Gerbes and Heisenberg’s Uncertainty Principle

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Abstract
We prove that a gerbe with a connection can be defined on classical phase space, taking the $U(1)$–valued phase of certain Feynman path integrals as Čech 2–cocycles. A quantisation condition on the corresponding 3–form field strength is proved to be equivalent to Heisenberg’s uncertainty principle.

Contents
1 Introduction 1
2 A gerbe on classical phase space 2
  2.1 The gerbe . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 2
  2.2 The steepest–descent approximation to the 2–cocycle . . . . . . . . . . . . 6
  2.3 The connection . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
  2.4 Symplectic area . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
  2.5 The field strength . . . . . . . . . . . . . . . . . . . . . . . . . . . 8
3 Outlook 10

1 Introduction
Feynman’s quantum–mechanical exponential of the classical action $S$,

$$\exp\left(\frac{iS}{\hbar}\right),$$

has an interpretation in terms of gerbes. The latter are geometrical structures developed recently, that have found interesting applications in several areas of geometry and theoretical physics. For the basics in the theory of gerbes the reader may want to consult the nice review. We have in ref. constructed a gerbe with a connection
over the configuration space $\mathcal{F}$ corresponding to $d$ independent degrees of freedom. Specifically, the $U(1)$–valued phase of the quantum–mechanical transition amplitude $\langle q_2 t_2 | q_1 t_1 \rangle$,

$$\frac{\langle q_2 t_2 | q_1 t_1 \rangle}{|\langle q_2 t_2 | q_1 t_1 \rangle|} = \exp \left( i \arg \langle q_2 t_2 | q_1 t_1 \rangle \right),$$

(2)
is closely related to the trivialisation of a gerbe on $\mathcal{F}$. This fact can be used in order to prove that the semiclassical vs. strong–quantum duality $S/\hbar \leftrightarrow \hbar/S$ of ref. [5] is equivalent to a Heisenberg–algebra noncommutativity [6] for the space coordinates. The connection on the gerbe is interpreted physically as a Neveu–Schwarz field $B_{\mu\nu}$ or, equivalently, as the magnetic background [7] that causes space coordinates to stop being commutative and close a Heisenberg algebra instead.

Now the transition amplitude $\langle q_2 t_2 | q_1 t_1 \rangle$ is proportional to the path integral

$$\int \mathcal{D}q \exp \left( \frac{i}{\hbar} \int_{t_1}^{t_2} dt \mathcal{L} \right).$$

(3)

Whenever the Hamiltonian $\mathcal{H}(q, p)$ depends quadratically on $p$, eqn. (3) is the result of integrating over the momenta in the path integral

$$\int \mathcal{D}q \int \mathcal{D}p \exp \left( \frac{i}{\hbar} \int_{t_1}^{t_2} dt \left[ p \dot{q} - \mathcal{H}(q, p) \right] \right).$$

(4)

In this sense the integral (4) over phase space $\mathcal{P}$ is more general than the integral (3) over configuration space $\mathcal{F}$.

On the other hand, Heisenberg’s uncertainty principle $\Delta Q \Delta P \geq \hbar/2$ can be derived from the Heisenberg algebra $[Q, P] = i\hbar$. In turn, the latter can be traced back to the corresponding classical Poisson brackets on $\mathcal{P}$. If, as shown in refs. [4, 7], a gerbe potential $B_{\mu\nu}$ on configuration space $\mathcal{F}$ is responsible for a Heisenberg algebra between space coordinates, then it makes sense to look for an interpretation of the uncertainty principle in terms of gerbes on classical phase space $\mathcal{P}$.

With this starting point, the purpose of this article is twofold:

i) To extend the formalism of ref. [4] from configuration space $\mathcal{F}$ to classical phase space $\mathcal{P}$, in order to construct a gerbe over the latter.

ii) To derive Heisenberg’s uncertainty principle from the 3–form field strength on the above gerbe.

