Differential Privacy of Dirichlet Posterior Sampling

Donlapark Ponnoprat*

Abstract

Besides the Laplace distribution and the Gaussian distribution, there are many more probability distributions that are not well-understood in terms of privacy-preserving property—one of which is the Dirichlet distribution. In this work, we study the inherent privacy of releasing a single draw from a Dirichlet posterior distribution (the Dirichlet posterior sampling). As our main result, we provide a simple privacy guarantee of the Dirichlet posterior sampling with the framework of Rényi Differential Privacy (RDP). Consequently, the RDP guarantee allows us to derive a simpler form of the $(\epsilon, \delta)$-differential privacy guarantee compared to those from the previous work. As an application, we use the RDP guarantee to derive a utility guarantee of the Dirichlet posterior sampling for privately releasing a normalized histogram, which is confirmed by our experimental results. Moreover, we demonstrate that the RDP guarantee can be used to track the privacy loss in Bayesian reinforcement learning.

1 Introduction

The Bayesian framework provides a way to perform statistical analysis by combining prior beliefs with real-life evidence. At a high level, the belief and the evidence are assumed to be described by probabilistic models. As we receive new data, our belief is updated accordingly via the Bayes’ theorem, resulting in the so-called posterior belief. The posterior tells us how much we are uncertain about the model’s parameters.

The Dirichlet distribution is usually chosen as the prior when performing Bayesian analysis on discrete variables, as it is a conjugate prior to the categorical and multinomial distributions. Specifically, Dirichlet distributions are often used in discrete mixture models, where a Dirichlet prior is put on the mixture weights [LW92, MMR05]. Such models have applications in NLP [PB98], biophysical systems [Hin15], accident analysis [de 06], and genetics [BHW00, PM01, CWS03]. All of these studies involve samplings from Dirichlet posteriors as parts of approximate Bayesian inferences.

Sampling from a Dirichlet posterior also appears in other Bayesian learning tasks. For example, in Bayesian active learning, it arises in Gibbs sampling, which is used to approximate the posterior of the classifier over the labeled sample [NLY+13]. In Thompson sampling for multi-armed bandits, one repeatedly draws a sample from the Dirichlet posterior of each arm, and picks the arm whose sample maximizes the reward [ZHG+20, AAFK20, NIK20]. And in Bayesian reinforcement learning, state-transition probabilities are sampled from the Dirichlet posterior over past observed states [Str00, ORR13].

In the above examples, the data that we integrate into these tasks might contain sensitive information. For example, Gibbs sampling could have been applied to private chat messages, or

*Chiang Mai University, Chiang Mai, Thailand. Email: donlapark.p@cmu.ac.th
genetic data of patients in a hospital. A Bayesian reinforcement learning can be employed as a movie recommender system, trained with each user’s watch history. However, watch histories are sensitive, as they can be used to identify the users’ viewpoints or political opinions. Thus it is important to ask: how much of the sensitive information is protected by sampling from a Dirichlet distribution? The goal of this study is to find an answer to this question.

The mathematical framework of differential privacy [DKM+06, DMNS06] allows us to quantify privacy guarantees of randomized algorithms. Differential privacy is a strong privacy model which assumes that the adversary knows the information of all but one users. Thus even count data of a sensitive attribute (e.g. cancer diagnosis) can reveal an individual’s record of that attribute if combined with side information. There are also several relaxed notions of differential privacy, such as approximate differential privacy [DKM+06, DMNS06], Rényi differential privacy [Mir17], and concentrated differential privacy [BS16]. It is natural to wonder if the Dirichlet posterior sampling satisfies any of these definitions.

1.1 Overview of Our results

This study focuses on the privacy and utility of Dirichlet posterior sampling. In summary, we provide a closed-form privacy guarantee of the Dirichlet posterior sampling, which allows us to effectively analyze its utility in various settings.

§3 Privacy. We study the role of the prior parameters in the privacy of the Dirichlet posterior sampling. Theorem 2 is our main result, where we provide a guaranteed upper bound for Rényi differential privacy (RDP) of the Dirichlet posterior sampling. In addition, we convert the RDP guarantee into an approximate differential privacy guarantee in Theorem 4.

§4 Utility. We use the Dirichlet posterior sampling for a private release of a normalized histogram. In this case, the accuracy is measured by the mean-squared error between the sample and the original normalized histogram. We compute the sample size that guarantees the desired level of accuracy. We also compare the Dirichlet posterior sampling to the Gaussian mechanism.

§5 Experiment on Bayesian Reinforcement Learning. To show a use case on more complex task, we apply our results to make Bayesian reinforcement learning differentially private by modifying the Dirichlet sampling for state transition probabilities. In this experiment, we compare between two variants of the Dirichlet posterior sampling, namely the diffuse sampling and the concentrated sampling.

1.2 Notations

We let \( \mathbb{R}_{\geq 0}^d \) be the set of \( d \)-tuples of non-negative real numbers and \( \mathbb{R}_{> 0}^d \) be the set of \( d \)-tuples of positive real numbers. We assume that all vectors are \( d \)-dimensional where \( d \geq 2 \). The notations for all vectors are always in bold. Specifically, \( \mathbf{x} := (x_1, \ldots , x_d) \in \mathbb{R}_{\geq 0}^d \) consists of sample statistics of the data and \( \mathbf{\alpha} := (\alpha_1, \ldots , \alpha_d) \in \mathbb{R}_{> 0}^d \) consists of the prior parameters. The vector \( \mathbf{p} := (p_1, \ldots , p_d) \) always satisfies \( \sum_i p_i = 1 \). The number of observations is always \( N \). We also denote \( x_0 := \sum_i x_i \) and \( \alpha_0 := \sum_i \alpha_i \). For any vectors \( \mathbf{x}, \mathbf{x}' \) and scalar \( r > 0 \), we write \( \mathbf{x} + \mathbf{x}' := (x_1 + x'_1, \ldots , x_d + x'_d) \) and \( r\mathbf{x} := (rx_1, \ldots , rx_d) \). For any positive reals \( x \) and \( x' \), the notation \( x \propto x' \) means \( x = Cx' \) for some constant \( C > 0 \), \( x \approx x' \) means \( cx' \leq x \leq Cx' \) for some \( c, C > 0 \), and \( x \lesssim x' \) means \( x \leq Cx' \) for some \( C > 0 \). Lastly, \( \|\mathbf{x}\|_\infty := \max_i |x_i| \) is the \( \ell_\infty \) norm of \( \mathbf{x} \).
2 Background and related work

2.1 Privacy models

We say that two datasets are neighboring if they differ on a single entry. Here, an entry can be a row of the datasets, or a single attribute of a row.

Definition 2.1 (Pure and Approximate differential privacy [DKM+06, DMNS06]). A randomized mechanism $M : \mathcal{X}^n \to \mathcal{Y}$ is $(\varepsilon,\delta)$-differentially private ($(\varepsilon,\delta)$-DP) if for any two neighboring datasets $x$ and $x'$ and all events $E \subset \mathcal{Y}$,

$$P[M(x) \in E] \leq e^\varepsilon P[M(x') \in E] + \delta.$$  

(1)

If $M$ is $(\varepsilon,0)$-DP, then we say that it is $\varepsilon$-differential private ($\varepsilon$-DP).

