MATCHING COMPLEXES OF SMALL GRIDS

TAKAHIRO MATSUSHITA

ABSTRACT. The matching complex $M(G)$ of a simple graph $G$ is the simplicial complex consisting of the matchings on $G$. It is known that the matching complex $M(G)$ is isomorphic to the independence complex of the line graph $L(G)$.

In this paper, we study the homotopy type of the matching complex of $(n \times 2)$-grid graph $\Gamma_n$. Braun and Hough introduced a family of graphs $\Delta_{m,n}$, which is a generalization of $L(\Gamma_n)$. In this paper, we show that the independence complex of $\Delta_{m,n}$ is a wedge of spheres. This gives an answer to a problem suggested by Braun and Hough.

1. Introduction

A matching on a simple graph $G = (V(G), E(G))$ is a subgraph of $G$ whose maximal degree is at most 1. A matching is identified with its edge set. The matching complex $M(G)$ of $G$ is the simplicial complex whose simplices are the matchings on $G$. In general, it is very difficult to determine the homotopy types of matching complexes (see [3], [8], and [10]). We refer to [7] for a concrete introduction to this subject.

In this paper, we study the homotopy types of the matching complexes of the $(n \times 2)$-grid graphs. For a pair $m$ and $n$ of positive integers, the grid graph $\Gamma(m, n)$ is defined by

$V(\Gamma(m, n)) = \{(i, j) \in \mathbb{Z}^2 \mid 1 \leq i \leq m, 1 \leq j \leq n\}$,

$E(\Gamma(m, n)) = \{(i, j), (i', j') \mid |i' - i| + |j' - j| = 1\}$.

In particular, we write $\Gamma_n$ instead of $\Gamma(n, 2)$.

Jonsson first studied the matching complex of grid graphs in his unpublished paper [6]. Recently, Braun and Hough [4] investigated the matching complex of $\Gamma_n$, and they use discrete Morse theory to derive some homological properties of $M(\Gamma_n)$. In fact, they studied more general simplicial complexes. To state it precisely, we need some preparation.

For a graph $G$, the independence complex $I(G)$ of $G$ is the simplicial complex whose simplices are the independent sets of $G$. The line graph $L(G)$ of $G$ is the graph whose vertex set is $E(G)$, and two distinct edges $e$ and $e'$ of $G$ are adjacent if and only if they have a common endpoint. Then the matching complex $M(G)$ coincides with the independence complex of the line graph $L(G)$. Figure 1 depicts the line graph of $\Gamma_n$. Here $e_i$, $f_i$, and $f'_i$ denote the edges $\{(i, 1), (i + 1, 1)\}$, $\{(i, 2), (i + 1, 2)\}$, and $\{(1, i), (2, i)\}$, respectively.

Key words and phrases. matching complex, independence complex.
For a pair $m$ and $n$ of positive integers, Braun and Hough [4] introduced the graph $\Delta^m_n$, which is a generalization of $L(\Gamma_n)$. The vertex set of $\Delta^m_n$ consists of $e_i$ for $i = 1, \ldots, n$ and $f^k_i$ for $i = 1, \ldots, n - 1$ and $k = 1, \ldots, m$. The adjacent relations are given as follows:

$$f^k_i \sim f^k_{i+1}, (i = 1, \ldots, n), e_i \sim f^k_i \sim e_{i+1}, (i = 1, \ldots, n - 1)$$

Figure 2 depicts the graph $\Delta^4_5$. Clearly, $\Delta^2_n$ and $L(\Gamma_n)$ are isomorphic, and hence $I(\Delta^2_n)$ and $M(\Gamma_n)$ are isomorphic.

Braun and Hough [4] actually studied the independence complexes of $\Delta^m_n$. The purpose of this paper is to determine the homotopy types of the independence complex of $\Delta^m_n$. The following two theorems are the main results in the present paper.

