Entanglement Detection Beyond Measuring Fidelities

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One of the most widespread methods to determine if a quantum state is entangled, or to quantify its entanglement dimensionality, is by measuring its fidelity with respect to a pure state. In this paper we find a large class of states whose (D-dimensional) entanglement cannot be detected in this manner: we call them (D-)unfaithful. We show that not only are most random bipartite states both entangled and unfaithful, but so are almost all pure entangled states when mixed with the right amount of white noise. We also find that faithfulness can be self-activated, i.e., there exist instances of unfaithful states whose tensor powers are faithful. To explore the entanglement dimensionality of D-unfaithful states, we additionally introduce a hierarchy of semidefinite programming relaxations that fully characterizes the set of states of Schmidt rank at most D.

Definitions. A bipartite state $\rho_{AB}$ (where we will omit the indices if they are clear from context) is separable if there exists a probability distribution $(p_i)_i$ and states $(|\psi_i\rangle, |\phi_i\rangle)_i$ such that

$$\rho_{AB} = \sum_i p_i |\psi_i\rangle \langle \psi_i| \otimes |\phi_i\rangle \langle \phi_i|.$$  

Otherwise, $\rho_{AB}$ is said to be entangled. We denote the set of separable quantum states by $S$.

The usefulness of a state for many information processing tasks is closely related to its entanglement dimensionality. A mixed bipartite state $\rho_{AB}$ is said to have Schmidt rank at most $D$ [27] if there exists a decomposition

$$\rho_{AB} = \sum_i p_i |\psi_i\rangle \langle \psi_i|,$$  

with $p_i \geq 0$, $\sum_i p_i = 1$ and all $|\psi_i\rangle$ have Schmidt rank at most $D$. We denote by $S_D$ the set of all such states. Note that all these sets are convex, $S_1 = S$, and $S_D \subset S_{D+1}$. If $\rho \notin S_{D-1}$, we say that $\rho$ is $D$-dimensionally entangled. If, additionally, $\rho \in S_D$ we say that it has Schmidt rank equal to $D$. Tying this back to the definition of simple entanglement, all entangled states are 2-entangled, and have Schmidt rank at least 2.

An entanglement witness $W$ is a Hermitian operator with the property that $\text{tr}[W\sigma] \geq 0$ for all separable states $\sigma \in S$. Hence, a measurement on a state $\rho$ having a negative expectation value $\text{tr}[W\rho] < 0$, implies that $\rho$ is entangled. Since $S$ is convex, the hyperplane separation theorem implies that for any entangled state $\rho$ there exists a witness $W$ such that $\text{tr}[W\rho] < 0$.

The same considerations apply when trying to witness entanglement dimension: for any state $\rho \notin S_{D-1}$ there exists a dimension-$D$ witness $W_D$ such that for all $\sigma \in S_{D-1}$, $\text{tr}[W_D\sigma] \geq 0$ and $\text{tr}[W_D\rho] < 0$. An entanglement witness is then just a dimension-2-witness.

One commonly used form of a witness, both for entanglement and D-dimensional entanglement, is the so-

Entanglement is a fundamental aspect of quantum information and one of the key dividing factors between the quantum and the classical worlds. This is shown by the wide range of protocols, such as teleportation [1], device independent quantum key distribution [2][4], one-way quantum computation [5], and metrology [6][7], in which it is a necessary resource. The amount of entanglement, which can be quantified, for example, by the entanglement dimension, is also a critical factor in a wide range of protocols [8][15].

As entanglement is such a useful resource, it is important to have experimental and theoretical tools to detect it [10][17], and for these to correctly identify entanglement even in very noisy states. Perhaps the most common used method to detect entanglement is via linear witnesses, but their characterization has been proven a hard problem [18][19], even though there are some general methods to tackle it [20][22]. The situation is substantially harder still when we want to detect the entanglement dimension of a state, as a practical characterization of the D-positive maps necessary for this task is still missing. In part due to these difficulties, it is common practice to detect the entanglement or entanglement dimension of a state via its fidelity with respect to a pure reference state [23][26]. The question we address in this letter is whether such a method misses out many instances of entangled states and, if so, what properties these states have.

