Topology and geometry of six-dimensional (1, 0) supergravity black hole horizons

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Received 31 October 2011, in final form 1 January 2012
Published 8 February 2012
Online at stacks.iop.org/CQG/29/055002

Abstract

We show that the supersymmetric near horizon black hole geometries of six-dimensional supergravity coupled to any number of scalar and tensor multiplets are either locally $\text{AdS}_3 \times \Sigma^3$, where $\Sigma^3$ is a homology 3-sphere, or $\mathbb{R}^{1,1} \times S^4$, where $S^4$ is a 4-manifold whose geometry depends on the hypermultiplet scalars. In both cases, we find that the tensorini multiplet scalars are constant and the associated 3-form field strengths vanish. We also demonstrate that the $\text{AdS}_3 \times \Sigma^3$ horizons preserve two, four and eight supersymmetries. For horizons with four supersymmetries, $\Sigma^3$ is in addition a non-trivial circle fibration over a topological 2-sphere. The near horizon geometries preserving eight supersymmetries are locally isometric to either $\text{AdS}_3 \times S^3$ or $\mathbb{R}^{1,1} \times T^4$. Moreover, we show that the $\mathbb{R}^{1,1} \times S^4$ horizons preserve one, two and four supersymmetries and the geometry of $S$ is Riemann, Kähler and hyper-Kähler, respectively.

PACS numbers: 04.65.+e, 04.50.Gh, 02.40.-k

1. Introduction

It is well known that the black hole uniqueness theorems in four dimensions [1–7] do not extend to five and higher dimensions. Specifically in five dimensions, apart from spherical black holes [8], there also exist black holes with near horizon topology $S^1 \times S^2$, the black rings [9, 10]. In more than five dimensions, it is expected that there are black holes with exotic horizon topologies [11–14].

In the context of supergravity, the recent progress made toward understanding of the geometry for all solutions to the Killing spinor equations (KSEs) raises the hope that all supersymmetric black hole solutions can be classified. So far this goal has not been attained, but some significant progress has been made toward classifying all near horizon black hole geometries. Results in this direction include the identification of all near horizon geometries of simple five- and six-dimensional supergravities [9, 15]. In addition, all near horizon geometries...
of four-dimensional, $\mathcal{N} = 1$ supergravity coupled to any number of vector and scalar multiplets have been classified [16]. Furthermore, the near horizon geometries of heterotic [17] and IIB with 5-form flux [18] supergravities have been identified and many examples have been constructed. Progress has also been made in 11 dimensions where all static near horizon geometries have been found [19].

In this paper, we shall investigate the topology and geometry of supersymmetric black hole horizons in six-dimensional $(1, 0)$ supergravity [20–22] coupled to any number of tensor, vector and scalar\footnote{The scalar multiplets are also referred to as hypermultiplets.} multiplets. For this, we adapt null Gaussian geodesic coordinates\footnote{The nature of the horizons captured by these coordinates is described in [23]. A discussion of the application of these coordinates to supergravity horizons with active form field strengths and the relation of the resulting solutions to asymptotic supersymmetry algebras can be found in [17, 16].} near the horizon [23] and then solve both the field and KSEs of the theory. The solution of the latter is facilitated by the identification of the geometry of all supersymmetric backgrounds of six-dimensional $(1, 0)$ supergravity in [24]. There is no general method to solve the field equations. However, in many cases of interest, the use of the results from the KSEs together with the maximal principle and the compactness of the horizon sections allow for the general solution to the field equations without imposing an ansatz on the form of near horizon geometries. To apply this technique in six dimensions, we shall restrict our attention to those horizons for which the vector multiplets can be consistently set to zero. This is because in the presence of active vectors the field equations of the theory cannot be put in a form that allows the application of the maximal principle\footnote{This is similar to what happens in the heterotic case and a more detailed explanation is given in [17].}

In particular, we find that there are two classes of near horizon geometries. The near horizon geometries of the first class are locally $\text{AdS}_3 \times \Sigma^3$ and so the horizon section is $\mathcal{S} = \mathbb{S}^1 \times \Sigma^3$. The proof of this product structure for $\mathcal{S}$ is key and utilizes the field and KSEs as well as the compactness of $\mathcal{S}$. Moreover, it turns out that $\Sigma^3$ is a three-dimensional manifold with a strictly positive Ricci tensor and a theorem of Gallot and Meyer, see e.g. [25], together with the Poincaré conjecture [26] implies that the universal cover of $\Sigma^3$ is diffeomorphic to $\mathbb{S}^3$. The scalars of the tensor multiplets are constant and the associated 3-form field strengths vanish. The scalars of the hypermultiplets depend only on the coordinates of $\Sigma^3$ and satisfy a natural first-order nonlinear differential equation which we describe.

We also demonstrate that the $\text{AdS}_3 \times \Sigma^3$ horizons preserve two, four or eight supersymmetries depending on the geometry of $\Sigma^3$. To prove this, we show that the horizons admit an isometry which commutes with all the KSEs. For a generic choice of $\Sigma^3$, the horizons preserve two supersymmetries. For horizons preserving four supersymmetries, the metric on $\Sigma^3$ is compatible with a non-trivial circle fibration over a topological 2-sphere $\Sigma^2$. Moreover, the scalars of the hypermultiplet depend only on the coordinates of $\Sigma^2$ and are pseudo-holomorphic types of maps into the hypermultiplet quaternionic Kähler manifold $\mathcal{Q}$. The $\text{AdS}_3 \times \Sigma^3$ horizons which preserve eight supersymmetries are locally isometric\footnote{Throughout this paper, $\mathbb{S}^n$ denotes the $n$-sphere equipped with the standard ‘round’ metric.} to $\text{AdS}_3 \times \mathbb{S}^3$.

The geometry of the horizon sections $\mathcal{S}$ in the $\mathbb{R}^{1,1} \times \mathcal{S}$ class of horizons depends only on the hypermultiplet scalars. In particular, as in the previous case, the tensor multiplet scalars are constant and all 3-form field strengths, including that of the gravity multiplet, vanish. The $\mathbb{R}^{1,1} \times \mathcal{S}$ horizons preserve one, two and four supersymmetries, and the geometry of $\mathcal{S}$ is Riemann, Kähler and hyper-Kähler, respectively. The first two cases require the existence of non-trivial hypermultiplet scalars which satisfy certain first-order nonlinear differential equations which we describe. In the last case, the hypermultiplet scalars are constant and
is locally isometric to either $K^3$ or $T^4$. The $R^{1,1} \times T^4$ horizons admit a supersymmetry enhancement to $N = 8$. The above results extend those of [15] on the near horizon geometries of simple six-dimensional supergravity.

This paper is organized as follows. In section 2, we set up our notation and solve the KSEs for $N = 1$ horizons. In section 3, we prove that all near horizon geometries with non-vanishing 'rotation' are products $\text{AdS}_3 \times \Sigma^3$ and preserve even number of supersymmetries. Moreover, we examine in detail the topology and geometry of $N = 2$ horizons. In section 4, we investigate the geometry of $R^{1,1} \times S$ horizons and show that they may preserve one, two and four supersymmetries. In the appendix, we present the solution of the KSEs for near horizon backgrounds preserving one supersymmetry.

