An arithmetic Lefschetz–Riemann–Roch theorem

Shun Tang

with an appendix by Xiaonan Ma

Abstract

In this article, we consider regular projective arithmetic schemes in the context of Arakelov geometry, any of which is endowed with an action of the diagonalizable group scheme associated to a finite cyclic group and with an equivariant very ample invertible sheaf. For any equivariant morphism between such arithmetic schemes, which is smooth over the generic fiber, we define a direct image map between corresponding higher equivariant arithmetic K-groups and we discuss its transitivity property. Then we use the localization sequence of higher arithmetic K-groups and the higher arithmetic concentration theorem developed in Tang (Math. Z. 290 (2018) 307–346) to prove an arithmetic Lefschetz-Riemann-Roch theorem. This theorem can be viewed as a generalization, to the higher equivariant arithmetic K-theory, of the fixed-point formula of Lefschetz type proved by Köhler and Roessler (Invent. Math. 145 (2001) 333–396).

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1. Introduction

The aim of this article is to prove an arithmetic Riemann–Roch theorem of Lefschetz type for the higher equivariant arithmetic K-theory of regular arithmetic schemes in the context of Arakelov geometry. This theorem is an arithmetic analogue of a special case of Köck’s Lefschetz theorem in higher equivariant K-theory (cf. [23]), and it also generalizes Köhler–Roessler’s Lefschetz fixed point formula [24, Theorem 4.4] to the case where higher arithmetic K-groups are concerned. To make things more explicit, let us first recall the study of such Lefschetz–Riemann–Roch problems.

Let $X$ be a smooth projective variety over an algebraically closed field $k$, and suppose that $X$ is endowed with an action of a cyclic group $(g)$ of finite order $n$ such that $n$ is prime to the characteristic of $k$. A $(g)$-equivariant coherent sheaf on $X$ is a coherent $O_X$-module $F$ on $X$ together with an automorphism $\varphi : g^*F \to F$ such that $\varphi^n$ is equal to the identity map. Then the classical Lefschetz trace formula gives an expression of the alternating sum of the trace of $H^i(\varphi)$ on the cohomology space $H^i(X, F)$, as a sum of the contributions from the components of the fixed-point subvariety $X_g$. For $k = \mathbb{C}$, the field of complex numbers, such a Lefschetz
trace formula was presented via index theory and topological K-theory in [1, III]. While for general $k$, a Grothendieck-type generalization to the scheme theoretic algebraic geometry is very natural to expect. Precisely, denote by $K_0(X,g)$ the Grothendieck group of the category of equivariant locally free coherent sheaves on $X$, then $K_0(Pt,g)$ is isomorphic to the group ring $\mathbb{Z}[g] \cong \mathbb{Z}[T]/(1 - T^n)$ and $K_0(X,g)$ has a natural $K_0(Pt,g)$-algebra structure (Pt stands for the point Spec(k)). Let $Y$ be another $(g)$-equivariant smooth projective variety, let $f : X \to Y$ be a projective morphism compatible with both $(g)$-actions on $X$ and on $Y$, then we have a direct image map $f_* : K_0(X,g) \to K_0(Y,g)$ given by

$$E \mapsto \sum_{i \geq 0} (-1)^i R^i f_*(E).$$

Unsurprisingly, the direct image map $f_*$ does not commute with the restriction map $\tau : K_0(\cdot, g) \to K_0(\cdot|_Y, g)$ from the equivariant $K_0$-group of an equivariant variety to the equivariant $K_0$-group of its fixed-point subvariety. Namely, the restriction map $\tau$ is not a natural transformation between the covariant functors $K_0(\cdot, g)$ and $K_0(\cdot|_Y, g)$. Like the other Riemann–Roch problems, the Lefschetz–Riemann–Roch theorem makes a correction of $\tau$ such that it becomes a natural transformation. In fact, for any $(g)$-equivariant smooth projective variety $X$, let $N_{X/Y}$ stand for the normal bundle associated to the regular immersion $X \leftarrow X$ and let $\lambda_{-1}(N_{X/Y})$ be the alternating sum $\sum (-1)^j \wedge^j N_{X/Y}$, then $\lambda_{-1}(N_{X/Y})$ is an invertible element in $K_0(X,g) \otimes_{\mathbb{Z}[g]} \mathcal{R}$, where $\mathcal{R}$ is any $\mathbb{Z}[g]$-algebra in which $1 - T^k$ is invertible for $k = 1, \ldots, n - 1$. We formally define $L_X : K_0(X,g) \to K_0(X,g) \otimes_{\mathbb{Z}[g]} \mathcal{R}$ as $\lambda_{-1}^*(N_{X/Y}) \cdot \tau$, the Lefschetz–Riemann–Roch theorem reads: the following diagram

$$\begin{align*}
K_0(X,g) &\xrightarrow{L_X} K_0(X,g) \otimes_{\mathbb{Z}[g]} \mathcal{R} \\
\downarrow f_* &\quad \downarrow f_* \\
K_0(Y,g) &\xrightarrow{L_Y} K_0(Y,g) \otimes_{\mathbb{Z}[g]} \mathcal{R}
\end{align*}$$

is commutative.

This commutative diagram (1) was presented by Donovan in [19], and later it was generalized to singular varieties by Baum, Fulton and Quart in [2]. Note that the settings in [2, 19] are more general than that in this introduction. The reasoning in the first paper runs similarly to the technique used in Borel–Serre’s paper [15], while the reasoning in the second paper relies on the deformation to the normal cone construction. These two processes are both traditional for producing the Grothendieck-type Riemann–Roch theorem.

After Quillen and other mathematicians’ work, algebraic K-groups are extended to higher degrees and the higher (equivariant) algebraic K-groups of $X$ are defined as the higher homotopy groups of the K-theory space associated to the category of (equivariant) locally free coherent sheaves on $X$. There are many methods to construct this ‘K-theory space’, but no matter which construction we choose, the tensor product of locally free coherent sheaves always induces a graded ring structure on $K_*(X,g)$. In particular, each $K_m(X,g)$ is a $K_0(X,g)$-module. Moreover, the functor $K_*(\cdot, g)$ is again covariant with respect to equivariant proper morphisms. Then, for any $m \geq 1$, the following diagram for higher algebraic K-groups which is similar to (1) does make sense:

$$\begin{align*}
K_m(X,g) &\xrightarrow{L_X} K_m(X,g) \otimes_{\mathbb{Z}[g]} \mathcal{R} \\
\downarrow f_* &\quad \downarrow f_* \\
K_m(Y,g) &\xrightarrow{L_Y} K_m(Y,g) \otimes_{\mathbb{Z}[g]} \mathcal{R}.
\end{align*}$$
The commutativity of diagram (2), which is named the Lefschetz–Riemann–Roch theorem for higher equivariant algebraic K-theory, was proved by K"ock in [23]. The main ingredient is an excess intersection formula whose proof also relies on the deformation to the normal cone construction. Moreover, it is worth indicating that the commutative diagram (2), combined with the Gillet’s Riemann–Roch theorem for higher algebraic K-theory (cf. [20]), implies a higher Lefschetz trace formula.

In the field of arithmetic geometry, one considers those Noetherian and separated schemes $f : X \to \text{Spec}(\mathbb{Z})$ over the ring of integers (actually over any excellent regular Noetherian domain). In this context, it is possible to produce an analogue of the commutative diagram (1), by endowing $X$ with an action of the diagonalizable group scheme $\mu_n = \text{Spec}(\mathbb{Z}[[\mathbb{Z}/n\mathbb{Z}]])$ of $n$th roots of unity rather than with the action of an automorphism of order $n$. Here, a $\mu_n$-action on $X$ is a morphism of schemes $m_X : \mu_n \times X \to X$ which satisfies the usual associativity property. The reason for this choice is that the fixed-point subscheme $X_{\mu_n}$ of a regular scheme $X$ equipped with an action of $\mu_n$ is still regular and the natural inclusion $i_X : X_{\mu_n} \to X$ is a regular immersion, while the fixed-point subscheme of a regular scheme under an automorphism of order $n$ can be very singular over the fibers lying above the primes dividing $n$. By a $\mu_n$-equivariant coherent sheaf $F$ on $X$, we understand a coherent $\mathcal{O}_X$-module $F$ together with an isomorphism

$$m_F : m_X^* F \to \text{pr}_X^* F$$

of $\mathcal{O}_{\mu_n \times X}$-modules which satisfies the following associativity property:

$$(\text{pr}_{2,3}^* m_F) \circ ((1 \times m_X)^* m_F) = (m_{\mu_n} \times 1)^* m_F.$$  

Here, $m_{\mu_n}$ denotes the multiplication $\mu_n \times \mu_n \to \mu_n$, $\text{pr}_X : \mu_n \times X \to X$ and $\text{pr}_{2,3} : \mu_n \times \mu_n \times X \to \mu_n \times X$ denote the obvious projections. Under this situation, Baum–Fulton–Quart method still works, so that the commutative diagram (1) holds for regular $\mu_n$-equivariant schemes over $\mathbb{Z}$.

In [35], Thomason used another way to do the same thing and he even got a generalization of the commutative diagram (2) for regular $\mu_n$-equivariant schemes. Thomason’s strategy was to use Quillen’s localization sequence for higher equivariant algebraic K-groups to show a concentration theorem. This theorem states that, after a suitable localization, the equivariant algebraic K-group $K_m(X_{\mu_n}, \mu_n)_p$ is isomorphic to $K_m(X, \mu_n)_p$ for any $m \geq 0$, and the inverse map is exactly given by $\lambda_{-1}^{-1}(N_{X/X_{\mu_n}}^\vee) \cdot i_X^\vee$. Here, $\rho$ is any prime ideal in $R(\mu_n) := K_0(\text{Spec} \mathbb{Z}, \mu_n) \cong \mathbb{Z}[T]/(1 - T^n)$ which does not contain the elements $1 - T^k$ for $k = 1, \ldots, n - 1$. For instance, $\rho$ can be chosen to be the kernel of the natural morphism $\mathbb{Z}[T]/(1 - T^n) \to \mathbb{Z}[T]/(\Phi_n)$, where $\Phi_n$ stands for the $n$th cyclotomic polynomial. Then the Lefschetz–Riemann–Roch theorem for regular $\mu_n$-equivariant schemes

$$K_m(X, \mu_n) \xrightarrow{L_X} K_m(X_{\mu_n}, \mu_n)_p$$

$$\xrightarrow{f_*} K_m(Y_{\mu_n}, \mu_n)_p$$

follows from the covariant functoriality of $K_*(\cdot, \mu_n)$ with respect to proper morphisms.

Now, let us turn to Arakelov geometry. Let $X$ be an arithmetic scheme over an arithmetic ring $(D, \Sigma, F_{\infty})$ in the sense of Gillet–Soulé (cf. [21]), then $X$ is quasi-projective over $D$ with smooth generic fiber. We denote $\mu_n := \text{Spec}(D[\mathbb{Z}/n\mathbb{Z}])$ the diagonalizable group scheme over $D$ associated to a cyclic group $\mathbb{Z}/n\mathbb{Z}$. By saying $X$ is $\mu_n$-projective, we understand that $X$ is endowed with a projective $\mu_n$-action. That means $X$ is projective and there exists a very ample invertible $\mu_n$-sheaf on $X$. 

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For each regular $\mu_n$-projective arithmetic scheme $X$, K. Köhler and D. Roessler have defined an equivariant arithmetic $K_0$-group $\hat{K}_0(X, \mu_n)$ in [24]. This arithmetic $K_0$-group is a modified Grothendieck group of the category of equivariant hermitian vector bundles on $X$, it contains some smooth form class on $X_{\mu_n}(\mathbb{C})$ as analytic datum. The same as the algebraic $K_0$-group $K_0(X, \mu_n)$, $\hat{K}_0(X, \mu_n)$ has a ring structure and it is an $R(\mu_n)$-algebra. Moreover, direct image maps between equivariant arithmetic $K_0$-groups can be defined for an equivariant morphism which is smooth over the generic fiber, by using Bismut–Köhler–Ma analytic torsion forms. Choose a Kähler metric for $X(\mathbb{C})$, and let $N_{X/X_{\mu_n}}$ be the normal bundle endowed with the quotient metric, then the main theorem in [24] reads: the element $\lambda_{-1}(\hat{N}_{X/X_{\mu_n}})$ is a unit in $\hat{K}_0(X_{\mu_n}, \mu_n)_\rho$ and the following diagram

\[
\begin{array}{c}
\hat{K}_0(X, \mu_n) \\
\downarrow f_*> \\
\hat{K}_0(D, \mu_n)
\end{array} \xrightarrow{\Lambda_R \tau} \begin{array}{c}
\hat{K}_0(X_{\mu_n}, \mu_n)_\rho \\
\downarrow f_{\mu_n*}
\end{array}
\]

is commutative, where $\rho$ is any prime ideal in $R(\mu_n)$ which does not contain the elements $1 - T^k$ for $k = 1, \ldots, n-1$, $\Lambda_R$ is defined as $(1 - R_{\rho}(N_{X/X_{\mu_n}})) \cdot \lambda_{-1}(\hat{N}_{X/X_{\mu_n}})$, and $R_{\rho}(\cdot)$ is the equivariant $R$-genus due to Bismut (see below).

Later, two refinements of (4) were presented by the author in [33] and in [32], respectively. In [33], $D$ was replaced by a general regular $\mu_n$-projective scheme $Y$. In [32], $X$ was allowed to have singularities on its finite fibers. The aim of this article is to show an Arakelovian analogue of a special case of (3), in which the higher equivariant algebraic $K$-groups are replaced by the higher equivariant arithmetic $K$-groups. Hence, our work is a generalization of Köhler–Roessler’s Lefschetz fixed-point formula to the higher equivariant arithmetic $K$-theory.

Let us describe the main result more precisely. First, note that we have constructed a group endomorphism $\otimes \lambda_{-1}(\hat{N}_{X/X_{\mu_n}}) : \hat{K}_m(X_{\mu_n}, \mu_n) \to \hat{K}_m(X_{\mu_n}, \mu_n)$ and its formal inverse $\otimes \lambda_{-1}(\hat{N}_{X/X_{\mu_n}}) : \hat{K}_m(X_{\mu_n}, \mu_n)_\rho \to \hat{K}_m(X_{\mu_n}, \mu_n)_\rho$ in [34, Section 5]. As what we stated before, $\rho$ is any prime ideal in $R(\mu_n) = K_0(\text{Spec}\mathbb{Z}, \mu_n) \cong \mathbb{Z}[T]/(1 - T^n)$ which does not contain the elements $1 - T^k$ for $k = 1, \ldots, n-1$. For instance, $\rho$ can be chosen to be the kernel of the natural morphism $\mathbb{Z}[T]/(1 - T^n) \to \mathbb{Z}[T]/(\Phi_n)$, where $\Phi_n$ stands for the $n$th cyclotomic polynomial. In this article, we shall further construct a group endomorphism $R_{\rho}(\hat{N}_{X/X_{\mu_n}}) : \hat{K}_m(X_{\mu_n}, \mu_n) \to \hat{K}_m(X_{\mu_n}, \mu_n)$ and we shall prove that this endomorphism $R_{\rho}(\hat{N}_{X/X_{\mu_n}})$ is independent of the choice of the metric over $N_{X/X_{\mu_n}}$ after tensoring by $\mathbb{Q}$. So, the expression $\Lambda_R = (1 - R_{\rho}(N_{X/X_{\mu_n}})) \cdot \lambda_{-1}(\hat{N}_{X/X_{\mu_n}})$ still makes sense as an endomorphism of $\hat{K}_m(X_{\mu_n}, \mu_n) \otimes \mathbb{Q}$. Moreover, for any equivariant morphism $f : X \to Y$ between regular $\mu_n$-projective arithmetic schemes, which is smooth over the generic fiber, we shall prove that there exists a reasonable direct image map $f_* : \hat{K}_m(X, \mu_n) \to \hat{K}_m(Y, \mu_n)$ with $m \geq 1$ and we discuss the transitivity property of the direct image maps up to torsion. Assume that the $\mu_n$-action on $Y$ is trivial and still use the notation $\tau$ to denote the morphism

\[
\hat{K}_m(\mu_n, \mu_n)_\rho \to \hat{K}_m(\mu_n, \mu_n)_\rho \otimes \mathbb{Q}
\]

\[
x \mapsto \tau(x) \otimes 1,
\]

Our main theorem reads: the following diagram

\[
\begin{array}{c}
\hat{K}_m(X, \mu_n) \\
\downarrow f_*
\end{array} \xrightarrow{\Lambda_R \tau} \begin{array}{c}
\hat{K}_m(X_{\mu_n}, \mu_n)_\rho \otimes \mathbb{Q} \\
\downarrow f_{\mu_n*}
\end{array}
\]

\[
\begin{array}{c}
\hat{K}_m(Y, \mu_n) \\
\downarrow \tau
\end{array} \to \begin{array}{c}
\hat{K}_m(Y, \mu_n)_\rho \otimes \mathbb{Q}
\end{array}
\]

\[
\tag{5}
\]
is commutative. In such a formulation, the equivariant $R$-genus again plays a crucial role.

To this aim, the definition of higher equivariant arithmetic K-groups and some reasonable technique that can be carried out for higher equivariant arithmetic K-theory should be clarified. We have settled these in [34]. In fact, we have defined the higher equivariant arithmetic K-groups via the simplicial description of the Beilinson’s regulators (cf. [17]) and we have developed a localization sequence as well as an arithmetic concentration theorem. So, principally, we shall follow Thomason’s approach to prove the commutativity of (5), but the fact that the direct image maps are only defined for the morphisms which are smooth over the generic fibers will lead to a big gap comparing with the purely algebraic case. Some highly non-trivial analytic machinery should be involved, such as the transitivity property of analytic torsion forms and the Bismut–Ma immersion formula.

The Köhler–Roessler arithmetic Lefschetz fixed-point formula has fruitful applications in number theory and in arithmetic geometry. One important reason is that the equivariant R-genus is closely related to the logarithmic derivative of certain $L$-functions. Köhler–Roessler and Maillot–Roessler have shown in [25] and in [29] that the Faltings heights and the periods of C.M. abelian varieties can be expressed as a formula in terms of the special value of logarithmic derivative of $L$-functions at 0. Further, in [28], Maillot–Roessler presented a series of conjectures about the relation between several invariants of arithmetic varieties and the special values of logarithmic derivative of Artin $L$-functions at negative integers. We hope that our Lefschetz–Riemann–Roch theorem for higher equivariant arithmetic K-groups would be helpful to understand these conjectures.

The structure of this article is as follows. In Section 2, we define the direct image maps between higher equivariant arithmetic K-groups. As an opportunity, we recall the analytic torsion for cubes of hermitian vector bundles introduced by Roessler in [30], actually our construction is slightly different to Roessler’s construction. In Section 3, we discuss certain transitivity property of the direct image maps, the relation of equivariant analytic torsion forms with respect to families of submersions will be presented. In Section 4, we formulate and prove the commutativity of the diagram (5), an accurate computation via the deformation to the normal cone construction is given. In the last section, we attach an Appendix on some properties of equivariant analytic torsion forms and immersion formula. These purely analytic properties are crucial for the main arguments in this article, the author is very grateful to Prof. Ma Xiaonan for writing this Appendix.

2. Higher equivariant arithmetic K-theory

2.1. Bott–Chern forms and arithmetic K-groups

Suggested by Soulé (cf. [31]), and also by Deligne (cf. [18]), the higher arithmetic K-groups of an arithmetic scheme $X$ can be defined as the homotopy groups of the homotopy fiber of Beilinson’s regulator map, so that one obtains a long exact sequence

$$\cdots \rightarrow \hat{K}_m(X) \rightarrow K_m(X) \xrightarrow{\text{ch}} \bigoplus_{p \geq 0} H_D^{2p-m}(X, \mathbb{R}(p)) \rightarrow \hat{K}_{m-1}(X) \rightarrow \cdots,$$

where $H_D^{2p}(X, \mathbb{R}(p))$ is the real Deligne–Beilinson cohomology and ch is the Beilinson’s regulator map. In order to do this, a simplicial description of Beilinson’s regulator map is necessary. In [17], such a simplicial description was given by Burgos and Wang by using the higher Bott–Chern forms. Recently, in [34], we followed Burgos–Wang approach to define the higher equivariant Bott–Chern forms and further the higher equivariant arithmetic K-theory. In this subsection, we shall recall some relevant constructions and definitions, for more details the reader is referred to [17, 34].
At first, let $X$ be a smooth algebraic variety over $\mathbb{C}$. In this subsection, we shall work with the analytic topology of $X$. Denote by $E^*_\log(X)$ the complex of differential forms on $X$ with logarithmic singularities along infinity (cf. [34, Definition 2.1]), then $E^*_\log(X)$ has a natural bigrading $E^n_{\log}(X) = \bigoplus_{p+q=n} E^{p,q}_\log(X)$ and this grading induces a Hodge filtration $F^pE^n_{\log}(X) = \bigoplus_{p' \geq p} E^{p',q}_\log(X)$. Write $E^*_\log,\mathbb{R}(X,p) := (2\pi i)^p E^*_\log(X)$ with $E^*_\log,\mathbb{R}(X)$ the subcomplex of $E^*_\log(X)$ consisting of real forms, then we have a decomposition $E^*_\log(X) = E^*_\log,\mathbb{R}(X,p) \oplus E^*_\log,\mathbb{R}(X,p-1)$ and the projection $\pi_p : E^*_\log(X) \to E^*_\log,\mathbb{R}(X,p)$ is given by $\pi_p(x) = \frac{1}{2}(x + (-1)^p x)$. Moreover, for any $x \in E^n_{\log}(X)$, we define two filtered functions
\[ F^{k,k}x = \sum_{i \geq k} x^{i,i'} \quad \text{and} \quad F^k x = \sum_{i \geq k} x^{i,i'} . \]
Then we set $\pi(x) := \pi_{p-1}(F^{n-p+1,n-p+1}x).$

The main result in [16, Section 2] states that the following Deligne complex
\[ D^n(E^*_\log(X),p) = \begin{cases} E^{n-1}_{\log,\mathbb{R}}(X,p-1) \cap \bigoplus_{p',q' < n, p' \geq p} E^{p',p'}(X) & n < 2p; \\ E^{n}_{\log,\mathbb{R}}(X,p) \cap \bigoplus_{p',q' < n, p' \geq p} E^{p',p'}(X) & n \geq 2p, \end{cases} \]
with differential
\[ d\varphi x = \begin{cases} \varphi(dx) & n < 2p-1; \\ -\pi(dx) & n = 2p-1; \\ dx & n > 2p-1. \end{cases} \]
computes the real Deligne–Beilinson cohomology of $X$. Namely, one has
\[ H^D_*(X,\mathbb{R}(p)) = H^n(D^*(E^*_\log(X),p)) . \]
We shall write $D^*(X,p) := D^*(E^*_\log(X),p)$ for short.

**Remark 2.1.** (i) According to the definition, the real Deligne–Beilinson cohomology of $X$ at degrees $2p$ and $2p-1$ are given by
\[ H^{2p}(D^*(E^*_\log(X),p)) = \{ x \in E^{p,p}_{\log}(X) \cap E^{2p,2p}_{\log,\mathbb{R}}(X,p) \mid dx = 0 \}/(\text{Im}(\partial \overline{\theta})) \]
and
\[ H^{2p-1}(D^*(E^*_\log(X),p)) = \{ x \in E^{p-1,p-1}_{\log}(X) \cap E^{2p-1,2p-1}_{\log,\mathbb{R}}(X,p-1) \mid \partial \overline{dx} = 0 \}/(\text{Im}(\partial + \text{Im} \overline{\theta})) . \]
(ii) Let $x \in D^n(X,p)$ and $y \in D^n(X,q)$, we write $l = n + m$ and $r = p + q$. Then
\[ x \bullet y = \begin{cases} (-1)^{s} r_p(x) \wedge y + x \wedge r_q(y) & n < 2p, m < 2q \\ \pi(x \wedge y) & n < 2p, m \geq 2q, l < 2r \\ F^{r,r}(r_p(x) \wedge y) + 2\pi_r \partial((x \wedge y)^{r-1,l-r}) & n < 2p, m \geq 2q, l \geq 2r \\ x \wedge y & n \geq 2p, m \geq 2q \end{cases} \]
induces a product on $\bigoplus D^*(X,p)$ which is graded commutative and is associative up to chain homotopy. Here, $r_p x = 2\pi_r(F^p dx)$ if $n \leq 2p - 1$ and $r_p x = x$ otherwise. At the level of cohomology groups, this product coincides with the product defined by Beilinson. Note that if $x \in D^{2p}(X,p)$ is a cocycle, then for all $y, z$ we have $x \bullet y = y \bullet x$ and $y \bullet (x \bullet z) = (y \bullet x) \bullet z = x \bullet (y \bullet z)$.

In order to introduce the higher Bott–Chern form, let us construct a new complex $\tilde{D}^*(X,p)$ using the cocubical structure of the cartesian product of projective lines ($\mathbb{P}^1$). This complex
\(\tilde{D}^*(X, p)\) has the same cohomology groups as \(D^*(X, p)\). First, one note that \(D^*(X \times (\mathbb{P}^1)^i, p)\) form a cubical complex with face and degeneracy maps

\[
d^j_i = (\text{Id} \times d^j_i)^* \quad \text{and} \quad s_i = (\text{Id} \times s^i_i)^*,
\]

where

\[
d^j_i : (\mathbb{P}^1)^k \to (\mathbb{P}^1)^{k+1}, \quad i = 1, \ldots, k, j = 0, 1,
\]

\[
s^i_i : (\mathbb{P}^1)^k \to (\mathbb{P}^1)^{k-1}, \quad i = 1, \ldots, k,
\]

which are given by

\[
d^j_i((x_1, \ldots, x_k)) = (x_1, \ldots, x_{i-1}, (0 : 1), x_i, \ldots, x_k),
\]

\[
d^j_i((x_1, \ldots, x_k)) = (x_1, \ldots, x_{i-1}, (1 : 0), x_i, \ldots, x_k),
\]

\[
s^i_i((x_1, \ldots, x_k)) = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_k)
\]

are the coface and the codegeneracy maps of \((\mathbb{P}^1)^k\). Then we write \(D_{\nu}^{r, k}(X, p) = D^r(X \times (\mathbb{P}^1)^{-k}, p)\) and denote by \(D_{\nu}^{r, \ast}(X, p)\) the associated double complex with differentials

\[
d' = d_D \quad \text{and} \quad d'' = \sum (-1)^{i+j-1}d_i^j.
\]

Next, let \((x : y)\) be the homogeneous coordinates of \(\mathbb{P}^1\), and let \(\omega = \partial \bar{\partial} \log \frac{x_1 + p}{x_2} \in (2\pi i)E^2_{\mathbb{P}^1, \mathbb{R}}\) be a Kähler form over \(\mathbb{P}^1\). We shall write \(\omega_i = p_i^* \omega \in E^r_\log (X \times (\mathbb{P}^1)^k)\), where \(p_i : X \times (\mathbb{P}^1)^k \to \mathbb{P}^1, i = 1, \ldots, k\) is the projection over the \(i\)th projective line. The complex \(\tilde{D}^*(X, p)\) is constructed by killing the degenerate classes and the classes coming from the projective spaces.

**Definition 2.2.** We define \(\tilde{D}^*(X, p)\) as the associated simple complex of the double complex \(\tilde{D}^{r, \ast}(X, p)\) which is given by

\[
\tilde{D}^{r, k}(X, p) = D_{\nu}^{r, k}(X, p)/\sum_{i=1}^{-k} s_i(D_{\nu}^{r, k+1}(X, p)) \oplus \omega_i \land s_i(D_{\nu}^{r, -2, k+1}(X, p - 1)).
\]

The differential of this complex will be denoted by \(d\).

A repetition of the proofs of [17, Proposition 1.2, Lemma 1.3] gives that the natural morphism of complexes

\[
t : D^*(X, p) = \tilde{D}^{r, 0}(X, p) \to \tilde{D}^*(X, p)
\]

is a quasi-isomorphism.

Now, let \(X\) be a smooth \(\mu_n\)-projective variety over \(\mathbb{C}\) and denote by \(U := \mathcal{P}(X, \mu_n)\) the exact category of \(\mu_n\)-equivariant vector bundles on \(X\) equipped with \(\mu_n\)-invariant smooth hermitian metrics. We consider the exact cubes in the category \(U\). By definition, an exact \(k\)-cube in \(U\) is a functor \(\mathcal{F}\) from \((-1, 0, 1)^k\), the \(k\)th power of the ordered set \((-1, 0, 1)\), to \(U\) such that for any \(\alpha \in (-1, 0, 1)^{k-1}\) and \(1 \leq i \leq k\), the 1-cube \(\partial_i^\alpha\) defined by

\[
\mathcal{F}_{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_k} \to \mathcal{F}_{\alpha_1, \ldots, \alpha_i, 0, \alpha_i, \ldots, \alpha_k} \to \mathcal{F}_{\alpha_1, \ldots, \alpha_i, 1, \alpha_i, \ldots, \alpha_k}
\]

which is called an edge of \(\mathcal{F}\) is a short exact sequence. From now on, we shall write cubes instead of exact cubes for short. Let \(\mathcal{F}\) be a \(k\)-cube in \(U\), for \(1 \leq i \leq k\) and \(j \in (-1, 0, 1)\), the \((k-1)\)-cube \(\partial_i^\alpha\mathcal{F}\) defined by \(\partial_i^\alpha\mathcal{F}_{\alpha_1, \ldots, \alpha_k} = \mathcal{F}_{\alpha_1, \ldots, \alpha_i, 1, \alpha_i, \ldots, \alpha_k}\) is called a face of \(\mathcal{F}\). On the other hand, for any \(1 \leq i \leq k + 1\), we denote by \(S_i^1\mathcal{F}\) the \((k + 1)\)-cube

\[
(S_i^1\mathcal{F})_{\alpha_1, \ldots, \alpha_k} = \begin{cases} 
0, & \alpha_i = 1; \\
\mathcal{F}_{\alpha_1, \ldots, \alpha_i, 0, \alpha_i, \ldots, \alpha_k}, & \alpha_i \neq 1,
\end{cases}
\]
such that the morphisms $(S_i^1 F)_{a_1,\ldots,a_i-1,0,a_{i+1},\ldots,a_k+1} \to (S_i^1 F)_{a_1,\ldots,a_i-1,0,a_{i+1},\ldots,a_k+1}$ are the identities of $(S_i^1 F)_{a_1,\ldots,a_i-1,0,a_{i+1},\ldots,a_k+1}$. Similarly, we have $(k+1)$-cube $S_i^{-1} F$.

