On nonlinear partial differential equations with an infinite-dimensional conditional symmetry

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Abstract

The invariance of nonlinear partial differential equations under a certain infinite-dimensional Lie algebra $\mathfrak{A}_N(z)$ in $N$ spatial dimensions is studied. The special case $\mathfrak{A}_1(2)$ was introduced in J. Stat. Phys. 75, 1023 (1994) and contains the Schrödinger Lie algebra $\mathfrak{so}_1$ as a Lie subalgebra. It is shown that there is no second-order equation which is invariant under the massless realizations of $\mathfrak{A}_N(z)$. However, a large class of strongly non-linear partial differential equations is found which are conditionally invariant with respect to the massless realization of $\mathfrak{A}_N(z)$ such that the well-known Monge-Ampère equation is the required additional condition. New exact solutions are found for some representatives of this class.

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1 Introduction

The maximal invariance algebra (MIA) of the free Schrödinger equation in $N$ spatial dimensions has been studied since a long time. This Lie algebra is known as the Schrödinger algebra \[20\] and will be denoted here by $\mathfrak{sch}_N$. An infinite-dimensional generalization of $\mathfrak{sch}_1$ was constructed in \[14\] and will be denoted here by $\mathfrak{A}_1(2)$. These algebras contain infinitesimal dilatations in ‘time’ and ‘space’ coordinates, respectively, and correspond to the finite transformations $t \mapsto b^2 t$, $x \mapsto b x$ where $b$ is an arbitrary real parameter and $x = (x_1, \ldots, x_N)$. While the transformations contained in $\mathfrak{sch}_N$ merely contain the projective transformation $t \mapsto (\alpha t + \beta) / (\gamma t + \delta)$ with real parameters satisfying the restriction $\alpha \delta - \beta \gamma = 1$, the transformations in $\mathfrak{A}_N(2)$ allow for arbitrary transformation in ‘time’, viz. $t \mapsto b(t)^2$, where $b$ is an arbitrary smooth function. Recently, this construction was extended to an arbitrary dynamical exponent $z$ defined by the dilatation

$$t \mapsto t' := b^z t, \quad x \mapsto x' := bx$$

An infinite-dimensional Lie algebra $\mathfrak{A}_N(z)$ can be obtained by allowing transformation in ‘time’ $t \mapsto b(t)^z$ \[15\]. In order to define it, consider the generators

$$X_n = z t^{n+1} \partial_t + (n+1) t^n x_a \partial_a + \lambda(n+1)t^n u \partial_u$$

and

$$Y_m^{(a)} = t^{m+1/z} \partial_a$$

$$J_{ab} = x_a \partial_b - x_b \partial_a \quad ; \quad a, b = 1, \ldots, N$$

where the summation convention over repeated indices is here and afterwards implied, $\partial_a = \partial / \partial x_a$, $\partial_t = \partial / \partial t$, $\partial_u = \partial / \partial u$ (where $u$ is the function the generators are supposed to act on). Finally $n \in \mathbb{Z}$ and $m + \frac{1}{z} \in \mathbb{Z}$ will be used throughout. The non-vanishing commutators of these generators are \[15\]

$$[X_n, X_{n'}] = z (n' - n) X_{n+n'}, \quad [X_n, Y_m^{(a)}] = (zm - n) Y_{n+m}^{(a)}, \quad [Y_m^{(a)}, J_{bc}] = \delta_{ab} Y_m^{(c)} - \delta_{ac} Y_m^{(b)}$$

In this paper we shall be concerned with two infinite-dimensional Lie algebras which are defined in terms of a basis of generators from \[12\] and \[13\] as follows.

$$\mathfrak{A}_N(z) := \left\{ X_n, Y_m^{(a)} \mid n \in \mathbb{Z}, m + \frac{1}{z} \in \mathbb{Z}, a \in \{1, \ldots, N\} \right\}$$

$$\mathfrak{B}_N(z) := \left\{ X_n, Y_m^{(a)} \mid n \in \{-1, 0\}, m + \frac{1}{z} \in \mathbb{Z}, a \in \{1, \ldots, N\} \right\}$$

Obviously, $\mathfrak{B}_N(z)$ is a Lie subalgebra of $\mathfrak{A}_N(z)$ and $\mathfrak{sch}_1$ is the maximal finite-dimensional Lie subalgebra of $\mathfrak{A}_1(2)$. Since with the definition \[13\] the commutator $[Y_m^{(a)}, Y_{m'}^{(b)}] = 0$ for all pairs $(m, m')$, the generators \[13\] are said to form a massless realization of the Lie algebras $\mathfrak{A}_N(z)$ or $\mathfrak{B}_N(z)$, respectively. For the massless realization \[13\] only the arguments of the function $u$ but not $u$ itself transform when acting with the generators $Y_m^{(a)}$.
Eqs. (1.2) and (1.3) can be extended to massive realizations of $\mathfrak{A}_N(2)$ as given in [14, 15] and it has been shown that these may be used as a dynamical symmetry in non-equilibrium statistical mechanics, in particular in relation with ageing phenomena [15]. Massive realizations of the infinite-dimensional Lie algebra $\mathfrak{A}_1(1)$ are also known [15]. For $\gamma \neq 1, 2$, massive realizations can no longer be constructed in terms of first-order differential operators, see [15] for details and for applications to physical ageing.

