Effective Action of QED in Electric Field Backgrounds II: Spatially Localized Fields

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We find the Bogoliubov coefficient from the tunneling boundary condition on charged particles in a static electric field \( E_0 \text{sech}^2(z/L) \) and, using the regularization scheme in Phys. Rev. D 78, 105013 (2008), obtain the exact one-loop effective action in scalar and spinor QED. It is shown that the effective action satisfies the general relation between the vacuum persistence and the mean number of produced pairs. We advance an approximation method for general electric fields and show the duality between the space-dependent and time-dependent electric fields of the same form at the leading order of the effective actions.

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I. INTRODUCTION

Understanding the vacuum structure of strong field backgrounds has been a challenging task in quantum field theory. Electromagnetic fields and spacetime curvatures provide a typical arena for strong field physics. The vacuum structure may be exploited by finding the effective action in these backgrounds. In quantum electrodynamics (QED), Sauter, Heisenberg and Euler, Weisskopf, and Schwinger obtained the effective action in a constant electromagnetic field several decades ago [1–4]. However, going beyond the constant electromagnetic field has been another long-standing problem in QED, and the effective actions have been carried out only for certain field configurations (for a review and references, see Ref. [5] and for physical applications, see also Ref. [6]). For instance, there has been an attempt to compute the effective action in a pulsed electric field of the form \( E_0 \text{sech}^2(t/\tau) \) in Refs. [7, 8].

The main purpose of this paper is to further develop the in- and out-state formalism based on the Bogoliubov transformation in Refs. [8] (hereafter referred to I) for a pulsed electric field and [9, 10] for a constant electric field to be applicable to the case of spatially localized electric fields. To quantize a charged particle in an electric field background is not trivial because the vacuum is unstable against pair production. Further, the boundary condition on the solution of the Klein-Gordon or Dirac equation distinguishes pulsed electric fields from spatially localized electric ones. In the former case of a pulsed electric field, the charged boson or fermion interacts for a finite period of time, and its positive frequency solution splits both into one branch of positive frequency solution and into another branch of negative frequency solution after completion of the interaction. In the second quantized field theory, the presence of negative frequency solution means that particle-antiparticle pairs of a given mode are created from the vacuum due to the external electric field.

In the latter case of a spatially localized electric field, charged bosons or fermions experience a tunneling barrier from the Coulomb gauge potential. Nikishov elaborated the Feynman method to find the pair-production rate in the spatially localized electric field \( E_0 \text{sech}^2(z/L) \) [11, 12]. In fact, the quantum field confronts the Klein paradox from the tunneling barrier. To resolve the paradox, one has to treat the field in the second quantized theory and impose a boundary condition different from that for the pulsed electric field [13], which is the scattering over the barrier. The tunneling probability through the barrier gives the probability for one-pair production [13], and the reflection probability leads to the vacuum persistence, that is, the probability for the vacuum-to-vacuum transition [14–17]. It is also shown that the tunneling and the reflection probabilities have an interpretation of instantons and anti-instantons [16, 17] and that the instanton action for the tunneling barrier provides the leading contribution to the pair-production rate [18]. The pair-production rate is approximately obtained in the worldline instanton method [19, 20] and in the WKB method [21].

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To calculate the effective action in $E_0 \operatorname{sech}^2(z/L)$, we first find the Bogoliubov coefficient as the ratio of the incident coefficient to the reflection coefficient of the flux, whose inverse magnitude square gives the vacuum persistence. This is consistent with the boundary condition from causality on signals (wave packets) of a particle [17]. We then employ the regularization scheme of Ref. I to calculate the effective action of scalar and spinor QED. To our knowledge, the vacuum polarization (real part) of the renormalized effective action is the first result for this spatially localized field configuration.

We also advance an approximation method based on the Liouville-Green transformation for general electric fields. The leading contributions to the effective action in general electric fields $E(z)$ and $E(t)$ are determined entirely by the instanton actions in Ref. [13]. In this sense the duality of effective actions approximately holds between any $E(z)$ and $E(t)$ of the same function form as well as the Sauter-type electric fields.

