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The $p$-adic $Z$-transform

Annales mathématiques Blaise Pascal, tome 2, n° 1 (1995), p. 131-146

<http://www.numdam.org/item?id=AMBP_1995__2_1_131_0>
THE P-ADIC Z-TRANSFORM

Lucien van Hamme

Abstract. Let $a + p^n \mathbb{Z}_p$ be a ball in $\mathbb{Z}_p$ and assume that $a$ is the smallest natural number contained in the ball. We define a measure $\mu_z$ on $\mathbb{Z}_p$ by putting $\mu_z(a + p^n \mathbb{Z}_p) = \frac{z^a}{1 - z^{p^n}}$ where $z \in \mathbb{C}_p, |z - 1|_p \geq 1$. Let $f$ be a continuous function defined on $\mathbb{Z}_p$. The mapping $f \mapsto \int_{\mathbb{Z}_p} f(x) \mu_z(x)$ is similar to the classical Z-transform. We use this transform to give new proofs of several known results: the Mahler expansion with remainder for a continuous function, the Van der Put expansion, the expansion of a function in a series of Sheffer polynomials. We also prove some new results.

1991 Mathematics subject classification: 46S10

1. Introduction

Let $\mathbb{Z}_p$ be the ring of p-adic integers, where $p$ is a prime. $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote, as usual, the field of the p-adic numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. $|\cdot|$ denotes the normalized p-adic valuation on $\mathbb{C}_p$.

We start by defining a measure on $\mathbb{Z}_p$.

Let $a + p^n \mathbb{Z}_p$ be a ball in $\mathbb{Z}_p$. We may assume that $a$ is the smallest natural number contained in the ball. Our measure will depend on a parameter $z \in \mathbb{C}_p$.

Put $\mu_z(a + p^n \mathbb{Z}_p) = \frac{z^a}{1 - z^{p^n}}$.

It is well-known that this defines a distribution on $\mathbb{Z}_p$.

Let $D$ denote the set $\{z \in \mathbb{C}_p | |z - 1| \geq 1\}$.

An easy calculation shows that if $z \in D$ then $\left|\frac{z^a}{1 - z^{p^n}}\right| \leq 1$.

Throughout this paper we will assume that $z \in D$. Hence $\mu_z$ is a measure.

Now let $f : \mathbb{Z}_p \rightarrow \mathbb{C}_p$ be a continuous function.
If we associate with $f$ the integral $F(z) = \int_{\mathbb{Z}_p} f(x) \mu_z(x)$ we get a transformation that we call the $p$-adic $Z$-transform since it is similar to the classical $Z$-transform used by engineers. The aim of this paper is to show how this transform can be used to obtain a number of results in $p$-adic analysis. In section 2 we start by studying the integral $F(z)$. In sections 3 and 4 we use the $p$-adic $Z$-transform to give new proofs of several known results: the Mahler expansion with remainder for a continuous function, the Van der Put expansion, the expansion of a function in a series of Sheffer polynomials. In section 5 we use the results of section 2 to find approximations to the $p$-adic logarithm of 2. We prove e.g. that the following congruence is valid in $\mathbb{Z}_p$

\[
2 \left(1 - \frac{1}{p}\right) \log 2 \equiv \sum_{k=1}^{p^n} \frac{(-1)^{k+1}}{k} \equiv 4(-1)^{n-\frac{n-1}{2}} \sum_{k=0}^{\frac{p^n-3}{2}} \frac{(-1)^k}{2k+1} \pmod{p^{2n}\mathbb{Z}_p}
\]

2. The integral $\int_{\mathbb{Z}_p} f(x) \mu_z(x)$

This integral has already been studied and used by Y. Amice and others in [1] and [4]. A fundamental property of this integral is

**Proposition**: $F(z)$ is an analytic element in $D$ (in the sense of Krasner).

This means that $F(z)$ is the uniform limit of a sequence of rational functions with poles outside $D$. But, by definition

\[
F(z) = \int_{\mathbb{Z}_p} f(x) \mu_z(x) = \lim_{n \to \infty} \sum_{k=0}^{p^n-1} f(k)z^k
\]

It is not difficult to show that the sequence in (1) is uniformly convergent. Since the zeroes of $1 - z^{p^n}$ are outside $D$, $F(z)$ is an analytic element in $D$.

**Corollary**: $F$ satisfies the "principle of analytic continuation" i.e. if $F(z)$ is zero on a ball in $D$ it is zero in the whole of $D$.

