1. Introduction

The instanton Floer homology for integral homology spheres was defined by Andreas Floer [4]. It has been making a comeback lately, in particular, in the work of Lim [10] and Kronheimer-Mrowka [8] on the instanton Floer homology of knots. The interest in this particular version of the theory, which first appeared in Floer [5], is explained by its conjectured relationship with the Seiberg–Witten and Heegaard Floer homologies. This is evidenced, for instance, by the fact that the Alexander polynomial of the knot can be expressed in terms of its instanton Floer homology; see [8] and [10].

In this paper we define instanton Floer homology for links of two components in an integral homology sphere by a slight variant of the construction of Floer [5] (which was later developed by Braam and Donaldson [1]). We show that the Euler characteristic of this homology theory is twice the linking number between the components of the link. A similar result for two-component links in the 3-sphere was announced by Braam and Donaldson [1, Part II, Example 3.13] with an outline of the proof that relied on the Floer exact triangle. Our approach is more direct and its main advantage is that it yields the result for links in arbitrary homology spheres.

We provide several examples of calculations of Floer homology groups of links. We also observe that our Floer homology can be viewed as a reduced version of Kronheimer and Mrowka’s instanton Floer homology for links.

Our interest in the Floer homology for two-component links was generated by the papers [8] and [10] and by our own study of the linking number in [6].
We are thankful to the referee for useful remarks which helped us improve the paper.

2. The instanton Floer homology $I_\ast(\Sigma, L)$

Let $L = \ell_1 \cup \ell_2$ be an oriented link of two components in an integral homology sphere $\Sigma$. Consider the link exterior $X = \Sigma - \text{int} \ N(L)$ and furl it up by gluing the boundary components of $X$ together via an orientation reversing diffeomorphism $\varphi : T^2 \to T^2$. The resulting closed orientable manifold will be denoted $X_\varphi$.

**Lemma 2.1.** The gluing map $\varphi$ can be chosen so that $X_\varphi$ has the integral homology of $S^1 \times S^2$.

**Proof.** An orientation reversing diffeomorphism $\varphi$ is determined up to isotopy by the homomorphism $\varphi_* : \mathbb{Z}^2 \to \mathbb{Z}^2$ it induces on the fundamental groups of the two boundary components of $X$. Let $\mu_1, \lambda_1$ be the canonical oriented meridian–longitude pair on one boundary component of $X$, and $\mu_2, \lambda_2$ on the other. With respect to this choice of bases, $\varphi_*$ is given by an integral matrix

$$\varphi_* = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \quad \text{with} \quad ad - bc = -1.$$  

Now $H_1(X_\varphi)$ has generators $t, \mu_1, \mu_2$ and relations

$$\mu_2 = a\mu_1 + b\lambda_1, \quad \lambda_2 = c\mu_1 + d\lambda_1,$$

where $\lambda_1 = n\mu_2$ and $\lambda_2 = n\mu_1$ with $n = \text{lk} (\ell_1, \ell_2)$. In particular, $X_\varphi$ has the integral homology of $S^1 \times S^2$ exactly when

$$\det \begin{pmatrix} a & bn - 1 \\ c - n & dn \end{pmatrix} = bn^2 - 2n + c = \pm 1. \quad (1)$$

It is clear that one can always find $\varphi$ such that this is the case. \qed
Remark. This construction appeared in Brakes [2] and Woodard [13] under the name of “sewing-up link exteriors”, and was generalized by Hoste in [7]. We thank Daniel Ruberman for drawing our attention to these papers.

