RADIALITY OF DEFINABLE SETS

JOHN WELLIAVEETIL

Abstract. In this article we use techniques developed by Hrushovski-Loeser to study certain metric properties of the Berkovich analytification of a finite morphism of smooth connected projective curves. In recent work, M. Temkin proved a radiality statement for the topological ramification locus associated to such finite morphisms. We generalize this result in two directions. We prove a radiality statement for a more general class of sets which we call definable sets. In another direction, we show that the result of Temkin can be obtained in families.

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1. Introduction

It is one of the many redeeming features of non-Archimedean geometry that the analytification of an algebraic curve is endowed with a structure that can be described in fairly explicit terms. For example, if \( k \) is an algebraically closed field complete with respect to a non-trivial non-Archimedean real valuation then every smooth curve \( C \) over \( k \) possesses a skeleton \( \Sigma^{1} \). In particular, \( C^{\text{an}} \prec \Sigma \) decomposes into the disjoint union of isomorphic copies of the Berkovich open unit disk and \( C^{\text{an}} \) admits a deformation retraction onto \( \Sigma \) (cf. [1, Lemma 3.4]).

Skeleta can also be used to shed light on the structure of a finite morphism \( f : C' \to C \) of smooth \( k \)-curves. For instance, it is a well known fact that there exists skeleta \( \Sigma \subset C^{\text{an}} \) and \( \Sigma' \subset C'^{\text{an}} \) such that \( \Sigma' = (f^{\text{an}})^{-1}(\Sigma) \) and the deformation retractions of \( C'^{\text{an}} \) onto \( \Sigma' \) and \( C^{\text{an}} \) onto \( \Sigma \) can be taken to be compatible for the

\footnote{Refer [4, §3.5.1] for a definition of skeleton.}
morphism \( f^{an} \) (cf. [11, §3]). Since \( f^{an} \) is clopen \(^2\), we see that if \( O' \) is a connected component of \( C^{an} \setminus \Sigma' \) then there exists a connected component \( O \) of \( C^{an} \setminus \Sigma \) such that the morphism \( f^{an} \) restricts to a surjective map \( O' \to O \). If we were to identify \( O' \) and \( O \) with the Berkovich open unit disk \( O(0, 1) \) then outside \( \Sigma' \), the morphism \( f^{an} \) reduces to finite endomorphisms of \( O(0, 1) \).

Let \( x \in O' \) be a point of type I, II or III. The point \( x \) must correspond to a closed sub-ball around a rigid point \( a \) of radius \( r \). We use the notation from [2, §1.2] and write \( x \) as \( \zeta_{a,r} \) where \( a \in O'(k), \ r \in [0, 1) \). Having identified \( O' \) with the Berkovich open unit disk, we define the map

\[
l_x : (0, -\log(r)) \to O'
\]

\[
t \mapsto \zeta_{a,\exp(-t)}
\]

where \( \zeta_{a,\exp(-t)} \) is the type II point corresponding to the closed disk around \( x \) of radius \( \exp(-t) \). By convention, we regard any interval \([t', t) \subset (0, \infty)\) as starting from \( t \) with terminal point \( t' \). We set \( |l_x| := -\log(r) \). We can extend these constructions to treat points of type IV as well.

Note that for every Zariski closed point \(^3\) \( x \in O'(k) \), the map \( l_x \) is an isometric embedding with respect to the metric on \( C^{an} \). It follows that \( f^{an} \) induces a map

\[
f_x^{an} := l_{f^{an}(x)}^{-1} \circ f^{an} \circ l_x : (0, \infty) \to (0, \infty).
\]

We deviate slightly from the notation introduced in [10] and refer to \( f_x^{an} \) as the profile function of \( f \) at \( x \). Note that the profile functions are dependent on the skeleton chosen. The morphism \( f_x^{an} \) is continuous, piecewise linear and its slopes are positive integers. It follows that there exists \( m_x \in \mathbb{N} \) and tuples

\[
T_x := ((t_0, d_0, a_0), \ldots, (t_{m_x}, d_{m_x}, a_{m_x})) \subset ((0, \infty] \times \mathbb{N} \times \operatorname{val}(k^*))^{m_x}
\]

such that \( t_0 = \infty \) and for every \( 0 \leq i \leq m_x, f_x^{an} \) restricted to \((t_{i+1}, t_i]\) coincides with the map \( s \mapsto d_i s + a_i \). By val, we mean \(-\log(|.|)\). The points \( t_i \) are the break points of \( f_x^{an} \) i.e. for every \( i, d_i \neq d_{i+1} \). Observe that \( f_x^{an} \) is completely determined by the integer \( m_x \) and the tuple \( T_x := ((t_0, d_0, a_0), \ldots, (t_{m_x}, d_{m_x}, a_{m_x})) \). Furthermore, \( f_x^{an} \) is independent of how we identified \( O' \) with the Berkovich unit disk since \( l_x \) is an isometric embedding.

Given the curve \( C' \) and the skeleton \( \Sigma' \), we introduce a function \( r_{\Sigma'} : C^{an} \to [0, \infty] \) as in [10, §3.2.2] which we require when we define the radiality of a set. If \( x \in C^{an} \setminus \Sigma' \) then we set \( r_{\Sigma'}(x) := |l_x| \). If \( x \in \Sigma'(k) \) then \( r_{\Sigma'}(x) := \infty \) and if \( x \in \Sigma' \) is not a \( k \)-point then \( r_{\Sigma'}(x) := 0 \). In [10], M. Temkin proved the following striking result. He showed that there exists a skeleton \((\Sigma', \Sigma) \) for the morphism \( f^{an} \) \(^4\) such that for every \( x \in C'(k) \), the tuple \( T_x \) depends only on its image in \( \Sigma' \) for the retraction map \( \tau_{\Sigma'} : C^{an} \to \Sigma' \). Equivalently, if \( x \) and \( y \) belong to \( C'(k) \times \Sigma' \) are such that \( \tau_{\Sigma'}(x) = \tau_{\Sigma'}(y) \) then \( m_x = m_y \) and \( T_x = T_y \). As a consequence, one deduces as in Theorem 3.2.10 in loc.cit. that for every \( d \in \mathbb{N} \), the set \( N_{f^{an},d} \) of points of multiplicity at least \( d \) (cf. [10, §2.1.3] is radial around \( \Sigma' \).

**Definition 1.1.** Let \( C \) be a smooth projective \( k \)-curve and \( \Sigma_0 \) be a skeleton of \( C^{an} \). We say a subset \( X \subset C^{an} \) is radial around the skeleton \( \Sigma_0 \subset C^{an} \) if there exists a function \( p_X : \Sigma_0 \to \mathbb{R}_\infty \) such that \( x \in C^{an} \) belongs to \( X \) if and only if \( r_{\Sigma_0}(x) \leq p_X(\gamma) \) where \( \gamma \in \Sigma_0 \) is the unique point that \( x \) retracts onto. We will say that \( X \) is piecewise affine radial if in addition, the function \( p_X \) is piecewise affine.

\(^2\)This is a particular instance of a more general statement (cf. [12, Lemma 2.17],[3, Proposition 3.4.7]).

\(^3\)These are also referred to as rigid points.

\(^4\)This means that \( \Sigma' \subset C^{an} \) and \( \Sigma \subset C^{an} \) are skeleton and the retraction maps \( C^{an} \to \Sigma' \) and \( C^{an} \to \Sigma \) are compatible with \( f^{an} \).
Observe that since the paths $l_γ$ are isometries, a radial set around a skeleton $Σ_0$ is a metric neighbourhood of $Σ_0$ where the distance to the skeleton is given by a function on $Σ_0$ (cf. [10, 3.2.4]).

1.1. Radiality of definable sets. Motivated by the structure of the multiplicity loci $N_{Γ_{an,d}}$, one asks if the phenomenon of radiality is displayed by members of a more general class of sets. To this end we introduce the notion of a definable subset of $C_{an}$ and show that such subsets are in fact radial around suitable skeleta. Furthermore, one can check easily that the multiplicity loci are examples of definable subsets of $C_{an}$.

Strictly speaking, one cannot discuss the definability in the model theoretic sense of a subset of $C_{an}$. However, the theory of non-Archimedean geometry as introduced by Hrushovski-Loeser allows us to overcome this hurdle. In order to better understand the homotopy types of the Berkovich analytification of quasi-projective varieties, Hrushovski and Loeser introduced a model theoretic analogue of the Berkovich space. To apply model theory to the study of non-Archimedean geometry, we work in the theory ACVF of algebraically closed valued fields. The Hrushovski-Loeser space associated to a $k$-curve $C$ is the space of stably dominated types that concentrate on $C$ and is denoted $\hat{C}$. A precise definition and detailed description of $\hat{C}$ can be found in [7, §2]. The space $\hat{C}$ is a Hausdorff topological space and can be seen as an enrichment of $C$.

Let us emphasize two important features that $\hat{C}$ is endowed with. Firstly, the space $\hat{C}$ is a $k$-definable set (cf. [7, §7.1]). We can relate $\hat{C}$ to the Berkovich space $C_{an}$ as follows. Let $k^{max}$ be a maximally complete, algebraically closed field extension of $k$ whose value group is $\mathbb{R}^*$ (when the value group is written multiplicatively). Such a field is unique up to isomorphism over $k$. There exists a canonical homeomorphism $π : \hat{C}(k^{max}) → C_{an}^{\max}$. This induces a map $π_{k,C} : \hat{C}(k^{max}) → C_{an}$ by composing $π$ with the projection $C_{an}^{\max} → C_{an}$. The map $π_{k,C}$ is closed and continuous.

**Definition 1.2.** A set $X ⊂ C_{an}$ is $k$-definable if there exists a $k$-definable subset $Y ⊂ \hat{C}$ such that $X = π_{k,C}(Y)$. We say that $X$ is closed if the set $Y$ is closed and $X$ is path-connected if $Y$ is path connected.

**Theorem 1.3.** Let $C$ be a smooth projective curve over $k$. We then have that every closed path connected $k$-definable set is piecewise affine radial.

Theorem 1.3 is the consequence of an analogous result for Hrushovski-Loeser curves. In order to discuss the radiality statement for definable subsets of $\hat{C}$, we introduce the notion of a definable homotopy.

A definable homotopy on the curve $C$ is a continuous function

$$h : [0, \infty] × C → \hat{C}.$$  

In [7, §7.5], Hrushovski and Loeser construct such homotopies such that the image $h(0,C) ⊂ \hat{C}$ is $Γ$-internal. One may think of a $Γ$-internal subset of $\hat{C}$ as the model theoretic analogue of a subspace of $C_{an}$ homeomorphic to a finite simplicial complex. Henceforth, we assume that all definable homotopies under consideration have $Γ$-internal subsets as image. We discuss the precise nature of these constructions briefly in §2.2.

Let $h$ be a definable homotopy of $C$ onto a $Γ$-internal set $Σ$. We associate to every definably path-connected definable subset of $\hat{C}$ a function $α : C → [0, \infty] × \{γ_1, γ_2\}$ where $γ_1 < γ_2 ∈ Γ_{an}[k]$. Although strictly speaking one cannot talk about metric properties of the curve $\hat{C}$, the function $α$ in some sense measures the distance from
the definable set to $\Sigma$. Theorem 2.5 shows that given a definably path connected subset of $\tilde{C}$, we can choose $h$ and $\Sigma$ so the associated function $\alpha$ is constant along the fibres of the retraction associated to $h$.

1.2. **Backward branching index.** An important ingredient in the proof of Theorem 1.3 is the finiteness property satisfied by the profile functions introduced above. By this we mean that if we are given a morphism $f : C' \to C$ then there exists skeleta $\Sigma'$ and $\Sigma$ compatible for $f^m$ and such that $f^m$ is constant along the fibres of the retractions i.e. if $x, y \in C'(k)$ retract to the same point on $\Sigma'$ then $f^m_x = f^m_y$. This is one of the main results of Temkin in [10]. Our approach however is different in that we introduce an analogue of the profile functions $f^m$ within the framework of the Hrushovski-Loeser curves. This new invariant is called the Backward branching index function associated to a definable homotopy. We reproduce the definition from §2.4. We suppose that the definable homotopy $h : [0, \infty] \times C \to \tilde{C}$ lifts uniquely to a a definable homotopy $h' : [0, \infty] \times C' \to C'$ via $\tilde{f}$. Let $\Sigma'$ denote the $\Gamma$-internal set which is the image of $h'$.

**Definition 1.4.** The backward branching index of a point $x \in C'$ is a function

$$BB_{h', f}(x) : [0, \infty] \to \mathbb{N} \times [0, \infty]$$

defined as follows. If $t \in [0, \infty]$, let $B_{h', f}(x)(t)$ denote the set of elements $z \in C'$ such that $f(z) = f(x)$, $h'(t, z) = h'(t, x)$ and $h'(t, z) \neq h'(0, x)$. Then

$$BB_{h', f}(x)(t) := (\text{card}(B_{h', f}(x)(t)), \Sigma_z \in B_{h', f}(x)(t) \text{val}(z)).$$

By Remark 2.8, we thus have a function

$$BB_{h', g} : C \to \text{Fn}([0, \infty], \mathbb{N} \times [0, \infty])$$

where $\text{Fn}([0, \infty], \mathbb{N} \times [0, \infty])$ is the set of definable functions $[0, \infty] \to \mathbb{N} \times [0, \infty]$.

In Theorem 2.9, we show that the backward branching index functions can be made to satisfy a strong finiteness property. More precisely, we show that we can choose $h'$ and $h$ so that if $x, y \in C'(k)$ retract to the same point in $\Sigma'$ then $BB_{h', f}(x) = BB_{h', f}(y)$. The proof of this fact exploits the explicit nature and flexibility with which Hrushovski and Loeser construct definable homotopies on $C'$ and $C$. We relate the Backward branching indices to the profile functions in Lemma 3.8 and proceed from there to deduce in Theorem 3.9 the finiteness property discussed above for the profile functions.

1.3. **Tameness of profile functions in families.** By introducing the Backward branching index invariants and proving that they determine the profile functions, we have set ourselves up to prove a generalization of Theorem 3.9. In Theorem 4.1, we study Theorem 3.9 in the context of families of morphisms of smooth projective curves. The Theorem 3.9 shows that considerable information about the metric properties of a finite morphism can be encoded in terms of combinatorial data i.e. the induced morphism of suitable skeleta and additional data on the skeleton of the source that controls the profile functions we introduced earlier. The goal of Theorem 4.1 is to show that given a family of morphisms of smooth projective curves the associated family of combinatorial data itself satisfies a nice finiteness property.

The framework within which we work is the following. Let $S$ be a quasi-projective $k$-variety. Let $\alpha_1 : X \to S$ and $\alpha_2 : Y \to S$ be smooth morphisms of quasi-projective $k$-varieties. Let $f : X \to Y$ be a morphism such that for every $s \in S$, $f_s : X_s \to Y_s$ is a finite separable morphism of smooth projective irreducible $k$-curves.
The results that we discussed previously, notably Theorem 3.9, imply that for every $s \in S(k)$, there exist deformation retractions

$$h_s^\text{an} : [0, \infty) \times X_s^\text{an} \to X_s^\text{an}$$

and

$$h_s^\text{an} : [0, \infty) \times Y_s^\text{an} \to Y_s^\text{an}$$

such that the following hold.

