UNIFORMITY AND SELF-NEGLECTING FUNCTIONS

N. H. BINGHAM and A. J. OSTASZEWSKI

Abstract. We relax the continuity assumption in Bloom’s uniform convergence theorem for Beurling slowly varying functions \( \varphi \). We assume that \( \varphi \) has the Darboux property, and obtain results for \( \varphi \) measurable or having the Baire property.

Keywords: Karamata slow variation, Beurling slow variation, Wiener’s Tauberian theorem, Beurling’s Tauberian theorem, uniform convergence theorem, Darboux property, Kestelman-Borwein-Ditor theorem, Baire’s category theorem, measurability, Baire property, affine group action.

Classification: 26A03; 33B99, 39B22, 34D05.

1. Definitions and motivation

The motivation for this paper may be traced back to two classic papers. First, in 1930, Karamata [Kar] introduced his theory of regular variation (in particular, of slow variation). Also in 1930 Karamata simplified the Hardy-Littlewood approach to Tauberian theorems; in 1931 he applied his theory of regular variation very successfully to Tauberian theory. For textbook accounts, see e.g. [BinGT] (BGT below), Ch. 4, Korevaar [Kor], IV. Secondly, in 1932 Wiener [Wie] transformed Tauberian theory by working with general kernels (rather than special kernels as Hardy and Littlewood had done); his method was based on Fourier transforms. There are two common forms for Wiener’s Tauberian theorem, one for the additive group of reals, one for the multiplicative group of positive reals. Both concern convolutions: additive convolutions for the first, with Fourier transforms and Haar measure being Lebesgue measure \( dx \); multiplicative convolutions for the second, with Mellin transforms and Haar measure \( dx/x \).

Theorem W (Wiener’s Tauberian theorem). For \( K \in L_1(\mathbb{R}) \) with the Fourier transform \( \hat{K} \) of \( K \) non-vanishing on the real line, and \( H \in L_\infty(\mathbb{R}) \):

if

\[
\int K(x - y)H(y)dy \rightarrow c \int K(y)dy \quad (x \rightarrow \infty),
\]

1
then for all $G \in L_1(\mathbb{R})$,

$$\int G(x-y)H(y)dy \to c \int G(y)dy \quad (x \to \infty).$$

The corresponding multiplicative form, with $\int_0^\infty K(x/y)H(y)dy/y$, is left to the reader. For textbook accounts, see e.g. Hardy [Har], XII, Widder [Wid], V, BGT Ch. 4, [Kor], II. Usually, in both regular variation (below) and Tauberian theory, the multiplicative form is preferred for applications, the additive form for proofs, a practice we follow here.

The classic summability methods of Cesàro and Abel fall easily into this framework. The next most important family is that of the Euler and Borel methods; these are tractable by Wiener methods, but are much less amenable to them; see e.g. [Har] VIII, IX, §12.15. Indeed, Tenenbaum [Ten], motivated by analytic number theory, gives an approach to Tauberian theory for the Borel method by Hardy-Littlewood rather than Wiener methods.

The Borel method is at the root of our motivation here. It is of great importance, in several areas: analytic continuation by power series ([Har] VIII, IX, Boas [Boa1] §5.5); analytic number theory [Ten]; probability theory ([Bin1] – [Bin4]).

In unpublished lectures, Beurling undertook the task of bringing the Borel method (and its numerous relatives) within the range of easy applicability of Wiener methods. His work was later published by Peterson [Pet] and Moh [Moh]. We state his result as Beurling’s Tauberian theorem below, but we must first turn to the two themes of our title.

A function $f : \mathbb{R} \to \mathbb{R}^+$ is regularly varying in Karamata’s sense if for some function $g$,

$$f(ux)/f(x) \to g(u) \quad (x \to \infty) \quad \forall \ u > 0.$$  \hfill (RV)

It turns out that to get a fruitful theory, one needs some regularity condition on $f$. Karamata himself used continuity. This was weakened to (Lebesgue) measurability by Korevaar et al. [KvAEdB] in 1949. Matuszewska [Mat] in 1962 showed that one could also use functions with the Baire property (briefly: Baire functions). Note that neither of the measurable and Baire cases contains the other. There are extensive and useful parallels between the measure and Baire (or category) cases – see e.g. Oxtoby [Oxt], [BinO3], [BinO4], [Ost3]; furthermore these have wide-ranging applications, see e.g. [BinO9], [Ost1], [Ost2] – particularly to the Effros Theorem, cf. [Ost4] and
§5.5.
Subject to a regularity condition ($f$ measurable or Baire, say), one has:
(i) the uniform convergence theorem (UCT): $(RV)$ holds uniformly on compact $u$-sets;
(ii) the characterization theorem: $g(u) = u^\rho$ for some $\rho$, called the index of regular variation.
The class of such $f$, those regularly varying with index $\rho$, is written $R_\rho$. One can reduce to the case $\rho = 0$, of the (Karamata) slowly varying functions, $R_0$. Then
\[ f(xu)/f(x) \to 1 \quad (x \to \infty) \quad \forall \ u \] (SV)
working multiplicatively, or
\[ h(x + u) - h(x) \to 0 \quad (x \to \infty) \quad \forall \ u \] (SV$_+$)
working additively; either way the convergence is uniform on compact $u$-sets (in the line for (SV), the half-line for (SV$_+$)).

Beurling observed in his lectures that the function $\sqrt{x}$ – known to be crucial to the Tauberian theory of the Borel method – has a property akin to Karamata’s slow variation. We say that $\varphi > 0$ is Beurling slowly varying, $\varphi \in BSV$, if $\varphi(x) = o(x)$ as $x \to \infty$ and
\[ \varphi(x + t\varphi(x))/\varphi(x) \to 1 \quad (x \to \infty) \quad \forall \ t. \] (BSV)
Using the additive notation $h := \log \varphi$ (whenever convenient):
\[ h(x + t\varphi(x)) - h(x) \to 0 \quad (x \to \infty) \quad \forall \ t. \] (BSV$_+$)
If (as in the UCT for Karamata slow variation) the convergence here is locally uniform in $t$, we say that $\varphi$ is self-neglecting, $\varphi \in SN$; we write (SN) for the corresponding strengthening of (BSV).

We may now state Beurling’s extension to Wiener’s Tauberian theorem (for background and further results, see §5.1 below and [Kor]).

