Unraveling-paired dynamical maps recover the input of quantum channels

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Abstract
We explore algebraic and dynamical consequences of unraveling general time-local master equations. We show that the ‘influence martingale’, the paramount ingredient of a recently discovered unraveling framework, pairs any time-local master equation with a one parameter family of Lindblad–Gorini–Kossakowski–Sudarshan master equations. At any instant of time, the variance of the influence martingale provides an upper bound on the Hilbert–Schmidt distance between solutions of paired master equations. Finding the lowest upper bound on the variance of the influence martingale yields an explicit criterion of ‘optimal pairing’. The criterion independently retrieves the measure of isotropic noise necessary for the structural physical approximation of the flow the time-local master equation with a completely positive flow. The optimal pairing also allows us to invoke a general result on linear maps on operators (the ‘commutant representation’) to embed the flow of a general master equation in the off-diagonal corner of a completely positive semi-group which in turn solves a time-local master equation that we explicitly determine. We use the embedding to reverse a completely positive evolution, a quantum channel, to its initial condition thereby providing a protocol to preserve quantum memory against decoherence. We thus arrive at a model of continuous-time error correction by a quantum channel.

1. Introduction
A consequential result of stochastic analysis is the representation of solutions of a large class of non-linear parabolic partial differential equations as Monte Carlo averages over random paths via Feynman–Kac type formulae [1]. The result has ubiquitous applications because both in classical and quantum physics, Monte Carlo methods often provide the only viable integration strategy when the number of involved degrees of freedom becomes large [2].

In the theory of open quantum systems, the representation of state operators as Monte Carlo averages over pure state operators generated by classical stochastic processes is called unraveling in quantum trajectories [3]. Different forms of unravellings have been discovered in connection to quantum measurement [4–9], possibly with delay and retrodiction [10–12], to microscopic state reduction [13–16], in addition to the scope of developing efficient computational algorithms in high dimensional Hilbert spaces [17–20]. We refer to [21, 22] for an overview of further recent applications.

A theoretical result on the dynamics of open system state operators on finite dimensional Hilbert spaces supports the generic existence of unravellings. Namely, it is possible to prove that solutions of the Nakajima–Zwanzig integro-differential equation [23] also satisfy a time-local master equation almost everywhere in time [24–29]. The proof requires the existence of a pseudo-inverse of the Nakajima–Zwanzig solution map and is non-constructive. Nonetheless, theoretical considerations permit to infer a canonical
form of the time-local master equation [30]. Explicit expressions are derived from microscopic dynamics typically by means of asymptotic expansions.

The most well-known, and mathematically rigorous derivation [31] relies on the weak-coupling scaling limit, and leads to the completely positive master equation. Indeed, Lindblad [32], Gorini, Kossakowski and Sudarshan [33] derived this master equation from the necessary and sufficient conditions to generate a completely positive semi-group on operators.

Beyond weak coupling, time-convolutionless perturbation theory [34–41] yields a systematic way to explicitly derive time-local master equations. In the presence only of regularity assumptions, a time-local master equation on a finite dimensional Hilbert space only generates a completely bounded group of self-adjoint and trace preserving linear maps. Particular solutions [42] may, however, describe positive and even completely positive evolution laws. This fact explains the physical relevance of the general canonical form of the time-local master equation which we call the completely bounded master equation.

Very recently [43], we introduced a framework for unraveling (solutions of) the completely bounded master equation. The basic idea is to implement the Wittstock–Paulsen canonical representation of a completely bounded map [44, 45] at statistical level. To this goal, we included in the Monte-Carlo average pure states compatible with the most general, non-linear with respect to the initial data, physically admissible Kraus-type form [46, 47]. A mean preserving martingale of the stochastic process generating the dynamics—the influence martingale, as we called it—attributes the proper weight and sign to each pure state in order to recover a completely bounded map, on average.

In this paper, we explore algebraic and dynamical consequences of unravelings based on the influence martingale and obtain the following main results

• Proposition 3.1 in section 3 states that there always exists an influence martingale which pairs the solution of a completely bounded master equation to an auxiliary completely positive master equation on the same finite dimensional Hilbert space \( \mathcal{H} \).

We reserve the adjectives completely bounded or completely positive to master equations in canonical form in the sense of [30]. Proposition 3.1 represents a refinement of the result of [43] consequential for physics applications which we illustrate.

• Proposition 4.1 in section 4 states that the pairing is unique (optimal) if the squared Hilbert–Schmidt distance between solutions of the assigned and auxiliary master equations is subject to an optimal upper bound.

• Proposition 5.1 in section 5 states that given a completely bounded master equation there exists a completely positive map on \( C^2 \otimes \mathcal{H} \) whose diagonal and off-diagonal blocks are respectively specified by completely positive and completely bounded maps paired by the influence martingale.

Proposition 5.1 allows us to construct an explicit protocol to time-reverse any evolution generated by a time-local master equation on a finite dimensional Hilbert space \( \mathcal{H} \) by means of a completely positive differentiable semi-group on \( C^2 \otimes \mathcal{H} \). The protocol, a further main result of the paper, is the subject of section 7.

The structure of the paper is as follows. In section 2 we set the stage for our results by recalling well established results on the time-local representation of the dynamics of an open quantum system. We draw mainly from [30]. Readers familiar with this reference or [29] may well skip this section.

Sections 3–5 contain the proofs of the results announced above and some complementary results. In section 4.1 we also discuss the physical interpretation of the information encoded in the influence martingale. In particular, we relate the growth rate of the influence martingale specified by proposition 4.1 to the minimum rate of quantum isotropic noise that must be added to a completely bounded flow over an infinitesimal time step to recover a completely positive evolution [30, 48] and to the structural physical approximation [49]. In subsection 5.1, we also contrast the optimal embedding of a completely bounded into a completely positive map implied by proposition 5.1 with the one interwoven with the definition of the ‘triple norm’ of a completely bounded map [50].

In section 6 we discuss time reversal of solutions of time-local master equations and their unraveling. The discussion serves to emphasize the significance of the inversion protocol by a completely positive operation that we introduce in section 7.

In section 7 we use the results of section 5 to construct the reversal protocol. The protocol can be also applied to a completely positive evolution due to the action of a completely bounded flow on a special initial data, and therefore not described by a Lindblad–Gorini–Kossakowski–Sudarshan master equation. Reversal is exact for any initial data. In this sense the result is more general than protocols based on the Petz recovery map that can only exactly reconstruct a reference state but have the advantage of not requiring an ancilla [51].
In the theory of continuous time error correction [52–56] (see e.g. [57] for a general overview of quantum error correction), decoherence operators in the master equation model unwanted interactions with the environment of a quantum computing device. We thus conclude section 7 by briefly discussing potential application of the recovery-by-embedding protocol to quantum error correction.

Finally, in section 8 we indicate how to apply the influence martingale to non-canonical forms of the time-local master equation. We recall some auxiliary results in the appendices.

2. Canonical form of the time-local master equation

Our starting point is the discussion of the unique canonical form of time-local master equations in section II of [30]. We summarize the points relevant to the present work. We suppose that the open system is defined on a Hilbert space \( \mathcal{H} \) of finite dimension \( d \). The space \( \mathcal{B}(\mathcal{H}) \) of bounded operators acting on \( \mathcal{H} \) then reduces to the set of \( d \times d \) dimensional matrices \( \mathcal{M}_d \). We regard \( \mathcal{M}_d \) as a Hilbert space with respect to the Hilbert–Schimdt inner product [58]. Any time-local master equation governing the evolution of the open system state operator \( \rho_t \) is amenable to the form

\[
\partial_t \rho_t = -i [H, \rho_t] + \sum_{\ell=1}^{\mathcal{X}} \omega_{\ell,t} \mathcal{D}_{\ell,t}(\rho_t)
\]

(1a)

\[
\mathcal{D}_{\ell,t}(\rho_t) = \frac{1}{2} \left( \left[ I_{\ell,t} \rho_t I_{\ell,t}^\dagger \right] + \left[ I_{\ell,t}^\dagger \rho_t I_{\ell,t} \right] \right).
\]

(1b)

The proof of the existence of (1) has a long history see e.g. [24–26]. It can be simply obtained from the Nakajima–Zwanzig equation (see e.g. [23]) under the hypothesis that the solution map from an initial time \( t_0 \) has a continuous inverse in an interval of finite duration [27–29]. The proof thus implies a parametric dependence of (1) upon \( t_0 \). Multi-scale perturbation theory typically involves a coarse-grain of time scales [59]. Hence, in explicit derivations by time convolutionless perturbation theory [34–41] \( t_0 \) can be set to zero.

