Operatorial characterization of Majorana neutrinos

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Abstract

The Majorana neutrino $\psi_M(x)$ when constructed as a superposition of chiral fermions such as $\nu_L + C \nu_L^T$ is characterized by $(CP)\psi_M(x)(CP)^\dagger = i\gamma^0\psi_M(t, -\vec{x})$, and the CP symmetry describes the entire physics contents of Majorana neutrinos. Further specifications of C and P separately could lead to difficulties depending on the choice of C and P. The conventional $C\psi_M(x)C^\dagger = \psi_M(x)$ with well-defined P is naturally defined when one constructs the Majorana neutrino from the Dirac-type fermion. In the seesaw model where the same number of left- and right-handed chiral fermions appear, it is possible to use the generalized Pauli-Gursey transformation to rewrite the seesaw Lagrangian in terms of Dirac-type fermions only; the conventional C symmetry then works to define Majorana neutrinos. In contrast, the “pseudo C-symmetry” $\nu_{L,R}(x) \to C\nu_{L,R}(x)^T$ (and associated “pseudo P-symmetry”), that has been often used in both the seesaw model and Weinberg’s model to describe Majorana neutrinos, attempts to assign a nontrivial charge conjugation transformation rule to each chiral fermion separately. But this common construction is known to be operatorially ill-defined and, for example, the amplitude of the neutrinoless double beta decay vanishes if the vacuum is assumed to be invariant under the pseudo C-symmetry.

1 Majorana as a superposition of chiral fermions

The construction of the chiral fermion from the Dirac fermion is well-known. Also the construction of the Majorana fermion from the Dirac fermion is straightforward. In these cases, the charge conjugation operator is basically common between the Dirac fermion and the chiral fermion or the Dirac fermion and the Majorana fermion, respectively. But the construction of the Majorana fermion by a superposition of chiral fermions is less well understood.
The Majorana fermion is generally defined as a fermion which is identical to its anti-particle \(^1\), and it is usually defined as an eigenstate of charge conjugation symmetry \(C\). This naive characterization becomes more involved when one constructs the Majorana fermion by a superposition of chiral fermions such as \(\nu_L + C\nu_L^T\), since the conventional charge conjugation is not defined for the left-handed chiral fermion \(\nu_L\) alone. We show that a more general definition of an anti-particle by the use of CP symmetry, which is a common practice in particle physics when C is ill-defined, characterizes the Majorana neutrino constructed from chiral fermions in a natural manner and that the CP symmetry describes the entire physics of Majorana neutrinos in an extension of the Standard Model. We also show that the separate specifications of C and P for the Majorana neutrino formed by a superposition of chiral fermions could lead to difficulties depending on the choice of C and P.

1.1 CP symmetry for a Majorana fermion

The conventional C and P are defined in textbooks on field theory \(^2\) as the symmetries of Dirac fermions with the convention \(C = i\gamma^2\gamma^0\) and using \(i\gamma^0\) for the parity \(^{1}\).

\[
C\psi(x)C^\dagger = C\psi(x)^T, \quad P\psi(x)P^\dagger = i\gamma^0\psi(t,-\vec{x}).
\]

The C and P symmetries translated to the chirally projected fields \(\psi_{L,R}(x)\) are used for charged leptons and quarks in gauge theory and thus set the convention to analyze discrete symmetries in the Standard Model; the analyses of C, P and CP symmetries are performed using the chiral basis. In the case of massive neutrinos, the transformation rules of the case of Dirac neutrinos, taken as a specific example, give the definitions of C and P for chirally projected components

\[
C\nu_L(x)C^\dagger = C\nu_R(x)^T, \quad C\nu_R(x)C^\dagger = C\nu_L(x)^T, \\
P\nu_L(x)P^\dagger = i\gamma^5\nu_R(t,-\vec{x}), \quad P\nu_R(x)P^\dagger = i\gamma^5\nu_L(t,-\vec{x}), \\
(P\mathcal{C})\nu_L(x)(P\mathcal{C})^\dagger = i\gamma^0C\nu_L(t,-\vec{x})^T, \quad (P\mathcal{C})\nu_R(x)(P\mathcal{C})^\dagger = i\gamma^0C\nu_R(t,-\vec{x})^T
\]

with

\[
\nu_{R,L}(x) = \left(\frac{1 \pm \gamma_5}{2}\right)\nu(x).
\]

\(^{1}\)We define the parity of a Dirac fermion by \(i\gamma^0\)-parity, \(P\psi(x)P^\dagger = i\gamma^0\psi(t,-\vec{x})\), instead of the conventional \(\gamma^0\)-parity, since the Majorana fermion satisfying the classical relation \(\psi(x) = C\psi(x)^T\) is invariant under the parity thus defined \(i\gamma^0\psi(t,-\vec{x}) = C\gamma^0\psi(t,-\vec{x})^T\). For consistency, we assign this convention to charged leptons also, although this extra phase is cancelled in the lepton number conserving terms. See \(^3\) for the phase freedom of parity operation.
These rules extracted from the Dirac fermion are mathematically consistent and often useful even if the given Lagrangian is not invariant under these rules. For example, one can check if the given seesaw Lagrangian is parity preserving or not using these rules. Good P naturally implies left-right symmetry, and P for the Dirac fermion is represented in the form of a doublet representation \( \{ \nu_R(x), \nu_L(x) \} \). The doublet representation of the charge conjugation is related to the absence of the Majorana-Weyl fermion in \( d = 4 \) dimensions; intuitively, the absence of the Majorana-Weyl fermion is related to the fact that the charge conjugation inevitably changes the signature of \( \gamma_5 \) in \( d = 4 \), namely, \( \gamma_5 \rightarrow -\gamma_5 \). The symmetry of the Lagrangian naturally specifies the symmetry properties of the given field. For example, these rules (2) imply that we assign only CP to the massless Weyl fermion

\[
\mathcal{L} = \int d^4x \overline{\psi} L(x) i \gamma^\mu \partial_\mu \psi_L(x) \quad (4)
\]

without specifying C and P separately.

It is also well-known that the Majorana fermions are defined in terms of the Dirac fermion \( \psi_D(x) \) (in \( d = 4 \) dimensions)

\[
\psi_{M\pm}(x) = \frac{1}{\sqrt{2}} [ \psi_D(x) \pm C \psi_D(x)^T ] \quad (5)
\]

which satisfy the classical Majorana conditions \( C\psi_{M\pm}(x)^T = \pm \psi_{M\pm}(x) \) identically, i.e., \( \psi_{M\pm}(x) \) are constrained variables. The operator definitions of the charge conjugation and parity are given using \( C \) and \( P \) defined for the Dirac fermion (1) by

\[
C \psi_{M\pm}(x) C^\dagger = C \overline{\psi}_{M\pm}(x)^T = \pm \psi_{M\pm}(x), \\
P \psi_{M\pm}(x) P^\dagger = i\gamma^0 \psi_{M\pm}(t, -\vec{x}). \quad (6)
\]

The C symmetry of the Dirac fermion is represented in the form of a doublet representation \( \{ \psi_{M+}(x), \psi_{M-}(x) \} \) of Majorana fermions, but in a diagonalized form. The relations (5) show that the Majorana fermion is defined on the same Hilbert space as the Dirac fermion with good C and P.

