Gaussian approximation of a finite space disturbance dynamics in the boundary layer

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Abstract. Approximate approach for the simple description of weakly nonlinear wave packet dynamics in laminar boundary layer is suggested. To this end time dependent integral nonlinear equation was obtained for the spectral amplitudes of disturbances in single-mode approximation with Gauss form of spectral distributions of every component in the state of 3-wave resonance together with a component concentrated near the origin of wave vector space. The dynamics of the wave packet and amplitudes of its components by time is demonstrated.

1. Introduction
Starting with the work [1] intense researches of nonlinear processes in transition range of boundary layer arose. There exists two ultimate routines of transition such as N- and K-regimes depending on the initial conditions. The research [2] suggest the hypothesis about the resonance (harmonic, parametric and 3-wave) mechanisms of disturbance development, and gives experimental proof. The dynamics of resonance triplet in the boundary layer near flat plate in incompressible fluid is considered in [3]. Amplitude envelope equations (Schrödinger type) in weakly nonlinear approximation in 3-wave and higher order resonances constructed in [4] with the help of the averaging method. It was suggested in these works that the ratio between the imaginary and real part of wave frequency is small (small parameter of the problem) and the wave vector tips are rather far from the coordinate origin. The wave dynamics in N and K transition regimes was considered with the help of these equations in space setting of the problem. A hypothesis is formed that the continuous low frequency spectrum in N-regime arises due to multiple 3-wave resonance, and many other effects have been obtained. In real flows a finite spectral size disturbances were detected as the initial condition. In [5, 6] the ways of wave packet creation are considered and its evolution in time has been watched. In this connection the necessity in the weakly nonlinear wave packet dynamics description in the transition range arises. Further Gauss’s approximation of weakly nonlinear phenomena is given [7, 8] including 3-wave and harmonic resonance, dispersion, interaction with mean field and other effects in boundary layer on the flat plate in incompressible fluid at zero angle of attack.

2. The integral equation for the vertical velocity component disturbance
Let us represent velocity \((u,v,w)\) and pressure \(p\) as \(u = U + u', v = V + v', w = w', p = p'\) where \(U\) and \(V\) satisfy the boundary layer equations on the plate at the pressure gradient absence. Let us look at the length \(L = U_\infty/\tilde{\omega}^J\) and the time \(T = L/U_\infty\) where \(U_\infty\) is
free stream velocity, $\tilde{\omega}^I$ is maximum increment value of Tollmien-Schlichting waves on the plate. Then the scale of thickness is $\delta = L/R_l^{1/2} = (\nu L/U_\infty)^{1/2} = \nu/\tilde{\omega}^{1/2}$ and the small scale of time is $\tau = \delta/U_\infty$. At the same time $e^2 = \delta/L = 1/R = \nu\tilde{\omega}^{-1/2}$, $R = U_\infty\delta/\nu$, $\tilde{\omega} = (\delta/U_\infty)\omega^0 + i(\delta/U_\infty)\omega^I = \tilde{\omega}^R + i\tilde{\omega}^I$, $\tilde{\omega}^I = \omega^I/\tilde{\omega}^I$. Let’s consider non-dimensional values: $\bar{U} = U/U_\infty$, $e^2\bar{V} = V/V_\infty$, $\epsilon(\bar{u}, \bar{v}, \bar{w}) = (u'/u_\infty, v'/u_\infty, \omega'/u_\infty)$, $\epsilon\phi = \phi/\nu U_\infty^2$. Values $\bar{U}$ and $\bar{V}$ change in scale $T$ and $L$ by time $t$ and $x$-coordinate and in scale $\delta$ by $y$. The corresponding scales of change for pulsation values are $\tau$ by $t$ and $\delta$ by $x,y,z$. It is easy to deduce two equations looking like Orr-Sommerfeld and Squire equations in linear part if we combine initial equations for $u', v', w', \phi'$, the first is for $v'$ and the second is for vertical vorticity component $\eta$. At the same time the components $\bar{u}, \bar{w}$, which are present at the rest members of the equations can be defined from the continuity equation and the expression of vertical vorticity component. Squire equation contain the member proportional to the vertical velocity component. So we can write, in Fourier representation, $\bar{\eta} = -\bar{v}'k\beta(\alpha \bar{U} - \alpha \bar{X} - \bar{\omega}_OSk)^{-1}d\bar{U}/dy + O(e^2)$. In this expression the value $\bar{\omega}_{OSk}$ is equal to the eigenvalue of the unstable mode of Orr-Sommerfeld equation, $X_0(t)$ is the coordinate center of the reference system moving together with the wave packet of harmonics, $k$ is the wave vector. Let $\bar{v}'_k = f^{(0)}(0)\varphi^{(0)}_k(\bar{y}) + \ldots$ where $\varphi^{(0)}_k(\bar{y})$ is the unstable mode of Orr-Sommerfeld equation. If we multiply Orr-Sommerfeld equation by the eigenfunction $\chi^{(0)}_k(\bar{y})$ of the adjoint equation and integrate by $y$ from 0 to $\infty$ we will get for the amplitude $f_k^{[7]}$