In (1a) we denote by \( H \) the self-adjoint Hamiltonian operator, eventually time dependent. In the absence of the ‘dissipator’ \( D \), the commutator in (1a) is the generator of an unitary dynamics. The dissipator encapsulates non-conservative interactions with the environment. In the canonical form (1b), \( D \) is manifestly trace and self-adjointness preserving. Distinguishing traits of the canonical form are [30]:

(i) the sum ranges over \( \mathcal{X} = d^2 - 1 \) addends corresponding to a collection \( \{ I_{\ell,t}, I_{\ell,t}^\dagger \}_{\ell=1}^{\mathcal{X}} \) of decoherence, or Lindblad’s, operators. At any time \( t \), the union of the collection together with the normalized identity operator, constitutes an orthonormal basis of \( \mathcal{M}_d \) with respect to the Hilbert–Schimdt inner product:

\[
\text{Tr} I_{\ell,t} = 0 \quad \& \quad \text{Tr} \left( I_{\ell,t}^\dagger I_{\ell,t} \right) = \delta_{\ell,\ell}
\]

(2)

(ii) the completeness relation in \( \mathcal{M}_d \) implies, that the decoherence operators satisfy the positive operator valued measurement type condition

\[
\sum_{\ell=1}^{\mathcal{X}} I_{\ell,t} I_{\ell,t}^\dagger = \mathbb{1}_\mathcal{H}
\]

(3)

with \( \mathbb{1} = (d^2 - 1)/d \). Since (3) is usually not emphasized in the literature but we use it in section 3, we prove it in appendix G;

(iii) the decoherence operators are unique modulo unitary transformations. The time dependence in general cannot be removed by unitary transformations unless \( d = 2 \);

(iv) the real-valued time dependent functions \( \{ \omega_{\ell,t} \}_{\ell=1}^{\mathcal{X}} \) in (1b) are uniquely determined by the canonical representation. They are referred to in [30] as canonical decoherence rates a terminology that we adopt here too at variance with [43] where we called them ‘weights’;

(v) the canonical decoherence rates in (1b) are not sign definite. Physically, a negative sign corresponds to information back-flow from the environment into the system [60].

In the present work we also surmise that canonical rates are sufficiently regular bounded functions. The latter assumption is expected to hold over a finite time interval \( \mathbb{1} \) in physically relevant contexts [61]. From the mathematical point of view, (1) is equivalent to a system of \( d^2 \) linear ordinary time non-autonomous equations by vectorization (see e.g. section 3.5 of [29]). Under the regularity assumptions we hypothesize,
existence and uniqueness theorems on ordinary differential equations ensure the existence of a flow $F: \mathbb{R} \times \mathbb{R} \times \mathbb{C}^d \mapsto \mathbb{C}^d$ [62] acting on vectorized state operators

$$\text{vec}(\rho_t) = F_{t,s} \text{vec}(\rho_s).$$

By construction, the flow is isomorphic to a two parameter family of linear maps

$$\mathcal{B}_{t,s}: \mathcal{M}_d \mapsto \mathcal{M}_d$$

enjoying as a consequence the group properties

$$\mathcal{B}_{t,s} = \mathcal{B}_{t,v} \mathcal{B}_{v,s} \quad \forall t, v, s \in \mathbb{I}$$

$$\mathcal{B}_{t,s} = \text{Id}_d \quad \forall s \in \mathbb{I}$$

(5)

$\text{Id}_d$ being here the identity map on $\mathcal{M}_d$. By construction, (5) preserves self-adjointness and trace of its argument. These properties are most conveniently exhibited by the Sudarshan-Mathews-Rau reshuffling operation of the flow matrix $F$ [29, 58]. Reshuffling is a linear non-spectrum preserving involution which turns the flow matrix into a self-adjoint matrix known as the dynamical or Choi-matrix:

$$\mathcal{D}_{a,t,s} := F_{t,s}.\$$

Any self-adjoint matrix can be written as the difference of two positive matrices. Hence upon expressing (4) in terms of $\mathcal{D}_{a,t,s}$ and using the isomorphism between $\mathbb{C}^d$-valued flows and linear maps on $\mathcal{M}_d$, we arrive at the Wittstock-Paulsen canonical representation of a 'completely bounded' linear map (5)

$$\rho_t = \mathcal{B}_{t,s}(\rho_s) = \sum_{a=1}^{s(+)\cdot} B_{a,t,s}^{(+)} \rho_s B_{a,t,s}^{(+)} + \sum_{a=1}^{s(-)\cdot} B_{a,t,s}^{(-)} \rho_s B_{a,t,s}^{(-)}$$

(6)

with

$$\sum_{a=1}^{s(+)\cdot} B_{a,t,s}^{(+)} B_{a,t,s}^{(+)\dagger} + \sum_{a=1}^{s(-)\cdot} B_{a,t,s}^{(-)} B_{a,t,s}^{(-)\dagger} = 1_{\mathcal{H}}$$

to ensure trace preservation.

The uniqueness of the canonical rates (property (iv)) yields a straightforward characterization of the Lindblad–Gorini–Kossakowski–Sudarshan master equation [32, 33]. Specifically, if the rates are positive-definite the restriction of (5) to times $0 \leq s \leq v \leq t$ always reduces (6) to the canonical form of a completely positive semi-group [63–65]; $B_{a,t,s}^{(-)} = 0$ for all $a = 1, \ldots, s\cdot$. The restriction to a semi-group composition law can be understood upon observing that the inverse of a completely positive map is itself completely positive if and only if the map is also unitary see e.g. [66].

The operator sum characterization (6) of completely bounded linear maps is all we need to discuss the influence martingale which we introduce in section 3. The upshot in the finite dimensional case is that completely bounded maps are the most general self-adjoint preserving linear maps between matrix spaces [67]. For completeness we recall in appendix A the abstract algebraic definitions of completely positive and completely bounded maps.

We conclude this section with a comment on the physical relevance of completely bounded master equations, i.e. (1) with non-sign definite canonical rates. Completely positive dynamical maps may come about as special solutions of a completely bounded master equation. This is a situation often encountered in models where the partial trace can be exactly evaluated see e.g. [38, 68, 69]. Complete positivity is in such cases a property enjoyed by the restriction of the dynamical map to a subset of initial data [42] and is not preserved under semi-group composition see e.g. section 3.3 of [66] and appendix B. It is, however, fair to add that from the foundational point of view, the interpretation of non-completely positive dynamical maps is in the general case contentious see e.g. [70, 71] although their phenomenological use is widely accepted [72].

We refer to [42, 58, 73] for further properties of completely bounded maps.
3. Unraveling-paired complete positive map

In [43] we prove that a state operator solution of (1) admits the unraveling in quantum trajectories

\[ \rho_t = E \left( \mu_t \psi_t \psi_t^\dagger \right) \]  

(7)

where the expectation value E is over classical piecewise-deterministic processes [37]. The logic leading to the proof in [43] is the following.

A state vector must take values on the Bloch hyper-sphere. Hence, the vector valued stochastic process \( \psi_t \) appearing in (7) must satisfy

\[ \| \psi_t \|^2 = 1 \]

with probability one. Next, trace and self-adjoint preservation impose that \( \mu_t \) must be a real-valued mean-preserving martingale adapted to the natural filtration \( \{ \mathcal{F}_t \}_{t \geq t_0} \) of the process \( \{ \psi_t, \psi_t^\dagger \}_{t \geq t_0} \). In other words, \( \mu_t \) is a functional of \( \psi_t, \psi_t^\dagger \) for any time \( s \) up to but no greater than \( t \). This means that at any instant of time it is always possible to define

\[ \mu_t^{(\pm)} = \max(0, \pm \mu_t) \]

and to couch (7) into the form

\[ \rho_t = E \left( \mu_t^{(\pm)} \psi_t \psi_t^\dagger \right) - E \left( \mu_t^{(-)} \psi_t \psi_t^\dagger \right). \]  

(8)

The result of [43] is the construction of Itô stochastic differential equations for \( \psi_t \) and \( \mu_t \). Furthermore, the expectations values on the right hand side of (8) are always individually amenable to Kraus form. These considerations show that the non-sign definite martingale \( \mu_t \)—the influence martingale—naturally enters the unraveling of the completely bounded master equation (1) in order to reproduce at statistical level the Wittstock-Paulsen decomposition (6).