The common model of Majorana neutrinos such as Weinberg’s model [4] or the seesaw model [5, 6, 7] constructs Majorana neutrinos from chiral fermions. To

\[2\text{The definition } \psi_{M-} = \frac{1}{\sqrt{2}} [ \psi_D(x) - C \psi_D(x)^T ] \text{ with an imaginary factor } i \text{ which satisfies classical relation } \psi_{M-} = C \overline{\psi}_{M-}^T \text{ is often used, instead of our } \psi_{M-}(x) = \frac{1}{\sqrt{2}} [ \psi_D(x) - C \psi_D(x)^T ] \text{, but this definition requires an anti-unitary } C \text{ to maintain } C \psi_{M-} C^\dagger = \psi_{M-}. \]
be explicit, Weinberg’s model of Majorana neutrinos \cite{4} is defined by an effective Lagrangian
\[
\mathcal{L} = \bar{\nu}_L(x) i \not\!\partial \nu_L(x) - (1/2) \{ \nu_L^T(x) C m_L \nu_L(x) + h.c. \} \tag{7}
\]
with a $3 \times 3$ complex symmetric mass matrix $m_L$. After the diagonalization of the symmetric complex mass matrix by the $3 \times 3$ Autonne-Takagi factorization using a unitary $U$ \cite{8}
\[
U^T m_L U = M \tag{8}
\]
with a real $3 \times 3$ diagonal matrix $M$, we define
\[
\nu_L(x) = U \tilde{\nu}_L(x) \tag{9}
\]
and thus transfer the possible CP breaking to the PMNS mixing matrix in the Standard Model. We then have (suppressing the tilde-symbol of $\tilde{\nu}_L(x)$)
\[
\mathcal{L} = \bar{\nu}_L(x) i \not\!\partial \nu_L(x) - (1/2) \{ \nu_L^T(x) C M \nu_L(x) + h.c. \}
\]
\[= (1/2) \{ \bar{\psi}(x) i \not\!\partial \psi(x) - \bar{\psi}(x) M \psi(x) \} \tag{10}
\]
where we defined
\[
\psi(x) \equiv \nu_L(x) + C \nu_L^T(x). \tag{11}
\]
The field $\psi(x)$ satisfies the classical Majorana condition identically (i.e., $\psi(x)$ is a constrained variable)
\[
\psi(x) = C \bar{\psi}(x)^T \tag{12}
\]
and thus one may define
\[
C_M \psi(x) C_M^\dagger = C \bar{\psi}(x)^T = \psi(x) \tag{13}
\]
with a suitable $C_M$ which we want to identify
\footnote{The difference between the classical Majorana condition and the operatorial characterization is that the classical Majorana condition specifies only the field $\psi(x)$ as in (12), while the operatorial definition requires a specification of transformation laws of its components $(\frac{1+i\gamma_5}{2}) \psi(x)$ also.}

An interesting complication is that the doublet representations of $C$ and $P$ for the chirally projected components \cite{2} induced by the Dirac fermion do not reproduce
the C and P symmetry transformation laws of the classical Majorana fermion \((11)\), of which transformation rules are also induced by the Dirac fermion,

\[
C\psi(x)C^\dagger = \nu_R(x) + C\overline{\nu_R}(x) \neq \psi(x),
\]

\[
P\psi(x)P^\dagger = i\gamma^0[\nu_R(t, -\vec{x}) + C\overline{\nu_R}(t, -\vec{x})] \neq i\gamma^0\psi(t, -\vec{x}).
\]  

(14)

Of course, this difficulty is related to the fact that we do not cover the left- and right-handed chiral fermion states symmetrically in \((10)\) and no \(\nu_R\) is defined in the present model. But this relation implies that both the transition from the Dirac to chiral fermions and the transition from the Dirac to Majorana fermions are straightforward, but the connection between the chiral and Majorana fermions is more subtle.

In the present case \((10)\), C and P are not specified for the field \(\nu_L\), but CP symmetry, \((PC)\nu_L(x)(PC)^\dagger = i\gamma^0C\overline{\nu_L}(t, -\vec{x})^T\), in \((2)\) is well-defined. We thus naturally characterize the Majorana fermion \((11)\) by CP symmetry

\[
(PC)[\nu_L(x) + C\overline{\nu_L}(x)](PC)^\dagger = i\gamma^0[C\overline{\nu_L}(t, -\vec{x}) + \nu_L(t, -\vec{x})],
\]  

(15)

namely,

\[
(PC)\psi(x)(PC)^\dagger = i\gamma^0C\overline{\psi}(t, -\vec{x})^T = i\gamma^0\psi(t, -\vec{x}).
\]  

(16)

The first equality in \((16)\) implies the operator relation while the second equality in \((16)\) implies the classical Majorana condition \((12)\) which holds identically in the sense that \((12)\) holds irrespective of the choice of \(\nu_L(x)\). This characterization \((16)\) implies that we specify the Majorana neutrino with emphasis on \(\nu_L(x)\). The chiral fermion \(\nu_L(x)\) appearing in \(\psi(x)\), which is generated by a smooth renormalization group flow starting with the massless Weyl fermion in an extension of the Standard Model \([4]\), for example, is assigned to have well-defined CP but no well-defined C nor P just as the starting massless Weyl fermion. The chiral component \(\nu_L(x)\) of \(\psi(x)\) describes the weak interaction of the Standard Model

\[
\int d^4x[(g/\sqrt{2})\overline{l}_L(x)\gamma^\mu W_\mu(x)U_{PMNS}\nu_L(x) + h.c.]
\]  

(17)

perfectly well, since the conventional C and P are broken in the chiral weak interaction and the propagator of \(\nu_L(x)\) is given by \((11)\).

The analysis of CP is described by the PMNS matrix combined with the conventional CP symmetries of the charged lepton \(l_L(x)\) and the chiral component \(\nu_L(x)\) of the neutrino. The absence of the \(U(1)\) phase freedom of \(\nu_L(x)\) in \((10)\) is important to count the correct number of possible CP phases \([9]\). The entire weak interaction is
thus described by the chiral component $\nu_L(x)$ using its CP property. The Majorana neutrinos characterized by CP symmetry retain certain information of their original chiral contents since the parity by itself is not specified in the present characterization of Majorana neutrinos. The fact that the free asymptotic field of the neutrino is a Majorana fermion is assured by $\nu_L(x) = [(1 - \gamma_5)/2]\psi(x)$ and which contains the classical identity (12). See also the discussion related to (19) below.

