LETTER

Derivation of the quantum-optical master equation based on coarse-graining of time

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Abstract
This is a derivation of the quantum-optical master equation using coarse-graining of time, which aims at exploring Markovian evolution from a microscopic perspective. Here, coarse-graining refers to finding an approximate evolution operator by integrating over a time step \( \Delta t \), and considering the field states to be approximated by a similarly coarse-grained basis. By using these ideas and keeping the field states coarse-grained throughout the derivation, I explore the past-future independence of the quantum white noise formalism explicitly. Namely, that a quantum-optical system interacting with a vacuum field may be viewed as sequentially interacting with a series of initially uncorrelated and temporally coarse-grained vacuum states—this process leads to correlations only between the system and the time bins which have interacted with the system in the past, along with a Markovian evolution of the reduced system state. The derivation here may also be viewed in terms of a type of quantum-mechanical path integral.

1. Introduction
A quantum-optical system often describes a low-dimensional Hamiltonian such as a cavity, two-level system, or Jaynes–Cummings system, locally coupled to a bath of field modes with infinite degrees of freedom [1, 2]. It is viewed as too difficult to compute the dynamics of the entire field plus local system \( |\Psi(t)\rangle \), and instead a Markovian evolution equation for the reduced density matrix of the system is desired. This evolution is derived by taking the trace over the total density matrix:

\[
\rho_{\text{sys}}(t) = \text{Tr}_{\text{field}}[|\Psi(t)\rangle \langle \Psi(t)|] \quad \text{or} \quad \rho_{\text{sys}}(t) = \text{Tr}_{\text{field}}[\rho(t)]
\]

in clever ways. Originally, a factorization approximation was made that \( \rho(t) \approx \rho_{\text{sys}}(t) \otimes \rho_{\text{field}}(t) \) in order to find the reduced dynamics of \( \rho_{\text{sys}}(t) \), based on the ability of a heat reservoir to quickly re-establish equilibrium conditions [3–6]. This approximation immediately implies a type of Markovian evolution, where there is no memory stored in correlations between the system and field (reviewed in [7]). On the other hand, the assumption of factorization between the system and bath is unnecessary in microscopic derivations such as the quantum white noise formalism based on a quantum stochastic calculus [7–11]. There, a Markovian property emerges where the field is thought of as a set of initially uncorrelated states sequentially interacting with the system. Consequently, the system becomes entangled with the field states that have interacted in the past yet remains unentangled with those to interact in the future—such a property is called past-future independence [7].

In either case, the reduced system evolution is Markovian and may be written as a completely positive and trace-preserving dynamical map [12]. Specifically, the reduced evolution is given by \( \rho_{\text{sys}}(0) \rightarrow \rho_{\text{sys}}(t) = V(t)\rho_{\text{sys}}(0) \) where the map is written in Lindblad form \( V(t) = e^{Lt} \) the Liouvillian is

\[
L\rho_{\text{sys}}(t) = -i[H_{\text{sys}}, \rho_{\text{sys}}(t)] + \sum_{[L]} L\rho_{\text{sys}}(t)L^\dagger - \frac{1}{2} \{L^\dagger L, \rho_{\text{sys}}(t)\}
\]
(with $\hbar = 1$) for a system Hamiltonian $H_{\text{sys}}$ and a set of $\{L\}$ operators. Additionally, there has been much recent progress on understanding the precise definitions and conditions of quantum Markovian behavior (see reviews [1, 7]), especially based on information-theoretic concepts such as information back flow [13–16].

Although the reduced dynamics are the same, the factorization approximation is a stronger condition on the Markovian nature of the total system-bath evolution than the quantum white noise formalism [7]. This is because it implies negligible correlation between the system and field, while quantum white noise allows for strong correlations between the system and field states in the past. To briefly give a relevant example demonstrating the build-up of system-field correlations, consider spontaneous emission. Suppose a quantum two-level system is prepared in its excited state $\rho_{\text{sys}}(0) = |e\rangle \langle e|$, and spontaneously decays at rate $\gamma$ to its ground state $|g\rangle$ via coupling to a bath prepared in vacuum. The Schrödinger-picture evolution for the system plus bath remains in a pure state the entire time, while the Lindblad evolution for the reduced density matrix of the system occurs with $L = \sqrt{\gamma} |g\rangle \langle e|$ whose solution is

$$\rho_{\text{sys}}(t) = |e\rangle \langle e| e^{-\gamma t} + |g\rangle \langle g| (1 - e^{-\gamma t}) .$$

