Exponential lower bounds on the size of approximate formulations in the natural encoding for Capacitated Facility Location*

Stavros G. Kolliopoulos† Yannis Moysoglou‡

Abstract

The metric capacitated facility location is a well-studied problem for which, while constant factor approximations are known, no efficient relaxation with constant integrality gap is known. The question whether there is such a relaxation is among the most important open problems of approximation algorithms [14].

In this paper we show that, if one is restricted to linear programs that use the natural encoding for facility location, at least an exponential number of constraints is needed to achieve a constant gap. Our proof does not assume any special property of the relaxation such as locality or symmetry.

1 Introduction

In recent years there has been an increasing interest in characterizing the strength of linear programming relaxations for approximating combinatorial optimization problems. In the seminal paper of Arora et al. [2] the integrality gap of general families of relaxations for the Vertex Cover problem was studied. These families include relaxations with local constraints, relaxations with low-defect inequalities and those relaxations obtained after $O(\log n)$ rounds of the Lovász-Schrijver hierarchy ([12]).

Subsequently, the idea of fooling local constraints was extended to deriving lower bounds on the number of levels of the Sherali-Adams hierarchy ([13]) starting with [7]. This framework has fueled several hierarchy-based gap bounds of the past years; it was given the name “from-local-to-global” in [5].

Recently, limitations on the approximation strength of extended formulations were shown for the maximum clique problem [4, 5]. In [6] it is proved that in terms of approximation, LPs

*This research has been co-financed by the European Union (European Social Fund – ESF) and Greek national funds through the Operational Program “Education and Lifelong Learning” of the National Strategic Reference Framework (NSRF) - Research Funding Program: “Thalis. Investing in knowledge society through the European Social Fund”.

†Department of Informatics and Telecommunications, National and Kapodistrian University of Athens, Panepistimiopolis Ilissia, Athens 157 84, Greece; (sgk@di.uoa.gr).

‡Department of Informatics and Telecommunications, National and Kapodistrian University of Athens, Panepistimiopolis Ilissia, Athens 157 84, Greece; (gmoys@di.uoa.gr).
of size $O(n^k)$ are exactly as powerful as $O(k)$-level relaxations of Sherali-Adams hierarchy for maximum constraint satisfaction problems.

The metric capacitated facility location problem (CFL) is a well-studied problem for which, while constant factor approximations are known \cite{3,1}, no efficient LP relaxation with constant integrality gap is known. The question whether such a relaxation exists is among the most important open problems in approximation algorithms \cite{14}. An instance $I$ of CFL is defined as follows. We are given a set $F$ of facilities and set $C$ of clients. We may open facility $i$ by paying its opening cost $f_i$ and we may assign client $j$ to facility $i$ by paying the connection cost $c_{ij}$. The latter costs satisfy the following variant of the triangle inequality: $c_{ij} \leq c_{ij'} + c_{i'j'} + c_{i'j}$ for any $i, i' \in F$ and $j, j' \in C$. We are asked to open a subset $F' \subseteq F$ of the facilities and assign each client to an open facility. The goal is to minimize the total opening and connection cost.

Apart from some previous work of the authors \cite{11,10}, no other progress towards the resolution of the question about the linear programming approximability of CFL has been made. In this paper we give further negative evidence for this notorious open problem by ruling out all polynomially-sized relaxations that use the natural encoding with facility opening and client assignment variables. This is a quite general family of relaxations that in the case of CFL, or network design problems in general, has been the focus of attention over the years.

### 1.1 Our results

In this paper we show unconditionally that at least an exponential number of constraints is necessary for a relaxation of CFL, that uses the natural encoding and has constant integrality gap. We do not make any assumptions on the structure of the constraints such as locality or symmetry.

Our proof, described at a high level, uses a simple yet insightful counting argument. We identify a large number of (fractional) vectors such that for each one of them, call it $s$, there is an admissible cost vector for which $s$ induces a cost that is $o(OPT)$, where OPT is the cost of the optimal integer solution with respect to the same cost vector. Then we show that an arbitrary valid inequality cannot be violated by more than a small number of such fractional vectors. Thus by using the union bound we get that a large number of inequalities is needed to separate those problematic points from the feasible region. A similar idea was independently used by Kaibel and Weltge in \cite{9} to derive lower bounds on the number of facets of a polyhedron which contains a given set $X$ of integer points and whose set of integer points is $\text{conv}(X) \cap \mathbb{Z}^d$. We note however that for problems such as facility location, the known polynomially-sized relaxations already have the aforementioned property. Our implementation of the counting argument is more general and allows proofs for bounds on approximate relaxations that achieve a given gap quality $g$.