The term pure differential privacy (pure DP) refers to $\varepsilon$-differential privacy, while approximate differential privacy (approximate DP) refers to $(\varepsilon,\delta)$-DP when $\delta > 0$.

In contrast to pure and approximate DP, the next definitions of differential privacy are defined in terms of the Rényi divergence between $M(x)$ and $M(x')$:

Definition 2.2 (Rényi Divergence [Rén61]). Let $P$ and $Q$ be probability distributions. For $\lambda \in (1, \infty)$ the Rényi divergence of order $\lambda$ between $P$ and $Q$ is defined as

$$D_\lambda(P||Q) = \frac{1}{\lambda - 1} \log \left( \mathbb{E}_{y \sim P} \left[ \frac{P(y)^{\lambda-1}}{Q(y)^{\lambda-1}} \right] \right).$$

Definition 2.3 (Rényi differential privacy [Mir17]). A randomized mechanism $M : \mathcal{X}^n \to \mathcal{Y}$ is $(\lambda,\varepsilon)$-Rényi differentially private ($(\lambda,\varepsilon)$-RDP) if for any two neighboring datasets $x$ and $x'$,

$$D_\lambda(M(x)||M(x')) \leq \varepsilon.$$

Intuitively, $\varepsilon$ controls the moments of the privacy loss random variable: $Z := \log \frac{P[M(x)=Y]}{P[M(x')=Y]}$, where $Y$ is distributed as $M(x)$, up to order $\lambda$. A smaller $\varepsilon$ and larger $\lambda$ correspond to a stronger privacy guarantee.

We can convert from RDP to approximate DP:

Lemma 1 (From RDP to Approximate DP [CKS20]). Let $\varepsilon > 0$. If $M$ is a $(\lambda,\hat{\varepsilon})$-RDP mechanism, then it also satisfies $(\varepsilon,\delta)$-DP with

$$\delta = \frac{\exp((\lambda - 1)(\hat{\varepsilon} - \varepsilon))}{\lambda - 1} \left( 1 - \frac{1}{\lambda} \right)^\lambda$$  

(2)

The composition property allow us to use the Dirichlet posterior sampling as a building block for more complex algorithms.

Lemma 2 (Composition Property [Mir17]). Let $M_1 : \mathcal{X}^n \to \mathcal{Y}$ be a $(\lambda_1,\varepsilon_1)$-RDP mechanism and $M_2 : \mathcal{X}^n \to \mathcal{Z}$ be a $(\lambda_2,\varepsilon_2)$-RDP mechanism. Then a mechanism $M : \mathcal{X}^n \to \mathcal{Y} \times \mathcal{Z}$ defined by $M(x) = (M_1(x), M_2(x))$ is $(\min(\lambda_1, \lambda_2), \varepsilon_1 + \varepsilon_2)$-RDP.
2.2 Dirichlet distribution

For $\alpha \in \mathbb{R}_d^+$, the Dirichlet distribution $\text{Dir}(\alpha)$ is a continuous distribution on $S_d = \{y \in \mathbb{R}_d^+ \mid \sum_i y_i = 1\}$. The density function of $\text{Dir}(\alpha)$ is given by:

$$p(y) = \frac{1}{B(\alpha)} \prod_{i=1}^d y_i^{\alpha_i - 1},$$

for any $y \in S_d$. Here, $B(\alpha)$ is the beta function, which can be written in terms of the gamma function:

$$B(\alpha) = \frac{\prod_i \Gamma(\alpha_i)}{\Gamma(\sum_i \alpha_i)}.$$  

(3)

2.3 Dirichlet posterior sampling

In Bayesian statistics, one specifies their prior beliefs to a model’s parameters $y$ through a prior distribution $p(y)$. After observing data $x$, its influence on the prior beliefs is associated with the likelihood $p(x|y)$. The beliefs on $y$ upon seeing $x$ are then updated via the Bayes’ rule: $p(y|x) = \frac{p(x|y)p(y)}{p(x)}$. Here, $p(y|x)$ is called the posterior distribution.

We consider the prior $p(y) \sim \text{Dir}(\alpha)$ and the likelihood of the form $p(x|y) \propto \prod_{i=1}^d y_i^{x_i}$ where $x \in \mathbb{R}_d^+$ consists of sample statistics of the dataset. The Dirichlet posterior sampling is a single draw from the Dirichlet posterior:

$$Y \sim \text{Dir}(x + \alpha).$$  

(4)

There is a modification of the sampling which introduces a concentration parameter $r > 0$, and instead we sample from $\text{Dir}(rx + \alpha)$ [GSC17, GWH+21]. Smaller values of $r$ make the sampling more private, and larger values of $r$ make $Y$ a closer approximation of $x$. Even though the case $r = 1$ is the main focus of this study, our main privacy results can be easily extended to other values of $r$ as we will see at the end of Section 3.3.

Consider a special case where $x$ is a normalized histogram. We can obtain a private approximation of $x$ by making a single draw $Y \sim \text{Dir}(rx + \alpha)$; this process is called the Dirichlet mechanism [GWH+21].

Dirichlet mechanism as an exponential mechanism. Let $x$ be a normalized histogram, $r > 0$ be the privacy parameter, $\text{Dir}(\alpha)$ be the prior, and the negative KL-divergence be the score function of the exponential mechanism [MT07]. Then the output $Y$ of this mechanism is distributed according to the following density function:

$$\frac{\exp(-r D_{KL}(x,y)) \prod_i y_i^{\alpha_i-1}}{\int \exp(-r D_{KL}(x,y)) \prod_i y_i^{\alpha_i-1} dy} \propto \exp \left( r \sum_{i,x_i\neq 0} x_i \log \left( \frac{y_i}{x_i} \right) \right) \prod_i y_i^{\alpha_i-1} \propto \prod_{i,x_i\neq 0} y_i^{rx_i} \prod_i y_i^{\alpha_i-1} = \prod_i y_i^{rx_i+\alpha_i-1},$$

which is exactly the density function of $\text{Dir}(rx + \alpha)$. 

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2.4 Polygamma functions

In most of this study, we take advantage of several nice properties of the log-gamma function and its derivatives. Specifically, \( \psi(x) := \frac{d}{dx} \log \Gamma(x) \) is concave and increasing, while its derivative \( \psi'(x) \) is positive, convex, and decreasing. In addition, \( \psi' \) can be approximated by the reciprocals:

\[
1 \frac{1}{x} + 1 \frac{1}{2x^2} < \psi'(x) < 1 \frac{1}{x} + \frac{1}{x^2},
\]

which implies that \( \psi'(x) \approx \frac{1}{x^2} \) as \( x \to 0 \) and \( \psi'(x) \approx \frac{1}{x} \) as \( x \to \infty \).

