Research Article

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On new stability results for composite functional equations in quasi-\(\beta\)-normed spaces

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Abstract: In this article, we prove the generalized Hyers-Ulam-Rassias stability for the following composite functional equation:

\[ f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y), \]

where \( f \) maps from a \((\beta, p)\)-Banach space into itself, by using the fixed point method and the direct method. Also, the generalized Hyers-Ulam-Rassias stability for the composite \(s\)-functional inequality is discussed via our results.

Keywords: additive functional equations, composite functional equations, composite \(s\)-functional inequalities, quasi-\(\beta\)-normed spaces

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1 Introduction and preliminaries

The stability problem of functional equations was first originated by Ulam [1] in 1940 as the following question concerning the stability of homomorphisms. Let \((G_1, \ast)\) be a group, \((G_2, \ast)\) be a metric group with the metric \(d\) and \(f: G_1 \rightarrow G_2\) be a mapping such that

\[ d(f(x \ast y), f(x) \ast f(y)) \leq \epsilon \]

for all \(x, y \in G_1\), where \(\epsilon > 0\). Do there exist \(\delta > 0\) and a unique homomorphism \(h: G_1 \rightarrow G_2\) such that

\[ d(f(x), h(x)) \leq \delta \]

for all \(x \in G_1\)? The above problem was partially answered by Hyers [2] in 1941 under the assumption that the function \(f\) maps between two Banach spaces.

In 1978, Rassias [3] considered an unbounded Cauchy difference which generalizes Hyers’s result as the following inequality:

\[ \|f(x + y) - f(x) - f(y)\| \leq \epsilon (\|x\|^p + \|y\|^p) \]

for all \(x, y \in X\), where \(f\) maps from a Banach space \(X\) into a Banach space \(Y\), \(\epsilon > 0\) and \(0 < p < 1\). Moreover, a generalization of this stability theorem was published by Găvruta [4] in 1994 by replacing the un-
bounded Cauchy difference with some condition involving a general control function in the spirit of Rassias approach.

In 1959, Goląb and Schinze [5] introduced one of the most important composite functional equations as follows:

\[ f(x + yf(x)) = f(x)f(y) \]

for all \( x, y \in \mathbb{R} \), where \( f \) maps from \( \mathbb{R} \) into \( \mathbb{R} \), which is called the Goląb-Schinzel functional equation. One of the solutions of the Goląb-Schinzel functional equation is a function \( f : \mathbb{R} \rightarrow \mathbb{R} \) defined by \( f(x) = 1 + ax \) for all \( x \in \mathbb{R} \), where \( a \) is a constant. In 2005, Chudziak [6] considered the stability problem concerning the following Goląb-Schinzel difference:

\[ f(x + yf(x)) - f(x)f(y) \leq \epsilon \]

for all \( x, y \in \mathbb{R} \), where \( f : \mathbb{R} \rightarrow \mathbb{R} \) is an unknown function and \( \epsilon > 0 \). Afterward, Charifi et al. [7] established the stability of the pexiderized Goląb-Schinzel functional equation:

\[ f(x + yf(x)) = g(x)h(y) \]

for all \( x, y \in E \) under the condition that \( \lim_{t \to 0} h(tx) \) exists and is nonzero for all \( x \in E \), where \( f, g, h : E \rightarrow K \) are unknown functions, \( K \) is a subfield of a set of all complex numbers \( \mathbb{C} \) and \( E \) is a \( K \)-vector space. Nowadays, the composite functional equation was employed by many mathematicians (see [8–13]).

In 2009, Fechner [14] discussed the following composite functional equation:

\[ f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y) \quad (1.1) \]

for all \( x, y \in G \), where \( f \) is a self-mapping on an abelian group \( G \), which is motivated from the following absolute value identity due to Tarski [15]:

\[ |x| - |y| = |x + y| + |x - y| - |x| - |y| \]

(1.2)

for all \( x, y \in \mathbb{R} \). Fechner [14] also solved the solution of the composite functional equation (1.1) as follows.

**Theorem 1.1.** [14] Let \( (G, +) \) be an abelian group uniquely divisible by 2. A function \( f : G \rightarrow G \) is a solution of (1.1) and

\[ f(G) \subseteq -f(G) \]

if and only if \( f \) is additive and \( f \circ f = f \).

**Theorem 1.2.** [14] Suppose that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous at zero. Then \( f \) is a solution of (1.1) if and only if one of the following possibilities holds:

1. \( f(x) = 0 \) for each \( x \in \mathbb{R} \);
2. \( f(x) = x \) for each \( x \in \mathbb{R} \);
3. \( f(x) = |x| \) for each \( x \in \mathbb{R} \);
4. \( f(x) = -|x| \) for each \( x \in \mathbb{R} \).

In 2010, Kochanek [16] investigated the problem of determining general solutions \( f : \mathbb{R} \rightarrow \mathbb{R} \) of the composite functional equation (1.1) under some assumptions upon \( f(\mathbb{R}) \). In the same year, Fechner [17] investigated three composite functional inequalities which are motivated from the composite functional equation (1.1) as follows:

\[
\begin{align*}
    f(f(x) - f(y)) & \leq f(x + y) + f(f(x - y)) - f(x) - f(y), \\
    f(f(x) - f(y)) & \leq f(f(x + y)) + f(x - y) - f(x) - f(y), \\
    f(f(x) - f(y)) & \leq f(f(x + y)) + f(f(x - y)) - f(f(x)) - f(y)
\end{align*}
\]

for all \( x, y \in \mathbb{R} \), where \( f \) is in the class of mappings from \( \mathbb{R} \) into itself.
One year later, Fechner [18] studied the approximated solutions of the composite functional equation (1.1), where \( f \) is a continuous function from a Banach space \( E \) into itself, that is, the stability result under the following assumption:

\[
\|f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y)\| \leq \varepsilon \|x\|^p \|y\|^p
\]

for all \( x, y \in \tilde{E} \), where \( \varepsilon > 0 \) is arbitrarily fixed, \( a, b \in \mathbb{R} \) with \( a + b \neq 1 \), \( \tilde{E} = E \) if \( a \geq 0 \) and \( b \geq 0 \) and \( \tilde{E} = E \setminus \{0\} \) otherwise. In 2013, Kenary [19] proved the generalized Hyers-Ulam-Rassias stability of the composite functional equation (1.1) in random normed spaces and non-Archimedean normed spaces by using the direct method and the fixed point method. It follows from the different behavior on the tool for measuring distance between points of abstract spaces that the results of Kenary [19] differ from the main results of Fechner [18]. Nowadays, many results concerning the composite functional equation (1.1) in various abstract spaces are given. Moreover, many mathematicians also investigated functional equations related to the Ulam stability theory in various forms of functional equations such as quintic, sextic, septic, octic, nonic, decic functional equations etc. (see [20–27] and references therein).

