Reliability sensitivity analysis using axis orthogonal importance Latin hypercube sampling method

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Abstract
In this article, a combined use of Latin hypercube sampling and axis orthogonal importance sampling, as an efficient and applicable tool for reliability analysis with limited number of samples, is explored for sensitivity estimation of the failure probability with respect to the distribution parameters of basic random variables, which is equivalently solved by reliability sensitivity analysis of a series of hyperplanes through each sampling point parallel to the tangent hyperplane of limit state surface around the design point. The analytical expressions of these hyperplanes are given, and the formulas for reliability sensitivity estimators and variances with the samples are derived according to the first-order reliability theory and difference method when non-normal random variables are involved and not involved, respectively. A procedure is established for the reliability sensitivity analysis with two versions: (1) axis orthogonal Latin hypercube importance sampling and (2) axis orthogonal quasi-random importance sampling with the Halton sequence. Four numerical examples are presented. The results are discussed and demonstrate that the proposed procedure is more efficient than the one based on the Latin hypercube sampling and the direct Monte Carlo technique with an acceptable accuracy in sensitivity estimation of the failure probability.

Keywords
Latin hypercube sampling, Halton, axis orthogonal importance sampling, spurious correlation reduction, parameter sensitivity, structural reliability

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Introduction
Reliability analysis and sensitivity analysis should be an important part of any analysis of engineering structures, with (1) reliability analysis providing the probabilities of failure or of unacceptable structural performance due to those uncontrolled random factors and (2) as an importance measure, sensitivity analysis identifying the contributions of random analysis inputs to those probabilities. In the framework of probability-based structural reliability analysis, an individual random input may be defined probabilistically by the probability density function (PDF) or cumulative distribution function (CDF) parameterized in terms of its probability characteristics such as mean (μ) and variance (σ), and if the correlation between inputs (e.g. x₁ and x₂) is considered, correlation coefficient (ρᵢⱼ), and

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Various reliability sensitivity analysis procedures have been proposed for use, including the most probability point (MPP)-based method,\(^1,^2\) differential analysis,\(^3,^4\) and sampling-based techniques.\(^5,^6\) MPP-based methods are simple, which gives the directions of the MPP in the standard normal space and provides sensitivities with respect to the MPP as its side product. However, it depends on the linearization of limit state function, and the nonlinear performance function will result in a poor precision in the reliability and the reliability sensitivity estimation.\(^2,^5\) Differential analysis is an indirect mathematical technique, whose precision is determined by the reliability analysis method itself. Sampling-based techniques are most adequate and employed for use in conjunction with Monte Carlo–type simulation methods including the standard Monte Carlo (SMC) simulation, importance sampling, quasi-random sampling\(^6\) such as Latin hypercube sampling (LHS), random sampling with the Sobol sequence, Niederreiter sequence, and Halton sequence, and so on. The SMC simulation is often criticized for its time consumption and expensive computation cost especially calculating those small failure probabilities. Wu and Mohanty\(^7\) suggested using a small number of random samples to compute sensitivities by classifying significant and insignificant random variables via acceptance limits derived from the test of hypothesis, without an approximation needed in either the form of the performance functions or the type of continuous distribution functions representing input variables. Lu et al.\(^8\) proposed analyzing reliability sensitivity of a structural system based on the point estimation method for evaluating the first few moments of the system performance function. In the reliability-based optimization design, the dynamic Kriging method is used for surrogate models to reduce the computation cost and a stochastic sensitivity analysis is introduced to compute the sensitivities of probabilistic constraints with respect to independent or correlated random variables in the study of Lee et al.\(^9\) A virtual support vector machine, as a classification method, is employed to evaluate a large number of Monte Carlo simulation (MCS) probabilistic sensitivity analyses for the same purpose in the study of Song et al.\(^10\) Lin et al.\(^11\) used the derivatives of response function incorporating with kernel density estimation for stochastic design sensitivity analysis. A local sampling method with variable radius is proposed to construct the Kriging model accurately and efficiently in the region of significance.\(^12\) Besides, other importance measures are also discussed in the literature. Wu et al.\(^13\) introduced an importance measure based on the component maintenance priority used to select components for preventive maintenance. An importance measure for both coherent and non-coherent systems is suggested defined on the change in mean time to failure caused by the failure (success) of a component in the study of Borgonovo et al.\(^14\). LHS is very popular for use with computationally demanding models but only slightly more efficient than the SMC for evaluating small probabilities.\(^15\) To solve this, Olsson et al.\(^16\) investigated a combined use of LHS and axis orthogonal importance sampling for reliability analysis, and the method performs better than simple importance sampling using SMC and proves to be an efficient importance sampling strategy. However, it seems that the combined use of LHS and axis orthogonal importance sampling for reliability sensitivity analysis has not yet been thoroughly evaluated.

In this article, we further focus on the numerical algorithm employing axis orthogonal importance Latin hypercube sampling (AOILHS) method to analyze the reliability sensitivities. As we know, the reliability sensitivities of the sampling-based method are evaluated as an expectation of the partial derivative of PDF with respect to the probability characteristics over the failure region. To verify the efficiency and accuracy of the proposed algorithm, it is compared to the other sampling-based methods in this article including the SMC method, axis orthogonal importance correlation Latin hypercube sampling (AOICLHS) method, and axis orthogonal importance sampling with the Halton sequence (AOIHalton) and illustrated by a number of numerical examples.

This article is structured as follows. The reliability analysis of AOILHS/AOICLHS is depicted in section “Basic idea of AOILHS/AOICLHS.” The reliability sensitivity estimator and its variance analysis based on AOILHS/AOICLHS are given in section “Reliability sensitivity estimation and variance analysis based on AOILHS/AOICLHS.” Numerical examples are shown and the efficiency and accuracy of the proposed algorithm are demonstrated in section “Numerical examples.” Section “Conclusion” concludes with a summary of the presented procedure.

**Basic idea of AOILHS/AOICLHS**

**Importance sampling**

Importance sampling proves to be an efficient variance reduction technique employed successfully in quite a few engineering fields. In the case of reliability analysis, the failure probability is calculated as the sum of a ratio \(\phi_0/\phi_i\) on all the realizations if a failure occurs.
\[ P_f = \frac{1}{N} \sum_{i=1}^{N} I_F(g(v_i)) \frac{\phi_\theta(v_i)}{\phi_\phi(v_i)} \]  

where \( v_i \) (\( i = 1, 2, \ldots, N \)) is a realization of the samples generated in terms of the importance sampling density function \( \phi_\theta, \phi_\phi \) is the original joint PDF of the random inputs, \( P_f \) is the estimation of the failure probability, and \( I_F(\cdot) \) is a indicator function which equals to one for the samples within the failure domain and zero elsewhere. The key for the successful implementation of importance sampling is to select an appropriate importance sampling density function \( \phi_\theta \). Many efficient guidelines for choice of importance sampling density function can be found in the literature. A simple and widely employed technique is used to define the sampling density function as multivariate normal distribution with standard deviations centered from the origin to the MPP on the limit state surface, and at the same time, the original joint PDF of the random inputs should be transformed into standard normal space.\(^{19}\) Rosenbluth’s transformations\(^{20}\) are available for the latter purpose. For the former MPP, it is evaluated by some gradient-based search algorithm that works in standard normal space such as the gradient projection method, Hasofer–Lind method,\(^{21}\) or importance sampling method, moving the sampling center according to information from a previous sampling without transformation to standard normal space or the limit state surface to be differentiable such as Markov chain Monte Carlo method.\(^{22,23}\) In particular, the axis orthogonal importance sampling method and even the AOILHS method exhibit more efficiency compared to the other importance sampling methods, whose sample is established on the tangent hyperplane of the limit state surface and centered at the MPP. The sampling dimension is thus reduced by one.\(^{24}\)

**LHS**

LHS is popularly employed for its efficient stratification properties with relatively lower computation while preserving the desirable probabilistic features of simple random sampling with a relatively small sample size.

