$f_K/f_\pi$ ratio from QCD sum rules

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Abstract

Using the correlation function of the axial vector mesons, we present a QCD sum rule calculation for the decay constants $f_\pi$ and $f_K$. Our calculations are only weakly dependent on the SU(3) breaking-parameter for the QCD vacuum and give the ratio $f_K/f_\pi = 1.11 \pm 0.02$.

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I. INTRODUCTION

Understanding the explicit SU(3) symmetry breaking effect in physical quantities, such as mass splitting, coupling constants, and decay constants, has been a subject of research in models of QCD for many years. Among those models, the QCD sum rule \[1\] provides a semi-direct calculation of QCD in that it relates, via the Borel transformed dispersion relation, the physical quantities to perturbative QCD supplemented by the non-perturbative nature of the QCD vacuum summarized systematically in non-vanishing condensates. Therefore, the main SU(3)-breaking effects are included systematically in perturbative quark-mass corrections (i.e., \(m_u = m_d \neq m_s\)) and in the different quark condensates (\(\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \neq \langle \bar{s}s \rangle\)). Using these prescriptions, the mass splittings within meson and baryon multiplets were calculated and it was found that the best fit was obtained with \(m_s \sim 150\) MeV and \(\gamma = \langle \bar{s}s \rangle / \langle \bar{u}u \rangle - 1 \sim -0.2\), where we assumed \(m_u = m_d \sim 0\) and \(\langle \bar{uu} \rangle = \langle \bar{dd} \rangle\) \[2\].

Despite all this success, it is not always possible to calculate all the physical quantities in the QCD sum rules, especially those related to Goldstone bosons. There are two reasons for this. First, the QCD sum rule is based on the operator product expansion (OPE) for which convergence is guaranteed only for large space-like momenta. This means, it is rather difficult to obtain information about the quantities carried by light Goldstone bosons for which information at rather small momentum transfer is needed. Second, for pseudo-scalar pseudo-scalar correlation functions, direct instantons can contribute, which spoils the convergence of the OPE. Therefore, it is not possible to calculate, for example, the pseudo-scalar meson masses in the QCD sum rules. Nonetheless, using suitable methods, it was found that low energy theorems, such as the Gell-Mann, Oakes, Renner relation \[3\] and the Goldberger-Treiman relation \[4\], could be derived or seen to hold within QCD sum rule approaches (see Appendices). Recently, it was shown further that even the chiral log behavior could be put in consistently by suitably modifying the continuum part \[5\]. These successes imply that by appropriately choosing the correlation function and improving the continuum part, we can estimate the effects of explicit chiral symmetry breaking, even for quantities related to the Goldstone bosons.\[6\]

In this work, we proceed along these line by presenting a QCD sum-rule calculation for the decay constants \(f_\pi\) and \(f_K\) and their ratio by using the correlation function of the axial vector currents for which no contamination from direct instantons is expected \[7\]. Our calculation for the ratio gives \(f_K/f_\pi = 1.11 \pm 0.02\). There have been many different models \[8\] to \[28\] to get \(f_\pi\), \(f_K\) and \(f_K/f_\pi\). Compared to those calculations, our value for the ratio lies at the lower end. However, the present calculations are only weakly dependent on the SU(3)-breaking parameter for the QCD vacuum and give values close to those of a recent lattice calculation.

In Sec. II, we present mass formula for the two axial vector mesons (\(a_1\) (non-strange axial vector meson) and \(K_a\) (strange axial vector meson)) by using the QCD sum rules. In Sec. III, we obtain the decay constants \(f_\pi\) and \(f_K\) and the couplings \(4\pi/g_{a_1}^2\) and \(4\pi/g_{K_a}^2\) from those sum rules which contain two SU(3) symmetry-breaking parameters, \(m_s\) and \(\gamma\).

\[1\] As an example, in Ref. \[8\], \(g_{K_{NN}}\) and \(g_{K_{N\Sigma}}\) are calculated, and they are compared to \(g_{\pi_{NN}}\).
We summarize our results in Sec. IV.

II. SUM RULES FOR $A_1$ AND $K_A$

Consider the T–product of the axial vector currents $[1, 29, 30]$

$$\Pi_{\mu\nu}(q^2) = i \int d^4x e^{ix\mu} (T(J^A_\mu(x)J^A_\nu(0))),$$

(1)

where $J^A_\mu(x) = \bar{q}(x)\gamma_\mu\gamma_5 q(x)$ and the $q'$s are either $u$ or $d$ quarks only. Then, this current couples to the $a_1 (1^{++})$ and $\pi (0^{-+})$ mesons, and $\Pi_{\mu\nu}(q^2)$ above can be decomposed as follows:

$$\Pi_{\mu\nu}(q^2) = (\frac{q_{\mu}q_{\nu}}{q^2} - g_{\mu\nu})\Pi_A(q^2) + \frac{q_{\mu}q_{\nu}}{q^2}\Pi_F(q^2),$$

