AUTOMORPHISMS OF REAL SEMISIMPLE LIE ALGEBRAS AND THEIR RESTRICTED ROOT SYSTEMS

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Abstract. We prove that every automorphism of the restricted root system of a real semisimple Lie algebra – when defined properly – can be lifted to an automorphism of that Lie algebra. In particular, this can be applied to automorphisms of the Dynkin diagram of the restricted root system. We also discuss some applications of this result to the theory of symmetric spaces of noncompact type.

1. Introduction

The correspondence between complex semisimple Lie algebras and reduced root systems is a classical and pivotal result in Lie theory. Its discovery allowed W. Killing and later E. Cartan to obtain the classification of complex simple Lie algebras by reducing it to the level of root systems. It also proves indispensable when one studies automorphisms of complex semisimple Lie algebras, as those can be – in a sense – reduced to automorphisms of the corresponding root system and its Dynkin diagram. One of the pillars of this theory is the Isomorphism Theorem, which basically says that isomorphisms between complex semisimple Lie algebras can be defined merely on the so-called canonical generators – provided that the Cartan matrix is preserved. This theorem can be used to lift any automorphism of the root system of $\mathfrak{g}$ to an automorphism of $\mathfrak{g}$ itself. As a consequence, one can show that the outer automorphism group of $\mathfrak{g}$ is isomorphic to the automorphism group of its Dynkin diagram.

Adjacent to the theory of complex semisimple Lie algebras is that of real semisimple Lie algebras. Those were classified by E. Cartan, but the classification is more intricate than in the complex case. Two standard classification routes are by means of Vogan or Satake diagrams: both are obtained by decorating the Dynkin diagram of the complexification of $\mathfrak{g}$ in a certain way. There is another diagram that one can associate to a real semisimple Lie algebra, namely the Dynkin diagram of its restricted root system. The restricted root system of $\mathfrak{g}$ loses all information about the compact ideals of $\mathfrak{g}$ and hence is not suitable for classification. However, when treated properly, it can recover the noncompact part of $\mathfrak{g}$, that is, the sum of all of its noncompact simple ideals. The main disadvantage of this theory when compared to the complex case is that there is no sensible analog of the Isomorphism Theorem – there are no canonical generators to begin with. Yet, by using the known classification of real semisimple Lie algebras, it is possible to prove that every automorphism of the restricted root system of $\mathfrak{g}$ – when defined properly – can be lifted to an automorphism of $\mathfrak{g}$. This is the main purpose of the current paper. In the proof, we deliberately eschew a case-by-case consideration. Instead, after reducing the problem
to simple real Lie algebras, we divide those into three different blocks – depending on the nature of $g$ as a real form of its complexification. This result appears to be somewhat folklore, as several authors state it without giving any reference (see, e.g., [2, p. 11] or [9, p. 111]).

We also give a reformulation of this result for symmetric spaces of noncompact type, as those are intimately related to noncompact semisimple Lie algebras.

This paper is organized as follows. In Section 2 we review some aspects of the theory of root systems and their relation to complex semisimple Lie algebras. We focus primarily on root system isomorphisms and the closely related notion of Dynkin diagram isomorphisms. In Section 3 we look at real semisimple Lie algebras and their restricted root systems. We introduce the notion of a weighted isomorphism and prove that every weighted root system isomorphism can, in a certain sense, be lifted to a Lie algebra isomorphism. In Section 4 we look at the results of Section 3 through the lens of symmetric spaces of noncompact type. We also establish some links between a symmetric space of noncompact type and its isometry Lie algebra.

2. Root system isomorphisms and complex semisimple Lie algebras

Practically everything discussed in this section can be found in [7, Ch. II] or [10, §1, 4]. We begin by reviewing some aspects of the theory of root systems. Let $(\mathfrak{V}, \Delta)$ be a root system. Here $\mathfrak{V}$ is a finite-dimensional Euclidean real vector space and $\Delta \subseteq \mathfrak{V}$ is the root system itself. We are not assuming $\Delta$ to be reduced or irreducible. First, we recall the notion of isomorphism of root systems.

Definition 1. Let $(\mathfrak{V}', \Delta')$ be another root system. A linear isomorphism $f : \mathfrak{V} \cong \mathfrak{V}'$ is called a (root system) isomorphism between $(\mathfrak{V}, \Delta)$ and $(\mathfrak{V}', \Delta')$ (or between $\Delta$ and $\Delta'$, for brevity) if the following two conditions are satisfied:

(i) $f(\Delta) = \Delta'$;

(ii) $f$ preserves the root integers, i.e. $n_{f(\alpha)f(\beta)} = n_{\alpha\beta}$ for all $\alpha, \beta \in \Delta$ (here $n_{\alpha\beta} = \frac{2(\alpha|\beta)}{||\beta||^2}$);

If $\mathfrak{V}' = \mathfrak{V}$ and $\Delta' = \Delta$, we call $f$ an automorphism of $(\mathfrak{V}, \Delta)$ (or of $\Delta$, for brevity). The (finite) group of all automorphisms of $\Delta$ is denoted by $\text{Aut}(\Delta) \subseteq \text{GL}(\mathfrak{V})$.

Note that, assuming (i), condition (ii) is automatically satisfied if $f$ is conformal (i.e., a scalar multiple of an isometry). Although a root system isomorphism does not have to be conformal in general, we are going to prove that it cannot stray too far from being one. To this end, we need the following simple lemma.

Lemma 2. Let $(\mathfrak{V}, \Delta)$ be a root system. There exists a unique (up to reordering) orthogonal decomposition $\mathfrak{V} = \bigoplus_{i=1}^{k} \mathfrak{V}_i$ such that $\Delta = \bigsqcup_{i=1}^{k} \Delta_i$, where $\Delta_i = \Delta \cap \mathfrak{V}_i$, and $(\mathfrak{V}_i, \Delta_i)$ is an irreducible root system. Two roots $\alpha, \beta \in \Delta$ lie in the same component $\Delta_i$ if and only if there exists a chain of roots $\lambda_0, \lambda_1, \ldots, \lambda_s \in \Delta$ with $\lambda_0 = \alpha, \lambda_s = \beta$, such that $\langle \lambda_{i-1}, \lambda_i \rangle \neq 0$ for $1 \leq i \leq s$.
Unsurprisingly, we call each \((V_i, \Delta_i)\) an **irreducible component of** \((V, \Delta)\) and the decomposition \(V = \bigoplus_{i=1}^k V_i, \Delta = \bigcup_{i=1}^k \Delta_i\) the **decomposition of** \((V, \Delta)\) into its irreducible components.

**Proof of the lemma.** Introduce an equivalence relation on \(\Delta\): two roots \(\alpha, \beta \in \Delta\) are equivalent if and only if they can be connected by a chain of roots \(\lambda_0, \lambda_1, \ldots, \lambda_s \in \Delta\) as above. This is clearly an equivalence relation, so we can write \(\Delta = \bigcup_{i=1}^k \Delta_i\) for the decomposition of \(\Delta\) into the equivalence classes. Define \(V_i\) to be the linear span of \(\Delta_i\). Since \(\Delta\) spans \(V\), we have \(V = \sum_{i=1}^k V_i\). By construction, given \(i, j \in \{1, \ldots, k\}, i \neq j\), every root \(\alpha \in \Delta_i\) is orthogonal to every root \(\beta \in \Delta_j\), so \(V_i \perp V_j\). Therefore, we have an orthogonal decomposition \(V = \bigoplus_{i=1}^k V_i\). In particular, this implies that \(\Delta_i = \Delta \cap V_i\) for each \(i \in \{1, \ldots, k\}\). Trivially, for every subspace \(W \subseteq V, (W, \Delta \cap W)\) is a root system, hence so is each \((V_i, \Delta_i)\). Note that each \(\Delta_i\) is irreducible by design. Let \(V = \bigoplus_{i=1}^{k'} V_i'\) be another decomposition of \(V\) as in the lemma. It follows from what we have already proven that all roots in \(\Delta \cap V_i'\) are equivalent to each other for each \(i \in \{1, \ldots, k'\}\). On the other hand, if \(i, j \in \{1, \ldots, k'\}, i \neq j\), no root in \(\Delta \cap V_j'\) can be equivalent to any root in \(\Delta \cap V_j'\). Consequently, the decomposition \(V = \bigoplus_{i=1}^{k'} V_i'\) coincides with our constructed decomposition up to reordering of the factors, which completes the proof. \(\square\)

Now we can prove the following result, which asserts that root system isomorphisms are “almost” conformal maps.

**Proposition 3.** Let \((V, \Delta)\) and \((V', \Delta')\) be root systems and \(f: V \to V'\) an isomorphism between them. Write \(V = \bigoplus_{i=1}^k V_i, \Delta = \bigcup_{i=1}^k \Delta_i\) and \(V' = \bigoplus_{i=1}^{k'} V_i', \Delta' = \bigcup_{i=1}^{k'} \Delta_i'\) for the decompositions of \((V, \Delta)\) and \((V', \Delta')\) into their irreducible components. Then \(k = k'\) and, after reordering \(V_i\)'s if needed, \(f(V_i) = V_i'\) and \(f(\Delta_i) = \Delta_i'\) for each \(i \in \{1, \ldots, k\}\). Moreover, for each \(i\), \(f|_{V_i}: V_i \to V_i'\) is a conformal map, i.e. there exists \(a_i > 0\) such that \(a_i f|_{V_i}: V_i \to V_i'\) is an isometry.

**Proof.** To begin with, observe that \(\alpha \perp \beta \iff n_{\alpha \beta} = 0\), so \(f\) must preserve root orthogonality. From this it easily follows that \(f\) preserves the equivalence relation on roots described in the proof of Lemma 2, which, in turn, implies the first assertion. For the remainder of the proof, we may assume that both \((V, \Delta)\) and \((V', \Delta')\) are irreducible, and we need to prove that \(f\) is conformal. Pick any \(\alpha_0 \in \Delta\) and define \(a = \frac{||f(\alpha_0)||}{||\alpha_0||} > 0\). We will prove that \(a^{-1} f\) is an isometry. Since \(a^{-1} f\) already preserves the root integers, it suffices to show that it preserves the length of each root. Note that \(\frac{n_{\alpha \beta}}{n_{\alpha_0}} = \frac{||\alpha||^2}{||\alpha_0||^2}\) whenever \(\langle \alpha | \beta \rangle \neq 0\). Hence, \(f\) preserves the length-ratio of any pair of non-orthogonal roots. Pick any \(\beta \in \Delta\). According to Lemma 2, there exists a chain of roots \(\lambda_0, \lambda_1, \ldots, \lambda_s\) with \(\lambda_0 = \alpha_0, \lambda_s = \beta\), such that \(\langle \lambda_{i-1} | \lambda_i \rangle \neq 0\) for \(1 \leq i \leq s\). We can compute:

\[
\frac{||f(\alpha_0)||}{||f(\beta)||} = \frac{||f(\lambda_0)||}{||f(\lambda_1)||} \cdots \frac{||f(\lambda_{s-1})||}{||f(\lambda_s)||} = \frac{||\lambda_0||}{||\lambda_1||} \cdots \frac{||\lambda_{s-1}||}{||\lambda_s||} = \frac{||\alpha_0||}{||\beta||},
\]
The chief example of diagram isomorphisms comes from root system isomorphisms. Suppose \( \text{Aut}(\Delta) \) is an irreducible root system, then \( \text{Aut}(\Delta) \subseteq O(V) \).

\textbf{Definition 4.} Let \(( V', \Delta' \) be another root system with a fixed choice of simple roots \( \Lambda' \subseteq \Delta'^+ \subseteq \Delta' \) and the corresponding Dynkin diagram DD'. A bijection \( s : \Lambda \cong \Lambda' \) is called a \textit{(diagram) isomorphism between DD and DD'} if it is a graph isomorphism that preserves edge directions, the number of lines an edge consists of, and the number of circles a vertex consists of. If \( V' = V, \Delta' = \Delta, \) and \( \Lambda' = \Lambda \), we call \( s \) an \textit{automorphism of DD}. The group of all automorphisms of DD is denoted by \( \text{Aut}(\Delta) \).

The chief example of diagram isomorphisms comes from root system isomorphisms. Suppose that \( f : V \cong V' \) is an isomorphism between \( \Delta \) and \( \Delta' \) such that \( f(\Lambda) = \Lambda' \). Then \( s = f|_\Lambda : \Lambda \cong \Lambda' \) is clearly an isomorphism between DD and DD'. This also explains why the Dynkin diagram of a root system is well-defined in the first place and does not depend on the choice of a Weyl chamber: if \( \Lambda_1 \subseteq \Delta \) is another set of simple roots, then there exists \( w \in W(\Delta) \subseteq \text{Aut}(\Delta) \) mapping \( \Lambda \) onto \( \Lambda_1 \), so the corresponding Dynkin diagrams DD and DD' are isomorphic. This construction \( (f \mapsto f|_\Lambda) \) actually exhausts\(^1\) all Dynkin diagram isomorphisms between DD and DD'. Although this is a standard fact in the theory of root systems (see, for example, \cite[Prop. 2.66]{7}), we will reprove it for our own purposes in Proposition 6 below.

\(^1\)This implies that a root system is fully determined by its Dynkin diagram up to isomorphism.

\[ i.e \quad ||f(\beta)|| = ||f(\alpha_0)|| = a \text{ or, in other words, } ||a^{-1}f(\beta)|| = ||\beta||. \] Consequently, \( a^{-1}f \) preserves the lengths of all the roots and is an isometry, which means that \( f \) is conformal. \( \square \)
Recall that for each \( r \geq 1 \) there exists only one irreducible nonreduced root system of rank \( r \) up to isomorphism (see [7, Prop. 2.92]). It is denoted by \((BC)_r\), and its Dynkin diagram looks like this:

\[ \text{Diagram} \]

**Remark 5.** Some authors who work only with reduced root systems ask \( s: \Lambda \rightarrow \Lambda' \) in Definition 4 to preserve the Cartan matrix instead. This is equivalent to our definition for reduced root systems, as the Cartan matrix and the Dynkin diagram encode the same amount of data for such systems. However, for nonreduced root systems, our definition is stronger because the Dynkin diagram carries more (in fact, all) information about the root system in this case. For instance, the Cartan matrices of \( B_r \) and \((BC)_r\) are the same, whereas their Dynkin diagrams are not – the difference is precisely the vertex represented by two concentrated circles.

