Theta Sectors and Thermodynamics of a Classical Adjoint Gas

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Abstract

The effect of topology on the thermodynamics of a gas of adjoint representation charges interacting via 1+1 dimensional $SU(N)$ gauge fields is investigated. We demonstrate explicitly the existence of multiple vacua parameterized by the discrete superselection variable $k = 1, \ldots, N$. In the low pressure limit, the $k$ dependence of the adjoint gas equation of state is calculated and shown to be non-trivial. Conversely, in the limit of high system pressure, screening by the adjoint charges results in an equation of state independent of $k$. Additionally, the relation of this model to adjoint QCD at finite temperature in two dimensions and the limit $N \to \infty$ are discussed.
1 Introduction

It has been known for a long time now that a gauge theory coupled to matter can have non-trivial vacuum structure. For our purposes this statement is taken to mean that in the theory there exists a family of isolated (vacuum) sectors of which the physical one must be chosen by a superselection rule. The classification of such ‘θ-vacua’ can be carried out in all cases by examining the topological structure of the effective gauge group which acts non-trivially on the matter content of the theory. Unfortunately, this classification has little to say about the consequences of the choice of vacuum on the physics of the system. Our purpose here is, in the case of two dimensional SU\((N)\) gauge fields coupled to a gas of classical adjoint charges, to explicitly determine the thermodynamics of the system as function of vacuum sector.

The SU\((N)\) non-Abelian Coulomb gas with adjoint charges in two dimensions is a useful model for investigating the topological and symmetric properties of gauge theories in higher dimensions. In the limit of infinite \(N\) this model has a first order phase transition which can be interpreted as a deconfining one separating a phase with tightly bound fundamental degrees of freedom and a plasma-like phase. This behaviour is analogous to the situation in higher dimensional Yang-Mills theory and adjoint QCD where the transition is characterized by a breaking of the center symmetry of the gauge group and the Polyakov loop serves an effective order parameter. Unfortunately, for the case of finite \(N\) which we will be studying here there is no such phase transition in two dimensions. However at finite \(N\) the vacuum structure due to the topological structure of the theory is clearly apparent. Moreover the model is explicitly solvable in the limits of high and low particle density and in these limits we will investigate the thermodynamics of the adjoint gas in each sector of the theory.

The standard method of classifying the multiplicity of vacua in a particular gauge theory with adjoint matter depends on identifying the effective gauge group. Here, since gauge transformations operate by adjoint action on all fields, a transformation which lies in the center \(Z\) is trivial. Consequently the true gauge group is the quotient of the gauge group and its center. This quotient is multiply connected, as can be seen from the relation,

\[
\pi_1(G/Z) = \pi_0(Z) = Z
\]

where \(G\) is any semi-simple gauge group. Thus, \(Z\) gives a classification of gauge fields which are constrained to be flat connections at infinity. In that case

\[
\lim_{|x| \to \infty} A_\mu(x) = ig^\dagger(x)\nabla_\mu g(x)
\]

where \(g(x)\) is a mapping of the circle at infinity to the gauge group \(G/Z\). Since \(G/Z\) is a symmetry of the Hamiltonian, we expect that all physical states carry a representation of \(\pi_1(G/Z)\). In the case where the center of the group is Abelian all of its irreducible representations are one dimensional and further, when \(Z \sim \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_j}\), we are lead to a classification of all physical states in terms of \(j\) generators of \(Z\), \(\{z_1, \cdots, z_j\}\). If \(Z\) is a unitary realization of \(Z\) and \(|\psi\rangle\) is a physical state we have

\[
Z|\psi\rangle = e^{i(z_1+\cdots+z_j)}|\psi\rangle
\]
The equivalence class of all states which have the same transformation properties under $\mathcal{Z}$ form an isolated (vacuum) sector of the theory which is typically called a $\theta$ sector. This somewhat abstract motivation for the existence of multiple sectors in a theory is in complete agreement with more concrete constructions of $\theta$ sectors in 1+1 dimensional Yang-Mills theory without matter content \[1,2,3\]. The objective of the current paper is to develop a framework for extending this analysis to systems with arbitrary matter content with particular attention given to the case of matter in the adjoint representation of the gauge group.

The thermodynamics of classical charges interacting via Abelian and non-Abelian electric forces in one spatial dimension has been considered previously \[4,5\]. Additionally the effect of multiple vacua in the $SU(2)$ adjoint gas has been considered previously \[6\]. We will consider constant pressure ensembles as these authors but we will use a different formalism to construct the partition function of the system that can be easily extended to configurations other than that of the open line.