**Theorem (Beurling’s Tauberian theorem).** If $\varphi \in BSV$, $K \in L_1(\mathbb{R})$ with $K$ non-zero on the real line, $H$ is bounded, and
\[ \int K\left(\frac{x-y}{\varphi(x)}\right)H(y)dy/\varphi(x) \to c \int K(y)dy \quad (x \to \infty), \]
then for all $G \in L_1(\mathbb{R})$,

$$
\int G\left(\frac{x-y}{\varphi(x)}\right)H(y)dy/\varphi(x) \to c \int G(y)dy \quad (x \to \infty).
$$

Notice that the arguments of $K$ and $G$ here involve both the additive group operation on the line and the multiplicative group operation on the half-line. Thus Beurling’s Tauberian theorem, although closely related to Wiener’s (which it contains, as the case $\varphi \equiv 1$), is structurally different from it. One may also see here the relevance of the affine group, already well used for regular variation (see e.g. BGT §8.5.1, [BinO2], and §3 below).

Analogously to Karamata’s UCT, the following result was proved by Bloom in 1976 [Blo]. A slightly extended and simplified version is in BGT, Th. 2.11.1.

**Theorem (Bloom’s theorem).** If $\varphi \in BSN$ with $\varphi$ continuous, then $\varphi \in SN$: (BSN) holds locally uniformly.

The question as to whether one can extend this to $\varphi$ measurable and/or Baire has been open ever since; see BGT §2.11, [Kor] IV.11 for textbook accounts. Our purpose here is to give some results in this direction. This paper is part of a series (by both authors, and by the second author, alone and in [MilO] with Harry I. Miller) on our new theory of topological regular variation; see e.g. [BinO1-11], [Ost1-4] and the references cited there.

One of the objects achieved was to find the common generalization of the measurable and Baire cases. This involves infinite combinatorics, in particular such results as the Kestelman-Borwein-Ditor theorem (KBD – see e.g. [MilO]), the category embedding theorem [BinO4] (quoted in §3 below) and shift-compactness [Ost3]. A by-product was the realization that, although the Baire case came much later than the measurable case, it is in fact the more important. One can often handle both cases together bitopologically, using the Euclidean topology for the Baire case and the density topology for the measurable case; see §3 below and also [BinO4]. Such measure-category duality only applies to qualitative measure theory (where all that counts is whether the measure of a set is zero or positive, not its numerical value). We thus seek to avoid quantitative measure-theoretic arguments; see §5.5.

Our methods of proof (as with our previous studies in this area) involve tools from infinite combinatorics, and replacement of quantitative measure theory by qualitative measure theory.
2. Extensions of Bloom’s Theorem: Monotone functions

We suggest that the reader cast his eye over the proof of Bloom’s theorem, in either [Blo] or BGT §2.11 – it is quite short. Like most proofs of the UCT for Karamata slow variation, it proceeds by contradiction, assuming that the desired uniformity fails, and working with two sequences, \( t_n \in [-T, T] \) and \( x_n \to \infty \), witnessing to its failure.

The next result, in which we assume \( \varphi \) monotone (\( \varphi \) increasing to infinity is the only case that requires proof) is quite simple. But it is worth stating explicitly, for three reasons:

1. It is a complement to Bloom’s theorem, and to the best of our knowledge the first new result in the area since 1976.
2. The case \( \varphi \) increasing is by far the most important one for applications. For, taking \( G \) the indicator function of an interval in Beurling’s Tauberian theorem, the conclusion there has the form of a moving average:

\[
\frac{1}{a \varphi(x)} \int_x^{x+a \varphi(x)} H(y)dy \to c \quad (x \to \infty) \quad \forall a > 0.
\]

Such moving averages are Riesz (typical) means and here \( \varphi \) increasing to \( \infty \) is natural in context. For a textbook account, see [ChaM]; for applications, in analysis and probability theory, see [Bin5], [BinG1], [BinG2], [BinT]. The prototypical case is \( \varphi(x) = x^\alpha \) \((0 < \alpha < 1)\); this corresponds to \( X \in L_{1/\alpha} \) for the probability law of \( X \).

3. Theorem 1 below is closely akin to results of de Haan on the Gumbel law \( \Lambda \) in extreme-value theory; see §5.8 below.

We offer three proofs (two here and a third after Theorem 2M) of the result, each is short and illuminating in its own way.

For the first, recall that if a sequence of monotone functions converges pointwise to a continuous limit, the convergence is uniform on compact sets. See e.g. Pólya and Szegő [PolS], Vol. 1, p.63, 225, Problems II 126, 127, Boas [Boa2], §17, p.104-5. (The proof is a simple compactness argument. The result is a complement to the better-known result of Dini, in which it is the convergence, rather than the functions, that is monotone; see e.g. [Rud], 7.13.)

**Theorem 1 (Monotone Beurling UCT).** If \( \varphi \in BSV \) is monotone, \( \varphi \in SN \): the convergence in \( (BSV) \) is locally uniform.
**First proof.** As in [Blo] or BGT §2.11, we proceed by contradiction. Pick $T > 0$, and assume the convergence is not uniform on $[-T, 0]$ (the case $[0, T]$ is similar). Then there exists $\varepsilon_0 > 0$, $t_n \in [-T, 0]$ and $x_n \to \infty$ such that

$$|\varphi(x_n + t_n \varphi(x_n))/\varphi(x_n) - 1| \geq \varepsilon_0 \quad \forall n.$$ 

Write

$$f_n(t) := \varphi(x_n + t \varphi(x_n))/\varphi(x_n) - 1.$$ 

Then $f_n$ is monotone, and tends pointwise to 0 by (BSV). So by the Pólya-Szegő result above, the convergence is uniform on compact sets. This contradicts $|f_n(t_n)| \geq \varepsilon_0$ for all $n$. □

The second proof is based on the following result, thematic for the approach followed in §4. We need some notation that will also be of use later. Below, $x > 0$ will be a continuous variable, or a sequence $x := \{x_n\}$ diverging to $+\infty$ (briefly, divergent sequence), according to context. We put

$$V^x_n(\varepsilon) := \{t \geq 0 : |\varphi(x_n + t \varphi(x_n))/\varphi(x_n) - 1| \leq \varepsilon\}, \quad H_k^x(\varepsilon) := \bigcap_{n \geq k} V^x_n(\varepsilon).$$

**Lemma 1.** For $\varphi > 0$ monotonic increasing and $\{x_n\}$ a divergent sequence, each set $V^x_n(\varepsilon)$, and so also each set $H_k^x(\varepsilon)$, is an interval containing 0.

