Superspace
Geometrical Representations of
Extended Super Virasoro Algebras

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ABSTRACT

Utilizing sets of super-vector fields (derivations), explicit expressions are obtained for; (a.) the 1D, $N$-extended superconformal algebra, (b.) the 1D, $N$-extended super Virasoro algebra for $N = 1, 2,$ and 4 and (c.) a geometrical realization ($\mathcal{GR}$) covering algebra that contains the super Virasoro algebra for arbitrary values of $N$. By use of such vector fields, the super Virasoro algebra is embedded as a geometrical and model-independent structure in 1D and 2D $\aleph_0$-extended superspace.

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I. Introduction

Spacetime symmetry groups possess many distinct representations. Consequently this is so for their supersymmetric extensions. Moreover when considering theories defined over extended manifolds, such symmetry groups continue to play important roles. In the realm of superstring theory, the role of the superconformal group is taken over by the super-Virasoro group \[1, 2\]. For spacetime symmetry groups, among the simplest faithful representations are those constructed from “derivations” or “vector fields” where the symmetry generators are represented in terms of functions of the coordinates of the manifold which multiply linear derivatives with respect to those coordinates. The set of all such derivations for a given algebra is sometimes referred to as a “D-module.”

The simplest of super manifolds are those associated with superspaces that possess a single bosonic dimension. For some time \[3\] we have been engaged in a study of 1D superspaces and the superfield representations that can be defined over such manifolds. One previous useful result found was the complete off-shell formulation of NSR spinning particle models for arbitrary numbers of extended supersymmetries on the worldline \[4\]. Although there are a number of features that remain to be explored in our previous work, it should be apparent that there is much to learn by studying such systems in great detail. A spectacular (but not obviously related) example of the utility of 1D supersymmetric models has been provided by the proposal that a microscopic description of M-theory is as a matrix theory of a 1D supersymmetric system \[5\].

Thus, a primary purpose of this letter is to present linear differential operators to represent the super Virasoro algebra for all values of \(N\), the degree of supersymmetric extension. Such constructions are intrinsically geometrical or kinematical having no \textit{a priori} relationships to properties of any given model and provide a basis for the study of super Virasoro algebras that is independent of the dynamical system in which they are realized.

II. 1D Superconformal Derivations

In the following, \(P, K, Q_1, S_1, \Delta\) and \(T_{1,4}\) are the momentum, special conformal, supersymmetry, s-supersymmetry, dilatation and SO\((N)\) generators. It is a simple exercise to find a set of derivations with respect to a one dimensional superspace with
coordinates \( \zeta^1 \) and \( \tau \) in terms of which these abstract operators may be realized

\[
P = i \partial_{\tau} , \quad K = i \left( \tau^2 \partial_{\tau} + \tau \zeta_1 \partial_t \right) ,
\]

\[
Q_1 = i \left( \partial_1 - 2i \zeta_1 \partial_{\tau} \right) , \quad S_1 = i \tau \partial_1 + 2 \tau \zeta_1 \partial_{\tau} + 2 \zeta_1 \zeta^1 \partial_J , \quad \Delta = i \left( \tau \partial_{\tau} + \frac{1}{2} \zeta^1 \partial_t \right) , \quad T_{IJ} = i \left( \zeta_I \partial_J - \zeta_J \partial_I \right) .
\]

The generic 1D, \( N \)-extended superconformal algebra is given by:

\[
[\Delta, P] = -iP , \quad [\Delta, Q_1] = -i\frac{1}{2}Q_1 , \quad [K, Q_1] = -iS_1 , \quad [\Delta, K] = iK , \quad [\Delta, S_1] = i\frac{1}{2}S_1 , \quad [P, S_1] = iQ_1 , \quad [Q_1, Q_J] = 4 \delta_{IJ} P , \quad [S_1, S_J] = 4 \delta_{IJ} K , \quad [P, K] = -i2\Delta , \quad [Q_1, S_J] = 4 \delta_{IJ} \Delta + 2T_{IJ} , \quad [T_{IJ}, Q_K] = -i\delta_{IK} Q_J + i\delta_{JK} Q_I , \quad [T_{IJ}, S_K] = -i\delta_{IK} S_J + i\delta_{JK} S_I , \quad [T_{IJ}, T_{KL}] = i\delta_{JK} T_{IL} - i\delta_{IL} T_{JK} + i\delta_{IL} T_{JK} - i\delta_{IK} T_{JL} .
\]

The quantity denoted by \( L \) above is the generator of 2D Lorentz rotations which is obviously absent in 1D. By an obvious modification of the above described replacement procedure, the generators of all 2D, \((p, q)\) superconformal algebras may be obtained.

