A higher order approximation method for jump-diffusion SDEs with discontinuous drift coefficient

Pawel Przybyłowicz  Verena Schwarz  Michaela Szölgyenyi

Preprint, December 2023

Abstract

We present the first higher-order approximation scheme for solutions of jump-diffusion stochastic differential equations with discontinuous drift. For this transformation-based jump-adapted quasi-Milstein scheme we prove $L^p$-convergence order $3/4$. To obtain this result, we prove that under slightly stronger assumptions (but still weaker than anything known before) a related jump-adapted quasi-Milstein scheme has convergence order $3/4$ – in a special case even order 1. Order $3/4$ is conjectured to be optimal.

Keywords: jump-diffusion stochastic differential equations, discontinuous drift, strong convergence rate, jump-adapted scheme, higher order scheme

Mathematics Subject Classification (2020): 60H10, 65C30, 65C20

1 Introduction

We consider time-homogeneous jump-diffusion stochastic differential equations (SDEs) of the form

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t + \rho(X_{t^-})\,dN_t, \quad t \in [0,T], \quad X_0 = \xi,$$

where $\xi \in \mathbb{R}$, $\mu, \sigma, \rho : \mathbb{R} \to \mathbb{R}$ are measurable functions, $T \in (0, \infty)$, $W = (W_t)_{t \geq 0}$ is a standard Brownian motion, and $N = (N_t)_{t \geq 0}$ is a homogeneous Poisson process with intensity $\lambda \in (0, \infty)$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We define the filtration $\mathcal{F} = (\mathcal{F}_t)_{t \geq 0}$ as the augmented natural filtration of $N$ and $W$, i.e. for all $t \geq 0$ we set $\mathcal{F}_t = \sigma\{(W_s, N_s) : s \leq t \} \cup \mathcal{N}$, where $\mathcal{N} = \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}$. With this, $(\Omega, \mathcal{F}, \mathbb{P})$ is a filtered probability space that satisfies the usual conditions. Existence and uniqueness of the solution of the above SDE under our assumptions is well settled by [50, 51].

This paper aims to provide a higher order scheme for SDE (1.1) for the case where the drift is allowed to be discontinuous. This setting is, for example, relevant for modeling energy prices or control actions on energy markets ([1, 56, 57]) as well as for modeling physical phenomena, cf. [58, 45].

In the case without jumps numerical approximation of SDEs with discontinuous drift has attracted a lot of attention in recent years, see, e.g., [13, 16, 14, 26, 40, 59, 58, 39, 3, 8, 36, 21]. In the case of presence of jumps, under standard assumptions, classical and jump-adapted Itô-Taylor approximations and Runge-Kutta methods are studied, e.g., in [11, 12, 45]. Approximation results for jump-diffusion SDEs under non-standard assumptions can be found, e.g., in [18, 19, 20, 7, 10, 8, 38, 30, 4, 37]. Asymptotically optimal approximation rates are proven in [17, 24, 43]. All above mentioned results assume continuity of the drift coefficient. In [50] they allow for finitely many discontinuities in the drift and prove $L^2$-convergence order $1/2$ for the Euler-Maruyama approximation.

We would like to highlight the contributions that were most influential for the current paper: In [45] jump-adapted schemes are studied. Transformation-based schemes have first been
introduced in \[26, 27\] for the case without jumps (\(\rho \equiv 0\)), where transformation-based Euler-Maruyama schemes as well as a proof technique have been introduced, which inspired many of the subsequent contributions. A transformation-based quasi-Milstein scheme for the case without jumps (\(\rho \equiv 0\)) has been introduced in \[34\]; it has convergence order 3/4 in \(L^p\), which is optimal for the class of all algorithms with deterministic grid points, cf. \[35\]. Approximation of discontinuous-drift-SDEs with jumps has been studied in \[50\].

In the current paper we construct the first higher-order scheme for jump-diffusion SDEs with discontinuous drift – a transformation-based jump-adapted quasi-Milstein scheme. Our main contribution is to prove convergence order 3/4 in \(L^p, p \in (1, \infty)\) for this scheme. This is proven to be optimal in \[54\]. As a side result we prove that under slightly stronger assumptions (but still weaker than anything known in the literature) a related jump-adapted quasi-Milstein scheme has convergence order 3/4; in a special case even order 1.

For the proof we proceed in two steps. First we introduce the jump-adapted quasi-Milstein scheme and prove convergence order 3/4 (respectively 1) under the stronger assumptions. Then we make use of a transformation and the previous result to prove our target result in Theorem 4.3.

2 Setting

In the following we denote by \(\tilde{N} = (\tilde{N}_t)_{t \geq 0}\) the compensated Poisson process, i.e. \(\tilde{N}_t = N_t - \lambda t\) for all \(t \in [0, \infty)\). It is well-known that \(\tilde{N}\) is a square integrable \(\mathbb{F}\)-martingale. We denote by \(\mathbb{L}_t\) the Lipschitz constant of a generic Lipschitz continuous function \(f\), and by \(\text{Id}\) the identity mapping. Furthermore, for \((A_1, A_1)\) and \((A_2, A_2)\) two measurable spaces, \(Y_1: (\Omega, \mathcal{F}) \to (A_1, A_1), Y_2: (\Omega, \mathcal{F}) \to (A_2, A_2)\) maps measurable functions, and \(\mathcal{G}\) a sub-\(\sigma\)-algebra of \(\mathcal{F}\), we denote by \(\mathbb{E}[f(Y_1, y)|\mathcal{G}]|_{y=Y_2}\) the random variable \(g(Y_2)\), where \(g: A_2 \to \mathbb{R}\) is for all \(y \in A_2\) defined by \(g(y) = \mathbb{E}[f(Y_1, y)|\mathcal{G}]\). Especially, we use this notation in case that \(\mathcal{G} = \{\emptyset, \Omega\}\), where we omit conditioning and write \(\mathbb{E}[f(Y_1, y)]|_{y=Y_2}\). For random variables \(Y_3, Y_4: \Omega \to \mathbb{R}\), denote by \(\mathbb{E}[Y_3|Y_4 = y]|_{y=Y_4}\) the random variable \(h(Y_4)\), where \(h: \mathbb{R} \to \mathbb{R}\) is for all \(y \in \mathbb{R}\) defined by \(h(y) = \mathbb{E}[Y_3|Y_4 = y]\).

Moreover, we recall the following definition.

**Definition 2.1 ([23, Definition 2.1]).** Let \(I \subseteq \mathbb{R}\) be an interval and let \(m \in \mathbb{N}\). We say a function \(f: I \to \mathbb{R}\) is piecewise Lipschitz, if there are finitely many points \(\zeta_1 < \ldots < \zeta_m \in I\) such that \(f\) is Lipschitz on each of the intervals \((-\infty, \zeta_1) \cap I, (\zeta_m, \infty) \cap I,\) and \((\zeta_k, \zeta_{k+1}), k = 1, \ldots, m - 1\).

**Assumption 2.1.** We assume on the coefficients of SDE (1.1) that there exist \(m \in \mathbb{N}\) and \(-\infty = \zeta_0 < \zeta_1 < \ldots < \zeta_m < \zeta_{m+1} = \infty\) such that:

(i) The drift coefficient \(\mu: \mathbb{R} \to \mathbb{R}\) is piecewise Lipschitz continuous with potential discontinuities in the points \(\zeta_1, \ldots, \zeta_m \in \mathbb{R}\).

(ii) The diffusion coefficient \(\sigma: \mathbb{R} \to \mathbb{R}\) is Lipschitz continuous and for all \(k \in \{1, \ldots, m\}\), \(\sigma(\zeta_k) \neq 0\).

(iii) The jump coefficient \(\rho: \mathbb{R} \to \mathbb{R}\) is Lipschitz continuous.

(iv) The drift coefficient \(\mu\) and the diffusion coefficient \(\sigma\) are differentiable with Lipschitz continuous derivatives on \((\zeta_{i-1}, \zeta_i)\) for all \(i \in \{1, \ldots, m\}\).

The following lemma is a direct consequence of Assumption 2.1.

**Lemma 2.2 ([54, Lemma 2]).** Let Assumption 2.1 hold. Then \(\mu, \sigma, \text{ and } \rho\) satisfy a linear growth condition, that is there exist constants \(c_\mu, c_\sigma, c_\rho \in (0, \infty)\) such that

\[
|\mu(x)| \leq c_\mu(1 + |x|), \quad |\sigma(x)| \leq c_\sigma(1 + |x|), \quad |\rho(x)| \leq c_\rho(1 + |x|).
\]
The existence and uniqueness of the strong solution of the SDE (1.1) is guaranteed by the following result.

**Theorem 2.3** ([46, Theorem 3.1]). Let Assumption 2.1 hold. Then the SDE (1.1) has a unique global strong solution.

### 3 Convergence of the jump-adapted quasi-Milstein scheme

We are going to introduce the jump-adapted quasi-Milstein scheme and prove under stronger assumptions than Assumption 2.1 convergence order 3/4, respectively order 1. This result in combination with a transformation technique will be used later to prove our main result. In this section we consider the time-homogeneous jump-diffusion SDE

\[
dZ_t = \tilde{\mu}(Z_t) \, dt + \tilde{\sigma}(Z_t) \, dW_t + \tilde{\rho}(Z_{t-}) \, dN_t, \quad t \in [0, T], \quad Z_0 = \tilde{\xi},
\]

where \( \tilde{\xi} \in \mathbb{R}, \tilde{\mu}, \tilde{\sigma}, \tilde{\rho}: \mathbb{R} \to \mathbb{R} \) are measurable functions, \( T \in (0, \infty) \), \( W = (W_t)_{t \geq 0} \), \( N = (N_t)_{t \geq 0} \), and \((\Omega, \mathcal{F}, \mathbb{P})\) are defined as in Section 2.

**Assumption 3.1.** We assume on the coefficients of SDE (3.1) that

(i) the drift coefficient \( \tilde{\mu}: \mathbb{R} \to \mathbb{R} \), the diffusion coefficient \( \tilde{\sigma}: \mathbb{R} \to \mathbb{R} \), and the jump coefficient \( \tilde{\rho}: \mathbb{R} \to \mathbb{R} \) are Lipschitz continuous;

(ii) there exists \( m_{\tilde{\mu}} \in \mathbb{N} \) and \(-\infty = \zeta_0 < \zeta_1 < \ldots < \zeta_{m_{\tilde{\mu}}} < \zeta_{m_{\tilde{\mu}}+1} = \infty \) such that the drift coefficient \( \tilde{\mu} \) is differentiable with Lipschitz continuous derivatives on \((\zeta_{k-1}, \zeta_k)\) for \( k \in \{1, \ldots, m_{\tilde{\mu}} + 1\} \);

(iii) there exists \( m_{\tilde{\sigma}} \in \mathbb{N} \) and \(-\infty = \eta_0 < \eta_1 < \ldots < \eta_{m_{\tilde{\sigma}}} < \eta_{m_{\tilde{\sigma}}+1} = \infty \) such that the diffusion coefficient \( \tilde{\sigma} \) is differentiable with Lipschitz continuous derivatives on \((\eta_{k-1}, \eta_k)\) for \( k \in \{1, \ldots, m_{\tilde{\sigma}} \} \);

(iv) for all \( k \in \{1, \ldots, m_{\tilde{\mu}}\}, \tilde{\sigma}(\zeta_k) \neq 0 \) and for all \( j \in \{1, \ldots, m_{\tilde{\sigma}}\}, \tilde{\sigma}(\eta_j) \neq 0 \).

Under Assumption 3.1 the existence of a strong unique solution is ensured by [46, p. 255, Theorem 6]. Further we define the functions \( d_f: \mathbb{R} \to \mathbb{R} \) for \( f \in \{\tilde{\mu}, \tilde{\sigma}\} \) by

\[
d_f(x) = \begin{cases} 
 f'(x) & \text{if } f \text{ differentiable at } x \\
 0 & \text{else.}
\end{cases}
\]

In the following lemma we state some direct consequences of Assumption 3.1.

**Lemma 3.1.** Let Assumption 3.1 hold and let \( f \in \{\tilde{\mu}, \tilde{\sigma}, \tilde{\rho}\}, \ g \in \{\tilde{\mu}, \tilde{\sigma}\} \).

(i) Then there exists a constant \( c_f \in (0, \infty) \) such that for all \( x \in \mathbb{R} \),

\[
|f(x)| \leq c_f(1 + |x|).
\]

(ii) It holds that

\[
\|d_g\|_{\infty} \leq L_g < \infty.
\]

(iii) There exists a constant \( b_g \in (0, \infty) \) such that for all \( x, y \in \mathbb{R} \) for which \( g \) is differentiable on \([x, y]\) it holds that

\[
|g(y) - g(x) - g'(x)(y - x)| \leq b_g|y - x|^2.
\]

3
Now we define our jump-adapted time discretisation. For this, let \( \delta = \frac{T}{M} \) for all \( M \in \mathbb{N} \) and define the equidistant time discretisation \( (s_m)_{m \in \{0, \ldots, M\}} \) by \( s_m = \delta m \) for \( m \in \{0, \ldots, M\} \). We add to these deterministic points the family \((\nu_i)_{i \in \mathbb{N}}\) of all jump times of the Poisson process. Then we order all points so that they are increasing, yielding the time discretisation \((\tau_n)_{n \in \mathbb{N}}\) with the maximum step size less or equal than \( \delta \). Formally, we define \((\tau_n)_{n \in \mathbb{N}}\) as follows:

\[
\tau_0 = 0 \\
\tau_{n+1} = \min\{\nu_i : i \in \mathbb{N}, \nu_i > \tau_n\} \wedge \min\{s_m : m \in \{0, \ldots, M\}, s_m > \tau_n\} \wedge T. 
\]

(3.2)

(3.3)

Given a time grid \((\tau_n)_{n \in \mathbb{N}}\) we define the time-continuous jump-adapted quasi-Milstein scheme \((Z^{(M)}_t)_{t \in [0,T]}\) by

\[
Z^{(M)}_0 = \tilde{\xi},
\]

(3.4)

and for all \( n \in \mathbb{N}_0 \) by

\[
Z^{(M)}_{\tau_{n+1}} = Z^{(M)}_{\tau_n} + \tilde{\mu}(Z^{(M)}_{\tau_n})(\tau_{n+1} - \tau_n) + \tilde{\sigma}(Z^{(M)}_{\tau_n})(W_{\tau_{n+1}} - W_{\tau_n}) \\
+ \frac{1}{2} \tilde{\sigma}(Z^{(M)}_{\tau_n}) d\tilde{\sigma}(Z^{(M)}_{\tau_n}) ((W_{\tau_{n+1}} - W_{\tau_n})^2 - (\tau_{n+1} - \tau_n))
\]

(3.5)

or equivalently

\[
Z^{(M)}_{\tau_{n+1}} = Z^{(M)}_{\tau_{n+1}^-} + \tilde{\rho}(Z^{(M)}_{\tau_{n+1}^-})(N_{\tau_{n+1}} - N_{\tau_n}).
\]

(3.6)

Between the points of the time discretisation, i.e. for \( t \in (\tau_n, \tau_{n+1}) \), \( n \in \mathbb{N}_0 \), we set

\[
Z^{(M)}_t = Z^{(M)}_{\tau_n} + \tilde{\mu}(Z^{(M)}_{\tau_n})(t - \tau_n) + \tilde{\sigma}(Z^{(M)}_{\tau_n})(W_t - W_{\tau_n}) \\
+ \tilde{\sigma}(Z^{(M)}_{\tau_n}) d\tilde{\sigma}(Z^{(M)}_{\tau_n}) \frac{1}{2} ((W_t - W_{\tau_n})^2 - (t - \tau_n)).
\]

Note that the following integral notation is equivalent to (3.4), (3.5), (3.6):

\[
Z^{(M)}_t = \tilde{\xi} + \int_0^t \sum_{n=0}^\infty \tilde{\mu}(Z^{(M)}_{\tau_n}) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) du \\
+ \int_0^t \sum_{n=0}^\infty (\tilde{\sigma}(Z^{(M)}_{\tau_n}) d\tilde{\sigma}(Z^{(M)}_{\tau_n})(W_u - W_{\tau_n})) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) dW_u \\
+ \int_0^t \tilde{\rho}(Z^{(M)}_{u^-}) dN_u.
\]

(3.7)

Similarly, we can express the solution on SDE (3.1) by

\[
Z_t = \tilde{\xi} + \int_0^t \sum_{n=0}^\infty \tilde{\mu}(Z_u) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) du + \int_0^t \sum_{n=0}^\infty \tilde{\sigma}(Z_u) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) dW_u + \int_0^t \tilde{\rho}(Z_{u^-}) dN_u.
\]

(3.8)

To simplify the notation we denote for given \( M \in \mathbb{N} \) and time point \( t \in [0, T] \), the largest value of the sequence \((s_m)_{m=0,\ldots,M}\) which is smaller or equal \( t \), by \( \ell_t \)

\[
\ell_t = \max\{s_m : m \in \{0, \ldots, M\}, s_m \leq t\} = \left\lfloor \frac{tM}{T} \right\rfloor \frac{T}{M}.
\]
3.1 Preparatory lemmas

In this section we provide some basic properties of the time discretisation, moment estimates, and Markov properties.

We start providing some basic properties of the time discretisation $\{\tau_n\}_{n \in \mathbb{N}}$. We consider two filtrations, which we both need later on. In addition to $\mathbb{F}$, we define the filtration $\tilde{\mathbb{F}} = (\tilde{\mathbb{F}}_t)_{t \geq 0}$ for all $t \in [0, \infty)$ by

$$\tilde{\mathbb{F}}_t = \sigma(\{W_s : s \leq t\} \cup \{N_s : s \geq 0\} \cup \mathcal{N}).$$

Recall that $\mathcal{N}$ is the set of all nullsets of $\mathbb{F}$. Because $W$ and $N$ are independent $W$ is also a Brownian motion with respect to the filtration $(\tilde{\mathbb{F}}_t)_{t \geq 0}$. Further it is obvious that $Z^{(M)}$ as well as $Z$ are $\tilde{\mathbb{F}}$-adapted càdlàg processes, since for all $t \in [0, \infty)$, $\mathbb{F}_t \subset \tilde{\mathbb{F}}_t$.

**Lemma 3.2.** Let $M \in \mathbb{N}$ and let the sequence $(\tau_n)_{n \in \mathbb{N}}$ be defined by (3.2) and (3.3). Then for all $n \in \mathbb{N}$,

1. $\tau_n$ is an $\mathbb{F}$-stopping time and also an $\tilde{\mathbb{F}}$-stopping time;
2. $Z^{(M)}_{\tau_n}$ is $\mathbb{F}_{\tau_n}$-measurable and also $\tilde{\mathbb{F}}_{\tau_n}$-measurable;
3. $(W_{\tau_n+s} - W_{\tau_n})_{s \geq 0}$ is a Brownian motion with respect to the filtration $(\mathbb{F}_{\tau_n+s})_{s \geq 0}$, $(N_{\tau_n+s} - N_{\tau_n})_{s \geq 0}$ is a Poisson process with respect to the filtration $(\mathbb{F}_{\tau_n+s})_{s \geq 0}$, both processes are independent of $\mathbb{F}_{\tau_n}$, and it holds that $(W_{\tau_n+s} - W_{\tau_n})_{s \geq 0}$ is independent of $(N_{\tau_n+s} - N_{\tau_n})_{s \geq 0}$;
4. $(W_{\tau_n+s} - W_{\tau_n})_{s \geq 0}$ is a Brownian motion with respect to the filtration $(\tilde{\mathbb{F}}_{\tau_n+s})_{s \geq 0}$, which is independent of $\tilde{\mathbb{F}}_{\tau_n}$.

**Proof.** For (i) we fix $t \in [0, \infty)$ and see that

$$\left\{ \tau_n \leq t \right\} = \left\{ N_t + \left\lfloor \frac{tM}{T} \right\rfloor \geq n \right\} = \left\{ N_t \geq n - \left\lfloor \frac{tM}{T} \right\rfloor \right\} \in \mathbb{F}_0.$$

Hence $\tau_n$ is an $\mathbb{F}$-stopping time and an $\tilde{\mathbb{F}}$-stopping time. By [46, p. 5, Theorem 6], item (ii) follows directly from item (i). Item (iii) is implied by [55, Theorem 40.10] by considering the two-dimensional Lévy process $(N, W)$. Item (iv) is proven in [22, Theorem 11.11].

Next we provide moment bounds for the jump-adapted quasi-Milstein scheme.

**Lemma 3.3.** Let $p \in \mathbb{N}$. Then there exists a constant $c_{W,p} \in (0, \infty)$ such that it holds for all stopping times $\tau$, all $M \in \mathbb{N}$, $\delta = \frac{T}{MT}$, and all $t \in [0, \infty)$ that

$$\mathbb{E}\left[ |W_{\tau+t} - W_\tau|^p \right] \leq c_{W,p} t^\frac{p}{2},$$

and

$$\mathbb{E}\left[ \sup_{s \in [0,\delta]} |W_{\tau+s} - W_\tau|^p \right] \leq c_{W,p} \delta^\frac{p}{2}. \tag{3.8}$$

These well known estimates can for example be obtained by combining [46, p. 299] and [44, p. 53, respectively p. 59, Proposition 1.88].

**Lemma 3.4.** Assume that $q \in [2, \infty)$, $a, b \in [0, T]$ with $a < b$, $Z \in \{\text{Id}, W, N\}$, and that $Y = (Y(t))_{t \in [a,b]}$ is a predictable stochastic process with respect to $(\mathbb{F}_t)_{t \in [a,b]}$ and with

$$\mathbb{E}\left[ \int_a^b |Y(t)|^q \, dt \right] < \infty.$$

Then there exists a constant $\hat{c} \in (0, \infty)$ such that for all $t \in [a,b]$,

$$\mathbb{E}\left[ \sup_{s \in [a,t]} \left| \int_a^s Y(u) \, dZ(u) \right|^q \right] \leq \hat{c} \int_a^t \mathbb{E}[|Y(u)|^q] \, du.$$
Proof. In the case that $Z = \text{Id}$ the claim is proven directly using Jensen’s inequality. In the case that $Z = W$, we know that $(\int_a^t Y(s) \, dW(s))_{c \in [a,b]}$ is a martingale. Hence we apply the Burkholder-Davis-Gundy inequality and afterwards Jensen’s inequality to obtain that there exists some $c \in (0, \infty)$ such that

$$E \left[ \sup_{s \in [a,t]} \left| \int_a^s Y(u) \, dW(u) \right|^q \right] \leq c \, E \left[ \left( \int_a^t (Y(u))^2 \, du \right)^{\frac{q}{2}} \right] \leq c \,(b-a)^{\frac{q}{2}-1} \, E \left[ \left| Y(u) \right|^q \right] \, du.$$

For the case that $Z = N$ the claim is proven by applying \cite{ref} Lemma 2.1.

We have not yet proven finiteness of moments of the approximation scheme, but this is not required for the following lemma; the lemma is proven in Appendix A.

**Lemma 3.5.** For all $p \in \mathbb{N}_0$, $q \in \mathbb{N}$, $u \in [0,T]$, $M \in \mathbb{N}$, $\delta = \frac{r}{M}$, $n \in \mathbb{N}$, $k \in \mathbb{N}$, and the constants $c_{W_q}$ as in Lemma 3.3 it holds that

$$E \left[ (1 + |Z(n,M)|^p) |W_n - W_{\tau_n}|^q \mathbb{I}_{(\tau_n,\tau_n+1)}(u) \mathbb{I}_{\{N_n = k\}} \right] \leq c_{W_q} \, \delta^{\frac{q}{2}} \, E \left[ (1 + |Z(n,M)|^p) \mathbb{I}_{(\tau_n,\tau_n+1)}(u) \mathbb{I}_{\{N_n = k\}} \right],$$

and

$$E \left[ \sum_{n=0}^{\infty} (1 + |Z(n,M)|^p) |W_n - W_{\tau_n}|^q \mathbb{I}_{(\tau_n,\tau_n+1)}(u) \right] \leq c_{W_q} \, \delta^{\frac{q}{2}} \, \sum_{n=0}^{\infty} (1 + |Z(n,M)|^p) \mathbb{I}_{(\tau_n,\tau_n+1)}(u).$$

(3.9)

In the next lemma we provide moment estimates for the jump-adapted quasi-Milstein scheme, which depend on the fineness of the discretisation scheme. We use these afterwards to prove moment estimates with constants independent of $M$. The proof can be found in Appendix A.

