Critical thickness ratio for buckled and wrinkled fruits and vegetables

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Abstract – This work aims at establishing the geometrical constraint for buckled and wrinkled shapes by modeling a fruit/vegetable with exocarp and sarcocarp as a hyperelastic layer-substrate structure subjected to uniaxial compression. A careful analysis on the derived bifurcation condition leads to the finding of a critical thickness ratio which separates the buckling and wrinkling modes, and remarkably, which is independent of the material stiffnesses. More specifically, it is found that if the thickness ratio is smaller than this critical value a fruit/vegetable should be in a buckled shape (under a sufficient stress); if a fruit/vegetable is in a wrinkled shape the thickness ratio is always larger than this critical value. To verify the theoretical prediction, we consider four types of buckled fruits/vegetables and four types of wrinkled fruits/vegetables with three samples in each type. The geometrical parameters for the 24 samples are measured and it is found that indeed all the data fall into the theoretically predicted buckling or wrinkling domains.

Background and model. – Many vegetables and fruits contain exocarp and sarcocarp and they have different morphologies, in particular wrinkled and globally buckled shapes are often observed, e.g. wrinkled pumpkins and buckled cucumbers (see figs. 6 and 7). Why a fruit/vegetable takes the final shape when they are ripe may be due to many different factors during the growing process. Yet, mechanical forces alone can play a very important role for determining the geometry of vegetables and fruits [1]. Now, it has been understood that the outer layer of plant meristems often expands faster than the inner one, leading to the whole structure under compression [1,2]. In fact, a newborn fruit or vegetable has a smooth surface, and only after a certain period of growth, the morphological features occur and remain since [3]. Over the past decades or so, many authors have used purely mechanical models to study patterns in biological objects, e.g., fingerprint formation by using the von Karman’s equations [4,5] and pattern formation in plants through shell instability [6] and shapes of sympetalous flowers through growth of a thin elastic sheet [7].

Actually, it has been known for some time that instabilities can lead to a variety of patterns in a mechanical system. For a thin layer coated to a compliant substrate or a core/shell system, various highly ordered patterns [8–14] can occur due to mismatched deformations. Since ordered motifs also appear in many plants, those works certainly give the motivations to use suitable mechanical structures which resemble the plants to capture those features in order to gain certain understanding of their formation. Yin et al. [3,15] approximated some fruits and vegetables as spheroidal core/shell systems. Under suitable stresses, they analyzed the post-bifurcation states to understand the different morphologies as the four dimensionless parameters (the shape factor, the thickness ratio of core/shell, the modulus ratio of shell/core and the growth stress ratio) vary. The numerical simulations successfully captured the similar morphologies of different fruits and vegetables by specifying the geometrical and material parameters within certain ranges. Their work made it clear that geometrical parameters have a great influence on the number of wrinkles in wrinkled fruits/vegetables. Here, we examine the influence of the geometrical parameters on the final shapes of fruits and vegetables with exocarp and sarcocarp from a different aspect. The aim is to show that there exists a critical thickness ratio of sarcocarp and...
exocarp, which separates a buckled shape and a wrinkled shape, independently of the stiffness of sarcocarp and exocarp. For a fruit/vegetable with exocarp and sarcocarp, we model it as a structure of a layer bonded to a substrate. As discussed before, the stress can play a vital role in the morphology of a fruit/vegetable. As an idealization, we suppose that the structure is subjected to a uniform compressive stress everywhere, which is equivalent to applying a uniaxial compression at both ends. We model the deformation as a two-dimensional generalization of the plane-strain problem, and the mechanical model is depicted in fig. 1, where \( \delta, \pi \) and \( l \) denote the thicknesses of the layer and substrate and the length, respectively. We point out that in this paper we only consider the onset to a buckled or wrinkled shape, not the morphology beyond. Of course, the surface of a fruit or vegetable is not planar, but it resembles the layer-substrate structure and it is expected that the theory can be applied with reasonable accuracy [16].

Both exocarp and sarcocarp are modelled as Saint-Venant materials, for which the strain energy takes the form

\[
\Phi = \frac{\lambda}{2} (\text{Tr} \mathbf{E})^2 + \mu (\text{Tr} \mathbf{E}^2),
\]

where \( \mathbf{E} = \mathbf{F}^T \mathbf{F} - \mathbf{I}/2 \) is the Green strain tensor (\( \mathbf{F} \) is the deformation gradient), and \( \lambda \) and \( \mu \) are the Lamé constants. It is supposed that the Lamé constants for the layer and substrate are different. For convenience, a bar on a quantity is referred to one for the layer and a quantity without a bar is referred to that for the substrate. For example, \( \lambda, \pi \) represent the Lamé constants of the layer. For further simplification, we assume that both layer and substrate have the same Poisson’s ratio and their Young’s moduli \( E \) and \( E \) are different.

