AN ANALYSIS OF A BROKEN $P_1$-NONCONFORMING FINITE ELEMENT METHOD FOR INTERFACE PROBLEMS

DO Y. KWAK†, KYE T. WEE†, AND KWANG S. CHANG†

Abstract. We study some numerical methods for solving a second order elliptic problem with interface. We introduce an immersed finite element method based on the “broken” $P_1$-nonconforming piecewise linear polynomials on interface triangular elements having edge averages as degrees of freedom. These linear polynomials are broken to match the homogeneous jump condition along the interface which is allowed to cut through the element. We prove optimal orders of convergence in the $H^1$- and $L^2$-norm. Next we propose a mixed finite volume method in the context introduced in [S. H. Chou, D. Y. Kwak, and K. Y. Kim, Math. Comp., 72 (2003), pp. 525–539] using the Raviart–Thomas mixed finite element and this “broken” $P_1$-nonconforming element. The advantage of this mixed finite volume method is that once we solve the symmetric positive definite pressure equation (without Lagrangian multiplier), the velocity can be computed locally by a simple formula. This procedure avoids solving the saddle point problem. Furthermore, we show optimal error estimates of velocity and pressure in our mixed finite volume method. Numerical results show optimal orders of error in the $L^2$-norm and broken $H^1$-norm for the pressure and in the $H(\text{div})$-norm for the velocity.

Key words. immersed finite element method, $P_1$-nonconforming finite element method, uniform grid, mixed finite volume method, average degrees of freedom

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1. Introduction. There are many physical problems where the underlying partial differential equations have an interface. For example, second order elliptic equations with discontinuous coefficients are often used to model problems in material sciences and porous media when two or more distinct materials or media with different conductivities, densities, or permeability are involved. The solution of these interface problems must satisfy interface jump conditions due to conservation laws.

If the interface is smooth enough, then the solution of the interface problem is also smooth in individual regions where the coefficient is smooth, but due to the jump of the coefficient across the interface, the global regularity is usually low, and the solution usually belongs to $H^{1+\alpha}(\Omega)$ for some $0 \leq \alpha < 1$. Because of the low global regularity, achieving accuracy is difficult with standard finite element methods (FEMs), unless the elements fit with the interface of general shape.

The immersed interface method using uniform grids has many advantages over the usual fitted grid method. Using uniform grid, one does not need to generate a grid. This is quite convenient in several aspects. First of all, the structure of the stiffness matrix is the same as that of the standard FEM, where many known efficient solvers can be exploited. Second, when a moving interface problem is involved one does not need to generate a new grid as time evolves. This saves considerable amount of time and storage.
The first attempt to avoid a fitted grid for the interface problem was made by LeVeque and Li [32], where they proposed an immersed interface method for the finite difference method where the jump condition was properly incorporated in the scheme. Cartesian grids are most natural in this case. They subsequently applied the same idea to other interface problems such as the Stokes flow problem, one-dimensional moving interface problem, and Hele–Shaw flow, etc. [26, 33, 34, 35]. For problems with sharp edges, Yu, Zhou, and Wei [47] have developed matched interface boundary methods which also handle jumps in the solutions. The resulting linear systems from these methods are nonsymmetric and indefinite even when the original problem is self-adjoint and uniformly elliptic. Although these methods were demonstrated to be very effective, convergence analysis of related finite difference methods are extremely difficult and are still open.

For FEMs, Li and Ito [37], Li et al. [38], Li, T. Lin, and Wu [39], and T. Lin et al. [40] recently studied an immersed FEM using uniform grid, and they proved the approximation property of the finite element space of their scheme. Their numerical examples demonstrated optimal orders of the error. Other related works in this direction can be found in [10, 24, 25, 31, 36, 41, 43, 46] and references therein. In particular, in [25], they studied matrix coefficient problems, while a three-dimensional finite volume scheme is introduced in [43]. Both cases show close to second order convergence.

On the other hand, the $P_1$-nonconforming FEM introduced in [19] for solving Stokes equation is being widely used in solving elliptic equations and is shown to be quite effective [29, 15, 19, 27]. Especially, it is extremely useful in solving the mixed FEM by hybridization [1, 2] or finite volume formulation [29, 15, 16, 18].

The mixed FEM based on the dual formulation is well known [5, 6, 7, 8, 12, 20, 22, 23, 45]. The motivation of the mixed method is to obtain an accurate approximation of the flow variable and has been widely used in the study of flow in porous media such as petroleum engineering, underground water flow, and electrodynamics, etc. (e.g. [13, 21, 44]). But this scheme leads to a saddle point problem for which many well-known fast iterative methods fail. To overcome this difficulty, mixed hybrid methods have been introduced [2, 8, 42] where the problem reduces to a symmetric positive definite system in Lagrange multiplier only. The flow and pressure variables are obtained via some postprocessing.

Recently, there has been some development of the mixed FEM in another direction: A mixed finite volume method was proposed in [18] and extended in [29, 15]. In this method, one uses Raviart–Thomas space and $P_1$-nonconforming space as trial spaces for velocity and pressure and integrates the mixed system of equation on each volume. Then one can eliminate the velocity variable and obtain the equation of pressure variable only (in terms of $P_1$-nonconforming FEM) directly from the formulation without using a Lagrange multiplier. The resulting linear system is again symmetric positive definite, and velocity can be recovered from pressure locally in a simple manner.

The purpose of this paper is twofold. First, we propose a FEM on a uniform triangular grid using “broken” $P_1$-functions having degrees of freedom on edges. This is a Galerkin-type $P_1$-nonconforming FEM with the basis functions having the average on edges as degrees of freedom, broken along the interface to match the flux condition. Then we show optimal error estimates in $H^1, L^2$-norms. Here, we emphasize that the meaning of “nonconforming” is different from the context of Li, Lin, and Wu [39] and Li and Ito [37] where the basis function has degrees of freedom at vertices, discontinuous along edges of interface elements. Meanwhile, the basis functions here
are Crouzeix–Raviart-type [19]. Hence it is discontinuous along all edges intrinsically. Furthermore, since we use the average of linear function (possibly broken) along edges as degrees of freedom, the overhead of dealing with nonconformity in the proof of error estimate is significantly reduced. (See section 3.)

Next, we propose a mixed finite volume method using Raviart–Thomas space and the immersed finite element introduced above. This is similar to the scheme studied in [29], but the usual nonconforming basis function is replaced by a “broken” one on the interface element. We provide an optimal error analysis of pressure and velocity.

The rest of the paper is organized as follows. In the next section, we will describe the model problem and some preliminaries. We construct an immersed $P_1$-nonconforming space with average degrees of freedom which preserves flux continuity weakly along the interface, and we prove an interpolation error estimate. In sections 3 and 4, we propose an immersed finite element scheme and prove $H^1$- and $L^2$-error estimates. In section 5, we propose a mixed finite volume method using a Raviart–Thomas mixed finite element and our $P_1$-nonconforming immersed FEM, where the problem reduces to a symmetric positive definite system in pressure variables. The velocity can be computed locally after pressure computation. Finally, in section 6, some numerical results involving variable coefficients and sharp edge are presented which indicate optimal orders convergence of our methods.

