Classical Integrable 2-dim Models Inspired by SUSY Quantum Mechanics.

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A class of integrable 2-dim classical systems with integrals of motion of fourth order in momenta is obtained from the quantum analogues with the help of deformed SUSY algebra. With similar technique a new class of potentials connected with Lax method is found which provides the integrability of corresponding 2-dim hamiltonian systems. In addition, some integrable 2-dim systems with potentials expressed in elliptic functions are explored.

1. Introduction

Construction of classical integrable systems with additional integrals of motion is of considerable interest in Mathematical Physics (see \cite{ref1} and references therein). Multidimensional integrable systems play an important role to describe the dynamics analogously to 1-dim manifestly integrable systems. In particular, they may serve as zero-approximations of perturbation theory in the case of weak, nonintegrable perturbations. Variety of traditional approaches to this problem exists starting from J.Kepler, S.Kowalewski till the Lax method. On the other hand a modern viewpoint on how to build classical integrable systems is based on the symmetries of related quantum systems \cite{ref2}. Recently the method for searching quantum integrable 2-dim systems was developed \cite{ref3,ref4} with the help of a deformed supersymmetry (SUSY) algebra formed by intertwining differential operators of finite order. Supersymmetry \cite{ref5}-\cite{ref7}, i.e. the construction of the isospectral pair of Hamiltonians, was proved \cite{ref6} to be in one-to-one correspondence to integrability of both Hamiltonians, i.e. to existence of a differential symmetry operator, which is polynomial in derivatives and which transforms solutions of 2-dim Schrödinger equation into other solutions with the same energy. Quasiclassical reduction of the deformed SUSY algebra \cite{ref3} gave the factorization of classical

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integrals of motion for the corresponding Hamiltonians \[8\]. As a result, the structure of analytically resolved integrals of motion became more clear, and new classes of integrable potentials were found \[8\],\[9\].

In our paper we continue our study \[9\] of classical systems which integrability is induced or inspired by a deformed SUSY algebra for the relevant quantum systems. The concise basic construction of systems possessing a dynamical symmetry with the help of Higher derivative SUSY algebra is essentially supplemented with algorithms of searching analytical solutions of related nonlinear equations for coefficients functions of symmetry operators and potentials.

In Sect.2 the intertwining relations between a pair of quantum Schrödinger-type Hamiltonians by general differential operators \(q^\pm\) of second order are investigated. The class of particular solutions of these relations is constructed for the cases of hyperbolic (Lorentz) \(g_{ik} = \text{diag}(1,-1)\) and degenerate \(g_{ik} = (1,0)\) metric structures of operators \(q^\pm\) in second derivatives. The differential operators of fourth order in derivatives, which are symmetry operators for intertwined Hamiltonians, are built. In Sect.3 the classical limit \(\hbar \to 0\) for the Hamiltonians is considered, and the class of systems with integrals of motion of fourth order in momenta is obtained. In Section 4 a new class of integrable systems with potentials connected to the Lax method is derived using ansatizes and technique taken from 2-dim SUSY Quantum Mechanics (Section 2). Last Section 5 is devoted to description of some integrable systems expressed in elliptic functions. We stress that quite a few of the obtained potentials do not allow the separation of variables in known coordinate systems and some of them so far have not been found.

2. Quantum integrable 2-dim systems.

In the 2-dimensional generalization \[3\], \[8\], \[4\] of Higher Order SUSY Quantum Mechanics \[10\] the intertwining relations of second order in derivatives are most essential:

\[
H^{(1)}q^+ = q^+ H^{(2)}, \quad q^- H^{(1)} = H^{(2)} q^-,
\]

\[
H^{(i)} = -\hbar^2 \Delta + V(x), \quad \Delta \equiv \partial_1^2 + \partial_2^2, \quad \partial_i \equiv \partial / \partial x_i,
\]

\[
q^+ = (q^-)^\dagger = \hbar^2 g_{ik}(x) \partial_i \partial_k + \hbar C_i(x, \hbar) \partial_i + B(x, \hbar),
\]

where \(\hbar\) is the Plank constant and all coefficient functions are real.
It means that, up to zero modes of \( q^\pm \), spectra of \( H^{(i)} \) coincide and their eigenfunctions:

\[ \Psi^{(2)} \sim q^- \Psi^{(1)}, \quad \Psi^{(1)} \sim q^+ \Psi^{(2)}. \] (2)

The intertwining relations (I) lead to existence of the symmetry operators \( R^{(1)}, R^{(2)} \) for the Hamiltonians \( H^{(1)}, H^{(2)} \), correspondingly:

\[ [R^{(i)}, H^{(i)}] = 0, \quad R^{(1)} = q^+ q^-, \quad R^{(2)} = q^- q^+, \quad i = 1, 2. \] (3)

In 1-dim case [10] analogous symmetry operators \( R^{(i)} \) become polynomials of \( H^{(i)} \) with constant coefficients. The distinguishing peculiarity of 2-dim case is existence [4] of nontrivial dynamical symmetry operators \( R^{(i)} \) which are not reduced to functions of the Hamiltonians \( H^{(i)} \).

It was shown in [3] that for the unit metrics \( g_{ik} = \delta_{ik} \) operators \( R^{(i)} \) can be written as second order differential operators (up to a function of \( H^{(i)} \)) and corresponding quantum systems allow separation of variables in parabolic, elliptic or polar coordinate systems. For all other metrics \( g_{ik} \) operators \( R^{(i)} \) are of fourth order in derivatives.

The intertwining relations (I) are equivalent to the following system of differential equations:

\[
\begin{align*}
\hbar \partial_i C_k + \hbar \partial_k C_i + \hbar^2 \Delta g_{ik} - (V^{(1)} - V^{(2)})g_{ik} &= 0; \\
\hbar^2 \Delta C_i + 2\hbar \partial_i B + 2\hbar g_{ik} \partial_k V^{(2)} - (V^{(1)} - V^{(2)})C_i &= 0; \\
\hbar^2 \Delta B + \hbar^2 g_{ik} \partial_k \partial_i V^{(2)} + hC_i \partial_i V^{(2)} - (V^{(1)} - V^{(2)})B &= 0.
\end{align*}
\] (4)

where the metrics \( g_{ik} \) is a quadratic polynomial in \( x_1, x_2 \):

\[
g_{11} = ax_2^2 + a_1 x_2 + b_1; \quad g_{22} = ax_1^2 + a_2 x_1 + b_2; \quad g_{12} = -\frac{1}{2} (2ax_1 x_2 + a_1 x_1 + a_2 x_2) + b_3.
\]

I. For the supercharges with Lorentz metrics \( (g_{ik} = diag(1, -1)) \):

\[ q^+ = \hbar^2 (\partial_1^2 - \partial_2^2) + \hbar C_k \partial_k + B, \] (5)

a solution of (I) can be reduced [3] to a solution of the system:

\[
\begin{align*}
\partial_-(C_- F) &= -\partial_+(C_+ F); \\
\partial_+^2 F &= \partial_-^2 F.
\end{align*}
\] (6) (7)
where \( C_1 \equiv C_2 \equiv C_\pm(x_\pm) \) depend only on \( x_\pm \), respectively. Eq.\((\mathbb{2})\) means that the function \( F \) can be represented as a sum \( F = F_1(x_+ + x_-) + F_2(x_+ - x_-) \). The potentials \( V^{(1,2)} \) and the function \( B \) are expressed in terms of solutions of system \((\mathbb{3}), (\mathbb{4})\):

\[
V^{(1,2)} = \pm \frac{\hbar}{2}(C'_+ + C'_-) + \frac{1}{8}(C^2_+ + C^2_-) + \frac{1}{4}(F_2(x_+ - x_-) - F_1(x_+ + x_-)),
\]

\[
B = \frac{1}{4}(C_+ C_- + F_1(x_+ + x_-) + F_2(x_+ - x_-)).
\]

The solutions for functions \( F \), which admit additionally the factorization \( F = F_+(x_+) \cdot F_-(x_-) \), were found in \([14]\). In the present paper other solutions of \((\mathbb{3}) - (\mathbb{4})\) will be built.

