THE INDEFINITE METRIC OF R. M. RUGALA AND THE GEOMETRY OF THE THERMODYNAMICAL PHASE SPACE

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Abstract. We study the indefinite metric $G$ in the contact phase space $(\mathcal{P};\mathcal{G})$ of a homogeneous thermodynamic system introduced by R. M. Rugala. We calculate the curvature tensor, Killing vector fields, second fundamental form of Legendre submanifolds of $\mathcal{P}$ - constitutive surfaces of different homogeneous thermodynamic systems. We establish an isomorphism of the space $(\mathcal{P};\mathcal{G})$ with the Heisenberg Lie group $H_n$ endowed with the right invariant contact structure and the right invariant indefinite metric. The lift $\mathcal{G}$ of the metric $G$ to the symplectization $\mathcal{P}'$ of contact space $(\mathcal{P};\mathcal{G})$, its curvature properties, and its Killing vector fields are studied. Finally we introduce the "hyperbolic projectivization" of the space $(\mathcal{P};\mathcal{G})$ that can be considered as the natural compactification of the thermodynamic phase space $(\mathcal{P};\mathcal{G})$.

1. Introduction.

Geometric methods in the study of homogeneous thermodynamic systems pioneered by J. Lee and C. Caratheodory. They were further developed in the works of R. Heilmann, R. M. Rugala, P. Salamon and their collaborators, in the dissertations of H. Heine and L. Benayoun to mention just a few. Thermodynamic metrics (TD-metrics) in the form of the Heessian of a thermodynamic potential were explicitly introduced by F. Weinhold and, from a different point of view, by G. Ruppeiner.

Deeper studies by P. Salamon and his collaborators, P. M. Rugala and H. Janyszek (see review papers) clarified principal properties of thermodynamic metrics, relations between different TD-metrics, and their relations to the contact structure of equilibrium thermodynamic phase space.

G. Ruppeiner (see review papers) has developed a covariant thermodynamic fluctuation theory based on the Riemannian metric defined by the second moment of entropy with respect to the fluctuations and related the curvature of this metric to the correlation volume near the critical point.

In his work R. M. Rugala introduced the pseudo-Riemannian (inde nite) metric $G$ of signature $(n + 1;n)$ in the thermodynamic contact space $(\mathcal{P};\mathcal{G})$ inducing TD-metrics on the constitutive surfaces defined by different thermodynamic potentials (Weinhold metric for the internal energy and Ruppeiner metric for the entropy).

In the present work we study geometric properties of the metric $G$ and its relation to the contact structure of the thermodynamic phase space (TPS) $(\mathcal{P};\mathcal{G})$.

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In Sec. 2 we introduce the model thermodynamical phase space $P$ with its contact structure. In Section 3 we represent TPS as the 1-jet bundle of the trivial line bundle (of 2-forms) over the manifold $X$ of extensive variables of TPS. This allows us to define Legendre submanifolds (constitutive surfaces of di erent thermodynamic systems with given TPS) corresponding to a thermodynamic potential $2 C^1(X)$, in their canonical representation (see).

In Section 4 we recall the de nitions of the Weil-Petersson and Kupkeiher metrics.

In Section 5 we introduce the Wigner-Mugnoli metric $G$, a canonical nonholonomic frame $(X;P)$ (with contact structure). Then we show the compatibility of the metric $G$ with the contact structure in a sense that is natural, though di erent from the conventional de nitions used for Riemannian metrics (see).

In Section 6 we determine the Levi-Civita connection of the metric $G$ and work out the formulas (6A) for covariant derivatives of the vector elds of the frame $(X;P;G)$.

In Section 7 we nd the Ricci tensor, scalar curvature $R(G) = \frac{1}{2} G$ of metric $G$, and the curvature transformation $R(X;Y)$ in terms of the frame $(X;P;G)$.

In Section 8 we determine the sectional curvatures of the planes generated by couples of vectors of the frame $(X;P;G)$ These curvatures (whenever they are ned) are all zero except for the plane $\Pi = \theta_i$, which has curvature $\frac{1}{2}$.

In Section 9 the Lie algebra $so_0$ of Killing vectors of metric $G$ is determined. It is shown that $so_0$ is the Lie algebra $gl(n;R)$, the semi-direct product of the linear Lie algebra $gl(n;R)$ embedded into the symplectic Lie algebra $sp(n;R)$ (with generators $f = p_i \theta_i; A_i = \theta_i; x^j \theta_i, g$) and of the Heisenberg Lie algebra $h_n$ with generators $f = \theta_i; A_i = \theta_i, x^j \theta_i; B_j = \theta_i, g$ and commutative relations (9.3).

In Section 10 we calculate the second fundamental form of constitutive surfaces (Legendre submanifolds) of contact manifold $P$.

In Section 11 the constitutive hypersurface $x^0 + \cdots + x^n = 0$, de ned by the homogeneity condition of thermodynamic potentials (see), is introduced and studied.

In Section 12 we establish the isomorphism of the PPS $(P);G)$ with the Heisenberg group $H_n$ endowed with the right invariant contact structure and right invariant nd metric $M$; in particular, we prove the following

Theorem 1. The di e omorphism de ned by

$$g = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & a & 0 & z \\ z & 0 & 1 & a \end{bmatrix}$$

determines an isomorphism of the "thermodynamic constituent contact manifold $(P);G)$ with the Heisenberg group $H_n$ endowed with the right invariant contact from $P$ and the right invariant metric $G$ of signature $(n + 1; n)$.

In Part II of the paper we study the symplectization $(\mathcal{P};G)$ of the contact manifold $(P);G)$ with the Wigner-Mugnoli metric $G$. We lift $G$ to the Wigner-Mugnoli (of signature $(n+1,n+1)$) metric $\tilde{G}$, calculate the Levi-Civita connection, the Ricci tensor, and the scalar curvature $R(\tilde{G}) = \frac{1}{2} (n+1)(n+2)$ of $\tilde{G}$. We get, in particular, that $Ric(\tilde{G}) = \frac{n+2}{2} G$, i.e. $\tilde{G}$ is an nd metric (pseudo-Riemannian) Einstein metric.

In Section 16 and in the Appendix we determine the Lie algebra $so_{\tilde{G}}$ of the Killing vector elds of metric $\tilde{G}$. We prove that $so_{\tilde{G}} \cong sl(n+2;R)$. 
In Section 19 we show that the contact metric space \((\mathbb{P}; ; ; \Theta)\) endowed with the almost contact structure given by the \((1,1)\)-tensor (see [21])
\[
\begin{pmatrix}
0 & p_1; \cdots; p_n \\
0 & 0_n & I_n \\
0 & I_n & 0_n
\end{pmatrix}
\]
is a Sasakian metric in that the almost complex structure \(J\) defined on the space \(\mathbb{P}\) by the formula
\[
J(X; \Theta p_i) = (X; f \Theta; X)\Theta_i
\]
for any function \(f \in C^1(\mathbb{P})\) is integrable.

In Section 20 we construct the symplectic form of the "positive quadrant"
\[
\mathbb{P}^+ = \{(p_1; x_i) | 2 \mathbb{P}; p_1 > 0; \Theta^x; \Theta^p\}
\]
of the manifold \(\mathbb{P}\) is isomorphic to the product \(\mathbb{Q}^{\infty} = \bigoplus_{i=0}^{n} A_1^i\) of \((n+1)\) copies of the affine group
\[
A_1^i = \left\{ e^{c} \Theta; c \right\}_{c \in \mathbb{R}}
\]
endowed with the symplectic structure generated by the right invariant 1-form
\[
\omega = \left( \mathbb{dz}_i - \mathbb{z}_i \mathbb{h} \Theta \mathbb{dz}_i \right)_{\Theta^x; \Theta^p}.
\]
Finally, in Section 21 we denote the "hyperbolic projectivization" \((\mathbb{P}'; \mathbb{P}^n_{2n+1}(\mathbb{R}); \tilde{\Theta}^\prime)\)
of the space \((\mathbb{P}'; \tilde{\Theta}^\prime)\) - a natural compactification of the TPS space \((\mathbb{P}; ; \Theta)\): This projectivization will be used for the study of geometric properties of thermodynamic systems in the continuation of this work.

Part I.

2. The contact structure of homogeneous thermodynamics.

A phase space of the Homogeneous Thermodynamics (thermodynamic phase space, or TPS) is the \((2n+1)\)-dimensional vector space \(\mathbb{P} = \mathbb{R}^{2n+1}\) endowed with the standard contact structure
\[
= dx^0 + \sum_{i=1}^{n} p_i dx^i; \quad (2.1)
\]
The horizontal distribution of this structure is generated by two families of vector fields: \(P_i; X_i\)
\[
D_m = \langle P_i; \Theta p_i; X_i = \Theta x_i; p_i \Theta x_i \rangle ;
\]
The 2-form
\[
! = \Theta = \sum_{i=1}^{n} dp_i \wedge dx^i
\]
is a nondegenerate, symplectic form on the distribution \(D\).
The Reeb vector field, uniquely defined as the generator of \(\ker(\Theta)\) satisfying \((\Theta) = 1\), is simply
\[
= \Theta x_i.
3. Gibbs space. Legendre surfaces of equilibrium.

Constitutive surfaces of concrete thermodynamic systems are determined by their "constitutive equations", which, in their fundamental form, determine the value of a thermodynamic potential \( x^i = E(x^i) \) as the function of extensive variables \( x^i \). Dual extensive variables are determined then as the partial derivatives of the thermodynamic potential by the extensive variables: \( p_i = \frac{\partial E}{\partial x^i} \).

Thus, a constitutive surface represents the Legendre submanifold (maximal integral submanifold) \( E \) of the contact form projecting ideomorphically to the space \( X \) of variables \( x^i \). Space \( Y \) of variables \( x^i; \dot{x}^i \), i = 1, ..., n is, sometimes, named the Gibbs space of the thermodynamic potential \( E(x^i) \). Thermodynamic phase space \( (\mathbb{P}; \omega) \) (or, more precisely, its open subset) appears as the first jet space \( J^1(\mathbb{P}; Y) \) of the (trivial) line bundle \( \gamma = Y \times X \). Projection of \( E \) to the Gibbs space \( Y \) is the graph \( E \) of the constitutive law \( E = E(x^i) \).

Another choice of the thermodynamic potential together with the n-tuple of extensive variables leads to another representation of an open subset of TPS \( P \) as the 1-jet bundle of the corresponding Gibbs space. It is known that any Legendre submanifold of a contact form can locally be presented in this form for some choice of the set of extensive variables and of thermodynamic potential as the function of these variables.

We will be using a local description of Legendre submanifolds which takes a slightly different form. Let \( P^{2n+1} \) be a contact manifold. The following result characterizes (locally) all Legendre submanifolds \( (V; \omega) \) (Arnold [1,2]).

Choose (local) Darboux coordinates \( (x^i; \dot{x}^i; p_j) \) in which \( \dot{x}^i = \frac{\partial E}{\partial x^i} ; p_j = \frac{\partial E}{\partial \dot{x}^j} \). Let \( I; J \) be a partition of the set of indices 1; ...; n, and consider any function \( (p_i; x^i); I \subset I; J \). Then the following equations define a Legendre submanifold:

\[
\dot{x}^i = \frac{p_i}{p_I}; \quad \dot{p}_j = \frac{\partial E}{\partial x^j}; \quad \dot{p}_j = \frac{\partial E}{\partial \dot{x}^j}; \quad i \in I; \quad j \in J; \quad \tag{3.1}
\]

Moreover, every Legendre submanifold is locally given by some choice of a splitting \( I; J \) and of a function \( (p_i; I \cap j \cap J) \).

In physics, the most common used thermodynamic potentials are: internal energy, entropy, free energy of Helmholtz, enthalpy and the free Gibbs energy.

On the intersection of the domains of these representations, corresponding points are related by a Legendre transformation (see [1,2]).

Example 1. As an example of such a thermodynamic system, consider the van der Waals gas—a system with two thermodynamic degrees of freedom. Space \( P \) is 5-dimensional (for a mole of gas) with the canonical variables \( (U; T; S); (p; V) \) (internal energy, temperature, entropy, pressure, volume) and the contact form

\[
dU = TdS + pdV;
\]

and the fundamental constitutive law

\[
U(S; V) = N \left( b \frac{a}{V^2} e^{\frac{a}{V}} - \frac{a}{V} \right);
\]

where \( R \) is the ideal gas constant, \( c_v \) is the heat capacity at constant volume, and \( a, b \) are parameters of the gas reflecting the interaction between molecules and the part of volume occupied by molecules respectively.
4. Thermodynamical metrics of Weinhold and Ruppeiner.

A thermodynamic metric \( g \) (the Weinhold metric) in the space \( X \) of extensive variables corresponding to the choice of internal energy \( U \) as the thermodynamical potential \( E \) was explicitly introduced by F. Weinhold (see [34] as the Hessian of the internal energy \( U(x^i) \))

\[
g_{ij} = \frac{\partial^2 U}{\partial x^i \partial x^j}; \quad (4.1)
\]

G. Ruppeiner's metric \( g \) corresponding to the choice of entropy \( S \) as the thermodynamical potential \( E \) was defined by the same formula and intensively studied by Ruppeiner in the framework of the fluctuation theory of thermodynamical systems.

Interest in these metrics is partly due to the fact that the definiteness of \( g \) (positive or negative) at a point \( x \in X \) of the of the constitutive surface delivers the local criteria of stability of the equilibria given by the corresponding point of the surface \( x \) or \( g \) (see [6]).

Later both of these metrics were studied by P. Salom and his collaborators and by R. Mrugala and H. Janyszek (see ref.). Geometrical properties of these metrics were studied for TD systems with small degree of freedom (small n). An interesting and important in application meaning was assigned to the length of curves ("processes") in the space \( X \). It was suggested that the curvature of these metrics is related to the interactions in the (micropscopical) system, and that singularities of the scalar curvature of these metrics were related to the properties of system near the phase transition and the triple point of the thermodynamical system.

5. Indefinite thermodynamical metric \( G \) of R. Mrugala.

In the paper, R. Mrugala defined a pseudo-Riemannian (inde nite) metric \( G \) of signature \( (n+1,n) \) in the thermodynamical contact space \( (P; ) \). It is given by the formula:

\[
G = 2dp_k \; dx^k + \quad ; \quad (5.1)
\]

where \( dp_k \; dx^k = \frac{1}{2}(dp_k \; dx^k + dx^k \; dp_k) \) is the symmetrical product of 1-forms.

Its physical motivation is two-fold. First, it was derived by means of statistical mechanics. Second, its reduction to the Legendre submanifold corresponding to the choice of entropy or internal energy as the TD potential coincides with the previously studied Ruppeiner and Weinhold metrics. Indeed, let \( \mathcal{E} = S \) or \( U \), and form the Legendre submanifold of the following type:

\[
x^0 = (x^1; : : : ; x^n) \quad p_k = \frac{\theta}{\partial x^k};
\]

In the coordinates \( x^k \), the restriction of \( G \) to has components

\[
g_{ij} = \frac{\theta^2}{\partial x^i \partial x^j};
\]

In coordinates \( x^0; p_i; x^j \), metric \( G \) is given by the following matrix:

\[
G = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & p_1 & 0 & \cdots & 0 \\
0 & 0 & I_m & A \\
0 & 0 & I_p & p_j \\
0 & 0 & 0 & 0 & p_j
\end{pmatrix}\quad (5.2)
\]
It is easy to see that
det\( G \) = \( (1)^n \); (5.3)
and therefore metric \( G \) is non-degenerate.