2. Preliminaries. Let $\Omega$ be a convex polygonal domain in $\mathbb{R}^2$ which is separated into two subdomains $\Omega^+$ and $\Omega^-$ by a $C^2$-interface $\Gamma = \partial \Omega^- \subset \Omega$, with $\Omega^+ = \Omega \setminus \Omega^-$. We consider the following elliptic interface problem

\[
\begin{cases}
-\text{div}(\beta \nabla p) = f & \text{in } \Omega \setminus \Gamma, \\
p = 0 & \text{on } \partial \Omega,
\end{cases}
\]

with the jump conditions on the interface

\[
[p] = 0, \quad \left[\beta \frac{\partial p}{\partial n}\right] = 0 \quad \text{across } \Gamma,
\]

where $f \in L^2(\Omega)$ and $p \in H^1_0(\Omega)$. We assume that the coefficient $\beta$ is a positive function bounded below and above by two positive constants.

We take as usual the weak formulation of the following interface problem: Find $p \in H^1_0(\Omega)$ such that

\[
\int_{\Omega} \beta \nabla p \cdot \nabla q \, dx = \int_{\Omega} f q \, dx \quad \forall q \in H^1_0(\Omega).
\]
Now we introduce the space
\[ \tilde{H}^m(\Omega) := \{ p \in H^{m-1}(\Omega) : p \in H^m(\Omega^s), s = +, - \} \]
equipped with the norm
\[ \| p \|_{\tilde{H}^m(\Omega)}^m := \| p \|_{H^{m-1}(\Omega)}^m + \| p \|_{H^m(\Omega^+)}^m + \| p \|_{H^m(\Omega^-)}^m \quad \forall p \in \tilde{H}^m(\Omega), \]
where for \( m = 1, 2, H^m(\Omega^s) = W^{2m}_0(\Omega^s) \) is the usual Sobolev space of order \( m \). By the Sobolev embedding theorem, for any \( p \in \tilde{H}^2(\Omega) \), we have \( p \in W^1_s(\Omega) \forall s > 2 \). Then we have the following regularity theorem for the weak solution \( p \) of the variational problem (2.3); see [4] and [30].

**Theorem 2.1.** The variational problem (2.3) has a unique solution \( p \in \tilde{H}^2(\Omega) \) which satisfies for some constant \( C > 0 \)
\[ \| p \| \tilde{H}^2(\Omega) \leq C \| f \| L^2(\Omega). \]

We now describe an immersed FEM with piecewise \( P_1 \)-nonconforming functions.

For the simplicity of presentation, we assume that \( \Omega \) is a rectangular domain. First we consider uniform rectangular partitions of mesh size \( h \). Then we obtain triangular partitions \( T_h \) by cutting the elements along diagonals. Thus we allow the interface \( \Gamma \) to cut through the elements. We assume the following situation: The interface

- meets the edges of an interface element at no more than two points;
- meets each edge at most once, except possibly it passes through two vertices.

These assumptions are reasonable if we choose \( h \) sufficiently small.

We call an element \( T \in T_h \) an interface element if the interface \( \Gamma \) passes through the interior of \( T \); otherwise we call \( T \) a noninterface element. (If one of the edges is part of the interface, then the element is a noninterface element.) Let \( \mathcal{E}_h \) be a collection of all edges of \( T_h \).

Let \( \overline{DE} \) be the line segment connecting the intersections of the interface and the edges of a triangle \( T \). This line segment divides \( T \) into two parts \( T^+ \) and \( T^- \) with \( T = T^+ \cup T^- \cup \overline{DE} \). There is a small region in \( T \) such that \( T^e = T - (\Omega^+ \cap T^+) - (\Omega^- \cap T^-) \) (see Figure 2.2). Since \( \overline{DE} \) can be considered as an approximation of the \( C^2 \)-curve \( \Gamma \cap T \), the interface is perturbed by an \( O(h^2) \) term. From [4, 14], one can see for the interpolation polynomial defined below, such a perturbation will affect only the interpolation error to the order of \( h^2 \).

As usual, we want to construct local basis functions on each element \( T \) of the partition \( T_h \). For a noninterface element \( T \in T_h \), we simply use the standard linear shape functions on \( T \) having degrees of freedom at the midpoints of the edges and use \( \mathcal{S}_h(T) \) to denote the linear spaces spanned by the three nodal basis functions on \( T \). Let \( m_i, i = 1, 2, 3 \) be the midpoints of edges of \( T \). Then
\[ \mathcal{S}_h(T) = \text{span}\{ \phi_i : \phi_i \text{ is linear on } T \text{ and } \phi_i(m_j) = \delta_{ij}, i, j = 1, 2, 3 \}. \]

Alternatively, we can use average values along edges \( e_j \) of \( T \) as degrees of freedom; i.e., \( \phi_i \) can be defined by \( \frac{1}{|e_j|} \int_{e_j} \phi_i ds = \delta_{ij}, i, j = 1, 2, 3 \).

For this space, we have the following well-known approximation property [17, 19]:
\[ \| p - I_h p \|_{L^2(T)} + h \| p - I_h p \|_{H^1(T)} \leq C h^2 \| p \|_{H^2(T)}, \]
where $I_h : H^2(T) \rightarrow S_h(T)$ is the interpolation operator. Finally, we use $S_h(\Omega)$ to denote the space of the standard piecewise $P_1$-nonconforming space with vanishing boundary nodal values.

2.1. Local basis functions on an interface element. We now consider a typical interface element $T$ whose geometric configuration is given in Figure 2.2 in which the curve between points $D$ and $E$ is part of the interface. Let $e_i, i = 1, 2, 3$, be the edges of $T$. For $\phi \in H^1(T)$, let $\bar{\phi}_{e_i}$ denote the average of $\phi$ along $e_i$, i.e.,

$$\bar{\phi}_{e_i} := \frac{1}{|e_i|} \int_{e_i} \phi \, ds.$$ 

We construct a piecewise linear function of the form

$$\phi(X) = \begin{cases} 
\phi^+(X) = a_0 + b_0x + c_0y, & X = (x, y) \in T^+, \\
\phi^-(X) = a_1 + b_1x + c_1y, & X = (x, y) \in T^-,
\end{cases}$$

(2.7) satisfying

$$\bar{\phi}_{e_i} = V_i, \quad i = 1, 2, 3,$$

(2.8)

$$\phi^+(D) = \phi^-(D), \quad \phi^+(E) = \phi^-(E), \quad \beta^+ \frac{\partial \phi^+}{\partial n_{\Gamma E}} = \beta^- \frac{\partial \phi^-}{\partial n_{\Gamma E}},$$

(2.9)

where $V_i, i = 1, 2, 3$, are given values, $n_{\Gamma E}$ is the unit normal vector on the line segment $DE$, and $\beta^+, \beta^-$ are averages along $DE$. This is a piecewise linear function on $T$ that satisfies the homogeneous jump conditions along $DE$.

Suppose that a typical reference interface element $T$ has vertices at $A(0, 0), B(1, 0), C(0, 1)$. We assume that the interface meets with the edges at $D(x_0, 0)$ and $E(0, y_0)$, where $0 < x_0, y_0 \leq 1$. Then the unit normal vector to the interface is $n_{\Gamma E} = (y_0, x_0)/\sqrt{x_0^2 + y_0^2}$.