We now show that the proof of the unraveling holds true even we impose stronger hypotheses than in [43] on the stochastic state vector. Specifically, we add the requirement that the expectation value

\[ \tilde{\rho}_t = E \psi_t \psi_t^\dagger \]  

(9)

simultaneously solves a Lindblad–Gorini–Kossakowski–Sudarshan master equation. To prove that (7) and (9) are indeed unravelings, we require \( \psi_t \) to satisfy exactly the same non-linear, Bloch hyper-sphere preserving Itô stochastic Schrödinger equation as in [6, 8, 17]

\[ \begin{align*}
\text{d}\psi_t &= \text{d} f_t + \sum_{\ell = 1}^{\mathcal{F}} \text{d} \nu_{\ell,t} \left( \left. \frac{L_{\ell,t} \psi_t}{\| L_{\ell,t} \psi_t \|} \right| - \psi_t \right) \\
- \text{d}\mu_t &= - \sum_{\ell = 1}^{\mathcal{F}} r_{\ell,t} \left( 1 - \| L_{\ell,t} \psi_t \|^2 \right) \mu_t \\
\psi_{t_0} &= z
\end{align*} \]  

(10a)

(10b)

(10c)

with \( z^\dagger z = 1 \) and

\[ r_{\ell,t} \geq 0 \quad \forall \ell = 1, \ldots, \mathcal{F} \]  

(11)

positive rate functions whose explicit value will be determined below. The dynamics of \( \psi_t^\dagger \) straightforwardly follows by applying the dual conjugation operation to (10). The influence martingale process \( \{ \mu_t \}_{t \geq t_0} \) satisfies the Itô stochastic differential equation

\[ \begin{align*}
\text{d}\mu_t &= \sum_{\ell = 1}^{\mathcal{F}} \left( \frac{\nu_{\ell,t}}{r_{\ell,t}} - 1 \right) \text{d}\nu_{\ell,t} \\
\text{d}r_{\ell,t} &= - r_{\ell,t} [\| L_{\ell,t} \psi_t \|^2] \text{d}t \\
\mu_{t_0} &= 1
\end{align*} \]  

(12a)

(12b)

(12c)
In (10a) and (12b) the \( \{ \text{d} \nu_{\ell,k} \}_{\ell,k=1}^{\mathcal{L}} \) denote the increments of the counting processes [21, 37]:

\[
\text{d} \nu_{\ell,k}; \text{d} \nu_{\ell',k'} = \text{d} \nu_{\ell,k}; \text{d} \nu_{\ell',k'} = \delta_{\ell,\ell'} \delta_{k,k'} \text{d} \nu_{\ell,k}; \text{d} \nu_{\ell',k'} = \delta_{\ell,\ell'} \delta_{k,k'} \text{d} \nu_{\ell,k}; \text{d} \nu_{\ell',k'}
\]  

\[\text{(13a)}\]

\[
E \left( \{ \psi_{t}, \psi_{t}^{+} \} \cap \delta \right) = r_{\ell}; \| L_{\ell}; \psi_{t} \| ^{2} \text{d} t
\]

\[\text{(13b)}\]

for \( \ell, k = 1, \ldots, \mathcal{L}. \) The stochastic differential in (12a) are the compensated increments of the counting process (13). This fact together with boundedness assumptions on the rates \( r_{\ell}; \) ensures that \( \{ \mu_{t} \}_{t \geq t_{0}} \) enjoys the martingale property.

As well known [3, 6, 17, 75, 76], the assumptions (10) and (13) imply that (9) solves the completely positive master equation

\[
\partial_{t} \tilde{\rho}_{t} = -i [H_{t}, \tilde{\rho}_{t}] + \sum_{\ell=1}^{\mathcal{L}} r_{\ell} L_{\ell}; (\tilde{\rho}_{t}).
\]

\[\text{(14)}\]

In order to ensure that (7) indeed satisfies (1) we impose the unraveling conditions

\[
\omega_{\ell}; = r_{\ell}; - c_{t}.
\]

\[\text{(15)}\]

The conditions determine the jump rates \( \{ r_{\ell}; \}_{\ell=1}^{\mathcal{L}} \) up to a positive function \( c_{t}, \) Self-consistency of (15) only requires that \( c_{t} \) is larger than the absolute value of the smallest negative canonical rate \( \omega_{\ell} \) at each time \( t, \) i.e.

\[
c_{t} > - \min_{\ell=1,\ldots,\mathcal{L}} \omega_{\ell}; \equiv | \omega_{\ell}; |.
\]

\[\text{(16)}\]

The reader concerned about the non-uniqueness of the unraveling conditions should consider that the state vector is not an observable process. We refer to [43] for an extensive discussion of this point. Figure 1(a) yields a graphical proof of the self-consistency of the unraveling conditions.

If we now insert (15) into the drift (10b) and use (3) on the Bloch hyper-sphere we recover the form of the drift hypothesized in [43]

\[
f_{t} = -i H_{t} \psi_{t} - \sum_{\ell=1}^{\mathcal{L}} \omega_{\ell}; L_{\ell}; - \frac{1}{2} \| L_{\ell}; \psi_{t} \| ^{2} \text{d} t
\]

\[\text{(17)}\]

A straightforward application of Itô lemma as done in [43] completes the proof. For readers’ convenience we reproduce the steps of the calculation in appendix D.

Some remarks are here in order. First, time dependence of decoherence rates and operators does not play any role in the proof. Second, the unraveling (10) models a continuous time record of indirect measurements of the system gathered by means of decoherence channels consistent with the orthonormal conditions (2).

We emphasize, however, that only the weaker condition (3) is needed for (7) and (9) to hold simultaneously true. Hence, the unraveling also holds when the time-local master (1) is not in canonical form (i.e. \( \mathcal{L} \) is arbitrary and the conditions (2) are not satisfied) if (3) is satisfied. In fact, in section 8 we show how to release (3) in the non-canonical setup. The drawback of non-canonical master equations is that the signs of the rates do not immediately characterize the properties of the flow as linear operator map [30].

We summarize the main result of this section in the following proposition

**Proposition 3.1.** The solution of the completely bounded master equation (1) i.e. canonical form enjoying properties (i)–(v)) always admits the unraveling representation (7). The stochastic state vector \( \psi_{t} \) and the influence martingale \( \mu_{t} \) respectively obey the Itô stochastic differential equations (10) and (12) driven by the counting processes (13). The canonical non-sign definite rates \( \{ \omega_{\ell} \}_{\ell=1}^{\mathcal{L}} \) in (1) and the auxiliary positive jump rates \( \{ r_{\ell} \}_{\ell=1}^{\mathcal{L}} \) are related by the unraveling conditions (15) up to a positive function \( c_{t} \) subject to the constraint (16). As a consequence, for each admissible choice of \( c_{t} \) there exists a completely positive master equation of the form (14) whose solution admits the unraveling representation (9).
4. An optimization criterion for the influence martingale

The Itô stochastic differential equation governing the influence martingale is enslaved to the stochastic Schrödinger equation (10) and exactly integrable given the solution of this latter equation. In particular, for any physical path of $\psi_t$ on the Bloch hyper-sphere we get

$$\mu_t = e^{\int_0^t dc_t \lambda_t}$$

where $\lambda_t$ is the step process satisfying the Itô stochastic differential equation

$$d\lambda_t = -c_t \lambda_t \sum_{\ell} \frac{d\nu_{\ell, t}}{w_{\ell, t} + c_t}$$

$$\lambda_0 = 1$$

having used (15) to exhibit the dependence upon the positive function $c_t$. So far, $c_t$ is subject only to the constraint (16) and otherwise arbitrary. The interpretation of the variance of the influence martingale suggests a criterion to resolve such indeterminacy. Namely, let us consider

$$\omega_t = (\mu_t - 1) \psi_t \psi_t^\dagger$$

and its variance

$$0 \leq \text{Tr} (\omega_t - E\omega_t)^2 = \text{Tr} E\omega_t^2 - \text{Tr}(E\omega_t)^2.$$ 

We readily arrive at the inequality

$$\text{Tr}(\rho_t - \bar{\rho}_t)^2 \leq E\mu_t^2 - 1$$

stating that the variance of the influence martingale yields an upper-bound on the squared Hilbert–Schmidt distance between the completely bounded and the unraveling-paired completely positive state operator. From the information-theoretic point of view we may interpret the variance of the influence martingale as a $\chi$-squared divergence [77] between bounded (pseudo-probability) measures.

The dynamics of the second moment of the influence martingale allows us to delve deeper on the consequences of the inequality (19). Itô lemma yields

$$E d\mu_t^2 = \sum_{\ell} \frac{c_t^2}{w_{\ell, t} + c_t} \left( E\mu_t^2 ||L_{\ell, t}^\dagger \psi_t||^2 \right) dt.$$ 

By (16) (see figure 1(b)) the right hand side is positive definite and admits an universal upper bound with respect to the state of the system. Namely, upon using (3) and the definition of the smallest negative decoherence rate (16), the sum reduces to

$$E d\mu_t^2 \leq \frac{2 c_t^2 E\mu_t^2}{|w_{\ell, t}| + c_t} dt.$$ 

The choice

$$c_t^* = 2|w_{\ell, t}|$$

yields the minimum upper bound in (20) and as a consequence an analytic estimate of the variance of the influence martingale. Notably, for a completely positive master equation (21) is consistent with the reduction of the martingale to a dispersion-less process and thus the vanishing of the $\chi$-squared divergence. We summarize our findings in the following proposition.