**Lemma 3.6.** Let Assumption 3.3 hold and let $p \in \mathbb{N}$, $p \geq 2$. Then for all $M \in \mathbb{N}$, $\delta = \frac{T}{M}$, and all $\xi \in \mathbb{R}$ there exists a constant $c_M \in (0, \infty)$ such that

$$E \left[ \sum_{n=0}^{N_T+M} (1 + |Z(n,M)|^p) \right] + E \left[ \sum_{n=0}^{N_T-1} (1 + |Z(n,M)|^p) \right] \leq c_M.$$

Next we provide several moment estimates for the jump-adapted quasi-Milstein scheme. Recall that $\tilde{\xi}$ denotes the initial value of $Z(n,M)$. Also this lemma is proven in Appendix A.

**Lemma 3.7.** Let Assumption 3.3 hold and let $p \in \mathbb{N}_0$, $p \geq 2$. Then there exist constants $c_1, c_2, c_3 \in (0, \infty)$ such that for all $M \in \mathbb{N}$, $\delta = \frac{T}{M}$, $\xi \in \mathbb{R}$, $s \in [0,T]$, and all $t \in [0,T]$ with $t \leq T - \delta$ it holds that

$$E \left[ \sup_{t \in [0,T]} |Z(n,M)|^p \right] \leq c_1 (1 + |\tilde{\xi}|^p).$$

(3.10)

$$E \left[ \sum_{n=0}^{\infty} |Z(n,M) - Z(n,M)|^p \mathbb{I}_{(\tau_n,\tau_n+1)}(s) \right] \leq c_2 (1 + |\tilde{\xi}|^p) \frac{\delta^p}{2}.$$

(3.11)

$$E \left[ \sup_{s \in [t,\delta]} |Z(n,M) - Z(n,M)|^p \right] \leq c_3 (1 + |\tilde{\xi}|^p) \delta.$$ 

(3.12)

Let $Z(n,M,\tilde{\xi})$ be the solution to SDE (3.7) with initial value $\tilde{\xi}$. 

6
Lemma 3.8. For all $\tilde{\xi} \in \mathbb{R}$, all $M \in \mathbb{N}$, and all $m \in \{0, \ldots, M-1\}$,

$$\mathbb{P}
\left(\int_{t \in [0,T-\Delta t]} Z^{(M)}_{s_m_{+}} \right|_{t \in [0,T-\Delta t]} = \mathbb{P}
\left(\int_{t \in [0,T-\Delta t]} Z^{(M)}_{s_m} \right|_{t \in [0,T-\Delta t]} = \mathbb{P}
\left(\int_{t \in [0,T-\Delta t]} Z^{(M)}_{s_m_{+}} \right|_{t \in [0,T-\Delta t]} \right).$$

Further it holds for all $\tilde{\xi} \in \mathbb{R}$, $M \in \mathbb{N}$, $m \in \{0, \ldots, M-1\}$, and $\mathbb{P}\{Z^{(M)}_{s_m} \}$-almost all $y \in \mathbb{R}$ that

$$\mathbb{P}
\left(\int_{t \in [0,T-\Delta t]} Z^{(M)}_{s_m_{+}} \right|_{t \in [0,T-\Delta t]} = \mathbb{P}
\left(\int_{t \in [0,T-\Delta t]} Z^{(M)}_{s_m} \right|_{t \in [0,T-\Delta t]} = \mathbb{P}
\left(\int_{t \in [0,T-\Delta t]} Z^{(M)}_{s_m_{+}} \right|_{t \in [0,T-\Delta t]} \right).$$

We denote by $\mathcal{D}_k = \{f: [0, T] \rightarrow \mathbb{R}^k : f \text{ is càdlàg}\}$ for all $k \in \mathbb{N}$, $T \in (0, \infty)$. Further, for a càdlàg stochastic process $Y$ on $[0, T]$ and a $\sigma$-algebra $\mathcal{G} \subset \mathcal{F}$ we denote by $\mathbb{P}^Y|\mathcal{G}$ the measure defined for all $A \in \mathcal{D}_1$ by

$$\mathbb{P}^Y|\mathcal{G}(A) = \mathbb{E}[\mathbb{1}_A(Y)|\mathcal{G}].$$

Additionally, for a càdlàg stochastic process $Y$, a random variable $Y_1$, and a real value $y_1 \in \mathbb{R}$ we define by $\mathbb{P}^Y|Y_1=y_1$ the measure defined for all $A \in \mathcal{B}(\mathbb{D}_1)$ by

$$\mathbb{P}^Y|Y_1=y_1(A) = \mathbb{E}[\mathbb{1}_A(Y)|Y_1 = y_1].$$

Proof. For fixed $m \in \{0, \ldots, M\}$ consider SDE (3.7) on the time interval $[0, T-s_m]$. This SDE satisfies [52, Assumption 2.1.]. Also the process $(Z^{(M)}_{s_m+t})_{t \in [0,T-\Delta t]}$ satisfies [52, Assumption 2.1.] with the same function in the integrand. Hence by [52, Theorem 3.1] there exists a Skorohod measurable function $\hat{\Psi}: \mathbb{R} \times \mathcal{D}_3 \rightarrow \mathbb{D}_1$ such that

$$(Z^{(M)}_{s_m+t})_{t \in [0,T-\Delta t]} = \hat{\Psi}(\tilde{\xi}, (t, W_t, N_t)_{t \in [0,T-\Delta t]}$$

and

$$(Z^{(M)}_{s_m+t})_{t \in [0,T-\Delta t]} = \hat{\Psi}(Z^{(M)}_{s_m}, (t + s_m - s_m, W_{t+s_m} - W_s, N_{t+s_m} - N_s)_{t \in [0,T-\Delta t]}).$$

Using these equalities the Markov property follows by direct computations.

\[ \Box \]

3.2 Occupation time estimates

In this section we provide occupation time estimates for our approximation scheme. For this we compose ideas of [34] and [50]. The novelty here lies in the additional complexity caused by the adapted scheme.

Lemma 3.9. Let Assumption [53] hold. Let $\zeta \in \mathbb{R}$ with $\tilde{\sigma}(\zeta) \neq 0$. Then there exists a constant $c_5 \in (0, \infty)$ such that for all $\tilde{\xi} \in \mathbb{R}$, all $M \in \mathbb{N}$, $\delta = \frac{1}{\mathcal{F}T}$, and all $\varepsilon \in (0, \infty)$ we have

$$\int_0^T \mathbb{P}
\left(\left|\frac{Z^{(M)}_{\tilde{\xi}} - \zeta}{\mathcal{F}T} \right| \leq \varepsilon \right) dt \leq c_5(1 + \tilde{\xi}^2)(\varepsilon + \delta^2).$$

Proof. Recall that we denote by $Z^{(M)}_{\tilde{\xi}}$ the solution of SDE (3.7) with initial value $\tilde{\xi} \in \mathbb{R}$. It holds by equation (3.13) that

$$Z^{(M)}_{\tilde{\xi}} = \tilde{\xi} + \sum_{n=0}^\infty \bar{\mu}(Z^{(M)}_{\tilde{\xi}})\mathbb{1}_{(\tau_n, \tau_{n+1})}(u) du$$

$$+ \sum_{n=0}^\infty (\tilde{\sigma}(Z^{(M)}_{\tilde{\xi}}) + \int_{\tau_n}^u \tilde{\sigma}(Z^{(M)}_{\tilde{\xi}}) d\tilde{\sigma}(Z^{(M)}_{\tilde{\xi}}) dW_u) \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) dW_u$$

$$+ \sum_{n=0}^\infty \tilde{\rho}(Z^{(M)}_{\tilde{\xi}}) dN_u.$$ 

\[ \text{Another way of proving this result is by using [32, Lemma 9.2] for the first statement and proceeding similar to [32] proof of Theorem 9.5 for the second statement.} \]
By [58, Lemma 158] we obtain for all \( a \in \mathbb{R} \) that
\[
|Z_t^{(M),\tilde{\xi}} - a| \\
= |\tilde{\xi} - a| + \int_0^t \text{sgn} \left( Z_{u^-}^{(M),\tilde{\xi}} - a \right) dZ_u^{(M),\tilde{\xi}} + L_t^a \left( Z^{(M),\tilde{\xi}} \right) \\
+ \int_0^t \left| \left( Z_{u^-}^{(M),\tilde{\xi}} + \tilde{\rho}(Z_{u^-}^{(M),\tilde{\xi}}) \right) - a \right| - \left| Z_{u^-}^{(M),\tilde{\xi}} - a \right| - \text{sgn} \left( Z_{u^-}^{(M),\tilde{\xi}} - a \right) \tilde{\rho}(Z_{u^-}^{(M),\tilde{\xi}}) \right) dN_u,
\]
(3.14)

where \( \text{sgn}(x) = \mathbb{1}_{(0,\infty)}(x) - \mathbb{1}_{(-\infty,0)}(x) \) for all \( x \in \mathbb{R} \) and \( L_t^a \left( Z^{(M),\tilde{\xi}} \right) \) is the local time of \( Z^{(M),\tilde{\xi}} \) in \( a \). By (3.14) we obtain
\[
L_t^a \left( Z^{(M),\tilde{\xi}} \right) = \left| L_t^a \left( Z^{(M),\tilde{\xi}} \right) \right|
\leq \left| Z_t^{(M),\tilde{\xi}} - a \right| - \left| \tilde{\xi} - a \right| + \left| \int_0^t \text{sgn} \left( Z_{u^-}^{(M),\tilde{\xi}} - a \right) dZ_u^{(M),\tilde{\xi}} \right|
+ \left| \int_0^t \left( \left| Z_{u^-}^{(M),\tilde{\xi}} + \tilde{\rho}(Z_{u^-}^{(M),\tilde{\xi}}) \right) - a \right| - \left| Z_{u^-}^{(M),\tilde{\xi}} - a \right| - \text{sgn} \left( Z_{u^-}^{(M),\tilde{\xi}} - a \right) \tilde{\rho}(Z_{u^-}^{(M),\tilde{\xi}}) \right) dN_u \right|.
\]
(3.15)

Now we estimate the expectation of each summand on the right hand side of (3.15) separately. For the first one we use Lemma [3.7] (3.10) to obtain that there exists \( c_1 \in (0, \infty) \) such that
\[
\mathbb{E} \left[ \left| Z_t^{(M),\tilde{\xi}} - a \right| - \left| \tilde{\xi} - a \right| \right] \leq \left( 1 + 2^T \right) (1 + \left| \tilde{\xi} \right|).
\]
(3.16)

Using (3.13) we get for the second summand of (3.15) that
\[
\left| \int_0^t \text{sgn} \left( Z_{u^-}^{(M),\tilde{\xi}} - a \right) dZ_u^{(M),\tilde{\xi}} \right|
\leq \left| \int_0^t \text{sgn} \left( Z_{u^-}^{(M),\tilde{\xi}} - a \right) \left( \sum_{n=0}^{\infty} \tilde{\mu}(Z_{\tau_n}^{(M),\tilde{\xi}}) \mathbb{1}_{(\tau_n,\tau_{n+1}]}(u) \right) du \right|
+ \left| \int_0^t \text{sgn} \left( Z_{u^-}^{(M),\tilde{\xi}} - a \right) \left( \sum_{n=0}^{\infty} \tilde{\sigma}(Z_{\tau_n}^{(M),\tilde{\xi}})
+ \int_{\tau_n}^u \tilde{\sigma}(Z_{\tau_n}^{(M),\tilde{\xi}}) d\tilde{\sigma}(Z_{\tau_n}^{(M),\tilde{\xi}}) dW_s \right) \mathbb{1}_{(\tau_n,\tau_{n+1}]}(u) \right) dW_u \right|
+ \left| \int_0^t \text{sgn} \left( Z_{u^-}^{(M),\tilde{\xi}} - a \right) \tilde{\rho}(Z_{u^-}^{(M),\tilde{\xi}}) dN_u \right|.
\]
(3.17)

We estimate the expectation of each summand of (3.17) separately. Lemma [3.7] (3.10) ensures
\[
\mathbb{E} \left[ \int_0^t \text{sgn} \left( Z_{u^-}^{(M),\tilde{\xi}} - a \right) \left( \sum_{n=0}^{\infty} \tilde{\mu}(Z_{\tau_n}^{(M),\tilde{\xi}}) \mathbb{1}_{(\tau_n,\tau_{n+1}]}(u) \right) du \right]
\leq c_{\tilde{\mu}} \mathbb{E} \left[ \int_0^t \left( \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M),\tilde{\xi}}|) \mathbb{1}_{(\tau_n,\tau_{n+1}]}(u) \right) du \right]
\leq c_{\tilde{\mu}} \int_0^t \mathbb{E} \left[ (1 + \sup_{\tau_n \in [0,T]} |Z_{\tau_n}^{(M),\tilde{\xi}}|) \right] du \leq c_{\tilde{\mu}} T (1 + c_1^T) (1 + \left| \tilde{\xi} \right|).
\]
(3.18)

Next we estimate the second summand of (3.17) using the Cauchy-Schwarz inequality, and the
Itô isometry, Lemma \[3.3\] (3.9), and Lemma \[3.7\] (3.10):
\[
\mathbb{E} \left[ \left| \int_0^t \text{sgn} \left( Z_u^{(M)} \xi \right) \left( \sum_{n=0}^{\infty} \left( \overline{\sigma} (Z_{\tau_n}^{(M)} \xi) + \int_{\tau_n}^u \overline{\sigma} (Z_{\tau_n}^{(M)} \xi) \, d\xi (Z_{\tau_n}^{(M)} \xi) \right) dN_u \right) \right| dW_u \right] \\
\leq \mathbb{E} \left[ \left| \int_0^t \text{sgn} \left( Z_u^{(M)} \xi \right) \left( \sum_{n=0}^{\infty} \left( \overline{\sigma} (Z_{\tau_n}^{(M)} \xi) \right) \right) dW_u \right| \right] \\
= \mathbb{E} \left[ \left| \int_0^t \sum_{n=0}^{\infty} \left( \overline{\sigma} (Z_{\tau_n}^{(M)} \xi) \right) dW_u \right| \right] \\
\leq \left( 4c_\sigma^2 \mathbb{E} \left[ \int_0^t \sum_{n=0}^{\infty} \left( 1 + \left| Z_{\tau_n}^{(M)} \xi \right|^2 \right) dW_u \right] \right) \frac{1}{2} \\
\leq \left( 4c_\sigma^2 T + 4c_\sigma^2 L_\sigma^2 c_{W_2} T^2 \right) \frac{1}{2} (1 + c_1) \left( 1 + \left| \xi \right| \right) \tag{3.19}
\]

For the third summand of (3.17) we use Cauchy-Schwarz inequality, Lemma \[3.4\] and Lemma \[3.7\] (3.10). This shows that there exists \( \tilde{c} \in (0, \infty) \) such that
\[
\mathbb{E} \left[ \left| \int_0^t \text{sgn} \left( Z_u^{(M)} \xi - a \right) \tilde{\rho} (Z_u^{(M)} \xi) \, dN_u \right| \right] \\
\leq \left( \tilde{c} \mathbb{E} \left[ \int_0^t \left| \text{sgn} \left( Z_u^{(M)} \xi - a \right) \right|^2 \tilde{\rho} (Z_u^{(M)} \xi) \, dW_u \right] \right) \frac{1}{2} \\
\leq \left( \tilde{c} \left( \mathbb{E} \left[ \int_0^t \left| \text{sgn} \left( Z_u^{(M)} \xi - a \right) \right|^2 \, dW_u \right] \right) \right) \frac{1}{2} \\
\leq \left( 2 \tilde{c} \mathbb{E} \left[ \int_0^t \left( 1 + \left| Z_u^{(M)} \xi \right|^2 \right) \, dW_u \right] \right) \frac{1}{2} \\
\leq \left( 2 \tilde{c} c_{\sigma}^2 \mathbb{E} \left[ \int_0^t \left( 1 + \left| Z_u^{(M)} \xi \right|^2 \right) \, dW_u \right] \right) \frac{1}{2} (1 + \left| \xi \right|) \tag{3.20}
\]

Combining (3.17), (3.18), (3.19), and (3.20) we obtain that there exists a constant \( \tilde{c}_1 \in (0, \infty) \) such that
\[
\mathbb{E} \left[ \left| \int_0^t \text{sgn} \left( Z_u^{(M)} \xi - a \right) \, dZ_u^{(M)} \xi \right| \right] \leq \tilde{c}_1 (1 + \left| \xi \right|) \tag{3.21}
\]

Next we consider the expectation of the third summand of (3.15). We use the Cauchy-Schwarz
inequality, Lemma 3.4 and Lemma 3.7 (3.10) to obtain that
\[
\mathbb{E}\left[\left|\int_0^t \left(\tilde{Z}_u^{(M)} - \tilde{Z}_u^{(M)}\right) - a| - |Z_u^{(M)} - a| - \operatorname{sgn} (Z_u^{(M)} - a) \tilde{\rho}(Z_u^{(M)}) \right) \, dN_u\right] \\
\leq \tilde{c} \mathbb{E}\left[\int_0^t \left|Z_u^{(M)} - \tilde{Z}_u^{(M)}\right| - a| - |Z_u^{(M)} - a| - \operatorname{sgn} (Z_u^{(M)} - a) \tilde{\rho}(Z_u^{(M)}) \right|^2 \, du \right]^{\frac{1}{2}} \\
\leq \tilde{c} \left(4 \int_0^t \mathbb{E}\left[\left|\tilde{Z}_u^{(M)}\right|^2\right]\, du \right)^{\frac{1}{2}} \leq \tilde{c} \left(8 \frac{c^2}{\rho} \int_0^t \mathbb{E}\left[\left|\tilde{Z}_u^{(M)}\right|^2\right]\, du \right)^{\frac{1}{2}} \\
\leq \tilde{c} \left(8 \frac{c^2}{\rho} T (1 + c_1) \right) \left(1 + \tilde{\xi}\right).
\]

Plugging (3.16), (3.21), and (3.22) into (3.15) we obtain that there exists a constant \(\tilde{c}_2 \in (0, \infty)\) such that
\[
\mathbb{E}\left[\left|L_t^2 (Z_t^{(M)})\right| \leq \tilde{c}_2 (1 + |\tilde{\xi}|). \right.
\]

Note that the continuous martingale part of (3.13) is
\[
\left(\int_0^t \sum_{n=0}^{\infty} \left(\tilde{\delta}(Z_t^{(M)}) + \int_{\tau_n}^u \tilde{\delta}(Z_t^{(M)}) \, d\tilde{\sigma}(Z_t^{(M)}) \right) \, dW_s \right) \mathbf{1}_{(\tau_n, \tau_{n+1})} \, (u) \, dW_u \right)_{t \in [0, T].}
\]

For this we get using [58, Theorem 88],
\[
\mathbb{E}\left[\mathbf{1}_{[\xi_0, \xi_0 + \tilde{\xi}]} (Z_t^{(M)}) \left(\sum_{n=0}^{\infty} \left(\tilde{\delta}(Z_t^{(M)}) + \int_{\tau_n}^u \tilde{\delta}(Z_t^{(M)}) \, d\tilde{\sigma}(Z_t^{(M)}) \right) \, dW_s \right) \mathbf{1}_{(\tau_n, \tau_{n+1})} \, (u) \right)^2 \, du \\
\leq \mathbb{E}\left[\int_0^t \mathbb{E}\left[\left|\tilde{Z}_u^{(M)}\right|^2\right]\, du \right] \leq 2\tilde{c}_2 (1 + |\tilde{\xi}|). \right.
\]

Since \(|a^2 - b^2| = |a - b| \cdot |a + b|\) for all \(a, b \in \mathbb{R}\), we get
\[
\mathbb{E}\left[\tilde{\sigma}^2 (Z_u^{(M)}) - \sum_{n=0}^{\infty} \left(\tilde{\delta}(Z_t^{(M)}) + \int_{\tau_n}^u \tilde{\delta}(Z_t^{(M)}) \, d\tilde{\sigma}(Z_t^{(M)}) \right) \, dW_s \right) \mathbf{1}_{(\tau_n, \tau_{n+1})} \, (u) \right]^2 \\
\leq \mathbb{E}\left[\tilde{\delta}(Z_u^{(M)}) - \sum_{n=0}^{\infty} \left(\tilde{\delta}(Z_t^{(M)}) + \int_{\tau_n}^u \tilde{\delta}(Z_t^{(M)}) \, d\tilde{\sigma}(Z_t^{(M)}) \right) \, dW_s \right) \mathbf{1}_{(\tau_n, \tau_{n+1})} \, (u) \right]^2 \right]^\frac{1}{2} \left(3.26\right)
\]

\[
\mathbb{E}\left[\tilde{\delta}(Z_u^{(M)}) - \sum_{n=0}^{\infty} \left(\tilde{\delta}(Z_t^{(M)}) + \int_{\tau_n}^u \tilde{\delta}(Z_t^{(M)}) \, d\tilde{\sigma}(Z_t^{(M)}) \right) \, dW_s \right) \mathbf{1}_{(\tau_n, \tau_{n+1})} \, (u) \right]^2 \right]^\frac{1}{2}. \right.
\]
For estimating the first factor in (3.26) we will use

\[
\left| \tilde{\sigma}(Z_u^{(M)}\tilde{\xi}) - \sum_{n=0}^{\infty} \left( \tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) + \int_{\tau_n}^{u} \tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \, d\tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \, dW_s \right) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right|
\]

\[
= \sum_{n=0}^{\infty} \left| \tilde{\sigma}(Z_u^{(M)}\tilde{\xi}) - \tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \right| - \int_{\tau_n}^{u} \tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \, d\tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \, dW_s \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u)
\]

\[
\leq \sum_{n=0}^{\infty} \left( L_{\tilde{\sigma}} |Z_{\tau_n}^{(M)}\tilde{\xi} - Z_{\tau_n}^{(M)}\tilde{\xi}| + L_{\tilde{\sigma}} c_2 (1 + |Z_{\tau_n}^{(M)}\tilde{\xi}|) |W_u - W_{\tau_n}| \right) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u).
\]

This, Lemma 3.30 and Lemma 3.37, 3.10, and 3.11 yield

\[
\mathbb{E} \left[ \left| \tilde{\sigma}(Z_u^{(M)}\tilde{\xi}) - \sum_{n=0}^{\infty} \left( \tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) + \int_{\tau_n}^{u} \tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \, d\tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \, dW_s \right) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right|^2 \right]^\frac{1}{2}
\]

\[
\leq 2 \left( L_{\tilde{\sigma}}^2 + \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}\tilde{\xi}|^2) |W_u - W_{\tau_n}|^2 \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u)
\]

\[
+ 2L_{\tilde{\sigma}}^2 c_2 \mathbb{E} \left[ \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}\tilde{\xi}|^2) |W_u - W_{\tau_n}|^2 \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right] \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u)
\]

\[
\leq 2 \left( L_{\tilde{\sigma}}^2 c_2 + L_{\tilde{\sigma}}^2 c_2 c_{W_2} (1 + c_1) \right) \left( 1 + |\tilde{\xi}| \right) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u)
\]

For estimating the second factor of (3.26) we will use

\[
\left| \tilde{\sigma}(Z_u^{(M)}\tilde{\xi}) + \sum_{n=0}^{\infty} \left( \tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) + \int_{\tau_n}^{u} \tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \, d\tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \, dW_s \right) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right|
\]

\[
\leq \sum_{n=0}^{\infty} \left( c_\tilde{\sigma} (1 + |Z_{\tau_n}^{(M)}\tilde{\xi}|) + c_\tilde{\sigma} (1 + |Z_{\tau_n}^{(M)}\tilde{\xi}|) + L_{\tilde{\sigma}} c_\tilde{\sigma} (1 + |Z_{\tau_n}^{(M)}\tilde{\xi}|) |W_u - W_{\tau_n}| \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right)
\]

This, Lemma 3.30 and Lemma 3.37, 3.10, show

\[
\mathbb{E} \left[ \left| \tilde{\sigma}(Z_u^{(M)}\tilde{\xi}) + \sum_{n=0}^{\infty} \left( \tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) + \int_{\tau_n}^{u} \tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \, d\tilde{\sigma}(Z_{\tau_n}^{(M)}\tilde{\xi}) \, dW_s \right) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right|^2 \right]^\frac{1}{2}
\]

\[
\leq \sqrt{3} \left( 2c_\tilde{\sigma}^2 \mathbb{E} \left[ \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}\tilde{\xi}|^2) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right] + 2c_\tilde{\sigma}^2 \mathbb{E} \left[ \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}\tilde{\xi}|^2) |W_u - W_{\tau_n}|^2 \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right] \right)
\]

\[
+ 2L_{\tilde{\sigma}}^2 c_\tilde{\sigma}^2 \mathbb{E} \left[ \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}\tilde{\xi}|^2) |W_u - W_{\tau_n}|^2 \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right] \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u)
\]

\[
\leq \sqrt{3} \left( 2c_\tilde{\sigma}^2 \mathbb{E} \left[ (1 + \sup_{t \in [0,T]} |Z_t^{(M)}\tilde{\xi}|^2) \right] + 2c_\tilde{\sigma}^2 \mathbb{E} \left[ (1 + \sup_{t \in [0,T]} |Z_t^{(M)}\tilde{\xi}|^2) \right] \right)
\]

\[
+ 2L_{\tilde{\sigma}}^2 c_\tilde{\sigma}^2 c_{W_2} \mathbb{E} \left[ \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}\tilde{\xi}|^2) \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u) \right] \mathbf{1}_{(\tau_n, \tau_{n+1}]}(u)
\]

\[
\leq \sqrt{3} \left( 4c_\tilde{\sigma}^2 + 2L_{\tilde{\sigma}}^2 c_\tilde{\sigma}^2 c_{W_2} T \right) \left( 1 + c_1 \right) \left( 1 + |\tilde{\xi}| \right).
\]

