Mathematical model and characteristics analysis of crossed-axis helical gear drive with small angle based on curve contact element

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Abstract
To improve load capacity and transmission characteristics of crossed-axis helical gear drive, a generation approach of the gear pair with small-angle based on the curve contact element is proposed. Contact principle based on spatial curve meshing relationships is introduced and geometric models of tooth profiles are developed according to a pair of mated conjugate curves. Furthermore, a mathematical model of crossed-axis helical gear drive with small-angle is established. Numerical examples are illustrated for this research using the 10° shaft angle, and the computerized simulation is also developed based on the solid models. According to gear geometry and finite element method, general characteristics including undercutting conditions, sliding ratios and contact stress for tooth profiles are analyzed. Comparisons with crossed-axis involute gears are also carried out. Finally, the gear prototype is processed using the gear milling method and a basic performance test is conducted. Analysis results show that the new gear pair has well contact characteristics. Further studies on the dynamic analysis and precision manufacturing method will be carried out.

Keywords
Gear transmission, crossed-axis helical gear, curve contact, mathematical model, characteristics analysis

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Introduction

More researchers have paid the attentions to the crossed-axis helical gears for current research involving the fields of automobile differential mechanism, helicopter speed reducer, robot constructions and conveyor driving systems, etc. Usually, tooth profiles of the crossed-axis helical gears mesh with point contact and have the large tooth flank gap.1 Series of works detailing their basic design principle, analysis, manufacturing and inspection had been developed and utilized to fulfill different application requirements. Litvin et al.2 proposed generation process and design approach of standard and non-standard involute crossed helical gears, respectively, using the two generating rack-cutters with a common normal section. Antal and Antal3 present the addendum modification method for design of crossed axes helical gears according to the engagement sliding conditions. Takahashi et al.4 performed endurance experiments for plastic crossed helical gears with grease lubrication. The failure mode of plastic crossed helical gears under grease lubrication conditions is tooth breakage. Additionally, an index for lifetime evaluation was also provided. Hsu and Su5 utilized the modified variable tooth thickness hob to reduce the tooth flank twisting of a longitudinal crowning gear under the conditions of unchanged center distance. Considering the modification of teeth, misalignment error, and machining error, Wang et al.6 put forward an approach for computerized simulation of double helical gears with crossed-axis. Meshing model and TCA procedure of the gears were given. Liu et al.7 investigated the influence of work holding equipment errors on meshing behavior and gear flank geometry of face-hobbed hypoid gear based on accurate mesh model which established from generated process. Song et al.8 analyzed the dynamic characteristics of a marine gearbox with crossed beveloid gears by finite element method.

Crossed-axis helical gears are not widely used for high power motion due to their lower load capacity because of load concentration and varied contact ratio. The authors proposed a new design theory for gear transmission based on curve contact element. A general principle, meshing characteristics analysis and manufacturing method of the gears with parallel axes or intersecting axes had been studied.9–16 Generation method and mathematical model of gear pair have been developed in terms of curve element. Related research conclusions can provide the theoretical basic and technical support for a new crossed-axis helical gear drive. To improve the load capacity and transmission characteristics of the crossed-axis helical gear drive, an approach for generation of the gear pair based on curve contact element is put forward. Mathematical model of the crossed-axis helical gear drive with small angle are established. General characteristics analysis of the new gear pair are also discussed.

Mathematical model of crossed-axis helical gear with small angle

The fixed coordinate systems $S(O−x, y, z)$, $S_p(O_p−x_p, y_p, z_p)$ and movable coordinate systems $S_1(O_1−x_1, y_1, z_1)$, $S_2(O_2−x_2, y_2, z_2)$ are established in Figure 1. The
variables $\omega^{(1)}$ and $\omega^{(2)}$ are angular velocities, $\phi_1$ and $\phi_2$ are rotational angles for the pinion 1 and gear 2, respectively. $\Sigma$ is shaft angle and $a$ is central distance. $P$ is the assumed contact point, and the coordinate transformation matrix $M_{21}$ is represented in equation (1) according to gear geometry.17

\[
M_{21} = \begin{bmatrix}
\cos \phi_1 \cos \phi_2 - \sin \phi_1 \sin \phi_2 \cos \Sigma & -\sin \phi_1 \cos \phi_2 - \cos \phi_1 \sin \phi_2 \cos \Sigma & -\sin \phi_1 \sin \Sigma & -\cos \phi_1 \sin \Sigma \\
\cos \phi_1 \sin \phi_2 + \sin \phi_1 \cos \phi_2 \cos \Sigma & -\sin \phi_1 \sin \phi_2 + \cos \phi_1 \cos \phi_2 \cos \Sigma & 0 & 0 \\
-\sin \phi_2 \sin \Sigma & -a \cos \phi_2 & 0 & 1 \\
\cos \Sigma & 0 & 1 & 0
\end{bmatrix}
\]

