Evolutionary forms:
The generation of differential-geometrical structures.  
(Symmetries and Conservation laws.)

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Evolutionary forms, as well as exterior forms, are skew-symmetric differential forms. But in contrast to the exterior forms, the basis of evolutionary forms is deforming manifolds (manifolds with unclosed metric forms). Such forms possess a peculiarity, namely, the closed inexact exterior forms are obtained from that.

The closure conditions of inexact exterior form (vanishing the differentials of exterior and dual forms) point out to the fact that the closed inexact exterior form is a quantity conserved on pseudostructure having the dual form as the metric form. We obtain that the closed inexact exterior form and corresponding dual form made up a conservative object, i.e. a quantity conserved on pseudostructure. Such conservative object corresponds to the conservation law and is a differential-geometrical structure.

Transition from the evolutionary form to the closed inexact exterior form describes the process of generating the differential-geometrical structures. This transition is possible only as a degenerate transformation, the condition of which is a realization of a certain symmetry.

Physical structures that made up physical fields are such differential-geometrical structures. And they are generated by material systems (medias). Relevant symmetries are caused by the degrees of freedom of material system.

**Exterior and evolutionary forms.**

The difference between exterior and evolutionary skew-symmetric differential forms is connected with the properties of manifolds on which skew-symmetric forms are defined.

It is known that the exterior differential forms [1] are skew-symmetric differential forms whose basis are differentiable manifolds or they can be manifolds with structures of any type. Structures, on which exterior forms are defined, have closed metric forms.

It has been named as evolutionary forms skew-symmetrical differential forms whose basis are deforming manifolds, i.e. manifolds with unclosed metric forms. The metric form differential, and correspondingly its commutator, are nonzero. The commutators of metric forms of such manifolds describe a manifold deformation (torsion, curvature and so on).

Lagrangian manifolds, manifolds constructed of trajectories of material system elements, tangent manifolds of differential equations describing physical processes and others can be examples of deforming manifolds.

A specific feature of the evolutionary forms, i.e skew-symmetric forms defined on deforming manifolds, is the fact that commutators of these forms include commutators of the manifold metric forms being nonzero. Such commutators
possess the evolutionary and topological properties. Just due to such properties of commutators the evolutionary forms can generate closed inexact exterior forms that correspond to the differential-geometrical structures.

It is known that the exterior differential form of degree \( p \) is written as \( \sum_{i_1 \ldots i_p} a_{i_1 \ldots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \quad 0 \leq p \leq n \) (1)

Here \( a_{i_1 \ldots i_p} \) are the functions of the variables \( x^{i_1}, x^{i_2}, \ldots, x^{i_p} \), \( n \) is the dimension of space, \( \wedge \) is the operator of exterior multiplication, \( dx^{i_1}, dx^{i_2}, \ldots, dx^{i_p} = dx^k, \ldots \) is the local basis which satisfies the condition of exterior multiplication.

The differential of the (exterior) form \( \theta_p \) is expressed as

\[
d\theta_p = \sum_{i_1 \ldots i_p} da_{i_1 \ldots i_p} \wedge dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_p} \quad (2)
\]

The evolutionary differential form of degree \( p \) is written similarly to exterior differential form. But the evolutionary form differential cannot be written similarly to that presented for exterior differential forms. In the evolutionary form differential there appears an additional term connected with the fact that the basis of the evolutionary form changes. For the differential forms defined on the manifold with unclosed metric form one has \( d(dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_p}) \neq 0 \). (For the differential forms defined on the manifold with closed metric form one has \( d(dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_p}) = 0 \).) For this reason the differential of the evolutionary form \( \theta^p \) can be written as

\[
d\theta^p = \sum_{\alpha_1 \ldots \alpha_p} da_{\alpha_1 \ldots \alpha_p} \wedge dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_p} + \sum_{\alpha_1 \ldots \alpha_p} a_{\alpha_1 \ldots \alpha_p} d(dx^{\alpha_1} \wedge dx^{\alpha_2} \wedge \ldots \wedge dx^{\alpha_p}) \quad (3)
\]

where the second term is a differential of unclosed metric form being nonzero.

[In further presentation the symbol of summing \( \sum \) and the symbol of exterior multiplication \( \wedge \) will be omitted. Summation over repeated indices is implied.]

The second term connected with the differential of the basis can be expressed in terms of the metric form commutator.