We find, surprisingly, that almost no bipartite entangled states can be detected via fidelities with pure states. We call such states unfaithful. Using a simple semidefinite programming (SDP) ansatz of unfaithful states, we prove that even states as innocent as a mixture between a pure entangled state and the maximally mixed state can be unfaithful. We also show that unfaithfulness can be self-activated, namely, there exist unfaithful states whose bipartite entanglement can be detected via fidelities when taking their tensor power. Going beyond separability, we extend the concept of unfaithful states to those states whose entanglement dimension cannot be certified with fidelity witnesses. Lacking general tools to determine their entanglement dimension, we introduce a complete hierarchy of semidefinite programming relaxations of the set of all states with Schmidt rank at most D.

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called pure fidelity witness:

\[ W_D := F \mathbb{I} - |\psi_W\rangle \langle \psi_W|, \]  

where \(|\psi_W\rangle\) is a fixed entangled state and \(F\) a real number. In order to witness as many entangled states as possible it is desirable to have \(F\) be as small as possible. The minimum value it can have while still satisfying \(\text{tr}[W_D \sigma] \geq 0\) for all \(\sigma \in S_{D-1}\) is \(F = \sum_{i=1}^{D-1} \lambda_i\), where \((\sqrt{\lambda_i})_i\) are the Schmidt coefficients: \(|\psi_W\rangle = \sum_i \sqrt{\lambda_i} |i\rangle_A |i\rangle_B\), in decreasing order.

Such a witness detects a state \(\rho\) as being \((D\text{-dimensionally})\) entangled if \(\langle \psi_W| \rho |\psi_W\rangle > F\). In other words, \(\rho\) is certified as being \((D\text{-dimensionally})\) entangled if the overlap with the target state \(|\psi_W\rangle\) is big enough. As intuition suggests, this works well when \(|\psi\rangle\) is close to \(\sigma_{\text{sep}}\). Hence, it is desirable to have \(\text{tr}(\rho - |\psi_W\rangle \langle \psi_W|)\) be as small as possible with respect to the bipartition \(A'B\). In order to witness as many entangled states as possible it is desirable to have \(\text{tr}(\rho - |\psi_W\rangle \langle \psi_W|)\) be as small as possible.

Our aim is to understand which \(D\text{-dimensionally}\) entangled states, for some \(D \geq 2\), are certifiable by \(D\text{-dimensional}D\text{-witnesses},\) that is those in \((U_D \setminus S_{D-1})_{D \geq 2}\). We do this by constructing semidefinite programs that form an outer approximation of \(S_{D-1}\) (SDP \([1]\)) and an inner approximation of \(U_D\) (SDP \([2]\)).

Let us begin with the former. For the set \(S_1\), we are concerned with detecting whether a state is entangled. There exist several complete criteria for entanglement detection (see, e.g., \([29]\)). We use the Doherty-Parrilo-Spedalieri (DPS) hierarchy \([20, 22]\), which is an infinite nested sequence of sets \(S^1 \supset S^2 \supset S^3 \supset \cdots\), with the property that \(\lim_{k \to \infty} S^k = S\). Furthermore, each of these sets is characterized via semidefinite programming. This means that, if \(\rho\) is entangled at all, then there exists an SDP that will detect this. In fact, for fixed local dimensions \(d_A, d_B\), there exists a strictly decreasing sequence \((\epsilon_k)_{k}\), with \(\lim_{k \to \infty} \epsilon_k = 0\), such that any state \(\sigma \in S^k\) can be expressed as

\[ \sigma = (1 - \epsilon_k)\sigma_{\text{sep}} + \epsilon_k \sigma', \]  

for some (separable) state \(\sigma'\) \((\sigma_{\text{sep}})\) \([30, 31]\). The first set \(S^1\) of the DPS hierarchy is the set of states with positive partial transpose \([22, 33]\), which we call PPT in the following.