2. Supersymmetric horizons

2.1. Fields and KSEs of six-dimensional supergravity

We consider a six-dimensional supergravity coupled to any number of tensor and scalar multiplets. The bosonic field content of the gravitational multiplet consists of the graviton and 3-form field strength $H$ of the gravitational multiplet. In addition, each tensor multiplet contains a 3-form field strength and a real scalar, while each scalar multiplet, or hypermultiplet, contains four real scalars. The action of the theory has been constructed progressively in [20–22]. We follow closely the description of the theory in [22] but we shall use the notation developed in [24] which differs from that in [22]. Apart from differences in the normalizations of fields, the ten-dimensional description of $(1, 0)$-supergravity spinors in [24] simplifies the solution of the KSEs.

Suppose that the $(1, 0)$-supergravity couples to $n_T$ tensor multiplets. In the absence of vector multiplets, the $n_T + 1$ 3-form field strengths of the supergravity theory are given in terms of 2-form potentials as

$$G_{\mu \nu \rho}^r = 3 \eta_{[\mu} B_{\nu \rho]}^r, \quad r = 0, \ldots, n_T.$$  \hspace{1cm} (2.1)

The gravitini, tensorini and hyperini KSEs of the theory are

$$D_\mu \epsilon \equiv \left( \nabla_\mu - \frac{1}{8} H_{\mu \nu \rho} \Gamma^{\nu \rho} + C^{\mu \rho}_{\nu} \right) \epsilon = 0,$$

$$\left( \frac{i}{2} \Gamma^\mu M^\nu - \frac{i}{24} H^M_{\mu \nu \rho} \Gamma^{\mu \nu \rho} \right) \epsilon = 0,$$

$$i \Gamma^{\nu} \phi^A_\mu \epsilon = 0,$$  \hspace{1cm} (2.2)

respectively, where the various fields and components of the KSEs are defined as

$$H_{\mu \nu \rho} = v^M_\mu C^r_{\mu \nu \rho}, \quad H^M_{\mu \nu \rho} = x^M_\mu C^r_{\mu \nu \rho}, \quad C^r_{\mu \rho} = \partial_\mu \phi^A \Lambda^A_\rho, \quad \eta_{\mu \nu} v^M_\rho = \eta_{\mu \nu} v_\rho = \sum_M x^M_\mu \Lambda^M_\rho = \eta_{\mu \nu} \Lambda^M_\rho = 0,$$  \hspace{1cm} (2.3)

and where

$$\eta_{\mu \nu} v^2 = 1, \quad v_\mu v_\nu - \sum_M x^M_\mu \Lambda^M_\nu = \eta_{\mu \nu}, \quad v^2 = 0.$$  \hspace{1cm} (2.4)

Both $v^2$ and $x^M_\mu$ depend on the tensor multiplet scalars $\phi$. For more notation details as well as the description of spinors, see [24]. Clearly $H^M_\mu$ are the $n_T$ 3-form field strengths of the tensor multiplets, and the tensor multiplet scalars $\phi$ parameterize the hyperbolic space $SO(n_T, 1)/SO(n_T)$. $\phi^A$ are the scalars of the hypermultiplets which take values on a
quaternionic Kähler manifold, \( Q \), \( F^{\mathbb{C}}_L \) is a frame of \( Q \), and \( A^L_{\mathbb{C}} \) is the \( Sp(1) \) part of the quaternionic Kähler connection. Note also that the 3-form field strengths satisfy the duality condition

\[
\zeta_{\mu} G^\mu_{\mu_1 \mu_2 \mu_3} = \frac{1}{3!} \epsilon_{\mu \mu_1 \mu_2 \mu_3}^\mu_1 \epsilon_{\nu_1 \nu_2 \nu_3}^\nu_1 G_{\nu_1 \nu_2 \nu_3},
\]

(2.5) where

\[
\zeta_{\mu} = \nu_{2} \nu_{\mu} + \sum_{M} \xi_{M}^\mu \xi_{M}^\mu.
\]

(2.6) i.e. \( H \) is anti-self-dual while \( H^\mu \) is self-dual.

The KSEs of \( (1, 0) \)-supergravity, including that of the vector multiplet which has not been given above, have been solved for all backgrounds preserving any number of supersymmetries in [24]. Here, we shall adapt the analysis to describe the topology and geometry of all supersymmetric horizons.

2.2. Near horizon geometry

The description of the fields of \( (1, 0) \) supergravity near the horizon of an extreme black hole using null Gaussian coordinates [23] is similar to that which has been given for the fields of heterotic supergravity in [17]. Because of this, we shall not present the details here. In particular, the near horizon metric, 3-form field strengths and scalars of \( (1, 0) \) supergravity can be written as

\[
ds^2 = 2 e^+ e^- + \delta_{ij} e^i e^j,
\]

\[
G^\mu = e^+ \wedge e^- \wedge (dS - N \wedge S' h) + r e^+ \wedge (h \wedge N^\mu - dN^\mu - S' d h) + dW^\mu, \tag{2.7}
\]

\[
\phi^\mu = \phi(y), \quad \varphi = \varphi(y),
\]

where

\[
e^+ = du, \quad e^- = dr + rh \Delta du, \quad e^i = e^i dy^i.
\]

(2.8) The spacetime has coordinates \((r, u, y^i)\). The black hole horizon section \( S \) is the co-dimension 2 subspace \( r = u = 0 \) and it is assumed to be compact, connected and without boundary. The dependence of fields on light-cone coordinates \((r, u)\) is explicitly given. In addition, \( W^\mu \) are 2-forms, \( h, N^\mu \) are 1-forms and \( S' \) are scalars on the horizon section \( S \) and depend only on the coordinates \( y^i \). \( e^i \) is a frame on \( S \) and depends only on \( y \) as well. Both the tensor and hypermultiplet scalars depend only on the coordinates of \( S \).

To find the supersymmetric horizons of six-dimensional \((1, 0)\) supergravity, one has to solve both the field and KSEs of the theory for the fields given in (2.7). We shall proceed with the solution of KSEs.

