How Does Casimir Energy Fall?

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(Dated: March 27, 2022)

PACS numbers: 03.70.+k, 04.20.Cv, 04.25.Nx, 03.65.Sq

The subject of quantum vacuum energy (the Casimir effect) dates from the same year as the discovery of renormalized quantum electrodynamics, 1948 [1]. It puts the lie to the naive presumption that zero-point energy is not observable. On the other hand, it continues to be surrounded by controversy, in large part because sharp boundaries give rise to divergences in the local energy density near the surface (see Refs. [2, 3, 4]). The most troubling aspect of these divergences is in the coupling to gravity. Gravity has its source in the local energy-momentum tensor, and such surface divergences promise serious difficulties. The gravitational implications of zero-point energy are an outstanding problem in view of our inability to understand the origin of the cosmological constant or dark energy [2, 3, 7].

As a prolegomenon to studying such issues, we here address a simpler question: How does the completely finite Casimir energy of a pair of parallel conducting plates couple to gravity? The question turns out to be surprisingly less straightforward than one might suspect! Previous authors [8, 9, 10, 11, 12] have given disparate answers, including gravitational forces, or gravitationally modified Casimir forces, that depend on the orientation of the Casimir apparatus with respect to the gravitational field of the earth. We will here resolve some of this confusion with a convincingly calculated result consistent with the equivalence principle. That is, the renormalized Casimir energy couples to gravity just like any other energy. In our opinion, this fact is evidence that vacuum energy couples to gravity just like any other energy. In gravity, the electromagnetic stress tensor is given by (1):

\[
\langle T^{\mu\nu} \rangle = \frac{\varepsilon_c}{a} \text{diag}(1, -1, -1, 3),
\]

where the third spatial direction is the direction normal to the plates. This is given in terms of the Casimir energy per unit area, \( \varepsilon_c = -\pi^2 \hbar c/(720a^3) \). Outside the plates, \( \langle T^{\mu\nu} \rangle = 0 \). Omitted here is a constant divergent term that is present both between and outside the plates, and also in the absence of plates, which cannot have any physical significance. Because the electromagnetic field respects conformal symmetry, there is no surface divergent term such as is present for a minimally coupled scalar field subject to Dirichlet conditions on the plates, or more generally for curved surfaces [14]. (Henceforth we will set \( \hbar = c = 1 \).)

Now we turn to the question of the gravitational interaction of this Casimir apparatus. It seems to us that this question can be most simply addressed through use of the gravitational definition of the energy-momentum tensor, as a variation of the matter part of the action,

\[
W_g \equiv \delta W_m = \frac{1}{2} \int (dx) \sqrt{-g} \delta g_{\mu\nu} T^{\mu\nu}. \tag{2}
\]

Following Schwinger (note the factor of 2 in the definition), for a weak field we define \( g_{\mu\nu} = g_{\mu\nu} + 2h_{\mu\nu} \). To first order we can ignore \( \sqrt{-g} \). The gravitational energy, for a static situation, is therefore given by \( \delta W = -\int dt \delta E \)

\[
E_g \equiv \delta E_m = -\int (dx) h_{\mu\nu} T^{\mu\nu}. \tag{3}
\]

We then replace \( T^{\mu\nu} \) here by the one-loop expectation value of the electromagnetic stress tensor [1].

Calloni et al. [9] and Bimonte et al. [12] use the Fermi metric

\[
g_{00} = -(1 + 2gz), \quad g_{ij} = \delta_{ij}, \tag{4}
\]

in terms of the gravitational acceleration \( g \). This is evidently appropriate for a constant gravitational field. We will discuss its relation to the field due to the earth below. Let the apparatus be oriented at an angle \( \alpha \) with respect to the direction of the gravitational field. The Cartesian
It is easy to verify that this gives the correct force on a mass point. For the constant field 44 the force on a Casimir apparatus is obtained from the change in the energy density $T^{00}$; that is, recalling that $z_0 = \zeta_0 \cos \alpha$,

$$
\delta T^{00} = \frac{\varepsilon_c \delta z_0}{a} \frac{1}{\cos \alpha} [\delta(\zeta_0 - a/2) - \delta(\zeta_0 + a/2)], \quad (9)
$$

which can again be checked to yield the correct force on a mass point. For the constant field 44 the force on a Casimir apparatus is obtained from the change in the energy density $T^{00}$; that is, recalling that $z_0 = \zeta_0 \cos \alpha$,

$$
\delta T^{00} = \frac{\varepsilon_c \delta z_0}{a} \frac{1}{\cos \alpha} [\delta(\zeta_0 - a/2) - \delta(\zeta_0 + a/2)], \quad (9)
$$

where the $\delta$ functions arise from the step functions at the boundaries. This yields from Eq. (9) the same result 44.

