Fracture mechanics analysis of cracked structures using weight function and neural network method

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Abstract. Stress intensity factors (SIFs) due to thermal-mechanical load has been established by using weight function method. Two reference stress states were used to determine the coefficients in the weight function. Results were evaluated by using data from literature and show a good agreement between them. So, the SIFs can be determined quickly using the weight function obtained when cracks subjected to arbitrary loads, and presented method can be used for probabilistic fracture mechanics analysis. A probabilistic methodology considering Monte-Carlo with neural network (MCNN) has been developed. The results indicate that an accurate probabilistic characteristic of the $K_I$ can be obtained by using the developed method. The probability of failure increases with the increasing of loads, and the relationship between is nonlinear.

1. Introduction

Many parameters in deterministic fracture mechanics evaluation are very conservative, actually many parameters in the evaluation process, such as fracture toughness and nil-ductility transition temperature, have statistical distribution characteristics. For example, a probabilistic model was developed by Rahman [1] for analysis of circumferential through wall cracks in pipes. Other researchers do some work on J-integral estimation [2-5]. Essentially, this presents a rational success for probabilistic analysis of some structures. When come to complex external loads, it becomes a great challenge to obtain the fracture parameters. The pipes often subject to thermal loads as well as mechanical loads, determining thermal-mechanical fracture parameters is difficult because of the nonlinearity of thermal stress. One needs to evaluate thermal-mechanical fracture problem by employing FEM or other numerical methods. To date, reports are found considering both thermal stress and mechanical stress, which results in a nonlinear stress problem.

A rapidly way has been presented to predict the stochastic properties of SIFs and failure probability of cracked structures subject to random thermal-mechanical loads. A general weight function was derived to evaluate the thermal stress intensity factors of a circumferential crack in cylinders. The accuracy of the analysis has been examined using the finite element method results.
2. Thermal stress

There is a circumferential crack on the internal surface of an axisymmetric cylinder, as shown in Figure 1. And the inner surface subjected to a sudden cooling, and there is an internal pressure inside.

Governing equation [6] of the temperature distribution along the thickness is as follows.

$$\frac{\partial^2 \theta}{\partial r^2} + \frac{1}{r} \frac{\partial \theta}{\partial r} = 0$$  \hspace{1cm} (1)

In which, $\theta$ is the body temperature in the cylinder. and $\theta = T - T_0$, $T_0$ is the ambient temperature. Considering the third kind boundary condition, boundary conditions on the outer and inner surfaces are as following.

$$-\lambda \frac{\partial \theta}{\partial n} = \lambda \frac{\partial \theta}{\partial r} = h(\theta - \theta_f) \text{ on } r = R_i$$  \hspace{1cm} (2)

$$\theta = 0 \text{ on } r = R_o$$

In which, $\lambda$ is coefficient of heat conductivity, $h$ is convection coefficient. $\theta_f$ is the temperature of the fluid in the pipe ($\theta_f = T_f - T_0$).

Solve equation (1) under the conditions of equation (2). The temperature distribution is as following:

$$\theta(r) = \frac{\theta_f \ln(R_o / r)}{1/B_i + \ln(R_o / R_i)}$$  \hspace{1cm} (3)

In which, $B_i = ah/\lambda$ is the Biot number.

The mechanical equilibrium equation in terms of displacement $u$ as

$$\frac{d}{dr} \left[ \frac{1}{r} \frac{d(ur)}{dr} \right] = \beta \frac{d\theta}{dr}$$  \hspace{1cm} (4)

The parameter $\beta$ is defined as $\beta = \alpha(1+\nu)/(1-\nu)$, in which $\alpha$ is the thermal expansion coefficient and $\nu$ is the Poisson’s ratio.

