Gradient flow on Finsler manifolds

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Abstract

studying various functionals and associated gradient flows are known problems in differential geometry. The purpose of this article is to provide a general overview of curvature functionals in Finsler geometry and use their information for introducing gradient flow on Finsler manifolds. For aiming this purpose, at first we prove space of Finslerian metrics is a Riemannian manifold, then we give some decompositions of tangent space of this manifold and finally we introduce gradient flow by using Akbar Zadeh curvature functional.

Keywords: Finsler manifold, Curvature functional, Ricci directional curvature, Berger-Ebin decomposition

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1 Introduction

Nonlinear heat flows first appeared in Riemannian geometry in 1964, when Eells and Sampson introduced the harmonic map heat flow as the gradient flow of the energy functional $E(u) = \int_M |\nabla u|^2 dV$ [16]. They used this flow as a tool to deform given maps $u : M \to N$ between two manifolds into extremal maps which are critical points in the sense of calculus of variations for $E(u)$. 

A fundamental problem in differential geometry is to find canonical metrics on Riemannian manifolds, i.e. metrics which are highly symmetrical, for example metrics with constant curvature in some sense. Hamilton used the idea of evolving an object to such an ideal state by a nonlinear heat flow for the first time and invented the Ricci flow in 1981 [23]. He proved that a Riemannian metric of strictly positive Ricci curvature on a compact 3-manifold can be deformed into a metric of positive constant curvature, using this idea G. Huisken [25], C. Margerin [26] and S. Nishikawa [29] proved that on a compact n-manifold, a Riemannian metric can be deformed into a metric of constant curvature, if it is sufficiently close to a metric of positive constant curvature [30].

The stationary metrics under the Ricci flow are Ricci flat metrics which are also the critical points of the Einstein -Hilbert functional $\mathcal{E}(g) = \int_M R dV$ but the Ricci flow is not exactly the gradient flow of functional $\mathcal{E}(g) = \int_M R dV$, it is just a part of Einstein -Hilbert functional’s gradient flow, $\partial_t g_{ij} = -R_{ij} + \frac{k}{n}g_{ij}$. If this functional is restricted to the class of conformal metrics then it has a strictly parabolic gradient flow which is called Yamabe flow. Hamilton proved that there is not any functional such that its gradient flow is Ricci flow. Perelman improved Einstein-Hilbert functional and introduced $F(g_{ij}, f) = \int_M (|\nabla f|^2 + R)e^{-f} dV$, this functional has a system of including two

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gradient flows which one of them is Ricci flow [27]. Since gradient flows have important role in global analysis on manifolds and in different branches of application sciences like image processing, biological problems and ..., it is an important branch of studying. So it might be good to extend this topic on Finsler manifolds. For the first attempt in this topic, we can mention the concept of Ricci flow by Bao [4] and as a bit more serious studying, Ohta and Sturm introduced heat flows on Finsler manifolds [21]. Since the classification of Finslerian manifolds with constant curvature is incomplete and furthermore, there are different kinds of curvature in Finsler geometry, we can not assure are gradient flows benefit tools for studying on Finsler manifolds like Riemannian ones? The main object of this paper is to introduce gradient flow of curvature functionals in Finsler geometry. For aiming this purpose, we start with studying space of Finsler metrics \( M_F \) and then we produced some decompositions for tangent space of \( M_F \) and in the last part of our paper we give an exact definition of variations of Finslerian metrics and using the calculate of variations for deriving some gradient flows.

2 Preliminaries

Let \( (M, g) \) be a connected, compact Finsler manifold. It means that there is a function \( F \) on tangent bundle \( TM \) with the following conditions:

- \( F \) is a smooth function on the entire slit tangent bundle \( TM_o \).
- \( F \) is a positive homogenous function on second variable, \( y \).
- The matrix \((g_{ij}), g_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^i \partial y^j}\) is nondegenerate.