1) After substitution of the general solution of \((\mathbb{3})\)

\[
F = L \left( \int \frac{dx_+}{C_+} - \int \frac{dx_-}{C_-} \right)/(C_+ C_-),
\]

into \((\mathbb{4})\), we obtain the functional-differential equation for functions \( L \) and \( A'_\pm \equiv 1/C_\pm(x_\pm) \):

\[
\left( \frac{A''_+}{A'_+} - \frac{A''_-}{A'_-} \right)L(A_+ - A_-) + 3(A''_+ + A''_-)L'(A_+ - A_-) + (A'^2_+ - A'^2_-)L''(A_+ - A_-) = 0,
\]

where \( L' \) denotes the derivative of \( L \) with respect to its argument. Eq.\((\mathbb{11})\) can be easily solved for functions \( A_\pm \) such that \( A''_\pm = \lambda^2 A_\pm \). Then

\[
L(A_+ - A_-) = \alpha(A_+ - A_-)^{-2} + \beta,
\]

for \( A_\pm = \sigma_\pm \exp(\lambda x_\pm) + \delta_\pm \exp(-\lambda x_\pm) \) with \( \sigma_+ \cdot \delta_+ = \sigma_- \cdot \delta_- \) and \( \alpha, \beta \) - real constants. For \( \lambda^2 > 0 \) we obtain (up to an arbitrary shift in \( x_\pm \)) two solutions:

1a) \( A_\pm = k \sinh(\lambda x_\pm), \quad 1b) \ A_\pm = k \cosh(\lambda x_\pm). \)

Then \((\mathbb{10})\) leads to:

1a) \( F_1(2x) = F_2(2x) = \frac{k_1}{\cosh^2(\lambda x)} + k_2 \cosh(2\lambda x), \quad C_\pm = \frac{k}{\cosh(\lambda x_\pm)}, \quad k \neq 0, \)

1b) \( F_1(2x) = -F_2(2x) = \frac{k_1}{\sinh^2(\lambda x)} + k_2 \sinh^2(\lambda x), \quad C_\pm = \frac{k}{\sinh(\lambda x_\pm)}, \quad k \neq 0. \)

For \( \lambda^2 < 0 \) hyperbolic functions must be substituted by trigonometric ones.

At last, in the limiting case of \( \lambda = 0 \) the solutions have the form:

\[
F_1(2x) = -F_2(2x) = k_1 x^{-2} + k_2 x^2, \quad C_\pm = \frac{k}{x_\pm}, \quad k \neq 0, \quad (14)
\]

\[
F_1(2x) = -F_2(2x) = k_1 x^2 + k_2 x^4, \quad C_\pm = \pm \frac{k}{x_\pm}, \quad k \neq 0. \quad (15)
\]
2) To find another class of solutions of the system (3), (7) it is useful to replace in (11) \( C_\pm \) by \( f_\pm \), such that \( C_\pm \equiv \pm f_\pm / f'_\pm \). Then \( F \) in (10) is represented in the form \( F = U( f_+ f_-) f'_+ f'_- \) with an arbitrary function \( U \). After substitution in (10) one obtains the equation:

\[
(f'^2 - f'_+ f'_-) U''(f) + 3f (f'' - f'_+) f'(f) + \left( f'' - f'_+ \right) U(f) = 0, \quad f \equiv f_+ f_-.
\]

One can check that \( f_\pm = \alpha_\pm \exp(\lambda x_\pm) + \beta_\pm \exp(-\lambda x_\pm) \) and \( U = a + 4bf_+ f_- \) are its particular solutions (\( a, b \)-real constants). Then functions

\[
F_1(x) = k_1(\alpha_+ \alpha_- \exp(\lambda x) + \beta_+ \beta_- \exp(-\lambda x)) + k_2(\alpha^2_+ \alpha^2_- \exp(2\lambda x) + \beta^2_+ \beta^2_- \exp(-2\lambda x)),
\]

\[
-F_2(x) = k_1(\alpha_+ \beta_- \exp(\lambda x) + \beta_+ \alpha_- \exp(-\lambda x)) + k_2(\alpha^2_+ \beta^2_- \exp(2\lambda x) + \beta^2_+ \alpha^2_- \exp(-2\lambda x)),
\]

\[
C_\pm = \pm \frac{\alpha_\pm \exp(\lambda x_\pm) + \beta_\pm \exp(-\lambda x_\pm)}{\lambda(\alpha_\pm \exp(\lambda x_\pm) - \beta_\pm \exp(-\lambda x_\pm))}
\]

are real solutions of the system (3), (7) if \( \alpha_\pm, \beta_\pm \) are real for the case \( \lambda^2 > 0 \) and \( \alpha_\pm = \beta_\pm^* \) for the case \( \lambda^2 < 0 \).

3) To find a third class of solutions it is useful to rewrite (3) in terms of the variables \( x_{1,2} \):

\[
2(F_1(x_1) + F_2(x_2)) \partial_1(C_+ + C_-) + F'_1(x_1)(C_+ + C_-) + F'_2(x_2)(C_+ - C_-) = 0.
\]

Its solutions are:

3a) \( C_+(x_+) = \sigma_1 \sigma_2 \exp(\lambda x_+) + \delta_1 \delta_2 \exp(-\lambda x_+) + c \),

\[
C_-(x_-) = \sigma_1 \delta_2 \exp(\lambda x_-) + \sigma_2 \delta_1 \exp(-\lambda x_-) + c,
\]

\[
F_1(x_1) = 0, \quad F_2(x_2) = \frac{1}{(\sigma_2 \exp(\lambda x_2) - \delta_2 \exp(-\lambda x_2))^2}; \tag{17}
\]

3b) \( C_+(x) = \sigma_1 \sigma_2 \exp(\lambda x) + \delta_1 \delta_2 \exp(-\lambda x) + c \), \( F_1(x_1) = 0, \quad F_2(x_2) = \frac{4b^2}{x_2^2}; \tag{18} \)

