A FUNCTIORIAL LOWER BOUND FOR THE ESSENTIAL MINIMUM OF VARIETIES IN A POWER OF AN ELLIPTIC CURVE

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ABSTRACT. A subvariety $V$ of an abelian variety is translate if it is the union of translates of proper algebraic subgroups. An irreducible $V$ is transverse if it is not contained in any translate variety. Effective sharp lower bounds for a transverse subvarieties of a power of an elliptic curve $E$ are known. Here, we prove a sharp lower bound for the essential minimum of non-translate subvarieties of $E^n$.

1. INTRODUCTION

In this work, variety means a variety defined over the algebraic numbers. Let $A$ be an abelian variety. To a symmetric ample line bundle $\mathcal{L}$ on $A$ we associate an embedding $i_{\mathcal{L}} : A \hookrightarrow \mathbb{P}^m$ defined by the minimal power of $\mathcal{L}$ which is very ample. Heights and degrees corresponding to $\mathcal{L}$ are computed via such an embedding. More precisely, $h_{\mathcal{L}}$ is the $\mathcal{L}$-canonical Néron-Tate height. The degree of a subvariety of $A$ is the degree of its image under $i_{\mathcal{L}}$. For $E$ an elliptic curve, we denote by $\mathcal{O}$ the line bundle on $E$ defined by the neutral element. The standard line bundle $\mathcal{O}_n$ on $E^n$ is the tensor product of the pull-back of $\mathcal{O}$ via the natural projections. In this paper, we consider irreducible subvarieties $V$ of $E^n$. We are going to analyse the sets of points of small height of $V$, with respect to different line bundles.

It is well-known that the set of torsion points of $A$ is a dense subset of $A$ and it coincides with the points of trivial height on $A$. A quite deep problem is to investigate lower bounds for the points of $A$ of positive height (ex. Baker and Silvermann [3, 19], David and Hindry [6], Masser [13]). A general question is to study the height of points on an algebraic subvariety $V$ of $A$ of positive dimension $d$. We say that $V$ is torsion if it is the union of some translates of algebraic subgroups
of $A$ by torsion points. Moreover an irreducible $V$ is transverse if it is not contained in any translate of an abelian subvariety. The Manin-Mumford conjecture (Raynaud [18]) ensures that the torsion points are dense in $V$ if and only if $V$ is torsion. The Bogomolov Conjecture (Ullmo [21] and Zhang [25]) is a statement on the points of $V$ of positive height.

**Essential minimum:** The essential minimum $\mu_\mathcal{L}(V)$ is the supremum of the reals $\theta$ such that the set $\{x \in V(\overline{\mathbb{Q}}) : h_\mathcal{L}(x) \leq \theta\}$ is non-dense in $V$.

**Bogomolov Conjecture:** The essential minimum $\mu_\mathcal{L}(V)$ is strictly positive if and only if $V$ is non-torsion.

The problem of giving explicit bounds for $\mu_\mathcal{L}(V)$ depending on the invariants of $V$ and of the ambient variety has been investigated in several deep works (for instance in the toric case Amorosi and David [1], Bombieri and Zannier [4], Schmidt [20] and in the abelian case David and Philippon [8], [9]). Different points of view can be assumed. On one side one can fix, once for all, the embedding of the ambient variety in a projective space. This corresponds to fixing a polarization on $A$. On the other hand one can vary the embedding and investigates the dependence of the essential minimum on the embedding.

Most of the known results are obtained after fixing a standard embedding. Amoroso and David [1] theorem 1.4 prove a quasi-optimal lower bound of the essential minimum of a transverse subvariety of a torus. A new and simplified approach is introduced by Amoroso and the author [2]. David and Philippon [7], [8] obtained several non-optimal lower bounds for the essential minimum of a subvariety of a general abelian variety and stronger bounds for a subvariety of a power of an elliptic curve. In a preprint, Galateau [10] proves a quasi-optimal lower bound in a product of elliptic curves.

If the polarization varies, we expect that the polarization influences the essential minimum in a natural way. This kind of problems are called ‘functorial Bogomolov’. For tori, a functorial conjecture has been introduced by Amoroso and David [1]. Not much is proven in this direction. Some results in the toric case and for $V$ a translate of a subtorus are proven by Sombra and Philippon [16]. A weak-functorial result has applications in the context of the so called Zilber-Pink conjecture, which is a generalization of a work of Bombieri, Masser and Zannier and of the Mordell-Lang plus Bogomolov problem. In our works [22] proposition 13.3, [23] proposition 4.3 and [24] proposition 4.6 we produce a functorial result in the very special case of an isogeny of $E^n$ or of an abelian variety. This motivated our interest. In [22], we
advice an isogeny functoriality. This is the first result of this paper. From the main result of Galateau [10], we deduce:

**Theorem 1.1.** Let $V$ be a transverse subvariety of $E^n$ of dimension $d$. Then, for any isogeny $\phi : E^n \to E^n$ and any $\eta > 0$,

$$\mu_{\mathcal{O}_n}(V) \geq c_1(E^n, \eta) \frac{(\deg_{\mathcal{O}_n} E^n)^{\frac{1}{n-d} - \eta}}{(\deg_{\mathcal{O}_n} V)^{\frac{1}{n-d} + \eta}},$$

where $c_1(E^n, \eta)$ is a positive constant depending on $E^n$ and $\eta$.

Since $\mu_{\mathcal{O}_n}(\phi V) = \mu_{\phi^*\mathcal{O}_n}(V)$ we can deduce relations of the several essential minima of the image of a variety via isogenies.

After this first result, we were induced to believe in a more general functorial principle. Let us clarify the setting from another point of view. Only under the geometric assumption that $V$ is transverse, there exist non trivial lower bounds for $\mu(V)$. However, an irreducible variety $V$ is always ‘relatively’ transverse, meaning that it is transverse in a minimal translate $H$ of an abelian subvariety of the ambient variety $E^N$. If on $E^N$ we consider the standard polarization, what can one say on the essential minimum of $V$? In other words, we consider on $H$ the restriction of the standard polarization of $E^N$. Such a restriction is in general not the standard polarization of $H$. Our main result is an elliptic analogue, up to a remainder term, of a toric conjecture of Amoroso and David.

**Theorem 1.2.** Let $H$ be the translate of an abelian subvariety of $E^N$ of dimension $n$. Let $V$ be a $d$-dimensional variety transverse in $H$. Then, for any $\eta > 0$, there exists a positive constant $c_2$ such that

$$\mu_{\mathcal{O}_N}(V) \geq c_2 \frac{(\deg_{\mathcal{O}_N} H)^{\frac{1}{n-d} - \eta}}{(\deg_{\mathcal{O}_N} V)^{\frac{1}{n-d} + \eta}}.$$

To prove the theorem, we first assume that $H$ is an abelian variety. This hypothesis is then removed with a simple trick, proposition 6.2. The main idea of the proof is to approximate $V \subset H$ with $V' \subset H'$ in a more convenient position, see section 4. For this we use a ‘small’ transformation, constructed with some technical steps in section 7. Then we prove that the restriction of $\mathcal{O}_N$ on $H$ is comparable to tensor products of pull backs of the standard polarization on $H$, proposition 5.1. We finally prove that essential minimum and degrees behave well with respect to such operations. For this we use, among other, theorem 1.1.

We would like to mention that what we actually prove is:

*if a quasi-optimal bound for the essential minimum holds for the standard polarization, then a natural analog holds for the restrictions of the standard polarization to an abelian subvariety.*
Finally, in the appendix we present a proposition of Patrice Philippon, which clarifies the relation between functorial conjectures.

In the next section we fix the notation and we prove some basic results. In section 3 we prove theorem 1.1. In section 4 we describe the geometric situation. Thanks to a technical result proven in section 7, we approximate our varieties with more convenient ones. For such varieties, we then prove an equivalence of line bundles which represents one of the key ingredients of the proof. In section 6 we conclude the proof of theorem 1.2. In the appendix we relate our result to other conjectures.