The geodesics of a Finsler structure \( F \) are characterized locally by \( \frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0 \) where \( G^i = \frac{1}{2}g^{ij}(\frac{\partial^2 F^2}{\partial y^i \partial y^j} y^j - \frac{\partial F^2}{\partial y^i}) \) and called geodesic spray coefficients. Set \( G^i_j = \frac{\partial G^i}{\partial y^j} \) which are the coefficients of nonlinear connection on \( TM \). By means of this nonlinear connection, tangent space of \( TM_o \) splits in two horizontal and vertical subspaces, which is spanned by \( \{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \} \), where \( \frac{\partial}{\partial x^i} := \frac{\partial}{\partial x^i} - G^i_j \frac{\partial}{\partial y^j} \) that are called Berwald bases and their dual bases are denoted by \( \{dx^i, \delta y^j\} \), where \( \delta y^j := dy^j + G^j_i dx^i \). Furthermore this nonlinear connection can be used to define a linear connection which is called Berwald connection and its one forms defined locally by \( \pi^i_j = G^i_j dx^k \) where \( G^i_j = \frac{\partial G^i}{\partial y^j} \). The one forms of cartan connection are defined by \( \nabla \frac{\partial}{\partial x^i} = \omega^k_i \frac{\partial}{\partial y^k} \), where for \( \Gamma^i_{jk} = \frac{1}{2}g^{im}(\frac{\partial g_{mk}}{\partial x^j} + \frac{\partial g_{mj}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^m}) - (C^i_{jk}G^s_k + C^i_{js}G^s_j - C_{jks}G^s_i) \) and \( C^i_{jk} = \frac{1}{2}g^{im}(\frac{\partial g_{oke}}{\partial y^j} + \frac{\partial g_{omk}}{\partial y^j} - \frac{\partial g_{ojk}}{\partial y^m}) \), \( \omega^i_j = \Gamma^i_{jk}dx^k + C^i_{jk}\delta y^k \). Now, by using definition of curvature tensor on Riemannian vector bundle \( \pi^*TM \), hh-curvature of Cartan and Berwald connections are related by [2],

\[
R^i_{jkl} = H^i_{jkl} + C^i_{jkr}R^r_{o kl} + \nabla_k \nabla_o C^i_{jk} - \nabla_k \nabla_o C^i_{jl} + \nabla_o C^i_{lr} \nabla_o C^r_{jl} - \nabla_o C^i_{lr} \nabla_o C^r_{jk}
\]

Indicatrix bundle of a Finsler structure is defined by \( SM := \bigcup_{x \in M} S_x M \), where \( S_x = \{y \in T_x M | F(x, y) = 1\} \), according to this definition, \( S_x M \) is the hypersurface in \( T_x M \). Indicatrix bundle, \( SM \) is always orientable and since we assume \( M \) is compact, \( SM \) is compact, too. These two properties help to define volume form and global inner product on \( SM \). The volume element of the indicatrix bundle is denoted by \((2n-1)-\text{form} \ \eta [2],

\[
\eta := \frac{(-1)^N}{(n-1)} \phi, \quad \phi = \omega \wedge (d\omega)^{(n-1)}, \quad N = \frac{n(n-1)}{2}
\]

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where $\omega$ is Hilbert form. On tensor spaces on $SM$, we have the canonical scalar product (pointwise) $<.|.>$ and on their sections, the global scalar product ($<.|.>$) = $\int_{SM} <.|.> \eta$. Furthermore, Akbarzadeh introduced the codifferential operator on the differentiable one forms defined on $SM$ in [2],

$$\delta a = -(\nabla^j a_j - a_j \nabla^j_0 )$$

where $a$ is a horizontal 1-form on $SM$. And,

$$\delta b = -F(\nabla^j b_j + b_j \pi^j) = -Fg^{ij}\partial_j b_i$$

where $b$ is a vertical 1-form on $SM$, $\nabla$ and $\tilde{\nabla}$ are horizontal and vertical coefficients of Finslerian(Cartan) connection $\tilde{\nabla} = \nabla + \tilde{\nabla}$, respectively. The Ricci tensor is introduced from different ways in Finsler geometry, in this paper we consider Akbarzadeh definition $\tilde{H}_{ij} = \frac{\partial^2 H}{\partial y^i \partial y^j} y^i y^j$ where $H_{ij} = y^k H_{ikj} y^s$ which is defined by hh-curvature. Another curvature which is defined by hh-curvature is Ricci-directional curvature, $H(u,u) = g^{ik} H_{ijkl} u^j u^l$. This curvature is related to second type scalar curvature $\tilde{H} = g^{ij} \tilde{H}_{ij}$, in the critical points of Akbarzadeh curvature functionals.

