THURSTON TYPE THEOREM FOR SUB-HYPERBOLIC RATIONAL MAPS

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Abstract. In 1980’s, Thurston established a combinatorial characterization for post-critically finite rational maps. This criterion was then extended by Cui, Jiang, and Sullivan to sub-hyperbolic rational maps. The goal of this paper is to present a new but simpler proof of this result by adapting the argument in the proof of Thurston’s Theorem.

1. Introduction

Let \( f : S^2 \to S^2 \) be an orientation-preserving branched covering map of degree \( d \geq 2 \). We denote by \( \deg_x f \) the local degree of \( f \) at \( x \). We will call
\[
\Omega_f = \{ x \in S^2 \mid \deg_f(x) \geq 2 \}
\]
the critical set of \( f \) and
\[
P_f = \bigcup_{k \geq 1} f^k(\Omega_f).
\]
the post-critical set. We say \( f \) is post-critically finite if \( P_f \) is a finite set. In 1980’s, Thurston established a combinatorial characterization for post-critically finite rational maps. The theorem says that if the associated orbifold \( O_f \) is hyperbolic, then \( f \) is combinatorially equivalent to a rational map if and only if it has no Thurston obstructions. The basic idea of the proof is as follows. Consider the Teichmüller space \( T_f \) modeled on \( (S^2, P_f) \). Then \( f \) induces an analytic operator \( \sigma_f : T_f \to T_f \). It turns out that the existence of a rational map which realizes \( f \) is equivalent to the existence of a fixed point of \( \sigma_f \). The proof is then reduced to showing that \( \sigma_f \) is a strictly contracting map. The reader may refer to [5] for a detailed proof of this theorem.

A natural question is that to what extent, Thurston’s theorem can be extended to rational maps with infinitely many post-critical points. It was proved by McMullen that having no Thurston obstruction is essentially true for any rational map with a hyperbolic orbifold — only trivial Thurston obstructions inside Siegel disks or Herman rings may occur for a rational map with a hyperbolic orbifold [7]. In 1994, Cui, Jiang, and Sullivan established a Thurston type theorem for sub-hyperbolic rational maps ([2], see also [4],[8]).

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The original proof of Cui-Jiang-Sullivan’s theorem is quite involved. The goal of this paper is to give a new but simpler proof of this theorem by adapting the argument used in the proof of Thurston’s theorem.

Before we present this theorem, let us introduce some definitions first. We say $f$ is geometrically finite if $P_f$ is an infinite set but with finitely many accumulation points. Suppose that $f$ is geometrically finite. Then it is not difficult to see that the accumulation set of $P_f$ consists of finitely many periodic cycles. We leave this to the reader as an exercise. Let $P_f'$ denote the set of all the accumulation points of $P_f$.

**Definition 1.1.** Let $f: S^2 \to S^2$ be a geometrically finite branched covering map of degree $d \geq 2$. We say $f$ is a sub-hyperbolic semi-rational branched covering if for any $a \in P_f'$ of period $p \geq 1$, there is an open neighborhood $U$ of $a$, such that $f$ is holomorphic in $U$, and moreover, if $\deg_a f^p = 1$, then

$$f^p(z) = a + \lambda(z - a) + o(|z - a|)$$

for $z \in U$ where $0 < |\lambda| < 1$ is some constant, and if $\deg_a f^p = k > 1$, then

$$f^p(z) = a + \alpha(z - a)^k + o(|z - a|^k)$$

for $z \in U$ where $\alpha \neq 0$ is some constant.

As in the post-critically finite case, one can define Thurston obstructions for a sub-hyperbolic semi-rational branched covering map $f$ in a similar way. If $\gamma$ is a simple closed curve in $S^2 \setminus P_f$, then the set $f^{-1}(\gamma)$ is a union of disjoint simple closed curves. If $\gamma$ moves continuously, so does each component of $f^{-1}(\gamma)$. A simple closed curve $\gamma$ is non-peripheral if each component of $S^2 \setminus \gamma$ contains at least two points of $P_f$. Consider a multi-curve

$$\Gamma = \{\gamma_1, \cdots, \gamma_n\}$$

of simple, closed, disjoint, non-homotopic, and non-peripheral curves in $S^2 \setminus P_f$. We say that $\Gamma$ is $f$-stable if for any $\gamma \in \Gamma$, every non-peripheral component of $f^{-1}(\gamma)$ is homotopic in $S^2 \setminus P_f$ to an element of $\Gamma$.

For each $f$-stable multi-curve $\Gamma$, define a linear transformation,

$$f_\Gamma : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$$

as follows: let $\gamma_{i,j,\alpha}$ denote the components of $f^{-1}(\gamma_j)$ homotopic to $\gamma_i$ in $S^2 \setminus P_f$ and $d_{i,j,\alpha}$ be the degree of $f|_{\gamma_{i,j,\alpha}} : \gamma_{i,j,\alpha} \to \gamma_j$. Define

$$f_\Gamma(\gamma_j) = \sum_i \left(\sum_\alpha \frac{1}{d_{i,j,\alpha}}\right)\gamma_i.$$ 

Since the matrix of $f_\Gamma$ is non-negative, there exists a largest eigenvalue $\lambda(\Gamma, f) \in \mathbb{R}_+$. We say that a multi-curve $\Gamma$ is a Thurston obstruction of $f$ if $\lambda(\Gamma, f) \geq 1$.

**Definition 1.2.** Suppose $f$ and $g$ are two sub-hyperbolic semi-rational branched coverings. We say that they are CLH-equivalent (combinatorially
and locally holomorphically equivalent) if there exist a pair of homeomorphisms $\phi : S^2 \to S^2$ and $\psi : S^2 \to S^2$ such that

- $\psi$ is isotopic to $\phi$ rel $P_f$,
- $\phi f = g \psi$,
- $\phi|U_f = \psi|U_f$ is holomorphic on some open set $U_f \supset P'_f$.

Now let us state the Thurston type theorem for sub-hyperbolic rational maps.

**Main Theorem.** Suppose $f$ is a sub-hyperbolic semi-rational branched covering. Then $f$ is CLH-equivalent to a rational map $R$ if and only if $f$ has no Thurston obstructions. In this case, the rational map $R$ is unique up to a Möbius conjugation of the Riemann sphere.

**Remark 1.1.** There are branched covering maps of the sphere which are geometrically finite and having no Thurston obstructions but are not combinatorially equivalent to rational maps. For the construction of such maps, see [3].

The proof of the "only if" part follows from a theorem of McMullen (see Appendix B of [7]). The main task of this paper is to prove the "if" part.

The essential difference between the post-critically finite case and the sub-hyperbolic case is that in the first case, the post-critical set is a finite set and the Thurston pull back induces an analytic operator defined on a finite-dimensional Teichmüller space, while in the latter case, the post-critical set is an infinite set and therefore, the induced operator is defined on an infinite-dimensional Teichmüller space. However, we observe in this paper that, in both cases, the following bounded geometry properties are similar. This allows us to prove the latter case by adapting the argument in the proof of the first case.

In the post-critically finite case, the base point of the Teichmüller space is the Riemann sphere minus the set of finite number of post-critical points. The branched covering induces a pull-back operator on this Teichmüller space. Iterations of this operator produce a sequence of sets of finite number of points in the Riemann sphere. The bounded geometry in this case means that there is a positive constant such that any two points in any element of this sequence have spherical distance greater than or equal to this constant.

In the sub-hyperbolic case, the base point of the Teichmüller space is the Riemann sphere minus the union of finitely many points and topological disks. Iterations of the pull-back operator produce a sequence of sets of finite number of points plus finite number of disks in the Riemann sphere. The bounded geometry in this case means that there is a positive constant such that in any element of this sequence, the spherical distance between any two points, any point and any disk, or any two disks is greater than or equal to this constant;
moreover, any disk in any element of this sequence contains another round disk of radius greater than or equal to this constant.

The paper is organized as follows. In §2, we prove the Shielding Ring Lemma. The proof is elementary but it is crucial in our construction of the Teichmüller space. In §3, we construct the Teichmüller space $T_f$. In §4, we introduce the pull back operator $\sigma_f : T_f \to T_f$. In §5, we introduce the concept of bounded geometry. In §6, we prove that bounded geometry implies the strictly contracting property of $\sigma_f$. In §7, we prove that no Thurston obstruction implies the bounded geometry. This completes the proof of the Main Theorem.

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2. Shielding Ring Lemma

We say an open annulus $A$ is attached to an open topological disk $D$ from the outside if $A$ and $D$ are disjoint but $\partial D$ is the inner boundary component of the annulus $A$. Then $D \cup A$ is a larger closed disk.

Suppose that $f$ is a sub-hyperbolic semi-rational branched covering. Let $P'_f = \{a_i\}$. The main purpose of this section is to prove the following lemma.

**Lemma 2.1 (Shielding Ring Lemma).** There is a collection $\{D_i\}$ of open disks and a collection of open annuli $\{A_i\}$ such that

- $a_i \in D_i$,
- $\overline{D}_i \cap \overline{D}_j = \emptyset$ for $i \neq j$,
- for each $i$, $A_i$ is an annulus attaching $D_i$ from the outside such that $A_i \cap P_f = \emptyset$,
- $f$ is holomorphic on $\overline{D}_i \cup A_i$,
- every $f(A_i)$ is contained in some $D_j$.

**Proof.** Since $P'_f$ consists of finitely many periodic cycles, it is sufficient to find $D_i$ and $A_i$ for each periodic cycle.

Suppose

$$\{a_1, \ldots, a_p\}$$

is a periodic cycle in $P'_f$ such that

$$f(a_i) = a_{i+1} \pmod{p}, \quad 1 \leq i \leq p.$$ 

This periodic cycle is either attracting or super-attracting. Let us assume that we are in the attracting case. That is, we can find a topological disk $W$ containing $a_1$ and a holomorphic isomorphism $\phi : W \to \Delta$ such that
\[ \phi \circ f^n \circ \phi^{-1} : \Delta \to \Delta \] is equal to \( \lambda z \) for some \( 0 < |\lambda| < 1 \). The superattracting case can be treated in a similar way by making minor changes.