1. The images $\Sigma' := h_s^\text{an}(0, X_s^\text{an})$ and $\Sigma_s = h_s^\text{an}(0, Y_s^\text{an})$ are skeleta of the curves $X_s^\text{an}$ and $Y_s^\text{an}$.

2. Given $x \in X_s(k)$, let $\text{prfl}(f_s^\text{an}, x)$ denote the profile function at $x$ associated to $f_s$. This temporary change in notation is to prevent any ambiguity that might arise since we regard $f_s^\text{an}$ as the analytification of the morphism $f_s : X_s \to Y_s$. We then have that for every $x_1, x_2 \in X_s(k)$, if $h_s^\text{an}(0, x_1) = h_s^\text{an}(0, x_2)$ then $\text{prfl}(f_s^\text{an}, x_1) = \text{prfl}(f_s^\text{an}, x_2)$.

Observe that (2) allows us to extend the function $\text{prfl}(f_s^\text{an}, -)$ to the skeleton $\Sigma'$. Indeed, given $p \in \Sigma'$, we set $\text{prfl}(f_s^\text{an}, -)$ to be the function $\text{prfl}(f_s^\text{an}, x)$ for any $x$ that retracts to $p$. By (2), $\text{prfl}(f_s^\text{an}, p)$ is well defined and we have a function

$$\text{prfl}(f_s^\text{an}, -) : \Sigma' \to \text{Fn}([0, \infty], [0, \infty]).$$

The principal assertion of Theorem 4.1 concerns a finiteness property for the family

$$\{f_s^\text{an} : \Sigma'_s \to \Sigma_s, \text{prfl}(f_s^\text{an}, -)|_{\Sigma'_s}\}_{s \in S}.$$ 

More precisely, we show that there exists a deformation retraction $g^\text{an} : I \times S^\text{an} \to S^\text{an}$ whose image $\Sigma(S) \subset S^\text{an}$ is homeomorphic to a finite simplicial complex and is such that the following holds. The tuple $(f_s^\text{an} : \Sigma'_s \to \Sigma_s, \text{prfl}(f_s^\text{an}, -))$ is constant along the fibres of the retraction of the deformation $g^\text{an}$ i.e. if $e$ denotes the end point of the interval $I$ and if $s_1, s_2 \in S(k)$ are such that $g^\text{an}(e, s_1) = g^\text{an}(e, s_2)$ then $\Sigma'_{s_1} = \Sigma'_{s_2}$ and $\Sigma_{s_1} = \Sigma_{s_2}$. Furthermore, with these identifications $(f_{s_1}^\text{an})_{|\Sigma'_{s_1}} = (f_{s_2}^\text{an})_{|\Sigma'_{s_2}}$ and $\text{prfl}(f_{s_1}^\text{an}, -)_{|\Sigma'_{s_1}} = \text{prfl}(f_{s_2}^\text{an}, -)_{|\Sigma'_{s_2}}$.

This theorem represents one of the primary reasons behind introducing the Backward branching indices as they allow us to use the Hrushovski-Loeser construction to control the combinatorial data as it varies in families. It should be noted that constructing deformations in higher dimensions is in general very difficult and the Hrushovski-Loeser construction is the only tool that allows for some flexibility.

1.4. Notation: We fix the field $k$ which is algebraically closed complete with respect to a non-Archimedean non-trivial real valuation. By a $k$-curve, we mean a connected variety of dimension 1.

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2. Definable subsets of $\hat{C}$

The theory ACVF is an $\mathcal{L}_\Gamma$-theory where $\mathcal{L}_\Gamma$ is the three sorted language consisting of the valued field sort VF, value group sort $\Gamma$ and residue field sort $k$. We add the functions, relations and constants necessary to make any structure of $\mathcal{L}_\Gamma$ a valued field. Within this framework, the theory ACVF consists of those sentences (formulae without quantifiers) such that any model of ACVF is an algebraically closed field. Although $\mathcal{L}_\Gamma$ is easily defined, we note that it does not eliminate imaginaries and hence we work in the extended language $\mathcal{L}_G$ (cf. [7, §2.7]) where we add the geometric sorts to $\mathcal{L}_\Gamma$. For simplicity, we write $\mathcal{L} := \mathcal{L}_G$. For an introduction to Model theory, we refer the reader to [8, §1].

2.1. Definable sets. We fix a large saturated model $\mathcal{U}$ of ACVF as in [7, §2.1]. We introduce the notion of a definable set as in [2.1].

Definable sets. $\hat{C}$ is a functor from the category whose objects are $\mathcal{L}_C$-models of ACVF and morphisms are elementary embeddings to the category of sets. Let $\{x_1, \ldots, x_m\}$ be the set of variables which occur in $\phi$. Given an $\mathcal{L}_C$-model $M$ of ACVF i.e. a model of ACVF that contains $C$, we set

$$Z_{\phi}(M) = \{\bar{a} \in M^n | M \models \phi(\bar{a})\}.$$ 

Clearly, $Z_{\phi}$ is well defined.

Definition 2.1. A $C$-definable set $Z$ is a functor from the category whose objects are $\mathcal{L}_C$-models of ACVF and morphisms are elementary embeddings to the category of sets such that there exists an $\mathcal{L}_C$-formula $\phi$ and $Z = Z_{\phi}$.

Example 2.2. Consider the following examples of three different classes of definable sets in ACVF. Observe that these objects appear naturally in the study of Berkovich spaces, tropical geometry and algebraic geometry.

(1) Let $K$ be a model of ACVF and $A \subset K$ be a set of parameters. Let $n \in \mathbb{N}$. A semi-algebraic subset of $K^n$ is a finite boolean combination of sets of the form

$$\{x \in K^n | \text{val}(f(x)) \geq \text{val}(g(x))\}$$

where $f, g$ are polynomials with coefficients in $A$. One verifies that semi-algebraic sets extend naturally to define $A$-definable sets in ACVF.

(2) Let $G = \Gamma(K)$ where $K$ is as above. A $G$-rational polyhedron is a finite Boolean combination of subsets of the form

$$\{(a_1, \ldots, a_n) \in G^n | \Sigma_i a_i \leq c\}$$

where $a_i \in \mathbb{Z}$ and $c \in G$. Such objects extend naturally to define a $K$-definable subset of $\Gamma^n$ where $\Gamma$ is the value group sort.

(3) Any constructible subset of $k^n$ gives a definable set in a natural way.

(4) Let $C$ be a $k$-curve. Then the space of stably dominated types - $\hat{C}$ is a $k$-definable set. This is unique to the one dimensional case i.e. the space $\hat{V}$ where $V$ is a $k$-variety of dimension atleast 2 is strictly pro-definable.

2.2. Homotopies on $\hat{C}$. Let $C$ be a projective $k$-curve. The space $\hat{C}$ is Hausdorff and definably path connected. Furthermore, it admits a definable deformation retraction

$$H : [0, \infty) \times \hat{C} \rightarrow \hat{C}$$

such that the image of $H$ is a $k$-definable $\Gamma$-internal set i.e. a $k$-definable subset of $\hat{C}$ that is in $k$-definable bijection with a $k$-definable subset of $\Gamma^n$ for some $n \in \mathbb{N}$.

\footnote{Here $\mathcal{L}_C$ denotes the extension of $\mathcal{L}$ obtained by adding constants corresponding to the parameter set $C$.}
We briefly explain the construction of the homotopy $H$ from [7, §7.5] as this will play a crucial role in the following sections.

We fix a system of coordinates on $\mathbb{P}^1$. Let $m : \mathbb{P}^1 \times \mathbb{P}^1 \to \Gamma_\infty$ be the standard metric as in [7, §3.10]. Let $O_0 := \{x \in \mathbb{P}^1|m(x, 0) \geq 0\}$ and $O_\infty := \{x \in \mathbb{P}^1|m(x, \infty) \geq 0\}$. Observe that $O_0$ and $O_\infty$ are definable subsets of $k$ and are the closed unit disk around 0 and $\infty$ respectively. Using $m$, we define a $v+g$-continuous homotopy, $\psi : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1$ by mapping $(t, a)$ to the generic type [7, Example 3.2.1] of the valuation ball

\[ \{x \in \mathbb{P}^1|m(x, a) \geq t\} \]

The map $\psi$ extends to define a map $\hat{\psi} : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1$ whose image is the generic type of the ball $O_0$ and is hence $\Gamma$-internal.

Let $D \subset \mathbb{P}^1(k)$ be a finite set. We define $m_D : \mathbb{P}^1 \to \Gamma_\infty$ by $x \mapsto \max\{m(x, d)|d \in D\}$. We can extend the homotopy $\psi$ above to a homotopy

\[ \psi_D : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1 \]

\[ (t, a) \mapsto \psi(\max\{t, m_D(a)\}, a) \]

As before, $\psi_D$ extends to a homotopy $\hat{\psi}_D : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1$ whose image $\Upsilon_D$ is the convex hull of $D$ in $\mathbb{P}^1$. We refer to $\hat{\psi}_D$ as the cut-off of $\psi$ with respect to the function $m_D$ (cf. [7, Lemma 10.1.6]).

A homotopy $h : [0, \infty] \times C \to \hat{C}$ is constructed as follows. Let $f : C \to \mathbb{P}^1$ be a finite morphism. By [7, §7.5.1], there exists a divisor $D \subset \mathbb{P}^1$ such that $\psi_D$ lifts uniquely to a definable map $h : [0, \infty] \times C \to \hat{C}$. This means that there exists a unique $h : [0, \infty] \times C \to \hat{C}$ such that the diagram below commutes.

\[ \begin{array}{ccc}
I \times C & \xrightarrow{H} & \hat{C} \\
\downarrow{id \times f} & & \downarrow{f} \\
I \times \mathbb{P}^1 & \xrightarrow{\psi_D} & \mathbb{P}^1 \\
\end{array} \]

where $I := [0, \infty]$. By [7, Lemma 10.1.1], the uniqueness of the map $h$ guarantees that the canonical extension (cf. §3.8 in loc. cit.) $H : [0, \infty] \times \hat{C} \to \hat{C}$ is a homotopy whose image is $f^{-1}(\Upsilon_D)$ and is $\Gamma$-internal. Note that we often say a map $g : [0, \infty] \times C \to \hat{C}$ is a homotopy. By this we mean that it is $v+g$-continuous and hence its canonical extension $G : [0, \infty] \times C \to \hat{C}$ is a homotopy.

Remark 2.3. We make the following observation that is crucial to relating the deformation retractions above to the metric structure of the Berkovich curve $\mathcal{C}^\text{an}$. Recall from §1, the field $k^{\text{max}}$ which is a maximally complete algebraically closed extension of $k$ whose value group is $\mathbb{R}^*$. We have a morphism $\pi_{k,C} : \hat{C}(k^{\text{max}}) \to \mathcal{C}^\text{an}$ that factors through the canonical homeomorphism $\pi : \hat{C}(k^{\text{max}}) \to C^{\text{an}}_{k^{\text{max}}}$.

The map $\pi_{k,\mathbb{P}^1}$ is such that if $a \in \mathbb{P}^1(k)$ and $\gamma \in \Gamma(k)$ then the generic type of the ball around $a$ of valuate radius $\gamma$ belongs to $\mathbb{P}^1(k^{\text{max}})$ and $\pi_{k,\mathbb{P}^1}$ maps this point to the point $\zeta_{a,-\log(\gamma)}$.

Let $D \subset \mathbb{P}^1(k)$ be a divisor. Let $x \in \mathbb{P}^1(k) \subset \mathbb{P}^1(k^{\text{max}})$. Let $t_1, t_2 \in [0, \infty](\mathbb{R})$ with $t_1 < t_2$ and such that the map $t \mapsto \hat{\psi}_D(t, x)$ is injective when restricted to $[t_1, t_2]$. Let $\rho$ denote the metric on the space $\mathbb{P}^1, \text{an}$ which we refer to as its skeletal metric or path distance metric (cf. [1, §5]). We then have that

\[ t_2 - t_1 = \rho(\pi_{k,\mathbb{P}^1}(t_1, x), \pi_{k,\mathbb{P}^1}(t_2, x)) \]
2.3. Radiality. One of our goals is to understand the topological nature of definable subsets. In general, a definable subset of \( \hat{C} \) does not necessarily have finitely many definably path connected components. For example, consider \( C \subset \hat{C} \). Hence we restrict our attention to definably path connected definable sets and use in a novel way the homotopies discussed in \S 2.2 to study them.

Let \( C \) be a projective \( k \)-curve. Let \( H : [0, \infty] \times \hat{C} \to \hat{C} \) be a homotopy of \( \hat{C} \) with image \( \Upsilon \) as constructed in \S 2.2. Let \( \mathfrak{B}(H) \) denote the set of definably path connected definable subsets of \( \hat{C} \) that intersect \( \Upsilon \) at least one point.

Let \( \gamma_1 < \gamma_2 < 0 \) be elements in \([0, \infty]\). Let \( \mathfrak{B}(H) \) denote the set of definable functions (cf. [7, \S 3.7]) \( \alpha : C \to [0, \infty] \times \{ \gamma_1, \gamma_2 \} \) such that the following properties hold.

1. For every \( x, y \in C \) if
   \[ t_{x,y} := \sup\{t \in [0, \infty]|H(t,x) = H(t,y)\} \]
   then
   \[ t_{x,y} \leq p_1(\alpha(x)) \implies t_{x,y} \leq p_1(\alpha(y)) \]
   and
   \[ t_{x,y} > p_1(\alpha(x)) \implies \alpha(x) = \alpha(y) \]
   where \( p_1 \) is the projection onto the first coordinate.
2. Given \( x \in C(k) \), let \( e(x) \) be the supremum of the set of elements \( t \in [0, \infty] \) such that \( H(\cdot,x) \) is constant on the interval \([0,t]\). We have that for every \( x \in C(k) \), \( \alpha(x) \in [e(x), \infty] \).
3. The complement in \( \Upsilon \) of the space \( \{H(0,x)|\alpha(x) = (e(x), \gamma_2)\} \subset \Upsilon \) is non-empty and definably path connected.

The properties required of the elements in \( \mathfrak{B}(H) \) is motivated by the following proposition.

**Proposition 2.4.** Let \( C \) be a projective \( k \)-curve. Let \( H : [0, \infty] \times \hat{C} \to \hat{C} \) be a homotopy of \( \hat{C} \) with image \( \Upsilon \) as constructed in Section \S 2.2. We then have a bijection \( \beta : \mathfrak{B}(H) \to \mathfrak{B}(H) \).

**Proof.** Let \( X \subset \hat{C} \) be a definably path connected definable subset. We define a function \( \beta^x_{X,H} : C \to [0, \infty] \) as follows. Let \( a \in C \). The homotopy \( H \) defines a path \([0, \infty] \to \hat{C} \) given by \( t \mapsto H(t,a) \). Suppose that \( H(t,a) \in X \) for some \( t \in [0, \infty] \).

