Pair frames
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\textbf{Abstract.} In this paper a new concept related to the frame theory is introduced: the notion of \textit{pair frame}. By investigating some properties of such frames, it is shown that pair frames are a generalization of ordinary frames. Some classes of them are introduced such as \((p, q)\)-pair frames and near identity pair frames.

\section{1. Introduction}

In 1946, Gabor \cite{14} introduced a method for reconstructing functions (signals) using a family of elementary functions. Later in 1952, Duffin and Schaeffer \cite{9} presented a similar tool in the context of nonharmonic Fourier series and this is the starting point of frame theory. After some decades, Daubechies, Grossmann and Meyer \cite{8} announced the definition of frame in the abstract Hilbert spaces.

In the past two decades, frame theory has become an interesting and fruitful field of mathematics with abundant applications in other branches of sciences. The main idea of the frame theory is reconstructing elements of a function space using some special subsets of it.

The interested readers are referred to \cite{4} and \cite{2} for more details. Some generalizations of frame significance have been presented such as fusion frames (frame of subspaces) \cite{3}, generalized frames \cite{21} and continuous frames \cite{13}.

The present paper is organized as follow: In section 2, some conditions equivalent to the concept of frames are introduced. These equivalent condition are
motivations for defining pair frames and near identity pair frames. Some other equivalent conditions for frames, based on the pair frame operator, are derived there. Also some frame-like inequalities for pair frames are given.

The \((p, q)\)-pair frames (Bessels) are introduced in section 3. The notion of near identity pair frame is presented in section 4 and some results related to this are proved.

Here we recall the definition of frames and some preliminary notations. \(\mathbb{I}\) denotes a countable index set and the subscripts \(i\) belong to \(\mathbb{I}\). \(\mathcal{H}\) is a Hilbert space with inner product \(\langle \cdot, \cdot \rangle\). A frame in \(\mathcal{H}\) is defined as below.

**Definition 1.1.** A subset \(\{f_i\} \subset \mathcal{H}\) is a frame for \(\mathcal{H}\) if there exist constants \(A, B > 0\) such that
\[
A \|f\|^2 \leq \sum_{i} |\langle f, f_i \rangle|^2 \leq B \|f\|^2 \quad \forall f \in \mathcal{H}.
\]
Constants \(A\) and \(B\) are called lower and upper frame bounds, respectively. If the upper inequality in (1.1) holds, \(\{f_i\}\) is said to be a Bessel sequence with bound \(B\).

If there is another Bessel sequence \(\{g_i\} \subset \mathcal{H}\) satisfying
\[
f = \sum_{i} \langle f, f_i \rangle g_i \quad \forall f \in \mathcal{H},
\]
then \(\{g_i\}\) is said to be a dual of \(\{f_i\}\). The above identity shows that for reconstructing \(f \in \mathcal{H}\) we need a sequence of scalars \(\{(f, f_i)\}\) and another sequence \(\{g_i\}\) of vectors of \(\mathcal{H}\). Sometimes in this paper, \(\{f_i\}\) in Definition 1.1 will be called an ordinary frame for \(\mathcal{H}\) and Definition 1.1 will be called the ordinary definition of frame, in contrast of the other types of frames which will be introduced.

For \(\{g_i\}, \{f_i\} \subset \mathcal{H}\) let the functionals
\[
\Lambda_i : \mathcal{H} \rightarrow \mathbb{C}, \quad \Gamma_i : \mathcal{H} \rightarrow \mathbb{C}.
\]
be such that
\[
\Lambda_i = \langle f, f_i \rangle \quad \text{and} \quad \Gamma_i f = \langle f, g_i \rangle, \quad \forall f \in \mathcal{H}.
\]
Then we get their adjoints as
\[
\Lambda_i^* \alpha = \alpha f_i, \quad \Gamma_i^* \alpha = \alpha g_i \quad \forall \alpha \in \mathbb{C}.
\]
Using these notations, (1.1) and (1.2) can be rewritten as
\[
A \|f\|^2 \leq \sum_{i} |\Lambda_i f|^2 \leq B \|f\|^2,
\]
and
\[
f = \sum_{i} \Gamma_i^* \Lambda_i f.
\]

From now on \(\mathcal{H}_i\) will denote a Hilbert space for every \(i \in \mathbb{I}\). \(\Lambda = \{\Lambda_i : \mathcal{H} \rightarrow \mathcal{H}_i\}\) and \(\Gamma = \{\Gamma_i : \mathcal{H} \rightarrow \mathcal{H}_i\}\) are sequences of bounded operators.
Definition 1.2. A sequence $\Lambda = \{\Lambda_i\}$ of bounded operators is called a generalize frame (or g-frame) for $\mathcal{H}$ if there exist constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_i \|\Lambda_i f\|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}. \quad (1.3)$$

Constants $A$ and $B$ are called lower and upper g-frame bounds, respectively. If the upper inequality in (1.3) holds, $\Lambda$ is said to be a g-Bessel sequence with bound $B$.

Remark 1.3. It can easily be seen that if we let $\mathcal{H}_i = \mathbb{C}$ for every $i \in \mathbb{I}$, the Definition 1.2 coincides with the definition of ordinary frames. So in what follows, the readers who are not familiar with g-frames, can treat with $\Lambda = \{\Lambda_i\}$ and $\Gamma = \{\Gamma_i\}$ as ordinary frame by putting $\mathcal{H}_i = \mathbb{C}$, $\Lambda_i f = \langle f, f_i \rangle$ and $\Gamma_i f = \langle f, g_i \rangle$ for every $i \in \mathbb{I}$ and $f \in \mathcal{H}$. Then

$$\Lambda = \{\Lambda_i\} = \{\langle , f_i \rangle\}, \quad \Gamma = \{\Gamma_i\} = \{\langle , g_i \rangle\}.$$

Our notations are similar to the notations used for g-frames, but for avoiding bustle in naming of notions we will not use the prefix ”g-“. All results in this paper are compatible with those, in the case of ordinary frames and g-frames. For this, one can put prefix ”g-“ in front of all definitions of Bessels and frames or not.

