ALMOST SURE SCATTERING FOR THE NONRADIAL
ENERGY-CRITICAL NLS WITH ARBITRARY REGULARITY IN
3D AND 4D CASES

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Abstract. In this paper, we study the defocusing energy-critical nonlinear Schrödinger
equations
\[ i\partial_t u + \Delta u = |u|^{\frac{4}{d-2}} u. \]
When \( d = 3, 4 \), we prove the almost sure scattering for the equations with non-
radial data in \( H^s \) for any \( s \in \mathbb{R} \). In particular, our result does not rely on any
spherical symmetry, size or regularity restrictions.

1. Introduction

1.1. Definition of randomization
1.2. Main result
1.3. Sketch of the proof
1.4. Organization of the paper

2. Preliminary

2.1. Notation
2.2. Useful lemmas
2.3. Probabilistic theory

3. Almost sure Strichartz estimates

3.1. Strichartz estimates
3.2. Linear estimates in 3D case
3.3. Linear estimates in 4D case

4. Global well-posedness and scattering in 3D case

4.1. Reduction to the deterministic problem
4.2. Local theory
4.3. Modified Interaction Morawetz
4.4. Almost conservation law
4.5. Perturbations
4.6. Proof of Proposition 4.1

5. Global well-posedness and scattering in 4D case

5.1. Reduction to the deterministic problem
5.2. Local theory
5.3. Modified Interaction Morawetz
5.4. Almost conservation law
5.5. Perturbations
5.6. Proof of Proposition 5.1

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theory.
1. INTRODUCTION

In this paper, we consider the nonlinear Schrödinger equations (NLS):
\[
\begin{aligned}
\begin{cases}
    i\partial_t u + \Delta u = \mu |u|^p u, \\
    u(0, x) = u_0(x),
\end{cases}
\end{aligned}
\]
(1.1)

where \( p > 0, \mu = \pm 1, \) and \( u(t, x) : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C} \) is a complex-valued function. The positive sign “+” in nonlinear term of (1.1) denotes defocusing source, and the negative sign “−” denotes the focusing one.

The equation (1.1) has conserved mass
\[
M(u(t)) := \int_{\mathbb{R}^d} |u(t, x)|^2 \, dx = M(u_0),
\]
(1.2)

and energy
\[
E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2} |\nabla u(t, x)|^2 \, dx + \mu \int_{\mathbb{R}^d} \frac{1}{p+2} |u(t, x)|^{p+2} \, dx = E(u_0).
\]
(1.3)

The class of solutions to equation (1.1) is invariant under the scaling
\[
u(t, x) \to u_\lambda(t, x) = \lambda^{\frac{2}{p}} u(\lambda^2 t, \lambda x) \quad \text{for} \quad \lambda > 0,
\]
(1.4)

which maps the initial data as
\[
u(0) \to u_\lambda(0) = \lambda^{\frac{2}{p}} u_0(\lambda x) \quad \text{for} \quad \lambda > 0.
\]
(1.5)

Denote
\[s_c = \frac{d}{2} - \frac{2}{p},\]
then the scaling leaves \( \dot{H}^s \) norm invariant, that is,
\[
\|u(0)\|_{\dot{H}^{s_c}} = \|u_\lambda(0)\|_{\dot{H}^{s_c}}.
\]

This gives the scaling critical exponent \( s_c \). Let
\[2^* = \infty, \text{ when } d = 1 \text{ or } d = 2; \quad 2^* = \frac{4}{d-2}, \text{ when } d \geq 3.\]
Therefore, according to the conservation law, the equation is called mass or \( \mathcal{L}^x_\mu \) critical when \( p = \frac{4}{d} \), and energy or \( \dot{H}^1_x \) critical when \( p = \frac{4}{d-2} \). Moreover, when \( \frac{4}{d} < p < 2^* \), we say that the equation is inter-critical.

Let us now recall the well-posedness and scattering theory of NLS (1.1). There are extensive studies about the subject, and we do not intend to mention all the results. Therefore in this paper, we mainly focus on the energy critical case. First, the equation is locally well-posed in \( H^s_x \) with \( s \geq 1 \), and ill-posed in the case \( s < 1 \), see [18, 20].

The first global well-posedness and scattering result for the energy critical NLS was established by Bourgain [8]. Then, the defocusing case was proved by Colliander, Keel, Staffilani, Takaoka, and Tao [22] in three dimensional case, and Ryckman and Visan [37, 44] in higher dimensional cases. For the focusing equations, Kenig and Merle [29] first studied the dynamics below the energy of ground state, and then the
result is extended by Killip and Visan [31] in five and higher dimensions, Dodson [24] in four dimension.

Now, we turn to the probability theory of NLS. Although there are ill-posedness results below the critical regularity for NLS due to the result of Christ, Colliander, and Tao [20], Bourgain [5, 6] first introduced a probabilistic method to study the well-posedness problem for periodic NLS for “almost” all the initial data in super-critical spaces. The probabilistic well-posedness result for super-critical wave equations on compact manifolds was also studied by Burq and Tzvetkov [15, 16]. There have been extensive studies about such subject since then, and we refer the readers to [4] for more complete overviews.

Next, we only review the study of random data theory mainly for NLS on $\mathbb{R}^d$. There are several ways of randomization for the initial data. We start with the Wiener randomization, which is related to a unit-scale decomposition in frequency. For NLS, the almost sure local well-posedness, small data scattering, and “conditional” global well-posedness were considered in [2, 3, 9, 36, 38].

The random data global well-posedness for the energy critical problems was first proved in the context of non-linear wave equations (NLW) by Pocovnicu [35]. See also [34] for the 3D result. As for the NLS, Oh, Okamoto, and Pocovnicu [33] studied the almost sure global well-posedness in the energy critical case when $d = 5, 6$.

The large data almost sure scattering was first obtained by Dodson, Lührmann, and Mendelson [25] for the 4D, defocusing, energy-critical NLW with randomized radial data in $H^s_x$ for $\frac{1}{2} < s < 1$, using a double bootstrap argument combining the energy and Morawetz estimates. The first almost sure scattering result for NLS was given by Killip, Murphy, and Visan [30]. They proved the result for 4D, defocusing, energy-critical case with almost all the radial initial data with $\frac{5}{6} < s < 1$. This result was then improved to $\frac{1}{2} < s < 1$ by Dodson, Lührmann, and Mendelson [26]. While all the above scattering results concerns the energy critical case, the authors [38] and Camps [17] independently proved the random data global well-posedness and scattering for the 3D defocusing cubic NLS, which is a typical model of inter-critical NLS.

We remark that the Wiener randomization is closely related to the modulation space introduced by Feichtinger [27]. Such space has been applied to non-linear evolution equations before the development of Wiener randomization, dating back to the results of Wang, Zhao, Guo, and Hudzik [45, 46].

There are also other kinds of randomization for NLS on $\mathbb{R}^d$. Burq, Thomann, and Tzvetkov [12, 14] introduced a randomization based on the invariant measure for NLS with harmonic potential, and proved almost sure $L^2$-scattering for 1D defocusing NLS. Such randomization relies on the countable eigen-basis of the Laplacian, while the Wiener randomization directly comes from the decomposition in frequency space. In addition, Murphy [32] introduced an another randomization based on the physical space unit-scale decomposition to study the almost sure wave operator problem.

Furthermore, other randomizations have been applied to the almost sure global well-posedness and scattering of energy critical models. In [11], Bringmann introduced a randomization based on wave packet decomposition to study the non-radial 4D NLW in $H^s_x$ with $s > \frac{11}{6}$. In [10], Bringmann introduced another randomization based on annuli decomposition, and observed that the smaller the scale of decomposition is, the more smoothing effect one can expect. Based on this, he proved
that the 3D energy-critical wave equation is almost surely scattering in $H^s, s > 0$ in the radial case. The related results on non-radial energy-critical nonlinear Klein-Gordon equations were studied by Chen and Wang \cite{19} for $d = 4$ and 5. In the very recently, we also learn that the almost sure scattering for the 4D non-radial NLS was obtained by Spitz \cite{39} in $H^s_x$ with $\frac{5}{7} < s < 1$, using a randomization similar to the one introduced by Burq and Krieger \cite{13}.

We remark that all the previous almost sure scattering results require the initial data in $H^s_x$ with $s \geq 0$. A nature question is whether the problem is almost surely well-posed in rougher space. In fact, the well known Gaussian white noise is almost surely in $H^s_x$ with $s < -\frac{d}{2}$, and the Brownian motion is almost surely in $H^s_x$ with $s < -\frac{1}{2}$, see for instance \cite{1, 43}. Therefore, this motivates us to consider the random data problem at arbitrary regularity setting.

In this paper, we are able to prove the almost sure global well-posedness and scattering for 3D and 4D energy critical NLS with non-radial data at arbitrary regularity.

1.1. Definition of randomization. To this end, we first introduce a new version of randomization based on rescaled cubes decomposition in frequency space, inspired by \cite{10} and the Wiener randomization. Before stating the main result, we give some definitions.

Let $a \in \mathbb{N}$. First, for $N \in 2^\mathbb{N}$, we denote the cube sets

$$O_N = \{ \xi \in \mathbb{R}^d : |\xi_j| \leq N, j = 1, 2, \cdots, d \},$$

and then we define

$$Q_N = O_{2N} \setminus O_N.$$

Next, we make a further decomposition of $Q_N$. Note that $a$ is a positive integer, we make a partition of the $Q_N$ with the essentially disjoint sub-cubes

$$\mathcal{A}(Q_N) := \{ Q : Q \text{ is a dyadic cube with length } N^{-a} \text{ and } Q \subset Q_N \}.$$  

Then, we have $\sharp \mathcal{A}(Q_N) = (2^d - 1) N^{d(a+1)}$ and $Q_N = \cup_{Q \in \mathcal{A}(Q_N)} Q$.

Now, we can define the final decomposition

$$Q := \{ O_1 \} \cup \{ Q : Q \in \mathcal{A}(Q_N) \text{ and } N \in 2^\mathbb{N} \}.$$  

By the above construction,

$$\mathbb{R}^d = O_1 \cup (\cup_{N \in 2^\mathbb{N}} Q_N) = O_1 \cup (\cup_{N \in 2^\mathbb{N}} \cup_{Q \in \mathcal{A}(Q_N)} Q) = \cup_{Q \in Q} Q.$$  

Therefore, $Q$ is a countable family of essentially disjoint caps covering $\mathbb{R}^d$. We can renumber the cubes in $Q$ as follows:

$$Q = \{ Q_j : j \in \mathbb{N} \}.$$

**Definition 1.1** (“Narrowed” Wiener Randomization). Let $d \geq 1$. Given $s \in \mathbb{R}$, we set the parameter $a \in \mathbb{N}$ such that

$$a > \max \{ 3 - 4s, 1 - 2s, 10 \}.$$  

Let $\tilde{\psi}_j \in C^\infty_0(\mathbb{R}^d)$ be a real-valued function such that $0 \leq \tilde{\psi}_j \leq 1$ and

$$\tilde{\psi}_j(\xi) = \begin{cases} 1, \text{ when } \xi \in Q_j, \\ \text{smooth, otherwise,} \\ 0, \text{ when } \xi \notin 2Q_j, \end{cases}$$

where $\tilde{\psi}_j = \tilde{\psi}_j(\xi)$ is the characteristic function of the cube $Q_j$. We define the random initial data $u_{\xi}$ as a white noise in $H^s_x$. By the randomization above, we see that

$$u_{\xi}(\xi) = \tilde{\psi}_j(\xi) u_{\xi}(\xi) = \tilde{\psi}_j(\xi) \tilde{\psi}_j(\xi) u_{\xi}(\xi) = \tilde{\psi}_j(\xi) \tilde{\psi}_j(\xi) u_{\xi}(\xi).$$

Therefore, the random initial data $u_{\xi}(\xi)$ is almost surely in $H^s_x$ with $s \geq 0$.
where we use the notation that $2Q_j$ is the cube with the same center as $Q_j$ and with $\text{diam}(2Q_j) = 2\text{diam}(Q_j)$. Now, let
\[
\psi_j(\xi) := \frac{\tilde{\psi}_j(\xi)}{\sum_{j' \in \mathbb{N}} \tilde{\psi}_{j'}(\xi)}.
\]
Then, $\psi_j \in C_0^\infty([0, +\infty))$ is a real-valued function, satisfying $0 \leq \psi_j \leq 1$ and for all $\xi \in \mathbb{R}^d$, $\sum_{j \in \mathbb{N}} \psi_j(\xi) = 1$.

Denote the Fourier transform on $\mathbb{R}^d$ by $\mathcal{F}$. Then, for the function $f$ on $\mathbb{R}^d$, we define
\[
\Box_j f = \mathcal{F}^{-1} (\psi_j(\xi) \mathcal{F} f(\xi)).
\]

1.2. Main result. In the following, we use the statement “almost every $\omega \in \Omega$, $PC(\omega)$ holds” to mean that
\[
\mathbb{P}\left(\{\omega \in \Omega : PC(\omega) \text{ holds}\}\right) = 1.
\]

Arguing similarly as in the previous works (see Appendix B. in [15]), we can show that this randomization does not improve the regularity of $f$, namely for any $s \in \mathbb{R}$, $f \notin H_x^s(\mathbb{R}^d)$, then $f^\omega \notin H_x^s(\mathbb{R}^d)$ almost surely.

Now, we study the defocusing, energy critical NLS with randomized initial data:
\[
\begin{cases}
    i\partial_t u + \Delta u = |u|^{\frac{4}{d-2}} u, \\
    u(0, x) = f^\omega(x).
\end{cases}
\]

Our main result is as follows:

**Theorem 1.2** (Global well-posedness and scattering). Let $d = 3$ or $d = 4$. Given any $s \in \mathbb{R}$ and $f \in H_x^s(\mathbb{R}^d)$. Suppose that the randomization $f^\omega$ is defined in Definition 1.1. Then, for almost every $\omega \in \Omega$, there exists a global solution $u$ of (1.7) such that
\[
u - e^{it\Delta} f^\omega \in C(\mathbb{R}; H_x^1(\mathbb{R}^d)).
\]
Moreover, the solution $u$ scatters, in the sense that there exist $u_\pm \in H_x^1(\mathbb{R}^d)$ such that
\[
\lim_{t \to \pm \infty} \|u - e^{it\Delta} f^\omega - e^{it\Delta} u_\pm\|_{H_x^1(\mathbb{R}^d)} = 0.
\]

As shown in Definition 1.1, for given $s \in \mathbb{R}$, we can find a randomization with the parameter $a = a(s)$, such that the theorem holds. This implies that our result has no regular restriction, and the equation could be solved globally no matter how rough the initial data is. Particularly, when $s < 0$, the initial data has infinite mass and energy. Hence, a new difficulty in this situation is that both the mass and energy conservation laws are not available.

Previously, there is no probabilistic global result for the 3D energy critical NLS with large data. As mentioned above, for the 4D case, the almost sure scattering
in the radial case was proved in [26, 30] and extended to the non-radial case in [39] very recently. All the above 4D results consider the initial data in $H^s$ with $s > \frac{1}{2}$, in which case the interaction Morawetz estimate holds for the original solution $u$, and more importantly, all the previous random data scattering results for both NLS and NLW are established in $H^s$ with $s > 0$. In this paper, we introduce a new method to prove the almost sure global well-posedness as well as the scattering for non-radial NLS in both 3D and 4D cases, without imposing any regularity restriction on the initial data.

Taking the 3D case for example, the main difficulty comes from that the non-linear part of solution does not have any $L^p_x$-estimate with $p > 6$. Therefore, we introduce a new strategy to control the energy increment, based on the $\Delta$-estimate of the linear flow and the modified interaction Morawetz estimate.

When considering the $L^2_x$ sub-critical data, it seems very difficult to establish the Strichartz estimates directly due to the lack of spherical symmetry. By introducing a new randomization, we are able to prove some $L^2_x$ super-critical estimates with good smoothing effect for the linear flow.

Finally, we remarks that our method also works for $d = 5$ and $d = 6$ cases when the non-linear term is not necessarily algebraic, by slightly modifying the argument. We do not pursue this issue here, since the argument would be technically complex when invoking the fractional calculus.

1.3. Sketch of the proof. The main ingredient of the proof is summarized as follows.

- **High-low frequency decomposition.** The high-low decomposition was introduced by Bourgain [7], and then first applied to the probabilistic setting by Colliander-Oh [23]. Here we use the framework in our previous paper in [38]. More precisely, in order to quantify the size of mass and energy, we first decompose the probability space $\Omega$ by setting

$$\tilde{\Omega}_M = \{ \omega \in \Omega : \|f^\omega\|_{H^s} + N_0^s\|P_{\leq N_0} f^\omega\|_{L^2_x}$$

$$+ N_0^{s-1}\|P_{\leq N_0} f^\omega\|_{H^1} + \|e^{it\Delta} f^\omega\|_{Y(\mathbb{R})} \leq M \|f\|_{H^s} \},$$

where the $Y$ is some required space-time norm.

Then we consider $\omega \in \tilde{\Omega}_M$ for each $M$ separately, and make the high-low frequency decomposition as

$$v = e^{it\Delta} P_{\geq N_0} f^\omega, \text{ and } w = u - v.$$  

Then, for any $\omega \in \tilde{\Omega}_M$, there exists a constant $C(M, \|f\|_{H^s}) > 0$ such that

$$M(0) \leq C(M, \|f\|_{H^s}) N_0^{-2s}, \text{ and } E(0) \leq C(M, \|f\|_{H^s}) N_0^{2(1-s)}.$$

The framework here has two benefits:

1. $\hat{v}$ is supported on $\{|\xi| \gtrsim N_0\}$.
2. We can explicitly keep track of the mass and energy increment by the large dyadic parameter $N_0$ that is independent of $\omega$.

- **Linear estimates with smoothing effect.**

In this paper, we consider the nonlinear Schrödinger equation in non-radial case in $H^s$ for any $s \in \mathbb{R}$. We propose a new kind of randomization based on the repartition of Wiener decomposition in the frequency space. According to the definition of $\Box_j$,
the following Bernstein-like estimate holds:

\[ \| \Box_j f \|_{L^\infty_x(\mathbb{R}^d)} \lesssim \| \nabla |^{\frac{d}{2}} \Box_j f \|_{L^2_x(\mathbb{R}^d)}, \quad \text{for each} \ j \in \mathbb{N}. \]

This implies that the smoothing effect only depends on the volume of support set of \( \psi_Q \) and one may gain the arbitrary regularity (\( |\nabla|^s \)) for non-radial function by choosing sufficiently large \( a > 0 \). Due to this, we obtain that for any fixed \( s \in \mathbb{R} \), any \( f \in H^s(\mathbb{R}^d) \), by choosing \( a \) suitably large, the following estimate is available:

\[ \| \Delta e^{it\Delta} f \|_{L^2_t L^\infty_x} < +\infty, \text{ a.e. } \omega \in \Omega. \]

**• Modified interaction Morawetz estimate.** The purpose here is to bound

\[ \| \nabla |^{-\frac{d-3}{4}} w \|_{L^4_t \mathbb{R}^d} \]

by the interaction Morawetz estimate. The key point is that we modify the remainder avoiding the terms containing \( w^{\frac{d-3}{2}} \nabla w \). More precisely, in the 3D case, the remainder includes the terms like

\[ \int_{\mathbb{R}^3} vw^4 \nabla \bar{w} \ dx. \]

Since we only have \( L^2_x \) estimate for \( \nabla w \), we are forced to use \( L^6_x \) for the remaining \( w \). However, we only have \( L^q_x \)-estimate with \( q \leq 6 \) for \( w \). This brings the difficulty to obtain the desired estimate. To overcome this difficulty, the main technique here is to transfer the derivative from \( w \) to \( v \), and reduce it to the easy term

\[ \int_{\mathbb{R}^3} |w|^4 w \nabla \bar{v} \ dx. \]

Note that it can not be obtained by integration-by-parts directly, instead it follows by using the structure of nonlinearity.

