MORPHISMS OF $A_\infty$-BIALGEBRAS AND APPLICATIONS

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Abstract. We define the notion of a relative matrad and construct a new family of polytopes $JJ = \{JJ_{m,n} = JJ_{m,n}\}_{m,n \geq 1}$, called bimultiplihedra, of which $JJ_{n,1} = JJ_{1,n}$ is the multiplihedron $J_n$. We realize the free relative matrad $r\mathcal{H}_\infty$ by extending the free $A_\infty$-bimodule structure on cellular chains of multiplihedra to a free $\mathcal{H}_\infty$-bimodule structure on cellular chains $C_\bullet(J_J)$. A morphism $G : A \Rightarrow B$ of $A_\infty$-bialgebras is the image of a map $C_\bullet(J_J) \rightarrow \text{Hom}(TA, TB)$ of relative matrads. We prove that the homology of every $A_\infty$-bialgebra over a commutative ring with unity (in particular, the bialgebra $S_\ast(\Omega X; Z)$ of singular chains on Moore base pointed loops) admits an induced $A_\infty$-bialgebra structure. We extend the classical Bott-Samelson isomorphism to an isomorphism of $A_\infty$-bialgebras and identify the $A_\infty$-bialgebra structure of $H_\ast(\Omega\Sigma X; Q)$.

1. Introduction

In [21] and [22] we constructed the objects in the category of $A_\infty$-bialgebras; in this paper we construct the morphisms. Given a module $H$ over a (graded or ungraded) commutative ring $R$ with unity, consider the bigraded module $M = \{M_{n,m} = \text{Hom}(H^\otimes_m, H^\otimes_n)\}$, the bisequence submodule $\mathcal{M} \subseteq TTM$ (the tensor module of $TM$), and the canonical upsilon product $\Upsilon : M \times M \rightarrow M$. Given an arbitrary family of $R$-multilinear operations

$$\omega = \{\omega^n_m \in M_{n,m} \mid |\omega^n_m| = m + n - 3\}_{m,n \geq 1},$$

the sum $\sum \omega^n_m$ extends uniquely to its biderivative $d_\omega \in \mathcal{M}$, and there is a (non-bilinear) product $\odot : M \times M \rightarrow \mathcal{M}$, which extends Gerstenhaber’s $\odot$-products on $\text{Hom}(TH, H) \oplus \text{Hom}(H, TH)$ [6]. The family $\omega$ defines an $A_\infty$-bialgebra structure on $H$ whenever $\omega \odot \omega = 0$, and the structure relations appear as bihomogeneous components of this equation. Indeed, the structure relation in bi-degree $(m, n)$ is modeled on the biassociahedron $KK_n$, and the cellular chains $C_\ast(KK)$ realize the free matrad $\mathcal{H}_\infty$. Thus $(H, \omega)$ is an $A_\infty$-bialgebra if $\omega$ is the image of a map $C_\ast(KK) \rightarrow M$ of matrads, i.e., $H$ is an algebra over $\mathcal{H}_\infty$.

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In this paper we define the notion of a relative matrad and construct a new family of polytopes $JJ = \{JJ_{n,m} = JJ_{m,n}\}_{m,n \geq 1}$, called bimultiplihedra, of which $JJ_{n,1} = JJ_1$ is the multiplihedron $J_n$. We realize the free relative matrad $r\mathcal{H}_\infty$ by extending the free $A_\infty$-bimodule structure on cellular chains of multiplihedra to a free $\mathcal{H}_\infty$-bimodule structure on cellular chains $C_*(JJ)$. On the other hand, given $A_\infty$-bialgebras $A$ and $B$, we consider $\text{Hom}(TA, TB)$ as an $\text{End}_T B - \text{End}_T A$-bimodule and define a morphism $G : A \Rightarrow B$ as the image of a map $C_*(JJ) \to \text{Hom}(TA, TB)$ of relative matrads, i.e., $G$ is a bimodule over $\mathcal{H}_\infty$.

Given an $A_\infty$-bialgebra $B$ and a homology isomorphism $g : H_*(B) \to B$, we apply our generalization of the Basic Perturbation Lemma to prove that under mild conditions, the $A_\infty$-bialgebra structure on $B$ pulls back along $g$ to an $A_\infty$-bialgebra structure on $H_*(B)$, and any two such structures so obtained are isomorphic. This is a special case of our main result:

**Theorem 2** Let $B$ be an $A_\infty$-bialgebra with homology $H = H_*(B)$, let $(RH, d)$ be a free $R$-module resolution of $H$, and let $h$ be a perturbation of $d$ such that $g : (RH, d + h) \to (B, dB)$ is a homology isomorphism. Then

(i) (Existence) $g$ induces an $A_\infty$-bialgebra structure $\omega_{RH}$ on $RH$ and extends to a map $G : RH \Rightarrow B$ of $A_\infty$-bialgebras.

(ii) (Uniqueness) $\omega_{RH}, G$ is unique up to isomorphism.

In particular, given a topological space $X$ and a field $F$, the bialgebra structure on simplicial singular chains $S_*(\Omega X; F)$ of Moore base pointed loops pulls back to an $A_\infty$-bialgebra structure on $H_*(\Omega X; F)$ and the induced $A_\infty$-(co)algebra substructures are exactly the ones observed earlier by Gugenheim and Kadeishvili. Furthermore, we show that the $A_\infty$-coalgebra structure on $H_*(X; F)$ extends to an $A_\infty$-bialgebra structure on the tensor algebra $T^*\tilde{H}_*(X; F)$, which is trivial if and only if the $A_\infty$-coalgebra structure on $H_*(X; F)$ is trivial. We extend the classical Bott-Samelson Isomorphism $t_* : T^*\tilde{H}_*(X; F) \xrightarrow{\cong} H_*(\Omega^\infty X; F)$ to an isomorphism of $A_\infty$-bialgebras and conclude that the $A_\infty$-bialgebra structure of $H_*(\Omega^\infty X; F)$ is trivial if and only if the $A_\infty$-coalgebra structure on $H_*(X; F)$ is trivial (Theorem 3).

This is the $A_\infty$-bialgebra structure of $H_*(\Omega^\infty X; \mathbb{Q})$ is the first nontrivial rational homology invariant for $\Omega^\infty X$ (Corollary 2). Finally, for each $n \geq 2$, we construct a space $X_n$ and a nontrivial operation $\omega^n_2 : H^*(\Omega X_n; \mathbb{Z}_2)^{\otimes 2} \to H^*(\Omega X_n; \mathbb{Z}_2)^{\otimes n}$ defined in terms of the action of the Steenrod algebra $A_2$ on $H^*(X_n; \mathbb{Z}_2)$. To our knowledge, $\omega^n_2$ is the first known example of a non-operadic operation on cohomology.

The paper is organized as follows: Section 2 reviews the theory of matrads and the related notation used throughout the paper; see [22] for details. We construct relative matrads in Section 3 and the bimultiplihedra in Section 4. We define morphisms of $A_\infty$-bialgebras in Section 5. We prove our generalization of the classical Basic Perturbation Lemma (Theorem 1) and our main result (Theorem 2) in Section 6. We conclude with various applications and examples in Section 7.

## 2. Matrads

Let $M = \{M_{n,m}\}_{m,n \geq 1}$ be a bigraded module over a (graded or ungraded) commutative ring $R$ with unity $1_R$. In particular, given a graded $R$-module $H$, we set $M_{n,m} = \text{Hom}(H^\otimes m, H^\otimes n)$ and think of $\alpha^m_n \in M_{n,m}$ as some composition of...
multilinear operations $\theta_i^j$ pictured as a possibly non-planar upward-directed graph with $m$ inputs and $n$ outputs.

Each pair of matrices $X = [x_{ij}], Y = [y_{ij}] \in \mathbb{N}^{q \times p}, p, q \geq 1$, uniquely determines a submodule
\[ M_{Y, X} = (M_{y_{11}, x_{11}} \otimes \cdots \otimes M_{y_{1p}, x_{1p}}) \otimes \cdots \otimes (M_{y_{q1}, x_{q1}} \otimes \cdots \otimes M_{y_{qp}, x_{qp}}) \subset \text{TTM}. \]

Fix a set of bihomogeneous module generators $G \subset M$. A monomial in $\text{TM}$ is an element of $G^\otimes p$ and a monomial in $\text{TTM}$ is an element of $(G^\otimes p)^\otimes q$. Thus $A \in (G^\otimes p)^\otimes q$ is naturally represented by the $q \times p$ matrix
\[
[A] = \begin{pmatrix}
\alpha_{11}^{y_{11}} & \cdots & \alpha_{1p}^{y_{1p}} \\
\vdots & & \vdots \\
\alpha_{q1}^{y_{q1}} & \cdots & \alpha_{qp}^{y_{qp}}
\end{pmatrix}
\]
with entries in $G$ and rows identified with elements of $G^\otimes p$. We refer to $A$ as a $q \times p$ monomial (we use the symbols $A$ and $[A]$ interchangeably). Consequently,
\[
(M^\otimes p)^\otimes q = \bigoplus_{X, Y \in \mathbb{N}^{q \times p}} M_{Y, X}
\]
and we refer to \( \mathbf{M} = \bigoplus_{X, Y \in \mathbb{N}^{q \times p}} M_{Y, X} \) and \( \nabla = \bigoplus_{X, Y \in \mathbb{N}^{1 \times p}, \mathbb{N}^{q \times 1}} M_{Y, X} \) as the matrix and vector submodules of $\text{TTM}$, respectively. The matrix transpose $A \mapsto A^T$ induces the permutation of tensor factors $\sigma_{p, q} : (M^\otimes p)^\otimes q \cong (M^\otimes q)^\otimes p$ given by
\[
\left(\alpha_{x_{11}}^{y_{11}} \otimes \cdots \otimes \alpha_{x_{1p}}^{y_{1p}}\right) \otimes \cdots \otimes \left(\alpha_{x_{q1}}^{y_{q1}} \otimes \cdots \otimes \alpha_{x_{qp}}^{y_{qp}}\right) \mapsto \left(\alpha_{x_{11}}^{y_{11}} \otimes \cdots \otimes \alpha_{x_{q1}}^{y_{q1}}\right) \otimes \cdots \otimes \left(\alpha_{x_{1p}}^{y_{1p}} \otimes \cdots \otimes \alpha_{x_{qp}}^{y_{qp}}\right).
\]

Let $\bar{X}$ and $\bar{Y}$ denote matrices in $\mathbb{N}^{q \times p}$ with constant columns $(x_i)^T$ and constant rows $(y_j)$, respectively, and consider $q \times p$ monomials
\[
A = \begin{pmatrix}
\alpha_{x_{11}}^{y_{11}} & \cdots & \alpha_{x_{1p}}^{y_{1p}} \\
\vdots & & \vdots \\
\alpha_{x_{q1}}^{y_{q1}} & \cdots & \alpha_{x_{qp}}^{y_{qp}}
\end{pmatrix} \in M_{\bar{X}, \bar{X}} \quad \text{and} \quad B = \begin{pmatrix}
\beta_{x_{11}}^{y_{11}} & \cdots & \beta_{x_{1p}}^{y_{1p}} \\
\vdots & & \vdots \\
\beta_{x_{q1}}^{y_{q1}} & \cdots & \beta_{x_{qp}}^{y_{qp}}
\end{pmatrix} \in M_{\bar{Y}, \bar{X}}.
\]
The row leaf sequence of $A$ is the $p$-tuple $\text{rls}(A) = (x_1, \ldots, x_p)$; the column leaf sequence of $B$ is the $q$-tuple $\text{cls}(B) = (y_1, \ldots, y_q)^T$. Define
\[
\bar{M}_{\text{row}} = \bigoplus_{\bar{X}, \bar{Y} \in \mathbb{N}^{q \times p}; \ p, q \geq 1} M_{\bar{X}, \bar{X}} \quad \text{and} \quad \bar{M}_{\text{col}} = \bigoplus_{\bar{X}, \bar{Y} \in \mathbb{N}^{1 \times p}; \ p, q \geq 1} M_{\bar{Y}, \bar{X}};
\]
then
\[
\mathbf{M} = \bar{M}_{\text{row}} \cap \bar{M}_{\text{col}}
\]
is the bisequence submodule of $\text{TTM}$, and a $q \times p$ monomial $A \in \mathbf{M}$ is represented as a bisequence matrix
\[
A = \begin{pmatrix}
\alpha_{x_{11}}^{y_{11}} & \cdots & \alpha_{x_{1p}}^{y_{1p}} \\
\vdots & & \vdots \\
\alpha_{x_{q1}}^{y_{q1}} & \cdots & \alpha_{x_{qp}}^{y_{qp}}
\end{pmatrix}.
\]
Unless otherwise stated, we shall assume that \( x \times y \in (x_1, \ldots, x_p) \times (y_1, \ldots, y_q)^T \in \mathbb{N}^{1 \times p} \times \mathbb{N}^{q \times 1} \), \( p, q \geq 1 \); we will often express \( y \) as a row vector. Let
\[
\text{M}_x^y = \langle A \in \text{M} \mid x = \text{rhs}(A) \text{ and } y = \text{cls}(A) \rangle ;
\]
then
\[
\text{M} = \bigoplus_{x \times y} \text{M}_x^y.
\]
By identifying \((H \otimes q)^{\otimes p} \approx (H \otimes p)^{\otimes q}\) with \((q, p) \in \mathbb{N}^2\), we think of a \( q \times p \) monomial \( A \in \text{M}_x^y \) as an operator on the positive integer lattice \( \mathbb{N}^2 \), pictured as an arrow \((|x|, q) \mapsto (p, |y|)\), where \(|u| = \Sigma u_i\); but unfortunately, this representation is not faithful. The \textit{bisequence vector submodule} is the intersection
\[
\text{V} = \nabla \cap \text{M} = \bigoplus_{s, t \in \mathbb{N}} \text{M}_x^y \oplus \text{M}_x^t.
\]
A submodule
\[
\text{W} = M \oplus \bigoplus_{x, y \in \mathbb{N}, s, t \in \mathbb{N}} \text{W}_s^y \oplus \text{W}_x^t \subseteq \text{V}
\]
is \textit{telescoping} if for all \( x, y, s, t \)
\begin{enumerate}
  \item \( \text{W}_s^y \subseteq \text{M}_x^y \) and \( \text{W}_x^t \subseteq \text{M}_x^t \);
  \item \( \alpha_y^{t_1} \otimes \cdots \otimes \alpha_y^{t_j} \in \text{W}_s^y \) implies \( \alpha_y^{t_1} \otimes \cdots \otimes \alpha_y^{t_j} \in \text{W}_s^{y_1 \cdots y_j} \) for all \( j < q \);
  \item \( \beta_x^{s_1} \otimes \cdots \otimes \beta_x^{s_i} \in \text{W}_x^t \) implies \( \beta_x^{s_1} \otimes \cdots \otimes \beta_x^{s_i} \in \text{W}_x^{s_1 \cdots s_i} \) for all \( i < p \).
\end{enumerate}
Thus the truncation maps \( \tau : \text{W}_s^{y_1 \cdots y_j} \rightarrow \text{W}_s^{y_1 \cdots y_{j-1}} \) and \( \tau : \text{W}_x^{s_1 \cdots s_i} \rightarrow \text{W}_x^{s_1 \cdots s_{i-1}} \) determine the following “telescoping” sequences of submodules:
\[
\tau \left( \text{W}_s^y \right) \subseteq \tau^2 \left( \text{W}_s^y \right) \subseteq \cdots \subseteq \tau^{q-1} \left( \text{W}_s^y \right) = \text{W}_s^{y_1}.
\]
\[
\tau \left( \text{W}_x^t \right) \subseteq \tau^2 \left( \text{W}_x^t \right) \subseteq \cdots \subseteq \tau^{p-1} \left( \text{W}_x^t \right) = \text{W}_x^{t_1}.
\]
In general, \( \text{W}_x^t \) is an \textit{additive} submodule of \( \text{M}_x^1 \otimes \cdots \otimes \text{M}_x^t \) and \text{does not necessarily} decompose as \( B_1 \otimes \cdots \otimes B_p \) with \( B_i \subseteq \text{M}_x^i \).

The \textit{telescopic extension} of a telescoping submodule \( \text{W} \subseteq \text{V} \) is the submodule of matrices \( \text{W} \subseteq \text{M} \) with the following properties: If \( A = \left[ \begin{smallmatrix} a_{y,1} & \cdots & a_{y,1+m} \\ \vdots & \ddots & \vdots \\ a_{y,n} & \cdots & a_{y,n+m} \end{smallmatrix} \right] \) is a \( q \times p \) monomial in \( \text{W} \) and
\begin{enumerate}
  \item \( a_{y,1} \cdots a_{y,1+m} \) is a string in the \( i \)th row of \( A \) such that \( y_{i,j} = \cdots = y_{i,j+m} = t \), then \( \alpha_{x_{i,j}}^{t_1} \otimes \cdots \otimes \alpha_{x_{i,j+m}}^{t_1} \in \text{W}_{x_{i,j}, \ldots, x_{i,j+m}} \);
  \item \( a_{x,1}^{t_1} \cdots a_{x,1+m}^{t_1} \) is a string in the \( j \)th column of \( A \) such that \( x_{i,j} = \cdots = x_{i,j+r} = s \), then \( \alpha_{x_{i,j}}^{s_1} \otimes \cdots \otimes \alpha_{x_{i,j+r}}^{s_1} \in \text{W}_{s^{y_{1},\ldots, y_{l+r}}} \).
\end{enumerate}
Thus if \( A \in \text{M}_x^y \cap \text{W} \), the \( i \)th row of \( A \) lies in \( \text{W}_s^{y_1} \) and the \( j \)th column of \( A \) lies in \( \text{W}_{x_j}^s \).

\textbf{Definition 1.} A pair \( A \otimes B = \left[ \begin{smallmatrix} a_{x_{1,k}}^{u_{1,k}} & \cdots & a_{x_{1,k}}^{u_{q,k}} \\ \vdots & \ddots & \vdots \\ a_{x_{1,k}}^{u_{k,1}} & \cdots & a_{x_{1,k}}^{u_{k,1}} \end{smallmatrix} \right] \otimes \left[ \begin{smallmatrix} \alpha_{x_{1,j}}^{s_1} & \cdots & \alpha_{x_{1,j}}^{s_q} \\ \vdots & \ddots & \vdots \\ \alpha_{x_{1,j}}^{s_1} & \cdots & \alpha_{x_{1,j}}^{s_q} \end{smallmatrix} \right] \in \text{M}_x^y \otimes \text{M}_x^y \) is a
\begin{enumerate}
  \item \textbf{Transverse Pair} \( (TP) \) if \( s = t = 1 \), \( u_{1,j} = q \), and \( v_{k,1} = p \) for all \( j \) and \( k \), \textit{i.e.}, setting \( x_j = x_{1,j} \) and \( y_k = y_{k,1} \) gives
  \[
  A \otimes B = \left[ \begin{smallmatrix} a_{x_{1,j}}^{u_{1,j}} & \cdots & a_{x_{1,j}}^{u_{q,j}} \\ \vdots & \ddots & \vdots \\ a_{x_{1,j}}^{u_{1,j}} & \cdots & a_{x_{1,j}}^{u_{q,j}} \end{smallmatrix} \right] \otimes \left[ \begin{smallmatrix} \beta_{x_{1,j}}^{s_1} & \cdots & \beta_{x_{1,j}}^{s_q} \\ \vdots & \ddots & \vdots \\ \beta_{x_{1,j}}^{s_1} & \cdots & \beta_{x_{1,j}}^{s_q} \end{smallmatrix} \right] \in \text{M}_x^y \otimes \text{M}_x^y.
  \end{enumerate}
(ii) **Block Transverse Pair** (BTP) if there exist \( t \times s \) block decompositions \( A = [A'_{i,l}] \) and \( B = [B'_{i,j}] \) such that \( A'_{i,l} \otimes B'_{i,j} \) is a TP for all \( i \) and \( l \).

A pair \( A \otimes B \in \text{Mat}_A \times \text{Mat}_B \) is a BTP if and only if \( x \times y \in \mathbb{N}^{1 \times |v|} \times \mathbb{N}^{1 \times 1} \) and if only if the initial point of arrow \( A \) coincides with the terminal point of arrow \( B \) as operators on \( \mathbb{N}^2 \).

