GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR OF SPHERICALLY SYMMETRIC SOLUTIONS FOR THE MULTI-DIMENSIONAL INFRARELATIVISTIC MODEL

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Abstract. In this paper, we establish the global existence, uniqueness and asymptotic behavior of spherically symmetric solutions for the multi-dimensional infrarelativistic model in $H^i \times H^i \times H^i \times H^{i+1}$ $(i = 1, 2, 4)$.

1. Introduction. As we know, the importance of thermal radiation in physical problems increases as the temperature is raised. Usually, the role of the radiation is one of transporting energy by radiative process at the moderate temperature, while the energy and momentum densities of the radiation field may become comparable to or even dominate the corresponding fluid quantities at the higher temperature. So the radiation field significantly affects the dynamics of the field. The theory of radiation hydrodynamics finds a wide range of applications, such as stellar atmospheres and envelopes, supernova explosions, stellar winds, physics of laser fusion, reentry of vehicles and many others. Therefore, the study of mathematical theory of radiation hydrodynamics is of great importance from both the mathematical theory and that of applications.

In this paper, we consider the motion of the compressible multi-dimensional viscous gas with radiation, which is a system of the Navier-Stokes equations coupled with a transport equation. We know that the energy in the radiation field to be carried by point, massless particles called photons, which are travelling at the speed $c$ of light, characterized by their frequency $\nu$, and their energy of each photon $E = h\nu$ (where $h$ is the Planck’s constant), the momentum $\vec{p} = \frac{h\nu}{c} \vec{\Omega}$, where $\vec{\Omega}$ is a unit vector and denotes the direction of travel of the photon (it requires two angular variables to specify $\vec{\Omega}$). In a radiative transfer, it is conventional to introduce the specific radiative intensity $I \equiv I(x, t, \nu, \vec{\Omega})$ driven by the so-called radiative
The transfer integro-differential equation introduced and discussed by Chandrasekhar [3]. Meanwhile, we can derive global quantities by integrating with respect to the angular and frequency variables: the specific radiative energy density $E_R(x,t)$ per unit volume is then $E_R(x,t) = \frac{1}{c} \int \int I(x,t,\nu,\vec{\Omega})d\vec{\Omega}d\nu$, and the specific radiative flux $F_R(x,t) = \int \int \vec{\Omega}I(x,t,\nu,\vec{\Omega})d\vec{\Omega}d\nu$. Under the consideration of the three basic interactions between photons and matter, namely, absorption, scattering and emission, we find the transfer in the conventional form (see, e.g., [17, 18, 19])

$$\frac{1}{c} \frac{\partial I(\nu, \vec{\Omega})}{\partial t} + \vec{\Omega} \cdot \nabla I(\nu, \vec{\Omega}) = S_e(\nu) - \sigma_a(\nu)I(\nu, \vec{\Omega}) + \int_0^\infty d\nu' \int_{S^{n-1}} \left( \frac{\nu}{\nu'} \times \sigma_s(\nu' \to \nu, \vec{\Omega}' \cdot \vec{\Omega})I(\nu', \vec{\Omega}') \right) d\vec{\Omega}', \quad (1)$$

where $I(\nu, \vec{\Omega}) \equiv I(x,t,\vec{\Omega},\nu), S_n^{-1}$ is the unit ball in $\mathbb{R}^n$, $S_e(\nu) \equiv S_e(x,t,\nu), \theta, \sigma_a(\nu) \equiv \sigma_a(x,t,\nu,\rho,\theta)$, and $\sigma_s(\nu) \equiv \sigma_s(x,t,\nu,\rho,\theta)$, respectively, denote the rate of energy emission due to spontaneous processes, the absorption coefficient and the scattering coefficient that also depend on the mass density $\rho$ and the temperature $\theta$ of the matter. The scattering interaction serves to change the photon’s characteristics $\nu'$ and $\vec{\Omega}'$ to a new set of characteristics $\nu$ and $\vec{\Omega}$. The sign $\nu' \to \nu$ stands for from $\nu'$ to $\nu$ and $\vec{\Omega}' \to \vec{\Omega}$ denotes the transfer from direction $\vec{\Omega}'$ to direction $\vec{\Omega}$ as an argument of $\sigma_s(\nu)$. Therefore, we can describe the scattering event by a probabilistic statement concerning this change as follows

$$\text{outscattering} = \int_0^\infty d\nu' \int_{S^{n-1}} \sigma_s(\nu \to \nu', \vec{\Omega}' \cdot \vec{\Omega})I(\nu, \vec{\Omega})d\vec{\Omega}',$$

$$\text{inscattering} = \int_0^\infty d\nu' \int_{S^{n-1}} \sigma_s(\nu' \to \nu, \vec{\Omega}' \cdot \vec{\Omega})I(\nu', \vec{\Omega}')d\vec{\Omega}'.$$

When the matter is in local thermodynamical equilibrium and radiation is present with coupling terms between matter and radiation, the coupled system can be read as (see, e.g., [18, 19])

$$\begin{align*}
\partial_t \rho + \text{div}(\rho U) &= 0, \\
\partial_t (\rho U) + \text{div}(\rho U \otimes U) &= -\text{div}\vec{\Pi} - \vec{S}_F, \\
\partial_t (\rho e) + \text{div}(\rho e U) &= -\text{div}Q - \vec{B} \cdot \vec{\Omega} - S_E, \\
-\frac{1}{c} \frac{\partial I(\nu, \vec{\Omega})}{\partial t} + \vec{\Omega} \cdot \nabla I(\nu, \vec{\Omega}) &= S_t(\nu, \vec{\Omega}), \tag{2}
\end{align*}$$

where $\rho = \rho(x,t), U = U(x,t), \theta = \theta(x,t), e = e(x,t), Q = Q(x,t)$ stand for the density, the velocity, the absolute temperature, the internal energy and the heat flux, respectively, $\vec{\Pi} = -P(\rho, \theta) \vec{I} + \vec{F}$ represents the material stress tensor for a Newtonian fluid with the viscous contribution $\vec{F} = 2\mu \vec{D} + \lambda \text{div} U \vec{I}$ with $\mu > 0$ and $n\lambda + 2\mu \geq 0$, and the strain tensor $\vec{D}$ such that $D_{ij} = \frac{1}{2} \left( \frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} \right)$, and the coupling terms are

$$S_t(\nu, \vec{\Omega}) \equiv S_t(x,t,\nu, \vec{\Omega}) = \sigma_a(\nu, \vec{\Omega})(B(\nu, \theta) - I(\nu, \vec{\Omega})) + \int_0^\infty d\nu' \int_{S^{n-1}} \sigma_s(\nu' \to \nu, \vec{\Omega}' \cdot \vec{\Omega})$$

$$\times \left( \frac{\nu}{\nu'} I(\nu', \vec{\Omega}') - \sigma_a(\nu \to \nu', \vec{\Omega}' \cdot \vec{\Omega})I(\nu, \vec{\Omega})I(\nu', \vec{\Omega}') \right) d\vec{\Omega}',$$
the radiative energy source
\[ S_{E}(x, t) := \int_{0}^{\infty} d\nu \int_{S_{n-1}} S_{t}(\nu, \Omega) d\Omega, \]
the radiative flux
\[ \overrightarrow{S}_{E}(x, t) := \frac{1}{c} \int_{0}^{\infty} d\nu \int_{S_{n-1}} \Omega S_{t}(\nu, \Omega) d\Omega, \]
and the Planck’s function \( B(\nu, \theta) \) describes the frequency-temperature black body distribution. The thermo-radiative flux \( Q \) satisfies the Fourier’s law
\[ Q = Q(\rho, \theta) = -\kappa(\rho, \theta) \nabla \theta, \quad (3) \]
where \( \kappa(\rho, \theta) \) is the heat conductivity coefficient.

Now we would like to mention some results on this system. First, in the inviscid case, Lowrie, Morel and Hittinger [16], Buet and Desprès [2] have investigated the asymptotic regimes, and Dubroca and Feugeas [4], Lin [14] and Lin, Coulombo and Guodon [15] have considered the numerical aspects. Second, Zhang and Jiang [29] has given a proof of local-in-time existence and blow-up of solutions. Then Golse and Perthame [8] have investigated a simplified version of the system. Third, for the Cauchy problem in multi-dimension case, Li and Zhu [13] have investigated the blow-up of smooth solutions under some sufficient conditions. For the one-dimensional initial-boundary value problem, Ducomet and Nečasová [5] have considered global existence and uniqueness of weak solutions in \( H^1 \times H^1 \) and Qin, Feng and Zhang [21] also proved the global existence and large-time behavior of solutions in \( H^i \times H^i \times H^{i+1} (i = 1, 2) \) for the infrarelativistic model. Furthermore, Ducomet and Nečasová [6] obtained the asymptotic behavior of global strong solutions. In the pure scattering case, Ducomet and Nečasová [7] first investigated the asymptotic behavior of a motion of a viscous heat-conducting one-dimensional gas with radiation, and then Qin, Feng and Zhang [22] also proved the large-time behavior of solutions in \( H^i \times H^j \times H^i \times H^{i-1} (i = 2, 3) \). Recently, Azevedo, Sauter and Thompson [1] also studied an approximation model of compressible radiative flow and established global classical solutions for this model in a slab under semi-reflective boundary conditions using energy-entropy estimates and a homotopic version of the Leray-Schauder fixed point theorem together with classical Friedman-Schauder estimates for linear second order parabolic equations in boundary Hölder spaces. Qin and Zhang [24] obtained global existence and asymptotic behavior of cylindrically symmetric solutions for the 3D infrarelativistic model with radiation.

In this paper, we shall consider the radial solution on this model and establish the global existence, uniqueness and asymptotic behavior of spherically symmetric solutions for the compressible viscous gas with radiation. For spherically symmetric Navier-Stokes equations, we would like to refer to [9, 10, 11, 12, 28] and the references therein.

It is worth pointing out some difficulties encountered in this paper. The first difficulty encountered here is to establish the uniform point-wise upper bound of the specific volume \( v(x, t) \). To overcome it, we construct a key estimate (48) on the temperature \( \theta^{1+s}(x, t) \) by Sobolev’s and Hölder’s inequalities. Then we can apply Gronwall’s inequality to obtain the uniform-in-time upper bound of \( v(x, t) \). The second difficulty is to establish the \( H^1 \) estimate of the velocity \( u(x, t) \). To do this, motivated by [6], we introduce the auxiliary function \( F(\xi) \) (see Lemma 3.6) and adopt the similar technique to construct the corresponding estimate. The third
difficulty is to deal with the radiation intensity $\mathcal{I}(x,t)$. From equation (244), we can obtain the expression (93) of the radiation density $I$ by solving the ordinary differential equation. By virtue of (93), we mainly make full use of Lemmas 3.1-3.8 to establish some important estimates for $\mathcal{I}(x,t)$, such as Lemmas 3.9-3.10, 4.3 and 5.5. The last difficulty is to construct the estimates of the radiative energy source $(S_E)_{R}$ in equation (243). To overcome it, we mainly apply the relation between $(S_E)_{R}$ and $I$. By using the estimates on $I$, we can obtain the estimates on $(S_E)_{R}$.

In addition, in order to derive our desired results, we mainly use the embedding theorem and interpolation technique, and some idea from Qin [20], Qin and Huang [23] and Umehara and Tani [26, 27]. Especially, compared with the 1D case, we have some essential new difficulties and techniques used in the proof. Firstly, the proof of (34) in Lemma 3.1 is obtained by the relation (38) and the integration by parts. Secondly, estimates on radiation term play a key role in our proofs. We construct these estimates by some new techniques, such as the Gronwall inequality in Lemma 3.9. Finally, high-order estimates can be established by some complicate calculations, such as (179), (219) and Lemma 5.5.

The notation in this paper will be as follows. Signs $L^p, 1 \leq p \leq +\infty$, and $H^1 = W^{1,2}, H^1_0 = W^{1,2}_0$ denote the usual Lebesgue spaces and Sobolev spaces on $(0, L)$; $\|\cdot\|_B$ denotes the norm in the space $B$, $\|\cdot\| := \|\cdot\|_{L^2}$. $\tilde{f}(t) = L^{-1} \int_0^L f(x,t)dx$. Letter $C$ will denote the general constant but may be different, and letters $C_i (i = 1, 2, 4)$ will denote the universal constants depending on the norms of initial datum $(v_0, u_0, \theta_0,I_0)$ in $\mathcal{H}^i$ (see below the definitions of $\mathcal{H}^i$) but being independent of $t$, respectively.

We organize our present paper as follows. In Section 2, we will induct the spherically symmetric infrarelativistic model and state our main results. Subsequently, we will complete the proofs of the global existence and asymptotic behavior to the generalized solutions in $\mathcal{H}^i (i = 1, 2, 4)$ in Sections 3, 4 and 5, respectively.

2. Main results. Now we assume $\Omega \subset \mathbb{R}^n$ is a bounded domain and consider the following boundary conditions

$$U|_{\partial \Omega} = 0, \quad Q|_{\partial \Omega} = 0, \quad I|_{\partial \Omega} = 0, \quad (4)$$

and the initial conditions

$$(\rho, U, \theta, I)(x,0) = (\rho_0, U_0, \theta_0, I_0)(x). \quad (5)$$

First, we construct the corresponding system for radial solutions in the Eulerian coordinates. Let $r = |x|$ and take

$$\rho(x,t) = \rho(r,t), \quad \rho U(x,t) = \rho u(r,t) \frac{x}{r}, \quad \theta(x,t) = \theta(r,t), \quad I(x,t) = I(r,t). \quad (6)$$
Then the system (2) can be written as by the direct calculations

\[
\begin{align*}
\rho_t + (\rho u)_r + \frac{n-1}{r} \rho u &= 0, \\
(\rho u)_t + (\rho u^2)_r + \frac{n-1}{r} \rho u^2 + P_r &= (2\mu + \lambda) \left( u_r + \frac{n-1}{r} u \right)_r - (S_F)_R, \\
(\rho e)_t + (\rho eu)_r - \frac{n-1}{r} \rho e u - \kappa r_{\theta r} - \kappa \left( \theta_{rr} + \frac{n-1}{r} \theta_r \right) &= 2\mu \left( u_r^2 + \frac{n-1}{r^2} u^2 \right) + \lambda \left( u_r + \frac{n-1}{r} u \right)^2 \\
&- P \left( u_r + \frac{n-1}{r} u \right) - (S_E)_R, \\
\frac{1}{c} I_t + \omega I_r &= S,
\end{align*}
\]

where the domain \( \Omega \) is given by \( \Omega = \{ x \in \mathbb{R}^n : a < |x| < b \} \) for some constants \( a \) and \( b \) with \( 0 < a < b < +\infty \). \( \omega \) denotes the cosine of the angular between position \( x \) and direction of travelling \( \mathbf{v} \),

\[
(S_E)_n(n, r, t) = \int_0^\infty d\nu \int_{S_{n-1}} S(\nu, \mathbf{\Omega}) d\mathbf{\Omega},
\]

(7) is invariant along the trajectory \( \{ \rho(s, t) : a \leq s \leq b, t \geq 0 \} \). Moreover, we derive from (13) that

\[
\partial_\xi r(\xi, t) = \left[ r^{n-1}(\xi, t) \rho(r(\xi, t), t) \right]^{-1}.
\]

Second, the Eulerian coordinates \( (r, t) \) are connected to the Lagrangian coordinates \( (\xi, t) \) by the relation

\[
r(\xi, t) = r_0(\xi) + \int_0^t \tilde{u}(\xi, \tau) d\tau,
\]

and \( \tilde{u}(\xi, t) = u(r(\xi, t), t) \) and

\[
r_0(\xi) = \eta^{-1}(\xi), \quad \eta(r) = \int_a^r s^{n-1} \rho_0(s) ds, \quad r \in (a, b).
\]

Thus we can transform the system (7)-(9) into Lagrangian coordinates conveniently.

It follows from equation (7) and boundary condition (8) that

\[
\frac{\partial}{\partial t} \int_a^{r(\xi, t)} s^{n-1} \rho(s, t) ds = 0.
\]

Thus,

\[
\int_a^{r(\xi, t)} s^{n-1} \rho(s, t) ds = \int_a^{r_0(\xi)} s^{n-1} \rho_0(s) ds = \xi,
\]

and \( \Omega \) is transformed to \( \Omega_n = (0, L) \) with

\[
L = \int_a^b s^{n-1} \rho_0(s) ds = \int_a^b s^{n-1} \rho(s, t) ds,
\]

which, together with (10)-(11) and (13), yields that \( L \) is invariant along the trajectory \( \{ \rho(s, t) : a \leq s \leq b, t \geq 0 \} \). Moreover, we derive from (13) that

\[
\partial_\xi r(\xi, t) = \left[ r^{n-1}(\xi, t) \rho(r(\xi, t), t) \right]^{-1}.
\]
In general, for any function \( \varphi(r, t) \), if we denote \( \tilde{\varphi}(\xi, t) := \varphi(r(\xi), t) \), then we have by the chain rule
\[
\partial_t \tilde{\varphi}(\xi, t) = \partial_t \varphi(r, t) + u \partial_r \varphi(r, t),
\]

(15)
\[
\partial_\xi \tilde{\varphi}(\xi, t) = \partial_r \varphi(r, t) \partial_\xi r(\xi, t) = \frac{\partial_r \varphi(r, t)}{r^{n-1} \rho(r, t)}.
\]

(16)

Finally, without danger of confusion, we still denote \( (\tilde{\rho}, \tilde{u}, \tilde{\theta}) \) by \( (\rho, u, \theta) \) and \( (\xi, t) \) by \( (x, t) \). Let \( \nu := \frac{1}{\rho} \) represent the specific volume. Thus by (10) and (14)-(16), equations (7) in the Eulerian coordinates can be written in the Lagrangian coordinates in the new variables \( (x, t) \), \( x \in \Omega_n, t \geq 0 \)

\[
\begin{align*}
v_t &= (r^{n-1}u)_x, \\
u_t &= r^{n-1} \left( \frac{\delta (r^{n-1}u)_x}{v} - P \right) - v(S_F)_R, \\
e_t &= \left( \frac{r^{2n-2} \kappa \theta_x}{v} + \left( \frac{\delta (r^{n-1}u)_x}{v} - P \right) (r^{n-1}u)_x \right), \\
&- 2\mu(n-1)(r^{n-2}u^2)_x - v(S_E)_R, \\
1/c I_t + (\omega - u/c) r^{n-1} I_x &= S,
\end{align*}
\]

(17)

with \( \delta = \lambda + 2\mu \) and (8)-(9) become

\[
u(x, t)|_{x=0,L} = 0, \quad \theta_x(x, t)|_{x=0,L} = 0, \quad t \geq 0,
\]

(18)
\[
I(0, t) = 0 \quad \text{for} \quad \omega \in (0, 1), \quad I(L, t) = 0 \quad \text{for} \quad \omega \in (-1, 0), \quad t \geq 0,
\]

(19)
\[
v(x, 0) = v_0(x), \quad u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x), \quad I(x, 0) = I_0(x), \quad x \in \Omega_n.
\]

(20)

Then by (10) and (14), \( r(x, t) \) is determined by

\[
r(x, t) = r_0(x) + \int_0^t u(x, s) ds, \quad r_0(x) := \left( a^n + n \int_0^x v_0(y) dy \right)^{\frac{1}{n}}, \quad n = 2, 3,
\]

that is,

\[
r_t = u, \quad r^{n-1} r_x = v, \quad r|_{x=0} = a, \quad r|_{x=L} = b.
\]

(21)

Thus, we can deduce from (21) by the same arguments as those in [20, 23] that for all \( (x, t) \in [0, L] \times [0, +\infty) \),

\[
0 < a = r(0, t) \leq r(x, t) \leq r(L, t) = b.
\]

(22)

In this symmetric model, the radiative energy source

\[
(S_E)_R \equiv (S_E)_R(x, t) = \int_0^\infty d\nu \int_{S^{n-1}} S(x, t; \nu, \overrightarrow{\Omega}) d\overrightarrow{\Omega},
\]

the radiative force

\[
(S_F)_R \equiv S_F(x, t) = \frac{1}{c} \int_0^\infty d\nu \int_{S^{n-1}} \omega S(x, t; \nu, \overrightarrow{\Omega}) d\overrightarrow{\Omega},
\]

the source term \( S \) is defined as

\[
S \equiv S(x, t; \nu, \overrightarrow{\Omega}) = S_{a,e}(x, t; \nu, \overrightarrow{\Omega}) + S_a(x, t; \nu, \overrightarrow{\Omega}),
\]

where the absorption-emission term is

\[
S_{a,e}(x, t; \nu, \overrightarrow{\Omega}) = \sigma_a(x, t; \nu, \overrightarrow{\Omega})(B(x, t; \nu) - I(x, t; \nu, \overrightarrow{\Omega})).
\]
and
\[ S_n(x, t; \nu, \sigma, \Omega) = \sigma_n(x, t; \nu) (\overline{I}(x, t; \nu) - I(x, t; \nu, \Omega)) \]
where \( \overline{I}(x, t; \nu) = \frac{1}{\Omega(x)} \int_{S_n-1} I(x, t; \nu, \Omega) d\Omega \), \( \Omega(x) \) is the area of unit sphere surface in \( \mathbb{R}^n \) and \( B(\theta; \nu) = B(x, t; \nu) \) is a function of temperature and frequency describing the equilibrium state.