Define

$$(\sum_i \bigoplus \mathcal{H}_i)_{\ell^2} = \{\{h_i\} \mid h_i \in \mathcal{H}_i, \forall i \in \mathbb{I}, \sum_i \|h_i\|^2 < \infty\}. \quad (1.4)$$

Proposition 1.4. A sequence $\Lambda = \{\Lambda_i\}$ of bounded operators is a Bessel sequence for $\mathcal{H}$ if and only if the operator

$$S_\Lambda : \mathcal{H} \to \mathcal{H}, \quad S_\Lambda f = \sum_i \Lambda_i^* \Lambda_i f, \quad (1.5)$$

is a well defined operator. In this situation $S_\Lambda$ is bounded.

Proof. Suppose $\Lambda = \{\Lambda_i\}$ is a Bessel sequence. From the definition of Bessel sequences it is obvious that the operator

$$\Lambda : \mathcal{H} \to (\sum_i \bigoplus \mathcal{H}_i)_{\ell^2}, \quad \Lambda f = \{\Lambda_i f\}, \quad (1.6)$$

is a well defined and bounded operator. Therefor its adjoint is

$$\Lambda^* : (\sum_i \bigoplus \mathcal{H}_i)_{\ell^2} \to \mathcal{H}, \quad \Lambda^*(\{h_i\}) = \sum_i \Lambda_i^* h_i.$$

Hence

$$S_\Lambda := \Lambda^* \Lambda : \mathcal{H} \to \mathcal{H}, \quad S_\Lambda f = \sum_i \Lambda_i^* \Lambda_i f,$$
is a well defined and bounded operator. For the converse, let the operator defined in (1.5) be a well defined and bounded operator. Then for every \( f \in \mathcal{H} \),

\[
0 \leq \sum_i \|\Lambda_i f\|^2 = \sum_i \langle \Lambda_i f, \Lambda_i f \rangle = \sum_i \langle \Lambda_i^* \Lambda_i f, f \rangle = \langle S_\Lambda f, f \rangle
\]

\[
= |\langle S_\Lambda f, f \rangle| \leq \|S_\Lambda\| \|f\|^2.
\]

By putting \( B = \|S_\Lambda\| \) we get

\[
\sum_i \|\Lambda_i f\|^2 \leq B \|f\|^2,
\]

for every \( f \in \mathcal{H} \).

In this paper \( \mathcal{B}(\mathcal{H}) \) denotes the set of all bounded operators on \( \mathcal{H} \) and for \( T \in \mathcal{B}(\mathcal{H}) \), \( \mathcal{N}(T) \) and \( \mathcal{R}(T) \) denote the kernel and the range of \( T \), respectively. A partial order on \( \mathcal{B}(\mathcal{H}) \) can be defined as follows. For \( T, U \in \mathcal{B}(\mathcal{H}) \) we write \( T \leq U \) (\( T < U \)) whenever

\[
\langle Tf, f \rangle \leq \langle Uf, f \rangle \quad (\langle Tf, f \rangle < \langle Uf, f \rangle, f \neq 0) \quad \forall f \in \mathcal{H}.
\]

A self adjoint operator \( T \in \mathcal{B}(\mathcal{H}) \) is called nonnegative (positive) if

\[
0 \leq \langle Tf, f \rangle \quad (0 < \langle Tf, f \rangle, f \neq 0) \quad \forall f \in \mathcal{H}.
\]

As it was seen in the Proposition 1.4 and its proof, \( \Lambda \) denotes simultaneously the sequence of bounded operators \( \{\Lambda_i\} \) and corresponding bounded operator from \( \mathcal{H} \) into \( (\sum_i \oplus \mathcal{H}_i)_{\ell^2} \).

### 2. Pair Frames

Next theorem present some equivalent conditions for frames and is a useful motivation for defining ”pair frames” and ”near identity pair frames”.

**Theorem 2.1.** For a sequence \( \Lambda = \{\Lambda_i\} \) of bounded operators and its corresponding operator \( S_\Lambda \) defined in (1.5), the following statments are equivalent:

1. \( \Lambda = \{\Lambda_i\} \) is a frame.
2. \( S_\Lambda \) is welldefined and there exist constants \( A, B > 0 \) such that

\[
A \leq S_\Lambda \leq B.
\]

3. \( S_\Lambda \) is welldefined, bounded and there exists \( \alpha \in (0, \infty) \) such that

\[
\|I - \alpha S_\Lambda\| < 1.
\]

4. \( S_\Lambda \) is welldefined, bounded and invertible.
5. \( S_\Lambda \) is welldefined, bounded and surjective from \( \mathcal{H} \) into \( \mathcal{H} \).
Proof. From Proposition 1.4 we know that the sequence \( \Lambda = \{ \Lambda_i \} \) is a Bessel sequence if and only if \( S_{\Lambda} \) is a welldefined and bounded operator. Using notations mentioned above and the proof of Proposition 1.4, it results that there is a constant \( B > 0 \) such that

\[
0 \leq \sum_i \| \Lambda_i f \|^2 \leq B \| f \|^2 \quad \forall f \in \mathcal{H}.
\]

Since

\[
\sum_i \| \Lambda_i f \|^2 = \langle S_{\Lambda} f, f \rangle \quad \forall f \in \mathcal{H},
\]

then

\[
0 \leq S_{\Lambda} \leq B.
\]

(1) \( \leftrightarrow \) (2). \( \Lambda = \{ \Lambda_i \} \) is a Bessel sequence if and only if \( S_{\Lambda} \) is welldefined. Additionally, \( \Lambda = \{ \Lambda_i \} \) is a frame with bounds \( A, B > 0 \) if and only if \( A \leq S_{\Lambda} \leq B \).