Denote by $C^k \zeta U$ the set of all $k$-cubes in $\zeta U$, then we have the face maps $\partial_i : C^k \zeta U \to C^{k-1} \zeta U$ and the degeneracy maps $S_i^j : C^k \zeta U \to C^{k+j} \zeta U$. The cubes in the image of $S_i^j$ are said to be degenerate. Let $\zeta C^k \zeta U$ be the free abelian group generated by $C^k \zeta U$ and $D_k$ be the subgroup of $\zeta C^k \zeta U$ generated by all degenerate $k$-cubes. Set $\zeta C^k \zeta U = \zeta C^k \zeta U/D_k$ and

$$d = \sum_{i=1}^{k} \sum_{j=-1}^{1} (1)^{i+j-1} \partial_i^j : \zeta C^k \zeta U \to \zeta C^{k-1} \zeta U.$$ 

Then $\zeta C^1 \zeta U = (\zeta C^1 \zeta U, d)$ is a homological complex.

Assume that $\zeta E$ is a hermitian $k$-cube in the category $\zeta U = \zeta \hat P(X, \mu_n)$. If $\zeta E$ is an emi-cube, namely the metrics on the quotient terms in all edges of $\zeta E$ are induced by the metrics on the middle terms (cf. [17, Definition 3.5]), one can follow [17, (3.7)] to associate a hermitian locally free sheaf $tr_k(\zeta E)$ on $X \times (\mathbb{P}^1)^k$. This $tr_k(\zeta E)$ is called the $k$-transgression bundle of $\zeta E$. If $k = 1$, as an emi-1-cube, $\zeta E$ is a short exact sequence

$$0 \longrightarrow \zeta E_{-1} \overset{i}{\longrightarrow} \zeta E_0 \longrightarrow \zeta E_1 \longrightarrow 0,$$

where the metric of $\zeta E_1$ is induced by the metric of $\zeta E_0$. Then $tr_1(\zeta E)$ is the cokernel with quotient metric of the map $\zeta E_{-1} \to \zeta E_{-1} \otimes \mathcal{O}(1) \oplus \zeta E_0 \otimes \mathcal{O}(1)$ by the rule $e_{-1} \to e_{-1} \otimes \sigma_0 \oplus i(e_{-1}) \otimes \sigma_0$. Here, $\sigma_0$ (respectively, $\sigma_\infty$) is the section of the tautological bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ which vanishes only at 0 (respectively, $\infty$), and $\mathcal{O}(1)$ is endowed with the Fubini-Study metric. If $k > 1$, suppose that the transgression bundle is defined for $k - 1$. Let $tr_{k}(\zeta E)$ be the emi-($k - 1$)-cube over $X \times \mathbb{P}^1$ given by $tr_k(\zeta E)_n = tr_{k-1}(\partial_i^j(\zeta E))$, then $tr_k(\zeta E)$ is defined as $tr_{k-1}(tr_1(\zeta E))$.

Moreover, according to [17, Proposition 3.6], for any hermitian cube $\zeta E$ in the category $\zeta U$, there is a unique way to change the metrics on $\zeta E_\alpha$ for $\alpha \neq 0$ such that the obtained new hermitian cube is emi. In fact, for $i = 1, \ldots, k$, define $\lambda_i^1 \zeta E$ to be

$$(\lambda_i^1 \zeta E)_\alpha = \begin{cases} (E_\alpha, h_\alpha), & \text{if } \alpha_i = -1, 0; \\ (E_\alpha, h'_\alpha), & \text{if } \alpha_i = 1, \end{cases}$$

where $h'_\alpha$ is the metric induced by $h_{a_1,\ldots,a_i-1,0,a_{i+1},\ldots,a_k}$. Thus, $\lambda_i^1 \zeta E$ has the same locally free sheaves as $\zeta E$, but the metrics on the face $\partial_i \zeta E$ are induced by the metrics of the face $\partial_i \zeta E$. To measure the difference between $\zeta E$ and $\lambda_i^1 \zeta E$, let $\lambda_i^2(\zeta E)$ be the hermitian $k$-cube determined by $\partial_i^{-1} \lambda_i^2(\zeta E) = \partial_i \zeta E$, $\partial_i \lambda_i^2(\zeta E) = \partial_i^j \lambda_i^1(\zeta E)$, and $\partial_i^j \lambda_i^2(\zeta E) = 0$. Set $\lambda_i = \lambda_i^1 + \lambda_i^2$, $\lambda = \lambda_0 \cdots \lambda_l$ if $k \geq 1$ and $\lambda = \text{Id}$ otherwise. Then the map $\lambda$ induces a morphism of complexes

$$\tilde\zeta C_\ast \zeta U \to \tilde\zeta C_\ast \text{emi} \zeta U$$

which is the quasi-inverse of the inclusion $\tilde\zeta C_\ast \text{emi} \zeta U \hookrightarrow \tilde\zeta C_\ast \zeta U$. To specify the $\mu_n$-equivariant variety $X$, we shall write $\tilde\zeta C_\ast(X, \mu_n) := \tilde\zeta C_\ast \zeta U$.

**Definition 2.3.** Fix a primitive $n$th root of unity $\zeta_n$, the restriction of an equivariant hermitian vector bundle $\overline{F}|_{X_{\mu_n}}$ over the fixed-point subvariety splits into a direct sum $\bigoplus_{l=1}^n \overline{F}_l$, where $F_l$ is the eigenbundle of $F|_{X_{\mu_n}}$ corresponding to the eigenvalue $\zeta_n^{-l}$. Let $K_l$ be the curvature form with respect to the unique connection on $F_l$ compatible with both the hermitian and the complex structure, the equivariant Chern–Weil form associated to $\overline{F}$ is defined as

$$\text{ch}_\ast^l(\overline{F}) := \sum_{l=1}^n \zeta^{-l}_n \text{Tr}(\exp(-K_l)).$$
Define $R_n = \mathbb{R}$ if $n = 1$ and $R_n = \mathbb{C}$ otherwise, denote $V \otimes_{\mathbb{R}} R_n$ by $V_{R_n}$ for any real vector space $V$, the equivariant higher Bott–Chern form associated to hermitian $k$-cube $\tilde{E}$ is defined as

$$
\text{ch}^k_g(\tilde{E}) := \text{ch}^0_g(\text{tr}_k(\lambda(\tilde{E}))) \in \bigoplus_{p \geq 0} \tilde{D}^*(X_{\mu_n}, p)[2p]_{R_n}.
$$

**Definition 2.4.** Let $\tilde{F} \mid_{X_{\mu_n}} = \bigoplus_{i=1}^n \tilde{F}_i$ be the restriction of an equivariant hermitian vector bundle over the fixed-point subvariety, where $F_i$ is the eigenbundle of $\tilde{F} \mid_{X_{\mu_n}}$ corresponding to the eigenvalue $\tilde{\zeta}_i$ and $K_i$ is the curvature form of $\tilde{F}_i$. The equivariant Todd form is defined as

$$
\text{Td}_g(\tilde{F}) = \det \left( \frac{-K_n}{1 - e^{K_n}} \right) \prod_{i \neq n} \det \left( \frac{1}{1 - \zeta_i e^{K_i}} \right).
$$

When $X$ is proper, Burgos and Wang gave in [17, Section 6] a quasi-inverse $\varphi : \tilde{D}^*(X, p) \to D^*(X, p)$ of the quasi-isomorphism $\iota : D^*(X, p) \to \tilde{D}^*(X, p)$. By means of this quasi-inverse, the equivariant higher Bott–Chern form has another expression with value in $\bigoplus_{p \geq 0} D^*(X_{\mu_n}, p)[2p]_{R_n}$. To see this expression, let us set $z = x/y$ which defines the coordinate map $\mathbb{C} \to \mathbb{P}_C^1$ by sending $z \to [z, 1]$. Then $\log |z|$ defines an $L^1$ function on $\mathbb{P}_C^1$, which can be considered as a current. We shall denote by $\log |z_1|, \ldots, \log |z_k|$ the corresponding currents on $(\mathbb{P}_C^1)^k$. These currents can be formally considered as elements in $D^1((\mathbb{P}_C^1)^k, 1)$, and they satisfy the following differential equation

$$
dp \log |z_j| = -2\partial \bar{\partial} \log |z_j| = -2i\pi(\delta_{\mathbb{P}_C^1}(\sum_{i=1}^{\infty} \cdots \mathbb{P}_C^1 - \delta_{\mathbb{P}_C^1}(\mathbb{P}_C^1 \times \cdots \times \mathbb{P}_C^1))
$$

where $\infty$ and $0$ stand at the $j$th place. Let $u_1, \ldots, u_k$ be $k$ elements in $\bigoplus_{p \geq 0} D^{2p-1}(-, p)$, we define an element in $\bigoplus_{p \geq 0} D^{2p-k}(-, p)$ by the formula

$$
C_k(u_1, \ldots, u_k) := \left(-\frac{1}{2}\right)^{k-1} \sum_{\sigma \in S_k} (-1)^{\sigma} u_{\sigma(1)} \cdot (u_{\sigma(2)} \cdot (\cdots u_{\sigma(k)}) \cdots),
$$

where $S_k$ stands for the $k$th symmetric group. Then we have

$$
d DC_k(u_1, \ldots, u_k) = \left(-\frac{1}{2}\right)^k k \sum_{j=1}^{k} (-1)^{j-1} dD(u_j) \cdot C_{k-1}(u_1, \ldots, \hat{u}_j, \ldots, u_k)
$$

$$
= \left(-\frac{1}{2}\right)^k k \sum_{j=1}^{k} (-1)^{j-1} dD(u_j) \wedge C_{k-1}(u_1, \ldots, \hat{u}_j, \ldots, u_k). \tag{6}
$$

We refer to [30, Lemma 2.9] for a proof of these identities. With the above notations, the equivariant higher Bott–Chern form associated to a hermitian $k$-cube $\tilde{E}$ with $k \geq 0$ is given by the expression

$$
\varphi \left( \text{ch}^k_g(\tilde{E}) \right) = \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}_C^1)^k} \text{ch}^k_g(\tilde{E}) \wedge C_k(\log |z_1|^2, \ldots, \log |z_k|^2).
$$

**Theorem 2.5.** The equivariant higher Bott–Chern forms induce a morphism of complexes

$$
\tilde{Z}C^*(X, \mu_n) \longrightarrow \tilde{Z}C_{\text{eml}}^*(X, \mu_n) \xrightarrow{\text{ch}^*_g \text{ext}} \bigoplus_{p \geq 0} \tilde{D}^*(X_{\mu_n}, p)[2p]_{R_n} \longrightarrow \bigoplus_{p \geq 0} D^*(X_{\mu_n}, p)[2p]_{R_n},
$$

which is denoted by $\text{ch}_g$. Here, $\tilde{Z}C^*(X, \mu_n)$ and $\tilde{Z}C_{\text{eml}}^*(X, \mu_n)$ are the (cohomological) complexes associated to the homological complexes $\tilde{Z}C^*_e(X, \mu_n)$ and $\tilde{Z}C_{\text{eml}}^*(X, \mu_n)$. 
Specify to the case $k = 1$, let $\bar{\varepsilon} : 0 \to \bar{E}_{-1} \to \bar{E}_0 \to \bar{E}_1 \to 0$ be a hermitian 1-cube, then
\[
d_P \text{ch}_g(\bar{\varepsilon}) = d_P \left( \frac{1}{4\pi i} \int_{\mathbb{P}^1} \text{ch}_g^0(\text{tr}_1(\lambda(\bar{\varepsilon}))) \log |z|^2 \right) = \text{ch}_g(\bar{E}_0) - \text{ch}_g(\bar{E}_{-1}) - \text{ch}_g(\bar{E}_1).
\]
If $\bar{\varepsilon}$ is split, by replacing $z$ by $1/z$, we know that
\[
\text{ch}_g(\bar{\varepsilon}) = \frac{1}{4\pi i} \int_{\mathbb{P}^1} \text{ch}_g^0(\text{tr}_1(\lambda(\bar{\varepsilon}))) \log |z|^2 = \frac{1}{4\pi i} \int_{\mathbb{P}^1} \text{ch}_g^0(\bar{E}_1(1)) \log |z|^2 + \frac{1}{4\pi i} \int_{\mathbb{P}^1} \text{ch}_g^0\left( \frac{\bar{E}_{-1}(1) \oplus \bar{E}_{-1}(1)}{\bar{E}_{-1}} \right) \log |z|^2 = 0.
\]

Let $\text{ch}_g'$ denote the usual equivariant Chern–Weil forms with the factor $2\pi i$ inside
\[
\text{ch}_g'(\bar{E}) = \sum_{l=1}^{n} \zeta_n^l \text{Tr}\left( \exp\left( \frac{-K_l}{2\pi i} \right) \right),
\]
and let $\Phi$ be an operator acting on $2n$-forms by $\Phi(\alpha) = (2\pi i)^{-n}\alpha$. Then
\[
\Phi(\text{ch}_g(\bar{E})) = \text{ch}_g'(\bar{E})
\]
and
\[
\frac{\bar{\partial} \partial}{2\pi i} (2\Phi(\text{ch}_g(\bar{\varepsilon}))) = \text{ch}_g'(\bar{E}_0) - \text{ch}_g'(\bar{E}_{-1}) - \text{ch}_g'(\bar{E}_1).
\]
This means, after a rescaling, $\text{ch}_g(\bar{\varepsilon})$ satisfies the axiomatic conditions for a theory of unique equivariant secondary Bott–Chern classes [24, Theorem 3.4] (see [9, §1, (f)] for the non-equivariant case). Note that in [9], the authors used the supertraces of Quillen’s superconnections to define the non-equivariant secondary Bott–Chern form $\text{ch}_\varepsilon$. Split $\bar{\varepsilon} |_{X_{n,n}}$ into a direct sum of short exact sequences of its eigenbundles $\oplus_{l=1}^{n} \bar{\varepsilon}_l$ and define
\[
\tilde{\text{ch}}_g(\bar{\varepsilon}) := \sum_{l=1}^{n} \zeta_n^l \tilde{\text{ch}}(\bar{\varepsilon}_l).
\]
Then we get another way, using the supertraces of Quillen’s superconnections, to define the equivariant secondary Bott–Chern form $\tilde{\text{ch}}_g(\bar{\varepsilon})$ which satisfies the equation
\[
\frac{\bar{\partial} \partial}{2\pi i} \tilde{\text{ch}}_g(\bar{\varepsilon}) = \text{ch}_g'(\bar{E}_0) - \text{ch}_g'(\bar{E}_{-1}) - \text{ch}_g'(\bar{E}_1).
\]
So, $2\Phi(\text{ch}_g(\bar{\varepsilon}))$ must be equal to $\tilde{\text{ch}}_g(\bar{\varepsilon})$ modulo $\text{Im} \partial + \text{Im} \bar{\partial}$. Let us write $2\Phi(\text{ch}_g(\bar{\varepsilon})) - \tilde{\text{ch}}_g(\bar{\varepsilon}) = \partial \Delta_\partial(\bar{\varepsilon}) + \bar{\partial} \Delta_{\bar{\partial}}(\bar{\varepsilon})$, the following theorem states that $\Delta_\partial(\bar{\varepsilon})$ and $\Delta_{\bar{\partial}}(\bar{\varepsilon})$ can be chosen to admit some functorial property.

**Theorem 2.6.** Let notations and assumptions be as above. There is a functorial choice of the differential forms $\Delta_\partial(\bar{\varepsilon})$ and $\Delta_{\bar{\partial}}(\bar{\varepsilon})$ such that
\[
2\Phi(\text{ch}_g(\bar{\varepsilon})) - \tilde{\text{ch}}_g(\bar{\varepsilon}) = \partial \Delta_\partial(\bar{\varepsilon}) + \bar{\partial} \Delta_{\bar{\partial}}(\bar{\varepsilon})
\]
and that $\Delta_\partial(j^* \bar{\varepsilon}) = j^* \Delta_\partial(\bar{\varepsilon})$, $\Delta_{\bar{\partial}}((j^*)^* \bar{\varepsilon}) = j^* \Delta_{\bar{\partial}}(\bar{\varepsilon})$ for any equivariant morphism $j : X' \to X$.

**Proof.** For hermitian 1-cube $\bar{\varepsilon} : 0 \to \bar{E}_{-1} \to \bar{E}_0 \to \bar{E}_1 \to 0$, we divide it into two emi-1-cubes
\[
\bar{\varepsilon}_1 : 0 \to \bar{E}_{-1} \to \bar{E}_0 \to \bar{E}_1' \to 0
\]
and

\[ \tilde{\varepsilon}_2 : 0 \to \tilde{E}_1 \to \tilde{E}_1' \to 0 \to 0, \]

where \( \tilde{E}_1' \) is \( E_1 \) endowed with the quotient metric. According to the definition of the morphism \( \lambda \), the higher Bott–Chern form is additive \( \text{ch}_g(\tilde{\varepsilon}) = \text{ch}_g(\tilde{\varepsilon}_1) + \text{ch}_g(\tilde{\varepsilon}_2) \). To study the secondary Bott–Chern form constructed by the supertraces of Quillen’s superconnections, we write down a double complex

\[
\begin{array}{cccc}
0 & 0 & 0 & \\
\downarrow & & & \\
0 & \tilde{E}_{-1} & \tilde{E}_0 & \tilde{E}_1 & 0 & \\
\downarrow & & & & & \\
0 & \tilde{E}_{-1} & \tilde{E}_0 & \tilde{E}_1' & 0 & \\
\downarrow & & & & & \\
0 & 0 & 0 & 0 & 0 & \\
\end{array}
\]

According to the definition of the morphism \( \lambda \), the higher Bott–Chern form is additive \( \text{ch}_g(\tilde{\varepsilon}) = \text{ch}_g(\tilde{\varepsilon}_1) + \text{ch}_g(\tilde{\varepsilon}_2) \). To study the secondary Bott–Chern form constructed by the supertraces of Quillen’s superconnections, we write down a double complex

\[
\begin{array}{cccc}
0 & 0 & 0 & \\
\downarrow & & & \\
0 & \tilde{E}_{-1} & \tilde{E}_0 & \tilde{E}_1 & 0 & \\
\downarrow & & & & & \\
0 & \tilde{E}_{-1} & \tilde{E}_0 & \tilde{E}_1' & 0 & \\
\downarrow & & & & & \\
0 & 0 & 0 & 0 & 0 & \\
\end{array}
\]

Restrict every bundle over \( X \times \mathbb{P}^1 \), and split it into the direct sum of eigenbundles, then one can immediately repeat the argument given in [9, Theorem 1.20] (where the non-equivariant bundles were dealt with) to write down a proof of the fact that \( \tilde{\text{ch}}_g(\tilde{\varepsilon}) = \text{ch}_g(\tilde{\varepsilon}_1) + \text{ch}_g(\tilde{\varepsilon}_2) \) modulo \( \text{Im} \bar{\partial} + \text{Im} \partial \). In the proof of [9, Theorem 1.20], the error terms were explicitly written down and were functorial (see [9, (1.71), (1.72), (1.78), (1.81) and (1.82)]). That means one can fix a functorial choice of differential forms \( \Delta'_{\bar{\partial}}(\tilde{\varepsilon}) \) and \( \Delta'_{\partial}(\tilde{\varepsilon}) \) such that

\[
\tilde{\text{ch}}_g(\tilde{\varepsilon}) - \left( \text{ch}_g(\tilde{\varepsilon}_1) + \text{ch}_g(\tilde{\varepsilon}_2) \right) = \partial \Delta'_{\bar{\partial}}(\tilde{\varepsilon}) + \bar{\partial} \Delta'_{\partial}(\tilde{\varepsilon}).
\]

So, we may reduce our proof to the case, where \( \tilde{\varepsilon} \) is an emi-1-cube.

Now, we consider the following exact sequence on \( X \times \mathbb{P}^1 \)

\[
\Psi : 0 \to \tilde{E}_{-1} \xrightarrow{\text{Id} \otimes \sigma_0 \oplus \iota \otimes \sigma_0} \tilde{E}_{-1}(1) \oplus \tilde{E}_0(1) \xrightarrow{\text{tr}_1(\tilde{\varepsilon})} 0,
\]

we compute, using the fact that \( \int_{\mathbb{P}^1} \text{ch}_g(\tilde{E}_{-1}) \log |z| = 0 \), \( \int_{\mathbb{P}^1} \text{ch}_g(\tilde{E}_{-1}(1) \oplus \tilde{E}_0(1)) \log |z| = 0 \) and the Stokes formula,

\[
2\Phi(\text{ch}_g(\tilde{\varepsilon})) = \int_{\mathbb{P}^1} \Phi(\text{ch}_g(\text{tr}_1(\tilde{\varepsilon}))) \log |z|^2
\]

\[
= - \int_{\mathbb{P}^1} \frac{\bar{\partial} \partial}{2\pi i} \text{ch}_g(\Psi) \log |z|^2
\]

\[
= \frac{1}{2\pi i} \left( \partial_X + \partial_z \right)(\bar{\partial}_X + \bar{\partial}_z) \text{ch}_g(\Psi) \log |z|^2
\]

\[
= \partial_X \bar{\partial}_X \left( \frac{1}{2\pi i} \int_{\mathbb{P}^1} \text{ch}_g(\Psi) \log |z|^2 \right) + \partial_X \left( \frac{1}{2\pi i} \int_{\mathbb{P}^1} \bar{\partial}_z \text{ch}_g(\Psi) \log |z|^2 \right)
\]

\[
- \partial_X \left( \frac{1}{2\pi i} \int_{\mathbb{P}^1} \partial_z \text{ch}_g(\Psi) \log |z|^2 \right) + \frac{1}{2\pi i} \int_{\mathbb{P}^1} \partial_z \bar{\partial}_z \text{ch}_g(\Psi) \log |z|^2
\]
\[ = \frac{1}{2\pi i} \int_{p_1} \hat{c}_h(\Psi) \partial_x \partial_z \log |z|^2 + \partial_X \partial_X \left( \frac{1}{2\pi i} \int_{p_1} \hat{c}_h(\Psi) \log |z|^2 \right) \]
\[
+ \partial_X \left( \frac{1}{2\pi i} \int_{p_1} \partial_z \hat{c}_h(\Psi) \log |z|^2 \right) - \partial_X \left( \frac{1}{2\pi i} \int_{p_1} \partial_x \hat{c}_h(\Psi) \log |z|^2 \right) .
\]

Set
\[ \Delta_\partial(\bar{\epsilon}) = \partial_X \left( \frac{1}{2\pi i} \int_{p_1} \hat{c}_h(\Psi) \log |z|^2 \right) + \frac{1}{2\pi i} \int_{p_1} \partial_z \hat{c}_h(\Psi) \log |z|^2 \]
and
\[ \Delta_\bar{\partial}(\bar{\epsilon}) = -\frac{1}{2\pi i} \int_{p_1} \partial_x \hat{c}_h(\Psi) \log |z|^2 . \]

We get
\[ 2\Phi(\hat{c}_h(\Psi)) = \hat{c}_h(\Psi |_{X \times \{\infty\}}) - \hat{c}_h(\Psi |_{X \times \{0\}}) + \partial \Delta_\partial(\bar{\epsilon}) + \bar{\partial} \Delta_\bar{\partial}(\bar{\epsilon}) . \]

By the construction, \( \Psi |_{X \times \{\infty\}} \) is split and \( \Psi |_{X \times \{0\}} \) is isometric to the direct sum of \( \bar{\epsilon} \) and a split exact sequence \( 0 \to 0 \to \bar{E}_- \to \bar{E}_{-1} \to 0 \), we finally have
\[ 2\Phi(\hat{c}_h(\Psi)) - \hat{c}_h(\bar{\epsilon}) = \partial \Delta_\partial(\bar{\epsilon}) + \bar{\partial} \Delta_\bar{\partial}(\bar{\epsilon}) . \]

Since the 1-transgression bundle construction \( \Psi \) is functorial, the differential forms \( \Delta_\partial(\bar{\epsilon}) \) and \( \Delta_\bar{\partial}(\bar{\epsilon}) \) associated to \( \hat{c}_h(\Psi) \) are also functorial, thus we complete the whole proof. \( \square \)

**Remark 2.7.** (i) According to Theorem 2.6, we can make a functorial choice of differential forms \( \Delta_\partial(\bar{\epsilon}) \) and \( \Delta_\bar{\partial}(\bar{\epsilon}) \) for any hermitian 1-cube \( \bar{\epsilon} \) such that
\[ 2\Phi(\hat{c}_h(\bar{\epsilon})) - \hat{c}_h(\bar{\epsilon}) = \partial \Delta_\partial(\bar{\epsilon}) + \bar{\partial} \Delta_\bar{\partial}(\bar{\epsilon}) . \]

Set \( \Delta(\bar{\epsilon}) = -\Phi^{-1}(\frac{\Delta_\partial(\bar{\epsilon}) + \Delta_\bar{\partial}(\bar{\epsilon})}{2}) \), then \( \Delta(\bar{\epsilon}) \) is functorial and by the definition of the Deligne complex \( D^*(E_{log}(X), p) \) we have
\[ \hat{c}_h(\bar{\epsilon}) - \Phi^{-1}\left( \frac{\hat{c}_h(\bar{\epsilon})}{2} \right) = -\pi((\partial + \bar{\partial})\Delta(\bar{\epsilon})) = d_D \Delta(\bar{\epsilon}) . \]

(ii) It is easily seen from the proof of Theorem 2.6 that if one uses another way to define the equivariant Bott–Chern form \( \hat{c}_h \) which satisfies the axiomatic conditions in [9, Theorem 1.29] (at the level of differential forms) and which is additive for direct sum of short exact sequences, then one can also make a functorial choice of element \( \Delta(\bar{\epsilon}) \) for any hermitian emi-1-cube \( \bar{\epsilon} \) such that
\[ \hat{c}_h(\bar{\epsilon}) - \Phi^{-1}\left( \frac{\hat{c}_h(\bar{\epsilon})}{2} \right) = d_D \Delta(\bar{\epsilon}) . \]

If \( X \) is a regular \( \mu_n \)-projective arithmetic scheme over an arithmetic ring \((D, \Sigma, F_\infty)\), we shall denote \( X_{\mathbb{R}} := (X(\mathbb{C}), F_\infty) \) the real variety associated to \( X \), where \( F_\infty \) is the antiholomorphic involution of \( X(\mathbb{C}) \) induced by the conjugate-linear involution \( F_\infty \) over \((D, \Sigma, F_\infty)\). For any sheaf of complex vector spaces \( V \) with a real structure over \( X_{\mathbb{R}} \), we denote by \( \sigma \) the involution given by
\[ \omega \mapsto F_\infty(\omega) . \]

Write \( D^*(X_{\mathbb{R}}, p) := D^*(X(\mathbb{C}), p)^\sigma \) for the subcomplex of \( D^*(X(\mathbb{C}), p) \) consisting of the fixed elements under \( \sigma \), we define the real Deligne–Beilinson cohomology of \( X \) as
\[ H_D^*(X, \mathbb{R}(p)) := H^*(D^*(X_{\mathbb{R}}, p)) . \]
Let us denote by \( \hat{P}(X, \mu_n) \) the exact category of \( \mu_n \)-equivariant hermitian vector bundles on \( X \), and by \( \hat{S}(X, \mu_n) \) the simplicial set associated to the Waldhausen \( S \)-construction of \( \hat{P}(X, \mu_n) \) (cf. [34, Section 2.3]). The forgetful functor (forget about the metrics) \( \pi : \hat{P}(X, \mu_n) \to \mathcal{P}(X, \mu_n) \) induces an equivalence of categories, so we have homotopy equivalence

\[
| \hat{S}(X, \mu_n) | \simeq | S(X, \mu_n) |
\]

and isomorphisms of abelian groups

\[
K_m(X, \mu_n) \cong \pi_{m+1} | \hat{S}(X, \mu_n) |, 0)
\]

for any \( m \geq 0 \). To give the simplicial description of the equivariant regulator maps, we associate to each element in \( S_k \hat{P}(X, \mu_n) \) a hermitian \( (k-1) \)-cube. First, note that an element \( A \) in \( S_k \hat{P}(X, \mu_n) \) is a family of injections \( A_{0,1} \to A_{0,2} \to \cdots \to A_{0,k} \) of \( \mu_n \)-equivariant hermitian vector bundles on \( X \) with quotients \( A_{i,j} \cong A_{0,j}/A_{0,i} \) for each \( i < j \).

For \( k = 1 \), we write \( \text{Cub}(A_{0,1}) = A_{0,1} \).

Suppose that the map Cub is defined for all \( l < k \), then Cub\( A \) is the \( (k-1) \)-cube with

\[
\partial_1^{-1} \text{Cub} A = s_{k-2}^1 \cdots s_1^1(A_{0,1}),
\]

\[
\partial_1^1 \text{Cub} A = \text{Cub}(\partial_0 A).
\]

Let \( \mathbb{Z}\hat{S}_*(X, \mu_n) \) be the simplicial abelian group generated by the simplicial set \( \hat{S}(X, \mu_n) \), and let \( \mathcal{N}(\mathbb{Z}\hat{S}_*(X, \mu_n)) \) be the Moore complex associated to \( \mathbb{Z}\hat{S}_*(X, \mu_n) \) with differential \( d = \sum_{i=0}^k (-1)^i \partial_i \), where \( \partial_i \) is the face map of \( \hat{S}(X, \mu_n) \). Then, according to [17, Corollary 4.8], the map Cub defined above extends by linearity to a morphism of homological complexes

\[
\text{Cub} : \mathcal{N}(\mathbb{Z}\hat{S}_*(X, \mu_n)) \to \mathbb{Z}C_*(X, \mu_n)[-1],
\]

and hence one gets a simplicial map

\[
\text{Cub} : \mathbb{Z}\hat{S}_*(X, \mu_n) \to \mathcal{K}(\mathbb{Z}C_*(X, \mu_n)[-1]),
\]

where \( \mathcal{K} \) is the Dold–Puppe functor.

