A CHARACTERIZATION OF HYPERBOLIC GEOMETRY
AMONG HILBERT GEOMETRY

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Abstract. In this paper we characterize hyperbolic geometry among Hilbert
geometry by the property that three medians of any hyperbolic triangle all
pass through one point.

We begin by recalling the Hilbert geometry of an open convex set. Let \( K \) be
a nonempty bounded open convex set in \( \mathbb{R}^n, n \geq 2 \). The Hilbert distance \( d_K \) on
\( K \) is introduced by David Hilbert as follows. For any \( x \in K \), let \( d_K(x, x) = 0 \);
for distinct points \( x, y \) in \( K \), assume the line passing through \( x, y \) intersects the
boundary \( \partial K \) at two points \( a, b \) such that the order of these four points on the line
is \( a, x, y, b \) as in Figure 1.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{hilbert_distance.png}
\caption{Hilbert distance}
\end{figure}

Denote the cross-ratio of the points by
\[
(a, x, y, b) = \frac{|y - a|}{|x - a|} \cdot \frac{|x - b|}{|y - b|}
\]
where \( | \cdot | \) is the Euclidean norm of \( \mathbb{R}^n \). Then the Hilbert distance is
\[
d_K(x, y) = \frac{1}{2} \ln(a, x, y, b).
\]

The metric space \((K, d_K)\) is called a Hilbert geometry. Notice that a straight line in
\((K, d_K)\) is a geodesic. When \( K \) is the unit open ball \( \{(x_1, \ldots, x_n) \in \mathbb{R}^n | \sum_{i=1}^n x_i^2 < 1 \} \), \((K, d_K)\) is the Klein model of the hyperbolic geometry.

Since the cross-ratio is invariant under any projective mapping \( P \), \((K, d_K)\) and
\((P(K), d_{P(K)})\) are isometric as Hilbert geometry. In particular, when \( K \) is an open
ellipsoid (ellipse when $n=2$), \( \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} \frac{x_i^2}{a_i^2} < 1 \} \) for some nonzero numbers $a_1, \ldots, a_n$, \((K, d_K)\) is isometric to the hyperbolic geometry.

A natural question is to characterize hyperbolic geometry among Hilbert geometry. By the argument above, it is equivalent to characterize ellipsoid among open convex set. In [3], the authors characterize ellipsoid by using the fact that every hyperbolic ideal triangle has a constant area. In this paper we will consider another elementary property. The following is a well-known fact in constant sectional curvature geometry (see [4] Chapter 7, especially problem K-19):

Three medians of any triangle all pass through one point.

That this property characterizes hyperbolic geometry among Hilbert geometry was conjectured by Feng Luo [7]. Our main result is the affirmative solution of it.

**Theorem 1.** Let $K$ be a nonempty bounded open convex set in $\mathbb{R}^n$ with a Hilbert distance $d_K$. If three medians of any triangle in $(K, d_K)$ all pass through one point (called property M), then $K$ is an open ellipsoid.

**Proof.** First we consider the case $n = 2$. If $K$ is not an open ellipse, we try to find a triangle which does not have the property M. We will use the following lemma which is a simplified version of a result due to Fritz John [6], see also [1] [5].

**Lemma 2.** Any nonempty bounded open convex set $K$ in $\mathbb{R}^2$ contains a unique open ellipse $E$ with maximal Euclidean area. Furthermore $\partial K \cap \partial E$ contains at least three points.

Continue the proof of Theorem 1. We consider the ellipse $E$ in $K$ with maximal Euclidean area. If the number of contact points of their boundary is 3 or 4. We extend the ellipse $E$ a little bit to obtain another ellipse $E'$ as in Figure 2.

![Figure 2. Extend ellipse E to E'](image)

More precisely, assume $E$ is defined by

\[
\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} < 1.
\]
We define ellipse $E'$ by

$$\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} < 1 + \varepsilon,$$

where $\varepsilon > 0$ and sufficiently small. Since $E$ is of maximal area, $\partial E'$ must intersect $\partial K$ at 6 or 8 points.