Introduce the radiative energy
\[ E_R = \int_0^\infty d\nu \int_{S_n-1} I(x, t; \nu, \Omega) d\Omega, \]
the radiative flux
\[ F_R = \int_0^\infty d\nu \int_{S_n-1} \omega I(x, t; \nu, \Omega) d\Omega, \]
and note pressure and energy of the matter have the thermodynamical relation
\[ e_v(\nu, \theta) = -P(\nu, \theta) + \theta P_0(\nu, \theta). \]

Especially, if we assume that the fluid motion is small enough with respect to the velocity of light \( c \) so that we can drop all the \( \frac{1}{c} \) factors in the previous formulation, then the system (17) can be rewritten as
\[
\begin{align*}
    v_t &= (r^{n-1}u)_x, \\
    u_t &= r^{n-1} \left( \delta \frac{(r^{n-1}u)_x}{v} - P \right), \\
    e_t &= \left( \frac{r^{2n-2}k\theta_x}{v} \right)_x + \left( \delta \frac{(r^{n-1}u)_x}{v} - P \right) (r^{n-1}u)_x \\
    &\quad - 2\mu(n-1)(r^{n-2}u^2)_x - v(S_E)R, \\
    r^{n-1} \omega I_x &= vS.
\end{align*}
\]

We assume that \( c, P, \sigma, \sigma_s, \kappa \) and \( B \) are \( C^{i+1} \) (\( i = 1, 2, 4 \)) functions on \( 0 < v < +\infty \) and \( 0 \leq \theta < +\infty \) and for any \( v \geq 0 \), we also suppose that the following growth conditions for any \( v \geq v \) and \( \theta \geq 0 \):

(A1) \( e(v, 0) \geq 0, \quad c_1^{-1}(1 + \theta^s) \leq e(v, \theta) \leq c_1(1 + \theta^s), \)

(A2) \( -c_2^{-1}v^{-2}(1 + \theta^1+s) \leq P(v, \theta) \leq P(v, \theta) \leq 0, \)

(A3) \( |P_0(v, \theta)| \leq c_3v^{-1}(1 + \theta^s), \)

(A4) \( c_4^{-1}(1 + \theta^1+s) \leq vP(v, \theta) \leq c_4(1 + \theta^1+s), \quad P(v, \theta) \leq 0, \)

(A5) \( 0 \leq P(v, \theta) \leq c_5(1 + \theta^1+s), \)

(A6) \( c_6^{-1}(1 + \theta^s) \leq \kappa(v, \theta) \leq c_6(1 + \theta^s), \)

(A7) \( |\kappa(v, \theta)| + |\kappa_\nu(v, \theta)| \leq c_7(1 + \theta^s), \)

(A8) \( v\sigma_a(v, \theta, \nu, \Omega) B^m(\theta; \nu) \leq c_8|\omega|^{\theta+\alpha}(\nu, \Omega), \quad \text{for} \ m = 1, 2, \)

(A9) \( 0 < \sigma_a(v, \theta, \nu, \Omega) \leq c_9|\omega|^{g}(\nu, \Omega), \)

(A10) \( \left( c_{10} + |(\sigma_a)_\nu| + |(\sigma_a)_\theta| \right) (v, \theta, \nu, \Omega)  \left( 1 + B(\theta; \nu) + |B_\theta(\theta; \nu)| + |B_\theta \theta(\theta; \nu)| \right) \)
\[
\leq c_{10}|\omega| h(\nu, \Omega),
\]

(A11) \( 0 < \sigma_a(v, \theta; \nu) \leq c_{11}|\omega|^{k}(\nu, \Omega), \)
we denote Planck and Boltzmann constants (see [17]). Obviously, this function satisfies our body radiation, emitted by a perfect radiator or black body, where are such that, for some $c_j (j = 1, \cdots, 13)$ are positive constants and the nonnegative functions $f, g, h, k, l, \mathcal{M}$ are such that, for some $\gamma > 0$,
\[
 f, g, h, k, l, \mathcal{M} \in L^{1+\gamma}(R^+ \times S^{n-1}) \bigcap L^\infty(R^+ \times S^{n-1}).
\]

Remark 1. The Planck function in thermal equilibrium $B(\theta; \nu) = 2\hbar^3 e^{-\frac{\hbar v}{kT}} - 1$ characterizes the radiation, usually called black-body radiation, emitted by a perfect radiator or black body, where $h$ and $k$ are the Planck and Boltzmann constants (see [17]). Obviously, this function satisfies our assumptions (A9), (A10) and (A12).

We assume that the viscosity coefficient $\mu$ is a positive constant. In the following, we denote
\[
 I(x, t) := \int_0^\infty d\nu \int_{S^{n-1}} I(x, t; \nu, \Omega) d\Omega
\]
for the integrated radiative intensity. In particular,
\[
 I(x, 0) = I_0 = \int_0^\infty d\nu \int_{S^{n-1}} I(x, 0; \nu, \Omega) d\Omega.
\]

We define some spaces as
\[
 \mathcal{H}^1 = \left\{ (v, u, \theta, I) \in H^1[0, L] \times H_0^1[0, L] \times H^1[0, L] \times H^2[0, L] : v(x) > 0, \theta(x) > 0, I(0) = 0 \text{ for } \omega \in (0, 1) \text{ and } I(L) = 0 \text{ for } \omega \in (-1, 0) \right\}
\]
and
\[
 \mathcal{H}^i = \left\{ (v, u, \theta, I) \in H^i[0, L] \times H^i[0, L] \times H^i[0, L] \times H^{i+1}[0, L] : v(x) > 0, \theta(x) > 0, u(0) = u(L) = 0, \theta'(0) = \theta'(L) = 0, I(0) = 0 \text{ for } \omega \in (0, 1) \text{ and } I(L) = 0 \text{ for } \omega \in (-1, 0) \right\}, \quad i = 2, 4,
\]
which become the metric spaces when equipped with the metrics induced from the usual norms. In the above, $H^i$ ($i = 1, 2, 3, 4, 5$) are the usual Sobolev spaces.

We are now in a position to state our main theorems.

Theorem 2.1. Assume that the initial data $(v_0, u_0, \theta_0, I_0) \in \mathcal{H}^1$ and the compatibility conditions hold. Then there exists a unique global solution $(v(t), u(t), \theta(t), I(t)) \in$
Theorem 2.2. Assume that the initial data \((v_0, u_0, \theta_0, I_0) \in H^2\) and the compatibility conditions hold. Then there exists a unique global solution \((v(t), u(t), \theta(t), I(t)) \in L^\infty([0, +\infty), H^2)\) to problem (24) with initial boundary value conditions (18)-(20) satisfying for all \((x, t) \in [0, L] \times [0, +\infty),\)

\[
0 < C^{-1}_1 \leq v(x, t) \leq C_1, \quad 0 < C^{-1}_1 \leq \theta(x, t) \leq C_1,
\]

and

\[
\|v(t) - \overline{v}\|_{H^1}^2 + \|u(t)\|_{H^1}^2 + \|\theta(t) - \overline{\theta}\|_{H^1}^2 + \|\mathcal{I}(t)\|_{H^5}^2 + \|r(t) - \overline{r}\|_{H^5}^2 + \|r_1(t)\|_{H^5}^2 + \int_0^t (\|v - \overline{v}\|_{H^1}^2 + \|u\|_{H^1}^2 + \|\theta - \overline{\theta}\|_{H^1}^2) \, d\tau
\]

\[
+ \int_0^t \int_0^\infty \int_{S^{n-1}} I_1^2 \, d\Omega \, d\nu \, d\tau \leq C_1.
\]

Moreover, we have, as \(t \to +\infty,\)

\[
\|v(t) - \overline{v}\|_{H^1} \to 0, \quad \|u(t)\|_{H^1} \to 0, \quad \|\theta(t) - \overline{\theta}\|_{H^1} \to 0, \quad \|\mathcal{I}(t)\|_{H^2} \to 0,
\]

where constant \(\overline{\theta} > 0\) is determined by \(e(\tau, \overline{\theta}) = \int_0^L \left(\frac{1}{2} u_0^2 + e(v_0, \theta_0) + r_0^{1-n} F_R(0)\right) \, dx,\)

\[
\overline{v} = \overline{v}_0 = L^{-1} \int_0^L v_0(x) \, dx
\]

\[
\overline{\tau} = (a^n + n\overline{v})^\frac{1}{n}.
\]

Theorem 2.3. Assume that the initial data \((v_0, u_0, \theta_0, I_0) \in H^4\) and the compatibility conditions hold. Then there exists a unique global solution \((v(t), u(t), \theta(t), I(t)) \in L^\infty([0, +\infty), H^4)\) to problem (24) with initial boundary value conditions (18)-(20) satisfying for all \((x, t) \in [0, L] \times [0, +\infty),\)

\[
\|v(t) - \overline{v}\|_{H^4}^2 + \|u(t)\|_{H^4}^2 + \|\theta(t) - \overline{\theta}\|_{H^4}^2 + \|\mathcal{I}(t)\|_{H^8}^2 + \|r(t) - \overline{r}\|_{H^8}^2 + \|r_1(t)\|_{H^8}^2 + \int_0^t (\|v - \overline{v}\|_{H^4}^2 + \|u\|_{H^4}^2 + \|\theta - \overline{\theta}\|_{H^4}^2) \, d\tau
\]

\[
+ \int_0^t \int_0^\infty \int_{S^{n-1}} I_1^2 \, d\Omega \, d\nu \, d\tau \leq C_4,
\]

where \(\overline{\theta} > 0\) and \(\overline{\tau}\) are also the same as those of Theorem 2.1.

Moreover, we have, as \(t \to +\infty,\)

\[
\|v(t) - \overline{v}\|_{H^4} \to 0, \quad \|u(t)\|_{H^4} \to 0, \quad \|\theta(t) - \overline{\theta}\|_{H^4} \to 0, \quad \|\mathcal{I}(t)\|_{H^5} \to 0.
\]

In order to derive our results of time asymptotic behavior of the global solutions, we may use the following basic inequality (Lemma 2.4) in analysis by Shen and Zheng [25].
Lemma 2.4. Let \( T \) be given with \( 0 < T \leq +\infty \). Suppose that \( y(t), h(t) \) are non-negative continuous functions defined on \([0, T]\) and satisfy the following conditions:

\[
\frac{dy(t)}{dt} \leq A_1 y^2(t) + A_2 + h(t),
\]

\[
\int_0^T y(t)dt \leq A_3, \quad \int_0^T h(t)dt \leq A_4
\]

where \( A_i \) (\( i = 1, 2, 3, 4 \)) are given non-negative constants. Then for any \( r > 0 \) with \( 0 < r < T \), the following estimates holds:

\[
y(t + r) \leq \left( \frac{A_3}{r} + A_2r + A_4 \right) e^{A_1 r^2}, \quad t \in (0, T - r).
\]

Furthermore, if \( T = +\infty \), then we have

\[
\lim_{t \to +\infty} y(t) = 0.
\]

3. Global existence and asymptotic behavior in \( \mathcal{H}^1 \). In this section, we establish the global existence and asymptotic behavior in \( \mathcal{H}^1 \) for the generalized solutions in \( \mathcal{H}^1 \) to problem (24) with initial boundary value conditions (18)-(20) and then complete the proof of Theorem 2.1 in terms of a series of lemmas.

Lemma 3.1. Under the assumptions in Theorem 2.1, there exists a constant \( C_1 > 0 \) such that the following estimates hold

\[
L^{-1} \int_0^L v(x,t)dx = L^{-1} \int_0^L v_0(x)dx := \overline{v_0}, \quad \forall t > 0,
\]

\[
\theta(x,t) > 0, \quad \forall (x,t) \in [0,L] \times [0, +\infty),
\]

\[
\int_0^L (\theta + \theta^{1+s})(x,t)dx \leq C_1, \quad \forall t > 0,
\]

\[
U(t) + \int_0^t V(\tau) d\tau \leq C_1, \quad \forall t > 0,
\]

where

\[
\begin{align*}
U(t) &= \int_0^L \left( (\theta - \log \theta - 1) + \theta^{1+s} + u^2 \right) (x,t)dx, \\
V(t) &= \int_0^L \left( \frac{u_x^2}{v} + \frac{(1 + \theta^2)\theta_x^2}{v^2} \right) (x,t)dx.
\end{align*}
\]

Proof. We can easily get (32) by integrating (24) over \( Q_t := (0, L) \times (0, t) \) and noting the boundary conditions. Equation (24) can be written as

\[
e_{0} \theta_t + \theta P_{0}(r^{n-1}u)_x = \left( \frac{r^{2n-2}k\theta_x}{v} \right)_x + \delta (r^{n-1}u)_x^2 - 2\mu(n-1)(r^{n-2}u^2)_x - v(S_{E})_R.
\]

We can deduce (33) from the maximal value principle and the positivity of \( \theta_0 \).

Multiplying (24) by \( u \), adding the resultant to (24), and then integrating it over \( Q_t \) and using the boundary conditions, we deduce

\[
\int_0^L (e + \frac{1}{2} u^2)dx + \int_0^t \int_0^L v(S_{E})_R dx dt = \int_0^L (e_0 + \frac{1}{2} u_0^2)dx.
\]

From the definitions of \( F_R \) and \( (S_{E})_R \), we can infer from (24) that

\[
v(S_{E})_R = r^{n-1}(F_R)_x
\]
which, together with (19), leads to
\[ \int_0^Lv^{n-1}(F_R)x\,dx = (r^{n-1}F_R)|_{x=0}^{x=L} - (n - 1) \int_0^Lv\,dF_R\]
\[ = \int_0^\infty \int_{S^{n-1}\cap \{\omega \in (0,1)\}} \omega \left( b^{n-1}I(L, t; \nu, \bar{\Omega}) - (n - 1) \int_0^Lv\,dI(x, t; \nu, \bar{\Omega}) \right) d\bar{\Omega} \, d
\nu \]
\[ - \int_0^\infty \int_{S^{n-1}\cap \{\omega \in (-1,0)\}} \omega \left( a^{n-1}I(0, t; \nu, \bar{\Omega}) + (n - 1) \int_0^Lv\,dI(x, t; \nu, \bar{\Omega}) \right) d\bar{\Omega} \, d\nu.\]

Thus, the contribution of the radiation term reads
\[ \int_0^t \int_0^L v(S_E)R\,dx\,d\tau = \int_0^t \int_0^L v^{n-1}(F_R)x\,dx\,d\tau \geq 0, \]
which, along with (37), implies
\[ \int_0^L (e + \frac{1}{2}u^2)\,dx \leq C_1. \tag{39} \]

Combining (39) with (23) and assumptions (A_1) – (A_3) yields (34).

Similarly to the proof of Lemma 2.1 in [21], estimate (35) can be shown, thus we omit it. The proof is now complete. \(\square\)

**Remark 2.** By Jensen’s inequality, the mean value theorem and (35), we can know that there exists a point \(a(t) \in [0, L]\) and two positive constants \(\alpha_1, \alpha_2\) such that
\[ 0 < \alpha_1 \leq \bar{\theta}(t) := L^{-1} \int_0^L \theta(x, t)\,dx = \theta(a(t), t) \leq \alpha_2, \tag{40} \]
where \(\alpha_1, \alpha_2\) are two roots of the equation \(y - \log y - 1 = C_1\).

Let
\[ \sigma(x, t) = \delta \frac{(r^{n-1}u_x)}{v} - P, \]
\[ \phi(x, t) = \int_0^t \sigma(x, \tau)\,d\tau + \int_0^x (v_0^{1-n}u_0)(y)\,dy + (n - 1) \int_0^t \int_x^L (r^{-n}u^2)(y, \tau)\,dy\,d\tau, \]
then, noting that \(r_t = u\), we can deduce from (24)_1, by the mean value theorem, that there exists a point \(x_0(t) \in [0, L]\) for any \(t \geq 0\) such that
\[ \int_0^L v\phi\,dx = \phi(x_0(t), t) \int_0^L v\,dx := v^*\phi(x_0(t), t), \]
i.e.,
\[ v^* = \frac{1}{\phi(x_0(t), t)} \int_0^L v\phi\,dx. \tag{41} \]

**Lemma 3.2.** For any \(t \geq 0\), we have the following expression
\[ v(x, t) = \frac{D(x, t)}{B(x, t)} \left( 1 + \delta^{-1} \int_0^t B(x, \tau)v(x, \tau)P(x, \tau)\,d\tau \right), \tag{42} \]
where
\[ B(x, t) = \exp \left[ \delta^{-1} \left( \frac{1}{v^*} \int_0^t \int_0^L \left( \frac{u^2}{n} + P\nu \right)(x, \tau)\,dx\,d\tau \right) \right]. \]
Under the assumptions in Theorem 2.1, the following estimate holds for all \((x, t) \in [0, L] \times [0, +\infty)\),

\[
0 < C_{i1}^{-1} \leq v(x, t) \leq C_1.
\]

**Proof.** The main idea of the proof is similar to that of Lemma 5.2.4 in [23]. But the different key point here is that \(B(x, t)\) in the expression of \(v(x, t)\) depends not only on \(t\) but also on \(x\). Thus we need a detailed analysis.

