Black holes in massive gravity as heat engines

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The paper at hand studies the heat engine provided by black holes in the presence of massive gravity. The main motivation is to investigate the effects of massive gravity on different properties of the heat engine. It will be shown that massive gravity parameters and graviton’s mass modify the efficiency of engine on a significant level. Furthermore, it will be shown that it is possible to have the heat engine for non-spherical black holes in massive gravity and we study the effects of topological factor on properties of the heat engine. Surprisingly, it will be shown that the highest efficiency for the heat engine belongs to black holes with hyperbolic horizon, while the lowest one belongs to spherical black holes.

I. INTRODUCTION

Einstein theory of gravity is one of the best theories which has introduced until now. This theory has some interesting predictions, such as the existence of gravitational waves which observed by advanced LIGO in 2016. But there are some phenomena in which such theory can not explain them, precisely. For example we refer the reader to the current acceleration of the universe and the cosmological constant problem. In addition, this theory predicts the existence of massless spin-2 gravitons in which they have two degrees of freedom. However, there have been some arguments regarding the possibility of the existence of massive spin-2 gravitons, such as the hierarchy problem and also brane-world gravity solutions (see Refs. [1, 2] for more details). These show that, despite its correctness, the general relativity is not the final theory of gravitation. On the other hand, the massive gravity includes some interesting properties. One of them is that this theory could explain the accelerated expansion of the universe without considering the dark energy. Also, the graviton behaves like a lattice excitation and exhibits a Drude peak in this theory of gravity. It is notable that, current experimental data from the observation of gravitational waves by advanced LIGO requires the graviton mass to be smaller than the inverse period of orbital motion of the binary system, that is \( m = 1.2 \times 10^{-22} \text{ev}/c^2 \) [3].

Another important reason in order to consider the massive gravity is related to the fact that possibility of the massive graviton help us to understand the quantum gravity effects. With this goal in mind, Fierz and Pauli introduced a massive theory of gravity in a flat background [6]. The problem with this theory was the fact that it suffers from vDVZ (van Dam-Veltman-Zakharov) discontinuity. To resolve this problem, Vainshtein introduced his well known mechanism requiring the theory being considered in a nonlinear framework. Although Vainshtein mechanism was a solution to vDVZ discontinuity, it reveals yet another profound problem of the Fierz and Pauli theory known as Boulware-Deser ghost which signals instability in the theory of interest. In order to avoid such instability, several models of massive theory are introduced by some authors. For example, Bergshoeff, Holm and Townsend proposed one of the ghost-free massive theories in three dimensional spacetime which is known as new massive gravity (NMG) [8]. This theory of massive gravity has been investigated in many literatures [9,10], however, this theory has some problems in higher dimensions. Another interesting class of massive gravity was introduced by de Rham, Gabadadze and Tolley (dRGT) [11,12]. This theory is valid in higher dimensions as well. It is notable that the mass terms in dRGT theory are produced by consideration of a reference metric. The stability of this massive theory was studied and it was shown that this theory enjoys absence of the Boulware-Deser ghost [13,14]. Black hole solutions and their thermodynamical properties with considering dRGT massive gravity have been investigated

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in Refs. 18–21. From the perspective of astrophysics, Katsuragawa et al in Ref. 22, studied the neutron star in the context of this theory and showed that, the massive gravity leads to small deviation from the general relativity (GR). In the cosmological context, phantom crossing and quintessence limit 23, bounce and cyclic cosmology 24, cosmological behavior 25, and other properties of this gravity have been studied by some authors 26–28.

Modification in the reference metric in dRGT theory provides the possibility of introduction of different classes of dRGT like massive theories. Among them, one can point out the one introduced by Vegh which has applications in gauge/gravity duality 29. This theory is similar to dRGT theory with a difference that its reference metric is a singular one. Considering this theory of massive gravity, Vegh in Ref. 29, showed that graviton may behave like a lattice and exhibits a Drude peak. Also, it was pointed out that for arbitrary singular metric, this theory of massive gravity is ghost-free and stable 30. Using this massive theory of gravity, different classes of the charged black hole solutions have been studied in Refs. 31–34. In addition, the existence of van der Waals like behavior in extended lattice and exhibits a Drude peak. Also, it was pointed out that for arbitrary singular metric, this theory of massive gravity leads to small deviation from the general relativity 35–37.