(2)

where the imaginary parts of $\Pi_A(q^2)$ and $\Pi_F(q^2)$ receive contributions from $1^{++}$ and $0^{-+}$ states, respectively. One can extend this argument to currents involving the $s$-quark. For example, we can take the current $J^A_\mu(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x)$ for the $a_1$ meson and $\bar{u}(x)\gamma_\mu\gamma_5 s(x)$ for the $K_a$ meson, where $K_a$ is the chiral partner of $a_1$. In the SU(3) symmetric limit the $3P_1$ and the $1P_1$ states do not mix, just like the $a_1$ and $b_1$ mesons. However, for the $s$-quark mass greater than the $u$ and the $d$-quark masses, the $3P_1$ and the $1P_1$ states mix to give the physical $K_1$ $(K_1'(1270)$ and $K_1'(1400))$ states $[31, 33, 30, 34–38]$. $\Pi_{\mu\nu}(q^2)$ can also be written as

$$\Pi_{\mu\nu} = -\Pi_1(q^2) g_{\mu\nu} + \Pi_2(q^2) q_\mu q_\nu.$$

(3)

We get $f_s$ and $f_K$ from $\Pi_2(q^2)$. On the OPE side, after the Borel transformation, we obtain the following for the $K_a$ meson to the leading order in $\alpha_s$:

$$\frac{1}{\pi} \int e^{-s/M^2} Im\Pi_2(s) ds = \frac{1}{4\pi^2} M^2 [1 + \frac{\alpha_s}{\pi} - \frac{3}{M^2} (m_u^2 + m_d^2) + \frac{\pi^2}{3M^4} (\frac{\alpha_s}{\pi} G^2)

+ \frac{4\pi^2}{M^4} (m_u \langle \bar{u}u \rangle + m_s \langle \bar{s}s \rangle) + \frac{64\pi^3 \alpha_s}{81M^6} (\langle \bar{q}q \rangle^2 + \langle \bar{s}s \rangle^2 + 9 \langle \bar{q}q \rangle \langle \bar{s}s \rangle)]$$

(4)

where we have assumed $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \equiv \langle \bar{q}q \rangle$. In the following we neglect the terms proportional to $m_u$ (and $m_d$ for the case of $a_1$) because they give corrections of less than 0.2% to the OPE side at the relevant Borel region. For the four quark-condensates, we assume the vacuum saturation hypothesis, i.e.,

$$\langle \bar{q}\gamma_\mu \lambda^a q \bar{q}\gamma_\mu \lambda^a q \rangle = -\frac{1}{N^2} Tr(\Gamma_i \Gamma_i) Tr(\frac{\lambda^a}{2}) \langle \bar{q}q \rangle^2$$

(5)

with $N=12$ being a normalization factor. For example,

$$\langle \bar{q}\gamma^\mu \lambda^a q \bar{q}\gamma^\mu \lambda^a q \rangle = -\frac{16}{9} \langle \bar{q}q \rangle^2,$$

$$\langle \bar{q}\gamma_5 \lambda^a q \bar{q}\gamma_5 \lambda^a q \rangle = -\frac{4}{9} \langle \bar{q}q \rangle^2.$$

(6)
This approximation has been criticized by some people [39]. However, it turns out to be adequate in the case of the \(a_1\) sum rules [30]. This is also true for the case of the \(K_a\) sum rules because \(K_a\) has the same structure. For the \(a_1\), we let \(m_s=0\) and replace \(\langle \bar{s}s \rangle\) with \(\langle \bar{q}q \rangle\) above. On the phenomenological side, the spectral density \(Im\Pi_2\) can be expressed as follows (a pole contribution plus a continuum contribution):

\[
\frac{1}{\pi} Im\Pi_2 = f_{\pi,K}^2 \delta(s - m_{\pi,K}^2) + \frac{m_{a_1,K_a}^2}{g_{a_1,K_a}^2} \delta(s - m_{a_1,K_a}^2)
\]

\[
+ \frac{1}{4\pi^2} (1 + \frac{\alpha_s}{\pi}) \theta(s - s_0),
\]

where the constants \(f_{\pi}, f_K, g_{a_1}, \) and \(g_{K_a}\) are defined in the usual way:

\[
\langle 0 | \bar{u} \gamma_\mu \gamma_5 d | \pi \rangle = i f_{\pi} p_\mu, \quad \langle 0 | \bar{u} \gamma_\mu \gamma_5 d | a_1 \rangle = \frac{m_{a_1}^2}{g_{a_1}} \epsilon_\mu,
\]

and

\[
\langle 0 | \bar{u} \gamma_\mu \gamma_5 s | K_a \rangle = i f_{K_a} p_\mu, \quad \langle 0 | \bar{u} \gamma_\mu \gamma_5 s | K_a \rangle = \frac{m_{K_a}^2}{g_{K_a}} \epsilon_\mu.
\]

\(m_{a_1}\) and \(m_{K_a}\) represent the masses of the \(a_1\) and the \(K_a\) mesons, respectively, and \(s_0\) is the continuum threshold.

After inserting Eq. (7) into Eq. (1), we can get the following expression:

\[
f_{\pi,K}^2 e^{-\frac{m_{\pi,K}^2}{M^2}} + \frac{m_{a_1,K_a}^2}{g_{a_1,K_a}^2} e^{-\frac{m_{a_1,K_a}^2}{M^2}}
\]

\[
= \frac{1}{4\pi^2} M^2 \left[ (1 + \frac{\alpha_s}{\pi}) (1 - e^{\frac{m_s^2}{M^2}}) + \frac{A}{M^2} + \frac{B}{M^4} + \frac{C}{M^6} \right],
\]

where for \(K_a\),

\[
A = -3m_s^2,
\]

\[
B = \frac{\pi^2}{3} \left( \frac{\alpha_s}{\pi} G^2 \right) + 4\pi^2 m_s \langle \bar{s}s \rangle,
\]

\[
C = \frac{64}{81} \pi^3 \alpha_s (\langle \bar{q}q \rangle^2 + \langle \bar{s}s \rangle^2 + 9 \langle \bar{q}q \rangle \langle \bar{s}s \rangle).
\]

For the \(a_1\) meson, we put \(m_s=0\) and replace \(\langle \bar{s}s \rangle\) by \(\langle \bar{q}q \rangle\) in Eq. (11).