### 3 Space of Finsler metrics

Endow compact manifold $M$, with a fixed Finslerian structure $F$ that making it a Finslerian manifold $(M,g)$. Since all coefficients of a Finslerian metric are zero homogenous, it is sufficient to consider them on $SM$. So without loss of generality, we can write $g \in S^2(\pi^*_s T^*M)$ where $\pi_s : SM \to M$ and $S^2_+$ denotes space of positive definite two forms. We note that Finsler metrics are special case of the GL-metrics on $TM$. In other words, a GL-metric $g_{ij}(x,y)$ is reducible to a Finsler metric if and only if the vertical coefficients of the cartan connection $C_{ijk}$, satisfied $C_{ijk} y^j = 0$ [7], but according to the Euler theorem and definition of cartan tensor, this property is always true when coefficients, $C_{ijk}$ are symmetric in all three indices.

**Proposition 1.** The space of all Finslerian metrics on a compact manifold $M$ is a Riemannian manifold $[\cdot]$.

**Proof.** The relation $C_{ijk} y^j = 0$ is simplifying to the linear differential equation $y^k \partial_{y^i} = 0$ on a Finsler manifold $M$. It is clear that this PDE is solvable. Let $D$ be a domain of $SM$ and we have a set of differential equations for a collection of functions

$$(g_{11}, \ldots, g_{1n}, \ldots, g_{n1}, \ldots, g_{nn}) : D \to \mathbb{R}^{n \times n}$$

Then solutions of the above differential equation are sections of $\mathbb{R}^{n \times n}$ fibered over $D$. The collection of these sections is an infinite dimensional vector space and so it is an infinite dimensional manifold which is represented by $\mathcal{M}_F$. For every $g \in \mathcal{M}_F$, the tangent space of this manifold is the space of all symmetric two tensors that are zero homogenous and symmetric in all three indices i.e, $T_g \mathcal{M}_F = \{ h \in S^2(\pi^*_s T^*M) | y^j \partial_{y^i} h = 0 \}$. we assume that $h$’s are squar integrable of order $s$ and define global inner product on $\mathcal{M}_F$ by

$$(a,b) := \sum_{s \geq 0} \int_{SM} \tilde{\nabla}^s a \cdot \tilde{\nabla}^s b \eta$$

where $a,b \in T_g \mathcal{M}_F$ and $\tilde{\nabla}$ is cartan connection. This inner product just depends on $x$. Therefore, the pair $(\mathcal{M}_F, (\cdot|\cdot))$ is a Riemannian manifold. \qed

**Definition 1.** The set of all Finslerian metrics on a compact manifold $M$ is defined as an answer set of linear partial differential equation $y^i C_{ijk} = 0$ and represented by $\mathcal{M}_F$. 

3
4 Different decompositions of tangent space of $\mathcal{M}_F$

Let $X = X^i \frac{\partial}{\partial x^i}$ be a section of $\Gamma(\pi^*TM)$. Define a unique associated horizontal vector field by $\hat{X} = X^i \delta_{x^i}$. Consider canonical linear mapping $\varphi: T_xM \rightarrow \pi^*T_xM$ that is $\varphi_x(\delta_{x^i}) = \frac{\partial}{\partial x^i}|_x$ and $\varphi_x(\frac{\partial}{\partial y^i}) = 0$ in local coordinates. Suppose $\hat{X}, \hat{Y}$ and $\hat{Z}$ are sections of $\Gamma(TM)$ so the Lie derivative of Finslerian metric $g$ is defined by $(L_{\hat{X}} g)(\varphi \hat{Y}, \varphi \hat{Z})$ by using Lie derivative and torsion definitions and properties of cartan connection, we obtain:

$$L_{\hat{X}} g(\varphi \hat{Y}, \varphi \hat{Z}) = L_{\hat{X}} g(\hat{Y}, \hat{Z})$$

$$= g(\text{symm}(\nabla X) \hat{Y}, \hat{Z}) + g(\hat{Y}, \text{symm}(\nabla X) \hat{Z})$$

$$+ 2g(T(\hat{X}, \hat{Z}), \hat{Y}) + g(T(\hat{X}, \hat{Z}), Y) + g(T(\hat{X}, Y), Z)$$

where $g(\text{symm}(\nabla X) \hat{Y}, \hat{Z}) := g(\nabla_{H \hat{Y}} X, \hat{Z}) + g(\hat{Y}, \nabla_{H \hat{Z}} X)$ and is defined for vertical connection similar to this.

Now, suppose vector field $\hat{X}$ is a complete lift of a vector field $X$ on $M$ by replacing this vector field in Lie derivative equation, and using $y^m \frac{\partial X_i}{\partial x^m} = y^m \delta_{X_i}^m$ and

$$\nabla_{(X^i G_i^l + y^m \frac{\partial X^l}{\partial x^m})} \frac{\partial}{\partial x^k} = (X^i G_i^l + y^m \frac{\partial X^l}{\partial x^m}) C_{ik}^m \frac{\partial}{\partial x^m}$$

$$= (y^m \frac{\partial X^l}{\partial x^m} + y^m X^i F_{im}^l) C_{ik}^m \frac{\partial}{\partial x^m}$$

$$= y^m \nabla X^i C_{ik}^m \frac{\partial}{\partial x^m}$$

we deduced that

$$L_{\hat{X}} g(Y, Z) = \nabla_i X_j + \nabla_j X_i + 2y^m \nabla_m X^i C_{kij}$$

(3)

By means of the global inner product, we define the adjoint of this operator.

**Lemma 1.** Let $(M, g)$ be a compact Finslerian manifold and $h$ be an arbitrary symmetric two form of $S^2\pi^*T^*M$, the adjoint of Lie derivative of $h$ in local coordinates is

$$\delta h = -(\nabla^i h_{ik} - h_{kj} \nabla_0 T^j + \dot{C}_{kij} h^{ij} + C_{kij} \nabla_0 h^{ij})$$

(4)

**Proof.**

$$\int_{SM} \frac{1}{2} (L_X g, h) \eta = \frac{1}{2} \int_{SM} (\nabla_i X_j + \nabla_j X_i + 2y^m \nabla_m X^k C_{ijk}) h^{ij} \eta$$

$$= \int_{SM} \nabla_i X_j h^{ij} \eta + \int_{SM} y^m \nabla_m X^k C_{kij} h^{ij} \eta$$

$$= \int_{SM} (h_{ik} \nabla_0 T^i - \nabla^i h_{ij} - (\nabla_0 C_{ijk}) h^{ij} - C_{ijk} \nabla_0 h^{ij}) X^k \eta$$

$$= -\int_{SM} (\nabla^i h_{ik} - h_{ik} \nabla_0 T^i + \dot{C}_{kij} h^{ij} + C_{kij} \nabla_0 h^{ij}) X^k \eta$$

$$= \int_{SM} (X, \delta h) \eta$$

□
Definition 2. Divergence of symmetric two forms in \( S^2 \pi^* T^* M \) is adjoint of \( L_X g \) with respect to global inner product and calculated by \( \partial v \) and represented by \( \delta \).