3c) \( C_+(x_+) = \sigma_1 \sigma_2 \exp(\lambda x_+) + \delta_1 \delta_2 \exp(-\lambda x_+), \)

\[
C_-(x_-) = \sigma_1 \delta_2 \exp(\lambda x_-) + \sigma_2 \delta_1 \exp(-\lambda x_-),
\]

\[
F_{1,2}(x_{1,2}) = \frac{\nu_{1,2}}{(\sigma_1 \exp(\lambda x_{1,2}) \pm \delta_1 \exp(-\lambda x_{1,2}))^2} \pm \gamma. \tag{19}
\]

Let us remark that two additional solutions, analogous to 3a) and 3b), can be obtained by replacing \( F_1(x_1) \) with \( F_2(x_2) \) and vice versa.
After inserting these solutions (12) - (19) into the general formulæ for potentials (8), we obtain, correspondingly, the following expressions for potentials (20)-(27):

\[
V^{(1,2)} = \pm \frac{\hbar k \lambda}{2} \left[ \frac{\sinh(\lambda x_+)}{\cosh^2(\lambda x_+)} + \frac{\sinh(\lambda x_-)}{\cosh^2(\lambda x_-)} \right] + \frac{k^2}{8} \left[ \frac{1}{\cosh^2(\lambda x_+)} + \frac{1}{\cosh^2(\lambda x_-)} \right] + \frac{1}{4} \left[ \frac{k_1}{\cosh^2(\lambda x_2)} - \frac{k_1}{\cosh^2(\lambda x_1)} + k_2 \cosh(2\lambda x_2) - k_2 \cosh(2\lambda x_1) \right]; \quad (20)
\]

\[
V^{(1,2)} = \pm \frac{\hbar k \lambda}{2} \left[ \frac{\cosh(\lambda x_+)}{\sinh^2(\lambda x_+)} + \frac{\cosh(\lambda x_-)}{\sinh^2(\lambda x_-)} \right] + \frac{k^2}{8} \left[ \frac{1}{\sinh^2(\lambda x_+)} + \frac{1}{\sinh^2(\lambda x_-)} \right] - \frac{1}{4} \left[ \frac{k_1}{\sinh^2(\lambda x_2)} + \frac{k_1}{\sinh^2(\lambda x_1)} + k_2 \cosh(2\lambda x_1) + k_2 \cosh(2\lambda x_2) \right]; \quad (21)
\]

\[
V^{(1,2)} = \pm \frac{\hbar k}{2} \left( \frac{1}{x_+^2} + \frac{1}{x_-^2} \right) + \frac{k^2}{8} \left( \frac{1}{x_+^2} + \frac{1}{x_-^2} \right) - \frac{1}{4} \left[ k_1 \left( x_1^2 + x_2^2 \right) + k_2 \left( x_1^4 + x_2^4 \right) \right]; \quad (22)
\]

Let us note that the potential (21) with \( k_2 = 0 \) and the potential (22) were investigated in the literature (c.f. for example [11]).

\[
V^{(1,2)} = \pm \frac{\hbar k}{2} \left( \frac{1}{x_+^2} - \frac{1}{x_-^2} \right) + \frac{k^2}{8} \left( \frac{1}{x_+^2} + \frac{1}{x_-^2} \right) - \frac{1}{4} \left[ k_1 \left( x_1^2 + x_2^2 \right) + k_2 \left( x_1^4 + x_2^4 \right) \right]; \quad (23)
\]

\[
V^{(1,2)} = \frac{2\alpha_+ \beta_+ (1 \mp 8\hbar \lambda^2) + \alpha_+^2 \exp(2\lambda x_+) + \beta_+^2 \exp(-2\lambda x_+)}{8\lambda^2(\alpha_+ \exp(\lambda x_+) - \beta_+ \exp(-\lambda x_+))^2} + \frac{2\alpha_- \beta_- (1 \pm 8\hbar \lambda^2) + \alpha_-^2 \exp(2\lambda x_-) + \beta_-^2 \exp(-2\lambda x_-)}{8\lambda^2(\alpha_- \exp(\lambda x_-) - \beta_- \exp(-\lambda x_-))^2} - \frac{1}{4} \left[ k_1 \left( \alpha_+ \beta_- \exp(2\lambda x_2) + \alpha_- \beta_+ \exp(-2\lambda x_2) \right) + k_2 \left( \alpha_+^2 \beta_-^2 \exp(4\lambda x_2) + \alpha_-^2 \beta_+^2 \exp(-4\lambda x_2) \right) + k_1 \left( \alpha_+ \alpha_- \exp(2\lambda x_1) + \beta_+ \beta_- \exp(-2\lambda x_1) \right) + k_2 \left( \alpha_+^2 \alpha_-^2 \exp(4\lambda x_1) + \beta_+^2 \beta_-^2 \exp(-4\lambda x_1) \right) \right]; \quad (24)
\]

\[
V^{(1,2)} = \pm \frac{\hbar \lambda}{2} \left( \sigma_1 \exp(\lambda x_1) - \delta_1 \exp(-\lambda x_1) \right)(\sigma_2 \exp(\lambda x_2) + \delta_2 \exp(-\lambda x_2)) + \frac{1}{8} \left[ \sigma_1^2 \exp(\lambda x_1) + \delta_1^2 \exp(-\lambda x_1) \right] \left( \sigma_2^2 \exp(\lambda x_2) + \delta_2^2 \exp(-\lambda x_2) \right) + 2c(\sigma_1 \exp(\lambda x_1) - \delta_1 \exp(-\lambda x_1))(\sigma_2 \exp(\lambda x_2) - \delta_2 \exp(-\lambda x_2)) \right] + \frac{1}{4(\sigma_2 \exp(2\lambda x_2) - \delta_2 \exp(-2\lambda x_2))^2}; \quad (25)
\]

\[
V^{(1,2)} = \pm 2\hbar ax_1 + \left[ \frac{a^2(x_1^4 + x_2^4 + 6x_1^2 x_2^2) + ac(x_1^2 + x_2^2)] + \frac{b^2}{x_2^2}}{4}; \quad (26)
\]

\[
V^{(1,2)} = \pm \frac{\hbar \lambda}{2} \left( \sigma_1 \exp(\lambda x_1) - \delta_1 \exp(-\lambda x_1) \right)(\sigma_2 \exp(\lambda x_2) + \delta_2 \exp(-\lambda x_2)) +
\]

6
\[
\frac{1}{8}(\sigma_1^2 \exp(\lambda x_1) + \delta_1^2 \exp(-\lambda x_1)) (\sigma_2^2 \exp(\lambda x_2) + \delta_2^2 \exp(-\lambda x_2)) + \nu_2 (\sigma_2 \exp(2\lambda x_2) - \delta_2 \exp(-2\lambda x_2))^2 - \frac{\nu_1}{(\sigma_1 \exp(2\lambda x_1) + \delta_1 \exp(-2\lambda x_1))^2}.
\]

(27)

The Hamiltonians with potentials \((21) - (27)\) possess the symmetry operators \(R^{(1)} = q^+ q^-, R^{(2)} = q^- q^+\), where \(q^\pm\) can be obtained inserting solutions \((12) - (19)\) into \((3)\).