**Theorem 1.1.** $\Delta^1_{2n} \simeq S^{n-1}$ and $\Delta^1_{2n-1} \simeq *$ for $n \geq 1$.

**Theorem 1.2.** For $n \geq 5$ and $m \geq 2$, we have

$$I(\Delta^m_n) \simeq \Sigma^2 I(\Delta^m_{n-3}) \lor \Sigma^m I(\Delta^m_{n-3}) \lor \Sigma^{m+1}(\Delta^m_{n-4}).$$

Here $\Sigma$ denotes the reduced suspension.

**Remark 1.3.** The equation among the Euler characteristics of $I(\Delta^m_n)$ obtained by Theorem 1.2 was known. See Corollary 4.4 of [4].

In particular, we have

$$M(\Gamma_n) \simeq \Sigma^2 M(\Gamma_{n-3}) \lor \Sigma^2 M(\Gamma_{n-3}) \lor \Sigma^3 M(\Gamma_{n-4}).$$
By Theorem 1.2, the homotopy type of $I(\Delta^m_n)$ is recursively determined by $I(\Delta^m_1), \ldots, I(\Delta^m_4)$. In Section 4, we show that all these complexes are wedges of spheres, and hence we have the following theorem:

**Theorem 1.4.** The independence complex $I(\Delta^m_n)$ of $\Delta^m_n$ is a wedge of spheres for positive integers $m$ and $n$. In particular, the matching complex $M(\Gamma_n)$ of $\Gamma_n$ is homotopy equivalent to a wedge of spheres.

In particular, the homology groups of the independence complex of $\Delta^m_n$ has no torsions. This gives an answer to a problem suggested by Braun and Hough (see the end of [4]).

This paper is organized as follows. In Section 2, we review some facts concerning independence complexes. Since Theorem 1.1 is easily deduced from known results, we discuss it in this section. Theorem 1.2 and Theorem 1.4 are proved in Section 3 and Section 4, respectively.

## 2. Preliminaries

We refer to [7] and [9] for fundamental terms and facts concerning simplicial complexes.

For a vertex $v$ of a simple graph $G$, let $N_G(v)$ denote the set of vertices adjacent to $v$. We write $N_G[v]$ to mean $N_G(v) \cup \{v\}$. For a subset $S$ of $V(G)$, the subgraph of $G$ induced by $V(G) \setminus S$ is denoted by $G \setminus S$. In particular, we write $G \setminus v$ instead of $G \setminus \{v\}$.

We first recall the following simple observation of independence complexes (see [1] for example). For a vertex $v$ of $G$, the link of $v$ in $I(G)$ coincides with $I(G \setminus N_G[v])$. Since $I(G) \setminus v = I(G \setminus v)$, we have that $I(G)$ is the mapping cone of the inclusion $I(G \setminus N_G(v)) \hookrightarrow I(G \setminus v)$. Here $I(G) \setminus v$ denotes the subcomplex of $I(G)$ whose simplices are the simplices of $I(G)$ not containing $v$. This observation clearly yields the following proposition:

**Proposition 2.1.** Let $v$ be a vertex of a graph $G$. If the inclusion $I(G \setminus N_G[v]) \hookrightarrow I(G \setminus v)$ is null-homotopic, then we have $I(G) \simeq I(G \setminus v) \vee \Sigma I(G \setminus N_G[v])$.

**Proposition 2.2** (Lemma 2.5 of [5]). Let $v$ and $w$ be a pair of distinct vertices of $G$ with $N_G(v) \subset N_G(w)$. Then the inclusion $I(G \setminus w) \hookrightarrow I(G)$ is a homotopy equivalence.

**Proof.** By the above observation, it suffices to see that $I(G \setminus N_G[w])$ is contractible. But this is clear since $G \setminus N_G[w]$ has an isolated vertex $v$. \qed

![Figure 3](image-url)
Here we give the proof of Theorem 1.1 since it easily follows from Proposition 2.2.

