Modular Curves and Mordell-Weil Torsion in F-theory

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Abstract

In this work we prove a bound for the torsion in Mordell-Weil groups of smooth elliptically fibered Calabi-Yau 3- and 4-folds. In particular, we show that the set which can occur on a smooth elliptic Calabi-Yau $n$-fold for ($n \geq 3$) is contained in the set of subgroups which appear on a rational elliptic surface, and is slightly larger for $n = 2$. The key idea in our proof is showing that any elliptic fibration with sufficiently large torsion has singularities in codimension 2 which do not admit a crepant resolution. We prove this by explicitly constructing and studying maps to a modular curve whose existence is predicted by a universal property. We use the geometry of modular curves to explain the minimal singularities that appear on an elliptic fibration with prescribed torsion, and to determine the degree of the fundamental line bundle (hence the Kodaira dimension) of the universal elliptic surface which we show to be consistent with explicit Weierstrass models. The constraints from the modular curves are used to bound the first fundamental group of any gauge group $G$ in a supergravity theory obtained from F-theory. We comment on the isolated 8-dimensional theories, obtained from extremal K3’s, that are able to circumvent lower dimensional bounds. These theories neither have a heterotic dual, nor can they be compactified to lower dimensional minimal SUGRA theories. We also comment on the maximal, discrete gauged symmetries obtained from certain Calabi-Yau threefold quotients.
Contents

1 Introduction  2

2 Overview  4
   2.1 Review of Modular Curves  .............................................. 4
   2.2 Summary of Main Arguments ............................................. 7

3 Modular Curves and Minimal Singularities  8
   3.1 Remarks about the $\mathbb{Z}_4$ Case ................................... 10

4 Bounds on Non-simply connected Gauge Groups from F-theory  11
   4.1 F-theory and the Role of Torsion ...................................... 11
   4.2 6D SUGRA with Putative Large Torsion .............................. 15
   4.3 Higher order and Non-Prime Torsion .................................. 17
   4.4 K3: 8-dimensional Exceptions ......................................... 20
   4.5 Bounds on Calabi-Yau Quotient Torsors ................................ 23

5 Technical Results  24
   5.1 Global Lemmas .............................................................. 24
   5.2 Local Lemmas ............................................................... 25
   5.3 Summary of Technical Portion ......................................... 26
   5.4 Application ................................................................. 27

6 Summary and Conclusions  28

A Universal Elliptic Curves with Points of Order $n \leq 8$.  30
   A.1 $n = 2, 3$ ................................................................. 30
   A.2 Universal Elliptic Curves with a Point of Order $n \geq 4$ .......... 30
   A.3 Torsion groups of type $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$ .......................... 33
   A.4 Examples ................................................................. 34

B Index of Congruence Subgroups  35
   B.1 $\Gamma(n)$ ................................................................. 36
   B.2 $\Gamma_1(n)$ ............................................................... 37
   B.3 $\Gamma_0(n)$ ............................................................... 38
   B.4 Non-classical congruence groups .................................... 38
1 Introduction

Calabi-Yau manifolds play a central role in the description of lower dimensional field and supergravity (SUGRA) theories in various dimensions through their use as compactification backgrounds of String/M/F-theory. Properties of the effective theories (EFT) such as matter and gauge theory content descend directly from geometric properties of the compactification spaces. This makes a full classification and study of EFTs obtained from string theory, known as the string landscape, in terms of topological properties of Calabi-Yau manifolds, an interesting program for both physics and mathematics. The string landscape is distinguished from the so-called swampland\textsuperscript{[1]}, the set of only seemingly consistent EFTs that do not have a consistent UV completion together with gravity (for a review see \textsuperscript{[2]}). The goal is thus to find the additional rules that allow to discriminate the swampland from the landscape. In the past, these quantum gravity conjectures existed often as folk theorems making use of heuristic black hole arguments, which by itself might not be fully understood.

In recent years a lot of a attention has been given to finding and connecting the set of quantum gravity conjectures and developing tools to put them on more solid grounds, as for example happened for weak gravity\textsuperscript{[3,4]} and distance conjectures\textsuperscript{[5–7]}. These tools on the other hand, make use of the geometric description of physics via Calabi-Yau compactifications and their geometric properties.

The framework of F-theory\textsuperscript{[66]} extends the geometrization of physics and allows to describe the largest patch of the string vacua (see e.g. \textsuperscript{[20]}) in a single framework to date. The key idea of F-theory is to focus on elliptically fibered Calabi-Yau manifolds $Y$ and to reformulate the monodromies of $[p,q]$-7 that wrap suitable cycles in the base $B$ acting on the IIB axio-dilaton, into the geometric action on the complex structure of the elliptic fiber. The gauge algebra living on the branes is encoded in singularity type of the elliptic fiber and goes beyond the limits of perturbative type IIB.

Certain global aspects of the gauge group $G$ of the F-theory compactifications are encoded in the Mordell-Weil (MW) group $MW(Y/B)$ of the fibration $Y$. For non-constant fibrations, the MW group of an elliptic fibration admits a decomposition:

$$MW(Y/B) = \mathbb{Z}^r \times T,$$

where either $T \cong \mathbb{Z}_n$ or $T \cong \mathbb{Z}_n \times \mathbb{Z}_m$ for some integers $n,m,r$. A priori\textsuperscript{[1]} there are no constraints on the possibilities for $n,m,r$: any combination can, in principle, be obtained by pulling back the fibration along a suitable map $B' \to B$. Classical results on elliptic surfaces show that assumptions about the canonical bundle of the total space constrain the possibilities for the Mordell-Weil group when $\dim Y = 2$, In that case, a complete classification of Mordell-Weil groups and configurations of singular fibers is known for rational elliptic surfaces and for elliptically fibered K3s is known, see\textsuperscript{[76,77]} for the case of rational surfaces and\textsuperscript{[78]} for elliptic K3 surfaces. A table with the torsion subgroups that occur on rational surfaces and K3 surfaces can be found in\textsuperscript{[32]}. There are no analogous classifications for smooth Calabi-Yau elliptic 3-folds and higher. The present work is a step in that direction.

\textsuperscript{1}That is, without making assumptions about the canonical bundle of the total space.
Within M- and F-theory, the free part of the MW group gives rise to $r$ U(1) gauge symmetry factors and has been extensively studied in F-theory \cite{8,10,31,47,51}. Abelian symmetries are in fact quite subtle. In order to constraint their (maximal) possible number even the strong 6d SUGRA anomalies are only of limited \cite{11} help and the range of Abelian singlet charges can not be constrained at all \cite{15}. In order to do so, the geometrization of Abelian symmetries, via the MW group allowed for a more systematic and consistent exploration of matter charges \cite{12,14,16,17} and maximal Abelian factors \cite{67}.

The finite part of the MW group on the other hand, has been given much less attention in the recent literature. In F-theory compactifications, its effect can be understood as a refinement of the co-weight lattice of representations \cite{30,53,54,57} geometrically induced by the torsion Shioda map. The appearing center charges of matter representation $R$ of the gauge algebra $G$ supported by the $[p,q]$-7 branes must be integral. Although in F-theory only the massless matter spectrum is directly visible, one might expect this constraint to apply to the massive sector as well. Such situations can be viewed as the gauging of $G$ by a discrete symmetry \cite{69} resulting in a non simply connected gauge group $G$ with first fundamental group $\pi_1(G) = T$. A similar effect can happen, although of slightly different origin, when additional U(1) factors embed non-trivially inside the center of other non-Abelian group factors. In the following however we want to focus purely on the non-Abelian case without additional Abelian gauge factors.

With a view on the swampland program, this begs the obvious question, which non-simply connected groups can be consistently realized in a quantum gravity, specifically in F-theory compactifications. In particular one might wonder ask for bounds on the first fundamental group $\pi_1(G)$, due to the expectation that the string landscape must be finite\cite{4}. In \cite{53} Aspinwall and Morrison have constructed various generic Weierstrass models with a wide variety of Mordell-Weil torsion subgroups, going up to $\mathbb{Z}_6$ for cyclic groups and $\mathbb{Z}_3 \times \mathbb{Z}_3$ in general. From pure 6D SUGRA considerations it is possible to set up models, whose massless spectrum is consistent with (almost) any putative first fundamental group $\pi_1(G) = \mathbb{Z}_n$ factor way beyond order six. However, the massless sector comprises only a subsector of the full theory, which might not allow conclusions about the full gauge group structure in the full theory. In F-theory, there exists such such a candidate entity in terms of the torsion sections. Indeed, the mere presence of torsion sections enforces the presence of certain minimal singularities whose structure often comes as a surprise, at least for higher order torsion.

In this paper, we want to prove that the list of MW torsion models appearing in Aspinwall and Morrison \cite{53} is in fact complete. This geometric result allows us to put sharp swampland constraints on the first fundamental group of gauge groups in SUGRA theories constructed from $[p,q]$-7 branes in F-theory. As a byproduct, we give a new perspective on elliptic fibrations with torsion sections by connecting them to the modular curve of the certain congruence subgroups of $SL(2,\mathbb{Z})$\footnote{The set of elliptically fibered CY threefolds, and therefore landscape of 6D F-theory compactifications in fact has been proven to be “bounded”, in the sense that there all such threefolds fit into one of finitely many families, see \cite{41}.}. The modular curve encodes all minimal singularities, and the degree of

\footnote{In a related F-theory context, fibrations with restricted monodromies have also been considered in \cite{23,24} and recently \cite{25}.}
the fundamental line bundle of the elliptic fiber, also for very large torsion points, which seize to have a smooth Calabi-Yau realization.

In a different context, torsion sections in the Mordell-Weil group can be used as a building block to construct the covering geometry of a specific class of smooth, non-simply connected Calabi-Yau quotient torsors \[22\]. Their associated F-theory physics admits discrete symmetries coupled to gauged superconformal matter and gravity \[18,19,21\], which are of the same order, as the torsion factor of the covering geometry. Therefore, our bounds also translate to bounds on manifolds that can be obtained using those constructions.

This paper is structured as follows: In Section 2 we give a review of the theory of modular curves and give a sketch of our main results. The full proof is deferred to Section 5 and aims at the mathematically inclined reader. In Section 3 we interpret the presence of singular fibers in torsion models directly from properties of the modular curves. In Section 4 we interpret our result in terms of swampland constraint on the order of non-simply connected groups within F-theory. We close with with a summary and outlook in Section 6.

2 Overview

In this section, we provide an overview over the main tools that we will be using in our analysis. Modular curves come up multiple times in this paper, so we start by giving a brief review of the theory. \[\text{4}\] We then give a summary of the main argument.

2.1 Review of Modular Curves

Modular curves arise as moduli spaces of elliptic curves, together with some extra torsion data. It is easiest to explain what we mean by "extra torsion data" by giving an example. An \(n\)-torsion pair is a pair \((E, P)\), where \(E\) is a smooth elliptic curve and \(P\) is a point on \(E\) of order \(n\) (in the Mordell-Weil group). An isomorphism of \(n\)-torsion pairs \((E, P) \rightarrow (E', P')\), is an isomorphism \(E \rightarrow E'\) mapping \(P \mapsto P'\). The (open) modular curve \(X_1(n)\) \(\text{5}\) parametrizes \(n\)-torsion pairs up to isomorphism of \(n\)-torsion pairs.

There are two ways of explicitly constructing modular curves in general: one algebraic and the other analytic. For a detailed exposition of the analytic perspective, see \[71\]. We describe both constructions for the “classical" modular curve before discussing general modular curves.

2.1.1 "The" Modular Curve

The modular curve \(X(1)\) parametrizes isomorphism classes of elliptic curves \(\mathbb{C}\).

- Algebraically, every elliptic curve admits a short Weierstrass equation:

\[
y^2 = x^3 + fx + g \quad (f, g \in \mathbb{C}, 4f^3 + 27g^2 \neq 0)
\]

\[\text{(2.1)}\]

\[\text{4}\]For a more detailed exposition, see \[71\].

\[\text{5}\]This notation is not standard. In the modular curves literature, the curve we are denoting as \(X_1(n)\) is usually denoted \(Y_1(n)\). We’ve deviated from the standard notation to avoid confusion with the Calabi-Yau total spaces, which we are denoting \(Y\).
Two such elliptic curves, with coefficients $f_i, g_i$, $i = 1, 2$, are isomorphic if and only if there exists $\lambda \in \mathbb{C}^{\times}$ such that $f_1 = \lambda^4 f_2$ and $g_1 = \lambda^4 g_2$.

Thus, $X(1)^o$ can be identified with the quotient:

$$X(1)^o = \{(f, g) \in \mathbb{C}^2 : 4f^3 + 27g^2 \neq 0\} / \{(f, g) \sim (\lambda^4 f, \lambda^6 g)\}$$  \hspace{1cm} (2.2)

Note that this quotient has two singular points, namely the orbits $[(1, 0)], [(0, 1)]$, corresponding to the two elliptic curves with complex multiplication.

- From the analytic perspective, Riemann’s uniformization theorem shows that every elliptic curve is isomorphic, as a Riemann surface, to $\mathbb{C}/\Lambda$, where $\Lambda$ is the lattice of periods of the elliptic curve. Two quotients give rise to isomorphic elliptic curves if and only if the corresponding lattices are homothetic, i.e. scalar multiples of one another.