It follows from Lemma 3.1 that

\[
\left| \int_{x_0(t)}^{x} (r^{1-n}u)(y, t) dy \right| \leq a^{1-n} \int_{0}^{L} |u| dy \leq a^{1-n} ||u|| \leq C_1
\]

which, together with the expression of \(D(x, t)\) in Lemma 3.2 and by Hölder’s inequality, leads to

\[
0 < C_{i1}^{-1} \leq D(x, t) \leq C_1.
\]

Noting that the assumptions (A4), we deduce from Lemma 3.1 and Remark 2 that, for any \(0 \leq \tau \leq t\),

\[
C_{i1}^{-1}(t - \tau) \leq \int_{\tau}^{t} \left( \int_{0}^{L} \left( \frac{u^2}{n} + P \nu \right) dx + \int_{0}^{L} r^{-n}u^2 dx \right) ds \leq C_1(t - \tau)
\]

which implies for any \(0 \leq \tau \leq t\),

\[
e^{-C_1(t-\tau)} \leq \frac{B(x, t)}{B(x, \tau)} \leq e^{-C_{i1}^{-1}(t-\tau)}.
\]

Noting by Hölder’s inequality, we have for any \(x \in [0, L]\),

\[
|\theta^{\frac{1}{1+s}}(x, t) - \theta^{\frac{1}{1+s}}(a(t), t)| \leq C_1 \left| \int_{a(t)}^{x} \theta^{\frac{1}{1+s}} \theta_x dy \right|
\]

\[
\leq C_1 \left( \int_{0}^{L} \frac{(1 + \theta^s)\theta_x^2}{v \theta^2} dx \right)^\frac{1}{2} \left( \int_{0}^{L} \frac{v \theta^{1+s}}{1 + \theta^s} dx \right)^\frac{1}{2}
\]

\[
\leq C_1 V_1(\tau)M_v(t) \quad (47)
\]

where \(V_1(t) = \int_{0}^{L} \frac{(1 + \theta^s)\theta_x^2}{v \theta^2} dx, M_v(\tau) = \sup_{x \in [0, L]} v(x, \tau)\) and \(a(t)\) is defined in Remark 2. Then we have

\[
C_{i1}^{-1} - C_{i1}^{-1} V_1(t) M_v(t) \leq \theta^{1+s}(x, t) \leq C_1 + C_1 V_1(t) M_v(t).
\]

Thus it follows from Lemma 3.2, assumption (A4), (46) and (48) that

\[
v(x, t) \leq C_1 e^{-C_1 t} + C_1 \int_{0}^{t} (1 + \theta^{1+s}) e^{-C_1(t-\tau)} d\tau
\]
Applying Gronwall’s inequality and Lemma 3.1, we can derive
\[ M_v(t) \leq C_1. \] (49)

From (45)-(46), we can deduce that
\[ v(x, t) \geq C_1^{-1} e^{-(1+C_1^{-1})t} \]
which implies that there exists a large time \( t_0 > 0 \) such that for any \( t \geq t_0 \),
\[ v(x, t) \geq C_1^{-1}. \] (50)

On the other hand, we also infer from (45)-(46) that for any \( 0 \leq t \leq t_0 \),
\[ v(x, t) \geq \frac{D(x,t)}{B(x,t)} \geq C_1^{-1} e^{-C_1^{-1}t} \]
which, together with (50), yields that for all \((x, t) \in [0, L] \times [0, +\infty)\),
\[ v(x, t) \geq C_1^{-1} > 0. \]

Thus we obtain the estimate (43). The proof is complete. \( \square \)

**Lemma 3.4.** Under the assumptions in Theorem 2.1, the following estimates hold for any \( t > 0 \),
\[
\int_0^t \|u(\tau)\|_{L^\infty}^2 d\tau \leq C_1, \quad (51)
\]
\[
\|v_x(t)\|^2 + \int_0^t \int_0^L (1 + \theta^{1+\sigma}) v_x^2(x, \tau) dx d\tau \leq C_1, \quad (52)
\]
\[
\|u(t)\|^2 + \int_0^t \|u_x(\tau)\|^2 d\tau \leq C_1. \quad (53)
\]

**Proof.** Estimate (51) has been obtained in Lemma 2.3 of [21]. The proofs of estimates (52)-(53) are similar to those of Lemma 2.4 in [21]. Thus we omit the details. \( \square \)

**Lemma 3.5.** Under the assumptions in Theorem 2.1, the following estimate holds for any \( t > 0 \),
\[
\int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_s I^2 d\Omega dxd\nu + \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_s (\tilde{I} - I)^2 d\Omega dxd\nu \\
+ \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_s t^2 d\Omega dxd\nu \leq C_1 \int_0^L \theta^{1+\alpha}(x,t) dx. \quad (54)
\]

**Proof.** Multiplying (24)_4 by \( I \) and then integrating the resultant over \([0, L] \times S^{n-1}\) and using the boundary conditions, we obtain
\[
\int_{S^{n-1}} \omega(\theta^{n-1} I^2(L,t; \tilde{\Omega}, \nu) - a^{n-1} I^2(0,t; \tilde{\Omega}, \nu)) d\tilde{\Omega}
\]
Noting the boundary condition (19), we know

\[ \text{Corollary 1. Under the assumptions in Theorem 2.1, the following estimate also} \]

\[ \geq 0. \]  

Integrating (55) with respect to \( \nu \) over \([0, +\infty)\) and using Young’s inequality, (56) and the assumption \((A_8)\), we find

\[ \int_0^\infty \int_{S^{n-1}} \omega (b^{n-1} I^2 (L, t; \Omega, \nu) - a^{n-1} I^2 (0, t; \Omega, \nu)) d\Omega d\nu \]

\[ + \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_a I^2 d\Omega dx d\nu + \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_s (I - I)^2 d\Omega dx d\nu \]

\[ \leq \frac{1}{2} \left( \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_a I^2 d\Omega dx d\nu + C_1 \int_0^\infty \int_0^L \int_{S^{n-1}} \theta^{1+\alpha} f(\nu, \Omega) d\Omega dx d\nu \right) \]

\[ \leq \frac{1}{2} \left( \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_a I^2 d\Omega dx d\nu + C_1 \int_0^L \theta^{1+\alpha} (x, t) dx \right) \]

which implies

\[ \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_a I^2 d\Omega dx d\nu + \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_s (I - I)^2 d\Omega dx d\nu \]

\[ \leq C_1 \int_0^L \theta^{1+\alpha} (x, t) dx. \]

Similarly, we also infer from (55) by Young’s inequality and the assumptions \((A_8)\) that

\[ \int_0^\infty \int_0^L \int_{S^{n-1}} v (\sigma_a + \sigma_s) I^2 d\Omega dx d\nu \leq C_1 \int_0^L \theta^{1+\alpha} (x, t) dx. \]

Therefore, we complete the proof of (54). \(\square\)

Obviously, we can obtain the following result from Lemmas 3.1 and 3.5.

**Corollary 1. Under the assumptions in Theorem 2.1, the following estimate also holds for any \( t > 0 \),

\[ \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_a I^2 d\Omega dx d\nu + \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_s (I - I)^2 d\Omega dx d\nu \]

\[ + \int_0^\infty \int_0^L \int_{S^{n-1}} v \sigma_s I^2 d\Omega dx d\nu \leq C_1 \sup_{x \in [0, L]} \theta^{\max(\alpha-s,0)} (x, t). \]
Lemma 3.6. Under the assumptions in Theorem 2.1, the following estimates hold for any $t > 0$,

\[
\begin{align*}
\int_0^t (\theta^2 + \theta^{2+2s} + u_x^2)(x,t)dx + \int_0^t \|u_x(\tau)\|_{L^\infty}^2 d\tau &\leq C_1, \\
\int_0^t \|u_{xx}(\tau)\|^2 d\tau + \int_0^t \int_0^L (1 + \theta^{2+s})\theta_x^2(x,\tau)dx d\tau &+ \int_0^t \int_0^L (1 + \theta^{2+2s})u_x^2(x,\tau)dx d\tau \leq C_1.
\end{align*}
\]  

(57)  

(58)

Proof. Here we adopt the technique from Lemma 7 in [6]. Noting the formula (40) in Remark 2, we can define the auxiliary function for any $\xi > 0$,

\[ F(\xi) = \int_\theta e_\theta(v,\tau)d\tau. \]

Thus it follows from the assumption $(A_1)$ that $F(\xi) \leq c_1|\xi - \tilde{\theta}|(1 + \xi^s)$. Multiplying (24) by $F(\theta)$ over $Q_t$, we can infer

\[
\begin{align*}
\int_0^t F(\theta)dx + \int_0^t \int_0^L e_\theta \frac{\theta_s^2}{v} \theta_2^2 dx d\tau &\leq \int_0^t \int_0^L |e_\theta + PF(\theta)|(r^{n-1}u)_x|dx d\tau + \int_0^t \int_0^L \delta |F(\theta)| \frac{(r^{n-1}u)_x^2}{v} dx d\tau \\
&+ \int_0^t \int_0^L |F(\theta)|(r^{n-2}u^2)_x|dx d\tau + \int_0^t \int_0^L |F(\theta)|v|(S_E)_R|dx d\tau.
\end{align*}
\]

Noting that

\[
\theta^{2+2s}(x,t) \leq (\theta^{1+s}(x,t) - \tilde{\theta}^{1+s})^2 + C\tilde{\theta}^{2+2s} \leq CV_1(t) \int_0^t \theta^{2+2s-q}(x,t)dx + C \leq CV_1(t) + C,
\]

and using the assumptions $(A_1)$ and $(A_5)$ - $(A_6)$, Lemma 3.3 and the Sobolev embedding theorem, we have for any $\varepsilon > 0$,

\[
\begin{align*}
\int_0^L (\theta^2 + \theta^{2+2s})dx + \int_0^t \int_0^L (1 + \theta^{2+s})\theta_x^2 dx d\tau &\leq C_\varepsilon + \varepsilon \int_0^t (1 + V_1(\tau))\|u_x\|^2 d\tau \\
&+ C_\varepsilon \int_0^t (\|u\|^2 + \|u_x\|^2)(\tau) \int_0^L \theta^{2+2s} dx d\tau + \int_0^t \int_0^L |F(\theta)|v|(S_E)_R|dx d\tau.
\end{align*}
\]

(59)

Using the Cauchy-Schwarz inequality and the Sobolev embedding inequality, Lemma 3.1 and Lemma 3.5, and noting that the assumption $\alpha \leq 1 + 2s$, we can obtain

\[
\begin{align*}
\int_0^t \int_0^L |F(\theta)|v|(S_E)_R|dx d\tau &\leq C_1 + C_1 \int_0^t \int_0^L (\theta - \tilde{\theta})^2(1 + \theta^{2s}) dx d\tau + C_1 \int_0^t \int_0^L v^2\theta^{1+s}(x,\tau)dx d\tau \\
&\leq C_1 + C_1 \int_0^t V_1(\tau) \int_0^L \theta^{2+2s} dx d\tau + C_1 \int_0^t \|v(\tau)\|_{L^\infty}^2 \int_0^L \theta^{2+2s} dx d\tau \\
&\leq C_1 + C_1 \int_0^t (V_1(\tau) + \|v_x(\tau)\|^2) \int_0^L \theta^{2+2s} dx d\tau.
\end{align*}
\]

(60)
Multiplying \((24)_2\) by \(-u_{xx}\) and integrating by parts over \(Q_t\), using \((22)\) and Lemma 3.3, we arrive at

\[
\begin{align*}
\|u_x(t)\|^2 + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \\
\leq C_1 + C_1 \int_0^t \int_0^L |u_{xx}(Pv_x + P\theta_x)| dx d\tau + C_1 \int_0^t \int_0^L |v_x(r^{n-1}u)_x u_{xx}| dx d\tau
\end{align*}
\]

where

\[
C_1 = C \int_0^t \int_0^L |v_x(r^{n-1}u)_x u_{xx}| dx d\tau
\]

and

\[
C_2 = C \int_0^t \int_0^L |v_x(r^{n-1}u)_x u_{xx}| dx d\tau.
\]

Thus, taking \(\varepsilon > 0\) small enough, inserting \((60)\) into \((59)\) and \((62)\) into \((61)\) and applying Gronwall’s inequality, we complete the proof of this lemma. \(\square\)

Now repeating the derivation of \((47)-(48)\) and applying \((57)\), we can conclude for \(0 \leq m \leq \frac{q+2s+2}{2}\),

\[
\int_0^L \frac{v\theta^{2m}}{\kappa(v, \theta)} dx \leq C_1 \int_0^L \frac{\theta^{2m}}{1+\theta} dx \leq C_1 \int_0^L (1+\theta)^{2m-q} dx
\]

Thus we readily obtain the next corollary.

**Corollary 2.** Under the assumptions in Theorem 2.1, the following estimate holds for any \((x, t) \in [0, L] \times [0, +\infty)\),

\[
C_1^{-1} - C_1 V_1(t) \leq \theta^{2m}(x, t) \leq C_1 + C_1 V_1(t)
\]

with \(0 \leq m \leq \frac{q+2s+2}{2}\).

**Corollary 3.** Under the assumptions in Theorem 2.1, the following estimate holds for any \((x, t) \in [0, L] \times [0, +\infty)\),

\[
\int_0^t \int_0^L (1+\theta)^{2m} u_x^2 dx d\tau \leq C_1
\]

with \(0 \leq m \leq \frac{q+2s+2}{2}\).

**Proof.** It follows from Corollary 2 and \((53)\) and \((57)\) that

\[
\int_0^t \int_0^L (1+\theta)^{2m} u_x^2 dx d\tau \leq C_1 \int_0^t \int_0^L u_x^2 dx d\tau + C_1 \int_0^t \int_0^L V_1(\tau) u_x^2 dx d\tau \leq C_1.
\]

\(\square\)

**Lemma 3.7.** Under the assumptions in Theorem 2.1, the following estimate holds for any \(t > 0\),

\[
\|(r^{n-1}u)_{x}(t)||^2 + \int_0^t \|u_v(\tau)||^2 d\tau \leq C_1.
\]

\(\text{(63)}\)
Proof. Multiplying \((24)_2\) by \(u_t\) over \(Q_t\) and using Young’s inequality, we have for any \(\varepsilon > 0\),

\[
\| (r^{n-1}u_x)(t) \|_2^2 + \int_0^t \| u_t(\tau) \|_2^2 d\tau \\
\leq C_1 + \frac{\varepsilon}{2} \int_0^t \| u_t(\tau) \|_2^2 d\tau + C_\varepsilon \int_0^t \int_0^L v_x^2(u^2 + u_x^2) dx dt \\
+ C_1 \int_0^t \int_0^L |u_t(P_v x + P_0 \theta_t)| dx dt \\
\leq C_1 + \varepsilon \int_0^t \| u_t(\tau) \|_2^2 d\tau + C_\varepsilon \int_0^t \| v_x \|_2^2 (\| u \|_{L^\infty}^2 + \| u_x \|_{L^\infty}^2) dx dt \\
+ C_\varepsilon \int_0^t \int_0^L (1 + \theta^{2+2s})v_x^2 dx dt + C_\varepsilon \int_0^t \int_0^L (1 + \theta^{q+s})\theta_t^2 dx dt
\]

which, by taking \(\varepsilon > 0\) small enough, along with Lemmas 3.4 and 3.6, leads to (63).

Lemma 3.8. Under the assumptions in Theorem 2.1, the following estimates hold that for any \(t > 0\),

\[
\int_0^L (1 + \theta)^{2q}\theta_t^2 dx + \int_0^t \int_0^L (1 + \theta)^{q+s}\theta_t^2 dx dt \leq C_1, \quad (64)
\]

\[
\sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty} \leq C_1. \quad (65)
\]

Proof. Let

\[
K(v, \theta) = \int_0^L \frac{r^{2n-2}\kappa(v, \xi)}{v} d\xi, \quad X(t) = \int_0^L (1 + \theta)^{q+s}\theta_t^2 dx dt,
\]

Then it is easy to verify that

\[
K_t = K_v (r^{n-1}u)_x + \frac{r^{2n-2}\kappa(v, \theta)}{v} \theta_t,
\]

\[
K_{xt} = \left( \frac{r^{2n-2}\kappa(v, \theta)}{v} \theta_x \right)_t + K_v (r^{n-1}u)_{xx} + K_v u_x (r^{n-1}u)_x \\
+ \left( \frac{r^{2n-2}\kappa(v, \theta)}{v} \right) v_x \theta_t.
\]

But we easily know from the assumptions \((A_6) - (A_7)\) that

\[
|K_v| + |K_{vv}| \leq C(1 + \theta^{1+q}). \quad (66)
\]

We rewrite \((24)_3\) as

\[
e_v \theta_t + \theta P_0 (r^{n-1}u)_x - \delta \frac{(r^{n-1}u)_x^2}{v} = \left( \frac{r^{2n-2}\kappa \theta_x}{v} \right)_x \\
- 2\mu(n-1)(r^{n-2}u_x^2)_x - v(S_E)_R. \quad (67)
\]