One may naturally want to describe the Majorana neutrino by a quantum operator of $C$. Apparently, a discontinuous deformation of discrete symmetry operators is required to describe the chiral fermion $\nu_L(x)$. One may think of the deformed symmetry operators
\[ C_M = 1, \quad P_M = CP \] (18)
which consist of well-defined operators 1 and $CP$ in (10); this shows that the vacuum of the chiral fermion and the vacuum of the Majorana fermion share the same CP symmetry. The choice (18) defines the Majorana fermion $\psi(x)$ in (11)
\[ C_M \psi(x) C_M^\dagger = \psi(x), \quad P_M \psi(x) P_M^\dagger = i\gamma^0 \psi(t, -\vec{x}) \] (19)
with $C_M P_M = CP$. We emphasize that the first relation in (19) is satisfied by any field, and thus only the CP operation contained in the second relation supplemented by the identity (12) is substantial. In this sense, the present characterization is identical to the relation (16); only difference is that we introduce the formal “charge conjugation operator” $C_M$. Since only CP acts effectively, this choice (19) may be regarded as a subset of the characterization of the Majorana by CP in (16).

The formulation (18) leads to a formal enhancement of discrete symmetries in (10) by assigning C and P to the chiral component $\nu_L(x) = (\frac{1-\gamma_5}{2})\psi(x)$,
\[ C_M \nu_L(x) C_M^\dagger = \nu_L(x) = C\nu_R^T(x), \quad P_M \nu_L(x) P_M^\dagger = i\gamma^0 C\nu_L(t, -\vec{x})^T = i\gamma^0 \nu_R(t, -\vec{x}) \] (20)
by defining a variable $\nu_R(x) \equiv (\frac{1+\gamma_5}{2})\psi(x) = C\nu_L^T(x)$, and
\[ C_M \nu_R(x) C_M^\dagger = C\nu_L^T(x), \quad P_M \nu_R(x) P_M^\dagger = i\gamma^0 \nu_L(t, -\vec{x}). \] (21)
These transformation rules are mathematically consistent and imply perfect left-right symmetry expected for a Majorana fermion $\psi = \nu_L + \nu_R$.

These rules (20) and (21) may be compared to (2). The physical degrees of freedom of a Majorana fermion $\psi = \nu_L + \nu_R$ are the same as either a left-handed
\[ \text{This deformation may be compared to the pseudo C-symmetry in subsection 1.2. Both retain the original CP symmetry but C and P separately are very different.}\]
chiral fermion or a right-handed chiral fermion but not both. If one measures the right-handed projection of $\psi$, for example, one obtains the chiral freedom $\nu_R$ and simultaneously the information of $\nu_L$ also.

The deformation (18) is a specific choice of the definition of C-symmetry with emphasis on the generated Majorana fermion by preserving CP symmetry. The trivial C may be natural from a point of view of the conventional Majorana fermion, but one may keep in mind that the fermion

$$\psi(x) = \nu_R(x) + C\nu_R^T(x)$$  \hspace{1cm} (22)

which has a completely different meaning from (11) in the context of chiral gauge theory, also defines a left-right symmetric Majorana fermion in the present formulation. The “Majorana-Weyl condition”

$$C_M\nu_L(x)C_M^\dagger = \nu_L(x)$$  \hspace{1cm} (23)

in $d = 4$ does not lead to a mathematical contradiction since $C_M$ is trivial; this relation is simply a chiral projection (i.e., a part) of the first relation in (19) which is satisfied by any field. The operators (18) imply an assignment of C, P and CP to a massless Weyl fermion in the vanishing mass limit in (10) and thus the equivalence of the massless Majorana neutrino and the massless Weyl neutrino, and both become Majorana fermions. This is a disturbing aspect of the use of the present charge conjugation operator [10].

The choice (18) shows that one can in principle define an operator $C_M$ which defines a Majorana fermion from a chiral fermion. But the physical significance of the use of this operator $C_M$ for $\psi = \nu_L + C\nu_L^T$ in SM is not very obvious; for example, it does not directly prohibit the $U(1)$ phase change $\nu_L(x) \rightarrow e^{i\alpha}\nu_L(x)$ in the analysis of CP in the weak interaction [17] since the Majorana condition $C_M e^{i\alpha}\nu_L(x)C_M = e^{i\alpha}\nu_L(x)$ allows it. In comparison, the simpler characterization by CP in (16) describes the entire weak interactions including the CP breaking without referring to C, as already explained.

### 1.2 Pseudo C-symmetry

One may still hope to define a nontrivial charge conjugation rule to each chiral component separately and define the Majorana fermion in a more direct manner. The “pseudo C-symmetry” (this naming was suggested in [11] to distinguish it from the conventional C-symmetry [2]) was invented as a result of such efforts. This scheme, which is commonly used to define Majorana neutrinos, in particular, in the seesaw model, thus attempts to assign a nontrivial charge conjugation rule to each
chiral component separately. To be precise, one starts with the definition of “pseudo C-symmetry” $\tilde{C}$ defined by the substitution rules (including $\nu_R$ to prepare for the analysis of the seesaw model later) \[12, 13, 14, 15\]

$$\tilde{C} : \begin{align*}
\nu_L(x) &\rightarrow C\nu_L(x)^T, \\
\nu_R(x) &\rightarrow C\nu_R(x)^T.
\end{align*}$$

(24)

One then has for the classical Majorana field (11) by noting $C\nu_L(x)^T \rightarrow \nu_L(x)$ by the above rule (24),

$$\tilde{C} : \psi(x) \rightarrow C\nu_L(x)^T + \nu_L(x) = \psi(x)$$

(25)

namely, the Majorana condition is maintained. The CP symmetry for the Majorana fermion is then defined by the composition rule

$$\tilde{C}\tilde{P} : \begin{align*}
\nu_L(x) &\rightarrow i\gamma^0C\nu_L(t, -\vec{x})^T, \\
\nu_R(x) &\rightarrow i\gamma^0C\nu_R(t, -\vec{x})^T
\end{align*}$$

(26)

if one defines the “pseudo P-symmetry” $\tilde{P}$ by

$$\tilde{P} : \begin{align*}
\nu_L(x) &\rightarrow i\gamma^0\nu_L(t, -\vec{x}), \\
\nu_R(x) &\rightarrow i\gamma^0\nu_R(t, -\vec{x})
\end{align*}$$

(27)

for each chiral component separately. The CP symmetry of chiral fermions is then defined by

$$\tilde{C}\tilde{P} : \begin{align*}
\nu_L(x) &\rightarrow i\gamma^0C\nu_L(t, -\vec{x})^T, \\
\nu_R(x) &\rightarrow i\gamma^0C\nu_R(t, -\vec{x})^T
\end{align*}$$

(28)

which agree with the conventional definition of CP symmetry \[2\]. One can also confirm that the action defined by \[10\] is formally invariant under $\tilde{C}$ and $\tilde{P}$, separately.