Let
\[ \beta^x_{X,H}(a) := \sup\{t \in [0, \infty]|H(t,a) \in X\}. \]
If \( H(\beta^x_{X,H}(a),a) \) belongs to \( X \) then we set
\[ \beta^x_{X,H}(a) := \gamma_1 \]
and if \( H(\beta^x_{X,H}(a),a) \) does not belong to \( X \) then we set
\[ \beta^x_{X,H}(a) := \gamma_2. \]

Let \( \beta_{X,H}(a) := (\beta^x_{X,H}(a),\beta^x_{X,H}(a)) \). Suppose on the other hand that for every \( t \in [0, \infty] \), \( H(t,a) \notin X \). We then define \( \beta_{X,H}(a) := (e(a), \gamma_2) \) where the notation is as introduced above.

Clearly, \( \beta_{X,H} \) defines a map \( C \to [0, \infty] \times \{ \gamma_1, \gamma_2 \} \) that is definable and one checks that \( \beta_{X,H} \in \mathfrak{B}(H) \). We verify the surjectivity of the map \( X \to \beta_{X,H} \) as follows. Suppose \( \alpha : C \to [0, \infty] \) belonged to the family \( \mathfrak{B}(H) \). We define \( X_\alpha \) to be the set of \( z \in \hat{C} \) such that there exists \( x \in C, t \in [0, \alpha(x)] \) and \( z = H(t,x) \). Clearly, \( X_\alpha \) is a definable set since \( \hat{C} \) is definable and \( \alpha \) is a definable function. Let \( S' \subset \Upsilon \) be the set \( \{H(0,x)|\alpha(x) = (e(x), \gamma_2)\} \) and let \( S \) denote its complement in \( \Upsilon \). By
condition (2) above, we see that $S$ is definably path connected and non-empty. We deduce that $X_\alpha$ intersects $Y$ non-trivially and must be definably path connected. One verifies directly that if $X \in \mathcal{A}(H)$ then $X_{\beta_{X,H}} = X$ and if $\alpha \in \mathcal{B}(H)$ then $\beta_{X_\alpha,H} = \alpha$.

**Theorem 2.5.** Let $C$ be a projective $k$-curve and let $f : C \to \mathbb{P}^1$ be a finite morphism. Let $X \subset C$ be a definably path connected definable subset. There exists a homotopy $H_1 : [0, \infty) \times \tilde{C} \to \tilde{C}$ whose image $Y$ is a $\Gamma$-internal subset of $\tilde{C}$ such that $X \in \mathcal{A}(H_1)$ and the function $\beta_{X,H_1}$ (as in Proposition 2.4) is constant along the fibres of the retraction. Furthermore, $H_1$ is the unique lift of a homotopy $\psi_{D_1}$, where $D_1 \subset C$ is a Zariski closed subset of $C$.

**Proof.** Let $h : [0, \infty) \times C \to \tilde{C}$ be a homotopy that is the unique lift of a homotopy $\psi_D : [0, \infty) \times \mathbb{P}^1 \to \mathbb{P}^1$ where $D \subset \mathbb{P}^1$ is a divisor. We use $H$ to denote the canonical extension $[0, \infty) \times C \to \tilde{C}$ of $h$. Let $\xi_1 := p_1 \circ \beta_{X,H}$ and $\xi_2 := p_2 \circ \beta_{X,H}$ where $p_i : \Gamma'_\infty \to \Gamma_\infty$ is the projection map to the $i$-th coordinate. By [7, Lemma 10.2.3], there exists a finite family $\{\xi_1, \ldots, \xi_r\}$ of definable functions $\mathbb{P}^1 \to \Gamma_\infty$ and a divisor $D_0 \subset \mathbb{P}^1$ such that any homotopy on $C$ that is the lift of a homotopy on $\mathbb{P}^1$ that preserves the functions $\xi_i$ and fixes the divisor $D_0$ must preserve the functions $\xi_i$. Since the functions $\xi_i$ are definable, we can enlarge the divisor $D_0$ so that for every $i$, there exists a definable partition $X_{i_1}, \ldots, X_{i_r}$ of $U_0 := \mathbb{P}^1 \setminus D_0$, $(\xi_1)_i\mid_{X_{i_1}}$ is of the form $\text{val}(g_{a_1})$ where $g_{a_1}$ is regular on $U_0$ and the set $X_{i_1}$ is given by the boolean combination of sets of the form $\{x \in U_0 | \text{val}(a_{i_1})(x) \geq \text{val}(b_{i_1})(x)\}$ where $a_{i_1}$ and $b_{i_1}$ are regular on $U_0$. Let $\{c_1, \ldots, c_r\}$ be the finite family of regular functions on $U_0$ consisting of the $g_{a_1}$’s and $a_{i_1}$’s and $b_{i_1}$’s as above. It can be verified that any homotopy of $U_0$ that preserves the functions $\text{val}(c_j)$ preserves $\xi_i$ for all $i$. Lastly, let $D_1$ be the union of $D_0$, $D$ and the zeros of the regular functions $c_i$. We have that $\psi_{D_1}$ preserves the functions $c_i$. Furthermore, $\psi_{D_1}$ is the cut-off $\psi_D|_{m_D}$ where $m_{D_1}$ is as defined in §2.2. Hence, we see that the cut-off $h[m_{D_1}, \circ f]$ is the unique lift of $\psi_{D_1} = \psi_D|_{m_{D_1}}$.

We simplify notation and write $h_1 := h|_{m_{D_1}, \circ f}$ and let $H_1$ be its canonical extension. Let $Y_1$ denote the image of the deformation $H_1$ i.e. $\{H_1(0,x)| x \in \tilde{C}\}$. By construction, the function $\beta_{X,H}$ is constant along the fibres of the retraction $H_1(0,\cdot) : \tilde{C} \to Y_1$. We check that this implies $\beta_{X,H}$ is also constant along the fibres of the retraction. Let $x \in Y_1$. Let $y_1, y_2 \in C$ such that $H_1(0,y_1) = H_1(0,y_2) = x$. We must show that $\beta_{X,H}(y_1) = \beta_{X,H}(y_2)$. Let $t_{y_1} := \sup \{t | H_1(t,y_1) \in Y_1\}$. We have that $t_{y_1} = t_{y_2}$. It suffices to verify this for the homotopy $\psi_{D_1}$, and in this case it is a consequence of the fact that $m_{D_1}$ is constant along the fibres of the retraction. Suppose, $p_1(\beta_{X,H}(y_1)) \leq t_{y_1}$ then $\beta_{X,H}(y_1) = \beta_{X,H}(y_2)$ and likewise, $\beta_{X,H}(y_2) = \beta_{X,H}(y_1)$. Since $\beta_{X,H}$ is constant along the fibres of the retraction of $H_1$, we verify the claim in this case. If $p_1(\beta_{X,H}(y_1)) > t_{y_1}$ then we get that $\beta_{X,H}(y_1) = \beta_{X,H}(y_2) = (t_{y_1}, \gamma_2) = (t_{y_1}, \gamma_2)$ (or $= (t_{y_1}, \gamma_1)$).

**Remark 2.6.** Let $C$ be a $k$-curve and let $X \subset C$ be a definably path connected definable subset. Let $H : [0, \infty) \times \tilde{C} \to \tilde{C}$ be a homotopy whose image $Y$ is a $\Gamma$-internal subset of $\tilde{C}$ such that $X \in \mathcal{A}(H)$ and the function $\beta_{X,H}$ (as in Proposition 2.4) is constant along the fibres of the retraction. Let us assume that $H$ is the canonical extension of a homotopy $h : [0, \infty) \times C \to \tilde{C}$. Let $r : C \to [0, \infty]$ be a $v + g$-continuous function such that the image of the cut-off homotopy $h[r]$ is $\Gamma$-internal as well. The arguments in the proof of Theorem 2.5 shows that $\beta_{X,H[r]}$.

---

6We say a homotopy $h$ preserves a function $\xi$ if the function $\xi$ is constant for the retraction map.
is constant along the fibres of the retraction where \( H[r] \) is the cut-off of \( H \) via the function \( r \). It is also the canonical extension of \( h[r] \).

### 2.4. Backward branching index

The homotopies \( \psi_D \) on \( \mathbb{P}^1 \) respect the metric structure of the curve \( \mathbb{P}^1_{k}^{\text{an}} \) as observed in Remark 2.3. However, lifts

\[
h : [0, \infty] \times C \to \hat{C}
\]

of \( \psi_D \) for a morphism \( f : C \to \mathbb{P}^1 \) no longer satisfy this property. Associated to \( h \) and the morphism \( f \), for every \( x \in C \), we introduce definable functions

\[
BB_{h,f}(x) : [0, \infty] \to \mathbb{N} \times [0, \infty]
\]

which will later help reconcile the homotopy \( H \) with the metric properties of \( C^{\text{an}} \).

We define the Backward branching index function more generally for any finite morphism \( g : C' \to C \) of smooth projective irreducible curves. Let \( h : [0, \infty] \times C \to \hat{C} \) be a deformation retraction of \( C \) onto a \( \Gamma \)-internal set \( \Sigma \). Let \( h' : [0, \infty] \times C' \to \hat{C}' \) be a lift of \( h \) and let \( \Sigma' \) denote the \( \Gamma \)-internal set which is the image of \( h' \). We will assume that \( h \) is obtained as in §2.2 by lifting a homotopy \( \psi_D \) on \( \mathbb{P}^1 \).

**Definition 2.7.** The backward branching index of a point \( x \in C' \) is a function

\[
BB_{h',g}(x) : [0, \infty] \to \mathbb{N} \times [0, \infty]
\]

defined as follows. If \( t \in [0, \infty] \), let \( B_{h',g}(x)(t) \) denote the set of elements \( z \in C' \) such that \( g(z) = g(x) \), \( h'(t,z) = h'(t,x) \) and \( h'(t,z) \neq h'(0,z) \). Then

\[
BB_{h',g}(x)(t) := (\text{card}(B_{h',g}(x)(t)),\Sigma_{z \in B_{h',g}(x)(t)} \text{val}(z)).
\]

By Remark 2.8, we thus have a function

\[
BB_{h',g} : C' \to \text{Fn}([0, \infty], \mathbb{N} \times [0, \infty])
\]

where \( \text{Fn}([0, \infty], \mathbb{N} \times [0, \infty]) \) is the set of definable functions \([0, \infty] \to \mathbb{N} \times [0, \infty] \).

**Remark 2.8.** The finiteness of \( g \) implies that the \( BB_{h',g}(x) \) takes on only finitely many values. It follows that there exists \( n_x \in \mathbb{N} \) such that the tuple

\[
S_x := (((s_0, e_0, \beta_0), \ldots, (s_{n_x}, e_{n_x}, \beta_{n_x}))) \subset ([0, \infty] \times \mathbb{N} \times [0, \infty])^{n_x}
\]

with \( s_0 = \infty \), completely determines \( BB_{h',g}(x) \). By this we mean that if \( s \in [s_i, s_{i+1}) \) then \( BB_{h',g}(x)(s) = (e_i, \beta_i) \) and if \( s \in [s_{n_x}, 0] \) then \( BB_{h',g}(x)(s) = (e_{n_x}, \beta_{n_x}) \) The \( s_i \) are the break points of the function i.e. \( e_i \neq e_{i+1} \).

**Theorem 2.9.** Let \( g : C' \to C \) be a finite morphism of smooth projective irreducible curves. Let \( h : [0, \infty] \times C \to \hat{C} \) be a deformation retraction of \( C \) onto a \( \Gamma \)-internal set \( \Sigma \). Let \( h' : [0, \infty] \times C' \to \hat{C}' \) be a lift of \( h \) and let \( \Sigma' \) denote the \( \Gamma \)-internal set which is the image of \( h' \). We will assume that \( h \) is obtained as in §2.2 by lifting a homotopy \( \psi_D \) on \( \mathbb{P}^1 \). There exists a \( \phi + g \)-continuous function \( \alpha : C' \to \Gamma_1 \) such that if \( h'_1 \) denotes the cut-off \( h'[\alpha] \) then the function \( BB_{h'_1,g} \) is constant along the fibres of the retraction associated to \( h'_1 \).

\(^7\)If \( X \) is a definable set and then a definable function \( a : X \to \mathbb{N} \) is a map with finite image and for every \( n \in \mathbb{N} \), \( a^{-1}(n) \) is a definable subset of \( X \).
Proof. We begin by showing that \( h'_1 \) can be chosen so that \( BB_{h'_1, \varphi} \) is constant along the retraction \( h'_1(0, -) \). Observe that \( h' \) is a lift of a homotopy \( \psi_D \) on \( \mathbb{P}^1 \). Using the arguments in the proof of Theorem 2.5, we observe that it suffices to show that there exists finitely many definable functions \( \{ \xi_1, \ldots, \xi_m \} \) such that if there exists a \( v + g \)-continuous function \( \alpha : C' \to \Gamma_\infty \) and \( \xi_i \) are constant along the retraction associated to \( h'[\alpha] \) then \( h'[\alpha] \) preserves \( BB_{h'_1, \varphi} \). Let \( N \in \mathbb{N} \) be the supremum of the set \( \{ \text{card}(g^{-1}(y)) | y \in C \} \). By definition, \( n_x \leq N \) for every \( x \in C \). Recall that if \( x \in C \) then
\[
S_x := ((s_0, e_0, \beta_0), \ldots, (s_{n_x}, e_{n_x}, \beta_{n_x}))
\]
is the tuple from Remark 2.8 that determines \( BB_{h'_1, \varphi}(x) \). Observe that the \( s_i \) are distinct and correspond to the set of \( t \in [0, \infty) \) such that there exists \( z \in C \), \( h(t, x) = h(t, z) \), \( g(t) = g(z) \) and \( h(t, z) \neq h(0, z) \). It follows that \( \{ s_1, \ldots, s_{n_x} \}_{x \in C} \) varies definably along \( C \) and hence \( x \mapsto n_x \) is a definable function. Let \( \gamma_1, \ldots, \gamma_N \) be \( N \) distinct points in \( \Gamma(k) \). We then have that the function \( \xi \) given by \( x \mapsto \gamma_{n_x} \) is definable.

Once again let
\[
S_x := ((s_0, e_0, \beta_0), \ldots, (s_{n_x}, e_{n_x}, \beta_{n_x}))
\]
be the tuple that determines \( BB_{h'_1, \varphi}(x) \). Let \( \gamma < 0 \in \Gamma(k) \). For every \( 1 \leq i \leq N \), we define a family \( \{ \xi_{i1}, \ldots, \xi_{ii} \} \) where for every \( x \in \xi^{-1}(\gamma_i) \), \( \xi_{ij}(x) := \beta_j \) and if \( x \notin \xi^{-1}(\gamma_i) \) then \( \xi_{ij}(x) := \gamma \). Hence we see that there exists a \( BB_{h'_1, \varphi} \) is uniformly definable in \( x \), we get that \( \{ S_x \}_{x \in C'} \) is also uniformly definable and hence the \( \xi_{ij} \) are definable functions.

We now encode the integers \( e_i \) that appear in the tuple \( S_x \) using definable functions. As before, we see that for every \( x \in C' \) if \( S_x = ((s_0, e_0, \beta_0), \ldots, (s_{n_x}, e_{n_x}, \beta_{n_x})) \) then for every \( i, e_i \leq N \). Let \( \theta_1 < \ldots < \theta_N \) be elements of \( \Gamma \). For every \( 1 \leq i \leq N \), we define a family \( \{ e_{i1}, \ldots, e_{ii} \} \) of functions from \( C' \) to \( \Gamma_\infty \) as follows. If \( x \notin \xi^{-1}(\gamma_i) \) then we set \( e_{ij}(x) := e_{ij} \) and if \( x \notin \xi^{-1}(\gamma_i) \) then we set \( e_{ij}(x) := \gamma \).