2 A gerbe on classical phase space

2.1 The gerbe

Classical phase space $\mathcal{P}$ is a $2d$–dimensional symplectic manifold endowed with the symplectic 2–form

$$\omega = \sum_{j=1}^{d} dq^j \wedge dp_j,$$

(5)
when expressed in Darboux coordinates. The canonical 1–form \( \theta \) on \( P \) defined as

\[
\theta := - \sum_{j=1}^{d} p_j dq^j
\]

satisfies

\[
d\theta = \omega. \tag{7}
\]

Let \( \{ U_\alpha \} \) be a good cover of \( P \) by open sets \( U_\alpha \). Pick any two points \( (q_{\alpha_1}, p_{\alpha_1}) \) and \( (q_{\alpha_2}, p_{\alpha_2}) \) on \( P \), respectively covered by the coordinate charts \( U_{\alpha_1} \) and \( U_{\alpha_2} \). The transition amplitude \( \langle q_{\alpha_2}, t_{\alpha_2} | q_{\alpha_1}, t_{\alpha_1} \rangle \) is proportional to the path integral

\[
\langle q_{\alpha_2}, t_{\alpha_2} | q_{\alpha_1}, t_{\alpha_1} \rangle \sim \int Dq \int dp \exp \left( -\frac{i}{\hbar} \int_{t_{\alpha_1}}^{t_{\alpha_2}} (\theta + \mathcal{H} dt) \right). \tag{8}
\]

The momenta \( p \) being integrated over in (8) are unconstrained, while the coordinates \( q \) satisfy the boundary conditions \( q(t_{\alpha_1}) = q_{\alpha_1} \) for \( j = 1, 2 \). Throughout, the \( \sim \) sign will stand for proportionality: path integrals are defined up to some (usually divergent) normalisation. However all such normalisation factors cancel in the ratios of path integrals that we are interested in, such as (11), (14) and (21) below. The combination \( \theta + \mathcal{H} dt \), which we will denote by \( \lambda \), is the integral invariant of Poincaré–Cartan:

\[
\lambda := \theta + \mathcal{H} dt. \tag{9}
\]

Let \( \mathbb{L}_{\alpha_1, \alpha_2} \subset P \) denote an oriented trajectory connecting \( (q_{\alpha_1}, p_{\alpha_1}) \) to \( (q_{\alpha_2}, p_{\alpha_2}) \), as time runs from \( t_{\alpha_1} \) to \( t_{\alpha_2} \). We define \( \tilde{a}_{\alpha_1, \alpha_2} \) as the following functional integral over all trajectories \( \mathbb{L}_{\alpha_1, \alpha_2} \) connecting \( (q_{\alpha_1}, p_{\alpha_1}) \) to \( (q_{\alpha_2}, p_{\alpha_2}) \):

\[
\tilde{a}_{\alpha_1, \alpha_2} \sim \int D\mathbb{L}_{\alpha_1, \alpha_2} \exp \left( -\frac{i}{\hbar} \int_{L_{\alpha_1, \alpha_2}} \lambda \right). \tag{10}
\]

The integral (10) differs from the transition amplitude (8) in that the momenta \( p \) in the latter are unconstrained, while the momenta \( p \) in (10) satisfy the same boundary conditions as the coordinates \( q \). With this proviso we will continue to call \( \tilde{a}_{\alpha_1, \alpha_2} \) a probability amplitude. Its \( U(1) \)–valued phase is

\[
a_{\alpha_1, \alpha_2} := \frac{\tilde{a}_{\alpha_1, \alpha_2}}{|\tilde{a}_{\alpha_1, \alpha_2}|}. \tag{11}
\]

Next assume that \( U_{\alpha_1} \cap U_{\alpha_2} \) is nonempty,

\[
U_{\alpha_1, \alpha_2} := U_{\alpha_1} \cap U_{\alpha_2} \neq \emptyset, \tag{12}
\]

and let \( (q_{\alpha_1, \alpha_2}, p_{\alpha_1, \alpha_2}) \in U_{\alpha_1, \alpha_2} \). For \( \alpha_1 \) and \( \alpha_2 \) fixed we define

\[
\tau_{\alpha_1, \alpha_2} : U_{\alpha_1, \alpha_2} \rightarrow U(1)
\]

\[
\tau_{\alpha_1, \alpha_2}(q_{\alpha_1, \alpha_2}, p_{\alpha_1, \alpha_2}) := a_{\alpha_1, \alpha_1, \alpha_2} a_{\alpha_1, \alpha_2}. \tag{13}
\]
Thus \( \tau_{\alpha_1 \alpha_2}(q_{12}, p_{12}) \) equals the \( U(1) \)-valued phase of the probability amplitude \( \tilde{a} \) for the particle to start at \((q_{11}, p_{11}) \in U_{\alpha_1}\), then pass through \((q_{12}, p_{12}) \in U_{\alpha_1 \alpha_2}\), and finally end at \((q_{22}, p_{22}) \in U_{\alpha_2}\). One readily verifies that (13) defines a gerbe trivialisation. We may rewrite the trivialisation (13) as

