Deterministic super-replication of unitary operations

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We show that one can deterministically generate out of $N$ copies of an unknown unitary operation up to $N^2$ almost perfect copies. The result holds for all operations generated by a Hamiltonian with an unknown interaction strength. This generalizes a similar result in the context of phase covariant cloning where, however, super-replication comes at the price of an exponentially reduced probability of success. We also show that multiple copies of unitary operations can be emulated by operations acting on a much smaller space, e.g., a magnetic field acting on a single $n$-level system allows one to emulate the action of the field on $n^2$ qubits.

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Introduction.— Quantum information can not be cloned. This simple statement, first manifested in [1], has far reaching consequences particularly in the context of quantum cryptography where the no-cloning principle ensures security [2]. A violation of the no-cloning principle would allow for super-luminal communication or the violation of Heisenberg’s uncertainty principle, illustrating its fundamental character.

However, imperfect replication of quantum information is possible and various works have derived optimal cloning devices under different circumstances [3] (see also [4]). Given $N$ copies of a system in some pure state $|\psi\rangle$, one can deterministically produce $M > N$ copies with a non-unit fidelity that depends on $N$ and $M$. Moreover, it was shown in [5] that for phase covariant states, i.e., states generated by a Hamiltonian with unknown interaction strength, up to $N^2$ almost perfect copies can be generated using a probabilistic replication processes. This super-replication of states comes at the price of a success probability that drops exponentially with $N$.

In this letter we show that a similar super-replication can be achieved for the cloning of unitary operations. Here the goal is to produce out of $N$ copies of an unknown unitary operation (given in the form of a black box that can be applied to arbitrary states) $M > N$ copies. This is in general a harder task than cloning of states, as one needs to replicate the action of the operation on all possible input states [6]. Nevertheless, we find that deterministic super-replication of unitary transformations of the form $U = e^{i\vartheta H}$, where $H$ is the Hamiltonian generating the unitary evolution and $\vartheta$ is an unknown interaction strength, is possible in contrast to super-replication of phase covariant states. We demonstrate this result by providing an explicit protocol that makes use of likely sequences as in Schumacher’s compression theorem [7].

We also consider the emulation of multiple copies of unitary operations by operations acting on a smaller space. Specifically, we find that a single unitary operation with a given interaction strength, $\vartheta$, acting on an $n$-dimensional system is sufficient to emulate $n^2$ copies of an operation with the same $\vartheta$ acting on a two-dimensional system. In addition, we show that if one can also intercept the evolution generated by the Hamiltonian with additional control operations, then one can generate an arbitrary number, $M$, of perfect copies from single instance of the unitary operation at the cost of a $\sqrt{M}$ reduction in interaction strength.

Background.— We start by specifying the set-up. We consider a class of $d$-dimensional unitary operations $U(\vartheta) = \exp(-i\vartheta H)$ generated by a Hamiltonian, $H$, and parametrized by $\vartheta$. The operations are provided in the form of an unknown black box, where $H = \sum_{k=0}^{d-1} \alpha_k |\varphi_k\rangle \langle \varphi_k|$ is known but $\vartheta$ is not. For instance, this may correspond to a situation where $\vartheta$ specifies the unknown interaction strength and the time for applying $H$ is fixed. For simplicity we will consider $d = 2$ in the following where $H = |1\rangle\langle 1|$ and $\vartheta \in [0, 2\pi)$, i.e., $U(\vartheta)$ is equivalent, up to an irrelevant global phase factor, to a rotation around the $z$-axis by an angle $\vartheta$. Generalization of the results to arbitrary $d$ and arbitrary Hamiltonians are straightforward.

The goal is to generate $M$ approximate copies of $U(\vartheta)$, i.e., $\tilde{V}(\vartheta) \approx U(\vartheta)^{\otimes M}$, given only $N$ copies, where $M \geq N$. To achieve this task we make use of a suitable number of auxiliary qubits and appropriate unitary operation $A$—to be applied before and after the application of $U(\vartheta)^{\otimes N}$—that yield an approximation of $U(\vartheta)^{\otimes M}$ on an arbitrary input state, see Fig. 1.