Given a family of maps \( \gamma = \{ M^{\otimes q} \otimes M^{\otimes p} \rightarrow M \}_{p,q \geq 1} \), let \( \gamma = \{ \gamma_x^y : M^y_x \otimes M^z_x \rightarrow M_{[x]}^{[y]} \} \). Then \( \gamma \) induces a global product \( \Upsilon : \text{Mat}_x \otimes \text{Mat}_z \rightarrow \text{Mat}_1 \) defined by

\[
\Upsilon (A \otimes B) = \begin{cases} 
\gamma \left( A'_{i,j} \otimes B'_{i,j} \right), & A \otimes B \text{ is a BTP} \\
0, & \text{otherwise},
\end{cases}
\]

where \( A'_{i,j} \otimes B'_{i,j} \) is the \((i,j)\)th TP in the BTP decomposition of \( A \otimes B \). Obviously \( \Upsilon \) is closed in both \( \text{Mat}_x \otimes \text{Mat}_z \); consequently, \( \Upsilon \) is closed in \( \text{Mat}_1 \). We denote the \( \Upsilon \)-product by \( " \gamma \) or juxtaposition; when \( A \otimes B = [\alpha^p_x]^{T} \otimes [\beta^q_z]^{T} \) is a TP, we write

\[
AB = \gamma (\alpha^1_x; \ldots , \alpha^{y_p}_x; \beta^1_z; \ldots , \beta^{y_q}_z).
\]

As an arrow, \( AB \) “transgresses” from the \( x \)-axis to the \( y \)-axis \( \mathbb{N}^2 \).

Let \( 1^{1 \times p} = (1, \ldots , 1) \in \mathbb{N}^{1 \times p} \) and \( 1^{q \times 1} = (1, \ldots , 1)^T \in \mathbb{N}^{q \times 1} \); we often suppress the exponents when the context is clear.

**Definition 2.** A **prematrad** \( (M, \gamma, \eta) \) is a bigraded \( R \)-module \( M = \{ M_{m,n} \}_{m,n \geq 1} \) together with a family of structure maps \( \gamma = \{ \gamma^x_y : M^y_x \otimes M^z_x \rightarrow M_{[x]}^{[y]} \} \) and a unit \( \eta : R \rightarrow M^1_1 \) such that

(i) \( \Upsilon \left( \Upsilon \left( A ; B \right) ; C \right) = \Upsilon \left( A ; \Upsilon \left( B ; C \right) \right) \) whenever \( A \otimes B \) and \( B \otimes C \) are BTPs in \( \text{Mat}_x \otimes \text{Mat}_z \);

(ii) the following compositions are the canonical isomorphisms:

\[
\begin{align*}
R^{\otimes b} \otimes M^a_x \xrightarrow{\eta^b \otimes \text{Id}} M^1_1 \otimes M^b_x & \xrightarrow{\gamma^b_1} M^b_x; \\
M^a_x \otimes R^{\otimes a} \xrightarrow{\text{Id} \otimes \eta^a} M^a_x \otimes M^1_1 & \xrightarrow{\gamma^1_{1 \times 1}} M^a_x.
\end{align*}
\]

We denote the element \( \eta(1_R) \) by \( 1_M \).

A **morphism of prematrad** \( (M, \gamma) \) and \( (M', \gamma') \) is a map \( \phi : M \rightarrow M' \) such that \( f \gamma^x_y = \gamma'^x_y (f^{\otimes q} \otimes f^{\otimes p}) \) for all \( x \times y \).

Although \( \Upsilon \) fails to act associatively on \( \text{Mat}_x \), Axiom (i) implies that \( (AB)C = A (BC) \) whenever \( A, B, C \in \text{Mat}_x \), \( AB \neq 0 \), and \( BC \neq 0 \). On the other hand, \( \Upsilon \) acts associatively on \( M \). Given a bisequence matrix \( A^{q \times p} \in \text{Mat}_x \), consider the constant (bisequence) matrices \( 1^{q \times 1} \) and \( 1^{1 \times p} \) whose entries are constantly \( 1_M \). Then \( \Upsilon \left( (1^{q \times 1}); A \right) = A = \Upsilon \left( (1^{1 \times p}); A \right) \) by Axiom (ii).

**Definition 3.** Let \( (M, \gamma, \eta) \) be a prematrad. A string of matrices \( A_1 \cdots A_1 \) is **basic in** \( M^n \) if

(i) \( A_1 \in \text{Mat}_x^b \), \( |x| = m \),

(ii) \( A_i \in \text{Mat}_x - \{ 1^{1 \times p} | p, q \in \mathbb{N} \} \) for all \( i \),

(iii) \( A_i \in \text{Mat}_x, \ |y| = n \), and

(iv) some association of \( A_1 \cdots A_1 \) defines a sequence of BTPs and non-zero \( \Upsilon \)-products.
Our next definition constructs a bigraded set $G_{\pre} = G_{\pre,1}^{\pre}$ in which $G_{\pre,1}^{\pre}$ is defined in terms of the intermediate set

$$G_{[n,m]} = \bigcup_{i \leq m, j \leq n, i+j < m+n} G_{j,i}^{\pre}.$$  

We denote the set of matrices over $G_{[n,m]}$ by $G_{[n,m]}$, the set of matrices over $G_{\pre}$ by $G$, and the subset of bisepOSE matrices in $G$ by $G$.

**Definition 4.** Let $\Theta = \{ \theta_{ij}^m | \theta_1^1 = 1 \}_{m,n \geq 1}$ be a free bigraded $R$-module generated by singletons $\theta_{ij}^m$ and set $G_{\pre,1}^{\pre} = 1$. Inductively, if $m + n \geq 3$ and $G_{j,i}^{\pre}$ has been constructed for $i \leq m$, $j \leq n$, and $i+j < m+n$, define

$$G_{n,m}^{\pre} = \theta_{i,j}^m \cup \{ \text{basic strings } A_i \cdots A_1 \text{ in } G_m \text{ with } s \geq 2 \}.$$  

Let $\sim$ be the equivalence relation on $G_{\pre} = G_{\pre,1}^{\pre}$ generated by $[A_{ij}B_{ij}] \sim [A_{ij}] [B_{ij}]$ if and only if $[A_{ij}] \times [B_{ij}] \in G \times G$ is a BTP, and let $F_{\pre}(\Theta) = \langle G_{\pre} / \sim \rangle$. The **free prematrad generated by** $\Theta$ is the prematrad $(F_{\pre}(\Theta), \gamma, \eta)$, where $\gamma$ is juxtaposition and $\eta$ is the prematrad unit axiom holds for $\Theta$.

Let $\mathcal{W}$ be the telescope extension of a telescoping submodule $\mathcal{W}$. If $A \otimes B$ is a BTP in $\mathcal{W} \otimes \mathcal{W}$, each TP $A' \otimes B'$ in $A \otimes B$ lies in $\mathcal{W}_x \otimes \mathcal{W}_y$ for some $x, y, p, q$. Consequently, $\gamma_{\mathcal{W}}$ extends to a global product $\mathcal{Y} : \mathcal{W} \otimes \mathcal{W} \rightarrow \mathcal{W}$ as in (2.1). In fact, $\mathcal{W}$ is the smallest module submodule containing $\mathcal{W}$ on which $\mathcal{Y}$ is well-defined.

**Definition 5.** Let $\mathcal{W}$ be a telescoping submodule of $TTM$ and let $\mathcal{W}$ be its telescopic extension. Let $\gamma_{\mathcal{W}} = \{ \gamma_{\mathcal{W}}^x : \mathcal{W}_x \otimes \mathcal{W}_y \rightarrow \mathcal{W}_{[xy]} \}$ be a structure map and let $\eta : R \rightarrow M$. The triple $(M, \gamma_{\mathcal{W}}, \eta)$ is a **local prematrad (with domain W)** if the following axioms are satisfied:

(i) $W_x^i = M_x^i$ and $W_y^j = M_y^j$ for all $x, y$;

(ii) $\mathcal{Y}(\mathcal{Y}(A;B);C) = \mathcal{Y}(A;\mathcal{Y}(B;C))$ whenever $A \otimes B$ and $B \otimes C$ are BTPs in $\mathcal{W} \otimes \mathcal{W}$;

(iii) the prematrad unit axiom holds for $\gamma_{\mathcal{W}}$.

Recall that each codimension $k$ face of the permuthahedron $P_m$ is identified with two planar rooted trees with levels (PLTs)–an up-rooted PLT with $m+1$ leaves and $k+1$ levels, and its down-rooted mirror image (see [14, 20]). Define the $(m,1)$-row descent sequence of the $m$-leaf up-rooted corolla $\lambda_m$ to be $m = (m)$. Given an up-rooted PLT $T$ with $k \geq 2$ levels, successively remove the levels of $T$ and obtain a sequence of subtrees $T = T^k, T^{k-1}, \ldots, T^1$, in which $T^i - T^{i-1}$ is the sequence of corollas $\lambda_{m_{i,j}} \cdots \lambda_{m_{i,r}}$. Recover $T$ inductively by attaching $\lambda_{m_{i,j}}$ to the $j^{th}$ leaf of $T^{i-1}$. Define the $i^{th}$ leaf sequence of $T$ to be the row matrix $m_i = (m_{i,1}, \ldots, m_{i,r_i})$ and the $(m,k)$-row descent sequence of $T$ to be the $k$-tuple of row matrices $(m_1, \ldots, m_k)$. Dually, define the $(n,1)$-column descent sequence of the $n$-leaf down-rooted corolla $\gamma^n$ to be $n = (n)$. Define the $(n,1)$-column descent sequence of the $n$-leaf down-rooted corolla $\lambda_n$ to be $n = (m)$. Given a down-rooted PLT $T$ with $l \geq 2$ levels, successively remove the levels of $T$ and obtain a sequence of subtrees $T = T^l, T^{l-1}, \ldots, T^1$, in which $T^i - T^{i-1}$ is the sequence of corollas $\gamma^{n_{i,1}} \cdots \gamma^{n_{i,s_i}}$. Recover $T$ inductively by attaching $\gamma^{n_{i,j}}$ to the $j^{th}$ leaf of $T^{i-1}$ for each $i$. Define the $i^{th}$ leaf sequence of $T$ to be the column matrix $n_i = (n_{i,1}, \ldots, n_{i,s_i})^T$ and the $(n,l)$-column descent sequence of $T$ to be the $l$-tuple of column matrices $(n_1, \ldots, n_l)$. 

Let $\mathcal{W}_{\text{row}} = \mathcal{W} \cap \mathcal{M}_{\text{row}}$ and $\mathcal{W}_{\text{col}} = \mathcal{W} \cap \mathcal{M}_{\text{col}}$.

**Definition 6.** Given a local prematrad $(M, \gamma_\mathcal{W})$ with domain $\mathcal{W}$, let $\zeta \in M_{*,m}$ and $\xi \in M_{n,*}$ be elements with $m, n \geq 2$.

(i) A **row factorization of $\zeta$ with respect to $\mathcal{W}$** is a $\gamma$-factorization $A_1 \cdots A_k = \zeta$ such that $A_j \in \mathcal{W}_{\text{row}}$ and $\gamma(A_j) \neq 1$ for all $j$. The sequence $\alpha = (\gamma(A_1), \ldots, \gamma(A_k))$ is the related $(m,k)$-row descent sequence of $\zeta$.

(ii) A **column factorization of $\xi$ with respect to $\mathcal{W}$** is a $\gamma$-factorization $B_1 \cdots B_l = \xi$ such that $B_i \in \mathcal{W}_{\text{col}}$ and $\gamma(B_i) \neq 1$ for all $i$. The sequence $\beta = (\gamma(B_1), \ldots, \gamma(B_l))$ is the related $(n,l)$-column descent sequence of $\xi$.

Given a local prematrad $(M, \gamma_\mathcal{W})$ and elements $A \in M_{s,*}$ and $B \in M_{t,*}$ with $s, t \geq 2$, choose a row factorization $A_1 \cdots A_k$ of $A$ with respect to $\mathcal{W}$ having $(s,k)$-row descent sequence $\alpha$, and a column factorization $B_1 \cdots B_l$ of $B$ with respect to $\mathcal{W}$ having $(l,t)$-column descent sequence $\beta$. Then $\alpha$ identifies $A$ with an up-rooted $s$-leaf, $k$-level PLT and a codimension $k - 1$ face $\hat{\gamma}_A$ of $P_{s-1}$. Similarly, $\beta$ identifies $B$ with a down-rooted $t$-leaf, $l$-level PLT and a codimension $l - 1$ face $\hat{\gamma}_B$ of $P_{t-1}$. Extending to Cartesian products, identify the monomials $A = A_1 \otimes \cdots \otimes A_q \in (M_{s,*})^{\otimes q}$ and $B = B_1 \otimes \cdots \otimes B_p \in (M_{t,*})^{\otimes p}$ with the product cells

$$\hat{\gamma}_A = \hat{\gamma}_{A_1} \times \cdots \times \hat{\gamma}_{A_q} \subset P_{s-1}$$

and

$$\hat{\gamma}_B = \hat{\gamma}_{B_1} \times \cdots \times \hat{\gamma}_{B_p} \subset P_{t-1}.$$
Definition 8. A local prematrad \((M, \gamma_W, \eta)\) is a (left) matrad if
\[
\Gamma^q_p(M, \gamma_W) \otimes \Gamma^r_s(M, \gamma_W) = W^q_p \otimes W^r_s
\]
for all \(p, q \geq 2\). A morphism of matrads is a map of underlying local prematrads.

The domain of the free prematrad \((M = F_{\text{prematrad}}(\Theta, \gamma, \eta))\) generated by \(\Theta = \langle \theta^a_{m,n} \rangle_{m,n \geq 1}\) is \(V = M \oplus \bigoplus_{x,y \in \mathbb{R}, s,t \in \mathbb{N}} M^a_x \oplus M^b_y\), whose submodules \(M, M^a_x, M^b_y\) are contained in the configuration module \(\Gamma(M)\). As above, the symbol “.” denotes the \(\gamma\) product.

Definition 9. Let \((M = F_{\text{prematrad}}(\Theta, \gamma, \eta))\) be the free prematrad generated by \(\Theta = \langle \theta^a_{m,n} \rangle_{m,n \geq 1}\); let \(F(\Theta) = \Gamma(M) \cdot \Gamma(M)\), and let \(\gamma_{F(\Theta)} = \gamma|_{\Gamma(M) \otimes \Gamma(M)}\). The free matrad generated by \(\Theta\) is the triple \((F(\Theta), \gamma_{F(\Theta)}, \eta)\).

The basis elements of \(F_{n,m}(\Theta)\) lie in one-to-one correspondence with the cells of the polytope \(KK_{n,m}\) mentioned in the introduction, and there is a differential \(\partial\) of degree \(-1\) on \(F_{n,m}(\Theta)\) that agrees with the cellular differential on the cellular chain complex \(C_* (KK_{n,m})\). We denote the differential object \((F(\Theta), \partial)\) by \(\mathcal{H}_\infty\), and define an \(A_\infty\)-infinity bialgebra as an algebra over the matrad \(\mathcal{H}_\infty\).

3. Relative Matrads

Let \(R\) be a commutative ring (graded or ungraded) with unity \(1_R\), let \((M, \gamma_M, \eta_M)\) and \((N, \gamma_N, \eta_N)\) be \(R\)-prematrads, and let \(E = \{E_{n,m}\}_{m,n \geq 1}\) be a bigraded \(R\)-module. Left and right actions \(\lambda = \{\lambda^a_x : M^a_x \otimes E^b_\ast \rightarrow E^b_\ast\}\) and \(\rho = \{\rho^a_x : E^b_\ast \otimes N^a_y \rightarrow E^b_\ast\}\) induce left and right upsilon products \(\Upsilon^{\lambda}_M : \mathbb{M} \otimes E \rightarrow E\) and \(\Upsilon^{\rho}_N : E \otimes \mathbb{N} \rightarrow E\) in the same way that \(\gamma_M = \{\gamma^a_x : M^a_x \otimes M^b_y \rightarrow M_\ast^{b \times c}\}\) induces an upsilon product \(\Upsilon_M : \mathbb{M} \otimes \mathbb{M} \rightarrow \mathbb{M}\) (see [22] and formula (2.1)).

Definition 10. A tuple \((M, E, N, \lambda, \rho)\) is a relative prematrad if

(i) Associativity holds:
\[
\begin{align*}
(\alpha) & \quad \Upsilon_p(\Upsilon_{\lambda}(\lambda_\ast \otimes 1)) = \Upsilon_{\lambda}(1 \otimes \Upsilon_{\rho}); \\
(\beta) & \quad \Upsilon_{\lambda}(\Upsilon_{\rho}(\rho_\ast \otimes 1)) = \Upsilon_{\lambda}(1 \otimes \Upsilon_{\lambda}); \\
(\gamma) & \quad \Upsilon_p(\Upsilon_{\rho}(\rho_\ast \otimes 1)) = \Upsilon_{\rho}(1 \otimes \Upsilon_{\rho}).
\end{align*}
\]

(ii) The units \(\eta_M\) and \(\eta_N\) induce the following canonical isomorphisms for all \(a, b \in \mathbb{N}\):
\[
R^b \otimes E^b_a \xrightarrow{\eta^b_a \otimes 1} M^b_1 \otimes E^b_a \xrightarrow{\lambda^b_a \otimes 1} E^b_a \\
E^b_a \times R^a \xrightarrow{1 \otimes \eta^a_b} E^b_a \otimes N^1_{1 \times x} \xrightarrow{\rho^1_{1 \times x} \otimes 1} E^b_a.
\]

We refer to \(E\) as an \(M-N\)-bimodule; when \(M = N\) we refer to \(E\) as an \(M\)-bimodule.

Definition 11. A morphism \(f : (M, E, N, \lambda, \rho) \rightarrow (M', E', N', \lambda', \rho')\) of relative prematrads is a triple \((f_M : M \rightarrow M', f_E : E \rightarrow E', f_N : N \rightarrow N')\) such that

(i) \(f_M\) and \(f_N\) are maps of prematrads;

(ii) \(f_E\) commutes with left and right actions, i.e., \(f_E \circ \lambda^a_x = \lambda'^a_x \circ (f_M \otimes f_E)\) and \(f_E \circ \rho^a_x = \rho'^a_x \circ (f_M \otimes f_E)\) for all \(x, y \in \mathbb{N}^{p \times 1} \times \mathbb{N}^{1 \times q}\).
Tree representations of $\lambda^x_\Phi$ and $\rho^x_\Phi$ are related to those of $\lambda^1_\Phi$ and $\rho^1_\Phi$ by a reflection in some horizontal axis. Although $\rho^x_\Phi$ agrees with Markl, Shnider, and Stasheff’s right module action over an operad [16], $\lambda^1_\Phi$ differs fundamentally from their left module action, and our definition of an "operadic bimodule" is consistent with their definition of an operadic ideal.

Given graded $R$-modules $A$ and $B$, let

$$U_A = \text{End}_{TA}, \quad U_{A,B} = \text{Hom}(TA, TB), \quad U_B = \text{End}_{TB}$$

and define left and right actions $\lambda$ and $\rho$ in terms of the horizontal and vertical operations $\times$ and $\odot$ analogous to those in the prematrad structures on $U_A$ and $U_B$ (see Section 2 above and [22]). Then the relative PROP $(U_B, U_{A,B}, U_A, \lambda, \rho)$ is the universal example of a relative prematrad.

We adopt the notation in Definition 11. $\mathcal{G}$ denotes the set of matrices over $G^\text{pre}$ and $G$ denotes the subset of bisequence matrices in $\mathcal{G}$. Given sets $G^\text{pre}_{j,i}$, $i \leq m$, $j \leq n$, $i + j < m + n$, let $\mathcal{G}_{[n,m]}$ denote the set of matrices over

$$G_{[n,m]} = \bigcup_{i \leq m, j \leq n, i + j < m + n} G^\text{pre}_{j,i}.$$ 

If $G^\text{pre}_{n,m}$ is given for each $m, n \geq 1$, then $\mathcal{G}$ denotes the set of matrices over $G^\text{pre} = G^\text{pre}_{*,*}$ and $\mathcal{C}$ denotes the subset of bisequence matrices in $\mathcal{G}$.