We begin in the next section with a short description of our methods for constructing the model of 1+1 dimensional non-Abelian Coulomb gas. Restricting ourselves to the case of the adjoint charges, we proceed with an analysis of the low density/pressure limit of the model. Here using group theoretic techniques the explicit dependence of the equation of state of the adjoint gas on the vacuum parameter $k$ is established. Converting to the Fourier domain we find the high density/pressure limit of the model is equivalent to solving a system of coupled quantum oscillators. In this limit the equation of state is shown to be independent of $k$. We conclude with a discussion of the limit of large rank groups, $N \to \infty$, and the connection of our model to adjoint two dimensional QCD.

## 2 The Classical Non-Abelian Coulomb Gas

The physical system which we will be investigating is that of a collection of static charges restricted to lie on a line, interacting with each other via 1+1 dimensional non-Abelian electric forces. Since we are interested in the thermodynamics of this system we will consider time to be a compactified coordinate with period equal to the inverse temperature $\beta = 1/T$. Consequently the topology of interest is that of a cylinder. In this section we demonstrate the construction of a 1+1-D Coulomb gas of non-Abelian charges using group theoretic ideas introduced by Migdal \[7\] and developed later by Rusakov \[8\]. From this point of view the propagator for colour electric charge along a spatial distance $L$ with boundary holonomies given by the unitary group elements $U_1$ and $U_2$ is,

$$K[U_1, U_2; A] = \sum_R \chi_R^*(U_1) \ e^{-A \ C_2(R)} \chi_R(U_2) \quad (2.1)$$

Here $\chi_R(U)$ is the group character of the element $U$ in representation $R$, $C_2(R)$ is the eigenvalue of the quadratic Casimir operator for the representation $R$, and $A \equiv g^2 \beta L/4$ where $g^2$ is the gauge field coupling constant. The characters form an orthonormal basis on the vector space of irreducible representations of the gauge group and hence the propagator (2.1) is seen
Figure 1: The propagator between two open ends on the cylinder transports irreducible representations with quadratic Casimir dependent exponential damping.

Figure 2: To insert matter into the theory the sewing property is frustrated by the introduction of a character of representation \( S \) at the sewing boundary. In the case of \( S \) being the trivial representation one recovers (2.3).

to be a diagonal operator on this space. In the following we will consider the propagator as an operator,

\[
K[A] = e^{-A C_2}
\]  
(2.2)

The convolution of two propagators to form a single propagator follows from the sewing property \( [7] \) (see fig.2),

\[
K[A_1] K[A_2] = K[A_1 + A_2]
\]  
(2.3)

This relation follows directly from the orthogonality of group characters, which has the effect of multiplying the exponents of (2.2) in the naive way. This sewing is fundamental to using cylinders to build models of spherical or toroidal topology and to include external charges.

The inclusion of a static charge of a particular irreducible representation \( S \) into the system \( \chi \) is accomplished by including character operator \( \chi_S \) at a sewing site (see fig.4). Here we will restrict ourselves to the case where all charges in the system are of the same representation but the extension for arbitrary configurations is straightforward.

Introducing \( n \), \( S \) representation loops each separated by a distance \( L_1 \cdots L_n \), the operator of interest is

\[
\prod_{i=1}^{n} \left[ e^{-g^2 \beta L_i C_2 / 4} \chi_S \right]
\]  
(2.4)

If one identifies the ends of the cylinder and integrates over each \( L_i \) with an appropriate restriction, we arrive at an expression for the partition function of \( n \) charges on a circle with
It is worth noting here that this last expression also has the interpretation as a gas of Wilson loops on a space-time torus and with a slight modification can be used as a starting point for investigating a gas of Wilson loops on the sphere. Taking the circumference of the circle, $L$ to infinity effectively reduces the configuration space to an open line. Carrying out the now unrestricted integrations over $\{L_i\}$ we find that the partition function takes on a rather simple transfer matrix form

$$ Z_n = \frac{1}{n!} \text{Tr} \prod_{i=1}^{n} \left[ \frac{1}{g^2 \beta C_2 / 4} \chi_S \right] \equiv \frac{1}{n!} \text{Tr} T^n $$

(2.6)

Hence, the calculation of the thermodynamics of $n$ static charges on the open line reduces to solving the eigenvalue problem for the operator $T$

$$ T \Psi = \frac{1}{g^2 \beta C_2 / 4} \chi_S \Psi = \lambda \Psi $$

(2.7)

For the purpose of finding thermodynamic quantities it is convenient to deal not with the constant volume ensemble as we have up to now, but rather a constant pressure ensemble. The change to a constant pressure ensemble can be carried out in a straightforward manner by introducing a $pV$ term into the energy of the system with the result of shifting the energy per unit length. The resulting eigenvalue problem reads

$$ T_p \Psi = \frac{1}{\beta (g^2 C_2 / 4 + p)} \chi_S \Psi = \lambda \Psi $$

(2.8)