**Theorem 2.2.** Given a reference interface triangle, the piecewise linear function $\phi(x, y)$ defined by (2.7)-(2.9) is uniquely determined by three conditions

$$\bar{\phi}_{e_i} = V_i, \quad i = 1, 2, 3.$$

**Proof.** Let $X = (x, y)^T \in T$. Since $\phi^+$ and $\phi^-$ are linear functions, we have

$$\phi(X) = \begin{cases} 
\phi^+(X) = a_0 + b_0x + c_0y, & X \in T^+, \\
\phi^-(X) = a_1 + b_1x + c_1y, & X \in T^-.
\end{cases}$$

(2.10)
The condition (2.8) gives the following three equations:

\begin{align}
\phi_{e_1} &= a_0 + \frac{1}{2}b_0 + \frac{1}{2}c_0 = V_1, \\
\phi_{e_2} &= \int_{e_2} \phi \, ds = \int_{AE} \phi^- \, ds + \int_{EC} \phi^+ \, ds \\
&= \left(a_1 + \frac{y_0}{2}c_1\right)y_0 + \left(a_0 + \frac{y_0 + 1}{2}c_0\right)(1 - y_0) = V_2, \\
\phi_{e_3} &= \left(1 + \frac{x_0}{2}b_1\right)x_0 + \left(a_0 + \frac{x_0 + 1}{2}b_0\right)(1 - x_0) = V_3.
\end{align}

where we used midpoint quadrature on \(AE\) and \(EC\). Similarly, we have

\begin{align}
\phi_{e_3} &= \left(1 + \frac{x_0}{2}b_1\right)x_0 + \left(a_0 + \frac{x_0 + 1}{2}b_0\right)(1 - x_0) = V_3.
\end{align}

From the continuity condition at \(D\) and \(E\), we have

\begin{align}
a_0 + b_0x_0 &= a_1 + b_1x_0, \\
a_0 + c_0y_0 &= a_1 + c_1y_0,
\end{align}

and the flux continuity condition along \(DE\) gives

\begin{align}
(b_0, c_0) \cdot (y_0, x_0) &= \rho(b_1, c_1) \cdot (y_0, x_0),
\end{align}

where \(\rho = \beta^- / \beta^+\) and we have used that the normal direction of the line segment \(DE\) is \((y_0, x_0)\).

Then the coefficient matrix of the above linear system for the unknowns \(a_0, b_0, c_0\) and \(a_1, b_1, c_1\) in this order is

\begin{align}
A = \begin{pmatrix}
1 & \frac{1}{2} & \frac{1}{2}(1 - y_0^2) & 0 & 0 & 0 \\
1 - y_0 & 0 & y_0 & 0 & \frac{1}{2}y_0^2 \\
1 - x_0 & \frac{1}{2}(1 - x_0^2) & 0 & x_0 & \frac{1}{2}x_0^2 \\
-1 & -x_0 & 0 & 1 & x_0 \\
-1 & 0 & -y_0 & 1 & 0 & y_0 \\
0 & -y_0 & -x_0 & 0 & \rho y_0 & \rho x_0
\end{pmatrix}.
\end{align}

Tedious calculation shows that the determinant of the matrix is

\begin{align}
det(A) = \frac{1}{4}(x_0^2 + y_0^2)\{\rho(x_0y_0 - 1) - x_0y_0\} < 0.
\end{align}

Thus the coefficients of (2.10) are uniquely determined.

Remark 2.1. If \(\phi_{e_1}, \phi_{e_2}, \text{ and } \phi_{e_3}\) have the same value, then the piecewise linear function \(\phi\) satisfying (2.8)–(2.9) reduces to a constant by uniqueness.

Now we can construct nodal basis functions on an interface element \(T\) in general position through affine mapping. We let \(S_h(T)\) denote the three-dimensional linear space spanned by these shape functions. We note that \(S_h(T)\) is a subspace of \(H^1(T)\). Finally, we define the immersed finite element space \(\hat{S}_h(\Omega)\) as the collection of functions such that

\begin{align}
\phi|_T &\in S_h(T) \text{ if } T \text{ is a noninterface element,} \\
\phi|_T &\in \hat{S}_h(T) \text{ if } T \text{ is an interface element,} \\
\int_{\partial T} \phi ds &= \int_{\partial T} \phi^+ ds \text{ if } T_1, T_2 \text{ are adjacent elements and } e \text{ is a common edge of } T_1 \text{ and } T_2, \\
\int_{\partial T} \phi ds &= 0 \text{ if } e \text{ is part of the boundary } \partial \Omega.
\end{align}
Although for functions in $\hat{S}_h(T)$ the flux jump condition is enforced on line segments, they actually satisfy a weak flux jump condition along the interface when $\beta$ is piecewise constant. This is stated in the following lemma [38], whose proof is a simple application of the divergence theorem.

**Lemma 2.3.** For an interface triangle $T$, every function $\phi \in \hat{S}_h(T)$ satisfies the flux jump condition on $\Gamma \cap T$ in the following weak sense:

\[
\int_{\Gamma \cap T} (\beta^- \nabla \phi^- - \beta^+ \nabla \phi^+) \cdot n_T \, ds = 0.
\]  

(2.19)

**Proof.** Let $\phi$ be any function in $\hat{S}_h(T)$. By the divergence theorem, we have

\[
\int_{\Gamma \cap T} (\beta^- \nabla \phi^- - \beta^+ \nabla \phi^+) \cdot n_T \, ds + \int_{DE} (\beta^- \nabla \phi^- - \beta^+ \nabla \phi^+) \cdot n_{DE} \, ds
\]

\[
= \int_T \text{div}(\beta^- \nabla \phi^- - \beta^+ \nabla \phi^+) \, dx = 0.
\]

By the flux continuity of $\phi$ on $DE$,

\[
\int_{DE} (\beta^- \nabla \phi^- - \beta^+ \nabla \phi^+) \cdot n_{DE} \, ds = 0,
\]

which completes the proof.  

\[\square\]

**2.2. Approximation property of nonconforming immersed space $\hat{S}_h(T)$.**

In this subsection, we would like to study the approximation property of $\hat{S}_h(T)$ by defining an interpolation operator. The difficulty lies in the fact that $\hat{S}_h(T)$ does not belong to $\tilde{H}^2(\Gamma \cap T)$, where $\tilde{H}^2(T) = H^1(T) \cap H^2(T \cap \Omega^+) \cap H^2(T \cap \Omega^-)$ (see Figure 2.3). To overcome the difficulty, we introduce a bigger space which contains both of these spaces.

For a given interface element $T$, we consider a function space $X(T)$ such that every $p \in X(T)$ satisfies

\[
\begin{align*}
\{ & p \in H^1(T) \cap H^2(T^+ \cap \Omega^+) \cap H^2(T^- \cap \Omega^-) \cap H^2(T_r^+) \cap H^2(T_r^-), \\
& \int_{\Gamma \cap T} (\beta^- \nabla p^- - \beta^+ \nabla p^+) \cdot n_T \, ds = 0,
\}
\end{align*}
\]

\[\text{Fig. 2.3. Regions of } H^2\text{-regularity.}\]
where $T^s_r = T_r \cap \Omega^s$, $s = +, -$. For any $p \in X(T)$, we define the following norms:

$$
\begin{align*}
|p|^2_{X(T)} &= |p|^2_{T - \cap \Omega^-} + |p|^2_{T^+ \cap \Omega^+} + |p|^2_{T_r^+}, \\
\|p\|^2_{\tilde{X}(T)} &= \|p\|^2_{T} + |p|^2_{\tilde{X}(T)}, \\
\|p\|_{2,T} &= \|p\|_{X(T)} + \sum_{i=1}^{3} |\bar{p}_{e_i}|,
\end{align*}
$$

where $\bar{p}_{e_i}$, $i = 1, 2, 3$, are the average on each edge $e_i$.