To the best of our knowledge, there is no discussion so far concerning the generalized Hyers-Ulam-Rassias stability of the composite functional equation (1.1) in \((\beta, p)\)-Banach spaces. The main aim of this article is to investigate the generalized Hyers-Ulam-Rassias stability of the composite functional equation (1.1), where \( f \) maps from a \((\beta, p)\)-Banach space into itself by using the fixed point method and the direct method. Moreover, the result on the composite \( s \)-functional inequality is discussed. Since each Banach space is also a \((\beta, p)\)-Banach space, we obtain the stability results of the composite functional equation (1.1) in the sense of Banach spaces from our main results which are generalizations of results of Fechner [18].

### 2 Preliminaries

In this section, we recall some basic concepts of quasi-\( \beta \)-normed spaces and many results which are needed in the main results.

**Definition 2.1.** [28] Let \( \beta \) be a real number with \( 0 < \beta \leq 1 \), and \( X \) be a vector space over a field \( K \) with \( K = \mathbb{R} \) or \( \mathbb{C} \). A function \( \| \cdot \| : X \to [0, \infty) \) is called a quasi-\( \beta \)-norm on \( X \) if it satisfies the following conditions:

1. \( \|x\| = 0 \) if and only if \( x = 0 \);
2. \( \|rx\| = |r|^\beta \|x\| \) for all \( r \in K \) and all \( x \in X \);
3. there is a constant \( K \geq 1 \) such that \( \|x + y\| \leq K(\|x\| + \|y\|) \) for all \( x, y \in X \).

Also, the pair \((X, \|\cdot\|)\) or \((X, \|\cdot\|, K)\) is called a quasi-\( \beta \)-normed space. The smallest possible \( K \) is called the modulus of concavity of \( \|\cdot\| \).

**Definition 2.2.** [28] A quasi-\( \beta \)-normed space \((X, \|\cdot\|, K)\) is called a \((\beta, p)\)-normed space if

\[
\|x + y\|^p \leq \|x\|^p + \|y\|^p
\]

for some \( 0 < p \leq 1 \), for all \( x, y \in X \) and \( \|\cdot\| \) is called a \((\beta, p)\)-norm on \( X \).

From the above definition, it is easy to see that a \((\beta, p)\)-norm is continuous.

**Definition 2.3.** [28] The sequence \( \{x_n\} \) in a quasi-\( \beta \)-normed space \( X \) is convergent to a point \( x \) if and only if \( \lim_{n \to \infty} \|x_n - x\| = 0 \).

**Definition 2.4.** [28] A sequence \( \{x_n\} \) in a quasi-\( \beta \)-normed space is called a Cauchy sequence if and only if the sequence \( \{x_{n+1} - x_n\} \) converges to zero.
Definition 2.5. [28] A quasi-β-normed $X$ is complete if every Cauchy sequence is convergent. In this case, $X$ is called a quasi-β-Banach space.

For more details of quasi-β-normed spaces, we refer to [28]. Next, we introduce a definition of $b$-metric spaces.

Definition 2.6. [29] Let $X$ be a nonempty set, $K \geq 1$ and $d : X \times X \rightarrow [0, \infty)$ be a function satisfying the following conditions for all $x, y, z \in X$:
1. $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. there is a constant $K \geq 1$ such that $d(x, y) \leq K (d(x, z) + d(z, y))$.

Then $d$ is called a $b$-metric with the coefficient $K$ and the pair $(X, d, K)$ is called a $b$-metric space.

Definition 2.7. [29] A sequence $(x_n)$ in a $b$-metric space $(X, d, K)$ is convergent to a point $x$ in $X$ if and only if $\lim_{n \to \infty} d(x_n, x) = 0$.

Definition 2.8. [29] A sequence $(x_n)$ in a $b$-metric space $(X, d, K)$ is called a Cauchy sequence if and only if $\lim_{n, m \to \infty} d(x_n, x_m) = 0$.

Definition 2.9. [29] A $b$-metric space $(X, d, K)$ is said to be complete if every Cauchy sequence is convergent.

To prove our main results, the following result is needed.

Lemma 2.10. [14] Let $G$ be an abelian group and $f : G \rightarrow G$ be a mapping satisfying (1.1). Then the following assertions hold:
- $f(f(x)) = f(x)$ for all $x \in G$;
- for each $n \in \mathbb{N}$, $f(nx) = nf(x)$ for all $x \in G$.

Next, we recall the classical fixed point theorem in generalized metric spaces which is the important tool for investigating many stability results.

Theorem 2.11. [30] Let $(X, d)$ be a complete generalized metric space and let $J : X \rightarrow X$ be a strictly contractive mapping with the Lipschitz constant $L$ with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers $n$ or there exists a positive integer $n_0$ such that the following assertions hold:
1. $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
2. the sequence $(J^n x)$ converges to a fixed point $y^*$ of $J$;
3. $y^*$ is the unique fixed point of $J$ in the set $Y = \{ y \in X | d(J^{n_0} x, y) < \infty \}$;
4. $d(y, y^*) \leq \frac{1}{1 - L} d(y, Jy)$ for all $y \in Y$.

A generalization of Theorem 2.11 was proved by Aydi and Czerwik [31] by considering the fixed point theorem in generalized $b$-metric spaces.

Theorem 2.12. [31] Let $(X, D, K)$ be a complete generalized $b$-metric space and $T : X \rightarrow X$ satisfies the condition
$$D(T(x), T(y)) \leq \varphi(D(x, y))$$
for all $x, y \in X$ with $D(x, y) < \infty$, where $\varphi : [0, \infty) \rightarrow [0, \infty)$ is nondecreasing and
$$\lim_{n \to \infty} \varphi^n (z) = 0$$
for all $z > 0$. Suppose that $x \in X$ is arbitrarily fixed. Then either for every nonnegative integer $n \in \mathbb{N}$.
or there exists an $k \in \mathbb{N}_0$ such that the following assertions hold:
1. $D(T^n(x), T^{n+k}(x)) < \infty$;
2. the sequence $\{T^n x\}$ is a Cauchy sequence in $X$;
3. there exists a point $u \in X$ such that
   \[
   \lim_{n \to \infty} D(T^n(x), u) = 0
   \]
   and $T(u) = u$;
4. $u$ is the unique fixed point of $T$ in the set $B = \{y \in X \mid D(T^k(x), y) < \infty\}$;
5. for every $y \in B$, we have
   \[
   \lim_{n \to \infty} D(T^n(y), u) = 0.
   \]
Moreover, if $D$ is continuous (with respect to one variable) and
\[
\sum_{k=1}^{\infty} K^k \varphi^k(t) < \infty
\]
for all $t > 0$, then for each $y \in B$, we have
\[
D(T^m y, u) \leq \sum_{k=0}^{\infty} K^{k+1} \varphi^{m+k}[D(y, T(y))]
\]
for $m \in \mathbb{N}_0$.

**Remark 2.13.** In Theorem 2.12, if $u$ is a fixed point of $T$ and a function $\varphi : [0, \infty) \to [0, \infty)$ defined by
\[
\varphi(t) = Lt
\]
for all $t \in [0, \infty)$, where $L \in [0, 1)$ with $KL < 1$, then for each $y \in X$, we have
\[
D(u, y) \leq K[D(u, Ty) + D(Ty, y)] = K[D(Tu, Ty) + D(Ty, y)] = K[LD(u, y) + D(Ty, y)].
\]
This implies that
\[
D(u, y) \leq \left(\frac{K}{1 - KL}\right)D(Ty, y)
\]
for all $y \in X$.

