Pascal algebra of matrices and Pascal map on jet bundles

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Abstract: We identify and study a matrix algebra consisting of Pascal-type matrices. The generator of the matrix algebra is shown to well define a canonical bundle map, called the Pascal map on jet bundles, and we use it to give an intrinsic definition of point-wise contact between Hermitian vector bundles in terms of unitary equivalence of the Pascal maps.

Key words: Pascal matrix; jet bundle; contact.

1 Introduction

An interesting topic in differential geometry is to formulate intrinsic concepts on vector bundles out of extrinsic ones for smooth maps(sub-manifolds) into ambient spaces. This note concerns geometric notion on high order behaviours carried by jet bundles. We formulate a canonical bundle map, called the Pascal map, on jet bundles and exhibit its use in the extrinsic-intrinsic transition by refining the extrinsic notion of point-wise contact between smooth maps into an intrinsic one that makes sense on vector bundles.

In Section 2 we identify and study a matrix algebra called the Pascal algebra as a preparation, which is of independent interest in extending the classical Pascal matrix as well as later works on generalized Pascal matrices. In Section 3 we show that the generator of the Pascal algebra represents a well-defined bundle map, called the Pascal map, on the jet bundle of a given vector bundle. In Section 4 we use the Pascal map to give an intrinsic definition of point-wise contact between Hermitian vector bundles. Both Section 3 and Section 4 are confined in the setting of vector bundles over a domain in $\mathbb{C}$, and in Section 5 we outline a several variable extension which works with domains in $\mathbb{C}^m$, $m > 1$.

2 Pascal algebra

Given a fixed positive integer $n$, we denote by $\Lambda^n$ the set of $(n+1) \times (n+1)$ matrices of the following form
That is, a matrix in \( \Lambda^n \) is determined by \( n+1 \) complex numbers \( a_0, a_1, \ldots, a_n \) lying in its first column, whose \((i, j)\) entry is \( \binom{i-1}{j-1} a_{i-j} \) if \( 1 \leq j \leq i \leq n+1 \) and 0 if \( j > i \).

Setting \( a_0 = a_1 = \cdots = a_n = 1 \) yields the classical Pascal matrix, the study of which starts from Call and Velleman [4]. Letting \( a_n \) vary with \( n \) according to different rules (for instance, set \( a_n = x^n \) for an indeterminate \( x \)), one gets various versions of “generalized Pascal matrices” ([1, 2, 9, 10, 11]). The class \( \Lambda^n \) we identified above allows arbitrary first column entries hence is the “biggest” class.

**Theorem 2.1.** Let \( P \) be the matrix in \( \Lambda^n \) whose first column is given by \( a_1 = 1 \) and \( a_k = 0 \), \( k \neq 1 \), that is,

\[
P = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
1 & 0 & \cdots & 0 \\
0 & 2 & 0 & \cdots \\
0 & 0 & 3 & 0 \\
\vdots & \vdots & & \ddots \\
0 & 0 & \cdots & n & 0
\end{pmatrix},
\]

then \( \Lambda^n \) is the commutant of \( P \).

**Proof.** To see \( \Lambda^n \subseteq P' \), let \( Q \) be in \( \Lambda^n \) and \( (a_0, a_1, \ldots, a_n)^T \) be its first column, then both \( PQ \) and \( QP \) are lower triangular with zero diagonals, and it remains to compare their \((i, j)\) entries for \( j \leq i-1 \). Observing that the \((i, j)\) entry of \( PQ \) is \( j \binom{i-1}{j} a_{i-j-1} \) and the corresponding entry of \( PQ \) is \( (i-1) \binom{i-2}{j-1} a_{i-j-1} \), the conclusion follows from the elementary identity \( j \binom{i-1}{j} = (i-1) \binom{i-2}{j-1} \).

For the other direction, let \( Q \) be any \((n+1) \times (n+1)\) matrix such that \( PQ = QP \), and we have to show \( Q \in \Lambda^n \). To this end, we view \( P \) and \( Q \) as linear maps acting on an \( n+1 \) dimensional space with a fixed base \( \{s_0, s_1, \ldots, s_n\} \). Then \( P \) corresponds to the action \( Ps_0 = 0 \) and \( Ps_k = ks_{k-1} \), \( k = 1, 2, \ldots, n \). Moreover, if \( (a_0, a_1, \ldots, a_n)^T \) is the first column of \( Q \), then for every \( 0 \leq k \leq n \), \( Qs_k = a_k s_0 + \) other terms involving \( s_1, \ldots, s_n \).