The fact that $F(z)$ is an analytic element in $D$ is very useful in proving properties of the integral (1). As an example we prove that

\[
\int_{\mathbb{Z}_p} f(x) \mu_z(x) = f(0) + z \int_{\mathbb{Z}_p} f(x+1) \mu_z(x) \quad \text{in } D
\]

**Proof**: For $|z| < 1$ formula (1) reduces to

\[
\int_{\mathbb{Z}_p} f(x) \mu_z(x) = \sum_{k=0}^{\infty} f(k)z^k
\]
The trivial identity

\[ \sum_{k=0}^{\infty} f(k)z^k = f(0) + z \sum_{k=0}^{\infty} f(k+1)z^k \quad (|z| < 1) \]

can be written as

\[ \int \mathcal{Z}_p f(x) \mu_z(x) = f(0) + z \int \mathcal{Z}_p f(x+1) \mu_z(x) \]

This is a priori valid for $|z| < 1$. By analytic continuation it is valid in $D$.

We now list some properties of the integral $\int \mathcal{Z}_p f(x) \mu_z(x)$. We only give a few indications about the proofs.

**P1** \[ \int \mathcal{Z}_p f(x) \mu_z(x) = \sum_{k=0}^{n-1} f(k)z^k + z^n \int \mathcal{Z}_p f(x+n) \mu_z(x) \quad \text{in } D \] (4)

**Proof**: This follows by iterating (2)

**P2** \[ \int \mathcal{Z}_p f(x) \mu_z(x) = -\sum_{k=1}^{n} \frac{f(-k)}{z^k} + \frac{1}{z^n} \int \mathcal{Z}_p f(x-n) \mu_z(x) \quad \text{in } D \]

\[ = -\sum_{k=1}^{\infty} \frac{f(-k)}{z^k} \quad \text{if } |z| > 1 \] (5)

**Proof**: Replace $f(x)$ by $f(x-1)$ in (2) to get

\[ \int \mathcal{Z}_p f(x) \mu_z(x) = -\frac{f(-1)}{z} + \frac{1}{z} \int \mathcal{Z}_p f(x-1) \mu_z(x) \]

Iteration of this formula yields (5).

**P3** \[ \int \mathcal{Z}_p f(x) \mu_z(x) = \sum_{k=0}^{n-1} (\Delta^k f)(0) \frac{z^k}{(1-z)^{k+1}} + \frac{z^n}{(1-z)^n} \int \mathcal{Z}_p (\Delta^n f)(x) \mu_z(x) \] (6)

\[ = \sum_{k=0}^{\infty} (\Delta^k f)(0) \frac{z^k}{(1-z)^{k+1}} \quad \text{in } D \]

Here $\Delta$ is the difference operator defined by $(\Delta f)(x) = f(x+1) - f(x)$. 

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Proof: Write (2) in the form
\[ \int_{\mathbb{Z}_p} f(x) \mu_z(x) = \frac{f(0)}{1 - z} + \frac{z}{1 - z} \int_{\mathbb{Z}_p} (\Delta f)(x) \mu_z(x) \]
then iterate.

Let \( E \) be the translation operator defined by \((Ef)(x) = f(x + 1)\) and put \( Q = \Delta E^{-1} \) then

\[ \int_{\mathbb{Z}_p} f(x) \mu_z(x) = \sum_{k=0}^{n-1} \frac{(Q^k f)(-1)}{(1 - z)^{k+1}} + \frac{1}{(1 - z)^n} \int_{\mathbb{Z}_p} (Q^n f)(x) \mu_z(x) \quad (7) \]

Proof: This follows from the obvious
\[ \int_{\mathbb{Z}_p} f(x) \mu_z(x) = \frac{f(-1)}{1 - z} + \frac{1}{1 - z} \int_{\mathbb{Z}_p} (Q f)(x) \mu_z(x) \]

Proof: Suppose first that \(|z| > 1\) and use (5) for the first integral and (3) for the second integral. The formula then reduces to the obvious identity.
\[ - \sum_{k=1}^{\infty} \frac{f(-k)}{z^k} + \sum_{k=0}^{\infty} \frac{f(-k)}{z^k} = f(0) \]
The formula is valid in \( D \) by analytic continuation.

If \( f \) is an even function then \( \int_{\mathbb{Z}_p} f(x) \mu_{-1}(x) = \frac{f(0)}{2} \quad (9) \)

Proof: Put \( z = -1 \) in (8).