$$\left(\frac{\partial}{\partial \tau} + i[\bar{\omega}^R(k) - \bar{X}\alpha]\right)f_k - \epsilon \int H_{k,k'} f_{k'} f_{k-k'} dk' - e^2 \frac{\partial}{\partial \alpha}(\bar{\omega}^R_0 f_k) - e^2\Omega_k f_k = 0 \quad (1)$$

Here $\Omega_k = \tilde{\omega}^I + Q_1 + Q_2$, $\bar{v}' = \text{Re}[f_k \psi^{(0)}_k \exp(ik \cdot r)]dr$, $k = (\alpha, \beta)$, $r = (\bar{x}, \bar{z})$. Values $H(k,k')$, $Q_l$, $l = 1, 2$, are defined by quadratures of $\varphi_k^{(0)}$, $\chi_k^{(0)}$ and $U, V$.

3. Equations for the wave packet envelope in Gauss’s approximation

We search the solution of (1) for spectral amplitude $Q$ in the form of compound wave packet which consists of the finite spectral size triplet, elements of which are concentrated near tips of wave vectors $Q$ in the state of 3-wave resonance and the range near coordinate origin in the wave vector space. In Gauss’s approximation it will be as follows:

$$f_k = \sum_{j=0}^3 (a_j(t)(\alpha\Delta_{\alpha j} + \Delta_{\beta j})^{-1} exp[-\Delta_{\alpha j}^{-2}(\alpha - \alpha_j)^2 - \Delta_{\beta j}^{-2}(\beta - \beta_j)^2 - i(\omega^R(k) - \alpha(\partial \omega^R(k)/\partial \alpha)_{k=k_j})t] + (c.c.), \quad \omega^R(k) = \alpha(\zeta_0 + \zeta_1 \sqrt{\alpha^2 + \beta^2})$$

Expression (c.c.) is equal to complex conjugate members to the first one, $j = 1, 2, 3$, $\zeta_1, \zeta_2$ are functions of Reynolds number. The envelope of the compound wave packet can be defined as

$$\varphi(x, z, t) = \text{Re}[\tilde{\phi}(2\pi)^{-2} \int_{-\infty}^{\infty} f_k \exp(i\alpha x + i\beta z) d\alpha d\beta]$$

Subexponential expressions can be decomposed in the series by $\Delta k = k - k_1$, $l = 1, 2, 3$ near tips of $k_1, k_2, k_3$ up to quadratic members in $\Delta \alpha, \Delta \beta$. The integral of such expression can be calculated in the explicit form if we shall do the shift in the space $(\Delta \alpha, \Delta \beta)$, exclude the linear terms in the quadratic form and rake into account the famous formula

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d\xi d\eta \exp(-Ay, y) = \pi \sqrt{\text{det}(A)}$$

which is correct in the case when the real part of coefficients at vector $y$ quadratic components are positive in the quadratic form $(Ay, y)$, $A = ((-1/\Delta_{\alpha}^2, a), (a, -1/\Delta_{\beta}^2))$ is a symmetric matrix, $y = (\xi, \eta)$, $(x, y)$ is the scalar product.
4. The equation for amplitudes, resonance triplet vector and widths of wave packets

