Fingering of Electron Droplets in Nonuniform Magnetic Fields

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A semiclassical analysis of a two-dimensional electron droplet in a high, nonuniform magnetic field predicts that the droplet will form “fingered” patterns upon increasing the number of electrons. We construct explicit examples of these patterns using methods first developed for the flow of two-dimensional viscous fluids. We complement our analytical results with Monte Carlo simulations of the droplet wavefunction, and find that at the point where the semiclassical analysis predicts a cusp on the interface, the droplet fissions—a type of “quantum breakup” phenomenon.

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The growth of many physical systems is limited by diffusion—e.g., the diffusion of heat for solids growing into supercooled liquids [1], the diffusion-limited aggregation (DLA) of colloids [2], or the diffusion of magnetic flux in a superconductor [3]. The unstable growth of all of these systems [1] produces beautiful patterns. An important paradigm for this class of pattern-forming systems is Laplacian growth (LG)—diffusion-limited growth in the limit of long diffusion lengths [4]. Agam et al. [5, 6] recently discovered an intriguing connection between LG and the growth of a two-dimensional (2D) electron droplet in a high, nonuniform, magnetic field. Their semiclassical analysis of the droplet wavefunction reveals that as the number of electrons in the droplet is increased the droplet maintains a uniform density while its boundary evolves according to the LG equations. The unstable growth leads to a “fingered” pattern!

We expand upon the work of Agam et al. in several important ways. First, we show that a simple magnetic field inhomogeneity outside the growing droplet yields tractable analytical results for the droplet growth which allow us to predict the onset and structure of interfacial singularities. Second, using an analogy between the probability density for the electrons and the Boltzmann weight for a fictitious 2D plasma in a background potential provided by the magnetic field inhomogeneity, we perform Monte Carlo (MC) simulations [7] of the droplet growth. The droplet breaks apart beyond the point where the semiclassical analysis predicts a cusp on the interface. The MC method allows us to predict the onset and structure of interfacial singularities. Second, using an analogy between the probability density for the electrons and the Boltzmann weight for a fictitious 2D plasma in a background potential provided by the magnetic field inhomogeneity, we perform Monte Carlo (MC) simulations [7] of the droplet growth. The droplet breaks apart beyond the point where the semiclassical analysis predicts a cusp on the interface. The MC method allows us to predict the onset and structure of interfacial singularities. Second, using an analogy between the probability density for the electrons and the Boltzmann weight for a fictitious 2D plasma in a background potential provided by the magnetic field inhomogeneity, we perform Monte Carlo (MC) simulations [7] of the droplet growth. The droplet breaks apart beyond the point where the semiclassical analysis predicts a cusp on the interface. The MC method allows us to predict the onset and structure of interfacial singularities. Second, using an analogy between the probability density for the electrons and the Boltzmann weight for a fictitious 2D plasma in a background potential provided by the magnetic field inhomogeneity, we perform Monte Carlo (MC) simulations [7] of the droplet growth. The droplet breaks apart beyond the point where the semiclassical analysis predicts a cusp on the interface. The MC method allows us to predict the onset and structure of interfacial singularities. Second, using an analogy between the probability density for the electrons and the Boltzmann weight for a fictitious 2D plasma in a background potential provided by the magnetic field inhomogeneity, we perform Monte Carlo (MC) simulations [7] of the droplet growth. The droplet breaks apart beyond the point where the semiclassical analysis predicts a cusp on the interface. The MC method allows us to predict the onset and structure of interfacial singularities.

Hele-Shaw flow provides a simple example of LG. Two glass plates confine air and water, with the air injected into the center at a constant rate and the displaced water extracted at the edge, see Fig. 1. The air domain is at constant pressure \( p = 0 \) and the velocity of the water is determined by Darcy’s law, \( \mathbf{v} = -\nabla p \); since \( \nabla \cdot \mathbf{v} = 0 \), the pressure in the water is a harmonic function of \( z = x + iy \) with a sink at infinity. On the interface \( v_n = -\partial_n p \), and in the idealized version of LG the surface tension is neglected and \( p = 0 \) on the interface. Conformal mapping methods [4,10] may be used to map the exterior of the unit circle in an auxiliary \( w \)-plane onto the water domain in the \( z \)-plane at each instant of time \( t \) using \( z = z(w, t) \) (see Fig. 1), with \( \text{Re}(w \partial_w z \partial_z z) = 1/2 \) on the interface.

Next, consider a flat interface and produce a sharp outward bump of the air into the water which locally compresses the water isobars and produces a large pressure gradient near the tip. Then \( v_n = -\partial_n p \) implies that outward bumps grow more rapidly [11] than the flat interface, and absent a stabilizing effect such as surface tension this bump develops into a singularity in finite time [4,10] for most initial conditions.