Integration of the equilibrium equation yields displacement as follows.

$$u = \frac{\beta}{r} \int_a^r r \partial \theta + C_i r + C_2$$  \hspace{1cm} (5)

The stress related to displacement and temperature can be obtained as follows.
\[
\sigma_{rr} = \frac{2\mu}{1-2\nu} \left[ (1-\nu) \frac{du}{dr} + \nu \frac{u}{r} - \alpha (1+\nu) \theta \right]
\]

\[
\sigma_{\phi\phi} = \frac{2\mu}{1-2\nu} \left[ (1-\nu) \frac{u}{r} + \nu \frac{du}{dr} - \alpha (1+\nu) \theta \right]
\]

\[
\sigma_{zz} = \nu (\sigma_{rr} + \sigma_{\phi\phi}) - 2\mu (1+\nu) \alpha \theta
\]

In which, \( \mu \) is shear modulus of the cylinder.

It is assumed that only the inner surface of the cylinder is subjected to pressure boundary condition. So, stress boundary conditions are described as follows.

\[
\sigma_{rr} = -P_i \quad \text{at} \quad r = R_i
\]

\[
\sigma_{rr} = 0 \quad \text{at} \quad r = R_o
\]

The boundary conditions above can be used to determine the unknown parameters \( C_1 \) and \( C_2 \). Then, the axial stress can be computed as follows.

\[
\sigma_{zz} = \frac{2\mu}{1-2\nu} \left[ \nu \left( \frac{du}{dr} + \frac{u}{r} \right) - \alpha (1+\nu) \theta \right]
\]

So, the axial stress, which is also the normal stress on the crack surface, can be rewritten as following.

\[
\sigma_{zz} = -4\mu \xi_1 \ln(r) + \phi_1
\]

In which,

\[
\xi_1 = -\frac{\beta \theta_f}{2(1/ R_i + \ln(R_o/ R_i))}
\]

\[
\phi_1 = 4\mu \xi_1 \left( \ln(R_o) + \nu \frac{R_i^2 \ln(R_o/ R_i)}{R_o^2 - R_i^2} - \frac{\nu}{2} \right) + \frac{2\nu P R_i^2}{R_o^2 - R_i^2}
\]

3. **Weight function method**

Geometry of a cracked cylinder is as shown in figure 2.

![Figure 2. Geometry of a cracked cylinder.](image)

The weight function method [7-9] can be calculated generally as follows.
\[ K_i = \int_0^a \sigma(r)m(r,a)dr \]  

(11)

In which, \( a \) is the crack depth, \( \sigma(r) \) is the stress distribution across the plane of the crack in the non-cracked body and \( m(r,a) \) is the weight function. The general expression of weight function can be written as follows [10].

\[ m(r,a) = \frac{2}{\pi} \left[ \frac{1}{R_i + a - r} + M_1 \frac{2}{\pi a} + M_2 \frac{1}{a} \sqrt{R_i + a - r} + M_3 \frac{2}{\pi a} (R_i + a - r) \right] \]  

(12)

In which \( M_i, i = 1,2,3 \) are constants. Two reference stress state are used to obtain the constant coefficients [11]. The third condition is that the second derivative of the weight function with respect to \( a \) is zero at inner surface \( r = R_i \), which results in \( M_1 = M_2 = 3 \) [12]. Two reference stress state, which are homogeneous and linear stress distributions, are suggested to derive the constants.

\[ \sigma_{1ref}(r) = \sigma_0 \]

\[ \sigma_{2ref}(r) = \sigma_0 \left( \frac{r - R_i}{t} \right) \]  

(13)

In which, \( t \) is the thickness. The SIFs due to the two stress distributions above in cylinders were found as.

\[ K_{1ref} = \sigma_0 \sqrt{\pi a} S_1 \]

\[ K_{2ref} = \sigma_0 \sqrt{\pi a} S_2 \]  

(14)

In which, \( S_1 \) and \( S_2 \) are boundary correction factors. Expressions for \( S_1 \) and \( S_2 \) are boundary correction factors, which can be expressed as following.

\[ S_j = \sum_{i=0}^{4} \left[ A_j e^{\alpha_i (r_i / R_i)} + C_j \left( \frac{a}{t} \right)^i \right], \quad j = 1, 2 \]  

(15)

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Figure 3. Finite element mesh of axisymmetric cylinder specimen with internal circumferential crack.