**Definition 12.** Given a free bigraded $R$-module $\Theta = \langle \theta^m_1 | \theta^1_1 = 1 \rangle_{m,n \geq 1}$ generated by singletons in each bidegree, let $(F^\text{pre}(\Theta), \gamma, \eta)$ be the free prematrad generated by $\Theta$. Let $\mathfrak{F} = \langle f^m_1 | m,n \geq 1 \rangle$ be a free bigraded $R$-module generated by singletons in each bidegree and set $G^\text{pre}_{1,1} = \{1\}$. Inductively, if $m + n \geq 3$ and $\mathcal{G}_{[n,m]}$ has been constructed, define

$$G^\text{pre}_{n,m} = f^m_1 \cup \{B_1 \cdots B_l \cdot C \cdot A_k \cdots A_l \mid k + l \geq 1; k,l \geq 0\},$$

where

(i) $A_i \in G^x_\Phi$ and $B_i \in G^y_\Phi$ for $(|x|, |y|) = (m,n)$;

(ii) $A_i, B_j \in \mathcal{G}_{[n,m]}$ for all $i, j$;

(iii) $C \in \mathcal{C}_{[n,m]}$; and

(vii) some association of $B_1 \cdots B_l \cdot C \cdot A_k \cdots A_l$ defines a sequence of BTPs.

Let $\sim$ be the equivalence relation on $G^\text{pre} = G^\text{pre}_{*,*}$ generated by

$$[X_{i,j}Y_{i,j}] \sim [X_{i,j}] [Y_{i,j}] \text{ iff } [X_{i,j}] \times [Y_{i,j}] \in \mathcal{G} \times \mathcal{G} \cup \mathcal{G} \times \mathcal{G} \cup \mathcal{G} \times \mathcal{G} \text{ is a BTP},$$

and let $F^\text{pre}(\Theta, \mathfrak{F}, \Theta) = (G^\text{pre}/\sim)$. The **free relative prematrad generated by** $\Theta$ and $\mathfrak{F}$ is the relative prematrad

$$(F^\text{pre}(\Theta), F^\text{pre}(\Theta, \mathfrak{F}, \Theta), F^\text{pre}(\Theta), \lambda^\text{pre}, \rho^\text{pre}),$$

where $\lambda^\text{pre}$ and $\rho^\text{pre}$ are juxtaposition.

**Example 1.** The module $F^\text{pre}_{2,2}(\Theta, \mathfrak{F}, \Theta)$ contains 25 module generators, namely, the indecomposable $f^1_2$ and the following $(\lambda^\text{pre}, \rho^\text{pre})$-decomposables:
More precisely, if $\Theta = 10$ then

$$[\theta_1^2] [ f_1 ] [ \theta_2^1 ] \quad \text{and} \quad [ \theta_1^1 ] [ f_1^1 ] [ \theta_2^1 ] [ f_1^2 ],$$

with

$$\theta_1^2 \theta_2^2 \quad \text{and} \quad \theta_1^1 \theta_2^1 \quad \text{respectively}.$$

Eleven of the form $CA_k \cdots A_1$:

$$[ f_1^2 ] [ \theta_2^1 ] [ f_1^1 ] [ \theta_1^2 ] [ f_1^1 ], \quad [ f_1^1 ] [ \theta_1^2 ] [ f_1^1 ] [ \theta_2^1 ] [ f_1^1 ], \quad [ \theta_1^2 ] [ f_1^1 ] [ \theta_2^1 ] [ f_1^1 ], \quad [ \theta_2^1 ] [ f_1^1 ] [ \theta_2^1 ] [ f_1^1 ], \quad [ \theta_2^1 ] [ f_1^1 ] [ \theta_2^1 ] [ f_1^1 ], \quad [ \theta_2^1 ] [ f_1^1 ] [ \theta_2^1 ] [ f_1^1 ],$$

And eleven respective duals of the form $B_1 \cdots B_k C$.

Example 2. Recall that the bialgebra prematrad $H_\text{pre}$ has two prematrad generators $c_{1,2}$ and $c_{2,1}$, and a single module generator $c_{n,m}$ in bidegree $(m,n)$ (see [22]). Consequently, the $H_\text{pre}$-bimodule $\mathcal{J}^\text{pre}$ has a single bimodule generator $f$ of bidegree $(1,1)$ and a single module generator in each bidegree satisfying the structure relations

$$\lambda(c_{n,m}; f_1, \ldots, f_m) = \rho(f_1, \ldots, f_m).$$

More precisely, if $\Theta = \langle \theta_1^1, \theta_2^1, \theta_2^2 \rangle$ and $\bar{f} = \langle f_1^1 \rangle$, then

$$\mathcal{J}^\text{pre} = F_\text{pre}(\Theta, \bar{f}, \Theta) / \sim$$

with $A \sim B$ if and only if $\text{bideg}(A) = \text{bideg}(B)$ for $A, B \in F_\text{pre}(\Theta, \bar{f}, \Theta)$.

A morphism $f : A \to B$ of bialgebras is the image of $f$ under a map $\mathcal{J}^\text{pre} \to U_{A,B}$ of relative prematras.

Example 3. Whereas $F_{1,*}^\text{pre}(\Theta)$ and $F_{*,1}^\text{pre}(\Theta)$ can be identified with the $A_\infty$-operad $A_\infty$ (see [22]), $F_{1,*}(\Theta, \bar{f}, \Theta)$ and $F_{*,1}(\Theta, \bar{f}, \Theta)$ can be identified with the $A_\infty$-bimodule $\mathcal{J}_\infty$ whose bimodule generators are in 1-1 correspondence with $\{ f_1^m \}_{m \geq 1}$ and $\{ f_2^n \}_{n \geq 1}$, respectively. Thus a morphism $f : A \to B$ of $A_\infty$-(co)algebras is the image of $\mathcal{J}^\text{pre}$-bimodule generators under a map of relative prematras $\mathcal{J}_\infty \to \text{Hom}(TA, B)$ (or $\mathcal{J}_\infty \to \text{Hom}(\bar{A}, \bar{T}B)$).

When $\Theta = \langle \theta_1^m \neq 0 \mid \theta_2^m \rangle_{m \geq 1}$ and $\bar{f} = \langle f_1^m \neq 0 \rangle_{m \geq 1}$, the canonical projections $\mathcal{P}_{\Theta}^\text{pre} : F_\text{pre}(\Theta) \to H_\text{pre}$ and $\bar{f}^\text{pre} : F_\text{pre}(\Theta, \bar{f}, \Theta) \to \mathcal{J}^\text{pre}$ give a map $(\mathcal{P}_{\Theta}^\text{pre}, \bar{f}^\text{pre}, \bar{\theta}_\Theta^\text{pre})$ of relative prematras. If $\theta_\Theta^\text{pre}$ is a differential on $F_\text{pre}(\Theta, \bar{f}, \Theta)$
such that $\mathcal{F}^{\text{pre}}$ is a free resolution in the category of relative prematrad, the induced isomorphism $\mathcal{F}^{\text{pre}} : H_\ast(\mathcal{F}^{\text{pre}}(\Theta, \mathfrak{S}, \Theta), \partial^{\text{pre}}) \approx \mathcal{J}^{\text{pre}}$ implies

$$\partial^{\text{pre}}(f_1^1) = 0$$

$$\partial^{\text{pre}}(f_2^1) = \rho(f_1^1; \theta_1^1) - \lambda(\theta_1^1; f_1^1, f_1^1)$$

$$\partial^{\text{pre}}(f_2^1) = \rho(f_1^1, f_1^1; \theta_1^1) - \lambda(\theta_1^1; f_1^1).$$

This gives rise to the standard isomorphisms

$$F_{1,2}^{\text{pre}}(\Theta, \mathfrak{S}, \Theta) = \langle [\theta_1^1] [f_1^1], [f_2^1], [f_1^1] [\theta_1^1] \rangle$$

$$\approx \uparrow \downarrow \uparrow \downarrow \downarrow$$

$$C_\ast(J_2) = \langle 1|2 \quad , \quad 12 \quad , \quad 2|1 \rangle$$

$$\approx \uparrow \downarrow \uparrow \downarrow \downarrow$$

$$F_{2,1}^{\text{pre}}(\Theta, \mathfrak{S}, \Theta) = \langle [\theta_2^1] [f_1^1], [f_2^1], [f_1^1] [\theta_2^1] \rangle$$

(see Figures 3 and 4). A similar application of $\partial^{\text{pre}}$ to $f_1^m$ and $f_1^m$ gives the isomorphisms

$$F_{n,1}^{\text{pre}}(\Theta, \mathfrak{S}, \Theta) \xrightarrow{\cong} J_\infty(n) = C_\ast(J_n)$$

and

$$F_{1,m}^{\text{pre}}(\Theta, \mathfrak{S}, \Theta) \xrightarrow{\cong} J_\infty(m) = C_\ast(J_m)$$

(see [23, 24, 20]). As in the absolute case, there is a differential $\partial$ on a canonical proper submodule $\mathcal{J}J_\infty \subset F^{\text{pre}}(\Theta, \mathfrak{S}, \Theta)$ such that the canonical projection $\varrho : \mathcal{J}J_\infty \to \mathcal{J}J^{\text{pre}}$ is a free resolution in the category of “relative matrads.”

**Definition 13.** Given local prematrad $(M, \gamma W_M, \eta_M)$ and $(N, \gamma W_N, \eta_N)$, a bigraded $R$-module $E = \{E_{m,n}\}_{m,n \geq 1}$, a telescoping submodule $W_E \subseteq V_E$, and actions $\lambda = \{\lambda^x : (W_M)_p^y \otimes (W_E)_p^y \to (W_E)_p^y\}$ and $\rho = \{\rho^x : (W_E)_p^y \otimes (W_N)_p^y \to (W_E)_p^y\}$, the tuple $(M, E, N, \lambda, \rho)$ is a relative local prematrad with domain $(W_M, W_E, W_N)$ if

1. $(W_E)_p^x = E^x_p$ and $(W_E)_p^y = E^y_p$ for all $x, y$;
2. $T_M, T_N, T_\lambda, T_\rho$ interact associatively on $W_E \cap E$;
3. the relative prematrad unit axiom holds for $\lambda$ and $\rho$.

Let

$$E_{\text{row}} = \bigoplus_{x, y \in \mathbb{N}^p; p, q \geq 1} E_{Y,x} \text{ and } E_{\text{col}} = \bigoplus_{x, y \in \mathbb{N} \times \mathbb{N}; p, q \geq 1} E_{Y,x},$$

where $x$ has constant columns and $y$ has constant rows. If $A$ is a $q \times p$ monomial in $E_{Y,x}$, we refer to a row of $x$ as the row leaf sequence of $A$ and write $x = \text{rls}A$; if $B$ is a $q \times p$ monomial in $E_{Y,x}$, we refer to a column of $y$ as the column leaf sequence of $B$ and write $y = \text{cls}B$. Given a telescoping submodule $W \subseteq V_E$ and its telescopic extension $W_E$, let $(W_E)_{\text{row}} = W_E \cap E_{\text{row}}$ and $(W_E)_{\text{col}} = W_E \cap E_{\text{col}}.$
Definition 14. Let \((M, E, N, \lambda, \rho)\) be a relative local prematrad with domain \((W_M, W_E, W_N)\), let \(m, n \geq 1\), let \(f \in E_{n,m}\) and \(g \in E_{n,*}\).

(i) A row factorization of \(f\) with respect to \((W_M, W_E, W_N)\) is an \((\Upsilon_\lambda, \Upsilon_\rho)\)-factorization

\[
A_1 \cdots A_s \cdot C \cdot A'_{s+1} \cdots A'_k = f
\]

such that
(a) \(A_i \in (W_M)_{\text{row}}\) and \(\text{rls}A_i \neq 1\) for all \(i\);
(b) \(C \in (W_E)_{\text{row}}\);
(c) \(A'_j \in (W_N)_{\text{row}}\) and \(\text{rls}A'_j \neq 1\) for all \(j\).

The \((k + 1)\)-tuple of row vectors

\[
m = (\text{rls}A_1, \ldots, \text{rls}A_s, (1, 0, \ldots, 0) + \text{rls}C, (1, \text{rls}A'_{s+1}), \ldots, (1, \text{rls}A'_k))
\]

is the related \((m + 1, k + 1)\)-row descent sequence of \(f\).

(ii) A column factorization of \(g\) with respect to \((W_M, W_E, W_N)\) is an \((\Upsilon_\lambda, \Upsilon_\rho)\)-factorization

\[
B'_1 \cdots B'_{t+1} \cdot D \cdot B_t \cdots B_1 = g
\]

such that
(a) \(B'_j \in (W_M)_{\text{col}}\) and \(\text{cls}B'_j \neq 1\) for all \(j\);
(b) \(D \in (W_E)_{\text{col}}\);
(c) \(B_j \in (W_N)_{\text{col}}\) and \(\text{cls}B_j \neq 1\) for all \(j\).

The \((l + 1)\)-tuple of column vectors

\[
n = \left( \begin{array}{c}
\text{cls}B'_1 \\
1
\end{array} \right), \ldots, \left( \begin{array}{c}
\text{cls}B'_{t+1} \\
1
\end{array} \right), \text{cls}D + \left( \begin{array}{c}
0 \\
\vdots \\
0 \\
1
\end{array} \right), \text{cls}B_t, \ldots, \text{cls}B_1
\]

is the related \((n + 1, l + 1)\)-column descent sequence of \(g\).

Column and row factorizations are not unique. Note that \(f \in E_{n,m}\) always has trivial row and column factorizations as a \(1 \times 1\) matrix \(C = [f]\).

Given elements \(f \in E_{n,m}\) and \(g \in E_{n,*}\) with \(m, n \geq 1\), choose a row factorization \(A_1 \cdots A_s \cdot C \cdot A'_{s+1} \cdots A'_k\) of \(f\) and a column factorization \(B'_1 \cdots B'_{t+1} \cdot D \cdot B_t \cdots B_1\) of \(g\). The related row descent sequence \(m\) identifies \(f\) with an up-rooted \((m + 1)\)-leaf, \((k + 1)\)-level planar rooted tree with levels (PLT) and hence with a codimension \(k\) face \(\hat{e}_f\) of \(P_m\). Dually, the related column descent sequence \(n\) identifies \(g\) with a down-rooted \((n + 1)\)-leaf, \((l + 1)\)-level PLT and hence with a codimension \(l\) face \(\hat{e}_g\) of \(P_n\). Extending to Cartesian products, identify the monomials \(F = f_1 \otimes \cdots \otimes f_g \in (E_{n,m})^{\otimes g}\) and \(G = g_1 \otimes \cdots \otimes g_p \in (E_{n,*})^{\otimes p}\) with the product cells

\[
\hat{e}_F = \hat{e}_{f_1} \times \cdots \times \hat{e}_{f_q} \subset P^{x q}_m \quad \text{and} \quad \hat{e}_G = \hat{e}_{g_1} \times \cdots \times \hat{e}_{g_p} \subset P^{x p}_n.
\]

As in the absolute case discussed in Section 2, the product cell \(\hat{e}_F\) either is or is not a subcomplex of \(\Delta^{(q-1)}(P_m) \subset P^{x q}_m\), and dually for \(\hat{e}_G\). Recall that \(x \times y \in N^{1 \times p} \times N^{q \times 1}\) and let

\[
r \Gamma(E) = E \oplus \bigoplus_{x, y \in \mathbb{N}^{1 \times p}, \, s, t \geq 1} r \Gamma^x(E) \oplus r \Gamma^x(E),
\]
where
\[ r \Gamma_x^r(E) = \left\{ F \in E^r_x \mid \hat{e}_F \subset \Delta^{(q-1)}(P_s) \right\}, \]
\[ r \Gamma_y^l(E) = \left\{ G \in E^l_y \mid \hat{e}_G \subset \Delta^{(p-1)}(P_t) \right\}. \]

**Definition 15.** The (left) configuration module of a relative local prematrad \((M, E, N, \lambda, \rho)\) with domain \((W_M, W_E, W_N)\) is the triple
\[ \left( \Gamma(M, \gamma_{W_M}), r \Gamma(E), \Gamma(N, \gamma_{W_N}) \right). \]

When \(s = t = 1\), the arguments in the absolute case carry over verbatim and give
\[ \bigoplus_x r \Gamma_x^r(E) = T^+(E_{1,s}) \quad \text{and} \quad \bigoplus_y r \Gamma_y^l(E) = T^+(E_{s,1}). \]

**Definition 16.** Let \((M, \gamma_{W_M})\) and \((N, \gamma_{W_N})\) be (left) matrads. A relative local prematrad \((M, E, N, \lambda, \rho)\) with domain \((W_M, W_E, W_N)\) is a relative (left) matrad if \(r \Gamma(E) = W_E\). When \(M = N\) we refer to \(r \Gamma(E)\) as a \(\Gamma(M)\)-bimodule. A morphism of relative matrads is a map of underlying relative local prematrad.

A relative prematrad \((M, E, N, \rho, \lambda)\) with domain \((W_M, W_E, W_N)\) restricts to a relative matrad structure with domain \((\Gamma(M), r \Gamma(E), \Gamma(N))\).

**Example 4.** The Bialgebra Morphism Matrad \(\mathcal{J}.\) The \(H^{\text{pre}}\)-bimodule \(\mathcal{J} \mathcal{J}^{\text{pre}}\) discussed in Example 2 satisfies
\[ r \Gamma^p_p(E) \otimes \Gamma^q_q(M) = E_p^r \otimes M_q^s \quad \text{and} \quad \Gamma^r_r(M) \otimes r \Gamma^l_l(E) = M^s_y \otimes E^u_u. \]

Consequently, \(\mathcal{J} \mathcal{J}^{\text{pre}}\) is also a relative matrad, called the bialgebra morphism matrad, and is henceforth denoted by \(\mathcal{J}\).

**Definition 17.** Let \(\Theta = \langle \theta^m_n \mid \theta^1_1 = 1 \rangle_{m,n \geq 1}\) and \(\hat{\Theta} = \langle \hat{p}^m_n \rangle_{m,n \geq 1}\). Let \(F^{\text{pre}} = F^{\text{pre}}(\Theta)\) and consider the free \(F^{\text{pre}}\)-bimodule \(F^{\text{pre}} = F^{\text{pre}}(\Theta, \hat{\Theta}, \Theta)\). The free relative matrad generated by \((\Theta, \hat{\Theta})\) with domain \((W_{F(\Theta)}, W_E, W_{F(\Theta)})\) is the tuple \((F(\Theta), F(\Theta, \hat{\Theta}, \Theta), F(\Theta), \lambda, \rho)\), where
\[ F(\Theta, \hat{\Theta}, \Theta) = \lambda^{\text{pre}} \left[ \Gamma^p_p(F^{\text{pre}}) \right] \otimes \rho^{\text{pre}} \left[ \Gamma^q_q(F^{\text{pre}}) \right] = \lambda^{\text{pre}} |_{W_{F(\Theta)}} \otimes_{W_E} \rho^{\text{pre}} |_{W_{F(\Theta)}}, \quad \text{and} \quad \mathcal{W} = \bigoplus_{p,q \in \mathbb{N}; x,y \in \mathbb{N}} F_{x,y}(\Theta, \hat{\Theta}, \Theta) \otimes \Gamma^p_p(E^{\text{pre}}) \otimes \Gamma^q_q(E^{\text{pre}}). \]

For example, the monomials in (3.1) are two of the 17 module generators in \(F_{2,2}(\Theta, \hat{\Theta}, \Theta)\) identified with the faces of the octagon \(JJ_{2,2}\) (see Figure 5). In general, \(F_{1,m}(\Theta, \hat{\Theta}, \Theta) = F_{1,m}^{\text{pre}}(\Theta, \hat{\Theta}, \Theta)\) and \(F_{n,1}(\Theta, \hat{\Theta}, \Theta) = F_{n,1}^{\text{pre}}(\Theta, \hat{\Theta}, \Theta)\) for all \(m, n \geq 1\).