Theorem 1. The Berger-Ebin decomposition for \( T_g \mathcal{M}_F \) is \( T_g \mathcal{M}_F = \{ h | h = L_X g \} \oplus S^T \) where \( S^T := \{ h | \delta_g h = 0 \} \)

Proof. We define the differential operator \( \tau_g \) for every \( g \in \mathcal{M}_F \) by \( \tau_g h := -\pi_g \delta_g h \) which is an operator from \( T_g \mathcal{M}_F \) to \( \Gamma TM \). Its adjoint is denoted by \( \tau^* \) and define by \( L_X g \) where \( X \) is a complete lift of \( \eta \). For an arbitrary one form \( t \) on \( SM \), the symbole of \( \tau \) is defined by \( \sigma_t(\tau) := g^* t \otimes X^h_t + X^h_t \otimes g^* t \) and it is injective so the Berger-Ebin decomposition of \( T_g \mathcal{M}_F \) is \( Im \tau \oplus ker \tau^* \).

Definition 3. Section \( X \) of the tangent bundle \( TM \) is a Finslerian killing vector field if its complete lift \( \hat{X} \) on \( TM \) is a killing vector field for Finslerian metric \( g \), that is \( L_X g = 0 \).

A point-wise conformal deformation of a Finslerian metric \( g \), is \( \tilde{g}(x,y) = f(x)g(x,y) \) where \( f \) is a smooth positive function on \( M \). Since there is a one to one corresponding between space of positive functions and space of exponential functions by \( f \rightarrow e^f \), we can write \( \tilde{g} = e^f g \). Therefore, let \( \mathcal{P} \) be the product group of positive function on \( M \) that acts on \( \mathcal{M}_F \) by function \( A \) as follows:

\[
A : \mathcal{P} \times \mathcal{M}_F \rightarrow \mathcal{M}_F
\]

\[
A(f,g) := f.g
\]

This action is free and smooth. The orbit of this action at \( g \in \mathcal{M}_F \) is defined by \( A_g = \{ f | f \in \mathcal{P} \} \) which is a submanifold of \( \mathcal{M}_F \). Tangent space of this manifold at \( g \) is \( \mathcal{F}g = \{ h = kg | k \in C^\infty(M) \} \) where \( \mathcal{F}g \) is a subspace of \( S^2 \pi^* T^* M \) at each point \( g \in \mathcal{M}_F \). Orthogonal subspace of \( \mathcal{F}g \) with respect to the global inner product is \( \{ h \in S^2 \pi^* T^* M | \int_{SM} kgh \eta = 0 \} = \{ h \in S^2 \pi^* T^* M | tr(h) = 0 \} \). On the other hand, from the variation of the volume forms [2], we have \( tr(h) = 0 \) is equivalent of being constant volume on \( SM \). So the orthogonal space of \( \mathcal{F}g \) is the space of two forms which preserve volume \( SM \) through metric variations. Thus there is a point wise decomposition like

\[
S^2 \pi^* T^* M = \mathcal{F}g \oplus S^T
\]

Let \( D \) be the group of infinitesimal diffeomorphism on \( M \) and \( \mathcal{P} \) be a one parameter group of positive function on \( M \). Put \( C = D \rtimes \mathcal{P} \) which is a semi-direct group with the following action:

\[
(\eta_1, f_1).(\eta_2, f_2) = (\eta_1 \circ \eta_2, f_1 \cdot f_2(\eta_1 \circ \eta_2))
\]

This group acts on \( \mathcal{M}_F \) by function \( \hat{A} \) as follows:

\[
\hat{A} : C \times \mathcal{M}_F \rightarrow \mathcal{M}_F
\]

\[
\hat{A}(\eta, f, g) = f.(\tilde{\eta}^* g)
\]

where \( \tilde{\eta} \) is the natural extention of \( \eta \) on \( TM \) which is defined by \( \tilde{\eta}_t : (x^i, y^i) \rightarrow (x^i + tv^i, y^i + ty^m \frac{\partial u^m}{\partial x^i}) \) such that \( v^i \)s are components of vector field \( V \) on \( M \) which is inducing infinitesimal point transformation \( \eta_t \). It is clear that \( \hat{V} := \frac{d}{dt}|_{t=0} \tilde{\eta}_t \) is a complete lift of the vector field \( V \) on \( TM \). The orbit of \( \hat{A} \) passing through \( g \in \mathcal{M}_F \) is

\[
\hat{A}_g : \mathcal{F} \rightarrow \mathcal{M}_F
\]

