Harmonic analysis and the Riemann-Roch theorem.

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1. Let $D$ be a smooth projective curve over a finite field $k$. It is known (see, e.g., [6, §3]) that the Poisson summation formula applied to the discrete subgroup $k(D)$ of the adelic space $\mathbb{A}_D$ implies the Riemann-Roch theorem on the curve $D$. This result is an important step in the application of harmonic analysis to the arithmetic of algebraic curves. In this note we show, how to solve the analogous problem for the case of dimension two. Namely, we will show how the Riemann-Roch theorem for invertible sheaves on a projective smooth algebraic surface $X$ over $k$ (in a variant without the Noether formula, see, e.g., [7]) is obtained from the two-dimensional Poisson formulas (see [3, §5.9] and [4, §13]).

First, we need some general proposition. Let $E = (I, F, V)$ be a $C_2$-space over the field $k$ (see [2]). Recall that for any $i, j \in I$ we have constructed in [3, §5.2] a one-dimensional $\mathbb{C}$-vector space of virtual measures $\mu(F(i) \mid F(j)) = \mu(F(i) F(l)) \otimes \mathbb{C} \mu(F(l) \mid F(j))$, where $l \in I$ such that $l \leq i$, $l \leq j$, and $\mu(H)$ is the space of $\mathbb{C}$-valued Haar measures on a $C_1$-space $H$. The space $\mu(F(i) \mid F(j))$ does not depend on the choice of $l \in I$ up to a canonical isomorphism.

Let $0 \rightarrow A \rightarrow E \rightarrow B \rightarrow 0$ be an admissible triple of $C_2$-spaces over $k$. Let $A = (J, G, W)$ as a $C_2$-space, and $W = F(j)$ for some $j \in I$. Then $A$ is a $cC_2$-space and $B$ is a $dC_2$-space (see [3, §5.1]). Let $o \in I$ and $\mu \in \mu(W/F(o) \cap W)$, $\nu \in \nu(F(o)/F(o) \cap W)^*$. Then in [3, form. (164)] we have constructed the characteristic element $\delta_{A, \mu \otimes \nu} \in \mathcal{D}_{F(o)}(E)$. We note that $\mu \otimes \nu \in \mu(F(o) \mid W)$. Therefore we can replace $\mu \otimes \nu$ by $\eta \in \mu(F(o) \mid W)$ and write $\delta_{A, \eta} \in \mathcal{D}_{F(o)}(E)$ instead of the previous notation.

Let $0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0$ be an admissible triple of $C_2$-spaces over $k$ such that $L$ is a $c f C_2$-space and $M$ is a $d f C_2$-space (see [3, §5.1]). In [3, form. (169)] we have constructed the characteristic element $\delta_L \in \mathcal{D}_{F(o)}(E)$. We note that for any $i, j \in I$ the space $L$ defines a non-zero element $\mu_{L,F(i),F(j)} \in \mu(F(i) \mid F(j))$ in the following way. Let $L = (K, T, U)$ as a $C_2$-space. Choose some $l \in I$ such that $l \leq i$, $l \leq j$. Then $\mu_{L,F(i),F(j)} = \mu_{L,F(l),F(i)} \otimes \mu_{L,F(l),F(j)}$, where for any $m \leq n \in I$ we define $\mu_{L,m,n} \in \mu(F(n)/F(m))$ as $\mu_{L,m,n}(U \cap F(n)/U \cap F(m)) = 1$. The element $\mu_{L,F(i),F(j)}$ does not depend on the choice of $l \in I$.

There is a natural pairing $\langle \cdot, \cdot \rangle: \mathcal{D}_{F(o)}(E) \times \mathcal{D}_{F(o)}(E) \rightarrow \mathbb{C}$. From the above definitions it is easy to prove the following proposition.