To compute the moduli space, we assume the lattice has been scaled so that one of the basis vectors is at 1 and the other basis vector lies in the (open) upper half plane, depicted in Figure 1. This allows us to identify points $\tau \in \mathcal{H}$ with isomorphism classes of elliptic curves: the point $\tau$ represents the elliptic curve $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}\tau)$. Two points $\tau, \tau' \in \mathcal{H}$ represent the same elliptic curve if and only if they are in the same $SL(2, \mathbb{Z})$-orbit (here $SL(2, \mathbb{Z})$ is acting on $\mathcal{H}$ by fractional linear transformations). The region:

$$\mathcal{D} = \left\{ \tau \in \mathcal{H} : 1 \leq |\tau|, -\frac{1}{2} \leq \text{Re}(\tau) < \frac{1}{2} \right\},$$  \hspace{1cm} (2.3)

is a fundamental domain for the quotient, so points in $\mathcal{D}$ are in bijection with points in $X(1)^o$. The points $i$ and $e^{2\pi i/3}$ are cone points on $\mathcal{H}/SL(2, \mathbb{Z})$, and correspond to the singular orbits of $(1, 0), (0, 1)$. They represent the square and hexagonal lattices, which have extra symmetries.

The celebrated $j$-function, which is defined using Eisenstein series in the analytic setting or simply as:

$$j([(f, g)]) = 1728 \cdot \frac{4f^3}{4f^3 + 27g^2},$$  \hspace{1cm} (2.4)

in the algebraic one, gives a bijection between $X(1)^o$ and $\mathbb{A}^1$.

We can compactify $X(1)^o$ by adding a single point to the moduli space representing the isomorphism class of an $I_1$ curve. Analytically, we achieve this by taking the quotient of the extended upper half plane $\mathcal{H}^* = \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$. The compactified modular curve is denoted $X(1)$, and is isomorphic to $\mathbb{P}^1$ as a Riemann surface.

2.1.2 General Modular Curves

We now explain how one constructs and studies modular curves in general.

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6The cone points represent the two isomorphism classes of elliptic curves with complex multiplication.
Figure 1: The Fundamental domain of $\tau$ for $X(1) = \mathcal{H}/\text{SL}(2, \mathbb{Z})$.

- To obtain an algebraic description, we start with the general equation for an elliptic curve with the coefficients treated as variables, and find relations between the coefficients that guarantee the elliptic curve has the desired torsion property. We view the modular curve as the vanishing set of the set of relations we just obtained.

For example, to construct $X_1(n)^o$, we start with the general equation for an elliptic curve containing the point $(0, 0)$. We can then compute multiples of $(0, 0)$ to obtain a relation between the coefficients that encodes the fact that the non-identity point has order exactly $n$. This gives a description of the modular curve as a variety. Along the way, we also obtain a universal elliptic curve with an $n$-torsion point, that we can view as an elliptic surface over the modular curve. This process is carried out in Appendix A.

More complicated methods have to be used to construct $X_0(n)^o$ and $X(n)^o$ and the Weierstrass models for the universal curves using algebraic methods.

- We can also construct modular curves analytically as quotients of the (open) upper half plane by an appropriate subgroup of $\text{SL}(2, \mathbb{Z})$. Explicitly, the subgroups associated to $X(n)^o$, $X_0(n)^o$, $X_1(n)^o$ are:

\[
\Gamma(n) = \left\{ \gamma \in \text{SL}(2, \mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}, \tag{2.5}
\]

\[
\Gamma_0(n) = \left\{ \gamma \in \text{SL}(2, \mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod{n} \right\}, \tag{2.6}
\]

\[
\Gamma_1(n) = \left\{ \gamma \in \text{SL}(2, \mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{n} \right\}. \tag{2.7}
\]

We can compactify these general modular curves using the same method as with $Y(1)$: the compactified modular curve $X(n)$ (resp. $X_0(n), X_1(n)$) is the quotient of the extended upper half plane by the appropriate congruence group, i.e. $X(n) = \mathcal{H}^*/\Gamma(n)$ (resp. $X_0(n) = \mathcal{H}^*/\Gamma_0(n), X_1(n) = \mathcal{H}^*/\Gamma_1(n)$.) The cusps of a modular curve are the points in $X(n)\backslash X(n)^o$ (resp. $X_0(n)\backslash X_0(n)^o, X_1(n)\backslash X_0(n)^o$).

Finally, we need to define the width of a cusp, as this idea plays an important role to discuss minimal singularities in Section 3. We need to introduce some more notation to define width.
For $\Gamma \subset \text{SL}(2, \mathbb{Z})$, we define:

$$\Gamma_\infty = \left\{ \gamma \in \Gamma : \gamma = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\}.$$  \hspace{2cm} (2.8)

Fix a congruence subgroup $\Gamma$. Each cusp of the associated modular curve corresponds to the orbit of some $s \in \mathbb{Q}$ under $\Gamma$. Choose such an $s$, and choose $\delta \in \text{SL}(2, \mathbb{Z})$ such that $\delta \cdot s = \infty$. The **width of the cusp** $x$, denoted $h(x)$, is abstractly defined as:

$$h(x) = [\text{SL}(2, \mathbb{Z})_\infty : (\delta \{ \pm I \} \Gamma \delta^{-1})_\infty]$$  \hspace{2cm} (2.9)

This number encodes the smallest integer $h$ such that $\begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} \in \delta \Gamma \delta^{-1}$; geometrically, we’ve acted on the upper half plane to put $s$ at the “top” of the picture, and we’re counting the number of vertical strips in the fundamental domain. We can also compute the width using the triangulation we pull back from $X(1)$, by dividing the number of triangles that meet at a given cusp by 2.

From the moduli perspective, the width of the cusp encodes the ramification over the $I_1$ point on $X(1)$, and thus determines the minimal singularities on the Néron model of the universal elliptic curve. Specifically, whenever there is a cusp of width $h$ on the modular curve, the associated elliptic surface should have an $I_h$ singularity (or worse). This will be discussed further in the minimal singularities section.

### 2.2 Summary of Main Arguments

The usefulness of modular curves in our analysis comes from the fact that they satisfy a universal property: elliptic fibrations with $n$-torsion over a fixed base $B$, for example, are classified by $\text{Hom}(B, X_1(n))$ (as long as $n \geq 4$). In fact, if we choose an elliptic fibration $Y \to B$ with an $n$-torsion section $B \to Y$, then there are (unique) rational maps making the following diagram commute:

$$
\begin{array}{ccc}
Y & \longrightarrow & S_2 \\
\downarrow \pi & & \downarrow p \\
B & \longrightarrow & C_1,
\end{array}
\hspace{2cm} (2.10)
$$

where $C \cong X_1(n)$ and $S \to C$ is the Néron model for the universal elliptic curve with an $n$-torsion section. The diagram above allows us to deduce many of the relevant properties of the geometry of $Y$ from the elliptic surface $S \to C$ and the map $\phi$. In particular:

- In Section [3] we use the notion of **width** of a cusp on a modular curve to explain why assumptions about Mordell-Weil torsion lead to certain minimal configurations of singularities on the fibration.

---

7These points must be of $I_h$ type, since $f$ and $g$ are non-vanishing, as they come from special points in the fundamental domain of $X(1)$, that are not mapped onto the real axis.
In Section 5, we prove the technical results needed to bound the torsion on smooth elliptically fibered Calabi-Yau 3-and 4-folds. We show that \( \phi \) can never be a morphism when \( Y \) is Calabi-Yau. Furthermore, the order of vanishing of \( (f, g, \Delta) \) over points in the indeterminacy locus of \( \phi \) is always \( (4d, 6d, 12d) \) for \( d \) a positive integer that depends only on the point, \( \phi \) and the fundamental line bundle of \( S \to C \). In particular, we show that \( d > 1 \) anytime the elliptic fibration is assumed to have a point of order at least 7. The presence of codimension 2 loci where \( (f, g, \Delta) \) vanish to order \( (8,12,24) \) means the Weierstrass models do not admit crepant resolutions, and thus there are no smooth Calabi-Yau fibrations with those torsion subgroups (except in the K3 case, since the codimension 2 locus we’ve described is empty).

This allows us to deduce our main theorem, namely that any non-constant, smooth elliptically fibered Calabi-Yau \( n \)-fold, with \( n \geq 3 \), has Mordell-Weil torsion isomorphic to one of the following groups:

\[
\mathbb{Z}_n : \quad (n = 1, 2, 3, 4, 5, 6),
\]
\[
\mathbb{Z}_2 \times \mathbb{Z}_{2m} : \quad (m = 1, 2),
\]
\[
\mathbb{Z}_3 \times \mathbb{Z}_3.
\]

In Appendix A, we show explicitly how to construct the map \( \phi : B \to X_1(n) \) from the Weierstrass equation of an elliptic fibration with a chosen \( n \)-torsion section.

### 3 Modular Curves and Minimal Singularities

When considering explicit Weierstrass models we find (e.g. see Section 4) that the presence of torsion in \( MW \) imposes a minimal number of singular fibers. In the context of F-theory this is surprising, as it forces a minimal gauge group. This is particularly true for larger order torsion, where not one but multiple singular fibers/gauge algebra factors are introduced. For example, any fibration with a section of order 5 necessarily has at least two fibers of type \( I_5 \) (or possibly \( I_{5d} \) for some integer \( d \), if the total space is not a rational elliptic surface). For fibrations with 7-torsion, the number of \( I_7d \) fibers has to be at least 3. The exact configuration of fibers which is imposed can be computed from the Weierstrass equation of the universal elliptic curve (e.g. see Appendix A), but obtaining such equations can often be tedious.

Fortunately, the degenerate fibers and the Euler characteristic of the universal elliptic surface \( S \to C \) can both be read off directly from the associated congruence subgroup. Geometrically, the singularities appear because of the cusps on the compactified modular curve. The information needed to determine which singular fibers appear in \( S \to C \) is encoded in the cusps of the modular curve:

- The points on the discriminant locus are in bijection with cusps on the compactified modular curve.

- The fiber over a point in the discriminant locus is of type \( I_d \), where \( d \) is the width of the associated cusp.

Both the number of cusps and their widths can easily be computed using the standard triangulations of the modular curves: the cusps are the points on the real line, and the width of each cusp is given by the denominator of the corresponding rational number.
The number of triangles in the triangulation is equal to twice the sum of the widths. As a result, it can be interpreted as twice the degree of the discriminant of an elliptic fibration with only the minimal singularities forced by the modular curve. The degree of the discriminant is useful, because it determines the degree of the fundamental line bundle of the universal surface, which is crucial to our analysis.

Since the triangulation is pulled back from the map $X \to X(1)$, the number of triangles is exactly the degree of the map, which is the index $[\text{SL}(2,\mathbb{Z}) : \Gamma]$.

In Appendix [B], we derive formulas for the index of all congruence subgroups. For example, the index of $\Gamma_1(n)$ is:

$$[\Gamma(1) : \Gamma_1(n)] = n^2 \prod_{p|n} \left(1 - \frac{1}{p^2}\right),$$

(3.1)

where the product is taken over all primes dividing $n$. One can check (see Appendix [B]) that this index is equal to 12 when $n = 4$ and is divisible by 24 for all $n > 4$. For $n > 4$, the index can be used to compute the degree of $\mathcal{L}_{S/C}$, where $S \to C$ is the Néron model of the universal elliptic curve:

$$[\Gamma(1) : \Gamma_1(n)] = 24 \deg \mathcal{L}_{S/C}$$

(3.2)

Thus, when $C = Y_1(n)$, we have $\deg \mathcal{L}_{S/C} = 1$ for $n = 5, 6$, $\deg \mathcal{L}_{S/C} = 2$ for $n = 7, 8$, $\deg \mathcal{L}_{S/C} = 3$ for $n = 9, 10$, etc. This matches up exactly with the tables in [32], and confirms that the $\mathbb{Z}_5, \mathbb{Z}_6$ rational elliptic surfaces are extremal, and similarly that the $\mathbb{Z}_7, \mathbb{Z}_8$ K3 surfaces are extremal, and any larger order has Kodaira dimension 1, hence not Calabi-Yau. E.g. the triangulation for $X_1(6)$ and $X_1(7)$ are depicted in Figure 2 using Sage [79, 80] and the first shows the following cusp points

$$S_{\text{cusp}}^{(6)} : \{0^{(6)}, 1/3^{(2)}, 1/2^{(3)}, \infty^{(1)}\},$$

(3.3)
with respective widths denoted as superscripts. This means the universal surface should have fibers of type $I_6, I_2, I_3, I_4$. These are exactly the discriminant loci, that we also find in the generic Weierstrass model that we discuss in Section 4.3.

Similarly, with $X_1(7)$, we find that there are 6 cusps (including $\infty$) at

$$S_{\text{cusp}}^{(7)} : \{0^{(7)}, 2/7^{(1)}, 1/3^{(7)}, 3/7^{(1)}, 1/2^{(7)}, \infty^{(1)}\},$$

(3.4)

three of which have width 7 and three of width 1, so the universal surface has 3 $I_7$ fibers and 3 $I_1$ fibers. This explains why we can’t construct a model with a 7-torsion section and with exactly one $I_7$, e.g. but predicts the right fibers in the K3, (see Section 4.4).

The congruence subgroup that leaves two $\mathbb{Z}_n$ torsion points, is $\Gamma(n)$ as defined above. We compute the index in Appendix B and repeat the result here as

$$[\Gamma(1) : \Gamma(n)] = n^3 \prod_{p|n} 1 - \frac{1}{p^2}.$$ (3.5)

For $n = 2, 3, 4, 5$ the indices are computed to as 6, 24, 48, 120. With the identification of the fundamental line bundle, this shows, that $\mathbb{Z}_3 \times \mathbb{Z}_3$ is the maximum allowed torsion possible in rational elliptic surfaces, three and fourfolds. The $\mathbb{Z}_4 \times \mathbb{Z}_4$ torsion extremizes K3 and is depicted in Figure 3 of Section 4.4, where we will come back to this topic. Higher order torsion points are not allowed.

