Curve network interpolation
by B-spline surfaces

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Abstract

In this paper we investigate the problem of interpolating a B-spline curve network, in order to create a surface satisfying such a constraint and belonging to the space of bivariate $C^1$ quadratic splines on criss-cross triangulations. We prove the existence and uniqueness of the surface, providing a constructive algorithm for its generation. We also present numerical and graphical results and comparisons with other methods.

Keywords: Interpolation; B-spline surface; Curve network

Subject classification AMS (MOS): 65D05, 65D07, 65D17

1 Introduction

Curve network-based design is an important problem in CAGD (Computer Aided Geometric Design). A curve network is often used to describe complicated 3D free-form surfaces. Indeed it is easier and more intuitive to create such a curve network than to manipulate the surface control points directly. For this reason, a curve network has been used in applications such as ship hull, car body design and character modeling in computer graphics. After designing a 3D shape using a curve network, that embodies the satisfaction of the design constraints, smooth surfaces are generated by interpolating it, without modification of the curves.

In the literature this interpolation problem is faced up by considering different approaches, based on either subdivision surfaces or the generation...
of patches in each region of the given curve network. In order to construct the final surface, in the latter case it is necessary to impose the required smoothness patch by patch and in both approaches its ready global parametric expression is not available.

Thus, the aim of this paper is to get such an expression for the final interpolating surface by B-spline techniques.

Tensor product B-spline surfaces have become a standard practice in CAGD, thanks to their many good properties as the convex hull and the affine transformation invariance ones. However, they can create unwanted oscillations and some inflection points on the surface, due to their high coordinate degree. Since B-spline surfaces on triangulations have the same good properties and the lower total degree is able to avoid the above drawbacks, they seem to be very useful in CAGD [15, 16, 17].

Therefore, in this paper we investigate the problem of interpolating a B-spline curve network, in order to create a surface satisfying such a constraint and belonging to the space $S^1_2(T_{mn})$ of all quadratic $C^1$ splines on a criss-cross triangulation $T_{mn}$. Indeed, this space has been widely studied, investigating dimension, local basis, approximation power, etc. and it has been used in many applications such as integration, function and data reconstruction, also in case of unequal smoothness and boundary conditions (see [4, 5, 6, 7, 9, 12, 13, 14, 15] and references therein).

The structure of this paper is the following. In Section 2 we define the curve network with its compatibility conditions. In Section 3 we give necessary and sufficient conditions for the existence of the B-spline interpolating surface in $S^1_2(T_{mn})$ and we prove its uniqueness. Moreover we propose a constructive algorithm for its generation and, for the sake of comparison, some remarks on box spline and tensor product B-spline surfaces. Finally, in Section 4 we present some numerical and graphical results.

## 2 The B-spline curve network

Let $\{P_j^{(r)}\}_{j=0}^{n+2}, r = 0, \ldots, m+1$ and $\{Q_i^{(s)}\}_{i=0}^{m+2}, s = 0, \ldots, n+1$ be the control points of the two sets of quadratic $C^1$ B-spline curves

$$
\begin{align*}
\phi_r(v) &= \sum_{j=0}^{n+2} P_j^{(r)} B_j(v) & r = 0, \ldots, m+1, \\
\psi_s(u) &= \sum_{i=0}^{m+2} Q_i^{(s)} B_i(u) & s = 0, \ldots, n+1.
\end{align*}
$$

We assume that the curves (1) satisfy the following compatibility conditions:
\[ C1. \text{as independent sets they are compatible in the B-spline sense, that is, all the } \psi_s(u) \text{ are defined on a common parametric knot vector} \]

\[ U = \{ u_{-2} \equiv u_{-1} \equiv u_0 = a; \ u_i = u_0 + ih, \ i = 1, \ldots, m, \ h = \frac{b - a}{m + 1}; \ u_{m+1} \equiv u_{m+2} \equiv u_{m+3} = b \} \]  
\[ (2) \]

and all the \( \phi_r(v) \) are defined on a common parametric knot vector

\[ V = \{ v_{-2} \equiv v_{-1} \equiv v_0 = c; \ v_j = v_0 + jk, \ j = 1, \ldots, n, \ k = \frac{d - c}{n + 1}; \ v_{n+1} \equiv v_{n+2} \equiv v_{n+3} = d \}. \]
\[ (3) \]