Finite-element analyses are carried out on a two-dimensional axisymmetric model. Figure 3 shows
the finite element model established by ANSYS 13.0 [13].

Parameters $A$, $B$ and $C$ in equation (15) are shown in table 1.

| Table 1. Coefficients: $A$, $B$ and $C$. |
|----------------------------------------|
| $i=0$  | $i=1$  | $i=2$  | $i=3$  | $i=4$  |
| $A_{ii}$  | 0.42536 | 6.16171 | 1378.9905 | 6405.4262 | 0.93451 |
| $B_{ii}$  | 0.04526 | -1.39118 | -6.5556 | -22.36137 | 0.36609 |
| $C_{ii}$  | 0.66445 | -1.16945 | 2.73214 | -4.02993 | 1.38286 |
| $A_{2i}$  | $2.50134 \times 10^4$ | 7.66246 | 395646.6828 | 826865.3 | -50828.7 |
| $B_{2i}$  | 2.69514 | -2.17119 | -1.78362 $\times 10^6$ | -12.3363 | -9.78166 |
| $C_{2i}$  | 0.58398 | -0.70693 | 2.24895 | -3.41457 | 2.48187 |

Substituting equations (12) and (13) into equation (11), and note that integrate region becomes $R_i$ to $R_i+a$, constants $M_i$, $i=1,3$ can be found as follows.

\[
M_1 = -\sqrt{2\pi S_i} + 3\sqrt{2\pi S_i}t / a - 4.8
\]

\[
M_3 = 3\sqrt{2\pi S_i} - 6\sqrt{2\pi S_i}t / a + 1.6
\]

4. Determination of the SIFs

Since the continuously distributed stress field in equation (9) and weight function in equation (12) are obtained, SIFs of the crack tip considering both mechanical and thermal loads can be determined as following.

\[
K_i(a) = \int_{R_i}^{R_i+a} \left( \sigma_{zz}(r) + P_t \right) m(r,a) dr
\]

Note that, internal pressure is considered when calculate the SIFs of the crack tip. The SIFs is as following.

\[
K_i(a) = \left( \phi_i + P_t \right) \frac{a}{2\pi} 4 + 2M_1 + \frac{4}{3} M_2 + M_3 - 4\mu\xi \sum_{i=1}^{3} \ln(R_i) \sqrt{ \frac{8a}{\pi} } \ln \left( \frac{R_i + a}{R_i} \right) + \frac{\sqrt{R_i + a}}{R_i} + \frac{\sqrt{a}}{R_i}
\]

\[
\frac{2}{\pi a} \frac{\ln(R_i + a)}{\ln(R_i)} \ln(R_i + a) \ln \left( \frac{R_i + a}{a} \right) \ln \left( \frac{R_i}{a} \right) + 2(\frac{R_i}{a} + a) \ln \left( \frac{R_i}{a} + a \right) \frac{1}{8\pi a} \frac{\ln(2a + 2R_i + a)}{\ln(R_i)} \ln(R_i)
\]

The SIFs for homogenous stress state is developed as following.

\[
K_i(a) = P_t \sqrt{ \frac{a}{2\pi} \left( 4 + 2M_1 + \frac{4}{3} M_2 + M_3 \right) }
\]

To verify the accuracy, consider a normal pressure $P_t = 10 MPa$. The material properties are assumed to as Shear modulus $\mu = 80$ Gpa, Possison’s ratio $\nu = 0.3$. The non-dimensional stress intensity factor is defined as $K_N = K_i / (P_t \sqrt{a})$. The results are indicted in figure 4. Different relative
depths ($a/t$) varied from 0.1 to 0.8 and ratio of outer radius to internal radius of the cylinder varied from 1.1 to 2.0.

