THE GLOBAL RIGIDITY OF A FRAMEWORK IS NOT AN AFFINE-ININVARIANT PROPERTY

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Abstract. It is known that the property of a bar-and-joint framework ‘to be infinitesimally rigid’ is preserved under projective transformations of ambient space. In this article, we prove that the property of a bar-and-joint framework ‘to be globally rigid’ is not preserved even under affine transformations of ambient space.

Keywords: bar-and-joint framework, globally rigid framework, Euclidean space, affine transformation, distance.

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1. Introduction

We recall basic definitions and notation from Global Rigidity (see, e.g., [4]).

Let $d \geq 1$ be an integer and $G = (V, E)$ be a graph. Here $V$ and $E$ are the sets of vertices and edges of $G$, respectively.

A bar-and-joint framework $G(p)$ in $\mathbb{R}^d$ (or a framework, for short) is a graph $G$ and a configuration $p$ which assigns a point $p_i \in \mathbb{R}^d$ to each vertex $i \in V$. For each $i \in V$, $p_i$ is called a joint of $G(p)$ and, for each edge $\{i, j\} \in E$, the straight-line segment with the endpoints $p_i$ and $p_j$ is called a bar.

Note that some authors use the term geometric realization of a graph $G$ instead of the term bar-and-joint framework $G(p)$ (see, e.g., [6, 9]).

We denote the Euclidean distance between $x, y \in \mathbb{R}^d$ by $|x - y|$.

Two frameworks $G(p)$ and $G(q)$ are equivalent to each other if $|p_i - p_j| = |q_i - q_j|$ for every $\{i, j\} \in E$ and are congruent to each other if $|p_i - p_j| = |q_i - q_j|$ for all $i, j \in V$.

A framework $G(p)$ is globally rigid in $\mathbb{R}^d$ if all frameworks $G(q)$ in $\mathbb{R}^d$ which are equivalent to $G(p)$ are congruent to $G(p)$.

Let $A : \mathbb{R}^d \to \mathbb{R}^d$ be an affine transformation and $G(p)$ be a framework in $\mathbb{R}^d$. We write $Ap$ for a configuration of the graph $G$ given by the formulas $(Ap)_k = A(p_k)$, $k \in V$. In the sequel, where this cannot cause misunderstanding, we write $Ax$ instead of $A(x)$ for $x \in \mathbb{R}^d$.

The main result of this article is the following

Theorem 1. For every $d \geq 2$, there is a framework $G_d(p)$ in $\mathbb{R}^d$ and an affine transformation $A : \mathbb{R}^d \to \mathbb{R}^d$ such that $G_d(p)$ is not globally rigid while $G_d(Ap)$ is globally rigid.
The problem of whether a given framework (or all frameworks from a given family) is globally rigid was raised both in mathematics and in its applications. As a purely mathematical problem, it was raised in distance geometry (see, e.g., [2, 3, 4, 5]), graph theory (see, e.g., [6, 12, 14]), matroid theory (see, e.g., [9]), etc. Since frameworks are a natural model for real-world mechanisms and molecules, this problem appeared also in classical mechanics of mechanisms (see, e.g., [10, 13]), the mechanics of microporous materials (see, e.g., [15]), stereochemistry (see, e.g., [7]), molecular biology (see, e.g., [1, 11]), etc.

In contrast, Theorem 1 reads that the property of a framework to be finitely rigid is preserved under projective transformations (see, e.g., [8, 16]).

Similarly, the property of a framework to be globally rigid is not preserved even under affine transformations. As far as we know, this difference previously was not mentioned in the literature.

2. Preliminary considerations

Let $G^* = (V^*, E^*)$ denote the graph such that $V^* = \{1, 2, 3, 4, 5\}$ and $E^* = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{3, 5\}, \{4, 5\}\}$. $G^*$ is shown in Fig. 1(a).

Let $p^*$ denote the configuration of $G^*$ in $\mathbb{R}^2$ which is given by the following formulas: $p^*_1 = (0, 2)$, $p^*_2 = (-1/4, 1/2)$, $p^*_3 = (21/20, 9/10)$, $p^*_4 = (-1, 0)$, and $p^*_5 = (1, 0)$. $G(p^*)$ is shown in Fig. 1(b).