2.3. Solution of KSEs

To continue, we substitute (2.7) into the KSEs (2.2) and assume that the backgrounds preserve at least one supersymmetry. Furthermore, we identify the stationary Killing vector field \( \partial_u \) of the near horizon geometry with the Killing vector constructed as a Killing spinor bilinear. A detailed analysis of these calculations has been presented in the appendix. The end result of this computation is that the Killing spinor can be chosen as

\[
\epsilon = 1 + \epsilon_{1234} \tag{2.9}
\]
and the fields can be rewritten as

\[ ds^2 = 2e^+ e^- + \delta_{ij} e^i e^j. \]

\[ H = e^+ \wedge e^- \wedge h + r e^+ \wedge \text{dh} - \frac{1}{3!} h e^i_{ijk} e^j \wedge e^k. \]

\[ \text{HM} = T M i e^- \wedge e^+ \wedge e^i - \frac{1}{3!} T M \ell \epsilon_{ij k} e^i \wedge e^j \wedge e^k, \]

\[ \phi^I = \phi^I(y), \quad \phi = \phi(y), \]

where we have used the duality relations of the 3-form field strengths. In particular,

\[ h \ell \epsilon_{ij k} = -v_r^i d W^{r i j k} \]

and similarly for \( \text{HM} \). In addition the anti-self-duality of \( H \) requires

\[ dh_{ij} = -\frac{1}{2} \epsilon_{ij} h_{k l} \text{dh}_{kl}. \]

It is clear that \( H \) is entirely determined in terms of \( h \) while \( \text{HM} \) is entirely determined in terms of the scalars \( \phi \) of the tensor multiplets.

Furthermore, the gravitino KSE along the horizon section directions requires that

\[ \tilde{D}_i (1 + e_{1234}) = 0, \]

where

\[ \tilde{D}_i = \tilde{\nabla}_i + C^r_{ij} \rho^r, \]

and \( \tilde{\nabla} \) is the connection on \( S \) with skew-symmetric torsion \(-\ast_4 h\). One can unveil the geometric content of this equation by considering the quaternionic-Hermitian 2-forms

\[ \omega_j = -i \delta_{ij} e^i \wedge e^j, \quad \omega_j = -e^1 \wedge e^2 - e^3 \wedge e^4, \quad \omega_k = i(e^1 \wedge e^2 - e^3 \wedge e^4) \]

on \( S \) which can be constructed as twisted Killing spinor bilinears, see [24]. In particular, setting \( \omega^1 = \omega_j, \omega^2 = \omega_j \) and \( \omega^3 = \omega_K \), the integrability condition of (2.12) can be expressed as

\[ -\tilde{R}_{mn} \epsilon_{ij} \omega^r_{k j} + \langle j, i \rangle + 2 \mathcal{F}^r_{mn} \epsilon_{rs} \omega^s_{ij} = 0, \]

where

\[ \mathcal{F}^r_{mn} = \partial_m \phi^r_{n} \partial_n \phi^r_{m} \].

The integrability condition identifies the \( Sp(1) \subset Sp(1) \cdot Sp(1) \) component of the curvature \( \tilde{R} \) of the four-dimensional manifold \( S \) with the pullback with respect to \( \phi \) of the \( Sp(1) \) component of the curvature of \( Q \). The restriction imposed on the geometry of \( S \) by (2.15) depends on the scalars \( \phi^I \). In particular, if \( \phi^I \) are constant, then \( \mathcal{F}^r_{mn} = 0 \) and (2.15) implies that \( S \) is an HKT manifold [28].

There are no additional conditions arising from the tensorini KSE. The hyperini KSE implies in addition that

\[ -V^M_1 + V^M_2 = 0, \quad V^M_2 + V^M_1 = 0. \]

We shall return to all the above conditions imposed by the KSEs after imposing the restrictions on the fields implied by the field equations of the theory and the compactness of \( S \).

3. Horizons with \( h \neq 0 \)

There are two classes of horizons to consider depending on whether or not \( h \) vanishes. First, we shall consider the case that \( h \neq 0 \).
3.1. Holonomy reduction

If $h \neq 0$, we shall demonstrate that, as in the heterotic case, the number of supersymmetries preserved by the near horizon geometries is always even. For this we shall use the results we have obtained from the KSEs for horizons preserving one supersymmetry and the field equations of the theory. The methodology we shall follow to prove this is to compute $\bar{\nabla}^2 h^2$ and apply the maximum principle utilizing the compactness of $S$.

The field equations of six-dimensional supergravity in the absence of vector multiplets are

$$
R_{\mu \nu} - \frac{1}{4} \varepsilon_{\alpha \beta} G^\alpha_m \varepsilon_{\mu \beta} G^\beta_n + \partial_\mu \varphi^I \partial_\nu \varphi_I - 2g_{IJ} \partial_\mu \varphi^I \partial_\nu \varphi_J = 0, \\
\nabla_\lambda (\varepsilon_{\alpha \beta} G^{\alpha \beta}_{\mu \nu}) = 0, \\
\nabla^\mu \partial_\mu v^I + \frac{1}{6} v^I G_{\mu \nu \rho} G^\mu_{\nu \rho} = 0, \\
D_\mu \partial^\mu \varphi^I = 0,
$$

(3.1)

where in the last equation it is understood that the Levi-Civita connections of both the spacetime and the hypermultiplet quaternionic Kähler manifold metrics have been used to covariantize the expression.

First, one finds that

$$
\bar{\nabla}^2 h^2 = 2\bar{\nabla} h^i \bar{\nabla} h^j + 2\bar{\nabla} (dh)_i h^j + 2\bar{\nabla} (dh)_j h^i + 2h^i \bar{\nabla} \bar{\nabla} h^i, \\
$$

(3.2)

where $\bar{\nabla}$ is the Levi-Civita connection of $\mathcal{S}$ with respect to $dx^2(\mathcal{S}) = \delta_{ij} e^i e^j$ and $\bar{R}$ is the associated Ricci tensor. The proof of this is given in [17]. To proceed, we shall utilize the field equations to rearrange the above expression in such a way that we can apply the maximum principle. Using the Einstein equation and

$$
\bar{\nabla}^2 h^i = -h^2 \partial_k v^k \partial^i v^k + 4\partial_i \varphi^I \partial_j \varphi_J g^{IJ} h^i - h^i \bar{\nabla} \bar{\nabla} h^i, \\
$$

(3.4)

The $\mu \nu = +-$ component of the field equation $\nabla_\lambda (\varepsilon_{\alpha \beta} G^{\alpha \beta}_{\mu \nu})$ together with $H^{i+-} = -h^i$ and $H^{Mij+} = T^{Mij}$ gives

$$
\partial_\mu v^+ \partial_\mu v^- + \bar{\nabla} \partial^\mu v^\mu = 0. \\
$$

(3.5)

Acting on the above expression with $v^\mu$, we find

$$
\bar{\nabla} h^i + v^i \bar{\nabla} v^\mu \partial^\mu v^\mu = 0, \\
$$

(3.6)

where we have used $v^i v^i = 1$.

