QUANTUM OBSERVABLES ON A COMPLETELY SIMPLE SEMIGROUP

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Abstract. Completely simple semigroups arise as the support of limiting measures of random walks on semigroups. Such a limiting measure is supported on the kernel of the semigroup. Forming tensor powers of the random walk leads to a hierarchy of the limiting kernels. Tensor squares lead to quantum observables on the kernel. Recall that zeons are bosons modulo the basis elements squaring to zero. Using zeon powers leads naturally to quantum observables which reveal the structure of the kernel. Thus asymptotic information about the random walk is related to algebraic properties of the zeon powers of the random walk.

1. Introduction

This work is based on the connection between graphs and semigroups developed by Budzban and Mukherjea [2] in the context of their study of the Road Coloring Problem [RCP]. We restrict throughout to finite transformation semigroups, that is, semigroups of functions acting on a finite set, $V$. In their interpretation of the RCP, a principal question is whether one can determine the rank of the kernel of a semigroup from its generators. Questions in general about the kernel of a semigroup are of interest. We will explain this terminology as we proceed.

The study of probability measures on semigroups by Mukherjea [6], Mukherjea-Högnas [5] in particular reveals the result that the convolution powers of a measure on a finite semigroup, $S$, converge (in the Cesàro sense) to a measure on the kernel, $\mathcal{K}$, the minimal ideal of the semigroup. The kernel is always completely simple (see Appendix A for precise details). In this context, a completely simple semigroup is described as follows. It is a union of disjoint isomorphic groups, called “local groups”. The kernel $\mathcal{K}$ can be organized as a two-dimensional grid with rows labelled by partitions of the underlying set $V$ and columns labelled by range classes, a family of equinumerous sets into whose elements the blocks of a given partition are mapped by the members
of $K$. The cardinality of a range class is the rank of the kernel. The idempotent $e$ determined by a pair $(\mathcal{P}, \mathcal{R})$, with $\mathcal{P}$ a partition of $V$ and $\mathcal{R}$ a range class, acting as the identity of the local group, fixes each of the elements of the range class. Each of the mutually isomorphic local groups is isomorphic to a permutation group acting on its range class. We think of the kernel as composed of cells each labelled by a pair $(\mathcal{P}, \mathcal{R})$ and containing the corresponding local group.

First we recall the basic theory needed about probability measures on semigroups, continuing with the related context of walks on graphs. Briefly, the adjacency matrix of the graph is rescaled to a stochastic matrix, $A$. A decomposition of the adjacency matrix into binary stochastic matrices, $C_i$, i.e., matrices representing functions acting on the vertices of the graph, is called a coloring. The coloring functions generate the semigroup $S$ and the question is to determine features of the kernel $K \subset S$ from the generators $C_i$. Here we present novel techniques by embedding the semigroup in a hierarchy of tensor powers. We will find particular second-order tensors to provide the operators giving us “quantum observables”. The constructions restrict to trace-zero tensors using zeon Fock space instead of the full tensor space.

For convenient reference, we put some notations here.

1.1. **Notations.** We treat vectors as “row vectors”, i.e., $1 \times n$ matrices, with $e_i$ as the corresponding standard basis vectors. The column vector corresponding to $v$ is $v^\dagger$. For a matrix $M$, $M^*$ denotes its transpose. And $\text{diag}(v)$ denotes the diagonal matrix with entries $v_i$ of the vector $v$.

The vector having components all equal to 1 is denoted $u$. And the matrix $J = u^\dagger u$ has entries all ones. We use the convention that the identity matrix, $I$, as well as $u$ and $J$, denote matrices of the appropriate size according to the context.

2. **Probability measures on finite semigroups**

Since we have a discrete finite set, a probability measure is given as a function on the elements. The semigroup algebra is the algebra generated by the elements $w \in S$. In general, we consider formal sums $\sum f(w)w$. The function $\mu$ defines a probability measure if

$$0 \leq \mu(w) \leq 1, \forall w \in S, \text{ and } \sum \mu(w) = 1.$$
The corresponding element of the semigroup algebra is thus $\sum \mu(w) w$. The product of elements in the semigroup algebra yields the convolution of the coefficient functions. Thus, for the convolution of two measures $\mu_1$ and $\mu_2$ we have

$$
\sum_{w \in S} \mu_1 * \mu_2(w) w = \left( \sum_{w \in S} \mu_1(w) w \right) \left( \sum_{w' \in S} \mu_2(w') w' \right) = \sum_{w, w' \in S} \mu_1(w) \mu_2(w') w w'
$$

Hence the convolution powers, $\mu^{(n)}$ of a single measure $\mu = \mu^{(1)}$ satisfy

$$
\sum_{w \in S} \mu^{(n)}(w) w = \left( \sum_{w \in S} \mu^{(1)}(w) w \right)^n
$$
in the semigroup algebra.

2.1. Invariant measures on the kernel. We can consider the set of probability measures with support the kernel of a finite semigroup of matrices (see Appendix A for more details). Given such a measure $\mu$, it is idempotent if $\mu * \mu = \mu$, i.e., it is idempotent with respect to convolution. An idempotent measure on a finite group must be the uniform distribution on the group, its Haar measure. In general, we have

**Theorem 2.1.** [5, Th. 2.2] An idempotent measure $\mu$ on a finite semigroup $S$ is supported on a completely simple subsemigroup, $K'$ of $S$. With Rees product decomposition $K' = \mathcal{X}' \times \mathcal{G}' \times \mathcal{Y}'$, $\mu$ is a direct product of the form $\alpha \times \omega \times \beta$, where $\alpha$ is a measure on $\mathcal{X}'$, $\omega$ is Haar measure on $\mathcal{G}'$, and $\beta$ is a measure on $\mathcal{Y}'$.

In our context, we will have the measure $\mu$ supported on the kernel $K$ of $S$. We identify $\mathcal{X}$ with the partitions of the kernel and $\mathcal{Y}$ with the ranges. The local groups are mutually isomorphic finite groups with the Haar measure assigning $1/|G|$ to each element of the local group $G$. Thus, if $k \in K$ has Rees product decomposition $(k_1, k_2, k_3)$ we have

$$
\mu(k) = \alpha(k_1) \beta(k_3)/|G|
$$

where $|G|$ is the common cardinality of the local groups, $\alpha(k_1)$ is the $\alpha$-measure of the partition of $k$, and $\beta(k_3)$ is the $\beta$-measure of the range of $k$.

**Remark.** Given a graph with $n$ vertices, we identify $V$ with the numbers $\{1, 2, \ldots, n\}$. Generally, given $n$, we are considering semigroups of transformations acting on the set $\{1, 2, \ldots, n\}$. 
For a function on \( \{1, 2, \ldots, n\} \), the notation \( f = [x_1x_2 \ldots x_n] \) means that \( f(i) = x_i, \ i \leq 1 \leq n \). And the rank of \( f \) is the number of distinct values \( \{x_i\}_{1 \leq i \leq n} \).

**Example.** Here is an example with \( n = 6 \). Take two functions \( r = [451314], \ b = [245631] \) and generate the semigroup \( S \). We find the elements of minimal rank, in this case it is 3. The structure of the kernel is summarized in a table with rows labelled by the partitions and columns labelled by the range classes. The entry is the idempotent with the given partition and range. It is the identity for the local group of matrices with the given partition and range.

\[
\begin{array}{|c|c|c|c|}
\hline
\{1, 3, 4\} & \{1, 4, 5\} & \{2, 3, 6\} & \{2, 5, 6\} \\
\hline
\{\{1, 2\}, \{3, 5\}, \{4, 6\}\} & [113434] & [115454] & [223636] & [225656] \\
\{\{1, 6\}, \{2, 4\}, \{3, 5\}\} & [143431] & [145451] & [623236] & [625256] \\
\hline
\end{array}
\]

The kernel has 48 elements, the local groups being isomorphic to the symmetric group \( S_3 \).

Each cell consists of functions with the given partition, acting as permutations on the range class. They are given by matrices whose nonzero columns are in the positions labelled by the range class. The entries in a nonzero column are in rows labelled by the elements comprising the block of the partition mapping into the element labelling that column. We label the partitions

\[
P_1 = \{\{1, 2\}, \{3, 5\}, \{4, 6\}\}, \quad P_2 = \{\{1, 6\}, \{2, 4\}, \{3, 5\}\}
\]

and the range classes

\[
R_1 = \{1, 3, 4\}, \quad R_2 = \{1, 4, 5\}, \quad R_3 = \{2, 3, 6\}, \quad R_4 = \{2, 5, 6\}.
\]

The local group with partition \( P_1 \) and range \( R_3 \), for example, are the idempotent \([223636]\) noted in the above table and the functions

\[
\{ [663232], [662323], [336262], [332626], [226363] \}
\]
isomorphic to \( S_3 \) acting on the range class \( \{2, 3, 6\} \). For the function \([332626]\), in the matrix semigroup we have the correspondence

\[
[332626] \leftrightarrow \begin{bmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

Using the methods developed below, one finds the measure on the partitions to be

\[ \alpha = \left[ \frac{1}{3}, \frac{2}{3} \right] \]

while that on the ranges is

\[ \beta = \left[ \frac{4}{9}, \frac{2}{9}, \frac{1}{9}, \frac{2}{9} \right] \]

as required by invariance.

3. Graphs, semigroups, and dynamical systems

We start with a regular \( d \)-out directed graph on \( n \) vertices with adjacency matrix \( A \). Number the vertices once and for all and identify vertex \( i \) with \( i \) and vice versa.

Form the stochastic matrix \( A = d^{-1} A \). We assume that \( A \) is irreducible and aperiodic. In other words, the graph is strongly connected and the limit

\[ \lim_{m \to \infty} A^m = \Omega \]

exists and is a stochastic matrix with identical rows, the invariant distribution for the corresponding Markov chain, which we denote by \( \pi = [p_1, \ldots, p_n] \). The limiting matrix satisfies

\[ \Omega = A\Omega = \Omega A = \Omega^2 \]

so that the rows and columns are fixed by \( A \), eigenvectors with eigenvalue 1.

Decompose \( A = \frac{1}{d} C_1 + \cdots + \frac{1}{d} C_d \), into binary stochastic matrices, “colors”, corresponding to the \( d \) choices moving from a given vertex to another. Each coloring matrix \( C_i \) is the matrix of a function \( f \) on the vertices as in the discussion in \S 2. Let \( S = S((C_1, C_2, \ldots, C_d)) \) be
the semigroup generated by the matrices $C_i, 1 \leq i \leq d$. Then we may write the decomposition of $A$ into colors in the form

$$A = \mu^{(1)}(1)C_1 + \cdots + \mu^{(1)}(d)C_d = \sum_{w \in \mathcal{S}} \mu^{(1)}(w) w$$

with $\mu^{(1)}$ a probability measure on $\mathcal{S}$, thinking of the elements of $\mathcal{S}$ as words $w$, strings from the alphabet generated by $\{C_1, \ldots, C_d\}$. We have seen in the previous section that

$$A^m = \sum_{w \in \mathcal{S}} \mu^{(m)}(w) w$$

where $\mu^{(m)}$ is the $m$th convolution power of the measure $\mu^{(1)}$ on $\mathcal{S}$.