(3.28)
Plugging (3.27) and (3.28) into (3.26) we obtain that there exists a constant $\tilde{c}_3 \in (0, \infty)$ such that

$$
\mathbb{E} \left[ \sigma^2(Z_u(M, \bar{\xi}) - \left( \sum_{n=0}^{N_T} \sigma(Z_{\tau_n}(M, \bar{\xi}) \right) + \int_{\tau_n}^{u} \sigma(Z_{\tau_n}(M, \bar{\xi}) d\sigma(Z_{\tau_n}(M, \bar{\xi}) dW_s) \mathbf{1}_{(\tau_n, \tau_{n+1})(u)}^2 \right] 
\leq \tilde{c}_3(1 + |\bar{\xi}|)^2 \delta^\frac{1}{2} \leq 2\tilde{c}_3(1 + |\bar{\xi}|^2)\delta^\frac{1}{2}.
$$

(3.29)

Since $\tilde{\sigma}(\zeta) \neq 0$, there exist $\varepsilon_0, \tilde{c}_4 \in (0, \infty)$ such that

$$
\inf_{|z - \bar{\xi}| < \varepsilon_0} |\tilde{\sigma}(z)|^2 > \tilde{c}_4.
$$

(3.30)

Hence (3.30), (3.28), and (3.29) ensure for all $\varepsilon \in (0, \varepsilon_0)$,

$$
\int_0^T \mathbb{P} \left( \left| Z_u^{(M, \bar{\xi})} - \zeta \right| < \varepsilon \right) du = \frac{1}{c_4} \int_0^T \mathbb{E} \left[ \mathbf{1}_{(\zeta- \varepsilon, \zeta+\varepsilon)} \left( |Z_u^{(M, \bar{\xi})}| \right) \tilde{c}_4 \right] du 
\leq \frac{1}{c_4} \mathbb{E} \left[ \int_0^T \mathbf{1}_{(\zeta- \varepsilon, \zeta+\varepsilon)} \left( |Z_u^{(M, \bar{\xi})}| \right) \sigma^2(Z_u^{(M, \bar{\xi})} du 
= \frac{1}{c_4} \mathbb{E} \left[ \int_0^T \mathbf{1}_{(\zeta- \varepsilon, \zeta+\varepsilon)} \left( |Z_u^{(M, \bar{\xi})}| \right) \left( \sum_{n=0}^{N_T+M} \tilde{\sigma}(Z_{\tau_n}(M, \bar{\xi}) 
+ \int_{\tau_n}^{u} \tilde{\sigma}(Z_{\tau_n}(M, \bar{\xi}) d\sigma(Z_{\tau_n}(M, \bar{\xi}) dW_s) \mathbf{1}_{(\tau_n, \tau_{n+1})(u)}^2 \right) du 
+ \frac{1}{c_4} \mathbb{E} \left[ \int_0^T \mathbf{1}_{(\zeta- \varepsilon, \zeta+\varepsilon)} \left( |Z_u^{(M, \bar{\xi})}| \right) \left( \sigma^2(Z_u^{(M, \bar{\xi})} - \left( \sum_{n=0}^{N_T+M} \tilde{\sigma}(Z_{\tau_n}(M, \bar{\xi}) 
+ \int_{\tau_n}^{u} \tilde{\sigma}(Z_{\tau_n}(M, \bar{\xi}) d\sigma(Z_{\tau_n}(M, \bar{\xi}) dW_s) \mathbf{1}_{(\tau_n, \tau_{n+1})(u)}^2 \right) du 
\leq \frac{1}{c_4} 2\varepsilon \tilde{c}_2 (1 + |\bar{\xi}|) + 2 \frac{1}{c_4} T \tilde{c}_3 (1 + |\bar{\xi}|^2)\delta^\frac{1}{2} \leq \frac{1}{c_4} (4\tilde{c}_2 + 2T\tilde{c}_3) (1 + |\bar{\xi}|^2)(\varepsilon + \delta^\frac{1}{2}),
$$

where in the last inequality we used that for $\bar{\xi} \in \mathbb{R}$, $|\bar{\xi}| \leq 1 + |\bar{\xi}|^2$. For $\varepsilon \in [\varepsilon_0, \infty)$ we have

$$
\int_0^T \mathbb{P} \left( \left| Z_u^{(M, \bar{\xi})} - \zeta \right| < \varepsilon \right) du \leq T = \frac{T}{\varepsilon_0} \leq \frac{T}{\varepsilon_0} (\varepsilon + \delta^\frac{1}{2}) \leq \frac{T}{\varepsilon_0} (1 + |\bar{\xi}|^2)(\varepsilon + \delta^\frac{1}{2}).
$$

Choosing $c_5 = \max \left\{ \frac{1}{c_4} (4\tilde{c}_2 + 2T\tilde{c}_3), \frac{T}{\varepsilon_0} \right\}$ proves the claim. \hfill $\Box$

**Lemma 3.10.** Let Assumption [A] hold. Let $q \in \mathbb{N}$ and $\zeta \in \mathbb{R}$ with $\tilde{\sigma}(\zeta) \neq 0$. Let

$$
S_\zeta = \{ (x, y) \in \mathbb{R}^2 : (x - \zeta)(y - \zeta) \leq 0 \}.
$$

Then there exists a constant $c_6 \in (0, \infty)$ such that for all $M \in \mathbb{N}$, $\delta = \frac{T}{M}$, all $s, t \in [0, T]$ with $t - s \geq \delta$, and all $\mathbb{F}_s$-measurable, non-negative, real-valued random variables $Y$,

$$
\mathbb{E} \left[ Y \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbf{1}_{S_\zeta}(Z_{\tau_n}^{(M, \bar{\xi})}, Z_{\tau_n}^{(M, \bar{\xi})}) \mathbf{1}_{(\tau_n, \tau_{n+1})(u)} \right]
\leq c_6 \left( \frac{\varepsilon_0^2}{c_4} \mathbb{E}[|Y|]^2 + \frac{\varepsilon_0^2}{c_4} \int \mathbb{E}[|Y|^q 1_{|Z_{\tau_n}^{(M, \bar{\xi})} - \zeta| \leq c_6 (1 + |\bar{\xi}|)^{\frac{3}{2}}}] e^{-\frac{z^2}{2}} dz \right).
$$

12
Proof. First, observe that
\[
\mathbb{E}\left[\sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbb{I}_{S^*_\delta}(Z_t^{(M)}, Z_{\tau_n}^{(M)}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(t)\right]
\]
\[
= \mathbb{E}\left[\sum_{n=0}^{N_T+M} |W_t - W_{\tau_n}|^q \mathbb{I}_{S^*_\delta}(Z_t^{(M)}, Z_{\tau_n}^{(M)}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(t) \mathbb{I}_{\{N_t - N_{\delta-\delta} > 0\}}\right] + \mathbb{E}\left[\sum_{n=0}^{N_T+M} |W_t - W_{\tau_n}|^q \mathbb{I}_{S^*_\delta}(Z_t^{(M)}, Z_{\tau_n}^{(M)}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(t) \mathbb{I}_{\{N_t - N_{\delta-\delta} = 0\}}\right].
\] (3.31)

We prove that \(Y\) and
\[
\sum_{n=0}^{N_T+M} \sup_{u \in [0,\delta]} |W_{u+\tau_n} - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1}]}(t) \mathbb{I}_{\{N_t - N_{\delta-\delta} > 0\}}
\] (3.32)
are independent. For this we rewrite (3.32) as
\[
\sum_{n=0}^{N_T+M} \sup_{u \in [0,\delta]} |W_{u+\tau_n} - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1}]}(t) \mathbb{I}_{\{N_t - N_{\delta-\delta} > 0\}}
\]
\[
= \sum_{n=N_{T-\delta}+M-\delta^{-1}+1}^{N_T+M} \sup_{u \in [0,\delta]} |W_{u+\tau_n} - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1}]}(t) \mathbb{I}_{\{N_t - N_{\delta-\delta} > 0\}}
\]
\[
= \sum_{n=0}^{N_T+M} \sup_{u \in [0,\delta]} \left|W_{u+\tau_n+N_{\delta-\delta}+2\delta^{-1}-1} - W_{\tau_n+N_{\delta-\delta}+2\delta^{-1}-1}\right|^q \mathbb{I}_{(\tau_n+N_{\delta-\delta}+2\delta^{-1}-1, \tau_n+N_{\delta-\delta}+2\delta^{-1}] \setminus \{0,\delta\}} \mathbb{I}_{\{N_t - N_{\delta-\delta} > 0\}}.
\]

Next we denote by \((\hat{\tau}_n)_{n \in \mathbb{N}}\) a sequence of points defined similar to \((\tau_n)_{n \in \mathbb{N}}\) but based on \((N_{T-\delta}+t - N_{\delta-\delta})_{t \in [0,\infty)}\) instead of \((N_t)_{t \in [0,\infty)}\). Then it holds that
\[
\{\tau_n+N_{\delta-\delta}+2\delta^{-1}-1 \leq t\} = \left\{N_t + \left\lfloor \frac{tM}{T} \right\rfloor \leq n + N_{\delta-\delta} + t\delta^{-1} - 1\right\}
\]
\[
= \left\{N_t - N_{\delta-\delta} + 1 \leq n\right\} \cup \left\{\hat{\tau}_n \leq t - \xi + \delta\right\} = \left\{\hat{\tau}_n + \xi - \delta \leq t\right\}.
\]

Hence,
\[
\sum_{n=0}^{N_T+M} \sup_{u \in [0,\delta]} |W_{u+\tau_n-N_{\delta-\delta}} - W_{\tau_n-N_{\delta-\delta}}|^q \mathbb{I}_{(\tau_n-N_{\delta-\delta}, \tau_n-N_{\delta-\delta}+1]}(t) \mathbb{I}_{\{N_t - N_{\delta-\delta} > 0\}}
\]
\[
= \sum_{n=0}^{N_T+M} \sup_{u \in [0,\delta]} \left|W_{u+\tau_n-N_{\delta-\delta}} - W_{\tau_n-N_{\delta-\delta}}\right|^q \mathbb{I}_{(\tau_n-N_{\delta-\delta}, \tau_n-N_{\delta-\delta}+1] \setminus \{0,\delta\}} \mathbb{I}_{\{N_t - N_{\delta-\delta} > 0\}},
\]

which shows that (3.32) is independent of \(F_{\delta-\delta}\). As \(Y\) is \(F_s \subset F_{\delta-\delta}\)-measurable, this proves the independence of \(Y\) and (3.32). Therefore the first summand of (3.31) becomes
\[
\mathbb{E}\left[\sum_{n=0}^{N_T+M} |W_t - W_{\tau_n}|^q \mathbb{I}_{S^*_\delta}(Z_t^{(M)}, Z_{\tau_n}^{(M)}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(t) \mathbb{I}_{\{N_t - N_{\delta-\delta} > 0\}}\right]
\]
\[
\leq \mathbb{E}\left[\sum_{n=0}^{N_T+M} \sup_{u \in [0,\delta]} |W_{u+\tau_n} - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1}]}(t) \mathbb{I}_{\{N_t - N_{\delta-\delta} > 0\}}\right]
\]
\[
= \mathbb{E}[Y] \mathbb{E}\left[\sum_{n=0}^{N_T+M} \sup_{u \in [0,\delta]} |W_{u+\tau_n} - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1}]}(t) \mathbb{I}_{\{N_t - N_{\delta-\delta} > 0\}}\right].
\] (3.33)
By the Cauchy-Schwarz inequality,
\[
\mathbb{E}\left[ \sum_{n=0}^{N_T+M} |W_t - W_{\tau_n}|^q \mathbb{I}_{S_t(Z^{(M)}_t, Z^{(M)}_{\tau_n}) I_{(\tau_n, \tau_{n+1})}(t) I_{\{N_t - N_{\zeta,\delta} > 0\}} \right] \\
\leq \mathbb{E}[Y] \mathbb{E}\left[ \left( \sum_{n=0}^{N_T+M} \sup_{u \in [0,\delta]} |W_{u+\tau_n} - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1})}(t) \right)^{\frac{q}{2}} \right]^{\frac{2}{q}} \mathbb{E}\left[ I_{\{N_t - N_{\zeta,\delta} > 0\}} \right].
\] (3.34)

Observe that
\[
\mathbb{E}\left[ I_{\{N_t - N_{\zeta,\delta} > 0\}} \right] = \mathbb{P}(N_t - N_{\zeta} > 0) = 1 - \mathbb{P}(N_t - N_{\zeta} = 0) = 1 - \exp\left( - (t - \tau + \delta) \lambda \right) \leq 1 - \exp\left( - 2\lambda \delta \right) \leq 2\lambda.
\] (3.35)

Lemma 3.5 equation (3.9) gives
\[
\mathbb{E}\left[ \left( \sum_{n=0}^{N_T+M} \sup_{u \in [0,\delta]} |W_{u+\tau_n} - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1})}(t) \right)^2 \right] \leq C W_{2q} \delta^q.
\] (3.36)

Plugging (3.35) and (3.36) into (3.34) we obtain
\[
\mathbb{E}\left[ \sum_{n=0}^{N_T+M} |W_t - W_{\tau_n}|^q \mathbb{I}_{S_t(Z^{(M)}_t, Z^{(M)}_{\tau_n}) I_{(\tau_n, \tau_{n+1})}(t) I_{\{N_t - N_{\zeta,\delta} > 0\}}} \right] \\
\leq \mathbb{E}[Y] \left( C W_{2q} \delta^q \right)^{\frac{q}{2}} (2\lambda \delta)^{\frac{q}{2}} = (2C W_{2q} \lambda)^{\frac{q}{2}} \delta^q \mathbb{E}[Y].
\] (3.37)

For estimating the second summand of (3.31) note that
\[
I_{(\tau_n, \tau_{n+1})}(t) I_{\{N_t - N_{\zeta,\delta} = 0\}} = I_{(\tau_n, \tau_{n+1})}(t) I_{\{N_t - N_{\zeta,\delta} = 0\}} \sum_{m=0}^{M} I_{\{\tau_n = s_m\}},
\] since assuming \( \tau_n \neq s_m \) for all \( m \in \{0, \ldots, M\} \) implies \( \tau_n < t < \tau_{n+1} \) and \( \Delta N_{\tau_n} = 1 \), which is a contradiction to \( I_{\{N_t - N_{\zeta,\delta} = 0\}} \). This ensures,
\[
\mathbb{E}\left[ \sum_{n=0}^{N_T+M} |W_t - W_{\tau_n}|^q \mathbb{I}_{S_t(Z^{(M)}_t, Z^{(M)}_{\tau_n}) I_{(\tau_n, \tau_{n+1})}(t) I_{\{N_t - N_{\zeta,\delta} = 0\}}} \right] \\
= \mathbb{E}\left[ \sum_{n=0}^{N_T+M} \sum_{m=0}^{M} |W_t - W_{\tau_n}|^q \mathbb{I}_{S_t(Z^{(M)}_t, Z^{(M)}_{s_m}) I_{(\tau_n, \tau_{n+1})}(t) I_{\{N_t - N_{\zeta,\delta} = 0\}} I_{\{\tau_n = s_m\}}} \right] \\
= \mathbb{E}\left[ \sum_{n=0}^{N_T+M} \sum_{m=0}^{M} |W_t - W_{s_m}|^q \mathbb{I}_{S_t(Z^{(M)}_t, Z^{(M)}_{s_m})} I_{\{s_m, s_{m+1} \wedge \min\{\nu_i : i \in \mathbb{N}, \nu_i > s_m\}\}} I_{\{N_t - N_{\zeta,\delta} = 0\}} I_{\{\tau_n = s_m\}} \right] \\
= \sum_{m=0}^{M} \mathbb{E}\left[ |W_t - W_{s_m}|^q \mathbb{I}_{S_t(Z^{(M)}_t, Z^{(M)}_{s_m})} I_{\{s_m, s_{m+1} \wedge \min\{\nu_i : i \in \mathbb{N}, \nu_i > s_m\}\}} I_{\{N_t - N_{\zeta,\delta} = 0\}} \right].
\] (3.38)

Since \( t \in (s_m, s_{m+1} \wedge \min\{\nu_i > s_m\}) \), \( s_m = t \). Moreover, the sum in the above calculation has at most one summand, since the indicator functions are disjoint. Hence (3.35) can be rewritten as follows:
\[
\mathbb{E}\left[ \sum_{n=0}^{N_T+M} |W_t - W_{\tau_n}|^q \mathbb{I}_{S_t(Z^{(M)}_t, Z^{(M)}_{\tau_n}) I_{(\tau_n, \tau_{n+1})}(t) I_{\{N_t - N_{\zeta,\delta} = 0\}}} \right] \\
= \mathbb{E}\left[ |W_t - W_{\tau}|^q \mathbb{I}_{S_t(Z^{(M)}_t, Z^{(M)}_{\tau_n}) I_{(\tau_n, \tau_{n+1})}(t) I_{\{N_t - N_{\zeta,\delta} = 0\}}} \right].
\] (3.39)
Next we define
\[ W_1 = \frac{W_t - W_s}{\sqrt{t - s}}, \quad W_2 = \frac{W_t - W_{t-(t-s)}}{\sqrt{t - s}}, \quad \text{and} \quad W_3 = \frac{W_{t-(t-s)} - W_s}{\sqrt{\delta - (t-s)}}. \]

It holds that \( W_1, W_2, \) and \( W_3 \) are standard normally distributed and independent of each other. Further they are all independent of \( F_s \), since \( s \leq t - \delta \), and \( W_1 \) and \( W_2 \) are independent of \( F_{t-(t-s)}. \)

In the following we set \( \bar{c} = \max\{c_\mu, c_\sigma\}, k = \bar{c}(1 + |\zeta|)(1 + L_\delta) \), and \( M_0 \in \mathbb{N} \setminus \{1, 2\} \) such that for all \( M \geq M_0, \delta = \frac{T}{M} \) it holds that \( \delta \leq 1, 8 \ln(T/\delta)\sqrt{\delta} \leq 1, \) and \( 20\kappa \sqrt{\delta} \ln(T/\delta) \leq \frac{1}{2}. \) Now let \( M \geq M_0. \) Then equation (3.39) assures
\[
\mathbb{E} \left[ Y \sum_{n=0}^{N_T + M} |W_t - W_{\tau_n}|^q \mathbb{I}_{S_\zeta}(Z_t^{(M)}, Z_{\tau_n}^{(M)}) \mathbb{I}(\tau_n, \tau_{n+1}) (t) \mathbb{I}_{\{N_t - N_{L-\delta} = 0\}} \right]
= \mathbb{E} \left[ Y |W_t - W_{\bar{c}}|^q \mathbb{I}_{S_\zeta}(Z_t^{(M)}, Z_{\bar{c}}^{(M)}) \mathbb{I}(\bar{c} + \delta) \mathbb{I}_{\min\{\nu_i: \nu_i \geq \bar{c}\}} (t) \right]
\cdot \mathbb{I}_{\{N_t - N_{L-\delta} = 0\}} \mathbb{I}_{\{\max_{\nu_i: |W_i| > 2 \sqrt{\ln(T/\delta)}} \}}
+ \mathbb{E} \left[ Y |W_t - W_{\bar{c}}|^q \mathbb{I}_{S_\zeta}(Z_t^{(M)}, Z_{\bar{c}}^{(M)}) \mathbb{I}(\bar{c} + \delta) \mathbb{I}_{\min\{\nu_i: \nu_i \geq \bar{c}\}} (t) \right]
\cdot \mathbb{I}_{\{N_t - N_{L-\delta} = 0\}} \mathbb{I}_{\{\max_{\nu_i: |W_i| \leq 2 \sqrt{\ln(T/\delta)}} \}}. \tag{3.40}
\]

For the first summand of (3.40) we have
\[
\mathbb{E} \left[ Y |W_t - W_{\bar{c}}|^q \mathbb{I}_{S_\zeta}(Z_t^{(M)}, Z_{\bar{c}}^{(M)}) \mathbb{I}(\bar{c} + \delta) \mathbb{I}_{\min\{\nu_i: \nu_i \geq \bar{c}\}} (t) \mathbb{I}_{\{N_t - N_{L-\delta} = 0\}} \mathbb{I}_{\{\max_{\nu_i: |W_i| > 2 \sqrt{\ln(T/\delta)}} \}} \right]
\leq \mathbb{E} \left[ Y \sup_{u \in [0, \bar{c}]} |W_t - W_{\bar{c}}|^q \mathbb{I}_{\{\max_{\nu_i: |W_i| > 2 \sqrt{\ln(T/\delta)}} \}} \right]
= \mathbb{E}[Y] \mathbb{E} \left[ \sup_{u \in [0, \bar{c}]} |W_t - W_{\bar{c}}|^q \mathbb{I}_{\{\max_{\nu_i: |W_i| > 2 \sqrt{\ln(T/\delta)}} \}} \right]
\leq \mathbb{E}[Y] \mathbb{E} \left[ \sup_{u \in [0, \bar{c}]} |W_t - W_{\bar{c}}|^{2q} \right]^{\frac{1}{2}} \mathbb{E}[\max_{\nu_i: |W_i| > 2 \sqrt{\ln(T/\delta)}}]^{\frac{1}{2}}
\leq \mathbb{E}[Y] c_{W_{\bar{c}}}^{\frac{1}{2}} \delta^{\frac{q}{2}} \mathbb{P} \left( \max_{\nu_i: |W_i| > 2 \sqrt{\ln(T/\delta)}} \right)^{\frac{1}{2}}. \tag{3.41}
\]

Next we observe
\[
\mathbb{P} \left( \max_{\nu_i: |W_i| > 2 \sqrt{\ln(T/\delta)}} \right) \leq 3 \mathbb{P} (|W_1| > 2 \sqrt{\ln(T/\delta)}) = \frac{\delta^2}{2 \sqrt{2\pi} \sqrt{\ln(T/\delta)} T^2}. \tag{3.42}
\]

Plugging (3.42) into (3.41) and using \( M \geq M_0 \) we obtain
\[
\mathbb{E} \left[ Y |W_t - W_{\bar{c}}|^q \mathbb{I}_{S_\zeta}(Z_t^{(M)}, Z_{\bar{c}}^{(M)}) \mathbb{I}(\bar{c} + \delta) \mathbb{I}_{\min\{\nu_i: \nu_i \geq \bar{c}\}} (t) \right]
\cdot \mathbb{I}_{\{N_t - N_{L-\delta} = 0\}} \mathbb{I}_{\{\max_{\nu_i: |W_i| > 2 \sqrt{\ln(T/\delta)}} \}} \leq c_{W_{\bar{c}}}^{\frac{1}{2}} \left( \frac{6}{\sqrt{2\pi} 2 T^2} \right)^{\frac{1}{2}} \delta^{\frac{q}{2} + 1} \mathbb{E}[Y]. \tag{3.43}
\]

For the second summand of (3.40), define the set
\[
A = \{ (Z_t^{(M)}, Z_{\bar{c}}^{(M)}) \in S_\zeta \} \cap \{ N_t - N_{L-\delta} = 0 \} \cap \{ \max_{\nu_i: |W_i| \leq 2 \sqrt{\ln(T/\delta)}} \} \cap \{ t \in (\bar{c}, (\bar{c} + \delta) \min\{\nu_i: \nu_i \geq \bar{c}\}) \},
\]
and choose \( \omega \in A \). With this,

\[
|Z^{(M)}_\perp - \zeta| \leq |Z^{(M)}_\perp - Z^{(M)}_\perp| \leq |\bar{\mu}(Z^{(M)}_\perp - Z^{(M)}_\perp)| + |\bar{\sigma}(Z^{(M)}_\perp - Z^{(M)}_\perp)| + \left| \frac{1}{2} \bar{\sigma}(Z^{(M)}_\perp - Z^{(M)}_\perp) ((W_{t'} - W_{\perp})^2 - (t - \perp)) \right|
\]

\[
\leq c_\mu (1 + |Z^{(M)}_\perp|) \delta + c_\sigma (1 + |Z^{(M)}_\perp|) \delta |\bar{\sigma}| |\bar{W}_1| + \frac{1}{2} \|d_{\sigma}\|_{\infty} c_\sigma (1 + |Z^{(M)}_\perp|) \delta (|\bar{W}_1|^2 + 1)
\]

\[
\leq c (1 + |Z^{(M)}_\perp|) \left( \delta + \sqrt{\delta} |\bar{W}_1| + \frac{1}{2} L_{\bar{\sigma}} \delta (|\bar{W}_1|^2 + 1) \right).
\]