**Remark 2.2.** If $p \in \tilde{H}^2(\Omega)$, then $p|_T \in X(T)$ and $|p|_{X(T)} = |p|_{\tilde{H}^2(T)}$, where $|p|^2_{\tilde{H}^2(T)} = |p|^2_{T - \cap \Omega^-} + |p|^2_{T^+ \cap \Omega^+}$.

**Lemma 2.4.** $\|\cdot\|_{2,T}$ is a norm in the space $X(T)$, which is equivalent to $\|\cdot\|_{X(T)}$.

**Proof.** Let $p \in X(T)$. If $\|p\|_{2,T} = 0$, then $|p|_{X(T)} = 0$ and $|\bar{p}_{e_i}| = 0$, $i = 1, 2, 3$. Hence $p$ is linear on each of the four regions $T^+ \cap \Omega^+$, $T^- \cap \Omega^-$, $T^+_r$, and $T^{0}_r$. Since $p \in H^1(T)$, $p$ is linear on each $T^s$, $s = +, -$. Since $p$ satisfies flux continuity condition, $\tilde{S}_h(T)$. Now we apply Theorem 2.2 to conclude $p = 0$.

We now show the equivalence of $\|\cdot\|_{2,T}$ and $\|\cdot\|_{X(T)}$ (cf. [3, p. 77], [40]). First, note that by the Sobolev embedding theorem, $H^2(T_i)$ is compactly embedded in $W^1_2(T_i)$ for any $s > 2$, where $T_i \subset T$. So we see that $X(T) \subset W^1_2(T) \subset C^0(T)$. If $p \in X(T)$, then $p$ is a continuous function on $T$ and $|\bar{p}_{e_i}| \leq C||p||_{X(T)}$, $i = 1, 2, 3$; thus

$$
\|p\|_{2,T} \leq C||p||_{X(T)},
$$

where $C$ is independent of $p$.

Now suppose that the converse

$$
\|p\|_{X(T)} \leq C\|p\|_{2,T} \quad \forall p \in X(T)
$$

fails for any $C > 0$. Then there exists a sequence $\{p_k\}$ in $X(T)$ with

$$
\|p_k\|_{X(T)} = 1, \quad \|p_k\|_{2,T} \leq \frac{1}{k}, \quad k = 1, 2, \ldots
$$

Since $W^1_2(T)$, ($s > 2$) is compactly imbedded in $H^1(T)$ by the Kondrasov theorem [17, p. 114], there exists a subsequence of $\{p_k\}$ which converges in $H^1(T)$. Without loss of generality, we can assume that the sequence itself converges. Then $\{p_k\}$ is a Cauchy sequence in $H^1(T)$. Noting that $|p_k|_{X(T)} \to 0$ and $||p_k - p||_{2,T} \leq \|p_k - p\|_{X(T)} + (|p_k|_{X(T)} + |p|_{X(T)})^2$, we see that $\{p_k\}$ is a Cauchy sequence in $X(T)$. By completeness, it converges to an element $p^* \in X(T)$, and (2.20), (2.21) gives

$$
\|p^*\|_{2,T} \leq \|p^* - p_k\|_{2,T} + \|p_k\|_{2,T} \leq C||p^* - p_k||_{X(T)} + \frac{1}{k} \to 0.
$$

But

$$
\|p^*\|_{X(T)} = 1 \quad \text{and} \quad \|p^*\|_{2,T} = 0.
$$

This is a contradiction, since $\|p^*\|_{2,T} = 0$ implies $p^* = 0$.

For any $p \in X(T)$, we define $\{h_p \in \tilde{S}_h(T)$ using the average of $p$ on each edge by

$$
(T_h p)_{e_i} = \bar{p}_{e_i}, \quad i = 1, 2, 3.
$$
and we call $I_h p$ the interpolant of $p$ in $\hat{S}_h(T)$. We then define $I_h p$ for $p \in \tilde{H}^2(\Omega)$ by $(I_h p)|_T = I_h(p)|_T$.

**Lemma 2.5.** Let $T$ be an interface element. Then for any $p \in X(T)$, we have

$$
\|p - I_h p\|_{m,T} \leq Ch^{2-m}\|p\|_{X(T)}, \quad m = 0, 1,
$$

where $h$ is the mesh size of $T$.

**Proof.** Let $\hat{T}$ be a reference interface element. Then for any $\hat{p} \in X(\hat{T})$

$$
\|\hat{p} - I_h \hat{p}\|_{2,\hat{T}} = |\hat{p} - I_h \hat{p}|_{X(\hat{T})} + \sum_{i=1}^{3} |(\hat{p} - I_h \hat{p})_{e_i}|
$$

$$
= |\hat{p} - I_h \hat{p}|_{X(\hat{T})},
$$

where we used the fact that $\hat{p}_{e_i} = (I_h \hat{p})_{e_i}$ on each edge and the $H^2$-seminorm of the piecewise linear function $I_h \hat{p}$ vanishes. Applying the scaling argument for $m = 0, 1$, we have

$$
\|p - I_h p\|_{m,T} \leq Ch^{1-m}\|\hat{p} - I_h \hat{p}\|_{m,\hat{T}} \leq Ch^{1-m}\|\hat{p} - I_h \hat{p}\|_{2,\hat{T}} \leq Ch^{1-m}\|\hat{p}\|_{X(\hat{T})} \leq Ch^{2-m}\|p\|_{X(T)}.
$$

By the above lemma, Remark 2.2, and (2.6), we obtain the following interpolation estimate.

**Theorem 2.6.** For any $p \in \tilde{H}^2(\Omega)$, there exists a constant $C > 0$ such that

$$
\|p - I_h p\|_{L^2(\Omega)} + h\|p - I_h p\|_{1,h} \leq Ch^2\|p\|_{\tilde{H}^2(\Omega)},
$$

where $\|\cdot\|_{1,h} := \sum_{T \in T_h} \|\cdot\|_{1,T}$.

3. **Immersed FEM with “broken” $P_1$-nonconforming elements.** We are now ready to define our immersed FEM based on a “broken” $P_1$-nonconforming element: Find $p_h \in \hat{S}_h(\Omega)$ such that

$$
a_h(p_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in \hat{S}_h(\Omega),
$$

where

$$
a_h(p, \phi) = \sum_{T \in T_h} \int_T \beta \nabla p \cdot \nabla \phi \, dx \quad \forall p, \phi \in H_h(\Omega),
$$

$$
H_h(\Omega) : = H_0^1(\Omega) + \hat{S}_h(\Omega).
$$

Here, $H_h(\Omega)$ is endowed with the piecewise $H^1$-norm $\|\cdot\|_{1,h}$. Note that if the discrete Poincaré inequality holds, then noting that the bilinear operator $a_h(\cdot, \cdot)$ is bounded and coercive on $\hat{S}_h(\Omega)$, the discrete problem (3.1) has a unique solution $p_h \in \hat{S}_h(\Omega)$.