**Proposition 4.1.** The choice of $c_t^*$ (21) optimizes the upper bound on the Hilbert–Schmidt distance (19) between the solutions of master equations paired according to proposition 3.1. The result is universal with respect to the initial value of the state operators and therefore it uniquely determines the pairing.
A further consequence of (21) is the confinement of the step process (18):

$$|\lambda_1| \leq 1 \quad \forall t \geq t_0.$$  

To prove confinement we observe that a jump of the $\ell$-counting processes on the right hand side of (18a) yields

$$\lambda_{t+\delta t} = \frac{\omega_{\ell, t}}{\omega_{\ell, t} + 2|\omega_{\ell, s}|} \lambda_t$$

where by definition (see figure 1(c))

$$\left| \frac{\omega_{\ell, t}}{\omega_{\ell, t} + 2|\omega_{\ell, s}|} \right| \leq 1.$$  

The confinement in the unit interval of the absolute value of the step process allows us to invoke the commutant representation of a completely bounded map [50] (see appendix E)

$$\mathcal{B}_{t,s}(\rho_s) = \sum_{a,b=1}^{\hat{\ell}} C_{a,b,t,s} \hat{B}_{a,t,s} \rho_s \hat{B}_{b,t,s}$$

to conclude that the evolution of the stochastic state operator in $\mathcal{M}_{2d}$

$$\xi_t = \frac{1 + \lambda_1 \sigma_i}{2} \otimes \psi_i \psi_i^\dagger$$

is governed by a completely positive trace preserving map. In (24) and below $\sigma_i, i = 1, 2, 3$ denote the Pauli matrices.

**4.1. On the relation between the influence martingale, the minimum rate of isotropic noise and the structural physical approximation**

We can interpret the optimal criterion (21) as the condition that determines, under the constraints imposed by the unraveling, the closest completely positive semi-group to that generated by a given completely bounded master equation. In this formulation, the optimization problem is reminiscent of the quantification of the distance of a snapshot of quantum evolution from a completely positive map studied in [48]. The quantifier, called the quantum isotropic noise, is obtained by considering the maximally mixed state generated by the depolarizing map. Adding a depolarizing channel to a positive map serves the purpose of engineering entanglement witness operators suited to laboratory implementation. The procedure is referred to as the structural physical approximation [49]. The authors of [48] introduce a version of the structural physical approximation [49] which leads to the definition of the minimum rate of isotropic noise $n^{\star}$. To exhibit the connection of these concepts with the influence martingale, we avail us of the derivation of the minimum rate of isotropic noise presented in [30]. Over an infinitesimal time interval the flow of (1) maps an arbitrary state operator $\rho$ as

$$\mathcal{B}_{t+dt, s}(\rho) = \rho + dt \mathcal{L}_s(\rho)$$
where $\mathcal{L}(\rho)$ denotes the right hand side of (1). In general, (25) only describes a completely bounded evolution. To extricate from (25) a completely positive map it is sufficient [49] to add a completely depolarizing channel with a rate large enough to offset the most negative eigenvalue of the Choi matrix of (25). Explicitly, this means to determine the minimum value of $n_t$ such that

$$\mathcal{R}_{\text{+d}t}[n_t](\rho) = (1 - dt n_t)\mathcal{R}_{\text{+d}t}[\rho] + dt n_t\frac{1}{d} I = \rho + dt \left( \mathcal{L}(\rho) + n_t \left( \frac{1}{d} I - \rho \right) \right)$$

is completely positive. The analysis of the Choi matrix of (26) detailed in appendix C of [30] proves that

$$n_t^* = d |\omega_{x,t}| = \frac{d}{2} c_t^*.$$

The optimization (26) corresponds to a structural physical approximation applied to the infinitesimal increment of a flow. We recall that in [49] Horodecki and Ekert introduce the structural physical approximation in connection with the positive map criterion stating that the state operator $\rho$ of a bipartite system is separable if and only if

$$(\text{Id} \otimes \Lambda)(\rho) \geq 0$$

i.e. $(\text{Id} \otimes \Lambda)(\rho)$ is positive for all maps $\Lambda$ positive but non-completely positive maps of operators acting on the Hilbert space of one of the constituents. The structural physical approximation corresponds to the minimal deformation of $(\text{Id} \otimes \Lambda)$ by a depolarizing map which occasion a completely positive maps. Hence, the structural physical approximation opens the way to laboratory realization of entanglement witness operations. We refer to [78] for a recent review of this subject.

5. ‘Embedding’ quantum channel induced by the influence martingale

The question naturally arises whether the expectation value of (24) specifies the solution of a Lindblad–Gorini–Kossakowski–Sudarshan master equation in $\mathcal{M}_{2d}$. The answer is positive. To see this, we start by observing that the operator

$$\varrho_t = E \lambda_t \rho_t \rho_t^\dagger = e^{\frac{1}{d} \int_0^t \sigma_\lambda \rho_s ds} \rho_t$$

satisfies the non-trace-preserving master equation

$$\partial_t \varrho_t = -it [H_t, \varrho_t] + \sum_{\ell=1}^d r_{\ell,t} D_{\ell,t}(\rho_t) - \sigma_\ell \sum_{\ell=1}^d L_{\ell,t} \varrho_t \sigma_\ell L_{\ell,t}^\dagger$$

implying

$$|Tr \varrho_t| \leq 1.$$

Next, we associate to each of the Lindblad operators in $\mathcal{M}_d$ two operators in $\mathcal{M}_{2d}$

$$V_{2\ell - 1,t} = I_2 \otimes L_{\ell,t}$$
$$V_{2\ell,t} = \sigma_3 \otimes L_{\ell,t}$$

with corresponding rates

$$\sigma_{2\ell - 1,t} = \sqrt{r_{\ell,t}} \cos \frac{\theta_{\ell,t}}{2}$$
$$\sigma_{2\ell,t} = \sqrt{r_{\ell,t}} \sin \frac{\theta_{\ell,t}}{2}.$$

It is straightforward to verify that the operators $\{V_{\ell,t}\}_{\ell=1}^{2d}$ satisfy the canonical relations (2) while the rates are positive for

$$0 \leq \theta_{\ell,t} \leq \pi.$$

(28)
As last step, we introduce the Lindblad–Gorini–Kossakowski–Sudarshan master equation for the state operator \( \gamma_i \) on \( \mathbb{C}^2 \otimes \mathcal{H} \)

\[
\partial_t \gamma_i = -i [1_2 \otimes H_t \gamma_i] + \sum_{\ell=1}^{2 L} \alpha_{\ell + i} \mathsf{D}_{\ell + i}(\gamma_i).
\]  

(29)

Our definitions imply that

\[
\sum_{\ell=1}^{2} \alpha_{2\ell - 2 + i \iota} V_{2\ell - 2 + i \iota}^\dagger V_{2\ell - 2 + i \iota} = \rho_{\ell + i} 1_2 \otimes \mathsf{L}_{\ell + i}^\dagger \mathsf{L}_{\ell + i}
\]

and

\[
\sum_{\ell=1}^{2} \alpha_{2\ell - 2 + i \iota} V_{2\ell - 2 + i \iota} = \rho_{\ell + i} \begin{bmatrix} \mathsf{L}_{\ell + i} \gamma_{1,1,i} \mathsf{L}_{\ell + i}^\dagger & \cos \theta_{\ell + i} \mathsf{L}_{\ell + i} \gamma_{1,2,i} \mathsf{L}_{\ell + i}^\dagger \\ \cos \theta_{\ell + i} \mathsf{L}_{\ell + i} \gamma_{2,1,i} \mathsf{L}_{\ell + i}^\dagger & \mathsf{L}_{\ell + i} \gamma_{2,2,i} \mathsf{L}_{\ell + i}^\dagger \end{bmatrix}.
\]

Hence, we are always entitled to relate diagonal blocks of \( \gamma_i \) to the solution of (14) e.g. by setting

\[
\gamma_{1,1,i} = \gamma_{2,2,i} = \frac{1}{2} \rho_i.
\]

Furthermore, the off-diagonal blocks satisfy (27)

\[
\gamma_{1,2,i} = \gamma_{2,1,i} = \frac{1}{2} \theta_i
\]

if the compatibility conditions

\[
\cos \theta_{\ell + i} = \frac{\rho_{\ell + i} - \rho_i}{\rho_{\ell + i} + \rho_i} = \frac{\omega_{\ell + i}}{\omega_{\ell + i} + \rho_i}
\]  

(30)

hold true. This is the case if \( \epsilon_i \) is equal to the ‘optimal’ value (21) which implies the confinement condition (25). Furthermore the compatibility conditions (30) admit unique solutions in the ‘principal branch’ specified by the requirement (28) of positive canonical rates.