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2. Preliminaries

2.1. Morphisms. Let $E$ be an elliptic curve defined over the algebraic numbers. We denote by $\text{End}(E)$ the ring of endomorphism of $E$, that is an order in a quadratic field. We fix an embedding of $\text{End}(E)$ in $\mathbb{C}$, and for $a \in \text{End}(E)$ we indicate by $|a|$ the standard absolute value in $\mathbb{C}$. Note that such a value does not depend on the choice of the embedding.

Let $n \leq N$ be positive integers. Recall that for a morphism $\psi : E^N \rightarrow E^n$, 

$$\ker \psi = \{x \in E^N : \psi(x) = 0\}.$$ 

To a matrix $\psi \in \text{Mat}_{n \times N}(\text{End}(E))$ we associate a morphism $\psi : E^N \rightarrow E^n$, $(x_1, \ldots, x_N) \mapsto (\psi_{11}x_1 + \cdots + \psi_{1N}x_N, \ldots, \psi_{n1}x_1 + \cdots + \psi_{nN}x_N)$.

Viceversa to a morphism $\psi : E^N \rightarrow E^n$, we associate the matrix defining its kernel

$$\ker \psi = \begin{cases} 
\psi_{11}x_1 + \cdots + \psi_{1N}x_N = 0 \\
\vdots \\
\psi_{n1}x_1 + \cdots + \psi_{nN}x_N = 0.
\end{cases}$$

Lemma 2.1 (Kernel relations). They hold:

i. For $\psi \in \text{Mat}_{N \times m}(\text{End}(E))$ and $\psi' \in \text{Mat}_{m' \times N}(\text{End}(E))$ with $m, m' \in \mathbb{N}^*$, 

$$\psi^{-1} \ker \psi' = \ker(\psi' \psi).$$

ii. For $a \in \text{Mat}_{1 \times N-n}(\text{End}(E))$, $b \in \text{Mat}_{1 \times n}(\text{End}(E))$,

$$\ker \begin{pmatrix} \text{Id}_{N-n} & 0 \\ a & b \end{pmatrix} = \ker \begin{pmatrix} \text{Id}_{N-n} & 0 \\ 0 & b \end{pmatrix}.$$
Proof. The first relation is proven by
\[ \psi^{-1} \ker \psi' = \psi^{-1} \{ x \in E^N : \psi'(x) = 0 \} = \{ y \in E^N : \psi'(y) = 0 \} = \ker \psi' \psi. \]

The second relation is a simple Gauss reduction. \(\square\)

We define the norm of a morphism \(\psi : E^N \to E^n\) as the maximum of the absolute value of the entries of its associated matrix
\[ ||\psi|| = \max_{ij} |\psi_{ij}|. \]

2.2. Basic relations of degrees. In the first instance, we recall basic relations of degrees. Let \(n\) and \(m\) be positive integers. Let \(L\) be a symmetric ample line bundle on \(E^n\) and let \(L^m\) be the tensor product of \(L\), \(m\)-times. Let \(V\) be an irreducible algebraic subvariety of \(E^n\) of dimension \(d\). Then,

\[ \text{deg}_{L^m} V = m^d \text{deg}_L V. \]

We now study some relations of degrees under the action of the multiplication morphism, or more in general under the action of an isogeny. For \(a \in \text{End}(E)\), we denote by \([a]\) the multiplication by \(a\) on \(E^n\). Hindry [11] Lemma 6 proves

\[ \text{deg}_L [a]^{-1} V = |a|^{2(n-d)} \text{deg}_L V \]

and

\[ \text{deg}_L [a] V = \frac{|a|^{2d}}{|\text{Stab} V \cap \ker[a]|} \text{deg}_L V. \]

Let \(\phi : E^n \to E^n\) be an isogeny. Projection’s Formula gives
\[ \text{deg}_\phi^* L V = \text{deg}_L \phi_* (V). \]

For a finite set \(S\) we denote by \(|S|\) its cardinality. By [12] Corollary (6.6) page 68
\[ \text{deg}_{\phi^* L} E^n = |\ker \phi| \text{deg}_L E^n. \]

We now recall a lemma which relates the degree of a variety and of its push-forward under an isogeny of the ambient variety. This relation will be used in several occasions.

Lemma 2.2 ([24] Lemma 4.2). Let \(\phi : E^n \to E^n\) be an isogeny. Let \(V\) be an irreducible algebraic subvariety of \(E^n\). Then
\[ \text{deg}_L \phi_* (V) = |\text{Stab} V \cap \ker \phi| \text{deg}_L \phi(V). \]
2.3. Basic relations of essential minima. We now investigate useful relations for the essential minimum. By definition, \( h_{\phi^*}(x) = h_V(\phi(x)) \), then
\[
\mu_{\phi^*}(V) = \mu_\mathcal{L}(\phi(V)) = \mu_\mathcal{L}(V).
\]
In addition
\[
\mu_{\mathcal{L}^m}(V) = m\mu_\mathcal{L}(V).
\]
A more interesting property is

**Lemma 2.3.** Let \( \phi_i \) be isogenies of \( E^n \). Then
\[
\mu_{\mathcal{L}}(V) \geq \sum_i \mu_{\phi_i^*}(V).
\]

**Proof.** The proof is worked out by contradiction. In addition it realizes on the height relation \( h_{\phi_i^*}(x) = \sum_i h_{\phi_i^*}(x) \) for every \( x \in E^n \).

Suppose, by contradiction, that the conclusion of the lemma does not hold. In other words that \( \mu_{\mathcal{L}}(V) < \sum_i \mu_{\phi_i^*}(V) \). Then there exists a dense subset \( U \) of \( V \) such that
\[
h(U) < \sum_i \mu_{\phi_i^*}(V),
\]
meaning that each element of \( U \) has height bounded by \( \sum_i \mu_{\phi_i^*}(V) \).

From the definition of \( \mu_{\phi_i^*}(V) \), the set of points of \( V \) such that \( h_{\phi_i^*}(x) < \mu_{\phi_i^*}(V) \) is contained in a closed subset \( V_i \subseteq V \). Since \( U \) is dense, \( U' = U \setminus \bigcup_i V_i \cap U \) is also dense in \( V \). In addition, for every \( x \in U' \), \( h_{\phi_i^*}(x) \geq \mu_{\phi_i^*}(V) \). Then, for all \( x \in U' \), \( h_{\phi_i^*}(x) = \sum_i h_{\phi_i^*}(x) \geq \sum_i \mu_{\phi_i^*}(V) \) and \( h(U) \geq \sum_i \mu_{\phi_i^*}(V) \). Which contradicts (5). \( \square \)

3. The isogeny functoriality

In this section we concentrate on the dependence of the essential minimum under isogenies. We deduce theorem 3.1 from a non-functorial result of Galateau.

**Theorem 3.1** (Galateau [10]). Let \( \mathcal{O}_n \) be the standard line bundle on \( E^n \). For any \( \eta > 0 \), there exists a positive constant \( c_0(E^n, \eta) \) such that for any transverse subvariety \( V \) of \( E^n \) of positive dimension \( d \), it holds
\[
\mu_{\mathcal{O}_n}(V) \geq c_0(E^n, \eta)(\deg \mathcal{O}_n V)^{-\frac{1}{n-d} - \eta}.
\]

His proof does not extend to other line bundles. In addition he needs to use a choice of a particular basis of global sections. The proof does not work for other choices.

For an isogeny \( \phi : E^n \to E^n \), we show that a lower bound like in theorem 3.1 holds for the polarization \( \phi^* \mathcal{O}_n \). An basic role in the proof is played by stabilizers. An isogeny is a group morphism, thus the stabilizer of the image or preimage of a variety can be easily understood. On one hand, the stabilizer of \( V \) is related to the fiber of \( \phi \) on \( V \). On the other hand, stabilizers
are related to the degree of the variety via the well-known formulas (3). We define a special variety depending on $V$ and $\phi$. Such a variety allows us to produce a kind of reverse degree formula. This will be a key ingredient to prove theorem 1.1. Let us prove a first elementary lemma.