The organization of this paper is as follows. In Sec. II, we introduce another method to directly find the out-vacuum from the Bogoliubov transformation without relying on the evolution operator of two-mode squeezed operator. In Sec. III, we find the Bogoliubov coefficient for the spatially localized electric field $E(z)$. Then, the effective action defined by the scattering amplitude as the Sauter-type electric field $E_0 \operatorname{sech}^2(z/L)$ and, then, compute the renormalized effective action using the regularization scheme of Ref. I. In Sec. IV, we put forth an approximation method to find the effective actions in general electric fields and discuss the duality between electric fields of the same form.

II. NEW DERIVATION OF EFFECTIVE ACTION

In this section we develop the in- and out-state formalism for the effective action in a background electric field. The in-state and the out-state are related through the Bogoliubov transformation

\[ a_{\omega k \sigma, \text{out}} = \mu_{\omega k \sigma} a_{\omega k \sigma, \text{in}} + \nu^*_{\omega k \sigma} b_{\omega k \sigma, \text{in}}, \]
\[ b_{\omega k \sigma, \text{out}} = \mu_{\omega k \sigma} b_{\omega k \sigma, \text{in}} + \nu^*_{\omega k \sigma} a_{\omega k \sigma, \text{in}}. \]  

(1)

Here, $a_{\omega k \sigma}$ and $b_{\omega k \sigma}$ denote the particle and the antiparticle operators with energy $\omega$, momentum $k$, transverse to the direction of the electric field, and spin $\sigma$ ($\sigma = 0$ for scalars and $\sigma = \pm 1/2$ for spin-1/2 fermions), respectively. The coefficients satisfy the Bogoliubov relation

\[ |\mu_{\omega k \sigma}|^2 - (-1)^{2|\sigma|} |\nu_{\omega k \sigma}|^2 = 1. \]  

(2)

In Ref. I, the Bogoliubov transformation is expressed by the evolution operator as

\[ a_{\omega k \sigma, \text{out}} = U_{\omega k \sigma} a_{\omega k \sigma, \text{in}} U^\dagger_{\omega k \sigma}, \]
\[ b_{\omega k \sigma, \text{out}} = U_{\omega k \sigma} b_{\omega k \sigma, \text{in}} U^\dagger_{\omega k \sigma}, \]  

(3)

and the out-vacuum is then given by $|0; \text{out}\rangle = U|0; \text{in}\rangle$.

However, one may find the out-vacuum without using the evolution operator. Indeed, the out-vacuum defined as

\[ a_{\omega k \sigma, \text{out}}|0; \text{out}\rangle = 0, \quad b_{\omega k \sigma, \text{out}}|0; \text{out}\rangle = 0, \]  

(4)

is given by

\[ |0; \text{out}\rangle = \prod_{\omega k \sigma} \left[ \frac{1}{\mu_{\omega k \sigma}} \sum_{n_{\omega k \sigma} = 0}^{\infty} \left(-\frac{\mu_{\omega k \sigma}}{\nu_{\omega k \sigma}}\right)^n_{\omega k \sigma} |n_{\omega k \sigma}, \bar{n}_{\omega k \sigma, \text{in}}\rangle \right] \]  

(5)

for scalar QED, and by

\[ |0; \text{out}\rangle = \prod_{\omega k \sigma} \left[ -\nu^*_{\omega k \sigma} |\omega k \sigma, \text{in}\rangle + \mu_{\omega k \sigma} |0_{\omega k \sigma, \text{in}}\rangle \right] \]  

(6)

for spinor QED. Here, the bar denotes the antiparticle number. The result is the same as obtained from the evolution operator of Ref. I. Note that particles and antiparticles are always produced in pairs.

