A combinatorial Fredholm module on self-similar sets built on $n$-cubes

Takashi Maruyama and Tatsuki Seto

Abstract. We construct a Fredholm module on self-similar sets such as the Cantor dust, the Sierpinski carpet and the Menger sponge. Our construction is a higher dimensional analogue of Connes’ combinatorial construction of the Fredholm module on the Cantor set. We also calculate the Dixmier trace of two operators induced by the Fredholm module.

Introduction

In the 1990s, A. Connes [3, Chapter IV] introduced the quantized calculus based on the Fredholm modules. A Fredholm module on an involutive algebra $\mathcal{A}$ is a pair $(H, F)$ of a Hilbert space $H$ and bounded operator $F$ such that $\mathcal{A}$ acts on $H$ and $a(F - F^*), a(F^2 - 1), [F, a] \in \mathcal{K}(H)$ for any $a \in \mathcal{A}$. The commutator $[F, a]$ is called a quantized differential of $a$. The notion and calculus of Fredholm modules provide many techniques in studying various spaces. Such examples are noncompact spaces, foliated spaces, noncommutative spaces, and fractal spaces, to name a few. In the present paper, we study Fredholm modules on a special class of fractal spaces called self-similar sets.

The first study of quantized calculus on self-similar sets is given by Connes [3, Chapter IV]. Connes defined the Fredholm module $(H, F)$ on $C(CS)$, where $CS$ is the Cantor set realized in the interval $[0, 1]$, by using vertices of the removed intervals. Specifically, he set $H_I = \ell^2(\{a\}) \oplus \ell^2(\{b\})$ for an open interval $I = (a, b)$ and $F_I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ on $H_I$, and constructed $(H, F)$ by taking a direct sum of $(H_I, F_I)$ on all removed open intervals which appear along the construction of $CS$. The Fredholm module $(H, F)$ defines an element in $K^0(C(CS))$. Connes also calculated the non-vanishing Dixmier trace $\text{Tr}_{\omega}(|[F, x]|^{\text{dim}_H(CS)})$. Here, $x$ is the coordinate function on $\mathbb{R}$ (we consider $x$ as a multiplication operator) and $\text{dim}_H(CS)$ is the Hausdorff dimension of $CS$. We call $|[F, x]|^{\text{dim}_H(CS)}$ the quantized volume measure on $CS$ and
Thus, the choice of $tordust with rotation angle $\pi$ specifies a unified choice of $K$ on rotation of the square. In fact, for a self-similar set square, we have the non-trivial value quantized differential of we mean by the generalization of Connes’ quantized volume to higher dimensional self-similar sets. For the generalization, we construct a $S$ subset the other hand, if we choose $w$ have
\[ [F_K, x^1] [F_K, x^2] \cdots [F_K, x^n] \] is a quantized volume measure of the volume measure $dx^1 dx^2 \cdots dx^n$ on $\mathbb{R}^n$. Here, $p \in \mathbb{R}$ is defined by a fractal dimension on $K$. Then the value
\[ \text{Tr}_\omega ([[F_K, x^1] [F_K, x^2] \cdots [F_K, x^n]]^p) \] may be also called as a generalization of Connes’ quantized volume on $K$.

Let us explain some examples that motivate us to conduct this work. We construct a Fredholm module on a self-similar set $K$ built on the square in $\mathbb{R}^2$. When we adopt a standard way to construct Fredholm modules on more generic $K$ (see [5, section 2]), it suffices to choose a subset $S \subset K$ for $(H_S, F_S)$. As constructed in [4], when we choose $S = \{a, b\}$ (2 points), we have the same Fredholm module $(H_S, F_S)$ as Connes’ one. The Fredholm module $(H_S, F_S)$ gives rise to a Fredholm module $(H_K, F_K)$ composed by the direct sum over all steps in the construction of $K$.

Then the commutator $[F_K, x]$ (resp. $[F_K, y]$) is essentially given by the length of the projection of a segment $ab$ to the $x$-axis (resp. $y$-axis), and we can calculate the value
\[ \text{Tr}_\omega ([[F_K, x] [F_K, y]]^p). \] However, the value may vanish: suppose that the vertices of the square are numbered counterclockwise in the order $v_0, v_1, v_2, v_3$. When $K$ is the Cantor dust (see Figure 5) and every edge of the square is parallel to either $x$- or $y$-axis, we have $[F_K, x] [F_K, y] = 0$ if $S = \{v_i, v_j\}$ is the boundary of an edge of the square. On the other hand, if we choose $S = \{v_0, v_2\}$ to be the boundary of a diagonal line of the square, we have the non-trivial value $\text{Tr}_\omega ([[F_K, x] [F_K, y]]^{\dim_K(n)})$. Therefore the subset $S = \{v_0, v_2\}$ may look like an appropriate choice for the Cantor dust. However, the value $\text{Tr}_\omega ([[F_K, x] [F_K, y]]^{\dim_K(n)})$ for $S = \{v_0, v_2\}$ is not preserved under the rotation of the square. In fact, for a self-similar set $K$ obtained by rotation of the Cantor dust with rotation angle $\pi/4$ around $v_0$, we have $[F_K, x] [F_K, y] = 0$ for $S = \{v_0, v_2\}$. Thus, the choice of $S = \{a, b\}$ giving a non-trivial $\text{Tr}_\omega ([[F_K, x] [F_K, y]]^p)$ depends on $K$. In this paper, we also present a way to construct a Fredholm module for $K$ that specifies a unified choice of $S$ (not necessarily 2 points) and show that the Fredholm
module induces a non-trivial higher dimensional quantized volume measure which is invariant under the Euclidean isometries in $\mathbb{R}^n$.