**Lemma 3.2.** Let $\phi : E^n \to E^n$ be an isogeny. Then,

i. $\text{Stab } \phi^{-1}(V) = \phi^{-1}(\text{Stab } V).

ii. Let $\hat{\phi}$ be an isogeny such that $\hat{\phi}\phi = \phi\hat{\phi} = [a]$. Then

$$|\text{Stab } \hat{\phi}^{-1}(V) \cap \ker[a]| = |\ker\hat{\phi}| |\text{Stab } V \cap \ker \phi|.$$

**Proof.**

i. Let $t \in \text{Stab } \phi^{-1}(V)$ then

$$\phi^{-1}(V) + t \subset \phi^{-1}(V).$$

Taking the image, $V + \phi(t) \subset V$ and $\phi(t) \in \text{Stab } V$. That gives $t \in \phi^{-1}(\text{Stab } V)$. Conversely, let $t \in \text{Stab } V$, then

$$V + t \subset V$$

and taking the preimage $\phi^{-1}(V + t) \subset \phi^{-1}V$. Then $\phi^{-1}(t) \subset \text{Stab } \phi^{-1}(V)$.

ii. By part i. applied to $\hat{\phi}$, we have $\text{Stab } \hat{\phi}^{-1}(V) = \hat{\phi}^{-1}(\text{Stab } V)$. As $\hat{\phi}\phi = [a]$, $\ker[a] = \hat{\phi}^{-1}(\ker \phi)$. Then

$$\text{Stab } \hat{\phi}^{-1}(V) \cap \ker[a] = \hat{\phi}^{-1}(\text{Stab } V \cap \ker \phi).$$

We are now ready to prove the isogeny functoriality. For the convenience of the reader we recall the statement.

**Theorem 1.1.** Let $V$ be a transverse subvariety of $E^n$ of dimension $d$. Then, for any isogeny $\phi : E^n \to E^n$ and any $\eta > 0$,

$$\mu_{\phi^*\mathcal{O}_n}(V) \geq c_1(E^n, \eta) \left(\frac{\deg_{\phi^*\mathcal{O}_n} E^n}{\deg_{\phi^*\mathcal{O}_n} V}\right)^\eta \left(\frac{\deg_{\phi^*\mathcal{O}_n} E^n}{\deg_{\phi^*\mathcal{O}_n} V}\right)^{-\eta},$$

where $c_1(E^n, \eta) = 3^{-\frac{n}{d}} c_0(E^n, 2(n-d)\eta)$ and $c_0(E^n, \eta)$ is as in theorem 3.7.

**Proof.** Let $a$ be an integer of minimal absolute value such that there exists an isogeny $\hat{\phi}$ with $\hat{\phi}\phi = \phi\hat{\phi} = [a]$. Then by definition of dual isogeny $|a| \leq |\det \phi|$. Define

$$W = \hat{\phi}^{-1}(V).$$

We have

$$[a]W = \phi\hat{\phi}W = \phi\hat{\phi}\hat{\phi}^{-1}(V) = \phi(V).$$

Then

(6)  $\mu_{\phi^*\mathcal{O}_n}(V) = \mu_{\mathcal{O}_n}(\phi(V)) = \mu_{\mathcal{O}_n}([a]W) = |a|^2 \mu_{\mathcal{O}_n}(W).$
In order to apply theorem 3.1 we estimate $\deg_{\mathcal{O}_n} W$. By formula (3),

$$
\deg_{\mathcal{O}_n} \phi(V) = \deg_{\mathcal{O}_n} [a] W = \frac{|a|^{2d}}{|\text{Stab} W \cap \ker[a]|} \deg_{\mathcal{O}_n} W
$$
or equivalently

$$
\deg_{\mathcal{O}_n} W = \frac{|\text{Stab} W \cap \ker[a]|}{|a|^{2d}} \deg_{\mathcal{O}_n} \phi(V).
$$

Using Lemma 3.2 ii. and Lemma 2.2 we obtain

$$
\deg_{\mathcal{O}_n} W = \frac{|\ker \hat\phi|}{|a|^{2d}} |\text{Stab} V \cap \ker \phi| \deg_{\mathcal{O}_n} \phi(V)
= \frac{|\ker \hat\phi|}{|a|^{2d}} \deg_{\mathcal{O}_n} \phi_*(V).
$$

We now estimate $\mu_{\mathcal{O}_n}(W)$ using theorem 3.1. For simplicity we denote $c_0 = c_0(E^n, \eta)$. Note that isogenies preserve dimensions and transversality, so $\dim W = \dim V = d$ and $W$ is transverse. We obtain

$$
\mu_{\mathcal{O}_n}(W) \geq c_0 (\deg_{\mathcal{O}_n} W)^{-\frac{1}{n-d} + \eta}
= c_0 \left( \frac{|a|^{2d}}{|\ker \hat\phi| \deg_{\mathcal{O}_n} \phi_*(V)} \right)^{-\frac{1}{n-d} + \eta} (\deg_{\mathcal{O}_n} E^n)^{-\frac{1}{n-d} - \eta}
= c_0 \left( \frac{|a|^{2d} \deg_{\mathcal{O}_n} E^n}{|\ker \phi| \deg_{\mathcal{O}_n} \phi_*(V)} \right)^{-\frac{1}{n-d} + \eta} (\deg_{\mathcal{O}_n} E^n)^{-\eta},
$$

indeed $\deg_{\mathcal{O}_n} E^n = 3^n$. We substitute this last estimate in (3), then

$$
\mu_{\phi^*\mathcal{O}_n}(V) = |a|^{2} \mu_{\mathcal{O}_n}(W)
\geq |a|^{2} 3^{\frac{n}{n-d}} c_0 \left( \frac{|a|^{2d} \deg_{\mathcal{O}_n} E^n}{|\ker \hat\phi| \deg_{\mathcal{O}_n} \phi_*(V)} \right)^{-\frac{1}{n-d} + \eta} (\deg_{\mathcal{O}_n} E^n)^{-\eta}
= 3^{\frac{n}{n-d}} c_0 \left( \frac{|a|^{2n} \deg_{\phi^*\mathcal{O}_n} E^n}{|\ker \hat\phi| \deg_{\mathcal{O}_n} \phi_*(V)} \right)^{-\frac{1}{n-d} + \eta} (\deg_{\mathcal{O}_n} E^n)^{-\eta}|a|^{-2(n-d)\eta}
= 3^{\frac{n}{n-d}} c_0 \left( \frac{\deg_{\phi^*\mathcal{O}_n} E^n}{\deg_{\phi^*\mathcal{O}_n} V} \right)^{-\frac{1}{n-d} + \eta} (\deg_{\mathcal{O}_n} E^n)^{-\eta}|a|^{-2(n-d)\eta},
$$
where $|\ker \hat{\phi} \cap \ker \phi| = |a|^{2n}$ because $\hat{\phi} \circ \phi = [a]$. In addition
\[ \deg_{\phi \circ \sigma} E^n = |\ker \phi| \deg_{\sigma} E^n = |\det \phi| \deg_{\sigma} E^n. \]
Since $|a| \leq |\deg \phi|$, $(\deg_{\sigma} E^n)^{-\eta} |a|^{-2(n-d)\eta} \geq (\deg_{\sigma} E^n |a|)^{-2(n-d)\eta} \geq (\deg_{\phi \circ \sigma} E^n)^{-2(n-d)\eta}$. Then
\[ \mu_{\phi \circ \sigma}(V) \geq 3^{\frac{n}{n-d} \epsilon_0} \left( \frac{\deg_{\phi \circ \sigma} E^n \left( \frac{1}{n-d} - 2(n-d)\eta \right) }{\deg_{\phi \circ \sigma}(V)^{\frac{1}{n-d} + \eta}} \right). \]
This easily implies the wished bound. \(\square\)

Note that, theorem 1.1 implies theorem 3.1, simply choosing $\phi = \text{id}$. So, we have proven

**Corollary 3.3.** Theorem 3.1 and theorem 1.1 are equivalent.

### 4. The minimal abelian subvariety containing $V$

In this section we assume that $H$ is an abelian subvariety of $E^N$ of dimension $n$. In proposition 6.2, we will see how to adapt our argument to the general case of a translate of an abelian subvariety.