On the other hand, massless realizations may also be of physical interest. For example, it is well-known that the 1D Burgers equation $u_t + uu_x - u_{xx} = f(t, x, U)$ has a massless realization of $\mathfrak{sch}_1$ (albeit different from (1.2) and (1.3)) for an external force $f = \text{constant}$ (and moreover $f = 0$ can be achieved by a non-trivial local substitution) [17, 16].

We shall attempt to look for non-linear partial differential equations (PDEs) which have the infinite-dimensional massless realization (1.2) and (1.3) of $\mathfrak{A}_N(z)$ or $\mathfrak{B}_N(z)$ as dynamical symmetry.

The so-called scaling-dimension $\lambda \neq 0$ can always be brought to $\lambda = 1$ by the local substitution $u \to u^\lambda$. Therefore, without loss of generality, we may write

$$X_n = z t^{n+1} \partial_t + (n+1)t^n x_a \partial_a + (n+1)t^n u \partial_u$$

instead of (1.2) and we shall do so throughout this paper. Note that the Lie algebra $\mathfrak{A}_N(z)$ contains the following subalgebra spanned by the generators

$$Y^{(a)}_{-1/z} = P_a = \partial_a , \quad Y^{(a)}_{1/z} = C_a = t \partial_a , \quad X_{-1} = P_t = \partial_t$$

with $a, b = 1, \ldots, N$. It is easy to see that these generators together with $J_{ab}$ (see (1.3)) span the Galilei algebra $AC_0^0(1, N)$ with a vanishing mass operator. The partial differential equations invariant with respect to the algebra $AC_0^0(1, N)$ and its subalgebra (1.8) were classified in [10, 5, 7].

Starting form this observation, we prove in section 2 that there is no second-order PDE invariant under the Lie algebra $\mathfrak{A}_N(z)$ but we do find PDEs with the Lie algebra $\mathfrak{B}_N(z)$ as a dynamical symmetry, for any given value of $z$. In addition, we construct a wide class of PDEs which admit $\mathfrak{A}_N(z)$ as a conditional dynamical symmetry and we show that the additional condition is the $N$-dimensional Monge-Ampère equation. Monge-Ampère equations are classical equations in differential geometry and analysis and arise in many different contexts. For recent reviews of their mathematical theory, see [4, 13, 24]. Among the physical applications, we quote the discussion of the equivalence of the mass-transport problem with a quadratic cost function to a (inhomogeneous 3D) Monge-Ampère equation and applications to cosmology in [3]. The two-dimensional Monge-Ampère equations also plays a central rôle arises in the description of pattern formation in convective systems modeled through the Cross-Newell equation [8, 19], as well as in M-theory extensions of quantum chromodynamics [25].

In section 3, we use the conditional symmetry to derive exact solutions for some of these invariant equations. In particular, we study the system of (1+2)-dimensional non-linear equations

$$\begin{vmatrix}
    u_t & u_1 & u_2 \\
    u_1 & u_{11} & u_{12} \\
    u_2 & u_{21} & u_{22}
\end{vmatrix} = \frac{\partial}{\partial x_1} (u^{-z}u_1) + \frac{\partial}{\partial x_2} (u^{-z}u_2) = 0$$

(1.9)

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$$\det \begin{bmatrix}
    u_t & u_1 & u_2 \\
    u_1 & u_{11} & u_{12} \\
    u_2 & u_{21} & u_{22}
\end{bmatrix} = \frac{\partial}{\partial x_1} (u^{-z}u_1) + \frac{\partial}{\partial x_2} (u^{-z}u_2) = 0$$

(1.10)
where \( u_t = \partial_t u \), \( u_a = \partial u/\partial x_a \) and \( u_{ab} = \partial^2 u/\partial x_a \partial x_b \) with \( a, b = 1, \ldots, N \). The Monge-Ampère equation associated to (1.9) is given by (1.10). Eq. (1.9) might be viewed as an example of a PDE conditionally invariant under \( \mathfrak{A}_2(z) \), but it might also be thought of as non-linear diffusion equation where the time-derivative \( u_t \) has been generalized. Our conclusions are presented in section 4.