**Definition 2.8.** Let notations and assumptions be as above. We denote by \( D^{2p-*}(X, \mu_n, p) \) the homological complex associated to the complex \( \tau_{\leq 0}(D^*(X, \mu_n, p)[2p]) \) which is the canonical truncation of \( D^*(X, \mu_n, p)[2p] \) at degree 0. We define a simplicial map

\[
\hat{\text{ch}}_g : \hat{S}(X, \mu_n) \xrightarrow{\text{Hu}} \mathbb{Z}\hat{S}_*(X, \mu_n)
\]

\[
\text{Cub}
\]

\[
\mathcal{K}(\mathbb{Z}C_*(X, \mu_n)[-1]) \xrightarrow{\mathcal{K}(\hat{\text{ch}}_g)} \mathcal{K} \left( \bigoplus_{p \geq 0} D^{2p-*}(X, \mu_n, p)[2p] \right),
\]

where \( \text{Hu} \) is the Hurewicz map.

**Definition 2.9.** Let \( X \) be a regular \( \mu_n \)-projective scheme over an arithmetic ring \( (D, \Sigma, F_{\infty}) \), and let \( X_{\mu_n} \) be the fixed-point subscheme. The higher equivariant arithmetic \( K \)-groups of \( X \) are defined as

\[
\hat{K}_m(X, \mu_n) := \pi_{m+1} \left( \text{homotopy fiber of } | \hat{\text{ch}}_g | \right) \quad \text{for } m \geq 1,
\]
and the equivariant regulator maps
\[ \text{ch}_g : \ K_m(X, \mu_n) \to \bigoplus_{p \geq 0} H^{2p-m}_D(X_{\mu_n}, \mathbb{R}(p))_{R_n} \]
are defined as the homomorphisms induced by \( \tilde{\text{ch}}_g \) at the level of homotopy groups.

**Remark 2.10.** (i). We have the long exact sequence
\[ \cdots \to \hat{K}_m(X, \mu_n) \to K_m(X, \mu_n) \to \bigoplus_{p \geq 0} H^{2p-m}_D(X_{\mu_n}, \mathbb{R}(p))_{R_n} \to \hat{K}_{m-1}(X, \mu_n) \to \cdots \]
ending with
\[ \cdots \to K_1(X, \mu_n) \to \bigoplus_{p \geq 0} H^{2p-1}_D(X_{\mu_n}, \mathbb{R}(p))_{R_n} \]
and the homotopy fiber of \( \tilde{\text{ch}}_g \)
\[ \pi_1(\text{homotopy fiber of } \tilde{\text{ch}}_g) \to K_0(X, \mu_n) \to \bigoplus_{p \geq 0} H^{2p}_D(X_{\mu_n}, \mathbb{R}(p))_{R_n}. \]

(ii) When \( n = 1 \), the equivariant higher Bott–Chern forms given in Definition 2.3 coincide with the higher Bott–Chern forms defined in [17] for non-equivariant proper varieties. So, in this case,
\[ \text{ch}_g : \ K_m(X, \mu_1) \to \bigoplus_{p \geq 0} H^{2p-m}_D(X, \mathbb{R}(p)) \]
is the Beilinson’s regulator map.

(iii) The higher equivariant arithmetic K-groups \( \hat{K}_m(X, \mu_n) \) can be defined for non-proper \( X \), for details, see [34, Section 2].

(iv) Let \( s(\text{ch}_g) \) denote the simple complex associated to the chain morphism
\[ \text{ch}_g : \tilde{\mathcal{Z}}C_*(X, \mu_n) \xrightarrow{\text{ch}_g} \bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)_{R_n}. \]
Then, for any \( m \geq 1 \), there is an isomorphism
\[ \hat{K}_m(X, \mu_n)_{Q} \cong H_m(s(\text{ch}_g), Q). \]

(v) A \( \mu_n \)-equivariant hermitian sheaf on \( X \) is a \( \mu_n \)-equivariant coherent sheaf on \( X \) which is locally free on \( X(\mathbb{C}) \) and is equipped with a \( \mu_n \)-invariant hermitian metric. To a \( \mu_n \)-equivariant hermitian sheaf, the higher equivariant Bott–Chern form can still be defined in the same way. Denote by \( \hat{P}'(X, \mu_n) \) the category of \( \mu_n \)-equivariant hermitian sheaves on \( X \), then instead of \( \hat{P}(X, \mu_n) \) one may define a new arithmetic K-theory \( \hat{K}'_m(X, \mu_n) \) which is called the equivariant arithmetic K’-theory. Since \( \hat{P}'(X, \mu_n) \) and \( \hat{P}(X, \mu_n) \) define the same algebraic K-theory when \( X \) is regular, it is easily seen from the Five-lemma that the natural inclusion \( \hat{P}(X, \mu_n) \subset \hat{P}'(X, \mu_n) \) induces isomorphisms \( \hat{K}_m(X, \mu_n) \cong \hat{K}'_m(X, \mu_n) \) for any \( m \geq 1 \).

2.2. Equivariant analytic torsion for hermitian cubes

In [12], Bismut and Köhler developed a theory of higher analytic torsion forms for holomorphic submersions of complex manifolds. The higher analytic torsion form solves a differential equation which gives a refinement of the Grothendieck–Riemann–Roch theorem at the level of characteristic forms. Later, in [27], Ma generalized Bismut and Köhler’s results to the equivariant case. Considering the higher K-theory and the Deligne-Beilinson cohomology, to make a refinement of the Riemann–Roch theorem at the level of higher Bott–Chern forms representing the regulator maps, one needs an extension of higher analytic torsion
for hermitian cubes, this has been done in [30]. In this subsection, we do the equivariant case by using Ma’s equivariant analytic torsion forms. Our construction is slightly different to Roessler’s construction.

Let $X, Y$ be two smooth $\mu_n$-projective varieties over $\mathbb{C}$, and let $f : X \to Y$ be an equivariant and smooth morphism. A Kähler fibration structure on $f$ is a real closed $(1,1)$-form $\omega$ on $X$ which induces Kähler metrics on the fibers of $f$ (cf. [12, Definition 1.1, Theorem 1.2]). For instance, we may fix a $\mu_n$-invariant Kähler metric on $X$ and choose corresponding Kähler form $\omega$ as a Kähler fibration structure on $f$. Let $(E, h_E)$ be a $\mu_n$-equivariant hermitian vector bundle on $X$ such that $E$ is $f$-acyclic, that is, the higher direct image $R^if_*E$ vanishes for $q > 0$. The equivariant analytic torsion form $T_g(f, \omega, h_E)$ is an element of $\bigoplus_{p \geq 0} D^{2p-1}(Y_{\mu_n}, p)_{R_n}$, which depends on $f, \omega$ and $(E, h_E)$ and satisfies the differential equation

$$d_T T_g(f, \omega, h_E) = ch_g(f_*E, f_*h_E) - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} Td_g(Tf, h^{Tf}) ch_g(E, h_E),$$

where $h^{Tf}$ is the hermitian metric induced by $\omega$ on the holomorphic tangent bundle $Tf$, $r$ is the rank of the bundle $Tf_{\mu_n}$, and $f_*h_E$ is the $L^2$-metric on $f_*E$ (see the end of [30, Section 2.2] for a definition). By definition, for elements $u, v \in (f_*E)_y$ of the fiber of $f_*E$ over a point $y \in Y$, the $L^2$-hermitian product is given by

$$\langle u, v \rangle_{L^2} = \frac{1}{(2\pi i)^r} \int_{f^{-1}y} \langle u, v \rangle_E \frac{\omega^b}{b!},$$

where $b$ is the relative dimension of $X$ over $Y$.

We would like to caution the reader that the equivariant analytic torsion form we use here coincides with Ma’s definition only up to a rescaling. If we denote by $T'_g(f, \omega, h_E)$ Ma’s equivariant torsion form, then the equality $2\Phi(T_g(f, \omega, h_E)) = T'_g(f, \omega, h_E)$ holds. From now on, we shall write $T_g(\omega, h_E)$ or $T_g(h_E)$ for $T_g(f, \omega, h_E)$, if there is no ambiguity about the underlying map or Kähler form. Now, let $\omega'$ be the form associated to another Kähler fibration structure on $f : X \to Y$ and let $h'^{Tf}$ be the metric on $Tf$ induced by this new fibration. Let $T\bar{\omega}(Tf, h'^{Tf}, h^{Tf})$ be the equivariant secondary Todd form used in the Appendix (Section A.1 (A.1)), and set $Td_g(Tf, h'^{Tf}, h^{Tf}) = \Phi^{-1}(\frac{T\bar{\omega}(Tf, h'^{Tf}, h^{Tf})}{2})$. So,

$$d_T Td_g(Tf, h'^{Tf}, h^{Tf}) = Td_g(Tf, h^{Tf}) - Td_g(Tf, h'^{Tf}).$$

The following anomaly formula is useful for our later discussion.

**Theorem 2.11.** Let notations and assumptions be as above. The following identity holds in $\bigoplus_{p \geq 0} (D^{2p-1}(Y_{\mu_n}, p)/\text{Im } d_T)$:

$$T_g(\omega, h_E) - T_g(\omega', h_E) = ch_g(f_*E, h'^{f_*E}, h^{f_*E}) - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} Td_g(Tf, h'^{Tf}, h^{Tf}) ch_g(E, h_E),$$

where $(f_*E, h'^{f_*E}, h^{f_*E})$ stands for the emi-1-cubes of hermitian vector bundles

\[
\begin{array}{ccc}
0 & \rightarrow & (f_*E, h'^{f_*E}) \\
\rightarrow & & \rightarrow \\
& \text{Id} & \\
& \rightarrow & (f_*E, h^{f_*E}) \\
& \rightarrow & 0
\end{array}
\]

\[
\begin{array}{ccc}
0 & \rightarrow & (f_*E, h^f) \\
\rightarrow & & \rightarrow \\
& & 0
\end{array}
\]

Proof. This is a translation of [27, Theorem 2.13], see also Theorem A.1. Considering the relation between the equivariant analytic torsion forms $T_g(\omega, h_E), T_g(\omega', h_E)$ and the ones used in Ma’s paper, we only need to show

$$ch_g(f_*E, h'^{f_*E}, h^{f_*E}) = \Phi^{-1}\left(\frac{ch_g(f_*E, h'^{f_*E}, h^{f_*E})}{2}\right) \in \bigoplus_{p \geq 0} (D^{2p-1}(Y_{\mu_n}, p)/\text{Im } d_T).$$

But this is the content of Remark 2.7. \qed
According to Remark 2.7 and Theorem A.2, there exists a functorial choice of the differential form which measures the difference

\[ T_g(\omega, h^E) - T_g(\omega', h^E) - \text{ch}_g(f_*E, h^{f_*E}, h^{f_*E}) + \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \text{Td}_g(Tf, h^{Tf}, h^{Tf})\text{ch}_g(E, h^E) \]

in Theorem 2.11. With the same notations as in Remark 2.7 and Theorem A.2, we set

\[ \Delta(f, E, \omega, \omega') := -\Phi^{-1}\left(\Delta^h(f, E, \omega, \omega') + \frac{\Delta_0(f, E, \omega, \omega')}{2}\right) + \Delta(f_*E, h^{f_*E}, h^{f_*E}), \]

it satisfies the differential equation

\[ d\Delta(f, E, \omega, \omega') = T_g(\omega, h^E) - T_g(\omega', h^E) - \text{ch}_g(f_*E, h^{f_*E}, h^{f_*E}) + \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \text{Td}_g(Tf, h^{Tf}, h^{Tf})\text{ch}_g(E, h^E). \]

We consider the following setting. Let \( Z \) be a compact Kähler manifold and let \( Z_1 \) be a closed submanifold of \( Z \). Choose a Kähler metric on \( Z \) and endow \( Z_1 \) with the restricted metric. Let \( f : X \times Z \to Y \times Z \) be the induced map and let \( \omega, \omega' \) be the Kähler forms of the product metrics on \( X \times Z \) with respect to two Kähler fibrations on \( f : X \to Y \). Similarly, let \( f_1 : X \times Z_1 \to Y \times Z_1 \) be the induced map and let \( \omega_1, \omega'_1 \) be the Kähler forms of the product metrics on \( X \times Z_1 \) with respect to the same two Kähler fibrations on \( f : X \to Y \). We shall denote by \( j \) (respectively, \( i \)) the natural embedding \( X \times Z_1 \to X \times Z \) (respectively, \( Y \times Z_1 \to Y \times Z \)). Then \( j^*\omega = \omega_1 \) and \( j^*\omega' = \omega'_1 \). Let \( E \) be an \( f_2 \)-acyclic hermitian bundle on \( X \times Z \), we have the following result.

**Lemma 2.12.** The identity \( i^*_{\mu_n} \Delta(f, E, \omega, \omega') = \Delta(f_1, j_! E, \omega_1, \omega'_1) \) holds.

**Proof.** This is a consequence of Theorem A.2. \( \square \)

**Definition 2.13.** By a chain homotopy of a diagram of homological complexes

\[
\begin{array}{ccc}
A_* & \xrightarrow{i} & B_* \\
\downarrow f & & \downarrow i \\
C_* & \xrightarrow{j} & D_*
\end{array}
\]

we understand a chain homotopy between the complex morphisms \( j \circ f \) and \( l \circ i \).

Roughly speaking, the equivariant analytic torsion for hermitian cubes is a chain homotopy of the following diagram:

\[
\begin{array}{ccc}
\tilde{Z}C_*^{f-\text{acyc}}(X, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-\bullet}(X_{\mu_n}, p)_{R_n} \\
\downarrow f_* & & \downarrow f_{\mu_n} \circ \text{Td}_g(\mathcal{T}^{\bullet\bullet}) \\
\tilde{Z}C_*(Y, \mu_n) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-\bullet}(Y_{\mu_n}, p)_{R_n}
\end{array}
\]

(8)

where \( \tilde{Z}C_*^{f-\text{acyc}}(X, \mu_n) \) is the subcomplex of \( \tilde{Z}C_*(X, \mu_n) \) made of \( f \)-acyclic bundles. Since the Waldhausen K-theory space of \( \tilde{P}(X, \mu_n) \) is homotopy equivalent to the Waldhausen K-theory space of the full subcategory of \( \tilde{P}(X, \mu_n) \) consisting of \( f \)-acyclic bundles, we shall always work with acyclic bundles.

Like the non-equivariant case treated in [30], the equivariant analytic torsion for hermitian cubes induces a commutative diagram at the level of homology groups and hence one gets an analytic proof of the equivariant version of Gillet’s Riemann–Roch theorem for higher algebraic K-theory.
To construct a chain homotopy of (8), let us move in two steps. Note that the equivariant higher Bott–Chern form factors as

\[ \tilde{Z}_{c_1}(X, \mu_n) \xrightarrow{\lambda} \tilde{Z}^{em}_c(X, \mu_n) \xrightarrow{ch^0 \circ tr} \bigoplus_{p \geq 0} D^{2p - s}(X_{\mu_n}, p)_{R_n}, \]

we first clarify the difference between \( f_\ast (tr \circ \lambda(\cdot)) \) and \( tr \circ \lambda(f_\ast(\cdot)) \). Let \( \tilde{E} \) be a \( f \)-acyclic hermitian \( k \)-cube in \( \tilde{\mathcal{P}}(X, \mu_n) \). The hermitian bundles \( f_\ast (tr_k \circ \lambda(\tilde{E})) \) and \( tr_k \circ \lambda(f_\ast(\tilde{E})) \) are canonically isomorphic as bundles, but carry in general different metrics. For instance, assume that \( \tilde{E} \) is a hermitian emi-1-cube, then \( f_\ast (tr_1(\tilde{E})) \) and \( tr_1(f_\ast(\tilde{E})) \) fit into the following two exact sequences:

\[ 0 \xrightarrow{f_\ast(p_X^\ast \tilde{E}_{-1})} f_\ast(p_X^\ast \tilde{E}_{-1})(1) \oplus f_\ast(p_X^\ast \tilde{E}_0)(1) \xrightarrow{f_\ast(tr_1(\tilde{E}))} 0, \]

and

\[ 0 \xrightarrow{p_Y^\ast(f_\ast(\tilde{E}_{-1}))} p_Y^\ast(f_\ast(\tilde{E}_{-1}))(1) \oplus p_Y^\ast(f_\ast(\tilde{E}_0))(1) \xrightarrow{tr_1(f_\ast(\tilde{E}))} 0. \]

Here, \( p_X \) (respectively, \( p_Y \)) stands for the obvious projection \( X \times \mathbb{P}^1 \to X \) (respectively, \( Y \times \mathbb{P}^1 \to Y \)). By the definition of the \( L \)-metric, over the point \((y, t)\) in \( Y \times \mathbb{P}^1 \), the hermitian product on \( f_\ast(tr_1(\tilde{E}))(y, t) \) relies on the integral of certain power of the Kähler form \( \omega_{X \times \mathbb{P}^1} \) over the fiber \( f_1 \) and hence relies on \( t \). But the pullback hermitian products on \( p_X^\ast(f_\ast(\tilde{E}_0))(y, t) \) and on \( p_Y^\ast(f_\ast(\tilde{E}_{-1}))(y, t) \) equal the hermitian products on \( f_\ast(\tilde{E}_0)_y \) and on \( f_\ast(\tilde{E}_{-1})_y \) which do not rely on \( t \), therefore the induced hermitian product on \( tr_1(f_\ast(\tilde{E}))(y, t) \) does not rely on \( t \) neither.

So, in general, \( f_\ast(tr_1(\tilde{E})) \) and \( tr_1(f_\ast(\tilde{E})) \) carry different metrics.

In the following, we shall write \( H(\tilde{E}) \) for the short exact sequence

\[ 0 \xrightarrow{f_\ast(tr_k \circ \lambda(\tilde{E}))} tr_k \circ \lambda(f_\ast(\tilde{E})) \xrightarrow{0} 0 \]

which is an emi-1-cube of hermitian bundles on \( Y \times (\mathbb{P}^1)^k \). The transgression bundle of \( H(\tilde{E}) \) is a hermitian bundle on \( Y \times (\mathbb{P}^1)^{k+1} = Y \times (\mathbb{P}^1)^k \times \mathbb{P}^1 \). But here we change the order of the \( \mathbb{P}^1 \), let \( p_1 \) be the first projection from \( Y \times \mathbb{P}^1 \times (\mathbb{P}^1)^k \) to \( Y \times (\mathbb{P}^1)^k \), we apply the transgression bundle construction to the short exact sequence \( H(\tilde{E}) \) with respect to the projection \( p_1 \) to get a hermitian bundle on \( Y \times (\mathbb{P}^1)^{k+1} \). With some abuse of notation, we still denote this hermitian bundle by \( tr_1(H(\tilde{E})) \) and it satisfies the following relations:

\[ \tr_1(H(\tilde{E}))|_{Y \times \{0\} \times (\mathbb{P}^1)^k} = tr_k \circ \lambda(f_\ast(\tilde{E})), \quad \tr_1(H(\tilde{E}))|_{Y \times \{\infty\} \times (\mathbb{P}^1)^k} = f_\ast(tr_k \circ \lambda(\tilde{E})) \]

and

\[ \tr_1(H(\tilde{E}))|_{Y \times (\mathbb{P}^1)^i \times \{0\} \times (\mathbb{P}^1)^{k-i}} = \tr_1(H(\partial_i^0 \tilde{E})), \]

\[ \tr_1(H(\tilde{E}))|_{Y \times (\mathbb{P}^1)^i \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} = \tr_1(H(\partial_i^{-1} \tilde{E})) \oplus \tr_1(H(\partial_i^1 \tilde{E})) \]

for \( i = 1, \ldots, k \). Now, we define

\[ \Pi_k(\tilde{E}) := \frac{(-1)^{k+1}}{2(k+1)! (2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} ch^0_g(\tr_1(H(\tilde{E}))) \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2). \]

The same reasoning as in [30, Lemma 3.3] proves that \( \Pi_k(\tilde{E}) \) vanishes on degenerate \( k \)-cubes, and hence we obtain a map \( \Pi_k' : \tilde{Z}C^{f-\text{ac}}_k(X, \mu_n) \to \bigoplus_{p \geq 0} D^{2p - k - 1}(Y_{\mu_n}, p)_{R_n} \) by linear extension.

**Proposition 2.14.** The equality

\[ d_D \circ \Pi_k'(\tilde{E}) + \Pi_{k-1}' \circ d(\tilde{E}) = \text{ch}_g(f_\ast(\tilde{E})) - \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} ch^0_g(tr_k(\lambda(\tilde{E}))) \wedge C_k(\log |z_1|^2, \ldots, \log |z_k|^2) \]

holds.
Proof. We compute
\[
d_D \circ \Pi_k'(E)
\]
\[
= \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(\text{tr}_1(H(E))) \wedge d_D C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2)
\]
\[
= \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(\text{tr}_1(H(E))) \wedge \left(\left(-\frac{1}{2}\right)(k+1) \sum_{j=1}^{k+1} (-1)^{j-1}(-4\pi i)^j \times (\delta_{z_j=\infty} - \delta_{z_j=0}) \land C_k(\log |z_1|^2, \ldots, \log |z_j|^2, \ldots, \log |z_{k+1}|^2)\right)
\]
\[
= \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(\text{tr}_1(H(E))) \wedge \left(\left(-\frac{1}{2}\right)(k+1) \sum_{j=2}^{k+1} (-1)^{j-1}(-4\pi i)^j \times (\delta_{z_j=\infty} - \delta_{z_j=0}) \land C_k(\log |z_1|^2, \ldots, \log |z_j|^2, \ldots, \log |z_{k+1}|^2)\right)
\]
\[
+ \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(\text{tr}_k \circ \lambda(f_s(E))) \wedge C_k(\log |z_1|^2, \ldots, \log |z_k|^2)
\]
\[
- \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(f_s(\text{tr}_k \circ \lambda(E))) \wedge C_k(\log |z_1|^2, \ldots, \log |z_k|^2)
\]
\[
= \frac{(-1)^{k+1}}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left(\sum_{j=2}^{k+1} (-1)^{j-1} \text{ch}_g^0(\text{tr}_1(H(\partial_j^{-1}E \oplus \partial_j^E))) - \text{ch}_g^0(\text{tr}_1(H(\partial_j^E)))\right) \wedge C_k(\log |z_1|^2, \ldots, \log |z_k|^2) + \text{ch}_g(f_s E)
\]
\[
- \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(f_s(\text{tr}_k \circ \lambda(E))) \wedge C_k(\log |z_1|^2, \ldots, \log |z_k|^2) + \text{ch}_g(f_s E)
\]
\[
= -\Pi_{k-1} \circ d(E) + \text{ch}_g(f_s E) - \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \text{ch}_g^0(f_s(\text{tr}_k \circ \lambda(E))) \wedge C_k(\log |z_1|^2, \ldots, \log |z_k|^2).
\]
So, we are done. \(\square\)

On the other hand, we equip \(X \times (\mathbb{P}^1)^k\) with the product metric and we define
\[
\Pi_k''(E) = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(T_g(\text{tr}_k \circ \lambda(E)), \log |z_1|^2, \ldots, \log |z_k|^2),
\]
where $T_g(tr_k \circ \lambda(\mathcal{E}))$ is the equivariant higher analytic torsion of the hermitian bundle $tr_k \circ \lambda(\mathcal{E})$ with respect to the fibration $f : X \times (\mathbb{P}^1)^k \to Y \times (\mathbb{P}^1)^k$. By [30, Lemma 3.5], the map $\Pi''_k$ vanishes on degenerate $k$-cubes and hence we obtain a map $\Pi''_k : \check{Z}C_k^{f-\text{ac}}(X, \mu_n) \to \bigoplus_{p \geq 0} D^{2p-k-1}(Y_{\mu_n}, p)_{R_n}$ by linear extension.

**Theorem 2.15.** Set $\Pi_k = \Pi'_k + \Pi''_k$, then $\Pi_k$ defines a chain homotopy of the diagram (8). This map $\Pi_k : \check{Z}C_k^{f-\text{ac}}(X, \mu_n) \to \bigoplus_{p \geq 0} D^{2p-k-1}(Y_{\mu_n}, p)_{R_n}$ is called the equivariant higher analytic torsion for hermitian cubes.

**Proof.** Let $\overline{E}$ be a hermitian $k$-cube in $\check{Z}C_k^{f-\text{ac}}(X, \mu_n)$, we compute

$$d_D \circ \Pi_k(\overline{E}) = d_D \circ \Pi'_k(\overline{E}) + d_D \circ \Pi''_k(\overline{E})$$

and

$$d_D \circ \Pi''_k(\overline{E})$$

$$= \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} d_DC_{k+1}(T_g(tr_k \circ \lambda(\mathcal{E})), \log | z_1 |^2, \ldots, \log | z_k |^2)$$

$$= \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left( -\frac{1}{2} \right) (k+1) \left( d_DT_g(tr_k \circ \lambda(\mathcal{E})) \right) \bullet C_k(\log | z_1 |^2, \ldots, \log | z_k |^2)$$

$$+ \sum_{j=1}^k (-1)^j (-4\pi i)(\delta z_j = \infty - \delta z_j = 0)$$

$$\wedge C_k \left( T_g(tr_k \circ \lambda(\mathcal{E})), \log | z_1 |^2, \ldots, \log | z_j |^2, \ldots, \log | z_k |^2 \right)$$

$$= \frac{(-1)^k}{k!(2\pi i)^{k-1}} \int_{(\mathbb{P}^1)^{k-1}} \sum_{j=1}^k (-1)^j \left( C_k(T_g(tr_{k-1} \circ \lambda(\partial_j^g(\mathcal{E}))), \log | z_1 |^2, \ldots, \log | z_{k-1} |^2)$$

$$- C_k(T_g(tr_{k-1} \circ \lambda(\partial_j^g(\mathcal{E}))) \wedge tr_{k-1} \circ \lambda(\partial_j^g(\mathcal{E}))), \log | z_1 |^2, \ldots, \log | z_{k-1} |^2)$$

$$+ \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left( \left( ch^0(f^*(\lambda(\mathcal{E})))) \right)$$

$$- \frac{1}{(2\pi i)^k} \int_{\mathbb{X}_{\mu_n} \times (\mathbb{P}^1)^k / Y_{\mu_n} \times (\mathbb{P}^1)^k} Td_g(T_{\mathcal{F}})ch^0(tr_k \circ \lambda(\mathcal{E})) \right)$$

$$\bullet C_k(\log | z_1 |^2, \ldots, \log | z_k |^2)$$
\[ \Pi_{k-1} \circ d(E) + \frac{(-1)^k}{2k!(2\pi i)^k} \int_{|p^1|} \text{ch}_g^0(\text{tr}_k \circ \lambda(E)) \wedge C_k(\log |z_1|^2, \ldots, \log |z_k|^2) \]

\[- \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \text{Td}_g(Tf) \cdot \text{ch}_g(E). \]

Combining these two computations, we finally get

\[ d_D \circ \Pi_k(E) + \Pi_{k-1} \circ d(E) = \text{ch}_g(f_*, E) - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \text{Td}_g(Tf) \cdot \text{ch}_g(E). \]

So, we are done. \( \square \)

If we are given another fibration structure \( \omega' \), then for any \( f \)-acyclic hermitian \( k \)-cube \( E \) in \( \hat{P}(X, \mu_n) \), the short exact sequence

\[ 0 \longrightarrow (f_* E, h^{f_*} E) \overset{\text{Id}}{\longrightarrow} (f_* E, h^{f'_*} E) \longrightarrow 0 \longrightarrow 0 \]

forms a hermitian \((k+1)\)-cube \( H_f(E) \) on \( Y \) such that the transgression bundle \( \text{tr}_{k+1}(\lambda(H_f(E))) \) satisfies the relations

\[ \text{tr}_{k+1}(\lambda(H_f(E))) \big|_{Y \times \{0\} \times (\mathbb{P}^1)^k} = \text{tr}_k(\lambda(f_* E, h^{f_*} E)), \]

\[ \text{tr}_{k+1}(\lambda(H_f(E))) \big|_{Y \times \{\infty\} \times (\mathbb{P}^1)^k} = \text{tr}_k(\lambda(f_* E, h^{f'_*} E)) \]

and

\[ \text{tr}_{k+1}(\lambda(H_f(E))) \big|_{Y \times (\mathbb{P}^1)^i \times \{0\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_k(\lambda(H_f(\partial^i_{\mathbb{P}^1} E))), \]

\[ \text{tr}_{k+1}(\lambda(H_f(E))) \big|_{Y \times (\mathbb{P}^1)^i \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_k(\lambda(H_f(\partial^{1-i}_{\mathbb{P}^1} E))) \oplus \text{tr}_k(\lambda(H_f(\partial^i_{\mathbb{P}^1} E))) \]

for \( i = 1, \ldots, k \). Therefore, the following map

\[ \Pi^{(1)}_k(E) = \frac{(-1)^{k+1}}{2(k+1)! (2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_{k+1} \circ \lambda(H_f(E))) \wedge C_k(\log |z_1|^2, \ldots, \log |z_{k+1}|^2) \]

which vanishes on degenerate cubes provides a chain homotopy of homological complexes between the maps \( \text{ch}_g \circ f_* \) and \( \text{ch}_g \circ f'_* \), where \( f'_*(E) := (f_* E, h^{f'_*} E) \) is the pushforward with respect to the new fibration \( \omega' \). Similarly, by projection formula, the map

\[ \Pi^{(3)}_k(E) := \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left( \left( \frac{1}{(2\pi i)^r} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Y_{\mu_n} \times (\mathbb{P}^1)^k} \text{Td}_g(Tf, h^{Tf}, h^Tf) \text{ch}_g^0(\text{tr} \circ \lambda(E)) \right) \right) \cdot C_k(\log |z_1|^2, \ldots, \log |z_k|^2) \]

gives a chain homotopy of homological complexes between the maps \( f_{\mu_n} \circ (\text{Td}_g(Tf, h^{Tf}) \bullet \text{ch}_g) \) and \( f_{\mu_n} \circ (\text{Td}_g(Tf, h^{Tf}) \bullet \text{ch}_g) \). Finally we write \( \Pi^{(2)}_k = \Pi^{(2)}_k + \Pi^{(3)}_k \) for the chain homotopy defined in Theorem 2.15 between the maps \( \text{ch}_g \circ f'_* \) and \( f_{\mu_n} \circ (\text{Td}_g(Tf, h^{Tf}) \bullet \text{ch}_g) \) with respect to the new fibration \( \omega' \). Then \( \Pi^{(1)}_k + \Pi^{(2)}_k - \Pi^{(3)}_k \) defines a chain homotopy between \( \text{ch}_g \circ f_* \) and \( f_{\mu_n} \circ (\text{Td}_g(Tf, h^{Tf}) \bullet \text{ch}_g) \). At the end of this subsection, we compare this homotopy \( \Pi^{(1)}_k + \Pi^{(2)}_k - \Pi^{(3)}_k \) with \( \Pi_k \) constructed in Theorem 2.15.