**Example 5.** The \(A_{\infty}\)-bialgebra Morphism Matrad. Let \(\Theta = \langle \theta^m_n \neq 0 \mid \theta^1_1 = 1 \rangle_{m,n \geq 1}\) and \(\hat{\Theta} = \langle \hat{p}^m_n \neq 0 \rangle_{m,n \geq 1}\), where \(\mid \hat{p}^m_n \mid = m + n - 2\). Following the construction in the absolute case, let \(r \mathcal{C}\) be the set indexing the module generators \(F(\Theta, \hat{\Theta}, \Theta)\), and let \(\mathcal{A} \mathcal{R}_{m,n} \subset C_{m,n} \times rC_{m,n}\) and \(\mathcal{Q} \mathcal{B}_{m,n} \subset rC_{m,n} \times C_{m,n}\) be the subsets of \(r \mathcal{C}\) that index the codimension 1 elements of \(F_{n,m}(\Theta, \hat{\Theta}, \Theta)\) of the form \(\lambda(-,-)\) and \(\rho(-,-)\), respectively. Let \(\{A^v_x\}_{x} \) and \(\{B^y_y\}_{y}\) be the bases defined in Example 19 of [22], and let \(\{Q_x^r\}_{x} \) and \(\{R^l_u\}_{u}\) be the analogous bases for \(r \Gamma_x^r(E)\) and
over \( \exists \) a unique BTP \( \Upsilon \)-factorization \( v \), and then each \( \hat{e}_Q \), is a subcomplex of \( \Delta^{(t-1)}(P_s) \) with associated sign \( (-1)^{s_r} \) and each \( \hat{v}_{R_m} \) is a subcomplex of \( \Delta^{(s-1)}(P_t) \) with associated sign \( (-1)^{s_u} \). Define a differential \( \partial : F(\Theta, \tilde{g}, \Theta) \to F(\Theta, \tilde{g}, \Theta) \) of degree \(-1 \) on generators by

\[
\partial(p^n_m) = \sum_{(a, \mu) \in AR_{m,n}} (-1)^{e_1 + e_\mu + e_\lambda} \left( (A^n_p)_a; (R^n_q)_{\mu} \right) + \sum_{(r, \beta) \in QB_{m,n}} (-1)^{e_2 + e_r + e_\beta} \rho \left[ (Q^n_p)_r; (B^n_q)_{\beta} \right],
\]

where \(( -1 )^{e_1} \) is the sign associated with the codimension 1 cell \( e_i = C[D] \subset P_{m+n-2} \) defined in line (6.1) of [22], and \( e_2 = \#D + e_\mu + 1 \). Extend \( \partial \) as a derivation of \( \rho \) and \( \lambda \); then \( \partial^2 = 0 \) follows from the associativity of \( \rho \) and \( \lambda \). The \( A_\infty \)-bialgebra morphism matrad is the DG \( H_\infty \)-bimodule \( JJ_\infty = (F(\Theta, \tilde{g}, \Theta), \partial) \).

We realize the \( A_\infty \)-bialgebra morphism matrad by the cellular chains of a new family of polytopes \( JJ = \bigsqcup_{m,n \geq 1} JJ_{n,m} \), called \( \text{bimultiplihedra} \), to be constructed in the next section. The standard isomorphisms \((3.2) \) and \((3.3) \) extend to isomorphisms

\[
(JJ_\infty)_{n,m} \cong C_*(JJ_{n,m}),
\]

and one recovers \( J_\infty \) by restricting the differential \( \partial \) to \((JJ_\infty)_{1,*} \) or \((JJ_\infty)_{*,1} \).

4. BIMULTIPLIHEDRA

In this section we construct the bimultiplihedron \( JJ_{n,m} \) as a subdivision of the cylinder \( KK_{n,m} \times I \). Whereas our construction of \( KK_{n+1,m+1} \) uses the combinatorics of \( P_{m,n} \), our construction of \( JJ_{n+1,m+1} \) uses the combinatorics of \( P_{m+n+1} \) thought of as a subdivision of \( P_{m+n} \times I \) (see [20]). And indeed, \( JJ_{n+1,m+1} \) is the geometric realization of a poset \( JJ_{n+1,m+1} \) whose construction resembles that of \( KK_{n+1,m+1} = PP_{n,m+1} / \sim \) given in [22], but with some technical differences.

Recall that faces of the permutahedron \( P_n \) are indexed by up-rooted (or down-rooted) planar rooted leveled trees (PLTs). In particular, the vertices of \( P_n \) are indexed by the set \( \wedge_n \) of all up-rooted (down-rooted) planar binary trees with \( n \) leaves and \( n \) levels. Since vertices of \( P_n \) are identified with permutations of \( \underline{n} = \{1, 2, \ldots, n\} \), the Bruhat partial ordering, generated by \( a_1 | \cdots | a_n < a_1 | \cdots | a_{i+1} | a_i | \cdots | a_n \) if and only if \( a_i < a_{i+1} \), induces natural poset structures on \( \wedge_n \) and \( \vee_n \).

In [22] we introduced the subposet \( X_n^m \subseteq \wedge_n^m \), which indexes the vertices of the subcomplex \( \Delta^{(n-1)}(P_m) \subseteq P_{m^\times n} \); an element \( x \in X_n^m \) is expressed as a column matrix \( x = [T_1 \cdots T_n]^T \) of \( n \) up-rooted binary trees with \( m+1 \) leaves and \( m \) levels. Let \( \wedge \) (\( \vee \)) denote the up-rooted (down-rooted) 2-leaf corolla. Now if \( a \in \wedge_n \), there exists a unique BTP \( T \)-factorization \( a = a_1 \cdots a_n \) such that \( a_j \) is a \( 1 \times j \) row matrix over \( \{1, \lambda\} \) containing the entry \( \lambda \) exactly once. Thus by factoring each \( \hat{T}_i \), we obtain the BTP \( \Upsilon \)-factorization \( x = x_1 \cdots x_m \) in which \( x_j \) is an \( n \times j \) matrix (see Example [6]).
Let \( \varrho(x, i) \) be the \emph{lowest level} of \( \hat{T}_i \) in which a branch is attached on the extreme left. Given positive integers \( \alpha \leq \beta \), consider the set of \emph{bitruncated elements}

\[
X^n_m(\alpha, \beta) = \left\{ \alpha x_\beta = x_\alpha \cdots x_\beta \mid x = x_1 \cdots x_\alpha \cdots x_\beta \cdots x_m \in X^n_m, \right. \\
\alpha = \min_{1 \leq i \leq n} \varrho(x, i) \text{ and } \beta = \max_{1 \leq i \leq n} \varrho(x, i) \bigg\}.
\]

Define \( \alpha x_\beta < \alpha' x'_\beta \) if and only if \( x < x' \) for some \( x = x_1 \cdots x_\alpha \cdots x_\beta \cdots x_m \) and \( x' = x'_1 \cdots x'_\alpha \cdots x'_\beta \cdots x'_m \) in \( X^n_m \); then \( X^n_m(\alpha, \beta) \) is the poset of \emph{bitruncated elements}.

Now replace the entries \( 1 \) and \( \lambda \) in each \( x_j \) with the integers \( 1 \) and \( 2 \), respectively. Then the \((m + 1, m)\)-row descent sequence \( C_i = (x_{i,1}, x_{i,2}, \ldots, x_{i,m}) \) of \( \hat{T}_i \) appears as the \( i^{th} \) rows of \( x_1, \ldots, x_m \), and \( \varrho = \varrho(x, i) \) is the largest integer such that \( x_{i,\varrho} = (2, 1, \ldots, 1) \) and \( \hat{x}_{i,j} = (1, 1, \ldots, 2, 1, \ldots) \) for \( j > \varrho \), and obtain the corresponding \((m + 1, m)\)-marked row descent sequence

\[
\hat{C}_i = (x_{i,1}, \ldots, x_{i,\varrho - 1}, \hat{x}_{i,\varrho}, \hat{x}_{i,\varrho + 1}, \ldots, \hat{x}_{i,m}).
\]

The term \( \hat{x}_{i,\varrho} \) represents the constant \( 1 \times \varrho \) matrix \([1 \cdots 1]\). Let \( \hat{x} \) denote the matrix string \( x = x_1 \cdots x_n \) with marked integer entries; then the poset of \emph{marked bitruncated elements} is

\[
\hat{X}^n_m(\alpha, \beta) = \{ \alpha \hat{x}_\beta \mid x_\beta \in X^n_m(\alpha, \beta) \}.
\]

Note that \( \varrho(x, i) \) is constant for all \( i \) if and only if \( \alpha = \beta \), in which case \( \hat{X}^n_m(\alpha, \alpha) \) is a singleton set containing the \( n \times \alpha \) matrix

\[
\begin{bmatrix}
\hat{x}_{1,\alpha} \\
\vdots \\
\hat{x}_{n,\alpha}
\end{bmatrix}
= \begin{bmatrix}
2 & 1 & \cdots & 1 \\
\vdots & \vdots & \ddots & \vdots \\
2 & 1 & \cdots & 1
\end{bmatrix},
\]

which represents the constant matrix \([1 \cdots 1]^{n \times \alpha}\).

Given \( \alpha \hat{x}_\beta \in \hat{X}^n_m(\alpha, \beta) \), consider the marked sequence of \( i^{th} \) rows

\[
(x_{i,\alpha}, \ldots, x_{i,\varrho - 1}, \hat{x}_{i,\varrho}, \hat{x}_{i,\varrho + 1}, \ldots, \hat{x}_{i,\beta}).
\]

Let \( x'_{i,k} \) denote the vector obtained from \( \hat{x}_{i,k} \) by deleting the marked entry \( \hat{1} \), and form the (unmarked) row-descent sequence

\[
(x''_{i,\alpha}, \ldots, x''_{i,\varrho - 1}) = (x_{i,\alpha}, \ldots, x_{i,\varrho - 1}, x'_{i,\varrho + 1}, \ldots, x'_{i,\beta}).
\]

Then \( \alpha x''_{\varrho - 1} \) denotes the bitruncated element whose \( i^{th} \) rows are \( x''_{i,\alpha}, \ldots, x''_{i,\varrho - 1} \).

**Example 6.** Consider the following element \( x \in X^2_6 \) and its BTP factorization:

\[
x = \left[ \begin{array}{ccc}
\flat & \flat & \flat \\
\flat & \flat & \flat \\
\flat & \flat & \flat \\
\flat & \flat & \flat \\
\flat & \flat & \flat \\
\flat & \flat & \flat
\end{array} \right] = \left[ \begin{array}{ccc}
\flat & \flat & \flat \\
\flat & \flat & \flat \\
\flat & \flat & \flat \\
\flat & \flat & \flat \\
\flat & \flat & \flat \\
\flat & \flat & \flat
\end{array} \right],
\]

where the dotted lines indicate the lowest levels in which branches are attached on the extreme left (\( \alpha = \varrho(x, 2) = 2 \) and \( \beta = \varrho(x, 1) = 4 \)). Then

\[
\hat{x} = \left[ \begin{array}{ccc}
2 & 2 & 1 \\
2 & 2 & 1 \\
1 & 1 & 1 \end{array} \right] \left[ \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array} \right] \left[ \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array} \right] \in \hat{X}^2_6.
\]
and
\[ 2\hat{x}_4 = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix} \in X^2_6(2, 4). \]

The projections
\[(2, 1), (1, 2, 1), (\hat{2}, \hat{1}, \hat{1}) \mapsto ((2, 1), (1, 2, 1)) \]

and
\[(\hat{2}, \hat{1}), (1, 2, 1), (\hat{1}, 1, 2, 1) \mapsto ((2, 1), (1, 2, 1)) \]
send \(2\hat{x}_4 \) to
\[ 2\hat{x}_3'' = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}. \]

Dually, there is the subposet \( Y^m_n \subseteq \vee^m_n \), which indexes the vertices of the subcomplex \( \Delta^{(m-1)}(P_n) \subseteq P^m_n \); an element \( y \in Y^m_n \) is expressed as a row matrix \( y = [T_1 \cdots T_m] \) of \( m \) down-rooted binary trees with \( n + 1 \) leaves and \( n \) levels. Now if \( b \in \vee_n \), there exists a unique BTP T-factorization \( b = b_n \cdots b_1 \) such that \( b_i \) is a \( i \times 1 \) column matrix over \( \{1, \gamma\} \) containing the entry \( \gamma \) exactly once. Thus by factoring each \( T_j \), we obtain the (unique) BTP T-factorization \( y = n_y \cdots y_1 \) in which \( y_j \) is an \( i \times m \) matrix (see Example [7]).

Let \( \kappa(y, j) \) be the highest level of \( T_j \) in which a branch is attached on the extreme right. Given positive integers \( \epsilon \geq \delta \), consider the set of bitruncated elements
\[ Y^m_n(\epsilon, \delta) = \{ y \in \bigcup_{1 \leq j \leq m} V^m, y = y_n \cdots y_1 \in [1, \gamma], \epsilon = \max_{1 \leq j \leq m} \kappa(y, j) \text{ and } \delta = \min_{1 \leq j \leq m} \kappa(y, j) \}. \]

Define \( \epsilon y \delta < \epsilon'y \delta' \) if and only if \( y < y' \) for some \( y = y_n \cdots y_1 \) and \( y' = y_n' \cdots y_1' \) in \( Y^m_n \); then \( Y^m_n(\epsilon, \delta) \) is the poset of bitruncated elements.

Now replace the symbols \( 1 \) and \( \gamma \) in each \( y_i \) with the integers 1 and 2, respectively. Then the \( (n + 1, n) \)-column descent sequence \( D_j = (y_{n,j}, \ldots, y_{1,j}) = (2) \) of \( T_j \) appears as the \( j \)-th columns of \( n_y, \ldots, y_1 \), and \( \kappa = \kappa(y, j) \) is the largest integer such that \( y_{\kappa,j} = (1, 2, 1, 2)^T \) and \( y_{i,j} = (\ldots, 1, 2, 1, 1)^T \) for \( i > \kappa \), and obtain the corresponding \( (n + 1, n) \)-marked column descent sequence
\[ \hat{D}_j = (\hat{y}_{n,j}, \ldots, \hat{y}_{\kappa+1,j}, \hat{y}_{\kappa,j}, y_{\kappa-1,j}, \ldots, y_{1,j}). \]

The term \( \hat{y}_{\kappa,j} \) represents the constant \( \kappa \times 1 \) matrix \( [1 \cdots 1]^T \). Let \( \hat{y} \) denote the matrix string \( y = y_n \cdots y_1 \) with decorated integer entries; then the poset of marked bitruncated elements is
\[ Y^m_n(\epsilon, \delta) = \{ \epsilon y \delta \mid \epsilon y \delta \in Y^m_n(\epsilon, \delta) \}. \]

Note that \( \kappa(y, j) \) is constant for all \( j \) if and only if \( \delta = \epsilon \), in which case \( Y^m_n(\delta, \delta) \) is a singleton set containing the \( \delta \times m \) matrix
\[ s \hat{y} \delta = [\hat{y}_{s,1} \cdots \hat{y}_{s,m}] = \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ 1 & \cdots & 1 \\ 2 & \cdots & 2 \end{bmatrix}, \]

which represents the constant matrix \( [1]_1^{\delta \times m} \).
Given \( \epsilon \check{y}_\delta \in \check{Y}_n^m(\epsilon, \delta) \), consider the marked sequence of \( j^{th} \) columns

\[
(\check{y}_{\epsilon,j}, \ldots, \check{y}_{\kappa+1,j}, \check{y}_{\kappa,j}, \check{y}_{\kappa-1,j}, \ldots, \check{y}_{\kappa,j}).
\]

Let \( \check{y}_{k,j} \) denote the vector obtained from \( \check{y}_{k,j} \) by deleting the marked entry \( \check{1} \), and form the (unmarked) column-descent sequence

\[
(\check{y}''_{\epsilon-1,j}, \ldots, \check{y}''_{\delta,j}) = (\check{y}'_{\epsilon-1,j}, \ldots, \check{y}'_{\kappa+1,j}, \check{y}_{\kappa-1,j}, \ldots, \check{y}_{\kappa,j}).
\]

Then \( \epsilon-1\check{y}''_\delta \) denotes the bitruncated element whose \( j^{th} \) columns are \( \check{y}''_{\epsilon,j}, \ldots, \check{y}''_{\delta,j} \).

**Example 7.** Consider the following element \( y \in Y_6^2 \) and its BTP factorization:

\[
y = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 1 & 1 & 1 & 1 \\
\end{bmatrix}
\]

where the dotted lines indicate the highest levels in which branches are attached on the extreme right (\( \epsilon = \kappa(y,1) = 4 \) and \( \delta = \kappa(y,2) = 2 \)). Then

\[
\check{y} = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 \\
2 & 2 & 1 & 1 & 1 \\
i & i & i & i & i \\
1 & 1 & 1 & 1 & 1 \\
i & i & i & i & i \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
i & i \\
i & i \\
i & i \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
i & i \\
i & i \\
i & i \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
i & i \\
i & i \\
i & i \\
i & i \\
\end{bmatrix}
\begin{bmatrix}
2 & 2 \\
1 & 1 \\
\end{bmatrix}
\in \check{Y}_6^2
\]

and

\[
4\check{y}_2 = \begin{bmatrix}
1 & i \\
2 & 1 \\
i & i \\
i & i \\
i & i \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
1 & 1 \\
1 & 1 \\
i & i \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
i & i \\
i & i \\
i & i \\
\end{bmatrix}
\begin{bmatrix}
2 & 2 \\
1 & 1 \\
\end{bmatrix}
\in \check{Y}_6^2(4,2).
\]

The projections

\[
\left(\begin{bmatrix}
1 \\
2 \\
\end{bmatrix}, \begin{bmatrix}
i \\
1 \\
\end{bmatrix}\right) \mapsto \left(\begin{bmatrix}
1 \\
2 \\
\end{bmatrix}, \begin{bmatrix}
i \\
2 \\
\end{bmatrix}\right)
\]

and

\[
\left(\begin{bmatrix}
i \\
1 \\
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
\end{bmatrix}\right) \mapsto \left(\begin{bmatrix}
i \\
1 \\
\end{bmatrix}, \begin{bmatrix}
1 \\
2 \\
\end{bmatrix}\right)
\]

send \( 4\check{y}_2 \) to

\[
3\check{y}_2'' = \begin{bmatrix}
1 & 1 \\
2 & 2 \\
1 & 1 \\
\end{bmatrix}
\begin{bmatrix}
1 & 1 \\
2 & 2 \\
\end{bmatrix}.
\]

Now extend the poset structures on these sets of bitruncated elements to

\[
\check{X}_m^j(\alpha, \beta) \cup \check{Y}_m^i(\epsilon, \delta).
\]
in the following way: Given $\alpha x_\beta \in X^I_{\delta}(\alpha, \beta)$ and $\epsilon y_\delta \in \tilde{Y}^I_m(\epsilon, \delta)$, define $\alpha x_\beta \leq \epsilon y_\delta$ if $\alpha \geq i$ and $j \leq \delta$, and $\epsilon y_\delta \leq \alpha x_\beta$ if $\alpha \leq i$ and $j \geq \delta$. Then $\alpha = \beta$ and $\epsilon = \delta$, in which case $\alpha x_\alpha = \delta y_\delta$. This equality reflects the correspondence

$$
\begin{bmatrix}
\dot{2} & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
\dot{2} & \ddots & 1
\end{bmatrix} \delta \times \alpha
\leftrightarrow
\begin{bmatrix}
\dot{1} & \ddots & \ddots \\
\vdots & \ddots & \ddots \\
1 & \ddots & \ddots
\end{bmatrix} \delta \times \alpha
$$

Let $A = [a_{ij}]$ be an $(n + 1) \times m$ matrix over $\{1, \lambda\}$, each row of which contains the entry $\lambda$ exactly once, and let $B = [b_{ij}]$ be an $n \times (m + 1)$ matrix over $\{1, \gamma\}$, each column of which contains the entry $\gamma$ exactly once. Recall that a BTP $(A, B)$ is an $(i, j)$-edge pair if $a_{ij} = a_{i+1,j} = \lambda$ and $b_{ij} = b_{i,j+1} = \gamma$. Note that $(x_m, y_n)$ is the only potential edge pair in a matrix string $x_1 \cdots x_m y_n \cdots y_1 \in X^{n+1}_m \times Y^{m+1}_n$.

Given an $(i, j)$-edge pair $(A, B)$, let $A^i$ and $B^{i+1}$ denote the matrices obtained by deleting the $i$th row of $A$ and the $j$th column of $B$. If $c = C_1 \cdots C_k C_{k+1} \cdots C_r$ is a matrix string in which $(C_k, C_{k+1})$ is an $(i, j)$-edge pair, the $(i, j)$-transposition of $c$ in position $k$ is the matrix string $T_{ij}^k(c) = C_1 \cdots C^*_{k+1} C^i_k \cdots C_r$.

