Retro-Prospective Differential Inclusions and their Control by the Differential Connection Tensors of their Evolutions: The trendometer

Jean-Pierre Aubin

December 10, 2013

Abstract

This study is motivated by two different, yet, connected, motivations. The first one follows the observation that the classical definition of derivatives involves prospective (or forward) difference quotients, not known whenever the time is directed, at least at the macroscopic level. Actually, the available and known derivatives are retrospective (or backward). They coincide whenever the functions are differentiable in the classical sense, but not in the case of non smooth maps, single-valued or set-valued. The later ones are used in differential inclusions (and thus, in uncertain control systems) governing evolutions in function of time and state. We follow the plea of some physicists for taking also into account the retrospective derivatives to study prospective evolutions in function of time, state and retrospective derivatives, a particular, but specific, example of historical of “path dependent” evolutionary systems. This is even more crucial in life sciences, in the absence of experimentation of uncertain evolutionary systems. The second motivation emerged from the study of networks with junctions (cross-roads in traffic networks, synapses in neural networks, banks in financial networks, etc.), an important feature of “complex systems”. At each junction, the velocities of the incoming (retrospective) and outgoing (prospective) evolutions are confronted. One measure of this confrontation (“jerkiness”) is provided by the product of the retrospective and prospective velocities, negative in “inhibitory” junctions, positive for “excitatory” ones, for instance. This leads to the introduction of the “differential connection tensor” of two evolutions, defined as the tensor product of retrospective and prospective derivatives, which can be used for controlling evolutionary systems governing the evolutions through networks with junctions.

Mathematics Subject Classification: 34A60, 90B10, 90B20, 90B99, 93C10, 93C30, 93C99,

Keywords Transport, networks, junction, impulse, viability, traffic control, jam, celerity, monad

1VIMADES (Viabilité, Marchés, Automatique, Décisions), 14, rue Domat, 75005, Paris, France
aubin.jp@gmail.com, http://vimades.com/aubin/

2Acknowledgments This work was partially supported by the Commission of the European Communities under the 7th Framework Programme Marie Curie Initial Training Network (FP7-PEOPLE-2010-ITN), project SADCO, contract number 264735.
Motivations

There are two different motivations of this study.

Retrospective-Prospective Differential Inclusions

The first motivation follows the plea of Efim Galperin in [21, 22, 23, 24] Galperin] for using “retrospective” derivatives instead of “prospective” derivatives, universally chosen since their introduction by Newton and Leibniz, at a time when physics became predictive and deterministic: the “prospective derivatives” $\frac{\Delta x(t)}{\Delta h}$ being (more or less weak) limits of prospective (future) difference quotients (on positive durations $h > 0$) $\frac{\Delta x(t)}{\Delta h} := \frac{x(t + h) - x(t)}{h}$ are “physically non-existent”, because they are not yet known at time $t$. Whereas the retrospective (past) difference quotients $\frac{\Delta x(t)}{\Delta h} := \frac{x(t) - x(t - h)}{h}$ may be known for some positive durations and should be taken into account.

This is an inescapable issue in life sciences, since the evolutionary engines evolve with time, under contingent and/or tychastic uncertainties and, in most cases, cannot be recreated (at least, for the time, since synthetic biology deals with this issue). Popper’s recommendations are valid for physical sciences, where experimentation is possible and renewable. However, the quest of the instant (temporal window with 0 duration) has not yet been experimentally created (the smallest measured duration is of the order of the yoctosecond ($10^{-24}$)). Furthermore, our brains deal with observations which are not instantaneous, but, in the best case, are perceived after a positive transmittal duration.

For overcoming this difficulty, Fermat, Newton, Leibniz and billions of human brains have invented instants and passed to the limit when duration of temporal windows goes to 0 to reach such an instant. This is actually an approximation of reality by clever mathematical constructions of objects belonging to an ever evolving “cultural world”. Derivatives are not perceived, but were invented, simplifying reality by passing to the limit in a mathematical paradise.

Therefore, for differentiable functions in the classical sense, the limits of retrospective and prospective difference quotients may coincide when we pass to the limit. But this is no longer

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3For evolutions (fonctions of one variable, retrospective derivatives are derivatives from the left and prospective derivatives are derivatives from the right. For fonctions of several variable, there is no longer left and right, so retrospective (or backward), prospective (or forward), are used instead.

4This has been pointed out by Jiri Buquoy, who in 1812, formulated the equation of motion of a body with variable mass, which retained only the attention of Poisson before being almost forgotten. See [16, Jiri Buquoy], [31, Mestschersky] and [32, Levi-Civita] among the precursors in this area.

5See for instance [34, Danchin et al.].

6Actually, an inductive approximation, whereas (deductive) application refers to approximate derivatives of the idealized world by difference quotients, which are closer to the actual perception of our brains and capabilities of the digital computers.
the case when evolutions are no longer differentiable in the classical sense, but derivatives may still exist for “weaker” limits, such as limits in the sense of distributions or graphical limits in set-valued analysis (see Section 18.9, p. 769, of Viability Theory. New Directions, [9, Aubin, Bayen & Saint-Pierre]). Even if we restrict our analysis to Lipschitz functions, the Rademacher’s Theorem states that Lipschitz maps from one finite dimensional vector space to another one are only almost everywhere differentiable. Although small, the set of elements where there are not differentiable is interesting because Lipschitz have always set-valued graphical derivatives. Hence we have to make a detour by recalling what are meant retrospective and prospective graphical derivatives of maps as well as set-valued maps and non differentiable (single-valued) maps.

Therefore, we devote the first part of this study to a certain class of viable evolutions governed by functional (or history-dependent) differential inclusions

\[ x'(t) \in G(t, x(t), \widehat{D}x(t)) \]

where \( \widehat{D}x(t) \) is the retrospective derivative (or derivative from the left since, at this stage, we consider evolutions defined on \( \mathbb{R} \)). Retrospective-prospective differential inclusions \( x'(t) \in G(t, x(t), \widehat{D}x(t)) \) describe predictions of evolutions based on the state and on the known retrospective velocity at each chronological time. As delayed differential equations or inclusions, they are particular cases of functional (or historical, path-dependent, etc.) differential equations. As for second-order differential equations, initial conditions \( x(t_0) \) at time \( t_0 \) must be provided, as well as (retrospective) initial velocities for selecting evolutions governed by retrospective-prospective differential equations.

**Differential Connection Tensors in Networks**

The second motivation emerged from the study of propagation through “junctions of a network”, such as cross-roads in road networks, banks in financial networks, synapses in neural network, etc. (see for instance [8, Aubin]).

**Neural Network : the Hebbian Rule**

If we accept that in formal neuron networks, “(evolving) knowledge” is coded as “synaptic weights” at each synapse, their collection defines a “synaptic matrix” which evolves, and, thus, becomes the “state of the network”. Donald Hebb introduced in 1949 in *The Organization of Behavior*, [30, Hebb], the Hebbian learning rule prescribing that the velocity of
the synaptic matrix is proportional to the tensor product\(^8\) of the “presynaptic activity” and “postsynaptic activity” described by the propagation of nervous influx in the neurons.

Hence, denoting the synaptic matrix \(W\) of synaptic weights, the basic question was to minimize a “matrix function” \(W \in \mathcal{L}(X, X) \mapsto E(Wx)\) where \(x \in X := \mathbb{R}^\ell\) and \(E : X \mapsto \mathbb{R}\) a differentiable function are given. Remembering\(^9\) that the gradient with respect to \(W\) is equal to the tensor product \(E'(Wx) \otimes x\), the gradient method leads to a differential equation of the form

\[
W'(t) = -\alpha E'(W(t)x) \otimes x
\]

which governs the evolution of the synaptic matrix (the “synapse \(x\) is fixed and does not evolve).

