$ is positive stable $\{I\}$, i.e. all the eigenvalues of $R$ have positive real part, so that all the normal modes actually decay. According to the theorem 2.4.7 of Ref. $\{I\}$, given a positive defined matrix $Q$, such that $QR + R^T Q$ is semi-positive defined, then $R$ is positive stable if and only if no eigenvector of $Q^{-1} (QR - R^T Q)$ lies in the null space of $QR + R^T Q$. In our case, we can simply choose $Q = I$ (the identity matrix), such that $QR + R^T Q = 2 \Gamma$, which is semi-positive defined, and $Q^{-1} (QR - R^T Q) = 2 i \mathcal{J}(S)$. The subspace orthogonal to the null space of $\Gamma$ is given by the single vector $v_0 = (1, 0, \cdots 0)^T$, corresponding to the auxiliary mode. Hence, when all the eigenvectors $w_j$ of $\mathcal{J}(S)$ are not orthogonal to $v_0$, i.e. the scalar product $w_j \cdot v_0 \neq 0$ for all $j$ [which is equivalent to the proposition $\{III.b\}$ of the main text], then the conditions of the theorem are fulfilled, so that $R$ is positive stable.

GAUSSIAN STATES AND THE BLOCH-MESSIAH DECOMPOSITION

In this work we consider $N$-modes, pure Gaussian states, which have zero average (no displacement). These states are given by

$$|\Psi\rangle = U |0\rangle$$ \hfill (S.2)

where $|0\rangle$ is the vacuum and $U$ is a unitary transformation which can be expressed in terms of the vector of bosonic operator $\mathbf{b} = (b_1 \cdots b_N, b_1^\dagger \cdots b_N^\dagger)^T$ and a $2N \times 2N$ complex symmetric matrix $S = S^T$ as

$$U = e^{-\frac{1}{2} \mathbf{b}^T S \mathbf{b}}.$$ \hfill (S.3)

The term $\mathbf{b}^T S \mathbf{b}$ is Hermitian. This entails that the matrix $S$ fulfills the relation $S = GS^* G$, with $G = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$ where the missing blocks are null matrices, and $S^*$ is the matrix whose entries are the complex conjugates of the entries of $S$. This means that $S$ has the block structure $S = \begin{pmatrix} \mathcal{Z} & \mathcal{K} \\ \mathcal{K}^* & \mathcal{Z} \end{pmatrix}$, where $\mathcal{Z}$ and $\mathcal{K}$ are $N \times N$ complex matrices which, since $S$ is symmetric, fulfill the relations $\mathcal{Z} = \mathcal{Z}^T$ and $\mathcal{K} = \mathcal{K}^T$.

The mode operators are transformed by the unitary $U$ according to a Bogoliubov matrix $B$ such as

$$U^\dagger \mathbf{b} U = B \mathbf{b}.$$ \hfill (S.4)

The matrix $B$ can be expressed in terms of the matrix $S$ as

$$B = e^{-i IS}$$ \hfill (S.5)

where

$$I = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$  

This can be shown using the Baker-Hausdorff formula

$$e^{A} B e^{-A} = \sum_{n=0}^{\infty} \frac{1}{n!} [A, B]^{(n)}$$  

with $[A, B]^{(0)} = B$ and $[A, B]^{(n)} = [A, [A, B]^{(n-1)}]$, such that

$$\{B \mathbf{b}\}_j = U^\dagger \mathbf{b}_j U = \mathbf{b}_j + \sum_{n=1}^{\infty} \frac{i^n}{2^n n!} \sum_{k,l} S_{k,l} \mathbf{b}_{k'} \mathbf{b}_{l'} \mathbf{b}_j^{(n)}$$ \hfill (S.6)

where $[\cdots, \cdots]^{(n)}$ indicates the $n$-fold commutator such that $[A, B]^{(1)} = [A, B]$, $[A, B]^{(2)} = [A, [A, B]]$, $[A, B]^{(3)} = [A, [A, [A, B]]]$ and so on. It is easy to show by induction, and using the bosonic commutation relations $\{\mathbf{b}_j, \mathbf{b}_k\} = i \delta_{jk}$, that

$$\sum_{k,l} S_{k,l} \mathbf{b}_k \mathbf{b}_l \mathbf{b}_j^{(n)} = (-2)^n [(IS)^n \mathbf{b}]_j$$ \hfill (S.7)