If $(x_m, y_n)$ is an $(i, j)$-edge pair in $u = x_1 \cdots x_m y_n \cdots y_1 \in X^{n+1}_m \times Y^{m+1}_n$, then $(x_{m-1}, y_n, y_n')$ are the potential edge pairs in $T_{ij}^m(u)$. If $(x_{m-1}, y_n')$ is a $(k, l)$-edge pair, then $(x_{m-2}, y_n'^{k+l})$ is a potential edge pair in $T_{kl}^{m-1} T_{ij}^m(u) = x_1 \cdots x_{m-2} y_n'^{k+l} x_{m-1} x_m y_n y_{n-1} \cdots y_1$, and so on. Clearly, $T_{ij}^{k_1} \cdots T_{ij}^{k_i} (u)$ uniquely determines a shuffle permutation $\sigma \in \Sigma_{m,n}$. On the other hand, $(A, B)$ can be an $(i, j)$-edge pair for multiple values of $i$ and $j$, in which case distinct collections $T_{ij}^{k_1} \cdots T_{ij}^{k_i}$ act on $u$ and determine the same $\sigma$. Thus $T_{Id} := \text{Id}$ and

$$
T_\sigma(u) := \left\{ T_{ij}^{k_1} \cdots T_{ij}^{k_i} (u) \mid T_{ij}^{k_1} \cdots T_{ij}^{k_i} \text{ determines } \sigma \in \Sigma_{m,n} \right\}.
$$

Recall that $PP_{n,m} = X^{n+1}_m \times Y^{m+1}_n \cup Z_{n,m}$, where

$$
Z_{n,m} = \{ T_\sigma(u) \mid u \in X^{n+1}_m \times Y^{m+1}_n \text{ and } \sigma \in \Sigma_{m,n} \setminus \{\text{Id}\}\}.
$$

The poset $rPP_{n,m}$ is built upon $PP_{n,m}$. Given $u = x_1 \cdots x_m y_n \cdots y_1$ and $a = T_\sigma(u)$, let $x_\#^i$ and $y_\#^j$ denote either $x_i$ or $y_j$ or their respective transpositions in $a$. Let

$$
X'_{n,m} = X'_{n,m} \cup X''_{n,m}, \quad m + n > 0, m, n \geq 0,
$$

where

$$
X'_{n,m} = \left\{ (a, \alpha \dot{x}_\alpha) \in \bigcup_{1 \leq \alpha \leq m+1} \bigcup_{1 \leq i \leq n+1} PP_{n,m} \times \tilde{X}^i_{m+1}(\alpha, \alpha) \mid \alpha x_\#^i \text{ has } i \text{ rows} \right\},
$$

and

$$
X''_{n,m} = \left\{ (a, \alpha \ddot{x}_\beta) \in \bigcup_{1 \leq \alpha \leq \beta \leq m+1} \bigcup_{1 \leq i \leq n+1} PP_{n,m} \times \tilde{X}^i_{m+1}(\alpha, \beta) \mid \alpha x_{\beta-1}'' \text{ is a substring of } a \right\}.
$$
Dually, let
\[ \mathcal{Y}_{n,m} = \mathcal{Y}'_{n,m} \cup \mathcal{Y}''_{n,m}, \quad m + n > 0, m, n \geq 0, \]
where
\[ \mathcal{Y}'_{n,m} = \left\{ (a, y) \in \bigcup_{1 \leq \delta \leq n+1} \mathcal{PP}_{n,m} \times \hat{Y}_{n+1}^j(\delta, \delta) \ \bigg| \ y^{\#} \ \text{has} \ j \ \text{columns} \right\}, \]

and
\[ \mathcal{Y}''_{n,m} = \left\{ (a, e) \in \bigcup_{1 \leq \delta \leq n+1} \mathcal{PP}_{n,m} \times \hat{Y}_{n+1}^j(\epsilon, \delta) \ \bigg| \ \epsilon^{-1}y'' \ \text{is a substring of} \ b \right\}. \]

Define
\[ r\mathcal{PP}_{n,m} = \mathcal{X}_{n,m} \cup \mathcal{Y}_{n,m} \]
with the poset structure generated by \((a, b) \leq (a', b')\) for
- \(a = a'\) and \(b \leq b'\);
- \(a \leq a'\) and \(b = b'\) such that \(u' = (\nu_x \times \nu_y)u\) for \(\nu_x \in S_{\alpha+1}^{x,n+1} \times S_{\beta+1}^{x,n+1} \subset S_{m+i}^{x,n+1} \times S_{m+i+1}^{x,n+1}, \nu_y \in S_{\alpha-i-1}^{x,n+1} \times S_{\alpha-i+1}^{x,n+1} \subset S_{n-m}^{x,n+1}, b \in X_{m+1}(\alpha, \beta), 1 \leq \alpha \leq \beta \leq m + 1, \) or \(\nu_x \in S_{\alpha-i-1}^{x,n+1} \times S_{\alpha-i+1}^{x,n+1} \subset S_{n-m}^{x,n+1}, \nu_y \in S_{\alpha-i}^{x,n+1} \times S_{m+i}^{x,n+1} \subset S_{n-m+1}^{x,n+1}, b \in Y_{n+1}(\epsilon, \delta), 1 \leq \delta \leq \epsilon \leq n + 1 \) with convention that \(S_0 \times S_k = S_k \times S_0 = S_k, k \geq 1.\)

Note that when \(a = m + 1\) in \(X'_{n,m}\) or \(\delta = n + 1\) in \(Y'_{n,m},\) in both cases \(a \in X_{n+1}^{m+1} \times Y_{n+1}^{m+1} \subset \mathcal{PP}_{n,m}\) for \(m, n \geq 1;\) hence, the poset structure automatically identifies the subset
\[ \mathcal{X}_{n,m} := (X_{m+1}^{n+1} \times Y_{m+1}^{n+1}) \times \hat{X}_{n+1}^n(m+1,m,1) \subset \mathcal{X}_{n,m} \]
with the subset
\[ \mathcal{Y}_{n,m} := (X_{m+1}^{n+1} \times Y_{m+1}^{n+1}) \times \hat{Y}_{n+1}^n(n+1,n+1) \subset \mathcal{Y}_{n,m}. \]

Note also that \(X'_{m,0} = X''_{m,0} = X''_{0,n} = Y''_{0,n} = Y''_{n,0} = \emptyset;\) thus \(r\mathcal{PP}_{0,m} = X'_{0,m}\) and \(r\mathcal{PP}_{n,0} = Y''_{n,0}.\)

The poset \(rJ_{n+1,m+1}\) is the image of the quotient map \(r\mathcal{PP}_{n,m} \to r\mathcal{PP}_{n,m}/\sim\) given by restricting the map \(\mathcal{PP}_{0,\alpha} \to \mathcal{KK}_{n+1} \times I\) to left-hand factors, and \(rJ_{n+1,m+1}\) is the geometric realization \(|rJ_{n+1,m+1}|.\)

The poset structure of \(r\mathcal{PP}_{n,m}\) corresponds to the “cylindrical poset” \(\mathcal{KK} \times I\) in the following way. Given \(a \in \mathcal{KK}_{n,m}\) and \(t \geq m + n + 1,\) consider a partition
\[(a, b_1) < \cdots < (a, b_t)\]
of the interval \(a \times I,\) where \((a, b_1) = a \times 0\) and \((a, b_t) = a \times 1.\) If \(a\) indexes a vertex \(a_1 | \cdots | a_n+m \in P_{n+m},\) and \(b_{t+1}\) denotes the single element of \(X_{n+1}^n(\alpha, \alpha),\) then \((a, b_{t+1})\) indexes the vertex
\[ a_1 | \cdots | a_{n+\alpha-j} | m+n+1 | a_{n+1+\alpha-j} | \cdots | a_{m+n} \in P_{m+n+1}; \]
and in particular,
\[(a, b_1) \leftrightarrow a_1 | \cdots | a_{m+n} | m+n+1 \text{ and } (a, b_t) \leftrightarrow m+n+1 | a_1 | \cdots | a_{m+n}.\]
Note that this correspondence agrees with the combinatorial representation of \( P_{m+n+1} \) as a subdivision of the cylinder \( P_m \times P_2 = P_{m+n} \times \mathbb{I} \) (see (20)), or equivalently, with the combinatorial join \( P_{m+n+1} = P_{m+n} \ast P_1 \) (see (22)). If in addition, \( b_\delta \in \mathcal{X}_{m+1}(\gamma,\beta) \) with \( \beta = \alpha + 1 \), then \( (a, b_\delta) \) subdivides the interval \([(a, b_{n+1}), (a, b_n)]\) (recall that \( b_{n+1} < b_n \)).

In the octagon \( JJ_{2,2} \) in Figure 5 and Example 8 below, we have
\[
(1|2, b_1) \prec (1|2, b_2) \prec (1|2, b_3) \prec (1|2, b_\delta) = 1|2|3 < v_1 < 1|3|2 < v_2 < 3|1|2,
\]
where \( v_1 \leftrightarrow b_2 \in \mathcal{X}_2^3(1,2) \) and \( v_2 \leftrightarrow b_4 \in \mathcal{X}_2^3(1,2) \); on the other hand,
\[
(2|1, b_1) \prec (2|1, b_2) \prec (2|1, b_\delta) = 2|1|3 < 2|3|1 < 3|2|1.
\]

Now consider an element \((a, b) \in \mathcal{P}P_{n,m} \times \mathcal{X}_{m+1}^3(\alpha, \beta) \subset r\mathcal{P}P_{n,m} \) and recall that \( a = a_1 \cdots a_{n+m} \) is represented by a piecewise linear path in \( \mathbb{N}^2 \) with \( m+n \) directed components. The element \((a, b)\) is represented by a piecewise linear path in \( \mathbb{N}^3 \) of the form \( BCA \) with \( m+n+1+\alpha-\beta \) directed components from \((m+1,1,0) \in \mathbb{N}^2 \times 0 \) to \((1,n+1,1) \in \mathbb{N}^2 \times 1 \). The component \( A \) is represented by the path from \((m+1,1,0) \) to \((\beta,j,0) \) in \( \mathbb{N}^2 \times 0 \) corresponding to \( a_{n+\beta-1} \cdots a_{n+m} \); the component \( C \) is represented by the path \((\beta,j,0) \rightarrow (\alpha,j,1) \); and the component \( B \) is represented by the path from \((\alpha,j,1) \) to \((1,n+1,1) \) in \( \mathbb{N}^2 \times 1 \) corresponding to \( a_1 \cdots a_{n+\alpha-1} \) (see Figure 1). Note that the arrow representing \( C \) is perpendicular to both integer lattices if and only if \( \alpha = \beta \), in which case the path has \( m+n+1 \) directed components. The case of \((a, b) \in \mathcal{X}_{n,m} \) is analogous with the arrow representing \( C \) lying in a vertical plane. For example, for \( m = n = 1 \), the singleton \((a, b) \in \mathcal{X}_{1,1}^3(= \mathcal{Y}_{1,1}^0) \) is expressed by \( DC'B \) in Figure 1 below.

![Figure 1. DECE \prec \mathcal{C}'' \prec \mathcal{C}'''' \prec \mathcal{C}''''''AB \prec \mathcal{C}''''''B'\mathcal{A}'\).](image-url)

Now suppose \( w \in \mathcal{J} \) is the projection of \( \tilde{w} = (a, b) \in r\mathcal{P}P_{n,m} \). Transform \( w \) into a 0-dimensional element of the \( A_\infty \)-bialgebra morphism matrad \( \mathcal{J} \) in the following way (see Example 8 below): If \( \tilde{w} \in \mathcal{X}_{n,m}^0 \) and \( a = u \times v \), replace \( a \) by \( u \cdot b \cdot v \); if \( \tilde{w} \in \mathcal{X}_{n,m}^\prime \), replace \( a = \cdots x_\alpha \cdots \) by \( \cdots \alpha \tilde{x}_\alpha \cdot x_\beta \cdots \); if \( \tilde{w} \in \mathcal{X}_{n,m}^\prime\prime \) and \( a = \cdots \alpha \tilde{x}_{\beta-1} \cdots \), replace \( \alpha \tilde{x}_{\beta-1} \) by \( \alpha \tilde{x}_\beta = x_\alpha \cdots x_\beta \); if \( \tilde{w} \in \mathcal{Y}_{n,m}^0 \), replace \( a = \cdots y_\# \cdots \) by \( \cdots y_\# \cdot \epsilon \tilde{y}_\# \cdots \); if \( \tilde{w} \in \mathcal{Y}_{n,m}^\prime \) and \( a = \cdots \epsilon_{-1} y_\beta \cdots \), replace \( \epsilon_{-1} y_\beta \) by \( \epsilon \tilde{y}_\beta = y_\epsilon \cdots y_\delta \). Now replace each row \((2,1,\ldots,1) \) by \((f_1^1, f_1^2, \ldots, f_1^n) \) and each column...
(1, ..., 1, 2) \rightarrow (f_1', ..., f_i', f_i')^T; delete \hat{i}'; replace each 1 by 1, replace each 2 in \( y \) by \( \theta_3 \) and replace each 2 in \( y \) by \( \theta_2 \). This transformation induces the bijection in (3.3) as in the absolute case.

**Example 8.** The labels \( v_1 \) and \( v_2 \) are the midpoints of the edges 1|23 and 13|2 of \( P_3 \), respectively (see Figure 5).

\[
\begin{align*}
1|2|3 & \leftrightarrow DEC = \left[ \begin{array}{c} \theta_2' \\ \theta_2'' \\ \theta_2'' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \end{array} \right] \\
1|3|2 & \leftrightarrow DC'B = \left[ \begin{array}{c} \theta_2' \\ \theta_2'' \\ \theta_2'' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \end{array} \right] \\
3|1|2 & \leftrightarrow C''AB = \left[ \begin{array}{c} f_1' \\ f_1' \\ \theta_2' \\ \theta_2'' \\ \theta_2'' \\ \theta_1' \\ \theta_1' \\ \theta_1' \end{array} \right] \\
2|1|3 & \leftrightarrow E'D'C = \left[ \begin{array}{c} \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \end{array} \right] \\
2|3|1 & \leftrightarrow E'C''A' = \left[ \begin{array}{c} \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \end{array} \right] \\
3|2|1 & \leftrightarrow C''B'A' = \left[ \begin{array}{c} f_1' \\ f_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \\ \theta_1' \end{array} \right].
\end{align*}
\]

The bijection in (3.3) can be described on the codimension 1 level in the following way: The components \( \left[ \theta_{m+1}' \right] \left[ f_1' \right] \left[ f_1' \right] \) and \( \left[ f_1' \right] \left[ f_1' \right] \) and \( \left[ \theta_{m+1}' \right] \left[ f_1' \right] \left[ f_1' \right] \) of \( (\mathcal{J}, J_{\infty})_{n+1,m+1} \) are assigned to the cells \( KK_{n+1,m+1} \times 0 \) and \( KK_{n+1,m+1} \times 1 \), which are the respective projections of \( n+m+1 \) and \( n+m+1 \) in \( P_{m+n+1} \), and labeled by the leaf sequences

\[
\begin{align*}
\begin{array}{c}
1 \\
\vdots \\
1 \\
\end{array} & \quad \begin{array}{c}
1 \\
\vdots \\
1 \\
\end{array} \\
\begin{array}{c}
1 \\
m+1 \\
\end{array} & \quad \begin{array}{c}
1 \\
m+1 \\
\end{array}
\end{align*}
\]

respectively. The other components \( A_p \mathcal{R}_x^y \) and \( Q_q \mathcal{B}_x^y \) of \( (\mathcal{J}, J_{\infty})_{n+1,m+1} \) are assigned to the cells \( e_1 \) and \( e_2 \) of \( JJ_{n+1,m+1} \) obtained by subdividing \( K_{n+1,m+1} \times I \) in the following ways: Let \( e_{(y,x)} = C|D \) be the codimension 1 cell of \( P_{m+n} \) defined in line (6.1) of [22]. Then

\[
C|\left( D \cup \{ m+n+1 \} \right) \cup \left( C \cup \{ m+n+1 \} \right) |D = C|D \times I \subset P_{m+n} \times I \approx P_{m+n+1},
\]

\( e_1 = (\theta_{n,m} \times 1)\left( C|D \cup \{ m+n+1 \} \right) \), and \( e_2 = (\theta_{n,m} \times 1)\left( C \cup \{ m+n+1 \} |D \right) \).

The leaf sequences

\[
\begin{array}{c}
y \\
x \end{array} \quad \begin{array}{c}
y \\\n\hat{x} \end{array}
\]

\[
label $e_1$ and $e_2$, respectively (see Figures 2–6 below).

Unlike the 3-dimensional biassociahedra, certain 2-cells of the 3-dimensional bi-multihedra are defined in terms of $\Delta P$. When constructing $JJ_{2,3}$, for example, we use $\Delta P(P_3)$ to subdivide the cell $124/3$ and label the faces in a manner similar to the labeling on $KK_{2,3}$ (see [22], Figure 19). Furthermore, it is convenient to think of $JJ_{2,3}$ as a subdivision of the cylinder $KK_{2,3} \times I$; then 2-faces of $(JJ_{\infty})_{2,3}$ are represented by a path $(3,1,0) \to (s, t, e) \to (1, 2, 1)$ in $\mathbb{N}^3$ with $e = 0, 1$. For example, the paths $(3, 1, 0) \to (2, 2, 1) \to (1, 2, 1)$ and $(3, 1, 0) \to (2, 2, 0) \to (1, 2, 1)$ represent the 2-faces $e_1 = 1|234$ and $e_2 = 14|23$, respectively. Finally, we remark that a general codimension 1 face of $(JJ_{\infty})_{n,m}$ is detected by the pair of leaf sequences $(x, y)$ and the corresponding path.

While $JJ_{1,n}$ and $JJ_{n,1}$ are combinatorially isomorphic to the multiplihedron $J_n$, their distinct combinatorial representations are related by the cellular isomorphism $JJ_{1,n} \rightarrow JJ_{n,1}$ induced by the reversing map $\chi : a_1 \cdots a_n \mapsto a_n \cdots a_1$ on $P_n$. Let $\pi_1 : P_n \rightarrow J_n$ and $\pi_2 : P_n \rightarrow J_n$ be the respective cellular projections $P_n = |rP_{n-1}| \rightarrow JJ_{1,n}$ and $P_n = |rP_{n-1,0}| \rightarrow JJ_{n,1}$, $n \geq 2$; then $\pi_1 = \pi_2 \circ \chi$.

**Example 9.** The bijection (3.7) is represented on the vertices of $J_4$ in $\pi_1(2|134 \cup 24|13)$ as follows:

\[
\begin{align*}
[\theta_2^2] \ [1 \theta_2^1] \ [\theta_2^1 \ 1 \ 1] \ [f_1 \ f_1 \ f_1 \ f_1] & \sim \ [\theta_2^2] \ [\theta_2^1 \ 1] \ [1 \ 1 \ \theta_2^1] \ [f_1 \ f_1 \ f_1 \ f_1] \\
\downarrow & \downarrow \\
((2), (12), (2111), (21111)) & \sim \ ((2), (21), (12), (11121)) \\
2|1|3|4 & \cong 2|3|1|4
\end{align*}
\]

\[
\begin{align*}
[\theta_2^2] \ [1 \ \theta_2^2] \ [f_1 \ f_1 \ f_1] \ [\theta_2^1 \ 1 \ 1] & \sim \ [\theta_2^2] \ [\theta_2^2 \ 1 \ 1] \ [f_1 \ f_1 \ f_1] \ [1 \ 1 \ \theta_2^2] \\
\downarrow & \downarrow \\
((2), (12), (2111), (12111)) & \sim \ ((2), (21), (211), (1112)) \\
2|1|4|3 & \cong 2|3|4|1
\end{align*}
\]

\[
\begin{align*}
[\theta_2^2] \ [\theta_2^2 \ 1 \ 1] \ [f_1 \ f_1] & \sim \ [\theta_2^1] \ [\theta_2^2 \ 1 \ 1] \ [\theta_2^2 \ 1 \ 1] \\
\downarrow & \downarrow \\
((2), (21), (112), (12111)) & \sim \ ((2), (21), (121), (1112)) \\
2|4|1|3 & \cong 2|4|3|1
\end{align*}
\]

\[
\begin{align*}
[\theta_2^2] \ [\theta_2^1 \ 1 \ 1] \ [f_1] & \sim \ [\theta_2^1] \ [\theta_2^2 \ 1 \ 1] \ [1 \ 1 \ \theta_2^1] \\
\downarrow & \downarrow \\
((2), (i2), (i12), (i1211)) & \sim \ ((2), (i2), (i21), (i112)) \\
4|2|1|3 & \cong 4|2|3|1.
\end{align*}
\]

*Note that the 2-cell $24|13$ and the edge $2|13|4$ of $P_4$ degenerate under $\pi_1$, whereas the 2-cell $13|24$ and the edge $4|13|2$ of $P_4$ degenerate under $\pi_2$. For comparison, the cellular map $\pi : P_n \rightarrow J_n$ defined in [20] differs from both $\pi_1$ and $\pi_2$: The 2-cell $24|13$ and the edge $1|24|3$ of $P_4$ degenerate under $\pi$.*
Combinatorial data For $JJ_{2,3}$:

\[
\begin{align*}
1234 & \leftrightarrow \frac{2}{111} = \theta_3^2/f_1 f_1 f_1 f_1 & 1|234 & \leftrightarrow \frac{11}{21} = \theta_2^3 \theta_2^2/f_2^2 (f_1 f_1 / \theta_1^2) \\
4|123 & \leftrightarrow \frac{1}{3} = f_1 f_1^2 / \theta_3^2 & 14|23 & \leftrightarrow \frac{11}{21} = [(f_1^2 / \theta_1^2) f_2 + f_1^2 (\theta_2^1 / f_1 f_1)]/ \theta_2^2 \theta_1^2 \\
13|24 & \leftrightarrow \frac{2}{21} = \theta_2^3 / f_2^2 f_1^2 & 2|134 & \leftrightarrow \frac{11}{12} = \theta_2^3 \theta_2^2 / (\theta_1^2 / f_1 f_1) f_2 \\
134|2 & \leftrightarrow \frac{2}{21} = f_2^2 / \theta_3^2 \theta_1^2 & 24|13 & \leftrightarrow \frac{11}{12} = [(f_1^2 / \theta_1^2) f_2 + f_1^2 (\theta_2^1 / f_1 f_1)]/ \theta_2^2 \theta_1^2 \\
3|124 & \leftrightarrow \frac{2}{3} = \theta_3^2 / f_3^2 & 23|14 & \leftrightarrow \frac{2}{12} = \theta_2^3 / f_1 f_2 \\
34|12 & \leftrightarrow \frac{2}{3} = f_3^2 / \theta_3^2 & 2341 & \leftrightarrow \frac{2}{12} = f_2^2 / \theta_1^1 \theta_2^2
\end{align*}
\]
Let $X$.