Therefore we always can find an ellipse, still denote it by $E$, such that the number of points in $\partial K \cap \partial E$ is at least 5.

Since we assume $K$ is not an ellipse, there exists an open set $U \subset K - E$. Now we choose a point $u \in U$ and five points $p_1, p_2, p_3, p_4, p_5$ from $\partial K \cap \partial E$. We arrange the subindex such that the order of these five points on $\partial E$ is $p_1, p_2, p_5, p_3, p_4$ and $u$ is between $p_1, p_4$ as in Figure 3. Notice that any three points do not lie on a straight line.

![Figure 3. Construction of triangle $\triangle ABC$](image)

Now join $p_1$ with $p_3$, $p_2$ with $p_4$ by line segments, denoted by $p_1p_3, p_2p_4$. Assume $p_1p_3 \cap p_2p_4 = A$. If the segment $p_5u$ passes through $A$, we can choose another point from $U$. Therefore we may assume $p_5u$ avoids $A$ and $p_5u \cap p_1p_3 = B, p_5u \cap p_2p_4 = C$.

Now we claim that the triangle $\triangle ABC$ in $K$ does not have the property M. Since $\triangle ABC$ is also in $(E, d_E)$ which is a hyperbolic geometry. Under the distance $d_E$, three medians of $\triangle ABC$ all pass through one point. We assume under the distance $d_E$, points $A', B', C'$ are middle points of edge $BC, AC, AB$ respectively. By the construction of $\triangle ABC$, we see under the distance $d_K$ the middle points of edge $BC, AC, AB$ are $A'', B'', C''$. Now we only need to check $A' \neq A''$ which will finish the proof. Since $AA', BB', CC'$ all pass through one point, $AA'', BB'', CC'$ can not all pass through one point.

Now we extend $p_5u$ such that it intersects $\partial E$ at $q$ and $\partial K$ at $q'$ as in Figure 3. Since the cross-ration is invariant under any projective mapping, consider a projection $P$ sending segment $p_5u$ to the coordinate line $\mathbf{R}$ such that $P(p_5) = 0, P(C) = 1, P(B) = b, P(q) = x, P(q') = x'$ as in Figure 4.

Notice $x < x'$. Furthermore $P(A') = m$ (respectively $P(A'') = m'$) which is the middle point of $1, b$ with two end points $1, x$ (respectively $1, x'$). In fact, since

$$\frac{m(x - 1)}{x - m} = d(1, m) = d(m, b) = \frac{b(x - m)}{m(x - b)},$$
we have
\[ m = \frac{\sqrt{bx}}{\sqrt{b} + \sqrt{(x-1)(x-b)}} =: f(x). \]
We define the right hand side to be a function \( f(x) \). And \( m' = f(x') \). Since the derivative \( f'(x) < 0 \), the function \( f(x) \) is monotonically decreasing in \( x \). Hence \( m = f(x) > f(x') = m' \) and \( A' \neq A'' \).

For general case \( n > 2 \). Let us recall the following result, see [2] (pp. 91).

**Lemma 3.** For a nonempty open convex set \( K \) in \( \mathbb{R}^n \), if for a fixed \( r, 2 \leq r \leq n-1 \), every \( r \) dimensional plain through a fixed point \( P \in K \) intersects \( K \) in an ellipsoid (ellipse when \( r=2 \)), \( K \) itself is an ellipsoid.

Continue the proof of Theorem [1] If every triangle in \( K \) has the property \( M \), by case \( n = 2 \), the intersection of \( K \) with any 2-plain is an ellipse. By the Lemma [3] \( K \) itself is an ellipsoid.

\[ \square \]

**Acknowledgement**

The author would like to thank his advisor, Feng Luo, for suggesting this problem and helpful discussions.

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