Borchers’ Commutation Relations for Sectors with Braid Group Statistics in Low Dimensions

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Abstract

Borchers has shown that in a translation covariant vacuum representation of a theory of local observables with positive energy the following holds: The (Tomita) modular objects associated with the observable algebra of a fixed wedge region give rise to a representation of the subgroup of the Poincaré group generated by the boosts and the reflection associated to the wedge, and the translations. We prove here that Borchers’ theorem also holds in charged sectors with (possibly non-Abelian) braid group statistics in low space-time dimensions. Our result is a crucial step towards the Bisognano-Wichmann theorem for Plektons in \(d = 3\), namely that the mentioned modular objects generate a representation of the proper Poincaré group, including a CPT operator. Our main assumptions are Haag duality of the observable algebra, and translation covariance with positive energy as well as finite statistics of the sector under consideration.

Introduction

Borchers has shown [3] that in a theory of local observables, which is translation covariant with positive energy, the modular objects associated with the observable algebra of a (Rindler) wedge region and the vacuum state have certain specific commutation relations with the representers of the translations. Namely, these commutation relations manifest that the corresponding unitary modular group implements the group of boosts which leave the wedge invariant, and that the corresponding modular conjugation implements the reflection about the edge of the wedge. Borchers’ theorem has profound consequences. For example in two-dimensional theories it means that the modular objects generate a representation of the proper Poincaré group, under which the observables behave covariant, and implies the CPT theorem. In higher dimensions, it is a crucial step towards the Bisognano-Wichmann theorem in the general context of local quantum physics [4, 7, 9, 23, 25, 30, 33]. This theorem asserts that a certain class of Poincaré covariant theories enjoys the property of

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modular covariance, namely that the mentioned unitary modular group coincides with the representers of the boosts, and that the modular conjugation is a CPT operator (where ‘PT’ means the reflection about the edge of the wedge).

The hypothesis under which Borchers’ theorem works is the double role played by the vacuum vector within a theory of local algebras: The vacuum is cyclic and separating for the local algebras, and it is invariant under the positive energy representation of the translation group under which these algebras are covariant. In a charged sector, i.e., a non-vacuum representation of the observables, this situation is not given. (This problem has been posed by Borchers in [5, Sect. VII.4].) In the case of permutation group statistics, one can use the field algebra instead of the observable algebra to recover the result. However, in low-dimensional space-time there may occur superselection sectors with braid group statistics [17, 20]. Then only in the Abelian case there is a field ($C^*$) algebra for which the vacuum is cyclic and separating. In the case of non-Abelian braid group statistics, there is no such field algebra. Due to this complication, a general result corresponding to Borchers’ theorem has not been achieved yet. In the present article, we prove an analogue of Borchers’ theorem for a superselection sector corresponding to a localizable charge. The implementers of the boosts and the reflection which we find are the relative modular objects associated with the observable algebra of the wedge, the vacuum state and some specific state in the conjugate sector. We assume that the observable algebra satisfies Haag duality, see Eq. (7), and that the sector under consideration has finite statistics and positive energy, and is irreducible. We also need a slightly stronger irreducibility property (12), which may be ensured by requiring for example Lorentz covariance or the split property. We consider charges which are localizable in space-like cones, and admit the case of non-Abelian braid group statistics which can occur in low space-time dimensions, $d = 2$ and 3.

It must be noted that in two dimensions, our result is already practically covered by the work of Guido and Longo [21]. Namely, they show how a certain condition of modular covariance in the vacuum sector allows, under the same hypothesis as in the present article, for the construction of a (ray) representation of the proper Poincaré group in charged sectors. But in two dimensions, their modular covariance condition is satisfied due to Borchers’ theorem (in the vacuum sector), so their analysis goes through, even in sectors with Braid group statistics. However, it must be noted that in $d = 2$ the assumption of Haag duality excludes some massive models with braid group statistics as e.g. the anyonic sectors of the CAR algebra [1], and together with the split property for wedges (expected to hold in massive models) excludes localizable charges altogether [28].

Our result shall be used to derive the CPT and Bisognano-Wichmann theorems for particles with braid group statistics in three-dimensional space-time [29]. It would be gratifying to extend our analysis to soliton sectors in 2 dimensions, which would extend the range (and simplify the proof) of Rehrens’ CPT theorem for solitons [31].

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1 A space-like cone is, in $d \geq 3$, a convex cone in Minkowski space generated by a double cone and a point in its causal complement, and in $d = 2$ the causal completion thereof, which is a wedge region.
1 General Setting, Assumptions and Results

We consider a theory of local observables, given by a family of von Neumann algebras $A_0(O)$ of operators acting in the vacuum Hilbert space $H_0$, indexed by the double cones $O$ in Minkowski space, and satisfying the conditions of isotony and locality:

$$A_0(O_1) \subset A_0(O_2) \text{ if } O_1 \subset O_2 \text{ and } A_0(O_1)^{'} \subset A_0(O_2)^{'} \text{ if } O_1 \subset O_2^{'},$$

where the prime denotes the commutant or the causal complement, respectively.

The vacuum Hilbert space $H_0$ carries a unitary representation $U_0$ of the group of space-time translations $\mathbb{R}^d$ with positive energy, i.e. its spectrum lies in the forward light cone. It has a unique, up to a phase, invariant vector $\Omega \in H_0$, corresponding to the vacuum state. The representation $U_0$ implements automorphisms under which the net $O \to A_0(O)$ is covariant:

$$AdU_0(x) A_0(O) = A_0(x + O)$$

for all $x \in \mathbb{R}^d$. (By $AdU$ we denote the adjoint action of a unitary $U$.)