(1)

Supposed that the spatial curve $\Gamma_1$: $r_1 = x_1(t)i_1 + y_1(t)j_1 + z_1(t)k_1$ is located in coordinate system $S_1$, the relative velocity at contact point $P$ can be derived using the established relationships between the gear pair in Figure 2.

\[
v^{(12)}_1 = [-y_1(1 + i_{21} \cos \Sigma) - z_1 i_{21} \cos \phi_1 \sin \Sigma - a i_{21} \sin \phi_1 \cos \Sigma]i_1 + [x_1(1 + i_{21} \cos \Sigma) + z_1 i_{21} \sin \phi_1 \sin \Sigma - a i_{21} \cos \phi_1 \cos \Sigma]j_1 + i_{21} \sin \Sigma(x_1 \cos \phi_1 - y_1 \sin \phi_1 - a)k_1,
\]

(2)
where \( t \) is curve parameter, \( i_{21} \) is transmission ratio and \( i_{21} = \phi_2/\phi_1 \).

The curve trihedron displayed in Figure 3 shows that the normal vector at contact point \( M \) can be represented by \( n_n = u n_\beta + v n_\gamma \). The variables \( n_\beta \) and \( n_\gamma \) denote the principal normal and binormal vectors, respectively.

Furthermore, the meshing equation for the designated contact position can be calculated as

\[
[ i_{21} \sin \Sigma (n_{nz1}x_1 - n_{nx1}z_1) - i_{21} \cos \Sigma n_{ny1}a] \cos \phi_1 \\
- [ -i_{21} \sin \Sigma (n_{ny1}z_1 - n_{nx1}y_1) + i_{21} \cos \Sigma n_{nx1}a] \sin \phi_1 \\
= (1 + i_{21} \cos \Sigma)(n_{nx1}y_1 - n_{ny1}x_1) + n_{nz1}ai_{21} \sin \Sigma.
\]

Figure 2. Spatial relationships between the gear pair.

Figure 3. Curve trihedron.

So the spatial curve \( \Gamma_2: r_2 = x_2(t)i_2 + y_2(t)j_2 + z_2(t)k_2 \) which is conjugated to spatial curve \( \Gamma_1 \) can be calculated as
Through the provided equidistant-enveloping method, action surfaces of tooth profiles are studied by the developed conjugate curves pair and the generation process is displayed in Figure 4. The respective equidistant curve $G_i$ is solved using $r_i G_i = \rho_i n_i \cos \phi_i \cos \alpha_i + \rho_i \cos \phi_i \cos \alpha_i$, where $n_i$ is the unit normal vector and $\rho_i$ is the equidistance.

So the general equations of tooth profiles are derived as

$$
\begin{align*}
\{ x_i & = r_i \pm \rho_i n_i^0 x_i + \rho_i \cos \phi_i \cos \alpha_i \\
y_i & = r_i \pm \rho_i n_i^0 y_i + \rho_i \cos \phi_i \cos \alpha_i \quad (i = 1, 2) \\
z_i & = r_i \pm \rho_i n_i^0 z_i + \rho_i \sin \phi_i \\
\Phi_i (r, \phi, \alpha) & = (r_{i\phi}, r_{i\alpha}, r_{i\alpha}) = 0,
\end{align*}
$$

where $r_i (\rho_i \cos \phi \cos \alpha, \rho_i \cos \phi \sin \alpha, \rho_i \sin \phi)$ denotes the sphere surface, $\phi \in [-0.5\pi, 0.5\pi]$ and $\alpha \in [0, 2\pi]$.
Specially, we developed the tooth profiles with convex-to-concave form according to the gear top and root surfaces. Meshing model of tooth profiles is displayed in Figure 5.