In our notation the B-splines \( B_i(u) \) and \( B_j(v) \) have supports \([u_{i-2}, u_{i+1}]\) and \([v_{j-2}, v_{j+1}]\), respectively [12] [13]. Moreover we assume \( (\sigma_i, v_s) \), with

\[ \sigma_i = \frac{u_{i-1} + u_i}{2}, \ i = 0, \ldots, m + 2 \]  
\[ (4) \]

and \( (u_r, \tau_j) \), with

\[ \tau_j = \frac{v_{j-1} + v_j}{2}, \ j = 0, \ldots, n + 2 \]  
\[ (5) \]

as the pre-image of \( Q^s_i \) for all \( s \) and \( P^r_j \) for all \( r \), respectively (Fig. [1]). We recall that \( \sigma_i \) and \( \tau_j \) are known as Greville abscissae;

\[ C2. \phi_r(v_s) = \psi_s(u_r), \quad r = 0, \ldots, m + 1, \quad s = 0, \ldots, n + 1. \]

We remark that a curve network satisfying the condition C2. always exists.

Indeed, from the property of B-spline local support, we notice that for \( r = 1, \ldots, m \) and \( s = 1, \ldots, n \), the condition C2. is equivalent to

\[ P^{(r)}_s B_s(v_s) + P^{(r)}_{s+1} B_{s+1}(v_s) = Q^{(s)}_r B_r(u_r) + Q^{(s)}_{r+1} B_{r+1}(u_r). \]  
\[ (6) \]

Since in \( U \) and \( V \) the inner knots are equally spaced, we have

\[ B_s(v_s) = B_{s+1}(v_s) = B_r(u_r) = B_{r+1}(u_r) = \frac{1}{2}. \]

Therefore, the condition (6) can be written as follows

\[ P^{(r)}_s + P^{(r)}_{s+1} = Q^{(s)}_r + Q^{(s)}_{r+1}. \]  
\[ (7) \]
By a similar argument, we can prove that, for $s = 0, n + 1$ and $r = 1, \ldots, m$, the condition C2. is equivalent to

$$Q^{(0)}_r + Q^{(0)}_{r+1} = 2P^{(r)}_0, \quad Q^{(n+1)}_r + Q^{(n+1)}_{r+1} = 2P^{(r)}_{n+2}$$

and, for $r = 0, m + 1$ and $s = 1, \ldots, n$, it is equivalent to

$$P^{(0)}_s + P^{(0)}_{s+1} = 2Q^{(s)}_0, \quad P^{(m+1)}_s + P^{(m+1)}_{s+1} = 2Q^{(s)}_{m+2}.$$  

Moreover

$$Q^{(0)}_0 = P^{(0)}_0, \quad Q^{(0)}_{m+2} = P^{(m+1)}_0, \quad Q^{(n+1)}_0 = P^{(n+1)}_{n+2}, \quad Q^{(n+1)}_{m+2} = P^{(m+1)}_{n+2}.$$  

Then, since from (11), the number of curve network control points is $(m + 2)(n + 3) + (m + 3)(n + 2)$ and, from (7)-(10), the number of constraints is $(m+2)(n+2)$, the statement is always verified because several control points can be arbitrarily chosen to define the curve network.

### 3 Construction of the B-spline surface interpolating the curve network

Let $\mathcal{T}_{mn}$ be the criss-cross triangulation of the parameter domain $\Omega = [a, b] \times [c, d]$, based on the knots $\{U \times V\}$, given in (2) and (3) (Fig. 2).

Let $\mathcal{B}_{mn} = \{B_{ij}(u,v), (i,j) \in K_{mn}\}$, with $K_{mn} = \{(i,j) : 0 \leq i \leq m+2, 0 \leq j \leq n+2\}$ be the collection of $(m+3)(n+3)$ bivariate quadratic B-splines $[4, 11, 12, 13]$, spanning the space $S^1_{2}(\mathcal{T}_{mn})$, i.e. the space of all quadratic $C^1$ splines whose restriction to each triangle of $\mathcal{T}_{mn}$ is a bivariate
polynomial of total degree two. It is well known that \( \dim S_2^1(\mathcal{T}_{mn}) = (m + 3)(n + 3) - 1 \).