Proposition 1

$$\langle \delta_L, \delta_{A, \eta} \rangle = \frac{\eta}{\mu_{L,F(o),W}}.$$
2. Let $X$ be a smooth projective algebraic surface over a finite field $k$. Let $|k| = q$. For any quasicoherent sheaf $\mathcal{F}$ on $X$ there is an adelic complex $\mathcal{A}_X(\mathcal{F})$ such that $H^*(\mathcal{A}_X(\mathcal{F})) = H^*(X, \mathcal{F})$. Let $C \in \text{Div}(X)$. For the sheaf $\mathcal{O}_X(C)$ on $X$ we will write this complex in the following way:

$$\mathcal{A}_{0,C} \oplus \mathcal{A}_{1,C} \oplus \mathcal{A}_{2,C} \longrightarrow \mathcal{A}_{01,C} \oplus \mathcal{A}_{02,C} \oplus \mathcal{A}_{12,C} \longrightarrow \mathcal{A}_{012,C},$$

where $\mathcal{A}_{*,C} = \mathcal{A}_{X,*}(\mathcal{O}_X(C))$ (see the corresponding notations and definitions in [I §14.1]), and we have omitted indication on $X$ in the notations of subgroups of the adelic complex, because we will work only with one algebraic surface $X$ during this note. We note that all the groups $\mathcal{A}_{*,C}$ are subgroups of the group $\mathcal{A}_{012,C}$. Besides, the following groups do not depend on $C \in \text{Div}(X)$:

$$\mathcal{A}_{0,C} = \mathcal{A}_0, \quad \mathcal{A}_{01,C} = \mathcal{A}_{01}, \quad \mathcal{A}_{02,C} = \mathcal{A}_{02}, \quad \mathcal{A}_{012,C} = \mathcal{A}_{012} = \mathcal{A}.$$

Moreover, $\mathcal{A} \subset \prod_{x \in D} K_{x,D}$, where $x \in D$ runs over all pairs with irreducible curve $D$ on $X$ and $x$ is a point on $D$. The ring $K_{x,D}$ is a finite product of two-dimensional local fields with the last residue field $k(x)$.

We fix a non-zero rational differential form $\omega \in \Omega^2_{k(X)/k}$. Let $(\omega) \in \text{Div}(X)$ be the corresponding divisor. The following pairing (which depends on $\omega$) is well-defined, symmetric and non-degenerate:

$$\mathbb{A} \times \mathbb{A} \longrightarrow k : \{f_{x,D}\} \times \{g_{x,D}\} \mapsto \sum_{x \in D} \text{Tr}_{k(x)/k} \circ \text{res}_{x,D}(f_{x,D} g_{x,D} \omega),$$

where $\text{res}_{x,D}$ is the two-dimensional residue. For any $k$-subspace $V \subset \mathbb{A}$ we will denote by $V^\perp$ the annihilator of $V$ in $\mathbb{A}$ with respect to the pairing [I]. Using the reciprocity laws for the residues of differential forms on $X$ (the reciprocity laws "around a point" and the reciprocity laws "along a curve") one can prove the following proposition.

**Proposition 2** We have the following properties.

$$\mathcal{A}_0^\perp = \mathcal{A}_{01} + \mathcal{A}_{02}, \quad \mathcal{A}_{1,C}^\perp = \mathcal{A}_{01} + \mathcal{A}_{12,(\omega) - C}, \quad \mathcal{A}_{2,C}^\perp = \mathcal{A}_{02} + \mathcal{A}_{12,(\omega) - C},$$

$$\mathcal{A}_{01}^\perp = \mathcal{A}_{01}, \quad \mathcal{A}_{02}^\perp = \mathcal{A}_{02}, \quad \mathcal{A}_{12,C}^\perp = \mathcal{A}_{12,(\omega) - C}.$$