For \( 0 < r < 1 \), let \( \mathbb{T}_r = \{ z \mid |z| = r \} \) and \( \Delta_r = \{ z \mid |z| < r \} \). Let \( U_r = \phi^{-1}(\Delta_r) \). Note that there are only countably many \( r \) such that

\[ \bigcup_{i \geq 0} f^i(\partial U_r) \cap P_f \neq \emptyset. \]

So we can take \( 0 < a < 1 \) such that

(1) \[ \bigcup_{i \geq 0} f^i(\partial U_a) \cap P_f \neq \emptyset. \]

Let \( b = a + \epsilon < 1 \) for some \( \epsilon > 0 \) small. Let

\[ H = \{ z \mid a < |z| < b \}. \]

From (1), it follows that by taking \( \epsilon > 0 \) small, we can assume

1. \( \bigcup_{0 \leq i \leq p-1} f^i(\phi^{-1}(\overline{H})) \cap P_f = \emptyset \), and
2. \( f^p(\phi^{-1}(\overline{H})) \subset U_a \).

Now divide the annulus \( H \) into \( p \) sub-annuli \( H_1, \ldots, H_p \) as follows. Take \( a = r_0 < r_1 < r_2 < \cdots < r_p = b \). Let \( H_i = \{ z \mid r_{i-1} < |z| < r_i \} \). Let \( E_i = \phi^{-1}(H_i) \). Define

\[ D_1 = U_a \quad \text{and} \quad A_1 = E_1, \]

and

\[ D_2 = f(U_a \cup E_1) \quad \text{and} \quad A_2 = f(E_2). \]

For \( 3 \leq i \leq p \),

\[ D_i = f^{i-1}(U_a \cup \bigcup_{1 \leq j \leq i-2} E_j \cup E_{i-1}) \quad \text{and} \quad A_i = f^{i-1}(E_i). \]

After we did for every periodic cycle in \( P_f \), we put those disks and annuli together to get a collection of open topological disks \( \{ D_i \} \) and a collection of open annuli \( \{ A_i \} \). By the construction, it is clear that they satisfy the properties in Lemma 2.1. This completes the proof of Lemma 2.1. \( \square \)

We call the disk \( D_i \) in Lemma 2.1 a holomorphic disk and the corresponding annulus \( A_i \) a shielding ring.

**Remark 2.1.** By our construction, the boundary of every \( D_i \) is a real-analytic curve.

### 3. The Teichmüller space \( T_f \)

Let us now fix a collection of holomorphic disks \( \{ D_i \} \) and a collection of shielding rings \( \{ A_i \} \) for \( f \). Let

\[ D_f = \bigcup_i D_i \quad \text{and} \quad P_1 = P_f \setminus D_f. \]
By taking $D_i$ smaller, we may assume that $\#(P_1) \geq 3$. We may further assume that \{0, 1, \infty\} \subset P_1. Define

$$Q_f = P_1 \cup \overline{D}_f \quad \text{and} \quad X_f = \partial Q_f = P_1 \cup \partial D_f.$$ 

**Definition 3.1.** The Teichmüller space $T_f$ is the Teichmüller space modeled on $(S^2 \setminus Q_f, X_f)$.

The Teichmüller space $T_f$ can be constructed as the space of all the Beltrami coefficients defined on $S^2 \setminus Q_f$ module the following equivalent relation: let $\mu$ and $\nu$ be two Beltrami coefficients defined on $S^2 \setminus Q_f$ and let $\phi_\mu : S^2 \setminus Q_f \to S$ and $\phi_\nu : S^2 \setminus Q_f \to R$ be two quasiconformal homeomorphisms which solve the Beltrami equations given by $\mu$ and $\nu$, respectively. we say $\mu$ and $\nu$ are equivalent to each other if there exists a holomorphic isomorphism $h : R \to S$ such that the map $\phi_\mu$ and $h \circ \phi_\nu$ are isotopic to each other rel $X_f$, that is, there is a continuous family of quasiconformal homeomorphisms $g_t : S^2 \setminus Q_f \to S$, $0 \leq t \leq 1$, such that

1. $g_0 = \phi_\mu$,
2. $g_1 = h \circ \phi_\nu$,
3. $g_t(z) = \phi_\mu(z) = (h \circ \phi_\nu)(z)$ for all $0 \leq t \leq 1$ and $z \in X_f$.

Now let us give a brief description of the relative background about the Teichmüller space $T_f$. The reader may refer to [6] for more knowledge in this aspect.

Let $\mu$ be a Beltrami coefficient defined on $S^2 \setminus Q_f$. Let

$$\phi_\mu : S^2 \setminus Q_f \to \phi_\mu(S^2 \setminus Q_f)$$

be a quasiconformal homeomorphism which solves the Beltrami equation given by $\mu$. Let

$$M_\mu = \{ \xi(z) \frac{d\sigma}{dz} \mid \xi(z) \text{ is measurable and } ||\xi||_\infty < \infty \}$$

be the linear space of all the Beltrami differentials defined on $\phi_\mu(S^2 \setminus Q_f)$. Let

$$A_\mu = \{ q(z)dz^2 \mid q(z) \text{ is holomorphic and } \int_{\phi_\mu(S^2 \setminus Q_f)} |q(z)|dz \wedge d\bar{z} < \infty \}$$

be the linear space of all the integrable holomorphic quadratic differentials defined on $\phi_\mu(S^2 \setminus Q_f)$.

A Beltrami differential $\xi(z) \frac{d\sigma}{dz} \in M_\mu$ is called *infinitesimally trivial* if

$$\int_{\phi_\mu(S^2 \setminus Q_f)} \xi(z)q(z)dz \wedge d\bar{z} = 0$$

holds for all $q(z)dz^2 \in A_\mu$. 


Let $N_\mu \subset M_\mu$ be the subspace of all the \textit{infinitesimally trivial} Beltrami differentials. Then the tangent space of $T_f$ at $[\mu]$ is isomorphic to the quotient space $M_\mu/N_\mu$.

Let $\mu$ be a Beltrami coefficient defined on $S^2 \setminus Q_f$. Let $\xi$ be a tangent vector of $T_f$ at $[\mu]$ which is identified with a Beltrami differential $\xi(z) \frac{dz}{d\bar{z}}$ defined on $\phi_\mu(S^2 \setminus Q_f)$.

**Definition 3.2.** The Teichmüller norm of the tangent vector $\xi$ is defined to be
\[
\| \xi \| = \sup \left\{ \left| \int_{\phi_\mu(S^2 \setminus Q_f)} q(z)\xi(z)dz \wedge d\bar{z} \right| \right\},
\]
where the sup is taken over all $q(z)dz^2 \in A_\mu$ with $\int_{\phi_\mu(S^2 \setminus Q_f)} |q(z)|dz \wedge d\bar{z} = 1$.

**Definition 3.3.** Let $[\mu], [\nu] \in T_f$. The Teichmüller distance $d_T([\mu],[\nu])$ is defined to be
\[
\frac{1}{2} \inf \log K(\phi_{\mu'} \circ \phi_{\nu}^{-1})
\]
where $\phi_{\mu'}$ and $\phi_{\nu}$ are quasi-conformal mappings with Beltrami coefficients $\mu'$ and $\nu'$ and the inf is taken over all $\mu'$ and $\nu'$ in the same Teichmüller classes as $\mu$ and $\nu$, respectively.

**Lemma 3.1.** Let $\mu$ and $\nu$ be two Beltrami coefficients defined on $S^2 \setminus Q_f$. Then
\[
d_T([\mu],[\nu]) = \inf \int_0^1 ||\tau'(t)||dt
\]
where $\inf$ is taken over all the piecewise smooth curves $\tau(t)$ in $T_f$ such that $\tau(0) = [\mu]$ and $\tau(1) = [\nu]$.

4. **The pull-back operator**

As in the post-critically finite case, we may assume that $f$ is a quasiconformal map (This is because except the finite holomorphic disks, there are only finitely many points in $P_f$, and therefore, the CLH-equivalent class of $f$ must contain a quasiconformal branched covering map of the sphere). From now on, we use $\mathbb{P}^1$ to denote the two sphere endowed with the standard complex structure.

Remind that for a Beltrami coefficient $\mu$ defined on the sphere $S^2$, the pull back of $\mu$ by $f$, which is denoted by $f^*(\mu)$, is defined to be
\[
(f^* \mu)(z) = \frac{\mu_f(z) + \mu(f(z))\theta(z)}{1 + \mu_f(z)\mu(f(z))\theta(z)}
\]
where $\theta(z) = \frac{f_z}{f_{\bar{z}}}$ and $\mu_f(z) = f_{\bar{z}}/f_z$. It is important to note that if $\mu$ depends complex analytically on $t$, then so does $f^*(\mu)$.
Now let $\mu(z)$ be a Beltrami coefficient defined on $S^2 \setminus Q_f$. Define the Beltrami coefficient $\text{Ext}(\mu)(z)$ on $S^2$ by setting

$$\text{Ext}(\mu)(z) = \begin{cases} 
\mu(z) & \text{for } z \in S^2 \setminus Q_f, \\
0 & \text{for otherwise}.
\end{cases}$$  

(3)

By (2), $f^*(\text{Ext}(\mu))$ is a Beltrami coefficient on the sphere $S^2$. Let us simply use $f^*(\mu)$ to denote the restriction of $f^*(\text{Ext}(\mu))$ on $S^2 \setminus Q_f$.

