Gradient estimates for some evolution equations on complete smooth metric measure spaces

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Abstract. In this paper, we consider the following general evolution equation

$$u_t = \Delta f u + au \log u + bu$$

on a smooth metric measure space $(M^n, g, e^{-f}dv)$. We give a local gradient estimate of Souplet–Zhang type for positive smooth solutions of this equation provided that the Bakry–Émery curvature is bounded from below. When $f$ is constant, we investigate the general evolution equation on compact Riemannian manifolds with nonconvex boundary satisfying an interior rolling $R$-ball condition. We show a gradient estimate of Hamilton type on such manifolds.

1. Introduction

Motivated by the understanding of HAMILTON’s Ricci flow ([8]), L. MA [14] introduced the following nonlinear evolution equation

$$\Delta u + au \log u + bu = 0$$

(1.1)

on a complete noncompact Riemannian manifold $(M, g)$ whose Ricci curvature is bounded from below. Here $a < 0$, $b$ are real constants. He obtained a gradient estimate for the positive solution of the evolution equation. Moreover, he pointed

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out that his result is almost optimal if we consider Ricci solitons. To see the relationship between equation (1.1) and gradient Ricci solitons, we recall that a Riemannian manifold \((M, g)\) is said to be a gradient Ricci soliton if there is a smooth function \(f\) on \(M\) and a constant \(\lambda \in \mathbb{R}\) such that

\[
\text{Ric} + \text{Hess} f = \lambda g.
\]

Here \(\text{Ric}\) is the Ricci tensor of the manifold and \(\text{Hess}\) is the Hessian with respect to the metric \(g\). Gradient Ricci solitons play an important role in Hamilton’s Ricci flow as they correspond to the self-similar solitons and arise as limits of dilations of singularities in the Ricci flow. A gradient Ricci soliton is said to be shrinking, steady, and expanding if \(\lambda\) is positive, zero, and negative, respectively.

Using the contracted Bianchi identity and letting \(u = e^f\), Ma proved that the equation \(\text{Ric} + \text{Hess} f = \lambda g\) can be deformed to

\[
\Delta u + 2\lambda u \log u = (A_0 - n\lambda)u,
\]

where \(A_0\) is a constant which can also be determined by choosing an extremal point of \(f\).

Ma also raised the interesting problem to consider the local gradient estimates for the positive solutions to the following evolution equation

\[
\frac{\partial u}{\partial t} = \Delta u + au \log u + bu \tag{1.2}
\]

in \(M\), where \(a, b\) are real constants.

Due to the nature of the equations (1.1), (1.2) and their potential in studying gradient Ricci solitons, (1.1) and (1.2) have been investigated in several papers. For example, in [22] Yang and in [4], Chen et al. proved gradient estimates of Li–Yau type for the positive solution of (1.2) on a Riemannian manifold \((M^n, g)\). When the Ricci curvature is non-negative, Yang showed that the positive smooth solutions of (1.4) must be bounded from below by \(e^{-n/4}\) (resp. from upper by \(e^{n/2}\)) if \(a < 0\) (resp. \(a > 0\)). (See [4] for similar results.) In [9], Huang et al. also pointed out local gradient estimates of Li–Yau type for the positive solutions of (1.2) on Riemannian manifolds with Ricci curvature bounded from below. As applications, they derived several parabolic Harnack inequalities. Later, in [2], Cao et al. considered gradient estimates for the positive smooth solutions of (1.2) and applied these to study Ricci flows and gradient Ricci solitons. They proved several differential Harnack inequalities and used them to derive bounds on gradient Ricci solitons. Recently, in [12], Jiang introduced a local Hamilton-type
gradient estimate for (1.2) and obtained a Liouville-type theorem for bounded smooth solutions of (1.2). We refer the reader to [2], [4], [9], [12], and [22] for further references.

It is worth to notice that gradient Ricci solitons are special cases of the so-called smooth metric measure spaces. Recall that an $n$-dimensional smooth metric measure space $(M^n, g, e^{-f} dv)$ is a complete Riemannian manifold $(M^n, g)$ endowed with a weighted measure $e^{-f} dv$ for some $f \in C^{-\infty}(M)$, where $dv$ is the volume element of the metric $g$. On $(M^n, g, e^{-f} dv)$, we have the Bakry–Émery Ricci tensor

$$\text{Ric}_f := \text{Ric} + \text{Hess} f.$$ 

Many geometric and topological properties for manifolds with Ricci tensor bounded below can be extended to smooth metric measure spaces with Bakry–Émery Ricci tensor bounded from below. In a smooth metric measure space, the $f$-Laplacian $\Delta_f$ is defined by

$$\Delta_f := \Delta - \nabla f \cdot \nabla.$$ 

This operator is self-adjoint with respect to the weighted measure. Observe that gradient Ricci solitons are smooth metric measure spaces in which $\text{Ric}_f = \lambda g$. Therefore, it is also natural to consider modified versions of (1.1), (1.2) on such spaces. Indeed, in [18], Wu studied the nonlinear equation

$$u_t = \Delta_f u + au \log u + bu$$

on smooth metric measure spaces $(M^n, g, e^{-f} dv)$ with $m$-dimensional Bakry–Émery curvature bounded from below. He obtained a local gradient estimate for the positive smooth solutions of (1.3). Then, Wu improved his results under the assumption that the Bakry–Émery curvature is bounded from below (see [19], [20]). Later, in [10], [11], and [7], HUANG and MA, KHANH and the first author studied gradient estimates of Hamilton type and Souplet–Zhang type for equation (1.3). As an application, they derived some Liouville-type theorems and Harnack inequalities for bounded smooth solutions of (1.3).

Recently, motivated by studying (1.2), in [23], ZHU and LI investigated the following nonlinear evolution equation on complete Riemannian manifolds (with or without boundary):

$$\left(\Delta - \frac{\partial}{\partial t}\right) u + au \log^o u + bu = 0,$$
where \( a, \alpha \) are constants and \( b \) is a \( C^2 \)-function on \( M \times (0, \infty) \). They proved a gradient estimate on complete compact Riemannian manifolds with convex boundary, which can be considered as a generalization of the famous Li–Yau gradient estimates (see [13]). Moreover, they were able to derive gradient estimates of Li–Yau type for the positive smooth solutions of (1.3) on complete non-compact Riemannian manifolds.

In this paper, motivated by studying of (1.2) and the gradient estimates of (1.3), we consider on a smooth metric measure space the following general evolution equation:

\[
\left( \Delta f - \frac{\partial}{\partial t} \right) u(x,t) + au(x,t)(\log(u(x,t)))^\alpha + bu(x,t) = 0,
\]

(1.4)

where \( a, b, \alpha \) are constants. We sometimes write \( u(x,t) \) as \( u \), also write \( \frac{\partial u}{\partial t} \) as \( u_t \).