3.1 Remarks about the $\mathbb{Z}_4$ Case

When $n \geq 5$, the singularities predicted by the widths of the cusps correspond exactly to the fibers observed on the universal surface. This is not the case when $n = 4$. We wish to briefly address this peculiarity in this section.

For $n > 4$, the sum of the widths adds up to an integer multiple of the Euler characteristic of the universal surface, allowing us to deduce the degree of the fundamental line bundle using Equation (3.2) and to determine the minimal singularities without needing a Weierstrass model. When $n = 4$, the index of the congruence subgroup is 12, so the formula breaks down. As a result, we see a discrepancy between the singularities predicted by the widths of the cusps observed on the modular curve and the singularities on the universal surface with a 4-torsion section: the modular curve only forces $I_4 + I_1 + I_1$ on the elliptic curve, but since the discriminant of any rational elliptic surface has degree 12, we need additional singularities to obtain a smooth elliptic surface over $\mathbb{P}^1$.

From the algebraic perspective, we have the equation for the universal elliptic curve:

$$y^2 + xy - ty = x^3 - tx^2$$ (3.6)

which extends to:

$$y^2 + t_1^2 xy - t_0 t_1^2 y = x^3 - t_0 t_1 x^2$$ (3.7)

There are other reasons to believe the $\mathbb{Z}_4$ group is special. For example, the group structure in the special fiber of an elliptic surface with an $I_{2k+1}^{*}$ is abstractly isomorphic to $\mathbb{Z}_4$, which means we can have non-semistable fibrations with 4 torsion.
Note that this equation has an $I^*_1$ singularity, which is the only type we see an additive fiber in an elliptic fibration with a torsion point of order at least 4. After a quadratic base change, we can also obtain a semistable rational surface which has a 4-torsion point. The short Weierstrass equations for the two models are given below:

\[ y^2 = x^3 + t_1^2 \frac{16t_0^3 - 16t_0t_1 + t_1^2}{48} x - t_1^3 \cdot \frac{64t_0^3 + 120t_0^2t_1 - 24t_0t_1^2 + t_1^3}{864}, \] (3.8)

\[ y^2 = x^3 + \frac{16t_0^4 - 16t_0^2t_1^2 + t_1^4}{48} x - \frac{64t_0^6 + 120t_0^4t_1^2 - 24t_0^2t_1^4 + t_1^6}{864}. \] (3.9)

We will come back to the $\mathbb{Z}_4$ model in Section 4.3 from a more general perspective.

4 Bounds on Non-simply connected Gauge Groups from F-theory

In this section we want to apply the bounds on torsion within the context of F-theory. Hence we first review F-theory and the role of Mordell-Weil torsion in its effective theory as first investigated in [53] and further explored in [52] which can be skipped by any expert in the field. The bounds on torsion, which were discussed in the Sections before and are proved more rigorously in Section 5 can be translated into bounds on non-simply connected gauge groups in SUGRA theories obtained from F-theory. As we show, those bounds are surprising from pure 6-dimensional SUGRA arguments and their massless spectra. Finally we also comment on bounds on smooth Calabi-Yau quotient torsors and their connection to discrete symmetries and superconformal matter.

4.1 F-theory and the Role of Torsion

This section is intended as a recap of Mordell-Weil torsion in F-theory models, based on [52]. By F-theory we mean to consider an elliptically fibered threefold

\[ \mathcal{E} \rightarrow Y_n \]
\[ \downarrow \pi \]
\[ B_{n-1} \] (4.1)

and treat the complex structure of the torus $\tau$ as the local axio-dilaton $\tau = C_0 + ig_{IIB}^{-1}$ of type IIB string theory. Being a fibration the axio-dilaton is allowed to vary over the complex base $B_{n-1}$. The singular fibers are located over the vanishing set of the discriminant:

\[ \Delta = 4f^3 + 27g^2. \] (4.2)

The codimension one components of the discriminant locus:

\[ \delta_i \subset V(\Delta), \] (4.3)

have the interpretation of generalized stacks of $[p,q]$-7-branes that host a local gauge algebra $\mathcal{G}_i$ according to their Tate-fiber type. The order of $\Delta$ can be enhanced, e.g. over the collision
of components at codimension two \( \delta_{i,j} = \delta_i \cap \delta_j \), which gives rise to matter multiplets, that form representations \( R \) of the algebra \( \mathcal{G} \). If the enhancement locus, is a minimal singularity, matter representations can be inferred from the local enhancement to the algebra \( \mathcal{G}_{i,j} \) and the decomposition of its adjoint representation into that of the two colliding factors \(^{68}\) as

\[
R_{i,j} = \text{adj}(\mathcal{G}_{i,j})/(\text{adj}(\mathcal{G}_i) \oplus \text{adj}(\mathcal{G}_j)).
\]  

(4.4)

This construction gives a powerful tool to construct and possibly classify a large class of consistent \( D = 10 - 2n \) dimensional SUGRA theories via geometric methods.

The physics of F-theory compactifications on CY n-folds \( Y_n \) with non-trivial (finite-) Mordell-Weil group \( \text{MW}(Y_n) = \mathbb{Z}_n \) is understood as a global gauging of the center of the gauge algebra \( \mathcal{G}_i \) resulting in a non-simply connected gauge group

\[
G = \prod_i \mathcal{G}_i / \mathbb{Z}_n \times \hat{G}, \quad \pi_1(G) = \mathbb{Z}_n.
\]  

(4.5)

The quotient factor induces a constraint matter spectrum by projecting out representations that have a non-trivial center charge \( q_{cen} \). In general, the restriction to center charges can be seen explicitly via the construction of the torsion Shioda map \( \Sigma \) in the resolved geometry. For any non-trivial torsion point \( S_k \) and zero-section \( Z \), the Shioda map defines a map from the elliptic curve to the Néron-Severi lattice of \( Y_n \)

\[
\Sigma(S_k) = S_k - Z + (S \cdot c_i)C_{i,j}^{-1}f_j = S - Z + \frac{1}{n} \sum_i a_j f_j.
\]  

(4.6)

The fractional linear contribution of the resolution divisors \( f_i \) above is determined by the \( i-th \) fibral curve \( c_i \) in the resolved geometry, that is intersected by the torsion section \( S \). The overall normalization \( n \) is inherited from the inverse determinant of the Cartan matrix \( C \), which is itself of the order of its center. Coming from a torsion section \( S_k \) implies the Shioda map to be a trivial divisor in \( H^2(Y) \) which therefore can be rewritten as

\[
\Xi(S_k) = S_k - Z = -\frac{1}{n} \sum_i a_j f_j,
\]  

(4.7)

and represents an n-torsion element in the quotient cohomology of \( H^{1,1}(Y)/[[f_i]]_{\mathbb{Z}} \). At the intersection \( \delta_{i,j} \), the fibral curves in the resolved model generically split further \( c_i \rightarrow c_{i,m} \) reflecting the nature of the enhanced singularity. The exact representation \( R_{i,j} \) and compatibility with the gauged \( \mathbb{Z}_n \) centers can be inferred from intersections with the resolution divisors \( f_i \) and the fractional element of the torsion Shioda map

\[
\mu_r = (c_{i,m} \cdot f_r) \quad q_{cen} = c_{i,m} \cdot \Xi(S_k).
\]  

(4.8)

All other weights can be obtained from adding fibral curves \( c_i \) which also shows, that the center charge is n-fractional and only well defined modulo 1. However, since the torsion Shioda map

\(^9\)We ignore contributions from vertical divisors at this point.
is a trivial divisor the center charge must be integral. Field theoretically one can understand this, by promoting the fractional piece of the torsion Shioda map to an projector of the form $\Phi = e^{2\pi i \Xi(S_k)}$. This projector removes representations with the wrong (combination) of weights of the gauge group $G$. Therefore the above element defines a generator of the n-refined (co-)weight lattice $^{10}$. Note that the spectrum computed above, is just the massless one and therefore comprises only a small sub-sector of the full theory. However it seems convincing, due to the geometric realization of the torsion Shioda map, that the constrained (co-) charge lattice extends to the full massive sector of the theory as well. In more in general it is expected that the non-simply connected gauge groups $G$, also affects the presence of dyonic line and surface operators in the theory $^{69}$. Similarly we note, that the presence of torsion affects the interpretation of $(p,q)$-strings as combined states of fundamental strings and $D1$ branes. A general $(p,q)$-string (with $p$ and $q$ being co-prime) that couples to the IIB $(B_2,C_2)$-fields with the respective quanta can always be rotated into the fundamental $(1,0)$ string picture by virtue of an $SL(2,\mathbb{Z})$ transformation. However under the reduced congruence subgroups we do not have the full transformation group at our disposal and therefore not every $(p,q)$ combination allows for such a transformation. E.g. for a parametrization of a $\Gamma_1(n)$ matrix consider the following action on a fundamental string

$$\begin{pmatrix} 1 + an & r \\ kn & 1 + bn \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 + an \\ kn \end{pmatrix}, \quad \text{with } a, b, r, k \in \mathbb{Z}. \quad (4.9)$$

One finds the $D1$ brane charge to necessarily be divisible by the order $n$ to allow for a rotation into the fundamental string picture. This might be interpreted in two ways: Either, must the $(p,q)$ string states, with odd $q$-$D1$ charge be fully absent from the spectrum, or secondly that these states can be present but come from a different sector of the theory, that can not be described as a fundamental string in an any given $\Gamma_1(n)$ frame. However, we take the non-trivial behavior of more general string states as evidence, that torsion also affects the massive sector of the theory in a non-trivial way. We would like to return to this interesting question in future work.

**An $su(2)/\mathbb{Z}_2$ example**

The simplest direct example is that of an $su(2)/\mathbb{Z}_2 \sim SO(3)$ group. As SO(3) does not admit two dimensional representations, it is directly evident, that the fundamental of the $su(2)$ covering algebra, must be absent from the spectrum. Let’s now engineer this model in F-theory, via an $\mathbb{Z}_2$ torsion point. There, the $su(2)$ gauge algebra is in fact forced upon us directly, as can be seen using the modular curve of the $X_1(2)$. Explicitly, the $\Gamma_1(2)$ congruence subgroup is generated by upper triangle matrices modulo two. To find the generic singularity(s) we have to find the cusp(s) of the modular curve and their widths. As reviewed in Section 3 we first have to find the triangulation of the fundamental domain into $n$ parts, given by the index that

$^{10}$Note that a similar effect can be resembled in the presence of a free Mordell-Weil group, that gives rise to a $U(1)$ symmetry that can embed non trivially inside the center of other non-Abelian group factors $^{49,50}$. Although having a similar origin, these factors can only be present in the presence of additional Abelian symmetries, which we do not want to focus on in this work.
can be computed using the Equation (3.2) as
\[ [\Gamma(1), \Gamma_1(2)] = 3. \] (4.10)
These correspond to the three coset elements
\[ \alpha_1 : \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \alpha_2 : \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_3 : \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}. \] (4.11)
The triangulation of the modular curve into three sub-regions is explicitly shown in Figure 3. There are two cusps at the points
\[ \mathcal{S}_{\text{cusp}}^{(2)} : \{0^{(2)}, \infty^{(1)}\} \] (4.12)
with the one at the origin, having width two. This is consistent with the requirement to have the appropriate gauge algebra \( G \) present on which the \( \mathbb{Z}_2 \) quotient can act. Lets now turn to the explicit Weierstrass model, which exactly reflects what we anticipated from the modular curve before
\[ y^2 = x(x^2 + a_2 x + a_4), \quad a_4 \in \mathcal{O}(K_b^{-4}), \quad a_2 \in \mathcal{O}(K_b^{-2}). \] (4.13)
\[ f = a_4 - \frac{1}{3} a_2^3, \quad g = \frac{1}{27} a_2(2a_2^2 - 9a_4), \quad \Delta = a_4^2(4a_4 - a_2^2). \] (4.14)
There is the expected \( I_2 \) locus over the \( a_4 = 0 \) locus associated with an \( \text{su}(2) \) algebra and an residual \( I_1 \) locus. The torsion factor restricts the algebra to \( \text{su}(2)/\mathbb{Z}_2 \) \( \sim \text{SO}(3) \). Hence, geometrically the enhancement from the \( I_2 \) to \( I_3 \) locus is forbidden, simply because of the incompatible center. Instead the collision point is of type \( III \), that has an associated \( \text{su}(2) \) algebra as well, compatible with the correct center, giving rise to no new matter multiplets.
In the resolved geometry (e.g. see [52]) it is possible to obtain the torsion Shioda map \( \Sigma(S_1) \) (modulo vertical divisors) as

\[
\Sigma(S_1) = S_1 - Z + \frac{1}{2} f_1, \tag{4.15}
\]

with \( f_1 \) being the \( su(2) \) resolution divisor. By abuse of notation, we denote with \( f_1 \) also the possible \( su(2) \) weight of some representation. At codimension two, any reducible matte curve must have vanishing (mod 1) intersection with \( \Sigma(S_1) \), which shows, that every representation with odd \( su(2) \) charge must be absent from the spectrum.