We must also add the generators that are missing from the above “oxidation” up to two dimensions. For this purpose we replace the coordinates and derivatives in (1) according to \( \tau \to \sigma^\pm , \quad \zeta^1 \to \zeta^{+1} \) and \( \partial_1 \to 0 \) and also simultaneously replace the 1D generators by 2D generators via

\[
P \to P^\pm , \quad K \to K^\pm , \quad \Delta \to \frac{1}{2}(\Delta + L) , \quad Q_1 \to Q_{+1} , \quad S_1 \to S_{-1} .
\]

We must also add the generators that are missing from the above “oxidation” up to two dimensions. For this purpose we replace the coordinates and derivatives in (1) according to \( \tau \to \sigma^\pm , \quad \zeta^1 \to 0 \) and \( \partial_1 \to 0 \) and also simultaneously replace the 1D generators by 2D generators via

\[
P \to P^\pm , \quad K \to K^\pm , \quad \Delta \to \frac{1}{2}(\Delta - L) , \quad Q_1 \to 0 , \quad S_1 \to 0 .
\]

The quantity denoted by \( L \) above is the generator of 2D Lorentz rotations which is obviously absent in 1D. By an obvious modification of the above described replacement procedure, the generators of all 2D, \((p, q)\) superconformal algebras may be obtained.

This approach also yields a new viewpoint on why 1D and 2D theories are singled out as special when considered from this geometrically based construction. The 1D theory may be regarded as the fundamental representation. The oxidation argument
above shows that the 2D theory is constructed essentially as two completely independent copies of the 1D theory. For no dimension \( D \geq 3 \) does the conformal group “factorize” into independent copies of the 1D group.

Another useful aspect of this representation is that it permits us to see that the case of \( N = 4 \) is exceptional and makes clear the geometrical origin of the “small” versus “large” \( N = 4 \) superconformal algebras. We observe that in the case of \( N = 4 \), there exists a Levi-Civita tensor \( \epsilon_{IJKL} \) which may be used to “deform” some of the derivations according to

\[
S_I(\ell) \equiv i\tau \partial_I + 2\tau \zeta_I \partial_{\tau} + 2\zeta_I \zeta^I \partial_J + \ell(\epsilon_{ijkl} \zeta^j \partial^l - i4\zeta^{(3)}_I \partial_{\tau}) ,
\]

\[
K(\ell) \equiv i \left[ \tau^2 \partial_{\tau} + \tau \zeta^I \partial_I - i2\ell(\zeta^{(3)}_I \partial_I + i4\zeta^{(4)}_I \partial_{\tau}) \right] ,
\]

\[
T_{IJ}(\ell) \equiv i \left[ \zeta_I \partial_J - \zeta_J \partial_I - \ell \epsilon_{IJKL} \zeta_K \partial_L \right] .
\]

Above the quantities \( \zeta^{(3)}_I \) and \( \zeta^{(4)}_I \) are defined by

\[
\zeta_I \zeta_J \equiv \epsilon_{IJKL} \zeta^{(3)}_L , \quad \zeta_I \zeta_J \zeta_K \zeta_L \equiv \epsilon_{IJKL} \zeta^{(4)}_I .
\]

The remaining generators of the deformed algebra remain unchanged from their definitions in (1). The exceptional role of the \( N = 4 \) theory is also apparent from the point of view of dimensional analysis. It is only for the value of \( N = 4 \) that terms of a proper dimension (-1/2 for \( S_I \), 0 for \( T_{IJ} \) and -1 for \( K \)) exist that can be used to modify some of the generators in an appropriate manner.

