Random Schrödinger operators
with a background potential

Hayk Asatryan and Werner Kirsch
Fakultät für Mathematik und Informatik
FernUniversität in Hagen, Germany

1 Notations, assumptions

We consider Schrödinger operators on $L^2(\mathbb{R})$ of the form

$$H_\omega = -\frac{d^2}{dx^2} + U + V_{\text{per}} + V_\omega.$$  \hspace{1cm} (1)

We assume that the background potential $U$ belongs to the space of real valued uniformly square integrable functions

$$L^2_{\text{loc, unif}} = \{ F : \mathbb{R} \to \mathbb{R} \mid \sup_{x \in \mathbb{R}} \int_{x-1}^{x+1} |F(x)|^2 \, dx < \infty \}$$ \hspace{1cm} (2)

and

$$U(x) \to a^- \text{ as } x \to -\infty, \quad U(x) \to a^+ \text{ as } x \to +\infty.$$ \hspace{1cm} (3)

Moreover, $V_{\text{per}}$ is a 1-periodic real valued function in $L^2_{\text{loc, unif}}$. $V_{\omega}$ is a random alloy-type potential of the form

$$V_{\omega}(x) = \sum_{k=-\infty}^{\infty} q_k(\omega)f(x-k) \quad (x \in \mathbb{R}),$$ \hspace{1cm} (4)

where $q_k$ are independent random variables with a common distribution $P_0$. We suppose that $f$, called the single site potential, is a real valued function satisfying

$$|f(x)| \leq C (1 + |x|)^{-\gamma} \quad (x \in \mathbb{R})$$ \hspace{1cm} (5)

for some $\gamma > 1$. We assume for simplicity that $\text{supp } P_0$ is a compact subset of $\mathbb{R}$. We remark that it would be sufficient that enough moments of $P_0$ exist. Moreover, $f$ may have local singularities.
Under the above assumptions the potentials $U, V_{\text{per}}, V_\omega$ and there sums belong to $L^2_{\text{loc, unif}}$, hence they are $H_0$-bounded by [8], Theorem XIII.96 and all operators are essentially self adjoint on $C_0^\infty(\mathbb{R})$.

We introduce the following notations:

$$H_0 = -\frac{d^2}{dx^2} \quad \text{(the free Hamiltonian)}, \quad (6)$$

$$H_U = H_0 + U \quad (7)$$

$$H_{\text{per}} = H_0 + V_{\text{per}}, \quad (8)$$

$$H_{U, \text{per}} = H_0 + U + V_{\text{per}}. \quad (9)$$

## 2 The essential spectra of $H_{U+V}$ and $H_{U,\text{per}}$

One of the main observations of this section is the following result.

**Theorem 2.1.** Let $U_1, U_2, V : \mathbb{R} \to \mathbb{R}$ be $H_0$-bounded measurable functions and

$$U_j(x) \xrightarrow{x \to -\infty} a^-, \quad U_j(x) \xrightarrow{x \to \infty} a^+ \quad (j = 1, 2)$$

for some $a^\pm \in \mathbb{R}$. Then

$$\sigma_{\text{ess}}(H_{U_1+V}) = \sigma_{\text{ess}}(H_{U_2+V}).$$

**Proof.** We need to prove that

$$\sigma_{\text{ess}}(H_{U_1+V}) \subset \sigma_{\text{ess}}(H_{U_2+V}),$$

$$\sigma_{\text{ess}}(H_{U_2+V}) \subset \sigma_{\text{ess}}(H_{U_1+V}).$$

We’ll prove the first inclusion (the proof of the second one is similar). Let

$$\lambda \in \sigma_{\text{ess}}(H_{U_1+V}).$$

By Weyl’s criterion and Theorem 3.11 in [3] we conclude that there is a Weyl sequence of functions $\varphi_n \in C_0^\infty(\mathbb{R})$ such that

$$\|\varphi_n\|_2 = 1 \quad (n \in \mathbb{N}),$$

$$\|(H_{U_1+V} - \lambda I) \varphi_n\|_2 \to 0 \quad \text{(10)}$$

such that either

$$\text{supp } \varphi_n \subset (-\infty, n) \quad \text{for all } n \quad \text{(11)}$$

or

$$\text{supp } \varphi_n \subset (n, \infty) \quad \text{for all } n \quad \text{(12)}$$
holds. Assume (11) is true, then

\[
\| (H_{U_1 + V} - \lambda I) \varphi_n \|_2 - \| (H_V - (\lambda - a^-) I) \varphi_n \|_2 \to 0,
\]

\[
\| (H_{U_2 + V} - \lambda I) \varphi_n \|_2 - \| (H_V - (\lambda - a^-) I) \varphi_n \|_2 \to 0
\]

and hence

\[
\| (H_{U_1 + V} - \lambda I) \varphi_n \|_2 - \| (H_{U_2 + V} - \lambda I) \varphi_n \|_2 \to 0.
\]