Given a link $L$ of two components in an integral homology sphere $\Sigma$, choose the gluing map $\varphi$ so that $H_*(X_\varphi) = H_*(S^1 \times S^2)$ and let

$$\mathcal{I}_*(\Sigma, L) = \mathcal{I}_*(X_\varphi).$$

Here, $\mathcal{I}_*(X_\varphi)$ is the instanton Floer homology defined in Floer [5], see also Braam–Donaldson [1], as follows. Let $E$ be a $U(2)$–bundle over $X_\varphi$ such that $c_1(E)$ is an odd element in $H^2(X_\varphi) = \mathbb{Z}$. Consider the space of $PU(2)$–connections in the adjoint bundle $\text{ad}(E)$ modulo the action of the gauge group consisting of automorphisms of $E$ with determinant one. The Floer homology arising from the Chern–Simons functional on this space is $\mathcal{I}_*(X_\varphi)$. It has a relative grading by $\mathbb{Z}/8$.

We will refer to $\mathcal{I}_*(\Sigma, L)$ as the instanton Floer homology of the two-component link $L \subset \Sigma$.

**Theorem 2.2.** Let $L = \ell_1 \cup \ell_2$ be an oriented two-component link in an integral homology sphere $\Sigma$. Then $\mathcal{I}_*(\Sigma, L)$ is independent of the choice of $\varphi$, and its Euler characteristic is given by

$$\chi(\mathcal{I}_*(\Sigma, L)) = \pm 2 \text{lk}(\ell_1 \cup \ell_2).$$

**Proof.** The first statement follows from the excision principle of Floer [5], see also [1] Part II, Proposition 3.5]. Since $X_\varphi$ is a homology $S^1 \times S^2$, we know that $\chi(\mathcal{I}_*(X_\varphi)) = \pm \Delta''(1)$, where $\Delta(t)$ is the Alexander polynomial of $X_\varphi$ normalized so that $\Delta(1) = 1$ and $\Delta(t) = \Delta(t^{-1})$ (a direct proof of this result can be found in [11]). To calculate $\Delta(t)$, let $\widetilde{X}_\varphi$ be the infinite cyclic cover of $X_\varphi$. Then $H_1(\widetilde{X}_\varphi)$, as a $\mathbb{Z}[t, t^{-1}]$–module, has generators $\mu_1$, $\mu_2$ and relations

$$t\mu_2 = a\mu_1 + b\lambda_1,$$

$$t\lambda_2 = c\mu_1 + d\lambda_1,$$

$$\Delta(t) = \frac{1}{(1-t)}.$$
where \( \lambda_1 = n\mu_2 \) and \( \lambda_2 = n\mu_1 \). Therefore,

\[
\Delta(t) = \det \begin{pmatrix} a & bm - t \\ c - nt & dn \end{pmatrix} = -nt^2 + (bn^2 + c)t - n,
\]

up to a unit in \( \mathbb{Z}[t, t^{-1}] \). After taking (1) into account, we obtain

\[
\Delta(t) = \pm(-nt + (2n \pm 1) - nt^{-1})
\]

so that

\[
\Delta''(1) = \pm 2n = \pm 2\text{lk}(\ell_1, \ell_2).
\]

\[\square\]

**Remark.** The requirement that \( X_\varphi \) have homology of \( S^1 \times S^2 \) was only needed to make the discussion more elementary, and in general can be omitted. Braam and Donaldson [1, Part II, Proposition 3.5] show that any choice of \( \varphi \) gives an admissible object \( X_\varphi \) in Floer’s category, and that the properly defined Floer homology of \( X_\varphi \) is independent of \( \varphi \).

### 3. Non-triviality

For two-component links \( L = \ell_1 \cup \ell_2 \) with linking number \( \text{lk}(\ell_1, \ell_2) \neq 0 \), the instanton Floer homology \( I_*(\Sigma, L) \) must be non-trivial by Theorem 2.2.

In this section, we will give a sufficient condition for \( I_*(\Sigma, L) \) to be non-trivial even when \( \text{lk}(\ell_1, \ell_2) = 0 \).