In the thermodynamical limit where $n \to \infty$ all information of the system is contained in the largest eigenvalue $\lambda_0$ of the operator $T_p$. The remainder of this paper will involve finding $\lambda_0$ for the case of a gas of adjoint charges.
3 Low Density Limit: Group Theory

The effective eigenvalue problem for the non-Abelian gas (2.8) was previously derived via a different approach by Nambu et al [5]. As in that case (2.8) is equivalent to the following linear equation,

\[ H \Psi \equiv (\alpha C_2 - q \chi_S) \Psi = -p \Psi \]  

which acts on the space of irreducible representations, so that \( \Psi = \sum R a_R \chi_R \); with \( \alpha = g^2/4 \) and \( q = 1/(\beta \lambda) \). The structure of this equation is the same as one would find in a quantum mechanics problem. The quadratic Casimir is a diagonal operator on the space of irreducible representations

\[ C_2 \chi_R = C_2(R) \chi_R \]  

which corresponds to the kinetic term. The role of the potential is played by the character \( \chi_S \) which mixes the eigenvectors of the kinetic term. This can be easily seen by the multiplication rule

\[ \chi_R \chi_S = \chi_{R \otimes S} = \sum_T N_{RS}^T \chi_T \]  

Here \( N_{RS}^T \) is the fusion number which enumerates the occurrence of the irreducible representation \( T \) in the Kronecker product of representations \( R \) and \( S \). The only difference between quantum mechanics and the current situation is that we would like to solve for the eigenvalue of the transfer matrix problem \( \lambda = 1/(\beta q) \) as a function of the pressure \( p \) as opposed to solving for the energy of the system as a function of the potential.

As in the case of quantum mechanics one can begin to solve the eigenvalue problem by considering the symmetries of the system which will lead to conserved quantities. Here we are most interested in the symmetric properties of the ‘Hamiltonian’, \( H \), under transformations which lie in the center of the gauge group. As we have seen, the presence of such a symmetry immediately leads to the phenomena of multiple vacua. The action of a transformation under the center of the gauge group is defined as

\[ Z \chi_R = z_R \chi_R \]  

Here \( z_R \) is a representation of the center of the gauge group. Since for all compact Lie groups the center forms an Abelian subgroup, we can take \( z_R \) to be a complex phase factor. The details of this phase factor depend on the structure of the center \( Z \). For \( U(N) \), \( Z \) is isomorphic to \( U(1) \) hence \( z_R = e^{iaC_1(R)} \) where \( a \in \mathbb{R} \) and \( C_1 \) is the integer valued linear Casimir operator. For the case of interest, \( SU(N) \), \( Z \sim \mathbb{Z}_N \) and consequently \( z_R = e^{2\pi i C_1(R)/N} \). This follows from the \( U(N) \) case with the restriction that \( z_R^N = 1 \). The question of whether \( H \) commutes with \( Z \) is reduced, since Casimir operators commute amongst themselves, to the question of whether

\[ [\chi_S, Z] = (1 - e^{2\pi i C_1(S)/N}) \chi_S = 0 \]  

The solutions to such a condition are clearly \( C_1(S) = 0 \) (mod \( N \)). In terms of irreducible representation \( S \) of the matter content of the theory this means that the full vacuum degeneracy is apparent only when \( C_1(S) = 0 \) (mod \( N \)). The simplest examples of such representations
are the trivial representation in which case the theory reduces to that of pure Yang-Mills in 1 + 1 dimensions and the case where all matter is in the adjoint representation. These are the cases we will consider for the remainder of this paper.

In the case of the adjoint gas where the center operator commutes with (3.1), in analogy with the conservation of eigenvalues of commuting operators in quantum mechanics we see that the eigenvalue of the linear Casimir is conserved (mod $N$) and is a good ‘quantum number’ (mod $N$). Consequently for the $SU(N)$ adjoint gas there exists a family of $N$ distinct solutions to the eigenvalue problem each of which we will label by $k = 0, \ldots, N - 1$. For each value of $k$ we have an isolated sector of the theory complete with a stable vacuum and an infinite tower of excited states. These are precisely the discrete ‘$\theta$-vacua’ of the model.

