MODELING OF MECHANICAL BEHAVIOUR OF A 1D, LINEAR
AND MICROPERIODIC COMPOSITE, WITH FAILURE, VIA THE
ASYMPTOTIC HOMOGENIZATION METHOD

MODELAGEM DO COMPORTAMENTO MECÂNICO DE UM
COMPÓSITO 1D, BIFÁSICO, MICROPERIÓDICO, COM FRATURA,
ATRÁVÉS DO MÉTODO DE HOMOGENEIZAÇÃO ASSINTÓTICA

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Abstract: In this paper the Asymptotic Homogenization Method is combined with a model of Hyperelasticity with softening (which models the mechanical failure of the material), with the purpose of estimating the mechanical behaviour of microperiodic composites in the way to reproduce better the behaviour obtained in experiments. This hybrid approach brings some satisfactory results, showing that the (hypothetical) composite studied presents effective behaviour very close to the expected from the Hyperelasticity model. That is a very important fact because can be used as a tool to predict the failure of composite materials, through its constituent materials information. This evaluating needs more steps, to be possible much bigger and strong conclusions.

Keywords: Composites, Asymptotic Homogenization Method, Hyperelasticity, Mechanical failure.

Abstract: O presente trabalho visa combinar o Método de Homogeneização Assintótica com um modelo de hiperelasticidade com amolecimento (que modela a falha mecânica do material), a fim de estimar o comportamento mecânico de compósitos microperiódicos de forma mais parecida com resultados experimentais. Essa abordagem híbrida se mostrou satisfatória, apontando que o compósito (hipotético) considerado apresenta comportamento efetivo muito próximo ao esperado através do modelo de hiperelasticidade citado. Um fato muito importante pois pode ser uma ferramenta na predição de falha de um compósito através de informações de suas fases constituintes. Essa avaliação precisa de mais etapas a fim de concluir-se algo maior.

Palavras-chave: Compósitos, Método de Homogeneização Assintótica, Hiperelasticidade, Falha mecânica.

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1 INTRODUCTION

Composite materials can be understood as heterogeneous material formed by the union of two or more homogeneous materials (these are the phases of the composite). From the material’s science and engineering point of view, the more interesting ones are those which present a significant proportion of each phase, to obtain a better combination of their properties (CALLISTER; RETHWISCH, 2018). Examples can be verified in (BANDIL et al., 2020) e (SINGH R.K.; TELANG, 2020).

On the applications context, the materials usually are submitted to some mechanical charges, making very important the study of its mechanical behaviour. That physical phenomenon is traditionally modelled by the Hooke’s law, a linear relation between stress and strain (the principal measures in the mechanical studies), also (but not so traditional) by nonlinear relations, like the potentials (PONTE-CASTAÑEDA; SUQUET, 1998), (DAI et al., 2018). However, these models shows an apparent idea of unlimited (strain) energy, for bigger values of strain - an idea that no has physical meaning. In fact, is interesting to refer (DÉCIO-JÚNIOR; PÉREZ-FERNÁNDEZ; BRAVO-CASTILLO, 2021), where even using potential relations and considering the imperfect adhesion of the composite’s phases, the mechanical behaviour shows the unlimited energy idea yet.

Thus, here is a nonlinear hyperelasticity with softening model, proposed by (VOLOKH, 2007), that represents a way to model the mechanical behaviour of a material in a more realistic form, in relation to experimental results (see (PERALTA; MOSALAM; LI, 2016)). On this model, a new formulation of the stress energy are considered, which bring the concept of a critical energy value, modelling the failure of the structure. It can be successfully applied in diverse context, for examples: (VOLOKH, 2013), (VOLOKH, 2015) and (VOLOKH, 2018).

In this work will be considered composite which distribution scale (of the phases) are much smaller than the macroscopic scale and much larger than the atomic - characterizing a microheterogeneous material. For materials who satisfy these hypothesis, the differential equations problems has rapidly oscillating coefficients, fact that difficult the direct application of usual numerical methods (PANASENKO, 2008). But, fortunately, the equivalent homogeneity hypothesis are satisfied too, and then the microheterogeneous material can be considered physically equivalent to a homogeneous one, which problems present constant coefficients. Besides that, the problem of the homogeneous equivalent material’s solution is close enough to the solution of the original problem (the one for the microheterogenous material). The obtaining of the homogeneous equivalent material are called Homogenization.