(2) \( \rightarrow \) (3). It is proved in the standard textbooks of frame theory (see for example [4]). But we restate the proof here. Since \( A \leq S_{\Lambda} \leq B \), then

\[
\frac{A}{B} \leq \frac{1}{B} S_{\Lambda} \leq 1.
\]

Therefore

\[
0 \leq 1 - \frac{1}{B} S_{\Lambda} = 1 - \frac{A}{B} \leq 1.
\]

By putting \( \alpha = \frac{1}{B} \) we get

\[
\| I - \alpha S_{\Lambda} \| \leq 1 - \frac{A}{B} \leq 1.
\]

(3) \( \rightarrow \) (2). Let \( \| I - \alpha S_{\Lambda} \| < 1 \) for some \( \alpha \in (0, \infty) \). Put \( C = 1/\alpha \). Then a positive number \( D \) can be found such that \( 0 < D < C \) and

\[
\| I - \alpha S_{\Lambda} \| < D/C < 1.
\]

Put \( A = C - D \). Then

\[
0 \leq \| I - \frac{1}{C} S_{\Lambda} \| \leq \frac{C - A}{C} < 1.
\]

On the other hand

\[
\frac{C - A}{C} = \langle \frac{C - A}{C} f, f \rangle \quad \forall f \in \mathcal{H}, \quad \| f \| = 1. \tag{2.1}
\]

Since \( S_{\Lambda} \) and hence \( I - \frac{1}{C} S_{\Lambda} \) are selfadjoint,

\[
\sup_{\| f \| = 1} | \langle (I - \frac{1}{C} S_{\Lambda}) f, f \rangle | = \| I - \frac{1}{C} S_{\Lambda} \| \leq \frac{C - A}{C}. \tag{2.2}
\]

Self adjointness of \( I - \frac{1}{C} S_{\Lambda} \) causes \( \langle (I - \frac{1}{C} S_{\Lambda}) f, f \rangle \) to be real for every \( f \in \mathcal{H} \). By keeping this in mind and using (2.1) and (2.2) we get

\[
\langle (I - \frac{1}{C} S_{\Lambda}) f, f \rangle \leq \langle \frac{C - A}{C} f, f \rangle \quad \forall f \in \mathcal{H}, \quad \| f \| = 1.
\]

A simple calculation implies that

\[
\langle Af, f \rangle \leq \langle S_{\Lambda} f, f \rangle \quad \forall f \in \mathcal{H}, \quad \| f \| = 1.
\]
Therefore
\[ A\|f\|^2 = \langle Af, f \rangle \leq \langle S_\Lambda f, f \rangle \quad \forall f \in \mathcal{H}. \tag{2.3} \]

Since
\[ \langle Af, f \rangle = A\|f\|^2 \geq 0 \quad \forall f \in \mathcal{H}, \tag{2.4} \]
the relation (2.3) yields that \( \langle S_\Lambda f, f \rangle \) is nonnegative for every \( f \in \mathcal{H} \). So
\[ \langle S_\Lambda f, f \rangle = |\langle S_\Lambda f, f \rangle| \quad \forall f \in \mathcal{H}. \tag{2.5} \]

Relations (2.3) and (2.5) together imply that
\[ A \leq S_\Lambda \leq B. \]

(3) → (4). It is a known result in the operator theory (see [7]).

(4) → (2). It is proved in Proposition 2.7 of [16], but for the sake of completeness we restate the proof. \( S \) is bounded and positive therefore \( \sigma(S) \subset [0, \infty) \), where \( \sigma(S) \) indicates the spectrum of \( S \). Invertibility of \( S \) implies that \( \sigma(S) \) does not contain zero. We know \( \sigma(S) \) is compact. Hence there are nonnegative numbers \( A \) and \( B \) such that \( 0 < A \leq B < \infty \) and \( \sigma(S) \subset [A, B] \). Therefore \( A \leq S \leq B \).

(4) → (5). It is obvious.

(5) → (4). It is enough to show that \( S_\Lambda \) is one to one. Since \( S_\Lambda \) is welldefined and onto then
\[ \mathcal{N}(S_\Lambda) = \mathcal{N}(S_\Lambda^*) = \mathcal{R}(S_\Lambda)^\perp = \{0\}. \]

Welldefinedness and invertibility of \( S_\Lambda \) in Theorem 2.1 is the essence of our new definitions. By \( m = \{m_i\} \) we mean a sequence of scalars in \( \mathbb{C} \).

**Definition 2.2.** For two sequences \( \Lambda = \{\Lambda_i\}, \Gamma = \{\Gamma_i\} \) of bounded operators and a scalar sequence \( m = \{m_i\} \), we say the triple \((m, \Gamma, \Lambda)\) is an \textbf{m-pair Bessel} for \( \mathcal{H} \) if the operator
\[ S_{m\Gamma\Lambda} : \mathcal{H} \to \mathcal{H}, \quad S_{m\Gamma\Lambda} f = \sum_i m_i \Gamma_i^* \Lambda_i f, \tag{2.6} \]
is welldefined, i.e. the series converges for every \( f \in \mathcal{H} \). If the series converges unconditionally we will call \((m, \Gamma, \Lambda)\) an \textbf{unconditional m-pair Bessel}. If \( m = \{1\} \), \((\Gamma, \Lambda)\) will be called a \textbf{pair Bessel}.

By principle of uniform boundedness, the operator \( S_{m\Gamma\Lambda} \) is bounded. Sometimes \( S_{m\Gamma\Lambda} \) will be written in the form of \( m\Gamma^*\Lambda \) in this paper.

