A GEOMETRICAL APPROACH TO TIME-DEPENDENT GAUGE-FIXING

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ABSTRACT

When a Hamiltonian system is subject to constraints which depend explicitly on time, difficulties can arise in attempting to reduce the system to its physical phase space. Specifically, it is non-trivial to restrict the system in such a way that one can find a Hamiltonian time-evolution equation involving the Dirac bracket. Using a geometrical formulation, we derive an explicit condition which is both necessary and sufficient for this to be possible, and we give a formula defining the resulting Hamiltonian function. Some previous results are recovered as special cases.

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In systems such as string theory or general relativity for which arbitrary time reparametrizations are symmetries, any complete gauge fixing must involve the imposition of conditions which are explicitly dependent on time. (Here ‘time’ means the evolution parameter entering in a canonical formulation of the system.) It was pointed out in [1] that the extension of the usual techniques first developed by Dirac [2-5] to the case of general time-dependent gauges is far from straightforward. A careful analysis of the resulting problems was given and some partial solutions were offered. In this sequel we show that a geometrical approach yields a complete solution to the problem in a sense made precise below. We begin by summarizing the problem in conventional, non-geometrical terms.

Consider a dynamical system consisting of (i) a phase space $\Gamma$ which can be parametrized locally by coordinates \( \{ z^\mu \} \) where \( \mu = 1, \ldots, 2d \) and which is equipped with a Poisson bracket, to be denoted by square brackets; (ii) a Hamiltonian function $H$ (possibly time-dependent) on $\Gamma$; and (iii) a set of second-class constraints $\psi^i(z^\mu, t)$ where $i = 1, \ldots, 2n$ which define the physical phase space $\Gamma^* \subset \Gamma$ by means of the local equations $\psi^i = 0$. The dynamics is specified by Hamilton’s equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]$$  \hspace{1cm} (1)

for any function $f(z^\mu, t)$. For consistency, the Hamiltonian $H$ must be such that the constraints are preserved in time

$$\frac{d\psi^i}{dt} = \frac{\partial \psi^i}{\partial t} + [\psi^i, H] = 0 \quad \text{when} \quad \psi^j = 0.$$  \hspace{1cm} (2)

We shall consider throughout systems with purely second-class constraints so that the matrix $c$ with elements

$$c^{ij} = [\psi^i, \psi^j]$$  \hspace{1cm} (3)

is non-singular. The situation dealt with in [1] of a completely gauge-fixed system which initially possesses only time-independent, first-class constraints then arises as a special case.

By virtue of (2), any trajectory which begins in $\Gamma^*$ stays in $\Gamma^*$ for all time, so we can attempt to re-formulate the dynamics intrinsically on the physical phase space. The first step is to define the Dirac bracket of any pair of functions $f$ and $g$ on $\Gamma$ by

$$[f, g]^* = [f, g] - [f, \psi^i] (c^{-1})^i_j [\psi^j, g].$$  \hspace{1cm} (4)

This obeys $[f, \psi^i]^* = 0$ by construction and therefore induces a well-defined bracket on $\Gamma^*$ by restriction. Although strictly the bracket on $\Gamma$ and the induced bracket on $\Gamma^*$ are different entities, we shall also refer to the latter as the Dirac bracket and we shall then understand the definition (4) to be supplemented by the condition $\psi^i = 0$. Note that the time dependence inherent in the definition of $\Gamma^* \subset \Gamma$ does not influence the definition of the Dirac bracket in any way.
To proceed we must choose functions $\xi^a(z^\mu, t)$ with $a = 1, \ldots, 2d-2n$ which will provide us with local coordinates on $\Gamma^*$ and which we shall call physical variables following [1]. More precisely, these functions must define a smooth change of coordinates $\{z^\mu\} \leftrightarrow \{\xi^a, \psi^i\}$ on $\Gamma$ such that the quantities $\{\xi^a\}$ parametrize $\Gamma^*$ on setting $\psi^i = 0$. To complete the gauge-fixing procedure we would like to write down a Hamilton’s equation on $\Gamma^*$ using the new bracket (4). In other words, we would like to find a Hamiltonian $H^*$ (possibly time-dependent) on $\Gamma^*$ such that

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + [f, H]^*$$

for any function $f(\xi^a, t)$. Notice that here $\partial/\partial t$ is defined with our chosen coordinates $\{\xi^a\}$ on $\Gamma^*$ held fixed, whereas in (1) the original coordinates $\{z^\mu\}$ are held fixed. Now the time dependence in the definition of $\Gamma^* \subset \Gamma$ is crucial: since the change of coordinates $\{z^\mu\} \leftrightarrow \{\xi^a, \psi^i\}$ on $\Gamma$ involves time explicitly we must allow $H^*$ to differ from $H$ (or more accurately from the restriction of $H$ to $\Gamma^*$).