The outline of our construction of the Fredholm module $(H_K, F_K)$ on $K$ is the following. Let $\gamma_n = [0, 1]^n$ be the $n$-cube and \( \{ f_s : \gamma_n \rightarrow \gamma_n \} \ (s = 1, 2, \ldots, N) \) be similitudes with the similarity ratio $0 < r_s < 1$. We note that we do not require the open set condition. We now have a decreasing sequence of compact sets $K_j = \bigcup_{(s_1, \ldots, s_j)} f_{s_1} \circ \cdots \circ f_{s_j}(\gamma_n)$ in which each $f_{s_1} \circ \cdots \circ f_{s_j}(\gamma_n)$ is a small copy of the $n$-cube. Then, the sequence gives rise to the limiting set $K = \bigcap_{j=0}^{\infty} K_j$. Our construction of $(H_K, F_K)$ is made of 2 steps: the first step is the construction of the Fredholm module $(H, F_n)$ on the $n$-cube; see subsection 1.1. In our construction, we use all vertices (instead of 2 points) of $n$-cubes, that is, we set $\mathcal{H} = \ell^2(\{\text{vertices}\})$ with a suitable $\mathbb{Z}_2$-grading. In the definition of $F_n$, we use induction on the dimension $n$. The resulting Fredholm module represents the Kasparov product ($n$-times) of Connes’ Fredholm module on an interval. The second step is taking the direct sum of $(\mathcal{H}, F_n)$ on all the copies of $n$-cubes; see subsection 2.1. Our Fredholm module $(\mathcal{H}_K, F_K)$ is defined over $C(V_K)$, where we denote by $V_K$ the closure of the vertices of all $n$-cubes $f_{s_1} \circ \cdots \circ f_{s_j}(\gamma_n)$. Note that, in general, $V_K$ includes $K$ properly, but $V_K$ coincides with $K$ for some important examples such as the Cantor dust, the Sierpinski carpet and the Menger sponge. Dividing by the length of edges of each $n$-cubes, we get the Dirac operator $D_K$ on $K$ and the spectral triple on $K$.

Main results in the paper are basically twofold: our first result is the construction of a higher dimensional analogue of the Connes’ Fredholm module. This Fredholm module is also non-trivial in $K^0$ group under additional assumptions, which is given in Theorem 2.5 as a part of other properties of the Fredholm module delved in Section 2. The second result is the derivation of concrete values for higher dimensional variants of the quantized volume measure and the quantized volume for some self-similar sets. The results are given in Section 3. The calculation is based on a Clifford algebra’s relation which the commutators $[F_n, x^\alpha] \ (\alpha = 1, \ldots, n)$ generally satisfy for the $\alpha$-th coordinate functions $x^\alpha$ on $\mathbb{R}^n$. The Clifford algebra’s relation is quantizaion of the relation of the exterior differentials $dx^\alpha$; see Proposition 2.8 and 2.9 for the details.

Fredholm modules on self-similar sets are constructed by various researchers and studied from various aspects. F. Cipirani-J. Sauvageot [2] constructed Fredholm modules on post critically finite fractals (p.c.f fractals) by regular harmonic structures. M. Ionescu-L. Rogers-A. Teplyaev [7] studied weakly summable Fredholm modules in the cases of some finitely and infinitely ramified fractals. As an unbounded picture of Fredholm modules, spectral triples on some self-similar sets have been also extensively investigated. E. Christensen-C. Ivan-L. Lapidus [1] defined a spectral triple on...
the Sierpinski gasket $\mathcal{S}_G$, which in turn defines an element in $K^1(C(\mathcal{S}_G))$, by using the Dirac operator on the circle. D. Guido-T. Isola [4] defined a spectral triple on self-similar sets with the open set condition in higher dimension by using Connes’ Fredholm module on an interval. Guido-Isola [5] also defined a spectral triple on nested fractals by using Connes’ Fredholm module on an interval. See Introduction in [5] for more related literatures.