It is necessary to note that there are some singular points of potentials \((21)-(27)\) on the plane \((x_1, x_2)\). Therefore both the asymptotics of corresponding wave functions and their behaviour under the action of supertransformation operators \(q^\pm\) \((5)\) and of symmetry operators \(R^{(1)}, R^{(2)}\) have to be investigated. In particular, for the potentials \((23) - (27)\) the operators \(q^\pm\) the preserve asymptotics of wave functions in the singular points, so the operators \(R^{(1)}, R^{(2)}\) are physical symmetry operators for these systems. For the potentials \((21), (23)\) symmetry properties were discussed before (see for example, \([11]\)).

II. For the supercharges with degenerate metrics \(g_{ik} = diag(1, 0):\)

\[
q^+ = \hbar^2 \partial_1^2 + \hbar C_k \partial_k + B
\]

(28)

Eqs.\((4)\) lead to:

\[
C_1(\vec{x}) = -x_2 F_1'(x_1) + G_1(x_1); \quad C_2(\vec{x}) = F_1(x_1);
\]

\[
V^{(1)} = \hbar (2G_1' - x_2 F_1''') + \frac{1}{4} x_2^2 (F_1'')'' - x_2 (F_1' G_1') + K_1(x_1) + K_2(x_2);
\]

\[
V^{(2)} = \hbar x_2 F_1''' + \frac{1}{4} x_2^2 (F_1'')'' - x_2 (F_1' G_1') + K_1(x_1) + K_2(x_2);
\]

\[
B = -\frac{\hbar}{2} (G_1' + x_2 F_1'') + \frac{1}{2} G_1^2 - \frac{1}{2} x_2^2 F_1'' + x_2 F_1 G_1' - K_1(x_1),
\]

(29)

where the real functions \(F_1(x_1), G_1(x_1), K_1(x_1)\) are solutions of the following system:

\[
-\frac{\hbar^2 G_1''}{2} + \frac{\hbar}{2} ((G_1'')^2 + 2G_1'^2) + G_1 K_1' + 2G_1' K_1 - F_1 (F_1' G_1') - G_1 G_1^2 = m_1 F_1; \quad (30)
\]

\[
\frac{\hbar^2}{2} F_1^{(IV)} - \hbar (F_1' G_1'' + 2G_1' F_1'') - G_1 (2G_1' F_1' + G_1'' F_1) - \frac{1}{4} G_1 (F_1'')'' + F_1'(F_1 G_1')'' + 3G_1' F_1'' = m_2 F_1;
\]

\[
\frac{1}{4} G_1 (F_1')''' + F_1' (F_1 G_1')'' + 3G_1' F_1'' = m_3 F_1;
\]

\[
\frac{1}{4} F_1' (F_1')''' + F_1 F_1'' = m_4 F_1,
\]

(33)
and \( K_2(x_2) \) is the polynomial of \( x_2 \) with constant coefficients:

\[
K_2(x_2) = m_0 - m_1 x_2 - \frac{1}{2} m_2 x_2^2 - \frac{1}{3} m_3 x_2^3 + \frac{1}{4} m_4 x_2^4.
\]

Several particular solutions of Eq.\((33)\) can be found. The constant function \( F_1 = k_1 \) is the solution of \((33)\) for \( m_4 = 0 \). To find other solutions we define the new function \( U(F_1) \):

\[
U(F_1) = F_1'(x_1),
\]

to decrease the order of the differential eq. \((33)\):

\[
U'' + \frac{3}{U} U'^2 + \frac{3}{F_1} U' - \frac{2 m_4}{U^3} = 0.
\]

Inserting its known solution \([17]\) into \((34)\), the following equation for \( F_1(x_1) \) is obtained:

\[
\int F_1^{1/2} (m_4 F_1^n + n F_1^2 + k)^{-1/4} dF_1 = x_1, \quad n, k = \text{Const.}
\]

The integral \((36)\) can be written as a finite combination of elementary functions only in the case when two of three constants \( m_4, n, k \) are zero. Thus the solutions of Eq.\((33)\) in elementary functions are: \( F_1 = k_1; F_1 = x_1/n; F_1 = (3/2)^{2/3} k^{1/6} x_1^{2/3}; F_1 = m_4^{1/2} x_1^2/4 \). Below, for simplicity, we shall consider the solutions with particular values of constants \( m_4, n, k \), while solutions with arbitrary values of these constants will differ by some of coefficients only. To solve Eqs.\((30) - (32)\) it is useful to consider separately two cases: \( G_1 \equiv 0 \) and \( G_1 \neq 0 \). In both cases solutions with \( F_1 = x_1 \) lead to potentials \( V^{(1,2)} \) with separation of variables. Below on such solutions will be ignored.

1) \( G_1 = 0 \).

In this case the potentials \( V^{(1,2)} \) for \( F_1 = k_1 \) correspond again to Schrödinger equations with separation of variables. More interesting choices \( F_1 = x_1^2 \) and \( F_1 = x_1^{2/3} \) lead, respectively, to potentials \( l-\) arbitrary real constant:

\[
V^{(1,2)} = \mp 2 h x_2 + l x_2^{-2} + \frac{1}{2} (x_1^4 + 6 x_1^2 x_2^2 + 8 x_2^4) - \frac{m_2}{8} (x_1^2 + 4 x_2^2); \quad (37)
\]

\[
V^{(1,2)} = \frac{7 h^2}{36} x_2^{-2} \pm \frac{2 h}{9} x_2 x_1^{-4/3} + \frac{1}{9} x_1^{-2/3} (x_2^2 + \frac{9}{2} x_1^2) 9 l x_1^{2/3} - \frac{m_2}{8} (9 x_1^2 + 4 x_2^2). \quad (38)
\]

2) \( G_1 \neq 0 \).

2a) For \( F_1 = k_1 \neq 0 \) Eq.\((31)\) leads to the following equation for \( G_1(x_1) \):

\[
\int \frac{G_1^2 dG_1}{\sqrt{k - \frac{1}{2} m_4 G_1^4}} = x_1,
\]

\[ \text{Page 8} \]
which has the solutions in terms of elementary functions in two cases: when \( k > 0, m_2 = 0 \) or \( k = 0, m_2 < 0 \). For the first one, after redefinition of constants and translation in \( x_2 \),

\[
V^{(1,2)} = -\frac{5h^2}{36}x_1^{-2} + \frac{\hbar k_2}{3}(1 \pm 1)x_1^{-2/3} - \frac{k_1 k_2}{3}x_2 x_1^{-2/3} + \frac{1}{4}(k_2^2 + 3k_1 m_1) x_1^{2/3} - m_1 x_2 + \frac{k_1^2}{2}.
\]

The second case \((k = 0, m_2 < 0)\) leads to separation of variables.