We thus conclude that for any initial condition

\[
\gamma_0 = \frac{1_2 + \sigma_i}{2} \otimes \rho_0
\]

the solution of (29) at time \( t \) is

\[
\gamma_t = \frac{1}{2} \left[ \begin{array}{c} \rho_t \\ \theta_t \end{array} \right] = \mathsf{E} \left( \frac{1_2 + \lambda_i \sigma_i}{2} \otimes \psi_t \psi_t^\dagger \right).
\]  

(31)

We can rephrase the result as follows

**Proposition 5.1.** The solutions of the completely bounded master equation (1) and that of its optimally paired in the sense of proposition 4.1 completely positive master equation (14) can be lifted to a differential family in \( \mathcal{M}_{2,2} \) which is itself solution of the completely positive master equation (29).

Physically, (31) describes the interaction of the original system with an ancillary qubit stylized in figure 2. An immediate corollary is thus

\[
\rho_t = \mathsf{Tr}_{\mathbb{C}^2} \left( \gamma_t \right)
\]

and

\[
\rho_t = e^{\int_0^t \omega_i dt} \mathsf{Tr}_{\mathbb{C}^2} \left( \sigma_1 \otimes \mathsf{1}_H \gamma_i \right).
\]

It is worth emphasizing that [79, 80] also introduce an embedding Hilbert space, namely, \( \mathbb{C}^3 \otimes \mathcal{H} \), where the solution \( \rho_t \) is realized by a Lindblad–Gorini–Kossakowski–Sudarshan master equation in the larger Hilbert space. The influence martingale has the advantage of requiring a smaller dimension of the embedding Hilbert space whilst diagonal and off diagonal blocks satisfy unraveling paired master equations.

One motivation to introduce the embedding representation is to provide an avenue for a continuous time measurement interpretation [81] even in the case when \( \rho_t \) is non positive [43, 80]. In the coming section, we analyze the application to time reversal of a completely positive evolution.
5.1. Remark on the Paulsen-Suen ‘triple’ norm and optimal definition of embedding

In [50] the examination of the order structure induced by completely bounded maps leads Paulsen-Suen to introduce a new norm, referred to as ‘triple’ in [82], for a completely bounded map \( \mathcal{B} \). The definition of the norm relies on the idea of finding an optimal auxiliary completely positive and trace preserving map \( \mathcal{P} \) permitting to embed \( \mathcal{B} \) in a completely positive map \( \mathcal{E} \) defined by

\[
\mathcal{E} \left( \begin{bmatrix} Y & X \\ W & Z \end{bmatrix} \right) = \begin{bmatrix} \mathcal{P}(Y) & \mathcal{B}(X) \\ \mathcal{B}^\dagger(W) & \mathcal{P}(Z) \end{bmatrix}.
\]

Paulsen and Suen triple norm is thus

\[
|||\mathcal{B}||| = \inf \{ ||\mathcal{P}|| \text{ such that } \mathcal{E} \text{ is completely positive} \}.
\]

In the definition the infimum is over the operator norm \( \|\cdot\|_{cb} \). In the finite dimensional case, the use of the canonical operator sum representation of \( \mathcal{P} : \mathcal{M}_d \rightarrow \mathcal{M}_d \) for some integer \( m \)

\[
\mathcal{P}(X) = \sum_{i=1}^{m} A_i X A_i^\dagger
\]
yields

\[
\|\mathcal{P}|| := \sup_{v \in \mathcal{H}, \|v\| \leq 1} \left\| \sum_{i=1}^{m} A_i A_i^\dagger v \right\|.
\]

A completely positive map satisfying \( |||\mathcal{B}||| = ||\mathcal{P}|| \) is called ‘dominating’ [50].

It has not escaped our notice that the optimization of the variance of the influence martingale yields a value \( (21) \) that is threshold to satisfy the embedding conditions \( (30) \). Larger values of \( \varepsilon_t \) would be consistent with the embedding at the price of larger values of the rates \( r_{\varepsilon,\ell} \) and consequently of the operator norm of the realizations of the completely positive map in the diagonal blocks. The question naturally arises whether restricting the optimization in the definition of the triple norm to flows of master equations is equivalent to the optimization we consider. We leave proving or disproving this conjecture to future work.

6. Recovery of an initial state operator

The influence martingale framework unravels any completely bounded master equations. An appealing application is time reversal of a completely positive evolution in a finite time horizon \([t_0, t_f]\). We start from the completely positive canonical master equation \( (\hbar_{\varepsilon,\ell} \geq 0) \)

\[
\dot{\chi}_t = -i [H_t, \chi_t] + \sum_{\ell=1}^{\ell} \hbar_{\varepsilon,\ell} D_{\ell,\varepsilon}(\chi_t) \quad (33a)
\]

\[
\chi_{t_0} = \chi_t \quad (33b)
\]

and consider the involution of the time parameter

\[
t' = t_f + t_0 - t. \quad (34)
\]

There are now two avenues to describe the evolution of the inverse of the flow of \( (33) \).
6.1. Genuine backward dynamics

We define the reverse of the solution of (33) as

\[ \chi_{\rho}^b = \chi_t. \]

We use the fluxion notation to indicate differentiation with respect to the explicit time dependence:

\[ \frac{d}{d\rho} \chi_{\rho}^b = \frac{d}{d\rho} \chi_t = -\chi_t. \]

We thus obtain the backward master equation

\[ \chi_{\rho}^b = t \left[ H_{\rho}^b, \chi_{\rho}^b \right] - \sum_{\ell = 1}^{\infty} \hat{H}_{\ell,\rho} ^b \mathcal{D}_{t,\rho} \left( \chi_{\rho}^b \right) \]

(35a)

\[ \chi_{t_j}^b = \chi_{t_j} \]  

(35b)

where

\[ H_{\rho}^b = H_t \]

\[ \hat{H}_{\ell,\rho} ^b = \hat{H}_{\ell,t}. \]

To unravel (35) we introduce the descending filtration \( \mathcal{F}_t \), of ‘future events’ \[ \mathcal{F}_t \] i.e. a sequence of \( \sigma \)-algebras increasing as the time \( \rho \) decreases from \( t_j \) to \( t_0 \). Correspondingly, we consider the backward stochastic differential equation \[ \mathcal{F}_t \] in the post-point prescription

\[ d^b \psi_{\rho}^b = \psi_{\rho}^b - \frac{\left[ H_{\rho}^b, \psi_{\rho}^b \right]}{2} dt - \sum_{\ell = 1}^{\infty} d\nu_{\ell,\rho} ^b \left( \frac{L_{\ell,\rho} \psi_{\rho}^b}{\| L_{\ell,\rho} \psi_{\rho}^b \|} - \psi_{\rho}^b \right). \]

(36)

The differentials

\[ d\nu_{\ell,\rho} ^b = \nu_{\ell,\rho} ^b - \nu_{\ell,\rho} ^{\rho \rightarrow t_0} dt \]

of the counting processes in (36) satisfy the same differential algebra relations of forward increments (13a). In consequence of the post-point prescription we require that counting process differentials are characterized by conditional expectations with respect to the descending filtration

\[ \mathbb{E} \left( d^b \nu_{\ell,\rho} ^b \left| \{ \psi_{\rho}^b, \psi_{\rho}^{\rho \rightarrow t_0} \} \cap \mathcal{F}_\rho \right. \right) = \hat{H}_{\ell,\rho} ^b \left\| L_{\ell,\rho} \psi_{\rho}^b \right\|^2 dt. \]

The backwards (post-point) Itô differential formula

\[ d\chi_{\rho}^b = \mathbb{E} \left( \chi_{\rho}^b \psi_{\rho}^{\rho \rightarrow t_0} \right) = \mathbb{E} \left( \left( d^b \psi_{\rho}^b \right) \psi_{\rho}^{\rho \rightarrow t_0} \right) \]

recovers (35a). The terminal condition (35a) is implemented by assigning the terminal condition on the state vector from a probability distribution such that

\[ \chi_{t_j} = \mathbb{E} \psi_{t_j} \psi_{t_j}^{\rho \rightarrow t_0}. \]