We next turn to the many-body wavefunction for 2D electrons in a perpendicular, nonuniform, magnetic field. If we ignore Coulomb interactions and scale magnetic fields by the average field \( B_0 \), lengths by \( \ell_0 \equiv \sqrt{\hbar c/e B_0} \), and energies by \( \hbar c B_0/2 m c \), then the Hamiltonian is

\[
H = \sum_{j=1}^{N} \left[ (-i \nabla_j + A_j)^2 + \frac{g}{2} \sigma_z B(r_j) \right].
\]

If \( g = 2 \) then the many-body ground state wavefunction for Eq. (1) can be found exactly even for a nonuniform magnetic field [12,13,14] for a spin-polarized, filled Landau level (\( \nu = 1 \)) [8,13,14] we have

\[
\Psi(z_1, \ldots, z_N) = \frac{1}{\sqrt{N!^N}} \prod_{i<j} (z_i - z_j)^{\nu} W(z_i),
\]

FIG. 1: Growth of a 2D electron droplet in an nonuniform magnetic field; on the left the interface is shown for successively larger electron number. In the Hele-Shaw analogy, the electron droplet corresponds to air which displaces water.
where $W(z) = -|z|^2/4 + V(z)$, with $V(z)$ a “potential” which solves $\nabla^2 V = -\delta B \equiv -|B(z) - B_0)/|B_0|$, and $\tau_N$ is a normalization integral. Following Ref. [2] we assume that $\delta B = 0$ in the region of the droplet; $V(z)$, however, is nonzero in this region and the electrons are influenced by distant field inhomogeneities—a manifestation of the Aharonov-Bohm effect [3]. In the region of the droplet $V(z)$ is an analytic function and has the expansion in terms of the harmonic moments $t_k$ \[ V(z) = \frac{1}{2} \text{Re} \sum_{k=1}^{\infty} t_k z^k, \quad t_k = \frac{1}{\pi k} \int \delta B(z) z^{-k} d^2z. \] (3)

Although the wavefunction [2] is exact for the noninteracting system, extracting physical observables such as the density is a formidable task. Agam et al. [3] perform a semiclassical analysis of $\tau_N$ for $N \gg 1$ and find that the integral is dominated by configurations in which the electrons are uniformly distributed in a domain of area $A = 2\pi N$ with moments $t_k$ given by Eq. (3). Further analysis establishes the connection with the LG problem and provides us with powerful techniques [10] for determining the density distribution.

The derivation of the wavefunction [2] requires two important assumptions—noninteracting electrons and $g = 2$—which deserve a brief comment. First, for a uniform magnetic field Laughlin’s incompressible liquid state does an excellent job of capturing correlation effects due to the Coulomb interaction [1]. However, this may not be correct when the magnetic field is Laughlin’s incompressible liquid state does not capture correlation effects due to distant field inhomogeneities—a manifestation of the Aharonov-Bohm effect [3], Ref. [18]; perhaps this approach could be merged with the Schwarz function method discussed in Refs. [11,12], the parameters in Eqs. (4) and (5) can be related as:

\[ t_{M+1} = \frac{u_M}{(M+1)^{3/2}}, \quad t = 2N = r^2 - M u_M^2. \] (6)

Maps of the form [5] have been studied [10] for the Hele-Shaw problem, and many of the results can be directly applied to the electron droplet. For $M = 1$, the interface is an ellipse which evolves smoothly, with constant eccentricity, for increasing $N$ [22]. The behavior for $M \geq 2$ is more interesting; the harmonic measure $w(z)$ may have poles which coincide with the interface at some critical value of the electron number $N^*$, resulting in singularities. A detailed analysis yields the following: (i) The interface develops $M + 1$ simultaneous cusps at $N^* = \frac{1}{4} (M+1) [((M+1) t_{M+1}]^{-2}/(M-1)$. (ii) If $x^* = x(N^*)$ is the position of the cusp and $x(N)$ is the position of the finger for $N \lesssim N^*$, then $x^* - x(N) = \frac{2}{\sqrt{M-1}} \sqrt{N^* - N + O(N^* - N)}$, so that the approach to the critical value is universal, with a “velocity” $v_N = dx/dN$ which diverges as $(N^* - N)^{-1/2}$. (iii) If we define the scaled variables $X \equiv (x^* - x)/x^*$ and $Y \equiv y/x^*$, then the shape of the cusp is universal and has the “3/2” form [10]: \[ Y(X) = \frac{2}{\sqrt{M}} - \frac{1}{3} X^{3/2} + O(X^2). \] (9)

These interfacial cusps arise from the neglect of surface tension in the idealized LG problem. For the Hele-Shaw problem the surface tension smooths the cusp, while for the electron droplet one expects that fluctuations beyond the semiclassical approximation will smooth the droplet [3]. However, the mere existence of the singularity suggests that something dramatic will happen to the droplet, and to investigate this we need to go beyond the semiclassical level and obtain an essentially exact evaluation of the particle density using the Monte Carlo method.