For example, let us consider the first-degree form \( \theta = a_{\alpha} dx^{\alpha} \). The differential of this form can be written as

\[
d\theta = K_{\alpha \beta} dx^{\alpha} dx^{\beta} \quad (4)
\]

where \( K_{\alpha \beta} = a_{\beta ; \alpha} - a_{\alpha ; \beta} \) are components of the commutator of the form \( \theta \), and \( a_{\beta ; \alpha}, a_{\alpha ; \beta} \) are covariant derivatives. If we express the covariant derivatives in terms of the connectedness (if it is possible), they can be written as \( a_{\beta ; \alpha} = \partial a_{\beta} / \partial x^{\alpha} + \Gamma_{\beta \alpha \sigma} a_{\sigma} \), where the first term results from differentiating the form coefficients, and the second term results from differentiating the basis. We arrive at the following expression for the commutator components of the form \( \theta \)

\[
K_{\alpha \beta} = \left( \frac{\partial a_{\beta}}{\partial x^{\alpha}} - \frac{\partial a_{\alpha}}{\partial x^{\beta}} \right) + (\Gamma_{\beta \alpha}^{\sigma} - \Gamma_{\alpha \beta}^{\sigma}) a_{\sigma} \quad (5)
\]
Here the expressions \((\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta})\) entered into the second term are just the components of commutator of the first-degree metric form.

If to substitute the expressions (5) for evolutionary form commutator into formula (4), we obtain the following expression for the differential of the first degree skew-symmetric form

\[
d\theta = \left(\frac{\partial a_{\alpha\beta}}{\partial x^\sigma} - \frac{\partial a_{\sigma\alpha}}{\partial x^\beta}\right) dx^\alpha dx^\beta + \left((\Gamma^\sigma_{\beta\alpha} - \Gamma^\sigma_{\alpha\beta})a_{\sigma}\right) dx^\alpha dx^\beta \tag{6}
\]

The second term in the expression for the differential of skew-symmetric form is connected with the differential of the manifold metric form, which is expressed in terms of the metric form commutator.

Thus, the differentials and, correspondingly, the commutators of exterior and evolutionary forms are of different types. In contrast to the exterior form commutator, the evolutionary form commutator includes two terms. These two terms have different nature, namely, one term is connected with coefficients of the evolutionary form itself, and the other term is connected with differential characteristics of manifold. Interaction between terms of the evolutionary form commutator (interactions between coefficients of evolutionary form and its basis) provide the foundation of evolutionary processes that lead to generation of closed inexact exterior forms to which the differential-geometrical structures are assigned.

**Closed inexact exterior forms. Conservation laws.**

**Differential-geometrical structures.**

From the closure condition of exterior form \(\theta^p\):

\[
d\theta^p = 0 \quad \tag{7}
\]

one can see that the closed exterior form \(\theta^p\) is a conserved quantity. This means that it can correspond to a conservation law, namely, to some conservative quantity.

If the form is closed only on pseudostructure, i.e. this form is a closed inexact one, the closure conditions are written as

\[
d_\pi \theta^p = 0 \quad \tag{8}
\]
\[
d_\pi^* \theta^p = 0 \quad \tag{9}
\]

where \(^* \theta^p\) is the dual form.

Condition (9), i.e. the closure condition for dual form, specifies a pseudostructure \(\pi\). (Cohomology (de Rham cohomology, singular cohomology), sections of cotangent bundles and so on may be regarded as examples of pseudostructures.) From conditions (8) and (9) one can see the following. The dual form (pseudostructure) and closed inexact form (conservative quantity) made up a conservative object that can also correspond to some conservation law. Conservative object, which corresponds to the conservation law, is a differential-geometrical structure. Such differential-geometrical structures are examples of
G-structures. The physical structures, which forms physical fields, and corresponding conservation laws are just such structures.

The properties of closed exterior forms reflect also the properties of differential-geometrical structures. The following properties should be emphasized.

1. **Invariance.**

It is known that the closed exact form is a differential of the form of lower degree:

\[ \theta^p = d\theta^{p-1} \]  

(10)

Closed inexact form is also a differential, and yet not a total one but an interior on pseudostructure

\[ \theta^p_\pi = d_\pi \theta^{p-1} \]

(11)

Since the closed exterior differential forms are differentials, they turn out to be invariant under all transformations that conserve the differential. Gauge transformations (the unitary transformations, tangent, canonical, and gradient ones) are examples of such nondegenerate transformations under which closed exterior forms, and hence, the differential-geometrical structures as well, turn out to be invariant.

2. **Conjugacy.**

Closure of exterior differential forms is a result of the conjugacy of elements of exterior or dual forms. The closure property of the exterior form means that any objects, namely, elements of the exterior form, components of elements, elements of the form differential, exterior and dual forms and others, turn out to be conjugated.

Conjugacy is possible if there is one or another type of symmetry.

Gauge symmetries are the symmetries of closed exterior differential forms. They are obtained as the result of conjugacy of any exterior form elements. The physical structures, which are differential-geometrical structures and correspond to conservation laws, are connected with gauge symmetries.