While it is highly improbable that the double cobar construction $\Omega^2 S(X)$ admits a coassociative coproduct, there is an $A_\infty$-coalgebra structure on $\Omega^2 S(X)$ that is compatible with the product, and $\Omega^2 S(X)$ is an $A_\infty$-bialgebra.

Let $H$ be a graded bialgebra with nontrivial product and coproduct, and let $\rho: RH \to H$ be a (bigraded) multiplicative resolution. Since $RH$ cannot be simultaneously free and cofree, it is difficult to introduce a strictly coassociative coproduct on $RH$ in such a way that $\rho$ is a bialgebra map. However, there is an $A_\infty$-bialgebra structure on $RH$ such that $\rho$ is a morphism of $A_\infty$-bialgebras.

If $A$ is an $A_\infty$-bialgebra over a field and $g: H(A) \to A$ is a cycle selecting homomorphism on homology, there is an $A_\infty$-bialgebra structure on $H(A)$ that

5. Morphisms Defined

In this section we define the morphisms of $A_\infty$-bialgebras. Our approach is to think of a morphism of $A_\infty$-(co)algebras as an $A_\infty$-bimodule; and analogously, to think of a morphism of $A_\infty$-bialgebras as an $H_\infty$-bimodule. But before we begin, we mention three settings in which $A_\infty$-bialgebras naturally appear:

(1) Let $X$ be a space and let $S_\ast(X)$ denote the simplicial singular chain complex of $X$. Then the cobar construction $\Omega S_\ast(X)$ is a DG bialgebra with coassociative coproduct [1], [4], [12]. While it is highly improbable that the double cobar construction $\Omega^2 S_\ast(X)$ admits a coassociative coproduct, there is an $A_\infty$-coalgebra structure on $\Omega^2 S_\ast(X)$ that is compatible with the product, and $\Omega^2 S_\ast(X)$ is an $A_\infty$-bialgebra.

(2) Let $H$ be a graded bialgebra with nontrivial product and coproduct, and let $\rho: RH \to H$ be a (bigraded) multiplicative resolution. Since $RH$ cannot be simultaneously free and cofree, it is difficult to introduce a strictly coassociative coproduct on $RH$ in such a way that $\rho$ is a bialgebra map. However, there is an $A_\infty$-bialgebra structure on $RH$ such that $\rho$ is a morphism of $A_\infty$-bialgebras.

(3) If $A$ is an $A_\infty$-bialgebra over a field and $g: H(A) \to A$ is a cycle selecting homomorphism on homology, there is an $A_\infty$-bialgebra structure on $H(A)$ that

Figure 6. The bimultiplihedron $JJ_{2,3}$ as a subdivision of $P_4$. 
is unique up to isomorphism, and a morphism of $A_\infty$-bialgebras $G : H(A) \Rightarrow A$ extending $g$ (see Theorem 2).

Recall the following equivalent definitions of an $A_\infty$-bialgebra:

**Definition 18.** A graded $R$-module $A$ together with an element
\[ \omega = \{ \omega_m^n \in \text{Hom}^{m+n-3}(A \otimes^m, A \otimes^n) \}_{m,n \geq 1} \in U_A \]
is an $A_\infty$-bialgebra if either

(i) $\omega \circ \omega = 0$ or
(ii) the map $\theta^n_m \mapsto \omega^n_m$ extends to a map $\mathcal{H}_\infty \rightarrow U_A$ of matrads.

There is an operation $\ominus : U_B \times U_{A,B} \times U_A \rightarrow U_{A,B}$ analogous to $\ominus$, which allows us to define a morphism of $A_\infty$-bialgebras in two equivalent ways. Given DG $R$-modules (DGMs) $(A, d_A)$ and $(B, d_B)$, let $d_A : TA \rightarrow TA$ and $d_B : TB \rightarrow TB$ be the free linear extensions of $d_A$ and $d_B$, and let $\nabla$ be the induced Hom differential on $U_{A,B}$, i.e., for $f \in U_{A,B}$ define $\nabla f = d_B \circ f - (-1)^{|f|} f \circ d_A$. Given $(\zeta, f, \eta) \in U_B \times U_{A,B} \times U_A$ and $m, n \geq 1$, obtain $X^{n,m} + Y^{m,n} \in (U_{A,B})_{n,m}$ by replacing $(\theta, f, \theta)$ in the right-hand side of formula (3.3) with $(\zeta, f, \eta)$. Then define
\[ \ominus(\zeta, f, \eta) = \{ X^{n,m} \}_{m,n \geq 1} + \{ Y^{n,m} \}_{m,n \geq 1} - \nabla f. \]

**Definition 19.** Let $(A, \omega_A)$ and $(B, \omega_B)$ be $A_\infty$-bialgebras. An element
\[ G = \{ g_m^n \in \text{Hom}^{m+n-2}(A \otimes^m, B \otimes^n) \}_{m,n \geq 1} \in U_{A,B} \]
is an $A_\infty$-bialgebra morphism from $A$ to $B$ if either

(i) $\ominus(\omega_B, G, \omega_A) = 0$ or
(ii) the map $f^n_m \mapsto g^n_m$ extends to a map $\mathcal{J}_\infty \rightarrow U_{A,B}$ of relative matrads.

The symbol $G : A \Rightarrow B$ denotes an $A_\infty$-bialgebra morphism $G$ from $A$ to $B$. An $A_\infty$-bialgebra morphism $\Phi = \{ \phi^n_m \}_{m,n \geq 1} : A \Rightarrow B$ is an isomorphism if $\phi^n_1 : A \rightarrow B$ is an isomorphism of underlying modules.

6. Transfer of $A_\infty$-Structure

If $A$ is a free DGM, $B$ is an $A_\infty$-algebra, and $g : A \rightarrow B$ is a homology isomorphism (weak equivalence) with a right-homotopy inverse, the Basic Perturbation Lemma (BPL) transfers the $A_\infty$-algebra structure from $B$ to $A$ (see [9], [15], for example). When $B$ is an $A_\infty$-bialgebra, Theorem 1 generalizes the BPL in two directions:

(1) We transfer the $A_\infty$-bialgebra structure from $B$ to $A$.
(2) Our transfer algorithm requires neither freeness nor the existence of a right-homotopy inverse.

Note that generalization (2) formulates the classical transfer of $A_\infty$-algebra structure in maximal generality (see Remark 2).

A chain map $g : A \rightarrow B$ of DGMs induces a cochain map $\tilde{g} : U_A \rightarrow U_{A,B}$ defined on $u \in \text{Hom}(A \otimes^m, A \otimes^n)$ by $\tilde{g}(u) = g_{\otimes^n} u$, which is a homology isomorphism if the hypotheses of our next proposition are satisfied (the proof is left to the reader):

**Proposition 1.** Let $(A, d_A)$ and $(B, d_B)$ be DGMs, let $g : A \rightarrow B$ be a chain map that is also a homology isomorphism. Then $\tilde{g} : U_A \rightarrow U_{A,B}$ is a homology isomorphism whenever either of the following conditions holds:

(i) $A$ is free as an $R$-module; or
(ii) for each \( n \geq 1 \), there is a DGM \( X(n) \) and a splitting \( B^\otimes n = A^\otimes n \oplus X(n) \) as a chain complex such that \( H^* \text{Hom} (A^\otimes k, X(n)) = 0 \) for all \( k \geq 1 \).

Since the hypotheses of Proposition 1 are satisfied when \( R \) is a field, there is the following generalization of the BPL:

**Theorem 1 (The Transfer).** Let \((A, d_A)\) be a DGM, let \((B, d_B, \omega_B)\) be an \( A_\infty \)-bialgebra, and let \( g : A \to B \) be a chain map that is also a homology isomorphism. If \( \tilde{g} : \mathcal{En}_{T_A} \to U_{A,B} \) is a homology isomorphism, then

(i) (Existence) \( g \) induces an \( A_\infty \)-bialgebra structure \( \omega_A = \{ \omega_A^{n,m} \} \) on \( A \) and extends to a map \( G = \{ g_m^n ; g_1^n = g \} : A \Rightarrow B \) of \( A_\infty \)-bialgebras.

(ii) (Uniqueness) \((\omega_A, G)\) is unique up to isomorphism, i.e., if \((\tilde{\omega}_A, \tilde{G})\) are induced by chain homotopic maps \( g \) and \( \tilde{g} \), there is an isomorphism \( \Phi : (A, \tilde{\omega}_A) \Rightarrow (A, \omega_A) \) and a chain homotopy \( T : G \cong \tilde{G} \circ \Phi \).

**Proof.** We obtain the desired structures by simultaneously constructing a map of matrads \( \alpha_A : C_* (KK) \to U_A \) and a map of relative matrads \( \beta : C_* (JJ) \to U_{A,B} \). Thinking of \( JJ_{n,m} \) as a subdivision of the cylinder \( KK_{n,m} \times I \), identify the top dimensional cells of \( KK_{n,m} \) and \( JJ_{n,m} \) with \( \theta_m^n \) and \( \theta_m^n \), and the faces \( KK_{n,m} \times 0 \) and \( KK_{n,m} \times 1 \) of \( JJ_{n,m} \) with \( \theta_m^n (\tilde{f}_1^n \otimes \theta_1^m) \) and \( \tilde{f}_1^n \otimes \theta_1^m \), respectively. By hypothesis, there is a map of matrads \( \alpha_B : C_* (KK) \to U_B \) such that \( \alpha_B (\theta_m^n) = \omega_B^{n,m} \).

To initialize the induction, define \( \beta : C_* (JJ_{1,1}) \to \text{Hom}^\circ (A, B) \) by \( \beta (\tilde{f}_1^1) = g_1^1 = g \) and extend \( \beta \) to \( C_* (JJ_{1,2}) \to \text{Hom}^1 (A^\otimes 2, B) \) and \( C_* (JJ_{2,1}) \to \text{Hom}^1 (A, B^\otimes 2) \) in the following way: On the vertices \( \theta_2^1 (\tilde{f}_1^n \otimes \tilde{f}_1^m) \in JJ_{1,2} \) and \( \theta_1^n \tilde{f}_1^m \in JJ_{2,1} \) define \( \beta (\theta_2^1 (\tilde{f}_1^n \otimes \tilde{f}_1^m)) = \omega_B^{1,2} (g \otimes g) \) and \( \beta (\theta_1^n \tilde{f}_1^m) = \omega_B^{2,1} g \). Since \( \omega_B^{1,2} (g \otimes g) \) and \( \omega_B^{2,1} g \) are \( \nabla \)-cocycles and \( \tilde{g}_n \) is an isomorphism, there exist cocycles \( \omega_A^{1,2} \) and \( \omega_A^{2,1} \) in \( U_A \) such that

\[ \tilde{g}_n [\omega_A^{1,2}] = [\omega_B^{1,2} (g \otimes g)] \quad \text{and} \quad \tilde{g}_n [\omega_A^{2,1}] = [\omega_B^{2,1} g]. \]

Thus \[ \nabla g_2^1 = \omega_B^{1,2} (g \otimes g) - g \omega_A^{1,2} \quad \text{and} \quad \nabla g_1^2 = \omega_B^{2,1} g - (g \otimes g) \omega_A^{2,1}. \]

For \( m = 1, 2 \) and \( n = 3 - m \), define \( \alpha_A : C_* (KK_{n,m}) \to \text{Hom} (A^\otimes m, A^\otimes n) \) by \( \alpha_A (\theta_m^n) = \omega_A^{n,m} \) and define \( \beta : C_* (JJ_{n,m}) \to \text{Hom} (A^\otimes m, B^\otimes n) \) by

\[
\beta (\tilde{f}_1^n) = g \omega_A^{1,2} \quad (m = 2) \quad \text{and} \quad \beta (\tilde{f}_1^n \theta_1^m) = (g \otimes g) \omega_A^{2,1} \quad (m = 1).
\]

Inductively, given \( (m, n) \), \( m + n \geq 4 \), assume that for \( i + j < m + n \) there exists a map of matrads \( \alpha_A : C_* (KK_{j,i}) \to \text{Hom} (A^\otimes i, A^\otimes j) \) and a map of relative matrads \( \beta : C_* (JJ_{j,i}) \to \text{Hom} (A^\otimes i, B^\otimes j) \) such that \( \alpha_A (\tilde{f}_1^n) = \omega_A^{i,j} \) and \( \beta (\tilde{f}_1^n) = g_1^n \). Thus we are given chain maps \( \alpha_A : C_* (\partial KK_{n,m}) \to \text{Hom} (A^\otimes m, A^\otimes n) \) and \( \beta : C_* (\partial JJ_{n,m} \setminus \text{int} KK_{n,m} \times 1) \to \text{Hom} (A^\otimes m, B^\otimes n) \); we wish to extend \( \alpha_A \) to the top cell \( \theta_m^n \) of \( KK_{n,m} \) and \( \beta \) to the codimension 1 cell \( \tilde{f}_1^n \otimes \theta_1^m \) and the top cell \( \theta_m^n \) of \( JJ_{n,m} \). Since \( \alpha_A \) is a map of matrads, the components of the cocycle

\[
z = \alpha_A (C_* (\partial KK_{n,m})) \in \text{Hom}^{m+n-4} (A^\otimes m, A^\otimes n)
\]
Hence $g_\phi$ may be chosen so that $\bar{\omega}_{\phi} \equiv 0$ and

$$\bar{\omega}_{\phi} = \sum \omega_{\phi} \in Hom_{n,m}(A^{\otimes m}, A^{\otimes n})$$

are expressed in terms of $\omega_B$, $\omega^i_A$ and $g^i$ with $i + j < m + n$. Clearly $\tilde{g}(z) = \nabla \varphi$; and $|z| = 0$ since $\tilde{g}$ is a homology isomorphism. Now choose a cochain $b \in Hom_{m+n-3}(A^{\otimes m}, A^{\otimes n})$ such that $\nabla b = z$. Then

$$\nabla (\tilde{g}(b - \varphi)) = \nabla \tilde{g}(b) - \tilde{g}(z) = 0.$$ 

Choose a class representative $u \in g^{-1}_s [\tilde{g}(b) - \varphi]$, set $\omega^m_{\phi} = b - u$, and define $\alpha_A(g^m_{\phi}) = \omega^m_{\phi}$. Then $[\tilde{g}(\omega^m_{\phi}) - \varphi] = [\tilde{g}(b - u) - \varphi] = [\tilde{g}(b) - \varphi] - [\tilde{g}(u)] = 0$. Choose a cochain $g^m_{\phi} \in Hom_{m+n-2}(A^{\otimes m}, B^{\otimes n})$ such that

$$\nabla g^m_{\phi} = g^{\otimes n} \omega_{\phi} - \varphi,$$

and define $\beta([g^m_{\phi}]) = g^m_{\phi}$. To extend $\beta$ as a map of relative matrads, define $\beta([g^m_{\phi}]) = g^{\otimes n} \omega_{\phi} - \varphi$. Passing to the limit we obtain the desired maps $\alpha_A$ and $\beta$.

Furthermore, if chain maps $\tilde{\alpha}_A$ and $\tilde{\beta}$ are defined in terms of different choices, beginning with a chain map $\tilde{g}$ chain homotopic to $g$, let $\omega_A = \text{Im} \tilde{\alpha}_A$ and $\tilde{G} = \text{Im} \tilde{\beta}$. There is an inductively defined isomorphism $\Phi = \sum \alpha_m : (A, \omega_A) \Rightarrow (A, \omega_A)$ with $\phi^1 = 1$, and a chain homotopy $T : \tilde{G} \simeq G \circ \Phi$. To initialize the induction, set $\phi^1 = 1$ and note that

$$\nabla g^1_A = g \omega^1_A - \omega^1_B(g \otimes g) \text{ and } \nabla g^1_B = g \omega^1_A - \omega^1_B(g \otimes g).$$

Let $s : \tilde{g} \simeq g$; then $c^1_s = \omega^1_B(s \otimes g + \tilde{g} \otimes s)$ satisfies

$$\nabla c^1_s = \omega^1_B(g \otimes g) - \omega^1_B(g \otimes \tilde{g}).$$

Hence

$$\nabla (g^1 - \tilde{g} + c^1_s) = g \omega^1_A - \tilde{g} \omega^1_A$$

and

$$\nabla (\omega_A - \omega^1_A) = \nabla (g^1 - \tilde{g} + c^1_s - s \omega^1_A).$$

Consequently, there is $\phi^1_2 : A^{\otimes 2} \rightarrow A$ such that $\nabla \phi^1_2 = \omega^1_A - \omega^1_A$; and, as above, $\phi^1_2$ may be chosen so that $\tilde{g} \phi^1_2 - (g^1 - \tilde{g} + c^1_s - s \omega^1_A)$ is cohomological to zero. Thus there is a component $t^1_2$ of $T$ such that

$$\nabla (t^1_2) = \tilde{g} \phi^1_2 - (g^1 - \tilde{g} + c^1_s + s \omega^1_A).$$

We shall refer to the algorithm in the proof of the Transfer Theorem as the Transfer Algorithm. Since $\tilde{g}$ is a homology isomorphism whenever $A$ is free (cf. Proposition I) we have:

**Corollary 1.** Let $(A, d_A)$ be a free DGM, let $(B, d_B, \omega_B)$ be an $A_\infty$-bialgebra, and let $g : A \rightarrow B$ be a chain map that is also a homology isomorphism. Then

(i) (Existence) $g$ induces an $A_\infty$-bialgebra structure $\omega_A$ on $A$ and extends to a map $G : A \Rightarrow B$ of $A_\infty$-bialgebras.