**Lemma 3.1** (discrete Poincaré inequality). There exists a constant $C > 0$ independent of $h$ such that for any $\phi \in \hat{S}_h(\Omega)$

$$
C\|\phi\|_{L^2(\Omega)}^2 \leq a_h(\phi, \phi).
$$

**Proof.** Let $e$ be the common edge of two adjacent elements $T_1$ and $T_2$. Note that since $\int_e \phi_1 \, ds = \int_e \phi_2 \, ds$, where $\phi_i = \phi|_{T_i}$, $i = 1, 2$, there exists a point $x_0 \in e$ such
that \( \phi_1(x_0) = \phi_2(x_0) \). Then a slight modification of Lemma 2.1 in [28] proves the inequality.

For the energy-norm error estimate, we need the well-known second Strang lemma, which is valid since \( a_h(\cdot, \cdot) \) is coercive.

**Lemma 3.2 (second Strang lemma).** If \( p \in \tilde{H}^2(\Omega) \), \( p_h \in \tilde{S}_h(\Omega) \) are the solutions of (2.3) and (3.1), respectively, then there exists a constant \( C > 0 \) such that

\[
(3.4) \quad ||p - p_h||_{1,h} \leq C \left\{ \inf_{q_h \in \tilde{S}_h(\Omega)} ||p - q_h||_{1,h} + \sup_{\phi_h \in \tilde{S}_h(\Omega)} \frac{|a_h(p, \phi_h) - (f, \phi_h)|}{\|\phi_h\|_{1,h}} \right\}.
\]

We shall need the following estimate: see Lemma 3 in [19].

**Lemma 3.3.** Let \( e \) be an edge of \( T \). Then there exists a constant \( C > 0 \) such that for all \( \phi, v \in H^1(T) \)

\[
\left| \int_e \phi(v - \overline{v}_e) \, ds \right| \leq Ch|\phi|_{1,T}|v|_{1,T},
\]

where \( \overline{v}_e := \frac{1}{h} \int_e v \, ds \).

**Remark 3.1.** This lemma also holds when \( \phi \) belongs to \( H^1(T^s), s = \pm \) with \( |\phi|_{1,T} \) understood as a sum of piecewise norm \( |\phi|_{1,T^{\pm}} \).

**Theorem 3.4.** Let \( p \in \tilde{H}^2(\Omega) \), \( p_h \in \tilde{S}_h(\Omega) \) be the solutions of (2.3) and (3.1), respectively. Then there exists a constant \( C > 0 \) such that

\[
(3.5) \quad ||p - p_h||_{1,h} \leq Ch||p||_{\tilde{H}^2(\Omega)}.
\]

**Proof.** We use the second Strang lemma. The first term is nothing but an approximation error. By Theorem 2.6, we have

\[
(3.6) \quad \inf_{q_h \in \tilde{S}_h(\Omega)} ||p - q_h||_{1,h} \leq Ch||p||_{\tilde{H}^2(\Omega)}.
\]

For the consistency error, we have from the definition of \( a_h(\cdot, \cdot) \) and Green’s formula

\[
a_h(p, \phi_h) - (f, \phi_h) = \sum_{T \in T_h} \int_T \beta \nabla p \cdot \nabla \phi_h \, dx - \int_\Omega f \phi_h \, dx \]
\[
= \sum_{T \in T_h} \int_T \beta \nabla p \cdot \nabla \phi_h \, dx \]
\[
- \left( \sum_{T \in T_h} \int_T \beta \nabla p \cdot \nabla \phi_h \, dx - \sum_{T \in T_h} \beta \frac{\partial p}{\partial n}, \phi_h > \partial T \right) \]
\[
= \sum_{T \in T_h} \beta \frac{\partial p}{\partial n}, \phi_h > \partial T,
\]

where \( \phi_h \in \tilde{S}_h(\Omega) \) and \( n \) is a unit outward normal vector on each \( \partial T \). Since \( \beta \frac{\partial p}{\partial n} \) belongs to \( H^1(T \cap \Omega^+) \cap H^1(T \cap \Omega^-) \) and \( \phi_h \in \tilde{S}_h(\Omega) \) has well-defined average value on the interior edges and vanishing average on the boundary, we have by Lemma 3.3.

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and Remark 3.1 that
\[
\sum_{T \in T_h} \beta \frac{\partial p}{\partial n}, \phi_h > \partial T = \sum_{T \in T_h} \sum_{e \in \partial T} \beta \frac{\partial p}{\partial n} - \left( \beta \frac{\partial p}{\partial n} \right)_e, \phi_h > e
\]
\[
\leq \sum_{T \in T_h} Ch |\beta \frac{\partial p}{\partial n}|_{1,T} |\phi_h|_{1,T}
\]
(3.8)
\[
\leq Ch \|p\|_{\tilde{H}^2(\Omega)} \|\phi_h\|_{1,h}.
\]
This completes the proof. \(\square\)

4. \(L^2\)-error estimate. We now apply the duality argument to obtain the \(L^2\)-norm estimate of the error. Let us consider an auxiliary problem: Given \(g \in L^2(\Omega)\), find \(\varphi \in \tilde{H}^2(\Omega)\) such that
\[
\begin{align*}
-\div(\beta \nabla \varphi) &= g & \text{in } \Omega \setminus \Gamma, \\
\varphi &= 0 & \text{on } \partial \Omega,
\end{align*}
\]
with jump conditions \([u] = 0, \ [\beta \frac{\partial u}{\partial n}] = 0\) across \(\Gamma\). Then we have
\[
\|\varphi\|_{\tilde{H}^2(\Omega)} \leq C \|g\|_{L^2(\Omega)}.
\]

Let \(\varphi_h \in \tilde{S}_h(\Omega)\) be the solution of the corresponding variational problem
\[
(a_h(v_h, \varphi_h) = (v_h, g) \ \forall v_h \in \tilde{S}_h(\Omega).
\]

Then
\[
(p - p_h, g) = \sum_{T \in T_h} \int_T \beta \nabla (p - p_h) : \nabla \varphi \, dx - \sum_{T \in T_h} \int_{\partial T} (p - p_h) \beta \frac{\partial \varphi}{\partial n} \, ds
\]
\[
= a_h(p - p_h, \varphi - \varphi_h) + a_h(p - p_h, \varphi_h) - \sum_{T \in T_h} \int_{\partial T} (p - p_h) \beta \frac{\partial \varphi}{\partial n} \, ds
\]
\[
= a_h(p - p_h, \varphi - \varphi_h) + \sum_{T \in T_h} \int_{\partial T} \beta \frac{\partial p}{\partial n} \varphi_h \, ds - \sum_{T \in T_h} \int_{\partial T} (p - p_h) \beta \frac{\partial \varphi}{\partial n} \, ds
\]
\[
=: I + II - III.
\]
By the continuity of \(a_h(\cdot, \cdot)\) and the \(H^1\)-error estimate of \(\varphi - \varphi_h\),
\[
|I| \leq C \|p - p_h\|_{1,h} \|\varphi - \varphi_h\|_{1,h} \leq Ch \|p - p_h\|_{1,h} \|\varphi\|_{\tilde{H}^2(\Omega)}
\]
\[
\leq Ch^2 \|p\|_{\tilde{H}^2(\Omega)} \|\varphi\|_{\tilde{H}^2(\Omega)}.
\]