This kind of estimate may also be of independent interest.

**• Mass increment.** Note that when \( s < 0 \), \( u_0 \notin L^\infty_t L^2_x \), which is much different from most of the previous papers. Hence, we need the following almost conserved \( L^2 \)-estimate firstly:

\[ \| w \|_{L^\infty_t L^2_x} \lesssim \| w_0 \|_{L^2_x}. \]

One may note that

\[ \| w(t) \|^2_{L^2_x} = \| w_0 \|^2_{L^2_x} + \int_I \int_{\mathbb{R}^d} O(w^{\frac{d+2}{2}} v) \ dx \ dt. \]

The cubic growth (when \( d = 3 \)) of the increment make the obstruction to close the estimate by Gronwall’s inequality. To cover the additional increment, we make use of the high-low frequency decomposition above, which allows us to get extra regularity of \( v \) which relies on the reciprocal of \( \| w(t) \|_{H^1} \).

**• Energy increment.** To prove the almost energy conservation law, which gives the bound of \( H^1 \), the main task is to control the energy increment

\[ \int_I \int_{\mathbb{R}^d} \Delta v |w|^{\frac{d-3}{2}} w \ dx \ dt. \]

Roughly speaking, based on the above estimates, the increment can be bounded by

\[ \| \Delta v \|_{L^2_t L^\infty_x} \| \nabla |^{-\frac{d-3}{4}} w \|_{L^4_t \mathbb{R}^d} \| \nabla |^{-\frac{d-3}{2}} (w^{\frac{6-d}{2}}) \|_{L^\infty_t L^2_x}. \]
In 3D case,
$$\|\nabla \frac{d-3}{2} (w^{\frac{d-2}{d}})\|_{L^\infty_t L^2_x} \lesssim \|w\|^3_{L^\infty_t L^2_x},$$
and in 4D case,
$$\|\nabla \frac{d-3}{2} (w^{\frac{d-2}{d}})\|_{L^\infty_t L^2_x} \lesssim \|\nabla \frac{1}{2} w\|_{L^\infty_t L^2_x}.$$ Again, we can cancel the additional energy increment using the $\|\Delta v\|_{L^2_t L^\infty_x}$ estimate. This close the bootstrap procedure.

- Stability. We also need to adopt the perturbation theory to approximate the perturbed equation (4.1). This idea of iterating the perturbation theory was first employed by [41], and then was extended in the probabilistic setting by [2] and [30]. This method was widely used in the almost sure scattering theory. In this paper, we are mainly inspired by the recent approach as in [30], which is tailored to the equation (4.1).

1.4. Organization of the paper. In Section 2, we give some notation and useful results. In Section 3, we prove the almost sure space-time estimates for the linear solution. Then, we prove the 3D case of Theorem 1.2 in Section 4, and the 4D case in Section 5.

2. Preliminary

2.1. Notation. For any $a \in \mathbb{R}$, $a \pm := a \pm \epsilon$ for arbitrary small $\epsilon > 0$. For any $z \in \mathbb{C}$, we define $\text{Re} z$ and $\text{Im} z$ as the real and imaginary part of $z$, respectively. For any set $A$, we denote $|A|$ as the cardinal number of $A$.

Let $C > 0$ denote some constant, and write $C(a) > 0$ for some constant depending on coefficient $a$. If $f \leq C g$, we write $f \lesssim g$. If $f \leq C g$ and $g \leq C f$, we write $f \sim g$. Suppose further that $C = C(a)$ depends on $a$, then we write $f \lesssim_a g$ and $f \sim_a g$, respectively. If $f \leq 2^{-5} g$, we denote $f \ll g$ or $g \gg f$.

Moreover, we write “a.e. $\omega \in \Omega$” to mean “almost every $\omega \in \Omega$”.

We use $\hat{f}$ or $\mathcal{F} f$ to denote the Fourier transform of $f$:
$$\hat{f}(\xi) = \mathcal{F} f(\xi) := \int_{\mathbb{R}^d} e^{-ix\cdot \xi} f(x) \, dx.$$
We also define
$$\mathcal{F}^{-1} g(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix\cdot \xi} g(\xi) \, d\xi.$$ Using the Fourier transform, we can define the fractional derivative $|\nabla| := \mathcal{F}^{-1} |\xi| \mathcal{F}$ and $|\nabla|^{s} := \mathcal{F}^{-1} |\xi|^{s} \mathcal{F}$.

We also need the usual inhomogeneous Littlewood-Paley decomposition for the dyadic number. Take a cut-off function $\phi \in C_0^\infty(0, \infty)$ such that $\phi(r) = 1$ if $r \leq 1$ and $\phi(r) = 0$ if $r > 2$. For dyadic $N \in \mathbb{N}$, when $N \geq 1$, let $\phi_{<N}(r) = \phi(N^{-1}r)$ and $\phi_{N}(r) = \phi_{<N}(r) - \phi_{<N/2}(r)$. We define the Littlewood-Paley dyadic operator
$$f_{<N} = P_{<N} f := \mathcal{F}^{-1}(\phi_{<N}(\xi) \hat{f}(\xi)),$$
and
\[ f_N = P_N f := F^{-1}(\phi_N(\xi) \hat{f}(\xi)). \]
We also define that
\[ f_{\geq N} = P_{\geq N} f := f - P_{< N} f, \quad f_{\leq N} = P_{\leq N} f, \quad f_{\geq N} := P_{\geq N} f, \quad f_{\leq N} := P_{\leq N} f. \]

Let \( S(\mathbb{R}^d) \) be the Schwartz space, \( S'(\mathbb{R}^d) \) be the tempered distribution space, and \( C^\infty_0(\mathbb{R}^d) \) be the space of all the smooth compact-supported functions.

Given \( 1 \leq p \leq \infty \), \( L^p(\mathbb{R}^d) \) denotes the usual Lebesgue space. We define the Sobolev space
\[ \dot{W}^{s,p}(\mathbb{R}^d) := \{ f \in S(\mathbb{R}^d) : \| f \|_{\dot{W}^{s,p}(\mathbb{R}^d)} := \| \| \nabla \|^s f \|_{L^p(\mathbb{R}^d)} < +\infty \}. \]
We denote that \( \dot{H}^s(\mathbb{R}^d) := \dot{W}^{s,2}(\mathbb{R}^d) \). The inhomogeneous spaces are defined by
\[ W^{s,p}(\mathbb{R}^d) = \dot{W}^{s,p} \cap L^p(\mathbb{R}^d), \quad H^s(\mathbb{R}^d) = \dot{H}^s \cap L^2(\mathbb{R}^d). \]
We often use the abbreviations \( H^s = H^s(\mathbb{R}^d) \) and \( L^p = L^p(\mathbb{R}^d) \). We also define \( \langle \cdot, \cdot \rangle \) as real \( L^2 \) inner product:
\[ \langle f, g \rangle = \text{Re} \int f(x) \overline{g}(x) \, dx. \]

For any \( 1 \leq p < \infty \), define \( l^p_N = l^p_{N \in 2^N} \) by its norm
\[ \| c_N \|^p_{l^p_N} := \sum_{N \in 2^N} |c_N|^p. \]
The space \( l^p_N = l^p_{N \in \mathbb{N}} \) is defined in a similar way.

We then define the mixed norms: for \( 1 \leq q < \infty \), \( 1 \leq r \leq \infty \), and the function \( u(t, x) \), we define
\[ \| u \|^q_{L^r_t L^q_x(\mathbb{R} \times \mathbb{R}^d)} := \int_\mathbb{R} \| u(t, \cdot) \|^q_{L^r_x} \, dt, \]
and for the function \( u_N(x) \), we define
\[ \| u_N \|^q_{L^r_t L^q_x(2^N \times \mathbb{R}^d)} := \sum_N \| u_N(\cdot) \|^q_{L^r_x}. \]
The \( q = \infty \) case can be defined similarly.

For any \( 0 \leq \gamma \leq 1 \), we call that the exponent pair \( (q, r) \in \mathbb{R}^2 \) is \( \dot{H}^\gamma \)-admissible, if \( \frac{2}{q} + \frac{d}{r} = \frac{d}{2} - \gamma \), \( 2 \leq q \leq \infty \), \( 2 \leq r \leq \infty \), and \( (q, r, d) \neq (2, \infty, 2) \). If \( \gamma = 0 \), we say that \( (q, r) \) is \( L^2 \)-admissible.

### 2.2. Useful lemmas.
In this subsection, we gather some useful results.

**Lemma 2.1** (Hardy’s inequality). For \( 0 < s < d/2 \), we have that
\[ \| |x|^{-s} u \|_{L^2(\mathbb{R}^d)} \lesssim \| u \|_{H^s(\mathbb{R}^d)}. \]

**Lemma 2.2** (Gagliardo-Nirenberg inequality). Let \( d \geq 1 \), \( 0 < \theta < 1 \), \( 0 < s_1 < s_2 \) and \( 1 < p_1, p_2, p_3 \leq \infty \). Suppose that \( \frac{1}{p_1} = \frac{\theta}{p_2} + \frac{1-\theta}{p_3} \) and \( s_1 = \theta s_2 \). Then, we have
\[ \| \nabla^{s_1} u \|_{L^{p_1}(\mathbb{R}^d)} \lesssim \| \nabla^{s_2} u \|_{L^{p_2}(\mathbb{R}^d)}^{\theta} \| u \|_{L^{p_3}(\mathbb{R}^d)}^{1-\theta}. \]  \hfill (2.1)

Then by Lemma 2.2, we can easily obtain
Lemma 2.3. Let $d \geq 1$, $0 < \theta < 1$, $0 < s_1 < s_2$ and $1 < p_1, p_2, p_3 \leq \infty$. Suppose that $\frac{1}{p_1} = \frac{\theta}{p_2} + \frac{1-\theta}{p_3}$ and $s_1 \leq \theta s_2$. Then, we have

$$\|\langle \nabla \rangle^{s_1} u\|_{L^p_t L^q_x(\mathbb{R}^d)} \lesssim \|\langle \nabla \rangle^{s_2} u\|_{L^p_t L^q_x(\mathbb{R}^d)}^{\theta} \|u\|_{L^p_t L^q_x(\mathbb{R}^d)}^{1-\theta}. \quad (2.2)$$

Lemma 2.4 (Strichartz estimate, [28]). Let $I \subset \mathbb{R}$. Suppose that $(q, r)$ and $(\tilde{q}, \tilde{r})$ are $L^2_q$-admissible. Then,

$$\|e^{it\Delta} \varphi\|_{L^q_t L^r_x(\mathbb{R}^d \times \mathbb{R}^d)} \lesssim \|\varphi\|_{L^2_x}, \quad (2.3)$$

and

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s) \, ds \right\|_{L^q_t L^r_x(\mathbb{R}^d \times \mathbb{R}^d)} \lesssim \|F\|_{L^{\tilde{q}}_x L^{\tilde{r}}_x(\mathbb{R}^d \times \mathbb{R}^d)}. \quad (2.4)$$

Lemma 2.5 (Littlewood-Paley estimates). Let $1 < p < \infty$ and $f \in L^p_x(\mathbb{R}^d)$. Then, we have

$$\|f_N\|_{L^{pN}_{t}} \approx_p \|f\|_{L^p_x}.$$

Next, we give some properties of $\Box_j$.

Lemma 2.6 (Orthogonality). Let $f \in L^2_x(\mathbb{R}^d)$. Then, we have

$$\|\Box_j f\|_{L^2_{t,j \in \mathbb{N}}} \sim \|f\|_{L^2_x}.$$

Proof. Since $Q$ is a set of essentially disjoint cubes, for any fixed $j \in \mathbb{N}$, let

$$B_j := \{j' \in \mathbb{N} : \text{supp } \psi_j \cap \text{supp } \psi_{j'} \neq \emptyset \},$$

then $2B_j \lesssim 1$. We also have

$$j' \in B_j \iff j \in B_{j'}.$$

Then by Plancherel’s identity,

$$\|f\|_{L^2_x(\mathbb{R}^d)}^2 = \sum_{j \in \mathbb{N}} \|\Box_j f\|_{L^2_x(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} \left| \sum_{j \in \mathbb{N}} \hat{\psi}_j(\xi) \hat{f}(\xi) \right|^2 d\xi = \sum_{j,j' \in \mathbb{N}; j' \in B_j} \int_{\mathbb{R}^d} \hat{\psi}_j(\xi) \hat{\psi}_{j'}(\xi) \left| \hat{f}(\xi) \right|^2 d\xi.$$

Hence, on one hand,

$$\|\Box_j f\|_{L^2_{t,j \in \mathbb{N}}} = \sum_{j \in \mathbb{N}} \int_{\mathbb{R}^d} \left| \hat{\psi}_j(\xi) \hat{f}(\xi) \right|^2 d\xi \leq \sum_{j,j' \in \mathbb{N}; j' \in B_j} \int_{\mathbb{R}^d} \hat{\psi}_j(\xi) \hat{\psi}_{j'}(\xi) \left| \hat{f}(\xi) \right|^2 d\xi = \|f\|_{L^2_x(\mathbb{R}^d)}^2.$$
On the other hand, by Cauchy-Schwartz’ inequality,
\[ \| f \|_{L^2_x(\mathbb{R}^d)}^2 = \sum_{j,j' \in \mathbb{N}, j' \in B_j} \int_{\mathbb{R}^d} \psi_j(\xi) \psi_{j'}(\xi) |\hat{f}(\xi)|^2 \, d\xi \]
\[ \lesssim \sum_{j,j' \in \mathbb{N}, j' \in B_j} \left( \int_{\mathbb{R}^d} |\psi_j(\xi)\hat{f}(\xi)|^2 \, d\xi + \int_{\mathbb{R}^d} |\psi_{j'}(\xi)\hat{f}(\xi)|^2 \, d\xi \right) \]
\[ \lesssim \sum_{j,j' \in \mathbb{N}, j' \in B_j} \int_{\mathbb{R}^d} |\psi_j(\xi)\hat{f}(\xi)|^2 \, d\xi \lesssim \| \Box_j f \|_{L^2_{x\in\mathbb{R}^d}}. \]

This finishes the proof. \(\square\)

We then prove a Bernstein-like estimate for \(\Box_j\). This provides the required smoothing effect, as long as the scale of cube \(N^{-a}\) is suitably small.

**Lemma 2.7** \((L^q-L^p)\ estimate). Let \(a > 0\) and \(2 \leq p \leq q \leq \infty\). Given any \(j \in \mathbb{N}\), then
\[
\| \Box_j f \|_{L^q_x(\mathbb{R}^d)} \lesssim \| \langle \nabla \rangle^{-a \left(\frac{q'}{q} - \frac{1}{2}\right)} \Box_j f \|_{L^p_x(\mathbb{R}^d)}.
\]

**Proof.** By the property of \(Q\), given any \(j \in \mathbb{N}\), there exists \(N \in 2^\mathbb{N}\) such that for all \(\xi \in \text{supp} \, \psi_j\),
\[
\langle |\xi| \rangle \sim N, \text{ and } |\text{supp} \, \psi_j| \sim N^{-da}.
\]

Let \(r \geq 2\) and \(0 \leq \theta \leq 1\) such that \(\frac{1}{r} = \frac{1}{q} - \frac{1}{2}\) and \(\frac{1}{p} = \frac{a}{2} + \frac{1-a}{q}\). Now, by Hausdorff-Young’s and Hölder’s inequalities,
\[
\| \Box_j f \|_{L^q_x(\mathbb{R}^d)} \lesssim \| \psi_j \hat{f} \|_{L^{q'}_x(\mathbb{R}^d)} \\
\lesssim \| 1_{\xi \in 2Q_j} \|_{L^r_x(\mathbb{R}^d)} \| \psi_j \hat{f} \|_{L^2_x(\mathbb{R}^d)} \\
\lesssim N^{-a} \| \Box_j f \|_{L^2_x(\mathbb{R}^d)}.
\]

Note that we have
\[
\frac{1}{r} = \frac{1}{2} - \frac{1}{q} \quad \text{and} \quad \frac{1}{p} = \frac{1}{q} + \frac{\theta}{r}.
\]
Then, interpolating with the trivial estimate \(\| \Box_j f \|_{L^2_x(\mathbb{R}^d)} \lesssim \| \Box_j f \|_{L^p_x(\mathbb{R}^d)}\), for \(2 \leq p \leq q\),
\[
\| \Box_j f \|_{L^q_x(\mathbb{R}^d)} \lesssim N^{-a \left(\frac{q}{p} - \frac{1}{2}\right)} \| \Box_j f \|_{L^p_x(\mathbb{R}^d)} \\
\lesssim N^{-a \left(\frac{q}{p} - \frac{1}{2}\right)} \| \Box_j f \|_{L^p_x(\mathbb{R}^d)} \\
\lesssim \| \langle \nabla \rangle^{-a \left(\frac{q}{p} - \frac{1}{2}\right)} \Box_j f \|_{L^p_x(\mathbb{R}^d)}.
\]

This completes the proof of this lemma. \(\square\)

We remark that this lemma is different from the usual Bernstein’ inequality for the Littlewood-Paley projection operator \(P_N\): for \(1 \leq p \leq q \leq \infty\) and \(N \in 2^\mathbb{N}\),
\[
\| P_N f \|_{L^q_x(\mathbb{R}^d)} \lesssim N^{\frac{q}{p} - \frac{1}{2}} \| P_N f \|_{L^p_x(\mathbb{R}^d)}.
\]

In fact, the cut-off \(\phi_N\) can be generated by rescaling, so this estimate follows by Young’s inequality. However, \(\psi_j\) does not have that property. Therefore, we need to use the Hausdorff-Young’s inequality. That’s the reason why we require \(p \geq 2\).
2.3. Probabilistic theory. We recall the large deviation estimate, which holds for the random variable sequence \( \{ \Re g_k, \Im g_k \} \) in the Definition 1.1.

**Lemma 2.8** (Large deviation estimate, [15]). Let \((\Omega, \mathcal{A}, P)\) be a probability space. Let \( \{g_n\}_{n \in \mathbb{N}^+} \) be a sequence of real-valued, independent, zero-mean random variables with associated distributions \( \{\mu_n\}_{n \in \mathbb{N}^+} \) on \( \Omega \). Suppose \( \{\mu_n\}_{n \in \mathbb{N}^+} \) satisfies that there exists \( c > 0 \) such that for all \( \gamma \in \mathbb{R} \) and \( n \in \mathbb{N}^+ \)

\[
| \int_{\mathbb{R}} e^{\gamma x} d\mu_n(x) | \leq e^{c\gamma^2},
\]

then there exists \( \alpha > 0 \) such that for any \( \lambda > 0 \) and any complex-valued sequence \( \{c_n\}_{n \in \mathbb{N}^+} \in l^2_n \), we have

\[
P(\{\omega : |\sum_{n=1}^{\infty} c_n g_n(\omega)| > \lambda\}) \leq 2 \exp \{-\alpha \lambda \|c_n\|_p^{-2}\}.
\]

Furthermore, there exists \( C > 0 \) such that for any \( 2 \leq p < \infty \) and complex-valued sequence \( \{c_n\}_{n \in \mathbb{N}^+} \in l^2_n \), we have

\[
\| \sum_{n=1}^{\infty} c_n g_n(\omega) \|_{L_p^\infty(\Omega)} \leq C \sqrt{p} \|c_n\|_{l^2_n}.
\] (2.5)

The following lemma can be proved by the method in [42], see also [25, 26].