Borchers’ theorem, which we wish to generalize to charged sectors, asserts that the representation $U_0$ has specific commutation relations with certain algebraic objects, the so-called modular group and conjugation, which suggest a geometric interpretation of the latter. Let us recall Borchers’ commutation relations in this setting. Let $W_1$ be the wedge defined as

$$W_1 := \{ x \in \mathbb{R}^d : |x^0| < x^1 \}.$$  

By the Reeh-Schlieder property, $\Omega$ is cyclic and separating for the von Neumann algebra $A_0(W_1)$ generated by all $A_0(O)$, $O \subset W_1$. This allows for the definition of the Tomita operator $[8]$, $S_0$, associated to $A_0(W_1)$: It is the closed anti-linear involution satisfying

$$S_0 A \Omega = A^\dagger \Omega, \quad A \in A_0(W_1).$$

Its polar decomposition, $S_0 = J_0 \Delta_0^{1/2}$, defines an anti-unitary involution $J_0$, the so-called modular conjugation, and a positive operator $\Delta_0$ giving rise to the so-called modular unitary group $\Delta_0^u$ associated to the wedge $W_1$. By Tomita’s Theorem, see e.g. [8], the adjoint action of $\Delta_0^u$ leaves $A_0(W_1)$ invariant, and the adjoint action of $J_0$ maps $A_0(W_1)$ onto its commutant $A_0(W_1)^{'}$. The mentioned theorem of Borchers now asserts that $\Delta_0^u$ and $J_0$, together with the representation $U_0$ of the translations, induce a representation of the subgroup of $P_+$ generated by the boosts $\lambda_t$ and the reflection $j$ associated to the wedge, and the translations. More precisely, let $\lambda_t$ be the (rescaled) 1-boosts, leaving $W_1$ invariant and acting on the coordinates $x^0, x^1$ as

$$
\begin{pmatrix}
\cosh(2\pi t) & \sinh(-2\pi t) \\
\sinh(-2\pi t) & \cosh(2\pi t)
\end{pmatrix},
$$

By spectrum of a representation of the translation group we mean the energy-momentum spectrum, namely the joint spectrum of the generators.
and let $j$ be the reflection about the edge of $W_1$, acting on the coordinates $x^0, x^1$ as $-1$ and leaving the other coordinates unchanged (if $d > 2$). Then Borchers’ theorem asserts that

\begin{align*}
\Delta_{0}^{it} U_{0}(x) \Delta_{0}^{-it} &= U_{0}(\lambda_{t} x), \\
J_{0} U_{0}(x) J_{0} &= U_{0}(j x)
\end{align*}

(5) (6)

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$. These relations implement the group relations of the translations with the boosts and reflections, respectively. Modular theory further implies that $J_{0}$ is an involution and commutes with the modular unitary group, implementing the group relations $j^2 = 1$ and $j \lambda_{t} j^{-1} = \lambda_{t}$. Altogether, $U_{0}(x)$, $\Delta_{0}^{it}$ and $J_{0}$ constitute a representation of the subgroup of the Poincaré group generated by the translations, the boosts $\lambda_{t}$ and the reflection $j$ (which is the direct product of the proper Poincaré group in the time-like $x^0$-$x^1$ plane and the translation group in the remaining $d - 2$ dimensions).

Our aim is to find a similar result in a charged sector, that is in a representation of the abstract $C^*$-algebra generated by the local algebras $A_{0}(O)$, which is inequivalent from the defining vacuum representation. We shall consider an irreducible representation $\pi$, which is localizable in space-like cones. That means that $\pi$ and the vacuum representation are unitarily equivalent in restriction to the observable algebra associated with the causal complement of any space-like cone. We assume that the observable algebra satisfies Haag duality for space-like cones and wedges, i.e., regions which arise by a proper Poincaré transformation from $W_1$. Namely, denoting by $\mathcal{K}$ the class of space-like cones, their causal complements, and wedges, we require

\begin{equation}
A_{0}(C') = A_{0}(C)', \quad C \in \mathcal{K}.
\end{equation}

A localizable representation can then be described by an endomorphism of the so-called universal algebra $A$ generated by isomorphic images $A(C)$ of the $A_{0}(C)$, $C \in \mathcal{K}$, see [18, 19, 21]. The family of isomorphisms $A(C) \cong A_{0}(C)$ extends to a representation $\pi_{0}$ of $A$, the vacuum representation. We then have

\begin{equation}
A_{0}(C) = \pi_{0} A(C),
\end{equation}

and the vacuum representation is faithful and normal on the local algebras $A(C)$. The adjoint action $\{\Pi\}$ of the translations on the local algebras lifts to a representation by automorphisms $\alpha_{x}$:

\begin{align*}
\text{Ad} U_{0}(x) \circ \pi_{0} &= \pi_{0} \circ \alpha_{x}, \\
\alpha_{x} A(C) &= A(x + C).
\end{align*}

(9) (10)

Our localizable representation $\pi$ is then equivalent [12, 18] with a representation of the form $\pi_{0} \circ \rho$ acting in $\mathcal{H}_{0}$, where $\rho$ is an endomorphism of $A$ localized in some
specific space-time region $C_0 \in K$ in the sense that
\[
\rho(A) = A \quad \text{if} \quad A \in \mathcal{A}(C'_0).
\] (11)

We shall take the localization region of $\rho$ to be properly contained in $W_1$, which implies by Haag duality (7) that $\rho$ restricts to an endomorphism of $\mathcal{A}(W_1)$. We shall require that this endomorphism of $\mathcal{A}(W_1)$ be irreducible, namely that
\[
\pi_0 \mathcal{A}(W_1) \cap \left( \pi_0 \rho \mathcal{A}(W_1) \right)' = \mathbb{C}1.
\] (12)