Mathematically these properties of closed exterior forms are written as identical relations. Since the conjugacy is a certain connection between two operators or mathematical objects, it is evident that, to express a conjugacy mathematically, it can be used relations. These are identical relations.

The identical relations express the fact that each closed exterior form is a differential of some exterior form (with a degree less by one). In general form such an identical relation can be written as

\[ d_\pi \varphi = \theta^p_\pi \]  

(12)

In this relation the form in the right-hand side has to be a closed one.

Identical relations of exterior differential forms are a mathematical expression of various types of conjugacy that leads to closed exterior forms.

Such relations like the Poincare invariant, vector and tensor identical relations, the Cauchi-Riemann conditions, canonical relations, the integral relations by Stokes, Gauss-Ostrogradskii, the thermodynamic relations, the eikonal relations and so on are examples of identical relations of closed exterior forms that have the form of relation (12) or its differential or integral analogs.
One can see that identical relations of closed exterior differential forms make it evident in various branches of physics and mathematics.

Below the mathematical and physical meaning of these relations and their role in generating differential-geometrical structures will be disclosed with the help of evolutionary forms.

Thus, one can see that the closure conditions of exterior inexact form and of corresponding dual form are a mathematical expression of the conservation law and differential-geometrical structures.

And here there arise the questions of: (a) how closed inexact exterior forms, which correspond to differential-geometrical structures and reflect the properties of conservation laws, are obtained; (b) what generates differential-geometrical structures; and (c) what is responsible for such processes?

The mathematical apparatus of evolutionary differential forms enables us to answer these questions.

**Properties of evolutionary forms.**

Above it has been shown that the evolutionary form commutator includes the commutator of the manifold metric form which is nonzero. Therefore, the evolutionary form commutator cannot be equal to zero. This means that the evolutionary form differential is nonzero. Hence, the evolutionary form, in contrast to the case of the exterior form, cannot be closed. This leads to that in the mathematical apparatus of evolutionary forms there arise new unconventional elements like nonidentical relations and degenerate transformations. Just such peculiarities allow to describe evolutionary processes.

Nonidentical relations can be written as

\[ d\phi = \eta^p \]  

Here \( \eta^p \) is the \( p \)-degree evolutionary form that is unclosed, \( \phi \) is some form of degree \( (p - 1) \), and the differential \( d\phi \) is a closed form of degree \( p \).

In the left-hand side of this relation it stands the form differential, i.e. a closed form that is an invariant object. In the right-hand side it stands the unclosed form that is not an invariant object. Such a relation cannot be identical one.

One can see a difference of relations for exterior forms and evolutionary ones. In the right-hand side of identical relation (see relation (12)) it stands a closed form, whereas the form in the right-hand side of nonidentical relation (13) is an unclosed one.

Nonidentical relations are obtained while describing any processes. A relation of such type is obtained while analyzing the integrability of the partial differential equation. An equation is integrable if it can be reduced to the form \( d\phi = dU \). However it appears that, if the equation is not subject to an additional condition (the integrability condition), it is reduced to the form (13), where \( \eta^p \) is an unclosed form and it cannot be expressed as a differential.

Nonidentical relations of evolutionary forms are evolutionary relations because they include the evolutionary form. Such nonidentical evolutionary relations appear to be selfvarying ones. A variation of any object of the relation
in some process leads to a variation of another object and, in turn, a variation of the latter leads to a variation of the former. Since one of the objects is a noninvariant (i.e. unmeasurable) quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot be completed.

The nonidentity of evolutionary relation is connected with a nonclosure of the evolutionary form, that is, it is connected with the fact that the evolutionary form commutator is nonzero. As it has been pointed out, the evolutionary form commutator includes two terms: one term specifies the mutual variations of the evolutionary form coefficients, and the second term (the metric form commutator) specifies the manifold deformation. These terms have a different nature and cannot make the commutator vanish. In the process of selfvariation of the nonidentical evolutionary relation it proceeds an exchange between the terms of the evolutionary relation and this is realized according to the evolutionary relation. The evolutionary form commutator describes a quantity that is a moving force of the evolutionary process and leads to generation of differential-geometrical structures.

The significance of the evolutionary relation selfvariation consists in the fact that in such a process it can be realized conditions under which the identical relation is obtained from the nonidentical relation. These are conditions of a degenerate transformation. The point of time at which such conditions are realized is that of originating the element of the differential-geometrical structure.

**Origination of differential-geometrical structures.**

To obtain the differential-geometrical structure, it is necessary to obtain a closed inexact exterior form, i.e. the form closed on pseudostructure. The condition of realization of the pseudostructure is vanishing the interior differential of the metric form. This is the closure condition for dual form and this leads to realization of closed inexact exterior form.

