Error Probability Analysis of Peaky Signaling over Fading Channels

Mustafa Cenk Gursoy
Department of Electrical Engineering
University of Nebraska-Lincoln, Lincoln, NE 68588
Email: gursoy@engr.unl.edu

Abstract—In this paper, the performance of signaling strategies with high peak-to-average power ratio is analyzed in both coherent and noncoherent fading channels. Two recently proposed modulation schemes, namely on-off binary phase-shift keying and on-off quaternary phase-shift keying, are considered. For these modulation formats, the optimal decision rules used at the detector are identified and analytical expressions for the error probabilities are obtained. Numerical techniques are employed to compute the error probabilities. It is concluded that increasing the peakedness of the signals results in reduced error rates for a given power level and hence improve the energy efficiency.

I. INTRODUCTION

In wireless communications, when the receiver and transmitter have only imperfect knowledge of the channel conditions, efficient transmission strategies have a markedly different structure than those employed over perfectly known channels. For instance, Abou-Faycal et al. [1] studied the noncoherent Rayleigh fading channel where the receiver and transmitter has no channel side information, and showed that the capacity-achieving input amplitude distribution is discrete with a finite number of mass points. It has also been shown that there always exists a mass point at the origin. Another key result for noncoherent channels is the requirement of transmission with high peak-to-average power ratio in the low signal-to-noise ratio (SNR) regime [2]. In [3], two types of signaling schemes are defined: on-off binary-shift keying (OOBPSK) and on-off quaternary phase-shift keying (OOQPSK). These modulations are obtained by overlaying on-off keying on phase-shift keying. The peakedness of these signals are controlled by changing the probability of no transmission. OOQPSK is shown to be an optimally efficient modulation scheme for transmission over noncoherent Rician fading channels in the low-SNR regime.

Motivated by the above-mentioned results, we study in this paper the error performance of using signals with high peak-to-average power ratios (PAR) over both coherent and noncoherent fading channels.

II. SYSTEM MODEL

We consider the following single-input, single-output flat fading channel

\[ y_i = h_i x_i + n_i \quad i = 1, 2, \ldots, \]  

where \( x_i \) is the complex channel input, \( y_i \) is the complex channel output, \( h_i \) is the channel fading coefficient, and \( n_i \) is the additive Gaussian noise. The fading process \( \{ h_i \} \) is assumed to be a stationary, ergodic, and proper complex random process. Moreover, \( \{ n_i \} \) is a sequence of independent and identically distributed (i.i.d.) zero-mean circularly symmetric complex Gaussian random variables with variance \( E[|n_i|^2] = N_0 \).

We consider that the transmitter employs OOBPSK or OOQPSK modulation for transmission. An OOBPSK signal, parametrized by \( 0 < \nu \leq 1 \), has the following constellation points together with their transmission probabilities:

\[
\begin{align*}
x_k &= \begin{cases} 
0 & \text{with prob. } (1 - \nu), \quad k = 0 \\
\sqrt{P} \nu & \text{with prob. } (\nu / 2), \quad k = 1 \\
-\sqrt{P} \nu & \text{with prob. } (\nu / 2), \quad k = 2 
\end{cases} 
\end{align*}
\]  

Similarly, an OOQPSK signal has the following constellation points with the corresponding transmission probabilities:

\[
\begin{align*}
x_k &= \begin{cases} 
0 & \text{with prob. } (1 - \nu), \quad k = 0 \\
\sqrt{P} \nu (\pm 1 \pm j) & \text{with prob. } (\nu / 4), \quad k = 1, 2, 3, 4 
\end{cases} 
\end{align*}
\]  

If the above modulations are adopted, the transmitter either sends no signal with probability \( 1 - \nu \) or one of the BPSK or QPSK constellation points with probability \( \nu \). Hence, \( \nu \) is the duty cycle of the transmission. Note that these definitions can straightforwardly be extended to phase-shift keying signals with larger constellation sizes. Note also that both signals have an average power of \( P \) and a peak power of \( \frac{P}{\nu^2} \), and hence a peak-to-average power ratio (PAR) of \( \frac{1}{\nu} \). Limitations on the PAR of the signaling scheme may be dictated by regulations or system component specifications.