If \( F(z) = \int_{\mathbb{Z}_p} f(x) \mu_z(x), G(z) = \int_{\mathbb{Z}_p} g(x) \mu_z(x) \)
then \( F(z)G(z) = \int_{\mathbb{Z}_p} (f \ast g)(x) \mu_z(x) \quad \text{in} \ D \quad (10) \)
where \( f \ast g \) the convolution of \( f \) and \( g \).
\( f \ast g \) is by definition the continuous function with value equal to \( (f \ast g)(n) = \sum_{k=0}^{n} f(k)g(n - k) \)
if \( n \) is a natural number.
Proof: For $|z| \leq 1$ the equality $F(z)G(z) = \int_{\mathbb{Z}_p} (f \ast g)(x) \mu_z(x)$ is simply
\[
\left( \sum_{k=0}^{\infty} f(k) z^k \right) \left( \sum_{k=0}^{\infty} g(k) z^k \right) = \sum_{k=0}^{\infty} (f \ast g)(k) z^k
\]
which is obvious. The formula is valid in $D$ by analytic continuation.

$$\text{P8} \quad \left| \int_{\mathbb{Z}_p} f(x) \mu_z(x) \right| \leq ||f||$$

where $||f||$ denotes the sup-norm.

Remark: It follows from (5) that $\lim_{z \to \infty} z F(z) G(z) = -(f \ast g)(-1)$. But $\lim_{z \to \infty} z F(z) G(z) = -f(-1) \lim_{z \to \infty} G(z) = 0$.

Hence we deduce the (known) fact that $(f \ast g)(-1) = 0$, i.e. the convolution of the two continuous functions is 0 at the point $-1$.

3. The $p$-adic $Z$-transform

Let $C(\mathbb{Z}_p)$ denote the Banach space of the all continuous functions from $\mathbb{Z}_p$ to $\mathbb{C}_p$, equipped with the sup-norm.

Let $(a_n)$ be a sequence in $\mathbb{C}_p$. A series of the form
\[
\sum_{k=0}^{\infty} a_k \frac{z^k}{(1-z)^{k+1}} \quad \text{with} \quad \lim_{k \to \infty} a_k = 0 \tag{12}
\]
is convergent in $D$.

Let $B$ be the set of all functions $F : D \to \mathbb{C}_p$ that are the sum of a series of the form (12) with $\lim_{k \to \infty} a_k = 0$.

If we define $||F|| = \sup_{z \in D} |F(z)|$ then $B$ is a Banach space.

Formula (6) shows that $F(z) = \int_{\mathbb{Z}_p} f(x) \mu_z(x)$ belongs to $B$ if $f \in C(\mathbb{Z}_p)$.

Hence it makes sense to consider the mapping
\[
T : C(\mathbb{Z}_p) \to B : f \to F(z) = \int_{\mathbb{Z}_p} f(x) \mu_z(x)
\]
We will call $F(z)$ the $p$-adic $z$-transform of $f$ for the following reason. If $|z| < 1$ then
$F(z) = \sum_{k=0}^{\infty} f(k)z^k$. In applied mathematics it is customary to call the "generating function" $F(z)$ the z-transform of $f$.

We now examine the properties of the z-transform. It is easily verified that $T$ is linear and continuous.

If $F(z)$ is identical 0 then $\sum_{k=0}^{\infty} f(k)z^k = 0$ for $|z| < 1$. Hence $f(x) \equiv 0$.

This proves that $T$ is injective.

$T$ is also surjective. To see this we start from a given $F(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{(1-z)^{k+1}}$ with

$$\lim_{k \to \infty} a_k = 0.$$ It follows from (6) that the z-transform of the function $f(x) = \sum_{k=0}^{\infty} a_k \binom{x}{k}$ is equal to the given $F(z)$ since $(\Delta^k f)(0) = a_k$.

Although we do not need it in the sequel we will also prove that $T$ is an isometry. For this we need a lemma.

**Lemma 1**

If $a = (a_k)$ is a sequence in $\mathbb{C}_p$, with $\lim_{k \to \infty} a_k = 0$, then

$$\sup |a_k| = \sup \{|a_0|, |a_0 + a_1|, |a_1 + a_2|, \ldots, |a_k + a_{k+1}|, \ldots\}.$$

**Proof:** Put $||a|| = \sup |a_k|$, then $||a|| = \sup \{|a_0|, |a_0 + a_1|, |a_1 + a_2|, \ldots, |a_k + a_{k+1}|, \ldots\}$. Since $|a_k + a_{k+1}| \leq \max\{|a_k|, |a_{k+1}|\} \leq ||a||$ we see that $||a|| \leq ||a||$.