**III. \(F_\pi, F_K, \) AND \(F_K/F_\pi\)**

In our formula (Eq. (11)), there are several sources of uncertainties in the OPE: these are the magnitude of the s-quark condensate, contributions of higher-dimensional operators, the effect of the running coupling constants \(\alpha_s (M)\). We examine \(f_\pi, f_K\) and the ratio \(f_K/f_\pi\) for four cases and find their changes case by case. These four cases are used to estimate the uncertainties on the OPE side.
A. Case I: \( \langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle \)

From the expression in Eq. (10), we can determine \( f_\pi \) (or \( f_K \)), \( s_0 \), and the coupling \( 4\pi/g_{a_1}^2 \) (or \( 4\pi/g_{K_\pi}^2 \)) by using the experimentally known \( a_1 \) (or \( K_\pi \)) mass. We use \( m_{a_1} = 1.230 \pm 0.040 \) GeV [40] and \( m_{K_\pi} \sim 1.340 \) GeV [30,36,38].

We have to determine \( f_{\pi,K}, g_{a_1,K}^2, s_0 \), and the Borel interval. To do this, we use a best-fit method. The equation has the following form:

\[
C_1 g_1(M^2) + C_2 g_2(M^2) = g_3(M^2),
\]

where \( C_1 = f_{\pi,K}^2 \) and \( C_2 = 1/g_{a_1,K_1}^2 \). We want to determine \( C_1 \) and \( C_2 \) by minimizing \( (C_1 g_1 + C_2 g_2 - g_3)^2 \) with a fixed \( s_0 \) and an appropriate Borel interval:

\[
\int_{M_1^2}^{M_2^2} (C_1 g_1 + C_2 g_2 - g_3)^2 dM^2 = \text{minimum}.
\]

The Borel interval \( M^2 \) is restricted by the following conditions: OPE convergence and pole dominance. The lower limit of \( M^2, M_1^2 \) is determined as the value at which the contribution of the power corrections on the OPE side is less than 30%. The upper limit \( M_2^2 \) is determined as the value where the continuum contribution on the right-hand sides of (Eq. (10)) is less than 50% of the total. After determining \( f_{\pi,K}, g_{a_1,K_1}^2 \), we repeat the procedure with a different threshold \( s_0 \) until the variation \( h(M^2) \equiv C_1 g_1(M^2) + C_2 g_2(M^2) - g_3(M^2) \) is minimized. When the variation is the least, we take those values of \( f_{\pi,K}, g_{a_1,K_1}^2, s_0 \) and the Borel interval as our results.

In Tables II and III we summarize our results. We choose two values for the quark condensate, \( \langle \bar{q}q \rangle = -(0.230 \) GeV\(^3 \) and \( \langle \bar{q}q \rangle = -(0.250 \) GeV\(^3 \). Table II is the result for the case of \( \langle \bar{q}q \rangle = -(0.230 \) GeV\(^3 \), and Table III is that for the case of \( \langle \bar{q}q \rangle = -(0.250 \) GeV\(^3 \). Throughout this paper, we take \( \langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle, \langle \bar{q}G^2 \rangle = 0.015 \) GeV\(^4 \), and \( \alpha_s = 0.5 \). If we take \( \langle \bar{s}s \rangle = 0.6 \langle \bar{u}u \rangle \), we obtain \( f_K = 0.146 \) GeV and \( \frac{4\pi}{g_{K_\pi}^2} = 0.411 \) for \( \langle \bar{q}q \rangle = -(0.230 \) GeV\(^3 \). For the case of \( \langle \bar{s}s \rangle = 1.0 \langle \bar{u}u \rangle, f_K = 0.142 \) GeV and \( \frac{4\pi}{g_{K_\pi}^2} = 0.429 \). Overall, the value of \( f_K \) is not sensitive (within a few percent) to any variations in the \( u \) and the \( d \) or the \( s \)-quark condensate within its expected range.

B. Case II: \( \langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle \) and Including Mixed Condensate

Here, we include the contributions from the dimension-6 mixed condensates, \( m_s \langle g_s \bar{s} \sigma \cdot Gs \rangle \) for \( K_\pi \), where \( \sigma \cdot G \equiv \sigma_{\mu\nu} \cdot G^{\mu\nu} \) and \( \sigma_{\mu\nu} = \frac{i}{2} [\gamma_\mu, \gamma_\nu] \), and examine the changes. If the mixed condensate is included, the constant \( C \) in Eq. (11) is changed as follows: \( C' = C - \frac{8\pi^2}{15} m_s \langle \bar{s}s \rangle \), where we let \( \langle g_s \bar{s} \sigma \cdot Gs \rangle \equiv 2 m_0^2 \langle \bar{s}s \rangle = 0.8 \langle \bar{s}s \rangle \) [1]. The value \( m_0^2 \) is quoted from the standard QCD sum-rule estimation, \( \sim 0.4 \) GeV\(^2 \). The results are in Tables IV and V. Of course, there are no changes in \( f_\pi \) and \( \frac{4\pi}{g_{a_1}^2} \), but there are small changes in \( f_K \) and the coupling \( \frac{4\pi}{g_{K_\pi}^2} \). A somewhat larger value of \( m_0^2 \) has been obtained from different approaches [41, 47]. However even with that value, the change in \( f_K \) is very small. If we take the value suggested in Ref. [17], \( m_0^2 \sim 1.2 \) GeV\(^2 \), we get \( f_K = 0.149 \) GeV and \( \frac{4\pi}{g_{K_\pi}^2} = \)
0.376 for $\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^3$. For $\langle \bar{q}q \rangle = -(0.250 \text{ GeV})^3$, we have $f_K = 0.150 \text{ GeV}$ and $\frac{4\pi}{g_{K\alpha}} = 0.365$.