In the above space, we want to define \( C^1 \) B-spline parametric surfaces of the form

\[
S(u, v) = \sum_{i=0}^{m+2} \sum_{j=0}^{n+2} C_{ij} B_{ij}(u, v),
\]  

(11)

interpolating the curve network (1). In order to do it, we have to get the control points \( C_{ij}, i = 0, \ldots, m + 2; j = 0, \ldots, n + 2 \), whose pre-images are the points \( (\sigma_i, \tau_j) \), with \( \sigma_i \) and \( \tau_j \) defined in (4) and (5), respectively (Fig. 2).

Thus, we interpret the curves (11) as isoparametric curves of \( S \), i.e.

\[
S(u_r, v) = \phi_r(v), \ r = 0, \ldots, m + 1,
\]  

(12)

\[
S(u, v_s) = \psi_s(u), \ s = 0, \ldots, n + 1
\]  

(13)

and in Theorem 3.1 we deduce the constraints that have to be satisfied by the surface control points \( \{C_{ij}\} \).

**Theorem 3.1.** The curves (11) are isoparametric curves of \( S \) if and only if the control points \( \{C_{ij}\} \) in (11) satisfy the following conditions:

\[
C_{r,j} + C_{r+1,j} = 2P_j^{(r)}, \ r = 1, \ldots, m; j = 1, \ldots, n + 1,
\]  

(14)

\[
C_{i,s} + C_{i,s+1} = 2Q_i^{(s)}, \ s = 1, \ldots, n; i = 1, \ldots, m + 1,
\]  

(15)
Proof. First we show the necessary condition.

From the locality of the bivariate B-splines of \( B_{mn} \), \( B_{ij}(u, v) \equiv 0 \) for \( i < r \), \( i > r + 1 \) and any \( j \). Then, from (11), we can write

\[
S(u, v) = \sum_{i=0}^{m+2} \sum_{j=0}^{n+2} C_{i,j} B_{i,j}(u, v) = \sum_{j=0}^{n+2} \left( C_{r,j} B_{rj}(u, v) + C_{r+1,j} B_{r+1,j}(u, v) \right). \tag{20}
\]

As in \( U \) and \( V \) the inner knots are equally spaced, considering the B-form of \( B_{rj}(u, v) \), \( B_{r+1,j}(u, v) \) and \( B_{j}(v) \), for \( r = 1, \ldots, m \) and \( j = 0, \ldots, n + 2 \) \([11]\), we get

\[
B_{rj}(u, v) = B_{r+1,j}(u, v) = \frac{1}{2} B_j(v). \tag{21}
\]

Then, from (20) and (21), we obtain

\[
S(u, v) = \sum_{j=0}^{n+2} \frac{C_{r,j} + C_{r+1,j}}{2} B_j(v). \tag{22}
\]

Therefore, from (12), (11) and (22), we get (14).

Now, by the same argument, since

\[
S(u_0, v) = \sum_{i=0}^{m+2} \sum_{j=0}^{n+2} C_{i,j} B_{i,j}(u_0, v) = \sum_{j=0}^{n+2} C_{0,j} B_{0j}(u_0, v) \tag{23}
\]

and \( B_{0j}(u_0, v) = B_j(v) \) \([11]\), from (12), (11) and (23), we get (18).

Similarly, by considering \( u = u_{m+1} \) in (11), (19) holds.

By using the same logical scheme, from (13), we can show (15), (16) and (17).

The sufficient condition can be shown in a similar way. Therefore, the theorem is proved.

From (16)-(19) we immediately note that

\[
C_{00} = Q_0^{(0)}, \quad C_{0,n+2} = Q_0^{(n+1)}, \quad C_{m+2,0} = Q_{m+2}^{(0)}, \quad C_{m+2,n+2} = Q_{m+2}^{(n+1)}.
\]
In Theorem 3.2 we show that the surface $S$ can be expressed as a linear combination of the $B_{ij}$’s, with coefficients depending only on the curve network control points $\{P_j^{(r)}\}_{j=0}^{n+2}$ and $\{Q_i^{(s)}\}_{i=0}^{m+2}$. Firstly we need the following lemma.