**Definition 2.3.** Suppose \( \Lambda = \{\Lambda_i\}, \Gamma = \{\Gamma_i\} \) and \( m = \{m_i\} \) are as in the Definition 2.2. Let \((m, \Gamma, \Lambda)\) be an \textit{m-pair Bessel} for \( \mathcal{H} \). We say that \((m, \Gamma, \Lambda)\) is an \textbf{m-pair frame} for \( \mathcal{H} \) if the operator \( S_{m\Gamma\Lambda} \) defined in (2.6) is invertible. In the case of \( m = \{1\} \), \((\Gamma, \Lambda)\) will be called a \textbf{pair frame}.
Similar to the $m$-pair Bessels, with respect to the type of convergence of the series in (2.6), we can define **unconditional $m$-pair frame**.

If $(m, \Gamma, \Lambda)$ is an $m$-pair Bessel(frame), then $(\overline{m}\Gamma, \Lambda)$ and $(\Gamma, m\Lambda)$ are pair Bessel(frame). Also, every pair Bessel(frame) is an $m$-pair Bessel (frame) by putting $m = \{1\}$. For this, sometimes instead of calling $(m, \Gamma, \Lambda)$ an $m$-pair Bessel(frame), we call it a pair Bessel(frame) simply.

In the case of $(m, \Gamma, \Lambda)$ being an $m$-pair Bessel, Balazs [1] called $S_{m\Gamma\Lambda}$ a **multiplier operator**. Also the invertibility of these operators are studied in [19].

There are examples for which $S_{m\Gamma\Lambda}^* \neq S_{m\Lambda\Gamma}$; even welldefinedness of $S_{m\Gamma\Lambda}$ does not imply the welldefinedness of $S_{m\Lambda\Gamma}$ in general [20, Remark 3.4]. The equality $S_{m\Gamma\Lambda}^* = S_{m\Lambda\Gamma}^*$ holds under some certain condition considered in Theorem 2.5. For proving that theorem, we need the following lemma.

**Lemma 2.4.** [15, 17, 18] For a sequence $\{f_i\} \subset \mathcal{H}$, the following statements are equivalent:

1. $\sum_i f_i$ converges unconditionally.
2. $\sum_i f_j$ converges for every $\{f_j\} \subset \{f_i\}$.
3. $\sum_i f_j$ converges weakly for every $\{f_j\} \subset \{f_i\}$.

**Theorem 2.5.** For two sequences $\Lambda = \{\Lambda_i\}$, $\Gamma = \{\Gamma_i\}$ of bounded operators and a scalar sequence $m = \{m_i\}$,

1. If $(m, \Gamma, \Lambda)$ and $(\overline{m}, \Lambda, \Gamma)$ are pair Bessels, then $S_{m\Gamma\Lambda}^* = S_{m\Lambda\Gamma}$. Additionally, in this case when $(m, \Gamma, \Lambda)$ is a pair frame, so is $(\overline{m}, \Lambda, \Gamma)$.
2. $(m, \Gamma, \Lambda)$ is an unconditional pair Bessel(frame) if and only if $(\overline{m}, \Lambda, \Gamma)$ is an unconditional pair Bessel(frame). In this case $S_{m\Gamma\Lambda}^* = S_{m\Lambda\Gamma}$.

**Proof.** (1). Since $(m, \Gamma, \Lambda)$ and $(\overline{m}, \Lambda, \Gamma)$ are pair Bessels, $S_{m\Gamma\Lambda}$ and $S_{m\Lambda\Gamma}$ are welldefined and for every $f, g \in \mathcal{H}$,

$$\langle S_{m\Lambda\Gamma}f, g \rangle = \left\langle \sum_i \overline{m_i}\Lambda_i^*\Gamma_i f, g \right\rangle = \sum_i \langle \overline{m_i}\Lambda_i^*\Gamma_i f, g \rangle = \sum_i \langle f, m_i\Gamma_i^*\Lambda_i g \rangle = \langle f, S_{m\Gamma\Lambda}g \rangle.$$ 

Thus $S_{m\Gamma\Lambda}^* = S_{m\Lambda\Gamma}$.

For the pair frame case, invertibility of $S_{m\Gamma\Lambda}$ results invertibility of $S_{m\Gamma\Lambda}^* = S_{m\Lambda\Gamma}$.

(2). Theorem 2.5 is proved in an informally published article [20]. For the sake of completeness the proof is stated completely here. $(m, \Gamma, \Lambda)$ is an unconditional pair Bessel if and only if for every $f \in \mathcal{H}$, $\sum_i m_i\Gamma_i^*\Lambda_i f$ converges unconditionally for every $f \in \mathcal{H}$. By Lemma 2.4 it is equivalent to the fact that $\sum_{i \in J} m_i\Gamma_i^*\Lambda_i f$ converges for every subset $J$ of $\mathbb{I}$ and $f \in \mathcal{H}$. Again by Lemma 2.4 this means that

$$\left\langle \sum_{i \in J} m_i\Gamma_i^*\Lambda_i f, g \right\rangle = \left\langle \sum_{i \in J} f, \overline{m_i}\Lambda_i^*\Gamma_i g \right\rangle.$$ 

for every subset $J$ of $\mathbb{I}$ and $f, g \in \mathcal{H}$. This means for every subset $J$ of $\mathbb{I}$ and $g \in \mathcal{H}$, $\sum_{i \in J} \overline{m_i}\Lambda_i^*\Gamma_i g$ converges weakly. So the above lemma implies that $S_{m\Lambda\Gamma}$
is unconditionally welldefined. In the other word \((m, \Lambda, \Gamma)\) is an unconditional pair Bessel.

The frame case is a consequence of the invertibility of the adjoint of an invertible operator.