In the limit where $q \to 0$, the eigenvalue problem in (3.1) is reduced to that of the free 1 + 1 dimensional Yang-Mills theory and we can easily identify the vacuum states of each sector. In this case the eigenvectors, $\Psi$ of the transfer matrix, $T_p$, are simply the irreducible representations of the gauge group. Since we are interested in the case with adjoint $SU(N)$ charges, the labeling of vacua introduced above with $k = 0, \ldots, N - 1$ will be followed although strictly speaking the free theory has a countably infinite vacuum degeneracy. We denote by $[k]$ the $k^{th}$ vacuum state which is the $k(k - 1)/2$ dimensional completely antisymmetric fundamental representation of $SU(N)$. Each of these fundamental representations is the lowest lying energy state of each of the $k$ sectors and will serve as a starting point for a perturbative calculation of the eigenvalue problem for small $q$, or equivalently, small $p$ in each sector.

Having identified the ground states of each sector of the theory all we need to calculate, to lowest order in the pressure $p$, the solution of the eigenvalue problem (3.1) are the fusion numbers $N^T_{RS}$ (3.3). As we have seen, these are pure group theoretic quantities which detail the mixing effect of the potential on irreducible representations and, in particular, the antisymmetric ground states $[k]$. With this information one can then proceed as in quantum mechanics and calculate with the unperturbed ground states the pressure $p$ up to third order in $q$. Leaving the details of the evaluation of the fusion numbers and the perturbative expansion of the eigenvalue problem to the Appendix, we record the results for the $SU(N)$ adjoint gas in the $k^{th}$ sector in Table 1.

Now we would like to develop the equation of state for system. As is familiar from more physical gauge theories, the number of microscopic degrees of freedom $n$ may not be the number of macroscopic degrees of freedom, $n^*$. For example it is believed that in QCD pairs and triples of quarks are bound into observable mesons and baryons, respectively. The equation of state per microscopic degree of freedom for the constant pressure ensemble is

$$\frac{p < V >}{n} \equiv p < v > = \frac{n^*}{n} T$$

(3.6)

where $< V >$ is the expectation value of the total volume of the system which is canonically conjugate to $p$. It can be determined by inverting the relation $p(q)$ and using the relationships
between the thermodynamic variables
\[ <V> = -T \frac{\partial \log Z}{\partial p} = -nT \frac{\partial \log \lambda}{\partial p} = nT \frac{\partial \log q(p)}{\partial p} \] (3.7)

It is convenient to define \( \rho \) as the ratio of macroscopic to microscopic degrees of freedom
\[ \rho = \frac{n^*}{n} = \frac{\partial \log q}{\partial \log p} \] (3.8)

Comparing with (3.6) we see the fundamental importance of the quantity \( \rho \). The dependence of \( \rho \) on \( p \) is tabulated for the different sectors of the \( SU(N) \) gas in Table 1.

The results of these calculations deserve some comment. The most striking between the different sectors of the theory is the configuration of adjoint charges in the limit of vanishing pressure. For the \( k = 0 \) sector we find that \( \rho = 1/2 \) and hence the adjoint charges in the system are bound pairwise in the low pressure limit. This behaviour is not surprising and is seen in both the \( U(1) \) and \( SU(N) \) one dimensional (fundamental representation) Coulomb gases. What is different here in the \( k = 0 \) sector is that the first corrections in pressure to this pair-wise binding come about with a negative sign and so the adjoint charges begin to form macroscopic configurations where the number of constituents is three or more. This is possible since adjoint charges are of course self-adjoint and an arbitrary number of them can form an observable charge singlet.

When one moves to the cases when \( k > 0 \) we see a distinct change in the vanishing pressure macroscopic structure of the theory. As explained previously for the Yang-Mills case 1, 2,
different sectors of the a $1 + 1$ dimensional gauge theory are equivalent to considering a the theory with different constant background colour electric fields. For $SU(N)$, each admissible background is given by one of the $N$ fundamental representations which we label by the parameter $k$. In Table 1 we see that for a non-trivial background ($k > 0$) the adjoint charges of the system can interact with the background electric field and form stable, colour singlet configurations where they are the macroscopic degrees of freedom. In other words they act as free particles.

4 High Density Limit: Fermions on the Circle

The eigenvalue problem of (3.1) can also be solved exactly in the limit of large values of $q$ which corresponds to the limit of high pressure. This is most conveniently carried out by converting the group theoretic equation of (3.1) to a linear differential equation with periodic coefficients. In this section the gauge group will be taken to be $U(N)$ as this will simplify calculations. Recovering the results for $SU(N)$ is a trivial step which will be noted at the appropriate point in the calculation.