2 Invariance of a class of PDEs under the Lie algebras \( \mathfrak{A}_N(z) \) and \( \mathfrak{B}_N(z) \)

Consider the following class of PDEs with first- and second-order derivatives

\[
F(t, x, u, u_t, u_{tt}, u_{11}, u_{111}, \ldots) = 0 \tag{2.1}
\]

in \((1 + N)\)-dimensional time-space. Here and afterwards, we use the notation \( x = (x_1, \ldots, x_N) \), \( u = (u_1, \ldots, u_N) \), \( u_t = (u_{t1}, \ldots, u_{tN}) \) and \( u = (u_{11}, \ldots, u_{1N}, \ldots, u_{NN}) \) and \( F \) is an arbitrary smooth function. Obviously, any PDE of the form (2.1) which admits \( \mathfrak{A}_N(z) \) as given by the generators (1.3,1.7) must also be invariant under the Galilei subalgebra \( \mathfrak{A}G^0(1, N) \) spanned by the generators (1.8). The PDEs of the form (2.1) which admit \( \mathfrak{A}G^0(1, N) \) as a dynamical symmetry have been classified long ago. The following statement holds true.

**Lemma.** [10, Theorem 11] If \( W^I \) and \( W^{II} \) are defined as follows

\[
W^I = \det \begin{bmatrix}
  u_t & u_1 & \cdots & u_N \\
  u_{t1} & u_{11} & \cdots & u_{1N} \\
  \vdots & \vdots & \cdots & \vdots \\
  u_{tN} & u_{1N} & \cdots & u_{NN}
\end{bmatrix} \tag{2.2}
\]

\[
W^{II} = \det \begin{bmatrix}
  u_{tt} & u_{t1} & \cdots & u_{tN} \\
  u_{t1} & u_{11} & \cdots & u_{1N} \\
  \vdots & \vdots & \cdots & \vdots \\
  u_{tN} & u_{1N} & \cdots & u_{NN}
\end{bmatrix} \tag{2.3}
\]

then PDEs of the form

\[
F_1 \left( W^I, W^{II}, u, u_t, u_{11} \right) = 0 \tag{2.4}
\]

where \( F_1 \) is an arbitrary smooth function, are most general PDEs of the form (2.1) which are invariant under the Lie algebra spanned by the generators (1.8).

In consequence, any PDE with \( \mathfrak{A}_N(z) \) as a Lie symmetry will be among the equations of the form (2.4). Furthermore, since the generators \( Y_m^{(a)} \) of (1.3) generate the following group continuous transformations

\[
x'_a = x_a + v_a t^{m+1/2}, \quad t' = t, \quad u' = u \tag{2.5}
\]
where \( v_1, \ldots, v_N \) are real (or complex) group parameters it is straightforward to check that \( W^I, u \) and \( u \) are absolute differential invariants of the Lie group \((2.5)\). On the other hand, \( W^{II} \) transforms non-trivially under the action of transformations from \((2.5)\) with \( m \neq -1/z, 1 - 1/z \). Therefore, any PDE which is invariant under the Lie algebra spanned by the generators \( \{ Y_m^{(a)} , X_{-1} = P_t \} \) as defined in \((1.8)\) must be of the form

\[
F_2(W^I, u, u_1, u_11) = 0 \tag{2.6}
\]

where \( F_2 \) is an arbitrary smooth function.

We now consider the continuous transformations generated from the \( X_n \). Their finite form is easily obtained from \((1.7)\)

\[
\begin{align*}
    t' &= |t^n - zn\varepsilon_n|^{-1/n} = t |1 - zn\varepsilon_n t^n|^{-1/n} \\
    x'_a &= x_a |1 - zn\varepsilon_n t^n|^{-(n+1)/(zn)} \\
    u' &= u |1 - zn\varepsilon_n t^n|^{-(n+1)/(zn)}
\end{align*} \tag{2.7}
\]

where the \( \varepsilon_n \), with \( n \in \mathbb{Z} \), are group parameters and \( a = 1, \ldots, N \).

In view of the quite lengthy calculations which have to be performed, it is useful to consider first the case \( N = 1 \). In this case equation \((2.6)\) takes the form

\[
W^I = f(u, u_x, u_{xx}) \tag{2.8}
\]

where

\[
W^I = \det \begin{bmatrix} u_t & u_x \\ u_{tx} & u_{xx} \end{bmatrix} \tag{2.9}
\]

and \( f \) is the inverse function of \( F_2 \) when solving for the variable \( W^I \). In addition, eq. \((2.7)\) reduces to

\[
\begin{align*}
    t' &= t |1 - zn\varepsilon_n t^n|^{-1/n} \\
    x' &= x |1 - zn\varepsilon_n t^n|^{-(n+1)/(zn)} \\
    u' &= u |1 - zn\varepsilon_n t^n|^{-(n+1)/(zn)}
\end{align*} \tag{2.10}
\]