A sample of size \( N \) in LHS is generated in the following manner from the target marginal distributions \( D_1, D_2, \ldots, D_K \) associated with the elements of \( x(x_1, x_2, \ldots, x_K) \). The sampling space of each \( x_j \) is exhaustively divided into \( N \) disjoint intervals of equal probability and one realization of \( x_j \) is randomly selected from one of these intervals. Thus, the sampling result can be represented by an \( N \times K \) matrix \( S \)

\[ S = \Theta^{-1} \left[ \frac{P + R}{N} \right] \]  

where \( P \) is an \( N \times K \) matrix, in which each column is a random permutation; \( R \) is an \( N \times K \) matrix of independent random numbers from the uniform \((0, 1)\) distribution; and \( \Theta^{-1} \) represents the inverse of the CDF from the target marginal distributions for each variable. Each row of \( S \) is a realization of \( x \).

Unfortunately, there is a risk that some spurious correlation will appear in the primary generated sampling matrix \( S \), usually introducing some bias in the following computation. It has been reported that such a spurious correlation can be reduced by several iterative modifications in the permutation matrix \( P \) or being designed as the Audze–Eglius uniform Latin hypercube design of experiments.\(^{25}\) The former method is employed in this article due to its simplicity, in which the covariance of sampling matrix \( S \) is first estimated by a mapping matrix \( Y \) from the permutation matrix \( P \) with the same shape. Each element of the mapping matrix \( Y \) is calculated by

\[ y_{ij} = \Phi^{-1} \left( \frac{p_{ij}}{N} \right) \]  

where \( \Phi^{-1} \) is the inverse of CDF from the standard normal distribution, and \( p_{ij} \) is the elements of \( P \). A new matrix \( Y^* \) is used to determine the elements of a new permutation matrix \( P^* \), which is computed as

\[ Y^* = Y(L^{-1})^T \]  

if the desired covariance equals to the identity, otherwise

\[ Y^* = Y(L^{-1})^T L^T \]  

where \( L \) is the lower triangular, and \( L_Y \) is the Cholesky decomposition defined as

\[ L_Y L_Y^T = \text{cov}(Y) \]  

\( L \) is the Cholesky decomposition of target covariance. The ranks of the elements in the columns of \( Y^* \) become the elements in the columns of the new permutation matrix \( P^* \). Several iterative modifications of the permutation matrix \( P(P^*) \) will improve the correlation especially when non-Gaussian variables are involved. As Melchers\(^{18}\) did, the sampling plan including the correlation-reduction procedure in this article will be called the correlation Latin hypercube sampling (CLHS) plan.

**Latin hypercubes in axis orthogonal importance sampling**

Latin hypercubes can be applied in importance sampling in different ways. In this article, for comparison,
only the simple Monte Carlo sampling method and the axis orthogonal importance sampling methods are considered.

A LHS (CLHS) sample of the axis orthogonal importance sampling in the transformed normal space is established in the directions of the approximating tangent hyperplane (Figure 1). If the number of stochastic variables is \( K \), the dimension of the tangent hyperplane is \( K - 1 \). The sampling dimension is thus reduced by one. In the LHS version, an \( N \times (K - 1) \) sampling matrix \( S \) in the standard normal space generated on its tangent hyperplane by a transformation \( \Phi^{-1}(\cdot) \) from \( S \) generated in the way as given in section “LHS” is depicted. One realization of the stochastic variables, namely, a row of the sampling matrix \( S \), is given in a local coordinate system with axes defining the hyperplane. A rotation transformation to the original coordinate system of the realization to determine the failure is performed as

\[
\tilde{S}_i = [b_i \tilde{s}_i]A
\]  

where \( \tilde{s}_i \) is the \( i \)th realization of \( S \) in the local coordinate system with axes defining the hyperplane, \( \tilde{S}_i^* \) is the corresponding \( i \)th realization of \( S^* \) in the original coordinate system, \( b_i \) is the coordinate value in the local coordinate system of the intersection point of the limit state surface with the line along the normal of tangent hyperplane and through the point defined by realization \( \tilde{s}_i \) (Figure 1(a)), and \( A \) is a square matrix, whose first row is given by the unit vector in the direction of the position vector of the design point, that is

\[
a_1 = \frac{m}{|m|}
\]

The remaining \( K - 1 \) directions are chosen to be orthogonal to each other and to \( a_1 \). A row of \( A \) contains a unit vector with the property

\[
A^T A = I
\]

More guidance on the \( K - 1 \) rows of \( A \) except \( a_1 \) can be found in the study of Hohenbichler and Rackwitz.\(^{26}\)

Afterwards, as the points corresponding to the rows of realization \( S^* \) lie on the limit state surface, \( b_i \) can be computed by finding the root of

\[
g(S_i^*) = 0
\]

The Newton–Raphson algorithm in one dimension can be applied to find the solution, and a suitable start value for \( b_i \) is the distance between the origin and the design point, namely, \( \beta \). Figure 1 exhibits the axis importance Monte Carlo method (AIMC), the axis importance Latin hypercube sampling (AILHS) method, and the axis importance correlation Latin hypercube sampling (AICLHS) method, respectively, for a problem with two stochastic variables.

Reliability sensitivity estimation and variance analysis based on AOILHS/AOICLHS

Reliability sensitivity estimation in the transformed standard normal space

The probability of failure with AIMC, AILHS, and AICLHS method is estimated by
\[
\hat{P}_f = \frac{1}{N} \sum_{i=1}^{N} P_{bi} = \frac{1}{N} \sum_{i=1}^{N} \Phi_{(0)}(-b_i) \tag{11}
\]

where

\[
P_{bi} = \Phi_{(0)}(-b_i)
\tag{12}
\]

Since the samples generated in transformed normal space are independent and identically distributed (i.i.d.), the variance of the probability of failure can be calculated by

\[
\text{var}(\hat{P}_f) = \frac{1}{N-1} \left( \frac{1}{N} \sum_{i=1}^{N} \Phi_{(0)}(-b_i) - \hat{P}_f^2 \right) \tag{13}
\]

The reliability sensitivity of \( \hat{P}_f \) with respect to \( \theta \) (where \( \theta \) denotes the mean \( \mu_u \), standard deviation \( \sigma_u \) of a normal variable \( u \), or other distribution parameters) can be evaluated by

\[
\frac{\partial \hat{P}_f}{\partial \theta} = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial \Phi_{(0)}(-b_i)}{\partial \theta} \cdot \frac{\partial b_i}{\partial \theta} \tag{14}
\]

where

\[
\frac{\partial \Phi_{(0)}(-b_i)}{\partial b_i} = \frac{e^{-b_i^2/2}}{\sqrt{2\pi}} \tag{15}
\]