We note that $\mathcal{A} = \lim_{C \in \text{Div}(X)} \mathcal{A}_{12,C}$, and $\mathcal{A}_{12,C} = \lim_{C' \leq C} \mathcal{A}_{12,C}/\mathcal{A}_{12,C'}$. For any $C' \leq C$ the $k$-space $\mathcal{A}_{12,C}/\mathcal{A}_{12,C'}$ has the natural structure of a complete $C_1$-space over the field $k$. Hence we obtain that the $k$-space $\mathcal{A}$ has the following structure of a complete $C_2$-space over $k$: $(\text{Div}(X), F, \mathcal{A})$, where $F(C) = \mathcal{A}_{12,C}$ for $C \in \text{Div}(X)$. For simplicity we will use the same notation $\mathcal{A}$ for this $C_2$-space, i.e. we will omit the partially ordered set $\text{Div}(X)$ and the function $F$. The subspaces $\mathcal{A}_{*,C}$ of $\mathcal{A}$ (and the factor-spaces by these subspaces) have induced structures of $C_2$-spaces, which we will also denote by the same notations $\mathcal{A}_{*,C}$ (by notations for factor-spaces).

From proposition 2 it follows that the $C_2$-dual space (see [I §5.1]) $\tilde{\mathcal{A}}$ coincides with the $C_2$-space $\mathcal{A}$ itself:

$$\tilde{\mathcal{A}} = \lim_{C \in \text{Div}(X)} \lim_{C' \leq C} \mathcal{A}_{12,C}/\mathcal{A}_{12,C'} = \lim_{C' \in \text{Div}(X)} \lim_{C \geq C'} \mathcal{A}_{12,(\omega) - C'}/\mathcal{A}_{12,(\omega) - C} = \mathcal{A}. $$

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3. For any \( E \in \text{Div}(X) \) we denote \( h^i(E) = \dim_k H^i(X, \mathcal{O}_X(E)) \), where \( 0 \leq i \leq 2 \). We fix any \( H, C \in \text{Div}(X) \). We consider the following admissible triple of complete \( C_2 \)-spaces over \( k \):

\[
0 \to A_0 \to A_{01} \to A_{01}/A_0 \to 0. \tag{2}
\]

The space \( A_0 \) is a \( cfC_2 \)-space, and the space \( A_{01}/A_0 \) is a \( dfC_2 \)-space. Therefore there is the characteristic element \( \delta_{A_0} \in \mathcal{D}_{k, H}(A_{01}) \).

Now we consider the following admissible triple of complete \( C_2 \)-spaces over \( k \):

\[
0 \to A_{1,C} \to A_{01} \to A_{01}/A_{1,C} \to 0. \tag{3}
\]

We note that the space \( A_{01} \) is a \( dfC_2 \)-space. Therefore for any \( H', C' \in \text{Div}(X) \) there is a natural element \( \delta_{H', C'} \in \mu(A_{1,H'} | A_{1,C'}) \) which is uniquely defined by the following two conditions: 1) \( \delta_{H', C'} \in \mu(A_{1,H'} | A_{1,C'}) \) is defined as \( \delta_{H', C'} = \delta_{H', C'} \) for any \( H', M', C' \in \text{Div}(X) \), and 2) \( \delta_{H', C'} \in \mu(A_{1,C'}) \) is defined as \( \delta_{H', C'}(0) = 1 \), where \( 0 \) is the zero subspace in the discrete \( C_1 \)-space \( A_{1,C'}/A_{1,H'} \). Besides, the space \( A_{1,C} \) is a \( cC_2 \)-space, and the space \( A_{01}/A_{1,C} \) is a \( dC_2 \)-space. Hence there is the characteristic element \( \delta_{A_{01}, C, H, C} \in \mathcal{D}_{k, H}(A_{01}) \).

**Lemma 1** We have the following equality:

\[
< \delta_{A_0}, \delta_{A_{1,C}, H, C} > = q^{\dim_k V}.
\]

**Proof** We will use proposition II. From this proposition it follows that it is enough to consider \( H \leq C \). In this case, by this proposition again, we have \( < \delta_{A_0}, \delta_{A_{1,C}, H, C} > = q^{\dim_k V} \), where \( k \)-vector space \( V = (A_0 \cap A_{1,C})/(A_0 \cap A_{1,H}) \). Now we use \( A_0 \cap A_{1,E} = H^0(X, \mathcal{O}_X(E)) \) for any \( E \in \text{Div}(X) \). The lemma is proved.

