The Energy Spectrum of the Membrane effective Model for Quantum Black Holes

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Abstract

We study first order fluctuations of a relativistic membrane in the curved background of a black hole. The zeroth-order solution corresponds to a spherical membrane tightly covering the event horizon. We obtain a massive Klein-Gordon equation for the fluctuations of the membrane’s radial coordinate on the 2+1 dimensional world-volume. We finally suggest that quantization of the fluctuations can be related to black hole’s mass quantization and the corresponding entropy is computed. This entropy is proportional to the membrane area and is related to the one-loop correction to the thermodynamical entropy $A_H/4$. With regards to the membrane model for describing effectively a quantum black hole, we connect these results with previous work on critical phenomena in black hole thermodynamics.

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1. Introduction  We will try to find an effective (or phenomenological) description of the black hole quantum degrees of freedom in the spirit of the black hole *complementary principle*. We shall consider the whole picture as described by a fiducial observer (an observer at rest with respect to the black hole as opposed to a free falling observer). The principle of black hole complementarity [1,2] will ensure us not to have any logical contradiction because of the non existence of “superobservers”. For fiducial observers the black hole appears well described classically by a membrane [3] sitting on the stretched horizon (to regularize infinite redshift factors).

At quantum level several authors [4–6] considered that the relevant black hole’s degrees of freedom to quantize were located practically on the horizon surface. For the sake of definiteness we will discuss the possibility of representing this black hole’s degrees of freedom, in an effective way, by a relativistic membrane. This model is an effective one in the sense that it is not fundamental (nor renormalizable), but will be eventually derivable (in the appropriate limit) from a consistent and finite quantum theory of gravitation.

To fix ideas we can consider the effective system to study at low energies as composed of the classical gravitational field (Einsteinian gravity, for instance) and the relativistic membrane (plus, eventually, matter fields) represented by the following Lagrangean density.

\[ L = L_{\text{grav}} + I_M \delta(X - X_M) + L_{\text{matter}} \]  

(1)

where the action of a relativistic bosonic membrane [7]

\[ I_M = -\frac{T}{2} \int d^3 \xi \sqrt{-\gamma} \left\{ g_{\mu\nu}(X) \gamma^{ij} \partial_i X^\mu \partial_j X^\nu - 1 \right\} . \]

(2)

Here \( T \) is the membrane tension. \( \xi = (\tau, \sigma, \rho) \) are the world-volume coordinates \((\mu, \nu = 0, 1, ..., D - 1.)\) and \( \partial_i = \partial/\partial \xi^i \) where the indices \( i, j \) take the values 0, 1, 2.

We can think of the above action as emerging from a coarse-graining process over very short distances from the black hole event horizon. Our ignorance about these short distances is hidden into a few phenomenological parameters such as the membrane tension, \( T \). One thus obtains, an effective lagrangian for the position of the “stretched horizon” (repre-
sent by collective variables) that, because of the reparametrization covariance, is just the
lagrangian of a relativistic membrane in a curved background.

The advantage of this approach is that, since we have a concrete dynamical model, we can
compute the energy levels and derive an associated entropy, and then check the consistence
of this results with what we already independently knew about quantum properties of black
holes, for finally to look for some new results.

It has already been observed by various authors, e.g. [8,2], that short distances play an
important role in the counting of the number of states near the horizon, and that such a
counting gives a divergent result unless one introduces a regulator such as a short distance
cutoff. Thus, we expect that also in our formulation the separation between membrane levels
depend on such a cutoff, and we should investigate up to what extent we can extract results
which are independent of short distance physics.

From action Eq. (2) we can derive the classical equations of motion

$$g_{\mu\rho}(\Box_\gamma X^\rho + \Gamma^\rho_{k\lambda}(g)\partial_i X^k \partial_j X^\lambda \gamma^{ij}) = 0 ,$$

and the constraint

$$\gamma_{ij} = \partial_i X^\mu \partial_j X^\nu g_{\mu\nu}(X)$$

by varying with respect to $X^\mu$ and $\gamma_{\mu\nu}$ respectively.