Multiplying (67) by \(K_t\) and integrating the resultant over \(Q_t\), we easily obtain

\[
\int_0^t \int_0^L \left( e_v \theta_t + \theta P_0 (r^{n-1}u)_x - \delta \frac{(r^{n-1}u)_x^2}{v} \right) K_t dx dt + \int_0^t \int_0^L \frac{r^{2n-2}\kappa \theta_x}{v} K_{tx} dx dt
\]
and applying Cauchy’s inequality, we can deduce
\[
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\]
\[
\int_0^t \int_0^L e_\theta t \frac{r^{2n-2} \kappa(v, \theta)}{v} \, dx \, d\tau \geq C^{-1} X(t),
\]
(69)
and applying Corollaries 2-3 and Lemma 3.6, we deduce for
\[
\int_0^t \int_0^L e_\theta t \kappa(v, \theta) \, dx \, d\tau \left| \leq \frac{1}{8} X(t) + C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty}^{2+s+q}) \right.
\]
and applying Cauchy’s inequality, we can deduce
\[
\left| \int_0^t \int_0^L \left( (1 + \theta)^{q+s+2} |(r^{n-1} u)x|^2 + (1 + \theta)^{q-s} |(r^{n-1} u)x|^4 \right) \, dx \, d\tau \right|
\]
\[
\leq \frac{1}{8} X(t) + C_1 \int_0^t \int_0^L \left( (1 + \theta)^{q+s+2} |u|^2 + (1 + \theta)^{q-s} (|u|^4 + |u|^2) \right) \, dx \, d\tau
\]
\[
\leq \frac{1}{8} X(t) + C_1 \int_0^t \int_0^L (1 + \theta)^{-s} (|u|^4 + |u|^2) \, dx \, d\tau.
\]
(71)
But
\[
\int_0^t \int_0^L (1 + \theta)^{-s} u^4 \, dx \, d\tau \leq (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty}^{-s}) \int_0^t \|u\|_{L^\infty}^2 \int_0^L u^2 \, dx \, d\tau
\]
\[
\leq C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty}^{-s}),
\]
(72)
and applying Corollaries 2-3 and Lemma 3.6, we deduce for \( q_1 = \max\left\{ \frac{q}{2} - 2s - 1, 0 \right\} \),
\[
\int_0^t \int_0^L (1 + \theta)^{-s} u_4^2 \, dx \, d\tau
\]
\[
\leq C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty}^{q_1}) \int_0^t \int_0^L (1 + \theta)^{q_1/2} u_4^2 \, dx \, d\tau
\]
\[
\leq C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty}^{q_1})
\]
\[
\times \left( \int_0^t \int_0^L u_4^2 \, dx \, d\tau + \int_0^t V_1^2(\tau) \int_0^L u_4^2 \, dx \, d\tau \right)
\]
\[
\leq C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty}^{q_1})
\]
\[
\times \left( \int_0^t \|u_x\|^3 \|u_{xx}\| d\tau + \int_0^t V_1^2(\tau) \|u_x\|^3 \|u_{xx}\| d\tau \right)
\]
\[
\leq C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty}^{q_1})
\]
\[
\times \left( \sup_{0 \leq \tau \leq t} \|u_x(\tau)\|^2 \left( \int_0^t \|u_x(\tau)\|^2 d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|u_{xx}(\tau)\|^2 d\tau \right)^{\frac{1}{2}} + \sup_{0 \leq \tau \leq t} \|u_x(\tau)\|^3 \left( \int_0^t V_1(\tau) d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|u_{xx}(\tau)\|^2 d\tau \right)^{\frac{1}{2}} \right)
\]
which, along with (71) and (72), gives

\[ L_3.7 \leq L_3.1 \leq \int_0^1 \leq \int_0^L \left( \theta P_\theta (r^{n-1} u)_x - \nu (r^{n-1} u)_x^2 \right) \frac{r^{2n-2} \kappa(v, \theta)}{v} \theta \, dx d\tau \]

\[ \leq \frac{1}{8} X(t) + C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty}^{q-s}). \]  

(74)

It follows from Lemma 3.6 and Corollaries 2-3 that

\[ \left| \int_0^t \int_0^L \left( \theta P_\theta (r^{n-1} u)_x - \nu (r^{n-1} u)_x^2 \right) K_v (r^{n-1} u)_x \, dx d\tau \right| \]

\[ \leq C_1 \int_0^t \int_0^L ((1 + \theta)^{q-s+2} |(r^{n-1} u)_x|^2 + (1 + \theta)^{q+1} |(r^{n-1} u)_x|^3) \, dx d\tau \]

\[ \leq C_1 \int_0^t \int_0^L (1 + \theta)^{q-s+2} |(r^{n-1} u)_x|^2 \, dx d\tau \]

\[ + C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{q+1} \int_0^t \int_0^L (|u|^3 + |u_x|^3) \, dx d\tau \]

\[ \leq C_1 + C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{q+1} \]

\[ \times \left( \int_0^t \int_0^L |u|^2 \, dx d\tau + \int_0^t \int_0^L |u_x|^3 \, dx d\tau \right) \]

\[ \leq C_1 + C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{q+1} \int_0^t \int_0^L \|u_x(\tau)\|^2 \, dx d\tau \]

\[ \leq C_1 + C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{q+1} \sup_{0 \leq \tau \leq t} \|u_x(\tau)\| \]

\[ \times \left( \int_0^t \int_0^L |u_x(\tau)|^2 \, dx d\tau \right)^{\frac{4}{7}} \left( \int_0^t \|u_{xx}(\tau)\|^2 \, dx d\tau \right)^{\frac{1}{2}} \]

\[ \leq C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{q+1}. \]  

(75)

Now let us consider the various contributions in the second integral of (68). By Lemmas 3.1-3.7 and Corollaries 1 and 3, we have

\[ \int_0^t \int_0^L \frac{r^{2n-2} \kappa \theta_x}{v} \left( \frac{r^{2n-2} \kappa \theta_x}{v} \right) \, dx d\tau \geq C_1^{-1} Y(t) - C_1, \]  

(76)

\[ \left| \int_0^t \int_0^L \frac{r^{2n-2} \kappa \theta_x}{v} K_v(r^{n-1} u)_{xx} \, dx d\tau \right| \leq C_1 \int_0^t \int_0^L (1 + \theta)^{1+2q} |\theta_x(r^{n-1} u)_{xx}| \, dx d\tau \]

\[ \leq C_1 \left( \int_0^t \int_0^L (1 + \theta)^{s+q} \theta_x^2 \, dx d\tau \right)^{\frac{1}{2}} \left( \int_0^t \int_0^L (1 + \theta)^{q-s+2} |(r^{n-1} u)_{xx}|^2 \, dx d\tau \right)^{\frac{1}{2}} \]

\[ \leq C_1 \left( \int_0^t \int_0^L (1 + \theta)^{q-s+2} (|u|^2 + |u_x|^2 + |u_{xx}|^2) \, dx d\tau \right)^{\frac{1}{2}} \]

\[ \leq C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{\frac{2q-s+2}{2}} \]  

(77)
and
\[
\left| \int_0^t \int_0^L \frac{r^{2n-2} \kappa \theta_x}{v} K_{uv} v_x r^{n-1} u_x \, dx \, d\tau \right| \\
\leq C_1 \int_0^t \int_0^L (1 + \theta)^{1 + 2q} |\theta_x v_x r^{n-1} u_x| \, dx \, d\tau \\
\leq C_1 \left( \int_0^t \int_0^L (1 + \theta)^{s+q} \theta_x^2 \, dx \, d\tau \right)^{\frac{1}{2}} \left( \int_0^t \int_0^L (1 + \theta)^{3q-s+2} |r^{n-1} u_x|^2 \, dx \, d\tau \right)^{\frac{1}{2}} \\
\leq C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty})^{\frac{3s-q+2}{2}} \left( \int_0^t \int_0^L u_x^2 (r^{n-1} u_x)^2 \, dx \, d\tau \right)^{\frac{1}{2}} \\
\leq C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty})^{\frac{s+q}{2}} \left( \int_0^t \int_0^L u_x^2 \, dx \, d\tau + \int_0^t \int_0^L v_x^2 \, dx \, d\tau \right)^{\frac{1}{2}} \\
\leq C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty})^{\frac{3s-q+2}{2}}. 
\] (78)

Noting the following facts
\[
\int_0^t \int_0^L (r^{n-1} u_x)_t^2 \, dx \, d\tau \leq C_1 \int_0^t \int_0^L (u^4 + u_x^2) \, dx \, d\tau \\
\leq C_1 + C_1 \int_0^t \|u_x(\tau)\|^3 \|u_{xx}(\tau)\| \, d\tau \\
\leq C_1 + C_1 \sup_{0 \leq \tau \leq t} \|u_x(\tau)\|^2 \left( \int_0^t \|u_x(\tau)\|^2 \, d\tau \right) \left( \int_0^t \|u_{xx}(\tau)\|^2 \, d\tau \right)^{\frac{1}{2}} \\
\leq C_1, \\
\int_0^t \int_0^L (r^{n-2} u_x^2)_t^2 \, dx \, d\tau \leq C_1 \int_0^t \int_0^L (u^4 + u_x^2) \, dx \, d\tau \\
\leq C_1 + C_1 \int_0^t \|u_x\|_{L^\infty}^2 \, d\tau \leq C_1, \\
\int_0^t \|v(S_E)_R\|^2 \, d\tau \leq \int_0^t \int_0^\infty \int_0^L \int_{\mathbb{S}^{n-1}} (u^2 \sigma^2 I^2 + v^2 \sigma^2 B^2 + v^2 \sigma^2 (I - I)^2) \, d\Omega \, dx \, d\tau \\
\leq C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\|_{L^\infty})^{\max\{\alpha-s,0\}}
\]
and from equation (24), we can deduce
\[
\int_0^t \left\| \frac{r^{2n-2} \kappa \theta_x}{v} \right\|_z^2 \, d\tau \\
\leq C_1 \int_0^t \left( \|e\theta_x\|^2 + \|P(r^{n-1} u)_x\|^2 + \|(r^{n-1} u)_x\|^2 + \|(r^{n-2} u_x)\|_{L^2}^2 + \|v(S_E)_R\|^2 \right) \, d\tau \\
\leq C_1 \int_0^t \int_0^L \left( (1 + \theta)^{2s} \theta_t^2 + (1 + \theta)^{2s+2} u^2 + (1 + \theta)^{2s+2} u_x^2 + (r^{n-1} u)_x^4 \right) \\
\]
Thus, by the Sobolev inequality, Lemmas 3.4 and 3.7, we can conclude

\[ \leq C_1 \int_0^t \int_0^L (1 + \theta)^{\gamma+s} \theta_t^2 dx d\tau + C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{\max\{\alpha-s,0\}} \]

Thus, by the Sobolev inequality, Lemmas 3.4 and 3.7, we can conclude

\[
\leq C_1 \int_0^t \int_0^L \left( \frac{r^{2n-2}K_x}{v} \right)_x^2 v_x \theta_t dx d\tau
\]

\[
\leq \left( \int_0^t \int_0^L (1 + \theta)^{2q} dx d\tau + \int_0^t \int_0^L (1 + \theta)^q \theta_x^2 \left( \frac{r^{2n-2}K_x}{v} \right)_x^2 dx d\tau \right)^{\frac{1}{2}}
\]

\[ \leq \frac{1}{16} X(t) + C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{\gamma-s}
\]

\[ \times \left( \int_0^t \int_0^L (1 + \theta)^{2q} \left( \frac{r^{2n-2}K_x}{v} \right)_x \left( \frac{r^{2n-2}K_x}{v} \right)_x \right)^{\frac{1}{2}}
\]

\[ \leq \frac{1}{16} X(t) + C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{\gamma-s}
\]

\[ \times \left( \int_0^t \int_0^L (1 + \theta)^{2q} \left( \frac{r^{2n-2}K_x}{v} \right)_x \left( \frac{r^{2n-2}K_x}{v} \right)_x \right)^{\frac{1}{2}}
\]

\[ \leq \frac{1}{8} X(t) + C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{\gamma-s}
\]

By Lemma 3.4 and Corollary 3, using Cauchy’s inequality, we have

\[
\left| \int_0^t \int_0^L (r^{n-2} u^2)_x \left( K_v (r^{n-1} u)_x + \frac{r^{2n-2}K_x}{v} \theta_t \right) dx d\tau \right|
\]

\[ \leq \frac{1}{8} X(t) + C_1 (1 + \sup_{0 \leq \tau \leq t} \| \theta(x, \tau) \|_{L^\infty})^{\gamma-s}.
\]

The last contribution in (68) can be estimated as follows,

\[
\left| \int_0^t \int_0^L v(S_E) R K_t dx d\tau \right| \leq \int_0^t \int_0^L \int_0^\infty v \sigma_a B d\Omega dv \left| K_t \right| dx d\tau
\]

\[ + \int_0^t \int_0^L \int_0^\infty v \sigma_a I d\Omega dv \left| K_t \right| dx d\tau
\]

\[ + \int_0^t \int_0^L \int_0^\infty v \sigma_a (I - I) d\Omega dv \left| K_t \right| dx d\tau
\]
\[ M_1 \leq C_1 \int_0^t \int_0^L (1 + \theta^\alpha)|K_1|dxd\tau \]
\[ \leq C_1 \int_0^t \int_0^L (1 + \theta^{q+\alpha+1})|(r^{n-1}u)_x|dxd\tau + C_1 \int_0^t \int_0^L (1 + \theta^{q+\alpha})|\theta_1|dxd\tau \]
\[ \leq \frac{1}{8} X(t) + C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\||_{L^\infty})^{q+2\alpha-s}. \] (83)

Using the assumption \((A_9)\), Cauchy’s inequality and Corollary 1, we have

\[ M_2 \leq C_1 \int_0^t \int_0^L \int_0^\infty \int_{S^{n-1}} \nu\sigma_d I_{K_v(r^{n-1}u)x + \frac{r^{2n-2}K}{\nu}\theta_1} d\Omega dxd\tau \]
\[ \leq C_1 \int_0^t \int_0^L \int_0^\infty \int_{S^{n-1}} \nu\sigma_d I_2 d\Omega dxd\tau \]
\[ + C_1 \int_0^t \int_0^L \int_0^\infty \int_{S^{n-1}} (1 + \theta^q)|I_{\theta_1}|d\Omega dxd\tau \]
\[ \leq \frac{1}{8} X(t) + C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\||_{L^\infty})^{\max\{q-s,\alpha-s\}}. \] (84)

Using the same technique, we also get

\[ M_3 \leq \frac{1}{8} X(t) + C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\||_{L^\infty})^{\max\{q-s,\alpha-s\}}. \] (85)

Inserting the estimates (83)-(85) into (82), we get

\[ \left| \int_0^t \int_0^L \nu(S_{E})R K_1 \right| \leq \frac{1}{8} X(t) + C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\||_{L^\infty})^{\max\{q-s,\alpha-s\}}. \] (86)

Inserting all previous estimates (69)-(71), (74)-(78), (80)-(81) and (86) into (68), we obtain

\[ X(t) + Y(t) \leq C_1 (1 + \sup_{0 \leq \tau \leq t} \|\theta(x, \tau)\||_{L^\infty})^\lambda \] (87)

with \(\lambda = \max\{q + s + 2, q + 2\alpha - s, \frac{3}{2}(q - s) + \alpha - s, 2q - s + 2, \frac{3q-s+2}{2}\}\).

By Lemmas 3.1, 3.4, 3.6 and the Hölder inequality, there exists a point \(b(t) \in [0, L]\) such that for any \(t > 0\),

\[ \theta^{q+s+2}(x, t) - \theta^{q+s+2}(b(t), t) = \int_{b(t)}^x (\theta^{q+s+2}(x, t))_x dx \leq C_1 \int_0^L \theta^{q+s+1}\theta_x dx \]
\[ \leq C_1 \left( \int_0^L \theta^{2s+2}dx \right)^{\frac{1}{2}} Y^{\frac{1}{2}}(t). \]

Thus we get

\[ \sup_{0 \leq \tau \leq t} \|\theta(\tau)\||_{L^\infty} \leq C_1 + C_1 Y^{\frac{1}{2q+s+2}}(t). \]
Using assumptions on $q, s$ and $\alpha$, we easily know that $\lambda < 2(q + s + 2)$. Thus, by Young’s inequality and (87), it follows that
\[
X(t) + Y(t) \leq C_1
\]
which yields
\[
\sup_{0 \leq \tau \leq t} \|\theta(\tau)\|_{L^\infty} \leq C_1.
\]
Therefore, we complete the proof. \hfill \Box

**Lemma 3.9.** The following estimates hold that for any $t > 0$,
\[
\begin{align*}
\int_0^\infty \int_{S^{n-1}} I(x, t; \nu, \vec{\Omega}) d\vec{\Omega} d\nu &\leq C_1, \\
\left\| \int_0^\infty \int_{S^{n-1}} I\vec{d}\vec{\Omega} d\nu \right\|_{L^\infty(Q_t)} &+ \left\| \int_0^\infty \int_{S^{n-1}} |I|\vec{d}\vec{\Omega} d\nu \right\|_{L^\infty(Q_t)} \leq C_1, \\
\left\| \int_0^\infty \int_{S^{n-1}} I^2\vec{d}\vec{\Omega} d\nu \right\|_{L^\infty(Q_t)} &+ \left\| \int_0^\infty \int_{S^{n-1}} |I|\vec{d}\vec{\Omega} d\nu \right\|_{L^\infty(Q_t)} \leq C_1, \\
\left\| \int_0^\infty \int_{S^{n-1}} |I|\vec{d}\vec{\Omega} d\nu \right\|_{L^\infty(Q_t)} &+ \int_{t}^{t+L} \int_{S^{n-1}} \int_{S^{n-1}} I^2(x, \tau; \nu, \vec{\Omega}) d\vec{\Omega} d\nu d\tau \leq C_1.
\end{align*}
\]

**Proof.** We consider the following integro-differential equation
\[
\begin{align*}
\frac{r^{n-1}}{r^n} \omega I_x &= vS = v(\sigma_\alpha B + \sigma_s \vec{I}) - v(\sigma_\alpha + \sigma_s)I, \\
I(0, t; \nu, \vec{\Omega}) &= 0 \text{ for } \omega \in (0, 1), \\
I(L, t; \nu, \vec{\Omega}) &= 0 \text{ for } \omega \in (-1, 0), \\
I(x, 0; \nu, \vec{\Omega}) &= I_0(x, \nu, \vec{\Omega}).
\end{align*}
\]
Solving the ordinary differential equation and using the boundary conditions, we arrive at
\[
I(x, t; \nu, \vec{\Omega})
= \begin{cases} 
\int_0^x \exp \left( \int_x^y \frac{v(\sigma_\alpha + \sigma_s)}{r^{n-1}} dz \right) \frac{v}{r^{n-1}} (\sigma_\alpha B + \sigma_s \vec{I}) dy, & \text{for all } \omega \in (0, 1), \\
- \int_x^L \exp \left( \int_x^y \frac{v(\sigma_\alpha + \sigma_s)}{r^{n-1}} dz \right) \frac{v}{r^{n-1}} (\sigma_\alpha B + \sigma_s \vec{I}) dy, & \text{for all } \omega \in (-1, 0).
\end{cases}
\]

