We prove that the tangent bundle of a manifold of $K3^{[2]}$-type is rigid.

1. Introduction

M. Verbitsky in the work [4] showed that for a hyperholomorphic vector bundle $F$ on a hyperkähler manifold $X$ there are no obstructions for stable deformations of $F$ besides the Yoneda pairing on $\text{Ext}^1(F, F)$. Moreover, he proved the existence of a canonical hyperkähler structure on the reduction of the coarse moduli space of stable deformations of $F$. If $S$ is a K3 surface then it is known that the Hilbert scheme $S^{[n]}$ is a hyperkähler manifold and the tangent bundle $T_{S^{[n]}}$ is a hyperholomorphic bundle on $S^{[n]}$. Thus, the investigation of the deformation space of the bundle $T_{S^{[n]}}$ is a very natural and interesting question from the point of view of hyperkahler geometry. This question also appeared in [1] in the context of the Lefschetz standard conjecture for hyperkähler manifolds. It was mentioned there without a proof that for $n = 2$ the tangent bundle might actually be rigid. In the present note we confirm this statement by proving the following theorem.

Theorem 1.1. Let $X$ be a manifold of $K3^{[2]}$-type. Then the tangent bundle $T_X$ is infinitesimally rigid, i.e. $H^1(X, \mathcal{E}nd(T_X)) = 0$.

The proof of this statement is given in sections 3 and 4. It follows from explicit computations in the case when $X$ is the second Hilbert scheme of a K3 surface and then from application of general results from the theory of hyperholomorphic bundles [4]. For $n > 2$ the question about deformations of $T_{S^{[n]}}$ seems to be much more difficult due to more complicated geometry of the corresponding Hilbert scheme and thus should be considered separately.

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2. Hilbert square

For a smooth projective surface $S$ the Hilbert scheme of length-2 subschemes of $S$ is denoted by $S^{[2]}$. Let $\Delta : S \hookrightarrow S \times S$ be the diagonal embedding, $p_1, p'_1 : S \times S \to S$ be the projections onto the first and the second component and $\sigma : Z \overset{\text{def}}{=} \text{Bl}_\Delta(S \times S) \to S \times S$ be the blowup of $S \times S$ in $\Delta$. The natural action of the symmetric group $\mathfrak{S}_2$ on $S \times S$ extends to an action on $Z$ and the Hilbert square $S^{[2]}$ is the quotient of $Z$ by this action. By $q_2$ we denote the corresponding quotient map $Z \to S^{[2]}$. Let $j : E \hookrightarrow Z$ be the exceptional divisor of $\sigma$. Recall that $E \cong \mathbb{P}(T_S)$ is a projective bundle over $S$ and we have the relative Euler exact sequence

$$0 \to \Omega_{E/S} \to \pi^*\Omega_S(-1) \to \mathcal{O}_E \to 0.$$  

Also, by $i : D \hookrightarrow S^{[2]}$ we denote the isomorphic image of $E$ by $q_2$. The divisor $D$ is precisely the locus parametrizing non-reduced subschemes of $S$ of length two.
Put $q_1 := p_1 \circ \sigma$ and $q'_1 := p'_1 \circ \sigma$. The following diagram depicts the relationship among all the natural maps between the varieties that we mentioned:

$$
\begin{array}{ccc}
E \cong \mathbb{P}(T_S) & \xrightarrow{j} & Z \\
\pi & \downarrow & \sigma \\
S \cong \Delta & \xrightarrow{q_1} & S_{[2]} \xleftarrow{i} D.
\end{array}
$$

(2)

Here $\pi : E \to S$ is the projective bundle map and the equality $\sigma \circ j = \Delta \circ \pi$ yields $q_1 \circ j = \pi$.

Note that $Z$ is isomorphic to the universal closed subscheme $Z := \{(x, \xi) \mid x \in \text{Supp}(\xi)\}$ in $S \times S_{[2]}$ and for any coherent sheaf $F$ over $S$ there is the incidence exact sequence

$$0 \to q_1^* F(-E) \to q_1^* F^{[2]} \to q_1^* F \to 0,$$

(3)

where $F^{[2]}$ is the image of $F$ under the tautological functor $q_2^* q_1^* : \text{Coh}(S) \to \text{Coh}(S_{[2]})$ (see [3, p. 193]).