This is a slightly stronger requirement than irreducibility of the representation $\pi_0 \rho$ of $\mathcal{A}$. It has been shown by Guido and Longo that irreducibility of $\pi_0 \rho$, together with finite statistics, imply irreducibility in the sense of Eq. (12) if $\rho$ is covariant under the (proper orthochronous) Poincaré group [22, Cor. 2.10] or if $\rho$ satisfies the split property [21, Prop. 6.3]. We further assume the representation $\pi \cong \pi_0 \rho$ to be translation covariant with positive energy. That means that there is a unitary representation $U_\rho$ of the translation group $\mathbb{R}^d$ with spectrum contained in the forward light cone such that
\[
\text{Ad} U_\rho(x) \circ \pi_0 \rho = \pi_0 \rho \circ \alpha_x, \quad x \in \mathbb{R}^d.
\] (13)

We finally assume that $\rho$ has finite statistics, i.e. that the so-called statistics parameter $\lambda_\rho$ [12] be non-zero. This holds automatically if $\rho$ is massive [16], and implies [13] the existence of a conjugate morphism $\bar{\rho}$ characterized, up to equivalence, by the fact that the composite sector $\pi_0 \bar{\rho} \rho$ contains the vacuum representation $\pi_0$ precisely once. Thus there is a unique, up to a factor, intertwiner $R_\rho \in \mathcal{A}(C_0)$ satisfying $\bar{\rho} \rho(A) R_\rho = R_\rho A$ for all $A \in \mathcal{A}$. The conjugate $\bar{\rho}$ shares with $\rho$ the properties of covariance (13), finite statistics, and localization (11) in some space-like cone which we choose to be $C_0$. Using the normalization convention of [13, Eq. (3.14)], namely $R_\rho^* R_\rho = |\lambda_\rho|^{-1} 1$, the positive linear endomorphism $\phi_\rho$ of $\mathcal{A}$ defined as
\[
\phi_\rho(A) = |\lambda_\rho| R_\rho^* \bar{\rho}(A) R_\rho
\] (14)
is the unique left inverse [10, 13] of $\rho$. In the low-dimensional situation, $d = 2, 3$, the statistics parameter $\lambda_\rho$ may be a complex non-real number, corresponding to braid group statistics. We admit the case when its modulus is different from one (namely when $\rho$ is not surjective), corresponding to non-Abelian braid group statistics.

The modular objects for which we shall prove Borchers’ commutation relations are defined as follows. Let $S_\rho$ be the closed anti-linear operator satisfying
\[
S_\rho \pi_0(A) \Omega := \pi_0 \bar{\rho}(A^*) R_\rho \Omega, \quad A \in \mathcal{A}(W_1),
\] (15)
and denote the polar decomposition of $S_\rho$ by $S_\rho = J_\rho \Delta_\rho^{1/2}$. $S_\rho$ is just the relative Tomita operator [34] with respect to a certain pair of (non-normalized) states. Namely, consider the vacuum state $\omega_0 := (\Omega, \pi_0(\cdot) \Omega)$, and the positive functional
\[
\phi_\rho := |\lambda_\rho|^{-1} \omega_0 \circ \phi_\rho = (R_\rho \Omega, \pi_0 \bar{\rho}(\cdot) R_\rho \Omega).
\]

\[\text{Although not explicitly mentioned in [22], the proof does not depend on covariance of } \rho \text{ under the full Moebius group. See also [27, Thm. 2.2].}\]
The restriction of \( \varphi_\rho \) to \( \mathcal{A}(W_1) \) is faithful and normal, and has the GNS-triple \((\mathcal{H}_0, \pi_0 \rho, R_\rho \Omega)\). Thus, \( S_\rho \) is the relative Tomita operator associated with the algebra \( \mathcal{A}(W_1) \) and the pair of states \( \omega_0 \) and \( \varphi_\rho \), see Appendix A. The motivation to consider these objects (instead of the modular objects associated with \( \mathcal{A}(W_1) \) and one suitable state, e.g. \( \varphi_\rho \)) is that the so-defined relative modular unitary group \( \Delta^it_\rho \) implements the modular automorphism group associated with \( \mathcal{A}(W_1) \) and \( \omega_0 \) in the same way as the representation \( U_\rho(x) \) implements the translations \( \alpha_x \), see Eq. (26) below. This opens up the possibility to lift Borchers’ commutation relations (5) in the vacuum representation to the representation \( \pi_0 \rho \). In fact, pursuing this strategy, we shall find the following result. Let \( G \) be the subgroup of the proper Poincaré group generated by the translations, the boosts \( \lambda_t \) and the reflection \( j \). Recalling that the representation \( U_\rho \) may be shifted to a representation \( e^{ik \cdot x} U_\rho(x) \) whose spectrum has a Lorentz invariant lower boundary \( [6,7] \) we show under the above-mentioned assumptions:

**Theorem 1 (Commutation Relations.)** Assume that the lower boundary of the spectrum of \( U_\rho \) is Lorentz-invariant. Then \( U_\rho(x), \Delta^it_\rho, J_\rho \) and the counterparts for \( \bar{\rho} \) constitute a continuous (anti-) unitary representation\(^8\) of \( G \). More specifically, there hold the commutation relations

\[
\Delta^it_\rho U_\rho(x) \Delta^-it_\rho = U_\rho(\lambda_t x),
\]

\[
J_\rho U_\rho(x) J^-1_\rho = U_\rho(j x),
\]

\[
J_\rho \Delta^it_\rho J^-1_\rho = \Delta^it_\bar{\rho},
\]

\[
J_\rho J_\bar{\rho} = \chi_\rho \mathbf{1},
\]

for all \( t \in \mathbb{R} \) and \( x \in \mathbb{R}^d \). The complex number \( \chi_\rho \) in Eq. (19) has modulus one, conjugate \( \bar{\chi}_\rho = \chi_\bar{\rho} \) and is a root of unity if \( \bar{\rho} = \rho \).

(Note that Eq. (18) corresponds to a standard property of modular objects, but needs to be proved for our relative modular objects.)