Similarly, for every \( 1 \leq i \leq N \), we define a family \( \{ \eta_{i1}, \ldots, \eta_{ii} \} \) where for every \( x \in \xi^{-1}(\gamma_i) \), \( \eta_{ij}(x) := \beta_j \) and if \( x \notin \xi^{-1}(\gamma_i) \) then \( \eta_{ij}(x) := \gamma \). As before, the definability of \( \xi_{ij} \) follows from the fact \( \{ S_x \}_{x \in C'} \) is uniformly definable.

Clearly, any homotopy that fixes the definable functions \( \xi_{ij}, \eta_{ij} \) and \( \xi \) must preserve \( BB_{h'_1, \varphi} \). Hence we see that there exists a \( v + g \)-continuous function \( \alpha : C' \to \Gamma_\infty \) such that the retraction associated to \( h'_1 := h'[\alpha] \) preserves \( BB_{h'_1, \varphi} \).

Let \( x \in \Sigma_1 \). Let \( y_1, y_2 \in \mathbb{C} \) such that \( h'_1(0, y_1) = h'_1(0, y_2) = x \). We must show that \( BB_{h'_1, \varphi}(y_1) = BB_{h'_1, \varphi}(y_2) \). Let \( t_{y_1} := \text{sup}(\{ t | h'_1(t, y_1) \in \Sigma_1 \}) \). As in Lemma 2.5, we have that \( t_{y_1} = t_{y_2} \). Since \( h'_1 \) is a cut-off of \( h' \), for every \( t > t_{y_1} \), \( BB_{h'_1, \varphi}(y_1)(t) = BB_{h'_1, \varphi}(y_2)(t) \) and by construction of \( h'_1 \), we get that \( BB_{h'_1, \varphi}(y_1)(t) = BB_{h'_1, \varphi}(y_2)(t) \) for \( t \geq t_{y_1} = t_{y_2} \). This shows that \( BB_{h'_1, \varphi}(y_1)(t_{y_1}, \infty) = BB_{h'_1, \varphi}(y_1)(t_{y_2}, \infty) \). Since we can choose \( \epsilon > 0 \) such that
\[
BB_{h'_1, \varphi}(y_1)(t_{y_1} + \epsilon) = BB_{h'_1, \varphi}(y_1)(t_{y_1} + \epsilon)
\]
and
\[
BB_{h'_1, \varphi}(y_2)(t_{y_2} + \epsilon) = BB_{h'_1, \varphi}(y_2)(t_{y_2} + \epsilon)
\]
we conclude that
\[
BB_{h'_1, \varphi}(y_1) = BB_{h'_1, \varphi}(y_1).
\]

Remark 2.10. Let \( f_1 : C' \to \mathbb{P}^1 \) and \( f_2 : C \to \mathbb{P}^1 \) be two finite morphisms between smooth projective irreducible \( k \)-curves. Let \( h_1 : [0, \infty] \times C' \to \mathbb{C} \) and \( h_2 : [0, \infty] \times C \to \mathbb{C} \) be two homotopies whose images are \( \Gamma \)-internal and which are the lifts of homotopies \( \psi_{D_1} : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1 \) and \( \psi_{D_2} : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1 \).
Recall that we have a map \( \pi : \mathbb{P}^1 \to \hat{\mathbb{P}}^1 \) respectively. By Theorem 2.9, there exists \( v + g \)-continuous functions \( \alpha_1 : C^* \to [0, \infty) \) and \( \alpha_2 : C \to [0, \infty] \) such that the cut-offs \( h_1[\alpha_1] \) and \( h_2[\alpha_2] \) preserve the functions \( BB_{h_1,f_1} \) and \( BB_{h_2,f_2} \) respectively. The proof of Theorem 2.9 uses arguments outlined in the proof of Theorem 2.5, from which we see that the functions \( \alpha_1 \) and \( \alpha_2 \) are the pull backs of \( v + g \)-continuous functions \( \beta_1 : \mathbb{P}^1 \to [0, \infty) \) and \( \beta_2 : \mathbb{P}^1 \to [0, \infty] \) via \( f_1 \) and \( f_2 \) respectively. Let \( \beta : \mathbb{P}^1 \to [0, \infty] \) be the function given by \( x \mapsto \max\{\beta_1(x), \beta_2(x), m_{D_1}, m_{D_2}\} \). The function \( \beta \) is \( v + g \)-continuous since it can be seen as the composition of \( v + g \)-continuous functions. One checks using the arguments in the proof of Theorem 2.9 that not only do \( \psi_{D_1}[\beta] \) lift to \( h_1[\beta \circ f_1] \) and \( \psi_{D_2}[\beta] \) lift to \( h_2[\beta \circ f_2] \) but more importantly that \( h_1[\beta \circ f_1] \) and \( h_2[\beta \circ f_2] \) respect the functions \( BB_{h_1,f_1} \) and \( BB_{h_2,f_2} \) respectively.

3. Path functions

3.1. Homotopies of \( C^{an} \). Let \( f : C \to \mathbb{P}^1 \) be a finite morphism. In §2, we discussed the construction of definable deformation retractions on \( \hat{C} \) via the morphism \( f \) and a suitable homotopy \( \psi_D : [0, \infty] \times \mathbb{P}^1 \to \hat{\mathbb{P}}^1 \). We now interpret these constructions in the analytic setting.

The homotopy \( \psi_D \) induces a homotopy \( \psi_D^{an} : [0, \infty] \times \mathbb{P}^{1,an} \to \mathbb{P}^{1,an} \) whose image is the closed subspace \( \Sigma_D := \pi_{k,P}(T_D) \) where \( \pi_{k,P} : \mathbb{P}(k^{max}) \to \mathbb{P}^{1,an} \) and \( T_D \) is the convex hull in \( \mathbb{P}^1 \) of the closed subset \( D \). If \( D \) is \( k \)-definable then one checks that for every \( x, y \in D \), the path \([x, y] \) in \( \mathbb{P}^1 \) is \( k \)-definable. We can describe these paths explicitly and one deduces that \( \Sigma_D \) is the convex hull in \( \mathbb{P}^{1,an} \) of the closed subset \( D \). Recall that \( \psi_D = \psi[m_D] \) and we describe \( \psi^{an} \) explicitly as follows. As before we choose a system of coordinates on \( \mathbb{P}^1 \). The standard metric \( m \) on \( \mathbb{P}^1 \) is defined by identifying \( \mathbb{P}^1 \) with the gluing of the closed unit ball around 0 and the closed unit ball around \( \infty \) along the annulus \( \{x \in \mathbb{P}^1 | T(x) = 1\} \) where \( T \) is the coordinate chosen.8 The homotopy \( \psi^{an} \) respects these two copies of the closed disk and hence it suffices to describe its restriction to the closed unit disk around 0.

Let \( B := \{x \in \mathbb{A}^{1,an} | |T(x)| \leq 1\} \) be the Berkovich closed unit disk around the origin. Let \( \zeta_{a,r} \) be a point of type I, II or III with \( a \in B(k) \) and \( r \in [0, 1] \). We have that \( \psi^{an}(t, \zeta_{a,r}) := \zeta_{a,max(\exp(-t), r)} \). Since \( \psi_D \) is by definition the cut-off \( \psi[m_D] \), we deduce that \( \psi_D^{an} = \psi[m_D] \) where \( \psi^{an}[m_D] \) is defined as for definable homotopies i.e. \( (t, x) \mapsto \psi^{an}(\max(t, m_D(x)), x) \).

Let \( h : [0, \infty] \times C \to \hat{C} \) be a homotopy that is the unique lift of a homotopy \( \psi_D : [0, \infty] \times \mathbb{P}^1 \to \hat{\mathbb{P}}^1 \). The results in [7, §14.1] show that the canonical extension \( H \) of \( h \) which is a homotopy on \( \hat{C} \) descends to a homotopy \( C^{an} \) via the map \( \pi_{k,C} \).

Since the homotopies of \( \hat{C} \) were all unique lifts of the homotopies on \( \hat{\mathbb{P}}^1 \), the induced homotopies on \( C^{an} \) will be the unique lifts of homotopies on \( \mathbb{P}^{1,an} \). i.e. we have the following commutative diagram.

\[
\begin{array}{ccc}
I \times C^{an} & \xrightarrow{h^{an}} & C^{an} \\
\downarrow{id \times f^{an}} & & \downarrow{f^{an}} \\
I \times \mathbb{P}^{1,an} & \xrightarrow{\psi_D^{an}} & \mathbb{P}^{1,an}
\end{array}
\]

where \( I = [0, \infty] \).

We are grateful to Jérôme Poineau for the proof of the following lemma.

**Lemma 3.1.** Recall that we have a map \( \pi_{k,C} : \hat{C}(k^{max}) \to C^{an} \). Let \( x \in C^{an} \) be a point of type I or II. Then \( \pi_{k,C}^{-1}(x) \) consists of precisely one point. If \( x \in C^{an} \) is

---

8See [12, Proposition 2.7] for a generalization of this construction.
of type III then \( \pi^{-1}_k(x) \) is homeomorphic to an annulus in \( \mathbb{A}^1_{k,an} \) whose skeleton consists of a single point. If \( x \) is of type IV then \( \pi^{-1}_k(x) \) is homeomorphic to a Berkovich closed disk contained in \( \mathbb{A}^1_{k,an} \).

Proof. The map \( \pi_{k,C} \) factors through \( \hat{C}(k^{\text{max}}) \to C^\text{an} \) and the projection \( \pi_{k,C}^\text{an} : C^\text{an} \to C^\text{an} \). By [7, Lemma 14.1.1], \( \hat{C}(k^{\text{max}}) \to C^\text{an} \) is a homeomorphism. Hence we restrict our attention to proving the analogous statements for the morphism \( \pi_{k,C}^\text{an} \). When \( x \) is of type I, it is clear that \( (\pi_{k,C}^\text{an})^{-1}(x) \) is a single point. When \( x \) is of type II, we have by [9, Theorem 2.15] that \( (\pi_{k,C}^\text{an})^{-1}(x) \) decomposes into a type II point \( x_{k^{\text{max}}} \) and the disjoint union of Berkovich open balls. Let \( O \) be one such open ball. Since the \( k^{\text{max}} \)-points are dense in \( C^\text{an} \), there exists \( z \in O(k^{\text{max}}) \). We must have that \( \hat{H}(x) \hookrightarrow \hat{H}(z) \). However this is not possible since \( \hat{H}(z) = k^{\text{max}} \) and \( \hat{k}^{\text{max}} = \hat{k} \).

Let \( x \in C^\text{an} \) be a point of type III. By [6, Theorem 4.3.5], there exists a neighbourhood \( X \subset C^\text{an} \) of \( x \) such that \( X \) is isomorphic to a Berkovich open annulus. We identify \( \Sigma := \pi_{k,C}(\Sigma^\text{max}) \) with \( \Sigma^\text{max} \) and see that there exists \( \hat{H}(x) \hookrightarrow \hat{H}(z) \). However this is not possible since \( \hat{H}(z) = k^{\text{max}} \) and \( \hat{k}^{\text{max}} = \hat{k} \).

Lemma 3.2. Let \( C \) be a smooth projective curve over \( k \) and \( f : C \to \mathbb{P}^1_\kappa \) be a finite morphism. Let \( D \subset \mathbb{P}^1_\kappa(k) \) be a divisor and let \( \psi_D : [0, \infty] \times \mathbb{P}^1 \to \hat{\mathbb{P}^1} \) be a \( k \)-definable homotopy whose image \( \Upsilon_D \) is \( \Gamma \)-integral. Let \( h : [0, \infty] \times C \to \hat{C} \) be a \( k \)-definable homotopy which is a lift of \( \psi_D \) and whose image is \( \Upsilon' \subset \hat{C} \). Let \( \Sigma' = \pi_{k,C}(\Upsilon')(k^{\text{max}}) \). We then have that \( \pi_{k,C}|_{\Upsilon'(k^{\text{max}})} : \Upsilon'(k^{\text{max}}) \to \Sigma' \) is a homeomorphism.

Proof. Let \( \Sigma := \pi_{k,C}(\Upsilon_D) \). We check using the explicit description of \( \Upsilon_D \) that the restriction \( \pi_{k,C}|_{\Upsilon'_D} : \Upsilon'_D(k^{\text{max}}) \to \Sigma \) is bijective. By [7, Lemma 14.1.2], it is closed as well and hence a homeomorphism.

By [7, Lemma 14.1.2] it suffices to show that \( \pi_{k,C}|_{\Upsilon'(k^{\text{max}})} \) is a bijection. Since \( \tilde{f} : \Upsilon' \to \Upsilon_D \) is a finite map and \( \pi_{k,C}|_{\Upsilon_D(k^{\text{max}})} \) is a bijection, we see that \( \pi_{k,C}|_{\Upsilon'(k^{\text{max}})} \) is finite.

We argue as in Lemma 3.1. By loc.cit., we know that the map \( \pi_{k,C}|_{\Upsilon'(k^{\text{max}})} \) is bijective over points of type I and II. Let \( x \in \Sigma' \) be a point of type III and let \( X \) be a neighbourhood in \( C^\text{an} \) which is homeomorphic to an open annulus in \( \mathbb{A}^1_{k,an} \). We identify \( \Upsilon'(k^{\text{max}}) \) with a subspace of \( C^\text{an} \) via the homeomorphism \( \hat{C}(k^{\text{max}}) \to C^\text{an} \). Recall the map \( \pi_{k,C}^\text{an} : C^\text{an} \to C^\text{an} \) and observe that \( (\pi_{k,C}^\text{an})^{-1}(x) \) is of type I point. Since \( \Upsilon'(k^{\text{max}}) \) is path connected, if \( z \neq x' \) and \( y \) is any other point in \( \Upsilon'(k^{\text{max}}) \) that doesn’t map to \( x \) then we must have a path from \( z \) to \( y \). However, this path must necessarily pass through to \( x' \) and contain an infinite number of preimages of \( x \). This is a contradiction and hence \( x' \) is the unique point in the fibre over \( x \) that is contained in \( \Upsilon'(k^{\text{max}}) \). □

3.2. Length of a definable path. Let \( f : C' \to C \) be a finite separable morphism of smooth projective connected curves. Let \( g : C \to \mathbb{P}^1_\kappa \) be a finite separable morphism. We deduce from [11, Lemma 3.3] that there exists a divisor \( D \subset \mathbb{P}^1 \)
exists a tuple $O$ from $\times \varepsilon v$ the tangent direction 

Let $x \in C'(k)$ and $y := f(x)$. Let $h^an_2 : [0, \infty] \rightarrow C^an$ denote the path defined by $t \mapsto h^an(t, x)$. Recall the isometric paths $l_z : (0, \infty] \rightarrow C^an$ and $l_y : (0, \infty] \rightarrow C^an$ from §1. The map $f^an$ induces a map $f^an_2 = l^{-1}_y \circ f^an \circ l_z : (0, \infty] \rightarrow (0, \infty]$. There exists a tuple $T_z := ((t_0 = \infty, d_0, \alpha_0), \ldots, (t_m, d_m, \alpha_m)) \subset ([0, \infty] \times \mathbb{N} \times [0, \infty])^{m_2}$ such that for every $i$, $f^an_2$ restricted to $(t_i, t_{i+1}]$ is the map $t \mapsto d_i t + \alpha_i$.