**Lemma 2.9.** Let \( F \) be a real-valued measurable function on a probability space \((\Omega, \mathcal{A}, P)\). Suppose that there exists \( C_0 > 0, K > 0 \) and \( p_0 \geq 1 \) such that for any \( p \geq p_0 \), we have

\[
\|F\|_{L_p^\infty(\Omega)} \leq \sqrt{p} C_0 K.
\]

Then, there exist \( c > 0 \) and \( C_1 > 0 \), depending on \( C_0 \) and \( p_0 \) but independent of \( K \), such that for any \( \lambda > 0 \),

\[
P(\{\omega : |F(\omega)| > \lambda\}) \leq C_1 e^{-c\lambda^2 K^{-2}}.
\]

Particularly, we have

\[
P(\{\omega : |F(\omega)| < \infty\}) = 1.
\]

3. Almost sure Strichartz estimates

3.1. Strichartz estimates.

**Lemma 3.1.** Let \( d \geq 1 \) and \( f \in L^2_\omega(\mathbb{R}^d) \). Suppose that the randomization \( f^\omega \) is defined in Definition 1.1. Then, we have the following estimates:

1. Given any \( 2 \leq q, r < \infty \) with \( \frac{2}{q} + \frac{d}{r} \leq \frac{d}{2} \) and let \( 2 \leq r_0 < \infty \) such that \( (q, r_0) \) is \( L^2_\omega \)-admissible. Then for any \( p \geq \max \{q, r\} \) and \( 0 \leq s \leq a\left(\frac{d}{r_0} - \frac{d}{2}\right) \),

\[
\left\| \langle \nabla \rangle^s e^{it\Delta} f^\omega \right\|_{L_t^p L_x^q L_x^r(\Omega \times \mathbb{R} \times \mathbb{R}^d)} \lesssim \sqrt{p} \|f\|_{L^2_\omega(\mathbb{R}^d)}.
\] (3.1)

2. For any \( p \geq 2 \),

\[
\| e^{it\Delta} f^\omega \|_{L_t^p L_x^q L_x^r(\Omega \times \mathbb{R} \times \mathbb{R}^d)} \lesssim \sqrt{p} \|f\|_{L^2_\omega(\mathbb{R}^d)}.
\] (3.2)
Lemma \( \leq \)

Then for \( a \) as above, by Minkowski’s inequality, Lemmas 3.1 and 3.2.

Proof. In the proof of this lemma, we restrict the variables on \( \omega \in \Omega \), \( t \in \mathbb{R} \), \( x \in \mathbb{R}^d \), and \( j \in \mathbb{N} \).

We first prove (3.1). By Minkowski’s inequality and Lemma 2.8, we have

\[
\| \langle \nabla \rangle^s e^{it\Delta} f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \| \langle \nabla \rangle^s e^{it\Delta} f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \sqrt{p} \| \langle \nabla \rangle^s e^{it\Delta} \Box_j f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \sqrt{p} \| \langle \nabla \rangle^s e^{it\Delta} \Box_j f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)}. \tag{3.3}
\]

Now, let \( 2 \leq r_0 \leq r \) such that \( (q, r_0) \) is \( L^2_2 \)-admissible. Then, for any \( k \in \mathbb{N}^+ \), by Lemma 2.7, we have

\[
\| \langle \nabla \rangle^s e^{it\Delta} \Box_j f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \| \langle \nabla \rangle^s e^{it\Delta} \Box_j f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \| e^{it\Delta} \Box_j f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)}. \tag{3.4}
\]

Then, by (3.5), (3.6), Lemmas 2.4, and 2.6, we have

\[
\| \langle \nabla \rangle^s e^{it\Delta} f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \| \langle \nabla \rangle^s e^{it\Delta} f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \| \langle \nabla \rangle^s e^{it\Delta} f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)}. \tag{3.7}
\]

This gives (3.1).

Next, we prove (3.2). By Plancherel’s identity,

\[
\| \langle \nabla \rangle^s e^{it\Delta} f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \| \langle \nabla \rangle^s f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \| \langle \nabla \rangle^s f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)}. \tag{3.8}
\]

Then for \( p \geq 2 \), by Minkowski’s inequality, Lemmas 2.8, and 2.6,

\[
\| \langle \nabla \rangle^s f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \| \langle \nabla \rangle^s f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \| \langle \nabla \rangle^s f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)}. \tag{3.9}
\]

Then, (3.7) and (3.8) imply (3.2).

We then prove (3.3). Let \( 0 < \varepsilon \leq \frac{d}{r_0} - a - s \) such that

\[
s + 2\varepsilon - a\left(\frac{d}{r_0} - \varepsilon\right) \leq 0.
\]

Using the Sobolev’s embedding \( W^{2s, \frac{d}{q}}_{2q} \hookrightarrow L^\infty_x \) in \( x \), we have

\[
\| \langle \nabla \rangle^s e^{it\Delta} f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)} \leq \| \langle \nabla \rangle^s e^{it\Delta} f \|_{L^p_t L^q_x(\Omega \times \mathbb{R}^d)}. \tag{3.9}
\]

Let \( p_0 = \max \left\{ q, \frac{d}{r_0} \right\} \) and \( 2 \leq r_0 \leq \frac{d}{q} \) such that \( (q, r_0) \) is \( L^2_2 \)-admissible. Then, similar as above, by Minkowski’s inequality, Lemmas 2.4, 2.7, 2.6, and 2.8, for any \( p \geq p_0 \),
Using Minkowski’s inequality, the Sobolev’s embeddings $\varepsilon$ choice of
with some
$C > \varepsilon$
By (3.9) and (3.10), we have that (3.3) holds.
Finally, we prove (3.4). We only consider the $r = \infty$ case. In fact, when $r < \infty$, we can prove it using interpolation between (3.2) and the $r = \infty$ case. Let $0 \leq s < \frac{d}{2} - a$ and some sufficiently small $0 < \varepsilon \leq \frac{1}{6 + 3a}(\frac{d}{2} - s)$ such that
\[ s + 6\varepsilon - a(\frac{d}{2} - \varepsilon) \leq 0. \]
Using Minkowski’s inequality, the Sobolev’s embeddings $W_{r,s}^{\frac{d}{r} - s} \hookrightarrow L_{r}^{\infty}$ in $t$, and $W_{x}^{\frac{d}{r} - s} \hookrightarrow L_{x}^{\infty}$ in $x$, we have
\[ \left\langle \langle \nabla \rangle \right\rangle^{s+2\varepsilon} e^{it\Delta} f^w \right\rangle_{L_{r}^{p}L_{t}^{q}} \lesssim \left\langle \langle \nabla \rangle \right\rangle^{s+2\varepsilon} e^{it\Delta} f^w \right\rangle_{L_{r}^{p}L_{t}^{q}} \]
\[ \lesssim \left\langle \langle \nabla \rangle \right\rangle^{s+2\varepsilon} \left( \partial_t \right)^{2\varepsilon} e^{it\Delta} f^w \right\rangle_{L_{r}^{p}L_{t}^{q}} \]
\[ \lesssim \left\langle \langle \nabla \rangle \right\rangle^{s+6\varepsilon} e^{it\Delta} f^w \right\rangle_{L_{r}^{p}L_{t}^{q}} \]
Let $p_{0} = \frac{2}{d}$ and $r_{0} = \frac{2d}{d+3a}$ such that $2 < r_{0} \leq \frac{d}{2}$ and $(\frac{1}{r_{0}}, r_{0})$ is $L_{x}^{2}$-admissible. By the choice of $\varepsilon$ and $r_{0}$, we have
\[ s + 6\varepsilon - a(\frac{d}{r_{0}} - \varepsilon) \leq 0. \]
Then, similar as above, for any $p \geq p_{0},$
\[ \left\langle \langle \nabla \rangle \right\rangle^{s+6\varepsilon} e^{it\Delta} f^w \right\rangle_{L_{r}^{p}L_{t}^{q}} \lesssim \left\langle \langle \nabla \rangle \right\rangle^{s+6\varepsilon} e^{it\Delta} f^w \right\rangle_{L_{r}^{p}L_{t}^{q}} \]
\[ \lesssim \sqrt{p} \left\langle \langle \nabla \rangle \right\rangle^{s+6\varepsilon} e^{it\Delta} \Box f \right\rangle_{L_{r}^{p}L_{t}^{q}} \]
By (3.11) and (3.12), we have that (3.4) holds.
We make a few remarks about this randomization. We prove the Strichartz estimates with smoothing effect, namely some gain of derivative quantified by $|\nabla|^C$ with some $C > 0$. Since we do not assume the radial condition, $L_{t}^{p}L_{x}^{q}$-estimates with
\[ \frac{2}{q} + \frac{d}{r} > \frac{d}{s} \] are unavailable. Moreover, when \( f \in H^s_x \) with \( s < 0 \), we are also lack of the \( L^2 \)-critical estimates in the form of
\[ \| e^{it\Delta} f^\omega \|_{L^1_t L^s_x} \]
with \( L^2 \)-admissible \((q, r)\). However, we can expect the \( L^2_t L^r_x \) estimates hold for \( L^2 \) super-critical scaling, by covering the \( s \)-order derivative by the smoothing effect.

Next, we gather all the space-time norms that will be used below, splitting into two cases.

### 3.2. Linear estimates in 3D case.

Define the \( Y(I) \) space by its norm in 3D case,
\[
\| v \|_{Y(I)} := \left\| \langle \nabla \rangle^{s+\frac{d}{2} - a} v \right\|_{L^1_t L^s_x(\mathbb{R}^3)} + \left\| v \right\|_{L^p_t L^r_x(\mathbb{R}^3)} + \left\| v \right\|_{L^4_t L^4_x(\mathbb{R}^3)}
\]
(3.13)

We also define the \( Z \)-norm by
\[
\| v \|_{Z(I)} := \left\| v \right\|_{L^\infty_t H^s_x(\mathbb{R}^3)} + \left\| \langle \nabla \rangle^{s+\frac{d}{2} - a} v \right\|_{L^\infty_t L^\infty_x(\mathbb{R}^3)}.
\]
(3.14)

**Corollary 3.2.** Let \( s \in \mathbb{R} \) and \( f \in H^s_x(\mathbb{R}^3) \). Suppose that the randomization \( f^\omega \) is defined in Definition 1.1. Then, there exist constants \( C, c > 0 \) such that for any \( \lambda \),
\[
\mathbb{P}\{ \omega \in \Omega : \| e^{it\Delta} f^\omega \|_{Y(\mathbb{R})} + \| e^{it\Delta} f^\omega \|_{Z(\mathbb{R})} > \lambda \} \leq C \exp \{-c\lambda^2 \| f \|_{H^s_x(\mathbb{R}^3)}^{-2} \},
\]
(3.15)

and we also have
\[
\| e^{it\Delta} f^\omega \|_{Y(\mathbb{R})} + \| e^{it\Delta} f^\omega \|_{Z(\mathbb{R})} < +\infty, \quad \text{a.e. } \omega \in \Omega.
\]
(3.16)

**Proof.** In the proof of this corollary, we restrict the variables on \( \omega \in \Omega, \ t \in \mathbb{R}, \ x \in \mathbb{R}^3 \). Fix a sufficiently large \( p_0 > 0 \), and let \( p \geq p_0 \). First, by (3.3),
\[
\| \langle \nabla \rangle^{s+\frac{d}{2} - a} e^{it\Delta} f^\omega \|_{L^p_t L^s_x} \lesssim \sqrt{p} \| f \|_{H^s_x},
\]
and by (3.1),
\[
\| \langle \nabla \rangle^{s+\frac{d}{2} - a} e^{it\Delta} f^\omega \|_{L^p_t L^s_x} \lesssim \sqrt{p} \| f \|_{H^s_x},
\]
\[
\| \langle \nabla \rangle^{s+\frac{d}{2} - a} e^{it\Delta} f^\omega \|_{L^p_t L^s_x} \lesssim \sqrt{p} \| f \|_{H^s_x},
\]
\[
\| \langle \nabla \rangle^{s+\frac{d}{2} - a} e^{it\Delta} f^\omega \|_{L^p_t L^s_x} \lesssim \sqrt{p} \| f \|_{H^s_x}.
\]
Since \( s > \frac{3}{4} - \frac{1}{2}a \), we have
\[
\| e^{it\Delta} f^\omega \|_{L^p_t L^s_x} + \| e^{it\Delta} f^\omega \|_{L^p_t L^s_x} + \| e^{it\Delta} f^\omega \|_{L^p_t L^s_x} \lesssim \sqrt{p} \| f \|_{H^s_x}.
\]
Therefore, we have
\[
\| e^{it\Delta} f^\omega \|_{L^p_t L^s_x} \lesssim \sqrt{p} \| f \|_{H^s_x}.
\]
By (3.2) and (3.4),
\[
\| e^{it\Delta} f^\omega \|_{L^p_t L^s_x} \lesssim \sqrt{p} \| f \|_{H^s_x}.
\]
Then, by Lemma 2.9, we have (3.15) and (3.16) hold. \( \square \)
3.3. Linear estimates in 4D case. Define the $Y(I)$ space by its norm in 4D case,
\[
\|v\|_{Y(I)} := \|\langle \nabla \rangle^{s+a-} v\|_{L^2_x L^{p_x}(I \times \mathbb{R}^4)} + \|\langle \nabla \rangle^{s+a} v\|_{L^2_y L^4_x(I \times \mathbb{R}^4)} + \|\langle \nabla \rangle^{- \frac{1}{2}} v\|_{L^2_{x,y}(I \times \mathbb{R}^4)},
\]
(3.17)
We also define the $Z$-norm by
\[
\|v\|_{Z(I)} := \|\langle \nabla \rangle^{s+2a-} v\|_{L^2_x H^2_y(\mathbb{R} \times \mathbb{R}^4)} + \|\langle \nabla \rangle^{s+2a} v\|_{L^2_y H^2_x(\mathbb{R} \times \mathbb{R}^4)}.
\]
(3.18)

**Corollary 3.3.** Let $s \in \mathbb{R}$ and $f \in H^s_x(\mathbb{R}^4)$. Suppose that the randomization $f^\omega$ is defined in Definition 1.1. Then, there exist constants $C,c > 0$ such that for any $\lambda$,
\[
\mathbb{P}\{ \{ \omega \in \Omega : \|e^{it\Delta} f^\omega\|_{Y(\mathbb{R})} + \|e^{it\Delta} f^\omega\|_{Z(\mathbb{R})} \gtrless C \exp \{ -c\lambda^2 \|f\|_{H^s(\mathbb{R}^4)}^2 \} \},
\]
and we also have
\[
\|e^{it\Delta} f^\omega\|_{Y(\mathbb{R})} + \|e^{it\Delta} f^\omega\|_{Z(\mathbb{R})} < +\infty, \quad \text{a.e. } \omega \in \Omega.
\]
(3.20)

**Proof.** In the proof of this corollary, we restrict the variables on $\omega \in \Omega$, $t \in \mathbb{R}$, $x \in \mathbb{R}^4$. Fix a sufficiently large $p_0 > 0$, and let $p \geq p_0$. First, by (3.3),
\[
\|\langle \nabla \rangle^{s+a-} e^{it\Delta} f^\omega\|_{L^p_x L^6_y L^4_t} \lesssim \sqrt{p} \|f\|_{H^s_x},
\]
and by (3.1),
\[
\|\langle \nabla \rangle^{s+a} e^{it\Delta} f^\omega\|_{L^p_x L^2_y L^6_t} \lesssim \sqrt{p} \|f\|_{H^s_x},
\]
\[
\|\langle \nabla \rangle^{s+\frac{1}{2}a} e^{it\Delta} f^\omega\|_{L^p_x L^4_y L^4_t} \lesssim \sqrt{p} \|f\|_{H^s_x},
\]
\[
\|\langle \nabla \rangle^{s+\frac{1}{2}a} e^{it\Delta} f^\omega\|_{L^p_x L^4_y L^4_t} \lesssim \sqrt{p} \|f\|_{H^s_x}.
\]
Since $s > \frac{1}{2} - \frac{1}{2}a$, we have
\[
\|e^{it\Delta} f^\omega\|_{L^p_x L^4_y L^6_t} + \|e^{it\Delta} f^\omega\|_{L^p_x L^6_y L^4_t} + \|\langle \nabla \rangle^{- \frac{1}{2}} e^{it\Delta} f^\omega\|_{L^p_x L^4_y L^4_t} \lesssim \sqrt{p} \|f\|_{H^s_x}.
\]
Therefore, we have
\[
\|e^{it\Delta} f^\omega\|_{L^p_x Y} \lesssim \sqrt{p} \|f\|_{H^s_x}.
\]
By (3.2) and (3.4),
\[
\|e^{it\Delta} f^\omega\|_{L^p_x Z} \lesssim \sqrt{p} \|f\|_{H^s_x}.
\]
Then, by Lemma 2.9, we have (3.19) and (3.20) hold. \qed

4. Global well-posedness and scattering in 3D case

4.1. Reduction to the deterministic problem. Suppose that $u = v + w$ with $u_0 = v_0 + w_0$, $v = e^{it\Delta} v_0$, and $w$ satisfying
\[
\begin{cases}
    i\partial_t w + \Delta w = |w|^4 u, \\
    w(0, x) = w_0(x).
\end{cases}
\]
(4.1)
Recall that
\[ \| v \|_{Y(I)} = \| (\nabla)^{(s+\frac{3}{4})} v \|_{L^2_t L^\infty_x(I \times \mathbb{R}^3)} + \| v \|_{L^a_t L^8_x(I \times \mathbb{R}^3)} + \| v \|_{L^4_t L^4_x(I \times \mathbb{R}^3)} \]
and
\[ \| v \|_{Z(I)} = \| v \|_{L^\infty_t H^s_x(\mathbb{R} \times \mathbb{R}^3)} + \| (\nabla)^{(s+\frac{3}{4})} v \|_{L^a_t L^8(\mathbb{R} \times \mathbb{R}^3)} \]
We define the energy as
\[ E(t) := \frac{1}{2} \int_{\mathbb{R}^3} |\nabla w(t, x)|^2 \, dx + \frac{1}{6} \int_{\mathbb{R}^3} |u(t, x)|^6 \, dx, \quad (4.2) \]
and the mass as
\[ M(t) := \int_{\mathbb{R}^3} |w(t, x)|^2 \, dx. \quad (4.3) \]
Now, we reduce the proof of Theorem 1.2 in 3D case when \( s < 0 \) to the following deterministic problem:

**Proposition 4.1.** Let \( a \in \mathbb{N}, \ a > 10, \ \frac{3}{4} - \frac{1}{4} a < s < 0, \) and \( A > 0. \) Then, there exists \( N_0 = N_0(A) \gg 1 \) such that the following properties hold. Let \( u_0 \in H^s_x(\mathbb{R}^3), \) \( v_0 \) satisfy that \( \text{supp} \, \hat{v}_0 \subset \{ \xi \in \mathbb{R}^3 : |\xi| \geq \frac{1}{2} N_0 \}, \) and \( w_0 = u_0 - v_0. \) Moreover, let \( v = e^{it\Delta} v_0 \) and \( w = u - v. \) Suppose that \( v \in Y \cap Z(\mathbb{R}), \) \( w_0 \in H^1(\mathbb{R}^3) \) such that
\[ \| u_0 \|_{H^s_x(\mathbb{R}^3)} + \| v \|_{Y \cap Z(\mathbb{R})} \leq A, \ M(0) \leq AN_0^{-2s}, \ \text{and} \ E(0) \leq AN_0^{2(1-s)}. \]
Then, there exists a solution \( u \) of (1.7) on \( \mathbb{R} \) with \( w \in C(\mathbb{R} ; H^s_x(\mathbb{R}^3)) \). Furthermore, there exists \( u_\pm \in H^s_x(\mathbb{R}^3) \) such that
\[ \lim_{t \to \pm \infty} \| u(t) - v(t) - e^{it\Delta} u_\pm \|_{H^s_x(\mathbb{R}^3)} = 0. \]
We will give the proof of Proposition 4.1 in Sections 4.2-4.6. Now we prove Theorem 1.2 in 3D case assuming that Proposition 4.1 holds.