Using Young’s inequality and Lemmas 3.5 and 3.8, we have for all $\omega \in (0, 1),
\[
\begin{align*}
\int_0^\infty \int_{S^{n-1}} I\vec{d}\vec{\Omega} d\nu \\
&= \int_0^\infty \int_{S^{n-1}} \int_0^x \exp \left( \int_x^y \frac{v(\sigma_\alpha + \sigma_s)}{r^{n-1}} dz \right) \frac{v}{r^{n-1}} (\sigma_\alpha B + \sigma_s \vec{I}) dy d\vec{\Omega} d\nu \\
&\leq \int_0^\infty \int_{S^{n-1}} \int_0^L \frac{v}{r^{n-1}} (\sigma_\alpha B + \sigma_s \vec{I}) dxd\vec{\Omega} d\nu \\
&\leq \int_0^L vr^{1-n} \int_{S^{n-1}} \frac{\sigma_\alpha B}{\omega} d\vec{\Omega} d\nu + \int_0^L vr^{1-n} \int_{S^{n-1}} \frac{\sigma_s \vec{I}}{\omega} d\vec{\Omega} d\nu d\tau \\
&\leq C_1 \int_0^L \theta^{\alpha + \alpha} d\nu + C_1 \int_0^L vr^{1-n} \int_{S^{n-1}} \frac{\sigma_s (\frac{1}{\omega^2} + (\vec{I} - \vec{I})^2 + I^2) d\vec{\Omega} d\nu}{\omega} \\
&\leq C_1.
\end{align*}
\]
Analogously, we have the same result for all \( \omega \in (-1, 0) \). Thus, using Lemma 3.8, we get
\[
\left\| \int_0^\infty \int_{S^{n-1}} 1 d\mathbf{\Omega} d\nu \right\|_{L^\infty(Q_1)} \leq C_1. \tag{95}
\]
Furthermore, we also get
\[
\left\| \int_0^\infty \int_{S^{n-1}} \tilde{d}\mathbf{\Omega} d\nu \right\|_{L^\infty(Q_1)} \leq C_1. \tag{96}
\]
Similarly to (94), using Young’s and Hölder’s inequalities, we derive
\[
\int_0^\infty \int_{S^{n-1}} t^2 d\mathbf{\Omega} d\nu \leq \int_0^\infty \int_{S^{n-1}} \left( \int_0^L \frac{v}{r^{n-1}\omega}(\sigma_a B + \sigma_s \bar{I}) dy \right)^2 d\mathbf{\Omega} d\nu
\]
\[
\leq C_1 \int_0^\infty \int_{S^{n-1}} \left( \int_0^L \frac{\sigma_a v^2}{r^{2n-2}\omega^2} dx \cdot \int_0^L \sigma_a B^2 dx \right) d\mathbf{\Omega} d\nu
\]
\[
+ C_1 \int_0^\infty \int_{S^{n-1}} \left( \int_0^L \frac{\sigma_s v^2}{r^{2n-2}\omega^2} dx \cdot \int_0^L \sigma_s \bar{I}^2 dx \right) d\mathbf{\Omega} d\nu
\]
\[
\leq C_1 + C_1 \int_0^\infty \int_{S^{n-1}} \int_0^L \sigma_s (\bar{I} - I)^2 dxd\mathbf{\Omega} d\nu
\]
\[
\leq C_1 \tag{97}
\]
which implies
\[
\left\| \int_0^\infty \int_{S^{n-1}} t^2 d\mathbf{\Omega} d\nu \right\|_{L^\infty(Q_1)} \leq C_1. \tag{98}
\]
It follows from (24)_4 that
\[
I_x = -\frac{r^{1-n}v}{\omega}(\sigma_a + \sigma_s)I + \frac{r^{1-n}v}{\omega}(\sigma_a B + \sigma_s \bar{I}).
\]
Integrating the above equality and using the assumptions (A_8) – (A_{11}), we can derive
\[
\int_0^\infty \int_{S^{n-1}} |I_x| d\mathbf{\Omega} d\nu \leq C_1 \int_0^\infty \int_{S^{n-1}} \frac{(\sigma_a + \sigma_s)|I|}{|\omega|} d\mathbf{\Omega} d\nu
\]
\[
+ C_1 \int_0^\infty \int_{S^{n-1}} \frac{(|\sigma_a|B + |\sigma_s|\bar{I})}{|\omega|} d\mathbf{\Omega} d\nu
\]
\[
\leq C_1 + C_1 \int_0^\infty \int_{S^{n-1}} |I| d\mathbf{\Omega} d\nu + C_1 \int_0^\infty \int_{S^{n-1}} |\bar{I}| d\mathbf{\Omega} d\nu
\]
which, along with (95) and (96), leads to
\[
\left\| \int_0^\infty \int_{S^{n-1}} |I_x| d\mathbf{\Omega} d\nu \right\|_{L^\infty(Q_1)} \leq C_1. \tag{99}
\]
It follows from (93) that for any \( \omega \in (0, 1) \),
\[
I_t = \int_0^x \exp \left( \int_x^y \frac{v(\sigma_a + \sigma_s)}{r^{n-1}\omega} dz \right) \left( \int_0^y \frac{v(\sigma_a + \sigma_s)}{r^{n-1}\omega} dz \right) \frac{v}{r^{n-1}\omega}(\sigma_a B + \sigma_s \bar{I}) dy
\]
\[
+ \int_0^x \exp \left( \int_x^y \frac{v(\sigma_a + \sigma_s)}{r^{n-1}\omega} dz \right) \frac{v}{r^{n-1}\omega}(\sigma_a B + \sigma_s \bar{I}) \left( \frac{v}{r^{n-1}\omega}(\sigma_a B + \sigma_s \bar{I}) \right) dy
\]
\[
=: N_1 + N_2. \tag{100}
\]
Using Young’s inequality, Lemmas 3.1-3.8 and the assumptions \((A_9) - (A_{11})\) and \((A_{13})\), we have

\[
\int_0^t \int_0^\infty \int_{S^{n-1}} N_2^2 d\Omega d\nu d\tau \\
\leq C_1 \int_0^t \int_0^\infty \int_{S^{n-1}} \left( \int_0^\infty \int_{S^{n-1}} \left( \frac{1}{\omega} (r^{1-n} v_t^2 (\sigma_a + \sigma_s) + r^{1-n} v (\sigma_a + \sigma_s) v_t + v (\sigma_a B + \sigma_s I)) d\nu d\tau \right)^2 d\Omega d\nu d\tau \\
\leq C_1 \int_0^t \int_0^\infty \int_{S^{n-1}} \left( \int_0^L \frac{(r^{1-n} u_z^2)}{\omega^2} (\sigma_a + \sigma_s)^2 d\nu d\tau \right)^2 d\Omega d\nu d\tau \\
\leq C_1 \int_0^1 \int_0^L \left( (r^{1-n} u_z^2 + \theta^2) d\nu d\tau \right) + C_1 \int_0^1 \int_0^\infty \int_{S^{n-1}} \int_0^L I_2^2 d\nu d\Omega d\tau \\
\leq C_1 (101)
\]

and

\[
\int_0^t \int_0^\infty \int_{S^{n-1}} N_2^2 d\Omega d\nu d\tau \\
\leq C_1 \int_0^t \int_0^\infty \int_{S^{n-1}} \left( \int_0^\infty \int_{S^{n-1}} \left( \frac{1}{\omega} (r^{1-n} v_t (\sigma_a B + \sigma_s I) + r^{1-n} v ((\sigma_a) v_t B + (\sigma_a) v_t (\sigma_a B + \sigma_s I)) d\nu d\tau \right)^2 d\Omega d\nu d\tau \\
\leq C_1 \int_0^t \int_0^L \left( (r^{1-n} u_z^2 + \theta^2) d\nu d\tau \right) + C_1 \int_0^t \int_0^\infty \int_{S^{n-1}} \int_0^L I_2^2 d\nu d\Omega d\tau \\
\leq C_1 + C_1 \int_0^t \int_0^\infty \int_{S^{n-1}} \int_0^L I_2^2 d\nu d\Omega d\tau \\
(102)
\]

which, together with (101), implies

\[
\int_0^t \int_0^\infty \int_{S^{n-1}} I_2^2 d\Omega d\nu d\tau \leq C_1 + C_1 \int_0^t \left( \int_0^\infty \int_{S^{n-1}} I_2^2 d\Omega d\tau \right) dy. \\
(103)
\]

Using the same technique, we have the above inequality for any \(\omega \in (-1, 0)\). By Gronwall’s inequality for any fixed \(t > 0\), we have

\[
\int_0^t \int_0^\infty \int_{S^{n-1}} I_2^2 d\Omega d\nu d\tau \leq C_1 e^{C_1 t} \leq C_1 e^{C_1 L} \leq C_1. \\
(104)
\]

Similarly, we can obtain

\[
\int_0^\infty \int_{S^{n-1}} |I_1| d\Omega d\nu \leq C_1. \\
(105)
\]

Thus, the estimate (91) follows from (104)-(105).

\[\square\]

**Lemma 3.10.** Under the assumptions in Theorem 2.1, the following estimates hold that for any \(t > 0\),

\[
\|I_2(t)\| \leq C_1, \quad \|I_{xx}(t)\| \leq C_1. \\
(106)
\]
Lemma 3.11. Under the assumptions in Theorem 2.1, we have, as 
and for all 
Using Lemma 3.9, we see that 

\[ \int_I \left| \frac{1}{\omega} \left( (r^{1-n}v_x)_x + (r^{1-n}V_s) \right) \right| d\nu = 0 \]

\[ \int_I \left| \frac{1}{\omega} \left( (r^{1-n}v_x)_x + (r^{1-n}V_s) \right) \right| d\nu = 0 \]

By virtue of the direct computation, we also have

\[ \int_I \left| \frac{1}{\omega} \left( (r^{1-n}v_x)_x + (r^{1-n}V_s) \right) \right| d\nu = 0 \]

The next two lemmas are aimed at showing the asymptotic behavior of solutions

\[ \| u(t) \|_{H^1} \to 0, \]

\[ \| u(t) \|_{H^1} \to 0, \]

\[ \| \theta(t) - \bar{\theta} \|_{H^1} \to 0, \]

\[ \| \theta(t) - \bar{\theta} \|_{L^\infty} \to 0 \]

and for all \((x, t) \in [0, L] \times [0, +\infty)\),

\[ 0 < C_1^{-1} \leq \theta(x, t) \leq C_1. \]
Proof. By Lemmas 3.4, 3.6 and 3.7, we can know
\[
\int_0^t (\|v_x\|^2 + \|u_x\|^2)(\tau)d\tau \leq C_1
\] (115)
and
\[
\int_0^t \left( \left| \frac{d}{dt} \|v_x\|^2 \right| + \left| \frac{d}{dt} \|u_x\|^2 \right| \right)(\tau)d\tau \leq C_1
\] (116)
which lead to (111)-(112).

Multiplying (67) by \(e_{\vartheta}^{-1}\theta_{xx}\), integrating the resultant over \((0, L)\) and using Young’s inequality, the interpolation inequality and Lemmas 3.1-3.4 and 3.6-3.8, we can conclude for any \(\varepsilon > 0\),
\[
\frac{d}{dt} \|\theta_x\|^2 + 2\int_0^L \frac{r^{2n-2K}}{e_{\vartheta}v} \theta_{xx}^2 dx
\]
\[= 2\int_0^L e_{\vartheta}^{-1}(\theta \theta_{\vartheta}(r^{n-1}u)_x - \delta \left(\frac{r^{n-1}u}{v}\right)_x^2 \theta_x + 2\mu(n-1)(r^{n-2}u^2)_x + v(S_E)_R) \theta_{xx} dx
\]
\[\leq \frac{\varepsilon}{2} \|\theta_{xx}\|^2 + C_1 (\|(r^{n-1}u)_x\|^2 +\| (r^{n-1}u)_x\|^2_{L^4} + \|u_x\theta_x\|^2
\]
\[+ \|\theta_x\|^2 + \|\theta_x\|^2_{L^4} + \|(S_E)_R\|^2)
\[\leq \varepsilon \|\theta_{xx}\|^2 + C_1 (\|u_x\|^2 +\|u_x\|^2 + \|\theta_x\|^2 + \|(S_E)_R\|^2).
\] (117)

Now we need to estimate the term \(\|(S_E)_R\|^2\) in (117) and derive from Young’s and Hölder’s inequalities that
\[
\|(S_E)_R\|^2 = \int_0^L \left( \int_0^\infty \int_{S^{n-1}} (\sigma_\alpha (B-I) + \sigma_\alpha (\tilde{I}-I)) d\tilde{\Omega} d\nu \right)^2 dx
\]
\[\leq C_1 \int_0^L \left( \int_0^\infty \int_{S^{n-1}} \sigma_\alpha (B-I) d\tilde{\Omega} d\nu \right)^2 dx
\]
\[+ C_1 \int_0^L \left( \int_0^\infty \int_{S^{n-1}} \sigma_\alpha (\tilde{I}-I) d\tilde{\Omega} d\nu \right)^2 dx
\]
\[\leq C_1 \int_0^L \left( \int_0^\infty \int_{S^{n-1}} \sigma_\alpha d\tilde{\Omega} d\nu \int_0^\infty \int_{S^{n-1}} \sigma_\alpha (B^2 + I^2) d\Omega d\nu \right) dx
\]
\[+ C_1 \int_0^L \left( \int_0^\infty \int_{S^{n-1}} \sigma_\alpha d\tilde{\Omega} d\nu \int_0^\infty \int_{S^{n-1}} \sigma_\alpha (\tilde{I}-I)^2 d\tilde{\Omega} d\nu \right) dx
\]
\[\leq C_1 \int_0^L \theta^{1+\alpha} dx + C_1 \leq C_1
\] (118)
which, together with (117), leads to
\[
\|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}(\tau)\|^2 d\tau \leq C_1.
\] (119)

Meanwhile, taking \(\varepsilon > 0\) small enough and using Lemmas 3.1-3.9, we deduce from (117) that
\[
\frac{d}{dt} \|\theta_x\|^2 + C_1 \int_0^L (1 + \theta^{\vartheta-x}) \theta_{xx}^2 dx \leq C_1 (1 + \|u_{xx}\|^2)
\] (120)
which, along with Lemma 2.4, yields
\[ \lim_{t \to +\infty} \| \theta_x(t) \|^2 = 0. \tag{121} \]

By Poincaré’s inequality, we infer
\[ \| \theta(t) - \bar{\theta} \|_{H^1} \leq C_1 \| \theta_x(t) \| \]
which, combined with (121), gives (113).

We derive from (113) that there exists a large time \( t_0 > 0 \) such that for any \( t \geq t_0 \),
\[ \theta(x,t) \geq \frac{1}{2} \bar{\theta} > 0. \tag{122} \]

On the other hand, if we put \( \Theta := \frac{1}{\theta} \), and take \( \hat{\delta} > 0 \) such that if \( n = 2, 3, 0 \leq 2(n-2)\mu < (n-1)\delta < \hat{\delta} < 1 \), then (67) becomes
\begin{align*}
\varepsilon_\partial \Theta_t &= \left( \frac{r^{2n-2} \kappa \Theta_x}{v} \right)_x - \left\{ \frac{2n \kappa r^{2n-2} \Theta_x^2}{v \Theta} + \frac{\Theta^2}{v} (n-1)\delta \bar{\delta} - 2(n-2)\mu \right. \\
&\quad \times \left( \frac{uv}{r} + \frac{(\delta \bar{\delta} - 2)\mu r^{n-1} u_x}{(n-1)\delta \bar{\delta} - 2(n-2)\mu} \right)^2 + \frac{2\mu(n\delta \bar{\delta} - 2(n-1)\mu) r^{2n-2} u_x^2 \Theta^2}{v((n-1)\delta \bar{\delta} - 2(n-2)\mu)} \\
&\quad + \frac{\delta(1-\delta) \Theta^2}{v} (r^{n-1} u_x - \frac{v P_\theta}{2\delta(1-\delta)\Theta} + v(S_E) r \Theta^2) \\
&\quad + \frac{vP_\theta^2}{4(1-\delta)\delta}, \tag{123} \end{align*}

which, together with Lemmas 3.1-3.8 and (118), implies that there exists a positive constant \( C_1 \) such that
\[ \Theta_t \leq \frac{1}{\varepsilon_\partial} \left( \frac{r^{2n-2} \kappa \Theta_x}{v} \right)_x + C_1. \]

Defining \( \bar{\Theta}(x,t) = C_1 t + \max_{x \in [0,L]} \frac{1}{\theta_0(x)} - \Theta(x,t) \) and a parabolic operator \( \mathcal{L} := -\frac{\partial}{\partial t} + \frac{1}{\varepsilon_\partial} \frac{\partial}{\partial x} \left( \frac{r^{2n-2} \kappa}{v} \frac{\partial}{\partial x} \right) \), we have a system
\[
\begin{cases}
\mathcal{L} \bar{\Theta} \leq 0, & \text{on } Q_{t_0}, \\
\bar{\Theta} \big|_{t=0} \geq 0, & \text{on } [0,L], \\
\bar{\Theta} \big|_{x=0,L} = 0, & \text{on } [0,t_0].
\end{cases}
\]

Thus the standard comparison argument implies
\[ \min_{(x,t) \in \overline{Q}_{t_0}} \bar{\Theta}(x,t) \geq 0 \]
which gives for any \( (x,t) \in \overline{Q}_{t_0} \),
\[ \theta(x,t) \geq \left( C_1 t + \max_{x \in [0,L]} \frac{1}{\theta_0(x)} \right)^{-1}. \]
Thus, for all $0 \leq t \leq t_0$,
\[ \theta(x,t) \geq \left( C_t t_0 + \max_{x \in [0,L]} \frac{1}{\theta_0(x)} \right)^{-1} \geq C_1^{-1} \]
which, along with (122), gives (114).