Suppose $x \notin \Sigma^\prime_2$. Let $O'$ be the connected component of $C^an \setminus \Sigma^\prime_2$ containing $x$ and $O$ be the connected component of $C^an \setminus \Sigma^\prime_2$ containing $y = f(x)$. We identify $O'$ and $O$ with the Berkovich open unit ball so that $x$ is the origin on $O'$ and $y$ is the origin on $O$. The function $f^an$ restricts to a map $O' \rightarrow O$. Let $F : O' \rightarrow \mathbb{R}_\infty$ be the function given by $p \mapsto -\log[T(f(p))]$ where $T$ is the coordinate on $O$. By Lemma 3.4, $F$ is piecewise affine as defined in [1, Definition 5.14]. Using the notation from the previous paragraph, observe that for every $i$, the integer $d_i$ is the outgoing slope (cf. [1, Definition 5.14]) of the piecewise affine function $F$ along the tangent direction $v$ towards the boundary of the ball containing $x$. We denote this $d_i F$.

In §2, we introduced the notion of the backward branching index $BB_{h', f}$ associated to the homotopy $h' : [0, \infty] \times C' \rightarrow \overline{C}'$ and the morphism $f : C' \rightarrow C$. Analogously, we define $BB_{h^an, f} : C'(k) \rightarrow \text{Fn}[0, \infty] \rightarrow \mathbb{N} \times [0, \infty)]$ where $\text{Fn}[0, \infty] \rightarrow \mathbb{N} \times [0, \infty)]$ denotes the set of definable functions from $[0, \infty]$ to $\mathbb{N} \times [0, \infty]$. Let $z \in C'(k)$ and $t \in [0, \infty]$. Let $BB_{h^an, g}(x)(t)$ denote the set of elements $z \in C'$ such that $g(z) = g(x)$ and $h^an(t, z) = h^an(t, x)$. Then

$$BB_{h^an, g}(x)(t) := (\text{card}(BB_{h^an, g}(x)(t)), \Sigma_z \in BB_{h^an, g}(x)(t) \text{val}(z)).$$

One checks using that if $z \in C'(k)$ then $BB_{h', f}(z) = BB_{h^an, f}(z)$. This follows from the following fact which can be deduced from Lemma 3.1. For every $z_1, z_2 \in C'(k)$ and $t \in [0, \infty](\mathbb{R})$, $h'(t, z_1) = h'(t, z_2)$ if and only if $h^an(t, z_1) = h^an(t, z_2)$. Hence $BB_{h^an, f}(z) \in \text{Fn}[0, \infty] \rightarrow \mathbb{N} \times [0, \infty]).$

We now show that the backward branching index associated to the homotopy $h^an$ coincides with the notion of outgoing slopes.

**Lemma 3.3.** We make use of the notation introduced above regarding the morphism $f : C' \rightarrow C$ and the homotopy $h' : [0, \infty] \times C' \rightarrow \overline{C}'$. Let $x \in C'(k) \setminus \Sigma^\prime_2$ and $p = l_z(t)$ where $t \in [0, \infty]$. Let $v$ be the tangent direction along $l_z$ towards $l_z(0)$. We have that

$$\text{card}(BB_{h^an, f}(x)(t)) = d_v F(p).$$

**Proof.** Let $y := f(x)$. Let $O'$ be the connected component of $C'(k) \setminus \Sigma^\prime_2$ containing $x$ and $O$ be the connected component of $C(k) \setminus \Sigma^\prime_2$ containing $y = f(x)$. We identify $O'$ and $O$ with the Berkovich open unit ball $O(0, 1)$ so $x$ and $y$ are the origins of the source and target respectively. We may hence identify the restriction of $f^an$ with an endomorphism of the Berkovich unit disk. The function $F$ was defined to be $p \mapsto -\log[T(f(p))]$ where $T$ is the coordinate on the target. Let $q := f^an(p)$. Let $b_0$ be the germ of the outgoing path from $q$ to the origin of $O$ and $b_\infty$ be the germ of the outgoing path from $q$ to the boundary of $O$. Observe that $T$ is constant along every outgoing path different from $b_0$ and $b_\infty$. It follows that if $T(p)$ is the tangent

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9One can go through the Hrushovski-Loeser construction to show that there is actually no need to enlarge $D$. 

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space at \( p \) i.e. the space of all outgoing paths from \( p \) then

\[
\Sigma_{z \in \tau(p)} d_F(p) = \Sigma_{w \in V_0} d_w F(p) + \Sigma_{u \in V_\infty} d_u F(p)
\]

where \( V_0 \) is the set of outgoing paths from \( p \) that map onto \( b_0 \) and \( V_\infty \) is the set of outgoing paths from \( p \) that map onto \( b_\infty \). By uniqueness of the lift, \( V_\infty \) consists of the single outgoing path from \( p \) to \( \infty \). Observe that this is the same path \( v \) in the statement of the lemma. Since \( F \) is harmonic, we see that

\[
d_v F(p) = -\Sigma_{w \in V_0} d_w F(p).
\]

Let \( p \) be of the form \( \zeta_0, r \) where \( r \in (0,1) \). By Lemma 3.4, we get that \( d_v F(p) \) is the sum of the order of vanishing of \( f \) at every zero in the closed ball of radius \( r \) around \( 0 \) i.e.

\[
d_v F(p) = \Sigma_{z \in \mathbb{Z}_f(r)} \text{ord}_z(f)
\]

where \( \mathbb{Z}_f(r) \) denotes the zeroes of \( f \) in the closed ball around \( 0 \) of radius \( r \).

To complete the proof, we show that

\[
\text{card}(B_{h^{an},f(x)}(t)(t)) = \Sigma_{z \in \mathbb{Z}_f(r)} \text{ord}_z(f).
\]

The homotopies \( h' \) and \( h \) are the unique lifts of a homotopy \( \psi_D \). The construction in [7, §7.5], implies that \( D \) must contain the image of the ramification loci of both \( g \) and \( g \circ f \). We deduce from this that \( f \) is unramified outside \( (g \circ f)^{-1}(D) \). It follows that for every \( z \in Z_f(r) \), \( \text{ord}_z(f) = 1 \). If \( z \) does not belong to the closed ball around \( 0 \) of radius \( r \) then \( h'(t,z) \neq p \). It follows that \( B_{h^{an},f(x)}(t) \subset Z_f(r) \) and hence

\[
\text{card}(B_{h^{an},f(x)}(t)(t)) \leq \Sigma_{z \in \mathbb{Z}_f(r)} \text{ord}_z(f)
\]

We show that if \( z \in Z_f(r) \) then \( h^{an}(t,z) = p \). Since \( h^{an} \) is a lift of the homotopy \( h^{an} \) and \( h^{an}(t,0) = q \), we get that \( f^{an} h^{an}(t,z) = q \). Furthermore, we know that \( f^{an}_z \) maps the image of \( l_z \) onto \( l_0 \) and hence the lift starting from \( z \) of the path \( h^{an}(-,0) : [0, \infty] \to C^{an} \) must be along \( l_z \). By assumption, \( z \) lies in the closed ball around \( 0 \) of radius \( r \) and hence we get that \( p \in l_z \). Since \( f^{an} \) is injective when restricted to the image of \( l_z \), we see that \( h^{an}(t,z) = p \). This proves that

\[
\Sigma_{z \in \mathbb{Z}_f(r)} \text{ord}_z(f) \leq \text{card}(B_{h^{an},f(x)}(t)(t))
\]

and by (1), we conclude the proof. \( \square \)

Recall from §1 that we used \( O(0,1) \) to denote the Berkovich open unit disk.

Lemma 3.4. Let \( f : O(0,1) \to O(0,1) \) be a clopen surjective analytic function that maps \( 0 \) to \( 0 \). Let \( F \) be the function \( p \mapsto -\text{log}|T(f(p))| \) where \( T \) is the coordinate on the target. The function \( F \) is piecewise affine and harmonic (cf. [1, Definition 5.14]).

Let \( r \in (0,1) \) and \( p := \zeta_0, r \). Let \( v \) be an outgoing branch from \( p \) that maps onto the outgoing branch from \( f(p) \) to the origin. We then have that the outgoing slope \( d_v F(p) \) is equal to the sum \( \Sigma_{z \in \mathbb{Z}_f(r,v)} - \text{ord}_z(f) \) where \( \mathbb{Z}_f(r,v) \) denotes those elements \( z \in Z_f(r) \) that belong to the open ball of radius \( r \) that intersects \( v \).

Proof. Let \( B' \subset O(0,1) \) be a Berkovich closed disk around \( 0 \) and \( B := f(B') \). By [11, Lemma 2.33], \( B \) is a Berkovich closed disk and we identify \( B' \) and \( B \) with the Berkovich closed unit disk around the origin. The function \( f : B' \to B \) corresponds to an element \( f \in k\{S\} \). By Weierstrass preparation, we get that \( f = c \prod_j (S-a_j)^{j_u} \) where \( u \in k\{S\} \) is invertible and \( |s(a)| = 1 \) for all \( s \in B' \). The map \( F_{\mid B'} \) coincides with the map \( s \mapsto \text{val}(c) + \sum_j -\text{log}(|(S-a_j)(s))| \) and this is well known to be
piecewise affine. Since $B'$ can be taken to be arbitrarily large in $O(0, 1)$, we get that $F$ is piecewise affine and harmonic on $O(0, 1)$.

We may assume without loss of generality that $v$ is the outgoing path from $p$ to the origin. By [1, Theorem 5.15], we get that

$$d_v F(p) = -\text{ord}_v(\tilde{f}_p)$$

where $\tilde{f}_p$ is the reduction as defined in §5.13 of loc.cit. We can calculate $\tilde{f}_p$ using the explicit description of $f$ to get the result. □

As before, let $x \in C'(k)$ and $y := f(x)$. Recall that the map $f^\text{an}$ induces a map $f^\text{an}_x : [0, \infty] \to \mathbb{R}$. There exists a tuple

$$T_x := ((t_0 = \infty, d_0, \alpha_0), \ldots, (t_{m_x}, d_{m_x}, \alpha_{m_x})) \subset ([0, \infty] \times \mathbb{N} \times [0, \infty])^{m_x}$$

such that for every $i$, $f^\text{an}_x$ restricted to $(t_i, t_{i+1})$ is the map $t \mapsto dt + \alpha_i$.

Recall from section §2.4, that the function $BB_{B^\text{an}} f(x) : [0, \infty] \to \mathbb{N} \times [0, \infty]$ is determined completely by the tuple $S_x := ((s_0 = \infty, e_0, \beta_0), \ldots, (s_{n_x}, e_{n_x}, \beta_{n_x}))$ where the $s_i$ are the break points i.e. for every $i$, $e_i \neq e_{i+1}$.

**Lemma 3.5.** Let $f : O(0, 1) \to O(0, 1)$ be a finite surjective analytic morphism such that $f(0) = 0$. There exists a real number $r_0 \in (0, 1)$ such that for every $r \geq r_0$, the restriction of $f$ to the ball $O(0, r)$ is of the form $T \mapsto [T - \beta_1] \ldots [T - \beta_k]$ where $\beta_1, \ldots, \beta_k$ are the preimages of $0$.

**Proof.** Let $r \in (0, 1)$ and $a \in k^*$ such that $|a| = r$. The function $T \mapsto f(aT)$ is an analytic function on the closed unit disk $B(0, 1)$. It follows from the Weierstrass preparation theorem that

$$f(aT) = c(r)(T - \alpha_1) \ldots (T - \alpha_s) u(T)$$

where $u$ is a unit on $B(0, 1)$ and $c(r) \in k^*$. By applying the transformation $T = T/a$, we see that if $0 \leq |T| \leq r$ then

$$f(T) = c(r)/a^s(T - a\alpha_1) \ldots (T - a\alpha_s) u(T/a).$$

Thus for $x \in O(0, r)$,

$$|f(T)(x)| = |c(r)/a^s||(T - a\alpha_1)(x)\ldots||(T - a\alpha_s)(x)||u(T/a)(x)|.$$  

Since $u(T)$ is a unit on $B(0, 1)$ and $T \mapsto T/a$ is an isomorphism from $O(0, r)$ to $O(0, 1)$ we get that $u(T/a)$ is a unit on $O(0, r)$ and hence

$$|f(T)(x)| = |c(r)/a^s||(T - a\alpha_1)(x)||\ldots||(T - a\alpha_s)(x)|.$$  

By [10, Lemma 2.2.5], $f_0 := l_0^{-1} f l_0$ is piecewise affine with integer slopes where $l_0$ is the isometric embedding of $(0, \infty)$ into $O(0, 1)$ as explained in §1. It should be noted that the notation employed in Temkin is multiplicative while we write the value group additively. It follows that there exists $t_0 \in [0, \infty]$ such that for $t \in [0, t_0]$, $f_0(t) = dt + \alpha$ for some $\alpha \in \text{val}(k^*)$. Since $f_0$ maps $O(0, 1)$ bijectively onto itself, we must have that $\alpha = 0$. Hence for $r > \exp(-t_0)$ and $r > |a\alpha_i|$ for every $i$, we must have that $|c(r)/a^s| = 1$. Setting $\beta_i = \alpha_i$ completes the proof of the lemma. □

**Corollary 3.6.** Let $f : O(0, 1) \to O(0, 1)$ be a finite surjective étale analytic morphism such that $f(0) = 0$. As before, let $f_0 := l_0^{-1} f o l_0$ where $l_0$ is the isometric embedding $(0, \infty) \to O(0, 1)$. Let

$$T_0 := ((t_0 = \infty, d_0, \alpha_0), \ldots, (t_{m_x}, d_{m_x}, \alpha_{m_x})) \subset ([0, \infty] \times \mathbb{N} \times [0, \infty])^m$$
be such that for every \( i, f_0 \) restricted to \( (t_i, t_{i+1}] \) is the map \( t \mapsto d_i t + \alpha_i \). Let \( \mathcal{B} := \{ \beta_0 = 0, \ldots, \beta_n \} \) be the preimages of 0 such that \( |\beta_0| < |\beta_1| < \ldots < |\beta_n| \) and \( |\beta_0| = 0 < |\beta_1| < \ldots < |\beta_n| \) be the distinct values of the set \( \{ |\beta_0|, \ldots, |\beta_n| \} \). We then have that \( m = t_i \) and for every \( 0 \leq j \leq m \), the following hold.

\[
\begin{align*}
(1) \quad & t_j = -\log |\beta_{i_j}|, \\
(2) \quad & d_j = \text{card} \{ \beta \in \mathcal{B} \mid |\beta| \leq |\beta_{i_j}| \}, \\
(3) \quad & \alpha_{j} = \sum_{\beta \in \mathcal{B} \mid |\beta| \leq |\beta_{i_j}|} (-\log |\beta|)
\end{align*}
\]

**Proof.** The corollary is deduced by explicit computation using the description provided by Lemma 3.5. \( \square \)

**Remark 3.7.** It might happen that for any \( x \in C' \), there exists \( t_0 \in [0, \infty) \) such that \( h'(−,x) \) is constant on \([0, t_0] \). As a consequence, \( s_i \geq t_0 \). The fact that the \( s_i \) take values in \((t_0, \infty)\) while the \( t_i \) take values in \([0, \infty)\) need to be taken into account when relating them. Hence we introduce an invariant that will help relate the Backward branching index with the profile functions. Given \( x \in C' \), let \( s_{n_x+1} \in [0, \infty) \) be the largest element such that \( h'(−,x) \) is constant when restricted to the interval \([0, s_{n_x+1}] \). By definition of \( BB_{h',x} \), we have that \( s_j \geq s_{n_x+1} \) for every \( 1 \leq j \leq n_x \).