The first ten results in the algebra of (2) remain unchanged for the \( \ell \)-deformed algebra. However, for the last three results we obtain

\[
[T_{IJ}, Q_K] = -i\delta_{IK} Q_J + i\delta_{JK} Q_I + i\ell \epsilon_{IJKL} Q_L ,
\]

\[
[T_{IJ}, S_K] = -i\delta_{IK} S_J + i\delta_{JK} S_I + i\ell \epsilon_{IJKL} S_L ,
\]

\[
[T_{IJ}, T_{KL}] = \frac{1}{2}(\ell^2 + 3) \left[ i\delta_{JK} T_{IL} - i\delta_{JL} T_{IK} + i\delta_{IL} T_{JK} - i\delta_{IK} T_{JL} \right] + \frac{1}{2}(\ell^2 - 1) \left[ i\delta_{JK} \mathcal{Y}_{IL} - i\delta_{JL} \mathcal{Y}_{IK} + i\delta_{IL} \mathcal{Y}_{JK} - i\delta_{IK} \mathcal{Y}_{JL} \right] ,
\]

where the operator \( \mathcal{Y}_{IJ} \) is defined by

\[
\mathcal{Y}_{IJ} \equiv i \left[ \zeta_I \partial_J - \zeta_J \partial_I + \ell \epsilon_{IJKL} \zeta_K \partial_L \right] .
\]

At the special values \( \ell = \pm 1 \), \( \mathcal{Y}_{IJ} \) does not appear in the \( \ell \)-modified superconformal \( N = 4 \) algebra. This has a dramatic effect. For all values \( N \neq 4 \), the \( T_{IJ} \) generators form a representation of \( O(N) \). For the \( N = 4 \) case, the deformed \( T_{IJ} \) generators form a representation of \( SU(2) \) if and only if \( \ell = \pm 1 \).
The appearance of this exceptional 1D, \( N = 4 \) theory may also be related to the appearance of \( N = 1 \) supersymmetry in four dimensional spacetime. A 4D spinor is equivalent to a 1D, \( N = 4 \) spinor. It is a simple matter to show that the \( K \)-generator of a four dimensional supersymmetric theory has the form of the \( K \)-generator in (5) with \( \ell \neq 0 \). We thus suspect this plays some role as the 1D worldline origin of 4D, \( N = 1 \) target space supersymmetry.

III. 1D, \( N = 1 \) Superspace and a Super D-module

If we set all Grassmann coordinates and their derivatives to zero, we are left with the bosonic \( N = 0 \) theory. It is well known that \( P, \Delta \) and \( K \) form a sub-group of the larger Virasoro algebra. We can define the generators

\[
L_m \equiv -\tau^{m+1}\partial_\tau \rightarrow L_{-1} \propto P , \quad L_0 \propto \Delta , \quad L_1 \propto K ,
\]

and for general values of \( m \) we see

\[
\left[ L_m , L_n \right] = (m - n) L_{m+n} ,
\]

the characteristic form of the Virasoro commutator algebra. The obvious question to ask is, “What are the explicit expressions for \( L_n^I \), \( F_n^I \) and \( G_r^I \) as derivations to represent the super Virasoro algebra?”

We will approach this question by first considering the case of \( N = 1 \). Here we define

\[
L_m \equiv -\left[ \tau^{m+1}\partial_\tau + \frac{1}{2}(m+1)\tau^m\zeta\partial_\zeta \right] , \quad F_m \equiv i\tau^{m+\frac{1}{2}}\left[ a_0 \partial_\zeta - ib_0 \zeta \partial_\tau \right] , \quad G_r \equiv i\tau^{r+\frac{1}{2}}\left[ \partial_\zeta - i2\zeta \partial_\tau \right] .
\]