Let $A : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the affine transformation given by the formula $A(x_1, x_2) = (x_1, x_2/2)$.

**Lemma 1.** Let the graph $G^*$, the framework $G^*(p^*)$ and the affine transformation $A$ be as above in this Section. Then $G^*(p^*)$ is not globally rigid in $\mathbb{R}^2$, while $G^*(Ap^*)$ is globally rigid in $\mathbb{R}^2$.

**Proof.** Let $q^*$ be the configuration of $G^*$ in $\mathbb{R}^2$ which is given by the following formulas: $q^*_1 = p^*_1 = (0, 2)$, $q^*_2 = (-21/20, 9/10)$, $q^*_3 = (1/4, 1/2)$, $q^*_4 = p^*_4 = (-1, 0)$, and $q^*_5 = p^*_5 = (1, 0)$. $G^*(q^*)$ is shown in Fig. 1(c).

We can say that $G^*(q^*)$ is obtained from $G^*(p^*)$ by replacing the joint $p^*_2$ by the joint $q^*_2$ and by replacing the joint $p^*_5$ by the joint $q^*_5$. Note that $q^*_2$ is chosen in such a way that it is symmetrical to $p^*_2$ with respect to the line $p^*_1p^*_4$ (here and subsequently $xy$ denotes the straight line passing through the points $x, y \in \mathbb{R}^2$).

Similarly, $q^*_5$ is chosen in such a way that it is symmetrical to $p^*_5$ with respect to the line $p^*_1p^*_5$.

Using the symmetry of some parts of the framework $G^*(p^*)$, it is easy to see that $|q^*_2 - q^*_3| = |p^*_2 - p^*_3|$. Hence, the frameworks $G^*(p^*)$ and $G^*(q^*)$ are equivalent to each other. However, we can arrive at the same conclusion by direct calculating the lengths of all bars of these frameworks.
THE GLOBAL RIGIDITY OF A FRAMEWORK IS NOT AFFINE-IN Variant

\[ (a) \]
\[ p_1^* = (0,2) \]
\[ p_2^* = (-\frac{1}{4}, 2) \]
\[ p_3^* = (-\frac{9}{10}, -\frac{1}{2}) \]
\[ p_4^* = (-1,0) \]
\[ p_5^* = (1,0) \]

\[ (b) \]
\[ q_1^* = p_1^* \]
\[ q_2^* = (-\frac{21}{20}, \frac{9}{10}) \]
\[ q_3^* = (\frac{1}{2}, -\frac{3}{4}) \]
\[ q_4^* = p_4^* \]
\[ q_5^* = p_5^* \]

\[ (c) \]
\[ Figure 1. (a): Graph \( G^* \). (b): Framework \( G^*(p^*) \). (c): Framework \( G^*(q^*) \). \]

\[ Figure 2. (a): Graph \( \tilde{G}^* \). (b): Framework \( \tilde{G}^*(p^*) \). (c): Framework \( \tilde{G}^*(A p^*) \). \]

On the other hand, \( G^*(p^*) \) and \( G^*(q^*) \) are not congruent to each other since
\[ |p_2^* - p_5^*| \neq |q_2^* - q_5^*|. \]
In fact, direct calculations show that
\[ |p_2^* - p_5^*| = \sqrt{29}/4 \]
\[ |q_2^* - q_5^*| = \sqrt{2005}/20. \]
Hence, \( G^*(p^*) \) is globally rigid in \( \mathbb{R}^2 \).

Let \( \tilde{G}^* = (\tilde{V}^*, \tilde{E}^*) \) denote the graph such that \( \tilde{V}^* = V^* \) and \( \tilde{E}^* = E^* \setminus \{\{2,3\}\} \).
\( \tilde{G}^* \) is shown in Fig. 2(a). The configuration \( p^* \) of the graph \( G^* \) and the affine transformation \( A : \mathbb{R}^2 \to \mathbb{R}^2 \) constructed above in this Section define also the frameworks \( G^*(p^*) \) and \( \tilde{G}^*(A p^*) \), which are shown in Fig. 2(b) and Fig. 2(c), respectively.