We are interested in studying the membrane dynamics in the curved background of a
static spherically symmetric black hole in $D = 4$ dimensions [In the following we work with
Minkowskian signature, and we use Planck units, $\hbar = c = G = 1$.]; its metric represented by

$$ds^2 = -a(r)dt^2 + b(r)^{-1}dr^2 + r^2d\Omega^2 , \quad d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2 .$$

For a Reissner-Nordström black hole

$$a(r) = b(r) = 1 - 2\frac{M}{r} + \frac{Q^2}{r^2} ,$$

and the classical event horizon is located at the radial coordinate $r_+ = M + \sqrt{M^2 - Q^2}$. 3
We next conveniently choose the coordinate system according to our description in terms of a fiducial observer outside the black hole. In fact, it is from the point of view of this observer that the black hole degrees of freedom can be represented by a membrane. We have thus the following set of coordinates

\[ X^0 = t, \quad X^1 = r, \quad X^2 = \theta, \quad X^3 = \phi \]  

(7)

To study the semiclassical quantization of a membrane sitting on the black hole event horizon we choose the, so called \cite{9}, \textit{rest gauge} \( X^0(\tau, \sigma, \rho) = \tau \).

Since we are thinking of studying the membrane fluctuations around a classical solution having spherical symmetry, we will fix the gauge by taking \( X^2(\tau, \sigma, \rho) = \sigma \) and \( X^3(\tau, \sigma, \rho) = \rho \) with \( 0 \leq \sigma \leq \pi \) and \( 0 \leq \rho \leq 2\pi \).

2. \textit{First Order Fluctuations} \hspace{1em} The classical equations of motion in the background of a Reissner-Nordström black hole have solutions \cite{5} corresponding to membranes which approach the background event horizon, \( r = r_+ \). If we try to quantize around this solution, however, we find that the membrane level actually form a continuum (see below), so that the number of states of the black hole would diverge. This divergence is nothing but the divergence first found by ‘t Hooft \cite{8}, and can be regulated with a short distance cutoff near the horizon. The physical origin of this cutoff, in the membrane approach, is quite clear. The membrane is only an effective theory, obtained performing a coarse graining over short distances, that is, over distances on the order of a few Planck lengths from the horizon (or a few string lengths, depending on what is the fundamental theory), and it is not legitimate to extrapolate the membrane equations of motion down to a distance smaller than a few Planck length from the classical horizon. Thus, we rather proceed as follows. We assume that the equations of motion for \( g_{\mu\nu} \), derived from the lagrangian \cite{1}, acquire corrections very close to the horizon, in such a way that the membrane, instead of approaching \( r = r_+ \) asymptotically, have an equilibrium position at \( X^1_M = R_M = r_H + \epsilon^2/r_H \) with \( \epsilon/r_H \ll 1 \), \( (\epsilon \simeq \text{a few Planck lengths}) \). Such corrections can produce a change in the dependence on \( r \) of the functions \( a(r) \) and \( b(r) \) which enter in the metric, see Eq. \cite{3}. We do not present
here any explicit expression indicating how $a(r)$ and $b(r)$ are modified, since it is clear that this is just a phenomenological way to take into account short distance effects, and we will be interested only in those results which have a rather general validity, and do not depend on the specific properties of the modification introduced.

Let us then study the first order fluctuations in the only independent coordinate left after our gauge choice

$$X^\mu = \left(X^0(\tau), X^1(\tau, \sigma, \rho), \theta(\sigma), \varphi(\rho)\right).$$ (8)

From expressions (3) and (4) it is easy to see that neglecting orders higher than the first, the equations of motion (3) become a Klein-Gordon equation for the radial coordinate, $X^1$, in the 2 + 1 world-volume of the membrane. In fact, $\gamma_{ij}$ has the following components

$$\gamma_{\tau\tau} = -a + b^{-1}(X^1)^2, \quad \gamma_{\sigma\sigma} = (X^1)^2 + b^{-1}(X^1)^2,$$

$$\gamma_{\sigma\tau} = \gamma_{\tau\sigma} = b^{-1}X^1\dot{X}^1, \quad \gamma_{\sigma\rho} = \gamma_{\rho\sigma} = b^{-1}X^1\dot{X}^1,$$

$$\gamma_{\tau\rho} = \gamma_{\rho\tau} = b^{-1}\dot{X}^1\ddot{X}^1, \quad \gamma_{\rho\rho} = (X^1)^2 \sin^2(X_2) + b^{-1}(\ddot{X}^1)^2.$$ (9)

where $\cdot = \partial_\tau$, $' = \partial_{\sigma}$ and $~ = \partial_{\rho}$.