A challenge in our proof is showing membership of a vector in the convex hull of integer solutions for CFL. We overcome this problem by building on the elegant probabilistic framework that we introduced in \cite{10}. We believe that our techniques apply to other problems as well.
2 The Method

Here we present in detail our methodology which we will subsequently use to derive results for metric capacitated facility location.

Let $Q = \{x \in [0, 1]^n \mid Ax \leq b\}$ be a linear relaxation and let $P = Q \cap \{0, 1\}^n$ be the set of integer solutions to $Q$ and $\text{conv}(P)$ be the convex hull of $P$ (we will also say that $\text{conv}(P)$ is the corresponding integer polytope). Our method consists of the following steps.

We design a family $I$ of instances parameterized by the dimension. For each instance $I \in \mathcal{I}$ of dimension $n$ we define a set of (exponentially many) points in $[0, 1]^n$ which we call the core of $I$. We denote the core of $I$ by $C_I$. We show that for each $s \in C_I$ there is an admissible cost function $w_s$ such that $w_s^T s = o(\text{Opt}_{I,w_s})$ where $\text{Opt}_{I,w_s}$ is the cost of the optimal integer solution with respect to $w_s$.

Then we prove that each inequality $\pi$ of $Q$ can separate at most $\lambda$ members of $C_I$ from $\text{conv}(P)$. We do so by the following argument: let $s_1 \in C_I$ be a vector that $\pi$ separates from $\text{conv}(P)$. Then we identify a set $U \subseteq C_I$ such that for any $s_2 \in U$ a convex combination $s'$ of $s_1, s_2$ is also a convex combination of integer solutions. Note here that $s_1, s_2$ themselves are not in $\text{conv}(P)$. Thus by validity and by selecting the size of $U$ to be independent from $s_1$ we have that $\pi$ cannot separate more that $\lambda = |C_I| - |U|$ members of $C_I$.

By the union bound we get that at least $|C_I| \lambda$ inequalities are needed to separate all members of $C_I$ from $\text{conv}(P)$. Thus at least that many inequalities are needed to acquire a relaxation of constant gap. To derive exponential bounds on the size of approximate relaxations, $\lambda$ is needed to be $\frac{|C_I|}{2^{O(n)}}$.

3 Preliminaries

In what follows, we use the definition of a $\rho$-approximate relaxation as given by [4]. We note that [4] is concerned with extended approximate relaxations that use the natural encoding, a direction that we do not pursue in this work.

Given a combinatorial optimization problem $T$, a linear encoding of $T$ is a pair $(L, O)$ where $L \subseteq \{0, 1\}^*$ is the set of feasible solutions to the problem and $O \subseteq \mathbb{R}^*$ is the set of admissible objective functions. An instance of the linear encoding is a pair $(d, w)$ where $d$ is a positive integer defining the dimension of the instance and $w \subseteq O \cap \mathbb{R}^d$ is the set of admissible cost functions for instances of dimension $d$. Solving the instance $(d, w)$ means finding $x \in L \cap \{0, 1\}^d$ such that $w^T x$ is either maximum or minimum, according to the type of problem under consideration. Let $P = \text{conv}(\{x \in \{0, 1\}^d \mid x \in L\})$ be the integer polytope of dimension $d$.