In order to determine \( \frac{\partial b_i}{\partial \theta} \) in equation (14), a hyperplane perpendicular to the normal of the approximating hyperplane and through the points defined by realization \( S_i^* \) (shown as hyperplane \( i \) in Figure 1(a)) is considered. The equation of hyperplane \( i \) can be written as follows

\[
g_i(u) = a_1 \cdot (S_i^* - u) = \sum_{j=1}^{K} a_{ij}(S_{ij}^* - u_j) = 0 \tag{16}
\]

where \( a_{ij}, S_{ij}^*, u_j \), and \( u_i \) are the \( j \)th component of vectors \( a_1, S_i^*, \) and \( u \), respectively. According to equation (12) and Figure 1, it is known that \( P_{bi} \) is equivalent to the probability of \( g_i(u) < 0 \), and the reliability index of linear limit state function \( g_i(u) = 0 \) equals \( b_i \). Since \( g_i(u) \) is the linear function of random variables \( u_1, u_2, \ldots, u_K \), \( b_i \) can be also evaluated by the first-order reliability method analytically as follows

\[
b_i = \frac{\mu_{g_i(u)}}{\sigma_{g_i(u)}} = \frac{1}{(\sum_{j=1}^{K} a_{ij}^2/\sigma_{u_j}^2)^{1/2}} \left( \sum_{j=1}^{K} a_{ij}^2/\sigma_{u_j}^2 \right)^{1/2} \tag{17}
\]

where \( \mu_{g_i(u)} \) and \( \sigma_{g_i(u)} \) are the mean and standard deviation of \( g_i(u) \), respectively. The partial derivative \( \frac{\partial b_i}{\partial \theta} \) can be obtained as

\[
\frac{\partial b_i}{\partial \mu_u} = - \frac{a_{ij}}{\sqrt{\sum_{j=1}^{K} a_{ij}^2/\sigma_{u_j}^2}} \tag{18}
\]

\[
\frac{\partial b_i}{\partial \sigma_u} = - \frac{a_{ij}^2/\sigma_{u_j}}{\sqrt{\sum_{j=1}^{K} a_{ij}^2/\sigma_{u_j}^2}} \tag{19}
\]

According to equations (14), (15), (18), (19), and considering \( \sum_{j=1}^{K} a_{ij}^2/\sigma_{u_j}^2 = 1 \), the reliability sensitivity of \( \hat{P}_f \) with respect to \( \theta \) can be calculated as

\[
\frac{\partial \hat{P}_f}{\partial \mu_u} = \frac{1}{N \sqrt{2\pi}} \sum_{i=1}^{N} e^{-b_i^2/2} a_{ij} \tag{20}
\]

\[
\frac{\partial \hat{P}_f}{\partial \sigma_u} = \frac{1}{N \sqrt{2\pi}} \sum_{i=1}^{N} e^{-b_i^2/2} a_{ij}^2 b_i \tag{21}
\]

Equations (20) and (21) indicate that the reliability sensitivities can be simply determined by \( b_i \) and the direction of MPP, \( a_1 \). Moreover, the sensitivity vector of \( \hat{P}_f \) with respect to the means of basic random variables is directly proportional to the direction of MPP, \( a_1 \), and the sensitivity vector with respect to their standard deviations is a certain multiple of the square of \( a_1 \).

**Reliability sensitivity estimation when non-normal random variables are not involved**

Usually, the parameter sensitivities with respect to \( \theta \) (where \( \theta \) denotes the mean \( \mu_u \), standard deviation \( \sigma_u \) of a random variable \( x_j \), or other distribution parameters), namely, \( \frac{\partial \hat{P}_f}{\partial \mu_u} \) and \( \frac{\partial \hat{P}_f}{\partial \sigma_u} \), are of most interest. If all the random variables are normal distribution and mutually independent (or linearly transformed from an originally correlated system of normal random variables), the reliability sensitivities can be easily estimated by

\[
\frac{\partial \hat{P}_f}{\partial \mu_u} = \frac{\partial \hat{P}_f}{\partial \mu_u} \cdot \frac{\partial \mu_u}{\partial x_i} \tag{22}
\]

\[
\frac{\partial \hat{P}_f}{\partial \sigma_u} = \frac{\partial \hat{P}_f}{\partial \sigma_u} \cdot \frac{\partial \sigma_u}{\partial x_i} \tag{23}
\]

where \( \frac{\partial \mu_u}{\partial x_i} \) and \( \frac{\partial \sigma_u}{\partial x_i} \) can be easily calculated from the transformation formula and are usually constants. Generally, when components of \( x = x_1, x_2, \ldots, x_K \) are mutually independent random
variables and assuming the mean and standard deviation of \( x_j \), a component of \( \mathbf{x} \), are \( \mu_j \) and \( \sigma_j \), respectively, as an example, \( x_j \) will be transformed into normal space by

\[
u_j = \frac{x_j - \mu_j}{\sigma_j}
\]  

(24)

Thus, \( \partial \mu_{u_j}/\partial \mu_{x_j} \) and \( \partial \sigma_{u_j}/\partial \sigma_{x_j} \) can be easily calculated by

\[
\frac{\partial \mu_{u_j}}{\partial \mu_{x_j}} = \frac{\partial \mu_{u_j}}{\partial \mu_{x_j}} = \frac{1}{\sigma_j}
\]  

(25)

**Reliability sensitivity estimation when non-normal random variables are involved**

There are no particular difficulties for the failure probability estimation using AOILHS/AOICLHS when non-normal random variables are involved, which is conducted in the same way as depicted in sections “Reliability sensitivity estimation in the transformed standard normal space” and “Reliability sensitivity estimation when non-normal random variables are not involved.” However, estimating the reliability sensitivities using AOILHS/AOICLHS is less straightforward. On the one hand, when non-normal random variables are involved, their transformations to the standard normal space applied in AOILHS/AOICLHS are usually non-linear, thus \( \partial \mu_{x_j}/\partial \mu_{x_j} \) and \( \partial \sigma_{x_j}/\partial \sigma_{x_j} \) cannot be easily obtained. On the other hand, the samples employed to estimate the failure probability \( P_f \) in AOILHS/AOICLHS are generated by the sampling density function centered at the MPP on the limit state surface. The MPP is the point maximizing the joint PDF \( \phi_0 \) in the failure domain. The reliability sensitivities calculated directly by differentiating \( P_f \) with respect to \( \theta \) (\( \theta \) denotes the mean, standard deviation of a random variable, or other distribution parameters) may produce big errors for the MPP which is usually not the point maximizing the derivative of joint PDF, namely, \( \partial \phi_0/\partial \theta \), and the samples employed to estimate the failure probability are not suitable to estimate the reliability sensitivities.

