AN IMBEDDING OF FRACTIONAL ORDER
SOBOLEV-GRAND LEBESGUE SPACES,
with constant evaluation.
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Abstract. We extend in this article the classical imbedding theorems for fractional
Lebesgue-Sobolev’s spaces into the so-called Grand Lebesgue spaces, with
sharp constant evaluation.

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1 Introduction. Notations. Problem Statement.

For the (measurable) numerical function $f : R^n \rightarrow R$ the Fourier transform
$F[f](\xi) = \hat{f}(\xi)$ is defined as ordinary:

$$\hat{f}(\xi) = F[f](\xi) := \int_{R^n} e^{-ix \cdot \xi} f(x) \, dx,$$

Hereafter $x \cdot \xi$ denotes the inner (scalar) product of two vectors $x, \xi \in R^d$ and
$|x|$ is ordinary Euclidean norm of the vector $x : |x| = \sqrt{x \cdot \bar{x}}.

The Fourier transform $F[f](\xi)$ is correctly defined, e.g. if $f \in \cup_{p \in [1,2]} L_p(R^n).
The norm of the function $f$ in the Lebesgue, more exactly, Lebesgue-Riesz space
$L_p = L_p(R^n), p \geq 1$ will be denoted for simplicity $|f|_p :$

$$|f|_p := \left( \int_{R^n} |f(x)|^p \, dx \right)^{1/p}.$$

Let $\Delta$ be the Laplacian. The fractional, in general case, power $\sqrt{-\Delta^s}$ may be defined
as a pseudo-differential operator through Fourier transform
The fractional Sobolev’s space $W^s_p = W^s_p(R^n)$ consists by definition on all the functions $f : R^n \rightarrow R$ with finite norm (more precisely, semi-norm)

$$
\| f \|_{W^s_p} \overset{def}{=} \| (-\Delta)^{s/2} [f] \|_p, \; p \geq 1,
$$

the Aronszajn-Gagliardo norm; which is equivalent to the Slobodetskii $\| \cdot \|_{S^s_p}$ semi-norm:

$$
\| f \|_{S^s_p} = \| f \|_{S^s_p(R^n)} \overset{def}{=} \left[ \int_{R^n} \int_{R^n} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \; dx \; dy \right]^{1/p}.
$$

More information about these spaces, in particular on the imbedding theorem see in the works [31], [32], [33], [35], [40], [41], [42], [43], [44], [45], [47], [48], [49], [50] etc.

**Remark 1.1.** In the definition (1.2) instead the whole space $R^n$ may be used arbitrary open set $\Omega \subset R^n$. In detail:

$$
\| f \|_{S^s_p(\Omega)} \overset{def}{=} \left[ \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} \; dx \; dy \right]^{1/p}.
$$

**Remark 1.2.** In the case when $s < 0$ the fractional Laplace operator $(-\Delta)^{-|s|/2}$ coincides with Riesz potential

$$
(-\Delta)^{-|s|/2}[f](x) = I_{|s|}[f](x) = C_R(n, s) \int_{R^n} \frac{f(y)}{|x - y|^{n-|s|}} \; dy.
$$

This case was detail investigated in many works, see e.g. [29], [34], [37], [39], [15], [16]. Therefore, we can assume further $s > 0$.

**Remark 1.3.** The ”complete” norm in the fractional Sobolev’s space may be introduced as follows.

$$
\| f \|_{V^s_p} \overset{def}{=} \left[ \| f \|_p^p + \| (-\Delta)^{s/2} [f] \|_p^p \right]^{1/p} = \\
\left[ \int_{R^n} |f(x)|^p \; dx + (\| f \| W^s_p) \right]^{1/p}, \; p \geq 1.
$$

**Remark 1.4.** The fractional Sobolev’s spaces are closely related with Besov spaces, see [1], page 330-341.

We will use further the following sharp imbedding theorem, see [37], [38], section 4.3. Define the following ”constant”

$$
K(n, s) := \pi^{s/2} \frac{\Gamma((n-s)/2)}{\Gamma((n+s)/2)} \left\{ \frac{\Gamma(n)}{\Gamma(n/2)} \right\}^{s/n},
$$