Let us compare our spectral triple with Guido-Isola’s triples. First, our Fredholm module cannot be constructed on self-similar sets on arbitrary subsets in $\mathbb{R}^n$, but on $n$-cubes. Our construction also does not require the open set condition. An example of the case for a self-similar set without the open set condition is given in subsection 4.5. Second, our triple and the triple in [4] are not constructed on the algebra $C(K)$ of the continuous functions on $K$. Our algebra $C(V_K)$ coincides with $C(K)$ for some important examples such as the Cantor dust, the Sierpinski carpet and the Menger sponge. The calculation of the value $\text{Tr}_{\omega}(D_K^{-p})$ for our Dirac operator is also given in subsection 3.1. The triple in [5] is defined on $C(K)$ for the class of nested fractals, but the examples mentioned above are not the case.

|            | Ours                        | G-I’s [4]                   | G-I’s [5]                   |
|------------|-----------------------------|-----------------------------|-----------------------------|
| space      | self-similar set on $n$-cube| self-similar set on $\mathbb{R}^n$ | nested fractal              |
| algebra    | $C(V_K)$                    | $C(C)$                      | $C(K)$                      |

We will study more noncommutative geometry of our Fredholm module $(\mathcal{H}_K, F_K)$ and the corresponding spectral triple $(\mathcal{H}_K, D_K)$ in future papers.

1. Fredholm module on $n$-cube

1.1. Definition of Fredholm module

In this subsection, we construct a “good” Fredholm module on $n$-cubes $\gamma_n$. For the simplicity, we set $\gamma_n = [0, e]^n$ in $\mathbb{R}^n$ with the length of edge $e > 0$; the following construction applies to any $n$-cubes.

Let $V$ be the set of vertices of $\gamma_n$:

$$V = \{(a_1, \ldots, a_n) \in \mathbb{R}^n ; a_i = 0 \text{ or } e \quad (i = 1, 2, \ldots, n)\}.$$  

We give a number of vertices in $V$ inductively. For $n = 1$, an interval $\gamma_1 = [0, e]$ has two vertices 0 and $e$. Set $v_0 = 0$ and $v_1 = e$. For an arbitrary $n$, we assume that we have a number of vertices of $\gamma_{n-1}$. Then a number of vertices of $\gamma_n$ is as follows:

$$(1) \quad v_i = (a_1, \ldots, a_{n-1}, 0) = (a_1, \ldots, a_{n-1}) \quad (0 \leq i \leq 2^{n-1} - 1) \text{ under the inclusion } \gamma_{n-1} \hookrightarrow \gamma_{n-1} \times \{0\} \subset \gamma_n.$$
(2) $v_{2n-1-i} = (a_1, \ldots, a_{n-1}, e)$ $(0 \leq i \leq 2^{n-1} - 1)$ if $v_i = (a_1, \ldots, a_{n-1}, 0)$.

Example 1.1. (1) When $n = 2$, the numbering of vertices given by $v_0 = (0, 0)$, $v_1 = (e, 0)$, $v_2 = (e, e)$, $v_3 = (0, e)$; see Figure 1.

(2) When $n = 3$, the numbering of vertices is given by $v_0 = (0, 0, 0)$, $v_1 = (e, 0, 0)$, $v_2 = (e, e, 0)$, $v_3 = (0, e, 0)$, $v_4 = (0, e, e)$, $v_5 = (e, e, e)$, $v_6 = (e, 0, e)$, $v_7 = (0, 0, e)$.

See Figure 2.

Set $V_0 = \{v_i : i = \text{even}\}$ and $V_1 = \{v_i : i = \text{odd}\}$, so we have $V = V_0 \cup V_1$. Set also

\[
\mathcal{H}^+ = \ell^2(V_0) = \ell^2(v_0) \oplus \ell^2(v_2) \oplus \cdots \oplus \ell^2(v_{2^{n-2}}),
\]

\[
\mathcal{H}^- = \ell^2(V_1) = \ell^2(v_1) \oplus \ell^2(v_3) \oplus \cdots \oplus \ell^2(v_{2^{n-1}})
\]

and $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$. The vector space $\mathcal{H}(\cong \mathbb{C}^{2^n})$ is a Hilbert space of dimension $2^n$ with an inner product

\[
\langle f, g \rangle = \sum_{i=0}^{2^n-1} f(v_i) \overline{g(v_i)}.
\]

We assume that $\mathcal{H}$ is $\mathbb{Z}_2$-graded with the grading $\epsilon = \pm 1$ on $\mathcal{H}^\pm$, respectively. The $C^*$-algebra $C(V)$ of continuous functions on $V$ acts on $\mathcal{H}$ by multiplication:

\[
\rho(f) = (f(v_0) \oplus f(v_2) \oplus \cdots \oplus f(v_{2^{n-2}})) \oplus (f(v_1) \oplus f(v_3) \oplus \cdots \oplus f(v_{2^{n-1}})).
\]
A Fredholm operator $F_n$ on $\mathcal{H}$ is also defined inductively. We set $X_1 = 1$ and $X_n = \begin{bmatrix} O & X_{n-1} \\ X_{n-1} & O \end{bmatrix} \in M_{2^{n-1}}(\mathbb{C})$ $(n \geq 2)$. We set also $G_1 = 1$, $G_n = \begin{bmatrix} G_{n-1} & -X_{n-1} \\ X_{n-1} & G_{n-1} \end{bmatrix} \in M_{2^{n-1}}(\mathbb{C})$ $(n \geq 2)$ and $U_n = \frac{1}{\sqrt{n}}G_n$ $(n \geq 1)$.