**Proof of Lemma 1.** Since $x + s \varphi(x) > x$ for $s > 0$, one has $1 \leq \varphi(x + t \varphi(x))/\varphi(x)$. Also if $0 < s < t$, then, as $\varphi(x) > 0$, one has $x + s \varphi(x) < x + t \varphi(x)$. So if $t \in V^x_n(\varepsilon)$, then

$$1 \leq \varphi(x_n + s \varphi(x_n))/\varphi(x_n) \leq \varphi(x_n + t \varphi(x_n))/\varphi(x_n) \leq 1 + \varepsilon,$$

and so $s \in V^x_n(\varepsilon)$. The remaining assertions now follow, because an intersection of intervals containing 0 is an interval containing 0. □

**Second proof of Theorem 1.** Suppose otherwise; then there are $\varepsilon_0 > 0$ and sequences $x_n := x(n) \to \infty$ and $u_n \to u_0$ such that

$$|\varphi(x_n + u_n \varphi(x_n))/\varphi(x_n) - 1| \geq \varepsilon_0, \quad (\forall n \in \mathbb{N}).$$

Since $\varphi$ is Beurling slowly varying the increasing sets $H_k^x(\varepsilon_0)$ cover $\mathbb{R}_+$ and so, being increasing intervals (by Lemma 1), their interiors cover the compact
set $K := \{u_n : n = 0, 1, 2, \ldots\}$. So for some integer $k$ the set $H_k^x(\varepsilon_0)$ already covers $K$, and then so does $V_k^x(\varepsilon_0)$. But this implies that
\[
|\varphi(x_k + u_k\varphi(x_k))/\varphi(x_k) - 1| < \varepsilon_0,
\]
contradicting (all) at $n = k$. □

Remark. Of course the uniformity property of $\varphi$ is equivalent to the sets $H_k^x(\varepsilon)$ containing arbitrarily large intervals $[0, t]$ for large enough $k$ (for all divergent $\{x_n\}$).

3. Combinatorial preliminaries

We work in the affine group $\text{Aff}$ acting on $(\mathbb{R}, +)$ using the notation
\[
\gamma_n(t) = c_n t + z_n,
\]
where $c_n \to c_0 = c > 0$ and $z_n \to 0$ as $n \to \infty$. These are to be viewed as (self-) homeomorphisms of $\mathbb{R}$ under either $\mathcal{E}$, the Euclidean topology, or $\mathcal{D}$, the Density topology. Recall that the open sets of $\mathcal{D}$ are measurable subsets, all points of which are (Lebesgue) density points, and that (i) Baire sets under $\mathcal{D}$ are precisely the Lebesgue measurable sets, (ii) the nowhere dense sets of $\mathcal{D}$ are precisely the null sets, and (iii) Baire’s Theorem holds for $\mathcal{D}$. (See Kechris [Kech] 17.47.) Below we call a set negligible if, according to the topological context of $\mathcal{E}$ or $\mathcal{D}$, it is meagre/null. A property holds for ‘quasi all’ elements of a set if it holds for all but a negligible subset. We recall the following definition and Theorem from [BinO4], which we apply taking the space $X$ to be $\mathbb{R}$ with one of $\mathcal{E}$ or $\mathcal{D}$.

Definition. A sequence of homeomorphisms $h_n : X \to X$ satisfies the weak category convergence condition (wcc) if:

For any non-meagre open set $U \subseteq X$, there is a non-meagre open set $V \subseteq U$ such that for each $k \in N$,
\[
\bigcap_{n \geq k} V \setminus h_n^{-1}(V) \text{ is meagre.}
\]

Theorem CET (Category Embedding Theorem). Let $X$ be a topological space and $h_n : X \to X$ be homeomorphisms satisfying (wcc). Then for any Baire set $T$, for quasi-all $t \in T$ there is an infinite set $M_t \subseteq \mathbb{N}$ such that
\[
\{h_m(t) : m \in M_t\} \subseteq T.
\]
From here we deduce:

**Lemma 2 (Affine Two-sets Lemma).** For $c_n \to c > 0$ and $z_n \to 0$, if $cB \subseteq A$ for $A, B$ non-negligible (measurable/Baire), then for quasi all $b \in B$ there exists an infinite set $M = M_b \subseteq \mathbb{N}$ such that

$$\{\gamma_m(b) = c_m b + z_m : m \in M\} \subseteq A.$$ 

**Proof.** It is enough to prove the existence of one such point $b$, as the Generic Dichotomy Principle (for which see [BinO7, Th. 3.3]) applies here, because we may prove existence of such a $b$ in any non-negligible $G_\delta$-subset $B'$ of $B$, by replacing $B$ below with $B'$. (One checks that the set of $b$s with the desired property is Baire, and so its complement in $B$ cannot contain a non-negligible $G_\delta$.)

Writing $T := cB$ and $w_n = c_n c^{-1}$, so that $c_n = w_n c$ and $w_n \to 1$, put

$$h_n(t) := w_n t + z_n.$$ 

Then $h_n$ converges to the identity in the supremum metric, so (wcc) holds by Th. 6.2 of [BinO6] (First Verification Theorem) and so Theorem CET above applies for the Euclidean case; applicability in the measure case is established as Cor. 4.1 of [BinO2]. (This is the basis on which the affine group preserves negligibility.) So there are $t \in T$ and an infinite set of integers $M$ with

$$\{w_m t + z_m : m \in M\} \subseteq T.$$ 

But $t = cb$ for some $b \in B$ and so, as $w_m c = c_m$, one has

$$\{c_m b + z_m : m \in M\} \subseteq cB \subseteq A. \quad \square$$

4. Extensions to Bloom’s theorem: Darboux property

In this section we generalize Bloom’s Theorem and simplify his proof. Bloom uses continuity only through the intermediate value property – that if a (real-valued) function attains two values, it must attain all intermediate values. This is the *Darboux property*. It is much weaker than continuity – it does not imply measurability, nor the Baire property. For measurability, see the papers of Halperin [Halp1,2]; for the Baire property, see e.g. [PorWBW]
and also §5.3 below.

We use Lemma 2 above to prove Theorem 3B below, which implies Bloom’s Theorem, as continuous functions are Baire and have the Darboux property. We note a result of Kuratowski and Sierpiński [KurS] that for a function of Baire class 1 (for which see §5.3) the Darboux property is equivalent to its graph being connected; so Theorem 2 goes beyond the class of functions considered by Bloom.

We begin with some infinite combinatorics associated with a positive function \( \varphi \in BSV \).

**Definitions.** Say that \( \{u_n\} \) with limit \( u \) is a *witness sequence at* \( u \) (for non-uniformity in \( \varphi \)) if there are \( \varepsilon_0 > 0 \) and a divergent sequence \( x_n \) such that for \( h = \log \varphi \)

\[
|h(x_n + u_n \varphi(x_n)) - h(x_n)| > \varepsilon_0 \quad \forall \ n \in \mathbb{N}.
\]

Say that \( \{u_n\} \) with limit \( u \) is a *divergent witness sequence* if also

\[
h(x_n + u_n \varphi(x_n)) - h(x_n) \to \pm\infty.
\]

Thus a divergent witness sequence is a special type of witness sequence, but, as we show, it is these that characterize absence of uniformity in the class \( BSV \).

