Determination of welding residual stresses by inverse approach with eigenstrain formulations of boundary integral equation

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Abstract. Based on the concept of eigenstrain, a straightforward computational model of the inverse approach is proposed for determining the residual stress field induced by welding using the eigenstrain formulations of boundary integral equations. The eigenstrains are approximately expressed in terms of low-order polynomials in the local area around welded zones. The domain integrals with polynomial eigenstrains are transformed into the boundary integrals to preserve the favourable features of the boundary-only discretization in the process of numerical solutions. The sensitivity matrices in the inverse approach for evaluating the eigenstrain fields are constructed by either the measured deformations (displacements) on the boundary or the measured stresses in the domain after welding over a number of selected measuring points, or by both the measured information. It shows from the numerical examples that the results of residual stresses from deformation measurements are always better than those from stress measurements but they are sensitive to the noises from experiments. The results from stress measurements can be improved by introducing a few deformation measuring points while reducing the number of points for stress measuring to reduce the cost since the measurement of deformation is easier than that of stresses in practice.

1. Introduction

The residual stresses, induced during the process of manufacturing or technological treatment of engineering structures, have a significant influence on the performance of related components in service [1]. When such a component is in service, the associated residual stresses may superimpose on the applied stress to influence the deformation behavior of components. Therefore, the residual stresses are generally regarded as detrimental. Whatever the impact of residual stresses on the component, their detection and measurement is an important technical issue. However, since the nature of residual stress is in self-equilibrium, the determination of it is not an easy task.

There are a great number of techniques to detect the residual stresses in a solid which can be classified as three major groups: the physical, the mechanical and the numerical techniques. In the mechanical techniques, since the direct detection is impossible, parts of the material have to be removed from the solid to disturb the stress balance while the response of the specimen is measured in terms of either strains or shape changes at some other locations on the surface of the body. Obviously, in addition to the great cost of the mechanical methods, all of them are more or less destructive to the measured component by the material removal. It is evident that the mechanical techniques always
provide a limited level of detail, due to the finite number of discrete data points that restricts the possibility of reconstructing full-field stress distributions.

Numerical methods present a supplementary but effective means for determining residual stresses. As is widely accepted, residual stresses in engineering components are caused by incompatible internal permanent strains, named originally as the inherent strains [2-3] and lately as the eigenstrains [4], induced by any inhomogeneous inelastic deformation, temperature gradients or phase transformations during the manufacturing and processing of components. On the other hand, the theory of inverse methods has been actively developed over the past decades. In particular, inverse problems in elasticity have gained great attention including that of determining residual stresses based on the concept of eigenstrain which makes use of the information observed from experiments [5-7]. With the aid of numerical methods, generally the finite element method (FEM) [3,5-7], the unknown eigenstrain distributions can be retrieved following the mathematical framework of inverse problem with eigenstrain theory to obtain the whole field of residual stresses. In spite of the inelastic origin of eigenstrains, the inherent state of residual stress fields falls really into elastic regime so that the boundary element method (BEM) would be the most efficient numerical means to deal with the residual stress problems [8-9]. With the aid of the BEM it is also possible to determine the residual stresses nondestructively utilizing the deformations measured after welding with a forward approach [10].

In the present work, based on the concept of eigenstrain, a straightforward computational model of inverse approach is proposed with the eigenstrain formulations of boundary integral equations to determine the welding residual stresses, together with the numerical examples verifying the proposed computational model.

2. Eigenstrain Formulations of Boundary Integral Equation

The boundary integral equations for displacements \( u_i \) and stresses \( \sigma_{ij} \), respectively [11], can be written as follows for an elastic domain \( \Omega \) with the boundary \( \Gamma \) containing eigenstrains \( \varepsilon_{ij}^0 \) in local area \( \Omega_i \) (\( \Omega_i \in \Omega \)):

\[
C u_i (p) + \int_{\Gamma} u_j (q) \tau_{ij}^*(p,q) d\Gamma(q) = \int_{\Gamma} \tau_{ij} (q) u_j^0 (p,q) d\Gamma(q) + \int_{\Omega_i} \varepsilon_{ij}^0 (q) \sigma_{ij}^* (p,q) d\Omega(q)
\]

\[
C \sigma_{ij} (p) = \int_{\Gamma} \tau_{ij} (q) u_{ij}^0 (p,q) d\Gamma(q) - \int_{\Gamma} u_{ij} (q) \tau_{ij}^* (p,q) d\Gamma(q) + \int_{\Omega_i} \varepsilon_{ij}^0 (q) \sigma_{ij}^* (p,q) d\Omega(q) + \varepsilon_{ij}^0 (p) O_{ijkl}
\]

where \( p \) and \( q \) are the source and the field points, \( u_{ij}^0 \), \( \tau_{ij}^* \), and \( \sigma_{ij}^* \) represent the fundamental solutions for displacement, traction and stress, respectively. \( u_{ij}^0 \), \( \tau_{ij}^* \), and \( \sigma_{ij}^* \) are the related derivatives. \( C \) is a conventional boundary shape coefficient. \( \Omega_i \) is a tiny \( \epsilon \) region around \( p \) when \( p \in \Omega_i \) and \( O_{ijkl} \) is the corresponding free term from the domain integral in (2).