Recall that $q_2^* O_Z \cong O_{S_{[2]}} \oplus L^{-1}$, where the line bundle $L^{-1}$ is the eigenspace to the eigenvalue $-1$ of the cover involution. Moreover, $L^{\otimes 2} \cong O_{S_{[2]}}(D)$ and $q_2^* L \cong O_Z(E)$. Note that $q_2^* j_* O_E \cong j_* O_D$ and $q_2^* j_* O_D \cong O_{E_{[2]}}$.

There is an exact sequence

$$0 \to q_1^* O_{S_{[2]}}(E) \to \Omega_Z(E) \to j_* O_E \to 0$$

(4)

and an isomorphism

$$\Omega_{S_{[2]}} \cong q_2^* (N_{Z/S \times S_{[2]}}^\vee(E)).$$

The sequence (4) implies that $\omega_{q_2} \cong \Omega_Z(E)$, hence the right adjoint functor to $q_2^*$ is $q_2^*(-) = q_2^*(\cdot) \otimes \Omega_Z(E)$.

Now we write down the exact sequence defining the cotangent bundle on $S_{[2]}$. Putting left non-zero arrow of the sequence (4) together with the conormal exact sequence of the embedding $Z \hookrightarrow S \times S_{[2]}$ twisted by $E$ into a commutative diagram

$$
\begin{array}{ccc}
0 & \to & q_2^* O_{S_{[2]}}(E) \xrightarrow{\sim} q_2^* \Omega_{S_{[2]}}(E) \\
\downarrow & & \downarrow \\
0 & \to & N_{Z/S \times S_{[2]}}^\vee(E) \to q_1^* \Omega_S(E) \oplus q_2^* \Omega_{S_{[2]}}(E) \to \Omega_Z(E)
\end{array}
$$

(5)

and applying the snake lemma we obtain the exact sequence

$$0 \to N_{Z/S \times S_{[2]}}^\vee(E) \to q_1^* \Omega_S(E) \to j_* O_E \to 0.$$  

(6)

After pushing forward (6) along $q_2$ we obtain the exact sequence

$$0 \to \Omega_{S_{[2]}} \to \Omega_{S_{[2]}}^2 \otimes L \to j_* O_D \to 0.$$  

(7)

### 3. Computation for the Hilbert square of K3 surface

From now on we assume that $S$ is a K3 surface. We fix some isomorphism $T_S \cong \Omega_S$. The isomorphism $\omega_S \cong \Omega_S$ yields $\omega_Z \cong \Omega_Z(E)$. From the Euler sequence (1) it follows that $\Omega_{E/S} \cong O_E(-2)$. From stability of $\Omega_S$ we have that $\text{Hom}(\Omega_S, \Omega_S) \cong \mathbb{C}$. The latter implies that $H^0(S, \text{Sym}^2(\Omega_S)) = 0$. Also, we will use the equality $H^0(S, \Omega_S) = 0$ which by [2, Remark 3.19] and by stability of $\Omega_S$ implies that $\text{Hom}(\Omega_S^{[2]}, \Omega_S^{[2]}) \cong \mathbb{C}$.

To prove the Theorem [11] in this case it is enough to show the following two equalities

$$\text{Ext}^1(\Omega_S^{[2]} \otimes L, \Omega_{S_{[2]}}) = 0,$$

(8)

$$\text{Ext}^2(j_* O_D, \Omega_{S_{[2]}}) = 0.$$  

(9)