C. Case III : $\langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle$ with Variation of $\alpha_s(M)$

We now take into account the running coupling constant $\alpha_s(M)$. We use the forms used in Ref. \[30\] :

$$\alpha_s(M) = \frac{4\pi}{9} \frac{1}{\ln(M^2/\Lambda^2)} \tag{14}$$

with $\Lambda \simeq 150 \text{ MeV}$ and

$$\langle \bar{q}q(M) \rangle = \left( \frac{\alpha_s(\mu)}{\alpha_s(M)} \right)^{4/9} \langle \bar{q}q(\mu) \rangle, \tag{15}$$

where $\mu$ is a normalization scale. For the four quark-condensates,

$$\alpha_s(M) \langle \bar{q}q(M) \rangle^2 = \alpha_s(M) \left( \frac{\alpha_s(\mu)}{\alpha_s(M)} \right)^{8/9} \langle \bar{q}q(\mu) \rangle^2 \simeq \alpha_s(\mu) \langle \bar{q}q(\mu) \rangle^2, \tag{16}$$

where we have assumed the vacuum saturation hypothesis, as before, and $\alpha_s(\mu) = 0.5$. Using this, we obtain new results from Eq. \[11\], which are given in Tables \[V\] and \[VI\]. As one can see, the decay constants $f_\pi$ and $f_K$ and the two couplings are smaller than before. However, the change is only within a few percent. Our results are not sensitive to the choice of $\alpha_s(\mu)$. If we take another $\alpha_s(\mu)$, i.e., 0.6 as usual, the change is less than 1% for the decay constants and 3% for the couplings.

D. Case IV : $\langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle$ and Including Mixed Condensate with Variation of $\alpha_s(M)$

In this case, we include the mixed condensate considered in Case III, and use the running coupling constant $\alpha_s(M)$. The anomalous dimension of the quark-gluon mixed operator $g_s \bar{q} \sigma \cdot G q$ is small and can be neglected \[48\]. The new results are in Table \[VII\] and \[VIII\]. These are our final results. Comparing our values to the experimental values ($f_\pi \sim 0.131 \text{ GeV}$, and $f_K \sim 0.160 \text{ GeV}$ \[10\]), $f_\pi$ is very similar, while $f_K$ is smaller than the experimental value. With $\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^3$, we get $f_\pi = 0.130 \pm 0.002 \text{ GeV}$ and $f_K = 0.144 \text{ GeV}$. The couplings are $4\pi/g_{a_1}^2 = 0.42 \pm 0.02$ and $4\pi/g_{K\alpha}^2 = 0.38$. With $\langle \bar{q}q \rangle = -(0.250 \text{ GeV})^3$, $f_\pi = 0.132 \pm 0.003 \text{ GeV}$ and $f_K = 0.144 \text{ GeV}$. Their couplings $4\pi/g_{a_1}^2$ and $4\pi/g_{K\alpha}^2$ are $0.40 \pm 0.02$ and 0.38, respectively. The error bars for $f_\pi$ and $4\pi/g_{a_1}^2$ come from the uncertainty in the $a_1$ mass, $m_{a_1} = 1.230 \pm 0.040 \text{ GeV}$ \[10\].

IV. DISCUSSION

In Tables \[IX\] and \[X\] we summarize the ratio $f_K/f_\pi$ for the four cases considered. One can see that there is not much of a difference in the ratio between those cases. This indicates
that the uncertainties coming from the OPE side in our formula are very small. For the sake of reference, we compare our ratio with those of other models in Table XI. One can see that our result $1.11 \pm 0.02$ for $\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^2$ is very similar to that from lattice gauge theory, but smaller than the experimental value, $1.22 \pm 0.02$. However, the error is within 10%. It should be noted that some models in the Table predict a ratio which is similar to the experimental value, but do not give the correct $f_\pi$ and $f_K$.

We also check that a different $K_a$ mass doesn’t change our result very much. For example, if we take the mass as $K_a = 1.270 \text{ GeV}$ (the same value as that of $K_1(1270)$), $f_K = 0.140 \text{ GeV}$. If we take the mass as $K_a = 1.400 \text{ GeV}$ (that of $K_1(1400)$), $f_K = 0.148 \text{ GeV}$. Among the parameters, the ratio is most sensitive to $m_s$. In the case with $\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^2$, if we let $m_s = 0.125 \text{ GeV}$, we get $f_K = 0.148 \text{ GeV}$ and $f_K/f_\pi = 1.14\pm 0.02$. In the case of $m_s = 0.175 \text{ GeV}$, $f_K = 0.140$ and $f_K/f_\pi = 1.08\pm 0.02$. Thus, a better determination of $m_s$ will pin down the value of $f_K$ with better accuracy. It seems contradictory at first that the result depends more sensitively on $m_s$ than on the mass of $K_a$. However, it should be noted that the mass of $K_a$ comes dominantly from chiral symmetry breaking, and the explicit SU(3) symmetry-breaking effect is rather small. This is evident from comparing its mass to that of the $a_1(1230)$ meson. On the other hand, in order to extract decay constants from experiments, we have to determine some elements of the CKM (Cabbibo-Kobayashi-Maskawa) matrix, such as $|V_{ud}|$ and $|V_{us}|$ for $f_\pi$ and $f_K$, respectively. These elements are also closely related to the current quark masses, $m_u, m_d, \text{ and } m_s$.