\[
\hat{A}_g(\eta, g) = f.(\tilde{\eta}^* g)
\]
which is a submanifold of $\mathcal{M}_F$. So we define $\tau_g := d\tilde{A}_g|_{e,1}$ as follows:

$$\tau_g : \Gamma(TM) \times F \to T_g\mathcal{M}_F$$

$$\tau_g(X, f) = L_{\hat{X}}g + kg$$

The adjoint of $\tau_g$ is denoted by $\tau^*_g$ and defined by:

$$\tau^*_g : T_g\mathcal{M}_F \to \Gamma(TM) \times F$$

$$h \to (\pi_{ss}(\sharp div h), \int_{SM} tr(h)\eta)$$

The kernel of this map is $S^{TT} = \{ h \in T_g\mathcal{M}_F | \sharp div h = 0, \int_{SM} tr(h)\eta = 0 \}$, and since the symbol of the map $\tau_g$ i.e. $\sigma_t(\tau_g)(X, f) = fg + t \otimes \pi^*_s X^s + \pi^*_s X^s \otimes t$ where $t$ is an arbitrary one form on $SM$ is injective so the Berger-Ebin decomposition is

$$S^2\pi^*_s T^*M = S^{TT} \oplus Im\tau_g$$

By corresponding this decomposition with point-wise decomposition [5], we have

$$S^2\pi^*_s T^*M = F_g \oplus S^{TT} \oplus (S^T \cap Im\tau_g)$$

The last term of the right hand side of the above equation indicates that every two form $h = L_{\hat{X}}g + fg$ preserve volume of $SM$ that is $tr(h) = 0$ so $h$ must be in the form $h = L_{\hat{X}}g - \frac{2}{n} div(\hat{X})g$.

All of the above discussion can be summarized in the following theorem.

**Theorem 2.** The Berger-Ebin decomposition for $T_g\mathcal{M}_F$ according to conformal deformation of metrics is $T_g\mathcal{M}_F = F_g \oplus S^{TT} \oplus (S^T \cap Im\tau_g)$.

## 5 Curvature functionals on $\mathcal{M}$

**Definition 4.** A variation of a Finslerian metric, $g_o$ is a one-parameter family of this metric i.e, $\{g_t\}_{t \in I}$ where $g_t = g_o + th : g_o \in \mathcal{M}_F$ and $h \in T_g\mathcal{M}_F$.

According to the above definition, a variation of Finslerian metric is a curve in infinite dimensional manifold $\mathcal{M}_F$, with tangent vector field, $h := \frac{dg_t}{dt}$. When a Finslerian metric has been deformed, geometric structures, like nonlinear coefficients, curvatures, volume forms and indicatrix will be changed. Variations of these objects are calculated in [2],

$$\eta' = (g'^{ij} - \frac{n}{2} u^i u^j) h_{ij} \eta$$

$$C'^{ij}_k = \frac{1}{2} (\nabla_k h^i_o + \nabla_o h^i_k - \nabla^i h_{ok}) - 2C^i_{ks} G'^{ks}$$

$$R'^{ki}_{jkl} = \nabla_k \Lambda^i_{jl} - \nabla_l \Lambda^i_{jk} + P^i_{jlr} \Lambda^r_{ok} - P^i_{jkr} \Lambda^r_{ol} + C'^{hi}_{jr} R'^{r}_{okl}$$

where

$$\Lambda^i_{jk} = \Gamma'^{hi}_{jk} + C'^{hi}_{jr} \Gamma'^{r}_{ok}$$

and

$$\Gamma'^{hi}_{jk} = \frac{1}{2} g^{im} (\nabla_k h_{mj} + \nabla_j h_{mk} - \nabla_m h_{jk})$$

$$- (C'^{si}_{jks} G^t_k + C'^{si}_{ks} G'^{t}_{js} - C'^{si}_{kjs} G^t_{ms} g^{im})$$
Taking derivative from both sides of above equation: for this equation? For answering this question, consider the linearization of this equation. Since \( H \) is a functional on the space \( F \) of point-wise conformal deformation and infinitesimal conformal deformation that the first one takes part in \( F \)-finsler killing vector fields. The variation of volume form with respect to the point-wise conformal variation is 