The field equation of the scalars of the tensor multiplet gives

$$
v^I \bar{\nabla} v^\mu \partial^\mu v^\mu = 0, \\
$$

(3.7)

which when combined with (3.6) implies that

$$
\bar{\nabla} h^i = 0. \\
$$

(3.8)

In addition, (3.7) and $v^i v^i = 1$ give

$$
\partial_\mu v^\mu v^\mu = 0. \\
$$

(3.9)

Thus substituting (3.4) into (3.2) and using (3.8) and (3.9), we find that

$$
\bar{\nabla}^2 h^2 + h^i \bar{\nabla} h^i = 2\bar{\nabla} h^i \bar{\nabla} h^j + 2\bar{\nabla} (dh)_i h^j + 4\partial_\mu \varphi^I \partial_\mu \varphi_J g^{IJ} h^i. \\
$$

(3.10)
This expression is close to the one required for the maximum principle to apply. It remains to determine $dh$. For this, consider the $jk$-component of the 3-form field equation to find

$$\nabla^i \left( v^i \mathcal{H}_{ijk} + s \mathcal{H}^l_{ijl} \right) = \epsilon_{ijkl} \partial^i v^j h^l + v^l \epsilon_{ijkl} \nabla^i h^l = 0,$$

which implies that

$$dh = 0.$$  \hspace{1cm} (3.12)

Substituting this into (3.10), we obtain

$$\tilde{\nabla}^2 h^2 + h^i \tilde{\nabla}_i h^2 = 2 \tilde{\nabla}^i h^j \tilde{\nabla}_j h^i + 4 \partial_i \phi j \partial j \phi g_{ij} h^i.$$  \hspace{1cm} (3.13)

Applying now the maximum principle using the compactness of $S$, we find that $h^2$ is constant and

$$\tilde{\nabla}_i h_j = 0,$$

$$h^i \partial_i \phi = 0.$$  \hspace{1cm} (3.14)

To establish the latter equation, we have used that the metric of the hypermultiplet quaternionic Kähler manifold is positive definite. Thus, $h$ is a parallel 1-form on $S$ with respect to the Lévi-Civitá connection and the scalars of the hypermultiplets are invariant under the action of $h$. Note also that $\tilde{\nabla} h = 0$ as $i^h \tilde{H} = 0$.

The existence of a parallel 1-form on the horizon section $S$ with respect to the Lévi-Civitá connection is a strong restriction. First, it implies that the holonomy of $\nabla$ is contained in $SO(3)$ and

$$\text{hol}(\tilde{\nabla}) \subset SO(3).$$  \hspace{1cm} (3.15)

Moreover, $S$ metrically (locally) splits into a product $S^1 \times \Sigma^3$, where $\Sigma^3$ is a three-dimensional manifold. In turn, as we shall see, the near horizon geometry is locally a product $\text{AdS}_3 \times \Sigma^3$.

More elegantly the near horizon geometry admits a supersymmetry enhancement from one supersymmetry to two.

### 3.2. Supersymmetry enhancement

To demonstrate supersymmetry enhancement for the backgrounds with $h \neq 0$, let us reinvestigate the KSEs for the fields given in (2.10). It is straightforward to see by substituting (2.10) into the KSEs and following the calculation in the appendix that the general form of a Killing spinor is

$$\epsilon = \eta_+ - \frac{u}{2} h \Gamma^i \eta_- + \eta_-,$$  \hspace{1cm} (3.16)

where $\eta_{\pm}$ depend only on the coordinates of $S$. In addition, the gravitino KSE requires that

$$\tilde{\nabla}_i \epsilon + C_i^l \partial_l \epsilon = 0,$$  \hspace{1cm} (3.17)

the tensorini KSE implies that

$$\left(1 \pm \frac{1}{2}\right) T_{ij}^M \Gamma^i \epsilon_{\pm} - \frac{1}{12} \mathcal{H}^M_{ijk} \Gamma^{ijk} \epsilon_{\pm} = 0$$  \hspace{1cm} (3.18)

and the hyperini KSE gives

$$i \Gamma^i \epsilon_{\pm} a V_{ij}^{ \pm \pm} = 0.$$  \hspace{1cm} (3.19)

Next we shall show that both

$$\epsilon_1 = 1 + e_{1234}, \quad \epsilon_2 = \Gamma_- h \Gamma^i (1 + e_{1234}) - uk^2 (1 + e_{1234})$$  \hspace{1cm} (3.20)

are Killing spinors, where we have set $k^2 = h^2$ for the constant length of $h$. Observe that the second Killing spinor is constructed by setting $\eta_+ = 0$ and $\eta_- = \Gamma_- h \Gamma^i (1 + e_{1234})$. 


We have already solved the KSEs for \( \epsilon_1 \). Next observe that \( \epsilon_2 \) solves the gravitino KSE as the Clifford algebra operation \( h_1 \Gamma^+ \Gamma^- \) commutes with the supercovariant derivative in (3.17) as a consequence of the reduction of holonomy demonstrated in the previous section. In addition, the same Clifford operation commutes with the hyperin KSE as a result of the second equation in (3.14) and (3.19).

It remains to show that \( \epsilon_2 \) solves the tensorini KSE as well. This is a consequence of (3.9). For this observe that the metric induced on \( SO(n_T, 1)/SO(n_T) \) by the algebraic equation \( \eta_{\omega_2} \nu \eta_2 = 1 \) is the standard hyperbolic metric. So it has Euclidean signature. As a result,

\[
\bar{d}_i \nu_2 = 0. \tag{3.21}
\]

Thus, we conclude that the scalar fields are constant and the 3-form field strengths of the tensorini multiplet vanish. This agrees with the classification results of [24] for solutions of the KSEs of six-dimensional supergravity preserving at least two supersymmetries whose Killing spinors have a compact isotropy group. Some of the results of this section are tabulated in table 1.

### 3.3. Geometry

To investigate the geometry of spacetime, one can compute the form bilinears associated with the Killing spinors (3.20). In particular, one finds that the spacetime admits 3 \( \hat{\nabla} \)-parallel 1-forms given by

\[
\lambda^- = e^- , \quad \lambda^+ = e^+ - \frac{1}{2} k^2 u^e e^- - u h , \quad \lambda^1 = k^{-1} (h + k^2 u^e) . \tag{3.22}
\]

Moreover, the Lie algebra of the associated vector fields closes in \( sl(2, \mathbb{R}) \). To verify this, see [17]. Since \( h \) is \( \hat{\nabla} \)-parallel, the spacetime is locally metrically a product \( SL(2, \mathbb{R}) \times \Sigma^3 \), i.e.

\[
\begin{align*}
\text{dx}^2 &= \text{dx}^2 (SL(2, \mathbb{R})) + \text{dx}^2 (\Sigma^3) , \\
H &= \text{dvol}(SL(2, \mathbb{R})) + \text{dvol}(\Sigma^3) , \\
\phi^L &= \phi^L (z) .
\end{align*}
\tag{3.23}
\]

where the scalars of the hypermultiplet depend only on the coordinates \( z \) of \( \Sigma^3 \).