Applying the analysis for the consistency error of the \(H^1\)-error estimate (3.8), we get
\[
|II| = \left| \sum_{T \in T_h} \int_{\partial T} \beta \frac{\partial p}{\partial n} \varphi_h \, ds \right| = \left| \sum_{T \in T_h} \int_{\partial T} \beta \frac{\partial p}{\partial n} (\varphi_h - \varphi) \, ds \right|
\]
\[
\leq Ch \|p\|_{\tilde{H}^2(\Omega)} \|\varphi_h - \varphi\|_{1,h} \leq Ch^2 \|p\|_{\tilde{H}^2(\Omega)} \|\varphi\|_{\tilde{H}^2(\Omega)}
\]
and
\[
|III| = \left| \sum_{T \in T_h} \int_{\partial T} (p - p_h) \beta \frac{\partial \varphi}{\partial n} \, ds \right| \leq Ch \|p - p_h\|_{1,h} \|\varphi\|_{\tilde{H}^2(\Omega)}
\]
\[
\leq Ch^2 \|p\|_{\tilde{H}^2(\Omega)} \|\varphi\|_{\tilde{H}^2(\Omega)}.
\]
Since \( \|\varphi\|_{H^2(\Omega)} \leq C\|g\|_{L^2(\Omega)} \), we see that

\[
\|p - p_h\|_{L^2(\Omega)} = \sup_{g \in L^2(\Omega)} \frac{\langle p - p_h, g \rangle}{\|g\|_{L^2(\Omega)}} \leq C h^2 \|p\|_{H^2(\Omega)}.
\]

Thus we obtain the following \( L^2 \)-error estimate.

**Theorem 4.1.** Let \( p \in \tilde{H}^2(\Omega) \), \( p_h \in \tilde{S}_h(\Omega) \) be the solutions of (2.3) and (3.1), respectively. Then there exists a constant \( C > 0 \) such that

\[
\|p - p_h\|_{L^2(\Omega)} \leq C h^2 \|p\|_{H^2(\Omega)}.
\]

5. Mixed finite volume method based on immersed interface FEM. In this section, we propose a new mixed finite volume method based on the “broken” \( P_1 \)-nonconforming FEM introduced in the previous section. Our method is similar to the mixed finite volume method studied in [29, 15, 18], but the usual nonconforming finite element space is replaced by our “broken” \( P_1 \)-nonconforming space.

Let us write the problem (2.3) in a mixed form by introducing the vector variable

\[
u = -\beta \nabla p
\]

as

\[
\begin{aligned}
\begin{cases}
\mathbf{u} + \beta \nabla p &= 0 & \text{in } \Omega, \\
\text{div } \mathbf{u} &= f & \text{in } \Omega, \\
p &= 0 & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
\]

The mixed FEM based on this dual formulation is well known [6, 7, 45]. The idea of the mixed method is to find a direct approximation of the flow variable \( \mathbf{u} \). For that purpose, we introduce \( \mathbf{V} = \mathbf{H}(\text{div}, \Omega) = \{ \mathbf{v} \in L^2(\Omega) : \text{div } \mathbf{v} \in L^2(\Omega) \} \) and use the local \( RT_h \) space to approximate the flow variable which is given by \( \mathbf{V}_h(T) = \{ \mathbf{v} : \mathbf{v} = (a + cx, b + cy), a, b, c \in \mathbb{R} \} \) for any triangle element \( T \). The global space \( \mathbf{V}_h \) is defined as

\[
\mathbf{V}_h = \{ \mathbf{v} : \mathbf{v}|_T \in \mathbf{V}_h(T); \mathbf{v} \cdot \mathbf{n} \text{ is continuous along interior edges} \}.
\]

This method gives a good approximation of the flow variable. However, it leads to a saddle point problem; that is, one obtains an indefinite matrix system when (5.1) is discretized. As mentioned earlier, a popular way to avoid this indefinite system is to use Lagrange multipliers [2]. Another possibility is to form a mixed finite volume method as in [29, 15, 18].

To define a mixed finite volume method for an interface problem, we use the well-known \( RT_h \) space \( \mathbf{V}_h \) for velocity and “broken” \( P_1 \)-nonconforming immersed space \( \tilde{S}_h \) for pressure variable. Note that every \( \mathbf{v} \in \mathbf{V}_h \) has continuous normal components across the edges of \( T_h \), which are constant.

We consider the following scheme: Find \( (\mathbf{u}_h, p_h) \in \mathbf{V}_h \times \tilde{S}_h \) which satisfies on each element \( T \in T_h \)

\[
\begin{aligned}
\begin{cases}
\int_T (\mathbf{u}_h + \beta \nabla p_h) \cdot \nabla \phi &= 0 & \forall \phi \in \tilde{S}_h(T), \\
\int_T \text{div } \mathbf{u}_h &= \int_T f.
\end{cases}
\end{aligned}
\]

Note that since \( \text{div } \mathbf{u}_h \) is constant, \( \text{div } \mathbf{u}_h|_T = \frac{1}{|T|} \int_T f \), where \( |T| \) denotes the area of \( T \). When the interface is not present, \( \tilde{S}_h(T) = \mathbf{S}_h(T) \), and this scheme
coincides with the one in [29, 18]. Since the numbers of unknowns and equations do not change, our scheme is a square linear system and has a unique solution. We refer to [29] for details.

Now since \( u_h \cdot n \) is constant on the edge and \( \phi \in \hat{S}_h \) has common average values on interior edges and vanishing boundary nodal values, we obtain

\[
\sum_{T \in T_h} \int_{\partial T} u_h \cdot \nabla \phi = \sum_{T \in T_h} \left[ \int_{\partial T} (u_h \cdot n)\phi - \int_T \text{div}u_h \phi \right] = -\int_{\Omega} \mathcal{F}\phi,
\]

where \( \mathcal{F} \in L^2(\Omega) \) is a simple function having value \( \mathcal{F}|_T \) for each \( T \). From (5.3), it immediately follows that

\[
\sum_{T \in T_h} \int_T \beta \nabla p_h \cdot \nabla \phi = \int_{\Omega} \mathcal{F}\phi \quad \forall \phi \in \hat{S}_h(\Omega).
\]

This is the immersed FEM introduced in the previous section, except that on the right-hand side \( f \) is replaced by \( \mathcal{F} \).

The velocity \( u_h \) can be computed directly from the solution \( p_h \) of (5.5) as follows. Let \( T \) be an element of \( T_h \) with the edges \( e_i, i = 1, 2, 3 \), and let \( \phi_i \in \hat{S}_h(T) \) be the “broken” \( P_1 \)-nonconforming basis function associated with the edge \( e_i \). Then the flux through the edge \( e_i \) is given by

\[
|e_i|(u_h \cdot n)_{e_i} = \int_{\partial T} (u_h \cdot n)\phi_i = \int_T \text{div}(u_h \phi_i) = \int_T (\text{div}u_h \phi_i + u_h \cdot \nabla \phi_i),
\]

where \( \phi_i \in \hat{S}_h(T) \) is a basis function on \( T \). Then it follows by (5.3) that

\[
|e_i|(u_h \cdot n)_{e_i} = \int_T \mathcal{F}\phi_i - \int_T \beta \nabla p_h \cdot \nabla \phi_i.
\]

Thus in order to compute the fluxes through the edges of an element \( T \), we need only to compute the local residual of the solution \( p_h \) on each \( T \).