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APPENDIX A:

Here, we derive the Goldberger-Trieman relation within the QCD sum rule approach. Consider the following nucleon correlation function:

$$\Pi(q; \pi(p)) = i \int d^4x e^{iqx} \langle 0 | T[\eta(x)\bar{\eta}(0)] | \pi(p) \rangle,$$  \hspace{1cm} \text{(A1)}

where $\pi(p)$ is a pion with momentum $p$, and $\eta$ is a nucleon interpolating field without any derivative. The nucleon interpolating field transforms as follows under the SU(2) axial rotation:

$$[Q_5^a, \eta] = -\gamma_5 \frac{\tau^a}{2} \eta,$$ \hspace{1cm} \text{(A2)}
where $Q_a^5$ is the axial charge. In the soft-pion limit, the OPE side is the commutator with the axial charge and, by using the relation in Eq. (A2), can be shown to be

$$\Pi_{OPE}(q) = \frac{2}{f_\pi} \Pi_{OPE}^1(q^2) \gamma_5.$$  \hfill (A3)

where $\Pi_{OPE}^1(q^2)$ is the OPE side of the following part of the nucleon-correlation function in vacuum:

$$\Pi(q; 0) \equiv \Pi_1(q^2) + \Pi_q(q^2) \gamma^\mu q_\mu.$$  \hfill (A4)

As for the phenomenological side, we assume an interaction lagrangian of the pion and the nucleon, $L^I = g_{\pi N} \bar{N} \gamma_5 N$. Then, we have,

$$\Pi_{Phen}(q; \pi) = \frac{\lambda_2^2 g_{\pi N} \gamma_5}{q^2 - m_N^2} + \frac{2}{f_\pi} \Pi_{cont}^1(q^2) \gamma_5.$$  \hfill (A5)

We also have the following vacuum sum rule:

$$\Pi_{OPE}^1(q^2) = \frac{\lambda_2^2 m_N}{q^2 - m_N^2} + \Pi_{cont}^1(q^2).$$  \hfill (A6)

Using this and comparing Eqs. (A3) and (A5), we have the Goldberger-Trieman relation

$$g_{\pi N} = \frac{2m_N}{f_\pi}.$$  \hfill (A7)

APPENDIX B:

Here, we show that the pion hadron T matrix $T_{\pi-H} \to 0$ (Adler Zero) in the chiral limit and the pion momentum $p \to 0$. Consider the correlation function

$$V(q; 2\pi(p)) = i \int d^4x e^{iqx} \langle \pi(p)|T[H(x)\bar{H}(0)]|\pi(p)\rangle.$$  \hfill (B1)

In the soft-pion limit, using the commutation relation with the axial current twice, the OPE side can be shown to be

$$V_{OPE}^\pi(q; 2\pi) = -\frac{c_1}{f_\pi^2} V_{1OPE}(q, 0) + \frac{c_1}{f_\pi^2} A_{1OPE}(q, 0),$$  \hfill (B2)

where $A_{1OPE}(q, 0)$ is the correlation function between the axial partner of the hadronic current $H$.

As for the phenomenological side, there will be a double pole, whose residue is the T matrix:

$$V_{Phen}^\pi(q; 2\pi) = \frac{T}{(q^2 - m_H^2)^2} + V_{pol}^\pi(q; 2\pi) + V_{cont}^\pi(q; 2\pi) + \frac{c_1}{f_\pi^2} [A_{pol}^\pi(q; 0) + A_{cont}^\pi(q; 0)].$$  \hfill (B3)
We note

\[ V^{pole}(q; 2\pi) \xrightarrow{q \to 0} -\frac{c_1}{f_\pi^2} V^{pole}(q; 0), \]
\[ V^{cont}(q; 2\pi) \xrightarrow{q \to 0} -\frac{c_1}{f_\pi^2} V^{cont}(q; 0). \]  \hspace{1cm} (B4)

Using the sum rules

\[ V(q; 0)^{OPE} = V(q; 0)^{pole} + V(q; 0)^{cont}, \]
\[ A(q; 0)^{OPE} = A(q; 0)^{pole} + A(q; 0)^{cont}. \] \hspace{1cm} (B5)

and comparing Eqs. (B2) and (B3), we have

\[ T = 0 \] \hspace{1cm} (B6)

in the chiral limit with zero incoming four momentum.
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TABLES

TABLE I. Case I: \( \langle q\bar{q} \rangle = -(0.230 \text{ GeV})^3 \), \( \langle s\bar{s} \rangle = 0.8 \langle q\bar{q} \rangle \).