In addition to the 1-forms given in (3.22), the spacetime admits three more twisted 1-form bilinears, see [24]. For the Killing spinors (3.20), these are given by

\[
e' = k^{-1} h J' \theta e' , \quad \tag{3.24}
\]

where \( J' \) is a quaternionic structure on \( S \) associated with the quaternionic-Hermitian 2-forms (2.14). As has been already mentioned, these quaternionic-Hermitian 2-forms are constructed from twisted spinor bilinears and so rotate to each other under patching conditions. Observe that the frame \( e' \) is orthogonal to \( h \) and the rotation between the \( e' \) and \( (h, e') \) is in \( SO(4) \). Therefore \( (k^{-1} h, e') \) is another frame on \( S \) with \( e' \) adapted to \( \Sigma^3 \). Thus, \( \text{dx}^2 (S) = k^{-2} h^2 + \text{dx}^2 (\Sigma^3) \) with \( \text{dx}^2 (\Sigma^3) = \delta_{e'} e'e' \).

| Iso(\( \eta_+ \)) | hol(\( \hat{\nabla} \)) | \( N \) | \( \eta_+ \) |
|-------------------|----------------|--------|-----------|
| \( Sp(1) \times Sp(1) \times \mathbb{H} \) | \( Sp(1) \) | 2 | \( 1 + e_{1234} \) |
| \( U(1) \times Sp(1) \times \mathbb{H} \) | \( U(1) \) | 4 | \( 1 + e_{1234} , i(1 - e_{1234}) \) |
| \( Sp(1) \times \mathbb{H} \times \mathbb{H} \) | \( \{1\} \) | 8 | \( 1 + e_{1234} , i(1 - e_{1234}) , e_{12} - e_{34} , i(1 + e_{34}) \) |

Table 1. Some of the geometric data used to solve the gravitino KSE are described. In the first column, we give the isotropy groups, Iso(\( \eta_+ \)), of \( \{ \eta_+ \} \) spinors in \( Spin(5, 1) \cdot Sp(1) \). In the second column, we state the holonomy of the supercovariant connection \( \hat{\nabla} \) of the horizon section \( S \) in each case. The holonomy of \( \hat{\nabla} \) is identical to that of \( \hat{\nabla} \). In the third column, we present the number of \( D \)-parallel spinors and in the last column we give representatives of the \( \{ \eta_+ \} \) spinors.
The metric on $\Sigma^3$ is restricted by the Einstein equation (3.1) and the integrability condition (2.15). The former gives

$$R^{(3)}_{r's'} - \frac{1}{2}k^2 \delta_{r's'} - 2 \partial_r \phi^l \partial_{r'} \phi^l g_{uu} = 0,$$

where $r'$ and $s'$ are indices of $\Sigma^3$ and $R^{(3)}$ is the Ricci tensor of $\Sigma^3$. This is an equation which determines the metric on $\Sigma^3$ in terms of $h$ and the hypermultiplet scalars $\phi$. The integrability condition (2.15) does not give an independent condition on the metric of $\Sigma^3$.

It remains to find the restriction imposed by supersymmetry on the scalars $\phi$ of the hypermultiplet. As we have shown, these depend only on the coordinates of $\Sigma^3$. A direct observation reveals that after an appropriate identification of the frame directions of $S$ with the Pauli matrices $\sigma_r$, the supersymmetry conditions can be rewritten as

$$\partial_r \phi^l = -\epsilon_r^{r'}(I_r)^l_2 \partial_r \phi^2,$$

where we have used that $(I_r)^l_2 \sigma_0 = -i \sigma_r^a \delta^a_2$. This is a rather natural condition constraining the maps $\phi$ from $\Sigma^3$ into $Q$. Constant maps are solutions.

The geometry on $\Sigma^3$ is determined by (3.25) and depends on the solutions of (3.26). For the constant map solutions of (3.26), $\Sigma^3$ is locally isometric to $S^3$ equipped with the round metric, and so the near horizon geometry is $AdS_3 \times S^3$.

Next suppose the existence of non-trivial solutions for equation (3.26), and upon substitution the existence of solutions for (3.25). A priori one expects that the geometry on $\Sigma^3$ depends on the choice of quaternionic Kähler manifold $Q$ for the hypermultiplets and the choice of a solution of (3.26). However, the differential structure on $\Sigma^3$ is independent of these choices. To show this first observe that the Ricci tensor $R^{(3)}$ is strictly positive. This turns out to be sufficient to determine the topology on $\Sigma^3$. To see this note that in three dimensions the Ricci tensor determines the curvature of a manifold. Next, the strict positivity of the Ricci tensor implies that the (reduced) holonomy of the Levi-Civita connection of $\Sigma^3$ is $SO(3)$. Then a result of Gallot and Meyer, see [25], implies that $\Sigma^3$ is a homology 3-sphere. A brief proof of this is as follows. Since the holonomy of the Levi-Civita connection of $\Sigma^3$ is $SO(3)$, the only parallel forms are the constant real maps and the volume form of the manifold. On the other hand, the positivity of the Riemann curvature tensor implies that all harmonic forms are parallel and the fundamental group is finite. Thus, de Rham cohomology of $\Sigma^3$ coincides with that of $S^3$ and so $\Sigma^3$ is a homology 3-sphere. In addition since the fundamental group is finite, the universal cover of $\Sigma^3$ is compact and so by the Poincaré conjecture [26] homeomorphic, and so diffeomorphic, to the 3-sphere. The above result implies that in the simply connected case and for non-constant solutions to (3.26), the geometry of the round sphere is deformed in such a way that the differential, and so topological, structure of $S^3$ is maintained.

The existence of non-trivial solutions to (3.26) is an open problem which may depend on the choice of quaternionic Kähler manifold $Q$ of the hypermultiplets. However, as we shall see, horizons that preserve eight supersymmetries require $\phi$ to be constant. This is compatible with the assertion made in the attractor mechanism, see [27] for the six-dimensional supergravity case, that all the scalars take constant values at the horizon. However, it is worth noting that the field and KSEs do not a priori imply that the scalars are constant for near horizon geometries which preserve a small number of supersymmetries. For this some further investigation is required, which may be case dependent.

\[5\] There is a unique differential structure on the topological 3-sphere.
4. \( N = 4 \) and \( N = 8 \) horizons

4.1. \( N = 4 \) horizons

We have shown that if \( h \neq 0 \), the near horizon geometries preserve two, four or eight supersymmetries. We have already investigated the case with two supersymmetries. The two additional Killing spinors of horizons with four supersymmetries can be chosen as

\[
e^3 = i(1 - e_{1234}), \quad e^4 = -i k^2 u(1 - e_{1234}) + i h_1 \Gamma^{1/1}(1 - e_{1234}).
\]

(4.1)

Observe that \( e^3 = \rho^1 e^1 \) and \( e^4 = \rho^1 e^2 \). Thus, the KSEs must commute with \( \rho^1 \). As a result \( \omega_1 \) is a well-defined Hermitian form on \( S \). The 1-form \( \bar{V} \)-parallel spinor bilinears are

\[
\lambda^- = e^- - \frac{1}{2} k^2 u^2 e^- - uh, \quad \lambda^1 = k^{-1}(h + k^2 u e^-),
\]

\[
\lambda^4 = e^1,
\]

where the first three bilinears are those of horizons with two supersymmetries and \( e^1 \) is given in (3.24). The associated vector fields are Killing and their Lie algebra is \( sl(2, \mathbb{R}) \oplus \mathfrak{u}(1) \).