2.5 Related work

There are several studies on the differential privacy of Bayesian posterior sampling. [WFS15] showed that any posterior sampling with the absolute value of the log-likelihood bounded above by \( B \) is \( 4B \)-differentially private. However, the likelihoods that we study are not bounded away from zero; they have the form \( \prod_i y_i \frac{1}{x_i} \) which becomes small when one of the \( y_i \)'s is close to zero. [DNZ+17] showed that if the log-likelihood \( \log p(x|y) \) is Lipschitz continuous in \( x \) with high probability in \( y \), then one can obtain the approximate DP. They proved that a single draw from the Beta distribution is \((0, \delta)-DP\), and the result cannot be improved unless the prior is assumed to be above a positive threshold. As a continuation of their work, we prove that, when the prior is bounded below by a fixed positive number, the Dirichlet posterior sampling is \((\varepsilon, \delta)-DP\) with \( \varepsilon > 0 \).

Let \( x \) be a sufficient statistic of an exponential family with finite \( \ell^1 \)-sensitivity. [FGWC16] showed that sampling \( Y \sim p(y|x) \), where \( \hat{x} = x + \text{Laplace noise} \), is differentially private and as asymptotically efficient as sampling from \( p(y|x) \). However, for a small sample size, the posterior over the noisy statistics might be too far away from the actual posterior. [BS18] thus proposed to approximate the joint distribution \( p(y, x, \hat{x}) \) using Gibbs sampling, which is then integrated over \( x \) to obtain a more accurate posterior over \( \hat{x} \).

[GSC17] were the first to study the posterior sampling with Rényi differential privacy (RDP; [Mir17]). Even though they provided a general framework to find \((\lambda, \varepsilon)-RDP\) guarantees for exponential families, explicit forms of \( \lambda \) and the upper bound of \( \lambda \) were not given. Specifically, they show that the Dirichlet posterior sampling is \((\lambda, \varepsilon)-RDP\) where

\[
\varepsilon = \lambda \Delta_2^2 \sup_x ||\nabla^2 \log B(x + \alpha)||,
\]

where \( ||\cdot|| \) is the spectral norm and \( \Delta_2 \) is the \( \ell^2 \)-sensitivity (see Section 3.1 below). This bound is cumbersome in practice due to the supremum and the spectral norm. In Section 3.3, we provide an RDP guarantee of the Dirichlet posterior sampling that has a simpler form of \( \varepsilon \), as well as an upper bound for \( \lambda \).

The privacy of data synthesis via sampling from Multinomial(\( Y \)), where \( Y \) is a discrete distribution drawn from the Dirichlet posterior, was first studied by [MKA+08]. They showed that the data synthesis is \((\varepsilon, \delta)-approximate DP\), where \( \varepsilon \) grows by the number of draws from Multinomial(\( Y \)). In contrast, we show that a single draw from the Dirichlet posterior is approximate DP, which by the post-processing property allows us to sample from Multinomial(\( Y \)) as many times as we want while retaining the same privacy guarantee.

The Dirichlet mechanism was first introduced by [GWH+21]. Originally, the Dirichlet mechanism takes a discrete distribution \( x := (x_1, \ldots, x_d) \) and draws one sample \( Y \sim \text{Dir}(r \mathbf{x}) \). Note
Figure 1: Left: the actual values of $\varepsilon = \frac{1}{2} \lambda D_X(P\|P')$ with $\lambda = 2$ and the $(2, \varepsilon)$-RDP guarantees as shown in (8) at $\Delta_2^2 = \Delta_\infty = 1$ and $\alpha = (\alpha, \ldots, \alpha)$. Right: $(\varepsilon, \delta)$-DP guarantees of the Dirichlet posterior samplings with three different $\alpha = (\alpha, \ldots, \alpha)$.

The absence of the prior parameters, which makes $Y$ an unbiased estimator of $x$. However, one has to deal with privacy violation that occurs when near-zero outputs are observed. The authors tackled this issue by letting $\delta$ be the probability that “bad” events happen: let $W \subset \mathbb{R}^d$ and, for any $\gamma \in (0, 1)$,

$$
\Omega^\gamma_W = \left\{ y \in \mathbb{R}^d_{\geq 0} \mid \sum_i y_i = 1, y_i \geq \gamma \text{ for all } i \in W \right\}.
$$

They showed that, for $\eta, \bar{\eta} \in (0, 1)$ such that $\eta + \bar{\eta} < \frac{1}{2}$ and $b \in (0, 1]$, drawing one sample $Y \sim \text{Dir}(r x)$ is $(\varepsilon, \delta)$-DP where

$$
\varepsilon = \log \left( \frac{\text{beta}(r \eta, r(1 - \eta - \bar{\eta}))}{\text{beta}(r(\eta + \frac{b}{2}), r(1 - \eta - \bar{\eta} - \frac{b}{2}))} \right) + \frac{rb}{2} \log \left( \frac{1}{\gamma} \left( 1 - (|W| - 1)^\gamma \right) \right),
$$

$$
\delta = 1 - \min_{x \in \Omega^\eta_{W}^b} \mathbb{P}_{Y' \sim \text{Dir}(rx)}[Y' \in \Omega^\gamma_W],
$$

where $\Omega^\eta_{W}^b$ is some convex subset of $\Omega^\eta_{W}$. This guarantee requires adjusting several parameters, as well as optimizing over a subset of probability $d$-simplex. In Section 3.4, we will provide an $(\varepsilon, \delta)$-DP guarantee in a relatively simpler form.

3 Main privacy results

3.1 Problem formulation

Given a dataset $D \in X^N$, we consider a statistic $x$ of $D$ with the following properties: (1) all components of $x$ are nonnegative, that is, $x \in \mathbb{R}^d_{\geq 0}$, (2) this statistic satisfies $\ell^2$- and $\ell^\infty$-sensitivity
bounds: there exist two constants $\Delta_2, \Delta_\infty > 0$ such that, for two such statistics $x$ and $x'$ of any two neighboring datasets,

$$\sum_i (x_i - x'_i)^2 \leq \Delta_2^2 \quad \text{and} \quad \max_i |x_i - x'_i| \leq \Delta_\infty.$$

With the statistic $x$ as above and $\alpha \in \mathbb{R}^d > 0$, we consider the Dirichlet posterior sampling: $Y \sim \text{Dir}(x + \alpha)$. Given $y \in \mathbb{R}^d_>$, we would like to see if the density $\mathbb{P}[Y = y]$ satisfies any of the definitions of differential privacy introduced in Section 2.1.

### 3.2 Dirichlet posterior sampling is not $\varepsilon$-differentially private

We start by showing that the Dirichlet posterior sampling does not satisfy the most fundamental notion of differential privacy—the pure differential privacy.

**Proposition 1.** For any $\varepsilon > 0$, the Dirichlet posterior sampling, as defined in (4), is not $\varepsilon$-differentially private.