**Differential Connection Tensors**

However, we take into account the evolution \(t \mapsto x(t) \in X\) of the propagation in networks (such as the propagation nervous influx, traffic, financial product, etc.). If the evolution is Lipschitz, retrospective and prospective derivatives exist at all times, so that we can define the tensor product \(\overrightarrow{D}x(t) \otimes \overleftarrow{D}x(t)\) of their retrospective and prospective velocities: we shall call it the differential connection tensor of the evolution \(x(\cdot)\) at time \(t\).

It plays the role of a “trendometer” measuring the trend reversal (or monotonicity reversals) at junctions: the differential connection tensor describes the trend reversal between the retrospective and prospective trends when they are strictly negative, the monotonicity congruence when they are strictly positive and the inactivity they vanish. In neural networks, for instance, *this an inhibitory effect or trend reversal in the first case, an excitatory or trend congruence in the second case, and inactivity of a synapse: one at least of the propagation of the nervous influx stops*. The absolute value of this product measures in some sense the jerkiness of the trend reversal at a junction of the network.

We are thus tempted to control (pilot, regulate, etc.) the evolution of propagation in the network governed by a system

\[
x'(t) = g(x(t), u(t)) \text{ where } u(t) \in U(\overrightarrow{D}x(t) \otimes \overleftarrow{D}x(t))
\]

controlled by differential connection tensors at junctions of the network. We recall that the evolutions governed by (Marchaud) controlled systems are Lipschitz under the standard assumption, *but not necessarily differentiable*. For example, in order to govern the viability of the propagation in terms of the inhibitory, excitatory and stopping behavior at the

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\(^8\)Recall that the tensor product \(p \otimes q\) of two vectors \(p := (p_i)_i \in \mathbb{R}^\ell\) and \(q := (q_j)_j \in \mathbb{R}^\ell\) is the rank one linear operator

\[
p \otimes q \in \mathcal{L}(\mathbb{R}^\ell \otimes \mathbb{R}^\ell) : x \mapsto \langle p, x \rangle q
\]

the entries of which (in the canonical basis) are equal to \((p_i q_j)_{i,j}\).

\(^9\)See Proposition 2.4.1, p. 37 and Chapter 2 of Neural Networks and Qualitative Physics: a Viability Approach, [\text{Aubin}].
junctions of the network, some constraints are imposed on the evolution of the differential connection tensors. Examples of retrospective-retrospective differential equations are provided by tracking or controlling differential connection tensors of the evolutions requiring that evolutions governed by differential equations $x'(t) = f(t, x(t))$ satisfy constraints of the form $\mathcal{D}x(t) \otimes \mathcal{D}x(t) \in C(t, x(t))$. These control systems are examples of retrospective-prospective differential inclusions.

These considerations extend to “multiple synapses” when we associate with each subset $S$ of branches $j$ meeting at a junction the tensor products $\otimes_{j \in S} x_j'(t)$ of the velocities at the junction.$^{10}$

### Organization of the Study

Section 4, p. 16, *Retrospective-Prospective Differential Inclusions*, defines retrospective and prospective (graphical) derivatives of tubes and evolutions, their differential connection tensor (Definition 1.1, p.6). They are the ingredients for introducing retrospective-prospective differential inclusions. The Viability Theorem (Theorem 1.2, p.7) is adapted for characterizing viable tubes under such differential inclusions using characterizations linking the retrospective and prospective derivatives of the tube. When these conditions are not satisfied, we restore the viability by introducing *retrospective-prospective viability kernel* of the tube under the retrospective-prospective differential inclusion (Subsection 1.3, p. 8).

Section 2, p. 9, *Control by Differential Connection Tensors*, studies the regulation of viable evolutions on tubes by imposing constraints on their differential connection tensors.

Section 3, p. 11, *Illustrations*, provides examples of differential connection tensors of vector evolutions in the framework of “technical analysis” of the forty prices series of the CAC 40 stock market index.$^{11}$

Section 4, p. 16, *Other Examples of Differential Connection Tensors*, defines differential connection tensors of set-valued maps (Subsection 4.1, p. 16, Prospective and Retrospective Derivatives of Set-Valued Maps) and gathers some other classes differential connection tensors than the ones of the evolutions $t \mapsto x(t)$ or tubes $t \mapsto K(t)$ from $\mathbb{R}$ to $\mathbb{R}^\ell$, which provided the first source of motivations for studying differential connection tensors. Other specific examples are the differential connection tensors of numerical functions $V : \mathbb{R}^\ell \mapsto \mathbb{R}$ (Subsection 4.2, p. 18), and *tangential connection tensors* of retrospective and prospective tangents (Subsection 4.3, p. 20). These issues are the topics of forthcoming studies.

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$^{10}$See [10] Aubin & Burnod and the literature on $\Sigma - \Pi$ neural systems, Section 12.2 of Viability Theory. New Directions, [9] Aubin, Bayen & Saint-Pierre, Analyse qualitative, [18] Dordan], as well as [3] [4] [5] [6] Aubin], [37] Vinogradova] and the literature on the regulation of networks.

$^{11}$See Chapter 2 of *Tychastic Viability Measure of Risk Eradication. A Viabilist Portfolio Performance and Insurance Approach*, [11] Aubin, Chen Luxi & Dordan], for a more detailed study.
1 Retrospective-Prospective Differential Inclusions

1.1 Prospective and Retrospective Derivatives of Tubes and Evolutions

A tube is the nickname of a set-valued map $K : t \in \mathbb{R} \rightarrow K(t) \subset X$. Since there are only two directions $+1$ and $-1$ in $\mathbb{R}$, the prospective (left) and retrospective (right) derivatives of a tube $K$ at a point $(t, x)$ of its graph are defined by

$$
\begin{align*}
\forall \nu \in D^{-}_{\nu} K(t, x) &\quad \text{if and only if} \quad \lim_{h \to 0^+} \text{inf} d(v, K(t + h) - x) = 0 \\
\forall \nu \in D^{+}_{\nu} K(t, x) &\quad \text{if and only if} \quad \lim_{h \to 0^+} \text{inf} d(v, x - K(t - h)) = 0 \\
\end{align*}
$$

(3)

(see Definition 4.1, p.16. in the general case).

**Definition 1.1 [Differential Connection Tensor of a Tube]** The differential connection tensor of a tube $K(\cdot)$ at $x \in K(t)$ is defined by

$$
\forall \nu \in D^{-}_{\nu} K(t, x), \ \forall \nu' \in D^{+}_{\nu'} K(t, x), \ a_{K(t, x)}[\nu', \nu] := \nu' \otimes \nu
$$

(4)

In particular, an evolution $x(\cdot)$ is a single-valued tube defined by $K(t) := \{x(t)\}$, so that we can define their graphical prospective derivative $\nabla K(t, x)$ (from the right) and retrospective derivatives $\nabla K(t, x)$ (from the left) respectively (see illustrations in Section 3, p. 11, Illustrations).

1.2 Retrospective-Prospective Differential Inclusions

Recall that whenever an evolution $t \mapsto x(t)$ is viable on a neighborhood of $t_0$ on a tube $K(t)$, then $\nabla K(t_0) \in \nabla K(t_0, t_0)$ and $\nabla K(t_0) \in \nabla K(t_0, t_0)$.