Given a linear encoding $(L, O)$ of a maximization problem, and $\rho \geq 1$, a $\rho$-approximate formulation that uses the natural encoding is a formulation $Ax \leq b$ with $x \in \mathbb{R}^d$ such that

\[
\max\{w^T x \mid Ax \leq b\} \geq \max\{w^T x \mid x \in P\} \quad \text{for all } w \in \mathbb{R}^d \text{ and } \\
\max\{w^T x \mid Ax \leq b\} \leq \rho \max\{w^T x \mid x \in P\} \quad \text{for all } w \in O \cap \mathbb{R}^d.
\]
For a minimization problem, we require
\[ \min \{ w^T x \mid Ax \leq b \} \leq \min \{ w^T x \mid x \in P \} \quad \text{for all } w \in \mathbb{R}^d \text{ and} \]
\[ \min \{ w^T x \mid Ax \leq b \} \geq \rho^{-1} \min \{ w^T x \mid x \in P \} \quad \text{for all } w \in O \cap \mathbb{R}^d. \]

4 Bounds for CFL

In the case of CFL, the linear encoding \((L, O)\) is defined as follows. For a CFL instance, given the number \(n\) of facilities, the number \(m\) of clients, the capacities \(K \in \mathbb{R}^n_+\) and the demands \(D \in \mathbb{R}^m_+\), we use the variables \(y_i, i = 1, \ldots, n, x_{ij}, i = 1, \ldots, n, j = 1, \ldots, m\) with the usual meaning of facility opening and client assignment respectively. The set of feasible solutions \((y, x)\) is defined in the obvious manner. Thus for dimension \(d = n + nm, L \cap \{0, 1\}^d\) is completely determined by the quadruple \((n, m, K, D)\). The set of admissible objective functions \(O \cap \mathbb{R}^{n+nm}\) is the set of pairs \((f, c)\) where \(f \in \mathbb{R}^n_+\) are the facility opening costs and \(c = \{c_{ij}\} \in \mathbb{R}^{n+m}_+\) are connection costs that satisfy \(c_{ij} \leq c_{ij'} + c_{ij''}\).

In our proof we will consider feasible sets of the form \((n, m, U, 1)\), i.e., with uniform capacities \(U > 0\), and unit demands. Therefore the triple \((n, m, U)\) is sufficient description. Furthermore, it will be convenient to deviate from the convention that the number of facilities is \(n\) – this is to simplify the expressions appearing through the proof. Let the number of facilities be \(n^2\), the number of clients be \(an^4\) for some integer \(a \geq 2\) and the capacity \(U\) of each facility be \(n^3\). Thus for a given \(n\), the feasible set is uniquely determined by the triple \((n^2, an^4, n^3)\). To avoid cumbersome expressions, we slightly abuse terminology and refer to such a triple as an instance \(I(n^2, an^4, n^3)\). We denote for the instance in question the set of facilities by \(F\) and the set of clients by \(C\).

We first describe the core \(C_I\) of the instance \(I(n^2, an^4, n^3)\).

**Definition 4.1** The core \(C_I\) of the instance \(I(n^2, an^4, n^3)\) is the following set of \((y, x)\) vectors. \(\forall k, l \subset F \text{ with } |k|, |l| = n \text{ and } k \cap l = \emptyset\) and for a set \(C_{k,l}\) of clients with \(|C_{k,l}| = Un + 1\) we define a vector \(s_{k,l}\) such that:

1. \(y_i = 1, \forall i \in k, y_i = \frac{10}{n^2}, \forall i \in l, y_i = 1, \forall i \notin k \cup l.\)
2. For a client \(j \in C_{k,l}\) we have \(x_{ij} = \frac{1 - n^2}{n^2}, \forall i \in k, x_{ij} = \frac{1}{n^2}, \forall i \in l \text{ and } x_{ij} = 0, \forall i \notin k \cup l.\)
3. For a client \(j \notin C_{k,l}\) we have \(\forall i \in k \cup l, x_{ij} = 0 \text{ and } \forall i \notin k \cup l, x_{ij} = \frac{1}{n^2-2n}.\)

We say that two vectors \(s_{k,l}, s_{k',l'} \in C_I\) collide with each other if \(l \setminus (k' \cup l') \neq \emptyset\) and \(l' \setminus (k \cup l) \neq \emptyset\). We proceed by proving that for each \(s \in C_I\) the ratio of the number of the members of \(C_I\) that do not collide with \(s\) to the number of the colliding members is exponentially small.

**Lemma 4.1** For each \(s_{k,l} \in C_I\) let \(U \subseteq C_I\) be the set of vectors in the core that collide with \(s_{k,l}\). Then \(\frac{|U| - |k|}{|U|} = 2^{-\Omega(n \log n)}\).