We also show that this representation of \( G \) acts geometrically correctly on the wedge algebras, namely for \( W \) in the family \( \mathcal{W}_1 \) of translates of \( W_1 \) and \( W'_1 \),

\[
\mathcal{W}_1 := \{ x + W_1, x \in \mathbb{R}^d \} \cup \{ x + W'_1, x \in \mathbb{R}^d \},
\]

there holds

\[
\text{Ad} \Delta^it_\rho : \pi_0 \rho A(W) \rightarrow \pi_0 \rho A(\lambda_t W),
\]

\[
\text{Ad} J_\rho : \pi_0 \rho A(W) \rightarrow \pi_0 \bar{\rho} A(j W).
\]

To this end, observe that modular theory \( [8] \) and the relation \( \pi_0^{-1}(\mathcal{A}_0(W_1)') = \mathcal{A}(W'_1) \) imply that \( \Delta^it_\rho \) and \( J_\rho \) implement an automorphism \( \sigma_t \) of \( \mathcal{A}(W_1) \) and \( \mathcal{A}(W'_1) \), and

\(^7\)This is automatically the case if \( \rho \) is localizable in double cones and \( d > 2 \) by a result of Borchers [2], which is applicable since in this case \( \rho \) is implemented by local charged field operators [14]. It is also the case of course if \( U_\rho \) extends to the Poincaré group.

\(^8\)Strictly speaking, a ray representation since \( J_\rho J_\bar{\rho} \) is only a multiple of unity.
an anti-isomorphism from $\mathcal{A}(W_1)$ onto $\mathcal{A}(W'_1)$ and vice versa, respectively, defined by

$$\text{Ad} \Delta^t_\rho \circ \pi_0 = \pi_0 \circ \sigma_t$$  \hspace{1cm} (22)

$$\text{Ad} J_\rho \circ \pi_0 = \pi_0 \circ \alpha_j$$  \hspace{1cm} (23)

on $\mathcal{A}(W_1) \cup \mathcal{A}(W'_1)$. By Borchers’ commutation relations, the same equations extend $\sigma_t$ and $\alpha_j$ to the family $\mathcal{A}(W)$, $W \in W_1$, acting in a geometrically correct way:

$$\sigma_t : \mathcal{A}(W) \rightarrow \mathcal{A}(\lambda^t W),$$  \hspace{1cm} (24)

$$\alpha_j : \mathcal{A}(W) \rightarrow \mathcal{A}(jW),$$  \hspace{1cm} (25)

$W \in W_1$, see [3, Lem. III.2]. But our representers $\Delta^t_\rho$ and $J_\rho$ implement these isomorphisms $\sigma_t$ and $\alpha_j$, respectively, in the direct product representation $\pi_0 \rho \oplus \overline{\pi_0 \rho}$, namely:

**Proposition 1 (Implementation Properties.)** There holds

$$\text{Ad} \Delta^t_\rho \circ \pi_0 \rho = \pi_0 \rho \circ \sigma_t$$  \hspace{1cm} (26)

$$\text{Ad} J_\rho \circ \pi_0 \rho = \pi_0 \rho \circ \alpha_j$$  \hspace{1cm} (27)

on the family of algebras $\mathcal{A}(W)$, $W \in W_1$.

Since $\sigma_t$ and $\alpha_j$ act geometrically correctly, c.f. Eqs. (24) and (25), this implies that $\Delta^t_\rho$ and $J_\rho$ act geometrically correctly, as claimed in Eqs. (20) and (21).

In two space-time dimensions, our group $G$ already coincides with the proper Poincaré group $P^+$, and our results therefore imply that the translations and the relative modular objects constitute an (anti-) unitary representation of the latter.

By our assumption of Haag duality (7) for wedges, the so-called dual net

$$\mathcal{A}^d(\mathcal{O}) := \bigcap_{W \supset \mathcal{O}} \mathcal{A}(W)$$

is still local. (One needs to intersect in fact only the algebras of one “right wedge” of the form $W_1 + x$ and one “left wedge” of the form $W'_1 + y$.) The modular (anti-) automorphisms $\sigma_t$ and $\alpha_j$ act on it in a geometrically correct way, see [3, Prop. III.3]. If the original net satisfies Haag duality also for double cones, it coincides with the dual net. Then the implementation properties (26) and (27) hold, and therefore the representation of $P^+$ constructed in Theorem 1 acts geometrically correctly, namely there holds for any double cone $\mathcal{O}$:

$$\text{Ad} U_\rho(g) : \pi_0 \rho \mathcal{A}(\mathcal{O}) \rightarrow \pi_0 \rho \mathcal{A}(g \mathcal{O}), \ \ g \in P^+_1,$$

$$\text{Ad} J_\rho : \pi_0 \rho \mathcal{A}(\mathcal{O}) \rightarrow \pi_0 \rho \mathcal{A}(j \mathcal{O}).$$  \hspace{1cm} (28)

Here we have written $U_\rho(a, \lambda_t) := U_\rho(a) \Delta^t_\rho$. In particular, $J_\rho$ is a CPT operator.\footnote{If the net does not satisfy Haag duality for double cones, it does not coincide with the dual net. Then our endomorphism $\rho$ has two generally distinct extensions $\rho_{R/L}$ to the dual net, according a choice of the right or left wedge [32]. (Each of them is localizable only in one type of wedges.) In this case, $J_\rho$ intertwines $\pi_0 \rho_{R/L}$ with $\pi_0 \rho_{L/R} \alpha_j$, and in Eq. (28) there appears $\rho_R$ on one side and $\rho_L$ on the other side.}

Again, it must be noted that these results (in $d = 2$) are already implicit in the work of Guido and Longo [21], and also that the split property would exclude any charged sectors in our sense.
2 Proofs