**Lemma 4.1.** The map $f^*$ induces a complex analytic operator $\sigma_f : T_f \rightarrow T_f$.

**Proof.** Suppose $\mu$ and $\nu$ are two Beltrami coefficients defined on $S^2 \setminus Q_f$ which are equivalent to each other. Let $\text{Ext}(\mu)$ and $\text{Ext}(\nu)$ be their extensions to $S^2$. Let $\phi_{\text{Ext}(\mu)}$ and $\phi_{\text{Ext}(\nu)}$ be the corresponding quasiconformal homeomorphisms of the sphere which fix 0, 1, and the infinity. Let $\phi_\mu$ and $\phi_\nu$ denote their restrictions to $S^2 \setminus Q_f$, respectively. Since $\mu$ is equivalent to $\nu$, we have a holomorphic isomorphism

$$h : \mathbb{P}^1 \setminus \phi_{\text{Ext}(\nu)}(Q_f) \rightarrow \mathbb{P}^1 \setminus \phi_{\text{Ext}(\mu)}(Q_f)$$

such that $\phi_\mu$ is isotopic to $h \circ \phi_\nu$ rel $X_f$. Now define a homeomorphism $\text{Ext}(h) : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ by setting

$$\text{Ext}(h)(z) = \begin{cases} 
h(z) & \text{for } z \in \mathbb{P}^1 \setminus \phi_{\text{Ext}(\nu)}(Q_f), \\
\phi_{\text{Ext}(\mu)} \circ \phi_{\text{Ext}(\nu)}^{-1}(z) & \text{for otherwise}.
\end{cases}$$  

(4)

It is clear that $\text{Ext}(h)$ is holomorphic everywhere except those points in $\phi_{\text{Ext}(\nu)}(X_f)$. Since $\phi_{\text{Ext}(\nu)}(X_f)$ is the union of finitely many points and finitely many quasi-circles (see Remark 2.1), it follows that $\text{Ext}(h)$ is a holomorphic homeomorphism of the sphere to itself, and therefore a Möbius map. By the normalization condition, $\text{Ext}(h)$ fixes 0, 1, and $\infty$ also. So $\text{Ext}(h) = \text{id}$. This implies that $\phi_\mu$ and $\phi_\nu$ are isotopic to each other rel $X_f$, and in particular, $\phi_\mu = \phi_\nu$ on $X_f$. Since $\phi_{\text{Ext}(\mu)}$ and $\phi_{\text{Ext}(\nu)}$ are holomorphic on $D_f$, it follows that $\phi_{\text{Ext}(\mu)} = \phi_{\text{Ext}(\nu)}$ on $Q_f$ and therefore are isotopic to each other rel $Q_f$. Since $f(Q_f) \subset Q_f$, we can therefore lift this isotopy and get an isotopy between $\phi_{f^*(\text{Ext}(\mu))}$ and $\phi_{f^*(\text{Ext}(\nu))}$ rel $Q_f$. It follows that $\phi_{f^*(\mu)}$ and $\phi_{f^*(\nu)}$, which are respectively the restrictions of $\phi_{f^*(\text{Ext}(\mu))}$ and $\phi_{f^*(\text{Ext}(\nu))}$ on $S^2 \setminus Q_f$, are isotopic to each other rel $X_f$. This implies that $[f^*(\mu)] = [f^*(\nu)]$. Let $\sigma_f([\mu]) = [f^*(\mu)]$.

Now let us show that $\sigma_f$ is complex analytic. Suppose that we have a curve $\tau(t)$ in $T_f$ such that $\tau(t)$ depends complex analytically on $t$ when $t$ varies in a small disk $\{ t \mid |t| < \epsilon \}$. We may assume that $\epsilon > 0$ is small enough so that the following arguments are valid. Let $[\mu] = \tau(0)$. Then the map $\phi_\mu$ induces an isometry between $T_f$ and the Teichmüller space modeled on $(\mathbb{P}^1 \setminus \phi_\mu(Q_f), \phi_\mu(X_f))$. This isometry maps the curve $\tau(t)$ to a complex analytic curve $\theta(t)$, $|t| < \epsilon$, which passes through the origin. Since $\epsilon > 0$ is
Lemma 4.2. Let \( \eta \) and \( \gamma \) be as above. Then

\[
\tilde{\xi}(w) = \frac{\partial}{\partial t} \bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_{\mu}^{-1}}(g(w)) \frac{g'(w)}{g'(w)}.
\]

Proof. Note that

\[
\tilde{\xi} = \frac{d}{dt} \bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ f \circ \phi_{\mu}^{-1}} = \frac{d}{dt} \bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_{\mu}^{-1} \circ f \circ \phi_{\mu}^{-1}} = \frac{d}{dt} \bigg|_{t=0} \mu_{\phi_{\gamma(t)} \circ \phi_{\mu}^{-1}} g.
\]

Since \( g \) is a rational map, by [2] we have

\[
\mu_{\phi_{\gamma(t)} \circ \phi_{\mu}^{-1}}(w) = \mu_{\phi_{\gamma(t)} \circ \phi_{\mu}^{-1}}(g(w)) \frac{g'(w)}{g'(w)}.
\]
Proposition 4.2. Let \( \tilde{q} = \tilde{q}(w)dw^2 \) be a non-zero integrable holomorphic quadratic differential defined on \( \mathbb{P}^1 \setminus \phi_{\mu}(Q_f) \). Define
\[
q(z) = \sum_{g(w)=z} \frac{\tilde{q}(w)}{|g'(w)|^2}.
\]

It is easy to see that \( q = q(z)dz^2 \) is a holomorphic quadratic differential defined on \( \mathbb{P}^1 \setminus \phi_{\mu}(Q_f) \).

**Proposition 4.1.**
\[
\int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} |q(z)|dz \wedge d\overline{z} \leq \int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} |\tilde{q}(w)|dw \wedge d\overline{w} - \int_{\cup_i \phi_{\mu}(A_i)} |\tilde{q}(w)|dw \wedge d\overline{w}.
\]

**Proof.**
\[
\int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} |q(z)|dz \wedge d\overline{z} = \int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} \left| \sum_{g(w)=z} \frac{\tilde{q}(w)}{|g'(w)|^2} \right| dz \wedge d\overline{z}
\leq \int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f) \setminus \cup_i \phi_{\mu}(A_i)} |\tilde{q}(w)|dw \wedge d\overline{w}
= \int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} |\tilde{q}(w)|dw \wedge d\overline{w} - \int_{\cup_i \phi_{\mu}(A_i)} |\tilde{q}(w)|dw \wedge d\overline{w}.
\]
The first inequality comes from the fact \( f(\cup A_i) \subset \cup D_i \). This completes the proof of Proposition 4.1. \( \Box \)

**Proposition 4.2.**
\[
\int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} \tilde{\xi}(w)\tilde{q}(w)dw \wedge d\overline{w} = \int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} \tilde{\xi}(z)q(z)dz \wedge d\overline{z}.
\]

**Proof.** Note that \( \phi_{\mu}(Q_f) \subset g^{-1}(\phi_{\mu}(Q_f)) \) and by \( \Box \), \( \tilde{\xi}(w) = 0 \) for all \( w \in g^{-1}(\phi_{\mu}(Q_f)) \setminus \phi_{\mu}(Q_f) \). We thus have
\[
\int_{\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)} \tilde{\xi}(w)\tilde{q}(w)dw \wedge d\overline{w} = \int_{\mathbb{P}^1 \setminus g^{-1}(\phi_{\mu}(Q_f))} \tilde{\xi}(w)\tilde{q}(w)dw \wedge d\overline{w}.
\]
Now Proposition 4.2 follows from \( \Box \), \( \Box(7) \), and the fact that
\[
dw \wedge d\overline{w} = \frac{dz \wedge d\overline{z}}{|g'(w)|^2}.
\]
\( \Box \)

As a direct consequence of Propositions 4.1 and 4.2 we have

**Corollary 4.1.** Let \( \tau \in T_f \). Then \( \|d\sigma_f\|_\tau \leq 1 \).

**Remark 4.1.** Corollary 4.1 also follows from the general fact that a complex analytic operator does not increase the Kabayashi’s metric. But this particular argument we used here will be established in the latter sections to prove a strict inequality (see Corollary 6.1).
The next lemma reduces the proof of the Main Theorem to showing that the pull back operator $\sigma_f$ has a unique fixed point in $T_f$.

**Lemma 4.3.** The map $f$ is CLH-equivalent to a unique rational map (up to Möbius conjugations) if and only if $\sigma_f$ has a unique fixed point in $T_f$.

**Proof.** If $\sigma_f$ has a fixed point $[\mu]$ in $T_f$, then $\mu = f^*\mu \sim \mu$. Let $\operatorname{Ext}(\mu)$ be the extension of $\mu$ to $S^2$. Let $\phi_{\operatorname{Ext}(\mu)}$ and $\phi_{f^*(\operatorname{Ext}(\mu))}$ be the corresponding quasiconformal homeomorphisms which fix 0, 1, and the infinity. Let $\phi_{\mu}$ and $\phi_{\tilde{\mu}}$ be their restrictions to $S^2 \setminus Q_f$, respectively. It follows that there is a conformal isomorphism

$$h : \mathbb{P}^1 \setminus \phi_{\mu}(Q_f) \to \mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)$$

such that $\phi_{\tilde{\mu}}$ and $h \circ \phi_{\mu}$ are isotopic to each other rel $X_f$. As in the proof of Lemma 4.1, one can show that such $h$ is actually equal to the identity map. In fact, we can again define a homeomorphism $\operatorname{Ext}(h) : \mathbb{P}^1 \to \mathbb{P}^1$ by setting

$$\operatorname{Ext}(h)(z) = \begin{cases} h(z) & \text{for } z \in \mathbb{P}^1 \setminus \phi_{\operatorname{Ext}(\mu)}(Q_f), \\ \phi_{f^*(\operatorname{Ext}(\mu))}^{-1} \circ \phi_{\operatorname{Ext}(\mu)}^{-1}(z) & \text{for otherwise.} \end{cases}$$

It is clear that $\operatorname{Ext}(h)$ is holomorphic everywhere except those points in $\phi_{\operatorname{Ext}(\mu)}(X_f)$. Since $\phi_{\operatorname{Ext}(\mu)}(X_f)$ is the union of finitely many points and finitely many quasi-circles (see Remark 2.1), it follows that $\operatorname{Ext}(h)$ is a holomorphic homeomorphism of the sphere to itself, and therefore a Möbius map. By the normalization condition, $\operatorname{Ext}(h)$ fixes 0, 1, and $\infty$ also. So $\operatorname{Ext}(h) = \text{id}$. This implies that $\phi_{\mu}$ and $\phi_{\tilde{\mu}}$ are isotopic to each other rel $X_f$. It follows that $\phi_{\operatorname{Ext}(\mu)}$ and $\phi_{f^*(\operatorname{Ext}(\mu))}$ are isotopic to each other rel $Q_f$. Note that when restricted to $D_f$, $\phi_{\operatorname{Ext}(\mu)}$ and $\phi_{f^*(\operatorname{Ext}(\mu))}$ are analytic and equal to each other. This implies that $f$ is CLH-equivalent to the rational map $g = \phi_{\operatorname{Ext}(\mu)} \circ f \circ \phi_{f^*(\operatorname{Ext}(\mu))}^{-1}$.