This, the fact that for all \( a, b \in \mathbb{R} \),

\[
(1 + |a|) \leq (1 + |a - b|)(1 + |b|),
\]

and

\[
\frac{1}{2} \sqrt{\delta} (|\bar{W}_1|^2 + 1) \leq \frac{1}{2} \sqrt{\delta} (|2 \sqrt{\ln(T/\delta)}|^2 + 1) = \frac{4 \ln(T/\delta) + 1}{2} \sqrt{\delta} \leq \frac{5 \ln(T/\delta)}{2} \sqrt{\delta} \leq 1
\]

give

\[
|Z^{(M)}_\perp - \zeta| \leq \frac{1}{2} \sqrt{\delta} (|2 \sqrt{\ln(T/\delta)}|^2 + 1) = \frac{4 \ln(T/\delta) + 1}{2} \sqrt{\delta} \leq \frac{5 \ln(T/\delta)}{2} \sqrt{\delta} \leq 1
\]

Analogously to the derivation of (3.44) we get

\[
|Z^{(M)}_\perp - Z^{(M)}_{\perp - \delta}| \leq \kappa (1 + |Z^{(M)}_\perp - \zeta|) (1 + |\bar{W}_2|) \sqrt{\delta}
\]

and

\[
|Z^{(M)}_\perp - Z^{(M)}_{\perp - (t - \perp)}| \leq \kappa (1 + |Z^{(M)}_\perp - \zeta|) (1 + |\bar{W}_2|) \sqrt{\delta}.
\]

Further it holds that

\[
|Z^{(M)}_\perp - \zeta| \leq |Z^{(M)}_\perp - Z^{(M)}_{\perp - \delta}| + |Z^{(M)}_\perp - \zeta|.
\]

For \( i \in \{1, 2, 3\} \) we estimate

\[
\kappa \sqrt{\delta} (1 + |W_i|) \leq \kappa \sqrt{\delta} (1 + 2 \sqrt{\ln(T/\delta)}) \leq \kappa \sqrt{\delta} 3 \sqrt{\ln(T/\delta)} \leq \frac{1}{2}.
\]

Combining (3.44) and (3.48) ensures

\[
|Z^{(M)}_\perp - \zeta| \leq \frac{\kappa (1 + |\bar{W}_1|) \sqrt{\delta}}{1 - \kappa (1 + |\bar{W}_1|) \sqrt{\delta}} \leq 2 \kappa (1 + |\bar{W}_1|) \sqrt{\delta}.
\]

By (3.45) and (3.48) we obtain that

\[
1 + |Z^{(M)}_\perp - \zeta| \geq 1 + |Z^{(M)}_\perp - \zeta| - |Z^{(M)}_\perp - Z^{(M)}_\perp|
\]

\[
\geq 1 + |Z^{(M)}_\perp - \zeta| - \kappa (1 + |Z^{(M)}_\perp - \zeta|) (1 + |\bar{W}_3|) \sqrt{\delta}
\]

\[
= (1 + |Z^{(M)}_\perp - \zeta|) (1 - \kappa (1 + |\bar{W}_3|) \sqrt{\delta}) \geq \frac{1}{2} (1 + |Z^{(M)}_\perp - \zeta|).
\]

Plugging (3.50) into (3.46) we get

\[
|Z^{(M)}_\perp - Z^{(M)}_{\perp - (t - \perp)}| \leq 2 \kappa (1 + |Z^{(M)}_\perp - \zeta|) (1 + |\bar{W}_2|) \sqrt{\delta}.
\]
Using \(3.47\), \(3.49\), and \(3.51\) we obtain
\[
|Z_{\mathcal{L}}^{(M)}(t_{1}) - \zeta| \leq |Z_{\mathcal{L}}^{(M)} - Z_{\mathcal{L}}^{(M)}(t_{1})| + |Z_{\mathcal{L}}^{(M)} - \zeta| \\
\leq 2\kappa (1 + |Z_{\mathcal{L}}^{(M)} - \zeta|) (1 + |\bar{W}_{2}|) \sqrt{\delta} + 2\kappa (1 + |\bar{W}_{1}|) \sqrt{\delta} \\
\leq 4\kappa \sqrt{\delta} (1 + |Z_{\mathcal{L}}^{(M)} - \zeta|) (1 + |\bar{W}_{1}| + |\bar{W}_{2}|).  
\]  
(3.52)

We further estimate
\[
4\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|) \leq 4\kappa \sqrt{\delta} (1 + 2\sqrt{\ln(T/\delta)} + 2\sqrt{\ln(T/\delta)}) \leq 20\kappa \sqrt{\delta} \sqrt{\ln(T/\delta)} \leq \frac{1}{2},
\]
(3.53)

Combining \(3.52\) and \(3.53\) we obtain
\[
|Z_{\mathcal{L}}^{(M)}(t_{1}) - \zeta| \leq \frac{4\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|)}{1 - 4\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|)} \leq 8\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|).
\]

Hence we have proven that
\[
A \subset \left\{ |Z_{\mathcal{L}}^{(M)}(t_{1}) - \zeta| \leq 8\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|) \right\}.
\]

Next we estimate the second summand of \((3.40)\). We use \(3.54\), the fact that \(Y\) is \(F_{\mathcal{L} - \delta} \subset F_{\mathcal{L} - (t - \delta)}\) measurable, \([13\), p. 33\), and the fact that \(\bar{W}_{1}\) and \(\bar{W}_{2}\) are independent of \(F_{\mathcal{L} - (t - \delta)}\) to obtain
\[
\begin{align*}
\mathbb{E} \left[ Y | \bar{W}_{t} - \bar{W}_{L} \right] & \mathbb{1}_{\mathcal{L}} \left( (Z_{\mathcal{L}}^{(M)}, Z_{\mathcal{L}}^{(M)}) \mathbb{I}_{\{\max_{i=1,2,3}(t_{i}) \leq \min_{i} \langle \nu_{i} > \mathcal{L} \rangle \} \right) (t) \\
& \leq \mathbb{E} \left[ Y | \bar{W}_{t} - \bar{W}_{L} \right] \mathbb{1}_{\{ |Z_{\mathcal{L}}^{(M)}(t) - \zeta| \leq 8\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|) \} \right] \\
& \leq \delta^{\frac{3}{2}} \mathbb{E} \left[ Y | \bar{W}_{1} \right] \mathbb{1}_{\{ |Z_{\mathcal{L}}^{(M)}(t_{1}) - \zeta| \leq 8\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|) \} | F_{\mathcal{L} - (t - \delta)} \right] \\
& = \delta^{\frac{3}{2}} \mathbb{E} \left[ Y | \bar{W}_{1} \right] \mathbb{1}_{\{ |\bar{W}_{1} - \zeta| \leq 8\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|) \} | F_{\mathcal{L} - (t - \delta)} \right] \\
& = \delta^{\frac{3}{2}} \mathbb{E} \left[ Y | \bar{W}_{1} \right] \mathbb{1}_{\{ |\bar{W}_{1} - \zeta| \leq 8\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|) \} | F_{\mathcal{L} - (t - \delta)} \right] \\
& = \delta^{\frac{3}{2}} \mathbb{E} \left[ Y | \bar{W}_{1} \right] \mathbb{1}_{\{ |\bar{W}_{1} - \zeta| \leq 8\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|) \} | F_{\mathcal{L} - (t - \delta)} \right]
\end{align*}
\]
(3.55)

Since \(\bar{W}_{1}\) and \(\bar{W}_{2}\) are independent, standard normally distributed random variables,
\[
\begin{align*}
\mathbb{E} \left[ |\bar{W}_{1}|^{q} \mathbb{1}_{\{ |\bar{W}_{1} - \zeta| \leq 8\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|) \} \right] \\
& = \frac{2}{\pi} \int_{0}^{\infty} \int_{0}^{\infty} \left| w_{1} \right|^{q} \mathbb{1}_{\{ |\bar{W}_{1} - \zeta| \leq 8\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|) \}} \exp \left( -\frac{1}{2} \left( w_{1}^{2} + w_{2}^{2} \right) \right) \, dw_{1} \, dw_{2}.
\end{align*}
\]
(3.56)

Plugging \(3.56\) into \(3.55\) we obtain
\[
\begin{align*}
\mathbb{E} \left[ Y | \bar{W}_{t} - \bar{W}_{L} \right] \mathbb{1}_{\mathcal{L}} \left( (Z_{\mathcal{L}}^{(M)}, Z_{\mathcal{L}}^{(M)}) \mathbb{I}_{\{\max_{i=1,2,3}(t_{i}) \leq \min_{i} \langle \nu_{i} > \mathcal{L} \rangle \} \right) (t) \\
& \leq \delta^{\frac{3}{2}} \frac{2^{\frac{q+1}{2}}}{\pi} \int_{\mathbb{R}^{2}} \mathbb{E} \left[ \left( \frac{w_{1} + w_{2}}{\sqrt{2}} \right)^{q} \mathbb{1}_{\{ |Z_{\mathcal{L}}^{(M)}(t) - \zeta| \leq 8\kappa \sqrt{\delta} (1 + |\bar{W}_{1}| + |\bar{W}_{2}|) \}} \right] e^{-\frac{1}{2}(w_{1}^{2} + w_{2}^{2})} \, dw_{1} \, dw_{2}.
\end{align*}
\]
(3.57)
Furthermore,
\[
\mathbb{E}
\left[
Y|W_t - W_u|^{q} \mathbb{I}_{S_{\mathcal{C}}(Z_{t}^{(M)}, Z_{t}^{(M)})}\mathbb{I}_{\{L(t) \leq \min\{\nu_i: i \in N, \nu_i > L\}\}}(t)
\right]
\cdot \mathbb{I}_{\{N_t - N_{t-1} = 0\}} \mathbb{I}_{\{\max_{s=1,2,3} |W_t| \leq 2\sqrt{T/\log(1+|w|)}\}}(t)
\leq \delta^2 \frac{2^q}{2} \int_{\mathbb{R}} \mathbb{E}_w \left[|Y|^{q} \mathbb{I}_{\{|Z_{t}^{(M)}(\mathcal{C}) - \zeta| \leq 2\sqrt{2e\sqrt{\delta}(1+|w|)}\}}\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw.
\]
(3.57)

Combining (3.31), (3.33), (3.34), (3.35), and (3.57) we obtain that there exists a constant \(\tilde{c} \in (0, \infty)\) such that for all \(M \geq M_0\),
\[
\mathbb{E}
\left[
Y \sum_{n=0}^{N_{t+1}} |W_t - W_{\tau_n}|^{q} \mathbb{I}_{S_{\mathcal{C}}(Z_{t}^{(M)}, Z_{t}^{(M)})}\mathbb{I}_{\{\tau_n, \tau_{n+1}\}}(t)
\right]
\leq \tilde{c}\left(\delta^2 + \mathbb{E}_w \left[|Y|^{q} \mathbb{I}_{\{|Z_{t}^{(M)}(\mathcal{C}) - \zeta| \leq 2\sqrt{2e\sqrt{\delta}(1+|w|)}\}}\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw \right).
\]
(3.58)

For all \(M < M_0\) similar calculations as in (3.33) and Lemma 3.5 yield
\[
\mathbb{E}
\left[
Y \sum_{n=0}^{N_{t+1}} |W_t - W_{\tau_n}|^{q} \mathbb{I}_{S_{\mathcal{C}}(Z_{t}^{(M)}, Z_{t}^{(M)})}\mathbb{I}_{\{\tau_n, \tau_{n+1}\}}(t)
\right] \leq \mathbb{E}\left[Y \sum_{n=0}^{N_{t+1}} |W_t - W_{\tau_n}|^{q} \mathbb{I}_{\{\tau_n, \tau_{n+1}\}}(t)\right]
\leq \mathbb{E}[Y] \mathbb{E}_w \delta^{\frac{q+1}{2}} \mathbb{E}_w \mathbb{E}_w \left[|Y|^{q} \mathbb{I}_{\{|Z_{t}^{(M)}(\mathcal{C}) - \zeta| \leq 2\sqrt{2e\sqrt{\delta}(1+|w|)}\}}\right] \frac{1}{\sqrt{2\pi}} e^{-\frac{w^2}{2}} dw.
\]

This and (3.58) prove the claim. \(\square\)

**Lemma 3.11.** Let Assumption 3.4 hold. Let \(q \in \mathbb{N}, \zeta \in \mathbb{R}\) with \(\delta(\zeta) \neq 0\), and
\[
S_{\mathcal{C}} = \{(x, y) \in \mathbb{R}^2 : (x - \zeta)(y - \zeta) \leq 0\}.
\]

Then there exist constants \(c_7, c_8 \in (0, \infty)\) such that for all \(M \in \mathbb{N}\), \(\delta = \frac{T}{M}\), all \(m \in \{0, \ldots, M-2\}\), all \(s \in [s_m, s_{m+1})\), and all \(\mathbb{F}_s\)-measurable, non-negative, real-valued random variables \(Y\),
\[
\int_{s_{m+\delta}}^{T} \mathbb{E} \left[\sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^{q} \mathbb{I}_{S_{\mathcal{C}}(Z_{t}^{(M)}, Z_{t}^{(M)})}\mathbb{I}_{\{\tau_n, \tau_{n+1}\}}(t)\right] dt
\leq c_7 \delta^{\frac{q+1}{2}} \left(\mathbb{E}[Y] + \mathbb{E}[Y|Z_{s_m+\delta}^{(M)} - \zeta|^2]\right)
\]
and
\[
\int_{s_{m+\delta}}^{T} \mathbb{E} \left[\sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^{q} \mathbb{I}_{S_{\mathcal{C}}(Z_{t}^{(M)}, Z_{t}^{(M)})}\mathbb{I}_{\{\tau_n, \tau_{n+1}\}}(t)\left(Z_{t+\delta}^{(M)} - \zeta\right)^2\right] dt
\leq c_8 \delta^{\frac{q+1}{2}} \left(\mathbb{E}[Y] + \mathbb{E}[Y|Z_{s_{m+\delta}}^{(M)} - \zeta|^2]\right).
\]

**Proof.** We start proving the first inequality. Lemma 3.10 yields that there exists \(c_6 \in (0, \infty)\) such that
\[
\int_{s_{m+\delta}}^{T} \mathbb{E} \left[\sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^{q} \mathbb{I}_{S_{\mathcal{C}}(Z_{t}^{(M)}, Z_{t}^{(M)})}\mathbb{I}_{\{\tau_n, \tau_{n+1}\}}(t)\right] dt
\leq c_6 \delta^{\frac{q+1}{2}} \mathbb{E}[Y] + c_6 \delta^{\frac{q}{2}} \int_{s_{m+\delta}}^{T} \mathbb{E} \left[|Y|^{q} \mathbb{I}_{\{|Z_{t+\delta}^{(M)}(\mathcal{C}) - \zeta| \leq c_6(1+|\zeta|)\sqrt{\delta}\}}\right] e^{-\frac{z^2}{2}} dz.
\]
(3.59)
Substitution \( u = s_k - (t - s_k) \) gives

\[
\int_{s_{m+2\delta}}^{T} \mathbb{E} \left[ \mathbf{1}_{\left\{ |Z_{t}^{(M)}| \leq c_6(1+|z|)\sqrt{\delta} \right\}} \right] dt = \sum_{k=m+2}^{M-1} \int_{s_k}^{s_{k+1}} \mathbb{E} \left[ \mathbf{1}_{\left\{ |Z_{t}^{(M)}| \leq c_6(1+|z|)\sqrt{\delta} \right\}} \right] dt = \sum_{k=m+2}^{M-1} \int_{s_k}^{s_{k+1}} \mathbb{E} \left[ \mathbf{1}_{\left\{ |Z_{t}^{(M)}| \leq c_6(1+|z|)\sqrt{\delta} \right\}} \right] du = \int_{s_{m+\delta}}^{T-\delta} \mathbb{E} \left[ \mathbf{1}_{\left\{ |Z_{t}^{(M)}| \leq c_6(1+|z|)\sqrt{\delta} \right\}} \right] du.
\]

By Lemma 3.3.3 we get

\[
\int_{s_{m+2\delta}}^{T} \mathbb{E} \left[ \mathbf{1}_{\left\{ |Z_{t}^{(M)}| \leq c_6(1+|z|)\sqrt{\delta} \right\}} \right] dt = \mathbb{E} \left[ Y \int_{s_{m+\delta}}^{T-\delta} \mathbb{E} \left[ \mathbf{1}_{\left\{ |Z_{t}^{(M)}| \leq c_6(1+|z|)\sqrt{\delta} \right\}} | Z_{s_{m+\delta}}^{(M)} \right] du \right].
\]

Lemma 3.3.9 assures

\[
\int_{s_{m+2\delta}}^{T} \mathbb{E} \left[ \mathbf{1}_{\left\{ |Z_{t}^{(M)}| \leq c_6(1+|z|)\sqrt{\delta} \right\}} \right] dt \leq c_5 (c_6 (1 + |z|) + 1) \sqrt{\delta} \mathbb{E} \left[ Y (1 + |Z_{s_{m+\delta}}^{(M)}|^2) \right].
\]

Plugging (3.60) into (3.59), yields

\[
\int_{s_{m+2\delta}}^{T} \mathbb{E} \left[ Y \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbf{1}_{S_\zeta}(Z_{t}^{(M)}, Z_{\tau_n}^{(M)}) \mathbf{1}_{(\tau_n, \tau_n+1)} (t) \right] dt 
\leq T c_6 \delta^\frac{2+q}{2} \mathbb{E} [Y] + c_6 c_5 \delta^\frac{2+q}{2} \mathbb{E} \left[ Y (1 + |Z_{s_{m+\delta}}^{(M)}|^2) \right] \int_{\mathbb{R}} |z|^q (c_6 (1 + |z|) + 1) e^{-\frac{z^2}{2}} dz.
\]

We observe that \( \int_{\mathbb{R}} |z|^q (c_6 (1 + |z|) + 1) e^{-\frac{z^2}{2}} dz =: \tilde{c}_1 < \infty \). Since for all \( a, b \in \mathbb{R} \), \( 1 + a^2 \leq 2(1 + (a - b)^2)(1 + b^2) \),

\[
\int_{s_{m+2\delta}}^{T} \mathbb{E} \left[ Y \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbf{1}_{S_\zeta}(Z_{t}^{(M)}, Z_{\tau_n}^{(M)}) \mathbf{1}_{(\tau_n, \tau_n+1)} (t) \right] dt 
\leq T c_6 \delta^\frac{2+q}{2} \mathbb{E} [Y] + 2 c_6 c_5 \delta^\frac{2+q}{2} \mathbb{E} \left[ Y (1 + |Z_{s_{m+\delta}}^{(M)}| - \zeta|^2)(1 + |\zeta|^2) \right] 
\leq T c_6 \delta^\frac{2+q}{2} \mathbb{E} [Y] + 2 c_6 c_5 \delta^\frac{2+q}{2} (1 + |\zeta|^2) \mathbb{E} [Y] + \mathbb{E} [Y |Z_{s_{m+\delta}}^{(M)} - \zeta|^2].
\]

This proves the first inequality.

For \( \omega \in \{ \tilde{\omega} \in \Omega : \sum_{n=0}^{\infty} \mathbf{1}_{S_\zeta}(Z_{t}^{(M)}(\tilde{\omega}), Z_{\tau_n}^{(M)}(\tilde{\omega})) \mathbf{1}_{(\tau_n, \tau_n+1)} (t) = 1 \} \) we have

\[
|Z_{s_{m+\delta}}^{(M)} - \zeta| \leq |Z_{s_{m+\delta}}^{(M)} - Z_{t}^{(M)}| + |Z_{t}^{(M)} - \zeta| \leq |Z_{s_{m+\delta}}^{(M)} - Z_{t}^{(M)}| + |Z_{t}^{(M)} - Z_{\tau_n}^{(M)}|. \tag{3.61}
\]
Since Y is $\mathbb{F}_t$-measurable, it is $\mathbb{F}_{s_m+1}$-measurable. Hence, using (3.61) and Hölder's inequality for the conditional expectation we get

$$
\mathbb{E}\left[ \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbb{I}_{S_{\delta}}(Z_t^{(M)}, Z_{\tau_n})(t) \left( Z_{L+\delta}^{(M)} - \zeta \right)^2 \right]
$$

$$
\leq \mathbb{E}\left[ \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1})}(t) \left( |Z_t^{(M)}| + |Z_{\tau_n}^{(M)}| \right)^2 \right]
$$

$$
\leq \mathbb{E}\left[ \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1})}(t) \times \left( |Z_t^{(M)}| - Z_{\tau_n}^{(M)} \right)^2 \right]
$$

$$
= \mathbb{E}\left[ \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1})}(t) \left( \frac{4}{4} \sup_{u \in [L+\delta]} \left| Z_u^{(M)} - Z_{\tau_n}^{(M)} \right| \right)^2 \right]
$$

$$
\leq \mathbb{E}\left[ \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1})}(t) \left( \frac{4}{4} \sup_{u \in [L+\delta]} \left| Z_u^{(M)} - Z_{\tau_n}^{(M)} \right| \right)^2 \right].
$$

For the first conditional expectation in (3.62), recalling that $t \geq s_m + \delta = s_{m+1}$, we get

$$
\mathbb{E}\left[ \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1})}(t) \left| \mathbb{F}_{s_{m+1}} \right. \right] = \mathbb{E}\left[ \sum_{n=N_{m+1}+m+1}^{N_{m+1}+M} |W_t - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1})}(t) \left| \mathbb{F}_{s_{m+1}} \right. \right]
$$

$$
= \mathbb{E}\left[ \sum_{n=m+1+m+1}^{N_{m+1}+M} |W_t - W_{\tau_n}|^q \mathbb{I}_{(\tau_n, \tau_{n+1})}(t) \right] \leq c_2 \mathbb{E}\left[ 1 + \left| Z_{s_{m+1}}^{(M)} - \zeta \right|^4 \right].
$$

For the second conditional expectation in (3.62), Lemma 3.8 and Lemma 3.7 equation (3.12) assure

$$
\mathbb{E}\left[ \left( \frac{4}{4} \sup_{u \in [L+\delta]} \left| Z_u^{(M)} - Z_{\tau_n}^{(M)} \right| \right)^2 \right] \leq c_2 \mathbb{E}\left[ 1 + \left| Z_{s_{m+1}}^{(M)} - \zeta \right|^4 \right].
$$

Then there exists a constant $\tilde{c}_2 \in (0, \infty)$ such that

$$
\mathbb{E}\left[ \left( \frac{4}{4} \sup_{u \in [L+\delta]} \left| Z_u^{(M)} - Z_{\tau_n}^{(M)} \right| \right)^4 \right] \leq \tilde{c}_2 \mathbb{E}\left[ 1 + \left| Z_{s_{m+1}}^{(M)} - \zeta \right|^4 \right].
$$

Plugging (3.63) and (3.64) into (3.62) we obtain

$$
\mathbb{E}\left[ \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbb{I}_{S_{\delta}}(Z_t^{(M)}, Z_{\tau_n})(t) \left( Z_{L+\delta}^{(M)} - \zeta \right)^2 \right]
$$

$$
\leq \mathbb{E}\left[ \frac{1}{2} c_{W_2q} \delta^\frac{3}{2} \left( \tilde{c}_2 \mathbb{E}\left[ 1 + \left| Z_{s_{m+1}}^{(M)} - \zeta \right|^4 \right] \right)^\frac{1}{2} \right] \leq \frac{1}{2} c_{W_2q} \tilde{c}_2^2 \delta^{\frac{3}{2} + \frac{3}{2}} \mathbb{E}\left[ Y \left( 1 + \left| Z_{s_{m+1}}^{(M)} - \zeta \right|^2 \right) \right].
$$

Hence, we obtain

$$
\int_{t_{s_{m+1}}}^{T} \mathbb{E}\left[ \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbb{I}_{S_{\delta}}(Z_t^{(M)}, Z_{\tau_n})(t) \left( Z_{L+\delta}^{(M)} - \zeta \right)^2 \right] dt
$$

$$
\leq T c_{W_2q} \tilde{c}_2^2 \delta^{\frac{3}{2} + \frac{3}{2}} \left( \mathbb{E}[Y] + \mathbb{E}[Y \left| Z_{s_{m+1}}^{(M)} - \zeta \right|^2 \right).}
Lemma 3.12. Let Assumption \([3.1]\) hold. Let \(\zeta \in \mathbb{R}\) such that \(\tilde{\sigma}(\zeta) \neq 0\) and
\[
S_\zeta = \{(x, y) \in \mathbb{R}^2 : (x - \zeta)(y - \zeta) \leq 0\}.
\]
Then for all \(p,q \in \mathbb{N}\) there exists a constant \(c_9 \in (0, \infty)\) such that for all \(M \in \mathbb{N}\), \(\delta = \frac{T}{M}\) it holds that
\[
\mathbb{E}\left[\left| \int_0^T \sum_{n=0}^{\infty} |W_t - W_{\tau_n}|^q \mathbb{1}_{S_\zeta}(Z_t^{(M)}, Z_{\tau_n}^{(M)}) \mathbb{1}_{[\tau_n, \tau_{n+1}]}(t) \, dt \right|^p\right] \leq c_9 \delta^{(q+1)\frac{p}{q}}.
\]

The proof of this lemma works analogously to the proof of \([34, equation (68)]\). The only changes are the notation due to the jump-adapted time grid and that we use Lemma 3.11 instead of \([34, Lemma 8]\).