The starting point for converting the eigenvalue problem of (3.1) to a differential equation is to consider the eigenvector $\Psi$ as a linear combination of irreducible representations each labeled by $N$ integers \{n$_1$, $\cdots$, n$_N$\}. These integers correspond to reduced row variables for the Young diagram associated with the irreducible representation, and satisfy the dominance condition

$$\infty > n_1 > n_2 > \cdots > n_N > -\infty \quad (4.1)$$

The quadratic Casimir operator is diagonal in this basis with the action

$$C_2\psi_{n_1, \cdots, n_N} = \frac{1}{2} \left( \sum_{i=1}^{N} n_i^2 - \frac{N(N^2 - 1)}{12} \right) \psi_{n_1, \cdots, n_N} \quad (4.2)$$

The action of the character $\chi_S$ on these states is, however, more complicated. This is partly due to the dominance restriction on the Young diagram. If one relaxes this restriction then the action of the character is simplified somewhat. In this case the action on a general state of a character in the adjoint representation is given by,

$$\chi_{Ad} \psi_{n_1, \cdots, n_N} = \sum_{r,s=1}^{N} \psi_{n_1+\delta_{r,1}-\delta_{s,1}, \cdots, n_N+\delta_{r,N}-\delta_{s,N}} \quad (4.3)$$

The result of this operation is to add unity to $n_r$ and then subtract unity from $n_s$ and sum over all $r$ and $s$. Consequently the eigenvalue problem of (3.1) is a difficult recurrence type equation. This type of equation is most successfully dealt with by introducing the function $\tilde{\psi}(x_1, \cdots, x_N)$, which is completely symmetric under the exchange of arguments, via a Fourier transform

$$\psi_{n_1, \cdots, n_N} = \int_0^{2\pi} dx_1 \cdots \int_0^{2\pi} dx_N e^{i \sum_n x_i \tilde{\psi}(x_1, \cdots, x_N)} \Delta(x_1, \cdots, x_N) \quad (4.4)$$
The factor $\Delta(\{x_i\})$ is a Vandermonde determinant defined as $\prod_{i<j}(x_i - x_j)$ and as such is completely antisymmetric under the exchange of any two $x_i$'s. This factor is included to force the integration to vanish identically for $n_i = n_j$, when $i \neq j$. In this way we can effectively impose the dominance condition (4.1) of the reduced Young diagram variables in the Fourier domain.

Acting with the Casimir and the adjoint character on (4.4) we find the transfer matrix problem of (3.1) is equivalent to the second order linear partial differential equation with periodic coefficients

$$
\left[ -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - \frac{N}{24}(N^2 - 1) - \frac{q}{\alpha} \sum_{i<j} \cos(x_i - x_j) \right] \Delta \tilde{\psi} = \frac{p}{\alpha} \Delta \tilde{\psi} \quad (4.5)
$$

In this form some of the features of the adjoint non-Abelian Coulomb gas are more apparent. For example, the center operator $Z$ has a simple interpretation in the Fourier domain

$$
Z = e^{iaC_1} = \exp \left[ ia \sum_{k=1}^{N} n_k \right] = \exp \left[ a \sum_{k=1}^{N} \frac{\partial}{\partial x_k} \right] \quad (4.6)
$$

This is exactly the same structure as the translation operator in quantum mechanics. Here $Z$ generates uniform shifts of the coordinates $\{x_i\} \rightarrow \{x_i + a\}$. It is easy to verify that (4.5) has this symmetry and from the arguments of the previous section we expect a continuum of vacua for the adjoint gas with $U(N)$ gauge group. Additionally with periodic coefficients and the restriction to completely anti-symmetrized wavefunctions, $\Delta \tilde{\psi}$, the eigenvalue problem is equivalent to that of non-relativistic fermions in a periodic potential. This correspondence is familiar from matrix models [10] and is exploited in the solution of the large $N$ non-Abelian Coulomb gas [11, 12, 13].

In the limit of high densities, or equivalently high pressure, the wave functions are localized about the minima of the potential. Expanding the potential about the local minimum at $\{x_i - x_j = 0\}$ leads to the coupled harmonic oscillator problem

$$
\left\{ -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} - \frac{N}{24}(N^2 - 1) + \frac{q}{\alpha} \sum_{i<j} (x_i - x_j)^2 - \frac{N^2 q}{\alpha} \right\} \Delta \tilde{\psi} = -\frac{p}{\alpha} \Delta \tilde{\psi} \quad (4.7)
$$

Performing the change of variables to the orthonormal basis $\{u_i\}$ given by

$$
u_{n} = \frac{(N-n)x_n - (x_{n+1} + x_{n+2} + \ldots + x_N))}{\sqrt{(N-n)(N-n+1)}} \quad n = 1 \ldots N-1$$

$$\nu_{N} = \frac{1}{\sqrt{N}}(x_1 + x_2 + \ldots + x_N)$$

diagonalizes the problem. In this basis the decoupled oscillator problem is

$$
\left[ -\frac{1}{2} \sum_{i=1}^{N} \frac{\partial^2}{\partial u_i^2} + \frac{1}{2} \left( \frac{2Nq}{\alpha} \right) \sum_{i=1}^{N-1} u_i^2 \right] \psi = E\psi \quad (4.9)
$$
where,

\[ E = \frac{N}{24} (N^2 - 1) - \frac{p - N^2 q}{\alpha} \]  