It is obvious from (1) and (2) that once the distributions of eigenstrains \( \varepsilon_{ij}^0 \) in domain integrals are known, the unknown boundary displacements can be solved using (1) and the total fields of stresses can be computed using (2). Considering the features of thermal cycles in welding, suppose that the distributions of eigenstrains can be approximately expressed in terms of low-order polynomials in the local area \( \Omega_i \) as follows:

\[
\varepsilon_{ij}^0 = \sum_{m=0, n=0}^{M} a_{ij}^{mn} x_i^m (q) x_j^n (q)
\]

where \( M \) is the number of terms of polynomials and \( a_{ij}^{mn} \) the coefficients to be identified. \( m \) and \( n \) are integers. The above equation gives a smooth constraint on the eigenstrain field. The domain integrals with polynomial eigenstrains in (1) and (2) can be transformed into the boundary integrals by introducing the two-point variables as follows

\[
x_i = x_i (q) - x_i (p)
\]

which are in fact the projection of the two-point distance, \( r_i \). With this definition, the domain integrals with the certain term of polynomials in (1) and (2), respectively, can be expressed in the form of two-
point polynomials and then the domain integrals at the right hand sides in (1) and (2) with eigenstrains can be transformed into the boundary integrals [12]. In this way, the favorable features of the boundary-only discretization are preserved.

3. Inverse Approach

In the inverse approach, the information from experiments is required to identify the unknown coefficients $\alpha_{ij}^{mn}$ in (3), including selected $M_U$ points for displacements measured on boundary and selected $M_S$ points for stresses measured in domain after welding. In the present work, since the residual stresses of welded plates are two-dimensional, the displacements are to be measured only in one direction, $x_1$ or $x_2$ for $u_1$ or $u_2$, at one point. The stresses are to be measured only for two normal components, $\sigma_{11}$ and $\sigma_{22}$, at one point. Therefore the numbers of known information from experiments are $M_U$ and $2M_S$, for displacements and stresses, respectively. Owing to the same reason, only normal components of eigenstrains, $\varepsilon_{011}$ and $\varepsilon_{022}$, are considered so that the number of unknown coefficients $\alpha_{ij}^{mn}$ in (3) is $2M$.

Using the boundary point method (BPM) [13] and noticed the traction-free condition of residual problems, the displacement equation (1) can be discretized into a system of algebraic equation in matrix form as

$$Hu = Ba$$

(5)

where $u$ is the vector of displacements at all the $N$ nodal points on the boundary $\Gamma$, $a$ the vector of unknown coefficients in (8). After $a$ is identified, the boundary displacements can be computed:

$$u = H^tBa$$

(6)

3.1. Residual stresses determined using measured displacements

Making use of the proposed algorithm with eigenstrain formulations, it is possible to determine the residual stresses nondestructively utilizing the deformations measured after welding [10]. Suppose that the displacements are measured on boundary $\Gamma$ at selected $M_U$ points from the experiment, the required sensitivity matrix $S_u$ can be obtained by reducing the matrix product $H^tB$ according to the displacement measuring points, that is

$$H^tB \rightarrow S_u$$

(7)

Then the unknown coefficients of eigenstrains can be obtained using the least square method by minimizing the object function, $\Phi$, defined as follows

$$\Phi = \frac{1}{2} \left\| S_u a - \bar{u} \right\|^2$$

(8)

where $\bar{u}$ represent the vector of measured displacements. The minimizing condition of (8) is

$$S_u^t (S_u a - \bar{u}) = 0$$

(9)

Therefore the unknown eigenstrain coefficients can be computed by

$$a = (S_u^t S_u)^{-1} S_u^t \bar{u}$$

(10)

It is obvious in practice that the number of measured displacement points is far less than that of the boundary nodal points ($M_U < N$). However, the number of measured points should not be less than that of eigenstrain coefficients, i.e. $M_U \geq 2M$.