so that

$$\{B \mathbf{b}\}_j = \left\{ \mathbf{1} + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} (IS)^n \right\} \mathbf{b}_j.$$ \hfill (S.8)
The Bogoliubov matrix fulfills the relation \( \mathcal{B} = \mathcal{G} \mathcal{B}^\ast \mathcal{G} \), and it can be expressed in block form as

\[
\mathcal{B} = \begin{pmatrix}
\mathcal{X} & \mathcal{Y} \\
\mathcal{Y}^\ast & \mathcal{X}^\ast
\end{pmatrix}
\]  
(S.9)

for some complex \( N \times N \) matrices \( \mathcal{X} \) and \( \mathcal{Y} \). Moreover, due to the standard bosonic commutation relation for the transformed operators, \( \mathcal{B} \) fulfill also the relation \( \mathcal{B} I \mathcal{B}^T = I \) which can be expressed in terms of the matrices \( \mathcal{X} \) and \( \mathcal{Y} \) as

\[
\mathcal{X} \mathcal{X}^\ast - \mathcal{Y} \mathcal{Y}^\ast = I
\]  
(S.10)

and

\[
\mathcal{X} \mathcal{Y}^T = \mathcal{Y} \mathcal{X}^T.
\]  
(S.11)

The Block-Messiah [2-6] reduction formula allows to decompose \( \mathcal{B} \) as the product of three Bogoliubov transformations

\[
\mathcal{B} = \begin{pmatrix}
\mathcal{V} & 0 \\
0 & \mathcal{V}^\ast
\end{pmatrix} \begin{pmatrix}
\mathcal{D}_s & \mathcal{D}_c \\
\mathcal{D}_c^\ast & \mathcal{D}_s^\ast
\end{pmatrix} \begin{pmatrix}
\mathcal{W}^\ast \\
\mathcal{W}
\end{pmatrix}
\]  
(S.12)

where \( \mathcal{D}_s \) and \( \mathcal{D}_c \) are semi-positive defined diagonal matrices and \( \mathcal{V} \) and \( \mathcal{W} \) are unitary matrices. They correspond to the singular value decomposition of the matrices \( \mathcal{X} \) and \( \mathcal{Y} \), such that

\[
\mathcal{X} = \mathcal{V} \mathcal{D}_s \mathcal{W}^{\ast T},
\]
\[
\mathcal{Y} = \mathcal{V} \mathcal{D}_c^\ast \mathcal{W}^T,
\]  
(S.13)

and the diagonal elements of \( \mathcal{D}_s \) and \( \mathcal{D}_c \) are the singular values of \( \mathcal{X} \) and \( \mathcal{Y} \) respectively. The first and third matrices in Eq. (S.12) describe passive multi-mode transformations, while the second one describes the single-mode-squeezing of all the modes. It is possible to include generic squeezing phases to the second transformation by defining these three matrices

\[
\mathcal{V} = \mathcal{V} e^{-i\Phi},
\]
\[
\mathcal{W} = \mathcal{W} e^{-i\Phi},
\]
\[
\mathcal{D}_s = \mathcal{D}_s e^{i\Phi},
\]
\[
\mathcal{D}_c = \mathcal{D}_c^\ast e^{i\Phi}
\]  
(S.14)

where \( \Phi \) is a real diagonal matrix, and where now \( \mathcal{D}_c \) have complex entries, such that we can write this decomposition

\[
\mathcal{B} = \mathcal{B}_s \mathcal{B}_D \mathcal{B}_W
\]  
(S.15)

with

\[
\mathcal{B}_s = \begin{pmatrix}
\mathcal{V} & 0 \\
0 & \mathcal{V}^\ast
\end{pmatrix},
\]
\[
\mathcal{B}_D = \begin{pmatrix}
\mathcal{D}_s & \mathcal{D}_c \\
\mathcal{D}_c^\ast & \mathcal{D}_s^\ast
\end{pmatrix},
\]
\[
\mathcal{B}_W = \begin{pmatrix}
\mathcal{W}^\ast \\
\mathcal{W}
\end{pmatrix}.
\]  
(S.16)

These three matrices correspond to three unitary transformations

\[
U_s = e^{-i\mathcal{b}^\ast \mathcal{S}_V \mathcal{b}},
\]
\[
U_D = e^{-i\mathcal{b}^\ast \mathcal{S}_D \mathcal{b}},
\]
\[
U_W = e^{-i\mathcal{b}^\ast \mathcal{S}_W \mathcal{b}},
\]  
(S.17)