It thus appears that the pseudo C-symmetry and the pseudo P-symmetry which satisfy $\tilde{C}\tilde{P} = CP$ can define the Majorana neutrinos consistently. But these symmetries, which look natural by assigning the common vacuum to Majorana fermions and chiral fermions, are ill-defined operatorially when carefully examined. If one assumes the existence of unitary operators which generate these symmetries, the pseudo C-symmetry $C\nu_L(x)^{\dagger} = C\nu_L^T$ gives a natural solution of the relation

$$\tilde{C}\psi(x)^{\dagger} = C\psi(x)^T,$$

namely,

$$\tilde{C}\nu_L(x)^{\dagger} + C\nu_L(x)^T = C\nu_L(x)^T + \nu_L(x)$$

(29)

and similarly, the pseudo P-symmetry $P\nu_L(x)^{\dagger} = i\gamma^0\nu_L(t, -\vec{x})$ is a natural solution of $P\psi(x)^{\dagger} = i\gamma^0\psi(t, -\vec{x})$, namely,

$$\tilde{P}\nu_L(x)^{\dagger} + P\nu_L(x)^T = i\gamma^0\nu_L(t, -\vec{x}) + i\gamma^0C\nu_L(t, -\vec{x})^T$$

(30)
with \(\tilde{C}\tilde{P} = CP\). But these relations are known to lead to disturbing results by noting 
\[\nu_L(x) = \left(\frac{1-\gamma_5}{2}\right)\nu_L(x)\] \[11\] \[16\] \[17\]

\[\tilde{C}\nu_L(x)\tilde{C}^\dagger = \left(\frac{1-\gamma_5}{2}\right)\tilde{C}\nu_L(x)\tilde{C}^\dagger = \left(\frac{1-\gamma_5}{2}\right)C\nu_L(x)^T = 0,\] (31)

and also

\[\tilde{P}\nu_L(x)\tilde{P}^\dagger = \left(\frac{1-\gamma_5}{2}\right)\tilde{P}\nu_L(x)\tilde{P}^\dagger = \left(\frac{1-\gamma_5}{2}\right)i\gamma^0\nu_L(t, -\vec{x}) = 0\] (32)

since both \(C\nu_L(x)^T\) and \(i\gamma^0\nu_L(t, -\vec{x})\) are right-handed. As an alternative to (31), one may start with \(\left(\frac{1-\gamma_5}{2}\right)\psi(x) = \nu_L(x)\) and obtain \(\tilde{C}\left(\frac{1-\gamma_5}{2}\right)\psi(x)\tilde{C}^\dagger = \left(\frac{1-\gamma_5}{2}\right)\tilde{C}\psi(x)\tilde{C}^\dagger = \left(\frac{1-\gamma_5}{2}\right)\psi(x)\) and \(\tilde{C}\nu_L(x)\tilde{C}^\dagger = C\nu_L(x)^T = \left(\frac{1+\gamma_5}{2}\right)\psi(x)\), namely, \(\left(\frac{1-\gamma_5}{2}\right)\psi(x) = \left(\frac{1+\gamma_5}{2}\right)\psi(x)\)
which implies

\[\left(\frac{1-\gamma_5}{2}\right)\psi(x) = \left(\frac{1+\gamma_5}{2}\right)\psi(x) = 0.\] (33)

In the level of substitution rules also, these symmetry operations are ill-defined. This is seen by considering the free action of the Majorana fermion using \(\psi(x) = \nu_L + C\nu_L^T\),

\[S_{\text{Majorana}} = \frac{1}{2} \int d^4x \bar{\psi}(x) [i \not{\partial} - M] \psi(x) = \int d^4x \left\{ \bar{\nu}_L \not{\partial} \nu_L - \frac{1}{2} \nu_L^T CM\nu_L - \frac{1}{2} \bar{\nu}_L M C\nu_L^T \right\} = \int d^4x \left\{ \bar{\nu}_L \not{\partial} \frac{1-\gamma_5}{2} \nu_L(x) - \frac{1}{2} \nu_L^T CM \frac{1-\gamma_5}{2} \nu_L + \text{h.c.} \right\}.\] (34)

We emphasize that these three expressions are identical. If one assumes the transformation rule of pseudo C-symmetry, \(\nu_L(x) \to C\nu_L(x)^T\), as in (24), it turns out that the first and second expressions in (34) are invariant under the transformation, while the last expression leads to a vanishing Lagrangian \[11\]. Similarly, one may assume a transformation rule of pseudo P-symmetry, \(\nu_L(x) \to i\gamma^0\nu_L(t, -\vec{x})\), as in (27), then the first and second expressions in (34) are invariant under the transformation, while the last expression leads to a vanishing Lagrangian. This implies that one cannot decide if the pseudo C-symmetry and pseudo P-symmetry are good symmetries of

\[\text{The relation } \left(\frac{1-\gamma_5}{2}\right)\psi(x) = \left(\frac{1+\gamma_5}{2}\right)\psi(x) \text{ means } \nu_L(x) = C\nu_L(x)^T \text{ which is the Majorana-Weyl condition on } \nu_L(x) \text{ in } d = 4 \text{ dimensions and thus no solution. This property is related to the failure of chirality conservation requirement by the pseudo C-symmetry } [11].\]
one cannot decide if the fermion $\psi(x)$ defined in (34) is a Majorana fermion when one uses the pseudo C-symmetry and pseudo P-symmetry.

We note that the puzzling aspects in (34) arise from the substitution rules (24) and (27), irrespective of the existence or non-existence of the quantum operators $\tilde{C}$ and $\tilde{P}$ (although in the framework of field theory, we assume the operator representations of valid substitution rules). Consequently, the example (34) shows that even as substitution rules, the pseudo C-symmetry and pseudo P-symmetry are ill-defined. In comparison, $\tilde{C}\tilde{P}$, which agrees with the conventional CP without referring to $\tilde{C}$ and $\tilde{P}$ separately, is consistent in every respect.

It has been shown in [11] that the pseudo C-symmetry is formally obtained from a truncation of the CP symmetry

$$\text{CP} : \nu_L(x) \to i\gamma^0 C\nu_L(t,-\vec{x})^T \Rightarrow \tilde{C} : \nu_L(x) \to C\nu_L(t,\vec{x})^T$$

in any CP invariant theory. Namely, the pseudo C-symmetry is obtained from CP symmetry by removing the prefactor $i\gamma^0$ and restoring the spatial inversion $-\vec{x} \to \vec{x}$, and it is still formally a symmetry of the CP invariant theory such as [10]. But the pseudo C-symmetry is operatorially ill-defined if one examines the transformation more carefully as in [34].