Here, we consider Vegh’s approach from the massive gravity. The action of

\[ X_{\mu \nu} = -c_1 \left( U_1 g_{\mu \nu} - K_{\mu \nu} \right) - c_2 \left( U_2 g_{\mu \nu} - 2U_1 K_{\mu \nu} + 2K_{\mu \nu} \right) - c_3 \left( U_3 g_{\mu \nu} - 3U_2 K_{\mu \nu} + 6U_1 K_{\mu \nu} - 6K_{\mu \nu} \right) - c_4 \left( U_4 g_{\mu \nu} - 4U_3 K_{\mu \nu} + 12U_2 K_{\mu \nu}^2 - 24U_1 K_{\mu \nu}^3 + 24K_{\mu \nu}^4 \right). \]
Black hole thermodynamics has been studied widely and intensively for a long time ever since the seminal work done by Hawking et al. \[56, 57\]. The amazing discovery of the thermodynamical property of black holes helps us to have a deeper understanding of gravity and realize that the gravitational systems have some profound relations to thermodynamical systems. The concept of extended phase space was proposed \[58\] by regarding the cosmological constant \(\Lambda\) as thermodynamical pressure \(P\) and its conjugate quantity as thermodynamic volume \(V\), as

\[
M = H = U + PV, \quad P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi c^2}, \quad V = \frac{\partial M}{\partial P} \bigg|_{s, Q}, \quad \Phi = \frac{\partial M}{\partial Q} \bigg|_{s, P}, \quad S = \int_0^{r+} T^{-1} \left( \frac{\partial M}{\partial r} \right)_{Q, P} dr, \quad (7)
\]

where \(Q\) is the electric charge and \(\Phi\) is its conjugate electric potential, \(S\) and \(T\) stand for the entropy and Hawking temperature on event horizon \(r_+\), respectively. The new first law of black hole thermodynamics in the extended phase space is written as

\[
dM = TdS + VdP + \Phi dQ + \Omega dJ, \quad (8)
\]

where the black hole mass \(M\) should be interpreted as enthalpy rather than internal energy of the gravitational system, which is indicated by the first law \[5\]. For the black hole in \(d = n + 2\) dimensional massive gravity, the first law reads

\[
dM = TdS + VdP + \Phi dQ + \frac{V_n c^2 m^2 r_+^n}{16\pi} dc_1 + \frac{nV_n c^2 m^2 r_+^{n-1}}{16\pi} dc_2 \\
+ \frac{n(n-1)V_n c^3 m^2 r_+^{n-2}}{16\pi} dc_3 + \frac{n(n-1)(n-2)V_n c^4 m^2 r_+^{n-3}}{8\pi} dc_4, \quad (9)
\]

where \(V_n\) is the volume of space spanned by space coordinates and we have viewed the coupling constants \(c_i\) as thermodynamic variables. By employing scaling method, the Smarr relation can be obtained as

\[
(n-1)M = nTS - 2PV + (n-1)\Phi Q - \frac{V_n c c m^2}{16\pi} r_+^n + \frac{n(n-1)V_n c^2 c m^2}{16\pi} r_+^{n-2} + \frac{n(n-1)(n-2)V_n c^3 c m^2}{8\pi} r_+^{n-3}. \quad (10)
\]

Inspired by the first law of black hole thermodynamics, one natural idea is to introduce the concept of traditional heat engines into the black hole thermodynamics with both the thermodynamic pressure and volume already defined in extended phase space \[59, 63\]. Treating AdS black holes as heat engines are thought as a possible way that the useful mechanical work of both static and stationary AdS black holes is allowed to be extracted from heat energy. While the Penrose Process, known as way to extract black hole energy, can only be used for rotating black holes, but Penrose Process can be exerted to black holes in both asymptotically AdS and flat spacetime. What’s more, the black hole energy can also be released and spread in spacetime in form of gravitational waves by the collision of two black holes, which has been detected recently \[3\].