Now we fix any \( P, Q \in \text{Div}(X) \). We consider the following admissible triple of complete \( C_2 \)-spaces over \( k \):

\[
0 \to A_{02}/A_0 \to A/A_{01} \to A/(A_{02} + A_{01}) \to 0, \tag{4}
\]

where we use that \( A_0 = A_{01} \cap A_{02} \). The space \( A_{02}/A_0 \) is a \( cfC_2 \)-space, and the space \( A/(A_{01} + A_{02}) \) is a \( dfC_2 \)-space. Therefore there is the characteristic element \( \delta_{A_{02}, A_0} \in \mathcal{D}_{k, P,A_{1,P}}(A/A_{01}) \).

Now we consider the following admissible triple of complete \( C_2 \)-spaces over \( k \):

\[
0 \to A_{12,Q}/A_{1,Q} \to A/A_{01} \to A/(A_{12,Q} + A_{01}) \to 0, \tag{5}
\]

where we use that \( A_{1,Q} = A_{01} \cap A_{12,Q} \). We note that the space \( A/A_{01} \) is a \( cC_2 \)-space. Therefore for any \( P', Q' \in \text{Div}(X) \) there is the following natural element \( 1_{P', Q'} \in \mu(A_{12,Q}/A_{1,H} | A_{12,Q}/A_{1,Q}) \) which is uniquely defined by the following two conditions: 1) \( 1_{P', Q'} \in \mu(A_{12,Q}/A_{1,H} | A_{12,Q}/A_{1,Q}) \) for any \( P', R', Q' \in \text{Div}(X) \), and 2) \( P' \leq Q' \) then \( 1_{P', Q'} \in \mu((A_{12,Q}/A_{1,H})/(A_{12,P}/A_{1,H})) \) is defined as \( 1_{P', Q'}((A_{12,Q}/A_{1,Q})/(A_{12,P}/A_{1,Q})) = 1 \), since \( (A_{12,Q}/A_{1,Q})/(A_{12,P}/A_{1,Q}) \) is a compact \( C_1 \)-space. Besides, the space \( A_{12,Q}/A_{1,Q} \) is a \( cC_2 \)-space, and the space \( A/(A_{12,Q} + A_{01}) \) is a \( dC_2 \)-space. Hence there is the characteristic element \( \delta_{A_{12,Q}/A_{1,Q}, P,Q} \in \mathcal{D}_{k, P,A_{1,P}}(A/A_{01}) \).
Lemma 2 We have the following equality:

\[ <\delta_{k_0}/k_0 : \delta_{k_{12},Q}/k_{1,2},1_{P,Q} > = q^{h_2(Q) - h_2(P)}. \]

Proof We will use proposition 1 By this proposition, it is enough to consider \( P \geq Q \). In this case, by this proposition again, we have \( <\delta_{k_0}/k_0 : \delta_{k_{12},Q}/k_{1,2},1_{P,Q} > = q^{\text{dim}_k \cdot W} \), where the \( k \)-vector space \( W = (A_{k_0} + A_{k_2} + A_{12,R})/(A_{k_0} + A_{k_2} + A_{12,Q}) \). Now we use that from the adelic complex \( \mathcal{A}_X(\mathcal{O}_X(E)) \) we have \( A/(A_{k_0} + A_{k_2} + A_{12,E}) = H^2(X, \mathcal{O}_X(E)) \) for any \( E \in \text{Div}(X) \). The lemma is proved.