Proposition 3.13. Let Assumption \([3.1]\) hold. Let \(\zeta \in \mathbb{R}\) such that \(\tilde{\sigma}(\zeta) \neq 0\) and
\[
S_\zeta = \{(x, y) \in \mathbb{R}^2 : (x - \zeta)(y - \zeta) \leq 0\}.
\]
Then for all \(p,q \in \mathbb{N}\) there exists a constant \(c_{10} \in (0, \infty)\) such that for all \(M \in \mathbb{N}\), \(\delta = \frac{T}{M}\) it holds that
\[
\mathbb{E}\left[\left| \int_0^T \sum_{n=0}^{\infty} |Z_t^{(M)} - Z_{\tau_n}^{(M)}|^q \mathbb{1}_{S_\zeta}(Z_t^{(M)}, Z_{\tau_n}^{(M)}) \mathbb{1}_{[\tau_n, \tau_{n+1}]}(t) \, dt \right|^p\right]^{\frac{1}{p}} \leq c_{10} \delta^{\frac{2+q}{q}}.
\]

Proof. We start calculating that there exist constants \(\tilde{c}_1, \tilde{c}_2 \in (0, \infty)\) such that
\[
\sum_{n=0}^{\infty} |Z_t^{(M)} - Z_{\tau_n}^{(M)}| \mathbb{1}_{S_\zeta}(Z_t^{(M)}, Z_{\tau_n}^{(M)}) \mathbb{1}_{[\tau_n, \tau_{n+1}]}(t)
\]
\[
\leq \sum_{n=0}^{\infty} \tilde{c}_1 (1 + |Z_{\tau_n}^{(M)}|) \left( \delta + |W_t - W_{\tau_n}| + |W_t - W_{\tau_n}|^2 \right) \mathbb{1}_{S_\zeta}(Z_t^{(M)}, Z_{\tau_n}^{(M)}) \mathbb{1}_{[\tau_n, \tau_{n+1}]}(t)
\]
\[
\leq \sum_{n=0}^{\infty} \tilde{c}_1 (1 + |Z_{\tau_n}^{(M)}|) \left( \delta + |W_t - W_{\tau_n}|^2 \right) \mathbb{1}_{[\tau_n, \tau_{n+1}]}(t)
\]
\[
+ \sum_{n=0}^{\infty} \tilde{c}_2 |W_t - W_{\tau_n}| \mathbb{1}_{S_\zeta}(Z_t^{(M)}, Z_{\tau_n}^{(M)}) \mathbb{1}_{[\tau_n, \tau_{n+1}]}(t)
\]
\[
+ \sum_{n=0}^{\infty} \tilde{c}_2 |Z_{\tau_n}^{(M)} - Z_t^{(M)}| |W_t - W_{\tau_n}| \mathbb{1}_{[\tau_n, \tau_{n+1}]}(t).
\]
Using this and Minkowski’s inequality we obtain that there exists a constant \(\tilde{c}_3 \in (0, \infty)\) such
that

\[
\mathbb{E}\left[ \left| \int_0^T \sum_{n=0}^{\infty} \left| Z_{t}^{(M)} - Z_{t_n}^{(M)} \right|^{q} \mathbb{I}_{S_{\tilde{c}_3}}(Z_{t}^{(M)}, Z_{t_n}^{(M)}) \mathbb{I}_{[\tau_n, \tau_{n+1})}(t) \right|^\frac{1}{q} \right] \leq \mathbb{E}\left[ \left| \int_0^T \sum_{n=0}^{\infty} \tilde{c}_1 \left( 1 + |Z_{t_n}^{(M)}| \right) \left( \delta + |W_{t} - W_{t_n}|^2 \right) \mathbb{I}_{[\tau_n, \tau_{n+1})}(t) \right|^{\frac{1}{q}} \right]
+ \sum_{n=0}^{\infty} \tilde{c}_2 |W_t - W_{t_n}| \mathbb{I}_{S_{\tilde{c}_3}}(Z_{t}^{(M)}, Z_{t_n}^{(M)}) \mathbb{I}_{[\tau_n, \tau_{n+1})}(t)
+ \sum_{n=0}^{\infty} \tilde{c}_2 |Z_{t_n}^{(M)} - Z_{t}^{(M)}| \left( \delta + |W_{t} - W_{t_n}|^2 \right) \mathbb{I}_{[\tau_n, \tau_{n+1})}(t) \right|^{\frac{1}{q}} \right].
\]

(3.65)

Next we consider each expectation of (3.65) separately. We start with the first one and apply Jensen’s inequality to obtain

\[
\mathbb{E}\left[ \left| \int_0^T \sum_{n=0}^{\infty} \left( 1 + |Z_{t_n}^{(M)}| \right) \left( \delta + |W_t - W_{t_n}|^2 \right) \mathbb{I}_{[\tau_n, \tau_{n+1})}(t) \right|^{\frac{1}{q}} \right]\]
\[
\leq 2^{pq-1}T^{p-1} \mathbb{E}\left[ \left| \int_0^T \sum_{n=0}^{\infty} \left( 1 + |Z_{t_n}^{(M)}| \right)^{pq} \delta^{pq} \mathbb{I}_{[\tau_n, \tau_{n+1})}(t) \right|^\frac{1}{p} \right]
\]
\[
+ 2^{pq-1}T^{p-1} \mathbb{E}\left[ \left| \int_0^T \sum_{n=0}^{\infty} \left( 1 + |Z_{t_n}^{(M)}| \right)^{pq} |W_t - W_{t_n}|^{2pq-1} \mathbb{I}_{[\tau_n, \tau_{n+1})}(t) \right|^{\frac{1}{p}} \right].
\]

(3.66)

For the first expectation in (3.66) we calculate using Lemma 3.7 equation (3.10),

\[
\mathbb{E}\left[ \left| \int_0^T \sum_{n=0}^{\infty} \left( 1 + |Z_{t_n}^{(M)}| \right)^{pq} \delta^{pq} \mathbb{I}_{[\tau_n, \tau_{n+1})}(t) \right|^{\frac{1}{p}} \right] \leq 2^{pq-1} \delta^{pq} \int_0^T 1 + \mathbb{E}\left[ \sup_{s \in [0,T]} |Z_{t_n}^{(M)}|^p \right] dt
\]
\[
\leq 2^{pq-1} \delta^{pq} T \left( 1 + c_1 \left( 1 + |\tilde{\xi}|^{pq} \right) \right).
\]

(3.67)

For the second expectation in (3.66), Lemma 3.5 and Lemma 3.7 give

\[
\mathbb{E}\left[ \left| \sum_{n=0}^{\infty} \left( 1 + |Z_{t_n}^{(M)}| \right)^{pq} |W_t - W_{t_n}|^{2pq-1} \mathbb{I}_{[\tau_n, \tau_{n+1})}(t) \right|^\frac{1}{p} \right] \leq 2^{pq-1} c_{W2pq} \delta^{pq} \int_0^T \mathbb{E}\left[ \sum_{n=0}^{\infty} \left( 1 + |Z_{t_n}^{(M)}|^{pq} \right) \mathbb{I}_{[\tau_n, \tau_{n+1})}(t) \right] dt
\]
\[
\leq 2^{pq-1} c_{W2pq} \delta^{pq} T \left( 1 + c_1 \left( 1 + |\tilde{\xi}|^{pq} \right) \right).
\]

(3.68)

Plugging (3.67) and (3.68) into (3.66) we obtain that there exists a constant \( \tilde{c}_4 \in (0, \infty) \) such that

\[
\mathbb{E}\left[ \left| \int_0^T \sum_{n=0}^{\infty} \left( 1 + |Z_{t_n}^{(M)}| \right) \left( \delta + |W_t - W_{t_n}|^2 \right) \mathbb{I}_{[\tau_n, \tau_{n+1})}(t) \right|^{q} \right] \leq \tilde{c}_4 \delta^{pq}.
\]

(3.69)
For the second expectation in (3.65), Lemma 3.12 gives
\[ E\left[ \left| \int_0^T \sum_{n=0}^{\infty} |W_t - W_{\tau_n}| I_{|\tau_n - \tau_{n+1}|}(t) \right|^q \, dt \right]^p \leq c_9 \delta^{(q+1)\frac{p}{2}}. \tag{3.70} \]

For the third expectation in (3.65) we use Jensen’s inequality, the Cauchy-Schwarz inequality, Lemma 3.12 and Lemma 3.7 to obtain that there exists a constant \( \bar{c}_3 \in (0, \infty) \) such that
\[ E\left[ \left| \int_0^T \sum_{n=0}^{\infty} |Z_{\tau_n}^{(M)} - Z_t^{(M)}| |W_t - W_{\tau_n}| I_{|\tau_n - \tau_{n+1}|}(t) \right|^q \, dt \right]^p \leq T^{p-1} E\left[ \left| \int_0^T \sum_{n=0}^{\infty} |Z_{\tau_n}^{(M)} - Z_t^{(M)}|^{pq} I_{|\tau_n - \tau_{n+1}|}(t) \right|^q \, dt \right]^p \leq T^{p-1} \int_0^T E\left[ \sum_{n=0}^{\infty} |Z_{\tau_n}^{(M)} - Z_t^{(M)}|^{pq} I_{|\tau_n - \tau_{n+1}|}(t) \right] \sup_{t \in [0, \delta]} |W_{s+\tau_i} - W_{\tau_i}| I_{|\tau_i - \tau_{i+1}|}(t) \, dt \leq T^{p-1} \int_0^T E\left[ \sum_{n=0}^{\infty} |Z_{\tau_n}^{(M)} - Z_t^{(M)}|^{pq} I_{|\tau_n - \tau_{n+1}|}(t) \right] \frac{p}{q} \, dt \leq \bar{c}_3 \delta^{pq}. \tag{3.71} \]
Combining (3.65), (3.69), (3.70), and (3.71) we obtain that there exists a constant \( c_{10} \in (0, \infty) \) such that
\[ E\left[ \left| \int_0^T N_T^{M} \sum_{n=0}^{N_T^{M} - 1} |Z_{\tau_n}^{(M)} - Z_t^{(M)}|^{pq} I_{|\tau_n - \tau_{n+1}|}(t) \, dt \right|^{\frac{p}{q}} \right] \leq \bar{c}_3 \left( \bar{c}_4 \delta^{pq} \right)^{\frac{1}{q}} + \bar{c}_3 \left( c_9 \delta^{(q+1)\frac{p}{2}} \right)^{\frac{1}{q}} + \bar{c}_3 \left( \bar{c}_5 \delta^{pq} \right)^{\frac{1}{q}} \leq c_{10} \delta^{\frac{q+1}{2}}. \]

\[ \square \]

### 3.3 Convergence result

In this section we provide the convergence rate of the jump-adapted quasi-Milstein scheme. For the proof we make use of ideas from [34].

**Theorem 3.14.** Let Assumption 3.1 hold. Then for all \( p \in [1, \infty) \) there exists a constant \( c_{11} \in (0, \infty) \) such that for all \( M \in \mathbb{N} \) and \( \delta = \frac{T}{M} \) it holds that
\[ E\left[ \sup_{t \in [0, T]} \| Z(t) - Z^{(M)}(t) \|^{\frac{p}{q}} \right] \leq c_{11} \delta^{\frac{q}{q+1}}. \]

If we additionally assume that \( m_2 = 0 \), then for all \( p \in [1, \infty) \) there exists a constant \( \hat{c}_{11} \in (0, \infty) \) such that for all \( M \in \mathbb{N} \) and \( \delta = \frac{T}{M} \) it holds that
\[ E\left[ \sup_{t \in [0, T]} \| Z(t) - Z^{(M)}(t) \|^{\frac{p}{q}} \right] \leq \hat{c}_{11} \delta. \]

**Remark 3.15.** We formulate our theorems for general \( p \in [1, \infty) \), while the lemmas we use are formulated, for simplicity, for \( p \in \mathbb{N} \). Extending the statements of the lemmas to \( p \in [1, \infty) \) is however straightforward and therefore omitted.
Proof. By Hölder’s inequality it is enough to consider \( p \in \mathbb{N}, p \geq 2 \). First denote

\[
S_{\mu} = \bigcup_{k=1}^{m_{\mu}} \{(x, y) \in \mathbb{R}^2 : (x - \zeta_k)(y - \zeta_k) \leq 0\},
\]

\[
S_{\sigma} = \bigcup_{j=1}^{m_{\sigma}} \{(x, y) \in \mathbb{R}^2 : (x - \eta_j)(y - \eta_j) \leq 0\}.
\]

For all \( s \in [0, T] \),

\[
\mathbb{E} \left[ \sup_{t \in [0, s]} |Z_t - Z_t^{(M)}|^p \right] \leq 3^{p-1} \left( \mathbb{E} \left[ \sup_{t \in [0, s]} \left| \int_0^t \sum_{n=0}^{\infty} (\bar{\mu}(Z_u) - \bar{\mu}(Z_u^{(M)})) \mathbb{I}_{\{\tau_n, \tau_{n+1}\}}(u) \ du \right|^p \right] 
\]

\[
+ \mathbb{E} \left[ \sup_{t \in [0, s]} \left| \int_0^t \sum_{n=0}^{\infty} (\bar{\sigma}(Z_u) - \bar{\sigma}(Z_u^{(M)})) - \int_0^u \bar{\sigma}(Z_u^{(M)})(Z_u^{(M)} - Z_u) \ du \right|^p \right] 
\]

\[
+ \mathbb{E} \left[ \sup_{t \in [0, s]} \left| \int_0^t (\bar{\rho}(Z_u) - \bar{\rho}(Z_u^{(M)})) \ dN_u \right|^p \right].
\]

(3.72)

For the first summand we get

\[
\mathbb{E} \left[ \sup_{t \in [0, s]} \left| \int_0^t \sum_{n=0}^{\infty} (\bar{\mu}(Z_u) - \bar{\mu}(Z_u^{(M)})) \mathbb{I}_{\{\tau_n, \tau_{n+1}\}}(u) \ du \right|^p \right] 
\]

\[
\leq 2^{p-1} \left( \mathbb{E} \left[ \sup_{t \in [0, s]} \left| \int_0^t \sum_{n=0}^{\infty} (\bar{\mu}(Z_u) - \bar{\mu}(Z_u^{(M)})) - \bar{\mu}(Z_u^{(M)})(Z_u^{(M)} - Z_u) \ du \right|^p \right] 
\]

\[
+ \mathbb{E} \left[ \sup_{t \in [0, s]} \left| \int_0^t \sum_{n=0}^{\infty} \bar{\sigma}(Z_u^{(M)})(Z_u^{(M)} - Z_u) \ du \right|^p \right] \right),
\]

(3.73)

Observe that for all \( u \in (\tau_n, \tau_{n+1}) \) by Lemma 3.1

\[
|\bar{\mu}(Z_u) - \bar{\mu}(Z_u^{(M)})| \leq |\bar{\mu}(Z_u) - \bar{\mu}(Z_u^{(M)})| + |\bar{\mu}(Z_u^{(M)}) - \bar{\mu}(Z_u^{(M)})| |Z_u^{(M)} - Z_u| \]

\[
+ |\bar{\mu}(Z_u^{(M)}) - \bar{\mu}(Z_u^{(M)}) - \bar{\mu}(Z_u^{(M)})| |Z_u^{(M)} - Z_u^{(M)}| |\mathbb{I}_{S_{\mu})(Z_u^{(M)}, Z_u)|
\]

\[
+ \left| \frac{d\bar{\mu}(Z_u^{(M)})}{d\sigma}(Z_u^{(M)})(u - \tau_n) + \frac{1}{2} \bar{\sigma}(Z_u^{(M)})(W_u - W_{\tau_n})^2 - (u - \tau_n)) \right|
\]

\[
\leq L_{\bar{\mu}}|Z_u - Z_u^{(M)}| + b_{\bar{\mu}}|Z_u^{(M)} - Z_u^{(M)}| |Z_u^{(M)} - Z_u^{(M)}| |\mathbb{I}_{S_{\mu})(Z_u^{(M)}, Z_u)|
\]

\[
+ \|d\bar{\mu}\|_{\infty} \left( c_{\bar{\mu}} + \frac{1}{2} \|d\sigma\|_{\infty} c_{\bar{\sigma}} \right) |Z_u^{(M)}| |u - \tau_n| + \frac{1}{2} \|d\bar{\mu}\|_{\infty} \|d\sigma\|_{\infty} c_{\bar{\sigma}} (1 + |Z_u^{(M)}|)|W_u - W_{\tau_n}|^2.
\]

For the first summand of (3.73) this and Jensen’s inequality ensure that there exists a constant
\[ \tilde{c}_1 \in (0, \infty) \) such that

\[
\begin{align*}
E \left[ \sup_{t \in [0, s]} \left| \int_0^t \sum_{n=0}^{\infty} (\tilde{\mu}(Z_u) - \tilde{\mu}(Z_{t_n}^{(M)}) - \tilde{\sigma}(Z_{t_n}^{(M)})d\tilde{p}(Z_{t_n}^{(M)})(W_u - W_{t_n})) \mathbb{I}_{(t_n, t_{n+1})} (u) du \right|^p \right] \\
\leq E \left[ \left( \int_0^s \sum_{n=0}^{\infty} |\tilde{\mu}(Z_u) - \tilde{\mu}(Z_{t_n}^{(M)}) - \tilde{\sigma}(Z_{t_n}^{(M)})d\tilde{p}(Z_{t_n}^{(M)})(W_u - W_{t_n})| \mathbb{I}_{(t_n, t_{n+1})} (u) du \right)^p \right] \\
\leq \tilde{c}_1 \left( E \left[ \int_0^s \sum_{n=0}^{\infty} |Z_u - Z_{t_n}^{(M)}|^p \mathbb{I}_{(t_n, t_{n+1})} (u) du \right] \\
+ E \left[ \int_0^s \sum_{n=0}^{\infty} |Z_u^{(M)} - Z_{t_n}^{(M)}|^{2p} \mathbb{I}_{(t_n, t_{n+1})} (u) du \right] \\
+ E \left[ \left( \int_0^s \sum_{n=0}^{\infty} (1 + |Z_{t_n}^{(M)}|)^p|u - \tau_n|^p \mathbb{I}_{(t_n, t_{n+1})} (u) du \right)^p \right] \\
+ E \left[ \int_0^s \sum_{n=0}^{\infty} (1 + |Z_{t_n}^{(M)}|)^p|W_u - W_{t_n}|^{2p} \mathbb{I}_{(t_n, t_{n+1})} (u) du \right] \right). 
\end{align*}
\]

Next we estimate each of the summands of (3.74) separately. For the first one we calculate

\[
E \left[ \int_0^s \sum_{n=0}^{\infty} |Z_u - Z_{t_n}^{(M)}|^p \mathbb{I}_{(t_n, t_{n+1})} (u) du \right] \leq \int_0^s E \left[ \sup_{v \in [0, u]} |Z_v - Z_{t_n}^{(M)}|^p \right] du. 
\] (3.75)

For the second summand of (3.74) we obtain using Lemma 3.7 equation (3.11) that there exists \( c_2 \in (0, \infty) \) such that

\[
E \left[ \int_0^s \sum_{n=0}^{\infty} |Z_u^{(M)} - Z_{t_n}^{(M)}|^{2p} \mathbb{I}_{(t_n, t_{n+1})} (u) du \right] \leq Tc_2 \left( 1 + |\tilde{\xi}|^{2p} \right) \delta^p. 
\] (3.76)

For the third summand of (3.74) we apply Proposition 3.13 and obtain that there exists \( c_{10} \in (0, \infty) \) such that

\[
E \left[ \left( \int_0^s \sum_{n=0}^{\infty} |Z_u^{(M)} - Z_{t_n}^{(M)}| \mathbb{I}_{S'_{\tilde{\mu}}(Z_u^{(M)}, Z_{t_n}^{(M)})} (u) du \right)^p \right] \leq c_{10} \delta^p. 
\] (3.77)

For the fourth summand of (3.74) we use Lemma 3.7 equation (3.10) to obtain

\[
E \left[ \int_0^s \sum_{n=0}^{\infty} (1 + |Z_{t_n}^{(M)}|)^p|u - \tau_n|^p \mathbb{I}_{(t_n, t_{n+1})} (u) du \right] \\
\leq 2^{p-1} \delta^p \int_0^s \left( 1 + E \left[ \sup_{v \in [0, T]} |Z_v^{(M)}|^p \right] \right) du \leq 2^{p-1}T \left( 1 + c_1 (1 + |\tilde{\xi}|^p) \right) \delta^p. 
\] (3.78)

For the fifth summand of (3.74) we use Lemma 3.5 and Lemma 3.7 equation (3.10) to get

\[
E \left[ \int_0^s \sum_{n=0}^{\infty} (1 + |Z_{t_n}^{(M)}|)^p|W_u - W_{t_n}|^{2p} \mathbb{I}_{(t_n, t_{n+1})} (u) du \right] \\
\leq 2^{p-1} c_{W2p} \delta^p \int_0^s \left( 1 + E \left[ \sup_{v \in [0, T]} |Z_v^{(M)}|^p \right] \right) du \leq 2^{p-1} c_{W2p} T \left( 1 + c_1 (1 + |\tilde{\xi}|^p) \right) \delta^p. 
\] (3.79)
Plugging (3.73), (3.76), (3.77), (3.78), and (3.79) into (3.74) we obtain that there exists $\bar{c}_2 \in (0, \infty)$ such that

$$
\begin{align*}
\mathbb{E} \left[ \sup_{t \in [0,s]} \left| \int_0^t \sum_{n=0}^{\infty} \left( \bar{\mu}(Z_u) - \bar{\mu}(Z^{(M)}_{t,n}) - \bar{\sigma}(Z^{(M)}_{t,n})d\bar{\mu}(Z^{(M)}_{t,n})(W_u - W_{\tau_n}) \right) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(u) \right|^p \right] \\
\leq \bar{c}_2 \int_0^s \mathbb{E} \left[ \sup_{v \in [0,u]} |Z_u - Z^{(M)}_u|^p \right] \, du + \bar{c}_3 \delta^p.
\end{align*}
$$

(3.80)

Next we estimate the second summand of (3.73). For this we define for all $M \in \mathbb{N}$ and $s \in [0,T]$,

$$
U_{M,s} = \int_0^s \sum_{n=0}^{\infty} \bar{\sigma}(Z^{(M)}_{\tau_n})d\bar{\mu}(Z^{(M)}_{\tau_n})(W_u - W_{\tau_n}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(u) \, du.
$$

(3.81)

It holds for all $M \in \mathbb{N}$, $m \in \{0, \ldots, M-1\}$, and $s \in [s_m, s_{m+1}]$ that

$$
U_{M,s} = U_{M,s_m} + \int_{s_m}^{s} \sum_{n=0}^{\infty} \bar{\sigma}(Z^{(M)}_{\tau_n})d\bar{\mu}(Z^{(M)}_{\tau_n})(W_u - W_{\tau_n}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(u) \, du.
$$