**Proposition 1.2.** $U_n$ is a unitary matrix.

**Proof.** Firstly, we have

$$X_nG_n^* - G_nX_n = \begin{bmatrix} O & X_{n-1} \\ X_{n-1} & O \end{bmatrix} \begin{bmatrix} G_{n-1}^* & X_{n-1} \\ -X_{n-1} & G_{n-1}^* \end{bmatrix} - \begin{bmatrix} G_{n-1} & -X_{n-1} \\ X_{n-1} & G_{n-1} \end{bmatrix} \begin{bmatrix} O & X_{n-1} \\ X_{n-1} & O \end{bmatrix}$$

$$= X_2 \otimes (X_{n-1}G_{n-1}^* - G_{n-1}X_{n-1})$$

$$= \cdots = X_n \otimes (X_1G_1^* - G_1X_1) = 0.$$

We prove $U_nU_n^* = E_{2^n}$ by induction. Clearly, $U_1 = 1$ is unitary. Assume that $U_{n-1}$ is a unitary matrix. Then we have

$$G_{n-1}G_{n-1}^* + X_{n-1}^2 = (n - 1)E_{2^{n-2}} + E_{2^{n-2}} = nE_{2^{n-2}}.$$

Thus we obtain

$$G_nG_n^* = \begin{bmatrix} G_{n-1} & -X_{n-1} \\ X_{n-1} & G_{n-1} \end{bmatrix} \begin{bmatrix} G_{n-1}^* & X_{n-1} \\ -X_{n-1} & G_{n-1}^* \end{bmatrix}$$

$$= \begin{bmatrix} G_{n-1}G_{n-1}^* + X_{n-1}^2 & G_{n-1}X_{n-1} - X_{n-1}G_{n-1}^* \\ X_{n-1}G_{n-1}^* - G_{n-1}X_{n-1} & X_{n-1}^2 + G_{n-1}G_{n-1}^* \end{bmatrix}$$

$$= \begin{bmatrix} G_{n-1}G_{n-1}^* + X_{n-1}^2 & (X_{n-1}G_{n-1}^* - G_{n-1}X_{n-1})^* \\ X_{n-1}G_{n-1}^* - G_{n-1}X_{n-1} & X_{n-1}^2 + G_{n-1}G_{n-1}^* \end{bmatrix} = nE_{2^{n-1}}.$$

Therefore, $U_n = \frac{1}{\sqrt{n}}G_n$ is a unitary matrix.

Set $F_n = \begin{bmatrix} U_n^* \\ U_n \end{bmatrix} \in M_{2^n}(\mathbb{C})$. By Proposition 1.2, we have $F_n^2 = E_{2^n}$ and $F_n^* = F_n$.

We consider that $F_n$ is a bounded operator on a finite dimensional Hilbert space $\mathcal{H} = (\ell^2(v_0) \oplus \ell^2(v_2) \oplus \cdots \oplus \ell^2(v_{2^n-2})) \oplus (\ell^2(v_1) \oplus \ell^2(v_3) \oplus \cdots \oplus \ell^2(v_{2^n-1})) \cong \mathbb{C}^{2^n}$ by the left multiplication of a matrix $F_n$. Because of $F_n\varepsilon + \varepsilon F_n = 0$, $(\mathcal{H}, F_n)$ is an even Fredholm module on $C(V)$.

**Example 1.3.**

1. When $n = 1$, we have $F_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, which is introduced by [3, Chapter IV. 3. ε].
(2) When \( n = 2 \), we have

\[
G_2 = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad U_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad F_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}.
\]

(3) When \( n = 3 \), we have

\[
G_3 = \begin{bmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad U_3 = \frac{1}{\sqrt{3}} G_3 \quad \text{and} \quad F_3 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}.
\]

**Remark 1.4.** The components of \( G_n \) correspond to the following orientation of edges, the correspondence is similar to adjacency matrices of oriented graphs. When \( n = 1 \), the orientation of the graph \( \gamma_1 = [0, e] \) is from \( v_0 = 0 \) to \( v_1 = e \); we denote such an orientation by \( v_0 \to v_1 \). Assume that we have the orientation of the edges of \( \gamma_{n-1} \).

