Real root finding for low rank linear matrices

Didier Henrion\textsuperscript{1,2,3} · Simone Naldi\textsuperscript{4} · Mohab Safey El Din\textsuperscript{5}

Received: 31 October 2017 / Revised: 15 July 2019 / Accepted: 17 July 2019 / Published online: 25 July 2019
© Springer-Verlag GmbH Germany, part of Springer Nature 2019

Abstract
We consider $m \times s$ matrices (with $m \geq s$) in a real affine subspace of dimension $n$. The problem of finding elements of low rank in such spaces finds many applications in information and systems theory, where low rank is synonymous of structure and parsimony. We design computer algebra algorithms, based on advanced methods for polynomial system solving, to solve this problem efficiently and exactly: the input are the rational coefficients of the matrices spanning the affine subspace as well as the expected maximum rank, and the output is a rational parametrization encoding a finite set of points that intersects each connected component of the low rank real algebraic set. The complexity of our algorithm is studied thoroughly. It is polynomial in $\binom{n + m(s-r)}{n}$. It improves on the state-of-the-art in computer algebra and effective real algebraic geometry. Moreover, computer experiments show the practical efficiency of our approach.

Keywords
Symbolic computation · Low rank matrices · Polynomial system solving · Real algebraic geometry

Mathematics Subject Classification
13-XX · 14Q20 · 12Y05 · 68W30

Simone Naldi
simone.naldi@unilim.fr

Didier Henrion
henrion@laas.fr

Mohab Safey El Din
Mohab.Safey@lip6.fr

\textsuperscript{1} CNRS, LAAS, 7 Avenue du colonel Roche, 31400 Toulouse, France
\textsuperscript{2} Université de Toulouse; LAAS, 31400 Toulouse, France
\textsuperscript{3} Faculty of Electrical Engineering, Czech Technical University in Prague, Prague, Czech Republic
\textsuperscript{4} CNRS, XLIM, UMR 7252, University of Limoges, 87000 Limoges, France
\textsuperscript{5} CNRS, Inria, Laboratoire d’Informatique de Paris 6, Sorbonne Université, LIP6, Equipe PolSys, 75252 Paris, France
1 Introduction

1.1 Problem statement

Let $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ be respectively the fields of rational, real and complex numbers. Let $s, m, n, r$ be positive integers with $0 \leq r < s \leq m$ and let $A_0, \ldots, A_n$ be $m \times s$ matrices with entries in $\mathbb{Q}$. Let $x = (x_1, \ldots, x_n)$ be variables. We consider the linear matrix $A(x)$ defined by

$$
(x_1, \ldots, x_n) \mapsto A(x) = A_0 + x_1A_1 + \cdots + x_n A_n.
$$

By abuse of notation, we denote the vector $(A_0, A_1, \ldots, A_n) \in (\mathbb{Q}^{m \times s})^{n+1}$ by $A$. Given $A$ as above, the set

$$
D_r = \{ x \in \mathbb{C}^n \mid \text{rank } A(x) \leq r \}.
$$

is called a determinantal variety. The goal of this paper is to design an efficient algorithm for deciding the emptiness of $D_r \cap \mathbb{R}^n$ and, if it is not empty, for computing at least one point in each connected component of $D_r \cap \mathbb{R}^n$.

Our algorithm is symbolic, that is its output is an exact encoding of finitely many points whose coordinates are algebraic numbers, given by a rational parametrization with coefficients in $\mathbb{Q}$. This is a vector $q = (q_0, q_1, \ldots, q_{n+1}) \in \mathbb{Q}[t]^{n+2}$ of univariate polynomials such that $q_{n+1}$ is square-free, $q_0 = \partial q_{n+1}/\partial t$ and $\deg(q_i) < \deg(q_{n+1})$ for $1 \leq i \leq n$, and such that the following set

$$
Z = \left\{ \left( \frac{q_1(t)}{q_0(t)}, \ldots, \frac{q_n(t)}{q_0(t)} \right) \in \mathbb{C}^n : q_{n+1}(t) = 0 \right\}.
$$

is contained in $D_r$ and contains at least one point in each connected component of $D_r \cap \mathbb{R}^n$. Such an encoding for finite algebraic sets goes back to the work of Macaulay and Kronecker [24,27] and has been extensively used and developed for computer algebra methods for solving polynomial systems (see e.g. [12,26,31]).

1.2 Motivations

The problem of finding low rank matrices in a given affine space has many applications in systems, signal and information engineering, where low rank typically corresponds to sparsity and structure requirements. For instance, in the context of semidefinite programming (SDP) hierarchies for polynomial optimization [25], low rank moment matrices provide guarantees of global optimality of a convex relaxation of a non-convex optimization problem. Similarly, the geometry of low rank structured matrices (e.g. Hurwitz, Hankel, Toeplitz, resultant matrices) is pervasive in algebraic approaches to information engineering (including systems control, signal processing, computer vision). In these cases, the given affine subspace lies in the linear space of symmetric (or more structured) matrices, while in this paper we address the problem from a more general point of view.
Example 1 The paper [23] studies distortion varieties, special algebraic varieties arising from computer vision. These have determinantal structure. For instance, the distortion variety in [23, Ex. 1.1] is the zero set of all $2 \times 2$ minors of the matrix

$$A(x_1, \ldots, x_{11}) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 & x_{10} & x_6 & x_{11} \end{pmatrix}.$$ 

The matrix $A$ defines a linear space of co-dimension 1 in the space of $2 \times 6$ matrices (indeed the $(1,6)\text{-entry}$ equals the $(2,5)\text{-entry}$), and we are interested in the locus of rank-one matrices of this form. The real points of this variety can be sampled with the algorithm developed in this paper, since the regularity assumptions needed by our algorithm (defined in Sect. 2.4) are satisfied by this example.

The specific geometry of low rank manifolds can be exploited to design efficient nonlinear local optimization algorithms [1]. Linear matrices and their loci of rank defects are the object of the so-called low rank approximation problem [28] and model problems arising in medical imagery [6].

In our paper, we are not after trying to solve approximately large-scale problem instances with floating point arithmetic. In contrast, our focus is on symbolic computation and rigorous algorithms. This means that we are not concerned with numerical scaling and conditioning issues. We provide mathematical guarantees of exactness of the output of our algorithm, under the assumption that the input is also exactly provided in rational arithmetic and satisfies some algebraic assumptions that are specified below. Obviously, these guarantees come with a price, and our algorithm complexity is exponential in the number of variables or problem size, and hence limited to “small” dimensions. But this is not specific to our algorithm, this limitation is shared with all symbolic computation methods: our algorithm should be applied to small-size problems for which it is absolutely crucial to find exact solutions.

Still, in this context, complexity issues are topical and can lead to practical improvements. The main difference with the state-of-the-art is that the complexity achieved by our algorithm is essentially quadratic in a multilinear Bézout bound on the maximum number of complex solutions encoded by the output. This bound is itself dominated by $\binom{n+m(s-r)}{n}^3$. Hence, for particular sub-classes of the problem, for example when the maximum dimension of the matrix is fixed, the multilinear bounds (and hence the complexity) are polynomial in the number of variables. We will see that this leads to faster implementations at the end of the paper.

1.3 State of the art

We distinguish in the state-of-the-art three subproblems. The first one is on computing sample points in each connected component of real algebraic sets, hence without taking care of the determinantal structure we consider here. Next, we review on previous work taking care of the determinantal structure but in the context of zero-dimensional algebraic sets. Finally, we consider real algebraic sets defined by rank constraints on matrices with polynomial entries.
Computing real solutions of systems of \( n \)-variate polynomial equations is a central question in computational geometry and effective real algebraic geometry. Since one typically deals with positive dimensional solution sets, one possible approach is to design algorithms computing a finite set intersecting each connected component of the real solution set. The complexity of Tarski’s algorithm [37] was not elementary recursive, while Collins’s Cylindrical Algebraic Decomposition [7] is doubly exponential in \( n \). Since Thom-Milnor bound for the maximum number of connected components of a real algebraic set [5, Theorem 7.23] is singly exponential in \( n \), tremendous efforts have been made to obtain optimal complexities.

Grigoriev and Vorobjov introduced in [15] the critical point method which culminates with the algorithms in [5, Ch.13] running in time singly exponential in \( n \). The algorithms in [4,33] also rely on the computation of critical points: On inputs of degree \( \leq d \), they lead to complexities which are essentially cubic (resp. quartic) in \( d^n \) for the smooth (resp. singular) case. These techniques have also been used in the context of polynomial optimization [14].

Dedicated algorithms in e.g. [3,13] or Gröbner bases [11] can be used to compute generic points in algebraic varieties in presence of determinantal structure. Observe that, by that, these are not sufficient to be applied to the problem of real root finding for positive dimensional real algebraic sets. In the context of determinantal varieties \( D_r \), the following cases have been already treated:

- \( m = s \) and \( r = s - 1 \): in [18], we designed a dedicated algorithm for computing sample points in each connected component of the studied real algebraic set under some genericity assumption on the input matrix pencil;
- \( m = s \) and the considered matrix is symmetric: we designed in [20] a dedicated algorithm for this situation without any other constraint on \( r \) than \( r \leq m - 1 \), again under some genericity assumption on the input matrix pencil. In [19], we also tackle the situation where the linear matrix is Hankel.

Observe that the cases \( m \neq s \) and arbitrary \( r \) were pending. In the current paper, we deal with the case \( m \neq s \) (assuming without loss of generality that \( m \geq s \)) and arbitrary \( r \leq m - 1 \). This paper builds on the previous work [18]: the spirit and the statement of our main result is rather close to this previous work but many of the techniques used in [18] cannot be applied mutatis mutandis to the more general setting we consider here and need to be adapted and generalized.

Moreover, we highlight that, contrarily to our previous contribution [18] concerning determinantal hypersurfaces, in this paper we explicitly describe the dependencies between the choice of parameters during the main algorithm.

1.4 Paper outline

The algorithm described in this paper, with input a \( m \times s \) linear matrix \( A(x) = A_0 + x_1 A_1 + \ldots + x_n A_n \), with \( m \geq s \), \( A_i \in \mathbb{Q}^{m \times s} \), \( i = 0, 1, \ldots, n \), and an integer \( r \leq s - 1 \), computes a rational parametrization of a finite set intersecting each connected component of \( D_r \cap \mathbb{R}^n \). The design of the algorithm is intended to take advantage of the special structure of the input problem and hence to behave better than algorithms
based on the critical point method that solve the same problem in a more general setting.

Since algebraic sets defined by minors of fixed size of a polynomial matrix are generically singular, the input of our algorithm does not satisfy regularity properties. Hence, the first step is to generate a second algebraic set $V_r$, defined by quadratic equations $A(x)Y(y) = 0$, where $Y(y)$ is a rectangular matrix whose columns generate the kernel of $A(x)$. The set we have obtained is a lifting of $D_r$, which is traditionally called an incidence variety.

We investigate properties of this incidence variety, proving that unlike $D_r$, the lifted set $V_r$ is regular (smooth and equidimensional) when the input matrices $A = (A_0, A_1, \ldots, A_n)$ lie outside a given algebraic hypersurface in $(Q^{m \times s})^{n+1}$. We show that our problem can be reduced to compute finitely many critical points of the restriction of a general linear projection to this lifted set. The system that defines these critical points has a special sparsity structure, namely it is bilinear in three groups of variables (the variables $x$ describing $D_r$, the variables $y$ encoding the kernel, and Lagrange multipliers $z$). Using the symbolic homotopy algorithm in [35] (which builds upon the one in [22]), one can compute a rational parametrization of these critical points by exploiting this sparsity structure. We establish a bound $\delta$ on the degree of the parametrization, and, using [35], we show that the complexity is essentially quadratic on $\delta$. This bound is dominated by $(n+m(s-r))^3$. Note that this complexity estimate does not take into account the cost of checking that the genericity assumption on the input is satisfied, which we suppose to be true.

Moreover, we provide computer experiments that show that our strategy allows to tackle problems that are unreachable by implementations of other generic algorithms based on the critical point method.

2 Definitions and notation

2.1 Basic notions

We denote by $Q^n$ (resp. $C^n$) the set of vectors of length $n$ with entries in $Q$ (resp. $C$). A subset $V \subset C^n$ is an affine algebraic variety (equivalently affine algebraic set) defined over $Q$ if it is the common zero locus of a system of polynomials $f = (f_1, \ldots, f_q) \in Q[x]^q$, with $x = (x_1, \ldots, x_n)$. We also write $V = f^{-1}(0) = Z(f)$. Algebraic varieties in $C^n$ define the closed sets of the so-called Zariski topology. Zariski open subsets of $C^n$ are sets whose complement are Zariski closed; they are either empty or dense in $C^n$.

The set of all polynomials vanishing on an algebraic set $V$ is an ideal and it is denoted by $I(V) \subset Q[x]$. This ideal is radical (i.e. $g^k \in I(V)$ for some integer $k$ implies that $g \in I(V)$) and it is generated by a finite set of polynomials, say $f = (f_1, \ldots, f_p)$. We also write $I(V) = \langle f_1, \ldots, f_p \rangle = \langle f \rangle$ when a set of generators is known.

Let $GL_n(C)$ (resp. $GL_n(Q)$) be the set of non-singular $n \times n$ matrices with entries in $C$ (resp. $Q$). The identity matrix is denoted by $I_n$. Given a matrix $M \in GL_n(Q)$ and a polynomial system $x \in C^n \mapsto f(x) \in C^p$ we denote by $f \circ M$ the polynomial.
system $x \in \mathbb{C}^n \mapsto f(Mx) \in \mathbb{C}^p$. If $\mathcal{V} = \mathcal{Z}(f)$, the image set $\mathcal{Z}(f \circ M) = \{x \in \mathbb{C}^n : f(Mx) = 0\} = \{M^{-1}x \in \mathbb{C}^n : f(x) = 0\}$ is denoted by $M^{-1}\mathcal{V}$. Given $q \leq n$ and $M \in \mathbb{C}^{n \times m}$, we denote by minors$(q, M)$ the set of determinants of $q \times q$ submatrices of $M$. The transpose of a matrix $M$ is denoted by $M^T$.

For $f \in \mathbb{Q}[x]^q$, we denote by $Df$ the Jacobian matrix of $f$, that is the $q \times n$ matrix $Df = \left(\frac{\partial f_j}{\partial x_i}\right)_{i,j}$. When $f$ generates a radical ideal, the codimension $c$ of $\mathcal{Z}(f)$ is the maximum rank of $Df$ evaluated at points in $\mathcal{Z}(f)$. Its dimension is $n - c$. The algebraic set $\mathcal{V} = \mathcal{Z}(f)$ is said irreducible, if it is not the union of two algebraic sets strictly contained in $\mathcal{Z}(f)$. If $\mathcal{V}$ is not irreducible, it is decomposable as the finite union of irreducible algebraic sets, called the irreducible components. If all the irreducible components have the same dimension, $\mathcal{V}$ is equidimensional. The dimension of $\mathcal{V}$ coincides with the maximum of the dimensions of its components.