The main limit theorem is the following

**Theorem 3.1.** [5, Th. 2.7] Let the support of $\mu^{(1)}$ generate the semigroup $\mathcal{S}$. Then the Cesàro limit

$$\lambda(w) = \lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} \mu^{(m)}(w)$$

exists for each $w \in \mathcal{S}$. The measure $\lambda$ is concentrated on $\mathcal{K}$, the kernel of the semigroup $\mathcal{S}$. It has the canonical decomposition

$$\lambda = \alpha \times \omega \times \beta$$

corresponding to the Rees product decomposition of $X \times G \times Y$ of $\mathcal{K}$.

**Remark.** See [1] for related material, including a proof of the above theorem.

We use the notation $\langle \cdot \rangle$ to denote averaging with respect to $\lambda$ over random elements $K \in \mathcal{K}$. Forming $M^{-1} \sum_{m=1}^{M}$ on both sides of (1), letting $M \to \infty$ we see immediately that

$$\Omega = \langle K \rangle$$

the average element of the kernel. We wish to discover further properties of the kernel by considering the behavior of tensor powers of the semigroup elements.

From the above discussion, we have the following.
Proposition 3.2. Let $\phi : S \to S'$ be a homomorphism of finite matrix semigroups. Let

$$A_{\phi} = \mu^{(1)}(1)\phi(C_1) + \cdots + \mu^{(d)}(d)\phi(C_d) = \sum_{w \in S} \mu^{(1)}(w) \phi(w)$$

then the Cesàro limit

$$\lim_{M \to \infty} \frac{1}{M} \sum_{m=1}^{M} A_{\phi}^m$$

exists and equals

$$\Omega_{\phi} = \langle \phi(K) \rangle$$

the average over the kernel $K$ with respect to the measure $\alpha \times \omega \times \beta$ as for $S$.

Proof. Checking that

$$A_{\phi}^m = \sum_{w \in S} \mu^{(m)}(w) \phi(w)$$

the result follows immediately. $\square$

$\Omega_{\phi}$ satisfies the relations

$$A_{\phi} \Omega_{\phi} = \Omega_{\phi} A_{\phi} = \Omega_{\phi}^2 = \Omega_{\phi}.$$ 

Proposition 3.3. The rows of $\Omega_{\phi}$ span the set of left eigenvectors of $A_{\phi}$ for eigenvalue 1. The columns of $\Omega_{\phi}$ span the set of right eigenvectors of $A_{\phi}$ for eigenvalue 1.

Proof. For a left invariant vector $v$, we have

$$v = v A_{\phi} = v A_{\phi}^m = v \Omega_{\phi} \quad \text{(taking the Cesàro limit)}$$

Thus, $v$ is a linear combination of the rows of $\Omega_{\phi}$. Similarly for right eigenvectors. $\square$

Now, as $\Omega_{\phi}$ is an idempotent, we have rank $\Omega_{\phi} = \text{tr} \Omega_{\phi}$ so

Corollary 3.4. The dimension of the eigenspace of right/left invariant vectors of $A_{\phi}$ equals $\text{tr} \Omega_{\phi}$.

Remark. We will only consider mappings $\phi$ that preserve nonnegativity. $A_{\phi}$ in general will be substochastic, i.e., it may have zero rows or rows that do not sum to one. As noted in Appendix B, the Abel limits exist for the $A_{\phi}$ in agreement with the Cesàro limits. Abel limits are suitable for computations, e.g. with Maple or Mathematica.
4. Tensor hierarchy

Starting with a set $V$ of $n$ elements, let $F(V) = \{f : V \to V\}$. For a field of scalars, take the rationals, $\mathbb{Q}$, as the most natural for our purposes and consider the vector space $\mathcal{V} = \mathbb{Q}^n$ with $\text{End}(\mathcal{V})$ the space of $n \times n$ matrices acting as endomorphisms of $\mathcal{V}$. We have the mapping

$$F(V) \xrightarrow{\text{Mt}} \text{End}(\mathcal{V})$$

taking $f \in F(V)$ to $\text{Mt}(f) \in \text{End}(\mathcal{V})$ defined by

$$(\text{Mt}(f))_{ij} = \begin{cases} 1, & \text{if } f(i) = j \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

Denote by $F_\circ(V)$ the semigroup consisting of $F(V)$ with the operation of composition, where we compose maps to the right: for $i \in V$, $i(f_1f_2) = f_2(f_1(i))$. The mapping $\text{Mt}$ gives a representation of the semigroup $F_\circ(V)$ by endomorphisms of $\mathcal{V}$, i.e., $\text{Mt}(f_1f_2) = \text{Mt}(f_1)\text{Mt}(f_2)$.

For $W$ of the form $\text{Mt}(f)$ corresponding to a function $f \in F(V)$, the resulting components of $W^{\otimes k}$ are components of the map on tensors given by the $k$th kronecker power of $W$. At each level $k$, there is an induced map

$$\text{End}(\mathcal{V}) \to \text{End}(\mathcal{V}^{\otimes k}) , \quad \text{Mt}(f) \to \text{Mt}(f)^{\otimes k}$$

satisfying

$$(\text{Mt}(f_1f_2))^{\otimes k} = (\text{Mt}(f_1)\text{Mt}(f_2))^{\otimes k} = \text{Mt}(f_1)^{\otimes k}\text{Mt}(f_2)^{\otimes k} \quad (3)$$

giving, for each $k$, a representation of the semigroup $F_\circ(V)$ as endomorphisms of $\mathcal{V}^{\otimes k}$.

What is the function, $f_k$, corresponding to $\text{Mt}(f)^{\otimes k}$, i.e., such that $\text{Mt}(f_k) = \text{Mt}(f)^{\otimes k}$? For degree $1$, we have from (2)

$$e_i\text{Mt}(f) = e_{f(i)} \quad (4)$$

And for the induced map at degree $k$, taking products in $\mathcal{V}^{\otimes k}$, for a multi-set $I$,

$$e_i\text{Mt}(f)^{\otimes k} = (e_{i_1}\text{Mt}(f)) \otimes (e_{i_2}\text{Mt}(f)) \otimes \cdots \otimes (e_{i_k}\text{Mt}(f))$$

$$= e_{f(i_1)} \otimes e_{f(i_2)} \otimes \cdots \otimes e_{f(i_k)}$$
We see that the degree $k$ maps are those induced on multi-subsets of $V$ mapping
\[\{i_1, \ldots, i_k\} \to \{f(i_1), \ldots, f(i_k)\}\]
with no further conditions, i.e., there is no restriction that the indices be distinct. This thus gives the second quantization of $\text{Mt}(f)$ corresponding to the induced map, the second quantization of $f$, extending the domain of $f$ from $V$ to $V^\otimes$, the space of all tensor powers of $V$.

Our main focus in this work is on the degree two component, where we have a natural correspondence of 2-tensors with matrices.

4.1. **The degree 2 component of $V^\otimes$.** Working in degree 2, we denote indices $I = (i, j)$, as usual, instead of $(i_1, i_2)$.

For given $n$, $X$, $X'$, etc., are vectors in $V^\otimes \cong \mathbb{Q}^n$. As a vector space, $V$ is isomorphic to $\mathbb{Q}^n$. Denote by $\text{End}(V)$ the space of matrices acting on $V$.

**Definition** The mapping
\[\text{Mat}: V^\otimes \to \text{End}(V)\]
is the linear isomorphism taking the vector $X = (x_{ij})$ to the matrix $\tilde{X}$ with entries
\[\tilde{X}_{ij} = x_{ij}.\]
We will use the explicit notation $\text{Mat}(X)$ as needed for clarity.

Equip $V^\otimes$ with the inner product
\[\langle X, X' \rangle = \text{tr} \, \tilde{X}^* \tilde{X}'\]
the star denoting matrix transpose.

Throughout, we use the convention wherein repeated Greek indices are automatically summed over. So we write
\[\langle X, X' \rangle = X_{\lambda \mu} X'_{\lambda \mu}.\]
4.2. **Basic Identities.** Multiplying $X$ with $u$, we note

$$ Xu^\dagger = \text{tr} \tilde{X}J = uX^\dagger \tag{5} $$

We observe

**Proposition 4.1.** *(Basic Relations)* We have

1. $\text{Mat}(XA^{\otimes 2}) = A^*\tilde{X}A$.

2. $\text{Mat}(A^{\otimes 2}X^\dagger) = A\tilde{X}A^*$.

*Proof.* We check #1 as #2 is similar. The matrix $A^{\otimes 2}$ has components

$$ A_{ab,cd} = A_{ac}A_{bd} $$

And we have

$$ (XA^{\otimes 2})_{cd} = \sum_{a,b} x_{ab}A_{ac}A_{bd} $$

$$ = (A^*\tilde{X}A)_{cd} \square $$

We see immediately

**Proposition 4.2.** For any $A$ and $X$,

$$ \tilde{X} = A^*\tilde{X}A \iff XA^{\otimes 2} = X $$

$$ \tilde{X} = A\tilde{X}A^* \iff A^{\otimes 2}X^\dagger = X^\dagger $$

4.3. **Trace Identities.** Using equation (5), we will find some identities for traces of these quantities.

**Proposition 4.3.** We have

1. $XA^{\otimes 2}u^\dagger = \text{tr}(\tilde{X}AJA^*)$.

2. $uA^{\otimes 2}X^\dagger = \text{tr}(\tilde{X}A^*JA)$.

3. If $A$ is stochastic, then $XA^{\otimes 2}u^\dagger = \text{tr}(\tilde{X}J)$. 
Proof. We have, using equation (5) and Basic Relation 1,
\[
XA^\otimes u^\dagger = \text{tr} \text{Mat}(XA^\otimes J)
= \text{tr}(A^*\tilde{X}AJ)
\]
and rearranging terms inside the trace yields \#1. Then \#3 follows since \( A \) stochastic implies \( AJ = J = JA^* \). And \#2 follows similarly, using the second Basic Relation in the equation \( uA^\otimes X^\dagger = \text{tr} \text{Mat}(A^\otimes X^\dagger J) \).