It is very straightforward to compute the group \( \text{Aut}(DD) \) for all irreducible root systems by looking at their classification:

\[
\text{Aut}(DD) \simeq \begin{cases} 
S_3, & \text{if } \Delta \simeq D_4; \\
\mathbb{Z}/2\mathbb{Z}, & \text{if } \Delta \simeq A_n \ (n \geq 2), D_n \ (n \geq 5), \text{ or } E_6; \\
\{e\}, & \text{otherwise}.
\end{cases}
\]

Since the set of simple roots forms a basis for the underlying space of a root system, every diagram isomorphism \( s: \Lambda \rightarrow \Lambda' \) between \( DD \) and \( DD' \) extends uniquely to a linear isomorphism \( V \rightarrow V' \), which we denote by the same letter. In particular, we have a natural group embedding \( \text{Aut}(DD) \subseteq \text{GL}(V) \) (once again, this embedding only makes sense after we fix the set of simple roots). Before we relate diagram isomorphisms to root system isomorphisms, we make a few observations. First off, note that \( \text{Aut}(\Delta) \) acts naturally on the set of Weyl chambers of \((V, \Delta)\). Second, let \( V = \bigoplus_{i=1}^k V_i, \Delta = \bigsqcup_{i=1}^k \Delta_i \) be the decomposition of \((V, \Delta)\) into its irreducible components. It is easy to see that for each \( i \in \{1, \ldots, k\} \), \( \Delta_i^+ = \Delta_i \cap \Delta^+ \) is a set of positive roots for \( \Delta_i \). Consequently, we have \( \Lambda = \bigcup_{i=1}^k \Lambda_i \) and \( D = \prod_{i=1}^k D_i \), where \( \Lambda_i = \Lambda \cap \Delta_i^+ \) is a set of simple roots for \( \Delta_i \) and \( D_i = D \cap V_i \) is the corresponding Weyl chamber. This implies that for each \( i \), the Dynkin diagram \( DD_i \) of \( \Delta_i \) is a connected component of \( DD \), and we have \( DD = \bigsqcup_{i=1}^k DD_i \). Finally, note that \( W(\Delta) = \prod_{i=1}^k W(\Delta_i) \).

**Proposition 6.** Let \((V, \Delta)\) and \((V', \Delta')\) be root systems with fixed choices of simple roots \( \Lambda \subseteq \Delta \) and \( \Lambda' \subseteq \Delta' \).

(1) Given any diagram isomorphism \( s: \Lambda \rightarrow \Lambda' \) between \( DD \) and \( DD' \), its linear extension \( s: V \rightarrow V' \) is an isomorphism between \( \Delta \) and \( \Delta' \). An isomorphism \( V \rightarrow V' \) between \( \Delta \) and \( \Delta' \) comes from a diagram isomorphism \( \Lambda \rightarrow \Lambda' \) precisely when it maps \( \Lambda \) onto \( \Lambda' \).
(2) $\text{Aut}(DD) \subseteq \text{Aut}(\Delta)$. In terms of the action of $\text{Aut}(\Delta)$ on the set of Weyl chambers, $\text{Aut}(DD)$ is the stabilizer of $D$.

(3) $\text{Aut}(DD) = W(\Delta) \rtimes \text{Aut}(DD)$.

Proof. Let $s: \Lambda \cong \Lambda'$ be a diagram isomorphism between $DD$ and $DD'$. Recall that the Weyl group of a root system is generated by the simple reflections with respect to any choice of simple roots: $W(\Delta)$ is generated by $\{s_\alpha \mid \alpha \in \Lambda\}$ and the same is true for $W(\Delta')$. Since $s(\Lambda) = \Lambda'$, we deduce that $sW(\Delta)s^{-1} = W(\Delta')$. On the other hand, it is well known that every root in a root system is simple (or double of a simple one) for a suitable choice of a Weyl chamber ([7, Prop. 2.62]). Since the Weyl group acts transitively on the set of Weyl chambers, we deduce that $\Lambda = W(\Delta) \cdot (\Lambda \cup (2\Lambda \cap \Delta))$ (the same is true for $\Lambda'$). We know that for any $\alpha \in \Lambda$, $2\alpha$ is a root if and only if $2s(\alpha)$ is one. Altogether, we have:

$$s(\Delta) = s(W(\Delta) \cdot (\Lambda \cup (2\Lambda \cap \Delta))) = s(W(\Delta))s^{-1} \cdot s(\Lambda \cup (2\Lambda \cap \Delta)) = W(\Delta') \cdot (\Lambda' \cup (2\Lambda' \cap \Delta')) = \Delta',$$

so $s$ satisfies condition (i) of Definition 1. What for condition (ii), observe that $s$ provides a bijection between the connected components of $DD$ and those of $DD'$. Thus, for each $i \in \{1, \ldots, k\}$, there exists $j \in \{1, \ldots, k'\}$ (clearly, $k' = k$) such that $s(DD_i) = DD'_j$, which means that $s(\Lambda_i) = \Lambda'_j$ and thus $s(V_i) = V'_j$ and $s(\Delta_i) = \Delta'_j$. Take any $\alpha \in \Lambda_i$ and let $a = \frac{|s(\alpha)|}{|\alpha|}$. We want to show that for every other $\beta \in \Lambda_i$, $\frac{|s(\beta)|}{|\beta|} = a$. Assume that $\beta$ is connected to $\alpha$ by an edge. Consider the root systems $\Delta \cap \text{span}_R \{\alpha, \beta\}$ and $\Delta' \cap \text{span}_R \{s(\alpha), s(\beta)\}$. They are both of rank 2 and we have an isomorphism between their Dynkin diagrams provided by $s$. Since there are just five root systems of rank 2 up to isomorphism, it is straightforward to see that two such root systems with isomorphic Dynkin diagrams are isomorphic. What it means for us is that $\frac{|s(\beta)|}{|\alpha|} = \frac{|s(\beta)|}{|s(\alpha)|} = a$. Since $DD_i$ is connected, it follows by induction that $s$ preserves the lengths of all simple roots in $\Lambda_i$ by the same factor of $a$. As we already know that it preserves the Cartan integers, we deduce that it is conformal on $V_i$ ($a^{-1}s: V_i \cong V'_j$ is an isometry). But this, together with condition (i) of Definition 1, implies that it preserves the root integers between all the roots in $\Delta_i$ (and not only between the simple ones). Since the root integers between roots lying in different components of $\Delta$ are all zero, we see that $s$ is a root system isomorphism, which was to be proven. The second assertion in part (1) of the proposition is trivial.

Part (2) follows from part (1), as $s \in \text{Aut}(\Delta)$ preserves $D$ if and only if it preserves $\Lambda$.

Part (3) hinges on the fact that $W(\Delta)$ acts simply transitively on the set of Weyl chambers. It is clear from (2) that $W(\Delta)$ and $\text{Aut}(DD)$ do not intersect. On the other hand, let $f \in \text{Aut}(\Delta)$. There exists $w \in W(\Delta)$ such that $w(f(D)) = D$. But then $s = wf$ fixes $D$ and thus lies in $\text{Aut}(DD)$. Therefore, we have a decomposition $f = w^{-1}s, w^{-1} \in W(\Sigma), s \in \text{Aut}(DD)$. This completes the proof of part (3).
Finally, we discuss some aspects of the correspondence between reduced root systems and complex semisimple Lie algebras. Let \( g \) be a (finite-dimensional) complex semisimple Lie algebra. Pick a Cartan subalgebra \( h \subset g \). We have the corresponding set of roots \( \Delta \subset h^* \) and the root space decomposition \( g = h \oplus \bigoplus_{\alpha \in \Delta} g_\alpha \). The restriction of the Killing form \( B \) of \( g \) to \( h \) is nondegenerate, so it induces a \( \mathbb{C} \)-linear isomorphism \( h \cong h^* \). Write \( h^*(\mathbb{R}) \subset h^* \) for the real span of \( \Delta \) and \( h^*(\mathbb{R}) \subset h \) for its preimage under \( h \cong h^* \). It is a standard fact that \( h^*(\mathbb{R}) = \{ h \in h \mid f(h) \in \mathbb{R} \ \forall f \in h^*(\mathbb{R}) \} \) (hence \( h^*(\mathbb{R}) \) is the real dual of \( h(\mathbb{R}) \)), and we have \( h = h(\mathbb{R}) \oplus \mathbb{i}h(\mathbb{R}) \) and \( h^* = h^*(\mathbb{R}) \oplus \mathbb{i}h^*(\mathbb{R}) \). The restriction of \( B \) to \( h^*(\mathbb{R}) \) is positive definite and we can carry it along the isomorphism \( h^*(\mathbb{R}) \cong h^*(\mathbb{R}) \) to an inner product on \( h^*(\mathbb{R}) \). This makes \( (h^*(\mathbb{R}), \Delta) \) into a reduced root system. Note that this inner product on \( h^*(\mathbb{R}) \) is natural and does not require any additional choices, for it comes from the Killing form, which is fully determined by Lie algebra structure of \( g \).

Now we make a choice of positive roots \( \Delta^+ \subset \Delta \) and let \( \Lambda = \{ \alpha_1, \ldots, \alpha_r \} \) be the corresponding set of simple roots. Write \( H_i \in h(\mathbb{R}) \) for the preimage of \( \alpha_i \) under the isomorphism \( h \cong h^* \) and let \( h_i = \frac{2}{|\alpha_i|^2} H_i \). Finally, make a choice of canonical generators \( e_i \in g_{\alpha_i}, f_i \in g_{-\alpha_i} \). It follows from the definition of \( h_i \)'s that

\[
[h_i, e_j] = n_{\alpha, \alpha} e_j, \quad [h_i, f_j] = -n_{\alpha, \alpha} f_j.
\]

The Isomorphism Theorem asserts that if \( g' \) is another complex semisimple Lie algebra with a fixed choice of \( h' \), \( \Lambda' = \{ \alpha'_1, \ldots, \alpha'_r \} \subset \Delta' \), and \( e'_i \in g_{\alpha'_i}, f'_i \in g_{-\alpha'_i} \), \( 1 \leq i \leq r \), such that the Cartan matrices \( A = (n_{\alpha, \alpha'})_{i,j} \) and \( A' = (n_{\alpha, \alpha'})_{i,j} \) coincide, then there exists a unique Lie algebra isomorphism \( g \cong g' \) sending \( h_i \) to \( h'_i \), \( e_i \) to \( e'_i \), and \( f_i \) to \( f'_i \) for \( 1 \leq i \leq r \).

Let \( F: g \cong g' \) be a Lie algebra isomorphism mapping \( h \) onto \( h' \) and write \( f = (F|_{h^*})^{-1}: h^* \cong h'^* \). It is a matter of simple computation that \( f(\Delta) = \Delta' \) and for any \( \alpha \in \Delta \), \( f(g_\alpha) = g'_{f(\alpha)} \).

In particular, we have \( f(h^*(\mathbb{R})) = h'^*(\mathbb{R}) \). We will slightly abuse the notation and use the same letter \( f \) for the restriction \( f|_{h^*(\mathbb{R})}: h^*(\mathbb{R}) \cong h'^*(\mathbb{R}) \). As \( F \) respects the Killing forms of \( g \) and \( g' \), it follows that \( f \) is an isometry and thus a root system isomorphism\(^2\) between \( \Delta \) and \( \Delta' \). The Isomorphism Theorem ensures that every isomorphism \( h^*(\mathbb{R}) \cong h'^*(\mathbb{R}) \) between \( \Delta \) and \( \Delta' \) arises in this way (this fact is essentially equivalent to the Isomorphism Theorem and is proven directly in [7, Th. 2.108]). As a consequence, every such isomorphism is an isometry.

If we let \( g' = g \) and \( h' = h \), we get a surjective Lie group homomorphism \( \Psi: N_{\text{Aut}(g)}(h) \to \text{Aut}(\Delta), \ F \mapsto f \), where \( N_{\text{Aut}(g)}(h) \) is the normalizer of \( h \) in \( \text{Aut}(g) \).

For each \( \alpha \in \Delta \), an automorphism \( \eta \in N_{\text{Aut}(g)}(h) \) such that \( \Psi(\eta) = s_\alpha \) can be constructed explicitly (here \( s_\alpha \) is the reflection of \( h^*(\mathbb{R}) \) in the hyperplane \( \alpha^\perp \)). Namely, if \( e_\alpha \in g_\alpha \) and

\(^2\)This construction also shows that different choices of a Cartan subalgebra of \( g \) lead to isomorphic root systems, as any two Cartan subalgebras differ by an inner automorphism of \( g \). The map sending \( g \) to \( (h^*(\mathbb{R}), \Delta) \) is a 1-to-1 correspondence between the isomorphism classes of complex semisimple Lie algebras on the one hand and of reduced root systems on the other.
Then every isomorphism $f: V \rightarrow V$ write $\phi \circ f$ and equals $B\phi$. It follows that $\phi$ is an isometry, which proves the claim. We call the form of $g$ a reducible root system. Take a complex semisimple Lie algebra $g$ such that the root system $(\phi \circ f)\Delta$ is isomorphic to $(V, \Delta)$. Pick any isomorphism $\phi: V \cong \phi_0(\mathbb{R})$ between $\Delta$ and $\Delta_\phi$ and carry the inner product from $\phi_0(\mathbb{R})$ to $V$ along $\phi$ (as we know from Proposition 3, it simply amounts to renormalizing the existing inner product on $V$ by some conformal factors on the irreducible components of $(V, \Delta)$). Suppose we have another pair $(g', \phi')$ and an isomorphism $\phi': V \cong \phi_0'(\mathbb{R})$ between $\Delta$ and $\Delta_{\phi'}$. We claim that the inner product on $V$ pulled back from $\phi_0'(\mathbb{R})$ along $\phi'$ is the same. Indeed, if we write $f = \phi' \circ \phi^{-1}: \phi_0'(\mathbb{R}) \cong \phi_0'(\mathbb{R})$, then $f$ is an isometry between $\Delta_{\phi'}$ and $\Delta_{\phi'}$. As we discussed above, each such isomorphism is an isometry, which proves the claim. We call the inner product on $V$ constructed above Killing.

**Corollary 7.** Let $(V, \Delta)$ and $(V', \Delta')$ be reduced root systems with Killing inner products. Then every isomorphism $V \cong V'$ between $\Delta$ and $\Delta'$ is an isometry. In particular, $\text{Aut}(\Delta) \subseteq \text{O}(V)$.

### 3. Real semisimple Lie algebras and their restricted root systems

In this section we discuss restricted root systems of real semisimple Lie algebras and prove the main results. See [10, §2, 3] for a detailed exposition of the theory of real semisimple Lie algebras.