**Proof.** Without loss of generality, let $x = (0, 0, \ldots, 0)$ and $x' = (1, 0, \ldots, 0)$. Let $\alpha$ be any vector in $\mathbb{R}^d_>$. Let $Y \sim \text{Dirichlet}(x + \alpha)$ and $Y' \sim \text{Dirichlet}(x' + \alpha)$. Let $y = (y_1, y_2, \ldots, y_d)$, where $y_1 \in (0, 1)$ which will be chosen later (then we can pick any combination of $y_2, \ldots, y_d \in (0, 1)$ that yields $\sum_i y_i = 1$). With $\delta = 0$ in (1), we check the ratio:

$$\frac{\mathbb{P}[Y = y]}{\mathbb{P}[Y' = y]} = \frac{B(x' + \alpha)}{B(x + \alpha)} \cdot \frac{\prod_i y_i^{x_i + \alpha_i}}{\prod_i y_i'^{x_i' + \alpha_i}} = \frac{B(x' + \alpha)}{B(x + \alpha)} \cdot \frac{1}{y_1}.$$

For any $\varepsilon > 0$, we can choose a sufficiently small $y_1 > 0$ so that the right-hand side is larger than $e^\varepsilon$. \qed

Since there is no hope for pure differential privacy, we turn our attention to one of relaxed notions of differential privacy. We shall see below that, with Rényi differential privacy (RDP), we can derive the privacy guarantee of the Dirichlet posterior sampling in a simple form.

### 3.3 Rényi differential privacy

Our RDP-guarantee depends on the $\ell_2$- and $\ell_\infty$-sensitivity.

**Theorem 2.** Let $\alpha \in \mathbb{R}^d_>$ and $\alpha_m := \min_i \alpha_i$. For any $\lambda \in (1, \alpha_m/\Delta_\infty + 1)$, define

$$g(\lambda) := (\lambda - 1)\Delta_\infty$$

Then, for any $x \in \mathbb{R}^d_>$, a single draw from $\text{Dir}(x + \alpha)$ is $(\lambda, \varepsilon)$-RDP, where

$$\varepsilon = \frac{1}{2} \lambda \Delta_2^2 \psi'(\alpha_m - g(\lambda)).$$

(8)
The proof of Theorem 2 is provided in Appendix A.1. Here, the RDP guarantee is simpler than (6) by [GSC17].

In practice, one first specifies a level of RDP guarantee \((\lambda, \varepsilon)\) that they would like to achieve. Then, as \(\psi'\) is strictly decreasing, any root finding algorithm can be used to find \(\alpha_m\) that satisfies (8).

Alternatively, we can upper bound the right-hand side of (8) using (5) and the fact that \(\frac{1}{x} + \frac{1}{x^2} < \frac{1}{x} - \frac{1}{x^2}\) for any \(x > 1\). For any \(\alpha_m > g(\lambda) + 1\), this results in a \((\lambda, \varepsilon)\)-RDP guarantee for any \(\lambda > 1\) and:

\[
\varepsilon = \frac{\lambda \Delta_2^2}{2(\alpha_m - g(\lambda) - 1)}.
\]

Thus, we can choose

\[
\alpha_m = \frac{\lambda \Delta_2^2}{2\varepsilon} + g(\lambda) + 1, \tag{9}
\]

to achieve \((\lambda, \varepsilon)\)-RDP.

Example. Let \(x = \{x_1, \ldots, x_d\}\) be a histogram of \(N\) observations, each of which has \(d\) possible values: \(\{1, \ldots, d\}\). We will use (8) to find an algorithm that privately releases \(x/N\) with \((\lambda, \varepsilon)\)-RDP guarantee, where \(\lambda = 2, \varepsilon = 1\). If we were to change the value of an observation from \(i\) to \(j\), then \(x_i\) and \(x_j\) would be altered by at most one. Therefore, \(\Delta_2^2 = 2, \Delta_\infty = 1\). Solving (8) with \(\varepsilon = 1\) and \(\lambda = 2\) is equivalent to solving \(\psi'(\alpha_m - 1) = 1\), which results in \(\alpha_m < 3.46\). Thus we can privately release \(x/N\) by sampling from Dir(\(x + \alpha\)), where \(\alpha = (3.46, \ldots, 3.46)\). Alternatively, we can use (9) which leads to \(\alpha_m = 4\).

The guaranteed upper bound (8) is independent of the sample statistics. As a result, the bound applies even in worst settings i.e., when \(x_i = 0\) and \(x'_i = \Delta_\infty\) for some \(i\). As we can see in Figure 1, the upper bound is a close approximation to the actual value of \(\varepsilon\) when \(x_6 = 0\) and \(x'_6 = 1\). However, being a sample independent bound, the difference becomes substantial when all \(x_i\)'s are large. There is one way to get around this issue: if there is no privacy violation in assuming that the sample statistics are always bounded below by some threshold \(\tau\), then we can incorporate the threshold into the prior.

Theorem 2 can be easily applied to sampling from Dir(\(rx + \alpha\)). Replacing \(x\) with \(rx\), \(\Delta_2\) is replaced by \(r \Delta_2\) and \(\Delta_\infty\) is replaced by \(r \Delta_\infty\). We thus have the following corollary:

**Corollary 3.** Let \(\alpha \in \mathbb{R}_{\geq 0}^d\) and \(\alpha_m := \min_i \alpha_i\). For any \(\lambda \in (1, \alpha_m/r \Delta_\infty + 1)\), define

\[
g(\lambda) := (\lambda - 1)r \Delta_\infty
\]

Then, for any \(x \in \mathbb{R}_{\geq 0}\) and \(r > 0\), a single draw from Dir(\(rx + \alpha\)) is \((\lambda, \varepsilon)\)-RDP, where

\[
\varepsilon = \frac{1}{2} \lambda r^2 \Delta_2^2 \psi'(\alpha_m - g(\lambda)). \tag{10}
\]

**Scaling between \(r\) and \(\alpha_m\).** From Corollary 3, there are now two parameters to be tuned: \(r\) and \(\alpha_m\). It might be fruitful to analyze how these parameters scale against each other at fixed privacy parameters \(\varepsilon\) and \(\lambda\).

- For \(\alpha_m < g(\lambda) + 1\), we have \(\psi'(\alpha_m - g(\lambda)) \approx 1/(\alpha_m - g(\lambda))^2\), so in view of (13), \(\alpha_m\) scales as \(r\).
• For $\alpha_m \geq g(\lambda) + 1$, we have $\psi'(\alpha_m - g(\lambda)) \approx 1/(\alpha_m - g(\lambda))$, so with $\alpha_m \gg g(\lambda)$, $\alpha_m$ scales as $r^2$.

In the case of Dirichlet mechanism (i.e. using a single draw from $\text{Dir}(x + \alpha)$ as a private release of $x/N$), adjusting $r$ and $\alpha_m$ leads to a bias-variance tradeoff situation: as $r$ increases, the variance of the draw decreases. However, if we would like to keep $\varepsilon$ the same, then $\alpha$ has to be increased in response, which in turn increases the bias of the draw.