It turns out that it is not always possible to find such a Hamiltonian for a given choice of physical variables, although it was shown in [1] how solutions could be obtained under certain assumptions on the gauge-fixing conditions. In this paper we solve the general problem by deriving necessary and sufficient conditions on the choice of physical variables $\xi^a(z^\mu, t)$ for the existence of a Hamiltonian $H^*$, as well as giving a formula defining $H^*$. The key is the use of geometrical methods which, as anticipated in [1], allow a much clearer formulation of the problem. The results of [1] will be recovered as special cases.

We must stress the importance of an equation of type (5). At the purely classical level, a system whose time evolution cannot be described in this way falls outside the realm of conventional Hamiltonian mechanics. Such an equation is also crucial in passing to the corresponding quantum theory, where it becomes the Heisenberg equation of motion and where its structure guarantees the existence of a unitary time evolution operator. If, in the absence of such an equation, one tries to specify the quantum dynamics in some other fashion then great care must be taken to ensure consistency. Such an alternative scheme has been proposed by Gitman and Tyutin [5].

It is also shown in [5] that for any set of time-dependent constraints there exists some canonical transformation to new variables such that some subset of these is equivalent to the original set of constraints. In this sense one can in principle always remove any time dependence from the constraints, but in practice the required canonical transformation is usually very difficult to find. The question which we address and solve here is quite different and concerns the existence of a Hamilton’s equation (5) for a prescribed set of physical variables. This is partly motivated by the fact that one often has strong physical prejudices as to how the physical variables should be chosen.

The relativistic point particle provides perhaps the simplest example of time-dependent gauge fixing; it is treated in detail in [1,5,8]. Examples of time-dependent gauge fixing in string theory can be found in [6]. Discussions of this
issue in general relativity, and of the role of time in general, can be found in [7] and references therein.

We now begin to translate our problem into the language of symplectic geometry, building up the necessary vocabulary in stages. For a dynamical system without constraints [9] the phase space $\Gamma$ is taken to be a symplectic manifold with coordinates $\{z^\mu\}$. This means that $\Gamma$ comes equipped with a symplectic, or non-degenerate and closed, two-form $\omega$. Let $\omega^{-1}$ be the corresponding antisymmetric contravariant tensor so that the components of these tensors are mutually inverse antisymmetric matrices. The Poisson bracket of two functions on $\Gamma$ is defined by

$$[f,g] = -\omega^{-1}(df, dg) = -(\omega^{-1})^{\mu\nu} \frac{\partial f}{\partial z^\mu} \frac{\partial g}{\partial z^\nu} \quad \text{where} \quad \omega = \frac{1}{2} \omega_{\mu\nu} d z^\mu \wedge d z^\nu. \quad (6)$$

This bracket is clearly antisymmetric and also satisfies the Jacobi identity because $\omega$ is closed. Darboux’s Theorem states that locally on $\Gamma$ there exist coordinates $\{\bar{z}^\mu\} = \{q^m, p_m\}$ with $\omega = dq^m \wedge dp_m$ and the expression in (6) then takes on the form familiar from non-geometrical treatments.

The time evolution of the system can be concisely specified as follows. Consider some trajectory on $\Gamma$ which is parametrized by time $t$ and which has tangent vector $v$ so that in components we can write

$$z^\mu = \gamma^\mu(t), \quad v^\mu = d\gamma^\mu / dt, \quad (7)$$

say. This trajectory is a solution of Hamilton’s equation (1) precisely when

$$i(v) \omega = dH. \quad (8)$$

Here $i(v)$ denotes interior multiplication of a form by the vector field $v$ so that in components $(i(v) \omega)_\nu = v^\mu \omega_{\mu\nu}$. To prove that (8) is equivalent to (1), note that along such a trajectory $df / dt - \partial f / \partial t = i(v) df = -\omega^{-1}(df, i(v) \omega) = -\omega^{-1}(df, dH) = [f, H]$ where the first and second equalities are identities, the third follows from (8) and the last follows from the definition (6).

A related approach, which will prove more useful for our purposes, involves the introduction of extended phase space. This is a manifold $\bar{\Gamma} = \Gamma \times \mathbb{R}$ with coordinates $\{\bar{z}^\mu\} = \{z^\mu, z^0 = t\}$, where $\{z^\mu\}$ are coordinates on $\Gamma$ and $t \in \mathbb{R}$ is time. The Poincaré-Cartan two-form on $\bar{\Gamma}$ is defined by

$$\Omega = \omega + dH \wedge dt, \quad (9)$$

where $\omega$ is now to be understood as a form on $\bar{\Gamma}$ in the obvious way. Using $\Omega$, Hamilton’s equation can be written even more compactly. Consider a trajectory on $\bar{\Gamma}$ which is a function of some auxiliary parameter $s$ and which has tangent vector $V$ so that

$$z^\mu = \bar{\gamma}^\mu(s), \quad V^\mu = d\bar{\gamma}^\mu / ds, \quad (10)$$

say.
say. Any trajectory on \( \Gamma \) parametrized by \( t \) is clearly equivalent to such a trajectory on \( \bar{\Gamma} \) and Hamilton’s equation (1) holds if and only if
\[
i(V) \Omega = 0 .
\]
(11)
This follows from (8) by observing that when the tangent vector \( v \) of a trajectory on \( \Gamma \) parametrized by \( t \) is regarded as a vector field on \( \bar{\Gamma} \) in the obvious way, then \( V = (dt/ds)(v + \partial/\partial t) \).