Let $g$ be a (finite-dimensional) real semisimple Lie algebra. Let us fix a Cartan involution $\theta$ of $g$ and write $g = \mathfrak{k} \oplus \mathfrak{p}$ for the corresponding Cartan decomposition. We also pick a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$. Having this fixed, we have the set of restricted roots $\Sigma \subseteq \mathfrak{a}^*$ and the restricted root space decomposition $g = g_0 \oplus \bigoplus_{\alpha \in \Sigma} g_\alpha$, where $g_0 = Z_\mathfrak{g}(\mathfrak{a}) = \mathfrak{t}_0 \oplus \mathfrak{a}$ and $\mathfrak{t}_0 = Z_\mathfrak{g}(\mathfrak{a}) = N_\mathfrak{t}(\mathfrak{a})$. Note also that $\theta(g_\alpha) = g_{-\alpha}$ for any $\alpha \in \Sigma$. If we write $B$ for the Killing form of $\mathfrak{g}$, then $B_\theta(X, Y) = -B(X, \theta Y)$ is an inner product on $g$ (which we choose as our default inner product on $g$ unless otherwise specified). The summands of the restricted root space decomposition are all pairwise orthogonal with respect to $B_\theta$, and $\mathfrak{k}$ and $\mathfrak{p}$ are mutually orthogonal with respect to both $B$ and $B_\theta$. Note that $B_\theta$ coincides with $B$ on $\mathfrak{p}$ and equals $-B$ on $\mathfrak{k}$. In a standard fashion, $B_\theta|_{\mathfrak{a} \times \mathfrak{a}} = B|_{\mathfrak{a} \times \mathfrak{a}}$ induces an isomorphism $\mathfrak{a} \cong \mathfrak{a}^*$, and we carry it along this isomorphism to an inner product on $\mathfrak{a}^*$. This way,
(a∗, Σ) becomes a root system, called the restricted root system of g. There are three main differences with the complex semisimple case (apart from the obvious difference in how the root system is constructed):

1. Σ does not have to be reduced.
2. Σ loses all information about the compact ideals of g.
3. The dimensions of the restricted root subspaces gα do not have to be equal to 1.

Let us address these points individually. To begin with, (1) is not really an issue, since we know the classification of all - not necessarily reduced - root systems up to isomorphism. As we mentioned in Section 2, the root systems (BC)r, r ≥ 1, exhaust the list of all irreducible nonreduced root systems up to isomorphism.

Regarding (2), we make the following observation. First of all, it is a standard fact about Cartan involutions that k is a maximal compact subalgebra of g. This immediately implies that g is compact ⇔ p = {0} ⇔ a = {0} ⇔ Σ = ∅.

Now, let g = ⊕i=1k gi be the decomposition of g into the sum of its simple ideals. Necessarily, θ = θ1 + ⋯ + θk, where each θi is a Cartan involution on gi. We then have k = ⊕i=1k ti, p = ⊕i=1k pi, where ti = k ∩ gi, pi = p ∩ gi. Furthermore, we necessarily have a = ⊕i=1k ai, ai = a ∩ pi, and hence a∗ = ⊕i=1k a∗ i, Σ = ⊔i=1k Σi, where Σi = Σ ∩ a∗ i is the restricted root system of gi. Observe also that the decomposition g = ⊕i=1k gi is orthogonal with respect to the inner product Bθ, and Bθ = Bθ1 + ⋯ + Bθk, where Bi is the Killing form of gi. Consequently, the decomposition a∗ = ⊕i=1k a∗ i is orthogonal as well. It is fairly easy to see from the properties of the restricted root space decomposition that a simple Lie algebra must have an irreducible restricted root system. We deduce that Σ = Σ1 ⊔ ⋯ ⊔ Σk is the decomposition of Σ into its irreducible components (except for the fact that some components Σi might be empty – precisely when the correspondent simple ideal gi is compact). We see that it is impossible to recover information about the compact ideals of g from Σ. We also arrive at the following

Corollary 8. Let g = ⊕i=1k gi as above. Then Σ is irreducible if and only if at most one simple ideal gi is noncompact. In particular, if g has no nonzero compact ideals and Σ is irreducible, then g is simple.

Let us now address (3) – arguably, the most important difference with the complex semisimple case. We make use of the fact that dimR(gα) might not be equal to 1 and call it the multiplicity of the root α (and denote it by mult(α)). The idea is that we incorporate this information into the root system Σ itself. To this end, we make the following definition: a weighted root system is a root system in which every root is assigned a positive integer,
called the multiplicity of the root\textsuperscript{4}. The assignment of multiplicities to the roots in $\Sigma$ is far from random, but since real semisimple Lie algebras seem to be the only context where root multiplicities arise, we do not make any additional assumptions in the definition. Since $\theta(g_\alpha) = g_{-\alpha}$, we know that $\text{mult}(\alpha) = \text{mult}(-\alpha)$ for any $\alpha \in \Sigma$. Below we will see that much more is true (see Theorem 15(2) and Proposition 16(5)).

**Definition 9.** Let $g'$ be another real semisimple Lie algebra with a Cartan involution $\theta'$ and a maximal abelian subspace $a' \subseteq p'$ fixed, and let $\Sigma' \subseteq a'^*$ be the corresponding restricted root system. We call a root system isomorphism $f: a^* \cong a'^*$ between $\Sigma$ and $\Sigma'$ **weighted** if it preserves the root multiplicities: $\text{mult}(f(\alpha)) = \text{mult}(\alpha)$ for every $\alpha \in \Sigma$. We say that $(a^*, \Sigma)$ and $(a'^*, \Sigma')$ are **weighted-isomorphic** if there exists a weighted isomorphism between them. Finally, we call a weighted isomorphism $a^* \cong a^*$ from $\Sigma$ to itself a **weighted automorphism** of $(a^*, \Sigma)$ (or of $\Sigma$, for short). The group of all weighted automorphisms of $(a^*, \Sigma)$ will be denoted by $\text{Aut}^w(\Sigma) \subseteq \text{Aut}(\Sigma)$.

We want to relate weighted root system isomorphisms to Lie algebra isomorphisms. Let $g$ and $g'$ be as above and suppose that $F: g \cong g'$ is an isomorphism such that $F \circ \theta = \theta' \circ F$ (hence $F(\mathfrak{k}) = \mathfrak{k}'$, $F(p) = p'$) and $F(\mathfrak{a}) = \mathfrak{a}'$. Consider $\left. F \right|_\mathfrak{a}: \mathfrak{a} \cong \mathfrak{a}'$ and define $f = (\left. F \right|_\mathfrak{a})^{-1}: a^* \cong a'^*$. Similarly to what we did in the complex semisimple case, it is easy to check that $f(\Sigma) = \Sigma'$ and $F(\mathfrak{g}_\alpha) = \mathfrak{g}'_{f(\alpha)}$ for every $\alpha \in \Sigma$. It is also clear that $F$ is an isometry with respect to the inner products $B_\theta$ and $B'_{\theta'}$, so $f$ is an isometry as well. All this implies that $f$ is a weighted isomorphism between $\Sigma$ and $\Sigma'$.

We can apply this construction to the situation when $g' = g$, $\theta' = \theta$, and $\mathfrak{a}' = \mathfrak{a}$. Consider the Lie group $\text{Aut}(g)$. We have a distinguished element of this group fixed, namely the Cartan involution $\theta \in \text{Aut}(g)$. Consider the closed subgroup $K = Z_{\text{Aut}(g)}(\theta)$ of automorphisms that commute with $\theta$. It can also be described as the fixed point subgroup $\text{Aut}(g)^\theta$ of the involutive automorphism $C_\theta: \eta \mapsto \theta \eta \theta^{-1} = \theta \eta \theta$ of $\text{Aut}(g)$ (strictly speaking, we should write $\text{Aut}(g)^{C_\theta}$, not $\text{Aut}(g)^\theta$, but this is a common abuse of notation). Yet another way to describe it is $K = N_{\text{Aut}(g)}(\mathfrak{k})$, the normalizer of $\mathfrak{k}$ in $\text{Aut}(g)$. Indeed, an automorphism $\eta$ commutes with $\theta$ if and only if it respects the Cartan decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$. Since every automorphism of $g$ is orthogonal with respect to the Killing form $B$ and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$ with respect to $B$, $\eta(\mathfrak{k}) = \mathfrak{k}$ automatically implies $\eta(\mathfrak{p}) = \mathfrak{p}$. As we will see later, $K$ is a maximal compact subgroup of $\text{Aut}(g)$ (see Corollary 26). Since $g$ is semisimple, we can identify it with the Lie algebra of $\text{Aut}(g)$ by means of the adjoint representation: $\text{ad}: g \cong \text{Der}(g) = \text{Lie}(\text{Aut}(g))$. Under this identification, we have

$$\text{Lie}(K) = \text{Der}(g)^{\text{Ad}(\theta)} \cong \{ X \in g \mid \text{ad}(X) \circ \theta = \theta \circ \text{ad}(X) \} = \{ X \in g \mid \text{ad}(X) = \text{ad}(\theta(X)) \} = \{ X \in g \mid \theta(X) = X \} = \mathfrak{k}.$$

\textsuperscript{4}In this paper, we are using the letter $\Delta$ for regular root systems and reserve $\Sigma$ for weighted ones.
One can show that $K = \text{Aut}(g) \cap O_{B_0}(g)$ (see [5, Lem. 2.2]). We are especially interested in the subgroup $N_K(a)$ of $K$. According to the previous paragraph, we can define a map $\Omega: N_K(a) \rightarrow \text{Aut}^w(\Sigma)$, $\varphi \mapsto (\varphi|^a)^{-1}$. This is easily seen to be a Lie group homomorphism.

Using some standard facts from the theory of real semisimple Lie algebras, we can prove the following:

**Proposition 10.** If $g$ and $g'$ are isomorphic real semisimple Lie algebras, then their restricted root systems are weighted-isomorphic for any choices of $\theta, \theta', a,$ and $a'$. In particular, the restricted root system of $g$ does not depend on the choice of $\theta$ and $a$ (up to a weighted isomorphism).

**Proof.** Let $F: g \cong g'$ be any isomorphism. Since any two Cartan involutions on $g$ are conjugate by an inner automorphism, we may assume without loss of generality that $F \circ \theta = \theta' \circ F$. Moreover, any two maximal abelian subspaces of $p$ differ by an element of $K$ (even of $K^0$; see [7, Th. 6.51]). Since, by definition, any element of $K$ commutes with $\theta$, we may assume, in addition, that $F(a) = a'$. But now the above construction implies that $F$ induces a weighted isomorphism between $\Sigma$ and $\Sigma'$.

In a similar vein to the complex semisimple case, one can show that $W(\Sigma) \subseteq \text{Im}(\Omega)$. In fact, if $\alpha \in \Sigma$ and $X \in g_\alpha$ is a vector of length $\frac{\sqrt{2}}{||\alpha||}$, then $\exp(\text{ad}_{\frac{\alpha}{2}}(X + \theta X)) \in N_K(a)$ and $\Omega(\exp(\text{ad}_{\frac{\alpha}{2}}(X + \theta X)))$ is the reflection $s_\alpha$ of $a^*$ in the hyperplane $\alpha^+$ (see [7, Prop. 6.52] for a proof). As a consequence, we see that $W(\Sigma) \subseteq \text{Aut}^w(\Sigma)$. This means that restricted roots lying in the same orbit of $W(\Sigma)$ have the same multiplicities.

We make a choice of positive roots $\Sigma^+ \subseteq \Sigma$ and write $\Lambda \subseteq \Sigma^+$ for the subset of simple roots and $D \subseteq a^*$ for the positive Weyl chamber. The sum $n = \bigoplus_{\alpha \in \Sigma^+} g_\alpha$ is a nilpotent subalgebra of $g$, and we have the Iwasawa decomposition $g = \mathfrak{h} \oplus a \oplus n$. Write $DD$ for the Dynkin diagram of $\Sigma$. Just like we did with $\Sigma$, we say that $DD$ is a weighted Dynkin diagram when we want to stress that each vertex has a positive integer assigned to it, namely its multiplicity. If a vertex is doubled, i.e. corresponds to a simple root $\alpha$ such that $2\alpha$ is also a root, we assign to it not just one number, but the ordered pair $(\text{mult}(\alpha), \text{mult}(2\alpha))$.

**Definition 11.** Let $g'$ be another real semisimple Lie algebra with a Cartan involution $\theta'$, a maximal abelian subspace $a' \subseteq p'$, and a choice of positive roots $\Sigma'^+ \subseteq \Sigma'$ fixed. We call a diagram isomorphism $s: \Lambda \cong \Lambda'$ between $DD$ and $DD'$ weighted if it preserves the vertex weights: $\text{mult}(s(\alpha)) = \text{mult}(\alpha)$ (and $\text{mult}(2s(\alpha)) = \text{mult}(2\alpha)$ in case $2\alpha$ is a root) for every $\alpha \in \Lambda$. We say that $DD$ and $DD'$ are weighted-isomorphic if there exists a weighted isomorphism between them. Finally, we call a weighted isomorphism $\Lambda \cong \Lambda$ from $DD$ to itself a weighted automorphism of $DD$. The group of all weighted automorphisms of $DD$ will be denoted by $\text{Aut}^w(DD) \subseteq \text{Aut}(DD)$.

Since $W(\Sigma) \subseteq \text{Aut}^w(\Sigma)$ and $W(\Sigma)$ acts transitively on the set of Weyl chambers in $a^*$, we immediately get the following:
Proposition 12. Let \( \mathfrak{g} \) and \( \mathfrak{g}' \) be real semisimple Lie algebras with restricted root systems \( \Sigma \) and \( \Sigma' \), respectively. If \( \Sigma \) and \( \Sigma' \) are weighted-isomorphic (in particular, if \( \mathfrak{g} \) and \( \mathfrak{g}' \) are isomorphic), then their Dynkin diagrams are weighted-isomorphic as well for any choices of \( \Sigma^+ \) and \( \Sigma'^+ \). In particular, the Dynkin diagram of \( \Sigma \) does not depend on the choice of \( \Sigma^+ \) (up to a weighted isomorphism).

Using the results of Section 2, we can prove the converse to Proposition 12.