**Lemma 3.8.** We make use of the notation introduced above. Suppose that \( C = \mathbb{P}^1 \) and \( g \) is the identity map. Let \( x \in C'(k) \). We then have that \( n_x = m_x \) and for every \( 0 \leq j \leq m_x = n_x \), the following equalities hold.

\[
\begin{align*}
(1) \quad & s_j - s_{n_x+1} = d_j t_j + \alpha_j, \\
(2) \quad & e_j = d_j, \\
(3) \quad & \alpha_j = \beta_{m_x} - \beta_j
\end{align*}
\]

Lastly, if \( h^\text{an}_x : [0, \infty] \to C'_{\text{an}} \) denotes the path defined by \( s \mapsto h^\text{an}_x(s, x) \) then for every \( a, b \in [s_i, s_{i+1}] \) with \( a < b \) and \( 1 \leq i \leq n_x \), we have that \( \rho(h^\text{an}_x(a), h^\text{an}_x(b)) = (1/d_i)(b - a) \) where \( \rho \) is the path distance metric on \( C'_{\text{an}} \).

**Proof.** Let \( T_x = ((t_0 = \infty, d_0, \alpha_0), \ldots, (l_{m_x}, d_{m_x}, \alpha_{m_x})) \). Recall that the function \( F \) given by \( t \mapsto -\log |T(f(p))| \) is piecewise affine and its breakpoints coincide with set \( \{ t_0, \ldots, t_{m_x} \} \). Lemma 3.3 relates the break points of the backward branching index \( BB_{h^\text{an}, f}(x) \) with the break points of \( F \) and we deduce that

\[
T_x = ((l_0^{-1} h^\text{an}_x(s_0), e_0, \alpha_0), \ldots, (l_{n_x}^{-1} h^\text{an}_x(s_{n_x}), e_{n_x}, \alpha_{n_x}))
\]

In particular, we see that \( n_x = m_x \) and verify assertion (2) i.e. \( e_i = d_i \). Let \( t = l_x^{-1} h^\text{an}_x(s) \) for \( s \in (s_{n_x+1}, \infty] \). We make the following sequence of deductions.

\[
\begin{align*}
(3) \quad & l_x(t) = h^\text{an}_x(s), \\
(4) \quad & f^\text{an} \circ l_x(t) = f^\text{an} \circ h^\text{an}_x(s).
\end{align*}
\]

Let \( y := f(x) \). By construction, the homotopy \( h' \) is the lift of a homotopy \( \psi_D : [0, \infty] \times \mathbb{P}^1 \to \overline{\mathbb{P}^1} \). The path \( \psi^\text{an}_{D,y} : [0, \infty] \to \mathbb{P}^1_{\text{an}} \) differs from the path \( l_y \) by a scaling factor. One checks that

\[
\psi^\text{an}_{D,y} \circ l_y(s) = \psi^\text{an}_{D,y}(s).
\]

Since \( h' \) is the lift of \( \psi_D \),

\[
f^\text{an} \circ h^\text{an}_x(s) = \psi^\text{an}_{D,y}(s) = l_y(s - s_{n_x+1}).
\]

One deduces from (4) above that for every \( s \in (s_{n_x+1}, \infty] \),

\[
(5) \quad s - s_{n_x+1} = f^\text{an}_x(t).
\]
It follows from (2) that $s_i - s_{n+1} = f^n(t_i)$. By definition of the invariants $d_i$ and $\alpha_i$, $f^n(t_i) = d_i + \alpha_i$. We may thus conclude a proof for assertion (1) of the lemma. Lastly, assertion (3) from the Lemma follows from Corollary 3.5.

We now prove the last part of the lemma. By Equation (2),
\[
\rho(h^\alpha_x(a), h^\alpha_x(b)) = \rho(t_x(f^n_x)^{-1}(a - s_{n+1}), t_x(f^n_x)^{-1}(b - s_{n+1}))
\]
\[
\rho(h^\alpha_x(a), h^\alpha_x(b)) = ||(f^n_x)^{-1}(a - s_{n+1}) - (f^n_x)^{-1}(b - s_{n+1})||.
\]

By Equation (5) and since $a, b \in [s_i, s_{i+1}]$, we must have that
\[
(f^n_x)^{-1}(a - s_{n+1}), (f^n_x)^{-1}(b - s_{n+1}) \in [t_i, t_{i+1}].
\]
Recall that if $t \in [t_i, t_{i+1})$ then $f^n(t) = d_i + \alpha_i$. It follows that
\[
(f^n_x)^{-1}(a - s_{n+1}) = (1/d_i)(a - s_{n+1}) - (1/d_i)(\alpha_i)
\]

and
\[
(f^n_x)^{-1}(b - s_{n+1}) = (1/d_i)(b - s_{n+1}) - (1/d_i)(\alpha_i).
\]

A simple calculation using Equation (7) now completes the proof.

\qed

3.3. Radiality. Lemma 3.8 tells us that the tuples $T_x$ and $S_x$ are equivalent as invariants of the morphism $f$ and the deformation $h'$. We use this fact along with Theorem 2.9 to show the following result which is also proved in [10, Theorem 3.3.11].

**Theorem 3.9.** Let $f : C' \to C$ be a finite separable morphism of smooth projective irreducible $k$-curves. There exists deformation retractions $h^\alpha : [0, \infty] \times C^\alpha \to C^\alpha$ and $h : [0, \infty] \times C^\alpha \to C^\alpha$ which are compatible with the morphism $f^\alpha$ and satisfy the following properties.

1. The images $\Sigma := h'(0, C^\alpha)$ and $\Sigma = h(0, C^\alpha)$ are skeleta of the curves $C^\alpha$ and $C^\alpha$.

2. We make use of the notation from §3.2. The functions $f^n_x$ are constant along the fibres of the retraction. Equivalently, for every $x, y \in C'(k)$, if $h^\alpha(x) = h^\alpha(y)$ then $T_x = T_y$. We write $T^f_x$ in place of $T_x$ to emphasize the role of the morphism $f$.

3. There exists a function $T^f : \Sigma' \to \bigcup_{M \in \mathbb{N}}((0, \infty] \times \mathbb{N})^M$ given by $p \mapsto T^f_p$ where if $T^f_p = ((t_0 = \infty, d_0, \alpha_0), \ldots, (t_m, d_m, \alpha_m))$ and $x$ retracts to $p$ via $h^\alpha$ then $T^f_x = T^f_p$. Furthermore, the function $T^f$ is definable by which we mean the following. If $\Sigma'_m$ denotes the locus of points $p \in \Sigma'$ where $T^f_p$ is an $m$-tuple then $\Sigma'_m$ is the union of finitely many segments whose end points are points of type I or II and for every $0 \leq i \leq m$, the function $p \mapsto d_i$ is constant on each such segment while $p \mapsto \alpha_i$ is piecewise affine on $\Sigma'_m$.

**Proof.** Let $g : C \to \mathbb{P}^1$ be a finite separable morphism. We choose a finite subset $D \subset \mathbb{P}^1(k)$ such that if $\Sigma_D \subset \mathbb{P}^{1,\text{an}}$ is the convex hull of $D$ then the following are satisfied.

1. Let $f' := g \circ f$. The preimages $\Sigma' := (f^n)^{-1}(\Sigma_D)$ and $\Sigma := (f^n)^{-1}(\Sigma_D)$ are skeleta of the curves $C^{\alpha}$ and $C^{\alpha}$ respectively.

2. The homotopy $\psi_D : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1$ lifts uniquely to homotopies $h' : [0, \infty] \times C' \to \mathbb{C}$ and $h : [0, \infty] \times C \to \mathbb{C}$ via the morphisms $f'$ and $f$ respectively.

3. The functions $BB_{h', f'} : C' \to \text{Fn}([0, \infty], \mathbb{N})$ and $BB_{h, g} : C \to \text{Fn}([0, \infty], \mathbb{N})$ are constant along the fibres of the retraction.
Assertion (1) follows from the [11, Lemma 3.3]. Assertion (2) is justified in §2.2. Assertion (3) is a consequence of Theorem 2.9 and Remark 2.10. By Lemma 3.8, we deduce from assertion (3) that if \( x', y' \in C^{an} \) are such that \( h^{an}(0, x') = h^{an}(0, y') \) then \( T^f_{x'} = T^f_{y'} \). Likewise, if \( x, y \in C^{an} \) are such that \( h^{an}(0, x) = h^{an}(0, y) \) then \( T^f_{x} = T^f_{y} \).

Let \( x' \in C' \) and \( x := f(x') \). Let

\[
T^f_{x'} := \{(t_0, d_0, \alpha_0), \ldots, (t_{n_1}, d_{n_1}, \alpha_{n_1})\}
\]

\[
T^f_{x} := \{(s_0, e_0, \beta_0), \ldots, (s_{n_2}, e_{n_2}, \beta_{n_2})\}
\]

and

\[
T^f_{x'} := \{(u_0, c_0, \gamma_0), \ldots, (u_{n_3}, c_{n_3}, \gamma_{n_3})\}.
\]

These tuples are such that if \( t \in (t_i, t_{i+1}] \) then we have that \( f'^{an}(t) = d_i t_i + c_i \). Likewise, if \( s \in (s_i, s_{i+1}] \) we have that \( g'^{an}(s) = e_i s_i + \beta_i \) and if \( u \in (u_i, u_{i+1}] \) we have that \( f^v(u) = c_i u_i + \gamma_i \).

Observe that if \( x'_1, x'_2 \in C' \) are such that \( h^{an}(0, x'_1) = h^{an}(0, x'_2) \) then \( T^f_{x'_1} = T^f_{x'_2} \). By construction, we have that \( T^f_{x'_1} = T^f_{x'_2} \) and if \( T^f_{y'_1} = T^f_{y'_2} \). By [10, Lemma 2.3.8], \( T^f_{x} \) is determined completely by \( T^f_{x'} \) and \( T^f_{x'_1} \).

To complete the proof, we must show part (3) of the theorem. By Lemma 3.8 and since \( S_2 \) is definable in \( x \), we get that part (3) is true for the functions \( T^g : \Sigma' \to \bigcup_{M \in \mathbb{N}} ([0, \infty] \times N)^{n_1} \) and \( T^{g \circ f} : \Sigma' \to \bigcup_{M \in \mathbb{N}} ([0, \infty] \times N)^{n_2} \).

\[ \square \]

**Remark 3.10.** We preserve our notation from Theorem 3.9. The proof above shows that \( T^f_{x} \) can be calculated from \( S^f_{x} \) and hence so can the functions \( p \mapsto m_p \) and for every \( i, p \mapsto d_i \) and \( p \mapsto c_i \).

We now prove Theorem 1.3 which was discussed in §1. Recall that our goal is to show that if given a closed definable subset \( X \) of the analytification of a smooth projective curve \( C^{an} \), there exists a skeleton \( \Sigma \) of \( C^{an} \) such that \( X \) is piecewise affine radial around \( \Sigma \) (cf. Definition 1.1). The key ingredient in the proof is Theorem 2.5 which proves a version of the radiality statement for Hrushovski-Looser curves.

**Proof.** (Theorem 1.3). Let \( X \subset C^{an} \) be a closed path connected \( k \)-definable set (cf. Definition 1.2). By definition, there exists a closed \( k \)-definable set \( X' \subset \mathcal{C} \) such that \( X' \) is closed and \( \pi_{k,c}(X') = X \). Let \( f : X \to \mathbb{P}^1 \) be a finite morphism. By Theorem 3.9, there exists a divisor \( D \subset \mathbb{P}^1 \) such that the deformation \( \psi_D \) lifts uniquely to a deformation retraction \( h : [0, \infty) \times C \to \mathcal{C} \), the image of the deformation \( h^{an} \) is a skeleton of \( C^{an} \) and for every \( x \in C(k) \), the function \( h^{an} \) depends only on the point \( h^{an}(0, x) \). Let \( \beta_X : C \to [0, \infty] \times \{\gamma_1, \gamma_2\} \) be as in §2.3. By Theorem 2.5, we can enlarge \( D \) and assume that \( \beta_X \) is constant along the fibres of the retraction \( h(0, -) \). By Theorem 2.9, we can further enlarge \( D \) and assume that the function \( BB_{h,f} \) is constant along the fibres of the retraction \( h \). Let \( \mathcal{Y} \subset \mathcal{C} \) denote the image of the deformation retraction \( h \). We have that \( \beta_X \) factors through a definable function \( \beta : \mathcal{Y} \to [0, \infty] \times \{\gamma_1, \gamma_2\} \). Recall from the construction of \( h^{an} \) and Lemma 3.1 that for \( x, y \in C(k) \), \( h(0, x) = h(0, y) \) if and only if \( h^{an}(0, x) = h^{an}(0, y) \). We deduce that \( \beta_X \) factors through the retraction \( h^{an}(0, -) \). Let \( \Sigma \) denote the image of the homotopy \( h^{an} \) and let \( \beta : \mathcal{Y}(k^{\text{max}}) \to \Sigma \to \Gamma_{\infty} \) be the function defined as follows. We set \( \beta(z) := p_1 \beta'(z) \) if \( \beta'(z) \neq (e(z), \gamma_2) \) and \( \beta(z) := \gamma_2 < 0 \) if \( \beta'(z) = (e(z), \gamma_2) \) where the notation \( e(z), \gamma_1, \gamma_2 \) is as in §2.3.

