Axion, photon-pair mixing in models of axion dark matter

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A system of light axions comprising a classical axion field, one candidate for dark matter, has an instability that would rapidly mix in photon pairs in a coherent fashion if the system were initially seeded by some tiny amount of such mixing. Analogy to other systems discussed in the literature suggests that in the case of the pure axion initial state, quantum effects will enable large mixing after a waiting time of the order of $((\text{the inverse of the classical growth rate}) \times \log N)$, where $N$ is the number of axions participating. We give the theory of the above for the axion case in very elementary terms, including the back reactions on the axions. But when multiple modes for the photons are taken into account we find that $N$, in the logarithm, gets replaced by a number that is many orders of magnitudes smaller. We also find that for extraordinary high axion densities, as may exist in axion stars, the usual limit to the free electron density, [axion mass < plasma frequency], is replaced by a more lenient limit.

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Cosmological models in which the dark matter is composed of light axions, in an essentially classical condensed state, have attracted attention recently [1] - [6]. Here we shall look again at the time evolution due to electromagnetic interactions of a piece of this matter, consisting of $N_a$ axions contained within a periodic box of volume, $V$, and over a time interval somewhat less than the light travel time over the box. We assume a standard interaction, $\mathcal{L}_I = g_a \mathbf{a} \cdot \mathbf{B}$, where $a$ is the axion field, and use coupling and mass parameters consistent with the literature, $g_a = 10^{-21} - 10^{-22} \text{eV}^{-1}$ and $10^{-22} \text{eV} < m_a < 10^{-4} \text{eV}$.

From early in the development of this subject, it has been known [3] - [6] that in some regions there can be an instability that could lead to exponential increase, with a term growing as $\exp[r_g t]$, where $r_g \approx g_a (\rho m_a)^{1/2}$ and $\rho = [\text{energy density}]$. In later literature [10] - [12] possible consequences of this instability have been explored but its seeding has remained obscure. This seeding is the focus of the present work, which will end with a picture not of a single growing classical mode but of a quantum superposition of such modes.

Our results apply only over time intervals in which the red shift is essentially constant, and in domains in which the electron density is sufficiently small. Generally we must demand $\omega_p < m_a$ [8], although we shall find that in cases of extraordinarily large axion density this condition is relaxed. The above references refer to the generating mechanism as a kind of “parametric amplification”, but our non-linear system is different in important ways from the linear ones that are exhaustively discussed in the quantum optics literature.

However beginning with a pure coherent axion state there is no exponentially increasing photon number, in our results. Instead there is a gestation time of order $r_g^{-1} \log(\rho m_a a)$ during which little happens that is apparent, followed by a sudden near-complete, somewhat transitory, transformation of axions into photons. “Sudden” here, means: on a time scale $r_g^{-1} \log N$ much smaller than the logarithmic gestation time. This type of behavior can be designated a “quantum break”, a term that has gained currency in describing a genre of actual and conjectured phenomena in several areas: in condensed matter literature describing, e.g. Bose condensates of atoms [13] - [15]; in polarization exchange processes in colliding photon beams [16] - [18]; in cosmology [10] - [21]. Finally, there is a close formal relation to “fast neutrino flavor exchange” in the neutrino-sphere region in the supernova [22] - [39], where the quantum term enabling the break is just the neutrino mass term. We mention the latter to emphasize that the underlying break dynamics are really not specific to Bose condensates, or even to bosons. In each case the initial state is taken to be stable in a mean field theory (MFT). In each case there is a well defined break-time. For a case with a large number, $N$, of particles it is generally found that the time waiting for the break is of the order of $r_g^{-1} \log N$. But in the axion case we will find a somewhat different answer for the argument of the logarithm.