The inverse (covariant) metric to \( G \) is given by
\[
G^{-1} = (G^{-1})_{ij} = \begin{bmatrix} 1 & p_i & 0 \\ p_i & 0 & I_n \end{bmatrix} : \tag{5.4}
\]

5.1. Non-holonomic frame \( \{ P_i; x_i \} \). It is convenient to introduce the following non-holonomic frame of the tangent bundle \( T(P) \) (see (5.5))
\[
= \xi_i; P_1 = \partial /\partial x_i; X_i = \partial /\partial x^i: \quad (5.5)
\]
whose only nonzero commutator relation is
\[
[K_1; x_i] = \varepsilon_{ij} : \tag{5.6}
\]
In this frame the metric \( G \) takes the simple form
\[
G = (G_{ij}) = \begin{bmatrix} 0 & 0 & I_n \end{bmatrix} : \tag{5.6}
\]
The dual coframe of the frame (5.5) is given by
\[
( ; dp_i; dx^i) : \tag{5.7}
\]
Using the frame (5.5) one can easily nd positive and negative sub-bundles of tangent bundle \( T(P) \) – they are generated, correspondingly, by the tangent vectors
\[
T_+ = < \xi_i; \xi_i; \partial /\partial x_i; p_i \xi_i >; \quad T_+ = < \xi_i; \xi_i; \partial /\partial x_i; p_i \xi_i > : \tag{5.8}
\]
In this basis, metric \( G \) takes the standard indefinite form
\[
G = (G_{ij}) = \begin{bmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} : \tag{5.9}
\]
In terms of this frame the light cone at each point \( m \) is given by the standard quadric, i.e. \( X = f^2 + \sum_{k} p_k h^k X_k \) belongs to the light cone at a point \( m \) if and only if
\[
f^2 + \sum_{k} p_k h^k = 0: \tag{5.10}
\]
5.2. Compatibility of contact structure and metric \( G \). The notion of compatibility between an almost contact structure and a Riemannian metric that was introduced by Sasaki has become a classical notion (see [5]). We recall here that an almost contact structure on a manifold \( M^{2n+1} \) is de ned by a triple \( (\xi, \eta, \phi) \) of a 1-form \( \eta \), a vector \( \xi \) and a \((1,1)\)-tensor \( \phi \) satisfying the conditions
\[
\phi^2 = 1 + \eta \eta = 0: \tag{5.11}
\]
From these properties the subsequent relations follow
\[
(\eta) = 0; \quad \phi(\xi) = 0; \quad \text{rank}(\phi) = 2n:
A Riemannian metric $g$ is said to be compatible with the almost contact structure $(\; ; \; )$ (or associated with it) if
\begin{equation}
g( X ; Y ) = g( X ; Y ) \quad (X) \quad (Y) \tag{5.11}
\end{equation}
for all tangent vectors $X ; Y$ at all points in $\mathbf{M}$ (see [5]).

It is known that any almost contact structure admits a (far from being unique) compatible Riemannian metric $g$.

For a contact manifold $(\mathbf{M} )$ the condition of compatibility above is equivalent to the following two conditions taken together: 1) on the contact distribution $D = \ker(\; )$ is a $g$-orthogonal transformation, and 2) $g$-orthogonal to $D$.

In a case of an indefinite metric $g$ on $\mathbf{M}^{2n+1}$, the compatibility condition (5.11) should be modified if we would like to incorporate even the most simple indefinite metric
\begin{equation}
g = 0 \quad 0 \quad 0
\end{equation}
de ned in standard 3D contact space $(\mathbb{R}^3 ; x^0 ; p_1 ; x^1)$ with the contact form $= dx^0 + p_1 dx^1$. It is sufficient to check the condition for two basic horizontal vectors $X = \theta_x ; \quad p_1 \theta_{x^1} ; \quad P = \theta_p ;$.

We have $g( X ; Y ) = 1$ and $(X) = X ; \quad (P) = P$, so that $g( X ; Y ) = 1$; Thus, condition (5.11) is not fulfilled.

For the MRugala metric, the $(1;1)$-tensor of the associated almost contact structure has the form (see [21])
\begin{equation}
= \theta_0 \quad 0 \quad 0 \quad 0_n \quad A : \quad 0 \quad 0 \quad n \quad 0_n \quad (5.12)
\end{equation}

We have
\begin{equation}
g( X ; Y ) = g( X ; Y ) \quad (X) \quad (Y) ; \tag{5.13}
\end{equation}
which is satisfied by both of these metrics.

6. Levi-Civita Connection.

In this section, we will compute the Christoffel connection coefficients and the covariant derivatives of vector fields of the frame (5.5) with respect to the vector fields of the same frame. For calculation of the Christoffel coefficients we define combinations
\begin{equation}
f ; \quad g = G ; \quad + G ; \quad G ;
\end{equation}
symmetrically by ( ).
With this notation, the formula for the connection coefficients is

\[ \frac{1}{2} G \overset{\theta}{=} f; s g \]

First, note that the only nonzero derivatives of the metric \( G \) are given by:

\[ G_{,x^0 p_1} = G_{0 x^1 p_1} = i_j; G_{x^1 x^1 p_k} = k_i p_j + k_j p_i; \]

This implies that the only nonzero combinations \( f; g \) are the following ones:

\[
\begin{align*}
0 x^i; p_j g &= i_j; f^0 p_j; x^i g = i_j; fx^i p_j; 0 g = i_j; \\
fx^i x^j; p_k g &= ( x^i p_j + x^j p_i); fx^i p_j; x^k g = i_j p_k + k_j p_i;
\end{align*}
\]

Now we will calculate the connection coefficients.

For any \( \theta ; 0 \theta = 0 \).

Next, \( \theta ; 0 = \frac{1}{2} p_k f^0 ; p_k g \). And this quantity is 0 unless \( \theta = x^i \), in which case, we obtain \( \theta ; x^i = \frac{1}{2} p_i \).

In the next case, we consider \( \theta \) 0 and \( \theta \) 0.

\[
0 = \frac{1}{2} [ f; 0 g; p f; p g ];
\]

So we obtain:

\[
\begin{align*}
0 x^i p_j &= \frac{1}{2} i_j; \\
0 x^i x^j &= \frac{1}{2} p_k ( x^i p_j + x^j p_i) = p_i p_j; \\
0 p_i p_j &= 0;
\end{align*}
\]

The next case to consider is when \( \theta = 0 \), and the other indices are nonzero. Then we have:

\[
0 = \frac{1}{2} [ f; p_k f^0 ; p_k g + G x^i f^0 ; x^i g ];
\]

If \( \theta = p_i \), then we get \( \frac{1}{2} \frac{1}{2} f^0 ; x^i g = \frac{1}{2} f^0 ; x^i g \). This is \( \frac{1}{2} \frac{1}{2} \) if \( p_i \) and is zero otherwise.

If \( \theta = x^i \), then we get \( \frac{1}{2} \frac{1}{2} f^0 ; p_k g = \frac{1}{2} f^0 ; p_k g \): If \( \theta = x^i \), then this is equal to \( \frac{1}{2} \frac{1}{2} \); otherwise it is zero.

The next case is when the upper index is \( p_i \), and the lower indices are nonzero.

\[
\theta = \frac{1}{2} [ p_i f; 0 g + f; x^i g ];
\]

The only nonzero term comes when \( \theta = x^i p_k \). So we obtain:

\[
\theta x^i p_k = \frac{1}{2} [ p_i j k + i_k p_j + i_j p_k ] = \frac{1}{2} i_k p_j ;
\]

The next case is the one in which the upper index is \( x^i \) and the lower indices are nonzero.

\[
x^i = \frac{1}{2} f; p g \]

This is only nonzero if \( \theta = x^i x^j \), and in this case, we get

\[
x^i x^j = \frac{1}{2} i_k p_j + k_j p_i ;
\]
To summarize, the non zero Christoffel coefficients are given by:

\[ 0_{\alpha x^i} = \frac{1}{2} p_{ij}, \quad 0_{x^i p_j} = \frac{1}{2} i_j, \quad 0_{x^i x^j} = p_{ij}; \quad p_{ij} = \frac{1}{2} i_j; \quad x^i_{\alpha x^j} = \frac{1}{2} i_j; \quad p_{ij} = \frac{1}{2} i_j. \]

Finally, we calculate the trace 1-form of the connection (see [26]), whose components in the coordinate coframe are given by:

\[ 0_{\alpha x} = p_{i0} x^i = \frac{1}{2} i_j 1 \frac{1}{2} i_j = 0 \]

\[ p_{ij} = 0; \quad p_{i} x^i + x^i p_{j} = 0 \]

\[ x^i = 0_{x^0} + p_{i} x^i + x^i_{x^j} = \frac{1}{2} p_{ij} + \frac{1}{2} i_j p_{i} + i_k p_{ij} = 0 \]

So in the coframe \( dx^{i} ; dp_{j} = 0 \) (recall that the form depends on the choice of a coordinate system - in other coordinates, the form differs from zero by the differential of a function (see [26]).

It is helpful for the future calculations to point out that the covariant derivative of the connection takes a particularly simple form when expressed in the nonholonomic basis:

\[ P_{x} = \theta_{x}; P_{i} = \theta_{x} x^i \theta_{x} = \theta_{x} p_{i} \]

defined above.

Let \( G \) be any pseudo-Riemannian metric. Then, for any vector fields \( X; Y; Z \), the Levi-Civita connection of metric \( G \) satisfies:

\[ Y G (X; Z) = G(r_y X; Z) + G(X; r_y Z); \]
\[ Z G (X; Y) = G(r_z X; Y) + G(X; r_z Y); \]
\[ X G (Y; Z) = G(r_x Y; Z) + G(Y; r_x Z); \]

If we add the first two equations and subtract the third, the result is:

\[ 2G (X; r_y Z) = Y G (X; Z) + Z G (X; Y) + X G (Y; Z); \]
\[ G (Y; X; Z) + G (Z; X; Y); \]

If we use the fact that the Levi-Civita connection is symmetric; that is, \( r_x Y + r_y X = [X; Y] \) for all \( X; Y \).

In taking \( X; Y; Z \) from the vectors of the frame fields \( P_{i}; X_{j} \), scalar products \( G (X; Y); G (Y; Z); G (Z; X) \) will all be constant; therefore, the first three terms on the right side of (6.2) will vanish, leaving:

\[ 2G (X; r_y Z) = G (Y; X; Z) + G (Z; X; Y); \]
\[ G (X; X; Y); \]

Among the basic vectors, the only pair of vectors with non-zero Lie-bracket is \( [P_{i}; X_{j}] = i_j \). It follows that if we substitute basic vectors into the equation above, the right side will equal zero unless two of the vectors are \( P_{i} \) and \( X_{j} \), respectively. Since their bracket is proportional to \( i_j \), which is orthogonal to the contact distribution, the third vector must be \( P_{i} \). In particular, we immediately obtain the following relations:

\[ r = 0; P_{i} P_{j} = 0; r_{X_{j}} X_{j} = 0; \]
Additionally, we see that the only nonzero component of $r_{P_i}X_j$ is the component, which is found by:

$$2G(r_{P_i}X_j) = G(r_{; ij}) = ij.$$ 

Therefore, $r_{P_i}X_j = \frac{1}{2}ij$. Note that interchanging $P_i$ with $X_j$ changes the sign in the right side: $r_{X_j}P_i = \frac{1}{2}ij$.

The next equation to consider is:

$$2G(P_i; rX_j) = G(r_{; iP}X_j) = ij.$$ 

It follows that $r X_j = \frac{1}{2}X_j$. Interchanging the roles of $P_i; rX_j$ yields $r P_i = \frac{1}{2}P_i$. By the symmetry of the connection, $r_{X_j} = r X_j$ and $r_{P_i} = r P_i$ since $[; iP] = 0$.

In summary, the curvature derivatives of the connection $r_{X_j}$ in the (canonical) frame (5.5) are given by the following equations:

$$r_{X_j} = 0;r_{P_i}P_j = 0;r_{X_j}X_j = 0;$$

$$r P_i = r_{P_i} = \frac{1}{2}P_i;$$

$$r X_j = r_{X_j} = \frac{1}{2}X_j;$$

$$r_{P_i}X_j = \frac{1}{2}ij = r_{X_j}P_i.$$ 

(6.4)

7. Ricci and scalar curvatures.

In this section, we calculate the Ricci tensor and the full curvature tensor in the form of a transformation of the tangent bundle.

Recall the formula for the components of the Ricci tensor:

$$R_{ij} = ;ij.$$ 

(7.1)

The Ricci tensor of a Riemannian space is symmetric. In addition to this, by the remarks above, the result of contracting an upper and lower index of the Christoffel coefficients $\frac{1}{2k}$ is zero for our metric $G$. Therefore the middle two terms in the right side of the above formula are zero. So for the present calculations, we may use the formula:

$$R_{ij} = ;ij.$$ 

$$R_{00} = 0_{00} = 0_{0p}0 + 0x^0 = \frac{1}{4}ij + \frac{1}{4}ij = \frac{n}{2};$$

$$R_{0p} = p_{00} = p_{00}0 + p_{0x^0}x^0 = 0;$$

$$R_{0x^1} = x^10 = p_{x^1}0 + x^1x^0 = \frac{1}{2}jkP_i\frac{1}{2}ij + x^1x^0 =$$

$$= \frac{1}{4}np_i + x^1x^0x_{ij} + x^1x^0x^0 = \frac{n}{4}p_i + \frac{1}{4}k\frac{1}{4}jk + \frac{1}{4}P_i + \frac{1}{4}jP_j =$$
curvature transformation with respect to other basic vectors of the same frame we get the action of the curvature transformation \( R(\xi' ; Y') = T(\xi' ; Y') \) on the couples \((\xi ; Y)\) of basic vectors on all the vectors of the frame \((\xi ; Y)\).

\[
R(\xi' ; Y') = T(\xi' ; Y') \cdot (\xi' ; Y')
\]

\( R \) is antisymmetric in \( \xi \) and \( Y \).

Using formulas (6.4) for the covariant derivatives of basic vectors of the frame \((\xi ; Y)\) with respect to other basic vectors of the same frame we get the action of the curvature transformation \( R(\xi' ; Y') \) on the couples \((\xi ; Y)\) of basic vectors on all the vectors of the frame \((\xi ; Y)\).

\[
R(\xi ; Y) = T(\xi ; Y) \cdot (\xi ; Y) = \xi \cdot \xi' \cdot Y \\
\]

and the constant scalar curvature

\[
R(G) = \frac{n}{2}
\]

To calculate the full curvature tensor recall that the curvature transformation \( R(\xi ; Y) = T(\xi ; Y) \cdot (\xi ; Y) \) of vector \( \xi \) is given by

\[
R(\xi ; Y) = T(\xi ; Y) \cdot (\xi ; Y) = \xi \cdot \xi' \cdot Y
\]

This completes the proof. 