Assuming \( z \neq 0 \) and that the parameters \( \varepsilon_n \) are sufficiently small, the absolute value in eqs. \((2.10)\) may be dropped and we find the following transformation laws of the derivatives

\[
\begin{align*}
    u'_x &= u_x \\
    u'_{xx} &= A(t)^{-1}u_{xx} \\
    u'_t &= A(t)^{-z}u_t + (u - xu_x) n(n + 1)\varepsilon_n t^{n-1} A(t)^{1+zn/(n+1)} \\
    u'_{tx} &= A(t)^{-z}u_{tx} - xu_{xx} n(n + 1)\varepsilon_n t^{n-1} A(t)^{zn/(n+1)}
\end{align*} \tag{2.11}
\]

where

\[
A(t) := (1 - zn\varepsilon_n t^n)^{-(n+1)/(zn)} \tag{2.12}
\]
Substituting the transformation formulas (2.10, 2.11) into eq. (2.8) for the function \( u'(t', x') \), we arrive at the equation

\[
A(t)^{-z} W^I + n(n+1)\varepsilon_n t^{n-1} A(t)^{zn/(n+1)} uu_{xx} = f \left( A(t)u, u_x, A(t)^{-1} u_{xx} \right)
\]  

(2.13)

Consider the function \( B(t) := (n+1)\varepsilon_n t^{n-1} \), \( n \in \mathbb{Z} \). It arises only in the second term on the left-hand side of eq. (2.13) and cannot be expressed as some power of \( A(t) \). Therefore, eq. (2.13) is not reducible to eq. (2.8) for any function \( f \) (and not even in the special case \( f = 0 \)). The only exceptions to this are when \( n = -1 \) and \( n = 0 \) when the second term on the left-hand side vanishes. The operators \( X_{-1} = P_1 \) and \( X_0 = D = zt \partial_t + x \partial_x + u \partial_u \) correspond to these cases.

It is easily checked that the same result is also obtained in the case when \( |1 - zn\varepsilon_n t^n| = zn\varepsilon_n t^n - 1 \) which arises when \( \varepsilon_n \) is not small.

Summarizing, we have just seen that there is no PDE belonging to the class (2.8) which admits the infinite-dimensional Lie algebra \( A_1(z) \) with generators given by eqs. (1.3, 1.7) and \( z \neq 0 \) as a dynamical symmetry. On the other hand, we can look for those PDEs which are invariant under the infinite-dimensional Lie algebra \( B_1(z) \). In this case eq. (2.13) simplifies into

\[
A(t)^{-z} W^I = f \left( A(t)u, u_x, A(t)^{-1} u_{xx} \right)
\]  

(2.14)

and we see that this can be reduced to (2.8) only if

\[
f = u^{-z} g \left( u_x, uu_{xx} \right)
\]  

(2.15)

where \( g \) is an arbitrary smooth function of two variables. Thus, we have proved the following theorem.

**Theorem 1.** (i) Any PDE belonging to the class (2.8) cannot be invariant under the Lie algebra \( A_1(z) \) generated by (1.3, 1.7) with \( N = 1 \) and \( z \neq 0 \).

(ii) Only non-linear PDEs of the form

\[
W^I = u^{-z} g \left( u_x, uu_{xx} \right)
\]  

(2.16)

where \( W^I \) is defined by (2.9) are invariant under the infinite-dimensional Lie algebra \( B_1(z) \) spanned by the basis of operators

\[
X_{-1} = P_1 = \partial_t, \quad X_0 = D = zt \partial_t + x \partial_x + u \partial_u, \quad Y_m = t^{m+1/z} \partial_x; \quad m + \frac{1}{z} \in \mathbb{Z}
\]

(2.17)

**Remark 1.** Consider the generators \( X_n \) in the case \( z = 0 \). Then we have the continuous transformations

\[
t' = t, \quad x' = x \exp(\varepsilon_n t^n), \quad u' = u \exp(\varepsilon_n t^n)
\]

with \( n \in \mathbb{Z} \). Simultaneously, we replace the series of the operators \( Y_m^{(a)} \) arising in (1.3) by the operators \( \tilde{Y}_m^{(a)} := t^m \partial_a \) with \( m \in \mathbb{Z} \). Then it is easy to check that the result of Theorem 1 remains valid also for the case \( z = 0 \).
We now treat the multidimensional case \( N > 1 \). It turns out that the transformations (2.7) lead to a simple generalization of the one-dimensional formulas Eqs. (2.11,2.13,2.15). Indeed, Eqs. (2.11) now take the form

\[
\begin{align*}
    u'_{a'} &= u_a \\
    u'_{a'b'} &= A(t)^{-1}u_{ab} \\
    u'_{t'} &= A(t)^{-z}u_t + (u - x_a u_a) n(n + 1) \varepsilon_n t^{n-1} A(t)^{1+z/n(n+1)} \\
    u'_{t'b'} &= A(t)^{-z}u_{tb} - x_a u_{ab} n(n + 1) \varepsilon_n t^{n-1} A(t)^{zn/n(n+1)}
\end{align*}
\]

(2.18)