1. Assume \( 0 \leq i, j \leq 2^{n-1} - 1 \). The orientation in \( \gamma_n \) is from \( v_i \) to \( v_j \); \( v_i \rightarrow v_j \), when the orientation in \( \gamma_{n-1} \) is from \( v_i \) to \( v_j \). Here, we consider that \( \gamma_{n-1} \) is a subset in \( \gamma_n \) under the inclusion \( \gamma_{n-1} \to \gamma_n \).
2. \( v_i \rightarrow v_{2^{n-1}-i} (0 \leq i \leq 2^{n-1} - 1) \), which means \( (a_1, \ldots, a_{n-1}, 0) \to (a_1, \ldots, a_{n-1}, e) \).
3. \( v_{2^{n-1}-i} \leftarrow v_{2^{n-1}-j} \) if \( v_i \rightarrow v_j \) \( (0 \leq i, j \leq 2^{n-1} - 1) \).

![Figure 3. orientation of edges of \( \gamma_2 \)](image)

![Figure 4. orientation of edges of \( \gamma_3 \)](image)

Then the \((i, j)\)-component \( g_{ij} (1 \leq i, j \leq 2^{n-1}) \) of \( G_n \) is as follows.

1. \( g_{ij} = 1 \) when \( v_{2j-2} \to v_{2i-1} \).
2. \( g_{ij} = -1 \) when \( v_{2j-2} \leftarrow v_{2i-1} \).
3. \( g_{ij} = 0 \) when \( v_{2j-2} \) and \( v_{2i-1} \) do not connect by an edge.
1.2. Calculation of quantized differential form

In this subsection we calculate an operator \( F_n - \alpha \) for the coordinate function \( x^\alpha \) on \( \mathbb{R}^n (\alpha = 1, 2, \ldots, n) \). We also show they satisfy a relation of the Clifford algebra on the Euclidean vector space of dimension \( n \).

Set \( d_n f = [F_n, f] = \begin{bmatrix} d^+_n f \\ d^-_n f \end{bmatrix} \), and we have

\[
\begin{align*}
  d^+_n f &= U f^+ - f^- U \\
  d^-_n f &= U^* f^- - f^+ U^* = -(U f^+ - f^- U)^* = -f^+_n f,
\end{align*}
\]

where \( f^+ = f|_{V_0} \) and \( f^- = f|_{V_1} \). Denote by \( A \circ B = [a_{ij} b_{ij}] \) the Hadamard product of two matrices \( A = [a_{ij}] \) and \( B = [b_{ij}] \) of the same size.

**Proposition 1.5.** For any \( f \in C(V) \), we set \( f_{a,b} = f(v_a) - f(v_b) \) and

\[
\Delta_n f = [f_{2j,2i+1}]_{i,j=0,1,\ldots,2n-1-1} \in \mathcal{B}(\ell^2(V_0), \ell^2(V_1)) \cong M_{2n-1}(\mathbb{C}).
\]

We have

\[
d_n f = \frac{1}{\sqrt{n}} \left[ \Delta_n f \circ G_n \right].
\]

**Proof.** As in Remark 1.4, we denote \( G_n = [g_{ij}] \). We have

\[
\begin{align*}
\sqrt{n} d^+_n f &= G_n \begin{bmatrix} f(v_0) \\ f(v_2) \\ \vdots \\ f(v_{2n-2}) \end{bmatrix} - G_n \begin{bmatrix} f(v_1) \\ f(v_3) \\ \vdots \\ f(v_{2n-1}) \end{bmatrix} \\
&= [g_{ij} f(v_{2j})] - [f(v_{2i-1}) g_{ij}] \\
&= [f_{2j,2i-1} g_{ij}] \\
&= \Delta_n f \circ G_n.
\end{align*}
\]

Thus an \((i, j)\)-component of \( d^+_n f \) is 0 if \( v_{2i-1} \) and \( v_{2j} \) do not connect by an edge.

**Proposition 1.6.** For the coordinate function \( x^\alpha \) on \( \mathbb{R}^n (\alpha = 1, 2, \ldots, n) \), we set \( e^\alpha_{(n)} = \frac{\sqrt{n}}{e} d_n x^\alpha \). We have

\[
e^\alpha_{(n)} = \frac{\sqrt{n}}{e} d_n x^\alpha = \begin{bmatrix} E_{2n-\alpha} & 1 \\ -E_{2n-\alpha} & -1 \end{bmatrix} \otimes X_\alpha. \tag{1.1}
\]

Here, $E_{1/2} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1$.

**Proof.** Firstly, by using $\Delta_n x^n \circ G_n = -eX_n$ and Proposition 1.5, we have $e^n_{(n)} = \begin{bmatrix} 1 & X_n \end{bmatrix}$.