### 3 Main results

In this section, we present the stability results of the composite functional equation (1.1) by using the fixed point method and the direct method. The first part is related to the fixed point method, and the results which are obtained by the direct method are given in the second part. Throughout this section, let $X$ be a $\beta$-$p$-Banach space and $f : X \to X$ be a mapping. For each $x, y \in X$, we will use the following symbol:
\[
Df(x, y) = f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y).
\]

#### 3.1 Stability of the composite functional equation by using the fixed point method

In this section, the stability of the composite functional equation (1.1) is proved in $\beta$-$p$-Banach spaces by using a fixed point approach.
Theorem 3.1. Let $X$ be a $(\beta, p)$-Banach space with the modulus of concavity $K$ and $\phi : X \times X \to [0, \infty)$ be a function such that
\begin{equation}
\phi(2x, 2y) \leq L\phi(x, y)
\end{equation}
for all $x, y \in X$, where $0 \leq L < 1$ and $KL < 2^\beta$. Suppose that $f : X \to X$ is a mapping satisfying
\begin{equation}
\|Df(x, y)\| \leq \phi(x, y)
\end{equation}
for all $x, y \in X$ and $f$ satisfies the following condition:
(A) if $C(a) = \lim_{n \to \infty} \frac{f(2^n a)}{2^n}$ exists for all $a \in X$, then
\begin{equation}
\lim_{n \to \infty} \frac{f(2^n x) - f(2^n y)}{2^n} = C(x) - C(y) \quad \text{for all } x, y \in X.
\end{equation}
Then there exists a unique composite mapping $A : X \to X$ such that
\begin{equation}
\|f(x) - A(x)\| \leq \left(\frac{K}{2^\beta - KL}\right)\phi(x, x)
\end{equation}
for all $x \in X$.

Proof. Let $\Omega = \{g : X \to X\}$. Define a generalized $b$-metric $d$ on $\Omega$ by
\begin{equation}
d(g, h) = \inf\{c \in \mathbb{R}^+ | \|g(x) - h(x)\| \leq c\phi(x, x) \text{ for all } x \in X\}.
\end{equation}

Since $X$ is a $(\beta, p)$-Banach space, we obtain $(\Omega, d)$ is a generalized complete $b$-metric space. Replacing $y$ by $x$ into (3.2), we have
\begin{equation}
\|2f(x) - f(2x)\| \leq \phi(x, x)
\end{equation}
and so
\begin{equation}
\left\|f(x) - \frac{f(2x)}{2}\right\| \leq \frac{\phi(x, x)}{2^\beta}
\end{equation}
for all $x \in X$. Define a mapping $J : \Omega \to \Omega$ by
\begin{equation}
(Jg)(x) = \frac{g(2x)}{2} \quad \text{for all } x \in X
\end{equation}
for all $g \in \Omega$. We want to show that
\begin{equation}
d(Jg, Jh) \leq \frac{L}{2^\beta}d(g, h)
\end{equation}
for all $g, h \in \Omega$.

Let $g, h \in \Omega$. If $d(g, h) = \infty$ for all $g, h \in \Omega$, then the above inequality is true. So we may assume that $d(g, h) < \infty$. Assume that
\begin{equation}
C = \{c \in \mathbb{R}^+ | \|g(x) - h(x)\| \leq c\phi(x, x) \text{ for all } x \in X\}.
\end{equation}
Since $d(g, h) < \infty$, we obtain $C \neq \emptyset$. Suppose that $c \in C$. For each $x \in X$, we have
\begin{equation}
\|(Jg)(x) - (Jh)(x)\| = \|2^{-1}g(2x) - 2^{-1}h(2x)\| = 2^{-\beta} \|g(2x) - h(2x)\| \leq \frac{c}{2^\beta}\phi(2x, 2x) \leq \frac{cL}{2^\beta}\phi(x, x)
\end{equation}
and so
\begin{equation}
d(Jg, Jh) \leq \frac{cL}{2^\beta}.
\end{equation}
By taking the infimum on $c \in C$, we obtain
\begin{equation}
d(Jg, Jh) \leq \frac{L}{2^\beta}d(g, h).
\end{equation}
Therefore, we can conclude that
\[ d(Jg, Jh) \leq \frac{L}{2^\beta} d(g, h) \]
for all \( g, h \in \Omega \). By taking a function \( \varphi : [0, \infty) \to [0, \infty) \) in Theorem 2.12 by
\[ \varphi(t) = \frac{L}{2^\beta} t \]
for all \( t \in [0, \infty) \), there is the unique fixed point \( A \) of \( J \) in \( \Omega \) such that \( \{J^n f\} \) converges to \( A \) in \( (\Omega, d) \). By Remark 2.13 and (3.6), we have
\[ d(Jf^n, f) \leq \frac{1}{2^\beta} \]
for all \( f \in \Omega \) and
\[ d(A, f) \leq \frac{K}{1 - KL2^\beta} d(Jf^n, f) \leq \frac{K}{2^\beta - KL} \]  
(3.8)
for all \( f \in \Omega \). By (3.2), we have
\[ \left\| \frac{Df_x(x, y)}{2^\beta} \right\|^p \leq \frac{1}{2^{np}} \left\| Df_x(2^nx, 2^ny) \right\|^p \leq \frac{1}{2^{np}} \left( \phi(2^nx, 2^ny) \right)^p \leq \left( \frac{Lp}{2^\beta} \right)^p \phi(x, y)^p \]
for all \( x, y \in X \). Letting \( n \to \infty \) in the last inequality and using the condition (A), we obtain
\[ DA(x, y) = 0 \]
(3.9)
for all \( x, y \in X \) and so \( A \) is a composite mapping. From (3.8), we obtain
\[ \|f(x) - A(x)\| \leq \left( \frac{K}{2^\beta - KL} \right) \phi(x, x) \]
(3.10)
for all \( x \in X \). To show the uniqueness of \( A \), suppose that \( A' : X \to X \) is a composite mapping and
\[ \|f(x) - A'(x)\| \leq \left( \frac{K}{2^\beta - KL} \right) \phi(x, x) \]
(3.11)
for all \( x \in X \). By Lemma 2.10, we have \( A(2^nx) = 2^nA(x) \) and \( A'(2^nx) = 2^nA'(x) \) for all \( x \in X \) and for all \( n \in \mathbb{N} \). From (3.1) and (3.11), we get
\[ \|A(x) - A'(x)\| = \left\| \frac{A(2^nx)}{2^\beta} - \frac{A'(2^nx)}{2^\beta} \right\| \]
\[ = \frac{1}{2^\beta} \|A(2^nx) - f(2^nx) - A'(2^nx) + f(2^nx)\| \]
\[ \leq \frac{K}{2^\beta} (\|A(2^nx) - f(2^nx)\| + \|A'(2^nx) - f(2^nx)\|) \]
\[ \leq \frac{2K}{2^\beta (2^\beta - KL)} \phi(2^nx, 2^nx) \]
\[ \leq \left( \frac{2K^2}{2^\beta - KL} \right) \left( \frac{Lp}{2^\beta} \right)^p \phi(x, x) \]
for all \( x \in X \). Since the right-hand side of the inequality tends to zero as \( n \to \infty \), we get \( A = A' \). □

**Corollary 3.2.** Let \( X \) be a \((\beta, p)\)-Banach space with the modulus of concavity \( K \) and \( f : X \to X \) be a mapping. Suppose that there are a positive real number \( \lambda \) and a negative real number \( s \) with \( K2^\beta < 2^\beta \) such that
\[ \|Df(x, y)\| \leq \lambda (\|x\|^p + \|y\|^p) \]
for all \( x, y \in X \setminus \{0\} \) and \( f \) satisfies the condition (A). Then there exists a unique composite mapping \( A : X \to X \) such that
for all \( x \in X \setminus \{0\} \).

