POINCARÉ-TYPE INEQUALITIES AND SAMPLING AND INTERPOLATION BY AVERAGE VALUES ON COMBINATORIAL GRAPHS

ISAAC Z. PESENSON

Abstract. In the setting of a weighted combinatorial finite or infinite countable graph $G$ we introduce functional Paley-Wiener spaces $PW_\omega(L)$, $\omega > 0$, defined in terms of the spectral resolution of the combinatorial Laplace operator $L$ in the space $L^2(G)$. It is shown that functions in certain $PW_\omega(L)$, $\omega > 0$, are uniquely defined by their averages over some families of "small" subgraphs. Reconstruction methods for reconstruction of an $f \in PW_\omega(L)$ from appropriate set of its averages are introduced. One method is using language of Hilbert frames. Another one is using average variational interpolating splines which are constructed in the setting of combinatorial graphs.

1. Introduction and main results

During the last decade signal processing on graphs was developed in a number of papers, for example, in [3, 6, 13-21]. Many of the papers on this list considered what can be called as a "point-wise sampling". The goal of the present article is to develop sampling on graphs which is based on averages over relatively small subgraphs. The idea to use local information (other than point values) for reconstruction of bandlimited functions on graphs was already explored in [20]. However, the results and methods of [20] and of our paper are very different. We also want to mention that methods of the present paper are similar to methods of our paper [12] in which sampling by average values was developed on Riemannian manifolds.

Let $G$ denote an undirected weighted graph, with a finite or countable number of vertices $V(G)$ and weight function $w : V(G) \times V(G) \rightarrow \mathbb{R}_0^+$. $w$ is symmetric, i.e., $w(u, v) = w(v, u)$, and $w(u, u) = 0$ for all $u, v \in V(G)$. The edges of the graph are the pairs $(u, v)$ with $w(u, v) \neq 0$. Our assumption is that for every $v \in V(G)$ the following finiteness condition holds

\[ w(v) = \sum_{u \in V(G)} w(u, v) < \infty. \]  
\[(1.1)\]

Let $\ell^2(G)$ denote the space of all complex-valued functions with the inner product

\[ \langle f, g \rangle = \sum_{v \in V(G)} f(v)\overline{g(v)} \]

and the norm

\[ \|f\|_G = \|f\| = \left( \sum_{v \in V(G)} |f(v)|^2 \right)^{1/2}. \]
Definition 1. The weighted gradient norm of a function $f$ on $V(G)$ is defined by

$$\|\nabla f\| = \left( \sum_{u,v \in V(G)} \frac{1}{2} |f(u) - f(v)|^2 w(u,v) \right)^{1/2}.$$  \hspace{1cm} (1.2)

The set of all $f : G \to \mathbb{C}$ for which the weighted gradient norm is finite will be denoted as $D(\nabla)$.

Remark 1.1. The factor $\frac{1}{2}$ makes up for the fact that every edge (i.e., every unordered pair $(u,v)$) enters twice in the summation. Note also that loops, i.e. edges of the type $(u,u)$, in fact do not contribute.

We intend to prove Poincaré-type estimates involving weighted gradient norm.

In the case of a finite graph and $\ell^2(G)$-space the weighted Laplace operator $L : \ell^2(G) \to \ell^2(G)$ is introduced via

$$Lf(v) = \sum_{u \in V(G)} (f(v) - f(u)) w(v,u).$$  \hspace{1cm} (1.3)

This graph Laplacian is a well-studied object; it is known to be a positive-semidefinite self-adjoint bounded operator.

According to Theorem 8.1 and Corollary 8.2 in [5] if for an infinite graph there exists a $C > 0$ such that the degrees are uniformly bounded

$$w(v) = \sum_{u \in V(G)} w(u,v) \leq C,$$  \hspace{1cm} (1.4)

then operator which is defined by (1.3) on functions with compact supports has a unique positive-semidefinite self-adjoint bounded extension $L$ which is acting according to (1.3).

Let every $S_j \subset V(G), j \in J$, be a finite and connected subset of vertices with more than one vertex. Our assumption is that a set of subsets of vertices $S = \{S_j\}_{j \in J}$ form a cover (not necessarily disjoint) of $V(G)$. With every $S_j, j \in J$, we associate a function $\psi_j \in \ell^2(G)$ whose support is in $S_j$ and introduce the functionals $\Psi_j$ on $\ell^2(G)$ defined by these functions

$$\Psi_j(f) = \langle f, \psi_j \rangle = \sum_{v \in V(S_j)} f(v) \psi_j(v), \quad f \in \ell^2(G).$$

Notation $\chi_j$ will be used for characteristic function of $S_j$. For a function $f \in \ell^2(G)$ the following notation will be adapted $f_j = f \chi_j$. Let $L_j$ and $\|\nabla_j f_j\|$ be the Laplacian and weighted gradient for the induced subgraph $S_j$. The first nonzero eigenvalue of the operator operator $L_j$ will be denoted as $\lambda_{1,j}$.

The two inequalities below (1.6) and (1.7) are essentially the main inequalities we prove in section 3. We call them generalized Poincaré-type inequalities since they contain an estimate of a size of a function through its smoothness. Namely, we show that if

$$\Psi_j(\chi_j) = \sum_{v \in S_j} \psi_j(v) \neq 0,$$

and

$$\theta_j = \frac{|S_j| \|\psi_j\|^2}{\|\Psi_j(\chi_j)\|^2}.\hspace{1cm} (1.5)$$

then the following inequalities hold for every $f \in \ell^2(G)$ and every $\epsilon > 0$
\begin{equation}
\|f\|^2 \leq (1 + \epsilon) \sum_{j \in J} \theta_j \lambda_{1,j} \|\nabla_j f_j\|^2 + \frac{1 + \epsilon}{\epsilon} \sum_{j \in J} \frac{|S_j|^2}{|\Psi_j(\chi_j)|^2} |\Psi_j(f_j)|^2,
\end{equation}

\begin{equation}
\|f\|^2 \leq (1 + \epsilon) \frac{\Theta_{\Xi}}{\Lambda_{\Xi}} \|L^{1/2}f\|^2 + \frac{1 + \epsilon}{\epsilon} \sum_{j \in J} \frac{|S_j|^2}{|\Psi_j(\chi_j)|^2} |\Psi_j(f_j)|^2.
\end{equation}

where \( \Xi = (S, \{\Psi_j\}_{j \in J}) \), \( S = \{S_j\}_{j \in J} \),

\begin{equation}
\Theta_{\Xi} = \sup_{j \in J} \theta_j < \infty,
\end{equation}

where \( \theta_j \) is computed according to \( 2.3 \) and

\begin{equation}
\Lambda_{\Xi} = \inf_{j \in J} \lambda_{1,j} > 0,
\end{equation}

Note, that the inequalities \( 1.6 \) and \( 1.7 \) are quite informative and capture highly irregular local structures of graphs. Indeed, in the case, say, of a Riemannian manifold a difference between two small neighborhoods \( S_i \) and \( S_j \) is essentially their diameter. However, in the case of a graph two small sets can have very different structures. These differences are better reflected by quantities like \( \lambda_{1,j} \) and \( \theta_j \). From the practical point of view, the averaging procedure (see, for example, Corollary 3.1) can be instrumental in reducing noise inherited into point wise measurements.