Since the evolutionary form differential is nonzero, whereas the closed exterior form differential is zero, the transition from the evolutionary form to the closed exterior form is allowed only as a degenerate transformation, i.e. a transformation that does not conserve the differential. The conditions of vanishing interior differential of the metric form (the additional condition) are the conditions of degenerate transformation.

As it has been already mentioned, the evolutionary differential form \( \eta^p \) involved into nonidentical relation (13) is an unclosed one. The commutator, and hence the differential, of this form is nonzero. That is,

\[
d\eta^p \neq 0 \quad (14)
\]

If the conditions of degenerate transformation are realized, then from the unclosed evolutionary form one can obtain a differential form closed on pseudostructure. The differential of this form equals zero. That is, it is realized the transition

\[
d\eta^p \neq 0 \to (\text{degenerate transformation}) \to d^*\pi \eta^p = 0, \ d_\pi \eta^p = 0.
\]

The relations obtained

\[
d_\pi \eta^p = 0, \ d^*\pi \eta^p = 0 \quad (15)
\]
are the closure conditions for exterior inexact form, and this points to realization of exterior form closed on pseudostructure, that is, this points to origination of the differential-geometrical structure.

The conditions of degenerate transformation that lead to origination of the differential-geometrical structures can be connected with any symmetries. (While describing material system, the symmetries can be conditioned, for example, by degrees of freedom of material system). Since the conditions of degenerate transformation are those of vanishing the interior differential of metric form, that is, vanishing the interior (rather then total) metric form commutator, the conditions of degenerate transformation can be caused by symmetries of coefficients of the metric form commutator (for example, it can be the symmetric connectedness).

To the conditions of degenerate transformation there correspond a requirement that some functional expressions become equal to zero. Such functional expressions are Jacobians, determinants, the Poisson brackets, residues, and others.

Under the degenerate transformation on pseudostructure the evolutionary form commutator vanishes, and this corresponds to a realization of the closed inexact exterior form. But in this case the total evolutionary form commutator is nonzero.

Vanishing on pseudostructure the external form differential (that is, vanishing on pseudostructure the interior commutator of the evolutionary form) points to that the exterior unclosed form is a conservative quantity in the direction of pseudostructure. However, in the direction normal to pseudostructure this quantity exhibits a discontinuity. The value of such discontinuity is defined by the value of the evolutionary form commutator being nonzero. This argues to discreteness of the differential-geometrical structures.

Thus, while selfvariation of the evolutionary nonidentical relation the interior metric form commutator can vanish (due to the symmetries of coefficients of the metric form commutator). This means that it is formed the pseudostructure on which the differential form turns out to be closed. The emergency of the form being closed on pseudostructure points out to origination of the differential-geometrical structures.

It has been already noted that the conditions of degenerate transformations, which lead to emergency of differential-geometrical structures, can just be realized under selfvariation of the nonidentical evolutionary relation.

**Obtaining identical relation from nonidentical one.**

On the pseudostructure $\pi$ evolutionary relation (13) converts to the relation

$$d_\pi \phi = \eta^\pi_p$$

which proves to be an identical relation. Indeed, since the form $\eta^\pi_p$ is a closed one, on the pseudostructure this form turns out to be a differential of some differential form. There are differentials in the left-hand and right-hand sides of this relation. This means that the relation is an identical one.
From evolutionary relation (13) it is obtained the identical on the pseudostructure relation. In this case the evolutionary relation itself remains to be nonidentical one. (At this point it should be emphasized that differential, which equals zero, is an interior one. Under degenerate transformation the evolutionary form commutator becomes zero only on the pseudostructure. The total evolutionary form commutator is nonzero. The evolutionary form remains to be unclosed.)

It can be shown that all identical relations of the exterior differential form theory are obtained from nonidentical relations (that contain the evolutionary forms) by applying degenerate transformations.

The degenerate transform is realized as a transition to nonequivalent coordinate system: a transition from the noninertial coordinate system to the locally inertial that. Evolutionary relation (13) and condition (14) relate to the system being tied to the deforming manifold, whereas condition (15) and identical relations (16) may relate only to the locally inertial coordinate system being tied to a pseudostructure.

Transition from nonidentical relation (13) to identical relation (16) means the following. Firstly, it is from such a relation that one can obtain the differential $d\pi\phi$ and find the desired function $\phi_\pi$. And, secondly, an emergence of the closed (on pseudostructure) inexact exterior form $\eta^p_\pi$ (right-hand side of relation (16)) points to an origination of the conservative object - the differential-geometrical structure. It occurs that the emergency of the differential-geometrical structure is connected with the realization of the differential $d\pi\phi$. Below it will be shown that the function $\phi_\pi$ is the state-function of the system described, which generates the differential-geometrical structures.