**Lemma 3.1.** The following equalities hold
(i) \( \text{Hom}(q_1^* \Omega_S(E), j_* \mathcal{O}_E) \cong \text{Hom}(q_1^* \Omega_S(E), q_1^* \Omega_S(E)|_E) \cong \text{Hom}(q_1^* \Omega_S(E)|_E, j_* \mathcal{O}_E) \cong \mathbb{C} \),
(ii) \( \text{Ext}^1(\iota_* \mathcal{O}_D, \iota_* \mathcal{O}_D) = 0 \),
(iii) \( \text{Hom}(\mathcal{O}_S^{[2]} \otimes L, \iota_* \mathcal{O}_D) \cong \mathbb{C} \),
(iv) \( \text{Ext}^2(\mathcal{O}_E, q_1^* \Omega_S) = 0 \),
(v) \( \text{Ext}^k(q_2^* \Omega_S^{[2]}, q_1^* \Omega_S(-E)) = 0 \) for \( k = 0, 1 \),
(vi) \( \text{Ext}^k(q_2^* \Omega_S^{[2]}, j_* \Omega_{E/S}(-E)) = 0 \) for \( k = 0, 1 \),
(vii) \( \text{Ext}^2(\Omega_{E/S}, j_* \Omega_{E/S}) = 0 \).

Proof. All listed equalities are straightforward consequences of standard adjunctions, properties of the blow-up and projective bundle map \( E \to S \), so we only give sketched proofs.

(i) We have
\[
\text{Hom}(q_1^* \Omega_S(E), j_* \mathcal{O}_E) \cong \text{Hom}(p_1^* \Omega_S, \sigma_* j_* \mathcal{O}_E(-E)) \\
\cong \text{Hom}(p_1^* \Omega_S, \Delta_* \pi_* \mathcal{O}_E(1)) \\
\cong \text{Hom}(p_1^* \Omega_S, \Delta_* \pi_* \mathcal{O}_E) \\
\cong \text{Hom}(\Omega_S, \mathbb{C}).
\]
By the projection formula we have that \( \text{Hom}(q_1^* \Omega_S(E), q_1^* \Omega_S(E)|_E) \cong \text{Hom}(\Omega_S, \Omega_S) \cong \mathbb{C} \). Finally, \( \text{Hom}(\pi^* \Omega_S(E), \mathcal{O}_E) \cong \text{Hom}(\Omega_S, \mathcal{O}_E) \cong \mathbb{C} \). From the fact that \( q_1 \circ j = \pi \) and since the functor \( j_* \) is fully faithful on the level of abelian categories, we get \( \text{Hom}(q_1^* \Omega_S(E)|_E, j_* \mathcal{O}_E) \cong \mathbb{C} \).

(ii) Using that \( \mathcal{L}_j \mathcal{O}_E \cong \mathcal{O}_E \oplus \mathcal{O}_E(-2E)[1] \) and \( \mathbb{R} \pi_* \mathcal{O}_E(-2) \cong \omega_S'[E] \), by the adjunction and the projection formula we have
\[
\text{Ext}^k(\iota_* \mathcal{O}_D, \iota_* \mathcal{O}_D) \cong \text{Ext}^k(\iota_* \mathcal{O}_D, q_2^* j_* \mathcal{O}_E) \\
\cong \text{Ext}^k(q_2^* \mathcal{O}_D, j_* \mathcal{O}_E) \\
\cong \text{Ext}^k(\mathcal{O}_E, \mathcal{O}_E) \\
\cong H^k(S, \mathcal{O}_S) \oplus H^k(S, \omega_S').
\]
Hence \( \text{Ext}^1(\iota_* \mathcal{O}_D, \iota_* \mathcal{O}_D) = 0 \).

(iii) Since \( \text{Hom}(q_1^* \Omega_S, j_* \mathcal{O}_E) = 0 \) and \( \text{Hom}(q_1^* \Omega_S, j_* \mathcal{O}_E(-E)) \cong \text{Hom}(\Omega_S, \mathbb{C}) \), from the exact sequence (3) with \( F = \Omega_S \) we have that \( \text{Hom}(q_2^* \Omega_S^{[2]}, j_* \mathcal{O}_E(-E)) \cong \mathbb{C} \). Then \( \text{Hom}(\Omega_S^{[2]} \otimes L, \iota_* \mathcal{O}_D) \cong \text{Hom}(q_2^* \Omega_S^{[2]}, j_* \mathcal{O}_E(-E)) \cong \mathbb{C} \).