The spacetime is locally metrically a product \( \text{AdS}_3 \times \Sigma^3 \), as for horizons preserving two supersymmetries. In addition, in this case, \( \Sigma^3 \) is a \( S^1 \) fibration over a two-dimensional manifold \( \Sigma^2 \). The fiber direction is spanned by \( \lambda^4 = e^1 \). Thus,

\[
ds^2(\Sigma^1) = (e^1)^2 + ds^2(\Sigma^2), \quad ds^2(S) = k^{-2}h^2 + (e^1)^2 + ds^2(\Sigma^2).
\]

(4.3)

Observe that \( de^1 \neq 0 \) as \( e^1 \wedge de^1 \) is proportional to \( \tilde{H} = d\text{vol}(\Sigma^1) \), and so the fibration is twisted.

It remains to specify the topology of \( \Sigma^2 \). For this first observe that from the results of \cite{24}, the hypermultiplet scalars depend only on the coordinates of \( \Sigma^2 \). To specify the topology of \( \Sigma^2 \), we compute the Ricci tensor \( R^{(2)} \) of \( \Sigma^2 \) using the Einstein equation and in particular (3.25) to find

\[
R^{(2)}_{\gamma \gamma} - \frac{1}{2} d\epsilon^{\gamma \gamma}_{\gamma \gamma'} (d\epsilon^1_{\gamma'} \delta_{\gamma'} - \frac{1}{2} k^2 \delta_{\gamma'} - 2 \partial_{\nu} \phi \partial_{\nu} \phi \delta_{\gamma'} \phi_{\gamma'} = 0,
\]

(4.4)

where now \( \gamma', \gamma', \alpha' \) and \( \nu ' \) are indices in \( \Sigma^2 \). It is clear that the Ricci tensor of \( \Sigma^2 \) is strictly positive and so \( \Sigma^2 \) is topologically a sphere irrespective of the properties of the maps \( \phi \).

We have already mentioned that the hypermultiplet scalars \( \phi \) depend only on the coordinates of \( \Sigma^2 \) as a consequence of the hyperini KSE. Thus, they are maps from \( \Sigma^2 \) into the quaternionic Kähler manifold of the hypermultiplets. In addition the hyperini KSE implies that

\[
V^{e_1}_{\Sigma^1} = 0, \quad V^{e_2}_{\Sigma^1} = 0,
\]

(4.5)

which is equivalent to (3.26) after additionally requiring that the scalars do not depend on the fiber direction \( \lambda^4 \). These conditions imply that \( \phi \) are pseudo-holomorphic maps from \( \Sigma^2 \) into the quaternionic Kähler manifold of the hypermultiplets. The analysis we have made for the existence of non-constant solutions to (3.26) applies to (4.5) as well.

4.2. \( N = 8 \) horizons

As in the cases with two and four supersymmetries, one can show that the spacetime is locally \( \text{AdS}_3 \times \Sigma^3 \). In addition, for horizons with eight supersymmetries, the hyperini KSE implies that the scalars of the hypermultiplet are constant \cite{24}. In such a case, the Einstein equation implies that \( \Sigma^3 \) is locally isometric to \( S^3 \). Thus, the only near horizon geometry preserving eight supersymmetries with \( h \neq 0 \) is \( \text{AdS}_3 \times S^3 \).
5. Horizons with $h = 0$

5.1. Geometry of $N = 1$ horizons

Let us now turn to horizons with $h = 0$. Clearly in such a case, the 3-form field strength of the gravitational multiplet vanishes, $H = 0$, and the near horizon geometry is a product $R^{1,1} \times \mathcal{S}$. It remains to determine the geometry of $\mathcal{S}$.

For this first observe that the tensor multiplet scalars are constant and the associated 3-form field strengths vanish. The proof for this is similar to that given for the horizons with $h \neq 0$. In particular, it utilizes the field equations of the tensor multiplet scalars as described in equations (3.6) and (3.7), with $h = 0$, and the argument developed around (3.21).

The Einstein equation expresses the Ricci tensor $\tilde{\mathcal{R}}$ of $\mathcal{S}$ in terms of the hypermultiplet scalars as

$$\tilde{\mathcal{R}}_{ij} = 2g_{ij} \partial_i \phi J^a \partial_j \phi^a. \quad (5.1)$$

The latter are also restricted by the KSEs as in (2.17). Observe that after an appropriate identification of frame directions of $\mathcal{S}$ with the matrices $$(\tau^i) = (1_{2 \times 2}, i \sigma^r)'$$, (2.17) can be written as

$$(K_j) \partial_j \phi^a = 0, \quad (5.3)$$

where $(K_j) = (1_{4\eta \times 4 \eta}, -I_r)$ and $1_{4\eta \times 4 \eta}$ is the identity tensor.

If the hypermultiplet scalars are constant, then the rhs of (5.1) vanishes and $\mathcal{S}$ is a hyper-Kähler manifold. So it is locally isometric to either $K_3$ or $T^4$. As we shall see, such horizons exhibit supersymmetry enhancement to at least $N = 4$.

The existence of horizons with strictly $N = 1$ supersymmetry depends on the existence of non-trivial solutions for (5.3) such that the rhs of (5.1) does not vanish. This in turn may depend on the choice of the 4-manifold $\mathcal{S}$ and that of the quaternionic Kähler manifold $Q$. As a result, this is a rather involved question, and possibly model dependent, which we shall not explore further here.

5.2. Geometry of $N = 2$ and $N = 4$ horizons

The second Killing spinor of $N = 2$ horizons with $h = 0$ can be chosen as

$$\epsilon_2 = i(1 - e_{1234}). \quad (5.4)$$

In such a case, and in agreement with the general classification results of [24], $\mathcal{S}$ is a Kähler manifold. In addition, the hypermultiplet scalars are holomorphic maps from $\mathcal{S}$ into the hypermultiplets quaternionic Kähler manifold. Again, the existence of such horizons with strictly two supersymmetries depends on the existence of such non-trivial holomorphic maps. The two remaining Killing spinors of $N = 4$ horizons with $h = 0$ can be chosen as

$$\epsilon_3 = e_{12} - e_{34}, \quad \epsilon_4 = i(e_{12} + e_{34}). \quad (5.5)$$

The general classification results of [24] imply that the hypermultiplet scalars are constant as a consequence of the hyperini KSEs. Therefore, $\mathcal{S}$ is hyper-Kähler and so locally isometric to either $K_3$ or $T^4$. In the latter case, there is supersymmetry enhancement to $N = 8$.