Our aim is to find gradient estimates of Souplet–Zhang type on complete Riemannian manifolds with or without boundary. We also look for applications of these estimates. Our first main theorem can be stated as follows.

**Theorem 1.1.** Let \((M,g,e^{-f}dv)\) be an \( n \)-dimensional complete smooth metric measure space. Suppose that for a fixed point \( x_0 \in M \) and for \( R \geq 2 \), we have \( \text{Ric}_f \geq -(n-1)K \) on the ball \( B(x_0,R) \). Let \( a, b \) be constants such that the evolution equation (1.4) has a smooth solution satisfying \( 1 \leq u \leq D \) with some constant \( D > 1 \) in \( Q_{R,T} \equiv B(x_0,R) \times [t_0-T,t_0] \), where \( t_0 \in \mathbb{R} \) and \( T > 0 \). If \((x,t) \in Q_{R/2,T} \) with \( t \neq t_0-T \), then

(i) if \( \alpha \geq 1 \), then there exists a constant \( c(n) \) such that

\[
\frac{|\nabla u|}{u} \leq c(n) \left( 1 + \log \frac{D}{u} \right) \left( \frac{1}{R} + \sqrt{\mu + \frac{1}{t-t_0+T}} + \sqrt{R} + \sqrt{P_1} \right),
\]

(1.5)

where \( P_1 = \max\{0,a\} |\alpha (\log D)^{\alpha-1} + (\log D)^\alpha| + \max\{0,b\} \);

(ii) if \( \alpha < 1 \), then there exists a constant \( c(n) \) such that

\[
\frac{|\nabla u|}{u} \leq c(n) \left( 1 + \log \frac{D}{u} \right) \left( \frac{1}{R} + \sqrt{\mu + \frac{1}{t-t_0+T}} + \sqrt{R} + \sqrt{P_2} \right),
\]

(1.6)

where \( P_2 = \max\{0,a\} |(\log u)^{\alpha-1}| + \max\{0,a\} |\log u|^\alpha + \max\{0,b\} \)

with \( |g|_\infty := \sup_{Q_{R,T}} |g| \), and \( \mu := \max_{\{x|d(x,x_0) = 1\}} \Delta f r(x) \), where \( r(x) \) is the distance function to \( x \) from the base point \( x_0 \).

Because of the appearance of the function \( \log^\alpha u \), when we say that \( u \) is a positive solution of the evolution equation (1.4), we mean that \( u \geq 1 \). As an application, we obtain the following Liouville property.
Theorem 1.2. Suppose that \((M, g)\) is a complete noncompact smooth metric measure space with nonnegative Bakry–Émery curvature. If \(a < 0\) and \(\alpha > 0\), then every positive smooth solution \(u\) of the equation
\[
\Delta fu + au \log \alpha u = 0
\]
is constant provided that \(u\) is bounded. Consequently, \(u \equiv 1\).

The second result of this paper is a gradient estimate of Hamilton type for the following evolution equation
\[
u_t = \Delta u + au \log u + bu
\]
on a compact Riemannian manifold with nonconvex boundary. Here we impose the Neumann boundary condition \(\frac{\partial u}{\partial \nu} = 0\), where \(\frac{\partial}{\partial \nu}\) is the outward-pointing unit normal vector field on \(\partial M\). To state the result, we recall a definition.

Definition 1.1. Let \(\partial M\) be the boundary of a compact Riemannian manifold \(M\). Then \(\partial M\) satisfies the interior rolling \(R\)-ball condition if for each point \(p \in \partial M\), there is a geodesic ball \(B_q(R/2)\), centered at some \(q \in M\) with radius \(R/2\), such that \(\{p\} = B_q(R/2) \cap \partial M\) and \(B_q(R/2) \subset M\).

Historically, the interior rolling \(R\)-ball condition is traced back to the paper [5] by R. Chen. In this paper, Chen gave a lower bound for the first Neumann eigenvalue. In his proof, the interior rolling \(R\)-ball condition is used to construct a good cut-off function near to the boundary. This condition is natural in studying compact Riemannian manifolds with nonconvex boundary. We refer the reader to an example in [5] for the necessity of the condition in estimating the first Neumann eigenvalue. This condition has also been in [6], [17] to derive estimates for the first Stekloff eigenvalue and global heat kernel. Now, we formulate our result.

Theorem 1.3. Let \((M^n, g)\) be a compact Riemannian manifold with boundary \(\partial M\) that satisfies the interior rolling \(R\)-ball condition. Let \(K\) and \(H\) be nonnegative constants such that the Ricci curvature \(\text{Ric}_M\) of \(M\) is bounded below by \(- (n - 1)K\), and the second fundamental form \(II\) of \(\partial M\) is bounded from below by \(-H\). Suppose that \(u \leq D\) is a positive smooth solution of the equation
\[
\begin{cases}
u_t = \Delta u + au \log u + bu; \\
\frac{\partial u}{\partial \nu} |_{\partial M} = 0,
\end{cases}
\]
on \(M \times (0, \infty)\), for some positive constant \(D\). By choosing \(R\) “small”, we have the following estimates.
(i) If \( a > 0 \), then
\[
\frac{\nabla u}{\sqrt{u}} \leq 3(1 + H) \left[ \frac{C_1}{R} + \sqrt{\frac{C_2}{R}} + \left( \frac{\sqrt{D}}{2t} + C_3 \right) (1 + H) \right].
\] (1.8)

(ii) If \( a < 0 \) and \( \delta \leq u(x, t) \leq D \) with some constant \( \delta > 0 \), then
\[
\frac{\nabla u}{\sqrt{u}} \leq 3(1 + H) \left[ \frac{C_1}{R} + \sqrt{\frac{C_2}{R}} + \left( \frac{\sqrt{D}}{2t} + C_4 \right) (1 + H) \right].
\] (1.9)

Here,
\[
C_1 = 4\sqrt{4374D^2(1 + H)^4 + \frac{2}{3}D^2(17H^2 + H)^2};
\]
\[
C_2 = \sqrt{\frac{2}{3} \left[ 2D(n-1)H(H+1)(3H+1) \right]^2};
\]
\[
C_3 = \frac{4}{3}D^2P^2, \quad P = \max \{2(n-1)K + a(2 + \log D) + b, 0\};
\]
\[
C_4 = \frac{4}{3}D^2S^2, \quad S = \max \{2(n-1)K + a(2 + \log \delta) + b, 0\}.
\]

We would like to mention that in the proofs of our theorems we mainly follow the methods in [1], [4], [16]–[17], [19]. In fact, these methods are standard and well-known. They are used in many works (see also [6]–[7], [10]–[11], [15], [20] and the references therein). More precisely, we first estimate the lower bound of the evolution operator acting on a suitable function in terms of the evolution solution. Next, we construct a good cut-off function and apply maximum principle to prove the desired results. It is also worth to notice that on compact manifolds with or without boundary, gradient estimates of Li–Yau type are well-studied in literature but up to our knowledge, gradient estimates of Hamilton type have been not investigated. This provides the novelty of our gradient estimates in Theorem 1.3.

