A review on development of rapid numerical quadrature in finite element method

V Sai Chandra and P V Jayakarthikeyan
Department of Mechanical Engineering, SRM Institute of Science & Technology, Kattankulathur, Tamil Nadu, India.
E-mail: jeyakarp@srmist.edu.in

Abstract. Typical problem areas of interest include structural analysis, heat transfer, fluid flow, mass transport, and electromagnetic potential. The analytical solution of these problems generally requires the solution to boundary value problems for partial differential equations. In particular, the focus will be on the Finite Element Method and its application to linear and nonlinear problems. Aim of the present proposal is to: initiate the investigation on the Universal Matrix method which is introduced presently for linear elastic and transient heat transfer problems when applied to non-linear elastic and inelastic problems. Multiple databases are searched for literature and limiting to last ten years. The keywords selected for the search were a combination of the nonlinear heat transient, inelastic problems. Universal Matrix Method applied to nonlinear heat transient & inelastic problems which a time is reducing than the existing methods like Gauss Quadrature methods, etc. Research on usage of various methods other than Gauss Quadrature method applied to linear problems extended to nonlinear heat transient & inelastic problems.

1. Introduction
Finite Element Method (FEM) is a powerful numerical tool generously applied to solve problems in Engineering and Mathematical Sciences. The method supposes that however complex the form of a solution may be over a domain, once it is viewed as an assemblage of several small sub-regions (elements), the solution forms over these individual sub-regions can be viewed as simplexes without loss of generality. The field variable function (distribution of displacement, temperature etc.) within an element is expressed in terms of nodal values and modest interpolation functions, often called shape functions. Also, the element shapes are usually simple (triangles, polygons and so on in two-dimensions (2D) and tetrahedrons, bricks etc. in 3D). But, when dealing with analysis domains of complex geometries, invariably, distorted shapes for elements would be needed at least to describe curved edges, surfaces and boundaries. Distortion of element shapes lead to distortion of shape functions affecting the performance of numerical quadrature. In practice, real world problems require both 2D and 3D modelling and analysis.

In heat transfer and stress analyses, conventional (1x1) GQ, if applicable, is often preferred, as it offers large savings in computation time. But then, it introduces the zero energy deformation modes; so called hourglass instability and is not desirable. In order to utilize (1x1) GQ and at the same time discount the effect of spurious hourglass issues different approaches are proposed, with the sole aim of re-establishing the rank of the element matrices. Primarily, by one of the following means viz. (i) Taylor series expansion (ii) alternate quadrature rules (iii) introducing stabilizing forces or (iv)
fluctuating the discrete gradient, the requisite rank of the element matrices is established for good accuracy, freedom from zero energy modes (hourglass issues) and better rates of convergence.

Wherever possible, closed form solution is recommended for reducing computation time required for numerical quadrature. If the element geometric transformation function (C0 continuous) is linear, closed form solution is possible as Jacobian becomes a matrix of constants. Conventionally, all element matrices are computed using Gauss numerical integration with optimum number of integration points. The element matrix computation time with any numerical integration scheme depends on the number and location of sampling points used and their weights, and also their type viz. if they are integers or floating-point variables.

Significantly, Weighted Richardson extrapolation is commonly used either to “control the size of the error formed in the solution by the chosen numerical method or to attempt to increase the accuracy of the solution calculated by the original numerical methods.” The one-point integration approach with hourglass control is appealing due to its computational effectiveness and the effortlessness of its implementation. It provides also accuracy of the final solution with appreciable rate of convergence in two- and three-dimensional finite element analyses.

2. Objective
Aim of the present proposal is to initiate the investigation on the Universal Matrix method which is introduced presently for linear elastic and transient heat transfer problems when applied to non-linear elastic and inelastic problems.

3. Methodology
For literature, multiple databases like IEEE, science direct were explored by limiting to last ten years from 2010 to 2019. The keywords selected for the search were a combination of Gauss Quadrature, Inelastic, Non-Linear Heat Transient, and Universal Matrix Method. The obtained papers are cut down to 40 by scrutinizing title, abstract, and conclusion and finalized with more appropriate and relevant work in papers.

4. Results and Discussion
All the forty papers are reviewed and divided in to four major categories based on work processed in the paper. The categories are trajectory of finite element methods with Galerkin approach, biomechanics applications using FEM, Software’s like ANSYS & ABACUS and linear heat transient. Galerkin approach gives mainly how FEM is done by having a multiple meshes with or without geometry consideration. Biomechanics applications gives FEA applied to knee, bone marrow, and etc. body parts design with material non linearity. Software’s like ANSYS & ABACUS gives the iterations compared to manual testing done in mat lab which reduces the time. Linear heat transient problems include even fluid mechanics, nonlinear problems which enhances the use of FEM in computational mechanics.