![FIG. 1. Illustration of the overall procedure to obtain $M$ approximate copies from $N$ applications of an (unknown) unitary operation $U(\vartheta)$. By applying the unitary basis change $A^\dagger$, $A$, before and after $U(\vartheta)^{\otimes N}$ we can obtain the operation $\tilde{V}(\vartheta)$ of Eq. 5 which is a good approximation of $U(\vartheta)^{\otimes M}$, i.e., $A^\dagger (1^{\otimes M} \otimes U^{\otimes N}) A \left( |\psi\rangle \otimes |0\rangle^{\otimes N} \right) \approx U^{\otimes M} |\psi\rangle \otimes |0\rangle^{\otimes N}$. Notice that $N$ auxiliary systems are used that are not affected by the transformation.]

We quantify the performance of $\tilde{V}(\vartheta)$ resulting from our protocol by the global Jamiołkowski Fidelity (process fidelity), $F_E$, averaged over all possible input operations $U(\vartheta)$ [8]. For an $n$-dimensional unitary operation, $X$, the process fidelity of a completely positive map $E$ is defined as

$$F_E(E, X) = \langle \psi_X | \rho_E | \psi_X \rangle$$ 

(1)
where $|\psi_X\rangle$, $\rho_E$ are the Choi-Jamiołkowski states associated to $X$, $E$ respectively via the Choi-Jamiołkowski isomorphism \[\Phi\]. The latter associates to the operations $X$, $E$ the states $|\psi_X\rangle = 1 \otimes X |\phi\rangle$ and $\rho_E = 1 \otimes \Phi (|\phi\rangle \langle \phi|)$ respectively, where $|\Phi\rangle = 1/\sqrt{n} \sum_n |n\rangle \otimes |n\rangle$ is a maximally entangled $n$-level state.

The process fidelity is closely related to the average fidelity, $\bar{F}(E, X) = \int d\psi |\langle U\psi | U^\dagger E U^\dagger |\psi\rangle|^2$, where the average is taken over all input states $|\psi\rangle$. It is known that $\bar{F}(E, X) = (F_E(E, X) + 1)/(n + 1)$ \[11\], meaning that a sufficiently large process fidelity ensures that the map $E$ provides a good approximation, on average, for all input states. Throughout this article we consider only unitary operations, where the process fidelity reduces to the overlap of the corresponding pure Jamiołkowski states.

**Faithful approximation of $U(\theta)^{\otimes M}$.**—Consider $M$ copies of an operation $U(\theta) = e^{-i\theta|1\rangle\langle 1|}$. We have that

$$U(\theta)^{\otimes M} = \sum_k e^{-i|k|\theta} |k\rangle \langle k|,$$

where we denote by $|k\rangle \in (C^2)^{\otimes M}$ the basis vectors of the $M$-qubit system using binary notation, i.e., $|0\rangle = |00\ldots 0\rangle$, and $|k\rangle$ denotes the Hamming weight of the vector $k$—the number of ones in binary notation. The corresponding Jamiołkowski state, $1 \otimes U(\theta)^{\otimes M} |\phi\rangle$, with $|\Phi\rangle = 2^{-M/2} \sum_k |k\rangle \otimes |k\rangle$ is given by

$$|\psi_{U(\theta)^{\otimes M}}\rangle = 2^{-M/2} \sum_k e^{-i|k|\theta} |k\rangle \otimes |k\rangle,$$

and all basis vectors with the same Hamming weight pick up the same phase factor.