We now progress to the geometrical description of a dynamical system with \textit{time-independent constraints}. We assume that the local constraint equations \( \psi^i = 0 \) define a submanifold \( X \subset \Gamma \). It is useful to define the \textit{physical phase space} \( \Gamma^* \) to be some abstract copy of this subset, rather than defining it to be the subset itself as was done in our introductory remarks. These spaces can be identified by an embedding map
\[
\varphi : \Gamma^* \to \Gamma
\]
(12)
which is a diffeomorphism onto its image \( X \). Let \( \omega^* \) be the pull back to \( \Gamma^* \) of the symplectic form \( \omega \) on \( \Gamma \) via this embedding so that with a choice of coordinates \( \{z^\mu\} \) on \( \Gamma \) and \( \{\xi^a\} \) on \( \Gamma^* \) we have
\[
\omega^* = \frac{1}{2} \omega^*_{ab} \, d\xi^a \wedge d\xi^b \quad \text{where} \quad \omega^*_{ab} = \partial z^\mu \frac{\partial z^\nu}{\partial \xi^a} \partial z^\nu \frac{\partial}{\partial \xi^b} \omega_{\mu\nu} .
\]
(13)
Since the exterior derivative commutes with the pull back, \( \omega^* \) is closed. One can also show that \( \omega^* \) is nondegenerate if and only if \( c^{ij} = [\psi^i, \psi^j] \) is nondegenerate. We assume this to be the case and then \( \omega^* \) makes \( \Gamma^* \) a symplectic manifold with an associated bracket defined just as in (6). This is the geometrical definition of the Dirac bracket on \( \Gamma^* \) and one can show that it is equivalent to the formula (4). Details are given in the appendix.

Consider a trajectory on \( \Gamma \) with tangent vector \( v \) as in (7). Suppose now that this trajectory corresponds to (i.e. is the image under \( \varphi \) of) a trajectory in \( \Gamma^* \) which has tangent vector \( v^* \), so that we can also write
\[
\xi^a = \gamma^* a(t) , \quad v^* a = d\gamma^* a / dt ,
\]
(14)
say. Applying the chain rule and using the fact that there is, by definition, no time dependence in the relationship between \( \{z^\mu\} \) and \( \{\xi^a\} \) we have
\[
v^\mu = \partial z^\mu \frac{\partial}{\partial \xi^a} v^* a .
\]
(15)
It follows from (13) and (15) that the equation of motion (8) is equivalent to
\[
i(v^*) \omega^* = dH
\]
(16)
\[\dagger\] Given a map between manifolds, we shall frequently regard coordinates or functions on its image as depending on coordinates on its domain, as in (13).
on the physical phase space $\Gamma^*$. This says precisely that (5) holds with $H^* = H$.

Similar considerations apply to the extended phase space formalism when the constraints are time-independent. We define the extended physical phase space to be $\tilde{\Gamma}^* = \Gamma^* \times \mathbb{R}$ with coordinates $\{\xi^a\} = \{\xi^a, \xi^0 = t\}$ where $\{\xi^a\}$ are coordinates on $\Gamma^*$ and $t \in \mathbb{R}$ is time. Given $\varphi$ in (12) there is a natural associated embedding

$$\tilde{\varphi} : \tilde{\Gamma}^* \to \tilde{\Gamma}, \quad \tilde{\varphi}(x, t) = (\varphi(x), t)$$

which is a diffeomorphism onto its image $X \times \mathbb{R}$. The Poincaré-Cartan two-form $\Omega$ on $\tilde{\Gamma}$ pulls back under $\tilde{\varphi}$ to a form $\Omega^*$ on $\tilde{\Gamma}^*$ given by

$$\Omega^* = \omega^* + dH \wedge dt,$$

where $\omega^*$, given by (13), is now to be interpreted as a form on $\tilde{\Gamma}^*$. It is easy to show that for a trajectory lying in $\tilde{\Gamma}^*$ with tangent vector $V^*$ the equation of motion (11) is equivalent to

$$i(V^*) \Omega^* = 0.$$  

(19)

The fact that $\omega^*$ has no $dt$ component again implies (by comparison of (13), (18), (19) with (6), (9), (11)) that (5) holds with $H^* = H$.