From this and (10) we obtain

\[
\| (H_{U_2 + V} - \lambda I) \varphi_n \|_2 \to 0,
\]

therefore

\[
\lambda \in \sigma_{ess}(H_{U_2 + V}).
\]

As a corollary to the proof of Theorem 2.1 we get

**Corollary 2.2.** Let \( U, V : \mathbb{R} \to \mathbb{R} \) be measurable, \( H_0 \)-bounded and

\[
U(x) \xrightarrow[x \to -\infty]{} a^-, \quad U(x) \xrightarrow[x \to \infty]{} a^+
\]

(in the usual sense), where \( a^\pm \in \mathbb{R} \). Then

\[
\sigma_{ess}(H_{U+V}) \subset (a^- + \sigma_{ess}(H_V)) \cup (a^+ + \sigma_{ess}(H_V)),
\]

(13)

**Remark 2.3.** The previous theorem shows that the knowledge of \( V, a^\pm \) is sufficient for unique determination of \( \sigma_{ess}(H_{U+V}) \). In fact,

\[
\sigma_{ess}(H_{U+V}) = \sigma_{ess}(H_{U_c+V}),
\]

where

\[
U_c = a^- \chi_{(-\infty, 0]} + a^+ \chi_{(0, \infty)}.
\]

In general equality in (13) does not hold. However, for the case of periodic potentials we have:

**Theorem 2.4.** Let \( U : \mathbb{R} \to \mathbb{R} \) be measurable, \( H_0 \)-bounded and satisfy the conditions

\[
U(x) \xrightarrow[x \to -\infty]{} a^- \quad \text{and} \quad U(x) \xrightarrow[x \to \infty]{} a^+
\]

and let \( W \) be a \( H_0 \)-bounded periodic potential, then

\[
\sigma_{ess}(H_0 + U + W) = (a^- + \sigma_{ess}(H_0 + W)) \cup (a^+ + \sigma_{ess}(H_0 + W)).
\]

**Remark 2.5.** It is well known that under the above assumptions on \( W \) we have \( \sigma_{ess}(H_0 + W) = \sigma(H_0 + W) \). See [8].
Proof. In the view of Corollary 2.2, we need to prove that
\[ a^- + \sigma_{\text{ess}}(H_0 + W) \subset \sigma_{\text{ess}}(H_0 + U + W), \tag{14} \]
\[ a^+ + \sigma_{\text{ess}}(H_0 + W) \subset \sigma_{\text{ess}}(H_0 + U + W). \tag{15} \]
We’ll prove (14) (the proof of (15) is similar). Let
\[ \lambda \in a^- + \sigma_{\text{ess}}(H_0 + W), \]
i.e. \( \lambda - a^- \in \sigma_{\text{ess}}(H_0 + W). \)
Then there is a Weyl sequence \( \varphi_n \in C_0^\infty(\mathbb{R}) \) with
1. \( \| \varphi_n^- \|_2 = 1 \) (\( n \in \mathbb{N} \)),
2. \( \| (H_0 + W - (\lambda - a^-)I) \varphi_n^- \|_2 \to 0, \)
Since \( W \) is periodic any shift of \( \varphi_n \) by an integer times the period of \( W \) is also a Weyl sequence for \( H_0 + W + a^- \). Thus we may assume that \( \text{supp} \varphi_n \subset (-\infty, -n) \). As in the previous proofs one easily sees that this sequence is also a Weyl sequence for \( H_0 + U + W \).

3 The essential spectrum of \( H_\omega \)

We turn to the spectrum of \( H_\omega \). To do so, we first describe the spectrum of \( H_0 + V_\omega \), i.e. the case \( U = 0 \).
We follow the investigation in [4].

Definition 3.1. A potential \( W(x) = \sum_{k \in \mathbb{Z}} \rho_k f(x - k) \) is called admissible, if \( \rho_k \in \text{supp} P_0 \) for all \( k \). Let us denote by \( \mathcal{P} \) the set of all admissible potentials, generated by \( \ell \)-periodic \( \rho_k \) for some \( \ell \in \mathbb{N} \).

Theorem 3.2. The spectrum \( \sigma(H_0 + V_\omega) \) is independent of \( \omega \) almost surely and is given (almost surely) by
\[ \Sigma := \sigma(H_0 + V_\omega) = \bigcup_{W \in \mathcal{P}} \sigma(H_0 + W) \tag{16} \]
For a proof we refer to [4].

In particular, the following result was proved in [4].