Then, the effective action defined by the scattering amplitude as

\[ e^{iS_{\text{eff}}} = e^{i \int dt d^2 x_\perp L_{\text{eff}}} = \langle \text{out}|0; \text{in}\rangle \]  

(7)
The boundary condition for a quantum field coupled to a space-dependent gauge field differs from that coupled to a time-dependent gauge field. In the latter case of the time-dependent gauge field, the in-state is a quantum state before onset of the interaction, which evolves to an out-state after completion of interaction. However, in the former case of the space-dependent gauge field, though the in-/out-state cannot be defined in the remote past/future, these may be defined analogously to the case of time-dependent gauge field. Indeed, Nikishov developed the scattering formalism for space-dependent gauge fields, where the incoming signal (wave packet) toward the barrier defines the in-state while the outgoing signal defines the out-state \[11\]. Here, a caveat is that the space-dependent gauge field confronts the Klein paradox, contrary to the time-dependent gauge field. The resolution from causality requirement is that the vacuum persistence (probability for the vacuum-to-vacuum transition) is given by the reflection probability \[13–17\]. Thus, the ratio of the incident coefficient to the reflection coefficient of the flux is the Bogoliubov coefficient for the vacuum persistence.

As a spatially localized electric field, we consider the Sauter-type field, \(E(z) = E_0 \text{sech}^2(z/L)\), which extends effectively over a length scale of \(L\). In the Coulomb gauge, the gauge potential is \(A_0(z) = -E_0 t \text{anh}(z/L)\). The Fourier component of the Klein-Gordon equation for scalar QED and the spin-diagonal component of the Dirac equation for spinor QED satisfy [in units with \(\hbar = c = 1\) and with metric signature \((+,-,-,-)\)]

\[
\left[\frac{\partial^2}{\partial z^2} - (m^2 + k_z^2) + (\omega - qE_0 L \text{tanh}(z/L))^2 + 2i\sigma qE(z)\right] \phi_{\omega k_{\perp} \sigma}(z) = 0,
\]

(11)

where \(\sigma = 0\) for scalar particles and \(\sigma = \pm 1/2\) for spin-1/2 fermions. The equation has two asymptotic momenta at \(z = \pm \infty\)

\[k_{\pm} = \sqrt{(\omega + qE_0 L)^2 - m^2 - k_z^2}.\]

(12)

Pairs are produced only when \(\omega + qE_0 L \geq m\) and \(\omega - qE_0 L \leq -m\), under which the particle- and antiparticle-state can be defined asymptotically and the transverse momentum can take the maximum \(k_{\perp \text{max}}^2 = \min\{(\omega + qE_0 L)^2 - m^2, (\omega - qE_0 L)^2 - m^2\}\).

Changing the variable as

\[\zeta = e^{-2z/L},\]

(13)

we find the solution in terms of the hypergeometric function as

\[
\phi_{\omega k_{\perp} \sigma}(z) = \frac{\zeta^{-iLk_{\perp}(z)/2}}{\sqrt{2k_{\pm}(+)^{1-2i\sigma}}} F(a_{\omega k_{\perp} \sigma}, b_{\omega k_{\perp} \sigma}; \gamma_{\omega k_{\perp} \sigma}; \zeta),
\]

(14)

where

\[\lambda_{\sigma} = \sqrt{(qE_0 L)^2 - \left(\frac{1 - 2|\sigma|}{2}\right)^2},\]

(15)
and

\[ \alpha_{\omega k,\sigma} = 1 - \frac{2\sigma}{2} - \frac{i}{2}(Lk_z(+) - Lk_z(-) - 2\lambda_\sigma), \]
\[ \beta_{\omega k,\sigma} = 1 - \frac{2\sigma}{2} - \frac{i}{2}(Lk_z(+) + Lk_z(-) - 2\lambda_\sigma), \]
\[ \gamma_{\omega k,\sigma} = 1 - iLk_z(+). \]

(16)

The solution is normalized to have the asymptotic form at \( z = \infty \),

\[ \phi_{\omega k,\sigma}(z) = \frac{e^{ik_z(+)z}}{\sqrt{2k_z(+)}}. \]

(17)

From the transformation formula [24], the other asymptotic form at \( z = -\infty \) is given by

\[ \phi_{\omega k,\sigma}(z) = A_{\omega k,\sigma}e^{ik_z(-)z} + B_{\omega k,\sigma}e^{-ik_z(-)z}. \]