For the axion cloud we take an initial state to be a coherent state $|\Psi_{N_a}\rangle$ with an average number of $N_a$ individual axions,

$$|\Psi_{N_a}\rangle = e^{\sqrt{N_a}|b^\dagger - b|}|0\rangle,$$

(1)

where $b$ is the annihilator of the axion at rest. and we have

$$b|\Psi_{N_a}\rangle = \sqrt{N_a}|\Psi_{N_a-1}\rangle.$$

(2)

We define $c_q^\dagger, c_q$ to create and annihilate photons with momentum $\mathbf{q}$. Next we write an effective interaction Hamiltonian that describes the mixing induced by the interaction in a lowest order calculation, and keep only terms that conserve momentum and kinetic energy exactly.

$$H_{\text{eff}} = V^{-1/2} \sum_{|\mathbf{q}| = m_a/2} \left[ b c_q^\dagger c_{-\mathbf{q}}^\dagger + b^\dagger c_q c_{-\mathbf{q}} \right] +$$
\[
\sum_{|\mathbf{q}| = \frac{m_a}{2}} \omega_p(c^\dagger_{\mathbf{q}} c_{\mathbf{q}} + c^\dagger_{-\mathbf{q}} c_{-\mathbf{q}}),
\]
(3)

where \( \lambda = g_v m_a^{1/2} \). Here \( \omega_p \) is the plasma frequency in the surrounding medium; we are assuming no other interactions of the very low energy photon with the medium. The constraint on the \( q \) values in the sum, making it a sum over angles, comes from taking the single particle axion decay reaction to conserve energy. This obviates the need for a kinetic term in the calculations below.

We economized in notation by leaving out photon polarization indices in the above; what is critical here is that for every direction in space \( \hat{q} \), there exists a state of two photons of momenta \( \pm \mathbf{q} \), for which polarizations can be chosen such that the total expectation value of \( \mathbf{E} \cdot \mathbf{B} \) is constant in space and periodic in time with frequency \( \lambda V \). Since we began with no photon field and expected the expectation of a product to be the product of expectations, for every direction in space \( \hat{q} \), we see that the system stays exactly where it began.

At the same time, if we explore the space with small initial \( \langle c_{\pm \mathbf{q}} \rangle = 0 \)'s we would find exponentially increasing modes. Thus we categorize the original system as being in unstable classical equilibrium. Our mission is to calculate "a quantum break time" as discussed above. We shall do this in two ways; first by creating an extended MFT based on operators that are quadratic in the original variables; second by just solving for the complete wave function, but in that case limited to a small number of axions. The basic agreement of the methods provides us with enough confidence in the extended MFT to proceed with predictions when, e.g., \( N = 10^{40} \).

We introduce three operators \( X, Y, Z \),
\[
Z = b; \quad Y = c_{\mathbf{q}} c_{-\mathbf{q}}; \quad X = c_{\mathbf{q}}^\dagger c_{\mathbf{q}} + c_{-\mathbf{q}}^\dagger c_{-\mathbf{q}},
\]
(6)
with the effective Hamiltonian for the mode,
\[
H = \lambda V^{-1/2} [Z^\dagger Y +ZY^\dagger],
\]
(7)
and introduce a scaled time variable \( s = t \lambda V^{-1/2} N_a^{1/2} = t \lambda N_a^{1/2} \), where \( N_a \) is the initial number density. In addition, we rescale the operators: \( X = N_a x, \quad y = N_a y, \quad Z = N_a^{1/2} z \). Then the equations of motion for the operators \( x, y, z \) under the interaction of the Hamiltonian \( (7) \), are,
\[
\begin{align*}
&i \frac{d}{ds} z = y, \\
&i \frac{d}{ds} y = (N_a^{-1} x) z, \\
&i \frac{d}{ds} x = 2(z y^\dagger - y z^\dagger).
\end{align*}
\]
(8)

The MFT replaces each variable in these equations by its expectation in the medium. The initial conditions for our problem are \( \langle z \rangle = 1; \quad \langle x \rangle = \langle y \rangle = 0 \). The term \( z \) in the \( \frac{d^2}{dz} y \) equation has, in effect, one more power of \( b \) than the \( x z \) term, having been produced from a final \( c, c^\dagger \) commutator, and it enables the evolution starting from the pure axion state. The break that it induces is therefore identified as a "quantum" break. In the dashed curves of fig. 1 we show the time dependence of the residual axion fraction \( \zeta(t) = N_a^{-1} \langle b^\dagger b \rangle \) derived from solutions of \( (8) \) using the above initial condition, for a sequence of values of \( N_a \), differing by a factor of two at each step. The equal spacings of the curves indicate a turnover time that increases as \( \log N_a \).