**Figure 5.** Generated tooth profiles with point contact.

A cylindrical helix curve is located on pinion 1 and it is described as

\[
\begin{align*}
    x_1 &= R \cos \theta \\
    y_1 &= R \sin \theta \\
    z_1 &= p \theta,
\end{align*}
\]

where \( R \) is the pitch circle radius, \( \theta \) is space curve parameter and \( p \) is helix parameter.

Substituting equation (6) to equations (3) and (4), the conjugated curve equation can be obtained as

\[
\begin{align*}
    x_2 &= R \cos \phi_2 \cos (\theta + \phi_1) + R \sin \phi_2 \cos \Sigma \sin (\theta + \phi_1) + p \theta \sin \phi_2 \sin \Sigma - a \cos \phi_2 \\
    y_2 &= R \sin \phi_2 \cos (\theta + \phi_1) + R \cos \phi_2 \cos \Sigma \sin (\theta + \phi_1) + p \theta \cos \phi_2 \sin \Sigma - a \sin \phi_2 \\
    z_2 &= -R \cos \theta \sin \phi_1 \sin \Sigma - R \sin \theta \cos \phi_1 \sin \Sigma + p \theta \cos \Sigma \\
    \phi_1 &= \arcsin \left( -\frac{\sqrt{B}}{C} \right) \\
    A &= \left[ -i_{21} \sin \Sigma (p \theta n_{y1} - R \sin \theta n_{z1}) + i_{21} \cos \Sigma n_{nx1} a \right] \\
    B &= \left[ (1 + i_{21} \cos \Sigma) (R \sin \theta n_{nx1} - R \cos \theta n_{y1}) + n_{nx1} a i_{21} \sin \Sigma \right]^2 \\
    C &= \left[ [i_{21} \sin \Sigma (R \cos \theta n_{z1} - p \theta n_{nx1}) - i_{21} \cos \Sigma n_{ny1} a \right]^2 \\
    &+ \left[ -i_{21} \sin \Sigma (p \theta n_{y1} - R \sin \theta n_{z1}) + i_{21} \cos \Sigma n_{nx1} a \right]^2 \\
    &+ \left[ (1 + i_{21} \cos \Sigma) (R \sin \theta n_{nx1} - R \cos \theta n_{y1}) + n_{nx1} a i_{21} \sin \Sigma \right]^2 \\
    &- \left[ c_{21} \sin \Sigma (R \cos \theta n_{z1} - p \theta n_{nx1}) - i_{21} \cos \Sigma n_{ny1} a \right]^2 \\
    &+ \left[ -i_{21} \sin \Sigma (p \theta n_{y1} - R \sin \theta n_{z1}) + i_{21} \cos \Sigma n_{nx1} a \right]^2 \\
\end{align*}
\]

(7)
Also the tooth profiles can be solved utilizing the developed equations as

\[
\begin{align*}
    x_{\Sigma 1} &= R \cos \theta + h_1 n_{nx1}^0 + h_1 \cos \phi_1 \cos \alpha_1 \\
    y_{\Sigma 1} &= R \sin \theta + h_1 n_{ny1}^0 + h_1 \cos \phi_1 \sin \alpha_1 \\
    z_{\Sigma 1} &= p \theta + h_1 n_{nz1}^0 + h_1 \sin \phi_1
\end{align*}
\]

\[
\text{Pinion 1 : } \left\{ \begin{array}{l} 
    x_{\Sigma 1} = R \cos \theta + h_1 n_{nx1}^0 + h_1 \cos \phi_1 \cos \alpha_1 \\
    y_{\Sigma 1} = R \sin \theta + h_1 n_{ny1}^0 + h_1 \cos \phi_1 \sin \alpha_1 \\
    z_{\Sigma 1} = p \theta + h_1 n_{nz1}^0 + h_1 \sin \phi_1 + (p + h_1 n_{nz1}^0) \tan \phi_1 = 0
\end{array} \right.
\] (8)