The reader will note \( F_m \) and \( G_r \) almost possess the same form. The conditions

\[
G_{-\frac{1}{2}} \propto Q , \quad G_{\frac{1}{2}} \propto S ,
\]

have been used to determine the would-be free constants appearing in \( G_r \). We can make \( F_m \) identical in form to \( G_r \) by picking \( a_0 = 1 \) and \( b_0 = 2 \). Since for this choice the two operators have identical forms, it is useful to clearly state why they are fundamentally different. In \( F_m \) (the Ramond generator) since \( m \in \mathbb{Z} \) its pre-factor \( \tau^{m+\frac{1}{2}} \) always contains a square root of \( \tau \). On the other hand in \( G_r \) (the Neveu-Schwarz...
generator) since \( r \in Z + \frac{1}{2} \), its pre-factor \( \tau^{r+\frac{1}{2}} \) never contains a square root of \( \tau \). So while the two Grassmann generators can possess the same form (by a choice of \( a_0 \) and \( b_0 \)) this appearance is deceptive. Furthermore, since the index \( m \) on the \( L_m \)-operators is integer valued, the choices of whether the pre-factor contains a square-root or not are the only allowed values.

It is a matter of direct calculation to show

\[
\begin{align*}
\{ F_m, F_n \} &= -i 2 a_0 b_0 L_{m+n}, \quad \{ G_r, G_s \} = -i 4 L_{r+s}, \\
\{ L_m, F_n \} &= (\frac{1}{2} m - n) F_{m+n}, \quad \{ L_m, G_r \} = (\frac{1}{2} m - r) G_{m+r}, \\
\{ L_m, L_n \} &= (m - n) L_{m+n}.
\end{align*}
\]

(13)

Let us call this the \( N = 1 \) super-Virasoro algebra. Only the first equation in (13) depends explicitly on the constants \( a_0 \) and \( b_0 \). Both of these must be nonzero, however. With these results we have constructed a realization of the classical (i.e. unquantized) version of the \( N = 1 \) super-Virasoro algebra that is implicitly carried by the coordinates of 1D, \( N = 1 \) superspace.

However, having written the usual super-Virasoro generators as derivations, we can also calculate \( \{ F_m, G_r \} \). We find

\[
\{ F_m, G_r \} = i (2a_0 + b_0) \left\{ \tau^{m+r+1} \partial_r + i \left[ \frac{1}{2} + \frac{(2a_0 m + b_0 r)}{(2a_0 + b_0)} \right] \tau^{m+r} \zeta \partial \zeta \right\}.
\]

(14)

Clearly, for the choice \( a_0 = 1 \) and \( b_0 = 2 \) we obtain

\[
\{ F_m, G_r \} = -i 4 H_{m+r}, \quad H_r \equiv -\left[ \tau^{r+1} \partial_r + \frac{1}{2} (m+1) \tau^r \zeta \partial \zeta \right].
\]

(15)

This new bosonic operator \( H_r \) is related to \( L_m \) in the same way that \( G_r \) is related to \( F_m \). We can use this to write things using a slightly different notation. We define \( L_A \equiv (L_m, H_r) \) and \( G_A \equiv (F_m, G_r) \). The commutation algebra in this new notation takes the form

\[
\begin{align*}
\{ L_A, L_B \} &= (A - B) L_{A+B}, \quad \{ G_A, G_B \} = -i 4 L_{A+B}, \\
\{ L_A, G_B \} &= (\frac{1}{2} A - B) G_{A+B}.
\end{align*}
\]

(16)

The set \( \{ L_A, G_A \} \) is closed under graded commutation. Above we have used the notation \( A \) to denote an index that may be valued in either \( Z \) or \( Z + \frac{1}{2} \). From these commutation relationships it can be seen that \( H_s \) is a primary (in the language of conformal field theory) operator that “rotates” the Neveu-Schwarz generator into the Ramond generator and vice-versa. This suggests that we identify this operator with a corresponding one known in conformal field theory [3]. Namely, the operator
$H_s$ seems similar to the purely chiral part of the generator of spectral flows which connects NS and R algebras. Although in the conformal field theoretical formulation, such an operator appeared only associated with the $N = 2$ theories.

The lesson is clear. If we represent the super Virasoro generators as derivations with respect to the coordinates of the simplest real 1D superspace, the minimal set which provides a representation of the super Virasoro algebra contains all of the generators in the set $\{L_A, G_A\}$. We will call this super-Lie algebra the 1D, $N = 1$ “super $\mathcal{GR}$-Virasoro” algebra (where $\mathcal{GR}$ stands for “geometrical realization”).