If $f$ is CLH-equivalent to $g$, then we have a Beltrami coefficient $\mu$ defined on $S^2 \setminus Q_f$ such that $g = \phi_{\operatorname{Ext}(\mu)} \circ f \circ \phi_{f^*(\operatorname{Ext}(\mu))}^{-1}$ and moreover, $\phi_{\operatorname{Ext}(\mu)}$ and $\phi_{f^*(\operatorname{Ext}(\mu))}$ are isotopic to each other rel $Q_f$. This implies that $\phi_{\mu}$ and $\phi_{\tilde{\mu}}$ are isotopic to each other rel $X_f$. It follows that $[f^*(\mu)] = [\mu]$ and thus $\sigma_f([\mu]) = [\mu]$.

It is clear that the fixed point $[\mu]$ is unique is equivalent to say that $g$ is unique up to Möbius conjugations. $\square$

### 5. Bounded geometry

Let $d(X, Y)$ denote the spherical distance between two subsets of the sphere. Recall that

$$D_f = \bigcup_i D_i, \ P_i = P_f \setminus D_f, \text{ and } P'_f = \{a_i\}.$$ 

**Definition 5.1.** Let $b > 0$ be a constant. Let $T_{f,b} \subset T_f$ be the subspace such that for every $[\mu] \in T_{f,b}$, the following conditions hold,
(1) for all \( z_i \neq z_i' \in \mathcal{P}_1 \),
\[
d(\phi_\mu(z_i), \phi_\mu(z_i')) \geq b;
\]
(2) for all \( z_j \in \mathcal{P}_1 \) and all \( D_i \),
\[
d(\phi_\mu(z_j), \phi_\mu(D_i)) \geq b;
\]
(3) for all \( D_i \neq D_i' \),
\[
d(\phi_\mu(D_i), \phi_\mu(D_i')) \geq b;
\]
(4) for every \( D_i \), \( \phi_\mu(D_i) \) contains a round disk of radius \( b \) centered at \( \phi_\mu(a_i) \),
where \( \phi_\mu : S^2 \to \mathbb{P}^1 \) is the quasiconformal homeomorphism which fixes 0, 1, and the infinity, and which solves the Beltrami equation given by \( \text{Ext}(\mu) \).

Let \( K > 1 \). Then the family of all the \( K \)-quasiconformal homeomorphisms of the sphere to itself, which fix 0, 1, and the infinity, is compact. We thus have

**Lemma 5.1.** Let \( K > 1 \). Then for every \( \delta > 0 \) depending only on \( K \) and \( \delta \) such that for every two points \( x, y \in \mathbb{P}^1 \) with \( d(x, y) > \delta \), and every \( K \)-quasiconformal homeomorphism \( \phi : \mathbb{P}^1 \to \mathbb{P}^1 \) which fixes 0, 1, and the infinity, we have \( d(\phi(x), \phi(y)) > \epsilon \).

By Definitions 3.3 and 5.1, and Lemma 5.1, we have

**Lemma 5.2.** Let \( b, D > 0 \). Then there is a \( b' > 0 \) depending only on \( b \) and \( D \) such that for any two Beltrami coefficients \( \mu \) and \( \nu \) defined on \( S^2 \setminus Q_f \), if \( d_T([\mu], [\nu]) < D \) and \( \mu \in T_{f,b} \), then \( \nu \in T_{f,b'} \).

**Definition 5.2.** Let \( Z \) be a subset of \( S^2 \) with \#(\( Z \)) \( \geq 4 \). Let \([\mu] \in T_f \) and \( \gamma \subset S^2 \setminus Z \) be a simple closed and non-peripheral curve. We use \( \|\gamma\|_{\mu,Z} \) to denote the hyperbolic length of the unique simple closed geodesic \( \xi \) which is homotopic to \( \phi_\mu(\gamma) \) in the hyperbolic Riemann surface \( \mathbb{P}^1 \setminus \phi_\mu(Z) \). We say \( \gamma \) is a \((\mu,Z)\)-simple closed geodesic if \( \phi_\mu(\gamma) \) is a simple closed geodesic in \( \mathbb{P}^1 \setminus \phi_\mu(Z) \).

For each holomorphic disk \( D_i \), fix a point \( b_i \) on the boundary \( \partial D_i \). Set
\[
E = \mathcal{P}_1 \cup \cup_i \{a_i, b_i \}.
\]

Note that \( \mathcal{P}_1 \) contains 0, 1, and the infinity by assumption. Since \( \mathcal{P}_1 \subset E \) and \( \phi_\mu \) fixes 0, 1, and the infinity, it follows that \( E \) and \( \phi_\mu(E) \) contain 0, 1, and the infinity also.

**Lemma 5.3.** Let \( a > 0 \). Then there is a \( b > 0 \) depending only on \( a \) such that for every Beltrami coefficient \( \mu \) defined on \( S^2 \setminus Q_f \) with \( \mu(z) = 0 \) on \( \cup_i A_i \), if every \((\mu,E)\)-simple closed geodesic \( \gamma \subset S^2 \setminus Q_f \) has hyperbolic length not less than \( a \), then \( \mu \in T_{f,b} \).
Proof. Note that \( \#(\phi_\mu(E)) = \#(E) \) is finite. Since \( \phi_\mu(E) \) contains 0, 1, and the infinity, it follows that the spherical distance between any two points in \( \phi_\mu(E) \) has a positive lower bound which depends only on \( a \) and \( \#(E) \).

Since \( \phi_\mu \) is holomorphic in every topological disk \( D_i \cup A_i \) and since \( \phi_\mu(D_i) \) contains \( \phi_\mu(a_i) \) and \( \phi_\mu(b_i) \), it follows from Koebe’s distortion theorem that every \( \phi_\mu(D_i) \) contains a round disk centered at \( \phi_\mu(a_i) \), the radius of which has a positive lower bound depending only on \( a \).

Since \( \{0, 1, \infty\} \notin \phi_\mu(D_i \cup A_i) \), it follows that the diameter of each component of \( \mathbb{P}^1 \setminus \phi_\mu(A_i) \) has a positive lower bound depending only on \( a \). Since \( \phi_\mu \) is analytic on every \( A_i \), we have

\[
\text{mod}(\phi_\mu(A_i)) = \text{mod}(A_i).
\]

It follows that every \( \phi_\mu(A_i) \) has definite thickness which depends only on \( a \).

All of these implies that there is a constant \( b > 0 \) depending only on \( a \) such that the four conditions in Definition 5.1 hold. The proof of the lemma is completed. \( \square \)

The next lemma is a direct consequence of Proposition 6.1 and Theorem 6.3 of [5].

**Lemma 5.4.** Let \( X \) be a hyperbolic Riemann surface and \( \gamma \subset X \) be a simple closed geodesic with hyperbolic length \( l \). Then there exists a topological annulus \( A \subset X \) such that

1. \( \gamma \) is the core curve of \( A \),
2. \( \frac{\pi}{2} - 1 < \text{mod}(A) < \frac{\pi}{2} \).

From the modulus inequality of Teichmüller extremal problem (for instance, see Chapter III of [1]), we have

**Lemma 5.5.** Let \( T \in \mathbb{P}^1 \setminus \{0, 1, \infty\} \). Let \( H \subset \mathbb{P}^1 \) be an annulus which separates \( \{0, 1\} \) and \( \{T, \infty\} \). Then

\[
\text{mod}(H) \leq \frac{1}{2\pi} \log 16(|T| + 1).
\]

**Lemma 5.6.** There exists an \( \eta > 0 \) such that for any Beltrami coefficient \( \mu \) defined on \( S^2 \setminus Q_f \) with \( \mu(z) = 0 \) on \( \cup_i A_i \) and any \( (\mu, E) \)-simple closed geodesic \( \gamma \subset S^2 \setminus E \) with \( \|\gamma\|_{\mu, E} < \eta \), we have \( \gamma \subset S^2 \setminus Q_f \). Moreover, for any \( \epsilon > 0 \), there is a \( \delta > 0 \) such that

\[
\|\gamma\|_{\mu, E} > (1 - \epsilon)\|\gamma\|_{\mu, Q_f}
\]

provided that \( \|\gamma\|_{\mu, E} < \delta \).

**Proof.** Let \( \gamma \subset S^2 \setminus E \) be a \( (\mu, E) \)-simple closed geodesic. By Lemma 5.4 there is an annulus \( A \subset \mathbb{P}^1 \setminus \phi_\mu(E) \) such that \( \phi_\mu(\gamma) \) is the core curve of \( A \) and

\[
\frac{\pi}{2\|\gamma\|_{\mu, E}} - 1 < \text{mod}(A) < \frac{\pi}{2\|\gamma\|_{\mu, E}}.
\]
We may assume that $A$ separates 0 and the infinity. Let $K_1$ and $K_2$ be the two components of $\mathbb{P}^1 \setminus A$ such that $0 \in K_1$ and $\infty \in K_2$. Let

$$r = \max \{|z| \mid z \in K_1\} \quad \text{and} \quad R = \min \{|z| \mid z \in K_2\}.$$ 

By Lemma 5.5 when $\|\gamma\|_{\mu,E}$ is small, $R/r$ is large. Consider the round annulus

$$H = \{z \mid r < |z| < R\}.$$ 

It follows that $H \subset A$ and that the core curve of $H$ is in the same homotopic class as $\gamma$. By Lemma 5.5 and (10), it follows that there is a uniform constant $0 < C < \infty$ such that

$$\text{mod}(H) \geq \text{mod}(A) - C$$

holds provided that $\|\gamma\|_{\mu,E}$ is small. Note that every pair $\{\phi_{\mu}(a_i), \phi_{\mu}(b_i)\}$ is contained either in $\{z \mid |z| < r\}$ or in $\{z \mid |z| > R\}$. Since $\phi_{\mu}$ is holomorphic in $D_i \cup A_i$ and $\{\phi_{\mu}(a_i), \phi_{\mu}(b_i)\} \subset D_i$, it follows from Koebe’s distortion theorem that there is an $1 < M < \infty$, which depends only on $\{D_i\}$ and $\{A_i\}$, such that every $\phi_{\mu}(D_i)$ is contained either in $\{z \mid |z| < Mr\}$ or in $\{z \mid |z| > R/M\}$. By (10) and (11), we have

$$R/M > Mr$$

provided that $\|\gamma\|_{\mu,E}$ is small enough. All of these implies that the annulus

$$H_M = \{z \mid Mr < |z| < R/M\}$$

is contained in $\mathbb{P}^1 \setminus \phi_{\mu}(Q_f)$ provided that $\|\gamma\|_{\mu,E}$ is small enough.

