A Riemannian geometrical method to classify tearing instabilities in plasmas

by

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Abstract

Riemannian geometrical tools, such as Ricci collineations and Killing symmetries, so often used in Einstein's general theory of gravitation are here applied to plasma physics to build magnetic surfaces from Einstein plasma metrics used in tokamak devices. It is shown that the Killing symmetries are constrains the Einstein magnetic surfaces while the Killing vectors are built in terms of the displacement of the toroidal surface. The pressure is computed by applying these constraints to the pressure equations in tokamaks. A method, based on the sign of the only nontrivial constant Riemann curvature component, is suggested to classify tearing instability. Throughout the computations two approximations are considered: The first is the small toroidality and the other is the small displacement of the magnetic surfaces as Einstein spaces.

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I Introduction

The tools of the Riemann geometry [1] so often used in other important areas of physics, such as Einstein theory of gravitation, have also been used by Mikhailovskii [2] to investigate the tearing and other sort of instabilities in confined plasmas [3], where the Riemann metric tensor played a dynamical role interacting with the magnetic field through the magnetohydrodynamical equations (MHD). More recently Garcia de Andrade [4] has applied Riemann metric to investigate magnetic flux tubes in superconducting plasmas. Earlier Thiffault and Boozer [5] have investigated the Riemann geometry in the context of chaotic flows and fast dynamos. In this paper we use the tools of Riemannian geometry, also used in other branches of physics as general relativity [6], such as Killing symmetries, Riemann and Ricci [7] collineations, shall be applied here to generate magnetic nested surfaces in Tokamaks through the built of Einstein spaces obtained from Tokamak plasma metric. This work is motivated by the fact that the magnetic surfaces are severely constrained in tokamaks [3] and Killing symmetries are well applied every time we have symmetries in the problem as in solutions of Einstein equations of general relativity. Equilibrium of these surfaces or their instabilities are fundamental in the constructio of tokamaks and other plasma devices such as stellarators where torsion is also present. The magnetic surfaces are more easily obtained when symmetries are present. This is our main motivation to apply the special Riemann geometrical techniques of Killing symmetries and Ricci collineation to obtain magnetic surfaces formed by Einstein spaces. To simplify matters we shall consider two usual approximations from plasma physics [3] which are the small toroidality or inverse aspect ratio $\epsilon = \frac{a}{R} << 1$ where here $R$ represents the external radius of the torus and $a$ is its internal radius, and the Shafranov displacement $\Delta << 1$ as well. The pressure of the tokamak is obtained from the tokamak Shafranov shift equation. Constant pressure closed ergodic nested surfaces in magnetohydrostatics have also been shown by Schief [8] to be generated by solitons. This is another mathematical technique, distinct from ours, is another use of mathematical theory to generate of nested magnetic surfaces in plasmas. The paper is organised as follows: In section 2 we review the Riemannian technique of Killing vector and Ricci and Riemann colineations not usually familiar to the plasma physicists and solve the Ricci tensor components from the plasma metric by considering that
nested surfaces are formed by Einstein spaces, where the Ricci tensor is proportional to the metric, and solve the Ricci collineation equations to find out the Killing vectors establishing a geometrical method for the classification of tearing instabilities. Conclusions are presented in section 3.

II Ricci collineations from plasma metrics and tearing instabilities

dynamos. Let us now start by considering the plasma metric given by Zakharov and Shafranov [9] to investigate the evolution of equilibrium of toroidal plasmas. The components $g_{ik}$ ($i,k=1,2,3$) and $(a, \theta, z)$ as coordinates, of their plasma metric are

$$g_{11} = 1 - 2\Delta' \cos \theta + \Delta'^2 \quad (\text{II.1})$$

$$g_{22} = a^2 \quad (\text{II.2})$$

$$g_{33} = (R - \Delta + acos \theta)^2 \quad (\text{II.3})$$

$$g_{12} = a\Delta' \sin \theta \quad (\text{II.4})$$

where $z = a\sin \theta$ and the dash represents derivation with respect to $a$. In our approximation the last term in the expression (II.1) may be dropped. The magnetic surface equations are tori with circular cross-section and equations

$$r = R - \Delta(a) + acos \theta \quad (\text{II.5})$$

$$z = a\sin \theta \quad (\text{II.6})$$

Let us now compute the Riemann space of constant curvature represented by the Riemann tensor components