More studies on this field are done in [11] where the concept of adjoint of pair frames is introduced in Banach space setting. Pair frames (Bessels) are generalizations of frames (Bessels sequences). If we use only a single sequence \(\Lambda\) in the pair frame (Bessel) definition instead two sequence \(\Lambda, \Gamma\) and putting \(m = \{1\}\) we get a frame (Bessels sequences). In fact:

**Corollary 2.6.** A sequence \(\Lambda = \{\Lambda_i\}\) of bounded operators is a frame (Bessel sequence) if and only if \((\Lambda, \Lambda)\) is a pair frame (Bessel).

**Proof.** It is an straightforward consequence of Theorem 2.1. □

Let \(T \in B(\mathcal{H})\). \(T\) is called bounded below if

\[
0 < \inf_{\|f\|=1} \|Tf\|
\]

For every \(T \in B(\mathcal{H})\) define

\[
||T|| = \inf_{\|f\|=1} \|Tf\|.
\]

Hence \(T\) is bounded below if and only if \(0 < ||T||\). For the constants \(A, B > 0\), we write

\[
A \leq ||T|| \leq B, \quad \forall f \in \mathcal{H}.
\]

**Lemma 2.7.** Let \(S \in B(\mathcal{H})\). The following statements are equivalent:

1. \(S\) is invertible.
2. \(S\) and \(S^*\) are bounded below.
3. \(S\) and \(S^*\) are injective and have closed ranges.

**Proof.** (1) \(\rightarrow\) (2). When \(S\) is invertible so is \(S^*\). Invertible operators are bounded below.

(2) \(\leftrightarrow\) (3). It is proved in [7, III.12.Ex5].

(3) \(\rightarrow\) (1). It suffices to show that \(S\) is onto. Since \(S^*\) is one to one and \(S\) has closed range

\[
\mathcal{R}(S) = \overline{\mathcal{R}(S)} = \mathcal{N}(S^*)^\perp = \mathcal{H}.
\]

□

We combine Theorem 2.1 and Lemma 2.7 to get some equivalent conditions for a Bessel sequence to be a frame based on its frame operator.

**Corollary 2.8.** Suppose \(\Lambda = \{\Lambda_i\}\) is a Bessel sequence. Then the following statements are equivalent:

1. \(\Lambda = \{\Lambda_i\}\) is a frame.
There exist constants $A, B > 0$ such that
$$A \leq |\lfloor S_\Lambda \rfloor| \leq B.$$  

(3) There exist constants $A, B > 0$ such that
$$A \leq S_\Lambda \leq B.$$  

(4) $S_\Lambda$ is injective and has closed range.

(5) $S_\Lambda$ is surjective from $\mathcal{H}$ into $\mathcal{H}$.

(6) $S_\Lambda$ is invertible.

**Theorem 2.9.** Let $\Lambda = \{\Lambda_i\}$ and $\Gamma = \{\Gamma_i\}$ be two sequences of bounded operators and $m = \{m_i\}$ be a scalar sequence. Assume $(m, \Gamma, \Lambda)$ is a pair Bessel.

1. If there exists a constant $A > 0$ such that
   $$A \|f\|^2 \leq \left| \sum_i \langle m_i \Lambda_i f, \Gamma_i f \rangle \right| \quad \forall f \in \mathcal{H},$$
   then $(m, \Gamma, \Lambda)$ is a pair frame. In this case we obtain a frame-like inequalities i.e. there will be a constant $B > 0$ such that
   $$A \|f\|^2 \leq \left| \sum_i \langle m_i \Lambda_i f, \Gamma_i f \rangle \right| \leq B \|f\|^2 \quad \forall f \in \mathcal{H}. \quad (2.7)$$

2. Additionally suppose $(\overline{m}, \Lambda, \Gamma)$ is a pair Bessel. $(m, \Gamma, \Lambda)$ (or $(\overline{m}, \Lambda, \Gamma)$) is a pair frame if and only if there are constants $A, B, A', B' > 0$ such that
   $$A \|f\|^2 \leq \left| \sum_i \langle m_i \Lambda_i f, \Gamma_i f \rangle \right| \leq B \|f\|^2 \quad \forall f \in \mathcal{H}. \quad (2.8)$$

   and
   $$A' \|f\|^2 \leq \left| \sum_i \langle \overline{m_i} \Gamma_i f, \Lambda_i f \rangle \right| \leq B' \|f\|^2 \quad \forall f \in \mathcal{H}. \quad (2.9)$$

**Proof.** Since $(m, \Gamma, \Lambda)$ is a pair Bessel, $S_{m\Gamma\Lambda}$ is welldefined and for $f \in \mathcal{H}$
$$\langle S_{m\Gamma\Lambda} f, f \rangle = \sum_i \langle m_i \Lambda_i f, \Gamma_i f \rangle.$$  

(1). By Cauchy-Schwarz inequality we have
$$A \|f\|^2 \leq |\langle S_{m\Gamma\Lambda} f, f \rangle| \leq \|S_{m\Gamma\Lambda} f\| \|f\| \quad \forall f \in \mathcal{H}. \quad (2.11)$$

Also
$$A \|f\|^2 \leq |\langle f, S_{m\Gamma\Lambda}^* f \rangle| \leq \|S_{m\Gamma\Lambda}^* f\| \|f\| \quad \forall f \in \mathcal{H}. \quad (2.12)$$

Relations (2.11) and (2.12) imply that $S_{m\Gamma\Lambda}$ and $S_{m\Gamma\Lambda}^*$ are bounded below. Therefore Lemma 2.7 shows that $S_{m\Gamma\Lambda}$ is invertible and hence $(m, \Gamma, \Lambda)$ is a pair frame. Now put $B = \|S_{m\Gamma\Lambda}\|$ to obtain (2.8).