Now we will show that the sequence $(U_{M,s_m})_{m \in \{0, \ldots, M\}}$ is a discrete martingale with respect to the filtration $(\mathbb{F}_{s_m})_{m \in \{0, \ldots, M\}}$. To prove this we first show integrability. By Lemma 3.3 and Lemma 5.7 for all $m \in \{0, \ldots, M\}$,

$$
\begin{align*}
\mathbb{E}[|U_{M,s_m}|] &= \mathbb{E} \left[ \left| \int_0^{s_m} \sum_{n=0}^{\infty} \bar{\sigma}(Z^{(M)}_{\tau_n})d\bar{\mu}(Z^{(M)}_{\tau_n})(W_u - W_{\tau_n}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(u) \right| \right] \\
&\leq L_{\bar{\mu}}c_{\bar{\sigma}} \int_0^T \mathbb{E} \left[ \sum_{n=0}^{\infty} (1 + |Z^{(M)}_{\tau_n}|) |W_u - W_{\tau_n}| \mathbb{I}_{(\tau_n, \tau_{n+1}]}(u) \right] \, du \\
&\leq L_{\bar{\mu}}c_{\bar{\sigma}}c_{W} \delta^{\frac{1}{2}} \int_0^T \left( 1 + \mathbb{E} \left[ \sup_{t \in [0,T]} |Z^{(M)}_t| \right] \right) \, du \leq L_{\bar{\mu}}c_{\bar{\sigma}}c_{W} \delta^{\frac{1}{2}} T \left( 1 + c_1(1 + |\xi|) \right) < \infty.
\end{align*}
$$

Further it is obvious by definition that $U_{M,s_m}$ is $\mathbb{F}_{s_m}$-measurable. Hence $(U_{M,s_m})_{m \in \mathbb{N}}$ is adapted. Last we prove the martingale property. For $m \in \{0, \ldots, M-1\}$,

$$
\begin{align*}
\mathbb{E} \left[ U_{M,s_{m+1}} - U_{M,s_m} \bigg| \mathbb{F}_{s_m} \right] &= \int_{s_m}^{s_{m+1}} \mathbb{E} \left[ \sum_{n=0}^{N_{u+1}+M} \bar{\sigma}(Z^{(M)}_{\tau_n})d\bar{\mu}(Z^{(M)}_{\tau_n})(W_u - W_{\tau_n}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(u) \bigg| \mathbb{F}_{s_m} \right] \, du \\
&= \int_{s_m}^{s_{m+1}} \sum_{k=0}^{\infty} \sum_{n=0}^{k+M} \mathbb{E} \left[ \bar{\sigma}(Z^{(M)}_{\tau_n})d\bar{\mu}(Z^{(M)}_{\tau_n})(W_u - W_{\tau_n}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(u) \mathbb{I}_{\{N_u=k\}} \bigg| \mathbb{F}_{s_m} \right] \, du \\
&= \int_{s_m}^{s_{m+1}} \sum_{k=0}^{\infty} \sum_{n=0}^{k+M} \mathbb{E} \left[ \bar{\sigma}(Z^{(M)}_{\tau_n})d\bar{\mu}(Z^{(M)}_{\tau_n})(W_u - W_{\tau_n}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(u) \mathbb{I}_{\{N_u=k\}} \bigg| \mathbb{F}_{s_m} \right] \, du.
\end{align*}
$$

(3.82)

Let $\hat{\tau} = \tau_n \lor s_m$. Then it holds $\mathbb{P}$-a.s. that

$$
\begin{align*}
\mathbb{E} \left[ \bar{\sigma}(Z^{(M)}_{\tau_n})d\bar{\mu}(Z^{(M)}_{\tau_n})(W_u - W_{\tau_n}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(u) \mathbb{I}_{\{N_u=k\}} \mathbb{I}_{\{\tau_n \geq s_m\}} \bigg| \mathbb{F}_{s_m} \right] &= \mathbb{E} \left[ \mathbb{E} \left[ \bar{\sigma}(Z^{(M)}_{\tau_n})d\bar{\mu}(Z^{(M)}_{\tau_n})(W_u - W_{\tau_n}) \mathbb{I}_{(\tau_n, \tau_{n+1}]}(u) \mathbb{I}_{\{N_u=k\}} \mathbb{I}_{\{\tau_n \geq s_m\}} \bigg| \hat{\tau} \right] \bigg| \mathbb{F}_{s_m} \right].
\end{align*}
$$
Next, we estimate the expectation of each summand separately. For the first one we apply the

$$E\left[\tilde{\sigma}(Z_{n}^{(M)})d_{\hat{\mu}}(Z_{n}^{(M)})(W_{u} - W_{\tau_{n}})\mathbb{1}_{(\tau_{n},\tau_{n+1}]}(u)\mathbb{1}_{\{N_{n} = k\}}\mathbb{1}_{\{\tau_{n} \geq s_{m}\}} \big| F_{s_{m}}\right]$$

$$= E\left[\tilde{\sigma}(Z_{\hat{\tau}}^{(M)})d_{\hat{\mu}}(Z_{\hat{\tau}}^{(M)})(W_{u} - W_{\hat{\tau}_{n}})\mathbb{1}_{\{\hat{\tau} < u\}}\mathbb{1}_{\{u \leq f(\hat{\tau},(N_{\hat{\tau}_{n}} - N_{\hat{\tau}_{n}})_{\geq 0})\}}\mathbb{1}_{\{N_{\hat{\tau}_{n}} = k\}}\mathbb{1}_{\{\tau_{n} \geq s_{m}\}} \big| F_{\hat{\tau}}\right] \big| F_{s_{m}}\right]$$

By [13, p. 33] and the independence of $W$ and $N$ it holds \( F_{\hat{\tau}} \)-a.s. that

$$\mathbb{1}_{\{\hat{\tau} < u\}} E\left[(W_{u} - W_{\hat{\tau}_{n}})\mathbb{1}_{\{u \leq f(\hat{\tau},(N_{\hat{\tau}_{n}} - N_{\hat{\tau}_{n}})_{\geq 0})\}} \big| F_{\hat{\tau}}\right] = 0.$$

Finally, we combine (3.84), (3.83), and (3.82) to obtain

$$E\left[U_{M,s_{m+1}} - U_{M,s_{m}} \big| F_{s_{m}}\right] = 0.$$

Hence, it holds that \((U_{M,s_{m}})_{m \in \{0,\ldots,M\}}\) is a discrete martingale with respect to \((F_{s_{m}})_{m \in \{0,\ldots,M\}}\).

Observe that

$$\sup_{0 \leq s \leq T} |U_{M,s}|^p$$

$$\leq 2^{p-1} \left( \sup_{0 \leq s \leq T} |U_{M,\hat{\tau}_{n}}|^p + \sup_{0 \leq s \leq T} \left| \int_{s}^{\infty} \tilde{\sigma}(Z_{n}^{(M)})d_{\hat{\mu}}(Z_{n}^{(M)})(W_{u} - W_{\tau_{n}})\mathbb{1}_{(\tau_{n},\tau_{n+1}]}(u) \, du \right|^p \right)$$

$$\leq 2^{p-1} \left( \max_{m = 0,\ldots,M} |U_{M,s_{m}}|^p + \max_{m = 0,\ldots,M-1} \left( \int_{s_{m}}^{s_{m+1}} \sum_{n=0}^{\infty} \tilde{\sigma}(Z_{n}^{(M)})d_{\hat{\mu}}(Z_{n}^{(M)})(W_{u} - W_{\tau_{n}})\mathbb{1}_{(\tau_{n},\tau_{n+1}]}(u) \, du \right)^p \right).$$

Next we estimate the expectation of each summand separately. For the first one we apply the
discrete Burkholder-Davis-Gundy inequality and Jensen’s inequality to obtain that there exists a constant \(c_{4} \in (0, \infty)\) such that

$$E\left[\max_{m = 0,\ldots,M} |U_{M,s_{m}}|^p \right] \leq c_{4} E\left[ \left( \sum_{k=0}^{M-1} |U_{M,s_{m+1}} - U_{M,s_{m}}|^2 \right)^{\frac{p}{2}} \right]$$

$$\leq c_{4} \left[ \left( \sum_{k=0}^{M-1} \delta \int_{s_{m}}^{s_{m+1}} \sum_{n=0}^{\infty} \tilde{\sigma}(Z_{n}^{(M)})d_{\hat{\mu}}(Z_{n}^{(M)})(W_{u} - W_{\tau_{n}})\mathbb{1}_{(\tau_{n},\tau_{n+1}]}(u)^2 \, du \right)^{\frac{p}{2}} \right]$$

$$\leq c_{4} \delta^{\frac{p}{2}} \left( \sum_{k=0}^{M-1} \int_{s_{m}}^{s_{m+1}} \sum_{n=0}^{\infty} |W_{u} - W_{\tau_{n}}|^2 \mathbb{1}_{(\tau_{n},\tau_{n+1}]}(u) \, du \right)^{\frac{p}{2}}.$$

We define \(c_{5} = c_{4}\delta^{\frac{p}{2}}L_{\mu}^{2}\) and apply the Cauchy-Schwarz inequality, the Minkowski inequality,
Jensen’s inequality, and Lemma 3.7, 3.10) to obtain
\[
\mathbb{E}
\left[
\max_{m=0,\ldots,M} |U_{M,m}|^p
\right]
\leq \tilde{c}_5 \frac{\delta^2}{2} \mathbb{E}
\left[
\sup_{t \in [0,T]}
(1 + |Z_t^{(M)}|)^{2p}
\right]^{\frac{1}{2}}
\left[
\left(\sum_{k=0}^{M-1} \sum_{n=0}^{\infty} W_{tn} - W_{tn}^2 \mathbb{I}_{(\tau_n,\tau_{n+1})}(u) \, du \right)^p
\right]^{\frac{1}{2}}
\leq 2^{2p-\frac{1}{2}} \tilde{c}_5 \delta^{p-\frac{1}{2}} \left(1 + c_1 \left(1 + |\tilde{c}|^{2p}\right)^{\frac{1}{2}} \left(\sum_{k=0}^{M-1} \left(\int_{s_m} \mathbb{E}
\left[
\sum_{n=0}^{\infty} |W_{tn} - W_{tn}^2 \mathbb{I}_{(\tau_n,\tau_{n+1})}(u) | \, du \right)^p
\right)^{\frac{1}{2}} \right)
\right).
\]

Next we apply Lemma 3.3 with \(p = 0\) and \(q = 2p\) to obtain
\[
\mathbb{E}
\left[
\max_{m=0,\ldots,M} |U_{M,m}|^p
\right]
\leq 4^p \tilde{c}_5 \delta^{p-\frac{1}{2}} \left(1 + c_1 \left(1 + |\tilde{c}|^{2p}\right)^{\frac{1}{2}} \left(\sum_{k=0}^{M-1} \left(\int_{s_m} \mathbb{E}
\left[
\sum_{n=0}^{\infty} |W_{tn} - W_{tn}^2 \mathbb{I}_{(\tau_n,\tau_{n+1})}(u) | \, du \right)^p
\right)^{\frac{1}{2}} \right)
\right)

\text{(3.86)}
\]

For the expectation of the second summand of (3.85), Jensen’s inequality, the Cauchy-Schwarz inequality, Lemma 3.7, (3.10), and Lemma 3.3 ensure
\[
\mathbb{E}
\left[
\max_{m=0,\ldots,M-1} \left(\sum_{n=0}^{\infty} \tilde{\sigma}(Z_{\tau_n}^{(M)})d_\mu(Z_{\tau_n}^{(M)}) (W_{tn} - W_{tn}) \mathbb{I}_{(\tau_n,\tau_{n+1})}(u) \, du \right)^p
\right]
\leq \delta^{p-1} \mathbb{E}
\left[
\max_{m=0,\ldots,M-1} \left(\sum_{n=0}^{\infty} \tilde{\sigma}(Z_{\tau_n}^{(M)})d_\mu(Z_{\tau_n}^{(M)}) (W_{tn} - W_{tn}) \mathbb{I}_{(\tau_n,\tau_{n+1})}(u) \, du \right)^p
\right]
\leq \delta^{p-1} \epsilon^2 \mu \mathbb{E}
\left[
\sup_{t \in [0,T]} (1 + |Z_t^{(M)}|)^p \max_{m=0,\ldots,M-1} \int_{s_m} \mathbb{E}
\left[
\sum_{n=0}^{\infty} |W_{tn} - W_{tn}^2 \mathbb{I}_{(\tau_n,\tau_{n+1})}(u) | \, du \right)^p
\right]
\leq \delta^{p-1} \epsilon^2 \mu \mathbb{E}
\left[
\sup_{t \in [0,T]} (1 + |Z_t^{(M)}|)^{2p}
\right]^{\frac{1}{2}}
\text{ (3.87)}
\]

Combining (3.85), (3.86), and (3.87) we obtain that there exists a constant \(\tilde{c}_6 \in (0, \infty)\) such that
\[
\mathbb{E}
\left[
\sup_{0 \leq s \leq T} |U_{M,s}|^p
\right] \leq \tilde{c}_6 \delta^p.
\text{(3.88)}
\]
For estimating the second summand of (3.72), observe that for $u \in (\tau_n, \tau_{n+1})$ by Lemma 3.1

\[
\begin{align*}
&\left|\sigma(Z_u) - \sigma(Z_{\tau_n}^{(M)}) - \int_{\tau_n}^{u} \sigma(Z_{\tau_n}^{(M)}) d\sigma(Z_{\tau_n}^{(M)}) dW_v \right| \\
&\leq \left|\sigma(Z_u) - \sigma(Z_{\tau_n}^{(M)}) + \sigma(Z_{u}^{(M)}) - \sigma(Z_{\tau_n}^{(M)}) - d\sigma(Z_{\tau_n}^{(M)})(Z_{u}^{(M)} - Z_{\tau_n}^{(M)})\right| 1_{S_{\sigma}^{(M)}}(Z_{u}^{(M)}, Z_{\tau_n}^{(M)}) \\
&\quad + \left|\sigma(Z_{\tau_n}^{(M)}) - \sigma(Z_{\tau_n}^{(M)}) - d\sigma(Z_{\tau_n}^{(M)})(Z_{u}^{(M)} - Z_{\tau_n}^{(M)})\right| 1_{S_{\sigma}^{(M)}}(Z_{\tau_n}^{(M)}, Z_{\tau_n}^{(M)}) \\
&\quad + \left|d\sigma(Z_{\tau_n}^{(M)}) \left(\tilde{\mu}(Z_{\tau_n}^{(M)}) (u - \tau_n) + \frac{1}{2} \sigma(Z_{\tau_n}^{(M)}) d\sigma(Z_{\tau_n}^{(M)}) ((W_u - W_{\tau_n})^2 - (u - \tau_n))\right)\right| \\
&\leq L_{\sigma} |Z_u - Z_{\tau_n}^{(M)}| + b_{\sigma} |Z_{\tau_n}^{(M)} - Z_{\tau_n}^{(M)}|^2 + (L_{\sigma} + \|\sigma\|_{\infty}) |Z_{u}^{(M)} - Z_{\tau_n}^{(M)}| 1_{S_{\sigma}^{(M)}}(Z_{\tau_n}^{(M)}, Z_{\tau_n}^{(M)}) \\
&\quad + \|d\sigma\|_{\infty} (c_{\mu} + \frac{1}{2} \|d\sigma\|_{\infty} c_{\sigma})(1 + |Z_{\tau_n}^{(M)}|)(u - \tau_n) + \frac{1}{2} \|d\sigma\|_{\infty} c_{\sigma}(1 + |Z_{\tau_n}^{(M)}|)|W_u - W_{\tau_n}|^2.
\end{align*}
\]

Using this, the Burkholder-Davis-Gundy inequality, and Jensen’s inequality we obtain for the second summand of (3.72) that there exist constants $\tilde{c}_{7}, \tilde{c}_{8} \in (0, \infty)$ such that

\[
\begin{align*}
&\mathbb{E}\left[ \sup_{t \in [0,u]} \left| \int_{0}^{t} \sum_{n=0}^{\infty} \left(\sigma(Z_u) - \sigma(Z_{\tau_n}^{(M)}) - \int_{\tau_n}^{u} \sigma(Z_{\tau_n}^{(M)}) d\sigma(Z_{\tau_n}^{(M)}) dW_v \right) 1_{(\tau_n, \tau_{n+1})}(u) dW_u \right|^{P} \right] \\
&\leq \tilde{c}_{7} \mathbb{E}\left[ \left( \int_{0}^{u} \sum_{n=0}^{\infty} \left|\sigma(Z_u) - \sigma(Z_{\tau_n}^{(M)}) - \int_{\tau_n}^{u} \sigma(Z_{\tau_n}^{(M)}) d\sigma(Z_{\tau_n}^{(M)}) dW_v \right| 1_{(\tau_n, \tau_{n+1})}(u) du \right)^{\frac{P}{2}} \right] \\
&\leq \tilde{c}_{8} \left( \int_{0}^{u} \sum_{n=0}^{\infty} |Z_u - Z_{\tau_n}^{(M)}| 1_{(\tau_n, \tau_{n+1})}(u) du \right) \\
&\quad + \mathbb{E}\left[ \int_{0}^{u} \sum_{n=0}^{\infty} \left|Z_{u}^{(M)} - Z_{\tau_n}^{(M)}\right|^{2P} 1_{(\tau_n, \tau_{n+1})}(u) du \right] \\
&\quad + \mathbb{E}\left[ \left( \int_{0}^{u} \sum_{n=0}^{\infty} \left|Z_{u}^{(M)} - Z_{\tau_n}^{(M)}\right|^{2P} 1_{S_{\sigma}^{(M)}}(Z_{\tau_n}^{(M)}, Z_{\tau_n}^{(M)}) 1_{(\tau_n, \tau_{n+1})}(u) du \right)^{\frac{P}{2}} \right] \\
&\quad + \mathbb{E}\left[ \int_{0}^{u} \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}|)^{P} |u - \tau_n|^{P} 1_{(\tau_n, \tau_{n+1})}(u) du \right] \\
&\quad + \mathbb{E}\left[ \int_{0}^{u} \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}|)^{P} |W_u - W_{\tau_n}|^{2P} 1_{(\tau_n, \tau_{n+1})}(u) du \right].
\end{align*}
\]

For the third summand of (3.89), Proposition 3.13 yields

\[
\begin{align*}
&\mathbb{E}\left[ \left( \int_{0}^{u} \sum_{n=0}^{\infty} \left|Z_{u}^{(M)} - Z_{\tau_n}^{(M)}\right|^{2P} 1_{S_{\sigma}^{(M)}}(Z_{\tau_n}^{(M)}, Z_{\tau_n}^{(M)}) 1_{(\tau_n, \tau_{n+1})}(u) du \right)^{\frac{P}{2}} \right] \leq c_{10} \delta^{\frac{2}{2P}}.
\end{align*}
\]

Plugging this, (3.75), (3.76), (3.78), and (3.79) into (3.89) that there exists a constant $\tilde{c}_{9} \in (0, \infty)$ such that

\[
\begin{align*}
&\mathbb{E}\left[ \sup_{t \in [0,u]} \left| \int_{0}^{t} \sum_{n=0}^{\infty} \left(\sigma(Z_u) - \sigma(Z_{\tau_n}^{(M)}) - \int_{\tau_n}^{u} \sigma(Z_{\tau_n}^{(M)}) d\sigma(Z_{\tau_n}^{(M)}) dW_v \right) 1_{(\tau_n, \tau_{n+1})}(u) dW_u \right|^{P} \right] \\
&\leq \tilde{c}_{9} \int_{0}^{u} \mathbb{E}\left[ \sup_{v \in [0,u]} \left|Z_u - Z_{u}^{(M)}\right|^{P} \right] du + \tilde{c}_{9} \delta^{\frac{2}{2P}}.
\end{align*}
\]

For the third summand of (3.72), Lemma 3.4 ensures

\[
\begin{align*}
&\mathbb{E}\left[ \sup_{t \in [0,u]} \left| \int_{0}^{t} \left(\tilde{\rho}(Z_{u-}) - \tilde{\rho}(Z_{u}^{(M)})\right) dN_u \right|^{P} \right] \leq \tilde{c} L_{p}^{u} \int_{0}^{u} \mathbb{E}\left[ \sup_{v \in [0,u]} \left|Z_v - Z_{v}^{(M)}\right|^{P} \right] du.
\end{align*}
\]

(3.91)
Combining (3.72), (3.73), (3.80), (3.81), (3.88), (3.90), and (3.91) we obtain that there exists a constant \( \tilde{c}_{10} \in (0, \infty) \) such that

\[
E \left[ \sup_{t \in [0,s]} |Z_t - Z_t^{(M)}|^p \right] \leq \tilde{c}_{10} \left( \int_0^s E \left[ \sup_{v \in [0,u]} |Z_v - Z_v^{(M)}|^p \right] du + \delta^{\frac{3p}{4}} \right).
\]

Note that \( E \left[ \sup_{v \in [0,T]} |Z_v - Z_v^{(M)}|^p \right] < \infty \) and that \( s \mapsto E \left[ \sup_{v \in [0,s]} |Z_v - Z_v^{(M)}|^p \right] \) is a Borel measurable mapping, because it is monotonically increasing. Hence we apply Gronwall’s inequality and obtain that there exists a constant \( c_{11} \in (0, \infty) \) such that

\[
E \left[ \sup_{t \in [0,T]} |Z_t - Z_t^{(M)}|^p \right] \leq c_{11} \delta^{\frac{3p}{4}}.
\]

This proves the first statement. For the second statement we assume \( S_\sigma = \emptyset \). This improves the estimate in (3.90). Hence, the claim follows from analogue calculations. \( \Box \)

4 Convergence of the transformation-based jump-adapted quasi-Milstein scheme

To get from SDE (1.1) satisfying Assumption (2.1) to SDE (3.1) satisfying Assumption (3.1) we use a certain transformation. It is based on the transformation from [28], we slightly modify it exactly as in [34]. The function \( G \) is constructed such that the coefficients of SDE (1.1) are adjusted locally around the potential discontinuities of the drift. For the convenience of the reader we recall the definition and its properties.

Let \( \alpha_1, \ldots, \alpha_m \) be defined for all \( i \in \{1, \ldots, m\} \) by

\[
\alpha_i = \frac{\mu(\zeta_i^-) - \mu(\zeta_i^+)}{2\sigma^2(\zeta_i)}.
\]

Set

\[
\varphi := \min \left\{ \min_{1 \leq i \leq m} \frac{1}{\min_{1 \leq i \leq m} \frac{\zeta_{i+1} - \zeta_i}{2}}, \min_{1 \leq i \leq m-1} \frac{\zeta_{i+1} - \zeta_i}{2} \right\},
\]

using the conventions that \( \min \emptyset = \infty \) and \( \frac{1}{0} = \infty \), fix \( \nu \in (0, \varphi) \) and define the bump function

\[
\phi(u) = \begin{cases} 
(1 - u^2)^4 & \text{if } |u| \leq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

With this we define the transformation function \( G : \mathbb{R} \to \mathbb{R} \) for all \( x \in \mathbb{R} \) by

\[
G(x) = x + \sum_{i=1}^k \alpha_i \phi \left( \frac{x - \zeta_i}{\nu} \right) (x - \zeta_i)|x - \zeta_i|.
\]  

(4.1)

The following lemma provides properties of the transformation \( G \).
Lemma 4.1 (\cite[Lemma 1]{34}). The function $G$ satisfies the following properties.

(i) The function $G$ is differentiable on $\mathbb{R}$.

(ii) The derivative $G'$ is Lipschitz continuous.

(iii) The derivative satisfies $G'((\zeta_i) = 1$ for all $i \in \{1, \ldots, m\}$.

(iv) The derivative satisfies $\inf_{x \in \mathbb{R}} G'(x) > 0$ and there exists a constant $c \in (0, \infty)$ such that for all $x \in \mathbb{R}$ with $|x| > c$, $G'(x) = 1$.

(v) The function $G$ has a global inverse $G^{-1} : \mathbb{R} \to \mathbb{R}$ which is Lipschitz continuous.

(vi) The function $G'' : \bigcup_{i=1}^{m} (\zeta_{i-1}, \zeta_i) \to \mathbb{R}$ is well defined. In the following we extend this mapping to $G'' : \mathbb{R} \to \mathbb{R}$ by defining

\[
G''(\zeta_i) = 2\alpha_i + 2\frac{\mu(\zeta_i) - \mu(\zeta_i)}{\sigma^2(\zeta_i)}, \quad i \in \{1, \ldots, m\}.
\]

Now we define the process $Z : [0, T] \times \Omega \to \mathbb{R}$ by $Z = G(X)$. Itô’s formula shows that $Z$ satisfies

\[
dZ_t = \tilde{\mu}(Z_t) \, dt + \tilde{\sigma}(Z_t) \, dW_t + \tilde{\rho}(Z_t) \, dN_t, \quad t \in [0, T], \quad Z_0 = \tilde{\xi},
\]

where

\[
\tilde{\mu} = (G' \cdot \mu + \frac{1}{2} G'' \cdot \sigma^2) \circ G^{-1},
\]

\[
\tilde{\sigma} = (G' \cdot \sigma) \circ G^{-1},
\]

\[
\tilde{\rho} = G(G^{-1} + \rho(G^{-1})) - \text{id},
\]

see \cite[Theorem 3.1]{50}.