(4.10)

At this point we note that there is no potential for the \( u_N \) coordinate which is to be expected as it describes a center of mass coordinate in the change of variables \((4.8)\) and the original potential in \((4.3)\) depends only on relative not absolute positions. Consequently the \( u_N \) dependence of the problem is only through a phase. In the case of \( SU(N) \) this center of mass coordinate is restricted but otherwise behaves exactly as in \( U(N) \), entering only as a phase. Regardless of the details of this phase we will see it does not contribute to the high pressure equation of state of the adjoint gas. The other modes corresponding to the coordinates \( \{u_1 \cdots u_{N-1}\} \) have degenerate frequencies which are easily read off the diagonal form

\[ \omega_N = \sqrt{\frac{2Nq}{\alpha}} \]  

(4.11)

Knowing the normal modes of the eigenvalue problem we are now in a position to find the ground state solution of \((4.5)\) which will correspond to the dominant eigenvalue of the transfer matrix problem. A solution which satisfies the requirement of antisymmetry with respect to permutation of the coordinates is given by,

\[ \Delta \tilde{\psi} \sim \prod_{i<j} (x_i - x_j) \exp \left\{ i \frac{M}{\sqrt{N}} \sum_i x_i \right\} \exp \left\{ -\frac{1}{2} \frac{\omega_N}{N} \sum_{i<j} (x_i - x_j)^2 \right\} \]  

(4.12)

The parameter \( M \) is an integer associated with the center of mass coordinate and contributes a constant to the energy eigenvalue. Notice that the potential exponentiated, this is a direct consequence of the normal modes having degenerate energy. The above state has \( N^2 - 1 \) quanta of energy,

\[ E = (N^2 - 1) \frac{\omega_N}{2} + \frac{1}{2} M^2 \]  

(4.13)

so that the pressure is given, up to an irrelevant constant by,

\[ \frac{p}{\alpha} = \frac{N^2 q}{\alpha} - (N^2 - 1) \sqrt{\frac{Nq}{2\alpha}} \]  

(4.14)

Inverting this relation to find \( q(p) \) and using the definition \((3.8)\) we find, to leading order, the ratio of macroscopic to microscopic degrees of freedom

\[ \rho = 1 - \frac{(N^2 - 1)}{2} \sqrt{\frac{\alpha}{2Np}} + O\left(\frac{1}{p}\right) \]  

(4.15)

Consequently, in this high pressure limit the adjoint charges are the macroscopic degrees of freedom and, as in the \( k > 0 \) low pressure cases, can be interpreted as being free. The most striking feature of \((4.13)\) is the absence of any \( k \) dependence. This follows from the fact that the information of the center of mass coordinate appears only as a phase contributing the additive constant \( M^2 \) to the eigenvalue problem which is negligible in the limit of large pressure. In terms of physics the high pressure adjoint gas effectively screens all colour electric fields over large distances and so the fundamental colour electric fields associated with the different vacuum sectors are washed out by the adjoint degrees of freedom.
5 Discussion and Conclusions

For any finite $N$ the $SU(N)$ non-Abelian Coulomb gas does not admit phase transitions, however, in the limit of infinite $N$ it has been shown [11, 12, 13] that a phase transition develops for the trivial vacuum sector. In those works, it was found that the adjoint ‘quarks’ remain bound strictly pairwise, to form ‘mesons’, up to some critical pressure at which a first order phase transition occurs and the mesons disassociate into free quarks. If one takes the infinite $N$ limit in the trivial vacuum sector ($k = 0$), for small pressures the ratio of macroscopic to microscopic degrees of freedom is (see table 3),

$$\rho = \frac{1}{2} - \sqrt{\frac{\tilde{p}}{2}} + O(\tilde{p}) \quad (5.1)$$

which appears to contradict the prediction of the large $N$ calculations. Equation (5.1) predicts that the quarks are bound pairwise only at zero pressure, and can form bound structures with more components slightly away from zero pressure. This discrepancy can be explained by noting that in the infinite $N$ limit all pressures are naturally measured in units of $N^2$. Consequently the statement that quarks are bound pairwise from zero pressure up to some critical pressure is misleading. The correct statement is: for pressures of order $N^2$ but less than the critical pressure, the quarks are bound strictly pairwise. In this large $N$ limit nothing can be said about pressures of order unity, where our current computations are valid. We see then that the infinite $N$ computations of [11, 12, 13] are complementary to our own analysis. To state it another way, our computations cannot say anything about the affect that multiple (vacuum) sectors in the theory have on the phase transition at large $N$. However in the limit of infinite $N$ our calculations are perfectly valid in the low pressure region. Taking the infinite $N$ limit of the density for $N > 3$, $k > 2$ one finds (see table 3),