**Lemma 4.1.** Let $H$ be an abelian subvariety of $E^N$ of dimension $n$. Then there exists an isogeny
\[ \varphi = \begin{pmatrix} \varphi_H \\ \varphi_{H'} \end{pmatrix} : E^N \to E^N. \]
such that:

i. The rank of $\varphi_H$ is $N - n$ and the rank of $\varphi_{H'}$ is $n$,
ii. $\varphi(H) = \{0\}^{N-n} \times E^n$.
iii. $|\det \varphi| \leq \kappa$, for $\kappa \leq 3^{2n}2^{8nN^2}$.

**Proof.** By a lemma of Bertand, (see the appendix of [17] proposition 5.1). We can find a complement $H'$ of $H$ such that $H + H' = E^N$ and the cardinality of $H \cap H'$ is bounded by a constant $\kappa_1$ depending only on $n, N$. A closer inspection of the proof of this lemma, shows that eventually $\kappa_1 \leq 2^{8nN^2}$.

By Masser and Wüstholz [15] Lemma 1.3, there exists a matrix $\varphi_H \in \text{Mat}_{(N-n) \times N}(\text{End}(E))$ of rank $N - n$ such that $\ker \varphi_H = H + \tau$ for $\tau$ a torsion group contained in $H'$ of cardinality bounded by $\deg_{\sigma} E^n = 3^N$. Similarly, let $\varphi_{H'} \in \text{Mat}_{n \times N}(\text{End}(E))$ be a matrix of rank $n$ such that $\ker \varphi_{H'} = H' + \tau'$ for $\tau'$ a torsion group contained in $H$ of cardinality bounded by $3^N$. Define the isogeny
\[ \varphi = \begin{pmatrix} \varphi_H \\ \varphi_{H'} \end{pmatrix} : E^N \to E^N. \]
Then
\[ \ker \varphi = (H + \tau) \cap (H' + \tau') = (H \cap H') + \tau + \tau' \leq \kappa_1 3^{2N}. \]
Using the isomorphism produced in the appendix, we are going to approximate $H$ with a $H$ in a convenient position in $E^N$.

**Lemma 4.2.** Let $\varphi$ be the isogeny defined in lemma 4.1. Then, there exists an isomorphism $T : E^N \to E^N$ such that:

i. $||T||, ||T^{-1}|| \leq \frac{1}{N} \binom{N}{n}$

ii. All $n \times n$ minors of the matrix consisting of the last $n$ columns of $(\varphi T)^{-1}$ are different from zero.

**Proof.** Apply proposition 7.2 to the matrix $\psi$ consisting of the last $n$ rows of $(\varphi^{-1})^t$, the transpose of the inverse of $\varphi$. Then there exists a permutation matrix $J$ and a matrix $T_0$ such that:

- $T_0 = \begin{pmatrix} \text{Id}_n & X \\ 0 & \text{Id}_{N-n} \end{pmatrix}$ and $|X_{ij}| \leq \frac{1}{N} \binom{N}{n}$,

- All the $n \times n$ minors of $(\varphi^{-1})^tJ T_0$ are different from zero.

Then all $n \times n$ minors of the last $n$ columns of $T_0^t J^t \varphi^{-1}$ are non zero. Note that $T_0^t J^t \varphi^{-1} = (\varphi^{-1})^t (T_0^{-1})^t$. In addition $T_0^{-1} = \begin{pmatrix} \text{Id}_n & -X \\ 0 & \text{Id}_{N-n} \end{pmatrix}$. Thus the lemma is proven for $T = (J^{-1})^t (T_0^{-1})^t$. 

This isomorphism is particularly nice. For our problem of estimating the essential minimum, we will see that it is equivalent to work in the domain or in the codomain of $T$, up to a constant depending on $N$ and $n$. This is a consequence of the fact that the entries of $T$ are bounded by a constant and of an estimate by Masser and Wüstholz. This estimate relates the degree of a variety and of its push-forward via an isogeny.

**Lemma 4.3 (I.4 Lemma 2.3).** Let $\psi : E^N \to E^N$ be an isogeny. Let $X$ be an irreducible algebraic subvariety of $E^N$ of dimension $d$. Then,

$$\deg_{L} \psi_{*}(X) \leq N^d(3N||\psi||)^{2d} \deg_{L} X.$$ 

We can then easily deduce the following:

**Lemma 4.4.** Let $T$ be as in lemma 4.2. Then

i. $(N||T||)^{-2} h_{L}(x) \leq h_{L}(T^{-1} x) \leq (N||T^{-1}||)^2 h_{L}(x)$, for every $x \in E^N$,

ii. $\deg_{L} T^{-1}(H) \geq (9N^3||T||^2)^{-n} \deg_{L} H$,

iii. $\deg_{L} T^{-1} V \leq (9N^3||T^{-1}||^2)^d \deg_{L} V$.

**Proof.** The second inequality of part i. is simply given by the triangle inequality. The triangle inequality also gives $h_{L}(x) = h_{L}(TT^{-1}(x)) \leq (N||T||)^2 h_{L}(T^{-1}(x))$. This is the first inequality of i.

To prove part ii. apply lemma 4.3 with $X = T^{-1} H$, then

$$\deg_{L} H = \deg_{L} TT^{-1}(H) \leq N^n(3N||T||)^{2n} \deg_{L} T^{-1}(H) = N(9N^3||T||^2)^n \deg_{L} T^{-1}(H).$$
Note that $T^{-1}$ is an isomorphism so $T^{-1}H$ is irreducible.

Part iii. is an immediate application of Lemma 4.3 with $X = V$. □

Thanks to the isomorphism $T$, we can give to the abelian subvariety $H$ a convenient position, in the sense that follows. We construct a matrix $\phi$, which has the property that all minors of the last $n$ columns are non zero. Moreover its entries are close to the entries of $\varphi$.

**Definition 4.5.** Let $\varphi$ be as in lemma 4.1 and let $T$ be as in lemma 4.2. We define the abelian subvariety

$$H = T^{-1}H$$

and the isogeny

$$\phi = \varphi T = \begin{pmatrix} \phi_H \\ \phi'_H \end{pmatrix},$$

where $\phi_H \in \text{Mat}_{N-n \times N}(\text{End}(E))$.

The kernel relation, immediately gives

$$H = T^{-1}H \subset \ker \phi_H.$$

Note that, the isogeny $\varphi : E^N \to E^N$ sends $H$ to the last $n$ factors,

$$\phi(H) = 0 \times \cdots \times 0 \times E^n.$$

Indeed $\phi(H) = \varphi(H)$, which has by construction such a property, see lemma 4.1 ii. We denote the immersion on the last $n$ factors by

$$i : E^n \to E^N, \quad (x_1, \ldots, x_n) \to (0, \ldots, 0, x_1, \ldots, x_n).$$

An immediate consequence of lemmas 2.2 and 4.1 is

**Corollary 4.6.** Let $L$ be a symmetric ample line bundle on $E^n$. Then, for $\phi$ as above,

$$\deg_L i^* \phi^*(H) = |\det \phi| \deg_L E^n \leq \kappa \deg_L E^n.$$

**Proof.** We first remark that $|\det \phi| \leq \kappa$. This simply follows by lemma 4.1 iii. and the fact that $T$ is an isomorphism. Now, apply lemma 2.2 to $\phi$ and $H$. Then $\deg_{i_*L} \phi^*(H) = |\det \phi| \deg_{i_*L} \phi(H)$. Note that the map $i$ preserves the degree of subvarieties of $\{0\}^{N-n} \times E^n$. In addition $\phi(H) = E^n$. So $|\det \phi| \deg_{i_*L} \phi(H) = |\det \phi| \deg_L E^n$. □

**Definition 4.7.** Let $\alpha$ be the minimal positive integer such that there exists an isogeny $\hat{\phi}$ satisfying $\hat{\phi} \hat{\phi} = \hat{\phi} \phi = [\alpha]$. We decompose $\hat{\phi} = |A|B$ with $A \in \text{Mat}_{N \times (N-n)}(\text{End}(E))$ and $B \in \text{Mat}_{N \times n}(\text{End}(E))$. We denote by $a_i$ the $i$-th row of $A$, similarly

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}.$$ 

Note that, by definition of dual isogeny $\alpha \leq |\det \phi|$. 