Our answer is consistent with the principle of equivalence, and with the second analysis of Jaekel and Reynaud 16, who state that the inertia of Casimir energy (at least in two dimensions) is $E_c/c^2$. However, it is only $\frac{1}{2}$ that found by Bimonte et al. 12, which is also the first force formula [Eqs. (7) and (8)] provided by Calloni et al. 9. Our Eq. (4) does, however, reproduce the second formula [Eq. (9)] given in Ref. 3, which those authors describe as the one that should be observable. We discuss this situation further below.

We now digress to consider whether the constant-field approximation 44 is adequate for an apparatus suspended above the earth or some other pointlike mass. Should we instead use the perturbation of the Schwarzschild metric? One might expect that the resulting curvature corrections are of order $\frac{\mu}{R} \ll 1$, at worst, relative to the main term, where $R$ is the earth’s radius. The point, however, is that naive attempts to do the calculation in curved space change the answer by factors like 2, and also differ among themselves, and the resolutions of the fallacies are sufficiently instructive to justify our belaboring the point.

The Schwarzschild metric in isotropic coordinates 17 and for weak fields $(GM/r \ll 1)$ is

$$
ds^2 = -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 + \frac{2GM}{r}\right)dr^2. \quad (10)
$$

If we expand a short distance $z$ above the earth’s surface, we find the nonzero components of the gravitational field to be $h_{00} = h_{11} = h_{22} = h_{33} = GM/R - g_z$, in terms of the acceleration of gravity, $g = GM/R^2$. (It is important to recognize that the constant $GM/R$ is irrelevant in the following, and that correspondingly the results do not depend on the origin of $z$.) The virtue of isotropic coordinates is that the spatial line element (apart from an overall factor) has the usual Cartesian form $dx^2 + dy^2 + dz^2$. Now when we compute the gravitational energy from Eq. (7) each component of the Casimir stress tensor contributes with equal weight:

$$
\delta E_g = gA \delta z_0 \int_{\zeta_0 - a/2}^{\zeta_0 + a/2} d\zeta (T^{00} + T^{11} + T^{22} + T^{33}), \quad (11)
$$

which gives the force

$$
-\frac{1}{A} \frac{\delta E_g}{\delta z_0} = \frac{F}{A} = -2gaT^{00} = -2g\varepsilon_c, \quad (12)
$$
since $T = T^\lambda_\lambda = 0$. This is twice the previous result \([3]\). Note that again the result is independent of $\alpha$. The same result is obtained if we start from Eq. (5).

We should be able to obtain the same result using the original Schwarzschild coordinates, where $h_{00} = -g_z$, $h_{0\theta} = -g_{\theta z}$, and all other components of $h_{\mu \nu}$ are zero. However, now if we use the first method \([7]\), the result is $F/A = -4g E_c \cos^2 \alpha$, so now the force depends on the orientation of the apparatus. Even if $\alpha = 0$, the magnitude differs from Eq. (12) by an additional factor of 2. (It thereby fortuitously agrees with the result in Ref. [12] for that angle.)

What is going on here? The reason we get different answers in different coordinate systems is that our starting point \([3]\) is not gauge-invariant. Under a coordinate redefinition, which for weak fields is a gauge transformation of $h_{\mu \nu}$ \([15]\), $h_{\mu \nu} \rightarrow h_{\mu \nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$, where $\xi_\mu$ is a vector field, Eq. (3) is invariant only if the stress tensor is conserved, $\partial_\mu T^{\mu \nu} = 0$ (in the weak-field context). Otherwise, there is a change in the action, $\Delta W_g = -2 \int (dx) \xi_\nu \partial_\mu T^{\mu \nu}$. Now in our case (where we make the finite size of the plates explicit, but ignore edge effects on $T^{\mu \nu}$ because $L \gg a$)

$$T^{\mu \nu} = \frac{E_c}{a} \text{diag}(1, -1, -1, 3) \theta(\zeta - \zeta_0 + a/2 - \zeta + \zeta_0) \theta(\eta + L/2) \theta(L/2 - \eta) \theta(\chi + L/2) \theta(L/2 - \chi). \quad (13)$$