**Proof of Theorem 1.2.** Now, we only need to prove the \( s < 0 \) case. In fact, in the case when \( 0 < s < 1, \) the mass conservation law is available, so the proof of related result in Proposition 4.1 is easier.
For any \( s < 0, \) by the Definition 1.1, we can find a randomization \( f^\omega \) with the parameter \( a. \) Furthermore, \( a \) and \( s \) satisfy the assumption of Proposition 4.1. Let \( N_0 \in 2^\mathbb{N} \) to be defined later, and make a high-low frequency decomposition for the initial data
\[ u(t) = e^{it\Delta} P_{\geq N_0} f^\omega + w(t), \]
then \( w \) satisfies the equation (4.1) with
\[ u_0 = f^\omega, \ v_0 = P_{\geq N_0} f^\omega, \ w_0 = P_{< N_0} f^\omega, \) and \( v = e^{it\Delta} P_{\geq N_0} f^\omega. \)
By Corollary 3.2 and boundedness of the operator \( P_{\geq N_0}, \) we have
\[ \mathbb{P}(\{ \omega \in \Omega : \| u_0 \|_{H^s_x} + \| v \|_{Y \cap Z(\mathbb{R})} > \lambda \}) \leq e^{-C \lambda^2 \| f \|_{H^s_x}^2}. \quad (4.4) \]
For any \( p \geq 2, \) by Lemmas 2.8 and 2.6,
\[ \| w_0 \|_{L^p_t L^2_x} \lesssim \sqrt{p} \| \Box_j P_{\geq N_0} f \|_{L^2_t L^2_x} \lesssim \sqrt{p} \| P_{\geq N_0} f \|_{L^2_x} \lesssim \sqrt{p} N_0^{-s} \| f \|_{H^s_x}. \]
Then, by Lemma 2.9, we have
\[ \mathbb{P}\left( \{ \omega \in \Omega : N_0^s \|w_0\|_{L^2_x} > \lambda \} \right) \leq e^{-C \lambda^2 \|f\|_{H^s_x}^2}. \] (4.5)

For any \( p \geq 2 \), by Lemmas 2.8 and 2.6,
\[ \|w_0\|_{L^p_x L^2_t} \leq \sqrt{p} \|\nabla P_{\leq N_0} \Box f\|_{L^2_x L^p_t} \leq \sqrt{p} \|\nabla P_{\leq N_0} f\|_{L^2_x L^p_t} \lesssim \sqrt{p} N_0^{1-s} \|f\|_{H^s_x}. \]

For any \( p \geq 6 \), since \( s > -\alpha \), by Minkowski’s inequality, Lemmas 2.8, 2.7, and 2.6,
\[ \|u_0\|_{L^p_x L^6_t} \lesssim \sqrt{p} \|\Box f\|_{L^6_x L^p_t} \lesssim \sqrt{p} \|\nabla f\|_{L^2_x} \lesssim \sqrt{p} \|f\|_{H^s_x}. \]

Note that \( N_0 \) only depends on \( M \) and \( \|f\|_{H^s_x} \). Then, by Lemma 2.9,
\[ \mathbb{P}\left( \{ \omega \in \Omega : N_0^{s-1} \|w_0\|_{H^s_x} + \|u_0\|_{L^6_x} \geq \lambda \} \right) \lesssim e^{-C \lambda^2 \|f\|_{H^s_x}^2}. \] (4.6)

For any \( M \geq 1 \), let \( \tilde{\Omega}_M \) be defined by
\[ \tilde{\Omega}_M = \left\{ \omega \in \Omega : \|u_0\|_{H^s_x} + \|v\|_{Y \cap Z(\mathbb{R})} < M \|f\|_{H^s_x} ; \
N_0^s \|w_0\|_{L^2_x} + N_0^{s-1} \|w_0\|_{H^s_x} + \|u_0\|_{L^6_x} < M \|f\|_{H^s_x} \right\}. \] (4.7)

Therefore, by (4.4) and (4.5), we have
\[ \mathbb{P}(\tilde{\Omega}_M^c) \lesssim e^{-CM^2}. \] (4.8)

For any \( \omega \in \tilde{\Omega}_M \), we have \( \|v\|_{Y \cap Z(\mathbb{R})} < M \|f\|_{H^s_x} \),
\[ M(0) \leq CN_0^{-2a} M^2 \|f\|_{H^s_x}^2, \quad \text{and} \quad E(0) \leq CN_0^{2(1-s)} M^2 \cdot \max \left\{ M^4 \|f\|_{H^s_x}^6, \|f\|_{H^s_x}^2 \right\}. \]

Therefore, for any \( M > 1 \) and any \( \omega \in \tilde{\Omega}_M \), let
\[ A = A(M, \|f\|_{H^s_x}) := \max \left\{ CM \|f\|_{H^s_x}, CM^2 \cdot \max \left\{ M^4 \|f\|_{H^s_x}^6, \|f\|_{H^s_x}^2 \right\} \right\}, \]
then we have for \( v = e^{it\Delta} P_{\geq N_0} f^\omega, \)
\[ \|u_0\|_{H^s_x} + \|v\|_{Y \cap Z(\mathbb{R})} \leq A, \quad M(0) \leq AN_0^{-2a}, \quad \text{and} \quad E(0) \leq AN_0^{2(1-s)}. \]

Therefore, we can apply Proposition 4.1. Let \( N_0 \) depend on \( A \) as in the statement of Proposition 4.1, and we obtain a global solution \( w \) that scatters. Then, for any \( \omega \in \Omega = \bigcup_{M > 1} \tilde{\Omega}_M \), we can also derive that (4.1) admits a global solution \( w \) that scatters. By (4.8), we have that \( \mathbb{P}(\tilde{\Omega}) = 1 \). Then for almost every \( \omega \in \Omega \), we obtain the global well-posedness and scattering for (4.1). This finishes the proof of Theorem 1.2 in 3D case. \( \square \)

4.2. Local theory. We define the space \( X(I) \) as
\[ \|w\|_{X(I)} := \|\nabla w\|_{L^2_x L^5_t(I \times \mathbb{R}^3)} + \|w\|_{L^8_x L^4(I \times \mathbb{R}^3)} + \|w\|_{L^6_x L^{12}(I \times \mathbb{R}^3)}. \]

Lemma 4.2 (Local well-posedness). Let \( a \in \mathbb{N}, a > 10, \frac{3}{4} - \frac{4}{3} a < s < 0, \) \( v \in Y \cap Z(\mathbb{R}) \), and \( w_0 \in H^2_x \). Then, there exists some \( T > 0 \) depending on \( a, w_0, \) and \( v \) such that there exists a unique solution \( w \) of (4.1) in some \( 0 \)-neighbourhood of \( C([0, T]; H^s_x(\mathbb{R}^3)) \cap X([0, T]). \)

Proof. First, we make the choices of some parameters:
(1) Let $C_0$ be the constant such that 
\[ \| e^{it\Delta} w_0 \|_{L_t^\infty (\mathbb{R}; H_x^1) \cap X(\mathbb{R})} \leq C_0 \| w_0 \|_{H_x^1}. \]

(2) Define 
\[ R := \max \left\{ C_0 \| w_0 \|_{H_x^1}, 1 \right\}. \]

(3) Let $\delta > 0$ be some small constant such that $C\delta^4 R^4 \leq \frac{1}{4}.$

(4) Let $T > 0$ satisfy the smallness condition 
\[ \| e^{it\Delta} w_0 \|_{X([0,T])} + \| v \|_{Y([0,T])} \leq \delta R. \]

Now, we define the working space as 
\[ B_{R,\delta,T} := \left\{ w \in C([0,T]; H_x^1) : \| w \|_{L_t^\infty H_x^1([0,T] \times \mathbb{R}^3)} + \delta^{-1} \| w \|_{X([0,T])} \leq 4R \right\}, \]

equipped with the norm 
\[ \| w \|_{B_{R,\delta,T}} := \| w \|_{L_t^\infty H_x^1([0,T] \times \mathbb{R}^3)} + \delta^{-1} \| w \|_{X([0,T])}. \]

Take the solution map as 
\[ \Phi_{w_0,v}(w) := e^{it\Delta} w_0 - i \int_0^t e^{i(t-s)\Delta} (|u|^4 u) \, ds. \]

Then, it suffices to prove that $\Phi_{w_0,v}$ is a contraction mapping on $B_{R,\delta,T}.$

Next, we prove that for any $w \in B_{R,\delta,T}, \Phi_{w_0,v}(w) \in B_{R,\delta,T}.$ By Lemma 2.4 and Hölder’s inequality, 
\[ \| \Phi_{w_0,v}(w) \|_{L_t^\infty H_x^1} \leq C_0 \| w_0 \|_{H_x^1} + C \| \nabla (|u|^4 u) \|_{L_t^1 L_x^2} + C \| |u|^4 u \|_{L_t^1 L_x^2} \]
\[ \leq R + C \| \nabla w u^4 \|_{L_t^1 L_x^2} + C \| \nabla v u^4 \|_{L_t^1 L_x^2} + C \| |u|^4 u \|_{L_t^1 L_x^2} \]
\[ \leq R + C \| \nabla w \|_{L_t^1 L_x^6} \| u \|_{L_t^4 L_x^2} + C \| \nabla v \|_{L_t^1 L_x^6} \| u \|_{L_t^4 L_x^2} \]
\[ + C \| w \|_{L_t^1 L_x^6} \| u \|_{L_t^4 L_x^2} + C \| v \|_{L_t^1 L_x^6} \| u \|_{L_t^4 L_x^2}. \]

Note that $v$ is high frequency part, then by the definition of $X$ and $Y$ norms, 
\[ \| (\nabla) w \|_{L_t^2 L_x^2} \leq \| w \|_{X} \lesssim \delta R, \]
\[ \| (\nabla) v \|_{L_t^2 L_x^2} \lesssim \| v \|_{Y} \lesssim \delta R, \]
\[ \| u \|_{L_t^4 L_x^2} \leq \| w \|_{L_t^4 L_x^2} + \| v \|_{L_t^4 L_x^2} \leq \| w \|_{X} + \| v \|_{Y} \lesssim \delta R, \]

and 
\[ \| u \|_{L_t^8} \leq \| w \|_{L_t^8} + \| v \|_{L_t^8} \leq \| w \|_{X} + \| v \|_{Y} \lesssim \delta R. \]

Therefore, by the choice of $\delta,$ 
\[ \| \Phi_{w_0,v}(w) \|_{L_t^\infty H_x^1} \leq R + C\delta^5 R^5 \leq 2R. \quad (4.9) \]

Similar as above, we also have 
\[ \| \Phi_{w_0,v}(w) \|_{X} \leq \| e^{it\Delta} w_0 \|_{H_x^1} + C \| \nabla (|u|^4 u) \|_{L_t^1 L_x^2} + C \| |u|^4 u \|_{L_t^1 L_x^2} \]
\[ \leq \delta R + C \delta^5 R^5 \leq 2\delta R. \quad (4.10) \]

Then, by (4.9) and (4.9), 
\[ \| \Phi_{w_0,v}(w) \|_{B_{R,\delta,T}} \leq \| \Phi_{w_0,v}(w) \|_{L_t^\infty H_x^1} + \delta^{-1} \| \Phi_{w_0,v}(w) \|_{X} \leq 4R. \]
This proves that $\Phi_{w_0,v}$ maps $B_{R,\delta,T}$ into itself.

Next, we are going to prove that $\Phi_{w_0,v}$ is a contraction mapping. Take $w_1$ and $w_2$ in $B_{R,\delta,T}$. By Lemma 2.4,

$$\|\Phi_{w_0,v}(w_1) - \Phi_{w_0,v}(w_2)\|_{L^\infty_t H^1_x} \leq C\|\langle \nabla \rangle(|w_1 + v|^4(w_1 + v) - |w_2 + v|^4(w_2 + v))\|_{L^1_t L^2_x}.$$  \hfill (4.11)

Note that we have an elementary inequality

$$|w_1 + v|^4(w_1 + v) - |w_2 + v|^4(w_2 + v)| \lesssim |w_1 - w_2||w_1|^4 + |w_2|^4 + |v|^4.$$  

Then, by Hölder’s inequality,

$$\|\nabla(|w_1 + v|^4(w_1 + v) - |w_2 + v|^4(w_2 + v))\|_{L^1_t L^2_x} \lesssim \|w_1 - w_2\|_{L^2_t L^\infty_x}^4 \|\nabla w_1\|_{L^6_t L^\infty_x}^4 + \|w_2\|_{L^6_t L^\infty_x}^4 \|\nabla w_2\|_{L^6_t L^\infty_x}^4 + \|v\|_{L^6_t L^\infty_x}^4 \|\nabla v\|_{L^6_t L^\infty_x}^4.$$

Note that we also have

$$\|\nabla(|w_1 + v|^4(w_1 + v) - |w_2 + v|^4(w_2 + v))\|_{L^1_t L^2_x} \lesssim \|w_1 - w_2\|_{L^2_t L^\infty_x}^4 \|\nabla w_1\|_{L^6_t L^\infty_x}^4 + \|w_2\|_{L^6_t L^\infty_x}^4 \|\nabla w_2\|_{L^6_t L^\infty_x}^4 + \|v\|_{L^6_t L^\infty_x}^4 \|\nabla v\|_{L^6_t L^\infty_x}^4.$$  \hfill (4.12)

Therefore, by (4.11), (4.12), and (4.13),

$$\|\Phi_{w_0,v}(w_1) - \Phi_{w_0,v}(w_2)\|_{L^\infty_t H^1_x} \lesssim C\delta^5 R^4 \|w_1 - w_2\|_{B_{R,\delta,T}}.$$  \hfill (4.14)

Then, by (4.14) and the choice of $\delta$,

$$\|\Phi_{w_0,v}(w_1) - \Phi_{w_0,v}(w_2)\|_{B_{R,\delta,T}} \leq C\delta^5 R^4 \|w_1 - w_2\|_{B_{R,\delta,T}} + C\delta^4 R^4 \|w_1 - w_2\|_{B_{R,\delta,T}} \leq \frac{1}{2} \|w_1 - w_2\|_{B_{R,\delta,T}}.$$

This proves that $\Phi_{w_0,v}$ is a contraction mapping on $B_{R,\delta,T}$.

4.3. Modified Interaction Morawetz. We need to prove a perturbation version of interaction Morawetz estimate as follows.

**Lemma 4.3** (Modified Interaction Morawetz). Given $T > 0$. Let $w \in C([0,T]; H^1_x)$ be the solution of perturbation equation (4.1). Then, we have

$$\|w\|_{L^4_t L^\infty_x}^4 \lesssim \|w\|_{L^1_t L^\infty_x}^4 \|w\|_{L^6_t H^\frac{1}{3}_x}^2 \|w\|_{L^2_t H^\frac{1}{2}_x}^2 + \|w\|_{L^6_t L^\infty_x}^2 \|w\|_{L^4_t L^\infty_x}^2 + \|w\|_{L^3_t L^6_x}^3 \|w\|_{L^3_t L^6_x}^3 + \|w\|_{L^3_t L^6_x}^3 \|w\|_{L^3_t L^6_x}^3,$$

where all the space-time norms are taken over $[0,T] \times \mathbb{R}^3$.  \hfill (4.15)
Proof. Recall that \( w \) satisfies
\[
i \partial_t w + \Delta w = |w|^4 w + e,
\]
where we denote \( e := |u|^4 u - |w|^4 w \). Denote that
\[
m(t, x) = \frac{1}{2} |w(t, x)|^2; \quad p(t, x) = \frac{1}{2} \text{Im}(\bar{w}(t, x) \nabla w(t, x)).
\]
Then, we have
\[
\partial_t m = -2 \nabla \cdot p + \text{Im} (e \bar{w}),
\]
and
\[
\partial_t p = -\text{Re} \nabla \cdot (\nabla \bar{w} \nabla w) - \frac{1}{6} \nabla \big(|w|^6\big) + \frac{1}{2} \nabla \Delta m + \text{Re} (\bar{v} \nabla w) - \frac{1}{2} \text{Re} \nabla (\overline{ve}).
\]
Moreover, we note that
\[
\partial_j \left( \frac{x_k}{|x|} \right) = \frac{\delta_{jk}}{|x|} \frac{x_j x_k}{|x|^3}; \quad \nabla \cdot \frac{x}{|x|} = \frac{2}{|x|}; \quad \Delta \nabla \cdot \frac{x}{|x|} = \delta(x).
\]
Let
\[
M(t) := \int \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot p(t, x) m(t, y) \, dx \, dy,
\]
then by (4.16) and (4.17), we have the interaction Morawetz identity
\[
\partial_t M(t) = \int \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot \partial_t p(t, x) m(t, y) \, dx \, dy
\]
\[
+ \int \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot p(t, x) \partial_t m(t, y) \, dx \, dy
\]
\[
= \int \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot \left( -\text{Re} \nabla \cdot (\nabla \bar{w} \nabla w) - \frac{1}{6} \nabla \big(|w|^6\big) \right)(t, x) m(t, y) \, dx \, dy
\]
\[
- 2 \int \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot p(t, x) \nabla \cdot p(t, y) \, dx \, dy
\]
\[
+ \frac{1}{2} \int \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot \nabla \Delta m(t, x) m(t, y) \, dx \, dy
\]
\[
+ \int \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot p(t, x) \text{Im} (e \bar{w})(t, y) \, dx \, dy
\]
\[
+ \int \int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot \text{Re} (\bar{v} \nabla w)(t, x) m(t, y) \, dx \, dy
\]
\[
+ \int \int_{\mathbb{R}^{3+3}} \frac{1}{|x - y|} \cdot \text{Re} (\bar{w})(t, x) m(t, y) \, dx \, dy.
\]
Note that by the classical argument in [21], we have
\[
(4.18a) + (4.18b) \geq 0,
\]
and
\[
(4.18c) \geq \|w(t)\|_{L^4}^4.
\]
Moreover,

\[
\sup_{t \in [0, T]} M(t) \lesssim \|w\|_{L^\infty_t L^2_x}^2 \|w\|_{L^\infty_t H^4_x}^2.
\]

Then, integrating over \([0, T]\), it holds that

\[
C \|w\|_{L^4_t L^4_x}^4 \leq M(T) - M(0) + \int_0^T \frac{|(4.18d)| + |(4.18e)| + |(4.18f)|}{dt},
\]

thus

\[
\|w\|_{L^4_t L^4_x}^4 \lesssim \|w\|_{L^\infty_t L^2_x}^2 \|w\|_{L^\infty_t H^4_x}^2 + \int_0^T \frac{|(4.18d)| + |(4.18e)| + |(4.18f)|}{dt}.
\]

Next, we estimate the terms containing \((4.18d), (4.18e),\) and \((4.18f)\). We first consider \((4.18d)\). By Hölder’s inequality,

\[
\int_0^T \frac{|(4.18d)|}{dt} \lesssim \int_0^T |\int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot p(t, x) \text{Im} (e^i \bar{w}) (t, y) dx dy| dt
\]

\[
\lesssim \int_0^T |\int_{\mathbb{R}^3} \text{Im} (e^i \bar{w}) (t, y) dy| sup \int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot p(t, x) dx| dt
\]

\[
\lesssim \|e\|_{L^6_t L^6_x}^6 \|w\|_{L^\infty_t L^6_x} \|w\|_{L^\infty_t H^6_x}^2.
\]

Note that

\[
\|e\|_{L^6_t L^6_x}^6 \lesssim \|v\|_{L^2_t L^\infty_x}^2 (\|w\|_{L^4_t L^4_x}^2 + \|v\|_{L^4_t L^4_x}^2) (\|w\|_{L^\infty_t L^6_x}^2 + \|v\|_{L^\infty_t L^6_x}^2).
\]