The paper is organized as follows. In Section 2, we give a proof of Theorem 1.1 and its corollaries. In Section 3, we prove gradient estimates on compact Riemannian manifolds with nonconvex boundary.

2. Gradient estimates on noncompact smooth metric measure spaces and their applications

In this section, we will give a proof of Theorem 1.1. First of all, we need two technical lemmas. We begin with some notations.
Suppose that $u(x,t)$ is a solution of (1.4) and $1 \leq u \leq D$ with some constant $D > 1$ in $Q_{R,T} := B(x_0, R) \times [t_0 - T, t_0]$, where $t_0 \in \mathbb{R}$ and $T > 0$. Let

$$h(x,t) = \log \frac{u(x,t)}{D}.$$ 

Then (1.4) can be written as

$$\left( \Delta f - \frac{\partial}{\partial t} \right) h + |\nabla h|^2 + a(h + \log D)^\alpha + b = 0,$$

where $a, b$ are real constants. Clearly, $h \leq 0$.

**Lemma 2.1.** Let $\omega = |\nabla \log (1 - h)|^2$, $(x,t) \in Q_{R,T}$.

(i) If $\alpha \geq 1$, then $\omega$ satisfies

$$\left( \Delta f - \frac{\partial}{\partial t} \right) \omega \geq \frac{2h}{1 - h} \langle \nabla h, \nabla \omega \rangle + 2(1 - h)\omega^2 - 2\omega \left[ (n - 1)K + P_1 \right],$$

where $P_1 := \max\{0, a\} \left( (\log D)^{\alpha - 1} + (\log D)^\alpha \right) + \max\{0, b\}$.

(ii) If $\alpha < 1$, then $\omega$ satisfies

$$\left( \Delta f - \frac{\partial}{\partial t} \right) \omega \geq \frac{2h}{1 - h} \langle \nabla h, \nabla \omega \rangle + 2(1 - h)\omega^2 - 2\omega \left[ (n - 1)K + P_2 \right],$$

where $P_2 := \max\{0, a\} \left( (\log u)^{\alpha - 1}\right)_\infty + \max\{0, a\} \left( (\log u)^\alpha\right)_\infty + \max\{0, b\}$.

**Proof.** By the Bochner–Weitzenböck formula and the assumption $\text{Ric}_f \geq -(n - 1)K$, we have

$$\Delta f \omega = |\nabla^2 \log (1 - h)|^2 + 2 \langle \nabla \Delta f \log (1 - h), \nabla \log (1 - h) \rangle + 2 \text{Ric}_f(\nabla \log (1 - h)), \nabla (\log (1 - h)) \rangle \geq -2K(n - 1)\omega + 2 \langle \nabla \Delta f \log (1 - h), \nabla \log (1 - h) \rangle.$$  

(2.2)

We calculate directly that

$$\Delta f \log (1 - h) = \frac{-\Delta f h}{1 - h} - \omega = \frac{|\nabla h|^2 + a(h + \log D)^\alpha + b - h_t}{1 - h} - \omega$$

$$= (1 - h)\omega + \frac{a(h + \log D)^\alpha + b}{1 - h} + (\log(1 - h))_t - \omega$$

$$= \frac{a(h + \log D)^\alpha + b}{1 - h} + (\log(1 - h))_t - h\omega.$$
This equality yields
\[
2 \langle \nabla \Delta f \log(1 - h), \nabla \log(1 - h) \rangle \\
= 2 \langle \nabla \left( \frac{a(h + \log D)^\alpha + b}{1 - h} + (\log(1 - h))_t - h \omega \right), \nabla \log(1 - h) \rangle \\
= -2a_0(h + \log D)^{\alpha - 1} \omega - 2 \left( \frac{a(h + \log D)^\alpha + b}{1 - h} \omega + \omega_t + 2(1 - h)\omega^2 \right) + \frac{2h}{1 - h} \nabla \omega \nabla h.
\]

Hence, inequality (2.2) implies
\[
\Delta f \omega - \omega_t \geq -2\omega \left( a_0(h + \log D)^{\alpha - 1} + \frac{a}{1 - h}(h + \log D)^\alpha + \frac{b}{1 - h} \right) \\
+ 2(1 - h)\omega^2 + \frac{2h}{1 - h} \nabla \omega \nabla h - 2(n - 1)K\omega.
\]

Denote \( S := a_0(h + \log D)^{\alpha - 1} + \frac{a}{1 - h}(h + \log D)^\alpha + \frac{b}{1 - h} \).

Since \( 1 - h \geq 1 \), we have
(i) if \( \alpha \geq 1 \), then \( S \leq \max\{0, a\} [\alpha(\log D)^{\alpha - 1} + (\log D)^\alpha] + \max\{0, b\} \); 
(ii) if \( \alpha < 1 \), then
\[
S \leq \max\{0, a\} |(h + \log D)^{\alpha - 1}|_{\infty} + \max\{0, a\} |(h + \log D)^\alpha|_{\infty} + \max\{0, b\}.
\]

Combining (2.1) and the above two cases, the proof of Lemma (2.1) is finished.

Next, we introduce a smooth cut-off function originated by Li–Yau ([13]).

**Lemma 2.2** ([16], [19]). Fix \( t_0 \in \mathbb{R} \) and \( T > 0 \). For any given \( \tau \in (t_0 - T, t_0] \), there exists a smooth function \( \psi : [0, +\infty) \times [t_0 - T, t_0] \to \mathbb{R} \) satisfying the following conditions:

(i) \( 0 \leq \tilde{\psi}(r, t) \leq 1 \) in \([0, R] \times [t_0 - T, t_0] \), and it is supported in a subset of \([0, R] \times [t_0 - T, t_0] \);

(ii) \( \tilde{\psi}(r, t) = 1 \) and \( \tilde{\psi}_r(r, t) = 0 \) in \([0, R/2] \times [\tau, t_0] \) and \([0, R/2] \times [t_0 - T, t_0] \), respectively;

(iii) \( |\tilde{\psi}_t| \leq \frac{C't_{1/2}}{\tau - t_0 + T} \) in \([0, +\infty) \times [t_0 - T, t_0] \) for some \( C > 0 \), and \( \tilde{\psi}(r, t_0 - T) = 0 \) for all \( r \in [0, +\infty) \);

(iv) \( -\frac{C'\tilde{\psi}}{R^2} \leq \tilde{\psi}_r \leq 0 \) and \( |\tilde{\psi}_{rrr}| \leq \frac{C_{\epsilon}\tilde{\psi}}{R^2} \) in \([0, +\infty) \times [t_0 - T, t_0] \) for every \( \epsilon \in (0, 1] \) with some constant \( C_{\epsilon} \) depending on \( \epsilon \).