Using equation (5) directly for \( X \), we have
\[
X(I - A^\otimes)u^\dagger = \text{tr}(\tilde{X}(J - AJA^*))
\]
\[
u(I - A^\otimes)X^\dagger = \text{tr}(\tilde{X}(J - A^*JA))
\]
And the first line reduces to
\[
X(I - A^\otimes)u^\dagger = 0
\]
for stochastic \( A \).

4.4. Convergence to tensor hierarchy.

Proposition 4.4. For \( \ell \geq 1 \), define
\[
A_{\otimes \ell} = \frac{1}{d}C_1^\otimes + \cdots + \frac{1}{d}C_d^\otimes.
\]
Then the Cesàro limit of the powers \( A_{\otimes \ell}^m \) exists and equals
\[
\Omega_{\otimes \ell} = \langle K^\otimes \rangle
\]
the average taken over the kernel \( K \) of the semigroup \( S \) generated by \( \{C_1, \ldots, C_d\} \).

We imagine the family \( \{K^\otimes \}_{\ell \geq 1} \), as a hierarchy of kernels corresponding to the family of semigroups generated by tensor powers \( C_i^\otimes \) in each level \( \ell \). Since each element \( k \in K \) is a stochastic matrix, in fact a binary stochastic matrix, as each corresponds to a function, we have \( kJ = J \) at every level. We thus have a mechanism to move down the hierarchy by the relation
\[
k^\otimes(J \otimes I^\otimes(\ell-1)) = J \otimes k^\otimes(\ell-1)
\]
which is composed of identical blocks \( k^\otimes(\ell-1) \).
Note that if \( \text{rank} \Omega \otimes \ell = 1 \), the invariant distribution can be computed as an element of the nullspace of \( I - A \otimes \ell \). Otherwise, \( \Omega \otimes \ell \) can be found computationally as the Abel limit
\[
\lim_{s \uparrow 1} (1 - s)(I - sA \otimes \ell)^{-1}.
\]

5. The principal observables: \( M \) and \( N \) operators

5.1. Graph-theoretic context. For the remainder of this paper, we work mainly in the context where the stochastic matrix \( A = \frac{1}{2} \mathcal{A} \), with \( \mathcal{A} \) the adjacency matrix of a 2-out regular digraph, assumed strongly connected and aperiodic. In other words, \( A \) is an ergodic transition matrix for the Markov chain induced on the vertices, \( V \), of the graph. \( A \) decomposes into two coloring matrices, which we call from now on \( R \) and \( B \), for “red” and “blue”, respectively. Thus
\[
A = \frac{1}{2}(R + B)
\]
where \( R \) and \( B \) are binary \( n \times n \) stochastic matrices.

Example. For a working example, consider
\[
A = \begin{bmatrix}
0 & 0 & 1/2 & 1/2 \\
0 & 0 & 1/2 & 1/2 \\
1/2 & 0 & 0 & 1/2 \\
0 & 1/2 & 1/2 & 0
\end{bmatrix}.
\]
The powers of \( A \) converge to the idempotent matrix
\[
\Omega = \begin{bmatrix}
1/6 & 1/6 & 1/3 & 1/3 \\
1/6 & 1/6 & 1/3 & 1/3 \\
1/6 & 1/6 & 1/3 & 1/3 \\
1/6 & 1/6 & 1/3 & 1/3
\end{bmatrix}.
\]
As one possible decomposition, we may take
\[
R = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}.
\]
In the notation used previously, we write
\[
R = [4312] \quad \text{and} \quad B = [3443].
\]
5.2. **Level 2 of the tensor hierarchy.** Let

$$A_{\otimes 2} = \frac{1}{2}(R_{\otimes 2}^* + B_{\otimes 2}^*)$$

Then fixed points $X$ of $A_{\otimes 2}$ are vectors corresponding to solutions to the matrix equations

$$\frac{1}{2}(R\tilde{X}R^* + B\tilde{X}B^*) = \tilde{X}$$

and

$$\frac{1}{2}(R^*\tilde{X}R + B^*\tilde{X}B) = \tilde{X}$$

5.2.1. **$M$ and $N$ operators.** In Section 4.1, we defined the map from vectors to matrices $X \to \tilde{X} = \text{Mat}(X)$. Now it is convenient to have a reverse map from matrices to vectors, say

$$Y \to Y_{vec}$$

where here we use $Y$ to denote a typical matrix to distinguish our conventional use of $X$ as a vector.

Denote by $\mathcal{N}$ the space of solutions to

$$\frac{1}{2}(RYR^* + BYB^*) = Y.$$

The nonnegative solutions denote by $\mathcal{N}_+$, and the space of nonnegative solutions with trace zero by $\mathcal{N}_0$.

Similarly, $\mathcal{M}$ denotes the space of solutions to

$$\frac{1}{2}(R^*YR + B^*YB) = Y$$

with $\mathcal{M}_+$ nonnegative solutions, and $\mathcal{M}_0$, nonnegative trace zero solutions.

Starting from the relation

$$\Omega_{\otimes 2} = \langle K_{\otimes 2}^* \rangle$$

We have solutions in the space $\mathcal{M}$ in the form

$$X\Omega_{\otimes 2} = \langle K^*\tilde{X}K \rangle = \sum' k^*\tilde{X}k$$

and solutions in $\mathcal{N}$ in the form

$$\Omega_{\otimes 2}X = \langle K\tilde{X}K^* \rangle = \sum' k\tilde{X}k^*$$

where the primed summation denotes an averaged sum, here with respect to the measure $\lambda$ on $\mathcal{K}$. 
5.2.2. Diagonal of $N \in \mathcal{N}$. For any $Y$, observe that 
\[(RYR^*)_{ij} = R_{i\lambda}Y_{\lambda\mu}R_{\mu j} = Y_{iRjR}\]
since $R$ acts as a function on the indices. Similarly for $B$.

**Proposition 5.1.** Let $N \in \mathcal{N}_+$. Then the diagonal of $N$ is constant.

**Proof.** Assume that $N_{11} = \max_i N_{ii}$, is the largest diagonal entry. Start with 
\[2N_{11} = N_{1R1R} + N_{1B1B}\]
Since $N_{11}$ is maximal, both terms on the right-hand side agree with $N_{11}$. Generally, if $N_{ii} = N_{11}$ then $N_{iRiR} = N_{iBiB} = N_{11}$. Proceeding inductively yields $N_{ii} = N_{11}$ for all $i$ since the graph is strongly connected. \qed

5.2.3. Diagonal of $M \in \mathcal{M}$. Define the diagonal matrix $\omega = \text{diag}(\pi)$, where $\pi$ is the invariant distribution for $A$.

**Proposition 5.2.** Let $M \in \mathcal{M}_+$. Then the diagonal of $M$ is a scalar multiple of $\omega$.

**Proof.** Denote the diagonal of $M$ by $D$, i.e., $M_{ii} = D_i$. Start with 
\[2M_{ij} = (R^*MR + B^*MB)_{ij} = R_{\lambda i}M_{\lambda \mu}R_{\mu j} + B_{\lambda i}M_{\lambda \mu}B_{\mu j}\]
Let $\bar{\delta}_{ij} = 1 - \delta_{ij}$, i.e., one if $i \neq j$, zero otherwise. Take $i = j$ and split off the diagonal terms on the right to get 
\[2D_i = R_{\lambda i}D_\lambda + B_{\lambda i}D_\lambda + R_{\lambda i}M_{\lambda \mu}R_{\mu i}\bar{\delta}_{\lambda \mu} + B_{\lambda i}M_{\lambda \mu}B_{\mu i}\bar{\delta}_{\lambda \mu}\]
using the fact that the entries of $R$ and $B$ are 0’s and 1’s. Thus, 
\[D_i - (DA)_i = \frac{1}{2} (R_{\lambda i}M_{\lambda \mu}R_{\mu i}\bar{\delta}_{\lambda \mu} + B_{\lambda i}M_{\lambda \mu}B_{\mu i}\bar{\delta}_{\lambda \mu})\]
Now sum both sides over $i$. Since $A$ is stochastic, $\sum (DA)_i = \sum D_i$ and the left-hand side vanishes. Since the right-hand side is a sum of nonnegative terms adding to zero, the right-hand side is identically zero. Thus, 
\[D = DA\]
is a left fixed point of $A$. By irreducibility, it must be a multiple of $\pi$. \qed
5.3. **Specification of operators $N$ and $M$.** In the following, we will use $M$ and $N$ to refer to the specific operators defined by

$$M = \text{Mat}(u\Omega \otimes \mathbb{J}) = \langle K^*JK \rangle$$

and

$$N = \text{Mat}(\Omega \otimes \mathbb{I}_{\text{vec}}) = \langle KK^* \rangle$$

Another special operator in $\mathcal{M}$ is defined as

$$\tilde{M} = \text{Mat}(\Omega_{\text{vec}} \Omega \otimes \mathbb{2}) = \langle K^*\Omega K \rangle$$

We can pass to trace-zero operators by subtracting off the diagonals defining

$$M_0 = M - \tau\omega$$

$$N_0 = J - N$$

(6)  

(7)

For now, $\tau$ is defined by the above relation. We will see that the diagonal of $N$ is $\text{diag}(u) = I$. The reason for complementing with respect to $J$, i.e. subtracting from 1 everywhere will become clear after Proposition 5.4.

Note that at level 1,

$$u\Omega = uu^\dagger\pi = n\pi = n\pi A$$

$$\Omega u^\dagger = u^\dagger = Au^\dagger$$

(8)  

(9)

So for level two, we define the fields

$$\pi_{\otimes 2} = u\Omega_{\otimes 2}$$

$$u_{\otimes 2}^\dagger = \Omega_{\otimes 2}I_{\text{vec}}$$

with analogous expressions for each level $\ell$. Indeed, at level two these satisfy

$$\pi_{\otimes 2}A_{\otimes 2} = \pi_{\otimes 2}$$

$$A_{\otimes 2}u_{\otimes 2}^\dagger = u_{\otimes 2}^\dagger$$

extending relations (8) and (9). The families $\{u_{\otimes \ell}\}$ and $\{\pi_{\otimes \ell}\}$ provide extensions to scalar fields the basic all-ones vector $u$ and invariant distribution $\pi$ of level one.
5.4. **Basic level two relations.** In detailing the structure of the kernel, we need notations corresponding to the range classes. Given an idempotent \(e \in \mathcal{K}\), define \(\tilde{\rho}(e)\) to be the *state vector* of the range of \(e\), namely, it is a 0-1 vector with \(\tilde{\rho}(e)_i = 1\) exactly when \(ie = e\). The corresponding matrix

\[\rho(e) = \text{diag}(\tilde{\rho}(e))\]

**Proposition 5.3.** We have the relations:

\[
\begin{align*}
\text{Mat}(\Omega_{\text{vec}} \Omega_{\otimes 2}) &= \tilde{M} = \frac{n}{r^2} \langle \rho^\dagger \rho \rangle \\
\text{Mat}(\Omega_{\otimes 2} \Omega_{\text{vec}}) &= \Omega \Omega^* = (\sum p_i^2) J \\
\text{Mat}(I_{\text{vec}} \Omega_{\otimes 2}) &= \langle K^* K \rangle = n \omega \\
\text{Mat}(J_{\text{vec}} \Omega_{\otimes 2}) &= M \\
\text{Mat}(\Omega_{\otimes 2} I_{\text{vec}}) &= N \\
\text{Mat}(\Omega_{\otimes 2} J_{\text{vec}}) &= J
\end{align*}
\]

Note we are writing \(J_{\text{vec}}\) for parallel formulation as an alternative for \(u\).