Lemma 4.8. For $I \in \mathbb{I} = \{(i_1, \ldots, i_n) : i_j \in \{1, \ldots, N\} \text{ and } i_j < i_{j+1}\}$ the morphism
\[
\phi_I = \begin{pmatrix} b_{i_1} \\ \vdots \\ b_{i_n} \end{pmatrix} : E^n \rightarrow E^n
\]
is an isogeny.

Proof. As $\alpha \hat{\phi} = \alpha \text{Id}_N$, lemma 4.2 ii. implies that all $n \times n$ minors of $B$ are non-zero. Then $\det \phi_I \neq 0$. This is equivalent to say that $\phi_I$ is an isogeny. \hfill \Box

5. An equivalence of line bundles

In this section, we work with the canonical line bundle $\mathcal{O}_N$ on the ambient variety $E^N$. For $H$ an abelian subvariety, we study $\mathcal{O}_{N|H}$. If $H$ has a sufficiently general position, we can express $\mathcal{O}_{N|H}$ as tensor products and pull-backs via isogenies of the canonical bundle $\mathcal{O}_n$ on $H$.

Let $\mathcal{L}$ and $\mathcal{M}$ be line bundles. We denote by $c_1(\mathcal{L})$ a representative of the first Chern-class of $\mathcal{L}$. By $\boxplus$ we mean the sum of cycles and by $\mathcal{L}^m$ we mean the tensor product of $\mathcal{L}$ $m$-times. Recall that $c_1(\mathcal{L} \otimes \mathcal{M}) = c_1(\mathcal{L}) \boxplus c_1(\mathcal{M})$. For $f$ a morphism $c_1(f^*\mathcal{L}) = f^{-1}c_1(\mathcal{L})$.

We recall that $\mathcal{O}$ is the line bundle on $E$ defined by the neutral element. The standard line bundle $\mathcal{O}_n$ on $E^n$ is the tensor product of the pull-back of $\mathcal{O}$ via the natural projections. Let $e_i : E^n \rightarrow E$ and $f_i : E^N \rightarrow E$ be the projections on the $i$-th factor. Note that $e_i$ and $f_i$ are the vectors of a standard basis of $\mathbb{Z}^n$ and $\mathbb{Z}^N$. By definition of standard line bundle,
\[
c_1(\mathcal{O}_n) = \boxplus_{i=1}^n \ker e_i,
\]
\[
c_1(\mathcal{O}_N) = \boxplus_{i=1}^N \ker f_i.
\]
Note that, for any integer $\alpha$, $\ker(\alpha f_i) = \ker(f_i[\alpha]) = [\alpha]^{-1}\ker f_i = [\alpha]^*\ker f_i$. In addition, by [12] page 34 corollary 3.6, for $\mathcal{L}$ a symmetric ample line bundle on $E^n$,
\[
c_1([\alpha]^*\mathcal{L}) = c_1(\mathcal{L}^{\alpha^2}).
\]
Then,
\[
c_1(\mathcal{O}_N^{\alpha^2}) = \boxplus_{i=1}^N \ker \alpha f_i.
\]
Let us first state a basic relation which proves useful.

Lemma 5.1. In the above notation, it holds
\[
c_1(\mathcal{O}_{N|H}^{\alpha^2}) = \boxplus_{i=1}^N \ker \left( \phi_H \phi_i \right).
\]
Proof. Note that,
\[
\begin{pmatrix}
\text{Id}_{N-n} & 0 \\
0 & b_i
\end{pmatrix}
\phi =
\begin{pmatrix}
\phi_H \\
\alpha f_i
\end{pmatrix},
\]
where \( \alpha \) is given in definition 4.7. From lemma 4.8, \( b_i \neq 0 \) for all \( 1 \leq i \leq N \). Thus the rank of \( \begin{pmatrix}
\text{Id}_{N-n} & 0 \\
0 & b_i
\end{pmatrix}
\phi \) is \( N - n + 1 \). Since \( \phi \) is invertible also \( \begin{pmatrix}
\phi_H \\
\alpha f_i
\end{pmatrix} \) has rank \( N - n + 1 \). This means that \( \text{ker} \alpha f_i \) intersects generically \( H \). Then, the restriction bundle \( \mathcal{O}_N^2 |_{H} \) satisfies
\[
c_1(\mathcal{O}_N^2 |_{H}) = H \cap c_1(\mathcal{O}_N^2) = H \cap (\bigoplus \text{ker} \alpha f_i) = \bigoplus_{i=1}^{N} \text{ker} \left( \phi_H \alpha f_i \right).
\]

We sum up the situation with the following diagram:

\[
\begin{array}{cccccc}
E^N \to & E^{N-n} \times E^n & \overset{i}{\to} & E^n & \overset{\phi}{\to} & E^m \\
H \to & 0 \times E^n & \leftarrow & E^n & \leftarrow & E^n
\end{array}
\]

Theorem 5.2. The following equivalence of line bundles holds
\[
\mathcal{O}_N^{\alpha^2(N-1)}|_{H} \cong \phi^* \bigotimes I \phi_I^* \mathcal{O}_n,
\]
where \( \phi \hat{=} = \hat{\phi} = [\alpha] \) is as in definition 4.7 and \( \phi_I \) is defined in lemma 4.8.

Proof. Two line bundles are equivalent if and only if they have the same Chern-class. Recall that
\[
c_1(\mathcal{O}_n) = \bigoplus_{i=1}^{n} \text{ker} e_i.
\]
For any isogeny \( \psi : E^n \to E^n \),
\[
c_1(\psi^* \mathcal{O}_n) = \psi^{-1} c_1(\mathcal{O}_n) = \bigoplus_{i=1}^{n} \text{ker}(e_i \psi) = \bigoplus_{i=1}^{n} \text{ker}(\psi_i),
\]
where \( \psi_i : E^n \to E \) is the \( i \)-th row of \( \psi \).

Apply this formula to each \( \phi_I \). Then, for \( I = (i_1, \ldots, i_n) \),
\[
(7) \quad c_1(\phi_I^* \mathcal{O}_n) = \text{ker} b_{i_1} \bigoplus \ldots \bigoplus \text{ker} b_{i_n}.
\]

Since the Chern class of the tensor product is the sum of the Chern classes, we obtain
\[
(8) \quad c_1 \left( \bigotimes_I \phi_I^* \mathcal{O}_n \right) = \bigoplus_{I \in \mathbb{I}} (\text{ker} b_{i_1} \bigoplus \ldots \bigoplus \text{ker} b_{i_n}) = \frac{n}{N} \binom{N}{n} \bigoplus_{i=1}^{N} \text{ker} b_i,
\]
where the last equality is justified from the fact that the cardinality of \( \mathbb{I} \) is \( \binom{N}{n} \), in addition each multi-index \( I \) consists of \( n \) coordinates and each of the \( N \) indices appears with the same recurrence, so each...
row $b_{ij}$ appears $\frac{n}{N}C(n, N)$-times. Note that, for a bundle on $E^n$, $c_1(i_*\mathcal{L}) = \{0\}^{N-n} \times c_1(\mathcal{L})$. In view of Lemma 2.1 ii., we deduce

$$c_1(i_* \bigotimes_I \phi_i^* \mathcal{O}_n) = \binom{N-1}{n-1} \otimes^{N}_{i=1} \ker (\text{Id}_{N-n} 0_{b_i})$$

$$= \binom{N-1}{n-1} \otimes^{N}_{i=1} \ker (\text{Id}_{N-n} 0_{a_i} b_i).$$

Recall that $\hat{\phi} = [\alpha]$. Using Lemma 2.1 i. and Lemma 5.1, we conclude

$$c_1(\phi^* i_* \bigotimes_I \phi_i^* \mathcal{O}_n) = \phi^{-1} c_1(i_* \bigotimes_I \phi_i^* \mathcal{O}_n)$$

$$= \binom{N-1}{n-1} \otimes^{N}_{i=1} \ker \left(\begin{array}{cc} \text{Id}_{N-n} & 0 \\ a_i & b_i \end{array}\right) \phi$$

$$= \binom{N-1}{n-1} \otimes^{N}_{i=1} \ker \left(\begin{array}{c} \phi_H \\ \alpha f_i \end{array}\right)$$

$$= \left(\begin{array}{c} N-1 \\ n-1 \end{array}\right) c_1 (\mathcal{O}_n^{\alpha^2(n-1)} |_H) = c_1 (\mathcal{O}_n^{\alpha^2(n-1)} |_H).$$

I am grateful to Gaël Rémond for suggesting me the following proof.