Our goal is to approximate the action of $U(\theta)^{\otimes M}$. To this aim, consider an operation $V(\theta)$ acting on $M$ qubits that only produces the appropriate phases for the majority of basis vectors. The underlying distribution of the basis vectors in $U(\theta)^{\otimes M}$ is binomial, centered at $k = |k| = M/2$, and in the limit of large $M$ approaches the Gaussian distribution of the same mean and standard deviation $\sigma = \sqrt{M/2}$ \[11\]. Hence, it suffices to reproduce phases for $k \in (k_-, k_+)$ with $k_\pm = M/2 \pm \alpha M^3$ for some $\alpha > 0$ and $1/2 < \beta < 1$. The operation

$$\tilde{V}(\theta) = \sum_{|k| \in (k_-, k_+)} e^{-i|k|\theta} |k\rangle \langle k| + \sum_{|k| \notin (k_-, k_+)} e^{-i\gamma_k} |k\rangle \langle k|,$$

with arbitrary $\gamma_k$ approximates $U(\theta)^{\otimes M}$, where the process fidelity $F_E(\tilde{V}(\theta), U(\theta)^{\otimes M}) = |\langle \psi_{\tilde{V}(\theta)} | \psi_{U(\theta)^{\otimes M}} \rangle|^2$ is bounded from below by $\Phi(2\alpha M^3/2)$ = \[11\], meaning that a sufficiently large process fidelity ensures that the map $E$ provides a good approximation, on average, for all input states. Notice that also for finite, moderate values of $M$ one obtains a faithful approximation, which can be checked by directly evaluating the sum of binomial coefficients. Using Stirling’s formula, one can approximate the binomial coefficients directly instead of invoking the Gaussian approximation, and arrives at the same conclusion, i.e., for our choice of $\alpha, \beta$, $F_E \rightarrow 1$ in the limit of large $M$.

**Cloning protocol.**—We now show how to obtain $\tilde{V}(\theta)$ from $U(\theta)^{\otimes N}$ whenever $N = M^3, \forall \beta > 1/2$. As mentioned above it is sufficient to obtain the proper phases on all basis states $|k\rangle$ with $|k| \in (k_-, k_+)$. The latter set contains $2\alpha M^3 + 1$ different phases, with values $k_- + m\theta$ where $0 < m < 2\alpha M^3$. Furthermore, we need only reproduce the phases $m\theta$ as the resulting operation is equivalent up to an irrelevant global phase factor $e^{-ik_- \theta}$. As $U(\theta)^{\otimes N}$ contains $N + 1$ distinct phases, $e^{i|k|\theta}, 0 < |k| \leq N$ (see Eq. \[3\]), choosing $N = 2\alpha M^3$ is sufficient to reproduce all the required phases of $\tilde{V}(\theta)$ in the interval $(k_-, k_+)$ (see Eq. \[3\]).

To properly approximate $U(\theta)^{\otimes M}$ each phase $e^{-ik|\theta}$ has to be reproduced on all the $\binom{M}{|k|}$ levels that lay in the multiplicity space for each $|k\rangle \in (k_-, k_+)$. To do so, we attach $M$ additional auxiliary systems and consider the operation $1^{\otimes M} \otimes U(\theta)^{\otimes N}$ (see Fig. \[2\]). As the largest multiplicity in $\tilde{V}(\theta)$ is $\binom{M}{M/2}$, $M$ auxiliary systems are sufficient as each eigenstate in $1^{\otimes M} \otimes U(\theta)^{\otimes N}$ is $2^M \otimes M$-degenerate.

To obtain $\tilde{V}(\theta)$ from $1^{\otimes M} \otimes U(\theta)^{\otimes N}$, all we need is to establish a basis change that maps the eigenstates with the appropriate phases onto each other. This is done as follows. Consider the $M + N$ qubit state $|k\rangle \otimes |0\rangle$ where $|k\rangle$ is an $M$-qubit state and $|0\rangle$ is the state of $N$ auxiliary qubits. We use the mapping