The error estimate of \( u_h \) would follow that of \( p_h \). In fact, we can relate the estimate \( \|u - u_h\|_{1,h} \) with \( \|p - p_h\|_{1,h} \). First, we show the following local formula.

**Lemma 5.1.** Let \( u_h, p_h \) be the solutions of (5.3); then

\[
u_h(x) = -\beta \nabla p_h + \frac{\mathcal{F}}{2}(x - x_B) \quad \forall x \in T,
\]

where \( \beta \nabla p_h \) denotes the average of \( \beta \nabla p_h \) on \( T \) and \( x_B \) is the center of \( T \).

**Proof.** Expanding \( u_h \) about \( x_B \), the barycenter of \( T \), we have

\[
u_h(x) = u_h(x_B) + \mathcal{D}u_h(x_B)(x - x_B), \quad x \in T,
\]

where \( \mathcal{D}u_h \) is the Jacobian matrix of \( u_h \). Let \( u_h = (a + cx, b + cy) \in \mathcal{V}_h(T) \). Then we have

\[
\mathcal{D}u_h(x_B)(x - x_B) = c(x - x_B) = \frac{\text{div}u_h}{2}(x - x_B) = \frac{\mathcal{F}}{2}(x - x_B),
\]
where we used the relation $\text{div}u_h = \overline{f}$. On the other hand, applying $\phi = (x,0)^T$ and $(0,y)^T$ in (5.3), we see that

$$
-\beta \nabla p_h = \frac{1}{|T|} \int_T u_h = u_h(x_B).
$$

Substituting these into (5.8), we obtain formula (5.7).

**Remark 5.1.** Our formula is different from the one in [29, 15] where they have

$$
u_h(x) = -\beta \nabla p + \frac{T}{2}(x - x_B) \quad \forall x \in T.
$$

The $p_h$ in our scheme is broken along a line segment contained in an interface element; hence we have taken the average of $\beta \nabla p_h$.

By the above lemma, we have

$$u(x)|_T - u_h(x) = -\beta \nabla p + \beta \nabla p_h - \frac{T}{2}(x - x_B).
$$

So

$$\|u - u_h\|_{0,T} \leq \|\beta \nabla p - \beta \nabla p_h\|_{0,T} + C|f| \|x - x_B\|_{0,T}.
$$

Since $\beta$ is piecewise constant and $u = -\beta \nabla p$, we have

$$\|\beta \nabla p - \beta \nabla p_h\|_{0,T} \leq \|\beta \nabla p - \beta \nabla p\|_{0,T} + \|\beta \nabla p - \beta \nabla p_h\|_{0,T}$$

$$\leq Ch\|u\|_{1,T} + Ch\|\beta \nabla p - \beta \nabla p_h\|_{0,T}$$

$$\leq Ch\|u\|_{1,T} + C|p - p_h|_{1,T},$$

provided $u \in H^1(T)$. Hence

$$\|u - u_h\|_{0,T} \leq C\{h\|u\|_{1,T} + |p - p_h|_{1,T} + h^2|f|\}$$

$$\leq C\{h\|u\|_{1,T} + |p - p_h|_{1,T} + h\|f\|_{0,T}\},$$

where we used $|f| \leq h^{-1}\|f\|_{0,T}$ and $\|x - x_B\|_{0,T} \leq Ch^2$.

Summing over every $T \in \mathcal{T}_h$, we have

$$
\|u - u_h\|_{L^2(\Omega)} \leq C\{h\|u\|_{H^1(\Omega)} + |p - p_h|_{1,h} + h\|f\|_{L^2(\Omega)}\}.
$$

Since $\text{div}u = f$ and $\text{div}u_h = \overline{f}$, we can easily obtain the estimate for $\|\text{div}u - \text{div}u_h\|_{L^2(\Omega)}$. This is summarized in the following.

**Theorem 5.2.** Let $u_h, p_h$ be the solutions of (5.3); then there exists a constant $C > 0$ such that

$$
\|u - u_h\|_{L^2(\Omega)} + \|\text{div}u - \text{div}u_h\|_{L^2(\Omega)}$$

$$\leq Ch\{h\|u\|_{H^1(\Omega)} + \|p\|_{H^1(\Omega)} + \|f\|_{H^1(\Omega)}\},$$

provided $u \in H^1(\Omega)$.

**Remark 5.2.** When the solution has nonzero jumps, i.e., $[p] = J_1, \ [\beta \frac{\partial p}{\partial n}] = J_2$ for some functions $J_1, J_2$ across $\Gamma$, we have suggested some numerical methods using immersed finite element with standard linear nodal basis functions which seem to show optimal order [11]. The corresponding algorithm for broken $P_1$-nonconforming bases will be investigated in the near future.
6. Numerical examples. In this section, we report numerical results for the schemes introduced previously. We solve problem (2.1) with the rectangular domain \( \Omega = [-1, 1] \times [-1, 1] \) partitioned into uniform right triangles having step size \( h \). We used the conjugate gradient method to solve the resulting discrete system. We present errors in the \( L^2 \), \( H^1 \)-norm for the pressure \( p \), while in the \( L^2 \) and \( H(\text{div}) \)-norm for the velocity \( \mathbf{u} \). In the first two columns of each table, we report the results of the “broken” \( P_1 \)-nonconforming immersed scheme introduced in section 3. It shows optimal order of convergence for the \( L^2 \)-norm and the \( H^1 \)-norm:

\[
\| p - p_h \|_0 \approx O(h^2), \quad \| p - p_h \|_{1, h} \approx O(h).
\]

We also present some results for the mixed finite volume method introduced in section 5. Again, this shows optimal order of convergence for the flow variable which is consistent with Theorem 5.2 (cf. last columns of the tables):

\[
\| \mathbf{u} - \mathbf{u}_h \|_{L^2(\Omega)} + \| \text{div}(\mathbf{u} - \mathbf{u}_h) \|_{L^2(\Omega)} \approx O(h).
\]

This is in good agreement with some fitted grid computation; when the jump of the coefficient is large, one usually has \( O(h) \) order accuracy; see [9, Problem 1, p. 310], for example.