**Definition 2.16.** Let \( f, l \) be two morphisms of homological complexes \( A \to B \), and let \( h_1, h_2 \) be two chain homotopies between \( f \) and \( l \). We say that \( h_1 \) is homotopic to \( h_2 \) if there exists a map \( H : A \to B_{s+2} \) satisfying the condition that \( H d - d H = h_1 - h_2 \).
Now, we denote by $H^p_f(\mathcal{E})$ the following emi-2-cube of hermitian bundles on $Y \times (\mathbb{P}^1)^k$

\[
\begin{align*}
f'_* (\text{tr}_k \circ \lambda(\mathcal{E})) & \xrightarrow{\text{Id}} \text{tr}_k \circ \lambda (f'_*(\mathcal{E})) \xrightarrow{0} \\
\text{Id} & \downarrow \quad \text{Id} \\
f_*(\text{tr}_k \circ \lambda(\mathcal{E})) & \xrightarrow{\text{Id}} \text{tr}_k \circ \lambda (f_*(\mathcal{E})) \xrightarrow{0} \\
0 & \downarrow \quad 0 \\
0 & \xrightarrow{0}. 
\end{align*}
\]

Changing the order of the $\mathbb{P}^1 \times \mathbb{P}^1$ in $(\mathbb{P}^1)^{k+2} = (\mathbb{P}^1)^k \times \mathbb{P}^1 \times \mathbb{P}^1$, so that $(\mathbb{P}^1)^{k+2} = \mathbb{P}^1 \times \mathbb{P}^1 \times (\mathbb{P}^1)^k$, we construct a hermitian bundle $\text{tr}_2(H^p_f(\mathcal{E}))$ on $Y \times (\mathbb{P}^1)^{k+2}$ as the second transgression bundle of $H^p_f(\mathcal{E})$ such that it satisfies the following relations:

\[
\begin{align*}
\text{tr}_2(H^p_f(\mathcal{E})) & \mid_{Y \times \{0\} \times (\mathbb{P}^1)^{k+1}} = \text{tr}_{k+1}(\lambda(H_f(\mathcal{E}))), \\
\text{tr}_2(H^{p'}_f(\mathcal{E})) & \mid_{Y \times \{\infty\} \times (\mathbb{P}^1)^{k+1}} = \text{tr}_1(H_f(\text{tr}_k \circ \lambda(\mathcal{E}))), \\
\text{tr}_2(H^p_f(\mathcal{E})) & \mid_{Y \times \mathbb{P}^1 \times \{0\} \times (\mathbb{P}^1)^k} = \text{tr}_1(H(\mathcal{E})), \\
\text{tr}_2(H^{p'}_f(\mathcal{E})) & \mid_{Y \times \mathbb{P}^1 \times \{\infty\} \times (\mathbb{P}^1)^k} = \text{tr}_1(H'/(\mathcal{E})), \\
\end{align*}
\]

and

\[
\begin{align*}
\text{tr}_2(H^p_f(\mathcal{E})) & \mid_{Y \times (\mathbb{P}^1)^{i+1} \times \{0\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_2(H^p_f(\partial_i^\mathcal{E})), \\
\text{tr}_2(H^{p'}_f(\mathcal{E})) & \mid_{Y \times (\mathbb{P}^1)^{i+1} \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_2(H^{p'}_f(\partial_i^{-1}\mathcal{E})) \oplus \text{tr}_2(H^{p'}_f(\partial_i^\mathcal{E})))
\end{align*}
\]

for $i = 1, \ldots, k$. We set

\[
\Pi^p_f(\mathcal{E}) := \frac{(-1)^{k+2}}{2(k+2)! (2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \text{ch}^p_f(\text{tr}_2(H^p_f(\mathcal{E}))) \wedge C_{k+2}(\log |z_1|^2, \ldots, \log |z_{k+2}|^2).
\]

Then $\Pi^p_f$ vanishes on degenerate $k$-cubes, and we obtain a map

\[
\Pi^{p'}_{f,k} : \tilde{ZC}_k^{l^{-ac}}(X, \mu_n) \to \bigoplus_{p \geq 0} D^{2p-k-2}(Y_{\mu_n}, p) R_n
\]

by linear extension.

**Proposition 2.17.** Let notations and assumptions be as above. Then the chain homotopy $\Pi_k$ is homotopic to the chain homotopy $\Pi_k^{(1)} + \Pi_k^{(2)} - \Pi_k^{(3)}$.

**Proof.** First, we set

\[
\Pi_k^{(3')} = \frac{(-1)^{k+1}}{(k+1)! (2\pi i)^{k}} \int_{(\mathbb{P}^1)^k} C_{k+1}
\]

\[
\left( \frac{1}{(2\pi i)^r} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Y_{\mu_n} \times (\mathbb{P}^1)^k} \text{Td}_g(T_f, h^{TF}, h^{TF}) \text{ch}^g_f(\text{tr}_k \circ \lambda(\mathcal{E})), \right.
\]

\[
\log |z_1|^2, \ldots, \log |z_k|^2.
\]
It also defines a chain homotopy between the maps \( f_{\mu_n} \circ (\text{Td}_g(Tf, h^{Tf}) \cdot ch_g) \) and \( f_{\mu_n} \circ (\text{Td}_g(Tf, h^{Tf}) \cdot ch_g) \). Since the product \( \cdot \) on Deligne complex is graded commutative and is associative up to homotopy, we claim that \( \Pi_k^{(3)}(E) \) is homotopic to \( \Pi_k^{(3)}(E) \), so that we are left to show that \( \Pi_k \) is homotopic to \( \Pi_k^{(1)} + \Pi_k^{(2)} - \Pi_k^{(3)} \). Actually, our claim follows from the fact that \( d_D \Pi_k^{(3)}(E) - d_D \Pi_k^{(3)}(E) = \Pi_{k-1}^{(3)}(-dE) - \Pi_{k-1}^{(3)}(-dE) \) and [34, Remark 2.4, Lemma 2.5].

Now, let \( \overline{E} \) be a hermitian \( k \)-cube in \( \hat{P}(\mathcal{X}, \mu_n) \) which is \( f \)-acyclic. We compute

\[
d_D \circ \Pi_{f,k}^{(3)}(E) = \frac{(-1)^{k+2}}{2(k+1)!(2\pi i)^{k+2}} \int_{(p_1)^{k+1}} \text{tr}_2 \left( H_f^{(3)}(E) \right) \wedge d_D C_{k+1}(\log | z_1 |^2, \ldots, \log | z_{k+2} |^2)
= \frac{(-1)^{k+2}}{2(k+1)!(2\pi i)^{k+2}} \int_{(p_1)^{k+1}} \text{tr}_2 \left( H_f^{(3)}(E) \right) \wedge \left( - \frac{1}{2} \right) (k + 2) \sum_{j=1}^{k+2} (-1)^{j-1}(4\pi i)
\times (\delta_{z_j=0} - \delta_{z_j=\infty}) \wedge C_{k+1}(\log | z_1 |^2, \ldots, \log | z_{j-1} |^2, \log | z_{j+1} |^2, \ldots, \log | z_{k+2} |^2)
\end{align*}

\[
= \Pi_{f,k-1}^{(3)}(E) + \Pi_{k}^{(2)}(E) + \Pi_{k}^{(1)}(E) - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \times \int_{(p_1)^{k+1}} \text{ch}_g \left( \text{tr}_1 \left( H_f(\text{tr}_k \circ \lambda(E)) \right) \right) \wedge C_{k+1}(\log | z_1 |^2, \ldots, \log | z_{k+1} |^2).
\]

On the other hand, according to the anomaly formula Theorem 2.11, we have

\[
\Pi_{k}^{(3)}(E) - \Pi_{k}^{(2)}(E) = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(p_1)^k} C_{k+1}(T_g(\text{tr}_k \circ \lambda(E)), \log | z_1 |^2, \ldots, \log | z_k |^2)
\]

\[
- \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(p_1)^k} C_{k+1}(T_g(\text{tr}_k \circ \lambda(E)), \log | z_1 |^2, \ldots, \log | z_k |^2)
\]

\[
= \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(p_1)^k} C_{k+1} \left( \frac{1}{4\pi i} \int_{Y_{\mu_n} \times (p_1)^k / Y_{\mu_n} \times (p_1)^k} \text{ch}_g^{(3)}(\text{tr}_1 \left( H_f(\text{tr}_k \circ \lambda(E)) \right)) \log | z_0 |^2, \right.
\]

\[
\log | z_1 |^2, \ldots, \log | z_k |^2 \bigg) - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(p_1)^k} C_{k+1} \left( \frac{1}{(2\pi i)^k} \int_{X_{\mu_n} \times (p_1)^k / Y_{\mu_n} \times (p_1)^k} \text{Td}_g(Tf, h^{Tf} \circ h^{Tf}) \text{ch}_g(\text{tr}_k \circ \lambda(E)), \right.
\]

\[
\int_{(p_1)^k} C_{k+1} \left( \frac{1}{(2\pi i)^k} \int_{X_{\mu_n} \times (p_1)^k / Y_{\mu_n} \times (p_1)^k} \text{Td}_g(Tf, h^{Tf} \circ h^{Tf}) \text{ch}_g(\text{tr}_k \circ \lambda(E)), \right.
\]
So, we are done.

We formally define a product $C_{k+1}(\Delta(f, \text{tr}_k \circ \lambda(E), \omega, \omega'), \log |z_1|^2, \ldots, \log |z_k|^2)$ in a similar way to $C_{k+1}(\ldots, \cdot)$ like follows.

$$C_{k+1}(\Delta(f, \text{tr}_k \circ \lambda(E), \omega, \omega'), \log |z_1|^2, \ldots, \log |z_k|^2)$$

$$= -\left(-\frac{1}{2}\right)^k \sum_{\sigma \in \mathcal{G}_k} (-1)^\sigma \Delta \star (\log |z_{\sigma(1)}|^2 \star (\log |z_{\sigma(2)}|^2 \star \cdots \log |z_{\sigma(k)}|^2))$$

$$- \left(-\frac{1}{2}\right)^k \sum_{\sigma \in \mathcal{G}_k} (-1)^\sigma \log |z_{\sigma(1)}|^2 \star (\Delta \star (\log |z_{\sigma(2)}|^2 \star \cdots \log |z_{\sigma(k)}|^2) \cdots)$$

$$\cdots$$

$$- \left(-\frac{1}{2}\right)^k \sum_{\sigma \in \mathcal{G}_k} (-1)^\sigma \log |z_{\sigma(1)}|^2 \star (\log |z_{\sigma(2)}|^2 \star \cdots \log |z_{\sigma(k)}|^2 \star \Delta) \cdots \right) \quad (9)$$

Then we set

$$\Delta_k(E) = \frac{(-1)^k}{(k+1)! (2\pi i)^k} \int_{(p_1)^k} C_{k+1}(\Delta(f, \text{tr}_k \circ \lambda(E), \omega, \omega'), \log |z_1|^2, \ldots, \log |z_k|^2),$$

and it is readily checked by Lemma 2.12 that

$$\Delta_{k-1}(dE) - dD \Delta_k(E) = \frac{(-1)^{k+1}}{(k+1)! (2\pi i)^k}$$

$$\times \int_{(p_1)^k} C_{k+1}(dD \Delta(f, \text{tr}_k \circ \lambda(E), \omega, \omega'), \log |z_1|^2, \ldots, \log |z_k|^2).$$

Combing all the above computations, we finally get

$$(\Pi_{f,k-1}'' + \Delta_{k-1}) \circ d(E) - dD \circ (\Pi_{f,k}'' + \Delta_k)(E)$$

$$= -\Pi_k''(E) + \Pi_k(E) - \Pi_k'(E) + \Delta_{k-1}(dE) - dD \Delta_k(E)$$

$$+ \frac{(-1)^{k+1}}{(k+1)! (2\pi i)^k \cdot (2\pi i)^{k+1}} \int_{(p_1)^{k+1}} \mathcal{G}_k^0(\text{tr}_1(\text{tr}_k \circ \lambda(E)))) \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2)$$

$$= -\Pi_k''(E) + \Pi_k(E) - \Pi_k'(E) - \Pi_k''(E) + \Pi_k'(E) + \Pi_k'''(E)$$

$$= \Pi_k(E) - \left(\Pi_k'(E) + \Pi_k''(E) - \Pi_k'''(E)\right).$$

So, we are done.
2.3. Direct image map between arithmetic K-groups

In this subsection, we define the direct image map between arithmetic K-groups of regular \( \mu_n \)-projective arithmetic schemes by means of the equivariant higher analytic torsion for hermitian cubes constructed in last subsection.

Let now \( X \) and \( Y \) be two regular \( \mu_n \)-projective schemes over an arithmetic ring \((D, \Sigma, F_\infty)\).
Assume that \( f : X \to Y \) is an equivariant and flat morphism from \( X \) to \( Y \) such that \( f \) is smooth over the generic fiber. Note that the chain homotopy

\[
\Pi_* : \tilde{\mathbb{Z}}C_*^{f-\text{ac}}(X(\mathbb{C}), \mu_n) \to \bigoplus_{p \geq 0} D^{2p-s-1}(Y(\mathbb{C})_{\mu_n}, p)_{R_n}
\]

is \( \sigma \)-invariant and the following diagrams:

\[
\begin{array}{ccc}
\tilde{\mathcal{S}}f^{\ast}(X, \mu_n) & \xrightarrow{\Pi_*} & \tilde{\mathbb{Z}}C_*^{f-\text{ac}}(X, \mu_n) \\
\tilde{\mathcal{S}}(Y, \mu_n) & \xrightarrow{\Pi_*} & \tilde{\mathbb{Z}}C_*^{f-\text{ac}}(Y, \mu_n)
\end{array}
\]

are commutative, the chain homotopy \( \Pi_* \) induces a simplicial homotopy between the maps \( \tilde{c}_g \circ f_* \) and \( f_{\mu_n}^{\ast} \circ \text{Td}_g(\overline{Tf}) \bullet (\cdot) \circ \tilde{c}_g \) in the following square

\[
\begin{array}{ccc}
\tilde{\mathcal{S}}f^{\ast}(X, \mu_n) & \xrightarrow{\tilde{c}_g} & \mathcal{K} \left( \bigoplus_{p \geq 0} D^{2p-s}(X_{\mu_n}, p)[-1]_{R_n} \right) \\
\tilde{\mathcal{S}}(Y, \mu_n) & \xrightarrow{\tilde{c}_g} & \mathcal{K} \left( \bigoplus_{p \geq 0} D^{2p-s}(Y_{\mu_n}, p)[-1]_{R_n} \right)
\end{array}
\]

To see the construction of this simplicial homotopy and general theory on homotopies in the category of simplicial abelian groups, the reader is referred to [22, Sections 2.1, 2.3 and 3.2], especially [22, p. 160, Proposition 3.8].

We remark that, according to the construction given in [22], the resulting simplicial homotopy is unique up to a homotopy in a strong sense: let \( h_1, h_2 \) be two simplicial homotopies arising from \( \Pi_* \), then there exists a homotopy

\[
\tilde{H} : \tilde{\mathcal{S}}f^{\ast}(X, \mu_n) \times \Delta^1 \times \Delta^1 \to \mathcal{K} \left( \bigoplus_{p \geq 0} D^{2p-s}(Y_{\mu_n}, p)[-1]_{R_n} \right)
\]

such that \( \tilde{H}(\cdot, 0, 0) = h_1, \tilde{H}(\cdot, 1, 1) = h_2, \tilde{H}(\cdot, 0, \cdot) \) is the constant homotopy on \( \tilde{c}_g \circ f_* \) and \( \tilde{H}(\cdot, 1, \cdot) \) is the constant homotopy on \( f_{\mu_n}^{\ast} \circ \text{Td}_g(\overline{Tf}) \bullet (\cdot) \circ \tilde{c}_g \) (cf. [22, Proposition 3.8]).

Thus, applying the geometric realization construction to the above simplicial square, we get a continuous map between homotopy fibers

\[
| f | : \text{homotopy fiber of } | \tilde{c}_g^X | \to \text{homotopy fiber of } | \tilde{c}_g^Y |
\]

which is unique up to a homotopy. So, we may have a well-defined direct image map between arithmetic K-groups as follows.

**Definition 2.18.** For \( m \geq 1 \), the direct image map \( f_* : \hat{K}_m(X, \mu_n) \to \hat{K}_m(Y, \mu_n) \) is defined as the homomorphism of abelian groups induced by the map \( | f | \) at the level of homotopy groups.
Remark 2.19. The condition ‘flatness’ of the map \( f \) is only used to guarantee that the direct image of a \( f \)-acyclic bundle is locally free. By introducing the arithmetic \( K' \)-theory and using the isomorphisms \( \tilde{K}_m(X, \mu_n) \cong \tilde{K}'_m(X, \mu_n) \) which hold for regular schemes, the condition ‘flatness’ can certainly be removed.

To study the direct image map up to torsion, we need the following lemma.

Lemma 2.20. Consider the following diagram of homological complexes

\[
\begin{array}{c}
A_* \\
\bigg\uparrow f_1 \\
f_2 \\
\bigg\uparrow l_1 \\
C_* \\
\bigg\uparrow j \\
D_*
\end{array}
\]

Assume that \( j \circ f_1 \) (respectively, \( j \circ f_2 \)) is homotopic to \( l_1 \circ i \) (respectively, \( l_2 \circ i \)) via the chain homotopy \( h_1 \) (respectively, \( h_2 \)), and that \( f_1 \) (respectively, \( l_1 \)) is homotopic to \( f_2 \) (respectively, \( l_2 \)) via the chain homotopy \( \pi_f \) (respectively, \( \pi_l \)). Suppose that the chain homotopy \( j \circ \pi_f + h_2 - \pi_l \circ i \) is homotopic to the chain homotopy \( h_1 \), then the morphism on simple complexes

\[
(f_1, l_1, h_1) : s_* (i : A_* \to B_*) \to s_* (j : C_* \to D_*)
\]

is chain homotopic to \((f_2, l_2, h_2)\).

Proof. Let \((a, b) \in A_* \bigoplus B_{k+1}\), the morphism \((f_1, l_1, h_1)\) (respectively, \((f_2, l_2, h_2)\)) sends \((a, b)\) to \((f_1(a), l_1(b) + h_1(a))\) (respectively, \((f_2(a), l_2(b) + h_2(a))\)). Let \( H : A_* \to D_{*+2} \) be the homotopy such that

\[
Hd - dH = h_1 - (j \circ \pi_f + h_2 - \pi_l \circ i),
\]

and we define \( \tilde{H}(a, b) = (\pi_f(a), -\pi_l(b) + H(a)) \). Then we compute

\[
d\tilde{H}(a, b) = d(\pi_f(a), -\pi_l(b) + H(a))
\]

\[
= (d\pi_f(a), j \circ \pi_f(a) + d\pi_l(b) - dH(a))
\]

\[
= (f_1(a) - f_2(a) - \pi_f(da), l_1(b) - l_2(b) - \pi_l(db) - H(a))
\]

\[
= (f_1(a), l_1(b) + h_1(a)) - (f_2(a), l_2(b) + h_2(a)) - (\pi_f d(a), \pi_l d(b) - \pi_l \circ i(a) + H(a))
\]

\[
= (f_1(a), l_1(b) + h_1(a)) - (f_2(a), l_2(b) + h_2(a)) - \tilde{H}(a, i(a) - db)
\]

\[
= (f_1(a), l_1(b) + h_1(a)) - (f_2(a), l_2(b) + h_2(a)) - \tilde{H}(a, b).
\]

So, we are done. \( \square \)

Corollary 2.21. Let notations and assumptions be as above, then the direct image map \( f_* : \tilde{K}_m(X, \mu_n) \to \tilde{K}_m(Y, \mu_n) \) without torsion is independent of the choice of the Kähler fibration structure.

Proof. This follows from Remark 2.10(iv), Theorem 2.17 and Lemma 2.20. \( \square \)

3. Transitivity of the direct image maps

Let \( f : X \to Y \), \( h : Y \to Z \) and \( l : X \to Z \) be three equivariant morphisms between regular \( \mu_n \)-projective schemes, which are all smooth over the generic fibers. Assume \( l = h \circ f \), in this
section, we shall compare the direct image map \( l_* \) with the composition \( h_* \circ f_* \). To this aim, we shall first discuss the functoriality of the equivariant analytic torsion forms with respect to a composition of submersions.

3.1. Analytic torsion forms and families of submersions

Let \( W, V \) and \( S \) be three smooth \( \mu_n \)-equivariant algebraic varieties over \( \mathbb{C} \) with \( S = S_{\mu_n} \). Suppose that \( f : W \to V \) and \( h : V \to S \) are two proper smooth morphisms, then passing to their analytifications the maps \( f : W(\mathbb{C}) \to V(\mathbb{C}) \) and \( h : V(\mathbb{C}) \to S(\mathbb{C}) \) are holomorphic submersions with compact fibers. Set \( l = h \circ f \), it is also a proper smooth morphism and \( l : W(\mathbb{C}) \to S(\mathbb{C}) \) is a holomorphic submersion with compact fiber as well.

Let \( \omega^W \) and \( \omega^V \) be two \( \mu_n \)-invariant Kähler forms on \( W \) and on \( V \). As before, \( \omega^W \) and \( \omega^V \) imply Kähler fibration structures on the morphisms \( f, h \) and \( l \) and they induce \( \mu_n \)-invariant hermitian metrics on relative tangent bundles \( Tf, Th \) and \( Tl \). Consider the following short exact sequence of hermitian vector bundles

\[
\mathcal{T}(f, h, h \circ f) : 0 \to \mathcal{T}f \to \mathcal{T}l \to f^*Th \to 0,
\]

denote by \( \text{Td}_g(\mathcal{T}(f, h, h \circ f)) = \Phi^{-1}(\mathcal{T}f(\mathcal{T}l, h^*\mu)) \) (see Section A.2 in the Appendix) the equivariant secondary Todd form such that

\[
d_{\mathcal{D}}\text{Td}_g(\mathcal{T}(f, h, h \circ f)) = \text{Td}_g(\mathcal{T}l) - f^*_{\mu_n} \text{Td}_g(Th) \text{Td}_g(\mathcal{T}f).
\]

Now, let \( E \) be a hermitian vector bundle on \( W \), we shall assume that \( E \) is \( f \)-acyclic and \( l \)-acyclic. Then the Leray spectral sequence \( E_2^{pq} = R^q h_* (R^p f_* E) \) degenerates at \( E_2 \), so that \( f_* E = R^0 f_* (E) \) is \( h \)-acyclic and \( l_* E \cong f_* f_* E \). Clearly, \( l_* E \) and \( h_* f_* E \) carry in general different \( L^2 \)-metrics (see Section A.2 in the Appendix). Consider the following short exact sequence of hermitian vector bundles

\[
\mathcal{E}(f, h, h \circ f) : 0 \to h_* f_* \mathcal{E} \to l_* \mathcal{E} \to 0 \to 0,
\]

can be regarded as an emi-1-cube of hermitian bundles on \( S \). Then the equivariant higher Bott–Chern form \( \text{ch}_g(\mathcal{E}(f, h, h \circ f)) \) satisfies the differential equation

\[
d_{\mathcal{D}}\text{ch}_g(\mathcal{E}(f, h, h \circ f)) = \text{ch}_g(l_* \mathcal{E}) - \text{ch}_g(h_* f_* \mathcal{E}).
\]

The main result in this subsection is the following.

**Theorem 3.1.** Let notations and assumptions be as above. Then the following identity holds in \( \bigoplus_{p \geq 0} (D^{2p-1}(S, p))/\text{Im} \, d_{\mathcal{D}} \):

\[
T_g(l_* \omega^W, h^E) - T_g(h_* \omega^V, h^f \omega^E) - \frac{1}{(2\pi i)^{r_h}} \int_{V_{\mu_n}/S} \text{Td}_g(Th) T_g(f_* \omega^W, h^E)
\]

\[
= \text{ch}_g(\mathcal{E}(f, h, h \circ f)) - \frac{1}{(2\pi i)^{r_l}} \int_{W_{\mu_n}/S} \text{Td}_g(\mathcal{T}(f, h, h \circ f)) \text{ch}_g(\mathcal{E}),
\]

where \( r_h \) and \( r_l \) are the relative dimensions of \( V_{\mu_n}/S \) and \( W_{\mu_n}/S \), respectively.

**Proof.** This is a translation of Theorem A.3.

\[\square\]

**Lemma 3.2.** With the same notations as in Remark 2.7 and Theorem A.4, we set

\[
\Delta(f, h, \omega^W, \omega^V, \mathcal{E}) := -\Phi^{-1} \left( \frac{\Delta^0(f, h, \omega^W, \omega^V, \mathcal{E}) + \Delta_0(f, h, \omega^W, \omega^V, \mathcal{E})}{2} \right) + \Delta(\mathcal{E}(f, h, h \circ f)).
\]
Then \( d_D \Delta(f, h, \omega^W, \omega^V, E) \) measures the difference

\[
T_g(l, \omega^W, h^E) - T_g(h, \omega^V, h^f_*E) - \frac{1}{(2\pi i)^r} \int_{V_{\mu_n}/S} T_{d_g}(Tf) T_g(f, \omega^W, h^E) - \text{ch}_g(E(f, h, h \circ f)) \]

\[
+ \frac{1}{(2\pi i)^r} \int_{W_{\mu_n}/S} T_{d_g}(T(f, h, h \circ f)) \text{ch}_g(E)
\]

in Theorem 3.1. Assume that we are in the same situation described before Lemma 2.12. Call \( l : S \times Z_1 \to S \times Z \) the natural inclusion, then similar to Lemma 2.12, we have

\[
l^* \Delta(f_Z, h_Z, \omega^W, \omega^V, E) = \Delta(f_{Z_1}, h_{Z_1}, \omega^W_1, \omega^V_1, j^*E).
\]

**Proof.** This is a consequence of Theorem A.4. \( \square \)

### 3.2. The transitivity property

In this subsection, we present certain transitivity property of direct image maps between equivariant higher arithmetic \( K \)-groups. To do this, we first write down the following diagram of homological complexes:

\[
\begin{array}{cccc}
\tilde{Z}C_*^{(f,j), \text{ac}}(X, \mu_n) & \xrightarrow{\text{ch}_j} & \bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)R_n & \\
& \xrightarrow{f_*} & f_{\mu_n *} Td_j(Tf)_*(\cdot) & \\
\tilde{Z}C_*^{j, \text{ac}}(Y, \mu_n) & \xrightarrow{\text{ch}_j} & \bigoplus_{p \geq 0} D^{2p-*}(Y_{\mu_n}, p)R_n & \\
& \xrightarrow{h_*} & h_{\mu_n *} Td_j(Th)_*(\cdot) & \\
\tilde{Z}C_*^*(Z, \mu_n) & \xrightarrow{\text{ch}_j} & \bigoplus_{p \geq 0} D^{2p-*}(Z_{\mu_n}, p)R_n & \\
\end{array}
\]

(10)

where \( l \) is \( h \circ f \) and \( \tilde{Z}C_*^{(f,j), \text{ac}}(X, \mu_n) \) is the subcomplex of \( \tilde{Z}C_*(X, \mu_n) \) made of those bundles which are \( f \)-acyclic and \( l \)-acyclic simultaneously.

Let \( E \) be a hermitian \( k \)-cube in \( P(X, \mu_n) \) which is \( f \)-acyclic and \( l \)-acyclic, the short exact sequence

\[
0 \longrightarrow h_* f_* E \xrightarrow{\text{id}} l_* E \longrightarrow 0 \longrightarrow 0
\]

can be regarded as a hermitian \((k + 1)\)-cube \( H_{hof}(E) \) on \( Z \) such that the transgression bundle \( \text{tr}_{k+1}(\lambda(H_{hof}(E))) \) satisfies the relations

\[
\text{tr}_{k+1}(\lambda(H_{hof}(E))) \mid_{Z \times \{0\} \times (\mathbb{P}^1)^k} = \text{tr}_k(\lambda(l_* E)),
\]

\[
\text{tr}_{k+1}(\lambda(H_{hof}(E))) \mid_{Z \times \{\infty\} \times (\mathbb{P}^1)^k} = \text{tr}_k(\lambda(h_* f_* E))
\]

and

\[
\text{tr}_{k+1}(\lambda(H_{hof}(E))) \mid_{Z \times (\mathbb{P}^1)^{k-i} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_k(\lambda(H_{hof}(\partial_i E))) \oplus \text{tr}_k(\lambda(H_{hof}(\partial_i^{-1} E)))
\]

for \( i = 1, \ldots, k \).
Proposition 3.3. The following map

\[ \Pi_k^{(1)}(E) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{([p^1]_{k+1})} \text{ch}_g^0(\text{tr}_{k+1} \circ \lambda(H_{h \circ f}(E))) \]

\[ \wedge C_{k+1}(\log | z_1 |^2, \ldots, \log | z_{k+1} |^2) \]

which vanishes on degenerate cubes provides a chain homotopy of homological complexes between the maps \( \text{ch}_g \circ l_* \) and \( \text{ch}_g \circ (h_* \circ f_*) \).