We make use of the notation introduced above Definition 1.1. More precisely, we refer to the function \( r_{\Sigma} : C^{an} \to [0, \infty] \) that measures the radius around the skeleton

\[ \text{The homeomorphism above is due to Lemma 3.2.} \]
Σ. Let \( x \in \mathbb{C}^{an} \) and \( \gamma := h^{an}(0, x) \). We must show that there exists \( p_X(\gamma) \in [0, \infty) \) such that \( x \in X \) if and only if \( r_\Sigma(x) \leq p_X(\gamma) \) where \( \gamma = h^{an}(0, x) \). Let us first assume that \( x \) is not of type IV. We see that there exists a point \( z \in C(k) \) such that \( x \) belongs to the image of the path \( h^{an}_z \) where \( h^{an}_z \) was defined in Lemma 3.8. By construction, \( h^{an}_z(t) \in X \) if and only if \( t \in [0, \beta(\gamma)] \). Lemma 3.8 allows us to calculate the distance \( \rho(\gamma) = h^{an}_z(0), h^{an}_z(\beta(\gamma)) \) using only the Backward branching index \( BB_{h^{an}_z}(z) \) or equivalently the tuple \( S_z \). Indeed, suppose

\[ S_z := \{(x_0, e_0, \beta_0), \ldots, (x_n, e_n, \beta_n)\} \]

and

\[ T_z := \{(t_0, d_0, a_0), \ldots, (t_n, d_n, a_n)\}. \]

Recall that the set \( S_{t_2} := \{s_1, \ldots, s_{n_1}\} \) is definable uniformly in the parameter \( z \). Let \( i(\gamma) \) be such that \( \beta(\gamma) \in [s_{i(\gamma)}, s_{i(\gamma)+1}] \). Using that \( S_{t_2} \) is uniformly definable, we deduce that \( \gamma \mapsto i(\gamma) \) is definable. Then by Lemma 3.8,

\[
\rho(\gamma, h^{an}_z(\beta(\gamma))) = t_{i(\gamma)+1} + (1/d_{i(\gamma)})(\beta(\gamma) - s_{i(\gamma)+1})
\]

\[
= [(1/d_{i(\gamma)+1})(s_{i(\gamma)+1} - s_{n_x+1}) - (1/d_{i(\gamma)+1})(\beta_{m_x} - \beta_{i(\gamma)+1})] + (1/d_{i(\gamma)})(\beta(\gamma) - s_{i(\gamma)+1})
\]

In the equation above, we have used the fact that

\[
\rho(h^{an}_{z}(s_{i(\gamma)+1}), \gamma) = \rho(l_{z}(t_{i(\gamma)+1}), \gamma)
\]

which follows from Equations (3) and (5) in the proof of Lemma 3.8. By construction, \( BB_{h^{an}_z}(z) \) depends only on \( \gamma = h^{an}(0, z) \). By Lemma 3.8, \( T_z \) depends only on \( h^{an}_z(0) \).

Let

\[
p_X : \Sigma \to [0, \infty]
\]

\[
p \mapsto [(1/d_{(p)+1})(s_{(p)+1} - s_{n_x+1}) - (1/d_{(p)+1})(\beta_{m_x} - \beta_{i(p)+1})] + (1/d_{i(p)})(\beta(\gamma) - s_{i(p)+1})
\]

One checks using the computations above that if \( x \in \mathbb{C}^{an} \) is a point not of type IV that retracts to \( \gamma \) then \( x \in X \) if and only if \( r_\Sigma(x) \leq p_X(\gamma) \). Given \( p \in \Sigma \), let \( S_p := S_z \) where \( z \in C \) is any point that retracts to \( p \). By construction, the tuple \( S_p \) is well defined. Furthermore, \( S_p \) varies definably along \( \Sigma \). Hence, we have that \( p_X \) is piecewise affine with rational slopes and parameters in \( \text{val}(k^*) \).

Note that \( X \) is closed and the points of type I, II and III are dense in

\[
x \in \mathbb{C}^{an}| |r_\Sigma(x)| \leq p_X(h^{an}(0, x))
\]

Hence it suffices to show that the points of type I, II and III are dense in \( X \). If this was not true then we must have that there exists a point \( q \in X \) of type IV such that \( q \) is isolated. This is impossible since \( X \) is path connected and intersects the skeleton \( \Sigma \), which does not contain any points of type IV, non trivially.

4. Tameness of families of profile functions

Let \( \phi : C' \to C \) be a finite morphism of smooth projective irreducible \( k \)-curves. Let \( h' : [0, \infty] \times C^{an} \to C^{an} \) and \( h : [0, \infty] \times C^{an} \to C^{an} \) be deformation retractions such that the images \( \Sigma' \) and \( \Sigma \) are skeleta of \( C^{an} \) and \( C^{an} \) respectively. In §1, we used the decompositions associated to \( \Sigma' \) and \( \Sigma \) to define a function \( \phi^{an}_x : [0, \infty] \to [0, \infty] \) for every \( x \in C'(k) \). Theorem 3.9 implies that we can assume the functions \( \phi^{an}_x \) are determined by the retraction associated to \( h' \). In other words, we can assume that there exists some data \( D_p \) for every \( p \in \Sigma' \) such that if \( x \) retracts to \( p \) then \( D_p \) determines \( \phi^{an}_x \) completely. Our goal in this section is to prove Theorem 4.1 which apart from being a relative version of Theorem 3.9 also implies a finiteness property for a family of morphisms \( f_s : X_s \to Y_s \) of smooth projective
irreducible $k$-curves parametrized by a quasi-projective $k$-variety $S$. More precisely, we show that there exists a family of tuples $\{(\Sigma'_s \subset X_s, \Sigma_s \subset Y_s)\}_{s \in S}$ such that the family and its associated profile functions are controlled by a finite simplicial complex embedded in $S^{an}$. Recall that we showed that profile functions can be controlled by skeleta by verifying its model theoretic analogue i.e. Theorem 2.9 where we proved that the Backward branching index can be controlled by skeleta. The primary advantage of this perspective is that it allows us to prove 4.1 using a relative version of 2.9 that shows the required finiteness property for the backward branching invariants as they vary in a family. It should be noted that constructing deformation retractions for the analytification of a general higher dimensional variety is extremely hard and the only available tool which allows for flexibility is the construction of Hrushovski-Loeser. This was one of the primary motivations behind rephrasing the results of Temkin in §2.

**Notation:** For reasons that will be clear from the statement of Theorem 4.1, we introduce the following notation. In place of $\varphi^{an}_\Sigma$ as introduced above, we write $\text{prfl}(\varphi^{an}_\Sigma, x)$.

**Theorem 4.1.** Let $\alpha_1 : X \to S$ and $\alpha_2 : Y \to S$ be smooth morphisms of quasi-projective $k$-varieties. Let $f : X \to Y$ be a morphism such that for every $s \in S$, $f_s : X_s \to Y_s$ is a finite separable morphism of smooth projective irreducible $k$-curves. There exists a deformation retraction $g^{an} : I \times S^{an} \to S^{an}$ whose image $\Sigma(S) \subset S^{an}$ is homeomorphic to a finite simplicial complex and is such that the following properties hold.

1. For every $s \in S(k)$, there exists deformation retractions
   
   \[ h^{an}_s : [0, \infty) \times X^{an}_s \to X^{an}_s \]
   
   \[ h^{an}_s : [0, \infty) \times Y^{an}_s \to Y^{an}_s \]
   
   which satisfy the assertions of Theorem 3.9. More precisely the following hold.
   a. The images $\Sigma'_s := h^{an}_s(0, X^{an}_s)$ and $\Sigma_s = h^{an}_s(0, Y^{an}_s)$ are skeleta of the curves $X^{an}_s$ and $Y^{an}_s$.
   b. We make use of the notation introduced above and Theorem 3.9. The functions $\text{prfl}(f^{an}_s, (-))$ are constant along the fibres of the retraction $h^{an}_s(0, -)$. Said otherwise, for every $x_1, x_2 \in X_s(k)$, if $h^{an}_s(0, x_1) = h^{an}_s(0, x_2)$ then $\text{prfl}(f^{an}_s, x_1) = \text{prfl}(f^{an}_s, x_2)$ (or equivalently $T^{f_s}_{x_1} = T^{f_s}_{x_2}$).
   c. There exists a function $T^{f_s} : \Sigma'_s \to \bigcup_{M \in \mathbb{N}}([0, \infty) \times \mathbb{N})^M$ given by $p \mapsto T^{f_s}_p$ where if $T^{f_s}_p = ((t_0, d_0, \alpha_0), \ldots, (t_m, d_m, \alpha_m))$ (cf. §3.2 and Theorem 3.9 (2) for notation) and $x$ retracts to $p$ via $h^{an}_s$ then $T^{f_s}_p = T^{f_s}_{x}$. Furthermore, the function $T^{f_s}$ is definable.\(^{11}\)

2. The tuple $(f^{an}_s : \Sigma'_s \to \Sigma_s, T^{f_s})$ is constant along the fibres of the retraction of the deformation $g^{an}$ i.e. if $e$ denotes the end point of the interval $I$ and if $s_1, s_2 \in S(k)$ are such that $g^{an}(e, s_1) = g^{an}(e, s_2)$ then $\Sigma'_s = \Sigma'_{s_1}$ and $\Sigma_s = \Sigma_{s_1}$. Furthermore, with these identifications $(f^{an}_{s_1})_{\Sigma'_s} = (f^{an}_{s_2})_{\Sigma'_s}$ and $T^{f_{s_1}} = T^{f_{s_2}}$.

\(^{11}\)By this we mean the following. We can divide $\Sigma'_s$ into the disjoint union of line segments and points such that on each such subspace, the functions $p \mapsto d_i$ and $p \mapsto m_p$ are constant while $p \mapsto \alpha_i$ is piecewise affine.
Proof. We use Lemma 4.2 to deduce that there exists a finite family \( \{ S_1, \ldots, S_m \} \) of sub-varieties of \( S \) such that \( S = \bigsqcup_{1 \leq i \leq m} S_i \) and if \( Y_i := Y \times S_i \) then \((\alpha 2)\gamma_y\) factors through a finite morphism \( \psi : Y_i \to \mathbb{P}^1 \times S_i \). We can now apply Theorem 4.5. It follows that there exists families of deformation retractions

\[ \{ h'_s : [0, \infty] \times X_s \to \widetilde{X}_s \}_{s \in S} \]

and

\[ \{ h_s : [0, \infty] \times Y_s \to \widetilde{Y}_s \}_{s \in S} \]

which are uniformly definable in the parameter \( s \) and for every \( s \in S \), the images \( Y'_s := h'_s(0, X_s) \) and \( Y_s = h_s(0, Y_s) \) are skeleta of the curves \( \widetilde{X}_s \) and \( \widetilde{Y}_s \). Observe from the construction in Theorem 4.5 that there exists a family of deformation retractions

\[ \{ h''_s : [0, \infty] \times \mathbb{P}^1 \times \{ s \} \to \mathbb{P}^1 \times \{ s \} \}_{s \in S} \]

which are uniformly definable in the parameter \( s \) and such that for every \( s \in S \), \( h'_s \) and \( h_s \) are the unique lifts of \( h''_s \). As in §3.1, we deduce that for every \( s \in S(k) \), there exists homotopies

\[ h^{\text{an}}_s : [0, \infty] \times X^{\text{an}}_s \to X^{\text{an}}_s \]

and

\[ h^{\text{an}}_s : [0, \infty] \times Y^{\text{an}}_s \to Y^{\text{an}}_s \]

whose images \( \Sigma'_s := h^{\text{an}}_s(0, X^{\text{an}}_s) \) and \( \Sigma_s = h^{\text{an}}_s(0, Y^{\text{an}}_s) \) are skeleta of the curves \( X^{\text{an}}_s \) and \( Y^{\text{an}}_s \).

Let \( s \in S_i \) for some \( i \in \{ 1, \ldots, m \} \). By (1b) of Theorem 4.5, we have that the functions \( \mathbb{B}B_{\psi, \phi, \alpha} \) and \( BB_{h_s, \psi, \phi} \) are constant along the fibres of the retractions \( h'_s(0, -) \) and \( h_s(0, -) \). It follows from Lemma 3.8 that if \( x_1, x_2 \in X_s(k) \) and \( h^{\text{an}}_s(0, x_1) = h^{\text{an}}_s(0, x_2) \) then \( T^{\psi_i, \alpha}_{x_1} = T^{\psi_i, \alpha}_{x_2} \). Likewise, if \( y_1, y_2 \in Y_s(k) \), and \( h^{\text{an}}_s(0, y_1) = h^{\text{an}}_s(0, y_2) \), then \( T^{\psi_s}_{y_1} = T^{\psi_s}_{y_2} \). We apply the arguments from the proof of Theorem 3.9 to get that if \( x_1, x_2 \in X_s(k) \) and \( h^{\text{an}}_s(0, x_1) = h^{\text{an}}_s(0, x_2) \) then \( T^{\phi}_{x_1} = T^{\phi}_{x_2} \).

Let \( g : I \times \widehat{S} \to \widehat{S} \) be the deformation retraction as provided by Theorem 4.5. We have that the image of \( g \) is a \( \Gamma \)-internal set. By [7, Corollary 14.1.6], the deformation retraction \( g \) induces a deformation retraction \( g^{\text{an}} : I \times S^{\text{an}} \to S^{\text{an}} \). The image \( \Sigma(S) \) of \( g^{\text{an}} \) is homeomorphic to a finite simplicial complex. By (2) of Theorem 4.5, Lemma 3.2 and Lemma 3.8 we deduce part (2). This concludes the proof.

\[ \square \]

Lemma 4.2. Let \( f : V' \to V \) be a projective morphism of \( k \)-varieties such that the fibres of \( f \) are pure of dimension \( m \). For every \( v \in V(k) \), there exists a Zariski open neighbourhood \( U \subset V \) of \( v \) such that the morphism \( f : f^{-1}(U) \to U \) factors through a finite \( k \)-morphism \( p : f^{-1}(U) \to \mathbb{P}^n \times U \). If \( u \in U \) is such that \( V'_u \) is generically reduced then the morphism \( p_u : V'_u \to \mathbb{P}^n \times \{ u \} \) is generically étale.

Proof. This is a relative version of [7, Lemma 11.2.1]. Let \( v \in V \). Let \( n \in \mathbb{N} \) be the smallest natural number greater than or equal to \( m \) such that there exists a Zariski open neighbourhood \( U \) of \( v \) and the morphism \( f \) factors through a finite morphism \( i : V'_U \to \mathbb{P}^m \times U \). The fact that there exists such an \( n \) is because \( f \) is projective. Let \( V'_U \) denote the fibre over \( v \). If \( m = n \) then we have nothing to prove. Suppose \( n > m \). Let \((z, v) \in (\mathbb{P}^n \times V)(k)\) be a point that is not contained in \( i(V'_U) \). Let \( C := i(V') \cap (\{ z \} \times V) \subset \mathbb{P}^n \times V \). Observe that \( C \) is a closed subset of \( \mathbb{P}^n \times V \) and hence \( p_2(C) \) is a closed subset of \( V \) where \( p_2 \) is the projection \( \mathbb{P}^n \times V \to V \). Furthermore, our choice of \( z \in V \) implies that \( v \notin p_2(C) \). Let \( U \) denote the Zariski open set which is the complement of \( C \) in \( V \). By construction, for every \( u \in U \), \((z, u) \notin i(V'_U) \). Let \( p : \mathbb{P}^n \setminus \{ z \} \to \mathbb{P}^{n-1} \) denote the projection through the point.
Proposition 4.3. Let \( S \) be a quasi-projective \( k \)-variety and \( Y := \mathbb{P}^1 \times S \). Let \( \alpha_2 : Y \to S \) be smooth morphisms of quasi-projective \( k \)-varieties. Let \( f : X_1 \to X_2 \) be an \( S \)-morphism such that for every \( s \in S \), \( f_s : X_{1s} \to X_{2s} \) is a finite morphism of smooth projective irreducible \( k \)-curves. Let \( g : X_2 \to Y \) be a \( S \)-morphism such that for every \( s \in S \), \( g_s : X_{2s} \to Y_s \) is a finite morphism. There exists a uniformly definable family \( \{ h_s : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1 \}_{s \in S} \) of deformation retractions where for every \( s \in S \), \( h_s \) is \( k(s) \)-definable and the family satisfies the following properties.