Substituting this into a multidimensional analogue of (2.13) we arrive at the expression

\[
A(t)^{1-z-N}W^I + n(n + 1) \varepsilon_n t^{n-1} A(t)^{zn/n(n+1)} uW^I_N = f \left( A(t)u, u_1, A(t)^{-1}u_{11} \right)
\]

(2.19)

where

\[
W^I_N = \det \begin{pmatrix}
    u_{11} & u_{12} & \cdots & u_{1N} \\
    u_{21} & u_{22} & \cdots & u_{2N} \\
    \vdots & \vdots & \ddots & \vdots \\
    u_{N1} & u_{N2} & \cdots & u_{NN}
\end{pmatrix}
\]

(2.20)

We can now analyze Eq. (2.19) along the same lines as before (and also its analogue for the case \( z = 0 \)) and arrive at the following result which generalizes Theorem 1.

**Theorem 2.** (i) Any partial differential equation of the class (2.1) cannot be invariant under the Lie algebra \( \mathfrak{A}_N(z) \) generated by (1.3,1.7).

(ii) Equations of the form

\[
W^I = u^{1-z-N}g \left( u_1, u_{11} \right)
\]

(2.21)

where \( g \) is an arbitrary smooth function of \( N(N + 3)/2 \) variables, are the only PDEs of the class (2.1) invariant under the infinite-dimensional Lie algebra \( \mathfrak{B}_N(z) \) (1.6).

**Remark 2.** The class (2.21) of PDEs contains some equations belonging to the class of reaction-diffusion-convection equations which were recently suggested and investigated in [7]. However, in general these are two different classes of equations.

To proceed further with equation (2.19), we need the concept of conditional invariance which we recall for the convenience of the reader.

**Definition.** [12] Section 5.7 A PDE of the form

\[
S \left( x, u, u_1, \ldots u_{m_1} \right) = 0
\]

(2.22)

(here \( u \) is the totality of \( k^{th} \)-order derivatives) is **conditionally invariant** under the operator

\[
Q = \xi^a(x_1, \ldots, x_N, u) \partial_a + \eta(x_1, \ldots, x_N, u) \partial_u
\]

(2.23)
where $\eta$ and $\xi^a$ with $a = 1, \ldots, N$ are smooth functions, if it is invariant (in Lie’s sense) under this operator only together with an additional condition of the form

$$S_Q \left( x, u, u_1, \ldots, u_{m_2} \right) = 0 \quad (2.24)$$

that is, the overdetermined system eqs. (2.22,2.24) is invariant under a Lie group generated by the operator $Q$.

If the additional condition (2.24) coincides with the equation $Qu = 0$ then a $Q$-conditional symmetry is obtained. The notion of $Q$-conditional symmetry coincides with the notion of the non-classical symmetry as introduced in [1] and the side condition as introduced in [22]. The notion of the non-classical symmetry was further developed in [18] where it is called ‘conditional symmetry’, these authors also considered linear second-order partial differential equations as an additional condition for the generation of conditional symmetries [18, p. 2922].

Following this definition, we impose the additional condition

$$W_{1}^{1} = 0 \quad (2.25)$$

and then eq. (2.19) reduces to the expression

$$A(t)^{1-z-N}W^{I} = f \left( A(t)u, A(t)^{-1}u_{11} \right) \quad (2.26)$$

Clearly, having (2.26) and the condition (2.25) we recover again eq. (2.21). Eq. (2.25) is a multidimensional extension of the well-known two-dimensional Monge-Ampère equation, see [13, 23, 24]

$$u_{11}u_{22} - u_{12}^2 = 0 \quad (2.27)$$

The MIA of the $N$-dimensional Monge-Ampère equation (2.25), which we denote by $\mathfrak{M}_N$, is well-known.

**Lemma.** [12, Theorem 1.10.1] Consider the $N$-dimensional Monge-Ampère equation (2.25) where $W_{1}^{1}$ is defined in (2.20) and also write $x_0 := u$. Then a basis of the MIA $\mathfrak{M}_N$ is given by the following operators

$$\partial_i, \ x_i \partial_j, \ x_i x_j \partial_j \quad (2.28)$$

where $i, j = 0, 1, \ldots, N$.

It is easily seen that $\mathfrak{N}_N(z)$ and $\mathfrak{A}_N(z)$ are subalgebras of $\mathfrak{M}_N$ because the ‘time’ $t$ is only a parameter in the Monge-Ampère equation (2.25). We therefore have our main result.

**Theorem 3.** A partial differential equation of the form (2.1) is conditionally invariant under the infinite-dimensional Lie algebra $\mathfrak{A}_N(z)$ if and only if (i) it is of the form (2.21) and (ii) the $N$-dimensional Monge-Ampère equation (2.25) holds.