**Proposition 2.3.** If \( n \geq 3 \), then \( I(\Delta^1_n) \simeq \Sigma I(\Delta^1_{n-2}) \).

**Proof.** Since \( N_{\Delta^1_n}(e_n) \subset N_{\Delta^1_{n-1}}(e_{n-1}) \) and \( N_{\Delta^1_n}(e_n) \subset N_{\Delta^1_{n-2}}(f_{n-2}^1) \), we have
\[
I(\Delta^1_n) \simeq I(\Delta^1_n \setminus \{e_{n-1}, f_{n-2}^1\}) = I(\Delta^1_{n-2}) \ast I(K_2) = \Sigma I(\Delta^1_{n-2}).
\]
See Figure 3. \( \square \)

**Proof of Theorem 1.1.** It is clear that \( I(\Delta^1_0) = * \) and \( I(\Delta^1_1) = I(P_3) \simeq S^0 \). Here \( P_3 \) denotes the path graph with 3-vertices. Thus Proposition 2.3 implies Theorem 1.1. \( \square \)

3. **Theorem 1.2**

In this section, we prove Theorem 1.2. Throughout this section, we assume that \( m \) is an integer greater than 1. Suppose \( n \geq 2 \), and put \( X_n = \Delta^m_n \setminus e_{n-1} \). Since \( N_{\Delta^m_n}(e_n) \subset N_{\Delta^m_{n-1}}(e_{n-1}) \), Proposition 2.2 implies the following:

**Lemma 3.1.** For \( n \geq 2 \) and \( m \geq 2 \), we have \( I(\Delta^m_n) \simeq I(X_n) \).

Next we consider the graph \( Y_n = X_n \setminus e_{n-2} \) (see Figure 5).

**Proposition 3.2.** For \( n \geq 4 \) and \( m \geq 2 \), we have \( I(X_n) \simeq I(Y_n) \lor \Sigma^2 I(\Delta^m_{n-3}) \).

**Proof.** We want to apply Proposition 2.1 to the vertex \( e_{n-2} \) of \( X_n \). Thus we need to show that \( I(X_n \setminus N_{X_n}[e_{n-2}]) \simeq \Sigma I(\Delta^m_{n-3}) \) and the inclusion \( I(X_n \setminus N_{X_n}[e_{n-2}]) \hookrightarrow I(X_n \setminus e_{n-2}) = I(Y_n) \) is null-homotopic.

By Figure 5 and Proposition 2.2, it is clear that \( I(X_n \setminus N_{X_n}[e_{n-2}]) \simeq I(\Delta^m_{n-3} \cup K_2) = \Sigma I(\Delta^m_{n-3}) \). To see that the inclusion \( I(X_n \setminus N_{X_n}[e_{n-2}]) \hookrightarrow I(X_n \setminus e_{n-2}) \) is null-homotopic, we first see that the inclusion
\[
I(X_n \setminus (N_{X_n}[e_{n-2}] \cup \{f_{n-1}^1\})) \hookrightarrow I(X_n \setminus e_{n-2}) = I(Y_n)
\]

\[\approx\]

![Figure 4](image-url)

Figure 4.
is a homotopy equivalence. Note that every vertex of $X_n \setminus (N_{X_n}[e_{n-2}] \cup \{f_{n-1}^1\})$ is not adjacent to $f_{n-2}^1$ in $Y_n$. Thus $I(X_n \setminus (N_{X_n}[e_{n-2}] \cup \{f_{n-1}^1\}))$ is contained in the star $\text{st}_{I(Y_n)}(f_{n-2}^1)$. This means that the composite

$$I(X_n \setminus (N_{X_n}[e_{n-2}] \cup \{f_{n-1}^1\})) \xrightarrow{\cong} I(X_n \setminus N_{X_n}[e_{n-2}]) \to I(Y_n)$$

is null-homotopic. Here we use the assumption $m \geq 2$ to show that the first map is a homotopy equivalence. Thus the inclusion $I(X_n \setminus N_{X_n}[e_{n-2}]) \to I(Y_n)$ is null-homotopic, and this completes the proof. □

Next we study the homotopy type of $I(Y_n)$

**Proposition 3.3.** For $n \geq 5$ and $m \geq 2$, we have $I(Y_n) \cong \Sigma^m I(\Delta_{n-3}^m) \vee \Sigma^{m+1} I(\Delta_{n-4}^m)$

*Proof.* We want to apply Proposition 2.1 to the vertex $e_n$ of $Y_n$. Namely, we must show the following:

1. The inclusion $I(Y_n \setminus N_{Y_n}[e_n]) \to I(Y_n \setminus e_n)$ is null-homotopic.
2. The homotopy type of $I(Y_n \setminus N_{Y_n}[e_n])$ is $\Sigma^m I(\Delta_{n-4}^m)$.
3. The homotopy type of $I(Y_n \setminus e_n)$ is $\Sigma^m I(\Delta_{n-3}^m)$.