Lemma 3.1. If curves (11) are isoparametric curves for the surface (14), then for any $C_{11} \in \mathbb{R}^3$

$$C_{ij} = \gamma_{ij} + (-1)^{i+j}C_{11},$$

\(i=1,\ldots,m+1, j=1,\ldots,n+1,\) with

$$\gamma_{ij} = 2\left[ \sum_{r=1}^{i-1} (-1)^{r+1}P_j^{(i-r)} + (-1)^j \sum_{s=1}^{i-1} (-1)^s Q_i^{(j-s)} \right]$$

and $\sum_{k=1}^{0} \cdot = 0$.

Proof. From (15), for any $j = 1, \ldots, n+1$, we can write (Fig. 3)

$$C_{1,j} = 2\sum_{s=1}^{j-1} (-1)^{s+1}Q_i^{(j-s)} + (-1)^j C_{11}.$$ (26)

Then, for any $i = 1, \ldots, m+1$, from (14) it results (Fig. 3)

$$C_{i,j} = 2\sum_{r=1}^{i-1} (-1)^{r+1}P_j^{(i-r)} + (-1)^{i+1}C_{1j}.$$ (27)

Substituting (26) into (27), we get (24).

Remark 3.1. We remark that the generation of the control points $C_{ij}$ by (24) can be not convenient, since many computations are repeated unnecessarily. Indeed, they can be obtained by the following scheme:

$$C_{1,j} = 2Q_i^{(j-1)} - C_{1,j-1}, \quad j = 2, \ldots, n+1,$$

$$C_{i,j} = 2P_j^{(i-1)} - C_{i-1,j}, \quad j = 1, \ldots, n+1, \quad i = 2, \ldots, m+1.$$

Remark 3.2. Now, we want to spend some words on the stability of the $C_{ij}$ generation process, looking at the round-off error growth. Suppose the $P_j^{(r)}$’s and $Q_i^{(s)}$’s are affected by some perturbations (random noise), say $\epsilon_P^{(r)}$ and $\epsilon_Q^{(s)}$. Then, instead of $\{C_{ij}\}$, the sequence $\{\overline{C}_{ij}\}$ is generated, where

$$\overline{C}_{1,j} = 2(Q_i^{(j-1)} + \epsilon_Q^{(j-1)}) - \overline{C}_{1,j-1},$$

$$\overline{C}_{i,j} = 2(P_j^{(i-1)} + \epsilon_P^{(i-1)}) - \overline{C}_{i-1,j}.$$
After \( i + j - 2 \) steps it is easy to see that the error growth is linear and not exponential. Therefore we can conclude the algorithm for \( C_{ij} \) generation is stable (\cite{1}, p.32).

**Theorem 3.2.** Given the B-spline curve network (1), satisfying the compatibility conditions C1. and C2., there is a unique surface (11), having (1) as isoparametric curves, and it can be written as follows:

\[
S(u, v) = S_b(u, v) + S_\gamma(u, v),
\]

where

\[
S_b(u, v) = \sum_{j=0}^{n+2} P_j^{(0)} B_{0j}(u, v) + \sum_{j=0}^{n+2} P_j^{(m+1)} B_{n+2,j}(u, v) + \sum_{i=1}^{m+1} Q_i^{(0)} B_{i0}(u, v) + \sum_{i=1}^{m+1} Q_i^{(n+1)} B_{i,n+2}(u, v),
\]

\[
S_\gamma(u, v) = \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} \gamma_{ij} B_{ij}(u, v),
\]

and \( \gamma_{ij} \) is defined in (25).

*Proof.* From Theorem 3.1 and Lemma 3.1 the surface (11) can be written...
as

\[ S(u, v) = \sum_{j=0}^{n+2} P_j^{(0)} B_{0j}(u, v) + \sum_{j=0}^{n+2} P_j^{(m+1)} B_{n+2,j}(u, v) \]
\[ + \sum_{i=1}^{m+1} Q_i^{(0)} B_{i0}(u, v) + \sum_{i=1}^{m+1} Q_i^{(n+1)} B_{i,n+2}(u, v) \]
\[ + \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} \gamma_{ij} B_{ij}(u, v) + C_{11} \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} (-1)^{i+j} B_{ij}(u, v). \]

(30)

We recall that, in \( S_2^2(T_{mn}) \), the \( B_{ij} \)'s are linearly dependent, the dependence relationship being [2]:

\[ \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} (-1)^{i+j} B_{ij}(u, v) = 0. \]

Therefore, from (30), we obtain (28). The interpolating surface \( S \) is unique and it is independent of \( C_{11} \) choice.