Now the first assertion of the lemma follows if we can show that

$$\phi_{\mu}(\gamma) \subset H_M$$

provided that $\|\gamma\|_{\mu,E}$ is small enough. Suppose this were not true. Then there are two cases. In the first case, there exist two points $z$ and $z'$ such that

1. $z \in K_2$ with $|z| = R$,
2. $|z'| = R/M$,
3. $\phi_{\mu}(\gamma)$ separates $\{0, z'\}$ and $\{z, \infty\}$.

In the second case, there exist two points $z$ and $z'$ such that

1. $|z| = Mr$,
2. $z' \in K_1$ and $|z'| = r$,
3. $\phi_{\mu}(\gamma)$ separates $\{0, z'\}$ and $\{z, \infty\}$.

Suppose we are in the first case. Note that the curve $\phi_{\mu}(\gamma)$ separates $A$ into two sub-annuli such that the modulus of each of them is equal to $\text{mod}(A)/2$. But on the other hand, the outer one separates $\{0, z'\}$ and $\{z, \infty\}$, and thus by Lemma 5.5 its modulus has an upper bound depending only on $M$. By (11) this is impossible when $\|\gamma\|_{\mu,E}$ is small enough. The same argument can be used to get a contradiction in the second case. This proves the first assertion of the Lemma.
Now let us prove the second assertion. Let $l$ denote the hyperbolic length of the core curve of $H_M$ with respect to the hyperbolic metric of $H_M$. Since $H_M \subset \mathbb{P}^1 \setminus \phi_{\mu}(Q_f)$ when $\|\gamma\|_{\mu,E}$ is small enough, it follows that $l > \|\gamma\|_{\mu,Q_f}$. Thus we have
\[
\text{mod}(H_M) = \frac{\pi}{2l} < \frac{\pi}{2\|\gamma\|_{\mu,Q_f}}.
\]
From (10) and (11), there is a constant $0 < C' < \infty$ such that
\[
\text{mod}(H_M) \geq \frac{\pi}{2\|\gamma\|_{\mu,E}} - C'
\]
holds provided that $\|\gamma\|_{\mu,E}$ is small enough. Thus we have
\[
\frac{\pi}{2\|\gamma\|_{\mu,Q_f}} \leq \frac{\pi}{2\|\gamma\|_{\mu,E}} \leq \frac{\pi}{2\|\gamma\|_{\mu,Q_f}} + C'.
\]
The second assertion follows. \qed

6. From Bounded geometry to strictly contracting

The main purpose of this section is to prove that bounded geometry implies the strict contracting property of the operator $\sigma_f : T_f \to T_f$. Let us first prove a technical lemma.

Lemma 6.1. Let $H = \{ z \mid 1 < |z| < R \}$ be an annulus. Let $F_n(w)$ be a sequence of integrable and holomorphic functions defined on $H$ such that
\[
(12) \quad \int_H |F_n(w)| dw \wedge d\overline{w} \to 0 \quad \text{as} \quad n \to \infty.
\]
Then for any $1 < r < R$,
\[
\int_{|w|=r} |F_n(w)| dw \to 0 \quad \text{as} \quad n \to \infty.
\]
Proof. Let $1 < r < R$ be fixed. Take $\delta > 0$ such that $1 + \delta < r < R - \delta$. Let $C(r, \delta) = \min\{r - 1 - \delta, R - \delta - r\}$. It follows that $C(r, \delta) > 0$. For any $\epsilon > 0$, by (12), there is an $N$ such that for every $n > N$, there exist $1 < R_1 < 1 + \delta$ and $R - \delta < R_2 < R$, such that
\[
\int_{|z|=R_1} |F_n(z)| dz < \epsilon
\]
and
\[
\int_{|z|=R_2} |F_n(z)| dz < \epsilon.
\]
For $|w| = r$, by Cauchy formula, we have
\[
|F_n(w)| \leq \left| \frac{1}{2\pi i} \int_{T_{R_1} \cup T_{R_2}} \frac{F_n(z)}{z - w} dz \right|.
\]
Note that $|z - w| \geq C(r, \delta)$ for $|w| = r$ and $z \in T_{R_1} \cup T_{R_2}$. This implies that

$$|F_n(w)| \leq \frac{\epsilon}{\pi C(r, \delta)}$$

holds for all $|w| = r$ and $n > N$. It follows that for all $n > N$,

$$\int_{|w|=r} |F_n(w)||dw| \leq \frac{2\epsilon r}{C(r, \delta)}.$$

The Lemma follows. □

For a Beltrami coefficient $\mu$ defined on $S^2 \setminus Q_f$, we use $\tilde{\mu}$ to denote $f^*(\mu)$.

**Lemma 6.2.** Let $b > 0$. Then there is a constant $0 < a < 1$ depending only on $b$ such that if both $[\mu]$ and $[\tilde{\mu}]$ belong to $T_{f,b}$, then

$$\int_{\cup_D(\cup A_i)} |\tilde{q}(w)|dw^2 \geq a$$

where $\tilde{q}(w)dw^2$ is any integrable holomorphic quadratic differential defined on $\mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)$ with

$$\int_{\mathbb{P}^1 \setminus \phi_{\tilde{\mu}}(Q_f)} |\tilde{q}(w)|dw^2 \Delta dw = 1.$$

**Proof.** Let us prove it by contradiction. By using a Möbius transformation which fixes 0 and 1, and maps $\phi_{\mu_n}(a_1)$ to the infinity, we may assume that $\infty \in D_1$. Since $\tilde{\mu} \in T_{f,b}$, such Möbius transformation lies in a compact family and therefore the assumption does not affect the validity of the proof.

Now let us suppose that there exist a sequence of pairs $(\tilde{\mu}_n, \mu_n)$ in $T_{f,b}$ and a sequence of holomorphic quadratic differentials $\tilde{q}_n$ over $\mathbb{P}^1 \setminus \phi_{\tilde{\mu}_n}(Q_f)$ such that

$$\int_{\mathbb{P}^1 \setminus \phi_{\tilde{\mu}_n}(Q_f)} |\tilde{q}_n(w)|dw^2 \Delta dw = 1,$$

and

$$\int_{\cup_D(\cup A_i)} |\tilde{q}_n(w)|dw^2 \Delta dw \to 0 \quad \text{as} \quad n \to \infty.$$

By Lemma 2.1 $f(\cup_i A_i) \subset \cup_i D_i$ and $f$ is holomorphic in $\overline{D_i} \cup A_i$. This, together with the fact that $\phi_{\mu_n}$ is holomorphic on $\cup_i D_i$, implies that $\phi_{\tilde{\mu}_n}$ is holomorphic and thus univalent on $\cup_i (\overline{D_i} \cup A_i)$.

Note that every ring $A_i$ is holomorphically isomorphic to some annulus

$$H_i = \{ z \mid 1 < |z| < R_i \}.$$

Let $\Phi_i : H_i \to A_i$ be a holomorphic isomorphism and let $T_r$ denote the circle $\{ z \mid |z| = r \}$. We claim that for every $1 < r < R_i$,

$$\int_{\phi_{\tilde{\mu}_n}(\Phi_i(T_r))} |\tilde{q}_n(w)||dw| \to 0 \quad \text{as} \quad n \to \infty.$$
In fact, from (14), we have
\[
\int_{H_i} |\tilde{q}_n((\phi_{\tilde{\mu}_n} \circ \Phi_i)(z))| |(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)|^2 \, dz \wedge d\sigma \to 0 \quad \text{as} \quad n \to \infty.
\]
By Lemma 6.1, we have
\[
\int_{T_r} |\tilde{q}_n((\phi_{\tilde{\mu}_n} \circ \Phi_i)(z))| |(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)|^2 |dz| \to 0 \quad \text{as} \quad n \to \infty.
\]
Since \(\phi_{\tilde{\mu}_n} \circ \Phi_i\) is univalent on \(H_i\), it follows from Koebe’s \(1/4\)-theorem that for every \(1 < r < R_i\), there is a \(C > 1\) depending only on \(r, R_i, b\) such that
\[
1/C < |(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)| < C \quad \text{holds for all} \quad z \in T_r.
\]
We thus have
\[
\int_{T_r} |\tilde{q}_n((\phi_{\tilde{\mu}_n} \circ \Phi_i)(z))| |(\phi_{\tilde{\mu}_n} \circ \Phi_i)'(z)|^2 |dz| \to 0 \quad \text{as} \quad n \to \infty.
\]
This implies (13) and the claim has been proved. Now for every \(A_i\), take an arbitrary \(1 < r_i < R_i\) and let
\[
\gamma_{i,n} = (\phi_{\tilde{\mu}_n} \circ \Phi_i)(T_{r_i}).
\]
For every \(n\), Let \(R_n\) denote the component of \(P^1 \setminus \cup_i \gamma_{i,n}\) such that
\[
\partial R_n = \cup_i \gamma_{i,n}.
\]
Recall that \(P_1 = \{z_j\}\) and \(P'_f = \{a_i\}\) are both finite sets and each \(\tilde{q}_n = \tilde{q}_n(w)dw^2\) has at most simple poles at the points in \(\{\phi_{\tilde{\mu}_n}(z_j)\}\). This implies that one can write
\[
\tilde{q}_n(w) = \sum_j \frac{b_{j,n}}{w - \phi_{\tilde{\mu}_n}(z_j)} + g_n(w)
\]
where \(g_n(w)\) is a holomorphic function on \(P^1 \setminus \phi_{\tilde{\mu}_n}(D_f)\).