Starting from any torsion model is a great way to systematically study and classify SUGRA theories in various dimensions with a non-trivial first fundamental group. In the following we show the full list of covering algebras and their centers that can occur

\[
\begin{array}{|c|c|}
\hline
\text{algebra} & \text{center} \\
\hline
A_{n-1} & \mathbb{Z}_n \\
B_n & \mathbb{Z}_2 \\
C_n & \mathbb{Z}_2 \\
D_{2n} & \mathbb{Z}_2 \times \mathbb{Z}_2 \\
D_{2n+1} & \mathbb{Z}_4 \\
E_6 & \mathbb{Z}_3 \\
E_7 & \mathbb{Z}_2 \\
\hline
\end{array}
\tag{4.16}
\]

For large torsion, the various options get more and more sparse, but still unbounded thanks to the \( su(n) \) factors. In the next section, we want to switch gears and see, weather anomalies can constrain the order \( n \) in a meaningful way.

### 4.2 6D SUGRA with Putative Large Torsion

From a field theory perspective, there is no direct reason why torsion of higher orders should not exist, given that there is always some \( su(n) \) covering algebra with the given center that can satisfy the strong 6d SUGRA anomalies. These constraints are given as

\[
\begin{align*}
H - V + 29T &= 273, \\
9 - T &= a \cdot a, & \text{(pure-gravitational)} \\
-\frac{1}{6} (A_{adjx} - \sum_\mathcal{R} x_\mathcal{R} A_\mathcal{R}) &= a \cdot b_\kappa, & \text{(Non-Abelian-gravitational)} \\
B_{adjx} - \sum_\mathcal{R} x_\mathcal{R} B_\mathcal{R} &= 0, & \text{(Pure non-Abelian)} \\
\frac{1}{3} (\sum_\mathcal{R} x_\mathcal{R} C_\mathcal{R} - C_{adjx}) &= b_\kappa^2,
\end{align*}
\tag{4.17}
\]

with \( H, V \) and \( T \) being the number of massless hyper-, vector-and tensor multiplets in the theory and \( x_\mathcal{R} \) the number of hypermultiplets in representations \( \mathcal{R} \). The anomaly coefficients \( a \) and \( b_\kappa \) transform as \( SO(1,T) \) vectors and the anomalies can be evaluated using the following group theory coefficients:\footnote{We do not consider \( su(2) \) and \( su(3) \) groups that have different \( C \) and vanishing \( B \) and \( E \) type coefficients.}

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Representation} & \text{Dimension} & A_\mathcal{R} & B_\mathcal{R} & C_\mathcal{R} & E_\mathcal{R} \\
\hline
\text{Fundamental} & n & 1 & 1 & 0 & 1 \\
\text{Adjoint} & n^2 - 1 & 2n & 2n & 6 & 0 \\
\hline
\end{array}
\tag{4.18}
\]
To match the F-theory geometry we make the following identification of base divisors $a \sim K_b$ and $b_\kappa \sim Z_\kappa$ and $su(n)_\kappa$ whereas $\cdot$ denotes the intersection in the base. When considering $su(n)/\mathbb{Z}_n$ groups, there are several massless representations, that can be exclude straight away due to their incompatible $\mathbb{Z}_n$ center charges. These include fundamentals and (single times) antisymmetric representations as those are associated to $su(n) \rightarrow su(n+1)$ and $su(n) \rightarrow so(2n)$ algebra enhancements at codimension two, that have incompatible centers for large enough $n$\footnote{The later ones have have at most an $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$}. Let’s exclude anti-symmetric matter directly and reformulate the anomalies as the following conditions on fundamental and adjoint hypermultiplets

\[ x_{\text{fund}} = 2n(x_{\text{adj}} - 1), \quad x_{\text{adj}} = g = 1 - \frac{1}{2}Z^2, \quad (4.19) \]

where $g$ is the genus of the divisor $Z$ in the base. Let’s consider solutions, compatible with a putative $su(n)/\mathbb{Z}_n$ group over the divisor $Z$. A first solution is that of theory, with only adjoint multiplets, realized by a genus one curve of self-intersection zero. This leaves only the pure gravitational anomaly, that can be solved, given the following amount of pure neutral hypermultiplets

\[ H_{\text{neutral}} = 272 + n - 29T. \quad (4.20) \]

Let’s try to engineer that model more concretely in F-theory. As a base we choose $dP_9$ and the $su(n)$ divisor to be $z = 0 : Z \in K_b^{-1}$ to guarantee it to be of genus $g = 1$. The factorization of a Tate model

\[ y^2 + x^3 + a_1yx + a_2x^2 + a_3y + a_4x + a_6, \quad [a_i] \in K_b^{-i}, \quad (4.21) \]

is given as

\[ a_2 \rightarrow b_1z \quad a_3 \rightarrow c_0z^3 \quad a_4 \rightarrow d_0z^4 \quad a_6 \rightarrow e_0z^6, \quad (4.22) \]

with the $c_0, d_0, e_0$ generic non-vanishing constants. The above factorization, gives an $su(6)$ over $z = 0$, without any further codimension two loci since $(K_b^{-1})^2 = 0$. By further setting $e_0$ and $c_0$ to zero, this singularity can be enhanced to $su(7)$ and $su(8)$ respectively or by further setting $b_1 = 0$, increases the singularity further to $su(9)$. This spectrum admits only a single adjoint representation in its massless spectrum and looks naively like a possible $su(n)/\mathbb{Z}_n$ model for values $n > 4$. Especially the later one admits a torsion point of order three as it is a variant of the Schoen \cite{40,62} but not of order nine as the massless spectrum might suggest.

A second solution to the anomaly constraints above is to put the $su(n)$ factor over a $\mathbb{P}^1$ of self-intersection -2 in the base, hosting no adjoint hypermultiplets but instead $2n$-fundamentals. In order to get that spectrum consistent with the enhancement rules we introduce another $su(n)$ gauge factor, over another -2 curve giving rise to the enhancement

\[ su(n)_1 \cap su(n)_2 \rightarrow su(2n), \quad (4.23) \]
consistent with the $\mathbb{Z}_n$ factor. This configuration can be continued with up to $k$ $su(n)$ factors, intersecting each other as

$$-su(n)_1 - su(n)_2 - \ldots su(n)_k - ,$$

forming the structure of an affine $su(k)$ Dynkin diagram in the base. The resulting gauge group, is of $su(n)^k$ type with $k$, bi-fundamental representations of type $(1, N_i, \bar{N}_{i+1}, \ldots, 1)$ consistent with an naive $\mathbb{Z}_n$ factor. Such a configuration, can cancel the gravitational anomaly, given the existence of

$$H_{\text{neutral}} = 273 - k - 29T , \quad (4.24)$$

neutral hyper and tensor multiplets present.

All the above models have a massless spectrum which is seemingly consistent with an $su(n)/\mathbb{Z}_n$ gauge group or multiple copies of those. It seems here that the pure gauge theory anomalies do not give much guidance, on what bounds on the the first fundamental group we might expect. However we note, that we mainly argued with the massless sub-sector of the theory and there is a priori no reason to assume, that massive modes must respect the putative $su(n)/\mathbb{Z}_n$ group as well. It is arguably more convincing, that the massive modes respect the (co-)charge refinement that is explicitly enforced by the torsion Shioda map and the reduced monodromy of $\Gamma_1(n)$, that also acts non-trivial on $(p,q)$-strings as argued in Section 4.1.

### 4.3 Higher order and Non-Prime Torsion

In this section we want to comment on torsion points and their field theory counterparts in the case, when the torsion affects only a sub-center of the gauge algebra as well when it is non-prime and how this is realized in the geometry itself.

Lets consider e.g. the case when only a $\mathbb{Z}_m$ sub-factor of some $su(n \cdot m)$ algebra becomes gauged. This is visible e.g. in the $\mathbb{Z}_2$ torsion model by further specializing $a_4 \rightarrow b_2^2$, which enhances the $su(2)$ to an $su(4)$. The spectrum however is still constraint to an $su(4)/\mathbb{Z}_2 \sim SO(6)$ sub-factor, consistent with the existence of antisymmetric 6-plet states . This is also consistent with the $\mathbb{Z}_2$ symmetry among 0-and $\mathbb{Z}_2$ torsion which dictates the torsion section to intersect the second $su(4)$ resolution divisor, leaving a torsion Shioda map of the form

$$\Sigma(S_1) = S_1 - Z + \frac{1}{2}(f_1 + 2f_2 + f_3) . \quad (4.25)$$

The factor $1/2$ shows, that the torsion Shioda map is indeed of order two, which gauges the $\mathbb{Z}_2$ sub-center of the $su(4)$ algebra. This does not allow for 4-plets with weight $(1,0,0)$ but the anti-symmetric 6-plets with $(0,1,0)$ weight, which can be seen in the associated Weierstrass model explicitly [52]. It is easy to extend this observation by noting, that globally, there is the freedom to redefine the zero-and order k-torsion section. This symmetry on the other hand, must extend to the appearing gauge algebra factors and hence an cyclic automorphism of the affine Dynkin diagram itself [50]. Indeed this can be geometrically seen by noting that each
order $k$– section can intersect an $su(n \cdot k)$ at the $k$-th-node. The inverse Cartan matrix of the $su(n \cdot k)$ then comes with a factor $\frac{1}{\det(C)} = \frac{1}{nk}$. However, the $k$-th row of $C^{-1}$ comes with a $k$-multiple and hence, the overall contribution of the $su(nk)$ is only $n$– fractional. Hence the discrete $\Sigma(S_k)$ Shioda map, effectively gauges only an $n$-sub-center.

Starting from the torsion forms, it is interesting to observe that, with order $MW(Y)_{tor} = \mathbb{Z}_n$ for $n > 3$, several additional singularities beyond the expected minimal $su(n)$ start appearing. To consider this effect in more detail, we go back to the $\mathbb{Z}_4$ example with general Weierstrass model

\[
y^2 + b_1 xy + b_1 b_2 y = x^3 + b_2 x^2, \quad [b_1] \in K_0^{-1}, [b_2] \in K_0^{-2},
\]

\[
f_4 = -\frac{1}{48} b_1^4 + \frac{1}{3} b_1^2 b_2 - \frac{1}{3} b_2^2, \quad g_4 = -\frac{1}{864} (b_1^2 - 8b_2)(b_1^4 - 16b_1^2b_2 - 8b_2^2),
\]

\[
\Delta = -\frac{1}{16} b_1^2 b_2^2 (b_1^2 - 16b_2).
\]

The above geometry suggests a minimal $(su(2) \times su(4))/\mathbb{Z}_4$ gauge group. The presence of the $su(2)$ factor is especially surprising, as it does not appear in the $\Gamma_1(4)$ modular curve, which we discussed in Section 3.1. From the argument above, it seems also puzzling how the torsion section can consistently intersect the local $su(2)$, although being of order 4. Therefore we will analyze the resolved model in more detail in the following, based on the codimension two resolution with PALP id $(3145,0)$ [28,57] and equation

\[
p_1 = f_{2,1} f_{1,1} f_{2,2} f_2^2 z_0^2 + b_1 f_{2,1} f_{2,2} f_2^2 z_1 z_2 + b_2 f_{1,1} z_2^2, \quad (4.27)
\]

\[
p_2 = f_{2,2} z_1^2 + f_{2,1} z_2^2 + f_{1,1} z_0 z_2, \quad (4.28)
\]

with $z_i$ toric sections and $f_i$ some resolution divisors that have Stanley-Reisner ideal

\[
SRI : \{ f_{2,2} z_2, z_0 z_2, z_1 z_2, f_{2,1} z_2, f_2 z_2, z_2 z_2, z_0 z_2, z_0 z_1, f_{1,1} z_1, f_{2,2} z_1, z_1 z_3, \\
\quad f_{2,2} z_3, z_0 z_3, f_{1,1} z_3, f_{2,3} z_3, f_{2,2} z_0, f_{1,1} f_{2,2}, f_{1,1} f_{2,3}, f_{2,1} z_0, f_{2,1} f_{1,1} \}.
\]

The resolved fibers are shown in Figure 4 together with the intersection of the four torsion sections $\{ z_0, z_1, z_2, z_3 \}$ which we call $s_i$ denoted by their order $i$ in the MW group law [50]. In the $su(4)$ it can be observed, that the section intersect in the same order as the Dynkin diagram whereas in the $su(2)$ case they intersect modulo two. This suggests, the presence of the $su(2)$ algebra, due to the non-trivial $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_4$. This can also be observed via the two different torsion Shioda maps $\Sigma(S_i)$, that give a trivial $i$-torsional divisor in the $NS$ lattice via the expressions

\[
\Sigma(S_1) = [s_1] - [s_0] - \frac{1}{2} f_1 + \frac{1}{4} (f_{2,1} + 2f_r + 3f_{2,2}),
\]

\[
\Sigma(S_2) = [s_2] - [s_0] + \frac{1}{2} (f_{1,1} + 2f_r + f_{2,2}),
\]

with $f_r = -(f_{2,2} + f_{2,3} + f_{2,1})$. We observe the first Shioda map to give a 4-torsion element whereas the second one is only two-torsion, that only sees the $\mathbb{Z}_2$ subgroup of $su(4)$ but not the
Figure 4: Depiction of the generic resolved $\mathbb{Z}_4$ torsion model and its intersection with the torsion sections $s_i$. The intersection pattern with the resolution divisors is compatible with the cyclic order four movement for the $su(4)$ fibers and an order two subgroup for $su(2)$ fibers.