II. THERMODYNAMIC CYCLE AND HEAT ENGINES

The entropy of these black holes is obtained through the area law \[66, 67\] in the following form

\[
S = \pi r_+^2. \quad (11)
\]

Using the new interpretation of the cosmological constant as thermodynamical pressure \((P \propto \Lambda)\) \[58, 68, 72\], one can replace the cosmological constant with following relation

\[
P = -\frac{\Lambda}{8\pi}, \quad (12)
\]

in which, the thermodynamic volume, \(V\) (conjugating to pressure), is given by

\[
V = \frac{4\pi}{3} r_+^3. \quad (13)
\]

Here, we are interested in classical heat engine. It is notable that a heat engine is a physical system that takes heat from warm reservoir and turns a part of it into the work while the remaining is dedicated to cold reservoir (see Fig. \[1\]). In order to calculate work done by the heat engine and given the equation of state, one can use the \(P - V\) diagram specified the heat engine which forms a closed path. For a thermodynamics cycle, one may extract mechanical work via the \(PdV\) term in the first law of thermodynamics as \(W = Q_H - Q_C\) (for a thermodynamic
cycle, the internal energy changes is zero ($\Delta U = 0$), so the first law of thermodynamics $\Delta U = \Delta Q - W$ reduces to $W = \Delta Q = Q_H - Q_C$, where $Q_H$ is a net input heat flow, $Q_C$ is a net output flow and $W$ is a net output work), and so we have

$$Q_H = W + Q_C, \quad (14)$$

where the efficiency of heat engine is defined as

$$\eta = \frac{W}{Q_H} = 1 - \frac{Q_C}{Q_H}, \quad (15)$$

It is known that the heat engine depends on the choice of path in the $P - V$ diagram and possibly the equation of state of the black hole in question. It is notable that, some of the classical cycles involve a pair of isotherms at temperatures $T_H$ and $T_C$, in which $T_H > T_C$. For example, in the Carnot cycle, there is a pair of isotherms with different temperatures in which this cycle has maximum efficiency and it is described as $\eta = 1 - \frac{T_C}{T_H}$. In this cycle, there is an isothermal expansion when the system absorbs some heat and an isothermal compression during expulsion of some heat of the system. Using different methods, one can connect these two systems to each other. The first method is isochoric path, like classical Stirling cycle, and the second one is adiabatic path, like classical Carnot cycle. Therefore, the form of path for the definition of cycle is important. As we know, for the static black holes, the entropy $S$ and the thermodynamic volume $V$ are related by Eqs. (11) and (13) as $S = \pi r_+^2 = \pi \left(\frac{3}{4\pi} V\right)^{2/3}$. It means that adiabatic and isochores are the same, so Carnot and Stirling methods coincide with each other. Therefore, the efficiency of cycle can be calculated easily.

So along the upper isotherm (Fig. (2)) and by using of Eqs. (11) and (13), we have the following heat flow

$$Q_H = T_H \nabla S_{1 \rightarrow 2} = \pi T_H \left(\frac{3}{4\pi}\right)^{2/3} \left(V_2^{2/3} - V_1^{2/3}\right), \quad (16)$$

and also along the lower isotherm (Fig. (2)) and by using Eqs. (11) and (13), isotherm the heat flow will be

$$Q_C = T_C \nabla S_{3 \rightarrow 4} = \pi T_C \left(\frac{3}{4\pi}\right)^{2/3} \left(V_3^{2/3} - V_4^{2/3}\right), \quad (17)$$

which according to Fig. (2), we have $V_1 = V_4$ and $V_2 = V_3$, so the efficiency becomes

$$\eta = 1 - \frac{Q_C}{Q_H} = 1 - \frac{T_C}{T_H}. \quad (18)$$

The heat engine for black holes was proposed by Johnson in 2014 [59]. Using the concepts introduced by Johnson, the heat engines provided by other types of black holes have been investigated. For example; the heat engine for Kerr AdS and dyonic black holes [60], Gauss-Bonnet [61], Born-Infeld AdS [62], dilatonic Born-Infeld [63], BTZ [64] and polytropic black holes [65] have been studied.
III. CHARGED BLACK HOLES IN MASSIVE GRAVITY AS HEAT ENGINES

In this section, the 4-dimensional static charged black holes in the context of massive gravity with adS asymptotes are introduced. For this purpose, we consider a metric of 4-dimensional spacetime in the following form

\[ ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) , \]  

(19)

with the following reference metric \[ f_{\mu\nu} = \text{diag}(0,0,c^2,c^2 \sin^2 \theta), \]  

(20)

where in the above equation, \( c \) is a positive constant. Using the reference metric introduced in Eq. (20) for 4-dimensional spacetime, \( U_i \)’s are in the following forms \[ U_1 = 2c r, \quad U_2 = 2c^2 r^2, \quad U_3 = 0, \quad U_4 = 0. \]

Using the gauge potential ansatz \( A_\mu = h(r) \delta_\mu^t \) in electromagnetic equation (5) and considering the metric (19), the metric function \( f(r) \) is obtained in Refs. [31, 32] as

\[ f(r) = 1 - \frac{m_0}{r} - \frac{\Lambda}{3} r^2 + \frac{q^2}{r^2} + m^2 \left( \frac{cc_1}{2} r + c^2 c_2 \right) , \]  

(21)

in which \( q \) and \( m_0 \) are integration constants related to the electrical charge and the total mass of black holes, respectively.