Proof. Using the above relations that the transgression bundle \( \text{tr}_{k+1}(\lambda(H_{h \circ f}(E))) \) satisfies and the expression of \( d_D C_{k+1} \), the proof is straightforward. This can be also seen from the fact that \( H_{h \circ f}(E) \) provides a chain homotopy between \( l_* \) and \( h_* \circ f_* \). \( \square \)

Proposition 3.4. The composition \( h_{\mu_n} \circ \text{Td}_g(\overline{T\theta}) \circ (f_{\mu_n} \circ \text{Td}_g(\overline{Tf}) \circ (\cdot)) \) is equal to \( l_{\mu_n} \circ f^*_{\mu_n} \text{Td}_g(\overline{T\theta}) \text{Td}_g(\overline{Tf}) \circ (\cdot) \). The following maps

\[ \Pi_k^{(3)}(E) := \frac{(-1)^k}{2k!(2\pi i)^k} \int_{([p^1]_k)} \left( \frac{1}{(2\pi i)^{k+1}} \int_{X_{\mu_n \times [p^1]_k} / Z_{\mu_n \times [p^1]_k}} \text{Td}_g(\overline{T(f,h,h \circ f)}) \text{ch}_g^0(\text{tr}_k \circ \lambda(E)) \right) \]

\[ \circ C_k(\log | z_1 |^2, \ldots, \log | z_k |^2) \]

and

\[ \Pi_k^{(3)}(E) := \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{([p^1]_{k+1})} C_{k+1} \left( \frac{1}{(2\pi i)^{k+1}} \int_{X_{\mu_n \times [p^1]_{k+1}} / Z_{\mu_n \times [p^1]_{k+1}}} \text{Td}_g(\overline{T(f,h,h \circ f)}) \text{ch}_g^0(\text{tr}_k \circ \lambda(E)) \right) \]

\[ \log | z_1 |^2, \ldots, \log | z_k |^2 \]

give two chain homotopies of homological complexes between the maps \( l_{\mu_n} \circ \text{Td}_g(\overline{T\theta}) \circ (\text{ch}_g(\cdot)) \) and \( l_{\mu_n} \circ f^*_{\mu_n} \text{Td}_g(\overline{T\theta}) \text{Td}_g(\overline{Tf}) \circ (\text{ch}_g(\cdot)) \). Moreover, \( \Pi_k^{(3)}(E) \) and \( \Pi_k^{(3')}(E) \) are homotopic to each other.

Proof. The first statement follows from the projection formula, the second statement follows from a straightforward computation and the third follows from [34, Remark 2.4, Lemma. 2.5]. \( \square \)

Now, we write \( \Pi_k^l = \Pi_k^{lf} + \Pi_k^{lf'} \) for the chain homotopy of the upper square in (10) and \( \Pi_k^h = \Pi_k^{dh} + \Pi_k^{dh'} \) for the chain homotopy of the lower square in (10). Then \( \Pi_k^{(1)} + h_{\mu_n} \circ (\text{Td}_g(\overline{T\theta}) \circ \Pi_k^{(3)}) \) defines a chain homotopy between maps \( \text{ch}_g \circ l_* \) and \( l_{\mu_n} \circ \text{Td}_g(\overline{T\theta}) \circ (\text{ch}_g(\cdot)) \). Suppose that the \( \mu_n \)-action on \( Z \) is trivial, it is the main result of this subsection that the chain homotopy \( \Pi_k^{(1)} + h_{\mu_n} \circ (\text{Td}_g(\overline{T\theta}) \circ \Pi_k^{(3)}) + \Pi_k^h \circ f_* - \Pi_k^{(3')} \) is homotopic to the chain homotopy \( \Pi_k^l = \Pi_k^{lf} + \Pi_k^{lf'} \) for the whole square in (10). According to Proposition 3.4, it is equivalent to show that \( \Pi_k^{(1)} + h_{\mu_n} \circ (\text{Td}_g(\overline{T\theta}) \circ \Pi_k^l) + \Pi_k^h \circ f_* - \Pi_k^{(3')} \) is homotopic to \( \Pi_k^l \).
To see this, we first denote by $H^1_{h.o.f}(E)$ the following emi-2-cube of hermitian bundles on $Z \times (\mathbb{P}^1)^k$

$$h_* f_* (\text{tr}_k \circ \lambda(E)) \xrightarrow{\text{id}} \text{tr}_k \circ \lambda (h_* f_* (E)) \longrightarrow 0$$

$$l_* (\text{tr}_k \circ \lambda(E)) \xrightarrow{\text{id}} \text{tr}_k \circ \lambda (l_* (E)) \longrightarrow 0$$

Then, like before, we construct a hermitian bundle $\text{tr}_2(H^1_{h.o.f}(E))$ on $(\mathbb{P}^1)^{k+2}$ as the second transgression bundle of $H^1_{h.o.f}(E)$ such that it satisfies the following relations:

$$\text{tr}_2(H^1_{h.o.f}(E)) |_{Z \times \{0\} \times (\mathbb{P}^1)^{k+1}} = \text{tr}_{k+1}(\lambda(H_{h.o.f}(E)))$$

$$\text{tr}_2(H^1_{h.o.f}(E)) |_{Z \times \{\infty\} \times (\mathbb{P}^1)^{k+1}} = \text{tr}_1(H_{h.o.f}(\text{tr}_k \circ \lambda(E)))$$

$$\text{tr}_2(H^1_{h.o.f}(E)) |_{Z \times \mathbb{P}^1 \times \{0\} \times (\mathbb{P}^1)^k} = \text{tr}_1(H(E,l_*))$$

$$\text{tr}_2(H^1_{h.o.f}(E)) |_{Z \times \mathbb{P}^1 \times \{\infty\} \times (\mathbb{P}^1)^k} = \text{tr}_1(H(E,h_* f_*))$$

and

$$\text{tr}_2(H^1_{h.o.f}(E)) |_{Z \times (\mathbb{P}^1)^i \times \{0\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_2(H^1_{h.o.f}(\partial^i E))$$

$$\text{tr}_2(H^1_{h.o.f}(E)) |_{Z \times (\mathbb{P}^1)^i \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} = \text{tr}_2(H^1_{h.o.f}(\partial^i E)) \oplus \text{tr}_2(H^1_{h.o.f}(\partial^i E))$$

for $i = 1, \ldots, k$. We set

$$H_{1,k}(E) := \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \chi^0_g(\text{tr}_2(H^1_{h.o.f}(E))) \wedge C_{k+2}(\log |z_1|^2, \ldots, \log |z_{k+2}|^2).$$

Then $H_{1,k}$ vanishes on degenerate $k$-cubes, and we obtain a map

$$H_{1,k} : \tilde{\mathbb{Z}}C^{(f,l)-ac}(X, \mu_n) \to \bigoplus_{p \geq 0} D^{2p-k-2}(Z, p) \cdot \mathbb{R}_n$$

by linear extension. This map satisfies the following differential equation:

$$d_D \circ H_{1,k}(E)$$

$$= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \chi^0_g(\text{tr}_2(H^1_{h.o.f}(E))) \wedge d_D C_{k+2}(\log |z_1|^2, \ldots, \log |z_{k+2}|^2)$$

$$= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \chi^0_g(\text{tr}_2(H^1_{h.o.f}(E))) \wedge \left( (-1)^{k+2} \sum_{j=1}^{k+2} (-1)^{j-1}(-4\pi i) \right)$$

$$\times (\delta_{z_j=\infty - \delta_{z_j=0}} \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_j|^2, \ldots, \log |z_{k+2}|^2))$$

$$= H_{1,k-1} \circ d(E) - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \left[ \left( \chi^0_g(\text{tr}_1(H(E,l_*))) - \chi^0_g(\text{tr}_1(H(E,h_* f_*))) \right) \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2) \right]$$
\[
\begin{align*}
&+ \frac{(-1)^{k+1}}{2(k+1)!} \int_{(p_1)^{k+1}} \left( ch^0_g(\text{tr}_{k+1}(H_{hol}(\mathcal{E}))) - ch^0_g(\text{tr}_{k}(H_{hol}(\text{tr}_{k} \circ \lambda(\mathcal{E}))))) \right) \\
&\wedge C_{k+1}(\log | z_1 |^2, \ldots, \log | z_{k+1} |^2) \\
= H_{1,k-1} \circ d(\mathcal{E}) - \Pi_{k}^{1}(\mathcal{E}) + \Pi_{k}^{1}(\mathcal{E}) \\
&+ \frac{(-1)^{k+1}}{2(k+1)!} \int_{(p_1)^{k+1}} ch^0_g(\text{tr}_{1}(H(\mathcal{E}, f_\ast))) \wedge C_{k+1}(\log | z_1 |^2, \ldots, \log | z_{k+1} |^2) \\
&- \frac{(-1)^{k+1}}{2(k+1)!} \int_{(p_1)^{k+1}} ch^0_g(\text{tr}_{2}(H_{hol}(\text{tr}_{k} \circ \lambda(\mathcal{E})))) \wedge C_{k+1}(\log | z_1 |^2, \ldots, \log | z_{k+1} |^2).
\end{align*}
\]

Second, we denote by \( H_{hol}^{d}(\mathcal{E}) \) the following emi-2-cube of hermitian bundles on \( Z \times (\mathbb{P}^1)^k \)

\[
\begin{array}{cccc}
\downarrow \text{Id} & \downarrow \text{Id} & \downarrow \text{Id} & \downarrow \text{Id} \\
0 & 0 & 0 & 0.
\end{array}
\]

Again, we construct a hermitian bundle \( \text{tr}_{2}(H_{hol}^{d}(\mathcal{E})) \) on \( Z \times (\mathbb{P}^1)^{k+2} \) as the second transgression bundle of \( H_{hol}^{d}(\mathcal{E}) \) such that it satisfies the following relations:

\[
\begin{align*}
\text{tr}_{2}(H_{hol}(\mathcal{E})) |_{Z \times \{0\} \times (\mathbb{P}^1)^{k+1}} &= \text{tr}_{1}(H(f, \mathcal{E}, f_\ast)), \\
\text{tr}_{2}(H_{hol}(\mathcal{E})) |_{Z \times \{\infty\} \times (\mathbb{P}^1)^{k+1}} &= \text{tr}_{1}(h_\ast f_\ast(\text{tr}_{k} \circ \lambda(\mathcal{E})) \to h_\ast f_\ast(\text{tr}_{k} \circ \lambda(\mathcal{E}))), \\
\text{tr}_{2}(H_{hol}(\mathcal{E})) |_{Z \times \mathbb{P}^1 \times \{0\} \times (\mathbb{P}^1)^{k}} &= \text{tr}_{1}(H(\mathcal{E}, f_\ast)), \\
\text{tr}_{2}(H_{hol}(\mathcal{E})) |_{Z \times \mathbb{P}^1 \times \{\infty\} \times (\mathbb{P}^1)^{k}} &= \text{tr}_{1}(h_\ast H(\mathcal{E}, f_\ast))
\end{align*}
\]

and

\[
\begin{align*}
\text{tr}_{2}(H_{hol}(\mathcal{E})) |_{Z \times (\mathbb{P}^1)^{i+1} \times \{0\} \times (\mathbb{P}^1)^{k-i}} &= \text{tr}_{2}(H_{hol}(\partial_i^{\mathcal{E}})), \\
\text{tr}_{2}(H_{hol}(\mathcal{E})) |_{Z \times (\mathbb{P}^1)^{i+1} \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} &= \text{tr}_{2}(H_{hol}(\partial_i^{-1} \mathcal{E})) \otimes \text{tr}_{2}(H_{hol}(\partial_i^{1} \mathcal{E}))
\end{align*}
\]

for \( i = 1, \ldots, k \). We set

\[
H_{2,k}(\mathcal{E}) := \frac{(-1)^{k+2}}{2(k+2)!} \int_{(p_1)^{k+2}} \text{ch}_{g}^{0}(\text{tr}_{2}(H_{hol}(\mathcal{E}))) \wedge C_{k+2}(\log | z_1 |^2, \ldots, \log | z_{k+2} |^2).
\]

Then \( H_{2,k} \) defines a map

\[
H_{2,k} : \tilde{ZC}^{(f,l)-at}(X, \mu_n) \to \bigoplus_{p \geq 0} D^{2p-k-2}(Z, p)_{R_n}
\]

which satisfies the following differential equation:

\[
\begin{align*}
d_{\mathcal{D}} \circ H_{2,k}(\mathcal{E}) \\
= \frac{(-1)^{k+2}}{2(k+2)!} \int_{(p_1)^{k+2}} \text{ch}_{g}^{0}\left(\text{tr}_{2}\left(H_{hol}(\mathcal{E})\right)\right) \wedge d_{\mathcal{D}} C_{k+2}(\log | z_1 |^2, \ldots, \log | z_{k+2} |^2)
\end{align*}
\]
forms an emi-1-cube of hermitian bundles on \( Z \times \mathbb{P}^1 \times (\mathbb{P}^1)^k \), we denote it by \( \tilde{H}_{ho}f(E) \). Using the same construction as before, we construct a transgression bundle \( tr_1(\tilde{H}_{ho}f(E)) \) on \( Z \times \mathbb{P}^1 \times \mathbb{P}^1 \times (\mathbb{P}^1)^k \) satisfying

\[
\begin{align*}
tr_1(\tilde{H}_{ho}f(E)) |_{Z \times \{0\} \times (\mathbb{P}^1)^{k+1}} &= tr_1(h_\ast H(E, f_\ast)), \\
tr_1(\tilde{H}_{ho}f(E)) |_{Z \times \{\infty\} \times (\mathbb{P}^1)^{k+1}} &= h_\ast tr_1(H(E, f_\ast)), \\
tr_1(\tilde{H}_{ho}f(E)) |_{Z \times \mathbb{P}^1 \times \{0\} \times (\mathbb{P}^1)^k} &= tr_1(h_\ast tr_{k} \circ \lambda(f_\ast E) \to h_\ast tr_{k} \circ \lambda(f_\ast E) \to 0), \\
tr_1(\tilde{H}_{ho}f(E)) |_{Z \times \mathbb{P}^1 \times \{\infty\} \times (\mathbb{P}^1)^k} &= tr_1(h_\ast f_\ast tr_{k} \circ \lambda(E) \to h_\ast f_\ast tr_{k} \circ \lambda(E) \to 0)
\end{align*}
\]

and

\[
\begin{align*}
tr_1(\tilde{H}_{ho}f(E)) |_{Z \times (\mathbb{P}^1)^{i+1} \times \{0\} \times (\mathbb{P}^1)^{k-i}} &= tr_1(\tilde{H}_{ho}f(\partial_i^0 E)), \\
tr_1(\tilde{H}_{ho}f(E)) |_{Z \times (\mathbb{P}^1)^{i+1} \times \{\infty\} \times (\mathbb{P}^1)^{k-i}} &= tr_1(\tilde{H}_{ho}f(\partial_i^{-1} E)) \oplus tr_1(\tilde{H}_{ho}f(\partial_i^{-1} E))
\end{align*}
\]

for \( i = 1, \ldots, k \). So, if we set

\[ H_{3,k}(E) := \frac{(-1)^{k+2}}{2(k+2)!((2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \text{ch}_g^0(\text{tr}_1(\tilde{H}_{ho}f(E))) \wedge C_{k+2}(\log |z_1|^2, \ldots, \log |z_{k+2}|^2), \]

it satisfies the differential equation
\[d_P \circ H_{3,k}(E)\]
\[= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \text{ch}_g^0(\text{tr}_1(\tilde{H}_{hof})) \wedge d_P C_{k+2}(\log |z_1|^2, \ldots, \log |z_{k+2}|^2)\]
\[= H_{3,k-1} \circ d(E) + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(h_*H(E,f_*)))\]
\[\wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_k|^2)\]
\[\quad - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(h_*\text{tr}_1(H(E,f_*))) \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2).\]

Finally, we set
\[H_{4,k}(E) := \frac{(-1)^{k+2}}{(k+2)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} C_{k+2}(T_g(h,h^{\text{tr}_1}(H(E,f_*))), \log |z_1|^2, \ldots, \log |z_{k+1}|^2),\]
then it satisfies
\[d_P \circ H_{4,k}(E)\]
\[= \frac{(-1)^{k+2}}{(k+2)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} C_{k+2}(T_g(h,h^{\text{tr}_1}(H(E,f_*))), \log |z_1|^2, \ldots, \log |z_{k+1}|^2)\]
\[= H_{4,k-1} \circ d(E) + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H(E,f_*)))\]
\[\wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2)\]
\[\quad - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k}} C_{k+1}(T_g(h,h^{\text{tr}_k}(f_*)), \log |z_1|^2, \ldots, \log |z_k|^2)\]
\[\quad + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^{k}} \int_{(\mathbb{P}^1)^{k}} C_{k+1}(T_g(h,h^{f_*},(f_*)), \log |z_1|^2, \ldots, \log |z_k|^2)\]
\[= H_{4,k-1} \circ d(E) - h_{\mu_n} \circ (\text{Td}_g(\mathcal{T}h) \bullet \Pi'_k(E))\]
\[\quad + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_g^0(\text{tr}_1(H(E,f_*))) \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2)\]
\[\quad - \Pi'_k(f_*,E) + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^{k}} \int_{(\mathbb{P}^1)^{k}} C_{k+1}(T_g(h,h^{f_*},(f_*)), \log |z_1|^2, \ldots, \log |z_k|^2).\]

**Proposition 3.5.** Let notations and assumptions be as above, then the chain homotopy
\[\Pi'_k = \Pi'_k + \Pi''_k is homotopic to \Pi'^{(1)} + h_{\mu_n} \circ (\text{Td}_g(\mathcal{T}h) \bullet \Pi'_k) + \Pi'_k \circ f_* - \Pi''_k^{(3')}.\]
Proof. Let $\overline{E}$ be a hermitian $k$-cube in $\overline{\mathcal{P}}(X, \mu_n)$ which is $f$-acyclic and $l$-acyclic. Using the above differential equations concerning $\mathbf{H}_{k,k}$, we obtain that

\[
(H_{1,k-1} + H_{2,k-1} - H_{3,k-1} - H_{4,k-1}) \circ d(\overline{E}) - d_D \circ (H_{1,k} + H_{2,k} - H_{3,k} - H_{4,k})(\overline{E})
\]

\[
= \Pi_k^{(1)}(\overline{E}) - \Pi_k^h(f, \overline{E}) - \Pi_k^h(f, \overline{E}) - h_{\mu_n} \circ (Td_g(\overline{T}h) \bullet \Pi_k^f(\overline{E}))
\]

\[
+ \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(p_{1})^{k+1}} c_{k+1}(T_g(h, h^f, \gamma), \log |z_1|^2, \ldots, \log |z_k|^2)
\]

\[
\wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2)
\]

\[
+ \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(p_{1})^k} c_{k+1}(d_D \Delta(f, h, \omega^X, \omega^Y, tr_k \circ \lambda(\overline{E})), \log |z_1|^2, \ldots, \log |z_k|^2).
\]

On the other hand, according to Theorem 3.1, we have

\[
\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(p_{1})^k} c_{k+1}(T_g(h, h^f, \gamma), \log |z_1|^2, \ldots, \log |z_k|^2)
\]

\[
- \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(p_{1})^k} c_{k+1}(T_g(h, h^f, \gamma), \log |z_1|^2, \ldots, \log |z_k|^2)
\]

\[
- h_{\mu_n} \circ (Td_g(\overline{T}h) \bullet \Pi_k^f(\overline{E})) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(p_{1})^{k+1}} c_{k+1}(T_g(h, h^f, \gamma), \log |z_1|^2, \ldots, \log |z_k|^2)
\]

\[
\wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2) - \Pi_k^{(3)}(\overline{E})
\]

\[
+ \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(p_{1})^k} c_{k+1}(d_D \Delta(f, h, \omega^X, \omega^Y, tr_k \circ \lambda(\overline{E})), \log |z_1|^2, \ldots, \log |z_k|^2).
\]

We then formally define a product

\[
C_{k+1}(\Delta(f, h, \omega^X, \omega^Y, tr_k \circ \lambda(\overline{E})), \log |z_1|^2, \ldots, \log |z_k|^2)
\]

in the same way as (9), and we set

\[
\Delta_k(\overline{E}) = \frac{(-1)^{k}}{(k+1)!(2\pi i)^k} \int_{(p_{1})^k} C_{k+1}(\Delta(f, h, \omega^X, \omega^Y, tr_k \circ \lambda(\overline{E})), \log |z_1|^2, \ldots, \log |z_k|^2).
\]

It is readily checked by Lemma 3.2 that

\[
\Delta_{k-1}(d\overline{E}) - d_D \Delta_k(\overline{E})
\]

\[
= \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(p_{1})^k} C_{k+1}(d_D \Delta(f, h, \omega^X, \omega^Y, tr_k \circ \lambda(\overline{E})), \log |z_1|^2, \ldots, \log |z_k|^2).
\]

Combing all the above computations, we finally get

\[
(H_{1,k-1} + H_{2,k-1} - H_{3,k-1} - H_{4,k-1} + \Delta_{k-1})(d\overline{E})
\]

\[
-d_D(H_{1,k} + H_{2,k} - H_{3,k} - H_{4,k} + \Delta_k)(\overline{E})
\]
So, we are done. □

Corollary 3.6. Let \( f : X \to Y, h : Y \to Z \) and \( l : X \to Z \) be three equivariant morphisms between regular \( \mu_n \)-projective schemes, which are all smooth over the generic fibers. Assume \( l = h \circ f \) and that the \( \mu_n \)-action on \( Z \) is trivial. Then the direct image map \( l_* \) is equal to the composition \( h_* \circ f_* \) from \( \hat{K}_m(X, \mu_n)_\mathbb{Q} \) to \( \hat{K}_m(Z, \mu_n)_\mathbb{Q} \) for any \( m \geq 1 \).

4. The Lefschetz–Riemann–Roch theorem

4.1. The statement

In order to formulate the Lefschetz–Riemann–Roch theorem for higher equivariant arithmetic K-groups, we need to introduce the equivariant \( R \)-genus due to Bismut. Let \( X \) be a \( \mu_n \)-equivariant smooth algebraic variety over \( \mathbb{C} \), and let \( E \) be a \( \mu_n \)-equivariant hermitian vector bundle on \( X \). For \( \zeta \in \mu_n(\mathbb{C}) \) and \( s > 1 \), we consider the following Lerch zeta function

\[
L(\zeta, s) = \sum_{k=1}^{\infty} \frac{\zeta^k}{k^s}.
\]

and its meromorphic continuation to the whole complex plane. Define a formal power series in the variable \( x \) as

\[
\tilde{R}(\zeta, x) := \sum_{n=0}^{\infty} \left( \frac{\partial L}{\partial s}(\zeta, -n) + L(\zeta, -n) \sum_{j=1}^{n} \frac{1}{2j} \right) \frac{x^n}{n!}.
\]

Definition 4.1. The Bismut’s equivariant \( R \)-genus of an equivariant hermitian vector bundle \( E \) with \( E|_{X_{\mu_n}} = \sum_{\zeta \in \mu_n(\mathbb{C})} E_\zeta \) is defined as

\[
R_g(E) := \sum_{\zeta \in \mu_n(\mathbb{C})} \left( \text{Tr} \tilde{R}(\zeta, -\Omega^E_\zeta) - \text{Tr} \tilde{R}(1/\zeta, \Omega^{E_\zeta}) \right),
\]

where \( \Omega^{E_\zeta} \) is the curvature form associated to \( E_\zeta \).

Now, let \( X \) be a regular \( \mu_n \)-projective arithmetic scheme over an arithmetic ring \((D, \Sigma, F_\infty)\) and we construct a naive commutative diagram of homological complexes

\[
\begin{array}{c}
\hat{\mathcal{C}}_*(X, \mu_n) \xrightarrow{ch_2} \bigoplus_{p \geq 0} D^{2p-1}(X_{\mu_n}, p)_{\mathbb{R}} \\
0 \downarrow 0 \\
\end{array}
\]

where 0 stands for the zero map. Let \( \mathcal{N} \) be a \( \mu_n \)-equivariant hermitian vector bundle on \( X \), we shall formally regard the \( R \)-genus \( R_g(\mathcal{N}) \) as an element in \( \bigoplus_{p \geq 0} D^{2p-1}(X, p) \). It is a \( d \)-
closed form. Denote by $p_0$ the projection from $X \times (P^1)^\ast$ to $X$. For any hermitian $k$-cube $\bar{E}$ in $\hat{\mathcal{P}}(X, \mu_n)$, we set
\[
\Pi_R(\bar{E}) = \frac{(-1)^k}{(k+1)!(2\pi i)^k} \int_{(P^1)^k} C_{k+1}(R_g(p_0^*N)ch_g^0(tr_k \circ \lambda(\bar{E}))) \log |z_1|^2, \ldots, \log |z_k|^2).
\]

It is clear that $\Pi_R(\bar{E})$ extends to be a map $\Pi_R : \hat{\mathbb{Z}}C_k(X, \mu_n) \to \bigoplus_{p \geq 0} D^{2p-k-1}(X_{\mu_n},p)R_n$ which provides a chain homotopy of the square (11). Therefore, we get an endomorphism of $\hat{K}_m(X, \mu_n)$ for any $m \geq 1$. This endomorphism will be denoted by $\otimes R_g(N)$.

Again, by [34, Remark 2.4, Lemma 2.5], the chain homotopy $\Pi_R$ is homotopic to the chain homotopy $\Pi'_R$ defined by
\[
\Pi'_R(\bar{E}) = \frac{(-1)^{k+1}}{2k!(2\pi i)^k} \int_{(P^1)^k} R_g(p_0^*N) \bullet ch_g^0(tr_k \circ \lambda(\bar{E})) \bullet C_k(\log |z_1|^2, \ldots, \log |z_k|^2),
\]
and hence is homotopic to $-R_g(N) \bullet ch_g(\bar{E})$ by the projection formula. Let $(x, \alpha)$ be an element in $\hat{K}_m(X, \mu_n)_Q$, then $dx = 0$ and $ch_g(x)$ is a $D$-closed form. Let $(0, \alpha)$ and $(0, \alpha')$ be two elements in $\hat{K}_m(X, \mu_n)_Q$, then $(0, \alpha) = (0, \alpha')$ if $\alpha$ and $\alpha'$ have the same cohomology class in $\bigoplus_{p \geq 0} H^2(X_{\mu_n}, \mathbb{R}(p))R_n$. Note that the product $\bullet$ on the Deligne-Beilinson complex induces the product on the real Deligne-Beilinson cohomology. Then, modulo torsion, the endomorphism $\otimes R_g(N)$ is independent of the choice of the metric on $N$ and it can be written as $\otimes R_g(N)$.

Assume that $\rho$ is any prime ideal in $\hat{R}(\mu_n) := K_0(\text{Spec} \mathbb{Z}, \mu_n) \cong \mathbb{Z}[T]/(1 - T^n)$ which does not contain the elements $1 - T^k$ for $k = 1, \ldots, n - 1$. For instance, $\rho$ can be chosen to be the kernel of the natural morphism $\mathbb{Z}[T]/(1 - T^n) \to \mathbb{Z}[T]/(\Phi_n)$, where $\Phi_n$ stands for the $n$th cyclotomic polynomial. Let $X_{\mu_n}$ be the fixed-point subscheme of $X$, and let $N_{X/X_{\mu_n}}$ be the normal bundle of $X_{\mu_n}$ in $X$ with some $\mu_n$-invariant hermitian metric. We set
\[
\Lambda_R := (\text{Id} - \otimes R_g(N_{X/X_{\mu_n}})) \otimes \lambda_{-1}^{-1}(N_{X/X_{\mu_n}}),
\]
it is a well-defined endomorphism of $\hat{K}_m(X_{\mu_n}, \mu_n)_\rho \otimes \mathbb{Q}$. Then the arithmetic Lefschetz–Riemann–Roch theorem for higher equivariant arithmetic K-groups can be formulated as follows.

**Theorem 4.2** (arithmetic Lefschetz–Riemann–Roch). Let $f : X \to Y$ be an equivariant morphism between two regular $\mu_n$-projective arithmetic schemes, which is smooth over the generic fiber. Suppose that the $\mu_n$-action on the base $Y$ is trivial. Then, for any $m \geq 1$, the following diagram
\[
\begin{array}{ccc}
\hat{K}_m(X, \mu_n) & \xrightarrow{\Lambda_R \circ \tau} & \hat{K}_m(X_{\mu_n}, \mu_n)_\rho \otimes \mathbb{Q} \\
\downarrow f & & \downarrow f_{\mu_n} \\
\hat{K}_m(Y, \mu_n) & \xrightarrow{\tau} & \hat{K}_m(Y, \mu_n)_\rho \otimes \mathbb{Q}
\end{array}
\]
where $\tau$ is the restriction map, is commutative.

The proof of Theorem 4.2 will be given in next two subsections.