So far we have considered Killing spinors annihilated by the light-cone projection operation $\Gamma_+$. As a result, they have a non-compact isotropy group in $Spin(5, 1) \cdot Sp(1)$. We could demand that the near horizon geometries $R^{1,1} \times \mathcal{S}$ admit Killing spinors with compact isotropy groups. In such a case, the only solution is $R^{1,1} \times T^4$, which preserves eight supersymmetries. Some of the results of this section are tabulated in table 2.
Table 2. Some geometric data of the horizon geometries with \( h = 0 \) are described. In the first column, we give the number of supersymmetries preserved. In the second column, we present the holonomy groups of the Lévi-Civitá connection of \( S \), and in the third we give the geometry of \( S \).

| \( N \) | \( \text{hol(\tilde{\nabla})} \) | Geometry of \( S \) |
|-------|-----------------|-----------------|
| 1     | \( Sp(1) \cdot Sp(1) \) | Riemann         |
| 2     | \( U(2) \)       | Kähler          |
| 4     | \( Sp(1) \)       | hyper-Kähler    |

Acknowledgments

We thank Jan Gutowski and Fabio Riccioni for many helpful discussions. MA is supported by the STFC studentship grant ST/F00768/1. GP is partially supported by the EPSRC grant EP/F069774/1 and the STFC rolling grant ST/G000/395/1.

Appendix. Supersymmetric horizons

A.1. Light-cone integrability of KSEs

The gravitino KSE is

\[
\mathcal{D}_\mu \epsilon = 0, \tag{A.1}
\]

where

\[
\mathcal{D}_\mu = \tilde{\nabla}_\mu + C^\rho_{\mu \nu} \rho_\nu = \partial_\mu + \frac{1}{2} \Omega_{\mu, \nu} \Gamma^{\nu \rho} - \frac{1}{8} H^{\mu \nu \rho} \Gamma^{\nu \rho} + C^\rho_{\mu \nu} \rho_\nu. \tag{A.2}
\]

We also identify the stationary Killing vector field of the black hole \( \partial_u \) with the Killing vector constructed as Killing spinor bilinear. Since the latter is null \[24\], \( \Delta = 0 \) in (2.7). As a result, the non-vanishing components of the frame connection associated with the Lévi-Civitá connection of the spacetime are

\[
\Omega_{+, -i} = -\frac{1}{2} h_i, \quad \Omega_{+, ij} = -\frac{1}{2} r (dh)_{ij}, \quad \Omega_{+, +i} = -\frac{1}{2} h_i, \quad \Omega_{i, +} = \frac{1}{2} h_i, \quad \Omega_{i, +j} = -\frac{1}{2} r (dh)_{ij}, \quad \Omega_{i, jk} = \Omega_{i, jk}. \tag{A.3}
\]

For later use, the anti-self-duality of \( H \) implies

\[
H_{+, -} = \frac{1}{3!} \epsilon_{+, -ijkl} H^{ijkl},
H_{+, 0} = \frac{1}{3!} \epsilon_{+, 0ijkl} v_{0} dW^{ijkl}. \tag{A.4}
\]

The KSEs can be integrated along the light-cone directions for the fields given in (2.7). For this, decompose the Killing spinor \( \epsilon \) as

\[
\epsilon = \epsilon_+ + \epsilon_-, \quad \Gamma_{\pm} \epsilon_{\pm} = 0. \tag{A.5}
\]

The \(-\) component of the gravitino KSE, (A.1), gives

\[
\partial_- \epsilon = \frac{1}{2} (v_2 dS^2 - N - (S + 1) h) \Gamma_-^\nu \epsilon_+ = 0, \tag{A.6}
\]

where we have used the expression for the frame connection stated above, the expression for the fields in (2.7) and that \( C'_\nu = 0 \) as the scalars do not depend on \((u, r)\) in the near horizon limit. We have also set \( N = v_2 N^2 \) and \( S = v_2 S^2 \). Noting that \( \partial_- = \partial_r \) and \( \partial_+ = \partial_u \) and upon integration, we find

\[
\epsilon_+ = \phi_+, \quad \epsilon_- = \phi_- + \frac{1}{2} r (v_2 dS^2 - N - (S + 1) h) \Gamma_-^\nu \phi_+, \tag{A.7}
\]

where \( \phi_{\pm} \) are independent of \( r \).
Similarly, the + component of the gravitino KSE gives
\[ \partial_+ \epsilon + \frac{1}{4}(v_\perp dS^2 - N - (S - 1)h_i)\Gamma^i \epsilon_+ - \frac{1}{8} r(h \wedge N - v_\perp dN^2 - (S - 1) dh)_{ij} \Gamma^{ij} \epsilon = 0. \]  
(A.8)

Substituting (A.7) into the + component of the gravitino KSE, we obtain
\[ \partial_+ (\phi + \frac{1}{4} r(v_\perp dS^2 - N - (S + 1)h_i)\Gamma^i \phi_+) \]
\[ + \frac{1}{2}(v_\perp dS^2 - N - (S - 1)h_i)\Gamma^i \Gamma_+ \]
\[ \times (\phi_+ + \frac{1}{4} r(v_\perp dS^2 - N - (S + 1)h_i)\Gamma^i \phi_+) \]
\[ - \frac{1}{8} r(h \wedge N - v_\perp dN^2 - (S - 1) dh)_{ij} \Gamma^{ij} \]
\[ \times (\phi + \frac{1}{4} r(v_\perp dS^2 - N - (S + 1)h_i)\Gamma^i \phi_+) = 0. \]  
(A.9)

This equation is valid in every order in \( r \). As a result \( O(r^0) \)-order term gives
\[ \partial_+ \phi + \frac{1}{4}(v_\perp dS^2 - N - (S - 1)h_i)\Gamma^i \epsilon_+ = 0, \]  
(A.10)

which can be solved to find
\[ \phi_+ = \eta_+ - \frac{1}{4} u(v_\perp dS^2 - N - (S - 1)h_i)\Gamma^i \eta_-, \]
\[ \phi_- = \eta-, \]  
(A.11)

where \( \eta_\pm \) is independent of \( r \) and \( u \). As a result, the components \( \epsilon_\pm \) of the Killing spinor can be written in terms of \( \eta_\pm \) as
\[ \epsilon_+ = \eta_+ - \frac{1}{4} u(v_\perp dS^2 - N - (S - 1)h_i)\Gamma^i \eta_-, \]
\[ \epsilon_- = \eta_- + \frac{1}{4} r(v_\perp dS^2 - N - (S + 1)h_i)\Gamma^i \eta_+, \]
\[ + \frac{1}{8} ur(v_\perp dS^2 - N - (S + 1)h_i) \Gamma^i \Gamma^j \eta_-. \]  
(A.12)

The remaining conditions implied by (A.9) are algebraic, which will be considerably simplified after the analysis of the following section. These are
\[ \alpha_i \beta_j \Gamma^i \Gamma^j \eta_+ + \gamma_i \Gamma^i \eta_+ = 0, \]  
(A.13)

\[ \gamma_i \beta_j \Gamma^i \Gamma^j \eta_+ = 0, \]  
(A.14)

\[ \beta_i \alpha_j \Gamma^i \Gamma^j \eta_- - \gamma_i \Gamma^i \eta_- = 0, \]  
(A.15)

\[ \alpha_i \beta_j \alpha_k \Gamma^i \Gamma^j \Gamma^k \eta_- + \gamma_i \alpha_k \Gamma^i \Gamma^k \eta_- = 0, \]  
(A.16)

\[ \gamma_i \beta_j \alpha_k \Gamma^i \Gamma^j \Gamma^k \eta_- = 0, \]  
(A.17)

where
\[ \alpha_i = (v_\perp dS^2 - N - (S - 1)h)_i, \]  
(A.18)
\[ \beta_i = (v_\perp dS^2 - N - (S + 1)h)_i, \]  
(A.19)
\[ \gamma_i = (h \wedge N - v_\perp dN^2 - (S - 1) dh)_{ij}. \]  
(A.20)