(1.4)
where $\Gamma(\cdot)$ denotes the usually Gamma-function. If

$$0 < s < n, \; 1 < p < n/s, \; u \in C_0^{\infty}(R^n), \; q = pn/(n - sp),$$

and $q = pn/(n - sp)$, then

$$|u|_q \leq K(n, s)\sqrt{-\Delta} u|_p = K(n, s)||u||W^s_p -$$

fractional Lebesgue-Sobolev’s imbedding theorem.

Note that the conditions (1.5) are also necessary for the inequality of a form (1.6) for some constant $K(n, s)$. This assertion may be proved by means of the well-known scaling method, see e.g. [28], [25].

Evidently, the inequality (1.6) holds true for all functions $u = u(x)$ belonging to the completion of the space $C_0^{\infty}$ relative the fractional semi-norm $|| \cdot ||W^s_p$.

Our aim in this article is to extrapolate the fractional Lebesgue-Sobolev’s imbedding theorem (1.6) into the so-called fractional Grand Lebesgu-Sobolev’s imbedding spaces, and as a particular case the - into the so-called Exponential Orlicz Spaces (EOS).

A particular (but weight) case was considered in the previous article [25].

2 Grand Lebesgue Spaces and Sobolev-Grand Lebesgue Spaces.

Now we will describe using Grand Lebesgue Spaces (GLS) and Sobolev’s Grand Lebesgue Spaces (SGLS).

1. Grand Lebesgue Spaces.

We recall in this section for reader conventions some definitions and facts from the theory of GLS spaces.

Recently, see [2], [3], [4], [5], [6], [7], [8], [10], [11], etc. appears the so-called Grand Lebesgue Spaces $GLS = G(\psi) = G\psi = G(\psi; A, B), \; A, B = \text{const}, A \geq 1, A < B \leq \infty$, spaces consisting on all the measurable functions $f : R^n \rightarrow R$, (or more generally $f : \Omega \rightarrow R$) with finite norms

$$||f||_{G(\psi)} \overset{def}{=} \sup_{p \in (A, B)} ||f|_p/\psi(p)||.$$

Here $\psi(\cdot)$ is some continuous positive on the open interval $(A, B)$ function such that
\[
\inf_{p \in (A,B)} \psi(p) > 0, \quad \psi(p) = \infty, \quad p \notin (A,B).
\]

We will denote
\[
\text{supp}(\psi) \overset{\text{def}}{=} (A,B) = \{ p : \psi(p) < \infty, \}
\]
(2.2)

The set of all \(\psi\) functions with support \(\text{supp}(\psi) = (A,B)\) will be denoted by \(\Psi(A,B)\).

This spaces are rearrangement invariant, see [1], and are used, for example, in the theory of probability [7], [10], [11]; theory of Partial Differential Equations [3], [6]; functional analysis [4], [5], [8], [11], [14]; theory of Fourier series [10], theory of martingales [11], mathematical statistics [12], [13]; theory of approximation [19] etc.

Notice that in the case when \(\psi(\cdot) \in \Psi(A,B)\), a function \(p \to p \cdot \log \psi(p)\) is convex, and \(B = \infty\), then the space \(G\psi\) coincides with some \textit{exponential} Orlicz space.

Conversely, if \(B < \infty\), then the space \(G\psi(A,B)\) does not coincides with the classical rearrangement invariant spaces: Orlicz, Lorentz, Marcinkiewicz etc.

\textbf{Remark 2.1.} If we define the \textit{degenerate} \(\psi_r(p), r = \text{const} \geq 1\) function as follows:
\[
\psi_r(p) = \infty, \quad p \neq r; \quad \psi_r(r) = 1
\]
and agree \(C/\infty = 0, C = \text{const} > 0\), then the \(G\psi_r(\cdot)\) space coincides with the classical Lebesgue space \(L_r\).