Lemma 4.2. Assume the coefficients $\mu$, $\sigma$, and $\rho$ satisfy Assumption 2.1. Then $\tilde{\mu}$, $\tilde{\sigma}$, and $\tilde{\rho}$ from (4.2) satisfy Assumption 3.1.

For $\tilde{\mu}$ and $\tilde{\sigma}$ this result is already proven in \cite[Lemma 2]{34}. For $\tilde{\rho}$ the proof works analogously to the respective part in \cite[proof of Theorem 3.1]{50}. Further we obtain the following lemma similar to \cite[Lemma 3]{34} and \cite[Theorem 3.1]{50}.

Lemma 4.3. Let Assumption 2.1 hold. Then the process $Z$ is the unique solution of SDE (3.1) with $\tilde{\mu}$, $\tilde{\sigma}$, and $\tilde{\rho}$ defined as in Lemma 4.2.

Proof. As in the proof of \cite[Theorem 3.1]{50} we can apply the Meyer-Itô formula of \cite[p. 221, Theorem 7]{46} to derive that $Z$ is a solution of SDE (3.1). Because by Lemma 4.2 $\tilde{\mu}$, $\tilde{\sigma}$, and $\tilde{\rho}$ are Lipschitz continuous, we know by \cite[p. 255, Theorem 6]{46} that the solution of the SDE is unique. \hfill \Box

Now we have all tools at hand to prove that there exists an approximation scheme with strong rate of convergence $3/4$. This scheme is the transformation-based jump-adapted quasi-Milstein scheme.
Theorem 4.4. Let Assumption 2.1 hold. Let $p \in [1, \infty)$, $G$ be the transformation defined in (1.1), let $Z : [0, T] \times \Omega \to \mathbb{R}$ be defined by $Z = G(X)$, and let $Z^{(M)}$ be the jump-adapted quasi-Milstein scheme for $Z$ as introduced in (3.3) and (3.6). Then there exists a constant $c \in (0, \infty)$ such that for all $M \in \mathbb{N}$ and $\delta = \frac{T}{M}$ it holds that

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t - G^{-1}(Z_t^{(M)})|^p \right]^{\frac{1}{p}} \leq c \delta^\frac{3}{4}.
$$

Proof. Using $X = G^{-1}(Z)$, the Lipschitz continuity of $G^{-1}$ (Lemma 4.1 (v)), the fact that the coefficients of $Z$ satisfy Assumption 3.1 (Lemma 4.3), and Theorem 3.14 we obtain that there exists a constant $c \in (0, \infty)$ such that

$$
\mathbb{E} \left[ \sup_{t \in [0, T]} |X_t - G^{-1}(Z_t^{(M)})|^p \right]^{\frac{1}{p}} \leq \mathbb{E} \left[ \sup_{t \in [0, T]} |G^{-1}(Z_t) - G^{-1}(Z_t^{(M)})|^p \right]^{\frac{1}{p}} 
\leq L_{G^{-1}} \mathbb{E} \left[ \sup_{t \in [0, T]} |Z_t - Z_t^{(M)}|^p \right]^{\frac{1}{p}} \leq L_{G^{-1}} c_1 \delta^\frac{3}{4} \leq c \delta^\frac{3}{4}.
$$

\[\square\]

Acknowledgements

V. Schwarz and M. Szölgyenyi are supported by the Austrian Science Fund (FWF): DOC 78. We would like to thank Thomas Müller-Gronbach, who at the Bedlewo workshop "Numerical analysis and applications of SDEs" pointed out to us that under Assumption [3.3] in the special case of $m_\sigma = 0$ we could possibly prove convergence order 1, which was successful and improved Theorem 3.14.

A Additional proofs

Proof of Lemma 3.3 We recall (3.3) to obtain

$$
\tau_{n+1} = (\tau_n + \inf\{s \geq 0 : N_{\tau_n+s} - N_{\tau_n} > 0\}) \wedge \min\{s_m : m \in \{0, \ldots, M\}, s_m > \tau_n\} \wedge T.
$$

Hence, $\tau_{n+1}$ can be expressed as a measurable function of $\tau_n$ and $(N_{\tau_n+s} - N_{\tau_n})_{s \geq 0}$. We denote this function by $f$ and compute

$$
\mathbb{E} \left[ (1 + |Z_{\tau_n}^{(M)}|^p) |W_u - W_{\tau_n}|^q \mathbbm{1}_{(\tau_n, \tau_n+1)}(u) \mathbbm{1}_{\{N_u = k\}} \right] 
= \mathbb{E} \left[ (1 + |Z_{\tau_n}^{(M)}|^p) |W_u - W_{\tau_n}|^q \mathbbm{1}_{(\tau_n, \tau_n+1)}(u) \mathbbm{1}_{\{N_u = k\}} \right] 
\leq \mathbb{E} \left[ (1 + |Z_{\tau_n}^{(M)}|^p) \sup_{s \in [0, \delta]} |W_{s+\tau_n} - W_{\tau_n}|^q \mathbbm{1}_{(\tau_n, \tau_n+1)}(u) \mathbbm{1}_{\{f(\tau_n, (N_{s+\tau_n} - N_{\tau_n})_{s \geq 0}) \geq u\}} \mathbbm{1}_{\{N_u = k\}} \right].
$$

Next we use the conditional expectation and [13, p. 33] to calculate

$$
\mathbb{E} \left[ (1 + |Z_{\tau_n}^{(M)}|^p) |W_u - W_{\tau_n}|^q \mathbbm{1}_{(\tau_n, \tau_n+1)}(u) \mathbbm{1}_{\{N_u = k\}} \right] 
\leq \mathbb{E} \left[ (1 + |Z_{\tau_n}^{(M)}|^p) \sup_{s \in [0, \delta]} |W_{s+\tau_n} - W_{\tau_n}|^q \mathbbm{1}_{(\tau_n, \tau_n+1)}(u) \mathbbm{1}_{\{f(\tau_n, (N_{s+\tau_n} - N_{\tau_n})_{s \geq 0}) \geq u\}} \mathbbm{1}_{\{N_u = k\}} |F_{\tau_n}| \right] 
= \mathbb{E} \left[ (1 + |Z_{\tau_n}^{(M)}|^p) \mathbbm{1}_{(\tau_n, \tau_n+1)}(u) \mathbbm{1}_{\{N_u = k\}} \mathbb{E} \left[ \sup_{s \in [0, \delta]} |W_{s+\tau_n} - W_{\tau_n}|^q \mathbbm{1}_{(\tau_n, \tau_n+1)}(u) \mathbbm{1}_{\{f(\tau_n, (N_{s+\tau_n} - N_{\tau_n})_{s \geq 0}) \geq u\}} \mathbbm{1}_{\{N_u = k\}} |F_{\tau_n}| \right] \right] 
= \mathbb{E} \left[ (1 + |Z_{\tau_n}^{(M)}|^p) \mathbbm{1}_{(\tau_n, \tau_n+1)}(u) \mathbbm{1}_{\{N_u = k\}} \mathbb{E} \left[ \sup_{s \in [0, \delta]} |W_{s+\tau_n} - W_{\tau_n}|^q \mathbbm{1}_{(\tau_n, \tau_n+1)}(u) \mathbbm{1}_{\{f(\tau_n, (N_{s+\tau_n} - N_{\tau_n})_{s \geq 0}) \geq u\}} \mathbbm{1}_{\{N_u = k\}} |F_{\tau_n}| \right] \right].
$$

32
By Lemma 3.2 it holds that \((W_{T_n+s} - W_{T_n})_{s \geq 0}\) and \((N_{T_n+s} - N_{T_n})_{s \geq 0}\) are independent of \(\mathbb{F}_{T_n}\) and that \((W_{T_n+s} - W_{T_n})_{s \geq 0}\) is independent of \((N_{T_n+s} - N_{T_n})_{s \geq 0}\). Using this, Lemma 3.3 and [13, p. 33] we obtain

\[
E\left[(1 + Z_{T_n}^{(M)} | u - W_{T_n}|^q 1_{\{T_n < u\} \cap (u \in \mathbb{F}_{T_n})}\right]_{z = T_n}
\]

\[
= cW_q \delta^2 \sum_{n=0}^{\infty} (1 + Z_{T_n}^{(M)} | u - W_{T_n}|^q 1_{\{T_n < u\} \cap (u \in \mathbb{F}_{T_n})}\right]_{z = T_n}
\]

\[
= cW_q \delta^2 \sum_{n=0}^{\infty} (1 + Z_{T_n}^{(M)} | u - W_{T_n}|^q 1_{\{T_n < u\} \cap (u \in \mathbb{F}_{T_n})}\right]_{z = T_n}
\]

which proves the first statement.

Using (A.1) we get

\[
E \left[ \sum_{n=0}^{N_T + M} (1 + Z_{T_n}^{(M)} | u - W_{T_n}|^q 1_{\{T_n < u\} \cap (u \in \mathbb{F}_{T_n})}\right]
\]

\[
= E \left[ \sum_{k=0}^{\infty} (1 + Z_{T_n}^{(M)} | u - W_{T_n}|^q 1_{\{T_n < u\} \cap (u \in \mathbb{F}_{T_n})}\right]
\]

\[
= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} E \left[ (1 + Z_{T_n}^{(M)} | u - W_{T_n}|^q 1_{\{T_n < u\} \cap (u \in \mathbb{F}_{T_n})}\right]
\]

This proves the second statement.

**Proof of Lemma 3.6** Observe that

\[
E \left[ \sum_{n=0}^{N_T + M} (1 + Z_{T_n}^{(M)} | u - W_{T_n}|^q 1_{\{T_n < u\} \cap (u \in \mathbb{F}_{T_n})}\right]
\]

\[
= \lambda T + M + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} E \left[ Z_{T_n}^{(M)} | u - W_{T_n}|^q 1_{\{T_n < u\} \cap (u \in \mathbb{F}_{T_n})}\right]
\]

Using (A.2) we get that for all \(n, k \in \mathbb{N}\) there exist constants \(\tilde{c}_1, \tilde{c}_2 \in (0, \infty)\) such that

\[
E \left[ Z_{T_n}^{(M)} | 1_{\{N_T = k\}} \right] \leq 2^{p-1} \left( E \left[ Z_{T_n}^{(M)} | 1_{\{N_T = k\}} \right] + E \left[ \rho Z_{T_n}^{(M)} | 1_{\{N_T = k\}} | N_T - N_{T_n-1} | 1_{\{N_T = k\}} \right] \right)
\]

\[
\leq 2^{p-1} \left( E \left[ Z_{T_n}^{(M)} | 1_{\{N_T = k\}} \right] + 2^{p-1} \tilde{c}_1 \left( E \left[ Z_{T_n}^{(M)} | 1_{\{N_T = k\}} \right] + E \left[ \rho Z_{T_n}^{(M)} | 1_{\{N_T = k\}} \right] \right) \right)
\]

\[
\leq \tilde{c}_1 \left( E \left[ Z_{T_n}^{(M)} | 1_{\{N_T = k\}} \right] + \tilde{c}_2 \left( E \left[ Z_{T_n}^{(M)} | 1_{\{N_T = k\}} \right] \right) \right)
\]

(A.3)
where we used that \( N_{\tau_n} - N_{\tau_{n-1}} \in \{0, 1\} \). Using (3.3) we obtain for all \( k, n \in \mathbb{N} \) that

\[
E\left[ |Z_{\tau_n}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right] \\
\leq 4^{p-1}\left( E\left[ |Z_{\tau_n-1}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right] + E\left[ |\tilde{\mu}(Z_{\tau_n-1})|^{p} \mathbb{1}_{\{N_T = k\}} \right] \right) \\
+ E\left[ \left| \tilde{\sigma}(Z_{\tau_n-1}) (W_{\tau_n} - W_{\tau_{n-1}}) \right|^p \mathbb{1}_{\{N_T = k\}} \right] \\
+ E\left[ \left( \frac{1}{2} \tilde{\sigma}(Z_{\tau_n-1}) d_{\tilde{\sigma}}(Z_{\tau_n-1}) \right) (W_{\tau_n} - W_{\tau_{n-1}})^2 \right] \\
\leq 4^{p-1}\left( E\left[ |Z_{\tau_n-1}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right] + E\left[ (1 + |\tilde{\mu}|^p) \mathbb{1}_{\{N_T = k\}} \right] \right) \\
+ 2^{p-1}E\left[ \left| \tilde{\sigma}(Z_{\tau_n-1}) \right|^p \mathbb{1}_{\{N_T = k\}} \right] \\
+ 2^{p-2}E\left[ \left| \tilde{\sigma}(Z_{\tau_n-1}) \right|^p \mathbb{1}_{\{N_T = k\}} \right] \\
+ 2^{p-2}E\left[ \left| \tilde{\sigma}(Z_{\tau_n-1}) \right|^p \mathbb{1}_{\{N_T = k\}} \right].
\]

(4.4)

Lemma 5.2 and Lemma 3.3 yield for all \( q \in \mathbb{N} \),

\[
E\left[ (1 + |Z_{\tau_n-1}^{(M)}|^p) \mathbb{1}_{\{N_T = k\}} \right] \\
\leq E\left[ (1 + |Z_{\tau_n-1}^{(M)}|^p) \sup_{t \in [0, \delta]} |W_{t+\tau_{n-1}} - W_{\tau_{n-1}}|^q \mathbb{1}_{\{N_T = k\}} \right] \\
= E\left[ (1 + |Z_{\tau_n-1}^{(M)}|^p) \mathbb{1}_{\{N_T = k\}} \right] E\left[ \sup_{t \in [0, \delta]} |W_{t+\tau_{n-1}} - W_{\tau_{n-1}}|^q \mathbb{1}_{\{N_T = k\}} \right] \\
\leq c_{W} \delta^q \left( \mathbb{P}(N_T = k) + E\left[ |Z_{\tau_n-1}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right] \right).
\]

(5.5)

Plugging (4.5) into (4.4) we get that there exist \( \bar{c}_3, \bar{c}_4 \in (0, \infty) \) such that

\[
E\left[ |Z_{\tau_n-1}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right] \\
\leq 4^{p-1}\left( E\left[ |Z_{\tau_n-1}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right] + 2^{p-1}E\left[ \left| \tilde{\mu} \right|^p \mathbb{1}_{\{N_T = k\}} \right] \\
+ 2^{p-1}E\left[ \left| \tilde{\sigma} \right|^p \mathbb{1}_{\{N_T = k\}} \right] \\
+ 2^{p-2}E\left[ \left| \tilde{\sigma} \right|^p \mathbb{1}_{\{N_T = k\}} \right] \right) \\
\leq \bar{c}_3 \mathbb{P}(N_T = k) + \bar{c}_4 E\left[ |Z_{\tau_n-1}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right].
\]

(6.6)

Combining this with (4.3) we get that there exist constants \( \tilde{c}_5, \tilde{c}_6, \tilde{c}_7, \tilde{c}_8 \in (1, \infty) \) such that

\[
E\left[ |Z_{\tau_n-1}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right] \leq \tilde{c}_5 \mathbb{P}(N_T = k) + \tilde{c}_6 E\left[ |Z_{\tau_n-1}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right]
\]

(7.7)

and

\[
E\left[ |Z_{\tau_n-1}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right] \leq \tilde{c}_7 \mathbb{P}(N_T = k) + \tilde{c}_8 E\left[ |Z_{\tau_n-1}^{(M)}|^p \mathbb{1}_{\{N_T = k\}} \right].
\]

(8.8)
Using (A.7) and the fact that ˜ξ is deterministic, we recursively obtain
\[
\begin{align*}
\mathbb{E}\left[|Z_{\tau_0}^{(M)}|^p \mathbbm{1}_{\{N_T = k\}}\right] & \leq \sum_{j=0}^{n-1} c_5 c_6^j \mathbb{P}(N_T = k) + c_6^n \mathbb{E}\left[\tilde{\xi} \mathbbm{1}_{\{N_T = k\}}\right] \\
\leq & \tilde{c}_6 (1 + |\tilde{\xi}|) \mathbb{P}(N_T = k) \sum_{j=0}^{n} \tilde{c}_6^j = \tilde{c}_6 (1 + |\tilde{\xi}|) \frac{1 - \tilde{c}_6^{n+1}}{1 - \tilde{c}_6} \mathbb{P}(N_T = k). \quad (A.9)
\end{align*}
\]

Analogously using (A.8) recursively, then (A.6), and defining ˜c_9 = ˜c_7 \lor (˜c_3 + ˜c_4|\tilde{\xi}|) we get
\[
\begin{align*}
\mathbb{E}\left[|Z_{\tau_0}^{(M)}|^p \mathbbm{1}_{\{N_T = k\}}\right] & \leq \sum_{j=0}^{n-2} \tilde{c}_7 \tilde{c}_8^j \mathbb{P}(N_T = k) + \tilde{c}_8^{n-1} \mathbb{E}\left[|Z_{\tau_1}^{(M)}|^p \mathbbm{1}_{\{N_T = k\}}\right] \\
\leq & \sum_{j=0}^{n-2} \tilde{c}_7 \tilde{c}_8^j \mathbb{P}(N_T = k) + \tilde{c}_8^{n-1} \left(\tilde{c}_3 \mathbb{P}(N_T = k) + \tilde{c}_4 \mathbb{E}\left[\tilde{\xi} \mathbbm{1}_{\{N_T = k\}}\right]\right) \\
\leq & (\tilde{c}_7 + (\tilde{c}_3 + \tilde{c}_4|\tilde{\xi}|))^p \mathbb{P}(N_T = k) \sum_{j=0}^{n-1} \tilde{c}_8^j = \tilde{c}_9 \frac{1 - \tilde{c}_8^{n+1}}{1 - \tilde{c}_8} \mathbb{P}(N_T = k). \quad (A.10)
\end{align*}
\]

Recalling ˜c_6 > 1 and plugging (A.9) into (A.2) we obtain
\[
\begin{align*}
\mathbb{E}\left[\sum_{n=0}^{N_T + M} (1 + |Z_{\tau_0}^{(M)}|^p)\right] & \leq \lambda T + M + \tilde{c}_6 (1 + |\tilde{\xi}|) \sum_{k=0}^{k+M} \frac{(\tilde{c}_6)^{n+1} - 1}{\tilde{c}_6} \mathbb{P}(N_T = k) \\
= & \lambda T + M + \frac{\tilde{c}_6 (1 + |\tilde{\xi}|)}{\tilde{c}_6 - 1} \left(\sum_{k=0}^{\infty} \mathbb{P}(N_T = k) \sum_{n=0}^{k+M} \tilde{c}_6^n - \sum_{k=0}^{\infty} \mathbb{P}(N_T = k) (k + M)\right) \\
\leq & \lambda T + M + \frac{\tilde{c}_6 (1 + |\tilde{\xi}|)}{\tilde{c}_6 - 1} \sum_{k=0}^{\infty} \mathbb{P}(N_T = k) \tilde{c}_6^k \frac{1 - \tilde{c}_6^{k+M+1}}{1 - \tilde{c}_6} \quad (A.11) \\
\leq & \lambda T + M + \frac{\tilde{c}_6^{M+2} (1 + |\tilde{\xi}|)}{(\tilde{c}_6 - 1)^2} \sum_{k=0}^{\infty} \mathbb{P}(N_T = k) \tilde{c}_6^k \\
= & \lambda T + M + \frac{\tilde{c}_6^{M+2} (1 + |\tilde{\xi}|)}{(\tilde{c}_6 - 1)^2} \exp(\lambda T(\tilde{c}_6 - 1)).
\end{align*}
\]

Similar steps as in (A.2) together with (A.10), and the fact that ˜c_8 > 1 ensure that there exists
a constant \( \tilde{c}_{10} \in (0, \infty) \) such that

\[
\mathbb{E} \left[ N_T^{p-1} \sum_{n=0}^{N_T + M} (1 + |Z_{\tau_n}^{(M)}|^p) \right] = \sum_{k=0}^{\infty} \mathbb{E} \left[ k^{p-1} \sum_{n=0}^{k} (1 + |Z_{\tau_n}^{(M)}|^p) \mathbf{1}_{\{N_T = k\}} \right]
\]

\[
= \sum_{k=0}^{\infty} \sum_{n=0}^{k+M} k^{p-1} \mathbb{P}(N_T = k) + \sum_{k=0}^{\infty} \sum_{n=0}^{k+M} k^{p-1} \mathbb{E} \left[ |Z_{\tau_n}^{(M)}|^p \mathbf{1}_{\{N_T = k\}} \right]
\]

\[
\leq \sum_{k=0}^{\infty} k^{p} \mathbb{P}(N_T = k) + M \sum_{k=0}^{\infty} k^{p-1} \mathbb{P}(N_T = k) + \sum_{k=0}^{\infty} \sum_{n=0}^{k+M} k^{p-1} \tilde{c}_9 \frac{1}{1 - \tilde{c}_8} \mathbb{P}(N_T = k)
\]

\[
\leq \mathbb{E}[N_T^p] + M \mathbb{E}[N_T^{p-1}] + \frac{\tilde{c}_9}{\tilde{c}_8 - 1} \sum_{k=0}^{\infty} k^{p-1} \mathbb{P}(N_T = k) \sum_{n=0}^{k+M} \tilde{c}_8^n
\]

\[
= \mathbb{E}[N_T^p] + M \mathbb{E}[N_T^{p-1}] + \frac{\tilde{c}_9}{(\tilde{c}_8 - 1)^2} \sum_{k=0}^{\infty} k^{p-1} \mathbb{P}(N_T = k) \tilde{c}_8^k
\]

\[
\leq \mathbb{E}[N_T^p] + M \mathbb{E}[N_T^{p-1}] + \frac{\tilde{c}_9 \tilde{c}_8^{M+1}}{(\tilde{c}_8 - 1)^2} \exp(\lambda T (\tilde{c}_8 - 1)).
\]

Here we set \( \tilde{c}_{10} \) as the \((p-1)\)-th moment of a Poisson distributed random variable with parameter \( \tilde{c}_8 \lambda T \). Choosing \( c_M \) as the sum of the constants derived in (A.11) and (A.12) finishes the proof.