Now the remaining effort is to estimate the reliability sensitivities directly upon the failure probability without taking derivatives, namely, the sensitivities of the failure probability to changes in the parameters. Considering the effect of changing one of the distribution parameters of a random variable, in the procedure of AOILHS/AOICLHS, small changes on the transformation of the random variable to normal space, the limit state function in normal space, and the failure probability associated with it will successively come into being. Thus, the reliability sensitivity can be calculated using finite difference method and written as

\[
\begin{align*}
\frac{\partial \bar{P}_f}{\bar{\theta}} &= \frac{\bar{P}_f|_{\theta + \Delta \theta} - \bar{P}_f|_{\theta}}{\Delta \theta} \\
&= \frac{1}{N} \sum_{i=1}^{N} \left[ \Phi(0,1)(-b_i|_{\theta + \Delta \theta}) - \Phi(0,1)(-b_i|_{\theta}) \right]
\end{align*}
\]  

(26)

where \( \Delta \theta \) is a small change of the parameter of interest \( \theta \), \( \bar{P}_f|_{\theta + \Delta \theta} \) and \( \bar{P}_f|_{\theta} \) are the estimated failure probabilities for the cases of the parameter \( \theta \) without a change and with a small change \( \Delta \theta \), namely, the parameter equal to \( \theta + \Delta \theta \), respectively. From equation (26), it can be seen that the reliability sensitivity is a ratio of the average of the failure probability changes of all the samples to the small change of the parameter \( \Delta \theta \). It must be noted that as the failure probabilities \( \bar{P}_f|_{\theta + \Delta \theta} \) and \( \bar{P}_f|_{\theta} \) are calculated on randomly generated samples, and considering only the variation of failure probability caused by the changes of a parameter are inspected, \( \bar{P}_f|_{\theta + \Delta \theta} \) and \( \bar{P}_f|_{\theta} \) should be estimated on the same group of generated samples. \( b_i|_{\theta + \Delta \theta} \) and \( b_i|_{\theta} \) are solved with the same sample but denotes different values.

**Reliability sensitivity variance analysis**

First considering the variance of reliability sensitivities in the transformed normal space and since the samples generated in normal space are i.i.d, the variance of reliability sensitivities can be estimated by

\[
\text{var} \left( \frac{\partial \bar{P}_f}{\partial \mu_{u_j}} \right) = \frac{1}{N-1} \left[ \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\partial \bar{P}_f}{\partial \mu_{u_j}} \right)^2 - \left( \frac{\partial \bar{P}_f}{\partial \mu_{x_j}} \right)^2 \right]
\]  

(27)

\[
\text{var} \left( \frac{\partial \bar{P}_f}{\partial \sigma_{u_j}} \right) = \frac{1}{N-1} \left[ \frac{1}{N} \sum_{i=1}^{N} \left( \frac{\partial \bar{P}_f}{\partial \sigma_{u_j}} \right)^2 - \left( \frac{\partial \bar{P}_f}{\partial \sigma_{x_j}} \right)^2 \right]
\]  

(28)

For those problems not involving non-normal random variables, variances of the reliability sensitivities can be calculated according to equations (22) and (23) and written as

\[
\begin{align*}
\text{var} \left( \frac{\partial \bar{P}_f}{\partial \mu_{x_j}} \right) &= \left( \frac{\partial \bar{P}_f}{\partial \mu_{x_j}} \right)^2 \text{var} \left( \frac{\partial \bar{P}_f}{\partial \mu_{u_j}} \right) \\
\text{var} \left( \frac{\partial \bar{P}_f}{\partial \sigma_{x_j}} \right) &= \left( \frac{\partial \bar{P}_f}{\partial \sigma_{x_j}} \right)^2 \text{var} \left( \frac{\partial \bar{P}_f}{\partial \sigma_{u_j}} \right)
\end{align*}
\]  

(29)

(30)
For those problems involving non-normal random variables, variances of the reliability sensitivities can be calculated according to equations (11) and (26) and written as:

\[
\begin{align*}
\var\left(\frac{\partial \hat{P}_f}{\partial \theta}\right) & = \var\left(\hat{P}_f|_{\theta} + \Delta \theta - \hat{P}_f|_{\theta}\right) \left(\frac{\Delta \theta}{s}\right)^2 \\
& = \frac{1}{(N-1)(\Delta \theta)^2} \left[ \frac{1}{N} \sum_{i=0}^{N-1} (\Phi(0,1)(-b_i|_{\theta} + \Delta \theta) - \Phi(0,1)(-b_i|_{\theta}))^2 - \left(\hat{P}_f|_{\theta} + \Delta \theta - \hat{P}_f|_{\theta}\right)^2 \right] \\
& \text{(31)}
\end{align*}
\]

**Methodology**

Now, the proposed procedure of AOILHS/AOICLHS for reliability sensitivity estimation can be summarized as follows:

1. Determine an approximate design point \( m \) in normal space using FORM or optimization algorithm.
2. Generate an \( N \times (K-1) \) sampling matrix \( S \) in the standard normal space.
3. Transform each sample of \( S \) into the original coordinate system using Rosenblatt’s transformation and obtain a realization \( S^* \) in the original coordinate system according to equation (7).
4. Each row of \( S^* \), as a point on the limit state surface, is substituted into the limit state function according to equation (10) and \( b_i \) can be found by the Newton–Raphson algorithm in one dimension.
5. For those problems in which non-normal random variables are not involved, calculate failure probability with equation (11) and reliability sensitivity according to equations (20)–(23). In the meantime, estimate the variances according to equations (27)–(30).
6. For those problems in which non-normal random variables are involved, calculate failure probability \( \hat{P}_f|_{\theta} \) with equation (11). For each parameter of interest \( \theta \), give a small change \( \Delta \theta \) (which is usually equal to \( 10^{-4} \) or \( 10^{-3} \) times the mean of \( \theta \)), repeat steps 3 and 4 and calculate failure probability \( \hat{P}_f|_{\theta} + \Delta \theta \) with equation (11) and reliability sensitivity with equation (26). In the meantime, estimate the variances with equation (31).

**Numerical examples**

Four examples are illustrated below to demonstrate the computational efficiency and accuracy of the proposed reliability sensitivity estimation method based on the AOILHS, AOICLHS, and a representative quasi-random sampling strategy AOIHalton. For comparison, the failure probabilities \( (P_f) \), the reliability indices \( (\beta) \), and sensitivities determined from the proposed method with different size of samples are presented, and the results calculated by AOICLHS with the largest samples are referred as exact and the results determined from the simple Monte Carlo by the other researchers or importance LHS for those high-reliability problem are provided to verify the accuracy of the proposed method. The largest relative errors of reliability sensitivity estimation with respect to the means and standard deviations employing LHS, CLHS, and sampling with the Halton sequence under different sample sizes are also shown in figures. Since all the non-normal or correlated random variables are transformed into independent standard normal ones in the proposed method, correlated random variables are not considered in this article.

**Example 1.** A non-linear limit state function with three independent normal random variables is considered. The limit state function, basic random variables, and their distribution parameters are presented in Table 1. The mean value sensitivities \( \partial P_f/\partial \mu_X \), \( \partial P_f/\partial \mu_Y \), and \( \partial P_f/\partial \mu_Z \), standard deviation sensitivities \( \partial \sigma_f/\partial \sigma_X \), \( \partial \sigma_f/\partial \sigma_Y \), and \( \partial \sigma_f/\partial \sigma_Z \), their standard deviations and coefficients of variations (COVs; the ratio of the standard deviation to the mean value of the estimator), as well as the relative errors of mean values of the estimator compared to the exact ones are calculated based on AOILHS/AOICLHS with 15, 100 samples (the sample size \( N = 15, 100 \)). The exact sensitivity analysis results are determined with 100,000 samples based on AOICLHS since all its COVs are less than 0.04%.