Now it suffices to show if \( PQ = QP \), then

\[
Qs_k = \sum_{i=0}^{k} \binom{k}{i} a_{k-i}s_i, \quad k = 0, 1 \cdots n
\]
Comparing \( PQ_{s_0} \) and \( QP_{s_0} \) one immediately sees that \( Q_{s_0} \) only has \( s_0 \) component, hence (2.3) holds for \( k = 0 \) and we verify (2.3) by induction on \( k \). Precisely, it suffices to show

\[
Q_{s_{k+1}} = a_{k+1}s_0 + \sum_{i=1}^{k+1} \binom{k+1}{i} a_{k+1-i}s_i
\]

assuming (2.3).

To this end, write

\[
Q_{s_{k+1}} = a_{k+1}s_0 + \sum_{i=1}^n b_is_i
\]

for some coefficients \( b_1, \ldots, b_n \), thus \( PQ_{s_{k+1}} = \sum_{i=1}^n ib_is_{i-1} \). While by (2.3) we have

\[
QP_{s_{k+1}} = (k+1)Q_{s_k} = (k+1) \sum_{i=0}^k \binom{k}{i} a_{k-i}s_i = (k+1) \sum_{i=1}^{k+1} \binom{k}{i-1} a_{k+1-i}s_{i-1}.
\]

Now \( PQ = QP \) implies \( b_i = 0 \) for \( k + 2 \leq i \leq n \) and \( b_i = \frac{(k+1)(k)}{i} a_{k+1-i} = \frac{(k+1)}{i} a_{k+1-i} \) for \( 1 \leq i \leq k + 1 \), which gives (2.4) as desired.

An immediate consequence of Theorem 2.1 is that \( \Lambda^n \) is an algebra, which justifies the following

**Definition 2.2.** The collection \( \Lambda^n \) of lower triangular matrices of the form (2.1) is called the Pascal algebra.

**Remark 2.3.** An alternative way to prove Theorem 2.1 is to show both \( \Lambda^n \) and \( P' \) equals the algebra of polynomials in \( P \) (so \( P \) is the generator of \( \Lambda^n \)) which involves arguments with minimal polynomials. The proof we present above is a straightforward “entry comparing” and is easily seen to work in the setting of block matrices. Precisely, replacing the scalers \( a_0, \ldots, a_n \) in (2.1) by \( l \times l \) matrices \( A_0, \ldots, A_n \), one gets a collection of \( (n+1) \times (n+1) \) block matrices. The set of these block matrices is still denoted by \( \Lambda^n \), which equals the commutant of

\[
P = \begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
I & 0 & 0 & \cdots & 0 \\
0 & 2I & 0 & \cdots & 0 \\
0 & 0 & 3I & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & nI \\
0 & 0 & \cdots & \cdots & 0
\end{pmatrix}.
\]

Later, \( \Lambda^n \subseteq P' \) will be needed in Theorem 3.2 and \( P' \subseteq \Lambda^n \) will be needed in Proposition 4.2.
3 Pascal map on jet bundles

In Section 3.1 we give a brief introduction on jet bundles and in Section 3.2 we introduce a canonical bundle map, called the Pascal map on the jet bundles.

3.1 Jet bundle

To quickly reveal the conceptual idea and make this note self-contained, we confine ourselves in an specific setting by focusing on vector bundles associated to holomorphic maps from a domain in $\mathbb{C}$ into a complex Grassmannian, which is elementary and facilitates our presentations in Section 4 as well. Readers familiar with abstract theory of jet bundles will find it routine to extend discussions in this paper to more general settings (see Remark 3.3).