Using inequalities \( 1.6 \) and \( 1.7 \) and their variations we develop a sampling theory of Paley-Wiener functions on finite and infinite graphs. We also outline a construction of variational interpolating splines which interpolate functions using their average values over subsets.

2. A Poincare-type inequality for finite graphs

The following lemma is important for us (see \( 7 \) for finite graphs, and \( 3 \) for infinite).

Lemma 2.1. If a graph \( G \) is finite or the condition \( 1.4 \) is satisfied then one has the equality

\begin{equation}
\|L^{1/2}f\| = \|\nabla f\|
\end{equation}

for all \( f \in \ell^2(G) \).

Proof. It is easy to verify that under assumption \( 1.4 \) the domain \( D(\nabla) \) coincides with \( \ell^2(G) \). Let \( d(u) = w_{V(G)}(u) \). Then we obtain

\[
\langle f, Lf \rangle = \sum_{u \in V(G)} f(u) \left( \sum_{v \in V(G)} (f(u) - f(v)) w(u, v) \right) = \sum_{u \in V(G)} \left( |f(u)|^2 d(u) - \sum_{v \in V(G)} f(u) f(v) w(u, v) \right).
\]
In the same way
\[ \langle f, Lf \rangle = \langle Lf, f \rangle = \sum_{u \in V(G)} \left( |f(u)|^2 d(u) - \sum_{v \in V(G)} f(u)f(v)w(u, v) \right). \]

Averaging these equations yields
\[ \langle f, Lf \rangle = \sum_{u \in V(G)} \left( |f(u)|^2 d(u) - \text{Re} \sum_{v \in V(G)} f(u)f(v)w(u, v) \right) \]
\[ = \frac{1}{2} \sum_{u, v \in V(G)} |f(u)|^2 w(u, v) + |f(v)|^2 w(u, v) - 2 \text{Re} f(u)f(v)w(u, v) \]
\[ = \sum_{u, v \in V(G)} \frac{1}{2} |f(v) - f(u)|^2 w(u, v) = \| \nabla f \|^2. \]

Lemma is proved.

For a finite connected graph \( G \) which contains more than one vertex let \( \Psi \) be a functional on \( \ell^2(G) \) which is defined by a function \( \psi \in \ell^2(G) \), i.e.
\[ \Psi(f) = \langle f, \psi \rangle = \sum_{v \in V(G)} f(v)\psi(v). \]

We will use notation \( \chi_G \) for the characteristic function: \( \chi_G(v) = 1 \) for all \( v \in V(G) \).

Using these notions we prove the following.

**Theorem 2.2.** Let \( G \) be a finite connected graph which contains more than one vertex and \( \Psi(\chi_G) \) is not zero. If \( f \in \text{Ker}(\Psi) \) then
\[ \| f \|^2 \leq \frac{\theta}{\lambda_1} \| \nabla f \|^2, \quad f \in \text{Ker}(\Psi), \]
where \( \lambda_1 \) is the first non zero eigenvalue of the Laplacian (1.3) and
\[ \theta = \frac{|G|\|\psi\|^2}{|\Psi(\chi_G)|^2}, \]
where \( |G| \) is cardinality of \( V(G) \).

**Proof.** If \( \lambda_0 < \lambda_1 \leq \ldots \lambda_{N-1} \), \( N = |G| \) is the set of eigenvalue and \( \varphi_0, \varphi_1, \ldots, \varphi_{N-1} \) is a set of orthonormal eigenfunctions then \( \{ c_k(f) = \langle f, \varphi_k \rangle \} \) is a set of Fourier coefficients. One has
\[ f = \sum_{k} c_k(f)\varphi_k \]
and if \( f \in \text{Ker}(\Psi) \) then
\[ 0 = \Psi(f) = \frac{1}{\sqrt{|G|}} c_0(f)\Psi(\chi_G) + \sum_{k=1}^{N-1} c_k(f)\Psi(\varphi_k). \]

From here
\[ c_0(f) = -\frac{\sqrt{|G|}}{\Psi(\chi_G)} \sum_{k=1}^{N-1} c_k(f)\Psi(\varphi_k), \]
and then using Parseval equality and Schwartz inequality we obtain
\[ \|f\|^2 = |c_0(f)|^2 + \sum_{k=1}^{N-1} |c_k(f)|^2 = \frac{|G|}{|\Psi(\chi_G)|^2} \left| \sum_{k=1}^{N-1} c_k(f) \Psi(\varphi_k) \right|^2 + \sum_{k=1}^{N-1} |c_k(f)|^2 \leq \begin{align*} (2.4) \quad \frac{|G|}{|\Psi(\chi_G)|^2} \sum_{k=1}^{N-1} |c_k(f)|^2 \sum_{k=1}^{N-1} |\Psi(\varphi_k)|^2 + \sum_{k=1}^{N-1} |c_k(f)|^2. \end{align*} \]

At the same time, since \( \varphi_0 = \frac{\chi_G}{\sqrt{|G|}} \) and \( \langle \psi, \varphi_k \rangle = \Psi(\varphi_k) \) we have

\[ \psi = \frac{1}{\sqrt{|G|}} \Psi(\chi_G) \varphi_0 + \sum_{k=1}^{N-1} \Psi(\varphi_k) \varphi_k, \]
and from Parseval formula

\[ \sum_{k=1}^{N-1} |\Psi(\varphi_k)|^2 = \|\psi\|^2 - \frac{|\Psi(\chi_G)|^2}{|G|}. \]

We plug the right-hand side of this formula into (2) and obtain the next inequality in which \( \theta \) is given by (2.3)

\[ \|f\|^2 \leq \theta \sum_{k=1}^{N-1} |c_k(f)|^2 \leq \frac{\theta}{\lambda_1} \sum_{k=1}^{N-1} |\lambda_k^{1/2} c_k(f)|^2 = \frac{\theta}{\lambda_1} \|L^{1/2}\|^2. \]

To finish the proof one has to apply Lemma 2.1. Theorem is proven. \( \square \)

When \( \psi \) equals to the eigenfunction \( \varphi_0 \) then for the corresponding functional \( \Psi_0 \) the condition \( f \in \text{Ker}(\Psi_0) \) is equivalent to \( \langle f, \varphi_0 \rangle = 0 \). It is easy to see that in this case \( \theta = 1 \) and then (2.2) gives the following Corollary.