The onset to a buckled shape or a wrinkled one can be determined by a linear bifurcation analysis of the current model. For a plane-strain deformation, the details for a linear bifurcation analysis were given in [17], in which we considered a similar structure with a Blatz-Ko substrate. For a generalized plane-strain problem, the pre-buckling state is given by \( x = m_1 X, y = m_2 Y \) and \( z = m_3 Z \), while for a plane-strain problem \( x = m_1 X, y = m_2 Y \) and \( z = Z \), where \( (x, y, z) \) is the position vector after deformation and \( m_1 \) and \( m_2 \) are the stretches. The bifurcation analysis in [17] can be carried through for a generalized plane-strain problem with the above modification with little difference, and here we omit the details. The final bifurcation condition can be written in the following form:

\[
f(r, Y, \nu, \pi_0, r_c, m_1, n) = 0, \quad (2)
\]

where \( Y = \frac{E}{\pi^2} \) is the ratio of Young’s moduli, \( \nu \) is the Poisson’s ratio, \( \pi_0 \) is the aspect ratio of the layer, \( r = \frac{\pi}{\pi_0} \) is the thickness ratio of substrate and layer, \( n \) is the wave number and \( m_1 \) is the critical stretch. Once the geometrical parameters \( \pi_0, r \) and material parameters \( Y \) and \( \nu \) are specified, for a given wave number this equation determines the critical stretch (the critical stress can then be easily determined). For this layer-substrate structure, the critical mode (for which the critical stretch is the largest, i.e. the critical compressive stress is smallest) can be either a buckling mode or a wrinkling mode. For the buckling modes, usually the critical stretch decreases as the wave number increases so the critical mode is for \( n = 1 \). However, for \( n = 1 \) the buckled shape is not symmetric but for buckled fruits/vegetables the shape is roughly symmetrical about the middle. Based on such a consideration, we only study \( n \geq 2 \) modes.

In fig. 2, we fix the values of \( \pi_0, \nu \) and \( \pi_0 \) and plot the critical stretch curves vs. the thickness ratio for \( n = 2, 3, 11 \) and 12. We point out that the curves for all other modes (which are not shown) are always below either \( n = 2 \) curve or \( n = 11 \) curve and thus the critical mode can only be the \( n = 2 \) mode or \( n = 11 \) mode, depending on whether the thickness ratio is smaller or larger than the value \( r_c \) shown in the figure. The bifurcation analysis also gives the eigenfunction for each mode, and based on which we can plot the eigenshape of the structure for a given mode. For the parameters chosen in fig. 2, \( r_c = 17.9348 \). Then, for \( r = 17.9 \) and \( r = 18 \), the critical mode is for \( n = 2 \) and \( n = 11 \), respectively. The eigenshapes of the structure for these two values of thickness ratio are shown in fig. 3. One can observe that for \( r < r_c \) the structure is in a buckling mode while for \( r > r_c \) it is in a wrinkling mode. Thus, the critical thickness ratio \( r_c \) separates buckling and wrinkling modes.

Remark: The definition of a buckling mode or a wrinkling mode can be found in [17] based on eigenfunctions.
More specifically, the vertical displacement scaled by the value at the interface is introduced

$$
\alpha(y) = \frac{v(y)}{v(0)},
$$

where $v(y)$ is the displacement of substrate along the thickness direction. A buckling mode or a wrinkling mode are classified as follows: 1) Buckling mode: The critical mode is for $n = 2$ and $\alpha(y)$ is always an $O(1)$ quantity as $y(\leq 0)$ decreases. 2) Wrinkling mode: The critical mode is for $n \geq 3$ and $\alpha(y)$ decays exponentially as $y(\leq 0)$ decreases. The asymptotic analysis in [17] showed that for a wrinkling mode one always has $\alpha(-a) = \alpha(0)$ so the lower surface of the substrate is almost flat (cf. fig. 3).

To further analyze $r_c$, we observe that $r_c$ corresponds to the intersection point of the $n = 2$ curve and an $n \geq 3$ curve, which has the largest stretch value (in fig. 2 it happens that this curve is $n = 11$). Therefore, $r_c$ can be determined from the following three equations:

$$
\begin{align*}
&f \left( r, Y, \nu, \frac{b}{l}, m_1, n \right) |_{n=2} = 0, \\
&f \left( r, Y, \nu, \frac{b}{l}, m_1, n \right) |_{n\geq 3} = 0, \\
&\frac{\partial f}{\partial n} \left( r, Y, \nu, \frac{b}{l}, m_1, n \right) |_{n\geq 3} = 0.
\end{align*}
$$