We point out that we deal here with a non-Lie invariance because the classical criterion of Lie invariance (see, e.g., [2], [21]) does not admit any side condition. Also, it is not the non-classical invariance in the Bluman-Cole sense [1] because that kind of invariance admits only an additional
condition in the form of a *quasilinear first-order* PDE, see (2.23), while the additional condition Eq. (2.25) is a *strongly non-linear second-order* PDE. Thus, we have found here a non-trivial example of a purely conditional invariance as introduced by Fushchych and collaborators [11, 12]. Of course, this could still turn out to be a Q-conditional symmetry with a higher-order symmetry operator, see e.g. [9, 26], and we hope to study this elsewhere.

**Remark 3.** One can easily check that Theorems 2 and 3 are still valid if one replaces the series of the operators $Y^{(a)}$ arising in the algebras $\mathfrak{B}_N(z)$ and $\mathfrak{A}_N(z)$ by the operators

$$Y^{(a)} = \varphi_a(t)\partial_a ; \quad a = 1, \ldots, N$$

(2.29)

where $\varphi_a(t)$ are arbitrary smooth functions and there is no summation over the index $a$ here.

## 3 Exact solutions of the conditionally invariant equation (1.9)

It is easily seen that Eq. (2.21) is a nonlinear PDE for any fixed function $g$. Even the simplest examples of this class, such as $W^I = 0$ and $W^I = u^{1-z-N}$ which were studied in [10, 5] are still non-linear equations. One possible point of view is to consider the determinant $W^I$ (see Eq. (2.2)) as a generalization of the ‘velocity’ $u_t$. In the context of such a physical interpretation it appears to be reasonable to study a generalized diffusion equation of the form

$$W^I = u^{2-z-N}\Delta u + \gamma u^{1-z-N}u_au_a$$

(3.1)

(where $\gamma$ is a constant) since it is the simplest example of (2.21) containing the Laplacian $\Delta = \partial_t\partial_a$. If we set $\gamma = 2 - z - N$, Eq. (3.1) takes the form

$$W^I = \frac{\partial}{\partial x_1}(u^{2-z-N}u_1) + \cdots + \frac{\partial}{\partial x_N}(u^{2-z-N}u_N)$$

(3.2)

which for $N = 2$ coincides with (1.9). In this section we shall be looking for exact solutions of this equation, under the condition $W^I_N = 0$.

The one-dimensional case $N = 1$ is trivial. The condition $W^I_N = 0$ reduces to $u_{xx} = 0$ which implies $u = p(t)x + q(t)$. Substituting this into (3.2) we arrive at the three different cases $z = 0$, $z = 1$ and $z \neq 0, 1$. The corresponding solutions are $u = cxe^t + q(t)$, $u = cx + q(t)$ and $u = q(t)$, respectively, where $q(t)$ remains arbitrary and $c$ is an arbitrary constant.

The first non-trivial case arises in two dimensions $N = 2$. The side condition (2.25) becomes the well-known two-dimensional Monge-Ampère equation. We want to use the conditional symmetry obtained in section 2 to find exact solutions to the pair (1.9, 1.10). According to Theorem 3 and using standard techniques, we can seek for exact solutions of (1.9) by solving the associated Lagrange system which reads

$$\frac{dt}{e_1z^{n+1}} = \frac{dx_1}{e_1(n+1)t^n x_1 + e_2^{m+1/z}} = \frac{dx_2}{e_1(n+1)t^n x_2 + e_3^{k+1/z}} = \frac{du}{e_1(n+1)t^nu}$$

(3.3)
where $e_1, e_2$ and $e_3$ are arbitrary parameters. In general, we might have used in finite series, viz. \( \sum_{i=n_1}^{n_2} e_{1i}(i+1)t^i, \sum_{i=m_1}^{m_2} e_{2i}t^{i+1/2} \) and \( \sum_{i=k_1}^{k_2} e_{3i}t^{i+1/2} \), respectively (where $n_1 < n_2 \in \mathbb{Z}$ and $m_1 < m_2, k_1 < k_2 \in \mathbb{Z} - 1/z$ and the $e_{ji}$ may be taken to be real). However, our aim is to show that even in the simplest case presented by (3.3) the conditional symmetry leads to non-trivial solutions. Indeed, setting $e_2 = e_3 = 0$ (the parameter $e_1 \neq 0$ since it corresponds to the conditional symmetry generator $X_n$) and solving the resulting linear system (3.3), we obtain the ansatz

\[
\begin{align*}
u &= t^{(n+1)/z} \varphi(\omega_1, \omega_2) \\
\omega_a &= x_a t^{-(n+1)/z} \quad ; \quad a = 1, 2
\end{align*}
\]

if $z \neq 0$ and

\[
u = x_1 \varphi(\omega, t) \quad , \quad \omega = \frac{x_1}{x_2}
\]

if $z = 0$.