Here, we introduce a hierarchy of SDPs that generalizes the DPS criterion by converging to \(S_D\) from the outside:

**SDP 1.** Let \(\sigma_{\text{AB}}\) be a bipartite state, with local dimensions \(d_A\) and \(d_B\) respectively. If there exists a positive matrix \(\omega_{\text{AA'}}|B'\rangle\langle B'|\) such that

\[ \omega_{\text{AA'}}|B'\rangle\langle B'| + \text{tr}(\omega) = D, \]  

\[ \Pi_D' \omega \Pi_D = \sigma \]  

where the dimensions of \(A'\) and \(B'\) are both \(D\) and \(|\psi^{\dagger}_{D'}\rangle_{A'B'} := \sum_{j=1}^{D} |j, j\rangle\) is the unnormalized maximally entangled state in this dimension, then we say that \(\sigma \in S^k_D\).

Note that for \(D = 1\) this coincides with the DPS hierarchy. From the above definition, it is straightforward that \(S^k_D \supset S^{k+1}_D\) and that each of these sets can be characterized by an SDP. Furthermore, \(S_D \subset S^k_D\) for all \(k, D\) and \(\lim_{k \to \infty} S^k_D = S_D\) (see below for a proof). For an implementation of the lowest order of this hierarchy in Python, we refer the reader to \([34]\).

**Proof.** To prove convergence of the hierarchy, note that any normalized pure state \(|\psi\rangle = \sum_{i} c_i |\phi_i\rangle |\Phi_i\rangle\) with Schmidt rank \(D\) can be written as \(\Pi_D' \langle\langle \alpha |_{A'} |\beta\rangle_{B'}\rangle\), with

\[ |\alpha\rangle = \sum_{i=1}^{D} c_i |\phi_i\rangle_A |i\rangle_{A'}, \]  

\[ |\beta\rangle = \sum_{i=1}^{D} |i\rangle_B |\Phi_i\rangle_B. \]  

Note that the vector \(|\alpha\rangle |\beta\rangle\) has norm \(\sqrt{D}\) and is separable with respect to the bipartition \(A'A'B\). By convexity, we thus have that any \(\sigma_{AB} \in S_D\) can be expressed as
\( \Pi^0_D \omega_{AA'BB'} \Pi_D \), for some positive semidefinite \( \omega_{A'A'B'B'} \) that is separable with respect to the bipartition \( AA'|B'B' \) and with \( \text{tr}(\omega) = D \).

The sets \( S^k_D \) are defined by relaxing the condition \( \omega / \text{tr}[\omega] \in S \) to \( \omega / \text{tr}[\omega] \in S^k \) in the relation above. If \( \sigma \in S^k_D \), we find, by Eq. (4), that

\[
\Pi^k_D \omega \Pi_D = (1 - \epsilon_k) \Pi^k_D \omega_{\text{sep}} \Pi_D + \epsilon_k \Pi^k_D \omega' \Pi_D.
\]

Define now the normalized states \( \sigma_D \propto \Pi^k_D \omega_{\text{sep}} \Pi_D, \sigma' \propto \Pi^k_D \omega' \Pi_D, \) and the parameter \( \epsilon_k := \text{tr}(\omega' \Pi_D) \). Then we have that \( \sigma = (1 - \epsilon_k) \sigma_D + \epsilon_k \sigma' \). From the relations \( \text{tr}(\omega') = D \) and \( ||\Pi_D \Pi_D'|| = D \) it follows that \( \epsilon_k \leq \epsilon_k D^2 \). Thus, as \( k \) grows, the separation between \( S^k_D \) and \( S_D \) tends to zero.

For the problem of certifying that a state is in \( \mathcal{U}_D \) we utilize an inner approximation that can be realized with an SDP. We do this by defining a new set, \( \mathcal{U}_D \), according to the following:

**SDP 2.** Let \( \rho_{AB} \) be a bipartite state. If there exists \( \mu \in [0,1] \) and positive semidefinite operators \( M_A, M_B \) such that

\[
M_A \otimes \mathds{1}_B + \mathds{1}_A \otimes M_B \geq \rho_{AB} \quad \mu(D - 1) = \text{tr}(M_A), \quad \mu \mathds{1}_A - M_A \geq 0, \\
(1 - \mu)(D - 1) = \text{tr}(M_B), \quad (1 - \mu) \mathds{1}_B - M_B \geq 0,
\]

then we say that \( \rho_{AB} \in \mathcal{U}_D \).