(iv) Applying \( \sigma_* \) to the exact sequence
\[
0 \to j_* \mathcal{O}_E \to \mathcal{O}_2(E) \to j_* \mathcal{O}_E(E) \to 0
\]
and using the equality \( \mathbb{R} \pi_* \mathcal{O}_E(-1) = 0 \) we obtain that \( \mathbb{R} \sigma_* \mathcal{O}_2(E) \cong \mathcal{O}_\Delta \). Thus \( \text{Ext}^2(q_1^* \Omega_S, \mathcal{O}_2(E)) \cong H^2(S, \mathcal{O}_S) = 0 \). The assertion then follows from the Serre duality.

(v) Applying adjunctions \( q_1^* \vdash q_1 \) and \( q_2^* \vdash q_2 \) we obtain \( \text{Ext}^k(q_2^* \Omega_S^{[2]}, q_1^* \Omega_S(-E)) \cong \text{Ext}^k(q_1^* \Omega_S, q_2^* \Omega_S^{[2]}) \).

Since \( \mathbb{R} q_1^* \mathcal{O}_S \cong H^1(S, \Omega_S) \otimes \mathcal{O}_S[-1] \) we have that \( \text{Ext}^k(q_1^* \Omega_S, q_1^* \Omega_S) = 0 \) for \( k = 0, 1 \). From the exact sequence
\[
0 \to I_\Delta \to \mathcal{O}_S \to \mathcal{O}_\Delta \to 0
\]
and the condition \( H^1(S, \mathcal{O}_S) = 0 \) we obtain \( \mathbb{R} p_1^* I_\Delta = \mathcal{O}_S[-2] \). This implies that \( \text{Ext}^k(q_1^* \Omega_S, q_1^* \Omega_S(-E)) \cong \text{Ext}^k(\Omega_S, \mathcal{O}_S \otimes \mathbb{R} p_1^* I_\Delta) = 0 \) for \( k = 0, 1 \). Now, applying \( \text{Hom}(q_1^* \Omega_S, -) \) to the incidence exact sequence (3) with \( F = \Omega_S \), we obtain the desired statement.

(vi) Applying \( \sigma_* \) to the sequence (10) twisted by \( E \), we obtain the isomorphism \( \mathbb{R} \sigma_* \mathcal{O}_2(E) \cong \mathcal{O}_\Delta[-1] \).

Together with the isomorphism \( \mathcal{O}_{E/S} \cong \mathcal{O}_{E}(-2) \) it gives
\[
\text{Ext}^k(q_2^* \Omega_S^{[2]}, j_* \mathcal{O}_{E/S}(-E)) \cong \text{Ext}^k(\Omega_S^{[2]} \otimes L, \mathcal{O}_D \otimes L)
\]
From assertions (v) and (vi) of Lemma 3.1 it follows that both maps in (14) are injective, thus \( \alpha_k : \text{Ext}^k(\Omega^2_S, \Omega^2_S) \rightarrow \text{Ext}^k(\Omega^2_S \otimes L, \iota_* \mathcal{O}_D) \), \( k = 0, 1 \), \( \beta_2 : \text{Ext}^2(\iota_* \mathcal{O}_D, \Omega^2_S \otimes L) \rightarrow \text{Ext}^2(\iota_* \mathcal{O}_D, \iota_* \mathcal{O}_D) \), coming from the exact sequence (7). By the adjunction \( q^2 \dashv q_2 \) and factorization (11) the map \( \alpha_k \) can be written as the composition

\[
\text{Ext}^k(q^2_1 \Omega_S, q^1_1 \Omega_S) \rightarrow \text{Ext}^k(q^2_1 \Omega_S \otimes \mathcal{O}_E, q^1_1 \Omega_S|_E \rightarrow j_* \mathcal{O}_E, \text{Ext}^k(q^2_1 \Omega_S \otimes \mathcal{O}_E, q^1_1 \Omega_S|_E \rightarrow j_* \mathcal{O}_E(-E)).
\]

From assertions (v) and (vi) of Lemma 3.1 it follows that both maps in (14) are injective, thus \( \alpha_0 \) and \( \alpha_1 \) are injective as well. Moreover, by Lemma 3.1(iii) we get that \( \alpha_0 \) is an isomorphism since it is a map between one-dimensional vector spaces. This implies the equality (8).