\[
\tau_{\alpha_1 \alpha_2} = \frac{\tilde{\tau}_{\alpha_1 \alpha_2}}{|\tilde{\tau}_{\alpha_1 \alpha_2}|},
\]

where \( \tilde{\tau}_{\alpha_1 \alpha_2} \) is defined as the path integral

\[
\tilde{\tau}_{\alpha_1 \alpha_2} \sim \int \text{DL}_{\alpha_1 \alpha_1}(\alpha_{12}) \exp \left( -\frac{i}{\hbar} \int L_{\alpha_1 \alpha_2}(\alpha_{12}) \lambda \right).
\]

In \( \tilde{\tau}_{\alpha_1 \alpha_2} \) one integrates over all trajectories that, connecting \( \alpha_1 \) to \( \alpha_2 \), pass through the variable midpoint \( \alpha_{12} \); the notation \( \mathbb{L}_{\alpha_1 \alpha_1}(\alpha_{12}) \) stresses this fact. Therefore \( \tilde{\tau}_{\alpha_1 \alpha_2} \), and hence also \( \tau_{\alpha_1 \alpha_2} \), is a function on \( U_{\alpha_1 \alpha_2} \).

Next consider three points

\[
(q_{11}, p_{11}) \in U_{\alpha_1}, \quad (q_{22}, p_{22}) \in U_{\alpha_2}, \quad (q_{33}, p_{33}) \in U_{\alpha_3}
\]

such that the triple overlap \( U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3} \) is nonempty,

\[
U_{\alpha_1 \alpha_2 \alpha_3} := U_{\alpha_1} \cap U_{\alpha_2} \cap U_{\alpha_3} \neq \emptyset.
\]

Once the trivialisation (13) is known, the 2–cocycle \( g_{\alpha_1 \alpha_2 \alpha_3} \) defining a gerbe on \( \mathbb{P} \) is given by

\[
 g_{\alpha_1 \alpha_2 \alpha_3} : U_{\alpha_1 \alpha_2 \alpha_3} \rightarrow U(1), \quad g_{\alpha_1 \alpha_2 \alpha_3}(q_{123}, p_{123}) := \tau_{\alpha_1 \alpha_2}(q_{123}, p_{123}) \tau_{\alpha_2 \alpha_3}(q_{123}, p_{123}) \tau_{\alpha_3 \alpha_1}(q_{123}, p_{123}), \tag{18}
\]

where all three \( \tau \)'s on the right–hand side are, by definition, evaluated at the same variable midpoint

\[
(q_{123}, p_{123}) \in U_{\alpha_1 \alpha_2 \alpha_3}. \tag{19}
\]

In this way, \( g_{\alpha_1 \alpha_2 \alpha_3}(q_{123}, p_{123}) \) equals the \( U(1) \)-phase of the probability amplitude \( \tilde{a} \) for the following transition (see figure\(^1\)): starting at \((q_{11}, p_{11})\) we pass through \((q_{123}, p_{123})\) on our way to \((q_{22}, p_{22})\); from here we cross \((q_{123}, p_{123})\) again on our way to \((q_{33}, p_{33})\); finally from \((q_{33}, p_{33})\) we once more pass through \((q_{123}, p_{123})\) on our way back to \((q_{11}, p_{11})\). The complete closed trajectory is

\[
\mathbb{L}_{\alpha_1 \alpha_2 \alpha_3}(\alpha_{123}) := \mathbb{L}_{\alpha_1 \alpha_2}(\alpha_{123}) + \mathbb{L}_{\alpha_2 \alpha_3}(\alpha_{123}) + \mathbb{L}_{\alpha_3 \alpha_1}(\alpha_{123}). \tag{20}
\]

Being \( U(1) \)-valued, we can write \( g_{\alpha_1 \alpha_2 \alpha_3} \) in eqn. (18) as the quotient

\[
g_{\alpha_1 \alpha_2 \alpha_3} = \frac{\tilde{g}_{\alpha_1 \alpha_2 \alpha_3}}{|\tilde{g}_{\alpha_1 \alpha_2 \alpha_3}|}, \tag{21}
\]