**Corollary 2.1.** If \( \langle f, \varphi_0 \rangle = 0 \) then

\[ \|f\|^2 \leq \frac{1}{\lambda_1} \|\nabla f\|^2. \]

Note also, that this inequality immediately follows from Lemma 2.1 and from the fact that norm of \( L^{-1/2} \) on the subspace of all functions which are orthogonal to \( \varphi_0 \) is \( 1/\sqrt{\lambda_1} \).

In another particular case when \( \psi = \chi_G \) (characteristic function of \( V(G) \)) then

\[ f_G = \frac{1}{|G|} \sum_{v \in V(G)} f(v), \]
and one has that \( f - f_G \chi_G \) belongs to the kernel of the corresponding functional \( \Psi \) and it gives the next Corollary.

**Corollary 2.2.** For every finite graph \( G \) and for every \( f \in \ell^2(G) \) the following holds

\[ \|f - f_G \chi_G\|^2 \leq \frac{1}{\lambda_1} \|\nabla f\|^2. \]
**Theorem 2.3.** Let $G$ be a finite graph and $\Psi$ be a functional on $\ell^2(G)$ such that $\Psi(\chi_G)$ is not zero. Then the following Poincaré inequality holds for every $f \in \ell^2(G)$ and every $\epsilon > 0$

\[
\|f\|^2 \leq \frac{\theta}{\lambda_1} (1 + \epsilon) \|\nabla f\|^2 + \frac{1 + \epsilon}{\epsilon} \frac{|G|^2}{|\Psi(\chi_G)|^2} |\Psi(f)|^2, \quad f \in \ell^2(G), \quad \epsilon > 0,
\]

where $\theta$ is defined in (2.3).

**Proof.** One has

\[
\|f\|^2 \leq \left\| f - \frac{\Psi(f)}{\Psi(\chi_G)} \chi_G \right\|^2 + \left\| \frac{\Psi(f)}{\Psi(\chi_G)} \chi_G \right\|^2.
\]

Next, we apply the inequality

\[
|A|^2 \leq (1 + \epsilon) |A - B|^2 + \frac{1 + \epsilon}{\epsilon} |B|^2,
\]

which holds for every positive $\epsilon > 0$. This inequality follows from two obvious inequalities

\[
|A|^2 \leq |A - B|^2 + 2|B||A - B| + |B|^2
\]

and

\[
2|B||A - B| \leq \epsilon |A - B|^2 + \epsilon^{-1} |B|^2, \quad \epsilon > 0.
\]

Choosing an $\epsilon > 0$ and using inequality (2.7) one obtains

\[
\|f\|^2 \leq (1 + \epsilon) \left\| f - \frac{\Psi(f)}{\Psi(\chi_G)} \chi_G \right\|^2 + \frac{1 + \epsilon}{\epsilon} \frac{|G|^2}{|\Psi(\chi_G)|^2} |\Psi(f)|^2.
\]

Note, that $f - \frac{\Psi(f)}{\Psi(\chi_G)} \chi_G$ belongs to $Ker(\Psi)$ because

\[
\Psi \left( f - \frac{\Psi(f)}{\Psi(\chi_G)} \chi_G \right) = \Psi(f) - \Psi(f) = 0.
\]

Since $L^{1/2} \chi_G = 0$ an application of Theorem 2.2 gives (2.6). Theorem is proved. \qed

In the case when $\Psi$ is defined by $\psi = \chi_G$ one has that

\[
\frac{\Psi(f)}{\Psi(\chi_G)} \chi_G = f_G \chi_G, \quad f_G = \frac{1}{|G|} \sum_{v \in V(G)} f(v).
\]

Since in this case $\theta$ in (2.3) is $1$, $|G|^2/|\Psi(\chi_G)|^2 = 1$, and $\Psi(f) = \sum_{v \in V(G)} f(v)$ we obtain

**Corollary 2.3.** For every connected and finite graph $G$ which contains more than one vertex the following Poincaré inequality holds

\[
\|f\|^2 \leq (1 + \epsilon) \frac{1}{\lambda_1} \|\nabla f\|^2 + \frac{1 + \epsilon}{\epsilon} \left\| \sum_{v \in V(G)} f(v) \right\|^2, \quad f \in \ell^2(G), \quad \epsilon > 0.
\]
3. A generalized Poincaré-type inequality for finite and infinite graphs

Let $G$ be a finite or infinite and countable connected graph and $S \subset V(G)$ is a finite and connected subset of vertices which we will treat as an induced graph and will denote by the same letter $S$. We remind that this means that the set of vertices of such graph, which will be denoted as $V(S)$, is exactly the set of vertices in $S$ and the set of edges are all edges in $G$ whose both ends belong to $S$. Let $L_S$ and $\|\nabla_S (f|_S)\|$ be the Laplace operator and the weighted gradient constructed according to \(^{(3.2)}\) and \(^{(3.1)}\) for the induced graph $S$. Let $w_S(u,v), u,v \in V(S)$, and

$$w_S(v) = \sum_{u \in V(S)} w_S(u,v), \ v \in V(S),$$

be the corresponding weight functions. We notice that for every induced subgraph $S$ one has the inequalities and every $u,v \in V(S)$ one has $w(u,v) = w_S(u,v)$. However, in general $w(u) \geq w_S(u)$.