4.2. Arithmetic K-theoretic form of Bismut–Ma immersion formula

Let $Y \hookrightarrow X$ be a $\mu_n$-equivariant closed immersion of regular $\mu_n$-projective arithmetic schemes over $(D, \Sigma, F_\infty)$. In [34, Section 4], we have proved an arithmetic purity theorem
\[
\hat{K}_m(Y, \mu_n) \cong \hat{K}_{Y,m}(X, \mu_n)
\]
for any integer \(m \geq 1\). As a byproduct, we get an embedding morphism \(\hat{K}_m(Y, \mu_n) \to \hat{K}_m(X, \mu_n)\). This embedding morphism is realized by constructing an explicit chain homotopy of the square

\[
\begin{array}{c}
\tilde{Z}C_*(Y, \mu_n) \xrightarrow{ch_g} \bigoplus_{p \geq 0} 'D^{2p-*}(Y, \mu_n, p)_{R_n} \\
\downarrow i_* \downarrow \downarrow i_{\mu_n!} \tau_{Y_1}^{-1}(\mathcal{N}_{X/Y})^* c \\
\tilde{Z}C_*(P, \mu_n) \xrightarrow{ch_g} \bigoplus_{p \geq 0} 'D^{2p-*}(P, \mu_n, p)_{R_n},
\end{array}
\]

(12)

where \('D^{2p-*}(\cdot, p)\) stands for the Deligne complex of currents computing the Deligne homology groups, \((i_{\mu_n!} T)(\eta) = T(i^*_{\mu_n} \eta)\) for a current \(T\) and a test form \(\eta\), \(i : Y \hookrightarrow P := \mathbb{P}(\mathcal{N}_{X/Y} \oplus \mathcal{O}_Y)\) is the associated zero section embedding with projection \(\pi : P \to Y\) and

\[
i_* : \tilde{Z}C_*(Y, \mu_n) \to \tilde{Z}C_*(P, \mu_n)
\]

is the complex morphism defined by sending a hermitian cube \(E\) to \(\sum_{j=0}^n (-1)^j \mathcal{Q}^j \otimes \pi^* E\) provided the Koszul resolution

\[
K(E, \mathcal{N}_{X/Y}) : 0 \to \wedge^n \mathcal{Q}^j \otimes \pi^* E \to \cdots \to \wedge^{n-1} \mathcal{Q}^j \otimes \pi^* E \to \pi^* E \to i_* E \to 0.
\]

For any hermitian \(k\)-cube \(E\), one chain homotopy \(H_k(E)\) of (12) is given by the formula

\[
H_k(E) = T_g(K(O_Y, \mathcal{N}_{X/Y})) \cdot ch_g(\pi^* E),
\]

where \(T_g(K(O_Y, \mathcal{N}_{X/Y}))\) is the equivariant Bott–Chern singular current associated to the Koszul resolution which satisfies

\[
d_D T_g(K(O_Y, \mathcal{N}_{X/Y})) = \sum_{j=1}^n (-1)^j ch_g(\wedge^n \mathcal{Q}^j) - i_{\mu_n!}(ch_g(O_Y) \tau_g^{-1}(\mathcal{N}_{X/Y})).
\]

For more details the reader is referred to [34, Section 4.2].

It is clear that if we choose another resolution

\[
0 \to F_n \to \cdots \to F_1 \to F_0 \to i_* \mathcal{O}_Y \to 0
\]

with respect to the zero section embedding \(i : Y \hookrightarrow \mathbb{P}(\mathcal{N}_{X/Y} \oplus \mathcal{O}_Y)\) such that the metrics on \(F\) satisfy the Bismut's assumption (A), we may construct a different homotopy of (12) and we shall get a different embedding morphism \(i_* : \hat{K}_m(Y, \mu_n) \to \hat{K}_m(P, \mu_n)\). Our first result in this subsection is the following.

**Proposition 4.3.** The embedding morphism over rational arithmetic \(K\)-groups

\[
i_* : \hat{K}_m(Y, \mu_n)_{\mathbb{Q}} \to \hat{K}_m(P, \mu_n)_{\mathbb{Q}}
\]

is independent of the choice of the resolution of \(i_* \mathcal{O}_Y\) on \(\mathbb{P}(\mathcal{N}_{X/Y} \oplus \mathcal{O}_Y)\) which satisfies the Bismut's assumption (A).
Proof. Since any two resolutions of $i_*\mathcal{O}_Y$ on $P(N_{X/Y} \oplus \mathcal{O}_Y)$ are dominated by a third one, we may assume that $\mathcal{F}$ and $\wedge^i\mathcal{Q}^\vee$ fit into the following diagram:

\[
\begin{array}{cccc}
0 & 0 & 0 & \downarrow \\
\downarrow & \downarrow & \downarrow & \\
0 & \mathcal{A}_n & \mathcal{F}_n & \mathcal{Q}_n \wedge^i \mathcal{Q}^\vee \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
0 & \mathcal{A}_1 & \mathcal{F}_1 & \mathcal{Q}_1 \wedge^i \mathcal{Q}^\vee \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
0 & \mathcal{A}_0 & \mathcal{F}_0 & \mathcal{Q}_0 \wedge^i \mathcal{Q}^\vee \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & i_*\mathcal{O}_Y & i_*\mathcal{O}_Y & 0
\end{array}
\]

where $\mathcal{A}_i$ is an exact sequence of hermitian vector bundles on $P$. We endow $A_i$ with the metrics coming from $\mathcal{F}$ via the natural inclusion. We split $\mathcal{A}_i$ into a family of short exact sequences of hermitian bundles from $j = 1$ to $n - 1$

\[
\chi_j : 0 \rightarrow \text{Ker } d_j \rightarrow \mathcal{A}_j \xrightarrow{d_j} \text{Ker } d_{j-1} \rightarrow 0.
\]

Moreover, we denote by $\varepsilon_j$ the short exact sequence

\[
\begin{array}{cccc}
0 & \mathcal{A}_j & \mathcal{F}_j & \wedge^i \mathcal{Q}^\vee \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
0 & \mathcal{A}_0 & \mathcal{F}_0 & \mathcal{Q}_0 \wedge^i \mathcal{Q}^\vee \rightarrow 0 \\
\downarrow & \downarrow & \downarrow & \\
\vdots & \vdots & \vdots & \\
\downarrow & \downarrow & \downarrow & \\
0 & i_*\mathcal{O}_Y & i_*\mathcal{O}_Y & 0
\end{array}
\]

from $j = 0$ to $n$. Write $i_*$ (respectively, $i'_*$) for the morphism $\tilde{\mathcal{Z}}C_*(Y, \mu_0) \rightarrow \tilde{\mathcal{Z}}C_*(P, \mu_0)$ with respect to the Koszul resolution $K(\mathcal{O}_Y, N_{X/Y})$ (respectively, the resolution $\mathcal{F}$). Then, for any hermitian $k$-cube $\mathcal{E}$ on $Y$, the assignment

\[
H_i(\mathcal{E}) := \sum_{j=0}^{n} (-1)^j \varepsilon_j \otimes \pi^*\mathcal{E} + \sum_{j=1}^{n-1} (-1)^j \chi_j \otimes \pi^*\mathcal{E} \in \tilde{\mathcal{Z}}C_{k+1}(P, \mu_0)
\]

provides a chain homotopy between $i'_*$ and $i_*$. Consequently, the formula

\[
H_k^{(1)}(\mathcal{E}) = \left( \sum_{j=0}^{n} (-1)^j \text{ch}_g(\varepsilon_j) + \sum_{j=1}^{n-1} (-1)^j \text{ch}_g(\chi_j) \right) \text{ch}_g(\pi^*\mathcal{E})
\]

defines a chain homotopy between $\text{ch}_g \circ i'_*$ and $\text{ch}_g \circ i_*$. We claim that there exists a homotopy of chain homotopies between $H'_k(\mathcal{E})$ and $H_k^{(1)}(\mathcal{E}) + H_k(\mathcal{E})$.

In fact, according to [24, Theorem 3.14, Corollary 3.10], we have

\[
- \sum_{j=1}^{n-1} (-1)^j \text{ch}_g(\chi_j) + T_g(\mathcal{F}) - T_g(K(\mathcal{O}_Y, N_{X/Y})) = \sum_{j=0}^{n} (-1)^j \text{ch}_g(\varepsilon_j)
\]
up to \( \text{Im} \, d_D \). We fix an element \( \Delta \) such that

\[
d_D \Delta = \sum_{j=0}^{n} (-1)^j \text{ch}_g(\varepsilon_j) + \sum_{j=1}^{n-1} (-1)^j \text{ch}_g(\chi_j) - T_g(\mathcal{F}) + T_g(K(\mathcal{O}_Y, \mathcal{N}_{X/Y}))
\]

and set

\[
\tilde{H}_k(\mathcal{E}) := \Delta \cdot \text{ch}_g(\pi^*\mathcal{E}).
\]

Then

\[
d_D \circ \tilde{H}_k(\mathcal{E}) = H_k^{(1)}(\mathcal{E}) + H_k(\mathcal{E}) - H_k(\mathcal{E}) + \tilde{H}_{k-1} \circ d(\mathcal{E}).
\]

So, we are done. \( \square \)

Note that the product \( P \times (\mathbb{P}^1)^l \) can be identified with the projective space bundle over \( Y \times (\mathbb{P}^1)^l \) with respect to the vector bundle \( p_0^* \mathcal{N}_{X/Y} \), and

\[
0 \to p_0^* \wedge^n \Omega_Y^* \to \cdots \to p_0^* \wedge \Omega_Y^* \to \mathcal{O}_{P \times (\mathbb{P}^1)^l} \to \mathcal{O}_Y \xrightarrow{i_*} \mathcal{O}_Y \xrightarrow{(P^1)^l} 0
\]

is the Koszul resolution, so that the corresponding Bott–Chern singular current is the pullback \( p_0^* T_g(K(\mathcal{O}_Y, \mathcal{N}_{X/Y})) \). We shall still write it as \( T_g(K(\mathcal{O}_Y, \mathcal{N}_{X/Y})) \) for the sake of simplicity. Then, like before, by the projection formula and [34, Remark 2.4, Lemma 2.5], \( H_k(\mathcal{E}) \) is homotopic to the following chain homotopy

\[
\frac{(-1)^{k+1}}{(k+1)!} \int_{(\mathbb{P}^1)^l} C_{k+1}(T_g(K(\mathcal{O}_Y, \mathcal{N}_{X/Y}))) \cdot \text{ch}_g^0(\text{tr}_k \circ \lambda(\mathcal{E})), \log |z_1|^2, \ldots, \log |z_k|^2),
\]

which will be still denoted by \( H_k(\mathcal{E}) \).

Now, let us recall the Bismut–Ma immersion formula which relates analytic torsion forms and the Bott–Chern singular current. Let \( X \) be a smooth \( \mu_n \)-equivariant algebraic variety over \( \mathbb{C} \) and let \( i : Y \hookrightarrow X \) be an equivariant closed smooth subvariety. Let \( S \) be a smooth algebraic variety with trivial \( \mu_n \)-action, and let \( f : Y \to S, \) \( l : X \to S \) be two equivariant proper smooth morphisms such that \( f = l \circ i \). Assume that \( \tilde{\eta} \) is an equivariant hermitian bundle on \( Y \) and \( \tilde{\xi} \), is a complex of equivariant hermitian bundles on \( X \) which provides a resolution of \( i_* \tilde{\eta} \) such that the metrics on \( \xi \), satisfy the Bismut’s assumption (A). Let \( \omega^X \), \( \omega^Y \) be two Kähler fibrations on \( f \) and on \( l \), respectively. We shall assume that \( \omega^Y \) is the pullback of \( \omega^X \), so that the Kähler metric on \( Y \) is induced by the Kähler metric on \( X \). Consider the following exact sequence

\[
\mathcal{N} : 0 \to Tf \to Tl \mid_{Y} \to \mathcal{N}_{X/Y} \to 0,
\]

where \( \mathcal{N}_{X/Y} \) is endowed with the quotient metric. Denote by \( \text{Td}_g(\mathcal{N}) = \Phi^{-1}(\text{Td}_g(\mathcal{N})) \) (see Section A.3 in the Appendix) the equivariant secondary Todd form of \( \mathcal{N} \) which satisfies the identity

\[
d_D \text{Td}_g(\mathcal{N}) = \text{Td}_g(Tl \mid_{Y}, h^{Tf}) - \text{Td}_g(Tf, h^{Tf}) \text{Td}_g(\mathcal{N}_{X/Y}).
\]

We suppose that in the resolution \( \xi, \xi_j \) are all -acyclic and moreover \( \eta \) is \( f \)-acyclic. Denote by \( h^{H(\xi)} \) the hermitian metric on \( f_* \eta \) corresponding to the \( L^2 \)-metric on the hypercohomology of \( \xi \), over the fiber of \( l : X \to S \) (see Section A.3 in the Appendix). By an easy argument of long exact sequence, we have the following exact sequence of hermitian vector bundles on \( S \)

\[
\Xi : 0 \to l_* (\tilde{\xi}_m) \to l_* (\tilde{\xi}_{m-1}) \to \cdots \to l_* (\tilde{\xi}_0) \to \left( f_* \eta, h^{H(\xi)} \right) \to 0.
\]

We may split \( \Xi \) into a family of short exact sequence of hermitian bundles from \( j = 1 \) to \( m \)

\[
\chi_j : 0 \to \text{Ker} \, d_j \to \Xi_j \xrightarrow{d_j} \text{Ker} \, d_{j-1} \to 0
\]
such that the kernel of every map \( d_{j-1} \) for \( j = 2, \ldots, m \) carries the metric induced by \( \Xi_j \) and \( \text{Ker} \, d_0 = \Xi_0 = (f_\ast \eta, h^H(\xi)) \), \( \text{Ker} \, d_m = \Xi_{m+1} = l_\ast (\xi_m) \). We regard \( \chi_f \) as a hermitian 1-cube on \( S \) and we set \( ch_g(\Xi) = \sum_{j=1}^m (-1)^j ch_g(\chi_j) \). Then it satisfies the differential equation
\[
d_D ch_g(\Xi) = ch_g(f_\ast \eta, h^H(\xi)) - \sum_{j=0}^m ch_g(l_\ast (\xi_j)).
\]
Set \( ch_g(\Xi, f, \eta) := ch_g(\Xi) + ch_g(f_\ast \eta, h^H(\xi), f_\ast h^v) \), it satisfies the differential equation
\[
d_D ch_g(\Xi, f, \eta) = ch_g(f_\ast \eta) - \sum_{j=0}^m ch_g(l_\ast (\xi_j)).
\]

With some abuse of notations, we still use \( \Xi \) to denote the long exact sequence
\[
0 \to l_\ast (\xi_m) \to l_\ast (\xi_{m-1}) \to \ldots \to l_\ast (\xi_0) \to f_\ast \eta \to 0
\]
and identify \( ch_g(\Xi, f, \eta) \) with \( ch_g(\Xi, f, \eta) \).

**Theorem 4.4** (Immersion formula). Let notations and assumptions be as above. Then the following identity holds in \( \bigoplus_{p \geq 0} (D^{2p-1}(S, p)/\text{Im} \, d_D) \).
\[
\sum_{i=0}^m (-1)^i T_g(\omega^X, h^{\xi_i}) - T_g(\omega^Y, h^v) + ch_g(\Xi) = \\
= - \frac{1}{(2\pi i)^{r_f}} \int_{X_{\mu_n}/S} Td_g(\overline{Tl}) T_g(\xi_i) - \frac{1}{(2\pi i)^{r_l}} \int_{Y_{\mu_n}/S} Td_g(\overline{N}) Td_g^{-1}(\overline{N}_{X/Y}) \text{ch}_g(\eta) \]
\[
+ \frac{1}{(2\pi i)^{r_f}} \int_{Y_{\mu_n}/S} Td_g(\overline{Tf}) R_g(\overline{N}_{X/Y}) \text{ch}_g(\eta),
\]
where \( r_f \) and \( r_l \) are the relative dimensions of \( Y_{\mu_n}/S \) and of \( X_{\mu_n}/S \), respectively. Note that the equivariant Bott–Chern singular current we use here differs from the one used in the appendix by a minus sign.

**Proof.** This is a translation of [14, Theorem 0.1 and 0.2] (see also Theorem A.5). \( \square \)

With the same notations as in Remark 2.7 and Theorem A.6, we set
\[
\Delta(f, l, i_\ast \eta, \xi) := -\Phi^{-1} \left( \frac{\Delta_0^0(f, l, i_\ast \eta, \xi)}{2} + \Delta_0(f, l, i_\ast \eta, \xi) \right) - \Delta(\Xi).
\]
Then \( d_D \Delta(f, l, i_\ast \eta, \xi) \) measures the difference
\[
\sum_{i=0}^m (-1)^i T_g(\omega^X, h^{\xi_i}) - T_g(\omega^Y, h^v) + ch_g(\Xi) + \frac{1}{(2\pi i)^{r_f}} \int_{X_{\mu_n}/S} Td_g(\overline{Tl}) T_g(\xi_i) \]
\[
+ \frac{1}{(2\pi i)^{r_f}} \int_{Y_{\mu_n}/S} Td_g(\overline{N}) Td_g^{-1}(\overline{N}_{X/Y}) \text{ch}_g(\eta) - \frac{1}{(2\pi i)^{r_f}} \int_{Y_{\mu_n}/S} Td_g(\overline{Tf}) R_g(\overline{N}_{X/Y}) \text{ch}_g(\eta)
\]
in Theorem 4.4. Let us go back to the same situation described before Lemma 2.12 and assume that the following diagrams
\[
Y \times Z \xrightarrow{i_Z} X \times Z \quad \text{and} \quad Y \times Z_1 \xrightarrow{i_{Z_1}} X \times Z_1
\]
are obtained by smooth base changes. Then \( Y \times Z \) and \( X \times Z_1 \) intersect transversely along \( Y \times Z_1 \) and the singular currents can be pulled back.

**Lemma 4.5.** The restriction of \( \Delta(f_Z, l_Z, i_Z \ast \eta, \xi) \) over \( S \times Z_1 \) is equal to the differential form \( \Delta(f_{Z_1}, l_{Z_1}, i_{Z_1} \ast \eta, \xi) \) over \( Y \times Z_1 \).

**Proof.** This is a consequence of Theorem A.6. \( \square \)

**Proposition 4.6.** Let \( Y \) be a regular \( \mu_\eta \)-projective arithmetic scheme over \( (D, \Sigma, F_\infty) \) and let \( \mathfrak{N} \) be a \( \mu_\eta \)-equivariant hermitian vector bundle on \( Y \). Suppose that the \( \mu_\eta \)-action on \( Y \) is trivial and consider the zero section embedding

\[
i : Y \hookrightarrow P := \mathbb{P}(N \oplus O_Y)
\]

with hermitian normal bundle \( \mathfrak{N} \) and the natural projection \( \pi : P \to Y \). Then for any element \( x \in \hat{K}_m(Y, \mu_\eta)_Q \) with integer \( m \geq 1 \), the following identity

\[
x \ast - R_g(N) \ast x = \pi_\ast i_\ast (x)
\]

holds in \( \hat{K}_m(Y, \mu_\eta)_Q \).

**Proof.** By the definition of the action of \( R_g(N) \) on \( \hat{K}_m(Y, \mu_\eta)_Q \), the map \( x \mapsto x \ast - R_g(N) \ast x \) is defined via the chain homotopy

\[
\Pi_{k}^{(0)}(E) = \frac{(-1)^{k+1}}{(k+1)! (2\pi)^k} \int_{(p1)^k} C_{k+1}(R_g(N) \ast \text{ch}_g^0(\text{tr}_k \circ \lambda(E)), \log |z_1|^2, \ldots, \log |z_k|^2)
\]

of the square

\[
\begin{array}{ccc}
\tilde{Z}C_\ast(Y, \mu_\eta) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-\ast}(Y_, \mu_\eta)_p R_n \\
\downarrow \text{Id} & & \downarrow \text{Id} \\
\tilde{Z}C_\ast(Y, \mu_\eta) & \xrightarrow{\text{ch}_g} & \bigoplus_{p \geq 0} D^{2p-\ast}(Y_, \mu_\eta)_p R_n .
\end{array}
\]

According to Proposition 4.3, to define the morphism \( i_\ast : \hat{K}_m(Y, \mu_\eta)_Q \to \hat{K}_m(P, \mu_\eta)_Q \), we may choose a resolution \( F \) of \( i_\ast O_Y \) on \( P \) such that every \( F_j \) is \( \pi \)-acyclic. We shall endow \( F \) with the metrics satisfying the Bismut’s assumption (A). Then we have an exact sequence of hermitian bundles on \( Y \)

\[
\Xi : 0 \to \pi_\ast(F_m) \to \pi_\ast(F_{m-1}) \to \ldots \to \pi_\ast(F_0) \to O_Y \to 0.
\]

Like before, splitting \( \Xi \) into a family of short exact sequence of hermitian bundles from \( j = 1 \) to \( m \)

\[
\chi_j : 0 \longrightarrow \text{Ker} d_j \longrightarrow \Xi_j \xrightarrow{d_j} \text{Ker} d_{j-1} \longrightarrow 0,
\]

we may construct a chain homotopy

\[
H_{\pi_\ast}(\Xi) := \sum_{j=1}^{m} (-1)^{j} \chi_j \otimes \Xi \in \tilde{Z}C_{k+1}(Y, \mu_\eta)
\]

between the maps Id and \( \pi_\ast \circ i_\ast : \tilde{Z}C_\ast(Y, \mu_\eta) \to \tilde{Z}C_\ast(Y, \mu_\eta) \). Consequently, the formula

\[
H_{k}^{(i)}(\Xi) = \frac{(-1)^{k+1}}{2(k+1)! (2\pi)^{k+1}} \int_{(p1)^k} \text{ch}_g^0(\text{tr}_{k+1} \circ \lambda(H_{\pi_\ast}(\Xi))) \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2)
\]
defines a chain homotopy between $ch_g \circ \text{Id}$ and $ch_g \circ \pi_{\ast} \circ i_{\ast}$.

Then $H^{(1)}_k + \Pi^r_{\ast} \circ \pi_{\ast} + \pi_{\ast} \circ (Td_g(T \pi) \bullet H_k)$ also defines a chain homotopy between $ch_g \circ \text{Id}$ and $\text{Id} \circ ch_g$. We compare it with $\Pi^r_k$.

First, denote by $Pr_P$ (respectively, $Pr_Y$) the projection from $P \times (\mathbb{P}^1)^k$ (respectively, $Y \times (\mathbb{P}^1)^k$) to $P$ (respectively, $Y$). Then, according to the functoriality of projective space bundle construction we have used before, $Pr_P F^\ast$ provides a resolution of $i_{\ast} \mathcal{O}_{Y \times (\mathbb{P}^1)^k}$ on $P \times (\mathbb{P}^1)^k$.

Hence, we have an exact sequence

$$\mathcal{E}: 0 \to \pi_{\ast}(Pr_P F_m) \to \pi_{\ast}(Pr_P F_{m-1}) \to \ldots \to \pi_{\ast}(Pr_P F_0) \to \mathcal{O}_{Y \times (\mathbb{P}^1)^k} \to 0$$

which can be split into a family of short exact sequence of hermitian bundles from $j = 1$ to $m$

$$\chi'_j: 0 \to \ker d_j \to \mathcal{E}'_j \xrightarrow{d_j} \ker d_{j-1} \to 0.$$ 

Furthermore, the short exact sequence of hermitian $1$-cube

$$H^{(1)}(\mathcal{E}): 0 \to \chi'_j \otimes \text{tr}_k \circ \lambda(\mathcal{E}) \xrightarrow{\text{Id}} Pr_Y \chi_j \otimes \text{tr}_k \circ \lambda(\mathcal{E}) \to 0 \to 0$$

forms a hermitian $2$-cube on $Y \times (\mathbb{P}^1)^k$. We set

$$\tilde{H}_k(\mathcal{E}) := \frac{(-1)^{k+2}}{2(k + 2)! (2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \sum_{j=1}^m (-1)^j \text{tr}_2 \circ \lambda \left( H^{(j)}(\mathcal{E}) \right)$$

$$\wedge C_{k+2}(\log | z_1 |^2, \ldots, \log | z_k |^2),$$

it satisfies the differential equation

$$d^D \circ \tilde{H}_k(\mathcal{E}) = \tilde{H}_{k-1} \circ d\mathcal{E} + \sum_{j=0}^m (-1)^j \Pi^r_k \left( F_j \otimes \pi^r_{\ast} \mathcal{E} \right)$$

$$+ \frac{(-1)^{k+1}}{2(k + 1)! (2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_0 \left( \text{tr}_{k+1} \circ \lambda \left( H_{\text{pair}}(\mathcal{E}) \right) \right) \wedge C_{k+1}(\log | z_1 |^2, \ldots, \log | z_{k+1} |^2)$$

$$- \frac{(-1)^{k+1}}{2(k + 1)! (2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \text{ch}_0 \left( \sum_{j=1}^m (-1)^j \text{tr}_1 \circ \lambda(\chi'_j) \otimes \text{tr}_k \circ \lambda(\mathcal{E}) \right)$$

$$\wedge C_{k+1}(\log | z_1 |^2, \ldots, \log | z_{k+1} |^2)$$

$$= \tilde{H}_{k-1} \circ d\mathcal{E} + H^{(1)}_k(\mathcal{E}) + \Pi^r_k \circ i_{\ast}(\mathcal{E})$$

$$- \frac{(-1)^{k+1}}{(k + 1)! (2\pi i)^{k}} \int_{(\mathbb{P}^1)^k} C_{k+1} \left( \text{ch}_g(\mathcal{E} \otimes \text{tr}_k \circ \lambda(\mathcal{E})), \log | z_1 |^2, \ldots, \log | z_k |^2 \right).$$

On the other hand, we apply the immersion formula to the resolution $Pr_P F^\ast \otimes \text{tr}_k \circ \lambda(\mathcal{E})$. We then have

$$\Pi^r_k \circ i_{\ast}(\mathcal{E})$$

$$= - \frac{(-1)^{k+1}}{(k + 1)! (2\pi i)^{k}} \int_{(\mathbb{P}^1)^k} C_{k+1} \left( \text{ch}_g(\mathcal{E} \otimes \text{tr}_k \circ \lambda(\mathcal{E})), \log | z_1 |^2, \ldots, \log | z_k |^2 \right)$$

$$- \frac{(-1)^{k+1}}{(k + 1)! (2\pi i)^{k}} \int_{(\mathbb{P}^1)^k} C_{k+1} \left( \frac{1}{(2\pi i)^r} \int_{\mathcal{P}_{\mu_{\ast}} / Y} Td_g(T \pi) T_g(Pr_P F^\ast \otimes \text{tr}_k \circ \lambda(\mathcal{E})), \right).$$
\[
\log |z_1|^2, \ldots, \log |z_k|^2 \\
+ \frac{(-1)^{k+1}}{(k+1)(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(R_{g}(\mathcal{N}) \bullet \text{ch}_g(\text{tr}_k \circ \lambda(\mathcal{E}))), \log |z_1|^2, \ldots, \log |z_k|^2')
\]

\[
+ \frac{(-1)^{k+1}}{(k+1)(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(d_D \Delta(\text{tr}_k \circ \lambda(\mathcal{E}))), \log |z_1|^2, \ldots, \log |z_k|^2')
\]

\[
= -\frac{(-1)^{k+1}}{(k+1)(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}\left(\text{ch}_g\left(\Xi \right) \otimes \text{tr}_k \circ \lambda(\mathcal{E})\right), \log |z_1|^2, \ldots, \log |z_k|^2')
\]

\[
- \pi_{\mu_n, \pi} \circ (\text{Td}_g(T\pi) \bullet H_k(\mathcal{E}))
\]

\[
+ \Pi_k(0)(\mathcal{E}) + \frac{(-1)^{k+1}}{(k+1)(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(d_D(\text{tr}_k \circ \lambda(\mathcal{E}))), \log |z_1|^2, \ldots, \log |z_k|^2')
\]

We then formally define a product \(C_{k+1}(\Delta(\text{tr}_k \circ \lambda(\mathcal{E}))), \log |z_1|^2, \ldots, \log |z_k|^2')\) in the same way as (9), and we set

\[
\Delta_k(\mathcal{E}) = \frac{(-1)^{k+1}}{(k+1)(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}(\Delta(\text{tr}_k \circ \lambda(\mathcal{E}))), \log |z_1|^2, \ldots, \log |z_k|^2')
\]

Again, it is readily checked by Lemma 4.5 that

\[
\Delta_{k-1}(d\mathcal{E}) - d_D \Delta_k(\mathcal{E}) = \frac{(-1)^k}{(k+1)(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}\left(d_D(\text{tr}_k \circ \lambda(\mathcal{E}))), \log |z_1|^2, \ldots, \log |z_k|^2')\right).
\]

Getting together all the above discussions, we see that \(\mathcal{H}_k + \Delta_k\) provides a homotopy between \(\Pi_k(0)^{\pi_1}\) and \(\Pi_k^{\pi_1} \circ \iota_+ + \pi_{\mu_n, \pi} \circ (\text{Td}_g(T\pi) \bullet H_k(\mathcal{E}))\) which implies that \(x - R_g(\mathcal{N}) \cdot x = \pi_\ast \iota_\ast(x)\) for any element \(x \in \hat{\mathcal{K}}_m(Y, \mu_n)\) with integer \(m \geq 1\).

**Corollary 4.7.** Let \(S\) be another regular \(\mu_n\)-projective arithmetic scheme with the trivial \(\mu_n\)-action. Let \(f : Y \to S\) and \(l = f \circ \pi : P \to S\) be two equivariant morphisms which are smooth over the generic fibers. Then the identity

\[
f_\ast(x) - f_\ast(R_g(\mathcal{N}) \cdot x) = l_\ast \circ i_\ast(x)
\]

holds in \(\hat{\mathcal{K}}_m(S, \mu_n)\) for any element \(x \in \hat{\mathcal{K}}_m(Y, \mu_n)\).

**Proof.** This is an immediate consequence of Proposition 4.6 and Corollary 3.6.

Now, we consider general situation. Let \(X, S\) be two regular \(\mu_n\)-projective arithmetic schemes over \((D, \Sigma, F_\infty)\), and let \(Y\) be a regular \(\mu_n\)-equivariant arithmetic closed subscheme of \(X\) with immersion \(i : Y \to X\). Let \(l : X \to S\) and \(f = l \circ i : Y \to S\) be two equivariant morphisms which are smooth over the generic fibers. We shall suppose that the \(\mu_n\)-actions on \(Y\) and on \(S\) are trivial (for example, \(Y = X_{\mu_n}, S = \text{Spec}D\)). Then the main result in this subsection is the following.