**Lemma 3.12.** Under the assumptions in Theorem 2.1, we have as $t \to +\infty$,
\[ \|Z(t)\|_{H^2} \to 0. \]

**Proof.** Similarly to the proof of Lemma 3.4 in [21], we can also obtain (124). Here we derive for any $\epsilon > 0$,
\[ \int_{t_0}^t (\|u_x(t)\|^2 + \|u_{xx}(t)\|^2) \, dt \leq C_2, \]
\[ \|\theta_t(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|\theta_x(t)\|^2_{L^\infty} + \int_0^t (\|\theta_{xt}\|^2 + \|\theta_{tx}\|^2) \, d\tau \leq C_2. \]

**Proof.** Estimate (125) has been obtained in Lemma 1.3.1 of [23].
Differentiating (24) with respect to $t$ and multiplying the result by $\theta_t$ over $[0,L]$, we have that for any $\epsilon > 0$,
\[ \frac{d}{dt} \sqrt{\epsilon \theta_t} \leq C_1 \left( \|\theta_t\|^2 + \|\theta_x\|^2 + \|\theta_{xx}\|^2 + \|\theta_{t\theta}\|^2 + \|\theta_{tt\theta}\|^2 + \|\theta_{t\theta x}\|^2 + \|\theta_{t\theta\theta x}\|^2 \right) \]
\[ \leq C_2 + \epsilon \int_0^t \|\theta_{t\theta}\|^2 \, d\tau + \int_0^t (\|\theta_t\|^2 + \|\theta_x\|^2 + \|\theta_{xx}\|^2 + \|\theta_{t\theta x}\|^2 + \|\theta_{t\theta\theta x}\|^2) \, d\tau \]
\[ \leq C_2 + \epsilon \int_0^t \|\theta_{t\theta}\|^2 + \sup_{0 \leq \tau \leq t} \|\theta_t(\tau)\|^2 + \|\theta_x(\tau)\|^2 + \|\theta_{xx}(\tau)\|^2 + \|\theta_{t\theta x}(\tau)\|^2 + \|\theta_{t\theta\theta x}(\tau)\|^2 + \|\theta_{t\theta\theta\theta x}(\tau)\|^2 \, d\tau. \]

Integrating (127) with respect to $t$ and using Lemmas 3.1-3.8 and Young’s inequality, we have for any $\epsilon > 0$,
\[ \int_0^t \|\theta_t(\tau)\|^2 + \int_0^t \|\theta_{t\theta}(\tau)\|^2 \, d\tau \leq C_2 + \epsilon \int_0^t \|\theta_{t\theta}\|^2 \, d\tau + \sup_{0 \leq \tau \leq t} \|\theta_t(\tau)\|^2 + \|\theta_x(\tau)\|^2 + \|\theta_{xx}(\tau)\|^2 + \|\theta_{t\theta x}(\tau)\|^2 + \|\theta_{t\theta\theta x}(\tau)\|^2 + \|\theta_{t\theta\theta\theta x}(\tau)\|^2 \, d\tau. \]

It follows from Lemmas 3.7-3.9 that
\[ \int_0^t \|\theta(\tau)\|^2 \, d\tau \leq C_2 + \epsilon \int_0^t \|\theta_{t\theta}\|^2 \, d\tau + \sup_{0 \leq \tau \leq t} \|\theta_t(\tau)\|^2 + \|\theta_x(\tau)\|^2 + \|\theta_{xx}(\tau)\|^2 + \|\theta_{t\theta x}(\tau)\|^2 + \|\theta_{t\theta\theta x}(\tau)\|^2 + \|\theta_{t\theta\theta\theta x}(\tau)\|^2 \, d\tau. \]
Under the assumptions of Theorem 2.2, the following estimate holds for any $t > 0$.

$$
\leq C_1 \int_0^t \int_0^L \left( \int_0^\infty \int_{S^{n-1}} \left( \sigma_\alpha (B - I) + \sigma_\alpha (\tilde{I} - I) \right) d\Omega d\nu \right)^2 dx d\tau \\
+ C_1 \int_0^t \int_0^L \left( \int_0^\infty \int_{S^{n-1}} \left( (\sigma_\alpha)_x (r^{n-1}u)_x + (\sigma_\alpha)_{\theta \theta_t} (B - I) + \sigma_\alpha (B_\theta \theta_t - I_t) \\
+ (\sigma_\alpha)_x (r^{n-1}u)_x + (\sigma_\alpha)_{\theta \theta_t} (\tilde{I} - I) + \sigma_\alpha (\tilde{I} - I)_t \right) d\Omega d\nu \right)^2 dx d\tau \\
\leq C_1 \int_0^t \left( \| (r^{n-1}u)_x \|^2 + \| \theta_t \|^2 \right) d\tau + C_1 \int_0^t \int_0^L \int_0^\infty \int_{S^{n-1}} I_t^2 d\Omega d\nu dx d\tau \\
\leq C_1.
$$

(129)

Inserting (129) into (128) and then taking $\varepsilon > 0$ small enough, we obtain

$$
\| \theta_t (t) \|^2 + \int_0^t \| \theta_{xx} (\tau) \|^2 d\tau \leq C_2.
$$

(130)

Using the Gagliardo-Nirenberg interpolation inequality and Young’s inequality, we derive from (24) that

$$
\| \theta_{xx} \| \leq C_1 (\| \theta_t \| + \| \theta_x \| + \| (r^{n-1}u)_x \| + \| v_x \| + \| v(S_E)_x \|) \leq C_2.
$$

(131)

Combining (130) and (131), we get (126).

\[ \square \]

**Lemma 4.2.** Under the assumptions of Theorem 2.2, the following estimates hold that for any $t > 0$,

$$
\| v_{xx} (t) \|^2 + \int_0^t \| v_{xx}(\tau) \|^2 d\tau \leq C_2,
$$

(132)

$$
\int_0^t (\| u_{xxx} \|^2 + \| \theta_{xxx} \|^2)(\tau) d\tau \leq C_2.
$$

(133)

**Proof:** Similarly to the proof of Lemma 1.3.2 in [23], we easily obtain (132).

It follows from (24) that

$$
\| u_{xxx} \| \leq C_2 (\| u_{xx} \| + \| u_{xx} \| + \| v_{xx} \| + \| \theta_{xx} \|).
$$

(134)

Similarly, we can infer from (24) that

$$
\| \theta_{xxx} \| \leq C_2 (\| \theta_{xx} \| + \| \theta_{xx} \| + \| u_{xx} \| + \| v_{xx} \| + \| (v(S_E)_x) \|).
$$

(135)

By the definition of $(S_E)_x$ and Lemma 3.9, it follows from that

$$
\int_0^t \| (v(S_E)_x)^2 \| d\tau
\leq C_1 \int_0^t (\| v_x \|^2 + \| \theta_x \|^2) d\tau + C_1 \int_0^\infty \int_{S^{n-1}} |I_x| d\Omega d\nu \leq C_1
$$

(136)

which, together with (134)-(135), implies (133).

\[ \square \]

**Lemma 4.3.** Under the assumptions of Theorem 2.2, the following estimate holds for any $t > 0$,

$$
\| \mathcal{I}_{xxx} (t) \| \leq C_2.
$$

(137)
Proof. It follows from (24) and the definitions of $I$ and $\mathcal{I}$ that
\begin{equation}
\|I_{xx}\|^2 = \int_0^L \left( \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega} (r^{1-n} v_S)_{xx} d\Omega d\nu \right)^2 dx
\leq C_1 \int_0^L \left( \int_0^\infty \int_{S_{n-1}} \frac{(r^{1-n})_{xx}}{\omega} S d\Omega d\nu \right)^2 dx
+ C_1 \int_0^L \left( \int_0^\infty \int_{S_{n-1}} \frac{(r^{1-n})_x}{\omega} S_x d\Omega d\nu \right)^2 dx
+ C_1 \int_0^L \left( \int_0^\infty \int_{S_{n-1}} \frac{r^{1-n} V}{\omega} S_{xx} d\Omega d\nu \right)^2 dx
=: J_1 + J_2 + J_3.
\end{equation}
Employing the Gagliardo-Nirenberg interpolation inequality and using Lemmas 3.9 and 4.2, we conclude
\begin{equation}
J_1 \leq C_1 \int_0^L (r^{1-n})_{xx}^2 \left( \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega} (\sigma_a(B - I) + \sigma_s(\mathcal{I} - I)) d\Omega d\nu \right)^2 dx
\leq C_1 \int_0^L (r^{1-n})_{xx}^2 dx \leq C_2,
\end{equation}
and
\begin{equation}
J_2 \leq C_1 \int_0^L \left( \int_0^\infty \int_{S_{n-1}} \frac{(r^{1-n})_x}{\omega} ((\sigma_a)v_x + (\sigma_a)\theta_x)(B - I) + \sigma_a(B \theta_x - I_x)
+ ((\sigma_s)v_x + (\sigma_s)\theta_x)(\mathcal{I} - I) + \sigma_s(\mathcal{I} - I)_x d\Omega d\nu \right)^2 dx
\leq C_1 \int_0^L (v_x^2 + v_x^4 + v_x^2 \theta_x^2) dx
\leq C_1 + C_1 \|v_x\|^2_{L^\infty} (\|v_x\|^2 + \|\theta_x\|^2)
\leq C_1 + C_1 (\|v_x\| \|v_{xx}\| + \|v_x\|^2) \leq C_2.
\end{equation}
Similarly, we also infer that
\begin{equation}
J_3 \leq C_1 \int_0^L \left( \int_0^\infty \int_{S_{n-1}} \frac{(r^{1-n})_x}{\omega} ((\sigma_a)v_x + (\sigma_a)\theta_x)(B - I) + \sigma_a(B \theta_x - I_x)
+ ((\sigma_s)v_x + (\sigma_s)\theta_x)(\mathcal{I} - I) + \sigma_s(\mathcal{I} - I)_x d\Omega d\nu \right)^2 dx
\leq C_1 \int_0^L (v_x^2 + v_x^4 + v_x^2 \theta_x^2 + \theta_x^4 + \theta_x^2 + \theta_{xx}^2) dx
\leq C_2 + C_1 (\|v_x\| \|v_{xx}\| + \|\theta_x\| \|\theta_{xx}\| + \|v_x\|^2 + \|\theta_x\|^2)(\|v_x\|^2 + \|\theta_x\|^2)
\leq C_2,
\end{equation}
which, along with (138)-(140), leads to (137).
\end{proof}

The next lemma concerns the asymptotic behavior of the global solution in $H^2$.

\textbf{Lemma 4.4.} Under the assumptions in Theorem 2.2, we have, as $t \to +\infty$,
\begin{align}
\|v(t) - v\|_{H^2} &\to 0, \\
\|u(t)\|_{H^2} &\to 0,
\end{align}
\[ \| \theta(t) - \bar{\theta} \|_{H^2} \to 0, \quad (144) \]
\[ \| I(t) \|_{H^3} \to 0. \quad (145) \]

**Proof.** Differentiating (24) with respect to \( x \) twice and multiplying the result by \( v_{xx} \) in \( L^2(0, L) \), we have
\[
\frac{d}{dt} \| v_{xx} \|^2 \leq C_1 \left( \| v_{xx} \|^2 + \| u_{xxx} \|^2 + \| u_{xx} \|^2 + \| u_x \|^2 \right) (146)
\]
which, together with Theorem 2.1 and Gronwall’s inequality, leads to (142).

It follows from (24) that
\[
\frac{d}{dt} \| u_{tt} \|^2 + \| u_{xt} \|^2 \leq C_1 \left( \| v_x \|^2 + \| u_x \|^2 + \| u_t \|^2 + \| \theta_t \|^2 \right). \quad (147)
\]
Applying Theorem 2.1 and Lemma 2.4 to (147), we get, as \( t \to +\infty \),
\[
\| u_{tt}(t) \| \to 0 \quad (148)
\]
which gives (143).

Similarly, we can derive from (24) that
\[
\frac{d}{dt} \| \theta_t \|^2 + \| \theta_{xt} \|^2 \leq C_2 \left( \| \theta_t \|^2 + \| u_x \|^2 + \| u_t \|^2 + \| \theta_x \|^2 + \| u_{xt} \|^2 + \| (S_E)_{R} \|^2 \right) \quad (149)
\]
which, along with Lemma 2.4, Lemma 4.1 and (129), gives, as \( t \to +\infty \),
\[
\| \theta_t(t) \| \to 0. \quad (149)
\]

We can deduce from Lemma 38 that
\[
\frac{d}{dt} \| (S_E)_{R} \|^2 = 2 \int_0^L \left( \int_0^\infty \int_{S^{n-1}} S d\Omega d\nu \right) \left( \int_0^\infty \int_{S^{n-1}} S d\Omega d\nu \right) dx \leq C_2 \left( 1 + \| u_x \|^2 + \| \theta_t \|^2 \right) \quad (150)
\]
which, together with (57), (64) and Lemma 2.4, implies, as \( t \to +\infty \),
\[
\| (S_E)_{R} \| \to 0. \quad (150)
\]

By (24), we see that
\[
\| \theta_{xx} \| \leq C_2 \left( \| \theta_t \| + \| \theta_x \| + \| u_x \| + \| v_x \| + \| (S_E)_{R} \| \right), \quad (151)
\]
which, along with (111)-(112), (121) and (149)-(150), gives, as \( t \to +\infty \),
\[
\| \theta_{xx}(t) \| \to 0. \quad (152)
\]

Thus, as \( t \to +\infty \),
\[
\| \theta(t) - \bar{\theta} \|_{H^2} \to 0. \quad (145)
\]

Similarly to the proof of Lemma 5.3 in [21], we can obtain the desired estimate (145).

Till now we have completed the proof of Theorem 2.2.
5. Global existence and asymptotic behavior in $\mathcal{H}^4$. In this section, we shall prove Theorem 2.3, that is, the global existence and asymptotic behavior of solutions in $\mathcal{H}^4$ to the problem (24) with the initial boundary conditions (18)-(20) under some relative assumptions.

**Lemma 5.1.** Under the assumptions of Theorem 2.3, there holds that for any $t > 0$ and $\varepsilon > 0$ small enough,

\[
\|u_{xx}(x, 0)\| + \|\theta_{xx}(x, 0)\| \leq C_4,
\]

\[
\|u_{tx}(x, 0)\| + \|\theta_{tx}(x, 0)\| + \|u_{xxx}(x, 0)\| + \|\theta_{txx}(x, 0)\| \leq C_4,
\]

\[
\|u_t(t)\|^2 + \int_0^t \|u_{txx}(\tau)\|^2 d\tau \leq C_4 + C_4 \int_0^t (\|\theta_{txx}\|^2 + \|u_{txx}\|^2)(\tau)d\tau,
\]

\[
\|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{txx}(\tau)\|^2 d\tau \leq C_4 \varepsilon^{-3} + C_2 \varepsilon^{-1} \int_0^t \|\theta_{txx}(\tau)\|^2 d\tau
\]

\[+ C_1 \varepsilon \int_0^t (\|u_{txx}\|^2 + \|u_{txx}\|^2)(\tau)d\tau.\]

**Proof.** Differentiating (24)$_2$ and (24)$_3$ with respect to $x$, respectively, using Theorem 2.1 and Lemmas 4.1 and 4.2, we can get

\[
\|u_{xx}(t)\| \leq C_2 (\|u_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{H^1}),
\]

\[
\|\theta_{xx}(t)\| \leq C_2 (\|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|u_x(t)\|_{H^1} + \|\theta_x(t)\|_{\mathcal{I}_x(t)}).\]

Combining (157)-(158), we easily obtain (153).

Similarly, differentiating (24)$_2$ and (24)$_3$ with respect to $x$ twice, respectively, using Theorem 2.1 and Lemmas 4.1 and 4.2, we can infer

\[
\|u_{xxx}(t)\| \leq C_2 (\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|u_{x}(t)\|_{L^\infty} \|v_{xx}(t)\|)
\]

\[+ \|v_x(t)\|_{L^\infty} \|u_{xxx}(t)\| + \|u_{xx}(t)\|_{L^\infty} \|v_{xx}(t)\|)
\]

\[\leq C_2 (\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2}),\]

\[
\|\theta_{xxx}(t)\| \leq C_2 (\|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|u_x(t)\|_{H^2} + \|\theta_x(t)\|_{\mathcal{I}_x(t)} + \|\theta_x(t)\|_{H^1}).
\]

or

\[
\|u_{xxxx}(t)\| \leq C_2 (\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|u_{xx}(t)\|),
\]

\[
\|\theta_{xxxx}(t)\| \leq C_2 (\|\theta_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|u_x(t)\|_{H^2} + \|\theta_{xx}(t)\|
\]

\[+ \|\theta_x(t)\|_{\mathcal{I}_x(t)}).\]

Differentiating (24)$_2$ with respect to $t$, we have

\[
\|u_{tt}(t)\| \leq C_2 (\|u_x(t)\|_{H^1} + \|v_x(t)\| + \|\theta_t(t)\| + \|\theta_{xt}(t)\| + \|u_{tx}(t)\|
\]

\[+ \|u_{tx}(t)\|)
\]

\[\leq C_2 (\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|\theta_x(t)\|_{\mathcal{I}_x(t)}).\]

Analogously, we can also deduce that

\[
\|\theta_{tt}(t)\| \leq C_2 (\|u_x(t)\|_{H^1} + \|v_x(t)\| + \|\theta_t(t)\|_{H^2} + \|\theta_x(t)\|_{H^2} + \|u_{tx}(t)\|
\]

\[+ \|\theta_x(t)\|_{\mathcal{I}_x(t)}(t))
\]

\[\leq C_2 (\|u_x(t)\|_{H^2} + \|v_x(t)\|_{H^2} + \|\theta_x(t)\|_{H^3} + \|\theta_x(t)\|_{\mathcal{I}_x(t)}_{H^1}
\]

\[+ \|\theta_x(t)\|_{\mathcal{I}_x(t)}).\]

Thus the estimate (154) follows from (159)-(160), (164) and (166).
Differentiating (24) with respect to \( t \) twice, multiplying the resultant by \( u_{tt} \) in \( L^2(0, L) \), performing an integration by parts, and using Theorem 2.2 and the embedding theorem and the Young inequality, we can deduce
\[
\frac{1}{2} \frac{d}{dt} \|u_{tt}\|^2 = -\int_0^L (r^{n-1}u_{tt})_x \left( \delta \frac{(r^{n-1})_x}{v} - P \right)_t dx - \int_0^L (r^{n-1})_t u_{tt} \left( \delta \frac{(r^{n-1})_x}{v} - P \right)_t dx - \int_0^L (r^{n-1})_{tt} u_{tt} + (r^{n-1})_t u_{tt} \left( \delta \frac{(r^{n-1})_x}{v} - P \right)_t dx \leq - \int_0^L \frac{2}{v} u_{tt}^2 dx + C_2 (\|u_{tt}\| + \|u_{tt} u_x\| + \|u_x^2\| + \|\theta_t u_x\| + \|u_{xt}\| + \|\theta_t\| + \|u_{tt}\|)^2 u_{tt} dx \leq - C_1^2 \|u_{tt}\|^2 + C_2 (\|u_x\|_H^2 + \|\theta_t\|^2 + \|u_{xt}\|^2 + \|\theta_t\|^2 + \|u_{tt}\|^2). \tag{167}
\]
Thus, by Theorem 2.2,
\[
\|u_{tt}(t)\|^2 + \int_0^t \|u_{tt}\|^2 d\tau \leq C_4 + C_4 \int_0^t (\|u_{tt}\|^2 + \|\theta_{tt}\|^2)(\tau) d\tau
\]
which, along with (163) and (165) and Lemma 3.9, gives estimate (155).