Hence, we get that \((4.18f)\). We first consider \((4.18d)\). By Hölder’s inequality,

\[
\int_0^T \frac{|(4.18d)|}{dt} \lesssim \|v\|_{L^2_t L^\infty_x}^2 (\|w\|_{L^4_t L^4_x}^2 + \|v\|_{L^4_t L^4_x}^2) (\|w\|_{L^\infty_t L^6_x}^2 + \|v\|_{L^\infty_t L^6_x}^2) \|w\|_{L^\infty_t H^4_x}^2.
\]

We then consider \((4.18e)\), where we need to modify the Morawetz estimate:

\[
\int_0^T \frac{|(4.18e)|}{dt} \lesssim \int_0^T |\int_{\mathbb{R}^{3+3}} \frac{x - y}{|x - y|} \cdot \text{Re} \left( (|u|^4 u - |w|^4 w) \nabla \bar{w} \right) (t, x) m(t, y) dx dy| dt.
\]

We note that

\[
(|u|^4 u - |w|^4 w) \nabla \bar{w}(t, x) = vw^4 \nabla \bar{w}(t, x) + \text{other terms}.
\]

However, it is difficult to estimate the piece \(vw^4 \nabla \bar{w}(t, x)\). To this end, we need the following equality;

\[
\int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot \text{Re} \left( (|u|^4 u - |w|^4 w) \nabla \bar{w} \right) (t, x) dx
\]

\[
= - \frac{1}{3} \int_{\mathbb{R}^3} \frac{1}{|x - y|} (|u(t, x)|^6 - |w(t, x)|^6) dx - \int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot \text{Re} \left( |u|^4 u \nabla \bar{v} \right) (t, x) dx.
\]
Indeed,

\[
\int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot \text{Re} \left( (|u|^4 u - |w|^4 w) \nabla \overline{v} \right) (t, x) \, dx
\]

\[
= \int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot \text{Re} \left( |u|^4 u \nabla \overline{v} \right) (t, x) \, dx - \int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot \text{Re} \left( |w|^4 w \nabla \overline{v} \right) (t, x) \, dx
\]

\[
= \int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot \text{Re} \left( |u|^4 u \nabla (\overline{w} - \overline{v}) \right) (t, x) \, dx - \frac{1}{6} \int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot \nabla \left( |w(t, x)|^6 \right) \, dx
\]

\[
= -\frac{1}{3} \int_{\mathbb{R}^3} \frac{1}{|x - y|} \left( |u(t, x)|^6 - |w(t, x)|^6 \right) \, dx - \int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot \text{Re} \left( |u|^4 u \nabla \overline{v} \right) (t, x) \, dx.
\]

This gives (4.20). Therefore, by (4.20) and Lemma 2.1, we have

\[
\int_0^T \left| (4.18e) \right| \, dt \lesssim \int_0^T \left| \int_{\mathbb{R}^3} \frac{1}{|x - y|} \left( |u(t, x)|^6 - |w(t, x)|^6 \right) \, dx \, dy \right| \, dt
\]

\[
+ \int_0^T \left| \int_{\mathbb{R}^3} \frac{x - y}{|x - y|} \cdot \text{Re} \left( |u|^4 u \nabla \overline{v} \right) (t, x) \, dx \, dy \right| \, dt
\]

\[
\lesssim \int_0^T \left| \int_{\mathbb{R}^3} \left( |u(t, x)|^6 - |w(t, x)|^6 \right) \, dx \right| \left( \sup_x \left| \frac{1}{|x - |x|^\frac{1}{2}} \right| \right)^2 \, dt
\]

\[
+ \int_0^T \left| \int_{\mathbb{R}^3} \left| |u|^4 u \nabla \overline{v}(t, x) \right| \, dx \right| \left( \sup_x \left| \frac{1}{|x - |x|^\frac{1}{2}} \right| \right)^2 \, dt
\]

\[
\lesssim \|v\|_{L_t^\infty L_x^\infty} \left( \|w\|_{L_t^4 L_x^4}^2 + \|v\|_{L_t^4 L_x^4}^2 \right) \left( \|w\|_{L_t^\infty L_x^5}^3 + \|v\|_{L_t^\infty L_x^5}^3 \right) \left( \|w\|_{L_t^\infty H_x^\frac{3}{2}} \right)^2
\]

\[
+ \|\nabla v\|_{L_t^2 L_x^\infty} \left( \|w\|_{L_t^4 L_x^4}^2 + \|v\|_{L_t^4 L_x^4}^2 \right) \left( \|w\|_{L_t^\infty L_x^5}^3 + \|v\|_{L_t^\infty L_x^5}^3 \right) \left( \|w\|_{L_t^\infty L_x^5}^2 \right)^\frac{1}{2}.
\]

Now, we consider the term (4.18f). Treated similarly as above, we have that

\[
\int_0^T \left| (4.18f) \right| \, dt \lesssim \int_0^T \left| \int_{\mathbb{R}^3} \frac{1}{|x - y|} \cdot c \overline{v}(t, x) \, dx \, dy \right| \, dt
\]

\[
\lesssim \int_0^T \left| \int_{\mathbb{R}^3} c \overline{v}(t, x) \, dx \sup_x \left| \frac{1}{|x - |x|^\frac{1}{2}} \right| \right| \, dt
\]

\[
\lesssim \|v\|_{L_t^\infty L_x^\infty} \left( \|w\|_{L_t^4 L_x^4}^2 + \|v\|_{L_t^4 L_x^4}^2 \right) \left( \|w\|_{L_t^\infty L_x^5}^3 + \|v\|_{L_t^\infty L_x^5}^3 \right) \left( \|w\|_{L_t^\infty L_x^5}^2 \right)^\frac{1}{2}.
\]

Combining the findings on (4.18), we have that

\[
\int_0^T \left| (4.18d) \right| + \left| (4.18e) \right| + \left| (4.18f) \right| \, dt
\]

\[
\lesssim \|v\|_{L_t^\infty L_x^\infty} \left( \|w\|_{L_t^4 L_x^4}^2 + \|v\|_{L_t^4 L_x^4}^2 \right) \left( \|w\|_{L_t^\infty L_x^5}^3 + \|v\|_{L_t^\infty L_x^5}^3 \right) \left( \|w\|_{L_t^\infty H_x^\frac{3}{2}} \right)^2
\]

\[
+ \|\nabla v\|_{L_t^2 L_x^\infty} \left( \|w\|_{L_t^4 L_x^4}^2 + \|v\|_{L_t^4 L_x^4}^2 \right) \left( \|w\|_{L_t^\infty L_x^5}^3 + \|v\|_{L_t^\infty L_x^5}^3 \right) \left( \|w\|_{L_t^\infty L_x^5}^2 \right)^\frac{1}{2}.
\]
Therefore, by (4.19) and (4.21),
\[
\|w\|_{L^2_t L^4_x} \lesssim \|w\|_{L^\infty_t L^8_x}^2 \|w\|_{L^\infty_t H^{1/2}_x}^2 + \|v\|_{L^2_t L^\infty_x}^2 (\|w\|^2_{L^4_t L^4_x} + \|v\|^2_{L^4_t L^4_x}) (\|w\|^3_{L^\infty_t L^6_x} + \|v\|^3_{L^\infty_t L^6_x}) \|w\|_{L^\infty_t H^{1/2}_x}^2
\]
\[
+ \|\nabla v\|_{L^2_t L^\infty_x} (\|w\|^2_{L^4_t L^4_x} + \|v\|^2_{L^4_t L^4_x}) (\|w\|^3_{L^\infty_t L^6_x} + \|v\|^3_{L^\infty_t L^6_x}) \|w\|_{L^\infty_t L^4_x}^2.
\]
This completes the proof of this lemma. \qed

4.4. Almost conservation law. Our main result in this subsection is

**Proposition 4.4.** Let \(a \in \mathbb{N}, a > 10, \frac{3}{4} - \frac{1}{a} < s < 0, A > 0, v = e^{i\Delta} v_0 \in Y \cap Z(\mathbb{R})\) and \(w\) be the solution of (4.1). Take some \(T > 0\) such that \(w \in C([0, T]; H^1_x)\). Then, there exists \(N_0 = N_0(A) \gg 1\) with the following properties. Assume that \(\tilde{v}_0\) is supported on \(\{\xi \in \mathbb{R}^3 : |\xi| \geq \frac{1}{2} N_0\}\),
\[
\|u_0\|_{H^1_x} + \|v\|_{Y \cap Z(\mathbb{R})} \leq A, M(0) \leq AN_0^{-2s}, \text{ and } E(0) \leq AN_0^{2(1-s)}.
\]
Then, we have
\[
\sup_{t \in [0, T]} M(t) \leq 2AN_0^{-2s}, \text{ and } \sup_{t \in [0, T]} E(t) \leq 2AN_0^{2(1-s)}.
\] (4.22)

**Proof.** Let \(N_0 = N_0(A)\) that will be defined later. We implement a bootstrap procedure on \(I \subset [0, T]\): assume an a priori bound
\[
\sup_{t \in I} M(t) \leq 2AN_0^{-2s}, \text{ and } \sup_{t \in I} E(t) \leq 2AN_0^{2(1-s)},
\] (4.23)
then it suffices to prove that
\[
\sup_{t \in I} M(t) \leq \frac{3}{2} AN_0^{-2s}, \text{ and } \sup_{t \in I} E(t) \leq \frac{3}{2} AN_0^{2(1-s)}.
\] (4.24)

From now on, all the space-time norms are taken over \(I \times \mathbb{R}^3\).

To start with, we collect useful estimates on \(I\). Now, we use the notation \(C = C(A)\) for short, and the implicit constants in "\(\lesssim\)" depend on \(A\). By Lemma 2.3, we have \(Z \subset L^\infty_t L^6_x\), then
\[
\|v\|_{L^\infty_t L^6_x} + \|v\|_{L^4_t L^4_x} \lesssim \|v\|_{Y \cap Z} \lesssim 1.
\] (4.25)
By the frequency support of \(v\), we have for any \(0 \leq l < \frac{1}{2} a + s\),
\[
\|\nabla^l v\|_{L^2_t L^\infty_x} \lesssim N_0^{-\frac{1}{2} a - s} \|v\|_{Y} \lesssim N_0^{-\frac{1}{2} a - s} \lesssim 1.
\] (4.26)

Note that we assume \(a > 10\) and \(s > \frac{3}{4} - \frac{1}{a}\), then \(\frac{1}{2} a + s > 2\). Therefore, this guarantees that \(\|\Delta v\|_{L^2_t L^\infty_x} \lesssim N_0^{-\frac{1}{2} a - s}\) is allowed. By bootstrap hypothesis (4.23),
\[
\|w\|_{L^\infty_t L^2_x} \lesssim N_0^{-s}, \|w\|_{L^\infty_t H^{1/2}_x} \lesssim N_0^{1-s}, \text{ and } \|w\|_{L^\infty_t L^4_x} \lesssim N_0^{2(1-s)}.
\] (4.27)
Then, by interpolation and (4.27), we have for any \(0 \leq l \leq 1\),
\[
\|w\|_{L^\infty_t H^l_x} \lesssim N_0^{l-s}.
\] (4.28)
Furthermore, by Lemma 4.3, (4.28), and (4.26),
\[ ||w||_{L^4_t L^2_x}^4 \lesssim ||w||_{L^2_t L^\infty_x}^2 ||w||_{L^6_t H^{\frac{1}{2}}_x}^2 + ||v||_{L^2_t L^\infty_x}^2 (||w||_{L^4_t L^2_x}^2 + ||v||_{L^4_t L^6_x}^2) (||w||_{L^2_t L^6_x}^3 + ||v||_{L^2_t L^6_x}^3) ||w||_{L^2_t L^6_x}^2 \]
\[ + ||\nabla v||_{L^2_t L^\infty_x}^2 (||w||_{L^4_t L^2_x}^2 + ||v||_{L^4_t L^6_x}^2) (||w||_{L^2_t L^6_x}^3 + ||v||_{L^2_t L^6_x}^3) ||w||_{L^2_t L^6_x}^2 \]
\[ \lesssim N_0^{1-4s} + N_0^{-\frac{1}{2}a-s} (||w||_{L^4_t L^2_x}^2 + 1)(N_0^{1-s} + 1)N_0^{1-2s} + N_0^{-\frac{1}{2}a-s} (||w||_{L^4_t L^2_x}^2 + 1)(N_0^{1-s} + 1)N_0^{-2s} \]
\[ \lesssim N_0^{1-4s} + N_0^{-\frac{1}{2}a-4s} + N_0^{-2-\frac{1}{2}a-4s} + N_0^{-1-4s}. \]

By \( s > \frac{3}{2} - \frac{1}{4}a \) and Young’s inequality,
\[ ||w||_{L^4_t L^\infty_x}^4 \lesssim N_0^{1-4s} + N_0^{-\frac{1}{2}a-4s} + N_0^{-2-\frac{1}{2}a-8s} \lesssim N_0^{-1-4s}. \]  

(4.29)

Now, we are prepared to give the proof of (4.24). To do this, we first need the following lemma.

**Lemma 4.5.** Assume that \( w \in C(I; H^1_x(\mathbb{R}^3)) \) solves (4.1). Let \( E(t) \) and \( M(t) \) be defined as in (4.2) and (4.3). Then, for any \( t \in I \),
\[ \left| \frac{d}{dt} M(t) \right| \leq 2 \int_{\mathbb{R}^3} (|u|^4 u - |w|^4 w) \bar{w} \, dx, \text{ and } \left| \frac{d}{dt} E(t) \right| \leq \int_{\mathbb{R}^3} |u|^4 u \Delta \bar{w} \, dx. \]

**Proof.** First, by (4.1) and integration-by-parts,
\[ \frac{d}{dt} \left( \int |w(t, x)|^2 \, dx \right) = 2 \text{Re} \int \bar{w} w_t \, dx \]
\[ = 2 \text{Re} \int \bar{w} (\Delta w - |u|^4 u) \, dx \]
\[ = -2 \text{Re} \int \bar{w} (|u|^4 u - |w|^4 w) \, dx. \]

Similarly by (4.1) and integration-by-parts,
\[ \frac{d}{dt} \left( \frac{1}{2} \int |\nabla w(t, x)|^2 \, dx \right) = - \Re \int \Delta w \bar{w}_t \, dx \]
\[ = \Re \int w_t \bar{w}_t \, dx - \Re \int |u|^4 u \bar{w}_t \, dx \]
\[ = - \Re \int |u|^4 u \bar{w}_t \, dx + \Re \int |u|^4 u \bar{w}_t \, dx \]
\[ = -\frac{d}{dt} \left( \frac{1}{6} \int |u|^6 \, dx \right) + \Re \int |u|^4 u \Delta \bar{w} \, dx, \]
then we have
\[ \frac{d}{dt} E(t) = \Re \int |u|^4 u \bar{w}_t \, dx = - \Re i \int |u|^4 u \Delta \bar{w} \, dx. \]

This finishes the proof of this lemma. \( \square \)
Now we continue to prove the proposition. We first consider the mass bound in (4.24). By Lemma 4.5 and Hölder’s inequality,

$$
sup_{t \in I} M(t) \leq M(0) + \int_I \left| \frac{d}{dt} M(t) \right| \, dt
$$

$$
\leq M(0) + \int_I \left| \int_I \mathfrak{W}(|u|^4 u - |w|^4 w) \, dx \right| \, dt
$$

$$
\leq A N_0^{-2s} + C \|v\|_{L_t^2 L_x^\infty} \|w\|_{L_t^6 L_x^4}(\|w\|_{L_t^{12} L_x^3} + \|v\|_{L_t^{12} L_x^3})(\|w\|_{L_t^6 L_x^6}^3 + \|v\|_{L_t^6 L_x^6}^3).
$$

Therefore, combining with (4.25), (4.26), (4.28), and (4.29),

$$
sup_{t \in I} M(t) \leq A N_0^{-2s} + C(A) N_0^{-\frac{1}{2}a + \frac{3}{2} - 2s +} \leq \frac{3}{2} A N_0^{-2s}, \tag{4.30}
$$

where we take $N_0 = N_0(A)$ such that $C(A) N_0^{-\frac{1}{2}a + \frac{3}{2} - 2s +} \leq \frac{1}{2} A$. This is allowed since $s > \frac{3}{4} - \frac{1}{4} a$.

We then prove the energy bound in (4.24). By Lemma 4.5 and Hölder’s inequality,

$$
sup_{t \in I} E(t) \leq E(0) + \int_I \left| \frac{d}{dt} E(t) \right| \, dt
$$

$$
\leq E(0) + \int_I \left| \int_I |u|^4 u \cdot \Delta \mathfrak{F} \, dx \right| \, dt
$$

$$
\leq A N_0^2 + C \|\Delta v\|_{L_t^2 L_x^\infty} (\|w\|_{L_t^{12} L_x^3}^2 + \|v\|_{L_t^{12} L_x^3}^2)(\|w\|_{L_t^6 L_x^6}^3 + \|v\|_{L_t^6 L_x^6}^3).
$$

Therefore, combining with (4.25), (4.26), (4.28), and (4.29),

$$
sup_{t \in I} E(t) \leq A N_0^{2(1-s)} + C(A) N_0^{-\frac{1}{2}a + \frac{3}{2} - 3s +} \leq \frac{3}{2} A N_0^{2(1-s)}, \tag{4.31}
$$

where we still need to take $N_0 = N_0(A)$ such that $C(A) N_0^{-\frac{1}{2}a + \frac{3}{2} - 2s +} \leq \frac{1}{2} A$. Therefore, (4.30) and (4.31) gives (4.24). This finishes the proof of this proposition. \qed

4.5. Perturbations. Now we consider the original energy critical equation:

$$
\begin{cases}
   i \partial_t \bar{w} + \Delta \bar{w} = |\bar{w}|^4 \bar{w}, \\
   \bar{w}(0, x) = \bar{w}_0,
\end{cases} \tag{4.32}
$$

where $\bar{w}(t, x) : \mathbb{R} \times \mathbb{R}^3 \rightarrow \mathbb{C}$. Let $g(t, x) := w(t, x) - \bar{w}(t, x)$. Then, the equation for $g$ is

$$
\begin{cases}
   i \partial_t g + \Delta g = F(g + v, \bar{w}), \\
   g(0, x) = w_0 - \bar{w}_0.
\end{cases} \tag{4.33}
$$

Here we denote that

$$
F(g, w) := |g + w|^4(g + w) - |w|^4 w.
$$

Recall that

$$
\|w\|_{X(I)} := \|\langle \nabla \rangle^{a} w\|_{L_t^2 L_x^6(I \times \mathbb{R}^3)} + \|w\|_{L_t^6 L_x^3(I \times \mathbb{R}^3)} + \|w\|_{L_t^{12} L_x^2(I \times \mathbb{R}^3)},
$$

and

$$
\|v\|_{Y(I)} = \|\langle \nabla \rangle^{a} v\|_{L_t^2 L_x^6(I \times \mathbb{R}^3)} + \|v\|_{L_t^6 L_x^3(I \times \mathbb{R}^3)} + \|v\|_{L_t^{12} L_x^2(I \times \mathbb{R}^3)}
$$

$$
+ \|v\|_{L_t^6 H_x^1(I \times \mathbb{R}^3)}.
$$
Lemma 4.6. Let $a \in \mathbb{N}$, $a > 10$, $\frac{2}{3} - \frac{1}{a} < s < 0$, $I \subset \mathbb{R}$, and $0 \in I$. Then, there exists $0 < \eta_1 \ll 1$ with the following properties. Let $\tilde{w} \in C(I; H^1_{x}(\mathbb{R}^3))$ be the solution of (4.32) on $I$, satisfying
\[ \|w_0 - \tilde{w}_0\|_{H^1(\mathbb{R}^3)} \leq \eta_1, \text{ and } \|\tilde{w}\|_{X(I)} \leq \eta_1. \]
For any $0 < \eta \leq \eta_1$, suppose that
\[ \|v\|_{Y(I)} \leq \eta, \]
then there exists a solution $w \in C(I; H^1_{x}(\mathbb{R}^3))$ of (4.1) with initial data $w_0$ such that
\[ \|w - \tilde{w}\|_{L^\infty_x H^1_x(I \times \mathbb{R}^3)} + \|w - \tilde{w}\|_{X(I)} \leq C_0(\|w_0 - \tilde{w}_0\|_{H^1_x(\mathbb{R}^3)} + \eta), \]
where $C_0 > 1$ is an absolute constant independent of $\eta$, $\eta_1$ and $I$.