Clearly paths solution of (36) are not reversed path of the unraveling of (33) but realization of a distinct stochastic process whose connection to (33) resides in the second order statistics:

\[ \chi_t = \mathbb{E} \psi_{\rho}^b \psi_{\rho}^{\rho \rightarrow t_0}. \]

Although mathematically straightforward, from the physical point of view the existence of the process (36) appears to be mostly of conceptual interest.
6.2. Forward implementation of the backward dynamics

In order to describe the reversed dynamics in terms of a time variable $t$ increasing from $t_0$ to $t_f$ we posit

$$X_t^\flat = X_{t_f + t_0 - t}.$$  \hspace{1cm} (37)

In such a case, we arrive at the completely bounded canonical master equation

$$\dot{X}_t^\flat = i [H_{t_f + t_0 - t}, X_t^\flat] - \sum_{\ell=1}^{L} \mathcal{H}_{\ell, t_f + t_0 - t} \mathcal{D}_{\mathcal{H}_{\ell, t_f + t_0 - t}}(X_t^\flat).$$  \hspace{1cm} (38a)

$$X_{t_0}^\flat = X_{t_f}.$$  \hspace{1cm} (38b)

The use of the influence martingale yields the identity

$$X_t^\flat = e^{\int_{t_0}^{t} ds \mathcal{E} \lambda_t \psi_t \psi_t^\dagger}$$  \hspace{1cm} (39)

where the state vector solves the forward Itô stochastic differential equation (10) once we perform in (10b) the replacement

$$H_t \rightarrow -H_{t_f + t_0 - t},$$

we impose the unraveling conditions

$$r_{\ell, t} = c_t - \mathcal{H}_{\ell, t_f + t_0 - t}$$

with

$$c_t = 2 \max_{\ell = 1, \ldots, L} \mathcal{H}_{\ell, t_f + t_0 - t}$$

and we sample the initial condition (10c) from a probability such that (38b) holds. These requirements entail that the process \{\lambda_t\}_{t \geq t_0} in (39) satisfies the equation

$$d\lambda_t = -\lambda_t \sum_{\ell=1}^{L} \left( \frac{\mathcal{H}_{\ell, t_f + t_0 - t}}{r_{\ell, t}} + 1 \right) d\nu_{\ell, t}$$

subject as usual to the initial condition $\lambda_{t_0} = 1$.

We emphasize that the jumping rates of the forward dynamics unraveling (38) in general differ from those unraveling the forward dynamics (33) we are reversing:

$$r_{\ell, t} \neq \mathcal{H}_{\ell, t}.$$  \hspace{1cm} (13)

If the microscopic dynamics yields

$$\mathcal{H}_{\ell, t} = 1 \quad \forall \ell = 1, \ldots, L \quad \& \quad t \in [t_0, t_f]$$

we get immediately

$$r_{\ell, t} = 1 \quad \forall \ell = 1, \ldots, L \quad \& \quad t \in [t_0, t_f].$$

This situation maybe encountered for an open system dynamics brought about by an environment described by an equilibrium ensemble in the high temperature limit. In this particular case, and for a purely dissipative dynamics ($H_t = 0$) the insertion of the martingale process

$$\mu_t = e^{2 g (t - t_0)} \lambda_t$$

in the average maps the unraveling of the forward master equation (33) into that of the forward representation of the reversed dynamics (38) according to (37).
7. Recovery by embedding in a quantum channel

Embedding the completely bounded flow (38) in a completely positive map as in section 5 allows us to design an operational protocol to reverse a quantum evolution. The protocol consists of the following steps:

- we wish to recover $\rho_t$ given

$$\rho_t = \mathcal{R}_{t_1, t_0}(\rho_{t_0})$$

where $\mathcal{R}_{t_1, t_0}$ is a completely positive map solving for $t \in [t_0, t_1]$ (33) or more generally (1);

- we couple the system to an ancillary qubit and define the tensor product state

$$\gamma_t = \frac{1 + \sigma_1}{2} \otimes \rho_t$$

- $\gamma_t$ specifies the input of a quantum channel of the type (29) such that the off-diagonal corners of the flow generated by the channel satisfy

$$\mathcal{R}_{t + t_0 - t_0, t_1} = \mathcal{R}_t^{-1} \quad \forall t \in [t_0, t_1]$$

- we obtain in output the initial value of the state operator by performing a measurement on the ancilla according to

$$\rho_t = e^{\int_{t_1}^{t_0} ds \mathcal{R}_t^{-1} \text{Tr}_1 \left( \sigma_1 \otimes 1_H \gamma_{t_0 - t} \right)}$$

where $\text{Tr}_1$ denotes the partial trace with respect to the first argument of the tensor product.

In order to illustrate the protocol with an actual example, we consider the completely positive master equation

$$\partial_t \chi_t = -i[H, \chi_t] + g D_{\sigma_+}(\chi_t) + g e^{i\omega D_{\sigma_-}(\chi_t)}$$

which models the evolution of a driven qubit in contact with a thermal reservoir stylized in figure 3. The ladder operators

$$\sigma_{\pm} = \frac{\sigma_1 \pm i \sigma_2}{2}$$

readily satisfy (3). The caption of figure 4 specifies the parameters necessary to reproduce the numerics.

We can also recover the initial state of the dynamics by unraveling the completely bounded dynamics describing the off-diagonal corner of the embedding flow with the influence martingale. Points of the reverse trajectory computed by means of the influence martingale are marked by diamonds in figure 4. From the numerical point of view, the use of the unravelling offers an advantage with respect to direct integration of the master equation only for systems with a sufficiently large number of states [43]. Convergence of the numerical simulations is in all cases guaranteed by standard results on the solutions of stochastic differential equations driven by counting processes [85]. We notice that, at least in principle, an operational implementation of the influence martingale algorithm is also possible. The implementation requires, however, post-selection by a classical apparatus. Namely, the protocol presumes storing the outcomes of continuous weak measurement records on a classical register. We suppose that each sequence of detected events in the record reconstructs a quantum trajectory. Hence each sequence must enter the average with a
Figure 4. Example of a forward evolution described by the master equation (40), with $g = 0.1$ and $\beta = \omega = 1$ and $H_t = \sigma_3/2 + 3\sin(15t)\sigma_1$ from $t = 0$ to $t = 1$. From $t = 1$ to $t = 2$ we recover the initial state. The full lines show the recovery using the embedding and performing an off-diagonal measurement. The diamonds show the recovery using the martingale.

Weigh determined by the corresponding realization of the influence martingale. This latter depends only on the measured detection rates which can be inferred from (15) and (21).

We conclude this section with some comments on the potential relevance of recovery protocol from the embedding for quantum error correction.

One of the main challenges in quantum computing is to efficiently protect quantum memory from decoherence effects, construed as errors see e.g. [21, 57]. In particular, it is desirable to implement any quantum error correction process by means of a quantum circuit without invoking any classical apparatus and using only few ancillary qubits. The recovery-by-embedding protocol only presumes completely positive operations and indirect measurement of an ancillary qubit. The full recovery the initial state of dynamics of a $d$-state system only calls for the addition of one ancillary qubit. The protocol also applies when the completely positive evolution in input is a particular solution of the canonical time local master equation. Over finite intervals of time, the canonical time-local master equation provides a description of open quantum system dynamics equivalent to the Nakajima–Zwanzig equation. In this sense, the recovery-by-embedding protocol does not require modeling decoherence-induced errors by means of the Lindblad–Gorini–Kossakowski–Sudarshan equation that was criticized in [86].

In summary, the influence martingale relates the recovery of the initial state operator evolved by a completely positive master equation to the unraveling of another completely positive master equation.

8. On the unraveling of non-canonical forms of the master equation

We now turn to describe how to proceed when (3) cannot be immediately invoked.