We assume that every $S_j \subset V(G), \ j \in J$, is a finite and connected subset of vertices with more than one vertex. Our assumption is that $S = \{S_j\}_{j \in J}$ and a set (possibly empty) of vertices $\{s_i\}_{i \in I}$ form a cover of $V(G)$ (not necessarily disjoint):

$$\left(\bigcup_{j \in J} S_j\right) \cup \left(\bigcup_{i \in I} \{s_i\}\right) = V(G).$$

With every $S_j, \ j \in J$, we associate a function $\psi_j \in \ell^2(G)$ whose support is in $S_j$ and introduce the functionals $\Psi_j$ on $\ell^2(G)$ defined by these functions

$$\Psi_j(f) = \langle f, \psi_j \rangle = \sum_{v \in V(S_j)} f(v)\psi_j(v), \ f \in \ell^2(G).$$

Notation $\chi_j$ will be used for characteristic function of $S_j$ and use $f_j$ for $f\chi_j, \ f \in \ell^2(G)$. Let $L_j$ and $\|\nabla_j f_j\|$ be the Laplacian and weighted gradient for the induced subgraph $S_j$. The first nonzero eigenvalue of the operator operator $L_j$ will be denoted as $\lambda_{1,j}$.

**Theorem 3.1.** Let $G$ be a connected finite or infinite and countable graph. Suppose that \(^{(3.1)}\) holds true. Let $L_j$ be the Laplace operator of the induced subgraph $S_j$ whose first nonzero eigenvalue is $\lambda_{1,j}$. The following inequality holds for every $f \in \ell^2(G)$ and every $\epsilon > 0$

$$\|f\|^2 \leq (1 + \epsilon) \sum_{j \in J} \frac{\theta_j}{\lambda_{1,j}} \|\nabla_j f_j\|^2 + \frac{1 + \epsilon}{\epsilon} \sum_{j \in I} \frac{|S_j|^2}{|\Psi_j(\chi_j)|^2} |\Psi_j(f_j)|^2 + \sum_{i \in I} |f(s_i)|^2,$$

where $\Psi_j(f) = \langle f, \psi_j \rangle$, function $\psi_j \in \ell^2(G)$ has support in $S_j$,

$$\Psi_j(\chi_j) = \sum_{v \in S_j} \psi_j(v) \neq 0,$$

and

$$\theta_j = \frac{|S_j||\psi_j|^2}{|\Psi_j(\chi_j)|^2}.$$
Proof. One has

\[
\|f\|^2 = \sum_{v \in V(G)} |f(v)|^2 = \sum_{j \in J} \left( \sum_{v \in V(S_j)} |f_j(v)|^2 \right) + \sum_{i \in I} |f(s_i)|^2.
\]

We apply Theorem 2.3 to have for every \( j \in J \) and every \( \epsilon > 0 \),

\[
\sum_{v \in V(S_j)} |f_j(v)|^2 \leq (1 + \epsilon) \frac{\theta_j}{\Lambda_{1,j}} \|\nabla_j f_j\|^2 + \frac{1 + \epsilon}{\epsilon} \frac{|S_j|^2}{\Psi_j(\chi_j)^2} |\Psi_j(f_j)|^2,
\]

and then we have for \( f \in \ell^2(G) \), \( \epsilon > 0 \),

\[
\|f\|^2 \leq (1 + \epsilon) \sum_{j \in J} \frac{\theta_j}{\Lambda_{1,j}} \|\nabla_j f_j\|^2 + \frac{1 + \epsilon}{\epsilon} \sum_{j \in J} \frac{|S_j|^2}{\Psi_j(\chi_j)^2} |\Psi_j(f_j)|^2 + \sum_{i \in I} |f(s_i)|^2.
\]

Theorem is proved. \( \square \)

As a consequence we obtain the following.

**Theorem 3.2.** If in addition to assumptions of Theorem 3.1 we have that

\[
\Theta_\Xi = \sup_{j \in J} \theta_j < \infty, \quad \Xi = (\{S_j\}_{j \in J}, \{\Psi_j\}_{j \in J}),
\]

where \( \theta_j \) is computed according to (2.3) and

\[
\Lambda_S = \inf_{j \in J} \lambda_{1,j} > 0, \quad S = \{S_j\}_{j \in J},
\]

then the following inequality holds for every \( f \in \ell^2(G) \) and every \( \epsilon > 0 \)

\[
\|f\|^2 \leq (1 + \epsilon) \frac{\Theta_\Xi}{\Lambda_S} \|L^{1/2}f\|^2 + \frac{1 + \epsilon}{\epsilon} \sum_{j \in J} \frac{|S_j|^2}{\Psi_j(\chi_j)^2} |\Psi_j(f_j)|^2 + \sum_{i \in I} |f(s_i)|^2.
\]

Proof of this statement follows from Theorem 3.1 and Lemma 2.1 according to which

\[
\sum_{j \in J} \|\nabla_j f_j\|^2 \leq \sum_{j \in J} \|\nabla_j f_j\|^2 \leq \|\nabla f\|^2 = \|L^{1/2}f\|.
\]

Let’s consider a few interesting cases.

**Corollary 3.1.** Suppose that all the notations and conditions of Theorems 3.1 and 2.2 are satisfied. If for every \( j \) the corresponding function \( \psi_j = \chi_j \) is the characteristic function of a subset of vertices \( \emptyset \neq U_j \subseteq S_j \) then

\[
\|f\|^2 \leq (1 + \epsilon) \sum_{j \in J} \frac{|S_j|}{|U_j|} \|\nabla_j f_j\|^2 + \frac{1 + \epsilon}{\epsilon} \left( \sum_{j \in J} \frac{|S_j|^2}{|U_j|^2} \right) \left( \sum_{v \in U_j} |f(v)|^2 \right)^2 + \sum_{i \in I} |f(s_i)|^2,
\]

and

\[
\|f\|^2 \leq (1 + \epsilon) \frac{1}{\Lambda_S} \sum_{j \in J} \frac{|S_j|}{|U_j|} \|L^{1/2}f\|^2 + \sum_{i \in I} |f(s_i)|^2.
\]
(3.11) \[
\frac{1 + \epsilon}{\epsilon} \left( \sum_{j \in J} \frac{|S_j|^2}{|U_j|^2} \right) \left| \sum_{v \in U_j} f(v) \right|^2 + \sum_{i \in I} |f(s_i)|^2.
\]