Proposition 13. Let \( \mathfrak{g} \) and \( \mathfrak{g}' \) be real semisimple Lie algebras with restricted root systems \( \Sigma \) and \( \Sigma' \) and Dynkin diagrams \( \mathrm{DD} \) and \( \mathrm{DD}' \), respectively. If \( \mathrm{DD} \) and \( \mathrm{DD}' \) are weighted-isomorphic, then so are \( \Sigma \) and \( \Sigma' \). More specifically, if \( s: \Lambda \cong \Lambda' \) is a weighted isomorphism between \( \mathrm{DD} \) and \( \mathrm{DD}' \), then its unique linear extension \( s: \mathfrak{a} \cong \mathfrak{a}' \) is a weighted isomorphism between \( \Sigma \) and \( \Sigma' \). In particular, \( \text{Aut}^w(\mathrm{DD}) \subseteq \text{Aut}^w(\Sigma) \).

Proof. We already know from Proposition 6(1) that \( s: \mathfrak{a} \cong \mathfrak{a}' \) is an isomorphism between \( \Sigma \) and \( \Sigma' \), so we only need to prove that it preserves the root multiplicities. We also know from the proof of Proposition 6 that \( sW(\Sigma)s^{-1} = W(\Sigma') \) and \( W(\Sigma) \cdot (\Lambda \cup (2\Lambda \cap \Sigma)) = \Sigma \). Let \( \alpha \in \Sigma \) be any root. Take \( w \in W(\Sigma) \) such that \( w(\alpha) \in \Lambda \cup (2\Lambda \cap \Sigma) \), and write \( w' = sws^{-1} \in W(\Sigma') \). We have:

\[
s(\alpha) = sw^{-1}(w(\alpha)) = w'^{-1}s(w(\alpha)).
\]

Since \( \text{mult}(s(w(\alpha))) = \text{mult}(w(\alpha)) \) and elements of the Weyl group preserve root multiplicities, we get \( \text{mult}(s(\alpha)) = \text{mult}(\alpha) \), so \( s \) is a weighted root system isomorphism. \( \square \)

Since both \( W(\Sigma) \) and \( \text{Aut}^w(\mathrm{DD}) \) are contained in \( \text{Aut}^w(\Sigma) \) and \( W(\Sigma) \) acts transitively on the set of Weyl chambers, we immediately get the following weighted analog of Proposition 6(3):

Corollary 13.1. \( \text{Aut}^w(\Sigma) = W(\Sigma) \rtimes \text{Aut}^w(\mathrm{DD}) \).

Remark 14. To recapitulate, we know that a weighted-isomorphism class of restricted root systems yields a weighted-isomorphism class of Dynkin diagrams, and it is determined by that class. Similarly, an isomorphism class of real semisimple Lie algebras yields a weighted-isomorphism class of restricted root systems. We know that it cannot be determined by that class though, since adding a compact semisimple summand to the Lie algebra would not change the restricted root system. But it turns out that this is the only obstacle: if \( \mathfrak{g} \) has no nonzero compact ideals, it is determined up to isomorphism by its (weighted) restricted root system – and thus by its (weighted) Dynkin diagram. The standard proof of this fact, however, is rather roundabout. One usually first classifies real semisimple Lie algebras – compact or not – by some other means like Satake or Vogan diagrams, and then computes explicitly the restricted root system of every Lie algebra in the classification list. It turns out that non-isomorphic real semisimple Lie algebras (without nonzero compact ideals) have non-weighted-isomorphic restricted root systems. The list of the (weighted) restricted root systems of all noncompact simple Lie algebras can be found in [1, pp. 336-340].
We need to account for two things: first, we will prove Theorem 15 by first reducing it to the simple case and then (mostly) to the theory of complex semisimple Lie algebras. For example, the restricted root system of \( \mathfrak{su}(r, r + n), n \geq 1 \), is isomorphic to \((BC)_r\), and its Dynkin diagram looks like this:

\[
\begin{aligned}
2 & \quad 2 \quad \cdots \quad 2 \quad 2 \quad (2n, 1) \\
\end{aligned}
\]

Here we write the simple root multiplicities near the corresponding vertices. Note that keeping track of the root multiplicities is crucial here: the Lie algebra \( \mathfrak{sp}(r, r + n), n \geq 1 \), also has \((BC)_r\) as its restricted root system, but the multiplicities are different.

Let us look at the homomorphism \( \Omega : N\mathfrak{K}(\mathfrak{a}) \to \text{Aut}^w(\Sigma) \) through the lens of the semidirect product decomposition \( \text{Aut}^w(\Sigma) = W(\Sigma) \rtimes \text{Aut}^w(\text{DD}) \). First off, note that \( \text{Ker}(\Omega) = Z_K(\mathfrak{a}) \). As we have already seen, \( W(\Sigma) \subseteq \text{Im}(\Omega) \). In fact, for each reflection \( s_\alpha \in W(\Sigma) \), we constructed an element in \( K^0 \) that preserves \( \mathfrak{a} \) and whose image under \( \Omega \) is \( s_\alpha \), which means that \( W(\Sigma) \subseteq \Omega(N_{K^0}(\mathfrak{a})) \). Here \( N_{K^0}(\mathfrak{a}) \) is a subgroup of \( N_K(\mathfrak{a}) \), and it can also be described as \( N_K(\mathfrak{a}) \cap K^0 = N_K(\mathfrak{a}) \cap \text{Inn}(\mathfrak{g}) \) (this equality follows from the fact that \( K \) is a maximal compact subgroup of \( \text{Aut}(\mathfrak{g}) \) and thus \( K^0 = K \cap \text{Aut}^w(\mathfrak{g}) = K \cap \text{Inn}(\mathfrak{g}) \)). In particular, \( N_{K^0}(\mathfrak{a}) \) is a normal subgroup of \( N_K(\mathfrak{a}) \). It is proven in [7, Prop. 6.52] that the image of \( N_{K^0}(\mathfrak{a}) \) under \( \Omega \) is precisely \( W(\Sigma) \) and thus \( N_{K^0}(\mathfrak{a})/Z_{K^0}(\mathfrak{a}) \cong W(\Sigma) \).

Consider another normalizer subgroup of \( K \) given as \( N_K(\mathfrak{n}) \). Let \( k \in N_K(\mathfrak{n}) \). As an element of \( K \), \( k \) commutes with \( \theta \) and thus preserves \( \theta \mathfrak{n} \). Since \( k \) is orthogonal with respect to \( B_\theta \), it must preserve \( \mathfrak{g} \oplus (\mathfrak{n} \oplus \theta \mathfrak{n}) = \mathfrak{g}_0 = \mathfrak{t}_0 \oplus \mathfrak{a} \). But \( \mathfrak{t}_0 \subseteq \mathfrak{t} \) and \( \mathfrak{t}_0 \perp \mathfrak{a} \), so \( k \) preserves \( \mathfrak{t}_0 = \mathfrak{g}_0 \cap \mathfrak{t} \) and thus \( \mathfrak{a} \). We conclude that \( N_K(\mathfrak{n}) \subseteq N_K(\mathfrak{a}) \). Since \( \text{Aut}^w(\text{DD}) \) consists precisely of those weighted automorphisms of \( \Sigma \) that preserve the set of positive roots and \( k(\mathfrak{g}_\alpha) = \mathfrak{g}_{\Omega(k)(\alpha)} \), we deduce that \( N_K(\mathfrak{n}) = \Omega^{-1}(\text{Aut}^w(\text{DD})) \).

**Theorem 15.** Let \( \mathfrak{g} \) be a real semisimple Lie algebra with \( \theta, \mathfrak{a} \), and \( \Sigma^+ \) fixed. Then:

1. \( \Omega(N_K(\mathfrak{n})) = \text{Aut}^w(\text{DD}) \), and hence \( \Omega \) is surjective.
2. If \( \mathfrak{g} \) is simple, then \( \text{Aut}^w(\text{DD}) = \text{Aut}(\text{DD}) \), and hence \( \text{Aut}^w(\Sigma) = \text{Aut}(\Sigma) \).

Informally, (1) means that every weighted automorphism of \( \Sigma \) can be lifted to an automorphism of \( \mathfrak{g} \). Also note that (2) might fail in case \( \mathfrak{g} \) is not simple. For instance, if \( \mathfrak{g} = \mathfrak{su}(r, r + n) \oplus \mathfrak{sp}(r, r + n), n \geq 1 \), then \( \Sigma = (BC)_r \sqcup (BC)_r \), so \( \text{Aut}(\text{DD}) = \mathbb{Z}/2\mathbb{Z} \). But the two connected components of DD are not weighted-isomorphic, which means that \( \text{Aut}^w(\text{DD}) \) is trivial.

We will prove Theorem 15 by first reducing it to the simple case and then (mostly) to the theory of complex semisimple Lie algebras.

We need to account for two things: first, \( \mathfrak{g} \) might have compact ideals, which make no contribution to the restricted root system, and second, \( \mathfrak{g} \) might have isomorphic noncompact simple ideals. To this end, let us write \( \mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{g}_1^{i_1} \oplus \cdots \oplus \mathfrak{g}_k^{i_k} \), where \( \mathfrak{g}_c \) is the sum of all
compact ideals of \( g \), each \( g_i \) is a noncompact simple ideal, \( g_i \not\cong g_j \) for \( i \neq j \), and \( g_i^l \) simply stands for \( \oplus_{j=1}^l g_i \). Write \( l = \sum_{i=1}^k l_i \) and let \( S_{l}^{1, \ldots, k} \) stand for the subgroup of the symmetric group \( S_l \) consisting of those permutations that permute the first \( l_1 \) elements with each other, the next \( l_2 \) elements with each other, etc. Clearly, \( S_{l}^{1, \ldots, k} \cong S_{l_1} \times \cdots \times S_{l_k} \). There is an obvious embedding \( S_{l}^{1, \ldots, k} \hookrightarrow \text{Aut}(g) \) given by the rule \( \sigma \cdot (X_c, (X_s)_{s=1}^{l}) = (X_c, (X_{\sigma(s)})_{s=1}^{l}) \) (to be precise, this map is an antihomomorphism of groups). We already know that any Cartan involution on \( g \) respects the decomposition \( g = g_c \oplus g_i^1 \oplus \cdots \oplus g_i^k \). Without loss of generality, we choose \( \theta \) so that it is the same on the isomorphic summands, i.e. \( \theta = \left( \text{Id}_{g_c}, \prod_{j=1}^{l_1} \theta_1, \ldots, \prod_{l_k} \theta_k \right) \). This way, we have \( \epsilon = g_c \oplus \epsilon_1^l \oplus \cdots \oplus \epsilon_k^l \) and \( p = p_1^l \oplus \cdots \oplus p_k^l \). For such a choice of \( \theta \), the image of the embedding \( S_{l}^{1, \ldots, k} \hookrightarrow \text{Aut}(g) \) actually lies in \( N_K(a) \). We also have a subgroup \( \text{Aut}(g_c) \times \text{Aut}(g_i^1) \times \cdots \times \text{Aut}(g_i^k) \subseteq \text{Aut}(g) \). Let \( K_i = \text{Aut}(g_i)^\theta_i \) for \( 1 \leq i \leq k \). We can choose \( a \) so that \( a = a_i^l \oplus \cdots \oplus a_k^l \), \( a_i \subseteq p_i \). We then have \( \Sigma = \Sigma_1 \sqcup \ldots \sqcup \Sigma_k \), where \( \Sigma_i \subseteq a_i \) is the restricted root system of \( g_i \) and \( \Sigma_i \) simply means \( \bigcup_{j=1}^{l_i} \Sigma_i \). In a similar vein, we have an obvious embedding \( S_{l}^{1, \ldots, k} \hookrightarrow \text{Aut}^w(\Sigma) \) and a subgroup \( \text{Aut}^w(\Sigma_1)^{l_1} \times \cdots \times \text{Aut}^w(\Sigma_k)^{l_k} \subseteq \text{Aut}^w(\Sigma) \). Any choice of positive roots for \( \Sigma \) is necessarily the union of those for its irreducible components. We may assume that \( \Sigma^+ = (\Sigma_1^+)^{l_1} \sqcup \cdots \sqcup (\Sigma_k^+)^{l_k} \); hence \( \lambda = \lambda_1^{l_1} \sqcup \cdots \sqcup \lambda_k^{l_k} \), \( D = D_1^{l_1} \times \cdots \times D_k^{l_k} \), and \( DD = DD_1^{l_1} \sqcup \cdots \sqcup DD_k^{l_k} \).

**Proposition 16.**

1. The group \( \text{Aut}(g) \) decomposes as a semidirect product

\[
\text{Aut}(g) = \left[ \text{Aut}(g_c) \times \text{Aut}(g_i^1)^{l_1} \times \cdots \times \text{Aut}(g_i^k)^{l_k} \right] \rtimes S_l^{1, \ldots, k}.
\]

In particular, we have \( \text{Inn}(g) = \text{Inn}(g_c) \times \text{Inn}(g_i^1)^{l_1} \times \cdots \times \text{Inn}(g_i^k)^{l_k} \).

2. The group \( K \) decomposes as a semidirect product

\[
K = \left[ \text{Aut}(g_c) \times K_1^{l_1} \times \cdots \times K_k^{l_k} \right] \rtimes S_l^{1, \ldots, k}.
\]

In particular, we have \( K_0 = \text{Inn}(g_c) \times (K_1^0)^{l_1} \times \cdots \times (K_k^0)^{l_k} \).

3. The group \( N_K(a) \) decomposes as a semidirect product

\[
N_K(a) = \left[ \text{Aut}(g_c) \times N_{K_1}(a_1)^{l_1} \times \cdots \times N_{K_k}(a_k)^{l_k} \right] \rtimes S_l^{1, \ldots, k}.
\]

In particular, we have \( N_{K_0}(a) = \text{Inn}(g_c) \times N_{K_1^0}(a_1)^{l_1} \times \cdots \times N_{K_k^0}(a_k)^{l_k} \).