**Proof.** We lower-bound the ratio in question by upper bounding the probability that a member of \(C_I\) picked uniformly at random does not collide with \(s_{k,l}\). Consider the event \(E_1\) that \(l' \setminus (k \cup l) = \emptyset\). It must be the case that \(l' \subseteq k \cup l\). The probability \(P[E_1]\) is at most \((\frac{2n}{n^2})^n = (2/n)^n\) – this is the probability that all members of \(l'\) are in \(k \cup l\) if we were to...
pick them with repetition and the probability of the actual \( E_1 \) is less since we do not allow repetitions in the set \( l \). Likewise the probability of the event \( E_2 \) that \( l \setminus (k' \cup l') = \emptyset \) is the same. So, by the union bound, the probability that a randomly picked element of \( C_I \) does not collide with \( s_{k,l} \) is \( P[E_1 \cup E_2] \leq 2(2/n)^n \).

Next we show that for any two colliding vectors \( s_{k,l} \) and \( s_{k',l'} \) in \( C_I \) there is a convex combination \( s' \) of them that is contained in the integer polytope.

**Lemma 4.2** For any two colliding vectors \( s_{k,l}, s_{k',l'} \in C_I \), \( \text{conv}(\{s_{k,l}, s_{k',l'}\}) \cap \text{conv}(P) \neq \emptyset \).

**Proof.** We will actually show that the average vector \( s' = \frac{s_{k,l} + s_{k',l'}}{2} \) is a convex combination of integer solutions. We will do show by giving a distribution \( D \) over integer solutions whose expected vector \((y^D, x^D)\) with respect to \( D \) is \( s' \). The intuition behind the proof is that each one of the vectors \( s_{k,l}, s_{k',l'} \), in order to become a convex combination of integer solutions, needs what the other has in abundance: some measure for the \( y \) variable of a facility in \( l \) and some measure for the \( y \) variable of a facility in \( l' \) respectively. With probability \( 1/2 \) we choose to perform experiment \( A \) and with probability \( 1/2 \) we perform experiment \( B \) described below.

Suppose that experiment \( A \) is chosen. We will describe the random solution in two steps: \( A_1 \) and \( A_2 \). We describe first step \( A_1 \). Let \( f \) be a member of the set \( l \setminus (k' \cup l') \), which is non-empty by the choice of \( k, l, k', l' \). We select exactly one facility to be opened from the set \( l \) according to the following probabilities: for \( i \in l \setminus \{f\} \) the probability is equal to \( y_i^{s_{k,l}} = \frac{10}{n} \), while for facility \( f \) the probability is equal to \( 1 - \sum_{i \in l \setminus \{f\}} y_i^{s_{k,l}} \). Facilities in \( k \) are always opened in the experiment step \( A_1 \). When some facility \( i \in l \setminus \{f\} \) is chosen we randomly select \( w_{A_1}^{\prime} = \frac{\sum_{j \in C_{k,l}} x_{ij}}{y_i^{s_{k,l}}} \) clients from \( C_{k,l} \) and assign them to \( i \) - we assume without loss of generality that \( w_{A_1}^{\prime} \) is an integer, in the Appendix we show how to handle fractional \( w \)’s. Assign the remaining clients in \( C_{k,l} \) randomly to the facilities \( i' \in k \) so that each one is assigned exactly \( w_{A_1}^{\prime} \). When facility \( f \) is chosen we randomly select \( w_{A_1}^{\prime} = \frac{\sum_{j \in C_{k,l}} x_{ij}}{1 - \sum_{i \in l \setminus \{f\}} y_i^{s_{k,l}}} \) clients from \( C_{k,l} \) and assign them to \( f \). Assign the remaining clients in \( C_{k,l} \) randomly to the facilities \( i' \in k \) so that each one is assigned exactly \( w_{A_1}^{\prime} = \frac{|C_{k,l}| - w_{A_1}^{\prime}}{n} \) clients (again we assume w.l.o.g. that the \( w \)’s are integers).