Since we know only retrospective derivatives, forecasting future evolution can be governed by prospective differential inclusion $\nabla K(t, x(t)) \in F(t, x(t))$ depending only on time and state, but also by the particular case of history-dependent evolutions $\nabla K(t, x(t)) \in G(t, x(t), \nabla K(t, x(t)))$ depending on time, state and the retrospective derivatives. This could be the case for system controlling the differential connection tensors of the evolutions, for instance (see Section 2, p. 9).

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12Actually, there is a third one, 0, where $\nabla K(t, x)(0)$ and $\nabla K(t, x)(0)$ are the retrospective and prospective tangent cones studied in Section 4.3, p. 20.
Theorem 1.2 [Viability Theory for Retrospective-Prospective Differential Inclusions] Let us assume that the map \((t, x, v) \in \mathbb{R} \times X \times X \rightarrow G(t, x, v) \subset X\) is Marchaud (closed graph, convex valued and linear growth) and that the tube \(t \sim K(t)\) is closed. Then the “tangential condition”

\[
\forall \; \overrightarrow{v} \in \overrightarrow{D}K(t, x), \; G(t, x, \overrightarrow{v}) \cap \overrightarrow{D}K(t, x) \neq \emptyset
\]  

(5)
is equivalent to the “viability property”: from any initial state \(x_0 \in K(t_0)\) and initial retrospective velocity \(\overrightarrow{v}_0 \in \overrightarrow{D}K(t_0, x_0)\), there exists at least one evolution \(x(\cdot)\) governed by the retrospective-prospective differential inclusion \(\overrightarrow{D}x(t) \in G(t, x(t), \overrightarrow{D}x(t))\) satisfying \(x(t_0) = x_0\) and \(\overrightarrow{D}x(t_0) = \overrightarrow{v}_0\) and viable in the tube \(K(\cdot)\).

Proof — The proof is an adaptation of the proof of the viability Theorem 19.4.2, p. 782, based on Theorems 11.2.7, p. 447, and 19.3.3, p. 777, of Viability Theory. New Directions, [9] Aubin, Bayen & Saint-Pierre. We just indicate the modifications to be made.

We construct approximate solutions by modifying Euler’s method to take into account the viability constraints, then deduce from available estimates that a subsequence of these solutions converges in some sense to a limit, and finally, check that this limit is a viable solution to the retrospective-prospective differential inclusion \((\overrightarrow{D}x(t) \in G(t, x(t), \overrightarrow{D}x(t)))\).

1. By assumption, there exists \(r > 0\) such that the neighborhood \(\mathcal{K}_r := \text{Graph}(K) \cap (t_0, x_0) + r([-1, +1]) \times B\) of the initial condition \((t_0, x_0)\) is compact. Since \(G\) is Marchaud, the set

\[
\mathcal{C}_r := \{F(t, x, \overrightarrow{v})\} + B, \quad \text{and} \quad T := r/\|\mathcal{C}_r\|
\]
is also compact. We next associate with any \(h\) the Euler approximation

\[
v^h_j := \frac{x^h_{j+1} - x^h_j}{h} \in G(jh, x^h_j, v^h_{j-1}) \quad \text{where} \quad v^h_{j-1} := \frac{x^h_j - x^h_{j-1}}{h}
\]
starting from \((t_0, x_0, \overrightarrow{v}_0)\).

2. Theorems 11.2.7, p. 447 of [9] Aubin, Bayen & Saint-Pierre] implies that for all \(\varepsilon > 0\),

\[
\begin{cases}
\exists \; \eta(\varepsilon) > 0 \text{ such that } \forall \; (t, x) \in \mathcal{K}_r, \; \forall \; h \in [0, \eta(\varepsilon)], \\
x^h_j + hG(jh, x^h_j, v^h_{j-1}) \in K(jh, x^h_j) + \varepsilon B
\end{cases}
\]

(7)

Since

\[
\|x^h_j - x_0\| \leq \sum_{i=0}^{i=j-1} \|x^h_{i+1} - x^h_i\| \leq \sum_{i=0}^{i=j-1} h \|v^h_j\| \leq \|\mathcal{C}_r\|
\]

the discrete evolution is viable in \(\mathcal{K}_r\) on the interval \([0, T]\). Denoting by \(x^h, \overrightarrow{v}^h\) and \(\overrightarrow{v}^h\) the linear interpolations of the sequences \(x^h_j, \overrightarrow{v}^h_j\) and \(\overrightarrow{v}^h_j\), we infer that there exists a constant \(\alpha > 0\) such that
\[
\begin{align*}
(t^h, x^h, \overleftarrow{v}^h, \overrightarrow{v}) & \in \text{Graph}(G) + \varepsilon\alpha \\
(t^h, x^h) & \in \text{Graph}(K) + \varepsilon\alpha
\end{align*}
\] (8)

and that there exists a constant \(\beta > 0\) such that the \textit{a priori} estimates

\[
\max(\|\overrightarrow{v}^h x^h\|_\infty, |\overrightarrow{v}^h x^h|_\infty) \leq \beta
\] (9)

are satisfied.

3. They imply the \textit{a priori} estimates of the Convergence Theorem 19.3.3, p. 777, of [9, Aubin, Bayen & Saint-Pierre], which states the limit of a converging subsequence is a solution to the retrospective-prospective differential inclusion, viable in Graph(K).

\[\Box\]

1.3 Retrospective-Prospective Viability Kernels

Naturally, the “tangential assumption” (5), p. 7, is not necessarily satisfied so that we have to adapt the concept of viability kernel to the retrospective-prospective case.

Definition 1.3 [Retrospective-Prospective Viability Kernel of a Tube] The viability kernel of the tube \(K(\cdot)\) is the set of initial conditions \((t_0, x_0, \overleftarrow{v}_0) \in \mathbb{R} \times K(t_0) \times \overrightarrow{DK}(t_0, x_0)\) from which starts at least one viable evolution \(t \mapsto x(t) \in K(t)\) to the retrospective-prospective differential inclusion in the sense that

\[
\begin{align*}
(i) & \quad \overrightarrow{D} x(t) \in G(t, x(t), \overrightarrow{D} x(t)) \\
(ii) & \quad \overleftarrow{D} x(t) \in \overrightarrow{DK}(t, x(t)) \text{ and } \overrightarrow{D} x(t) \in \overrightarrow{DK}(t, x(t))
\end{align*}
\] (10)

We provide a viability characterization of retrospective-prospective viability kernel tubes:

Proposition 1.4 [Viability Characterization of Retrospective-Prospective Viability Kernel] Let us consider the control system

\[
\begin{align*}
(i) & \quad \tau'(t) = 1 \\
(ii) & \quad x'(t) \in G(\tau(t), x(t), \overrightarrow{v}(t)) \\
(iii) & \quad \|\overrightarrow{v}(t)\| \leq c \|G(t, x, \overrightarrow{v})\| \\
& \quad \text{where } \overrightarrow{v}(t) \in \overrightarrow{DK}(\tau(t), x(t))
\end{align*}
\] (11)