Let $\text{Gr}(l, \mathcal{H})$ be the Grassmannian of $l$ dimensional subspaces in a complex vector space $\mathcal{H}$ and $f$ be a map from a domain $\Omega \subseteq \mathbb{C}$ to $\text{Gr}(l, \mathcal{H})$. The map $f$ is holomorphic in the sense that for any point $z_0$ in $\Omega$, there exists a neighborhood $\Delta$ of $z_0$ and holomorphic $\mathcal{H}$-valued functions $s_1, \ldots, s_l$ on $\Delta$ such that

$$f(z) = \bigvee \{s_1(z), \ldots, s_l(z)\}$$

for every $z \in \Delta$. The vector bundle

$$E := \{(h, z) \in \mathcal{H} \times \Omega | h \in f(z)\}$$

associated to $f$ is then a holomorphic vector bundle over $\Omega$ of rank $l$ and $s = \{s_1, \ldots, s_l\}$ implements a local holomorphic frame of $E$.

Fix a positive integer $n$ and a point $z \in \Omega$, we set

$$E^n(z) := \bigvee_{1 \leq i \leq l, 0 \leq k \leq n} \{s_i^{(k)}(z)\}, \quad (3.1)$$

where $s = \{s_1, \ldots, s_l\}$ is any holomorphic frame of $E$ around $z$.

**Definition 3.1.** The vector space $E^n(z)$ does not depend on choice of $s$ hence

$$E^n := \{(h, z) \in \mathcal{H} \times \Omega | h \in E^n(z)\},$$

is a well-defined vector bundle, called the $n$-jet bundle of $E$.

In fact, suppose $t = \{t_1, \ldots, t_l\}$ is another holomorphic frame around $z$, then there is an invertible holomorphic matrix function $A$ such that $s = At$ (here $s$ and $t$ are interpreted as
column vectors so $A$ acts from the left). A differentiation gives
\[
\begin{pmatrix}
  s \\
  s' \\
  s'' \\
  \vdots \\
  s^{(n)}
\end{pmatrix} = 
\begin{pmatrix}
  A & A' & A \\
  A' & 2A' & A \\
  \vdots & \vdots & \ddots \\
  A^{(n)} & (n)A^{(n-1)} & \cdots & (n)!A' & A
\end{pmatrix}
\begin{pmatrix}
  t \\
  t' \\
  t'' \\
  \vdots \\
  t^{(n)}
\end{pmatrix},
\]
(3.2)

hence $\bigvee_{1 \leq i \leq l,0 \leq k \leq n} \{ s^{(k)}_i (z) \}$ and $\bigvee_{1 \leq i \leq l,0 \leq k \leq n} \{ t^{(k)}_i (z) \}$ is the same vector space as they just differs by an invertible block matrix.

The high order derivatives $\{ s^{(k)}_i (z), 1 \leq i \leq l, 0 \leq k \leq n \}$ (or $\{ s, s', \ldots, s^{(n)} \}$ for short) implements a canonical local holomorphic frame for $E^n$. An important fact as one immediately finds is that the transition matrix (3.2) between two such frames lies in the Pascal algebra $\Lambda^n$, which is determined by its upper block $A$ and we denote it by $\Lambda^n_A$ in the sequel.

### 3.2 Pascal map

Now we are ready to introduce the Pascal bundle map on $E^n$. Recall that a bundle map on a vector bundle maps each fiber linearly to itself, and given a collection of frames $\{ s_\alpha \}$ as local trivializations, the standard way to construct an unambiguously defined bundle map $\Phi$ is to give a collection of matrix functions $\{ \Phi (s_\alpha) \}$ such that the compatibility condition
\[
\Phi (s_\alpha) = A_{\alpha\beta} \Phi (s_\beta) A^{-1}_{\alpha\beta}
\]
(3.3)

holds, where $A_{\alpha\beta}$ is the transition function between $s_\alpha$ and $s_\beta$ (so different matrices represents the same linear map). The following theorem asserts that on the $n$-jet bundle of a holomorphic vector bundle, a single constant matrix will do, which gives the promised Pascal map on $E^n$.

**Theorem 3.2.** Let $E$ be a holomorphic vector bundle over $\Omega \subseteq \mathbb{C}$ and $n$ be a positive integer, the constant block matrix (2.3) represents a well defined bundle map, called the Pascal map, on $E^n$.