Now we suppose that \( Q = (\omega) - C \) and \( P = (\omega) - H \). From proposition 2 it follows that triple \( [1] \) is a \( C_2 \)-dual sequence to triple \( [2] \), and triple \( [3] \) is a \( C_2 \)-dual sequence to triple \( [4] \). We have also the two-dimensional Fourier transforms \( F : D_{k_1,1}(A_{k_0}) \to D_{k_{12},P}/k_{1,2} \cdot (A_{k_0}) \) and \( F : D_{k_1,1}(A_{k_0}) \to D_{k_{12},P}/k_{1,2} \cdot (A_{k_0}) \) (see \([3, \text{th. 2}]\)) we have \( F(\delta_{k_0}) = \delta_{k_0}/k_0 \). By the two-dimensional Poisson formula II (see \([3, \text{th. 3}]\)) we have \( F(\delta_{k_0}) = \delta_{k_0}/k_0 \). (We used that according to \([3, \text{form. (103)}]\) we have \( F(\omega) = g \) for \( g = \delta_{k_0} \) or \( g = \delta_{k_{1,2}} \), and the maps \( F \) are conjugate with respect to each other (see \([3, \text{prop. 24}]\)). Now since \( F \circ F(\omega) = g \) for \( g = \delta_{k_0} \) or \( g = \delta_{k_{1,2}} \), and the maps \( F \) are conjugate with respect to each other (see \([3, \text{prop. 24}]\)), we have that \( <\delta_{k_0}, \delta_{k_{1,2}},1_{P,Q} > = F(\delta_{k_0}) = F(\delta_{k_{1,2}}) > 0 \). Hence and from lemmas \([1,2] \) we obtain for any \( H, C \in \text{Div}(X) \) the following equality:

\[ h_0(C) - h_0(H) = h^2((\omega) - C) - h^2((\omega) - H). \]

4. For any \( E \in \text{Div}(X) \) we denote the Euler characteristic \( \chi(E) = h_0(E) - h_1(E) + h_2(E) \). We fix any \( R, S \in \text{Div}(X) \). We consider the following admissible triple of complete \( C_2 \)-spaces over \( k \):

\[ 0 \to A_{k_0} \to A \to A/k_{1,2} \to 0. \]

The space \( A_{k_0} \) is a \( cfC_2 \)-space, and the space \( A/k_{1,2} \) is a \( dfC_2 \)-space. Therefore there is the characteristic element \( \delta_{k_0} \in D_{k_{12},2}(A_{k_0}) \).

Now we consider the following admissible triple of complete \( C_2 \)-spaces over \( k \):

\[ 0 \to A_{12,S} \to A \to A/k_{12,S} \to 0. \]

The subspace \( A_{k_0} \) uniquely defines an element \( \nu_{R,S} \in \mu(A_{12,R} | A_{12,S}) \) for any \( R, S \in \text{Div}(X) \) in the following way. If \( R' \leq S' \), then we consider the following admissible triple of \( C_1 \)-spaces:

\[ 0 \to A_{k_0}/k_{1,R} \to A_{12,S}/A_{12,R} \to A_{12,S}/(A_{12,S} + A_{12,R}) \to 0, \]

where \( A_{12,S}/A_{12,R} \) is a discrete \( C_1 \)-space, and \( A_{12,S}/(A_{12,S} + A_{12,R}) \) is a compact \( C_1 \)-space. Now \( \nu_{R,S} \in \mu(A_{12,S}/A_{12,R}) \) is equal to \( \delta_0 \otimes 1 \), where \( \delta_0((0)) = 1 \), \( \delta_0 \in \mu(A_{12,S}/A_{12,R}) \), and \( 1(A_{12,S}/A_{12,R}) = 1 \). For arbitrary \( R', S' \) the element \( \nu_{R',S'} \) is defined by the following rule: \( \nu_{R',S'} = \nu_{R',T'} \otimes \nu_{T',S'} \), where \( T' \in \text{Div}(X) \) is any. The space \( A_{12,S} \) is a \( cC_2 \)-space, and the space \( A_{12,S}/A_{12,R} \) is a \( dC_2 \)-space. Hence there is the characteristic element \( \delta_{12,S}/\nu_{R,S} \in D_{k_{12},2}(A_{k_0}) \).
Lemma 3 We have the following equality:

\[ <\delta_{h_{02}}, \delta_{h_{12},S,\nu_{R,S}} >= q^{x(S) - x(R)}. \]

Proof We will use proposition 1. From this proposition it follows that it is enough to consider \( R \leq S \). In this case, by this proposition again, we have \( <\delta_{h_{02}}, \delta_{h_{12},S,\nu_{R,S}} >= a^{\chi}, \)

where \( a \) is equal to the Euler characteristic of the following complex, which has the finite-dimensional over \( k \) cohomology groups:

\[ A_{11,2}/A_{11,1} \oplus A_{22,2}/A_{22,1} \rightarrow A_{12,2}/A_{12,1}. \]  

(9)

Complex (9) is the factor-complex of the adelic complex \( A_{X}(O_{X}(S)) \) by the adelic complex \( A_{X}(O_{X}(R)) \). Therefore the Euler characteristic of complex (9) is the difference of the Euler characteristics of corresponding adelic complexes. The lemma is proved.