Hence, the factorization approximation for the system and bath can easily be checked from a calculation of the von-Neumann entropy. Specifically, the von-Neumann entropy characterizes the degree of entanglement between the system and the bath

$$S(t) = - \text{Tr} \left[ \rho_{\text{sys}}(t) \ln \rho_{\text{sys}}(t) \right]$$

$$= \gamma t e^{-\gamma t} - (1 - e^{-\gamma t}) \ln (1 - e^{-\gamma t}) .$$

From this expression, factorization of the system-bath state occurs when $S = 0$ and thus only holds at $t = 0$ when $\rho(0) = |e\rangle \langle e| \otimes \text{vac} \langle \text{vac}|$ and $t \to \infty$ when $\rho(t \to \infty) = |g\rangle \langle g| \otimes |1\rangle \langle 1|$ (where $|1\rangle$ denotes a single photon in the bath mode $\xi$). For $\gamma t = \ln 2$ the entanglement entropy reaches the maximum possible value of $S = \ln 2$ for a two-level system, showing deviation from the factorization approximation.

The purpose of this work is to investigate the Markovian nature of quantum-optical systems from a microscopic perspective, including to see the way past field states do not influence future evolution of the system. In particular, the bath states are coarse-grained in time to allow for a computationally tractable form of the system-bath interaction operator. This process is comparable to an Itô quantum stochastic calculus, however, the main contribution in this work is to reveal a new method for deriving the master equation. The reduced evolution of the system is written in terms of a finite quantum mechanical path integral in the coarse-grained basis. By appropriately ordering the operators in the path integral, the Fock-state computational basis of the coarse-grained field states is used to resolve the identity between operators, which is shown to always be the vacuum state. This procedure enables a very clear perspective on the Markovian behavior of the system and bath together, showing directly how the past scattering events with the field do not enter into the future evolution of the system.

2. Schrödinger-picture Hamiltonian

With the advent of nanophotonics [17], it has become increasingly possible to explore the interaction between a bath of photonic modes and a small local system. For example, it is now routine to prepare single- and few-photon states in nanophotonic waveguides by exciting a local quantum system and letting it decay into the waveguide(s) [18–20]. Hence, it is becoming increasingly important to have a good physical understanding of these processes to engineer nonclassical states of the quantum light field. In this section, I present the most basic Hamiltonian model for this type of process.

To begin, consider a chiral (unidirectional) waveguide with a single spatial mode profile, supporting a bath of harmonic oscillators. In the quasi-monochromatic regime, where the frequency content of the excitations in the waveguide is narrowband [21, 22], its Hamiltonian is approximately

$$H_0 = I_{\text{sys}} \otimes \int_{-\infty}^{\infty} d\omega b_{\omega}^\dagger b_{\omega} .$$

If the waveguide is linearly coupled to a local quantum system, with the Hamiltonian $H_{\text{sys}}$, then in the rotating wave approximation the interaction part of the Hamiltonian is

$$H_I = H_{\text{sys}} \otimes I_{\text{field}} + i \sqrt{\gamma} \left( \sigma \otimes \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} b_{\omega}^\dagger \sigma^\dagger \otimes \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} b_{\omega} \right) .$$

I make an assumption of weak-coupling between the system and field, which is standard for quantum-optical problems, i.e. $\gamma$ is much weaker than any of the natural frequencies of the system. The field mode operators obey the commutation relation

$$[b_{\omega^\prime}, b_{\omega}^\dagger] = \delta (\omega - \omega^\prime),$$

From this expression, factorization of the system-bath state occurs when $S = 0$ and thus only holds at $t = 0$ when $\rho(0) = |e\rangle \langle e| \otimes \text{vac} \langle \text{vac}|$ and $t \to \infty$ when $\rho(t \to \infty) = |g\rangle \langle g| \otimes |1\rangle \langle 1|$ (where $|1\rangle$ denotes a single photon in the bath mode $\xi$). For $\gamma t = \ln 2$ the entanglement entropy reaches the maximum possible value of $S = \ln 2$ for a two-level system, showing deviation from the factorization approximation.