3.2. Residual stresses determined using measured stresses

Suppose that the stresses are measured in domain $\Omega$ at selected $M_S$ points from the experiment, the sensitivity matrix $S_{\sigma}$ can be obtained by expressing the stresses using the stress equation (2) in matrix form and then by substituting (6) into it as follows

$$\sigma = Fu + Da = \left( FH^tB + D \right) a = S_{\sigma} a$$

(11)
where $F$ and $D$ stands for the corresponding coefficient matrices in the stress equation (2). Similarly, the unknown coefficients of eigenstrains can be obtained using the least square method by minimizing the object function, $\Phi$, defined as follows:

$$\Phi = \frac{1}{2} \| S_\alpha \mathbf{a} - \bar{\sigma} \|^2$$

(12)

where $\bar{\sigma}$ represent the vector of measured stresses. The unknown eigenstrain coefficients can be computed by the minimizing condition of (12) and the unknown $\alpha$ can be obtained

$$S_0^T (S_\alpha \mathbf{a} - \bar{\sigma}) = 0$$

(13)

$$\mathbf{a} = (S_0^T S_\alpha)^{-1} S_0^T \bar{\sigma}$$

(14)

The number of measured points for stresses should obey the inequality $2M_\sigma \geq 2M$.

3.3. Residual stresses determined using both of the measured displacements and stresses

In order to cut down the cost of stress measurements by reducing the number of measuring points, the hybrid procedure can be used by combining the measured displacement and stress data. Suppose that the displacements be measured on boundary $\Gamma$ at selected $M_U$ points and the stresses be measured in domain $\Omega$ at selected $M_S$ points from the experiment. The object function $\Phi$ can be defined as follows using the combined sensitivity matrix:

$$\Phi = \frac{1}{2} \left\| \begin{bmatrix} S_0^T \\ S_\alpha \end{bmatrix} \mathbf{a} - \begin{bmatrix} \bar{\mathbf{u}} \\ \bar{\sigma} \end{bmatrix} \right\|^2$$

(15)

The unknown eigenstrain coefficients $\mathbf{a}$ can then be computed in a way similar to those described above by minimizing (15):

$$\left( S_0^T, S_\alpha^T \right) \begin{bmatrix} S_0^T \\ S_\alpha \end{bmatrix} \mathbf{a} - \begin{bmatrix} \bar{\mathbf{u}} \\ \bar{\sigma} \end{bmatrix} = 0$$

(16)

$$\mathbf{a} = \left( S_0^T S_\alpha + S_\alpha^T S_\alpha \right)^{-1} \left( S_0^T S_\alpha \right) \begin{bmatrix} \bar{\mathbf{u}} \\ \bar{\sigma} \end{bmatrix}$$

(17)

The number of measured points in the hybrid procedure should satisfy the inequality $M_U + 2M_\sigma \geq 2M$. With the known eigenstrains, the total stress fields in the domain can be computed using the stress equation (2), after the boundary displacements are solved using (1).

4. Numerical examples

In the numerical examples, two square weld plates ($h=w$) are considered with a center weld bead (figure 1) and a restrained weld bead (figure 2), respectively. The latter is usually used as the test plate for weld cracking. The width of local area $\Omega_I$ where the eigenstrains distribute is set as $w_I=0.2w$ for the two plates, which is somewhat wider than that of the HAZ according to the parameters of material and
welding [10]. The length of local area $\Omega_I$ is set as $h_I=0.4h$ for the plate with restrained weld bead (figure 2). The distribution of eigenstrains in local region is obviously incompatible.

Table 1. The coefficients of eigenstrains employed

| $\alpha_{11}$ | $\alpha_{11}$ | $\alpha_{11}$ | $\alpha_{22}$ | $\alpha_{22}$ | $\alpha_{22}$ |
|---------------|---------------|---------------|---------------|---------------|---------------|
| $-c_1$        | $4c_1w_I$    | $3c_1h_I$    | $-c_2$        | $3c_2w_I$    | $4c_2h_I$    |

Table 2. The parameters in table 1

| | Plate | Centre bead | Constrained bead |
|---|-------|-------------|-------------------|
| $c_1$ | 4     | 4           |                   |
| $c_2$ | 1.1   | 0.8         |                   |

The ‘exact’ solution of residual stresses can be obtained as controls using the BPM with the values of eigenstrains listed in tables 1 and 2. The measuring points for stresses and displacements are shown in figure 3.

The idealized measuring data for stress and displacement are computed using the BPM at the measuring points with the values of eigenstrains listed in tables 1 and 2. With these idealized data, the residual stress can be reconstructed after solving the eigenstrains using the inverse approaches stated above. However, as there are always errors in the experimental measurements, 10% noises are introduced into the idealized data as follows

\[
\begin{align*}
\bar{\sigma}_{\text{noise}} &= (1 \pm 0.1 \text{ran}) \bar{\sigma} \\
\tilde{\sigma}_{\text{noise}} &= (1 \pm 0.1 \text{ran}) \tilde{\sigma}
\end{align*}
\]

where ran is the random function varying between 0 and 1. With these noise introduced data, the residual stress can also be reconstructed after solving the eigenstrains using inverse approaches.

