Relativistic Resonances, Relativistic Gamow Vectors and Representations of the Poincaré Semigroup

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Abstract

The foundations of time asymmetric quantum theory are reviewed and are applied to the construction of relativistic Gamow vectors. Relativistic Gamow vectors are obtained from the resonance pole of the $S$-matrix and furnish an irreducible representation of the Poincaré semigroup. They have all the properties needed to represent relativistic quasistable particles and can be used to fix the definition of mass and width of relativistic resonances like the $Z$-boson. Most remarkably, they have only a semigroup time evolution into the forward light cone—expressing time asymmetry on the microphysical level.

1 Time Asymmetric Quantum Mechanics

In classical physics one has time symmetric dynamical equations with time asymmetric boundary conditions [1, 2]. These time asymmetric boundary conditions come in pairs: given one time asymmetric boundary condition, its time reversed boundary condition can also be formulated mathematically. For example in classical electrodynamics one has retarded and advanced solutions of the time symmetric dynamical (Maxwell) equations or in general relativity one has time asymmetric big bang and

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big crunch solutions of Einstein’s time symmetric equation. Except for a few prominent cases of pedagogical importance (e.g. stationary states or cyclic evolutions), the physics of our world is predominantly time-asymmetric. Somehow nature chooses one of the pair of time asymmetric boundary conditions.

The standard quantum mechanics in Hilbert space \([3]\) does not allow time asymmetric boundary conditions for the Schrödinger or von Neumann equation \([4]\). However this is a consequence of the \textit{mathematical} properties of the Hilbert space and need not imply that quantum \textit{physics} is strictly time symmetric. It would be incredible if classical electrodynamics had a radiation arrow of time and quantum electrodynamics did not also have an arrow of time.

In quantum physics Peierls and Siegert considered many years ago time asymmetric solutions with purely outgoing boundary conditions \([5]\). The choice of appropriate dense subspaces \(\Phi_+\) and \(\Phi_-\) of the (complete) Hilbert space \(\mathcal{H}\) allows the formulation of time asymmetric boundary conditions:

\[
\Phi_+ \subset \mathcal{H} \quad \text{for the out-states } \{\psi^-\} \text{ of scattering theory which are actually observables as defined by the registration apparatus (detector), and}
\]

\[
\Phi_- \subset \mathcal{H} \quad \text{for the in-states } \{\phi^+\} \text{ which are prepared states as defined by the preparation apparatus (accelerator).}
\]

Time asymmetric quantum theory distinguishes meticulously between states \(\{\phi^+\}\) and observables \(\{\psi^-\}\). Two different dense subspaces of the Hilbert space \(\mathcal{H}\) are chosen, \(\Phi_- = \{\phi^+\}\) and \(\Phi_+ = \{\psi^-\}\). The standard Hilbert space quantum theory uses \(\mathcal{H}\) for both, \(\{\psi^-\} = \{\phi^+\} = \mathcal{H}\) and as a result is time symmetric with a reversible unitary group time evolution.

In the theory of scattering and decay, a pair of time asymmetric boundary conditions can be heuristically implemented by choosing in- and out-plane wave “states” \(|E^\pm\rangle\) which are solutions of the Lippmann-Schwinger equation \([3]\)

\[
|E^\pm\rangle = |E\rangle + \frac{1}{E - H \pm i\delta} V |E\rangle = \Omega^\pm |E\rangle, \quad (1)
\]

where \((H - V)|E\rangle = E|E\rangle\). The energy distribution of the incident beam is given by \(|\langle E|\phi^+\rangle|^2 = |\langle E|\phi^{in}\rangle|^2\) and the energy resolution of the detector (which for perfect efficiency is the energy distribution of the detected “out-states”) is measured as \(|\langle -E|\psi^-\rangle|^2 = |\langle E|\psi^{out}\rangle|^2\).