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and $\sigma$ is a lowering operator in the Hilbert space of the system. This gives a total Hamiltonian

$$H = H_0 + H_1. \quad (8)$$

### 3. Interaction-picture Hamiltonian

Due to the coupling between the local system and all frequency modes of the waveguide, integrating Schrödinger’s equation for this Hamiltonian is a bit unwieldy. The first step to getting a Hamiltonian that is easier to work with is to transform into an interaction picture to remove the free evolution of the waveguide $H_0$. Specifically, I choose the state vector

$$|\Psi(t)\rangle = e^{iH_0t}|\Psi(t)\rangle. \quad (9)$$

Then, Schrödinger’s equation for $|\Psi(t)\rangle$ becomes

$$\frac{\partial}{\partial t}|\Psi(t)\rangle = H(t)|\Psi(t)\rangle \quad (10)$$

with

$$H_1(t) = e^{iH_0t}H_1e^{-iH_0t} \quad (11a)$$

$$H_1(t) = H_{sys} \otimes 1_{field} + i\sqrt{\gamma}\left(\sigma \otimes \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{i\omega t}b^\dagger_\omega - \sigma^\dagger \otimes \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega t}b_\omega\right) \quad (11b)$$

Here, I used the relationship

$$e^{iH_0t}b_\omega e^{-iH_0t} = e^{-i\omega t}b_\omega, \quad (12)$$

which follows from the commutation in equation (7). Next, I define a new operator, which naturally appeared as a Fourier transform of the $b_\omega$’s

$$b(t) = \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega t}b_\omega, \quad (13)$$

I then rewrite the interaction-picture Hamiltonian as

$$H_1(t) = H_{sys} \otimes 1_{field} + V(t), \quad (14)$$

where

$$V(t) = i\sqrt{\gamma}\left(\sigma \otimes b^\dagger(t) - \sigma^\dagger \otimes b(t)\right). \quad (15)$$

The operator $b(t)$ obeys commutations similar to the frequency mode operators, but in time

$$[b(t), b^\dagger(t')] = \left[\int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega t}b_\omega \int_{-\infty}^{\infty} \frac{d\omega'}{\sqrt{2\pi}} e^{i\omega' t'}b^\dagger_\omega\right] \quad (16a)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{d\omega'}{\sqrt{2\pi}} e^{-i\omega t}e^{i\omega' t'}[b_\omega, b^\dagger_\omega]\quad (16b)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} \frac{d\omega'}{\sqrt{2\pi}} e^{-i\omega t}e^{i\omega' t'}\delta(\omega - \omega')\quad (16c)$$

$$= \int_{-\infty}^{\infty} \frac{d\omega}{\sqrt{2\pi}} e^{-i\omega(t - t')}\quad (16d)$$

$$= \delta(t - t'). \quad (16e)$$

### 4. Coarse-grained time evolution operator

The goal of this section is to present a method to integrate over the singularities in the Schrödinger equation caused by the delta commutation of equation (16c). Specifically, the integration of a time step $\Delta t$ is referred to as a coarse-graining [23]. To proceed, the Schrödinger equation can be rewritten by iteratively applying its integral form, yielding a Dyson series

$$|\Psi(t_f)\rangle = |\Psi(t_0)\rangle + \int_{t_0}^{t_f} dt \frac{\partial}{\partial t}|\Psi(t)\rangle \quad (17a)$$

$$= |\Psi(t_0)\rangle - i \int_{t_0}^{t_f} dt H(t)|\Psi(t)\rangle \quad (17b)$$
\[ V \] are assigned an\( \hat{\mathcal{H}}_t \) which occur on a much faster timescale\( \mathcal{H}_t \). On the second-order term it is defined by\( \mathcal{H}_t \) and the map becomes exact in the weak coupling approximation, which avoids incommensurate systematic errors. Nevertheless, in the limit of\( \Delta t \rightarrow 0 \) the error vanishes in the limit of\( \Delta t \rightarrow 0 \) [21]. Hence, I choose
\[
U[k + 1, k] \approx \exp \left[ -i \int_{k \Delta t}^{(k + 1) \Delta t} dt H_t(t) \right]
\] (21)
This map differs from the correct map via time-ordering,
\[
\text{Error} = U[k + 1, k] - U(k + 1, k) \Delta t
\]
(22a)
\[
= \exp \left[ -i \int_{k \Delta t}^{(k + 1) \Delta t} dt H_t(t) \right] - \mathcal{T} \exp \left[ -i \int_{k \Delta t}^{(k + 1) \Delta t} dt H_t(t) \right]
\] (22b)
\[
= \int_{k \Delta t}^{(k + 1) \Delta t} dr \int_{r \Delta t}^{(k + 1) \Delta t} dr' [H_t(t), H_t(t')] + \cdots
\] (22c)
where the limits of integration are only over the upper half of the coordinate plane for\( t < t' \). Here, the operators\( H_t(t) \) and\( H_t(t') \) need to be reordered, which gives their commutator in equation (14).