**Proof.** From the construction of $E^n$ one sees that if a collection of frames $\{ s_\alpha \}$ gives local trivializations for $E$, then $\{ s_\alpha, s'_\alpha, \ldots, s^{(n)}_\alpha \}$ gives local trivializations for $E^n$. If $s$ and $t$ are any two overlapping holomorphic frames of $E$ with transition matrix $A$, then the transition matrix for $\{ s, s', \ldots, s^{(n)} \}$ and $\{ t, t', \ldots, t^{(n)} \}$ is $\Lambda^n_A$ (see (3.2) above). Now it suffices to verify the compatibility condition $P = \Lambda^n_A P (\Lambda^n_A)^{-1}$, or equivalently, $P \Lambda^n_A = \Lambda^n_A P$. As $\Lambda^n_A$ lies in the Pascal algebra, this follows from the block matrix version of Theorem 2.1.

Explicitly, for any holomorphic frame $s = \{ s_1, \ldots, s_l \}$ of $E$, the Pascal map (still denoted by $P$) acts on the local frame $\{ s(z), s'(z), \ldots, s^{(n)}(z) \}$ of $E^n$ by
\[
P s^{(k)}_i = k s^{(k-1)}_i, 1 \leq k \leq n, \text{ and } P s_i = 0
\]
(3.4)
Remark 3.3. On a general holomorphic vector bundle where it is not so straightforward to make sense of high order derivative of its sections, $E^n$ is standardly defined via transition functions with compatibility conditions. Precisely, let $\{U_\alpha\}$ be local trivializations of $E$ and $\{A_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \emptyset}$ be the corresponding set of transition matrix functions, then it is easy to check that the matrix functions $\{\Lambda^n_{\alpha\beta}\}_{U_\alpha \cap U_\beta \neq \emptyset}$ satisfy the compatibility condition $\Lambda^n_{\alpha\beta} \Lambda^n_{\beta\gamma} \Lambda^n_{\gamma\alpha} = I$ when $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ hence well defines a vector bundle which is the n-jet bundle $E^n$. In particular, our construction of the Pascal map above remains valid since it only depends on the fact that $\Lambda^n_{\alpha\beta}$ commutes with (2.5).

Like many familiar geometric notions (such as curvature or torsion), the intrinsically defined Pascal map also has an “extrinsic predecessor”. Precisely, if there exists a bounded linear operator $T$ on the ambient space $\mathcal{H}$ extending “coordinate multiplication” on $E$, that is, $Ts_i(z) = zs_i(z), 1 \leq i \leq l$, then differentiating it $k$ times yields $(T - z)s_i^{(k)}(z) = ks_i^{(k-1)}(z)$, which is exactly the action (3.4) of the Pascal map. Boundedness of $T$ validating this differentiation argument is a nontrivial extrinsic condition and this operator is historically called the Cowen-Douglas operator\[3]. Our study in the next section will involve unitary equivalence of Pascal maps, which also has an extrinsic counterpart as unitary equivalence of operators restricted to generalized eigen-spaces (see Sec 2.[3]), hence our study can be somehow regarded as an “intrinsic Cowen-Douglas theory”.

4 Contact between vector bundles

Contact order between smooths maps is a useful extrinsic invariant. Historically it implements a universal criteria for the congruence problem of determining if two maps can be identified up to a rigid motion in the ambient spaces (Sec.5 [6]). In particular, the congruence problem for holomorphic maps into complex Grassmannians had been extensively studied over years (5 [6], 7, 8). In this section we refine this classical notion into a frame-independent version so that it makes sense on the associated vector bundles. To make sense of “rigid motion”, we assume in this section that the ambient space $\mathcal{H}$ admits an inner product (Hilbert space) and correspondingly $E$ and $E^n$ are Hermitian vector bundles.

Given two holomorphic maps $f$ and $\tilde{f}$ from a domain $\Omega \subseteq \mathbb{C}$ into $Gr(l, \mathcal{H})$, point-wise contact between them are defined to be order $n$ agreement up to a rigid motion (isometry) of the ambient space (see Sec.5 [6] or Sec.2.3):

**Definition 4.1.** Let $\mathcal{H}$ be a Hilbert space. Two holomorphic maps $f$ and $\tilde{f}$ from a domain $\Omega \subseteq \mathbb{C}$ into $Gr(l, \mathcal{H})$ are said to have contact of order $n$ at a point $z_0$ if there exist
holomorphic frames $s = \{s_1, \cdots, s_l\}$ and $\tilde{s} = \{\tilde{s}_1, \cdots, \tilde{s}_l\}$ for $f$ and $\tilde{f}$ around $z_0$ such that the linear map defined by

$$s_i^{(k)}(z_0) \mapsto \tilde{s}_i^{(k)}(z_0), 1 \leq i \leq l, 0 \leq k \leq n$$

is isometric.