Let \( Y \) be a closed, irreducible, orientable 3-manifold, and let \( 0 \neq v \in H^2(Y; \mathbb{Z}/2) \). In the proof of [9, Theorem 3], Kronheimer and Mrowka show that the instanton Floer homology of \( Y \) constructed from \( PU(2) \) connections in the bundle \( \text{ad}(E) \) with \( w_2(\text{ad}(E)) = v \) must be non-trivial. This implies that \( I_*(\Sigma, L) \) is non-trivial whenever \( X_\varphi \) is irreducible. A standard 3–manifold topology argument can be used to show that the irreducibility of \( X \) implies that of \( X_\varphi \), which leads to the following theorem.

**Theorem 3.1.** Let \( L = \ell_1 \cup \ell_2 \) be an oriented two-component link in an integral homology sphere \( \Sigma \) such that the link exterior is irreducible. Then \( I_*(\Sigma, L) \) is non-trivial.
Since link exteriors are irreducible for non-split links in the 3-sphere, we conclude the following:

**Corollary 3.2.** For all non-split, two-component links in $S^3$, the Floer homology $I_* (S^3, L)$ is non-trivial.

On the other hand, for any split link $L \subset \Sigma$, we have $I_* (\Sigma, L) = 0$ since $w_2$ must evaluate non-trivially on $S^2 \subset X_\varphi$, but there are no non-trivial flat $SO(3)$ connections on $S^2$ due to the fact that $\pi_1(S^2) = 1$.

4. **A surgery description**

Let $L \subset \Sigma$ be an oriented two-component link in a homology 3-sphere. In what follows we will give a surgery description of a manifold $Y$ such that $I_* (\Sigma, L) = I_* (Y)$. This will allow us to calculate $I_* (\Sigma, L)$ for several examples.

Attach a band from one component of $L$ to the other matching orientations, and call the resulting knot $k$. Introduce a small circle $\gamma$ going once around the band with linking number zero. Frame $\gamma$ by zero and $k$ by an integer $m = \pm 1$ such that $m$–surgery along $k$ results in a homology sphere $\Sigma + m \cdot k$. Any manifold obtained from $\Sigma$ by performing surgery on the framed link $k \cup \gamma$ will be called $Y$. According to [1, Part II, Proposition 3.5], see also [7], the manifold $Y$ is diffeomorphic to $X_\varphi'$ for a choice of gluing map $\varphi'$. Since $Y$ has the integral homology of $S^1 \times S^2$, the map $\varphi'$ must be as in Lemma 2.1. The independence of the choice of $\varphi$ then implies that

$$I_* (Y) = I_* (X_{\varphi'}) = I_* (X_\varphi) = I_* (\Sigma, L).$$

To calculate $I_* (Y)$, we will use the Floer exact triangle of [5], see also [1]. Let $\gamma$ be a knot in an integral homology sphere $M$. Denote by $M - \gamma$ the integral homology sphere obtained by $(-1)$–surgery along $\gamma$, and by $Y = M + 0 \cdot \gamma$ the homology $S^1 \times S^2$ obtained by 0–surgery along $\gamma$. The instanton Floer homology groups of the three manifolds are then related by the Floer exact triangle of total degree $-1$:
Example. Let $L_n \subset S^3$ be the Hopf link with linking number $n$ as in Figure 1 (where $n = 2$). Then $\mathcal{I}_*(S^3, L) = I_*(Y)$ for the manifold $Y$ as in Figure 2.

Apply the Floer exact triangle with $M = S^3$ viewed as the manifold obtained by $(-1)$–surgery on the trivial knot framed by $-1$ in Figure 2. Let $\gamma$ be the zero framed circle in Figure 2. Since $I_*(M) = 0$, we have an isomorphism $I_*(Y) = I_*(M - \gamma)$, where the manifold $M - \gamma$ has surgery description as shown in Figure 3.