(2). Since $(\overline{m}, \Lambda, \Gamma)$ is a pair Bessel, $S_{\overline{m}\Lambda\Gamma}$ is welldefined and for $f \in \mathcal{H}$
$$\langle S_{\overline{m}\Lambda\Gamma} f, f \rangle = \sum_i \langle \overline{m_i} \Gamma_i f, \Lambda_i f \rangle.$$  

Assume that $(m, \Gamma, \Lambda)$ is a pair frame. By Theorem 2.5 it can be concluded that $(\overline{m}, \Lambda, \Gamma)$ is also a pair frame. Hence the operators $S_{m\Gamma\Lambda}$ and $S_{\overline{m}\Lambda\Gamma}$ are invertible and therefore bounded below by Lemma 2.7. By this and the fact that $S_{m\Gamma\Lambda}$ and
are bounded we conclude that there are constants $A, B, A', B' > 0$ such that relations (2.9) and (2.10) holds.

Conversely assume that (2.9) and (2.10) holds for some constants $A, B, A', B' > 0$. Then both $S_m\Gamma\Lambda$ and $S_m\pi\Lambda\Gamma$ are bounded below and therefore by Lemma 2.7 they are invertible. Hence $(m, \Gamma, \Lambda)$ and $(\bar{m}, \Lambda, \Gamma)$ are pair frames. □

The frame inequality (1.3) can be written in the form
\[ A\|f\|^2 \leq \left| \sum_i \langle \Lambda_i f, \Lambda_i f \rangle \right| \leq B\|f\|^2 \quad \forall f \in \mathcal{H}. \] (2.13)

Then it can be seen that the inequalities in Theorem 2.9 are similar to the inequalities in ordinary frame definition (see (2.13)). In fact if instead of using the pairs $\Gamma$ and $\Lambda$, one put $\Lambda = \Gamma$ and $m = \{1\}$, then all of the relations (2.8), (2.9) and (2.10) coincide with the inequalities in the definition of frames (see (2.13)).

If $T \in B(\mathcal{H})$, for the real constants $A, B$, let we write
\[ A \leq |\langle T \rangle| \leq B, \] (2.14)
whenever
\[ A\|f\|^2 \leq |\langle Tf, f \rangle| \leq B\|f\|^2 \quad \forall f \in \mathcal{H}. \]

When $T = T^*$ and $A, B \geq 0$,
\[ A \leq T \leq B, \]
if and only if
\[ A \leq |\langle T \rangle| \leq B. \]

Because
\[ 0 \leq \langle Tf, f \rangle = |\langle Tf, f \rangle|, \quad \forall f \in \mathcal{H}. \]

Namely, when $T = T^*$ and $A, B \geq 0$, the notations $A \leq T \leq B$ and $A \leq |\langle T \rangle| \leq B$ are the same. For example in the case of $\Lambda$ being a frame, put $T = S_{\Lambda}$. With this notations, we can summarize the above results in the following corollary.

**Corollary 2.10.** Let $\Lambda = \{\Lambda_i\}$ and $\Gamma = \{\Gamma_i\}$ be two sequences of bounded operators and $m = \{m_i\}$ be a scalar sequence. Then the following statements are equivalent:

1. $(m, \Gamma, \Lambda)$ and $(\bar{m}, \Lambda, \Gamma)$ are pair frames.

2. There exist constants $A, B, A', B' > 0$ such that
\[ A \leq |\langle S_m\Gamma\Lambda \rangle| \leq B, \quad A' \leq |\langle S_m\pi\Lambda\Gamma \rangle| \leq B'. \]

3. There exist constants $A, B, A', B' > 0$ such that
\[ A \leq |\langle S_m\Gamma\Lambda \rangle| \leq B, \quad A' \leq |\langle S_m\pi\Lambda\Gamma \rangle| \leq B'. \]

4. $S_m\Gamma\Lambda$ and $S_m\pi\Lambda\Gamma$ are injective and have closed ranges.

5. $S_m\Gamma\Lambda$ and $S_m\pi\Lambda\Gamma$ are surjective from $\mathcal{H}$ into $\mathcal{H}$.

6. $S_m\Gamma\Lambda$ and $S_m\pi\Lambda\Gamma$ are invertible.
3. \((p, q)\)-Pair Frames

In this section we will introduce an important class of the pair Bessels and pair frames.

**Definition 3.1.** Let \(\Lambda = \{\Lambda_i\}\) be a sequences of bounded operators and \(1 \leq p < \infty\). \(\Lambda\) is called a \(p\)-frame for \(\mathcal{H}\) if there are constants \(A, B > 0\) such that
\[
A \|f\|^p \leq \sum_i \|\Lambda_i f\|^p \leq B \|f\|^p \quad \forall f \in \mathcal{H}.
\] (3.1)

\(A\) and \(B\) are called upper and lower \(p\)-frame bounds, respectively. If the right hand inequality of (3.1) is satisfied for some constant \(B > 0\), \(\Lambda\) is called a \(p\)-Bessel sequence for \(\mathcal{H}\) with bound \(B\).

\(p\)-frames in Banach spaces are considered in [6].

**Definition 3.2.** Let \(\Lambda = \{\Lambda_i\}\) and \(\Gamma = \{\Gamma_i\}\) be two sequences of bounded operators and \(m = \{m_i\}\) be a scalar sequence. If \(1 \leq p, q < \infty\) with \(1/p + 1/q = 1\), we say that \((m, \Gamma, \Lambda)\) is an \(m\)-(\(p, q\))-pair Bessel for \(\mathcal{H}\) if \(m \in \ell^\infty\) and \(\Gamma, \Lambda\) are \(p\)-Bessel sequence and \(q\)-Bessel sequence, respectively. When \(m = \{1\}\), \((\Gamma, \Lambda)\) will be said to be a \((p, q)\)-pair Bessel.