Put $b_0 = a_0, b_1 = a_0 + a_1, \ldots, b_k = a_{k-1} + a_k, \ldots$. Then $a_k = b_k - b_{k-1} + b_{k-2} - \ldots \pm b_0$.

Hence $|a_k| \leq \max\{|b_0|, |b_1|, \ldots, |b_k|\} \leq ||a||$.

Thus $||a|| \leq ||a||$ and the lemma is proved.

**Proposition:** $T$ is an isometry.

**Proof:** Let $F(z) = \sum_{k=0}^{\infty} a_k \frac{z^k}{(1-z)^{k+1}}$ be the z-transform of $f(x) = \sum_{k=0}^{\infty}(\Delta^k f)(0) \binom{x}{k}$.

$$||f|| = \sup_k ||(\Delta^k f)(0)||$$ since the polynomials $\binom{x}{k}$ form an orthogonal base for $C(\mathbb{Z}_p)$

$$= \sup_k |a_k|$$

$$= \sup \{|a_0|, |a_0 + a_1|, \ldots, |a_k + a_{k+1}|, \ldots\}$$ by lemma 1

Writing $u = \frac{z}{1-z}$ we observe that $z \in D$ if and only if $|u + 1| \leq 1$. Now
\[ ||f|| = \sup \{ |a_0|, |a_0 + a_1|, \ldots, |a_k + a_{k+1}|, \ldots \} \]
\[ = \sup_{|u| \leq 1} \{ a_0 + (a_0 + a_1)u + \ldots + (a_{k-1} + a_k)u^k + \ldots \} \]
\[ = \sup_{|u+1| \leq 1} \{ a_0 + (a_0 + a_1)u + \ldots + (a_{k-1} + a_k)u^k + \ldots \} \]
\[ = \sup_{z \in D} |F(z)| = ||F|| \]

We now show how the z-transform can be used in p-adic analysis.

**Application 1** Mahler's expansion with remainder

We start from formula (6)

\[ F(z) = \sum_{k=0}^{n-1} (\Delta^k f)(0) \frac{z^k}{(1-z)^{k+1}} + \frac{z^n}{(1-z)^n} \int_{\mathbb{Z}_p} (\Delta^n f)(x) \mu_z(x) \]  

(6)

If \( f(x) = \left( \begin{array}{c} x \\ n-1 \end{array} \right) \) all terms on the R.H.S. vanish except the term \( \frac{z^{n-1}}{(1-z)^n} \). This means that the z-transform of \( \left( \begin{array}{c} x \\ n-1 \end{array} \right) \) is \( \frac{z^{n-1}}{(1-z)^n} \).

Hence every term of (3) is the transform of a function in \( C(\mathbb{Z}_p) \). Taking the inverse transform we get something of the form

\[ f(x) = \sum_{k=0}^{n-1} (\Delta^k f)(0) \binom{x}{k} + r_n(x) \]

where \( r_n(x) \) is the inverse transform of

\[ z \cdot \frac{z^{n-1}}{(1-z)^n} \int_{\mathbb{Z}_p} (\Delta^n f)(x) \mu_z(x) \]

(13)

Using (10) we see that \( r_n(x) = \left\{ \left( \begin{array}{c} x \\ n-1 \end{array} \right) \ast \Delta^n f \right\} (x-1) \).

The presence of the first factor \( z \) in the product (13) makes it necessary to evaluate the convolution of \( \left( \begin{array}{c} x \\ n-1 \end{array} \right) \) and \( \Delta^n f \) at the point \( x-1 \) instead of \( x \).

This gives Mahler's expansion with an expression for the remainder

\[ f(x) = \sum_{k=0}^{n-1} (\Delta^k f)(0) \binom{x}{k} + \left\{ \left( \begin{array}{c} x \\ n-1 \end{array} \right) \ast \Delta^n f \right\} (x-1) \]

This was obtained in [5] by a different method.
Remark: Until now we have assumed that the functions of $C(\mathbb{Z}_p)$ take their values in $\mathbb{C}_p$. If we replace $\mathbb{C}_p$ by a field that is complete for a non-archimedean valuation containing $\mathbb{Q}_p$, the method still works. The only restriction is that we can no longer use any property whose proof uses analytic continuation.

**Application 2** Van der Put’s expansion

Notation: If $n = a_0 + a_1p + \ldots + a_sp^s$ with $a_s \neq 0$ then we put $m(n) = s$ and $n_- = a_0 + a_1p + \ldots + a_{s-1}p^{s-1}$.