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^q$ generate a radical ideal, and let $\mathcal{V} = \mathcal{Z}(f)$ be equidimensional of dimension $d$. A point $x \in \mathcal{V}$ such that the rank of $Df$ is equal to $n - d$ is a regular point, otherwise is a singular point. We denote by $\text{reg} \ V$ and $\text{sing} \ V$ respectively the set of regular and singular points of $\mathcal{V}$.

Let $f : \mathbb{C}^n \rightarrow \mathbb{C}^q$ generate a radical ideal, and let $\mathcal{V} = \mathcal{Z}(f)$ be equidimensional of dimension $d$. Let $g : \mathbb{C}^n \rightarrow \mathbb{C}^p$. A point $x \in \text{reg} \ V$ is a critical point of the restriction of $g$ to $\mathcal{V}$ if the minors of size $n - d + p$ of the extended Jacobian matrix $D(f,g)$ vanish at $x$. The Zariski-closure of the set of critical points is denoted by $\text{crit}(g, \mathcal{V})$. Let $\pi_1 : \mathbb{C}^n \rightarrow \mathbb{C}$ be the projection $\pi_1(x) = x_1$ and let $D_1f$ be the matrix obtained by deleting the first column of $Df$. Then $\text{crit}(\pi_1, \mathcal{V})$ is equivalently defined by the zero set of the polynomials in $f$ and the maximal minors of $D_1f$. The set $\text{crit}(\pi_1, \mathcal{V})$ is also called a polar variety when $\mathcal{V}$ is smooth.

### 2.2 Incidence variety

Let $A = (A_0, A_1, \ldots, A_n)$ be $m \times s$ matrices $(m \geq s)$ with entries in $\mathbb{Q}$, and $A(x) = A_0 + x_1A_1 + \cdots + x_nA_n$ the associated linear matrix. If $x \in D_r = \{x \in \mathbb{C}^n : \text{rank} \ A(x) \leq r\}$, the right kernel of $A(x)$ is a subspace of dimension $\geq s - r$ in $\mathbb{C}^s$ by linear algebra.

We introduce $s(s - r)$ variables $y = (y_{1,1}, \ldots, y_{s,s-r})$, stored in a $s \times (s - r)$ linear matrix

$$Y(y) = \begin{pmatrix} y_{1,1} & \cdots & y_{1,s-r} \\ \vdots & \ddots & \vdots \\ y_{s,1} & \cdots & y_{s,s-r} \end{pmatrix}$$

and, for $U \in \mathbb{Q}^{(s-r)\times s}$, we define the incidence variety associated to $(A, U)$ as

$$\mathcal{V}_r(A, U) := \{(x, y) \in \mathbb{C}^n \times \mathbb{C}^{s(s-r)} : A(x)Y(y) = 0, UY(y) - I_{s-r} = 0\}. \quad (2)$$
Remark that the matrix \( Y(y) \) has full rank \( s - r \) if and only if there exists \( U \in \mathbb{Q}^{(s-r)\times s} \) of full rank such that \( UY(y) - \mathbb{I}_{s-r} = 0 \). For \( A \in (\mathbb{C}^{m \times s})^{n+1}, U = (u_{i,j})_{1 \leq i \leq s-r, 1 \leq j \leq s} \in \mathbb{Q}^{(s-r)\times s} \), and \( c := (m + s - r)(s - r) \), define

\[
f(A, U) : \mathbb{C}^{n+s(s-r)} \to \mathbb{C}^c
\]

\[
(x, y) \mapsto (A(x)Y(y), UY(y) - \mathbb{I}_{s-r})
\]

Remark that \( \mathcal{V}_r(A, U) = \mathcal{Z}(f(A, U)) \) and that the projection of \( \mathcal{V}_r(A, U) \) over the \( x \)-space is contained in the determinantal variety \( \mathcal{D}_r \), by definition. We denote this projection map by \( \Pi_X \).

2.3 Data representation

The input is a \( m \times s \) linear matrix \( A(x) = A_0 + x_1A_1 + \cdots + x_nA_n \), with \( m \geq s \), encoded by the vector of defining matrices \( A = (A_0, A_1, \ldots, A_n) \), with coefficients in \( \mathbb{Q} \), and an integer \( r \) such that \( r \leq s - 1 \). The vector \( A \) is understood as a point in \( (\mathbb{Q}^{m \times s})^{n+1} \).

The output is a finite set sampling the connected components of \( \mathcal{D}_r \cap \mathbb{R}^n \). Indeed, the initial problem is reduced to isolating the real solutions of an algebraic set \( \mathcal{Z} \subset \mathbb{C}^n \) of dimension at most 0, represented by a rational parametrization \( q = (q_0, q_1, \ldots, q_n, q_{n+1}) \in \mathbb{Q}[t]^{n+2} \), that is with the representation as in (1).

2.4 Genericity assumptions

Our algorithm works under some assumptions on the input \( A \) (that we denote with the letters \( G_1 \) and \( G_2 \)). These are natural regularity assumptions about the singular locus of the locus of low-rank matrices and about smoothness of algebraic sets, that are often satisfied in practice. We will prove below in Sect. 3 that these are generic. We recall that the parameter \( r \) is fixed and it holds \( 0 \leq r < s \leq m \).

**Property \( G_1 \)**. A \( m \times s \) linear matrix \( A \) satisfies \( G_1 \) if, for all \( 0 \leq p \leq r \), \( \mathcal{D}_p \subset \mathbb{C}^n \) is either empty or \( n - (m - p)(s - p) \)-equidimensional, \( \text{sing}(\mathcal{D}_p) = \mathcal{D}_{p-1} \) and the ideal generated by the \((p + 1, p + 1)\) minors of \( A(x) \) is radical.

Our algorithm takes as input a linear matrix \( A(x) \) assuming that \( A \) satisfies \( G_1 \); we will prove that \( G_1 \) holds generically in the sequel. The second property, which we often refer to as a regularity property, is defined for any polynomial system.

**Property \( G_2 \)**. A polynomial sequence \( h = (h_1, \ldots, h_k) \in \mathbb{Q}[x_1, \ldots, x_n]^k \) satisfies \( G_2 \) if

- the ideal \((h)\) is a radical ideal of co-dimension \( k \), and
- the algebraic set \( \mathcal{Z}(h) \subset \mathbb{C}^n \) is either empty or smooth and equidimensional.

3 Algorithm: description, correctness, complexity

In this section, we describe the algorithm \textsf{LowRank}, prove its correctness and estimate its arithmetic complexity.
3.1 Formal description

The input of LowRank is a couple \((A, r)\), where \(A\) is a tuple of \(n + 1\) matrices \(A_0, A_1, \ldots, A_r\), of size \(m \times s\) \((m \geq s)\), with entries in \(\mathbb{Q}\), and \(r \leq s - 1\) is an integer. The algorithm is probabilistic and, upon success, its output is a rational parametrization encoding a finite set of points intersecting each connected component of the real algebraic set \(\{x \in \mathbb{R}^n : \text{rank } A(x) \leq r\}\) as described in Sect. 2.3.

3.1.1 Notation

Recall that given \(A \in (\mathbb{C}^{m \times s})^{n+1}\) and \(U \in \mathbb{C}^{(s-r) \times s}\), the polynomial system \(f(A, U)\) (of cardinality \(c = (m+s-r)(s-r)\)) and its zero locus \(\mathcal{V}_r(A, U)\) have been defined in Sect. 2.2.

Change of variables Let \(M \in \text{GL}_n(\mathbb{C})\). As already explained in Sect. 2, we denote by \(A \circ M\) the affine map \(x \mapsto A(Mx)\) obtained from \(A\) by applying a change of variables induced by the matrix \(M\). In particular \(A = A \circ \mathbb{I}_n\). For \(M \in \text{GL}_n(\mathbb{C})\), and for all \(A \in (\mathbb{C}^{m \times s})^{n+1}\), \(U \in \mathbb{C}^{(s-r) \times s}\), we consequently denote by \(f(A \circ M, U)\) the polynomial system \(f(A, U)\) applied to \((Mx, y)\), and by \(\mathcal{V}_r(A \circ M, U) = \mathcal{Z}(f(A \circ M, U))\).

Fibers Given \(w \in \mathbb{C}^n\), we introduce the notation \(\pi_w\) for the map \(\pi_w : \mathbb{C}^n \to \mathbb{C}\), \(\pi_w(x) = w^T x\), and \(\Pi_w : \mathbb{C}^{n+s(s-r)} \to \mathbb{C}\), \(\Pi_w(x, y) = w^T x\), that is \(\Pi_w = \pi_w \circ \Pi_X\). For \(w \in \mathbb{C}^n\) and \(t \in \mathbb{C}\), we define

\[
f_{w, t : \mathbb{C}^{n+s(s-r)} \to \mathbb{C}^{c+1}} (x, y) \mapsto (f(A, U), \Pi_w(x, y) - t)
\]

and denote by \(\mathcal{V}_{r, w, t}(A, U) = \mathcal{Z}(f_{w, t}) \subset \mathbb{C}^{n+s(s-r)}\) the section of \(\mathcal{V}_r\) with the linear space defined by \(\Pi_w(x, y) - t = 0\). When parameters are clear from the context, we use the shorter notation \(\mathcal{V}_{r, w, t}\).

For \(A \in (\mathbb{C}^{m \times s})^{n+1}\), and \(w \in (\mathbb{C} \setminus \{0\})^n\) we denote by \(A|_{w, t} \in (\mathbb{C}^{m \times s})^n\) the linear matrix obtained by eliminating one variable (up to renaming variable, \(x_1\)) from \(A\) using the affine equation \(w^T x - t = 0\).

Lagrange systems Given \(w \in \mathbb{C}^n\), we define

\[
\ell(A, U, w : \mathbb{C}^{n+s(s-r)+c} \to \mathbb{C}^{n+s(s-r)+c}) (x, y, z) \mapsto (f(A, U), z^T Df - (w, 0)^T)
\]

where \(z = (z_1, \ldots, z_c)\) is the column vector of Lagrange multipliers and \((w, 0) \in \mathbb{C}^{n+s(s-r)}\). Let \(\mathcal{Z}(A, U, w) = \mathcal{Z}(\ell(A, U, w)) \subset \mathbb{C}^{n+s(s-r)+c}\).

3.1.2 Subroutines

The algorithm LowRank uses different subroutines, described as follows.

IsReg: inputs parameters \(A, U\) and outputs true if \(A\) satisfies \(G_1\) and \(f(A, U)\) satisfies \(G_2\), false otherwise;

\(\mathcal{G}\) Springer
RatPar: inputs a polynomial system \( f \); returns either the empty list if \( Z(f) = \emptyset \), or an error message if \( Z(f) \) is not zero-dimensional, otherwise a rational parametrization of \( Z(f) \);

Project: inputs a rational parametrization of a finite set \( Z \subset \mathbb{C}^N \) and a subset of the variables \( x_1, \ldots, x_N \), and outputs a rational parametrization of the projection of \( Z \) on the space generated by this subset;

Lift: inputs a rational parametrization of a finite set \( Z \subset \mathbb{C}^N \) and a number \( t \in \mathbb{C} \), and outputs a rational parametrization of the set \( \{(t, x) \in \mathbb{C}^{N+1} : x \in Z\} \), that is the pre-image of \( Z \) under the projection on the last \( N \) variables;

Union: inputs rational parametrizations encoding finite sets \( Z_1, Z_2 \) and outputs a rational parametrization of \( Z_1 \cup Z_2 \).

Moreover, a routine \( \text{nvar}(A) \) outputs the number of variables of a linear matrix \( A \), and a routine \( \text{minors}(A, d) \) outputs the list of \( d \times d \) minors of \( A \).

3.1.3 The algorithm

This is the formal description of the algorithm. The main routine \( \text{LowRank} \) calls the recursive routine \( \text{LowRankRec} \); the recursion is on the number of variables of the \( m \times s \) linear matrix \( A \) (which is always denoted by \( n \), and computed by a subroutine \( \text{nvar} \)).

**Algorithm** LowRank\((A, r)\):

1. Choose \( U \in \mathbb{Q}^{(s-r) \times s} \)
2. If \( \text{IsReg}(A, U) = \text{false} \) then return('error: input data not generic')
3. return LowRankRec\((A, U, r)\)

**Algorithm** LowRankRec\((A, U, r)\):

1. \( n = \text{nvar}(A) \). If \( n \leq (m - r)(s - r) \) then return\((\text{RatPar}(\text{minors}(A, r + 1)))\)
2. Choose \( w \in \mathbb{Q}^n \setminus \{0\} \), \( P = \text{Project}(\text{RatPar}(\ell(A, U, w)), x) \)
3. Choose \( t \in \mathbb{Q} \), \( Q = \text{Lift}(\text{LowRankRec}(A|_{w, t}, U, r), t) \)
4. return Union\((Q, P)\).

**Example 2** Consider the \( 4 \times 3 \) linear matrix \( A = (x_{ij}), i = 1 \ldots 4, j = 1 \ldots 3 \) with unknown entries. One of the incidence varieties encoding one rank defect in \( A \) is given as the zero set of the polynomials \( f = (f_1, f_2, f_3, f_4) \) defined by

\[
\begin{pmatrix}
  f_1 \\
  f_2 \\
  f_3 \\
  f_4
\end{pmatrix} = \begin{pmatrix}
  x_{11}y_1 + x_{12}y_2 + x_{13}y_3 \\
  x_{21}y_1 + x_{22}y_2 + x_{23}y_3 \\
  x_{31}y_1 + x_{32}y_2 + x_{33}y_3 \\
  x_{41}y_1 + x_{42}y_2 + x_{43}y_3
\end{pmatrix} = \begin{pmatrix}
  x_{11} & x_{12} & x_{13} \\
  x_{21} & x_{22} & x_{23} \\
  x_{31} & x_{32} & x_{33} \\
  x_{41} & x_{42} & x_{43}
\end{pmatrix} \begin{pmatrix}
  y_1 \\
  y_2 \\
  y_3
\end{pmatrix}
\]

together with an affine equation of the form \( u_1y_1 + u_2y_2 + u_3y_3 = 1 \), ensuring that a non-zero kernel vector is computed. The Lagrange system \( \ell(A, U, w) \) is generated by the critical equations

\[
f_1 = 0, \ f_2 = 0, \ f_3 = 0, \ f_4 = 0, \ u^T y - 1 = 0, \ \sum_{i=1}^4 z_i \nabla f_i + z_5(u, 0)^T = (w, 0)^T 
\]
whose solutions are the local minima and maxima of the restriction of a linear function \( x \mapsto w^T x \) to the incidence variety. Once the solutions of this system (that are finitely many, for general choices of \( u \) and \( w \)) are computed (with a rational parametrization) a generic fiber is considered by adding an affine constraint \( w^T x - t = 0 \), one variable is eliminated and the algorithm is called recursively.