Thus, the mathematical apparatus of evolutionary forms describes the process of generation of the closed inexact exterior forms, and this discloses the process of origination of the differential-geometrical structure, namely, a new conjugated object.

The above described process of generating the differential-geometrical structures discloses the process of conjugating any elements or operators.

The evolutionary differential form is an unclosed form, that is, it is a form whose differential is not equal to zero. The differential of the exterior differential form equals zero. To the closed exterior form there correspond conjugated operators, whereas to the evolutionary form there correspond nonconjugated operators. A transition from the evolutionary form to the closed exterior form and origination of the differential-geometrical structures is a transition from nonconjugated operators to conjugated ones. This is expressed mathematically as a transition from a nonzero differential (the evolutionary form differential is nonzero) to a differential that equals zero (the closed exterior form differential equals zero).

It can be seen that the process of generating the differential-geometrical structures is a mutual exchange between the quantities of different nature (for example, between the algebraic and geometric quantities, between the physical and spatial quantities) and an exchange while realizing any symmetry.
Characteristics of differential-geometrical structures.

Since the closed exterior form, which corresponds to the differential-geometrical structure emerged, was obtained from the evolutionary form, it is evident that characteristics of this structure has to be connected with those of the evolutionary form and of the manifold on which this form is defined, with the conditions of degenerate transformation and with the values of commutators of the evolutionary form and the manifold metric form.

The conditions of degenerate transformation, as it was said before, determine the pseudostructures. The first term of the evolutionary form commutator determines the value of the discrete change (the quantum), which the quantity conserved on the pseudostructure undergoes when transition from one pseudostructure to another. The second term of the evolutionary form commutator specifies a characteristics that fixes the character of the initial manifold deformation, which took place before the differential-geometrical structure emerged. (Spin is an example of such a characteristics).

The connection of the differential-geometrical structures with the skew-symmetric differential forms allows to introduce a classification of these structures in dependence on parameters that specify the skew-symmetric differential forms and enter into nonidentical and identical relation. To determine these parameters one has to consider the problem of integration of the nonidentical evolutionary relation.

Under degenerate transformation from the nonidentical evolutionary relation one obtains a relation being identical on pseudostructure. Since the right-hand side of such a relation can be expressed in terms of differential (as well as the left-hand side), one obtains a relation that can be integrated, and as a result he obtains a relation with the differential forms of less by one degree.

The relation obtained after integration proves to be nonidentical as well.

By sequential integrating the nonidentical relation of degree $p$ (in the case of realization of the corresponding degenerate transformations and forming the identical relation), one can get closed (on the pseudostructure) exterior forms of degree $k$, where $k$ ranges from $p$ to 0.

In this case one can see that after such integration the closed (on the pseudostructure) exterior forms, which depend on two parameters, are obtained. These parameters are the degree of evolutionary form $p$ in the evolutionary relation and the degree of created closed forms $k$.

In addition to these parameters, another parameter appears, namely, the dimension of space. If the evolutionary relation generates the closed forms of degrees $k = p$, $k = p - 1$, ..., $k = 0$, to them there correspond the pseudostructures of dimensions $(N - k)$, where $N$ is the space (formed) dimension.

Forming fields and manifolds.

The pseudostructures, on which the closed inexact forms are defined, form the pseudomanifolds. (Integral surfaces, pseudo-Riemann and pseudo-Euclidean spaces are the examples of such manifolds). In this process the dimensions of the manifolds formed are connected with the evolutionary form degree.
To transition from pseudomanifolds to metric manifolds it is assigned a transition from closed \emph{inexact} differential forms to \emph{exact} exterior differential forms. (Euclidean and Riemann spaces are examples of metric manifolds).

Since the closed metric form is dual with respect to some closed exterior differential form, the metric forms cannot become closed by themselves, independently of the exterior differential form. This proves that manifolds with closed metric forms are connected with the closed exterior differential forms. This indicates that the fields of conservative quantities are formed from closed exterior forms at the same instant of time when the manifolds are created from the pseudostuctures. (The specific feature of the manifolds with closed metric forms that have been formed is that they can carry some information.) That is, the closed exterior differential forms and manifolds, on which they are defined, are mutually connected objects.

\emph{Symmetries, conservation laws, differential-geometrical structures.}

As it has been noted, the closure conditions of exterior inexact form and of corresponding dual form are a mathematical expression of the conservation law and differential-geometrical structures.

The closure of the exterior differential forms and corresponding dual form is the result of the conjugacy of elements of exterior or dual forms.

Conjugacy is possible if there is one or other type of symmetry.