We begin with a lemma that yields simplifications later; it implies a Beurling analogue of the Bounded Equivalence Principle in the Karamata theory, first noted in [BinO1]. As it shifts attention to the origin, we call it the Shift Lemma. Below uniform near a point \( u \) means ‘uniformly on sequences converging to \( u \)’ and is equivalent to local uniformity at \( u \) (i.e. on compact neighbourhoods of \( u \)).

**Lemma 3 (Shift Lemma).** For any \( u \), convergence in \( (BSV_+) \) is uniform near \( t = 0 \) iff it is uniform near \( t = u \).

**Proof.** Take \( z_n \to 0 \). For any \( u \) write \( y_n := x_n + u \varphi(x_n) \), \( \gamma_n := \varphi(x_n)/\varphi(y_n) \) and \( w_n = \gamma_n z_n \). Then \( \gamma_n \to 1 \), so \( w_n \to 0 \). But

\[
h(x_n + (u + z_n) \varphi(x_n)) - h(x_n)
= [h(x_n + u \varphi(x_n) + z_n \varphi(x_n)) - h(x_n + u \varphi(x_n))] + [h(x_n + u \varphi(x_n)) - h(x_n)]
= [h(y_n + z_n \gamma_n \varphi(y_n)) - h(y_n)] + [h(y_n) - h(x_n)],
\]
i.e.
\[ h(x_n + (u + z_n)\varphi(x_n)) - h(x_n) = [h(y_n + w_n\varphi(y_n)) - h(y_n)] + [h(y_n) - h(x_n)]. \]

The result follows, since \( h(y_n) - h(x_n) \to 0 \), as \( \varphi \in BSV \) and \( y_n \to \infty \). \( \square \)

**Theorem 2B (Divergence Theorem – Baire version).** If \( \varphi \in BSV \) has the Baire property and \( u_n \) with limit \( u \) is a witness sequence, then \( u_n \) is a divergent witness sequence.

**Proof.** As \( u_n \) is a witness sequence, for some \( x_n \to \infty \) and \( \varepsilon_0 > 0 \) one has \( (\varepsilon_0) \), with \( h = \log \varphi \), as always. By the Shift Lemma (Lemma 3) we may assume that \( u = 0 \). So (as in the Proof of Lemma 2) we will write \( z_n \) for \( u_n \). If \( z_n \) is not a divergent witness sequence, then \( \{\varphi(x_n + z_n\varphi(x_n))/\varphi(x_n)\} \) contains a bounded subsequence and so a convergent sequence. W.l.o.g. we thus also have

\[
c_n := \varphi(x_n + z_n\varphi(x_n))/\varphi(x_n) \to c \in (0, \infty). \tag{lim}
\]

Write \( \gamma_n(s) := c_n s + z_n \) and \( y_n := x_n + z_n\varphi(x_n) \). Then \( y_n = x_n(1 + z_n\varphi(x_n)/x_n) \to \infty \) and

\[
|h(y_n) - h(x_n)| \geq \varepsilon_0. \tag{\varepsilon_0}
\]

Now take \( \eta = \varepsilon_0/3 \) and amend the notation of §2 to read

\[
V_n^x(\eta) := \{s \geq 0 : |h(x_n + s\varphi(x_n)) - h(x_n)| \leq \eta\}, \quad H^x_k(\eta) := \bigcap_{n \geq k} V_n^x(\eta).
\]

These are Baire sets, and

\[
\mathbb{R} = \bigcup_k H^x_k(\eta) = \bigcup_k H^y_k(\eta), \tag{cov}
\]

as \( \varphi \in BSV \). The increasing sequence of sets \( \{H^x_k(\eta)\} \) covers \( \mathbb{R} \). So for some \( k \) the set \( H^x_k(\eta) \) is non-negligible. Furthermore, as \( c > 0 \), the set \( c^{-1}H^x_k(\eta) \) is non-negligible and so, by (cov), for some \( l \) the set

\[
B := (c^{-1}H^x_k(\eta)) \cap H^y_l(\eta)
\]

is also non-negligible. Taking \( A := H^x_k(\eta) \), one has \( B \subseteq H^y_l(\eta) \) and \( cB \subseteq A \) with \( A, B \) non-negligible. Applying Lemma 2 to the maps \( \gamma_n(s) = c_n s + z_n \), there exist \( b \in B \) and an infinite set \( \mathcal{M} \) such that

\[
\{c_m b + z_m : m \in \mathcal{M}\} \subseteq A = H^x_k(\eta).
\]
That is, as \( B \subseteq H^y_i(\eta) \), there exist \( t \in H^y_i(\eta) \) and an infinite \( M_t \) such that
\[
\{ \gamma_m(t) = c_m t + z_m : m \in M_t \} \subseteq H^x_k(\eta).
\]
In particular, for this \( t \) and \( m \in M_t \) with \( m > k, l \) one has
\[
t \in V^y_m(\eta) \text{ and } \gamma_m(t) \in V^x_m(\eta).
\]
Fix such an \( m \). As \( \gamma_m(t) \in V^x_m(\eta) \),
\[
|h(x_m + \gamma_m(t) \varphi(x_m)) - h(x_m)| \leq \eta. \tag{*}
\]
But \( \gamma_m(t) = c_m t + z_m = z_m + t \varphi(y_m)/\varphi(x_m) \), so
\[
x_m + \gamma_m(t) \varphi(x_m) = x_m + z_m \varphi(x_m) + t \varphi(y_m) = y_m + t \varphi(y_m),
\]
‘absorbing’ the affine shift \( \gamma_m(t) \) into \( y \). So, by \( (*) \),
\[
|h(y_m + t \varphi(y_m)) - h(x_m)| \leq \eta.
\]
But \( t \in V^y_m(\eta) \), so
\[
|h(y_m + t \varphi(y_m)) - h(y_m)| \leq \eta.
\]
Using the triangle inequality and combining the last two inequalities, we have
\[
|h(y_m) - h(x_m)| \leq |h(y_m + t \varphi(y_m)) - h(y_m)| + |h(y_m + t \varphi(y_m)) - h(x_m)|
\]
\[
\leq 2 \eta < \varepsilon_0,
\]
contradicting \( (\varepsilon'_0) \).

**Theorem 2M (Divergence Theorem – Measure version).** If \( \varphi \in BSV \) is measurable and \( u_n \) with limit \( u \) is a witness sequence, then \( u_n \) is a divergent witness sequence.