All these obtained results are listed in Table 2. The results obtained from the study of Melchers and

| Variable | Mean | Standard deviation | Distribution |
|----------|------|--------------------|--------------|
| \( X_1 \) | 40.0 | 5.0 | Normal |
| \( X_2 \) | 50.0 | 2.5 | Normal |
| \( X_3 \) | 1000.0 | 200.0 | Normal |
Ahammed\textsuperscript{2} by the simple Monte Carlo method are also listed in the last line of the table. The largest relative errors of reliability sensitivity estimation with the strategy of LHS, CLHS, and the Halton sequence under the sample sizes 15, 50, 100, 1000, and 5000 are also shown in Figure 2. In fact, since the sensitivities are determined by the same group of $b_i$ and the same direction of MPP, $a_1$, the sensitivities with respect to the means of different random variables are identical to each other (Table 2). The same holds for the standard deviation. From these, we can see that as the sample size increases, the AOI/LHS/AOICLHS/AOIH gives better and better estimates. The AOIH obtains a satisfactory estimate even with the fewest 15 samples. The AOIH performs better than AOICLHS, and AOICLHS performs better than AOI/LHS under the same size of samples. The relative errors of the standard deviation sensitivities are almost equal to those of the mean sensitivities with the same sampling strategy. For this problem, neglecting the computational costs of establishing the sampling plans and several evaluations of the limit state function to find $b_i$ in equation (10) by the Newton–Raphson algorithm, only 15 samples or 15 evaluations of the limit state function based on the AOI/LHS/AOICLHS produce less than 1% relative errors of the mean values of the estimator.

**Example 2.** A roof truss structural reliability problem is considered as shown in Figure 3.\textsuperscript{27} The top chords and compression bars are made of steel-reinforced concrete, and the bottom chords and tension bars are made of steel. According to structural mechanics, the uniformly distributed load $q$ applied on the roof truss can be equivalent to the nodal load $P = ql/4$. The perpendicular deflection of peak node $C$, $\Delta_C$ not exceeding 3 cm, gives the following limit state function

$$g = 0.03 - \Delta_C = 0.03 - \frac{q l^2}{2} \left( \frac{3.81}{A_C E_C} + \frac{1.13}{A_S E_S} \right)$$

where $A_C, A_S$ are the cross-sectional areas of reinforced concrete and steel bars, respectively; $E_C, E_S$ are their corresponding elastic modulus; and $l$ is the span length of the truss. The basic independent normal random variables and their distribution parameters are presented in Table 3. The mean value sensitivities ($\partial g / \partial \mu_q$, $\partial g / \partial \mu_l$, $\partial g / \partial \mu_{A_C}$, $\partial g / \partial \mu_{A_S}$, $\partial g / \partial \mu_{E_C}$, and $\partial g / \partial \mu_{E_S}$) and standard deviation sensitivities ($\partial g / \partial \sigma_q$, $\partial g / \partial \sigma_l$, $\partial g / \partial \sigma_{A_C}$, $\partial g / \partial \sigma_{A_S}$, $\partial g / \partial \sigma_{E_C}$, and $\partial g / \partial \sigma_{E_S}$), their standard deviations and COVs, as well as the relative errors of mean values of the estimator compared to the exact ones are calculated based on AOI/LHS/AOICLHS with 50, 1000 samples. The exact sensitivity analysis results are determined with 1,000,000 samples based on AOICLHS since all its COVs are less than 0.04%. All these obtained results are listed in Table 4. The results obtained from the study of Song et al.\textsuperscript{27} based on the simple Monte Carlo method are listed in the last line of the table. The largest relative errors of reliability sensitivity estimation with the strategy of LHS, CLHS, and the Halton sequence under the sample sizes 50, 100, 1000, 5000, and 10,000 are also shown in Figure 4. From these, we can see that as the sample size increases, the AOI/LHS/AOICLHS/AOIH gives better and better estimates. The AOICLHS obtains a satisfactory estimate with the fewest 50 samples. The AOICLHS performs better than AOIH.

![Figure 2](image-url) Relative errors of sensitivities with different sample sizes in example 1.

![Figure 3](image-url) Schematic diagram of a roof truss: (a) under uniformly distributed load on roof truss and (b) under equivalent nodal loads.
Table 2. Sensitivities, failure probability, and reliability indices obtained from the AOILHS/AOICLHS.

| Methods      | \( \partial p_f / \partial \mu_{X_1} \times 10^{-4} \) | \( \partial p_f / \partial \mu_{X_2} \times 10^{-4} \) | \( \partial p_f / \partial \mu_{X_3} \times 10^{-5} \) | \( \partial p_f / \partial r_{X_1} \times 10^{-3} \) |
|--------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|--------------------------------------------------|
| AOICLHS      | \( N = 10^5 \)                                     |                                                   |                                                   |                                                   |
| Estimator    | -5.9071                                           | -3.3090                                          | 1.2104                                           | 1.3574                                           |
| SD           | 0.0017273                                         | 0.0009676                                        | 0.00035396                                       | 0.00034917                                       |
| COV (%)      | 0.03                                              | 0.03                                             | 0.03                                             | 0.026                                            |
| AOILHS       | \( N = 15 \)                                       |                                                   |                                                   |                                                   |
| Estimator    | -5.7821                                           | -3.2391                                          | 1.1849                                           | 1.3323                                           |
| SD           | 0.053628                                          | 0.030042                                         | 0.01099                                          | 0.011018                                         |
| COV (%)      | 0.93                                              | 0.93                                             | 0.93                                             | 0.83                                             |
| Error (%)    | -2.12                                             | -2.11                                            | -2.11                                            | -1.85                                            |
| AOICLHS      | \( N = 15 \)                                       |                                                   |                                                   |                                                   |
| Estimator    | -5.8576                                           | -3.2814                                          | 1.2004                                           | 1.3473                                           |
| SD           | 0.014355                                          | 0.080417                                         | 0.029417                                         | 0.029493                                         |
| COV (%)      | 2.45                                              | 2.45                                             | 2.45                                             | 2.19                                             |
| Error (%)    | -0.84                                             | -0.83                                            | -0.83                                            | -0.74                                            |
| AOILHS       | \( N = 100 \)                                      |                                                   |                                                   |                                                   |
| Estimator    | -5.9174                                           | -3.3149                                          | 1.2126                                           | 1.3594                                           |
| SD           | 0.061526                                          | 0.034467                                         | 0.012608                                         | 0.012467                                         |
| COV (%)      | 1.04                                              | 1.04                                             | 1.04                                             | 0.92                                             |
| Error (%)    | 0.17                                              | 0.18                                             | 0.18                                             | 0.15                                             |
| AOICLHS      | \( N = 100 \)                                      |                                                   |                                                   |                                                   |
| Estimator    | -5.8956                                           | -3.3027                                          | 1.2082                                           | 1.3553                                           |
| SD           | 0.046089                                          | 0.025818                                         | 0.0094447                                        | 0.0093648                                        |
| COV (%)      | 0.78                                              | 0.78                                             | 0.78                                             | 0.69                                             |
| Error (%)    | -0.19                                             | -0.19                                            | -0.18                                            | -0.15                                            |
| SMC          | \( N = 10^7 \)                                     |                                                   |                                                   |                                                   |
| Estimator    | -5.95                                             | -3.48                                            | 1.13                                             | 1.31                                             |