**Theorem 4.8.** For any element \(x \in \hat{\mathcal{K}}_m(Y, \mu_n)\) with integer \(m \geq 1\), the identity

\[
f_\ast(x) - f_\ast(R_g(\mathcal{N}/\mathcal{Y}) \cdot x) = l_\ast \circ i_\ast(x)
\]

holds in \(\hat{\mathcal{K}}_m(S, \mu_n)\).
To prove Theorem 4.8, we use the deformation to the normal cone construction. Denote by $W$ the blowing up of $X \times \mathbb{P}^1$ along $Y \times \{0\}$, and denote by $q_W : W \to \mathbb{P}^1$ the composition of the blow-down map $W \to X \times \mathbb{P}^1$ with the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$. For any point $t \in \mathbb{A}^1 \subset \mathbb{P}^1$, $t$ is called a $\mathbb{Z}$-point if it corresponds to a prime ideal $(x - a)$ in $D[x]$ with $a \in \mathbb{Z}$. Then for any $\mathbb{Z}$-point $t \in \mathbb{P}^1$ we have

$$q_W^{-1}(t) \cong \begin{cases} X \times \{t\}, & \text{if } t \neq 0, \\ P \cup \tilde{X}, & \text{if } t = 0, \end{cases}$$

where $\tilde{X}$ is isomorphic to the blowing up of $X$ along $Y$ and $P$ is the projective space bundle $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$. Let $j : Y \times \mathbb{P}^1 \to W$ be the closed immersion induced by $i \times \text{Id}$, then the component $\tilde{X}$ does not meet $j(Y \times \mathbb{P}^1)$ and the intersection of $j(Y \times \mathbb{P}^1)$ with $P$ is exactly the image of $Y$ under the zero section embedding. Moreover, denote by $s_t$ the obvious section $Y \cong Y \times \{t\} \hookrightarrow Y \times \mathbb{P}^1$ for every $\mathbb{Z}$-point $t$ and denote by $u_t$ the natural inclusion $q_W^{-1}(t) \hookrightarrow W$.

We have two Tor-independent squares

$$
\begin{array}{ccc}
Y \times \mathbb{P}^1 & \xrightarrow{j} & W \\
\downarrow s_t & & \downarrow u_t \\
Y & \xrightarrow{i} & X
\end{array}
$$

with $t \neq 0$ and

$$
\begin{array}{ccc}
Y \times \mathbb{P}^1 & \xrightarrow{j} & W \\
\downarrow s_0 & & \downarrow u_0 \\
Y & \xrightarrow{i_0} & \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y).
\end{array}
$$

Note that the complement $X \setminus Y$ is contained in $W \setminus Y \times \mathbb{P}^1$, we have pullback morphism $u_t^* : \tilde{K}_{Y \times \mathbb{P}^1,m}(W,\mu_n) \to \tilde{K}_{Y,m}(X,\mu_n)$.

**Lemma 4.9.** For any $\mathbb{Z}$-point $t \neq 0$, the diagram

$$
\begin{array}{ccc}
\tilde{K}_{m}(Y \times \mathbb{P}^1,\mu_n) & \xrightarrow{c_t^*} & \tilde{K}_{Y \times \mathbb{P}^1,m}(W,\mu_n) \\
\downarrow \cong \quad & \quad & \downarrow \cong \\
\tilde{K}_{m}(Y,\mu_n) & \xrightarrow{i_*} & \tilde{K}_{Y,m}(X,\mu_n)
\end{array}
$$

is commutative.

**Proof.** The commutativity of the algebraic prototype of this diagram follows from the Tor-independence of the deformation diagrams, but for arithmetic K-theory it is more complicated because the morphisms $j_*$ and $i_*$ are defined via another deformation to the normal cone construction according to the $\mathbb{A}^1$-homotopy invariance of the K-theory and the Deligne–Beilinson cohomology.

Write $c_t^* : \tilde{K}_m(Y \times \mathbb{P}^1,\mu_n) \to \tilde{K}_m(Y,\mu_n)$ for the composition $i_*^{-1} \circ u_t^* \circ j_*$. We need to show that $c_t^* = s_t^*$. The morphism $s_t^*$ is induced by the commutativity between $s_t^*$ and $\text{ch}_Y$, while the morphism $c_t^*$ is induced by the homotopy defining $j_*$ and the homotopy defining $i_*$. Again, using the $\mathbb{A}^1$-homotopy invariance of the K-theory and the Deligne–Beilinson cohomology, we may consider the pullbacks of $s_t^*$ and $c_t^*$ to $\tilde{K}_m(Y \times \mathbb{P}^1 \times \mathbb{A}^1,\mu_n) \to \tilde{K}_m(Y \times \mathbb{A}^1,\mu_n)$ and restrict
them to \( \{0\} \hookrightarrow \mathbb{A}^1 \), then the statement in this lemma will follows from the commutativity of the diagram

\[
\begin{array}{c}
\hat{K}_m(Y \times \mathbb{P}^1, \mu_n) \xrightarrow{\sim} \hat{K}_{Y \times \mathbb{P}^1}(P', \mu_n) \\
\downarrow s^*_t \quad \downarrow u^*_t \\
\hat{K}_m(Y, \mu_n) \xrightarrow{\sim} \hat{K}_{Y,m}(P, \mu_n)
\end{array}
\tag{13}
\]

where \( P' = \mathbb{P}((N_{X/Y} \boxtimes \mathcal{O}(1)) \oplus \mathcal{O}_{Y \times \mathbb{P}^1}) \) is the projective completion of \( N_{W/Y \times \mathbb{P}^1} \) over \( Y \times \mathbb{P}^1 \). It is equivalent to show that the following diagram

\[
\begin{array}{c}
\hat{K}_m(Y \times \mathbb{P}^1, \mu_n) \xrightarrow{j_0} \hat{K}_m(P, \mu_n) \\
\downarrow s^*_t \quad \downarrow u^*_t \\
\hat{K}_m(Y, \mu_n) \xrightarrow{i_0} \hat{K}_m(P, \mu_n)
\end{array}
\tag{14}
\]

is commutative because the morphism \( i_0 : \hat{K}_m(Y, \mu_n) \to \hat{K}_m(P, \mu_n) \) is injective. We endow \( N_{X/Y} \boxtimes \mathcal{O}(1) \) with the product metric coming from the metric on \( N_{X/Y} \) and the Fubini–Study metric on \( \mathcal{O}(1) \), then the pullback of \( N_{W/Y \times \mathbb{P}^1} \) along \( s_t \) is isometric to \( N_{X/Y} \), so that the pullback along \( s_t \) of the Koszul resolution and of the corresponding Bott–Chern singular current with respect to \( j_0 \) is exactly the Koszul resolution and the corresponding Bott–Chern singular current with respect to \( i_0 \). According to the construction of the homotopies defining \( j_0 \) and \( i_0 \), we get the commutativity of the diagram (14) and hence of (13). So, we are done.

\[\square\]

**Corollary 4.10.** For any \( \mathbb{Z} \)-point \( t \neq 0 \), the diagram

\[
\begin{array}{c}
\hat{K}_m(Y \times \mathbb{P}^1, \mu_n) \xrightarrow{j^*} \hat{K}_m(W, \mu_n) \\
\downarrow s^*_t \quad \downarrow u^*_t \\
\hat{K}_m(Y, \mu_n) \xrightarrow{i^*} \hat{K}_m(X, \mu_n)
\end{array}
\]

is commutative.

**Remark 4.11.** Using the same argument as in Lemma 4.9, we know that the diagram

\[
\begin{array}{c}
\hat{K}_m(Y \times \mathbb{P}^1, \mu_n) \xrightarrow{j^*} \hat{K}_m(W, \mu_n) \\
\downarrow s^*_t \quad \downarrow u^*_t \\
\hat{K}_m(Y, \mu_n) \xrightarrow{i_0^*} \hat{K}_m(P, \mu_n)
\end{array}
\]

is also commutative.

Next, we consider the commutative diagram

\[
\begin{array}{c}
W \\
\downarrow u_t \\
X \xrightarrow{f} S
\end{array}
\]

\[\text{shun_tang@tajima.math.kyoto-u.ac.jp}\]
with $\mathbb{Z}$-point $t \neq 0$ and we compare the map $f_{\ast} \circ u_{\ast}^t$ with the map $l_{\ast}$ from $\bar{K}_{m}(W, \mu_{n})\mathbb{Q}$ to $\bar{K}_{m}(S, \mu_{n})\mathbb{Q}$.

First, for any $\mu_{n}$-invariant Kähler metric $\omega^{X}$ on $X$ which induces an invariant Kähler metric $\omega^{Y}$ on $Y$, there exists a $\mu_{n}$-invariant Kähler metric $\omega^{W}$ on $W$ such that the restrictions of $\omega^{W}$ over $X \cong X \times \{t\}$ with $t \neq 0$ and to $Y \cong Y \times \{0\}$ are exactly $\omega^{X}$ and $\omega^{Y}$. This fact follows from [32, Lemma 3.5]. Actually, such a metric is constructed via the Grassmannian graph construction. In this construction, we have an embedding $W \to X \times \mathbb{P}^{r} \times \mathbb{P}^{1}$ and the metric $\omega^{W}$ is the $\mu_{n}$-average of the restriction of the metric on $X \times \mathbb{P}^{r} \times \mathbb{P}^{1}$. We fix such an invariant Kähler metric $\omega^{W}$ on $W$ and endow all submanifolds of $W$ with the induced metrics. Moreover, all normal bundles appearing in the construction of the deformation to the normal cone will be endowed with the quotient metrics.

Second, to the three divisors $u_{t}(X)$, $u_{0}(P)$ and $u_{0}(\tilde{X})$ in $W$, we have the following result.

**Lemma 4.12.** Over $W$, there are $\mu_{n}$-invariant hermitian metrics on $\mathcal{O}(X)$, $\mathcal{O}(P)$ and $\mathcal{O}(\tilde{X})$ such that the isometry $\mathcal{O}(X) \cong \mathcal{O}(P) \otimes \mathcal{O}(\tilde{X})$ holds and such that the restriction of $\mathcal{O}(X)$ over $X$ yields the metric of $N_{W/X}$, the restriction of $\mathcal{O}(\tilde{X})$ over $\tilde{X}$ yields the metric of $N_{W/\tilde{X}}$ and the restriction of $\mathcal{O}(P)$ over $P$ yields the metric of $N_{W/P}$.

**Proof.** Choose metric on $\mathcal{O}(P)$ in a small neighborhood of $P$ such that the restriction of $\mathcal{O}(P)$ over $P$ yields the metric of the normal bundle. Do the same for $\mathcal{O}(\tilde{X})$. Since $X$ is closed and disjoint from $\tilde{X}$ and $P$, we can extend these metrics via a partition of unity to metrics defined on $W$, so that the restriction of the metric that $\mathcal{O}(X)$ inherits from the isomorphism $\mathcal{O}(X) \cong \mathcal{O}(P) \otimes \mathcal{O}(\tilde{X})$ yields the metric of the normal bundle $N_{W/X}$. We then take the $\mu_{n}$-averages of these metrics to make them $\mu_{n}$-invariant. Since the metrics on $N_{W/X}$, $N_{W/P}$ and $N_{W/\tilde{X}}$ are already $\mu_{n}$-invariant, the $\mu_{n}$-invariant metrics on $\mathcal{O}(X)$, $\mathcal{O}(P)$ and $\mathcal{O}(\tilde{X})$ obtained as above have the properties that we require.\hfill $\square$

Now, consider the Koszul resolution

$$0 \to \mathcal{O}(-X) \to \mathcal{O}_{W} \to u_{t^*}\mathcal{O}_{X} \to 0.$$  

The associated equivariant singular Bott–Chern current $\mathcal{B}_{g}(W/X)$ satisfies the identity

$$d\mathcal{B}_{g}(W/X) = c_{0}\mathcal{O}_{W} - c_{0}\mathcal{O}(-X)) - u_{t^*}[c_{0}\mathcal{O}_{X}]Td_{g}^{-1}(N_{W/X}).$$

We claim the following result.

**Lemma 4.13.** For any element $x \in \bar{K}_{m}(W, \mu_{n})\mathbb{Q}$ with integer $m \geq 1$, the identity

$$f_{\ast} \circ u_{t^*}(x) - f_{\ast}(R_{g}(N_{W/X}) \cdot u_{t^*}x) = l_{\ast}(x) - l_{\ast}(\mathcal{O}(-X) \otimes x)$$

hold in $\bar{K}_{m}(S, \mu_{n})\mathbb{Q}$.

**Proof.** Let $\mathcal{E}$ be a $l$-acyclic hermitian $k$-cube in $\bar{P}(W, \mu_{n})$. Since $W$ admits a very ample invertible $\mu_{n}$-sheaf which is relative to the morphism $l : W \to S$ (cf. [32, Lemma 3.9]), we may assume that $\mathcal{O}(-X) \otimes \mathcal{E}$ is also $l$-acyclic and $u_{t^*}\mathcal{E}$ is $f$-acyclic. Then we have a short exact sequence of hermitian $k$-cubes in $\bar{P}(S, \mu_{n})$

$$\chi(\mathcal{E}) : 0 \to l_{\ast}(\mathcal{O}(-X) \otimes \mathcal{E}) \to l_{\ast}(\mathcal{E}) \to f_{\ast}(u_{t^*}\mathcal{E}) \to 0,$$
which will be regarded as a hermitian \((k+1)\)-cube and as a chain homotopy between the maps 
\(l_* - l_*(\overline{O}(-X) \otimes)\) and \(f_* \circ u_*^*\). Consequently, the formula

\[
H_k^{(1)}(E) = \frac{(-1)^{k+1}}{2(k+1)! (2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^k} c_0^0 (tr_{k+1} \circ \lambda(E)) \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2)
\]

defines a chain homotopy between \(ch_g \circ l_* - ch_g \circ l_*(\overline{O}(-X) \otimes)\) and \(ch_g \circ f_* \circ u_*^*\).

On the other hand, for any element \(\alpha \in \bigoplus_{p \geq 0} D^{2p-\bullet}((W_{\mu_0}, p) \cdot R_\alpha)\), the formula

\[
\frac{1}{(2\pi i)^r} \int_{W_{\mu_0}/S} T_g(W/X) \cdot Td_g(Tg) \cdot \alpha + \frac{1}{(2\pi i)^r} \int_{X_{\mu_0}/S} Td_g(N) \cdot Td_g^{-1}(N_{W/X}) \cdot \alpha
\]
gives a chain homotopy between the maps \(l_{\mu_0} \circ (Td_g(Tg) \cdot \bullet) - l_{\mu_0} \circ (Td_g(Tg)ch_g^0(\overline{O}(-X)) \cdot \bullet)\) and \(f_{\mu_0} \circ (Td_g(Tg) \cdot \bullet) - u_*^* \circ \bullet\). Hence, it defines a chain homotopy between \(l_{\mu_0} \circ (Td_g(Tg) \cdot \bullet) - l_{\mu_0} \circ (Td_g(Tg)ch_g^0(\overline{O}(-X)) \cdot \bullet)\) and \(f_{\mu_0} \circ (Td_g(Tg) \cdot \bullet) - u_*^* \circ \bullet\). Like before, using the projection formula and the fact that the deformation to the normal cone construction is base-change along smooth morphisms, we write the induced homotopy as

\[
H_k^{(2)}(E) = \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left( \left( \frac{1}{(2\pi i)^r} \int_{W_{\mu_0} \times (\mathbb{P}^1)^k/S \times (\mathbb{P}^1)^k} T_g(W/X) \cdot Td_g(Tg)ch_g^0(tr_k \circ \lambda(E)) \right) \wedge C_{k}(\log |z_1|^2, \ldots, \log |z_k|^2) \right) + \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left( \left( \frac{1}{(2\pi i)^r} \int_{X_{\mu_0} \times (\mathbb{P}^1)^k/S \times (\mathbb{P}^1)^k} Td_g(N) \cdot Td_g^{-1}(N_{W/X})ch_g^0(tr_k \circ \lambda(u_*^*E)) \right) \wedge C_k(\log |z_1|^2, \ldots, \log |z_k|^2) \right).
\]

Now, we denote by \(H_\chi(E)\) the following 2-cube of hermitian bundles on \(S \times (\mathbb{P}^1)^k\)

\[
\begin{align*}
&l_* (tr_k \circ \lambda(\overline{O}(-X) \otimes E)) \xrightarrow{\text{Id}} tr_k \circ \lambda (l_* (\overline{O}(-X) \otimes E)) \rightarrow 0 \\
&l_* (tr_k \circ \lambda(E)) \xrightarrow{\text{Id}} tr_k \circ \lambda (l_* (E)) \rightarrow 0 \\
&f_* (tr_k \circ \lambda(u_*^*E)) \xrightarrow{\text{Id}} tr_k \circ \lambda (f_* (u_*^*E)) \rightarrow 0 \\
\end{align*}
\]

and we set

\[
\tilde{H}_k(E) := \frac{(-1)^{k+2}}{2(k+2)! (2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} ch_0^0 (tr_2 \circ \lambda(H_\chi(E))) \wedge C_{k+2}(\log |z_1|^2, \ldots, \log |z_{k+2}|^2),
\]

it satisfies the differential equation

\[
d_D \circ \tilde{H}_k(E) = \tilde{H}_{k-1} \circ dE + \frac{(-1)^{k+1}}{2(k+1)! (2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} ch_0^0 (tr_{k+1} \circ \lambda(\chi(E))) \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2)
\]
which hold in $\{x\}$ and a homotopy between then our lemma follows from the Bimut–Ma immersion formula and the fact that there exists $x$ and corresponding Koszul resolutions $S, \mu \to O$ and a way to have $W \leftarrow P \to S$, $W \leftarrow X \to S$ and $W \leftarrow P \cap X \to S$ and corresponding Koszul resolutions

Now, we are ready to give the proof of Theorem 4.8. 

\[ \Phi_{k+1}^{\pm 1} \left( \frac{1}{(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \operatorname{ch}_g^0(\lambda(\lambda(\lambda(\lambda(\lambda(\lambda)))))) \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2) \right) \]

\[ - \Pi_k^O(\overline{\mathcal{O}}(-X) \otimes O) + \Pi_k^O(u_1^* \mathcal{E}) \]

\[ = \Phi_{k-1} \circ dE + \Phi_k^O(\overline{\mathcal{O}}(-X) \otimes O) + \Pi_k^O(u_1^* \mathcal{E}) \]

\[ - \left( -1 \right)^{k+1} \frac{1}{(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \operatorname{ch}_g^0(\lambda(\lambda(\lambda(\lambda(\lambda(\lambda)))))) \wedge C_{k+1}(\log |z_1|^2, \ldots, \log |z_{k+1}|^2) \]

Similar to the tricks that we used frequently before, we set

\[ \Phi_{k}^{(2)}(\mathcal{E}) = \left( -1 \right)^{k+1} \frac{1}{(k+1)! (2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1} \left( \frac{1}{(2\pi i)^{\tau_1}} \int_{W_{\mu_n} \times (\mathbb{P}^1)^k \otimes \mathcal{O}(\mathbb{P}^1)^k} T_g(W/X) \right) \]

\[ \bullet \operatorname{Td}_g(\overline{\mathcal{O}}(-X)) \operatorname{ch}_g^0(\lambda(\lambda(\lambda(\lambda(\lambda(\lambda)))))), \log |z_1|^2, \ldots, \log |z_k|^2 \]

then our lemma follows from the Bimut–Ma immersion formula and the fact that there exists a homotopy between $\Phi_{k}^{(2)}(\mathcal{E})$ and $\Phi_{k}^{(2)}(\mathcal{E})$. So, we are done. 

Remark 4.14. Similar to Lemma 4.13, we consider other three divisors $W \leftarrow P \to S$, $W \leftarrow X \to S$ and $W \leftarrow P \cap X \to S$ and corresponding Koszul resolutions

\[ 0 \to \overline{\mathcal{O}}(-P) \to \overline{\mathcal{O}}(W) \to u_0^* \mathcal{O}_P \to 0, \]

\[ 0 \to \overline{\mathcal{O}}(-X) \to \overline{\mathcal{O}}(W) \to u_0^* \mathcal{O}_X \to 0, \]

and

\[ 0 \to \overline{\mathcal{O}}(-X) \otimes \overline{\mathcal{O}}(-P) \to \overline{\mathcal{O}}(-X) \otimes \overline{\mathcal{O}}(-P) \to \overline{\mathcal{O}}(W) \to u_0^* \mathcal{O}_{X \cap P} \to 0. \]

Then, for any element $x \in \widehat{K}_m(W, \mu_n)_\mathbb{Q}$, we have

\[ p_* \circ u_0^*(x) - p_* (R_g(N_{W/P}) \cdot u_0^* x) = l_*(x) - l_*(\overline{\mathcal{O}}(-P) \otimes x), \]

\[ h_{1*} \circ u_0^*(x) - h_{1*} (R_g(N_{W/X}) \cdot u_0^* x) = l_*(x) - l_*(\overline{\mathcal{O}}(-X) \otimes x), \]

and

\[ h_{2*} \circ u_0^*(x) - h_{2*} (R_g(N_{W/P \cap X}) \cdot u_0^* x) = l_*(x) - l_*(\overline{\mathcal{O}}(-P) \otimes x) - l_*(\overline{\mathcal{O}}(-X) \otimes x) + l_*(\overline{\mathcal{O}}(-P) \otimes \overline{\mathcal{O}}(-X) \otimes x) \]

which hold in $\widehat{K}_m(S, \mu_n)_\mathbb{Q}$. 

Now, we are ready to give the proof of Theorem 4.8.
Proof. Let \( x \) be an element in \( \hat{K}_m(Y, \mu_n)_\mathbb{Q} \), we consider the following two diagrams

\[
\begin{array}{c}
Y \times \mathbb{P}^1 \xrightarrow{j} W \\
\downarrow s_t \quad \quad \quad \downarrow u_t \\
Y \quad \quad x \quad \quad X \\
\downarrow i \quad \quad \quad \downarrow h \\
S
\end{array}
\]

with \( \mathbb{Z} \)-point \( t \neq 0 \) and

\[
\begin{array}{c}
Y \times \mathbb{P}^1 \xrightarrow{j} W \\
\downarrow s_t \quad \quad \quad \downarrow u_t \\
Y \quad \quad x_0 \quad \quad \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y) \\
\downarrow i_0 \quad \quad \quad \downarrow h \\
S
\end{array}
\]

By Corollary 4.10 and the fact that \( s_t \) is a section of the obvious projection \( Pr \) from \( Y \times \mathbb{P}^1 \) to \( Y \), we have \( i_*(x) = u_t^* \circ j_* \circ Pr^*(x) \) and hence \( l_* \circ i_*(x) = l_* \circ u_t^* \circ j_* \circ Pr^*(x) \). According to Lemma 4.13,

\[
l_* \circ u_t^*(j_*Pr^*x) = h_*(j_*Pr^*x) - h_*(\overline{O}(-X) \otimes j_*Pr^*x) + l_*(R_g(N_{W/X}) \cdot i_*x).
\]

Similarly, we have

\[
l_* \circ u_0^*(j_*Pr^*x) = h_*(j_*Pr^*x) - h_*(\overline{O}(-P) \otimes j_*Pr^*x) + p_*(R_g(N_{W/P}) \cdot i_*x).
\]

Note that the image \( j(Y \times \mathbb{P}^1) \) does not meet \( \tilde{X} \), the localization sequence of the higher equivariant arithmetic K-groups implies that \( u_0^*(j_*Pr^*x) \) vanishes in \( \hat{K}_m(\tilde{X}, \mu_n)_\mathbb{Q} \) and in \( \hat{K}_m(P \cap \tilde{X}, \mu_n)_\mathbb{Q} \), so that

\[
h_*(\overline{O}(-X) \otimes j_*Pr^*x) = h_*(\overline{O}(-P) \otimes j_*Pr^*x).
\]

This can be seen from the several identities mentioned in Remark 4.14. On the other hand,

\[
R_g(N_{W/X}) \cdot i_*x = R_g(N_{W/X})ch_g(i_*x) = R_g(N_{W/X})i_*(\text{Td}^{-1}_g(N_{X/Y})ch_g(x))
\]

\[
= i_*(i^*R_g(N_{W/X})\text{Td}^{-1}_g(N_{X/Y})ch_g(x)) = 0.
\]

The same reasoning gives that \( R_g(N_{W/P}) \cdot i_*x = 0 \) also. So, \( l_* \circ i_*(x) \) is actually equal to \( p_\circ i_0(x) \). Therefore, the statement in Theorem 4.8 follows from Corollary 4.7.

\[\square\]

4.3. Proof of the statement

In this subsection, we give a complete proof of Theorem 4.2. Denote by \( i \) the closed immersion \( X_{\mu_n} \rightarrow X \), then the arithmetic concentration theorem (cf. [34, Theorem 5.2]) tells us that

\[
i_* : \hat{K}_m(X_{\mu_n}, \mu_n)_\mathbb{P} \cong \hat{K}_m(X, \mu_n)_\mathbb{P}
\]

with inverse map \( \otimes \lambda_{-1}^-(N_{X/X_{\mu_n}}) \circ \tau \).

Then let \( x \) be any element in \( \hat{K}_m(X, \mu_n) \), we apply Theorem 4.8 to the morphisms \( i, f \) and \( f_{\mu_n} = f \circ i \) and we compute

\[
f_*(x) = f_*(i_* \circ \otimes \lambda_{-1}^-(N_{X/X_{\mu_n}}) \circ \tau(x))
\]

\[
= f_* \circ i_*(\otimes \lambda_{-1}^-(N_{X/X_{\mu_n}}) \circ \tau(x))
\]

\[
= f_{\mu_n}(\otimes \lambda_{-1}^-(N_{X/X_{\mu_n}}) \circ \tau(x)) - f_{\mu_n}(\otimes R_g(N_{X/X_{\mu_n}}) \circ \otimes \lambda_{-1}^-(N_{X/X_{\mu_n}}) \circ \tau(x))
\]

\[
= f_{\mu_n}(\Lambda_R \circ \tau(x))
\]

which holds in \( \hat{K}_m(Y, \mu_n)_\mathbb{P} \otimes \mathbb{Q} \). This completes the proof of Theorem 4.2.
Appendix. Remarks on the equivariant analytic torsion forms and the immersion formula
by Xiaonan Ma

A.1. Anomaly formula for the equivariant analytic torsion forms

Let $W, V$ be two $\mu_n$-projective complex manifolds, and let $f : W \to V$ be an equivariant, holomorphic submersion with fiber $X$. Fix a $\mu_n$-invariant Kähler metric on $W$ and choose corresponding Kähler form $\omega$ as a Kähler fibration structure on $f$. We fix a primitive $n$th root of unity $g$ as a generator of $\mu_n(\mathbb{C})$. In the following, $\text{ch}_g$ and $\text{Td}_g$ should stand for the usual Chern–Weil forms with the factor $2\pi i$ in their definitions. Note that they are denoted by $\text{ch}'_g$ and $\text{Td}'_g$ in the text.

Let $(E, h^E)$ be a $\mu_n$-equivariant hermitian vector bundle on $W$ such that $E$ is $f$-acyclic. Let $T_g(f, \omega, h^E) \in \bigoplus_{p \geq 0} A^{p,p}(V_{\mu_n})$ be the equivariant analytic torsion form [27, (2.27)] which satisfies the differential equation

$$\frac{\bar{\partial}\partial}{2\pi i} T_g(f, \omega, h^E) = \text{ch}_g(f_* E, f_* h^E) - \int_{W_{\mu_n}/V_{\mu_n}} \text{Td}_g(T_f, h^{T_f}) \text{ch}_g(E, h^E),$$

where $h^{T_f}$ is the hermitian metric induced by $\omega$ on the holomorphic tangent bundle $T_f$. We shall write $T_g(\omega, h^E)$ for $T_g(f, \omega, h^E)$, if there is no ambiguity about the underlying map. The following result is [27, Theorem 2.13] which extends [7, Theorem 2.5; 11, Theorem 1.23; 12, Theorem 3.10].