Next we calculate $e^{n-1}_{(n)} = \sqrt{n}d_n x^{n-1}$. By the definition of the numbering of vertices and the orientation of edges of $\gamma_n$, for $0 \leq i, j \leq 2^n - 1$, "$v_i \rightarrow v_j$ is positive (resp. negative) with $x^{n-1}$ direction" if and only if "$v_{i+2^n-1} \rightarrow v_{j+2^n-1}$ is negative (resp. positive) with $x^{n-1}$ direction". So we have $e^{n-1}_{(n)} = \begin{bmatrix} e^{n-1}_{(n-1)} & -e^{n-1}_{(n-1)} \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix} \otimes (-X_{n-1})$. This implies

$$e^{n-1}_{(n)} = \begin{bmatrix} E_1 \otimes \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \end{bmatrix} \otimes X_{n-1}.$$

We calculate $e^\alpha_{(n)} (\alpha = 1, 2, \ldots, n - 2)$ by induction on $n \geq 3$. Note that the calculation of $e^\alpha_{(n)}$ for $n = 1, 2$ is already done. Namely, the beginning of induction is the following:

$$e^1_{(1)} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad e^1_{(2)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad e^2_{(2)} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

Assume that equation (1.1) holds for $n - 1$. By the definition of the numbering of vertices and the orientation of edges of $\gamma_n$, for $1 \leq \alpha \leq n - 2$, "$i \rightarrow j$ is positive (resp. negative) with $x^\alpha$ direction" if and only if "$v_{i+2^n-1} \rightarrow v_{j+2^n-1}$ is positive (resp. negative) with $x^\alpha$ direction". So we have

$$e^\alpha_{(n)} = \begin{bmatrix} e^{\alpha+}_{(n-1)} & e^{\alpha+}_{(n-1)} \end{bmatrix} = E_2 \otimes e^{\alpha+}_{(n-1)} = -E_2 \otimes \left( E_{2n-1-a-1} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \otimes X_\alpha \right)$$

$$= -E_{2n-a-1} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \otimes X_\alpha.$$

Therefore we have

$$e^\alpha_{(n)} = \begin{bmatrix} E_{2n-a-1} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \otimes X_\alpha \end{bmatrix} (\alpha = 1, 2, \ldots, n - 2).$$
We have equation (1.1) by the above calculations for any \( n \) and \( \alpha = 1, 2, \ldots n \).

By the explicit formula of \( e^{\alpha}_{(n)} \) in Proposition 1.6, we have a Clifford relation of \( d_{n}x^{\alpha} \).

**Proposition 1.7.** We have

\[
e^{\alpha}_{(n)}e^{\beta}_{(n)} = \begin{cases} -e^{\beta}_{(n)}e^{\alpha}_{(n)} & (\alpha \neq \beta) \\ -E_{2n} & (\alpha = \beta) \end{cases}.
\]

By \( d_{n}x^{\alpha} = \frac{e}{\sqrt{n}}e^{\alpha}_{(n)} \), we have

\[
d_{n}x^{\alpha}d_{n}x^{\beta} = \begin{cases} -d_{n}x^{\beta}d_{n}x^{\alpha} & (\alpha \neq \beta) \\ -\frac{e}{n}E_{2n} & (\alpha = \beta) \end{cases}.
\]

**Proof.** Firstly, we have

\[
e^{\alpha}_{(n)}e^{\alpha}_{(n)} = \begin{bmatrix} -E_{2n-\alpha-1}^{2} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{2} \\ -E_{2n-\alpha-1}^{2} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix}^{2} \end{bmatrix} \otimes X^{2}_{\alpha} = -E_{2n}.
\]

Set \( k = \alpha - \beta > 0 \), then we have \( X^{\alpha}_{\alpha} = X_{k+1} \otimes X_{\beta} \). We can rewrite \( e^{\alpha}_{(n)} \) and \( e^{\beta}_{(n)} \) as follows:

\[
e^{\alpha}_{(n)} = \begin{bmatrix} E_{2n-\alpha-1} \otimes \begin{bmatrix} X_{k+1} \\ -X_{k+1} \end{bmatrix} \\ -E_{2n-\alpha-1} \otimes \begin{bmatrix} X_{k+1} \\ -X_{k+1} \end{bmatrix} \end{bmatrix} \otimes X_{\beta},
\]

\[
e^{\beta}_{(n)} = \begin{bmatrix} E_{2n-\alpha-1} \otimes E_{2k} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ -E_{2n-\alpha-1} \otimes E_{2k} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \end{bmatrix} \otimes X_{\beta}.
\]

Now, we set \( \epsilon_{1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \), and we have

\[
\begin{bmatrix} X_{k+1} \\ -X_{k+1} \end{bmatrix} \left( E_{2k} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} X_{k} \otimes (X_{2}\epsilon_{1}) \\ -X_{k} \otimes (X_{2}\epsilon_{1}) \end{bmatrix}
\]

and

\[
\left( E_{2k} \otimes \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} X_{k+1} \\ -X_{k+1} \end{bmatrix} = \begin{bmatrix} X_{k} \otimes (\epsilon_{1}X_{2}) \\ -X_{k} \otimes (\epsilon_{1}X_{2}) \end{bmatrix}.
\]
Thus the relation $e^{\alpha}_{(n)} e^{\beta}_{(n)} = -e^{\alpha}_{(n)} e^{\beta}_{(n)}$ ($\alpha \neq \beta$) holds since we have $X_2 \varepsilon_1 + \varepsilon_1 X_2 = O$. 