Now, we are ready to prove Theorem 1.1.
Proof of Theorem 1.1. We only prove the case $\alpha \geq 1$, the case $\alpha < 1$ is proved similarly. Pick any fixed number $\tau \in (t_0 - T, t_0]$ and choose a cut-off function $\bar{\psi}(r, t)$ specified by Lemma 2.2. We shall show that inequality (1.5) holds at every point $(x, \tau)$ in $Q_{R/2, T}$.

To do this, we introduce a cut-off function $\psi: M \times [t_0 - T, t_0] \to \mathbb{R}$ such that

$$\psi(x, t) = \bar{\psi}(d(x, x_0), t),$$

where $x_0 \in M$ is a fixed point. Consider the function $\psi \omega$ in $Q_{R, T} = \{(x, t) \in M \times [t_0 - T, t_0] : d(x, x_0) \leq R\}$. Since $\psi \omega$ is continuous, it attains its maximum in $Q_{R, T}$. Let $(x_1, t_1)$ be a maximum for $\psi \omega$ in the set $Q_{R, T}$. We may suppose that $(\psi \omega)(x_1, t_1) > 0$; otherwise, if $(\psi \omega)(x_1, t_1) \leq 0$, then $(\psi \omega)(x, \tau) \leq 0$ for all $x \in B_{x_0}(R)$. However, by the definition of $\psi$, we have $\psi(x, \tau) = 1$ for all $x \in M$ satisfying $d(x, x_0) \leq R/2$. This implies $\omega(x, \tau) \leq 0$ when $x \in B_{x_0}(R)$. Since $\tau$ is arbitrary, the conclusion then follows.

Since $(\psi \omega)(x_1, t_1) > 0$, we infer $t_1 \neq t_0 - T$. Due to a standard argument of Calabi [3], we may also assume that $\psi \omega$ is smooth at $(x_1, t_1)$. Therefore, at $(x_1, t_1)$, we have

$$\nabla(\psi \omega) = 0, \quad \Delta_f(\psi \omega) \leq 0, \quad (\psi \omega)_t \geq 0. \quad (2.4)$$

By Lemma 2.1, it follows that

$$0 \geq \left(\Delta_f - \frac{\partial}{\partial t}\right)(\psi \omega) - \left(\frac{2h}{1-h} \nabla h + 2 \frac{\nabla \psi \omega}{\psi}, \nabla(\psi \omega)\right)$$

$$= -2[(n-1)K + P_1](\psi \omega) + 2\psi(1-h)\omega^2 - \frac{2h}{1-h} \langle \nabla \psi, \nabla h \rangle \omega$$

$$\quad + \omega \Delta_f \psi - \omega \psi_t - 2\frac{|
abla \psi|^2}{\psi} \omega.$$  

At $(x_1, t_1)$, the inequality can be simplified as

$$2\psi(1-h)\omega^2$$

$$\leq \frac{2h}{1-h} \langle \nabla \psi, \nabla h \rangle \omega - \omega \Delta_f \psi + 2\frac{|
abla \psi|^2}{\psi} \omega + \omega \psi_t + 2[(n-1)K + P_1] \psi \omega. \quad (2.5)$$

Here we used (2.4) to obtain (2.5).
Case 1. If \( x_1 \in B(x_0, R/2) \), then by our assumption, \( \psi \) is constant in space direction in \( B(x_0, R/2) \), where \( R \geq 2 \). So at \((x_1, t_1)\), inequality (2.5) yields

\[
2\psi \omega^2 \leq 2\psi(1-h)\omega^2 \leq 2[(n-1)K + P_1]\psi \omega + \omega \psi_t.
\]

Here we used \( h \leq 0 \). Note that \( \psi(x, \tau) = 1 \) when \( d(x, x_0) < R/2 \). Hence, using Lemma 2.2 for any \( x \in B_{x_0}(R) \), the above estimate indeed implies that

\[
\omega(x, \tau) = \psi(x, \tau)\omega(x, \tau) \leq (\psi^1/2)(x_1, t_1) \leq (\psi^1/2)\omega(x_1, t_1)
\]

\[
\leq \frac{\psi_t}{2\psi^{1/2}} + [(n-1)K + P_1] \psi^{1/2} \leq \frac{C}{\tau - t_0 + T} + [(n-1)K + P_1]
\]

for all \( x \in B(x_0, R/2) \). Since \( \tau \) can be chosen arbitrarily, we completed the proof.

Case 2. Now, we assume \( x_1 \notin B(x_0, R/2) \) where \( R \geq 2 \). Since \( \text{Ric}_f \geq -(n-1)K \) and \( r(x_1, x_0) \geq 1 \) in \( B(x_0, R) \), the \( f \)-Laplacian comparison theorem in [1] gives

\[
\Delta_f r(x_1) \leq \mu + (n-1)K(R-1),
\]

where \( \mu := \max_{x \in B(x_0, 1)} \Delta_f r(x) \). In the following computations, \( c \) denotes a constant depending only on \( n \) whose value may change from line to line.

Using the above \( f \)-Laplacian comparison theorem and Lemma 2.2, we have

\[
-\omega \Delta_f \psi = -[\psi_r \Delta_f r + \psi_{rr} |\nabla r|^2] \omega \leq -[\psi_r (\mu + (n-1)K(R-1)) + \psi_{rr}] \omega
\]

\[
\leq \omega \psi^{1/2} \left| \frac{\psi_t}{\psi^{1/2}} \right| + |\mu| \psi^{1/2} \left| \frac{\psi_r}{\psi^{1/2}} \right| + \frac{(n-1)K(R-1) |\psi_r|}{\psi^{1/2}} \omega
\]

\[
\leq \frac{1}{8} \psi \omega^2 + c \left( \left( \frac{\psi_{rr}}{\psi^{1/2}} \right)^2 + \left( \frac{|\mu| |\psi_r|}{\psi^{1/2}} \right)^2 + \left( \frac{(K(R-1) |\psi_r|)}{\psi^{1/2}} \right)^2 \right)
\]

\[
\leq \frac{1}{8} \psi \omega^2 + c \left( \frac{1}{R^4} + \frac{|\mu|^2}{R^2} + K^2 \right). \tag{2.6}
\]