Define the induced subgraphs $Z_n$, $Z'_n$, and $Z''_n$ of $Y_n$ as follows:

$$Z_n = Y_n \setminus \{(f_{n-4}^i, f_n^i) \mid i = 1, \cdots, m\} \cup \{e_{n-3}\} \cup N[e_n],$$

$$Z'_n = Y_n \setminus \{(f_{n-4}^i, f_n^i) \mid i = 1, \cdots, m\} \cup \{e_{n-3}, e_n\},$$

$$Z''_n = Y_n \setminus (N_{Y_n}[e_{n-3}] \cup \{e_n\})$$

Figure 6 depicts the graphs $Z_n$, $Z'_n$, and $Z''_n$ in the case $m = 4$.

By Proposition 2.2, $I(Y_n \setminus N[e_n])$ is homotopy equivalent to $I(Z_n)$. Clearly, we have $I(Z_n) \cong \Sigma^m I(\Delta_{n-4}^m)$, which implies (2). By Proposition 2.2, the inclusions $I(Z_n) \to I(Z'_n)$ and $I(Z''_n) \to I(Z'_n)$ are homotopy equivalences. Since $I(Z''_n)$ is contained in the star $\text{st}_{I(Y_n)}(e_{n-3})$, we have that the inclusion $I(Z''_n) \to I(Y_n)$ is null-homotopic. It follows from
the commutative diagram

\[ I(Z_n) \xrightarrow{\sim} I(Z'_n) \xleftarrow{\sim} I(Z''_n) \xrightarrow{} I(Y_n) \]

that the inclusion \( I(Z_n) \hookrightarrow I(Y_n) \) is null-homotopic. By the sequence

\[ I(Z_n) \xrightarrow{\sim} I(Y_n \setminus N_{Y_n}[e_{n-3}]) \rightarrow I(Y_n), \]

of inclusions, we have that \( I(Y_n \setminus N_{Y_n}[e_{n-3}]) \hookrightarrow I(Y_n) \) is null-homotopic. This completes the proof of (1).

Finally, we prove (3). By Proposition 2.2, it is easy to see that \( I(Y_n \setminus e_n) \) is homotopy equivalent to \( I(W_n) \) (see Figure 6). Here \( W_n \) is defined by

\[ W_n = Y_n \setminus (\{f_{n-3}^k \mid k = 1, \cdots, m\} \cup \{e_n\}). \]

Clearly, \( I(W_n) \) is homotopy equivalent to \( \Sigma^m I(\Delta^m_{n-3}) \). This completes the proof of (3). \( \square \)

Theorem 1.2 is deduced from Lemma 3.1, Proposition 3.2, and Proposition 3.3.

4. Theorem 1.4

In this section, we prove Theorem 1.4, which asserts that \( I(\Delta^m_n) \) is a wedge of spheres. The case \( m = 1 \) is proved by Theorem 1.1, and hence we assume \( m \geq 2 \) in the rest of this section. It follows from Theorem 1.2 that for \( n \geq 5 \), \( I(\Delta^m_n) \) is a wedge of spheres if both \( I(\Delta^m_{n-3}) \) and \( I(\Delta^m_{n-4}) \) are wedges of spheres. Thus to prove Theorem 1.4, it suffices to show that \( I(\Delta^m_1), \cdots, I(\Delta^m_4) \) are wedges of spheres. This follows from the next proposition.
Proposition 4.1. For $m \geq 2$, the complexes $I(\Delta^m_1), \ldots, I(\Delta^m_n)$ are described as follows:

$I(\Delta^m_1) = *, I(\Delta^m_2) \simeq S^0, I(\Delta^m_3) \simeq S^1 \vee S^{m-1}, I(\Delta^m_4) \simeq S^m$

Proof. Note that $I(\Delta^m_1)$ is a point. It clearly follows from Proposition 2.2 that $I(\Delta^m_2) \simeq I(K_2) = S^0$.