$$R_{ijkl} = \Lambda(g_{ik}g_{jl} - g_{il}g_{jk}) \quad (\text{II.7})$$

where $\Lambda$ ia constant which is called de Sitter cosmological constant. Contraction of expression (II.7) in two non-consecutive indices, otherwise the symmetry of the Riemann curvature tensor $R_{ijkl} = -R_{jikl} = R_{jilk}$ would make them vanish, yields the Einstein space Ricci relation

$$R_{ik} = 2\Lambda g_{ik} \quad (\text{II.8})$$
The Ricci collineations equations are given by

\[ [\partial_l R_{ik}]\eta^l + R_{il}\partial_k\eta^l + R_{kl}\partial_i\eta^l = 0 \]  

(II.9)

where \( \eta^l \) are the components of the Killing vector \( \vec{\eta} \) which defines the symmetries of the associated space, and \( \partial_l := \frac{\partial}{\partial x^l} \) are the components of the partial derivative operator. This equation is obtained from the more elegant definition in terms of the Lie derivative \( \mathcal{L}_\eta \) as

\[ \mathcal{L}_\eta R_{ik} = 0 \]  

(II.10)

In the next section we shall solve the Ricci collineation equations in terms of the plasma metric above.

### III Nested surfaces as Einstein spaces in plasmas

Let us now consider the application of the above plasma metric into the Ricci collineation equation, which yields

\[ [\partial_l R_{11}]\eta^l + 2R_{1l}\partial_1\eta^l = 0 \]  

(III.11)

\[ [\partial_l R_{22}]\eta^l + 2R_{2l}\partial_2\eta^l = 0 \]  

(III.12)

We shall consider now just two independent coordinates \( (x^1 = a, x^2 = \theta) \) since nested surfaces are bidimensional in the case of plasmas, Let us now compute the Riemann tensor components in the linear approximation

\[ R_{1212} = + \frac{\partial^2 g_{12}}{\partial x^1\partial x^2} - \frac{\partial^2 g_{11}}{\partial x^2\partial x^1} - \frac{\partial^2 g_{22}}{\partial x^1\partial x^1} \]  

(III.13)

Substitution of the plasma metric above into expression (III.13) yields the expression

\[ R_{1212} = [2 + (a\sin\theta + 2\cos\theta)] \]  

(III.14)

It is easy to show that the components \( R_{1313} \) and \( R_{2323} \) both vanishes within our approximations. At this point we consider that \( \theta \) is so small that \( \sin\theta \) vanishes and \( \cos\theta = 1 \) this simplifies extremely our metric and turns it into a diagonal metric where \( g_{12} = 0 \) and \( g^{bb} = (g_{bb})^{-1} \) \( (b = 1, 2) \) and this allows us to compute the components of the Ricci tensor.
from the Riemann component. But before that let us compute the use the condition that the nested surface is an Einstein space to compute the Riemann component again

\[ R_{1212} = \Lambda a^2 [1 - 2 \Delta' \cos \theta] \]  

(III.15)

Since both expressions for the Riemann component \( R_{1212} \) must coincide, equating expressions (III.15) and (III.14) yields an expression for the derivative of the Shafranov shift \( \Delta \) as

\[ \Delta' = [1 + \frac{\Lambda a^2}{2}] \]  

(III.16)

Integration of this expression yields the value of the shift in terms of the radius \( a \) as

\[ \Delta = [a + \frac{\Lambda a^3}{6}] \]  

(III.17)

Since the radius \( a \) is assumed to be small we may neglect terms of the order \( a^3 \) which yields

\[ \Delta = a \]  

(III.18)

which satisfies the well-known boundary condition \( \Delta(0) = 0 \). From these expressions one may also compute \( \Delta'' = \Lambda a \). An important result in plasma physics is that tearing instabilities coming from ion or electron currents possess the shift condition \( \Delta' < 0 \). This condition would be clearly violated from expression (III.16) unless the \( \Lambda \) curvature constant would be negative and in modulus \( |\frac{\Lambda a^2}{2}| > 1 \). Note that this situation is very similar to the condition of favorable or unfavorable curvature for the instabilities in plasmas [3]. The main difference is that here we are referring to Riemann curvature and not to Frenet curvature of the magnetic lines in plasmas. This suggests another method to classify geometrically tearing instabilities. Actually, since has been shown [11] recently that the Riemann tensor in plasmas can be expressed in terms of the Frenet curvature both methods seems to be equivalent. Now let us compute the Ricci components \( R_{11} \) and \( R_{22} \) from the component \( R_{1212} \) by tensor contraction with metric components \( g^{11} \) and \( g^{22} \), which results in the expressions

\[ R_{11} = -\frac{2}{a^2} [1 + \Delta'] \]  