For the relatively small values of \( N_a \) used in these plots we can instead do a complete quantum calculation of the wave function, based on the Hamiltonian \( (3) \) (but still with a single value of \( \mathbf{q} \)), beginning with the pure axion state. This requires the solution of \( N_a + 1 \) simultaneous, coupled, first-order, linear equations. At laptop Mathematica level we can solve the system for values of \( N_a \leq 1000 \). Results are shown as the solid curves of fig. 1, for the same set of \( N_a \) as used in the MFT model. The agreement of the two calculations is good only up to the inflection point midway through the break. On the other hand, the equal spacings of the minima, as we repeatedly double \( N_a \), are remarkably similar in the two calculations. The bounce at about \( \zeta = .2 \) for the complete quantum case, while the MFT result goes all
the way to zero, is mysterious. There is a further qualitative difference in that the quantum solution does not return to $\zeta = 1$ in the finite $N_a$ solutions. Indeed, when extended to longer times, it appears to experience very irregular jagged oscillations around a value $\zeta = .6$, while the mean field solutions are periodic. In any case, all of our arguments for physical relevance will be based on the location of the first break and a large mixing at time, 

$$T \approx (\lambda n_a^{1/2})^{-1}\log_{10}[N_a],$$

where the base 10 is a rough fit to the spacings shown in fig. 1. In fig. 2 we show the continuation of the log $N_a$ behavior in the MF solution for larger $N_a$, successively higher by factors of 100. Note that the “break”

of each these plots has exactly the same apparent shape in scaled time. The duration $\Delta T$ of the break itself is $\Delta T \sim T/\log_{10}[N_a]$, the logarithmic part of the total time going entirely into the nearly imperceptible simmering stage.

2. Many angles

An issue that was implicitly raised in arriving at (9), was the fact that we chose one direction in space for the final photon pair, thinking “when something is equally unstable in multiple directions, one in particular will be favored by variations in the environment, and will take over from all others.” But that is getting too classical too soon. The wave-function for the system, in its quantum simmering phase before the break, is perfectly able to run away in many directions simultaneously. The calculation in a box of side $L$ puts a limit on the number of allowable directions for the photons, which at the order-of-magnitude level is $N_d \approx m_a^2L^2$. In the MF approach we introduce the notations $c_i$, $\tilde{c}_i$ as the respective annihilation operators for photons with momenta $q_i$, $-q_i$ and define operators,

$$Z = b : Y_i = c_i\tilde{c}_i ; X_i = c_i^\dagger c_i + c_i^\dagger\tilde{c}_i,$$

with the Hamiltonian

$$H = g(Z \sum_{i=1}^{N_d} \lambda_i Y_i^\dagger + Z^\dagger \sum_{i=1}^{N_d} \lambda_i Y_i).$$

The equations for the rescaled $x_i, y_i, z_i$ are

$$i\frac{d}{ds}y_i = (N_a^{-1} + \lambda_i x_i) z,$$

$$i\frac{d}{ds}x_i = 2\lambda_i (z y_i^\dagger - y_i z^\dagger),$$

where we now can verify that random changes at the $20\%$ level of individual couplings, away from from a universal value $\lambda_i = 1$, make almost no difference to the axion disappearance plot.

Ideally we would have checked the agreement of the mean-field approach with the complete solutions over a wide range of $N_a$, and $N_d$, but in the complete case we could only afford $N_d=2$. In fig. 3 we show the comparison of the results of MFT calculation of (12) to the complete quantum calculation of the Schrodinger wave-function.

The agreement of the mean-field approach with the complete solution is even improved somewhat over the single angle case. This emboldens us to use the MF approach when $N_a$ and $N_d$ are both large. Our conclusion after many MF solutions for different values of $N_a$ and $N_d$ is that the turnover time, in our basic unit $T_0 = \lambda^{-1} n_a^{-1/2}$, is now approximately $T \sim T_0 \log[N_a/N_d]$, so long as $N_d < N_a$. The effect of the additional final channels is the mitigation of the logarithmic factor. Putting in the above estimate of $N_d$, we now have $T \sim T_0 \log[Lm_a^{-2}n_a]$.