Having seen this occurrence for the case of $N = 1$, it is natural to ask if this may be extended to higher values of $N$. In fact, the $N = 1, 2$ cases have been discussed in a textbook by West previously [7]. The important point that was overlooked in his presentation, however, is that if the set of generators $\{L_m, F^m, G_r\}$ are represented as derivations, then it is not a closed set. Of course, in string theory customarily one does not simultaneously discuss NS and R sector generators.

IV. New 1D, $N = 1$ Super D-module Operators

In the last section, we saw that the simplest geometrical realization of the super-Virasoro algebra leads to an enlarged algebra. However, our method of construction also points to the existence of another derivation that is not contained in the super-Virasoro algebra. If we return to (11) and choose $a_0 = -i, b_0 = i2$, this defines a new derivation via

\[ D_r \equiv \tau^r + \frac{1}{2} \left[ \partial_\zeta + i 2 \zeta \partial_r \right] \rightarrow [D_r, D_s] = -i 4 L_{r+s}, \]  

(17)

so that

\[ [D_r, G_s] = i 2 (r - s) d_{r+s}, \quad d_m \equiv \tau^m \zeta \partial_\zeta. \]  

(18)

Before we calculate its graded commutators with $L_m, F_m, G_r$, and $H_r$, let us note that setting $r = -1/2$,

\[ D_{-\frac{1}{2}} = D_\zeta, \]  

(19)

where $D_\zeta$ is the world line supersymmetry covariant derivative.

The commutation of $d_A$ with $L_A, D_A$ and $G_A$ yields:

\[ [L_A, d_B] = -B d_{A+B}, \quad [G_A, d_B] = G_{A+B}, \quad [D_A, d_B] = D_{A+B}, \]  

(20)

so that the set $\{L_A, G_A, D_A, d_A\}$ is closed under graded commutation. This complete set of generators is what we call the “covering algebra.” It is larger than the
super Virasoro algebra which it contains as a proper subgroup. The usefulness of the covering algebra can be seen by recalling a past example of such a structure. Within the context of 4D, $N = 1$ superspace, Sokatchev [8] used precisely a covering algebra to provide the first definitions of the irreducible superfield projection operators. The interpretation of this new infinite set of generators is pretty clear. These provide the infinite dimensional extension to the superspace covariant derivative just as the the usual super Virasoro generators provide an infinite dimensional extension of the superconformal generators.

V. 1D, $\aleph_0$-extended Superspace and the Super Covering $\mathcal{GR}$ D-module

We have previously introduced the notion of 1D, $\aleph_0$-extended superspace [9] which is the limit of the union of all possible 1D superspaces with a finite number of supersymmetries $N$ as this parameter approaches infinity. The ideas we have discussed so far in this letter permit additional structures to be embedded in $\aleph_0$-extended superspace. We introduce the set of derivations (by taking linear combinations of these, other bases are also possible as we shall see in the exceptional $N = 4$ case)

\begin{align}
G_A^I &\equiv i\tau^{A+\frac{1}{2}}\left[\partial^I - i2\zeta^I \partial_r\right] + 2(\mathcal{A} + \frac{1}{2})\tau^A\zeta^K \partial_K , \\
L_A &\equiv -\left[\tau^{A+1}\partial_r + \frac{1}{2}(\mathcal{A} + 1)\tau^A\zeta^I \partial_I\right], \\
T_A^{IJ} &\equiv \tau^A \left[\zeta^I \partial^J - \zeta^J \partial^I\right], \quad d_A^{IJ} \equiv \tau^A \left[\zeta^I \partial^J + \zeta^J \partial^I\right], \\
D_A^I &\equiv \tau^{A+\frac{1}{2}} \left[\partial^I + i2\zeta^I \partial_r\right] + i2(\mathcal{A} + \frac{1}{2})\tau^A\zeta^K \partial_K , \\
R_A^{1\cdots p} &\equiv (i\tau^A)^{\left(p - \frac{1}{2}\right)} \partial_{\mathcal{A}} \zeta_{1\cdots p} \partial_r , \quad p = 2, \ldots , N , \\
U_A^{1\cdots q} &\equiv i(i\tau^A)^{\left(q - \frac{1}{2}\right)} \zeta_{1\cdots q-1} \partial_{\mathcal{A}} , \quad q = 3, \ldots , N + 1 ,
\end{align}