\(^1\)Figure available upon request.
where
\[ \tilde{g}_{\alpha_1\alpha_2\alpha_3} \sim \int DL_{\alpha_1\alpha_2\alpha_3}(\alpha_{123}) \exp \left( \frac{-i}{\hbar} \int L_{\alpha_1\alpha_2\alpha_3}(\alpha_{123}) \lambda \right). \] (22)

This functional integral extends over all trajectories described in (20). The notation \( L_{\alpha_1\alpha_2\alpha_3}(\alpha_{123}) \) stresses the fact that all such paths traverse the variable midpoint \((q_{\alpha_{123}}, p_{\alpha_{123}})\). Therefore \( \tilde{g}_{\alpha_1\alpha_2\alpha_3} \) and hence also \( g_{\alpha_1\alpha_2\alpha_3} \), is a function on \( U_{\alpha_1\alpha_2\alpha_3} \).

Consider now the first half of the leg \( \mathbb{L}_{\alpha_1\alpha_2}(\alpha_{123}) \), denoted \( \frac{1}{2} L_{\alpha_1\alpha_2}(\alpha_{123}) \). The latter runs from \( \alpha_1 \) to \( \alpha_{123} \). Consider also the second half of the leg \( \mathbb{L}_{\alpha_3\alpha_1}(\alpha_{123}) \), denoted \( \frac{1}{2} L_{\alpha_3\alpha_1}(\alpha_{123}) \), with a prime to remind us that it is the second half: it runs back from \( \alpha_{123} \) to \( \alpha_1 \) (see figure). The sum of these two half legs,
\[ \frac{1}{2} L_{\alpha_1\alpha_2}(\alpha_{123}) + \frac{1}{2} L_{\alpha_3\alpha_1}(\alpha_{123}), \] (23)
completes one roundtrip and it will, as a rule, enclose an area \( S_{\alpha_1}(\alpha_{123}) \), unless the path from \( \alpha_{123} \) to \( \alpha_1 \) happens to coincide exactly with the path from \( \alpha_1 \) to \( \alpha_{123} \):
\[ \partial S_{\alpha_1}(\alpha_{123}) = \frac{1}{2} L_{\alpha_1\alpha_2}(\alpha_{123}) + \frac{1}{2} L_{\alpha_3\alpha_1}(\alpha_{123}). \] (24)

Although the surface \( S_{\alpha_1}(\alpha_{123}) \) is not unique, for the moment any such surface will serve our purposes. Analogous considerations apply to the other half legs \( \frac{1}{2} L_{\alpha_2\alpha_3}(\alpha_{123}), \frac{1}{2} L_{\alpha_3\alpha_1}(\alpha_{123}) \) and \( \frac{1}{2} L_{\alpha_2\alpha_3}(\alpha_{123}) \) under cyclic permutations of 1, 2, 3 in the Čech indices \( \alpha_1, \alpha_2 \) and \( \alpha_3 \):
\[ \partial S_{\alpha_2}(\alpha_{123}) = \frac{1}{2} L_{\alpha_2\alpha_3}(\alpha_{123}) + \frac{1}{2} L_{\alpha_1\alpha_2}(\alpha_{123}), \] (25)
\[ \partial S_{\alpha_3}(\alpha_{123}) = \frac{1}{2} L_{\alpha_3\alpha_1}(\alpha_{123}) + \frac{1}{2} L_{\alpha_2\alpha_3}(\alpha_{123}). \] (26)

The boundaries of the three surfaces \( S_{\alpha_1}(\alpha_{123}), S_{\alpha_2}(\alpha_{123}) \) and \( S_{\alpha_3}(\alpha_{123}) \) all pass through the variable midpoint \( (q_{\alpha}, p_{\alpha}) \) (19). We define their connected sum
\[ S_{\alpha_1\alpha_2\alpha_3} := S_{\alpha_1} + S_{\alpha_2} + S_{\alpha_3}. \] (27)

In this way we have
\[ \mathbb{L}_{\alpha_1\alpha_2\alpha_3} = \partial S_{\alpha_1\alpha_2\alpha_3} = \partial S_{\alpha_1} + \partial S_{\alpha_2} + \partial S_{\alpha_3}. \] (28)