$$|k\rangle \otimes |0\rangle \rightarrow |k\rangle \otimes |0\rangle$$

if $|k| \notin (k_-, k_+)$

$$|k\rangle \otimes |0\rangle \rightarrow |k\rangle \otimes |k_-\rangle$$

if $|k| \in (k_-, k_+), \ (6)$$

where $|k_-\rangle = |0\rangle \otimes N - (|k_-\rangle \otimes |1\rangle)^{\otimes N - |k_-\rangle}$ is a specific $N$-qubit state upon which $U(\theta)^{\otimes N}$ acts and $|k\rangle$ is an $M$-qubit state upon which the identity acts (see Fig. \[2\]). Notice that for $|k| \in (k_-, k_+), |k| \otimes |k_-\rangle$ picks up the phase $e^{-i|k|\theta}$, which is the correct phase up to an overall phase factor $e^{i\theta}$. Moreover, the number of states with this phase factor corresponds to all $M$-bit strings $|k\rangle$ with Hamming weight $|k|$, which is precisely the multiplicity of $e^{-ik|\theta}$ for $|k| \in (k_-, k_+)$ in Eq. \[3\]. All other states outside the bulk do not obtain a phase \[12\]. For all other states $\{|k\rangle \otimes |l\rangle\}$ we can choose an arbitrary mapping to one of the other basis states such that the overall operator, $A$, is unitary \[13\]. After application of $1^{\otimes N}$ to the last $N$ qubits, one only needs to undo the basis change by applying $A^\dagger$, see Fig. \[1\]. The choice of $N = 2\alpha M^3$ for $\beta > 1/2$ ensures that the Jamiołkowski fidelity is close to 1 in the limit of large $N, M$, and hence super-replication with a rate of $O(N^2)$ is achieved. Note that one can indeed show that this rate is optimal. It is known that in state super-replication, the Heisenberg limit, i.e., a replication rate of $N^2$, is optimal \[5\]. As this also applies to the Choi-Jamiołkowski state—which can be obtained deterministically from the unitary—any higher replication rate for unitaries would imply a corresponding higher rate for the state which is impossible \[14\].

Notice that in contrast to state super-replication our protocol works deterministically. This also holds when we apply the protocol to input states $|+\rangle^{\otimes M}$, which corresponds to the case of phase-covariant state cloning. The difference is that in our case the information on the unknown parameter, $\theta$, is encoded in the unitary operation and not in a particular state as is the case in \[5\]. Whereas
standard cloning protocols deal with input states that are of tensor product structure, here it is possible to apply the unitary operations to general (entangled) states, which is effectively achieved by the mapping \( A \). One can also directly adapt the protocol of [5] to accomplish deterministic state super-replication if we incorporate the filter into the state preparation procedure, prior to the application of the unitaries—which imprint the state information—and the cloning protocol.

We remark that our result can be generalized to arbitrary \( d \)-dimensional unitary operations generated by a Hamiltonian with unknown interaction strength, \( W(\vartheta) = \exp(-i\vartheta H) \) where \( H = \sum \lambda_i |\varphi_i \rangle \langle \varphi_i | \). For \( d > 2 \), the relevant, likely subspace of \( W(\vartheta) \) is not a multinomial, rather than a binomial, distribution that converges to a multivariable Gaussian distribution centered at \( p_k = \lambda_k N \). As long as the Gaussian has a width of \( O(\sqrt{N}) \) in each dimension, the approximation is faithful. It follows that one can generate an approximation of \( W(\vartheta)^{\otimes n} \) from \( W(\vartheta) \) in this case as well, where the required protocol is a direct generalization of the one presented for \( d = 2 \). The key ingredient is again the unitary operation \( A \) where now the tensor product of eigenstates \( |\varphi_k \rangle \), belonging to the likely subspace, are appropriately mapped so that they pick up the correct phase factor when \( W(\vartheta)^{\otimes n} \) is applied. As the spectral properties of the Hamiltonian have no bearing in our argument, super-replication is possible for arbitrary Hamiltonians as well.