III. ERROR PROBABILITY OVER COHERENT FADING CHANNELS

In this section, we study the error probability of uncoded OOBPSK and OOQPSK signaling schemes. Here, it is assumed that the receiver perfectly knows the instantaneous realizations of the fading coefficients \( \{ h_k \} \) whereas the transmitter has no such knowledge.

For the detection of the signals, maximum a posteriori probability (MAP) decision rule, which minimizes the probability of error, is employed at the receiver. It is assumed that symbol-by-symbol detection is performed. Using the property that phase-shift keying signals have the same energy, the detection
rule can be simplified as follows. The signal $x_k$ for $k \neq 0$ is the detected signal if
\[ \Re(yx_k^*) > \Re(yx_l^*) \quad \forall l \neq k, 0 \quad \text{and} \quad \Re(yx_k^*) > T \] (3)
where
\[ T = \frac{1}{2} \frac{|h|P}{\nu} + \frac{N_0}{|h|} \ln \left( \frac{\xi(1 - \nu)}{\nu} \right) \] (4)
is a threshold value with $\xi = 2$ for OOBPSK signaling and $\xi = 4$ for OOQPSK signaling. In the above formulation, $\Re(z)$ denotes the real part of the complex scalar $z$, and $z^*$ is the complex conjugate of $z$. The signal point at the origin, $x_0$, is the detected signal if $\Re(yx_0^*) < T \quad \forall l \neq 0$.

A. OOBPSK Signaling

The error probability of OOBPSK signaling as a function of the instantaneous realization of the fading coefficient is given by
\[ P_e|_{|h|} = \left\{ \begin{array}{ll} (1 - \nu)Q\left(T_b - \frac{\sqrt{|h|^2P}}{\nuN_0} - \frac{T_q}{\sqrt{\nuN_0/2}}\right) & T_b > 0 \\ (1 - \nu) + \nu Q\left(\frac{\sqrt{|h|^2P}}{\nuN_0} - T_q\right) & T_b \leq 0 \end{array} \right. \] (5)
where $Q(\cdot)$ is the Gaussian Q-function and
\[ T_b = T \sqrt{\frac{\nu}{P}} = \frac{1}{2} \frac{|h|^2P}{\nu} + \frac{1}{2} \frac{N_0^2\nu}{|h|^2P} \ln \left( \frac{2(1 - \nu)}{\nu} \right) . \]

The average probability of error is obtained from
\[ P_e = \int_0^\infty P_e|_{|h|} dF_{|h|}(|h|) \] (6)
where $F_{|h|}$ is the distribution function of the fading magnitude. Figure 1 plots the error probability curves for OOBPSK signaling in the Rayleigh fading channel with $E\{|h|^2\} = 1$.

It is observed that if the peakedness of the input signals is increased sufficiently (e.g., $\nu = 0.2, 0.1, 0.05$), significant improvements in error performance are achieved over ordinary BPSK (i.e., OOBPSK with $\nu = 1$) performance.

B. OOQPSK Signaling

In this case, it is more convenient to initially obtain the correct detection probabilities. When $x_1$ is the transmitted signal point, the correct detection probability is
\[ P_e|_{x_1, |h|} = \left\{ \begin{array}{ll} 1 - Q\left(\frac{|h|^2P}{\nuN_0}\right) + T_q\left(\frac{|h|^2P}{\nuN_0/2}\right) & T_q > 0 \\ (1 - \nu) + \nu Q\left(\frac{|h|^2P}{\nuN_0/2} - \frac{T_q}{\sqrt{\nuN_0/2}}\right) & T_q \leq 0 \end{array} \right. \] (7)
where
\[ T_q = T \sqrt{\frac{2\nu}{P}} = \sqrt{\frac{|h|^2P}{2\nu}} + \frac{N_0^2\nu}{2|h|^2P} \ln \left( \frac{4(1 - \nu)}{\nu} \right) . \] (8)