As a result, the components of the symmetric Ricci tensor are given by:

\[
R_{00} = \frac{n}{2}; \quad R_{10} = \frac{n}{2}; \quad R_{20} = \frac{n}{2}; \quad R_{11} = \frac{n}{2}; \quad R_{22} = \frac{n}{2}; \quad R_{01} = \frac{n}{2}; \quad R_{02} = \frac{n}{2}; \quad R_{12} = \frac{n}{2};
\]

To calculate the scalar curvature \( G \), we contract the Ricci tensor

\[
R = G \quad R_{00} = \frac{n}{2} \quad R_{02} = \frac{n}{2} \quad R_{11} = \frac{n}{2} \quad R_{12} = \frac{n}{2} \quad R_{20} = \frac{n}{2} \quad R_{22} = \frac{n}{2}
\]

In summary, we have the following results:

**Proposition 1.** Metric \( G \) has the Ricci Tensor

\[
R\left(\frac{n}{2}\right) = \begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

and the constant scalar curvature

\[
R(G) = \frac{n}{2}
\]

To calculate the full curvature tensor recall that the curvature transformation \( R(\xi ; Y) = T(\xi ; Y) \cdot (\xi ; Y) \) of vector \( \xi \) is given by

\[
R(\xi ; Y) = T(\xi ; Y) \cdot (\xi ; Y) = \xi \cdot \xi' \cdot Y
\]

Note that \( R \) is antisymmetric in \( \xi \) and \( Y \).

Using formulas (6.4) for the covariant derivatives of basic vectors of the frame \((\xi ; Y)\) with respect to other basic vectors of the same frame we get the action of the curvature transformation \( R(\xi ; Y) \) on the couples \((\xi ; Y)\) of basic vectors on all the vectors of the frame \((\xi ; Y)\).
R (\(;X_1\)) = \(-\frac{1}{4}X_1; P_j \) \(=\) \(\frac{1}{4}ij; X_j \) \(=\) 0

R (\(P_i; P_j\)) = \(-\frac{1}{4}\) \(P_k \) \(=\) 0; \(X_k \) \(=\) \(\frac{1}{4}(\mu X_j \mu X_k)\)

R (\(P_i; X_j\)) = \(-\frac{1}{4}\) \(P_k \) \(=\) \(\frac{1}{4}(\mu X_j \mu X_k)\)

By the antisymmetry properties of the curvature tensor \(R\), this determines the full tensor \(R_{ijkl}\).

8. Sectional curvatures.

Using the above calculations of the curvature transformation \(R (X;Y)\) we calculate, in the frame \((P_i; \theta_{\alpha}); \theta_{\alpha}^\dagger\)), the sectional curvatures of the nondegenerate planes generated by couples of these vectors.

Given tangent vectors \(A\) and \(B\) spanning a surface in \(T_m (P)\), the sectional curvature they determine is given by the formula:

\[
R (\theta; A \wedge B) = G_{mn} (R(A \wedge B)B; A))
\]

where \(\theta \wedge B \theta^\dagger = G (A; A)G (B; B) G (A; B) G (B; A)\).

We calculate the sectional curvatures corresponding to the planes generated by the following pairs of tangent vectors at a point \(m \in \mathbb{P} (P_i; \theta_{\alpha}); \theta_{\alpha}^\dagger; \theta_{\alpha}^\dagger)\):

1. \((P_i)

R (\(;P_i\)) \(P_i \) \(=\) 0; \(R \theta \wedge ; P_i \) \(=\) 0;

2. \((\theta_{\alpha})

R (\(;\theta_{\alpha}\)) \(\theta_{\alpha}^\dagger = R (\;X_1 \wedge p_1 \) \(\theta_{\alpha}^\dagger = R (\;X_1 \wedge p_1 \) \(=\) \(-\frac{3}{4}X_1\)

Therefore,

\[
G (R (\theta; \theta_{\alpha}) \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger) \theta_{\alpha}^\dagger = 0 \)

3. \(P_i; P_j\)

R (\(P_i; P_j\)) \(P_j \) \(=\) 0; \(R \theta \wedge P_i \wedge P_j \) \(=\) 0;

4. \(\theta_{\alpha}^\dagger; \theta_{\alpha}^\dagger\)

R (\(\theta_{\alpha}; \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger \) \(\theta_{\alpha}^\dagger = R (\;X_1 \wedge p_1 \) \(\theta_{\alpha}^\dagger = R (\;X_1 \wedge p_1 \)

\[= p_j R (\theta; X_1 \wedge \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger) \theta_{\alpha}^\dagger = \]

\[= \frac{p_j^2}{4}X_1 + \frac{p_j^2}{4}X_j\]

Therefore,

\[
G (R (\theta; \theta_{\alpha}) \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger) \theta_{\alpha}^\dagger = 0 \)

5. \(P_i; \theta_{\alpha}^\dagger\)

R (\(P_i; \theta_{\alpha}^\dagger \) \(\theta_{\alpha}^\dagger = R (P_i; \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger \) \(\theta_{\alpha}^\dagger = R ( \)

\[= p_j R (P_i; x_j \wedge p_j \) \(x_j \wedge p_j \)

\[= \frac{p_j^2}{4}X_1 + \frac{p_j^2}{4}X_j\]

Therefore,

\[
G (R (\theta; \theta_{\alpha}) \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger \theta_{\alpha}^\dagger) \theta_{\alpha}^\dagger = 0 \)

For the denominator we have:

\[
G (P_i; P_i) G (\theta; \theta_{\alpha}^\dagger) \theta_{\alpha}^\dagger P_i = \frac{3}{4} ij.
\]

\[
G (P_i; \theta_{\alpha}^\dagger) G (\theta; P_i) = \frac{3}{4} ij.
\]
Therefore, if \( i = j \), the sectional curvature equals to 3/4:

\[
R \in \mathcal{P}_1 \wedge \mathcal{E}_{x_1} = \frac{3}{4}:
\]

When \( i \neq j \), the sectional curvature is not defined since the metric on the corresponding surface is degenerate (zero).

Remark 1. Notice that the only nonzero (positive!) sectional curvature in this basis happens to be exactly at the planes corresponding to the couples of conjugate variables \((p_i; x^i)\).

9. Killing vector fields, isotropy Lie algebra.

In this section we calculate Killing vector fields for the metric \( G \) and, correspondingly, determine the Lie algebra of the isotropy group of metric \( G \).

Recall that such vector fields are defined by the condition \( L_X G = 0 \). On the other hand for all \( Y; Z \),

\[
(L_X G)(Y; Z) = X G (Y; Z) \quad G (X; Y; Z) = G (Y; X; Z) :\]

From Section 7, formula (1.2), we have:

\[
2G (X; Y; Z) = Y G (Z; X) + Z G (X; Y) \quad X G (Y; Z) + G (X; Y; Z) + G (X; Y; Z)
\]

\[
= Y G (Z; X) + Z G (Y; X) \quad (L_X G)(Y; Z) + G (X; Y; Z)
\]

Since the connection is symmetric, \( L_X G = 0 \) if and only if:

\[
Y G (X; Y) + Z G (X; Y) = G (X; Y; Z + r Z Y)
\]

for all \( Y; Z \). Notice that the condition \( L_X G (Y; Z) = 0 \) is linear in \( Y; Z \). Therefore, it is sufficient to ensure the fulfillment of this condition for couples of vector fields \((Y; Z)\) from some frame on the manifold \( P \).

To determine Killing vector fields \( X \), let \( X = f + \sum P_i \mathcal{E}_i + \sum h^k X_k \).

Make the following substitutions into the formula (9.1):

\[
Y = \mathcal{E}_i; Z = \mathcal{E}_j; Y = \mathcal{E}_i; \quad Y = \mathcal{E}_j; Y = \mathcal{E}_i + X_i; Z = \mathcal{E}_j; \quad Y = P_i; Z = X_j
\]

Using the formulas (6.4) for covariant derivatives of basic vectors along other basic vectors of basis \( \{ \mathcal{E}_i; P_j \} \) we get the system of equations for coefficients of the vector field \( X \):

1. \( f = 0 \);
2. \( h^i = P_i f + h^i; \)
3. \( g^i = X_i f + g^i; \)
4. \( P_i h^j + P_j h^i = 0; \)
5. \( X_i g^j + X_j g^i = 0; \)
6. \( P_i g^j + X_j h^i = 0; \)
We begin by examining the case when \( f = (X) = 0 \).

Since \( \{ ; \} \) commutes with both \( P_i \) and \( X_j \), we may apply it to equation 6 to obtain
\[ P_i \, g^j + X_j \, h^i = 0. \]
By equations 2 and 3 this reduces to \( P_i g^j \cdot X_j h^i = 0 \). Together with equation 6, this shows:
\[\text{7. } P_i g^j = 0; \quad X_j h^i = 0;\]

Now apply \( X_k \) to equation 4. Using the relation \( [X_k; P_i] = k_i \) and equations 7, we see that \( 0 = x_k h^j \cdot x_k h^i \). By equation 2, this gives \( 0 = x_k h^j + x_k h^i \). Setting \( i = j = k \), we obtain \( h^i = 0 \).

Similarly, if we apply \( P_k \) to equation 5 and make the same reductions, the result is \( g^i = 0 \). Therefore, when \( (X) = 0 \), we may conclude that \( X = 0 \).

Now consider the general case. It is easy to see that \( \{ ; \} \) satisfies the equations for a Killing vector eld. Therefore, the Lie bracket \( [ ; ] \) must also be a Killing vector eld. But by equation 1, \( [; X] = f = 0 \). So by the previous calculations, \( [; X] = 0 \). But this proves that \( H = 0 \), and \( g^i = 0 \).

Hence equations 2 and 3 give explicit formulas for \( h^i \) and \( g^i \) in terms of \( f \) (a generating function). If we substitute these expressions into equations 4, 5, and 6, then 6 is satisfied automatically, while 4 and 5 yield:
\[\text{8. } P_i P_j f = 0; \quad X_i X_j f = 0;\]

Thus we see that \( f \) must have the form:
\[ f = a + b x^i + c^k p_k + d^k i x^i p_k \]
with constant coefficients \( a; b_i; c^k; d^i k \).

From equations 2 and 3, we obtain \( g^i \) and \( h^i \):
\[ g^i = b_i + d^i k p_k; \]
\[ h^i = (c^i + d^i k x^k); \]

As a result, the Lie algebra is generated by the vectors:
\[\text{9. } A_i = x^i + P_i \theta p_i + x^i \theta x^i; \quad B_j = p_j \theta x_j; \quad Q_k^i = x^k p_i + p_k x^k X_i = p_i \theta p_i + x^k \theta x_i; \]

These vector elds satisfy the following nonzero commutator relations:
\[\text{10. } [A_i; B_j] = \delta_{ij} ; \quad [Q_i; A_j] = u A_k; \quad [Q_i^k; B_j] = k_j B_i; \]

As a result, we get the following description of the Lie algebra \( \text{iso}_G \) of the isometry group \( \text{iso}(G) \) of the metric \( G \).
Proposition 2. The Lie algebra iso$_0$ of the isometry group Iso($G$) of the metric $G$ is the Lie algebra gl(n;R) under the semi-direct product of linear Lie algebra gl(n;R) embedded into the symplectic Lie algebra sp(n;R) (with generators $fQ^T g$) and of the Heisenberg Lie algebra $h_n$ with generators $\frac{\partial}{\partial x^i}$ and commutative relations (9.3-4). All these vector fields are contact with the Hamiltonians $H = 1_p H_A = p_j H_B = x^i H_{Q^1} = x^i p_i$ respectively (see 4).

10. Second fundamental form of Legendre surfaces.

Here we will calculate the second fundamental form $II(X;Y)$ of a Legendre submanifold at the points where metric $g = G$ is nondegenerate.

Let $I$ be any subset of the indices from 1 to $n$, and let $J$ be the complementary subset. Consider a function $(p; x^i)$ as in Sec.3 determining a Legendre submanifold. Throughout the calculation, indices $i; j$ will be assumed to be in $I$ and indices $jr$ to be in $J$. Any other indices will be assumed to run through $1; \cdots; n$. Also we will be using the following conventions in notation:

\[ i = \theta_p_i; j = \theta_{x^j} \]

The Legendre submanifold is given, in these notations by the equations (3.1):

\[
\begin{align*}
&\partial x^i = p_i; \\
&\partial p_i = j; \\
&x^i = i;
\end{align*}
\]

The metric $G = 2dp_i dx^j + 2$ restricted to this submanifold is given by $g = 2 d\theta_p_i d\theta_{x^j} + 2 \partial_{x^j} dx^k + dx^j$. Under the assumption that is nondegenerate at a given point (and, therefore, at some neighborhood of this point), the square matrices $\frac{\partial}{\partial x^i}$, $\frac{\partial}{\partial p_i}$ must be nonsingular. Let their inverses be given by $\frac{\partial}{\partial p^j}$, $\frac{\partial}{\partial x^j}$, respectively. Denote, for this section, the entries $i$ to be 0.

Correspondingly, the tangent space $T(\cdot)$ to the surface at each point is generated by the vectors:

\[
\begin{align*}
Y_1 = (\partial_p_i) &= p_i \frac{\partial}{\partial x^j} + \theta_p_i \theta_{x^j} = \theta_p_i \frac{\partial}{\partial x^j} + \theta_{x^j} x^j \theta_{x^j} \theta_{x^k} \theta_{x^l} = \\
&= p_i \frac{\partial}{\partial x^j} + \theta_{x^j} x^j \theta_{x^j} \theta_{x^l}; \quad (10.1) \\
Y_j = (\partial_{x^j}) &= (j p_i \frac{\partial}{\partial x^j} + \theta_{x^j}) = j \theta_{p_i} \frac{\partial}{\partial x^j} + \theta_{x^j} = \theta_{x^j} x^j \theta_{x^j} \theta_{x^l} = \\
&= X_j + i_j X_i \frac{\partial}{\partial x^j} \theta_{x^j} \theta_{x^l}; \quad (10.2)
\end{align*}
\]

We may simplify these expressions by defining vectors $V_k$ and $W_k$ by the following rules.
We also have the following useful relations for the scalar products of these functions
\[ G(\mathbf{v}_k;\mathbf{v}_l) = G(\mathbf{w}_k;\mathbf{w}_l) = 0; \quad G(\mathbf{v}_k;\mathbf{w}_l) = k_s; \quad \text{if } k; s \in I; \]
\[ G(\mathbf{v}_k;\mathbf{w}_l) = 0; \quad \text{if } k; s \in J; \]
\[ G(\mathbf{v}_k;\mathbf{w}_l) = 0; \quad \text{if } k; s \in \text{otherwise}. \]

It is then easy to check that the following vectors are orthogonal and complementary to \( \mathbf{v}_k \):
\[ \mathbf{w}_k = \mathbf{w}_k = \frac{1}{2} k_1 \mathbf{v}_k; \quad (10.4) \]

To prove this we notice that is obviously orthogonal \( \mathbf{v}_k \) being orthogonal to \( \mathbf{p}_i;\mathbf{x}_j \) Presenting an arbitrary vector in D orthogonal to \( \mathbf{v}_k \) in the form \( \mathbf{Z}_1 = a_i \mathbf{v}_i + b_k \mathbf{w}_k \) we nd, using conditions of orthogonality, relations between coefficients:
\[ a_i = i^s i^s + i^t i^t; \quad b_k = j^s j^s + s^t s^t; \]
and from this the basic orthogonal vectors
\[ \mathbf{Z}_1 = \mathbf{Y}_1 = 2 \mathbf{w} \mathbf{W} \mathbf{W}; \quad \mathbf{Z}_1 = \mathbf{Y}_1 = 2 \mathbf{j} \mathbf{j} \mathbf{W} \mathbf{W}; \]
Applying \( \frac{1}{2} \) to the right side of \( \mathbf{Z}_1 \) (and using the agreement that \( i^s = 0 \)) we get from \( \mathbf{Z}_1 \) the other basic vectors \( \frac{1}{2} \mathbf{Y}_1 \mathbf{W} \mathbf{W} \mathbf{W} \mathbf{W} \). Changing sign and renaming these vectors \( \mathbf{Z}_1 \) we get the second set of orthogonal vectors in (10.4). In the same way we get \( \mathbf{v}_j \) of the form (10.4).