and

\[
\begin{align*}
    x_{\Sigma 2} &= R \cos \phi_2 \cos (\theta + \phi_1) + R \sin \phi_2 \cos \Sigma \sin (\theta + \phi_1) + p \theta \sin \phi_2 \sin \Sigma \\
    y_{\Sigma 2} &= R \sin \phi_2 \cos (\theta + \phi_1) + R \cos \phi_2 \cos \Sigma \sin (\theta + \phi_1) + p \theta \cos \phi_2 \sin \Sigma \\
    z_{\Sigma 2} &= -R \cos \theta \sin \phi_1 \sin \Sigma - R \sin \theta \cos \phi_1 \sin \Sigma + p \theta \cos \Sigma - h_2 n_{nz2}^0 \\
    &+ h_2 \sin \phi_2
\end{align*}
\]

\[
\text{Gear 2 : } \left\{ \begin{array}{l} 
    x_{\Sigma 2} = R \cos \phi_2 \cos (\theta + \phi_1) + R \sin \phi_2 \cos \Sigma \sin (\theta + \phi_1) + p \theta \sin \phi_2 \sin \Sigma \\
    y_{\Sigma 2} = R \sin \phi_2 \cos (\theta + \phi_1) + R \cos \phi_2 \cos \Sigma \sin (\theta + \phi_1) + p \theta \cos \phi_2 \sin \Sigma \\
    z_{\Sigma 2} = -R \cos \theta \sin \phi_1 \sin \Sigma - R \sin \theta \cos \phi_1 \sin \Sigma + p \theta \cos \Sigma - h_2 n_{nz2}^0 \\
    &+ h_2 \sin \phi_2 \\
    &- R \sin [(i_{21} + 1)\phi_1 + \theta][(i_{21} + 1)\phi_1 + \theta] + ai_{21} \phi_1 \theta \sin (i_{21} \phi_1) - h_2 n_{nz2}^0 \\
    &+ \{R \cos [(i_{21} + 1)\phi_1 + \theta][(i_{21} + 1)\phi_1 + \theta] - ai_{21} \phi_1 \theta \cos (i_{21} \phi_1) - h_2 n_{nz2}^0 \} \cos \alpha_2 \\
    &+ (p - h_2 n_{nz2}^0) \tan \phi_2 = 0.
\end{array} \right.
\] (9)

Based on the design parameters in Table 1, we can calculate the data points results by MATLAB software. The schemes of cylindrical helix conjugate curves and tooth surfaces with 10° angle are shown in Figure 6.

Furthermore, the solved data results of tooth profiles to the Pro/E software and the three-dimensional solid models of crossed-axis helical gear pair with 10° angle are established according to Boolean operations. The drawing images are shown in Figure 7.

Computerized simulation for engagement motion of tooth profiles is carried out. The meshing process of gear pair is shown in Figure 8. The results show that gear pair rotates with a fixed transmission ratio and continuous motion. For the axial direction, tooth profiles mesh in point contact and there is no engagement interference during the mated gear pair.

**Characteristics analysis of the established tooth profiles**

**Undercutting conditions**

The undercutting of tooth profiles usually occurs due to the appearance of a singular point. For the proposed crossed-axis helical gear pair with small angle, the pinion is easy to happen undercutting because of the few numbers of teeth. We discuss the undercutting conditions and supposed that the contact region equation is shown as
Table 1. Basic parameters of crossed-axis helical gear pair with 10° angle.

| Parameters                             | Values       |
|----------------------------------------|--------------|
| Shaft angle $\Sigma$ (degrees)         | 10           |
| Pitch circle radius for pinion 1 $R$ (mm) | 35.5         |
| Central distance $a$ (mm)              | 135          |
| Normal module $m_n$ (mm)               | 5            |
| Pressure angle $\alpha_a$ (degrees)    | 30           |
| Contact ratio $i_{21}$                 | 31/11        |
| Teeth number of pinion 1 $Z_1$         | 11           |
| Teeth number of gear 2 $Z_2$           | 31           |
| Helix parameter $\beta$                | 39.28        |
| Convex tooth profile radius $h_1$ (mm) | 5            |
| Concave tooth profile radius $h_2$ (mm) | 5.5          |
| Tooth width $B$ (mm)                   | 30           |
| Curve parameter $\theta$ (rad)         | $[0,0.745]$  |
| Coefficient $u$                        | −0.57        |
| Coefficient $\nu$                      | −0.82        |

Figure 6. Simplified engagement model: (a) spatial curve pair and (b) mated tooth surfaces.