**Proposition 5.3.** The following equivalence of degrees holds

$$\deg \bigotimes_{I \in \mathcal{I}} \phi_i^* \mathcal{O}_n = \left(\begin{array}{c} N-1 \\ n-1 \end{array}\right)^n \sum_{I \in \mathcal{I}} \deg \phi_i^* \mathcal{O}_n.$$ 

**Proof.** We compute the degrees as intersection numbers. By relation (7), we have

$$\deg \phi_i^* \mathcal{O}_n = n! \prod_{i,j \in I} \ker b_{ij},$$

where the product has the sense of intersection number. Similarly, by formula (8), we obtain

$$\deg \otimes_I \phi_i^* \mathcal{O}_n = n! \left(\begin{array}{c} N-1 \\ n-1 \end{array}\right)^n \prod_{i_1 < \ldots < i_n} \ker b_{i_1 j} = \left(\begin{array}{c} N-1 \\ n-1 \end{array}\right)^n \sum_{I} \deg \phi_i^* \mathcal{O}_n.$$

**6. THE PROOF OF THEOREM 1.2: THE CONCLUSION**

We first prove a weak form of Theorem 1.2. We then remove the restrictive hypothesis.

**Theorem 6.1.** Theorem 1.2 holds for $H$ an abelian subvariety and an explicit positive constant $c'(E, n, N, \eta)$. 
Proof. First we prove the theorem for
\[ H = T^{-1}H = \ker \phi_H \]
and
\[ V = T^{-1}V, \]
where \( T \) is as in lemma 4.2. The isomorphism \( T \) preserves transversality and dimensions. So \( \dim H = n, \dim V = d \) and \( N \) is the dimension of the ambient variety. Since \( V \) is transverse in \( H \), \( i^*\phi_*(V) \) is transverse in \( E^n \).

By lemma 2.3, we know that
\[ \mu_{\mathcal{O}_n}(i^*\phi_*(V)) \geq \sum_I \mu_{\mathcal{O}_n}(i^*\phi_*(V)). \]

We apply Theorem 1.1 to each \( \Phi_I \) on \( E^n \) and \( \mathcal{O}_n \). For simplicity we denote \( c_1 = c_1(E^n, \eta) \). We deduce that for each \( I \),
\[ \mu_{\mathcal{O}_n}(i^*\phi_*(V)) \geq c_1 \frac{(\deg_{\mathcal{O}_n} E^n)^{\frac{1}{n-d-\eta}}}{(\deg_{\mathcal{O}_n} i^*\phi_*(V))^{\frac{1}{n-d+\eta}}}. \]

We obtain
\[ \mu_{\mathcal{O}_n}(i^*\phi_*(V)) = \sum_I \mu_{\mathcal{O}_n}(i^*\phi_*(V)) \]
\[ \geq c_1 \sum_I \frac{(\deg_{\mathcal{O}_n} E^n)^{\frac{1}{n-d-\eta}}}{(\deg_{\mathcal{O}_n} i^*\phi_*(V))^{\frac{1}{n-d+\eta}}}. \]

Since each bundle is ample, for every variety \( X \), we have \( \deg_{\mathcal{O}_n} X \leq \deg_{\mathcal{O}_n} X \). Recall that, for \( x_i \geq 1, (\sum_{x_i})^{1/m} \leq \sum x_i^{1/m} \). Using then Proposition 5.3, we deduce
\[ \sum_I \left( \frac{(\deg_{\mathcal{O}_n} E^n)^{\frac{1}{n-d-\eta}}}{(\deg_{\mathcal{O}_n} i^*\phi_*(V))^{\frac{1}{n-d+\eta}}} \right) \geq \left( \frac{N-1}{n-1} \right)^{-\frac{n}{n-d+n\eta}} \left( \frac{(\deg_{\mathcal{O}_n} E^n)^{\frac{1}{n-d-\eta}}}{(\deg_{\mathcal{O}_n} i^*\phi_*(V))^{\frac{1}{n-d+\eta}}} \right). \]

Therefore
\[ \mu_{\mathcal{O}_n}(i^*\phi_*(V)) \geq c_1 \left( \frac{N-1}{n-1} \right)^{-\frac{n}{n-d+n\eta}} \left( \frac{(\deg_{\mathcal{O}_n} E^n)^{\frac{1}{n-d-\eta}}}{(\deg_{\mathcal{O}_n} i^*\phi_*(V))^{\frac{1}{n-d+\eta}}} \right). \]
Note that $E^a = i^*\phi(H)$. Moreover, by corollary 4.6 $\deg E^n \leq \kappa^{-1} \deg i^*\phi_*(H)$.

Define $c_2 = c_1\kappa^{-1}$.

We deduce

$$
\mu \otimes \phi_! \phi_* (i^* \phi_*(V)) \geq c_2 \left( \frac{N - 1}{n - 1} \right)^{\frac{n}{n-d} + \eta} \left( \frac{\deg \phi_i \phi_! \phi_* (i^* \phi_*(H))}{\deg \phi_i \phi_! \phi_* (i^* \phi_*(V))} \right)^{\frac{1}{n-d} - \eta}.
$$

By theorem 5.2 and relations (11) and (14) we obtain

$$
\deg \phi_i \phi_! \phi_* (i^* \phi_*(H)) = \deg \phi_i \phi_! \phi_* (i^* \phi_*(V)) = \alpha^2 \left( \frac{N - 1}{n - 1} \right) \deg H,
$$

$$
\deg \phi_i \phi_! \phi_* (i^* \phi_*(V)) = \deg \phi_i \phi_! \phi_* (i^* \phi_*(V)) = \alpha^2 \left( \frac{N - 1}{n - 1} \right) \deg V,
$$

$$
\mu \otimes \phi_! \phi_* (i^* \phi_*(V)) = \mu \phi_! \phi_* (i^* \phi_*(V)) = \alpha^2 \left( \frac{N - 1}{n - 1} \right) \mu \phi_! (V).
$$

Then

$$
\alpha^2 \left( \frac{N - 1}{n - 1} \right) \mu \phi_! (V) \geq c_2 \left[ \alpha^2 \left( \frac{N - 1}{n - 1} \right) \right]^{1-(n+d)\eta} \left( \frac{N - 1}{n - 1} \right)^{\frac{n}{n-d} + \eta} \frac{\deg H}{\deg V}^{\frac{1}{n-d} - \eta}.
$$

In conclusion

$$
\mu \phi_! (V) \geq c_2 \alpha^{-2(n+d)\eta} \left( \frac{N - 1}{n - 1} \right)^{\frac{1}{n-d} - \eta} \frac{\deg H}{\deg V}^{\frac{1}{n-d} + \eta}.
$$

By Lemma 4.1 $\alpha \leq |\det \phi| \leq \kappa$. Then, for $c_3 = c_2\kappa^{2(n+d)\eta} \left( \frac{N - 1}{n - 1} \right)^{\frac{n}{n-d} - \eta}$,

$$
\mu \phi_! (V) \geq c_3 \frac{\deg H}{\deg V}^{\frac{1}{n-d} - \eta}.
$$

So we just proved the theorem for $V$ and $H$. We are now going to show it for $V$ and $H$. This is a consequence of the fact that the isomorphism $T$ does not change much degrees and heights (see lemma 4.3).