Every framework in \( \mathbb{R}^2 \), which is equivalent to \( \tilde{G}^*(A p^*) \), is congruent to one of the frameworks shown in Fig. 3 where \( x_2 = (-3/4, 3/4) \) and \( x_3 = (11/20, -1/20) \). Note that the points \( x_2 \) and \( x_3 \) are chosen in such a way that \( x_2 \) and \( A p_3^* \) are symmetrical to each other with respect to the line \( A p_1^* A p_4^* \), and \( x_3 \) and \( A p_3^* \) are symmetrical to each other with respect to the line \( A p_1^* A p_5^* \).

Direct calculations show that
\[ |A p_2^* - A p_3^*| = \sqrt{173}/10 \] (see Fig. 3(a)),
\[ |A p_2^* - x_3| = \sqrt{73}/10 \] (see Fig. 3(b)),
\[ |x_2 - x_3| = \sqrt{233}/10 \] (see Fig. 3(c)), and
\[ |x_2 - A p_3^*| = 3\sqrt{37}/10 \] (see Fig. 3(d)). Since among these numbers there are no two
equal, the framework $G^*(Ap^*)$ is globally rigid in $\mathbb{R}^2$. For the convenience of the reader $G^*(Ap^*)$ is shown in Fig. 4(a).

3. Proof of Theorem 1

In the proof of Theorem 1 given below, we mainly use the same ideas that were used in Section 2 in the proof of Lemma 1. The novelty is in details that are needed for transition from the plane to space.

Proof. Let $\tilde{G}_d = (\tilde{V}_d, \tilde{E}_d)$ denote the graph such that $\tilde{V}_d = \{1, 2, 3, 4, \ldots, 2^{d-1} + 2, 2^{d-1} + 3\}$ and an unordered pair of non-coincident vertices $i, j$ is an edge of $\tilde{G}_d$ (i.e. $\{i, j\} \in \tilde{E}_d$) if and only if one of the conditions (i)–(iii) is fulfilled:

(i) $i, j \in \{1, 4, 5, \ldots, 2^{d-1} + 2, 2^{d-1} + 3\}$ (i.e. the complete graph with the vertices $1, 4, 5, \ldots, 2^{d-1} + 2, 2^{d-1} + 3$ is contained in $\tilde{G}_d$);

(ii) the unordered pair $\{i, j\}$ coincides with either $\{1, 2\}$ or $\{2, 2k\}$, where $2 \leq k \leq 2^{d-2} + 1$ (i.e. the vertex 2 is connected by an edge to each of the vertices $1, 4, 6, \ldots, 2k, \ldots, 2^{d-1} + 2$);

(iii) the unordered pair $\{i, j\}$ coincides with either $\{1, 3\}$, or $\{3, 2k + 1\}$, where $2 \leq k \leq 2^{d-2} + 1$ (i.e. the vertex 3 is connected by an edge to each of the vertices $1, 5, 7, \ldots, 2k + 1, \ldots, 2^{d-1} + 3$).

Let $G_d = (V_d, E_d)$ denote the graph such that $V_d = \tilde{V}_d$ and $E_d = \tilde{E}_d \cup \{\{2, 3\}\}$. $G_2$ is shown in Fig. 1(a), $\tilde{G}_2$ is shown in Fig. 2(a), and $G_3$ is shown in Fig. 4(b).
The points \((\pm 1, \pm 1, \ldots, \pm 1) \in \mathbb{R}^{d-2}\) are the vertices of a cube with edge length 2. We enumerate them in an arbitrary way using the numbers \(1 \leq k \leq 2^{d-2}\) and denote them by \(w_k\). So, \(w_k = (\pm 1, \pm 1, \ldots, \pm 1)\) with the proper selection of plus and minus signs.

Let \(p\) denote the configuration of the graphs \(G_d\) and \(\tilde{G}_d\) in \(\mathbb{R}^d\) that is given by the following formulas: \(p_1 = (0, 2, 0, \ldots, 0), p_2 = (-1/4, 1/2, 0, \ldots, 0), p_3 = (21/20, 9/10, 0, \ldots, 0)\), \(p_{2j} = (-1, 0, w_{2j-1}), p_{2j+1} = (1, 0, w_{2j-1})\), where \(2 \leq j \leq 2^{d-2} + 1\).