We go back to the beginning, for a moment, and relax the decision to restrict our set of states to those that
exactly conserve energy. We add in the effects of $N_t$ new $2\gamma$ modes, each with equal additions $q_T$ to the two transverse photon momenta, constrained by $q_T^2/m_a < m_a$ in order to maintain near-coherence. Counting these states we find $N_t = L q_T^{\text{max}} = L m_a$. Then the final result for the turnover time is
\[
T \sim T_0 \log \frac{N_a}{N_a N_t} \approx T_0 \log [n_a m_a^{-3}]. \tag{13}
\]

3. Plasma effects

It is true that if the plasma frequency originating from a background of free electrons is greater than the axion mass there can be no actual axion $\rightarrow 2\gamma$ decay. But in any case we are interested in long-term fluctuation behavior, rather than a decay. Now including the $\omega_p$ term in (13), we calculate evolution over the same time scales as shown in the plots of fig. 1. When $0 < \omega_p < \lambda n_a^{1/2}$ we see mixings that are nearly the same as in that plot. As $\omega_p$ is increased we see the dips slowly disappear. When axion densities are very high, as in proposed axion stars [41]-[47], this criterion can supersede the $\omega_p < m_a$ condition, and allow more free electrons in the background.

4. More complex initial condensate

We used the simplest initial coherent state axion wave-function to describe the state at the beginning of the calculation. One could worry that the state in a physical cosmological system will be much more complex. We are not able to address this issue in any general way. But we can report one calculation that sheds some light. We divide the $N_a$ axions into two groups of $N_a/2$. For simplicity we will think of the axions in the second group as having a small space momentum $k$ such that $L k^2/m_a << 1$. This is physically plausible since, over the eons, axion clouds have drifted and consolidated. Now for each mode we take an initial state of the form $|\uparrow\rangle$, for each cloud, with $N_a \rightarrow N_a/2$, and each axion from either group coupled to the same set of photons as before. Again, for small numbers $N_a$ we can do the complete wave-function calculation at a later time to see that the result for the sum of the occupancies is almost identical to that of the single beam case. What matters is not the detailed construction of the state, but rather the the number density of axions.

5. Discussion

We have given an argument that axions in dense clouds, with the usual form of electromagnetic coupling, can mix strongly in a coherent way with photon pairs. Do current models of axion dark matter ever yield the combination of axion density and cloud size that makes our calculation relevant? Note that the predicted phenomena are very much an all-or-nothing affair; inconsequential until we get to a critical time, then a rapid change of everything.

The ideal locale for our phenomena could well be in a “soliton” of axion field in heart of a galaxy. Here the axion density would be many orders of magnitude greater than the average axion density in the universe. But for now we instead look back in time at the evolution of a cloud when the DM density was nearly uniform, but where this density was higher by a factor of $z^5$ where $z$ is the red shift. For $z = 10^5$, the beginning of the “dark era” (just post recombination) the energy density of dark matter is $\rho \approx .04$ (eV)$^4$. For the case $g = 10^{-21}$ eV$^{-1}$ the basic distance scale for turn-over is a few light-years. This era is a promising one because of its high axion density and low electron density. It is also plausible to assume the pure axion initial state at this point in time; prior coherent mixings with photons being suppressed by the much higher free electron density before recombination. The plasma frequency in this dark era could be as low as $\omega_p \approx 10^{-11}$ eV (taking H ionization of $10^{-5}$). This axion cloud would be a venue for the application of the results of this paper only in a domain $m_a > 10^{-11}$ eV, and therefore excludes application to the fuzzy models with $m_a = 10^{-22}$ eV. But axion stars [41]-[47] with enormously higher densities are a promising site, and here the considerations of sec. 3 could make the free electron densities a less critical factor.

We do not know what the exact consequences would be if there were substantial coherent mixing of the dark matter with photons, say in an over-dense region early on in the dark era. The gravitational physics within the over-dense region would be changed immediately, and we may have been overoptimistic that our time scales were short enough to justify leaving it out. Subject to this caveat, one might envision a bleeding away of the over-
density when there is conversion into photons.

The present paper contains the following new material:
1. A modified mean-field approach that gives the quantum break in the two photon channel. Such a break is necessary for the development of large mixing.
2. A complete solution of the Schrödinger equation for values \( N \leq 1000 \) supporting the mean-field method in this region.
3. A demonstration that many modes of the electromagnetic field enter in an essential way in determining the argument of the inevitable logarithm. The terms “laser” or “parametric amplifier” then appear to be somewhat ill-suited to the situation.
4. In solutions not explicitly plotted, a demonstration that for sufficiently high initial axion number densities the requirement \( m_a > \omega_p \) is replaced by a more lenient bound on the free electron density.

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