Finally we can analyze the case of interest in which the system has explicitly time-dependent constraints. To begin we must discuss carefully the various spaces which arise in the problem. At each fixed $t$ we have an instantaneous physical phase space $X_t \subset \Gamma$ defined by the constraints. We assume that the constraints are such that the $X_t$ are all diffeomorphic to one another and that the collection of all these instantaneous physical phase spaces is a submanifold $\tilde{X} = \{(x, t) : x \in X_t\} \subset \tilde{\Gamma}$. As before it is convenient to introduce a standard copy $\Gamma^*$ of each $X_t$ which we call simply the physical phase space. With the assumptions above, we can find a family of embeddings

$$\varphi_t : \Gamma^* \to \Gamma$$

parametrized smoothly by time, such that each $\varphi_t$ is a diffeomorphism from $\Gamma^*$ onto its image $X_t$. We also define $\tilde{\Gamma}^* = \Gamma^* \times \mathbb{R}$ to be the extended physical phase space and we notice that the smooth family of embeddings $\varphi_t$ is equivalent to a single embedding

$$\tilde{\varphi} : \tilde{\Gamma}^* \to \tilde{\Gamma}, \quad \tilde{\varphi}(x, t) = (\varphi_t(x), t)$$

(21)

which acts as the identity on the time factor and which is a diffeomorphism onto its image $\tilde{X}$. There exist many inequivalent ways to choose the related embeddings $\varphi_t$ and $\tilde{\varphi}$, and the important point is that each of these constitutes a distinct way of identifying $\tilde{X}$ as a product manifold $\Gamma^* \times \mathbb{R}$. The situation is summarized in the figure.

We can now give a more precise description of the physical variables which will clarify their significance. Consider a fixed set of coordinates $\{\xi^a\}$ on $\Gamma^*$. For some choice of the embedding $\tilde{\varphi}$ (or the family $\varphi_t$) we can use the property that
this map is a diffeomorphism onto its image $\bar{X}$ to express the coordinates on the physical phase space as functions on $\bar{X}$. These functions, suitably extended to some patch in $\bar{\Gamma}$, are precisely the physical variables $\xi^a(z^\mu, t)$. A choice of physical variables therefore corresponds to a specific choice of the embedding $\bar{\varphi}$ and hence to a specific way of diffeomorphically identifying $\bar{X}$ as a product manifold $\Gamma^* \times \mathbb{R}$. Only the values of the physical variables on $\bar{X}$ are important in this respect, since all suitably smooth extensions to $\bar{\Gamma}$ clearly define the same identification.

In order to identify $\bar{X}$ as a product manifold globally, and not just locally, the physical variables $\xi^a(z^\mu, t)$ chosen in each coordinate patch must be related by time-independent transformations of the coordinates $\{\xi^a\}$ where these patches overlap. A related point is that the Hamiltonian $H^*$ in (5) is clearly insensitive to such transformations of the $\{\xi^a\}$, but that the Hamiltonian might cease to exist or at least would require modification in general after a time-dependent transformation of the set $\{\xi^a\}$. We shall elaborate on these issues later. The above discussion should also be contrasted with the case of time-independent constraints in which there is a preferred way to identify $\bar{X}$ as a product manifold because $X_t = X$ is actually constant in time. This corresponds to the existence of preferred embeddings $\varphi$ and $\bar{\varphi}$ in (12) and (17) for which the associated physical variables can be chosen as time-independent functions $\xi^a(z^\mu)$.

We noted earlier that the time dependence of the constraints has essentially no effect on the way the Dirac bracket is constructed. In geometrical terms, although $\varphi_t$ depends on time we can use it at each fixed $t$ to pull back the symplectic form $\omega$ on $\Gamma$ to a symplectic form $\omega^*$ on $\Gamma^*$, which defines a bracket by an equation exactly like (6). The formulas (13) for the components of $\omega^*$ and (4) for the Dirac bracket are clearly unaltered. What is new, however, is that both $\omega^*$ and the Dirac bracket can now change with time so that the physical phase space has a time-varying symplectic structure. We shall return to this issue below.

We are now faced with the central question of how to describe the dynamics. The important point is that the equation of motion (8) no longer leads to an equation of the same form (16) on the physical phase space as it did in the time-independent case. One way to understand this is to consider a trajectory in $\Gamma^*$ and to use the coordinate expressions given in (7) and (14). The chain rule tells us that the components of the tangent vectors are now related by

$$v^\mu = \frac{\partial z^\mu}{\partial \xi^a} v^* a + \frac{\partial z^\mu}{\partial t},$$

instead of by (15), so that $v$ is no longer tangent to $X_t$ in general. The new inhomogeneous term complicates the structure of the resulting equation on $\Gamma^*$ and from this point of view it is unclear how best to proceed.