4.1. Galerkin approach
Generally for finite element analysis Galerkin method is frequently used for differential equation to convert into a discrete problem. This method gives weighted residuals for calculating global stiffness matrix for various problems, which simplifies the grid mesh and so on.

D.V. KUBAIR, B. BHANU-CHANDAR [1] gives the stress concentration factor(SCF) having material property inhomogeneity is numerically investigated, which is characterized by Power law index, length scale and modulus ratio. The finite element formulation activated by much functional form. Modulus ratio is negligible for SCF and due to circular effect stiffness increased away from hole.

K.M. LIEW, S. RAJENDRAN, J. WANG [2] analyzed the basics of US-QUAD8 broad class of Petrov-Galerkin formulation influenced by test and trail functions which develops quadratic plane triangular elements. The element is mainly read for typical static and free vibration problems having
undistorted meshes compared to classical isoperametric quadratic triangular elements extends to 6-noded elements. The element gives good results for free vibration analysis.

H. T. Rathod [3] Presented about axisymmetric loading of regular triangular family basically having stiffness matrices for cubic, linear and quadratic elements. Lagrange interpolation function occurs for stiffness coefficients occurs when (1) one side if triangle is parallel to axis of symmetry, (2) one or more nodes on axis of symmetry and may be combined to each other.

G. SWR and G. Voyiadjis [4] Here presented about geometrically nonlinear analysis of plates in which four noded quadrilateral is reduced to three noded triangular element based on Lagrangian formulation, where element tangent stiffness matrix is constant.

R. Levy, E. Gal [5] Paper gives a brief idea that due to load perturbing the linear equilibrium equations gives geometric nonlinear analysis by deriving geometric stiffness matrix. Results give pure elastic deformation in which rotations are to be taken as rigid and finite motion.

D. Sengupta and S. Dasgupta [6] Paper worked on formulation for 20-DOF triangular plane element condensed to obtain 18-DOF higher order element internal node by having an expression for stiffness matrix which is more applicable to microcomputers.

M. S. Islama, G. Sahaa [7] Investigated the stiffness matrix which is divided into two groups non-diagonal and diagonal of quadrilateral element in closed form using n x n. One term is computed from each group and the other 14 terms are computed by proposed Gaussian quadrature n x n rule.

P. S. Shiakolas, K. L. Lawrence [8] This paper gives reduced form of element stiffness expression using closed form finite element code which reduces time required as compared to Gaussian quadrature, were the matrix is material and geometry dependent.

W. A. Rocer, R. V. Nambiar [9] Explains tetrahedral elements of quadratic strain straight edge of equivalent loads by closed-form representation using symbolic algebra. The result element stiffness matrix is compared with use of Gauss quadrature.

Sara E, McCaslin [10] In this work closed-from stiffness matrix is increased to p-level 4 element. Resulted matrices were tested by floating-point operations. Results which are found for p-level 2 through 4, used to calculate upper triangular stiffness matrix.

C. E. Zhou, M. Asce [11] Presents requirement of nonlinear analysis reinforced concrete membrane structure for four node quadrilateral element. The developed material stiffness matrix is inserted to nonlinear finite element algorithm.

H.K. Ching, S.C. Yen [12] Obtained thermal and mechanical loads given for 2D FGM is proved by Petrov-Galerkin (MLPG method). Out of two considerations one is micro mechanics model and the other is continuum model in which numerical and analytical solutions are obtained.

4.2. Biomechanics applications using FEM
Several dental rehabilitations, knee problems and many other biomechanics problems are easily soled now a days with the help of FEM. To date, finite element analysis is a common tool to virtually analyse mechanical strength as well as stress shielding areas in mechanics and in biomechanics.

Tawakol A. Enab [13] had developed potential tibia tray material of TKRc to improve capability of stress distribution FGM caused by mechanical stress holding in the bone. Achieves full integration of implant to increase biocompatibility by use of FGM.

Tawakol, A. Enab [14] Studies behaviour of knee implant biomechanics FEA including FGM tibia tray and artificial knee. To find optimal gradation direction elastic modulus is changed horizontally and vertically.

4.3. Software’s (ANSYS & ABACUS) with FGM
By utilizing FGM the derivation of a finite element model for the static analysis of functionally graded (FG) plates integrated with a layer of piezoelectric fiber reinforced composite (PFRC) material and many other applications were done. With these composites and FGM extended to find with the help of many software’s like ANSYS, ABACUS, etc.
JEONG-HO KIM, G. H. PAULINO [15] Studied on FGM which have various material property like Poisson’s ratio (\(v\)), Young’s modulus (\(E\)). Here for Four nodes quadrilateral elements (Q4) may not provide accurate local stress. The framework described further continued to dynamic and thermal problems of graded materials.