From proposition 2 it follows that triple (7) itself is a \( C_{2} \)-dual sequence to triple (7), and triple (8) is a \( C_{2} \)-dual sequence to triple (8) when \( S \rightarrow (\omega) - S \). We have also the two-dimensional Fourier transforms \( F : D_{h_{12},R}(A_{1}) \rightarrow D_{h_{12},(\omega)-R}(A_{1}) \) and \( F : D_{h_{12},R}(A_{1}) \rightarrow D_{h_{12},(\omega)-R}(A_{1}). \) By the two-dimensional Poisson formulas (see [3, th. 2-th. 3]) we have \( F(\delta_{h_{02}}) = \delta_{h_{02}} \) and \( F(\delta_{h_{12},S,\nu_{R,S}}) = \delta_{h_{12},(\omega)-S,\nu_{(\omega),(\omega)-S}}. \) (We used that from proposition 2 it follows that \( \nu_{R,S} \rightarrow \nu_{(\omega)-R,S} \) under the natural isomorphism \( \mu(A_{12,1,\omega}-R, A_{12,1,\omega}-S) \) from \( A_{12,1,\omega}-R, A_{12,1,\omega}-S \) we have that \( \chi(S) = \chi(\omega) - S \) - \( \chi((\omega) - R) \). If we put \( R = (\omega) - S \), then for any \( S \in \text{Div}(X) \) we obtain from the previous formula the following equality:

\[ \chi(S) = \chi((\omega) - S). \]  

(10)

5. In section 1 we introduced the element \( \mu_{L,F(i),F(j)}(F(i),F(j)) \) for the admissible monomorphism of \( C_{2} \)-spaces \( L \rightarrow E \). When \( L = A_{02}, E = A, F(i) = A_{12,1}, F(j) = A_{12,2} \) for \( R, S \in \text{Div}(X) \) we will denote this element by \( \mu_{R,S} \). From the proof of lemma 3 it follows that

\[ q^{x(S) - x(R)} = \frac{\nu_{R,S}}{\mu_{R,S}}. \]  

(11)

For any \( g \in A^{*} \) and any \( R, S \in \text{Div}(X) \) we have a natural action: \( g^{*} : \mu(A_{12,1} | A_{12,1}) \rightarrow \mu(gA_{12,1} | gA_{12,1}) \). Hence we obtain a central extension (see also [3, §5.5.3]):

\[ 1 \rightarrow C^{*} \rightarrow \tilde{A}^{*} \rightarrow A^{*} \rightarrow 1, \]

where \( \tilde{A}^{*} = \{(g, \phi) : g \in A^{*}, \phi \in \mu(A_{12,0} | gA_{12,0}), \phi \neq 0\}, \) and \( (g_{1}, \phi_{1})(g_{2}, \phi_{2}) = (g_{1}g_{2}, \phi_{1} \otimes g_{1}^{*}(\phi_{2})). \) (Here \( A_{12,0} \) is the group connected with the zero divisor on \( X \).) For any \( g_{1}, g_{2} \in A^{*} \) we denote \( \langle g_{1}, g_{2} \rangle = [\tilde{g}_{1}, \tilde{g}_{2}] \in C^{*}, \) where \( \tilde{g}_{i} \in \tilde{A}^{*} \) are any such that \( \pi(\tilde{g}_{i}) = g_{i}. \) The element \( \langle g_{1}, g_{2} \rangle \) does not depend on the choice of appropriate elements \( \tilde{g}_{i}. \) From [1] it follows the following equality:

\[ \langle g_{1}, g_{2} \rangle = \prod_{x \in D} q^{-(x) \cdot k(1)(g_{1x}, D, g_{2x}, D)x_{x}, D}, \]  

(12)

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where $(\cdot, \cdot)_{x,D}$ is the composition of the maps: $K_{x,D}^* \times K_{x,D}^* \to K_2(K_{x,D}) \xrightarrow{\partial} \mathbb{R}_{x,D}^* \xrightarrow{\partial} \mathbb{Z}$.