Table 1: Summary of massless 6D F-theory spectra with $\mathbb{Z}_4$ and $\mathbb{Z}_6$ first fundamental group, geometrically realized by the Schoen manifold.
former one is given as
\[ y^2 + a_1xy + \frac{1}{32}(a_1 - b_1)(3a_1 + b_1)(a_1 + b_1) = x^3 + \frac{1}{8}(a_1 - b_1)(a_1 + b_1)x^2 \]
\[ f = \frac{1}{192}b_1(3a_1^3 - 3a_1^2b_1 - 3a_1b_1^2 - b_1^3) \]
\[ g = \frac{1}{110592}(3a_1^2 - 6a_1b_1 - b_1^2)(9a_1^4 - 6a_1^2b_1^2 - 24a_1^3b_1 - 11b_1^4) \]
\[ \Delta = \frac{1}{2^3}(a_1 - 5b_1)(3a_1 + b_1)^2(a_1 + b_1)^3(a_1 - b_1)^6, \quad a_1, b_1 \in \mathcal{O}(K_{\kappa}^{-1}), \] (4.32)

and is also naturally described by a Schoen manifold, in order to keep all (4,6,12) points absent. Note that the generic \( I_6 \), \( I_3 \) and \( I_2 \) singular fibers expected from the cusp points of the modular curve, as discussed in Section 3 and depicted in Figure 2 are present. This model gives rise to a good F-theory vacuum in six dimension with \((su(6) \times su(3) \times su(2))/\mathbb{Z}_6\) gauge group, whose massless spectrum is summarized in Table 1.

The modular curve perspective, gives a good understanding, which minimal singularities are to be expected in certain torsion models. Let’s go to a higher \( n \)–torsion point say \( \mathbb{Z}_8 \) e.g. over a \( \mathbb{F}_0 \sim \mathbb{P}^1_{[t_0,t_1]} \times \mathbb{P}^1_{[s_0,s_1]} \) base which we derive explicitly in Appendix A and analyze the associated Weierstrass model directly. We repeat the model here, as
\[ y^2 + (s_0s_1t_0t_1 + (s_0t_0 - s_1t_1)(2s_1t_1 - s_0t_0))xy + s_0s_1^3t_0t_1^3(s_0t_0 - 2s_1t_1)(s_1t_1 - s_0t_0)y \]
\[ = x^3 + s_1^2t_1^2(s_0t_0 - 2s_1t_1)(s_1t_1 - s_0t_0)x^2, \] (4.33)

with discriminant
\[ \Delta = \frac{-1}{16}s_0^2s_1^8t_0^2t_1^8(s_0t_0 - 2s_1t_1)^4(s_0t_0 - s_1t_1)^8(s_0^2t_0^2 - 8s_0s_1t_0t_1 + 8s_1^2t_1^2). \] (4.34)

This model admits two \( I_2 \), one \( I_4 \) and three \( I_8 \) fibers as well as two (8,12,24) codimension two singularities. Therefore, the \((su(8))^3 \times su(4) \times su(2)^2)/\mathbb{Z}_8\) six dimensional SUGRA model can not exist. In the next section we show, however that a related model in eight dimensions can exist.

### 4.4 K3: 8-dimensional Exceptions

The codimension two (8,12) singularities are pathological and can not be avoided over any base of dimension at least 2. This does not rule out these models fully, as one can avoid them in one dimensional bases, i.e restricting to K3s. This allows some additional possible higher order torsion models that exist exclusively as two-folds, as opposed to three and fourfolds. Hence their underlying 8D SUGRA theories, obtained from F-theory comprise of isolated theories as we discuss in the following. The \( \mathbb{Z}_8 \) model is one of the few exceptional torsion models, only\textsuperscript{13} present in elliptic K3’s. The groups and their fiber configurations are summarized in Table 2 that is was adapted from a table in [32]. As can be found, all elliptic K3’s are of extremal

\textsuperscript{13}As opposed to Calabi-Yau fibrations of higher dimension.
Table 2: Summary of large torsion models, that can not be realized in three and fourfolds and their fibers. All of them are extremal K3’s.

| MW$_{tor}$ | Fibers       |
|------------|--------------|
| $\mathbb{Z}_7$ | $3I_7$       |
| $\mathbb{Z}_8$ | $2I_8 + I_4 + I_2$ |
| $\mathbb{Z}_4 \times \mathbb{Z}_2$ | $3I_6 + 3I_2$ |
| $\mathbb{Z}_4 \times \mathbb{Z}_4$ | $6I_4$       |

type which means that their Néron-Severi group has maximal possible rank 20. In short, no elements of $h^{1,1}(Y)$ can contribute to complex structure deformations which results in a rigid geometry. This rules out the possibility to construct K3-fibered Calabi-Yau three-or four-folds.

In fact, the $T = \mathbb{Z}_8$ threefold given in (4.33) can be viewed as an enforced K3-fibration over a base $\mathbb{P}^1$ in terms of $[t_0, t_1]$ coordinates. Deleting the $\mathbb{P}^1$ by effectively setting the coordinates to one, gives the consistent K3 with the correct fibers.

In this regard, also their associated 8D F-theory vacua are special. Their rigidity, does not allow for a minimal SUSY preserving compactification to lower dimension\textsuperscript{14}. Moreover, the rigidity, also does not allow for a stable degeneration limit at finite distance in the moduli space, as those require at least one free complex structure modulus to assign the proper scaling. Therefore, these models do not have a (geometric)\textsuperscript{15} heterotic dual either. This is consistent with the fact, that the groups do not fit into $E_8 \times E_8$, which is also reflected by the fact, that none of the torsion factors, are discrete finite subgroups of $E_8$.

The same conclusions can be obtained, by simply considering the modular curves of the higher torsion models which naturally explains the absence of even larger torsion in K3. In Figure 5 we depicted the three modular curves that correspond to $\mathbb{Z}_4 \times \mathbb{Z}_4$, $\mathbb{Z}_8$ and $\mathbb{Z}_9$ torsion model explicitly. The $\mathbb{Z}_4 \times \mathbb{Z}_4$ modular curve, admits six cusp points at

$$S_{\text{cusp}}^{(4\times 4)} : \{0^{(4)}, 1/2^{(4)}, 1^{(4)}, 2^{(4)}, 3^{(4)}, \infty^{(4)}\},$$

(4.35)

where we highlighted their widths as superscripts. The associated fibers are consistent with the expectation from the K3 classification. Similarly, the $\mathbb{Z}_8$ curve, discussed earlier admits six cusps

$$S_{\text{cusp}}^{(8)} : \{0^{(8)}, 1/4^{(2)}, 1/3^{(8)}, 3/8^{(1)}, 1/2^{(4)}, \infty^{(1)}\},$$

(4.36)

consistent with K3 construction as well. Finally depict also the $\mathbb{Z}_9$ case in Figure 5. There are eight cusps in total at

$$S_{\text{cusp}}^{(9)} : \{0^{(9)}, 2/9^{(1)}, 1/4^{(9)}, 1/3^{(3)}, 4/9^{(1)}, 1/2^{(9)}, 2/3^{(3)}, \infty^{(1)}\}.$$  

(4.37)

Collecting all singular fibers, we end up with three $I_9$ and two $I_3$ fibers in total. Resolving these singularities while adding the class of the generic fiber and base required 30 independent Kähler

\textsuperscript{14}Note that trivial, e.g. torus compactifications are still possible.

\textsuperscript{15}In \textsuperscript{73} non-geometric heterotic backgrounds have been proposed to be dual to certain F-theory backgrounds, that also non-crepant singularities.
Figure 5: Triangulations of $X(4)$, $X_1(8)$ and $X_1(9)$. Colors highlight identified lines, allowing to read off the cusp points and their widths.
forms. This clearly overshots the maximum of 20 that $K3$ has at its disposal by large. With view on F-theory, this also shows explicitly why there can be no eight dimensional SUGRA theories, with a non-simply connected gauge group larger eight, or $\mathbb{Z}_4 \times \mathbb{Z}_4$.

### 4.5 Bounds on Calabi-Yau Quotient Torsors

In more in general, the above limits, also constraint the construction of Calabi-Yau torsors that are smooth, non-simply connected genus-one fibered threefolds $\hat{Y}_3$. This construction has relevance for heterotic $[22,44,45]$ and F-theory compactifications $[19,21]$, on which we want to focus on in the following. The non-simply connected manifolds in question, can be constructed by starting with a smooth elliptic threefold $X_3$ that admits a free and finite, automorphism $\Gamma_n$.

To preserve the fibration, $\Gamma_n$ must be decomposable into a base action

$$\Gamma_{n,b} : B_2 \rightarrow B_2,$$

that preserves the section $\sigma_0$ and a fiber translation $\Gamma_{n,f}$ that acts on a local coordinate $x_b = \pi^{-1}(b)$ for $b \in B_2$ as a translation

$$\Gamma_{n,f} : x_b + \sigma_n(b)$$

with $\sigma_n$ being a point of order $n$, with respect to the MW addition law, that is a point of order $n$. Requiring the shift symmetry to be globally defined results in a non-trivial Mordell-Weil group $\text{MW}(X_3)_{\text{tor}} = \mathbb{Z}_n$. The resulting quotient geometries have two central properties that we want to comment on.

1. The new fibration no longer has a global section, due to the free action on the zero-section:

$$\Gamma_{n,f} : \sigma_0 \sim \sigma_i, \text{ with } i = 1 \ldots n,$$

In the quotient, the zero section is glued to the torsion section to become a multi-section.

2. Although $Y_3$ is itself smooth, the base has isolated singularities over the fixed points of the $\Gamma_{n,b}$ action. The fibration has multiple fibers over the singular points. $[55]$ over it $[17]$.

In the physics of F-theory, the n-sections have been shown to lead to discrete $\mathbb{Z}_n$ gauge symmetries in the 6D SUGRA theory. The fixed points on the other hand carry discrete gauged superconformal matter theories coupled to gravity. In $[19,21]$ consistency of such theories for all $\mathbb{Z}_n$ quotients have been shown.

The bounds on the MW torsion groups therefore also constraint the admissible Calabi-Yau torsors to not be larger than $\mathbb{Z}_6, \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_3 \times \mathbb{Z}_3$. In the physics of F-theory, these constraints can be translated into bounds on discrete symmetries and gaugings of superconformal matter. In particular the large order quotients, $n > 4$ turn out to be all quotients of the Schoen

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16Different possibilities can be found in $[21]$.

17In the context of F-theory, these fibers also appeared in the duality to the CHL string $[37]$ and classification of little string theories $[40]$. 

23
manifolds, classified in [40] whose F-theory physics has been investigated in [21]. Therefore, the orders of the quotients can not exceed the symmetries of rational elliptic surface, which are bounded by the orders above, seem to be all torsors possible via that method.

5 Technical Results

Our main result is the bound on torsion groups in smooth Calabi-Yau $n$-folds, $n \geq 3$. The key idea in the proof is the fact that the diagram (2.4) forces $(f, g, \Delta)$ to vanish to order $(4d, 6d, 12d)$ over points in the indeterminacy locus of $\phi$ which obstruct the existence of a flat crepant resolution of the total space. To ease exposition, we will prove the theorem for 3-folds specifically and comment on its validity for four-or five folds at the end.

To avoid having to constantly refer to the diagram, we introduce the phrase “special fibration”. An elliptic fibration $Y_n \to B_{n-1}$ is special if it fits into a commutative diagram:

$$
\begin{array}{cccc}
Y_n & \xrightarrow{\phi} & S_2 \\
\downarrow \pi & & \downarrow p \\
B_{n-1} & \xrightarrow{\phi} & C_1,
\end{array}
$$

satisfying the following conditions:

- $S_2 \to C_1$ is a smooth, minimal elliptic surface.
- The horizontal maps are non-constant, rational maps.
- The vertical maps are proper.
- $\phi$ is flat.

5.1 Global Lemmas

**Lemma 5.1.** Let $B$ be a rational variety and $\phi : B \to C$ be a non-constant rational map. Then $C$ has genus 0 and $\phi$ is flat.

*Proof.* Since $\phi$ is non-constant, we can find points $b_1, b_2 \in B$ such that $\phi(b_1) \neq \phi(b_2)$. Since $B$ is rationally connected, there is a regular map $\mathbb{P}^1 \to B$ taking 0 to $b_1$ and $\infty$ to $b_2$. The composition $\mathbb{P}^1 \to C$ is a non-constant rational map, so it extends to a surjective morphism. This is only possible if $C$ has genus 0.

To prove flatness of $B \to \mathbb{P}^1$, it suffices to prove the map is flat over each point in $\mathbb{P}^1$. Any proper open neighborhood of $\mathbb{P}^1$ has the form $\text{Spec} R_0$ for $R_0$ a principal integer domain (PID), so we can determine whether the map is flat by studying the morphism of algebras $R_0 \to K(B)$. Since $R_0$ is a PID, flatness is equivalent to $K(B)$ being torsion-free, which follows immediately from the fact that the map $B \to \mathbb{P}^1$ is non-constant and $K(B)$ is purely transcendental.

---

\footnote{Higher order quotients of type $(K3 \times T^k)/\mathbb{Z}_n$ [25,37] might still be viable though as well torsion, that do not make use of MW torsion in the covering geometry.}
The fundamental line bundle of an elliptic fibration \( \pi : Y \to B \) is \( R^1\pi_*\mathcal{O}_Y \). We denote it by \( \mathcal{L}_{Y/B} \). We write \( \omega_Y \) (resp. \( \omega_B \)) to denote the canonical bundle of \( Y \) (resp. \( B \)).

**Lemma 5.2.** Let \( B \) be a rational surface and \( \pi : Y \to B \) a special elliptic 3-fold. Then \( \mathcal{L}_{Y/B} \cong \phi^*(\mathcal{L}_{S/P^1}) \).