QIN QING-HUA, ZHANG Qr [16] studied that a plate having novel element is given. For this a simple interpolation function is taken and degree of freedom for corner node should be unique as of others are not. For the element model results are simple ad reliable. The quadrant of square plate is taken by 2x2, 4x4 or 8x8 meshes which are distributed uniformly.

H. A. SLVPER [17] The finite element having linear thickness variation is more accurate for application compared to rectangular and triangular elements used to determine vibration and static deflections characteristics in transverse bending to find out natural frequencies of cantilever square plan form having triangular cross section. The results are mostly applicable for shells and plates of non-uniform thickness.

GAUTAM DASGUPTA [18] To compute element stiffness matrices using object oriented (C++) and FORTRAN with a side node for convex triangle and quadrilateral in \( \eta \) and \( \xi \) co-ordinates.

CH. ZHANG, M. CUI [19] Paper presents mainly linear elastic functionally graded materials and structures of three-dimensional isotropic and continuously nonhomogeneous electrostatic crack analysis by applying boundary domain integral equation. In the place of interior cells, nodes are required to develop mesh less scheme.

X.W. GAO, CH. ZHANG [20] Paper solves nonlinearly varying material parameters using boundary element method (BEM) for FGM. Results give field quantities with no gradients, so that for multi region problems a unified equation can be developed.

SEONG HYEOK SONG [21] Paper explains evaluation of Dynamic stress intensity factors (DSIF) by knowing fracture behavior of a cracked body using bench mark problems.

MOHAMED M. EL-AWAD [22] Paper adopts bilinear quadrilateral elements by comparing analytical integration with one-point quadrature. The solution can be done by two point quadrature estimated with square domain of one point quadrature.

LUCIA DELLA CROCE [23] analyzed behavior of rectangular plates using governing equations based on Reissner-Mindlin plate theory varies power law in study of volume fraction.

TAWAKOL A. ENAB [24] For analysis of biaxial and uniaxial loads root of an elliptic hole is unidirectional and are noted by using ANSYS and APDL. Shows SCF in finite element plate reduced by proper distribution of FGM.

ALI O. AYHAN [25] For given solutions of surface cracks in an edge cracks and finite-thickness FGM, computes stress intensity factors for three dimensional cracks based on fracture analysis program (FRAC3D).

ANDREW J. GOUPEE, SENTHIL S. VEL [26] A methodology is proposed on optimization of material distribution for 2-dimensional simulation using element free Galerkin method for thermo elasticity and quasi static heat conduction to minimize mass of a graded component.

G. N. PRA VEEN and J. N. REDDY [27] Investigates functionally graded ceramic-metal plates for transverse shear strains in von karman type. Response for deflection and stress under thermal and mechanical loading are compared with homogeneous isotropic plates.

J. D. RICHARDSON, G. H. PAULINO [28] Work involves homogeneous media which concludes primarily boundary integral analysis. Results show Green’s function implemented to FGM for getting standard boundary equation.

4.4. Transient heat and Fluid Mechanics
In the application of finite element techniques to especially fluid flow and heat transfer problems should be able to perform quick analysis of small problems using the finite element method. Heat transfer is the area of engineering science which describes the energy transport between material bodies due to a difference in temperature. The three different modes of heat transport are conduction, convection and radiation and so on.
L. S. D. MORLEY and B. C. MERRIFIELD [29] Details are given for cubically varying normal displacements to plate bending triangular finite element gives rational function to ten term cubic displaced surface. For circular and square plates problems element is efficient and derivatives are considered for rectilinear and true boundary.

A. KAVEHA, G.R. ROOSTAB [30] Presented a new graph model for partitioning unstructured finite element meshes for developing connectivity properties, which is used to balance number of interface nodes for desired values.

K. M. RAO and Y. U. MAHHWARA RAO [31] Prepared a computer program having an arbitrary number of layers for hybrid stress finite element method, which results compares between three-dimensional elastic solution and simply supported square laminated plate.

H.T. RATHOD, MD. SHAFIQUL ISLAM [32] Investigates using linear convex quadrilateral elements to generate determinant of Jacobian matrix. Some arrays are developed to apply analytical integration for element matrices developed.

D N VENKATESH, U SHRINIVASA [33] This article traces about generalization of 2-dimensional finite elements. To develop them various hard techniques are used for lower order finite elements using interpolation functions.