**Emulation of multi-qubit operations.** — In a similar way one can also consider emulation of operations that depend on the same (unknown) parameter, \( \vartheta \), but act on different systems. For example, consider the operation \( V(\vartheta) = \exp(-i\vartheta H_V) \), where \( H_V = \sum_{j=0}^{N-1} j |j \rangle \langle j | \) is the Hamiltonian acting on an \( n \)-level system, and the unitary operation \( U(\vartheta) \) of Eq. (2) acting on a qubit. The above operations describe a spin-(\( n-1 \))/2 and a spin-1/2 particle coupled to the same magnetic field of unknown strength \( \vartheta \). Using the techniques established in the previous section it is straightforward to show that a single use of \( V(\vartheta) \) is sufficient to approximate \( M \) uses of \( U(\vartheta) \) whenever \( n = 20M^2 \) and \( \alpha > 0, \beta > 1/2 \). To see this first note that \( V(\vartheta) \) and \( U(\vartheta)^{\otimes n} \) have the same spectrum; only the multiplicities of the various eigenvalues differ. By attaching \( M \) auxiliary qubits, on which the identity acts, one can construct a similar unitary operator to \( A \) above and obtain \( U(\vartheta)^{\otimes n} \) exactly [15]. Using the scheme described in the previous section we can now obtain an approximation of \( U(\vartheta)^{\otimes n^2} \) from \( n \) uses of \( U(\vartheta) \).

The above result highlights an important equivalence between higher dimensional systems and the number of uses of a unitary operator on a two-level system. One can trade a single use of a unitary acting on an \( n \)-level system for an approximate \( n^2 \) uses of a unitary operator acting on qubits.

So far we have considered that additional control is available only before and after the application of the unitary operations. However, in many physically relevant situations, where \( U(\vartheta) \) is generated by a Hamiltonian with unknown interaction strength that is applied for a fixed time, additional control is available. In these cases one can interject the Hamiltonian evolution with ultrafast control pulses thus modifying the effective evolution [16]. This technique, also known as “bang-bang control”, allows one to generate an effective Hamiltonian with a modified spectrum. The use of bang-bang control techniques allows for more advanced emulation schemes. For example, consider the \( n \)-fold degenerate Hamiltonian with eigenvalues 0,1. Such a Hamiltonian describes, for example, the spin and motional degrees of freedom of an electron, where the spin degrees of freedom are acted upon by the Hamiltonian \( H = \vartheta |1 \rangle \langle 1 | \) —the same Hamiltonian that generates \( U(\vartheta) \) in Eq. (2)—and \( n \) motional degrees of freedom are acted on by the identity. Intermediate control pulses allow one to modify the spectrum of the effective Hamiltonian such that it contains \( n \) eigenstates, whose eigenvalues are evenly gapped, and all but the ground state level are non-degenerate. Up to a multiplicative factor of \( n \), this is the same spectrum as for the Hamiltonian \( H_N \) above and the multiplicative factor leads to an evolution \( V(\vartheta/n) \) instead of \( V(\vartheta) \). Hence, one can use the same technique as before to obtain multiple single-qubit operations. In fact, as \( n \) can be freely chosen, we have that from a single application of \( H \) for time \( t = 1 \), a single qubit operation \( U(\vartheta) \), one can generate up to \( n^2 \) copies of an operation with reduced strength \( \vartheta/n \), i.e., \( U(\vartheta) \to U(\vartheta/n)^{\otimes n^2} \).

**Links to quantum metrology.** — We now discuss connections between the super-replication of unitary operations established above and quantum metrology. The latter deals with optimally estimating an unknown parameter, \( \vartheta \), by choosing an optimal input state on which \( \vartheta \) is imprinted, and reading out the desired information by means of an optimal measurement [17]. When the input state is a product state of \( N \) qubits and the parameter, \( \vartheta \), is imprinted by applying the operation \( U(\vartheta) \) of Eq. (2) on each qubit, then the achievable precision, \( \delta \vartheta \), in the estimation of \( \vartheta \) is bounded by \( \delta \vartheta \geq O(1/N) \), the standard quantum limit. When the \( N \) qubits are prepared in an entangled state, however, an accuracy of \( \delta \vartheta = O(1/N^2) \) can be achieved.