We now convert RDP to approximate DP.

### 3.4 Approximate differential privacy

Let $\varepsilon > 0$. Using Lemma 1, the Dirichlet posterior sampling, with $\text{Dir}(\alpha)$ as the prior, is $(\varepsilon, \delta)$-DP with

$$
\delta = \exp((\lambda - 1)(\hat{\varepsilon}(\lambda) - \varepsilon)) \left(1 - \frac{1}{\lambda}\right)^\lambda
$$

where $\hat{\varepsilon}(\lambda) = \frac{1}{2} \lambda \Delta_2^2 \psi'(\alpha_m - g(\lambda))$.

We try to minimize $\delta$ by adjusting $\lambda$. First, we take the logarithm of (11).

$$
f(\lambda) := (\lambda - 1)(\hat{\varepsilon}(\lambda) - \varepsilon) + (\lambda - 1) \log(\lambda - 1) - \lambda \log(\lambda).
$$

Recall that $g(\lambda) = (\lambda - 1)\Delta_\infty$. Since $\psi'$ is a strictly convex function, $\hat{\varepsilon}$ is also strictly convex. Also, the second order derivative of $(\lambda - 1) \log(\lambda - 1) - \lambda \log(\lambda)$ is $1/(\lambda - 1) - 1/\lambda > 0$ on $(1, \infty)$. Thus, $f(\lambda)$ is a sum of two strictly convex functions on $(1, \alpha_m/\Delta_\infty + 1)$, and hence is itself strictly convex.

As $\lambda \to 1$ from the right, the derivative of the first term in $f$ is a small positive number, while that of the second term is a large negative number. Moreover, $f(\lambda) \to \infty$ as $\lambda \to \alpha_m/\Delta_\infty + 1$. Therefore, $f$ has a unique minimizer on $(1, \alpha_m/\Delta_\infty + 1)$, which can be found by a root finding method.

**Theorem 4.** Let $\alpha \in \mathbb{R}_d^d$ and $\alpha_m := \min_i \alpha_i$. For any $\lambda \in (1, \alpha_m/\Delta_\infty + 1)$, define

- $g(\lambda) := (\lambda - 1)\Delta_\infty$,
- $\hat{\varepsilon}(\lambda) := \frac{1}{2} \lambda \Delta_2^2 \psi'(\alpha_m - g(\lambda))$.

Then, for any $x \in \mathbb{R}_d^d$ and $\varepsilon > 0$, a single draw from $\text{Dir}(x + \alpha)$ is $(\varepsilon, \delta)$-DP, where $\delta$ is the minimum of the following function on $(1, \alpha_m/\Delta_\infty + 1)$:

$$
h(\lambda) := \exp((\lambda - 1)(\hat{\varepsilon}(\lambda) - \varepsilon)) \left(1 - \frac{1}{\lambda}\right)^\lambda.
$$

Figure 1 shows how $\delta$ decays as a function of $\varepsilon$ at three different values of $\alpha_m$. In contrast to (7) which involves several parameters and optimization over a subset of probability $d$-simplex, our $(\varepsilon, \delta)$-DP guarantee is relatively simpler.
4 Utility for private normalized histograms

Using the results from the previous section, we analyze the Dirichlet posterior sampling’s utility for privately releasing a normalized histogram.

Let \( x = (x_1, \ldots, x_d) \) be a histogram of \( N \) observations and \( p := x/N \). We can privatize \( p \) by sampling a probability vector: \( Y \sim \text{Dirichlet}(x + \alpha) \). We measure the accuracy of this sample with the \( \ell^2 \)-loss between \( Y \) and \( p \) and take advantage of the following bias-variance-type upper bound:

\[
\|Y - p\|_2 \leq \|Y - \mathbb{E}[Y]\|_2 + \|\mathbb{E}[Y] - p\|_2.
\]

For the first term on the right-hand side, we apply the following tail inequality:

**Lemma 3 ([HKZ12]).** For any \( \sigma^2 \)-sub-Gaussian \( d \)-dimensional variable \( Y \) and any \( t > 0 \),

\[
P \left[ \sum_i (Y_i - \mathbb{E}[Y_i])^2 > \sigma^2(2d + 3t) \right] \leq e^{-t}.
\]

For the second term, we apply the triangle inequality:

\[
\left( \sum_i (\mathbb{E}[Y_i] - p_i) \right)^{\frac{1}{2}} = \left( \sum_i \left( \frac{x_i + \alpha_i}{N + \alpha_0} - \frac{x_i}{N} \right)^2 \right)^{\frac{1}{2}}
\]

\[
= \left( \sum_i \left( \frac{N\alpha_i - x_i\alpha_0}{N(N + \alpha_0)} \right)^2 \right)^{\frac{1}{2}}
\]

\[
\leq \frac{\alpha_0}{N(N + \alpha_0)} \left( \sum_i x_i^2 \right)^{\frac{1}{2}} + \frac{1}{N + \alpha_0} \left( \sum_i \alpha_i^2 \right)^{\frac{1}{2}}
\]

Figure 2: The \( \ell^2 \)-loss, as a function of \( N \), of three \((2, \epsilon)\) algorithms for private normalized histograms, where \( \epsilon \in \{0.01, 0.1, 1\} \) and \( d \in \{10, 1000\} \): the Dirichlet posterior sampling \((\lambda = 2)\), the Gaussian mechanisms and the Laplace mechanism \((\Delta_2^2 = 2 \text{ and } \Delta_\infty = 1)\).
which leads to the main utility result of this section:

**Theorem 5.** For any $\beta \in (0, 1)$, the following inequality holds:

$$\|Y - p\|_2 \leq \sqrt{\frac{3\log(1/\beta) + 2d}{4(N + \alpha_0 + 1)}} + \frac{2\alpha_0}{N + \alpha_0} \quad \text{w.p. } 1 - \beta.$$  \hspace{1cm} (15)

Armed with this, we divide our analysis of the utility of the Dirichlet posterior sampling into two cases.