1. For every \( s \in S \), the image of the retraction associated to \( h_s \) is a \( \Gamma \)-internal subset of \( \mathbb{P}^1 \times \{ s \} \).
2. For every \( s \in S \), the deformation retraction \( h_s \) lifts uniquely to deformation retractions \( h'_{1s} : [0, \infty] \times X_{1s} \to \bar{X}_{1s} \) and \( h'_{2s} : [0, \infty] \times X_{2s} \to \bar{X}_{2s} \).
3. For every \( s \in S \), the functions 

\[
BB_{h'_{1s}, g_s} : X_{1s} \to \text{Fn}([0, \infty], \mathbb{N} \times [0, \infty])
\]

and 

\[
BB_{h'_{1s}, g_s \circ f_s} : X_{1s} \to \text{Fn}([0, \infty], \mathbb{N} \times [0, \infty])
\]

(cf. \( \S 2.4 \)) are constant along the fibres of the retraction associated to \( h'_{1s} \) and \( h'_{1s} \), respectively.

Proof. The proof of [7, Lemma 11.3.1] shows that there exists a constructible set \( D \subset Y \) such that for every \( s \in S \), \( D_s \) is finite, \( k(s) \)-definable and its convex hull \( \overline{D_s} \subset \mathbb{P}^1 \times \{ s \} \) contains the set of forward branching points of the morphism \( g_s \) and \( g_s \circ f_s \). We fix a definable metric \( m : \mathbb{P}^1 \times \mathbb{P}^1 \to \Gamma_\infty \) as in [7, \S 3.10]. Let \( \kappa : Y \to \Gamma_\infty \) be the function \( (y, s) \mapsto m_{D_s}(y) \) where \( m_{D_s} \) is as defined in \( \S 2.2 \).

By [7, \S 7.4], if \( s \in S \) then any homotopy on \( \mathbb{P}^1 \times \{ s \} \) that respects the function \( m_{D_s} \) lifts uniquely to homotopies on \( \bar{X}_{1s} \) and \( \bar{X}_{2s} \). By [7, Theorem 11.7.1], there exists a uniformly definable family \( \{ h_{0s} : [0, \infty] \times \mathbb{P}^1 \to \mathbb{P}^1 \}_{s \in S} \) of deformation retractions such that for every \( s \in S \), \( h_{0s} \) lifts uniquely to deformation retractions \( h'_{01s} : [0, \infty] \times X_{1s} \to \bar{X}_{1s} \) and \( h'_{02s} : [0, \infty] \times X_{2s} \to \bar{X}_{2s} \) whose images are \( \Gamma \)-internal.

Let \( s \in S \). In Theorem 2.9, we showed that there exists a finite family of definable functions \( \{ \zeta_{1s} : X_{1s} \to \Gamma_\infty, \ldots, \zeta_{ms} : X_{1s} \to \Gamma_\infty \} \) such that if \( \alpha_{1s} : X_{1s} \to [0, \infty] \) is \( v + g \)-continuous and \( h_{01s}[\alpha_{1s}] \) preserves the levels of the functions \( \zeta_{is} \) then the function \( BB_{h_{01s}[\alpha_{1s}], g_s \circ f_s} \) is constant along the retraction associated to \( h'_{01s}[\alpha_{1s}] \). Similarly, there exists a finite family of definable functions \( \{ \zeta'_{1s} : X_{2s} \to \Gamma_\infty, \ldots, \zeta'_{ms} : X_{2s} \to \Gamma_\infty \} \) such that if \( \alpha_{2s} : X_{2s} \to [0, \infty] \) is \( v + g \)-continuous and \( h_{02s}[\alpha_{2s}] \) preserves the levels of the functions \( \zeta'_{is} \) then the function \( BB_{h_{02s}[\alpha_{2s}], g_s} \) is constant along the retraction associated to \( h'_{02s}[\alpha_{2s}] \).

Since the families \( \{ h'_{01s} \}_{s \in S} \) and \( \{ h'_{02s} \}_{s \in S} \) are uniform, we can take the functions \( \{ \zeta_{1s}, \ldots, \zeta_{ms} \} \) and \( \{ \zeta'_{1s}, \ldots, \zeta'_{ms} \} \) to be uniform in \( s \). The arguments in the proof of Theorem 2.5 shows that for every \( s \), there exists definable functions \( \{ \alpha_{1s} : \mathbb{P}^1 \to \Gamma_\infty, \ldots, \alpha_{ns} : \mathbb{P}^1 \to \Gamma_\infty \} \) and \( \{ b_{1s} : \mathbb{P}^1 \to \Gamma_\infty, \ldots, b_{ns} : \mathbb{P}^1 \to \Gamma_\infty \} \) such that if \( \alpha_s : \mathbb{P}^1 \to [0, \infty] \) is \( v + g \)-continuous and \( h_{0s}[\alpha_s] \) preserves the levels of
the functions \( a_{is} \) and \( b_{is} \) then the lifts \( h'_{2s} := h'_{1s} [\alpha_s \circ g_s] \) and \( h'_{1s} := h'_{1s} [\alpha_s \circ g_s \circ f_s] \) preserve the levels of the functions \( \zeta_{is} \) and \( \kappa'_{is} \) and consequently \( BB_{\zeta'_{ks}, g_s} \) and \( BB_{\kappa'_{ks}, g_s \circ f_s} \) will be constant along the fibres of the retractions \( h'_{1s} \) and \( h'_{2s} \) respectively. Since the given data is uniform in the parameter \( s \), we get that the functions \( a_{1s}, \ldots, a_{ns}, b_{1s}, \ldots, b'_{ns} \) are uniformly definable as well and as a consequence, we can suppose that the family \( \alpha_s \) is uniformly definable in the parameter \( s \). The homotopies \( h_s := h_{0s}[\alpha_s], h'_{1s} \) and \( h'_{2s} \) satisfy the assertions of the proposition. \( \square \)

Remark 4.4. Let \( f : C' \rightarrow C \) be a finite morphism between smooth irreducible projective \( k \)-curves. Let \( h' : [0, \infty) \times C' \rightarrow \overline{C} \) be a homotopy whose image \( \overline{Y} \) is \( \Gamma \)-internal and the function \( BB_{h', f} \) is constant along the fibres of the retraction \( h'(0, -) \). In this case, we can define
\[
BB_{h', f} : Y \rightarrow \text{Fn}([0, \infty), \mathbb{N} \times [0, \infty])
\]
where \( \text{Fn}([0, \infty), \mathbb{N} \times [0, \infty]) \) is the set of definable functions from \([0, \infty)\) to \( \mathbb{N} \times [0, \infty] \).

Let \( p \in Y \) and \( x \in C \) be a point that retracts to \( p \). We define \( BB_{h', f}(p) := BB_{h', f}(x) \). Our assumption that \( BB_{h', f} \) is constant along the fibres of the retraction associated to \( h' \) implies that the function on \( Y \) is well defined.

Theorem 4.5. Let \( S \) be a quasi-projective \( k \)-variety. Let \( \phi_1 : X \rightarrow S \) and \( \phi_2 : Y \rightarrow S \) be projective morphisms of quasi-projective varieties. Let \( \phi : X \rightarrow Y \) be a morphism such that for every \( s \in S \), \( \phi : X_s \rightarrow Y_s \) is a finite morphism of smooth projective irreducible \( k \)-curves. Let \( \{S_1, \ldots, S_m\} \) be a family of disjoint sub-varieties of \( S \) such that \( S = \bigsqcup_{1 \leq i \leq m} S_i \) and for every \( i \), there exists a finite morphism \( \psi_i : Y_i := Y \times_S S_i \rightarrow \mathbb{P}^1_{S_i} \).

We then have that there exists a deformation retraction \( g : I \times S \rightarrow \hat{S} \) whose image \( \overline{Y}(S) \subset \hat{S} \) is \( \Gamma \)-internal and is such that the following properties hold.

1. There exists families of \( k(s) \)-definable deformation retractions
\[
\{h'_s : [0, \infty) \times X_s \rightarrow \overline{X}_s\}_{s \in S}
\]
and
\[
\{h_s : [0, \infty) \times Y_s \rightarrow \overline{Y}_s\}_{s \in S}
\]
which are uniformly definable in the parameter \( s \) and which satisfy the following conditions.
   (a) For every \( s \in S \), the images \( Y'_s := h'_s(0, X_s) \) and \( T_s = h_s(0, Y_s) \) are skeletons of the curves \( \overline{X}_s \) and \( \overline{Y}_s \).
   (b) For every \( i \in \{1, \ldots, m\} \) and \( s \in S_i \), the functions \( BB_{h'_s, \psi_i, o\phi_i} \) and \( BB_{h_s, \psi_s} \) are constant along the fibres of the retractions \( h'_s(0, -) \) and \( h_s(0, -) \).

2. The tuple \( (\phi_s |_{Y'_s} : Y'_s \rightarrow T_s, (BB_{h'_s, \psi_i, o\phi_i})|_{Y'_s}, (BB_{h_s, \psi_s})|_{T_s}) \) (cf. Remark 4.4) is constant along the fibres of the retraction of the deformation \( \tilde{q} \) i.e. if \( e \) denotes the end point of the interval \( I \) and if \( s_1, s_2 \in S \) are such that \( g(e, s_1) = g(e, s_2) \) then \( Y'_{s_1} = Y'_{s_2} \) and \( T_{s_1} = T_{s_2} \). Furthermore, with these identifications
\[
(\phi_{s_1})|_{Y'_{s_1}} = (\phi_{s_2})|_{Y'_{s_2}}
\]
\[
(BB_{h'_{s_1}, \psi_i, o\phi_i})|_{Y'_{s_1}} = (BB_{h'_{s_2}, \psi_i, o\phi_i})|_{Y'_{s_2}}
\]
\[
(BB_{h_{s_1}, \psi_{s_1}})|_{T_{s_1}} = (BB_{h_{s_2}, \psi_{s_2}})|_{T_{s_2}}
\]

Proof. By applying Proposition 4.3 to \( X \times_S S_i \rightarrow Y_i \rightarrow \mathbb{P}^1_{S_i} \) for every \( i \in \{1, \ldots, m\} \), we deduce that there exists uniformly definable families \( \{h'_s : [0, \infty) \times X_s \rightarrow \overline{X}_s\}_{s \in S} \)
and \( \{ h_s : [0, \infty] \times Y \to \widehat{Y}_s \}_{s \in S} \) which satisfy conditions 1(a) and 1(b) above. In particular, we see that \( Y' := \bigcup_s \Gamma' \) and \( Y := \bigcup_s \Upsilon_s \) are iso-definable and relatively \( \Gamma \)-internal subsets of \( X/S \) and \( Y/S \) respectively. By [7, Theorem 6.4.2], there exists a finite pseudo-Galois covering \( \alpha : S' \to S \) such that if \( X' := X \times_S S' \) and \( Y' := Y \times_S S' \) then there exists definable morphisms \( \lambda : X' \to S' \times \Gamma_\infty^M \) for some \( N \in \mathbb{N} \) and \( \lambda : Y' \to S' \times \Gamma_\infty^M \) for some \( M \in \mathbb{N} \) such that \( \lambda : X' \to S' \times \Gamma_\infty^M \) and \( \lambda : Y' \to S' \times \Gamma_\infty^M \) are continuous morphisms that are injective when restricted to \( \Upsilon'_1 \) and \( \Upsilon_1 \). Observe that for every \( s \in S, \Upsilon'_1 \) and \( \Upsilon_1 \) are definably compact since they are the images via continuous maps of \( \widehat{X}_s \) and \( \widehat{Y}_s \) which are definably compact. Since \( S' \to S \) is a finite morphism, we deduce that for every \( s \in S' \), \( \Upsilon'_1 \) and \( \Upsilon_1 \) are definably compact. It follows that for every \( s \in S', \widehat{X}_s \) and \( \widehat{Y}_s \) restrict to homeomorphisms from \( \Upsilon'_1 \) and \( \Upsilon_1 \) onto their images in \( \{ s \} \times \Gamma_\infty^N \) and \( \{ s \} \times \Gamma_\infty^M \). Identifying \( \Upsilon'_1 \) and \( \Upsilon_1 \) with their images via \( \lambda'_s \) and \( \lambda_s \), we have uniformly definable families \( \{ \Upsilon'_1 \}_{s \in S'} \) and \( \{ \Upsilon_1 \}_{s \in S'} \) of subsets of \( \Gamma_\infty^N \) and \( \Gamma_\infty^M \) respectively. The functions \( \BB_{h'_s, \psi_s, \phi_s} \BB_{h_s, \psi_s} \) lift via pull back to functions \( \alpha'_s \) and \( \beta'_s \) on \( \Upsilon'_1 \) and \( \Upsilon_1 \) (cf. Lemma 4.4). Furthermore, since \( \BB_{h'_s, \psi_s, \phi_s} \BB_{h_s, \psi_s} \) are uniformly definable, we see that \( \alpha'_s \) and \( \beta'_s \) are uniformly definable for \( s \in S' \). Recall that for every \( s \in S, \BB_{h'_s, \psi_s, \phi_s} \) is completely determined by a finite collection of \( \Gamma_\infty \)-valued definable functions which can be assumed to be uniform in the parameter \( s \). The pull back of these functions to \( \Upsilon'_1 \) must then completely determine \( \alpha_s \). The same holds for the collection \( \beta_s \).

We use the arguments in [7, Theorem 6.4.4], to get a finite family of \( \Gamma_\infty \)-valued definable functions \( \xi_j \) on \( S' \) such that for every \( s \in S' \), \( \{ \xi_j(s) \} \), completely determine \( \Upsilon'_1 \), \( \Upsilon_1 \), and the functions \( \alpha_s \) and \( \beta_s \) and the morphism \( \Upsilon'_1 \to \Upsilon_1 \). More precisely, if for \( s, s' \in S', \xi_j(s) = \xi_j(s') \) for every \( j \) then \( \Upsilon'_1 = \Upsilon'_{1,s'}, \Upsilon_1 = \Upsilon_{1,s'} \), and with these identifications \( \Upsilon'_{1,s'} \to \Upsilon_{1,s'} \) coincides with \( \Upsilon'_1 \to \Upsilon_1 \), \( \alpha_{s'} = \alpha_s \) and \( \beta_{s'} = \beta_s \). By [7, Theorem 11.1.1], there exists a deformation retraction \( g' : I \times S' \to \widehat{S} \) whose image is \( \Gamma \)-internal, the functions \( \xi_j \) are constant along the fibres of the retraction and \( g' \) is \( \text{Aut}(S'/S) \)-invariant. It follows that \( g' \) descends to a deformation retraction \( g : I \times S \to \widehat{S} \) whose image is \( \Gamma \)-internal. One checks that condition (2) of the statement is satisfied for the retraction associated to \( g \).

\[ \square \]

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John Welliaveetil
Department of Mathematics, Imperial College London,
180 Queens Gate,
SW7 2AZ,
London, UK.

*email*: welliaveetil@gmail.com