Proof. Let $g$ be the solution of (4.33), $0 = \inf I$, and we restrict the time interval on $I$. Then, we have
\[ g = e^{it \Delta} g(0) - i \int_0^t e^{i(t-s) \Delta} F(g + v, \tilde{w}) \, ds. \]
Note that we have the pointwise estimate
\[ |F(g + v, \tilde{w})| = |g + v + \tilde{w}|^4 (g + v + \tilde{w}) - |\tilde{w}|^4 \tilde{w} \]
\[ \lesssim (|g| + |v|)(|g|^4 + |v|^4 + |\tilde{w}|^4), \]
and
\[ |\nabla F(g + v, \tilde{w})| = |\nabla ((g + v + \tilde{w})^4 (g + v + \tilde{w}) - |\tilde{w}|^4 \tilde{w})| \]
\[ \lesssim (|\nabla g| + |\nabla v|)(|g|^4 + |v|^4 + |\tilde{w}|^4) + |\nabla \tilde{w}|(|g|^4 + |v|^4) \]
\[ + |\nabla \tilde{w}|(|g| + |v|) |\tilde{w}|^3. \]
Note that by the definition of $a$ and $s$, we have $s + \frac{1}{2}a > 1$. Therefore, since $v$ is high-frequency,
\[ \|v\|_{L^2_x L^\infty_t} + \|\nabla v\|_{L^2_x L^\infty_t} \lesssim \|v\|_Y. \]
Then, using the similar argument in Lemma 4.2,
\[ \|F(g + v, \tilde{w})\|_{L^1_t L^2_x} \lesssim \|g\|_{L^4_x L^2_t} (\|g\|_{L^4_x L^2_t}^4 + \|v\|_{L^4_x L^2_t}^4 + \|\tilde{w}\|_{L^4_x L^2_t}^4) \]
\[ + \|v\|_{L^4_x L^2_t} (\|g\|_{L^4_x L^2_t}^4 + \|v\|_{L^4_x L^2_t}^4 + \|\tilde{w}\|_{L^4_x L^2_t}^4) \]
\[ \lesssim (\|g\|_X + \|v\|_Y)(\|g\|_{L^4_x L^2_t}^4 + \|v\|_{L^4_x L^2_t}^4 + \|\tilde{w}\|_{L^4_x L^2_t}^4), \]
and
\[ \|\nabla F(g + v, \tilde{w})\|_{L^1_t L^2_x} \lesssim \|\nabla g\|_{L^2_t L^6_x} (\|g\|_{L^4_x L^2_t}^4 + \|v\|_{L^4_x L^2_t}^4 + \|\tilde{w}\|_{L^4_x L^2_t}^4) \]
\[ + \|\nabla v\|_{L^2_t L^6_x} (\|g\|_{L^4_x L^2_t}^4 + \|v\|_{L^4_x L^2_t}^4 + \|\tilde{w}\|_{L^4_x L^2_t}^4) \]
\[ + \|\nabla \tilde{w}\|_{L^2_t L^6_x} (\|g\|_{L^4_x L^2_t}^4 + \|v\|_{L^4_x L^2_t}^4 + \|\tilde{w}\|_{L^4_x L^2_t}^4) \]
\[ \lesssim (\|g\|_X + \|v\|_Y)(\|g\|_{L^4_x L^2_t}^4 + \|v\|_{L^4_x L^2_t}^4 + \|\tilde{w}\|_{L^4_x L^2_t}^4) \]
\[ + \|\tilde{w}\|_X (\|g\|_{L^4_x L^2_t}^4 + \|v\|_{L^4_x L^2_t}^4 + \|g\|_X \|\tilde{w}\|_{L^4_x L^2_t}^4 + \|v\|_Y \|\tilde{w}\|_{L^4_x L^2_t}^4). \]
Therefore, by Lemma 2.4,
\[
\|g\|_{L_t^\infty H_x^1 \cap X} \lesssim \|g(0)\|_{H_x^1} + \| (\nabla) F(g + v, \tilde{w})\|_{L_t^1 L_x^2} \\
\lesssim \|g(0)\|_{H_x^1} + (\|g\|_X + \|v\|_Y)(\|g\|_{X} + \|v\|_{X} + \|\tilde{w}\|_{X}) \\
+ \|\tilde{w}\|_X (\|g\|_{X} + \|v\|_{X} + \|g\|_X \|\tilde{w}\|_X + \|v\|_Y \|\tilde{w}\|_X) \\
\lesssim \|g(0)\|_{H_x^1} + (\|g\|_X + \|g\|_X \|\tilde{w}\|_X + \|v\|_X \|\tilde{w}\|_X) \\
+ \eta_1 (\|g\|_{X} + \eta_1^4 + \|g\|_X \eta_1^3 + \eta_1^4) \\
\lesssim \|g(0)\|_{H_x^1} + \|g\|_{X} \eta_1^4 + \|g\|_X + \|g\|_X \eta_1 + \eta_1 \|g\|_X,
\]
then we have
\[
\|g\|_{L_t^\infty H_x^1 \cap X} \lesssim \|g(0)\|_{H_x^1} + (\|g\|_X + \eta_1) \|g\|_{X} \eta_1^4 + \|g\|_X + \eta_1 \|g\|_X.
\]

Lemma 4.7. Suppose that \(a \in \mathbb{N}, a > 10, \frac{3}{4} \frac{3}{4} \frac{3}{4} a < s < 0, I \subset \mathbb{R}, M_0 > 1, I \subset \mathbb{R},\)
and \(0 \in I.\) Let \(\tilde{w} \in C(I; H_x^1(\mathbb{R}^3))\) be the solution of (4.32) on \(I\) with 
\[
\tilde{w}_0 = w_0, \quad \text{and} \quad \|\tilde{w}\|_{X(I)} \leq M_0.
\]

Let \(0 < \eta_1 \ll 1\) and \(C_0 > 1\) be defined as in Lemma 4.6. Then, there exists \(\eta_2 = \eta_2(C_0, M_0, \eta_1) > 0\) such that if \(v\) satisfies
\[
\|v\|_{Y(I)} \leq \eta_2,
\]
then there exists a solution \(w \in C(I; \dot{H}_x^1(\mathbb{R}^3))\) of (4.1) with initial data \(w_0\) such that
\[
\|w - \tilde{w}\|_{L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^3)} + \|w - \tilde{w}\|_{X(I)} \leq C(C_0, M_0, \eta_1) \eta_2. \quad (4.34)
\]

Proof. First, we divide the time interval \(I\) as consecutive sub-intervals \(I = \bigcup_{j=1}^J I_j,\) such that 
\[
\frac{1}{2} \eta_1 \leq \|\tilde{w}\|_{X(I_j)} \leq \eta_1,
\]
where \(\eta_1\) is defined in Lemma 4.6. Let \(t_{j-1} = \inf I_j\) and assume without loss of generality that \(t_0 = 0.\) Then, we have 
\[
J = J(M_0, \eta_1).
\]

From now on, we set another parameter \(\eta_2 = \eta_2(C_0, M_0, \eta_1) > 0\) such that 
\[
(2C_0)^J \eta_2 \leq \eta_1. \quad (4.35)
\]

We take \(v\) such that 
\[
\|v\|_{S(I)} \leq \eta_2.
\]

We start from the first interval \(I_1.\) In this case, \(w(0) - \tilde{w}(0) = 0.\) Then, applying Lemma 4.6 with \(\eta = \eta_2\) on \(I = [t_0, t_1],\) we obtain the existence of \(w\) on \(I_1,\) and
\[
\|w - \tilde{w}\|_{L_t^\infty \dot{H}_x^1(I_1 \times \mathbb{R}^3)} \cap X(I_1) \leq C_0 \eta \leq 2C_0 \eta_2. \quad (4.36)
\]

Now, we can start the induction procedure. Our aim is to prove for all \(k = 1, \ldots, J,\)
\[
\|w - \tilde{w}\|_{L_t^\infty \dot{H}_x^1(I_k \times \mathbb{R}^3)} \cap X(I_k) \leq (2C_0)^k \eta_2. \quad (4.37)
\]
(4.36) shows that (4.37) holds when \( k = 1 \). We assume that (4.37) holds for \( k = j < J \), namely
\[
\|w - \tilde{w}\|_{L_t^{\infty} H^1_x((t_j, t_{j+1}))} \leq (2C_0)^j \eta_2. \tag{4.38}
\]
It suffices to prove (4.37) holds for \( k = j + 1 \). By (4.35) and (4.38), we have
\[
\|w(t_j) - \tilde{w}(t_j)\|_{H^1_x(\mathbb{R}^3)} \leq (2C_0)^j \eta_2 \leq (2C_0)^{j+1} \eta_2.
\]
Then, we can apply Lemma 4.6 with \( \eta = \eta_2 \) on \( I_{j+1} \) after translation in \( t \) starting from \( t_j \), and obtain the existence of \( w \) on \( I_{j+1} \), and
\[
\|w - \tilde{w}\|_{L_t^{\infty} H^1_x((t_{j+1}, t_{j+2}))} \leq C_0 \left( \|w(t_j) - \tilde{w}(t_j)\|_{H^1_x(\mathbb{R}^3)} + \eta_2 \right)
\leq C_0 \left( (2C_0)^j \eta_2 + \eta_2 \right) \leq (2C_0)^{j+1} \eta_2.
\]
This gives (4.37) for \( k = j + 1 \).

Then, we have the existence of \( w \) on \( I \) and for all \( k = 1, ..., J \),
\[
\|w - \tilde{w}\|_{L_t^{\infty} H^1_x((t_k, t_{k+1}))} \leq (2C_0)^k \eta_2.
\]
Therefore, (4.34) follows by summation over \( k \). \( \square \)

4.6. Proof of Proposition 4.1. We need to use the following classical result:

**Lemma 4.8.** Suppose that \( \tilde{w}_0 \in H^1(\mathbb{R}^3) \). Then, the equation (4.32) is globally well-posed and scatters, and the solution \( \tilde{w} \in C(\mathbb{R}; H^1(\mathbb{R}^3)) \) satisfies
\[
\|\tilde{w}\|_{X(\mathbb{R})} \leq C(\|\tilde{w}_0\|_{H^1(\mathbb{R}^3)}).
\]

**Proof.** Using the result in [22], we have the global well-posedness and the space-time bound
\[
\|\tilde{w}\|_{L_{t,x}^{\infty}(\mathbb{R})} \leq C(\|\tilde{w}_0\|_{H^1(\mathbb{R}^3)}).
\]
Then, given \( 0 < \varepsilon \ll 1 \), we can split \( \mathbb{R} = \bigcup_{j=1}^J I_j \) such that
\[
\|\tilde{w}\|_{L_{t,x}^{10}(I_j)} \leq \varepsilon.
\]
Using the equation (4.32),
\[
\|\tilde{w}\|_{X(I_j)} \leq C(\|\tilde{w}_0\|_{H^1(\mathbb{R}^3)}).
\]
Therefore, the lemma follows by summation over \( I_j \). \( \square \)

**Proof of global well-posedness:** We first prove the global well-posedness and space-time norm estimate by iterating the perturbation theory. We consider only the forward time interval \([0, +\infty)\) case. By Proposition 4.4, we have that if \( w \in C([0, T]; H^1_x) \) solves (4.1) for some \( T > 0 \), then
\[
\sup_{t \in [0, T]} \|w(t)\|_{H^1(\mathbb{R}^3)} \leq C(A, N_0) =: E_0. \tag{4.39}
\]
Combining Lemma 4.8, it holds that for any \( t' \in [0, \infty) \), there exists a solution \( \tilde{w}(t, x) = \tilde{w}(t')(t, x) \) of
\[
\begin{align*}
\left\{ \begin{array}{l}
\imath \partial_t \tilde{w} + \Delta \tilde{w} = |\tilde{w}|^4 \tilde{w}, \\
\tilde{w}(t', x) = w(t', x),
\end{array} \right.
\end{align*}
\]
such that
\[
\|\tilde{w}(t')\|_{L_t^{\infty} H^1_x(\mathbb{R})} + \|\tilde{w}(t')\|_{X(\mathbb{R})} \leq C(\|w(t')\|_{H^1(\mathbb{R}^3)}).
\tag{4.40}
\]
By (4.39) and (4.40), we have that if \( w \in C([0, T]; H^1_\epsilon) \) solves (4.1) for some \( T > 0 \), then
\[
\sup_{t' \in [0, T]} (\| \tilde{w}(t') \|_{L^\infty H^1_\epsilon(\mathbb{R})} + \| \tilde{w}(t') \|_{X(\mathbb{R})}) \leq C(E_0) =: M_0, \tag{4.41}
\]
where \( M_0 \) depends only on \( A \) and \( N_0 \).

By the assumption in Proposition 4.1, we have \( \| v \|_{Y([0, \infty))} \leq A \). Next, we split \([0, \infty) = \cup_{l=1}^L I_l \) such that
\[
\frac{1}{2} \eta_2 \leq \| v \|_{Y(I_l)} \leq \eta_2,
\]
where \( \eta_2 = \eta_2(M_0) \) is defined in Lemma 4.7. Then, \( L \) may depend on \( M_0, A \) and \( \eta_2 \).

Let \( s_{l-1} = \inf I_l \) and \( s_0 = 0 \). We can start from \( I_1 \). By (4.41), we have
\[
\| \tilde{w}(s_0) \|_{S(\mathbb{R})} \leq M_0.
\]
Then, we can apply Lemma 4.7 on \( I_1 \) to obtain the existence of \( w \in C([s_0, s_1]; H^1(\mathbb{R}^d)) \).

Furthermore, by Proposition 4.4, we can get
\[
\sup_{t \in I_1} \| w(t) \|_{H^1(\mathbb{R}^3)} \leq E_0, \quad \| w \|_{X(I_1)} \leq M_0.
\]
Particularly for \( \tilde{w}(s_1) \), we have \( \| \tilde{w}(s_1) \|_{H^1(\mathbb{R}^3)} = \| w(s_1) \|_{H^1(\mathbb{R}^3)} \leq E_0 \). Using (4.41) again,
\[
\| \tilde{w}(s_1) \|_{X(\mathbb{R})} \leq M_0.
\]
Then, we can apply Lemma 4.7 on \( I_2 \) after translation in \( t \) from the starting point \( s_1 \). Therefore, we obtain the existence of \( w \in C([s_1, s_2]; H^1(\mathbb{R}^d)) \), and
\[
\sup_{t \in I_2} \| w(t) \|_{H^1(\mathbb{R}^3)} \leq E_0, \quad \| w \|_{X(I_2)} \leq M_0.
\]
Inductively, for all \( l = 1, 2, \ldots, L \), we can obtain that \( w \in C(I_l; H^1(\mathbb{R}^3)) \), and also
\[
\| w \|_{X(I_l)} \leq M_0.
\]
Therefore, we have \( w \in C([0, \infty); H^1(\mathbb{R}^3)) \), and
\[
\| w \|_{X([0, \infty))} \leq LM_0 = C(A, M_0, \eta_2) = C(A).
\]

**Proof of scattering:** Next, we prove the scattering statement. We only consider the \( t \to +\infty \) case, and it suffices to prove that
\[
\| \langle \nabla \rangle \int_0^\infty e^{-i\Delta t} |u|^4 u \, dx \|_{L^2} \leq C(A). \tag{4.42}
\]
In fact, since the global well-posedness already holds, we do not care the explicit expression of \( A \). Now, all the space-time norms are taken over \([0, +\infty) \times \mathbb{R}^3\). From previous argument,
\[
\| w \|_{X([0, +\infty))} \leq C(A).
\]
Recall also that
\[
\| v \|_{Y(\mathbb{R})} \leq A.
\]
Now, we can prove (4.42) using the argument in Lemma 4.2,

\[
\text{L.H.S. of (4.42)} \lesssim \left\| \int_0^\infty e^{-is\Delta} \left| u \right|^4 u \, ds \right\|_{L^2_x} + \left\| \int_0^\infty e^{-is\Delta} (\nabla wu^4) \, ds \right\|_{L^2_x} \\
+ \left\| \int_0^\infty e^{-is\Delta} (\nabla vu^3) \, ds \right\|_{L^2_x} \\
\lesssim \left\| u^4 \right\|_{L^1_t L^2_x} + \left\| \nabla wu^4 \right\|_{L^1_t L^2_x} + \left\| \nabla vu^3 \right\|_{L^1_t L^2_x} \\
\lesssim \left\| (\nabla) w \right\|_{L^6_t L^6_x} \left\| u \right\|_{L^4_t L^4_x} + \left\| (\nabla) v \right\|_{L^2_t L^\infty_x} \left\| u \right\|_{L^4_t L^4_x} \\
\leq C(A).
\]

This finishes the proof of scattering statement.

5. Global well-posedness and scattering in 4D case

Now, we give the proof of Theorem 1.2 in 4D case. The argument is parallel to the 3D case, so we only give a sketch of the proof and highlight the different part.

5.1. Reduction to the deterministic problem. Suppose that \( u = v + w \) with \( u_0 = v_0 + w_0 \), \( v = e^{it\Delta} v_0 \), and \( w \) satisfying

\[
\begin{cases}
i\partial_t w + \Delta w = |u|^2 u, \\
w(0, x) = w_0(x).
\end{cases}
\]

(5.1)

Recall that

\[
\| v \|_{Y(I)} := \left\| (\nabla)^{s+a-} v \right\|_{L^4_t L^8_x(I \times \mathbb{R}^4)} + \left\| v \right\|_{L^4_t L^8_x(I \times \mathbb{R}^4)} + \left\| v \right\|_{L^4_t L^8_x(I \times \mathbb{R}^4)} \\
+ \left\| (\nabla) \right\|_{L^6_t L^6_x(I \times \mathbb{R}^4)},
\]

and

\[
\| v \|_{Z(I)} := \| v \|_{L^\infty_t H^2_x(R \times \mathbb{R}^4)} + \| (\nabla)^{s+2a-} v \|_{L^\infty_t L^\infty_x(R \times \mathbb{R}^4)}.
\]

We define the energy as

\[
E(t) := \frac{1}{2} \int_{\mathbb{R}^4} |\nabla w(t, x)|^2 \, dx + \frac{1}{4} \int_{\mathbb{R}^4} |u(t, x)|^4 \, dx,
\]

(5.2)

and the mass as

\[
M(t) := \int_{\mathbb{R}^4} |w(t, x)|^2 \, dx.
\]

(5.3)

**Proposition 5.1.** Let \( a \in \mathbb{N}, a > 10, \frac{1}{2} - \frac{1}{2}a < s < 0 \), and \( A > 0 \). Then, there exists \( N_0 = N_0(A) \gg 1 \) such that the following properties hold. Let \( u_0 \in L^2_x(\mathbb{R}^4) \), \( v_0 \) satisfy that \( \sup \tilde{v}_0 \subset \{ \xi \in \mathbb{R}^4 : |\xi| \geq \frac{1}{2} N_0 \} \), and \( w_0 = u_0 - v_0 \). Moreover, let \( v = e^{it\Delta} v_0 \) and \( w = u - v \). Suppose that \( v \in Y \cap Z(\mathbb{R}) \), \( w_0 \in H^1(\mathbb{R}^4) \) such that

\[
\| u_0 \|_{L^2_x} + \| v \|_{Y \cap Z(\mathbb{R})} \leq A, \text{ and } E(w_0) \leq AN_0^2.
\]

Then, there exists a solution \( u \) of (1.7) on \( \mathbb{R} \) with \( w \in C(\mathbb{R}; H^1_x(\mathbb{R}^4)) \). Furthermore, there exists \( u_\pm \in H^1_x(\mathbb{R}^4) \) such that

\[
\lim_{t \to \pm \infty} \| u(t) - v(t) - e^{it\Delta} u_\pm \|_{H^1_x(\mathbb{R}^4)} = 0.
\]
We will give the proof of Proposition 5.1 in Sections 5.2-5.6. Now, we can prove
Theorem 1.2 in 4D case assuming that Proposition 5.1 holds, using the same argument
in 3D case.