If $x_0$ is sent, the correct detection probability is
\[ P_e|_{x_0, |h|} = \left\{ \begin{array}{ll} 4 \int_0^{T_q} \frac{1}{2} - Q\left(\frac{T_q - x}{\sqrt{\nuN_0/2}}\right) \frac{\sqrt{x}}{\sqrt{\nuN_0}} dx & T_q > 0 \\ 0 & T_q \leq 0 \end{array} \right. . \]

Now, the error probability as a function of the fading coefficients and the average error probability are given by
\[ P_e|_{|h|} = 1 - ((1 - \nu)P_e|_{x_0, |h|} + \nu P_e|_{x_1, |h|}) \] (9)
and
\[ P_e = \int_0^\infty P_e|_{|h|} dF_{|h|}(|h|) \] (10)
respectively, where $F_{|h|}$ is the distribution function of the fading magnitude. Figure 2 plots the error probability curves for OOQPSK signaling again as a function of the SNR per bit (i.e., SNR normalized by the entropy of OOQPSK signals) in the coherent Rayleigh fading channel with $E\{|h|^2\} = 1$. It is seen that error performance improves if $\nu \leq 0.5$ and significant gains are obtained when $\nu = 0.1$ which requires a 10-fold increase in the peak-to-average power ratio when compared to ordinary QPSK signaling. It is interesting to note that if $\nu < 0.8$, the threshold value $T_q \rightarrow \infty$ as $P \rightarrow 0$. Therefore, for sufficiently small $P$, the zero signal, $x_0$, is always chosen as the detected signal and the error probability becomes $\nu$. Indeed, it is observed in Fig. 2 that the error curves corresponding to OOQPSK signaling with $\nu < 0.8$ approach to $\nu$ as SNR decreases. If $\nu > 0.8$, $T_q \rightarrow -\infty$ as $P \rightarrow 0$. When $T_q \leq 0$, the zero signal is never detected. Note that this behavior is also exhibited if OOBPSK modulation is used.

IV. ERROR PROBABILITY OVER NONCOHERENT FADING CHANNELS

In this section, we consider the scenario in which neither the transmitter nor the receiver knows the instantaneous realizations of the fading coefficients. We consider a fast Rician fading environment and hence $\{h_k\}$ is a sequence of i.i.d. proper complex Gaussian random variables with mean $E\{h_k\} = m$ and variance $E\{|h_k|^2\} = \gamma^2$. It is assumed that channel statistics and hence the values of $m$ and $\gamma^2$ are assumed to be known at the transmitter and receiver. The
conditional output probability density function given the input is
\[ f_{y|x}(y|x) = \frac{1}{\pi(\gamma^2|x|^2 + N_0)} e^{-\frac{|y-mx|^2}{\gamma^2|x|^2 + N_0}}. \] (9)

The MAP detection yields the following decision rule:
\[ \Re(yx_k^*) > \Re(yx_l^*) \quad \forall l \neq k, 0 \] (10)
and
\[ \frac{\gamma^2P}{\nu}|y|^2 + 2N_0|m|\Re(yx_k^*) > T \] (11)
where
\[ T = N_0\frac{P}{\nu}|m|^2 + N_0 \left( \frac{\gamma^2P}{\nu} + N_0 \right) \ln \left( \frac{\xi(1-\nu)}{\nu} \left( \frac{\gamma^2P}{\nuN_0} + 1 \right) \right) \] (12)
with \( \xi = 2 \) for OOBPSK signaling and \( \xi = 4 \) for OOQPSK signaling. The signal point \( x_0 \) is the detected signal if
\[ \frac{\gamma^2P}{\nu}|y|^2 + 2N_0|m|\Re(yx_l^*) < T \quad \forall l \neq 0. \] (13)

For brevity, we only discuss OOQPSK signaling in the following. The analysis of the OOBPSK signaling can be found in the extended version of this paper [5].