(for some matrices \( \mathcal{S}_V, \mathcal{S}_D \) and \( \mathcal{S}_W \) which are specified below) which can be used to decompose the \( U \) as

\[
U = U_s U_D U_W,
\]  
(S.18)

such that

\[
U^\dagger \mathcal{b} U = \mathcal{B} \mathcal{b} = \mathcal{B}_s \mathcal{B}_D \mathcal{B}_W \mathcal{b} = \mathcal{B}_s \mathcal{B}_D U_W^\dagger \mathcal{b} U_W = \mathcal{B}_s \mathcal{B}_D \mathcal{b} U_W = U_W^\dagger \mathcal{U}_W \mathcal{b} U_V U_D U_W.
\]

By means of these operators we find that a general \( N \)-modes, zero-average, pure Gaussian states can be expressed as

\[
|\Psi\rangle = U_V U_D |0\rangle,
\]  
(S.19)

where the vacuum is not changed by the passive transformation \( U_W \). This corresponds to the decomposition introduced in the main text with

\[
U^{(S)} = U_D
\]
\[
U^{(p)} = U_V.
\]  
(S.20)

Since \( \mathcal{V} \) is a unitary matrix, it can be expressed as

\[
\mathcal{V} = e^{i\mathcal{K}_V},
\]  
(S.21)

for a \( N \times N \) hermitian matrices \( \mathcal{K}_V \) so that

\[
\mathcal{S}_V = \begin{pmatrix}
\mathcal{K}_V & 0 \\
0 & \mathcal{K}_V^T
\end{pmatrix}
\]  
(S.22)

(similar considerations hold also for \( \mathcal{S}_W \)). Moreover the matrices \( \mathcal{D}_s \) and \( \mathcal{D}_c \) are diagonal and have to fulfill a condition analogous to Eq. (S.10). This implies that they can be rewritten in terms of a diagonal matrix \( \mathcal{D} \) as

\[
\mathcal{D}_s = \cosh (\mathcal{D})_s,
\]
\[
\mathcal{D}_c = \sinh (\mathcal{D})_c e^{i\Phi},
\]  
(S.23)

and, in turn, the corresponding unitary transformation \( U_D \) [see Eq. (S.17)], which describes a batch of single-mode squeezing transformations, is expressed in terms of the matrix

\[
\mathcal{S}_D = \begin{pmatrix}
-i \mathcal{D}_c e^{-i\Phi} & i \mathcal{D}_c e^{i\Phi} \\
i \mathcal{D}_c e^{i\Phi} & i \mathcal{D}_c e^{-i\Phi}
\end{pmatrix},
\]  
(S.24)

where the non-zero entries of the the diagonal matrix \( \Phi = \varphi_j \), with \( \varphi_j \) the squeezing phases introduced in the main text.

In the main text we have shown that with our approach it is possible to generate any state of the form (S.19) where \( U_D = U^{(S)} \) describes the equal squeezing for all the modes, namely states for which \( \mathcal{D}_c = z \mathbb{I} \) for some real non-negative \( z \), so that

\[
\mathcal{S}_D = \begin{pmatrix}
-i z e^{-i\Phi} & i z e^{i\Phi} \\
i z e^{i\Phi} & i z e^{-i\Phi}
\end{pmatrix},
\]  
(S.25)

In other terms we can prepare states for which the singular values of the blocks that constitute the corresponding Bogoliubov transformation, \( \mathcal{X} \) and \( \mathcal{Y} \), are all equal, i.e. \( \mathcal{D}_s = \cosh(z) \mathbb{I} \) and \( \mathcal{D}_c = \sinh(z) \mathbb{I} \). Since the singular values of a generic matrix \( \mathcal{M} \) are the square roots of the eigenvalues of \( \mathcal{M}^T \mathcal{M} \), this means that the matrices \( \mathcal{X} \) and \( \mathcal{Y} \) are proportional to unitary matrices, i.e. \( \mathcal{X} X^T = \cosh^2(z) \mathbb{I} \) and \( \mathcal{Y} Y^T = \sinh^2(z) \mathbb{I} \).
PROOF OF THE MAIN TEXT

It is straightforward to prove the lemma by noting that, on the one hand, on site energy terms \( b_j^\dagger b_j \) result in non-passive single mode squeezing terms under the transformation \( U_j \), and that, on the other hand, interaction terms \( h_{jk} = \hbar \left( J^{(S)}_{jk} b_j^\dagger + J^{(S)}_{jk} b_k \right) \), with \( j \neq k \), are invariant under the effect of the transformation \( U_j U_k \), namely \( U_k^\dagger U_j^\dagger h_{jk} U_j U_k = h_{jk} \), if and only if the proposition (ii) is true; different squeezing strengths or phases result instead in non-passive two-mode squeezing terms in the transformed Hamiltonian.