For the second step \( A_2 \) of the experiment, let \( g \) be a facility in \( l' \setminus (k \cup l) \). We select facility \( g \) to be opened with a probability \( \sum_{i \in l} y_i^{s_{l',l}} \), the other facilities in \( F - (k \cup l) \) are always opened in the experiment step \( A_2 \). If \( g \) is opened, it is assigned \( w_{A_2}^{g} = \frac{\sum_{j \in C_{k,l}} x_{gj}}{\sum_{i \in l} y_i^{s_{k,l}}} \) clients randomly chosen from \( C - C_{k,l} \) and the remaining clients of \( C - C_{k,l} \) are assigned randomly to the facilities \( i' \) in \( F - (k \cup l) \) \( \setminus \{g\} \) so that each one is assigned exactly \( w_{A_2}^{\prime} = \frac{|C - C_{k,l}| - w_{A_2}^{g}}{|F - (k \cup l) \setminus \{g\}|} \). If \( g \) is not opened, all the clients in \( C - C_{k,l} \) are assigned randomly to the facilities \( i' \) in \( F - (k \cup l) \) \( \setminus \{g\} \) so that each one is assigned exactly \( w_{A_2}^{\prime} = \frac{|C - C_{k,l}|}{|F - (k \cup l) \setminus \{g\}|} \).

Now suppose that experiment \( B \) is chosen. This case is symmetric to the previous experiment by exchanging sets \( k, l \) with \( k', l' \) respectively but we give the full description
for the sake of completeness. Again we will describe the random solution in two steps $B_1$ and $B_2$.

We describe first step $B_1$. Let $g$ be the member of $l' \setminus (k \cup l)$ that we used in step $A_2$. We select exactly one facility to be opened from the set $l'$ with respect to the following probabilities: for $i' \in l' - \{g\}$ the probability is equal to $y_{i',g}^{y_{i',g}} = \frac{10}{n'}$, while for facility $g$ the probability is equal to $1 - \sum_{i' \in l' - \{g\}} y_{i',g}^{y_{i',g}}$. Facilities in $k'$ are always opened in the experiment step $B_1$. When some facility $i' \in l' - \{g\}$ is chosen we randomly select $w_{B_1}^{i'} = \frac{\sum_{i \in C_{k,l}'} x_{i,i'}^{y_{i,i'}}}{y_{i',g}^{y_{i',g}}}$ clients from $C_{k,l}'$ and assign them to $i'$ – assume again w.l.o.g. that $w_{B_1}^{i'}$ is an integer. Assign the rest of the clients in $C_{k,l}'$ randomly to the facilities $i'' \in k'$ so that each one is assigned exactly $w_{B_1}^{i''} = \frac{|C_{k,l}'| - w_{B_1}^{i'}}{n}$ clients (again we assume w.l.o.g. that this is an integer). When facility $g$ is chosen we randomly select $w_{B_1}^{g} = \frac{\sum_{i \in C_{k,l}'} x_{i,g}^{y_{i,g}}}{1 - \sum_{i' \in l' - \{g\}} y_{i',g}^{y_{i',g}}}$ clients from $C_{k,l}'$ and assign them to $g$. Assign the rest of clients in $C_{k,l}'$ randomly to the facilities $i'' \in k'$ so that each one is assigned exactly $w_{B_1}^{i''} = \frac{|C_{k,l}'| - w_{B_1}^{g}}{n}$ clients (again we assume w.l.o.g. that the $w$'s are integers).