Since \(\tilde{\mu}_n \in T_{f,b}\), it follows by taking a subsequence if necessary, that we can assume that for every \(a_i\), the sequence
\[
a_{i,n} = \phi_{\tilde{\mu}_n}(a_i)
\]
converges to a point \(e_i\) with respect to the spherical distance as \(n\) goes to \(\infty\).
Since \(\phi_{\tilde{\mu}_n}\) is holomorphic in \(D_i \cup A_i\), similarly, we can assume that for every \(D_i\), the sequence
\[
D_{i,n} = \phi_{\tilde{\mu}_n}(D_i)
\]
converges to a topological disk \(E_i\) with respect to the Hausdorff metric. It follows that each \(E_i\) contains a round disk of radius \(b\) centered at \(e_i\). Note that by taking each \(A_i\) thinner, we may assume that \(\phi_{\tilde{\mu}_n}\) is univalent in a
larger disk containing \( D_i \cup A_i \) in its interior. So by taking a subsequence if necessary, we can also assume that
\[
A_{i,n} = \phi_{\tilde{\mu}_n}(A_i)
\]
converges to a topological annulus \( B_i \) with respect to the Hausdorff metric. It is clear that
\[
\text{mod}(B_i) = \text{mod}(A_i).
\]
Note that \( \gamma_{i,n} = \left( \phi_{\tilde{\mu}_n} \circ \Phi_i \right)(T_{r_i}) \). Since \( \phi_{\tilde{\mu}_n} \circ \Phi_i \) maps \( H_i \) univalently into \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \) and since \( \tilde{\mu}_n \in T_{f,b} \), it follows again by taking a subsequence if necessary, that we may assume that \( \phi_{\tilde{\mu}_n} \circ \Phi_i \) converges to some univalent function \( \Lambda_i \) defined on \( H_i \), and moreover,
\[
(20) \quad (\phi_{\tilde{\mu}_n} \circ \Phi_i)(z) \to \Lambda_i(z) \text{ uniformly in any compact set of } H_i.
\]
Let \( \gamma_i = \Lambda_i(T_{r_i}) \).

It is not difficult to see that every \( \gamma_i \) is a real analytic and simple closed curve which is homotopic to the core curve of \( B_i \).

Again by taking a subsequence if necessary, we may assume that as \( n \to \infty \), for every \( z_j \in P_1 \),
\[
w_{j,n} = \phi_{\tilde{\mu}_n}(z_j)
\]
converges to some \( w_j \) in the spherical distance. It is important to note that the objects in \( \{E_i\} \) and \( \{w_j\} \) still satisfy the bounded geometry properties in Definition 5.1. Let
\[
R = \mathbb{P}^1 \setminus (\bigcup_i E_i \cup \{w_j\})
\]
Since \( g_n(w) \) is a holomorphic function on \( \mathbb{P}^1 \setminus \phi_{\tilde{\mu}_n}(Q_f) \), it follows that for any compact set \( W \subset R \), the function \( g_n(w) \) is defined on \( W \) provided \( n \) is large enough. Moreover, from (18), for any such compact set \( W \), we can always take \( r_i \) close to 1 or \( R_i \) such that
\[
W \subset R_n.
\]
For any \( w \in W \), from (19) and Cauchy formula, we have
\[
g_n(w) = \frac{1}{2\pi i} \int_{\cup_i \gamma_i,n} \frac{g_n(\xi)}{\xi - w} d\xi
\]
\[
= \frac{1}{2\pi i} \int_{\cup_i \gamma_i,n} \frac{\hat{q}_n(\xi)}{\xi - w} d\xi - \frac{1}{2\pi i} \sum_j \int_{\cup_i \gamma_i,n} \frac{b_{j,n}}{(\xi - w_j,n)(\xi - w)} d\xi
\]
Note that by assumption \( \infty \in D_1 \) and hence \( \infty \notin R_n \). It follows that
\[
\frac{b_{j,n}}{(\xi - w_j,n)(\xi - w)}
\]
is holomorphic in $R_n$ and the residues at the two simple poles are equal to each other. It follows that its integral along $\cup \gamma_i,n$ is zero. We thus have

$$g_n(w) = \frac{1}{2\pi i} \int_{\cup \gamma_i,n} \tilde{q}_n(\xi) \frac{d\xi}{\xi - w}.$$ 

By (15) and the fact that $d(W, \cup \gamma_i,n) > 0$, it follows that $g_n(w) \to 0$ uniformly in $W$ as $n \to \infty$. In particular, since $\cup \gamma_i,n$ is a compact subset of $\mathcal{R}$, it follows that $g_n(w) \to 0$ uniformly for $w \in \cup \gamma_i,n$. This, together with (15) and (19), implies

$$\int_{\cup \gamma_i,n} \left| \sum_j \frac{b_{j,n}}{w - w_{j,n}} \right| dw = 0, \quad n \to \infty.$$ 

We claim that $b_{j,n} \to 0$ as $n \to \infty$ for each $j$. Let us prove the claim by contradiction. Let $\beta_n = \max_j \{ |b_{j,n}| \}$. By taking a subsequence we may assume that there is an $\epsilon > 0$ such that $\beta_n \geq \epsilon$ for all $n \geq 0$. Let

$$h_{j,n} = b_{j,n}/\beta_n.$$ 

Then $\max_j \{ |h_{j,n}| \} = 1$. By (21), we have

$$\int_{\cup \gamma_i,n} \left| \sum_j \frac{h_{j,n}}{w - w_{j,n}} \right| dw = 0 \quad n \to \infty.$$ 

By taking a convergent subsequence again, we may assume that every $h_{j,n}$ converges to a number $h_j$ as $n$ goes to infinity. We thus have

$$\max_j \{ |h_j| \} = 1.$$ 

From (20) and (22), we have

$$\int_{\cup \gamma_i} \left| \sum_j \frac{h_j}{w - w_j} \right| dw = 0.$$ 

This implies that

$$\sum_j \frac{h_j}{w - w_j} = 0 \quad \text{for all } w \in \cup \gamma_i \text{ and thus equal to zero everywhere.}$$

Since all $w_j$ are distinct with each other, it follows by computing the residue at each $w_j$ that all $h_j$ are equal to zero. This contradicts with (23) and the claim has been proved.

Since $g_n(z) \to 0$ uniformly on any compact set of $\mathcal{R}$ and $b_{j,n} \to 0$ as $n \to \infty$ for every $j$, it follows from (19) that

$$\int_{R_n} |\tilde{q}_n(w)| dw \wedge \overline{dw} \to 0 \quad n \to \infty.$$
This, together with (14), implies
\[\int_{\mathbb{P}^1 \setminus \Phi_n(Q_f)} |\tilde{q}_n(w)| dw \wedge dw \to 0 \text{ as } n \to \infty.\]
This contradicts with the assumption (13) and completes the proof of the lemma.

By Propositions 4.1, 4.2 and Lemma 6.2, we have

**Corollary 6.1.** Let \( b > 0 \). Then there is a constant \( 0 < \delta < 1 \) depending only on \( b \) such that
\[
\|d\sigma_f|_{\tau}\| \leq \delta
\]
for all \( \tau \in T_{f,b} \).

Given any \([\mu_0] \in T_f\). Let \( [\mu_n] = \sigma_f^n([\mu_0]) = [(f^*)^n\mu_0] \) for \( n \geq 0 \).

**Lemma 6.3.** Suppose that there exist a \( b > 0 \) and a point \([\mu_0] \in T_f\) such that \( \{[\mu_n]\}_{n=0}^{\infty} \subset T_{f,b} \). Then \( \sigma_f \) has a unique fixed point in \( T_f \).

**Proof.** From Corollary 6.1 and Lemma 3.1, it follows that \( \{[\mu_n]\}_{n=0}^{\infty} \) is a Cauchy sequence. Since \( T_f \) is complete, \([\mu_n]\) converges to a limit point \([\mu]\) in \( T_f \), that is,
\[
\lim_{n \to \infty} [\mu_n] = [\mu].
\]
It follows that \( \sigma_f([\mu]) = [\mu] \). The uniqueness of the fixed point follows also from Corollary 6.1. \( \square \)

7. No Thurston obstruction implies bounded geometry

**Lemma 7.1.** Suppose that \( f \) has no Thurston obstructions. Then there is an integer \( k > 0 \) such that for every \( f \)-stable multi-curve \( \Gamma = \{\gamma_i\} \) with \( \gamma_i \subset S^2 \setminus Q_f \) and the associated linear transformation matrix \( A_{\Gamma} \), we have
\[
\max_j \sum_i b_{i,j} < 1/2
\]
where \( A_k^k = (b_{ij}) \).

**Proof.** Let \( \Gamma = \{\gamma_i\} \) be a \( f \)-stable multi-curve with \( \gamma_i \subset S^2 \setminus Q_f \). It is clear that the number of the elements in \( \Gamma \) has an upper bound which depends only on \#(E). This implies that there can be only finitely many distinct \( A_{\Gamma} \). The lemma follows. \( \square \)

Let \( Z \subset S^2 \) be a subset with \#(Z) \( \geq 4 \) and \( \gamma \subset S^2 \setminus Z \) be a non-peripheral simple closed curve. For \([\mu]\) \( \in T_f \), define
\[
w_Z(\gamma, [\mu]) = -\log \|\gamma\|_{\mu,Z}.
\]
By using the same argument as in the proof of Proposition 7.2 of [5], we have
Lemma 7.2. Let \( Z \subset S^2 \) be a subset with \( \#(Z) \geq 4 \) and \( \gamma \subset S^2 \setminus Z \) be a non-peripheral simple closed curve. Then the function
\[
[\mu] \to w_Z(\gamma, [\mu]) : T_f \to \mathbb{R}
\]
is Lipschitz with Lipschitz constant 2.