2b) If \( F_1 = x_1^{2/3} \) and \( F_1 = x_1^2 \) the general solutions of Eq. \((32)\) can be found and after substitution into \((30), (31)\) give the function \( K_1(x_1) \). Corresponding potentials are:

\[
V^{(1,2)} = -\frac{7h^2}{36}x_1^{-2} + \frac{2\hbar}{9} x_2 x_1^{-4/3} \pm \hbar k_1 + \frac{1}{9} x_1^{-2/3} (x_2^2 + \frac{9}{2} x_1^2) - \frac{5k_1}{3} x_2 x_1^{-2/3} + \frac{3m_1}{8k_1} x_1^{2/3} + \frac{k_1}{4} x_1^2 - m_1 x_2 + \frac{k_1}{9} (1 + 15k_1) x_2^2,
\]

\[
V^{(1,2)} = (\frac{3h^2}{4} \mp \hbar k_1 + \frac{k_1}{4}) x_1^{-2} \mp 2\hbar x_2 + \frac{1}{2} (x_1^4 + 6x_2^2 x_1^2 + 8x_2^4) - \frac{1}{8} (12k_2^2 + m_2) (x_1^2 + 4x_2^2) - \frac{km_2 + 6k_1^3 + 4k_1}{4} x_2.
\]

Because all potentials \((37) - (41)\) are singular at the \( x_2 \) axis, similar to the case of Lorentz metrics, it is necessary to investigate separately the behaviour of wave functions at \( x_1 \to 0 \). Straightforward though cumbersome calculations show that the fourth order differential operators \( R^{(1)}, R^{(2)} \) (see \((3)\)) preserve the asymptotics of wave functions for systems \((37) - (41)\) and play the role of true dynamical symmetry operators.

3. Construction of 2-dim integrable classical systems

by the limit \( \hbar \to 0 \)

Quantum dynamical symmetries which were found by the intertwining method in Sect.2 have their natural analogs - integrals of motion - in the corresponding classical systems. These integrals of motion are polynomials of fourth order in momenta. Similar to Sect.2, it is useful to consider separately the classical limits for Lorentz and degenerate metrics.

I. For the Lorentz metrics in the limit \( \hbar \to 0 \) classical supercharge functions become:

\[
q_{cl}^+ = -4p_+ p_- \pm i(C_-(x_-)p_+ + C_+(x_+)p_-) + B(x_-, x_+).
\]
From Eqs. (8) and (3) we find that the classical Hamiltonian

\begin{equation}
    h_{cl} = 2(p_+^2 + p_-^2) + \frac{1}{8}(C_+^2 + C_-^2) + \frac{1}{4}[F_2(x_+ - x_-) - F_1(x_+ + x_-)],
\end{equation}

has the additional integral of motion:

\begin{equation}
    I = 16p_1^2p_-^2 + C_+^2p_-^2 + C_-^2p_+^2 - 2(F_1 + F_2)p_+p_- + B^2.
\end{equation}

Such sort of classical systems was considered in the literature (see [13] and references therein). Usually the functional equation, which provides existence of integrals of motion for corresponding classical systems, is solved by the Lax method. Comparison of (42) and (43) with notations in [13] leads to the relations:

\begin{equation}
    v_1 \equiv \frac{1}{8}C_+^2(x_1), \quad v_2 \equiv \frac{1}{8}C_-^2(x_2), \quad v_3(x_-) \equiv \frac{1}{16}F_2(x_-), \quad v_4(x_+) \equiv -\frac{1}{16}F_1(x_+),
\end{equation}

and functions \( v_k \) must satisfy [13] a certain functional equation (see Eq. (47) below). In the next Section we prove that (44), where \( C_\pm, F_{1,2} \) are solutions of the system (3), (5), satisfy also Eq. (47). Moreover some additional solutions of this equation will be found in Sect. 4.

II. Let us study the integrable classical systems which can be obtained in the limit \( \hbar \to 0 \) from SSQM systems with degenerate metrics in \( q^\pm \). The classical supercharges have the form \( q_{cl}^+ = (q_{cl}^-)^* = -p_1^2 + iC_k(\bar{x})p_k + B(\bar{x}) \) and the Hamiltonian

\begin{equation}
    H_{cl} = p_k^2 + \frac{1}{2}x_2^2\partial_1^2F_1^2 - x_2(F_1G_1)' + K_1(x_1) + K_2(x_2)
\end{equation}

has the integral of motion of fourth order in momenta:

\begin{equation}
    I \equiv q_{cl}^+q_{cl}^- = p_1^4 + (C_1^2 - 2B)p_1^2 + C_2^2p_2^2 + 2C_1C_2p_1p_2 + B^2.
\end{equation}

All functions in (45), (46) were defined in the previous Section, where we have to put \( \hbar = 0 \).

Thus in the case of degenerate metrics the following classical integrable systems are obtained (new definitions of constants were used for some of these systems):

1) \( V = \frac{1}{2}(x_1^4 + 6x_1^2x_2^2 + 8x_2^4) + m(x_1^2 + 4x_2^2) + lx_1^{-2}, \)

\( I = p_1^4 + (6x_1^2x_2^2 + mx_1^2 + x_1^4 + 2lx_1^{-2})p_1^2 + x_1^4p_2^2 - 4x_1^3x_2p_1p_2 + (x_1^2x_2 + mx_1^2 + \frac{x_1^4}{2} + lx_1^{-2})^2; \)
2) \( V = \frac{1}{9} x_1^{-2/3} \left( \frac{9}{2} x_1^2 + x_2^2 \right) + m(9x_1^2 + 4x_2^2) + kx_1^{2/3}, \)
\[
I = p_1^4 + \left( \frac{2}{9} x_2 x_1^{-2/3} + 2kx_1^{-2/3} + 18mx_1^2 + x_1^{4/3} \right)p_1^2 + x_1^{4/3} p_2^2 - \frac{4}{3} x_2 x_1^{1/3} p_1 p_2 + \left( \frac{1}{9} x_2 x_1^{-2/3} - kx_1^{-2/3} - 9mx_1^2 - \frac{1}{2} x_1^{4/3} \right)^2;
\]

3) \( V = -\frac{k_1 k_2}{3} x_2 x_1^{-2/3} + \frac{1}{4} (k_2^2 + \frac{3k_1 m_1}{k_2}) x_1^{2/3} - m_1 x_2, \)
\[
I = p_1^4 + \left( \frac{1}{2} (k_2^2 + \frac{3k_1 m_1}{k_2}) x_1^{2/3} - \frac{2k_1 k_2}{3} x_2 x_1^{-2/3} + k_1^2 \right) p_1^2 + k_1^2 p_2^2 + 2k_1 k_2 x_1^{1/3} p_1 p_2 + \left( \frac{k_1 k_2}{3} x_2 x_1^{-2/3} + \frac{1}{4} (k_2^2 - \frac{3k_1 m_1}{k_2}) x_1^{2/3} - \frac{k_2^2}{2} \right)^2;
\]

4) \( V = \frac{1}{9} x_1^{-2/3} \left( \frac{9}{2} x_2^2 + x_2^2 \right) - \frac{5k}{3} x_2 x_1^{2/3} + \frac{3m k}{8k} x_1^{2/3} + \frac{k_2^2}{4} x_1^2 - mx_2 + \frac{k}{9} (1 + 15k) x_2^2, \)
\[
I = p_1^4 + \left( \frac{2}{9} x_2 x_1^{-2/3} - \frac{10k}{3} x_2 x_1^{2/3} + \frac{k_2^2}{2} x_1^2 + \frac{3 m k}{4 k} x_1^{4/3} \right) p_1^2 + x_1^{4/3} p_2^2 + 2x_1^{2/3} \left( -\frac{2}{3} x_2 x_1^{-1/3} + k x_1 \right) p_1 p_2 + \left( \frac{1}{9} x_2 x_1^{-2/3} + k x_2 x_1^{2/3} - \frac{3 m k}{8k} x_1^{2/3} - \frac{1}{2} x_1^{4/3} + \frac{k_2^2}{4} x_1^2 \right)^2;
\]