By lemma 4.3 i. we deduce that $\mu \phi_! (V) \geq \frac{\mu \phi_! (V)}{(H||T||)^2}$, indeed isomorphisms preserve non-dense subsets of varieties. Then

$$
\mu \phi_! (V) \geq c_3 (N||T||)^{-2} \frac{\deg H}{\deg V}^{\frac{1}{n-d} - 4(n+d+1)\eta}.
$$
By Lemma 4.4 ii. and iii. we obtain
\[
\mu_{O_N}(V) \geq c_3(N||T||)^{-2} (9N^3||T||) - \frac{n}{n-d} - \eta (9N^3||T^{-1}||^2) - \frac{d}{n-d} - n\eta (\deg_{O_N} H)^{\frac{1}{n-d} - \eta} (\deg_{O_N} V)^{\frac{1}{n-d} + \eta}.
\]
In view of lemma 4.2 i., \( ||T||, ||T^{-1}|| \leq \frac{1}{N} \binom{N}{n} \). Thus
\[
\mu_{O_N}(V) \geq c_4 \frac{(\deg_{O_N} H)^{\frac{1}{n-d} - \eta}}{(\deg_{O_N} V)^{\frac{1}{n-d} + \eta}},
\]
where
\[
c_4 = c_3 (9N)^{\frac{n-d}{n-d} + \eta} (N)^{-n-d} \eta (n-d)^{-2} (n-d)^{\eta}.
\]
This directly gives the wished result. \( \square \)

We now relax the hypothesis on \( H \). We prove theorem 1.2 for \( V \) transverse in a translate of an abelian subvariety.

**Proposition 6.2.** Theorem 1.2 holds for \( H \) a translate of an abelian subvariety and a positive constant \( c' = \frac{1}{2} c' \), where \( c' \) is as in theorem 6.1.

**Proof.** Suppose that \( V \) is transverse in a translate \( H \) of an abelian subvariety. Define
\[
\theta = c' \frac{(\deg_{O_N} H)^{\frac{1}{n-d} - \eta}}{(\deg_{O_N} V)^{\frac{1}{n-d} + \eta}}.
\]
If the set of points of \( V \) of height at most \( \frac{1}{2} \theta \) is empty then \( \mu(V) \geq \frac{1}{2} \theta \). If not, choose a point \( \xi \in V \) such that \( h(\xi) \geq \frac{1}{2} \theta \). Then \( V - \xi \subset H \). Translations preserve transversality and degrees. By theorem 6.1 for \( V - \xi \),
\[
\mu(V - \xi) \geq \theta.
\]
If \( x \in V \) and \( h(x) \leq \frac{1}{2} \theta \), then \( x - \xi \in V - \xi \) and \( h(x - \xi) = h(x) + h(\xi) \leq \theta \leq \mu(V - \xi) \). This shows that
\[
\mu(V) \geq \frac{1}{2} \theta.
\]
\( \square \)

7. A Matrix Transformation

The method used to prove theorem 1.2 works for matrices which have certain minors different from zero. Such a condition is an open condition. Therefore, any matrix can be approximate with a matrix satisfying such a condition via a ‘small’ rotation. As rational numbers are dense in the reals, one can assume that the rotation is rational. We want to ensure that the rotation is integral and that the absolute value of the entries is controlled by an absolute constant. We explicitly
construct the transformation. As usual, the most complicated part is to control the size of its entries (see proposition [7.2]).

Let $1 \leq n \leq N$ be integers. We denote by $\text{Id}_N$ the identity matrix of size $N$. For a matrix $\psi \in \text{Mat}_{n \times N}(\mathbb{R})$ we denote by $\psi_i$ the $i$-th column of $\psi$. For a multi-index $I = (i_1, \ldots, i_n)$ with $i_j \in \{1, \ldots, N\}$, we define the associated minor

$$M_I(\psi) = \det(\psi_{i_1} \ldots \psi_{i_n}).$$

For $1 \leq i, j \leq N$ and $\lambda \in \mathbb{R}$, we denote by $E_{i,j}(\lambda)$ the matrix such that the entry at the $i$-row and $j$-column is equal to $\text{Id}_{ij} + \lambda$ and all other entries are equal to the corresponding entry of the identity. Note that, for a matrix $X$, the multiplication $X E_{i,j}(\lambda) = (X_1, \ldots, X_j + \lambda X_i, \ldots, X_N)$.

**Lemma 7.1.** Let $\phi \in \text{Mat}_{n \times N}(\text{End}(E))$ be a matrix. Assume that all the $n \times n$ minors of the matrix consisting of the first $N-1$ columns of $\phi$ are non zero. Then, there exist integers $\lambda_1, \ldots, \lambda_n$ and a permutation $i_1, \ldots, i_n$ of $1, \ldots, n$ such that:

i. $|\lambda_i| \leq \frac{1}{N}(\frac{N}{n})$,

ii. For $\Lambda = E_{i_1,N}(\lambda_1) \ldots E_{i_n,N}(\lambda_n)$, all the $n \times n$ minors of $\phi \Lambda$ are non zero.

**Proof.** To prove the lemma is equivalent to prove the following claim.

**Claim**

For $0 \leq r \leq n$, there exists a set $\rho_r \subset \{1, \ldots, n, N\}$, an index $i_r \in \{1, \ldots, n, N\}$ and an integer $\lambda_r$, such that

i. $|\lambda_r| \leq \frac{1}{N}(\frac{N}{n})$,

ii. $|\rho_r| = r + 1$,

iii. Define $\Lambda^r = E_{i_0,N}(0)E_{i_1,N}(\lambda_1) \ldots E_{i_r,N}(\lambda_r)$, $\phi^r = \phi \Lambda^r$.

Then, for any multi-index $I$ such that $\rho_r \not\in I$,

$$M_I(\phi^r) \neq 0.$$  

First we clarify that the claim for $r = n$ proves the lemma. Indeed, the cardinality of $\rho_n$ is $n + 1$, so no index $I$ contains $\rho_n$. Then all the $n \times n$ minors of $\phi \Lambda^r$ are non zero. In addition $\Lambda = \Lambda^r$, because $E_{i,k}(0) = \text{Id}_N$.

Then, we prove the claim by induction on $r$.

Let $r = 0$. Define $\rho_0 = \{N\}$, $i_0 = N$ and $\lambda_0 = 0$. Note that $E_{N,N}(0) = \text{Id}_N$ and $\phi^1 = \phi$. By assumption all $n \times n$ minors of the
first \(N - 1\) columns of \(\phi\) are non zero. Equivalently, If \(N \not\in I\) then \(M_I(\phi^1) \neq 0\). So the claim is satisfied for \(r = 0\)

Let \(r \geq 1\). Suppose to have proven the claim for \(r - 1\), we prove it for \(r\). If all minors of \(\phi^{r-1}\) are non zero, define \(\lambda_r = 0\), choose any element \(i_r \not\in \rho_{r-1}\) with \(1 \leq i_r \leq n\) and define \(\rho_r = \rho_{r-1} \cup i_r\). Otherwise, choose a multi-index \(I_r\) such that \(M_{I_r}(\phi^{r-1}) = 0\). By claim iii. for \(r - 1\), \(N \in \rho_{r-1} \in I_r\). Decompose \(I_r = (I_0,N)\) where \(I_0\) is a multi-index which has \(n - 1\) entries. Then, for a question of cardinality, there exists \(1 \leq i_r \leq n\) and \(i_r \not\in I_0\). So \(i_r \not\in \rho_{r-1}\). Define \(\rho_r = \rho_{r-1} \cup i_r\). Then claim ii. is satisfied for \(r\).