Using the formalism of extended phase space, however, the situation is simpler in one important respect. Because time is reduced to the status of a coordinate on $\bar{\Gamma}$, with trajectories being parametrized by an auxiliary quantity $s$, the time dependence of the embedding $\bar{\varphi}$ does not influence the reduction of the dynamical
equations to \( \bar{\Gamma}^* \). We can therefore immediately write
\[
i(V^*)\Omega^* = 0 \tag{23}
\]
as the correct equation of motion on \( \bar{\Gamma}^* \), where as before \( \Omega^* \) is the pull-back of \( \Omega \) under \( \bar{\varphi} \). But our problem now manifests itself in the fact that \( \Omega^* \) no longer has the structure exhibited in (18) in general. We find instead
\[
\Omega^* = \omega^* + dH \wedge dt + A \wedge dt \tag{24}
\]
where \( \omega^* \) is given by (13) and where
\[
A = -\frac{\partial z^\mu}{\partial t} \frac{\partial z^\nu}{\partial \xi^a} \omega_{\mu\nu} d\xi^a. \tag{25}
\]
Note that \( \omega^* \) is the pull-back to \( \Gamma^* \) of \( \omega \) on \( \Gamma \) using \( \varphi_t \) whereas \( \omega^* + A \wedge dt \) is the pull back to \( \Gamma^* \) of \( \omega \) on \( \Gamma \) using \( \bar{\varphi} \). The decomposition of this last pulled-back form into two terms \( \omega^* \) and \( A \wedge dt \) is clearly independent of the choice of the coordinates \( \{\xi^a\} \) on \( \Gamma^* \), depending only on the chosen embeddings \( \varphi_t \) and \( \bar{\varphi} \). The decomposition would change under a general time-dependent coordinate transformation on \( \bar{\Gamma}^* \) however.

Once the problem is presented in this way, the solution is straightforward. To have a Hamilton’s equation (5) it is necessary and sufficient that
\[
\Omega^* = \omega^* + dH^* \wedge dt \tag{26}
\]
by comparison with (18). But from (24) this is true precisely when
\[
A = dK \mod dt \quad \Rightarrow \quad H^* = H + K \tag{27}
\]
for some function \( K \). Here \( \mod dt \) means that equality holds up to terms proportional to \( dt \). It is not difficult to prove that the Poincaré Lemma [9] holds \( \mod dt \), so that a form which contains no \( dt \) terms (such as \( A \)) is closed \( \mod dt \) if and only if it is locally exact \( \mod dt \). One therefore has the following concrete condition
\[
dA = 0 \mod dt \quad \iff \quad \frac{\partial}{\partial \xi^b} \left\{ \frac{\partial z^\nu}{\partial \xi^a} \frac{\partial z^\mu}{\partial t} \omega_{\mu\nu} \right\} = 0 \tag{28}
\]
for the existence locally of a Hamiltonian \( H^* \), and an explicit expression for the Hamiltonian is then derivable from (27). There is also a potential global obstruction if \( A \) defines a non-trivial cohomology class in \( H^1(\Gamma^*) \). There are a number of simple situations where this can be ruled out – if \( \Gamma^* \) is simply connected for example – but a more detailed discussion is beyond the scope of the present paper.

Equation (27) has one more important consequence. We emphasized above that the time dependence of the constraints implies that the symplectic form \( \omega^* \)
and hence the Dirac bracket on $\Gamma^*$ will also be explicitly time-dependent in general. There seem to be no obvious grounds for objecting to this, and it might appear that we are forced to accept this new feature despite the fact that it runs counter to our experience with conventional classical mechanics. It turns out, however, that if (27) holds then $\omega^*$ is automatically time-independent. This can be deduced straightforwardly by working in components, comparing the time derivative of $\omega^*$ with the condition (28) above and using the fact that $\omega^*$ is closed on $\Gamma^*$. Thus (28) is actually a necessary and sufficient condition both for the existence of a Hamilton’s equation (5) and also for the time independence of the Dirac bracket occurring in this equation.

The problem we posed is now solved, but it is instructive nevertheless to take the analysis one step further and to frame things more systematically. As we have formulated it here, the problem of specifying the dynamics of a system in conjunction with time-dependent gauge fixing is the problem of how the structure of the Poincaré-Cartan two-form is affected by pulling back under the embedding

$$\bar{\phi} : \Gamma^* \times \mathbb{R} \to \Gamma \times \mathbb{R}.$$  \hspace{1cm} (29)

This embedding specifies a particular way of identifying the submanifold $\bar{X} \subset \Gamma \times \mathbb{R}$ with the product $\Gamma^* \times \mathbb{R}$ and it is defined in local coordinates by a particular choice of physical variables.

Whenever a manifold has a product structure $M = M' \times M''$ there is an induced decomposition of the space of differential forms of a given degree $\Lambda^p(M) = \bigoplus_{r+s=p} \Lambda^{(r,s)}$ where $\Lambda^{(r,s)} = \Lambda^r(M') \otimes \Lambda^s(M'')$, and an associated decomposition of the exterior derivative $d = d' + d''$ where $d' : \Lambda^{(r,s)} \to \Lambda^{(r+1,s)}$ and $d'' : \Lambda^{(r,s)} \to \Lambda^{(r,s+1)}$. For each of the extended phase spaces $\bar{\Gamma}$ and $\bar{\Gamma}^*$ the time factor is one-dimensional and so the decomposition of a general $p$-form $\alpha$ involves just two terms; let us write $\alpha = \alpha' + \alpha''$ where $\alpha' \in \Lambda^{(p,0)}$ and $\alpha'' \in \Lambda^{(p-1,1)}$. Thus for the Poincaré-Cartan form on $\bar{\Gamma}$ we have

$$\Omega' = \omega , \quad \Omega'' = dH \wedge dt,$$  \hspace{1cm} (30)

while for its pull-back to $\bar{\Gamma}^*$ we have

$$\Omega'^* = \omega^* , \quad \Omega''^* = (A + dH) \wedge dt.$$  \hspace{1cm} (31)