(18)

where the incident and the reflection coefficients are

\[ A_{\omega k,\sigma} = \frac{\Gamma(\gamma_{\omega k,\sigma})\Gamma(\beta_{\omega k,\sigma} - \alpha_{\omega k,\sigma})}{\Gamma(\beta_{\omega k,\sigma})\Gamma(\gamma_{\omega k,\sigma} - \alpha_{\omega k,\sigma})}, \]
\[ B_{\omega k,\sigma} = \frac{\Gamma(\gamma_{\omega k,\sigma})\Gamma(\alpha_{\omega k,\sigma} - \beta_{\omega k,\sigma})}{\Gamma(\alpha_{\omega k,\sigma})\Gamma(\gamma_{\omega k,\sigma} - \beta_{\omega k,\sigma})}. \]

(19)

The Bogoliubov coefficient, \( \mu_{\omega k,\sigma} = A_{\omega k,\sigma}/B_{\omega k,\sigma} \), from the group velocity can be written as

\[ \mu_{\omega k,\sigma}^* = \frac{\Gamma(1/2) + i\Delta_{\omega k,\sigma}(-)}{\Gamma(1/2) + i\Omega_{\omega k,\sigma}(-)} \times \frac{\Gamma(1/2) + i\Delta_{\omega k,\sigma}(+)}{\Gamma(1/2) + i\Omega_{\omega k,\sigma}(+)}, \]

(20)

where

\[ \Omega_{\omega k,\sigma}^{(+)} = Lk_z(+) + Lk_z(-) \pm 2\lambda_\sigma, \]
\[ \Delta_{\omega k,\sigma}^{(+)} = Lk_z(-) - Lk_z(+) \pm 2\lambda_\sigma. \]

(21)

Here, we have deleted the term, \( \Gamma(-iLk_z(-))/\Gamma(iLk_z(-)) \), which is independent of the interaction with the electric field and is removed through normalization. Note that \( \Omega_{\omega k,\sigma}^{(+)} \Omega_{\omega k,\sigma}^{(+)} > 0 \) and \( \Omega_{\omega k,\sigma}^{(-)} \Omega_{\omega k,\sigma}^{(-)} < 0 \).

Now, we compute the effective action in Eq. 8. For that purpose, we follow the method of Ref. I, where we use the integral representation of the gamma function [25], sum over two spin states, \( \sigma = \pm 1/2 \), do the contour integral of the first term in the first quadrant and that of the second term in the fourth quadrant, and subtract the divergent terms, which is equivalent to renormalizing the vacuum energy and the charge. Finally, we obtain the exact one-loop effective action of scalar QED per unit time and per unit cross-sectional area

\[ \mathcal{L}_{\text{eff}}^\text{sc} = \frac{1}{2} \int \frac{d\omega d^2k_{\perp}}{(2\pi)^3} \int_0^{\infty} \frac{ds}{s} \left( e^{\Omega_{\omega k,\sigma}^{(-)}s} - e^{\Delta_{\omega k,\sigma}^{(-)}s} + e^{-\Omega_{\omega k,\sigma}^{(+)}s} - e^{-\Delta_{\omega k,\sigma}^{(+)}s} \right) \left( \frac{1}{\sin(s)} - \frac{1}{s} - \frac{s}{6} \right), \]

(22)

and that of spinor QED

\[ \mathcal{L}_{\text{eff}}^\text{sp} = -\frac{1}{2} \int \frac{d\omega d^2k_{\perp}}{(2\pi)^3} \int_0^{\infty} \frac{ds}{s} \left( e^{\Omega_{\omega k,\sigma}^{(+)}s} - e^{\Delta_{\omega k,\sigma}^{(+)}s} + e^{-\Omega_{\omega k,\sigma}^{(-)}s} - e^{-\Delta_{\omega k,\sigma}^{(-)}s} \right) \left( \cot(s) - \frac{1}{s} + \frac{s}{3} \right), \]

\[ -i \int \frac{d\omega d^2k_{\perp}}{(2\pi)^3} \ln \left[ \frac{\sinh(\pi\Omega_{\omega k,\sigma}^{(+)}/2) \sinh(\pi\Omega_{\omega k,\sigma}^{(-)}/2)}{\sinh(\pi\Delta_{\omega k,\sigma}^{(+)}/2) \sinh(\pi\Delta_{\omega k,\sigma}^{(-)}/2)} \right]. \]