| axial meson | mass (GeV) | \( s_0 \) (GeV\(^2\)) | \( M^2 \) (GeV\(^2\)) | \( f_\pi, f_K \) (GeV) | \( \frac{4\pi}{g_{a_1,K_a}} \) |
|-------------|------------|-----------------|-----------------|-----------------|-----------------|
| \( a_1 \)   | 1.190      | 2.15            | 1.35–3.10       | 0.130           | 0.410           |
|             | 1.230      | 2.35            | 1.35–3.40       | 0.132           | 0.427           |
|             | 1.270      | 2.55            | 1.35–3.65       | 0.134           | 0.441           |
| \( K_a \)   | 1.340      | 2.85            | 0.85–4.00       | 0.143           | 0.425           |

TABLE II. Case I: \( \langle q\bar{q} \rangle = -(0.250 \text{ GeV})^3 \), \( \langle s\bar{s} \rangle = 0.8 \langle q\bar{q} \rangle \).

| axial meson | mass (GeV) | \( s_0 \) (GeV\(^2\)) | \( M^2 \) (GeV\(^2\)) | \( f_\pi, f_K \) (GeV) | \( \frac{4\pi}{g_{a_1,K_a}} \) |
|-------------|------------|-----------------|-----------------|-----------------|-----------------|
| \( a_1 \)   | 1.190      | 2.10            | 2.25–3.05       | 0.132           | 0.392           |
|             | 1.230      | 2.25            | 2.25–3.25       | 0.135           | 0.397           |
|             | 1.270      | 2.45            | 2.25–3.55       | 0.137           | 0.413           |
| \( K_a \)   | 1.340      | 2.80            | 1.00–3.90       | 0.143           | 0.417           |
TABLE III. Case II: $\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^3$, $\langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle$, and including a mixed condensate.

| axial meson | mass (GeV) | $s_0$ (GeV$^2$) | $M^2$ (GeV$^2$) | $f_\pi$, $f_K$ (GeV) | $\frac{4\pi}{g_{a_1,K_a}^2}$ |
|-------------|------------|----------------|----------------|-------------------|-------------------|
| $a_1$       | 1.190      | 2.15           | 1.35–3.10      | 0.130             | 0.410             |
|             | 1.230      | 2.35           | 1.35–3.40      | 0.132             | 0.427             |
|             | 1.270      | 2.55           | 1.35–3.65      | 0.134             | 0.441             |
| $K_a$       | 1.340      | 2.75           | 1.05–3.85      | 0.146             | 0.402             |

TABLE IV. Case II: $\langle \bar{q}q \rangle = -(0.250 \text{ GeV})^3$, $\langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle$, and including a mixed condensate.

| axial meson | mass (GeV) | $s_0$ (GeV$^2$) | $M^2$ (GeV$^2$) | $f_\pi$, $f_K$ (GeV) | $\frac{4\pi}{g_{a_1,K_a}^2}$ |
|-------------|------------|----------------|----------------|-------------------|-------------------|
| $a_1$       | 1.190      | 2.10           | 2.25–3.05      | 0.132             | 0.392             |
|             | 1.230      | 2.30           | 2.25–3.30      | 0.134             | 0.411             |
|             | 1.270      | 2.45           | 2.25–3.55      | 0.137             | 0.413             |
| $K_a$       | 1.340      | 2.70           | 1.20–3.80      | 0.146             | 0.392             |
TABLE V. Case III: $\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^3$ and $\langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle$ with variation of $\alpha_s(M)$.

| axial meson | mass (GeV) | $s_0$ (GeV)$^2$ | $M^2$ (GeV)$^2$ | $f_\pi$, $f_K$ (GeV) | $\frac{4\pi}{g_{a_1,K_a}^2}$ |
|-------------|------------|----------------|----------------|-------------------|-----------------------------|
| $a_1$       | 1.190      | 2.30           | 1.35–3.30      | 0.128             | 0.406                       |
|             | 1.230      | 2.50           | 1.35–3.60      | 0.130             | 0.420                       |
|             | 1.270      | 2.70           | 1.35–3.90      | 0.132             | 0.431                       |
| $K_a$       | 1.340      | 2.95           | 0.85–4.15      | 0.142             | 0.403                       |

TABLE VI. Case III: $\langle \bar{q}q \rangle = -(0.250 \text{ GeV})^3$ and $\langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle$ with variation of $\alpha_s(M)$.

| axial meson | mass (GeV) | $s_0$ (GeV)$^2$ | $M^2$ (GeV)$^2$ | $f_\pi$, $f_K$ (GeV) | $\frac{4\pi}{g_{a_1,K_a}^2}$ |
|-------------|------------|----------------|----------------|-------------------|-----------------------------|
| $a_1$       | 1.190      | 2.30           | 2.25–3.30      | 0.129             | 0.404                       |
|             | 1.230      | 2.45           | 2.25–3.55      | 0.132             | 0.405                       |
|             | 1.270      | 2.65           | 2.25–3.80      | 0.134             | 0.417                       |
| $K_a$       | 1.340      | 2.90           | 1.00–4.05      | 0.142             | 0.395                       |
TABLE VII. Case IV: $\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^3$, $\langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle$, and including a mixed condensate with variation of $\alpha_s(M)$.