To be specific, Given the squeezing operator \( U_j = e^{\frac{i\hbar}{\lambda}(\varphi_j b_j^\dagger - \varphi_i b_j)} \) we find \( U_j^\dagger h_j U_j = c_j b_j + s_j e^{i\varphi_j} b_j^\dagger \), with \( c_j = \cosh(\varphi_j) \) and \( s_j = \sinh(\varphi_j) \). Thus, given the Hamiltonian \( H^{(S)} \), which we rewrite as \( H^{(S)} = \hbar \sum_{j=0}^N J^{(S)}_{jk} b_j^\dagger b_k + \hbar \sum_{j<k} \left( J^{(S)}_{jk} b_j^\dagger b_k + J^{(S)*}_{jk} b_k^\dagger b_j \right) \) we find

\[
\tilde{H} = U_N^\dagger \cdots U_0^\dagger H^{(S)} U_0 \cdots U_N
\]

\[= \hbar \sum_{j=0}^N J^{(S)}_{jk} \left[ c_j^2 b_j^\dagger b_j + s_j^2 b_j^\dagger b_j + c_j s_j \left( e^{i\varphi_j} b_j^\dagger b_j^\dagger + e^{-i\varphi_j} b_j^\dagger b_j \right) \right] + \hbar \sum_{j<k} \left[ J^{(S)}_{jk} c_j c_k + J^{(S)*}_{jk} s_j s_k e^{i(\varphi_j - \varphi_k)} \right] b_j^\dagger b_k
\]

\[+ \left[ J^{(S)}_{jk} c_j s_k e^{i\varphi_j} + J^{(S)*}_{jk} s_j c_k e^{i\varphi_j} \right] b_j^\dagger b_k^\dagger + h.c. \]. \hspace{1cm} (S.26)

This Hamiltonian is passive if and only if

\[ J^{(S)}_{jk} c_j s_j = 0 \] \hspace{1cm} (S.27)

\[ J^{(S)}_{jk} c_j s_k e^{i\varphi_j} + J^{(S)*}_{jk} s_j c_k e^{i\varphi_j} = 0 \] \hspace{1cm} (S.28)

for all \( j < k \). Finally, we note that, Eq. (S.27) is equivalent to the proposition (i) of the lemma, and Eq. (S.28) is equivalent to \( \frac{d}{dt} \eta = -\frac{d}{dt} e^{i(\varphi_j - \varphi_k)} \), which is equivalent to the proposition (ii) of the lemma. In particular, this case

\[ \tilde{H} = \hbar \sum_{j<k} \left[ J^{(S)}_{jk} c_j^2 + J^{(S)*}_{jk} s_j^2 e^{i(\varphi_j - \varphi_k)} \right] b_j^\dagger b_k + h.c. \]

\[= H^{(S)} \]. \hspace{1cm} (S.29)

RELATION BETWEEN THE CHIRAL SYMMETRY OF REF. [7] AND THE PRESENT RESULT

The chiral symmetry of Ref. [7] and the Hamiltonians \( H^{(S)} \) and \( H \)

Here we show that the Hamiltonians \( H^{(S)} \) and \( H \) of the main text satisfy the chiral symmetry discussed in Ref. [7].

 According to the lemma of the main text, the Hamiltonian \( H^{(S)} \) can be expressed as

\[ H^{(S)} = \hbar \sum_{j<k=0}^N J^{(S)}_{jk} b_j^\dagger b_k \] \hspace{1cm} (S.30)