**Remark 1.8.** When we take the limit as the length of edges tends to 0, that is $e \rightarrow 0$, we have

$$d_n x^\alpha d_n x^\alpha = -\frac{e^2}{n} E_{2n} \rightarrow O.$$  

Thus we regard $d_n x^\alpha$ as a quantization of the ordinal exterior differential $d x^\alpha$ on $\mathbb{R}^n$.

**Remark 1.9.** For any unitary matrix $U \in U(2^{n-1})$, since an odd matrix $F = \begin{bmatrix} U^* \\ U \end{bmatrix}$ defines an operator on $\mathcal{H}$, $F$ defines a Fredholm module on $C(V)$. Moreover, since any $F$ is homotopic to $F_n$, it defines a same $K$-homology class in $K^0(C(V))$. However, the general $F$ sometimes does not have good properties. For example, we have $[F, x^\alpha] = O$ for $\alpha = 2, 3, \ldots, n$ when we assume $U = E_{2n-1}$. Thus, in this case, we cannot regard $[F, x^\alpha]$ as a quantization of the ordinal exterior differential $d x^\alpha$ on $\mathbb{R}^n$.

By Proposition 1.7, we have the volume element $\omega_n = e^1_{(n)} e^2_{(n)} \cdots e^n_{(n)}$ in the Clifford algebra. We can easily calculate its absolute value $|\omega_n|$. We do not use $|\omega_n|$ directly, but we use $|d_n x^1 d_n x^2 \cdots d_n x^n|$, which is a constant multiple of $|\omega_n|$; see also Section 3.2.

**Proposition 1.10.** We have $|[F_n, x^1] \cdots [F_n, x^n]| = \frac{e^n}{n^{n/2}} E_{2n}$. By the definition of $e^{\alpha}_{(n)}$, we also have $|\omega_n| = E_{2n}$.

**Proof.** Because of $[F_n, x^\alpha]^*[F_n, x^\alpha] = \frac{e^2}{n} e^{\alpha*}_{(n)} e^{\alpha}_{(n)} = -\frac{e^2}{n} (e^{\alpha}_{(n)})^2 = \frac{e^2}{n} E_{2n}$, we have

$$|[F_n, x^1] \cdots [F_n, x^n]|^2 = ([F_n, x^1] \cdots [F_n, x^n])^*[F_n, x^1] \cdots [F_n, x^n]$$

$$= [F_n, x^n]^* \cdots [F_n, x^1]^*[F_n, x^1] \cdots [F_n, x^n]$$

$$= \left(\frac{e^2}{n}\right)^n E_{2n}.$$

This implies

$$|[F_n, x^1] \cdots [F_n, x^n]| = \frac{e^n}{n^{n/2}} E_{2n}.$$
2. Fredholm module on self-similar sets built on \( n \)-cubes

2.1. Fredholm module and spectral triple

In this subsection, we construct a Fredholm module and a spectral triple on self-similar sets built on any \( n \)-cubes \( \gamma_n \). For the simplicity, we assume that the length of edges of \( \gamma_n \) equals 1. Let \( f_s : \gamma_n \to \gamma_n \) \((s = 1, \ldots, N)\) be similitudes. We define the similarity ratio of \( f_s \) to be

\[
r_s = \frac{||f_s(x) - f_s(y)||_{\mathbb{R}^n}}{||x - y||_{\mathbb{R}^n}} \quad (x \neq y).
\]

An iterated function system (IFS) \((\gamma_n, S = \{1, \ldots, N\}, \{f_s\}_{s \in S})\) defines the unique non-empty compact set \( K = K(\gamma_n, S = \{1, \ldots, N\}, \{f_s\}_{s \in S}) \) called the self-similar set such that \( K = \bigcup_{s=1}^{N} f_s(K) \). We use \( \text{dim}_S(K) \) to denote the similarity dimension of \( K \), that is, the number \( s \) that satisfies

\[
\sum_{s=1}^{N} r_s^s = 1.
\]

If an IFS \((\gamma_n, S, \{f_s\}_{s \in S})\) satisfies the open set condition, \( \text{dim}_S(K) \) turns out to be equal to the Hausdorff dimension \( \text{dim}_H(K) \) of \( K \).