Take $f \in C(\mathbb{Z}_p)$ and let $f_r$ denote the locally constant function defined by

$$f_r(k) = f(k) \quad \text{for } k = 0, 1, \ldots, p^r - 1$$

$$f_r(x) = f_r(x + p^r)$$

By induction on $r$ we can verify that

$$\sum_{0 \leq n < p^r} (f(n) - f(n_-)) \frac{z^n}{1 - z^{m(n)}} = \sum_{n=0}^{p^r-1} f(n)z^n \quad \frac{1}{1 - z^{p^r}} \quad (14)$$

Using the definition (1) we see that the R.H.S. of (14) is the $z$-transform of $f_r$. In the same way we can verify that $\frac{z^n}{1 - z^{m(n)}}$ is the $z$-transform of the function

$$e_n(x) = 1 \quad \text{if } |x - n| < \frac{1}{n}$$

$$e_n(x) = 0 \quad \text{if } |x - n| \geq \frac{1}{n}$$

The inverse transform of (8) gives the identity

$$\sum_{0 \leq n < p^r} [f(n) - f(n_-)]e_n(x) = f_r(x)$$

If $r \to \infty$ we recover the Van der Put expansion of $f(x)$.

**Application 3**

If we put $f(x) = \binom{x + n}{n}$ in (7) we see that $z$-transform of $\binom{x + n}{n}$ is $\frac{1}{(1-z)^{n+1}}$. The inverse of (7) yields

$$f(x) = \sum_{k=0}^{n} (Q^k f)(-1) \binom{x + k}{k} + \left\{ \binom{x + n}{n} * Q^{n+1} f \right\}(x) \quad Q = \Delta E^{-1}$$
4. The expansion of a continuous function in a series of Sheffer polynomials

In this section we will use the p-adic z-transform to generalize the main theorem of [6]. We first recall a few elements of the p-adic umbral calculus developed in [6].

Let $R$ be a linear continuous operator on $C(\mathbb{Z}_p, K)$, where $K$ is a field containing $\mathbb{Q}_p$ that is complete for a non archimedean valuation. If $R$ commutes with $E$ it can be written in the form $R = \sum_{i=0}^{\infty} b_i \Delta^i$ where $(b_i)$ is a bounded sequence in $K$. The result that we want to generalize is the following.

**Proposition [6]**

If $Q = \sum_{i=0}^{\infty} b_i \Delta^i$ is a linear continuous operator on $C(\mathbb{Z}_p, K)$ such that $b_0 = 0$, $|b_1| = 1$, $|b_i| \leq 1$ for $i \geq 2$ then

a) there exists a unique sequence of polynomials $p_n(x)$ such that

$$Qp_n = p_{n-1}, \quad \text{deg } p_n = n, \quad p_n(0) = 0 \quad \text{for } n \geq 1 \text{ and } p_0 = 1$$

b) every continuous function $f: \mathbb{Z}_p \to K$ has a uniformly convergent expansion of the form

$$f(x) = \sum_{n=0}^{\infty} (Q^n f)(0)p_n(x)$$

(15)

With an operator $R = \sum_{i=0}^{\infty} b_i \Delta^i$ we can associate a measure on $\mathbb{Z}_p$ by means of the functional sending a $f \in C(\mathbb{Z}_p, K)$ to $(Rf)(0)$.

**Example**: Take $R = \frac{1}{1-Ez}$ with $z \in D$. Then

$$R = \frac{1}{1 - z + \Delta z} = \sum_{k=0}^{\infty} \frac{\Delta^k z^k}{(1 - z)^{k+1}}$$

Formula (6) shows that the measure obtained in this way is the measure introduced in section 1.

Now let $Q = \sum_{i=0}^{\infty} b_i \Delta^i$ and $S = \sum_{i=0}^{\infty} s_i \Delta^i$ be two operators commuting with $E$ where $S$ is invertible.
If \( b_0 = 0 \), any operator \( R \), commuting with \( E \), can be written in the form

\[
R = \sum_{n=0}^{\infty} r_n Q^n, \quad r_n \in K
\]

We can see this as an equality between operators or as an identity between formal power series in \( \Delta \). If we take \( R = \frac{S}{1 - Ez} \) the coefficients \( r_n \) will depend on \( z \). Let us write it in the form

\[
\frac{S}{1 - Ez} = \sum_{n=0}^{\infty} \frac{T_n(z)}{(1 - z)^{n+1}} Q^n \tag{16}
\]

Writing out everything as a powerseries in \( \Delta \) and comparing the coefficient of \( \Delta^n \) we see that \( T_n(z) \) is a polynomial of degree \( n \) in \( z \). If, moreover, \( |b_1| = 1 \) the sequence \( \frac{T_n(z)}{(1 - z)^{n+1}} \) is bounded.