We can use box splines instead of B-splines on \( T_{mn} \) as blending functions in (11), since the first ones are all equal and they are obtained as scaled translates of the ZP-element [3, 14]. Therefore, we define the curve network (1) on the two sets of simple knots

\[ U = \left\{ u_i = a + ih, \ i = -2, -1, \ldots, m+3, \ h = \frac{b-a}{m+1} \right\} \]

and

\[ V = \left\{ v_j = c + jk, \ j = -2, -1, \ldots, n+3, \ k = \frac{d-c}{n+1} \right\}. \]

Then, in this case, the following changes have to be considered:

1. the conditions (7)-(10) reduce to (7) for \( r = 0, \ldots, m+1, \ s = 0, \ldots, n+1 \);

2. the constraints that have to be satisfied by the \( C_{ij} \)'s are given only by (14) and (15), with \( r = 0, \ldots, m+1, \ j = 0, \ldots, n+2 \) and \( s = 0, \ldots, n+1, \ i = 0, \ldots, m+2 \), respectively;

3. Lemma 3.1 has to be formulated as follows, taking into account Fig. 4 instead of Fig. 3.
Lemma 3.2. If curves (11) are isoparametric curves for the surface (11), then for any $C_{00} \in \mathbb{R}^3$

$$C_{ij} = \gamma_{ij} + (-1)^{i+j} C_{00},$$

$i=0, \ldots, m+2$, $j=0, \ldots, n+2$, with

$$\gamma_{ij} = 2\sum_{r=0}^{i-1} (-1)^r P_j^{(i-1-r)} + (-1)^i \sum_{s=0}^{j-1} (-1)^s Q_0^{(j-1-s)}$$

(31)

and $\sum_{k=0}^{i-1} \cdot = 0$.

4. The surface $S$ assumes the simpler form

$$S(u, v) = \sum_{i=0}^{m+2} \sum_{j=0}^{n+2} \gamma_{ij} B_{ij}(u, v),$$

where $\gamma_{ij}$ is defined in (31) and the $B_{ij}$’s are the quadratic box splines on $T_{mn}$.

Remarks 3.1 and 3.2 also hold for box spline case, with suitable index modifications.

Remark 3.3. For the sake of comparison, we can consider alternative methods to those above considered. Such methods are based on classical tensor product B-splines.
Let $R_{mn}$ be the rectangular partition of $\Omega$, based on the knots $\{U \times V\}$, defined in (2), (3), and let $S_{2,2}^{1,1}(R_{mn})$ be the space of all biquadratic $C^1$ tensor product splines, whose restriction to each subrectangle of $R_{mn}$ is a bivariate polynomial of coordinate degree two. It is well known that $\dim S_{2,2}^{1,1}(R_{mn}) = (m + 3)(n + 3)$ and its basis is the set $\{B_{ij}(u, v), (i, j) \in K_{mn}\}$, where $B_{ij}(u, v) = B_i(u)B_j(v)$, with $B_i(u)$ and $B_j(v)$ defined in Section 2.

In this case, in order to construct the corresponding surface (11), Theorem 3.1 and Lemma 3.1 still hold, while we have to modify Theorem 3.2. Since for tensor product quadratic B-splines

$$S_1(u, v) = \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} (-1)^{i+j} B_{ij}(u, v)$$

(32)

is different from zero, then, from the expression (30), the surface is not unique, because it depends on the choice of $C_{11}$.

As in most papers a surface is said to be “smooth” if the energy of a thin plate described by the surface is minimal, following [8], we can consider an approximation of the thin plate energy given by

$$\Pi = \int_a^b \int_c^d [(S_{uu}(u, v))^2 + 2(S_{uv}(u, v))^2 + (S_{vv}(u, v))^2] \, du \, dv,$$

with $S_{uu}(u, v) = \frac{\partial^2 S(u, v)}{\partial u^2}$, $S_{uv}(u, v) = \frac{\partial^2 S(u, v)}{\partial u \partial v}$, $S_{vv}(u, v) = \frac{\partial^2 S(u, v)}{\partial v^2}$ and choose $C_{11}$ by minimizing $\Pi$ with respect to $C_{11}$.