In particular, if \( U_j = S_j \) for every \( j \in J \) then

(3.12) \[
\|f\|^2 \leq (1 + \epsilon) \sum_{j \in J} \frac{1}{\lambda_{1,j}} \|\nabla_j f_j\|^2 + \frac{1 + \epsilon}{\epsilon} \sum_{j \in J} \left| \sum_{v \in S_j} f(v) \right|^2 + \sum_{i \in I} |f(s_i)|^2,
\]

and

(3.13) \[
\|f\|^2 \leq (1 + \epsilon) \frac{1}{\Lambda_S} \|L^{1/2} f\|^2 + \frac{1 + \epsilon}{\epsilon} \sum_{j \in J} \left| \sum_{v \in S_j} f(v) \right|^2 + \sum_{i \in I} |f(s_i)|^2.
\]

Indeed, it follows from the fact that in this situation \( \|\psi_j\|^2 = |U_j|, \ |\Psi_j(\chi_j)|^2 = |U_j|^2 \) and

\[\theta_j = \frac{|S_j|}{|\Psi_j(\chi_j)|^2} = \frac{|S_j|}{|U_j|}.\]

The condition (3.7) boils down to \( \sup_{j \in J} |S_j| < \infty \).

**Corollary 3.2.** Suppose that all the notations and conditions of Theorems 3.1 and 3.2 are satisfied. If for every \( j \) the corresponding function \( \psi_j \) is a Dirac measure \( \delta_{v_j} \) at a vertex \( v_j \in S_j \) then

(3.14) \[
\|f\|^2 \leq (1 + \epsilon) \sum_{j \in J} \frac{|S_j|}{\lambda_{1,j}} \|\nabla_j f_j\|^2 + \frac{1 + \epsilon}{\epsilon} \sum_{j \in J} |S_j|^2 \|f(v_j)\|^2,
\]

and

(3.15) \[
\|f\|^2 \leq (1 + \epsilon) \frac{\sup_{j \in J} |S_j|}{\Lambda_S} \|L^{1/2} f\|^2 + \frac{1 + \epsilon}{\epsilon} \sum_{j \in J} |S_j|^2 \|f(v_j)\|^2.
\]

**Proof.** In this case one has \( \|\psi_j\| = 1, \ \Psi_j(f) = f(v_j), \ \Psi_j(\chi_j) = 1, \ \theta_j = |S_j| \) for every \( j \in J \).

\qed

The next corollary is about functions which annihilate all the functionals \( \Psi_j, j \in J \).

**Corollary 3.3.** If all the notations and conditions of Theorems 3.1 and 3.2 are satisfied and for a function \( f \in \ell^2(G) \) one has that

\[ f \in \bigcap_{j \in J} \text{Ker} \Psi_j \]

then

(3.16) \[
\|f\|^2 \leq (1 + \epsilon) \sum_{j \in J} \frac{\Theta_j}{\lambda_{1,j}} \|\nabla_j f_j\|^2, \quad f \in \bigcap_{j \in J} \text{Ker} \Psi_j,
\]

and

(3.17) \[
\|f\|^2 \leq (1 + \epsilon) \frac{\Theta_j^2}{\Lambda_S} \|L^{1/2} f\|^2, \quad f \in \bigcap_{j \in J} \text{Ker} \Psi_j.
\]
Remark 3.3. If \( J_0 \subset J \) and \( G_0 = \bigcup_{j \in J_0} G_j \), then every inequality in this section can be replaced by a similar one in which the term \( \| f \|^2 \) on the left is replaced by
\[
\| f \|^2_{G_0} = \sum_{v \in G_0} \| f \|^2,
\]
and summation over \( J \) on the right is replaced by summation over \( J_0 \). For example, the last two inequalities (3.16) and (3.17) would take the form
\[
\| f \|^2_{G_0} \leq (1 + \epsilon) \sum_{j \in J_0} \frac{\Theta_{j}}{\Lambda_{j}} \| \nabla_j f_j \|^2, \quad f \in \bigcap_{j \in J_0} \text{Ker} \Psi_j,
\]
and
\[
\| f \|^2_{G_0} \leq (1 + \epsilon) \frac{\Theta_{j}}{\Lambda_{j}} \| L_{G_0}^{1/2} f_0 \|^2, \quad f_0 = f|_{G_0}, \quad f_0 \in \bigcap_{j \in J_0} \text{Ker} \Psi_j,
\]
where \( L_{G_0} \) is the Laplacian of the induced graph \( G_0 \).

Note, that in the case when \( \{ \Psi_j \} \) is a set of "uniformly" distributed Dirac functions the last inequality (3.19) is called sometimes "the inequality for functions with many zeros".

4. Paley-Wiener vectors in \( \ell^2(G) \)

Our next goal is to introduce the so-called Paley-Wiener functions (bandlimited functions) for which a sampling theory will be developed in the setting of combinatorial graphs. We use for this the self-adjoint positive definite operator \( L \) in a Hilbert space \( \ell^2(G) \). In the case when \( L \) has discrete spectrum (which is always the case with finite graphs) then the Paley-Wiener space \( PW_{\omega}(L) \) is simply the span of eigenfunctions of \( L \) whose corresponding eigenvalues are not greater \( \omega \). However, when graph is infinite and spectrum of \( L \) is continuous it takes a bigger effort to define spaces \( PW_{\omega}(L) \).

Consider a self-adjoint positive definite operator \( L \) in a Hilbert space \( \ell^2(G) \). According to the spectral theory \( \mathbf{1} \) for self-adjoint non-negative operators there exists a direct integral of Hilbert spaces \( \mathcal{H} = \int \mathcal{H}(\lambda) dm(\lambda) \) and a unitary operator \( \mathcal{F} \) from \( \ell^2(G) \) onto \( \mathcal{H} \), which transforms the domains of \( L^k, k \in \mathbb{N} \), onto the sets \( \mathcal{H}_k = \{ x \in \mathcal{H} | \lambda^k x \in \mathcal{H} \} \) with the norm
\[
\| x(\lambda) \|_{\mathcal{H}_k} = \langle x(\lambda), x(\lambda) \rangle_{\mathcal{H}(\lambda)}^{1/2} = \left( \int_0^{\infty} \lambda^{2k} \| x(\lambda) \|^2_{\mathcal{H}(\lambda)} dm(\lambda) \right)^{1/2}.
\]
and satisfies the identity \( \mathcal{F}(L^k f)(\lambda) = \lambda^k (\mathcal{F} f)(\lambda) \), if \( f \) belongs to the domain of \( L^k \). We call the operator \( \mathcal{F} \) the Spectral Fourier Transform. As known, \( \mathcal{H} \) is the set of all \( m \)-measurable functions \( \lambda \mapsto x(\lambda) \in \mathcal{H}(\lambda) \), for which the following norm is finite:
\[
\| x \|_{\mathcal{H}} = \left( \int_0^{\infty} \| x(\lambda) \|^2_{\mathcal{H}(\lambda)} dm(\lambda) \right)^{1/2}.
\]
For the characteristic function \( 1_{[0, \omega]} \) one can introduce the projector \( 1_{[0, \omega]}(L) \) by using the formula
\[
1_{[0, \omega]}(L) f = \mathcal{F}^{-1} 1_{[0, \omega]}(\lambda) \mathcal{F} f, \quad f \in \mathcal{H}.
\]
Definition 2. The Paley-Wiener space $\text{PW}_\omega(L) \subset \ell^2(G)$ is defined as the image space of the projection operator $1_{[0, \omega]}(L)$.