Now, we use the Young inequality to estimate the terms of the right-hand side of (2.5). First, we estimate the first term of the right-hand side of (2.5):

\[
\frac{2h}{1 - h} \langle \nabla \psi, \nabla h \rangle \omega \leq 2h|\nabla \psi| \omega^{3/2} = 2[\psi(1-h)\omega^2]^{3/4} \left[ \frac{|h||\nabla \psi|}{[\psi(1-h)]^{3/4}} \right] \omega^{3/2}
\]

\[
\leq \psi(1-h)\omega^2 + c \left( \frac{(h|\nabla \psi|)^4}{[\psi(1-h)]^3} \right) \leq \psi(1-h)\omega^2 + c \frac{h^4}{R^3(1-h)^3}. \tag{2.7}
\]
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For the third term of the right-hand side of (2.5):
\[
2 \frac{\lvert \nabla \psi \rvert^2}{\psi} \omega = 2 \psi^{1/2} \omega \frac{\lvert \nabla \psi \rvert^2}{\psi^{3/2}} \leq \frac{1}{8} \psi \omega^2 + \frac{c}{R^4}. \tag{2.8}
\]

For the fourth term of the right-hand side of (2.5):
\[
\omega \psi_t = \psi^{1/2} \omega \frac{\psi_t}{\psi^{1/2}} \leq \frac{1}{8} \psi \omega^2 + \frac{c}{(\tau - t_0 + T)^2}. \tag{2.9}
\]

Finally, we estimate the last term as
\[
2 [(n - 1)K + P_1] \psi \omega \leq \frac{1}{8} \psi \omega^2 + c(K^2 + (P_1)^2). \tag{2.10}
\]

Substituting (2.6)–(2.10) into (2.5), we get
\[
2 \psi (1 - h) \omega^2 \leq \psi (1 - h) \omega^2 + \frac{h^4 c}{R^4 (1 - h)^3} + \frac{c}{R^4} + \frac{\mu^2 c}{R^2} + \frac{c}{(\tau - t_0 + T)^2} + cK^2 + c(P_1)^2. \tag{2.11}
\]

Recall that $1 - h \geq 1$, so (2.11) implies
\[
\psi \omega^2 \leq c \left( \frac{h^4}{R^4 (1 - h)^3} + \frac{1}{R^4} + \frac{\mu^2}{R^2} + \frac{1}{(\tau - t_0 + T)^2} + K^2 + (P_1)^2 \right).
\]

Moreover, since $\psi(x, \tau) = 1$ with $x$ in $B(x_0, R/2)$, and $h^4/(1 - h)^4 \leq 1$, we have
\[
\omega^2(x, \tau) \leq \psi \omega^2(x_1, t_1) \leq c \left( \frac{1}{R^4} + \frac{\mu^2}{R^2} + \frac{1}{(\tau - t_0 + T)^2} + K^2 + (P_1)^2 \right).
\]

Since $\omega(x, \tau)$ is defined for any $\tau$ in $(t_0 - T, t_0]$, we get for all $(x, t)$ in $Q_{R/2, T}$
\[
\frac{\lvert \nabla h \rvert}{(1 - h)} (x, t) \leq c \left( \frac{1}{R^2} + \frac{|\mu|}{R} + \frac{1}{\sqrt{t - t_0 + T}} + \sqrt{K} + \sqrt{P_1} \right).
\]

Since $h = \log(u/D)$, the proof is completed. \qed

**Proof of Theorem 1.2.** Suppose that $u$ is a positive smooth solution of the equation
\[
\Delta_t u + au \log^\alpha u = 0,
\]
and $u \leq D$ for some $D > 1$. Note that if $b = 0$, $\alpha > 0 > a$, then $P_1 = P_2 = 0$.

Since $u$ is independent of $t$, we may let $t$ tending to infinity in Theorem 1.1. Thus, letting $R \to \infty$, we obtain for $K = 0$ that
\[
|\nabla u| = 0.
\]

Hence $u$ is constant. This forces $u \equiv 1$. \qed
Corollary 2.3 (Harnack-type inequality). Let \((M, g, e^{-f} dv)\) be an \(n\)-dimensional complete smooth metric measure space with \(\text{Ric}_f \geq -(n-1)K\) for some constant \(K \geq 0\) in \(B(x_0, R)\), with fixed \(x_0 \in M\) and \(R \geq 2\). Suppose that \(u\) is a smooth solution to equation (1.4) satisfying \(1 < u \leq D\) with some constant \(D > 1\). Let \(\rho = \rho(x_1, x_2)\) be the geodesic distance between \(x_1\) and \(x_2\) for all \(x_1, x_2 \in M\).

(i) If \(\alpha \geq 1\), then
\[
 u(x_2, t) \leq u(x_1, t)^{\beta_1} (eD)^{1-\beta_1},
\]
where \(\beta_1 = \exp \left( -\frac{c(n)\rho}{\sqrt{t-t_0 + R}} - c(n)\sqrt{K}\rho - c(n)\sqrt{P_1}\rho \right)\), \(c(n)\) is a constant depending on \(n\).

(ii) If \(\alpha < 1\), then
\[
 u(x_2, t) \leq u(x_1, t)^{\beta_2} (eD)^{1-\beta_2},
\]
where \(\beta_2 = \exp \left( -\frac{c(n)\rho}{\sqrt{t-t_0 + R}} - c(n)\sqrt{K}\rho - c(n)\sqrt{P_2}\rho \right)\), \(c(n)\) is a constant depending on \(n\).

Proof of Corollary 2.3. We consider only the case \(\alpha \geq 1\), when \(\alpha < 1\), the proof is similar. Using the estimates (1.5) and tending with \(R\) to infinity, we obtain
\[
 \frac{|\nabla u|}{u(1 + \log D)} \leq c(n) \left( \frac{1}{\sqrt{t-t_0 + T}} + \sqrt{K} + \sqrt{P_1} \right).
\]
Let \(\gamma : [0, 1] \to M\) be a minimal geodesic joining \(x_1\) and \(x_2\) satisfying \(\gamma(0) = x_2\) and \(\gamma(1) = x_1\). Then
\[
 \log \frac{1 - h(x_1, t)}{1 - h(x_2, t)} = \int_0^1 \frac{d\log (1 - h(\gamma(s), t))}{ds} ds \leq \int_0^1 |\dot{\gamma}| \frac{|\nabla u|}{u(1 + \log D)} ds 
\]
\[
 \leq c(n) \left( \frac{1}{\sqrt{t-t_0 + T}} + \sqrt{K} + \sqrt{P_1} \right) \rho.
\]
Let \(\beta_1 = \exp \left( -\frac{c(n)\rho}{\sqrt{t-t_0 + T}} - c(n)\sqrt{K}\rho - c(n)\sqrt{P_1}\rho \right)\). Then we have
\[
 \frac{1 - h(x_1, t)}{1 - h(x_2, t)} \leq \frac{1}{\beta_1}.
\]
After some further calculations, we find
\[
 u(x_2, t) \leq u(x_1, t)^{\beta_1} (eD)^{1-\beta_1},
\]
as wanted. \(\square\)
3. Gradient estimates on compact Riemannian manifolds with non-convex boundary