Then, from (30), the surface $S$ can be written as

$$S(u, v) = S_b(u, v) + S_{\gamma}(u, v) + C_{11}S_1(u, v),$$

with $S_b$ and $S_{\gamma}$ defined in (29) and $S_1$ in (32). By setting $\frac{\partial \Pi}{\partial C_{11}} = 0$, we obtain

$$C_{11} = -\frac{\int_a^b \int_c^d [(S_{bb} + S_{\gamma b})S_{1u} + 2(S_{bb} + S_{\gamma b})S_{1v} + (S_{\gamma b} + S_{\gamma \gamma})S_{1\gamma}] \, du \, dv}{\int_a^b \int_c^d [(S_{1u})^2 + 2(S_{1v})^2 + (S_{1\gamma})^2] \, du \, dv}.$$  

(33)

We can exactly evaluate the integrals appearing in (33), for instance by using a composite tensor-product Gauss-Legendre quadrature formula with 3 nodes.

Remarks 3.1 and 3.2 also hold in this case.

A further possible tensor product approach is the construction of Gordon-type B-spline surface proposed in [10, Chap. 10]. However, in this case the bivariate resulting spline space is not defined on the rectangular grid generated by the knot vectors of the curve network, since knot refinement is required.
4 Numerical and graphical results

In this section we present some numerical and graphical results.

4.1 Test 1

We consider the function

\[ f(u, v) = (1 + 2 \exp \left(-3(10\sqrt{u^2 + v^2} - 6.7)\right))^{\frac{1}{2}}, \]

in \( \Omega = [0, 1] \times [0, 1] \) and, for given integers \( m \) and \( n \), we define two knot vectors \( U \) and \( V \) as in (2) and (3), respectively. We assume as control points of the B-spline curve network (1) the following ones

\[
P_j^{(0)} = (u_0, \tau_j, f(u_0, \tau_j)), \quad P_j^{(m+1)} = (u_{m+1}, \tau_j, f(u_{m+1}, \tau_j)),
\]

\[
P_j^{(r)} = \left( u_r, \tau_j, \frac{f(\sigma_r, \tau_j) + f(\sigma_{r+1}, \tau_j)}{2} \right), \quad j = 0, \ldots, n + 2, \quad r = 1, \ldots m,
\]

\[
Q_i^{(0)} = (\sigma_i, v_0, f(\sigma_i, v_0)), \quad Q_i^{(n+1)} = (\sigma_i, v_{n+1}, f(\sigma_i, v_{n+1})),
\]

\[
Q_i^{(s)} = \left( \sigma_i, v_s, \frac{f(\sigma_i, \tau_s) + f(\sigma_i, \tau_{s+1})}{2} \right), \quad i = 0, \ldots, m + 2, \quad s = 1, \ldots n,
\]

where \( \sigma_i \) and \( \tau_j \) are defined in (4) and (5), respectively. It is easy to see that such curves satisfy conditions C1. and C2.

In Fig. 5(a) we show the curve network and in Fig. 5(b) the corresponding control polygons for \( m = 4 \) and \( n = 5 \).

![Figure 5: (a) B-spline curve network and (b) control polygons of Test 1](image)

Considering such \( m \) and \( n \), we construct the interpolating B-spline surfaces both in \( S^1_2(T_{4,5}) \) and \( S^1_{2,2}(R_{4,5}) \), for different values of \( C_{11} \). We present
their graphs, with the corresponding control nets, in Fig. 6 for \( C_{11} = (1.5, 1, 1) \) and in Fig. 7 for the value of \( C_{11} \) obtained by solving the minimization problem described Remark 3.3.

In the latter case the abscissa and the ordinate of \( C_{11} \) coincide with \( \sigma_1 \) and \( \tau_1 \), defined in (1) and (5), respectively. The tensor product surface has not oscillations and it has a pleasant shape.

As proved in Section 3, we can notice that the interpolating surface in \( S_{1}^{2}(T_{4,5}) \) is unique. Indeed, modifying the control point \( C_{11} \), only the control net changes, but the surface is always the same. In case of \( S_{1}^{1,1}(R_{4,5}) \) the surface depends on \( C_{11} \), i.e. if we change \( C_{11} \), we get different surfaces.