Among the homogenization methods is the asymptotic homogenization method (AHM)
(BAKHTALOV; PANASENKO, 1989), which consider a asymptotic expansion in the form of a potential series (of a small positive parameter $\varepsilon$, which characterizes the microscale) as an approach for the solution of the original problem. The AHM presents the main advantages: lower computational cost for the application of numerical methods and give good approach of the problem of interesting’s solution (see (DONG Q.; CAO, 2014) and (DÉCIO-JÚNIOR; PÉREZ-FERNÁNDEZ; BRAVO-CASILLERO, 2019)). Besides that, the AHM shows very good results in determining the effective behaviour of microheterogenous materials, including with imperfect contact between its phases (see (BORDES et al., 2018) and (DÉCIO-JÚNIOR; PÉREZ-FERNÁNDEZ; BRAVO-CASILLERO, 2021)).

In light of the above mentioned, this paper evaluate the effective mechanical behaviour of a onedimensional, biphasic, linear and microperiodic composite, which both phases are modeled by the hyperelasticity with softening model (that will be called by "Volokh’s model" several times). This combination (AHM and Volokh’s model) is original of this work, in the sense that are not found similar works in current and available literature.

2 PROBLEM’S FORMULATION

A onedimensional composite can be interpreted as a bar. Without generality lost, will be considered a $\varepsilon$-periodic and biphasic bar, with unitary length, represented by $[0, 1]$. More exactly, the interval $[0, 1]$ can be represented by the finite union of $N \in \mathbb{N}$ subintervals (the periodic cells), which ones will be formed by the union of domains $\Omega_1$ and $\Omega_2$ (the phases of the composite). Besides that, the boundaries $\partial\Omega_1$ and $\partial\Omega_2$ will be the interface points in the periodic cells, denoted by the $x_j$, and $\Gamma^\varepsilon = \{x_j\}_{j=1,...,N}$ is the set of these interface points.

The problem that models the static diffusion phenomenon in this bar, which solution is $u^\varepsilon$, are formed as follows:

\[
\frac{d}{dx} \left[ \sigma^\varepsilon \left( \frac{du^\varepsilon}{dx} \right) \right] = f(x), \quad x \in (0, 1) / \Gamma^\varepsilon \tag{1}
\]
\[
\sigma^\varepsilon \left( \frac{du^\varepsilon}{dx} \right) = \begin{cases} \sigma_1^\varepsilon \left( \frac{du^\varepsilon}{dx} \right), & x \in \Omega_1 \\ \sigma_2^\varepsilon \left( \frac{du^\varepsilon}{dx} \right), & x \in \Omega_2 \end{cases} \tag{2}
\]
\[
\left[ \sigma^\varepsilon \left( \frac{du^\varepsilon}{dx} \right) \right]_{x=x_j} = 0 \tag{3}
\]
\[
[u^\varepsilon(x)]_{x=x_j} = 0 \tag{4}
\]
\[
u^\varepsilon|_{x=0} = g_1, \quad u^\varepsilon|_{x=1} = g_2, \tag{5}
\]
where the index $\varepsilon$ indicates the variable’s dependence on the small parameter. In fact, it also indicates the dependence on the microscale, defined by the position variable $y = x/\varepsilon$, being $x$ the macroscale position variable.