\[
[H_t(t), H_t(t')] = [H_{\text{sys}}(t), H_{\text{sys}}(t')] + [H_{\text{sys}}(t), V(t')] + [V(t), H_{\text{sys}}(t')] + [V(t), V(t')]
\]
(23a)
\[
= \mathcal{O}(1) + \mathcal{O}(1/\sqrt{\Delta t}) + \mathcal{O}(1/\sqrt{\Delta t}) + 0.
\]
(23b)
The commutators between\( H_{\text{sys}}(t) \) and\( V(t) \) are assigned an\( \mathcal{O}(1/\sqrt{\Delta t}) \) because the singular operators in\( V(t) \) (equation (15)) obey the commutation
\[
\left[ \int_{k \Delta t}^{(k + 1) \Delta t} dt b(t), \int_{k \Delta t}^{(k + 1) \Delta t} dt' b'(t') \right] = \Delta t.
\]
(24)
Combining the orders of\( [H_t(t), H_t(t')] \) in equation (22a), I see that my choice of map is correct to\( \mathcal{O}(\Delta t) \), with a leading error\( \mathcal{O}(\Delta t^{3/2}) \) from the commutation of\( H_{\text{sys}}(t) \) and\( V(t) \). This approximation amounts to a coarse-graining of the system-bath-interaction dynamics to a timescale of\( \Delta t \), which occur on a much faster timescale than the dynamics generated by the system evolution. Nevertheless, in the limit of\( \Delta t \rightarrow 0 \) the error vanishes and the map becomes exact in the weak coupling approximation, which avoids influence from any secular terms in the Dyson series.

Writing the coarse-grained map out explicitly
\[
U[k + 1, k] = \exp \left[ -i H_{\text{sys}} \Delta t \otimes I_{\text{field}} + \sqrt{\gamma} \Delta t \left( \sigma \otimes \int_{k \Delta t}^{(k + 1) \Delta t} dt \frac{b^\dagger(t)}{\sqrt{\Delta t}} - \sigma^\dagger \otimes \int_{k \Delta t}^{(k + 1) \Delta t} dt \frac{b(t)}{\sqrt{\Delta t}} \right) \right]
\]
(25a)
\[ \exp[-iH_{\text{sys}} \Delta t \otimes \mathbb{1}_{\text{field}} + \sqrt{\Delta t} (\sigma \otimes \Delta B^\dagger[k] - \sigma^\dagger \otimes \Delta B[k])], \tag{25b} \]

where I defined the coarse-grained operator

\[ \Delta B[k] = \int_{k \Delta t}^{(k+1) \Delta t} dt \frac{b(t)}{\sqrt{\Delta t}}. \tag{26} \]

This operator obeys the commutation

\[ [\Delta B[j], \Delta B^\dagger[k]] = \delta_{jk} \tag{27} \]

and in the limit

\[ \lim_{\Delta t \to 0} \frac{\Delta B[t/\Delta t]}{\sqrt{\Delta t}} = b(t). \tag{28} \]

(A very nice alternative to this derivation is based on using a wavelet expansion of \( b(t) \), which yields an approximate Hamiltonian directly rather than just a coarse-grained map [24].)

Now, the accessible Hilbert space of the field is also coarse-grained. Specifically, the map can only create or remove excitations from the field in time bins of width \( \Delta t \). Owing to the commutation in equation (27), the relevant Hilbert space of the field is now a product of a bunch of harmonic oscillator spaces, each labeled by the time-bin number \( n \)

\[ \mathcal{H}_{\text{field}}^{\text{coarse}} = \bigotimes_{n=-\infty}^{+\infty} \mathcal{H}_n. \tag{29} \]

The vacuum state with zero excitations is \( |0\rangle = |0\rangle \otimes |0\rangle \otimes \cdots \), the field with a single excitation is in a state

\[ \Delta B^\dagger[j]|0\rangle = \cdots \otimes |0\rangle \otimes |0\rangle \otimes |1\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots, \tag{30} \]

or with multiple excitations in the same \( j \)-th bin

\[ \frac{(\Delta B^\dagger[j])^m}{\sqrt{m!}}|0\rangle = \cdots \otimes |0\rangle \otimes |0\rangle \otimes |m_j\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots. \tag{31} \]

5. Quantum-optical master equation for spontaneous emission

Consider the initial state of the local system to be a mixed state, with the field in the vacuum state

\[ \rho(t_0 = 0) = \rho_{\text{sys}}(0) \otimes |0\rangle \langle 0|. \tag{32} \]

To obtain the density matrix of the system at time \( t_1 = \Delta t \), we need to evolve the total system with the unitary map \( U[1,0] \) and then trace out the field states

\[ \rho_{\text{sys}}[1] = \text{Tr}_{\text{field}}[U[1,0]|\rho_{\text{sys}}[0] \otimes |0\rangle \langle 0|] U[0,1]]. \tag{33} \]