### 3.2 Correctness

We start by stating intermediate results which will be used to prove the correctness of the algorithm. The proof of the first result below is given in Sect. 4.

**Proposition 1** Let \( m, s, n, r \in \mathbb{N} \), with \( 0 \leq r < s \leq m \). The following holds.

1. There exists a non-empty Zariski open set \( A \subset (\mathbb{C}^m \times \mathbb{C}^s)^{n+1} \) such that for \( A \in A \), \( A \) satisfies Property \( G_1 \).
2. Let \( A \) satisfy \( G_1 \). There exists a non-empty Zariski open set \( U \subset \mathbb{C}^{(s-r) \times s} \) such that, for \( U \in U \), the following holds:
   
   (a) \( f(A, U) \) satisfies Property \( G_2 \);
   
   (b) letting \( \mathcal{N}_r(A, U) \) be the Zariski closure of \( \mathcal{D}_r - \Pi_X(\mathcal{V}_r(A, U)) \) and \( Z \) an irreducible component of \( \mathcal{D}_p \) for \( 0 \leq p \leq r \), \( Z \cap \mathcal{N}_r(A, U) \) has co-dimension at least 1 in \( Z \).
3. Let \( w \in \mathbb{Q}^n \setminus \{0\} \). There exists a non-empty Zariski open set \( T \subset \mathbb{C} \) such that if \( t \in T \cap \mathbb{Q}^n \), then \( A \mid_{w,t} \) satisfies Property \( G_1 \).

The second result is proved in Sect. 5.

**Proposition 2** Let \( A \) be in the non-empty Zariski open set \( \mathcal{A} \subset (\mathbb{C}^m \times \mathbb{C}^s)^{n+1} \) and \( U \) in the non-empty Zariski open set \( \mathcal{U}_A \subset \mathbb{C}^{(s-r) \times s} \) defined in Proposition 1. There exists a non-empty Zariski open set \( \mathcal{V}_{A,U} \subset \mathbb{C}^n \) such that for \( w \in \mathcal{V}_{A,U} \cap \mathbb{Q}^n \) the following holds.

1. \( Z(A, U, w) \) is finite and \( \ell(A, U, w) \) satisfies Property \( G_2 \);
2. the projection of \( Z(A, U, w) \) on \((x, y)\) contains the set of critical points of the restriction of \( \Pi_w : (x, y) \to w^T x \) to \( \mathcal{V}_r \).

The following proposition will be proved in Sect. 6.

**Proposition 3** Let \( \mathcal{A} \subset (\mathbb{C}^m \times \mathbb{C}^s)^{n+1} \) and \( \mathcal{U}_A \subset \mathbb{C}^{(s-r) \times s} \) be the non-empty Zariski open sets, and let \( \mathcal{N}_r(A, U) \subset \mathcal{D}_r \) be the Zariski closed set, defined in Proposition 1. Let \( A \in \mathcal{A} \) and \( U \in \mathcal{U}_A \).

Let \( C \subset \mathbb{R}^n \) be a connected component of \( \mathcal{D}_r \cap \mathbb{R}^n \).

There exists a non-empty Zariski open set \( \mathcal{V}_r' \subset \mathbb{C}^n \) such that, for \( w \in \mathcal{V}_r' \cap \mathbb{Q}^n \), the following holds:

1. \( \pi_w(C) \) is closed
2. for \( t \in \mathbb{R} \) in the boundary of \( \pi_w(C) \), there exists \((x, y, z)\) in \( Z(A, U, w) \) such that \( \Pi_w(x, y) = t \) and \( x \notin \mathcal{N}_r(A, U) \).
Observe that in the above statements, the defined non-empty Zariski open sets (except for the set $\mathcal{A}$) have subscripts indicating which data they depend on. Hence, starting with $A$ satisfying $G_1$, we highlight the following facts:

- the non-empty Zariski open set $\mathcal{U}_A$ (Proposition 1) depends on $A$;
- the non-empty Zariski open sets $\mathcal{W}_A, U$ (Proposition 2) and $\mathcal{W}_A', U'$ (Proposition 3) depend on $A$ and $U$.

**Hypothesis** $H_1$. In the sequel, $A$ (resp. $U$) is assumed to belong to the non-empty Zariski open set $\mathcal{A}$ (resp. $\mathcal{U}_A$) defined in Proposition 1.

One also has to ensure that the parameters $w \in \mathbb{Q}^n$ and $t \in \mathbb{Q}$ chosen, respectively, at steps 2 and 3 belong to the non-empty Zariski open sets defined in Proposition 2 and 3 at each call of LowRankRec. The choices of random parameters can be stored in an array

\[
\left(\left(w^{(n)}, t^{(n)}\right), \ldots, \left(w^{((m-r)(s-r))}, t^{((m-r)(s-r))}\right)\right)
\]

where the superscript represents the number of variables at the given recursion step (hence $n$ here is the number of variables of $A$ at the input of LowRank). We also denote by $\mathcal{T}(j)$, $\mathcal{W}(j)$ and $\mathcal{W}(j)'$ the non-empty Zariski open sets defined by Propositions 1, 2 and 3, at the $(n - j + 1)$-th recursion call (here we avoid the dependency on the Zariski open sets).

**Hypothesis** $H_2$. Given $A$ and $U$ satisfying $H_1$, the parameters (3) satisfy:

- $w^{(j)} \in \mathcal{W}(j) \cap \mathcal{W}(j)' \cap \mathbb{Q}^n \setminus \{0\}$ for $j = (m-r)(s-r), \ldots, n$;
- $t^{(j)} \in \mathcal{T}(j) \cap \mathbb{Q}$ for $j = (m-r)(s-r), \ldots, n$.

**Theorem 1** If $H_1$ and $H_2$ hold, algorithm LowRank returns a rational parametrization whose set of solutions intersects each connected component of $\mathcal{D}_r \cap \mathbb{R}^n$.

**Proof** Suppose first that $n \leq (m-r)(s-r)$. Since $H_1$ holds, then the variety $\mathcal{D}_r$ is empty or finite. Hence the algorithm returns the correct output, that is either the empty list or a rational parametrization of the finite set $\mathcal{V}_r$. Thereafter, we proceed by induction on $n$.

Let $n > (m-r)(s-r)$ and suppose that for any $(n-1)$-variate linear matrix, algorithm LowRank returns the expected output when $H_1$ and $H_2$ hold, namely one point per connected component of $\mathcal{D}_r \cap \mathbb{R}^n$. Let $A$ be a $n$-variate $m \times s$ linear matrix, let $r$ be an integer such that $0 \leq r \leq s - 1$ and let $\mathcal{C}$ be a connected component of $\mathcal{D}_r \cap \mathbb{R}^n$. Let $U$ be the matrix chosen at Step 1 of LowRank. Let $w \in \mathbb{Q}^n$ be the vector chosen at Step 2 of LowRankRec with input $A$, $U$, and $r$. Consider the projection $\pi_w: (x_1, \ldots, x_n) \to w^T x$ restricted to $\mathcal{V}_r(A, U)$. Since Property $H_1$, $H_2$ hold, by Proposition 3, $\pi_w(\mathcal{C})$ is closed, and so either $\pi_w(\mathcal{C}) = \mathbb{R}$ or $\pi_w(\mathcal{C}) \subseteq \mathbb{R}$ is a closed set with non-empty boundary. We claim that, in both cases, LowRank with input $(A, r)$ returns a point which lies in the connected component $\mathcal{C}$. This is proved next.

**First case** Suppose first that $\pi_w(\mathcal{C}) = \mathbb{R}$. In particular, for $t \in \mathbb{Q}$ chosen at Step 3 of LowRankRec with input $A$, $U$, $r$, the set $\pi_w^{-1}(t)$ intersects $\mathcal{C}$, so $\pi_w^{-1}(t) \cap \mathcal{C} \neq \emptyset$. Let $A|_{w,t}$ be the $(n-1)$-variate $m \times s$ linear matrix obtained from $A$ by substituting $x_1 = (1/w_1)(t - \sum_{j=2}^{n} w_j x_j)$ obtained from the linear constraint $w^T x = t$ (indeed,
since \( w \neq 0 \), one can suppose \( w_1 \neq 0 \) up to permutation of indices). Remark that 
\[
\pi_w^{-1}(t) \cap C = \bigcup_{i \in \{1, \ldots, m\}} \pi_w^{-1}(t_{\pi_i}) \cap C = \bigcup_{i \in \{1, \ldots, m\}} \pi_w^{-1}(t_{\pi_i}) \cap C.
\]

Since \( H_1 \) holds, then \( A|_{w,t} \) satisfies \( \text{Proposition } 4 \); we deduce by the induction hypothesis (since \( A|_{w,t} \) is \((n-1)-\text{variate}) that the subroutine \( \text{LowRankRec} \) computes one point in each connected component of \( D_r(n-1) \cap \mathbb{R}^{n-1} \), and so at least one point in \( C \).

Second case Suppose now that \( \pi_w(C) \neq \mathbb{R} \). By \text{Proposition } 3, \( \pi_w(C) \) is closed. Since \( C \) is connected, \( \pi_w(C) \) is a closed interval, and since \( \pi_w(C) \neq \mathbb{R} \) there exists \( t \) in the boundary of \( \pi_w(C) \) such that \( \pi_w(C) \subset [t, +\infty) \) or \( \pi_w(C) \subset (-\infty, t] \). Suppose without loss of generality that \( \pi_w(C) \subset [t, +\infty) \), so that \( t \) is the minimum value attained by \( \pi_w \) on \( C \).

By \text{Proposition } 3, there exist \( x_2, \ldots, x_n \) in \( \mathbb{R} \), \( y \in \mathbb{C}^{s(s-r)} \) and \( z \in \mathbb{C}^c \) such that, for \( x = (t, x_2, \ldots, x_n) \), it holds that \( (x, y, z) \in Z(A, U, w) \), and \( x \notin N_r(A, U) \). Then, we conclude that the point \( x \in C \) appears among the solutions of the rational parametrization \( P \) obtained at Step 2 of \( \text{LowRankRec} \).

Both cases considered in the above proof are important to be considered. For instance, the projection of the set defined by the rank defect of the matrix 
\[
\begin{bmatrix}
x_1 + x_2 & 1 \\
1 & x_1 - x_2
\end{bmatrix}
\]
is the whole line. The second case is well illustrated with the set of points at which the matrix 
\[
\begin{bmatrix}
x_1 & 1 - x_2 \\
x_2 & x_1
\end{bmatrix}
is rank defective: since it is compact over \( \mathbb{R}^2 \), its projection on a line cannot be surjective.

### 3.3 Complexity analysis

In this section we provide an analysis of the complexity of algorithm \( \text{LowRank} \). We also give bounds for the maximum number of complex solutions computed by \( \text{LowRank} \). We suppose that \( A \) satisfies Property \( G_1 \) and that \( f(A, U) \) satisfies Property \( G_2 \). Recall that the complexity of checking these properties is not evaluated here.

In order to bound the complexity of \( \text{LowRank} \), it is essentially sufficient to bound the complexity of \( \text{LowRankRec} \). This latter quantity mainly depends on the subroutine \( \text{RatPar} \) computing the rational parametrization, whose complexity is computed in Sect. 3.3.2. We rely on routines described in [35], which consists in a symbolic homotopy algorithm taking advantage of the sparsity structure of the input polynomial system.

Finally, complexity bounds for the subroutines \( \text{Project}, \text{Lift}, \text{Image} \) and \( \text{Union} \) are provided in Sect. 3.3.3 and refer to results of [29].

#### 3.3.1 Bounds on the degree of the output of \( \text{RatPar} \)

We consider the subroutine \( \text{RatPar} \) at the first recursion step of \( \text{LowRank} \). Its input consists in either the generators \( f(A \circ M, U, S) \) of the incidence variety (if \( n = (m-r)(s-r) \)) or the Lagrange system \( \ell(A \circ M, U, S, \nu) \) (if \( n > (m-r)(s-r) \)). In both cases, we provide below in \text{Proposition } 4 a bound on the degree of the rational parametrization returned by \( \text{RatPar} \).
We recall that if $x^{(1)}, \ldots, x^{(p)}$ are $p$ groups of variables, and $f \in \mathbb{Q}[x^{(1)}, \ldots, x^{(p)}]$, we say that the multidegree of $f$ is $(d_1, \ldots, d_p)$ if its degree with respect to the group of variables $x^{(j)}$ is $d_j$ for $j = 1, \ldots, p$.

**Proposition 4** Let $A$ be an $n \times m$ matrix, $0 \leq r < s \leq m$ and let $U$ and $w$ be respectively the parameters chosen at step 1 of LowRank and at step 2 of LowRankRec. Suppose that $H_1$ and $H_2$ hold. Then:

1. if $n = (m-r)(s-r)$, the degree of the output of RatPar at step 1, is bounded from above by $(m(s-r))$;
2. if $n > (m-r)(s-r)$, the degree of the output of RatPar at step 2, with input $\ell(A, U, w)$, is bounded from above by

\[
\delta(m, n, r) := \sum_{k \in \mathcal{F}_{m, n, r}} \binom{m(s-r)}{n-k} \binom{n-1}{k} \binom{r(s-r)}{k},
\]

with $\mathcal{F}_{m, n, r} = \{ k : \max\{0, n-m(s-r)\} \leq k \leq \min\{n-(m-r)(s-r), r(s-r)\} \}$.

**Proof (of Assertion 1)** If $n = (m-r)(s-r)$, since $H_1$ holds, the dimension of $\mathcal{D}_r$ is zero. Consequently, the degree of the rational parametrization returned by RatPar is the degree of $\mathcal{D}_r$. We bound this degree by the degree of $\mathcal{V}_r$, which is a finite set by Proposition 1 (indeed, in the zero-dimensional case, the set $\mathcal{N}_r(A, U)$ is empty). Since the entries of $f(A, U)$ have a natural bilinear structure in $x, y$, one takes advantage in using the Multilinear Bézout bound (see [34, Chapter 11]) to bound the degree of the set it defines.

From $UY(y) - I_{s-r}$ one can eliminate $(s-r)^2$ variables $y_i, j$. Indeed, recall that $U$ has full rank $s-r$. Without loss of generality, we suppose that the last $s-r$ columns of $U$ are linearly independent, and hence we eliminate the variables $y_i, j$ corresponding to the last $s-r$ rows of $Y(y)$. Abusing notation, we denote by the same symbol $f \subset \mathbb{Q}[x, y_1, 1, \ldots, y_r, s-r]$ the polynomial system obtained after this elimination. It is constituted by $m(s-r)$ polynomials of multidegree bounded by $(1, 1)$ with respect to $x = (x_1, \ldots, x_n)$ and $y = (y_1, 1, \ldots, y_r, s-r)$.