The temperature of these black holes could be obtained by employing definition of Hawking temperature which is based on the surface gravity on the outer horizon, \( r_+ \), and by considering Eq. (21), it will be

\[ T = \frac{1}{4\pi \sqrt{r_+}} \left[ 1 - \Lambda r_+^2 + \frac{q^2}{r_+^2} + m^2 \left( \frac{cc_1}{2} r + c^2 c_2 \right) \right] , \]  

(22)

which by using Eqs. (11)-(13), we can rewrite the temperature in terms of \( S \) and \( P \) as

\[ T = \frac{1}{4\sqrt{\pi S}} \left[ 1 + 8SP - \frac{\pi q^2}{S} + m^2 cc_1 \sqrt{\frac{S}{\pi}} + m^2 c^2 c_2 \right] . \]  

(23)

Considering the Eq. (23), one can write the pressure as a function of \( T \) and \( S \) in the following form

\[ P = \frac{\sqrt{\pi T}}{2\sqrt{S}} - \frac{m^2 c^2 c_2 + 1}{8S} + \frac{\pi q^2}{8S^2} + \frac{m^2 cc_1}{8\sqrt{\pi S}}. \]  

(24)
On the other hand, there are two different heat capacities for a system, the heat capacity at constant pressure and the heat capacity at constant volume. The heat capacity can be calculated by the standard thermodynamic relations, as

\[ C_V = T \frac{\partial S}{\partial T} \mid_V \quad \& \quad C_P = T \frac{\partial S}{\partial T} \mid_P. \] (25)

Considering the fact that entropy is a regular function of the thermodynamic volume \( S = \pi r^2 = \pi \left( \frac{3V}{4\pi} \right)^{2/3} \), the heat capacity at constant volume will vanish, \( C_V = 0 \). The heat capacity at constant pressure is calculated as

\[ C_P = T \frac{\partial S}{\partial T} \mid_P = \frac{2S \left[ 8S^2 P + \left(m^2 c^2 c_2 + 1\right) S - \pi q^2 + \frac{m^2 c_1 c_2}{\sqrt{\pi}} S^{3/2} \right]}{8S^2 P - \left(m^2 c^2 c_2 + 1\right) S + 3\pi q^2}. \] (26)

Evidently, due to presence of the massive gravity’s parameter, obtained \( C_P \) differs from the usual Reissner-Nordström one. Using the relation between \( S \) and \( V \), we can rewrite the equation (24) versus \( T \) and \( V \) as

\[ P = \frac{T - \frac{m^2 c c_1}{4\pi}}{V^{1/3}} \left( \frac{\pi}{6} \right)^{1/3} - \frac{\left(m^2 c^2 c_2 + 1\right)}{2\pi V^{2/3}} \left( \frac{\pi}{6} \right)^{2/3} + \frac{2q^2}{\pi V^{4/3}} \left( \frac{\pi}{6} \right)^{4/3}. \] (27)

It is notable that, in the absence of massive gravity \( (m^2 = 0) \), the above equation reduces to the following form \[59\]

\[ P = \frac{T}{V^{1/3}} \left( \frac{\pi}{6} \right)^{1/3} - \frac{1}{2\pi V^{2/3}} \left( \frac{\pi}{6} \right)^{2/3} + \frac{2q^2}{\pi V^{4/3}} \left( \frac{\pi}{6} \right)^{4/3}. \] (28)

In order to highlight the similarity between usual the heat engine and our system under consideration, we plot \( P - V \) diagrams for the above obtained equation in Fig. [3], by considering fixed quantities for \( m^2 \) (the massive parameter), \( c, c_1, c_2 \) and \( q \).

Left panel of Fig. [3] shows that there are three areas for black holes which encounter with a phase transition. The first area is related to high pressure (continuous line in left panel in Fig. [3]). The black hole in this area has small radius and it is called small black hole (SBH) region. The second area is related to unstable phase (dotted line in left panel in Fig. [3]). The third area is related to low pressure case (dashed line in left panel of Fig. [3]). In this area, black holes have large radius, and it is known as large black hole (LBH) region. On the other hand, for the temperatures more than critical temperature \((T > T_c)\), the phase transition and the second area are disappeared (see right panel of Fig. [3] for more details). In phase transition point, the transition takes place between SBH to LBH.