CH. ZHANG, M. CUI [21] Paper presents mainly linear elastic functionally graded materials and structures of three-dimensional isotropic and continuously nonhomogeneous electrostatic crack analysis by applying boundary domain integral equation. In the place of interior cells, nodes are required to develop mesh less scheme.

X.W. GAO, CH. ZHANG [34] In this paper 2-Dimensional crack analysis for boundary-domain integral equation is studied. Having radial integration method displacements are given by polynomials and basic functions in terms of global coordinates.

SUNIL C. JOSHI, H.W. NG [35] Investigation on nickel-zirconia for lowest uniform stress field is presented. Nickel substrate with coating is used in thermo-mechanical finite element analysis gives number of FGC distribution profiles.

ARTELLE ANTHOINE [36] For case of Quasi-periodic composite media 2-dimensional second-order homogenization should be considered for higher number of dimensions. Concept of Representative Volume Element (RVE) is extended to graded medium.

M. STEIGEMANN, M. SPECOVIUS-NEUGEBAUER [37] Using Griffith energy principle fracture mechanics crack propagation process to be calculated for energy release. The influence of inhomogeneity on crack path to change potential energy is derived.

ALOK SUTRADHAR, GLAUCIO H. PAULINO [38] For transient heat conduction a simple boundary element method is developed for specific heat and thermal conductivity into standard homogenous diffusion problem.

A.I. Abreu, A. Canelas, W.J. Mansur [39] Study of non-homogeneous media properties proves that they differ according to smooth functions. For time-dependent problems, integral equation is taken by adopting convolution quadrature method.