We now prove Theorem 1 and Proposition 1. Instead of proving Borchers’ commutation
relations directly (e.g. paralleling Florig’s nice proof [15]), we show how they lift
from the vacuum sector to our charged sector. We shall use some well-known facts
about relative modular objects, which we recall in the Appendix for the convenience
of the reader, see also [34] for a review. Namely, the operator $\Delta_{\rho}^{it} \Delta_0^{-it}$ is in $\pi_0 \mathcal{A}(W_1)$
for $t \in \mathbb{R}$, and we define

$$Z_{\rho}(t) := \pi_0^{-1}(\Delta_{\rho}^{it} \Delta_0^{-it}) \in \mathcal{A}(W_1). \quad (29)$$

This family of observables coincides with the Connes cocycle $(D_{\varphi_{\rho}} : D_{\omega_0})_t$ with
respect to the pair of weights $\omega_0$ and $\varphi_{\rho}$, see Eq. (A.2). In the present context, it satisfies

$$\text{Ad}Z_{\rho}(t) \circ \sigma_t \circ \rho = \rho \circ \sigma_t \quad \text{on } \mathcal{A}(W_1), \quad (30)$$

see Proposition 1.1 in [26]. The definition (29) and Eq. (30) are analogous to well-
known properties of the translation cocycles which we shall use in the sequel. Ob-
serve that for $a \in W_1^-$, the closure of $W_1$, we have $W_1 + a \subset W_1$ and $W_1' - a \subset W_1'$. Since $\rho$ acts trivially on $W_1'$, this implies that the operator $U_{\rho}(a)U_0(-a)$ is in $\pi_0 \mathcal{A}(W_1)'$ which coincides with $\pi_0 \mathcal{A}(W_1)$ by Haag duality. This gives rise to the translation cocycle

$$Y_{\rho}(a) := \pi_0^{-1}(U_{\rho}(a)U_0(-a)) \in \mathcal{A}(W_1), \quad a \in W_1^+. \quad (31)$$

By virtue of Eqs (30) and (13), it satisfies the intertwiner relation

$$\text{Ad}Y_{\rho}(x) \circ \alpha_x \circ \rho = \rho \circ \alpha_x, \quad x \in \mathbb{R}^d. \quad (32)$$

The definitions of the cocycles $Z_{\rho}(t)$ and $Y_{\rho}(x)$, the intertwiner relations (30) and
(32), and invariance of $\Omega$ under $\Delta_{\rho}^{it}$ and $U_0(x)$ imply the identities

$$\Delta_{\rho}^{it} \pi_0(A) \Omega = \pi_0(Z_{\rho}(t)\sigma_t(A)) \Omega, \quad A \in \mathcal{A}(W_1), \quad (33)$$

$$U_{\rho}(x) \pi_0(A) \Omega = \pi_0(Y_{\rho}(x)\alpha_x(A)) \Omega, \quad A \in \mathcal{A}, \quad (34)$$

which we shall frequently use in the sequel. We shall also use the fact that Borchers’
theorem applied to the observable algebra implies that

$$\sigma_t \alpha_{\lambda \cdots a \sigma-t} = \alpha_x \quad (35)$$

holds as an isomorphism from $\mathcal{A}(W)$ onto $\mathcal{A}(W + x)$, $x \in \mathbb{R}^d$, $W \in \mathcal{W}_1$. Finally, we
make the interesting observation that $S_{\rho}$ is the relative Tomita operator associated
not only with the pair of states $(\omega_0, \varphi_{\rho})$, but also with pair of states $(\varphi_{\tilde{\rho}}, \omega_0)$:

**Lemma 1** The span, $D$, of vectors of the form $\pi_0[\rho(A)R_{\rho}]\Omega$, $A \in \mathcal{A}(W_1)$, is a core
for the relative Tomita operator $S_{\rho}$, and $S_{\rho}$ acts on $D$ as

$$S_{\rho} \pi_0 \rho(A)R_{\rho} \Omega = \chi_\rho \pi_0(A^*)\Omega, \quad A \in \mathcal{A}(W_1), \quad (36)$$

where $\chi_\rho$ is a complex number of modulus one, with $\bar{\chi}_\rho = \chi_{\bar{\rho}}$, and is a root of unity
if $\bar{\rho} = \rho$. 

Proof. Eq.s (30), (33) and (41) imply that for $A \in \mathcal{A}(W_1)$ there holds
\[
\Delta_\rho^{it} \pi_0[\rho(A)R_\rho]\Omega = \pi_0[\rho\sigma_t(АЗ\rho(-t))R_\rho]\Omega.
\]
Thus, the domain $D$ is invariant under the unitary group $\Delta_\rho^{it}$. It is therefore a core for $\Delta_\rho^{1/2}$ and hence for $S_\rho$. On this core, we have by definition
\[
S_\rho \pi_0[\rho(A)R_\rho]\Omega = \pi_0[\bar{\rho}(R_\rho^*)R_\rho A^*]\Omega.
\]
But $\bar{\rho}(R_\rho^*)R_\rho$ is a self-intertwiner of $\rho$, hence a multiple of unity, $\chi_\rho \mathbb{1}$. This proves Eq. (36). For the stated properties of $\chi_\rho$, see [19, Eq. (3.2)]. □

Since $(\mathcal{H}_0, \pi_0, R_\rho\Omega)$ is the GNS triple for the (non-normalized) state $\phi_\rho$ and $\chi_\rho\Omega$ is a GNS vector for $\omega_0$, the Lemma implies that $S_\rho$ is the relative Tomita operator associated with the pair of states $(\phi_\rho, \omega_0)$.