\[
\begin{align*}
    x &= \rho_a \sin \alpha_a - e_a \\
    y &= - (\rho_a \cos \alpha_a - l_a) \cos \beta + u_a \sin \beta \\
    z &= (\rho_a \cos \alpha_a - l_a) \sin \beta + u_a \cos \beta,
\end{align*}
\]

(10)

where $\rho_a$ is convex tooth profile radius, $e_a$ and $l_a$ is the movement and offset distances of circle center, respectively. $\alpha_a$ is pressure angle and $\beta$ is helix angle. $u_a$ is motion displacement with axial direction.
Considering that surface $\Sigma_1$ is tool action surface and it is expressed in double-parameter form, tooth surface $\Sigma_2$ is generated using the proposed surface $\Sigma_1$. Singular points on surface $\Sigma_2$ mean that this surface may happen to undercut during the generation process. Its mathematical descriptions is expressed by equation $\nu_r^{(2)} = 0$ and it has $\nu_r^{(2)} = \nu_r^{(1)} + \nu^{(12)} = 0$, where $\nu^{(12)}$ is the sliding velocity. According to equation (5), utilizing the differentiated geometry, it has

$$
\frac{d}{ds} [\Phi(t, \varphi, \alpha)] = 0,
$$

where $s$ is arc length parameter, $\Phi(t, \varphi, \alpha)$ is enveloping conditions and it has $\Phi(t, \varphi, \alpha) = (r_t, r_\varphi, r_\alpha) = 0$. The line on surface $\Sigma_1$ which generating singular points on surface $\Sigma_2$ can be determined. The following relationship is derived as

$$
\begin{align*}
\frac{\partial r_1}{\partial t} \frac{dt}{ds} + \frac{\partial r_1}{\partial \varphi} \frac{d\varphi}{ds} &= -\nu^{(12)}_r, \\
\frac{\partial \Phi}{\partial t} \frac{dt}{ds} + \frac{\partial \Phi}{\partial \varphi} \frac{d\varphi}{ds} &= -\frac{\partial \Phi}{\partial \alpha} \frac{d\alpha}{ds},
\end{align*}
$$

where $\nu^{(12)}_r$ is the sliding velocity.
where $\partial r_1/\partial t$, $\partial r_1/\partial \varphi$, and $v_1^{(12)}$ are three-dimensional vectors for spatial gearing.

There is two unknowns $dt/ds$ and $d\varphi/ds$ in equation (12), while $d\alpha/ds$ is considered as given. It has a certain solution if the matrix

$$R = \begin{bmatrix} \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \\ \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \\ \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \end{bmatrix}$$

has two rank. And further it can be written as

$$\Delta_1 = \begin{bmatrix} \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \\ \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \\ \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \end{bmatrix}; \quad \Delta_2 = \begin{bmatrix} \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \\ \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \\ \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \end{bmatrix}; \quad \Delta_3 = \begin{bmatrix} \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \\ \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \\ \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \end{bmatrix};$$

$$\Delta_4 = \begin{bmatrix} \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \\ \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \\ \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial \varphi} & -v_1^{(12)} \end{bmatrix},$$

where $v_1^{(12)}$, $v_1^{(12)}$, and $v_1^{(12)}$ are the coordinate components of $v_1^{(12)}$. $\Delta_1$, $\Delta_2$, and $\Delta_3$ are used to form the singularity conditions and singularity of surface $\Sigma_2$ is represented as

$$\Delta_1^2 + \Delta_2^2 + \Delta_3^2 = F(t, \varphi, \alpha) = 0$$

(14)

Substituting equation (10) into above equations, the undercutting conditions of tubular tooth surfaces are obtained as

$$\Delta_1 = -\rho_a \cos \alpha_a \sin \beta \frac{\partial \Phi}{\partial \alpha} - (r_1 \varphi_1 + y) \rho_a \sin \alpha_a \cos \beta \frac{\partial \Phi}{\partial \varphi}$$

$$+ (r_1 \varphi_1 + y) \frac{\partial \Phi}{\partial t} \sin \beta + (z - x) \rho_a \sin \alpha_a \frac{\partial \Phi}{\partial \varphi} = 0,$$