5. Conclusion
From the literature survey a clear idea about the basic principles of all the schemes can be explained readily by referring to the strain displacement matrix of quadrilateral elements. The calculation of strain displacement matrix (usually identified as B-matrix) is one expressed in terms of geometry (x and y coordinates) and field variable function (“u and v displacements in structural mechanic problems along x and y-axis respectively”). At the same time, closed form formulations using Universal matrices means that “it samples the values only of the geometric transformation function (defined by B2 matrix) and material functions (D matrix) at its origin (0,0) and the EL-ST-MAT is generated explicitly” in terms of derivatives of the original shape functions. Importantly, closed form formulations using “Universal matrices for a quadrilateral renders the geometric transformation linear” and the associated errors in the EL-ST-MAT coefficients as of order h-squared.
6. References
[1] Kubair D V and Bhanu-Chandar B 2010 Stress concentration factor due to a circular hole in functionally graded panels under uniaxial tension International Journal of Mechanical Sciences 50 732-742
[2] Liew K M, Rajendran S. and Wang J 2010 A quadratic plane triangular element immune to quadratic mesh distortions under quadratic displacement fields Computer methods in applied mechanics and engineering 195 1207-1223
[3] Rathod H.T 2010 Explicit stiffness matrices for axisymmetric triangular elements Computers & structures 30 1091-1100
[4] Shi G and Voyiadjis G 2010 Geometrically nonlinear analysis of plates by assumed strain element with explicit tangent stiffness matrix Computers & structures 41 757-763
[5] Levy, R. and Gal, E., 2011 Geometrically nonlinear three-noded flat triangular shell elements Computers & Structures 79 2349-2355
[6] Sengupta D and Dasgupta S 2010 Stiffness matrix for a quadratic strain triangle using area coordinates Computers & structures 36 963-970
[7] Islam M S, Saha G and Akter N 2011 Gauss-Legendre Numerical Integrations over a Quadrilateral Element in Closed Form Bangladesh Journal of Scientific and Industrial Research 46 399-405
[8] Shiakolas P S, Lawrence K L and Nambiar RV, 2011 Closed-form expressions for the linear and quadratic strain tetrahedral finite elements Computers & structures 50 743-747
[9] Shiakolas PS, Nambiar, R V, Lawrence, K L and Rogers W.A 2012 Closed-form stiffness matrices for the linear strain and quadratic strain tetrahedron finite elements Computers & structures 45 237-242
[10] McCaslin, S E, Shiakolas, P S, Dennis, B H and Lawrence K L 2012 Closed-form stiffness matrices for higher order tetrahedral finite elements Advances in Engineering Software 44 75-79
[11] Zhou C E and Vecchio FJ 2013 Closed-form stiffness matrix for the four-node quadrilateral element with a fully populated material stiffness Journal of engineering mechanics 132 1392-1395
[12] Ching H K and Yen S C 2015 Meshless local Petrov-Galerkin analysis for 2D functionally graded elastic solids under mechanical and thermal loads Composites Part B: Engineering 36 223-240
[13] Enab T A and Bondok N E 2013 Material selection in the design of the tibia tray component of cemented artificial knee using finite element method Materials & Design 44 454-460
[14] Enab TA 2012 A comparative study of the performance of metallic and FGM tibia tray components in total knee replacement joints Computational materials science 53 94-100
[15] Kim J H and Paulino GH 2012 Isoparametric graded finite elements for nonhomogeneous isotropic and orthotropic materials. J. Appl. Mech. 69 502-514
[16] Qing-Hua Q and Qing-Jie 2012 A triangular element of a plate with ten degrees of freedom. Computers & structures 45 795-797
[17] Slyper H.A 2010 Development of explicit stiffness and mass matrices for a triangular plate element International Journal of Solids and Structures 5 241-249
[18] Dasgupta G 2010 Stiffness matrices of isoparametric four-node finite elements by exact analytical integration Journal of Aerospace Engineering 21 45-50
[19] Zhang C H, Cui M, Wang J, Gao, X W, Sladek J and Sladek V 2011 3D crack analysis in functionally graded materials Engineering Fracture Mechanics 78 585-604
[20] Gao XW, Zhang C and Guo L 2012 Boundary-only element solutions of 2D and 3D nonlinear and nonhomogeneous elastic problems Engineering analysis with boundary element. 31 974-982
[21] Song S H and Paulino G.H 2013 Dynamic stress intensity factors for homogeneous and smoothly heterogeneous materials using the interaction integral method International Journal of Solids and Structure., 43 4830-4866
[22] El-Awad, M M 2011 Efficient Formation of Element Stiffness Matrices on Personal Computers, using Finite-Element Methods University Of Khartoum Engineering Journal
[23] Della Croce L and Venini P 2014 Finite elements for functionally graded Reissner–Mindlin plates Computer Methods in Applied Mechanics and Engineering 193 705-725
[24] Enab T A 2014 Stress concentration analysis in functionally graded plates with elliptic holes under biaxial loadings Ain Shams Engineering Journal 5 839-850
[25] Ayhan A O 2017 Stress intensity factors for three-dimensional cracks in functionally graded materials using enriched finite elements International Journal of Solids and Structures 44 8579-8599
[26] Goupee A J and Vel S S 2016 Two-dimensional optimization of material composition of functionally graded materials using meshless analyses and a genetic algorithm Computer methods in applied mechanics and engineering 195 5926-5948
[27] Praveen G N and Reddy J N 2010 Nonlinear transient thermoelastic analysis of functionally graded ceramic-metal plates International journal of solids and structures 35 4457-4476
[28] Gray L J, Kaplan T, Richardson D and Paulino G.H 2013 Green’s functions and boundary integral analysis for exponentially graded materials: heat conduction J. Appl. Mech 70 543-549
[29] Morley L.S.D and Merrifield B C 2015 On the conforming cubic triangular element for plate bending Computers & Structures 2 875-892
[30] Kaveh A and Roosta G R 2013 An algorithm for partitioning of finite element meshes Advances in Engineering Software 30 857-865
[31] Rao M and Rao Y M 2013 Computer program for the stiffness matrix of laminated plates using the hybrid-stress finite element Computers & structures 43 351-363
[32] Rathod H T and Islam M S 2010 Some pre-computed universal numeric arrays for linear convex quadrilateral finite elements Finite elements in analysis and design 38 113-136
[33] Venkatesh D N and Shrinivasa U 2012 Solid finite elements through three decades Sadhana. 19 271-287
[34] Gao X W, Zhang C, Sladek J and Sladek V 2018 Fracture analysis of functionally graded materials by a BEM Composites Science and Technology 68 1209-1215
[35] Joshi SC and Ng H W 2011 Optimizing functionally graded nickel–zirconia coating profiles for thermal stress relaxation Simulation Modelling Practice and Theory 19 586-598
[36] Anthoine A 2010 Second-order homogenisation of functionally graded materials International Journal of Solids and Structures 47 1477-1489
[37] Steigemann M, Specovius-Neugebauer M, Fulland M. and Richard H A 2010 Simulation of crack paths in functionally graded materials Engineering fracture mechanics 77 2145-2157
[38] Sutradhar A and Paulino G H 2014 The simple boundary element method for transient heat conduction in functionally graded materials Computer Methods in Applied Mechanics and Engineering 193 4511-4539
[39] Abreu A I, Canelas A and Mansur W J 2013 A CQM-based BEM for transient heat conduction problems in homogeneous materials and FGMs Applied Mathematical Modelling 37 776-792
[40] Pedersen P 2014 Analytical stiffness matrices for tetrahedral elements Computer methods in applied mechanics and engineering 196 261-278