Let us suppose that only some of the canonical rates $\{\omega_{t,e}\}$ in (1) are non-sign definite. In [43] we consider an example of this situation. It may turn out to be computationally convenient to construct an influence martingale enslaved only to the counting processes corresponding the ‘non-positive’ decoherence channels. The apparent drawback is that we cannot expect that the corresponding operators satisfy condition (3). There is, however, a straightforward workaround to the problem. For simplicity of presentation, we illustrate the workaround surmising that the master equation has not been cast in canonical form and involves $\mathcal{D}'$ decoherence operators that do not satisfy (3). The extension to other related cases is straightforward. Indeed, we can always include in the drift of the stochastic Schrödinger equation an additional operator $L_0$ such that

$$\sum_{t = 0}^{\mathcal{D}'} L_{t,e} L_{t,e} = \tilde{\mathcal{G}}' 1_H$$

(41)

and, correspondingly, a counting process with increments $d\nu_{0,t}$ also satisfying the differential algebra (13). Next, we associate to the Itô stochastic Schrödinger equation
\[ \text{d}\psi_t = \mathbf{g}_t \text{d}t + \sum_{\ell=0}^{\mathcal{G}'} \text{d}\nu_{\ell,t} \left( \frac{L_{\ell,t} \psi_t}{\|L_{\ell,t} \psi_t\|} - \psi_t \right) \]  

\[ \mathbf{g}_t = -\varepsilon_t \mathbf{H}_t \psi_t - \sum_{\ell=0}^{\mathcal{G}'} r_{\ell,t} \left( \frac{L_{\ell,t} L - \|L_{\ell,t} \psi_t\|^2}{2} \right) \psi_t \]  

the influence martingale equation

\[
\text{d}\mu_t = \mu_t \left( \sum_{\ell=1}^{\mathcal{G}'} \left( \frac{\omega_{\ell,t}}{r_{\ell,t}} - 1 \right) \text{d}\nu_{\ell,t} - \text{d}\mu_0,t \right) \\
\text{d}\nu_{\ell,t} = \text{d}\nu_{\ell,t} - r_{\ell,t} \|L_{\ell,t} \psi_t\|^2 \text{d}t.
\]

In words, by enforcing the condition \( \omega_{0,t} = 0 \) the influence martingale suppresses paths solution of (42) whenever a jump of the counting process \( \{\nu_{0,t}\}_{t \geq t_0} \) occurs. It thus remain to verify that the unraveling conditions

\[
r_{\ell,t} = \omega_{\ell,t} + \varepsilon_t \geq 0 \\
\varepsilon_t = 2 \max_{\ell, t} (-\omega_{\ell,t})
\]

ensure that the drift (42b) recovers the form (17) hypothesized in the proof [43] of the unraveling applies. The dynamics (42) preserve by construction the Bloch hyper-sphere as (10) does. On the Bloch hyper-sphere the identity the chain of identities

\[
\mathbf{g}_t = \sum_{\ell=0}^{\mathcal{G}'} (\omega_{\ell,t} + \varepsilon_t) L_{\ell,t} L_{\ell,t} - \|L_{\ell,t} \psi_t\|^2 \frac{1}{2} L_{\ell,t} \psi_t \\
= \varepsilon_t \sum_{\ell=0}^{\mathcal{G}'} L_{\ell,t} L_{\ell,t} - \|L_{\ell,t} \psi_t\|^2 \frac{1}{2} \psi_t + \sum_{\ell=1}^{\mathcal{G}'} \omega_{\ell,t} L_{\ell,t} - \|L_{\ell,t} \psi_t\|^2 \frac{1}{2} \psi_t \\
= \sum_{\ell=1}^{\mathcal{G}'} \omega_{\ell,t} L_{\ell,t} L_{\ell,t} - \|L_{\ell,t} \psi_t\|^2 \frac{1}{2} \psi_t = \mathbf{f}_t
\]

hold true as \( \omega_{0,t} \) is zero by hypothesis. Hence, we recover (17) and the proof of the unraveling by a completely positive stochastic state vector dynamics is complete. We thus arrive at

**Proposition 8.1.** The solution of every time-local master equation admits an unraveling in quantum trajectory of the form (7). Furthermore, the influence martingale can be always chosen such to optimally pair in the sense of proposition 4.1 the solution to that of a master equation with positive rates although not necessarily in canonical form.

Finally, we describe a physically relevant application of the workaround. As well known, the master equation (1) is invariant if we replace Hamilton and decoherence operators with

\[
\tilde{L}_{\ell,t} = L_{\ell,t} + \varepsilon_{\ell,t} 1\mathcal{H} \\
\tilde{H}_t = H_t - \frac{1}{2} \sum_{\ell=1}^{\mathcal{G}'} \omega_{\ell,t} \left( \varepsilon_{\ell,t} L_{\ell,t} - L_{\ell,t} \varepsilon_{\ell,t} \right).
\]

The unraveling is, however, not invariant under the transformation. In particular, the \( \{\tilde{L}_{\ell,t}\}_{\ell=1}^{\mathcal{G}} \) do not satisfy the positive operator valued measurement type condition (3). Nevertheless, the resulting stochastic Schrödinger equation models a measurement setup which in the completely positive case can be experimentally realized by homodyne detection [21].
9. Discussion and outlook

The influence martingale [43] provides a general framework to unravel in quantum trajectories solutions of the canonical master equation to which any time-local open quantum system dynamics is always reducible [30]. Results such as those of [27, 28] prove the equivalence of the time-local and the time-non-local Nakajima–Zwanzig descriptions of a quantum open system dynamics, at least on finite time intervals [61]. Hence, the influence martingale offers a general and in principle exact unraveling of any open system dynamics. In practice, however, application of the method generically relies on time convolutionless perturbation theory.

From the mathematical point of view, the unraveling consists in solving a system of $d + 1$ ordinary stochastic Itô differential equations. It is thus computationally equivalent to the unraveling of the completely positive master equation [17] because the statistics of state-vector dependent counting processes is usually reconstructed from Poisson processes by means of Girsanov’s change of measure formula [76].

The statistics generated by a stochastic Schrödinger equation models a weak measurement record. The physical interpretation of the measurement record requires the statistics to be non-anticipating (i.e. the present record cannot be affected by events in the future see discussion e.g. in [9]) and to establish a correspondence with an instrument i.e. a completely positive unital map see e.g. [6, 8, 81]. The influence martingale unraveling framework is by construction non-anticipating. Together, the first two main results of the present work imply that there are two ‘natural’ Hilbert spaces where we can relate a completely bounded or completely positive master equation. The second is the embedding Hilbert space $\mathcal{C}^2 \otimes \mathcal{H}$, that can always be chosen to be the tensor product of the Hilbert space of the system with that of an ancillary qubit. The second avenue, embedding, is not new [79, 80]. The embedding induced by the influence martingale may be regarded as, in some sense, minimal [50] and enjoys the property that diagonal an non-diagonal blocks of the completely positive semi-group are themselves semi-groups of master equations in the original Hilbert space. This latter property paves the way to applications to recovery of an initial state of a quantum open system evolution.

A criticism [86] to continuous-time error correction theory, is that modeling error build-up by the Lindblad–Gorini–Kossakowski–Sudarshan master equation may be inaccurate in realistic situations. We refer to [87] for a recent quantitative appraisal of the criticism. The recovery protocol that we propose does not necessarily require that the completely positive evolution to be reversed be the solution of a Lindblad–Gorini–Kossakowski–Sudarshan master equation. It may well be a particular solution of the completely bounded master equation (see also discussion in appendix B). The recovery-by-embedding protocol requires a detailed knowledge of the decoherence channels. This is, however, the case for any particular physical implementation of quantum computation (see e.g. supplementary information to [88]) and is therefore not a limitation specific of the protocol.

In conclusion, the influence martingale unraveling framework provides a numerically efficient and conceptually ductile tool to analyze maps on operators and their ‘matrix block’ structure. A particularly interesting development is the quantum state recovery protocol that the influence martingale naturally brings about in view of potential applications to quantum error correction.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix A. Definitions

A linear map $\mathcal{B}$ between operator spaces is said to be completely bounded when the supremum over the norms of extensions $\operatorname{Id}_n \otimes \mathcal{B}$ for any $n$ is bounded:

$$\|\mathcal{B}\|_{cb} := \sup_{n \geq 1} \|\operatorname{Id}_n \otimes \mathcal{B}\| < \infty.$$  \hspace{1cm} (A.1)
Here $\text{Id}_n$ denotes the identity map on operators on $C^n$ and
\begin{equation}
\|B\| = \sup_{\|O\| \leq 1} \|B(O)\|
\end{equation}
for $O$ any operator in the domain of $B$ acting on the Hilbert space $H$ and with norm
\begin{equation}
\|O\| := \sup_{v \in H \|v\| \leq 1} \|Ov\|.
\end{equation}
Finally, in the finite dimensional case $B : M_d \rightarrow M_d$ the bound
\begin{equation}
\|B\|_{cb} \leq d \|B\|
\end{equation}
reduces the requirement (A.1) to that of bounded operator norm (A.2). We refer to [67] for further details.

A linear map $\mathcal{P}$ between operator spaces is said to be a 
positive map if it takes positive operators to positive operators.
A linear map completely positive when its extensions $\text{Id}_n \otimes \mathcal{P}$ for any $n$ are positive. In the case of a map on operators over a $d$ dimensional Hilbert space the requirement of complete positivity corresponds to the requirement that $\text{Id}_d \otimes B$ be positive [58]. This latter property can be equivalently stated by saying that the Choi (dynamical) matrix has positive spectrum.