| Methods      | \( \partial p_f / \partial r_{X_1} \times 10^{-3} \) | \( p_f \times 10^{-3} \) | \( \beta \) |
|--------------|--------------------------------------------------|--------------------------|-------------|
| AOICLHS      | \( N = 10^5 \) (continued)                       |                          |             |
| Estimator    | 2.1298                                           | 2.2801                   | 1.1769      | 3.0415     |
| SD           | 0.00054787                                        | 0.00058652               | 0.0003802   |             |
| COV (%)      | 0.026                                            | 0.026                    | 0.032       |             |
| AOILHS       | \( N = 15 \) (continued)                         |                          |             |
| Estimator    | 2.0905                                           | 2.2380                   | 1.1493      | 3.0487     |
| SD           | 0.017287                                          | 0.018507                 | 0.01165     |             |
| COV (%)      | 0.83                                             | 0.83                     | 1.01        |             |
| Error (%)    | -1.85                                            | -1.85                    | -2.35       | 0.24       |
| AOICLHS      | \( N = 15 \) (continued)                         |                          |             |
| Estimator    | 2.1139                                           | 2.2630                   | 1.1661      | 3.0443     |
| SD           | 0.046276                                          | 0.049540                 | 0.03118     |             |
| COV (%)      | 2.19                                              | 2.19                     | 2.67        |             |
| Error (%)    | -0.75                                            | -0.75                    | -0.92       | 0.092      |
| AOILHS       | \( N = 100 \) (continued)                        |                          |             |
| Estimator    | 2.1329                                           | 2.2833                   | 1.1793      | 3.0409     |
| SD           | 0.019562                                          | 0.020942                 | 0.01351     |             |
| COV (%)      | 0.92                                              | 0.92                     | 1.15        |             |
| Error (%)    | 0.15                                              | 0.14                     | 0.20        | -0.020     |
| AOICLHS      | \( N = 100 \) (continued)                        |                          |             |
| Estimator    | 2.1265                                           | 2.2765                   | 1.1742      | 3.0422     |
| SD           | 0.14694                                          | 0.015730                 | 0.01010     |             |
| COV (%)      | 0.69                                              | 0.69                     | 0.86        |             |
| Error (%)    | -0.15                                            | -0.16                    | -0.23       | 0.02       |
| SMC          | \( N = 10^7 \) (continued)                       |                          |             |
| Estimator    | 2.60                                              | 2.12                     | 1.175       | 3.042      |

AOILHS: axis orthogonal importance Latin hypercube sampling; AOICLHS: axis orthogonal importance correlation Latin hypercube sampling; SD: standard deviation; COV: coefficient of variation; SMC: standard Monte Carlo simulation.
and AOIHalton performs better than AOILHS under the same size of samples. The relative errors of the standard deviation sensitivities are less than those of the mean sensitivities with the same sampling strategy. For this problem, more basic random variables than example 1 need more samples to achieve the same precision. Neglecting the other computational costs, 50 samples or 50 evaluations of the limit state function based on the AOICLHS produce less than 5% relative errors of the mean values of the estimator.

Example 3. A highly non-linear limit state function involving both normal and non-normal basic random variables is considered. The limit state function, basic random variables, and their distribution parameters are presented in Table 5.2. The mean value sensitivities \( \frac{\partial \psi}{\partial \mu(X_i)} \), \( \frac{\partial \psi}{\partial \sigma(X_i)} \), \( \frac{\partial \psi}{\partial \mu(X_j)} \), \( \frac{\partial \psi}{\partial \mu(X_k)} \), and \( \frac{\partial \psi}{\partial \mu(X_l)} \) and standard deviation sensitivities \( \frac{\partial \psi}{\partial \sigma(X_i)} \), \( \frac{\partial \psi}{\partial \sigma(X_j)} \), \( \frac{\partial \psi}{\partial \sigma(X_k)} \), \( \frac{\partial \psi}{\partial \sigma(X_l)} \), \( \frac{\partial \psi}{\partial \sigma(X_m)} \), \( \frac{\partial \psi}{\partial \sigma(X_n)} \), \( \frac{\partial \psi}{\partial \sigma(X_o)} \), and \( \frac{\partial \psi}{\partial \sigma(X_p)} \), their standard deviations and COVs, as well as the relative errors of mean values of the estimator compared to the exact ones are calculated based on AOILHS/AOICLHS with 100, 1000 samples. The exact sensitivity analysis results are determined with 1,000,000 samples based on AOICLHS since all its COVs are less than 1%. All these obtained results are listed in Table 6. The results obtained from Melchers and Ahammed\(^2\) by the simple Monte Carlo method are listed in the last line of the table. The errors of reliability sensitivity estimation with the strategy of LHS, CLHS, and the Halton sequence under the sample sizes 100, 500, 1000, 10,000, and 100,000 are also shown in Figure 5. From these, we can see that as the sample size increases, the AOILHS/AOICLHS/AOIHalton also gives better and better estimates. The AOICLHS obtains a satisfactory estimate with the fewest 100 samples in the mean sensitivities but not in the standard deviation sensitivities. The AOICLHS performs better than AOILHS, and AOILHS performs better than AOIHalton under the same size of samples. The

| Random variable | \( q (N/m) \) | \( l (m) \) | \( A_s (m^2) \) | \( A_c (m^2) \) | \( E_s (N/m^2) \) | \( E_c (N/m^2) \) |
|-----------------|---------------|-------------|----------------|----------------|----------------|----------------|
| Mean            | 20,000        | 12          | \( 9.82 \times 10^{-4} \) | 0.04           | \( 1 \times 10^{11} \) | \( 2 \times 10^{10} \) |
| COV             | 0.07          | 0.01        | 0.06           | 0.12           | 0.06           | 0.06           |
| SD              | 1400          | 0.12        | \( 5.892 \times 10^{-5} \) | 0.0048         | \( 6 \times 10^9 \)  | \( 1.2 \times 10^9 \) |

COV: coefficient of variation; SD: standard deviation.