where $N$ is an arbitrary integer. This set of vector fields is closed under graded commutation and as well contains the super Virasoro-like sub-algebra for all values of $N$. We shall call this algebra the super “$\mathcal{GR}$ covering algebra.” Additionally, the notation in the exponents of the factors of $i$ includes the “greatest integer in” function. So $[\frac{p}{2}]$ denotes the greatest integer in $\frac{p}{2}$, etc. All of these derivations possess engineering dimensions (not to be confused with “scale weight”) of $(\text{mass})^{-A}$ power.

These definitions do not depend on a specific value of $N$ and are thus appropriate for the entire 1D, $\aleph_0$ superspace that we have introduced before. Of course, for low values of $N$, not all of the generators appear. For example, $T_A^{IJ}$ and $R_A^{1\cdots p}$ only appear for superspaces with $N \geq 2$. Generically, $U_A^{1\cdots q}$ only appears for superspaces
with $N \geq 3$. This set of derivations forms a closed algebra under the action of the graded commutator. Explicitly for some of the graded commutators we find
\[
[ L_A, L_B ] = (A - B) L_{A+B}, \quad [ L_A, U_{A+B}^{1:1-m} ] = -[B + \frac{1}{2} m A] U_{A+B}^{1:1-m},
\]
\[
[ G_A, G_B ] = -i 4 \delta^{1J} L_{A+B} - i 2(A - B) [ T_{A+B}^{1J} + 2(A + B) U_{A+B}^{1:1K} ],
\]
\[
[ L_A, G_B ] = (\frac{1}{2} A - B) G_{A+B}, \quad [ L_A, D_B ] = (\frac{1}{2} A - B) D_{A+B},
\]
\[
[ D_A, D_B ] = -i 4 \delta^{1J} L_{A+B} - i 2(A - B) [ T_{A+B}^{1J} - 2(A + B) U_{A+B}^{1:1K} ],
\]
\[
[ D_A, G_B ] = i 2(A - B) \left\{ d_{A+B}^{1J} - \delta^{1J} d_{A+B}^{K K} + 2(A + B) U_{A+B}^{1:1K} \right\},
\]
\[
[ L_A, R_{A+B}^{1:1-m} ] = -[B + \frac{1}{2} (m - 2) A] R_{A+B}^{1:1-m},
\]
\[
G_A, R_{B}^{1:1-m} = 2(i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+1}{2} \right]} [ B + (m - 1) A + \frac{1}{2} ] R_{A+B}^{1:1-m},
\]
\[
+ i (i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+1}{2} \right]} \sum_{r=1}^{m} (-1)^{r-1} \delta^{1J_r} R_{A+B}^{1:1-J_r,r+1:1-m},
\]
\[
- (-1)^m (i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+1}{2} \right]} [ A + \frac{1}{2} ] U_{A+B}^{1:1-m I},
\]
\[
+ i 2(i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+3}{2} \right]} [ A^2 - \frac{1}{4} I ] U_{A+B}^{1:1-J_m K K},
\]
\[
D_A, R_{B}^{1:1-m} = i 2(i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+1}{2} \right]} [ B + (m - 1) A + \frac{1}{2} ] R_{A+B}^{1:1-m},
\]
\[
+ (i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+1}{2} \right]} \sum_{r=1}^{m} (-1)^{r-1} \delta^{1J_r} R_{A+B}^{1:1-J_r,r+1:1-m},
\]
\[
+ i (-1)^m (i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+1}{2} \right]} [ A + \frac{1}{2} ] U_{A+B}^{1:1-m I},
\]
\[
- 2(i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+3}{2} \right]} [ A^2 - \frac{1}{4} I ] U_{A+B}^{1:1-J_m K K},
\]
\[
R_{A}^{1:1-m}, R_{B}^{1:1-n} = -i(i)^{\left[ \frac{m}{2} \right]+\left[ \frac{n}{2} \right]} [ A - B + \frac{1}{2} (m - n) ] R_{A+B}^{1:1-m},
\]
\[
G_A, U_{B}^{1:1-m} = 2(i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+1}{2} \right]} [ B + (m - 2) A ] U_{A+B}^{1:1-m},
\]
\[
- 2(-1)^m (i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+1}{2} \right]} [ A + \frac{1}{2} ] \delta^{1J_m} U_{A+B}^{1:1-J_m K K},
\]
\[
+ i (i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+1}{2} \right]} \sum_{r=1}^{m-1} (-1)^{r-1} \delta^{1J_r} U_{A+B}^{1:1-J_r,r+1:1-m},
\]
\[
- i 2(-1)^m (i)^{\left[ \frac{m}{2} \right]-\left[ \frac{m+1}{2} \right]} \delta^{1J_m} R_{A+B}^{1:1-J_m},
\]
\[
[ d_A^{1J}, d_B^{KL} ] = \delta^{1K} T_{A+B}^{1L} + \delta^{1L} T_{A+B}^{1K} + \delta^{1J} T_{A+B}^{1K} + \delta^{1L} T_{A+B}^{1K},
\]