In section 6, we computed the connection in the basis \( \mathbf{p}_i;\mathbf{x}_j \) (see (6.4)). Referring to these calculations, it is simple to see the following relations:
\[ \mathbf{r}_{\mathbf{v}_i} \mathbf{V}_1 = 0; \quad \mathbf{r}_{\mathbf{w}_i} \mathbf{W}_1 = 0; \]
\[ \mathbf{r}_{\mathbf{v}_i} \mathbf{W}_1 = \mathbf{r}_{\mathbf{w}_i} \mathbf{V}_1 = \frac{1}{2} k_1 \]

Now we can easily calculate covariant derivatives of tangent vector fields \( \mathbf{v}_k \) with respect to \( \mathbf{v}_s \):
\[ \mathbf{r}_{\mathbf{v}_s} \mathbf{Y}_1 = \mathbf{r}_{\mathbf{v}_s} \mathbf{v}_s (\mathbf{v}_1 + \mathbf{w}_s) = \]
\[ = \mathbf{v}_k (\mathbf{v}_s) \mathbf{w}_s + \mathbf{v}_s \mathbf{v}_s \mathbf{w}_s + \mathbf{w}_s \mathbf{v}_s (\mathbf{v}_1 + \mathbf{w}_s) = \]
\[ = \mathbf{v}_k (\mathbf{v}_s) \mathbf{w}_s + \frac{1}{2} \mathbf{k}_s \mathbf{w}_s + \frac{1}{2} \mathbf{k}_s \mathbf{w}_s + \mathbf{w}_s \mathbf{v}_s (\mathbf{v}_1 + \mathbf{w}_s) = \]
\[ = \mathbf{v}_k (\mathbf{v}_s) \mathbf{w}_s + \mathbf{w}_s \mathbf{v}_s (\mathbf{v}_1 + \mathbf{w}_s) \]
\[ = \mathbf{v}_k (\mathbf{v}_s) \mathbf{w}_s = \mathbf{w}_s \mathbf{v}_s ; \]

Expressed in the \( \mathbf{Y}_k;\mathbf{Z}_1 \) basis, the result is:
\[ \mathbf{r}_{\mathbf{y}_s} \mathbf{y}_1 = \frac{1}{2} \mathbf{w}_s \mathbf{v}_s + \frac{1}{2} \mathbf{w}_s \mathbf{v}_s = \]
\[ = \frac{1}{2} \mathbf{v}_s \mathbf{v}_s \mathbf{v}_s + \frac{1}{2} \mathbf{v}_s \mathbf{v}_s \mathbf{v}_s ; \]
The rst term in the last line represent the covariant derivative $r_{Y_k}Y_1$ on the subm anifold with respect to the induced (thermodynamical) metric while the second term represents the second fundamental form of with respect to the M rugal metric.

Nam ely we have proved the following

Proposition 3. Let a Legendre subm anifold of the contact mani fold ($P$) be defined by the equations

$\begin{align*}
8 \ x^0 &= P_i I_i \\
I_{ij} &= \delta_{ij}; \quad i2 I_{j2} J_i;
\end{align*}$

Then the second fundamental form $II(X;Y)$ of the subm anifold is given by the expression

$II(Y_k;Y_l) = \ y_{ks}Z_s;$

where vector elds

$Y_k = V_k + k_iW_i$

form the basis of tangent bundle of the subm anifold (see above), vector elds

$Z_k = W_k + \frac{1}{2} k_iY_i$ form the basis of the orthogonal bundle of the subm anifold in the distribution $D$ and $y_{ks} = Y_k(\ l_k).$

Example 2. Consider the special case where $I = \ ;$ so $= (x^j); \ j = 1;\ldots; n$ and the tangent bundle to the surface is generated by the tangent vectors

$Y_j = X_j + \ f_pP_{p}.\ The\ Nom\ al (orthogonal)\ subspace\ of\ the\ tangent\ space\ $T_m(P)$ at the points of is generated by

$Z_j = W_j + \frac{1}{2} j_pY_p = P_j + \frac{1}{2} j_p X^{\ j}_p = \frac{1}{2} \ (P_j + j_p X^{\ j}_p);$

We now calculate the coe cients $y_{ks}$ of the second fundamental form ($k = j$ in this case)

$y_{ks} = Y_j\ l_k = (X_j \ f_pP_{p})\ _{x^s} = \theta_{kl};$

As a result, the second fundamental form of the surface has the form

$II(Y_j;Y_l) = \ _{x^s}Z_s; \quad (10.5)$

carrying information about all the third derivatives of a thermodynamical potential, or, equivalently, of the rst derivatives of the thermodynamical metric.

Second fundamental form (10.5) of a surface is zero if the metric is constant with respect to the variables $x^i.$ Only in such case (quite improbable in real TD systems) metric is at and subm anifold is totally geodesic in $P.$

11. Constitutive hypersurface.

By the reasons of dimensions, the fundamental thermodynamical constitutive equation (law) $x^0 = (x^i)$ of any materials is homogeneous of order one, i.e. the
Thus, the intersection is satisfying the condition

\[ (x^1) = (x^1) \]  \hspace{1cm} (11.1) \]

for all \( \theta \neq 0 \). In other words, the action of the one-parameter group of transformations

\[ D : (x^0; \theta; x^1)! \rightarrow (x^0; \theta; x^1); \quad 2R \]  \hspace{1cm} (11.2) \]

leaves the constitutive Legendre surface of a real material invariant.

As a result, the surface lies in the canonical quadratic (hyperbolic paraboloid)

\[ C = \sum_{i=1}^{n} (x^0; \theta; x^1) \]  \hspace{1cm} (11.3) \]

The intersection of the contact distribution \( D \) with the fibers of the tangent bundle \( T(C) \) determines in \( T(C) \) the subbundle \( D_c \).

Along the hypersurface \( C \) one has

\[ 0 = \sum_{i=1}^{n} (x^0; \theta; x^1) \]  \hspace{1cm} (11.4) \]

as a result, on the distribution \( D_c \) we have

\[ x^i = 0 \]  \hspace{1cm} (11.5) \]

which represents the abstract Gibbs-Duhem equation.

Using these relations it is easy to see that the subbundle \( D_c \) contains and is generated by the following vector fields

\[ D_c = < x_1 = \theta_x + x^1 \theta_{x^1}; \quad P_{ij} = x^j \theta_{p_i} + x^1 \theta_{p_j} > \]  \hspace{1cm} (11.6) \]

at all points of \( C \) except the point of the plane \( X = f x^1 = 0; \ i = 0; 1; \ldots; n \) where \( D = T(C) \). To see this we recall that \( D \) is generated by the vector fields \( x_1 = \theta_x + x^1 \theta_{x^1}; P = \theta_{p} \). The tangent space to the quadric \( C \) is formed by vectors satisfying the condition

\[ x^i = 0; \]  \hspace{1cm} (11.7) \]

Thus, the intersection \( D \setminus T(C) \) is formed by vectors from \( D \) satisfying the Gibbs-Duhem equation. This is true for vectors \( X \) at all points. Now, consider the partition of the index interval \( 1; \ldots; n = 1 \) \( J \) and denote by \( C_I \) the set of points

\[ C_I = f(x; p) \]  \hspace{1cm} (11.8) \]

Let now \( J \neq ? \). Then, form 2 \( C_I \) vectors \( \theta_{p}; I \) \( 2 \) \( I \) belong to \( T_m(C) \) as do all the vectors \( P_{ij} = x^i \theta_{p_i} - x^j \theta_{p_j} \); \( i; j \in 2J \) choose index \( k \in 2J \) and consider vectors \( x^i \theta_{p_i} \) \( x^1 \theta_{x^1} \); \( \theta_{p_k}; \) \( \theta_{p_j}; \) \( \theta_{p_l} \) \( j \neq k \). These vectors belong to \( T_m(C) \), and, together with \( \theta_{p_k}; \) \( \theta_{p_l} \) \( I \) \( 1 \) \( I \) and \( x_1 = \theta_x + x^1 \theta_{x^1} \), they form the 2n \( 1 \) dimensional subspace of \( T_m(C) \) which is the intersection \( D \setminus T(C) \).

The only points where these arguments fail to work are the points of \( X = C_{\{i,p\}} \), where \( J = ? \). In this case, the Gibbs-Duhem condition is empty and \( D_m = T_m(C) \).

Notice that physically it would mean that all the extensive variables of the system are zero—quite an improbable case.
12. The Heisenberg Group as the thermodynamical phase space.

In this section we establish isomorphism of the TPS $\left(P; G\right)$ (with its contact structure and metric $G$) with the Heisenberg Lie group $H_n$ with the right invariant contact structure and right invariant indefinite metric. This isomorphism is locally suggested by the commutativity relations of vector fields of the frame (5.5).

Recall that the Heisenberg group $H_n$ is the nilpotent Lie group of $n$ $n$ real matrices

\begin{equation}
\begin{pmatrix}
0 & 1 & 0 \\
1 & a & c \\
0 & 0 & 1
\end{pmatrix}
\end{equation}

with the product

\begin{equation}
gg = \begin{pmatrix}
0 & 1 & 0 \\
1 & a & c \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & 1 & 0 \\
1 & a & c \\
0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 \\
1 & a + a_i & c + c_i + \langle a_i; b_i \rangle \\
0 & 0 & 1
\end{pmatrix}
\end{equation}

where $\langle a_i; b_i \rangle$ is the Euclidean scalar product of two vectors from $\mathbb{R}^n$ (see, for instance).

The Lie group $H_n$ is the central extension of the abelian group $R^{2n} = \mathbb{R}_a \times \mathbb{R}_b$ with the local parameters $a; b$ respectively by the 1-dim abelian group $R_z$ with the local parameter $z$:

\begin{equation}
1 ! R_z ! H_n ! R^{2n} ! 1:
\end{equation}

The Lie algebra $h_n$ of the Heisenberg group is formed by the matrices

\begin{equation}
\begin{pmatrix}
0 & 1 & 0 \\
0 & a & z \\
0 & 0 & 0
\end{pmatrix}
\end{equation}

with the conventional matrix bracket as the Lie algebra operation. The Lie algebra $h_n$ is mapped diemorphically onto $H_n$ by the exponential mapping

\begin{equation}
\exp(\begin{pmatrix}
0 & 1 & 0 \\
0 & a & z \\
0 & 0 & 0
\end{pmatrix}) = \begin{pmatrix}
0 & 1 & 0 \\
0 & a + a_i & c + c_i + \langle a_i; b_i \rangle \\
0 & 0 & 1
\end{pmatrix}
\end{equation}

Lie algebra $h_n$ is the central extension

\begin{equation}
0 ! R_z ! h_n ! R^{2n} ! 0
\end{equation}

de ned by the 2-cocycle $\chi : R^{2n} \times R^{2n} \times R$

\begin{equation}
\chi(\begin{pmatrix}
0 & a & b \\
0 & 0 & 0
\end{pmatrix}; \begin{pmatrix}
0 & a & b \\
0 & 0 & 0
\end{pmatrix}) = \frac{1}{2} \langle a; b \rangle < a_i; b_i >
\end{equation}

of the canonical symplectic form in $R^{2n}$ see.

We construct the diemorphic mapping

\begin{equation}
\phi : H_n \rightarrow \mathbb{P}
\end{equation}

by requiring

\begin{equation}
\begin{pmatrix}
0 & 1 & 0 \\
0 & x & x^6 \\
0 & 0 & 1
\end{pmatrix}
\end{equation}

\begin{equation}
\phi = \begin{pmatrix}
0 & 1 & 0 \\
0 & x & x^6 \\
0 & 0 & 1
\end{pmatrix} \begin{pmatrix}
0 & 1 & 0 \\
0 & b & A \\
0 & 0 & 1
\end{pmatrix}
\end{equation}
The action of the group $H_n$ on itself by left translation: $L_g : g_0 \mapsto g_0 g$, defines the corresponding left action of $H_n$ on the space $P$ as $T_g : m \mapsto (L_g^{-1}(m))$, or

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}x + a = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1
\end{pmatrix}x + \begin{pmatrix}
1 & a & z & 1 & x & x^0 & c & a & p & 1
\end{pmatrix}
$$

$T_g @ p \cdot A = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & x + a
\end{pmatrix}$

Let us nd, in these terms, generators of the left action of the basic one-parameter subgroups of the group $H_n$ corresponding to the elements $A_i = X(0; b; 0); B_j = X(0; b; 0); Z = X(0; 0; 1)$ of the Lie algebra $h_n$ of the vectors of $P$. For any element $X_2 h_n$ denote by $x$ (respectively by $X$ the right invariant (respectively left invariant) vector field on $H_n$ generated by the left (respectively right) translations by $e^x(X)$.

We have in coordinates $(A; b; c)$

$$z = \theta_c; A_i = \theta_{a_i} + \theta_{b_i}; B_j = \theta_{b_j}$$

Applying the Lie inomorphism to these vector fields we get the correspondence

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}x + \begin{pmatrix}
1 & x & x^0 & t
\end{pmatrix}
$$

$\theta_0 I \cdot 0 A \cdot 1 (x) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & x
\end{pmatrix}$

$$
\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0
\end{pmatrix}x + \begin{pmatrix}
1 & x & x^0 & t
\end{pmatrix}
$$

$\theta_0 I \cdot 0 A \cdot 1 (x) = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 & x
\end{pmatrix}$

The pullback of the contact form $\omega$ as $dx^0 + p dx^1$ from $P$ to $H_n$ is the 1-form $\tilde{\omega}$ on $H_n$

$$H = (\omega) = dc + h da$$

Reeb vector field of this form is

$$H = c = \theta_c$$

and we have

$$\omega =$$

for the Reeb vectors of contact manifolds $(H_n; \omega)$ and $(P; \omega)$.

The kernel, $D_{\omega}$, of this 1-form (a distribution of codimension 1 on $H_n$) is, at each point $g$, generated by the values of vector fields $A_i; B_j$ of left translations, and is therefore right invariant. As a result, the right invariant contact structure on $H_n$ (given as the kernel of the form $\omega$).

Considering the right translations on the group $H_n$, corresponding to the 1-dim Lie subalgebras of $h_n$ with generators $Z; A_i; B_j$ we nd that their generators have the form

$$c = \theta_c; A_i = \theta_{a_i}; B_j = \theta_{b_j} + a_i \theta_c$$

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Considering the right translations on the group $H_n$, corresponding to the 1-dim Lie subalgebras of $h_n$ with generators $Z; A_i; B_j$ we nd that their generators have the form

$$c = \theta_c; A_i = \theta_{a_i}; B_j = \theta_{b_j} + a_i \theta_c$$
and it is easy to check that the form is invariant under the action of these (therefore contact) vector fields. Thus, the form $\omega$ is right invariant. The di emorphism send these vectors into

\[
(\mathbf{c}) = \mathbf{e}_1; \quad (\mathbf{a}_i) = \mathbf{e}_{2+ix_i}; \quad (\mathbf{b}_j) = \mathbf{e}_{2+jx_j};
\]

Comparing this result with the description of the Killing vector field of the metric $G$ we see that these vector fields form the nilradical of the Lie algebra of the Killing vector fields of the metric $G$.