This set is an inner approximation to \( \mathcal{U}_D \). That is, \( \mathcal{U}_D \subset \mathcal{U}_D \).

**Proof.** Consider an arbitrary bipartite pure state \( |\psi\rangle \), and define \( \Lambda_{AB} := |\psi\rangle \langle \psi| \). Next, multiply the first line of Eq. (8) by \( \Lambda_{AB} \) and take the trace. The result is:

\[
\text{tr}(M_A \Lambda_A) + \text{tr}(M_B \Lambda_B) \geq \mu \rho_{AB} \Lambda_A \Lambda_B.
\]

Now, let \( \Lambda_A = \sum_i \lambda_i |\psi_i\rangle \langle \psi_i| \) be the spectral decomposition of \( \Lambda_A \) (note that \( \lambda_i \) are the Schmidt coefficients squared of \( |\psi\rangle \)). Then, \( \text{tr}(M_A \Lambda_A) = \sum_i \lambda_i p_i \), where \( p_i := \langle \psi_i| M_A |\psi_i\rangle \). By the positivity of \( M_A \) and the second line of Eq. (8), we have that \( 0 \leq p_i \leq \mu \) and \( \sum_i p_i = (D - 1) \mu \).

For \( j \geq D \) we have that

\[
r_j := \mu p_j \frac{\sum_{i=1}^{D-1}(\mu - p_i)\lambda_i}{\sum_{k=1}^{D-1}(\mu - p_k)} - p_j \lambda_j \geq 0.
\]

Hence,

\[
\sum_i \lambda_i p_i \leq \sum_i \lambda_i p_i + \sum_j r_j = \mu \sum_i \lambda_i.
\]
certify to be at least 3-dimensionally entangled but unfaithful, and refer the reader to [34]. A different method to numerically produce states with high Schmidt rank but which are at the same time PPT (and therefore unfaithful) is presented in [35].

For $D \geq 2$, there are furthermore examples of states that are faithful but 3-unfaithful. Consider, for instance,
\[ \frac{1}{2} |Ψ_3⟩⟨Ψ_3| + \frac{1}{2} \left( \frac{23 + |32⟩}{\sqrt{2}} \right) \left( \frac{23 + |32⟩}{\sqrt{2}} \right), \] (12)
where $|Ψ_3⟩ := \frac{1}{\sqrt{3}} (|00⟩ + |11⟩ + |22⟩)$. This state belongs to $U_3 \setminus (S^2_1 \cup U_2)$. That is, it is a 3-unfaithful state that is indeed of Schmidt rank 3, but that is 2-faithful. Note that for the last claim we cannot use our SDP approximation because it is inner. However, it is easy to find a fidelity witness that certifies its entanglement; namely
\[ W_2 := \frac{1}{3} I - |Ψ_3⟩⟨Ψ_3|. \]
For the certification of the Schmidt rank as well as the 2-unfaithfulness of the state using SDP 1 and SDP 2, we refer the reader to [34].

**Noisy pure states are unfaithful.** Having established that most states are unfaithful, we now investigate if this holds for states that are of commonly used and important in quantum information. Consider the Bell state $|Ψ_2⟩ = 1/\sqrt{2} (|00⟩ + |11⟩)$ embedded in a $d \times d$ dimensional bipartite Hilbert space and define the mixed state
\[ ρ_{AB} = p \frac{I_{AB}}{d^2} + (1 - p) |Ψ_2⟩⟨Ψ_2|. \] (13)
For $d > 2$, there exists a large parameter regime, illustrated in Fig 1 where this state is entangled but unfaithful. This is not a specific property of $|Ψ_2⟩$: we found numerically (with $10^6$ random Haar-distributed pure states for each of $d = 3, 4, 5$) that all pure entangled states are unfaithful and entangled, for a certain range of white noise. The only exception we know of is the maximally entangled state $1/\sqrt{d} \sum_{i=0}^{d-1} |ii⟩$ subjected to white noise in a Hilbert space of dimension $d \times d$.