$$\rho = 1 - \left(\frac{1}{\theta(1-\theta)} + 1\right) \tilde{p} + \left(\frac{1 + 6\theta - 6\theta^2}{(1-\theta)^2\theta^2} - 1\right) \tilde{p}^2 + O\left(\tilde{p}^3\right) \quad (5.2)$$

where $k \equiv \theta N$. This demonstrates the explicit dependence on vacuum sector in the small pressure region.

In the high pressure limit the picture is much simpler. The infinite $N$ limit of the exact expression for the ratio of macroscopic to microscopic degrees of freedom $\rho$ is given by

$$\rho = 4\tilde{p} \left(1 + (1 + 8\tilde{p})^{-\frac{1}{2}}\right) \quad (5.3)$$

Here we have taken the pressure to be of order $N^2$ by setting $\tilde{p} = \tilde{p} N^2$ before taking the infinite $N$ limit. This ensures that the expression for the density is in the correct pressure scale. Taking the pressure to be of order $N^2$ allows us to probe near the phase transition region, but not the transition itself, since our high pressure approximation of the last section has neglected the topology of the problem which drives this transition. For instance we find

\[3\text{In this limit the product } \alpha N \text{ is kept at order unity, and we use } \tilde{p} \text{ to denote the reduced pressure } \frac{\tilde{p}}{\alpha N}\]
that at a pressure of $\hat{p} = \frac{3}{8}$ the density of the gas is exactly one half. This gives a lower bound on $\hat{p}_{crit}$, since the phase transition must occur before the gas confines. Also at very large pressures we have the prediction

$$\rho = 1 - \frac{1}{\sqrt{8\hat{p}}} + O\left(\frac{1}{\hat{p}}\right)$$ (5.4)

which agrees with the large $N$ computations.

Finally, some comments on the application of our results to the case of two dimensional $U(N)$ Yang-Mills theory coupled to adjoint matter. For adjoint representation fields with mass $m$ the high temperature effective potential can be calculated \[14, 15, 16\]

$$V(x_i - x_j) \sim \sum_{k=1}^{\infty} \frac{(-1)^k}{k} K_1(m\beta k) \cos[k(x_i - x_j)]$$ (5.5)

Here $K_1$ is a modified Bessel function. In the limit of large mass such that $m\beta$ is large it can be shown that the effective potential (5.4) reduces to the form of the transfer matrix problem (4.3) we have considered previously. For the case of finite mass, the full high temperature potential is difficult to deal with but it clearly has a symmetry under translations of the form $\{x_i\} \rightarrow \{x_i + a\}$. As we have seen previously, such a symmetry is synonymous with invariance of the model under center transformations (4.6). This strongly suggests that our results for the classical adjoint gas carry forward to massive adjoint two dimensional QCD at high temperature and we should expect multiple vacua to exist in that theory. Moreover, the physics in each sector of the theory will depend on the superselection parameter which labels the sectors.

In conclusion, from consideration of the thermodynamics of a system of static adjoint representation charges interacting via $SU(N)$ colour electric fields in $1 + 1$ dimension we have shown that the physics depends on the discrete vacuum index $k$. We have solved the model in two regions: low and high pressure. In the low pressure regime, which is equivalent to low particle density, the constant pressure equation of state was shown to have strong dependence on $k$. In the limit of high pressures/particle densities the dependence on $k$ was shown to become trivial, and does not enter into the equation of state for the adjoint gas. This is attributed to the screening nature of the high pressure limit, which washes out any global structure like a vacuum index.

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Appendix

Here we present the details of the perturbative calculation of the lowest lying eigenvalue of equation (3.1) for the adjoint gas. This can be accomplished via the familiar time independent perturbation theory of quantum mechanics. The problem we are faced with is

\[(\alpha C_2 - q\chi_{Ad})\Psi = -p\Psi \quad (A.1)\]

As noted previously the eigenvectors of the Casimir operator $C_2$ are the irreducible representations of the gauge group so a convenient basis in the limit as $q \to 0$ is $\Psi = \sum a_R \chi_R$. Also we have seen that the ground state of each sector of the theory is given by one of the $N$ fundamental representations, $\{ [k] \}$ of $SU(N)$. Consequently our objective is to calculate the perturbations to the pressure $p$ for small $q$ for $(A.1)$ about each ground state $[k]$. In order to carry out this calculation we need to know the matrix elements of the potential in the basis of the irreducible representations. This information follows from the Kronecker product of the adjoint representation with our chosen basis

\[\chi_{Ad} \chi_R = \chi_{Ad \otimes R} = \sum_T N_{Ad}^{T \ R \ T} \quad (A.2)\]