4. The group \( N_K(n) \) decomposes as a semidirect product

\[
N_K(n) = \left[ \text{Aut}(g_c) \times N_{K_1}(n_1)^{l_1} \times \cdots \times N_{K_k}(n_k)^{l_k} \right] \rtimes S_l^{1, \ldots, k}.
\]

5. The group \( \text{Aut}^w(\Sigma) \) decomposes as a semidirect product

\[
\text{Aut}^w(\Sigma) = \left[ \text{Aut}^w(\Sigma_1)^{l_1} \times \cdots \times \text{Aut}^w(\Sigma_k)^{l_k} \right] \rtimes S_l^{1, \ldots, k}.
\]
(6) The group $W(\Sigma)$ decomposes as a product
\[ W(\Sigma) = W(\Sigma_1)^{l_1} \times \cdots \times W(\Sigma_k)^{l_k}. \]

(7) The group $\text{Aut}^w(DD)$ decomposes as a semidirect product
\[ \text{Aut}^w(DD) = [\text{Aut}^w(DD_1)^{l_1} \times \cdots \times \text{Aut}^w(DD_k)^{l_k}] \rtimes S_{l_1,\ldots,l_k}. \]

(8) With respect to the decompositions (3) and (5), the homomorphism $\Omega: N_K(a) \to \text{Aut}^w(\Sigma)$ decomposes as
\[ \Omega = \left( E, \Omega_1^{l_1}, \ldots, \Omega_k^{l_k}, \text{Id}_{S_{l_1,\ldots,l_k}} \right), \]
where $E$ is the trivial homomorphism $\text{Aut}(g_c) \to \{e\}$, $\Omega_i: N_{K_i}(a_i) \to \text{Aut}^w(\Sigma_i)$, and the last component $\text{Id}_{S_{l_1,\ldots,l_k}}$ formally means that the following diagram commutes:

```
  N_K(a)  \Omega
   ↓       ↓
  S_{l_1,\ldots,l_k}  \Omega
   ↓       ↓
  Aut^w(\Sigma)
```

We omit the proof, as it simply boils down to the fact that any automorphism of $g$ must preserve $g_c$ and permute the remaining noncompact simple ideals, and the same is true for weighted automorphisms of $\Sigma = \Sigma_1^{l_1} \sqcup \cdots \sqcup \Sigma_k^{l_k}$.

Part (8) of Proposition 16 implies that if $\Omega_i(N_{K_i}(n_i)) = \text{Aut}^w(DD_i)$ for each $i$, then $\Omega(N_K(n)) = \text{Aut}^w(DD)$. Consequently, in order to prove Theorem 15, we may restrict to the case when $g$ is simple and noncompact. We will actually show that $\Omega(N_K(n)) = \text{Aut}(DD)$ in this case, thus proving both parts (1) and (2) of the theorem.

We will consider three different scenarios. To begin with, we can immediately cast aside all those simple Lie algebras where $\text{Aut}(DD)$ (and hence $\text{Aut}^w(DD)$) is trivial. This leaves us with those Lie algebras where $\Sigma = A_n (n \geq 2), D_n (n \geq 4)$, or $E_6$.

Each complex semisimple Lie algebra $g$ gives rise to at least two noncompact real ones: the realification and the split real form of $g$. These are going to be our first two scenarios. As a matter fact, here we do not require the Lie algebra to be simple, and we will only use that assumption in the third scenario. Let $g$ be a complex semisimple Lie algebra, and let $h, \Delta, \Lambda$, and $\{h_i, e_i, f_i\}_{i=1}^r$ be as on page 7. It follows from the Isomorphism Theorem that there exists a unique automorphism $\Theta$ of $g$ as of a real Lie algebra that is $\mathbb{C}$-antilinear and
satisfies
\[ \theta(h_i) = -h_i, \quad \theta(e_i) = -f_i, \quad \theta(f_i) = -e_i. \tag{1} \]

This automorphism is involutive and is in fact a compact real structure and a Cartan involution\(^5\) (hence the notation). In particular, \( p = i \mathfrak{k} \). Every Cartan involution on \( \mathfrak{g} \) is of this form (for some choice of \( \mathfrak{h}, \Lambda, \) and canonical generators). We can introduce two more involutive automorphisms of \( \mathfrak{g} \): the Weyl involution \( \omega \) and the split real structure \( \tau \). The Weyl involution is given on the canonical generators by the same formula (1) but is \( \mathbb{C} \)-linear, whereas the split real form fixes all the canonical generators but is \( \mathbb{C} \)-antilinear. Once again, the existence and uniqueness of both of these automorphisms follow from the Isomorphism Theorem. Clearly, the three automorphisms commute pairwise and the product of any two of them equals the third one. The fixed point (real) subalgebra \( \mathfrak{g}^r \) is the split real form of \( \mathfrak{g} \), and every split real form is of this form (for some choice of \( \mathfrak{h}, \Lambda, \) and canonical generators).

**Scenario 1: the realification.** Here we assume that our real semisimple Lie algebra is the realification of a complex one and use the notation established in the previous paragraph. In particular, the Cartan involution \( \theta \) is given on the canonical generators by (1). Write \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) for the corresponding Cartan decomposition. Since \( \theta \) is \( \mathbb{C} \)-antilinear and \( \theta(h_i) = -h_i \) for each \( i \), we have \( \mathfrak{h} \cap \mathfrak{p} = \mathfrak{h}(\mathbb{R}), \mathfrak{h} \cap \mathfrak{k} = i\mathfrak{h}(\mathbb{R}) \). We claim that \( \mathfrak{h}(\mathbb{R}) \) is a maximal abelian subspace of \( \mathfrak{p} \). Indeed, let \( \mathfrak{b} \subset \mathfrak{p} \) be an abelian subspace containing \( \mathfrak{h}(\mathbb{R}) \). If we think of \( \mathfrak{g} \) as a real Euclidean vector space (with respect to the inner product \( B_\theta \)), then all operators of the form \( \text{ad}(X), X \in \mathfrak{p} \), are self-adjoint (this is true for any real semisimple Lie algebra). Such operators are also \( \mathbb{C} \)-linear, they are also self-adjoint with respect to the Hermitian inner product \( \langle X \mid Y \rangle = B_\theta(X, Y) + iB_\theta(X, iY) \) and hence diagonalizable over \( \mathbb{C} \). It follows that \( \mathfrak{b}_\mathbb{C} = i\mathfrak{b} \oplus \mathfrak{b} \) (here \( i\mathfrak{b} \subseteq i\mathfrak{p} = \mathfrak{k} \)) is an abelian complex subalgebra of \( \mathfrak{g} \) consisting of semisimple elements and containing \( \mathfrak{h} \), hence we must have \( \mathfrak{b}_\mathbb{C} = \mathfrak{h} \) and thus \( \mathfrak{b} = \mathfrak{h}(\mathbb{R}) \). So we can write \( \mathfrak{a} = \mathfrak{h}(\mathbb{R}) \). In this case, \( \mathfrak{a}^* = \mathfrak{h}^*(\mathbb{R}) \), and the root system \( \Delta \) of \( \mathfrak{g} \) as of a complex semisimple Lie algebra coincides with the restricted root system \( \Sigma \) of \( \mathfrak{g} \) as of a real semisimple Lie algebra (for our specific choice of \( \theta \) and \( \mathfrak{a} \)). Moreover, the root space decomposition and the restricted root space decomposition coincide as well. Note that \( \mathfrak{g}_0 = \mathfrak{h} = i\mathfrak{a} \oplus \mathfrak{a} \) and \( \mathfrak{k}_0 = i\mathfrak{a} \). Each restricted root subspace \( \mathfrak{g}_\alpha, \alpha \in \Sigma \), thus has real dimension 2, i.e. all of the root multiplicities equal 2. As we know from Section 2, every (not necessarily weighted) automorphism \( s \in \text{Aut}(\mathfrak{DD}) \) can be lifted to a (complex) automorphism \( \hat{s} \) of \( \mathfrak{g} \) given by the rule \( \hat{s}(h_i) = h_{s(i)}, \hat{s}(e_i) = e_{s(i)}, \hat{s}(f_i) = f_{s(i)} \). This automorphism satisfies \( \hat{s}(\mathfrak{a}) = \mathfrak{a} \) and \( \Psi(\hat{s}) = (\hat{s}|_\mathfrak{a})^{-1} = s \). It is easily seen on the canonical generators that \( \hat{s} \) commutes with \( \theta \) and preserves \( \mathfrak{n} \). Consequently, \( \hat{s} \in N_K(\mathfrak{n}) \) and \( \Omega(\hat{s}) = \Psi(\hat{s}) = 1 \), which finishes the proof in this scenario.

**Scenario 2: split real form.** Now suppose that our real semisimple Lie algebra is a split real form of its complexification. We denote the latter by \( \mathfrak{g} \) and use the same notation as above. Then our real semisimple Lie algebra can be described as \( \mathfrak{g}^r \). One can easily see that \( \mathfrak{g}^r \) is the real Lie subalgebra of \( \mathfrak{g} \) generated by \( e_i, f_i, \) and \( h_i, 1 \leq i \leq r \). The automorphism

\(^5\)The latter two notions coincide for any complex semisimple Lie algebra, see [10, Sec. 5]
\( \theta \) of \( \mathfrak{g} \) clearly preserves \( \mathfrak{g}^\tau \). Since \( \mathfrak{g}^\tau \) is a real form of \( \mathfrak{g} \), the Killing form \( B \) of \( \mathfrak{g} \) is the \( \mathbb{C} \)-bilinear extension of the Killing form \( B^\tau \) of \( \mathfrak{g}^\tau \). This implies that the restriction of \( \theta \) to \( \mathfrak{g}^\tau \) is a Cartan involution on \( \mathfrak{g}^\tau \). Moreover, if we write the corresponding Cartan decomposition as \( \mathfrak{g}^\tau = \mathfrak{k} \oplus \mathfrak{p} \), then \( \mathfrak{h}(\mathbb{R}) \) lies in \( \mathfrak{p} \) and is its maximal abelian subspace. Indeed, for each \( X \in \mathfrak{p} \), the operator \( \text{ad}_{\mathfrak{g}^\tau}(X) \) is diagonalizable over \( \mathbb{R} \), hence its \( \mathbb{C} \)-linear extension \( \text{ad}_{\mathfrak{g}}(X) \) is diagonalizable over \( \mathbb{C} \) as an operator on \( \mathfrak{g} \). Therefore, in the same fashion as above, the existence of a larger abelian subspace of \( \mathfrak{p} \) would lead to a toral subalgebra of \( \mathfrak{g} \) larger than \( \mathfrak{h} \), hence a contradiction. Once again, we can take \( \mathfrak{a} = \mathfrak{h}(\mathbb{R}) \), in which case \( \mathfrak{a}^* = \mathfrak{h}^*(\mathbb{R}) \) and the restricted root system \( \Sigma \) of \( \mathfrak{g}^\tau \) coincides with \( \Delta \). Just as before, every diagram automorphism \( s \in \text{Aut}(\text{DD}) \) can be lifted to the complex automorphism \( \hat{s} \in \text{N}_{\text{Aut}(\mathfrak{g})}(\mathfrak{h}) \) of \( \mathfrak{g} \) such that \( \Psi(\hat{s}) = s \), and it can be seen from the defining formula for \( \hat{s} \) that it commutes with both \( \theta \) and \( \tau \). In particular, it preserves \( \mathfrak{g}^\tau \) and the restriction \( \hat{s}|_{\mathfrak{g}^\tau} \) lies in \( \text{N}_K(\mathfrak{n}) \). We have \( \Omega(\hat{s}|_{\mathfrak{g}^\tau}) = \Psi(\hat{s}) = s \), which finishes the proof in scenario 2.

**Scenario 3: the rest.** Now we go back to our assumption that \( \mathfrak{g} \) is simple and \( \text{Aut}(\text{DD}) \) is nontrivial. An examination of the list of all real simple noncompact Lie algebras ([1, pp. 336-340]) reveals that if \( \mathfrak{g} \) is neither split nor complex, it has to be isomorphic to either \( \mathfrak{sl}(n, \mathbb{H}) \) \((n \geq 3)\) or \( \mathfrak{c}_6^{−26} \). The restricted root systems of these Lie algebras are \( A_{n−1} \) and \( A_2 \), respectively. In both cases, \( \text{Aut}(\text{DD}) \cong \mathbb{Z}/2\mathbb{Z} \), and there is only one nontrivial diagram automorphism that we want to lift to \( N_K(\mathfrak{n}) \). Recall that we have a distinguished automorphism \( \theta \) of \( \mathfrak{g} \) fixed. Plainly, \( \theta \in N_K(\mathfrak{a}) \) and \( \Omega(\theta) = −\text{Id}_{\mathfrak{a}^*} \). The weighted root system automorphism \( −\text{Id}_{\mathfrak{a}^*} \) can be decomposed as \( −\text{Id}_{\mathfrak{a}^*} = w_0s \), where \( w_0 \in \text{W}(\Sigma) \) and \( s \in \text{Aut}^w(\text{DD}) \). Here \( s(D) = D \) and \( −\text{Id}_{\mathfrak{a}^*}(D) = −D \), so \( w_0(D) = −D \) (this uniquely determines \( w_0 \) and also shows that it is the longest element of \( \text{W}(\Sigma) \) with respect to the system of generators \( s_{\alpha_1}, \ldots, s_{\alpha_n} \)). The diagram automorphism \( s = −w_0 \) may or may not be trivial, depending on \( \Sigma \). Note that this construction does not really rely on \( \mathfrak{g} \), nor does it use root multiplicities, so it can be carried out for any root system \( (V, \Delta) \): pick \( \Delta^\tau \) and decompose \( −\text{Id}_{\mathfrak{V}} \in \text{Aut}(\Delta) \) as \( −\text{Id}_{\mathfrak{V}} = w_0s \) with respect to the semidirect product decomposition \( \text{Aut}(\Delta) = \text{W}(\Delta) \rtimes \text{Aut}(\text{DD}) \). It was shown in [10, §4, Prop. 4] that, in case \( \Delta \) is irreducible, \( s \) is a nontrivial diagram automorphism precisely when \( \Delta = A_n \) \((n \geq 2)\), \( D_{2n+1} \) \((n \geq 2)\), or \( E_6 \). This covers both of our cases \( \mathfrak{g} = \mathfrak{sl}(n, \mathbb{H}) \) \((n \geq 3)\) and \( \mathfrak{g} = \mathfrak{c}_6^{−26} \). Now, we know from the discussion after Proposition 10 that there exists \( \varphi \in N_K(\mathfrak{a}) \) such that \( \Omega(\varphi) = w_0 \). We have \( \varphi \theta \in N_K(\mathfrak{n}) \) and \( \Omega(\varphi \theta) = w_0^2s = s \). In other words, the only nontrivial element of \( \text{Aut}(\text{DD}) \) lies in the image of \( \Omega \) and so \( \Omega(N_K(\mathfrak{n})) = \text{Aut}(\text{DD}) \), which completes the proof of Theorem 15.

**Remark 17.** Part (2) of Theorem 15 can also be proven simply by examining the classification of simple noncompact Lie algebras and the list of their (weighted) Dynkin diagrams ([1, pp. 336-340]). Indeed, for any such diagram, if there are two vertices that differ by a (not necessarily weighted) diagram automorphism, then they happen to have the same multiplicity, which implies that every diagram automorphism is weighted.