**Proof.** The argument above applies, with the density topology \( D \) in place of the Euclidean topology \( E \) (the real line is still a Baire space, as remarked earlier).

As an immediate corollary we have:
Third Proof of Theorem 1. If not, then there exists a witness sequence \( u_n \) with limit \( u \). By Lemma 3, w.l.o.g. \( u > 0 \). Let \( v > u > w > 0 \). Since \( \varphi \in BSV \),

\[
\varphi(x_n + v\varphi(x_n))/\varphi(x_n) \to 1 \quad \text{and} \quad \varphi(x_n + w\varphi(x_n))/\varphi(x_n) \to 1,
\]

so there is \( N \) such that both \((1/2)\varphi(x_n) < \varphi(x_n + w\varphi(x_n)) \) and \( \varphi(x_n + v\varphi(x_n)) < 2\varphi(x_n) \) for all \( n > N \). By increasing \( N \) if necessary we may assume that \( w < u_n < v \) for \( n > N \). But then

\[
(1/2)\varphi(x_n) < \varphi(x_n + u_n\varphi(x_n)) < \varphi(x_n + v\varphi(x_n)) < 2\varphi(x_n)
\]

implies that \( 1/2 < \varphi(x_n + u_n\varphi(x_n))/\varphi(x_n) < 2 \), contradicting Theorem 2B/2M, as \( \varphi \) is Baire/measurable. \( \blacksquare \)

We now deduce

**Theorem 3B (Beurling-Darboux UCT: Baire version).** If \( \varphi \in BSV \)
has the Baire and Darboux properties, then \( \varphi \in SN \): (BSV) holds locally uniformly.

*Proof.* Suppose the conclusion of the theorem is false. Take \( h = \log \varphi \). Then there exists a witness sequence \( v_n \) with limit \( v \) and in particular for some \( x_n \to \infty \) and \( \varepsilon_0 > 0 \) one has the inequality \((\varepsilon_0)\) above modified so that \( v_n \) replaces \( u_n \).

We construct below a convergent sequence \( u_n \), with limit \( u \) say, such that

\[
c_n := h(x_n + u_n\varphi(x_n)) - h(x_n) \to c \in (-\infty, \infty), \quad \text{ (lim+)}
\]

and also the unmodified \((\varepsilon_0)\) holds. This will contradict Theorem 2B.

The proof here splits according as \( h(x_n + v_n\varphi(x_n)) - h(x_n) \) are bounded.

Case (i) The differences \( h(x_n + v_n\varphi(x_n)) - h(x_n) \) diverge to \( \pm \infty \). Here we appeal to the Darboux property to replace the sequence \( \{v_n\} \) with another sequence \( \{u_n\} \) for which the corresponding differences are convergent.

Now \( f_n(t) = h(x_n + t\varphi(x_n)) - h(x_n) \) has the Darboux property and \( f_n(0) = 0 \). Either \( f_n(v_n) \geq \varepsilon_0 \) and so there exists \( u_n \) between 0 and \( v_n \) with \( f_n(u_n) = \varepsilon_0 \), or \( -f_n(v_n) \geq \varepsilon_0 \), and so there exists \( u_n \) with \( -f_n(u_n) = \varepsilon_0 \). Either way

\[
|f_n(u_n)| = \varepsilon_0.
\]

W.l.o.g. \( \{u_n\} \) is convergent with limit \( u \) say, since \( \{v_n\} \) is so, and now (lim+) and \((\varepsilon_0)\) hold, the latter as in fact

\[
|h(x_n + u_n\varphi(x_n)) - h(x_n)| = \varepsilon_0.
\]
Case (ii) $h(x_n + v_n\varphi(x_n)) - h(x_n)$ are bounded. In this case we can get (lim+) by passing to a subsequence.

In either case we contradict Theorem 2B. ■

**Theorem 3M (Beurling-Darboux UCT - Measure version).** If $\varphi \in BSV$ is measurable and has the Darboux property, then $\varphi \in SN$: (BSV) holds locally uniformly.

*Proof.* The argument above applies, appealing this time to Theorem 2M. ■

**Remarks.** 1. The Darboux property in Theorems 3 above may be replaced with a weaker local property. It is enough to require that $\varphi$ be *locally range-dense* – i.e. that at each point $t$ there is a bounded open neighbourhood $I_t$ such that the range $\varphi[I_t]$ is dense in the interval $(\inf \varphi[I_t], \sup \varphi[I_t])$ – or be in the class $A_0$ of [BruC, §2], cf. also [BruCW].

2. The proofs of Theorems 3B and 3M begin as Bloom’s does, but only in the case (i) of the first step, and even then we appeal to the Darboux property rather than the much stronger assumption of continuity. Thereafter, we are able to use Theorem CET to base the rest of the proof on Baire’s category theorem. This enables us to handle Theorems 3B and 3M together, by qualitative measure theory; see the end of §1 and §5.5 below. By contrast, the proofs of Bloom’s theorem in [Blo] and BGT §2.11 use quantitative measure theory; see §5.5.

5. **Complements**

1. **Beurling’s Tauberian theorem: approximation form.** Recall (see e.g. [Kor] II.8) that Wiener’s Tauberian theorem is a consequence of Wiener’s approximation theorem: that for $f \in L_1(\mathbb{R})$ the following are equivalent:
   (i) linear combinations of translates of $f$ are dense in $L_1(\mathbb{R})$,
   (ii) the Fourier transform $\hat{f}$ of $f$ has no real zeros.

   The result is the key to Beurling’s Tauberian theorem ([Kor] IV Th. 11.1). Rate of convergence results (Tauberian remainder theorems) are also possible; see e.g. [FeiS], [Kor] VII.13).

   The theory extends to Banach algebras (indeed, played a major role in their development). In this connection ([Kor], V.4) we mention weighted versions of $L_1$: for *Beurling weights* $\omega$ – positive measurable functions on $\mathbb{R}$ with subadditive logarithms, $\omega(t + u) \leq \omega(t)\omega(u)$ – define $L_{\omega} = L_{1,\omega}$ to be
the set of $f$ with

$$||f|| = ||f||_{1,\omega} := \int_{\mathbb{R}} |f(t)|\omega(t)dt < \infty.$$ 

Then (Beurling’s approximation theorem): Wiener’s approximation theorem extends to the weighted case when $\omega$ satisfies the nonquasi-analyticity condition

$$\int_{\mathbb{R}} \frac{|\log \omega(t)|}{1+t^2} dt < \infty.$$ 

This condition has been extensively studied (see e.g. [Koo]) and is important in probability theory (work by Szegö – see e.g. [Bin6]).

