A Bayesian level set method for inverse source scattering problems with multi-frequencies

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Abstract. This paper concerns the reconstruction of the support of source function based on the Bayesian level set approach. The unknown source to be reconstructed is assumed to be piecewise constant with a known value. In this setting, the support of the source function can be characterized by the level set functions. In the Bayesian level set inversion, the solution of the inverse problem is posterior distribution. The Markov Chain Monte Carlo (MCMC) algorithm is applied to generate the samples of the posterior distribution. The numerical results show the effectiveness of the proposed method and the dependence of the posterior samples on the flexible and proper smoothness priors with the Whittle-Matérn Gaussian random fields.

1. Introduction
Inverse source problems (ISP) have many important applications, such as antenna synthesis, biomedical engineering, and sound source localization [1-3], which provide motivation for the study of these problems. The ISP have been extensively investigated on uniqueness [4-7], stability analysis [8-10] and numerical approaches [4, 8, 11, 12].

For the computational methods, we can roughly divide into two categories for solving the ISP: iterative methods [4, 8, 12, 13] and direct methods [14, 15]. The iterative method transforms the inverse problem into a nonlinear optimization problem. For each iteration, a sequence of direct problems and the corresponding adjoint equations need to be solved. Direct methods need no direct solvers and characterize the geometry by designing an imaging function that peaks near the target boundary.

In this paper, we consider a statistical approach solving the ISP, especially from the Bayesian level set inversion. We characterize the source function by using the level set map. Our aim is to determine the support of the source function which is represented by the level set function from the multi-frequency data. The classical level set method is originally designed to track the wave front interface [16]. Recently, it has been extensively used to solve inverse problems involving obstacles [17-23]. Although the level set method provides large degrees of flexibility in shape reconstruction, there are numerical concerns associated with the level set equations. Besides, these studies involving shape reconstruction problems rarely analyze the uncertainty of the unknown variables. The uncertainty is often characterized by some statistical techniques. To quantify the uncertainty, the Bayesian approach is coupled with the level set method [24-28]. The proposed method needs no implementation of the Fréchet derivative of the forward map as well as the corresponding adjoint operator. And moreover the solution in Bayesian level set inversion is posterior distribution, which can provide the estimation for the unknown variables and their reliability information. To our best knowledge, there are few studies
of the Bayesian level set method solving inverse scattering problems [29, 30]. In [31], it establishes the mathematical foundations of the Bayesian level set method, and its hierarchical extension is studied [32]. In [29, 30], the Bayesian method and the ensemble Kalman filter approach based on level set parameterization are introduced for acoustic source identification by using multiple frequency information, respectively.

In Bayesian level set inversion, all unknowns are treated as random variables. According to the Bayes’ formula, we marry data with prior beliefs to produce the posterior distribution. Here, we use the Whittle-Matérn Gaussian random fields as the prior of the level set function [33, 34]. Draws from these Gaussian random fields can be obtained by the use of Karhunen-Loève expansion or a related stochastic differential equation. The Karhunen–Loève expansion exploits knowledge of the eigenfunctions and eigenvalues of the covariance operator to construct series with random coefficients [31, 32, 35]. The stochastic differential equation can be solved by the finite difference method or the finite element approach [34]. In our problem, we will focus primarily on solving the stochastic differential equation. In order to generate the samples of the posterior distribution, we employ a computationally expedient algorithm, the preconditioned Crank-Nicolson Markov chain Monte Carlo (pCN-MCMC) algorithm [36, 37].

The rest of the paper is organized as follows. In section 2, we provide a description of the forward model. The open domain needs to be truncated into a bounded domain. Based on the Dirichlet-to-Neumann (DtN) map [38], we can reduce the original scattering problem to a nonlocal boundary value problem on a bounded domain. Section 3 is devoted to the Bayesian level set approach to the inverse source scattering problem. In section 4, some numerical results are shown to illustrate the effectiveness of the proposed method.

2. Inverse source problem
In this section, we introduce the ISP that determine the support of the source function from the noisy multi-frequency scattering data. Let \( B_R = \{ x \in \mathbb{R}^2 : |x| \leq R \} \) with radius \( R \). Denote by \( \Gamma_R \) the boundary of \( B_R \). Let \( H^2(\mathbb{R}^2) \) be the standard Sobolev space.