The first step to evaluating this expression is to expand the trace as a series of partial traces over the field time bins, with the partial trace over the \( j \)-th bin as

\[ \text{tr}_{\text{bin}}[\cdots] = \sum_m \langle 0| \frac{(\Delta B[j])^m}{\sqrt{m!}} \cdots \frac{(\Delta B^\dagger[j])^m}{\sqrt{m!}} |0\rangle, \tag{34} \]

so

\[ \rho_{\text{sys}}[1] = \text{tr}_{\text{bin} \to \infty} \cdots \text{tr}_{\text{bin} \to 1} |U[1,0]|\rho_{\text{sys}}[0] \otimes |0\rangle \langle 0| U[0,1]|. \tag{35} \]

Then, I note that \( U[k + 1, k] \) only acts on \( \mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{n=k} \) so

\[ [\Delta B[j], U[k, j+1]] = 0 \quad \text{and} \quad [\Delta B[j], U[j, l]] = 0, \tag{36} \]

where it is assumed that \( j > k + 1 \) and \( l < k \). Thus, the creation and annihilation operators from partial traces over bins other than the 0th one in equation (35) commute with the unitary evolution operators and evaluate to zero in their expectations. This leaves only the first partial trace and the zero elements from the other partial traces—hence

\[ \rho_{\text{sys}}[1] = \cdots \otimes |0\rangle \otimes |0\rangle |U[1,0]| \rho_{\text{sys}}[0] \otimes |0\rangle \langle 0| U[0,1]| \otimes |0\rangle \otimes \cdots \tag{37a} \]

\[ = \sum_m \langle 0| \frac{(\Delta B[0])^m}{\sqrt{m!}} U[1,0]|0\rangle \rho_{\text{sys}}[0] |0\rangle U[0,1] |(\Delta B^\dagger[0])^m}{\sqrt{m!}} |0\rangle. \tag{37b} \]
Notably this map is of the operator-sum representation [25], i.e.

\[ \rho_{\text{sys}}[1] = \sum_m K_m[0] \rho_{\text{sys}}[0] K_m^*[0], \]

(38)

where \( K_m[k] \) are the so-called Kraus operators (which are typically associated with quantum measurements, though I do not invoke any measurement theory in my derivation).

In this specific scenario, the Kraus operators for the first time step \( \Delta t \) in evolution are

\[ K_m[0] = \langle 0 | \frac{(\Delta B[0]^m)}{\sqrt{m!}} U[1, 0] | 0 \rangle. \]

(39)

To evaluate these operators, I expand the discrete map to \( \Delta t \)

\[ U[1, 0] = \exp[-i H_{\text{sys}} \Delta t \otimes 1_{\text{field}} + \sqrt{\gamma \Delta t} \left( \sigma \otimes \Delta B[0] - \sigma^\dagger \otimes \Delta B[0] \right)] \]

(40a)

\[ \Delta t \left[ -i H_{\text{sys}} \otimes 1_{\text{field}} + \frac{1}{2} \gamma (\sigma \otimes \Delta B[0] - \sigma^\dagger \otimes \Delta B[0])^2 \right] + \mathcal{O}(\Delta t^{3/2}). \]

(40b)

Then, I evaluate these Kraus operators for zero photon emissions into the field

\[ K_0[0] = \langle 0 | U[1, 0] | 0 \rangle = 1_{\text{sys}} + \Delta t \left( -i H_{\text{sys}} - \frac{1}{2} \sigma^\dagger \sigma \right) + \mathcal{O}(\Delta t^{3/2}), \]

(41a)

one photon emission into the field

\[ K_1[0] = \langle 0 | \Delta B[0] U[1, 0] | 0 \rangle = \sqrt{\gamma \Delta t} \sigma + \mathcal{O}(\Delta t^{3/2}), \]

(41b)

and two photon emissions into the field

\[ K_2[0] = \langle 0 | \frac{(\Delta B[0])^2}{\sqrt{2}} U[1, 0] | 0 \rangle = \mathcal{O}(\Delta t). \]

(42a)

(43a)

(43b)

Operators representing more photon emissions are higher order in \( \Delta t \) and hence are taken \( K_{m > 2}[0] \approx 0 \). Then, keeping all terms \( \mathcal{O}(\Delta t) \) in the density matrix map

\[ \rho_{\text{sys}}[1] = \sum_m K_m[0] \rho_{\text{sys}}[0] K_m^*[0] \]