The sets \(||E^\pm\rangle\) are the basis system that is used for the Dirac basis vector expansion of the in-states \(\phi^+ \in \Phi_-\) and the out-states (observables) \(\psi^- \in \Phi_+\):

\[
\psi^- = \sum_b \int_0^\infty dE |E, b^-\rangle \langle -E, b|\psi^-\rangle
\]

\[
\phi^+ = \sum_b \int_0^\infty dE |E, b^+\rangle \langle +E, b|\phi^+\rangle, \quad (2)
\]
where $b$ are the degeneracy labels. The Dirac kets of the Lippmann-Schwinger equation $|E^\pm\rangle$ are in our time asymmetric quantum theory antilinear functionals on the spaces $\Phi_\mp$, i.e. they are elements of the dual space $|E^+\rangle \in \Phi_+^\times$.

This leads to two rigged Hilbert spaces (RHS) and the following new hypothesis for time asymmetric quantum theory:

2. pure registered observables or so-called “out-states” are described by the vectors

$$\psi^- \in \Phi_+ \subset \mathcal{H} \subset \Phi_+^\times$$

and pure prepared in-states are described by the vectors

$$\phi^+ \in \Phi_- \subset \mathcal{H} \subset \Phi_-^\times. \quad (3)$$

This new hypothesis—with the appropriate choice for the spaces $\Phi_+$ and $\Phi_-$ given below in (4)—is essentially all by which our time asymmetric quantum theory differs from the standard Hilbert space quantum mechanics which imposes $\{\psi^-\} = \{\phi^+\} = \mathcal{H}$ (or $\{\psi^-\} = \{\phi^+\} \subset \mathcal{H}$).

In addition to the Dirac Lippmann-Schwinger kets $|E^\pm\rangle \in \Phi_+^\times$, the dual spaces $\Phi_-^\times$ of the RHS’s also contain Gamow kets $|E_R \pm i\Gamma_R/2^\pm\rangle \in \Phi_+^\times$, which are generalized eigenvectors of the (self-adjoint) Hamiltonian with complex eigenvalue $(E_R \pm i\Gamma_R/2)$. We use these Gamow kets to describe quasistable particles.

We shall now mathematically define $\Phi_+$ and $\Phi_-$, and therewith the RHS’s (3). From a mathematical formulation of causality expressed by the truism “a state must be prepared before an observable can be measured (registered) in it”, one can argue that the energy wave functions $\langle -E | \psi^- \rangle$ are the boundary values of analytic functions in the upper half energy plane (second sheet of the $S$-matrix) and the $\langle +E | \phi^+ \rangle$ are the same for the lower half plane.

$$\phi^+ \in \Phi_- \iff \langle +E | \phi^+ \rangle \in \mathcal{S} \cap H_2^-$$
$$\psi^- \in \Phi_+ \iff \langle -E | \psi^- \rangle \in \mathcal{S} \cap H_2^+. \quad (4)$$

where $\mathcal{S}$ is the Schwartz space and $\mathcal{S} \cap H_2^\pm$ are well-behaved Hardy functions in the lower (upper) half plane $\mathbb{C}^\pm$ of the second Riemann sheet for the $S$-matrix $S(E)$. The disparity between the labels $\pm$ for the vectors and the spaces (e.g. $\phi^+ \in \Phi_-$) now makes sense. The superscripts of the vectors is the standard notation of scattering theory while the subscripts of the spaces comes from their mathematical definition.

This correspondence between the physical state vectors and the mathematical spaces is a wonderful example of what Wigner calls “the unreasonable effectiveness of mathematics in the natural sciences.”

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2General states are described by density operators $W^+ = \sum_i w_i |\phi^+_i\rangle \langle \phi^+_i|$ and general observables by (positive definite) operators $\Lambda^+ = \sum_i \lambda_i |\psi^-_i\rangle \langle \psi^-_i|$. 

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2 Gamow Vectors and Resonance States

Stable states are described by bound state poles or by eigenvectors of the self-adjoint Hamiltonian $H$ with real eigenvalue $E_n$:

\begin{align*}
H |E_n\rangle &= E_n |E_n\rangle, \text{ or equivalently } \\
(H f |E_n\rangle &= E_n (f |E_n\rangle \text{ for all } f \in \Phi \text{ dense in } \mathcal{H}.
\end{align*}  \tag{5}

Quantum mechanical resonances are most commonly defined by the pair of resonance poles in the second Riemann sheet of the analytically continued $S$-matrix at the position $E_R \mp i \Gamma_R / 2$; a decaying state is associated to the pole at $z_R = E_R - i \Gamma_R / 2$. Vector representations of quasistable particles like the $K^0$ mesons in the Lee-Oehme-Yang theory [10] use approximate methods such as the Weisskopf-Wigner approximation [11] for which “there does not exist...a rigorous theory to which these various methods can be considered approximations” [12].