*Proof.* The conditions in the definition of a special 3-fold allow us to use Prop. III.9.3 in [70] to compute:

\[
\mathcal{L}_{Y/B} = R^1\pi_*\mathcal{O}_Y = R^1\pi_*\Phi^*\mathcal{O}_S = \phi^*R^1\pi_*\mathcal{O}_S = \phi^*\mathcal{L}_{S/B}.
\]

(5.2)

**Lemma 5.3.** Let \( \pi : Y \to B \) be as above and let \( d \) be the degree of \( \mathcal{L}_{S/P^1} \). Then \( \omega_Y \cong \omega_B \otimes \phi^*(\mathcal{O}_{P^1}(1))^\otimes d \).

*Proof.* This follows from the canonical bundle formula for elliptic fibrations, together with the computation from the previous lemma.

**Proposition 5.4.** Let \( \pi : Y \to B \) be a special elliptic 3-fold. Assume that \( B \) is a minimal rational surface and \( \omega_Y \) is trivial. Then \( \phi \) is not a morphism.

*Proof.* There are no non-constant morphisms \( \mathbb{P}^2 \to \mathbb{P}^1 \), so we may assume \( B \cong \mathbb{F}_n \) for \( 0 \leq n \leq 12 \). Note that in all cases \( NS(B) \cong \mathbb{Z} \oplus \mathbb{Z} \), the class of the canonical bundle is \((-2, -n)\), and \([K_B] \cdot [K_B] = 8 \).

If \( \phi^{-1}(\mathcal{O}(1)) = \omega_B \), then the number of points in the locus of indeterminacy (counted appropriately) is \([K_B] \cdot [K_B] = 8 \). Otherwise, \( n \) is necessarily even and \( \phi^{-1}(\mathcal{O}(1))^\otimes 2 \cong \omega_B \). In this case, the number of points in the locus of indeterminacy is \( \frac{1}{2} \cdot 8 = 2 \).

In all cases, the requirement \( \phi^{-1}(\mathcal{O}(1)) \cong \omega_B \) forces the existence of at least one base point.

### 5.2 Local Lemmas

In this section \( k \) is an algebraically closed field, \( R \) a factorial, finitely generated \( k \)-algebra and \( K \) is the fraction field of \( R \).

**Definition 5.5.** Let \( \mathfrak{m} \subset R \) be a maximal ideal and \( I \subset R \) an ideal. The order of vanishing of \( I \) at \( \mathfrak{m} \) is the largest integer \( m \) such that \( I \subset \mathfrak{m}^m \).

**Definition 5.6.** Let \( \phi = \frac{p}{q} \in K \), with \( p, q \in R \) relatively prime. We can think of \( \phi \) as a map \( \text{Spec} R \to \mathbb{P}^1 \). The **locus of indeterminacy** of \( \phi \) is \( V(p) \cap V(q) \subset \text{Spec} R \).

**Definition 5.7.** Let \( \phi \) be as above, and assume the locus of indeterminacy of \( \phi \) contains a closed point \( b \). Let \( \mathfrak{m}_b \subset R \) be the corresponding ideal. We define \( m_\phi(b) \) to be the multiplicity of the ideal \( (p, q) \) at \( \mathfrak{m}_b \).

Note that \( m_\phi(b) \) makes sense whenever we have a rational map \( B \to \mathbb{P}^1 \) from a normal scheme \( B \), since the property is local on \( B \).
Proposition 5.8. Let $\phi : B \to \mathbb{P}^1$ be a non-constant rational map and $f$ a global section of $\mathcal{O}_{\mathbb{P}^1}(d)$ for some $d > 0$. If $b \in B$ is in the indeterminacy locus of $\phi$, then $\phi^*(f)$ vanishes to order $m_\phi(b)d$ at $b$.

Proof. The claim is local on $B$, so we assume $B$ is affine, say $B = \text{Spec} R$. Choosing coordinates $[x_0 : x_1]$ on $\mathbb{P}^1$, we can express $f(x_0, x_1)$ as a homogeneous polynomial of degree $d$ in $x_0, x_1$. Furthermore, we can write the map $\phi : B \to \mathbb{P}^1$ as $b \mapsto [p(b) : q(b)]$, where $p, q \in R$ have no common factors. In this notation, we have $\phi^*(f)(b) = f(p(b), q(b))$.

Since $f$ is a homogenous polynomial of degree $d$, $f \in (x_0, x_1)^d \subset k[x_0, x_1]$. If $b \in B$ is in the locus of indeterminacy of $\phi$, then $m_\phi(b)d \supset (p(b), q(b)) = \phi^* ((x_0, x_1))$ so:

$$\phi^* f \in (p(b), q(b))^d \subset m_\phi(b)^d$$

We now easily deduce the following:

Corollary 5.9. Let $\pi : Y \to B$ be a special fibration, let $d$ be the degree of the fundamental line bundle of $L_{S/\mathbb{P}^1}$, and let $b \in B$ be a point in the indeterminacy locus of $\phi$. Then the Weierstrass coefficients $f, g$ of $Y$ vanish to order $(4n, 6n)$, where $n = dm_\phi(b)$.

Proof. Commutativity of the square tells us $f_B = \phi^* (f_B^2)$ and $g_B = \phi^* (g_B^2)$. The Weierstrass coefficients of the elliptic surface are homogenous polynomials of degree $4d, 6d$ respectively, proving the claim.

Corollary 5.10. Let $\pi : Y \to B$ be special elliptic fibration and suppose the order of vanishing of $(f, g)$ does not exceed $(4, 6)$ over any point $b \in B$. Then either $\phi$ is a morphism, or $S$ is rational and the locus of poles and the locus of zeros of $\phi$ intersect transversely.

Proof. Assume $\phi$ is not a morphism.

An elliptic surface is rational if and only if the fundamental line bundle has degree 1. If the fundamental line bundle has degree $d > 1$, then the order of vanishing over all points in the locus of indeterminacy is at least $(4d, 6d)$. Thus the condition on $(f, g)$ forces $d = 1$, hence rationality of $S$. If the locus of poles of and zeros meet non-transversely at some point $b$, then $m_\phi(b) > 1$ so $(f, g)$ vanish to order at least $(8, 12)$ over $b$.

5.3 Summary of Technical Portion

The results of the previous two sections are summarized in the following theorem:

Theorem 5.11. Let $\pi : Y \to B$ be an elliptic 3-fold satisfying:

1. $Y$ has trivial canonical bundle.
2. $B$ is a minimal rational surface.
3. The order of vanishing of the Weierstrass coefficients \((f, g)\) does not reach \((8, 12)\) over any point \(b \in B\).

If the fibration is special, then:

- \(C \cong \mathbb{P}^1\).
- \(S\) is rational, or equivalently \(\mathcal{L}_{S/C} \cong \mathcal{O}(1)\).
- \(\phi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \omega_B^\vee\)
- \(m_\phi(b) = 1\) for all points in the locus of indeterminacy of \(\phi\).

We make some remarks before proving the main theorem.

- The condition on the order of vanishing of \((f, g)\) is precisely the condition needed to guarantee that the Weierstrass model admits a proper, flat crepant resolution, see e.g. [64].
- Using the birational classification of algebraic surfaces, we can be more precise: \(\phi\) has 8 or 9 points of indeterminacy, with 9 occurring if and only if \(B \cong \mathbb{P}^2\). If we assume that \(B = \mathbb{P}^2\) e.g., it is easy to see that resolving indeterminacy of \(\phi\) means blowing up \(\mathbb{P}^2\) at the 9 points in the base locus of a pair of cubics, so the new map \(\tilde{\phi} : \tilde{B} \to \mathbb{P}^1\) is itself an elliptic fibration. Commutativity of (5.1) gives us a natural map \(\tilde{Y} \to \tilde{B} \times_{\mathbb{P}^1} S\). Minimality of the fibration \(\tilde{Y} \to \tilde{B}\) then forces that map to be an isomorphism, showing that any special \(Y \to B\) is birational to a Schoen manifold.

### 5.4 Application

**Theorem 5.12.** Let \(\pi : Y \to B\) be an elliptic 3-fold satisfying the hypotheses of the previous theorem. Then \(\text{MW}(Y/B)_{\text{tors}}\) is isomorphic to one of the following groups:

\[
\begin{align*}
\mathbb{Z}_n : & \quad (n = 1, 2, 3, 4, 5, 6), \\
\mathbb{Z}_2 \times \mathbb{Z}_{2m} : & \quad (m = 1, 2), \\
\mathbb{Z}_3 \times \mathbb{Z}_3 .
\end{align*}
\]

Note that this list of groups is exactly the list studied in [53].

**Proof.** The map \(B \to X_1(n)\) shows that \(Y \to B\) is special. The degree of \(\mathcal{L}_{S/C}\), where \(C \cong X_1(n)\), can be found in [32] or computed using the formulae in Appendix B. To obtain a smooth Calabi-Yau 3-fold, we need the degree to be exactly 1, which occurs precisely for the torsion groups listed above.

In particular, we note that this argument can be applied to bound Mordell-Weil torsion in Calabi-Yau fibrations of larger dimension: the map \(\phi : B \to C\) always exists, and is guaranteed to have nontrivial indeterminacy locus if we assume the fundamental line bundle has positive degree. The order of vanishing of \((f, g, \Delta)\) over points in the indeterminacy locus is at least \((8, 12, 24)\) as soon as \(\deg \mathcal{L}_{S/C} > 1\). Thus, it is impossible to crepantly resolve singularities of the Weierstrass model and obtain a smooth Calabi-Yau total space. We devote Appendix A to the explicit construction Weierstrass models beyond the list above.
6 Summary and Conclusions

In this work we prove that the Mordell-Weil torsion group $T$ of elliptically fibered $n$-folds, with $n > 2$ can not exceed $T = \mathbb{Z}_6$ and $T = \mathbb{Z}_3 \times \mathbb{Z}_3$ while being crepant resolvable. This shows, that the list of generic Weierstrass models with torsion points, constructed in [53] is complete. We show explicitly, that higher order torsion points lead to (multiple) minimal singularities, that mutually intersect in codimension two, leading to (8,12) fiber singularities. We present a novel perspective on those minimal singularities as a direct feature of the congruence groups $\Gamma_1(n)$ and $\Gamma(n)$ of $SL(2, \mathbb{Z})$. There, the amount and type of minimal singularities can directly be read off from cusps in the fundamental domain of the modular curve of the respective congruence subgroup. We find the order of torsion groups bound by those of rational elliptic surfaces. On the other hand, K3 manifolds can avoid (8,12) non-crepant codimension two singularities and allow for more possibilities. By connecting the index of the congruence subgroups, with the degree of the fundamental line bundle, we can show however that also K3 cannot exceed $\mathbb{Z}_8$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$ as classified in [32]. These K3 manifolds however are all of extremal type and do not admit any free complex structure deformation. Therefore they neither have a stable degeneration limit, nor can they be used as building blocks for K3 fibered three- or four-folds. Secondly we interpret these bounds within the physics of F-theory, where torsion realizes the first fundamental group of the global gauge group as a \textit{swampland} constraint. We argue that, from pure field theory considerations, the above constraints are surprising as it is possible to construct rather large rank $su(n)$ gauge theories where the massless spectrum respects a putative $\mathbb{Z}_n$ quotient factor. This might point towards the possibility, that it is the massive spectrum that does not respect the putative first fundamental group, which is geometrically enforced when torsion is present.

We observe a variety of surprising effects, when large torsion is present, that have no field theory explanation yet. Especially the necessity for multiple gauge factors does not have an immediate field theory explanation. A possible explanation of this effect might be to view the torsion as a kind orbifold in the torus $SL(2, \mathbb{Z})$ of the torus to an $\Gamma_1(n)$ subgroup requiring the presence of additional gauge theory sectors for consistency of the modularity. It would also be interesting to explore, weather the subtle Dai-Freed anomalies [75] could be the right framework, to give a better field theory answer why we see these specific gauge algebra factors for a given large fundamental group. These bounds however can partially be avoided in isolated 8D SUGRA vacua. These vacua admit at most trivial circle compactifications, but none that preserve only a minimal amount of SUSY. They also do not admit a heterotic dual, due to their rigidity as well as because their gauge group does not fit into $E_8 \times E_8$ (or \text{SO}(32)) . It would be interesting, to explain this eight dimensional exception from a field theoretical perspective, maybe in the spirit of [74]. Further, since all admissible torsion groups must be embeddable into a rational elliptic surface (and into its $E_8$ lattice) it seems plausible, that the heterotic string plays a similar prominent role as in [67].
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A Universal Elliptic Curves with Points of Order $n \leq 8$.

In this appendix we give explicit constructions for the universal elliptic curves with a point of order $n$, for $2 \leq n \leq 8$, and we show how to extend the natural rational maps $B \to \mathbb{P}^1$ over codimension 1 components of the discriminant when $n = 7, 8$. The construction for modular curves is well-known in the arithmetic theory of elliptic curves, see [59]. See [32,34] for the generalization to elliptic surfaces. We are including the explicit constructions since these maps have not been used to study higher dimensional elliptic fibrations and the information we need is in the locus of indeterminacy.

The "main" construction applies only to elliptic curves with points of order $n \geq 4$. For completeness, we also explain what happens when we try to parametrize elliptic curves with a 2 or 3 torsion point.

The construction can be carried out over any ground field - for simplicity, we assume our ground $K$ has characteristic 0 throughout.