Consider the following two-dimensional Helmholtz equation

\[
\Delta u^s + k^2 u^s = f \quad \text{in} \; \mathbb{R}^2,
\]

where \( k \) is the wavenumber of the radiated scalar field \( u^s \), the source function \( f \) is assumed to have a compact support contained in \( B_R \). The radiated field \( u^s \) is required to satisfy the Sommerfeld radiation condition

\[
\lim_{r \to \infty} r^{-\frac{1}{2}} \left( \frac{\partial u^s}{\partial r} - ik u^s \right) = 0 \quad r = |x|,
\]

uniformly along all directions \( \hat{x} = \frac{x}{|x|} \).

To solve the problem (2.1) - (2.2), we reduce the original scattering problem on a bounded domain by introducing a transparent boundary conditions (TBC) on an artificial boundary \( \Gamma_R \). Here, we formulate the TBC by the DtN operator.

In the domain \( \mathbb{R}^2 \setminus B_R \), the solution of the (2.1) has the Fourier series expansions in the polar coordinates:

\[
u^s(r, \theta) = \sum_{n \in \mathbb{Z}} \frac{H_n^{(1)}(kr)}{\mu_n^{(1)}(kr)} \tilde{u}^s_n e^{i n \theta}, \quad \tilde{u}^s_n = \frac{1}{2 \pi} \int_0^{2 \pi} u^s(R, \theta) e^{i n \theta} d\theta,
\]

where \( H_n^{(1)} \) is the Hankel function of the first kind with order \( n \).

Let \( T : H^2(\Gamma_R) \to H^1(\Gamma_R) \) be the DtN operator defined as follows: for any \( u^s \in H^2(\Gamma_R) \),

\[
Tu^s = \sum_{n \in \mathbb{Z}} \frac{H_n^{(1)}(kr)}{\mu_n^{(1)}(kr)} \tilde{u}^s_n e^{i n \theta}.
\]

Using the DtN operator, we get the transparent boundary condition

\[
\frac{\partial u^s}{\partial n} = Tu^s \quad \text{on} \; \Gamma_R,
\]

where \( n \) is the unit outward normal on \( \Gamma_R \).
Based on the transparent boundary condition (2.5), the original scattering problem is equally reduced to the following problem defined on the bounded domain (13)

\[
\begin{align*}
\Delta u^s + k^2 u^s &= f & \text{in } B_R, \\
\frac{\partial u^s}{\partial n} &= T u^s & \text{on } \Gamma_R.
\end{align*}
\] (2.6)

Then the variation formulation of (2.6) reads as follow:

\[
a(u^s, v) = b(v),
\] (2.7)

where the bilinear form \( a: H^1(B_R) \times H^1(B_R) \rightarrow \mathbb{C} \) is defined by

\[
a(u^s, v) = \int_{B_R} \nabla u^s \cdot \nabla \bar{v} dx - k^2 \int_{B_R} u^s \bar{v} dx - \int_{\Gamma_R} T u^s \bar{v} ds,
\]

and the linear function is on \( H^1(B_R) \)

\[
b(v) = -\int_{B_R} f \bar{v} dx.
\]

Given the source function \( f \), the direct problem is to find the radiated field \( u^s \). The ISP is to determine the support of the source. In order to overcome the ill-posedness and non-uniqueness of the ISP, we take measurements with multiple frequencies [13].

3. Bayesian level set inversion

For the problem (2.6), define the forward operator \( G:X \rightarrow Y \), we have the observed data \( y \) given by

\[
y = G(f) + \eta.
\] (3.1)

where \( X = L^2(B_R), Y = \mathbb{C}^l \) and \( \eta \sim \mathcal{N}(0,2) \), \( \Sigma \in \mathbb{R}^{l \times l} \), is additive Gaussian noise. We obtain that the ISP is to find \( f \) from \( y \).

Assuming that the source function \( f \) has the form (27)

\[
f = \sum_{i=1}^{n} a_l \mathbb{I}_{D_l},
\] (3.2)

where \( \mathbb{I}_{D_l} \) denotes the indicator function of subset \( D_l \subset \mathbb{R}^2 \), \( \{D_l\}_{l=1}^n \) are subsets of \( B_R \) such that \( \bigcup_{l=1}^{n} D_l = B_R \) and \( D_l \cap D_l = \emptyset \), \( l \neq j \), and \( \{a_l\}_{l=1}^n \) are known positive constants. In this setting, the unknown is the domain \( D_l \). It is natural to represent the shape of the domains by using level set functions.