(III.19)

and

\[ R_{22} = [2 + 6 \Delta'] \]  

(III.20)
which in turn yields the expressions

\[ \partial_1 R_{11} = -\frac{4}{a^3}[1 + \Delta'] - \frac{2}{a^2}[\Delta''] = \frac{8}{a^3} \]  
(III.21)

and

\[ \partial_2 R_{11} = \frac{-\Delta'}{a} = -\frac{(1 + \frac{\Delta a^2}{2})}{a} \]  
(III.22)

\[ \partial_1 R_{22} = \Delta' \]  
(III.23)

\[ \partial_2 R_{22} = a\Delta' \]  
(III.24)

Substitution of these derivatives of the Ricci tensor components into the Ricci collineations equations one obtains the following set of PDE equations

\[ [a^{-1} + \frac{\Lambda}{4}\partial_1 \eta^1 - \eta^1 \frac{1}{a^3} = 0 \]  
(III.25)

which yields

\[ \eta^\theta = \exp[\frac{1}{a}] \]  
(III.26)

\[ \frac{16}{a} \partial_2 \eta^2 + \eta^2 + 6\Lambda a \exp[\frac{1}{a}] = 0 \]  
(III.27)

which by considering the gauge \( \partial_2 \eta^2 = 0 \) one obtains the constraint

\[ \eta^\theta = -6\Lambda \exp[\frac{1}{a}] \]  
(III.28)

for the Killing vector field poloidal component. The relation

\[ \frac{\eta^\phi}{\eta^\theta} = -\frac{1}{6}\Lambda^{-1} \]  
(III.29)

allows us finally to write down the Killing vector as

\[ \vec{\eta} = -\frac{1}{6}\Lambda^{-1}[1, -6\Lambda]\eta^\theta \]  
(III.30)

which depends again upon the curvature scalar \( \Lambda \) which is central in the classification of tearing instabilities.
IV Conclusions

In conclusion, we have investigated a method of classification and identification of tearing instability, allowing for example to distinguish between tearing instabilities that come from ions and electron currents or not, based on the Riemann curvature constant submanifolds as nested surfaces in Einstein spaces. The Killing symmetries are shown also to be very useful in the classification of plasma metrics in the same way they were useful in classifying general relativistic solutions of Einstein's gravitational equations in four-dimensional spacetime [6]. Other interesting examples of the utility of this method is the Ricci collineations investigations of the twisted magnetic flux tubes and the Arnolds metric for the fast dynamo [10, 11, 12].

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References

[1] E. Cartan, Riemannian geometry in an orthonormal Frame, (2001) Princeton University Press.

[2] A. Mikhailovskii, Instabilities in a Confined Plasma, (1998) IOP.

[3] R. White, The theory of toroidally confined Plasmas, revised second edition (2006) Imperial College Press.

[4] L. C. Garcia de Andrade, Curvature and Torsion effects on carrying currents twisted solar loops, (2006) Phys of Plasmas nov issue.

[5] J. Thiffault and A.H. Boozer, The Onset of Dissipation in the Kinematic Dynamo, Los Alamos arXiv: nlin.CD/0209042v1. L.C. Garcia de Andrade, Physics of Plasmas 13, 022309 (2006).

[6] R. Penrose and W. Rindler, Spinors and spacetime vol1, Oxford University Press (1984).

[7] G. Ricci, Tensor Analysis, Boston.

[8] W.K. Schief, J. Plasma Physics 69 (2003)465.

[9] L.E. Zakharov and V. D. Shafranov, Evolution of Equilibrium Toroidal Plasmas in Plasma Physics, MIR physics series, Moscow (1981).

[10] V. Arnold and B. Khesin, Topological Methods in Hydrodynamics, Applied Mathematics Sciences 125 (1991). Imperial College Press.

[11] S. Childress and A. Gilbert, Stretch, Twist and Fold: The Fast Dynamo (1996)(Springer).

[12] B. Khesin, Topology Bounds Energy, in reference 1.