So far we have emphasized the disturbing aspects of the pseudo C-symmetry based on the statements in (31), (32) and (34) which are mathematical facts. On the other hand, the pseudo C-symmetry has been used in many papers on Majorana neutrinos in the seesaw model, for example, without any obvious contradictions. One may thus want to understand the basic reason of the apparent phenomenological success of the formulation with the pseudo C-symmetry. The pseudo C-symmetry $\tilde{C}\nu_L(x)\tilde{C}^\dagger = C\nu_L^T$ as a natural solution of the relation

$$\tilde{C}\psi(x)\tilde{C}^\dagger = \tilde{C}\nu_L(x)\tilde{C}^\dagger + \tilde{C}C\nu_L(x)^T \tilde{C}^\dagger = \psi(x)$$

is an interesting finding. We have also shown the existence of a pseudo P-symmetry $\tilde{P}\nu_L(x)\tilde{P}^\dagger = i\gamma^0 \nu_L(t,-\vec{x})$ as a solution of

$$\tilde{P}\psi(x)\tilde{P}^\dagger = \tilde{P}\nu_L(x)\tilde{P}^\dagger + \tilde{P}C\nu_L(x)^T \tilde{P}^\dagger = i\gamma^0 \psi(t,\vec{x})$$

which satisfies the condition $\tilde{C}\tilde{P} = CP$. If one uses the pseudo C-symmetry only for the purpose of the identification of a Majorana neutrino [36] (and the pseudo P-symmetry only in the form [37]), and if one uses the conventional CP symmetry to

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The pseudo C-symmetry is thus related to the use of CP symmetry we suggest; the difference is that $\tilde{C}$ fails to act consistently on the component $(\frac{1-\gamma^5}{2})\psi(x)$ of $\psi(x)$, $\tilde{C}(\frac{1-\gamma^5}{2})\psi(x)\tilde{C}^\dagger = \tilde{C}(\frac{1-\gamma^5}{2})\nu_L(x)\tilde{C}^\dagger = 0$ as in [31] which is related to the failure of the chirality conservation [11], while the CP symmetry is operatorially consistent for both $\psi(x)$ and $(\frac{1-\gamma^5}{2})\psi(x)$. 

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analyze the weak interactions, one can analyze the weak interaction phenomenology successfully without encountering any contradictions. We have emphasized in (10) that the proper use of CP symmetry describes all the physics aspects of Majorana neutrinos in an extension of SM successfully.

In conclusion of this subsection, the secret of the practical success of the formulation with the pseudo C-symmetry is that people used the pseudo C-symmetry only to identify the Majorana fermion (36) and simply did not use the problematic aspects of the pseudo C-symmetry (and the pseudo P-symmetry) to analyze weak interactions, which can be described well by the conventional CP symmetry without referring to the pseudo C-symmetry. A problematic aspect of the pseudo C-symmetry, if one should use it directly in weak interaction phenomenology, shall be illustrated in Section 3.

2 Majorana neutrino from Dirac-type fermions

2.1 Seesaw model

We have already mentioned that Majorana fermions are defined in terms of the Dirac fermion $\psi_D(x)$, $\psi_{M\pm}(x) = \frac{1}{\sqrt{2}} [\psi_D(x) \pm C\psi_D(x)^T]$, as in in (5). The operator definitions of the charge conjugation and parity are naturally given using nontrivial $C$ and $P$ defined for the Dirac fermion (1) in the manner (6), since $\psi_D(x)$ and $\psi_{M\pm}(x)$ are defined on the same left-right symmetric state space. This option is not available for the Majorana fermions defined by chiral fermions in Weinberg’s model. But we shall show that this option is available in the seesaw model, which contains the same number of left- and right-handed chiral fermions, if one uses the generalized Pauli-Gursey transformation which is a canonical transformation. We thus have more options, (6) and the use of CP symmetry (16), to define Majorana neutrinos in the seesaw model, in addition to the use of trivial C (18) which may be regarded as a subset of (16). This analysis is useful to illustrate the fact that the vacua for the Majorana neutrino and the chiral neutrino are identical from the point of view of CP symmetry but they are very different from the point of view of charge conjugation symmetry; these different vacua are, however, smoothly connected by a canonical transformation.

We first recapitulate the basic aspects of the seesaw model. The seesaw model for the three generations of neutrinos starts with

$$
\mathcal{L} = \overline{\nu}_L(x) i \gamma^\mu \partial_\mu \nu_L(x) + \overline{\nu}_R(x) i \gamma^\mu \partial_\mu \nu_R(x) - \overline{\nu}_L(x) m_D \nu_R(x) - (1/2) \nu_L^T(x) C m_L \nu_L(x) - (1/2) \nu_R^T(x) C m_R \nu_R(x) + \text{h.c.},
$$

\[38\]
where \( m_D \) is a complex \( 3 \times 3 \) Dirac mass matrix, and \( m_L \) and \( m_R \) are \( 3 \times 3 \) complex Majorana-type matrices. The anti-symmetry of the matrix \( C \) and Fermi statistics imply that \( m_L \) and \( m_R \) are symmetric. This is the Lagrangian of neutrinos with Dirac and Majorana mass terms. For \( m_L = 0 \), it represents the classical seesaw Lagrangian of type I. In the following, we shall call the expression as the seesaw Lagrangian for the sake of generality.

We start with writing the mass term as

\[
(-2) \mathcal{L}_{\text{mass}} = \left( \begin{array}{c} \nu_R^C, \bar{\nu}_R^C \end{array} \right) \left( \begin{array}{cc} m_R & m_D \\ m_D^T & m_L \end{array} \right) \left( \begin{array}{c} \nu_L^C \\ \nu_L \end{array} \right) + \text{h.c.}, \tag{39}\]

where

\[
\nu_L^C \equiv C \nu_R^T, \quad \nu_R^C \equiv C \bar{\nu}_L^T. \tag{40}\]

Note that \( \nu_L^C \) and \( \nu_R^C \) are left-handed and right-handed, respectively. Since the mass matrix appearing is complex and symmetric, we can diagonalize it by a \( 6 \times 6 \) unitary transformation \( U \) (Autonne-Takagi factorization \[8\]) as

\[
U^T \left( \begin{array}{cc} m_R & m_D \\ m_D & m_L \end{array} \right) U = \left( \begin{array}{cc} M_1 & 0 \\ 0 & -M_2 \end{array} \right), \tag{41}\]

where \( M_1 \) and \( M_2 \) are \( 3 \times 3 \) real diagonal matrices (characteristic values). We choose one of the eigenvalues as \(-M_2\) instead of \( M_2 \) since it is a natural choice in the case of a single generation model.