Recall that \( E = P_1 \cup \cup_i \{a_i, b_i\} \). Let \([\mu] \in T_f\) and \( b \) be a real number. Define
\[
\Gamma^b_{\mu} = \{ \gamma \mid \gamma \text{ is a } (\mu, E)\text{-simple closed geodesic with } \wedge_{E}(\gamma, [\mu]) \geq b \},
\]
and
\[
L_{\mu} = \{ \wedge_{E}(\gamma, [\mu]) \mid \gamma \text{ is a } (\mu, E)\text{-simple closed geodesic} \}.
\]

Lemma 7.3. There exists an \( A > -\log \log \sqrt{2} \) such that for any \([\mu] \in T_f\) and any real numbers \( a < b \), if
1. \( a > A \),
2. \( b - a \geq \log d + 2d_T([\mu], [f^*\mu]) + 1 \),
3. \( [a, b] \cap L_{\mu} = \emptyset \),
4. \( \Gamma^b_{\mu} \neq \emptyset \),
then \( \Gamma^b_{\mu} \) is a \( f \)-stable multi-curve in \( S^2 \setminus Q_f \).

Proof. Let \( \gamma \in \Gamma^b_{\mu} \). By the first assertion of Lemma 5.6, \( \gamma \) is a non-peripheral and simple closed curve in \( S^2 \setminus Q_f \) provided that \( A \) is big and thus \( \|\gamma\|_{\mu, E} \) is small. By the second assertion of Lemma 5.6 we have
\[
\wedge_{Q_f}(\gamma, [\mu]) > \wedge_{E}(\gamma, [\mu]) - 1
\]
provided that \( A \) is big and thus \( \|\gamma\|_{\mu, E} \) is small. Now suppose that \( \gamma' \) is a non-peripheral component of \( f^{-1}(\gamma) \). Since \( f \) is a degree \( d \) branched covering map of the sphere, it follows that
\[
\wedge_{f^{-1}(Q_f)}(\gamma', [f^*\mu]) \geq \wedge_{Q_f}(\gamma, [\mu]) - \log d.
\]
Since \( E \subset f^{-1}(Q_f) \), it follows that
\[
\wedge_{E}(\gamma', [f^*\mu]) > \wedge_{f^{-1}(Q_f)}(\gamma', [f^*\mu]).
\]
By Lemma 7.2 we have
\[
\wedge_{E}(\gamma', [\mu]) \geq \wedge_{E}(\gamma', [f^*\mu]) - 2d_T([\mu], [f^*\mu]).
\]
This implies that
\[
\wedge_{E}(\gamma, [\mu]) - \wedge_{E}(\gamma', [\mu]) < \log d + 2d_T([\mu], [f^*\mu]) + 1 \leq b - a.
\]
Since \( \wedge_{E}(\gamma, [\mu]) > b \) and \( [a, b] \cap L_{\mu} = \emptyset \), it follows that \( \wedge_{E}(\gamma', [\mu]) > b \). This implies that \( \gamma' \) is homotopic to some element in \( \Gamma^b_{\mu} \). The Lemma follows. \( \square \)

Let \( k \geq 0 \) be the integer in Lemma 7.1. Let
\[
P_2 = E \cup f^k(E) \cup \bigcup_{1 \leq j \leq k} f^j(\Omega_f).
\]
Lemma 7.4. There exists an $\epsilon_0 > 0$ which is independent of $\mu$ such that for any $(\mu, P_2)$-simple closed geodesic $\eta$, if $\|\eta\|_{\mu, P_2} \leq \epsilon_0$, then there is a $(\mu, E)$-simple closed geodesic $\gamma$ such that $\eta$ is homotopic to $\gamma$ in $S^2 \setminus P_2$.

Proof. Suppose $\eta$ is not homotopic to any $(\mu, E)$-simple closed geodesic in $S^2 \setminus P_2$. Then there is at least one holomorphic disk $D_i$, such that $\gamma$ separates the points in $D_i \cap P_2$. Let $x, y \in D_i \cap P_2$ which are separated by $\gamma$. Let $z \in P_2 \setminus D_i$. Let $\phi : S^2 \to \mathbb{P}^1$ be the homeomorphism which solves the Beltrami equation given by $\mu$ and which maps $x$, $y$, and $z$ respectively to 0, 1, and the infinity. Then there are two cases.

In the first case, $\phi(\gamma)$ is contained in $\phi(D_i \cup A_i)$. Note that $A_i$ is the shielding ring attached to the outside of $D_i$. Then $\phi(\gamma)$ must enclose the $\phi$-images of at least two points in $D_i \cap P_2$. Since $\phi$ is univalent in $D_i \cup A_i$, it follows from Koebe’s distortion theorem that there exist $R > 0$ and $D > 0$ independent of $\mu$ such that $\phi(\gamma) \cap \{z \mid |z| \leq R\} \neq \emptyset$ and the Euclidean diameter of $\phi(\gamma)$ is greater than $D$. This, together with Koebe’s distortion theorem, implies that the hyperbolic length of $\phi(\gamma)$ in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and thus $\|\gamma\|_{\mu, P_2}$, has a positive lower bound independent of $\mu$.

In the second case, $\phi(\gamma)$ is not contained in $\phi(D_i \cup A_i)$. Since $\phi(\gamma)$ separates $\phi(x)$ and $\phi(y)$, it follows that $\phi(\gamma)$ must cross through the annulus $\phi(A_i)$. By Koebe’s distortion theorem, the annulus $\phi(A_i)$ has definite thickness. This again implies that there exist $R > 0$ and $D > 0$ independent of $\mu$ such that $\phi(\gamma) \cap \{z \mid |z| \leq R\} \neq \emptyset$ and the Euclidean diameter of $\phi(\gamma)$ is greater than $D$. Therefore, the hyperbolic length of $\phi(\gamma)$ in $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, and thus $\|\gamma\|_{\mu, P_2}$, has a positive lower bound independent of $\mu$. The proof of the lemma is completed. \hfill \Box

Note that

\begin{equation}
(26) \quad f^k : S^2 \setminus f^{-k}(P_2) \to S^2 \setminus P_2
\end{equation}

is a covering map of degree $d^k$. Let $A > -\log \log \sqrt{2}$ be the constant in Lemma 7.3.

Lemma 7.5. Let $B > A$. Then there exists a constant $M > 0$ depending only on the numbers $k, B, \#(E), \epsilon_0$, and the degree $d$ of $f$, such that for any $[\mu] \in T_f$ and any real numbers $a < b$, if

1. $A < a < B$,
2. $b - a \geq \log d + 2d r([\mu], [f^*\mu]) + 1$,
3. $[a, b] \cap L_\mu = \emptyset$,
4. $F_\mu^b \neq \emptyset$,

then

$$
\sum_{\gamma \in F_\mu^b} \frac{1}{\|\gamma\|_{\nu, E}} \leq \frac{1}{2} \sum_{\gamma \in F_\mu^a} \frac{1}{\|\gamma\|_{\mu, E}} + M,
$$

where $\nu = (f^k)^*(\mu)$ and $k \geq 0$ is the integer in Lemma 7.1.
Proof. By Lemma 7.3, $\Gamma^b_\mu$ is a $f$-stable multi-curve in $S^2 \setminus Q_f$. For each $\gamma_j \in \Gamma^b_\mu$, let $\gamma_{i,j,\alpha}$ be any component of $f^{-k}(\gamma_j)$ homotopic to $\gamma_i$ in $S^2 \setminus Q_f$. Then $\gamma_{i,j,\alpha}$ is also homotopic to $\gamma_i$ in $S^2 \setminus E$.

Let $g = \phi_\mu \circ f^k \circ \phi_\nu^{-1}$. Then $g$ is a rational map. It follows from (26) that $g: \mathbb{P}^1 \setminus \phi_\nu(f^{-k}(P_2)) \to \mathbb{P}^1 \setminus \phi_\mu(P_2)$ is a holomorphic covering map, and therefore,

$$
\| \gamma_{i,j,\alpha} \|_{\nu,f^{-k}(P_2)} = d_{i,j,\alpha} \| \gamma_j \|_{\mu,P_2}
$$

where $d_{i,j,\alpha} \leq d^k$ is the degree of $f^k: \gamma_{i,j,\alpha} \to \gamma_j$.

Thus

$$
\sum_\alpha \frac{1}{\| \gamma_{i,j,\alpha} \|_{\nu,f^{-k}(P_2)}} = \left( \sum_\alpha \frac{1}{d_{i,j,\alpha}} \right) \frac{1}{\| \gamma_j \|_{\mu,P_2}} = b_{i,j} \frac{1}{\| \gamma_j \|_{\mu,P_2}}
$$

Since $E \subset P_2$ by (25), it follows that $\| \gamma_j \|_{\mu,P_2} > \| \gamma_j \|_{\mu,E}$, and therefore

$$
\frac{1}{\| \gamma_j \|_{\mu,P_2}} < \frac{1}{\| \gamma_j \|_{\mu,E}}.
$$

This implies

$$
(27) \quad \sum_\alpha \frac{1}{\| \gamma_{i,j,\alpha} \|_{\nu,f^{-k}(P_2)}} < b_{i,j} \frac{1}{\| \gamma_j \|_{\mu,E}}
$$

Note that $E \subset f^{-k}(P_2)$ by (25). Let $p$ denote the number of the points in $f^{-k}(P_2) \setminus E$. It follows from (25) that there is a constant

$$
0 < C(k,d,\#(E)) < \infty
$$

depending only on $d$, $k$, and $\#(E)$ such that $p \leq C(k,d,\#(E))$.