**Corollary 15.1.** Let \( \mathfrak{g} \) be a real semisimple Lie algebra. Then we have \( N_K(\mathfrak{a})/Z_K(\mathfrak{a}) \cong \text{Aut}^w(\Sigma) \) and \( N_K(\mathfrak{n})/Z_K(\mathfrak{n}) \cong \text{Aut}^w(\text{DD}) \).
The results of the previous section have a useful interpretation in terms of the theory of symmetric spaces of noncompact type. It is well known that the isometry group $G \rtimes M$ is precisely the Lie subalgebra $\mathfrak{g}_c$ of the (transitive) action $G \rtimes M$, that sends $X$ to the corresponding fundamental vector field $X^*$. This map is an injective anti-homomorphism of Lie algebras and its image is precisely the Lie subalgebra $\mathfrak{K}(M) \subseteq \mathfrak{X}(M)$ of Killing vector fields. Denote $G = \overline{G}^0$. Since $M$ is connected, the action $G \rtimes M$ is transitive as well.

Pick any $o \in M$, write $\overline{K}$ for the isotropy subgroup of $\overline{G}$ at $o$, and denote $K = \overline{K} \cap G$. The action $\overline{G} \rtimes M$ (as well as $G \rtimes M$) is proper (see [3]), which implies that $\overline{K}$ is a compact subgroup of $\overline{G}$. Moreover, $\overline{G}/\overline{K} = M$ is contractible, so $\overline{K}$ is a maximal compact subgroup of $\overline{G}$, $K$ is a maximal compact subgroup of $G$, and $K = \overline{K}^0$ is connected. We have the geodesic symmetry $s_o \in \overline{G}$ at $o$, which gives rise to an involutive automorphism

**Corollary 15.2.** Let $\mathfrak{g}, \mathfrak{g}'$ be real semisimple Lie algebras with $\theta, \theta', a, a', \Sigma, \Sigma'^+$, and $\Sigma'^+$ fixed.

1. Every weighted isomorphism $f : a \supseteq a^*$ between $\Sigma$ and $\Sigma'$ is an isometry. In particular, $\text{Aut}^w(\Sigma) \subseteq \text{O}(a^*)$.

2. Now assume that neither $\mathfrak{g}$ nor $\mathfrak{g}'$ have nonzero compact ideals. Then for every weighted isomorphism $f : a \supseteq a^*$ between $\Sigma$ and $\Sigma'$, there exists a Lie algebra isomorphism $F : \mathfrak{g} \rightarrow \mathfrak{g}'$ such that $F \circ \theta = \theta' \circ F$ and $F(a) = a'$ and the induced weighted isomorphism $a^* \supseteq a'^*$ between $\Sigma$ and $\Sigma'$ coincides with $s$. In particular, for every weighted diagram isomorphism $s : \Lambda \supseteq \Lambda'$ between $\mathfrak{D}$ and $\mathfrak{D}'$, there exists such $F : \mathfrak{g} \rightarrow \mathfrak{g}'$ that the induced diagram isomorphism $f|_{\Lambda} : \Lambda \supseteq \Lambda'$ coincides with $s$.

**Proof.** For part (2), we know from Remark 14 that there exists some Lie algebra isomorphism $\overline{F} : \mathfrak{g} \rightarrow \mathfrak{g}'$. As explained in the proof of Proposition 10, we may assume $\overline{F} \circ \theta = \theta' \circ \overline{F}$ and $\overline{F}(a) = a'$. Let $\tilde{f} : a^* \supseteq a'^*$ be the induced weighted isomorphism between $\Sigma$ and $\Sigma'$. Then $f \circ \tilde{f}^{-1} \in \text{Aut}^w(\Sigma')$. According to Theorem 15, there exists $\varphi' \in N_{\mathfrak{k}'}(\mathfrak{a}')$ such that $\Omega'(\varphi') = f \circ \tilde{f}^{-1}$. Then the weighted root system isomorphism between $\Sigma$ and $\Sigma'$ induced by $F = \varphi' \circ \overline{F}$ is $\Omega'(\varphi') \circ \tilde{f} = f$. For part (1), write $\mathfrak{g} = \mathfrak{g}_c \oplus \mathfrak{g}_{nc}$, where $\mathfrak{g}_c$ is the maximal compact ideal and $\mathfrak{g}_{nc}$ is the complementary noncompact ideal, and do the same for $\mathfrak{g}' : \mathfrak{g}' = \mathfrak{g}'_c \oplus \mathfrak{g}'_{nc}$. The restricted root systems of $\mathfrak{g}$ and $\mathfrak{g}'$ coincide with those of $\mathfrak{g}_{nc}$ and $\mathfrak{g}'_{nc}$, respectively, so every weighted isomorphism between them is an isometry by part (2) and our observation after Definition 9. 

4. **Symmetric spaces of noncompact type**

The results of the previous section have a useful interpretation in terms of the theory of symmetric spaces of noncompact type, morally because those are 'the same' as noncompact real semisimple Lie algebras. See [6, Ch. IV-VI] for a background on symmetric spaces.

Let $M$ be a symmetric space of noncompact type. It is well known that the isometry group $I(M)$ is a Lie group in the compact-open topology and its action on $M$ is smooth. We denote it by $\overline{G}$ and its Lie algebra by $\mathfrak{g}$. We have a map $\mathfrak{g} \rightarrow \mathfrak{X}(M)$, called the infinitesimal generator of the (transitive) action $\overline{G} \rtimes M$, that sends $X$ to the corresponding fundamental vector field $X^*$. This map is an injective anti-homomorphism of Lie algebras and its image is precisely the Lie subalgebra $\mathfrak{K}(M) \subseteq \mathfrak{X}(M)$ of Killing vector fields. Denote $G = \overline{G}^0$. Since $M$ is connected, the action $G \rtimes M$ is transitive as well.
\( \Theta = C_{s_o} : g \mapsto s_og s_o^{-1} = s_og s_o \) of \( \widetilde{G} \) and thus to an involutive automorphism \( \Theta = (C_{s_o})_* = \text{Ad}(s_o) \) of \( \mathfrak{g} \). Since \( M \) is of noncompact type, \( \widetilde{G} \) (and thus \( \mathfrak{g} \)) is semisimple and \( \Theta \) is a Cartan involution on \( \mathfrak{g} \), so we can write \( \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \) for the corresponding Cartan decomposition\(^6\). One can show that \( \widetilde{K} \) is an open subgroup of \( \widetilde{G}^\Theta \), which means that \( \text{Lie}(\widetilde{K}) = \mathfrak{k} \) and \( \text{Ad}_{\widetilde{G}}(\widetilde{K}) \subseteq \text{Aut}(\mathfrak{g})^\Theta \). In particular, the adjoint representation of \( \widetilde{K} \) on \( \mathfrak{g} \) is orthogonal with respect to \( B_o \) and preserves \( \mathfrak{k} \) and \( \mathfrak{p} \).

Consider the orbit map \( \widetilde{G} \twoheadrightarrow M, g \mapsto g \cdot o \). Its differential at \( e \) is a surjective linear map \( \pi : \mathfrak{g} \to T_oM \) with \( \text{Ker}(\pi) = \mathfrak{k} \), so it maps \( \mathfrak{p} \) isomorphically onto \( T_oM \). We always identify \( \mathfrak{p} \) and \( T_oM \) by means of this isomorphism. One can easily show that this is an isomorphism of \( \widetilde{K} \)-representations: between the adjoint representation of \( \widetilde{K} \) in \( \mathfrak{p} \) and its isotropy representation in \( T_oM \).

**Remark 18.** Note that if we were to pick another point \( o' = g \cdot o \in M \), we would have \( s_{o'} = g s_o g^{-1} \) and hence \( \Theta' = \text{Ad}(s_{o'}) = \text{Ad}(g) \Theta \text{Ad}(g)^{-1} \). Consequently, the corresponding Cartan decompositions differ by \( \text{Ad}(g) : \text{Ad}(g)(\mathfrak{k}) = \mathfrak{k}', \text{Ad}(g)(\mathfrak{p}) = \mathfrak{p}' \).

If we denote the Riemannian metric on \( M \) by \( g \), then \( g_o \) is a \( \widetilde{K} \)-invariant inner product on \( T_oM \), which is the same as an \( \text{Ad}_{\widetilde{G}}(\widetilde{K}) \)-invariant inner product on \( \mathfrak{p} \). From the previous section, we know that another such inner product is \( B_o|_{\mathfrak{p} \times \mathfrak{p}} = B|_{\mathfrak{p} \times \mathfrak{p}} \). In general, these two inner products may not coincide. However, the difference between them allows an explicit description.

Let \( M = M_1 \times \cdots \times M_k \) be the de Rham decomposition of \( M \) (see \([8, \text{Ch. IV, Sec. 5, 6}]\) for more on the de Rham decomposition). We say that \( M_i \) and \( M_j \) are homothetic and write \( M_i \sim M_j \) if there exists a diffeomorphism \( \varphi : M_i \cong M_j \) such that \( \varphi^* g_j = a g_i \) for some \( a > 0 \) (i.e., \( \varphi \) is ‘almost’ an isometry: it is conformal with a constant conformal factor). This is an equivalence relation on the set of de Rham factors, and it is in general weaker than \( M_i \simeq M_j \) (being isometric). The isometry group of \( M \) allows a description similar to that of \( \text{Aut}(\mathfrak{g}) \) (established in Proposition 16(1)). We will present it in a slightly different form without grouping isometric de Rham factors together. We define:

\[
S^\sim_k = \{ \sigma \in S_k \mid M_i \simeq M_{\sigma(i)} \forall i \in \{1, \ldots, k\} \},
\]

\[
S^\approx_k = \{ \sigma \in S_k \mid M_i \approx M_{\sigma(i)} \forall i \in \{1, \ldots, k\} \}.
\]

These are subgroups of the symmetric group \( S_k \), and we have \( S^\approx_k \subseteq S^\sim_k \). Note that these definitions only make sense after we fix the (order in the) de Rham decomposition of \( M \). For any pair of indices \( i, j \in \{1, \ldots, k\} \) such that \( M_i \simeq M_j \), pick an isometry \( \varphi_{ij} : M_i \cong M_j \) in such a way that if we have \( M_i \simeq M_j \simeq M_l \), then \( \varphi_{jl} \circ \varphi_{ij} = \varphi_{il} \). This gives an embedding

\[
S^\approx_k \hookrightarrow I(M), \sigma \mapsto \varphi_\sigma, \text{ where}
\]

\[
\varphi_\sigma(p_1, \ldots, p_k) = (\varphi_{\sigma(1)}(p_{\sigma(1)}), \ldots, \varphi_{\sigma(k)}(p_{\sigma(k)}))
\]

\(^6\)In this section, we are no longer using the letter \( K \) to denote \( \text{Aut}(\mathfrak{g}) \).
(to be precise, this is an injective group anti-homomorphism). Surely, this embedding
depends on the choice of \( \varphi_{ij} \)'s. We also have an obvious embedding \( I(M_1) \times \cdots \times I(M_k) \subseteq I(M) \). The following result is proven in [11] (in much greater generality and in a slightly
different notation more akin to the one we used in Proposition 16):

**Fact 19.** Let \( M \) be a symmetric space of noncompact type and \( M = M_1 \times \cdots \times M_k \) its de
Rham decomposition. Then the group \( I(M) \) decomposes as a semidirect product

\[
I(M) = [I(M_1) \times \cdots \times I(M_k)] \rtimes S_k^\infty.
\]

In particular, we have \( I^0(M) = I^0(M_1) \times \cdots \times I^0(M_k) \).

More generally, if \( M' \) is another symmetric space of noncompact type with the de Rham
decomposition \( M' = M'_1 \times \cdots \times M'_k \) and \( \varphi: M \cong M' \) is an isometry, then there exists a
permutation \( \sigma \in S_k \) and a collection of isometries \( \varphi_i: M_i \cong M'_{\sigma(i)}, 1 \leq i \leq k \), such that

\[
\varphi(p_1, \ldots, p_k) = (\varphi_{\sigma(1)}(p_{\sigma(1)}), \ldots, \varphi_{\sigma(k)}(p_{\sigma(k)})).
\]

As a consequence, the de Rham decomposition of \( M \) is essentially unique up to reordering
of the factors.

Let us write \( \tilde{G}_i = I(M_i), G_i = \tilde{G}_i^0, \mathfrak{g}_i = \text{Lie}(\tilde{G}_i) \). Denote \( o = (o_1, \ldots, o_k) \) and write \( \tilde{K}_i \)
for the stabilizer subgroup of \( \tilde{G}_i \) at \( o_i \) and \( K_i = \tilde{K}_i \cap G_i = \tilde{K}_i^0 \). We necessarily have \( s_o = (s_{o_1}, \ldots, s_{o_k}), \Theta = (\Theta_1, \ldots, \Theta_k), \) and \( \theta = (\theta_1, \ldots, \theta_k), \) where \( \theta_i = (\Theta_i)_* = \text{Ad}(s_{o_i}). \)

The latter decomposition of \( \theta \) implies that we have \( \mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i \) and \( \mathfrak{p} = \bigoplus_{i=1}^k \mathfrak{p}_i, \) just
like we had on page 14. It is well know from the theory of symmetric spaces that \( M \)
is (de Rham) irreducible if and only if its isometry Lie algebra \( \mathfrak{g} \) is simple (this is no
longer true for symmetric spaces of compact type). In particular, each \( \mathfrak{g}_i \) is simple
(and noncompact). We stress that \( \mathfrak{g} \) has no nonzero compact ideals. The image of the
restricted isotropy representation \( K_i \hookrightarrow O(T_{o_i}M_i) \) coincides with the restricted holonomy
group \( \text{Hol}^0(M_i, o_i) \) (this is true for all semisimple symmetric spaces). In particular, this
representation is irreducible and thus any two \( K_i \)-invariant inner products on \( \mathfrak{p}_i \cong T_{o_i}M_i \)
are proportional. We already have two such inner products: \( B|_{\mathfrak{p}_i \times \mathfrak{p}_i} \) and \( (g_i)_{o_i} \) (here \( g_i \) is
the Riemannian metric on \( M_i \)). Therefore, we have \( (g_i)_{o_i} = \lambda_i B|_{\mathfrak{p}_i \times \mathfrak{p}_i} \), and we can write
\( g_o = \lambda_1 B|_{\mathfrak{p}_1 \times \mathfrak{p}_1} + \cdots + \lambda_k B|_{\mathfrak{p}_k \times \mathfrak{p}_k}. \) We call \( \lambda_1, \ldots, \lambda_k \) the normalizing constants of \( M \).