Then the viability kernel of the graph Graph(\(DK(\cdot)\)) of the derivative tube \(K(\cdot)\) coincides with the retrospective-prospective viability kernel of the tube.
Proof — The viability kernel of the control system (11), p. 8 is the set of initial triple \((t_0, x_0, \leftarrow v_0)\) such that \(x_0 \in K(t_0)\) and \(\leftarrow v_0 \in \overrightarrow{DK}(t_0, x_0)\) from which starts an evolution \(t \mapsto (t_0 + t, x(t), \leftarrow v(t))\) of the control system such that \(x(t) \in K(\tau(t))\) and \(\leftarrow v(t) \in \overrightarrow{DK}(\tau(t), x(t))\). Setting \(x_*(t) := x(t - t_0)\) and \(\leftarrow v_*(t) := \leftarrow v(t - t_0)\), we observe that \(x_*(t) \in G(t, x_*(t), \leftarrow v_*(t))\), \(\leftarrow v_*(t) \in \overrightarrow{DK}(t, x_*(t))\) and \(x_*(t) \in K(t)\). We thus infer that \(\overrightarrow{Dx_*(t)} \in \overrightarrow{DK}(t, x_*(t))\). Since \(x(t)\) is viable in the tube, we also infer that \(\overrightarrow{Dx(t)}\) actually belongs to \(\overrightarrow{DK}(t, x(t))\). Hence \((t_0, x_0, \leftarrow v_0)\) belongs to the retrospective-prospective viability kernel of the tube \(K(\cdot)\).

Therefore, it remains to provide sufficient conditions for the viability kernel of the graph of \(K(\cdot)\) under the control system is Marchaud.

Theorem 1.5 [Properties of the Retrospective-Prospective Viability Kernel] Let us assume that the set-valued map \(G : (t, x, \leftarrow v) \mapsto G(t, x, \leftarrow v)\) is Marchaud. Then the retrospective-prospective viability kernel of the tube \(K(\cdot)\) under the \(\overrightarrow{Dx(t)} \in G(t, x(t), \overrightarrow{Dx(t)})\) is closed and inherits all properties of viability kernels.

2 Control by Differential Connection Tensors

We study the tracking at each date \(t\) of the differential connection tensor \(\overrightarrow{Dx(t)} \otimes \overrightarrow{Dx(t)}\) of evolutions governed by a differential inclusion \(x'(t) \in F(t, x(t))\).

For that purpose, we introduce a connection map \((t, x) \mapsto C(t, x) \subset \mathcal{L}(X, X)\). We are looking for evolutions \(x(\cdot)\) governed by the differential inclusion satisfying the constraints on the differential connection tensors

\[
\forall t \geq 0, \quad \overrightarrow{Dx(t)} \otimes \overrightarrow{Dx(t)} \in C(t, x(t))
\]  

(12)

This is a problem analogous to the search of the slow evolutions governed by control systems (solutions governed by controls of the regulation map with minimal norm): see [13 Aubin & Frankowska] or Theorem 6.6.3, p. 229, of [2, Viability Theory].

We follow the same strategy by introducing the set-valued map \(G\) defined by

\[
G(t, x, \overrightarrow{v}) := \{w \in F(t, x) \mid \overrightarrow{v} \otimes w \in C(t, x)\}
\]  

(13)

Theorem 2.1 [Control of Differential Connection Tensors] We assume that \(F\) is Marchaud, that the tube \(K(\cdot)\) is closed and that
\[
\left\{
\begin{array}{l}
(i) \quad \text{the graph of } (t, x) \sim C(t, x) \subset \mathcal{L}(X, X) \text{ is closed and its images are convex}

(ii) \quad \forall (t, x) \in \text{Graph}(K), \quad \forall \overrightarrow{v} \in \overrightarrow{D}K(t, x), \quad \exists w \in F(t, x) \in \overrightarrow{D}K(t, x)
\end{array}
\right.
\]

For any \( t_0 \), for any \( x_0 \in K(t_0) \), for any \( \overrightarrow{v}_0 \in \overrightarrow{D}K(t_0, x_0) \), there exists at least an evolution \( x(\cdot) \) governed by the differential inclusion \( x'(t) \in F(t, x(t)) \) starting at \( x_0 \) viable in the tube \( K(\cdot) \) such that \( \overrightarrow{v}_0 \otimes \overrightarrow{D}x(t_0) \in C(t_0, x_0) \) and satisfying the differential connection tensor constraints

\[
\forall t \geq t_0, \quad \overrightarrow{D}x(t) \otimes \overrightarrow{D}x(t) \in C(t, x(t))
\]

and the retrospective-prospective viability property

\[
\forall t \geq t_0, \quad \overrightarrow{D}x(t) \otimes \overrightarrow{D}x(t) \in \overrightarrow{D}K(t, x(t)) \otimes \overrightarrow{D}K(t, x(t))
\]

**Proof** — The set-valued map \( G \) satisfies the assumptions of Theorem 1.2, p.7, in such a way that there exists one evolution \( x(\cdot) \) governed by \( \overrightarrow{D}x(t) \in G(t, x(t), \overrightarrow{D}x(t)) \) viable in the tube \( K(\cdot) \). Therefore, \( \overrightarrow{D}x(t) \in \overrightarrow{D}K(t, x(t)) \) for all \( t \geq t_0 \). Consequently,

\[
\overrightarrow{D}x(t) \otimes \overrightarrow{D}x(t) \in C(t, x(t))
\]

and since the evolution is viable in the tube \( K(\cdot) \), that

\[
\overrightarrow{D}x(t) \in \overrightarrow{D}K(t, x(t)) \text{ and } \overrightarrow{D}x(t) \in \overrightarrow{D}K(t, x(t))
\]

The theorem ensues. \( \blacksquare \)

For instance, we can choose

\[
C(t, x, \overrightarrow{v}) := \left\{ \overrightarrow{v} \text{ such that } \sup_{w \in F(t, x)} \sup_{(i, j)} \overrightarrow{v}_i (\overrightarrow{v}_j - w_j) \leq 0 \right\}
\]

In other words, the entries \( \overrightarrow{v}_i \overrightarrow{v}_j \) minimize the entries \( \overrightarrow{v}_i w_j \) of the differential connection tensors when the velocities \( w \in F(t, x) \).

Proposition 6.5.4, p. 226, of *Set-valued analysis*, [12, Aubin & Frankowska], implies that the connection constraint map has a closed graph and convex values whenever the set-valued map \( F \) is lower semicontinuous with convex compact images.

We could as well require that the entries of the differential connection tensor maximize the entries \( \overrightarrow{v}_i \overrightarrow{v}_j \) minimize the entries \( \overrightarrow{v}_i w_j \) of the differential connection tensors when the velocities \( w \in F(t, x) \) or that for some pairs \( (i, j) \), the entries \( \overrightarrow{v}_i \overrightarrow{v}_j \) minimize \( \overrightarrow{v}_i w_j \) and for the other pairs, that they maximize \( \overrightarrow{v}_i w_j \) when the velocities \( w \in F(t, x) \).
3 Illustrations

The question arises whether it is possible to detect the connection dates when the monotonicity of a series of a family of temporal series is followed by the reverse (opposite) monotonicity of other series, in order to detect the influence of each series on the dynamic behavior of other ones. When the two functions are the same, we obtain their reversal dates when the series achieve their extrema. The differential connection tensor measures the jerkiness between two functions, smooth or not smooth (temporal series) providing the trend reversal dates of the differential connection tensor.

This matrix plays for time series a dynamic rôle analogous to the static rôle played by the correlation matrix of a family of random variable measuring the covariance entries between two random coefficients. In other words, we add in our analysis the dependence on random events of variables their dependence on time.