The key property of $\Omega$ which ensures that (11) leads to Hamilton’s equation (1) is that it has the structure exhibited in (30) above with $d'\omega = 0$. This property can be exactly characterized locally by saying that both $\Omega'$ and $\Omega''$ are separately closed with respect to $d'$ on $\bar{\Gamma}$ (using the generalized Poincaré Lemma mentioned above). Now the criterion for the existence of a Hamilton’s equation (5) on physical phase space can be similarly expressed as the condition that both $\Omega'^*$ and $\Omega''^*$ are separately closed with respect to $d'$ on $\bar{\Gamma}^*$:

\begin{align}
d'\Omega'^* &= 0 \quad \text{(32a)} \\
d'\Omega''^* &= 0. \quad \text{(32b)}
\end{align}
But since \( \omega \) in (9) is independent of time, we know also that \( d\Omega = 0 \) implying
\[
d'\Omega^* = 0 \quad \text{(33a)}
\]
\[
d''\Omega^* + d'\Omega'' = 0 . \quad \text{(33b)}
\]

Because these last equations hold automatically, only (32b) actually has any content and moreover it is equivalent to the condition
\[
d''\Omega^* = 0 . \quad \text{(34)}
\]

To relate this to our previous work, it is convenient to employ an abuse of notation and to write the exterior derivatives on \( \bar{\Gamma}^* \) as \( d' = d\xi^a(\partial/\partial\xi^a) \) and \( d'' = dt(\partial/\partial t) \). This allows us to read off, using (31), the content of equations (32-34). Clearly (32a) says just that \( \omega^* \) is closed on \( \bar{\Gamma}^* \) and (32b) yields the condition (28) for \( \bar{A} \) found earlier. But we have also found that (32b) is equivalent to (34) which says precisely that \( \omega^* \) is independent of time, as discussed above.

We stated earlier that the physical variables \( \xi^a(z^\mu,t) \) defined in each coordinate patch should be related on overlaps by time-independent transformations of the set \( \{\xi^a\} \) if we wish to identify \( \bar{X} \) as a product manifold globally. But the subsequent analysis and the criterion (27) we derived for the existence of a Hamiltonian’s equation were purely local. We are therefore free to adopt a more flexible approach and to look for solutions to (27) independently in each coordinate patch. Under these circumstances, the identification of \( \bar{X} \) as a product will hold only locally, being defined by the particular choice of \( \xi^a(z^\mu,t) \) specific to each patch. This choice will also define, via \( d\xi^a \) and \( dt \), local decompositions of the spaces of differential forms on each patch, as described above. The \( \xi^a(z^\mu,t) \) will be related by time-dependent transformations of the set \( \{\xi^a\} \) on overlaps and the associated symplectic forms \( \omega^* \) and Hamiltonians \( H^* \) will differ in these regions. See also the remarks following (25).

It remains for us to demonstrate how the results of [1] can be recovered within the present geometrical framework. This will also serve to illustrate how the techniques we have developed work in practice.

We first treat case (B) of [1]. Suppose that on \( \Gamma \) we have coordinates \( z^\mu = \{q^m, p_m\} \) in which \( \omega = dq^m \wedge dp_m \). Let \( \{Q^I, P_I\} \) with \( I = 1, \ldots, n \) and \( \{q^A, p_A\} \) with \( A = 1, \ldots, d-n \) be disjoint subsets of these coordinates and suppose that the constraints have the form \( \{\psi^I\} = \{\chi^I, \phi^I\} \) where
\[
\chi^I = Q^I - \zeta^I(q^A, t) \quad \text{(35)}
\]
for certain functions \( \zeta^I \). Result (B) of [1] states that for physical variables \( \{\xi^a\} = \{q^*^A, p^*_A\} \) there exists a Hamiltonian \( H^* \) where
\[
q^{*A} = q^A , \quad p^{*_A} = p_A + \frac{\partial \zeta^I}{\partial q^A} P_I , \quad H^* = H - \frac{\partial \zeta^I}{\partial t} P_I \quad \text{(36)}
\]
and this last expression is to be thought of as a function of the physical variables.