In the same manner, differentiating (24) with respect to \( t \) twice, multiplying the resultant by \( \theta_{tt} \) and performing an integration by parts over \( L^2(0, L) \), and using the embedding theorem and the Young inequality, we have
\[
\frac{1}{2} \frac{d}{dt} \int_0^L e_\theta \theta_{tt}^2 dx = - \int_0^L \frac{2 n - 2 \theta e_\theta^2}{v} \theta_{tt} dx - \int_0^L \left( e_\theta + e_{tt} (r^{n-1})_x \right) \theta_{tt} dx - \frac{3}{2} \int_0^L e_\theta \theta_{tt}^2 dx - \int_0^L \left( e_v + P - \frac{(r^{n-1})_x}{v} \right) (r^{n-1})_{tt} \theta_{tt} dx + \int_0^L \left( \frac{(r^{n-1})_x}{v} - P \right)_t (r^{n-1})_{tt} \theta_{tt} dx - 2 \int_0^L \left( e_{tt} + \left( P - \frac{(r^{n-1})_x}{v} \right)_t \right) (r^{n-1})_{tt} \theta_{tt} dx - 2 \mu (n - 1) \int_0^L (r^{n-2})_{tt} \theta_{tt} dx - \int_0^L (v(S_E)_E)_{tt} \theta_{tt} dx
\]
\[
= \sum_{i=1}^8 A_i. \tag{168}
\]
By virtue of Theorems 2.1-2.2 and Lemmas 4.1-4.3, and using the embedding theorem, we deduce that for any \( \varepsilon \in (0, 1) \),
\[
A_1 \leq - C_1 \|\theta_{tt}\|^2 + C_2 (\|\theta_x\|_L^\infty \|u_x\| + \|u_{xi}\|_L^\infty \|\theta_{xi}\| + \|u_{x\theta}^2\|_L^\infty \|\theta_x\|)
+ \|\theta_x\|_L^\infty \|\theta_t\| + \|\theta_x\|_L^\infty \|\theta_{xt}\|) \|\theta_{tt}\|
\leq - (2C_1)^{-1} \|\theta_{tt}\|^2 + C_2 (\|\theta_{xt}\|^2 + \|u_{xt}\|^2 + \|u_{x\theta}^2\|_L^\infty + \|\theta_{tt}\|^2), \tag{169}
\]
\[
A_2 \leq C_1 \int_0^L \left( (\|u_x\| + |\theta_t|)^2 + |u_x| + |\theta_{tt}|) (|u_x| + |\theta_{tt}|) \right) dx \leq C_1 \|\theta_{tt}\|_L^\infty (\|u_x\| + |\theta_t|) \left( (\|u_x\|_L^\infty + |\theta_t|_L^\infty) (\|u_x\| + |\theta_t|) + \|u_{xt}\| + |\theta_{tt}|) \right)
\]
\[ \leq C_2(\|\theta_{tt}\| + \|\theta_{tx}\|)(\|u_x\|_{L^1} + \|\theta_t\| + \|\theta_{xt}\| + \|u_{xt}\| + \|\theta_{tt}\|) \]
\[ \leq \varepsilon \|\theta_{tt}\|^2 + C_2 \varepsilon^{-1}(\|u_x\|_{L^2}^2 + \|\theta_t\|^2 + \|\theta_{xt}\|^2 + \|u_{xt}\|^2 + \|\theta_{tt}\|^2), \quad (170) \]

\[ A_3 \leq C_1 \int_0^L (|u_x| + |\theta_t|)^2 \, dt \leq C_1 \|\theta_{tt}\|_{L^\infty}(\|u_x\| + \|\theta_t\|) \|\theta_{tt}\| \]
\[ \leq C_1(\|\theta_{tt}\| + \|\theta_{tx}\|)(\|u_x\| + \|\theta_t\|)\|\theta_{tt}\| \leq \varepsilon \|\theta_{tt}\|^2 + C_2 \varepsilon^{-1}\|\theta_{tt}\|^2, \quad (171) \]

\[ A_4 \leq \varepsilon \|u_{ttx}\|^2 + C_2 \varepsilon^{-1}\|\theta_{tt}\|^2, \quad (172) \]

\[ A_5 \leq C_2 \|u_x\|_{L^\infty}\|\theta_{tt}\|((\|u_x\|_{L^\infty} + \|\theta_t\|_{L^\infty})(\|u_x\| + \|\theta_t\|) + \|u_{xt}\| + \|\theta_{tx}\| \]
\[ + \|\theta_{tt}\| + \|\theta_{tx}\| + \|u_{xt}\| + \|\theta_{tt}\| + \|u_{tt}\| + \|u_x\|) \]
\[ \leq C_2\|\theta_{tt}\|((\|u_x\|_{H^1} + \|\theta_t\| + \|\theta_{tx}\| + \|u_{xt}\| + \|\theta_{tt}\| + \|u_{tt}\|) \]
\[ \leq \varepsilon \|u_{ttx}\|^2 + C_2 \varepsilon^{-1}(\|\theta_{tt}\|^2 + \|u_x\|_{H^1}^2 + \|\theta_t\|^2 + \|\theta_{tx}\|^2 + \|u_{xt}\|^2), \quad (173) \]

\[ A_6 \leq C_1 \int_0^L (|u_x| + |\theta_t| + |u_{xt}| + |u_x|^2 + |\theta_t||u_{xt}| + |u_{tt}|)|\theta_{tt}| \, dx \]
\[ \leq C_2 \|u_{ttx}\| \|\theta_{tt}\|((\|u_x\| + \|\theta_t\| + \|u_{xt}\|)\|\theta_{tt}\|, \quad (174) \]

\[ A_7 \leq C_2(\|\theta_{tt}\||\|u_{tt}\| + \|u_{tt}\| + \|u_x\| + \|u_x\|_{L^\infty} \|\theta_t\| + \|u_{xt}\|_{L^\infty} \|\theta_{tt}\| + \|u_{tt}\| + \|u_{tt}\| + \|u_{tt}\|) \]
\[ \leq \varepsilon \|u_{ttx}\|^2 + C_2 \varepsilon^{-1}(\|u_x\|_{H^1}^2 + \|u_{xt}\|^2 + \|u_{tt}\|^2 + \|\theta_{tt}\|^2). \quad (175) \]

It follows from (174) by the Hölder inequality that
\[ \int_0^t A_6 \, d\tau \leq C_2 \sup_{0 \leq \tau \leq t} \|\theta_{tt}(\tau)\| \left( \int_0^t \|u_{txx}(\tau)\|^2 \, d\tau \right)^{\frac{1}{2}} \left( \int_0^t \|u_{tx}(\tau)\|^2 \, d\tau \right)^{\frac{1}{2}} \]
\[ \times \left( \int_0^t (\|u_x\|^2 + \|\theta_t\|^2 + \|u_{tx}\|^2)(\tau) \, d\tau \right)^{\frac{1}{2}} \]
\[ \leq \varepsilon \left( \sup_{0 \leq \tau \leq t} \|\theta_{tt}(\tau)\|^2 + \int_0^t \|u_{txx}(\tau)\|^2 \, d\tau \right) + C_2 \varepsilon^{-3}. \quad (176) \]

Now we can estimate \( A_8 \) as
\[ A_8 \leq C_1 \int_0^L (r^{n-1}u)^2 \, dx + C_1(r^{n-1}u_x)^2 L^\infty \int_0^L ((S(E))^2 R_t^2 \, dx \]
\[ + C_1 \int_0^L \|\theta_{xx}\|^2 \, dx + C_1 \int_0^L ((S(E))^2 R_t^2 \, dx \]
\[ \leq C_1(\|u_x\|^2 + \|\theta_{tt}\|^2 + \|\|S(E)_R\|_t^2\|^2) + C_1 \int_0^L ((S(E))^2 R_t^2 \, dx. \quad (177) \]

By the induction of (129), we can easily obtain
\[ ||(S(E)_R)_{tt}||^2 \leq C_1(\|u_x\|^2 + \|\theta_t\|^2 + \|I_t\|^2). \quad (178) \]

Using Hölder’s inequality and the interpolation theorem, we can deduce from Theorems 2.1.2.2 that
\[ \int_0^L ((S(E))^2 R_{tt}^2 \, dx = \int_0^L \left( \int_0^L \int_{S^{n-1}} (\sigma_u (B - I) + \sigma_u (I - I) d\Omega d\nu) \right)^2 \, dx \]
Differentiating (93) with respect to $t$ and after the lengthy calculation, we can deduce
\[
\int_0^t \int_0^\infty I_{s+1}^2 d\hat{\Omega} d\tau \leq C_1 \int_0^t \int_0^\infty I_{s+1}^2 d\hat{\Omega} d\tau \leq C_2 + C_2 \int_0^t \int_0^\infty I_{s+1}^2 d\hat{\Omega} d\tau dy.
\] (180)

Applying Gronwall’s inequality to (180), we can obtain
\[
\int_0^t \int_0^\infty I_{s+1}^2 d\hat{\Omega} d\tau \leq \left( C_2 + C_2 \int_0^t \int_0^\infty \|\dot{\theta}_t(\tau)\|^2 d\tau \right) e^{C_2t} \leq C_2 + C_2 \int_0^t \int_0^\infty \|\dot{\theta}_t(\tau)\|^2 d\tau.
\] (181)

Inserting (178)-(179) and (181) into (177), we have
\[
A_8 \leq C_1 (\|u_x\|^2 + \|\theta_t\|^2 + \|u_x\|^2 + \|\theta_t\|^2 + \|\theta_{tt}\|^2).
\] (182)

Thus we infer from (168)-(176) and (182) that for any $\varepsilon \in (0, 1)$ small enough,
\[
\|\theta_t(t)\|^2 + \int_0^t \int_0^\infty I_{s+1}^2 d\hat{\Omega} d\tau \leq C_4 \varepsilon^{-3} + C_2 \varepsilon^{-1} \int_0^t \|\dot{\theta}_t(\tau)\|^2 d\tau
\] + $C_1 \varepsilon \left( \sup_{0 \leq \tau \leq t} \|\theta_t(\tau)\|^2 + \int_0^t (\|u_x\|^2 + \|\theta_x\|^2)(\tau) d\tau \right)$. (183)

Therefore taking supremum in $t$ on the left-hand side of (183), picking $\varepsilon \in (0, 1)$ small enough, and using (165), we can derive estimate (156). The proof is thus complete.

**Lemma 5.2.** Under the assumptions of Theorem 2.3, the following estimates hold that for any $t > 0$ and $\varepsilon > 0$ small enough,
\[
\|u_x(t)\|^2 + \int_0^t \|u_{xx}(\tau)\|^2 d\tau \leq C_4 + C_2 \varepsilon^2 \int_0^t (\|u_{xx}\|^2 + \|\theta_{xx}\|^2)(\tau) d\tau,
\] (184)
\[
\|\theta_x(t)\|^2 + \int_0^t \|\theta_{xx}(\tau)\|^2 d\tau \leq C_4 + C_2 \varepsilon^2 \int_0^t \|u_{xx}(\tau)\|^2 d\tau.
\] (185)

**Proof.** Similarly to Lemma 4.2.2 in Qin [20], we can obtain estimate (184).

We derive from (24) that
\[
\frac{1}{2} \frac{d}{dt} \int_0^L e\theta_x^2 dx =: \sum_{i=1}^5 D_i(t)
\] (186)
where
\[ D_1(t) = - \int_0^L \left( \frac{\rho^{2n-2} \theta_x}{v} \right)_x \theta_{txx} dx, \]
\[ D_2(t) = - \int_0^L \left( \eta_x + P - \delta \frac{\rho^{n-1} u_x}{v} \right) (n-1) u_x \right)_x \theta_{tx} dx, \]
\[ D_3(t) = 2\mu(n-1) \int_0^L (\rho^{n-2} u^2)_{tx} \theta_{txx} dx, \]
\[ D_4(t) = - \int_0^L \left( \eta_{tx} \theta_t + \frac{1}{2} \eta_{tt} \right) \theta_{txx} dx, \]
\[ D_5(t) = \int_0^L (v(S_E) R_t) \theta_{tx} dx. \]

By virtue of Theorems 2.1, 2.2, Lemma 5.1 and (184), and using the embedding theorem and the Young inequality, we deduce that for any \( \varepsilon \in (0, 1), \)
\[ D_1(t) \leq -(2C_1)^{-1} \| \theta_{txx} \|_2^2 + C_2(\| u_x \|_{H^1}^2 + \| \theta_x \|_{H^1}^2 + \| \theta_t \|_{H^1}^2), \]  
\[ D_2(t) \leq \varepsilon^2 \| u_{txx} \|_2^2 + C_2\varepsilon^{-2}(\| u_x \|_{H^2}^2 + \| \theta_x \|_{H^2}^2 + \| u_{xx} \|_{H^2}^2), \]  
\[ D_3(t) \leq \varepsilon^2 \| \theta_{txx} \|_2^2 + C_2\varepsilon^{-2}(\| u_x \|_{H^1}^2 + \| \theta_x \|_{H^1}^2 + \| u_{xx} \|_{H^1}^2), \]  
\[ D_4(t) \leq \varepsilon^2 \| \theta_{txx} \|_2^2 + C_2\varepsilon^{-2}(\| u_x \|_{H^1}^2 + \| \theta_x \|_{H^1}^2 + \| u_{xx} \|_{H^1}^2), \]  
\[ D_5(t) \leq \varepsilon^2 \| \theta_{tx} \|_2^2 + C_1\varepsilon^{-2}(\| v(S_E) R_t \|_1^2). \]

Inserting (187)-(190) into (186) and using (178), we obtain estimate (185). \( \square \)

**Lemma 5.3.** Under the assumptions of Theorem 2.3, we have, for any \( t > 0, \)
\[ \| u_{tt}(t) \|^2 + \| u_{xx}(t) \|^2 + \| \theta_{tt}(t) \|^2 + \| \theta_{xx}(t) \|^2 \]
\[ + \int_0^t \left( \| u_{txx}(t) \|^2 + \| u_{tx}(t) \|^2 + \| \theta_{txx}(t) \|^2 \right) d\tau \leq C_4, \]
\[ \| v_{xxx}(t) \|^2_{H^1} + \| u_{xxx}(t) \|^2_{H^1} + \| \theta_{xxx}(t) \|^2_{H^1} + \| u_{txx}(t) \|^2 + \| \theta_{txx}(t) \|^2 \]
\[ + \int_0^t \left( \| u_{tx}(t) \|^2 + \| u_{xx}(t) \|^2_{H^1} + \| \theta_{tx}(t) \|^2 + \| \theta_{xx}(t) \|^2_{H^1} \right) d\tau \leq C_4, \]
\[ \int_0^t \left( \| v_{xxx}(t) \|^2_{H^1} + \| u_{xxx}(t) \|^2_{H^1} + \| \theta_{xxx}(t) \|^2_{H^1} \right) d\tau \leq C_4. \]

**Proof.** Adding up (184)-(185) and choosing \( \varepsilon \) small enough, we conclude
\[ \| u_{tt}(t) \|^2 + \| u_{xx}(t) \|^2 + \| \theta_{tt}(t) \|^2 + \| \theta_{xx}(t) \|^2 \]
\[ \leq C_4 + C_2\varepsilon^2 \int_0^t \| u_{tx}(t) \|^2 d\tau. \]  
Now multiplying (155) and (156) by \( \varepsilon \) and \( \varepsilon^2 \), respectively, then adding up the resultant to (195) and picking \( \varepsilon \) small sufficiently, we obtain (192).

Applying Lemma 3.9, Theorems 2.1-2.2 and Lemmas 5.1-5.2 and using Hölder’s inequality, we can deduce from equation (24) that
\[ \int_0^t \int_0^L \int_{S^{n-1}} \rho^2 d\Omega d\nu dx d\tau = \int_0^t \int_0^L \int_{S^{n-1}} \frac{\rho^{2-2n} v^2}{\Omega^2} S^2 d\Omega d\nu dx d\tau \]
\[ \leq C_1, \]  
(196)
Obviously, we can infer from Theorem 2.2 and Lemmas 5.1-5.2 that

\[
\|E_1(t)\| \leq C_2(\|v\|_{H^1} + \|u\|_{H^2} + \|\theta_u\|_{H^2} + \|u_{txx}\|)
\]

and

\[
\int_0^t \|E_1(\tau)\|^2 d\tau \leq C_4. \tag{202}
\]

Multiplying (200) by \(\frac{v_{xxx}}{v}\) over \(L^2(0, L)\), we can obtain

\[
\frac{d}{dt} \left( \frac{v_{xxx}}{v} \right)^2 + C_1^{-1} \left( \frac{v_{xxx}}{v} \right)^2 \leq C_1 \|E_1(t)\|^2
\]

which, combined with (202), gives

\[
\|v_{xxx}(t)\|^2 + \int_0^t \|v_{xxx}(\tau)\|^2 d\tau \leq C_4. \tag{204}
\]
Thus, noting (196)-(198), using the embedding theorem, Theorem 2.2 and Lemmas 5.1-5.2, we can deduce for any \( t > 0 \),
\[
\| u_{xxx} \|^2 + \| \theta_{xxx} \|^2 + \| u_{xx} \|^2_{L^\infty} + \| \theta_{xx} \|^2_{L^\infty} + \int_0^t (\| u_{xxx} \|^2_{H^1} + \| \theta_{xxx} \|^2_{H^1}) + \| u_{xx} \|^2_{W^{1,\infty}} + \| \theta_{xx} \|^2_{W^{1,\infty}}(\tau)d\tau \leq C_4.
\] (205)

Differentiating (24) with respect to \( t \) and using (196)-(198), Theorem 2.2 and Lemmas 5.1-5.2, we can conclude
\[
\| u_{txx} \| \leq C_1 \| u_{tt} \| + C_2 (\| u_x \|_{H^1} + \| v_x \| + \| \theta_x \| + \| \theta_t \| + \| u_{xt} \|)
\leq C_4,
\] (206)
\[
\| \theta_{txx} \| \leq C_1 (\| \theta_{tt} \| + \| \theta_t \|) + C_2 (\| u_x \|_{H^1} + \| v_x \| + \| \theta_x \|_{H^2} + \| \theta_t \|_{H^1} + \| u_{xt} \|)
\leq C_4.
\] (207)

Combining with Theorems 2.1-2.2, (206)-(207) and Lemmas 5.1-5.2, we arrive at
\[
\| u_{xxxx}(t) \|^2 + \| \theta_{xxxx}(t) \|^2
+ \int_0^t (\| u_{txx} \|^2 + \| \theta_{txx} \|^2 + \| u_{xxx} \|^2 + \| \theta_{xxx} \|^2)(\tau)d\tau \leq C_4.
\] (208)

Therefore,
\[
\| u_{xxx}(t) \|^2_{L^\infty} + \| \theta_{xxx}(t) \|^2_{L^\infty} + \int_0^t (\| u_{xxx} \|^2_{L^\infty} + \| \theta_{xxx} \|^2_{L^\infty})(\tau)d\tau \leq C_4.
\] (209)

Now differentiating (200) with respect to \( x \), we find
\[
\nu \frac{\partial}{\partial t} \left( \frac{v_{xxxx}}{v} \right) - P_x v_{xxxx} = E_2(x, t)
\] (210)

where
\[
E_2(x, t) = E_{1x}(x, t) + P_{xx} v_{xxx} + \delta \left( \frac{v_{xxxx} v_x}{\nu^2} \right)_x.
\]

Using the embedding theorem, Lemmas 5.1-5.2 and (204)-(209), we derive
\[
\| E_{xx}(t) \| \leq C_4 (\| u_x \|_{H^3} + \| \theta_x \|_{H^3} + \| v_x \|_{H^2}),
\| E_{1x}(t) \| \leq C_1 (\| E_{xx}(t) \| + \| u_{xxx} \| + \| u_{xx} \| + \| (P_{xx} v_{xxx})_x \| + \| v_x u_{xx} \|
+ \| v_x u_t \| + \| v_{xx} u_t \| + \| (\frac{v_x v_{xx}}{\nu^2})_x \|)
\leq C_4 (\| u_{xxx} \| + \| \theta_{xxx} \|_{H^3} + \| v_x \|_{H^2})
\|
whence
\[
\| E_2(t) \| \leq C_1 \| u_{xxx} \| + C_4 (\| u_x \|_{H^3} + \| \theta_x \|_{H^3} + \| v_x \|_{H^2}).
\] (211)