![Figure 4](image1.png)

**Figure 4.** Relative errors of sensitivities with different sample sizes in example 2.

![Figure 5](image2.png)

**Figure 5.** Relative errors of sensitivities with different sample sizes in example 3.
Table 4. Sensitivities, failure probability, and reliability indices obtained from the AOILHS/AOICLHS.

| Methods | $\partial pf / \partial \mu_q (\times 10^{-6})$ | $\partial pf / \partial \mu_I (\times 10^{-2})$ | $\partial pf / \partial \mu_{A_C} (\times 10^2)$ | $\partial pf / \partial \mu_{E_C} (\times 10^{-12})$ | $\partial pf / \partial \mu_{E_C} (\times 10^{-12})$ | $pf / \partial \sigma_q (\times 10^{-5})$ |
|---------|----------------------|----------------------|----------------------|----------------------|----------------------|----------------------|
| AOILHS  | $N = 10^6$           |                      |                      |                      |                      |                      |
| Estimator | 9.9576               | 4.3845               | -1.8543              | -2.4242              | -1.8030              | -4.4142              | 1.2980               |
| SD      | 0.0032015            | 0.0014096            | 0.00059620           | 0.00077945           | 0.00057968           | 0.0014192            | 0.00031988           |
| COV (%) | 0.032                | 0.032                | 0.032                | 0.032                | 0.032                | 0.032                | 0.025                |
| AOILHS  | $N = 50$             |                      |                      |                      |                      |                      |
| Estimator | 10.130               | 4.4604               | -1.8864              | -2.4663              | -1.8342              | -4.4907              | 1.3186               |
| SD      | 0.41389              | 0.18224              | 0.077076             | 0.10077              | 0.074941             | 0.18348              | 0.042317             |
| COV (%) | 4.09                 | 4.09                 | 4.09                 | 4.09                 | 4.09                 | 3.21                 |
| Error (%) | 1.73                 | 1.73                 | 1.73                 | 1.73                 | 1.73                 | 1.59                 |
| AOILHS  | $N = 1000$           |                      |                      |                      |                      |                      |
| Estimator | 9.8190               | 4.3234               | -1.8285              | -2.3905              | -1.7779              | -4.3528              | 1.2881               |
| SD      | 0.33296              | 0.14661              | 0.062004             | 0.081063             | 0.060287             | 0.14760              | 0.035839             |
| COV (%) | 3.89                 | 3.89                 | 3.89                 | 3.89                 | 3.89                 | 2.78                 |
| Error (%) | 1.39                 | 1.39                 | 1.39                 | 1.39                 | 1.39                 | 0.76                 |
| AOILHS  | $N = 1000$           |                      |                      |                      |                      |                      |
| Estimator | 9.8518               | 4.3379               | -1.8346              | -2.3986              | -1.7838              | -4.3674              | 1.2877               |
| SD      | 0.094955             | 0.041810             | 0.017683             | 0.023118             | 0.017193             | 0.042094             | 0.0096986            |
| COV (%) | 0.96                 | 0.96                 | 0.96                 | 0.96                 | 0.96                 | 0.75                 |
| Error (%) | -1.06                | -1.06                | -1.06                | -1.06                | -1.06                | -0.79                |
| AOILHS  | $N = 5 \times 10^7$  |                      |                      |                      |                      |                      |
| Estimator | 9.9539               | 4.3829               | -1.8537              | -2.4234              | -1.8023              | -4.4126              | 1.2981               |
| SD      | 0.098425             | 0.043338             | 0.018329             | 0.023963             | 0.017821             | 0.043632             | 0.009566             |
| COV (%) | 0.99                 | 0.99                 | 0.99                 | 0.99                 | 0.99                 | 0.77                 |
| Error (%) | -0.037               | -0.037               | -0.037               | -0.037               | -0.039               | 0.0077               |
| SMC     | $N = 5 \times 10^7$  | 11.059               | 4.068                | -1.86262             | -2.1299              | -1.8265              | 3.7592               |

(continued)
relative errors of the mean sensitivities are much less than those of the standard deviation sensitivities with the same sampling strategy especially under the small sample sizes. Thus, we can also obtain the same conclusions as the previous two examples about the performance of AOILHS/AOICLHS/AOIHalton when they are applied to estimate the reliability sensitivities, except that for this problem, basic random variables with normal and non-normal distributions both involved needs more samples to achieve the same precision due to its high nonlinearity. Neglecting the other computational costs, 100 samples or 100 evaluations of the limit state function based on the AOICLHS give less than 1% and less than 10% relative errors of the mean values of the estimator on the mean value sensitivities and standard deviation sensitivities, respectively, except variable $X_1$, namely, $\partial P_f / \partial \sigma_{X_1}$, with nearly 20% relative error of the mean values of the estimator.

Example 4. A more complex 23-bar truss with 30 independent normal random variables is considered (Figure 6), including cross-sectional area for members, applied loads, and modulus of elasticity, whose distribution parameters are listed in Table 7. The limit state function is given based on vertical displacement of middle node $D(X)$ as follows

$$G(X) = 0.14 - D(X)$$

Due to so many basic variables, only the relative errors of the mean value sensitivities and standard deviation sensitivities of the estimator with 50, 100, 500, 1000, and 10,000 samples compared to the exact ones determined with 1,000,000 samples based on AOICLHS are shown in Figure 7. The sensitivities with respect to the means and standard deviations of odd-numbered areas via different sampling methods are presented in Table 8. The failure probability and reliability index calculated by the FORM are $1.44 \times 10^{-4}$ and 3.6266, respectively. Using the simple Monte Carlo method, the failure probability and reliability index obtained are $1.82 \times 10^{-4}$ and 3.5648 with 1,000,000 samples.
Table 6. Sensitivities, failure probability, and reliability indices obtained from the AOILHS/AOICLHS.

| Methods | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-3}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-7}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-2}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-4}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-4}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-4}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-4}$ |
|---------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| AOILHS  | $N = 10^6$                       |                                |                                |                                |                                |                                |                                |
| Estimator | -4.4548                         | -7.3439                        | -2.1454                        | -8.9977                        | 1.7756                          | 2.2904                          | 6.4306                          |
| SD      | 0.0024682                       | 0.0038877                     | 0.0013030                     | 0.0048062                     | 0.0010073                      | 0.0012916                      | 0.051994                        |
| COV (%) | 0.055                           | 0.053                          | 0.061                          | 0.053                          | 0.057                          | 0.056                          | 0.81                            |
| AOILHS  | $N = 100$                       |                                |                                |                                |                                |                                |                                |
| Estimator | -4.5514                         | -7.4480                        | -2.2033                        | -9.2060                        | 1.8156                          | 2.3289                          | 5.1484                          |
| SD      | 0.29269                         | 0.44897                        | 0.15970                        | 0.57204                        | 0.12006                         | 0.14844                         | 5.3684                          |
| COV (%) | 6.43                            | 6.03                           | 7.25                           | 6.21                           | 6.61                           | 6.37                           | 104.27                          |
| Error (%) | 2.17                            | 1.42                           | 2.70                           | 2.31                           | 2.25                           | 1.68                           | -19.94                          |
| AOILHS  | $N = 1000$                      |                                |                                |                                |                                |                                |                                |
| Estimator | -4.4704                         | -7.3614                        | -2.1465                        | -9.0266                        | 1.7795                          | 2.2972                          | 7.7070                          |
| SD      | 0.23671                         | 0.37035                        | 0.12277                        | 0.45558                        | 0.095734                        | 0.12358                         | 5.7165                          |
| COV (%) | 5.30                            | 5.03                           | 5.72                           | 5.05                           | 5.38                           | 5.38                           | 74.17                           |
| Error (%) | 0.29                            | 0.24                           | 0.051                          | 0.32                           | 0.22                           | 0.30                           | 19.85                           |
| AOILHS  | $N = 10000$                     |                                |                                |                                |                                |                                |                                |
| Estimator | -4.4571                         | -7.3409                        | -2.1444                        | -9.0005                        | 1.7760                          | 2.2923                          | 6.9634                          |
| SD      | 0.077069                        | 0.12059                        | 0.040209                       | 0.14980                        | 0.031379                        | 0.040512                        | 1.6701                          |
| COV (%) | 1.80                            | 1.71                           | 1.96                           | 1.73                           | 1.84                           | 1.82                           | 25.08                           |
| Error (%) | 0.35                            | 0.18                           | 0.26                           | 0.38                           | 0.34                           | 0.23                           | 2.92                            |
| SMC     |                                | -5.20                          | -8.70                          | -2.49                          | N = $10^6$                      | 1.62                           | 2.18                            | 5.80                            |