To prove this we shall calculate the one-form $A$. This illustrates one general practical approach to dealing with the physical variables: we can take the explicit expressions $\xi^a(z^\mu, t)$ and the equations $\psi^i(z^\mu, t) = 0$ and solve them to express $z^\mu(\xi^a, t)$, thus reducing to the physical phase space. For the case at hand we have

$$Q^I = \zeta^I(q^*A, t), \quad P_I = \eta_I(q^*A, p^*_A, t)$$

$$q^A = q^{*A}, \quad p_A = p^*_A - \frac{\partial \zeta^I}{\partial q^{*a}} \eta_l$$

where the functions $\eta_l$ depend in detail upon the entire set of constraints. Now because $(\partial q^A/\partial t)_{\xi^a} = 0$ we have

$$A = \frac{\partial p_A}{\partial t} \frac{\partial q^A}{\partial \xi^a} d\xi^a - \frac{\partial \zeta^I}{\partial t} \frac{\partial \eta_I}{\partial \xi^a} d\xi^a + \frac{\partial \eta_I}{\partial t} \frac{\partial \zeta^I}{\partial \xi^a} d\xi^a.$$ (37)

By making further use of (37), and in particular the fact that $(\partial \zeta^I/\partial p^*_A)_{q^*A, t} = 0$, the first term in this expression for $A$ can be written

$$\frac{\partial p_A}{\partial t} \frac{\partial q^A}{\partial \xi^a} d\xi^a = - \frac{\partial}{\partial t} \left\{ \frac{\partial \zeta^I}{\partial q^{*a}} \eta_I \right\} dq^*_A = - \frac{\partial}{\partial t} \left\{ \frac{\partial \zeta^I}{\partial \xi^a} \eta_I \right\} d\xi^a.$$ (38)

Combining these expressions gives

$$A = - \frac{\partial}{\partial \xi^a} \left\{ \frac{\partial \zeta^I}{\partial t} \eta_I \right\} d\xi^a = - d \left\{ \frac{\partial \zeta^I}{\partial t} \eta_I \right\} \mod dt$$ (40)

and so, as claimed, the chosen physical variables admit a Hamiltonian

$$H^* = H - \frac{\partial \zeta^I}{\partial t} \eta_I.$$ (41)

The other case to be discussed is result (A) of [1]. We assume the constraints take the form $\{\psi^i\} = \{\chi^I, \phi^J\}$ where

$$(\partial \phi^J/\partial t)_{z^\mu} = 0, \quad [\phi^I, \phi^J] = 0 \quad \text{when} \quad \phi^K = 0.$$ (42)

The physical variables $\xi^a(z^\mu, t)$ are defined by the requirements that they are time-independent and gauge-invariant

$$(\partial \xi^a/\partial t)_{z^\mu} = 0, \quad [\phi^I, \xi^a] = 0 \quad \text{when} \quad \phi^J = 0.$$ (43)

Result (A) of [1] then states that (5) holds with $H^* = H$.

To prove this we use another approach of general applicability, namely we consider a coordinate transformation $\{z^\mu, t\} \rightarrow \{\xi^a, \psi^i, t\}$ on $\Gamma$, view (13), (25).
and (28) as defined on $\bar{\Gamma}$, then reduce to the physical phase space by setting $\psi^i = 0$ (and then (28) has to hold only after so doing). In the present case it is convenient to introduce as a shorthand $\{\hat{z}^\mu\} = \{\xi^a, \chi^I, \phi^I\}$ and to let $\hat{\omega}_{\mu\nu}$ denote the components of the symplectic form in this coordinate system. Equation (6) tells us that $[\hat{z}^\mu, \hat{z}^\nu] = -\hat{\omega}^{-1}_{\mu\nu}$ and therefore conditions (42) and (43) above imply that when $\psi^i = 0$ we have the block forms

$$\hat{\omega}^{-1} = \begin{pmatrix} a & x & 0 \\ -x^T & b & y \\ 0 & -y^T & 0 \end{pmatrix} \Rightarrow \hat{\omega} = \begin{pmatrix} \alpha & 0 & -\lambda^T \\ 0 & 0 & -\mu^T \\ \lambda & \mu & \beta \end{pmatrix}. \quad (44)$$

Here $a$ and $b$ are antisymmetric, $a$ and $y$ are invertible, with

$$\alpha = a^{-1}, \quad \mu = y^{-1}, \quad \lambda = y^{-1}x^Ta^{-1}, \quad \beta = y^{-1}(b + x^Ta^{-1}x)(y^T)^{-1}, \quad (45)$$

although actually only the block structure of these matrices is important for our purposes. Now when we regard the coordinates $\hat{z}^\mu$ as functions of the original coordinates $\{z^\mu, t\}$ on $\bar{\Gamma}$, only the quantities $\chi^I$ depend on time explicitly. Consequently we have the identity

$$\left(\frac{\partial z^\mu}{\partial t}\right)_{\hat{z}^\mu} + \left(\frac{\partial \chi^I}{\partial t}\right)_{z^\mu} \left(\frac{\partial z^\mu}{\partial \chi^I}\right)_{\xi^a, \phi^I, t} = 0 \quad (46)$$

and on substituting this in (25) we find

$$A = \frac{\partial \chi^I}{\partial t} \left(\frac{\partial z^\mu}{\partial \chi^I} \frac{\partial z^\nu}{\partial \xi^a} \omega_{\mu\nu}\right) d\xi^a. \quad (47)$$

But the factor in curly brackets is one of the block entries in $\hat{\omega}$ which vanishes when $\psi^i = 0$ according to (44). Hence $A = 0$ on $\bar{\Gamma}^*$ and the result is established.