5) \( V = \frac{1}{2} (x_1^4 + 6x_1^2 x_2^2 + 8x_2^4) + m(x_1^2 + 4x_2^2) + \frac{k_2^2}{4} x_1^{-2} - (k_1 - 2mk - \frac{3}{2} k^3) x_2, \)
\[
I = p_1^4 + \left( 6x_2 x_1^2 + 2(m + k^2) x_1^2 - 2k_1 x_2 - 4k_2 x_1 x_2 + x_1^4 + \frac{k_2^2}{2} x_1^{-2} + k k_1 \right) p_1^2 + x_1^4 p_2^2 + 2x_1 (k_1 - 2x_1^2) p_1 p_2 + \left( x_1^2 x_2 + (m + \frac{k_2^2}{2}) x_1^2 + \frac{1}{2} x_1^4 + k_1 x_2 - \frac{k_1^2}{4} x_1^{-2} - \frac{k_1 k_2}{2} \right)^2.
\]

These potentials are not new: they were found by other methods in [14] - [16].

In conclusion of this Section we formulate the procedure of construction of integrals of motion in terms of Classical Mechanics objects. For any classical Hamiltonians \( H_{cl} \) and a complex function \( q_{cl}^+(\vec{x}, \vec{p}) = (q_{cl}^+)^* \), polynomial in momenta, such that:

\[
\{q_{cl}^+ \cdot H_{cl}\} = i f(\vec{x}, \vec{p}) q_{cl}^+, \quad \{(q_{cl}^+)^* \cdot H_{cl}\} = -i f(\vec{x}, \vec{p})(q_{cl}^+)^*;
\]

with arbitrary real function \( f(\vec{x}, \vec{p}) \), the classical factorizable integral of motion \( I = q_{cl}^+ \cdot q_{cl}^- \) exists (\( \{,\} \) – Poisson brackets).
4. Integrable systems connected with the Lax method.

Let us consider classical systems with potentials of the form:

\[ V(x_1, x_2) = v_1(x_1) + v_2(x_2) + v_3(x_1 - x_2) + v_4(x_1 + x_2). \]

It is known [13] that these systems have the integrals of motion of fourth order in momenta:

\[ I = \frac{1}{2}p_1^2p_2^2 + v_2(x_2)p_1^2 - (v_3(x_1 - x_2) - v_4(x_1 + x_2))p_1p_2 + v_1(x_1)p_2^2 + f, \]

if the functions \( v_1, v_2, v_3, v_4 \) satisfy the basic functional equation:

\[
\begin{align*}
[v_4(x_1 + x_2) - v_3(x_1 - x_2)][v_2''(x_2) - v_1''(x_1)] &+ 2[v_4''(x_1 + x_2) - v_3''(x_1 - x_2)] \cdot \\
[v_2(x_2) - v_1(x_1)] &+ 3v_4'(x_1 + x_2)[v_2'(x_2) - v_1'(x_1)] + \\
3v_4'(x_1 - x_2)[v_2'(x_2) + v_1'(x_1)] &= 0.
\end{align*}
\]

(47)

There is a list of known particular solutions of Eq.(47) in the book [13]. To search for new solutions of this equation it is useful to rewrite it in the equivalent form:

\[
\partial_2(vv_2' + 2v_2\partial_2v) = \partial_1(vv_1' + 2v_1\partial_1v),
\]

(48)

where

\[ v \equiv v_4(x_1 + x_2) - v_3(x_1 - x_2). \]

(49)

The general solution of Eq.(48) is:

\[
v = \left[ G\left( \int \frac{dx_1}{\sqrt{v_1}} + \int \frac{dx_2}{\sqrt{v_2}} \right) + L\left( \int \frac{dx_1}{\sqrt{v_1}} - \int \frac{dx_2}{\sqrt{v_2}} \right) \right] / \sqrt{v_1v_2}; \quad v_1 \neq 0; \quad v_2 \neq 0,
\]

(50)

where \( G \) and \( L \) are arbitrary functions of their arguments. Thus the problem is reduced to searching for the functions \( v \) of the form (50), which satisfy the condition

\[
(\partial_1^2 - \partial_2^2)v = 0.
\]

(51)

In particular, it is easy to check that all solutions which were found from SSQM in \( \hbar \to 0 \) limit in the Sect.3 (see Eq.(44)) are particular solutions of Eq.(47) and have the form (50) with \( G \equiv 0 \).

Let us apply the technique, which was used in investigation of the system (8), (7) in the framework of SSQM, to find the new particular solutions of eq. (18).

12
1) If the function $v$ is factorizable $v = u_1(x_1) \cdot u_2(x_2)$, Eq. (48) admits separation of variables and its solutions have the form:

$$v_k = \frac{n_k [a_k \exp(\sqrt{\lambda} x_k) + b_k \exp(-\sqrt{\lambda} x_k)] + l_k}{(a_k \exp(\sqrt{\lambda} x_k) - b_k \exp(-\sqrt{\lambda} x_k))^2}, \quad k = 1, 2,$$

$$v_3 = a_1 b_2 \exp(\sqrt{\lambda} \cdot x_+) + a_2 b_1 \exp(-\sqrt{\lambda} \cdot x_+),$$

$$v_4 = a_1 a_2 \exp(\sqrt{\lambda} \cdot x_+) + b_1 b_2 \exp(-\sqrt{\lambda} \cdot x_+),$$

where for $\lambda > 0$ all constants are real and for $\lambda < 0$ $a_k = b_k^*$ and $n_k, l_k$ are real.

2) Let us introduce new functions

$$v_1 \equiv (W_1')^{-2}, \quad v_2 \equiv (W_2')^{-2}. \quad (53)$$

Then Eq. (51) means that (derivatives of $G$ and $L$ are taken in their arguments):

$$(G + L) \left[ \frac{W''_1}{W'_1} - \frac{W''_2}{W'_2} \right] + 3 \left[ (W''_1 - W''_2)G' + (W''_1 + W''_2)L' \right] + (W''_1^2 - W''_2^2)(G'' + L'') = 0. \quad (54)$$

It is possible to construct several particular solutions of (54).

2a)

$$W_{1,2} = \sigma_{1,2} \exp(\lambda x_{1,2}) + \delta_{1,2} \exp(-\lambda x_{1,2}), \quad (55)$$

where constant $\lambda^2$ is real and $\sigma_k, \delta_k$ are complex. If these constants satisfy the condition

$$\sigma_1 \delta_1 = \sigma_2 \delta_2, \quad (56)$$

then substitution of (53) into (54) leads to an equation with separable variables. Its solutions are:

$$G(W_1 + W_2) = \frac{\alpha_1}{(W_1 + W_2)^2} + \alpha (W_1 + W_2)^2 + \beta_1; \quad (57)$$

$$L(W_1 - W_2) = \frac{\alpha_2}{(W_1 - W_2)^2} + \alpha (W_1 - W_2)^2 + \beta_2, \quad (58)$$

with arbitrary constants $\alpha, \alpha_i, \beta_i (i = 1, 2)$.