To first order in the fluctuations of $X^1$ around $R_M$ we are left with a diagonal 3-metric

$$\gamma_{ij}^{(1)} = \begin{pmatrix} -a & 0 & 0 \\ 0 & R_M^2 & 0 \\ 0 & 0 & R_M^2 \sin^2(\theta) \end{pmatrix}.$$ (10)

The field equations (3) for the $r$-component in this case read

$$\frac{1}{b(R_M)} \left\{ \frac{-\partial^2_\tau \delta X^1}{a(R_M)} + \frac{1}{R_M^2 \sin \theta} \left[ \partial^2_\sigma \delta X^1 + \partial_\rho (\sin \theta \partial_\rho \delta X^1) \right] \right\}$$

$$- \left( \frac{2}{R_M^2} + \frac{2a'(R_M)}{R_M a(R_M)} - \frac{1}{4} \left( \frac{a'(R_M)}{a(R_M)} \right)^2 + \frac{a''(R_M)}{2a(R_M)} \right) \delta X^1 = \frac{2}{R_M} + \frac{a'(R_M)}{2a(R_M)},$$ (11)

where $\delta X^1 = X^1 - R_M$. 5
For the $t$-component
\[
\left( \frac{2}{R_M} + \frac{a'(R_M)}{2a(R_M)} \right) \partial_t \delta X^1 = 0 ,
\]
for the $\theta$-component
\[
\left( \frac{2}{R_M} + \frac{a'(R_M)}{2a(R_M)} \right) \partial_\theta \delta X^1 = 0 ,
\]
and for the $\varphi$-component
\[
\frac{1}{\sin \theta} \left( \frac{2}{R_M} + \frac{a'(R_M)}{2a(R_M)} \right) \partial_\varphi \delta X^1 = 0 .
\]

From Eqs. (12)-(14) we see that to have a non trivial solution for $\delta X^1$ it must be
\[
\frac{2}{R_M} + \frac{a'(R_M)}{2a(R_M)} = 0 .
\]

This condition is in fact equivalent to the existence of a zeroth-order solution representing a spherical membrane at rest at $X^1 = \text{constant} = R_M$, and, precisely, determines $R_M$ given the background metric (5). With Eq. (15) fulfilled, Eqs. (12)-(14) are satisfied automatically and we are left with Eq. (11) as the only relevant equation for $\delta X^1(\tau, \sigma, \rho)$. It can be written as a massive 2+1 dimensional Klein-Gordon equation on the sphere of radius $R_M$, (considering the proper time $\tau'$ of an observer at rest at $R_M$, i.e. $\tau' = a(R_M)^{1/2}\tau$.) Thus we have
\[
\Box_{\tau'} \delta X^1 - \mu^2 \delta X^1 = 0 ,
\]
where
\[
\mu^2 = \left[ \frac{2}{R_M^2} + \frac{2a'(R_M)}{R_M a(R_M)} - \frac{1}{4} \left( \frac{a'(R_M)}{a(R_M)} \right)^2 + \frac{a''(R_M)}{2a(R_M)} \right] b(R_M)
\]
is the effective mass squared of the first order fluctuations.

3. Energy Spectrum and Mass Quantization

Membrane theories are intrinsically nonlinear and therefore much more difficult to quantize than string theories. Thus, next, we will perform a semiclassical quantization expanding around a classical solution [10]. Working
at first order and in a physical gauge we obviate the problem of the constrained system quantization.

Let us decompose the solution to Eq. (16) in spherical modes

$$\delta X^1(\tau, \sigma, \rho) = \sum_{l,m} \delta X^1_{lm}(\tau) Y_{lm}(\sigma, \rho), \quad (18)$$

where $Y_{lm}(\theta, \phi)$ are the usual spherical harmonics.

Plugging Eq. (18) into Eq. (16) we obtain the following expression for the modes $\delta X^1_{lm}(\tau)$ of first order fluctuations

$$\dddot{\delta X^1_{lm}} + \omega_l^2 \delta X^1_{lm} = 0, \quad \omega_l^2 = a(R_M) \left( \mu^2 + \frac{l(l+1)}{R_M^2} \right). \quad (19)$$

This is just the classical equation of motion of an harmonic oscillator with frequency $\omega_l$ for each mode labeled by $lm$. As in any system with spherical symmetry $\omega_l$ does not depend on $m$, the azimuthal angular momentum. This implies a degeneracy of states of $d_l = 2l + 1$.