(ii) (Uniqueness) $(\omega_A, G)$ is unique up to isomorphism.
Given a chain complex $B$ of (not necessarily free) $R$-modules, there is always a chain complex of free $R$-modules $(A, d_A)$ and a homology isomorphism $g : A \to B$. To see this, let $\left(RH : \cdots \to R_1H \to R_0H \xrightarrow{\partial} H, d\right)$ be a free $R$-module resolution of $H = H_* (B)$. Since $R_0H$ is projective, there is a cycle-selecting homomorphism $g_0 : R_0H \to Z (B)$, which lifts the map $\partial$ through the projection $Z (B) \to H$ and extends to a chain map $g_0 : (RH, 0) \to (B, d_B)$. If $RH : 0 \to R_1H \to R_0H \to H$ is a short $R$-module resolution of $H$, then $g_0$ extends to a homology isomorphism $g : (RH, d + h) \to (B, d_B)$ with $(A, d_A) = (RH, d)$. Otherwise, there is a perturbation $h$ of $d$ such that $g : (RH, d + h) \to (B, d_B)$ is a homology isomorphism with $(A, d_A) = (RH, d + h)$ (see [3], [17]). Thus an $A_\infty$-structure on $B$ always transfers to an $A_\infty$-structure on $(RH, d + h)$ via Corollary 1 and we obtain our main result concerning the transfer of $A_\infty$-structure to homology:

**Theorem 2.** Let $B$ be an $A_\infty$-bialgebra with homology $H = H_* (B)$, let $(RH, d)$ be a free $R$-module resolution of $H$, and let $h$ be a perturbation of $d$ such that $g : (RH, d + h) \to (B, d_B)$ is a homology isomorphism. Then

(i) (Existence) $g$ induces an $A_\infty$-bialgebra structure $\omega_{RH}$ on $RH$ and extends to a map $G : RH \Rightarrow B$ of $A_\infty$-bialgebras.

(ii) (Uniqueness) $(\omega_{RH}, G)$ is unique up to isomorphism.

**Remark 1.** Note that $A_\infty$-bialgebra structures induced by the Transfer Algorithm are isomorphic for all choices of the map $g : (RH, d + h) \to (B, d_B)$, and we obtain an isomorphism class of $A_\infty$-bialgebra structures on $RH$.

**Remark 2.** When $H = H_* (B)$ is a free module, we can set $RH = H$ and recover the classical results of Kadeishvili [11], Markl [15], and others, which transfer a DG (co)algebra structure to an $A_\infty$-(co)algebra structure on homology. For an example of a DGA structure that cannot be transferred using classical techniques, but to which our method applies see [20]. Furthermore, any pair of $A_\infty$-(co)algebra structures $\{\omega^n_H\}_{m \geq 1}$ and $\{\omega^1_H \}_{m \geq 1}$ on $H$ induced by the same cycle-selecting map $g : H \to B$ extend to an $A_\infty$-bialgebra structure $(H, \omega_{H, m}^n)$, by the proof of Theorem 2.

7. **Applications and Examples**

The applications and examples in this section apply the Transfer Algorithm given by the proof of Theorem 1. Three kinds of specialized $A_\infty$-bialgebras $(A, \{\omega^n\})$ are relevant here:

1. $\omega^1_m = 0$ for $m \geq 3$ (the $A_\infty$-algebra substructure is trivial).
2. $\omega^m_n = 0$ for $m, n \geq 2$ (all higher order structure is concentrated in the $A_\infty$-algebra and $A_\infty$-coalgebra substructures).
3. Conditions (1) and (2) hold simultaneously.

Of these, $A_\infty$-bialgebras of the first and third kind appear in the applications.

Structure relations defining $A_\infty$-bialgebras of the second and third kind are expressed in terms of the S-U diagonal on associahedra $\Delta_K$ [20] and have especially nice form. Structure relations of the second kind were derived in [25]. Structure relations in an $A_\infty$-bialgebra $(A, \omega)$ of the third kind with $\omega^1_1 = 0$, $\mu = \omega^1_2$ and $\psi^n = \omega^n_1$ are a special case of those derived in [25] and are given by the formula

(7.1) $\{\psi^n \mu = \mu^\otimes n \psi^n\}_{n \geq 2}$.
where the \(n\)-ary \(A_\infty\)-coalgebra operation

\[
\Psi^n = (\sigma_{n,2})_* \iota \left( \xi \otimes \xi \right) \Delta_K \left( e^{n-2} \right) : A \otimes A \to (A \otimes A)^{\otimes n}
\]

is defined in terms of

- a map \(\xi : C_*(K) \to \text{Hom}(A, TA)\) of operads sending the top dimensional cell \(e^{n-2} \subseteq K_n\) to \(\psi^n\),
- the canonical isomorphism

\[
\iota : \text{Hom} \left( A, A^{\otimes n} \right)^{\otimes 2} \to \text{Hom} \left( A^{\otimes 2}, (A^{\otimes n})^{\otimes 2} \right),
\]

- and the induced isomorphism

\[
(\sigma_{n,2})_* : \text{Hom} \left( A^{\otimes 2}, (A^{\otimes n})^{\otimes 2} \right) \to \text{Hom} \left( A^{\otimes 2}, (A^{\otimes 2})^{\otimes n} \right).
\]

Structure relations defining a morphism \(G = \{g^n\} : (A, \omega_A) \Rightarrow (B, \omega_B)\) between \(A_\infty\)-bialgebras of the third kind are expressed in terms of the S-U diagonal on multiplihedra \(\Delta_j\) [20] by the formula

\[
\{g^n \mu_A = \mu_B \{g^n\} \}_{n \geq 1},
\]

where

\[
g^n = (\sigma_{n,2})_* \iota (v \otimes v) \Delta_J (e^{n-1}) : A \otimes A \to (B \otimes B)^{\otimes n},
\]

and \(v : C_*(J) \to \text{Hom}(A, TB)\) is a map of relative prematras sending the top dimensional cell \(e^{n-1} \subseteq J_n\) to \(g^n\) (the maps \(\{g^n\}\) define the tensor product morphism \(G \otimes G : (A \otimes A, \Psi_{A \otimes A}) \Rightarrow (B \otimes B, \Psi_{B \otimes B})\)).

Given a simply connected topological space \(X\), consider the Moore loop space \(\Omega X\) and the simplicial singular cochain complex \(S^*(\Omega X; R)\). Under the hypotheses of the Transfer Theorem, the DG bialgebra structure of \(S^*(\Omega X; R)\) transfers to an \(A_\infty\)-bialgebra structure on \(H^*(\Omega X; R)\). Our next two theorems apply this principle and identify some important \(A_\infty\)-bialgebras of the third kind on loop space (co)homology.

**Theorem 3.** If \(X\) is simply connected, \(H^*(\Omega X; \mathbb{Q})\) admits an induced \(A_\infty\)-bialgebra structure of the third kind.

**Proof.** Let \(A_X\) be a free DG commutative algebra cochain model for \(X\) over \(\mathbb{Q}\) (e.g., Sullivan’s minimal or Halperin-Stasheff’s filtered model); then \(H^* (A_X) \approx H^*(X; \mathbb{Q})\). The bar construction \((B = BAX, d_B, \Delta_B)\) with shuffle product is a cofree DG commutative Hopf algebra cochain model for \(\Omega X\), and \(H = H^*(B, d_B)\) is a Hopf algebra with induced coproduct \(\psi^1 = \omega_1^2\) and free graded coproduct \(\mu = \omega_2^1\) (by a theorem of Hopf). Since \(H\) is a free commutative algebra, there is a multiplicative cocycle-selecting map \(g_1^1 : H \to B\). Consequently, we may set \(\omega_1^1 = 0\) for all \(n \geq 3\) and \(g_1^1 = 0\) for all \(n \geq 2\) and obtain a trivial \(A_\infty\)-algebra structure \((H, \mu)\) induced by \(g_1^1\). There is an induced \(A_\infty\)-coalgebra structure \((H, \psi^n)_{n \geq 2}\) and an \(A_\infty\)-coalgebra map \(G = \{g^n | g^1 = g_1^1\}_{n \geq 1} : H \Rightarrow B\) constructed as follows: For \(n \geq 2\), assume \(\psi^n\) and \(g^{n-1}\) have been constructed, and apply the Transfer Algorithm to obtain candidates \(\omega_1^{n+1}\) and \(g_1^{n+1}\). Restrict \(\omega_1^{n+1}\) to generators and let \(\psi^{n+1}\) be the free extension of \(\omega_1^{n+1}\) to all of \(H\) using Formula (7.1). Similarly, restrict \(g_1^{n+1}\) to generators and let \(g^{n+1}\) be the free extension of \(g_1^{n+1}\) to all of \(H\) using Formula (7.2).

To complete the proof, we show that all other \(A_\infty\)-bialgebra operations may be trivially chosen. Refer to the Transfer Algorithm and note that the Hopf relation...
identifies the Bott-Samelson Isomorphism $H_\omega, \mu$.

Since $\omega$ and $\mu$ are both torsion free, $\omega_2 = \mu_2 = 0$ and $\omega_3 = \mu_3 = 0$. Then the Bott-Samelson Theorem [4] asserts that $\omega$ and $\mu$ are isomorphisms of Hopf algebras ([10], [12]), and $\omega$ is a map of underlying bialgebras and $\mu$ is a map of underlying bialgebras. Furthermore, restricting $\omega$ to the multiplicative generators $H_s(X; R)$ recovers the $A_\infty$-coalgebra operations on $H_s(X; R)$. Thus if $\{\psi^\alpha\}_{\alpha \geq 1}$ is an $A_\infty$-coalgebra of the third kind with respect to the free extension of each $\psi^n$ via Formula [7.1] and $G$ is a map of $A_\infty$-bialgebras.

### Theorem 4
Let $R$ be a PID and let $X$ be a connected space such that $H_\ast(X; R)$ is torsion free. Then the Bott-Samelson Theorem [4] asserts that $H_\ast(\Omega\Sigma X; R)$ is isomorphic as an algebra to the tensor algebra $T^n\hat{H}_\ast(X; R)$ generated by the reduced homology of $X$, and the adjoint $t : X \to \Omega\Sigma X$ induces the canonical inclusion $i_\ast : \hat{H}_\ast(X; R) \to T^n\hat{H}_\ast(X; R) \approx H_\ast(\Omega\Sigma X; R)$. This isomorphism is Hopf algebras ([10], [12]), and $T^n\hat{S}_\ast(X; R)$ is a free DG Hopf algebra chain model for $\Omega\Sigma X$.

### Proof
Since $H_\ast(X; R)$ is free as an $R$-module, we may choose a cycle-selecting map $\tilde{g} = \tilde{g}_1 : H_\ast(X; R) \to S_\ast(X; R)$ and apply the Transfer Algorithm to obtain an induced $A_\infty$-bialgebra structure $\tilde{g} = \{\tilde{g}_n\}_{n \geq 2}$ on $H_\ast(X; R)$ and a corresponding map $\hat{G} = \{\hat{g}_n\}_{n \geq 1} : H_\ast(X; R) \to S_\ast(X; R)$. Let $\hat{H} = T^n\hat{H}_\ast(X; R)$, let $\hat{B} = T^n\hat{S}_\ast(X; R)$, and consider the free (multiplicative) extension $\hat{g} = T(\tilde{g}) : \hat{H} \to \hat{B}$. As in the proof of Theorem [3] use formulas [7.1] and [7.2] to freely extend $\tilde{g}$ and $\hat{G}$ to families $\omega = \{\omega^n\}$ and $G = \{g^n | g_1 = g\}_{n \geq 1}$ defined on $\hat{H}$, and choose all other $A_\infty$-bialgebra operations to be zero. Then $\omega$ lifts to an $A_\infty$-bialgebra structure $(H, \omega, \mu)$ of the third kind with free product $\mu$, and $G$ lifts to a map $G : H \Rightarrow B$ of $A_\infty$-bialgebras. Furthermore, restricting $\omega$ to the multiplicative generators $H_\ast(X; R)$ recovers the $A_\infty$-coalgebra operations on $H_\ast(X; R)$. Thus $A_\infty$-bialgebra structure of $H$ is trivial if and only if the $A_\infty$-coalgebra structure of $H_\ast(X; R)$ is trivial. Finally, since $B$ is a free DG Hopf algebra chain model for $\Omega\Sigma X$, the Bott-Samelson Isomorphism $t_\ast$ extends to an isomorphism of $A_\infty$-bialgebras and identifies the $A_\infty$-bialgebra structure of $H_\ast(\Omega\Sigma X; R)$ with $(H, \omega, \mu)$. \[\square\]
It is important to note that prior to this work, all known rational homology invariants of $ΩΣX$ are trivial for any space $X$. However, we now have the following:

**Corollary 2.** A nontrivial $A_∞$-coalgebra structure on $H_∞(X; \mathbb{Q})$ induces a nontrivial $A_∞$-bialgebra structure on $H_∞(ΩΣX; \mathbb{Q})$. Thus the $A_∞$-bialgebra structure of $H_∞(ΩΣX; \mathbb{Q})$ is a nontrivial rational homology invariant.

**Proof.** First, $H = H_∞(ΩΣX; \mathbb{Q})$ admits an induced $A_∞$-bialgebra structure of the third kind by Theorem[1], which is trivial if and only if the $A_∞$-coalgebra structure of $H_∞(X; \mathbb{Q})$ is trivial. Second, the dual version of Theorem[3] imposes an induced $A_∞$-bialgebra structure on $H$ whose $A_∞$-coalgebra substructure is trivial, and whose $A_∞$-algebra substructure is trivial if and only if the $A_∞$-coalgebra structure of $H_∞(X; \mathbb{Q})$ is trivial. \hfill $\square$

The two $A_∞$-bialgebras identified in the proof of Corollary[2]—one with trivial $A_∞$-coalgebra substructure and the other with trivial $A_∞$-algebra substructure—are in fact isomorphic, and represent the same isomorphism class of $A_∞$-bialgebra structures on $H_∞(ΩΣX; \mathbb{Q})$ (cf. Remark 1). Indeed, choose a pair of isomorphisms for the two $A_∞$-(co)algebra substructures (their component in bidegree $(1,1)$ is $1 : H \rightarrow H$). Since $ω^{i}_j = 0$ for $i, j \geq 2$, these isomorphisms clearly determine an isomorphism of $A_∞$-bialgebras.

Our next example exhibits an $A_∞$-bialgebra of the first but not the second kind. Given a $1$-connected DGA $(A, d_A)$, the bar construction of $A$, denoted by $BA$, is the cofree DGC $T^e(\frac{1}{\partial} A)$ whose differential $d$ and coproduct $\Delta$ are defined as follows: Let $[x_1, \cdots, x_n]$ denote the element $\downarrow x_1 \cdots \downarrow x_n \in BA$ and let $e$ denote the unit $\uparrow$. Then

$$d [x_1, \cdots, x_n] = \sum_{i=1}^{n} \pm [x_1, \cdots, d_A x_i, \cdots, x_n] + \sum_{i=1}^{n-1} \pm [x_1, \cdots, x_i, x_{i+1}, \cdots, x_n];$$

$$\Delta [x_1, \cdots, x_n] = e \otimes [x_1, \cdots, x_n] + [x_1, \cdots, x_n] \otimes e + \sum_{i=1}^{n} [x_1, \cdots, x_i] \otimes [x_i, \cdots, x_n].$$

Given an $A_∞$-coalgebra $(C, Δ^n)_{n \geq 1}$, the tilde-cobar construction of $C$, denoted by $ΔC$, is the free DGA $T^a(\uparrow \frac{1}{\partial} C)$ with differential $d_{\partial A}$ given by extending $\sum_{i \geq 1} Δ^n$ as a derivation. Let $[x_1, \cdots, x_n]$ denote $\uparrow x_1 \cdots \uparrow x_n \in ΔH$.

**Example 10.** Consider the DGA $A = \mathbb{Z}_2[a, b]/(a^4, ab)$ with $|a| = 3$, $|b| = 5$ and trivial differential. Define a homotopy Gerstenhaber algebra (HGA) structure $\{E_{p,q} : A^\otimes_p \otimes A^\otimes_q \rightarrow A\}_{p,q \geq 0}$ with $E_{p,q}$ acting trivially except $E_{1,0} = E_{0,1} = 1$ and $E_{1,1}(b; b) = a^3$ (cf. [7, 12]). Form the tensor coalgebra $BA \otimes BA$ with coproduct $ψ = σ_{2,2}(Δ \otimes Δ)$, and consider the induced map

$$φ = E'_{1,0} + E'_{0,1} + E'_{1,1} : BA \otimes BA \rightarrow A$$

of degree $+1$, which acts trivially except for $E'_{1,0}(x \otimes e) = E'_{0,1}(e \otimes [x]) = x$ for all $x \in A$, and $E'_{1,1}([b] \otimes [b]) = a^3$. Since $E_{p,q}$ is an HGA structure, $φ$ is a twisting cochain, which lifts to a chain map of DG coalgebras $μ : BA \otimes BA \rightarrow BA$ defined by

$$μ = \sum_{k=0}^{∞} \downarrow^k φ^k \otimes e^k(φ^k),$$
where \( \tilde{\psi}^{(0)} = 1, \tilde{\psi}^{(k)} = (\tilde{\psi} \otimes 1^{\otimes k-1}) \cdots (\tilde{\psi} \otimes 1) \tilde{\psi} \) for \( k > 0 \), and \( \tilde{\psi} \) is the reduced coproduct on \( BA \otimes BA \). Then, for example, \( \mu \left( [b] \otimes [b] \right) = [a^3] \) and \( \mu \left( [b] \otimes [a[b]] \right) = [a[a^3]] + [b[a]b] \). It follows that \( (BA, d, \Delta, \mu) \) is a DG Hopf algebra. Let \( \mu_H \) and \( \Delta_H \) be the product and coproduct on \( H = H^* (BA) \) induced by \( \mu \) and \( \Delta \); then \( (H, \Delta_H, \mu_H) \) is a graded bialgebra. Let \( \alpha = \text{cls} [a] \) and \( z = \text{cls} [a[a^3]] \) in \( H \), and note that \( [a^3] = d [a[a^3]] \). Let \( g : H \to BA \) be a cycle-selecting map such that \( g (\text{cls} [x_1] \cdots [x_n]) = [x_1] \cdots [x_n] \). Then

\[
\tilde{\Delta}_H (z) = \text{cls} \tilde{\Delta} [a[a^3]] = \text{cls} \{ [a] \otimes [a^3] \} = 0
\]

so that

\[
\{ \Delta g + (g \otimes g) \Delta_H \} (z) = [a] \otimes [a^3].
\]

By the Transfer Theorem, we may choose a map \( g^2 : H \to BA \otimes BA \) such that \( g^2 (z) = [a] \otimes [a[a^3]] \); then

\[
\nabla g^2 (z) = \{ \Delta g + (g \otimes g) \Delta_H \} (z).
\]

Furthermore, note that

\[
\{ (g^3 \otimes g + g \otimes g^3) \Delta_H + (\Delta \otimes 1 + 1 \otimes \Delta) g^3 \} (z) = [a] \otimes [a] \otimes [a^2].
\]

Since \( [a^2] = d [a[a]] \), there is an \( A^{\infty} \)-coalgebra operation \( \Delta^3_H : H \to H^{\otimes 3} \) and a map \( g^3 : H \to (BA)^{\otimes 3} \) satisfying the general relation on \( J_3 \) such that \( \Delta^3_H (z) = 0 \) and \( g^3 (z) = [a] \otimes [a] \otimes [a[a]] \). In fact, we may choose \( \Delta^3_H \) to be identically zero on \( H \) so that

\[
\nabla g^3 = (\Delta \otimes 1 + 1 \otimes \Delta) g^2 + (g^2 \otimes g + g \otimes g^3) \Delta_H.
\]

Now the potentially non-vanishing terms in the image of \( J_4 \) are

\[
(g^3 \otimes g + g^2 \otimes g^2 + g \otimes g^3) \Delta_H + (\Delta \otimes 1 + 1 \otimes \Delta) g^3,
\]

and evaluating at \( z \) gives \( [a] \otimes [a] \otimes [a] \otimes [a] \). Thus there is an \( A^{\infty} \)-coalgebra operation \( \Delta^4_H \) and a map \( g^4 : H \to (BA)^{\otimes 4} \) satisfying the general relation on \( J_4 \) such that \( \Delta^4_H (z) = \alpha \otimes \alpha \otimes \alpha \otimes \alpha \) and \( g^4 (z) = 0 \). Now recall that the induced \( A^{\infty} \)-coalgebra structure on \( H \otimes H \) is given by

\[
\Delta_{H \otimes H} = \sigma_{2,2} (\Delta_H \otimes \Delta_H)
\]

\[
\Delta^4_{H \otimes H} = \sigma_{4,2} (\Delta^4_H \otimes (1^{\otimes 2} \otimes \Delta_H)(1 \otimes \Delta_H) \Delta_H + (\Delta_H \otimes 1^{\otimes 2})(\Delta_H \otimes 1) \Delta_H \otimes \Delta^4_H)
\]

Let \( \beta = \text{cls} [b], u = \text{cls} [a[b]], v = \text{cls} [b[a]], \) and \( w = \text{cls} [b[a][b]] \) in \( H \), and consider the induced map of tilde cobar constructions

\[
\hat{\mu}_H = \sum_{n \geq 1} \left( \uparrow \mu_H \downarrow \right)^{\otimes n} : \hat{\Omega} (H \otimes H) \to \hat{\Omega} H.
\]