For the second step $B_2$ of the experiment, let $f$ be the facility in $l \setminus (k' \cup l')$ used in step $A_1$. We select facility $f$ to be opened with a probability $\sum_{i \in l} y_{i,f}^{y_{i,f}}$, the other facilities in $F - (k' \cup l')$ are always opened in the experiment step $B_2$. If $f$ is opened, it is assigned $w_{B_2}^{f} = \frac{\sum_{i \in l} x_{i,f}^{y_{i,f}}}{\sum_{i \in C_{k,l}'} y_{i,f}^{y_{i,f}}}$ clients randomly chosen from $C - C_{k,l}'$ and the remaining clients of $C - C_{k,l}'$ are assigned randomly to the facilities $i'' \in F - (k' \cup l') - \{f\}$ so that each one is assigned exactly $w_{B_2}^{i''} = \frac{|C - C_{k,l}'| - w_{B_2}^{f}}{|C - (k' \cup l') - (f)|}$. If $f$ is not opened, all the clients in $C - C_{k,l}'$ are assigned randomly to the facilities $i'' \in F - (k' \cup l') - \{f\}$ so that each one is assigned exactly $w_{B_2}^{i''} = \frac{|C - C_{k,l}'|}{|F - (k' \cup l') - (f)|}$. It is easy to see that the outcome of each experiment is always a feasible integer solution, since all clients are assigned to opened facilities and the capacities are respected by the choice of $w$'s. It is also easy to verify that $s'$ is the expected vector of the distribution $D$ defined above. A facility $i \in F - (k \cup l \cup k' \cup l')$ is always opened in both experiments and thus $y_{f}^{D} = y_{f}^{f'} = 1$. A facility $i'' \in (k \cup l \cup k' \cup l') - (\{f, g\})$ is opened in experiment $A$ a fraction $y_{i''}^{s,1}$ of the time and is opened in experiment $B$ a fraction $y_{i''}^{s,2}$ of the time, and since each experiment is selected with $1/2$ probability, we have $y_{f}^{D} = \frac{y_{f}^{s,1} + y_{f}^{s,2}}{2} = y_{f}^{f'}$. For facility $f$ we have that in experiment $A$ it is opened $1 - \sum_{i \in (\{f\})} y_{i,f}^{y_{i,f}}$ while in experiment $B$ it is opened $\sum_{i \in l} y_{i,f}^{y_{i,f}} = \sum_{i \in l'} y_{i',f}^{y_{i',f}}$ of the time and so $y_{f}^{D} = 1/2 + y_{f}^{s,1}/2 = y_{f}^{f'}$. Similarly for facility $g$. By similar arguments the desired properties can be shown for the assignment variables. Let us consider for example, for facility $f \in F - (k' \cup l')$ we have that in $A_1$ the expected total demand assigned to it is $w_{A_1}^{f} P[y_{f} = 1] = \sum_{i \in C_{k,l}'} x_{i,f}^{y_{i,f}}$ and by the symmetric way that clients in $C_{k,l}$ are assigned to $f$ in $A_1$ we have that $x_{f}^{A_1} = x_{f}^{s,1}$ for all $j$. In experiment step $B_2$, the expected total demand assigned to $f$ is $w_{B_2}^{f} P[y_{f} = 1] = \sum_{i \in C - C_{k,l}'} x_{i,f}^{y_{i,f}}$ and by the symmetric way that clients in $C - C_{k,l}'$ are assigned to $f$ in $B_2$ we have that $x_{f}^{B_2} = x_{f}^{s,2}$ for all $j$. Thus $x_{f}^{D} = \frac{x_{f}^{s,1} + x_{f}^{s,2}}{2}$.
Theorem 4.1 Every approximate formulation for metric CFL that uses the natural encoding and has integrality gap at most $g$ for some constant $g > 0$, has $2^{\Omega(n \log n)}$ constraints.

Proof. We first prove that for every vector $s_{k,l} \in C_I$ there is an admissible cost function $w_{k,l}$ such that $w_{k,l}^T s_{k,l} = o(w_{k,l}^T s_{k,l}^{opt})$ where $s_{k,l}^{opt}$ is an optimal integer solution of $I(n^2, an^4, n^3)$ with respect to $w_{k,l}$. Consider two points $p_1, p_2$ in some Euclidean space at distance 1 from each other. At the first point $p_1$, the facilities of $k \cup l$ and the clients of $C_{k,l}$ are co-located and the remaining facilities and clients are all co-located at $p_2$. Additionally the facilities in $l$ have all opening cost of 1 and the rest have 0 opening cost. It is easy to see that every integer solution has a cost of at least 1: either some client $j \in C_{k,l}$ is assigned to some facility located at $p_2$ and thus incurs a connection cost of 1, or some costly facility in $l$ must be opened at $p_1$, incurring a facility cost of 1. On the other hand $w_{k,l}^T s_{k,l} = o(1)$.