**Proof.** Define a mapping \( \phi : X \times X \to [0, \infty) \) by

\[
\phi(x, y) = \begin{cases} 
0, & \text{if } x = 0 \text{ or } y = 0; \\
\lambda (\|x\|^p + \|y\|^q), & \text{otherwise}.
\end{cases}
\]

We will show that \( \phi(2x, 2y) \leq L \phi(x, y) \) for all \( x, y \in X \), where \( 0 \leq L < 1 \). We can easily see that it holds for \( x = 0 \) or \( y = 0 \). Suppose that \( x \neq 0 \) and \( y \neq 0 \). Then we get

\[
\phi(2x, 2y) = 2^{p+q} \phi(x, y) = 2^{p+q} \phi(x, y) = L \phi(x, y),
\]

where

\[
L = 2^{p+q} \in (0, 1).
\]

So we have

\[
\phi(2x, 2y) \leq L \phi(x, y)
\]

for all \( x, y \in X \), where \( 0 \leq L < 1 \). By Theorem 3.1, there exists a unique composite mapping \( A : X \to X \) such that

\[
\|f(x) - A(x)\| \leq \left( \frac{K}{2^\beta - KL} \right) \phi(x, x) = \left( \frac{2K\lambda}{2^\beta - K2^\beta} \right) \|x\|^p
\]

for all \( x \in X \setminus \{0\} \).

**Corollary 3.3.** Let \( X \) be a \((\beta, p)\)-Banach space with the modulus of concavity \( K \) and \( f : X \to X \) be a mapping. Suppose that there are a positive real number \( \lambda \) and real numbers \( p, q \) with \( p + q < 0 \) and \( K2^{p+q} < 2^\beta \) such that

\[
\|Df(x, y)\| \leq \lambda (\|x\|^p\|y\|^q)
\]

for all \( x, y \in X \setminus \{0\} \) and \( f \) satisfies the condition (A). Then there exists a unique composite mapping \( A : X \to X \) such that

\[
\|f(x) - A(x)\| \leq \left( \frac{K\lambda}{2^\beta - K2^{p+q}} \right) \|x\|^{p+q}
\]

for all \( x \in X \setminus \{0\} \).

**Proof.** Define a mapping \( \phi : X \times X \to [0, \infty) \) by

\[
\phi(x, y) = \begin{cases} 
0, & \text{if } x = 0 \text{ or } y = 0; \\
\lambda (\|x\|^p\|y\|^q), & \text{otherwise}.
\end{cases}
\]

We will show that \( \phi(2x, 2y) \leq L \phi(x, y) \) for all \( x, y \in X \), where \( 0 \leq L < 1 \). We can easily see that it holds for \( x = 0 \) or \( y = 0 \). Suppose that \( x \neq 0 \) and \( y \neq 0 \). Then we get

\[
\phi(2x, 2y) = 2^{p+q} \phi(x, y) = 2^{p+q} \phi(x, y) = L \phi(x, y),
\]

where

\[
L = 2^{p+q} \in (0, 1).
\]

So we have

\[
\phi(2x, 2y) \leq L \phi(x, y)
\]
for all $x, y \in X$, where $0 \leq L < 1$. By Theorem 3.1, there exists a unique composite mapping $A : X \to X$ such that

$$
\|f(x) - A(x)\| \leq \left( \frac{K}{2^\beta - KL} \right) \phi(x, x) = \left( \frac{K\lambda}{2^\beta - K 2^{p+q} \beta} \right) \|x\|^{p+q}
$$

for all $x \in X \setminus \{0\}$.

\[ \square \]

**Corollary 3.4.** Let $X$ be a $(\beta, p)$-Banach space with the modulus of concavity $K$ and $f : X \to X$ be a mapping. Suppose that there are a positive real number $\lambda$ and a negative real number $s$ with $K 2^\beta < 2^\beta$ such that

$$
\|Df(x, y)\| \leq \lambda (\|x\|^s \|y\|^s + \|x\|^s + \|y\|^s)
$$

for all $x, y \in X \setminus \{0\}$ and $f$ satisfies the condition (A). Then there exists a unique composite mapping $A : X \to X$ such that

$$
\|f(x) - A(x)\| \leq \left( \frac{K\lambda}{2^\beta - K 2^\beta} \right) (\|x\|^{2\beta} + 2 \|x\|^s)
$$

for all $x \in X \setminus \{0\}$.

**Proof.** Define a mapping $\phi : X \times X \to [0, \infty)$ by

$$
\phi(x, y) = \begin{cases} 
0, & \text{if } x = 0 \text{ or } y = 0; \\
\lambda (\|x\|^s \|y\|^s + \|x\|^s + \|y\|^s), & \text{otherwise},
\end{cases}
$$

for all $x, y \in X$. We will show that $\phi(2x, 2y) \leq L\phi(x, y)$ for all $x, y \in X$, where $0 \leq L < 1$. We can easily see that it holds for $x = 0$ or $y = 0$. Suppose that $x \neq 0$ and $y \neq 0$, then we get

$$
\phi(2x, 2y) = \lambda (\|2x\|^s \|2y\|^s + \|2x\|^s + \|2y\|^s)
$$

$$
= \lambda (2^{2\beta}\|x\|^s \|y\|^s + 2^{\beta}\|x\|^s + 2^{\beta}\|y\|^s)
$$

$$
\leq 2^{\beta}\lambda (\|x\|^s \|y\|^s + \|x\|^s + \|y\|^s)
$$

$$
= 2^\beta\phi(x, y)
$$

$$
= L\phi(x, y),
$$

where

$$
L = 2^\beta.
$$

So we have

$$
\phi(2x, 2y) \leq L\phi(x, y)
$$

for all $x, y \in X$, where $0 \leq L < 1$. By Theorem 3.1, there exists a unique composite mapping $A : X \to X$ such that

$$
\|f(x) - A(x)\| \leq \left( \frac{K}{2^\beta - KL} \right) \phi(x, x) = \left( \frac{K\lambda}{2^\beta - K 2^\beta} \right) (\|x\|^{2\beta} + 2 \|x\|^s)
$$

for all $x \in X \setminus \{0\}$.

\[ \square \]

Next, we present the following lemma in order to give the final stability result of the composite functional equation (1.1) in this part.