**THEOREM A.1 (Anomaly formula).** Let $\omega'$ be the form associated to another Kähler fibration structure on $f : W \to V$. Let $h'^{T_f}$ be the metric on $T_f$ induced by $\omega'$. Then the following identity holds in $\bigoplus_{p \geq 0} A^{p,p}(V_{\mu_n})/(\text{Im } \partial + \text{Im } \bar{\partial})$:

$$T_g(\omega, h^E) - T_g(\omega', h^E) = \text{ch}_g(f_* E, h^{f,*} E, h'^{f,*} E)$$

$$+ \int_{W_{\mu_n}/V_{\mu_n}} \text{Td}_g(T_f, h^{T_f}, h'^{T_f}) \text{ch}_g(E, h^E),$$

where $(f_* E, h^{f,*} E, h'^{f,*} E)$ and $(T_f, h^{T_f}, h'^{T_f})$ stand for the exact sequences of hermitian vector bundles

$$0 \to (f_* E, h^{f,*} E) \xrightarrow{\text{Id}} (f_* E, h'^{f,*} E) \to 0 \to 0$$

and

$$0 \to (T_f, h^{T_f}) \xrightarrow{\text{Id}} (T_f, h'^{T_f}) \to 0 \to 0.$$  

We shall see that there is a natural way to write down explicitly some differential forms $\Delta^0(f, E, \omega, \omega')$, $\Delta_0(f, E, \omega, \omega')$ such that they are functorial in certain sense and they measure the difference of the anomaly formula.

$$\Delta = \partial \Delta^0(f, E, \omega, \omega') + \bar{\partial} \Delta_0(f, E, \omega, \omega')$$

$$= T_g(\omega, h^E) - T_g(\omega', h^E) + \text{ch}_g(f_* E, h^{f,*} E, h'^{f,*} E) - \int_{W_{\mu_n}/V_{\mu_n}} \text{Td}_g(T_f, h^{T_f}, h'^{T_f}) \text{ch}_g(E, h^E).$$

To do so, we need to fix the construction of $\text{ch}_g(f_* E, h^{f,*} E, h'^{f,*} E)$, $\text{Td}_g(T_f, h^{T_f}, h'^{T_f})$ at the differential form level, that is., without modulo $\text{Im } \partial + \text{Im } \bar{\partial}$. Let us fix the definition of $\text{ch}_g(f_* E, h^{f,*} E, h'^{f,*} E)$ as the left side of [7, (2.42)], and the definition $\text{Td}_g(T_f, h^{T_f}, h'^{T_f})$ as the integral for 0 to 1 for the parameter $c$ of the differential form as the part $\frac{\partial}{\partial b} \cdots$ via the last
term of [12, (3.67)], note that we can also fix the path of the metric as the segment direct connecting two metrics. Thus, we can write them under the notation in [27, (2.34), (2.56)] (cf. also the convention before Theorem 2.5 of this paper),

\[ \tilde{\text{ch}}_b(f_\ast E, h_{I-E}, h'_{I-E}) = \int_0^1 \Phi \text{Tr}_s[gQ_c^{H(X,E|X)} \exp(-(\nabla_c^{H(X,E|X)}))^2]dc, \]

\[ \tilde{\text{Td}}_g(Tf, h^Tf, h'^Tf) = \int_0^1 \frac{\partial}{\partial b} \left[ \text{Td}\left( \frac{-R_c^{TXg}}{2i\pi} - b(h^{TXg})^{-1} \frac{\partial h^{TXg}}{\partial c} \right) \right] \]

\[ \times \prod_{j=1}^q \text{Td} \left( \frac{-R_c^{Xg/X}}{2i\pi} - b(h^{N_{Xg/X}})^{-1} \frac{\partial h^{N_{Xg/X}}}{\partial c} + i\theta_j \right) \bigg|_{b=0} dc. \] (A.1)

Let \( V_1 \) be an equivariant closed submanifold of \( V \), and let \( W_1 = f^{-1}(V_1) \subset W \) be the closed submanifold of \( W \) with restricted Kähler metric. Then \( f_1 : W_1 \to V_1 \) is also an equivariant holomorphic submersion with compact fiber. Denote by \( j \) (respectively, \( i \)) the natural embedding \( W_1 \to W \) (respectively, \( V_1 \to V \)) and by \( \omega, \omega' \) the induced Kähler forms \( j^* \omega, j^* \omega' \). Let \( \overline{E} \) be an \( f \)-acyclic hermitian bundle on \( W \).

**Theorem A.2.** There is a natural way to write down explicitly differential forms \( \Delta^0(f, \overline{E}, \omega, \omega') \) such that \( \Delta = \partial \Delta^0(f, \overline{E}, \omega, \omega') + \bar{\partial} \Delta^0(f, \overline{E}, \omega, \omega') \) and they are functorial in the following sense.

\[ i_{\mu_n}^* \Delta^0(f, \overline{E}, \omega, \omega') = \Delta^0(f_1, j^* \overline{E}, \omega_1, \omega'_1) \] (A.2)

and

\[ i_{\mu_n}^* \Delta_0(f, \overline{E}, \omega, \omega') = \Delta_0(f_1, j^* \overline{E}, \omega_1, \omega'_1). \] (A.3)

**Proof.** By the equivariant extension of [12, Definition 3.14, Theorems 3.16 and 3.17] (cf. [27, (2.34)]), there exist differential forms \( \vartheta^1, \vartheta^2 \) and \( \vartheta^3 \) such that

\[ \Delta + d\varrho = \partial \vartheta^1 + \bar{\partial} \vartheta^2 + \bar{\partial} \vartheta^3 \] (A.4)

and \( d\varrho \) is from the last term of [12, (3.38)], in particular, \( \varrho \) is a local term from the small time heat kernel asymptotics of Bismut superconnection, \( \vartheta^k (k = 1, 2, 3) \) have universal expression in terms of \( g, \omega, \omega' \) and \( h^\overline{E} \) via the Bismut superconnection. Thus, from [12, Definition 3.14, Theorem 3.16] and [27, (2.34)], we know that if \( i : V_1 \to V \) is a complex submanifold of \( V \), when we consider the corresponding objects for the submersion \( f_1 \), each above term is the restriction of the corresponding term for the global submersion \( f \). Thus, let \( \Delta_1, \vartheta^k_1 \) be the corresponding terms associated to the fibration \( f_1 : W_1 \to V_1 \), then we have \( \Delta_1 = i_{\mu_n}^* \Delta_1 \) and \( \vartheta^k_1 = i_{\mu_n}^* \vartheta^k \). So, write \( \Delta^0(f, \overline{E}, \omega, \omega') = \vartheta^1 - \varrho \) and \( \Delta_0(f, \overline{E}, \omega, \omega') = \vartheta^1 + \bar{\partial} \varrho \), we are done. \( \square \)

**A.2. Functoriality of the equivariant analytic torsion forms**

Let \( W, V \) and \( S \) be three \( \mu_n \)-equivariant projective complex manifolds with \( S = S_{\mu_n} \). Suppose that \( f : W \to V \) and \( h : V \to S \) are two holomorphic submersions with compact fibers \( X, Y \). Then \( h \circ f \) is also a holomorphic submersion with compact fiber \( Z \).

Let \( \omega^W \) and \( \omega^V \) be two \( \mu_n \)-invariant Kähler forms on \( W \) and on \( V \). As usual, \( \omega^W \) and \( \omega^V \) decide Kähler fibration structures on the morphisms \( f, h \) and \( g \) and they induce \( \mu_n \)-invariant hermitian metrics associated with the Kähler forms \( \omega^X, \omega^Y \) and \( \omega^Z \) on relative tangent
bundles $Tf, Th$ and $T(h \circ f)$. Consider the following short exact sequence of hermitian vector bundles
\[ \mathcal{T}(f, h, h \circ f) : 0 \to Tf \to \mathcal{T}(h \circ f) \to f^*Th \to 0. \]

Denote by $\operatorname{Td}_g(\mathcal{T}(f, h, h \circ f))$ the equivariant secondary Todd form, it satisfies the differential equation
\[ \frac{\partial \vartheta}{2\pi i} \operatorname{Td}_g(\mathcal{T}(f, h, h \circ f)) = \operatorname{Td}_g(\mathcal{T}(h \circ f)) - f^* \operatorname{Td}_g(Th) \operatorname{Td}_g(Tf). \]

Now, let $\mathcal{E}$ be a hermitian vector bundle on $W$, we shall assume that $E$ is $f$-acyclic and $h \circ f$-acyclic. Then the Leray spectral sequence $E_2^{p,q} = R^p h_* (R^q f_* E)$ degenerates at $E_2$, so that $f_* E = R^0 f_* (E)$ is $h$-acyclic and $(h \circ f)_* E \cong h_* f_* E$. Clearly, $(h \circ f)_* E$ and $h_* f_* E$ carry in general different $L^2$ metrics (Note that for $\sigma \in ((h \circ f)_* E)_b$, $b \in S$, $\| \sigma \|^2_{(h \circ f)_* E} = (2\pi)^{-\dim Z} \int_{Z_b} |\sigma|^2 (\Omega^Z)^{\dim Z} (\dim Z)!$, (A.5)
\[ \| \sigma \|^2_{{h_* f_* E}} = (2\pi)^{-\dim Z} \int_{Y_b} \left( \int_X |\sigma|^2 (\Omega^X)^{\dim X} (\dim X)! \right) (\Omega^Y)^{\dim Y} (\dim Y)!, \]
thus they are different in general). Consider the following short exact sequence of hermitian vector bundles
\[ \mathcal{E}(f, h, h \circ f) : 0 \to h_* f_* \mathcal{E} \to (h \circ f)_* \mathcal{E} \to 0 \to 0. \]

The equivariant secondary Bott–Chern form $\tilde{\operatorname{ch}}_g(\mathcal{E}(f, h, h \circ f))$ satisfies the differential equation
\[ \frac{\partial \vartheta}{2\pi i} \tilde{\operatorname{ch}}_g(\mathcal{E}(f, h, h \circ f)) = \operatorname{ch}_g((h \circ f)_* \mathcal{E}) - \operatorname{ch}_g(h_* f_* \mathcal{E}). \]

**THEOREM A.3.** Let notations and assumptions be as above. Then the following identity holds in $\bigoplus_{p \geq 0} A^{p,p}(S)/(\operatorname{Im} \partial + \operatorname{Im} \bar{\partial})$
\[ T_g(h \circ f, \omega^W, h^E) - T_g(h, \omega^V, h^L_* E) - \int_{V_{\mu_n}/S} \operatorname{Td}_g(Th) T_g(f, \omega^W, h^E) \]
\[ = \tilde{\operatorname{ch}}_g(\mathcal{E}(f, h, h \circ f)) - \int_{W_{\mu_n}/S} \tilde{\operatorname{Td}}_g(T(f, h, h \circ f)) \operatorname{ch}_g(\mathcal{E}). \]

**Proof (a sketch).** This is a natural extension of [26, Theorem 3.5] to the equivariant case, or the family extension of [27, Theorem 3.1] which is an equivariant extension of [4, Theorem 3.1]. To prove this extension, one may follow the same approach as in [26, Sections 4–9]. In fact, as a purely functional analysis argument, the [26, Theorems 4.5–4.7] can be extended formally to the equivariant case by introducing in the right place the operator $g$. The reason one can do this formal extension has been given in [27, Section 5]. For the equivariant extensions of [26, Theorems 4.8–4.11], one can show that their proofs are local on $f^{-1}(V_{\mu_n})$ and certain rescaling on Clifford variables which does not effect the action of $g$ can be made (cf. [26, Section 7 b]). Replacing the equivariant local index technique in [27, Sections 7–9] by its equivariant relative local index, one gets the desired identity.

To help the readers, we will use directly the notation in [26, Section 4]. By the anomaly formula Theorem 2.11, we only need to establish Theorem A.3 for a special couple of Kähler forms, thus we will assume that $\omega^W = \tilde{\omega}^W + f^* \omega^V$ with $\tilde{\omega}^W$ a Kähler form on $W$. 

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Let $\Delta$ be the rectangular domain in $\mathbb{R}^2$ with coordinates $(u, T)$, defined by the four vertices $(1, \varepsilon), (T_0, \varepsilon), (T_0, A), (1, A)$, following [26, (4.7)], set
\begin{align*}
\theta_1^0 &= (2\pi i)^{-1/2} \int_\Delta \frac{2}{u} \frac{\partial}{\partial b} \left\{ \varphi \text{Tr}_s \left[ g \left[ B^s_{3,u^2,T}, N_{3,u^2,T} \right] \exp(-B^2_{3,u^2,T} - bM_{3,u^2,T}) \right] \right\}_{b=0} dt du,
\theta_2^0 &= (2\pi i)^{-1/2} \int_\Delta \frac{2}{u} \frac{\partial}{\partial b} \left\{ \varphi \text{Tr}_s \left[ g \left[ B''_{3,u^2,T}, N_{3,u^2,T} \right] \exp(-B^2_{3,u^2,T} - bM_{3,u^2,T}) \right] \right\}_{b=0} dt du, \\
\theta_3^0 &= (2\pi i)^{-1} \int_\Delta \frac{2}{u} \frac{\partial}{\partial b} \left\{ \varphi \text{Tr}_s \left[ g N_{3,u^2,T} \exp(-B^2_{3,u^2,T} - bM_{3,u^2,T}) \right] \right\}_{b=0} dt du.
\end{align*}
The only difference comparing with [26, (4.7)] is that in (A.7), we add the operator $g$ as the first term in $\text{Tr}_s[\cdots]$ in [26, (4.7)], that is, replace $\text{Tr}_s[\cdots]$ by $\text{Tr}_s[g \cdots]$. Note that $B_{3,u^2,T}$ is the Bismut superconnection associated with the submersion $h \circ f$ and the form $\omega_1^W = \frac{1}{4\pi} \tilde{\omega}^W + f^*\omega^V$, and $B''_{3,u^2,T}, B_{3,u^2,T}$ are holomorphic and anti-holomorphic part of $B_{3,u^2,T}$. Moreover, $N_{3,u^2,T}$ is a generalized number operator associated with $\omega_1^W$.

The boundary of $\Delta$ composes as four oriented segments $\Gamma_1, \ldots, \Gamma_4$. Let $I_k^0$ be the integral of the one form on $\mathbb{R}^2$ with values in $\Lambda^\bullet(T^*_\mathbb{R} S)$ defined by replacing $\text{Tr}_s[\cdots]$ by $\text{Tr}_s[g \cdots]$ in [26, Definition 4.2], then we have the $g$-analogue of [26, (4.8)]:
\begin{equation}
\sum_{k=1}^4 I_k^0 = \partial \theta_1^0 - \partial \theta_2^0 - \partial \theta_3^0. \tag{A.8}
\end{equation}
We study the terms $I_k^0$ and $\theta_j^0$ in succession as $A \to +\infty, T_0 \to +\infty, \varepsilon \to 0$: roughly, we get
\begin{itemize}
  \item the term $-T_g(h, \omega^V, h^f, h)$ from $I_1^0$;
  \item a differential form version of $-\text{ch}_g(E(f, h, h \circ f))$ (via [9, (1.58)] or [26, (4.17)]) by replacing $\text{Tr}_s[\cdots]$ by $\text{Tr}_s[g \cdots]$ from $I_2^0$;
  \item $T_g(h \circ f, \omega^W, h^E)$ from $I_3^0$;
  \item $- \int_{W_{\varepsilon, \varepsilon}/S} T_{d_g} (\overline{T} h) T_g(f, \omega^W, h^E) + \int_{W_{\varepsilon, \varepsilon}/S} \overline{T}_{d_g}(\overline{T}(f, h, h \circ f)) \text{ch}_g(E)$ (here we should use the differential form version of $\overline{T}_{d_g}(\overline{T}(f, h, h \circ f))$ from the term $\int_1^\infty \cdots$ in [4, (4.72)] by replacing $Td$ therein by $T_{d_g}$) from $I_4^0$.
\end{itemize}

Let $\theta_3^j (j = 1, 2, 3)$ be the differential forms on $S$ obtained from $\theta_j^0$ by the above procedure, then the difference of two sides in (A.6) (by using the differential form versions of $\text{ch}_g(E(f, h, h \circ f)$ and $\overline{T}_{d_g}(\overline{T}(f, h, h \circ f)$) as above) is
\begin{equation}
\Delta + d^\delta \Theta = \partial \theta_1^3 - \partial \theta_2^3 - \partial \theta_3^3 \tag{A.9}
\end{equation}
and $\theta_k^3 (k = 1, 2, 3)$ have universal expressions via the Bismut superconnection $B_{3,u^2,T}$, $\Theta$ is a combination of local terms from the small time heat kernel asymptotics of the Bismut superconnection for the fibration $h$ and $h \circ f$ (cf. [26, (4.27) and (4.29); 27, (2.24) and (2.27)]).

Let $\theta_k^3 (k = 1, 2, 3)$ be the form in (A.9) associated with the couple $\omega^M = \tilde{\omega}^W + f^*\omega^V, \omega^V$. Set
\begin{equation}
\Delta^0(f, h, \omega^W, \omega^V, E) = -\theta_3^3 - \Theta \quad \text{and} \quad \Delta_0(f, h, \omega^W, \omega^V, E) = \theta_1^3 - \theta_3^3 - \Theta. \tag{A.10}
\end{equation}
Then when we fix the differential form versions of \( \tilde{\chi}_g(\mathcal{E}(f,h,h \circ f)) \) and \( \tilde{T}_d_g(\mathcal{T}(f,h,h \circ f)) \) as above, (A.10) measure the difference of the formula (A.6) at the differential form level from (A.9):

\[
\Delta = \partial \Delta^0(f,h,\omega^W,\omega^V,\mathcal{E}) + \bar{\partial} \Delta^0(f,h,\omega^W,\omega^V,\mathcal{E})
\]

\[
= T_g(h \circ f,\omega^W,h^E) - T_g(h,\omega^V,h^{f,E}) - \int_{W_{\mu_n}/S} T_d_g(\mathcal{TT})T_g(f,\omega^W, h^E)
\]

\[-\tilde{\chi}_g(\mathcal{E}(f,h,h \circ f)) + \int_{W_{\mu_n}/S} \tilde{T}_d_g(\mathcal{T}(f,h,h \circ f))\chi_g(\mathcal{E}). \quad (A.11)\]

Let \( S_1 \) be a closed submanifold of \( S \), and let \( V_1 = h^{-1}(S_1) \subset V \) (respectively, \( W_1 = (h \circ f)^{-1}(S_1) \subset W \)) be the closed submanifold of \( V \) (respectively, \( W \)) with restricted Kähler metric. Then \( f_1 : W_1 \to V_1, h_1 : V_1 \to S_1 \) and \( h \circ f_1 : W_1 \to S_1 \) also form a triple of equivariant holomorphic submersions with compact fibers. Denote by \( j \) (respectively, \( i \)) the natural embedding \( W_1 \to W \) (respectively, \( V_1 \to V \)) and by \( \omega^{W_1}, \omega^{V_1} \) the induced Kähler forms \( j^*\omega^W, i^*\omega^V \). Denote by \( l \) the embedding \( S_1 \to S \). Let \( \mathcal{E} \) be an \( f \)-acyclic and \( h \circ f \)-acyclic hermitian bundle on \( W \).

**Theorem A.4.** The forms \( \Delta^0(f,h,\omega^W,\omega^V,\mathcal{E}) \) and \( \Delta_0(f,h,\omega^W,\omega^V,\mathcal{E}) \) are functorial in the following sense that

\[
l^* \Delta^0(f,h,\omega^W,\omega^V,\mathcal{E}) = \Delta^0(f_1,h_1,\omega^{W_1},\omega^{V_1}, j^* \mathcal{E})
\]

and

\[
l^* \Delta_0(f,h,\omega^W,\omega^V,\mathcal{E}) = \Delta_0(f_1,h_1,\omega^{W_1},\omega^{V_1}, j^* \mathcal{E}).
\]

**Proof.** Note that the square of the Bismut superconnection is a second-order fiberwise elliptic operator with differential form coefficients [5, Theorem 3.6] (cf. also [3, Theorem 10.17]), in particular, its heat kernel along the fibers is well defined, and in (A.7), the terms \([B_{3,u^2},T], [N_{3,u^2},T] \) are first-order differential operators along the fiber, the terms \( N_{3,u^2}, T, M_{3,u^2}, T \) are tensors, thus we see clearly that when we consider the corresponding objects for the submersion \( h_1 \circ f_1 \), each above term is the restriction of the corresponding term for the global submersion \( h \circ f \).

We obtain that if \( l : S_1 \to S \) is a complex submanifold of \( S \), and \( \theta_{0,1}^0, \Theta_1 \) are the corresponding terms associated to the relevant fibrations, then we have

\[
\Theta_1 = l^* \Theta, \quad \theta_{k,1}^0 = l^* \theta_k^0 \quad (k = 1, 2, 3). \quad (A.12)
\]

Now, we make the procedure as \( A \to +\infty, T_0 \to +\infty, \varepsilon \to 0 \), to get \( \theta_{k,1}^0 \), then from (A.12), we get \( \theta_{k,1}^0 = l^* \theta_k^0 \) \((k = 1, 2, 3)\). Combining it with (A.10), we get Theorem A.4. \( \Box \)

Now, for a general \( \omega^W \), as we can use the anomaly formula for the trip \((h \circ f, \omega^W, \omega^W + f^* \omega^V) \), in particular, its differential form version as in Section A.1, we can still define \( \Delta^0(f,h,\omega^W,\omega^V,\mathcal{E}) \) and \( \Delta_0(f,h,\omega^W,\omega^V,\mathcal{E}) \) such that Theorem A.4 and (A.11) still hold, again we need to fix a differential form version of \( \tilde{T}_d_g(\mathcal{T}(f,h,h \circ f)) \).

**A.3. Immersion formula**

Let \( V, W \) be two \( \mu_n \)-equivariant projective complex manifolds and let \( i : W \to V \) be an equivariant closed immersion. Let \( S \) be a compact complex manifold with trivial \( \mu_n \)-action, and let \( f : W \to S, l : V \to S \) be two equivariant holomorphic submersions with fibers \( Y, X \) such that \( f = l \circ i \). Assume that \( \mathfrak{e} \) is an equivariant hermitian bundle on \( W \) and \((\tilde{\xi}, v)\) is a
complex of equivariant hermitian bundles on $V$ which provides a resolution of $i_*\eta$ such that the metrics on $\xi$, satisfy the Bismut’s assumption (A). Let $\omega^W$, $\omega^V$ be two Kähler fibrations on $f$ and on $l$, respectively. We shall assume that $\omega^W$ is the pullback of $\omega^V$, so that the Kähler metric on $W$ is induced by the Kähler metric on $V$. Consider the following exact sequence

$$\overline{\mathcal{N}}: \quad 0 \to T\overline{f} \to T\overline{l} \mid_W \to \overline{\mathcal{N}}_{X/Y} \to 0,$$

where $\overline{\mathcal{N}}_{X/Y}$ is endowed with the quotient metric. Then the equivariant secondary Todd form of $\overline{\mathcal{N}}$ satisfies the identity

$$\frac{\overline{\partial}\overline{\partial}}{2\pi i} T_d_g(\overline{\mathcal{N}}) = T_d_g(Tl \mid_W, h^Tl) - T_d_g(Tf, h^Tf)T_d_g(\overline{\mathcal{N}}_{X/Y}).$$

We suppose that in the resolution $\xi_\eta$, $\xi_j$ are all $l$–acyclic and moreover $\eta$ is $f$–acyclic.

Let $T_g(\omega^V, h^\xi)$ be the equivariant analytic torsion forms associated with the family of relative Dolbeault double complexes $(\Omega(X, \xi |_X), \overline{\partial}^X + v)$. Let $h^{H(X, \xi |_X)}$ be the corresponding $L_2$ metric on the hypercohomology $H(X, \xi |_X)$ of $\xi |_X$.

Note that under our assumption, $H(X, \xi |_X) \simeq f_*\eta$. And we have the following exact sequence of hermitian vector bundles on $S$

$$\Xi: \quad 0 \to l_* (\overline{\xi}_m) \to l_*(\overline{\xi}_{m-1}) \to \ldots \to l_*(\overline{\xi}_0) \to H(X, \xi |_X) \to 0.$$  

We can split $\Xi_i$ into a family of short exact sequence of hermitian bundles from $j = 1$ to $m$

$$\chi_j: \quad 0 \longrightarrow \text{Ker } d_j \longrightarrow \Xi_j \xrightarrow{d_j} \text{Ker } d_{j-1} \longrightarrow 0$$

such that the kernel of every map $d_{j-1}$ for $j = 2, \ldots, m$ carries the metric induced by $\Xi_j$ and Ker $d_0 = \Xi_0 = H(X, \xi |_X)$, Ker $d_m = \Xi_{m+1} = l_*(\overline{\xi}_m)$.

Then it satisfies the differential equation

$$\frac{\overline{\partial}\overline{\partial}}{2\pi i} \tilde{c}_g(\Xi_j) = c_h_g(H(X, \xi |_X)) - \sum_{j=0}^{m} (-1)^{j} c_h_g(l_*(\xi_j)).$$

The following result is the combination of [14, Theorems 0.1 and 0.2] which is an equivariant extension of [8, Theorems 0.1 and 0.2], and a families extension of [7, Theorem 0.1; 13, Theorem 0.1].

Let $R_g$ be the equivariant R-genus of Bismut [6].

**Theorem A.5** (Immersion formula). The following identity holds in $\bigoplus_{p \geq 0} A_p(S)/(\text{Im } \partial + \text{Im } \bar{\partial})$.

$$T_g(\omega^V, h^\xi) - T_g(\omega^W, h^\eta) + \tilde{c}_g(f_*\eta, h^{H(X, \xi |_X)}, h^{f_*\eta}) = \int_{\nu_\mu / S} T_d_g(T\overline{f})T_g(\overline{\xi})$$

$$- \int_{\nu_\mu / S} \tilde{T}_d_g(\overline{\mathcal{N}})T_d_g^{-1}(\overline{\mathcal{N}}_{X/Y})c_h_g(\overline{\eta}) + \int_{\nu_\mu / S} T_d_g(T\overline{f})R_g(\overline{\mathcal{N}}_{X/Y})c_h_g(\overline{\eta}), \quad (A.13)$$

$$T_g(\omega^V, h^\xi) - \sum_{i=0}^{m} (-1)^i T_g(\omega^V, h^\xi_i) - \tilde{c}_g(\Xi_i) = 0. \quad (A.14)$$

Again to understand (A.13) at the differential form level, that is, without modulo $\text{Im } \partial + \text{Im } \bar{\partial}$, then we need to fix first $\tilde{c}_g(f_*\eta, h^{H(X, \xi |_X)}, h^{f_*\eta})$ and $\tilde{T}_d_g(\overline{\mathcal{N}})$ as differential
forms, and $T_g(\xi)$ as a current. The natural and nice way is that we use [7, (7.33)] to replace $-\mathrm{Td}_g(\nabla)\mathrm{Td}_g^{-1}(\nabla_{X/Y}) + \mathrm{Td}_g(\mathcal{T})R_g(\nabla_{X/Y})$ by the differential form $B_g(\nabla)$ in [7, (7.24)]. Then we use the current $T_g(\xi)$ defined in [7, (6.30)] and $\tilde{\mathrm{ch}}_g(f,\eta, h^{H(X,\xi|x)}, h^{f-n})$ as the integral $\int_{f^{-\infty}}$ in [14, (3.24)].

Let $\Delta^0(f, l, i, \eta, \xi)$ and $\Delta_0(f, l, i, \eta, \xi)$ be the differential forms such that

$$\Delta := \partial \Delta^0(f, l, i, \eta, \xi) + \partial \Delta_0(f, l, i, \eta, \xi)$$

measures the difference

$$T_g(\omega^V, h^\zeta) - T_g(\omega^W, h^n) + \tilde{\mathrm{ch}}_g(f,\eta, h^{H(X,\xi|x)}, h^{f-n})$$

$$- \int_{V_{\mu_n}/S} T_g(\mathcal{T})T_g(\xi) - \int_{W_{\mu_n}/S} B_g(\nabla)\mathrm{ch}(\eta).$$

We claim that $\Delta^0(f, l, i, \eta, \xi)$ and $\Delta_0(f, l, i, \eta, \xi)$ can be written down explicitly and they admit certain functoriality.

Let $S_1$ be a closed submanifold of $S$, and let $W_1 = f^{-1}(S_1) \subset W$ (respectively, $V_1 = l^{-1}(S_1) \subset V$) be the closed submanifold of $W$ (respectively, $V$) with restricted Kähler metric. Then $i_1 : W_1 \to V_1, l_1 : V_1 \to S_1$ and $f_1 : W_1 \to S_1$ also form a triple of equivariant morphisms such that $f_1 = l_1 \circ i_1$. Denote by $j$ the embedding $S_1 \to S$.

**Theorem A.6.** There is a natural way to write down explicitly differential forms $\Delta^0(f, l, i, \eta, \xi)$ and $\Delta_0(f, l, i, \eta, \xi)$ such that $\Delta := \partial \Delta^0(f, l, i, \eta, \xi) + \partial \Delta_0(f, l, i, \eta, \xi)$ and they are functorial in the following sense.

$$j^* \Delta^0(f, l, i, \eta, \xi) = \Delta^0(f_1, l_1, i_1, \eta | V_1, \xi | V_1)$$

and

$$j^* \Delta_0(f, l, i, \eta, \xi) = \Delta_0(f_1, l_1, i_1, \eta | V_1, \xi | V_1).$$

**Proof.** By the equivariant extension of [8, (6.109), (6.110), (6.158) and (6.170)] in [14, Definition 3.4], there exist universal smooth forms $\gamma^3, \delta^3$ on $S$ such that

$$\Delta + dS \beta = \partial \gamma^3 + \partial \delta^3.$$ 

Again $\beta$ is a combination of local terms from the small time heat kernel asymptotics of the Bismut superconnection for the fibrations $h$ and $h \circ f$ (cf. [8, Theorem 6.4, (6.36), (6.55); 27, (2.24), (2.27)]). More precisely, before we make the procedure as $A \to +\infty, T_0 \to +\infty, \varepsilon \to 0$, the forms $\gamma, \delta$ defined in [14, (3.13)] are double integrals of certain supertrace of the heat kernel of the square of the Bismut superconnection as in (A.7). Note that the square of the Bismut superconnection is a second-order fiberwise elliptic operator with differential form coefficients and when we consider the corresponding objects for the submersion $l_1$, each above term is the restriction of the corresponding term for the global submersion $l$, thus if $\Delta_1, \gamma_1^3, \delta_1^3, \beta_1$ are corresponding terms associated to the relevant fibrations $i_3, l_1$ and $f_1$, we have

$$\Delta_1 = j^* \Delta, \gamma_1^3 = j^* \gamma^3, \delta_1^3 = j^* \delta^3, \beta_1 = j^* \beta,$$

So, write $\Delta^0(f, l, i, \eta, \xi) = \gamma^3 - \beta$ and $\Delta_0(f, l, i, \eta, \xi) = \delta^3 - \beta$, we are done. \qed

We can do the same analysis for (A.14).

Note that we can relax our condition on $f : V \to S$ as follows: $S$ is a (possible noncompact) complex manifold and $f : V \to S$ is a Kähler fibration in the sense of Bismut–Gillet–Soule [10, Definition 1.4].
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Shun Tang
Beijing Advanced Innovation Center for Imaging Theory and Technology
Academy for Multidisciplinary Studies
School of Mathematical Sciences
Capital Normal University
West 3rd Ring North Road 105
Beijing 100048
P.R. China
shun.tang@outlook.com

Xiaonan Ma
Institut de Mathématiques de Jussieu-Paris Rive Gauche
Université de Paris
CNRS
Paris 75013
France
xiaonan.ma@imj-prg.fr