It has been shown that the differential-geometrical structures originate when the conditions of degenerate transformation are realized. These conditions relate to the symmetries of coefficients of metric form of the manifold on which the evolutionary form is defined. These symmetries of the dual form are exterior symmetries. Symmetries of the closed exterior form that is formed on the pseudostucture are interior symmetry.

Whereas the exterior symmetries of the dual forms are connected with degenerate transformations, the interior symmetries, i.e. those of the closed exterior forms, are connected with nonidentical transformations.

Degenerate and nondegenerate transformations are mutually connected. Degenerate transformations lead to formatting the differential-geometrical structures and nondegenerate transformations execute a transition from one differential-geometrical structure to another.

One of the problems in the theory of symmetry is a searching for symmetries of differential equations. A knowledge of symmetries enables one to get a solution of differential equations that corresponds to the conservation law and defines the differential-geometrical structures. The dependence of symmetries on any parameter, i.e. the dependence of the nondegenerate transformations on a given parameter allows to study the “evolution” of the differential-geometrical structures in given parameter. But in this case the question of how these structures emerge is not posed (that is, the evolutionary process of originating these structures is not considered). As it has been shown in present paper, the answer to this question gives the theory of evolutionary forms.

It has been already noted that the evolutionary forms which generate the closed exterior forms corresponding to the differential-geometrical structures ap-
appear while describing any processes by differential equations. It can be shown that the tangent manifold of differential equation is nondifferentiable manifold (the Lagrangian manifold is an example) and the derivatives satisfying the differential equation do not make up a differential (closed exterior forms). However, under degenerate transformation it occurs a transition from the tangent manifold to cotangent one (the manifold of Hamiltonian systems is an example) on which the differential made up by derivatives of differential equation becomes a closed form. This points to originating differential-geometrical structures. That is, the origination of the differential-geometrical structure is connected with degenerate transformation that executes the transition from tangent space to cotangent one. And nondegenerate transformation executes the transition into tangent space from any differential-geometrical structure to another.

A peculiarity of the degenerate transformation can be considered by the example of Hamiltonian systems. Here the degenerate transformation is a transition from the tangent space \((q_j, \dot{q}_j)\), which is a tangent (Lagrangian) manifold, to the cotangent characteristic (Hamiltonian) manifold \((q_j, p_j)\). On the other hand, the nondegenerate canonical transformation is a transition from one characteristic manifold \((q_j, p_j)\) to another characteristic manifold \((Q_j, P_j)\). {The formula of nondegenerate canonical transformation can be written as \(p_j dq_j = P_j dQ_j + dW\), where \(W\) is the generating function.}

**Physical meaning of the differential-geometrical structures.**

The differential-geometrical structures obtained may carry a physical meaning.

It has been already noted that the evolutionary forms appear under description of a certain process. In particular, they are obtained while describing physical processes in material media (material systems). Analysis of the evolutionary forms obtained and nonidentical relations shows that material media generate differential-geometrical structures, which are physical structures forming physical fields. {Material system is a variety of elements that have internal structure and interact to one another. As examples of material systems it may be thermodynamic, gas dynamical, cosmic systems, systems of elementary particles and others.}

Evolutionary forms, as well as the closed exterior forms, correspond to conservation laws. But closed exterior forms correspond to conservation laws that can be named as exact ones, whereas evolutionary forms correspond to the balance conservation laws.

*The balance conservation laws.*

The balance conservation laws are conservation laws that establish a balance between the variation of physical quantity and the corresponding external action. They are described by differential equations. The balance conservation laws for material systems are conservation laws for energy, linear momentum, angular momentum, and mass. From the equations, which describe the balance conservation laws, one can obtain a relation that is nonidentical relation since it contains the evolutionary forms.
Let us analyze the equations that describe the balance conservation laws for energy and linear momentum.

In the accompanying reference system (this system is connected with the manifold built by the trajectories of the material system elements) the energy equation is written in the form

$$\frac{\partial \psi}{\partial \xi^1} = A_1$$

Here $\psi$ is the functional specifying the state of material system (the action functional, entropy, wave function can be regarded as examples of the functional), $\xi^1$ is the coordinate along the trajectory, $A_1$ is the quantity that depends on specific features of material system and on external energy actions onto the system.

In a similar manner, in the accompanying reference system the equation for linear momentum appears to be reduced to the equation of the form

$$\frac{\partial \psi}{\partial \xi^\nu} = A_\nu, \quad \nu = 2, \ldots$$

where $\xi^\nu$ are the coordinates in the direction normal to the trajectory, $A_\nu$ are the quantities that depend on the specific features of material system and on external force actions.

Eqs. (17) and (18) can be convoluted into the relation

$$d\psi = A_\mu \, d\xi^\mu, \quad (\mu = 1, \nu)$$

where $d\psi$ is the differential expression $d\psi = (\partial \psi/\partial \xi^\mu) d\xi^\mu$.