Once $\frac{b}{l}$, $Y$ and $\nu$ are given, $r_c$, the wave number $n$ and the critical stretch $m_1$ can be found. It turns out that the value of Poisson’s ratio has little influence on $r_c$ and thus from now on we fix its value to be $\nu = 0.1$. Then, for a given $\frac{b}{l}$, the above three equations yield a relation between $r_c$ and the ratio of Young’s moduli. For $\frac{b}{l} = 0.01$, we plot the $r_c - Y$ curve in fig. 4. For a given $Y$, if the thickness ratio is above or below this curve, the structure is in a wrinkling mode or a buckling mode. Another intrinsic feature is that this curve has a global minimum at $r_c = r_m$. Thus, under the condition that there is a sufficient compressive stress, if $r < r_m$ the structure is always in a buckling mode and if the structure is in a wrinkling mode one must have $r > r_m$. Another important aspect of the existence of $r_m$ is that it is independent of the ratio of Young’s moduli. Actually, for $r_m$, we should have $\frac{\partial f}{\partial n} |_{r=r_m} = 0$, which leads to

$$
\begin{align*}
&\frac{\partial f}{\partial Y} \left( r_m, Y, \nu, \frac{b}{l}, m_1, n \right) |_{n=2} + \frac{\partial f}{\partial m_1} \left( r_m, Y, \nu, \frac{b}{l}, m_1, n \right) |_{n=2} \frac{\partial m_1}{\partial Y} = 0, \\
&\frac{\partial m_1}{\partial Y} = - \left( \frac{\partial^2 f}{\partial n \partial Y} \frac{\partial f}{\partial n} + \frac{\partial^2 f}{\partial n^2} \frac{\partial f}{\partial n} \right) / \left( \frac{\partial^2 f}{\partial n^2} \frac{\partial f}{\partial m_1} + \frac{\partial^2 f}{\partial n \partial m_1} \frac{\partial f}{\partial m_1} \right).
\end{align*}
$$

This equation together with the previous three equations in (4) determine a relation between $r_m$ and $\frac{b}{l}$. We point out that this relation is purely geometrical and is independent of the material parameters of the layer and substrate. Probably, the existence of this critical ratio $r_m$, which is only related to the aspect ratio of the layer, is the major finding of the present theoretical analysis.

We plot the $r_m$ curve as $\frac{b}{l}$ varies in fig. 5. This $r_m$ curve divides the whole $\frac{b}{l} - r$ plane into two parts. Based on the discussions above (5), we can conclude that, no matter what the material parameters are, for a given aspect ratio of the layer, if the thickness ratio is below this $r_m$ curve the structure is in a buckling mode (with a sufficient compressive stress) and if the structure is in a wrinkling mode the thickness ratio must be above this $r_m$ curve.

**Measured data and discussions.** – The theoretical analysis on a layer-substrate structure demonstrates the criticalness of the thickness ratio $r_m$ in determining...
Table 1: The geometrical data of buckled fruits and vegetables.

| Name              | Thickness of exocarp (cm) \(\delta\) | Thickness of sarcocarp (cm) \(\varpi\) | Length (cm) \(\ell\) |
|-------------------|--------------------------------------|----------------------------------------|----------------------|
| Banana1           | 0.40                                 | 1.50                                   | 18.0                 |
| Banana2           | 0.30                                 | 1.20                                   | 21.2                 |
| Banana3           | 0.32                                 | 1.83                                   | 21.5                 |
| Emperor banana1   | 0.150                                | 1.20                                   | 8.0                  |
| Emperor banana2   | 0.178                                | 0.86                                   | 7.0                  |
| Emperor banana3   | 0.110                                | 0.87                                   | 8.5                  |
| Zucchini1         | 0.040                                | 2.21                                   | 25.1                 |
| Zucchini2         | 0.030                                | 1.96                                   | 21.1                 |
| Zucchini3         | 0.034                                | 2.23                                   | 24.1                 |
| Cucumber1         | 0.046                                | 1.60                                   | 24.0                 |
| Cucumber2         | 0.048                                | 1.96                                   | 29.5                 |
| Cucumber3         | 0.046                                | 1.88                                   | 33.2                 |

Since our model is two-dimensional, we shall take certain cross-section of a fruit/vegetable for measurement. For those buckled fruits/vegetables, we take a longitudinal cross-section and then cut the half-part averagely along the latitudinal direction (see fig. 7). Then, this half cross-section can be considered as a layer-substrate structure. It is pretty straightforward to measure the thicknesses of the exocarp and sarcocarp and the length (although for the thin exocarp the measurement may have some error and we assume that our measurement has an error of 10% for the thickness of exocarp). For those wrinkled fruits/vegetables, we take a latitudinal cross-section and also cut it into a half (see fig. 6). It is not difficult to measure the thicknesses of exocarp and sarcocarp. But, since the cross-section is not flat, the corresponding length of the structure in the mechanical model is not obvious. Here, as a simplification we take it as the average of the outer and inner boundary lengths. We show the definition of the geometrical parameters of fruits and vegetables in fig. 8. The data for these buckled and wrinkled fruits/vegetables are given in tables 1 and 2.
Table 2: The geometrical data of wrinkled fruits and vegetables.