Let $M$ be a compact Riemannian manifold with nonconvex boundary. Let $\partial/\partial \nu$ be the outward-pointing unit normal vector field on $\partial M$, and let $II$ be the second fundamental form of $\partial M$ with respect to $\partial/\partial \nu$. In their seminal paper [13], Li and Yau proved that if $M$ is a compact Riemannian manifold with nonnegative Ricci curvature and its boundary $\partial M$ is convex in the sense that $II \geq 0$, then any nonnegative solution $u$ of the evolution equation $\Delta u - \partial_t u = 0$ on $M \times (0, +\infty)$ with Neumann boundary condition $\partial_\nu u = 0$ satisfies

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}$$

on $M \times (0, +\infty)$. Later, the Li–Yau gradient estimates were generalized to manifolds with non-convex boundary by Chen [5] and Wang [17]. Here the boundary $\partial M$ is said to be non-convex if there exists a positive constant $H$ such that the second fundamental form $II$ is bounded from below by $-H$. As in [5] and [17], we need to construct a good cut-off function near to $\partial M$, thus the interior rolling $R$-ball condition is necessary, because we have to do some technical estimates near to $\partial M$ involving non-convexity of the boundary. Furthermore, we would like to note that the interior rolling $R$-ball condition is a geometric condition on the boundary $\partial M$ to ensure that the first Neumann eigenvalue is bounded away from zero (see [5]) and that the second fundamental form is bounded from above (see [6]).

Assume that $u$ is a positive solution of the evolution equation

$$u_t = \Delta u + au \log u + bu,$$  \hspace{1cm} (3.1)

where $a, b \in \mathbb{R}$ are fixed constants. To prove Theorem 1.3, let, as in [12] and [19],

$$h(x, t) := u^{1/3}(x, t).$$

Then equation (3.1) becomes

$$h_t = \Delta h + 2h^{-1}|\nabla h|^2 + ah \log h + \frac{b}{3}h.$$  \hspace{1cm} (3.2)

Using the same strategy as in the proof of Theorem 1.1, we first derive a technical result.
Lemma 3.1. Let $\omega := h|\nabla h|^2$. For any $(x,t) \in M \times (0, \infty)$,

(i) if $a \geq 0$, then $\omega$ satisfies

$$(\Delta - \partial_t)\omega \geq \frac{9}{2} h^{-3/2} - 3h^{-1} \langle \nabla \omega, \nabla h \rangle - [2(n-1)K + a \log D + 2a + b] \omega;$$

(ii) if $a < 0$ and $0 < \delta \leq u(x,t) \leq D$ for some constant $\delta > 0$, then $\omega$ satisfies

$$(\Delta - \partial_t)\omega \geq \frac{9}{2} h^{-3/2} - 3h^{-1} \langle \nabla \omega, \nabla h \rangle - [2(n-1)K + a \log \delta + 2a + b] \omega.$$

Proof. For any smooth function $v$, the Bochner–Weitzenböck formula gives

$$\Delta |\nabla v|^2 \geq 2 |\nabla^2 v|^2 + 2 \text{Ric} \langle \nabla v, \nabla v \rangle + 2 \langle \nabla \Delta v, \nabla v \rangle.$$ 

If $v := \frac{2}{3} h^{3/2}$, then $\omega = |\nabla v|^2$. The assumption $\text{Ric} \geq -(n-1)K$ and the Bochner–Weitzenböck formula imply

$$\Delta \omega \geq -2(n-1)K \omega + 2 \left\langle \nabla \Delta \left(\frac{2}{3} h^{3/2}\right), \nabla \left(\frac{2}{3} h^{3/2}\right) \right\rangle.$$ 

By (3.2), a direct calculation shows that

$$\Delta \left(\frac{2}{3} h^{3/2}\right) = h^{1/2} \Delta h + \frac{1}{2} h^{-1/2} |\nabla h|^2$$

$$= h^{1/2} h_t - \frac{3}{2} h^{-3/2} \omega - ah^{3/2} \log h - \frac{b}{3} h^{3/2}.$$ 

This equality yields

$$2 \left\langle \nabla \Delta \left(\frac{2}{3} h^{3/2}\right), \nabla \left(\frac{2}{3} h^{3/2}\right) \right\rangle$$

$$= 2 \left\langle \nabla \left(h^{1/2} h_t - \frac{3}{2} h^{-3/2} \omega - ah^{3/2} \log h - \frac{b}{3} h^{3/2}\right), h^{1/2} \nabla h \right\rangle$$

$$= \omega_t + \frac{9}{2} h^{-3/2} \omega - 3h^{-1} \langle \nabla \omega, \nabla h \rangle - a \omega (2 + 3 \log h) - b \omega. \quad (3.3)$$

Thus, we obtain

$$\Delta \omega - \omega_t \geq \frac{9}{2} h^{-3/2} \omega - 3h^{-1} \langle \nabla \omega, \nabla h \rangle - \omega [2(n-1)K + a (2 + 3 \log h) + b].$$

To conclude the proof, observe that

- if $a \geq 0$ and $0 < h \leq D^{1/3}$, then $\log h \leq 1/3 \log D$;
- if $a < 0$ and $\delta^{1/3} \leq h \leq D^{1/3}$, then $1/3 \log \delta \leq \log h \leq 1/3 \log D$.

Combining these observations and inequality (3.3), we finally obtain the desired results. \qed
Proof of Theorem 1.3. We only consider the case $a \geq 0$, the case $a < 0$ is similar.

Suppose $r(x)$ is the distance from $x \in M$ to $\partial M$. As in [5], we define a function on $M$ by $\phi(x) = \psi\left(\frac{r(x)}{R}\right)$, where $\psi$ is a nonnegative $C^2$-function defined on $[0, \infty]$ such that

\begin{align*}
\psi(r) &\leq H \quad \text{if } 0 \leq r < 1/2, \\
\psi(r) &= H \quad \text{if } 1 \leq r < \infty,
\end{align*}

with $\psi(0) = 0$, $0 \leq \psi'(r) \leq 2H$, $\psi'(0) = H$ and $\psi''(r) \geq -H$.

Put $\chi(x) = (1 + \phi(x))^2$, and let $F(x,t) = t\chi \omega = t\chi |\nabla h|^2$.

For any fixed $T < \infty$, $F(x,t)$ is continuous on $\mathcal{M} \times [0,T]$. We can suppose that $(x_0,t_0) \in \mathcal{M} \times [0,T]$ is a maximum point for the function $F$. If $F(x_0,t_0) = 0$, then the right-hand side of (1.8) and of (1.9) is zero at $(x,T) \in M \times (0,T]$. Since $T$ is arbitrary, the inequalities in Theorem 1.3 are obvious. Hence, we can assume that $F(x_0,t_0) > 0$. Consequently, $t_0 \neq 0$.