It must be borne in mind that \( \mathbb{L}_{\alpha_1\alpha_2\alpha_3} \) is a function of the variable midpoint \( \alpha_{123} \in U_{\alpha_1\alpha_2\alpha_3} \), even if we no longer indicate this explicitly. Eventually one, two or perhaps all three of \( S_{\alpha_1}, S_{\alpha_2} \) and \( S_{\alpha_3} \) may degenerate to a curve connecting the midpoint \( (q_{\alpha_1}, p_{\alpha_1}) \) with \( \alpha_1, \alpha_2 \) or \( \alpha_3 \), respectively. Whenever such is the case for all three surfaces, the closed trajectory \( L_{\alpha_1\alpha_2\alpha_3} \) cannot be expressed as the boundary of a 2-dimensional surface \( S_{\alpha_1\alpha_2\alpha_3} \). In what follows we will however exclude this latter possibility, so that at least one of the three surfaces on the right-hand side of (27) does not degenerate to a curve.
One further comment is in order. The gerbe we have constructed is defined on phase space $\mathbb{P}$. If $\mathbb{R}$ denotes the time axis, we have the natural inclusion $\iota: \mathbb{P} \to \mathbb{P} \times \mathbb{R}$. The 1–form $\lambda$ is defined on $\mathbb{P} \times \mathbb{R}$, but all the line integrals we have considered here in fact involve its pullback $\iota^* \lambda$ to $\mathbb{P}$, rather than $\lambda$ itself, even if this has not been denoted explicitly. Moreover, the term $\mathcal{H}dt$ within $\lambda$ will drop out of our calculations, as we will see in section 2.4. An equivalent statement of this fact is that we are working on $\mathbb{P}$ at a fixed value of the time.

2.2 The steepest–descent approximation to the 2–cocycle

We can approximate the path–integral (22) by the method of steepest descent [9]. We are given a path integral

$$\int \mathcal{D}f \exp \left(\mathcal{F}[f]\right),$$

(29)

where the argument of the exponential contains a 1–dimensional integral

$$\mathcal{F}[f] := \int dt \, f(u_i(t), \dot{u}_i(t), t), \quad i = 1, \ldots, r.$$  

(30)

Consider the diagonal $r \times r$ matrix $M$ whose $i$–th entry $m_i$ equals

$$m_i := \frac{\partial^2 f}{\partial u_i^2}, \quad i = 1, \ldots, r.$$  

(31)

If the extremals $u_i^{(0)}$, $i = 1, \ldots, r$, make the integral $\mathcal{F}$ a minimum, then all the $m_i$, evaluated at the extremals $u_i^{(0)}$, are nonnegative [10]. Hence

$$\det M^{(0)} = \prod_{i=1}^r m_i^{(0)} \geq 0,$$  

(32)

the superindex $(0)$ standing for “evaluation at the extremal”. We will assume that $\det M^{(0)} > 0$. Then the steepest descent approximation to (29) yields

$$\int \mathcal{D}f \exp \left(\mathcal{F}[f]\right) \sim \left(-\det M^{(0)}\right)^{-1/2} \exp \left(\mathcal{F}[f^{(0)}]\right).$$  

(33)

In our case (22), the saddle point is given by those closed paths $L_{\alpha_1\alpha_2\alpha_3}^{(0)}$ that minimise the integral

$$\int_{L_{\alpha_1\alpha_2\alpha_3}} \lambda$$  

(34)

for fixed $\alpha_1$, $\alpha_2$ and $\alpha_3$. The $u_i(t)$ of eqns. (30)–(33) are replaced by the pullbacks $q_j(t)$, $p^j(t)$, to the path $L_{\alpha_1\alpha_2\alpha_3}$, of the Darboux coordinates $q_j$, $p^j$ on phase space $\mathbb{P}$. In particular we have $r = 2d$. Altogether, the steepest descent approximation (33) to the path integral (29) leads to

$$\tilde{g}_{\alpha_1\alpha_2\alpha_3}^{(0)} \sim \left(\frac{1}{\hbar} \det M^{(0)}\right)^{-1/2} \exp \left(-\frac{i}{\hbar} \int_{L_{\alpha_1\alpha_2\alpha_3}^{(0)}} \lambda\right).$$  

(35)
Now $\det M^{(0)} > 0$ so, by eqn. (21), it does not contribute to the 2–cocycle. After dropping an irrelevant $e^{-i\pi/4}$ we finally obtain

$$g^{(0)}_{\alpha_1 \alpha_2 \alpha_3} = \exp \left( -\frac{i}{\hbar} \int_{L^{(0)}_{\alpha_1 \alpha_2 \alpha_3}} \lambda \right). \quad (36)$$

Eqn. (36) gives the steepest–descent approximation $g^{(0)}_{\alpha_1 \alpha_2 \alpha_3}$ to the 2–cocycle $g_{\alpha_1 \alpha_2 \alpha_3}$ defining the gerbe on phase space $P$. As already remarked before eqn. (28), $g^{(0)}_{\alpha_1 \alpha_2 \alpha_3}$ is a function of the variable midpoint (19) through the integration path $L^{(0)}_{\alpha_1 \alpha_2 \alpha_3}$, even if we no longer indicate this explicitly.