Often one hears the opinion that resonances and decaying states are complicated objects and cannot be represented by simple (exponentially decaying) state vectors. This opinion differs from the common practice to classify quasistable particles along with stable particles [13] and also contradicts empirical facts: quasistable particles are not qualitatively different from stable particles but only quantitatively by a non-zero value of the width $\Gamma$ which one takes to be the inverse lifetime, $\Gamma = \hbar / \tau$. Stability or the value of the lifetime is not a criterion for elementarity. A particle decays if it can and remains stable if selection rules for some quantum numbers prevent it from decaying.

Stable and quasistable states should be described on the same footing, e.g. both defined by a pole of the $S$-matrix at the position $E_n$ for stable particles and at the position $z_R = E_R - i \Gamma_R / 2$ for quasistable particles. Then in analogy to the vector description (2) for stable particles, there should also be a description of a quasistable particle by a generalized eigenvector with eigenvalue $z_R$. A vector description is also needed to express the initial decay rate by the Golden Rule as in [14]. Therefore, in analogy to (2) resonances and quasistable particles will be described as generalized eigenvectors of the self-adjoint Hamiltonian with complex eigenvalue $(E_R - i \Gamma_R / 2)$, where $E_R$ represents the resonance energy and $\hbar / \Gamma_R$ is the lifetime:

\begin{align*}
\langle H \psi^- | E_R - i \frac{\Gamma_R}{2} \rangle &\equiv \langle \psi^- | H^\times | E_R - i \frac{\Gamma_R}{2} \rangle = (E_R - i \frac{\Gamma_R}{2}) \langle \psi^- | E_R - i \frac{\Gamma_R}{2} \rangle, \text{ or } \\
H^\times | z_R^- \rangle &= z_R | z_R^- \rangle, \tag{6}
\end{align*}

where the only distinction from (2) is that $\langle \psi^- | z_R^- \rangle$ are bra-kets ($\Phi_+\text{-continuous functionals}$) and $(f | E_n)$ are scalar products ($\mathcal{H}\text{-continuous functionals}$). We call the vectors $| z_R^- \rangle \in \Phi_+^\times$ Gamow vectors.

To obtain this vector representation of quasistable states one starts with the $S$-
matrix element

$$\langle \psi_{out}, \phi_{out} \rangle = \langle \psi_{out}, S\phi_{in} \rangle = \langle \phi^-, \phi^+ \rangle = \sum_{bb'} \int_0^\infty dE \langle \psi^-, b, E^- \rangle \langle b, S(E)|b \rangle \langle +b', E|\phi^+ \rangle.$$

(7)

and analytically continues the integral to the resonance pole $z_R = E_R - i\Gamma/2$ of the $S$-matrix $\langle b|S(z)|b \rangle$ as shown in Fig. 1. For this to be possible, the set of in-states $\{\phi^+\} \equiv \Phi_-$ and out-states $\{\psi^-\} \equiv \Phi_+$ must have certain analyticity properties. In order to get a Breit-Wigner energy distribution for the pole term of (7), the energy wave functions $(-E|\psi^-)$ and $(+E|\phi^+)$ must be well-behaved Hardy class functions of the upper and lower half plane, respectively, in the second sheet of the energy surface of the $S$-matrix. This is the same mathematical condition as provided by causality and stated by (4).

Using the Cauchy formula, the analytically continued kets $|E^-\rangle \in \Phi_+^\times$ become the Gamow kets $\psi^G = |z_R^-\rangle \in \Phi_+^\times$ at the resonance pole $z_R$. For the Gamow kets one can prove the equations (3). The first equality of (3) is the definition of the conjugate operator $H^\times$ in $\Phi_+^\times$, which is a unique extension of the Hamiltonian $H |E^-\rangle = E |E^-\rangle$, which in Dirac’s notation is written $H|E^-\rangle = E|E^-\rangle$.