Multiplying (16) with \( S^{-1} \) and applying the operators on both sides to a function \( f \in C(\mathbb{Z}_p, K) \) we get the series

\[
F(z) = \sum_{n=0}^{\infty} (S^{-1}Q^n f)(0) \frac{T_n(z)}{(1 - z)^{n+1}} \tag{17}
\]

This series is uniformly convergent since \( \lim_{n \to \infty} (S^{-1}Q^n f)(0) = 0 \).

The idea is now to take the inverse z-transform of (17).

Now the z-transform of \( \binom{x^n}{n} \) is \( \frac{z^n}{(1 - z)^{n+1}} \). Hence the z-transform of a polynomial of degree \( n \) is of the form \( \frac{P_n(z)}{(1 - z)^{n+1}} \) where \( P_n(z) \) is also a polynomial of degree \( n \).

Taking the inverse transform of (17) we get

\[
f(x) = \sum_{n=0}^{\infty} (S^{-1}Q^n f)(0)t_n(x) \tag{18}
\]

where \( t_n(x) \) is a polynomial of degree \( n \).

This is the expansion we wanted to obtain.

To see that (18) is a generalization of (15) take \( S \) equal to the identity operator and take \( f \) equal to the polynomial \( p_n \) in (15). (18) then reduces to \( p_n(x) = t_n(x) \).

In the general case the polynomials \( t_n(x) \) are called "Sheffer polynomials" in umbral calculus.
Remark

It is possible to work in an even more general situation. Let $Q_1, Q_2, \ldots, Q_n, \ldots$ be a sequence operators satisfying the same conditions as the operator $Q$ above. There exists a sequence of polynomials $T_n(z), \deg T_n = n$, such that

$$ \frac{S}{1 - Ez} = \sum_{n=0}^{\infty} \frac{T_n(z)}{(1 - z)^{n+1}} Q_1 Q_2 \ldots Q^n $$

5. A formula for $\lg 2$

The formula

$$ 2(1 - \frac{1}{p})\lg 2 = \lim_{n \to \infty} \sum_{\substack{k=1 \atop (k, p) = 1}}^{p^n} \frac{(-1)^{k+1}}{k}, \quad p \neq 2 $$

is proved in [2] p. 180 and [3] p. 38. Here $\lg 2$ is the $p$-adic logarithm.

In this section we show that it is possible to refine this result using the properties of the integral studied in section 2.

Let $f(x) = 0$ for $|x| < 1$

$$ = \frac{1}{x} \quad \text{for } |x| = 1 $$

In [1] (lemma 6.4, chapter 12) it is proved that, for $z \in D$,

$$ \int_{\mathbb{Z}_p} f(x) \mu_z(x) = \frac{1}{p} \lg \frac{1 - z^p}{(1 - z)^p} \quad (19) $$

If $U_p = \mathbb{Z}_p \setminus p\mathbb{Z}_p$ denotes the group of units of $\mathbb{Z}_p$ the integral can be written as

$$ \int_{U_p} \frac{\mu_z(x)}{x} = \frac{1}{p} \lg \frac{1 - z^p}{(1 - z)^p} $$

Putting $z = -1$ we get

$$ \int_{U_p} \frac{\mu_{-1}(x)}{x} = -(1 - \frac{1}{p})\lg 2 \quad (20) $$
The idea is to construct approximations for the integral on the LHS of (20). This will yield the following theorem.

**Theorem**: If $p \neq 2$ then

\[ a) \quad 2(1 - \frac{1}{p}) \log 2 \equiv \sum_{k=1, (k, p) = 1}^{p^n} \frac{(-1)^{k+1}}{k} \pmod{p^{2n}} \]

\[ b) \quad 2(1 - \frac{1}{p}) \log 2 \equiv 4 \varepsilon_n \sum_{k=0}^{\frac{p^n-3}{2}} \frac{(-1)^{k+1}}{2k+1} \pmod{p^{2n}} \]

where $\varepsilon_n = (-1)^{n+\frac{p-1}{2}}$

\[ c) \quad -2(1 - \frac{1}{p}) \log 2 \equiv \sum_{k=1, (k, p) = 1}^{p^n} \frac{(-1)^{k+1}}{k} - 8 \varepsilon_n \sum_{k=0}^{\frac{p^n-3}{2}} \frac{(-1)^{k+1}}{2k+1} \pmod{p^{4n}} \]

For the proof we need the value of a few integrals. We collect these results in the following lemma. $i$ denotes a squareroot of $-1$.