**Proof of Lemma 3.7.** By (3.7) we have for all \( s \in [0, T] \),

\[
\mathbb{E} \left[ \sup_{t \in [0, s]} |Z_t^{(M)}|^p \right]
\]

\[
\leq 4^{p-1} \left( \tilde{c}_s^p + \mathbb{E} \left[ \sup_{t \in [0, s]} \left| \int_0^t \sum_{n=0}^{\infty} \tilde{\mu}(Z_{\tau_n}^{(M)}) \mathbf{1}_{(\tau_n, \tau_{n+1})}(u) \, du \right|^p \right] \right)^{1/p} + \mathbb{E} \left[ \sup_{t \in [0, s]} \left| \int_0^t \sum_{n=0}^{\infty} (\tilde{\sigma}(Z_{\tau_n}^{(M)}) + \tilde{\sigma}(Z_{\tau_n}^{(M)}) d\tilde{\sigma}(Z_{\tau_n}^{(M)})(W_u - W_{\tau_n}) \mathbf{1}_{(\tau_n, \tau_{n+1})}(u) \, dW_u \right|^p \right]^{1/p} + \mathbb{E} \left[ \sup_{t \in [0, s]} \left| \int_0^t \tilde{\rho}(Z_{u-}^{(M)}) \, dN_u \right|^p \right]^{1/p} \]  

(A.13)

In the following we estimate each summand of (A.13) separately. For the first one we apply Jensen’s inequality, use the fact that \( u \) lies in at most one interval \((\tau_n, \tau_{n+1}]\), and apply Lemma...
Next we estimate the second expectation of (A.15) using Lemma 3.2, Lemma 3.3, (3.8), and
\[ E \left[ \sup_{t \in [0,s]} \left| \int_0^t \sum_{n=0}^{\infty} \tilde{\mu}(Z^{(M)}_{\tau_n}) \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \, du \right|^p \right] = E \left[ \sup_{t \in [0,s]} \left| \int_0^t \sum_{n=0}^{N_T+M} \tilde{\mu}(Z^{(M)}_{\tau_n}) \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \, du \right|^p \right] 
\leq E \left[ \sup_{t \in [0,s]} T^{p-1} \int_0^t \sum_{n=0}^{N_T+M} \left| \tilde{\mu}(Z^{(M)}_{\tau_n}) \right|^p \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \, du \right] 
\leq 2^{p-1} T^{p-1} c^p \delta \left[ \sum_{n=0}^{N_T+M} (1 + |Z^{(M)}_{\tau_n}|^p) \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \right] 
\leq 2^{p-1} T^{p-1} c^p \delta c_M < \infty. \tag{A.14} \]

Next we consider the second summand of (A.13) and apply Lemma 3.4 to show that there exists \( \hat{c} \in (0, \infty) \) so that
\[
E \left[ \sup_{t \in [0,s]} \left| \int_0^s \sum_{n=0}^{\infty} \tilde{\sigma}(Z^{(M)}_{\tau_n}) + \tilde{\sigma}(Z^{(M)}_{\tau_n}) \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \, dW_u \right|^p \right] 
\leq \hat{c} E \left[ \int_0^s \sum_{n=0}^{\infty} \left| \tilde{\sigma}(Z^{(M)}_{\tau_n}) + \tilde{\sigma}(Z^{(M)}_{\tau_n}) \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \right|^p \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \, du \right] 
\leq 2^{p-1} \hat{c} \left( E \left[ \int_0^s \sum_{n=0}^{\infty} |\tilde{\sigma}(Z^{(M)}_{\tau_n})|^p \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \, du \right] 
+ E \left[ \int_0^s \sum_{n=0}^{\infty} |\tilde{\sigma}(Z^{(M)}_{\tau_n})| \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \, du \right] \right). \tag{A.15} \]

Estimating the first expectation of (A.15) using Lemma 3.4, we obtain that there exists a constant \( c_M \in (0, \infty) \) such that
\[
E \left[ \int_0^s \sum_{n=0}^{\infty} \tilde{\sigma}(Z^{(M)}_{\tau_n}) \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \, du \right] \leq 2^{p-1} c^p \delta \left[ \sum_{n=0}^{N_T+M} (1 + |Z^{(M)}_{\tau_n}|^p) \mathbb{1}_{(\tau_n, \tau_{n+1})}(u) \right] 
= 2^{p-1} c^p \delta \left[ \sum_{n=0}^{N_T+M} (1 + |Z^{(M)}_{\tau_n}|^p) \, du \right] \leq 2^{p-1} c^p \delta c_M. \tag{A.16} \]

Next we estimate the second expectation of (A.15) using Lemma 3.2, Lemma 3.4, (3.8), and
Lemma 3.6 We get that there exist $c_M, c_W \in (0, \infty)$ such that
\[
E \left[ \int_0^s \sum_{n=0}^{\infty} \left| \delta(Z_{t_n}^{(M)}) d\delta(Z_{t_n}^{(M)}) (W_u - W_{t_n}) \right|^p 1_{(t_n, t_{n+1})}(u) \, du \right] \\
\leq 2^{p-1} c_p \rho^p \delta \sum_{k=0}^{N_T + M} \left( 1 + |Z_{t_n}^{(M)}|^p \right) \sup_{v \in [0, \delta]} |W_{v + t_n} - W_{t_n}|^p \\
= 2^{p-1} c_p \rho^p \delta \sum_{k=0}^{N_T + M} \E \left[ \left( 1 + |Z_{t_n}^{(M)}|^p \right) \sup_{v \in [0, \delta]} |W_{v + t_n} - W_{t_n}|^p 1_{(N_T = k)} \right] \\
= 2^{p-1} c_p \rho^p \delta \sum_{k=0}^{N_T + M} \E \left[ \left( 1 + |Z_{t_n}^{(M)}|^p \right) 1_{(N_T = k)} \E \left[ \sup_{v \in [0, \delta]} |W_{v + t_n} - W_{t_n}|^p \right] \right] \\
\leq 2^{p-1} c_p \rho^p \delta c_M \leq 2^{p-1} c_p \rho^p \delta^{\frac{p}{2} + 1} c_M.
\]

Plugging (A.16) and (A.17) into (A.15) we obtain
\[
E \left[ \sup_{t \in [0, s]} \left| \int_0^t \sum_{n=0}^{\infty} \delta(Z_{t_n}^{(M)}) d\delta(Z_{t_n}^{(M)}) (W_u - W_{t_n}) \right|^p 1_{(t_n, t_{n+1})}(u) \, dW_u \right] \\
\leq 2^{p-1} c_p \rho^p \delta c_M + 2^{p-1} c_p \rho^p \delta^{\frac{p}{2} + 1} c_M < \infty.
\]

Next we consider the last summand of (A.13). We recall that by $\nu_i$ we denote the $i$-th jump time of $N$ and use Lemma 3.6. This ensures that there exists $c_M \in (0, \infty)$ such that
\[
E \left[ \sup_{t \in [0, s]} \left| \int_0^t \tilde{\nu}(Z_{t_n}^{(M)}) d\tilde{\nu}(N_{t_n}^{(M)}) \right|^p \right] = E \left[ \sup_{t \in [0, s]} \left| \sum_{i=1}^{N_t} \tilde{\nu}(Z_{\nu_i}^{(M)}) \right|^p \right] \\
\leq 2^{p-1} c_p \rho^p E \left[ N_T^{p-1} \sum_{i=1}^{N_T} \left( 1 + |Z_{\nu_i}^{(M)}|^p \right) \right] \\
\leq 2^{p-1} c_p \rho^p E \left[ N_T^{p-1} \sum_{n=0}^{N_T + M} \left( 1 + |Z_{t_n}^{(M)}|^p \right) \right] \\
\leq 2^{p-1} c_p \rho^p c_M < \infty.
\]

Combining (A.13), (A.14), (A.18), and (A.19) we obtain for all $s \in [0, T]$, all $M \in \mathbb{N}$, and all $p \in \mathbb{N}$,
\[
E \left[ \sup_{t \in [0, s]} |Z_{t}^{(M)}|^p \right] < \infty.
\]

Next we estimate (A.13) again with the goal to to apply the Gronwall inequality. We once more estimate all summands separately and start with the first one. Similar as in (A.14) we obtain
\[
E \left[ \sup_{t \in [0, s]} \left| \int_0^t \sum_{n=0}^{\infty} \mu(Z_{t_n}^{(M)}) 1_{(t_n, t_{n+1})}(u) \, du \right|^p \right] \\
\leq 2^{p-1} T^{p-1} c_p \rho^p \mu \left[ \int_0^s \sum_{n=0}^{\infty} \left( 1 + \sup_{v \in [0, \mu]} |Z_{v}^{(M)}|^p \right) 1_{(t_n, t_{n+1})}(u) \, du \right] \\
\leq 2^{p-1} T^{p-1} c_p \rho^p + 2^{p-1} T^{p-1} c_p \mu \left[ \int_0^s \left( \sup_{v \in [0, \mu]} |Z_{v}^{(M)}|^p \right) \, du \right].
\]
expectation of \((A.15)\) we obtain

\[
E \left[ \int_0^s \sum_{n=0}^\infty |\bar{\sigma}(Z_{\tau_n}^{(M)})|^p \mathbf{1}_{(\tau_n, \tau_{n+1})}(u) \, du \right]
\]

\[
\leq 2^{p-1}c_\sigma^p \frac{\tau_{n+M}}{\tau_n} \left( 1 + \sup_{v \in [0,u]} |Z_v^{(M)}|^p \right) \mathbf{1}_{(\tau_n, \tau_{n+1})}(u) \, du \tag{A.22}
\]

\[
\leq 2^{p-1}c_\sigma^p T + 2^{p-1}c_\sigma^p \int_0^s E \left[ \sup_{v \in [0,u]} |Z_v^{(M)}|^p \right] \, du.
\]

Combining \((A.15), (A.22),\) and \((A.23)\) we obtain

\[
E \left[ \sup_{t \in [0,s]} \int_0^t \sum_{n=0}^\infty \left( \bar{\sigma}(Z_{\tau_n}^{(M)}) + \bar{\sigma}(Z_{\tau_n}^{(M)}) d\bar{\sigma}(Z_{\tau_n}^{(M)}) (W_u - W_{\tau_n}) \right) \mathbf{1}_{(\tau_n, \tau_{n+1})}(u) \, dW_u \right]^p
\]

\[
\leq 2^{p-1}c_\sigma^p T \int_0^s E \left[ \sup_{v \in [0,u]} |Z_v^{(M)}|^p \right] \, du \tag{A.24}
\]

\[
+ 2^{p-1}c_\sigma^p L_\sigma^p c_\sigma \delta_\sigma^2 \int_0^s E \left[ \sup_{v \in [0,u]} |Z_v^{(M)}|^p \right] \, du.
\]

For the last summand of \((A.13)\) Lemma \(3.4\) yields that there exists \(\hat{c} \in (0, \infty)\) such that

\[
E \left[ \sup_{t \in [0,s]} \int_0^t \bar{\rho}(Z_{u-}^{(M)}) \, dN_u \right]^p \leq \hat{c} \int_0^s E \left[ |\bar{\rho}(Z_u^{(M)})|^p \right] \, du \tag{A.25}
\]

\[
\leq 2^{p-1} \hat{c} c_\rho^p T + 2^{p-1} \hat{c} c_\rho^p \int_0^s E \left[ \sup_{v \in [0,u]} |Z_v^{(M)}|^p \right] \, du.
\]

Plugging \((A.21), (A.24),\) and \((A.25)\) into \((A.13)\) we obtain that there exists a constants \(\tilde{c}_1 \in (0, \infty)\) such that

\[
E \left[ \sup_{t \in [0,s]} |Z_t^{(M)}|^p \right] \leq \tilde{c}_1 \left( 1 + |\tilde{\xi}|^p \right) + \tilde{c}_1 \int_0^s E \left[ \sup_{v \in [0,u]} |Z_v^{(M)}|^p \right] \, du.
\]

Because \([0,T] \ni s \mapsto E \left[ \sup_{v \in [0,s]} |Z_v^{(M)}|^p \right] \) is finite due to \((A.20)\) and Borel measurable since it is increasing, we conclude by Gronwall’s inequality that there exists a constant \(c_1 \in (0, \infty)\) such that

\[
E \left[ \sup_{t \in [0,T]} |Z_t^{(M)}|^p \right] \leq c_1 \left( 1 + |\tilde{\xi}|^p \right).
\]

This proves the first inequality.
Hence, we obtain that there exists a constant \( c \in (0, \infty) \) such that

\[
\mathbb{E}\left[ \sum_{n=0}^{\infty} \left| Z^{(M)}_s - Z^{(M)}_t \right|^p \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \right] 
\leq \mathbb{E}\left[ \sum_{n=0}^{\infty} \tilde{c}_2 \left( \left| \tilde{\mu}(Z^{(M)}_{\tau_n})(s - \tau_n) \right|^p + \left| \tilde{\sigma}(Z^{(M)}_{\tau_n})(W_s - W_{\tau_n}) \right|^p \right. \right.
\left. + \left| \frac{1}{2} \tilde{\sigma}(Z^{(M)}_{\tau_n})d_\delta(Z^{(M)}_{\tau_n})((W_s - W_{\tau_n})^2 - (s - \tau_n)) \right|^p \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \right]
\leq \tilde{c}_2 \left( 2^{p-1} \tilde{c}_2^p \delta^p \mathbb{E}\left[ \sum_{n=0}^{\infty} (1 + |Z^{(M)}_{\tau_n}|^p) \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \right] \right.
\left. + 2^{p-1} \tilde{c}_2^p \mathbb{E}\left[ \sum_{n=0}^{\infty} (1 + |Z^{(M)}_{\tau_n}|^p) W_s - W_{\tau_n} |^p \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \right] \right.
\left. + \frac{1}{2} \tilde{c}_2^p L_\sigma^p \mathbb{E}\left[ \sum_{n=0}^{\infty} (1 + |Z^{(M)}_{\tau_n}|^p) W_s - W_{\tau_n} |^p \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \right] \right.
\left. + \frac{1}{2} \tilde{c}_2^p L_\sigma^p \mathbb{E}\left[ \sum_{n=0}^{\infty} (1 + |Z^{(M)}_{\tau_n}|^p) \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \right] \right) .
\]

Applying Lemma 3.3 to the second and third summand we obtain that there exists a constant \( \tilde{c}_3 \in (0, \infty) \) such that

\[
\mathbb{E}\left[ \sum_{n=0}^{\infty} \left| Z^{(M)}_s - Z^{(M)}_t \right|^p \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \right] 
\leq \tilde{c}_3 \left( \delta^p + \delta^p \cdot \delta^p + \delta^p \right) \mathbb{E}\left[ \sum_{n=0}^{\infty} (1 + |Z^{(M)}_{\tau_n}|^p) \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \right] .
\]

By the first claim it holds that

\[
\mathbb{E}\left[ \sum_{n=0}^{\infty} (1 + |Z^{(M)}_{\tau_n}|^p) \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \right] \leq 1 + \mathbb{E}\left[ \sup_{v \in [0, T]} |Z^{(M)}_v|^p \right] \leq 1 + c_1 (1 + |\tilde{\xi}|^p). 
\]

Hence, we obtain that there exists a constant \( c_2 \in (0, \infty) \) such that

\[
\mathbb{E}\left[ \sum_{n=0}^{\infty} \left| Z^{(M)}_s - Z^{(M)}_t \right|^p \mathbb{1}_{[\tau_n, \tau_{n+1})}(s) \right] \leq c_2 (1 + |\tilde{\xi}|^p) \delta^p, 
\]

which proves the second statement.

For the third statement let \( t \in [0, T - \delta] \). Then by Lemma 3.4 there exists a constant \( \hat{c} \in (0, \infty) \) such that

\[
\mathbb{E}\left[ \sup_{s \in [t, t+\delta]} \left| Z^{(M)}_s - Z^{(M)}_t \right|^p \right] 
\leq 3^{p-1} \hat{c} \left( \int_t^{t+\delta} \mathbb{E}\left[ \sum_{n=0}^{\infty} \tilde{\mu}(Z^{(M)}_{\tau_n}) \mathbb{1}_{[\tau_n, \tau_{n+1})}(u) \right]^p \right) du 
\leq \int_t^{t+\delta} \mathbb{E}\left[ \sum_{n=0}^{\infty} \left( \tilde{\sigma}(Z^{(M)}_{\tau_n})d_\delta(Z^{(M)}_{\tau_n})(W_s - W_{\tau_n}) \right) \mathbb{1}_{[\tau_n, \tau_{n+1})}(u) \right]^p \right) du 
\leq \int_t^{t+\delta} \mathbb{E}\left[ \tilde{\rho}(Z^{(M)}_{\tau_n}) \right]^p \right) du ,
\]

(A.26)
Next we consider each expectation separately. For the first one we obtain

$$\begin{align*}
E \left[ \sum_{n=0}^{\infty} \tilde{\mu}(Z_{\tau_n}^{(M)}) I_{[\tau_n,\tau_{n+1})}(u)^p \right] & \leq 2^{p-1} c_\rho^p E \left[ \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}|^p) I_{[\tau_n,\tau_{n+1})}(u) \right] \\
& \leq 2^{p-1} c_\rho^p (1 + E \left[ \sup_{t \in [0,T]} |Z_t^{(M)}|^p \right]) \leq 2^{p-1} c_\rho^p (1 + 1 + |\tilde{\xi}|^p). \tag{A.27}
\end{align*}$$

For the second one we use Lemma 3.5 to obtain that there exists $c_{W_p} \in (0, \infty)$ such that

$$\begin{align*}
E \left[ \sum_{n=0}^{\infty} \tilde{\sigma}(Z_{\tau_n}^{(M)}) + \tilde{\sigma}(Z_{\tau_n}^{(M)}) d_{\tilde{\sigma}}(Z_{\tau_n}^{(M)})(W_u - W_{\tau_n}) I_{[\tau_n,\tau_{n+1})}(u)^p \right] \\
& \leq E \left[ \sum_{n=0}^{\infty} 2^{p-1} \left( |\tilde{\sigma}(Z_{\tau_n}^{(M)})|^p + |\tilde{\sigma}(Z_{\tau_n}^{(M)}) d_{\tilde{\sigma}}(Z_{\tau_n}^{(M)})(W_u - W_{\tau_n})|^p \right) I_{[\tau_n,\tau_{n+1})}(u) \right] \\
& \leq 4^{p-1} c_\sigma^p E \left[ \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}|^p) I_{[\tau_n,\tau_{n+1})}(u) \right] \\
& \quad + 4^{p-1} c_\sigma^p L_{\sigma}^P E \left[ \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}|^p)|W_u - W_{\tau_n}|^p I_{[\tau_n,\tau_{n+1})}(u) \right] \\
& \leq 4^{p-1} c_\sigma^p (1 + E \left[ \sup_{t \in [0,T]} |Z_t^{(M)}|^p \right]) + 4^{p-1} c_\sigma^p L_{\sigma}^P c_{W_p}^p \beta_\tilde{\sigma} \beta_\tilde{\rho} \left[ \sum_{n=0}^{\infty} (1 + |Z_{\tau_n}^{(M)}|^p) I_{[\tau_n,\tau_{n+1})}(u) \right] \\
& \leq \left( 4^{p-1} c_\sigma^p + 4^{p-1} c_\sigma^p L_{\sigma}^P c_{W_p}^p \beta_\tilde{\rho} \right) (1 + c_1(1 + |\tilde{\xi}|^p)). \tag{A.28}
\end{align*}$$

For the third term we calculate

$$\begin{align*}
E \left[ \tilde{\rho}(Z_{\tau_n}^{(M)})^p \right] & \leq 2^{p-1} c_\rho^p (1 + E \left[ \sup_{t \in [0,T]} |Z_t^{(M)}|^p \right]) \leq 2^{p-1} c_\rho^p (1 + c_1(1 + |\tilde{\xi}|^p)). \tag{A.29}
\end{align*}$$

Plugging (A.27), (A.28), and (A.29) into (A.26) we obtain that there exists a constant $\tilde{c}_4 \in (0, \infty)$ such that

$$\begin{align*}
E \left[ \sup_{s \in [t,t+\delta]} |Z_s^{(M)} - Z_t^{(M)}|^p \right] & \leq \tilde{c}_4 \int_t^{t+\delta} \left( 1 + c_1(1 + |\tilde{\xi}|^p) \right) du \\
& = \tilde{c}_4 \left( 1 + c_1(1 + |\tilde{\xi}|^p) \right) \delta \leq \tilde{c}_4 (1 + c_1)(1 + |\tilde{\xi}|^p) \delta.
\end{align*}$$

Defining $c_3 := \tilde{c}_4 (1 + c_1)$ finishes the proof. \qed

References

[1] F. E. Benth, C. Klüppelberg, G. Müller, and L. Vos. Futures pricing in electricity markets based on stable CARMA spot models. Energy Economics, 44:392–406, 2014.
[2] E. Buckwar and M. Riedler. Runge-Kutta methods for jump-diffusion differential equations. Journal of Computational and Applied Mathematics, 236:1155–1182, 2011.
[3] O. Butkovsky, K. Dareiotis, and M. Gerencsér. Approximation of SDEs: a stochastic sewing approach. Probability theory and related fields, 181(4):975–1034, 2021.
[4] O. Butkovsky, K. Dareiotis, and M. Gerencsér. Strong rate of convergence of the Euler scheme for SDEs with irregular drift driven by Lévy noise. arXiv:2204.12926, 2022.
[5] Z. Chen and S. Gan. Convergence and stability of the backward Euler method for jump-diffusion SDEs with super-linearly growing diffusion and jump coefficients. Journal of Computational and Applied Mathematics, 363:350–369, 2020.
[6] K. Dareiotis and M. Gerencsér. On the regularisation of the noise for the Euler–Maruyama scheme with irregular drift. Electronic Journal of Probability, 25:1–18, 2020.
[7] K. Dareiotis, C. Kumar, and S. Sabanis. On tamed Euler approximations of SDEs driven by Lévy noise with applications to delay equations. SIAM Journal on Numerical Analysis, 54(3):1840–1872, 2016.
[38] A. Neuenkirch and M. Szölgyenyi. The Euler–Maruyama scheme for SDEs with irregular drift: convergence rates via reduction to a quadrature problem. IMA Journal of Numerical Analysis, 41(2):1164–1196, 2021.
[39] A. Neuenkirch, M. Szölgyenyi, and L. Szpruch. An adaptive Euler-Maruyama scheme for stochastic differential equations with discontinuous drift and its convergence analysis. SIAM Journal on Numerical Analysis, 57(1):378–403, 2019.
[40] H. L. Ngo and D. Taguchi. Strong rate of convergence for the Euler-Maruyama approximation of stochastic differential equations with irregular coefficients. Mathematics of Computation, 85(300):1793–1819, 2016.
[41] H. L. Ngo and D. Taguchi. On the Euler-Maruyama approximation for one-dimensional stochastic differential equations with irregular coefficients. IMA Journal of Numerical Analysis, 37(4):1864–1883, 2017.
[42] H. L. Ngo and D. Taguchi. Strong convergence for the Euler-Maruyama approximation of stochastic differential equations with discontinuous coefficients. Statistics & Probability Letters, 125:55–63, 2017.
[43] O. M. Pamen and D. Taguchi. Strong rate of convergence for the Euler-Maruyama approximation of SDEs with Hölder continuous drift coefficient. Stochastic Processes and their Applications, 127(8):2542 – 2559, 2017.
[44] E. Pardoux and A. Rascanu. Stochastic Differential Equations, Backward SDEs, Partial Differential Equations, volume 69 of Stochastic Modelling and Applied Probability. Springer, 2014.
[45] E. Platen and N. Bruti-Liberati. Numerical Solution of Stochastic Differential Equations with Jumps in Finance. Springer Verlag, Berlin, Heidelberg, 2010.
[46] P. Protter. Stochastic Integration and Differential Equations. Stochastic Modelling and Applied Probability. Springer, Berlin-Heidelberg, 2005.
[47] P. Przybyłowicz. Optimal global approximation of stochastic differential equations with additive Poisson noise. Numerical Algorithms, 73(2):323–348, 2016.
[48] P. Przybyłowicz. Optimal sampling design for global approximation of jump diffusion stochastic differential equations. Stochastics, 91(2):235–264, 2019.
[49] P. Przybyłowicz. Efficient approximate solution of jump-diffusion SDEs via path-dependent adaptive step-size control. Journal of Computational and Applied Mathematics, 350:396–411, 2019.
[50] P. Przybyłowicz and M. Szölgyenyi. Existence, uniqueness, and approximation of solutions of jump-diffusion sdes with discontinuous drift. Applied Mathematics and Computation, 403:126191, 2021.
[51] P. Przybyłowicz, M. Szölgyenyi, and F. Xu. Existence and uniqueness of solutions of SDEs with discontinuous drift and finite activity jumps. Statistics & Probability Letters, 174(109072), 2021.
[52] P. Przybyłowicz, V. Schwarz, and M. Szölgyenyi. A Skorohod measurable universal functional representation of solutions to semimartingale SDEs. Working paper, 2022.
[53] P. Przybyłowicz, M. Sobieraj, and Ł. Stepień. Efficient approximation of SDEs driven by countably dimensional Wiener process and Poisson random measure. SIAM Journal on Numerical Analysis, 60(2):824–855, 2022.
[54] P. Przybyłowicz, V. Schwarz, and M. Szölgyenyi. Lower error bounds and optimality of approximation for jump-diffusion sdes with discontinuous drift. arXiv:2303.05945, 2023.
[55] K. Sato. Lévy Processes and Infinitely Divisible Distributions. Cambridge Studies in Advanced Mathematics. Cambridge University Press, second edition, 2013.
[56] A. A. Shardin and M. Szölgyenyi. Optimal control of an energy storage facility under a changing economic environment and partial information. International Journal of Theoretical and Applied Finance, 19(4):1–27, 2016.
[57] A. A. Shardin and R. Wunderlich. Partially observable stochastic optimal control problems for an energy storage. Stochastics, 89(1):280–310, 2017.
[58] R. Situ. Theory of Stochastic Differential Equations with Jumps and Applications. Mathematical and Analytical Techniques with Applications to Engineering. Springer, 2005.
[59] A. Tambue and J. D. Mukam. Strong convergence and stability of the semi-tamed and tamed Euler schemes for stochastic differential equations with jumps under non-global Lipschitz condition. International Journal of Numerical Analysis and Modeling, 16:847–872, 2019.
[60] L. Yaroslavtseva. An adaptive strong order 1 method for SDEs with discontinuous drift coefficient. Journal of Mathematical Analysis and Applications, 513(2):126180, 2022.

Paweł Przybyłowicz
Faculty of Applied Mathematics, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Krakow, Poland
pprzybyl@agh.edu.pl

43