By the Multilinear Bézout theorem [34, Prop. I.1], $\deg \mathcal{Z}(f)$ is bounded by the sum of the coefficients of

\[
(s_x + s_y)^{m(s-r)} \mod \left\{ s_x^{n+1}, s_y^{r(s-r)+1} \right\} \subset \mathbb{Z}[s_x, s_y].
\]

Since $n+1 = m(s-r)$, and $(s_x + s_y)^{m(s-r)}$ is homogeneous of degree $m(s-r)$, the aforementioned bound equals the coefficient of $s_x^{n}s_y^{r(s-r)}$ in the expansion of $(s_x + s_y)^{m(s-r)}$, that is exactly $\binom{m(s-r)}{m-r(s-r)}$.

**Proof (of Assertion 2)** In this case, the input of RatPar is the Lagrange system $\ell(A, U, w)$. Let $f$ be the equivalent system defined in the proof of Assertion 1. We apply a similar reduction to $\ell(A, U, w)$. We introduce Lagrange multipliers $z = [1, z_2, \ldots, z_{m(s-r)}]$ (we put $z_1 = 1$ w.l.o.g., since $\ell(A, U, w)$ is defined over the Zariski open set $z \neq 0$) and we consider polynomials $(g, h) = z^TD_1f$. Hence the new equivalent system $\ell = (f, g, h)$ is constituted by:
\begin{itemize}
    \item $m(s - r)$ polynomials of multidegree bounded by $(1, 1, 0)$;
    \item $n - 1$ polynomials of multidegree bounded by $(0, 1, 1)$;
    \item $r(s - r)$ polynomials of multidegree bounded by $(1, 0, 1)$.
\end{itemize}
Moreover, by Proposition 2, $\mathcal{Z}(f, g, h)$ has dimension at most zero and $(f, g, h)$ satisfies $G_2$. As above, $\deg \mathcal{Z}(f, g, h)$ is bounded by the sum of the coefficients of
\[(s_x + s_y)_{m(s-r)}(s_y + s_z)_{n-1}(s_x + s_z)_{r(s-r)} \mod \{s_x^{n+1}, s_y^{r(s-r)+1}, s_z^{m(s-r)}\} \subseteq \mathbb{Z}[s_x, s_y, s_z].\]

As in the proof of Assertion 1, by homogeneity of the polynomial and by counting the degrees, the previous sum is given by the coefficient of the monomial $s_x^n s_y^{r(s-r)} s_z^{m(s-r)-1}$ in the expansion
\[
\sum_{i=0}^{m(s-r)-1} \sum_{j=0}^{r(s-r)} \sum_{k=0}^{n-1} \binom{m(s-r)}{i} \binom{n-1}{j} \binom{r(s-r)}{k} s_x^i s_y^j s_z^{m(s-r)-i-j} s_x^{n-1-j+r(s-r)-k}.
\]

The coefficient is obtained by setting the equalities $i + k = n$, $m(s - r) - i + j = r(s - r)$ and $n - 1 - j + r(s - r) - k = m(s - r) - 1$. These equalities imply $i + k = n = j + k + (m - r)(s - r) = j + k + i - j = i + k$ and consequently one deduces the claimed expression. \qed

Proposition 4 implies straightforwardly the following estimate.

**Corollary 1** Suppose that the hypotheses of Proposition 4 are satisfied. Then LowRank returns a rational parametrization whose degree is less than or equal to
\[
\binom{m(s-r)}{(m-r)(s-r)} + \sum_{j=(m-r)(s-r)+1}^{\min\{n, (m+r)(s-r)\}} \delta(m, j, r).
\]

**Proof** Since $H_1$ holds, for $n < (m - r)(s - r)$ the algorithm returns the empty list. For $m, s, j, r$ let $F_{m, s, j, r}$ be the set of indices defined in Proposition 4. Observe that $F_{m, s, j, r} = \emptyset$ if and only if $j > (m + r)(s - r)$. Hence, the thesis is deduced straightforwardly from bounds given in Proposition 4. \qed

One can also deduce the following bound on $\delta(m, s, n, r)$.

**Lemma 1** For all $m, s, n, r$, with $r < s \leq m$, $\delta(m, s, n, r) \leq \left(\binom{n+m(s-r)}{n}\right)^3$.

**Proof** This comes straightforwardly from the formula
\[
\binom{a+b}{a}^3 = \sum_{i_1, i_2, i_3 = 0}^{\min(a, b)} \binom{a}{i_1} \binom{b}{i_2} \binom{a}{i_3} \binom{b}{i_2}
\]
applied with $a = n$ and $b = m(s - r)$, and from the expression of $\delta(m, s, n, r)$ computed in Proposition 4. \qed
3.3.2 Complexity of RatPar

The computation of the rational parametrization by the subroutine RatPar is done via the symbolic homotopy algorithm [35]. In this section, we analyze the complexity of the algorithm in [35] for our special case.

We suppose that \( n > (m - r)(s - r) \) and that the input of RatPar is the equivalent Lagrange system \( \ell = \ell(A, U, w) \in \mathbb{Q}[x, y, z]^{n-1+(m+r)(s-r)} \) built in the proof of Assertion 2 of Proposition 4. First, the strategy consists in building a second polynomial system \( \tilde{\ell} \subset \mathbb{Q}[x, y, z] \), such that:
- the length of \( \tilde{\ell} \) equals that of \( \ell \), that is \( n - 1 + (m + r)(s - r) \);
- for \( i = 1, \ldots, n - 1 + (m + r)(s - r) \), the support of \( \tilde{\ell}_i \) equals that of \( \ell_i \);
- the solutions of \( \tilde{\ell} \) can be computed efficiently (see below).

Indeed, we remind that by construction, \( \ell \) contains three groups of quadratic polynomials in \( \mathbb{Q}[x, y, z] \), of multidegree respectively bounded by \((1, 1, 0)\), \((0, 1, 1)\) and \((1, 0, 1)\). We denote by \( \Delta_1 \subset \mathbb{Q}[x, y], \Delta_2 \subset \mathbb{Q}[y, z] \) and \( \Delta_3 \subset \mathbb{Q}[x, z] \) the supports of the three groups, so that for example \( \Delta_1 = \{1, x_i, y_j, x_i y_j : 1 \leq i \leq n, 1 \leq j \leq r(s - r)\} \), or, equivalently, \( \Delta_1 \) can be seen as the subset of \( \mathbb{Z}^{n+r(m-r)} \) made by the exponents of its monomials. Let \( \ell_i \) be with support in \( \Delta_1, 1 \leq i \leq m(s - r) \). Hence we generate two linear forms \( g_{i,1} \in \mathbb{Q}[x] \) and \( g_{i,2} \in \mathbb{Q}[y] \) and we define \( \tilde{\ell}_i(x, y) = g_{i,1}(x) g_{i,2}(y) \). We equivalently generate polynomials \( \tilde{\ell}_j(y, z) = g_{j,1}(y) g_{j,2}(z), m(s - r) + 1 \leq j \leq m(s - r) + n - 1 \) and \( \tilde{\ell}_k(x, z) = g_{k,1}(x) g_{k,2}(z), m(s - r) + n \leq k \leq n - 1 + (m + r)(s - r) \).

We deduce straightforwardly that \( \tilde{\ell} \) satisfies the above properties. Indeed, the set \( \mathbb{Z}(\tilde{\ell}) \) can be computed by solving systems of linear equations. When the affine polynomials \( g_{i,1}, g_{i,2}, 1 \leq i \leq n - 1 + (m + r)(s - r) \), are chosen generically, the number of linear systems to be solved equals the multilinear Bézout bound \( \delta(m, s, n, r) \), computed in Proposition 4. Hence the complexity of solving the starting system is in \( \mathcal{O}((n + (m + r)(s - r))^{\omega} \delta(m, s, n, r)) \), where \( \omega \) is the exponent of linear algebra.

In [35], the authors build a homotopy path between \( \ell \) and \( \tilde{\ell} \), such as

\[
 t \ell + (1 - t) \tilde{\ell} \subset \mathbb{Q}[x, y, z, t]
\]

where \( t \) is a new variable. The system (4) defines a 1-dimensional algebraic set, that is a curve. We deduce by [35, Theorem 1, Corollary 2 and Proposition 5] that, if the solutions of \( \tilde{\ell} \) are known, one can compute a rational parametrization of the solution set of system (4) within \( \mathcal{O}((\tilde{n} N \log Q + \tilde{n}^3)dd') \) arithmetic operations over \( \mathbb{Q} \), where:
- \( \tilde{n} \) is the number of variables in \( \ell \);
- \( N = m(s - r) \# \Delta_1 + (n - 1) \# \Delta_2 + r(s - r) \# \Delta_3 \) (\( \# \) is the cardinality);
- \( Q = \max_{i=1,2,3} \{ \|q\| : q \in \Delta_i \} \);
- \( d \) is the number of isolated solutions of \( \ell \);
- \( d' \) is the degree of the curve \( \mathbb{Z}(t \ell + (1 - t) \tilde{\ell}) \);

**Lemma 2** Let \( \mathcal{F}_{m,s,n,r} \) and \( \delta(m, s, n, r) \) be the set and the bound defined in Proposition 4, and suppose \( \mathcal{F}_{m,s,n,r} \neq \emptyset \). Then the degree of \( \mathbb{Z}(t \ell + (1 - t) \tilde{\ell}) \) is in

\[
 \mathcal{O} \left( (n + (m + r)(s - r)) \min\{n, m(s - r)\} \delta(m, s, n, r) \right).
\]
Proof We exploit the multilinear structure of $t \ell + (1-t) \tilde{\ell}$. By the Multilinear Bézout theorem, $\deg \mathcal{Z}(t \ell + (1-t) \tilde{\ell})$ is bounded by the sum of the coefficients of

$$q(s_x, s_y, s_z, s_t) = (s_x + s_y + s_t)^m(s-r)(s_y + s_z + s_t)^n-1(s_x + s_z + s_t)^r(s-r)$$

modulo $I = (s_{x}^{n+1}, s_{y}^{r(s-r)+1}, s_{z}^{m(s-r)}, s_{t}^{2}) \subset \mathbb{Z}[s_{x}, s_{y}, s_{z}, s_{t}]$. It is easy to check that $q = q_1 + s_t(q_2 + q_3 + q_4) + g$ with $s_t^2$ that divides $g$ and

$$q_1 = (s_x + s_y)^{m(s-r)}(s_y + s_z)^{n-1}(s_x + s_z)^r(s-r)$$
$$q_2 = m(s-r)(s_x + s_y)^{m(s-r)-1}(s_y + s_z)^{n-1}(s_x + s_z)^r(s-r)$$
$$q_3 = (n - 1)(s_x + s_y)^{m(s-r)}(s_y + s_z)^{n-2}(s_x + s_z)^r(s-r)$$
$$q_4 = r(s-r)(s_x + s_y)^{m(s-r)}(s_y + s_z)^{n-1}(s_x + s_z)^r(s-r)-1,$$

and hence that $q \equiv q_1 + s_t(q_2 + q_3 + q_4) \mod I$. Below, we bound the contribution of $q_i, i = 1 \ldots 4$. The stated bound is given by the sum of the contributions and follows straightforwardly.

Contributions of $q_1$ The contribution of $q_1$ is the sum of its coefficients modulo the ideal $I' = (s_{x}^{n+1}, s_{y}^{r(s-r)+1}, s_{z}^{m(s-r)})$. This has been computed in Proposition 4, and coincides with $\delta(m, s, n, r).

The contribution of $q_2$ Write $q_2 = m(s-r)\tilde{q}_2$ with $\tilde{q}_2 \in \mathbb{Z}[s_{x}, s_{y}, s_{z}]$. Consequently the contribution is given by the sum of the coefficients of $\tilde{q}_2$, modulo $I'$, multiplied by $m(s-r)$. Now, observe that $\deg \tilde{q}_2 = n - 2 + (m + r)(s-r)$ and that maxima powers admissible modulo $I'$ are $s_{x}^{n}, s_{y}^{r(s-r)}, s_{z}^{m(s-r)-1}$. Hence, three configurations give a contribution.

(A) The coefficient of the monomial $s_{x}^{n-1} s_{y}^{r(s-r)} s_{z}^{m(s-r)-1}$ in $\tilde{q}_2$, that is

$$\Sigma_A = \sum_{k=0}^{r(s-r)} \binom{m(s-r)-1}{n-1\ n-1\ k\ k} \binom{n-1}{k-1+(m-r)(s-r)} \binom{r(s-r)}{k}.$$  

(B) The coefficient of the monomial $s_{x}^{n} s_{y}^{r(s-r)-1} s_{z}^{m(s-r)-1}$ in $\tilde{q}_2$, that is

$$\Sigma_B = \sum_{k=0}^{r(s-r)} \binom{m(s-r)-1}{n-1\ n-1\ k\ k} \binom{n-1}{k-1+(m-r)(s-r)} \binom{r(s-r)}{k}.$$  

(C) The coefficient of the monomial $s_{x}^{n} s_{y}^{r(s-r)} s_{z}^{m(s-r)-2}$ in $\tilde{q}_2$, that is

$$\Sigma_C = \sum_{k=0}^{r(s-r)} \binom{m(s-r)-1}{n-1\ n-1\ k\ k} \binom{n-1}{k-2+(m-r)(s-r)} \binom{r(s-r)}{k}.$$  

So the contribution of $q_2$ equals $m(s-r)(\Sigma_A + \Sigma_B + \Sigma_C).$  

\[ Springer \]
One easily deduces that $\Sigma_A \leq \delta(m, s, n, r)$ and $\Sigma_B \leq \delta(m, s, n, r)$. Remember that we suppose $F_{m,s,n,r} \neq \emptyset$, that is $\delta(m, s, n, r) > 0$. We claim that $\Sigma_C \leq (1 + \min(n, m(s - r))) \delta(m, s, n, r)$. Consequently, we conclude that the contribution of $q_2$ is $m(s - r) (\Sigma_A + \Sigma_B + \Sigma_C) \in \mathcal{O}(m(s - r) \min(n, m(s - r)) \delta(m, s, n, r))$.

Let us prove this claim. First, denote by

$$
\chi_1 = \max\{0, n - m(s - r)\} \quad \chi_2 = \min\{r(s - r), n - (m - r)(s - r)\}
$$

$$
\alpha_1 = \max\{0, n + 1 - m(s - r)\} \quad \alpha_2 = \min\{r(s - r), n + 1 - (m - r)(s - r)\}
$$

the indices such that $\delta(m, s, n, r)$ sums over $\chi_1 \leq k \leq \chi_2$ and $\Sigma_C$ over $\alpha_1 \leq k \leq \alpha_2$. Remark that $\chi_1 \leq \alpha_1$ and $\chi_2 \leq \alpha_2$. Finally, denote by $\varphi(k)$ the $k$–th term in the sum defining $\Sigma_C$, and by $\gamma(k)$ the $k$–th term in the sum defining $\delta(m, s, n, r)$.