A.1 $n = 2, 3$

Let $E/K$ be an elliptic curve and $P \in E(K)$ a 2-torsion point. We can choose coordinates so:

$$E : \quad y^2 = x(x^2 + ax + b), \quad P = (0, 0), \quad (A.1)\n$$

for some $a, b \in K$. The isomorphism class of the pair $(E, P)$ is determined by $a, b$, but the choice of $a, b$ are not unique. Two choices $a, b$ and $a', b'$ give rise to isomorphic pairs if and only if there exists $\lambda \in K^\times$ such that $\lambda^2 a = a'$ and $\lambda^4 b = b'$. Thus, the modular curve parametrizing pairs $(E, P)$, where $P$ is a 2-torsion point, is:

$$X_1(2) = \{(a, b) \in K^2\} / (a, b) \sim (\lambda^2 a, \lambda^4 b), \quad (A.2)\n$$

Similarly, any pair $(E, P)$ with $P$ a point of order 3 is isomorphic to:

$$y^2 + axy + by = x^3, \quad P = (0, 0), \quad (A.3)\n$$

for some $a, b \in K$. Two choices $a, b$ and $a', b'$ give rise to isomorphic pairs if and only if there exists $\lambda \in K^\times$ such that $\lambda a = a'$ and $\lambda^3 b = b'$. Thus:

$$X_1(3) = \{(a, b) \in K^2\} / (a, b) \sim (\lambda a, \lambda^3 b). \quad (A.4)\n$$

Both of these quotients are singular - this is due to the presence of finite order points in $\Gamma_1(2), \Gamma_1(3)$. For $n \geq 4$, there are no more finite order points in the congruence subgroup, so we obtain smooth modular curves. As a result, the moduli spaces are better behaved. This is why we study elliptic curves with a point of order at least 4 separately.

A.2 Universal Elliptic Curves with a Point of Order $n \geq 4$

Let $E/K$ be an elliptic curve and let $P \in E(K)$ be a point which has order at least 4 in the MW group. We can do a change of variable that takes the tangent at $P$ to the line $y = 0$. Since
the order of $P$ is not 2, the equation of the tangent has the form $y = \lambda x + \nu$, so the equation we obtain after this change of variables only contains "Weierstrass monomials". Explicitly, if the tangent at $P$ is $y = \lambda x + \nu$, then we set $y' = y - (\lambda x + \nu)$; setting $y' = 0$ is then the same as requiring $y = \lambda x + \nu$. Thus, in the new coordinates, the equation of $E$ has the form:

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2,$$  \hspace{1cm} (A.5)

since the polynomial on the RHS is the equation of $E$ restricted to the tangent at $(0,0)$, and therefore has to vanish twice at $(0,0)$. Finally, multiplying through by $a_3^2/a_2^3$ and replacing $x, y$ by $a_3^2/a_2^3, a_3^2$ gives a Weierstrass equation:

$$y^2 + \frac{a_1a_2}{a_3}xy + \frac{a_3^3}{a_2^2}y = x^3 + \frac{a_3^3}{a_2^3}x^2$$  \hspace{1cm} (A.6)

The main things to notice about the last equation are that only the coefficients $a_1, a_2, a_3$ are nonzero, and that $a_2 = a_3$.

Thus, for any elliptic curve $E$ with a point $P$ of order at least 4, there exist unique $u, v \in K$ such that the pair $E, P$ is isomorphic to:

$$y^2 + (1 - u)xy - vy = x^3 - vx^2.$$  \hspace{1cm} (A.7)

We note that we are using parameters $1 - u, -v$ instead of just $u, v$ to simplify future computations.

Viewing this as an elliptic curve over $K(u, v)$, we can compute multiples of $P = (0, 0)$:

$$P = (0, 0), \quad 2P = (v, uv), \quad 3P = (u, u - v),$$

$$-P = (0, v), \quad -2P = (v, 0), \quad -3P = (u, u^2),$$

$$4P = \left( \frac{(v-u)v}{u^2}, \frac{v^2(u^2 + u - v)}{u^3} \right), \quad 5P = \left( \frac{uv(u^2 + u - v)}{(u - v)^3}, \frac{u^2v(u^3 + uv - v^2)}{(u - v)^3} \right),$$

$$-4P = \left( \frac{(v-u)v}{u^2}, \frac{(v-u^2)v}{u^3} \right), \quad -5P = \left( \frac{uv(u^2 + u - v)}{(u - v)^3}, \frac{v^2(u^2 + u - v)^2}{(u - v)^3} \right).$$

We can construct the universal elliptic curve with an $n$-torsion point for $n = 4, 5, 6$ by setting $P = -3P, 3P = -2P, 3P = -3P$, respectively.

$$4P = O \iff P = -3P \iff u = 0,$$

$$5P = O \iff 3P = -2P \iff u = v,$$

$$6P = O \iff 3P = -3P \iff u^2 + u = v,$$

This gives us the universal elliptic curves:

$$y^2 + xy - ty = x^3 - tx^2, \quad (0, 0) \in E[4],$$  \hspace{1cm} (A.8)

$$y^2 + (1-t)xy + ty = x^3 - tx^2, \quad (0, 0) \in E[5],$$  \hspace{1cm} (A.9)

$$y^2 + (1-t)xy - (t^2+t)y = x^3 - (t^2+t)x^2, \quad (0, 0) \in E[6],$$  \hspace{1cm} (A.10)
Next, we compute the universal elliptic curves for \( n = 7, 8 \). For these fibrations, the relation between \( u, v \) will define a singular curve, and we will compute the normalization of the curve explicitly to find the universal curve.

For \( n = 7 \), we set \( 5P = -2P \) to obtain:

\[
7P = O, \quad \iff u^3 - uv + v^2 = 0.
\]

The cubic is clearly nodal. The normalization of the curve is:

\[
\text{Spec}\mathbb{C}[t] \to C' = \text{Spec}\mathbb{C}[u, v]/u^3 - uv + v^2, \quad t \mapsto (t^2 - t, t^3 - t^2),
\]

and has a rational inverse given by \((u, v) \mapsto \frac{v}{u}\). Thus, the universal elliptic curve is:

\[
y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2.
\]

For \( n = 8 \), we set \( 4P = 4P \) to obtain:

\[
v(u^2 + u - v) = (v - u)^2.
\]

The normalization of this curve is:

\[
t \mapsto \left(\frac{(2t - 1)(t - 1)}{t}, (2t - 1)(t - 1)\right).
\]

Note that the inverse of the normalization map is again \((u, v) \mapsto \frac{v}{u}\).

### A.2.1 Extending \( \phi \)

Let \( \pi : Y \to B \) be an elliptic fibration with an \( n \)-torsion section \( P, 4 \leq n \leq 8 \). Let \( S \to \mathbb{P}^1 \) be the Néron model of one of the following universal elliptic curves over \( \mathbb{C}(t) \):

\[
y^2 + (1 + t - t^2)xy + (t^2 - t^3)y = x^3 + (t^2 - t^3)x^2, \quad (A.11)
y^2 + (t^2 - (t - 1)(2t - 1))xy - t^3(t - 1)(2t - 1)y = x^3 - t^2(t - 1)(2t - 1)x^2. \quad (A.12)
\]

Note that \((0, 0)\) has order \( n \) where \( n = 7 \) for the first equation and \( n = 8 \) for the second. Let \( U = B - V(\Delta) \) and \( Y_U = Y \times_B U \). We have a commutative diagram:

\[
\begin{array}{ccc}
Y_U & \xrightarrow{\phi} & S \\
\downarrow{\pi} & & \downarrow{\pi} \\
B_U & \xrightarrow{\phi} & \mathbb{P}^1,
\end{array} \quad (A.13)
\]

with \( B \) flat and non-constant. We can take the closure of the graph to extend to a diagram:

\[
\begin{array}{ccc}
Y_0 & \xrightarrow{\phi} & S \\
\downarrow{\pi} & & \downarrow{\pi} \\
B_0 & \xrightarrow{\phi} & \mathbb{P}^1.
\end{array} \quad (A.14)
\]
The map $B_0 \to \mathbb{P}^1$ is obtained by composing the map to the singular modular curve (in $u, v$ coordinates):

$$b \mapsto \left( \frac{a_3 - a_1 a_2}{a_3}(b), -\frac{a_2^3}{a_3^2}(b) \right),$$

(A.15)

with the inverse of the normalization map. The inverse of the normalization map is the same for both $n = 7, 8$, and the composite map is:

$$b \mapsto \left[ \frac{a_3(a_3 - a_1 a_2)}{a_3^2}(b) : 1 \right].$$

(A.16)

### A.3 Torsion groups of type $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$

In this section we derive the Weierstrass equation for the elliptic surface with torsion group $\mathbb{Z}_2 \times \mathbb{Z}_6$.

Any elliptic curve with torsion group $\mathbb{Z}_2 \times \mathbb{Z}_6$ has a point of order 6, and therefore admits a Weierstrass equation of the form:

$$y^2 + (1 - \psi(t))xy - (\psi(t) + \psi(t)^2)y = x^3 - (\psi(t) + \psi(t)^2)x^2,$$

(A.17)

with $\psi(t)$ a function on the base. To find the “universal $\psi$", we use the following facts from algebra:

- Let $G$ be a group. Then $G$ contains a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_6$ if and only if it contains a subgroup isomorphic to $\mathbb{Z}_6$ and a subgroup isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

- The 2-torsion points of an elliptic curve:

$$y^2 = x^3 + fx + g,$$

(A.18)

are in bijection with roots of $x^3 + fx + g$. The discriminant of the elliptic curve is the same (up to a unit) as the discriminant of the cubic.

- All roots of $x^3 + fx + g$ are defined over a field $K$ if and only if the cubic has at least one root in the field, and the discriminant of the cubic is a perfect square.

Thus, to find the Weierstrass equation, we need to find $\psi$ so that the equation above is a perfect square. The choice $\psi(t) = \frac{10 - 2t}{t^2 - 9}$ gives us a Weierstrass equation for the elliptic surface:

$$y^2 + \left( 1 - \frac{10 - 2t}{t^2 - 9} \right) xy + \frac{2(t - 1)^2(t - 5)}{(t^2 - 9)^2} y = x^3 + \frac{2(t - 1)^2(t - 5)}{(t^2 - 9)^2} x^2,$$

(A.19)

with discriminant:

$$\Delta = \frac{(t - 9)^2(t - 1)^6}{(t^2 - 9)^2}.$$  

(A.20)

This model has the correct torsion group.
We now perform a standard change of variable to obtain an integral model:

\[ y^2 + (t^2 + 2t - 19)xy + 2(-5 + t)(-1 + t)^2(-9 + t^2)y = x^3 + 2(-5 + t)(-1 + t)^2x^2. \] (A.21)

The degree of \( a_1 \) is 2 and the degree of \( a_i \) is bounded above by \( 2i \) for \( i = 2, 3 \); thus the fundamental line bundle of the elliptic surface over \( \mathbb{P}^1 \) has degree 2, as expected from Miranda’s Table 2.

### A.4 Examples

In this section, we give Weierstrass models of singular Calabi-Yau 3-folds with Mordell-Weil torsion in \( \{ \mathbb{Z}_7, \mathbb{Z}_8, \mathbb{Z}_2 \times \mathbb{Z}_6 \} \). To construct these 3-folds, we used the method described in the introduction:

- To obtain the Weierstrass model of the modular surface \( S \to \mathbb{P}^1 \), we use the construction in the universal elliptic curve section.

- We chose the map \([s_0 : s_1] \times [t_0 : t_1] \to [s_0t_0 : s_1t_1]\) as \( \phi \), although any other bi-degree \((1,1)\) map would have given an equivalent 3-fold. Note that we need a bi-degree \((1,1)\) function because the \( \mathcal{L}_{S/\mathbb{P}^1} = \mathcal{O}(2) \) for both 7 and 8 torsion. Note that for our choice of \( \phi \), the locus of indeterminacy is \([1 : 0] \times [0 : 1] \) and \([0 : 1] \times [1 : 0] \).

To obtain the equation for the 3-fold, we pulled back the sections of the elliptic surface by \( \phi \) to obtain the coefficients shown below.

Note that both 3-folds have \((f, g, \Delta)\) vanishing to order exactly \((8, 12, 24)\) over \([1 : 0] \times [0 : 1] \) and \([0 : 1] \times [1 : 0] \), as predicted by our lemma.

### A.4.1 Explicit Equations

Explicit equations for the 3-folds over \( \mathbb{P}^1 \times \mathbb{P}^1 \) are given below. We use coordinates \([s_0 : s_1] \times [t_0 : t_1]\) for the base.