2.3 The connection

On a gerbe determined by the 2–cocycle $g_{\alpha_1 \alpha_2 \alpha_3}$, a connection is specified by forms $A, B, H$ satisfying [1]

$$H|_{U_\alpha} = dB_\alpha \quad (37)$$

$$B_{\alpha_2} - B_{\alpha_1} = dA_{\alpha_1 \alpha_2} \quad (38)$$

$$A_{\alpha_1 \alpha_2} + A_{\alpha_2 \alpha_3} + A_{\alpha_3 \alpha_1} = g_{\alpha_1 \alpha_2 \alpha_3}^{-1} dg_{\alpha_1 \alpha_2 \alpha_3}. \quad (39)$$

The gerbe is called flat if $H = 0$.

We can use eqn. (36) in order to compute the connection, at least to the same order of accuracy as the 2–cocycle $g_{\alpha_1 \alpha_2 \alpha_3}$ itself:

$$A^{(0)}_{\alpha_1 \alpha_2} + A^{(0)}_{\alpha_2 \alpha_3} + A^{(0)}_{\alpha_3 \alpha_1} = \left( g^{(0)}_{\alpha_1 \alpha_2 \alpha_3} \right)^{-1} dg^{(0)}_{\alpha_1 \alpha_2 \alpha_3}. \quad (40)$$

We will henceforth drop the superindex $(0)$, with the understanding that all our computations have been done in the steepest–descent approximation. We find

$$A_{\alpha_1 \alpha_2} = -\frac{i}{\hbar} \lambda_{\alpha_1 \alpha_2} = -\frac{i}{\hbar} (\theta + \mathcal{H} dt)_{\alpha_1 \alpha_2}. \quad (41)$$

Therefore

$$B_{\alpha_2} - B_{\alpha_1} = dA_{\alpha_1 \alpha_2} = -\frac{i}{\hbar} (\omega + d\mathcal{H} \wedge dt)_{\alpha_1 \alpha_2}. \quad (42)$$

On constant–energy submanifolds of phase space the above simplifies to

$$B_{\alpha_2} - B_{\alpha_1} = -\frac{i}{\hbar} \omega_{\alpha_1 \alpha_2}. \quad (43)$$

We will henceforth assume that we are working on constant–energy submanifolds of phase space.

2.4 Symplectic area

Let $S \subset P$ be a 2–dimensional surface with the boundary $\partial S = L$. By Stokes’ theorem and eqns. (7), (9),

$$\int_{L} \lambda = \int_{\partial S} \lambda = \int_{S} d\lambda = \int_{S} (\omega + d\mathcal{H} \wedge dt). \quad (44)$$
Let us pick $S$ such that it is a constant–energy surface, or else a constant–time surface. Then
\[ \int_L \lambda = \int_S \omega. \] (45)
The right–hand side of eqn. (45) does not depend on the particular surface $S$ chosen because
\[ d\omega = 0. \] (46)
Next pick $S$ as $S_{\alpha_1 \alpha_2 \alpha_3}$ in eqn. (27). By eqn. (45), the 2–cocycle (36) reads
\[ g_{\alpha_1 \alpha_2 \alpha_3} = \exp \left( -\frac{i}{\hbar} \int_{S_{\alpha_1 \alpha_2 \alpha_3}} \omega \right). \] (47)
The above can be given a nice quantum–mechanical interpretation. The integral
\[ \frac{1}{\hbar} \int_{S_{\alpha_1 \alpha_2 \alpha_3}} \omega \] (48)
equals the symplectic area of the surface $S_{\alpha_1 \alpha_2 \alpha_3}$ in units of $\hbar$. In the WKB approximation [9], the absolute value of (48) is proportional to the number of quantum states contributed by the surface $S_{\alpha_1 \alpha_2 \alpha_3}$ to the Hilbert space of quantum states. Now the steepest descent approximation used here is a rephrasing of the WKB method. We conclude that the 2–cocycle $g_{\alpha_1 \alpha_2 \alpha_3}$ equals the exponential of ($-i$ times) the number of quantum states contributed by any surface $S_{\alpha_1 \alpha_2 \alpha_3}$ bounded by the closed loop $L_{\alpha_1 \alpha_2 \alpha_3}$. The constant–energy condition on the surface translates quantum–mechanically into the stationarity of the corresponding states. The steepest–descent approximation minimises the symplectic area of the open, constant–energy surface $S_{\alpha_1 \alpha_2 \alpha_3}$.