Similarly, we now decompose \( \beta_2 \) as

\[
\text{Ext}^2(\mathcal{O}_E, q^1_1 \Omega_S(E)) \rightarrow \text{Ext}^2(\mathcal{O}_E, q^1_1 \Omega_S(E)|_E \rightarrow \text{Ext}^2(\mathcal{O}_E, j_* \mathcal{O}_E). \]

Lemma 3.1(iv) implies the injectivity of the first map in (15). The injectivity of the second map follows from Lemma 3.1(vii). This shows that \( \beta_2 \) is injective, which together with Lemma 3.1(ii) gives the vanishing (9).

4. General case

Let \( X \) be an irreducible holomorphic symplectic manifold and \( \mathcal{H} = (I, J, K) \) be the corresponding hyperkähler structure. For any triple \( a, b, c \in \mathbb{R} \) such that \( a^2 + b^2 + c^2 = 1 \) the operator \( L := aI + bJ + cK \) defines a complex structure on \( X \). Such a complex structure \( L \) is called induced by the hyperkähler structure. The space \( Q_\mathcal{H} \) of all induced complex structures of \( \mathcal{H} \) is isomorphic to \( \mathbb{C}P^1 \) and is called the twistor line of \( \mathcal{H} \). Denote by \( \text{Comp}_X \) the coarse moduli space of complex structures on \( X \). Then for each hyperkähler structure we have an embedding \( Q_\mathcal{H} \subset \text{Comp}_X \).

Definition 4.1. A twistor path in \( \text{Comp}_X \) is a collection of consecutively intersecting twistor lines \( Q_0, ..., Q_n \subset \text{Comp}_X \). Two points \( I, I' \in \text{Comp}_X \) are called equivalent if there exists a twistor path \( \gamma = Q_0, ..., Q_n \) such that \( I \in Q_0 \) and \( I' \in Q_n \). The path \( \gamma \) is then called a connecting path of \( I \) and \( I' \).

Theorem 4.2. [5, Theorem 3.2] Any two points \( I, I' \in \text{Comp}_X \) are equivalent.

Now we recall the definition of a hyperholomorphic bundle over \( X \).

Definition 4.3. Let \( F \) be a holomorphic vector bundle over \( (X, L) \) with a Hermitian connection \( \nabla \) on \( F \). The connection \( \nabla \) is called compatible with a holomorphic structure if \( \nabla_v(\xi) = 0 \) for any holomorphic section \( \xi \in F \) and any antiholomorphic tangent vector \( v \). If there exists a holomorphic structure compatible with the given Hermitian connection \( \nabla \), then this connection is called integrable. The connection \( \nabla \) is called hyperholomorphic if it is integrable for any complex structure induced by the hyperkähler structure. Then \( F \) is called a hyperholomorphic bundle.

For an induced complex structure \( L \) denote by \( H^*_L(X, F) \) the holomorphic cohomologies of \( F \) with respect to \( L \). We mention the following important property of hyperholomorphic bundles.

Theorem 4.4. [6, Corollary 8.1] Let \( F \) be a hyperholomorphic vector bundle. Then for any \( i \geq 0 \) the dimension of the space \( H^*_L(X, \text{End}(F)) \) is independent of an induced complex structure \( L \).
Note that the tangent bundle $T_X$ equipped with the Levi-Civita connection is always hyperholomorphic (see [6, Example 2.9(i)]). By Theorem 4.2, for any deformation $X' = (X, I')$, $I' \in \text{Comp}_X$ of $(X, I)$ there exists a twistor path $\gamma$ connecting $I'$ and $I$. Since $T_X$ is hyperholomorphic, the dimension of the cohomology space $H^1(X, \text{End}(T_X))$ is constant along $\gamma$ by Theorem 4.4. In the case when $X$ is a manifold of $K3^{[2]}$-type this dimension is equal to zero by the result of Section 3. This proves Theorem 1.1.

5. References

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