In above definition one have to assume that the holomorphic maps are $n$-nondegenerate, that is, $\{s_i^{(k)}(z), 1 \leq i \leq l, 0 \leq k \leq n\}$ and $\{\tilde{s}_i^{(k)}(z), 1 \leq i \leq l, 0 \leq k \leq n\}$ are linearly independent sets. We keep this assumption in the sequel.

**Proposition 4.2.** Let $H$ be a Hilbert space. Let $f$ and $\tilde{f}$ be holomorphic maps from $\Omega \subseteq \mathbb{C}$ to $Gr(l, H)$ with associated holomorphic Hermitian vector bundles $E$ and $\tilde{E}$. Fix a point $z_0$ in $\Omega$, the followings are equivalent:

(i) $f$ and $\tilde{f}$ have contact of order $n$ at $z_0$.

(ii) There is a linear isometric map $\Phi$ from $E^n(z_0)$ to $\tilde{E}^n(z_0)$ such that $\Phi P = \tilde{P}\Phi$, where $P$ and $\tilde{P}$ are Pascal maps on $E^n$ and $\tilde{E}^n$.

**Proof.** (i)$\Rightarrow$(ii)Let $\Phi$ be the isometric linear map as in Definition 4.1, then $\Phi P = \tilde{P}\Phi$ holds trivially since $P$ and $\tilde{P}$ are both represented by (2.5) while $\Phi$ is represented by the identity matrix.

(ii)$\Rightarrow$(i) Fix holomorphic frames $s$ and $\tilde{s}$ for $E$ and $\tilde{E}$, $\Phi P = \tilde{P}\Phi$ implies that the representing matrix of $\Phi$ commutes with (2.5) with respect to $\{s(z_0), s'(z_0), \cdots, s^{(n)}(z_0)\}$ and $\{\tilde{s}(z_0), \tilde{s}'(z_0), \cdots, \tilde{s}^{(n)}(z_0)\}$. By Theorem 2.1, the matrix has to be in $\Lambda^n$ hence there exists matrices $A_0, A_1, \cdots, A_n$ such that the linear isometric map $\Phi$ is represented by

$$\begin{pmatrix} s(z_0) \\ s'(z_0) \\ s''(z_0) \\ \vdots \\ s^{(n)}(z_0) \end{pmatrix} \mapsto \begin{pmatrix} A_0 & A_0 \\ A_1 & A_0 \\ A_2 & 2A_1 & A_0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ A_n & (n)_{n-1}A_{n-1} & \cdots & \cdots & (n)_{n-2}A_{n-2} & \cdots & \cdots & (n)_{n-1}A_1 & A_0 \end{pmatrix} \begin{pmatrix} \tilde{s}(z_0) \\ \tilde{s}'(z_0) \\ \tilde{s}''(z_0) \\ \vdots \end{pmatrix}.$$ 

Set $A(z) = \sum_{k=0}^n \frac{1}{k!}(z - z_0)^k A_k$, then $A(z)$ is a holomorphic matrix function around $z_0$ with $A^{(k)}(z_0) = A_k$, $k = 0, 1 \cdots n$. Let $\tilde{t} = A(z)\tilde{s}$, then $\tilde{t}$ is also a holomorphic frame for $\tilde{E}$ (over a sufficiently small neighborhood of $z_0$) and the frames $s$ and $\tilde{s}$ meets Definition 4.1.

Now we see that condition (ii) of Proposition 4.2 as a frame-independent criteria is compatible with the original extrinsic Definition 4.1, hence is eligible to be the intrinsic condition that defines point-wise contact between holomorphic Hermitian vector bundles over a one dimensional domain.
Definition 4.3. Two holomorphic Hermitian vector bundles $E$ and $\tilde{E}$ over $\Omega \subseteq \mathbb{C}$ are said to have contact of order $n$ at a point $z_0$ if there is a linear isometric map $\Phi$ from $E^n(z_0)$ to $\tilde{E}^n(z_0)$ such that $\Phi P = \tilde{P} \Phi$, where $P$ and $\tilde{P}$ are Pascal maps on $E^n$ and $\tilde{E}^n$.