**Large dataset:** $N \gg d$. Given $\epsilon > 0$, we find a lower bound for $N$ that yields $\|Y - p\|_2 < \epsilon$, for a given $\epsilon > 0$, when $Y$ is sampled with $\epsilon$-RDP. For simplicity, we consider a uniform prior: $\alpha_i = \alpha > 0$ for all $i$. We also assume that $\alpha - g(\lambda) > 1$ (i.e. our algorithm is privacy preserving). Then $\psi'(\alpha - g(\lambda)) \approx 1/(\alpha - g(\lambda))$, which gives $\epsilon = \frac{1}{2} \lambda \Delta^2 \psi'(\alpha_m - g(\lambda)) \approx \frac{1}{2} \lambda \Delta^2/(\alpha - g(\lambda))$. It follows that $\alpha \approx \frac{1}{2} \lambda \Delta^2/\epsilon + g(\lambda) \approx \lambda/\epsilon$. Replacing $\alpha_0$ by $d\alpha$ in (15) yields the sample size that attains $\|Y - p\|_2 < \epsilon$ w.p. $1 - \beta$:

$$N = \Omega\left(\log\left(\frac{1}{\beta}\right) + \frac{d\lambda}{\epsilon^2}\right). \hspace{1cm} (16)$$

Let us compare this result to the Gaussian mechanism, which adds a noise $Z \sim N(0, \sigma^2 I_d)$ to the normalized histogram $p$ directly. Thus the $\ell_2$-sensitivity in this case is $\Delta^2/N$. As a result, the Gaussian mechanism is $(\lambda, \epsilon)$-RDP where $\epsilon = \frac{\lambda \Delta^2}{2N^2 \sigma^2}$. Applying (14) to the Gaussian mechanism yields

$$\|x + Z - p\|_2 \leq \sqrt{\frac{(3\log(1/\beta) + 2d)\lambda \Delta^2}{2N^2 \epsilon}} \quad \text{w.p. } 1 - \beta.$$  \hspace{1cm} (17)

Hence, the sample size $N = \Omega\left(\sqrt{(\log(1/\beta) + d)\lambda/\epsilon^2}\right)$ guarantees the desired accuracy. Comparing this to (16), if we assume $\epsilon < 1$, the AM-GM inequality tells us that

$$\frac{\log(1/\beta) + d}{\epsilon^2} + \frac{d\lambda}{\epsilon \epsilon} > \frac{\log(1/\beta) + d}{\epsilon^2} + \frac{\lambda}{\epsilon} \geq 2\sqrt{\frac{(\log(1/\beta) + d)\lambda}{\epsilon^2}}.$$  \hspace{1cm} (18)

The inequality above suggests that the Gaussian mechanism requires less observations than the Dirichlet mechanism to obtain the same accuracy.

**Small dataset or high dimensional data:** $N = O(d)$. Notice that the bound in (17) grows with $d$, while that in (15) has the term

$$N + \alpha_0 = N + d\alpha \approx N + d\left(\frac{\lambda \Delta^2}{2\epsilon} + g(\lambda)\right)$$

in the denominators. This observation suggests that, when $\alpha_0$ dominates the denominator i.e. when $N = O(d\lambda/\epsilon)$, the $\ell_2$-loss of the Dirichlet mechanism is smaller than that of the Gaussian mechanism. This conclusion is supported by our simulation with $\lambda = 2$ in Figure 2. We see that the $\ell_2$-loss of the Dirichlet mechanism is smaller than that of the Gaussian mechanism for when $N = O(d/\epsilon)$.
5 Experiment on Bayesian Reinforcement Learning

To demonstrate the utility of the Dirichlet posterior sampling in a complex task, we apply it to Posterior Sampling for Reinforcement Learning (PSRL) [Str00, ORR13]. Consider an episodic fixed horizon Markov decision process $M = \langle S, A, R, P, H, \varepsilon \rangle$, where $S$ is a set of states, $A$ a set of actions, $R_a(s)$ a probability distribution over reward upon selecting action $a$ in state $s$, $P_a(s'|s)$ the probability of transitioning to $s'$ after selecting action $a$ in state $s$, $H$ the episodic horizon, and $\varepsilon$ the initial state distribution. In PSRL, we put a normal-gamma prior on $R$, and Dir($\alpha$) as a prior on $P$. After episode $t$ (which consists of $H$ iterations), $P_t$ is sampled from Dir($x_t + \alpha$) where $x_t$ consists of $x_{s,s',a}$, the total number of transitions from $s$ to $s'$ after performing action $a$ after episode $t$.

Our goal is to privatize the agent’s state $s$ and action $a$ at one iteration $i$, which can be observed from the transition at iteration $i-1$ (a transition to $s$) and the transition at iteration $i$ (a transition from $s$ after selecting action $a$). Thus, two neighboring sequences of transitions differ at two entries, resulting in the following sensitivities of $x_t$ for all $t$: $\Delta_2^2 = 4$ and $\Delta_\infty = 1$.

In this experiment, we simulate the RiverSwim environment with $S = \{1, 2, 3, 4, 5, 6\}$, $|A| = \{1, 2\}$ and $H = 30$. We run the algorithm for 3000 episodes with the following two variants of private sampling:

- Diffuse sampling: $P_t \sim \text{Dir}(rx + \alpha)$, where $r > 0$ and $\alpha$ is the non-private prior parameter.
- Concentrated sampling: $P_t \sim \text{Dir}(x + \alpha')$, where $\alpha' > \alpha$.

We set $\alpha = (10, \ldots, 10)$ and $\lambda = 2$. We use (13) to solve for $r$ and $\alpha'$ so that a Dirichlet draw from each variant is $(2, \varepsilon/3000)$-RDP. Using the composition property (Lemma 2), the sequence of 3000 Dirichlet samplings is $(2, \varepsilon)$-RDP. We also privatize rewards with a Gaussian mechanism that gives $(\infty, 0.5)$-RDP. Figure 4 compares the total (non-private) rewards for $\varepsilon \in \{0.01, 0.1, 1, 10\}$. The result shows that the private PSLR can attain the same total reward as that of the original algorithm if $\varepsilon$ is not too small, and that the diffuse sampling outperforms the concentrated sampling at the same privacy guarantee.

6 Conclusion

We study Dirichlet posterior sampling for its privacy-preserving properties. Even though the sampling is not $\varepsilon$-DP, it has a simple RDP guarantee that allows us to analytically derive the tradeoff between privacy and utility. As an example, using the RDP guarantee, we analyze the utility of the Dirichlet posterior sampling for privately releasing a normalized histogram, and compare it with the Gaussian mechanism. The analysis proves to be useful as its results agree with the experimental results: the Dirichlet posterior sampling has better utility (measured in $l^2$-loss) than the Gaussian mechanism and the Laplace mechanism when dealing with high-dimensional data.
Experimental results also show that our privacy guarantee can be used to build private Bayesian reinforcement learners with trackable privacy losses. We envision that future work will be able to build on our study for large private Bayesian systems that use Dirichlet posterior sampling as a part of the systems.

Limitations. Even though we have discussed and analyzed the Diffuse sampling (i.e. sampling from $\text{Dir}(r\mathbf{x} + \alpha)$) to some extent, there is still much more to be explored regarding its utility and the privacy-utility trade-offs between $r$ and $\alpha$. The results from Section 5 suggest that we should keep $\alpha$ small and focus on tuning the value of $r$.

As the results from Section 4 suggest, the Dirichlet posterior sampling is inferior to the Gaussian mechanism for large data, and the former outperforms the latter for high-dimensional data. We might be able to combine both methods, and obtain an algorithm that performs well in both regimes. Nonetheless, we cannot add Gaussian noises to the Dirichlet parameters directly, as some of the parameters might become negative.

Societal impact. It is desirable that differentially private algorithms are accurate for the task at hand, especially when the data is used for important decision-making. Thus, one needs to make sure that there is enough sample to achieve the desired level of accuracy. For a large differentially private system, privacy budgets need to be allocated to the parts that require accurate outputs.