(23)
Here, the integration is restricted to \( \int dw = \int_{qE_0 L - m}^{qE_0 L - m} dw \) and \( \int d^2k_\perp = 2\pi \int_0^{k_{\perp,\text{max}}} k_{\perp,\text{max}} dk_{\perp,\text{max}} \). It can be shown that the general relation between the vacuum persistence (twice of the imaginary part) and the mean number of produced pairs holds in scalar and spinor QED

\[
2\text{Im}(\mathcal{L}_\text{eff}) = (-1)^{2|\sigma|} \int \frac{d\omega d^2 k_\perp}{(2\pi)^3} \ln(1 + (-1)^{2|\sigma|} N_{\omega k_\perp \sigma}),
\]

where

\[
N_{\omega k_\perp \sigma} = \frac{2 \sinh(\pi L k_\perp^{(+)}) \sinh(\pi L k_\perp^{(-)})}{\cosh(2\pi \lambda_\rho) + (-1)^{2\sigma} \cosh(\pi L k_\perp^{(+)} - \pi L k_\perp^{(-)})}.
\]

A few comments are in order. First, the mean number of produced pairs, Eq. (25), agrees with the exact result of Refs. \[11, 13\] and also Ref. \[26\] for scalar QED. The effective action in a constant electric field can be obtained by taking \( L = \infty \). In fact, the term \( e^{i\omega k_\perp^{(+)} \cdot z} \) yields the constant field limit while all the other terms vanish. The leading term of Eq. (25), \( N_{\omega k_\perp \sigma} \approx e^{-\pi(2\lambda_\rho - Lk_\perp^{(+)} - Lk_\perp^{(-)})} \), agrees with Eq. (36) of Ref. \[18\] from the instanton action. Second, it would be interesting to compare the effective actions \[22\] and \[23\] with Eqs. (66) and (80) for \( E(t) = E_0 \text{sech}(t/\tau) \) in Ref. I. The kinetic momenta \( k_{\perp,\sigma} \) along the direction of the electric field at spatial infinities now correspond to the kinetic energy \( \omega k_{\perp,\sigma} \) at the remote past and future. However, the different boundary conditions select different contours so that the mean number of produced pairs, Eq. (25), becomes the inverse of Eqs. (68) and (83) of Ref. I. This point will be further discussed in the next section.

### IV. APPROXIMATE EFFECTIVE ACTIONS IN GENERAL ELECTRIC FIELDS

We now put forth an approximation method for the effective action in general electric fields, which cannot be solved exactly. In a general Coulomb potential \( A_0(z) \), the Fourier component of the field equation \[11\] is given by

\[
\left[ \frac{d^2}{dz^2} + Q_{\omega k_\perp \sigma}(z) \right] \varphi_{\omega k_\perp \sigma}(z) = 0,
\]

where

\[
Q_{\omega k_\perp \sigma}(z) = (\omega + qA_0(z))^2 - (m^2 + k_\perp^2) + 2i\sigma qE(z).
\]

Here \( |\omega + qA_0(\pm \infty)| \geq m \) and the maximum transverse momentum \( k_{\perp,\text{max}} = \min\{(\omega + qA_0(\pm \infty))^2 - m^2\} \). Our stratagem is to transform Eq. \[26\] into the differential equation whose solution can be found approximately. In Ref. \[2\] the uniform semiclassical approximation is used for general time-dependent electric fields. The uniform semiclassical approximation \[27\] is an extension of the Liouville-Green transformation \[28\] to the following form

\[
\left[ \frac{d^2}{d\eta^2} + \eta^2 - \frac{S_{\omega k_\perp \sigma}}{\pi} \right] \varphi_{\omega k_\perp \sigma}(\eta) = 0,
\]

where \( \varphi_{\omega k_\perp \sigma}(\eta) = \sqrt{d\eta/dz} \varphi_{\omega k_\perp \sigma}(z) \) and

\[
\left( \eta^2 - \frac{S_{\omega k_\perp \sigma}}{\pi} \right) \left( \frac{d\eta}{dz} \right)^2 = Q_{\omega k_\perp \sigma}(z).
\]