For all indices $k$ admissible for both $\delta(m, s, n, r)$ and $\Sigma_C$, that is for $\alpha_1 \leq k \leq \chi_2$, one gets, by basic properties of binomial coefficients (we apply \(\binom{a}{b} = \frac{b}{a-b+1} \binom{a}{b-1}\)), that

$$
\varphi(k) = \Psi(k) \gamma(k) \quad \text{with} \quad \Psi(k) = \frac{k - 1 + (m - r)(s - r)}{n - k - (m - r)(s - r) - 1}.
$$

When $k$ runs over all admissible indices, the rational function $\Psi(k)$ is non-decreasing monotone, and its maximum is attained in $\Psi(\chi_2)$ and is bounded by $\min(n, m(s - r))$.

Three possible cases can hold:

1. $\alpha_1 = 0$. Hence $\chi_1 = 0$, $\alpha_2 = r(s - r)$ and $\chi_2 = r(s - r)$. We deduce straight-forwardly from the above discussion that $\Sigma_C \leq \min(n, m(s - r)) \delta(m, s, n, r)$;
2. $\alpha_1 = n - m(s - r) + 1$ and $\chi_1 = n - m(s - r)$. We deduce that $\chi_2 = \alpha_2 = r(s - r)$ and that $\Sigma_C = \sum_{k=\alpha_1}^{\chi_2} \varphi(k) \leq \varphi(\alpha_1) + \min(n, m(s - r)) \delta(m, s, n, r) \leq (1 + \min(n, m(s - r))) \delta(m, s, n, r)$;
3. $\chi_1 = 0$ and $\alpha_1 = n - m(s - r) + 1$. Hence, we deduce the chain of inequalities $0 \leq n - m(s - r) + 1 \leq 1$. Hence, either this case coincides with case 2 (if $n = m(s - r)$) or we deduce that $n = m(s - r) - 1$, and we fall into case 1.

*The contribution of $q_3$ and $q_4$* Following exactly the same path as in the case of $q_2$, one respectively deduces that the contribution of $q_3$ is in $\mathcal{O}(n \min(n, m(s - r)) \delta(m, s, n, r))$ and that of $q_4$ is in $\mathcal{O}(r(s - r) \min(n, m(s - r)) \delta(m, s, n, r))$. $\Box$

We can now state the main result of this paragraph. We denote by $\mathcal{O}(D) = \mathcal{O}(D \log^d(D))$, that is linear complexity up to logarithmic factors.

**Theorem 2** Let $n > (m - r)(s - r)$. Let $A$ be an $n$–variate $m \times s$ linear matrix, $0 \leq r < s \leq m$ and let $U$ be the matrix chosen in step 1 of $\text{LowRank}$. Let $\delta = \delta(m, s, n, r)$ be the bound defined in Proposition 4. Then, $\text{RatPar}$ returns a rational parametrization within

\[
\mathcal{O}\left((n + (m + r)(s - r))^{6 \delta^2}\right)
\]

arithmetic operations.
Proof Following the notation introduced above, \( \tilde{n} = n - 1 + (m + r)(s - r) \). The bound for \( d \) is \( \delta \) and is given in Proposition 4 and a bound for \( d' \) is given in Lemma 2, and is in \( O(\tilde{n}^2 \delta) \). Moreover, \( N \in O(mnr(s - r)^2) \), and hence \( N \in O(\tilde{n}^3) \). The proof follows from [35, Proposition 5], since the maximum diameter of \( \Delta_1, \Delta_2, \Delta_3 \) is bounded above by \( \tilde{n} \), that is \( Q \leq \tilde{n} \) in the notation above. \( \square \)

3.3.3 Complexity of subroutines

For these complexity bounds, we refer to those given in [29, Lemmas 3 and 4] (see [34, Lemma J.3, J.5 and J.6] for a unified treatment of these algorithms) from which they are obtained straightforwardly.

Proposition 5 Let \( \delta(m, s, n, r) \) be the bound defined in Proposition 4. At the first recursion step of LowRankRec, the following holds:

- the complexity of Project is in \( O((n + (m + r)(s - r))^2 \delta(m, s, n, r))^2) \);
- the complexity of Lift is in \( O((n + (m + r)(s - r)) \delta(m, s, n, r))^2) \);
- the complexity of Union is in \( O((n + (m + r)(s - r))) \delta(m, s, n, r))^2) \).

4 Regularity of the incidence variety

The goal of this section is to prove Proposition 1. We introduce the notation \( \mathcal{B} \) representing a \( m \times s \) matrix, with \( m \geq s \), whose entries are indeterminates \( b = (b_{i,j}) \); similarly, the \( (s - r) \times s \) matrix \( \mathcal{U} \) whose entries are indeterminates \( u = (u_{i,j}) \), and we use the notation \( a = (a_{i,j}) \) to denote generic entries of the linear matrix \( A(x) \).

All the projection maps will be denoted by \( \pi \) when source and target spaces are clear from the context.

Proof (of Assertion 1 of Proposition 1) By [30, Prop 3.1], there exists a non-empty Zariski open set \( \mathcal{A}_1 \subset \mathbb{C}^{m \times s} \) such that for \( A \in \mathcal{A}_1 \) and all \( 0 \leq p \leq r, D_p \subset \mathbb{C}^n \) is either empty or \( n - (m - p)(s - p) \)-equidimensional, and \( \text{sing}(D_p) = D_{p-1} \). It remains to prove that there exists a non-empty Zariski open set \( \mathcal{A}_2'' \subset \mathbb{C}^{m \times s} \) such that, for \( A \in \mathcal{A}_2'' \) and all \( 0 \leq p \leq r \), the ideal generated by the \( (p + 1, p + 1) \) minors of \( A(x) \) is radical. Defining \( \mathcal{A}_1 \) as the intersection of \( \mathcal{A}_1' \) and \( \mathcal{A}_2'' \) leads to the following conclusion: there exists a non-empty Zariski open set \( \mathcal{A}_1 \subset \mathbb{C}^{m \times s} \) such that for \( A \in \mathcal{A}_1, A \) satisfies \( G_1 \).

By [16, Prop. 12.2], the ideal \( I \subset \mathbb{C}[b] \) generated by the \( (p + 1, p + 1) \) minors of \( \mathcal{B} \) is radical and \( \mathcal{Z}(I) \) is prime, of co-dimension \( (m - p)(s - p) \). We deduce that the Jacobian matrix of the set of generators of \( I \) has full rank \( (m - p)(s - p) \) when instantiated at a smooth point (having rank exactly \( p \)) of \( \mathcal{Z}(I) \). A simple dimension count shows that one can apply Bertini’s theorem [16, Theorem 17.16] when adding generic linear forms \( \{L_{i,j} = b_{i,j} - \sum \alpha_{i,i,j} x_i \} \) (where we put by convention \( x_0 = 1 \)) to the ideal \( I' = I + \langle L_{i,j} \rangle \) to deduce that one obtains a prime ideal. Using the Jacobian crierion [8, Theorem 16.19], we deduce that the rank of the Jacobian matrix of \( I' \) equals the rank of the Jacobian matrix of \( I \) (which is \( (m - p)(s - p) \)) plus \( ms = \#\{L_{i,j}\} \), since an identity submatrix will appear in correspondence with the
derivatives with respect to $a_{0,i,j}$. We consider now the restriction to $\mathcal{Z}(I')$ of the projection $\pi(b, a, x) = a$ eliminating variables $b, x$. By Sard’s theorem the singular values of $\pi$ lie in a Zariski-closed set of the image of $\pi$. We deduce that there exists a non-empty and Zariski-open set $\mathcal{A}_1'' \subset (\mathbb{C}^{m \times s})^{n+1}$ such that if $A \in \mathcal{A}_1''$, the ideal $I'' = I + (b_i,j - \sum \alpha_{t,i,j}x_\ell) \subset \mathbb{C}[b, x]$ is radical. Thus the intersection $I'' \cap \mathbb{C}[x]$ (that eliminates the variables $b$) still yields a radical ideal in $\mathbb{C}[x]$. Finally, note that this elimination ideal coincides with the ideal generated by the $(p+1, p+1)$ minors of $A(x)$ (indeed, the elimination is performed by substituting the generic entries of $\mathcal{B}$ with the entries of $A$). We conclude that if $A \in \mathcal{A}_1''$, the ideal of $(p+1, p+1)$ minors of $A(x)$ is radical.

**Proof (of Assertion 2 of Proposition 1)** We prove now that there exists a non-empty Zariski open set $\mathcal{A}_2 \subset (\mathbb{C}^{m \times s})^{n+1}$ such that the following holds. For $A \in \mathcal{A}_2$, there exists a non-empty Zariski open set $\mathcal{U}_A \subset \mathbb{C}^{(s-r) \times s}$ such that for $U \in \mathcal{U}_A$, Assertions (2a) and (2b) of Proposition 1 are satisfied. Taking $\mathcal{A}$ as the intersection of $\mathcal{A}_1$ and $\mathcal{A}_2$ will end the proof.

Let $\mathfrak{F} \subset \mathbb{C}[a, b, u, y]$ denote the vector of polynomials consisting of: the generic linear forms $\{L_{i,j} = b_{i,j} - \sum \alpha_{t,i,j}x_\ell\}$ as in the proof of Assertion 1, and the entries of $\mathcal{B}Y$ and $\mathfrak{F}Y - \mathbb{I}_{s-r}$. First, remark that all solutions $(A, B, U, y)$ of $\mathfrak{F} = 0$ satisfy $\text{rank } U = \text{rank } Y = s - r$ by the classical condition $\text{rank } MN \leq \min\{\text{rank } M, \text{rank } N\}$. The Jacobian matrix $D\mathfrak{F}$ of $\mathfrak{F}$ has full rank when restricted to $\mathcal{Z}(F)$, since we can construct a non-singular block sub-matrix of $D\mathfrak{F}$ made by the following blocks:

- the derivatives of forms $L_{i,j}$ with respect to $a_{0,i,j}$, a $ms \times ms$ block equal to $\mathbb{I}_{ms}$;
- the derivatives of $\mathcal{B}Y$ with respect to $b$, a $m(s - r) \times ms$ block-diagonal matrix with $m$ blocks equal to $Y^T$;
- the derivatives of $\mathfrak{F}Y - \mathbb{I}_{s-r}$ with respect to $u$, a $(s-r)^2 \times s(s-r)$ block-diagonal matrix with $s-r$ blocks equal to $Y^T$.

By the Jacobian criterion, $\mathfrak{F}$ satisfies $G_2$, hence $\mathcal{Z}(\mathfrak{F})$ is smooth and equidimensional. We consider the restriction of $\pi(a, b, u, x, y) = a$ to $\mathcal{Z}(\mathfrak{F})$. Applying Sard’s theorem, we obtain a non-empty and Zariski-open set $\mathcal{A}_2$ such that, for $A \in \mathcal{A}_2$, the ideal generated by $\mathfrak{F}'$ (obtained from $\mathfrak{F}$ by instantiating $a$ to the entries of $A$) satisfies $G_2$.

Let us fix $A \in \mathcal{A}_2$. Considering the new projection $\pi : (b, u, x, y) \to u$ restricted to $\mathcal{Z}(\mathfrak{F}')$, and applying Sard’s theorem implies that there exists $\mathcal{U}_A' \subset \mathbb{C}^{(s-r) \times s}$, non-empty and Zariski-open, such that if $U \in \mathcal{U}_A'$, instantiating $u$ to $U$ yields a radical ideal $I \subset \mathbb{C}[b, x, y]$, with the Jacobian matrix of $I$ full rank at every solution. Now, $f(A, U)$ with $A \in \mathcal{A}_2$ and $U \in \mathcal{U}_A'$ generates the elimination ideal $I \cap \mathbb{C}[x, y]$, hence it is still radical. Since $f(A, U)$ is obtained from $I$ by instantiating $b$ to the entries of $A$, the Jacobian matrix $Df(A, U)$ is a submatrix of the Jacobian matrix of a set of generators of $I$, which has full rank. Since the polynomials in $f(A, U)$ do not depend on variables $b$, it is easily seen that $Df(A, U)$ has full rank too. Hence we deduce that, for $A \in \mathcal{A}_2$ and $U \in \mathcal{U}_A'$, $f(A, U)$ satisfies $G_2$, as claimed.
It remains to prove that there exists a non-empty Zariski open set $\mathcal{U}_A'' \subset \mathbb{C}^{(s-r) \times s}$ such that for $U \in \mathcal{U}_A''$, assertion (2b) holds. Finally taking the intersection of $\mathcal{U}_A'$ and $\mathcal{U}_A''$ to define $\mathcal{U}_A$ ends the proof. Let $A \in \mathcal{A}_2$, $p \leq r$ and let $Z$ be one of the (finitely-many) irreducible components of $\mathcal{D}_p$, and let $d$ be its dimension (all such components have the same dimension, since we proved that $\mathcal{D}_p$ is empty or equidimensional). Intersecting $Z$ with $d$ general hyperplanes, we get a finite number of smooth points in $Z$. For every such point $x \in Z$, we first build a Zariski-open set $\mathcal{U}_{A,p,Z,x} \subset \mathbb{C}^{(s-r) \times s}$, as follows.

The rank of $A(x)$ is $p$ since $x$ is a smooth point of $Z$ (because $Z$ is an irreducible component of the Zariski closure of the set of points at which $A$ has rank $p$). The polynomial system $y \mapsto f(A, U)$ is linear in $y$. Since $\text{rank } A(x) = p$, the condition $A(x)Y(y) = 0$ defines a linear space $V = \{Y(y) \in \mathbb{C}^{s \times (s-r)} : A(x)Y(y) = 0\}$ of dimension $(s-p)(s-r)$. Since $p \leq r$, remark that $(s-r)^2 \leq (s-p)(s-r)$. For a generic $U \in \mathbb{C}^{(s-r) \times s}$, the $(s-r)^2$ affine equations $UY(y) - \mathbb{I}_{s-r} = 0$ define a linear space intersecting $V$. Hence there exists a non-empty Zariski open set $\mathcal{U}_p'' \subset \mathbb{C}^{(s-r) \times s}$ such that, if $U \in \mathcal{U}_A''$, the linear system $A(x)Y(y) = 0$, $UY(y) - \mathbb{I}_{s-r} = 0$ has at least one solution.