For 3-unfaithfulness, we observe an analogous behaviour: the embedded state $|Ψ_3⟩ := 1/\sqrt{3} (|00⟩ + |11⟩ + |22⟩)$ mixed with noise as in Eq. (13) belongs to $U_3 \setminus S^2_1$ (meaning that it is certified to have Schmidt rank 3 but that this cannot be detected with a dimension-3-witness) for $p$ in the following ranges: in $d = 4$ for $p \in (0.364, 0.449)$; in $d = 5$ for $p \in (0.357, 0.493)$. In $d = 3$, the state becomes 3-unfaithful at the same point as we cease to certify (using $S^2_2$) that it is 3-dimensionally entangled.

**Faithfulness can be self-activated.** Note that, if $ρ_{AB}$ is faithful, then so is $ρ^{⊗n}_{AB}$. Indeed, let $|ψ⟩$ be such that $⟨ψ| ρ_{AB} |ψ⟩ > λ_1 (|ψ⟩)$. Then,
\[ ⟨ψ|⊗n ρ^{⊗n}_{AB} |ψ⟩⊗n > λ_1 (|ψ⟩) ⊗n = λ_1 (|ψ⟩⊗n). \] (14)

The property of being faithful is thus preserved under tensor powers. It is therefore natural to ask if unfaithfulness is also preserved this way, or if faithfulness can be self-activated.

Such a self-activation effect is impossible for states in $R$. Indeed, whenever two matrices satisfy $A \geq B \geq 0$ we have $A ⊗ A \geq B ⊗ B \geq 0$, so that
\[ ρ_A ⊗ I_B - ρ_{AB} \geq 0 \Rightarrow (ρ_A ⊗ I_B)^{⊗n} - ρ^{⊗n}_{AB} \geq 0 \] (15)

Nevertheless, it is possible to self-activate the faithfulness of certain states. Let us introduce the states
\[ |φ_1⟩ := 0.628 |11⟩ - 0.778 |22⟩ \] (16)
\[ |φ_2⟩ := 0.807 |01⟩ - 0.185 |02⟩ - 0.102 |10⟩ - 0.027 |11⟩ + 0.011 |12⟩ + 0.551 |20⟩ - 0.024 |21⟩ - 0.022 |22⟩. \]
Then, the state
\[ ρ_{AB} := 0.999 (0.50179 |φ_1⟩⟨φ_1| + 0.49821 |φ_2⟩⟨φ_2|) + 0.001 \frac{1}{9} \] is unfaithful, but $ρ^{⊗2}_{AB}$ is faithful over the partition $A/B$. This is proved, and the witness showing faithfulness given, in [34]. Thus, fidelity witnesses are sometimes useful, even if the target state is unfaithful.

**Conclusion.** In this paper we have introduced the set of unfaithful states, namely, those states whose entanglement cannot be detected via fidelities to pure states.
We analyzed different properties of such states: their frequency, their robustness and the phenomenon of activation. As we discovered, the set of unfaithful states is large and comprises quantum states which are very relevant in quantum information theory. We argued, based on our studies for small $D$, that the sets $U_2$ and $U_D$ have a similar behavior. Our work is therefore to be understood as a warning towards the blind application of fidelity-type dimension witnesses.

While there exist several general methods to detect entanglement beyond state fidelities, one could not say the same regarding the quantification of the entanglement dimensionality. In this regard, our work also provides methods to construct general dimension-$D$-witnesses, by means of a complete hierarchy of semidefinite programs. We expect this hierarchy to play a significant role in the quantification of entanglement in noisy experimental setups.

There are, naturally, some open questions about unfaithful states that could be studied in future work. One avenue is to extend the results from bipartite entanglement to multi-partite entanglement, which has a much richer and more elaborate structure. A different path is to investigate activation further. We have shown that some (but not all) unfaithful states become faithful when we take a tensor power of it; this leads to the hypothesis that for any entangled and unfaithful state, there exist another unfaithful state such that the tensor product of them is faithful. If this is true, then pure-state fidelities could always detect entanglement provided that we have access to the correct auxiliary state.

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