Lemma 3.3. If \( W \) is a periodic admissible potential and \( \lambda \in \sigma(H_0 + W) \) then there are sequences \( \varphi_n^+, \varphi_n^- \in L^2(\mathbb{R}) \) in the domain of \( H_0 + W \), such that
1. \( \| \varphi_n^+ \| = \| \varphi_n^- \| = 1 \)
2. The supports of \( \varphi_n^+ \) and \( \varphi_n^- \) are compact and satisfy

\[ \text{supp} \varphi_n^+ \subset [n, \infty) \text{ and } \text{supp} \varphi_n^- \subset (-\infty, -n] \]
3. For almost all $\omega$

$$\| (H_0 + V_\omega - \lambda) \varphi^+_n \| \to 0 \text{ and } \| (H_0 + V_\omega - \lambda) \varphi^-_n \| \to 0$$

From this we conclude

**Theorem 3.4.** Almost surely

$$\sigma(H_\omega) = \sigma(H_0 + V_\omega + a^-) \cup \sigma(H_0 + V_\omega + a^+) \quad (17)$$

**Proof.** By Corollary 2.2 we know that

$$\sigma(H_\omega) \subset \sigma(H_0 + V_\omega + a^-) \cup \sigma(H_0 + V_\omega + a^+) \quad (18)$$

To prove the converse we observe that for any $W \in \mathcal{P}$

$$\sigma(H_0 + W + a^\pm) \subset \sigma_{ess}(H + U + W) \quad (19)$$

by Theorem 2.4. It is easy to see (e. g. as in [4]) that almost surely for $W \in \mathcal{P}$

$$\sigma_{ess}(H + U + W) \subset \sigma_{ess}(H + U + V_\omega) \quad (20)$$

We conclude that

$$\bigcup_{W \in \mathcal{P}} \sigma(H_0 + W + a^+) \cup \bigcup_{W \in \mathcal{P}} \sigma(H_0 + W + a^-) \subset \sigma_{ess}(H + U + V_\omega). \quad (21)$$

Since the righthand side is a closed set we infer from Theorem 3.2 that almost surely

$$\sigma(H_0 + V_\omega + a^-) \cup \sigma(H_0 + V_\omega + a^+) \subset \sigma(H_\omega). \quad (22)$$

\[\square\]

4 The Integrated Density of States

In this section we investigate the integrated density of states of the operators $H_\omega$.

**Definition 4.1.** Let $A$ be a self adjoint operator bounded below and with (possibly infinite) purely discrete spectrum $\lambda_1(A) \leq \lambda_2(A) \leq \lambda_3(A) \leq \ldots$ where we count eigenvalues according to their multiplicities. Then we set

$$N(A, E) := \# \{j \mid \lambda_j(A) \leq E \}. \quad (23)$$

For $H = H_0 + W$ (with $W \in L^2_{\text{loc,unit}}$) and $a, b \in \mathbb{R}$, $a < b$ we define $H_{a,b}^D$ to be the operator $H$ restricted to $L^2([a,b])$ with Dirichlet boundary conditions both at $a$ and $b$. Similarly, $H_{a,b}^N$ has Neumann
boundary conditions at \(a\) and \(b\), \(H_{a,b}^{D,N}\) has Dirichlet boundary condition at \(a\) and Neumann boundary condition at \(b\), \(H_{a,b}^{N,D}\) has Neumann boundary condition at \(a\) and Dirichlet one at \(b\).

If for \(H = H_0 + W\) the limit

\[
\mathcal{N}(E) = \mathcal{N}(H, E) := \lim_{L \to \infty} \frac{1}{2L} N(H_{-L,L}^D, E)
\]

exists for all but countably many \(E\), we call \(\mathcal{N}(E)\) the integrated density of states for \(H\).

It is well known that under our assumptions the integrated density of states for \(H = H_0 + V_\omega\) exists, more precisely:

**Theorem 4.2.** If \(V_\omega\) satisfies the assumptions of Section 1, then the integrated density of states for \(\mathcal{N}(H, E)\) exists and for all but countably many \(E\) the following equalities hold:

\[
\mathcal{N}(H, E) = \lim_{L \to \infty} \frac{N(H_{-L,L}^N, E)}{2L} = \lim_{L \to \infty} \frac{\mathbb{E} \left( N(H_{-L,L}^D, E) \right)}{2L} = \lim_{L \to \infty} \frac{\mathbb{E} \left( N(H_{-L,L}^N, E) \right)}{2L}.
\]

(\(\mathbb{E}\) denotes expectation with respect to \(\mathbb{P}\).)