Taking the divergence of Eq. (13) gives corresponding $\delta$ functions on the surfaces and leads immediately to

$$\Delta E_o = \frac{6E_c}{a} \int d\eta d\chi \left[ \xi_\zeta (\zeta_0 - a/2, \eta, \chi) - \xi_\zeta (\zeta_0 + a/2, \eta, \chi) \right] - \frac{2E_c}{a} \int d\xi d\chi \left[ \xi_\eta (\zeta, -L/2, \chi) - \xi_\eta (\zeta, L/2, \chi) \right] - (\eta \leftrightarrow \chi). \quad (14)$$

This transformation entirely accounts algebraically for the difference between the forces in isotropic and Schwarzschild coordinates, but it does not yet explain physically why there are two different answers, nor tell us which, if either, is correct.

There seem to be two possible ways to proceed. First, it is clear that the energy-momentum tensor of the complete physical situation must be conserved, and therefore the expression \([3]\) would be gauge-invariant if we included a physical mechanism holding the plates apart against the Casimir attraction. That road probably leads to complicated, model-dependent calculations. The alternative is to find a physical basis for believing that one coordinate system is more realistic than another. Fortunately, that problem apparently has a natural solution. The crux of the difficulty is that the relations between coordinate increments and physical distances depend upon the distance from the gravitating center in the most common coordinate systems.

Of course, a perfect coordinate system is not possible in a curved space, but the kind that comes closest to representing distances accurately all along a timelike worldline is a Fermi coordinate system, the general-relativistic extrapolation of an inertial coordinate frame. Such a system has been given by Marzlin \([18]\) for a resting observer in the field of any static mass distribution. Here we give a simple rederivation for the case at hand. Starting from the isotropic metric \([10]\), first eliminate the constant term by rescaling the coordinates, $t \rightarrow (1 + \frac{GM}{R}) \ t$, $r \rightarrow (1 - \frac{GM}{R}) \ r$, and expand to first order:

$$ds^2 = -(1 + 2 gz) dt^2 + (1 - 2 gz) dr^2. \quad (15)$$

But we need $r$ to measure physical displacements even when $z \neq 0$, so we write

$$x = x' + g x' z', \quad y = y' + g y' z', \quad z = z' + \frac{g}{2} (z'^2 - x'^2 - y'^2). \quad (16)$$

Then to first order in coordinates we obtain the Fermi metric \([1]\). The corresponding gravitational force is therefore given by Eq. (6), after all!

Now we can use the method described above to transform the energy in isotropic coordinates to that in Fermi coordinates. We use Eq. (14) to compute the additional gravitational energy, in terms of the gauge field $\xi_\mu$ that carries us from isotropic coordinates to Fermi coordinates,

$$h^F_{\mu \nu} = h^I_{\mu \nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \quad (17)$$

Here $h^I_{00} = -g_z$, $h^I_{0 \theta} = -g_{\theta z}$, $h^I_{\theta \theta} = -g_{\theta \theta}$, $h^I_{0 \theta} = 0$, $h^I_{\theta \theta} = 0$. The gauge field turns out to be

$$\xi_\zeta = \frac{1}{2} g \left( \frac{1}{2} \zeta^2 \cos \alpha + \zeta \eta \sin \alpha \right) + f(\eta, \chi),$$

$$\xi_\eta = \frac{1}{2} g \left( \zeta \eta \cos \alpha + \frac{1}{2} \eta^2 \sin \alpha \right) + g(\zeta, \chi),$$

$$\xi_\chi = \frac{1}{2} g \left( \zeta \cos \alpha + \eta \sin \alpha \right) \chi + h(\zeta, \eta). \quad (18)$$
where the functions $f$, $g$, and $h$ are irrelevant. Now from Eq. (14) we obtain a $\Delta E_{g}(s_{0})$ that yields an additional force, $\Delta F / A = g\mathcal{E}_{c}$. When this is added to the isotropic force $\langle 12 \rangle$, we obtain the result given in Eq. (19)

$$\frac{F^{I} + \Delta F}{A} = -2g\mathcal{E}_{c} + g\mathcal{E}_{c} = -g\mathcal{E}_{c} = \frac{F^{F}}{A}. \quad (19)$$

The conceptual reason why other coordinates give different answers is that under the virtual displacement involved in defining $\frac{\partial}{\partial s_{0}}$ one is stretching the apparatus as well as moving it. Correcting for the spurious changes in $L$ and $a$ restores the Fermi result in all cases. The importance of distinguishing $a$ from the physical gap was noted by Sorge $\langle 11 \rangle$ in studying a related problem.