**Lemma 3.5.** Let $X$ be a quasi-$\beta$-normed space and $f : X \to X$ be a mapping satisfying (1.1). If $f$ is odd, then $f$ is additive.
Proof. Letting $x = y$ in (1.1), we obtain

$$f(2x) = 2f(x)$$

(3.12)

for any $x \in X$. Putting $x = 0$ in (3.12), we get $f(0) = 0$. Letting $y = 0$ in (1.1), we have

$$f(f(x)) = f(x)$$

(3.13)

for any $x \in X$. Substituting $y$ by $-y$ into (1.1), by the oddness of $f$ yields that

$$f(f(x) + f(y)) = f(x + y) + f(x - y) - f(x) + f(y)$$

(3.14)

for any $x, y \in X$. From (1.1) and (3.14), we obtain

$$f(f(x) - f(y)) - f(f(x) + f(y)) = -2f(y)$$

(3.15)

for any $x, y \in X$. By (3.13) and (3.15), we get

$$f(f(x) - f(y)) - f(f(x) + f(y))) = f(-2f(y))$$

(3.16)

for any $x, y \in X$. From (1.1), (3.14) and (3.15), we obtain

$$f(2f(x)) + f(-2f(y)) - f(f(x) - f(y)) - f(f(x) + f(y)) = f(-2f(y)),$$

$$f(2f(x)) + f(-2f(y)) - f(x + y) - f(x - y) + f(x) + f(y)$$

$$- f(x + y) - f(x - y) + f(x) - f(y) = f(-2f(y)),$$

$$f(2f(x)) - 2f(x + y) - 2f(x - y) + 2f(x) = 0,$$

$$f(x + y) + f(x - y) = 2f(x)$$

(3.17)

and so

$$f(2f(x)) = f(-2f(y)) - f(f(x) - f(y)) - f(f(x) + f(y)) = f(-2f(y)),$$

$$f(2f(x)) + f(-2f(y)) - f(x + y) - f(x - y) + f(x) + f(y)$$

$$- f(x + y) - f(x - y) + f(x) - f(y) = f(-2f(y)),$$

$$f(2f(x)) - 2f(x + y) - 2f(x - y) + 2f(x) = 0,$$

$$f(x + y) + f(x - y) = 2f(x)$$

(3.18)

for any $x, y \in X$. Interchanging $x$ into $y$ in (3.18) and using the oddness of $f$, we get

$$f(x + y) - f(x - y) = 2f(y)$$

(3.19)

for all $x, y \in X$. From (3.18) and (3.19), we obtain

$$f(x + y) = f(x) + f(y)$$

(3.20)

for any $x, y \in X$, and so $f$ is additive. □

Theorem 3.6. Let $X$ be a $(\beta, p)$-Banach space with the modulus of concavity $K$ and $\phi : X \times X \to [0, \infty)$ be a function such that

$$\phi(2x, 2y) \leq L\phi(x, y)$$

(3.21)

for all $x, y \in X$, where $0 \leq L < 1$ and $KL < 2^\beta$. Suppose that $f : X \to X$ is a mapping satisfying

$$\|Df(x, y)\| \leq \phi(x, y)$$

(3.22)

for all $x, y \in X$ and $f$ satisfies the condition (A). Then there exists a unique composite mapping $A : X \to X$ such that

$$\|f(x) - A(x)\| \leq \left(\frac{K}{2^\beta - KL}\right)\phi(x, x)$$

(3.23)

for all $x \in X$. Moreover, if $A$ is odd, then $A$ is additive.

Proof. The assertion immediately follows from Theorem 3.1 combined with Lemma 3.5. □
3.2 Stability of the composite functional equation by using the direct method

In this section, we consider the generalized Hyers-Ulam-Rassias stability of the composite functional equation (1.1) in \((\beta, p)\)-Banach spaces by using the direct method.

**Theorem 3.7.** Let \(X\) be a \((\beta, p)\)-Banach space with the modulus of concavity \(K\) and \(\phi : X \times X \to [0, \infty)\) be a function such that

\[
\sum_{i=1}^{\infty} \frac{\phi^p(2^i x, 2^i y)}{2^{i\beta p}} < \infty \tag{3.24}
\]

for all \(x, y \in X\). Suppose that \(f : X \to X\) is a mapping satisfying

\[
\|Df(x, y)\| \leq \phi(x, y) \tag{3.25}
\]

for all \(x, y \in X\) and \(f\) satisfies the condition (A). Then there exists a unique composite mapping \(A : X \to X\) such that

\[
\|f(x) - A(x)\| \leq \frac{1}{2^\beta} \left( \sum_{i=0}^{\infty} \phi^p(2^i x, 2^i x) \right)^\frac{1}{p} \tag{3.26}
\]

for all \(x \in X\). Moreover, if \(A\) is odd, then \(A\) is additive.

**Proof.** Replacing \(y\) by \(x\) into (3.25), we get

\[
\|f(2x) - 2f(x)\| \leq \phi(x, x). \tag{3.27}
\]

It yields that

\[
\left\| \frac{f(x) - f(2x)}{2} \right\| \leq \frac{\phi(x, x)}{2^\beta p} \tag{3.28}
\]

for all \(x \in X\). Replacing \(x\) by \(2x\) and dividing by \(2^\beta p\) into (3.28), we have

\[
\left\| \frac{f(2x)}{2} - \frac{f(2^2 x)}{2^2} \right\| \leq \frac{\phi^p(2x, 2x)}{2^{2\beta p}} \tag{3.29}
\]

for all \(x \in X\). From (3.28) and (3.29), we obtain

\[
\left\| \frac{f(x) - f(2^2 x)}{2^2} \right\| \leq \frac{1}{2^\beta} \left( \frac{\phi^p(2x, 2x)}{2^{2\beta p}} + \phi(x, x) \right) \tag{3.30}
\]

for all \(x \in X\). By the mathematical induction, (3.30) extends to

\[
\left\| \frac{f(x) - f(2^n x)}{2^n} \right\| \leq \frac{1}{2^\beta} \sum_{i=0}^{n-1} \phi^p(2^i x, 2^i x) \tag{3.31}
\]

for all \(x \in X\). To show that \(\left\{ \frac{f(2^n x)}{2^n} \right\}\) is a Cauchy sequence, for \(m, n \in \mathbb{N}\), we have

\[
\left\| \frac{f(2^{n+m} x)}{2^{n+m}} - \frac{f(2^m x)}{2^m} \right\| = \frac{1}{2^{m\beta p}} \left\| f(2^{n+m} x) - f(2^m x) \right\| \leq \frac{1}{2^{m\beta p}} \sum_{i=0}^{n-1} \phi^p(2^i x, 2^i x) \tag{3.32}
\]

for all \(x \in X\). From (3.24), the right-hand side of the above inequality tends to zero as \(m \to \infty\). This implies that \(\left\{ \frac{f(2^n x)}{2^n} \right\}\) is a Cauchy sequence. Since \(X\) is complete, we can define a function \(A : X \to X\) by