The symbol $[\ ]_{x=x_j}$ indicates the continuity (or contact) condition in the interface points, defined as follows:

$$ [F^e \varepsilon \varepsilon \varepsilon (x)] = \lim_{x \to x_j^+} F^e \varepsilon \varepsilon (x) - \lim_{x \to x_j^-} F^e \varepsilon \varepsilon (x) \equiv F^e \varepsilon \varepsilon (x_j^+) - F^e \varepsilon \varepsilon (x_j^-).$$

### 3 THE AHM METHODOLOGY

Will be taking a two-scale asymptotic expansion in the form:

$$u^{(2)} (x, y) = v_0 (x) + \varepsilon u_1 (x, y) + \varepsilon^2 u_2 (x, y),$$

(6)
as an approach of the original solution $u^\varepsilon$ of the $P_O$ problem. Are also considered that $v_0(x), u_k (x, y) \in C^2 ([0, 1])$ for $k = 1, 2, 1$—periodics in $y$ scale.

Replacing $u^{(2)} (x, y)$ in equation (1) equation, and considering the flux $\sigma$ linearized by a Taylor polynomial in the point $\zeta = dv_0/dx + \partial u_1/\partial y$, are obtained that:

$$\frac{d}{dx} \left[ \sigma \left( \frac{x}{\varepsilon}, \frac{du^{(2)}}{dx} \right) \right] - f(x) = \varepsilon^{-1} \left[ \frac{\partial}{\partial y} \left( y, \zeta \right) + \varepsilon^0 \left[ \frac{\partial}{\partial x} \left( y, \zeta \right) + \frac{\partial}{\partial y} \left[ \left( \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} \right) \frac{\partial}{\partial \varepsilon} \left( y, \zeta \right) \right] - f(x) \right] + O(\varepsilon).$$

(7)

Notice that from here, are considered the scales separation ($x$ and $y$). Following, in order to satisfy:

$$\left| \frac{d}{dx} \left[ \sigma \left( \frac{x}{\varepsilon}, \frac{du^{(2)}}{dx} \right) \right] - f(x) \right| = O(\varepsilon),$$

has taking $u_2 \equiv 0$, to obtain the follow equation for $\varepsilon^{-1}$ from equation (7):

$$\frac{\partial}{\partial y} \left( y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y} \right) = 0.$$

(8)

By considering $u^{(2)} (x, y)$ also in the (3), (4) and (5), amongst (8), are obtained one-parametric family of problems, denoted by $P_{\varepsilon}^{dv_0}$, which has as solution $u_1 (x, y)$, with the equations:
where $c_1$ represents the interface point of the periodic cell on the microscale. The existence and uniqueness of the 1-periodic solution of $P_{L}^{\text{eq}}$, are guaranteed by the lemma as follows:

**Lemma:** Let $\varepsilon$ a parameter and $\sigma(y, \varepsilon)$ a piecewise and continuously differentiable function in $[0, 1]$. Then, for all $x$ fixed, there are $N_1(y, \varepsilon)$ 1-periodic in $y$ functions, which solve the one-parametric family of problems $P_L^\varepsilon$ with the parameter $\varepsilon$, defined as follows:

\[
\frac{\partial \sigma}{\partial y} \left( y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y} \right) = 0, \quad y \in (0, 1)/\{c_1\}
\]

(9)

\[
\left[ \sigma \left( y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y} \right) \right]_{y=c_1} = 0
\]

(10)

\[
[u_1]_{y=c_1} = 0
\]

(11)

\[
u_1(0,0) = u_1(l,0) = 0,
\]

(12)

The condition $N_1(0,\varepsilon) = 0$ guarantees the solution’s uniqueness.

The Lemma’s demonstration are found in (BAKHVALOV; PANASENKO, 1989).

Thus, the uniqueness and existence of the solution of the family of problems $P_L^{\text{eq}}$ in (9)-(12) equations guaranteed by the Lemma. So, now is possible to obtain of the effective law (which means the effect physical behaviour) for the composite of interest.

To this, the equation (9) are integrated and with the condition in (10), follows that

\[
\sigma \left( y, \frac{dv_0}{dx} + \frac{\partial u_1}{\partial y} \right) = \sigma,
\]

(17)

and the integration of $\frac{\partial u_1}{\partial y}$ from (17) equation, gives:

\[
\frac{\partial u_1}{dy} = \varphi(y, \sigma) - \frac{dv_0}{dx},
\]

(18)

where $\varphi(y, \sigma)$ is the inverse function of the flux $\sigma$ in (17), which is guaranteed by the implicit
function theorem.