Remark 2. The distribution $D_H$ is the direct sum of two $n$-dimensional distributions

\[
D = D_A \oplus D_B
\]

in obvious notations, distributions $D_A \oplus D_B$ are integrable having as the basis at each point values of pair wise commuting vector fields. Denote by $A$ (respectively by $B$) the abelian subgroup of $H_n$ of matrices of the form (12.1) with $c = b = 0$ (respectively $c = a = 0$). Then integrable anisn distribution $D_A$ (respectively of distribution $D_B$) are orbits of the left translations by the subgroup $A$ (respectively, by the Lie subgroup $B$).

Metric $G$ is transferred under the di emorphism into the metric $G_H$ on the Heisenberg group. This metric is constant in the right invariant (non-holonomic) frames $(\mathbf{c}; \mathbf{a}_i; \mathbf{b}_j)$ and is, therefore, right invariant by itself.

As a result we've proved the following

Theorem 2. The di emorphism defined by

\[
\begin{align*}
0 & : g = \mathbf{e}_0 \quad pA = \mathbf{e}_0 \quad m = \mathbf{e}_0 \\
1 & : x^1 \quad x^2 \quad x^3 \\
0 & : 0 \quad 0 \quad 1
\end{align*}
\]

determines an isomorphism of the "thermodynamical metric contact manifold" $(P; \omega)$ with the Heisenberg group $H_n$ endowed with the right invariant contact form $\omega_H$ and the right invariant metric $G_H$ of signature $(n+1; n)$.

Remark 3. Recall (see Remark 3) that the automorphism group $\text{Aut}(H_n)$ of the Heisenberg group has, as its connected component of unity the Lie group $\text{Aut}_0(H_n) = \text{Sp}(n; \mathbb{R}) \times A^{2n+1}$, where $A^{2n+1}$ is the abelian group of dimension $2n+1$. This group acts on the space of all right invariant contact 1-forms on the group $H_n$. Since right invariant one-forms on $H_n$ are defined by their values at the unit of the group $e \in H_n$ and since the automorphism group $\text{Aut}(G)$ leaves $e \in G$ fixed, it is sufficient to study action of this group at the set of elements $e \in H_n$.

The following result for the left invariant contact structures (with the sketch of the proof) was sent to the authors in a letter by M. Goze. We reformulate this result for the right invariant contact structures due to the obvious duality between left and right translations.

Let $\{X_i, Y_j, Z\}$ be a standard basis of $H_n$ with the only nontrivial brackets being $[X_i, Y_j] = \mathbf{e}_i$, $i = 1, \ldots, n$. Let $f^i \wedge !$ be the dual basis in $h_n$. Then, extending this basis to the coframe of right invariant vector fields we get relations

\[
X_i \wedge ! = d_i = d_j = 0; \quad ! = 1 \wedge X_i.
\]

Then $\wedge ! = 0$ and, therefore, one-form $!$ defines the right invariant contact structure on $H_n$. 


Proposition 4. Let \( \! = a! + \sum_{i} a_{i} + \sum_{j} b_{j} \) be a contact form in \( h_{n} \). The group \( \text{Aut}(h_{n}) \) acts transitively on the set of right invariant contact structures with the isotropy group of \( \! \) being the intersection of the group \( \text{Aut}(h_{n}) \) with the group of \( \! \)-conformally contact di eomorphism s of \( h_{n} \).

It follows from this that the contact structure \( \! \) of the them odynam ic phase space is the typical representative of the \( \text{Aut}(h_{n}) \)-conjugacy class of right invariant contact structures on the Heisenberg group defined by a choice of the canonical basis \( FX_{i};Y_{j};Z \) of the Lie algebra \( h_{n} \) and, therefore, unique, up to an automorphism of the group \( H_{n} \).

Remark 4. After the isomorphism of the TPS \( (P; \! ; G) \) with \( (H_{n}; \! ; \text{Aut}(H_{n}) \) is established, any properties of metric \( G \) can be obtained from the corresponding results for invariant metrics on Lie groups. Further use of this isomorphism for the study of them odynam ic system s will be a subject of future work.

Part II.

13. Symplectization of manifold \( (P; \! ; G) \).

Let \( P \) be the standard \( (2n+2) \)-dim real vector space \( R^{2n+2} \) with the coordinates \( (p_{i};x^{i}) \); \( i,j = 0;\ldots;n \), endowed with the 1-form
\[
\sim = \sum_{i=0}^{n} p_{i} dx^{i};
\]
and the standard symplectic structure
\[
! = d \sum_{i} X^{i} dp_{i} \wedge dx^{i}:
\]

We consider the embedding of the space \( (P; \! ; \text{Aut}(H_{n}) \) into \( P \)
\[
J : (x^{0};x^{1};p_{0}) \to (x^{0};x^{1};p_{0} = 1)p_{1}; j = 1;\ldots;n)
\]
as the affine subspace \( p_{0} = 1 \):

It is easy to see that

Proposition 5. 1. The pullback by \( J \) of the 1-form \( \sim \) coincides with the contact form .

\[
J (\sim) = =
\]

2. The symplectic manifold \( (P; \! ; G) \) is the standard symplectization of \( (P; \! ; \text{Aut}(H_{n}) \) (see [1,18]) and \( J \) is the section of the symplectization bundle :\( P! P \):

3. The symmetrical tensor
\[
G = (G_{ij}) = \begin{pmatrix}
0 & \ldots & 0 & n+1 & \ldots & n+1 \\
0 & \ldots & 0 & p_{0} & \ldots & p_{0}
\end{pmatrix}
\]
determines in \( P \) the pseudo-Riemannian metric of signature \( (n+1;n+1) \).

4. The restriction of metric \( G \) to the image of the embedding \( J \) coincides with the metric \( G \)
\[
J G = G .
\]
The relations show, in particular, that the couples of vector fields $\mathcal{F}_i; \mathcal{F}_j$ from the 2-dim solvable Lie algebras of vector fields commuting between them satisfy (see below, Sec.19).

Scalar products of the introduced vector fields are calculated as follows

$$G(\mathcal{F}_i;\mathcal{F}_j) = 0; \ G(L_i;L_j) = 1; \ G(\mathcal{F}_i;L_j) = i_j;$$

$$G(\mathcal{F}_i;X_j) = G(p_1\partial_{p_1};p_j^{-1}\partial_{x_j};\mathcal{F}') = p_ip_j^{-1}G(\partial_{p_1};\partial_{x_j}) = p_ip_j^{-1}i_j = i_j;$$

$$G(\mathcal{X}_i;\mathcal{F}') = \frac{1}{2} X_i; \ G(\mathcal{X}_i;\mathcal{F}_s) = \frac{1}{2} X_i; \ G(\mathcal{X}_s;\mathcal{F}_s) = 1;$$

$$G(\mathcal{X}_i;X_j) = G(p_1^{-1}\partial_{x_i};\mathcal{F}';p_j^{-1}\partial_{x_j};\mathcal{F}') = p_1^{-1}p_j^{-1}G(\partial_{x_i};\partial_{x_j}) = p_1^{-1}G(\partial_{x_i};\partial_{x_j});$$

$$p_j^{-1}G(\partial_{x_i};\mathcal{F}') + G(\mathcal{F}';\mathcal{F}') = 1 - \frac{1}{2} p_1^{-1}X_s p_s G(\partial_{x_i};\partial_{p_s}) - \frac{1}{2} p_j^{-1}X_s p_s G(\partial_{x_j};\partial_{p_s}) + 0 =$$

$$1 - \frac{1}{2} p_1^{-1}X_s p_s is - \frac{1}{2} p_j^{-1}X_s p_s js = 1 \ 1 = 0 : (14.3)$$
As a result in the basis \((F_i;X^i)\), the matrix of metric \(G\) has in \(F\) the following canonical form
\[
(G_{ij}) = \begin{pmatrix}
0_{n+1} & I_{n+1} \\
I_{n+1} & 0_{n+1}
\end{pmatrix}.
\tag{14.4}
\]

The positive and negative distributions of metric \(G\) are
\[
T^+ = \begin{pmatrix}
\frac{1}{2} (F_1 + X_1)
\end{pmatrix}, \quad T^- = \begin{pmatrix}
\frac{1}{2} (F_1 - X_1)
\end{pmatrix}.
\tag{14.5}
\]

The zero cone is given in the form \(F_1;X^i\) for \(X = f_1F_1 + g_jX^j\)
\[
G(X;X) = 0, \quad f_i g_i = 0:
\tag{14.6}
\]

15. Levi-Civita connection of metric \(G\).

To calculate curvature of the metric \(G\) we start with the combinations \(f_{ijk}g = G_{jki} + G_{jik}\). It is easy to see that the only nonzero combinations are
\[
f_{ijk}g = \begin{pmatrix}
\frac{1}{2} (p_j + p_i) \\
\frac{1}{2} (p_j + p_i)
\end{pmatrix}.
\tag{15.1}
\]

Using these combinations we calculate the Christoffel coefficients \(G_{ijk}\). We notice that to be nonzero, the Christoffel coefficients should have at least two \(x^i\) between the three indices in one of the terms \(f_{ijk}g\). Using this it is easy to see that the only nonzero Christoffel coefficients are
\[
x^i x^j x^k = \frac{1}{2} (p_j + p_i); \quad p_j x^j = p_i p_x x^i; \quad p_i x^i = \frac{1}{2} (p_i + p_j) - \frac{1}{2} p_i p_j.
\tag{15.2}
\]

Next, we calculate the Ricci tensor. \(R_{ij} = k_{jik} \frac{k_{jks}}{k_{jki}}\). We have
\[
R_{jik} = \begin{pmatrix}
p_{j} p_{k} \frac{p_{i}}{p_{k}} & n + 1 \frac{p_{i} p_{j}}{p_{k}} & n + 1 \frac{p_{i} p_{j}}{p_{k}}
\end{pmatrix}.
\tag{15.3}
\]

Finally, since \(p_{x} x^i x^j x^k = \frac{1}{2} \frac{p_{j} p_{k}}{p_{i}}\), we have
\[
R_{jik} = \begin{pmatrix}
p_{j} p_{k} \frac{p_{i}}{p_{k}} & n + 1 \frac{p_{i} p_{j}}{p_{k}} & n + 1 \frac{p_{i} p_{j}}{p_{k}}
\end{pmatrix}.
\tag{15.4}
\]

Thus,
\[
Ric(G) = \begin{pmatrix}
0_{n+1} & I_{n+1} \\
I_{n+1} & 0_{n+1}
\end{pmatrix}.
\tag{15.5}
\]
Thus, metric $\mathcal{G}$ is pseudo-Riemannian Einstein metric, see [24].

Lifting an index of $R_{ij}$ with

$$(G^{ij}) = \begin{pmatrix} p_i p_j & I_{n+1} \\ I_{n+1} & 0 \end{pmatrix}$$

we get $G^{ik} R_{kj} = (n+2)$. Taking the trace, we find the scalar curvature to be

$$R(\mathcal{G}) = \text{Tr}(G^{-1} G) = (n+1)(n+2): \quad (15.6)$$

Thus, we have

Proposition 6. Metric $\mathcal{G}$ is the indefinite Einstein metric of scalar curvature $R(\mathcal{G}) = (n+1)(n+2)$.

16. Killing vector fields of metric $\mathcal{G}$.

It is natural to find the form of Killing vector fields of metric $\mathcal{G}$, in infinitesimal isometries of $\mathcal{G}$. The details of the calculations are given in the Appendix. Here we formulate the final result.

Theorem 3. The Lie algebra $\text{iso}_\mathcal{G} = \text{sl}(n+2; \mathbb{R})$ of Killing vector fields of the metric $\mathcal{G}$ is (as the vector space) the linear sum

$$\text{iso}_\mathcal{G} = q \oplus d \oplus x$$

of Lie subalgebras

1) $q = \langle Q^i_j = x^i \partial_{x^j}, p_j \partial_{p_i} \rangle$;

with the commutator relations

$$[Q^i_j; Q^p_k] = \frac{p}{n+2} Q^i_k \cdot \frac{i}{n+2} Q^p_j;$$

Subalgebra $q$ is isomorphic, thus, to $\text{sl}(n+1; \mathbb{R})$, the abelian subalgebra

2) $d = \langle D^i = \frac{x^i}{2} Q + (1 - \frac{1}{2} (x^i p_j) \partial_{p_i}) \partial_{p_i} \rangle$;

where $Q = \frac{p}{n+2} Q^i_i = \frac{p}{n+2} (x^i \partial_{x^i}; p \partial_{p_i})$ is the generator of hyperbolic rotation $H_t: (p; x) \mapsto (e^{tp}; e^{-tp} q)$.

and abelian subalgebra

3) $x = \langle X_s = \frac{\partial}{\partial x^s} \rangle$;

Generators $Q^i_j; X_j; D^j$ satisfy to the following commutator relations

$$[Q^i_j; X_s] = \frac{1}{2} X_j; [Q^i_j; D^s] = \frac{1}{2} D^j; [X_s; D^j] = \frac{1}{2} Q^j_s + \frac{1}{2} \frac{1}{2} Q^i_j; \quad (16.1)$$

Vector fields $Q^i_j; X_j; D^j$ are Hamiltonian with Hamiltonian functions

$$H_{Q^i_j} = x^i p_j; H_{X_s} = p_s; H_{D^j} = x^j \left(1 - \frac{x^i p_i}{2} \right);$$
17. Hypersurface $C$.

There exists a natural lift of the constitutive hyperquadric $C$ to the space $P$ as the homogeneous hypersurface

$$C = f(p_i; x^i) x^i p_i + p_0 x^1 = 0; \quad (17.1)$$

The hyperquadric $C$ is invariant under the action of $R = (x^i; p_i) \rightarrow (x^i; -p_i)$ as well as under the hyperbolic rotations $(x^i; p_i) \rightarrow (x^i; p_i)$.

Polarizing the coordinates $(x^i; p_i)$ - introducing new coordinates $i = x^i + p_i$; $j = x^j$ $p_j$ - we rewrite the equation of $C$ as follows

$$X^n = \begin{pmatrix} \hat{z} \\ \hat{i} \end{pmatrix} = 0.$$

From this we see that the hyperquadric $C$ is a cone in the space $P$ of signature $(n + 1; n + 1)$.

18. $(P; \; ; ; )$ as the indefinite Sasakian manifold.

Recall that an almost contact manifold $(M^{2n+1}; \; ; )$ is called Sasakian if the almost complex structure on the manifold $M^{2n+1}$ defined by

$$J \left( X \circ f g \right) = \left( (X) \ f ; (X) g \right) \quad (18.1)$$

is integrable. Here $t$ is the coordinate on the factor $R$ of the product, $f$ the $C^1$ manifold $R$, see Chapter 6.

Sasakian manifolds are considered to be the natural odd-dimensional analog of Kähler manifolds, Chapter 6.

A necessary and sufficient condition for the integrability of an almost complex structure is the vanishing of the Nijenhuis tensor of the $(1,1)$-tensor $J$:

$$N_J \left( X ; Y \right) = J^2 \left( X \right) \left( Y \right) + J \left( X ; JY \right) \left( JX ; Y \right) \quad J \left( X ; JY \right).$$

In the situation where $(M; \; )$ is the contact manifold and the almost contact structure $(\; ; )$ is associated with the contact structure, so that in particular, the $1$-form $\omega = \theta$ is the same in both structures and $\theta$ is the Reeb vector of the contact structure, it is natural to study the integrability of the almost complex structure defined by (18.1) on the symplectization of the manifold $(M; \; )$:

In the case of the standard contact structure $(P; \; )$, the symplectization of $P$ is naturally embedded in the symplectic vector space $(P; J)$ which can be considered as the product manifold $P \times R$.