(15)

$$\Delta_2 = (r_1 \varphi_1 + y) \rho_a \sin \alpha_a \sin \beta \frac{\partial \Phi}{\partial \varphi} - \rho_a \cos \alpha_a \cos \beta \frac{\partial \Phi}{\partial \alpha}$$

$$+ (r_1 \varphi_1 + y) \cos \beta \frac{\partial \Phi}{\partial t} + (r_1 \varphi_1 + y) \rho_a \cos \alpha_a \frac{\partial \Phi}{\partial \varphi} = 0,$$

(16)

and
\[ \Delta_1 = (z-x)\rho_a \sin \alpha_a \sin \beta \frac{\partial \Phi}{\partial \varphi} + \rho_a \sin \alpha_a \cos \beta \frac{\partial \Phi}{\partial \alpha} \]
\[ + ( - r_1 \varphi_1 + y) \sin \beta \frac{\partial \Phi}{\partial t} + (z-x) \cos \beta \frac{\partial \Phi}{\partial t} \]
\[ + ( - r_1 \varphi_1 + y) \rho_a \sin \alpha_a \cos \beta \frac{\partial \Phi}{\partial \varphi} = 0, \]  

where

\[ \varphi_1 = \frac{1}{r_1} \left[ \cos \alpha \cos \beta (x + z) + y \right]; \]
\[ \frac{\partial \Phi}{\partial \alpha} = -r_1 (\sin \alpha + \cos \alpha \sin \beta); \]
\[ \frac{\partial \Phi}{\partial \varphi} = \sin \alpha \sin \beta + \cos \alpha; \]
\[ \frac{\partial \Phi}{\partial t} = \rho \sin \alpha \cos \beta (\sin \alpha + \cos \alpha \sin \beta) + (x + z) \sin \alpha \cos \beta \]
\[ + (\cos \alpha - \sin \alpha \sin \beta)(-r_1 \varphi_1 + y + \rho \cos \alpha \cos \beta). \]

Considering the singularity condition of formed tooth profiles, and avoiding undercutting of generated tooth profiles, the equations are given as

\[
\begin{cases}
  r_1 = r_1(t, \varphi) \\
  \Phi(t, \varphi, \alpha) = 0 \\
  F(t, \varphi, \alpha) = 0
\end{cases}
\]  

The equations determine a line which has to limit the generating surface \( P_1 \). In many cases, the undercutting can be avoided by choosing appropriate settings for surface \( P_2 \) that generates \( P_1 \).

**Sliding ratios**

The sliding ratios calculation of tooth profiles based on curve element is studied by Liang et al.\(^{18}\) Similarly, supposing a driving gear with original curve \( \Gamma_1 \) transmits movement to a driven gear with its conjugated curve \( \Gamma_2 \), they contact at point \( K \) as displayed in Figure 9.

\( \Delta S_1 \) and \( \Delta S_2 \) denote respectively the traveling arcs of conjugate curves \( \Gamma_1 \) and \( \Gamma_2 \) in a period of time \( \Delta t \) which approaches to zero during the meshing process. Assuming the relative sliding exists, the length of arc \( MM_1 \) is not equal to that of arc \( MM_2 \), and the difference between \( \Delta S_1 \) and \( \Delta S_2 \) is called the sliding arc. The sliding coefficient is analyzed as a ratio of the length of sliding arc relative to length of the corresponding arc in meshing area. So calculation formulas of sliding ratios of the crossed-axis helical gear pair with small shaft angle are expressed as
sliding ratios of the new gear pair are also calculated and listed in Figure 10(b) for comparison. Obviously, the sliding ratios of the new gear pair are smaller than 0.5 with the growth of parameter \(u\). Sliding ratios of gear pair only pass through zero at pitch point and their symbols change due to the various direction of sliding velocity when the nearby contact points on both sides begin to mesh. The maximum absolute values occur at the tooth root where the gear teeth mesh in and out. However, because of the contact of convex and concave tooth profiles, the absolute values of sliding ratios are smaller than 0.5 with the growth of parameter \(u\). The sliding ratios of the corresponding crossed-axis involute gear drive are also calculated and listed in Figure 10(b) for comparison. Obviously, the sliding ratios of the new gear pair are