2.5 The field strength

By eqns. (43) and (46) it follows that $dB_{\alpha_1} = dB_{\alpha_2}$. This implies that the 3–form field strength $H$, contrary to the 2–form potential $B$, is globally defined on $P$. Consider now a 3–dimensional volume $V \subset P$ whose boundary is a 2–dimensional closed surface $S$. If $V$ is connected and simply connected we may, without loss of generality, take $V$ to be a solid ball, so $S = \partial V$ is a sphere. Let us cover $S$ by stereographic projection. This gives us two coordinate charts, respectively centred around the north and south poles on the sphere. Each chart is diffeomorphic to a copy of the plane $\mathbb{R}^2$. Each plane covers the whole $S$ with the exception of the opposite pole. The intersection of these two charts is the whole sphere $S$ punctured at its north and south poles. The situation just described is perfect for a discussion of eqn. (43). Let us embed the chart $\mathbb{R}^2_{\alpha_1}$, centred at the north pole within the open set $U_{\alpha_1}$, i.e., $\mathbb{R}^2_{\alpha_1} \subset U_{\alpha_1}$, if necessary by means of some diffeomorphism. Analogously, for the south pole we have $\mathbb{R}^2_{\alpha_2} \subset U_{\alpha_2}$. There is also no loss of generality in assuming that only two points on the sphere $S$ (the north and south poles) remain outside the 2–fold overlap $U_{\alpha_1} \cap U_{\alpha_2}$. By Stokes’ theorem,
\[ \int_V H = \int_V dB = \int_{\partial V} B = \int_S B = \int_{\mathbb{R}^2_{\alpha_2}} B - \int_{\mathbb{R}^2_{\alpha_1}} B. \] (49)
and, by eqn. \((43)\),
\[
\int_{\mathbb{V}} H = -\frac{i}{\hbar} \int_{\mathbb{R}^2 - \{0\}} \omega,
\]
where \(\mathbb{R}^2 - \{0\}\) denotes either one of our two charts, punctured at its corresponding origin. Since \(\mathbb{R}^2 - \{0\}\) falls short of covering the whole sphere \(\mathbb{S}\) by just two points (the north and south poles), and the latter have zero measure, we may just as well write
\[
\int_{\mathbb{V}} H = -\frac{i}{\hbar} \int_{\mathbb{S}} \omega.
\]

Eqn. \((51)\) is analogous to the Gauss law in electrostatics, with \(H\) replacing the electric charge density 3–form and \(\omega/\hbar\) replacing the corresponding surface flux 2–form. If our gerbe is nonflat, then \(H\) may be regarded as a source term for the quantum states arising from a nonvanishing flux of \(\omega/\hbar\) across the closed surface \(\mathbb{S}\). On the contrary, the gerbe is flat if and only if every closed surface \(\mathbb{S} \subset \mathbb{P}\) (satisfying the above requirements concerning \(\mathbb{V}\)) contributes no quantum states at all to the Hilbert space. This is tantamount to the statement that every closed surface \(\mathbb{S} \subset \mathbb{P}\) (satisfying the above requirements concerning \(\mathbb{V}\)) has zero symplectic area. In other words, the gerbe is flat if, and only if, open surfaces \(\mathbb{S}\) are the unique sources of quantum states. Then the mechanism responsible for the generation of quantum states is a nonvanishing symplectic area of the open surface \(\mathbb{S}\). Equivalently, by eqn. \((45)\), this mechanism is a nonvanishing circulation of the Poincaré–Cartan 1–form \(\lambda\) along its boundary \(\mathbb{L}\).