We thus have

\[
(-2) \mathcal{L}_{\text{mass}} = \left( \begin{array}{c} \bar{\nu}_R \end{array} \right) \left( \begin{array}{cc} M_1 & 0 \\ 0 & -M_2 \end{array} \right) \left( \begin{array}{c} \bar{\nu}_L \end{array} \right) + \text{h.c.}, \tag{42}\]

with

\[
\left( \begin{array}{c} \nu_L^C \\ \nu_L \end{array} \right) = U \left( \begin{array}{c} \bar{\nu}_L^C \\ \bar{\nu}_L \end{array} \right), \quad \left( \begin{array}{c} \nu_R^C \\ \nu_R \end{array} \right) = U^* \left( \begin{array}{c} \bar{\nu}_R^C \\ \bar{\nu}_R \end{array} \right). \tag{43}\]

Hence we can write

\[
\mathcal{L} = (1/2) \{ \bar{\nu}_L(x) \partial_0 \nu_L(x) + \bar{\nu}_R^C(x) \partial_0 \nu_L^C(x) + \bar{\nu}_R(x) \partial_0 \nu_R(x) + \bar{\nu}_R^C(x) \partial_0 \nu_R^C(x) \} \\
- (1/2) \left( \begin{array}{cc} \bar{\nu}_R & \bar{\nu}_R^C \end{array} \right) \left( \begin{array}{cc} M_1 & 0 \\ 0 & -M_2 \end{array} \right) \left( \begin{array}{c} \bar{\nu}_L^C \\ \bar{\nu}_L \end{array} \right) + \text{h.c.}. \tag{44}\]

12
In the present transformation (43) in terms of a unitary matrix, one can confirm that the conditions of canonical transformation

\[ \tilde{\nu}_C^L = C \tilde{\nu}_R^T, \quad \tilde{\nu}_R^C = C \tilde{\nu}_L^T \]  

(45)

hold after the transformation. See (53) below. This canonical transformation with a unitary \( U \) has been identified as a special case of the generalized Pauli-Gursey transformation [10]. Phenomenologically, this unitary transformation transfers the possible CP breaking in the neutrino sector to the PMNS weak mixing matrix in the seesaw model.

The Lagrangian (38) is then written in the form (by suppressing the tilde symbol for the chiral states \( \tilde{\nu}_{R,L} \) which diagonalize the mass terms)

\[
\mathcal{L} = \frac{1}{2} \left\{ \bar{\psi}_+^+(x) i \not{\partial} \psi_+(x) + \bar{\psi}_-^-(x) i \not{\partial} \psi_-(x) \right\} - \frac{1}{2} \left\{ \psi_+ M_1 \psi_+ + \psi_- M_2 \psi_- \right\} 
\]

(46)

where

\[
\psi_+(x) = \nu_R + C \tilde{\nu}_R^T, \quad \psi_-(x) = \nu_L - C \tilde{\nu}_L^T 
\]

(47)

which satisfy the classical Majorana conditions identically (i.e., \( \psi_\pm(x) \) are constrained variables)

\[
C \psi_+^T(x) = \psi_+(x), \quad C \psi_-^T(x) = -\psi_-(x). 
\]

(48)

But the charge conjugation operation defined for the chirally projected Dirac fermion (2) does not work

\[
C \psi_+^T(x) C^\dagger = \nu_L + C \tilde{\nu}_L^T \neq \psi_+(x), \\
C \psi_-^T(x) C^\dagger = -\nu_R + C \tilde{\nu}_R^T \neq -\psi_-(x). 
\]

(49)

Our suggestion is thus to characterize the Majorana neutrinos using CP, namely, using the CP transformation laws of chiral fermions (2) which are the good symmetry of (11),

\[
(PC)\psi_+^+(x)(PC)^\dagger = i \gamma^0 C \psi_+^+(t,-\vec{x})^T = i \gamma^0 \psi_+(t,-\vec{x}), \\
(PC)\psi_-^-(x)(PC)^\dagger = i \gamma^0 C \psi_-^-(t,-\vec{x})^T = -i \gamma^0 \psi_-(t,-\vec{x}) 
\]

(50)

that are consistent in every respect and describe all the physics aspects of the seesaw model.
In contrast, in most of the common treatments of the seesaw model \cite{12, 13, 14, 15}, one adopts the “pseudo C-symmetry” \cite{24} (and “pseudo P-symmetry” \cite{27}, although not often mentioned), which formally appear to work (using the operator notation)

\[
\tilde{C}\psi(x)\tilde{C}^\dagger = C\nu_R(x)^T + \nu_R(x) = \psi_+(x),
\]

\[
\tilde{C}\psi_-(x)\tilde{C}^\dagger = C\nu_L(x)^T - \nu_L(x) = -\psi_-(x),
\]

\[
\tilde{P}\psi_+(x)\tilde{P}^\dagger = i\gamma^0[\nu_R(t, -\vec{x}) + C\nu_R(t, -\vec{x})^T] = i\gamma^0\psi_+(t, -\vec{x}),
\]

\[
\tilde{P}\psi_-(x)\tilde{P}^\dagger = i\gamma^0[\nu_L(t, -\vec{x}) - C\nu_L(t, -\vec{x})^T] = i\gamma^0\psi_-(t, -\vec{x})
\]

by assigning both the charge conjugation and parity transformation rules to Majorana fermions. One can also confirm that the actions constructed from (44) and (46) are formally invariant under \(\tilde{C}\) and \(\tilde{P}\). However, one encounters operatorial ill-definedness for these symmetry transformation rules in both quantum and substitution-rule levels when carefully examined, as already analyzed in \(31\), \(32\) and \(34\), respectively.

### 2.2 Generalized Pauli-Gursey transformation

A way to resolve the complications associated with the definition of Majorana neutrinos using the C symmetry in the seesaw model, which are briefly summarized above, has been discussed in detail using a relativistic analogue of the Bogoliubov transformation \cite{16, 17, 22}. The generalized Pauli-Gursey transformation, which is closely related to the Bogoliubov transformation, is more transparent in the treatment of CP symmetry \cite{10}. The generalized Pauli-Gursey transformation is defined by the transformation \cite{13} but now with arbitrary \(U(6)\) \cite{10}

\[
\begin{pmatrix}
\nu_L \\
\nu_C \\
\nu_R \\
\nu_C \\
\end{pmatrix} = U
\begin{pmatrix}
\tilde{\nu}_L \\
\tilde{\nu}_C \\
\tilde{\nu}_R \\
\tilde{\nu}_C \\
\end{pmatrix},
\]

\[
\begin{pmatrix}
\nu_R \\
\nu_C \\
\end{pmatrix} = U^* \begin{pmatrix}
\tilde{\nu}_R \\
\tilde{\nu}_C \\
\end{pmatrix}
\]

which still satisfies the conditions of canonical transformation (namely, the anticommutation relations are preserved after the transformation) 7

\[
\tilde{\nu}_L^C = C\tilde{\nu}_R^T, \quad \tilde{\nu}_R^C = C\tilde{\nu}_L^T.
\]