| axial meson | mass (GeV) | $s_0$ (GeV$^2$) | $M^2$ (GeV$^2$) | $f_\pi$, $f_K$ (GeV) | $\frac{4\pi}{g_{a_1,Ka}}$ |
|-------------|------------|----------------|----------------|---------------------|--------------------------|
| $a_1$       | 1.190      | 2.30           | 1.35–3.30      | 0.128               | 0.406                    |
|             | 1.230      | 2.50           | 1.35–3.60      | 0.130               | 0.420                    |
|             | 1.270      | 2.70           | 1.35–3.90      | 0.132               | 0.431                    |
| $K_a$       | 1.340      | 2.85           | 1.05–4.00      | 0.144               | 0.381                    |

TABLE VIII. Case IV: $\langle \bar{q}q \rangle = -(0.250 \text{ GeV})^3$, $\langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle$, and including a mixed condensate with variation of $\alpha_s(M)$.

| axial meson | mass (GeV) | $s_0$ (GeV$^2$) | $M^2$ (GeV$^2$) | $f_\pi$, $f_K$ (GeV) | $\frac{4\pi}{g_{a_1,Ka}}$ |
|-------------|------------|----------------|----------------|---------------------|--------------------------|
| $a_1$       | 1.190      | 2.30           | 2.25–3.30      | 0.129               | 0.404                    |
|             | 1.230      | 2.45           | 2.25–3.55      | 0.132               | 0.405                    |
|             | 1.270      | 2.65           | 2.25–3.80      | 0.134               | 0.417                    |
| $K_a$       | 1.340      | 2.85           | 1.20–4.00      | 0.144               | 0.382                    |
TABLE IX. $f_K/f_\pi$ ratio and couplings for various cases ($\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^3$).

| cases          | ratio | $\frac{f_\pi}{g_{\rho_1}}$ | $\frac{f_\pi}{g_{K_0}}$ |
|----------------|-------|-----------------------------|---------------------------|
| Case I: $\langle \bar{s}s \rangle = 0.8\langle \bar{q}q \rangle$ | 1.08±0.02 | 0.43±0.02                   | 0.43                      |
| Case II: $\langle \bar{s}s \rangle = 0.8\langle \bar{q}q \rangle + \text{mixed con.}$ | 1.11±0.02 | 0.43±0.02                   | 0.40                      |
| Case III: $\langle \bar{s}s \rangle = 0.8\langle \bar{q}q \rangle + \alpha_s(M)$  | 1.09±0.02 | 0.42±0.02                   | 0.40                      |
| Case IV: $\langle \bar{s}s \rangle = 0.8\langle \bar{q}q \rangle + \text{mixed con.} + \alpha_s(M)$ | 1.11±0.02 | 0.42±0.02                   | 0.38                      |

TABLE X. $f_K/f_\pi$ ratio and couplings for various cases ($\langle \bar{q}q \rangle = -(0.250 \text{ GeV})^3$).

| cases          | ratio | $\frac{f_\pi}{g_{\rho_1}}$ | $\frac{f_\pi}{g_{K_0}}$ |
|----------------|-------|-----------------------------|---------------------------|
| Case I: $\langle \bar{s}s \rangle = 0.8\langle \bar{q}q \rangle$ | 1.06±0.02 | 0.40±0.02                   | 0.42                      |
| Case II: $\langle \bar{s}s \rangle = 0.8\langle \bar{q}q \rangle + \text{mixed con.}$ | 1.09±0.02 | 0.40±0.02                   | 0.39                      |
| Case III: $\langle \bar{s}s \rangle = 0.8\langle \bar{q}q \rangle + \alpha_s(M)$ | 1.08±0.02 | 0.41±0.02                   | 0.40                      |
| Case IV: $\langle \bar{s}s \rangle = 0.8\langle \bar{q}q \rangle + \text{mixed con.} + \alpha_s(M)$ | 1.09±0.02 | 0.41±0.02                   | 0.38                      |
TABLE XI. $f_K/f_\pi$ ratio from various models

| models                           | ratio      | references |
|----------------------------------|------------|------------|
| Experiment                       | 1.22 ± 0.02| Ref. [40]  |
| SU(3) $\sigma$ Model             | 1.31       | Ref. [3]   |
| Current Algebra                  | 1.3 ± 0.3  | Ref. [3]   |
| Chiral Perturbation Theory       | 1.21       | Ref. [10]  |
| Bag Model I                      | 1.02       | Ref. [11]  |
| Bag Model II                     | 1.2        | Ref. [12]  |
| Potential Model I                | 1.27       | Ref. [13]  |
| Potential Model II               | 1.27       | Ref. [14]  |
| Relativistic Quark Model         | 1.25       | Ref. [15]  |
| Finite $Q^2$ Sum Rule (FQSR)     | 1.15       | Ref. [16]  |
| Effective Lagrangian Method I    | 1.15       | Ref. [17]  |
| Effective Lagrangian Method II   | 1.13       | Ref. [18]  |
| Electroweak Theory I             | 1.22 ± 0.01| Ref. [19]  |
| Electroweak Theory II            | 1.23 ± 0.02| Ref. [20]  |
| Lattice Gauge Theory I           | 1.35       | Ref. [21]  |
| Lattice Gauge Theory II          | 1.10       | Ref. [22]  |
| Lattice Gauge Theory III         | 1.16 ± 0.07| Ref. [23]  |
| Lattice Gauge Theory IV          | 1.16       | Ref. [24]  |
| Nambu–Jona-Lasinio Model I       | 1.03       | Ref. [25]  |
| Nambu–Jona-Lasinio Model II      | 1.05       | Ref. [26]  |
| Nambu–Jona-Lasinio Model III     | 1.12       | Ref. [27]  |
| QCD Sum Rules I                  | 1.22       | Ref. [28]  |
| QCD Sum Rules II$^a$             | 1.11 ± 0.02| present work |

$^a$ Case IV and $\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^3$ (see text).
Erratum : $f_K/f_\pi$ ratio from QCD sum rules

In this erratum we correct an error in the previous calculation and present a more detailed analysis including the variations of the decay constants $f_\pi$ and $f_K$ on input parameters.