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A Proof of Theorem 2

For notational convenience, let \( u := x + \alpha \) and \( u' := x' + \alpha \). Let \( P(y) \) be the density of Dirichlet\((u)\) and \( P'(y) \) be the density of Dirichlet\((u')\). To compute the Rényi divergence between \( P(y) \) and \( P'(y) \), we start with:

\[
\mathbb{E}_{y \sim P(y)}\left[ \frac{P(y)^{\lambda-1}}{P'(y)^{\lambda-1}} \right] = \frac{B(u')^{\lambda-1}}{B(u)^{\lambda-1}} \mathbb{E}_{y \sim P(y)}\left[ y^{(\lambda-1)(u-u')} \right]
\]

\[
= \frac{B(u')^{\lambda-1}}{B(u)^{\lambda-1}} \cdot \frac{B(u + (\lambda - 1)(u - u'))}{B(u)}.
\]  \( \quad \text{(19)} \)

The ratio can be expressed in terms of gamma functions:

\[
\frac{B(u')}{B(u)} = \frac{\prod_i \Gamma(u'_i)/\Gamma(\sum_i u'_i)}{\prod_i \Gamma(u_i)/\Gamma(\sum_i u_i)} = \frac{\Gamma(u_0)}{\Gamma(u'_0)} \prod_i \frac{\Gamma(u'_i)}{\Gamma(u_i)},
\]

where \( u_0 := \sum_i u_i \) and \( u'_0 := \sum_i u'_i \). Similarly,

\[
\frac{B(u + (\lambda - 1)(u - u'))}{B(u)} = \frac{\Gamma(\sum_i u_i)}{\Gamma(\sum_i u_i + (\lambda - 1)\sum_i (u_i - u'_i))} \prod_i \frac{\Gamma(u_i + (\lambda - 1)(u_i - u'_i))}{\Gamma(u_i)}.
\]

Taking the logarithm on both side of (19), we need to find an upper bound of:

\[
\log \mathbb{E}_{y \sim P(y)}\left[ \frac{P(y)^{\lambda-1}}{P'(y)^{\lambda-1}} \right] = \sum_i \left( G(u_i, u'_i) + H(u_i, u'_i) \right) - G(u_0, u'_0) - H(u_0, u'_0),
\]  \( \quad \text{(20)} \)

where

\[
G(u_i, u'_i) := (\lambda - 1)(\log \Gamma(u'_i) - \log \Gamma(u_i))
\]

\[
H(u_i, u'_i) := \log \Gamma(u_i + (\lambda - 1)(u_i - u'_i)) - \log \Gamma(u_i),
\]

and similarly for \( G(u_0, u'_0) \) and \( H(u_0, u'_0) \). Using the second-order Taylor expansion, there exist there exist \( \xi \) between \( u_i + (\lambda - 1)(u_i - u'_i) \) and \( u_i \), and \( \xi' \) between \( u_i \) and \( u'_i \) such that

\[
G(u_i, u'_i) = -(\lambda - 1)(x_i - x'_i)\psi(u_i) + \frac{1}{2}(\lambda - 1)(x_i - x'_i)^2\psi'(\xi')
\]

\[
H(u_i, u'_i) = (\lambda - 1)(x_i - x'_i)\psi(u_i) + \frac{1}{2}(\lambda - 1)^2(x_i - x'_i)^2\psi'(\xi').
\]

We will try to find an upper bound of both \( \psi'(\xi') \) and \( \psi'(\xi') \). Note that \( \psi' \) is increasing. If \( x_i > x'_i \), then \( u'_i < u_i < u_i + (\lambda - 1)(u_i - u'_i) \). Thus both \( \xi \) and \( \xi' \) are bounded below by \( u_i \geq \alpha_m \). On the
Therefore, \( \psi'(\xi) \) and \( \psi'(\xi') \) are both bounded above by \( \psi'(\alpha_m - g(\lambda)) \). Consequently,

\[
G(u_i, u_i') + H(u_i, u_i') \leq \frac{1}{2}((\lambda - 1) + (\lambda - 1)^2)(x_i - x_i')^2\psi'(\alpha_m - g(\lambda))
= \frac{1}{2}\lambda(\lambda - 1)(x_i - x_i')^2\psi'(\alpha_m - g(\lambda)).
\]

The same argument can be used to show that, there exist \( \xi_0 \) and \( \xi'_0 \) such that:

\[
G(u_0, u'_0) + H(u_0, u'_0) = \frac{1}{2}(\lambda - 1)(u_0 - u'_0)^2\psi'(\xi'_0) + \frac{1}{2}(\lambda - 1)^2(u_0 - u'_0)^2\psi'(\xi_0) > 0.
\]

Therefore, continuing from (20),

\[
D_\lambda(P(y)\|P'(y)) = \frac{1}{\lambda - 1}\left(\sum_i (G(u_i, u_i') + H(u_i, u_i')) - G(u_0, u'_0) - H(u_0, u'_0)\right)
\leq \frac{1}{\lambda - 1}\sum_i (G(u_i, u_i') + H(u_i, u_i'))
\leq \frac{1}{2}\lambda \sum_i (x_i - x_i')^2\psi'(\alpha_m - g(\lambda))
\leq \frac{1}{2}\lambda \Delta_2^2\psi'(\alpha_m - g(\lambda)).
\]

In conclusion, the Dirichlet posterior sampling is \((\lambda, \frac{1}{2}\lambda \Delta_2^2\psi'(\alpha_m - g(\lambda)))\)-RDP.

**B Multinomial-Dirichlet sampling**

There are many Bayesian methods that involve repeated sampling from \( P_X \sim \text{Dirichlet}(\mathbf{x} + \alpha) \), where \( x_i, 1 \leq i \leq d \) are count data; \( \sum x_i = N \), for example, Gibbs sampling, Thompson sampling and Bayesian reinforcement learning. Suppose that we are employing one of these methods, but the count data must be kept private. So we instead sample from \( Q_X \sim \text{Dirichlet}(\mathbf{x} + \alpha') \) where \( \alpha'_i > \alpha_i \) for all \( i \).

In this setting, the relative quality of samples from \( Q_X \) compared to those from \( P_X \) is measured by a statistical distance between \( P_X \) and \( Q_X \). Our choice of distance is the KL-divergence, since there is a closed form for the KL-divergence between two Dirichlet densities (22).