Example 6.1 (from Li [38]). Take a circle with radius \( r_0 = 0.5 \) as an interface, and choose the exact solution

\[
p = \begin{cases} 
\frac{r^3}{\beta^-} & \text{in } \Omega^-, \\
\frac{r^3}{\beta^+} + \left( \frac{1}{\beta^-} - \frac{1}{\beta^+} \right) \frac{r^3}{r_0^3} & \text{in } \Omega^+.
\end{cases}
\]

In this example, two cases \( \beta^+/\beta^- = 1/1000 \) and \( \beta^+/\beta^- = 1000 \) are reported in Tables 6.1 and 6.2.

| \( 1/h \) | \( \| p - p_h \|_0 \) | Order | \( \| p - p_h \|_{1, h} \) | Order | \( \| \mathbf{u} - \mathbf{u}_h \|_0 \) | Order | \( \| \text{div}(\mathbf{u} - \mathbf{u}_h) \|_0 \) | Order |
|---|---|---|---|---|---|---|---|---|
| 8 | 9.576e-3 | 1.208e-1 | 2.945e-1 | 1.053e+0 |
| 16 | 2.666e-3 | 1.845 | 6.744e-2 | 0.841 | 1.702e-1 | 0.791 | 5.292e-1 | 0.993 |
| 32 | 6.488e-4 | 2.039 | 3.341e-2 | 1.015 | 8.966e-2 | 0.934 | 2.666e-1 | 0.998 |
| 64 | 1.408e-4 | 2.212 | 1.573e-2 | 1.012 | 4.290e-2 | 1.054 | 1.326e-1 | 0.999 |
| 128 | 3.716e-5 | 1.914 | 8.422e-3 | 1.008 | 2.015e-2 | 1.090 | 6.629e-2 | 1.000 |
| 256 | 8.953e-6 | 2.050 | 4.176e-3 | 1.001 | 9.865e-3 | 1.030 | 3.315e-2 | 1.000 |

| \( 1/h \) | \( \| p - p_h \|_0 \) | Order | \( \| p - p_h \|_{1, h} \) | Order | \( \| \mathbf{u} - \mathbf{u}_h \|_0 \) | Order | \( \| \text{div}(\mathbf{u} - \mathbf{u}_h) \|_0 \) | Order |
|---|---|---|---|---|---|---|---|---|
| 8 | 1.447e-2 | 6.576e-1 | 3.361e-1 | 1.053e+0 |
| 16 | 3.497e-3 | 2.049 | 3.312e-1 | 0.989 | 1.657e-1 | 1.020 | 5.292e-1 | 0.993 |
| 32 | 8.286e-4 | 1.986 | 1.661e-1 | 0.996 | 8.165e-2 | 1.021 | 2.650e-1 | 0.998 |
| 64 | 2.210e-4 | 1.998 | 8.311e-2 | 0.999 | 4.075e-2 | 1.003 | 1.326e-1 | 0.999 |
| 128 | 5.537e-5 | 2.005 | 4.157e-2 | 0.999 | 1.939e-2 | 1.005 | 6.629e-2 | 1.000 |
| 256 | 1.370e-5 | 2.001 | 2.079e-2 | 1.000 | 9.638e-3 | 1.020 | 3.315e-2 | 1.000 |

Table 6.1 Nonconforming immersed FEM: \( \beta^- = 1, \beta^+ = 1000 \).

Table 6.2 Nonconforming immersed FEM: \( \beta^- = 1000, \beta^+ = 1 \).
Nonconforming immersed FEM: \( \beta^-=1 + 0.5(x^2 - xy + y^2), \beta^+=1 \).

| \(1/h\) | \(\|p-p_h\|_0\) Order | \(\|p-p_h\|_{L^\infty}\) Order | \(\|u-u_h\|_0\) Order | \(\|\text{div}(u-u_h)\|_0\) Order |
|---|---|---|---|---|
| 8 | 2.871e-1 | 4.742e-0 | 3.988e+0 | 5.499e-1 |
| 16 | 7.343e-2 | 1.970 | 1.722e+0 | 3.802e-1 |
| 32 | 1.842e-2 | 1.995 | 8.646e-1 | 1.908e-1 |
| 64 | 4.608e-3 | 1.999 | 4.326e-1 | 8.894e-2 |
| 128 | 1.152e-3 | 2.000 | 2.165e-1 | 4.409e-2 |
| 256 | 2.881e-4 | 2.000 | 1.083e-1 | 2.208e-2 |

Table 6.4

Nonconforming immersed FEM sharp edge, \( \theta = 10^\circ, \beta^- = 1, \beta^+ = 1000 \).

| \(1/h\) | \(\|p-p_h\|_0\) Order | \(\|p-p_h\|_{L^\infty}\) Order | \(\|u-u_h\|_0\) Order | \(\|\text{div}(u-u_h)\|_0\) Order |
|---|---|---|---|---|
| 8 | 2.685e-2 | 3.164e-1 | 2.709e-1 | 2.195e-2 |
| 16 | 5.349e-3 | 1.944 | 1.381e-1 | 1.096e-2 |
| 32 | 1.422e-3 | 1.911 | 7.263e-2 | 5.489e-3 |
| 64 | 3.564e-4 | 1.997 | 4.158e-2 | 2.746e-3 |
| 128 | 8.756e-5 | 2.025 | 1.938e-2 | 1.373e-3 |
| 256 | 2.164e-5 | 2.017 | 9.999e-3 | 6.868e-4 |

Example 6.2 (variable coefficient). We take the level set of \( L = x^2/(0.5)^2 + y^2/(0.25)^2 - 1.0 \) as an interface. The exact solution is chosen as \( p = L(x,y)/\beta \), where

\[
\beta = \begin{cases} 
1 + 0.5(x^2 - xy + y^2) & \text{on } \Omega^-, \\
1 & \text{on } \Omega^+. 
\end{cases}
\]

It can be easily checked that this solution indeed satisfies the jump condition (2.2). The results are reported in Table 6.3.

Example 6.3 (sharp edge). In this example, we consider an interface with sharp edge. With \( L = -y^2 + ((x-1)\tan \theta)^2 x \), the level set has a sharp corner of angle \( 2\theta \) at the point \( (1,0) \) (see Figure 6.1). The exact solution is chosen as \( p = L(x,y)/\beta \), where \( \beta^- = 1, \beta^+ = 1000 \). We have tested two cases: \( \theta = 10^\circ \) and \( \theta = 40^\circ \), and the results are reported in Tables 6.4 and 6.5.
Finite Element Method for Interface Problems

Table 6.5

| $1/h$ | $\|p - p_h\|_0,0$ | Order | $\|p - p_h\|_{1,h}$ | Order | $\|u - u_h\|_0,0$ | Order | $\|\text{div}(u - u_h)\|_0$ | Order |
|-------|------------------|-------|------------------|-------|------------------|-------|------------------|-------|
| 8     | 1.389e−3         |       | 1.476e−1         |       | 1.308e−1         |       | 2.106e−1         |       |
| 16    | 3.765e−4         | 0.748 | 2.106e−1         | 1.308 | 2.106e−1         | 0.988 | 2.106e−1         | 0.988 |
| 32    | 8.606e−4         | 1.303 | 1.060e−1         | 1.031 | 1.240e−1         | 0.994 | 1.240e−1         | 0.994 |
| 64    | 2.192e−4         | 1.303 | 5.117e−2         | 1.051 | 6.211e−2         | 0.998 | 6.211e−2         | 0.998 |
| 128   | 5.510e−5         | 1.093 | 2.481e−2         | 1.062 | 3.108e−2         | 0.999 | 3.108e−2         | 0.999 |
| 256   | 1.359e−5         | 1.064 | 1.196e−2         | 1.035 | 1.555e−2         | 0.999 | 1.555e−2         | 0.999 |

Order | 1.097 | 1.097 | 1.097 | 1.097 | 1.097 | 1.097 | 1.097 | 1.097 |

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