These are in fact the same constraints as those found for the heterotic horizons in [17].
A.2. Stationary and spinor bilinear vector fields

Additional restrictions on $\eta_\pm$ can be derived for horizons preserving one supersymmetry arising from the identification of stationary black hole Killing vector field $\partial_u$ with that constructed as a Killing spinor bilinear. This identification implies that the components of the 1-form associated with the latter are

$$X_+ = 0, \quad X_- = 1, \quad X_i = 0.$$  \hspace{1cm} (A.21)

The $\eta_\pm$ spinors can be expanded in basis of symplectic Majorana–Weyl spinors as

$$\eta_+ = a_1 (1 + e_{1234}) + a_2 (e_{15} + e_{2345}) + a_3 (e_{15} - e_{2345}) + a_4 i (e_{15} + e_{2345}),$$  \hspace{1cm} (A.22)

$$\eta_- = b_1 (e_{15} + e_{2345}) + b_2 i (e_{15} - e_{2345}) + b_3 (e_{25} - e_{1345}) + b_4 i (e_{15} + e_{2345}),$$  \hspace{1cm} (A.23)

where all components depend on the coordinates $y$ of $S$. The field data (2.7) are covariant under local $Spin(4) \cdot Sp(1)$ gauge transformations of $S$. So these can be used to choose $\eta_\pm$ as

$$\eta_+ = a(y) (1 + e_{1234}),$$

$$\eta_- = b(y) (e_{15} + e_{2345}).$$  \hspace{1cm} (A.23)

The next step is to consider the spinor bilinear

$$Y_A e^A = \langle B \epsilon^\star, \Gamma_A \epsilon \rangle e^A,$$  \hspace{1cm} (A.24)

where $B = \Gamma_{06789}$. In order to satisfy the relations in (A.21), we require the + component for the spinor bilinear to vanish. This in particular means that $Y_+ |_{r=0} = 0$, and as a consequence we find

$$\eta_- = 0.$$  \hspace{1cm} (A.25)

Therefore, we can write the spinor in (A.12) as

$$\epsilon = \eta_+ + \frac{1}{2} r (v_2 dS^c - N - (S + 1) h)i \Gamma_- \eta_+.$$  \hspace{1cm} (A.26)

Since the bilinear components on the horizon are independent of $r$, the next requirement we impose is for the $O(r)$ term in the bilinear to vanish. This means

$$\langle B (1 + e_{1234}), \Gamma_A (v_2 dS^c - N - (S + 1) h)i \Gamma_- \eta_+ \rangle = 0,$$  \hspace{1cm} (A.27)

from which we obtain the condition

$$v_2 dS^c - N - (S + 1) h = 0.$$  \hspace{1cm} (A.28)

This simplifies the Killing spinor as

$$\epsilon = \eta_+ = a(x) (1 + e_{1234}).$$  \hspace{1cm} (A.29)

Finally calculating $Y_-$ and comparing this to $X_-$, we find

$$-2 \sqrt{2} a^2 = 1,$$  \hspace{1cm} (A.30)

i.e. $a$ is a constant, which without loss of generality can be set to 1. This means

$$\epsilon = 1 + e_{1234}.$$  \hspace{1cm} (A.31)

This choice of a Killing spinor for the horizon geometries is the same as that for general solutions of the KSEs of six-dimensional supergravity preserving one supersymmetry. This will be used to simplify the analysis of near horizon geometries.
A.2.1. Further analysis of the gravitino KSE. Revisiting the \( + \) component of the gravitino KSE for \( \epsilon = 1 + e_{1234} \), one finds that

\[
(h \wedge N - v_\perp \, dN^\perp - (S - 1) \, dh)_j \Gamma^{ij} \eta_+ = 0. \tag{A.32}
\]

As a consequence all algebraic conditions (A.13)–(A.17) are also satisfied.

Next consider the \( i \)-component of the gravitino KSE. After separating the various orders in \( r \), one finds that

\[
\tilde{D}_i \eta_+ = \left( \partial_i + \frac{1}{3} \tilde{\Omega}_{i,jk} \Gamma^{jk} - \frac{1}{5} v_\perp (dW^\perp)_{jk} \Gamma^{jk} + C'_r \rho_r \right) \eta_+ = 0 \tag{A.33}
\]

and

\[
(h \wedge N - v_\perp \, dN^\perp - (S + 1) \, dh)_j \Gamma^j \eta_+ = 0. \tag{A.34}
\]

Using \( \eta_+ = 1 + e_{1234} \) in the last equation, we find that

\[
h \wedge N - v_\perp \, dN^\perp = (S + 1) \, dh = 0. \tag{A.35}
\]

As a result, the 3-form \( H \) simplifies as

\[
H = e^+ \wedge e^- \wedge h + re^+ \wedge dh + v_\perp \, dW^\perp. \tag{A.36}
\]

A.3. Tensorini KSEs

Now consider the tensorini KSEs

\[
\left( T^M_{\mu} \Gamma^{\mu} - \frac{1}{11} H^M_{\mu\nu\rho} \Gamma^{\mu\nu\rho} \right) \epsilon = 0, \tag{A.37}
\]

where \( \epsilon = \eta_+ = 1 + e_{1234} \). This has been solved in [24] where we have found

\[
T^M_{\perp} = 0, \quad H^M_{+\alpha} = H^M_{+\alpha\beta} = 0, \quad T^M_{\sigma} - \frac{1}{2} H^M_{+\sigma} - \frac{1}{2} H^M_{+\beta} = 0. \tag{A.38}
\]

These together with the self-duality of \( H^M_{+\sigma} \) imply that

\[
H^M_{+\sigma} = 0, \quad T^M_{\sigma} = H^M_{+\sigma}. \tag{A.39}
\]

Comparing this with the expression of \( H^M_{+\sigma} \) in (2.7), we obtain the expression for \( H^M_{+\sigma} \) in (2.10). In particular, we have

\[
\chi^M_+ \, d\Sigma^\perp - N^M + S^M \, dh = - T^M, \quad h \wedge N^M - \chi^M_+ \, dN^\perp - S^M \, dh = 0, \quad \tag{A.40}
\]

where \( N^M = \chi^M_+ \, N^\perp \) and \( S^M = \chi^M_+ \, \Sigma^\perp \).

A.4. Hyperini KSEs

A direct application of the results in [24] and using that the scalars of the hypermultiplet do not depend on the coordinates \((u, r)\) reveal that the hyperini KSEs imply that

\[
-V^M_1 + V^M_2 = 0, \quad V^M_2 + V^M_1 = 0, \tag{A.41}
\]

for \( \epsilon = 1 + e_{1234} \).
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