\[
A(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n} \tag{3.33}
\]

for all \(x \in X\). From (3.31), we arrive to the inequality
\[ \|f(x) - A(x)\|^p \leq \frac{1}{2^{\beta p}} \sum_{i=0}^{\infty} \frac{\phi^p(2^i x, 2^i x)}{2^{\beta i p}} \]  
(3.34)

and so

\[ \|f(x) - A(x)\| \leq \frac{1}{2^{\beta p}} \left( \sum_{i=0}^{\infty} \frac{\phi^p(2^i x, 2^i x)}{2^{\beta i p}} \right)^{\frac{1}{p}} \]  
(3.35)

for all \( x \in X \). To show that \( A \) satisfies (1.1), by using (3.24) and (3.25), we get

\[ \left\| Df(2^n x, 2^n y) \right\|^p = \frac{1}{2^{\beta p}} \left\| Df(2^n x, 2^n y) \right\|^p \leq \left[ \frac{\phi(2^n x, 2^n y)}{2^{\beta n}} \right]^p \]

for all \( x, y \in X \). Letting \( n \to \infty \) in the last inequality and using the condition (A), we obtain

\[ DA(x, y) = 0 \]  
(3.36)

for all \( x, y \in X \) and so \( A \) is a composite mapping. To show the uniqueness of \( A \), suppose that \( A' : X \to X \) is a composite mapping and it satisfies (3.26). By Lemma 2.10, we have \( A(2^n x) = 2^n A(x) \) and \( A'(2^n x) = 2^n A'(x) \) for all \( x \in X \) and \( n \in \mathbb{N} \). From (3.24) and (3.26), we get

\[ \|A(x) - A'(x)\|^p \leq \left( \frac{2K}{2^{\beta p + \beta d}} \sum_{i=0}^{\infty} \frac{\phi^p(2^{i+d} x, 2^{i+d} x)}{2^{\beta i p}} \right)^p \]

for all \( x \in X \). Since the right-hand side of the inequality tends to zero as \( n \to \infty \), we have \( A = A' \). If \( A \) is odd, by Lemma 3.5, then \( A \) is additive. \( \Box \)

**Corollary 3.8.** Let \( X \) be a \((\beta, p)\)-Banach space with the modulus of concavity \( K \) and \( f : X \to X \) be a mapping. Suppose that there are a positive real number \( \lambda \) and a real number \( s \) with \( 2^{p(1-s)} > 1 \) such that

\[ \left\| Df(x, y) \right\| \leq \lambda (\|x\|^s + \|y\|^s) \]

for all \( x, y \in X \setminus \{0\} \) and \( f \) satisfies the condition (A). Then there exists a unique composite mapping \( A : X \to X \) such that

\[ \left\| f(x) - A(x)\right\| \leq \frac{2\lambda \|x\|^s}{2^p \left( 1 - \frac{1}{2^{p(1-s)}} \right)^{\frac{1}{p}}} \]

for all \( x \in X \setminus \{0\} \).

**Proof.** Define a mapping \( \phi : X \times X \to [0, \infty) \) by

\[ \phi(x, y) = \begin{cases} 0, & \text{if } x = 0 \text{ or } y = 0; \\ \lambda (\|x\|^s + \|y\|^s), & \text{otherwise}. \end{cases} \]

If \( x = 0 \) or \( y = 0 \), we get

\[ \sum_{i=0}^{\infty} \frac{\phi^p(2^i x, 2^i y)}{2^{\beta i p}} = 0. \]

Suppose that \( x \neq 0 \) and \( y \neq 0 \). Since \( 2^{p(1-s)} > 1 \), we get
By Theorem 3.7, there exists a unique composite mapping \( A : X \to X \) such that
\[
\|f(x) - A(x)\| \leq \frac{1}{2^\beta} \left( \sum_{i=0}^{\infty} \frac{\phi^p(2^i x, 2^i y)}{2^{i \beta}} \right)^{\frac{1}{\beta}} = 2\lambda \|x\|^s \left( \sum_{i=0}^{\infty} \left( \frac{1}{2^\beta(1-r-s)} \right)^i \right)^{\frac{1}{\beta}} = \frac{2\lambda \|x\|^s}{2^\beta} \left( 1 - \frac{1}{2^\beta(1-r-s)} \right) < \infty.
\]
for all \( x \in X \setminus \{0\} \).

\[\Box\]

Corollary 3.9. Let \( X \) be a \((\beta, p)\)-Banach space with the modulus of concavity \( K \) and \( f : X \to X \) be a mapping. Suppose that there are a positive real number \( \lambda \) and real numbers \( r, s \) with \( 2^\beta(1-r-s) > 1 \) such that
\[
\|Df(x, y)\| \leq \lambda (\|x\|^r + \|y\|^s)
\]
for all \( x, y \in X \setminus \{0\} \) and \( f \) satisfies the condition (A). Then there exists a unique composite mapping \( A : X \to X \) such that
\[
\|f(x) - A(x)\| \leq \frac{\lambda \|x\|^r}{2^\beta} \left( 1 - \frac{1}{2^\beta(1-r-s)} \right) \|y\|^s
\]
for all \( x \in X \setminus \{0\} \).

\[\Box\]

Proof. Define a mapping \( \phi : X \times [0, \infty) \to [0, \infty) \) by
\[
\phi(x, y) = \begin{cases} 
0, & \text{if } x = 0 \text{ or } y = 0; \\
\lambda (\|x\|^r + \|y\|^s), & \text{otherwise}.
\end{cases}
\]

If \( x = 0 \) or \( y = 0 \), we get
\[
\sum_{i=0}^{\infty} \frac{\phi^p(2^i x, 2^i y)}{2^{i \beta}} = 0.
\]

Suppose that \( x \neq 0 \) and \( y \neq 0 \). Since \( 2^\beta(1-r-s) > 1 \), we get
\[
\sum_{i=0}^{\infty} \frac{\phi^p(2^i x, 2^i y)}{2^{i \beta}} = \lambda \|x\|^r \|y\|^s \left( \sum_{i=0}^{\infty} \left( \frac{1}{2^\beta(1-r-s)} \right)^i \right)^{\frac{1}{\beta}} = \frac{\lambda \|x\|^r \|y\|^s}{2^\beta} \left( 1 - \frac{1}{2^\beta(1-r-s)} \right)^{\frac{1}{\beta}} < \infty.
\]

By Theorem 3.7, there exists a unique composite mapping \( A : X \to X \) such that
\[
\|f(x) - A(x)\| \leq \frac{\lambda \|x\|^r}{2^\beta} \left( \sum_{i=0}^{\infty} \left( \frac{1}{2^\beta(1-r-s)} \right)^i \right)^{\frac{1}{\beta}} = \frac{\lambda \|x\|^r}{2^\beta} \left( 1 - \frac{1}{2^\beta(1-r-s)} \right) \|y\|^s
\]
for all \( x \in X \setminus \{0\} \).

\[\Box\]

Corollary 3.10. Let \( X \) be a \((\beta, p)\)-Banach space with the modulus of concavity \( K \) and \( f : X \to X \) be a mapping. Suppose that there are a positive real number \( \lambda \) and a real number \( s \) with \( 2^\beta(1-s) > 1 \) such that
\[
\|Df(x, y)\| \leq \lambda (\|x\|^r + \|y\|^s + \|y\|^s)
\]
for all \( x, y \in X \setminus \{0\} \) and \( f \) satisfies the condition (A). Then there exists a unique composite mapping \( A : X \to X \) such that
\[ \|f(x) - A(x)\| \leq \frac{\lambda (\|x\|^2s + 2\|x\|^s) \left( \frac{1}{2} \right)}{1 - 2^{\beta p(s-1)}} \]

for all \( x \in X \setminus \{0\} \).