After the blow down, we see that $M - \gamma$ is the result of $(-1)$–surgery on a twist knot, hence is diffeomorphic to the Brieskorn homology sphere $\Sigma(2, 3, 6n + 1)$ with reversed orientation; see for instance [12, Figure 3.19]. Therefore (cf. Fintushel–Stern [3])

\[
\mathcal{I}_*(S^3, L_n) = (\mathbb{Z}^{n/2}, 0, \mathbb{Z}^{n/2}, 0, \mathbb{Z}^{n/2}, 0) \text{ if } n \text{ is even, and } \\
\mathcal{I}_*(S^3, L_n) = (\mathbb{Z}^{(n-1)/2}, 0, \mathbb{Z}^{(n+1)/2}, 0, \mathbb{Z}^{(n-1)/2}, 0, \mathbb{Z}^{(n+1)/2}, 0) \text{ if } n \text{ is odd.}
\]
5. Relation with the instanton Floer homology of knots

In this section, we will show that the Floer homology $I_\ast(\Sigma, L)$ can be viewed as a reduced version of the instanton knot Floer homology $KHI$ of Kronheimer and Mrowka.

The Floer homology $KHI$ is defined as follows; see for instance [3]. Let $L = \ell_1 \cup \ldots \cup \ell_r$ be an oriented link in a homology sphere $\Sigma$, and $F_r$ a genus-one surface with $r$ boundary components $\delta_1, \ldots, \delta_r$. Form a closed manifold $Z$ by attaching manifolds $F_r \times S^1$ and $\Sigma - \text{int} N(L)$ to each other along their boundaries in such a fashion that each $\delta_i$ matches the canonical longitude of the component $\ell_i$ of $L$, and the $S^1$ factor matches the meridians.

Let $E$ be a $U(2)$–bundle over $Z$ with $w = c_1(E)$ dual to a curve $\nu \subset F_r$ representing a generator of the first homology of the closed genus-one surface obtained by closing up the $r$ boundary components of $F_r$ by discs. The instanton Floer homology with complex coefficients arising from the space of $PU(2)$–connections in $\text{ad}(E)$ modulo the group of automorphisms of $E$ with determinant one will be denoted $I_\ast(Z)_w$. It has a relative grading by $\mathbb{Z}/8$.

If $z \in Z$ is a point, the $\mu$–map gives us a degree-four operator $\mu(z) : I_\ast(Z)_w \to I_\ast(Z)_w$ with eigenvalues $\pm 2$. The space $I_\ast(Z)_w$ is then a sum of two subspaces of equal dimensions,

$$I_\ast(Z)_w = I_\ast(Z)_{w,2} \oplus I_\ast(Z)_{w,-2}.$$
namely, the eigenspaces of $\mu(z)$ corresponding to the eigenvalues 2 and $-2$. By definition,

$$KHI(\Sigma, L) = I_*(Z)_w.$$  

Similarly, given a two-component link $L \subset \Sigma$, the Floer homology $I_*(\Sigma, L)$ splits into a sum of the eigenspaces of the degree-four operator $I_*(\Sigma, L) \to I_*(\Sigma, L)$ corresponding to the eigenvalues 2 and $-2$. These eigenspaces have equal dimension. The $(+2)$–eigenspace will be denoted by $I'_*(\Sigma, L)$.

**Theorem 5.1.** Let $L = \ell_1 \cup \ell_2$ be an oriented two-component link in a homology sphere $\Sigma$. Then

$$\chi(I'_*(\Sigma, L)) = \pm \text{lk}(\ell_1, \ell_2)$$  

and

$$KHI(\Sigma, L) = I'_*(\Sigma, L) \otimes H_*(T^2, \mathbb{C}).$$  

**Proof.** The first statement is a direct consequence of Theorem 2.2. To prove the second statement, we will use a different description of the manifold $Z$. Remember that $Z$ is obtained by attaching $F_2 \times S^1$ to the exterior of the link $L$. We claim that $Z$ can be obtained by first attaching $F_1 \times S^1$ to the knot $\ell_1 \# \ell_2$, which is a band-sum of the knots $\ell_1$ and $\ell_2$, and then performing 0-surgery on a small loop $\gamma$ going once around the band with linking number zero.