The opposite could also take place in the case black holes horizon shrinking (the LBH to the SBH). This enables us to define a classical cycle for black holes. It is notable that the LBH lose \( Q_C \) amount of heat along isothermal
contraction and the SBH absorbs $Q_H$ amount of heat along isothermal expansion. Such transition also exists at high pressure. On the other hand, an explicit expression for $C_P$ would suggest that there should be a new engine which includes two isobars and two isochores/adiabatic similar to Fig. 4. For this purpose, we can consider a rectangle cycle in the $P-V$ plane in which this rectangle consists of two isobars ($1 \to 2$ and $3 \to 4$) and two isochores ($2 \to 3$ and $4 \to 1$), see Fig. 4 for more details. Also, a possible scheme for this heat engine involves specifying values of temperature where $T_2 = T_H$ and $T_4 = T_C$, in which $T_H$ and $T_C$ are temperatures of the warm and the cold reservoirs, respectively. According to the fact that the paths of $1 \to 2$ and $3 \to 4$ are isobars, we find $P_1 = P_2$ and $P_3 = P_4$. Now, we can calculated the work which is done in this cycle as

$$W = \oint P dV$$  \hspace{1cm} (29) $$W_{total} = W_{1\to2} + W_{2\to3} + W_{3\to4} + W_{4\to1} = W_{1\to2} + W_{3\to4} = P_1 (V_2 - V_1) + P_4 (V_4 - V_3).$$ \hspace{1cm} (30) $$\text{It is notable that the works which are done in the paths of } 2 \to 3, \text{ and } 4 \to 1 \text{ are isochores, therefore, these terms are zero. Using Eqs. (11) and (13), we have}$$ $$W_{total} = \frac{4}{3\sqrt{\pi}} (P_1 - P_4) \left( S_{3/2}^2 - S_{1/2}^2 \right).$$ \hspace{1cm} (31)$$ $$\text{Also, the upper isoobar (Fig. 4) will give the net inflow of heat which is } Q_H, \text{ so we have}$$ $$Q_H = \int_{T_1}^{T_2} C_P (P_1, T) dT.$$ \hspace{1cm} (32)$$ $$\text{Here, we want to obtain the efficiency of this cycle, so we use two approximations:}$$ Case I : In limit of high pressure and considering temperature (Eq. (23) and the heat engine (Eq. (26)), we have

$$T \sim 2P \sqrt{\frac{S}{\pi}},$$ \hspace{1cm} (33)$$ $$C_P \sim 2S,$$ \hspace{1cm} (34)$$ $$\text{which by using Eq. (33), the entropy will be obtained as } S = \frac{\pi T^2}{4P^2}, \text{ and by replacing it in Eq. (34), we can find}$$ $$C_P = \frac{\pi T^2}{2P^2}.$$ \hspace{1cm} (35)$$ $$\text{This leads to obtain the } Q_H \text{ as}$$ $$Q_H = \frac{\pi}{6P_1^2} (T_2^3 - T_1^3) = \frac{4}{3\sqrt{\pi}} P_1 \left( S_{3/2}^2 - S_{1/2}^2 \right).$$ \hspace{1cm} (36)
Considering Eqs. (31) and (36), we can calculate the efficiency of this cycle as

$$\eta = \frac{W}{Q_H} = 1 - \frac{P_4}{P_1},$$  \hspace{1cm} (37)$$
so we have

$$\eta = 1 - \frac{T_C}{T_H}.$$  \hspace{1cm} (38)

**Case II:** In limit of high pressure and considering the path 1 → 2 (remembering that in the path 1 → 2, the entropy increases, we can omit terms in which the entropy is in the denominator) in order to obtain $Q_H$, the temperature will be

$$T = 2P\sqrt{\frac{S}{\pi} + \frac{m^2c_1}{4\pi}},$$  \hspace{1cm} (39)$$
which by using the above equation, we have

$$S = \frac{\pi T^2}{4P^2} + \frac{m^2c_1T}{8P^2} + \frac{m^4c_1^2}{64\pi P^2}. $$ \hspace{1cm} (40)$$