If \(\Lambda = \{\Lambda_i\}\) is a Bessel sequence, it is a \((2, 2)\)-pair Bessel sequence. Hence Bessel sequences form a subclass of the pair Bessels. In what follows we show that \(m\)-(\(p, q\))-pair Bessels are really pair Bessels; in fact, they are unconditionally pair Bessels.

**Theorem 3.3.** Let \(\Lambda = \{\Lambda_i\}\), \(\Gamma = \{\Gamma_i\}\) and \(m = \{m_i\}\) be as above. If \((m, \Gamma, \Lambda)\) is an \(m\)-(\(p, q\))-pair Bessel then it is an unconditionally \(m\)-pair Bessel and
\[
\|S_{m \Gamma \Lambda}\| \leq \sqrt{\|m\| \infty B^{1/p} B'^{1/q}},
\]
where \(B, B'\) are Bessel sequence bounds of \(\Gamma\) and \(\Lambda\), respectively.

**Proof.** According to the definition, \(\Gamma\) and \(\Lambda\) are \(p\)-Bessel sequence and \(q\)-Bessel sequence, respectively. For a finite set \(J \subset \mathbb{I}\) and \(f \in \mathcal{H}\) put \(g = \sum_{i \in J} m_i \Gamma_i^* \Lambda_i f\). Then
\[
\left\| \sum_{i \in J} m_i \Gamma_i^* \Lambda_i f \right\|^2 = |\langle g, g \rangle| = \left| \sum_{i \in J} \langle m_i \Gamma_i^* \Lambda_i f, g \rangle \right|
\]
\[
\leq \sum_{i \in J} |m_i| \langle \Lambda_i f, \Gamma_i g \rangle \leq \sum_{i \in J} |m_i| \|\Lambda_i f\| \|\Gamma_i g\|
\]
\[
\leq (\sup_i |m_i|) \left( \sum_{i \in J} \|\Lambda_i f\|^q \right)^{1/q} \left( \sum_{i \in J} \|\Gamma_i f\|^p \right)^{1/p}
\]
\[
\leq (\sup_i |m_i|) B'^{1/q} \|f\| \left( \sum_{i \in J} \|\Gamma_i f\|^p \right)^{1/p}.
\]
This implies that \( \sum_{i \in J} m_i \Gamma_i^* \Lambda_i f \) converges unconditionally for every \( f \in \mathcal{H} \). Thus \( S_{m\Gamma\Lambda} \) is welldefined unconditionally. The above computations imply that

\[
\| S_{m\Gamma\Lambda} f \| = \left\| \sum_i m_i \Gamma_i^* \Lambda_i f \right\| \leq \sqrt{\| m \|_{\infty} B_{1p} B'_{1q} \| f \|}.
\]

\[\square\]

In the case that \((m, \Gamma, \Lambda)\) is an \(m-\(p, q\))-pair Bessel, \((m, \Gamma, \Lambda)\) will be called an \(m-\(p, q\)\)-pair frame if \(S_{m\Gamma\Lambda}\) is invertible. In the other contexts \(p\)-Bessel sequence and \(q\)-Bessel sequence are called \(\ell^p\)-Bessel sequence and \(\ell^q\)-Bessel sequence. Theorem 3.3 shows that \(\ell^p\)-Bessel sequences and \(\ell^q\)-Bessel sequences are pairable i.e. a pair of sequences of these types can construct a pair Bessel. If we let \( \ell = \ell^p \), then \( \ell^* = \ell^q \) and \( \ell, \ell^* \) are pairable. For a more general scalar sequence space \( \ell \), it is proved that \( \ell \) and \( \ell^* \) are pairable [11].

### 4. Near Identity Pair Frames

Christenson and Laugesen [5] introduced the notion of ”approximately dual frames” in the context of ordinary Bessel sequences. Here we present a new notion similar to ”approximately dual frames” which extends that notion. Also some results in [5] are generalized in this section.

**Definition 4.1.** Let \( \Lambda = \{\Lambda_i\} \) and \( \Gamma = \{\Gamma_i\} \) be two sequences of bounded operators and \( m = \{m_i\} \) be a scalar sequence. A pair Bessel \((m, \Gamma, \Lambda)\) will be called a near identity pair frame if there exists a nonzero \( \alpha \in \mathbb{C} \) such that

\[
\| I - \alpha S_{m\Gamma\Lambda} \| < 1.
\]

\((m, \Gamma, \Lambda)\) will be said to be a positively near identity pair frame if \( \alpha \in (0, \infty) \) and \( S_{m\Gamma\Lambda} \) is self adjoint.

In Theorem 4.3(1), it is proved that every near identity pair frame is really a pair frame. The operator \( S_{m\Gamma\Lambda} \) corresponding to a near identity pair frame \((m, \Gamma, \Lambda)\) can be regarded as a perturbation of the identity operator. With our definition, ”approximately dual frames” are ”positively near identity (2, 2)-pair frame”. In the other words approximately dual frames are a very special case of pair frames or even near identity pair frames. Also every frame is a positively near identity (2, 2)-pair frame; see 2.1(3). Figure (1) shows the relations between some different kinds of frames proposed here. The following result shows that the nature of positively near identity pair frames are very similar to ordinary frames.

**Proposition 4.2.** Let \( \Lambda = \{\Lambda_i\} \) and \( \Gamma = \{\Gamma_i\} \) be two sequences of bounded operators and \( m = \{m_i\} \) be a scalar sequence. \((m, \Gamma, \Lambda)\) is a positively near identity pair frame if and only if there are constants \( A, B > 0 \) such that

\[
A \leq S_{m\Gamma\Lambda} \leq B.
\]
Proof. Since $S_{m\Gamma\Lambda}$ is selfadjoint and $\alpha$ is positive, a reasoning like the proof of the equivalence $(2) \leftrightarrow (3)$ in Theorem 2.1, establishes the claim. It is enough to replace $S_\Lambda$ with $S_{m\Gamma\Lambda}$. \qed

Figure 1.