The differential connection tensor softwares provides at each date the coefficients of the differential connection tensor.

We use the tensor trendometer for detecting the dynamic correlations between the forty price series of the CAC 40. For instance, on August 6, 2010, the prices are displayed in the following figure:

At each date, it provides the $40 \times 40$ matrix displaying the qualitative jerkiness for each pair of series when the trend of the first one is followed by the opposite trend of the second one. At each entry, the existence of a trend reversal by a circles:
The quantitative version replaces the circles by the values of the jerkiness:

In order to analyse further the evolutionary behavior of the CAC 40, we present the analysis of the CAC 40 index only, but over the period from 03/01, 1990 to 09/25, 2013. The first figure displays the series of the CAC 40 indexes (closing prices). The vertical bars indicate the reversal dates and their height displays their jerkiness.

The 2000 Internet crisis (around May 4, 2000) and the 2008 “subprime” crisis (around October 10, 2008) are detected and measured by the trendometer:
The next figure displays the velocities of the jerkiness between two consecutive trend reversal dates, a ratio involving the variation of the jerkiness and the duration of the congruence period (bull and bear):

The following one displays the classification of trend speeds and absolute value of the accelerations by decreasing jerkiness:
The analysis of this series shows that often the jerkiness at minima (bear periods) is higher than the ones at maxima (bull periods). For the CAC 40, the proportion of “bear jerkiness” (57%) over “bull jerkiness” (43%).

The next table provides the first dates by decreasing jerkiness. The most violent are those of the subprime crisis (in bold), then the ones of the year 2006 and, next, the dates of the Internet crisis (in italics).
| Date     | Jerkiness | Date     | Jerkiness | Date     | Jerkiness |
|----------|-----------|----------|-----------|----------|-----------|
| 10/10/2008 | 94507.21  | 03/01/2001 | 15153.31  | 17/02/2000 | 10025.57  |
| 23/01/2008 | 57315.90  | 11/09/2002 | 15111.43  | 28/10/2002 | 9962.69   |
| 07/05/2010 | 53585.50  | 10/03/2000 | 15055.45  | 01/09/1998 | 9917.22   |
| 05/12/2008 | 44927.23  | 10/08/2011 | 15011.24  | 15/02/2008 | 9905.51   |
| 03/10/2008 | 43319.41  | 27/08/2002 | 14958.41  | 19/04/1999 | 9887.67   |
| 19/09/2008 | 37200.13  | 22/11/2000 | 14768.91  | 26/10/2001 | 9556.17   |
| 05/04/2000 | 34609.80  | 03/04/2000 | 14280.35  | 29/06/2000 | 9470.44   |
| 21/01/2008 | 34130.42  | 03/04/2001 | 14003.47  | 25/02/2000 | 9438.07   |
| 16/10/2008 | 29794.42  | 18/07/2002 | 13813.67  | 27/03/2001 | 9436.84   |
| 21/11/2008 | 28840.69  | 19/12/2000 | 13743.01  | 15/05/2000 | 9411.84   |
| 04/12/2000 | 27861.03  | 12/03/2003 | 13707.93  | 04/10/2011 | 9409.14   |
| 12/11/2001 | 26039.07  | 12/09/2008 | 13682.85  | 17/01/2000 | 9398.39   |
| 22/03/2001 | 25128.11  | 01/12/2008 | 13207.66  | 11/08/1998 | 9320.83   |
| 27/04/2000 | 24577.70  | 29/10/1997 | 13085.95  | 20/11/2007 | 9291.91   |
| 17/03/2008 | 24416.22  | 04/03/2009 | 12845.84  | 05/10/1998 | 9277.96   |
| 14/10/2008 | 24007.60  | 14/03/2007 | 12801.09  | 29/07/1999 | 9253.97   |
| 05/08/2002 | 22021.61  | 24/06/2002 | 12658.98  | 04/12/2007 | 9200.48   |
| 14/09/2001 | 21658.15  | 02/08/2012 | 12628.14  | 04/02/2000 | 9093.25   |
| 10/08/2007 | 21252.50  | 24/05/2000 | 12456.94  | 02/10/2002 | 8959.94   |
| 13/11/2000 | 20662.32  | 10/05/2000 | 12411.27  | 13/09/2000 | 8987.37   |
| 22/01/2008 | 20184.96  | 28/07/2000 | 12145.83  | 10/05/2010 | 8877.39   |
| 14/08/2002 | 20052.16  | 23/02/2001 | 11960.59  | 30/09/2002 | 8845.61   |
| 28/10/1997 | 19720.61  | 04/11/2008 | 11904.50  | 04/11/1998 | 8843.75   |
| 14/06/2002 | 19114.56  | 08/06/2006 | 11773.65  | 09/08/2011 | 8833.20   |
| 06/11/2008 | 18900.51  | 30/10/2001 | 11733.86  | 11/06/2002 | 8832.22   |
| 03/08/2000 | 18621.37  | 15/10/2001 | 11630.50  | 07/07/2000 | 8797.60   |
| 29/10/2002 | 18550.19  | 24/03/2003 | 11294.44  | 16/01/2001 | 8778.74   |
| 08/10/1998 | 18307.12  | 15/03/2000 | 11232.52  | 27/04/1998 | 8721.52   |
| 02/05/2000 | 18087.38  | 17/09/2007 | 10948.51  | 19/02/2008 | 8327.20   |
| 21/09/2001 | 17771.78  | 13/08/2007 | 10933.30  | 20/11/2000 | 8299.90   |
| 11/09/2001 | 17660.29  | 25/08/2001 | 10809.42  | 03/07/2002 | 8289.95   |
| 16/08/2007 | 17398.86  | 02/10/2008 | 10720.31  | 28/06/2000 | 8258.67   |
| 16/05/2000 | 17228.62  | 23/10/2002 | 10675.86  | 28/06/2010 | 8137.05   |
| 04/04/2000 | 16958.95  | 25/08/1998 | 10673.02  | 31/01/2000 | 8093.58   |
| 18/10/2000 | 16761.07  | 30/03/2009 | 10672.64  | 21/11/2000 | 8074.23   |
| 29/09/2008 | 16502.34  | 24/01/2008 | 10352.96  | 28/01/2009 | 8049.26   |
| 08/08/2007 | 16048.09  | 20/03/2001 | 10294.67  | 26/02/2007 | 8038.76   |
| 21/03/2003 | 15703.11  | 14/12/2001 | 10253.40  | 31/01/2001 | 8033.95   |
| 18/09/2008 | 15506.17  | 31/07/2007 | 10134.80  | 26/11/2002 | 7933.90   |
| 22/05/2006 | 15470.19  | 26/04/2000 | 10093.65  | 08/08/2011 | 7821.87   |
| 05/09/2008 | 15406.87  | 02/09/1999 | 10080.12  | 18/05/2010 | 7793.80   |
4 Other Examples of Differential Connection Tensors

4.1 Prospective and Retrospective Derivatives of Set-Valued Maps

We summarize the concept of graphical derivatives.