Now we claim that for any $(\nu,f^{-k}(P_2))$-simple closed geodesic $\gamma$ which is homotopic to $\gamma_i$ in $S^2 \setminus E$, either $\gamma$ is homotopic to some $\gamma_{i,j,\alpha}$ in $S^2 \setminus f^{-k}(P_2)$, or

$$
\| \gamma \|_{\nu,f^{-k}(P_2)} > \min \{ e^{-B}, \epsilon_0 \}.
$$

Let us prove the claim. In fact, if $\gamma$ is not homotopic in $S^2 \setminus f^{-k}(P_2)$ to some $\gamma_{i,j,\alpha}$, then $f^k(\gamma)$ is a $(\mu,P_2)$-simple closed geodesic which is not homotopic to any $\gamma_j$ in $S^2 \setminus P_2$. There are two cases. In the first case, $f^k(\gamma)$ is homotopic in $S^2 \setminus P_2$ to some $(\mu,E)$-simple closed geodesic $\xi$ which does not belong to $\Gamma^b_\mu$. By the assumption that $L_\mu \cap [a,b] = \emptyset$, we have

$$
\| f^k(\gamma) \|_{\mu,P_2} > \| f^k(\gamma) \|_{\mu,E} = \| \xi \|_{\mu,E} > e^{-a} > e^{-B}.
$$

In the second case, $f^k(\gamma)$ is not homotopic in $S^2 \setminus P_2$ to any $(\mu,E)$-simple closed geodesic. By Lemma 7.4 we have

$$
\| f^k(\gamma) \|_{\mu,P_2} > \epsilon_0.
$$
We thus have
\[ \| \gamma \|_{\nu, f^{-k}(P_2)} \geq \| f^k(\gamma) \|_{\mu, P_2} > \min\{e^{-B}, \epsilon_0\}. \]

Now from the left hand of the inequality given by (c) in Theorem 7.1 of [5], we have
\[ \frac{1}{\| \gamma \|_{\nu, E}} \leq \sum_j \sum_{\alpha} \frac{1}{\| \gamma_{i,j,\alpha} \|_{\nu, f^{-k}(P_2)}} + \frac{2}{\pi} + \frac{C(k, d, \#(E)) + 1}{\min\{e^{-B}, \epsilon_0\}}. \]
Let
\[ M' = \frac{2}{\pi} + \frac{C(k, d, \#(E)) + 1}{\min\{e^{-B}, \epsilon_0\}}. \]
Thus
\[ \sum_{\gamma \in \Gamma_\mu} \frac{1}{\| \gamma \|_{\nu, E}} \leq \sum_j \sum_{\alpha} \frac{1}{\| \gamma_{i,j,\alpha} \|_{\nu, f^{-k}(P_2)}} + K M'. \]
where \( K \) is the number of the curves in \( \Gamma \) which is bounded above by \( \#(E) - 3 \). Let
\[ M = (\#(E) - 3) M'. \]
By (27), we have
\[ \sum_{\gamma \in \Gamma_\mu} \frac{1}{\| \gamma \|_{\nu, E}} \leq \sum_j \left( \sum_i b_{ij} \right) \frac{1}{\| \gamma_j \|_{\mu, E}} + M \leq \frac{1}{2} \sum_{\gamma \in \Gamma_\mu} \frac{1}{\| \gamma \|_{\mu, E}} + M. \]
This completes the proof of the Lemma.

The following is a technical lemma from Calculus.

**Lemma 7.6.** Let \( b_0 > 1, \ c_0, M_0 > 0, \) and integer \( m_0 > 1 \) be given. Then for any sequence \( \{x_n\}_{n=0}^\infty \) of positive numbers satisfying
1. \( x_0 \leq c_0, \)
2. \( x_{n+1}/x_n \leq b_0, \)
3. \( \text{if } x_n \geq M_0, \text{ then } x_{n+m_0} \leq x_n, \)
one has
\[ x_n \leq \max\{b_0^{m_0-1} c_0, b_0^{m_0} M_0\}, \forall n \geq 0. \]

**Proof.** Let \( C = \max\{b_0^{m_0-1} c_0, b_0^{m_0} M_0\}. \) It is sufficient to prove that
\[ x_{i+l m_0} \leq C \]
for all \( 0 \leq i \leq m_0 - 1 \) and \( l \geq 0. \) Take an arbitrary integer \( 0 \leq i \leq m_0 - 1. \) Let us prove that
\[ x_{i+l m_0} \leq C \]
for all \( l \geq 0 \) by induction. For \( l = 0, \) we have
\[ x_i \leq b_0^i x_0 \leq b_0^{m_0-1} c_0 \leq C. \]
Now assume that
\[ x_{i+km_0} \leq C \]
(28)
for some integer \( k \geq 0 \). Let us prove that

\[ x_{i+(k+1)m_0} \leq C. \]

In fact, there are two cases by assumption (28). In the first case, \( x_{i+km_0} < M \).

In this case, we have

\[ x_{i+(k+1)m_0} \leq b_{0m_0} x_{i+km_0} < b_{0m_0} M \leq C. \]

In the second case, \( x_{i+km_0} \geq M \). Then we have

\[ x_{i+(k+1)m_0} \leq x_{i+km_0} \leq C. \]

This proves that \( x_{i+lm_0} \leq C \) for all \( l \geq 0 \). Since this holds for any \( 0 \leq i \leq m_0 - 1 \), the lemma follows.

Lemma 7.7. If \( f \) has no Thurston obstructions, then for any \([\mu_0] \in T_f\), there exists a constant \( b > 0 \) such that for all \( n \geq 1 \),

\[ [\mu_n] \in T_{f,b}, \]

where \( \mu_n = (f^*)^n(\mu_0) \).

Proof. Since \( f \) is holomorphic on \( \cup A_i \) and \( f(\cup A_i) \subset \cup D_i \), it follows that for all \( n \geq 1 \), \( \mu_n(z) = 0 \) on \( \cup A_i \). By Lemma 5.3, it is equivalent to prove that there is a uniform positive lower bound of the length of all the \((\mu_n, E)\)-simple closed geodesics.

Let \( D = d_T([\mu_0], [\mu_1]) \). Then by Lemma 3.1 and Corollary 6.1, we have

\[ d_T([\mu_n], [\mu_{n+1}]) \leq D \quad \text{for all } n \geq 0. \]

Let \( K = \#(E) - 3 \geq 1 \) and \( k \geq 1 \) be the integer in Lemma 7.1. Let \( l_0 \geq 1 \) be the least integer such that

\[ (29) \quad K < 2^{l_0-1}. \]

Now it is sufficient to prove that there exist positive constants \( c_0, M_0 > 0 \), \( b_0 > 1 \), and an integer \( m_0 > 0 \), such that the sequence

\[ x_n = \max_{\gamma} \{ \|\gamma\|^{-1}_{\mu_n, E} \}, \]

where max is taken over all the \((\mu_n, E)\)-simple closed geodesics, satisfies the three conditions in Lemma 7.6.

By Corollary 6.6 of [5], there are at most \( K \) \((\mu_n, E)\)-simple closed geodesics which has hyperbolic length less than \( \log(\sqrt{2} + 1) \). This implies that we can have \( c_0 > 0 \) such that

\[ x_0 \leq c_0. \]

It is the first condition in Lemma 7.6. From Lemma 7.2 we can take \( b_0 = e^{2D} \).

Recall that we use \( d \) to denote the degree of \( f \). Let \( k_0 = \log d + 2D \) and \( m_0 = kl_0 \). Let

\[ (30) \quad k_1 = k_0 + 4m_0D + 1. \]
In particular, $k_1 > \log d + 2D + 1$. Let $A > -\log \log(\sqrt{2} + 1)$ be the constant in Lemma 7.3. In Lemma 7.3, take
\[ B = A + (K + 1)k_1 \]
and let $M$ denote the corresponding constant there. Let
\[ M_0 = \max\{e^B, 2^{l_0 + 1}M\}. \]

It remains to prove that if $x_n > M_0$, then $x_{n+m_0} < x_n$. To see this, suppose that $x_n > M_0$. It follows that there is a $(\mu_n, E)$-simple close geodesic such that $w_E(\gamma, [\mu_n]) > B$. Then by the choice of the numbers $k_1$ and $B$, and the fact that there are at most $K (\mu_n, E)$-simple closed geodesics which have hyperbolic length less than $\log(\sqrt{2} + 1)$, one can take an interval $[a, b]$ such that
1. $A < a < b < B$,
2. $b - a = k_1$,
3. $[a, b] \cap L_{\mu_n} = \emptyset$.

It follows that $\Gamma_{\mu_n}^b \neq \emptyset$ and therefore is a $f$-stable multicurve by Lemma 7.3.

Now for each $i = 0, 1, \cdots, l_0$, let
\[ [a_i, b_i] = [a + 2kiD, b - 2kiD]. \]

By Lemma 7.2, the gap condition $b - a = k_1$, and (30), it follows that each family $\Gamma_{\mu_n}^i$, $0 \leq i \leq l_0$, contains the same set of homotopy classes of simple closed curves as $\Gamma_{\mu_n}^b$. Let us simply denote each of them by $\Gamma$. Now for each $0 \leq i \leq l_0 - 1$, let $\mu = \mu_n + ki$ and $\nu = \mu_n + k(i + 1)$, and let $[a_i, b_i]$ be the corresponding gap interval. Then the conditions in Lemma 7.5 are satisfied with the constants $A$ and $B$ given as above. By Lemma 7.5, we have
\[ \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n + k(i + 1), E}} \leq \frac{1}{2} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n + k, E}} + M \]
for $0 \leq i \leq l_0 - 1$. It follows from $m_0 = kl_0$ that
\[ \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n + m_0, E}} \leq \frac{1}{2l_0} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n, E}} + 2M. \]

Since
\[ \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n, E}} \geq x_n > M_0 \geq 2^{l_0 + 1}M, \]
it follows that
\[ M < \frac{1}{2l_0 + 1} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n, E}}. \]

From (31) and (32), we have
\[ \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n + m_0, E}} < \frac{1}{2^{l_0 - 1}} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n, E}}. \]
Since the number of the elements in $\Gamma$ is at most $K$, it follows that

$$\sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n,E}} \leq Kx_n.$$ 

From (29) and (33), we have

$$x_{n+m_0} \leq \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n+m_0,E}} < \frac{1}{2^{l_0-1}} \sum_{\gamma \in \Gamma} \frac{1}{\|\gamma\|_{\mu_n,E}} \leq \frac{K}{2^{l_0-1}} x_n < x_n.$$

□

The Main Theorem now follows from Lemmas 4.3, 6.3, and 7.7.

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