2. Representation. As with Karamata slow variation, Beurling slow variation has a representation theorem: $\varphi \in SN$ iff $\varphi > 0$ and

$$\varphi(x) = c(x) \int_0^x e(u)du,$$

with $e(.) \to 0, c(.) \to c \in (0, \infty)$; as in BGT §2.11, [BinO5, Part II] we may take $e(.) \in C^\infty$ (so the integral is smooth), and then $c(.)$ has the same degree of regularity (Baire/measurable, descriptive character, etc.) as $\varphi(.)$. The treatment of BGT §2.11 goes over to the setting here without change. So too does the drawback that the representation on the right above does not necessarily imply that $\varphi$ is positive – this has to be assumed, or to be given from context.

3. Functions of Baire class 1. Recall that Baire class 1 functions – briefly, Baire-1 functions – are limits of sequences of continuous functions, and then the Baire hierarchy is defined by successive passages to the limit. See e.g. Bruckner and Leonard [BruL] §2, and the extensive bibliography given there, [Sol]. Compare [Kech] §24.B. The union of the classes in the Baire hierarchy gives the Borel functions; see e.g. [Nat] Ch. XV. Since Borel functions are (Lebesgue) measurable and Baire (have the Baire property)$^1$, the Baire-1 functions are both measurable and Baire (see e.g. [Kur] §11).

Lee, Tang and Zhao [LeeTZ] define a concept of weak separation involving neighbourhood assignments. They show that for real-valued functions on a Polish space this is equivalent to being of Baire class 1. Their result is greatly

$^1$In general one needs to distinguish between Borel and Baire measurability (cf. [Halm, §51] and [BinO5, §11]), but the two coincide for real analysis, our context here – see [Kech, 24.3].
generalized by Bouziad [Bou].

4. **Darboux functions of Baire class 1.** We recall that a Darboux function need be neither measurable nor (with the property of) Baire – hence the need to impose Darboux-Lebesgue or Darboux-Baire as double conditions in our results.

While Darboux functions in general may be badly behaved, Darboux functions of Baire class 1 are more tractable; recall the Kuratowski-Sierpiński theorem of §4. See e.g. [BruL, §5], [BruC, §6], [CedP], [EvH] for Darboux functions of Baire class 1, and Marcus [Mar], [GibN1], [GinN2] for literature and illuminating examples in the study of the Darboux property.

5. **Qualitative versus quantitative measure theory.** Bloom’s proof of his theorem used quantitative measure theory. Our proof replaces this by qualitative measure theory, thus allowing use of measure-category duality.

The application of Theorem CET above requires the verification of (wcc), and in the measure case this calls for just enough of the quantitative aspects to suffice – see [BinO6, §6]. The Baire and measure cases come together here via the coincidence between measure and metric for real intervals, cf. §5.11.

6. **Beyond the reals.** Theorem CET above was conceived to capture topologically the embedding properties enjoyed by non-negligible sets under translation as typified by the Steinhaus Theorem, or Sum-Set Theorem (that $A - A$ has 0 in its interior). Thus CET refers to the underlying group of homeomorphisms of a space. A more general setting involves the apparatus of group action on a topological space (see [MiO]). Here the central result is the Effros Theorem, which may be deduced from CET-like theorems (see [Ost3]).

The context in our results here is real analysis, as in BGT and [Blo]. But the natural setting is much more general. One such setting is the normed groups of [BinO6] (where one has the dichotomy: normed groups are either topological, or pathological); see [BinO5, Part I] for a development of slow variation in that context. Other possible settings include semitopological groups, paratopological groups, etc.; see e.g. [ElfN].

7. **Monotone rearrangements.** The theory of monotone rearrangements is considered in the last chapter of Hardy, Littlewood and Pólya [HarLP] Ch. X. For a function $f$ its distribution function $\{u : f(u) \leq x\}$ is non-decreasing and so has a non-decreasing inverse function, called the non-decreasing (briefly, increasing) rearrangement $f_\uparrow$ (thus $f$ and $f_\uparrow$ have the same distribution function). Such rearrangements are of great interest, and use, in a variety of contexts, including

(i) **probability** (Barlow [Bar], Marcus and Rosen [MarR] §6.4);
(ii) statistics: estimation under monotonicity constraints [JanW];
(iii) optimal transport: in transport problems with $f$ a strategy $f_\uparrow$ gives the optimal strategy [Vil];
(iv) analysis: [Graf, §1.4.1], [HorW], [BerLR].

As we have seen, the uniformity result for the monotone case is quite simple – much simpler than for the general case – and also, $\varphi$ monotone will typically be clear from context. In the general case we may aim to replace $\varphi$ by $\varphi_\uparrow$; for specific $\varphi$, replacing $e$ (in §5.2) by $e_+$ may well suffice.

8. The class $\Gamma$ (BGT §3.10) consist of functions $f: \mathbb{R} \to \mathbb{R}$, non-decreasing right-continuous which, for some measurable $g: \mathbb{R} \to (0, \infty)$, the auxiliary function of $f$,

$$f(x + u g(x))/f(x) \to e^u \quad x \to \infty \quad (\forall u \in \mathbb{R}).$$

It turns out that the convergence here is uniform on compact $u$-sets (from $f$ being monotone – as in Th. 1), and hence that $g$ is self-neglecting.

The class $\Gamma$ originates in extreme-value theory (EVT) in probability theory, in connection with de Haan’s work on the domain-of-attraction problem for the Gumbel (double-exponential) extremal law $\Lambda$ ($\Lambda(x) := \exp\{-e^{-x}\}$). See BGT §8.13, [BalE].

9. Beurling regular variation. In a sequel [BinO10] we explore the consequences of the Beurling regular variation property

$$f(x + u \varphi(x))/f(x) \to g(u) \quad x \to \infty \quad (\forall u \in \mathbb{R}). \quad (BRV)$$

We obtain, in particular, a characterization theorem

$$g(u) = e^{\rho u}$$

for some $\rho$. We also relax the condition above that $f$ be monotone.

Just as Beurling slow variation can be developed in contexts more generic than the reals (§5.6), so too can Beurling regular variation, a theme that we develop elsewhere [BinO11].

10. Continuous and sequential aspects. The reader will have noticed that Beurling slow variation is (like Karamata slow variation) a continuous-variable property, while the proofs here are by contradiction, and use sequences (bearing witness to the contradiction). This is a recurrent theme; see e.g. BGT §1.9, [BinO6].