Definition 4.1 [Retrospective and Prospective Graphical Derivatives] Consider a set-valued map \( F : X \rightrightarrows Y \) from a finite dimensional vector space \( X \) to another one \( Y \). Let \((x, y) \in \text{Graph}(F)\) an element of its graph. We denote in this study by

1. retrospective derivative \( \overleftarrow{DF}(x, y) : X \rightrightarrows Y \) associating with any direction \( u \in X \) the set of elements \( v \in Y \) satisfying

\[
\liminf_{h \to 0^+, u_h \to u} d \left( v, \frac{y - F(x - hu_h)}{h} \right) = 0 \tag{19}
\]

2. prospective derivative \( \overrightarrow{DF}(x, y) : X \rightrightarrows Y \) associating with any direction \( u \in X \) the set of elements \( v \in Y \) satisfying

\[
\liminf_{h \to 0^+, u_h \to u} d \left( v, \frac{F(x + hu_h) - y}{h} \right) = 0 \tag{20}
\]

The retrospective and prospective difference quotients of \( F \) at \((x, y) \in \text{Graph}(F)\) are defined by

\[
\overleftarrow{\nabla}_h F(x, y)(\overleftarrow{u}) := \frac{y - F(x - hu)}{h} \quad \text{and} \quad \overrightarrow{\nabla}_h F(x, y)(\overrightarrow{u}) := \frac{F(x + hu) - y}{h}.
\]

We can reformulate the definition of the (contingent) derivative by saying that it is the upper Painlevé-Kuratowski limit of the difference quotients,

\[
\forall \overleftarrow{u}, \overrightarrow{u}, \overleftarrow{DF}(x, y)(\overleftarrow{u}) = \text{Limsup}_{h \to 0^+, u_h \to \overleftarrow{u}} \overleftarrow{\nabla}_h F(x, y)(u_h) \tag{21}
\]

i.e., the retrospective (resp. prospective) derivatives are the cluster points \( \overleftarrow{v} \) of \( \overleftarrow{\nabla}_h F(x, y)(u_h) \) (resp. of i.e., the cluster points of \( \overrightarrow{\nabla}_h F(x, y)(u_h) \)). Whenever the set-valued map \( F \) is Lipschitz, the retrospective and prospective difference quotients are bounded, and thus, relatively compact set since the dimension of the vector spaces is finite. In this case, the prospective and retrospective derivatives are not empty.

Taking the tensor product of both the retrospective and prospective derivatives allows us to define the differential connection tensor:
Definition 4.2 [Differential Connection Tensor] The differential connection tensor $a_F(x,y)[(\overrightarrow{u}, \overrightarrow{v}),(\overrightarrow{u}, \overrightarrow{v})]$ of retrospective and prospective derivatives of $F$ at $(x,y) \in \text{Graph}(F)$ is defined by

$$a_F(x,y)[(\overrightarrow{u}, \overrightarrow{v}),(\overrightarrow{u}, \overrightarrow{v})] := \overrightarrow{v} \otimes \overrightarrow{v}$$

(22)

Remark — A normalized version of the differential connection tensor is defined by

$$a_F(x,y)[(\overrightarrow{u}, \overrightarrow{v}),(\overrightarrow{u}, \overrightarrow{v})] := \frac{\overrightarrow{v} \otimes \overrightarrow{v}}{||\overrightarrow{v}||^2}$$

(23)

The normalized version is not that useful whenever we are interested to the signs of the entries of the connection matrix.

Remark — One can associate with the prospective difference quotient $\overrightarrow{\nabla} h F(x,y)(\overrightarrow{u})$ and retrospective difference quotient $\overleftarrow{\nabla} h F(x,y)(\overrightarrow{u})$ their difference quotient

$$\nabla^2 F(x,y)(\overrightarrow{u}, \overrightarrow{v}) := \frac{\overrightarrow{\nabla} h F(x,y)(\overrightarrow{u}) - \overleftarrow{\nabla} h F(x,y)(\overrightarrow{u})}{h} = \frac{F(x + h \overrightarrow{v}) + F(x - h \overrightarrow{v}) - 2y}{h^2}$$

(24)

The Painlevé-Kuratowski upper limit of $\nabla^2 F(x,y)(\overrightarrow{u}, \overrightarrow{v})$ defines the retrospective-prospective second order graphical derivative of $F$ at $(x,y) \in \text{Graph}(F)$ by:

$$D^2 F(x,y)(\overrightarrow{u}, \overrightarrow{v}) := \text{Limsup}_{h \to 0+} \text{, } \overrightarrow{\nabla} h F(x,y)(\overrightarrow{u}, \overrightarrow{v})$$

(25)

The differential connection tensor replaces the difference between the retrospective and prospective derivatives by their tensor products. We refer to Section 5.6, p. 315, of Set-valued analysis, [12] Aubin & Frankowska, for other approaches of higher order graphical derivatives of set-valued maps.

Remark — In 1884, Giuseppe Peano proved in Giuseppe Peano See [33] Applicazioni geometriche del calcolo infinitesimale] that continuous derivatives are the limits

$$\forall t \in ]a,b[, \lim_{h \to 0} \frac{x(t + h) - x(t - h)}{2h} = \frac{1}{2} \left( \lim_{h \to 0+} \frac{x(t) - x(t) - h}{h} + \lim_{h \to 0} \frac{x(t + h) - x(t)}{h} \right)$$

of both the retrospective and prospective average velocities (difference quotients) at time $t$. We follow his suggestion by taking the average of the prospective difference quotient $\overrightarrow{\nabla} h F(x,y)(\overrightarrow{u})$ and retrospective difference quotient $\overleftarrow{\nabla} h F(x,y)(\overrightarrow{u})$ their difference quotient

$$\overrightarrow{\nabla} 2h F(x,y)(\overrightarrow{u}) + \overleftarrow{\nabla} h F(x,y)(\overrightarrow{u})$$

(26)
and taking their Painlevé-Kuratowski limits

\[
\limsup_{h \to 0^+} \overrightarrow{\nabla}_h F(x, y)(\overrightarrow{u}_h) + \limsup_{h \to 0^+} \overleftarrow{\nabla}_h F(x, y)(\overleftarrow{u}_h)
\]

in order to define Peano graphical derivatives of \( F \) at \((x, y) \in \text{Graph}(F)\) depending on pairs \((\overrightarrow{u}, \overleftarrow{u})\) of directions.

\[\square\]

### 4.2 Differential Connection Tensors of Numerical Functions

When \( V : x \in X \mapsto V(x) \in \{ -\infty \} \cup \mathbb{R} \cup \{ +\infty \} \) is an extended numerical function on \( \mathbb{R} \), it can also be regarded as a set-valued map (again denoted by) \( V : X \leadsto \mathbb{R} \) defined by

\[
V(x) := \begin{cases} 
\{ V(x) \} & \text{if } V(x) \in \mathbb{R} \text{ (i.e., } x \in \text{Dom}(V)) \\
\emptyset & \text{if not}
\end{cases}
\]

(28)

A slight modification of Theorem 6.1.6, p. 230 of Set-valued analysis, [12, Aubin & Frankowska], states that

\[
\begin{cases}
\overrightarrow{D} V(x)(\overrightarrow{u}) = [\overrightarrow{D}_\uparrow V(x)(\overrightarrow{u}), \overrightarrow{D}_\downarrow V(x)(\overrightarrow{u})] \\
\overleftarrow{D} V(x)(\overleftarrow{u}) = [\overleftarrow{D}_\uparrow V(x)(\overleftarrow{u}), \overleftarrow{D}_\downarrow V(x)(\overleftarrow{u})]
\end{cases}
\]

where

\[
\begin{align*}
\overrightarrow{D}_\uparrow V(x)(\overrightarrow{u}) &:= \liminf_{h \to 0^+} \frac{V(x + h\overrightarrow{u}) - V(x)}{h} \quad \text{(epiderivative of } V) \\
\overrightarrow{D}_\downarrow V(x)(\overrightarrow{u}) &:= \limsup_{h \to 0^+} \frac{V(x + h\overrightarrow{u}) - V(x)}{h} \quad \text{(hypoderivative of } V) \\
\overleftarrow{D}_\uparrow V(x)(\overleftarrow{u}) &:= \liminf_{h \to 0^+} \frac{V(x) - V(x - h\overleftarrow{u})}{h} = -\overrightarrow{D}_\downarrow V(x)(-\overleftarrow{u}) \\
\overleftarrow{D}_\downarrow V(x)(\overleftarrow{u}) &:= \limsup_{h \to 0^+} \frac{V(x) - V(x - h\overleftarrow{u})}{h} = -\overrightarrow{D}_\uparrow V(x)(-\overleftarrow{u})
\end{align*}
\]

