We generalize the Kuramoto model of coupled oscillators to allow time-delayed interactions. New phenomena include bistability between synchronized and incoherent states, and unsteady solutions with time-dependent order parameters. We derive exact formulas for the stability boundaries of the incoherent and synchronized states, as a function of the delay, in the special case where the oscillators are identical. The experimental implications of the model are discussed for populations of chirping crickets, where the finite speed of sound causes communication delays, and for physical systems such as coupled phase-locked loops or lasers.

\begin{equation}
\dot{\theta}_i(t) = \omega_i + \xi_i(t) + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j(t-\tau) - \theta_i(t) - \alpha),
\end{equation}

for \( i = 1, \ldots, N \). Here \( \theta_i(t) \) is the phase of the \( i \)th oscillator at time \( t \), and \( \omega_i \) is its intrinsic frequency, randomly drawn from a probability density \( g(\omega) \) with mean \( \omega_0 \). The white noise \( \xi_i(t) \) represents frequency fluctuations at an effective temperature \( D \geq 0 \), and is defined by the ensemble averages \( < \xi_i(t) > = 0, < \xi_i(s)\xi_j(t) > = 2D\delta_{ij}\delta(s-t) \). In the global coupling term, \( K \geq 0 \) is the coupling strength, \( \tau > 0 \) is the delay, and \( \alpha \) is a phase frustration parameter. This model reduces to the Kuramoto model if \( \tau = 0, \alpha = 0, \) and \( D = 0 \), and to the mean-field XY model if \( \tau = 0, \alpha = 0, \) and the oscillators are identical, i.e., \( g(\omega) = \delta(\omega - \omega_0) \). For \( \tau = 0 \), the separate effects of frustration \( \alpha \) and noise \( D \) have been studied by Sakaguchi and Kuramoto.

As the one-parameter family of rotating-frame transformations \( \theta_i(t) \rightarrow \theta_i(t) - \Omega t, \omega_i \rightarrow \omega_i - \Omega, \alpha \rightarrow \alpha + \Omega \tau \) leave Eq. (1) invariant for any \( \Omega \), we may assume \( \alpha = 0 \) without loss of generality — except if \( \tau = 0 \), which we forbid. (This restriction is merely for convenience. All our results are well-behaved as \( \tau \rightarrow 0 \) and converge to those obtained by setting \( \tau = 0 \) from the start.) Moreover, since Eq. (1) is invariant under the reflection \( \omega_i \rightarrow -\omega_i, \theta_i \rightarrow -\theta_i, \alpha \rightarrow -\alpha \), it suffices to consider \( \omega_0 \geq 0 \).

It is often helpful to describe the macroscopic state of the system in terms of the complex order parameter
\[
R(t) e^{i\psi(t)} = \frac{1}{N} \sum_{j=1}^{N} e^{i\theta_j(t)} \introduced by Kuramoto. Here \( R(t) \) measures the system’s phase coherence. In particular, \( R = 1 \) if all the oscillators are in phase, whereas \( R = 0 \) if the oscillators are scattered around the unit circle with their centroid at the origin.

Our first analytical result concerns the stability of the incoherent state for the infinite-\( N \) limit of Eq. (1). We
rewrite the model as a Fokker-Planck equation for the density \( \rho(\theta, \omega, t) \) of oscillators currently at phase \( \theta \), with intrinsic frequency \( \omega \). Because the method is standard \([6]\), we omit the details. The only new feature here is that the drift velocity field inherits the time delay in Eq. (3). When the Fokker-Planck equation is linearized about the incoherent state (the stationary density \( \rho \equiv 1/(2\pi) \)), we find \([7]\) that its continuous spectrum is \( \{ -D - i\omega \mid \omega \in \text{supp}\(g\) \}. Hence for \( D > 0 \), the continuous spectrum corresponds to damped modes and therefore the stability of the incoherent state is determined solely by the discrete eigenvalues. But when \( D = 0 \), the continuous spectrum is pure imaginary and corresponds to neutrally stable rotating waves in the full system. In this case we will say “all eigenvalues are stable”, for brevity).

Figure 1 shows that this analytical result agrees with numerical simulations, even for as few as \( N = 12 \) oscillators. Although the incoherent state is neutrally stable in the grey region, we observe numerically that \( R(t) \to 0 \) exponentially fast, as in Landau damping \([8]\). In this sense, incoherence is stable in the grey region.

Continuing with the instructive case of noiseless, identical oscillators, we now consider the possibility of perfect synchrony: \( \theta_i(t) = \theta(t) \) for all \( i \). We restrict our attention to a particular class of such solutions, namely uniform rotations: \( \theta(t) = \Omega + \beta \). Self-consistency then requires

\[
\Omega = \omega_0 - K \sin(\Omega \tau)
\]

for such solutions to exist, and linearization \([7]\) imposes

\[
\cos(\Omega \tau) > 0
\]

as the condition for their orbital stability.