![Figure 6: The surfaces \( S(u, v) \) interpolating the curve network of Test 1 in (a) \( S_{2}(T_{4,5}) \) and (b) \( S_{2,2}^{1,1}(R_{4,5}) \), with \( C_{11} = (1.5, 1, 1) \)](image)

### 4.2 Test 2

We consider \( m = 17, n = 3 \), the two knot vectors \( U = \{u_i\}_{i=-2}^{20}, V = \{v_j\}_{j=-2}^{6} \), defined as in (2) and (3) with \( a = 0, b = 18, c = 0, d = 4 \) and the B-spline curve network of type (1) shown in Fig. 8. The interpolating quadratic B-spline surface in \( S_{2}(T_{17,3}) \) is shown in Fig. 9(a) with \( C_{11} = (1.1.5, 20) \). In Figs. 9(b) and (c), we present the two B-spline surfaces in \( S_{2}(T_{17,3}) \) and \( S_{2,2}^{1,1}(R_{17,3}) \), respectively, with \( C_{11} = (1.5, 1.5, 32) \), obtained by solving the minimization problem of Remark 3.3.
4.3 Test 3

In the last test, we interpolate the B-spline curve network shown in Fig. 10(a), where three-sided and four-sided regions are present. Considering $m = 4$, $n = 2$, the two knot vectors $U = \{u_i\}_{i=-2}^7$, $V = \{v_j\}_{j=-2}^5$, defined as in (2) and (3) with $a = 0$, $b = 5$, $c = 0$, $d = 3$, we construct the interpolating B-spline surface in $S^1_2(\mathcal{T}_{4,2})$ and the tensor product one in $S^{1,1}_{2,2}(\mathcal{R}_{4,2})$, that are
Figure 9: The surfaces $S(u, v)$ interpolating the curve network of Test 2 in (a) $S_2^1(\mathcal{T}_{17,3})$ with $C_{11} = (1, 1.5, 20)$, (b) $S_2^1(\mathcal{T}_{17,3})$ with $C_{11} = (1.5, 1.5, 32)$ and (c) $S_2^1(\mathcal{R}_{17,3})$ with $C_{11} = (1.5, 1.5, 32)$ shown in Figs. (b) and (c), respectively, with $C_{11}$ obtained by solving the minimization problem of Remark 3.3.

Figure 10: (a) The curve network of Test 3. The surfaces $S(u, v)$ interpolating the curve network in (b) $S_2^1(\mathcal{T}_{4,2})$ and (c) $S_2^1(\mathcal{R}_{4,2})$, with $C_{11}$ obtained by solving the minimization problem of Remark 3.3.

We can remark that, in all tests, the shape of the surfaces generated in $S_2^1(\mathcal{T}_{m,n})$ are comparable with those in $S_2^1(\mathcal{R}_{m,n})$, in case of the “optimal” choice of $C_{11}$. However for tensor product splines, the construction is more expensive from a computational point of view, due to the evaluation of $C_{11}$ in (33).
5 Final remarks

In this paper we have investigated the problem of interpolating a B-spline curve network, in order to create a surface satisfying such a constraint and belonging to the space $S^1_2(\mathcal{T}_{mn})$. We have proved the surface existence and uniqueness, providing a constructive algorithm for its generation.

We have also presented numerical and graphical results and comparisons with tensor product B-spline surfaces.

According to [10], we have restricted the curves of the network to be nonrational. Theoretically, the construction, discussed in this paper, can be carried out in homogeneous space for rational curves, but it is generally not practical to do so and the three-dimensional results can be unpredictable. If rational curves are involved, it is recommended using constrained approximation techniques to obtain nonrational approximations of the rational curves to within necessary tolerances.

Finally, we remark that an open problem could be the construction of B-spline surfaces interpolating a curve network of the following form

$$
\phi_r(v) = \sum_{j=0}^{n+2} P_j^{(r)} B_j(v) \quad r = 0, \ldots, R + 1,
$$

$$
\psi_s(u) = \sum_{i=0}^{m+2} Q_i^{(s)} B_i(u) \quad s = 0, \ldots, S + 1,
$$

with $m \geq R$ and $n \geq S$, i.e. not all surface isoparametric curves, corresponding to the knots in $U$ and $V$, are curves of the network. In this case we would have free parameters that could be managed, in order to model the interpolating surface, for example by optimization techniques.

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