VI. The Exceptional 1D, \( N = 4 \) Super Virasoro D-Module

The exceptional \( N = 4 \) case for the superconformal algebra arose due to the possibility of modifying the generic set of generators in (1) and replacing a subset of them by the “deformed” subset of (4). We can attempt the same type of approach to the construction of an exceptional \( N = 4 \) super Virasoro D-module. This essentially follows the lines of the superconformal case. We define \( \ell \)-deformed vector fields,

\[
G_A^I \equiv i \tau^{A+\frac{1}{2}} \left[ \partial^I - i 2 \zeta^I \partial_\tau \right] + 2(\mathcal{A} + \frac{1}{2}) \tau^{A+\frac{1}{2}} \zeta^K \partial_K \\
+ \ell (\mathcal{A} + \frac{1}{2}) \tau^{A-\frac{1}{2}} \left[ \epsilon^{IJLK} \zeta_L \partial_K - i 4 \zeta^{(3)} \partial_\tau \right] + i 4 \ell (\mathcal{A}^2 - \frac{1}{4}) \tau^{A-\frac{3}{2}} \zeta^{(4)} \partial^I,
\]

(22)
\[ L_A \equiv - \left[ \tau^{A+1} \partial_\tau + \frac{1}{2}(A + 1) \tau^A \zeta^I \partial_I \right] 
+ i \ell A (A + 1) \tau^{A-1} \left[ \zeta^{(3)I} \partial_I + i 4 \zeta^{(4)} \partial_\tau \right] , \]
\[ T_A^{I J} \equiv \tau^A \left[ \zeta^I \partial^J - \zeta^J \partial^I - \ell \epsilon^{IJKL} \zeta^K \partial_L \right] 
- i 2 \ell A \tau^{A-1} \left[ \zeta^{(3)I} \partial^J - \zeta^{(3)J} \partial^I - \ell \epsilon^{IJKL} \zeta^{(3)K} \partial_L \right] . \]  

It is a matter of simple but long and intricate calculations to show that these derivations provide a representation of an \( N = 4 \) super Virasoro algebra. We find the closure of the \( \ell \)-dependent set \( \{ L_A, G_A^I, T_A^{I J} \} \) under commutation compelling given the number of identities required to achieve it. It should be clear from our method that only in the extended cases of \( N = 2 \) and 4, does the set \( \{ L_A, G_A^I, T_A^{I J} \} \) close. For all other extended values of \( N \), additional operators (contained in the super covering \( \mathcal{GR} \) D-module) are required for closure. Another implication of our construction is that there exists a distinct “large” \( N = 4 \) super \( \mathcal{GR} \) Virasoro algebra that appears as a subset of the operators in (21).