If we graph both sides of Eq. (8) as functions of \( \Omega \), we see that for all sufficiently large \( K \), there exist multiple stable synchronized states \([3]\), as Eq. (8) has non-unique solutions satisfying \([3]\). We can also see that stable synchrony is impossible for certain combinations of \( \tau \) and \( K \). The problem reduces to characterizing the two-parameter family of lines of negative slope that intersect the sine function on its descending limbs. We find that stable synchronized states do not exist if and only if

\[
K < \frac{\omega_0}{2m - 1} \quad \text{and} \quad \frac{(4m - 3)\pi}{2\omega_0 - K} < \tau < \frac{(4m - 1)\pi}{2\omega_0 + K},
\]

with \( m \) being an arbitrary positive integer.
These zones of forbidden synchrony are shown in black in Fig. 2. For comparison, they are overlaid on top of the earlier grey regions (Fig. 1) where incoherence is stable. The black regions fit neatly inside the grey; they have the same base and half the height. The exposed parts of the grey regions correspond to bistability: stable incoherence coexists with stable synchrony, and hysteresis can occur.

Numerical simulations reveal windows in the bistable regions where R can be time-periodic (Fig. 3). In the example shown, period doubling occurs as K increases, but seems to be truncated beyond period 16 (not shown); after that, a synchronized state apparently takes over, suppressing further period doubling (Fig. 3(d)). Such unsteady behavior is a consequence of the delay; in the standard Kuramoto model, numerical experiments show that R(t) always approaches a constant value if g(ω) is unimodal and symmetric (although this has never been proven). Oscillator configurations with two or more clusters cause the unsteady behavior seen here. All the clusters move with the same average velocity, but their separation is periodically modulated.

So far we have concentrated on identical oscillators. To check how the results would be modified for other frequency distributions, we have considered the Lorentzian distribution q(ω) = (γ/π)(γ^2 + (ω - ω_0)^2)^{-1}. Then by a remarkable coincidence Eq. (3) can again be reduced to Eq. (3), but now with p = -(γ + D)τ, r = -ω_0τ, q = Kτ/2, and z = λτ. The critical coupling is given by

\[ K_c = 2(\gamma + D) \sec(\Omega_c \tau), \]

where \( \Omega_c = \omega_0 - (\gamma + D) \tan(\Omega_c \tau) \). Figure 3 plots the corresponding region where incoherence is stable. It resembles Fig. 1 with a series of evenly spaced peaks (at \( \omega_0 \tau = (2n + 1)\pi \)) that decrease in height. The main difference is that the distributed frequencies produce some rounding of the boundary, and lift it off the \( \tau \)-axis so that it now has minima \( 2(\gamma + D) \) at \( \omega_0 \tau = 2n\pi \).

The bistability found earlier also has a counterpart in the Lorentzian case (Fig. 3(a)). Partially locked states (which may not be unique) replace the earlier in-phase states, but otherwise the story is unchanged [23]. Thus, the case of identical oscillators captures the essential features introduced by delay.

Our final result concerns the bifurcation at \( K = K_c \), where the incoherent state becomes unstable. We have adapted the two-timing method of Ref. 6 to handle the delay-differential equations (1). We find [17] that generically, for \( D \geq 0 \) and arbitrary \( g(\omega) \), a Hopf bifurcation occurs at \( K_c \) [24], giving rise to a partially locked state, or in the density description, a rotating wave with constant coherence \( R = O(\sqrt{K - K_c}) \). This bifurcation may be subcritical (Fig. 3(a)) or supercritical (Fig. 3(b)).

Experimental tests of the model may be possible in arrays of phase-locked loops [25], relativistic magnetrons [26], or solid-state lasers [27], as they are all approximately governed by coupled Adler equations [28], similar in form to Eq. (4). The delay and the coupling strength are both natural control parameters, and perhaps one could try to map out the stability boundaries, look for hysteresis between incoherence and synchrony, etc. Our model may also help to explain how crickets can synchronize their chirps [3], despite the time delays caused by the speed of sound. Crickets listen to each other’s chirps and adjust their own timing according to a phase response curve [2]. The propagation delay between two crickets 3 meters apart is about 10 msecs. This is short compared to the chirp period (300-500 msecs). Our results suggest that delay effects become significant only in the first peak in Fig. 2, i.e., for delays near half the period of the oscillation. Thus the delays that crickets actually encounter in the field are probably negligible as far as synchrony is concerned. It would be interesting to try lab experiments on crickets interacting via chirp signals whose delay and
amplitude can be electronically manipulated.

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