Let us consider the case with $\lambda^2 > 0$. Constants $\sigma_1, \delta_1$ must be both real or both positive because of requirement that functions $v_n(x_n)$ for $n = 1, 2$ should be real. Analogous arguments work for the pair $\sigma_2, \delta_2$. The condition (56) for real $(\sigma_i, \delta_i)$ leads to two options:

$$W_1(x) = W_2(x) = k \cosh(\lambda x), \quad (59)$$

$$W_1(x) = W_2(x) = k \sinh(\lambda x), \quad k \in \mathbb{R}, \quad (60)$$
and the function \( v \) is real for real constants \( \alpha, \alpha_i, \beta_i (i = 1, 2) \).

In the case of real \((\sigma_1, \delta_1)\) and imaginary \((\sigma_2, \delta_2)\) the solutions take the form:

\[
W_1 = k \sinh(\lambda x_1), \quad W_2 = ik \cosh(\lambda x_2), \\
W_1 = k \cosh(\lambda x_1), \quad W_2 = ik \sinh(\lambda x_2), \quad k \in \mathbb{R}.
\]

(61) \hspace{1cm} (62)

Let us remark that in this case the arguments of the functions \( G \) and \( L \) are complex conjugated and in order to have a real function \( v \) the following relation must be fulfilled:

\[
L^*(W_1 + W_2) = -G(W_1 + W_2).
\]

Therefore \( \alpha_2 = -\alpha_1^*; \quad \beta_2 = -\beta_1^*; \quad \alpha = \alpha^* \) in (57), (58).

2b)

\[
W_1 = W_2 = g x^2; \quad g^2 \in \mathbb{R}.
\]

(63)

It follows from Eq.(54) that:

\[
G = a_0 + a_1(W_1 + W_2) + a(W_1 + W_2)^2; \quad L = b_0 - \frac{a}{4}(W_1 - W_2)^2 + \frac{b}{(W_1 + W_2)^2},
\]

where all constants are real.

Thus the functions \( v_n \) \((n = 1, \ldots, 4)\) for solutions \((53) - (63)\) are, correspondingly:

\[
v_1(x) = v_2(x) = \frac{k}{\sinh^2(\lambda x)}, \\
v_3(x) = v_4(x) = \frac{k_1}{\sinh^2(\lambda x)} + \frac{k_2}{\sinh^2(\frac{\lambda x}{2})} + k_3 \cosh(2\lambda x) + k_4 \cosh(\lambda x);
\]

(64)

\[
v_1(x) = v_2(x) = \frac{k}{\cosh^2(\lambda x)}, \\
v_3(x) = \frac{k_1}{\sinh^2(\lambda x)} + \frac{k_2}{\sinh^2(\frac{\lambda x}{2})} + k_3 \cosh(2\lambda x) + k_4 \cosh(\lambda x); \\
v_4(x) = \frac{k_1 + 4k_2}{\sinh^2(\lambda x)} - \frac{k_2}{\sinh^2(\frac{\lambda x}{2})} + k_3 \cosh(2\lambda x) - k_4 \cosh(\lambda x);
\]

(65)

\[
v_1(x) = \frac{k}{\cosh^2(\lambda x)}, \quad v_2(x) = -\frac{k}{\sinh^2(\lambda x)}; \\
v_3(x) = v_4(x) = \frac{k_1 + k_2 \sinh(\lambda x)}{\cosh^2(\lambda x)} + k_3 \cosh(2\lambda x) - k_4 \sinh(\lambda x);
\]

(66)
In this Section we formulate the method of construction of new integrable systems from \( v \) of Eq. (47), which were found in previous Section. If we define new functions:

\[
\begin{align*}
  v_1(x) &= \frac{k}{\sinh^2(\lambda x)}, \quad v_2(x) = -\frac{k}{\cosh^2(\lambda x)}; \\
  v_3(x) &= \frac{k_1 + k_2 \sinh(\lambda x)}{\cosh^2(\lambda x)} + k_3 \cosh(2\lambda x) + k_4 \sinh(\lambda x); \\
  v_4(x) &= \frac{k_1 - k_2 \sinh(\lambda x)}{\cosh^2(\lambda x)} + k_3 \cosh(2\lambda x) - k_4 \sinh(\lambda x); \\
  v_1(x) &= v_2(x) = k x^{-2}; \quad v_3(x) = v_4(x) = k_1 x^{-2} + k_2 x^2 + k_3 x^4 + k_4 x^6,
\end{align*}
\]

\( (67) \)

The solutions \((55) - (57)\) are absent in the list of \([13]\) and to our knowledge they are novel. As to expressions \((69)\), they are present in \([13]\) only for \( \sigma_1 \delta_1 > 0, \sigma_2 \delta_2 > 0 \).

In conclusion, let us note that one can easily check some invariance properties of Eq. (47).

In particular, from arbitrary solutions \((52), \ (64) - (69)\) one derives new ones with:

\[
\begin{align*}
  v_4(2x) &\rightarrow v_1(x); \quad v_3(2x) \rightarrow v_2(x); \quad v_1(x) \rightarrow -v_3(x); \quad v_2(x) \rightarrow -v_4(x).
\end{align*}
\]

\( (70) \)

Eq. (47) is invariant if \( v_{1,2} \rightarrow v_{1,2} + c \) and \( v_{3,4} \rightarrow v_{3,4} + \tilde{c} \) with arbitrary constants \( c, \tilde{c} \). It is also invariant under dilatation of all arguments \( x_i \rightarrow \Lambda x_i \).

5. Integrable systems with potentials, expressed in elliptic functions.

In this Section we formulate the method of construction of new integrable systems from solutions \( v \) of Eq. (47), which were found in previous Section. If we define new functions:

\[
W_1'^2 = f_1(W_1), \quad W_2'^2 = f_2(W_2),
\]

\( (71) \)

Eq. (54) takes the form:

\[
\begin{align*}
  &G(W_1 + W_2) - L(W_1 - W_2)[f_2''(W_2) - f_1''(W_1)] + \\
  &2[G''(W_1 + W_2) - L''(W_1 - W_2)][f_2(W_2) - f_1(W_1)] + \\
  &3G'(W_1 + W_2)[f_2'(W_2) - f_1'(W_1)] + 3L'(W_1 - W_2)[f_2'(W_2) + f_1'(W_1)] = 0.
\end{align*}
\]

\( (72) \)
This equation has the same structure as Eq.(47), thus from (71) we obtain:

\begin{align}
W_1'^2 &= v_1(W_1); \\
W_2'^2 &= v_2(W_2); \\
G(W_1 + W_2) &= v_4(W_1 + W_2); \\
L(W_1 - W_2) &= -v_3(W_1 - W_2),
\end{align}

(73)

where we can use solutions (52), (64) - (69) for \( v_n(x) \) \((n = 1, ..., 4)\) with arguments \(x_{1,2}\) replaced, respectively, by \(W_{1,2}\). The first pair of equations in (73) defines functions \(W_1(x_1), W_2(x_2)\) and last two equations define new functions \(G, L\). Substituting these set of functions into (53) and (50), we find some new solutions \(V_n\) of the same Eq.(47) from already known solutions \(v_n\) (see (52), (64) - (69)).