Here $N_{Ad \ R}^{T \ R \ T}$ is the fusion number enumerating the occurrence of the irreducible representation $T$ in the product of the adjoint representation $Ad$ and $R$. As in quantum mechanics we can easily calculate the corrections to $p$ up to third order in $q$ using the unperturbed basis of irreducible representations.

\[p_{[k]} = qN_{Ad \ [k]}^{[k]} + q^2 \sum_{R \neq [k]} \frac{\left( N_{Ad \ [k]}^{R \ [k]} \right)^2}{C_2([k]) - C_2(R)} \]

\[+ q^3 \left[ \sum_{R,S \neq [k]} \frac{N^{[k]}_{Ad \ S} N^{S}_{Ad \ R} N_{Ad \ [k]}^{R \ [k]}}{C_2([k]) - C_2(R)} \frac{C_2([k]) - C_2(S)}{C_2([k]) - C_2(R)} \right] + O(q^4) \quad (A.3)\]

It should be noted that we have left out a constant, sector dependent, background contribution to the pressure.

In order to give the details of the calculation we need a notation to label the irreducible representations. The one we will use here is given by the column variables $[m_1, m_2, \ldots]$ of the Young diagram associated with the representation. For example the antisymmetric combination of $k, N$ dimensional fundamental representations in $SU(N)$ - the ground state of the $k^{th}$ sector - corresponds to a Young diagram with a single column of $k$ boxes: $[k]$. Another example which appears in all calculations is that of the adjoint representation $Ad$ which in column variables is given by: $[N - 1, 1]$. In this notation the quadratic Casimir for
a representation \([m_1, m_2, \ldots]\) is given by

\[
C_2(R) = V \left( \begin{array}{cccc}
1 - \frac{1}{N} & 1 - \frac{2}{N} & 1 - \frac{3}{N} & \ldots & 1 - \frac{N-1}{N} \\
1 - \frac{2}{N} & 2(1 - \frac{2}{N}) & 2(1 - \frac{3}{N}) & \ldots & 2(1 - \frac{N-1}{N}) \\
1 - \frac{3}{N} & 2(1 - \frac{3}{N}) & 3(1 - \frac{3}{N}) & \ldots & 3(1 - \frac{N-1}{N}) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 - \frac{N-1}{N} & 2(1 - \frac{N-1}{N}) & 3(1 - \frac{N-1}{N}) & \ldots & (N-1)(1 - \frac{N-1}{N})
\end{array} \right) V^T - \frac{N}{12}(N^2 - 1)
\]

where

\[
V = (m_1 + 1, m_2 + 1, \ldots, m_{N-1} + 1)
\]

The remaining task is to compute the relevant fusion numbers. We begin by presenting the results of the calculations in Table 2. Here we record the fusion numbers \(N_{Ad R}^S = N_{Ad S}^R\) for the representations \(R\) and \(S\) of importance in the calculation (A.3) of the pressure. Each sub-table corresponds to a different background \(k\) for \(SU(N)\) since the details of the calculation of fusion numbers in general depends on \(k\) and \(N\). These results are only good for \(k \leq N/2\) where the remainder of the cases can be found via the symmetry of the eigenvalue problem under conjugation \(k \to N - k\). For completeness we present the details of the second table for \(k = 1\) and \(N = 3\). This is the familiar case of \(SU(3)\) and via common tensor or Young diagram methods the Kronecker products of the 8-dimensional adjoint representation (Ad) with the lowest lying representations can be calculated. In dimension notation we have

\[
\begin{align*}
8 \otimes 3 &= 3 \oplus 6 \oplus 15 \\
8 \otimes 6 &= 3 \oplus 6 \oplus 15 \oplus \cdots \\
8 \otimes 15 &= 3 \oplus 6 \oplus 2 \times 15 \oplus \cdots
\end{align*}
\]

In the last two products we have ignored higher representations which do not contribute to \(O(q^3)\) in (A.3). Converting to our column notation

\[
\begin{align*}
3 &\equiv [1] \\
6 &\equiv [2, 2] \\
8 &\equiv [2, 1] \\
15 &\equiv [2, 1, 1]
\end{align*}
\]

we have the results of the second sub-table in Table 2.
Table 2: Table of relevant fusion numbers $N_{Ad\ R}^S = N_{Ad\ S}^R$ for the calculation of the pressure of the adjoint gas. Note these results hold only for $k \leq N/2$ with the other cases given by the symmetry $k \rightarrow N - k.$
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