Define the level set function by the following way

\[
D_l = \{ x \in B_R | c_{l-1} \leq \phi(x) < c_l \} \quad l = 1,2,\cdots,n
\] (3.3)

where \(-\infty = c_0 < c_1 < \cdots < c_n = \infty \) are constants. Then we define the level set map \( F: X \rightarrow \mathbb{R} \) by

\[
F(\phi) = f
\] (3.4)

where \( X = C(\mathbb{B}_R) \). Combining (3.1) with (3.4), we reformulate the ISP in terms of the level set function \( \phi \): find \( \phi \) from \( y \) where

\[
y = G(\phi) + \eta = G \circ F(\phi) + \eta.
\] (3.5)

In the Bayesian framework, the qualities \( \phi, y, \eta \) in (3.5) are viewed as random variables. Assume that \( \phi \sim \mu_0 \) (with density \( \pi_0 \)) is independent of \( \eta \). The solution of the inverse problem is the posterior distribution \( \mu \) (with density \( \pi_\mu \)) of \( \phi | y \). According to the Bayes’ formula, we get

\[
\pi_\mu = \frac{\pi(y|\phi)\pi_\phi}{\int_X \pi(y|\phi)\pi_\phi d\phi} \propto \pi(y|\phi)\pi_\phi,
\] (3.6)

where \( \pi(y|\phi) \) is the likelihood function, \( \pi_\phi \) denotes the prior density, and \( \int_X \pi(y|\phi)\pi_\phi d\phi \) is a normalizing constant. Since the noise is the additive Gaussian, the likelihood function can be written as

\[
\pi(y|\phi) = \exp(-\frac{1}{2} \| y - G(\phi) \|^2_2) = \exp(-\Phi(\phi; y))
\] (3.7)

where \( \| \cdot \|^2_2 = \| \Sigma^{\frac{1}{2}} \cdot \| \) denotes the weighted norm, \( \Phi(\phi) \) is the negative log-likelihood.

In this paper, we take the Whittle-Matérn Gaussian random field as the prior. The covariance function has the form of (32)

\[
c(x,y) = \sigma^2 \frac{\Gamma^{1-a+d/2}}{\Gamma(a-d/2)} \left( \frac{|x-y|}{l} \right)^{a-d/2} K_{a-d/2} \left( \frac{|x-y|}{l} \right)
\] (3.8)
where \( K \) is the modified Bessel function of the second kind, \( l \) is the length scale, \( \sigma^2 > 0 \) is the variance, \( \alpha \) controls the sample regularity, and \( \Gamma(\cdot) \) denotes the Gamma function. On \( \mathbb{R}^d \), we generate the samples from the Gaussian random field prior by solving the stochastic partial differential equation

\[
(I - l^2 \Delta)^{\frac{\alpha}{2}} \phi = \sqrt{\xi} l^d W,
\]

where \( \xi \) denotes the Gaussian white noise on \( \mathbb{R}^2 \), and

\[
\xi = \sigma^2 z^D \frac{d}{\Gamma(\alpha - d/2)}.
\]

From (3.9), we have

\[
C_{\alpha, \tau}^{-1} \phi = w
\]

where \( \tau = \frac{1}{l} > 0 \), and the covariance operator \( C_{\alpha, \tau} \) has the form \( C_{\alpha, \tau} = \tau^{2\alpha - d} \xi (\tau^2 I - \Delta)^{-\alpha} \). Choosing \( \xi = \tau^{-2\alpha + d} \), we obtain

\[
C_{\alpha, \tau} = (\tau^2 I - \Delta)^{-\alpha}.
\]

We apply the finite element method with suitable boundary conditions to solve (3.9) ([34]). In Figure 1, we show the random samples from (3.9) with imposed Neumann boundary conditions.

**Figure 1.** Samples from the prior with inverse length scales \( \tau = 20, 10, 5, \alpha = 4 \).

4. Numerical experiments

We generate a sequence of samples from the posterior distribution given by (3.6). The preconditioned Crank-Nicolson (pCN) MCMC method is applied to explore the information of posterior distribution.

**Algorithm 1 (pCN-MCMC)**

1. Set \( s = 0 \). Choose an initial state \( \phi^{(0)} \in \mathcal{X} \).
2. Fix jump parameter \( \beta \in (0, 1] \), and propose \( \psi^{(s)} = \sqrt{1 - \beta^2} \phi^{(s)} + \beta \psi^{(s)} \), \( \xi \sim N(0, \tilde{C}_{\alpha, \tau}) \).
3. Set \( \phi^{(s+1)} = \psi^{(s)} \) with probability

\[
\gamma(\phi^{(s)}, \psi^{(s)}) = \min\{1, \exp(\Phi(\psi^{(s)}) - \Phi(\phi^{(s)}))\},
\]

independently of \( (\phi^{(s)}, \psi^{(s)}) \).
4. Set \( \phi^{(s+1)} = \phi^{(s)} \) otherwise.
5. Set \( s \to s + 1 \) and return to 2.