Even in the case of a general contact manifold this seems to be a natural modification of the definition of "normality" of an almost contact structure.

Proposition 7. The Nijenhuis tensor $N_J$ of the almost complex structure defined by the formula (18.1) on the symplectization $P$ of the manifold $(P; ; ; )$ with given in (5.12) is identically zero. As a result, $(P; J)$ is a complex manifold and $(P; ; ; )$ is an "indefinite Sasakian manifold".

Proof. With a slight abuse of notation we will use the coordinate $t$ instead of $p_0$ for the $n + 2$-th coordinate in $P$. By the linearity of the condition $N_J (X; Y) = 0$, it is
sufficient to only check vectors from some frame. We will use the frame \((i; X; i; P; j; \theta_i)\) for the calculations.

Recall that

\[
(X_i = P_i; \ P_i) = X_i; ( ) = 0;
\]

With this we get

\[
J (X) = (X) + (X)@; X 2 T (P); J ( ) = \theta_i; J (\theta_i) = :;
\]

Now we calculate

\[
N_j ( i; \theta_i) = J^2 [ \theta_i; \theta_i] + [ \theta_i; \theta_i] J [ \theta_i; \theta_i] = 0+ \theta_i; \ J [ \theta_i; \theta_i] J ( ) = 0;
\]

Now we calculate

\[
N_j ( X; i; \theta_i) = J^2 [ X; i; \theta_i] + [ X; i; \theta_i] J [ X; i; \theta_i] = \ P_i; \ J [ X; i; \theta_i] \ J ( ) = 0;
\]

\[
N_j ( X; i; \theta_i) = J^2 [ X; i; \theta_i] + [ X; i; \theta_i] J [ X; i; \theta_i] \ J [ X; i; \theta_i] = 0;
\]

\[
N_j ( i; \theta_i) = J^2 [ i; \theta_i] + [ X; i] J [ X; i; \theta_i] \ J [ X; i; \theta_i] = 0;
\]

\[
N_j ( i; \theta_i) = J^2 [ i; \theta_i] + [ X; i] J [ X; i; \theta_i] \ J [ X; i; \theta_i] = 0;
\]

\[
N_j ( i; \theta_i; X_j) = J^2 [ i; \theta_i; X_j] + [ P_i; P_j] J [ P_i; X_j] J [ P_i; P_j] = J ( i_j) J ( i_j) = 0;
\]

\[
N_j ( i; \theta_i; X_j) = J^2 [ i; \theta_i; X_j] + [ P_i; P_j] J [ P_i; X_j] J [ P_i; P_j] = J ( i_j) + J ( i_j) = 0;
\]

\[
N_j ( i; \theta_i; X_j) = J^2 [ i; \theta_i; X_j] + [ P_i; X_j] J [ P_i; P_j] J [ X; i; X_j] = J^2 [ i_j] + 0 + 0 = 0;
\]

Remark 5. Notice that the manifold \((P; ; ; )\) is not cosymplectic in sense of the definition of D. Blair (see Sec. 6.5). More specifically, the \((1,1)\)-tensor \(\alpha_{ij}\) is not parallel. To see this we notice that the formula

\[
2g (i X; ) Y; Z) = g (N^1 (Y; Z); X) + 2d ( Y; X) (Z) = 2d ( Z; X) (Y)
\]

for the covariant derivatives of the \((1,1)\)-tensor \(\alpha\), proved in Corollary 6.1, is true for an indefinite metric as well. In order to see that it is not parallel, substitute in this formula \(X = Z = X_i; Y = 1\). Then \(N^1 = 0\) by the previous Proposition, the second term on the right side of this formula is zero since \(X_i\) is horizontal, and

\[
2d ( X_i; X_i) ( ) = 2! \theta_i; X_i = 2;
\]

Moreover, since \(\alpha\) is not a closed form, \((P; ; ; )\) is not cosymplectic in sense of P. Libermann.

Remark 6. In the case of a contact metric manifold \((M; ; g)\) one has \(\text{Ric}( ) = 2n \ 2i g(\gamma, \omega)^2\) for the tensor \(\text{Ric}\) with the component \(N^1 (3)\) of the Nijenhuis Tensor \(N^1\), see Corollary 7.1. Unlike the Riemannian case, we have for the metric \(G\) \(\text{Ric}( ) = R_{ij} \ 1 \ 1 = R_{0i} = \frac{\partial}{\partial i}\).
19. Group action of $A_1^\sharp$.

Denote by $A_1$ the Lie group of affine transformations of the real line $\mathbb{R}$. In this section, we define an action of the product $A_1^\sharp$ on the space $\mathbb{P}$ of symplectization which is similar to the action of the Heisenberg group $H_n$ on $\mathbb{P}$ (see [25]).

The Lie group $A_1$ of affine transformations of the real line $\mathbb{R}$ can be identified with the group of $2 \times 2$ real matrices of the form

$$g = \begin{pmatrix} h & z \\ 0 & 1 \end{pmatrix}.$$ 

The Lie algebra $a_1$ of the group $A_1$ in this representation consists of matrices

$$Y(a; z) = \begin{pmatrix} a & z \\ 0 & 0 \end{pmatrix},$$

and the exponential mapping $\exp: A_1 \to A_1$ takes the form

$$\exp \begin{pmatrix} a & z \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^a & e^a z \\ 0 & 1 \end{pmatrix}.$$ 

Left translations by the elements of the basic one-parameter group $\exp(t Y(1; 0))$:

$$e^t \begin{pmatrix} 0 & h \\ 0 & 1 \end{pmatrix} = e^t h \begin{pmatrix} e^t & e^t z \\ 0 & 1 \end{pmatrix}$$

generate the basic right invariant vector field

$$a = h\partial_h + z\partial_z;$$

while a similar action of the one-parameter group $\exp(t Y(0; 1))$ produces the right invariant vector field

$$z = \partial_z;$$

We have $[a; z] = z$.

For the right translations we have, respectively

$$\begin{pmatrix} h & z \\ 0 & 1 \end{pmatrix} e^t \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = e^t h \begin{pmatrix} e^t & e^t z \\ 0 & 1 \end{pmatrix}, \quad a = h\partial_h;$$

$$\begin{pmatrix} h & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} h & z + h t \\ 0 & 1 \end{pmatrix}, \quad z = h\partial_h: \quad (19.1)$$

We have $[a; z] = z$.

Consider now the identification of the Lie group $A_1$ with the upper half-space $R^2_+ = \{ (p; x) \mid 2 \mathbb{R}^2 \setminus \mathbb{R}^2 \}$

$$\sim: h \begin{pmatrix} x & p \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h \\ x \end{pmatrix}, \quad \sim^{-1}: p \begin{pmatrix} x & p \\ h & h \end{pmatrix} \begin{pmatrix} x \\ 0 \end{pmatrix}.$$ 

Under this identification

$$\begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} h \begin{pmatrix} e^t h \\ 0 \end{pmatrix} = \begin{pmatrix} e^t h \\ 0 \end{pmatrix},$$

so that

$$\begin{pmatrix} a \\ p \end{pmatrix} = p\partial_p;$$

Since

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} h \begin{pmatrix} h \\ 0 \end{pmatrix} = \begin{pmatrix} h \\ (z + t)h \end{pmatrix},$$
we get
\( (c) = p^i \partial_x^i : \)

We also calculate
\( (pdx) = dz \quad zh^1 dh; \quad (dp^i dx^i) = h^1 dh \wedge dz: \)

Now we apply these considerations to the product mapping
\[
\begin{array}{cccc}
\psi^n & \psi^n \\
\cap & 0 & 0 \\
\imath: & A^i_1, & F_2 = f(p_i;x_i) & 2 F_p > 0 \forall y \\
\end{array}
\]

We see that this mapping is the diemorphism satisfying
\[
(1.a) = p_i \partial_{p_i} = P_1; \quad (1.2) = p^i \partial_{x^i} = L_1;
\]

In addition to this,
\[
\begin{align*}
X & (p_i dx^i) = X (dz_i z_i h^1_i dh_i); \\
X & (dp^i_i dx^i) = p^1_i dp^i_i \wedge dz_i;
\end{align*}
\]

Dual to the right invariant frame \( \gamma ; z \) is the coframe

\[
!_a = h^1_i dh; \quad !_z = dz;
\]

Thus,
\[
\begin{align*}
X & \left( dp^1_i \wedge dx^i \right) = p^1_i dp^i_i \wedge dz_i \\
\end{align*}
\]

is a right invariant symplectic structure on the group \( A^{n+1}_1: \)

**Proposition 8.** The symplectic structure on the group \( A^{n+1}_1: \)

Proposition 8. The map
\[
\begin{array}{cccc}
\psi^n & \psi^n \\
\cap & 0 & 0 \\
\imath: & A^i_1, & F_2 = f(p_i;x_i) & 2 F_p > 0 \forall y \\
\end{array}
\]

defined by
\[
\begin{align*}
\psi^n & = h_i z_i \quad 0 \quad 1 \\
\cap & 0 & 0 \\
\imath: & \psi_i = 0; \quad h_i z_i g \\
\end{align*}
\]

defines a symplectomorphism of the symplectic space \( F_2; \quad F_p \cap \psi_i = 0 p_i dx^i \) with the product of \( n+1 \) copies of a one group \( A^i_1 \) endowed with the symplectic structure generated by the right invariant 1-form
\[
\begin{align*}
\frac{1}{n+1} (dz_i z_i h^1_i dh_i): \\
\end{align*}
\]

20. **Hyperbolic Rotations and the projectivization of \( F \).**

In this section we construct a natural compactification of the TPS \( F \) endowed with the extension of the contact structure and that of the indefinite metric \( G \).

Consider the action of the one-parameter group \( R \) in the space \( F \) acting by the one-parameter group \( H R \) of hyperbolic rotations
\[
g^p : (p_i x^i) ! (e^p \partial_p e^{-p} x^i): \quad (20.1)
\]
We have obviously

Lemma 1. (1) The 1-form $\omega$ is invariant under this action of the group $H \cdot R$.
(2) The metric $G$ is invariant under the action of the group $H \cdot R$.

Proposition 9. The space $\mathcal{O}$ of orbits of the points $P \setminus 0$ under the action of the group $H \cdot R$ is canonically isomorphic to the projective space $P_{2n+1}(R)$.

Proof. Cover the space $R^{2n+2}$ with the open subsets of two types:

Sets of the first type are
\[ \mathcal{U}_j = \{ \text{points } x \in R^{2n+1} \} \]
and associate with these sets the affine domains
\[ U_j \cap R^{2n+1} \]
of the projective space $P_{2n+1}(R)$ with the coordinates
\[ (x^i p_j; p_j) \]

Sets of the second type are
\[ \mathcal{V}_k = \{ \text{points } x \in R^{2n+1} \} \]
and associate with these sets the affine domains
\[ V_k \cap R^{2n+1} \]
of the projective space $P_{2n+1}(R)$ with the coordinates
\[ (x^i p_j; p_j) \]

Notice that on the intersections $U_j \setminus U_j$, we have relations between the corresponding affine coordinates
\[ x^i p_j = x^i p_j, \quad p_k = p_k \]

On the intersections $U_j \setminus V_k$, we have relations between the corresponding affine coordinates
\[ x^i p_j = x^i p_j, \quad p_k = p_k \]

Finally, on the intersections $V_j \setminus V_j$, we have relations between the corresponding affine coordinates
\[ x^i p_j = x^i p_j, \quad p_k = p_k \]

This shows that affine coordinates of all the affine charts are related by the transition functions invariant under the action of hyperbolic rotations. Thus, they are glued into the standard projective space $P_{2n+1}(R)$.

Combining Lemma 2 and the previous construction we get the following

Proposition 10. (1) The projections $\pi$ of the 1-form $\omega$ and that of the metric $G$ of $\mathcal{O}$ endow the projective space $\mathcal{O}$ with the contact structure and the metric of signature $(n+1; n)$.
(2) The composition $J$ of the embeddings $j : P \to \mathcal{O}$ and the projection $P \to \mathcal{O}$ defines the compactification $\mathcal{O}$ of the TPS $(P; \omega, G)$ with the contact structure and Möbius metric $G$. 
21. Group action of $H_n$ and the "partial orbit structure" of $\mathcal{P}$.

In this section we consider the lift to the space $\mathcal{P}$ of the action of the group $H_n$ on $P$ discussed in Sec. 12 and the action of subgroups of $H_n$ on the cells of an arbitrary structure of the standard CW-structure of the projective space $\mathcal{P}$.

The differential operators $X_{i;P_j}$ of the canonical frame (6.5) act also in the space $\mathcal{P}$ with the same commutator relations. This action generates the action of the Lie group $H_n$ on the space $\mathcal{P}$ leaving hyperplanes $p_0 = \text{const}$ invariant.

Introduce the sequence $L_n; k = 1,\ldots,n$ of subgroups of the Heisenberg group $H_n$ defined by the condition

$$L_n; k = fg(a;bc)p_0 = \cdots = b_k = 0g; \quad (21.1)$$

These subgroups form the series

$$H_n \quad L_n; 1 \quad L_n; 2 \quad \cdots \quad L_n; n;$$

It is easy to see that

$$L_n; k' R^k H_n; k \quad (21.2)$$

is the product of the $k$-dim. abelian group $R^k$ and the Heisenberg group $H_n; k$.

The right invariant vector fields on $H_n$ tangent to (and generated by) the subgroup $L_n; k$ are (in terms of the isomorphism of Sec. 12) $;X_{i;P_j} = 1,\ldots,n; P_{j;P_0} = k + 1,\ldots,n$.

In the space $\mathcal{P}$, consider the affine planes $V_k$ defining the cells of the standard cell structure of the projective space $\mathcal{P} = P_{2n+1}(R)$ with respect to the (hyperbolically) homogeneous coordinates of $\mathcal{P}$

$$V_k = f(x^1;p_0) = p_1;\ldots;p_k = 0; p_k = 1g; \quad (21.3)$$

with $k = 0,1,\ldots,n$. It is clear that the projective space $\mathcal{P}$ is obtained by gluing to the cell $V_0 = (P)$ the smaller cells $V_1;V_2;V_3;\cdots$; and finally, by gluing in $n$-dim. projective space $P_n(R)$ obtained by the action of hyperbolic rotations (usual dilatations here) on the subspace $V_{n+1} = f(x^1;p_0 = 0) = 0; p_1;\ldots;p_n$.

It is easy to see now that each cell $V_k$ is canonically diomorphic to the group $L_n; k$ whose action on $V_k$ is induced by the action of the Heisenberg group $H_n$ on the space $\mathcal{P}$ considered above.

So, even though the action of $H_n$ on $P$ cannot be extended to the compactification $\mathcal{P}$, a coherent action of the subgroups of series (21.1) produces the partial cell structure of $\mathcal{P}$ starting with the projective subspace $V_{n+1} = P_{n}(R)$.

The restriction of the 1-form $\tilde{\gamma}_k$ to the cell $V_k$ has the form

$$\tilde{\gamma}_k = x^0 + \sum_{i=k+1}^n p_idx^i.$$ 

Therefore, this form determines the canonical contact structure on the Heisenberg factor of the cell $V_k^* \sim L_n; k' R^k H_n; k$ and is zero on the rest factor.