VII. Conclusions

Since the super \( \mathcal{GR} \) covering algebra is embedded as an algebra of derivations, it is a geometrical property of 1D, \( N \)-extended superspace itself and does not depend on the existence of any dynamical quantities. Alternately, we can now undertake investigations of arbitrarily extended super Virasoro algebras using super differential operators and super functions as opposed to the traditional use of Hilbert spaces and objects contained therein. It is clear (by the same argument that we used to oxidize the 1D, \( N \)-extended superconformal algebra to 2D (1,0) superspace) that there exist super 2D \( (p, q) \) \( \mathcal{GR} \) covering algebras. Our approach shows how the 2D generators of super Virasoro algebras can be constructed from the same vector fields used for 1D theories by following the oxidation procedure described below (4) in section two. In any event, our approach offers a field-independent way to study arbitrary \( N \)-extensions of super-Virasoro algebras in 1D superspace as well as arbitrary \( (p, q) \)-extensions in 2D superspace.

The behavior of \( T_A^{I J} \) as the value of \( N \) changes is very characteristic and reminiscent of another system we have previously investigated [4]. There it was shown that there exist matrices denoted by \( f^{I J} \) which occur in the off-shell description of \( N \)-extended NSR spinning particles. For all values of \( N \), except four, these matrices were representations of \( \text{O}(N) \). For the case of \( N = 4 \), these were representations of \( \text{SU}(2) \). We are thus led to suspect that the \( f^{I J} \)’s provide a matrix representation of
the $T^{IJ}_0$ generators. If this suggestion is accepted, then the vector spaces upon which the $f^{IJ}$-matrices act can be interpreted as the spinor representations of the super Virasoro algebra. The $\mathcal{GR}(d, N)$ algebras discussed in ref. [4] would then be related to the super Virasoro algebra in exactly the same way that the Spin($d$) algebra is related to the conformal algebra. With this new interpretation, we may say that the $\aleph_0$-extended off-shell NSR spinning particle models we constructed required the use of “Virasoro spinor representations.”

As we noted, the super $\mathcal{GR}$ covering algebra is larger than the super Virasoro algebra. Here the most interesting operator seems to be $D_{A \lambda}$. It bares a striking similarity (even in form) to the covariant Green-Schwarz superstring operator $D_{\alpha}(\sigma)$, first introduced by Siegel [10]. In turn the zero mode of $D_{\alpha}(\sigma)$ corresponds to the usual superspace “supercovariant derivative.” We still have before us the task of fully understanding the significance of $D_{A \lambda}$. The most naive expectation is that it is an NSR analog to $D_{\alpha}(\sigma)$.

Other interesting future avenues to study are the behaviors of the realizations of the super Virasoro algebra that are induced by our vector fields on restricted super functions of the form

$$\mathcal{F}(\tau, \zeta^I) = \sum_A \tau^A a_{(A)}(\zeta^I) \ ,$$

in the 1D case and

$$\mathcal{F}(\tau, \sigma, \zeta^{+1L}, \zeta^{-1L}) = \sum_{A, B} (\sigma^+)^A (\sigma^-)^B a_{(A, B)}(\zeta^{+1L}, \zeta^{-1L}) \ ,$$

in the 2D case. Clearly such restricted super functions form a closed set under the operators in (21) as well as ordinary multiplication.

Looking back at the $N = 0$ truncation of our results, we see that there remains only the operator $L_A$. This again raises the logical question of the import of the operator $H_r$? It remains present in the purely bosonic limit. We have successfully reached our goal of a model-independent representation of the super Virasoro algebras for all values of $N$. An important next step is to investigate whether any of the new purely “kinematical” features (e.g. $H_r, D_A$, etc.) we have proposed yield new insights into dynamical systems such as $N$-extended supergravity as well as super and heterotic strings. Apothegmatically, do our model-independent reformulations of super Virasoro algebras matter?

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3It is a simple step to append a space-time index to these in order to introduce quantities that bare a resemblance to particles and strings $\mathcal{F} \rightarrow X^m$. 

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“There is geometry in the humming of the strings.” – Pythagoras

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