(44a)

\[ \approx \rho_{\text{sys}}[0] + \Delta t \left[ -i H_{\text{sys}} - \frac{1}{2} \sigma^\dagger \sigma \right] \rho_{\text{sys}}[0] + \rho_{\text{sys}}[0] \left( i H_{\text{sys}} - \frac{1}{2} \sigma^\dagger \sigma \right) + \gamma \sigma \rho_{\text{sys}}[0] \sigma^\dagger. \]

(44b)

Rearranging, I show this is in the standard Lindblad form

\[ \frac{\rho_{\text{sys}}[1] - \rho_{\text{sys}}[0]}{\Delta t} = -i[H_{\text{sys}}, \rho_{\text{sys}}[0]] + \gamma \left( \sigma \rho_{\text{sys}}[0] \sigma^\dagger - \frac{1}{2} \{\sigma^\dagger, \sigma \} \rho_{\text{sys}}[0] \right). \]

(45)

It is further important to show that this map holds for all time steps. For example, consider the second time step, i.e. from \( t = 0 \rightarrow \Delta t \rightarrow 2 \Delta t \).

\[ \rho_{\text{sys}}[2] = T_{\text{field}}[U[2, 0] \{ \rho_{\text{sys}}[1] \otimes |0\rangle \langle 0| \} U[0, 2]] \]

(46a)

\[ = \text{tr}_{\text{bin}} 0 \text{tr}_{\text{bin}} 1 \cdots \text{tr}_{\text{bin}} \infty [U[2, 0] \{ \rho_{\text{sys}}[0] \otimes |0\rangle \langle 0| \} U[0, 2]] |0\rangle \langle 0| \cdots \]

(46b)

\[ = \sum_m \sum_{m'} (\Delta B[1])^m (\Delta B[0])^{m'} U[2, 0] |0\rangle \rho_{\text{sys}}[0] \langle 0| U[0, 2] (\Delta B[1])^{m'} (\Delta B[0])^m |0\rangle \]

(46c)

\[ = \sum_{m'} \sum_m (\Delta B[0])^m (\Delta B[1])^m U[2, 0] |0\rangle \rho_{\text{sys}}[0] \langle 0| U[0, 2] (\Delta B[1])^{m'} (\Delta B[0])^m |0\rangle. \]

(46d)

Again, I have used equation (36) to reduce the complexity of the trace by commuting away operators that have no effect on the expectations. The amplitudes in the summation of equation (46d) represent a type of finite path integral over all possible field states.

Next, I chronologically order and group all operators based on their action on the time bins and insert a field identity operator \( 1_{\text{field}} \) between the groups. Then, I note that all the elements of the identity with nonzero photon number can commute past the other operators, either to the left or right depending on their time bin, and annihilate to zero. Hence, the identity is resolved to \( |0\rangle \langle 0| \). The left expectation in equation (46d) then becomes
\[
\langle 0 | (\Delta B [1])^{m'} U [2, 0] | 0 \rangle = \langle 0 | (\Delta B [1])^{m'} U [1, 1] | 0 \rangle = \langle 0 | (\Delta B [0])^{m} U [1, 0] | 0 \rangle
\]
and using the commutations from equation (36). This expectation value represents the quantum-mechanical amplitude that \( m \) photons scatter into the 0th time bin and \( m' \) photons scatter into the 1st time bin. The insertion of a complete set of field states at intermediate times, allowing computation in terms of Kraus operators, bears conceptual resemblance towards using path integrals in deriving the master equation [26].

Applying to all of equation (46d), I obtain

\[
\rho_{\text{sys}}[2] = \sum_{m} \sum_{m'} \langle 0 | (\Delta B [1])^{m'} U [2, 1] | 0 \rangle \langle 0 | (\Delta B [0])^{m} U [1, 0] | 0 \rangle \mathcal{L}_{\rho_{\text{sys}}} (\rho_{\text{sys}} [0] | 0 \rangle U [0, 1] \rho_{\text{sys}} [0] \rangle U [0, 1] \rho_{\text{sys}} [0] \rangle U [0, 1],
\]

by inspection of equation (37b). Note that equation (48) looks just like equation (33), but with all indices stepped by one and hence all times stepped by \( \Delta t \). This pattern (or recursion relation) then holds for all future time steps as well, based on the ability to insert \( | 0 \rangle \langle 0 | \) between operators acting on different time bins (see appendix B).

Another way to state this result is that the Kraus operators are time-independent, i.e. \( \mathcal{K}_{\text{sys}} [0] = \mathcal{K}_{\text{sys}} [k] \equiv \mathcal{K}_{\text{sys}} \) and hence the quantum system has a type of Markovian evolution. Physically, the intuition is that the local system interacts with each temporal bin only once and thus the state of the bath modes after interaction are irrelevant if we consider only the reduced dynamics of the system. Here, we are free to choose the interacted modes to be vacuum again and get the same reduced dynamics as if we kept track of their entire state.