Many properties of Paley-Wiener spaces for general self-adjoint operators in Hilbert spaces can be found in our papers [10]. The most important for us is the following.

Theorem 4.1. A function $f \in \ell^2(G)$ belongs to the spaces $\text{PW}_\omega(L)$ if and only if the following Bernstein inequalities holds true
\begin{equation}
\|L^s f\| \leq \omega^s \|f\| \quad \text{for all } s \in \mathbb{R}_+;
\end{equation}

5. A sampling theorem and a reconstruction methods using frames

5.1. A sampling theorem. Let’s remind that a set of vectors $\{\xi_\nu\}$ in a Hilbert space $H$ is called a Hilbert frame if there exist constants $A, B > 0$ (frame bounds) such that for all $f \in H$
\begin{equation}
A \|f\|^2 \leq \sum_\nu |\langle f, \xi_\nu \rangle|^2 \leq B \|f\|^2.
\end{equation}

What is remarkable about frames is the fact that one can perfectly reconstruct a vector $f$ from its projections $\langle f, \xi_\nu \rangle$. Namely, according to the general theory of Hilbert frames [2], [4] the frame inequality (5.1) implies that there exists a dual frame $\{\Omega_\nu\}$ (which is not unique in general) for which the following reconstruction formula holds
\begin{equation}
f = \sum_\nu \langle f, \xi_\nu \rangle \Omega_\nu.
\end{equation}

In general it is not easy to find a dual frame. For this reason one can resort to the following frame algorithm (see [4], Ch. 5) which performs reconstruction by iterations. Given a relaxation parameter $0 < \rho < \frac{2}{A}$, set $\eta = \max\{|1 - \rho A|, |1 - \rho B|\} < 1$. Let $f_0 = 0$ and define recursively
\begin{equation}
f_n = f_{n-1} + \rho \Phi(f - f_{n-1}),
\end{equation}
where $\Phi$ is the frame operator which is defined on $H$ by the formula $\Phi f_1 = \sum_\nu \langle f, \xi_\nu \rangle \xi_\nu$. In particular, $f_1 = \rho \Phi f = \rho \sum_\nu \langle f, \xi_\nu \rangle \xi_\nu$. Then $\lim_{n \to \infty} f_n = f$ with a geometric rate of convergence, that is,
\begin{equation}
\|f - f_n\| \leq \eta^n \|f\|.
\end{equation}

In particular, for the choice $\rho = \frac{2}{\lambda + B}$ the convergence factor is
\begin{equation}
\eta = \frac{B - A}{A + B}.
\end{equation}

Let $\delta_{s_i}, i \in I$ be the Dirac delta concentrated at the vertex $s_i$.

Theorem 5.1. If all the notations and conditions of Theorems 3.1 and 3.2 hold then the set of functionals $\{\Psi_j\}_{j \in J} \cup \{\delta_{s_i}\}_{i \in I}$ is a frame in any space $\text{PW}_\omega(L)$ as long as
\begin{equation}
0 < \omega < \frac{\Lambda S}{(1 + \epsilon) \Theta}, \quad \epsilon > 0,
\end{equation}

\begin{equation}
\Lambda S (1 + \epsilon) \Theta
\end{equation}
(2) there exists a constant \( c = c(\{S_j\}, \{\Psi_j\}) \) such that for every \( j \in J \) the following inequality holds

\[
\frac{|S_j|^2}{|\Psi_j(\chi_j)|^2} \leq c,
\]

(3) there exists a constant \( C = C(\{S_j\}, \{\Psi_j\}) \) such that for every \( j \in J \) one has

\[
\|\psi_j\|^2 \leq C.
\]

In other words, if for an \( \epsilon > 0 \) the following inequality holds

\[
\gamma = (1 + \epsilon) \frac{\Theta_{\omega}}{\lambda_S} < 1, \quad \epsilon > 0,
\]

then

\[
\frac{(1 - \gamma)\epsilon}{(1 + \epsilon)c} \|f\|^2 \leq \sum_{j \in J} |\Psi_j(f)|^2 + \sum_{i \in I} |\delta_i(f)|^2 \leq C\|f\|^2.
\]

Proof. We notice that since support of \( \psi_j \) is in \( S_j \) we have

\[
\Psi_j(f) = \langle f, \psi_j \rangle = \Psi_j(f).
\]

Now, if \( f \in PW_{\omega}(L) \) then by the Bernstein inequality (4.3) the (3.9) can be rewritten as

\[
\|f\|^2 \leq (1 + \epsilon) \frac{\Theta_{\omega}}{\lambda_S} \|f\|^2 + \frac{1 + \epsilon}{\epsilon} \sum_{j \in J} \frac{|S_j|^2}{|\Psi_j(\chi_j)|^2} |\Psi_j(f_j)|^2 + \sum_{i} |f(s_i)|^2.
\]

If (5.6) and (5.8) hold then one obtains the left-hand side of (5.9). On the other hand, we have

\[
\sum_{j \in J} |\Psi_j(f)|^2 + \sum_{i} |f(s_i)|^2 = \sum_{j \in J} \left| \sum_{v \in S_j} f_j(v) \psi_j(v) \right|^2 + \sum_{i} |f(s_i)|^2 \leq
\]

\[
\sum_{j \in J} \|\psi_j\|^2 \|f_j\|^2 + \sum_{i} |f(s_i)|^2 \leq C\|f\|^2.
\]

Theorem is proven. \( \Box \)

Note, that for the classical Paley-Wiener spaces on the real line the inequalities similar to (5.9) in the case when \( \{\psi_j\} \) are delta functions were proved by Plancherel and Polya. Today they are better known as the frame inequalities. Now we can formulate sampling theorem based on average values.