**Lemma 2**

1. $\int_{\mathcal{U}_p} \frac{\mu_{-1}(x)}{x^2} = \int_{\mathcal{U}_p} \frac{\mu_{-1}(x)}{x^4} = 0$
2. $\int_{\mathcal{U}_p} \frac{\mu_i(x)}{x^2} + \int_{\mathcal{U}_p} \frac{\mu_{-i}(x)}{x^2} = 0$
3. $\int_{\mathcal{U}_p} \frac{\mu_i(x)}{x^4} + \int_{\mathcal{U}_p} \frac{\mu_{-i}(x)}{x^4} = 0$
4. $\int_{\mathcal{U}_p} \frac{\mu_i(x)}{x} = \int_{\mathcal{U}_p} \frac{\mu_{-i}(x)}{x} = -\frac{1}{2} (1 - \frac{1}{p}) \log 2$ for $p \neq 2$
5. $\int_{\mathcal{U}_p} \frac{\mu_i(x)}{x^3} = \int_{\mathcal{U}_p} \frac{\mu_{-i}(x)}{x^3} = \frac{1}{8} \int_{\mathcal{U}_p} \frac{\mu_{-1}(x)}{x^3}$

**Proof of the lemma**

1. These are special cases of formula (9).
2. These are special cases of (8) with $z = i$. 
(3) Suppose first that \( p \equiv 1 \pmod{4} \). Then \( i^p = i \), hence

\[
\int_{U_p} \frac{\mu_i(x)}{x} = \frac{1}{p} \log \frac{1-i}{(1-i)^p} = -(1 - \frac{1}{p}) \log(1-i)
\]

Since \((1-i)^2 = -2i\) and \( \log i = 0 \) we see that \( \log(1-i) = \frac{1}{2} \log 2 \) and the assertion is proved. If \( p \equiv 3 \pmod{4} \) we have \( i^p = -i \) and we get

\[
\int_{U_p} \frac{\mu_i(x)}{x} = \frac{1}{p} \log \frac{1+i}{(1-i)^p}
\]

Since \( \frac{1+i}{1-i} = i \) and \( \log i = 0 \) we conclude that

\[
\frac{1}{p} \log \frac{1+i}{(1-i)^p} = -(1 - \frac{1}{p}) \log(1-i) = -\frac{1}{2} (1 - \frac{1}{p}) \log 2
\]

The integral \( \int_{U_p} \frac{\mu_i(x)}{x} \) is calculated in the same way.

(4) Let \( k \) be a natural number and let \( \zeta(s) \) be the Riemann zeta function. It is well-known that the numbers \( \zeta(-k) \) are rational and that the sequence \( k \to (1 - p^k) \zeta(-k) \) can be interpolated p-adically. This can be deduced from the following formula (see [1] p. 295).

\[
(1 - p^k) \zeta(-k) = \frac{1}{q^{k+1} - 1} \sum \int_{U_p} x^k \mu_\theta(x)
\]

The sum is extended over all primitive \( q \)-th roots of unity \( \theta \) with \( \theta \neq 1 \). \( q \) is an integer prime to \( p \).

In [1] the author supposes that \( q \) is a prime but this restriction is not necessary.

Clearly the LHS of (21) is independant of \( q \). Taking respectively \( q = 2 \) and \( q = 4 \) we get

\[
\frac{1}{2^{k+1} - 1} \int_{U_p} x^k \mu_{-1}(x) = \frac{1}{4^{k+1} - 1} \left\{ \int_{U_p} x^k \mu_{-1}(x) + \int_{U_p} x^k \mu_i(x) + \int_{U_p} x^k \mu_{-i}(x) \right\}
\]

or

\[
2^{k+1} \int_{U_p} x^k \mu_{-1}(x) = \int_{U_p} x^k \mu_i(x) + \int_{U_p} x^k \mu_{-i}(x)
\]
If $k$ remains in a fixed residue class mod $(p - 1)$ the LHS of (21) is a continuous function of $k$. Hence (21) and (22) remain valid for negative integers (except possibly for $k = -1$). Taking $k = -3$ we get

$$4 \int_{U_p} \frac{\mu_{-1}(x)}{x^3} = \int_{U_p} \frac{\mu_1(x)}{x^3} + \int_{U_p} \frac{\mu_{-3}(x)}{x^3}$$

Since (8) implies that $\int_{U_p} \frac{\mu_1(x)}{x^3} = \int_{U_p} \frac{\mu_{-3}(x)}{x^3}$ the last assertion of lemma 2 is proved.