**Lemma 20.** The normalizing constants of \( M \) are well defined up to reordering.

**Proof.** Formally, the lemma means that if we are given another de Rham decomposition
\( M = M'_1 \times \cdots \times M'_k \) and base point \( o' = (o'_1, \ldots, o'_k) \) with the corresponding normalizing
constants \( \lambda'_1, \ldots, \lambda'_k, \) then there exists a permutation \( \sigma \in S_k \) such that \( M_i \cong M'_{\sigma(i)} \) and
\( \lambda_i = \lambda'_{\sigma(i)} \) for each \( i \in \{1, \ldots, k\} \). According to Fact 19, there exists \( \sigma \in S_k \) satisfying
the former condition: \( M_i \cong M'_{\sigma(i)} \). Since each de Rham factor is a symmetric space in its
own right, it is in particular a Riemannian homogeneous space, so we can pick isometries
\( \varphi_i : M_i \cong M'_{\sigma(i)} \), \( 1 \leq i \leq k \), such that \( \varphi_i(o_i) = o'_{\sigma(i)} \). We have a Lie group isomorphism \( F_i : \widetilde{G}_i \cong \widetilde{G}'_{\sigma(i)} \), \( g \mapsto \varphi_i \circ g \circ \varphi_i^{-1} \), such that \( F_i(s_{o_i}) = s_{o'_{\sigma(i)}} \), and thus the induced Lie algebra isomorphism \( f_i : \mathfrak{g}_i \cong \mathfrak{g}'_{\sigma(i)} \) satisfies \( f_i(e_i) = e'_{\sigma(i)} \) and \( f_i(p_i) = p'_{\sigma(i)} \). Plainly, \( f_i : p_i \cong p'_{\sigma(i)} \) is an isometry with respect to the inner products \( B_i|_{p_i \times p_i} \) and \( B'_{\sigma(i)}|_{p'_{\sigma(i)} \times p'_{\sigma(i)}} \), while \( d(\varphi_i)_{o_i} : T_o M_i \cong T'_{\sigma(i)} o M'_{\sigma(i)} \) is an isometry with respect to \( g_o \) and \( g'_{\sigma(i)} \). We have a commutative diagram

\[
\begin{array}{ccc}
p_i & \xrightarrow{f_i} & p'_{\sigma(i)} \\
\sim & \downarrow & \sim \\
T_o M_i & \xrightarrow{d(\varphi_i)_{o_i}} & T'_{\sigma(i)} o M'_{\sigma(i)}
\end{array}
\]

which implies that \( \lambda_i = \lambda'_{\sigma(i)} \). \( \square \)

**Proposition 21.** Given \( 1 \leq i, j \leq k \), consider the following conditions:

(i) \( M_i \) is isometric to \( M_j \).

(ii) \( M_i \) is homothetic to \( M_j \).

(iii) \( \mathfrak{g}_i \cong \mathfrak{g}_j \).

Then (i) \( \Rightarrow \) (ii) \( \iff \) (iii). Moreover, (ii) \( \Rightarrow \) (i) if and only if \( \lambda_i = \lambda_j \).

**Proof.** Clearly, (i) \( \Rightarrow \) (ii) and, since rescaling the Riemannian metric does not change the isometry Lie algebra, (ii) \( \Rightarrow \) (iii). To show (iii) \( \Rightarrow \) (ii), recall that \( M_i \) can be recovered from \( \mathfrak{g}_i \) up to an isometry by taking a simply connected Lie group \( \widetilde{G}_i \) with \( \text{Lie}(\widetilde{G}_i) \cong \mathfrak{g}_i \), a maximal compact subgroup \( \widetilde{K}_i \subseteq \widetilde{G}_i \), and endowing the quotient \( \widetilde{G}_i / \widetilde{K}_i \) with a suitable \( \widetilde{G}_i \)-invariant Riemannian metric (there is only one up to rescaling). If \( \mathfrak{g}_i \cong \mathfrak{g}_j \), we can always find a Lie group isomorphism \( F : \widetilde{G}_i \cong \widetilde{G}_j \) such that \( F(\widetilde{K}_i) = \widetilde{K}_j \), which induces a homothety \( M_i \rightarrow M_j \).

The proof of the last assertion is very similar to that of Lemma 20. Assume (ii) and start with any homothety \( \varphi : M_i \rightarrow M_j \). Composing it with a suitable isometry of \( M_j \) if needed, we can replace \( \varphi \) with a homothety \( \varphi' : M_i \rightarrow M_j \) mapping \( o_i \) to \( o_j \), and \( \varphi' \) is an isometry if and only if \( \varphi \) is. We have an isomorphism \( F : \widetilde{G}_i \cong \widetilde{G}_j \), \( \varphi \mapsto \varphi' \circ g \circ \varphi'^{-1} \), mapping \( s_{o_i} \) to \( s_{o_j} \). If we write \( f = F' : \mathfrak{g}_i \cong \mathfrak{g}_j \), then \( f(p_i) = p_j \), and the following diagram commutes:

\[
\begin{array}{ccc}
p_i & \xrightarrow{f} & p_j \\
\sim & \downarrow & \sim \\
T_o M_i & \xrightarrow{d(\varphi_{o_i}'} & T_o M_j
\end{array}
\]
The top arrow is an isometry with respect to the inner product $B\big|_{p_i \times p_i}$ and $B\big|_{p_j \times p_j}$. Consequently, the bottom arrow is an isometry (i.e., $\varphi'$ is an isometry) with respect to the inner product $(g_i)_{o_i} = \lambda_i B\big|_{p_i \times p_i}$ on $T_{o_i}M_i$ and $(g_j)_{o_j} = \lambda_j B\big|_{p_j \times p_j}$ on $T_{o_j}M_j$ if and only if $\lambda_i = \lambda_j$, which completes the proof. □

As we already know, condition (iii) in Proposition 21 is also equivalent to $\Sigma_i$ and $\Sigma_j$ (or $\text{DD}_i$ and $\text{DD}_j$) being weighted-isomorphic. Thus, the symmetric space $M$ is fully determined up to an isometry by the (weighted) Dynkin diagram $\text{DD}$ of $g$ together with the normalizing constants $\lambda_1, \ldots, \lambda_k$ (which we could assign as weights to the connected components $\text{DD}_1, \ldots, \text{DD}_k$ of $\text{DD}$). Proposition 21 has the following immediate

**Corollary 21.1.** The following conditions are equivalent:

(i) $\lambda_i = \lambda_j$ whenever $M_i \sim M_j$.

(ii) $S_{\text{k}^-} = S_{\text{k}^-}$.

If these conditions are satisfied, we call the Riemannian metric $g$ on $M$ **almost Killing**. If, moreover, $\lambda_1 = \ldots = \lambda_k = 1$ (i.e. if $g_o = B\big|_{p \times p}$), we call $g$ **Killing**.

Note that the Riemannian metric of $M$ is automatically almost Killing if $M$ is irreducible. More generally, if no two distinct de Rham factors of $M$ are homothetic, than its Riemannian metric is almost Killing.

Since the isometry group is not affected by constant rescaling of the metric, Fact 19 tells us that by rescaling the metric on the de Rham factors of $M$ – that is, by adjusting the normalizing constants – we might ‘gain’ or ‘lose’ some connected components of $I(M)$ (whereas $I^0(M)$ always stays the same). From this perspective, the almost Killing condition on the Riemannian metric ensures precisely that the isometry group is as large as possible, namely that $I(M) \simeq [I(M_1) \times \cdots \times I(M_k)] \rtimes S_{\text{k}^-}$.

Having defined the normalizing constants, we can now formulate precisely the correspondence between symmetric spaces of noncompact type and noncompact real semisimple Lie algebras that we mentioned at the beginning of the section. Let $(M, g)$ be a symmetric space of noncompact type with the de Rham decomposition $M = M_1 \times \cdots \times M_k$. Note that if we rescale the Riemannian metric on each de Rham factor $M_i$ by some constant conformal factor $a_i > 0$ and denote the resulting metric on $M$ by $\tilde{g}$, then $(M, \tilde{g})$ is still a symmetric space of noncompact type and it has the same isometry Lie algebra. We say that $(M, \tilde{g})$ is obtained from $(M, g)$ by **rescaling the normalizing constants**. If $M'$ is another symmetric space of noncompact type, we say that $M$ and $M'$ are equivalent if they become isometric after a suitable rescaling of their normalizing constants. Note that this notion of equivalence is weaker than being homothetic. We can now formulate the aforementioned correspondence:
If we start with a real semisimple Lie algebra \( g \) without nonzero compact ideals, one way to get a symmetric space \( M \) of noncompact type with \( \text{Lie}(I(M)) \cong g \) is as follows. Take a simply connected Lie group \( G \) with \( \text{Lie}(G) \cong g \), take a maximal compact subgroup \( K \subset G \), write \( M = G/K \), and endow \( M \) with any \( G \)-invariant Riemannian metric. One can show that \( I(M) \cong G/Z(G) \) and thus \( \text{Lie}(I(M)) \cong g/\mathfrak{z}(g) = g \). We already used this construction in the proof of Proposition 21.

We can reformulate part (1) of Proposition 16 in a similar fashion to Fact 19. For any pair of indices \( i, j \in \{1, \ldots, k\} \) such that \( g_i \cong g_j \), pick an isomorphism \( f_{ij} : g_i \rightarrow g_j \) in such a way that if we have \( g_i \cong g_j \cong g_l \), then \( f_{jl} \circ f_{ij} = f_{il} \). It follows from Proposition 21 that this gives an embedding

\[
S_k^\sim \hookrightarrow \text{Aut}(g), \, \sigma \mapsto f_\sigma, \text{ where } f_\sigma(X_1, \ldots, X_k) = (f_\sigma(1)(X_{\sigma(1)}), \ldots, f_\sigma(k)(X_{\sigma(k)}))
\]

(again, this is an injective group anti-homomorphism). Proposition 16(1) now asserts that \( \text{Aut}(g) = [\text{Aut}(g_1) \times \cdots \times \text{Aut}(g_k)] \rtimes S_k^\sim \). We have an open subgroup

\[
\text{Aut}(g)_M = [\text{Aut}(g_1) \times \cdots \times \text{Aut}(g_k)] \rtimes S_k^\sim \subseteq \text{Aut}(g),
\]

which may or may not be a proper subgroup, depending on the normalizing constants\(^7\). Note that this subgroup does not depend on the choice of \( f_{ij} \)'s (although the embedding \( S_k^\sim \hookrightarrow \text{Aut}(g) \) surely does depend on this choice). By passing from \( \text{Aut}(g) \) to \( \text{Aut}(g)_M \), we are prohibiting those automorphisms of \( g \) that permute isomorphic simple ideals whose corresponding normalizing constants do not coincide.

We are now in a position to prove the following result, which characterizes \( I(M) \) intrinsically in terms of the Lie algebra \( g \) – at least when the metric is almost Killing:

**Proposition 22.** Let \( M \) be a symmetric space of noncompact type. The adjoint map \( \text{Ad} : I(M) \rightarrow \text{Aut}(g) \) is an open embedding of Lie groups with image \( \text{Aut}(g)_M \). Moreover, \( \text{Ad} \) is an isomorphism if and only if the Riemannian metric of \( M \) is almost Killing. In particular, we always have \( \text{Ad} : I^0(M) \Rightarrow \text{Inn}(g) \).

**Proof.** To begin with, observe that \( \text{Ad} \) is a local isomorphism. Indeed, its induced morphism of Lie algebras is \( \text{ad} : g \Rightarrow \text{Der}(g) = \text{Lie} \left( \text{Aut}(g) \right) \).

\(^7\)It may be preferable to write \( \text{Aut}(g)_{(M,g)} \) to avoid ambiguity in case one has another metric \( \tilde{g} \) obtained from \( g \) by rescaling the normalizing constants.
Next we prove that $\text{Ad}$ is injective. Assume that $\varphi \in \ker(\text{Ad})$. We first show that $\varphi(o) = o$. We have $(C_\varphi)_s = \text{Id}_g$, i.e. $C_\varphi|_o = \text{Id}_g$, which is the same as to say that $\varphi$ commutes with every element of $G$. In particular, it commutes with every element of $K$, which implies that $K$ stabilizes $\varphi(o)$. Assume that $\varphi(o) \neq o$. Then $K$ fixes every point of a geodesic $\gamma$ emanating from $o$ and passing through $\varphi(o)$. Let $v = \gamma(0) \in T_oM$. We see that $v$ is an invariant of the (restricted) isotropy representation of $K$ in $T_oM$. But it is well known that the subspace of invariants of the (restricted or full) isotropy representation of a symmetric space is contained in (the tangent space to) the flat factor, which $M$ does not have by definition. We deduce that $\varphi \in K$. But then $d\varphi_o = (\text{Ad}(\varphi))|_p = \text{Id}_p$, so $\varphi = e$. This, together with the previous paragraph, implies that $\text{Ad}$ embeds $I(M)$ into $\text{Aut}(g)$ as an open subgroup.

Finally, we prove that $\text{Im}(\text{Ad}) = \text{Aut}(g)_M$ (note that the rest will follow, as this subgroup equals the whole $\text{Aut}(g)$ if and only if $S^\infty_K = S^\infty$, i.e. if and only if the metric is almost Killing). Recall that if $V$ is a (finite-dimensional) vector space, then the representation of $\text{GL}(V)$ in $V$ admits a unique extension to a representation in the full tensor algebra $TV = \bigoplus_{p,q=0}^\infty T^{(p,q)}V$ by algebra automorphisms such that on $V^*$ it coincides with the dual representation. We will make use of the following description of the isotropy representation (see exercise A.6 on p. 227 in [6] for the statement and p. 564 there for a solution):

**Fact 23.** Let $M$ be a simply connected symmetric space, $o \in M$, and $K \subseteq I(M)$ the full isotropy subgroup at $o$. Then $T \in \text{GL}(T_oM)$ lies in the image of the isotropy representation $\tilde{K} \hookrightarrow \text{GL}(T_oM)$ if and only if it preserves the inner product\(^\text{8}\) $g_o \in T^{(0,2)}T_oM$ and the curvature tensor $R_o \in T^{(1,3)}T_oM$.