Then

\[
\hat{\mu}_H \left[ \beta \otimes u \right] = [\mu_H (\beta \otimes u)] = [\text{cls} \mu ([b] \otimes [a[b]])] = [z + w]
\]

so that

\[
d_{\Omega H \hat{\mu}_H} [\beta \otimes u] = d_{\hat{\Omega} H} [z + w] = [\alpha[a][a] \alpha] + [\beta u] + [v \beta].
\]
But on the other hand,

\[ d_{\Omega(H \otimes H)} [\beta \otimes u] = [\Delta_{H \otimes H} (\beta \otimes u)] \]

\[ = [e \otimes u | \beta \otimes e] + [\beta \otimes (\beta \otimes e) \otimes e \otimes u] \]

\[ + [\beta \otimes \alpha | e \otimes \beta] + [e \otimes \alpha | \beta \otimes \beta] \]

so that

\[ \hat{\mu}_H d_{\Omega(H \otimes H)} [\beta \otimes u] = [\beta u] + [v | \beta] . \]

Although \( \hat{\mu}_H \) fails to be a chain map, the Transfer Theorem implies there is a chain map \( \hat{\mu}_H^2 : \Omega(H \otimes H) \rightarrow \Omega H \) such that \( \hat{\mu}_H^2 [e \otimes \alpha | \beta \otimes \beta] = [\alpha | \alpha | \alpha | \alpha] \), which can be realized by defining

\[ \hat{\mu}_H^2 = \sum_{n \geq 1} (\uparrow \mu_H \downarrow + \uparrow \otimes^3 \omega_3 \downarrow)^n , \]

where \( \omega_3 (\beta \otimes \beta) = \alpha \otimes \alpha \otimes \alpha \). Indeed, to see that the required equality holds, note that \( \mu_H (\beta \otimes \beta) = 0 \) since \( [a^3] = d [a^2] \). Thus there is a map \( g_2 : H \otimes H \rightarrow BA \) such that \( g_2 (\beta \otimes \beta) = [a | a^2] \) and \( \nabla g_2 = g_2 \mu + \mu (g \otimes g) \). Furthermore, there is the following general relation on \( JJ_{3,2} \):

\[ \nabla g_2^2 = \omega_{BA}^2 (g \otimes g) + (\mu \otimes \mu) \sigma_{2,2} (\Delta g \otimes g^2 + g^2 \otimes (g \otimes g) \Delta_H) + g_2^2 \mu_H \]

\[ + (\mu (g \otimes g) \otimes g_2 + g_2 \otimes g \mu_H) \sigma_{2,2} (\Delta_H \otimes \Delta_H) + \Delta g_2 + (g \otimes g) \omega_2^2 . \]

The first expression on the right-hand side vanishes since \( BA \) has trivial higher order structure and the next two expressions vanish since \( \mu_H (\beta \otimes \beta) = 0 \) and \( g^2 (\beta) = 0 \) (\( \beta \) is primitive). However,

\[ \{(\mu (g \otimes g) \otimes g_2 + g_2 \otimes g \mu_H) \sigma_{2,2} (\Delta_H \otimes \Delta_H) + \Delta g_2\} (\beta \otimes \beta) \]

\[ = \Delta g_2 (\beta \otimes \beta) = [a] \otimes [a^2] . \]

Since \( d [a | a] = [a^2] \), there an operation \( \omega_3^2 : H^{\otimes 2} \rightarrow H^{\otimes 2} \) and a map \( g_2^2 : H^{\otimes 2} \rightarrow (BA)^{\otimes 3} \) satisfying relation \((7.3)\) such that \( \omega_3^2 (\beta \otimes \beta) = 0 \) and \( g_2^2 (\beta \otimes \beta) = [a] \otimes [a | a] \). Similarly, there is an operation \( \omega_3^3 : H^{\otimes 3} \rightarrow H^{\otimes 3} \) and a map \( g_3^3 : H^{\otimes 3} \rightarrow (BA)^{\otimes 3} \) satisfying the general relation on \( JJ_{3,3} \) such that \( \omega_3^3 (\beta \otimes \beta) = \alpha \otimes \alpha \otimes \alpha \) and \( g_3^3 (\beta \otimes \beta) = 0 \). Thus \( (H, \mu_H, \Delta_H, \Delta_H^4, \omega_3^3, ... \) is an \( A_\infty \)-bialgebra of the first kind.

One can think of the algebra \( A \) in Example \( \text{[10]} \) as the singular \( \mathbb{Z}_2 \)-cohomology algebra of a space \( X \) with the Steenrod algebra \( A_2 \) acting nontrivially via \( S_q b = a^3 \) (recall that \( S_q : H^n (X; \mathbb{Z}_2) \rightarrow H^{2n-1} (X; \mathbb{Z}_2) \)) is a homomorphism defined by \( S_q \{ x \} = [x \sim_1 x] \). Recall that a space \( X \) is \( \mathbb{Z}_2 \)-formal if there exists a DGA \( B \) and cohomology isomorphisms \( C^* (X; \mathbb{Z}_2) \leftrightarrow B \rightarrow H^* (X; \mathbb{Z}_2) \). Thus when \( X \) is \( \mathbb{Z}_2 \)-formal, \( H^* (BA) \cong H^* (\Omega X; \mathbb{Z}_2) \) as graded coalgebras. Now consider a \( \mathbb{Z}_2 \)-formal space \( X \) whose cohomology \( H^* (X; \mathbb{Z}_2) \) is generated multiplicatively by \( \{ a_1, \ldots, a_{n+1}, b \} \), \( n \geq 2 \). Then Example \( \text{[10]} \) suggests the following conditions on \( X \), which if satisfied, give rise to a nontrivial operation \( \omega_2^2 \) with \( n \geq 2 \), on the loop cohomology \( H^* (\Omega X; \mathbb{Z}_2) \):

1. \( a_1 b = 0 \);
2. \( a_1 \cdots a_{n+1} = 0 \);
3. \( a_{i_1} \cdots a_{i_k} \neq 0 \) whenever \( k \leq n \) and \( i_p \neq i_q \) for all \( p \neq q \);
4. \( Sq(b) = a_2 \cdots a_{n+1} \).
To see this, consider the non-zero classes \( \alpha_i = \text{cls} \{ a_i \} \), \( \beta = \text{cls} \{ b \} \), \( u = \text{cls} \{ a_1 \} \), \( w = \text{cls} \{ b a_1 \} \), and \( z = \text{cls} \{ a_1 a_2 \cdots a_{n+1} \} \) in \( H = H^*(BA) \). Conditions (2) and (3) give rise to an induced \( A_\infty \)-coalgebra structure \( \{ \Delta^k_H : H \to H^{ \otimes k} \} \) such that \( \Delta^k_H(z) = 0 \) for \( 3 \leq k \leq n \) and \( \Delta^k_H(z) = \alpha_1 \cdots \alpha_{n+1} \) with \( g^k_1(z) = [a_1] \otimes \cdots \otimes [a_{k-1}] \otimes [a_k a_{k+1} \cdots a_{n+1}] \) for \( 2 \leq k \leq n \) and \( g^{n+1}_1(z) = 0 \). Next, condition (4) implies \( \beta \overset{\sim}{\to} u = w + z \), and we can define \( \omega^k_2(\beta \otimes \beta) = 0 \) for \( 2 \leq k < n \) and \( \omega^k_2(\beta \otimes \beta) = \alpha_2 \otimes \cdots \otimes \alpha_{n+1} \) with \( g^k_1(\beta \otimes \beta) = [a_2 a_3 \cdots a_{n+1}] \), \( g^k_1(\beta \otimes \beta) = [a_2] \otimes \cdots \otimes [a_k] \otimes [a_{k+1} a_{k+2} \cdots a_{n+1}] \) for \( 2 \leq k \leq n - 1 \) and \( g^k_1(\beta \otimes \beta) = 0 \). Indeed, the Transfer Theorem implies the existence of an \( A_\infty \)-bialgebra structure in which \( \omega^k_2 \) satisfies the required structure relation on \( J_{n,2} \).

Note that the \( \mathbb{Z}_2 \)-formality assumption is in fact superfluous here, as it is sufficient for \( \alpha_i, \beta \), and \( u \) to be non-zero.

Spaces \( X \) with \( \mathbb{Z}_2 \)-cohomology satisfying conditions (1)-(4) abound.

**Example 11.** Given an integer \( n \geq 2 \), choose positive integers \( r_1, \ldots, r_{n+1} \) and \( m \geq 2 \) such that \( r_2 + \cdots + r_{n+1} = 4m - 3 \). Consider the “thick bouquet” \( S^{r_1} \vee \cdots \vee S^{r_{n+1}} \), i.e., \( S^{r_1} \times \cdots \times S^{r_{n+1}} \) with top dimensional cell removed, and generators \( a_i \in H^r(S^r; \mathbb{Z}_2) \). Also consider the suspension of complex projective space \( \Sigma \mathbb{C}P^{2m-2} \) with generators \( b \in H^{2m-1}(\Sigma \mathbb{C}P^{2m-2}; \mathbb{Z}_2) \) and \( Sq^1 b \in H^{3m-3}(\Sigma \mathbb{C}P^{2m-2}; \mathbb{Z}_2) \). Let \( Y_n = S^{r_1} \vee \cdots \vee S^{r_{n+1}} \vee \Sigma \mathbb{C}P^{2m-2} \), and choose a map \( f : Y_n \to K(\mathbb{Z}_2, 4m - 3) \) such that \( f^*(\tau_{4m-3}) = \bar{a}_2 \cdots \bar{a}_{n+1} + Sq^1 \bar{b} \). Finally, consider the pullback \( p : X_n \to Y_n \) of the following path fibration:

\[
K(\mathbb{Z}_2, 4m - 4) \rightarrow X_n \rightarrow \mathcal{L}K(\mathbb{Z}_2, 4m - 3) \\
p \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\
Y_n \quad \rightarrow \quad K(\mathbb{Z}_2, 4m - 3) \\
\bar{a}_2 \cdots \bar{a}_{n+1} + Sq^1 \bar{b} \leftrightarrow f_* \tau_{4m-3}
\]

Let \( a_i = p^*(\bar{a}_i) \) and \( b = p^*(\bar{b}) \); then \( a_1, \ldots, a_{n+1}, b \) are multiplicative generators of \( H^*(X_n; \mathbb{Z}_2) \) satisfying conditions (1) - (4) above. We remark that one can also obtain a space \( X'_2 \) with a non-trivial \( \omega^2_2 \) on its loop cohomology by setting \( Y'_n = (S^2 \times S^3) \vee \Sigma \mathbb{C}P^2 \) in the construction above (see [26] for details).

Finally, we note that the cohomology of Eilenberg-MacLane spaces and Lie groups fail to satisfy all of (1)-(4), and it would not be surprising to find that the operations \( \omega^k_2 \) vanish in their loop cohomologies for all \( n \geq 2 \). In the case of Eilenberg-MacLane spaces, recent work of Berciano and the second author seems to support this conjecture. Indeed, each tensor factor \( A = E(v; 2n + 1) \otimes \Gamma(w; 2n p + 2) \subset H_*(K(\mathbb{Z}_n; \mathbb{Z}_p)) \), \( n \geq 3 \), and \( p \) an odd prime, is an \( A_\infty \)-bialgebra of the third kind of the form \( (A, \Delta^2, \Delta^p, \mu) \) (see [2]).

HGs with nontrivial actions of the Steenrod algebra \( A_2 \) were first considered by the first author in [18] and [19]. In general, the Steenrod \( \smile_1 \)-cochain operation (together with the other higher cochain operations) induces a nontrivial HGA structure on \( S^*(X) \). However, the failure of the differential to be a \( \smile_1 \)-derivation prevents an immediate lifting of the HGA structure to cohomology. Nevertheless, such liftings are possible in certain situations, as we have seen in Example [10]. Here is another such example.

**Example 12.** Let \( g : S^{2n-2} \to S^n \) be a map of spheres and let \( Y_{m,n} = S^m \times (e^{2n-1} \cup_g S^n) \). Let \( * \) be the wedge point of \( S^m \vee S^n \subset Y_{m,n} \), let \( f : S^{2m-1} \to \)
$S^m \times *$, and let $X_{m,n} = e^{2m} \cup f Y_{m,n}$. Then $X_{m,n}$ is $\mathbb{Z}_2$-formal for each $m$ and $n$ (by a dimensional argument), and we may consider $A = H^*(X_{m,n}; \mathbb{Z}_2)$ and $H = H^*(BA) \approx H^*(\Omega X_{m,n}; \mathbb{Z}_2)$. Below we prove that:

(i) The $A_{\infty}$-coalgebra structure of $H$ is nontrivial if and only if the Hopf invariant $h(f) = 1$, in which case $m = 2, 4, 8$.

(ii) If $h(f) = 1$, the $A_{\infty}$-coalgebra structure on $H$ extends to a nontrivial $A_{\infty}$-bialgebra structure on $H$. Furthermore, let $a \in A^n$ and $c \in A^{2n-1}$ be multiplicative generators; then there is a perturbed multiplication $\varphi$ on $H$ if and only if $Sq_1 a = c$, in which case $n = 3, 5, 9$; otherwise $\varphi$ is induced by the shuffle product on $BA$.

Proof. Suppose $h(f) = 1$. Then $A$ is generated multiplicatively by $a \in A^n, b \in A^m$, and $c \in A^{2n-1}$ subject to the relations

$$a^2 = e^2 = ac = ab^2 = b^2 c = b^3 = 0.$$ 

Let $\alpha = \text{cls} [a], \beta = \text{cls} [b], \gamma = \text{cls} [c]$, and $z = \text{cls} [b^2] \in H = H^* (BA)$. Given $x_i = \text{cls} [u_i] \in H$ with $u_i u_{i+1} = 0$, let $x_1 \cdots x_n = \text{cls} [u_1 \cdots u_n]$. Note that $x = \alpha z = z \alpha$ and $y = \gamma z = z \gamma$. Let $\Delta_H$ denote the coproduct in $H$ induced by the cofree coproduct $\Delta$ in $BA$. Then $x$ and $y$ are primitive, and $\Delta_H (\alpha | z | \alpha) = e \otimes \alpha | z \otimes \alpha + \alpha \otimes (z | \alpha) | \alpha \otimes e$. Define $g (x) = [a] [b^2]$ and $g^2 (x) = [a] \otimes [b b]$; define $g (\alpha | z | \alpha) = [a] [b^2] [a]$ and $g^2 (\alpha | z | \alpha) = [a] \otimes ([a] [b] b + [b] [a] b + [b] [b] a)$. There is an induced $A_{\infty}$-coalgebra operation $\Delta_H^3 : H \rightarrow H \otimes H \otimes H$, which vanishes except on elements of the form $\cdots \otimes z \otimes \cdots$, and may be defined on the elements $x, y$, and $\alpha | z | \alpha$ by

$$\Delta_H^3 (x) = \alpha \otimes \beta \otimes \beta,$$
$$\Delta_H^3 (y) = \gamma \otimes \beta \otimes \beta,$$
$$\Delta_H^3 (\alpha | z | \alpha) = \alpha \otimes (\alpha | \beta \otimes \beta | \alpha + \beta \otimes (\alpha | \beta \otimes \beta | \alpha)).$$

Then $\{ \Delta_H, \Delta_H^3 \}$ defines an $A_{\infty}$-coalgebra structure on $H$. Furthermore, if $Sq_1 a = c$, which can only occur when $n = 3, 5, 9$, the induced HGA structure on $A$ is determined by $Sq_1$ and induces a perturbation of the shuffle product $\mu : BA \otimes BA \rightarrow BA$ with $\mu ([a] \otimes [a]) = [c]$. The product $\mu$ lifts to a perturbed product $\varphi$ on $H$ such that

$$\varphi (\alpha \otimes \alpha | z) = \alpha | z | \alpha + \gamma | z,$$

and the $A_{\infty}$-coalgebra structure $(H, \Delta_H, \Delta_H^3)$ extends to an $A_{\infty}$-bialgebra structure $(H, \Delta_H, \Delta_H^3, \varphi)$ as in Example 11. On the other hand, if $Sq_1 a = 0$, then is induced by the shuffle product on $BA$ and $\varphi (\alpha \otimes \alpha | z) = \alpha | z | \alpha$. Conversely, if $h(f) = 0$, then $b^2 = 0$ so that $\Delta_H^k = 0$, for all $k \geq 3$.

We conclude with an investigation of the $A_{\infty}$-bialgebra structure on the double cobar construction. To this end, we first prove a more general fact, which follows our next definition:

**Definition 20.** Let $(A, d, \psi, \varphi)$ be a free DG bialgebra, i.e., free as a DGA. An acyclic cover of $A$ is a collection of acyclic DG submodules

$$C (A) = \{ C^a \subseteq A \mid \text{a is a monomial of } A \}$$

such that $\psi (C^a) \subseteq C^a \otimes C^a$ and $\varphi (C^a \otimes C^b) \subseteq C^{ab}$.

**Proposition 2.** Let $(A, d, \psi, \varphi)$ be a free DG bialgebra with acyclic cover $C (A)$.

(i) Then $\varphi$ and $\psi$ extend to an $A_{\infty}$-bialgebra structure of the third kind.
(ii) Let \((A', d', \psi', \varphi')\) be a free DG bialgebra with acyclic cover \(C(A')\), and let \(f : A \to A'\) be a DGA map such that \(f(C^a) \subseteq C'^{(a)}\) for all \(C^a \in C(A)\).

Then \(f\) extends to a morphism of \(A_\infty\)-bialgebras.

Proof. Define an \(A_\infty\)-coalgebra structure as follows: Let \(\psi^2 = \psi\); arbitrarily define \(\psi^3\) on multiplicative generators and extend \(\psi^3\) to decomposables via

\[
\psi^3 \mu_A = \mu_A^3 \sigma_{3,2}(\psi^3 \otimes \psi^3).
\]

Inductively, if \(\{\psi^i\}_{i<n}\) have been constructed, arbitrarily define \(\psi^n\) on multiplicative generators and extend \(\psi^n\) to decomposables. Since each \(\psi^n\) preserves \(C(A)\) by hypothesis, \(\{\varphi, \psi^2, \psi^3, \ldots\}\) is an \(A_\infty\)-bialgebra structure of the third kind, as desired. The proof of part (ii) is similar. \(\square\)

Given a space \(X\), choose a base point \(y \in X\). Let \(\text{Sing}^2 X\) denote the Eilenberg 2-subcomplex of \(\text{Sing} X\) and let \(C_*(X) = C_*(\text{Sing}^2 X)/C_{>0}(\text{Sing} y)\). In \([20]\) we constructed an explicit (non-coassociative) coproduct on the double-cobar construction \(\Omega^2 C_*(X)\), which turns it into a DG bialgebra. Let \(\Omega^2\) denote the functor from the category of (2-reduced) simplicial sets to the category of permutahedral sets \([20], [13]\) such that \(\Omega^2 C_*(X) = C^\varphi_3(\Omega^2 \text{Sing}^2 X)\), where \(C^\varphi_3(Y) = C_*(\text{Sing}^3 Y)\) (degeneracies), where \(\text{Sing}^3 Y\) is the multipermutahedral singular complex of \(Y\) (see Definition 15 in \([20]\); cf. \([1], [5]\)). Now consider the monoidal permutahedral set \(\Omega^2 \text{Sing}^2 X\), and let \(V_\cdot\) be its monoidal (non-degenerate) generators. For each \(a \in V_n\), let

\[
C^a = R \{r\text{-faces of } a \mid 0 < r < n\}.
\]

Then \(\{C^a\}\) is an acyclic cover, and by Proposition \([2]\) we immediately have:

**Theorem 5.** The DG bialgebra structure on double cobar-construction \(\Omega^2 C_*(X)\) extends to an \(A_\infty\)-bialgebra of the third kind.

**Conjecture 1.** Given a 2-connected space \(X\), the chain complex \(C^\varphi_3(\Omega^2 X)\) admits an \(A_\infty\)-bialgebra structure extending the DG bialgebra structure constructed in \([20]\).

Moreover, there exists a morphism of \(A_\infty\)-bialgebras

\[
G = \{g^a_m\} : \Omega^2 C_*(X) \Rightarrow C^\varphi_3(\Omega^2 X)
\]

such that \(g^1_1\) is a homology isomorphism.

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