Upon quantization of these modes one obtains a discrete spectrum of energies [11]

$$E_{nl} = (n + 1/2)\omega_l. \quad (20)$$

In $\omega_l$, the mass term, $\mu$, is background dependent, but of less relevance than the dependence on $\sqrt{a(R_M)}$ which is a non-trivial and a largely model-independent feature. Physically, $a(R_M)^{1/2}$ is just the redshift factor: the excitation of the membrane gives a contribution to the energy at infinity of the black hole which is suppressed by a factor equal to the redshift from the membrane location to infinity. It is clear therefore that, if we remove the cutoff and send naively $a(R_M) \rightarrow 0$, we get a continuum of energy levels.

One can interpret Eq. (20) as indicative of a discrete spectrum of black hole masses. The spacing between two neighbour levels is given by

$$\Delta E_n = \frac{E_{nl}}{n + 1/2}, \quad \text{for fixed } l. \quad (21)$$

And

$$\Delta E_l = \frac{2a(R_M)(n + 1/2)^2(l + 1)}{R_M^2(E_{nl} + E_{n,l+1})} \approx \frac{E_{nl}}{l}, \quad \text{for fixed } n. \quad (22)$$
We observe that both level separations are of the same order and tend to zero as \( l \) or \( n \) goes to infinity (continuum limit). [In the brick wall model \[8\] one would have \( \Delta E \sim \sqrt{a(R_M)}/R_M \sim \epsilon/M^2 \).] This result, interpreted as a black hole mass quantization gives much thinner level separation than earlier estimates [12–17]. Suppose that the black hole mass \( E_{nl} \) is large but not astronomically large compared to the Planck mass \( M_{Pl} \), say \( E_{nl} \sim 10^3 M_{Pl} \); then the separation \( \Delta E_{nl} \sim 10^{-6} M_{Pl} \) is small compared to \( M_{Pl} \), but still not small enough to allow a particle with ordinary energy, say a few GeV, to be absorbed or emitted by the black hole. Thus, the emission line-spectrum would still be significantly different from the semiclassical (continuum) result, even for black holes with mass \( M \gg M_{Pl} \), for which we would rather expect that the semiclassical approximation works well.

4. Entropy

The contribution to the entropy of the energy levels (20) can be obtained from the partition function

\[
Z = \prod_{l=0}^{\infty} \prod_{m=-l}^{l} \sum_{n=0}^{\infty} e^{-\beta E_{nl}},
\]

where \( \beta = 1/T \) and the sum over \( n \) is essentially a geometric series. Then,

\[
\ln Z = -\sum_{l=0}^{\infty} (2l + 1) \ln \left[ \frac{e^{-\beta \omega_l/2}}{1 - e^{-\beta\omega_l}} \right]
\]

(24)

[Note that we have taken the sum over \( n \) up to infinity, although the expression (20) for \( E_{nl} \) was derived in an approximation that makes it valid in the regime \( E_{nl} \ll M \). However the big \( n \) contribution to \( Z \) is negligible.]

The mean energy per mode is

\[
\langle E \rangle = \partial_\beta \ln Z = \sum_{l=0}^{\infty} (2l + 1) \left[ \frac{\beta \omega_l}{e^{\beta \omega_l} - 1} + \frac{\beta \omega_l^2}{2} \right]_{\beta = T_{BH}^{-1}},
\]

(25)

wherefrom we see that the spectrum of radiation of this system will be essentially planckian. [The \( \beta \omega_l/2 \) term is the usual vacuum polarization contribution to be renormalized away upon operator ordering.]

From the expression for the entropy \( S = [\ln Z - \beta \partial_\beta \ln Z]_{\beta = T_{BH}^{-1}} \) we obtain

\[
S_M = -\sum_{l=0}^{\infty} (2l + 1) \left[ \ln \left( 1 - e^{-\beta \omega_l} \right) - \frac{\beta \omega_l}{e^{\beta \omega_l} - 1} \right]_{\beta = T_{BH}^{-1}}.
\]

(26)
To perform the sum above we approximate it by an integral over $l$. We then change to the variable $x = \beta \omega_l$ and obtain

$$S_M = -\frac{2R_M^2}{\beta^2 a(R_M)} \int_{\beta \omega_0}^{\infty} dx \left\{ \ln \left( 1 - e^{-x} \right) - \frac{x}{e^x - 1} \right\} \bigg|_{\beta = T_{BH}^{-1}}. \quad (27)$$

Since $\beta \omega_0 = \beta \sqrt{a(R_M)} \mu \ll 1$ the integral gives

$$S_M = 6\zeta(3) \left( \frac{T_{BH}^2}{a(R_M)} \right) R_M^2 + (\mu R_M)^2 \left( \ln \left[ \frac{\mu \sqrt{a(R_M)}}{T_{BH}} \right] + 3/2 \right), \quad \zeta(3) = 1.20206. \quad (28)$$

The first term in this expression is the leading term. It is proportional to the membrane (or stretched horizon) area and to the square of the local temperature. The logarithmic term here appears due to the $l = 0$ mode contribution to the energy levels. It can be taken as a sample of what we would expect from non-leading corrections to the one-loop entropy. [Note that in Eq. (28) the same dependence on the redshift factor would appear had we considered observations at $R_M$ instead of at infinity, since $\beta E_{nl}|_{\infty} = \beta E_{nl}|_{R_M}$.]