Hence,

\[
\frac{\rho_{\text{sys}} [k + 1] - \rho_{\text{sys}} [k]}{\Delta t} = -i \mathcal{H}_{\text{sys}} \rho_{\text{sys}} [k] + \gamma \left( \sigma \rho_{\text{sys}} [k] \sigma - \frac{1}{2} [\sigma^\dagger \sigma, \rho_{\text{sys}} [k]] \right)
\]

and in the continuum limit

\[
\frac{\partial \rho_{\text{sys}} (t)}{\partial t} = -i \mathcal{H}_{\text{sys}} \rho_{\text{sys}} (t) + \gamma \left( \sigma \rho_{\text{sys}} (t) \sigma - \frac{1}{2} [\sigma^\dagger \sigma, \rho_{\text{sys}} (t)] \right)
\]

This is often written with the Liouvillian superoperator as \( \partial \rho_{\text{sys}} (t) / \partial t = \mathcal{L}_{\rho_{\text{sys}}} (t) \). Notably, the entire derivation holds for other system operators like \( \sigma^\dagger \sigma \) to yield a dephasing rate instead of spontaneous emission. See reference [25] for a detailed explanation of how thermal bath states drive the system, and their corresponding Lindblad superoperators.

We have now seen clearly that although the total density matrix does not factorize between the system and field, the Markovian nature of the evolution means that the reduced system dynamics are agnostic to the way photon emission alters the bath state [7]. To state this point mathematically, we define the factorized density matrix \( \chi (t) = \rho_{\text{sys}} (t) \otimes | 0 \rangle \langle 0 | \), which in general does not equal the actual system plus bath density matrix \( \rho (t) = U (t, 0) \rho (0) U (0, t) = \chi (t) \). Then, as long as \( \rho (0) = \chi (0) \), the two generate the same evolution in the reduced system density matrix

\[
d\rho_{\text{sys}} (t) = \mathcal{L}_{\rho_{\text{sys}}} (t) \, dt
\]

In summary, I derived the quantum-optical master equation from a microscopic model of linear system-bath interactions. The derivation was based on an explicit choice of coarse-grained basis for the field states, and can be viewed as a quantum mechanical path integral. This choice provides a different perspective on the Markovian nature of the evolution by showing why future system evolution is independent of past scattering events at the microscopic level.

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Appendix A. Relationship to quantum stochastic calculus

In this appendix, I briefly provide the relationship between coarse-graining of time and operators from quantum stochastic calculus explicitly. The continuum limit of the coarse-grained field operator is called the quantum noise increment

$$dB(t) = \int_t^{t+dt} dsb(s)$$

$$= \lim_{\Delta t \to dt} \sqrt{\Delta t} \Delta B[t/\Delta t].$$

The Ito quantum stochastic differential equation is given by

$$dU(t) = U(t + dt, 0) - U(t, 0)$$

$$= \left( \left( -iH - \frac{1}{2}\gamma \sigma^\dagger \sigma \right) dt \otimes I_{\text{field}} + \sqrt{\gamma} \sigma \otimes dB^\dagger(t) - \sqrt{\gamma} \sigma^\dagger \otimes dB(t) \right) U(t),$$

and defining $U(t) \equiv U(t, 0)$, which is related to the coarse-grained dynamical map via

$$dU(t) = \lim_{\Delta t \to dt} U[t/\Delta t + 1, 0] - U[t/\Delta t, 0]$$

$$= \left( \lim_{\Delta t \to dt} U[t/\Delta t + 1, t/\Delta t] - I_{\text{sys}} \otimes I_{\text{field}} \right) U(t).$$

This equivalence holds because the terms that differ between the Ito increment and the dynamical map are $O((\Delta t)^{3/2})$ or smaller. The master equation is typically derived by using the Ito calculus \cite{9} for noise increments to compute

$$d\rho_{\text{sys}}(t) = \text{Tr}_{\text{field}}[U(t + dt)\rho(0)U^\dagger(t + dt) - U(t)\rho(0)U^\dagger(t)]$$

$$= \text{Tr}_{\text{field}}[U(t)\rho(0)dU(t)U^\dagger(t) + dU(t)\rho(0)U^\dagger(t) + dU(t)\rho(0)dU^\dagger(t)]$$

$$= \mathcal{L}_{\text{sys}}(t)\rho_{\text{sys}}(t)dt,$$

given $\rho(0) = \rho_{\text{sys}}(0) \otimes |0\rangle \langle 0|$, which only requires factorization of the system and field at some reference time $t = 0$. Equivalently, in terms of the dynamical map developed here

$$d\rho_{\text{sys}}(t) = \lim_{\Delta t \to dt} \text{Tr}_{\text{field}}[U[t/\Delta t + 1, 0]\rho(0)U[0, t/\Delta t + 1] - U(t)\rho(0)U^\dagger(t)]$$

$$= \mathcal{L}(t)\rho_{\text{sys}}(t)dt.$$

Appendix B. Insertion of $|0\rangle \langle 0|$ in a sequence of temporally ordered operators