It is necessary to use the equation for $f_k$ to deduce equations for amplitudes, resonance triplet vectors and widths of wave packets. Let us multiply (1) by 1, $(\alpha, \beta)$, $\alpha^2$, $\beta^2$ in consecutive order and integrate along concentration range of the components of compound wave packet. Then for amplitudes of resonance triplet we shell have:

$$
\frac{d\alpha_l}{dt} = \sum_{l=1}^{3} |a_i|^2 h_{l0} + \epsilon \Omega_0 a_0, \\
\frac{d\alpha_l}{dt} = -i[\vec{\omega}^R(k_l^0, X_0) - (\vec{X} a_0)] a_l + a_l \alpha_0 h_{l0} + a_m a_n h_{mn} + \epsilon \Omega_k a_l; \\
l = 1, 2, 3; \quad m, n \in \{1, 2, 3\} \setminus \{l\}, \quad m < n;
$$

for resonance triplet vectors (Hamilton equations):

$$
\frac{d\alpha_l}{dt} = -\epsilon \partial \vec{\omega}^R(k_l) / \partial X_0, \quad \frac{d\beta_l}{dt} = 0, \quad l = 1, 2, 3;
$$

for widths of Gauss's distributions:

$$
\frac{d(\alpha_l^2 / 2)}{dt} = \Delta \alpha_0^2 / 2(\alpha_0) h_0 + \sum_{l=1}^{3} \Delta \alpha_0^2 |a_l|^2 h_0 + \epsilon \left< \Delta \alpha^2 \right>_{k=0} a_0, \\
\frac{d(\alpha_l^2 / 2)}{dt} = h_{l0}(\Delta \alpha^2 / 2) a_0 a_l + h_{mn}(\Delta \alpha^2 + \Delta \beta^2) a_m a_n / 2 + \epsilon \left< \Delta \alpha^2 \right>_{k_l} - (\Delta \alpha^2 / 2) \left< \Omega \right>_{k_l},
$$

$\left< g(k) \right>_{k_l} = \int g(\Delta k + k_l) f_{\Delta k + k_l} d(\Delta k)$, $\Delta k = k - k_l$, $l = 1, 2, 3; \quad m, n \in \{1, 2, 3\} \setminus \{l\}, \quad m < n, g$ is a function of $k$. In the similar manner we can write equations for $\beta$-direction. To get the solution to (2),(3),(4) we must do the following actions: 1) to solve the equation (3) for wave vectors $k_l^0$, $l = 1, 2, 3$; 2) to define $k' = k / \sqrt{X}$, $c = c(k')$ and $\varphi(\eta, R_{\beta l}(X), k')$, $\eta = \sqrt{\eta / \sqrt{X}}$ for these wave vectors, to find the matrix elements; 3) after that we can solve the equations for amplitudes (2) and widths (4) of the wave packets. The initial conditions or the fundamental harmonics are $\alpha_1(0) = \alpha_{10}$, $\beta_1(0) = 0$, $X_1(0) = 1, Z_1(0) = 0$, for subharmonics are $\alpha_2(0) = \alpha_1(0) / 2$, $\beta_2(0) = \alpha_1(0) / \sqrt{2}$, $X_2(0) = 1, Z_2(0) = 0$. Further $\alpha_{1,0} = 0.15$. The coefficients $(\zeta_0(R), (\zeta_1(R))$, $R$ is Reynolds number, are defined from the Orr-Sommerfeld spectral problem solution. As a result we have the following expressions:

$$
\zeta_0 \cong 0.2583 - 0.00000657786 R + 1.10714 \cdot 10^{-8} R^2, \quad \zeta_1 \cong 1.1.
$$
The solution to Hamilton equations is shown in Figure 1, where $t_2 = \epsilon^2 t_0$, $t_0 = U_\infty t/\delta$. The results show that the centers of the wave packets move in a linear manner by time practically, the detailed comparison shows that the value of the longitudinal wave number $\alpha_3 = \alpha_1 - \alpha_2$ of the second subharmonics is bit behind the wave number $\alpha_2$ of the first subharmonics. This fact can be interpreted as the spreading of the subharmonics supporters. Numerical simulation shows that ”zero” component is created first (Figure 2). At the same time resonance triplet amplitudes decrease by the absolute value. Then all components increase. At rather long time the amplitudes reveal singular behavior. Within this region by time weakly nonlinear approximation becomes inapplicable. The wave packet evolution is on Figure 3 ($t_1 = \epsilon t_0$). Origination of the wake from the wave packet can be seen after some time of the beginning (Figure 3).

![Figure 3. The evolution of compound wave packet by time (contour lines of $\varphi(x, y, z)$).](image)

5. References

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