**Proof.** We will give the proof of the first relation later, as it relies on Friedman’s Theorem. Consider

\[
\text{Mat}(\Omega_{\otimes 2} \Omega_{\text{vec}}) = \langle K \Omega K^* \rangle = \langle \Omega K^* \rangle \\
= \Omega \Omega^* = u^\dagger \pi \pi^\dagger u = (\pi \pi^\dagger) u^\dagger u
\]

where we use the fact that elements of \(\mathcal{K}\) are stochastic matrices. Next, observe that \(k^* k\) is always diagonal. Indeed

\[(k^* k)_{ij} = k_{\lambda i} k_{\lambda j}\]

indicates a summation over \(\lambda\) such that \(k\) maps \(\lambda\) to both \(i\) and \(j\). In fact it is the size of the block of the partition that \(k\) maps into \(i\) when \(i = j\). Furthermore, for any vector \(v\), we have the tautology

\[\text{diag}(v) = \text{diag}(\text{diag}(v) u^\dagger)\]

We have

\[\text{Mat}(I_{\text{vec}} \Omega_{\otimes 2}) = \langle K^* K \rangle\]

and we compute

\[\langle K^* K u^\dagger \rangle = \langle K^* u^\dagger \rangle = \Omega^* u^\dagger = n \pi^\dagger.
\]

hence the result. And

\[\text{Mat}(\Omega_{\otimes 2} I_{\text{vec}}) = \langle KK^* \rangle = N\]

by our definition. We have

\[\text{Mat}(J_{\text{vec}} \Omega_{\otimes 2}) = \langle K^* JK \rangle\]
the definition of $M$. And finally,
\[
\text{Mat}(\Omega \otimes I_{\text{vec}}) = \langle KK^* \rangle = J
\]
immediately. \qed

For some trace calculations, note that since $\text{tr} \omega = 1$, equation (6) effectively says
\[
\tau = \text{tr} M
\]
Now, we note that
\[
\text{tr} NJ = \text{tr} \langle KK^* J \rangle = \text{tr} \langle K^* J K \rangle = \text{tr} M = \tau
\]
as well.

5.4.1. Interpretation of $\tau$. In describing the kernel, we have two basic constants, the rank $r$ and $\tau$. We want to interpret $\tau$ as describing features of the kernel. Start with

Proposition 5.4. Consider an idempotent $e \in \mathcal{K}$, determined by a partition and range $(\mathcal{P}, \mathcal{R})$. For all $k$ in the local group $G_e$ for which $e$ acts as the identity,
\[
k^* k = ee^*
\]
is a 0-1 matrix such $(kk^*)_{ij} = 1$ if $i$ and $j$ are in the same block of $\mathcal{P}$.

Proof. Writing out $k_{i\lambda} k_{j\lambda}$ shows that this sum counts 1 precisely when both $i$ and $j$ map to the same range element, i.e., they are in the same block of $\mathcal{P}$. \qed

We have an interpretation of the entries $N_{ij}$ as the probability that vertices $i$ and $j$ appear in the same block of a partition. Thus, the entry $(J - N)_{ij}$ gives the probability that $i$ and $j$ are split, i.e., they never appear together in the same block of a partition.

Remark. Note that $ee^*$ is the same for all cells in a given row of the kernel. And is independent of $k$ in a given cell. Thus,
\[
N = \langle KK^* \rangle = \langle EE^* \rangle
\]
where it is sufficient in this last average to consider only the idempotents in any single column of the kernel.

The column sum $(uee^*)_i$ is the size of the block of $\mathcal{P}$ containing $i$. Now $\text{tr} J ee^* = \text{tr} J kk^*$ sums all the ones in $kk^*$, and for all local $k$,
\[
\text{tr} k^* J k = \text{tr} J kk^* = \text{equals the sum of the squares of the blocksize of } \mathcal{P}.
\]
It follows that
Proposition 5.5. \( \tau = \text{tr} M \) is the average sum of squares of the block-sizes over the partitions \( \mathcal{P} \) of \( K \).

We continue with some relations useful for deriving and describing properties of the kernel.

5.5. Useful relations between an idempotent and its range matrix. We make some useful observations regarding an idempotent \( e \) and its range matrix \( \rho(e) \).

Proposition 5.6. For any idempotent \( e \),

1. \( \rho(e)e = \rho(e), \quad e^*\rho(e) = \rho(e), \quad e\rho(e) = e \).
2. \( ee^*\rho(e) = \rho(e) \).

Proof. For \#1, the first and last relations follow since \( e \) fixes its range. Transposing the first yields the second. And \#2 follows directly. \( \square \)

For \( k \in K \), let \( \varrho(k) \) denote the state vector of the range of \( k \) and \( \rho(k) \) the corresponding diagonal matrix.

Corollary 5.7. 1. For any \( k \in K \), \( k\rho(k) = k \).
2. \( e^*\varrho(e)^\dagger = \varrho(e)^\dagger \).
3. \( e\varrho(e)^\dagger = u^\dagger \).

Proof. For \#1, write \( k = ke \) with \( \rho(k) = \rho(e) \). Then
\[
k\rho(k) = ke\rho(e) = ke = k
\]
as in the Proposition. And \#2 and \#3 follow from the Proposition via \( \text{diag}(v)u^\dagger = v^\dagger \). \( \square \)

6. Projections

We can compute the row and column projections in the kernel using our operators \( M \) and \( N \).

The idempotents having the same range \( \mathcal{R} \), say, form a left-zero semigroup. That is, if \( e_1 \) and \( e_2 \) have the same range, then \( e_1e_2 = e_1 \), since \( e_1 \) determines the partition and the range is fixed by both idempotents. Correspondingly, the idempotents having the same partition \( \mathcal{P} \), say, form a right-zero semigroup. That is, if \( e_1 \) and \( e_2 \) have the same partition, then \( e_1e_2 = e_2 \), since the partition for both is the same and an element in the range of \( e_2 \) is mapped by \( e_1 \) into an element in the
same block of the common partition, which in turn is mapped back to the original element by $e_2$.

**Remark.** Note that in this terminology Proposition 5.6 says that the pair $\{e, \rho(e)\}$ form a left-zero semigroup.

**Notation.** It is convenient to fix an enumeration of the partitions $\{P_1, P_2, \ldots\}$ and ranges $\{R_1, R_2, \ldots\}$. We set $\rho_j$ to be the matrix $\text{diag}(\rho_j)$ where $\rho_j$ is the state vector for $R_j$.

6.1. **Column projections.** Denote $e_{ij}$ the idempotents for range $R_j$. Define the column projection to be the average

$$P_j = \alpha_\lambda e_{\lambda j}.$$  

The measure $\alpha$ is the component of $\lambda$ on the partitions.

**Proposition 6.1.**

1. $P_j^2 = P_j$ is an idempotent.
2. $P_j = N\rho_j$.

**Proof.** For #1, with $u$ a vector of all ones,

$$P_j^2 = \alpha_\lambda \alpha_\mu e_{\lambda j}e_{\mu j} = \alpha_\lambda \alpha_\mu u_{\mu j} = \alpha_\lambda \alpha_\mu R_j = P_j$$

using the fact that the idempotents with a common range form a left-zero semigroup and that the $\alpha$’s are a probability distribution.

For #2, recall the relation $e^*\rho(e) = \rho(e)$ from Proposition 5.6, and we have

$$N\rho_j = \alpha_\lambda e_{\lambda j}e_{\lambda j}^{*}\rho_j = \alpha_\lambda e_{\lambda j}\rho_j = P_j$$

with $e_{ij}\rho_j = e_{ij}$ as each $e_{ij}$ fixes the common range.

6.2. **Row projections.** Analogously, define $Q_i = \beta_\mu e_{i\mu}$, the average idempotent with common partition $P_i$. As in #1 of Prop. 6.1, $Q_i^2 = Q_i$, where now the idempotents form a right-zero semigroup.

We need the fact [to be proved later] that the average of the range matrices is $r$ times $\omega = \text{diag}(\pi)$. We state this in the form

$$\omega = r^{-1}\beta_\mu \rho_\mu$$  

where $\beta$ is the part of the measure $\lambda$ on the range classes.
Proposition 6.2. 
\[ Q_i = r e_i e_i^* \omega \]
where \( e_i \) is any idempotent with partition \( \mathcal{P}_i \).

Proof. First, recall that \( e_ij e_i^j = e_i e_i^* \) for any idempotent \( e_ij \) with partition \( \mathcal{P}_i \). And \( e_i e_i^* \rho(e_i) = e_i \). Using equation (10) we have
\[
\begin{align*}
  r e_i e_i^* \omega &= e_i e_i^* \beta_\mu \rho_\mu \\
  &= \beta_\mu e_{i\mu} e_{i\mu}^* \rho_\mu \\
  &= \beta_\mu e_{i\mu} = Q_i
\end{align*}
\]
as required. □

6.3. Average idempotent. We now compute the average idempotent \( \langle E \rangle \).

Theorem 6.3. The average idempotent satisfies
\[ \langle E \rangle = rN \omega \]

Proof. From Proposition 6.2, we need to average \( Q_i \) over the partitions. We get
\[
\langle E \rangle = \alpha_\lambda Q_\lambda = r \alpha_\lambda e_{\lambda 1} e_{\lambda 1}^* \omega = r N \omega
\]
using the idempotents with common range \( \mathcal{R}_1 \). □

7. Equipartitioning

In the current framework, as in [2], we express Friedman’s Theorem this way:

Theorem 7.1 ([2, 4], version 2). For \( k \in \mathcal{K} \), let \( \rho(k) \) denote the 0-1 vector with support the range of \( k \). Then
\[ \pi k = \frac{1}{r} \rho(k) \]
where \( r \) is the rank of the kernel.

We will refer to this theorem as BM/F.