To prove this claim, we will resort to a 4-dimensional picture. Choose a compact oriented 4-manifold $W$ with boundary $\Sigma$. Then the manifold $Z$ will be the boundary of the 4-manifold obtained from $W$ by attaching $F_2 \times D^2$ to $\partial W$ along the two solid tori $\partial F_2 \times D^2$. This operation can be done in two steps. First, we choose a properly embedded arc $J \subset F_2$ connecting the two boundary components of $F_2$. Let $N(J)$ with a tubular neighborhood of $J$ in $F_2$, and attach the 1-handle $N(J) \times D^2$ to $\partial W$. What is left to attach is $F_1 \times D^2$. It is attached along $\partial F_1 \times D^2$ to the band-sum $\ell_1 \# \ell_2$, the band running geometrically once over the 1-handle. Since we are only interested
in the boundary of the resulting 4-manifold, we trade the 1-handle for a 2-handle to complete the proof.

With the above description of the manifold $Z$ in place, the second statement of the theorem follows from Proposition 3.11 (2) of [1, Part II]. □

Let $k_+$ and $k_-$ be knots in $\Sigma$ related by a single crossing change, and let $k_0$ be the corresponding two-component link, see Figure 4.

![Figure 4](image_url)

The instanton Floer homology of these can be included into the following Floer exact triangle, see [8, Theorem 3],

$$KHI(\Sigma, k_0) \xrightarrow{\alpha} KHI(\Sigma, k_+) \xrightarrow{\beta} KHI(\Sigma, k_-)$$

Observe that $\chi(KHI_*(k)) = 1$ for all knots $k \subset \Sigma$, see Kronheimer–Mrowka [8, Theorem 1.1] and also Braam–Donaldson [1, Part II, Example 3.13]. This is consistent with the above exact triangle because

$$\chi(KHI(\Sigma, k_0)) = \chi(I^*_\Sigma(\Sigma, k_0) \otimes H_*(T^2)) = \chi(I^*_\Sigma(\Sigma, k_0)) \cdot \chi(T^2) = 0$$

by Theorem 5.1.

**References**

[1] P.J. Braam, S.K. Donaldson, *Floer’s work on instanton homology, knots, and surgery*. In: The Floer memorial volume, 195 – 256, Progr. Math. **133**, Birkhäuser, 1995.
[2] W. Brakes, Sewing-up link exteriors. Low-dimensional topology (Bangor, 1979), 27–37, London Math. Soc. Lecture Note Ser., 48, Cambridge Univ. Press, 1982

[3] R. Fintushel, R. Stern, Instanton homology of Seifert fibred homology three spheres, Proc. London Math. Soc. 61 (1990), 109 – 137

[4] A. Floer, An instanton-invariant for 3-manifolds, Comm. Math. Phys. 118 (1988), 215 – 240

[5] A. Floer, Instanton homology, surgery, and knots. In: Geometry of low-dimensional manifolds, 1 (Durham, 1989), 97–114, London Math. Soc. Lecture Note Ser., 150, Cambridge Univ. Press, 1990

[6] E. Harper, N. Saveliev, A Casson-Lin type invariant for links, Pacific J. Math. 248 (2010), 139–154

[7] J. Hoste, Sewn-up r-link exteriors, Pacific J. Math. 112 (1984), 347–382

[8] P. Kronheimer, T. Mrowka, Instanton Floer homology and the Alexander polynomial, Algebr. Geom. Topol. 10 (2010), 1715–1738

[9] P. Kronheimer, T. Mrowka, Witten’s conjecture and Property P. Geom. Top. 8 (2004), 295 – 310

[10] Y. Lim, Instanton homology and the Alexander polynomial, Proc. Amer. Math. Soc. 138 (2010), 3759–3768

[11] K. Masataka, Casson’s knot invariant and gauge theory, Topology Appl. 112 (2001), 111–135

[12] N. Saveliev, Lectures on the topology of 3-manifolds. An introduction to the Casson invariant. Walter de Gruyter, 1999

[13] M. Woodard, The Rochlin invariant of surgered, sewn link exteriors. Proc. Amer. Math. Soc. 112 (1991), 211–221

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