| Methods | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-3}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-7}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-2}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-4}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-4}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-4}$ | $\frac{\partial p_f}{\partial \mu_X} \times 10^{-4}$ |
|---------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| AOILHS  | $N = 10^6$ (continued)          |                                |                                |                                |                                |                                |                                |
| Estimator | 3.0182                         | 1.0226                         | 2.3960                         | 1.7756                         | 1.1570                          | 3.4252                          | 2.7040                          |
| SD      | 0.0074955                       | 0.0021001                      | 0.0099550                      | 0.0010073                      | 0.0025039                      | 0.0021327                      | -                              |
| COV (%) | 0.25                            | 0.21                           | 0.42                           | 0.057                          | 0.22                           | 0.062                          | -                              |
| AOILHS  | $N = 100$ (continued)           |                                |                                |                                |                                |                                |                                |
| Estimator | 2.3228                         | 1.1304                         | 2.5674                         | 1.8156                         | 1.3778                          | 3.5384                          | 2.6932                          |
| SD      | 0.87881                         | 0.25832                        | 1.1188                         | 0.12006                        | 0.26457                        | 0.26774                        | -                              |
| COV (%) | 37.83                           | 22.85                          | 43.58                          | 6.61                           | 19.20                          | 7.57                           | -                              |
| Error (%) | -23.04                          | 10.54                          | 7.15                           | 2.25                           | 19.08                          | 3.30                           | -0.40                          |
| AOILHS  | $N = 100$ (continued)           |                                |                                |                                |                                |                                |                                |
| Estimator | 3.0420                         | 0.99386                        | 2.5834                         | 1.7795                         | 1.1322                          | 3.4213                          | 2.7044                          |
| SD      | 0.74880                         | 0.21645                        | 1.0192                         | 0.095734                       | 0.25408                        | 0.19835                        | -                              |
| COV (%) | 24.62                           | 21.78                          | 39.45                          | 5.38                           | 22.44                          | 5.79                           | -                              |
| Error (%) | 0.79                            | -2.81                          | 7.82                           | 0.22                           | -2.14                          | -0.11                          | 0.015                          |
samples, respectively, while it can be easily figured out from the symmetry of loads and geometry of the structure that the derived sensitivities are inaccurate (see column 6 in Table 8). The failure probability and reliability index calculated by the importance LHS around the MPP are \( \frac{1}{8709} \times 10^{-3} \) and 3.5577 with 1,000,000 samples. The sensitivities of mean values calculated by the axis orthogonal importance LHS/CLHS/Halton are very close to those by the importance LHS, but the sensitivities of standard deviations obtained by the importance LHS are obviously inaccurate for the same reason (Table 8). From these, we can also see that as the sample size increases, the AOILHS/AOICLHS/AOILHalon gives better and better estimates. The AOICLHS obtains a satisfactory estimate even with the fewest 50 samples. The AOICLHS performs better than AOILHS, and AOILHS performs better than AOIHalton under the same size of samples. For this problem, neglecting the computational costs of establishing the sampling plans and several evaluations of the limit state function to find \( b_i \) in equation (10) by the Newton–Raphson algorithm, only 50 samples or 50 evaluations of the limit state function based on the AOIHalton/AOICLHS produce less than 1% relative errors of the mean values of the estimator.

### Discussion

The proposed procedure for reliability sensitivity estimation based on the axis orthogonal importance sampling methods (AOILHS, AOICLHS, and AOIHalton) is, as the previous examples indicate, considerably more efficient than the simple Monte Carlo sampling methods with relatively little computational cost. The variance-reduction effect of using the AOICLHS instead of the AOILHS, even the AOIHalton, results in better parameter sensitivity estimation especially under small sample size. The results obtained from the AOICLHS, AOIHalton, and AOILHS have a smaller difference as the sample size increases to some extent.

The proposed method can be employed to estimate reliability sensitivities with an acceptable degree of accuracy, which is demonstrated by a variety of the above nonlinear limit state functions including a normal and non-normal random variables involved case. The sensitivity estimation is closely similar to the results

| Methods          | Estimator | SD COV (%) | Error (%) | N = 1000 (continued) | AOILHS: axis orthogonal importance Latin hypercube sampling; AOICLHS: axis orthogonal importance correlation Latin hypercube sampling; SMC: standard Monte Carlo sampling. |
|------------------|-----------|------------|-----------|----------------------|---------------------------------------------------------------------------------------------------------------------------------|
| AOILHS           | 2.8977    | 0.32775    | 1.0100    | 0.24175              | 0.052                                                                                                                          |
| AOICLHS          | 2.4600    | 0.31329    | 1.0098    | 0.23735              | 0.052                                                                                                                          |
| SMC              | 2.866     | 0.125      | 1.0100    | 0.24175              | 0.035                                                                                                                          |

### Table 7. Random variables and their parameters in example 4.

| Variable | Unit   | Mean   | SD      | Distribution |
|----------|--------|--------|---------|--------------|
| \( A_1 \) – \( A_{23} \) | \( m^2 \) | 0.0014 | 0.00014 | Normal       |
| E        | GPa    | 200    | 20      | Normal       |
| \( P_1 \) – \( P_6 \) | kN | 40 | 4 | Normal       |
obtained from the direct simple Monte Carlo or simple Monte Carlo–based analysis. The involved non-normal random variables have a significant effect on reliability sensitivity estimation especially for some varieties of non-normal distributions and may produce more errors, for example, the mean value sensitivity estimation \( \partial p_f / \partial \mu_A \) (Weibull distribution, the two-parameter extreme value type distribution for minima), \( \partial p_f / \partial \mu_{X_2} \) (lognormal distribution), \( \partial p_f / \partial \mu_{X_3} \) (uniform distribution), and standard deviation sensitivity estimation \( \partial p_f / \partial \sigma_{X_1} \) (Weibull distribution) and \( \partial p_f / \partial \sigma_{X_3} \) (uniform distribution) in example 3.

**Conclusion**

This article demonstrates the axis orthogonal Latin hypercube importance sampling as an efficient and applicable tool for reliability sensitivity analysis. The Latin hypercube sample is established around near the MPP and in the directions of the approximating tangent hyperplane of limit state surface. The Newton–Raphson algorithm in one dimension is applied to find a coordinate value in the local coordinate system of the intersection point with limit state surface for each realization of a sampling plan, which is used in the
following reliability analysis. Two versions of axis orthogonal Latin hypercube importance sampling with and without reduction in spurious correlation (AOICLHS and AOILHS) and an AOIHalton are applied and discussed in the parameter sensitivity estimation. The results obtained from the numerical examples show that the presented procedure is superior in computation cost with a relatively small sample size compared with the LHS and Monte Carlo method while preserving an acceptable accuracy.

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