(30)

Definition 4.2, p 17 implies that

\[
\begin{cases}
\forall (\overrightarrow{u}, \overrightarrow{v}), (\overleftarrow{u}, \overleftarrow{v}) \in \overrightarrow{D} V(x)(\overrightarrow{u}), \overleftarrow{D} V(x)(\overleftarrow{u}) \\
\text{a}_V(x, y)((\overrightarrow{u}, \overrightarrow{v}), (\overleftarrow{u}, \overleftarrow{v})) := \overrightarrow{v} \overleftarrow{v}
\end{cases}
\]

(31)

since tensor products of real numbers boil down to their multiplication.

Therefore, for any pair \((\overrightarrow{u}, \overrightarrow{v})\), the subset of differential connection tensors of retrospective and prospective directions is equal to
\[
\begin{align*}
\left\{ \begin{array}{l}
\overrightarrow{D}V(x)(\overleftarrow{u}) \otimes \overrightarrow{D}V(x)(\overrightarrow{u}) := \\
\{(\overrightarrow{v},\overrightarrow{u}) \in [\overrightarrow{D}_x V(x)(\overrightarrow{u}), \overrightarrow{D}_x V(x)(\overrightarrow{u})] \times [\overleftarrow{D}_x V(x)(\overleftarrow{u}), \overleftarrow{D}_x V(x)(\overleftarrow{u})] \}
\end{array} \right. \\
\end{align*}
\] (32)

**Definition 4.3 [Reversal Direction Pair]** A pair \((\overleftarrow{u}, \overrightarrow{u})\) of directions \(\overleftarrow{u} \in X\) and \(\overrightarrow{u} \in X\) is a reversal direction pair of \(V\) at \(x \in \text{Dom}(V)\) if

\[
\overrightarrow{D}_x (\overleftarrow{u}) \overrightarrow{D}_x (\overrightarrow{u}) = \overrightarrow{D}_x (-\overrightarrow{u}) \overleftarrow{D}_x (\overrightarrow{u}) < 0
\] (33)

A direction \(u \in X\) is a reversal direction of \(V\) at \(x\) if the diagonal pair \((u, u)\) is reversal direction pair.

This means that a positive (resp. negative) retrospective epiderivative of \(V\) at \(x\) in the direction \(\overleftarrow{u}\) is followed by a negative (resp. positive) prospective epiderivative in the direction \(\overrightarrow{u}\), or, respectively, that a positive (resp. negative) retrospective hypoderivative in the direction \(-\overrightarrow{u}\) is followed by a negative (resp. positive) prospective hypoderivative in the direction \(-\overleftarrow{u}\).

Recall that if \(V\) achieves a local minimum at \(x\), the Fermat rule states that

\[
\forall \overrightarrow{u} \in X, \overrightarrow{D}_x V(x)(\overrightarrow{u}) \geq 0 \quad \text{and} \quad \forall \overleftarrow{u} \in X, \overleftarrow{D}_x V(x)(\overleftarrow{u}) \leq 0
\] (34)

and if it achieves a local maximum at \(x\), that

\[
\forall \overrightarrow{u} \in X, \overrightarrow{D}_x V(x)(\overrightarrow{u}) \leq 0 \quad \text{and} \quad \forall \overleftarrow{u} \in X, \overleftarrow{D}_x V(x)(\overleftarrow{u}) \geq 0
\] (35)

These conditions are not sufficient for characterizing local extrema: convexity or many second order conditions provide sufficient conditions (see *Set-valued analysis*, [12, Aubin & Frankowska], *Variational Analysis*, [35, Rockafellar & Wets] and an important literature on set-valued and variational analysis).

Recall that the prospective epidifferential (or prospective epiderivative subdifferential) \(\overrightarrow{D}_x V(x)\) of a function \(V\) at \(x\) is the set of elements \(\overrightarrow{\rho}_x \in X^*\) such that for any \(v \in X\),

\[
\langle \overrightarrow{\rho}_x, v \rangle \leq \overrightarrow{D}_x V(x)(v).
\]

In the same way, we define the retrospective epidifferential (or retrospective epiderivative subdifferential) \(\overleftarrow{D}_x V(x)\) of a function \(V\) at \(x\) as the set of elements \(\overleftarrow{\rho}_x \in X^*\) such that for any \(v \in X\),

\[
\langle \overleftarrow{\rho}_x, v \rangle \leq \overleftarrow{D}_x V(x)(v).
\]

It is equal to prospective hypodifferential (or prospective superdifferential) \(\overrightarrow{D}_x V(x)\), the set of elements \(\overrightarrow{\rho}_x \in X^*\) such that for any \(v \in X\),

\[
\langle \overrightarrow{\rho}_x, v \rangle \geq \overrightarrow{D}_x V(x)(v).
\]

For instance, the trendometer detects the local extrema of numerical functions, such as the function \(t \mapsto 1 - \cos(2t) \cos(3t)\):
4.3 Tangential Connection Tensors

The tangent spaces to differentiable manifolds being vector spaces, directions arriving at a point (we may call them retrospective) and directions starting from this point (prospective) belong to the same vector space. This is no longer the case when the subset is any (closed) subset $K \subset X$ of a finite dimensional vector space $X$. However, we may replace vector spaces by cones.

We are indebted to the historical studies [17, Dolecki & Greco] (in which the authors quote Maurice Fréchet stating that “Cette théorie des “contingents et paratingents” dont l’utilité a été signalée d’abord par M. Beppo Levi, puis par M. Severi, mais dont on doit à M. Bouligand et ses élèves d’en avoir entrepris l’étude systématique.”) and [25, Greco, Mazzucchi & Pagani]. Francesco Severi and Georges Bouligand, a whole menagerie of tangent cones, the definitions of which depend upon the limiting process, have been proposed (among many monographs, see [12, Set-valued analysis] and Variational Analysis, [35, Rockafellar & Wets] for instance). At some points, the tangent cones are not vector spaces, and the opposite of some tangent directions may no longer be tangent.

We suggest to regard the (contingent) tangent cone$^{13}$ as the prospective tangent cone to $K$ at $x \in K$ defined by the Painlevé-Kuratowski upper limits.