Let \(q\) denote the configuration of the graphs \(G_d\) and \(\tilde{G}_d\) in \(\mathbb{R}^d\) that is given by the following formulas: \(q_1 = p_1, q_2 = (-21/20, 9/10, 0, \ldots, 0), q_3 = (1/4, 1/2, 0, \ldots, 0)\), and \(q_k = p_k\), where \(k = 4, 5, \ldots, 2^{d-1} + 2, 2^{d-1} + 3\).

Note that \(q_2\) is chosen in such a way that it is symmetrical to \(p_2\) with respect to the hyperplane in \(\mathbb{R}^d\) passing through the points \(p_1, p_4, p_6, \ldots, p_{2^{d-1}}, p_{2^{d-1}+2}\). Similarly, \(q_3\) is chosen in such a way that it is symmetrical to \(p_3\) with respect to the hyperplane in \(\mathbb{R}^d\) passing through the points \(p_1, p_5, p_7, \ldots, p_{2^{d-1}+1}, p_{2^{d-1}+3}\).

As in the proof of Lemma 1, a direct verification shows that the frameworks \(G_d(p)\) and \(G_d(q)\) are equivalent to each other, but not congruent. Hence, \(G_d(p)\) is not globally rigid.

Let \(A : \mathbb{R}^d \to \mathbb{R}^d\) be the affine transformation given by the formula \(A(x_1, x_2, x_3, \ldots, x_d) = (x_1, x_2/2, x_3, \ldots, x_d)\) (i.e. \(A\) is the contraction with the factor of 2 along the second axis of \(\mathbb{R}^d\)).

Let \(r\) denote the configuration of the graphs \(G_d\) and \(\tilde{G}_d\) in \(\mathbb{R}^d\) that is given by the following formulas: \(r_1 = Ap_1, r_2 = Ap_2, r_3 = (11/2, -1/20, 0, \ldots, 0)\), and \(r_k = Ap_k\), where \(k = 4, 5, \ldots, 2^{d-1} + 2, 2^{d-1} + 3\). Note that \(r_3\) is symmetrical to \(Ap_3\) with respect to the hyperplane in \(\mathbb{R}^d\) passing through the points \(Ap_1, Ap_5, Ap_7, \ldots, Ap_{2^{d-1}+1}, Ap_{2^{d-1}+3}\).

Let \(s\) denote the configuration of the graphs \(G_d\) and \(\tilde{G}_d\) in \(\mathbb{R}^d\) that is given by the following formulas: \(s_1 = Ap_1, s_2 = (-3/4, 3/4, 0, \ldots, 0), s_3 = r_3, s_k = Ap_k\), where \(k = 4, 5, \ldots, 2^{d-1} + 2, 2^{d-1} + 3\). Note that \(s_2\) is symmetrical to \(Ap_2\) with respect to the hyperplane in \(\mathbb{R}^d\), passing through the points \(Ap_1, Ap_4, Ap_6, \ldots, Ap_{2^{d-1}}, Ap_{2^{d-1}+2}\).

Finally, let \(t\) denote the configuration of the graphs \(G_d\) and \(\tilde{G}_d\) in \(\mathbb{R}^d\) that is given by the following formulas: \(t_1 = Ap_1, t_2 = s_2, t_3 = r_3, t_k = Ap_k\), where \(k = 4, 5, \ldots, 2^{d-1} + 2, 2^{d-1} + 3\).

As in the proof of Lemma 1, we make sure that any framework equivalent to \(\tilde{G}_d(Ap)\) in \(\mathbb{R}^d\) is congruent to one of the frameworks \(\tilde{G}_d(Ap), \tilde{G}_d(r), \tilde{G}_d(s),\) or \(\tilde{G}_d(t)\). Direct calculations show that \(|Ap_2 - Ap_3| = \sqrt{173}/10, |r_2 - r_3| = \sqrt{73}/10, |s_2 - s_3| = \sqrt{233}/10, |t_2 - t_3| = 3\sqrt{37}/10\). Since among these numbers there are no two equal, none of the frameworks \(G_d(Ap), G_d(r), G_d(s),\) and \(G_d(t)\) are equivalent to each other. Consequently, \(G_d(Ap)\) is globally rigid. \(\square\)
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