Consider the case of $n = 3$. By Lemma 3.1, we have that $I(\Delta^m_3) \simeq I(X_3)$. Braun and Hough determined the homotopy types of the independence complexes of $X_3$ (see Lemma 3.2 of [4]), but we give an alternative proof of this result for self-containedness. First Proposition 2.2 implies that $I(X_3 \setminus e_3)$ and $I(X_3 \setminus \{e_1, e_3\})$ are homotopy equivalent. Since $X_3 \setminus \{e_1, e_3\}$ is the $m$-copies of $K_2$, we have

$I(X_3 \setminus e_3) \simeq I(X_3 \setminus \{e_1, e_3\}) = S^{m-1}$.

On the other hand, applying Proposition 2.2 again, we have that $I(X_3 \setminus N_{X_3}[e_3])$ and $I(K_2) = S^0$ are homotopy equivalent. Since every map from $S^0$ to $S^{m-1}$ is null-homotopic, the inclusion $I(X_3 \setminus N_{X_3}[e_3]) \hookrightarrow I(X_3 \setminus e_3)$ is null-homotopic. Thus Proposition 2.1 implies $I(X_3) = S^1 \vee S^{m-1}$.

Finally we consider the case $n = 4$. By Proposition 3.2 and $I(\Delta^m_1) = *$, we have that $I(X_4) \simeq I(Y_4)$. By Proposition 2.2, $I(Y_4 \setminus e_4)$ is homotopy equivalent to the independence complex of the disjoint union of one isolated vertex and $m$-copies of $K_2$, and hence contractible. In particular, the inclusion $I(Y_4 \setminus N_{Y_4}[e_4]) \hookrightarrow I(Y_4 \setminus e_4)$ is null-homotopic, and hence Proposition 2.1 implies $I(Y_4) \simeq \Sigma I(Y_4 \setminus N_{Y_4}[e_4])$. Since $Y_4 \setminus N_{Y_4}[e_4] \cong X_3 \setminus e_3$, we have that $I(Y_4 \setminus N_{Y_4}[e_4]) = S^{m-1}$ by the previous paragraph. Thus we conclude that

$I(\Delta^m_4) \simeq I(Y_4) \simeq \Sigma I(Y_4 \setminus N_{Y_4}[e_4]) = S^m$.

This completes the proof. \qed

References

[1] M. Adamazek, A note on independence complexes of chordal graphs and dismantling, The Electronic Journal of Combinatorics, Volume 24, Issue 11, (2017).

[2] J.A. Barmak, Star clusters in independence complexes of graphs, Adv. Math., 241, 33-57, 2013.

[3] A. Björner, L. Lovász, S.T. Vrečica, Živaljević, Chessboard complexes and matching complexes, Journal of the London Mathematical Society, 49(1), (1994) 25-39

[4] B. Braun, W.K. Hough, Matching and independence complexes related to small grids, The Electronic Journal of Combinatorics, Volume 24, Issue 4, (2017)

[5] Engström, Complexes of directed trees and independence complexes, Discrete Mathematics, 309 (2009) 3299-3309.

[6] J. Jonsson, Matching complexes on grids, http://www.math.kth.se/~jakobj/doc/thesis/grid.pdf, unpublished manuscript

[7] J. Jonsson, Simplicial complexes of graphs, Volume 1928 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2008.

[8] J. Jonsson, More torsion in the homology of matching complex, Experiment. Math. Volume 19, Issue 3, (2010) 363-383.

[9] D.N. Kozlov, Combinatorial Algebraic Topology, volume 21 of Algorithms and Computation in Mathematics, Springer, Berlin, 2008.
[10] J. Shareshian and M. L. Wachs, *Torsion in the matching and chessboard complexes*, Adv. Math. 212 (2007), 525570

Department of Mathematical Sciences, University of the Ryukyu, Nishihara-cho, Okinawa 903-0213, Japan

E-mail address: mtst@sci.u-ryukyu.ac.jp