**Proof of Theorem 1.** To prove Eq. (16) of Theorem 1 let $A \in \mathcal{A}(W_1)$ and $a \in W_1^\perp$. Using Eq.s (33), (34) and (35), we then have
\[
\Delta_\rho^{it} U_\rho(\lambda - t a) \Delta_\rho^{-it} \pi_0(A) \Omega = \pi_0(\tilde{Y}_\rho(a, t)\alpha_a(A)) \Omega,
\]
where
\[
\tilde{Y}_\rho(a, t) := Z_\rho(t)\sigma_t(\alpha_{\lambda - t a}(Z_\rho(-t))).
\]
The intertwiner relations (30) and (32) imply that on $\mathcal{A}(W_1)$ there holds
\[
\text{Ad}\tilde{Y}_\rho(a, t) \circ \alpha_a \circ \rho = \text{Ad}Z_\rho(t) \circ \sigma_t \circ \text{Ad}Y_\rho(\lambda - t a) \circ \alpha_{\lambda - t a} \circ \text{Ad}Z_\rho(-t) \circ \sigma_{-t} \circ \rho
\]
\[
= \rho \circ \sigma_t \circ \alpha_{\lambda - t a} \circ \sigma_{-t} = \rho \circ \alpha_a.
\]
That is, $\tilde{Y}_\rho(a, t)$ satisfies the same intertwiner relation (32) on $\mathcal{A}(W_1)$ as $Y_\rho(a)$. On the other hand, $\tilde{Y}_\rho(a, t)$ is also contained in $\mathcal{A}(W_1)$. Therefore $\tilde{Y}_\rho(a, t)Y_\rho(a)^*$ is in $(\rho \mathcal{A}(W_1))' \cap \mathcal{A}(W_1)$ which is trivial by our assumption (12) of irreducibility. Thus $\tilde{Y}_\rho(a, t)$ coincides with $Y_\rho(a)$ up to a scalar function $c(a, t)$. Hence Eq. (37) reads
\[
\Delta_\rho^{it} U_\rho(\lambda - t a) \Delta_\rho^{-it} \pi_0(A) \Omega = c(a, t) \pi_0(Y_\rho(a)\alpha_a(A)) \Omega
\]
\[
= c(a, t) U_\rho(a) \pi_0(A) \Omega.
\]
Since the vacuum is cyclic for $\pi_0 \mathcal{A}(W_1)$ by the Reeh-Schlieder property, this shows that
\[
\Delta_\rho^{it} U_\rho(\lambda - t a) \Delta_\rho^{-it} = c(a, t) U_\rho(a)
\]
for $a \in W_1^\perp$. By adjoining, we get an analogous equation for $-a \in W_1^\perp$. Since the closures of $W_1$ and $-W_1$ span the whole Minkowski space, this shows that there is a function $c(a, t)$ such that Eq. (38) holds for all $a \in \mathbb{R}^d$. It remains to show that $c(a, t) \equiv 1$. Eq. (38) gives us a ray representation of the group $G$ generated by the boosts $\lambda_t$ and the translations in the $0, 1$-plane, defined by
\[
U(a, \lambda_t) := U_\rho(a) \Delta_\rho^{it}.
\]
(The group $G$ is a subgroup of $P^1_\mathbb{Z}$ in $d = 3$ and coincides with $P^1_\mathbb{Z}$ in $d = 2$. The product in $G$ is $(a, \lambda t) \cdot (a', \lambda' t') = (a + \lambda a', \lambda t + t')$. Now $G$ is simply connected, and its second cohomology group is known to be trivial. Therefore there exists a function $\nu$ from $G$ into the unit circle such that $\hat{U}(g) := \nu(g) U(g)$ is a true representation of $G$. Eq. (38) then implies that

$$c(a, t) = \nu(a, 1) \nu(\lambda^{-t} a, 1)^{-1}. \quad (39)$$

Since $U_\rho$ is a true representation of the translations, the restriction of $\nu$ to the translations is a one-dimensional representation, that is of the form $\nu(a, 1) = e^{ik \cdot a}$. Therefore, the spectra of the representations $\hat{U} = \nu \otimes U$ and $U$ differ by a translation about a vector $k$. But the spectrum of $\hat{U}$ is invariant under the 1-boosts since $\hat{U}$ extends to a true representation of the (2-dimensional) Poincaré group $G$, and the lower boundary of the spectrum of $U$ is also Lorentz invariant since it coincides with the spectrum of $U_\rho$. This implies that $k = 0$ and hence, by Eq. (39), that $c(a, t) \equiv 1$. This completes the proof of Eq. (16) of the Theorem.

We now prove Eq. (18) of the Theorem. For $A \in \mathcal{A}(W_1)$, we have by Eq. (33) and the intertwiner relation (30)

$$\Delta^{it}_\rho S_\rho \Delta^{-it}_\rho \pi_0(A) \Omega = \pi_0(\check{\rho}(A^*) Z_\rho(t) \sigma_t [\check{\rho}(Z_\rho(-t)^*) R_\rho]) \Omega. \quad (40)$$

We shall now use a result of Longo [26]. Namely, we are in the situation where Propositions 1.3 and 1.4 in [26] apply, yielding

$$R^x_\rho \check{\rho}(Z_\rho(-t)) Z_\rho(-t) = \sigma_{-t}(R^x_\rho).$$

Applying $\sigma_t$, adjoining, and using the cocycle identity $Z_\rho(t) \sigma_t (Z_\rho(-t)) = 1$, see Eq. (A.1), yields

$$Z_\rho(t) \sigma_t [\check{\rho}(Z_\rho(-t)^*) R_\rho] = R_\rho. \quad (41)$$

Hence Eq. (40) reads

$$\Delta^{it}_\rho S_\rho \Delta^{-it}_\rho \pi_0(A) \Omega = \pi_0(\check{\rho}(A^*) R_\rho) \Omega \equiv S_\rho \pi_0(A) \Omega.$$

Since $\Delta^{it}_\rho$ maps the core $\pi_0 \mathcal{A}(W_1) \Omega$ of $S_\rho$ onto itself by Eq. (33), this shows that

$$\Delta^{it}_\rho S_\rho \Delta^{-it}_\rho = S_\rho,$$

which implies Eq. (18) of the Theorem. For the proof of Eq. (17) we need the following Lemma.