Our super-replication procedure establishes an equivalence between different resources namely, \( N \) uses of \( U(\vartheta) \) on an entangled input state of \( N \) qubits, \( N^2 \) uses of \( U(\vartheta) \) on the optimal product state of \( N^2 \) qubits, and a single use of \( V(\vartheta) = \exp(-i\vartheta H_V) \), where \( H_V = \sum_{j=0}^{N-1} j |j \rangle \langle j | \)
acts on a single $N$-dimensional spin.

In particular, consider the case of quantum metrology where the input state comprises multiple qubits in the state $|0\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$. This set-up corresponds precisely to phase covariant cloning [5], where the information on the phase is contained in the unitary $U(\vartheta)$, and, in this case, the global fidelity of the cloned states is equivalent to the process fidelity. In [5] it was shown that the optimal super-replication strategy for states can produce at most $M = N^2$ copies, and saturates the Heisenberg limit. Using our super-replication procedure for unitary operations, and applying it to the product input state $|0\rangle^\otimes N$, one achieves the same, optimal, precision in quantum metrology as when $U(\vartheta)$ acts directly on the optimal entangled input state of $N$ qubits.

However, this does not guarantee that a high fidelity is achieved for all input states. The figure of merit for the super-replication procedure is the process fidelity, which provides a bound on the average state fidelity averaged over all possible input states. We stress that the process fidelity is the standard way of measuring the accuracy of operations and processes, and a high process fidelity implies a good approximation of the process [6] [8]. On the one hand, there exist states where the fidelity exceeds the process fidelity, e.g. for any state of the form $|\psi\rangle = \sum_{k} |k\rangle \alpha_k |k\rangle$ the fidelity is one. On the other hand, there are also several input states for which the achievable fidelity is smaller than the process fidelity.

In fact, it turns out that the action of $U(\vartheta)^{\otimes M}$ is not appropriately mimicked for input states that are themselves useful for parameter estimation, i.e., have a quantum Fisher information that is of $O(M^2)$. This is to be expected, as otherwise the Heisenberg limit for metrology would be violated by combining the super-replication of unitary operations as established here, and letting the protocol act on entangled input states.

Indeed, the protocol will not work for the optimal input state for quantum metrology, $(|0\rangle^M + |1\rangle^M)/\sqrt{2}$. Only random phases will be imprinted on both, $|0\rangle^M$ and $|1\rangle^M$, rather than the required phases $0$ and $M \vartheta$. By construction, our super-replication protocol yields a faithful approximation only for the bulk of states where the Hamming weight is approximately $M/2$, i.e., only for energy eigenstates with energy approximately $M/2 \pm \sqrt{M}$. All states with a large support on this subspace have quantum Fisher information that scales only as $O(M)$. States with a quantum Fisher information scaling as $O(M^2)$ are superpositions of eigenstates where the eigenvalues differ by $O(M)$ [13]. For all these states the proper phases are not reproduced by our super-replication protocol, however the relative volume of those states goes to zero with increasing $M$.

Finally, a single use of $V(\vartheta)$ on a $N$-dimensional spin also allows to mimic the action of $U(\vartheta)$ on $N^2$ product states $|0\rangle$, and hence to achieve the same precision in the estimation of $\vartheta$. One can trade between the number of levels and the number of copies of a two-level system.

Conclusion and outlook.—We have demonstrated the deterministic super-replication of unknown unitary operations. For all operations generated by a Hamiltonian with unknown interaction strength, one can produce up to $N^2$ copies of the operation using the operation only $N$ times. This surprising result is in perfect agreement with similar effects in state super-replication and quantum metrology. Whether a similar improvement can be obtained for arbitrary unitary operations of the group $SU(2)$ remains an open question.

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