Consider the case $z \neq 0$. Substituting the ansatz (3.4) into eqs. (3.9,10), we arrive at the system

\[
\begin{align*}
\varphi \left( \frac{\partial^2 \varphi}{\partial \omega_1^2} + \frac{\partial^2 \varphi}{\partial \omega_2^2} \right) &= z \left( \frac{\partial \varphi}{\partial \omega_1} \right)^2 + z \left( \frac{\partial \varphi}{\partial \omega_2} \right)^2 \\
\frac{\partial^2 \varphi}{\partial \omega_1^2} \frac{\partial^2 \varphi}{\partial \omega_2^2} &= \left( \frac{\partial^2 \varphi}{\partial \omega_1 \partial \omega_2} \right)^2
\end{align*}
\]

It turns out that the general solution of this system can be written down explicitly. Indeed, the local substitution

\[
\varphi = \begin{cases}
\exp \phi & ; \quad z = 1 \\
\phi^{1/(1-z)} & ; \quad z \neq 1
\end{cases}
\]

with $\phi = \phi(\omega_1, \omega_2)$ reduces the first equation of (3.6) to the Laplace equation $\Delta \phi = 0$ with the general solution

\[
\phi = f(\omega_1 + i\omega_2) + g(\omega_1 - i\omega_2)
\]

where $f$ and $g$ are arbitrary smooth functions and $i^2 = -1$. When reinserting this into the second equation (3.6), we obtain an equation which can be separated into two ordinary differential equations for the functions $f$ and $g$ which are elementarily integrable. We merely give the final result. For $z = 1$ we find

\[
u = \left( c_1(x_1 + ix_2) + d_1 t^{n+1} \right)^\alpha \left( c_2(x_1 - ix_2) + d_2 t^{n+1} \right)^{1-\alpha}
\]

where $\alpha$ is constant. For $z \neq 1$ the result is

\[
u = \left( c_1(x_1 + ix_2) + d_1 t^{n+1} \right)^{1-z} + \left( c_2(x_1 - ix_2) + d_2 t^{n+1} \right)^{1-z} \right)^{1/(1-z)}
\]

and where $c_{1,2}$ and $d_{1,2}$ are arbitrary integration constants.

In general, these solutions are complex. In order to make them real, we must have $c_1 = c_2 = c^2 > 0$, $d_1 = d_2^* = c^2(1 + ic_2)$ and $\alpha = 1/2$, where $c > 0$ and $e_{1,2} \in \mathbb{R}$. Under these restrictions, the above solutions become

\[
u = c \sqrt{(x_1 + e_1 t^{n+1})^2 + (x_2 + e_2 t^{n+1})^2}
\]
for $z = 1$ and
\[
\begin{align*}
    u &= c \left[ \left( x_1 + e_1 t^{(n+1)/z} \right) + i \left( x_2 + e_2 t^{(n+1)/z} \right) \right]^{1/z} + \left[ \left( x_1 + e_1 t^{(n+1)/z} \right) - i \left( x_2 + e_2 t^{(n+1)/z} \right) \right]^{1/z} \left( 1/(1-z) \right) \\
    &= 2c \sqrt{\frac{\left( x_1 + e_1 t^{(n+1)/z} \right)^2 + \left( x_2 + e_2 t^{(n+1)/z} \right)^2}{\cos \left[ (1-z) \arctan \frac{x_2 + e_2 t^{(n+1)/z}}{x_1 + e_1 t^{(n+1)/z}} \right]}} \left( 1/(1-z) \right)
\end{align*}
\] (3.12)
for $z \neq 1$, respectively. From the second line in Eq. (3.12) the reality of the solution is explicit and we also see that the solution (3.11) obtained for $z = 1$ can be recovered by taking the limit $z \to 1$ in (3.12). Therefore, Eq. (3.12) gives the general solution of the system (3.6) and we have obtained a four-parameter family of functions, depending on ‘time’ and on two ‘space’ coordinates, which simultaneously solve the generalized diffusion equation (1.9) and the Monge-Ampère equation (2.27). Of these four parameters, $c$ merely is a trivial normalization constant and $e_{1,2}$ control the scaling between ‘time’ and ‘space’ directions. The functional form of the solutions is controlled by the dynamical exponent $z$.

Finally, the integer parameter $n$ selects the rescaling between ‘time’ and ‘space’ coordinates.

Different exact solutions of (1.9) which are not a solution of a Monge-Ampère equation might be found by using only the Lie symmetries of the algebra $\mathfrak{B}_N(z)$ obtained in Theorem 2 but we shall not go into this here.