For $T \in \mathcal{B}(\mathcal{H})$ and a sequences of bounded operators $\Lambda = \{\Lambda_i\}$, let $\Lambda T = \{\Lambda_i T\}$.

**Theorem 4.3.** Let $\Lambda = \{\Lambda_i\}$ and $\Gamma = \{\Gamma_i\}$ be two sequences of bounded operators and $m = \{m_i\}$ be a scalar sequence. Also suppose that $V, W \in \mathcal{B}(\mathcal{H})$. Then

1. If $(m, \Gamma, \Lambda)$ is a near identity pair frame, then it is a pair frame.
2. If $(m, \Gamma, \Lambda)$ is a pair Bessel, then $(m, \Gamma V, \Lambda W)$ is a pair Bessel.
3. If $(m, \Gamma, \Lambda)$ is a pair frame and $V, W$ are invertible, then $(m, \Gamma V, \Lambda W)$ is a pair frame.

Proof. Recall that when $(m, \Gamma, \Lambda)$ is a pair Bessel, then the operator $S_{m\Gamma\Lambda}$ is welldefined.

(1). If $\|I - \alpha S_{m\Gamma\Lambda}\| < 1$, for some nonzero $\alpha \in \mathbb{C}$, then $\alpha S_{m\Gamma\Lambda}$ is invertible, [7, VII.2.1]. Since $\alpha$ is nonzero, then $S_{m\Gamma\Lambda}$ is invertible and hence $(m, \Gamma, \Lambda)$ is a pair frame.

(2). Clearly the equation $V^* S_{m\Gamma\Lambda} W = \sum_i m_i (\Gamma_i V)^* \Lambda_i W = m (\Gamma V)^* \Lambda W = S_{m,\Gamma V,\Lambda W}$ proves (2).

(3). We know that the invertibility of $V, S_{m\Gamma\Lambda}$ and $W$ imply the invertibility of $V^* S_{m\Gamma\Lambda} W = S_{m,\Gamma V,\Lambda W}$. This fact and (2) prove (3). \qed
The importance of the "near identity pair frames" is that the inverse of its corresponding operator $S_{m\Gamma\Lambda}$, can be written in "Neumann series" as

$$S_{m\Gamma\Lambda}^{-1} = \alpha \sum_{n=0}^{\infty} (I - \alpha S_{m\Gamma\Lambda})^n,$$

which is very useful in the computational aspects. Next theorem gives a sequence of operators $\{J^{(N)}\}_{N=0}^{\infty}$ which converges to the identity operator.

**Proposition 4.4.** Let $\Lambda = \{\Lambda_i\}$ and $\Gamma = \{\Gamma_i\}$ be two sequences of bounded operators and $m = \{m_i\}$ be a scalar sequence. Suppose $(m, \Gamma, \Lambda)$ is a near identity pair frame for some nonzero $\alpha \in \mathbb{C}$. For every $N \in \mathbb{N}$, define

$$(S_{m\Gamma\Lambda}^{-1})_N = \alpha \sum_{n=0}^{N} (I - \alpha S_{m\Gamma\Lambda})^n,$$

and let

$$J^{(N)} = \sum_{i=1}^{\infty} m_i (S_{m\Gamma\Lambda}^{-1})_N \Gamma_i^* \Lambda_i.$$

Then

$$\|I - J^{(N)}\| \to 0, \quad \text{as} \quad N \to \infty,$$

and

$$\|I - J^{(N)}\| \leq \|I - \alpha S_{m\Gamma\Lambda}\|^{N+1}.$$

for every $N \in \mathbb{N}$.

**Proof.** By the definition of near identity pair frames we have

$$\|I - \alpha S_{m\Gamma\Lambda}\| < 1.$$

Therefore

$$S_{m\Gamma\Lambda}^{-1} = \alpha \sum_{n=0}^{\infty} (I - \alpha S_{m\Gamma\Lambda})^n,$$

and its partial sum is

$$(S_{m\Gamma\Lambda}^{-1})_N = \alpha \sum_{n=0}^{N} (I - \alpha S_{m\Gamma\Lambda})^n,$$
for every \( N \in \mathbb{N} \). So
\[
J^{(N)} = \sum_{i=1}^{\infty} m_i (S^{-1}_{m\Gamma\Lambda})^N \Gamma_i^* \Lambda_i
\]
\[
= (S^{-1}_{m\Gamma\Lambda})^N \sum_{i=1}^{\infty} m_i \Gamma_i^* \Lambda_i
\]
\[
= (\alpha \sum_{n=0}^{N} (S^{-1}_{m\Gamma\Lambda})^n) S_{m\Gamma\Lambda}
\]
\[
= (\sum_{n=0}^{N} (I - \alpha S_{m\Gamma\Lambda})^n) \alpha S_{m\Gamma\Lambda}
\]
\[
= (\sum_{n=0}^{N} (I - \alpha S_{m\Gamma\Lambda})^n) (I - (I - \alpha S_{m\Gamma\Lambda}))
\]
\[
= I - (I - \alpha S_{m\Gamma\Lambda})^{N+1}.
\]

Hence
\[
\|I - J^{(N)}\| \leq \|(I - \alpha S_{m\Gamma\Lambda})^{N+1}\| \leq \|I - \alpha S_{m\Gamma\Lambda}\|^{N+1}.
\]

By 4.1, \( \|I - J^{(N)}\| \to 0 \), as \( N \to \infty \). \( \square \)

The authors have considered pair frames in Banach spaces and have defined Banach pair frames [10]. It is shown that this notion generalizes some various types of frames. Some characterizations of Banach pair frames are presented in [12]. The concept of adjoint of pair frames is considered by the authors in [11].

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Figure 1.