- **\( \mathbb{Z}/7 \):**

\[
y^2 + \left(s_0^2t_0^2 + s_0s_1t_0t_1 - s_1^2t_1^2\right)xy + s_0s_1^2t_0t_1^2(s_0t_0 - s_1t_1)y = x^3 + s_0t_0s_1^2t_1^2(s_0t_0 - s_1t_1)x^2, \]

(A.22)

The coefficients \( f, g \) and the discriminant \( \Delta = 4f^3 + 27g^2 \) are given below:

\[
f = \frac{1}{48} - (s_0^2t_0^2 - s_0s_1t_0t_1 + s_1^2t_1^2) \times
\]

\[
\left(s_0^6t_0^6 + 5s_0s_1^2t_0^2t_1^2 - 10s_0^4s_1^4t_0^4 + 15s_0^3s_1^2t_0^2t_1^2 + 30s_0^2s_1t_0t_1^2 - 11s_0s_1^2t_0t_1^2 + s_1^6t_1^6\right),
\]

\[
g = \frac{1}{864} \left(256s_0^2t_0^2(t_0^{10}t_1^{10} + 6s_0^9s_1t_0^9t_1^9 - 15s_0^8s_1^2t_0^8t_1^8 + 24s_0^7s_1^2t_0^7t_1^7 + 174s_0^6s_1^2t_0^6t_1^6 - 22s_0s_1^2t_0t_1^6 + 27s_0^6s_1^2t_0^6t_1^6 - 48s_0^5s_1^2t_0^5t_1^5 - 23s_0^4s_1^2t_0^4t_1^4 - 11s_0^3s_1^2t_0^3t_1^3 - 48s_0^2s_1t_0^2t_1^2 - 57s_0s_1^2t_0t_1^2 + 174s_0s_1^2t_0^2t_1^2 + 117s_0^2t_0^2t_1^2 \right),
\]

(A.23)

\[
\Delta = -\frac{1}{16} s_0^7s_1^7t_0^7t_1^7(s_0t_0 - s_1t_1)^7(s_0^3t_0^3 + 5s_0^2s_1t_0^2t_1 - 8s_0s_1^2t_0t_1^2 + s_1^3t_1^3). \]

(A.24)
In this section, we explain how to compute the indices of congruence subgroups defined earlier. Further details can be found in [71].

The coefficients $f, g$ and $\Delta$ are:

$$f = \frac{1}{48}(-s_0^8 + 16s_0^7s_1t_0^3 + 96s_0^6s_1^2t_0^2 + 288s_0^5s_1^3t_0^3 - 480s_0^4s_1^4t_0^4 + 448s_0^3s_1^5t_0^5 - 224s_0^2s_1^6t_0^6 + 64s_0^1s_1^7t_0^7 - 16s_1^8),$$

$$g = \frac{1}{864}(-s_0^4 - 8s_0^3s_1^3t_0^3 + 16s_0^2s_1^2t_0^6 - 16s_0s_1^3t_0^8 + 8s_1^4t_0^8 - 16s_0^7s_1^7t_0^7 + 96s_0^6s_1^6t_0^6 - 288s_0^5s_1^5t_0^5 + 456s_0^4s_1^4t_0^4 - 352s_0^3s_1^3t_0^3 + 80s_0^2s_1^2t_0^2 + 32s_0^1s_1t_0^1 - 8s_1^8),$$

$$\Delta = -\frac{1}{16}s_0^2s_1^2(s_0t_0 - 2s_1t_1)(s_0t_0 - s_1t_1)^4(s_0^2t_0^2 - 8s_0s_1t_0t_1 + 8s_1^2t_1^2).$$

The coefficients $f, g$ and $\Delta$ are:

$$f = \frac{1}{48}(-119761s_0^8 + 36920s_0^7s_1^7t_0^7 + 15700s_0^6s_1^6t_0^6 + 11432s_0^5s_1^5t_0^5 - 11958s_0^4s_1^4t_0^4 - 1992s_0^3s_1^3t_0^3 + 180s_0^2s_1^2t_0^2 + 40s_0s_1t_0 - s_1^2),$$

$$g = \frac{1}{864}(-41545241s_0^8 + 19809780s_0^7s_1^7t_0^7 + 4915350s_0^6s_1^6t_0^6 + 7207028s_0^5s_1^5t_0^5 - 9699039s_0^4s_1^4t_0^4 + 2418984s_0^3s_1^3t_0^3 + 172820s_0^2s_1^2t_0^2 - 12216s_0s_1t_0 - 1),$$

$$\Delta = 2s_0^4t_0^4(s_0t_0 - s_1t_1)^4(5s_0t_0 - s_1t_1)^4(9s_0^2t_0^2 - s_1^2t_1^2).$$

B  Index of Congruence Subgroups

In this section, we explain how to compute the indices $[\text{SL}(2, \mathbb{Z}) : \Gamma]$ where $\Gamma$ is one of the congruence subgroups defined earlier. Further details can be found in [71].
We start by computing the index of $[\text{SL}(2, \mathbb{Z}) : \Gamma(n)]$ for $n > 1$. The short exact sequence:

$$0 \longrightarrow \Gamma(n) \longrightarrow \text{SL}(2, \mathbb{Z}) \longrightarrow \text{SL}_2(\mathbb{Z}_n) \longrightarrow 0,$$

(B.1)

shows that $[\text{SL}(2, \mathbb{Z}) : \Gamma(n)] = |\text{SL}_2(\mathbb{Z}_n)|$. Furthermore, by the Chinese Remainder Theorem, we have isomorphisms:

$$\text{SL}_2(\mathbb{Z}_{nm}) \rightarrow \text{SL}_2(\mathbb{Z}_n) \times \text{SL}_2(\mathbb{Z}_m),$$

(B.2)

whenever $n, m$ are relatively prime. Thus, to compute $[\text{SL}(2, \mathbb{Z}) : \Gamma(n)]$ for general $n$, it suffices to find a formula for $|\text{SL}_2(\mathbb{Z}_n)|$ when $n$ is a prime power.

We derive that formula in the following. Assume $n = p^k$ for some prime $p$ and some positive integer $k$. We will count the 4-tuples $(a, b, c, d) \in \mathbb{Z}_n^4$ with $ad - bc = 1$.

- We count 4-tuples where $a$ is a unit first. There are $p^k - p^{k-1}$ units in $\mathbb{Z}_n$, so we have that many choices for $a$. Having chosen $a$, we can choose $b, c$ freely, and take $d = \frac{1+bc}{a}$ to guarantee the corresponding matrix has determinant 1. Altogether, there are:

$$\left(p^k - p^{k-1}\right) \cdot p^k \cdot p^k \cdot 1 = p^{3k} \left(1 - \frac{1}{p}\right),$$

(B.3)

such 4-tuples.

- Next, we count 4-tuples where $a$ is not a unit. We can choose $a$ in one of $p^{k-1}$ ways. Since $a$ is not a unit, $b, c$ necessarily have to be units. We can choose $b$ freely in the set of units, and we can choose $d$ freely in $\mathbb{Z}_n$. We then solve for $c$ as $c = \frac{ad - 1}{b}$. This means the number of 4-tuples is:

$$p^{k-1} \cdot (p^k - p^{k-1}) \cdot p^k \cdot 1 = p^{3k} \left(\frac{1}{p} - \frac{1}{p^2}\right),$$

(B.4)

- Altogether, this means we have:

$$p^{3k} \left(\left(1 - \frac{1}{p}\right) + \left(\frac{1}{p} - \frac{1}{p^2}\right)\right) = p^{3k} \left(1 - \frac{1}{p^2}\right),$$

(B.5)

elements in $\text{SL}_2(\mathbb{Z}_n)$.

Finally, if $n = n_1 \cdots n_r$, with $n_i = p_i^{e_i}$, $p_i \neq p_j$ for $i \neq j$, then:

$$[\text{SL}(2, \mathbb{Z}) : \Gamma(n)] = \prod_i [\text{SL}(2, \mathbb{Z}) : \Gamma(n_i)] = n^3 \prod_i 1 - \frac{1}{p_i^2}. $$

(B.6)
Next we use the computation of $[\text{SL}(2, \mathbb{Z}) : \Gamma(n)]$ to obtain formulas for the index of $\Gamma_1(n)$. Recall that elements of $\Gamma_1(n)$, after passing to the quotient, have the form $(1 a 0 1)$ for some $a \in \mathbb{Z}_n$, and that:

$$
\begin{pmatrix}
1 & a \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & a' \\
0 & 1
\end{pmatrix} = 
\begin{pmatrix}
1 & a + a' \\
0 & 1
\end{pmatrix}.
$$

(B.7)

Thus, we have a short exact sequence:

$$
0 \rightarrow \Gamma(n) \rightarrow \Gamma_1(n) \rightarrow \mathbb{Z}_n \rightarrow 0,
$$

(B.8)

where the map $\Gamma_1(n) \rightarrow \mathbb{Z}_n$ is the projection onto the top-right factor. This shows that $[\Gamma_1(n) : \Gamma(n)] = n$, so we obtain:

$$
[\text{SL}(2, \mathbb{Z}) : \Gamma_1(n)] = \frac{1}{n}[\text{SL}(2, \mathbb{Z}) : \Gamma(n)] = n^2 \prod_{p|n} 1 - \frac{1}{p^2}.
$$

(B.9)

We can use this formula to check one of the claims made in the main body - namely, that $24|[\text{SL}(2, \mathbb{Z}) : \Gamma_1(n)]$ for $n \geq 5$. By multiplicativity of the index, it suffices to prove the divisibility claim when $n$ is a prime power.

- First, assume $n = p^k$ for $p \geq 5$ and $k \geq 1$. The formula above shows that:

$$
[\text{SL}(2, \mathbb{Z}) : \Gamma_1(n)] = p^{3k} - p^{3k-2} = p^{3k-2}(p + 1)(p - 1).
$$

(B.10)

Since $3 \nmid p$, $p^2 - 1$ is necessarily divisible by 3. Furthermore, since $p$ is odd, both $p+1$, $p-1$ are even and exactly one of them is divisible by 4. Thus, $p^2 - 1$ is divisible by 3 and 8, so it is divisible by 24.

- Next, assume $n = 3^k$ for $k \geq 2$. Since 3 is odd, $3^{2k} - 1$ is divisible by 8 for the same reason as above. Furthermore, since $k > 1$, $3k - 2 \geq 1$, so the index is divisible by 3, so again we see that the index is divisible by 24.

- Finally, assume $n = 2^k$. Then:

$$
[\text{SL}(2, \mathbb{Z}) : \Gamma_1(n)] = 2^{3k} - 2^{3k-2} = 2^{3k-2} \cdot 3
$$

(B.11)

In particular, the index is divisible by 24 precisely when $3k - 2 > 3$, i.e. $k \geq 3$.

The observations above prove the claim for almost all $n$. We can check the claim by hand for $n = 6, 12$, since they are not divisible by 8, 9 or any larger prime, to complete the argument.
B.3 $\Gamma_0(n)$

We compute $[\text{SL}(2, \mathbb{Z}) : \Gamma_0(n)]$ using a similar argument. A general element of $\Gamma_0(n)$ reduces to a matrix of the form

\[
\begin{pmatrix}
a & b \\
0 & d
\end{pmatrix}
\]

in $\text{SL}_2(\mathbb{Z}_n)$. The diagonal entries are necessarily units after we reduce mod $n$. Now, observe that:

\[
\begin{pmatrix}
a & b \\
0 & d
\end{pmatrix}
\begin{pmatrix}
a' & b' \\
0 & d'
\end{pmatrix} =
\begin{pmatrix}
aa' & ab' + bd' \\
0 & dd'
\end{pmatrix}.
\] (B.12)

Thus, we can write down a group homomorphism $\Gamma_0(n) \rightarrow \mathbb{Z}_n^\times$ whose kernel is exactly $\Gamma_1(n)$, so $[\Gamma_0(n) : \Gamma_1(n)] = \phi(n)$. Using the identity $\phi(n) = n \prod_{p|n} 1 - \frac{1}{p}$, we obtain:

\[
[\text{SL}(2, \mathbb{Z}) : \Gamma_0(n)] = \frac{1}{\phi(n)} [\text{SL}(2, \mathbb{Z}) : \Gamma_1(n)] = n \prod_{p|n} 1 + \frac{1}{p}.
\] (B.13)

B.4 Non-classical congruence groups

For completeness, we include a formula for the index of the group that parametrizes $(E, (P, Q))$ for $E$ an elliptic curve, $P$ a point of order $nd$ and $Q$ a point of order $d$ for integers $n, d$.

The idea is to take the fiber product of $X_1(nd)$ and $X(d)$. At the level of groups, this corresponds to taking the intersection of the groups. To compute the index of the intersection, we use the identity:

\[
[\text{SL}(2, \mathbb{Z}) : \Gamma_1(nd) \cap \Gamma(d)] = [\text{SL}(2, \mathbb{Z}) : \Gamma_1(nd)][\Gamma_1(nd) : \Gamma_1(nd) \cap \Gamma(d)].
\] (B.14)

An element of $\Gamma_1(nd)$ has the form $\begin{pmatrix}1 & a \\ 0 & 1\end{pmatrix} \mod nd$. In order for it to be an element of $\Gamma(d)$, $a$ has to be a multiple of $d$. Thus, the index of the intersection in $\Gamma_1(nd)$ is $n$ so:

\[
[\text{SL}(2, \mathbb{Z}) : \Gamma_1(nd) \cap \Gamma(d)] = n^3d^2 \prod_{p|nd} 1 - \frac{1}{p^2}.
\] (B.15)

Thus, the index of the group associated to the Mordell-Weil torsion $\mathbb{Z}_2 \times \mathbb{Z}_4$ has index 24, $\mathbb{Z}_2 \times \mathbb{Z}_6$ has index 48, $\mathbb{Z}_3 \times \mathbb{Z}_6$ has index 72 and $\mathbb{Z}_2 \times \mathbb{Z}_8$ has index 96. Applying the formula for the degree of the fundamental line bundle for $\Gamma_1(n)$, this predicts the corresponding universal surface has a fundamental line bundle of degree 1,2,3,4, respectively. These numbers match up exactly with the table in [32].

We have not discussed torsion groups of type $\mathbb{Z}_m \times \mathbb{Z}_n$ where $n \neq m$ and neither divides the other, but the formulas above cover all of the 19 subgroups with a quotient of genus 0.

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