In the utility analysis, we assume that \( \mathbf{x} \) is a single draw from \( \text{Multinomial}(\mathbf{p}) \) where \( \mathbf{p} \) is a probability vector. The following Theorem tells us that, on average, the KL-divergence is small when \( N = \sum x_i \) is large and \( \mathbf{p} \) is evenly distributed, and it is relative large when \( \mathbf{p} \) is sparse.
Theorem 6. Let \( p := (p_1, \ldots, p_d) \) where \( p_i > 0 \) for all \( i \) and \( \sum_i p_i = 1 \). Define a random variable \( X \sim \text{Multinomial}(p) \). Let \( P_X \sim \text{Dirichlet}(X + \alpha) \) and \( Q_X \sim \text{Dirichlet}(X + \alpha') \) where \( \alpha' \geq \alpha \geq 1 \) for all \( i \). The following estimate holds:

\[
\mathbb{E}_X[D_{KL}(P_X\|Q_X)] \leq \frac{1}{N+1} \sum_i (\alpha'_i - \alpha_i)^2 \cdot \frac{1}{p_i^2}.
\]  

(21)

Proof. For notational convenience, define \( \alpha_0 := \sum_i \alpha_i, \alpha'_0 := \sum_i \alpha'_i \) and \( X_0 = \sum_i X_i \). Moreover, we define \( \delta_i = \alpha'_i - \alpha \) and \( \delta_0 = \sum_i \delta_i \). The KL-divergence can be explicitly written as follows [Pen01]:

\[
D_{KL}(P_X\|Q_X) = \log \Gamma(X_0 + \alpha_0) - \sum_i \log \Gamma(X_i + \alpha_i)
\]

\[
- \log \Gamma(X_0 + \alpha'_0) + \sum_i \log \Gamma(X_i + \alpha'_i)
\]

\[
- \sum_i (\alpha'_i - \alpha_i)(\psi(X_i + \alpha_i) - \psi(X_0 + \alpha_0)).
\]

Using the second-order Taylor approximation, there exists \( C \in [X_0 + \alpha_0, X_0 + \alpha'_0] \) such that

\[
\log \Gamma(X_0 + \alpha_0) - \log \Gamma(X_0 + \alpha'_0) + (\alpha'_0 - \alpha_0)\psi(X_0 + \alpha_0)
\]

\[
= -\frac{1}{2} (\alpha'_0 - \alpha_0)^2 \psi'(C) \leq 0,
\]

since \( \psi' \) is positive. Hence, \( D_{KL}(P_X\|Q_X) \) is bounded by the rest of the terms in (22). In other words,

\[
D_{KL}(P_X\|Q_X) \leq \sum_i \left( \log \Gamma(X_i + \alpha'_i) - \log \Gamma(X_i + \alpha_i) - \delta_i \psi(X_i + \alpha_i) \right).
\]

We apply the Taylor approximation again and use \( \psi'(X) \leq \frac{1}{X} + \frac{1}{X^2} \). With \( X_i + \alpha_i \geq 1 \), we have

\[
\log \Gamma(X_i + \alpha'_i) - \log \Gamma(X_i + \alpha_i) - \delta_i \psi(X_i + \alpha_i)
\]

\[
\leq \frac{1}{2} \delta_i^2 \psi'(X_i + \alpha_i)
\]

\[
\leq \frac{\delta_i^2}{2(X_i + \alpha_i)} + \frac{\delta_i^2}{2(X_i + \alpha_i)^2}
\]

\[
\leq \frac{\delta_i^2}{X_i + \alpha_i},
\]

from which we take the sum in \( i \) to obtain the upper bound of the KL-divergence. Since \( \alpha_i \geq 1 \) for all \( i \), we have

\[
\mathbb{E}[D_{KL}(P_X\|Q_X)] \leq \sum_i \delta_i^2 \cdot \mathbb{E} \left[ \frac{1}{X_i + \alpha_i} \right] \leq \sum_i \delta_i^2 \cdot \mathbb{E} \left[ \frac{1}{X_i + 1} \right].
\]

Here, \( X_i \) is distributed as \( \text{Binomial}(N, p_i) \). The desired estimate follows from the following inequality regarding the reciprocal of a binomial variable from [CS72]:

\[
\mathbb{E} \left[ \frac{1}{X_i + 1} \right] = \frac{1 - (1 - p_i)^N}{(N + 1)p_i} < \frac{1}{(N + 1)p_i}.
\]

\[\square\]
Let us consider a simple privacy scheme where we fix $s > 0$ and let $\alpha'_i = \alpha_i + s$ for all $i$. Thus (21) becomes:

$$
\mathbb{E}_x[D_{KL}(P_x || Q_x)] \leq \frac{G(p)s^2}{N + 1},
$$

where $G(p) := \sum_i 1/p_i$. Now we take into account the privacy parameters. Let $\varepsilon = \frac{1}{2} \Delta_2^2 \psi'(\alpha_m - g(\lambda))$ and $\varepsilon' = \frac{1}{2} \Delta_2^2 \psi'(\alpha'_m - g(\lambda))$, where $\alpha_m = \min_i \alpha_i$, $\alpha'_m = \min_i \alpha'_i$, and $\alpha_m > g(\lambda)$. Here, we approximate the values of $\psi'(\alpha_m - g(\lambda))$ and $\psi'(\alpha'_m - g(\lambda))$ under two regimes:

**High-privacy regime:** $\alpha'_m - g(\lambda) > 1$. We have $\psi'(\alpha'_m - g(\lambda)) \approx 1/(\alpha'_m - g(\lambda))$, which implies $\Delta_2^2 / \varepsilon'$. We also have $\alpha_m - g(\lambda) \approx \Delta_2^2 / \varepsilon$ for $\alpha_m - g(\lambda) \geq 1$ and $\alpha_m - g(\lambda) > (\alpha_m - g(\lambda))^2 \approx \Delta_2^2 / \varepsilon$ for $\alpha - g(\lambda) < 1$. Thus we have the following bound for the right-hand side of (23):

$$
\frac{G(p)s^2}{N + 1} \approx \frac{G(p)(\alpha'_m - \alpha_m)^2}{N + 1} \leq \frac{\Delta_2^4 G(p)}{N + 1} \left( \frac{1}{\varepsilon'} - \frac{1}{\varepsilon} \right)^2 < \frac{\Delta_2^4 G(p)}{\varepsilon'2(N + 1)}.
$$

Consequently, we have $D_{KL}(P||Q) < \varepsilon$ for $N = \Omega\left( \frac{G(p)}{\varepsilon'^2} \right)$.

**Low-privacy regime:** $1 > \alpha'_m - g(\lambda) > 0$. This is similar as above, except we have $\alpha'_m - g(\lambda) \approx \Delta_2 / \varepsilon'^{1/2}$ and $\alpha_m - g(\lambda) \approx \Delta_2 / \varepsilon^{1/2}$. Similar computation as (24) shows that $D_{KL}(P||Q) < \varepsilon$ when $N = \Omega\left( \frac{G(p)}{\varepsilon^2} \right)$.

In both regimes, the sample size scales with $G(p)$, which measures the concentration of the probability vector $p$: $G(p)$ is large when $p$ is sparse, and $G(p)$ is small when $p$ is evenly distributed. Moreover, for small $\varepsilon'$ the sample size scales as $1/\varepsilon'^2$, while for large $\varepsilon'$ the sample size scales as $1/\varepsilon'$. This conclusion agrees with the result of our simulation in Figure 5.