Consider now Eq. (1.9) for $z = 0$, when it becomes
\[
    \det \begin{bmatrix}
        u_t & u_1 & u_2 \\
        u_{t1} & u_{11} & u_{12} \\
        u_{t2} & u_{21} & u_{22}
    \end{bmatrix} = \Delta u, \quad u = u(t, x_1, x_2).
\] (3.13)

Substituting the ansatz (3.5) into Eq. (3.13), we arrive at the equation
\[
    -\omega^2 \frac{\partial \varphi}{\partial t} \left( \omega \frac{\partial^2 \varphi}{\partial \omega^2} + 2 \frac{\partial \varphi}{\partial \omega} \right) = (1 + \omega^2) \left( \omega \frac{\partial^2 \varphi}{\partial \omega^2} + 2 \frac{\partial \varphi}{\partial \omega} \right).
\] (3.14)

Simultaneously we substitute one into condition (2.27) and see that it simply leads to the identity. Since Eq. (3.14) decomposes into two independent equations
\[
    -\omega^2 \frac{\partial \varphi}{\partial t} = 1 + \omega^2, \quad \omega \frac{\partial^2 \varphi}{\partial \omega^2} + 2 \frac{\partial \varphi}{\partial \omega} = 0
\] (3.15)
we easily obtain its general solution. Inserting into (3.5), we arrive at two families of exact solutions of Eq. (3.13)
\[
\begin{align*}
    u &= \sqrt{2t(x_1^2 + x_2^2) + \psi(x_1/x_2)x_1^2} \\
    u &= x_1 \psi_1(t) + x_2 \psi_2(t)
\end{align*}
\] (3.16)-(3.17)
where $\psi, \psi_1, \psi_2$ are arbitrary smooth functions.

Finally, we point out how Remark 3 can be used for generalization of the exact solutions obtained. Operators (2.29) with $N = 2$ generate the invariance transformations
\[
    t' = t, \quad x'_1 = x_1 + e_1 \varphi_1(t), \quad x'_2 = x_2 + e_2 \varphi_2(t), \quad u' = u
\] (3.18)
where \( e_1 \) and \( e_2 \) are real parameters. Hence, we can transform any known solution \( u = u^0(t, x_1, x_2) \) of Eq. (1.9) into a new solution of the form \( u = u^0(t, x_1 + e_1\varphi_1(t), x_2 + e_2\varphi_2(t)) \) using formulæ (3.18). In the particular case, the solution (3.12) can be generalized to the form
\[
 u = 2c\sqrt{(x_1 + e_1\varphi_1(t))^2 + (x_2 + e_2\varphi_2(t))^2} \left( \cos \left[ (1 - z) \arctan \frac{x_2 + e_2\varphi_2(t)}{x_1 + e_1\varphi_1(t)} \right] \right)^{1/(1-z)} . \tag{3.19}
\]
By an appropriate choice of two arbitrary functions \( \varphi_1(t) \) and \( \varphi_2(t) \) in the solution (3.19), one can satisfy a wide range of boundary conditions that can be given for Eq. (1.9).

**Remark 4.** A long list of exact solutions of the Monge-Ampère equation is given in [12, Section 1.10]. However, the solutions (3.16) and (3.19) are not contained in this list and appear to be new. Considering the 'time' \( t \) only as a parameter and taking into account the space-translation invariance of the Monge-Ampère equation, these solutions can be united to the form \( u = x_1\varphi(x_1/x_2) \). Moreover, one can check that the obvious generalization of this, namely
\[
 u = x_1\varphi\left(\frac{x_1}{x_2}, \ldots, \frac{x_1}{x_N}\right) \tag{3.20}
\]
where \( \varphi \) is an arbitrary function of \( N - 1 \) arguments, is a new solution of the \( N \)-dimensional Monge-Ampère equation (2.25).

### 4 Conclusions

In this paper, we have given in Theorems 2 and 3 a complete description of first- and second-order PDEs of the form (2.1) which are (conditionally) invariant under the infinite-dimensional Lie algebras \( \mathfrak{A}_N(z) \) and \( \mathfrak{B}_N(z) \). While there is a wide class of PDEs (2.21) which are invariant under Lie algebras \( \mathfrak{B}_N(z) \), there is no equations with the \( \mathfrak{A}_N(z) \) Lie symmetry. However, we have shown that the same class (2.21) contains all possible PDEs which are conditionally invariant under the Lie algebra \( \mathfrak{A}_N(z) \) and that the well-known Monge-Ampère equation is the required additional condition. Both theorems can be extended on the case of the Lie algebras \( \mathfrak{A}_N(z), J_{ab} \) and \( \mathfrak{B}_N(z), J_{ab} \), where the operators \( J_{ab} \) are given in (1.3). We applied these general results to the non-linear diffusion equation with the generalized time-derivative (1.9). Using the conditional invariance, some families of exact solutions of this strongly non-linear equation have been found. Simultaneously, a new exact solution of the two-dimensional Monge-Ampère equation was obtained.

In a wider perspective, we note that the result obtained here cannot be directly extended on the case of the Lie algebras \( \mathfrak{A}_N(z) \) and \( \mathfrak{B}_N(z) \) with ‘mass’ operators because of the well-known fact that classes of PDEs invariant under the Galilei algebra with the ‘mass’ operator and the massless Galilei algebra have essentially different structures. We hope to be able to return to this open problem elsewhere.

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