In the case of the $K_a$ sum rule, there is no contribution of the mixed condensate, i.e., $m_s\langle g_s \bar{s} \sigma \cdot G s \rangle$. Therefore, the argument on the mixed condensate in Sec. III B and D becomes irrelevant. When we include the effects from the anomalous dimension, Eq. (10) is rewritten as

$$f_{\pi,K}^2 e^{-\frac{m_{\pi,K}^2}{M^2}} + \frac{m_{a_1,K_a}^2}{g_{a_1,K_a}^2} e^{-\frac{m_{a_1,K_a}^2}{M^2}} = \frac{1}{4\pi^2} M^2 \left[ \left( 1 + \frac{\alpha_s(M)}{\alpha_s(\mu)} \right) \left( 1 - e^{-\frac{\mu^2}{M^2}} \right) + \frac{A}{M^2} L^{-\frac{2}{3}} + \frac{B}{M^4} + \frac{C}{M^6} L^{-\frac{4}{3}} \right],$$

where $L$ is defined by

$$L \equiv \frac{\alpha_s(\mu)}{\alpha_s(M)}$$

and

$$\alpha_s(M) = \frac{4\pi}{9} \frac{1}{\ln(M^2/\Lambda^2)}.$$ (3)

Here, $\mu$ is the renormalization point taken to be 1 GeV and $\Lambda$ the QCD scale parameter, 0.25 GeV.

Using the methods described in the paper, we get 0.133 GeV and 0.145 GeV for $f_\pi$ and $f_K$, respectively, for the basic input values given in Tables I and II. We also include variations of these decay constants for other inputs, which are coming from the uncertainty of the basic inputs as discussed in the paper. For example, the first line in Table I shows that $f_\pi = 0.131$ GeV (or 0.135 GeV) if we change the quark condensate to $\langle \bar{q}q \rangle = -(0.210 \text{ GeV})^3$ (or $-(0.250 \text{ GeV})^3$) while other basic inputs are fixed. $f_K$ is most sensitive to the strange quark mass $m_s$, while $f_\pi$ to the $a_1$ mass. Including the variations, we get the maximum ratio of $f_K/f_\pi = 1.16$. Here, we do not include the variation of $f_K$ on $m_{K_a}$.

Some references use the renormalization point $\mu = 0.5$ GeV and the QCD scale parameter $\Lambda = 0.15$ GeV. If we take these values of $\mu$ and $\Lambda$, we get 0.131 GeV and 0.145 GeV for $f_\pi$ and $f_K$, respectively, for the same basic inputs. In the tables we also present the variations coming from these changes.

Here, $\mu$ is the renormalization point taken to be 1 GeV and $\Lambda$ the QCD scale parameter, 0.25 GeV.
TABLES

TABLE I. $f_\pi$ and its variations. Other inputs means other possible inputs discussed in the paper. [⋯] corresponds to taking $\mu = 0.5$ GeV.

| basic inputs | other inputs | variations (MeV) |
|--------------|--------------|-----------------|
| $\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^3$ | $-(0.210 \text{ GeV})^3, -(0.250 \text{ GeV})^3$ | $\pm 2 [\mp 3]$ |
| $\langle \frac{\alpha_s}{\pi} G^2 \rangle = 0.015 \text{ GeV}^4$ | 0.012 $\text{ GeV}^4$ | $-2 [-2]$ |
| $\Lambda = 0.25 \ [0.15] \text{ GeV}$ | 0.20 $[0.10] \text{ GeV}$ | $-2 [-3]$ |
| $m_{a_1} = 1.230 \text{ GeV}$ | 1.190, 1.270 $\text{ GeV}$ | $\mp 3 [\mp 3]$ |

TABLE II. $f_K$ and its variations. Other inputs means other possible inputs discussed in the paper. [⋯] corresponds to taking $\mu = 0.5$ GeV.

| basic inputs | other inputs | variations (MeV) |
|--------------|--------------|-----------------|
| $\langle \bar{q}q \rangle = -(0.230 \text{ GeV})^3$ | $-(0.210 \text{ GeV})^3, -(0.250 \text{ GeV})^3$ | $\pm 1 [\pm 1]$ |
| $\langle \frac{\alpha_s}{\pi} G^2 \rangle = 0.015 \text{ GeV}^4$ | 0.012 $\text{ GeV}^4$ | $-2 [-2]$ |
| $\Lambda = 0.25 \ [0.15] \text{ GeV}$ | 0.20 $[0.10] \text{ GeV}$ | $-3 [-4]$ |
| $m_{K_s} = 1.340 \text{ GeV}$ | 1.270, 1.400 $\text{ GeV}$ | $\pm 4 [\mp 4]$ |
| $\langle \bar{s}s \rangle = 0.8 \langle \bar{q}q \rangle$ | 0.7, 0.9 $\langle \bar{q}q \rangle$ | $\pm 1 [\pm 1]$ |
| $m_s = 0.150 \text{ GeV}$ | 0.120, 0.180 $\text{ GeV}$ | $\pm 7 [\pm 6]$ |