$^{13}$See [12, Set-valued analysis]. The (adjacent) Peano-Severi-Bouligand tangent cone is defined by the
\[ \widetilde{T}_{K}(x) := \limsup_{h \to 0^+} \frac{K - x}{h} := \left\{ \nu \in X \text{ such that } \liminf_{h \to 0^+} \frac{d_{K}(x + h \nu)}{h} = 0 \right\} \] (37)

with which we associate (adjacent) retrospective tangent cone

\[ \check{T}_{K}(x) := \limsup_{h \to 0^+} \frac{x - K}{h} := \left\{ \nu \in X \text{ such that } \liminf_{h \to 0^+} \frac{d_{K}(x - h \nu)}{h} = 0 \right\} \] (38)

satisfying \( \check{T}_{K}(x) := -\check{T}_{K}(x) \). It is natural to consider their tensor product \((x - h \nu) \otimes (x + h \nu)\). The signs of its entries detect the “blunt” and “sharp” elements of the boundary in the same directions (trend congruence) or in opposite directions (trend reversal).

References

[1] Aubin J.-P. (1983) Slow and heavy trajectories of controlled problems: smooth viability domains, In \textit{Multifunctions and Integrands}, 105-116 Lecture Notes in Mathematics, #1091, Ed. Salinetti G., Springer-Verlag

[2] Aubin J.-P. (1991) \textit{Viability Theory}, Birkhäuser

[3] Aubin J.-P. (1996) \textit{Neural Networks and Qualitative Physics: a Viability Approach}, Cambridge University Press

[4] Aubin J.-P. (1998) Connectionist Complexity and its Evolution, in \textit{Equations aux dérivées partielles, Articles dédiés à J.-L. Lions}, 50-79, Elsevier

[5] Aubin J.-P. (2003) Regulation of the Evolution of the Architecture of a Network by Connectionist Tensors Operating on Coalitions of Actors, \textit{J. Evolutionary Economics}, 13, 95-124

Painlevé-Kuratowski lower limits instead of upper limits

\[ \liminf_{h \to 0^+} \frac{K - x}{h} := \left\{ \nu \in X \text{ such that } \lim_{h \to 0^+} \frac{d_{K}(x + h \nu)}{h} = 0 \right\} \] (36)

The smaller \textit{adjacent} tangent cone is used whenever more regularity is required. An element \( x \in K \) is said to be \textit{regular} in \( K \) at \( x \) if both contingent and adjacent tangent cones coincide, i.e., when \( T_{K}(x) \) is the Painlevé-Kuratowski limit of \( \frac{K - x}{h} \).

\textsuperscript{14}Backward evolutions and negative tangents have been introduced in [19, 20, Frankowska] for characterizing lower semicontinuous (viscosity) solutions to Hamilton-Jacobi-Bellman equations.
[6] Aubin J.-P. (2010) Macroscopic Traffic Models: Shifting from Densities to “Celerities”, *Applied Mathematics and Computation*, 217, 963-971, \url{http://dx.doi.org/10.1016/j.amc.2010.02.032}

[7] Aubin J.-P. (2013) Chaperoning State Evolutions by Variable Durations, *SIAM Journal of Control and Optimization*, DOI. 10.1137/120879853

[8] Aubin J.-P. (submitted) Transports Regulators of Networks with Junctions Detected by Durations Functions,

[9] Aubin J.-P., Bayen A. & Saint-Pierre P. (2011) *Viability Theory. New Directions*, Springer

[10] Aubin J.-P. & Burnod Y. (1998) Hebbian Learning in Neural Networks with Gates, *Cahiers du Centre de Recherche Viabilité, Jeux, Contrôle* # 981

[11] Aubin J.-P., Chen Lx & Dordan O. (2014) *Tychastic Measure of Viability Risk. A Viabilist Portfolio Performance and Insurance Approach*

[12] Aubin J.-P. & Frankowska H. (1990) *Set-valued analysis*, Birkhäuser

[13] Aubin J.-P. & Frankowska H. (1985) Heavy viable trajectories of controlled systems, *Proceedings of Dynamics of Macrosystems*, IIASA, September 1984,, Ed. Aubin J.-P., Saari D.& Sigmund K., Springer-Verlag,148-167

[14] Aubin J.-P. & Haddad G. (2001) Path-dependent impulse and hybrid control systems, in *Hybrid Systems: Computation and Control*, 119-132, Di Benedetto & Sangiovanni-Vincentelli Eds, Proceedings of the HSCC 2001 Conference, LNCS 2034, Springer-Verlag

[15] Aubin J.-P. & Haddad G. (2002) History (Path) Dependent Optimal Control and Portfolio Valuation and Management, *J. Positivity*, 6, 331-358

[16] Buquoy G. (1815) *Exposition d’un nouveau principe général de dynamique, dont le principe des vitesses virtuelles n’est qu’un cas particulier*, V. Courtier

[17] Dolecki S. & Greco G. H. (2007) Towards Historical Roots of Necessary Conditions of Optimality: Regula of Peano, *Control and Cybernetics*, 36, 491-518

[18] Dordan O. (1995) *Analyse qualitative*, Masson

[19] Frankowska H. (1991) Lower semicontinuous solutions to Hamilton-Jacobi-Bellman equations, in *Proceedings of the 30th IEEE Conference on Decision and Control*, Brighton, UK,
[20] Frankowska H. (1993) Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equations, *SIAM J. Control Optim.*, 31, 257-272.

[21] Galperin E. A. (2009) Information transmittal, time uncertainty, and special relativity, *Computers and Mathematics with Applications*, 57, 1554-1573, doi: 10.1016/j.camwa.2008.09.048

[22] Galperin E. A. (2011) Left time derivatives in mathematics, mechanics and control of motion, *Computers and Mathematics with Application*, 62 4742–4757

[23] Galperin E. A. (submitted) Information Transmittal, Causality, Relativity and Optimality,

[24] Galperin E. A. (submitted) Time And Relativity In Dynamical Systems,

[25] Greco G. H., Mazzucchi S. & Pagani E. M. (2010) Peano on derivative of measures: strict derivative of distributive set functions, *Rend. Lincei Mat. Appl.*, 21, 305-339 DOI 10.4171/RLM/575

[26] Haddad G. (1981) Monotone trajectories of differential inclusions with memory, *Isr. J. Math.*, 39, 83-100

[27] Haddad G. (1981) Monotone viable trajectories for functional differential inclusions, *J. Diff. Eq.*, 42, 1-24

[28] Haddad G. (1981) Topological properties of the set of solutions for functional differential inclusions, *Nonlinear Anal. Theory, Meth. Appl.*, 5, 1349-1366

[29] Hale J. K. (1993) *Introduction to Functional Differential Equations*, Springer

[30] Hebb D. (1949) *The Organization of Behavior*, Wiley

[31] Mestschersky I.V. (1897) Dynamics of point with variable mass. In I.V. Mestschersky, *Works on Mechanics of Bodies with Variable Mass*, Edition, Gostechizdat, Moscow, 1952, 37-188

[32] Levi-Civita (1928) Sul moto di un corpo di massa variabile, *Rendiconti dei Lincei*, 329-333

[33] Peano G. (1887) *Applicazioni geometriche del calcolo infinitesimale*, Fratelli Bocca Editori, http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?id=00610002&seq=

[34] Porcar M., Danchin A. de Lorenzo V., dos Santos V., Krasnogor N., Rasmussen S. & Moya A. (2011), The ten grand challenges of synthetic life, *Syst Synth Biol*, 5, 1–9, doi: 10.1007/s11693-011-9084-5
[35] Rockafellar R.T. & Wets R. (1997) *Variational Analysis*, Springer-Verlag

[36] Tallos P. (1991) Viability problems for nonautonomous differential inclusions, *SIAM J. on Control and Optimization*, 29 253-263

[37] Vinogradova G. (2012) Correction of Dynamical Network’s Viability by Decentralization by Price, *Complex Systems*, 21, 37-55
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