As noted above, Calloni et al. $\langle 9 \rangle$ find a result 4 times ours, which is the only result from Ref. $\langle 9 \rangle$ cited in the later paper $\langle 12 \rangle$ with which it shares authors. However, Ref. $\langle 9 \rangle$ states that that force formula has two terms, in the ratio of 3/1, and that only the smaller one is “Newtonian,” or “to be tested against observation.” Our understanding of what that means is the following. Start from Eq. (22) and consider a general coordinate transformation, $x'^{\mu} = x^{\mu} + \delta x^{\mu}$, so that $g_{\mu\nu}(x') = g_{\mu\nu}(x) + \delta g_{\mu\nu}(x)$, where

$$\delta g_{\mu\nu} = \delta x^{\lambda} \partial_{\lambda} g_{\mu\nu} + g_{\alpha\nu} \frac{\partial \delta x^{\alpha}}{\partial x^{\mu}} + g_{\mu\beta} \frac{\partial \delta x^{\beta}}{\partial x^{\nu}}. \quad (20)$$

For a rigid translation, $\delta x^{\lambda}$ is a constant, so only the first term in Eq. (20) is present, which gives the result $\langle 9 \rangle$. However, if we do not make this restriction, we obtain from Eq. (22) (after integration by parts) a surface-term correction to the force:

$$\int_{\Omega} (dx) \sqrt{-g} f_{\lambda} = \frac{\delta W_{m}}{\delta x^{\lambda}} - \int_{\partial \Omega} d\Sigma_{\nu} \sqrt{-g} T^{\nu}_{\lambda}, \quad (21)$$

where the force vector density is $\delta W_{m} = -\nabla_{\nu} T^{\nu}_{\lambda}$ or

$$\sqrt{-g} f_{\lambda} = -\partial_{\nu} (\sqrt{-g} T^{\nu}_{\lambda}) + \frac{1}{2} \sqrt{-g} T^{
u\nu} \partial_{\lambda} g_{\mu\nu}. \quad (22)$$

Note that the surface term identically cancels the first term in $f_{\lambda}$. Now if $\Omega$ refers to all space, the surface term vanishes (as we have shown explicitly). But if $\Omega$ is just the space volume between the plates, and we include this correction for the Fermi metric $\langle 13 \rangle$ for which $\sqrt{-g} = 1 + gz$, we obtain an additional term $-3g\mathcal{E}_{c} \cos \alpha$. Adding this to the previous result $\langle 10 \rangle$, we obtain the result of Ref. $\langle 12 \rangle$ if $\alpha = 0$. However, in general the result depends on the angle between the apparatus and the vertical. Is this consistent with the equivalence principle (the scalar nature of mass)? A similar angle dependence will now occur with the isotropic Schwarzschild metric. Why should one trust the formula $\langle 22 \rangle$ over the more fundamental variational principle when boundaries are present? Omission of the surface term resolves the discrepancy, giving the equivalence-principle result $\langle 11 \rangle$.

After circulation of the present paper as arXiv:hep-th/0702091, Bimonte et al. $\langle 19 \rangle$ responded that (i) our result is implicit in their earlier analysis; (ii) our Eq. (2) implicitly assumes the equivalence principle, which they identify with the conservation law $\nabla_{\nu} T^{\mu\nu} = 0$. We reply: (i) Ref. $\langle 12 \rangle$ states without qualification that the force is 4 times the correct value. (ii) By “equivalence principle” we merely mean that gravity tends to enforce geodesic motion. This is not obviously equivalent to conservation, as shown by the existing uncertainty in the literature. Contrary to statements in Ref. $\langle 12 \rangle$, it is generally accepted in quantum field theory in curved space-time that the renormalized total $T^{\mu\nu}$ must be conserved.

This work is supported in part by Collaborative Research Grants from the US National Science Foundation and in part by the US Department of Energy. It was completed while S.A.F. enjoyed partial support at the I. Newton Institute for Mathematical Sciences, Cambridge. We thank Gabriel Barton, Michael Bordag, Robert Jaffe, and Ron Kantowski for helpful conversations and emails.

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