Consider some inequality $\pi$ of a $g$-approximate relaxation $Q$, where $g > 0$ is a constant. (In fact the proof holds for $g = o(n)$). Suppose there is some $s_{k,l} \in C_I$ that violates $\pi$. Then, for every $s'_{k',l'} \in C_I$ which collides with $s_{k,l}$, $\pi$ must be satisfied otherwise by Lemma 4.2 we have violation of validity. By Lemma 4.1 we have that $\pi$ eliminates $2^{-\Omega(n \log n)}|C_I|$ members of the core, and by using the union bound the theorem is proved. We note that for the sake of simplicity the parameters are not optimized – by using a different core we can get tighter bounds.

References

[1] Ankit Aggarwal, Anand Louis, Manisha Bansal, Naveen Garg, Neelima Gupta, Shubham Gupta, and Surabhi Jain. A 3-approximation algorithm for the facility location problem with uniform capacities. To appear in Mathematical Programming, Ser. A. Extended abstract in Proc. IPCO 2010.

[2] Sanjeev Arora, Béla Bollobás, László Lovász, and Iannis Tourlakis. Proving integrality gaps without knowing the linear program. Theory of Computing, 2(1):19–51, 2006.

[3] Manisha Bansal, Naveen Garg, and Neelima Gupta. A 5-approximation for capacitated facility location. In Leah Epstein and Paolo Ferragina, editors, Algorithms ESA 2012, volume 7501 of Lecture Notes in Computer Science, pages 133–144. Springer Berlin Heidelberg, 2012.

[4] Gábor Braun, Samuel Fiorini, Sebastian Pokutta, and David Steurer. Approximation limits of linear programs (beyond hierarchies). In FOCS, pages 480–489, 2012.

[5] Mark Braverman and Ankur Moitra. An information complexity approach to extended formulations. Electronic Colloquium on Computational Complexity (ECCC), 19:131, 2012. To appear in Proc. STOC 2013.

[6] Siu On Chan, James R. Lee, Prasad Raghavendra, and David Steurer. Approximate constraint satisfaction requires large lp relaxations. CoRR, abs/1309.0563, 2013.

[7] Wenceslas Fernandez de la Vega and Claire Kenyon-Mathieu. Linear programming relaxations of maxcut. In Proceedings of the eighteenth annual ACM-SIAM symposium
A Appendix to Section \[4\]

Here we explain how to handle fractional bin capacities in the proof of Lemma 4.2.

To handle the case where the \(w\)'s are not integers (which is actually always the case), we simply do the following. We will give the proof for step \(A_1\) – the proofs for the other steps are similar. Each time facility \(f\) (or \(i \in l - \{f\}\)) is selected to be opened, the number of the clients that are randomly selected to be assigned to it is \(\lfloor w_{A_1}^f \rfloor \cdot \left(\lfloor w_{A_1}^i \rfloor \right)\) with probability \(1 - (w_{A_1}^f - \lfloor w_{A_1}^f \rfloor) (1 - (w_{A_1}^i - \lfloor w_{A_1}^i \rfloor))\), otherwise the number of clients is \(\lceil w_{A_1}^f \rceil \cdot \left(\lceil w_{A_1}^i \rceil \right)\). If the number of clients assigned to \(f\) (\(i\)) is selected to be \(\lfloor w_{A_1}^f \rfloor \cdot \left(\lfloor w_{A_1}^i \rfloor \right)\) then we randomly select \(n\left(C_{k,l} - \lfloor w_{A_1}^f \rfloor \right)\) facilities in \(k\) at random and set the number of clients assigned to them \(\lfloor C_{k,l} - \lfloor w_{A_1}^f \rfloor \rfloor\) and set the number of of clients assigned to the remaining facilities in \(k\) to \(\lceil C_{k,l} - \lceil w_{A_1}^f \rceil \rceil\). Otherwise select some \(n\left(C_{k,l} - \lfloor w_{A_1}^f \rfloor \right) - \left(\lfloor C_{k,l} - \lfloor w_{A_1}^f \rfloor \rfloor\right)\) facilities in \(k\) at random and set the number of clients assigned to them to \(\lfloor C_{k,l} - \lfloor w_{A_1}^f \rfloor \rfloor\) and set the number of of clients assigned
to the rest of them $\left\lfloor \left( 1 - \left\lceil \frac{w_{A_1}}{n} \right\rceil \right) \left( 1 - \left\lceil \frac{w_{A_1}}{n} \right\rceil \right) \right\rfloor$. Note that the expected vector is as in the proof of Lemma 4.2.