**Theorem 5.2.** Under the same conditions and notations as in Theorem 5.1 every function \( f \in PW_{\omega}(L) \) is uniquely determined by the set of numbers \( \{\langle f, \psi_j \rangle\}_{j \in J} \) and can be reconstructed from this set of values in a stable way using dual frames (5.2) or the iterative frame algorithm (5.3).
5.2. Important particular cases.

1. (Sampling by averages-I). If for every $j$ the corresponding function $\psi_j = \chi_j$ is the characteristic function of a subset of vertices $U_j \subset S_j$ then inequalities (5.5)-(5.7) take the form respectively

\[0 < \omega < \frac{\Lambda S}{(1 + \epsilon) \sup_{j \in J} |S_j|}, \quad \frac{|S_j|^2}{|U_j|^2} \leq c, \quad |U_j| \leq C,\]

and the Plancherel-Polya inequalities (5.9) hold with the same constants $c$ and $C$. In particular, if $U_j = S_j$ for every $j \in J$ then (5.5) takes the form

\[0 < \omega < \Lambda S, \quad 1 + \epsilon > 0, \quad |S_j|^2 \leq c, |U_j| \leq C,\]

the condition (5.6) is trivially satisfied with $c = 1$, and (5.7) becomes $|S_j| \leq C$. The (5.9) holds true with the corresponding constants $C$ and $c = 1$.

2. (Sampling by averages-II). In the case $U_j = S_j$ and

\[\psi_j = \frac{1}{\sqrt{|S_j|}} \chi_j,\]

every $\theta_j$ in (3.3) is one and it gives that $\Theta \Xi$ in (3.7) is also one. Thus (5.5) takes the form

\[0 < \omega < \frac{\Lambda S}{1 + \epsilon}, \quad \epsilon > 0.\]

Moreover, in this case $C = c = 1$. After all the Plancherel-Polya inequality (5.9) becomes

\[\frac{(1 - \gamma)\epsilon}{(1 + \epsilon)} \|f\|^2 \leq \sum_{j \in J} |\Psi_j(f)|^2 + \sum_{i \in I} |f(s_i)|^2 \leq \|f\|^2, \quad f \in PW_\omega(G),\]

where

\[\gamma = \frac{1 + \epsilon}{\Lambda S} \omega < 1, \quad \epsilon > 0.\]

3. (Point wise sampling). If for every $j$ the corresponding function $\psi_j$ is a Dirac measure $\delta_{v_j}$ at a vertex $v_j \in S_j$ then the condition (5.5) takes the form (5.10), the condition (5.6) will have form $|S_j|^2 \leq c$, the condition (5.7) is trivially satisfied with $C = 1$. The (5.9) holds true with these constants.

5.3. Reconstruction algorithms in terms of frames. What we just proved in the previous section is that under the same assumptions as above the set of functionals $f \to \langle f, \psi_j \rangle$ is a frame in the subspace $PW_\omega(L)$. This fact allows to apply the well known result of Duffin and Schaeffer [2] which describes a stable method of reconstruction of a function $f \in PW_\omega(L)$ from a set of samples $\{\langle f, \psi_j \rangle\}$.

**Theorem 5.3.** If all the conditions of Theorem 3.1 are satisfied then there exists a dual frame $\{\Omega_j\}$ in $PW_\omega(L)$ such that

\[f = \sum_j \langle f, \psi_j \rangle \Omega_j = \sum_j \langle f, \Omega_j \rangle \mathcal{P}_\omega \psi_j\]

where $\mathcal{P}_\omega$ is the orthogonal projection of $\ell^2(G)$ onto $PW_\omega(L)$. 


Another possibility for reconstruction is to use frame algorithm (see section 5).

6. AVERAGE VARIATIONAL SPLINES AND A RECONSTRUCTION ALGORITHM

6.1. Variational interpolating splines. As in the previous sections we assume that $G$ is a connected finite or infinite graph, $S = \{S_j\}_{j \in J}$, is a disjoint cover of $V(G)$ by connected and finite subgraphs $S_j$ and every $\psi_j \in \ell^2(S_j)$, $j \in J$, has support in $S_j$.

For a given sequence $\mathbf{a} = \{a_j\} \in l_2$ the set of all functions in $\ell^2(G)$ such that $\Psi_j(f) = \langle f, \psi_j \rangle = a_j$ will be denoted by $Z_\mathbf{a}$. In particular, $Z_0 = \bigcap_{j \in J} \text{Ker}(\Psi_j)$ corresponds to the sequence of zeros. We consider the following optimization problem:

For a given sequence $\mathbf{a} = \{a_j\} \in l_2$ find a function $f$ in the set $Z_\mathbf{a} \subset \ell^2(G)$ which minimizes the functional $u \rightarrow \| L^{k/2} u \|$, $u \in Z_\mathbf{a}$.

**Theorem 6.1.** Under the above assumptions the optimization problem has a unique solution for every $k$.

**Proof.** Using Theorem 3.1 one can justify the following algorithm (see [9], [11]):

1. Pick any function $f \in Z_\mathbf{a}$.
2. Construct $P_0 f$ where $P_0$ is the orthogonal projection of $f$ onto $Z_0$ with respect to the inner product $\langle f, g \rangle_k = \sum_j \langle f, \psi_j \rangle \langle g, \psi_j \rangle + \langle L^{k/2} f, L^{k/2} g \rangle$.
3. The function $f - P_0 f$ is the unique solution to the given optimization problem.

$\square$

**Definition 3.** For $f \in \ell^2(G)$ the interpolating variational spline is denoted by $s_k(f)$ and it is the solution of the minimization problem such that $s_k(f) - f \in Z_0$.

Clearly, "interpolation" is understood in the sense that

$$\Psi_j(s_k(f)) = \Psi_j(f).$$

One can easily prove the following characterization of variational splines.