For a proof see [5]. The proof there uses the method of Dirichlet-Neumann bracketing (see [8]), in particular it is used:

**Theorem 4.3.** If \(a < c < b\) and \(X, Y \in \{D, N\}\), then

\[
N(H_{a,c}^{X,D}, E) + N(H_{c,b}^{D,Y}, E) \leq N(H_{a,c}^{X,Y}, E) \leq N(H_{a,c}^{N,N}, E) + N(H_{c,b}^{N,Y}, E).
\]

For the integrated density of states of the operator \(H_\omega = H_0 + U + V_{\text{per}} + V_\omega\) we have the following result.

**Theorem 4.4.** The integrated density of states \(\mathcal{N}(H_\omega, E)\) exists and can be expressed in terms of \(\mathcal{N}_0(E)\), the integrated density of states of \(H_0 + V_\omega\) by:

\[
\mathcal{N}(H_\omega, E) = \frac{1}{2} \mathcal{N}_0(E - a^-) + \frac{1}{2} \mathcal{N}_0(E - a^+).
\]

To prove this result we need the following lemma:

**Lemma 4.5.** For the integrated density of states \(\mathcal{N}_0\) of \(H_0 + V_\omega\) we have for any fixed \(M\) with \(M < L\) and any \(X, Y \in \{D, N\}:\)

\[
\mathcal{N}_0(E) = \lim_{L \to \infty} \frac{1}{L} \mathbb{E} \left( N(H_0 + V_\omega)_{X,Y}^{M,L} \right) \quad (28)
\]

\[
= \lim_{L \to \infty} \frac{1}{L} \mathbb{E} \left( N(H_0 + V_\omega)_{X,Y}^{M,-L} \right) \quad (29)
\]

**Proof.** By the stationarity of the potential we have

\[
\mathbb{E} \left( N(H_0 + V_\omega)_{X,Y}^{M,L} \right) = \mathbb{E} \left( N(H_0 + V_\omega)_{X,Y}^{M,-L} \right). \quad (30)
\]
Thus, the lemma follows from Theorem 4.2.

We now prove Theorem 4.4.

**Proof.**

\[
\mathbb{E}\left( N\left( (H_0 + U + V_\omega)^{X,Y}_{-L,L} \right) \right) \leq \mathbb{E}\left( N\left( (H_0 + U + V_\omega)^{X,N}_{-L,-M} \right) \right) + \\
+ \mathbb{E}\left( N\left( (H_0 + U + V_\omega)^{N,N}_{-M,L} \right) \right) + \mathbb{E}\left( N\left( (H_0 + U + V_\omega)^{N,Y}_{M,L} \right) \right).
\]  

(31)

We take \( M > 0 \) so large that \( |U(x) - a^-| < \varepsilon/2 \) for \( x \leq -M \) and \( |U(x) - a^+| < \varepsilon/2 \) for \( x \geq M \).

Let us divide inequality (31) by \( 2L \). Then the middle term goes to zero as \( L \to \infty \). Moreover in the limit the first term on the right hand side can be bounded by \( \frac{1}{2} \mathcal{N}_0(E - a_-) + \varepsilon/2 \). Similarly the third term can be bounded by \( \frac{1}{2} \mathcal{N}_0(E - a_+) + \varepsilon/2 \). Since \( \varepsilon > 0 \) was arbitrary we proved

\[
\mathbb{E}\left( N\left( (H_0 + U + V_\omega)^{X,Y}_{-L,L} \right) \right) \leq \frac{1}{2} \mathcal{N}_0(E - a^-) + \frac{1}{2} \mathcal{N}_0(E - a^+).
\]  

(32)

The inverse inequality follows if we use Dirichlet, instead of Neumann boundary conditions for the inequalities (31).

**References**

[1] Adams R. A., Fournier J. J. F. *Sobolev Spaces.* – Academic Press, 2003

[2] Eastham M. S. P. *The Spectral Theory of Periodic Differential Equations.* – Belfast, Scottish Academic Press, 1973

[3] Cycon H. L., Froese R. G., Kirsch W., Simon B. *Schrödinger Operators.* – Springer, 2008

[4] Kirsch W., Martinelli F. *On the spectrum of Schrödinger operators with a random potential.* Comm. Math. Phys. 85, 329–350, 1982

[5] Kirsch W., Martinelli F. (1982): *On the density of states of Schrödinger operators with a random potential.* J. Phys. A: Math. Gen. 15 (7), 2139–2156, 1982

[6] Naimark M. A., *Linear differential operators. Part II.* – London, George G. Harrap & Co., 1968.

[7] Reed M., Simon B. *Methods of Modern Mathematical Physics, vol. 2. Fourier Analysis, Self-Adjointness.* – Academic Press, 1975

[8] Reed M., Simon B. *Methods of Modern Mathematical Physics, vol. 4. Analysis of Operators.* – Academic Press, 1978