On the other hand, according the second approximation, we can use of the first term of the heat capacity in which it will be the similar to Eq. (31). Using Eqs. (32) and (34), one can obtain $Q_H$ in the following form

$$Q_H = \frac{\pi(T_2^2 - T_1^2)}{6P^2} + \frac{m^2c_1(T_2^2 - T_1^2)}{8P^2} + \frac{m^4c_1^2}{32\pi P^2}(T_2 - T_1).$$ \hspace{1cm} (41)$$

Now, we can investigate the efficiency of cycle by using Eqs. (31) and (41) which leads to

$$\eta = \frac{W}{Q_H} = \left( 1 - \frac{P_4}{P_1} \right) \left\{ 1 - \frac{3m^2c_1}{4\sqrt{\pi P_1}(\sqrt{S_2} - \sqrt{S_1})} + \frac{3m^4c_1^2}{8\pi P_1^2(S_2 - S_1)} + \mathcal{O}\left( \frac{1}{P_1^3} \right) \right\},$$ \hspace{1cm} (42)$$

$$= \left( 1 - \frac{T_C}{T_H} \right) \left\{ 1 - \frac{3m^2c_1}{4\sqrt{\pi T_H}(\sqrt{S_2} - \sqrt{S_1})} + \frac{3m^4c_1^2}{8\pi T_H^2(S_2 - S_1)} + \mathcal{O}\left( \frac{1}{T_H^3} \right) \right\}. $$ \hspace{1cm} (42)$$

Evidently, the massive parameter affects the efficiency of cycle for the obtained black holes in massive gravity. It is notable that, when $m = 0$, the efficiency of cycle reduces to the efficiency of cycle for obtained black holes in Einstein gravity.

**IV. EXACT EFFICIENCY FORMULA**

Accordingly, if the thermodynamical cycle does not include the phase transition area in the $P - V$ plane which leads to every temperature at the fixed pressure only corresponds to one positive horizon radius $r_+$ (or $V$) such that we can get the exact efficiency formula. In this case, we do not have to calculate the approximate efficiency formula by appealing to high temperature and high pressure limits. However when the phase transition included in the cycle, the approximation method is necessary. For simplicity, we just put the thermodynamical cycle in the third area, i.e. stable large black hole region in Fig. 3 to avoid the phase transition region, and we can obtain $Q_H$ as

$$Q_H = \int_{T_1}^{T_2} C_P(P_1, T)dT = \int_{r_1}^{r_2} C_P(P_1, T)\frac{dT}{dr}dr = Q_{H2} - Q_{H1},$$ \hspace{1cm} (43)$$
where

$$Q_{H1} = \frac{6\pi^{3/2}q^2 + 3cc_1m_2S_1^{3/2} + 2\sqrt{\pi}S_1(3 + 3c^2c_1m^2 + 8P_1S_1)}{12\pi \sqrt{S_1}},$$ \hspace{1cm} (44)$$
$$Q_{H2} = \frac{6\pi^{3/2}q^2 + 3cc_1m_2S_2^{3/2} + 2\sqrt{\pi}S_2(3 + 3c^2c_1m^2 + 8P_1S_2)}{12\pi \sqrt{S_2}}.$$

(45)
The efficiency is calculated as
\[ \eta = \frac{W}{Q_H} = \left(1 - \frac{P_4}{P_1}\right) \times \frac{1}{1 + \frac{3(\sqrt{S_1 S_2} - \pi q^2)}{8 \sqrt{S_1 S_2 (1 + S_2 + \sqrt{S_1 S_2}) P_1} + \frac{3c m^2 (2\sqrt{c c_2 \sqrt{S_2}} + c_1 (S_2 + \sqrt{S_1 S_2}))}{16 \sqrt{\pi S_2 (1 + S_2 + \sqrt{S_1 S_2}) P_1}}. \] (46)

Here, we focus on the large volume branch of solutions and therefore neglect \( q \) to leading order. This leads to
\[ \eta = \left(1 - \frac{P_4}{P_1}\right) \left(1 - \frac{1}{P_1} \left(\frac{3c m^2 (2\sqrt{c c_2 \sqrt{S_2}} + c_1 (S_2 + \sqrt{S_1 S_2}))}{16 \sqrt{\pi S_2 (1 + S_2 + \sqrt{S_1 S_2})}}\right) - \frac{3}{8 P_1} \left(\frac{S_2^2 - S_1^2}{S_2^2 - S_1^2}\right) + O \left(\frac{1}{P_1^2}\right) \right). \] (47)