\[
U_1 = \lim_{\Delta S_i \to 0} \frac{\Delta S_1 - \Delta S_2}{\Delta S_1} = \frac{[R \cos \phi_2 \cos (\theta + \phi_1) + R \sin \phi_2 \cos \Sigma \sin (\theta + \phi_1) + p \theta \sin \phi_2 \sin \Sigma - a \cos \phi_2]^2}{\sqrt{R^2 + p^2} - \sqrt{R^2 + p^2}} + \frac{[R \sin \phi_2 \cos (\theta + \phi_1) + R \cos \phi_2 \cos \Sigma \sin (\theta + \phi_1) + p \theta \cos \phi_2 \sin \Sigma - a \sin \phi_2]^2}{\sqrt{R^2 + p^2} - \sqrt{R^2 + p^2}} + \frac{[-R \cos \theta \sin \phi_1 \sin \Sigma - R \sin \theta \cos \phi_1 \sin \Sigma + p \theta \cos \Sigma]^2}{\sqrt{R^2 + p^2} - \sqrt{R^2 + p^2}}
\]

and

\[
U_2 = \lim_{\Delta S_i \to 0} \frac{\Delta S_2 - \Delta S_1}{\Delta S_2} = \frac{[R \cos \phi_2 \cos (\theta + \phi_1) + R \sin \phi_2 \cos \Sigma \sin (\theta + \phi_1) + p \theta \sin \phi_2 \sin \Sigma - a \cos \phi_2]^2}{\sqrt{R^2 + p^2} - \sqrt{R^2 + p^2}} + \frac{[R \sin \phi_2 \cos (\theta + \phi_1) + R \cos \phi_2 \cos \Sigma \sin (\theta + \phi_1) + p \theta \cos \phi_2 \sin \Sigma - a \sin \phi_2]^2}{\sqrt{R^2 + p^2} - \sqrt{R^2 + p^2}} + \frac{[-R \cos \theta \sin \phi_1 \sin \Sigma - R \sin \theta \cos \phi_1 \sin \Sigma + p \theta \cos \Sigma]^2}{\sqrt{R^2 + p^2} - \sqrt{R^2 + p^2}}
\]
smaller than that of crossed-axis involute gear drive, which can help to improve the transmission performance. The meshing process can realize limit position and the approximate pure rolling contact may be accomplished in theory.

**Stress analyses**

Stress analysis of tooth profiles can show the mechanics properties of the new gear drive. Finite element model of conjugate gear pair with 10° shaft angle is shown in Figure 11 and the model is analyzed with ANSYS Workbench. Considering the actual engagement conditions, gear pair is plotted with hexahedron unit Solid 185. Tooth profiles of the pinion and gear are defined as the contact surface and target surface, respectively, which are also corresponding to contact unit CONTA 173.
and TARGE 170. Hertz model is applied to analysis process of contact stress. The extended Lagrange algorithm is regarded as the calculation method and MPC 184 constraint unit is also conducted to the whole process. The selected material for the simulation was 20CrMnTi steel, with Poisson coefficient of 0.25 and Young modulus of 205 GPa. A torque value of 200 Nm is applied to the pinion.

Analysis results of the proposed new gear pair are displayed in Figure 12. The maximum contact stress is 1266.5 MPa. The maximum stress occurs at contact point, which locates on the middle of tooth profile. It has regular elliptical distribution along the direction of tooth width, and the distribution area has the trend of expanding to the tooth root direction. With the increase of the contact area, the contact stress will gradually decrease. The maximum von Mises stress of the pinion is 793.69 MPa.

Under the same settings, the involute gear pair is used as the contrast object, and its contact state and finite element analysis results are shown in Figure 13. The maximum contact stress is 1639 MPa, which occurs at the meshing position, and the stress is distributed along the contact line. The maximum von Mises stress of the pinion is 1233.8 MPa. Obviously, contact stress results of the new gear pair are smaller than that of crossed-axis involute gear drive, which show the well contact capacity and strength.

**Gear prototype and experimental study**

Gear milling method is used to achieve the gear pair due to the special tooth profiles form. According to an established solid model, the simulation motion process of machine tool with ball milling cutter can be obtained. The program of machining codes considering milling cutter and workbench motions is developed by CNC (Computer Numerical Control) five-axis machining center DMU60 in Figure 14(a). Generally, based on fast moving and feeding function of ball milling cutter, rotation function of working axis and workbench, the gear pair can be
manufactured. The final generated pinion and gear are depicted in Figure 14(b) and (c), respectively.