Corollary 7.2. For any \( k \in \mathcal{K} \), we have
\[ \pi kk^* = \frac{1}{r} u \]

Proof. By Corollary 5.7, we have
\[ k \rho(k) = k \quad \text{which implies} \quad k \rho(k)^\dagger = u^\dagger \]
Now transpose. □
This corollary can be rephrased as stating that each block of any partition has \( \pi \)-probability equal to \( 1/r \). Here we will give an independent proof with a functional analytic flavor.

7.1. \( A \) and \( \Delta \). Working with two colors \( R \) and \( B \), it is convenient to introduce the operator \( \Delta \)

\[
\Delta = \frac{1}{2}(R - B)
\]

Then we check that:

\[
A \otimes \Delta = \frac{1}{2}(R \otimes 2 + B \otimes 2) = A \otimes 2 + \Delta \otimes 2
\]

Hence, elements \( Y \in N \) satisfy

\[
AYA^* + \Delta Y \Delta^* = Y
\]

and those in \( M \) satisfy

\[
A^*YA + \Delta^*Y \Delta = Y
\]

In particular, \( N \) and \( M \) satisfy these equations accordingly:

\[
ANA^* + \Delta N \Delta^* = N \tag{11}
\]

\[
A^*MA + \Delta^*M \Delta = M \tag{12}
\]

7.2. Friedman’s Theorem d’après Budzban-Mukherjea. First, some notation

**Notation.** Let \( \mathcal{E}(\mathcal{K}) \) denote the set of all idempotents of the kernel \( \mathcal{K} \).

Start with

**Proposition 7.3.** For every \( k \in \mathcal{K} \), \( \pi \Delta k = 0 \). Equivalently, \( \pi Rk = \pi Bk = \pi k \).

**Proof.** Multiplying eq. (11) on the left by \( \pi \) and on the right by \( \pi^\dagger \), we have, using \( \pi A = \pi \), \( A^* \pi^\dagger = \pi^\dagger \),

\[
\pi N \pi^\dagger + \pi \Delta N \Delta^* \pi^\dagger = \pi N \pi^\dagger
\]

Since \( N = \langle KK^* \rangle \), we have

\[
\pi \Delta N \Delta^* \pi^\dagger = 0
\]

\[
= \langle \pi \Delta KK^* \pi^\dagger \rangle = \langle \pi \Delta K \rangle^2
\]

All terms are nonnegative so that \( \pi \Delta k = 0 \) for every \( k \). From the definition of \( \Delta \) the alternative formulation results after averaging \( \pi Rk = \pi Bk \) to \( \pi Ak = \pi k \). \( \square \)
And directly from the definition of $N$

**Corollary 7.4.** We have $\pi \Delta N = 0$.

We extend to words in the semigroup generated by $R$ and $B$.

**Proposition 7.5.** For every $w \in S$, and $k \in K$, $\pi wk = \pi k$.

*Proof.* This follows by induction on $l$, the length of $w$. Proposition 7.3 is the result for $l = 1$. Let $w_{l+1}$ be a word of length $l + 1$, then $w_{l+1} = \text{one of } Rw_l, Bw_l$, where $w_l$ has length $l$. Since $w_lk \in K$ for $k \in K$, we have, by Proposition 7.3,

\[
\pi w_{l+1}k = \pi Rw_lk = \pi Bw_lk
\]

(averaging)

\[
= \pi wk
\]

(by induction)

\[
= \pi k
\]

\[
\square
\]

**Proposition 7.6.** For any $e \in \mathcal{E}(K)$, $w \in S$,

\[
\pi we = (1/r) \rho(e) \quad \text{and} \quad \pi wee^* = (1/r) u
\]

*Proof.* From Corollary 7.4, $\pi \Delta N = 0$. Equivalently, $N\Delta^*\pi^\dagger = 0$. Hence, multiplying equation (11) on the right by $\pi^\dagger$, using $A^*\pi^\dagger = \pi^\dagger$, we find

\[
AN\pi^\dagger = N\pi^\dagger
\]

So $N\pi^\dagger$ is fixed by $A$, hence is a constant vector. Transposing back, say $\pi N = au$ for some constant $a$. Now let $e \in \mathcal{E}(K)$ and take the $e_i$ with the same range as $e$ in the averaging defining $N$ as $\langle EE^* \rangle$. Then from Proposition 7.5 we have $\pi e_i = \pi e_i$. Multiplying by $e_i^*$ and averaging yields $\pi e N = \pi N$, hence $\pi e N e^* = \pi N e^*$. Since the idempotents of a column are a left-zero semigroup, $ee_i = e$, $e_i^* e^* = e^*$. Thus,

\[
e N e^* = e \alpha_\mu e_\mu e_\mu^* e^* = ee^*
\]

Thus on the one hand $\pi e N e^* = \pi N e^*$ and on the other $\pi e N e^* = \pi e e^*$. Or $\pi e e^* = \pi N e^* = a e e^* = au$ for any idempotent $e$. Thus, by Prop. 5.6, #2, $\pi e = a \rho(e)$ and $\pi e e^* = 1 = ar$ yields $a = 1/r$, where $r$ is the rank of the kernel. So, for any $w \in S$, $\pi we = \pi e = (1/r) \rho(e)$ and Cor. 5.7, #3 yields $\pi wee^* = (1/r) u$, as required.

Now taking $w \in K$ yields $\pi k = (1/r) \rho(k)$, the theorem of Budzban-Mukherjea/Friedman.

Averaging the second relation in the above Proposition yields

**Corollary 7.7.** For any $w \in S$, $\pi w N = \frac{1}{r} u$. 

7.2.1. More on projections. Referring to the column projections $P_j$ of §6, we have

$$
\pi P_j = \pi N \rho_j = \frac{1}{r} u \rho_j = \frac{1}{r} \rho_j
$$

And for row projections $Q_j$, via Proposition 7.6,

$$
\pi Q_j = r \pi e_i e_i^* \omega = u \omega = \pi
$$

Since $Q_j$ is a stochastic matrix, $Q_j u^\dagger = u^\dagger$, we have the result

**Proposition 7.8.** Each of the row projections $Q_j$ commutes with $\Omega$.

7.2.2. Local groups. Let $G_{ij}$ be the local group with partition $P_i$ and range $R_j$. For convenience, when averaging over $G_{ij}$, we use the abbreviated form

$$
\sum' = \frac{1}{|G_{ij}|} \sum.
$$

An important fact is

**Lemma 7.9.** The average group element is $u^\dagger \rho / r$, specifically,

$$
\sum'_{G_{ij}} k = r^{-1} u^\dagger \rho_j
$$

**Proof.** From the structure of the kernel, for a fixed $e$, $eK e$ is the local group having $e$ as local identity. So take $e = e_{ij}$, the identity for $G_{ij}$. Since $\Omega = \langle K \rangle$, we have, via Prop. 7.6,

$$
\langle e K e \rangle = e \Omega e = e u^\dagger \pi e = u^\dagger \pi e = r^{-1} u^\dagger \rho(e)
$$

\[ \Box \]

Now we see that to average over the kernel, we need only average the above relation over the ranges. This yields

$$
\Omega = r^{-1} u^\dagger \beta \mu \rho \mu = r^{-1} u^\dagger u \beta \mu \rho \mu = r^{-1} J \beta \mu \rho \mu
$$

(13)

since $\rho(e) = u \rho(e)$. Comparing diagonals, we thus derive equation (10)

$$
\omega = r^{-1} \beta \mu \rho \mu
$$

used in computing the row projections, finding as well

$$
\pi = r^{-1} \beta \mu \rho \mu
$$

We are now in a position to prove the first relation of Proposition 5.3:

**Theorem 7.10.**

$$
\text{Mat}(\Omega \vec{\Omega} \otimes 2) = \frac{n}{r^2} \langle \rho^\dagger \rho \rangle = \frac{n}{r^2} \langle \rho J \rho \rangle
$$
Proof.

\[ \langle K^*\Omega K \rangle = \langle K^* u^\dagger \pi K \rangle = r^{-1} \langle K^* u^\dagger \rho(K) \rangle \]

\[ = r^{-1} \sum_{i,j} k^* u^\dagger \rho_j \]

summing over the cells of the kernel, the primes indicating averaging. By Lemma 7.9, we have

\[ r^{-1} \sum_{i,j} k^* u^\dagger \rho_j \]

\[ = r^{-2} \sum_{i,j} \rho_j^\dagger uu^\dagger \rho_j = nr^{-2} \sum_{i,j} \rho_j^\dagger \rho_j \]

\[ = nr^{-2} \sum_{i,j} \rho_j^\dagger u \rho_j = nr^{-2} \langle \rho J \rho \rangle \]

\[ \square \]

8. Properties of $M$, $N$, and $\Omega$

Here we look at some relations among the main operators of interest, including $M$.

Start with

**Proposition 8.1.**

1. $\Omega N = r^{-1} J$.
2. $M\Omega = n\Omega^*\Omega$.

**Proof.** $\Omega N = u^\dagger \pi N = r^{-1} u^\dagger u = r^{-1} J$ as required. And, noting that $J\Omega = n\Omega$,

\[ M\Omega = \langle K^* JK \Omega \rangle = \langle K^* J \Omega \rangle = n\langle K^* \rangle \Omega = \Omega^*\Omega \]

via $k\Omega = \Omega$, for all $k \in \mathcal{K}$.

And consequently,

**Corollary 8.2.** $NM$ commutes with $\Omega$.

**Proof.** First,

\[ NM\Omega = nN\Omega^*\Omega = n(\Omega N)^*\Omega = (n/r)J\Omega = (n^2/r)\Omega \]

And second,

\[ \Omega NM = r^{-1} J N = r^{-1} \langle JK^*JK \rangle = r^{-1} \langle (KJ)^*JK \rangle \]

\[ = r^{-1} \langle J^2K \rangle = r^{-1} n J \Omega = (n^2/r)\Omega \]

as required. $\square$
Observe in the proof the relation

\[ JM = n^2 \Omega \]

The case of doubly stochastic \( A \) turns out to be particularly interesting. Recall the basic fact that two symmetric matrices commute if and only if their product is symmetric.

**Theorem 8.3.** If \( A \) is doubly stochastic, then \( \{M, N, J\} \) generate a commutative algebra.

**Proof.** If \( A \) is doubly stochastic, then \( u = n\pi \) and \( n\Omega = J. \) Thus \( JM = n^2\Omega = nJ \) is symmetric. And \( JN = n\Omega N = (n/r)J \) is symmetric. So \( J \) commutes with \( M \) and \( N. \) We check that \( M \) and \( N \) commute:

\[
MN = \langle K^* u^\dagger uK \rangle N
\]

\[
= n^2 \langle K^* \pi \dagger \pi KN \rangle
\]

\[
= (n^2/r) \langle K^* \pi \dagger u \rangle
\]

\[
= (n^2/r) \langle \Omega K \rangle^*
\]

\[
= (n^2/r) (\Omega^*)^2 = (n/r) J. \quad \square
\]

which is symmetric as well.