These Gamow kets $|z_R^-\rangle$ have all the properties required of a vector representing the “state” of an unstable particle or (together with the ket $|z_R^+\rangle = |E_R+i\Gamma_R/2^+\rangle$ for the $S$-matrix pole at $z_R^*$) of a resonance in non-relativistic quantum physics. Gamow vectors have the following features:

1. They are derived as functionals of the resonance pole term at $z_R = E_R - i\Gamma_R/2$.
(and at $z^*_R = E_R + i\Gamma_R/2$) in the second sheet of the analytically continued $S$-matrix.

2. The Gamow vectors have a Breit-Wigner energy distribution:
\[
\langle -E|\psi^G \rangle = \frac{i\sqrt{\Gamma_R/2\pi}}{E - (E_R - i\Gamma_R/2)}, \quad -\infty < E < \infty. \tag{8}
\]

3. They have an asymmetric time evolution and obey an exponential law:
\[
\psi^G(t) = e^{-iH^*t}|z_R^-\rangle = e^{-iE_R t}e^{-\Gamma_R t/2}|z_R^-\rangle, \quad \text{only for } t \geq 0. \tag{9}
\]

4. The decay probability of $\psi^G$ into the final non-interacting decay products described by $\Lambda$, $P(t) = \text{Tr}(\Lambda|\psi^G\rangle\langle\psi^G|)$, can be calculated as a function of time for $t \geq 0$, and from this the decay rate $R(t) = dP/dt$ is obtained by differentiation. This leads to an exact Golden rule (with the natural lineshape given by a Breit-Wigner) which in the Born approximation ($\psi^G \to f^D; \Gamma_R/E_R \to 0; E_R \to E_0$, where $f_D$ is an eigenvector of $H_0 = H - V$ with eigenvalue $E_0$) goes into Fermi’s Golden rule No. 2 of Dirac.

3 Relativistic Resonances, Relativistic Gamow Kets and Poincaré Semigroup Representations

Stable relativistic particles are defined and described by unitary irreducible representations $\mathcal{H}(s,j)$ of the Poincaré group $\mathcal{P}$ [14] where $s$ is the eigenvalue of $P^\mu P_\mu$ and $j$ is the spin. In the irreducible representation space one commonly uses the Wigner basis vectors $|p,j_3;(s,j)\rangle$, but one could just as well use any other complete basis system, e.g. the 4-velocity basis $|\hat{p},j_3;(s,j)\rangle$ where $\hat{p}_\mu = p_\mu/\sqrt{s}$ and $|\hat{p},j_3\rangle = U(L(\hat{p}))|\hat{0},j_3\rangle$ with
\[
L^\mu_\nu(p) = L^\mu_\nu(\hat{p}) = \begin{pmatrix}
\frac{p_0}{m} & -\frac{p_3}{m} \\
\frac{p_3}{m} & \frac{-\eta_{03}}{m} & \frac{-\eta_{0\mu}}{m} & \frac{-\eta_{3\mu}}{m} \\
\delta_{ij} & \frac{-\eta_{i\mu}}{m} & \frac{-\eta_{j\mu}}{m} & \frac{-\eta_{ij}}{m} \end{pmatrix}. \tag{10}
\]

This means every vector
\[
\phi^\pm \in \Phi_\mp \subset \mathcal{H}(s,j) \subset \Phi_+^\times \tag{11}
\]
can be written as (Dirac basis vector expansion):
\[
\phi^\pm = \sum_{j_3} \int \frac{d^3\hat{p}}{2E} |\hat{p},j_3^\pm\rangle\langle^\pm\hat{p},j_3|\phi^\pm\rangle. \tag{12}
\]
Stable relativistic particles are characterized in addition to the charge quantum numbers or particle labels by the value of mass $m$ and spin $j$. They are therefore described by the vectors $\phi^+ (s = m^2, j)$ or $\psi^- (s = m^2, j)$ of the unitary irreducible representation spaces $\mathbf{11}$. They lead to poles of the $j$th partial $S$-matrix $S_j(s)$ at the value $s = m^2$ on the real axis of the first sheet $\mathbf{15}$.