Relation (19) can be written as

$$d\psi = \omega$$

here $\omega = A_\mu \, d\xi^\mu$ is the skew-symmetrical differential form of the first degree.

Since the balance conservation laws are evolutionary ones, the relation obtained is also an evolutionary relation.

Relation (20) was obtained from the equation of the balance conservation laws for energy and linear momentum. In this relation the form $\omega$ is that of the first degree. If the equations of the balance conservation laws for angular momentum be added to the equations for energy and linear momentum, this form in the evolutionary relation will be the form of the second degree. And in combination with the equation of the balance conservation law of mass this form will be the form of degree 3.

Thus, in general case the evolutionary relation can be written as

$$d\psi = \omega^p$$

where the form degree $p$ takes the values $p = 0, 1, 2, 3, \ldots$ (The evolutionary relation for $p = 0$ is similar to that in the differential forms, and it was obtained from the interaction of energy and time.)
Let us show that relation obtained from the equation of the balance conservation laws proves to be nonidentical.

To do so we shall analyze relation (20).

In the left-hand side of relation (20) there is a differential that is a closed form. This form is an invariant object. The right-hand side of relation (20) involves the differential form $\omega$, that is not an invariant object because in real processes, as it is shown below, this form proves to be unclosed. The commutator of this form is nonzero. The components of commutator of the form $\omega = A_\mu d\xi^\mu$ can be written as follows:

$$K_{\alpha\beta} = \left( \frac{\partial A_\beta}{\partial \xi^\alpha} - \frac{\partial A_\alpha}{\partial \xi^\beta} \right)$$

(here the term connected with the manifold metric form has not yet been taken into account).

The coefficients $A_\mu$ of the form $\omega$ have been obtained either from the equation of the balance conservation law for energy or from that for linear momentum. This means that in the first case the coefficients depend on the energetic action and in the second case they depend on the force action. In actual processes energetic and force actions have different nature and appear to be inconsistent. The commutator of the form $\omega$ consisted of the derivatives of such coefficients is nonzero. This means that the differential of the form $\omega$ is nonzero as well. Thus, the form $\omega$ proves to be unclosed and cannot be a differential like the left-hand side.

This means that the relation (20) cannot be an identical one.

What is a physical meaning of such a relation?

This relation obtained from the equations of the balance conservation laws involves the functional that specifies the material system state. However, since this relation turns out to be not identical, from this relation one cannot get the differential $d\psi$ that could point out to the equilibrium state of material system. The absence of differential means that the system state is nonequilibrium. That is, in material system the internal force acts. This leads to distortion of trajectories of material system. A manifold made up by the trajectories (the accompanying manifold) turns out to be a deforming manifold. The differential form $\omega$, as well as the forms $\omega^p$, appear to be evolutionary forms. Commutators of these forms will contain an additional term connected with the commutator of unclosed metric form of manifold.

The availability of two terms in the commutator of the form $\omega^p$, as it has been already shown, leads to that the nonidentical evolutionary relation turns out to be a selfvarying relation.

Selfvariation of the nonidentical evolutionary relation points to the fact that the nonequilibrium state of material system turns out to be selfvarying. It is evident that this selfvariation proceeds under the action of internal force whose quantity is described by commutator of the unclosed evolutionary form $\omega^p$. (If the commutator be zero, the evolutionary relation would be identical, and this would point to the equilibrium state, i.e. the absence of internal forces.)
Everything that gives a contribution into the commutator of the form $\omega^p$ leads to emergency of internal force.

Above it has been shown that under degenerate transformation from non-identical evolutionary relation it can be obtained the identical relation

$$d_\pi \psi = \omega^p_\pi$$

(22)

From such a relation one can obtain the state function and this corresponds to equilibrium state of the system. But identical relation can be realized only on pseudostructure (which is specified by the condition of degenerate transformation). This means that a transition of material system to equilibrium state proceeds only locally. In other words, it is realized a transition of material system from nonequilibrium state to locally equilibrium one. In this case the general state of material system remains to be nonequilibrium.

It has been already noted that the symmetries of coefficients of the metric form commutator are conditions of degenerate transformation. They are conditioned by degrees of freedom of material system.

As one can see from the analysis of nonidentical evolutionary relation, the transition of material system from nonequilibrium state to locally-equilibrium state proceeds spontaneously in the process of selfvarying nonequilibrium state of material system under realization of any degrees of freedom of this system. (Translational degrees of freedom, internal degrees of freedom of the system elements, and so on can be examples of such degrees of freedom).