So far, in Eq. (28) does not appears explicitly any cut-off. If we introduce 't Hooft’s brick wall \[8\], $R_M = 2M + \epsilon^2/(2M)$, we would obtain $S_M \sim C_1 M^2/\epsilon^2 + C_2 \ln(M/\epsilon)$. The leading term corresponds to the one obtained by other methods such as the study of four-dimensional quantum fields in the Schwarzschild background \[8,13,\ 24\]. [We note that this result is non-trivial in our case, since it was obtained from the study of a Klein–Gordon field in 2+1 dimensions.]

For further comparison it is interesting to recall here that in Ref. [25] it was studied the back reaction effect of the Hawking radiation on the background Schwarzschild black hole metric. It was thus obtained a correction to the Bekenstein–Hawking temperature of the following form

$$T_{BH} = \frac{1}{8\pi M} \left[ 1 + \frac{\alpha}{M^2} \right], \quad (29)$$

where $\alpha$ is a constant depending on the field content of the Hawking radiation. From the above equation we obtain for the entropy
\[ S_{BH} = \int \frac{dM}{T} = 4\pi M^2 - 8\pi \alpha \ln M . \] (30)

We thus see that the effect of the one-loop corrections to the entropy found above can be interpreted, respectively, as producing a renormalization of the gravitational constant \( G \) and the higher order coupling constants (Back reaction terms in our case or purely vacuum polarization ones in the case studied in Ref. [26]).

5. Discussion

In this paper we have taken the relativistic membrane as a phenomenological model to describe the quantum degrees of freedom of a black hole. The main results we have obtained are the energy spectrum, i.e. Eq. (20) and its associated entropy, i.e. Eq. (28). The main approximations in arriving to these results have been the linearization of the system (and considering \( l \) as a continuum variable in the computation of the entropy). It is clear from the lagrangian formulation (1) that the membrane contributions are perturbative with respect to the classical (or tree-level) expressions. Then, in particular, the total entropy will be given by \( S_{BH} = A_H/4 + S_M \) and thus, the membrane corrections can be associated to the one-loop level. Within this approach we did not address the questions of the stability of the classical membrane solution at a given radius, \( R_M \), near the black hole event horizon as well as the origin of the Hawking radiation at a black hole temperature \( T_{BH} \), but rather consider them as external parameters provided by the curved background. Also covariant higher order fluctuations and careful quantization of the constrained system should be considered. All these points deserve a particular study and will indeed be the line of research to follow in obtaining a completely self consistent picture of which the present work must be considered only a first step.

In Refs. [27,13,17,28] it was shown strong evidence for that four-dimensional, rotating and charged black holes undergo critical phenomena. Under critical conditions their characteristic behaviour is as if they had an effective dimension equal to two. In fact, critical exponents of correlation functions and quite general arguments coming from the Renormalization Group theory assign an effective (spatial) dimension, \( d = 2 \), to the system. Also, by comparison of the black hole critical exponents with those of the Gaussian model in
$d$-dimensions, complete agreement is only found for $d = 2$. We can think of Eq. (16) as coming from an action for the “field” $\delta X^1(\tau, \sigma, \rho)$

$$I_M(\delta X^1) = -\frac{T}{2} \int d\tau d\theta d\phi R_M^2 a(R_M)^{-1/2} \sin \theta \times$$

$$\left\{ -\frac{(\partial_\tau \delta X^1)^2}{a(R_M)} + \frac{1}{R_M^2} \left[ (\partial_\theta \delta X^1)^2 + \frac{1}{\sin \theta} (\partial_\phi \delta X^1)^2 \right] - \mu^2 (\delta X^1)^2 \right\} . \quad (31)$$

which generates essentially a Gaussian integral and we can [17], then, reproduce in the membrane approach all the black hole critical exponents since $\delta X^1(\tau, \sigma, \rho)$ is now a Gaussian field living on a spatially two-dimensional sphere of radius $R_M$.

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