Now Heisenberg’s uncertainty principle implies a discretisation, or quantisation, of symplectic area in units of \(\hbar\). To begin with let us consider closed surfaces \(\mathbb{S}\) inside phase space. Then, within the WKB approximation \([9]\),
\[
\frac{1}{\hbar} \int_{\mathbb{S}} \omega = 2\pi n, \quad n \in \mathbb{Z}, \quad \partial \mathbb{S} = 0,
\]
which, by eqn. \((51)\), is tantamount to quantising the volume integral of \(H/2\pi\). In turn, this can be recast as the quantisation condition \([1, 11]\)
\[
\frac{1}{2\pi} \int_{\mathbb{V}} H \in \mathbb{Z}, \quad \partial \mathbb{V} = \mathbb{S},
\]
for all 3–dimensional, connected and simply connected volumes \(\mathbb{V} \subset \mathbb{P}\). Starting from Heisenberg’s principle we have obtained the quantisation condition \((53)\). Conversely, assume taking \((53)\) above as our starting point on phase space, and let us derive Heisenberg’s principle. Given a 3–dimensional volume \(\mathbb{V} \subset \mathbb{P}\) such that \(\partial \mathbb{V} = \mathbb{S}\), eqns. \((51)\) and \((53)\) imply that symplectic area is quantised on closed surfaces. This is an equivalent rendering of the uncertainty principle, at least on closed surfaces.

Now open surfaces within phase space have their symplectic area quantised according to the WKB rule \([9]\)
\[
\frac{1}{\hbar} \int_{\mathbb{S}} \omega = 2\pi \left( n + \frac{1}{2} \right), \quad n \in \mathbb{Z}, \quad \partial \mathbb{S} \neq 0.
\]
Notice the additional \(1/2\) in \((54)\) (open surfaces) as opposed to \((52)\) (closed surfaces). Consider now two open surfaces \(\mathbb{S}_1\) and \(\mathbb{S}_2\) such that \(\partial \mathbb{S}_1 = -\partial \mathbb{S}_2\). We can glue them
along their common boundary to produce a closed surface to which the quantisation condition \( \text{(52)} \) applies, hence \( \text{(53)} \) follows. Conversely, if we start off from a gerbe on \( \mathbb{P} \) satisfying eqn. \( \text{(53)} \), let us prove that symplectic area is quantised on open surfaces as well. Consider a fixed open surface \( S_1 \), plus a family of open surfaces \( S_2^{(i)} \) parametrised by a certain index \( i \), such that \( \partial S_2^{(i)} = -S_1 \) for all \( i \). Glue each \( S_2^{(i)} \) on to \( S_1 \) along the common boundary, in order to obtain a family of closed surfaces \( S^{(i)} \). Symplectic area is quantised on all of the latter. Now \( S_1 \) is fixed while the \( S_2^{(i)} \) are varied. As the index \( i \) is arbitrary, the variations in the \( S_2^{(i)} \), hence in the \( S^{(i)} \), can be made arbitrary. Meanwhile the symplectic area of the \( S^{(i)} \), which is the sum of the areas of \( S_1 \) and \( S_2^{(i)} \), remains quantised as per eqn. \( \text{(52)} \). This can only be the case if symplectic area is quantised on open surfaces as well. Strictly speaking, this argument only establishes that the symplectic area of open surfaces is quantised as \( 2\pi k \), where \( 2k \in \mathbb{Z} \). The additional \( 1/2 \) present in \( \text{(54)} \) follows when \( k \notin \mathbb{Z} \).

To summarise, eqn. \( \text{(53)} \) is an equivalent statement of Heisenberg’s uncertainty principle.

### 3 Outlook

A number of challenging questions arise.

We have worked in the WKB approximation; it would be interesting to compute higher quantum corrections to our results. Such corrections will generally depend on the dynamics. In this respect one could consider the approach of ref. [12], where Planck’s constant \( \hbar \) is regarded as a dynamically–generated quantum scale. What modifications of the uncertainty principle this may bring about in our setup remains to be clarified. Current field–theoretic and string models certainly do lead to such modifications.

According to conventional folklore, “the uncertainty principle prohibits quantum mechanics on phase space”. Here we have shown that endowing phase space with a gerbe and a connection is a way of quantising classical mechanics. In fact, phase space is becoming increasingly popular as a natural arena for quantum mechanics [13]. Our conclusions also contribute towards a modern geometric view of quantum mechanics, a beautiful presentation of which has been given in ref. [14]. Last but not least, the ideas explored here are connected, not as remotely as it may on first sight appear, with quantum theories of gravity [15].

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