| Name            | Thickness of exocarp (cm) | Thickness of sarcocarp (cm) | Length (cm) |
|-----------------|---------------------------|-----------------------------|-------------|
| Large Pumpkin1  | 0.03                      | 3.02                        | 20.9        |
| Large Pumpkin2  | 0.03                      | 2.87                        | 20.6        |
| Large Pumpkin3  | 0.03                      | 3.04                        | 19.0        |
| Small Pumpkin1  | 0.03                      | 2.29                        | 12.3        |
| Small Pumpkin2  | 0.03                      | 1.85                        | 11.8        |
| Small Pumpkin3  | 0.03                      | 2.20                        | 11.7        |
| Korea melon1    | 0.030                     | 1.90                        | 9.48        |
| Korea melon2    | 0.036                     | 1.73                        | 8.25        |
| Korea melon3    | 0.030                     | 1.64                        | 9.10        |
| Chayote1        | 0.05                      | 2.30                        | 8.45        |
| Chayote2        | 0.05                      | 2.50                        | 8.40        |
| Chayote3        | 0.05                      | 2.60                        | 8.90        |

We plot the measured data in the $b_l - r_m$ plane together with the critical $r_m$ curve. The results for the buckled fruits/vegetables are shown in fig. 9. Due to the possible 10% measurement error for the thickness of exocarp, for each sample the data is represented as a line segment. For example, for the three banana samples there are three line segments. For the four types of buckled fruits/vegetables, there are in total twelve line segments, which, as can be seen, are all below the critical $r_m$ curve. Thus, indeed, the thickness ratio of sarcocarp and exocarp and aspect ratio of exocarp is located in the buckling mode as predicted by the theoretical analysis. In particular, for the samples of banana, emperor banana and cucumber, those nine line segments are well below the $r_m$ curve, which implies that even with a bigger measurement error the theory provides the correct prediction. The data for four types of wrinkled fruits/vegetables are shown in fig. 10. As one can see, all the twelve line segments are located well above the critical $r_m$ curve. In this case, it implies that even the measurement error is bigger than 10% the data still fall into the theoretically predicted wrinkling mode. In summary, the existence of the critical thickness ratio of sarcocarp and exocarp, which separates a buckled shape and a wrinkled shape, is supported by the 24 samples of buckled and wrinkled fruits/vegetables.

Remark: Our model is an idealized two-dimensional one (of course, in reality, all fruits are three-dimensional). As a result, we can only focus on its transversal or longitudinally directions separately when applying our results. For a fruit buckled in the longitudinal direction and wrinkled in the circumferential direction, we need to consider its two different cross-sections, i.e. along circumferential or longitudinally directions, respectively, and define the corresponding geometrical parameters according to fig. 8.

Conclusion. – It should be pointed out that the growth of fruits and vegetables is really complicated, combining mechanical, biological and biochemical processes together [18–20]. Nevertheless, a purely mechanical model may still provide useful insights, as demonstrated here that the samples indeed support the existence of a critical thickness ratio for buckled and wrinkled fruits/vegetables. At least, the work, once again, shows the importance of geometrical parameters in determining the shapes of fruits/vegetables. As a side product, the results could help in choosing a fruit/vegetable in our daily life. For example, if a pumpkin has no wrinkles, the thickness ratio of sarcocarp and exocarp should be relatively small (less than $r_m$), which could imply a relatively thick exocarp or thin sarcocarp. On the other hand, if a pumpkin has wrinkles, the thickness ratio should be larger than $r_m$, which could imply a relatively thick sarcocarp or thin exocarp. Actually, a further analysis of eq. (2) shows that in the wrinkling mode the mode number $n$ (i.e., wrinkle number) is a decreasing function of the aspect ratio of the layer. Thus, choosing a pumpkin with more wrinkles is more desirable as its exocarp is relatively thin.
We also mention that if one models this problem as a plane-strain problem, qualitatively, there are no essential differences with the generalized plane-strain model used here. Finally, we pointed out that it was shown in [3] and [15] that the curvature can affect the critical stress and wave number when bifurcation takes place, with no mentioning of the effects on the bifurcated shape (buckled or wrinkled). Here, the main focus is on the separation of buckled or wrinkled shape (rather than the critical stress and wave number) due to the critical thickness ratio. Of course, the underlying assumption is that the curvature does not play an essential role in determining the mode types.

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