\[ \text{Lemma 2.} \quad \text{The variation of volume form with respect to the point-wise conformal variation is} \quad \eta' = \frac{2}{n} \text{tr}_g(h) \eta. \]

\[ \text{Proof.} \quad \text{The point-wise conformal variation of a metric} \ g \ is \ \tilde{g}_{ij} = e^{2f(t,x)} g_{ij} \ so \ h_{ij} = \varrho(t,x) g_{ij} \ where \ \varrho(t,x) = f'(t,x)e^{f(t,x)} = \frac{1}{n} \text{tr}_g(h) \ by \ replacing \ this \ equation \ in \ [6] \ we \ obtain} \quad \eta' = \frac{1}{2n} \text{tr}_g(h) \eta. \]

\[ \text{Theorem 3.} \quad \text{Let} \ M \ be \ a \ closed \ and \ connected \ Finslerian \ manifold \ with \ dim \geq 3. \ A \ metric} \ g \ is \ critical \ for} \ I(g_t) \ \text{under all pointwise conformal variations if and only if the Finslerian manifold is Ricci directional flat.} \]
Proof. The derivative of functional $I(g)$ in usual direction is

$$\left(\tilde{H}_{jk} - \lambda H(u,u)u_ju_k - (n\tau - \phi)u_ju_k - \frac{1}{2} \dot{H} g_{jk}\right)h^{jk} = 0 \quad (9)$$

We product two side of equation by $u^i u^j$ and obtain

$$\tilde{H}(u,u) - \lambda H(u,u) - (n\tau - \phi) - \frac{n}{2} \dot{H} = 0 \quad (10)$$

Since $t^{jk} = \frac{\text{tr}_g(t)}{n} g^{jk}$, so we product both side of (9) by $g^{jk}$,

$$\tilde{H} - \lambda H(u,u) - (n\tau - \phi) - \frac{n}{2} \dot{H} = 0 \quad (11)$$

Now from equations (10) and (11) we deduced that

$$\frac{n - 1}{2} \dot{H} = - \frac{n - 1}{2} a + \tilde{H} - \dot{H}(u,u) \quad (12)$$

and

$$\lambda H(u,u) + (n\tau - \phi) = \frac{n}{n - 1} \tilde{H}(u,u) - \frac{1}{n - 1} \dot{H} \quad (13)$$

Replacing two last equations in (9). By simplifying the equation and product it to $u^i u^j$, we have

$$(n - 2) \dot{H}(u,u) = 0$$

so $H(u,u) = 0 \quad \Box$

This functional is not invariant under scaling. For eliminating this problem, we use a normal factor $\psi = \psi(t)$, and put $\tilde{g} = \psi(t)g(t)$ such that $\int_{SM} \tilde{\eta} = 1$. So we deduced that $\eta = \psi_{\frac{n}{2}} \tilde{\eta}$ and by replacing it in volume formula, we have $\psi = (V(t))^{\frac{2}{n}}$. Now, we rewrite the functional $I(g_t)$ with respect to this normal factor;

$$\tilde{I}(g) = I(\tilde{g}) = \int_{SM} (H(\tilde{g}) - \lambda H(u,u)(\tilde{g}))\tilde{\eta}$$

$$= \int_{SM} \psi^{-1}(H(g) - \lambda H(u,u)(g))\psi^{\frac{2}{n}} \eta$$

$$= \psi^{\frac{2-n}{2}} I(g)$$

$$= (V(t))^{\frac{2-n}{n}} I(g)$$

**Theorem 4.** Let $M$ be a closed and connected Finslerian manifold with $\text{dim} \geq 3$. A metric $g$ is critical for $\tilde{I}(g_t)$ under all pointwise conformal variations if and only if the Finslerian manifold is of constant Ricci-directional curvature.