Another feature involves \( \tilde{M}. \)

**Theorem 8.4.**

\[ N\tilde{M} = \frac{r}{n} \Omega \]

**Proof.** We have

\[ N\tilde{M} = nr^{-2} N \langle \rho J \rho \rangle \]

Note that for any \( P_j, P_j J = J. \) Fixing a range, we drop subscripts for convenience. And with \( P = P_j \) the corresponding column projection,

\[ N\rho J \rho = P J \rho = J \rho \]

By equation (13), averaging yields \( r\Omega. \) Hence the result. \( \square \)

9. Zeon hierarchy

Now we pass from the tensor hierarchy to the zeon hierarchy by looking at representations of the semigroup \( F_0(V) \) acting on zeon Fock space. We begin with the basic definitions and constructions. Then we proceed with statements paralleling those of §4, including proofs when they illustrate important differences between the two systems.

**Remark.** The introductory material in this section is largely taken from [3].
Consider the exterior algebra generated by a chosen basis \( \{ e_i \} \subset V \), with relations \( e_i \wedge e_j = -e_j \wedge e_i \). Denoting multi-indices by roman capital letters \( I = (i_1, i_2, \ldots, i_k) \), \( J \), \( K \), etc., at level \( k \), a basis for \( V^\wedge k \) is given by
\[
e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}
\]
with \( I \) running through all \( k \)-subsets of \( \{1, 2, \ldots, n\} \); i.e., \( k \)-tuples with distinct components. For \( \text{Mt}(f), f \in F(V) \), define the matrix
\[
(\text{Mt}(f)^{\vee k})_{IJ} = |(\text{Mt}(f)^\wedge k)_{IJ}|
\]
taking absolute values entry-wise. It is important to observe that we are not taking the fully symmetric representation of \( \text{End}(V) \), which would come by looking at the action on boson Fock space, spanned by symmetric tensors. However, note that the fully symmetric representation is given by maps induced by the action of \( \text{Mt}(f) \) on the algebra generated by commuting variables \( \{e_i\} \). We take this viewpoint as the starting point of the construction of the zeon Fock space, \( Z \), to be defined presently.

**Definition.** A **zeon algebra** is a commutative, associative algebra generated by elements \( e_i \) such that \( e_i^2 = 0, \ i \geq 1 \).

For a standard zeon algebra, \( Z \), the elements \( e_i \) are finite in number, \( n \), and are the basis of an \( n \)-dimensional vector space, \( V \approx \mathbb{Q}^n \). We assume no further relations among the generators \( e_i \). Then the \( k^{th} \) zeon tensor power of \( V \), denoted \( V^\vee k \), is the degree \( k \) component of the graded algebra \( Z \), with basis
\[
e_I = e_{i_1} \cdots e_{i_k}
\]
analogously to the exterior power except now the variables commute. The assumptions on the \( e_i \) imply that \( V^\vee k \) is isomorphic to the subspace of symmetric tensors spanned by elementary tensors with no repeated factors. As vector spaces,
\[
V^\vee k \approx V^\wedge k
\]
The **zeon Fock space** is \( Z \) presented as a graded algebra
\[
Z = \mathbb{Q} \oplus (\bigoplus_{k \geq 1} V^\vee k)
\]
Since \( V \) is finite-dimensional, \( k \) runs from 1 to \( n = \dim V \).

A linear operator \( W \in \text{End}(V) \) extends to the operator \( W^\vee k \in \text{End}(V^\vee k) \). The **second quantization** of \( W \) is the induced map on \( Z \).
For the exterior algebra, the $IJ^{th}$ component of $W^\wedge k$ is the determinant of the corresponding submatrix of $W$, with rows indexed by $I$ and columns by $J$. Having dropped the signs, the $IJ^{th}$ component of $W^\vee k$ is the permanent of the corresponding submatrix of $W$.

For $W$ of the form $\text{Mt}(f)$ corresponding to a function $f \in F(V)$, the resulting components of $W^\vee k$ are exactly the absolute values of the entries of $W^\wedge k$, as we wanted. At each level $k$, there is an induced map

$$\text{End}(\mathcal{V}) \to \text{End}(\mathcal{V}^\vee k), \quad \text{Mt}(f) \to \text{Mt}(f)^\vee k$$

satisfying

$$(\text{Mt}(f_1 f_2))^\vee k = (\text{Mt}(f_1)\text{Mt}(f_2))^\vee k = \text{Mt}(f_1)^\vee k \text{Mt}(f_2)^\vee k$$

(14)

giving, for each $k$, a representation of the semigroup $F_\circ(V)$ as endomorphisms of $\mathcal{V}^\vee k$. However, for general $W_1, W_2$, the homomorphism property, (14), no longer holds, i.e., $(W_1 W_2)^\vee k$ does not necessarily equal $W_1^\vee k W_2^\vee k$. It is not hard to see that a sufficient condition is that $W_1$ have at most one non-zero entry per column or that $W_2$ have at most one non-zero entry per row. For example, if one of them is diagonal, as well as the case where both correspond to functions.

What is the function, $f_k$, corresponding to $\text{Mt}(f)^\vee k$, i.e., such that $M(f_k) = \text{Mt}(f)^\vee k$ ? For degree 1, we have from (2)

$$e_i \text{Mt}(f) = e_{f(i)}$$

(15)

And for the induced map at degree $k$, taking products in $\mathcal{Z}$,

$$e_i \text{Mt}(f)^\vee k = (e_{i_1} \text{Mt}(f)) (e_{i_2} \text{Mt}(f)) \cdots (e_{i_k} \text{Mt}(f)) = e_{f(i_1)} e_{f(i_2)} \cdots e_{f(i_k)}$$

We see that the degree $k$ maps are those induced on $k$-subsets of $V$ mapping

$$\{i_1, \ldots, i_k\} \to \{f(i_1), \ldots, f(i_k)\}$$

with the property that the image in the zeon algebra is zero if $f(i_l) = f(i_m)$ for any pair $i_l, i_m$. Thus the second quantization of $\text{Mt}(f)$ corresponds to the induced map, the second quantization of $f$, extending the domain of $f$ from $V$ to the power set $2^V$.

Now we proceed to the degree 2 component in the zeon case.
9.1. **The degree 2 component of \( Z \).** Working in degree 2, we denote indices \( I = (i, j) \), as usual, instead of \( (i_1, i_2) \).

For given \( n, X, X' \), etc., are vectors in \( V^{v^2} \approx \mathbb{Q}^{(n)} \).

As a vector space, \( V \) is isomorphic to \( \mathbb{Q}^n \).

Denote by \( \text{Sym}(V) \) the space of symmetric matrices acting on \( V \).

**Definition** The mapping 
\[
\text{Mat}: \ V^{v^2} \rightarrow \text{Sym}(V)
\]

is the linear embedding taking the vector \( X = (x_{ij}) \) to the symmetric matrix \( \hat{X} \) with components

\[
\hat{X}_{ij} = \begin{cases} 
  x_{ij}, & \text{for } i < j \\
  0, & \text{for } i = j
\end{cases}
\]

and the property \( \hat{X}_{ji} = \hat{X}_{ij} \) fills out the matrix.

We will use the explicit notation \( \text{Mat}(X) \) as needed for clarity.

Equip \( V^{v^2} \) with the inner product

\[
\langle X, X' \rangle = \frac{1}{2} \text{tr} \hat{X} \hat{X}' = X_{\lambda\mu}X'_{\lambda\mu}
\]

9.2. **Basic Identities.** Multiplying \( X \) with \( u \), we observe that

\[
Xu^\dagger = \frac{1}{2} \text{tr} \hat{X}J = uX^\dagger
\]

Observe also that if \( D \) is diagonal, then \( \text{tr} D = \text{tr} DJ \).

**Proposition 9.1. (Basic Relations)** We have

1. \( \text{Mat}(XA^{v^2}) = A^*\hat{X}A - D^+ \), where \( D^+ \) is a diagonal matrix satisfying 
   \( \text{tr} D^+ = \text{tr} A^*\hat{X}A \).

2. \( \text{Mat}(A^{v^2}X^\dagger) = A\hat{X}A^* - D^- \), where \( D^- \) is a diagonal matrix satisfying 
   \( \text{tr} D^- = \text{tr} A\hat{X}A^* \).

3. If \( A \) and \( X \) have nonnegative entries, then \( D^+ \) and \( D^- \) have nonnegative entries. In particular, in that case, vanishing trace for \( D^\pm \) implies vanishing of the corresponding matrix.
Proof. The components of $X A^{v2}$ are

$$\theta_{ij} \theta_{\lambda \mu} (x_{\lambda \mu} A_{\lambda i} A_{\mu j} + x_{\lambda \mu} A_{\mu i} A_{\lambda j})$$

$$= \theta_{ij} (A^* \hat{X} A)_{ij}$$

with the theta symbol for pairs of single indices

$$\theta_{ij} = \begin{cases} 1, & \text{if } i < j \\ 0, & \text{otherwise} \end{cases}$$

Note that the diagonal terms of $\hat{X}$ vanish anyway. And $A^* \hat{X} A$ will be symmetric if $\hat{X}$ is. Since the left-hand side has zero diagonal entries, we can remove the theta symbol and compensate by subtracting off the diagonal, call it $D^+$. Taking traces yields #1. And #2 follows similarly. \qed

Remark. Observe that $D^+$ and $D^-$ may be explicitly given by

$$D^+_{ii} = 2 x_{\lambda i} A_{\lambda i} A_{\mu i}$$

$$D^-_{ii} = 2 x_{\lambda i} A_{i \lambda} A_{i \mu}$$

where for $D^+$ the $A$ elements are taken within a given column, while for $D^-$, the $A$ elements are in a given row.

Connections between zeons and nonnegativity that we develop here give some indication that their natural place is indeed in probability theory and quantum probability theory.

Paralleling Proposition 4.2, we see the role of nonnegativity appearing.

Proposition 9.2. Let $X$ and $A$ be nonnegative. Then

$$\hat{X} = A^* \hat{X} A \Rightarrow X A^{v2} = X$$

$$\hat{X} = A \hat{X} A^* \Rightarrow A^{v2} X^\dagger = X^\dagger$$

Proof. We have

$$D^+ = A^* \hat{X} A - \text{Mat}(X A^{v2})$$

If $\hat{X} = A^* \hat{X} A$, then since $\hat{X}$ has vanishing trace, $\text{tr} A^* \hat{X} A = 0$. So $\text{tr} D^+ = 0$, hence $D^+ = 0$, and $\hat{X} = A^* \hat{X} A = \text{Mat}(X A^{v2})$. The second implication follows similarly. \qed
9.3. **Trace Identities.** We continue with trace identities for zeons. The proofs follow from the above relations much as they do in the tensor case.