In conclusion: we have derived an explicit condition (28) on a set of physical variables $\xi^a(z^\mu, t)$ which is necessary and sufficient both for the existence (locally) of a Hamiltonian $H^*$ in (5) – which is then determined by (27) – and for the constancy in time of the Dirac bracket (4) on the physical phase space. It follows from the work of Gitman and Tyutin [5] that for any set of constraints there exists some set of physical variables which satisfy these conditions. The proof is of no practical help in finding these, however, hence the utility of our result. Note also that in the special case where there are no constraints, our analysis determines when there exists a Hamilton’s equation for some new set of coordinates $\{\xi^\mu, t\}$ on $\bar{\Gamma}$ which are related by a time-dependent transformation to the original set $\{z^\mu, t\}$.

The geometrical techniques used here have proved far more efficient than the original approach of [1]. There the emphasis was also placed on finding an expression on the full phase space $\Gamma$ which gave the desired equation on restriction, rather than on working directly on the physical phase space $\Gamma^*$. Such an approach
can also be expressed in geometrical terms although we have not pursued the
details here. In the future it would be interesting to apply our results to specific
elements such as those in [6,7]. It would also be very interesting to study the
extension of this work to the quantum case and particularly its relationship to
gometric quantization.

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APPENDIX: THE DIRAC BRACKET

Here we show how the geometrical definition (6) is related to formula (4). We
can ignore all questions of time dependence and work throughout at some fixed
instant. Take coordinates \( \{ \hat{z}^\mu \} = \{ \xi^a, \psi^i \} \) on \( \Gamma \) and let the components of \( \omega \) and \( \omega^{-1} \) have the corresponding block forms

\[
-\hat{\omega} = \begin{pmatrix} \alpha & -\beta^T \\ \beta & \gamma \end{pmatrix}, \quad -\hat{\omega}^{-1} = \begin{pmatrix} a & b \\ -b^T & c \end{pmatrix}
\]

(A.1)

so that

\[
a\alpha + b\beta = 1, \quad -b^T \alpha + c\beta = 0, \quad c\gamma + b^T \beta^T = 1.
\]

(A.2)

If \( c \) is invertible the first two equations imply that \( \alpha \) is invertible with

\[
\alpha^{-1} = a + bc^{-1}b^T,
\]

(A.3)

whilst if \( \alpha \) is invertible the last two equations imply that \( c \) is invertible with

\[
c^{-1} = \gamma + \beta\alpha^{-1}\beta^T.
\]

(A.4)

Now from (6) we have \(- (\hat{\omega}^{-1})^{\mu\nu} = [\hat{z}^\mu, \hat{z}^\nu] \) or in detail

\[
a^{ab} = [\xi^a, \xi^b], \quad b^{ai} = [\xi^a, \psi^i], \quad c^{ij} = [\psi^i, \psi^j].
\]

(A.5)

Also, taking \( \{ \xi^a \} \) as coordinates on \( \Gamma^* \), the components of \( \omega^* \) are

\[
\omega_{ab}^* = -\alpha_{ab}.
\]

(A.6)

The remarks above therefore prove that \( \omega^* \) is non-degenerate if and only if \([\psi^i, \psi^j]\)
is non-degenerate, as stated in the text. Furthermore, the bracket on \( \Gamma \) defined by

\[
[f, g]^* = \frac{\partial f}{\partial \xi^a} \frac{\partial g}{\partial \xi^b} (\alpha^{-1})^{ab}
\]

(A.7)

clearly restricts to the bracket on \( \Gamma^* \) defined by (6) by virtue of (A.6). But (A.3)
and (A.5) allow us to express the right-hand side in terms of Poisson brackets:

\[
[f, g]^* = \frac{\partial f}{\partial \xi^a} \frac{\partial g}{\partial \xi^b} \left( [\xi^a, \xi^b] + [\xi^a, \psi^i] (c^{-1})_{ij} [\xi^b, \psi^j] \right)
\]

\[
= [f, g] - [f, \psi^i] (c^{-1})_{ij} [\psi^j, g]
\]

(A.8)

which reproduces (4). The equivalence of the definitions is therefore established.
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Extended phase space is $\tilde{\Gamma} = \Gamma \times \mathbb{R}$. Extended physical phase space is $\tilde{\Gamma}^* = \Gamma^* \times \mathbb{R}$. $\varphi$ is an embedding from $\tilde{\Gamma}^*$ to $\tilde{\Gamma}$ with image $\tilde{X}$. It is equivalent to the family of embedding maps $\varphi_t$ from $\Gamma^*$ to $\Gamma$ with images $X_t$. 