**Proof of the theorem**

Starting from (1) we have

$$\int_{U_p} \frac{\mu_z(x)}{x} = \sum_{k=1}^{p^n} \frac{z^k}{k} + z p^n \int_{U_p} \frac{\mu_z(x)}{x + p^n}$$

Now

$$\frac{1}{x + p^n} = \frac{1}{x} - \frac{p^n}{x^2} + \frac{p^{2n}}{x^3} - \frac{p^{3n}}{x^4} + \frac{p^{4n}}{x^4(x + p)}$$

Integrating this over $U_p$ and observing that (11) implies

$$\left| \int_{U_p} \frac{\mu_z(x)}{x^i(x + p^n)} \right| \leq 1$$

we see that the $(p$-adic$)$ value of

$$(1 - z p^n) \int_{U_p} \frac{\mu_z(x)}{x} - \sum_{\substack{k=1 \\ (k, p) = 1}}^{p^n} k + z p^n \left[ p^n \int_{U_p} \frac{\mu_z(x)}{x^2} - p^{2n} \int_{U_p} \frac{\mu_z(x)}{x^3} + p^{3n} \int_{U_p} \frac{\mu_z(x)}{x^4} \right]$$

is $\leq \frac{1}{p^4}$.

For $z = -1$ the first assertion of lemma 2 implies that two of these integrals are zero. Since the other integrals clearly lie in $\mathbb{Z}_p$ we obtain the following congruence in $\mathbb{Z}_p$

$$2 \int_{U_p} \frac{\mu_{-1}(x)}{x} \equiv \sum_{\substack{k=1 \\ (k, p) = 1}}^{p^n} \frac{(-1)^k}{k} - p^{2n} \int_{U_p} \frac{\mu_{-1}(x)}{x^3} \quad (\text{mod } p^{4n}) \quad (24)$$
If we compare this with (20) we see that point (a) of the theorem is proved.

In order to prove (b) note that \( i^p = (-1)^{\frac{p-1}{2}} \) and hence \( i^n = \varepsilon_n i \).

Now put \( z = i \) in (23). This gives

\[
(1 - \varepsilon_n i) \int_{U_p} \frac{\mu_i(x)}{x} - \sum_{k=1}^{p^n} i^k + p^n \varepsilon_n i \int_{U_p} \frac{\mu_i(x)}{x^2} - p^{2n} \varepsilon_n i \int_{U_p} \frac{\mu_i(x)}{x^3} + p^{3n} \varepsilon_n i \int_{U_p} \frac{\mu_i(x)}{x^4}
\]

\[
\leq \frac{1}{p^4}
\]

Replace \( i \) by \(-i\) and subtract. When the integrals are replaced by their values given in lemma 2 we obtain the congruence

\[
\varepsilon_n i (1 - \frac{1}{p}) \log 2 \equiv 2i \sum_{k=0}^{\frac{k^n}{2}} \frac{(-1)^k}{2k+1} + \frac{\varepsilon_n i p^{2n}}{4} \int_{U_p} \frac{\mu_{-1}(x)}{x^3} \quad (\text{mod } p^4)
\]

(25)

Neglecting the last term we see that (b) is proved.

To obtain (c) it is sufficient to take a linear combination of (24) and (25) such that the integral \( \int_{U_p} \frac{\mu_{-1}(x)}{x^3} \) disappears.

We can deduce the following purely arithmetical result from the theorem.

**Corollary**

For \( p \neq 2 \)

\[
2 \cdot \frac{2^{(p-1)} - 1}{p^2} \equiv 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots - \frac{1}{p-1} \quad (\text{mod } p^2)
\]

\[
\equiv 4(-1)^{\frac{p-1}{2}}(1 - \frac{1}{3} + \frac{1}{5} - \ldots + \frac{1}{p-2}) \quad (\text{mod } p^2)
\]

**Proof:** Since \( 2^{(p-1)p} \equiv 1 \) (mod \( p^2 \)) we have

\[
p(p - 1) \log 2 = \log(2^{(p-1)p} - 1 + 1) \equiv 2^{(p-1)p} - 1 \quad (\text{mod } p^4)
\]

and hence
\[(1 - \frac{1}{p^2}) \log 2 \equiv \frac{2^{(p-1)p} - 1}{p^2} \pmod{p^4}\]

Combining this with the congruences (a) and (b) of the theorem (for \(n = 1\)) we see that the required congruences are established.

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