In this appendix, I justify the ability to insert the vacuum state $|0\rangle \langle 0|$ in a sequence of temporally ordered operators when the initial and final states are additionally vacuum. This type of relation appears in, e.g., equation \eqref{47}. More generally, suppose we have the amplitude

$$|0\rangle A_1 A_2 \cdots A_m |0\rangle$$

where each $A_k$ acts on the system and the $k$-th waveguide bin at a time (on $\mathcal{H}_{\text{sys}} \otimes \mathcal{H}_{\text{field}}$), and the operators are all ordered based on which waveguide bin they act on. Then,

$$|0\rangle A_1 A_2 \cdots A_m |0\rangle = (|0\rangle A_1 |0\rangle) (|0\rangle A_2 |0\rangle) \cdots (|0\rangle A_m |0\rangle).$$

To see this, consider explicitly inserting the field identity

$$I_{\text{field}} = |0\rangle \langle 0| + \sum_j \Delta B^\dagger [j] |0\rangle \langle 0| \Delta B [j] + \sum_{j,l} \frac{\Delta B^\dagger [j] \Delta B^\dagger [l]}{1[l \neq j] + \sqrt{2}[l = j]} |0\rangle \langle 0| \frac{\Delta B[j] \Delta B[l]}{1[l \neq j] + \sqrt{2}[l = j]} + \cdots,$$

which sums over the vacuum projector, then all single-photon state projectors, then all two-photon projectors, etc., into

$$|0\rangle A_1 A_2 \cdots A_m |0\rangle = (|0\rangle A_1 I_{\text{field}} A_2 \cdots A_m |0\rangle$$
\[\begin{align*}
&= \langle 0 | A_1 | 0 \rangle \langle 0 | A_2 \cdots A_m | 0 \rangle \\
&+ \sum_j \langle 0 | A_1 | \Delta B[j] | 0 \rangle \langle \Delta B[j] | A_2 \cdots A_m | 0 \rangle \\
&+ \sum_{j,l} \langle 0 | A_1 \left[ \frac{\Delta B[j] \Delta B[l]}{1[l = j] + \sqrt{2} [l = j]} \right] \langle 0 | \frac{\Delta B[j] \Delta B[l]}{1[l = j] + \sqrt{2} [l = j]} | A_2 \cdots A_m | 0 \rangle \\
&+ \cdots.
\end{align*}\]

Importantly, the intermediate field states with more than one photon evaluate to zero. For example, take the amplitudes \( \langle 0 | A_1 | \Delta B[j] | 0 \rangle \langle \Delta B[j] | A_2 \cdots A_m | 0 \rangle \): for \( j > 1 \) then \( B[j] \) can be commuted to the left to annihilate with \( | 0 \rangle \) and for \( j \leq 1 \) then \( B[j] \) can be commuted to the right to annihilate with \( | 0 \rangle \) (using equation (36)). Critically, this relies on the ordering of the operators \( \{ A_k \} \) based on their action on the time bins of the field. The same argument holds for states with higher photon number as well. Therefore, the only allowed intermediate field state is vacuum and

\[\langle 0 | A_1 A_2 \cdots A_m | 0 \rangle = \langle 0 | A_1 | 0 \rangle \langle 0 | A_2 \rangle \cdots \langle 0 | A_m | 0 \rangle.\]  

This process can be repeated to show

\[\langle 0 | A_1 A_2 \cdots A_m | 0 \rangle = \langle 0 | A_1 | 0 \rangle \langle 0 | A_2 | 0 \rangle \cdots \langle 0 | A_m | 0 \rangle.\]

For negative ordering of the operators \( \{ A_k \} \), this relation can also be shown to hold through a similar manner. The insertion here is comparable to a path integral approach [26], except in this work the complete set of intermediate basis states is fully described by the vacuum state.

Finally, I note this is not equivalent to saying that the physical state of the field contains zero photons at all times. The operators \( \{ A_k \} \) contain combinations of creation or annihilation operators of the field time bins, whose expectations over vacuum become the Kraus operators corresponding to any number of possible photon scattering events.
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