Appendix B. Memory effects as parametric dependence of the generator

The derivation of (1) in e.g. [28] surmises the existence of an instant of time $t_0$ when the state operator of the microscopic bipartite system is the tensor product of the state operators of the constituents: system and environment. Upon tracing out the environment the system evolves from $t_0$ according to a completely positive operator evolution map $\Lambda$ solution of an integro-differential equation given by the Nakajima–Zwanzig projection operator. This time-non-local equation is equivalent to a time-local master equation of the form
\begin{equation}
\frac{d\Lambda_{t,t_0}}{dt} = \Lambda_{t-t_0}(\Lambda_{t,t_0})
\end{equation}
\begin{equation}
\Lambda_{t_0,t_0} = \text{Id}
\end{equation}
in a time interval $[t_0, t_f]$ where $\Lambda$ admits a continuous inverse. Upon assuming that the generator $L_t$ is well defined for all in $[0, t_f - t_0]$, we may then regard (B.1) as a special case of the equation
\begin{equation}
\frac{d\mathcal{B}_{t,s}}{dt} = L_{t-t_0}(\mathcal{B}_{t,s})
\end{equation}
\begin{equation}
\mathcal{B}_{s,s} = \text{Id}
\end{equation}
defining a flow $\mathcal{B}_{s,t}$ for any $s, t \in [0, t_f - t_0]$. We thus recover [28]
\begin{equation}
\Lambda_{t,t_0} = \mathcal{B}_{t-t_0,0}(\rho_{t_0}).
\end{equation}
Whilst $\Lambda_{t,t_0}$ is by construction completely positive, the flow is generically completely bounded. To see this, we may consider any $v \leq t$ and use the group properties of the flow to establish the chain of identities
\begin{equation}
\Lambda_{t,t_0} = \mathcal{B}_{t-t_0,v-t_0}\mathcal{B}_{v-t_0,0} = \mathcal{B}_{t-t_0,v-t_0}\Lambda_{v,t_0}.
\end{equation}
We arrive at the identity
\begin{equation}
\mathcal{B}_{t-t_0,v-t_0}(\rho) = (\Lambda_{t,t_0}\Lambda_{v,0}^{-1})(\rho) \quad \forall \rho \in M_d.
\end{equation}
The inverse of completely positive operator evolution map is completely positive if and only if the map is unitary. Hence $\mathcal{B}_{t-t_0,v-t_0}$ is generically the composition of a completely positive with a completely bounded map and is therefore only completely bounded.
Appendix C. Proof of (3) if the master equation is in canonical form

If (3) is in canonical form then \( \{ L_{\ell',\ell}; \} \ell = 1 \) are related by a unitary transformation to the elements of an orthonormal basis of \( \mathcal{M}_d \) whose \( L_{\ell,\ell} \) is proportional to the identity \([37]\). Namely, upon writing the completeness relation in matrix components \( (A)_{ik} \) extracts the \( l, k \) entry from \( A \)

\[
\sum_{\ell}^d (L_{\ell',\ell})_{ij} (L_{\ell',\ell}^\dagger)_{lk} = \delta_{ik} \delta_{jl}
\]

we get

\[
\sum_{\ell}^d \sum_{\ell'}^d (L_{\ell',\ell})_{ii} (L_{\ell',\ell}^\dagger)_{kk} = \sum_{\ell}^d \delta_{ik} \delta_{jj} = d \delta_{ik}
\]

which readily implies

\[
\sum_{\ell}^d L_{\ell',\ell} (L_{\ell',\ell}^\dagger) = \left( d - \frac{1}{d} \right) 1_{\mathcal{H}}.
\]

The result can also be read as a consequence of Schur’s lemma applied to the generators of \( su(d) \). Furthermore the result is not affected by any unitary transformation of the basis elements.

Appendix D. Proof of the unraveling

Itô lemma for stochastic processes with finite quadratic variation \([74]\) implies that

\[
d \left( \mu, \psi, \psi^\dagger \right) = (d\mu_i) \psi_i \psi_i^\dagger + (\mu_i + d\mu_i) d \left( \psi_i \psi_i^\dagger \right)
\]

and

\[
d \left( \psi, \psi^\dagger \right) = (d\psi_i) \psi_i^\dagger + \psi_i d\psi_i^\dagger + (d\psi_i) d\psi_i^\dagger.
\]

The explicit expressions of the differentials \((10a)\) and \((12)\) and the telescopic property of expectation values in the Itô prescription e.g.

\[
E \left( d\nu_{\ell',\ell} \psi_i \psi_i^\dagger \right) = E \left( d\nu_{\ell',\ell} | \mathcal{F}_i \right) \psi_i \psi_i^\dagger
\]

which immediately imply

\[
E \left( d\mu_i \psi_i \psi_i^\dagger \right) = E \left( d\nu_{\ell',\ell} | \mathcal{F}_i \right) \psi_i \psi_i^\dagger = 0 \quad \forall \ell
\]

prove that \((7)\) satisfies \((1)\).

Appendix E. Commutant representation

We recall that the adverb completely in reference to a property enjoyed by a linear map \( \mathcal{B} \) on \( B(\mathcal{H}) \) indicates that that the same property is enjoyed by the extension

\[
\text{Id}_k \otimes \mathcal{B} : \begin{bmatrix} O_{11} & \cdots & O_{1k} \\ \vdots & \ddots & \vdots \\ O_{k1} & \cdots & O_{kk} \end{bmatrix} \mapsto \begin{bmatrix} \mathcal{B}(O_{11}) & \cdots & \mathcal{B}(O_{1k}) \\ \vdots & \ddots & \vdots \\ \mathcal{B}(O_{k1}) & \cdots & \mathcal{B}(O_{kk}) \end{bmatrix}
\]

for any \( k \in \mathbb{N} \) and \( O_{ij} \)’s in \( B(\mathcal{H}) \).

We now summarize some of the results of \([50]\) about the representation of completely bounded maps also drawing from the pedagogic presentations \([67, 82]\). Restricting for simplicity the discussion to finite dimensional spaces, the commutant representation of a completely bounded map \( \mathcal{B} : \mathcal{M}_d \mapsto \mathcal{M}_d \) consists in
the fact that there exists a collection of \( \tilde{d} \times d \) rectangular matrices \( A_i \), \( i = 1, \ldots, m \leq d \times \tilde{d} \) satisfying the positive operator value measurement condition

\[
\sum_{i=1}^{m} A_i A_i^\dagger = I_d
\]

and a matrix \( T \in \mathcal{M}_m \) such that for any \( X \in \mathcal{M}_d \) we can write

\[
\mathcal{B}(X) = [A_1 \ldots A_m] (T \otimes 1_H)(1_m \otimes X) \begin{bmatrix} A_1^\dagger \\ \vdots \\ A_m^\dagger \end{bmatrix}.
\]  

(E.1)

The name ‘commutant’ stems from the observation

\[
(T \otimes 1_H)(1_m \otimes X) = (1_m \otimes X)(T \otimes 1_H).
\]

In particular if \( T = 1_m \) we recover the Choi-Stinespring representation of a completely positive map. It is then convenient to write (E.1) in the Krauss operator product form

\[
\mathcal{B}(X) = \sum_{i,j=1}^{m} T_{i,j} A_i X A_j^\dagger.
\]

A consequence \([50]\) of the existence of the commutant representation is that the embedding linear map \( \mathcal{E}: \mathbb{C}^2 \otimes \mathcal{M}_d \rightarrow \mathbb{C}^2 \otimes \mathcal{M}_d \) defined by

\[
\mathcal{E} \left( \begin{bmatrix} Y & X \\ W & Z \end{bmatrix} \right) = \begin{bmatrix} \sum_{i=1}^{m} T_{i,j} A_i X A_j^\dagger & \sum_{i,j=1}^{m} T_{i,j} A_i X A_j^\dagger \\ \sum_{i,j=1}^{m} T_{i,j} A_i X A_j^\dagger & \sum_{i=1}^{m} T_{i,j} A_i X A_j^\dagger \end{bmatrix}
\]

is completely positive if and only if the \( 2m \times d \times 2m \times d \) squared matrix

\[
M = \begin{bmatrix} 1_n \otimes 1_H & T \otimes 1_H \\ T^\dagger \otimes 1_H & 1_n \otimes 1_H \end{bmatrix}
\]

is positive. The completely positive map

\[
\mathcal{P}(Y) = \sum_{i} A_i Y A_i^\dagger
\]

appearing in the embedding map is called the ‘associated’ completely positive map. Finally, \([50]\) addresses the problem of the uniqueness of the representation and proves that the commutant representation is unique up to similarity under the same conditions as the Choi-Stinespring representation in the completely positive case \([73]\).

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