Fix $o \in M$ as before and consider the Cartan involution $\theta = \text{Ad}(s_o)$ and the subgroup $\text{Aut}(g)^\theta \subseteq \text{Aut}(g)$ (we used to denote it by $K$ in Section 3). We want to show that $\text{Aut}(g)^\theta$ intersects every connected component of $\text{Aut}(g)$. Take any $\eta \in \text{Aut}(g)$. The automorphism $\eta \theta \eta^{-1}$ is also a Cartan involution. Since all Cartan involutions are conjugate by inner automorphisms, there exists $\delta \in \text{Inn}(g) = \text{Aut}^0(g)$ such that $\delta \eta \theta \eta^{-1} \delta^{-1} = \theta$, i.e. $\delta \eta \in \text{Aut}(g)^\theta$. As $\delta \eta$ and $\eta$ lie in the same connected component of $\text{Aut}(g)$, we are done.

It then suffices to show that $\text{Im}(\text{Ad})$ contains\(^\text{9}\)

$$\text{Aut}(g)_M^\theta = \text{Aut}(g)^\theta \cap \text{Aut}(g)_M = \left[\text{Aut}(g_1)^\theta_1 \times \cdots \times \text{Aut}(g_k)^\theta_k\right] \rtimes S^\infty_k.$$  

Take any element $\eta$ of this subgroup. It preserves the Cartan decomposition $g = \mathfrak{k} \oplus \mathfrak{p}$, so we can write $T = \eta|_p \in \text{GL}(\mathfrak{p}) \cong \text{GL}(T_oM)$. We claim that $T$ lies in the image of the isotropy representation $\tilde{K} \hookrightarrow \text{O}(T_oM)$. Due to Fact 23, it suffices to show that it is orthogonal and preserves the curvature tensor. As any other automorphism of $g$, $\eta$ is orthogonal with respect to $B$, so $T$ is orthogonal with respect to $B|_{\mathfrak{p} \times \mathfrak{p}}$. By construction, for every

\(^{8}\)This first condition cuts out precisely $\text{O}(T_oM)$.

\(^{9}\)In order for the RHS in this expression to make sense, we need to put an additional requirement on the choice of $f_{ij}$'s, namely $f_{ij} \circ \theta_i = \theta_j \circ f_{ij}$ for any $i, j$. 
$i \in \{1, \ldots, k\}$, if we write $T(p_i) = p_j$, then $\lambda_i = \lambda_j$, so $T|_{p_i} : p_i \cong p_j$ is an isometry with respect to the inner products $(g_i)_{o_i}$ and $(g_j)_{o_j}$, which implies that $T$ is orthogonal with respect to $g_o$ as well.

The fact that $T$ preserves the curvature tensor at $o$ can be readily seen from the well-known expression for $R_o$ under the identification $T_oM \cong p$ if $X, Y, Z \in p$ then $R_o(X, Y)Z = -[[X, Y], Z]$. We deduce that there exists $k \in \tilde{K}$ such that $\text{Ad}(k)$ and $\eta$ coincide on $p$. Since they are both Lie algebra automorphisms, they have to coincide on the subspace $[p, p] \subseteq \mathfrak{k}$ as well. The rest follows from the following

**Lemma 24.** Let $\mathfrak{g}$ be a real semisimple Lie algebra without nonzero compact ideals, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition. Then $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$.

**Proof of the lemma.** If $\mathfrak{g}$ is simple, this is the content of [7, Ch. VI, problems 22-24] (see p. 735 for solutions). If $\mathfrak{g}$ is the sum of several noncompact simple ideals, namely $\mathfrak{g} = \bigoplus_{i=1}^k \mathfrak{g}_i$, then $\mathfrak{k} = \bigoplus_{i=1}^k \mathfrak{k}_i$, $\mathfrak{p} = \bigoplus_{i=1}^k \mathfrak{p}_i$, where $\mathfrak{k}_i = \mathfrak{k} \cap \mathfrak{g}_i$, $\mathfrak{p}_i = \mathfrak{p} \cap \mathfrak{g}_i$, so we have $[\mathfrak{p}_i, \mathfrak{p}_i] = \mathfrak{k}_i$ and the assertion follows. □

Looking at the proof of Proposition 22, we have the following

**Corollary 22.1.** In the notation of Proposition 22, the map $\text{Ad}_{\tilde{G}} : \tilde{K} \hookrightarrow \text{Aut}(\mathfrak{g})^\theta$ is an open embedding of Lie groups with image $\text{Aut}(\mathfrak{g})^\theta_M$. Moreover, its image is the whole $\text{Aut}(\mathfrak{g})^\theta$ if and only if the Riemannian metric is almost Killing. In particular, we always have $\text{Ad}_{\tilde{G}} : \tilde{K} \cong \text{Inn}(\mathfrak{g})^\theta$ and $\tilde{K} = \tilde{G}^\theta$ (and thus $K = G^\Theta$).

**Proof.** We need only prove the very last assertion. We compute:

\[
\text{Ad}_{\tilde{G}}(\tilde{G}^\Theta) = \text{Ad}_{\tilde{G}}(\tilde{G})^\theta = \text{Ad}(\tilde{G}) \cap \text{Aut}(\mathfrak{g})^\theta = \text{Aut}(\mathfrak{g})_M \cap \text{Aut}(\mathfrak{g})^\theta = \text{Aut}(\mathfrak{g})^\theta_M = \text{Ad}_{\tilde{G}}(\tilde{K}) \text{ (by the first assertion),}
\]

so $\tilde{G}^\Theta = \tilde{K}$. □

We can use Proposition 22 to prove the following simple result, which tells how one can recover $M$ from its isometry Lie algebra $\mathfrak{g}$ in an invariant fashion. Given $\mathfrak{g}$ real semisimple, note that the set $C \subseteq \text{Aut}(\mathfrak{g})$ of all Cartan involutions on $\mathfrak{g}$ is an immersed submanifold of $\text{Aut}(\mathfrak{g})$, since it is an orbit of the adjoint action of $\text{Aut}(\mathfrak{g})$ on itself.

**Proposition 25.** Let $M$ be a symmetric space of noncompact type and $\mathfrak{g} = \text{Lie}(I(M))$. Then the map $\Xi : M \to C, p \mapsto \text{Ad}(s_p)$, is a diffeomorphism.
Note that we do not assume the Riemannian metric to be almost Killing here.

**Proof.** We know that each Ad(s_p) is indeed a Cartan involution. If we identify I(M) with a subgroup of Aut(g) by means of Ad, Ξ is easily seen to be I(M)-equivariant. As both M and C are smooth homogeneous I(M)-spaces, Ξ is smooth and surjective. Since we already know that Ad is injective, it suffices to show that the map M → I(M), p ↦ s_p, is injective. Given p ∈ M, observe that p is the only fixed point of the symmetry s_p (this is specific to symmetric spaces of noncompact type and utterly fails for symmetric spaces of compact type). Therefore, the symmetries s_p and s_q cannot coincide unless p = q, which completes the proof. □

Combining Propositions 16(1) and 22, we obtain the following:

**Corollary 26.** Let g be a real semisimple Lie algebra. For any Cartan involution θ on g, Aut(g)θ is a maximal compact subgroup of Aut(g) and Inn(g)θ is a maximal compact subgroup of Inn(g).

**Proof.** Let us write g = g_c ⊕ g_{nc}, where g_c is the sum of all compact ideals of g and g_{nc} is the complementary noncompact semisimple ideal. We know from Section 3 that θ respects this decomposition and is the identity on g_c, so we can write θ = Id_g_c + θ_{nc}. According to Proposition 16(1), Aut(g) = Aut(g_c) × Aut(g_{nc}), so Aut(g)θ = Aut(g_c) × Aut(g_{nc})θ_{nc}. Take a symmetric space M of noncompact type with Lie(I(M)) ≅ g_{nc} and o ∈ M such that Ad(s_o) = θ_{nc} (the latter is possible by Proposition 25). Rescale the normalizing constants of M if necessary (this would change neither the isometry Lie algebra nor s_o) so that the metric becomes almost Killing. By Proposition 22 and Corollary 22.1, we have Ad : I(M) ∼= Aut(g_{nc}) and Ad_{G_c}(K) = Aut(g_{nc})θ_{nc}, which implies that Aut(g_{nc})θ_{nc} is a maximal compact subgroup of Aut(g_{nc}). We are left to prove that Aut(g_c) is compact. This simply follows from the fact that it is a closed subgroup of O_{B_c}(g_c), and the latter is compact because the Killing form B_c of g_c is negative-definite. The last assertion follows easily, for (Aut(g)θ)^0 = Aut^0(g)θ = Inn(g)θ has to be a maximal compact subgroup of Aut^0(g) = Inn(g). □

Eventually, we can reformulate the results of Section 3 in the language of symmetric spaces. Let M be a symmetric space of noncompact type, and let us keep all the notation as above. Just as we did in Section 3, take a ⊆ p a maximal abelian subspace, Σ ⊆ a* the corresponding restricted root system, and DD its Dynkin diagram (for some choice Σ+). Let M = M_1 × ⋯ × M_k be the de Rham decomposition of M. In a similar fashion to Aut(g)_M, we can define:

\[
\text{Aut}^w(\Sigma)_M := \prod_{i=1}^{k} \text{Aut}(\Sigma_i) \rtimes S_k^\sim \subseteq \prod_{i=1}^{k} \text{Aut}(\Sigma_i) \rtimes S_k^\sim \simeq \text{Aut}^w(\Sigma),
\]

\[
\text{Aut}^w(DD)_M := \prod_{i=1}^{k} \text{Aut}(DD_i) \rtimes S_k^\sim \subseteq \prod_{i=1}^{k} \text{Aut}(DD_i) \rtimes S_k^\sim \simeq \text{Aut}^w(DD).
\]
We are implicitly using Theorem 15(2) here by writing \( \text{Aut}(\Sigma_i) \) and \( \text{Aut}(DD_i) \) instead of \( \text{Aut}^w(\Sigma_i) \) and \( \text{Aut}^w(DD_i) \), respectively.

Just as \( \text{Aut}(g)_M \) can be described as \( \text{Im}(\text{Ad}_{\tilde{G}}) \) due to Proposition 22, the groups \( \text{Aut}^w(\Sigma)_M \) and \( \text{Aut}^w(DD)_M \) allow a neat alternative description as well. Indeed, note that we could endow \( a^* \) with an alternative inner product by considering \( g_o |_{a \times a} \) and carrying it to \( a^* \) along the induced isomorphism \( a \cong a^* \). Let us denote the corresponding orthogonal group by \( O_{g_o}(a^*) \). It follows by a straightforward computation that:

\[
\begin{align*}
\text{Aut}^w(\Sigma)_M &= \text{Aut}^w(\Sigma) \cap O_{g_o}(a^*), \\
\text{Aut}^w(DD)_M &= \text{Aut}^w(DD) \cap O_{g_o}(a^*).
\end{align*}
\]

It easily follows from Theorem 15 and Proposition 16 that:

\[
\begin{align*}
W(\Sigma) &\subseteq \text{Aut}^w(\Sigma)_M, \\
\text{Aut}^w(\Sigma)_M &= W(\Sigma) \rtimes \text{Aut}^w(DD)_M, \\
\Omega(N_{\text{Aut}(g)}^\theta(a)) &= \text{Aut}^w(\Sigma)_M, \\
\Omega(N_{\text{Aut}(g)}^\theta(n)) &= \text{Aut}^w(DD)_M.
\end{align*}
\]

Consider the adjoint representation of \( \tilde{K} \) in \( g \) and the normalizer \( N_{\tilde{K}}(a) \) together with its subgroups \( N_K(a) \) and \( N_{\tilde{K}}(n) \). It easily follows from Corollary 22.1 that:

\[
\begin{align*}
\text{Ad}(N_{\tilde{K}}(a)) &= N_{\text{Aut}(g)}^\theta(a), \\
\text{Ad}(N_K(a)) &= N_{\text{Inn}(g)}^\theta(a), \\
\text{Ad}(N_{\tilde{K}}(n)) &= N_{\text{Aut}(g)}^\theta(n).
\end{align*}
\]

We arrive at the following result, which can be regarded as the geometric version of Theorem 15(1):

**Corollary 27.** Let \( M \) be a symmetric space of noncompact type. For every \( f \in \text{Aut}^w(\Sigma)_M \), there exists an isometry \( k \in N_{\tilde{K}}(a) \) such that \( \text{Ad}(k)|_{a^*} = f \). If \( f \in \text{Aut}^w(DD)_M \), then \( k \) necessarily lies in \( N_{\tilde{K}}(n) \), and if \( f \in W(\Sigma) \), then \( k \) can be chosen in \( N_K(a) \).

We finish off with the following application. Corollary 27 proves useful when one studies submanifolds of \( M \) and wants to understand whether two given submanifolds are isometrically congruent, i.e. one can be mapped onto the other by some global isometry of \( M \). One particular class of submanifolds that is especially important consists of so-called boundary components of \( M \). We will give only the necessary definitions and will not go into detail as it would require an exposition of the theory of parabolic subgroups and subalgebras.

Let \( \Lambda \) be a choice of simple roots for \( \Sigma \), and let \( \Phi \subseteq \Lambda \) be any subset. Write \( \Sigma_\Phi \) for the root subsystem of \( \Sigma \) spanned by \( \Phi \), and let \( g_\Phi \) be the Lie subalgebra of \( g \) generated by all the root subspaces \( g_\alpha \) as \( \alpha \) runs through \( \Sigma_\Phi \). If we let \( G_\Phi \) be the connected Lie subgroup of \( G \) corresponding to \( g_\Phi \), then the orbit \( B_{\Phi} = G_\Phi \cdot o \) is a totally geodesic submanifold...
of $M$ called a boundary component. It is itself a symmetric space of noncompact type of smaller rank ($\text{rank}(B_\Phi) = |\Phi|$). In a sense, boundary components of $M$ are its nicest possible totally geodesic submanifolds.

**Proposition 28.** Let $M$ be a symmetric space of noncompact type. Let $\Phi_1, \Phi_2 \subseteq \Lambda$, and assume that there exists $s \in \text{Aut}^w(DD)_M$ such that $s(\Phi_1) = \Phi_2$. Then the boundary components $B_{\Phi_1}$ and $B_{\Phi_2}$ are isometrically congruent.

**Proof.** According to Corollary 27, there exists some $k \in N_{\hat{K}}(n)$ such that $\Omega(\text{Ad}(k)) = (\text{Ad}(k)|_a)^{-1} = s$. Since $\text{Ad}(k)(g_\alpha) = g_{\Omega(\text{Ad}(k))(\alpha)} = g_{s(\alpha)}$, we must have $\text{Ad}(k)(g_{\Phi_1}) = g_{\Phi_2}$, which implies $kG_{\Phi_1}k^{-1} = G_{\Phi_2}$ and thus $k(B_{\Phi_1}) = B_{\Phi_2}$, which completes the proof. \qed

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