The probability density for the electron droplet can be written as $|\Psi|^2 = \exp(-\beta U_{cl})/N! \tau_N$, with

\[ U_{cl} = -\sum_{i<j} \ln |z_i - z_j| + \sum_{i=1}^{N} \left[ \frac{1}{4} |z_i|^2 - V(z_i) \right], \] (10)
which is the Boltzmann weight for a classical 2D plasma in a background potential $-V(z)$, with a partition function $\tau_N$ and $\beta = 2$. The dominant contributions to $\tau_N$ can be determined using the Metropolis algorithm, analogous to Laughlin’s studies of the quantum Hall effect in a uniform field [i.e., $V(z) = 0$]. An initially random distribution of $N$ particles is equilibrated by moving each particle a distance of 2.0 in a random direction, with moves accepted or rejected according to the detailed balance condition; the average acceptance rate for the moves is close to 0.5. After an equilibration period of about $10^3$ moves per particle we compile a histogram of particle positions in bins of size $0.3 \times 0.3$ for about $4 \times 10^3$ moves per particle, and then calculate the average density in each bin. The electrons were confined by the walls of an impenetrable “simulation box” whose dimensions were chosen to be large compared to the linear dimension of the droplet. We accurately reproduce known results for the droplet size, density, and energy for a uniform field. For a nonuniform field the density forms a complicated pattern, but it still has a uniform value $\rho_{ave}$ well away from the interface. We determine the interface by finding the locus of points whose density is between, say, $0.3\rho_{ave}$ and $0.4\rho_{ave}$ (other definitions give similar results).

We have performed numerous MC simulations for ordered and random magnetic field configurations, with the results shown in Figs. 2–4. Let’s start with the potential $V(z) = (t_3/2)z^3$ for $M = 2$, which may be thought of as the leading term in a multipole expansion for an arrangement of six thin solenoids with alternating flux $\pm \Phi$ placed on the vertices of a hexagon a large distance $R$ from the center of the droplet, so that $t_3 = 4\Phi/R^3$. The semiclassical analysis predicts that three cusps will appear at the critical electron number $N^* = 1/(144\pi_3^2)$, and we have chosen $N^* = 500$ for these simulations. The excellent agreement between the semiclassical and MC results can be seen in Fig. 2; for $N = 494$ the nascent MC cusp is slightly rounded compared to the semiclassical prediction due to the discrete nature of the particles. We find equally impressive agreement with the position of one of the tips, Eq. (5), and the shape of the cusp, Eq. (5). When $N > N^*$ the electrons in excess of $N^*$ move toward the edges of the simulation box as shown in Fig. 3, so that the central droplet fissions. In the 2D plasma language, the incompressible fluid of particles saturates the potential minimum and additional particles which are “poured” into the potential “spill out” and accumulate at the edge of the simulation box. Finally, Fig. 4 shows a simulation for a random distribution of thin solenoids, with a highly ramified pattern for the averaged electron density. Determining whether such a pattern is fractal requires a study of the scaling law $N \sim R^{D_f}$, with $R$ the radius of gyration and $D_f$ the fractal dimension (with $D_f = 2$ for compact and $D_f < 2$ for fractal patterns). An unambiguous determination of $D_f$ requires simulations over many decades in $N$; this is currently beyond our computational resources and we leave its resolution as an interesting open problem.

In summary, we have constructed several explicit examples of magnetic field inhomogeneities which cause a noninteracting 2D electron droplet to finger. The semiclassical results (idealized LG) are in excellent agreement with our Monte Carlo simulations of the droplet wavefunction. An important implication of this result is that if there is an effective “surface tension” in the non-
FIG. 4: Average electron density for 280 electrons in a random array of 400 solenoids; blue is low density and red is high density. In this simulation 50 “attractive” solenoids were placed far from the droplet, with 350 “repulsive” solenoids placed closer to the droplet. The total solenoid flux is zero.

Interacting electron droplet problem it must be extremely small, and the regularization of the cusps appears to be controlled by the discreteness of the particles. Beyond the semiclassical singularity we find that the droplet fissions. Recent advances in imaging techniques for the 2D electron gas [24] may make it possible to observe these fingered structures; an easily tuned field inhomogeneity could be provided by placing the electron gas in close proximity to a type-II superconductor in the vortex state [13]. Our results also raise the interesting question of whether equilibrium Monte Carlo methods using a random $V(z)$ can be used to efficiently simulate nonequilibrium growth processes such as DLA [2], in other words, what can the quantum Hall effect tell us about DLA?

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[1] J. S. Langer, Rev. Mod. Phys. 52, 1 (1980).