In the following, we illustrate the effectiveness of the above algorithm by performing three examples. We assume that the domain \( B_R = \{ x \in \mathbb{R}^2, |x| \leq 1 \} \). For the forward solver, the finite element method is applied with a DtN technique. The observational data \( y \) is measured on the boundary \( \chi_j \in \Gamma_R, \ j = 1, \cdots, 128 \), and the wavenumber \( k_l = \frac{\pi}{\delta} + ia, i = 1, \cdots, 25 \). The noise \( \eta \) is Gaussian distribution, \( \eta \sim N(0, \delta^2 I) \), where \( \delta = 0.005 \). We use a mesh of 16512 elements to create the data. In order to avoid an inverse crime, a mesh of 4128 elements is used for the inversion via the MCMC approach. The jump parameter is \( \beta = 0.007 \).

**Example 1** The true source function has the form of

\[
\{ 1, \ x \in D, \}
\]

where the true \( D = \{ x \in \mathbb{R}^2 | |x| \leq 0.25 \} \) is a circle. We parameter the source function in terms of the level set function given by \( D = \{ x \in B_R | \phi(x) \geq 0 \} \). In Figure 2, we show the true source function,
and we use the mentioned above algorithm to generate the samples of posterior distribution with \( \tau = 20, 10, 5 \). The results displayed in Figure 2 show the effect that the selection of the prior has on the posterior distribution. Fixing \( \tau = 10 \) and \( \alpha = 4 \), we display some statistical information that shows the trace plot and the corresponding Auto-correlation function (ACF) for the negative log-likelihood \( \Phi(\phi) \) (or ‘data-misfit’), as seen in Figure 3. From the trace plot of data misfit, we can see the pCN-MCMC needs several iterations to reach the stationary stage. The auto-correlation functions of selected samples decay with lag.

**Figure 2.** The top row is the true source function, and the bottom is the reconstruction of the source function with \( \tau = 20, 10, 5, \alpha = 4 \) (from left to right).

**Figure 3.** The trace plot of data misfit function and the corresponding ACF function.

**Example 2** The true source function can be written as

\[
    f = \begin{cases} 
        2, & x \in \mathcal{D}, \\
        0, & x \in \mathcal{B}_R \setminus \mathcal{D}.
    \end{cases}
\]

The parameterization of \( \mathcal{D} \) in terms of the level set function is given by \( \mathcal{D} = \{ x \in \mathcal{B}_R | \phi(x) \geq 0 \} \). In Figure 4, we present the true source function and the numerical results with different priors corresponding to the aforementioned choices of \( \tau \). It shows that the capability of the proposed approach to properly identify a shape \( \mathcal{D} \). Figure 5 shows the trace plot and corresponding
autocorrelation functions for the negative log-likelihood $\Phi(\phi)$ with $\tau = 5$ and $\alpha = 4$. The trace plot of data misfit has a stable chain after some iterations. And the ACF decays with the lag.

![Figure 4](image1.png)

**Figure 4.** The top row is the true source function, and the bottom is the reconstruction of the source function with $\tau = 20, 10, 5, \alpha = 4$ (from left to right).

![Figure 5](image2.png)

**Figure 5.** The trace plot of data misfit function and the corresponding ACF function.

**Example 3** The source function is defined by

$$f = \begin{cases} 1, & x \in D, \\ 0, & x \in B_r \setminus \bar{D}, \end{cases}$$

where $D = \{(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2 | (x_1 - 0.3)^2 + (y_1 + 0.3)^2 \leq 0.09, \ (x_2 + 0.3)^2 + (y_2 - 0.3)^2 \leq 0.09 \}$. Similar to the previous example, we use a prior of the form (3.10) for the level set function. In Figure 6, we display samples from experiments with $\tau = 20, 10, 5, \alpha = 4$. Taking $\tau = 10$ and $\alpha = 4$, we display the trace plot of data misfit and the corresponding Auto-correlation function (ACF) for the negative log-likelihood in Figure 7. Our results offer evidence that this uncertainty can be properly captured with our Bayesian level set framework.
Figure 6. The top row is the true source function, and the bottom row is the reconstruction of the source function with $\tau = 20, 10, 5, \alpha = 4$ (from left to right).

Figure 7. The trace plot of data misfit function and the corresponding ACF function.

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