Let \(S_r\) be the set of values \(-\frac{M_{I_r}(\phi^{r-1})}{M_{J_r}(\phi^{r-1})}\), for \(I\) ranging over all indeces \(I = (I_1,N)\) with \(i_r \not\in I_1\) and \(J = (I_1,i_r)\). Note that \(M_{J_r}(\phi^{r-1}) = M_{I_r}(\phi) \neq 0\), by assumption. The cardinality of \(S_r\) is at most \(\frac{N}{n}\).

For a question of cardinality, there exists an integer \(\lambda_r \not\in S_r\) such that \(|\lambda_r| \leq \frac{1}{N}(N\ n)\). Note that \(\lambda_r \neq 0\) because \(M_{I_r}(\phi^{r-1}) = 0\) is a value in \(S_r\).

Using the linearity of the determinant on the rows, we show claim iii. for \(r\). Suppose that \(\rho_r \not\in I\).

- If \(N \not\in I\) or \(i_r \in I\), then \(\rho_{r-1} \not\in I\) and \(M_I(\phi^r) = M_I(\phi^{r-1}) \neq 0\), because of claim iii. for \(r - 1\).
- If \(N \in I\) and \(i_r \not\in I\), then \(I = (I_1,N)\) and \(M_I(\phi^r) = M_I(\phi^{r-1}) + \lambda_r M_{(I_1,i_r)}(\phi^{r-1}) \neq 0\) because \(\lambda_r \not\in S_r\).

\[\Box\]

**Proposition 7.2.** Let \(\psi \in \text{Mat}_{n \times N}(\text{End}(E))\) be a matrix of rank \(n\). There exists a permutation matrix \(J\) and an upper triangular integral matrix \(T \in SL_N(\mathbb{Z})\) such that

i. \(T = \begin{pmatrix} \text{Id}_n & X \\ 0 & \text{Id}_{N-n} \end{pmatrix}\) and \(|X_{ij}| \leq \frac{1}{N}(N\ n)\),

ii. All the \(n \times n\) minors of \(\psi JT\) are non zero.

**Proof.** The rank of \(\psi\) is \(n\). Then, up to a permutation of the columns given by a matrix \(J\), we can assume that the first \(n\) columns of \(\psi\) have rank \(n\).

We proceed by induction on \(N\). The basis of the induction is \(n\).

For \(N = n\) the proposition is clearly satisfied with \(T = \text{Id}_N\).

Let \(N > n\). Suppose the proposition holds for \(N - 1\), we show that it holds for \(N\).

Let \(\psi \in \text{Mat}_{n \times N}(\text{End}(E))\) be such that the first \(n\) columns have rank \(n\). Recall that \(\psi_i\) is the \(i\)-th column of \(\psi\). Define \(\phi = (\psi_1,\ldots,\psi_{N-1})\). By inductive hypothesis there exists an upper triangular integral matrix \(R \in SL_{N-1}(\mathbb{Z})\) such that

i. \(R = \begin{pmatrix} \text{Id}_n & Y \\ 0 & \text{Id}_{N-n-1} \end{pmatrix}\) and \(|Y_{ij}| \leq \frac{1}{N-1}(N\ n)\),

ii. All the \(n \times n\) minors of \(\phi R\) are different from zero.
Define $T' = \begin{pmatrix} R & 0 \\ 0 & 1 \end{pmatrix}$. Then $\psi' = \psi T' = (\phi R | \psi_N)$ is such that all minors of the first $N - 1$ columns are non-zero. Apply lemma 7.1 to $\psi'$. Then for $|\lambda_i| \leq \frac{1}{N} \binom{N}{n}$ and $\Lambda = E_{i_1,N}(\lambda_1) \ldots E_{i_n,N}(\lambda_n)$, all the $n \times n$ minors of $\psi' \Lambda$ are non zero. Define $T = T' \Lambda$. Note that $T = \begin{pmatrix} \text{Id}_n & X \\ 0 & \text{Id}_{N-n} \end{pmatrix}$ where $X = (Y | \lambda)$ and $\lambda$ is the column vector given by a permutation of $(\lambda_1, \ldots, \lambda_n)$. This proves the proposition for $N$. \hfill \Box

8. Appendix I: A conjectural implication

F. Amoroso, S. David and P. Philippon related the essential minimum of a transverse variety with the relative obstruction index. Let $H$ be the translate of an abelian subvariety of $E^N$ containing $V$. The relative obstruction index is

$$\omega_L(V, H) = \min_{V \subset Z} \deg_L Z,$$

for $Z$ varying over all divisors of $H$ containing $V$.

The following conjecture can be deduced form work of Amoroso, David and Philippon.

**Conjecture 8.1.** Let $H$ be the translate of an abelian subvariety of $E^N$ of dimension $n$. Let $V$ be transverse in $H$. For any symmetric ample line bundle $L$ on $E^N$, there exists a positive constant $c_1$ depending on $E^N$ and $L$ such that

$$\mu_L(V) \geq c_1 \frac{\deg_L H}{\omega_L(V, H)}.$$

In view of our theorem 1.2 we can suggest

**Conjecture 8.2.** Let $H$ be the translate of an abelian subvariety of $E^N$ of dimension $n$. Let $V$ be a $d$-dimensional variety transverse in $H$. Then, for any polarization $L$ on $E^N$, there exists a positive constant $c'_1$ depending on $E^N$ and $L$ such that

$$\mu_L(V) \geq c'_1 \left( \frac{\deg_L H}{\deg_L V} \right)^{\frac{1}{n-d}}.$$

We conclude our work clarifying the relation, pointed out by Philippon, between the conjecture 8.1 and our conjecture 8.2.

**Proposition 8.3.** Conjecture 8.1 implies Conjecture 8.2

**Proof.** We denote by $\ll$ an inequality up to a multiplicative constant, depending only on irrelevant parameters. Let $H$ be the translate of an abelian subvariety of dimension $n$ and let $L$ be any line bundle on $H$. Let $V$ be transverse in $H$ of dimension $d$. We shall prove that

$$\frac{\omega_L(V, H)}{\deg_L H} \ll \left( \frac{\deg_L V}{\deg_L H} \right)^{\frac{1}{n-d}}.$$
The main result of M. Chardin [5] gives the following upper bound for the Hilbert function
\[ \mathcal{H}_L(V, \nu) \ll \frac{\deg L V}{d!} \nu^d. \]
The remark 1 of [5], implies the lower bound for the Hilbert function
\[ \mathcal{H}_L(H, \nu) \gg \frac{\deg L H}{n!} \nu^n, \]
for \( \nu \gg \deg L H \).
Choose \( \nu \) minimal so that \( \mathcal{H}_L(H, \nu) \leq \mathcal{H}_L(V, \nu) \) and \( \nu \gg \deg L H \).
Then
\[ \nu \ll \left( \frac{n! \deg L V}{d! \deg L H} \right)^{d \over n-d}. \]
Equivalently, there exists a hyperplane of \( \mathbb{P}^n \) containing \( V \) but not \( H \) of degree \( \leq \nu \). By Bezout’s theorem, this hypersurface defines a divisor of \( H \) containing \( V \) of degree \( \leq \nu \deg L H \). It follows
\[ \omega_L(V, H) \leq \nu \deg L H \ll \deg L H \left( \frac{\deg L V}{\deg L H} \right)^{1 \over n-d}. \]
This concludes the proof.

Finally we remark that in [9], David and Philippon manage to obtain a lower bound for the essential minimum of a transverse variety in a power of an elliptic curve where \( h(E) \) is at the numerator. Unfortunately they lost in the depends on \( \deg_{\mathcal{O}_N} V \). This is not strong enough to apply the method presented in this work and to extend their result to other polarizations. However, they announce a strong conjecture, [9] conjecture 1.5 ii. Using our method, we can conclude that if [9] conjecture 1.5 ii. holds for \( \mathcal{O}_N \), then it holds for the restrictions of the standard line bundle to translates of abelian subvarieties.
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