Decaying states or resonances are defined by poles of the $j$th partial $S$-matrix $S_j(s)$ on the second sheet at

$$s = s_R \equiv (m_R - i\Gamma_R/2)^2 = m_R^2 - i\Gamma_R.$$

(13a)

Where $(m_R, \Gamma_R)$ and $(m_\rho, \Gamma_\rho)$ are different parameterizations of the complex value $s_R$. Here

$$s \equiv (p_1 + p_2 + \cdots + p_n)^2 = E_R^2 - \mathbf{p}_R^2$$

(13b)

is the invariant mass squared, i.e. the eigenvalue of the total mass operator $P_\mu P^\mu$, $P^\mu = P_1^\mu + P_2^\mu + \cdots + P_n^\mu$. The spin $j$ is the total angular momentum of the $n$ decay products and is equal to the resonance spin $j = j_R$.

Quasistable relativistic particles are thus characterized by $w_R \equiv \sqrt{s_R} = m_R - i\Gamma_R/2$ and by spin $j_R$, where we call the resonance mass $m_R$ and the resonance width $\Gamma_R$ for reason seen below in (19). More common is to call $m_\rho$ and $\Gamma_\rho$ of (13a) the resonance mass and width $\mathbf{15}$. Some examples are:

1. Relativistic hadron resonances such as the $\rho$ meson in $e^+e^- \to \rho \to \pi^+\pi^-$ with $(\Gamma/m) \sim 10^{-1}$.

2. Relativistic (weakly) decaying states such as $K_s^0 \to \pi^+\pi^-$ with $(\Gamma/m) \sim 10^{-15}$.

3. The $Z$-bosons in $e^+e^- \to Z \to \pi^+\pi^-$ with $(\Gamma/m) \sim 10^{-2}$.

For these resonance scattering and decay processes one uses in the $S$-matrix elements $\mathbf{17}$, the relativistic Dirac Lippmann-Schwinger “scattering states” $|E, b^-\rangle = |\hat{p}, j_3; (s, j)^-\rangle$ for “physical” values of $s$, or in other words for values of $s$ with $(m_1 + m_2 + \cdots + m_n)^2 \leq s < \infty$. To obtain the relativistic Gamow kets one analytically continues these relativistic Dirac Lippmann-Schwinger kets in the contour deformation for the integral of $\mathbf{17}$ in the following way: the real $s = p^2 = (p_1 + p_2)^2$ becomes a complex $s$ but the $\hat{p}_\mu = p_\mu/\sqrt{s}$ remain real. At the pole position $s = s_R$ one obtains the relativistic Gamow kets:

$$|\hat{p}, j_3; (s_R, j)^-\rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} ds |\hat{p}, j_3; (s, j)^-\rangle \frac{1}{s - s_R}.$$  

(14)

These Gamow kets are basis vectors of a “minimally complex” semigroup representation of $\mathcal{P}$ which is characterized by $(s_R, j)$. This is the analogy of the representation
for non-relativistic Gamow vectors \[8\]:

\[
|z_R^{-}\rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dE E^\prime E^{-} \frac{1}{E - z_R^{-}}.
\]  

(15)

The relativistic Gamow vectors have according to (14) a “relativistic Breit-Wigner” \(s\)-distribution given by \((s - (M_R - i\Gamma/2)^2)^{-1}\). They can be shown to be generalized eigenvectors of the invariant mass-squared operator \(M^2 = P_{\mu}P_{\mu}\) \[16\], i.e. they fulfill

\[
\langle \psi^{-} | (M^2)^{x} | j, s_R; b^{-}\rangle = \frac{i}{2\pi} \int_{-\infty}^{+\infty} ds s \langle \psi^{-} | j, s; b^{-}\rangle \frac{1}{s - s_R}
\]

\[
= s_R \langle \psi^{-} | j, s; b^{-}\rangle
\]