**Proposition 9.3.** We have

1. \( XA^\vee u = \frac{1}{2} \text{tr}(\hat{X} A(J - I)A^*) \).

2. \( uA^\vee X = \frac{1}{2} \text{tr}(\hat{X} A^*(J - I)A) \).

3. If \( A \) is stochastic, then \( XA^\vee u = \frac{1}{2} \text{tr}(\hat{X}(J - AA^*)) \).

Using equation (16) directly for \( X \), we have

\[
X(I - A^\vee)u = \frac{1}{2} \text{tr}(\hat{X}(J - AJA^* + AA^*)) \tag{17}
\]

\[
u(I - A^\vee)X = \frac{1}{2} \text{tr}(\hat{X}(J - A^*JA + A^*A)) \tag{18}
\]

For zeons a new feature appears. For stochastic \( A \), equation (17) yields

**Lemma 9.4** ("integration-by-parts for zeons").

\[
X(I - A^\vee)u = \frac{1}{2} \text{tr}(A^*\hat{X} A) \tag{19}
\]

9.4. **Zeon hierarchy.** \( M \) and \( N \) operators. Consider the semigroup generated by the matrices \( C_i^\vee \), corresponding to the action on \( \ell \)-sets of vertices. The map \( w \to \hat{w}^\vee \) is a homomorphism of matrix semigroups. We now have our main working tool in this context.

**Proposition 9.5.** For \( 1 \leq \ell \leq n \), define

\[
A_\ell = \frac{1}{d} C_1^\vee + \cdots + \frac{1}{d} C_d^\vee .
\]

Then the Cesàro limit of the powers \( A_\ell^m \) exists and equals

\[
\Omega_\ell = \langle K^\vee \rangle
\]

the average taken over the kernel \( K \) of the semigroup \( S \) generated by \( \{C_1, \ldots, C_d\} \).

With \( r \) the rank of the kernel, we see that \( \Omega_\ell \) vanishes for \( \ell > r \).

Here we have \( A_\ell \) in general substochastic, i.e., it may have zero rows or rows that do not sum to one. As noted in the Appendix B, the Abel limits exist for the \( A_\ell \). They agree with the Cesàro limits.
9.5. **M and N operators via zeons.** Here we will show how the operators $M$ and $N$ appear via zeons. The main tool are the trace identities. Note that we automatically get $N_0$ while the diagonal of $M$ is cancelled out in the inner product with $\hat{X}$.

**Proposition 9.6.** For any $X$,

$$X\Omega_2 u^\dagger = Xu_2^\dagger = \langle X, u_2 \rangle = \frac{1}{2} \text{tr} \hat{X} \hat{u}_2 = \frac{1}{2} \text{tr} \hat{X} N_0$$

where $N_0 = \langle J - KK^* \rangle$, taken over the kernel.

And

$$u\Omega_2 X^\dagger = \pi_2 X^\dagger = \langle \pi_2, X \rangle = \frac{1}{2} \text{tr} \hat{\pi}_2 \hat{X} = \frac{1}{2} \text{tr} \hat{X} M$$

where $M = \langle K^* JK \rangle$, taken over the kernel.

**Proof.** We have $\Omega_2 = \langle K^{\vee 2} \rangle$. Using Proposition 9.3 for stochastic matrices,

$$X\Omega_2 u^\dagger = \langle XK^{\vee 2} u^\dagger \rangle = \frac{1}{2} \text{tr} (\hat{X} (J - KK^*)) = \frac{1}{2} \text{tr} \hat{X} N_0$$

Similarly,

$$u\Omega_2 X^\dagger = \langle uk^{\vee 2} X^\dagger \rangle = \frac{1}{2} \text{tr} (\hat{X} K^* (J - I) K) = \frac{1}{2} \text{tr} \hat{X} K^* JK$$

where, since $K^* K$ is diagonal, $\text{tr} \hat{X} K^* K = 0$ as the diagonal of $\hat{X}$ vanishes. □

From these relations we see that

$$u_2^\dagger = \Omega_2 u^\dagger \Rightarrow N_0 = \langle J - KK^* \rangle = \text{Mat}(u_2)$$

and

$$\pi_2 = u\Omega_2 \Rightarrow M_0 = \langle K^* JK \rangle - \tau \omega = \text{Mat}(\pi_2)$$

Cf., Proposition 5.3. Here $M$ and $N$ only require applying $\Omega_2$ to $u$.

9.6. **The spaces $M_0$ and $N_0$.** Now we have

$$A_2 = \frac{1}{2} (R^{\vee 2} + B^{\vee 2}) = A^{\vee 2} + \Delta^{\vee 2}.$$  

Define the mappings $F, G: \text{Sym}(\mathcal{V}) \to \text{Sym}(\mathcal{V})$ by

$$F(Y) = \text{AYA}^* + \Delta Y \Delta^* \quad \text{and} \quad G(Y) = \text{A}^* Y A + \Delta^* Y \Delta$$

consistent with

$$F(Y) = \frac{1}{2} (RYR^* + BYB^*) \quad \text{and} \quad G(Y) = \frac{1}{2} (R^* Y R + B^* Y B).$$

First, some observations

**Proposition 9.7.** For any vector $v$, $G(\text{diag}(v)) = \text{diag}(vA)$. 

Proof. Let $Y = G(\text{diag}(v))$. Then $2Y_{ij} = R_{\lambda i} v_{\lambda} R_{\lambda j} + B_{\lambda i} v_{\lambda} B_{\lambda j}$. A term like $R_{li}$ means that $R$ maps $l$ to $i$. Since $R$ is a function, $l$ can only map to both $i$ and $j$ if $i = j$. So $Y$ is diagonal. If $i = j$, then we get, using $R_{ij}^2 = R_{ij}, B_{ij}^2 = B_{ij},$

$$2Y_{ii} = R_{\lambda i} v_{\lambda} + B_{\lambda i} v_{\lambda} = 2v_{\lambda} A_{\lambda i}$$

as required. \qed

For example, taking $v = u$, we see that $G(I) = I$ if and only if $A$ is doubly stochastic.

First, observe

**Proposition 9.8.** $F(J) = J$ and $G(\omega) = \omega$. That is, $J \in \mathcal{N}_+$ and $\omega \in \mathcal{M}_+$. 

**Proof.** We have $F(J) = AJA^* + \Delta J A^*$. With $AJ = J = JA^*$ and $\Delta J = 0$, this reduces to $J$. Second, from the previous Proposition, we have $G(\omega) = G(\text{diag}(\pi)) = \text{diag}(\pi A) = \text{diag}(\pi) = \omega$. \qed

Now recall Propositions 5.1 and 5.2. We see that an element of $\mathcal{M}_+$ differs by a multiple of $\omega$ from an element of $\mathcal{M}_0$. For an element $N \in \mathcal{N}_+$ with entries bounded by 1, with diagonal entries all equal to one, we see that $J - N$ gives a corresponding element in $\mathcal{N}_0$. We have seen the special operators $M, N, M_0$ and $N_0$ as interesting examples.

Note that equation (19) applied to $R$ and $B$ and then averaged yields

$$X(I - A_2)u^\dagger = \frac{1}{2} \text{tr} G(\hat{X})$$

(20)

The next theorem shows the general relation between fixed points of $A_2$ and fixed points of $F$ and $G$. Note that we are not restricting a priori to nonnegative solutions except in case 2 of the theorem. This allows us to associate trace-zero $M$ and $N$ operators with averages over zeon powers of the kernel elements.

**Theorem 9.9.**

1. $F(\hat{X}) = \hat{X}$ if and only if $A_2 X^\dagger = X^\dagger$.

2. For nonnegative solutions $X$, $G(\hat{X}) = \hat{X}$ if and only if $XA_2 = X$.

**Proof.** First, from Proposition 9.1, applying the Basic Relations for $R$ and $B$ and averaging, we get

$$\text{Mat}(XA_2) = G(\hat{X}) - \frac{1}{2} (D_R^+ + D_B^+)$$

(21)

$$\text{Mat}(A_2 X^\dagger) = F(\hat{X}) - \frac{1}{2} (D_R^+ + D_B^+)$$

(22)
where the subscripts on the $D$'s indicate the diagonals corresponding to $R$ and $B$ respectively. Recalling the remark following Proposition 9.1, note that for $D_R^-$, say, we have terms like $R_i\lambda x_{\lambda\mu}R_{i\mu}$. Since $R$ is a function, we must have $\lambda = \mu$, but then $x_{\lambda\mu} = 0$. In other words, $D_R^-$ and $D_B^-$ vanish so that, in fact, $\text{Mat}(A_2X^\dagger) = F(\hat{X})$. And #1 follows immediately.

For #2, if $G(\hat{X}) = \hat{X}$, then the trace of the diagonal terms in equation (21) vanish, so under the assumption of nonnegativity, they must vanish. And replacing $G(\hat{X})$ by $\hat{X}$, we have $XA_2 = X$.

Assume $XA_2 = X$. Taking traces in equation (21), the trace of the left-hand side vanishes and the trace of the diagonal terms equals $\text{tr}G(\hat{X})$. Now, use the integration-by-parts formula for $A_2$, equation (20). The left-hand side vanishes, hence $\text{tr}G(\hat{X}) = 0$, implying the vanishing of the terms $D_R^-$ and $D_B^-$ as well. Then equation (21) reads $\text{Mat}(X) = G(\hat{X})$ as required. $\square$

9.7. Quantum observables via zeons in summary. We have $\Omega_2$ as the Cesàro limit of the powers $A_2^m$. Then with $u$ here denoting the all-ones vector of dimension $\binom{n}{2}$, we have

$$u_2^\dagger = \Omega_2 u_2^\dagger \quad \text{and} \quad \pi_2 = u\Omega_2$$

1. $N_0 = \langle J - KK^* \rangle = \text{Mat}(u_2)$ satisfies $AN_0A^* + \Delta N_0\Delta^* = N_0$. It is a nonnegative solution with $\text{tr}N_0 = 0$. $N = \langle KK^* \rangle$ and $J = u^\dagger u$ are also nonnegative solutions, with trace equal to $n$.

2. $M_0 = M - \tau\omega$. From Proposition 5.3, we have $\langle K^*K \rangle = n\omega$. This allows us to express

$$M_0 = \langle K^*JK \rangle - \tau/n \langle K^*K \rangle .$$

$M_0 = \text{Mat}(\pi_2)$ satisfies $A^*M_0A + \Delta^*M_0\Delta = M_0$. It is a nonnegative solution with $\text{tr}M_0 = 0$. $M = \langle K^*JK \rangle$ and $\omega = \text{diag}(\pi)$ are also nonnegative solutions, with trace equal to $\tau$ and 1 respectively.