where \( J^{(S)} \) is a \((N+1) \times (N+1)\) Hermitian matrix with entries \( J^{(S)}_{jk} = 0 \) and \( J^{(S)}_{jk} = i \left[ J^{(S)}_{jk} e^{i(\varphi_j - \varphi_k)/2} \right] \), for \( j < k \). It can be decomposed as \( J^{(S)} = e^{i\Phi} \overline{J^{(S)}} e^{-i\Phi} \), where \( \Phi \) is the diagonal matrix with entries \( \Phi_{jj} = \varphi_j/2 \), and \( \overline{J^{(S)}} \) is an Hermitian matrix with imaginary entries. The matrices \( J^{(S)} \) and \( \overline{J^{(S)}} \) have the same eigenvalues \( \lambda_j \) and the eigenvectors \( w_j \) of \( J^{(S)} \) are related to the eigenvectors \( \overline{w}_j \) of \( \overline{J^{(S)}} \) by the relation \( w_j = e^{i\Phi} \overline{w}_j \). Given an eigenvalue \( \lambda_j \) and the corresponding eigenvector \( \overline{w}_j \), if we take the complex conjugate of \( \overline{J^{(S)}} \overline{w}_j = \lambda_j \overline{w}_j \), we find that \( (\overline{J^{(S)}} \overline{w}_j) \) is an eigenvalue \( -\lambda_j \) the eigenvector corresponding to the eigenvector \( \overline{w}_j \). And finally, this means that \( H^{(S)} \) fulfills the chiral symmetry of Ref. [7]. Namely, the normal modes of \( H^{(S)} \) [i.e. the eigenvectors of \( J^{(S)} \) and \( \overline{J^{(S)}} \)] come in pairs, with opposite frequencies, such that, by proper reordering of the normal modes, \( \lambda_j = -\lambda_{j+1} \) (for and odd number of modes there is also a zero-frequency mode, \( \lambda_0 = 0 \); and, moreover, the overlap between the auxiliary mode, described by the vector \( v_0 = (1,0,\cdots,0)^T \), and the normal mode \( w_j \) is equal in modulus to the overlap between \( v_0 \) and the normal mode with opposite frequency \( w_{j+1} \) (which, as we have seen, is given by \( w_{j+1} = w'_j \), such that \( v_0 \cdot w_j = (v_0 \cdot w_{j+1})^\ast \).

Correspondingly, the passive Hamiltonian \( H \) of the model (1) of the main text can be expressed in terms of a \((N+1) \times (N+1)\) Hermitian matrix \( J \), as

\[ H = \hbar \sum_{j<k=0}^N J_{jk} b_j^\dagger b_k \]. \hspace{1cm} (S.31)

It is related to \( H^{(S)} \) by the unitary passive transformation \( U^{(p)} \) [see the proposition[7] of the theorem of the main text], which does not act on the auxiliary mode, and which can be expressed in terms of a \( N \times N \) Hermitian matrix \( K^{(p)} \) as \( U^{(p)} = e^{-i\Sigma_{k=1}^N K^{(p)}_k b_k^\dagger} \). Therefore the matrices \( J^{(S)} \) [see Eq. (S.30)] and \( J \) [see Eq. (S.31)] are related by a \((N+1)\times(N+1)\) unitary matrix \( U \), according to

\[ J = U^\dagger J^{(S)} U \], \hspace{1cm} (S.32)

where \( U \) can be constructed in terms of the \( N \times N \) matrix \( K^{(p)} \) which enters into the definition of \( U^{(p)} \) as

\[
U = \begin{pmatrix}
1 & \cdots \\
\vdots & \ddots \\
0 & \cdots & e^{-iK^{(p)}_N}
\end{pmatrix},
\] \hspace{1cm} (S.33)

where the missing entries are all zeros. This means that, on the one hand, the spectrum of \( J \) is equal to the spectrum of \( J^{(S)} \), and that, on the other, given an eigenvector \( w_j \) of \( J^{(S)} \), the corresponding eigenvector of \( J \) is \( U^\dagger w_j \). In particular we
find that the overlap with the auxiliary mode is equal for the eigenvectors of $\mathcal{J}$ and for the corresponding eigenvectors of $\mathcal{J}$, i.e. $\Psi_0 \cdot U \mathbf{w}_j = \mathbf{v}_j \cdot U \mathbf{w}_j$. And, in turn, this entails that also $H$, fulfills the chiral symmetry of Ref. [7].

The chiral symmetry of Ref. [7] and the transformation which generates the steady state

Here we show that the passive transformation $U^{(p)}$ (defined in the theorem of the main text) which is part of the transformation which generates the steady state, is related to the passive unitary transformation $U^{(T)}$ which diagonalize the Hamiltonian $H$ [such that $U^{(T)} \dot{H} U^{(T)^\dagger} = \hbar \sum_{j=0}^N \lambda_j b_j^\dagger b_j$], according to the relation

$$U^{(p)} = U^{(T)} \bar{U},$$

(S.34)

where $\bar{U}$ is the product of many beam splitter interactions and phase shifts between the normal modes at opposite frequency, the specific form of which is specified below.