\textbf{Remark 2.2.} Let \(\xi : \Omega \to R\) be some (measurable) function from the set \(L(p_1,p_2), 1 \leq p_1 < p_2 \leq \infty\). We can introduce the so-called \textit{natural} choice \(\psi_\xi(p)\) as as follows:
\[
\psi_\xi(p) \overset{\text{def}}{=} |\xi|_p; \quad p \in (p_1,p_2).
\]

\section{Sobolev-Grand Lebesgue (SGL) spaces.}

Let \(\psi = \psi(p)\) be the function described above. We will say that the function \(f : R^d \to R\) belongs to the Sobolev-Grand Lebesgue space \(\text{SGL}\psi\), iff the following it semi-norm is finite:
\[
||u||_{\text{SGL}\psi} \overset{\text{def}}{=} \sup_{p \in \text{supp}(\psi)} \left[ \frac{||u||_{W^s_p}}{\psi(p)} \right] .
\]
(2.3)

This notion (up to equivalence) for the integer values \(s\) appeared at first (presumably) in an article [26] (2010); the near definition see in [9] (2013).

\section{Main result.}

\textbf{Theorem 2.1.} Let \(\psi(\cdot) \in \Psi(1,n/s)\) where \(0 < s < n, \ 1 < p < n/s\). Define the function
\[
\nu(q) := \psi \left( \frac{qn}{n + qs} \right),
\]
(2.4)
so that
\[ \text{supp } \nu(\cdot) = ((n/(n-s), \infty)). \]

Let also
\[ u \in C_0^\infty(R^n) \cap SGL\psi_s. \quad (2.4) \]

**Proposition:**
\[ ||u||_{G\nu} \leq K(n,s) \cdot ||u||_{SGL\psi_s}, \quad (2.5) \]
where the constant \( K(n,s) \) is the best possible.

**Proof.** Let \( u \in C_0^\infty(R^n) \cap SGL\psi_s \); we can and will suppose without loss of generality \( ||u||_{SGL\psi_s} = 1 \). It follows from direct definition of Sobolev-Grand Lebesgue spaces
\[ ||u||_{W^s_p} \leq \psi(p). \]
It follows from inequality (1.6)
\[ |u|_q \leq K(n,s)||u||_{W^s_p} \leq K(n,s)\psi(p). \quad (2.6) \]
Since \( p = qn/(n+qs) \), we deduce from (2.6) for the values \( q > n/(n-s) \)
\[ |u|_q \leq K(n,s)\psi(qn/(n+qs)) = K(n,s)\nu(q) = K(n,s) \nu(q) \cdot ||u||_{SGL\psi_s}, \quad (2.7) \]
or equally
\[ ||u||_{G\nu} \leq K(n,s) \cdot ||u||_{SGL\psi_s}, \]
The exactness of the constant \( K(n,s) \) in (2.5) follows immediately from the one of main results, namely, theorem 2.1, of an article [27].

Note that in the case when \( \psi(p) = \psi_r(p) \), \( r = \text{const} \geq 1 \) we obtain as a particular case the ordinary fractional Sobolev’s imbedding theorem.

### 3 Boundedness of fractional Laplacian in DGLS

Let us return to the fractional Sobolev’s inequality (1.6):
\[ |u|_q \leq K(n,s)||\nabla^s u||_p = K(n,s)||u||_{W^s_p}. \quad (1.6) \]
Assume now that the inequality (1.6) is true for some interval of values \( s : \ s \in (s_-, s_+), \ 0 < s_- < s_+ < 1. \)
More detail, let \( Q = Q(s_-, s_+) \) be some (measurable) set in the plane \((p, s), p \geq 1, s \in (0, 1)\), \( Q = \{(p, s)\} \) such that
\[
\forall s \in (s_-, s_+) \Rightarrow \exists p \geq 1, \ u \in W_p^s.
\] (3.1)

If for some values \((p, s)\) \( u \notin W_p^s\), we denote formally \( ||u||_{W_p^s} = \infty \).