It is notable that when graviton mass is \( m = 0 \), the efficiency calculated in (47) reduces to efficiency obtained in Ref. [59]. The Carnot efficiency is
\[ \eta_c = 1 - \frac{T_4(S_1, P_1)}{T_2(S_2, P_1)} = 1 - \frac{1}{4\sqrt{\pi S_1}} \left[1 + 8 S_1 P_4 - \frac{\pi c^2}{S_1} + m^2 c c_1 \sqrt{\frac{S_2}{\pi} + m^2 c^2 c_2}\right], \] (48)
where \( T_2 \) and \( T_4 \) refer to the highest and lowest temperatures in the thermodynamical cycle, respectively. This fact holds if the thermodynamical cycle is not putted in the second area (i.e. unstable black hole phase) in Fig. (5). From the efficiency formula obtained above, we can see that the graviton mass can affect the efficiency, but whether the mass \( m \) will improve the efficiency or reduce it, depends on the choice of \( c, c_1 \) and \( c_2 \) parameters. In order to probe how these parameters influence the efficiency, we plot some figures in the following discussion. We plot \( \eta \) and \( \eta/\eta_c \)

![FIG. 5: The two figures above are plotted at 3 different parameters \( c_2 \), here we take \( P_1 = 4, P_4 = 1, S_1 = 1, q = 1, c = c_1 = 1 \) and graviton mass \( m = 1 \).

in Fig. (5) under the change of \( S_2 \) which shows that with the growth of \( S_2 \), the efficiency will decrease to a minimal value and then monotonically increase to a maximum value obtained by
\[ \lim_{S_2 \to \infty} \eta = \lim_{S_2 \to \infty} \frac{\eta}{\eta_c} = 1 - \frac{P_4}{P_1}, \] (49)
and increasing \( c_2 \) will lead to a lower efficiency, \( \eta \). The behavior of curves of \( \eta \) is different from the figures plotted in Ref. [78] which demonstrates that the efficiency influenced by quintessence field would only monotonically decrease with the growth of \( S_2 \). From the right panel of Fig. (5), we can see that the ratio between efficiency \( \eta \) and Carnot efficiency \( \eta_c \) will monotonically decrease with the growth of \( S_2 \), and increasing \( c_2 \) parameter corresponds to a higher ratio, although for bigger \( c_2 \), the \( \eta \) will be lower.

In Fig. (6), we can see that both the efficiency \( \eta \) and \( \eta/\eta_c \) will monotonously increase with the growth of pressure \( P_1 \), and \( \eta \) will infinitely approach the maximum efficiency, i.e. the Carnot efficiency allowed by thermodynamics laws. In fact, in the high pressure limit, we have
\[ \lim_{P_1 \to \infty} \eta = \lim_{P_1 \to \infty} \eta/\eta_c = 1. \] (50)
FIG. 6: The two figures above are plotted at 3 different parameters \( c_2 \), here we take \( P_1 = 1, S_1 = 8, S_2 = 12, q = 1, c = c_1 = 1 \) and graviton mass \( m = 1 \).

Previously, it was shown that it is possible to define van der Waals like phase transition for non-spherical black hole solutions, and hence horizon-flat and hyperbolic black holes \(^{[38]}\). Since the topology of the horizon of black hole in massive gravity could be sphere, Ricci flat or hyperbolic, corresponding to \( k = 1, 0 \) or \(-1\), respectively, it would be interesting to investigate the efficiency calculated on the horizon with these three different topologies and then make a comparison. The temperature for this topological black hole is \[^{[38]}\] 

\[
T = \frac{1}{4\pi r_+} \left[ k - \Lambda r_+^2 - \frac{q^2}{r_+^2} + m^2 c c_1 r_+ + m^2 c^2 c_2 \right].
\] (51)

Based on Eq. (51), we can get the efficiency

\[
\eta_k = \frac{W}{Q_H} = \left( 1 - \frac{P_4}{P_1} \right) \times \frac{1}{1 + \frac{3(k\sqrt{S_1 S_2} - \pi q^2)}{8\sqrt{S_1 S_2(S_1 + S_2 + \sqrt{S_1 S_2})P_1}} + \frac{3cm^2(2c c_2 \sqrt{\pi S_2} + c_1(S_2 + \sqrt{S_1 S_2}))}{16\sqrt{\pi S_2(S_1 + S_2 + \sqrt{S_1 S_2})P_1}}},
\] (52)

It is obvious that when all the other variables are fixed, we have

\[
\eta_{-1} > \eta_0 > \eta_1,
\] (53)

which means that the efficiency of black hole engines in massive gravity with hyperbolic horizon is higher than that of black holes with flat horizon. In addition, one finds the spherical black holes have the lowest efficiency.