Thus, applying the average operator \( \langle . \rangle \) to (18), which is defined by:

\[
\langle . \rangle = \int_0^1 (. )dy, \tag{19}
\]
yields:

\[
u_1|_{y=1} - \left[ u_1 \right]_{y=c_1} - u_1|_{y=0} = \langle \varphi(y, \sigma) \rangle - \frac{dv_0}{dx}. \tag{20}
\]

From the contact condition in (11) and the 1-periodicity in relation to \( y \) of \( u_1 \) (that is guaranteed by the Lemma), (20) equation turns to:

\[
\langle \varphi \rangle(\sigma) = \frac{dv_0}{dx}. \tag{21}
\]

The effective law, which will denoted by \( \hat{\sigma} \left( \frac{dv_0}{dx} \right) \), is obtained by the relation between \( \sigma \) and \( dv_0/dx \), which is implicit in (21). In practice, is necessary to solve the equation like (21) for \( \sigma \), which resolution methods will depend of the form of the constitutive relations in each phase of composite. In the case of this work, this resolution is hindered with the exponential terms in the Volokh’s model.

4 THE HYPERELASTICITY WITH SOFTENING (VOLOKH’S) MODEL

When a body is submitted to mechanical charges (internal or externally), these forces does work over it, that store the energy involved in the form of strain energy. This one will be denoted by \( W(\epsilon) \) and is related to stress as follows:

\[
W(\epsilon) = \int_0^\epsilon \sigma(s)ds. \tag{22}
\]

The Hyperelasticity with softening model proposed by (VOLOKH, 2007) brings a new formulation for the strain energy, which one reproduces the mechanical behaviour experimentally observed more exactly. This new formulation is denoted by \( \Psi(W(\epsilon)) \), and defined as:

\[
\Psi(W(\epsilon)) = \Phi \left( 1 - \exp \left( -\frac{W(\epsilon)}{\Phi} \right) \right), \tag{23}
\]
where \( \Phi \) is the critical energy of failure, a characteristic of the material (like its conductivity or...
elasticity).

A comparison of the behaviour for a elastic material that follows the Hooke’s law in the classical way and in the Volokh’s model way (equation (23)), is illustrated in figure 1.

Figure 1: The strain energy (left) and of the stress-strain curve (right) compared with the classical approach and the Volokh’s model.

Source: From the author.

The figure 1 shows the idea of unlimited energy that occurs in the classical formulation, and the more realistic behaviour obtained with the Volokh’s model. Furthermore, the stress-strain curve are also more realistic when the model in discuss are considered. It’s important to also observe that the classical approach looks a good estimate of the behaviour just for smaller values of strain, and the Volokh’s model are good to reproduce this fact too, as can be checked in Figure 1, where the curve for this model is close to the linear one.

In experiments, the material presents a limit value of strain (called the tensile strength), which after it occurs the material failure. It suggest a limit value of energy that the body support without failure, in agreement with the critical energy $\Phi$ of the Volokh’s model. These experimental aspects can be checked in (PERALTA; MOSALAM; LI, 2016).

5 RESULTS FROM A NUMERICAL EXPERIMENT AND DISCUSSION

From the elasticity theory, are considered that $\sigma(\epsilon) = dW(\epsilon)/d\epsilon$ (see equation (22)). So, in the approach of the proposed model, are considered that:

$$\sigma(\epsilon) = \frac{d\Psi(\epsilon)}{d\epsilon}.$$  

(24)
Thus, deriving the equation (23) in relation to $\epsilon$, are obtained the following constitutive relation (for each phase of the linear composite considered):

$$\sigma_i(\epsilon) = K_i \epsilon \exp\left(-\frac{K_i \epsilon^2}{2\Phi_i}\right), \; i = 1, 2,$$

(25)

where $K_i$ is the elasticity coefficient (Hooke’s Law) and $\Phi_i$ is the critical energy of failure, for each $i$ phase.