Denote
\[
Q_s = \{p, p \geq 1, (p, s) \in Q\};
\]
the "section" of the set \( Q \) on the \( s \) - level. Then \( \forall s \in (s_-, s_+) \Rightarrow Q_s \neq \emptyset \).

The inequality (1.6) may be rewritten as follows:
\[
|u|_q \leq K(n, s) ||u||_{W_q^n/(n+qs)}, \ s \in (s_-, s_+),
\]
therefore
\[
|u|_q \leq \inf_{s \in (s_-, s_+)} \left[ K(n, s) ||u||_{W_q^n/(n+qs)} \right]. \tag{3.2}
\]

The last inequality be reformulated on the language of Grand Lebesgue spaces as follows. Denote
\[
\zeta(q) = \inf_{s \in (s_-, s_+)} \left[ K(n, s) ||u||_{W_q^n/(n+qs)} \right], \ q \in (n/(n-s_+), \infty),
\]
then \( ||u||_{G\zeta} \leq 1 \).

**Definition of Derivative Grand Lebesgue spaces.**

Let \( \tau = \tau(p, s), p > 1, s \in (s_-, s_+) \) be continuous function such that 
\[
\inf_{p,s} \tau(p,s) = 1.
\]
By definition, the function \( u = u(x), x \in \mathbb{R}^n \) (or \( x \in \Omega \)) belongs to the Derivative Grand Lebesgue space \( DGL(\tau) \), if it has a finite semi - norm
\[
||u||_{DGL\tau} := \sup_{p>1} \sup_{s \in (s_-, s_+)} \left[ \frac{||u||_{W_p^s}}{\tau(p, s)} \right]. \tag{3.3}
\]

We can now formulate the imbedding theorems in Derivative Grand Lebesgue spaces.

**Theorem 3.1.** Let \( u(\cdot) \in DGL(\tau); \) then
\[
|u|_q \leq \inf_{s \in (s_-, s_+)} \left[ K(n, s) \tau(qn/(n+qs), s) \right] \cdot ||u||_{DGL\tau}. \tag{3.4}
\]

**Proof** is alike one in the theorem 2.1. Indeed, let \( ||u||_{DGL\tau} = 1 \); then
\[
||u||_{W_p^s} \leq \tau(p, s), \ p = qn/(n+qs).
\]
It remains to use the inequality (3.2) and take the minimum over \( s \).
Corollary 3.1. The inequality (3.4) may be reformulated on the language of GL spaces as follows. Denote

$$\lambda(q) = \inf_{s \in (s-, s_+)} \left[ K(n, s) \tau(qn/(n + qs), s) \right];$$

then

$$|u|_q \leq \lambda(q) \cdot |||u|||_{DGL\tau},$$
or equally

$$|||u|||_{G\lambda} \leq |||u|||_{DGL\tau}. \quad (3.5)$$

4 Weight generalization

Let \( \Omega \) be an open convex subset of a whole space \( R^n \). Introduce after R. L. Frank and R. Seiringer [33] (case \( \Omega = R^n_+ \)) and M. Loss and C. Sloane [39] (general case) the following functions, measures and operators:

$$d_\alpha(x) := \inf_{y \notin \Omega} |x - y|^\alpha,$$

$$D_{\alpha,n}(p) := 2\pi^{(n-1)/2} \frac{\Gamma((1 + \alpha)/2)}{\Gamma((n + \alpha)/2)} \int_0^1 \frac{|1 - r^{(\alpha-1)/p}|^p}{(1 - r)^{1+\alpha}} \, dr, \ \alpha = \text{const} \in (1, p);$$