The restriction of the Maurer-Cartan metric $\mathcal{G}$ to the cell $V_k$ has, in variables $(p_0;\ldots;p_n;x^0;\ldots;x^k;\ldots;x^n)$, the form

$$\mathcal{G}_k = \mathcal{G} \tilde{\gamma}_k = \begin{pmatrix} 0 & 0_{(n, k)} & 0_{(n, k)} & I_n & 1 \\ 0_{(n, k)} & 0_{k & n} & 0_{k & n} & 0_{k & n} & A \\ I_n & 0_{(n, k)} & 0_{(n, k)} & 0_{(n, k)} & 0_{(n, k)} \end{pmatrix} \quad (21.4)$$

with

$$A = \begin{pmatrix} p_0 & p_1 & \cdots & p_k \\ 0 & p_0 & \cdots & p_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p_0 \end{pmatrix},$$

where $p_0, p_1, \ldots, p_k$ are the basis elements of the right invariant vector fields on $H_n$ tangent to the subgroup $L_n; k$. The determinant $\text{det} A$ is equal to $p_0\cdots p_k$. Conditions $p_0 < 0$ and $p_k > 0$ are satisfied.
Thus, this metric is zero on the rst abelian factor of the cell $V_k \backslash L_n \backslash k$ and coincides with the M rugala metric on the Heisenberg factor $H_n \backslash k$ of the cell $V_k$.

Combining these arguments we get the following

Theorem 4. (1) The restriction of the action of the Heisenberg group $H_n$ on the space $P \backslash V_0$ (embedded in $P^k$) to the subgroup $L_n \backslash k \backslash R^k \backslash H_n \backslash k$ of the form (23.1) extends to the action of this subgroup on the cell $(V_k) \backslash P^k$ and determines the di eomorphism of the group $L_n \backslash k$ with $V_k$ and with its image $(V_k) \backslash P^k$.

(2) Restrictions of the 1-form $\sim$ and metric $G$ to the cell $V_k$ endow the Heisenberg factor $H_n \backslash k$ of $V_k$ with the contact structure $k$ and the M rugala metric $G_k$ and are both zero on the abelian factor $R^k$.

Remark 7. Every cell $V_k$ represents the therm odynamical phase space of an abstract thermodynamical system with $n$ extensive and $k$ intensive variables. This corresponds to a situation where the thermodynamical potential $(x^i)$ depends on $x_k+1,\ldots,x^n$ but not on the $k$ intensive variables $x^i$. As a result the participation of factors $x^i; i=0,\ldots,k$ in the processes is "switched out" and they become parameters only.

Example 3. Consider the case $n=2$, i.e. take $P^5$ to be ve-dimensional with the contact form $dU \otimes (s dt + pdV)$ (a one-component homogeneous system, per 1 mol). Hypersurface $C$ (see Sec.11) has, in this case, the well known form

$$U^*ST + pdV = 0.$$ Its lift to $P^* - C$ has the form $p_0U \otimes ST + pdV = 0$.

Intersection of this quadratic with the plane $p_0 = 0$ is the (degenerate) quadratic $C_1: pV = ST$. Fixing value of $S$, say, taking $S = R$ constant determines the cell $V_1$ that projects onto the cell $V_1$ of the compact space $P$. In age of the quadratic $C_1$ under this projection determines in the $3$-dim $H_1$-factor of the cell $V_1$ the surface $pV = RT$ given by the equation of one-atom ideal gas.

Hypersurface $C$ is the submanifold containing all the constitutive (equilibrium) surfaces of all thermodynamical systems with the TPS $P^5$. Closures of these surfaces in $P^5$ contains points from cells of smaller dimension $V_1; V_2$: Thus, equation of a manifold ideal gas appears here as the equation of the surface $pV = RT$ by the limit points in $V_1$ of all possible constitutive surfaces in $P^5$.

Remark 8. The construction of a compact manifold $F^k$ in terms of a series of subgroups (23.1) of a Lie group $H$ represents a way to represent a manifold in terms of a Lie group with the open dense orbit isomorphic to the group $H$ itself and a natural Whitney stratification in terms of extensions of subgroup actions on the cells of smaller dimension. Removed generators (here $P; i=1,\ldots,k$) determine the projections from cells of higher dimension to the cells in their closure (see Sec.

22. Conclusion

In this work we've studied basic properties of the indefinite metric $G$ of R M rugala defined on the contact $(2n+1)$-dimensional phase space $(P; )$ of a homogeneous thermodynamical system. We have calculated the curvature tensor, Killing vector fields, and the second fundamental form of the Legendre submanifolds of $P$ - constitutive surfaces of different homogeneous thermodynamical systems, and established...
an isomorphism of the TPS \( (\mathcal{P}; L; \mathcal{G}) \) with the Heisenberg Lie group \( H_n \) endowed with the right invariant contact structure and the right invariant indefinite metric.\(^{23}\) We lifted the metric \( \mathcal{G} \) to a metric \( \mathcal{G}' \) of signature \((n + 1; n + 1)\) in the symplectization \( \mathcal{F} \) of the contact space \( (\mathcal{P}; \mathcal{L}) \) and studied curvature properties and Killing vector fields of this metric. Finally, we introduced the "hyperbolic projection" of the space \( (\mathcal{P}; \mathcal{L}; \mathcal{G}) \) that can be considered as the natural compactification (with the contact structure and the indefinite metric) of the TPS space \( (\mathcal{P}; \mathcal{L}; \mathcal{G}) \).

Many interesting questions were left outside of this paper—study of the geodesics of the metric \( \mathcal{G} \), the relation of the metric properties of \( (\mathcal{P}; \mathcal{L}; \mathcal{G}) \) with the contact transformation (see), the characterization of the submanifolds of signature changes of the thermodynamic world metrics on the Legendre submanifolds (related to the phase transitions in the corresponding homogeneous thermodynamic systems) in terms of the Grassmannian of the Legendre n-dimensional subspaces in \( \mathcal{F} \), the use of the geometry of Heisenberg group \( H_n \) to the study of Legendre submanifolds of \( (\mathcal{P}; \mathcal{L}; \mathcal{G}) \), and others. Some of these questions will be considered in the continuation of this work.

In the conclusion we would like to thank Professor M. Goze for the useful information about contact and metric structures on the Heisenberg group.

### 23. Appendix: Killing vector fields for \( \mathcal{G} \).

In this Appendix we provide the details of calculations of Killing vector fields of metric \( \mathcal{G} \).

Conditions for a vector \( \mathbf{X} \in \mathcal{F} \) on a manifold \( M \) with a (pseudo-Riemannian) metric \( g \) to be a Killing vector field have the form \( L_{\mathbf{X}} g = 0 \), or, in local coordinates \((p_i, x^i)\),

\[
\begin{align*}
\frac{\partial g_{ij}}{\partial x^k} + g_{ik} \frac{\partial g_{kj}}{\partial x^j} + g_{jk} \frac{\partial g_{ki}}{\partial x^i} &= 0 \quad \text{for all } i, j, k; \quad (23.1)
\end{align*}
\]

We take a vector field \( \mathbf{X} \) in \( \mathcal{F} \) in the form

\[
X = p_i \partial_{p_i} + x^i \partial_{x^i} \quad \text{(23.2)}
\]

and consider cases of different pairs of indices \((ij)\).

**Case** \((ij) = (p_i, p_j)\):

\[
0 + p_i \partial_{p_i} \partial_{x^j} + p_j \partial_{p_j} \partial_{x^i} = 0
\]

or

\[
\partial_{p_i} x^i + \partial_{p_j} x^j = 0; \quad (23.3)
\]

**Case** \((ij) = (p_i, x^j)\):

\[
0 + \frac{1}{k} \partial_{x^k} x^i + (\frac{1}{k} \partial_{x^k} p_k + p_k \partial_{p_k} \partial_{x^i} x^j) = 0;
\]

or

\[
\partial_{x^j} x^i + \partial_{p_i} p_k + p_k \partial_{p_k} \partial_{x^i} x^j = 0; \quad (23.4)
\]

**Case** \((ij) = (x^i, x^j)\):

\[
(p_i \partial_{p_i} + p_j \partial_{p_j} + (\frac{1}{k} \partial_{x^k} p_k + p_k \partial_{p_k} \partial_{x^i} x^j) + (\frac{1}{k} \partial_{x^k} p_k + p_k \partial_{p_k} \partial_{x^i} x^j) = 0;
\]

or

\[
(p_i \partial_{p_i} + p_j \partial_{p_j} + (\partial_{x^j} p_i + p_j \partial_{x^j} \partial_{x^i} x^j) + (\partial_{x^i} p_j + p_j \partial_{x^j} \partial_{x^i} x^j) = 0. \quad (23.5)
\]

We will need the following
Lemma 2. Let $f^{i}(y^{i};:::;y^{n})$ be $n$ functions of $n$ variables $x^{i}$ such that for all $i,j$

\[ \theta_{y^{i}}f^{j} + \theta_{y^{j}}f^{i} = 0; \]

then

\[ f^{i} = a^{i}_{j}y^{j} + b^{i} \]

with some constant vector $b^{i}$ and skew-symmetric constant matrix $(a^{i}_{j})$: $a^{i}_{j} = a^{j}_{i}$.

Proof. Apply $\theta_{y^{k}}$ to the condition of Lemma. We get

\[ \theta_{y^{k}}\theta_{y^{i}}f^{j} + \theta_{y^{j}}\theta_{y^{k}}f^{i} = 0; \]

cyclic permutation of indices $ijk$ give us two more equalities. Add last two equations and subtract the third one. We get

\[ 2\theta_{y^{k}}\theta_{y^{l}}f^{k} = 0; \]

This being true for all triples of indices $ijk$ shows that all second derivatives of all functions $f^{k}$ are zero.

Therefore

\[ f^{i} = \sum_{j}a^{i}_{j}y^{j} + b^{i} \]

with constant coefficients.

Writing down condition of Lemma for these linear functions we end that the matrix $a^{i}_{j}$ is skew-symmetric.

Applying this Lemma to the functions $x^{i}$ with $p_{i}$ as arguments in Lemma we have due to the equality (23.3)

\[ x^{i} = a^{i}_{j}(x)p_{j} + b^{i}(x) \quad (23.6) \]

with skew-symmetric matrix function $a^{i}_{j}(x)$ and scalar functions $b^{i}(x)$. This solves equations (23.3).

Substituting these expressions into the other two families of equations we present these equations in the form

\[ \theta_{x^{i}}a^{k}_{j}(x)p_{k} + \theta_{x^{k}}b^{j}(x) + \theta_{p^{i}}p^{j} + p_{p}p_{k}a^{i}_{k}(x) = 0; \]

\[ p_{k}p^{j} + p^{j}p^{k} + \theta_{p^{j}}p^{k} + p_{p}p_{k}(a^{i}_{j}(x)p_{j} + b^{i}(x)) + p_{p}p_{k}(a^{j}_{k}(x)p_{j} + b^{j}(x))) = 0; \quad (23.7) \]

Rewrite 1st equation in the form

\[ \theta_{p^{i}}p^{j} = \theta_{x^{i}}a^{k}_{j}(x)p_{k} + p_{p}p_{k}a^{i}_{k}(x) = \theta_{x^{i}}b^{j}; \quad (23.8) \]

where we have used skew-symmetry of a matrix $a$ in the last term, and apply $\theta_{p^{k}}$ to this formula. We get

\[ \theta_{p^{k}}\theta_{p^{i}}p^{j} = \theta_{x^{i}}a^{k}_{m}(x) + p_{p}a^{k}_{m}(x) + a^{i}_{m}p_{k}; \]

Swiching $m$ and $k$ we get

\[ \theta_{p^{k}}\theta_{p^{i}}p^{j} = \theta_{x^{i}}a^{k}_{m}(x) + p_{p}a^{k}_{m}(x) + a^{i}_{m}p_{k}; \]

Equating terms in the right side that do not depend on $p$ we get

\[ \theta_{x^{i}}a^{k}_{m}(x) = \theta_{x^{i}}a^{k}_{m}(x) \quad \theta_{x^{i}}(2a^{i}_{m}(x)) = 0; \]

Thus, matrix $(a^{i}_{m})$ is constant.
Equating terms linear by $p$ we get
\[ p_a a_1^m (x) + s a_k^m p_k = p_a a_1^m (x) + s a_k^m p_k : \]
For $s \neq 1$ this gives $a_1^m = a_2^m$ and, due to the skew-symmetry of matrix $a$, $a_1^m = 0$.
For $s = 1$ we get $p_a a_1^m = p_a a_1^m + a_1^m p_k$. From this it follows that $a_k^m = 0$ for all $k \neq 1$ and that $a_1^m = 0$ as well. Finally, if $s = 1 = m$ we get by the skew-symmetry of $a_1^m$ that $a_1^m = 0$. Therefore, matrix $a$ is zero ($a_1^2 = 0$ and $x^* = b^s (x)$).

As a result, rst of the equations (23.7) reads now
\[ \Theta_{p_i} p_s = \Theta_{x^*} b^1 (x) : \]

This solves the rst of equations (23.7).

The second equation in (23.7) can now be written in the form
\[ p_1 \left( \Theta_{b^1 (x)} \right) (p_1 + h^1 (x)) + p_2 \left( \Theta_{b^2 (x)} \right) (p_2 + h^2 (x)) + \Theta_{b^1 (x)} \left( \Theta_{x^*} b^1 (x) p_1 + h^1 (x) \right) + \Theta_{b^2 (x)} \left( \Theta_{x^*} b^2 (x) p_2 + h^2 (x) \right) = 0; \]

or
\[ p_1 h^1 (x) + p_2 h^2 (x) + \Theta_{b^1 (x)} h^1 (x) + \Theta_{b^2 (x)} h^2 (x) = 0; \]

Tensors independent on $p$ give us
\[ \Theta_{x^*} h^i (x) = \Theta_{x^*} h^i (x) = 0; \]

for all $i; j$. The lemma above gives us
\[ h^i (x) = X = q_j^i x^j + k_i^i \]
with constant coefficients and $q_j^i = q_j^i$.

Linear by $p$ part of equality (23.12) has the form
\[ p_1 h^1 (x) + p_2 h^2 (x) = \Theta_{x^*} \left( \Theta_{b^1 (x)} p_1 + h^1 (x) \right); \]

Taking derivatives by $p_1$ and then by $x^1$ we get,
\[ \Theta_{p_1} : \ h^1 (x) + \frac{1}{2} h^1 (x) = 2 \Theta_{x^*} \Theta_{b^1 (x)} \]
and then as the core of the $(b^1 (x))$ (using (23.13))
\[ \Theta_{x^*} : \ q_j^i + \frac{1}{2} q_j^i = 2 \Theta_{x^*} \Theta_{b^1 (x)} \]
Since the right side in the second equation (s) is symmetrically (by $i; j$), one has
\[ q_j^i + \frac{1}{2} q_j^i = q_j^i + \frac{1}{2} q_j^i \]
no summation.