Doing a contour integral exterior to the branch cut \[29\], \( S_{\omega k_\perp \sigma} \) turns out to be the instanton action in \( E(z) \) \[18\]:

\[
S_{\omega k_\perp \sigma} = -i \int dz \sqrt{Q_{\omega k_\perp \sigma}(z)}.
\]

The electric field studied in this paper is either constant or spatially localized such that \( A_0(z) \propto z^{1-c} \) with \( c \geq 0 \) for \( |z| \gg 1 \) and less singular than \( 1/z \) at finite \( z \). Then the correction term, the last term in Eq. \[28\], is asymptotically proportional to \( 1/\eta^2 \) and may be neglected in this approximation scheme. Thus, we approximately find the transmitted wave in terms of the parabolic cylinder function

\[
\varphi_{\omega k_\perp \sigma}(\eta) = D_p(\sqrt{2}\eta^{1/4}), \quad p = -\frac{1}{2} + \frac{i}{2\pi} S_{\omega k_\perp \sigma},
\]
which has the asymptotic form $D_p(\sqrt{2}e^{i\pi/4}\eta) \propto \eta^p e^{-in^2/2}$ for $\eta \gg 1$. On the other hand, for $\eta \ll -1$ the solution \ref{31} has another asymptotic form
\begin{equation}
\varphi_{\omega k_\perp \sigma}(\eta) = \frac{\sqrt{2\pi}e^{-i(p+1)\pi/2}}{\Gamma(-p)} D_{-(p+1)}(\sqrt{2}e^{i3\pi/4}\eta) + e^{-ip\sigma}D_p(\sqrt{2}e^{i5\pi/4}\eta),
\end{equation}
and the ratio of the reflection coefficient to the flux is the Bogoliubov coefficient
\begin{equation}
\mu_{\omega k_\perp \sigma} = \frac{\sqrt{2\pi}}{\Gamma(-p)} e^{i(p-1)\pi/2}.
\end{equation}

Following the procedure in Sec. III and doing the contour integral in the fourth quadrant, we obtain the approximate effective action
\begin{equation}
\mathcal{L}_{\text{eff}} = \frac{(-1)^{2|\sigma|}}{2} \sum_\sigma \int \frac{d\omega d^2k_\perp}{(2\pi)^3} \left[ \mathcal{P} \int_0^\infty \frac{ds}{s} \frac{e^{-sS_{\omega k_\perp \sigma}/\pi}}{\sin(s)} - i \sum_{n=1}^\infty \frac{(-1)^n}{n} e^{-nsS_{\omega k_\perp \sigma}} \right].
\end{equation}
Here dots denote the terms to regularize the vacuum energy and the charge. For a constant electric field, $A_0(z) = -E_0 z$ and $S_{\omega k_\perp \sigma} = \pi(m^2 + k_\perp^2 - 2i\sigma q E_0)/q E_0$, so the effective action \ref{33} recovers the Heisenberg-Euler effective action. For $E(z) = E_0 \text{sech}^2(z/L)$, $S_{\omega k_\perp \sigma} \approx -\pi \Omega_{\omega k_\perp \sigma}^{(\perp)} = 1/2 + 2\pi i \sigma$ with $\Omega_{\omega k_\perp \sigma}^{(\perp)} < 0$, and Eq. \ref{33} becomes for scalar QED
\begin{equation}
\mathcal{L}_{\text{eff}} = \frac{1}{2} \int \frac{d\omega d^2k_\perp}{(2\pi)^3} \left[ \mathcal{P} \int_0^\infty \frac{ds}{s} e^{-\frac{1}{2} S_{\omega k_\perp \sigma}} \left( \frac{1}{\sin(s)} - \frac{1}{s} - \frac{s}{6} \right) + i \ln(1 + e^{\frac{1}{2} \pi i S_{\omega k_\perp \sigma}}) \right],
\end{equation}
and for spinor QED
\begin{equation}
\mathcal{L}_{\text{eff}} = -\frac{1}{2} \int \frac{d\omega d^2k_\perp}{(2\pi)^3} \left[ \mathcal{P} \int_0^\infty \frac{ds}{s} e^{-\frac{1}{2} S_{\omega k_\perp \sigma}} \left( \cot(s) - \frac{1}{s} + \frac{s}{3} \right) + i \ln(1 - e^{-\frac{1}{2} \pi i S_{\omega k_\perp \sigma}}) \right].
\end{equation}
The results, \ref{35} and \ref{36}, are consistent with the leading terms of Eqs. \ref{22} and \ref{23}, respectively.