**Proof.** Define a mapping \( \phi : X \times X \to [0, \infty) \) by
\[
\phi(x, y) = \begin{cases} 
0, & \text{if } x = 0 \text{ or } y = 0; \\
\lambda (\|x\|^s \|y\|^s + \|x\|^s + \|y\|^s), & \text{otherwise}.
\end{cases}
\]

If \( x = 0 \) or \( y = 0 \), we get
\[
\sum_{i=0}^{\infty} \frac{\phi^i(x, y)}{2^{i\beta p}} = 0.
\]

Suppose that \( x \neq 0 \) and \( y \neq 0 \). Since \( 2^{\beta p(1-s)} > 1 \), we get
\[
\sum_{i=0}^{\infty} \frac{\phi^i(x, y)}{2^{i\beta p}} = \lambda \left( \|x\|^s \|y\|^s + \|x\|^s + \|y\|^s \right) \sum_{i=0}^{\infty} \left( \frac{1}{2^{i(1-s)}} \right) < \infty.
\]

By Theorem 3.7, there exists a unique composite mapping \( A : X \to X \) such that
\[
\|f(x) - A(x)\| \leq \frac{1}{2^\beta} \left( \sum_{i=0}^{\infty} \frac{\phi^i(x, y)}{2^{i\beta p}} \right) \leq \frac{\lambda}{\left( \frac{1}{2^\beta} \right)} \left( \sum_{i=0}^{\infty} \frac{1}{2^{i(1-s)}} \right) \leq \frac{\lambda}{\left( 1 - 2^{\beta p(s-1)} \right)} \]

for all \( x \in X \setminus \{0\} \). \( \square \)

**Corollary 3.11.** Let \( X \) be a \( (\beta, p) \)-Banach space with the modulus of concavity \( K \) and \( f : X \to X \) be a mapping. Suppose that there is a positive real number \( \lambda \) such that
\[
\|Df(x, y)\| \leq \lambda
\]

for all \( x, y \in X \) and \( f \) satisfies the condition \((A)\). Then there exists a unique composite mapping \( A : X \to Y \) such that
\[
\|f(x) - A(x)\| \leq \frac{\lambda}{\left( 1 - 2^{\beta p(s-1)} \right)}
\]

for all \( x \in X \).

**Proof.** Define a mapping \( \phi : X \times X \to [0, \infty) \) by
\[
\phi(x, y) = \lambda
\]

for all \( x, y \in X \). Since \( 2^{\beta p} > 1 \), we get
\[
\sum_{i=0}^{\infty} \frac{\phi^i(x, y)}{2^{i\beta p}} = \sum_{i=0}^{\infty} \frac{\lambda^i}{2^{i\beta p}} = \lambda^p \sum_{i=0}^{\infty} \frac{1}{2^{i\beta p}} < \infty
\]
for all $x, y \in X$. By Theorem 3.7, there exists a unique composite mapping $A : X \to X$ such that
\[
\|f(x) - A(x)\| \leq \frac{1}{2^\beta} \left( \sum_{j=0}^{\infty} \phi^p(2^j, 2^j) \right)^{\frac{1}{p}} = \frac{\lambda}{2^\beta} \left( \frac{1}{1 - 2^\beta} \right)\]
for all $x \in X$. \hfill \Box

\section{Composite $s$-functional inequality}

In 2019, Park et al. [32] introduced the new idea of an additive $s$-functional inequality in Banach spaces. Also, they solved its solution and proved the Hyers-Ulam stability of an additive $s$-functional inequality in Banach spaces by using the fixed point method and the direct method.

Next, we solve the composite $s$-functional inequality in quasi-$\beta$-normed spaces.

\begin{theorem}
Let $X$ be a quasi-$\beta$-normed space, $f : X \to X$ be a mapping and $s$ be a fixed nonzero complex number with $|s|^{\beta} < 1$. If $f$ is odd and satisfies the composite $s$-functional inequality
\[
\|f(f(x) - f(y)) - f(x + y) + f(x) + f(y)\| \\
\leq \|sf(f(x) + f(y)) - f(x - y) - f(x + y) + f(x) + f(y)\| \tag{4.37}
\]
for all $x, y \in X$, then $f$ is additive.
\end{theorem}

\begin{proof}
Putting $y$ by $-y$ in (4.37) and using the oddness of $f$, we have
\[
\|f(f(x) + f(y)) - f(x + y) - f(x) - f(y)\| \\
\leq \|sf(f(x) - f(y)) - f(x + y) - f(x) + f(y)\| \tag{4.38}
\]
for all $x, y \in X$. It follows from (4.37) and (4.38) that
\[
\|f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y)\| \\
\leq \|sf(f(x) + f(y)) - f(x - y) - f(x) - f(y)\| \\
= |s|^{\beta} \|f(f(x) + f(y)) - f(x - y) - f(x + y) + f(x) - f(y)\| \\
\leq |s|^{\beta} \|sf(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y)\| \\
= |s|^{\beta} \|f(f(x) - f(y)) - f(x + y) - f(x - y) + f(x) + f(y)\|
\]
for all $x, y \in X$. Since $|s|^{\beta} < 1$, we obtain from the above inequality that
\[
f(f(x) - f(y)) = f(x + y) + f(x - y) - f(x) - f(y)
\]
for all $x, y \in X$. By Lemma 3.5, we obtain $f$ is an additive mapping. \hfill \Box

By using the fixed point method and the direct method, the Hyers-Ulam stability results of the composite $s$-functional inequality (4.37) can also be proved in quasi-$\beta$-normed spaces (see more details in [32]).

\section{Limitations of the method in this article}

In Section 3, many stability results of the composite functional equation (1.1) in $(\beta, p)$-Banach spaces can be obtained from two main results by choosing several suitable control functions for the upper bound of the norm of the difference between both sides of the composite functional equation (1.1). This is the advantage of the method used in this article. Even though this is the main advantage of the method used in
this article, it is also the disadvantage at the same time. In the case of the upper bound of the norm of the difference between both sides of the composite functional equation (1.1) is controlled by some positive constant, the first main result in Section 3 (Theorem 3.1) cannot be applied in this case. It is still open for the researcher to seek the suitable or the generalized control function covering this case and all results in this article.