11. The Weil topology and logarithms: lines and half-lines. Recall that in
connection with Theorem W in §1 we mentioned the additive form with Haar measure $dx$ on the line and the multiplicative form with Haar measure $dx/x$ on the half-line. The relevant background here is the Weil topology ([Halm, §62], [Wei], cf. [BinO6, Th. 6.10], see also [HewR]). The relevant metrics involve the logarithmic mapping ($\log x = \int_{[1,x]} du/u$ and $|a,b|_W = \log b - \log a$). This theme underlies the passage between $\varphi$ and $h = \log \varphi$, and is important in the sequel, [BinO10].

12. Questions. We close with two questions.

1. Does Bloom’s theorem extend to measurable/Baire functions – that is, can one omit the Darboux requirement? Does it even extend to Baire-1 functions?

2. Are the classes $BSV$, $SN$ closed under monotone rearrangement?

References

[BalE] A. A. Balkema and P. Embrechts, *High risk scenarios and extremes: A geometric approach*. European Math. Soc., Zürich, 2007.

[Bar] M. T. Barlow, Necessary and sufficient conditions for the continuity of local times of Levy processes. *Ann. Prob.* 16 (1988), 1389-1427.

[BerLR] H. Berestycki and T. Lachand-Robert, Some properties of monotone rearrangements with applications to elliptic equations in cylinders. *Math. Nachrichten* 266 (2004), 3-19.

[Bin1] N. H. Bingham, Tauberian theorems and the central limit theorem. *Ann. Probab.* 9 (1981), 221-231.

[Bin2] N. H. Bingham, On Euler and Borel summability. *J. London Math. Soc.* (2) 29 (1984), 141-146.

[Bin3] N. H. Bingham, On Valiron and circle convergence. *Math. Z.* 186 (1984), 273-286.

[Bin4] N. H. Bingham, Tauberian theorems for summability methods of random-walk type. *J. London Math. Soc.* (2) 30 (1984), 281-287.

[Bin5] N. H. Bingham, Moving averages. *Almost Everywhere Convergence I* (ed. G.A. Edgar & L. Sucheston) 131-144, Academic Press, 1989.

[Bin6] N. H. Bingham, Szegő’s theorem and its probabilistic descendants. *Probability Surveys* 9 (2012), 287-324.

[BinG1] N. H. Bingham and C.M. Goldie, On one-sided Tauberian conditions. *Analysis* 3 (1983), 159-188.

[BinG2] N. H. Bingham and C. M. Goldie, Riesz means and self-neglecting functions. *Math. Z.* 199 (1988), 443-454.
[BinGT] N. H. Bingham, C. M. Goldie and J. L. Teugels, Regular variation. 2nd ed., Cambridge University Press, 1989 (1st ed. 1987).

[BinO1] N. H. Bingham and A. J. Ostaszewski, Infinite combinatorics and the foundations of regular variation. J. Math. Anal. Appl. 360 (2009), 518-529.

[BinO2] N. H. Bingham and A. J. Ostaszewski, Infinite combinatorics in function spaces: category methods. Publ. Inst. Math. (Beograd) (N.S.) 86 (100) (2009), 55-73.

[BinO3] N. H. Bingham and A. J. Ostaszewski, Beyond Lebesgue and Baire: generic regular variation. Colloq. Mathematicum 116 (2009), 119-138.

[BinO4] N. H. Bingham and A. J. Ostaszewski, Beyond Lebesgue and Baire II: bitopology and measure-category duality. Colloq. Math. 121 (2010), 225-238.

[BinO5] N. H. Bingham and A. J. Ostaszewski, Topological regular variation. I: Slow variation; II: The fundamental theorems; III: Regular variation. Topology and its Applications 157 (2010), 1999-2013, 2014-2023, 2024-2037.

[BinO6] N. H. Bingham and A. J. Ostaszewski, Normed groups: Dichotomies and duality. Dissertationes Math. 472 (2010), 138p.

[BinO7] N. H. Bingham and A. J. Ostaszewski, Kingman, category and combinatorics. Probability and Mathematical Genetics (Sir John Kingman Festschrift, ed. N. H. Bingham and C. M. Goldie), 135-168, London Math. Soc. Lecture Notes in Mathematics 378, CUP, 2010.

[BinO8] N. H. Bingham and A. J. Ostaszewski, Dichotomy and infinite combinatorics: the theorems of Steinhaus and Ostrowski. Math. Proc. Cambridge Phil. Soc. 150 (2011), 1-22.

[BinO9] N. H. Bingham and A. J. Ostaszewski, Steinhaus theory and regular variation: De Bruijn and after. Indagationes Mathematicae (N. G. de Bruijn Memorial Issue), to appear.

[BinO10] N. H. Bingham and A. J. Ostaszewski, Uniformity and self-neglecting functions: II. Beurling regular variation and the class $\Gamma$, preprint (available at: http://www.maths.lse.ac.uk/Personal/adam/).

[BinO11] N. H. Bingham and A. J. Ostaszewski, Beurling regular variation in Banach algebras: asymptotic actions and cocycles, in preparation.

[BinT] N. H. Bingham and G. Tenenbaum, Riesz and Valiron means and fractional moments. Math. Proc. Cambridge Phil. Soc. 99 (1986), 143-149.

[Blo] S. Bloom, A characterization of B-slowly varying functions. Proc. Amer. Math. Soc. 54 (1976), 243-250.

[Boa1] R. P. Boas, Entire functions. Academic Press, 1954.
[Boa2] R. P. Boas, *A primer of real functions*. 3rd ed. Carus Math. Monographs 13, Math. Assoc. America, 1981.

[Bou] A. Bouziad, The point of continuity property, neighbourhood assignments and filter convergence. *Fund. Math.* 218 (2012), 225-242.

[BruC] A. M. Bruckner and J. G. Ceder, *Darboux continuity*. Jahresber. d. DMV 67 (1965), 93-117. (Available at: http://gdz.sub.uni-goettingen.de/dms/load/img/?PPN=PPN37721857X_0067&DMDID=DMDLOG_0011&LOGID=LOG_0011&PHYSID=PHYS_0101)

[BruCW] A. M. Bruckner, J. G. Ceder and M. Weiss, *On uniform limits of Darboux functions*. Coll. Math. 15 (1966), 65-77.

[BruL] A. M. Bruckner and J. L. Leonard, Derivatives. *Amer. Math. Monthly*, 37 (1966), 24-56.

[CedP] J. Ceder and T. Pearson, A survey of Darboux Baire 1 functions. *Real Analysis Exchange* 9 (1983-84), 179-194.

[ChaM] K. Chandrasekharan and S. Minakshisundaram, *Typical means*. Oxford University Press, 1952.