If $x_0 \in \partial M$, then $\frac{\partial F}{\partial \nu}(x_0,t_0) \geq 0$. Let $e_1, e_2, \ldots, e_n$ be an orthonormal frame at $x_0$ with $e_n = \nu$. We have

$$0 \leq \frac{\partial F}{\partial \nu}(x_0,t_0) = t_0 \left( \frac{\partial \chi(x_0)}{\partial \nu} |\nabla h|^2 + \chi(x_0) h_{\nu} |\nabla h|^2 + \chi(x_0) 2h \sum_{i=1}^{n} h_i h_{i\nu} \right).$$

Further, $h_n = h_\nu = \frac{\partial \alpha}{\partial \nu} = 0$ on $\partial M$. Since $t_0 > 0$, we get

$$\frac{\partial \chi(x_0)}{\partial \nu} = \frac{1}{\chi(x_0)} + \frac{2}{|\nabla h|^2} \geq 0.$$

On the other hand, by a direct computation, we obtain

$$\sum_{i=1}^{n} h_i h_{i\nu} = \langle \nabla h, (\nabla h)_\nu \rangle = -II \langle \nabla h, \nabla h \rangle \leq H|\nabla h|^2.$$
and
\[
\frac{\partial \chi(x_0)}{\partial \nu} \frac{1}{\chi(x_0)} = -\frac{2H}{R}.
\]

If we choose \( R < 1 \), then
\[
\frac{\partial \chi(x_0)}{\partial \nu} \frac{1}{\chi(x_0)} + 2 \sum_{i=1}^{n} h_i h_{i\nu} \leq -\frac{2H}{R} + 2H < 0.
\]

Thus we arrived at a contradiction.

Now, we can assume that \( F \) achieves its maximum at \((x_0, t_0) \in (\overline{M} \setminus \partial M) \times [0, T]\). Then \( \nabla F(x_0, t_0) = 0, \frac{\partial F}{\partial t}(x_0, t_0) \geq 0, \Delta F(x_0, t_0) \leq 0 \). We infer at \((x_0, t_0)\)
\[
0 \geq \Delta F - F_t = t_0 \omega \Delta \chi + t_0 \chi \Delta \omega + 2t_0 \langle \nabla \chi, \nabla \omega \rangle - t_0 \chi \omega_t - \chi \omega = t_0 \chi (\Delta \omega - \omega_t) + t_0 \omega \Delta \chi + 2t_0 \langle \nabla \chi, \nabla \omega \rangle - \chi \omega.
\]

By Lemma 3.1, we obtain
\[
0 \geq \Delta F - F_t \geq t_0 \chi \left( \frac{9}{2} h^{-3} \omega^2 - 3h^{-1} \langle \nabla h, \nabla \omega \rangle - \omega P \right) + t_0 \omega \Delta \chi + 2t_0 \langle \nabla \chi, \nabla \omega \rangle - \chi \omega.
\]

(3.4)

Since \( 0 = \nabla F = t_0 \omega \nabla \chi + t_0 \chi \nabla \omega \), we have at \((x_0, t_0)\)
\[
\begin{align*}
\langle \nabla \chi, \nabla \omega \rangle &= -\frac{\|\nabla \chi\|^2}{\chi} \omega \geq -16 \frac{H^2}{R^2} \omega,
\langle \nabla \omega, \nabla h \rangle &= -\left( \frac{\nabla \chi}{\chi}, \nabla h \right) \omega.
\end{align*}
\]

(3.5)

We let \( M(R) = \{ x \in M | r(x) \leq R \} \). By using an index comparison theorem as in [21] (see also [5]), we have
\[
\Delta r(x) \geq -(n-1)(3H+1)
\]
for \( x \in M(R) \). Therefore,
\[
\Delta \phi = \frac{1}{R} \psi' \Delta r + \frac{1}{R^2} \psi'' |\nabla r|^2 \geq \frac{2(n-1)H(3H+1)}{R} - \frac{H}{R^2}.
\]

With the help of this Laplacian comparison, we can estimate
\[
\Delta \chi = 2(1+\phi) \Delta \phi + 2|\nabla \phi|^2 \geq 2(1+H) \left( \frac{2(n-1)H(3H+1)}{R} - \frac{H}{R^2} \right).
\]

(3.6)
Plugging (3.5), (3.6) into (3.4), we obtain

\[
0 \geq \frac{9}{2} t_{0}\omega^2 \chi h^{-3} + 3t_{0}\omega h^{-1} \langle \nabla \chi, \nabla h \rangle - P(1 + H)^2 t_{0}\omega
+ t_{0}\omega (1 + H) \left( - \frac{2(n - 1)H(3H + 1)}{R} - \frac{H}{R^2} \right) - 32t_{0}\omega \frac{H^2}{R^2} - (1 + H)^2 \omega
\geq \frac{9}{2} t_{0}\omega^2 h^{-3} + 3t_{0} \omega h^{-1} \langle \nabla \chi, \nabla h \rangle - (1 + H)^2 \omega
- 2t_{0}\omega \left[ \frac{P(1 + H)^2}{2} + (1 + H) \left( \frac{2(n - 1)H(3H + 1)}{R} + \frac{H}{R^2} \right) + 16H^3 \right]
\geq \frac{9}{2} t_{0}\omega^2 h^{-3} + 3t_{0} \omega h^{-1} \langle \nabla \chi, \nabla h \rangle - (1 + H)^2 \omega - 2t_{0}\omega Q,
\]

where

\[
Q := \frac{P(1 + H)^2}{2} + \frac{2(n - 1)H(3H + 1)}{R} + 17H^2 + H.
\]

Multiplying both sides of the above inequality with \(h^3\), we have

\[
9t_{0}\omega^2 \leq -6t_{0}\omega h^2 \langle \nabla \chi, \nabla h \rangle + 2(1 + H)^2 \omega h^3 + 4Q t_{0}\omega h^3. \tag{3.7}
\]

Applying the Young inequality, we obtain the following two estimates:

(i)

\[
-6t_{0}\omega^2 \langle \nabla \chi, \nabla h \rangle \leq 6t_{0}h^3 |\nabla \chi| \omega^3 / 2 \leq \frac{8}{3} t_{0} \left[ \omega^3 / 2 \frac{9}{4} D^{1/2} |\nabla \chi| \right].
\]

\[
\leq 2t_{0}\omega^2 + 4374 \frac{t_{0}D^2 H^4 (1 + H)^4}{R^4}; \tag{3.8}
\]