This can be shown as follows. Let us, first, introduce the unitary matrix

$$\mathcal{T} = [\mathbf{w}_0 \mathbf{w}_1 \cdots \mathbf{w}_N],$$

(S.35)

which diagonalize $\mathcal{J}$ (i.e. $\mathbf{w}_j$ are the eigenvectors of $\mathcal{J}$ and $\mathcal{J} \mathcal{T} = \mathcal{T} \Lambda$, with $\Lambda$ the diagonal matrix with entries $\Lambda_{jj} = \lambda_j$), and which can be expressed in terms of a hermitian matrix $\mathcal{K}_T$ as

$$\mathcal{T} = e^{-i \mathcal{K}_T}.$$  

(S.36)

The density matrix $\rho^{(T)} = U^{(T)^\dagger} \rho U^{(T)}$, with

$$U^{(T)} = e^{-i \sum_{j=0}^N \mathcal{K}_T b_j^\dagger b_j},$$

(S.37)

fulfills the master equation $\dot{\rho}^{(T)} = -iH^{(T)} \rho^{(T)} + \mathcal{L}^{(T)} \rho^{(T)}$, where $H^{(T)} = \hbar \sum_{j=0}^N \lambda_j b_j^\dagger b_j$ and $\mathcal{L}^{(T)} \rho^{(T)} = U^{(T)^\dagger} \left[ H^{(T)} \rho^{(T)} U^{(T)} \right] U^{(T)^\dagger}$, and has been shown in Ref. [7] that in this representation, the steady state is characterized by many two-mode squeezed pairs, corresponding to the normal modes with opposite frequency $\lambda_j$ and $\lambda_{j+1} = -\lambda_j$ (in the case of an odd number of modes, the mode with zero frequency, that is the one with index $j = 0$, is in a single-mode squeezed state), generated by the transformations

$$U^{(2)}_j = e^{-i \xi_j \left( e^{i \varphi_0} b_j^\dagger b_{j+1} + e^{-i \varphi_0} b_{j+1} b_j \right)}$$

(S.38)

(with $\varphi_0$ the phase of the squeezed reservoir defined in the main text), where for an even number of modes ($N$ odd) $j$ takes even values $j \in \{0, 2, 4 \cdots (N - 1)\}$, instead, for an odd number of modes ($N$ even) $j$ takes odd values $j \in \{1, 3, 5 \cdots N - 1\}$. Thus, we introduce the transformation which generates all the entangled pairs, that is $U^{(TMS)} = U^{(2)}_0 U^{(2)}_2 U^{(2)}_4 \cdots$ for and even number of modes, and $U^{(TMS)} = U^{(2)}_0 U^{(2)}_1 U^{(2)}_3 \cdots$ for and odd number of modes (where $U_0$ is the single mode squeezing transformation defined in the main text), and we find that, in this representation, the steady state is $U^{(TMS)} |0\rangle$. Correspondingly, in the original representation, the steady state can be written as

$$|\Psi_{tot}\rangle = U^{(T)} U^{(TMS)} |0\rangle.$$  

(S.39)

Let us, now, consider the 50/50 beam splitter transformations between all the entangled pairs

$$U^{(BS)}_j = e^{-i \xi_j \left( b_j^\dagger b_{j+1} + b_{j+1}^\dagger b_j \right)}$$

(S.40)

where $j$ is even (odd) for an even (odd) number of modes) which realizes the transformations

$$U^{(BS)}_j b_j U^{(BS)}_j = \frac{1}{\sqrt{2}} (b_j - i b_{j+1})$$

$$U^{(BS)}_j b_{j+1} U^{(BS)}_j = \frac{1}{\sqrt{2}} (-i b_j + b_{j+1})$$

(S.41)

and the phase shifts for all the modes

$$U^{(\phi)}_j b_j U^{(\phi)}_j = e^{-i \xi_j} b_j$$

$$U^{(\phi)}_j b_{j+1} U^{(\phi)}_j = b_{j+1} e^{-i \xi_{j+1}}$$

(S.42)

with

$$\xi_j = \frac{\varphi_j - \varphi_0}{2}$$

(S.43)

(where $\varphi_j$ are the squeezing phases introduced in the proposition [7] of the theorem of the main text), which realizes the transformations

$$U^{(\phi)}_j b_j U^{(\phi)}_j = b_j e^{-i \xi_j}$$

$$U^{(\phi)}_j b_{j+1} U^{(\phi)}_j = b_{j+1} e^{-i \xi_{j+1}}$$

(S.44)

We find

$$U^{(\phi)}_j U^{(BS)}_j U^{(2)}_j U^{(BS)}_j U^{(\phi)}_j U^{(2)}_j U^{(BS)}_j \cdots = U^{(TMS)}$$