$$g_{\alpha,n}(p) = [D_{\alpha,n}(p)]^{-1/p},$$

$$\mu_\alpha(A) := \int_A \frac{dx}{d_\alpha(x)}, \ \ A \subset \Omega;$$

$$\nu_\alpha(B) := \int \int_B \frac{dx \, dy}{|x - y|^{n+\alpha}}, \ \ B \subset \Omega \times \Omega.$$  

$$\delta[f](x, y) := f(x) - f(y), \ \ f : R^n \rightarrow R. \quad (4.1)$$

The fractional weight Sobolev’s type inequality

$$|f|_{L_p(R^n, \mu_\alpha)} \leq g_{\alpha,n}(p) \cdot \delta[f] |L_p(R^n \times R^n, \nu_\alpha) \quad (4.2)$$

was proved by R. L. Frank and R. Seiringer [33] (case \( \Omega = R^n_+ \)) and M.Loss and C.Sloane [39] (general case). See also [32].

Theorem 4.1. Let the function \( \delta[f](\cdot, \cdot) \) belongs to some space \( G\psi \) on the set \( \Omega \times \Omega \) relative the measure \( \nu_\alpha \). Put
\[ \theta(p) = g_{\alpha,n}(p) \cdot \psi(p). \]

**Proposition:**

\[
\|f\|_{G\theta(\Omega, \mu_\alpha)} \leq 1 \cdot \|\delta[f]\|_{G\psi(\Omega \times \Omega, \nu_\alpha)},
\]

(4.3)

where the constant ”1” in (4.3) is the best possible.

**Proof** is at the same as in theorem 2.1 and may be omitted.

## 5 Auxiliary results

**Constants** \( K(n, s) \).

As long as \( \Gamma(\epsilon) \sim 1/\epsilon, \ \epsilon \to 0^+ \), we deduce at \( s \to n - 0 \)

\[
K(n, s) \sim \frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{1}{n-s}.
\]

The point \( s = n - 0 \) is unique point of singularity for the function \( s \to K(n, s) \); for instance, \( K(n, 0) = K(n, 0+) = 1 \).

**Constants** \( D_{\alpha,n}(p) \).

Denote for the values \( \alpha = \text{const} > 1 \)

\[
L_\alpha(p) = \int_0^{1} \frac{|1 - r^{(\alpha-1)/p}|^p}{(1-r)^{1+\alpha}} \, dr, \ p \in (\alpha, \infty).
\]

Recall that

\[
D_{\alpha,n}(p) := 2\pi^{(n-1)/2} \frac{\Gamma((1+\alpha)/2)}{\Gamma((n+\alpha)/2)} L_\alpha(p).
\]

The extreme points \( p = \alpha + 0 \) and \( p \to \infty \) are points of singularity for the function \( p \to L_\alpha(p) \).

**A. Case** \( p \to \alpha + 0 \).

We have:

\[
r^{(\alpha-1)/p} = e^{\ln r^{(\alpha-1)/p}} \sim 1 - \frac{\ln r(\alpha - 1)}{p},
\]

therefore
\[ L_\alpha(p) \sim \frac{(\alpha - 1)^p}{p^p} \int_0^1 \frac{|\ln r|^p}{(1-r)^{1+\alpha}} dr \sim \frac{(\alpha - 1)^p}{p^p} \int_0^1 (1-r)^{p-\alpha} dr = \frac{(\alpha - 1)^p}{p^p} \cdot (p - \alpha)^{-1}. \]

**B. Case \( p \to \infty. \)**

We find:

\[ L_\alpha(p) \sim \frac{(\alpha - 1)^p}{p^p} \int_0^1 \frac{|\ln r|^p}{(1-r)^{1+\alpha}} \sim \frac{(\alpha - 1)^p}{p^p} \int_0^1 \frac{\ln r}{(1-r)^{1+\alpha}} + \frac{\ln r}{(1-r)^{-1+\alpha}} \frac{1}{(1-\Delta)^{1+\alpha}} \cdot \Gamma(p+1). \]