5.2. Local theory. We define the space \( X(I) \) as
\[
\|w\|_{X(I)} := \|\nabla w\|_{L^2(I;X^4)} + \|w\|_{L^2(I;L^4_x(1 \times \mathbb{R}^4))} + \|w\|_{L^4(I;L^2_x(1 \times \mathbb{R}^4))}.
\]

**Lemma 5.2** (Local well-posedness). Let \( a \in \mathbb{N}, a > 10, \frac{1}{2} - \frac{1}{2}a < s < 1, v \in \mathbb{Y} \cap \mathbb{Z} (\mathbb{R}), \) and \( w_0 \in H^1_x \). Then, there exists some \( T > 0 \) depending on \( a, w_0, \) and \( v \) such that there exists a unique solution \( w \) of (4.1) in some 0-neighbourhood of
\[
C([0, T]; H^1_x(\mathbb{R}^4)) \cap X([0, T]).
\]

**Proof.** First, we make the choices of some parameters:

1. Let \( C_0 \) be the constant such that
\[
\|e^{it\Delta}w_0\|_{L^\infty(\mathbb{R};H^1_x)} \leq C_0 \|w_0\|_{H^1_x}.
\]

2. Define
\[
R := \max\left\{ C_0 \|w_0\|_{H^1_x}, 1 \right\}.
\]

3. Let \( \delta > 0 \) be some small constant such that \( C\delta^2R^2 \leq \frac{1}{2} \).

4. Let \( T > 0 \) satisfy the smallness condition
\[
\|e^{it\Delta}w_0\|_{X([0,T])} + \|v\|_{Y([0,T])} \leq \delta R.
\]

Now, we define the working space as
\[
B_{R, \delta, T} := \left\{ w \in C([0, T]; H^1_x) : \|w\|_{L^\infty(\mathbb{R};H^1_x)} + \delta^{-1}\|w\|_{X([0,T])} \leq 4R \right\},
\]
equipped with the norm
\[
\|w\|_{B_{R, \delta, T}} := \|w\|_{L^\infty(\mathbb{R};H^1_x)} + \delta^{-1}\|w\|_{X([0,T])}.
\]

Take the solution map as
\[
\Phi_{w_0, v}(w) := e^{it\Delta}w_0 - i \int_0^t e^{i(t-s)\Delta}(|u|^2u) \, ds.
\]

Then, it suffices to prove that \( \Phi_{w_0, v} \) is a contraction mapping on \( B_{R, \delta, T} \).

Now, we only prove that for any \( w \in B_{R, \delta, T}, \Phi_{w_0, v}(w) \in B_{R, \delta, T} \). By Lemma 2.4 and Hölder’s inequality,
\[
\|\Phi_{w_0, v}(w)\|_{L^\infty(\mathbb{R};H^1_x)} \leq C_0 \|w_0\|_{H^1_x} + C\|\nabla(|u|^2u)\|_{L^1L^2} + C\|u|^2u\|_{L^1L^2_x} \\
\leq R + C\|\nabla w\|_{L^2L^2_x} \|u\|^2_{L^2L^2_x} + C\|\nabla v\|_{L^2L^2_x} \|u\|^2_{L^2L^2_x} + C\|u\|_{L^2L^2_x} \|u\|^2_{L^2L^2_x} \\
\leq R + C\delta^3R^3 \leq 2R.
\]

Similar as above, we have
\[
\|\Phi_{w_0, v}(w)\|_{X} \leq \|e^{it\Delta}w_0\|_{H^1_x} + C\|\nabla(|u|^2u)\|_{L^1L^2} + C\|u|^2u\|_{L^1L^2} \\
\leq \delta R + C\delta^3R^3 \leq 2\delta R.
\]
Therefore, we have
\[ \| \Phi_{w_0,v}(w) \|_{B_{R,\delta,T}} \leq \| \Phi_{w_0,v}(w) \|_{L_{t,x}^\infty H_x^1} + \delta^{-1} \| \Phi_{w_0,v}(w) \|_X \leq 4R. \]

This shows that \( \Phi_{w_0,v} \) maps \( B_{R,\delta,T} \) into itself. Since we already establish the non-linear estimates, the contraction mapping statement follows by similar argument in 3D case. \( \square \)

5.3. Modified Interaction Morawetz.

**Lemma 5.3** (Modified Interaction Morawetz). Given \( T > 0 \). Let \( w \in C([0,T]; H_x^1) \) be the solution of perturbation equation (4.1). Then, we have
\[
\left\| \| \nabla \left| -\frac{i}{2} w \right|^{\frac{1}{4}} \right\|_{L_{t,x}^4}^4 \leq \left\| w \right\|_{L_{t,x}^7 L_x^2}^2 \left\| w \right\|_{L_{t,x}^4 H_x^\frac{1}{2}}^2 \left( \left\| \nabla \left| \frac{i}{2} w \right| \right\|_{L_{t,x}^2}^2 + \left\| \nabla w \right\|_{L_{t,x}^2} \left\| w \right\|_{L_{t,x}^2} \right) \left( \left\| \nabla \right\|_{L_{t,x}^2} \left\| v \right\|_{L_{t,x}^2} + \left\| v \right\|_{L_{t,x}^2}^3 \right),
\]

where all the space-time norms are taken over \( [0,T] \times \mathbb{R}^4 \).

**Proof.** Recall that \( w \) satisfies
\[ i\partial_t w + \Delta w = |w|^2 w + e, \]
where we denote \( e := |u|^2 u - |w|^2 w. \) Denote that
\[ m(t,x) = \frac{1}{2} |w(t,x)|^2; \quad p(t,x) = \frac{1}{2} \text{Im}(\bar{w}(t,x) \nabla w(t,x)). \]

Then, we have
\[ \partial_t m = -2\nabla \cdot p + \text{Im}(e\bar{w}), \quad (5.5) \]
and
\[ \partial_t p = -\text{Re} \nabla \cdot (\nabla \bar{w} \nabla w) - \frac{1}{4} \nabla (|w|^4) + \frac{1}{2} \nabla \Delta m + \text{Re} (e \nabla w) - \frac{1}{2} \text{Re} \nabla (\bar{w} e). \quad (5.6) \]

Let
\[ M(t) := \int_{\mathbb{R}^{4+4}} \frac{x-y}{|x-y|} \cdot p(t,x) m(t,y) \, dx \, dy, \]
then by (5.5) and (5.6), we have the interaction Morawetz identity

\[
\partial_t M(t) = \int_{\mathbb{R}^{4+4}} \frac{x - y}{|x - y|} \cdot \partial_t p(t, x) \, m(t, y) \, dx \, dy
+ \int_{\mathbb{R}^{4+4}} \frac{x - y}{|x - y|} \cdot p(t, x) \, \partial_t m(t, y) \, dx \, dy
= \int_{\mathbb{R}^{4+4}} \frac{x - y}{|x - y|} \cdot \left( - \text{Re} \nabla \cdot (\nabla \bar{w} \nabla w) - \frac{1}{4} \nabla (|w|^4) \right)(t, x) \, m(t, y) \, dx \, dy
- 2 \int_{\mathbb{R}^{4+4}} \frac{x - y}{|x - y|} \cdot p(t, x) \, \nabla \cdot p(t, y) \, dx \, dy
+ \frac{1}{2} \int_{\mathbb{R}^{4+4}} \frac{x - y}{|x - y|} \cdot \nabla \Delta m(t, x) \, m(t, y) \, dx \, dy
+ \int_{\mathbb{R}^{4+4}} \frac{x - y}{|x - y|} \cdot p(t, x) \, \text{Im} (\bar{w} \bar{w})(t, y) \, dx \, dy
+ \int_{\mathbb{R}^{4+4}} \frac{x - y}{|x - y|} \cdot \text{Re} (\nabla \bar{w})(t, x) \, m(t, y) \, dx \, dy
+ \int_{\mathbb{R}^{4+4}} \frac{1}{|x - y|} \cdot \text{Re} (\bar{w})(t, x) \, m(t, y) \, dx \, dy.
\]  

(5.7a)

(5.7b)

(5.7c)

(5.7d)

Different from the 3D case, we have that

\[
(5.7a) \gtrsim \| \nabla |^{-\frac{1}{2}} w(t) \|_{L^4_x}^4.
\]

Then treating similar as in the proof of Lemma 4.3, we obtain that

\[
\| \nabla |^{-\frac{1}{2}} w \|_{L^4_{t,x}}^4 \lesssim \| w \|_{L^\infty_t L^2_x}^2 \| w \|_{L^\infty_t H^\frac{1}{2}_x}^2 + \int_0^T \| (5.7b) \| + \| (5.7c) \| + \| (5.7d) \| \, dt.
\]

Next, we estimate the terms containing (5.7b), (5.7c), and (5.7d). We first consider (5.7b). By Hölder’s inequality,

\[
\int_0^T \| (5.7b) \| \, dt \lesssim \int_0^T \left( \int_{\mathbb{R}^{4+4}} \frac{x - y}{|x - y|} \cdot p(t, x) \, \text{Im} (\bar{w} \bar{w})(t, y) \, dx \, dy \right) \, dt
\lesssim \int_0^T \left( \int_{\mathbb{R}^4} \text{Im} (\bar{w} \bar{w})(t, y) \, dy \right) \, dt \sup_{y, t} \left( \int_{\mathbb{R}^4} \frac{x - y}{|x - y|} \cdot p(t, x) \, dx \right)
\lesssim \| v \|_{L^2_t L^\infty_x} \| w \|_{L^2_t L^2_x} \left( \| w \|_{L^2_t L^2_x}^2 + \| v \|_{L^2_t L^2_x}^2 \right) \| w \|_{L^\infty_t H^\frac{1}{2}_x}^2.
\]

Note that by Gagliardo-Nirenberg’s inequality, we have that

\[
\| w \|_{L^2_{x}(\mathbb{R}^4)}^2 \lesssim \| w \|_{H^\frac{1}{2}_x(\mathbb{R}^4)} \cdot \| \nabla |^{-\frac{1}{2}} w \|_{L^4_{t}(\mathbb{R}^4)}^2.
\]

(5.8)

This gives that

\[
\int_0^T \| (5.7b) \| \, dt \lesssim \| v \|_{L^2_t L^\infty_x} \| w \|_{L^\infty_t H^\frac{1}{2}_x} \left( \| \nabla |^{-\frac{1}{2}} w \|_{L^4_{t,x}}^2 + \| w \|_{L^\infty_t H^\frac{1}{2}_x}^2 \right)
+ \| v \|_{L^2_t L^\infty_x} \| v \|_{L^2_t L^2_x} \| w \|_{L^\infty_t H^\frac{1}{2}_x}^2.
\]
We then consider (5.7c), where we need to modify the Morawetz estimate.

\[
\int_0^T |(5.7c)| \, dt \leq \int_0^T \left| \int_{\mathbb{R}^+} \frac{x-y}{|x-y|} \cdot \text{Re} \left( (|u|^2 u - |w|^2 w) \nabla \bar{w} \right) (t, x) m(t, y) \, dx \, dy \right| \, dt.
\]

Similar as in the proof of (4.20), we have that by integration-by-parts,

\[
\int_{\mathbb{R}^4} \frac{x-y}{|x-y|} \cdot \text{Re} \left( (|u|^2 u - |w|^2 w) \nabla \bar{w} \right) (t, x) \, dx = -\frac{3}{4} \int_{\mathbb{R}^4} \frac{1}{|x-y|} \left( |u(t,x)|^4 - |w(t,x)|^4 \right) \, dx - \int_{\mathbb{R}^4} \frac{x-y}{|x-y|} \cdot \text{Re} \left( |u|^2 u \nabla \bar{w} \right) (t, x) \, dx.
\]

Therefore, by Lemma 2.1 and (5.8), we have

\[
\int_0^T |(5.7c)| \, dt \leq \int_0^T \left| \int_{\mathbb{R}^4} \frac{1}{|x-y|} \left( |u(t,x)|^4 - |w(t,x)|^4 \right) m(t, y) \, dx \, dy \right| \, dt \\
+ \int_0^T \left| \int_{\mathbb{R}^4} \frac{x-y}{|x-y|} \cdot \text{Re} \left( |u|^2 u \nabla \bar{w} \right) (t, x) m(t, y) \, dx \, dy \right| \, dt \\
\approx \int_0^T \left| \int_{\mathbb{R}^4} \left( |u(t,x)|^4 - |w(t,x)|^4 \right) \, dx \left( \sup_x \left\| \frac{1}{|x - \cdot|^\frac{1}{2}} w(t, \cdot) \right\|_{L^6}^2 \right) \, dt \\
+ \int_0^T \left| \int_{\mathbb{R}^4} |u|^2 u \nabla \bar{w}(t, x) \, dx \left( \left\| \nabla w(t) \right\|_{L^6}^2 \right) \, dt \right|
\]
\[
\lesssim \left\| v \right\|_{L^2 L^\infty_x} \left( \left\| w \right\|_{L^3_t L^3_x}^3 + \left\| v \right\|_{L^3_t L^3_x}^3 \right) \left\| w \right\|_{L^\infty_t H^\frac{1}{2}_x}^2 \\
+ \left\| \nabla v \right\|_{L^2_t L^\infty_x} \left( \left\| w \right\|_{L^3_t L^3_x}^3 + \left\| v \right\|_{L^3_t L^3_x}^3 \right) \left\| w \right\|_{L^\infty_t L^2_x}^2 \\
\lesssim \left\| v \right\|_{L^2_t L^\infty_x} \left\| w \right\|_{L^3_t H^\frac{1}{2}_x}^2 \left\| \nabla^{-\frac{1}{4}} w \right\|_{L^4_t L^4_x}^2 \left\| w \right\|_{L^\infty_t L^2_x}^2 \\
+ \left\| v \right\|_{L^2_t L^\infty_x} \left\| v \right\|_{L^3_t L^3_x}^2 \left\| w \right\|_{L^3_t H^\frac{1}{2}_x}^2 \\
+ \left\| \nabla v \right\|_{L^2_t L^\infty_x} \left\| w \right\|_{L^3_t H^\frac{1}{2}_x}^2 \left\| \nabla^{-\frac{1}{4}} w \right\|_{L^4_t L^4_x}^2 \left\| w \right\|_{L^\infty_t L^2_x}^2 \\
+ \left\| v \right\|_{L^2_t L^\infty_x} \left\| v \right\|_{L^3_t L^3_x}^2 \left\| w \right\|_{L^3_t L^3_x}^2 \left\| w \right\|_{L^\infty_t L^2_x}^2.
\]

Now, we consider the term (5.7d). By Hölder’s inequality and (5.8),

\[
\int_0^T |(5.7d)| \, dt \leq \int_0^T \left| \int_{\mathbb{R}^4} \frac{1}{|x-y|} |\bar{q}(t,x)| m(t, y) \, dx \, dy \right| \, dt \\
\leq \int_0^T \int_{\mathbb{R}^4} \left| \frac{1}{|x-y|^\frac{1}{2}} w(t, \cdot) \right| \, dx \, \left( \sup_x \left\| \frac{1}{|x - \cdot|^\frac{1}{2}} w(t, \cdot) \right\|_{L^6}^2 \right) \, dt \\
\approx \left\| v \right\|_{L^2_t L^\infty_x} \left\| w \right\|_{L^\infty_t H^\frac{1}{2}_x}^2 \left\| \nabla^{-\frac{1}{4}} w \right\|_{L^4_t L^4_x}^2 \left\| w \right\|_{L^\infty_t L^2_x}^2 \\
+ \left\| v \right\|_{L^2_t L^\infty_x} \left\| v \right\|_{L^3_t L^3_x}^2 \left\| w \right\|_{L^3_t L^3_x}^2 \left\| w \right\|_{L^\infty_t L^2_x}^2.
\]
Combining the three findings on (5.7), we get
\[
\int_0^T (|5.7b| + |5.7c| + |5.7d|) \, dt \\
\lesssim \|v\|_{L^4_t L^\infty_x} \|w\|_{L^4_t L^\infty_x} \left\| \nabla \right\|_{L^1_t L^4_x} \left\| \frac{1}{4} w \right\|_{L^4_t L^8_x}^2 + \|v\|_{L^6_t L^\infty_x} \|w\|_{L^4_t L^2_x} \|v\|_{L^2_t L^4_x}^2 + \|v\|_{L^2_t L^\infty_x} \left\| v \right\|_{L^6_t L^4_x}^3 \\
+ \left\| \nabla v \right\|_{L^4_t L^\infty_x} \left\| w \right\|_{L^4_t L^8_x} \left\| \nabla \right\|_{L^1_t L^4_x} \left\| \frac{1}{4} w \right\|_{L^4_t L^8_x}^2 + \left\| \nabla v \right\|_{L^4_t L^\infty_x} \left\| w \right\|_{L^4_t L^2_x}^2 + \left\| v \right\|_{L^2_t L^4_x}^2 + \left\| v \right\|_{L^6_t L^4_x}^3 .
\]
Therefore, we have
\[
\|w\|_{L^4_t L^2_x}^4 \lesssim \|w\|_{L^4_t L^\infty_x} \|w\|_{L^2_t L^4_x}^2 \left\| \nabla \right\|_{L^1_t L^4_x} \left\| \frac{1}{4} w \right\|_{L^4_t L^8_x}^2 + \left\| \nabla v \right\|_{L^4_t L^\infty_x} \left\| w \right\|_{L^4_t L^2_x}^2 + \left\| v \right\|_{L^2_t L^4_x}^2 + \left\| v \right\|_{L^6_t L^4_x}^3 .
\]
This completes the proof of this lemma. \(\square\)

5.4. Almost conservation law.

**Proposition 5.4.** Let \(a \in \mathbb{N}, a > 10, \frac{1}{2} - \frac{1}{a} < s < 0, A > 0, v = e^{i\Delta} v_0 \in Y \cap Z(\mathbb{R})\) and \(w\) be the solution of (4.1). Take some \(T > 0\) such that \(w \in C([0,T]; H_x^1)\). Then, there exists \(N_0 = N_0(A) \gg 1\) with the following properties. Assume that \(\tilde{v}_0\) is supported on \(\{\xi \in \mathbb{R}^4 : \|\xi\| \geq \frac{1}{2} N_0\}\),

\[
\|u_0\|_{H_x^s} + \|v\|_{Y \cap Z(\mathbb{R})} \leq A, M(0) \leq A N_0^{-2s}, \text{ and } E(0) \leq A N_0^{2(1-s)}.
\]

Then, we have
\[
\sup_{t \in [0,T]} M(t) \leq 2 A N_0^{-2s}, \text{ and } \sup_{t \in [0,T]} E(t) \leq 2 A N_0^{2(1-s)}. \tag{5.9}
\]

**Proof.** Let \(N_0 = N_0(A)\) that will be defined later. We implement a bootstrap procedure on \(I \subset [0,T]\): assume an a priori bound
\[
\sup_{t \in I} M(t) \leq 2 A N_0^{-2s}, \text{ and } \sup_{t \in I} E(t) \leq 2 A N_0^{2(1-s)}, \tag{5.10}
\]
then it suffices to prove that
\[
\sup_{t \in I} M(t) \leq \frac{3}{2} A N_0^{-2s}, \text{ and } \sup_{t \in I} E(t) \leq \frac{3}{2} A N_0^{2(1-s)}. \tag{5.11}
\]
From now on, all the space-time norms are taken over \(I \times \mathbb{R}^4\).