For any $E \in \text{Div}(X)$ we choose an element $j_{1,E} \in \mathbb{A}_{01}^*$ such that $\mathbb{A}_{1,E} = j_{1,E}\mathbb{A}_{2,0}$, and an element $j_{2,E} \in \mathbb{A}_{02}^*$ such that $\mathbb{A}_{2,E} = j_{2,E}\mathbb{A}_{2,0}$, where we take the product inside the ring $A$. Now from [3] §2.2 and from [12] it follows the following formula for any $C, H \in \text{Div}(X)$ ($(C, H)$ means the intersection index of divisors $C$ and $H$ on $X$):

$$\langle j_{2,C}, j_{1,H} \rangle = q^{- (C, H)}.$$  \hfill (13)

Since we can take $j_{1,E_1+E_2} = j_{1,E_1}j_{2,E_2}$ and $j_{2,E_1+E_2} = j_{2,E_1}j_{2,E_2}$, we obtain $j_{1,E_1}\mathbb{A}_{1,E_2} = \mathbb{A}_{1,E_1+E_2}$ and $j_{2,E_1}\mathbb{A}_{2,E_2} = \mathbb{A}_{2,E_1+E_2}$ for any $E_1, E_2 \in \text{Div}(X)$. Hence we have $j_{1,E}(\nu_{R,S}) = \nu_{R+E,S+E}$ and $j_{2,E}(\mu_{R,S}) = \mu_{R+E,S+E}$ for any $R, S \in \text{Div}(X)$. For any $C \in \text{Div}(X)$ we choose $\widehat{j_{2,C}} = (j_{2,C}, \nu_{0,C}) \in \mathbb{A}_{02}^*$ and $\widehat{j_{1,(\omega)-C}} = (j_{1,(\omega)-C}, \mu_{0,(\omega)-C}) \in \mathbb{A}_{02}^*$. We have

$$\langle \widehat{j_{2,C}}, \widehat{j_{1,(\omega)-C}} \rangle = \frac{\nu_{0,C} \ast j_{2,C}(\mu_{0,(\omega)-C})}{j_{1,(\omega)-C} \ast j_{2,C}} \quad = \frac{\nu_{0,C} \ast \mu_{C,(\omega)}}{\mu_{0,(\omega)-C} \ast \nu_{(\omega)-C}(\mu_{C,(\omega)})} \quad = \frac{\nu_{0,C} \ast \mu_{C,(\omega)-C} \ast \nu_{(\omega)-C}(\mu_{C,(\omega)})}{\mu_{0,C} \ast \mu_{C,(\omega)-C} \ast \nu_{(\omega)-C}(\mu_{C,(\omega)})}. \quad \hfill (14)$$

From (11) and (10) we obtain $\frac{\nu_{0,C}}{\mu_{0,C}} = \frac{\mu_{C,(\omega)-C} \ast \nu_{(\omega)-C}(\mu_{C,(\omega)})}{\mu_{0,C} \ast \nu_{(\omega)-C}(\mu_{C,(\omega)})} = q^{\chi(C) - \chi(0)}$. Therefore from (14) and (13) we have $2(\chi(C) - \chi(0)) = -(C, (\omega) - C)$ for any $C \in \text{Div}(X)$. From the last equality and formula (6) we obtain the Riemann-Roch theorem in the following form.

**Theorem 1** For any $C \in \text{Div}(X)$ and $\omega \in \Omega_{k(X)}^2, \omega \neq 0$ we have the following equality

$$h^0(C) - h^1(C) + h^0(\omega - C) = h^0(0) - h^1(0) + h^0(\omega) - \frac{1}{2} (C, (\omega) - C).$$

**References**

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