We infer from (163)-(166) that
\[
\int_0^t (\| u_{tt} \|^2 + \| \theta_{tt} \|^2)(\tau)d\tau \leq C_4.
\] (212)

On the other hand, differentiating (24) with respect to \( x \) and \( t \), and using Theorem 2.2 and Lemma 5.1, we can derive
\[
\| u_{xxxx}(t) \| \leq C_1 \| u_{xxtt} \| + C_4 (\| u_x \|_{H^2} + \| \theta_x \|_{H^1} + \| v_x \|_{H^1} + \| \theta_t \|_{H^2}).
\] (213)
which, together with (192) and (212), gives
\[
\int_0^t \|u_{xxxt}(\tau)\|^2 d\tau \leq C_4. \tag{214}
\]
Thus it follows from (205), (208), (211), (214), Theorem 2.2 and Lemmas 5.1-5.2 that
\[
\int_0^t \|E_2(\tau)\|^2 d\tau \leq C_4. \tag{215}
\]
Multiplying (210) by \(|v|^{\alpha} \) in \(L^2(0, L)\), we can get
\[
\frac{d}{dt} \left\| \frac{v_{xxxx}}{v} \right\|^2 + C_1^{-1} \left\| \frac{v_{xxxx}}{v} \right\|^2 \leq C_1 \|E_2(t)\|^2 \tag{216}
\]
whence by (215),
\[
\|v_{xxxx}(t)\|^2 + \int_0^t \|v_{xxxx}(\tau)\|^2 d\tau \leq C_4. \tag{217}
\]
Differentiating (24) with respect to \(x\) and \(t\), and using Theorem 2.2 and Lemmas 5.1-5.2, we can derive
\[
\|\theta_{txx}\| \leq C_1 \|\theta_{tx}\| + C_2(\|u_x\|_{H^3} + \|v_x\|_{H^2} + \|\theta_x\|_{H^3} + \|\theta_{tx}\| + \|(v(S_E)_{xt})\|). \tag{218}
\]
We know that
\[
\int_0^L (v(S_E)_{xt})^2 dx \leq C_1 \int_0^L \left( (r^{n-1} u)^2_{xx} (S_E)_R^2 + (r^{n-1} u)^2_x (S_E)_x^2 \right.
\]
\[
\left. + v_x^2 ((S_E)_R^2 + ((S_E)_x)_x^2) \right) dx
\]
\[
=: \sum_{i=1}^4 B_i. \tag{219}
\]
Using Hölder’s inequality and (196)-(198), we have
\[
B_1 \leq C_1 \int_0^L (r^{n-1} u)^2_{xx} \left( \int_0^\infty \int_{S^{n-1}} S^2 d\Omega d\nu \right) dx \leq C_1 \|u_x\|_{H^1}^2, \tag{220}
\]
\[
B_2 \leq C_1 \int_0^L (r^{n-1} u)^2_x \left( \int_0^\infty \int_{S^{n-1}} S^2 d\Omega d\nu \right) dx
\]
\[
\leq C_1 \int_0^L ((r^{n-1} u)^2_x + (r^{n-1} u)^2_x v_x^2 + (r^{n-1} u)^2_x \theta_x^2)
\]
\[
\leq C_1 (\|v_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|u_x\|_2^2), \tag{221}
\]
\[
B_3 \leq C_1 \int_0^L v_x^2 \left( \int_0^\infty \int_{S^{n-1}} ((S_E)_R)_x^2 d\Omega d\nu \right) dx
\]
\[
\leq C_1 \int_0^L (v_x^2 + (r^{n-1} u)_x v_x^2 + \theta_x v_x^2)
\]
\[
\leq C_1 (\|u_x\|_{H^1}^2 + \|v_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2), \tag{222}
\]
\[
B_4 \leq C_1 \int_0^L \left( \int_0^\infty \int_{S^{n-1}} ((S_E)_R)_{xt}^2 d\Omega d\nu \right) dx
\]
\[
\leq C_1 (\|u_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2 + \|\theta_t\|_{H^1}^2 + \|v_x\|_{H^1}^2). \tag{223}
\]
\[ + C_1 \int_0^L \int_0^\infty \int_{S^{n-1}} I_x^2 d\Omega d\tau dx. \]  

(223)

Meanwhile, it follows from (24) and Lemma 3.9 that

\[ \int_0^t \int_0^L \int_0^\infty \int_{S^{n-1}} I_x^2 d\Omega d\tau dx \]

\[ = \int_0^t \int_0^L \int_0^\infty \int_{S^{n-1}} \left( \frac{r^{1-n} S}{\omega} \right)_t^2 d\Omega d\tau dx \]

\[ \leq C_1 \int_0^t \int_0^L \int_0^\infty \int_{S^{n-1}} I_x^2 d\Omega d\tau dx \]

\[ \leq C_1. \]  

(224)

Thus we deduce from (218)-(224) that

\[ \int_0^t \| \theta_{xxxx}(\tau) \|^2 d\tau \leq C_4, \text{ } \forall t > 0. \]  

(225)

Differentiating (24) around with respect to \( x \) three times, using Lemmas 5.1-5.2 and Theorem 2.2 and applying Poincaré’s inequality, we deduce

\[ \| u_{xxxx} \| \leq C_1 \| u_{xxx} \| + C_2 (\| u_x \|_{H^3} + \| v_x \|_{H^3} + \| \theta_x \|_{H^3}). \]  

(226)

Thus,

\[ \int_0^t \| u_{xxxx}(\tau) \|^2 d\tau \leq C_4, \text{ } \forall t > 0. \]  

(227)

Using the same technique, we can deduce from (24) around with

\[ \| \theta_{xxxx} \| \leq C_1 (\| \theta_{xxx} \| + \| \theta_{xxxx} \|) + C_2 (\| u_x \|_{H^3} + \| v_x \|_{H^3} + \| \theta_x \|_{H^3}) \]  

(228)

which, together with (133), (161)-(162), (192), (198), (204) and (214), leads to

\[ \int_0^t \| \theta_{xxxx}(\tau) \|^2 d\tau \leq C_4, \text{ } \forall t > 0. \]  

(229)

Hence, we complete the proofs of (193) and (194).

By Lemmas 5.1-155, we have proved the global existence of solutions in \( \mathcal{H}^4 \) to the problem (24) in \( \mathcal{H}^4 \) with arbitrary initial datum \((u_0, v_0, \theta_0) \in \mathcal{H}^4\) and the uniqueness of solution follows from that of solution in \( \mathcal{H}^3 \) or in \( \mathcal{H}^2 \). Next, we shall show the asymptotic behavior of solutions.

Lemma 5.4. Under the assumptions in Theorem 2.3, we have, as \( t \to +\infty \),

\[ \| v(t) - \overline{v} \|_{H^4} \to 0, \text{ } \| u(t) \|_{H^4} \to 0, \text{ } \| \theta(t) - \overline{\theta} \|_{H^4} \to 0. \]  

(230)

Proof. Differentiating (24) around with respect to \( x \) three times and then multiplying the resultant by \( v_{xxx} \) over \( L^2(0, L) \), we have

\[ \frac{d}{dt} \| v_{xxx} \|^2 \leq C_2 (1 + \| u_{xxx} \|^2 + \| u_{xxxx} \|^2 + \| v_{xxx} \|^2) \]  

(231)

which, along with (208) and Theorems 2.1-2.2 and Lemma 2.4, leads to, as \( t \to +\infty \),

\[ \| v_{xxx}(t) \| \to 0. \]  

(232)

Similarly, differentiating (24) around with respect to \( x \) four times, we can obtain

\[ \frac{d}{dt} \| v_{xxxx} \|^2 \leq C_2 (1 + \| u_{xxx} \|^2 + \| u_{xxxx} \|^2 + \| u_{xxxxx} \|^2 + \| v_{xxxx} \|^2) \]  

(233)
which, along with (227) and Theorems 2.1-2.2 and Lemma 2.4, yields, as \( t \to +\infty \),
\[ \|v_{xxx}(t)\| \to 0, \]  
(234)
which, combined with Poincaré’s inequality, gives, as \( t \to +\infty \),
\[ \|v(t) - \varphi\|_{H^4} \to 0. \]  
(235)

From Lemma 5.2, we can easily get
\[
\frac{d}{dt} \|\theta_{xt}(t)\|^2 + \|\theta_{xxt}(t)\|^2 \\
\leq \varepsilon (\|\theta_{xxt}(t)\|^2 + \|v_{xxt}(t)\|^2) + C_2(\|v_{xxx}(t)\|^2 + \|v_{xt}(t)\|^2 + \|\theta_{xxt}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|\mathcal{I}_x(t)\|^2),
\]  
(236)
which, combined with Lemma 2.4, leads to, as \( t \to +\infty \),
\[ \|\theta_{xt}(t)\| \to 0. \]  
(237)

Noting that, as \( t \to +\infty \),
\[ \|\mathcal{I}(t)\|_{H^3} \to 0 \]  
(238)
and using (135), we have, as \( t \to +\infty \),
\[ \|\theta_{xxx}(t)\| \to 0. \]  
(239)
From Lemma 5.1, we can easily obtain, for any \( \varepsilon > 0 \),
\[
\frac{d}{dt} \|\theta_{tt}(t)\|^2 + \|\theta_{xt}(t)\|^2 \\
\leq \varepsilon (\|\theta_{xxt}(t)\|^2 + \|v_{xxt}(t)\|^2) + C_2(\|v_{xxx}(t)\|^2 + \|v_{xt}(t)\|^2 + \|\theta_{t}(t)\|^2 + \|\theta_{xt}(t)\|^2 + \|\theta_{xx}(t)\|^2 + \|\mathcal{I}_x(t)\|^2),
\]  
(240)
which, by Lemma 2.4, leads to, as \( t \to +\infty \),
\[ \|\theta_{tt}(t)\| \to 0. \]  
(241)

On the other hand, we have
\[ \|\theta_{xxt}(t)\| \leq C_2(\|\theta_{t}(t)\| + \|v_x(t)\| + \|u_x(t)\| + \|\theta_{t}(t)\| + \|\theta_{xt}(t)\| + \|\theta_{xx}(t)\| + \|\mathcal{I}_x(t)\|), \]  
which, combined with Lemma 3.12 and (237)-(240), yields, as \( t \to +\infty \),
\[ \|\theta_{xxx}(t)\| \to 0. \]  
(242)
Thus it follows from (241) and (162) that, as \( t \to +\infty \),
\[ \|\theta_{xxxx}(t)\| \to 0, \]  
(243)
which, obviously, along with Lemma 2.4 and Poincaré’s inequality, leads to our desired estimate
\[ \|\theta(t) - \overline{\varphi}\|_{H^4} \to 0, \]  
as \( t \to +\infty \).
Analogously, applying Lemmas 5.1-5.2 and Lemma 2.4, we can deduce from (24) that \( \|u(t)\|_{H^4} \to 0. \) Here we omit it.

**Lemma 5.5.** Under the assumptions in Theorem 2.3, the following estimate holds for any \( t > 0 \),
\[ \|\mathcal{I}(t)\|_{H^5} \leq C_4. \]  
(244)
Moreover, we have, as \( t \to +\infty \),
\[ \|\mathcal{I}(t)\|_{H^5} \to 0. \]  
(245)
Proof. By equation (24), we have

\[
\|I_{xxx}(t)\|^2 = \int_0^L \left( \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega} (r^{1-n} v S)_{xxx} d\Omega d\nu \right)^2 dx \\
\leq C_1 \int_0^L \left( \left( \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega} (v^3 + v_x + v_x^2 + v_{xx} + v_{xxx}) d\Omega d\nu \right)^2 \\
+ \left( \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega} (v^2 + v_x) d\Omega d\nu \right)^2 \\
+ \left( \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega} (S_{xxx}) d\Omega d\nu \right)^2 \right) dx \\
=: \sum_{i=1}^4 F_i.
\]  

(246)

Applying Holder’s inequality and the interpolation inequality, we can estimate each term in (246) as follows,

\[
F_1 \leq C_1 \int_0^L (v^3 + v_x^2 + v_x^2 + v_{xx}^2 + v_{xxx}) \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega^2} S^2 d\Omega d\nu dx \\
\leq C_1 \int_0^L (v^2 + v_x^2 + v_x^2 + v_{xx}^2 + v_{xxx}) dx \leq C_1 \|v_{xxx}\|^2 \leq C_4,
\]  

(247)

\[
F_2 \leq C_1 \int_0^L (v^3 + v_x^2 + v_x^2) \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega^2} S^2 d\Omega d\nu dx \\
\leq C_1 \int_0^L (v^2 + v_x^2 + v_{xx}^2) \left( v_x^2 + \theta_x^2 + \left( \int_0^\infty \int_{S_{n-1}} (\sigma_a + \sigma_s) I_{xx} d\Omega d\nu \right)^2 \right) dx \\
\leq C_1 (\|v_x\|^2 + \|\theta_x\|^2) \leq C_4,
\]  

(248)

\[
F_3 \leq C_1 \int_0^L (v^3 + v_x^2) \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega^2} S^2 d\Omega d\nu dx \\
\leq C_1 \int_0^L (v^2 + v_x^2) \left( v_{xx}^2 + \theta_{xx}^2 + v_x^2 + \theta_x^2 + v_x^2 \theta_x^2 \\
+ \left( \int_0^\infty \int_{S_{n-1}} (\sigma_a + \sigma_s) I_{xx} d\Omega d\nu \right)^2 \right) dx \\
\leq C_1 (\|v_x\|^2 + \|\theta_x\|^2) \leq C_4,
\]  

(249)

\[
F_4 \leq C_1 \int_0^L \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega^2} S^2 dx d\Omega d\nu \\
\leq C_1 (\|v_x\|^2 + \|\theta_x\|^2) + C_1 \int_0^L \left( \int_0^\infty \int_{S_{n-1}} (\sigma_a + \sigma_s) I_{xxx} d\Omega d\nu \right)^2 dx \\
\leq C_1 (\|v_x\|^2 + \|\theta_x\|^2) \leq C_4.
\]  

(250)

Inserting (247)-(250) into (246), we have

\[
\|I_{xxx}(t)\|^2 \leq C_4, \quad \forall t > 0.
\]  

(251)

Analogously, we also deduce from equation (24) that

\[
\|I_{xxxxx}(t)\|^2
\]
Naturally, we can also get the following estimates

\[
G_1 \leq C_1 \int_0^L (v^{10} + v_x^2 + v_x^4 + v_{xx}^2 + v_{xxx}^2 + v_x^2 v_{xx}^2 + v_{xxx}^2) \times \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega^2} S^2 d\Omega d\nu dx
\]
\[
\leq C_1 \|v_{xxx}\|^2 \leq C_4,
\]
\[
G_2 \leq C_1 \int_0^L (v^2 + v_x^2 + v_x^4 + v_{xx}^2 + v_{xxx}^2) \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega^2} S^2 d\Omega d\nu dx
\]
\[
\leq C_1 \int_0^L (v^2 + v_x^2 + v_x^4 + v_{xx}^2 + v_{xxx}^2)(v_x^2 + \theta_x^2
\]
\[
+ \left( \int_0^\infty \int_{S_{n-1}} (\sigma_a + \sigma_s) I_x d\Omega d\nu \right)^2 \right) dx
\]
\[
\leq C_1 (\|v_x\|^2_{H^3} + \|\theta_x\|^2) \leq C_4,
\]
\[
G_3 \leq C_1 \int_0^L (v^2 + v_x^2 + v_{xx}^2) \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega^2} S^2 d\Omega d\nu dx
\]
\[
\leq C_1 \int_0^L (v^2 + v_x^2 + v_{xx}^2)(v_x^2 + \theta_x^2 + v_x^2 + \theta_x^2 + v_x^2 \theta_x^2
\]
\[
+ \left( \int_0^\infty \int_{S_{n-1}} (\sigma_a + \sigma_s) I_{xx} d\Omega d\nu \right)^2 \right) dx
\]
\[
\leq C_1 (\|v_x\|^2_{H^2} + \|\theta_x\|^2_{H^1}) \leq C_4,
\]
\[
G_4 \leq C_1 \int_0^L (v^2 + v_x^2) \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega^2} S^2 d\Omega d\nu dx
\]
\[
\leq C_1 (\|v_x\|^2_{H^1} + \|\theta_x\|^2_{H^1}) + C_1 \int_0^L \left( \int_0^\infty \int_{S_{n-1}} (\sigma_a + \sigma_s) I_{xxx} d\Omega d\nu \right)^2 dx
\]
\[
\leq C_1 (\|v_x\|^2_{H^1} + \|\theta_x\|^2_{H^1}) \leq C_4,
\]
\[
G_5 \leq C_1 \int_0^L \int_0^\infty \int_{S_{n-1}} \frac{1}{\omega^2} S^2 d\Omega d\nu dx
\]
\[ \leq C_1(\|v_{xx}\|_{H^2}^2 + \|\theta_{xx}\|_{H^2}^2) + C_1 \int_0^L \left( \int_0^\infty \int_{S^{n-1}} (\tau_a + \tau_s) I_{xxxx} d\Omega d\nu \right)^2 \, dx \]
\[ \leq C_1(\|v_{xx}\|_{H^2}^2 + \|\theta_{xx}\|_{H^2}^2) \leq C_4. \tag{257} \]
Inserting (253)-(257) into (252), we get
\[ \|I_{xxxx}(t)\|^2 \leq C_4, \forall t > 0, \tag{258} \]
which, along with (251), leads to (244).

We can derive from the inductions of (247)-(250) that
\[ F_1 \leq C_1 \|v_{xxx}\|^2, \tag{259} \]
\[ F_2 \leq C_1(\|v_x\|_{H^1}^2 + \|\theta_x\|_{H^1}^2), \tag{260} \]
\[ F_3 \leq C_1(\|v_x\|_{H^2}^2 + \|\theta_x\|_{H^2}^2 + \|I_{xx}\|^2), \tag{261} \]
\[ F_4 \leq C_1(\|v_x\|_{H^2}^2 + \|\theta_x\|_{H^2}^2 + \|I_x\|^2 + \|I_{xx}\|^2). \tag{262} \]
Applying (246), (259)-(262) and Theorems 2.1-2.2, we have, as \( t \to +\infty \),
\[ \|I_{xxxx}(t)\|^2 \to 0. \tag{263} \]
Similarly, combining (252)-(257), (263) and using Theorems 2.1-2.2, we also have, as \( t \to +\infty \),
\[ \|I_{xxxxx}(t)\|^2 \to 0. \tag{264} \]
Therefore, we complete the proof.

Till now we have completed the proof of Theorem 2.3.

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