The approximation method can also be applied to the time-dependent electric fields $E(t)$. The Fourier component of the field equation in the gauge field $A_\sigma(t)$ takes the form
\begin{equation}
\left[ \frac{d^2}{dt^2} + Q_{k\sigma}(t) \right] \phi_{k\sigma}(t) = 0,
\end{equation}
where
\begin{equation}
Q_{k\sigma}(t) = (k_z + q A_\sigma(t))^2 + (m^2 + k_\perp^2) + 2i\sigma q E(t).
\end{equation}
Changing the variable
\begin{equation}
(\xi^2 + \frac{S_{k\sigma}}{\pi}) \left( \frac{d\xi}{dt} \right)^2 = Q_{k\sigma}(t),
\end{equation}
and introducing the instanton action \ref{18}
\begin{equation}
S_{k\sigma} = i \int dt \sqrt{Q_{k\sigma}(t)},
\end{equation}
and finally doing the contour integral in the first quadrant, we approximately obtain the effective actions in Sec. III of Ref. I. There the only modification is the parameter $p = -1/2 - i S_{k\sigma}/2\pi$. Thus, the leading contribution to the effective action is the same as Eq. \ref{33} with $S_{k\sigma}$ replacing $S_{\omega k_\perp \sigma}$.

A passing remark is that the approximation method based on the Liouville-Green transformation not only provides the effective action \ref{33} but also explains how the instanton actions \ref{30} and \ref{40} determine the mean number of the produced pairs either in spatially localized electric fields or in pulsed electric fields, as shown in Ref. \ref{18}. The different boundary conditions for electric fields $E(z)$ and $E(t)$, which are imprinted in the parameters $p = -1/2 \pm i S_{k\sigma}/2\pi$, requires contours in the fourth and first quadrant, respectively. As a consequence, the duality approximately holds between $E(z)$ and $E(t)$ for the same form. Further, the mean numbers of produced pairs for $E(z) = E_0 \text{sech}^2(z/L)$ and $E(t) = E_0 \text{sech}^2(t/\tau)$ are inverse to each other in the form.
V. CONCLUSION

In this paper we have further developed the regularization scheme using the Bogoliubov coefficient in Ref. I to obtain the effective action in $E(z) = E_0 \text{sech}^2(z/L)$. The Klein paradox due to tunneling barrier makes the boundary condition on the field equation in a space-dependent gauge differ from that in a time-dependent gauge. This is resolved by the causality argument, which requires the reflection and the transmission probabilities not by the flux but by the group velocity or signal. The Bogoliubov coefficient is then given by the ratio of the incident coefficient to the reflection one with respect to the flux, which is the reason why the mean number of produced pairs in $E(z) = E_0 \text{sech}^2(z/L)$ is inverse of that in $E(t) = E_0 \text{sech}^2(t/\tau)$.

We have also introduced an approximation method for general electric fields, which is based on the Liouville-Green transformation that changes the field equation into a solvable one, for instance, the parabolic equation with correction terms. The method can be applied both to spatially localized electric fields $E(z)$ and to pulsed electric fields $E(t)$. Remarkably, the leading contributions are determined by the instanton actions, which confirm the mean number of produced pairs in the phase-integral approximation [18]. Further, the leading contribution to the effective actions show duality between the space-dependent and time-dependent Sauter type electric fields and the duality seems to be generic for $E(z)$ and $E(t)$ of the same form at this approximation. However, whether this exists the exact duality remains an open question.

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