[ElfN] A. S. Elfard and P. Nickolas, On the topology of free paratopological groups. *Bull. London Math. Soc.* 44 (2012), 1103-1115.

[EvH] M. J. Evans and P. D. Humke, Revisiting a century-old characterization of Baire one Darboux functions. *Amer. Math. Monthly* 116 (2009), 451-455.

[FeiS] H. G. Feichtinger and H. J. Schmeisser, Weighted versions of Beurling’s slowly varying functions. *Math. Ann.* 275 (1986), 353-363.

[GibN1] R. G. Gibson and T. Natkaniec, Darboux like functions. *Real Analysis Exchange* 22.2 (1996/97), 492-453.

[GibN2] R. G. Gibson and T. Natkaniec, Darboux-like functions. Old problems and new results. *Real Analysis Exchange* 24.2 (1998/99), 487-496.

[Graf] L. Grafakos, *Classical Fourier Analysis*, 2nd ed., Graduate Texts in Mathematics 249, Springer, 2008.

[Halm] P. R. Halmos. *Measure theory*, (1955) Van Nostrand, 1950 (Grad. Texts in Math. 18, Springer, 1970).

[Halp1] I. Halperin, On the Darboux property. *Pacific J. Math.* 5 (1955), 703-705.

[Halp2] I. Halperin, Discontinuous functions with the Darboux property. *Canad. Math. Bull.* 2 (1959), 111-118.

[Har] G. H. Hardy, *Divergent series*. Oxford University Press, 1949.

[HarLP] G. H. Hardy, J. E. Littlewood and G. Pólya. *Inequalities*. 2nd ed.,
CUP, 1952 (1st. ed. 1934)

[HorW] A. Horsley and A. J. Wróbel, The Mackey continuity of the monotone rearrangement. *Proc. Amer. Math. Soc.* 97 (1986), 626-628.

[HewR] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis, I Structure of topological groups, integration theory, group representations.* Grundl. math. Wiss. 115, Springer, 1963.

[JanW] H. K. Jankowski and J. A. Wellner, Estimation of discrete monotone distributions. *Electronic J. Stat.* 3 (2009), 1567-1605.

[Kar] J. Karamata, Sur un mode de croissance régulière des fonctions. *Mathematica (Cluj)* 4 (1930), 38-53.

[Kech] A. S. Kechris, *Classical Descriptive Set Theory.* Grad. Texts in Math. 156, Springer, 1995.

[Koo] P. Koosis, *The logarithmic integral.* I 2nd ed. CUP, 1998 (1st ed. 1988), II, CUP, 1992.

[Kor] J. Korevaar, *Tauberian theorems: A century of development.* Grundl. math. Wiss. 329, Springer, 2004.

[KorvAEdB] J. Korevaar, T. van Aardenne-Ehrenfest and N. G. de Bruijn: A note on slowly oscillating functions. *Nieuw Arch. Wiskunde* 23 (1949), 77-86.

[Kur] C. Kuratowski, *Topologie.* Monografie Mat. 20 (4th. ed.), PWN Warszawa 1958 [K. Kuratowski, *Topology.* Translated by J. Jaworowski, Academic Press-PWN 1966].

[KurS] K. Kuratowski and W. Sierpiński, Sur les fonctions de classe I et les ensembles connexes ponctiformes. *Fund. Math.* 3, (1922) 303-313.

[LeeTZ] P.-Y. Lee, W.-K. Tang and D. Zhao, An equivalent definition of functions of first Baire class. *Proc. Amer. Math. Soc.* 129 (2001), 2273-2275.

[Mar] S. Marcus, Functions with the Darboux property and functions with connected graphs. *Math. Annalen.* 141 (1960), 311-317.

[MarR] M. B. Marcus and J. Rosen, *Markov processes, Gaussian processes and local time.* Cambridge Studies in Adv. Math., 100, CUP, 2006.

[Mat] W. Matuszewska, On a generalisation of regularly increasing functions. *Studia Math.* 24 (1962), 271-276.

[MilO] H. I. Miller and A. J. Ostaszewski, Group actions and shift-compactness. *J. Math. Anal. Appl.* 392 (2012), 23-39.

[Moh] T. T. Moh, On a general Tauberian theorem. *Proc. Amer. Math. Soc.* 36 (1972), 167-172.

[Nat] I. P. Natanson, *Theory of functions of a real variable.* Vol. I,II, Fred-
erick Ungar, 1960.

[Ost1] A. J. Ostaszewski, Analytic Baire spaces. Fund. Math. 217 (2012), 189-210.

[Ost2] A. J. Ostaszewski, Almost completeness and the Effros Theorem in normed groups. Topology Proceedings 41 (2013), 99-110.

[Ost3] A. J. Ostaszewski, Shift-compactness in almost analytic submetrizable Baire groups and spaces, invited survey article. Topology Proceedings 41 (2013), 123-151.

[Ost4] A. J. Ostaszewski, On the Effros Open Mapping Theorem – separable and non-separable, preprint (available at: www.maths.lse.ac.uk/Personal/adam).

[Oxt] J. C. Oxtoby, Measure and category. 2nd ed., Grad. Texts Math. 2, Springer, 1980.

[Pet] G. E. Petersen, Tauberian theorems for integrals II. J. London Math. Soc. 5 (1972), 182-190.

[PolS] G. Pólya and G. Szegő, Aufgaben und Lehrsätze aus der Analysis Vol. I. Grundl. math. Wiss. XIX, Springer, 1925.

[PorWBW] W. Poreda, E. Wagner-Bojakowska and W. Wilczyński, A category analogue of the density topology. Fund. Math. 125 (1985), 167-173.

[Rud] W. Rudin, Functional analysis. 2nd ed. McGraw-Hill, 1991 (1st ed. 1973).

[Sol] S. Solecki, Decomposing Borel sets and functions and the structure of Baire class 1 functions. J. Amer. Math. Soc. 11.3 (1998), 521-550.

[Ten] G. Tenenbaum, Sur le procédé de sommation de Borel et la répartition des facteurs premiers des entiers. Enseignement Math. 26 (1980), 225-245.

[Vil] C. Villani, Topics in optimal transportation. Grad. Studies in Math. 58, Amer. Math. Soc., 2003.

[Wei] A. Weil, L’intégration dans les groupes topologiques et ses applications. Actual. Sci. Ind. 689, Hermann, Paris, 1940 (republished, Princeton Univ. Press, 1941).

[Wid] D. V. Widder, The Laplace Transform. Princeton, 1941.

[Wie] N. Wiener, Tauberian theorems. Acta Math. 33 (1932), 1-100 (reprinted in N. Wiener, Generalized harmonic analysis and Tauberian theorems. MIT Press, 1964).