Similar to Remark 3.3 in Definition 4.3, one does not need to assume the vector bundles are associated to holomorphic maps. In fact, if $E$ a general Hermitian vector bundle where $\{H_\alpha\}$ is the Gram matrix for the local frame on $U_\alpha$, then one can check that $\{[\partial^p \overline{\partial^q} H_\alpha]_{0 \leq p,q \leq n}\}$ also glue to a well defined Hermitian form on $E^n$ hence it makes sense to talk about isometric bundle maps on the jet bundles.

5 Several variable case

In this section we outline how to extend discussions in Section 3 and Section 4 with $\Omega \subseteq \mathbb{C}^m$, $m > 1$. Such a several variable extension is not trivial but given the idea in previous sections on $m = 1$, this is essentially a technical work and we omit the details.

We begin with jet bundles and the Pascal map. Fix a point $z = (z_1, \ldots, z_m) \in \Omega$ and a holomorphic frame $s = \{s_1 \cdots s_l\}$ of $E$ around $z$, set

$$E^n(z) := \bigvee_{1 \leq i \leq l, 0 \leq |I| \leq n} \partial^I s_i(z),$$

where $I = (i_1, \ldots, i_m)$ is an multi-index and $|I| = i_1 + \cdots + i_m$.

For any $1 \leq k \leq m$, we define a linear map on $E^n(z)$ by

$$\partial^I s_i(z) \mapsto \begin{cases} i_k \partial_{z_1}^{i_1} \cdots \partial_{z_k}^{i_k-1} \cdots \partial_{z_m}^{i_m} s_i(z), & i_k \geq 1 \\ 0, & i_k = 0 \end{cases} \tag{5.2}$$

where $\partial^I = \partial_{z_1}^{i_1} \cdots \partial_{z_m}^{i_m}$, $1 \leq i \leq l$.

One needs to check two issues:

(i) if $t = \{t_1, \ldots, t_l\}$ is another holomorphic frame of $E$ around $z$, then $\bigvee_{1 \leq i \leq l, 0 \leq |I| \leq n} \partial^I s_i(z)$ and $\bigvee_{1 \leq i \leq l, 0 \leq |I| \leq n} \partial^I t_i(z)$ is the same vector space so the $n$-jet bundle $E^n$ of $E$ is well-defined;

(ii) with respect to either $\{\partial^I s_i(z), 1 \leq i \leq l, 0 \leq |I| \leq n\}$ or $\{\partial^I t_i(z), 1 \leq i \leq l, 0 \leq |I| \leq n\}$, the rule (5.2) represents the same linear map on $E^n(z)$, which is the several variable analogue of Theorem 3.2.

With these issues checked, one sees that for every $1 \leq k \leq m$, (5.2) gives a well defined bundle map, called the $k$-th Pascal map on $E^n$(denoted by $P_k$).

Finally, one can prove the following analogue of Proposition 4.2 where condition (ii) implements the intrinsic definition of point-wise contact between Hermitian holomorphic vector bundles in the several variable case.
Proposition 5.1. Let $\mathcal{H}$ be a Hilbert space, $f$ and $\tilde{f}$ be holomorphic maps from $\Omega \subseteq \mathbb{C}^m$ to $\text{Gr}(l, \mathcal{H})$ with associated holomorphic Hermitian vector bundles $E$ and $\tilde{E}$. The followings are equivalent

(i) $f$ and $\tilde{f}$ have contact of order $n$ at $z_0$, that is, there exists holomorphic frames $s = \{s_1, \cdots s_l\}$ and $\tilde{s} = \{\tilde{s}_1, \cdots \tilde{s}_l\}$ around $z_0$ such that the linear map defined by

$$\partial^I s_i(z_0) \mapsto \partial^I \tilde{s}_i(z_0), 1 \leq i \leq l, 0 \leq |I| \leq n$$

is isometric.

(ii) There is a linear isometric map $\Phi$ from $E^n(z_0)$ to $\tilde{E}^n(z_0)$ such that $\Phi P_k = \tilde{P}_k \Phi$ for all $1 \leq k \leq m$, where $P_k$ and $\tilde{P}_k$ are $k$-th Pascal maps on $E^n$ and $\tilde{E}^n$.

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