Furthermore, performance experiment of gear prototype is carried out and the trial site is shown in Figure 15. The equipments are linked by spring coupling. Input and output torques are measured by torque and rotational speed transducer. Oil temperature in the box is measured by the temperature transducer. Rotational speed is controlled by the variable speed electric motor, and the gear pair is loaded by the loading motor.

The contact point between the conjugate tooth profiles will spread over a small area under the load due to elastic deformations. To achieve a better performance, it is necessary to carry out the running-in process. It can expand the contact area for increasing the load capability and modify the gear tooth surfaces for reducing noise and vibration. During the test process, transmission efficiency of gear prototype can be obtained and displayed in the screen through the backstage calculation formula \( \eta = n_o T_o / n_i T_i \), where \( T_i \) and \( n_i \) are the input torque and input shaft speed,
To and no are the output torque and output shaft speed. The setting speeds are 250, 500, 750, 1000, and 1250 rpm, respectively. The load applied to tooth profiles are 200, 300, 400, 500, and 600 Nm, respectively. Oil temperature at each stage is also recorded.

Transmission efficiency and oil temperature results under given work conditions are shown in Figures 16 and 17. It can be concluded that the transmission efficiency will increase by increasing rotational speed and keeping torque constant. Similarly, it will also increase by increasing torque and keeping rotational speed constant. The maximum efficiency may be up to 95.9% at the load of 600 Nm and the whole efficiency of gear prototype is in the range of 91.2%–95.9%. The oil temperature arrives at balance when the time is 70–80 min and the highest value of oil temperature is about 69.7°C with respect to the room temperature.
Conclusion

1. Contact principle and generation method of the gear pair with small angle based on curve contact element is proposed. The mated tooth profiles are developed through the given spatial conjugate curves. Mathematical model of the presented gear drive is established. Numerical example is illustrated using the $10^\circ$ shaft angle, and the three-dimensional solid models of crossed-axis helical gear pair with $10^\circ$ angle are established by MATLAB and Pro/E software. Computerized meshing motion is simulated and developed to verify the transmission conditions.

2. Considering the singularity condition of formed tooth profiles, and avoiding undercutting of generated tooth profiles, the general equations are derived. In additional, calculation method of sliding ratios of tooth profiles is provided based on the developed conjugate curves. Under the given parameters, sliding ratios of the new gear pair are calculated. The results are smaller than that of crossed-axis involute gear drive, which can help to improve the transmission performance.

3. Stress analysis of the new gear drive is carried out by ANSYS software. The maximum contact stress is 1266.5 MPa, occurring at contact point position, which locates on the middle of tooth profile. It has regular elliptical distribution along the direction of tooth width, and the distribution area has the trend of expanding to the tooth root direction. With the increase of the contact area, the contact stress will gradually decrease. The maximum von Mises stress of the pinion is 793.69 MPa. The involute gear pair is used as the contrast object, and the maximum contact stress is 1639 MPa, which occurs at the meshing position, and the stress is distributed along the contact line. The maximum von Mises stress of the pinion is 1233.8 MPa. Obviously, contact stress results of the new gear pair are smaller than that of crossed-axis involute gear drive, which show the well contact capacity and strength.

Figure 17. Oil temperature results of the new gear pair.
4. Gear milling method is used to achieve the gear pair due to the special tooth profiles form. According to performance experiment of gear prototype, the transmission efficiency will increase by increasing rotational speed and keeping torque constant. The maximum efficiency may be up to 95.9% at the load of 600 Nm and the whole efficiency of gear prototype is in the range of 91.2%–95.9%. The oil temperature arrives at balance when the time is 70–80 min and the highest value of oil temperature is about 69.7°C with respect to the room temperature.

5. The further study on the dynamic analysis and precision manufacturing technology will be carried out.

Declaration of conflicting interests
The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

Funding
The author(s) disclosed receipt of the following financial support for the research, authorship, and/or publication of this article: This research was supported by National Natural Science Foundation of China (Grant No. 51975078), Fundamental Research and Frontier Exploration Program of Chongqing City (Grant No. cstc2018jcyjAX0029), Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No. KJQN201900736) and Chongqing Key Laboratory of Urban Rail Transit System Integration and Control Open Fund (Grant No. CKLURTSIC-KFKT-202005).

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