(S.45)

(with $U_j$ the single-mode squeezing transformation defined in the proposition [7] of the theorem of the main text). So, if we define the passive unitary transformation

$$\bar{U} = U^{(\phi)}_0 U^{(BS)}_0 U^{(\phi)}_1 U^{(BS)}_1 \cdots$$

(S.46)

for an even (odd) number of modes, we find

$$\bar{U}^\dagger U^{(TMS)} \bar{U} = U^{(S)} U_0,$$

(S.47)

where $U^{(S)}$ is defined in the main text, and therefore

$$|\Psi_{tot}\rangle = U^{(T)} \bar{U} \bar{U}^\dagger U^{(TMS)} \bar{U} \bar{U}^\dagger |0\rangle = U^{(T)} \bar{U} U^{(S)} |0\rangle,$$

(S.48)

where in the last step we have used Eq. (S.47) and the fact that $\bar{U}^\dagger$ is a passive transformation that does not change the vacuum. In the main text, instead, we have shown that $|\Psi_{tot}\rangle = U^{(p)} U^{(S)} U_0 |0\rangle$, and therefore

$$U^{(p)} = U^{(T)} \bar{U}.$$  

(S.49)
FIG. S.1. Graph corresponding to the symmetric adjacency matrix $A$ of the cluster state of $N = 25$ modes in a square lattice discussed in the main text. The non-zero entries of $A$ are equal to one and correspond to the edges of the graph.

FIG. S.2. (a) Real and (b) Imaginary parts of the interaction coefficients $J_{jk}$ of the Hamiltonian (4) of the main text used for the results of Fig. 2 of the main text.

FIG. S.3. (a) Real and (b) Imaginary parts of the interaction coefficients $J_{jk}$ of the Hamiltonian (4) of the main text used for the results of Fig. 2 of the main text.

CLUSTER STATES THAT CAN BE PREPARED WITH THE PRESENT APPROACH

Given a $N \times N$ real symmetric adjacency matrix $A$, the corresponding cluster states is the zero eigenstate of the collective operators (called nullifiers)

$$x_j = p_j - \sum_{k=1}^{N} A_{jk} q_k, \quad \text{for } j \in \{1, \ldots, N\}.$$ \hspace{1cm} (S.50)

with $p_j = -i(b_j - b_j^\dagger)$ and $q_j = b_j + b_j^\dagger$. In other terms, these collective quadratures are infinitely squeezed for a cluster state with adjacency matrix $A$. To be specific a cluster state can be written as $|\Psi_{\text{cluster}}\rangle = e^{-\frac{1}{2}\sum_j A_{jk} q_j q_k} |0\rangle_p$, where $|0\rangle_p$ is the infinitely squeezed states that is the zero eigenstate of the operators $p_j$, i.e $p_j |0\rangle_p = 0 \forall j$.

For realistic, approximated cluster states, these operators are squeezed by a finite amount. In general an approximated cluster state $|\Psi_z\rangle$ can be defined in terms of a finite squeezing parameter $z$ and the adjacency matrix $A$, such that the covariance matrix of the nullifiers $C_z = \langle \Psi_z | \mathbf{x} \mathbf{x}^T + (\mathbf{x} \mathbf{x}^T)^T |\Psi_z\rangle/2$ approaches the null matrix in the limit $z \to \infty$.

An example is given by the state generated by the unitary transformation

$$U_z = e^{\frac{1}{2}\sum_j A_{jk} q_j q_k} e^{\frac{i}{2} \sum_j (b_j^\dagger - b_j^2)}.$$ \hspace{1cm} (S.51)
that is \(|\psi_z\rangle = U_z |0\rangle\), where \(|0\rangle\) is the vacuum. In this case we find that the corresponding Bogoliubov matrix has the structure of Eq. (S.9) with

\[
X_z = \mathbb{I} \cosh(z) + \frac{i}{2} e^z \mathcal{A}_z
\]
\[
Y_z = \mathbb{I} \sinh(z) + \frac{i}{2} e^z \mathcal{A}_z.
\] (S.52)

It is possible to check that the covariance matrix of the nullifiers (S.50) approaches the null matrix in the limit of large \(z\). To be specific in this case \(C_z = e^{-2z} \mathbb{I}\). In our approach we can construct states for which the singular values of the matrices \(Z\) are all equal. In other terms the matrices \(X_z X_z^* = \mathbb{I} \cosh^2(z) + \frac{e^{2z}}{4} \mathcal{A}^2\) and \(Y_z Y_z^* = \mathbb{I} \sinh^2(z) + \frac{e^{2z}}{4} \mathcal{A}^2\) have to be proportional to the identity. This implies that with our approach we can construct cluster states given by Eq. (S.51) for which the adjacency matrix is proportional to a self-inverse matrix \(\mathcal{A}^2 = \alpha \mathbb{I}\) for some positive real \(\alpha\).