One concludes by defining

$$\mathcal{U}_A'' = \bigcap_{p \leq r} \bigcap_{Z \subset \mathcal{D}_p \cap \mathbb{R}^n} \bigcap_{x \in Z} \mathcal{U}_{A,p,Z,x}$$

which is non-empty and Zariski open by the finiteness of the number of irreducible components of $\mathcal{D}_p \cap \mathbb{R}^n$ and of the set of points $x$ in $Z$. \hfill \Box

**Proof (of Assertion 3 of Proposition 1)** By Sard’s Theorem, the critical values of the projection $\pi_w(x) = w^T x$ are finitely many, hence the regular values of this map define a non-empty Zariski open set $\mathcal{T}_A \subset \mathbb{C}$. For $w$ as in the hypothesis, and $t \in \mathcal{T}_A$, we denote by $\mathcal{D}_p' = \{x \in \mathbb{C}^{n-1} : \text{rank } A|_{w,t}^\prime(x) \leq p\}$. As in the proof of Assertion 1, since $A \in \mathcal{A}$, we deduce that if $t \in \mathcal{T}_A$, then the ideal $I = \{(p+1) \times (p+1) \text{ minors of } A\}$, $\pi_w(x) - t$ is still radical and $\mathcal{Z}(I)$ has co-dimension $(m-p)(s-p) + 1$. Remark that $I \cap \mathbb{R}[x_2, \ldots, x_n]$ is generated by the $(p+1) \times (p+1)$ minors of $A|_{w,t}$, it is still radical and $\mathcal{D}_p' = \mathcal{Z}(I \cap \mathbb{R}[x_2, \ldots, x_n])$ has co-dimension $(m-p)(s-p)$ in $\mathbb{C}^{n-1}$. Moreover, always for $t \in \mathcal{T}_A$, a point in $\mathcal{D}_p$ is regular if and only if it is regular in $\mathcal{D}_p \cap \mathcal{Z}(\pi_w(x) - t)$. Hence we deduce that for $t \in \mathcal{T}_A$, the matrix $A|_{w,t}'$ satisfies Property $G_1$, as claimed. \hfill \Box

**5 Dimension of Lagrange systems**

The goal of this section is to prove Proposition 2. We first need to give a local description of the incidence variety $\mathcal{V}$, and of the solution set $\mathcal{Z}(A, U, w)$ of the Lagrange system $\ell(A, U, w)$.
5.1 Local description of the incidence variety

As in our previous work [18], we need to compute equations for the incidence sets lifting the determinantal varieties. Here we generalize the equations in [18, Sect. 4] to the case of low-rank rectangular matrices.

Let \( A = A_0 + x_1A_1 + \cdots + x_nA_n \) be a \( n \)-variate \( m \times s \) linear matrix with coefficients in \( \mathbb{Q} \), and let \( r \leq s - 1 \). From now on, for \( g \in \mathbb{Q}[x] \), we denote by \( \mathbb{Q}[x]_g \) the localization of the ring \( \mathbb{Q}[x] \) at \( \langle g \rangle \), see [8]. We recall that the polynomial system defining \( V_r \) is given by \( f(A, U) \), which contains the entries of \( A(x)Y(y) \) and \( UY(y) - I_{s-r} \). For \( p \leq r \), let \( N \) be the upper-left \( p \times p \) submatrix of \( A \), so that

\[
A = \begin{pmatrix} N & Q \\ P^T & R \end{pmatrix}
\]

with \( P \in \mathbb{Q}[x]^{p \times (m-p)} \), \( Q \in \mathbb{Q}[x]^{p \times (s-p)} \) and \( R \in \mathbb{Q}[x]^{(m-p) \times (s-p)} \). The next Lemma computes the equations of \( V_r \) in the local ring \( \mathbb{Q}[x, y]_{\det N} \).

**Lemma 3** Let \( A, N, Q, P, R \) be as above, and \( U \) be any full-rank matrix. Then there exist \( \{q_{i,j}\}_{1 \leq i \leq p, 1 \leq j \leq s-p}, \{q'_{i,j}\}_{1 \leq i \leq m-p, 1 \leq j \leq s-p} \subset \mathbb{Q}[x]_{\det N} \) such that the constructible set \( \mathcal{V}_r \cap \{(x, y) : \det N(x) \neq 0\} \) is defined by the equations

\[
\begin{align*}
y_{i,j} - q_{i,j}y_{p+1,j} - \cdots - q_{i,s-p}y_{s,j} &= 0 \quad 1 \leq i \leq p, 1 \leq j \leq s-r \\
q'_{i,j}y_{p+1,j} + \cdots + q'_{i,s-p}y_{s,j} &= 0 \quad 1 \leq i \leq m-p, 1 \leq j \leq s-r \\
UY(y) - I_{s-r} &= 0.
\end{align*}
\]

**Proof** We denote by \( Y^{(1)} \) and \( Y^{(2)} \) the submatrices of \( Y(y) \) containing respectively the first \( p \) rows and the last \( s - p \) rows. We also use the block-division of \( A \) as in (5). We claim that in \( \mathbb{Q}[x, y]_{\det N} \) the \( m(s - r) \) equations \( A(x)Y(y) = 0 \) are equivalent to the \( m(s - r) \) equations:

\[
\begin{pmatrix} \mathbb{I}_p Y^{(1)} + N^{-1}Q Y^{(2)} \\ \Sigma(N) Y^{(2)} \end{pmatrix} = 0
\]

where \( \Sigma(N) = R - P^T N^{-1}Q \) is the Schur complement of \( N \) in \( A \). Renaming the entries of \( N^{-1}Q \) and \( \Sigma(N) \) concludes the proof. To prove the claim, remark that since \( \det N \neq 0 \), \( A(x)Y(y) = 0 \) if and only if

\[
\begin{pmatrix} \mathbb{I}_p & 0 \\ -P^T & \mathbb{I}_{m-p} \end{pmatrix} \begin{pmatrix} N^{-1} & 0 \\ 0 & \mathbb{I}_{m-p} \end{pmatrix} \begin{pmatrix} N & Q \\ P^T & R \end{pmatrix} Y(y) = 0.
\]

\[\square\]
5.2 The rank at a critical point

Given $A$, $N$, $P$, $Q$, $R$, $\Sigma(N)$ as above, let

$$\tilde{A} = \begin{pmatrix} I_p & N^{-1}Q \\ 0 & \Sigma(N) \end{pmatrix}.$$ 

Lemma 3 implies that the equations of $V_r$ in the open set $\{(x, y) : \det N \neq 0\}$ can be rewritten as $\tilde{A}(x)Y(y) = 0$ and $UY(y) - I_{s-r} = 0$: the polynomial entries of the above expressions are elements of the local ring $\mathbb{Q}[x]_{\det N}$. Now, from the first group of relations $\tilde{A}(x)Y(y) = 0$ one eliminates variables $\{y_{i,j}\}_{1 \leq i \leq p, 1 \leq j \leq s-r}$, which can be expressed as polynomial functions of $x$ and $\{y_{i,j}\}_{p+1 \leq i \leq s, 1 \leq j \leq s-r}$. That is, using the notations introduced in Lemma 3, we can express the entries of $Y^{(1)}$ as polynomials in $x$ and in the entries of $Y^{(2)}$.

Now, consider the relations $UY(y) - I_{s-r} = 0$ where the entries of $Y^{(1)}$ have been eliminated. This is a linear system in the entries of $Y^{(2)}$ with coefficients in $\mathbb{Q}[x]_{\det N}$. Since $I_{s-r}$ is full-rank, then $U$ is full-rank and hence $UY(y) - I_{s-r} = 0$ consists of $(s-r)^2$ independent relations. Finally one can eliminate $(s-r)^2$ among the $(s-p)(s-r)$ entries of $Y^{(2)}$ (suppose the first $(s-r)$ rows) and re-write $\Sigma(N)Y^{(2)} = 0$ as $(m-p)(s-r)$ relations in $x$ and in the last $(r-p)(s-r)$ entries of $Y^{(2)}$.

Let us call $F$ this polynomial system, and consider a vector of Lagrange multipliers $z = (z_1, \ldots, z_{(m-p)(s-r)})$ and the polynomial system

$$(g_1, \ldots, g_n) = z^TD_xF - (w_1, \ldots, w_n).$$

The solutions to the above polynomial system contain the critical points of the projection $\pi_w : \mathbb{C}^n \rightarrow \mathbb{C}, \pi_w(x) = w^Tx := w_1x_1 + \cdots + w_px_p$, restricted to $V_r \cap \{(x, y) : \det N \neq 0\}$. The next Lemma shows that, when $w$ is generic in $\mathbb{C}^n$, the solutions to the Lagrange systems project on points of $D_r$ with rank exactly $r$ (namely in $D_r \setminus D_{r-1}$).

**Lemma 4** Let $A, U$ be as above and suppose that $A$ satisfies $G_1$. Let $p \leq r - 1$ and let $g = (g_1, \ldots, g_n)$ be the polynomial system defined above. Then there exists a non-empty Zariski open set $\mathcal{W}_{A,U} \subset \mathbb{C}^n$ such that the following holds: if $w \in \mathcal{W}_{A,U}$ then the system $g(x, y, z, w) = 0$ has no solutions in $x, y, z$.

**Proof** Let $C \subset \mathbb{C}^{2n+(r-p)(s-r)+(m-p)(s-r)}$ be the constructible set defined by $g = 0$ and by $\det N \neq 0$ and rank $A(x) = p$, where the coordinates of $w$ are variables. Let $\overline{C}$ be the Zariski closure of $C$. Let $\pi_x : (x, y, z, w) \rightarrow x$ be the restriction of the projection on the first $n$ variables to $C$. The image $\pi_x(C)$ is dense in $D_p$. Hence, since $A$ satisfies $G_1$, it has dimension at most $n - (m-p)(s-p)$. The fiber of $\pi_x$ given a generic point $x \in D_p$ is the graph of the polynomial function $w = z^TD_xF$, and so it has codimension $n$ and dimension $(r-p)(s-r) + (m-p)(s-r) = (m+r-2p)(s-r)$. By the Theorem of the Dimension of Fibers [36, Sect. 6.3, Th. 7] one deduces that the dimension of $C$ (and of $\overline{C}$) is at most

& Springer
\[ n - (m - p)(s - p) + (m + r - 2p)(s - r) = n - (r - p)(m - p + r - s) < n. \]

We deduce that the projection of \( C \) onto the space \( \mathbb{C}^n \) of \( w \) is a constructible set of dimension at most \( n - 1 \), and it is included in a hypersurface \( H \subset \mathbb{C}^n \). Defining \( \mathcal{W}_{A,U} = \mathbb{C}^n \setminus H \) ends the proof. \( \square \)

## 5.3 Local description of the Lagrange system

We consider the incidence variety \( \mathcal{V}_r = \mathcal{V}_r(A, U) \) and the restriction of the projection \( \Pi_w(x, y) = w^T x \) to \( \mathcal{V}_r \), with \( w \in \mathbb{C}^n \). Under the hypothesis that \( A \) satisfies \( G_1 \) and that \( f(A, U) \) satisfies \( G_2 \), the set \( \mathcal{V}_r \) is either empty or smooth and equidimensional of codimension \( c := (m + s - r)(s - r) \). The set of critical points of the restriction of \( \Pi_w \) to \( \mathcal{V}_r \) is the projection on the \((x, y)\)-space of the solutions of the Lagrange system \( \ell(A, U, w) \):

\[
f(A, U) = 0, \quad (g, h) := z^T \begin{pmatrix} D_x f & D_y f \\ w^T & 0 \end{pmatrix} = 0, \tag{6}
\]

where \( z = (z_1, \ldots, z_c, 1) \). The polynomial system (6) consists of \( n + c + s(s - r) \) polynomials in \( n + c + s(s - r) \) variables. We show that it can be re-written in a local form when we consider the local description of the incidence variety \( \mathcal{V}_r \) as in Sect. 5.1.

We use the block-division of matrix \( A \) as in (5) with \( p = r \) and without loss of generality one can assume to work in the open set \( \det N \neq 0 \), with \( N \) the upper-left \( r \times r \) submatrix of \( A \). We deduce by Lemma 3 that the local equations of \( \mathcal{V}_r \) are

\[
Y^{(1)} = -N^{-1} Q Y^{(2)}, \quad \Sigma(N) Y^{(2)} = 0, \quad U^{(1)} Y^{(1)} + U^{(2)} Y^{(2)} = \mathbb{I}_{s-r},
\]

where \( Y^{(1)}, Y^{(2)} \) is the row-subdivision of the matrix \( Y(y) \) as in Lemma 3 and \( U^{(1)}, U^{(2)} \) is the corresponding column-subdivision of \( U \). From the first and third groups of equations one obtains that \( \mathbb{I}_{s-r} = U^{(1)} (-N^{-1} Q Y^{(2)}) + U^{(2)} Y^{(2)} = (-U^{(1)} N^{-1} Q + U^{(2)}) Y^{(2)}. \) Since \( \mathbb{I}_{s-r} \) is full-rank, then \( Y^{(2)} \) and \( -U^{(1)} N^{-1} Q + U^{(2)} \) are non-singular, and so:

- the second group of equations can be re-written as \( \Sigma(N) = 0 \);
- the third group of equations can be re-written as \( Y^{(2)} = (-U^{(1)} N^{-1} Q + U^{(2)})^{-1}. \)

The entries of \( \Sigma(N) \) in the local ring \( \mathbb{Q}[x]_{\det N} \) are exactly the \((m - r)(s - r)\) minors of \( A(x) \) obtained as determinants of the \((r + 1) \times (r + 1)\) submatrices of \( A(x) \) containing \( N \) (see for example the proof of [34, Proposition 3.2.7]). Since \( A \) satisfies \( G_1 \), the Jacobian \( D_y \Sigma(N) \) of the vector of entries of \( \Sigma(N) \) has full-rank at each point \( x \) such that \( \det A(x) = r \).

We call \( f' = (f'_1, \ldots, f'_{\ell}) \) the local equations represented by the entries of \( \Sigma(N) \), \( Y^{(1)} + N^{-1} Q Y^{(2)} \) and \( Y^{(2)} - (-U^{(1)} N^{-1} Q + U^{(2)})^{-1}. \) The Jacobian matrix of \( f' \) has the form

\[
Df' = (D_x f' \quad D_y f') = \begin{pmatrix}
D_x \Sigma(N)_{i,j} & 0_{(m-r)(s-r) \times s(s-r)} \\
\ast & \mathbb{I}_{r(s-r)} \\
\ast & 0_{(m-r)(s-r)}
\end{pmatrix}
\]
We consider the polynomials
\[
(g'_1, \ldots, g'_n, h'_1, \ldots, h'_{s(s-r)}) = (z_1, \ldots, z_c, 1) \left( \begin{array}{ccc} D_x f' & D_y f' \\ w_1 \ldots w_n & 0 \end{array} \right).
\]

Polynomials in \( h' = (h'_1, \ldots, h'_{s(s-r)}) \) give the relations \( z_i = 0 \), for \( i = (m-r)(s-r) + 1, \ldots, c \), and can be eliminated together with variables \( z_i, i = (m-r)(s-r) + 1, \ldots, c \). So the local equations of the Lagrange system (6) are:
\[
f' = 0, \quad g' = 0.
\]

This is a square system consisting of \( n + c \) polynomials in \( n + c \) variables.