**Lemma 2** For $a$ in the closure of $W_1$, there holds

$$U_\rho(a)^{-1} S_\rho U_\rho(a) \subset S_\rho. \quad (42)$$

**Proof.** First recall from [10, 16] that the representation $\hat{U}_\rho$ defined by

$$\hat{U}_\rho(x) \pi_0[\check{\rho}(A) R_\rho] \Omega := \pi_0[\check{\rho}(\alpha_x(A) Y_\rho(x)^*) R_\rho] \Omega \quad (43)$$
implements $\alpha_x$ in the representation $\pi_0\hat{\rho}$, i.e. $\text{Ad}\hat{U}_\rho(x) \circ \pi_0\hat{\rho} = \pi_0\hat{\rho} \circ \alpha_x$. The representation $\hat{U}_\rho$ therefore coincides with $U_\rho$ up to a one-dimensional representation $c(\cdot)$. We now have, for $a$ in the closure of $W_1$ and $A \in \mathcal{A}(W_1 - a)$,

$$S_\rho U_\rho(a) \pi_0(A)\Omega = \pi_0[\hat{\rho}(\alpha_a(A^*)Y_\rho(a)^*)R_\rho] \Omega = \hat{U}_\rho(a) \pi_0[\hat{\rho}(A^*)R_\rho] \Omega = \hat{U}_\rho(a) S_\rho \pi_0(A)\Omega.$$

Since $\hat{U}_\rho$ and $U_\rho$ coincide up to the character $c$ as discussed above, we therefore have

$$U_\rho(-a) S_\rho U_\rho(a) = c(a) S_\rho \quad \text{on} \quad D := \mathcal{A}_0(W_1 - a)\Omega.$$

Applying $\Delta^{it}_\rho \cdot \Delta^{-it}_\rho$ to this equation and using the by now established Eq.s 16 and 18 of the Theorem, yields $c(\lambda_t a) = c(a)$ or $c((1 - \lambda_t)a) = 1$. By the representation property of $c$, the same holds for $-a \in W_1^-$. Since $W_1^-$ and $-W_1^-$ span the whole Minkowski space and $1 - \lambda_t$ is invertible for $t \neq 0$, this shows that $c$ is trivial. Since $D$ is a core for the left hand side of relation (12), this completes the proof. □

We are now ready to prove Eq. (17) of the Theorem. To this end, let $a \in W_1^-$ and $\phi \in D := \mathcal{A}_0(W_1 - a)\Omega$. By Eq. 16, we have for all $t \in \mathbb{R}$

$$\Delta^{it}_\rho U_\rho(a) \phi = U_\rho(\lambda_t a) \Delta^{it}_\rho \phi. \quad (44)$$

Now by Lemma 2 the vector $U_\rho(a) \phi$ is in the domain of the operator $\Delta^{1/2}_\rho$, hence the left hand side is bounded for $t$ in the strip $\mathbb{R} - i[0, 1/2]$ and analytic in its interior. The same holds for the vector valued function $t \mapsto \Delta^{it}_\rho \phi$ on the right hand side. Further, for $a \in W_1^-$ the operator valued function $t \mapsto U_\rho(\lambda_t a)$ is norm-bounded on the strip $\mathbb{R} - i[0, 1/2]$ and analytic in its interior, and at $t = -i/2$ has the value $U_\rho(ja)$, see e.g. [24, Section V.4.1]. Therefore, Eq. (44) implies that

$$\Delta^{1/2}_\rho U_\rho(a) \phi = U_\rho(ja) \Delta^{1/2}_\rho \phi.$$

Multiplying with $J_\rho$ and using relation 12 of Lemma 2 yields

$$U_\rho(a) S_\rho \phi = J_\rho U_\rho(ja) J^{-1}_\rho S_\rho \phi. \quad (45)$$

Since $S_\rho$ has dense range, this shows $U_\rho(a) S_\rho \phi = J_\rho U_\rho(ja) J^{-1}_\rho S_\rho \phi$ in the closure of $jW_1$ and, by adjoining, also for arbitrary $x$. This completes the proof of Eq. (17) of the Theorem. To prove Eq. (19), note that Lemma 1 implies that $S_\rho = \chi_\rho S_\rho^{-1}$. Using that $\Delta^{1/2}_\rho J^{-1} = J^{-1} \Delta^{1/2}_\rho$ by Eq. (18) and that $J_\rho$ is anti-linear, one gets Eq. (19). This completes the proof of the Theorem.\footnote{We recall the argument in the present setting. The endomorphism $\alpha_{-x} \circ \phi_\rho \circ \beta_x$, where $\beta_x := \text{Ad}Y_\rho(x) \circ \alpha_x$, is a left inverse of $\rho$ and therefore coincides with $\hat{\rho}$ by uniqueness. This implies that the state $\varphi_\rho$ is invariant under the automorphism group $\beta_x$ and hence that $U_{\rho}(x)\pi_0[\hat{\rho}(A)R_\rho]\Omega = \pi_0[\hat{\rho}\beta_x(A)R_\rho]\Omega$ defines a unitary representation of the translations. But $\hat{U}_\rho(x)$ defined above coincides with $\pi_0[\hat{\rho}(Y_\rho(x)^*)U_{\rho}(x)]$, hence is a well-defined unitary operator. The implementation property is checked directly from the definition 13, and implies in turn the representation property.}
Proof of Proposition 1. We now turn to Eq. (26) of Proposition 1. On $\mathcal{A}(W_1)$, this equation follows from Eq. (30) by applying $\pi_0$ to the latter. Further, the fact that $\pi_0^{-1}(\Delta_t^0 \Delta_0^{-it})$ is in $\mathcal{A}(W_1)$ and hence commutes with $\mathcal{A}(W'_1)$ implies that $\text{Ad} \Delta_t^0 \circ \pi_0 = \pi_0 \circ \sigma_t$ on $\mathcal{A}(W'_1)$. Since $\rho$ acts as the identity on $\mathcal{A}(W'_1)$, this implies Eq. (26) on $\mathcal{A}(W'_1)$. For translates of $W_1$ or $W'_1$, the equation follows from Borchers’ commutation relations, Eqs. (16) and (35). Before proving Eq. (27) of the proposition, we establish the following intertwiner properties of the relative modular conjugation.