Another example is given by a multi-mode squeezed state generated by the transformation

\[
U_z = e^{-i \frac{z}{2} \mathcal{S}_z \mathcal{S}_b}
\] (S.53)

with

\[
\mathcal{S}_z = \begin{pmatrix} \mathcal{Z} & \mathcal{Z} \\ \mathcal{Z}^* & \mathcal{Z} \end{pmatrix},
\] (S.54)

where \(\mathcal{Z}\) is a complex symmetric, non-singular matrix. In Ref. [8] we have shown that these states are approximated cluster states, which can be realized using many equally squeezed modes, when the matrix \(\mathcal{Z}\) is related to the adjacency matrix \(\mathcal{A}\) by the relation

\[
\mathcal{Z} = -i \frac{\mathcal{A} - i \mathbb{I}}{\mathcal{A} + i \mathbb{I}},
\] (S.55)

such that it is unitary. In this case

\[
X = \cosh(z) \mathbb{I}
\]
\[
Y = -i \sin(z) \mathcal{Z}
\] (S.56)

and the Bloch-Messiah decomposition (S.15) is given by

\[
D_z = \cosh(z) \mathbb{I}
\]
\[
D_z^* = \sinh(z) \mathbb{I}
\]
\[
V^p = (-i \mathcal{Z})^{1/2}
\]
\[
W^p = (-i \mathcal{Z}^*)^{-1/2},
\] (S.57)

where the last two matrices are found by the Autonne–Takagi factorization [5, 6, 9] of the symmetric unitary \(-i \mathcal{Z}\), such that \(-i \mathcal{Z} = (-i \mathcal{Z})^{1/2} (-i \mathcal{Z})^{-1/2}\). In the main text we have studied the preparation of a state of this form where the adjacency matrix \(\mathcal{A}\) represents the square lattice depicted in Fig. S.1. The decomposition of the corresponding unitary transformation \(U_z = U^{(0)} U^{(3)}\) (see the proposition [7] in the theorem of the main text) can be found as discussed in Sec. 5. See in particular Eqs. (S.17), (S.20), (S.22), and (S.25), where in this case the matrix \(V^p\) is given in Eq. (S.57).

In Figs. S.2 and S.3 we report the coefficients of the system Hamiltonians that we have used in the result presented in the main text. In particular, in the main text, we have shown that the steady state of Eq. (5) of the main text, with the Hamiltonian represented in Figs. S.2 and S.3 approximates the cluster state with the adjacency matrix represented in Fig. S.1.

[1] Roger A. Horn and Charles R. Johnson, *Topics in Matrix Analysis* (Cambridge University Press, 1991).
[2] Samuel L. Braunstein, “Squeezing as an irreducible resource,” Phys. Rev. A 71, 055801 (2005).
[3] Peter van Loock, Christian Weedbrook, and Mile Gu, “Building Gaussian cluster states by linear optics,” Phys. Rev. A 76, 032321 (2007).
[4] Mile Gu, Christian Weedbrook, Nicolas C. Menicucci, Timothy C. Ralph, and Peter van Loock, “Quantum computing with continuous-variable clusters,” Phys. Rev. A 79, 062318 (2009).
[5] Gianfranco Cariolaro and Gianfranco Pierobon, “Reexamination of Bloch-Messiah reduction,” Phys. Rev. A 93, 062115 (2016).
[6] Gianfranco Cariolaro and Gianfranco Pierobon, “Bloch-Messiah reduction of Gaussian unitaries by Takagi factorization,” Phys. Rev. A 94, 062109 (2016).
[7] Yariv Yanay and Aashish A. Clerk, “Reservoir engineering of bosonic lattices using chiral symmetry and localized dissipation,” Phys. Rev. A 98, 043615 (2018).
[8] Stefano Zippilli and David Vitali, “Any Gaussian cluster state can be generated by a multi-mode squeezing transformation,” arXiv: 2007.12772 (2020).
[9] Roger A. Horn and Charles R. Johnson, *Matrix Analysis* (Cambridge University Press, 2013).