A. OOQPSK Signaling

As before, we first express the correct detection probabilities. If a nonzero signal point is sent and \( T > 0 \), we have
\[ P_{c|x} = \left( 1 - Q \left( \frac{\sqrt{m^2P}}{\sqrt{\frac{\gamma^2P}{\nu} + N_0}} \right) \right) Q \left( \frac{D_q}{\sqrt{\frac{\gamma^2P}{\nu} + N_0}} \right) \]
\[ + \int_0^{D_q} Q \left( \frac{C_q-x+A_q}{\sqrt{\frac{\gamma^2P}{2\nu} + N_0}} \right) \frac{e^{-\frac{x^2}{2(N_0+1)}}}{\sqrt{\pi(N_0+1)}} dx \]
where
\[ A_q = \frac{N_0|m|}{\gamma^2} \sqrt{\frac{\nu}{2P}}, \quad C_q = \frac{\nu}{\gamma^2P} + 2A_q, \quad D_q = \sqrt{C_q - A_q^2 - A_q} \] (14)
with \( T \) given in (12). If \( T < 0 \),
\[ P_{c|x} = \left( 1 - Q \left( \frac{\sqrt{m^2P}}{\sqrt{\frac{\gamma^2P}{2\nu} + N_0}} \right) \right)^2. \] (15)

Finally, the error probability is obtained from
\[ P_e = 1 - ((1-\nu)P_{c|x} + \nuP_{c|x}). \] (16)

Fig.3 plots the error probability curves for OOQPSK signaling as a function of SNR per bit in the noncoherent Rician fading channel with Rician factor \( K = |m|^2/\gamma^2 = 5 \). Similarly as before, it is seen that error performance improves for sufficiently small duty factors. It is again noted that for \( \nu < 0.8, C_q \rightarrow \infty \) as \( P \rightarrow 0 \). Hence, for sufficiently small \( P \), the zero signal is always declared as the detected signal and the error probability becomes equal to \( \nu \). Another interesting observation in Fig.3 is the existence of error floors for sufficiently high values of SNR. This is due to the fact that even if the additive noise vanishes, the performance is limited by the multiplicative noise introduced by unknown fading. Note that in the correct detection probability expressions (14) and (15), the arguments of the Q-function have the term \( P \) in both the numerator and denominator. Hence, letting \( P \rightarrow \infty \) or \( N_0 \rightarrow 0 \) does not drive the correct detection probabilities to 1.
V. Conclusion

We have studied the error performance of peaky signaling over fading channels. We have considered two modulation formats: OOBPSK and OOQPSK. We have found the optimal MAP decision rules and obtained analytical error probability expressions. Through numerical examples, we have seen that error performance improves if the peakedness of the signaling schemes are sufficiently increased. For fixed error probabilities, substantial gains in terms of SNR per bit are realized, making the peaky signaling schemes energy efficient and hence desirable modulation formats for wireless sensor networks where low-data rate transmissions with low energy consumption are required [4].

References

[1] I. Abou-Faycal, M. D. Trott, and S. Shamai (Shitz), “The capacity of discrete-time memoryless Rayleigh fading channels,” IEEE Trans. Inform. Theory, vol. 47, pp. 1290-1301, May 2001.
[2] S. Verdú, “Spectral efficiency in the wideband regime,” IEEE Trans. Inform. Theory, vol. 48, pp. 1319-1343, June 2002.
[3] M. C. Gursoy, H. V. Poor, and S. Verdú, “The noncoherent Rician fading channel – Part II : Spectral efficiency in the low-snr regime,” IEEE Trans. Wireless Commun., vol. 4, no. 5, pp. 2207 - 2221.
[4] A. Y. Wang, S. Cho, C. G. Sodini, and A. P. Chandrakasan, “Energy efficient modulation and MAC for asymmetric RF microsensor systems,” Proc. Int. Symp. Low Power Electronics and Design, pp. 106-111, 6-7 August 2001.
[5] M. C. Gursoy, “Error performance of peaky signaling over fading channels,” available at http://www.ee.unl.edu/people/gursoy.html