3. We have $\tilde{M} = \langle K^*\Omega K \rangle = nr^{-2} \langle \rho^\dagger \rho \rangle$. The diagonal of $\rho^\dagger \rho$ is $\rho$ so the diagonal of $\tilde{M}$ is $r\omega$. Thus, $\tilde{M}_0 = \tilde{M} - r\omega$ has zero diagonal. Hence $\tilde{M}_0$ and $M_0$ are nonnegative, symmetric, with vanishing trace, elements of $\mathcal{M}_0$, corresponding to solutions to $XA_2 = X$. Notice that $\text{tr}\Omega_2 = 1$ would imply that the space of such solutions equals 1 so in that case $\tilde{M}_0$ and $M_0$ are proportional. Thus, $M_0$ could be computed using knowledge of the range classes.
9.8. **Levels in the zeon hierarchy.** At level 1, we have left and right eigenvectors of $A$: $\pi_1 = \pi$ and $u_1^+ = u^+$, respectively. All of the components of $u$ are equal to 1. And $\Omega = \Omega_1 = u^+\pi$.

At every level $\ell$, $1 \leq \ell \leq r$, we have an Abel limit

$$\Omega_\ell = \lim_{s \uparrow 1} (1 - s)(I - sA_\ell)^{-1}$$

When $\pi_\ell$ and $u_\ell$ are unique, we have $\Omega_\ell = u_\ell^+\pi_\ell$.

At level $r$, as at level 1, we have $\Omega_r = u_r^+\pi_r$.

9.8.1. **Interpretation.** Start at level $r$. Then the nonzero entries of $\pi_r$ are in the columns corresponding to the ranges of the kernel, $K$. The nonzero entries of $u_r^+$ are in rows corresponding to cross-sections of the partition classes of $K$. I.e., for each partition class, find all possible cross-sections weighted by one, take the total, and you get a lower bound on the corresponding entries of $u_r^+$. In particular, for a right group, all cross-sections appear with the same weight, which can be scaled to 1.

At each level, if unique, $\pi_\ell$ and $u_\ell$ give you information about the recurrent class and corresponding cross-sections.

A sequence of pairs $(\pi_\ell, u_\ell)$ can be constructed inductively from $(\pi_r, u_r)$ as consecutive marginals. Thus, for $\ell < r$, let

$$\pi_\ell(I) = \sum_{J \supset I} \pi_{\ell+1}(J)$$

Note that in the sum each $J$ differs from $I$ by a single vertex. Starting from level $r$, we inductively move down the hierarchy and determine a fixed point of $A_\ell$, $\pi_\ell$, for each level $\ell$.

For $u$’s, proceed as follows:

$$u_\ell(I) = \sum_i p_i u_{\ell+1}(I \cup \{i\})$$

where the weights $p_i$ are the components of the invariant measure $\pi$.

With this construction, one recovers at level one, a multiple of $\pi_1 = \pi$ and a constant vector $u_1$, a multiple of $u$.

Note that the rank $r$ is the maximum level $\ell$ such that $\Omega_\ell \neq 0$. 
For our closing observation, we remark that by introducing an absorbing state at each level, $R_{\ell}$, $B_{\ell}$, and $A_{\ell}$, can be extended to stochastic matrices as at level one.

**Remark.** For further information as to the structure of the zeon hierarchy see [1].

10. **Conclusion**

At this point the interpretation of the eigenvalues and eigenvectors of the operators $M$ and $N$ is unavailable. It would be quite interesting to discover their meaning. For further work, it would be nice to have similar constructions for at least some classes of infinite semigroups, e.g. compact semigroups. Another avenue to explore would be to develop such a theory for semigroups of operators on Hilbert space.

**Appendix A. Semigroups and kernels**

A finite semigroup, in general, is a finite set, $\mathcal{S}$, which is closed under a binary associative operation. For any subset $T$ of $\mathcal{S}$, we will write $\mathcal{E}(T)$ to refer to the set of idempotents in $T$.

**Theorem A.1.** [5, Th. 1.1] Let $\mathcal{S}$ be a finite semigroup. Then $\mathcal{S}$ contains a minimal ideal $K$ called the kernel which is a disjoint union of isomorphic groups. In fact, $K$ is isomorphic to $\mathcal{X} \times \mathcal{G} \times \mathcal{Y}$ where, given $e \in \mathcal{E}(\mathcal{S})$, then $eK e$ is a group and

$$
\mathcal{X} = \mathcal{E}(Ke), \quad \mathcal{G} = eKe, \quad \mathcal{Y} = \mathcal{E}(eK)
$$

and if $(x_1, g_1, y_1), (x_2, g_2, y_2) \in \mathcal{X} \times \mathcal{G} \times \mathcal{Y}$ then the multiplication rule has the form

$$
(x_1, g_1, y_1)(x_2, g_2, y_2) = (x_1, g_1\phi(y_1, x_2)g_2, y_2)
$$

where $\phi: \mathcal{Y} \times \mathcal{X} \rightarrow \mathcal{G}$ is the sandwich function.

The product structure $\mathcal{X} \times \mathcal{G} \times \mathcal{Y}$ is called a Rees product and any semigroup that has a Rees product is completely simple. The kernel of a finite semigroup is always completely simple.

**A.1. Kernel of a matrix semigroup.** An extremely useful characterization of the kernel for a semigroup of matrices is known.

**Theorem A.2.** [5, Props. 1.8, 1.9] Let $\mathcal{S}$ be a finite semigroup of matrices. Then the kernel, $K$, of $\mathcal{S}$ is the set of matrices with minimal rank.
Suppose $S = S((C_1, C_2, \ldots, C_d))$ is a semigroup generated by binary stochastic matrices $C_i$. Then $S$ is a finite semigroup with kernel $K = \mathcal{X} \times \mathcal{G} \times \mathcal{Y}$, the Rees product structure of Theorem A.1. Let $k, k'$ be elements of $K$. Then with respect to the Rees product structure $k = (k_1, k_2, k_3)$ and $k' = (k'_1, k'_2, k'_3)$. We form the ideals $Kk, k'K$, and their intersection $k'Kk$:

1. $Kk = \mathcal{X} \times \mathcal{G} \times \{k_3\}$ is a minimal left ideal in $K$ whose elements all have the same range, or nonzero columns as $k$. We call this type of semigroup a left group.

2. $k'K = \{k'_1\} \times \mathcal{G} \times \mathcal{Y}$ is a minimal right ideal in $K$ whose elements all have the same partition of the vertices as $k'$. A block $B_j$ in the partition can be assigned to each nonzero column of $k'$ by

$$B_j = \{i : k'_{i,j} = 1\}$$

A semigroup with the structure $k'K$ is a right group.

3. $k'Kk$, the intersection of $k'K$ and $Kk$, is a maximal group in $K$ (an $H$-class in the language of semigroups). It is best thought of as the set of functions specified by the partition of $k'$ and the range of $k$, acting as a group of permutations on the range of $k$. The idempotent of $k'Kk$ is the function which is the identity when restricted to the range of $k$.

**Appendix B. Abel limits**

We are working with stochastic and substochastic matrices. We give the details here on the existence of Abel limits. Let $P$ be a substochastic matrix. The entries $p_{ij}$ satisfy $0 \leq p_{ij} \leq 1$ with $\sum_j p_{ij} \leq 1$. Inductively, $P^n$ is substochastic for every integer $n > 0$, with $P^n$ stochastic if $P$ is.

**Proposition B.1.** The Abel limit

$$\Omega = \lim_{s \searrow 1} (1 - s)(I - sP)^{-1}$$

exists and satisfies

$$\Omega = \Omega^2 = P\Omega = \Omega P$$

**Proof.** We take $0 < s < 1$ throughout.

For each $i, j$, the matrix elements $\langle e_i, P^n e_j \rangle$ are uniformly bounded by 1. Denote

$$Q(s) = (1 - s)(I - sP)^{-1}$$
So we have
\[
\langle e_i, Q(s)e_j \rangle = (1 - s) \sum_{n=0}^{\infty} s^n \langle e_i, P^n e_j \rangle \leq (1 - s) \sum_{n=0}^{\infty} s^n = 1
\]
so the matrix elements of \(Q(s)\) are bounded by 1 uniformly in \(s\) as well.

Now take any sequence \(\{s_j\}, s_j \uparrow 1\). Take a further subsequence \(\{s_{j11}\}\) along which the 11 matrix elements converge. Continuing with a diagonalization procedure going successively through the matrix elements, we have a subsequence \(\{s'\}\) along which all of the matrix elements converge, i.e., along which \(Q(s')\) converges. Call the limit \(\Omega\).

Writing \((I - sP)^{-1} - I = sP(I - sP)^{-1}\), multiplying by \(1 - s\) and letting \(s \uparrow 1\) along \(s'\) yields
\[
\Omega = P\Omega = \Omega P
\]
writing the \(P\) on the other side for this last equality. Now, \(s\Omega = sP\Omega\) implies \((1 - s)\Omega = (I - sP)\Omega\). Similarly, \((1 - s)\Omega = \Omega(I - sP)\), so
\[
Q(s)\Omega = \Omega Q(s) = \Omega
\]
(23)
Taking limits along \(s'\) yields \(\Omega^2 = \Omega\).

For the limit to exist, we check that \(\Omega\) is the only limit point of \(Q(s)\). From (23), if \(\Omega_1\) is any limit point of \(Q(s)\), \(\Omega_1\Omega = \Omega\Omega_1 = \Omega\). Interchanging roles of \(\Omega_1\) and \(\Omega\) yields \(\Omega_1\Omega = \Omega\Omega_1 = \Omega_1\) as well, i.e., \(\Omega = \Omega_1\).

If \(P\) is stochastic, it has \(u^+\) as a nontrivial fixed point. In general we have

**Corollary B.2.** Let \(\Omega = \lim_{s \uparrow 1} (1 - s)(I - sP)^{-1}\) be the Abel limit of the powers of \(P\).

\(P\) has a nontrivial fixed point if and only if \(\Omega \neq 0\).

**Proof.** If \(\Omega \neq 0\), then \(P\Omega = \Omega\) shows that any nonzero column of \(\Omega\) is a nontrivial fixed point. On the other hand, if \(Pv = v \neq 0\), then, as in the above proof, \(Q(s)v = v\) and hence \(\Omega v = v\) shows that \(\Omega \neq 0\).

Note that since \(\Omega\) is a projection, its rank is the dimension of the space of fixed points.

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