**C. Non-asymptotical approach.**

We will use the following elementary estimate:

\[ 1 - \sinh(1) \leq e^{-\epsilon} \leq 1 - \epsilon, \quad \epsilon \in [0, 1]. \]

Let \( \Delta = \text{const} \in (0, 1); \) for example, \( \Delta = \Delta_0 = 1/2. \) We calculate:

\[ J := \int_0^1 \frac{|\ln r|^p}{(1-r)^{1+\alpha}} dr = \int_1^\Delta dr + \int_\Delta^1 dr = J_1 + J_2; \]

\[ J_1 \leq (1-\Delta)^{-1-\alpha} \int_0^\Delta |\ln r|^p dr \leq (1-\Delta)^{-1-\alpha} \int_0^1 |\ln r|^p dr = \frac{\Gamma(p+1)}{(1-\Delta)^{1+\alpha}}; \]

\[ J_2 = \int_\Delta^1 \frac{|\ln r|^p}{(1-r)^{1+\alpha}} dr \leq \frac{|\ln \Delta|^p}{((1-\Delta)^p)} \frac{1}{p - \alpha}, \]

so

\[ J \leq \frac{\Gamma(p+1)}{(1-\Delta)^{1+\alpha}} + \frac{|\ln \Delta|^p}{((1-\Delta)^p)} \frac{1}{p - \alpha}, \]

following

\[ L_\alpha(p) \leq \frac{(\alpha - 1)^p}{p^p} \cdot \left[ \frac{\Gamma(p+1)}{(1-\Delta)^{1+\alpha}} + \frac{|\ln \Delta|^p}{((1-\Delta)^p)} \frac{1}{p - \alpha} \right]. \]

If we choose \( \Delta = 1/2, \) then
\[ L_\alpha(p) \leq \left( \frac{\alpha - 1}{p^p} \right) \cdot \left[ 2^{1+\alpha} \Gamma(p + 1) + (2 \ln 2)^p \frac{1}{p - \alpha} \right], \]

and analogously

\[ L_\alpha(p) \geq C^p(\alpha) \cdot \left( \frac{\alpha - 1}{p^p} \right) \cdot \left[ \Gamma(p + 1) + \frac{1}{p - \alpha} \right], \quad C(\alpha) \in (0, 1). \]

D. Constant of L. Cafarelli, E. Valdinoci, O. Savin.

There exists a "constant" \( Z = Z(n, s, p), \ s \in (0, 1), \ p \geq 1 \) such that for all measurable set \( E \subset \mathbb{R}^n \) with positive finite measure \( |E| \)

\[ \int_{\mathbb{R}^n \setminus E} \frac{dy}{|x - y|^{n+sp}} \geq Z(n, s, p) |E|^{-sp/n}, \]

see, e.g. [30], [46]. This constant play very important role in the theory of imbedding of fractional Sobolev’s spaces [43].

We will understand as a capacity of the value \( Z(n, s, p) \) its maximal value, i.e.

\[ Z(n, s, p) \overset{def}{=} \inf_{x \in E} \inf_{|E| \in (0, \infty)} \left\{ \int_{\mathbb{R}^n \setminus E} \frac{dy}{|x - y|^{n+sp}} : |E|^{-sp/n} \right\}. \]

Denote also as usually \( \omega_n = \frac{2\pi^{n/2}}{\Gamma(n/2)} \)

the area of surface of unit sphere \( \mathbb{R}^n \); recall that the volume of unit ball in this space is equal to \( \omega_n/n. \)

**Proposition 5.1.**

\[ (sp)^{-1} \omega_n^{1+sp/n} n^{-1-sp/n} \leq Z(n, s, p) \leq (sp)^{-1} \omega_n^{1+sp/n} n^{-sp/n}. \]

The left - hand side follows immediately from lemma 6.1 in the article [43] after simple calculations; the right - hand side may be obtained by choosing \( x = 0 \) and \( E = \{ y : |y| \leq 1. \} \)

Obviously, the upper bound in the last inequality is attainable.

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