In the sense of obtain the inverses $\sigma^{-1}_i(\sigma)$ and $\sigma^{-1}_2(\sigma)$, because the relations in equation (25) are not biunivocal, the resolution of the equation (21) for the effective law obtaining has to be making by observing the injective domains for both phases. Moreover, to solve the equation (21) the following criteria are established in the numerical implementation:

- $\sigma(i) > 0$;
- $|\sigma(i) - \sigma(i-1)| \leq H|\sigma(i) - \sigma(i-1)|$, with $H > 0$;

the second one is a version of the Lipschitz condition, where $H$ are a fixed constant (Lipschitz’s constant), which has the purpose of assuring the continuity of the effective law obtained.

To illustrate the effect of taking the AHM and Volokh’s model together to determinate the effective behaviour of a composite, are considered a hypothetic composite with the follow characteristics: $K_1 = 0.3$ and $\Phi_1 = 0.1$ for the phase 1, $K_2 = 2.5$ and $\Phi_2 = 0.3$ for the phase 2. In a physical meaning, the phase 1 is the less stiff, presenting failure with a smaller tensile strength, when is compared to the material of phase 2. The effective behaviour of this composite is shown in the figure 2:

Can be observed in figure 2 that the global (effective) behaviour of the composite follows qualitatively the behaviour of its constituent phases. Also, as the phase 1 concentration (given by $c_1$) increases to 1 (which means just the material of the phase 1), the effective behaviour tends to the phase 1 behaviour. And, like was mentioned before for the Volokh’s model, the effective behaviour are practically linear for small values of $\epsilon$ too.

Besides that, the figure 2 indicates that the effective behaviour of the composite are conditioned to the tensile strength (maximum value of stress) of the phase 1. This fact has physical meaning, in the sense that the phase 1 is the less rigid, so, the composite material can’t be submitted to a stress bigger than that the phase less rigid can stand without failure.

Now, will be discussed quantitative aspects of the results obtained in figure 2. A main question is important to evaluate: the effective behaviour of the composite which phases follow the Volokh’s model, in the sense of equation (25), follows it too? More exactly, is desired to
Figure 2: Effective law for the mechanical behaviour for the hypothetical composite considered, for different values of concentration.

Source: From the author.

know if there are $\hat{K}$ and $\hat{\Phi}$ that correspond to equation (25) and the effective behaviour observed in figure 2.

Applying the AHM to a case with a composite biphasic linear, in a classical approach, are obtained (DÉCIO-JÚNIOR; PÉREZ-FERNÁNDEZ; BRAVO-CASTILLERO, 2021):

$$\hat{K} = \left( \frac{c_1}{K_1} + \frac{c_2}{K_2} \right)^{-1},$$  \hspace{1cm} (26)

where $c_2 = 1 - c_1$.

To determinate the value of $\hat{\Phi}$, one way is trough the effective energy curve, considering that (as in the Volokh’s model) the effective energy tends to a value critical when are taken bigger values of $\epsilon$, which will be the $\hat{\Phi}$. This curve are obtained by integrating the effective law (equation (24)) using Simpson’s 1/3 rule e analyzing its asymptotic behaviour. In figure 3 are shown the results for the cases where $c_1 = 0.3$ and $c_1 = 0.8$.

The figure 3 shows that are a strong relation between the effective behaviour obtained numerically and the one obtained "theoretically" using directly the Volokh’s model with the $\hat{K}$ and $\hat{\Phi}$ considered. The fact that the curves are not identical can be result of the numerical methods applied to obtain $\hat{\Phi}$, or there is no match between the relations. This questions and others must to be more accurately studied, to be possible make more solid conclusions. The study seems like very promissory.
6 FINAL CONSIDERATIONS

This work shows that the combination of the AHM and the Volokh’s model is an interesting tool to evaluate the effective behaviour of microperiodic composites. Was observed that the behaviour of composite determined by the AHM follows the Volokh’s model.

Furthermore, this approach shows good insights that it can be used successfully to predict the failure of composites, through its phase’s characteristics. Ultimately, this study looks like a promising one in the research of composites field.

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