Proof. Taking derivative from both sides of equation $\tilde{I}(g_t) = (V(t))^{\frac{2-n}{n}} I(g)$ and calculate it at $t = 0$:

$$\tilde{I}'(g_t)|_{t=0} = \frac{2-n}{n} V(t)|_{t=0} (V(0))^{\frac{2-n}{n}} - 1 I(g_0) + v(0) \frac{2-n}{n} I'(g_t)|_{t=0}$$

$$= V(0)^{\frac{2-n}{n}} \left\{ \frac{2-n}{2n} \frac{I(g_0)}{V(0)} \int_{SM} \text{tr}(h) \eta + \int_{SM} A_{ij} h^{ij} \eta \right\}$$
Put $Ave := \frac{I(g_0)}{V(0)}$ which is a constant value. With restricted to point-wise conformal deformation, we have:

$$0 = \dot{I}(g_t)|_{t=0} = V(0) \frac{2-n}{n} \int_{SM} \left( \frac{2-n}{n} Ave + A_{ij}g^{ij} \right) tr_g(h) \eta$$

Now we try to simplify parentheses equation:

$$0 = \frac{2-n}{n} Ave + A_{ij}g^{ij} = \frac{2-n}{n} Ave - \tilde{H} + \lambda H(u, u) + (n\tau - \varphi) + \frac{n}{2} \tilde{H}$$  \hspace{1cm} (15)

By replacing $12$ and $13$ in $15$ we deduced

$$H(u, u) = -\frac{(n-2)(n-1/2)}{8n} Ave$$

\[\square\]

Corollary 1. Under assumption of above theorem, second type scalar curvature is constant, too.

**Proof.** In the stationary points of curvature functional $I(g_t)$ based on constant indicatrix volume, we have $nH(u, u) = \tilde{H}[2]$. So $\tilde{H}$ is constant, too. \[\square\]

**Definition 5.** The normalized gradient flow of functional $I(g_t)$ with restricted to the point-wise conformal deformation is

$$\frac{\partial}{\partial t}g_{ij} = -\tilde{H}^{ij}$$

where $\tilde{H}$ is constant value. It is clear that unnormalized gradient flow is defined by

$$\frac{\partial}{\partial t}g_{ij} = -H(u, u)g_{ij}$$

**Corollary 2.** Unnormalized gradient flow $\frac{\partial}{\partial t}g_{ij} = -H(u, u)g_{ij}$ is a weakly parabolic equation.

**Proof.** The derivative of functional $I(g)$ with above assumptions is $I'(g_t)|_{t=0} = \int_{SM} H(u, u)tr_g(t)\eta = \int_{SM} H(u, u)g_{jk}t^{jk}\eta = 0$ so its Euler-lagrange equation is $H(u, u)g_{ij} = 0$ and its associated gradient flow is $\frac{\partial}{\partial t}g_{ij} = -H(u, u)g_{ij}$. The linearization of this equation is

$$\tilde{H}(u, u) = F^{-2}\xi_i\xi_j g_{ij} + \tilde{F}^{-2}t_{00}\xi_i\xi_j g_{ij}t_{00} + \text{lower order terms}.$$  

Put $\xi_1 = 1$ and $\xi_j = 0$ for all $j \neq 1$. To evaluate the symbol of this equation, we take an orthonormal frame $(e_i)$ at $x \in M$ such that $u^\alpha = \frac{u^n}{1} = 1$ and $u^\alpha = 0$ for all $\alpha \neq n$, it is clear that

$$(\sigma D(E)(g_{jk})(\xi)(\tilde{g}))(j) = -\frac{F^{-2}}{4}t_{00}j = k = 1$$  and for all other case is zero

So it is a weakly parabolic equation. \[\square\]

Since scalar form of both equations are same so it seems that scalar form is not such a good form for studying flows. On the other hand, you saw there is not any difference between choosing second part of definition GEM or first part of it for introducing Ricci flow but with last proposition, we can say that the tensor form of Ricci flow is $\frac{\partial}{\partial t}g_{ij} = -\tilde{H}_{ij}$ since we use the second term for introducing a new version of flow in Finsler geometry.
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