Let us consider several examples of this method of reproducing new solutions.

1) The first attempt to start our procedure from the simplest solutions (68) leads to discouraging result: we obtain the same solution. But we can firstly transform \(v_{1,2} \iff v_{3,4}\) in (68), using the invariance property (70) mentioned at the very end of Sect.4. As follows from (73), the functions \(W_{1,2}(x_{1,2})\) are defined from the equations (we omit indices \(i = 1, 2\)):

\[W_2'(x) = k_1 W^{-2} + k_2 W^2 + k_3 W^4 + k_4 W^6 + k_0,\]

with constant \(k_i\). It is useful to rewrite them in terms of functions \(U(x) \equiv \frac{1}{2} W^2(x)\):

\[U'^2(x) = k_1 + k_0 U + k_2 U^2 + k_3 U^3 + k_4 U^4.\]

(74)

From (50) and (53) we obtain new solutions:

\[V_1 = g_1 \frac{U(x_1)}{U'^2(x_1)}, \quad V_2 = g_1 \frac{U(x_2)}{U'^2(x_2)},\]

\[V_4(x_1 + x_2) - V_3(x_1 - x_2) = g \frac{U'(x_1) U'(x_2)}{(U(x_1) - U(x_2))^2},\]

(75)

(76)

where arbitrary constants \(g_1, g\) appear because solutions \(v_n\) are defined from Eq.(17) up to constant factors. We can check directly that for \(U(x)\), which satisfy Eq.(74), r.h.s. of (76) is the solution of Eq.(51).

When r.h.s. polynomial in Eq.(74) has degenerate roots, \(U(x)\) can be expressed through elementary functions and corresponds to solutions (52), (64) - (69), found above. When all roots are simple, \(U(x)\) can be given in terms of elliptic functions.

1a) \(U(x) = \wp(x) + b\),

where \(b\) is constant and \(\wp(x)\) is the Weierstrass function with semiperiods \(\omega_1, \omega_2\) (their values
depend on constants $k_i$. Eqs. (73), (74) lead to new solutions:

$$V_1(x) = V_2(x) = a_1\varphi(x + \omega_1) + a_2\varphi(x + \omega_2) + a_3\varphi(x + (\omega_1 + \omega_2)/2);$$
$$V_3(x) = V_4(x) = a\varphi(x),$$

where $a$ is arbitrary constant and constants $a_k (k = 1, 2, 3)$ satisfy the following condition:

$$\sum_k a_k^2 - \sum_{i \neq j} a_ia_j = 1.$$

1b) $U(x) = \text{sn} \ x + b$, where $b$ is constant and $\text{sn} \ (x)$ is the Jacobi function with modulus $k$ (it depends again on constants $k_i$). In this case new solutions are:

$$V_1(x) = V_2(x) = a \frac{\text{sn} \ x + b}{\text{cn}^2 x \cdot \text{dn}^2 x};$$
$$V_3(x) = c\left(\frac{1}{\text{sn}^2(x/2)} - k^2\text{cn}^2(x/2) - k^4\right), \quad V_4(x) = c(1 - k^2)\left(\frac{1}{\text{cn}^2(x/2)} - \frac{1}{\text{sn}^2(x/2)}\right).$$

1c) $U(x) = \text{dn} \ x + b$.

Correspondingly, the new solution takes the form:

$$V_1(x) = V_2(x) = a \frac{\text{dn} x + b}{\text{sn}^2 x \cdot \text{dn}^2 x}; \quad V_3(x) = V_4(x) = c\left(\frac{1}{\text{sn}^2(x/2)} + (1 - k^2)\text{cn}^2(x/2)\right).$$

2) The second solution of (47), which we can take as the starting point of proposed procedure, is one of solutions in Eq. (52):

$$v_1(x) = v_2(x) = a \cos x + c, \quad a > 0, \quad c > 0,$$
$$v_3(x) = k_1(\sin(x/2))^{-2} + k_2(\sin(x/4))^{-2}, \quad v_4(x) = k_3(\sin(x/2))^{-2} + k_4(\sin(x/4))^{-2}.$$

Then according to Eq. (73), functions $W(x)$ must be found from equation:

$$W'^2(x) = a \cdot \cos W(x) + c,$$

which solution can be expressed through the Jacobi function with modulus $k$ ($k^2 = 2a/(a + c)$):

$$W(x) = \arccos(1 - 2(k \cdot \text{sn} y)^{-2}), \quad y = \frac{1}{2}\sqrt{(a + c)} \cdot x. \quad (77)$$

Thus the new solution of Eq. (47) is:

$$V_1(x + x_2) = a_1\text{sn}^2(y_+/2) + a_2\frac{\text{cn}^2(y_+/2)}{\text{dn}^2(y_+/2)} + a_3\frac{\text{dn}^2(y_+/2)}{\text{sn}^2(y_+/2)\text{cn}^2(y_+/2)} + a_4\frac{1}{\text{cn}^2(y_+/2)},$$
$$V_3(x_1 - x_2) = a_2\text{sn}^2(y_-/2) + a_1\frac{\text{cn}^2(y_-/2)}{\text{dn}^2(y_-/2)} + a_3\frac{\text{dn}^2(y_-/2)}{\text{sn}^2(y_-/2)\text{cn}^2(y_-/2)} + a_4\frac{1}{\text{sn}^2(y_-/2)},$$
$$V_1(x) = V_2(x) = \frac{a_0}{\text{cn}^2 y},$$
where $a_0, a_k$ are arbitrary constants.

In conclusion, we mention briefly the analogous method of construction of new integrable systems in Quantum Mechanics (see Sect.2). In this case the procedure is based on Eq.(11) of Sect.2. Similarly to (71), we introduce in (11) new functions $M_\pm(A_\pm)$:

\[ A_+^2(x_+) = M_+(A_+), \quad A_-^2(x_-) = M_-(A_-). \] (78)

Then Eq.(11) takes the form:

\[
[M_+(A_+'' - M_-(A_-''))]L(A_+ - A_-) + 3[M_+(A_+') + M_-(A_-')]L'(A_+ - A_-) + 2[M_+(A_+') - M_-(A_-')]L''(A_+ - A_-) = 0.
\]

The general solution of this equation can be found in [17] and the corresponding functions $M_\pm(A_\pm)$, $L(A_+ - A_-)$ can be used here to find $A_\pm(x)$ from (78). Substitution of these functions $A_\pm(x)$ into $M_\pm(A_\pm)$, $L(A_+ - A_-)$ and Eq.(11) leads to new solutions of the system Eqs.(6), (7) for the quantum case.

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