7The fundamental condition \((53)\), which is essential to define a canonical transformation, is satisfied after the change of variables \((52)\), if one notes \(\tilde{\nu}_L = (U^\dagger)_{21}\nu_L + (U^\dagger)_{22}\nu_L\) and \(\tilde{\nu}_R = (U^\dagger)_{21}\nu_R + (U^\dagger)_{22}\nu_R\) using \(3 \times 3\) submatrices defined by

\[
U^\dagger = \begin{pmatrix}
(U^\dagger)_{11} & (U^\dagger)_{12} \\
(U^\dagger)_{21} & (U^\dagger)_{22}
\end{pmatrix}.
\]
Recall that $\tilde{\nu}_C^L$ and $\tilde{\nu}_R^L$ in our definition are left-handed and right-handed, respectively. The generalized Pauli-Gursey transformation with an arbitrary unitary transformation $U(6)$ in (52) mixes fermions and anti-fermions, and thus changes the definition of the vacuum together with C and P symmetries defined on each vacuum. Historically, the Pauli-Gursey transformation was defined for a single generation with $U(2)$ [18, 19]. See also [20, 21] for related analyses.

A general strategy is then to choose a suitable “Pauli frame” defined by the generalized Pauli-Gursey transformation, analogously to the “Lorentz frame” in the terminology of Lorentz transformation, such that the seesaw Lagrangian is expressed in terms of Dirac-type variables only, which then allows us to define the Majorana neutrinos in a natural manner using the conventional C and P symmetries (1). For this purpose, we consider a further $6 \times 6$ real generalized Pauli-Gursey transformation $O(6)$ (which is of course included in $U(6)$) in addition to (44), that is orthogonal and thus preserves CP [10], by

$$
\begin{pmatrix}
\tilde{\nu}_C^L \\
\tilde{\nu}_R^L
\end{pmatrix} = O \begin{pmatrix} N_C^L \\
N_L^C
\end{pmatrix}, \quad
\begin{pmatrix}
\tilde{\nu}_R \\
\tilde{\nu}_C^R
\end{pmatrix} = O \begin{pmatrix} N_R \\
N_C^R
\end{pmatrix}.
$$

(54)

The exact solution (44) is then rewritten, by choosing a specific $6 \times 6$ orthogonal transformation

$$
O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\
-1 & 1
\end{pmatrix}
$$

(55)

where 1 stands for a $3 \times 3$ unit matrix, in the form

$$
\mathcal{L} = \begin{pmatrix} 1/2 \{ \overline{N}(x)i \not\partial N(x) + \overline{N}^C(x)i \not\partial N^C(x) \} \\
- \begin{pmatrix} 1/4 \{ \overline{N}(M_1 + M_2)N + \overline{N}^C(M_1 + M_2)N^C \} \\
- \begin{pmatrix} 1/4 \{ \overline{N}(M_1 - M_2)N^C + \overline{N}^C(M_1 - M_2)N \}.
\end{pmatrix}
\end{pmatrix}
$$

(56)

This Lagrangian is expressed in terms of Dirac-type variables $N(x)$ and $N^C(x)$ only (noting that $N_C^L(x) = C\overline{N}_R^T(x)$ and $N^C_L(x) = C\overline{N}_L^T(x)$, respectively) and invariant under the conventional C, P and CP defined by

$$
C : \quad N(x) \leftrightarrow N^C(x) = C\overline{N}_L^T(x),
$$

$$
P : \quad N(x) \rightarrow i\gamma^0 N(t, -\vec{x}), \quad N^C(x) \rightarrow i\gamma^0 N^C(t, -\vec{x}),
$$

$$
CP : \quad N(x) \rightarrow i\gamma^0 N^C(t, -\vec{x}), \quad N^C(x) \rightarrow i\gamma^0 N(t, -\vec{x}).
$$

(57)

The weak mixing angles are now determined by $U' = UO$ instead of $U$ in (44). This is also the essence of a relativistic analogue of the Bogoliubov transformation [16].
It is significant that only the Dirac-type particles \( N(x) \) and \( N^C(x) \) with the conventional C, P and CP transformation properties appear in this specific Pauli frame where the Lagrangian becomes C and P invariant and the chiral structure disappears, although the lepton number is violated.

We now make a renaming of variables

\[
\psi_+ (x) = \frac{1}{\sqrt{2}} (N(x) + N^C(x)), \quad \psi_- (x) = \frac{1}{\sqrt{2}} (N(x) - N^C(x)),
\]

which satisfies the classical Majorana conditions identically, \( C\psi_+ (x) = \psi_+ (x) \) and \( C\psi_- (x)^T = -\psi_- (x) \), and we obtain

\[
\mathcal{L} = \frac{1}{2} \{ \psi_+(x)i \not\partial \psi_+(x) + \psi_-(x)i \not\partial \psi_-(x) \} - \frac{1}{2} \{ \psi_+ M_1 \psi_+ + \psi_- M_2 \psi_- \}.
\]

After this renaming of variables, we find the transformation laws of \( \psi_\pm (x) \) induced by those of \( N \) and \( N^C \) in (57),

\[
\begin{align*}
C &: \psi_+ (x) \rightarrow \psi_+(x), \quad \psi_- (x) \rightarrow -\psi_- (x), \\
P &: \psi_+ \rightarrow i\gamma^0 \psi_+(t, -\vec{x}), \quad \psi_- (x) \rightarrow i\gamma^0 \psi_- (t, -\vec{x}), \\
CP &: \psi_+ (x) \rightarrow i\gamma^0 \psi_+(t, -\vec{x}), \quad \psi_- (x) \rightarrow -i\gamma^0 \psi_- (t, -\vec{x})
\end{align*}
\]

which naturally keep the Lagrangian invariant. When one defines a nontrivial unitary charge conjugation operator

\[
C_N N(x) C_N^\dagger = N^C(x) = C N^T(x),
\]

the operator \( C_M = C_N \) gives rise to

\[
C_M \psi_\pm (x) C_M^\dagger = \pm \psi_\pm (x)
\]

which is an analogue of the conventional definition of the Majorana fermion in terms of a Dirac fermion in (5) and (6) (in the actual operator construction, it is easier to construct \( C_M \) first). We can also define parity consistently for Majorana fermions

\[
P_M \psi_\pm (x) P_M^\dagger = i\gamma^0 \psi_\pm (t, -\vec{x})
\]

if one defines \( P_M = P_N \) with \( P_N N(x) P_N^\dagger = i\gamma^0 N(t, -\vec{x}) \) and thus \( P_N N^C(x) P_N^\dagger = i\gamma^0 N^C(t, -\vec{x}) \). We thus determine 6 Majorana fermions \( \psi_\pm (x) \) (each contains 3

\[\text{It is interesting that the notion of multiple vacua was considered at about the same time independently by Pauli and by BCS and Bogoliubov around 1957.}\]
flavor freedom) in the conventional manner after a suitable choice of generalized Pauli-Gursey transformation as above.