### 5.4 Proof of Proposition 2

**Proof (of Assertion 1 of Proposition 2)** Let \( \mathcal{W}_{A,U} \subset \mathbb{C}^n \) be the set defined by Lemma 4, and \( w \in \mathcal{W}_{A,U} \). Then one has that all solutions \((x, y, z)\) to (6) (hence of the local version (7)) satisfy \( A(x) = r \). We deduce that there exists a \( r \times r \) submatrix \( N \) of \( A(x) \) such that \( \det N \neq 0 \). We prove below that there exists a non-empty Zariski open set \( \mathcal{W}_{N,A,U} \subset \mathbb{C}^n \) such that, for \( w \in \mathcal{W}_{N,A,U} \), the statement of Assertion 1 holds locally. Hence, to retrieve the global property, it is sufficient to define \( \mathcal{W}_{A,U} \) as the (finite) intersection of sets \( \mathcal{W}_{A,U} \cap \mathcal{W}_{N,A,U} \), where \( N \) varies in the collection of \( r \times r \) submatrices of \( A \).

We suppose without loss of generality that \( N \) is the upper-left \( r \times r \) submatrix of \( A \). Let \((f', g')\) be the local Lagrange system defined in (7). Consider the polynomial map
\[
\varphi : \mathbb{C}^{n+c} \times \mathbb{C}^n \longrightarrow \mathbb{C}^{n+c} \\
(x, y, z, w) \longmapsto (f', g')
\]
and, for a fixed \( w \in \mathbb{C}^n \), its section map
\[
\varphi_w : \mathbb{C}^{n+c} \longrightarrow \mathbb{C}^{n+c} \\
(x, y, z) \longmapsto \varphi(x, y, z, w).
\]

If \( \varphi^{-1}(0) = \emptyset \), then for all \( w \in \mathbb{C}^n \), \( \varphi_w^{-1}(0) = \emptyset \), and the claim is proved by taking \( \mathcal{W}_{N,A,U} = \mathcal{W}_{A,U} \) (see Lemma 4).

Suppose now that \( \varphi^{-1}(0) \neq \emptyset \) and let \((x, y, z, w) \in \varphi^{-1}(0) \). We claim that the Jacobian matrix of \( \varphi \) at \((x, y, z, w)\) has maximal rank. Hence, \( 0 \) is a regular value for \( \varphi \) and by Thom’s Weak Transversality Theorem [34, Proposition B.3] there exist a non-empty Zariski open set \( \mathcal{W}_{N,A,U} \subset \mathbb{C}^n \) such that for \( w \in \mathcal{W}_{N,A,U}, 0 \) is a regular value of \( \varphi_w \). This implies that, by the Jacobian criterion, the set \( \mathcal{Z}(A, U, w) \cap \{(x, y, z) : \det N(x) \neq 0\} \) is empty or zero-dimensional. We prove below this claim by exhibiting a non-singular submatrix of \( D\varphi \).

We remark that, since \( f(A, U) \) satisfies \( S_2 \), the Jacobian matrix \( Df' \) has maximal rank at \((x, y)\) and consider the submatrix of \( D\varphi \) obtained by isolating:

\( \mathbb{C} \) Springer
– a non-singular maximal submatrix of $Df'$;
– the derivatives of $g'_1, \ldots, g'_n$ with respect to $w_1, \ldots, w_n$, giving the identity block $I_n$.

The previous blocks define a submatrix of $D\varphi(x, y, z, w)$ of size $(n + c) \times (n + c)$ whose determinant does not vanish at $(x, y, z, w)$. □

**Proof (of Assertion 2 of Proposition 2)** Let $w \in \mathbb{C}^n$, and let $(x, y) \in \text{crit}(\Pi_w, \mathcal{V}_r)$. Since $A \in \mathcal{A}$, $f(A, U)$ satisfies $G_2$ and $\mathcal{V}_r(A, U)$ is smooth and equidimensional. Hence $(x, y) \in \text{reg}(\mathcal{V}_r(A, U)) = \mathcal{V}_r(A, U)$. In particular $Df(x, y)$ has full rank. Since $(x, y) \in \text{crit}(\Pi_w, \mathcal{V}_r)$, the extended Jacobian matrix $D(f, \Pi_w)$ has a rank defect. Hence, there exists $z = (z_1, \ldots, z_{c+1}) \neq 0$ with $z^T D(f, \Pi_w)(x, y) = 0$.

If $z_{c+1} = 0$, then $0 = z^T D(f, \Pi_w) = (z_1, \ldots, z_c) Df(x, y)$, which is a contradiction since $Df(x, y)$ has full rank. Then we can assume $z_{c+1} = 1$, and hence that $(x, y, z) \in \mathcal{Z}(A, U, w)$. We conclude that $\text{crit}(\Pi_w, \mathcal{V}_r)$ is contained in the projection of $\mathcal{Z}(A, U, w)$ on $(x, y)$, as claimed. □

## 6 Closure properties

The goal of this section is to prove Proposition 3. We use notation of [18, Sect. 5], which we recall below.

**Notations** For $M \in \text{GL}_n(\mathbb{C})$, and $\mathcal{Z} \subset \mathbb{C}^n$ any set, we define

$$M^{-1} \mathcal{Z} := \{ x \in \mathbb{C}^n : Mx \in \mathcal{Z} \} = \{ M^{-1} x : x \in \mathcal{Z} \}.$$  

Remark that, if $w \neq 0$ and if $M \in \text{GL}_n(\mathbb{C})$ with $w = M^{-1}(1)$ (the first row of $M^{-1}$), then

$$\pi_1(M^{-1} \mathcal{Z}) = M^{-1}(1) \mathcal{Z} = \pi_{M^{-1}}(\mathcal{Z}) = \pi_w(\mathcal{Z}).$$  

Let $\mathcal{Z} \subset \mathbb{C}^n$ be an algebraic variety of dimension $d$. The $i$–equidimensional component of $\mathcal{Z}$ is denoted by $\Omega_i(\mathcal{Z})$, $i = 0, \ldots, d$. We denote by $\mathcal{J}(\mathcal{Z})$ the union of the following sets:

– $\Omega_0(\mathcal{Z}) \cup \cdots \cup \Omega_{d-1}(\mathcal{Z})$
– the set $\text{sing}(\Omega_d(\mathcal{Z}))$ of singular points of $\Omega_d(\mathcal{Z})$

and by $\mathcal{C}(\pi_i, \mathcal{Z})$ the Zariski closure of the union of the following sets:

– $\Omega_0(\mathcal{Z}) \cup \cdots \cup \Omega_{i-1}(\mathcal{Z})$;
– the union for $k \geq i$ of the sets $\text{crit}(\pi_i, \text{reg}(\Omega_k(\mathcal{Z})))$ of critical points of the restriction of $\pi_i$ to the regular locus of $\Omega_k(\mathcal{Z})$.

For $M \in \text{GL}_n(\mathbb{C})$ we recursively define the collection $\{ \mathcal{O}_i(M^{-1} \mathcal{Z}) \}_{0 \leq i \leq d}$ as follows:

– $\mathcal{O}_d(M^{-1} \mathcal{Z}) = M^{-1} \mathcal{Z}$;
– $\mathcal{O}_i(M^{-1} \mathcal{Z}) = \mathcal{J}(\mathcal{O}_{i+1}(M^{-1} \mathcal{Z})) \cup \mathcal{C}(\pi_{i+1}, \mathcal{O}_{i+1}(M^{-1} \mathcal{Z})) \cup \mathcal{C}(\pi_{i+1}, M^{-1} \mathcal{Z})$ for $i = 0, \ldots, d - 1$. 

© Springer
Property $P(Z)$ Let $Z \subset \mathbb{C}^n$ be an algebraic set of dimension $d$. We say that $M \in \text{GL}_n(\mathbb{C})$ satisfies $P(Z)$ when for all $i = 0, 1, \ldots, d$
1. $O_i(M^{-1}Z)$ has dimension $\leq i$;
2. $O_i(M^{-1}Z)$ is in Noether position with respect to $x_1, \ldots, x_i$.

Property $Q(Z)$ Let $Z \subset \mathbb{C}^n$ be an algebraic set of dimension $d$ and $1 \leq i \leq d$. We say that $Q_i(Z)$ holds if for any connected component $\mathcal{C}$ of $Z \cap \mathbb{R}^n$ the boundary of $\pi_i(\mathcal{C})$ is contained in $\pi_i(O_{i-1}(Z) \cap \mathcal{C})$. We say that $Q(Z)$ holds if $Q_i(Z)$ holds for all $1 \leq i \leq d$.

In [18] the authors proved that given any algebraic variety $Z$ of dimension $d$, Property $P(Z)$ holds generically in $\text{GL}_n(\mathbb{C})$ (Proposition 17) and that if $M \in \text{GL}_n(\mathbb{C})$ satisfies $P(Z)$, then $Q(M^{-1}Z)$ holds (Proposition 18). We use these results in the following proof of Proposition 3.

**Proof (of Assertion 1 of Proposition 3)** Let $A$ and $U$ be as in the hypothesis. Let $\mathcal{M}_{A,U} \subset \text{GL}_n(\mathbb{C})$ be the non-empty Zariski open set defined in [18, Proposition 17] for $Z = D_r$. This set might depend on the choice of $A$ and $U$, as explicited by the indices. One obtains that every $M \in \mathcal{M}_{A,U} \cap \text{GL}_n(Q)$ satisfies $P(D_r)$. Remark that since $M \in \text{GL}_n(Q)$, there is a natural bijective correspondence between the set of connected components of $D_r \cap \mathbb{R}^n$ and the ones of $M^{-1}D_r \cap \mathbb{R}^n$ given by $C \leftrightarrow M^{-1}C$. Fix a connected component $M^{-1}C \subset M^{-1}D_r \cap \mathbb{R}^n$ and consider the projection $\pi_i$ restricted to $M^{-1}D_r \cap \mathbb{R}^n$. Since $M \in \mathcal{M}_{A,U}$, by [18, Proposition 18] the boundary of $\pi_i(M^{-1}C)$ is contained in $\pi_i(O_{i-1}(M^{-1}D_r) \cap M^{-1}C)$ and in particular in $\pi_i(M^{-1}C)$. This implies that $\pi_i(M^{-1}C)$ is closed for all $i$. Let $\varphi : \mathbb{C}^{n \times n} \to \mathbb{C}^n$ be the map sending $M \in \mathbb{C}^{n \times n}$ to its first row $M(1)$. The map $\varphi$ is a projection, hence a morphism. Applying [17, 9.1, Ex.III], one gets that $\varphi(\mathcal{M}_{A,U}) := \mathcal{W}_{A,U}^{(1)}$ is Zariski-open in $\mathbb{C}^n$. In particular, for $i = 1$ and applying (8), one gets that $\pi_w(\mathcal{C})$ is closed, for $w \in \mathcal{W}_{A,U}^{(1)}$. \hfill $\square$

**Proof (of Assertion 2 of Proposition 3)** Let $A$ and $U$ be as in the hypothesis. Fix $0 \leq p \leq r$. Since $A \in \mathcal{A}$, by Proposition 1 the set $D_p$ has the expected co-dimension, hence $c = (m - p)(s - p)$. Let $\eta_1, \ldots, \eta_c$ be local equations for $D_p$.

We consider the following constructible set:

$$ J = \left\{ (x, z, w) : x \in D_p \cap N_r(A, U), w_i = \sum_j z_j \frac{\partial \eta_j}{\partial x_i}, z \neq 0 \right\} $$

and the projection $\pi : \mathbb{C}^N \to \mathbb{C}^n$, $\pi(x, y, z, w) = w$, with $N = 2n + c$, where $c$ is the co-dimension of $D_p$. We claim that $\pi(J)$ is a constructible subset of $\mathbb{C}^n$ of positive co-dimension. Hence there exists a non-empty Zariski open set $\mathcal{W}_{A,U}^{(2)} \subset (\mathbb{C}^n \setminus \pi(J))$. \hfill $\Box$
such that Assertion 2 holds for $w \in \mathcal{W}^{(2)}$. We conclude the proof by defining $\mathcal{W}'_{A,U} = \mathcal{W}^{(1)}_{A,U} \cap \mathcal{W}^{(2)}_{A,U}$ (where $\mathcal{W}^{(1)}_{A,U}$ has been defined in the proof of Assertion 1).

We prove now our claim. By Assertion (2b) of Proposition 1, the set $D_p \cap N_r(A, U)$ has positive codimension in $D_p$, hence it has codimension at least $c + 1$. Moreover, the equations $w_i = \sum_j z_j \frac{\partial \eta_j}{\partial x_i}$ define an algebraic set of co-dimension $n$ (indeed, their Jacobian matrix has full rank, having a $n \times n$ identity matrix corresponding to the derivatives w.r.t. $w_i$). We deduce that $J$ is a constructible set of dimension at most $N - (c + 1 + n) = n - 1$ in $\mathbb{C}^N$. Hence its projection $\pi(J)$ has dimension at most $n - 1$ in $\mathbb{C}^n$, as claimed. 

\[\Box\]

7 Experiments

This section reports on experiments made with a first implementation of our algorithm. Note that for computing rational parametrizations, we use Gröbner bases and change of ordering algorithms [10]. Our experiments are done using the C library FGb, developed by Faugère [9] and interfaced with MAPLE.

We start by comparing our implementation with implementations of general algorithms based on the critical point method in RAGlib [32]. Next, we comment the behaviour of our algorithm on special examples that are well-known by the research community working on linear matrices.

7.1 Comparison with RAGLIB

We have generated randomly linear matrices for various values of $m = s$ (for simplicity we perform computations on square matrices) and $n$ and run our implementation for different values of $r$. By randomness of rational numbers we mean that we generate couples of integers chosen with uniform distribution in a fixed interval (here $[-100, 100]$).

Our implementation is in MAPLE and computations have been performed on an Intel(R) Xeon(R) CPU E7540@2.00GHz 256 Gb of RAM. We report in Table 1 numerical data of our tests. The reader can find an implementation of our algorithm and the details of these computations at the following permanent link

https://www.unilim.fr/pages_perso/simone.naldi/lowrank.html

For any choice of $m = s$, $2 \leq r \leq m - 1$ and $n$, we generate a random dense linear matrix $A$ and we let LowRank run with input $(A, r)$. We compare our algorithm (timings in seconds reported in column “LowRank”) with the function PointsPerComponents (column “PPC”) of RAGlib [32]. The input of PointsPerComponents is the list of all $(r + 1) \times (r + 1)$ minors of the matrix $A(x)$ (generating the ideal of the set of matrices of rank $\leq r$ in the pencil $A(x)$).

The symbol $\infty$ means that no result has been returned after 4 days of computation. In column deg we report the degree of $q_{n+1}$ in the output parametrization. Finally, in column maxdeg we report the maximum of the degrees of the partial rational
parametrizations (corresponding to the different recursive steps of \texttt{LowRank}). When \texttt{deg} is “–”, we mean that the empty list is returned.

We make the following remarks about Table 1.