(ii)

\[
2(1 + H)^2 \omega h^3 = 4 \left[\sqrt{t_{0}\omega} \frac{h^3 (1 + H)^2}{2} \right] \leq 2t_{0}\omega^2 + \frac{(1 + H)^4 h^6}{2t_{0}}
\leq 2t_{0}\omega^2 + \frac{(1 + H)^4 D^2}{2t_{0}}. \tag{3.9}
\]

Finally, we use the Cauchy’s inequality to obtain

\[
4t_{0}\omega Q h^3 \leq 2t_{0}\omega^2 + 2t_{0}Q^2 D^2. \tag{3.10}
\]

Plugging (3.8)–(3.10) into (3.7), we infer at \((x_0, t_0)\)

\[
9t_{0}\omega^2 \leq 8t_{0}\omega^2 + 4374 \frac{t_{0}D^2 (1 + H)^4 H^4}{R^4} + \frac{(1 + H)^4 D^2}{2t_{0}} + 2t_{0}Q^2 D^2.
\]
Consequently,

\[
\omega^2 \leq 4374 \frac{D^2(1 + H)^4 H^4}{R^4} + \frac{D^2(1 + H)^4}{2l_0^2} + 2Q^2 D^2
\]

\[
eq \frac{1}{R^4} \left[ 4374D^2(1 + H)^4 H^4 + \frac{2}{3} D^2(17H^2 + H)^2 \right] + \frac{D^2(1 + H)^4}{2l_0^2}
\]

\[
+ \frac{2}{3} D^2 P^2(1 + H)^4 + \frac{2}{3R^2} \left[ 2D(n - 1)H(H + 1)(3H + 1) \right]^2
\]

\[
= \left( \frac{C_1}{R^2} \right)^4 + \left( \frac{C_2}{R^2} \right)^2 + \frac{D^2(1 + H)^4}{2l_0^2} + (C_3)^4(1 + H)^4.
\]

Therefore, for any \( x \in \overline{M} \),

\[
T \omega(x, T) \leq T(1 + \phi(x))^2 w(x, T) \leq t_0(1 + \phi(x_0))^2 w(x_0, t_0)
\]

\[
\leq (1 + H)^2 \left[ t_0 \left( \frac{(C_1)^2}{R^2} + \frac{C_2}{R} \right) + \frac{D(1 + H)^2}{\sqrt{2}} + t_0(C_3)^2(1 + H)^2 \right]
\]

\[
\leq (1 + H)^2 \left[ T \left( \frac{(C_1)^2}{R^2} + \frac{C_2}{R} \right) + \frac{D(1 + H)^2}{\sqrt{2}} + T(C_3)^2(1 + H)^2 \right].
\]

This implies that

\[
w(x, T) \leq (1 + H)^2 \left[ \left( \frac{(C_1)^2}{R^2} + \frac{C_2}{R} + \frac{D(1 + H)^2}{\sqrt{2T}} + (C_3)^2(1 + H)^2 \right) \right].
\]

Since \( T \) is arbitrary, the proof is complete. \( \square \)

Remark 3.1. As in [5] and [17], in our estimate, the statement “\( R \) is small” means that \( R \) is chosen to be a positive constant less than 1. Moreover, \( R \) depends on the upper bound of the sectional curvature of the manifold near to the boundary. More precisely, the upper bound of \( R \) is determined by

\[
\sqrt{K} \tan(R \sqrt{K}) \leq \frac{H}{2} + \frac{1}{2}
\]

and

\[
\frac{H}{\sqrt{K}} \tan(R \sqrt{K}) \leq \frac{1}{2},
\]

where \( K \) is the upper bound of the sectional curvature on the set \( M_R = \{ x \in M : r(x) \leq R \} \).

As an application, first we deduce the following gradient estimates.
\textbf{Theorem 3.2.} Let \((M^n, g)\) be a compact Riemannian manifold with convex boundary \(\partial M\) that satisfies the interior rolling \(R\)-ball condition. Let \(K\) be a non-negative constant such that the Ricci curvature \(\text{Ric}_M\) of \(M\) is bounded below by \(-K\). Suppose that \(u \leq D\) is a positive smooth solution of the equation

\[
\begin{aligned}
  &u_t = \Delta u + au \log u + bu; \\
  &\frac{\partial u}{\partial \nu}|_{\partial M} = 0,
\end{aligned}
\]  

(3.11)
on \(M \times (0, \infty)\), for some positive constant \(D\). By choosing \(R\) “small”, we have the following estimates.

(i) If \(a > 0\), then

\[
\frac{\nabla u}{\sqrt{u}} \leq 3 \left( \sqrt{\frac{D}{2t}} + C_3 \right).
\]  

(3.12)

(ii) If \(a < 0\) and \(\delta \leq u(x, t) \leq D\) for some constant \(\delta > 0\), then

\[
\frac{\nabla u}{\sqrt{u}} \leq 3 \left( \sqrt{\frac{D}{2t}} + C_4 \right).
\]  

(3.13)

Here

\[
C_3 = \sqrt[3]{\frac{2}{3} D^2 P^2}, \quad P = \max\{2K + a(2 + \log D) + b, 0\};
\]

\[
C_4 = \sqrt[3]{\frac{2}{3} D^2 S^2}, \quad S = \max\{2K + a(2 + \log \delta) + b, 0\}.
\]

**Proof.** Since \(\partial M\) is convex, we have \(H = 0\). Therefore, \(C_1 = C_2 = 0\). The proof of Theorem 3.2 follows directly by appealing to Theorem 1.3. \(\square\)

\textbf{Corollary 3.3.} Let \((M^n, g)\) be a compact Riemannian manifold with convex boundary \(\partial M\) that satisfies the interior rolling \(R\)-ball condition. Suppose that the Ricci curvature is nonnegative and \(u\) is a positive smooth solution of the equation

\[
\begin{aligned}
  &\Delta u + au \log u + bu = 0 \\
  &\frac{\partial u}{\partial \nu}|_{\partial M} = 0,
\end{aligned}
\]  

(3.14)
on \(M \times (0, \infty)\). By choosing \(R\) “small”, we have the following estimates.
(i) If $a > 0 \geq b$, then $u \geq e^{-2}$.

(ii) If $a < 0$, $b \leq 0$ and $e^{-2} \leq u(x,t) \leq D$, then $u$ is a constant function. Consequently, $u = e^{-b/a}$.

Proof. Since $u$ does not depend on $t$, we may let $t \to \infty$ in Theorem 3.2 to have the following two cases:

1. If $a > 0 \geq b$, then by Theorem 3.2, we have that $|\nabla u| = 0$. Consequently, $u = e^{-b/a} > 1$. This is a contradiction, since $u \leq e^{-2}$.

2. If $a < 0, b \leq 0$ and $e^{-2} \leq u(x,t) \leq D$, then by Theorem 3.2, we infer $|\nabla u| = 0$. Then, $u$ is constant, namely $u = e^{-b/a}$.

The proof is complete. □

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