The normalized longitudinal and the transverse stresses, $\sigma_{22}\sigma_S^{-1}$ and $\sigma_{11}\sigma_S^{-1}$, for the two plates are compared with the ‘exact’ solutions in figures 4 and 5, respectively, with the measured stresses at a

\[M_S=6 \quad M_U=0\]

\[\sigma_{22}\sigma_S^{-1} \quad \sigma_{11}\sigma_S^{-1}\]
few points ($M_S=6, M_U=0$), indicating the feasibility and effectiveness of the present algorithm where $\sigma_S$ is the yield strength of material. The present eigenstrain formulations of the BIE are considered to be in the most consistent with the concept of eigenstrain.

Table 3. Errors for the plate with centre weld bead using idealized measuring data

| $M_S$ | $M_U$ | $\varepsilon_{\text{MAX}}(\sigma_{11})$ | $\varepsilon_{\text{MAX}}(\sigma_{22})$ | $\varepsilon_{\text{RMS}}(\sigma_{11})$ | $\varepsilon_{\text{RMS}}(\sigma_{22})$ |
|-------|-------|-------------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| 0     | 8     | 0                                   | 0                                   | 0                                   | 0                                   |
| 6     | 0     | 0.0516                              | 0.0194                              | 0.008590                            | 0.000066                            |
| 9     | 0     | 0.0153                              | 0.0146                              | 0.002480                            | 0.000667                            |
| 12    | 0     | 0.0153                              | 0.0143                              | 0.002482                            | 0.000650                            |
| 2     | 8     | 0.0019                              | 0.0086                              | 0.000320                            | 0.000050                            |
| 4     | 8     | 0.0038                              | 0.0096                              | 0.000637                            | 0.000010                            |

Table 4. Errors for the plate with centre weld bead using noise introduced measuring data

| $M_S$ | $M_U$ | $\varepsilon_{\text{MAX}}(\sigma_{11})$ | $\varepsilon_{\text{MAX}}(\sigma_{22})$ | $\varepsilon_{\text{RMS}}(\sigma_{11})$ | $\varepsilon_{\text{RMS}}(\sigma_{22})$ |
|-------|-------|-------------------------------------|-------------------------------------|-------------------------------------|-------------------------------------|
| 0     | 8     | 0.0835                              | 0.6114                              | 0.015592                            | 0.003765                            |
| 6     | 0     | 0.0793                              | 0.0492                              | 0.013604                            | 0.000183                            |
| 9     | 0     | 0.0253                              | 0.0325                              | 0.003887                            | 0.000714                            |
| 12    | 0     | 0.0254                              | 0.0328                              | 0.003882                            | 0.000698                            |
| 2     | 8     | 0.0312                              | 0.0430                              | 0.006563                            | 0.004127                            |
| 4     | 8     | 0.0309                              | 0.0409                              | 0.006051                            | 0.004006                            |

The maximum errors, $\varepsilon_{\text{MAX}}$, and the root mean square errors, $\varepsilon_{\text{RMS}}$, for the plate with the center weld bead using the idealized data and the noise introduced data are compared in tables 3 and 4, respectively, showing that the results of the inverse approach with measured displacements are satisfactory with the idealized data but very sensitive to the noises. In contrast, the results of the inverse approach with measured stresses are relatively insensitive to the noises, which can be improved further by introducing a few deformation measuring points while reducing the number of stress measuring points.

It is demonstrated in practice that the welding residual stresses can be determined nondestructively with the measuring deformations [10] using a forward approach with iterations. With the inverse approach, however, it cannot be employed independently due to its high sensitivity to the experimental noises. The inverse approach with measured displacements should be used in combination with the measured stresses for the purpose of cutting down the cost in practice.

5. Conclusions
(1) A straightforward computational model of the inverse approach is proposed for determining the welding residual stress field with the eigenstrain formulations of boundary integral equations, which are considered to be in most consistent with the concept of eigenstrain.
(2) The sensitivity matrices in the inverse approach for evaluating the eigenstrain fields can be constructed by either the measured deformations or the measured stresses or by both the measured information over a number of selected measuring points.
(3) Using the boundary point method, the numerical examples verify the feasibility and the effectiveness of the present algorithm. The results from stress measurements can be improved by introducing a few deformation measuring points while reducing the number of stress measuring points to cut down the cost.

Acknowledgments
The work was supported by the National Natural Science Foundation of China (Grant No. 10972131) and the Graduate Innovative Foundation of Shanghai University (Grant No. SHUCX102351).
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