The essence of the generalized Pauli-Gursey transformation is that we mix fermions and anti-fermions as in (52) and thus leading to the existence of multiple vacua. This change of the vacuum allows us to define the Majorana neutrinos in the seesaw model consistently using the standard definitions of C and P for Dirac-type fermions on a suitably defined new vacuum. The fact that the orthogonal transformation $O(6)$, which preserves CP but modifies C and P [10], works in (56) shows that the chiral fermions and the Majorana fermions share the same CP symmetry; in the presence of left- and right-components, the chiral fermions are re-arranged to be Dirac-type fermions with this CP kept in tact.

3 Neutrinoless double beta decay

To illustrate the phenomenological implications of the present formal analyses, we comment on the neutrinoless double beta decay [23], which is described by the weak interaction Lagrangian

$$\int d^4x L_{\text{Weak}} = \int d^4x [(g/\sqrt{2})\bar{l}_L(x)\gamma^\mu W_\mu(x)U_{PMNS}\nu_L(x) + h.c.] = \int d^4x [(g/\sqrt{2})\bar{l}_L(x)\gamma^\mu W_\mu(x)U_{PMNS}\frac{1-\gamma_5}{2}\psi_M(x) + h.c.]$$ (64)

with the charged lepton triplet $l_L(x)$ and the PMNS mixing matrix $U_{PMNS}$ in the case of Weinberg’s model $\psi_M(x) = \nu_L(x) + C\bar{\nu}_L^T(x)$ in (11). A similar analysis is valid for the seesaw model when expressed in terms of chiral fermions as in (47). A necessary condition of the neutrinoless double beta decay is that not all the time-ordered correlations of the neutrino mass eigenstates

$$\langle 0|T^*\nu_L(x)\nu_L(y)|0\rangle = \langle 0|T^*\frac{1-\gamma_5}{2}\psi_M(x)\frac{1-\gamma_5}{2}\psi_M(y)|0\rangle$$ (65)

vanish in the second order perturbation in $L_{\text{Weak}}$ [23]. It is interesting that the neutrinoless double beta decay is neatly characterized by the vacuum expectation value of the product of two neutrino fields without referring to charged leptons. We suppress the hadronic sector.

If a unitary operator $\tilde{C}$ which generates the pseudo C-symmetry [24] exists and if the (neutrino) vacuum $|0\rangle$ should be invariant

$$\tilde{C}|0\rangle = |0\rangle$$ (66)
and thus $\langle \mathcal{C} = \langle 0 \mid 0 \rangle$, one can prove that all of the above correlations vanish

\[
\langle 0 \mid T^* \nu_L(x) \nu_L(y) \rangle 0 = \langle 0 \mid T^* [(1 - \gamma_5^5^2) \nu_L(x) \nu_L(y) \rangle 0 \\
= \langle 0 \mid C T^* [(1 - \gamma_5^5^2) \nu_L(x) \nu_L(y) \rangle ^\dagger 0 \\
= \langle 0 \mid T^* [(1 - \gamma_5^5^2) C \nu_L^T] (x) [C \nu_L^T] (y) \rangle 0 \\
= 0
\]

\text{(67)}

where we used $\nu_L(x) = (1 - \gamma_5^5^2) \nu_L(x)$ and $\tilde{C} \nu_L(x) \tilde{C}^\dagger = C \nu_L(x)^T$ and the fact that $C \nu_L^T(x)$ is right-handed. Namely, no neutrinoless double beta decay would take place in the second order of perturbation in weak interactions. The same conclusion holds if one assumes that the substitution rule $\nu_L(x) \rightarrow C \nu_L(x)$ $\tilde{C}$ in (24) is a good symmetry of the action of the neutrino sector [10]. The conclusion (67) is a consequence of the operatorially ill-defined pseudo C-symmetry (31), and it illustrates a problematic aspect of the pseudo C-symmetry when it is used directly in weak interaction phenomenology except for the identification of the Majorana neutrino. In contrast, one can confirm that CP invariance $(PC)^\dagger |0\rangle = |0\rangle$ of the vacuum with $(PC) \nu_L(x)(PC)^\dagger = i \gamma^0 C \nu_L(x)^T(-x)$ as in [16] is consistent and leads to the neutrinoless double beta decay in general, as the explicit evaluation of (65) indicates.

One can also confirm that other choices of C-symmetry operators such as C invariance $C_N^\dagger |0\rangle = |0\rangle$ of the vacuum after the generalized Pauli-Gursey transformation in (60) with $C_N \nu_L(x) C_N^\dagger = \pm \nu_L(x)$ where

\[
\nu_L(x) = (1 - \gamma_5^5^2) (N_L(x) \pm C N_R^T (x)) = (1 - \gamma_5^5^2) \psi_\pm (x),
\]

(68)

does not lead to any constraint and gives the conventional result for the above correlation of neutrino fields. Note that one of $\nu_L(x)$ with a smaller mass is physically relevant in the seesaw model. Also, the trivial $\mathcal{C}$ in [15], which may be regarded as a subset of (16), does not give rise to complications in the analysis of (65).

\footnote{Alternatively, by recalling that the pseudo C-symmetry implies $(1 - \gamma_5^5^2) \psi_M(x) \rightarrow C (1 - \gamma_5^5^2) \psi_M(x)$, one may conclude $\langle 0 \mid T^* (1 - \gamma_5^5^2) \psi_M(x) \psi_M(y) \rangle 0 = \langle 0 \mid T^* (1 - \gamma_5^5^2) \psi_M(x) \psi_M(y) \rangle 0 \rightarrow 0$, which is consistent with (33).}
4 Conclusion

We have shown that the Majorana neutrinos constructed by a superposition of chiral fermions are naturally and succinctly characterized by CP symmetry in both Weinberg’s model and the seesaw model. The CP symmetry thus formulated describes all the physics aspects of the Majorana neutrinos in an extension of SM. For the seesaw model, there is an additional theoretically attractive possibility to use the generalized Pauli-Gursey transformation and, after a suitable transformation, one can adopt the conventional definition of Majorana neutrinos using the Dirac-type fermions and the conventional C and P symmetries on a suitably chosen vacuum. This illustrates the fact that the vacuum for the chiral fermion and the vacuum for the Majorana fermion in the seesaw model are different with respect to the charge conjugation symmetry, but they are smoothly connected by a canonical transformation.

I thank A. Tureanu for helpful discussions on Majorana neutrinos. I also thank T. Maskawa for an informative conversation on the Pauli-Gursey transformation and J. Arafune for clarifying the phenomenological implications of $i\gamma^0$-parity. The present work is supported in part by JSPS KAKENHI (Grant No.18K03633).

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