The results of $R_o/R_i=1.1$ are compared with the literature [14] and the results for $R_o/R_i=1.25$ and 2.0 are compared with those of Mettu [15]. From figure 4 we can see that the predicted $K_i$ is reasonable at a wide range of radius ratio.

It is assumed that sudden cooling ($B_i = \infty$) happened at internal surface of the cylinder. Thermal expansion coefficient of the cylinder is $\alpha = 14.2 \times 10^{-6}$. The results of the normalized thermal stress intensity factors are shown in figure 5. The results show an excellent agreement.

5. Reliability analysis
There are several criteria to determine the failure. $K_i$ based criteria is often used in engineering, described as follows.

$$K_i = K_{ic}$$ (20)

Structure fails when the driving force ($K_i$) is greater than the fracture toughness ($K_{ic}$), failure criterion due to crack initiation is widely used [16].

Consider a cracked cylinder subject to random loads and the structure fails when $K_i > K_{ic}$, in which, $K_i$ depends on input geometric parameters and loads which are random and $K_{ic}$ itself is a random variable. The probability of failure $P_f$, can defined as following.
\[ P_F = \Pr \{ g(X) < 0 \} \]  
(21)

Where \( g(x) \) is as following.

\[ g(x) = K_{Ic} - K_I \]  
(22)

Monte-Carlo with neural network (MCNN) was used to compute these probabilities. Firstly, we produce many SIFs using the presented weight function method. The parameters associated with the weight function method are sampled form its own statistical distribution. Then, a comparison between the random SIF and the critical SIF has been done, and if the random SIF is greater than the critical SIF, make the tag equals 1, otherwise 0. We input all of the random parameters and the result tag in the neural network, and train the neural network until we get a stable neural network. At last, we can use the neural network to get the failure of the structure by input a large number of random parameters and a large number of result tags also can be obtained. The failure probability is obtained by total number of tag which equals 1 divided by total number of simulations. The cylinder with 360-degree circumferential crack subjected to internal pressure with \( P_i \), the inner surface cooling with temperature \( \theta_s \). The internal radius is 100mm, Biot number is \( B_i = \infty \). The input statistical variables are as shown in Table 2. The probabilistic characteristics of \( P_i \) were chosen arbitrarily.

| Statistical Variable       | Mean   | COV\(^a\) | Probability distribution |
|----------------------------|--------|-----------|-------------------------|
| Elastic modulus (\( E \))  | 208GPa | 0.05      | Gaussian                |
| Internal pressure (\( P_i \)) | 10MPa  | 0.1       | Gaussian                |
| Fluid temperature (\( \theta_f \)) | -100°C  | 0.1       | Gaussian                |
| Crack size (\( a/t \))    | Variable\(^b\) | 0.2       | Gaussian                |
| Initiation fracture toughness (\( K_{Ic} \)) | 10MPa\( \sqrt{m} \) | 0.5       | Lognormal               |

\(^{a}\) COV=coefficient of variation.
\(^{b}\) varies from 0.1 to 0.8.

Figure 6 shows \( P_F \) vs \( a/t \), as obtained using standard reliability methods. No significant differences are found between them. Outer to internal radius ratio varied from 1.1 to 2.0.

![Graph](image_url)

**Figure 6.** Probability of failure for cylinder specimen.

6. **Summary and conclusions**

An accurate weight function method to achieve thermal-mechanical SIFs for cracked cylinder has been established. Reference stress state is used to determine the coefficient. Results show a good agreement
with the documents. The SIFs can be determined quickly using the weight function obtained when cracks subjected to arbitrary loads, and can be used for probabilistic fracture mechanics analysis.

A probabilistic fracture mechanics analysis methodology has been obtained for cracked cylinder. Computational reliability methods of the $K_I$ based fracture have been presented to analysis the statistical characteristics. The results from the example show that a precise statistical characteristics of the $K_I$ can be determined using the developed methodology. The failure probability of the structure increases with the increasing of loads, and the relationship between is not linear.

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