As it has been already said above, the transition from nonidentical relation (21) obtained from the balance conservation laws to identical relation (22) means the following. Firstly, an existence of the state differential (left-hand side of relation (22)) points to a transition of the material system from nonequilibrium state to the locally-equilibrium state. And, secondly, an emergency of the closed (on pseudostructure) inexact exterior form (right-hand side of relation (22)) points to an origination of the physical structure.

Thus one can see that the transition of material system from nonequilibrium state to locally-equilibrium state is accompanied by originating differential-geometrical structures, which are physical structures. The emergency of physical structures in the evolutionary process reveals in material system as an emergency of certain observable formations, which develop spontaneously. In this manner the causality of emerging various observable formations in material media is explained. Such formations and their manifestations are fluctuations, turbulent pulsations, waves, vortices, creating massless particles and others. The intensity of such formations is controlled by a quantity accumulated by the evolutionary form commutator at the instant in time of originating physical structures.

Physical structures that are generated by material systems made up physical fields.

The availability of physical structures points out to the fulfilment of conservation laws. These are conservation laws for physical fields. The process of generating physical fields demonstrates a connection of these conservation laws,
which can be referred to as exact ones, with the balance (differential) conservation laws for material media. Closed inexact exterior forms that correspond to physical structures and exact conservation laws are obtained from equations describing the balance conservation laws.

Since the closed exterior forms corresponding to physical structures are obtained from the evolutionary forms describing material systems, the characteristics of physical structures are determined by characteristics of the material system generating these structures, and the parameters of evolutionary forms and closed exterior forms enables one to classify physical structures.

As it has been shown above, the type of differential-geometrical structures, and hence of physical structures (and, accordingly, of physical fields) generated by the evolutionary relation, depends on the degree of differential forms \( p \) and \( k \) and on the dimension of original inertial space \( n \) (here \( p \) is the degree of the evolutionary form in the nonidentical relation that is connected with a number of interacting balance conservation laws, and \( k \) is the degree of closed form generated by the nonidentical relation). By introducing the classification by numbers \( p, k, n \) one can understand the internal connection between various physical fields. Since physical fields are the carriers of interactions, such classification enables one to see a connection between interactions. It can be shown that the case \( k = 0 \) corresponds to strong interaction, \( k = 1 \) corresponds to weak interaction, \( k = 2 \) corresponds to electromagnetic interaction, and \( k = 3 \) corresponds to gravitational interaction.

Mathematical apparatus of closed exterior forms that corresponds to physical structures and conservation laws for physical fields lies at the basis of field theory.

It can be shown that the equations of existing field theories are those obtained on the basis of the properties of the exterior form theory. The Hamilton formalism is based on the properties of closed exterior and dual forms of the first degree, quantum mechanics does on the forms of zero degree, the electromagnetic field equations are based on the forms of second degree. The third degree forms are assigned to the gravitational field.

The gauge symmetries, which are interior symmetries of the field theory equations, are symmetries of closed (inexact) forms.

The internal symmetries in field theory are those of closed exterior differential forms, whereas the external symmetries in field theory are symmetries of relevant dual forms.

The gauge transformations of field theory, which are nondegenerate transformations of the closed exterior differential forms, are connected with the gauge symmetries. Since the closed exterior differential form is a differential (a total one if the form is exact, or an interior one on pseudostructure if the form is inexact), it is obvious that the closed form proves to be invariant under all transformations that conserve the differential. The unitary transformations (0-form), the tangent and canonical transformations (1-form), the gradient and gauge transformations (2-form) and so on are examples of such transformations. These are gauge transformations for spinor, scalar, vector, and tensor (3-form) fields.
It has been shown that the closed exterior forms and relevant dual forms, which correspond to the conservation laws for physical fields, are obtained from the evolutionary forms, which describe the balance conservation laws for material media. This proceeds under degenerate transformation, which is connected with the degrees of freedom of material system. The conditions of degenerate transformation defines a closed dual form. From this it follows that the external symmetries, namely, the symmetries of dual forms, are due to the degrees of freedom of material system. It is for this reason the exterior symmetries are spatial-temporal symmetries.

The realization of the closed dual form, which proceeds due to the degrees of freedom of material system, leads to realization of the closed exterior form, that is, to the conjugacy of the differential form elements, and emergency of internal symmetries. From this one can see a connection between internal and external symmetries.

Whereas the internal symmetries are connected with the conservation laws for physical fields, the external symmetries caused by the degrees of freedom of material media are connected with the balance conservation laws for material media.

The nondegenerate transformations are connected with internal symmetries, and the degenerate transformations of evolutionary forms are connected with external symmetries.

**Conclusion**

As it has been shown the theory of evolutionary forms explains the process of generation of differential-geometrical structures. This cannot be carried out within the framework of any other formalisms.

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