V. CONCLUSIONS

In this paper, we have considered charged black holes in the presence of massive gravity. It was shown that the temperature and pressure of these black holes are functions of the massive gravity. This indicated the dependency of thermal phase transition points on the massive gravity and its parameters as well.

Next, by using the phase transition points, it was shown that it is possible to define a cyclic thermodynamical behavior consisting two isobars and two isochores. This cycle could be interpreted as a heat engine. In other words, it was possible to show that charged black holes in the massive gravity admits a heat engine. Interestingly, it was shown that contrary to other cases, it is possible to define heat engine for non-spherical black holes as well. In other words, due to contributions of the massive gravity, the heat engine could be constructed for horizon flat and hyperbolic black holes and it is not limited only to spherical black holes.

The expressions extracted for efficiency and heat were shown to be massive gravity dependent. The effects of massive gravity were highly dependent on the sign and values of the massive coefficients, \( c_i \)'s. Especially for \( c_2 \), it was shown that efficiency is a decreasing function of this parameter. If we consider only positive and non-zero values of the \( c_i \), one can conclude that efficiency of heat engine in the massive gravity is smaller comparing to the absence of the massive gravity. This highlights the contributions of massive gravity on the properties of heat engine.
To have a better understanding of the massive gravity’s impacts on the heat engine efficiency, we plotted Figs. 5 and 6 of the efficiency under the change of entropy $S_2$ (or larger black hole with thermodynamical volume $V_2$) and pressure $P_1$ at the fixed parameters $c_i$’s which can be chosen to promote or reduce efficiency, respectively. As it is shown by Fig. 5, under the chosen parameters $c_i$, we find that with the grow of volume difference $\Delta V = V_2 - V_1$ or $\Delta S = S_2 - S_1$ between the smaller black hole with thermodynamical volume $V_1$ (or $S_1$) and the larger black hole with volume $V_2$ (or $S_2$), the efficiency $\eta$ of the thermodynamics cycle will decrease to a minimal value at first and then gradually increase from the minimal $\eta$ to the limit of $\eta = 1 - \frac{P_1}{\eta c}$ when $\Delta V$ (or $\Delta S$) goes to infinity. For the ratio $\frac{\Delta V}{\eta c}$, it will monotonously decrease with the grow of $\eta$ and it is interesting to note that the limits of $\frac{\Delta V}{\eta c}$ is equal to the limits of $\eta$ which depends on the pressures $P_1$ and $P_2$ as it was shown by Eq. (19). On the other hand, the efficiency $\eta$ is lower but the ratio $\frac{\Delta S}{\eta c}$ is higher when the thermodynamic cycle has a bigger $c_2$. Furthermore, Fig. 6 showed that for the bigger pressure difference $\Delta P = P_1 - P_2$, the efficiency $\eta$ and ratio $\frac{\Delta S}{\eta c}$ will be larger. When $\Delta P \to \infty$, $\eta$ and $\frac{\Delta S}{\eta c}$ will infinitely approach value 1 which is not allowed to be exceeded.

One of the possibilities provided for black holes in the presence of massive gravity is existence of van der Waals like behavior for non-spherical black holes. Such possibility was not reported for other black holes in the presence of different matter fields and gravities. Using this possibility, we were able to have a heat engine for non-spherical black holes which was not observed before. We also conducted a study regarding the effects of topological structure of the black hole on efficiency of the heat engine. Interestingly, it was shown that the smallest efficiency for heat engine belongs to spherical black holes while the highest one was provided for hyperbolic black holes.

At last but not least, according to the AdS/CFT correspondence, it would also be of interest to have a deeper holographic understanding of the black hole heat engines in AdS space. It has been argued that such heat engines may have interesting holographic implications because the engine cycle represents a journey through a family of holographically dual large $N$ field theories as explained in Ref. [61]. On the other hand, considering that the energy of the black hole can be extracted by the way of transferring heat to mechanical work, the black hole heat engine may be regarded as a possible energy source for the high energy astrophysical phenomena near the black holes. We should point out that the topological black holes as heat engines in higher dimensional massive gravity are also worthy to be investigated, which we leave this issue for future work.

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