(16)

for every \(\psi^{-} \in \Phi_{+} \subset \mathcal{H}(s_R, j) \subset \Phi^{\times}_{+}\). The Lorentz transformations \(\Lambda\) in these minimally complex representations are represented by a group of unitary operators \(U(\Lambda)\) and act in the well-known way:

\[
U(\Lambda)|\hat{p}, j_3; (s_R, j)^{-}\rangle = \sum_{j_3'} D_{j_3 j_3'}^{j} (\mathcal{R}(\Lambda, \hat{p})) |\Lambda \hat{p}, j_3'; (s_R, j)^{-}\rangle.
\]  

(17)

However the space-time translations can no more be represented by a group of operators \(U(a, 1)\) in

\[
\Phi_{+} \subset \mathcal{H}(s_R, j) \subset \Phi_{+}.
\]

(18)

The rigged Hilbert spaces of Hardy class have to be employed in rigorously obtaining (14) and (16) and then one can see that only the semigroup of space-time translations into the forward light cone can be represented in the space (18). Much like in the RHS theory of non-relativistic Gamow vectors, one can show that the time translation of the decaying state in the rest frame \(\hat{p} = 0\) is given by

\[
e^{-iHt} |0, j_3; (s_R, j)^{-}\rangle = e^{-im_R t} e^{-\Gamma_R t/2} |0, j_3; (s_R, j)^{-}\rangle, \quad t \geq 0 \text{ only},
\]

(19)

where \(t\) is the time in the rest system.

Thus, the relativistic Gamow vectors have a semigroup time evolution and obey the exponential law. From (19) it follows that the lifetime of the particle represented by the relativistic Gamow vector is given by \(\tau_R = \hbar/\Gamma_R\) where the width \(\Gamma_R\) is the \(\text{Im}\sqrt{s_R}\) given by the resonance pole position \(s_R\) of the relativistic \(S\)-matrix. The real resonance mass \(m_R\) in (19) is the \(\text{Re}\sqrt{s_R}\) and therefore not exactly the same as the peak position \(m_{\rho} \equiv m_R \sqrt{1 - 1/4(\Gamma_R/m_R)^2}\) of the modulus of the relativistic Breit-Wigner amplitude of (14).
4 Summary

Our fundamental assertion is that stable and unstable particles are not fundamentally different. The theoretical treatment of stable and unstable particles should therefore be the same. Within the Hilbert space (HS) quantum theory, time asymmetric boundary conditions cannot be formulated and a theory of decaying states within the HS framework can only be approximate [12]. Within the RHS, time asymmetric quantum theory can be introduced by the new hypothesis (3,4) which allows a distinction between states and observables.

Within this theory, Gamow vectors, both relativistic and non-relativistic, satisfy all necessary properties of representing quasi-stable states and resonances. They are associated to poles of the S-matrix, have Breit-Wigner energy distribution and obey an exponential semigroup time evolution. All of these properties can be formulated mathematically precisely, and a particular property of the Gamow vector is that it relates the width $\Gamma_R$ of the Breit-Wigner energy distribution and the lifetime $\tau$ of the exponential decay $\tau = \hbar/\Gamma_R$. Without the Gamow vectors, the width and inverse lifetime only can be shown to be approximately equal by use of some approximate method based on Weisskopf-Wigner [11]. Consequently, one did not know whether to choose $(m_R, \Gamma_R)$ or $(m_\rho, \Gamma_\rho)$ of (13a) or any other $(m, \Gamma)$ as the mass and width of relativistic resonances.

The relativistic Gamow vectors unify in a fundamental picture both stable and unstable relativistic particles. Both are given by irreducible representations of Poincaré transformations. Stable particles are unitary representations characterized by a real mass and have unitary group time evolution, quasi-stable particles are “minimally complex” semigroup representations characterized by a complex mass and have semigroup time evolution with an arrow of time.

That relativistic Gamow states $|z_R^-\rangle$ possess only semigroup transformations into the forward light cone (and that others possess only those into the backward light cone) and no space-like translations (for either case) is a result whose interpretation is not yet clear. It is the relativistic analogue of the semigroup time evolution in non-relativistic quantum mechanics which can be understood as the causality condition that a state must be first prepared before an observable can be measured in it [4]. This time asymmetry on the microphysical level—irreversibility without entropy increase and without violation of time reversal invariance—was the most surprising and remarkable property of the non-relativistic and relativistic Gamow vectors.

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