To start with, we collect useful estimates on \(I\). Now, we use the notation \(C = C(A)\) for short, and the implicit constants in \(\lesssim\) depend on \(A\). By interpolation, we have
\[
\|v\|_{L^\infty_t L^s_x} + \|\langle \nabla \rangle^{-\frac{1}{4}} v\|_{L^4_t L^4_x} + \|v\|_{L^6_t L^4_x} \lesssim \|v\|_{Y \cap Z} \lesssim 1. \tag{5.12}
\]
By the frequency support of $v$, we have for any $0 \leq l < a + s$,
\[
\|\nabla^l v_N\|_{L_{t,x}^{2}} \lesssim N_0^{l-a-s+} \|v\|_{Y} \lesssim N_0^{l-a-s} \lesssim 1. \tag{5.13}
\]
Note that we assume $a > 10$ and $s > \frac{1}{2} - \frac{1}{2}a$, then $a + s > 2$. Therefore, this guarantees that $\|\Delta v\|_{L_{t}^{2}L_{x}^{\infty}} \lesssim N_0^{3-a-s}$ is allowed. By bootstrap hypothesis (5.10),
\[
\|w\|_{L_{t}^{\infty}L_{x}^{2}} \lesssim N_0^{-s}, \text{ and } \|w\|_{L_{t}^{\infty}\dot{H}_{x}^{2}} \lesssim N_0^{1-s}. \tag{5.14}
\]
Then, by interpolation and (5.14), we have for any $0 \leq l \leq 1$,
\[
\|w\|_{L_{t}^{\infty}\dot{H}_{x}^{l}} \lesssim N_0^{l-s}. \tag{5.15}
\]
Furthermore, by Lemma 5.3, (5.15), and (5.13),
\[
\|\nabla^{-\frac{1}{2}}w\|_{L_{t,x}^{4}}^{4} \lesssim \|w\|_{L_{t}^{2}L_{x}^{2}}^{2} \|w\|_{L_{t}^{2}H_{x}^{\frac{1}{2}}}^{2}
\]
\[
+ \|w\|_{L_{t}^{\infty}H_{x}^{2}} \|\nabla^{-\frac{1}{2}}w\|_{L_{t,x}^{4}}^{2} \left( \|v\|_{L_{t}^{2}L_{x}^{\infty}} \|w\|_{L_{t}^{2}H_{x}^{\frac{1}{2}}}^{2} + \|\nabla v\|_{L_{t}^{2}L_{x}^{\infty}} \|w\|_{L_{t}^{2}L_{x}^{2}}^{2} \right)
\]
\[
+ \|v\|_{L_{t}^{2}L_{x}^{2}} \|w\|_{L_{t}^{2}H_{x}^{\frac{1}{2}}}^{2} \left( \|w\|_{L_{t}^{\infty}L_{x}^{2}} \|v\|_{L_{t,x}^{2}}^{2} + \|v\|_{L_{t,x}^{2}}^{2} \right)
\]
\[
\lesssim N_0^{-1+4s} + N_0^{\frac{1}{2}-s} \|\nabla^{-\frac{1}{2}}w\|_{L_{t,x}^{4}}^{2} \left( N_0^{-a-s} + N_0^{-2s} \right)
\]
\[
+ N_0^{1-2s} \left( N_0^{-s} + 1 \right)
\]
\[
\lesssim N_0^{-1+4s} + N_0^{\frac{3}{2}-a-4s} + N_0^{3-2a-8s} \lesssim N_0^{-1+4s}. \tag{5.16}
\]
Since $s > \frac{1}{2} - \frac{1}{2}a$, by Young’s inequality, we have
\[
\|w\|_{L_{t,x}^{4}}^{4} \lesssim N_0^{-1+4s} + N_0^{3-2a-8s} \lesssim N_0^{-1+4s}. \tag{5.16}
\]
Now, we are prepared to give the proof of (5.11). We first consider the mass bound in (5.11). Note that similarly to Lemma 4.5,
\[
\left| \frac{d}{dt} M(t) \right| \leq 2 \left| \int_{\mathbb{R}^{d}} (|u|^2 u - |w|^2 w) \bar{w} \ dx \right|.
\]
Then by Hölder’s inequality and (5.8), we have that
\[
\sup_{t \in I} M(t) \leq M(0) + \int_{I} \frac{d}{dt} M(t) \ dt
\]
\[
\leq M(0) + \int_{I} \left| \int_{\mathbb{R}^{d}} \bar{w} (|u|^2 u - |w|^2 w) \ dx \right| \ dt
\]
\[
\leq AN_0^{-2s} + C \|v\|_{L_{t}^{2}L_{x}^{\infty}} \left( \|w\|_{L_{t}^{2}L_{x}^{2}}^{3} + \|v\|_{L_{t}^{2}L_{x}^{2}}^{3} \right)
\]
\[
\leq AN_0^{2s} + C \|v\|_{L_{t}^{2}L_{x}^{\infty}} \left( \|w\|_{L_{t}^{\infty}H_{x}^{\frac{1}{2}}} \|\nabla^{-\frac{1}{2}}w\|_{L_{t,x}^{4}}^{2} + \|v\|_{L_{t}^{2}L_{x}^{2}} \|v\|_{L_{t}^{2}H_{x}^{\frac{1}{2}}}^{2} \right).
\]
Therefore, combining with (5.12), (5.13), (5.15), and (5.16),
\[
\sup_{t \in I} M(t) \leq AN_0^{-2s} + C(A)N_0^{-a-s+1-3s} \leq \frac{3}{2} A N_0^{-2s}, \tag{5.17}
\]
where we take $N_0 = N_0(A)$ such that $C(A)N_0^{-a+1-2s} \leq \frac{1}{2} A$. This is allowed since $s > \frac{1}{2} - \frac{1}{2}a$. 

We then consider the energy bound in (5.11). Note that similarly to Lemma 4.5,
\[ \left| \frac{d}{dt} E(t) \right| \leq \left| \int_{\mathbb{R}^4} |u|^2 u \Delta \mathbf{v} \, dx \right|, \]
then by Hölder’s inequality and (5.8),
\[ \sup_{t \in I} E(t) \leq E(0) + \int_{I} \left| \frac{d}{dt} E(t) \right| \, dt \]
\[ \leq E(0) + \int_{I} \left| \int |u|^2 u \cdot \Delta \mathbf{v} \, dx \right| \, dt \]
\[ \leq AN_0^2 + C \left\| \Delta v \right\|_{L_t^2 L_x^\infty} \left( \left\| w \right\|_{L_t^3 L_x^6}^3 + \left\| v \right\|_{L_t^3 L_x^6}^3 \right) \]
\[ \leq AN_0^2 + C \left\| \Delta v \right\|_{L_t^2 L_x^\infty} \left( \left\| \nabla \right\|_{L_t^6 L_x^3}^{-\frac{1}{2}} \left\| w \right\|_{L_t^4 L_x^4}^2 + \left\| v \right\|_{L_t^2 L_x^2} \left\| \nabla \right\|_{L_t^\infty L_x^2}^2 \right). \]
Therefore, combining with (5.12), (5.13), (5.15), and (5.16),
\[ \sup_{t \in I} E(t) \leq AN_0^{2(1-s)} + C(A)N_0^{-2a-s+1-3s^+} \leq \frac{3}{2} AN_0^{2(1-s)}, \] (5.18)
where we still need to take \( N_0 = N_0(A) \) such that \( C(A)N_0^{-a+1-2s^+} \leq \frac{1}{2} A \). Therefore, (5.17) and (5.18) gives (5.11). This finishes the proof of this proposition. \( \square \)

5.5. Perturbations. Now, we consider the original energy critical equation:
\[
\begin{cases}
  i\partial_t \tilde{w} + \Delta \tilde{w} = |\tilde{w}|^2 \tilde{w}, \\
  \tilde{w}(0, x) = \tilde{w}_0,
\end{cases}
\]
(5.19)
where \( \tilde{w}(t, x) : \mathbb{R} \times \mathbb{R}^4 \to \mathbb{C} \). Let \( g(t, x) := w(t, x) - \tilde{w}(t, x) \). Then, the equation for \( g \) is
\[
\begin{cases}
  i\partial_t g + \Delta g = F(g + v, \tilde{w}), \\
  g(0, x) = w_0 - \tilde{w}_0.
\end{cases}
\]
(5.20)
Here we denote that
\[ F(g, w) := |g + w|^2(g + w) - |w|^2 w. \]
Recall that
\[ \left\| w \right\|_{X(I)} = \left\| \langle \nabla \rangle w \right\|_{L_t^2 L_x^2(I \times \mathbb{R}^4)} + \left\| w \right\|_{L_t^4 L_x^6(I \times \mathbb{R}^4)} + \left\| w \right\|_{L_t^\infty L_x^\infty(I \times \mathbb{R}^4)}, \]
and Recall that
\[ \left\| v \right\|_{Y(I)} := \left\| \langle \nabla \rangle^{s+a-\frac{1}{2}} v \right\|_{L_t^2 L_x^\infty(I \times \mathbb{R}^4)} + \left\| v \right\|_{L_t^4 L_x^6(I \times \mathbb{R}^4)} + \left\| v \right\|_{L_t^\infty L_x^2(I \times \mathbb{R}^4)} \]
\[ + \left\| \langle \nabla \rangle^{-\frac{1}{2}} v \right\|_{L_t^4 L_x^4(I \times \mathbb{R}^4)}. \]

Lemma 5.5. Let \( a \in \mathbb{N}, a > 10, \frac{1}{2} - \frac{1}{2}a < s < 0, I \subset \mathbb{R}, \) and \( 0 \in I \). Then, there exists \( 0 < \eta_1 \ll 1 \) with the following properties. Let \( \tilde{w} \in C(I; H_x^1(\mathbb{R}^4)) \) be the solution of (4.32) on \( I \), satisfying
\[ \left\| w_0 - \tilde{w}_0 \right\|_{H_x^1(\mathbb{R}^3)} \leq \eta_1, \text{ and } \left\| \tilde{w} \right\|_{X(I)} \leq \eta_1. \]
For any \( 0 < \eta \leq \eta_1 \), suppose that
\[ \left\| v \right\|_{Y(I)} \leq \eta, \]
then there exists a solution \( w \in C(I; H^1(x; R^4)) \) of (4.1) with initial data \( w_0 \) such that
\[
\|w - \tilde{w}\|_{L_t^\infty H_x^1(I \times R^4)} + \|w - \tilde{w}\|_{X(I)} \leq C_0(\|w_0 - \tilde{w}_0\|_{H^1(R^4)} + \eta),
\]
where \( C_0 > 1 \) is an absolute constant independent of \( \eta, \eta_1 \) and \( I \).

**Proof.** Let \( g \) be the solution of (4.33), \( 0 = \inf I \), and we restrict the time interval on \( I \). Then, we have
\[
g = e^{it\Delta} g(0) - i \int_0^t e^{i(t-s)\Delta} F(g + v, \tilde{w}) \, ds.
\]
Note that we have the pointwise estimate
\[
|F(g + v, \tilde{w})| = |g + v + \tilde{w}|^2 (g + v + \tilde{w}) - |\tilde{w}|^2 \tilde{w}|
\lesssim (|g| + |v|)(|g|^2 + |v|^2 + |\tilde{w}|^2),
\]
and
\[
|\nabla F(g + v, \tilde{w})| = |\nabla (|g + v + \tilde{w}|^2 (g + v + \tilde{w}) - |\tilde{w}|^2 \tilde{w})|
\lesssim (|\nabla g| + |\nabla v|)(|g|^2 + |v|^2 + |\tilde{w}|^2) + |\nabla \tilde{w}| (|g|^2 + |v|^2)
\]
and
\[
+ |\nabla \tilde{w}| (|g| + |v|) |\tilde{w}|.
\]
Note that by the definition of \( a \) and \( s \), we have \( s + a > 1 \). Therefore, since \( v \) is high-frequency,
\[
\|v\|_{L^2_t L^\infty_x} + \|\nabla v\|_{L^2_t L^\infty_x} \lesssim \|v\|_Y.
\]
Then, using the similar argument in Lemma 5.2,
\[
\|F(g + v, \tilde{w})\|_{L^1_t L^2_x} \lesssim \|g\|_{L^2_t L^2_x} (\|g\|_{L^2_t L^2_x}^2 + \|v\|_{L^2_t L^2_x}^2 + \|\tilde{w}\|_{L^2_t L^2_x}^2)
+ \|v\|_{L^2_t L^\infty_x} (\|g\|_{L^2_t L^2_x}^2 + \|v\|_{L^2_t L^2_x}^2 + \|\tilde{w}\|_{L^2_t L^2_x}^2)
\lesssim (\|g\|_X + \|v\|_Y) (\|g\|_X^2 + \|v\|_Y^2 + \|\tilde{w}\|_X^2),
\]
and
\[
\|\nabla F(g + v, \tilde{w})\|_{L^1_t L^2_x} \lesssim \|\nabla g\|_{L^2_t L^2_x} (\|g\|_{L^2_t L^2_x}^2 + \|v\|_{L^2_t L^2_x}^2 + \|\tilde{w}\|_{L^2_t L^2_x}^2)
+ \|\nabla v\|_{L^2_t L^\infty_x} (\|g\|_{L^2_t L^2_x}^2 + \|v\|_{L^2_t L^2_x}^2 + \|\tilde{w}\|_{L^2_t L^2_x}^2)
+ \|\nabla \tilde{w}\|_{L^2_t L^2_x} (\|g\|_{L^2_t L^2_x}^2 + \|v\|_{L^2_t L^2_x}^2 + \|\tilde{w}\|_{L^2_t L^2_x}^2)
\lesssim (\|g\|_X + \|v\|_Y) (\|g\|_X^2 + \|v\|_Y^2 + \|\tilde{w}\|_X^2)
+ \|\tilde{w}\|_X (\|g\|_X^2 + \|v\|_Y^2 + \|g\|_X \|\tilde{w}\|_X + \|v\|_Y \|\tilde{w}\|_X).
\]
Therefore, by Lemma 2.4,
\[
\|g\|_{L^\infty_t H^1_x} \lesssim \|g(0)\|_{H^1_x} + \langle \langle \nabla \rangle \rangle F(g + v, \tilde{w})\|_{L^1_t L^2_x}
\lesssim \|g(0)\|_{H^1_x} + (\|g\|_X + \|v\|_Y)(\|g\|_X^2 + \|v\|_Y^2 + \|\tilde{w}\|_X^2)
+ \|\tilde{w}\|_X (\|g\|_X^2 + \|v\|_Y^2 + \|g\|_X \|\tilde{w}\|_X + \|v\|_Y \|\tilde{w}\|_X)
\lesssim \|g(0)\|_{H^1_x} + \left((\|g\|_X + \eta)(\|g\|_X^2 + \|v\|_Y^2 + \eta^2)\right)
+ \eta \left((\|g\|_X^2 + \eta^2 + \|g\|_X \eta + \eta \eta_1)\right)
\lesssim \|g(0)\|_{H^1_x} + \|g\|_X^2 + \eta^2 \|g\|_X + \eta \|g\|_X^2 + \eta \eta_1^2 + \eta_1 \|g\|_X^2.
\]
then we have
\[ \|g\|_{L_t^\infty H_x^1(I \times \mathbb{R}^4)} \lesssim \|g(0)\|_{H_x^1} + (\|g\|_X + \eta_1) \|g\|_X^2 + \eta_1^2 \|g\|_X + \eta_1^2 \eta. \]

Then, this lemma follows by the standard continuity argument. \(\square\)

**Lemma 5.6.** Suppose that \(a \in \mathbb{N}, a > 10, \frac{1}{2} - \frac{1}{2}a < s < 0, M_0 > 1, I \subset \mathbb{R}, \) and \(0 \in I. \) Let \(\tilde{w} \in C(I; H_x^1(\mathbb{R}^4))\) be the solution of (4.32) on \(I\) with \(\tilde{w}_0 = w_0, \) and \(\|\tilde{w}\|_{X(I)} \leq M_0. \)

Let \(0 < \eta_1 \ll 1\) and \(C_0 > 1\) be defined as in Lemma 5.5. Then, there exists \(\eta_2 = \eta_2(C_0, M_0, \eta_1) > 0\) such that if \(v\) satisfies
\[ \|v\|_{Y(I)} \leq \eta_2, \]
then there exists a solution \(w \in C(I; \dot{H}_x^1(\mathbb{R}^4))\) of (4.1) with initial data \(w_0\) such that
\[ \|w - \tilde{w}\|_{L_t^\infty H_x^1(I \times \mathbb{R}^4)} + \|w - \tilde{w}\|_{X(I)} \leq C(C_0, M_0, \eta_1)\eta_2. \] (5.21)

The proof of Lemma 5.6 is the same as Lemma 4.7, so we omit the details.

5.6. **Proof of Proposition 5.1.** Using the classical result in [37], we can obtain

**Lemma 5.7.** Suppose that \(\tilde{w}_0 \in H^1(\mathbb{R}^4).\) Then, the equation (4.32) is globally well-posed and scatters, and the solution \(\tilde{w} \in C(\mathbb{R}; H^1(\mathbb{R}^4))\) satisfies
\[ \|\tilde{w}\|_{X(\mathbb{R})} \leq C(\|\tilde{w}_0\|_{H_x^1(\mathbb{R}^4)}). \]

Then, we are able to prove the global well-posedness of \(w \in C(\mathbb{R}; H_x^1(\mathbb{R}^4))\), using the same argument in Section 4.6. Moreover, we have
\[ \|w\|_{L_t^\infty(\mathbb{R}; H_x^1(\mathbb{R}^4))} + \|w\|_{X(\mathbb{R})} \leq C(A). \]

Next, we prove the scattering statement. We only consider the \(t \to + \infty\) case, and it suffices to prove that
\[ \|\langle \nabla \rangle \int_0^\infty e^{-isA}(|u|^2 u) \, dx\|_{L_x^2} \leq C(A). \] (5.22)

Now, all the space-time norms are taken over \([0, +\infty) \times \mathbb{R}^4.\) From previous argument,
\[ \|w\|_{X(\mathbb{R}, [0, +\infty))} \leq C(A). \]

Recall also that
\[ \|v\|_{Y(\mathbb{R})} \leq A. \]

Now, we can prove (5.22) using the argument in Lemma 5.2,
\[
\text{L.H.S. of (5.22)} \lesssim \left\| \int_0^\infty e^{-isA}\left(\|u|^2 u\right) \, ds \right\|_{L_x^2} + \left\| \int_0^\infty e^{-isA}\left(\nabla w u^2\right) \, ds \right\|_{L_x^2} \\
+ \left\| \int_0^\infty e^{-isA}\left(\nabla v u^2\right) \, ds \right\|_{L_x^2} \\
\lesssim \|u|^2 u\|_{L_t^1 L_x^2} + \|\nabla w u^2\|_{L_t^1 L_x^2} + \|\nabla v u^2\|_{L_t^1 L_x^2} \\
\lesssim \|\langle \nabla \rangle w\|_{L_t^1 L_x^4} \|u\|_{L_t^4 L_x^4}^2 + \|\langle \nabla \rangle v\|_{L_t^1 L_x^{11/4}} \|u\|_{L_t^{22/9} L_x^{11/9}}^2 \\
\leq C(A).
\]

This finishes the proof of scattering statement.
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