1. We first observe that our algorithm is most of the time faster than \textsc{RAGlib} and it allows to tackle examples that are out of reach of \textsc{RAGlib}.

2. The growth in terms of timings with respect to \(n\) seems to respect the corresponding growth in terms of degrees of output parametrizations; in particular note that we have established that for \(r\) and \(m\) fixed, the sum of the degrees of parametrizations we need to compute stabilizes when \(n\) grows. This is observed in practice of course and is reflected in our timings compared to those of \textsc{RAGlib}.

3. Accordingly to the related Multilinear Bézout Bounds computed in Sect. 3.3.1, the degrees of rational parametrizations stabilize when \(n\) grows, since when \(n > \ldots\)
\[(m + r)(s - r)\) and the input is generic, LowRank does not compute critical points at first calls. This fact is remarkable, since:

- it is known (see [2, Ch. II]) that a natural geometric invariant associated to \(D_r\), its degree as complex algebraic set, does not depend on the dimension \(n\) of the affine section (one can prove easily that generically this degree is given by the Thom–Porteous–Giambelli formula, cf. [2, Ch. II, § 4]);
- an algebraic invariant naturally associated to the output-size (the degree of \(q_{n+1}\)) is constant in \(n\), coherently with the aforementioned geometric invariant.

Finally, we give a final remark on potential a posteriori verification of the correctness of the output of LowRank. Deciding whether a finite set, encoded by a rational parametrization, meets every connected component of a given real algebraic set, is a hard problem, far from being solved, both from a theoretical and computational viewpoint. As far as the authors know, there are no symbolic or numerical algorithms able to perform this task. Also, producing such a certificate seems to be hard to imagine, but this was not among the goals of this paper. In the recent paper [34], an algorithm to address connectivity queries (for instance, deciding whether 2 points of a smooth compact real algebraic set lie on the same connected component) has been developed.

7.2 Examples

In this last section, we consider some examples of linear matrices coming from the literature, and we test the behavior of LowRank. We consider examples of symmetric linear matrices since, as observed in Sect. 1.2, the main motivation for solving the real root finding problem is to obtain dedicated algorithms for spectrahedra and semidefinite programming. All these tests can be replicated via the Maple library Spectra via the function SolveLMI (cf. the documentation of Spectra and [21] for details).

Example 3 (The Cayley cubic) We consider the \(3 \times 3\) linear matrix

\[
A(x) = \begin{pmatrix}
1 & x_1 & x_2 \\
x_1 & 1 & x_3 \\
x_2 & x_3 & 1
\end{pmatrix}.
\]

The convex region \(\{x \in \mathbb{R}^3 \mid A(x) \succeq 0\}\) is the Cayley spectrahedron. We run our algorithm with input \((A, r)\) with \(r = 2\) and \(r = 1\) (the case \(r = 0\) is trivial since \(A(x)\) is always non-zero and hence \(D_0\) is empty). In both cases, the algorithm first chooses a random matrix \(U\), then verifies that the genericity assumptions are satisfied.

Let us first analyze the case \(r = 2\). Our algorithm runs 3 recursive steps. Its output is a rational parametrization of degree 14 with 12 real solutions and 2 complex solutions. We give below details of each recursive call of LowRankRec. At the first, at step 2, a rational parametrization of degree 5 is computed, with the following 5 real solutions:

\[
\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 18.285118452 \\ 164.322822823 \\ 4.552268485 \end{pmatrix} \right\}.
\]
The coordinates of the fifth point are approximated to 9 certified digits and such approximation can be computed by isolating the coordinates in intervals of rational numbers as:

\[
x_1 \in \left[ \frac{21081306277346124211}{1152921504606846976}, \frac{21081306277346754459}{1152921504606846976} \right] \approx 18.285118452
\]

\[
x_2 \in \left[ \frac{5920353629066611305}{36028797018963968}, \frac{2368141516266799197}{144115188075855872} \right] \approx 164.322822823
\]

\[
x_3 \in \left[ \frac{10496816461511385723}{2305843009213693952}, \frac{2624204115377866059}{576460752303423488} \right] \approx 4.552268485.
\]

Remark that it also computes the 4 singular points of \( D_2 \), where the rank of \( A \) is \( 1 \). At the second (resp. third) recursive call, it computes a rational parametrization of degree 6 (resp. of degree 3) with 4 (resp. 3) real solutions.

In the case \( r = 1 \), step 1 of \( \text{LowRankRec} \) returns a rational parametrization of degree 4 which encodes the 4 singular points of \( D_2 \cap \mathbb{R}^3 \), that is \( D_1 \cap \mathbb{R}^3 \). At the second and third recursions, \( \text{LowRankRec} \) returns empty lists.

We finally remark that the above results are typical, in the sense that the 4 singular points contained in \( D_1 \cap \mathbb{R}^3 \) are always computed at the first recursion step, both in case \( r = 2 \) and \( r = 1 \). Conversely, the coordinates of the other real solutions depend on the choice of random parameters (while their number is constant). Moreover, all computations end after a few seconds (\(< 5 \text{ s})\).

**Example 4** Let

\[
A(x) = \begin{pmatrix}
a_1 & x_1 & x_2 & x_3 \\
x_1 & a_2 & x_3 & x_4 \\
x_2 & x_3 & a_3 & x_5 \\
x_3 & x_4 & x_5 & a_4
\end{pmatrix},
\]

where \( x = (x_1, x_2, x_3, x_4, x_5) \) are variables and \( (a_1, a_2, a_3, a_4) \) are parameters. This example shows that the genericity assumptions of our algorithm are satisfied up to small perturbation of the input data.

We let \( (a_1, a_2, a_3, a_4) \) vary randomly in \( \mathbb{Q}^4 \). For all random instances, we observe that the inputs \( (A, 3), (A, 2) \) and \( (A, 1) \) verify the genericity assumptions, and that the degrees of the rational parametrizations returned at each recursion step are constant, while the number of real solutions changes with parameters. We summarize our results in Table 2.

| Table 2 | Degrees and timings for Example 4 with generic parameters |
|---------|----------------------------------------------------------|
|         | \( r = 3 \) | \( r = 2 \)   | \( r = 1 \)   |
| Partial degrees | [12 24 24 12 4] | [12 20 8 0 0] | [0 0 0 0 0] |
| Total degree | 76       | 40           | 0             |
| Time (s)    | 768      | 21.5         | 4.6           |
Example 5 (The pillow) Let

$$A(x) = \begin{pmatrix} 1 & x_1 & 0 & x_1 \\ x_1 & 1 & x_2 & 0 \\ 0 & x_2 & 1 & x_3 \\ x_1 & 0 & x_3 & 1 \end{pmatrix}.$$ 

The spectrahedron $S = \{ x \in \mathbb{R}^3 : A(x) \succeq 0 \}$ is known as the pillow.

The Zariski closure of its boundary is the real trace of the complex hypersurface defined by the vanishing of

$$\det A(x) = 1 - x_3^2 - x_2^2 - 2x_1^2 + x_1^2x_3^2 - 2x_1x_2x_3 + x_2^2x_3^2.$$ 

We tested our algorithm with input $(A,2)$. We obtain that at the first recursion, at step 1 a rational parametrization $q = (q_0, q_1, q_2, q_3, q_4)$ of degree 4 (with only real roots) is computed. By isolating the 4 real roots of $q_4$ as in Example 3, one gets the following rational approximation of one of the singular points:

$$x_1 \in \left[ -\frac{6521908912666475339}{9223372036854775808}, -\frac{1304381782533264843}{18446744073709551616} \right] \approx -\sqrt{2}/2$$

$$x_2 \in \left[ \frac{26087635650665343561}{36893488147419103232}, \frac{6521908912666428733}{9223372036854775808} \right] \approx \sqrt{2}/2$$

$$x_3 \in \left[ -\frac{6521908912666412349}{9223372036854775808}, -\frac{13043817825332731855}{18446744073709551616} \right] \approx -\sqrt{2}/2.$$ 

Acknowledgements The second author is supported by the FMJH Program PGMO and would like to thank LAAS-CNRS and the Mathematics Department of Technische Universität Dortmund where he was working when a significant part of this work was designed. The third author is supported by Institut Universitaire de France. We thank the reviewers for their help in improving the first version of this paper.

References

1. Absil, P.-A., Mahony, R., Sepulchre, R.: Optimization Algorithms on Matrix Manifolds. Princeton University Press, Princeton (2009)
2. Arbarello, E., Cornalba, M., Griffiths, P.A.: Geometry of Algebraic Curves: Volume II with a Contribution by Joseph Daniel Harris, vol. 267. Springer, Berlin (2011)
3. Bank, B., Giusti, M., Heintz, J., Lecerf, G., Matera, G., Solernó, P.: Degeneracy loci and polynomial equation solving. Found. Comput. Math. 15(1), 159–184 (2015)
4. Bank, B., Giusti, M., Heintz, J., Mbakop, G.-M.: Polar varieties and efficient real elimination. Math. Z. 238(1), 115–144 (2001)
5. Basu, S., Pollack, R., Roy, M.-F.: Algorithms in Real Algebraic in Mathematics. Algorithms and Computation in Mathematics, vol. 10, 2nd edn. Springer, Berlin (2006)
6. Bonnard, B., Faugère, J.-C., Jacquemard, A., Safey El Din, M., Verron, T.: Determinantal sets, singularities and application to optimal control in medical imagery. In: Proceedings of the ACM on International Symposium on Symbolic and Algebraic Computation, ISSAC ’16, pp. 103–110, ACM, New York (2016)
7. Collins, G.E.: Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In: Automata Theory and Formal Languages (Second GI Conf., Kaiserslautern, 1975), pp. 134–183. Lecture Notes in Computer Science, vol. 33. Springer, Berlin (1975)
8. Eisenbud, D.: Commutative Algebra with a View Toward Algebraic Geometry. Graduate Texts in Mathematics, vol. 150. Springer, Berlin (1995)
9. Faugère, J.-C.: Fgb: a library for computing Gröbner bases. In: International Congress on Mathematical Software, pp. 84–87. Springer, Berlin (2010)
10. Faugère, J.-C., Gianni, P., Lazard, D., Mora, T.: Efficient computation of zero-dimensional Gröbner bases by change of ordering. J. Symb. Comput. 16(4), 329–344 (1993)
11. Faugère, J.-C., Safey El Din, M., Spaenlehauer, P.-J.: On the complexity of the generalized minrank problem. J. Symb. Comput. 55, 30–58 (2013)
12. Gianni, P., Mora, T.: Algebraic solution of systems of polynomial equations using Groebner bases. In: Applied Algebra, Algebraic Algorithms and Error-Correcting Codes, vol. 356 of Lecture Notes in Computer Science, pp. 247–257. Springer, Berlin (1989)
13. Giusti, M., Lecerf, G., Salvy, B.: A Gröbner-free alternative for polynomial system solving. J. Complex. 17(1), 154–211 (2001)
14. Greuet, A., Safey El Din, M.: Probabilistic algorithm for the global optimization of a polynomial over a real algebraic set. SIAM J. Optim. 24(3), 1313–1343 (2014)
15. Grigoriev, D., Vorobjov, N.: Solving systems of polynomial inequalities in subexponential time. J. Symb. Comput. 5(1/2), 37–64 (1988)
16. Harris, J.: Algebraic Geometry: A First Course, vol. 133. Springer, Berlin (1992)
17. Hartshorne, R.: Algebraic Geometry, vol. 52. Springer, Berlin (2013)
18. Henrion, D., Naldi, S., Safey El Din, M.: Real root finding for determinants of linear matrices. J. Symb. Comput. 74, 205–238 (2015)
19. Henrion, D., Naldi, S., Safey El Din, M.: Real root finding for rank defects in linear Hankel matrices. In: Proceedings of the 40th International Symposium on Symbolic and Algebraic Computation, Bath (UK), pp. 221–228 (2015)
20. Henrion, D., Naldi, S., Safey El Din, M.: Exact algorithms for linear matrix inequalities. SIAM J. Optim. 26(4), 2512–2539 (2016)
21. Henrion, D., Naldi, S., Safey El Din, M.: Spectra: a Maple library for solving linear matrix inequalities in exact arithmetic. Optim. Methods Softw. 34(1), 62–78 (2019)
22. Jeronimo, G., Matera, G., Solernó, P., Waisbein, A.: Deformation techniques for sparse systems. Found. Comput. Math. 9(1), 1–50 (2009)
23. Kileel, J., Kukelova, Z., Pajdla, T., Sturmfels, B.: Distortion varieties. Found. Comput. Math. 18(4), 1043–1071 (2018)
24. Kronecker, L.: Grundzüge einer arithmetischen theorie der algebraischen Grössen. J. für die Reine und angewandte Math. 92, 1–122 (1882)
25. Lasserre, J.-B.: Moments, Positive Polynomials and Their Applications. Optimization Series, vol. 1. Imperial College Press, London (2010)
26. Lecerf, G.: Computing an equidimensional decomposition of an algebraic variety by means of geometric resolutions. In: Proceedings of the 2000 International Symposium on Symbolic and Algebraic Computation, ISSAC ’00, pp. 209–216, New York, NY. ACM (2000)
27. Macaulay, F.S.: The Algebraic Theory of Modular Systems. Cambridge University Press, Cambridge (1916)
28. Ottaviani, G., Spaenlehauer, P.-J., Sturmfels, B.: Exact solutions in structured low-rank approximation. SIAM J. Matrix Anal. Appl. 35(4), 1521–1542 (2014)
29. Poteaux, A., Schost, É.: On the complexity of computing with zero-dimensional triangular sets. J. Symb. Comput. 50, 110–138 (2013)
30. Ranestad, K.: Algebraic degree in semidefinite and polynomial optimization. In: Handbook on Semidefinite, Conic and Polynomial Optimization, pp. 61–75. Springer, Berlin (2012)
31. Rouillier, F.: Solving zero-dimensional systems through the rational univariate representation. Appl. Algebra Eng. Commun. Comput. 9(5), 433–461 (1999)
32. Safey El Din, M.: Raglib (Real Algebraic Geometry library), Maple package (2007)
33. Safey El Din, M., Schost, É.: Polar varieties and computation of one point in each connected component of a smooth real algebraic set. In: Proceedings of the 2003 International Symposium on Symbolic and Algebraic Computation, ISSAC ’03, pp. 224–231, New York, NY. ACM (2003)
34. Safey El Din, M., Schost, E.: A nearly optimal algorithm for deciding connectivity queries in smooth and bounded real algebraic sets. J. ACM 63(48), (2017)
35. Safey El Din, M., Schost, É.: Bit complexity for multi-homogeneous polynomial system solving: application to polynomial minimization. J. Symb. Comput. 87, 176–206 (2018)
36. Shafarevich, I.: Basic Algebraic Geometry 1. Springer, Berlin (1977)
37. Tarski, A.: A Decision Method for Elementary Algebra and Geometry. University of California Press, California (1951)

Publisher's Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.