**Lemma 3 (Intertwiner Properties of $J_{\rho'}$)** The unitary operators $J_{\rho} J_0$ and $J_0 J_{\rho}$ have the intertwiner properties

$$\pi_0 \hat{\rho}(A) J_{\rho} J_0 = J_{\rho} J_0 \pi_0(A), \quad (46)$$

$$\pi_0(A) J_0 J_{\rho} = J_0 J_{\rho} \pi_0 \rho(A) \quad (47)$$

for $A \in \mathcal{A}(W_1)$.

**Proof.** These are consequences of a standard result [34] which relates the conjugations of relative Tomita operators, see Eq. (A.3) in the Appendix. Here, in Eq. (46) $S_{\rho}$ is being considered as the relative Tomita operator associated with the pair of states $(\omega, \varphi_{\rho})$, characterized by Eq. (15), and in Eq. (47) as the relative Tomita operator associated with the pair $(\varphi_{\hat{\rho}}, \omega_0)$, characterized by Eq. (36) of Lemma 1.

We are now ready to prove Eq. (27) of Proposition 1. By Eqs. (47) and Eq. (23) we have on $\mathcal{A}(W_1)$

$$\text{Ad} J_{\rho} \circ \pi_0 \rho \equiv \text{Ad} J_0 \circ \text{Ad}(J_{\rho} J_0) \circ \pi_0 \rho = \pi_0 \circ \alpha_j = \pi_0 \hat{\rho} \circ \alpha_j,$$

since $\hat{\rho}$ acts as the identity on $\alpha_j \mathcal{A}(W_1) \equiv \mathcal{A}(W'_1)$, while by Eq. (46) and Eq. (23) we have on $\mathcal{A}(W'_1)$

$$\text{Ad} J_{\rho} \circ \pi_0 \rho = \text{Ad} J_{\rho} \circ \pi_0 \equiv \text{Ad}(J_{\rho} J_0) \circ \text{Ad} J_0 \circ \pi_0 = \pi_0 \hat{\rho} \circ \alpha_j.$$

This shows that Eq. (27) holds on $\mathcal{A}(W_1) \cup \mathcal{A}(W'_1)$. Borchers’ commutation relations then imply that it holds on $\mathcal{A}(W), W \in W_1$, completing the proof of Proposition 1.

**A Relative Tomita Operators**

We recall the relevant notions from relative Tomita theory, following [34]. (For the standard Tomita theory, see e.g. [8] and Eq. (3) above.) Let $\mathcal{M}$ be a von Neumann algebra and $\varphi_1, \varphi_2$ two faithful normal positive functionals on $\mathcal{M}$, and denote by $\sigma^1_t$ and $\sigma^2_t$ the respective modular automorphism groups. Then there exists a family of unitaries $Z_{21}(t) \in \mathcal{M}$ satisfying the intertwiner and cocycle properties

$$\sigma^2_t(A) Z_{21}(t) = Z_{21}(t) \sigma^1_t(A), \quad (A.1)$$

$$Z_{21}(t + s) = Z_{21}(t) \sigma^1_t(Z_{21}(s)).$$
respectively, and characterized by a certain KMS property. These facts have been shown by Connes [11] and are reviewed in [34, Sect. I.3.1]. The family $Z_{21}(t)$ is called the Connes-cocycle associated with the pair $\varphi_1$ and $\varphi_2$ and usually denoted by $(D\varphi_1 : D\varphi_2)_t$. This cocycle may be expressed in terms of the corresponding GNS representations as follows [34, Sect. I.3.11]. Let $(H_i, \pi_i, \xi_i)$ be the GNS triples of $\varphi_i$, $i = 1, 2$. Then the operator $S_{21}$ from $H_1$ to $H_2$ defined by

$$S_{21} \pi_1(A)\xi_1 := \pi_2(A^*)\xi_2, \quad A \in \mathcal{M},$$

is closable. We denote its closure by the same symbol, and its polar decomposition by

$$S_{21} = J_{21} \Delta_{21}^{1/2}.$$ 

These operators are called the relative Tomita modular objects associated with the pair $\varphi_1$ and $\varphi_2$. Let now $\Delta_{1t}$ denote the unitary modular group of $\pi_1(\mathcal{M})$ and $\xi_1$. Then $\Delta_{21t} \Delta_{1t}^{-1}$ is in $\pi_1(\mathcal{M})$ and coincides with $\pi_1(Z_{21}(t))$, i.e. there holds [34, Sect. I.3.11]

$$Z_{21}(t) = \pi_1^{-1}(\Delta_{21t} \Delta_{1t}^{-1}). \quad (A.2)$$

Finally, as shown in [34, Sect. I.3.16], the unitary operator

$$V_{21} := J_{21} J_1 \equiv J_2 J_{21},$$

where $J_i$ is the modular conjugation of $\pi_i(\mathcal{M})$ and $\xi_i$, $i = 1, 2$, is an intertwiner from $\pi_1$ to $\pi_2$, that means it satisfies

$$\pi_2(A) V_{21} = V_{21} \pi_1(A), \quad A \in \mathcal{M}. \quad (A.3)$$

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