For $i = j$ last equality gives (since the matrix $q$ is skew-symmetric)
\[ 2 q_j^i = q_j^i = q_j^i ; \]
so that for all $i; j q_j^i = 0$ and therefore,
\[ h^i (x) = k^i : \]

\[ \text{(23.9)} \]

\[ \text{(23.10)} \]

\[ \text{(23.11)} \]

\[ \text{(23.12)} \]

\[ \text{(23.13)} \]

\[ \text{(23.14)} \]

\[ \text{(23.15)} \]

\[ \text{(23.16)} \]

\[ \text{(23.17)} \]
Substituting this back to (23.14) we get
\[ p_i k^j + p_j k^i = 2\theta_x \theta_x \theta^j(x) p_i \]  
(23.18)
Differentiating by \( p_i \); \( i \ne j \) we get
\[ \theta_x \theta_x \theta^j(x) = 0 \]
As a result, \( b^j(x) \) is linear by all variables except \( x^i \)
\[ b^j(x) = \sum_{k \ne l} X^{k} x^k + ^1(x^l) \]
(23.19)
Using \( h^k(x) = k^l \) in (23.15) we get
\[ \sum_{k \ne l} \theta_x \theta_x b^j(x) = ^1 x^r \]
(23.20)
Taking here \( r = l \), then \( i = l \ne j = r \) we nd
\[ \theta_x^2 b^l(x) = k^l \; \theta_x \theta_x b^l(x) = \frac{1}{2} k^r \]
Substituting here the expression for \( b^l(x) \) from (23.9) we get the 1st equation in the form
\[ k^l = \sum_{r \ne l} X^{r} x^r + ^1(x^l) \]
Therefore, \( ^1 \) is linear by \( x^i \) while
\[ ^1 = \frac{1}{2} k^l (x^l)^2 + ^1 x^l + ^1 \]
The second equation, valid for \( i \ne l \), gives \( ^2 x^l = \theta_x ^1 (x^l) \) and therefore
\[ \frac{1}{r} = \frac{1}{2} k^r x^l + ^1 \]
Combining yields the coefficients of \( b^l \) and using them in (23.19) we nd
\[ x^l = \sum_{r \ne l} X^{r} x^r + \frac{1}{2} k^l (x^l)^2 + ^1 x^l + ^1 = \sum_{r \ne l} X^{r} x^r + \frac{1}{2} k^r x^l + ^1 + ^1 \]
(23.21)
or, renaming coefficients \( ^1 = \frac{1}{l} \)
\[ x^l = \sum_{r \ne l} X^{r} x^r + \frac{1}{2} k^r x^l \]
(23.22)
The equality (23.10) with the found values of the coefficients gives us
\[ p_i = \sum_{l = 1}^{X \theta_x} x^l p_1 + \sum_{k = 1}^{k^s} X^{k} \left( \frac{1}{s} + \frac{1}{2} \frac{1}{s} \frac{1}{s} \right) \left( k^r x^r \right) + \frac{1}{2} x^l k^s p_1 = \]
\[ \begin{array}{c}
\sum_{l = 1}^{X \theta_x} X^{l} p_1 + \sum_{r = 1}^{X} \frac{1}{r} \frac{1}{s} \frac{1}{s} \left( x^l p_1 \right) \frac{1}{2} x^l p_1 \\
\end{array} \]
(23.23)
As a result we get for a Killing vector field $X$, the following representation with arbitrary scalar coefficients $k^r$; $s^i_r$:

$$X = x^s \theta_s^{(s)} + p^r \theta_p^r = X^{r} \left( \frac{1}{2} k^r x^r \right) x^s + X^{r} s^i_r \theta_s^{(r)} + p^r \theta_p^r$$

$$+ \frac{1}{2} \left( \frac{1}{2} x^r \right) \left( \frac{1}{2} x^{s^i_r} \right) \left( \frac{1}{2} x^p \right) \left( \frac{1}{2} x^s \right) \theta_p^r \theta_p^r + \left( \frac{1}{2} x^p \right) \left( \frac{1}{2} x^s \right) \theta_p^r \theta_p^r$$

Splitting these expressions in accordance with the different independent parameters we get the following basis of the Lie algebra of Killing vector fields:

$s^i_r$-term

$$X_{s^i_r} = \theta_s^{(s)}$$

$q^r$-term

$$Q^r_s = x^r \theta_s^{(r)} + p^r \theta_p^r$$

$k^i$-term $s$

$$D^i = \frac{x^i}{2} \left[ x^s \theta_s^{(s)} + p^r \theta_p^r \right] = \left( \frac{1}{2} \left( \frac{1}{2} x^r \right) \right) \theta_p^r = \frac{x^i}{2} Q + \left( \frac{1}{2} \left( \frac{1}{2} x^i \right) \right) \theta_p^r$$

where

$$Q = X^{s} Q_s^{r} = x^i \theta_p^r$$

Vector fields $X^i$ form the abelian Lie subalgebra $\mathbf{x}$ of Lie algebra of Killing vector fields $k^i$.

It is easy to see that $(n + 1)^2$ vector fields $Q^r_s$ form the Lie subalgebra $q$ of the type $\mathfrak{gl}(n + 1; \mathbb{R})$ since

$$[Q^i_j; Q^k_p] = \frac{1}{2} Q^i_j Q^k_p$$

similar to the commutator relations of the basic matrices $E^i_j$ of $\mathfrak{gl}(n + 1; \mathbb{R})$. Vector fields $Q$ generate the center of this Lie subalgebra.

We also have

$$[Q^i_j; X_s] = \frac{1}{2} X^i_j [Q; X_s] = X_s$$

so that adjoint action of $q$ on $x$ is isomorphic to the standard action of $\mathfrak{gl}(n + 1; \mathbb{R})$ on $\mathbb{R}^{n+1}$.

Calculate commutator $[D^i; D^j]$

$$[D^i; D^j] = \frac{x^i}{2} Q + \left( \frac{1}{2} \left( \frac{1}{2} x^r \right) \right) \theta_p^r + \frac{x^j}{2} Q + \left( \frac{1}{2} \left( \frac{1}{2} x^r \right) \right) \theta_p^r$$

$$+ \frac{x^i}{2} \left( \frac{1}{2} x^p \right) \theta_p^r + \frac{x^j}{2} \theta_p^r + \left( \frac{1}{2} \left( \frac{1}{2} x^p \right) \right) \theta_p^r + \left( \frac{1}{2} \left( \frac{1}{2} x^p \right) \right) \theta_p^r$$

$$+ \frac{x^i}{2} \frac{1}{2} x^j + \frac{x^j}{2} x^i \theta_p^r + \frac{x^j}{2} x^i \theta_p^r + \left( \frac{1}{2} \left( \frac{1}{2} x^p \right) \right) \theta_p^r + \left( \frac{1}{2} \left( \frac{1}{2} x^p \right) \right) \theta_p^r$$

$$= \frac{1}{4} Q x^i x^j + \frac{1}{4} x^i x^j \left( \frac{1}{2} x^s \theta_s^{(s)} + \frac{x^j}{2} \theta_p^r + \left( \frac{1}{2} \left( \frac{1}{2} x^p \right) \right) \theta_p^r \right) + \left( \frac{1}{2} \left( \frac{1}{2} x^p \right) \right) \theta_p^r + \left( \frac{1}{2} \left( \frac{1}{2} x^p \right) \right) \theta_p^r = 0$$

(23.28)
Thus, vector elds $D^i$ also form the abelian Lie subalgebra $\mathfrak{d}$ of $k$.

Next we calculate

\[
P^s; Q^i = \frac{x^s}{2} Q + (1 - \frac{1}{2} (x^p D_p, x^l \partial l) \theta_p, x^l \theta_l, p_j \theta_p) = \frac{x^s}{2} f \ x^m \ m_i \theta_i, + \ p_j \ j \theta_j, g^+ \\
+ f (1 - \frac{1}{2} (x^l \partial l, j \partial l) g \ x^i [\frac{s}{2} Q + \frac{x^s}{2} \theta_s] + (1 - \frac{1}{2} (x^p D_p, x^l \theta_l, p_j \theta_p) = \\
x^s f \ x^l \theta_l, + p_j \theta_p, g (1 - \frac{x^s}{2} r \ s \partial s \ l \theta l] \ x^i [\frac{s}{2} Q + \frac{x^s}{2} \theta_s] + \frac{x^s}{2} \theta_s, p_j \theta_p] \ x^s p_j \theta_p, x^l p_l \theta_l] = \\
= (1 - \frac{x^s}{2} r \ s \partial s \ l \theta l] \ x^i \ j \theta_j, + \ j \theta_j, Q = \ j \theta j, D^i; (23.29)
\]

or

\[
P^s; Q^i = \ j \theta j, D^i; (23.30)
\]

so, abelian subalgebra $\mathfrak{d}$ is invariant under the adjoint action of $g$ with the same standard action as for $x$.

Notice that

\[
Q^j = \text{Id}_j; Q^j = \text{Id}_j;
\]

Finally,

\[
[x^s; D^i] = [x^s; \frac{i}{2}], Q + (1 - \frac{1}{2} (x^p D_p, x^l \theta_l, p_j \theta_p) = \frac{i}{2} x^s \ x^l \theta_l, \ p_j \theta_p, + \frac{i}{2} \ p_j \theta_p = \\
= \frac{i}{2} (x^s \theta_l, + p_j \theta_p) + \frac{i}{2} j \theta j, Q = \frac{i}{2} Q^j + \frac{i}{2} j \theta j, D^i; (23.31)
\]

Next step is to prove that the Lie algebra isos of Killing vector elds is isomorphic to the Lie algebra $\mathfrak{sl}(n + 2; R)$.

Consider the Lie algebra $\mathfrak{gl}(n + 2; R)$ of real matrices and introduce several subspaces of this Lie algebra:

The Lie subalgebra $\mathfrak{g}_{n+1}$ of matrices

\[
g = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} ; \ A \ \mathfrak{gl}(n + 1; R)
\]

with the basis form ed by matrices $E^i_j$ having the entry 1 at the $ij$ place and zero otherwise,

the subspace $\mathfrak{d}$ of matrices of the form

\[
0 \ f \\
0 \ 0
\]

with the basis $f^l g$ of matrices having entry 1 at the place $l(n + 2)$ and zero at all other places,

the subspace $\mathfrak{x}$ of matrices of the form

\[
0 \ 0 \\
e \ 0
\]

with the basis $f^l j g$ of matrices having entry 1 at the place $(n + 2)j$ and zero at all other places,
one-dimensional subspace of matrices

\[
\begin{pmatrix}
0 & 0 \\
0 & t
\end{pmatrix}; t \in \mathbb{R}
\]

with the basis \( E \) formed by the matrix \( E = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \).

The Lie algebra \( \mathfrak{gl}(n+2;\mathbb{R}) \) is (as the vector space) the direct sum of these four subspaces. The commutator relations between the basis introduced above are

\[
[E_j^i; E_k^j] = [E_k^i; E_j^k]; E_j^i; 1^k] = \frac{i}{j} 1^j; E_j^i; 1_k] = \frac{j}{i} 1_j;
\]

\[
[E; 1^k] = 1^k; [E; 1_k] = 1_k; [1^i; 1_j] = E_j^i \frac{i}{j} E : (23.32)
\]

In the complexification \( \mathfrak{gl}(n+2;\mathbb{C}) \) of this Lie subalgebra \( \mathfrak{gl}(n+1;\mathbb{R}) \) of matrices

\[
A \text{ if } \lambda = t
\]

where \( A \in \mathfrak{gl}(n+1;\mathbb{R}); f; e \in \mathbb{R}^n; t \in \mathbb{R} \). The center of this Lie algebra is one-dimensional, formed by the matrices proportional to the unit matrix \( I_{n+2} \). Factorization by this center defines the epimorphism \( \mathfrak{gl}(n+1;\mathbb{R}) \to \mathfrak{sl}(n+1;\mathbb{R}) \) onto the Lie subalgebra of traceless matrices of the form (23.33).

Define the embedding \( j: \mathfrak{iso}_G \to \mathfrak{gl}(n+2;\mathbb{C}) \) as follows

\[
P^{-X}k! 1l_k; P^{-D} 1^i; 1^i; Q_1^i \neq E_j^i.
\]

It is easy to see that image of \( \mathfrak{iso}_G \) under this embedding is the subspace of \( \mathfrak{gl}(n+1;\mathbb{R}) \): The commutator relations between the basic elements \( X; D; Q_1 \) are preserved except for the last one. We have

\[
\begin{align*}
\hat{j}(P^{-D} 1^i; P^{-X}k) &= j(Q_1^i 1^i; Q_1^k) = E_k^i I_{n+1}; E_j^i 1^j E + I_{n+2} = \\
&[1^i; 1_k] \hat{I}_{n+2} = j(P^{-D} 1^i); j(P^{-X}k) \hat{I}_{n+2}: (23.34)
\end{align*}
\]

Combining the embedding \( j \) with the projection \( \mathfrak{sl}(n+1;\mathbb{R}) \to \mathfrak{sl}(n+1;\mathbb{R}) \) we get the isomorphism of Lie algebras

\[
\mathfrak{iso}_G \to \mathfrak{sl}(n+1;\mathbb{R})
\]

The Lie algebras \( \mathfrak{sl}(n+1;\mathbb{R}) \) and \( \mathfrak{sl}(n+1;\mathbb{C}) \) are dual to one another in the complex Lie algebra \( \mathfrak{sl}(n+1;\mathbb{C}) \) relative to the involution

\[
A_f \to \mathbb{C} \quad \mathbb{C} \to \mathbb{C}
\]

and isomorphic as Lie algebras (see ). Combining all the calculations above we get the proof of the following

Theorem 5. The Lie algebra \( \mathfrak{iso}_G \to \mathfrak{sl}(n+1;\mathbb{R}) \) of Killing vector fields of metric \( G \) is (as the vector space) the linear sum

\[
\mathfrak{iso}_G = x^i \delta_{ij}; \quad p_j \partial_{p_i} > ;
\]

1) \( q = \langle Q_j^i = x^i \delta_{ij}; \quad p_j \partial_{p_i} > ; \)
with the commutator relations

$$[\mathbf{Q}_i^j;\mathbf{Q}_k^p] = \frac{i}{2} \mathbf{Q}_k^i \mathbf{Q}_j^p;$$

Subalgebra $\mathfrak{g}$ is isomorphic thus, to $\mathfrak{gl}(n+1;\mathbb{R})$,
the abelian subalgebra

$$2) \quad [x;\mathbf{Q}_i^j] = \frac{i}{2} \mathbf{Q}_i^j$$

where $\mathbf{Q} = \mathbf{P} \mathbf{Q} = \mathbf{P} (\mathbf{x}^i \mathbf{p}_i; \mathbf{p}_i \mathbf{p}_j)$ is the generator of hyperbolic rotation $\mathbf{H}_c : (\mathbf{p};\mathbf{x}) \rightarrow (\mathbf{e}^\mathbf{p};\mathbf{e}^\mathbf{x})$.

and abelian subalgebra

$$3) \quad [x;\mathbf{Q}_i^j] = \frac{i}{2} \mathbf{Q}_i^j$$

Generators $\mathbf{Q}_i^j X_\mathfrak{g}^j$ satisfy the following commutator relations

$$[\mathbf{Q}_i^j;\mathbf{X}_\mathfrak{g}^j] = \frac{i}{2} \mathbf{X}_\mathfrak{g}^j; [\mathbf{Q}_i^j;\mathbf{D}_\mathfrak{g}^j] = \frac{i}{2} \mathbf{D}_\mathfrak{g}^j; [\mathbf{X}_\mathfrak{g}^j;\mathbf{D}_\mathfrak{g}^j] = \frac{i}{2} \mathbf{X}_\mathfrak{g}^j + \frac{i}{2} \mathbf{D}_\mathfrak{g}^j.$$ (23.35)

Vector fields $\mathbf{Q}_i^j X_\mathfrak{g}^j$ are Hamilatonian with Hamiltonian functions

$$H_{\mathbf{Q}_i^j} = \mathbf{x}^i \mathbf{p}_j; H_{\mathbf{X}_\mathfrak{g}^j} = \mathbf{p}_k; H_{\mathbf{D}_\mathfrak{g}^j} = x^2 (\frac{\mathbf{p}_k}{2});$$

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