6 Conclusions and recommendations

Based on the fixed point method and the direct method, we proved the generalized Hyers-Ulam stability results of the composite functional equation (1.1) in \((\beta, p)\)-Banach spaces. Also, we gave many results which are obtained by choosing the suitable mapping \(\phi\) like the sum or the multiplication of the power of norms. Moreover, the generalized Hyers-Ulam stability results of the composite \(s\)-functional inequality is solved and the generalized Hyers-Ulam stability result of this inequality is discussed. To recommend the way for making the research, the reader can use the main results in this article to provide the stability results for the pexiderized composite functional equation and the pexiderized composite \(s\)-functional inequality. Furthermore, our results can be extended to the generalized hyperstability results of the composite functional equation and the composite \(s\)-functional inequality by using the fixed point theorem in quasi-\(\beta\)-Banach spaces (see [33,34]).

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References

[1] S. M. Ulam, Problems in Modern Mathematics, Wiley, New York, 1964.
[2] D. H. Hyers, On the stability of the linear functional equation, Proc. Nat. Acad. Sci. 27 (1941), no. 4, 222–224.
[3] Th. M. Rassias, On the stability of the linear mapping in Banach spaces, Proc. Amer. Math. Soc. 72 (1978), no. 2, 297–300.
[4] P. Găvruta, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings, J. Math. Anal. Appl. 184 (1994), 431–436.
[5] S. Golab and A. Schinzel, Sur l'equation fonctionnelle \(f(x + f(x) y) = f(x)f(y)\), Publ. Math. Debrecen 6 (1959), 113–125.
[6] J. Chudziak, On a functional inequality related to the stability problem for the Gołąb-Schinzel equation, Publ. Math. Debrecen 67 (2005), no. 1–2, 199–208.
[7] A. Charifi, B. Bouikhalene, S. Kabbaj, and J. M. Rassias, On the stability of a pexiderized Gołąb-Schinzel equation, Comput. Math. Appl. 59 (2010), 3193–3202.
[8] N. Briloué-Belluot and J. Brzdęk, On continuous solutions and stability of a conditional Gołąb-Schinzel equation, Publ. Math. Debrecen 72 (2008), 441–450.
[9] J. Chudziak, Approximate solutions of the Gołąb-Schinzel equation, J. Approx. Theory 136 (2005), 21–25.
[10] J. Chudziak, Stability of the generalized Gołąb-Schinzel equation, Acta Math. Hungar. 113 (2006), 133–144.
[11] J. Chudziak, Approximate solutions of the generalized Gołąb-Schinzel equation, J. Inequal. Appl. 2006 (2006), 89402, DOI: https://doi.org/10.1155/JIA/2006/89402.
[12] J. Chudziak, Stability problem for the Gołąb-Schinzel type functional equations, J. Math. Anal. Appl. 339 (2008), 454–460.
[13] J. Chudziak and J. Tabor, On the stability of the Gołąb-Schinzel functional equation, J. Math. Anal. Appl. 302 (2005), 196–200.
[14] W. Fechner, On a composite functional equation on Abelian groups, Aequationes Math. 78 (2009), 185–193.
[15] A. Tarski, Problem no. 83, Parametr 1 (1930), no. 6231; Solution, Mielży Matematyk 1 (1931), no. 190.
[16] T. Kochanek, On a composite functional equation fulfilled by modulus of an additive function, Aequationes Math. 80 (2010), 155–172.
[17] W. Fechner, On some composite functional inequalities, Aequat. Math. 79 (2010), 307–314.
[18] W. Fechner, Stability of a composite functional equation related to idempotent mappings, J. Approx. Theory 163 (2011), 328–335.
19. H. A. Kenary, Hyers-Ulam-Rassias stability of a composite functional equation in various normed spaces, Bull. Iranian Math. Soc. 39 (2013), 383–403.

20. S. Alshybani, S. M. Vaezpour, and R. Saadati, Stability of the sextic functional equation in various spaces, J. Inequal. Spec. Funct. 9 (2018), no. 4, 8–27.

21. Y. Ding, T. Z. Xu, and J. M. Rassias, On Ulam-Hyers stability of decic functional equation in non-Archimedean spaces, J. Comput. Anal. Appl. 26 (2019), no. 4, 671–677.

22. A. Maghshe, R. Veera Sivaji, and A. Ponmana Selvan, Stability of quintic functional equation in matrix normed apaces: A fixed point approach, Int. J. Sci. Eng. Sci. 1 (2017), no. 7, 65–69.

23. M. Nazariampoor, J. M. Rassias, and Gh. Sadeghi, Stability and nonstability of octadecic functional equation in multi-normed spaces, Arab. J. Math. 7 (2018), 219–228, DOI: https://doi.org/10.1007/s40065-017-0186-0.

24. J. M. Rassias, K. Ravi, and B. V. Senthil Kumar, A fixed point approach to Ulam-Hyers stability of duodecic functional equation in quasi-β-normed spaces, Tbilisi Math. J. 10 (2017), no. 4, 83–101.

25. J. M. Rassias, S. S. Kim, and S. H. Kim, Solution and stability of nonic functional equations in non-Archimedean normed spaces, J. Comput. Anal. Appl. 24 (2018), no. 6, 1162–1174.

26. K. Ravi, J. M. Rassias, and B. V. Senthil Kumar, Ulam-Hyers stability of undecic functional equation in quasi-β-normed spaces: Fixed point method, Tbilisi J. Math. 9 (2016), no. 2, 83–103, DOI: https://doi.org/10.1515/tmj-2016-0022.

27. Y. Shen and W. Chen, On the stability of septic and octic functional equations, J. Comput. Anal. Appl. 18 (2015), no. 2, 277–290.

28. S. Rolewicz, Metric Linear Spaces, PWN-Polish Sci. Publ., Warszawa, Reidel, Dordrecht, 1984.

29. S. Czerwik, Nonlinear set-valued contraction mappings in b-metric spaces, Atti Sem. Math. Fis. Univ. Modena 46 (1998), 263–276.

30. J. B. Diaz and B. Margolis, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc. 74 (1968), 305–309.

31. H. Aydi and S. Czerwik, Fixed point theorems in generalizes b-metric spaces, in: N. Daras, T. Rassias (eds.), Modern Discrete Mathematics and Analysis, Springer Optimization and Its Applications, vol. 131, Springer, Cham, 2018, pp. 1–9, DOI: https://doi.org/10.1007/978-3-319-74325-7_1.

32. C. Park, J. R. Lee, and X. H. Zhang, Additive s-functional inequality and hom-derivations in Banach algebras, J. Fixed Point Theory Appl. 21 (2019), 18, DOI: https://doi.org/10.1007/s11784-018-0652-0.

33. N. V. Dung and V. T. L. Hang, The generalized hyperstability of general linear equations in quasi-Banach spaces, J. Math. Anal. Appl. 462 (2018), 131–147, DOI: https://doi.org/10.1016/j.jmaa.2018.01.070.

34. Iz-I. EL-Fassi, On the general solution and hyperstability of the general radical quintic functional equation in quasi-β-Banach spaces, J. Math. Anal. Appl. 466 (2018), 733–748, DOI: https://doi.org/10.1016/j.jmaa.2018.06.024.