**Theorem 6.2.** A function $u \in \ell^2(G)$ is a variational spline if and only if $L^k u$ is orthogonal to $L^k Z_0$.

6.2. Reconstruction using splines. The following Lemma was proved in [9], [11].

**Lemma 6.3.** If $A$ is a self-adjoint non-negative operator in a Hilbert space $X$ and for an $\varphi \in X$ and a positive $a > 0$ the following inequality holds true

$$\| \varphi \| \leq a \| A \varphi \|,$$

then for the same $\varphi \in H$, and all $k = 2^l, l = 0, 1, 2, \ldots$ the following inequality holds

$$\| \varphi \| \leq a^k \| A^k \varphi \|.$$
By using the same reasoning as in [9], [11] one can prove the following reconstruction theorem. Below we are keeping notations of Theorem 5.1.

**Theorem 6.4.** Let’s assume that $G$ is a connected finite or infinite graph, $\{S_j\}_{j \in J}$ is a disjoint cover of $V(G)$ by connected and finite subgraphs $S_j$ and every $\psi_j \in \ell^2(S_j)$, $j \in J$, has support in $S_j$. If the assumptions of Theorem 5.1 are satisfied and in particular, $0 < \omega < \frac{\Lambda_S}{(1 + \epsilon)\Theta S}$,

$$\Theta S = \sup_{j \in J} \theta_j = \theta_j = \frac{|S_j|}{\|\psi_j\|^2},$$

(6.2)

$$\Lambda_S = \inf_{j \in J} \lambda_{1,j}, \quad \epsilon > 0,$$

then any function $f$ in $PW_\omega(L)$, $\omega > 0$, can be reconstructed from a set of values $\{(f, \psi_j)\}$ using the formula

$$f = \lim_{k \to \infty} s_k(f), \quad k = 2^l, \ l = 0, 1, \ldots,$$

and the error estimate is

$$\|f - s_k(f)\| \leq 2\gamma^k\|f\|, \quad k = 2^l, \ l = 0, 1, \ldots,$$

(6.3)

where

$$\gamma = (1 + \epsilon)\frac{\Theta S}{\Lambda_S} \omega < 1.$$

**Proof.** For a $k = 2^l$, $l = 0, 1, 2, \ldots$ apply to the function $f - s_k(f)$ inequality (3.17) to obtain

$$\|f - s_k(f)\|^2 \leq (1 + \epsilon)\frac{\Theta S}{\Lambda_S}\|L^{1/2}(f - s_k(f))\|^2, \quad \epsilon > 0,$$

and use Lemma 6.3 to get

$$\|f - s_k(f)\|^2 \leq \left(1 + \epsilon\frac{\Theta S}{\Lambda_S}\right)^k \|L^{k/2}(f - s_k(f))\|^2.$$

Using minimization property of $s_k(f)$ and the Bernstein inequality (4.3) for $f \in PW_\omega(L)$ one obtains (6.3). Theorem is proved. \hfill $\square$

One can formulate similar statements adapted to particular cases listed in subsection 5.2.

**7. Example. Average sampling on Z**

Let us consider a one-dimensional infinite lattice $Z = \{\ldots, -1, 0, 1, \ldots\}$ as an unweighted graph. The dual group of the commutative additive group $Z$ is the one-dimensional torus. The corresponding Fourier transform $\mathcal{F}$ on the space $\ell^2(Z)$ is defined by the formula

$$\mathcal{F}(f)(\xi) = \sum_{k \in Z} f(k)e^{ik\xi}, \ f \in \ell^2(Z), \xi \in [-\pi, \pi).$$

It gives a unitary operator from $\ell^2(Z)$ on the space $L_2(T) = L_2(T, d\xi/2\pi)$, where $T$ is the one-dimensional torus and $d\xi/2\pi$ is the normalized measure. One can verify the following formula

$$\mathcal{F}(Lf)(\xi) = 4\sin^2\frac{\xi}{2}\mathcal{F}(f)(\xi).$$
The next result is obvious.

**Theorem 7.1.** The spectrum of the Laplace operator $L$ on the one-dimensional lattice $\mathbb{Z}$ is the interval $[0, 4]$. A function $f$ belongs to the space $PW_\omega(\mathbb{Z})$, $0 \leq \omega \leq 4$, if and only if the support of $Ff$ is a subset of $[-\pi, \pi)$ on which $4\sin^2 \frac{\xi}{2} \leq \omega$.

We consider the cover $\Xi = \{S_j\}$ of $\mathbb{Z}$ by disjoint sets $S_j = \{j - 1, j, j + 1\}$ where $j$ runs over all integers divisible by 3: $\{..., -3, 0, 3, ...\} = 3\mathbb{Z}$. We treat every $S_j$ as an induced graph whose set of vertices is $V(S_j) = \{j - 1, j, j + 1\}$, $j \in 3\mathbb{Z}$, and which has two edges $(j - 1, j)$ and $(j, j + 1)$. Let’s introduce functionals $\Psi_j$ as

\begin{equation}
\Psi_j(f) = \langle f, \psi_j \rangle = \frac{1}{\sqrt{3}} (f(j - 1) + f(j) + f(j + 1)), \quad j \in 3\mathbb{Z}, \quad f \in l^2(\mathbb{Z}).
\end{equation}

One can check that spectrum of the Laplace operator $L_j$ on $S_j$ defined by (1.3) contains just three values $\{0, 2, 4\}$. Thus $\Lambda_S = 2$. For an $0 < \omega < 4$ and $\epsilon > 0$ condition (6.4) takes form

\begin{equation}
\gamma = (1 + \epsilon)\frac{\omega}{2} < 1.
\end{equation}

Note, that since $1 + \epsilon$ can be arbitrary close to 1 the condition (7.2) implies that $0 < \omega < 2$. As an application of Theorem 5.2 we obtain the following result.

**Theorem 7.2.** If $0 < \omega < 2$ then every $f \in PW_\omega(\mathbb{Z})$ is uniquely determined by its average values $\{\langle f, \psi_j \rangle\}$ defined in (7.1) and can be reconstructed from them in a stable way.

In particular, if instead of infinite graph $\mathbb{Z}$ one would consider a path graph $\mathbb{Z}_N$ whose eigenvalues are given by formulas $2 - 2 \cos \frac{k\pi}{N}$, $k = 0, 1, ..., N - 1$, the last Theorem would mean that any eigenfunction with eigenvalue from a lower half of the spectrum is uniquely determined and can be reconstructed from averages (7.1).

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DEPARTMENT OF MATHEMATICS, TEMPLE UNIVERSITY, PHILADELPHIA, PA 19122

E-mail address: pesenson@temple.edu