Set \( f_S = f_{s_1} \circ \cdots \circ f_{s_j} \) for \( s = (s_1, \ldots, s_j) \in S^\infty = \bigcup_{j=0}^{\infty} S^j \) and \( f_0 = \text{id} \). For the simplicity, we will use \( t \) to express the vertex \( f_S(v_t) \) of an \( n \)-cube \( f_S(\gamma_n) \) and write \( V_S \) as the vertices of an \( n \)-cube \( f_S(\gamma_n) \). We also denote the length of the edge of \( f_S(\gamma_n) \) by \( e_S \). As introduced in subsection 1.1, we define the Hilbert space \( H_S = \ell^2(V_S) \) on an \( n \)-cube of the length \( e_S \) that consists of the positive part \( H_S^+ \) and the negative part \( H_S^- \). By taking the direct sum on all \( n \)-cubes, we define the following data:

\[
H_K = \bigoplus_{s \in S^\infty} H_S, \quad F_K = \bigoplus_{s \in S^\infty} F_n, \quad D_K = \bigoplus_{s \in S^\infty} \frac{1}{e_S} F_n.
\]

Let \( V_K \) be the closure of the set of vertices of all \( n \)-cubes \( f_S(\gamma_n) \subset \mathbb{R}^n \). That is, \( V_K \) is the closure of \( \bigcup_{s \in S^\infty} V_S \). Then, if \( V \subset \bigcup_{s=1}^{N} f_s(V) \) holds, we have \( V_K = K \). If not, \( V_K \) equals the union of \( \bigcup_{s=1}^{N} V_S \) and \( K \). We also let \( \mathcal{A}_K \) be the Banach algebra of Lipschitz functions \( \text{Lip}(V_K) \) on \( V_K \) with the norm \( ||a||_{\mathcal{A}_K} = ||a||_{\infty} + \text{Lip}(a) \), where the second term is the Lipschitz constant of a Lipschitz function \( a \). The Banach algebra \( \mathcal{A}_K \) acts on \( H_K \) by

\[
\rho_K : \mathcal{A}_K \to \mathcal{B}(H_K); \quad \rho_K(a)(\oplus \xi_S) = \oplus(a|V_S) \cdot \xi_S.
\]
Lemma 2.1. Define
\[
\mathcal{H}_K^1 = \left\{ \bigoplus_{s \in S^n} \xi_s \in \mathcal{H}_K ; \| \bigoplus_{s \in S^n} \xi_s \|_{H_K}^2 = \sum_{s \in S^n} 1 \sum_{i=0}^{2^n-1} |\xi_s(i)|^2 < \infty \right\}.
\]

Then, an operator \(D_K\) is a self-adjoint operator of \(\text{dom}(D_K) = \mathcal{H}_K^1\).

Proof. By inclusions \(\{ \bigoplus_{s \in S^n} \xi_s \in \mathcal{H}_K ; \xi_s = 0 \text{ except finite} s \} \subset \mathcal{H}_K^1 \subset \mathcal{H}_K\), \(\mathcal{H}_K^1\) is a dense subset in \(\mathcal{H}_K\).

On each \(n\)-cubes \(f_s(\gamma_n)\), we have
\[
\|F_n \xi_s\|_{\ell^2}^2 = \|U_n \xi_s^+\|_{\ell^2}^2 + \|U_n \xi_s^-\|_{\ell^2}^2 = \|\xi_s^+\|_{\ell^2}^2 + \|\xi_s^-\|_{\ell^2}^2 = \sum_{i=0}^{2^n-1} |\xi_s(i)|^2
\]
for any function \(\xi_s\) on \(V_s\), where \(\xi_s^\pm\) denote the \(\mathcal{H}_s^\pm\) parts of \(\xi_s\), respectively. Then, we have
\[
\|D_K (\bigoplus_{s \in S^n} \xi_s)\|_{\mathcal{H}_K}^2 = \sum_{s \in S^n} 1 \sum_{i=0}^{2^n-1} |\xi_s(i)|^2 = \| \bigoplus_{s \in S^n} \xi_s \|_{\mathcal{H}_K}^2
\]
for \(\bigoplus_{s \in S^n} \xi_s \in \mathcal{H}_K\). Thus we have \(D_K (\mathcal{H}_K^1) \subset \mathcal{H}_K\), and \(D_K\) is a symmetric operator with domain \(\mathcal{H}_K^1\).

On the other hand, we set \(\bigoplus_{s \in S^n} \eta_s = \bigoplus_{s \in S^n} F_n \xi_s\) for any \(\bigoplus_{s \in S^n} \xi_s \in \mathcal{H}_K\). Then, \(\bigoplus_{s \in S^n} \eta_s \in \mathcal{H}_K^1\) since
\[
\| \bigoplus_{s \in S^n} \eta_s \|_{\mathcal{H}_K}^2 = \sum_{s \in S^n} \|F_n \xi_s\|_{\ell^2}^2 = \sum_{s \in S^n} \|\xi_s\|_{\ell^2}^2 = \| \bigoplus_{s \in S^n} \xi_s \|_{\mathcal{H}_K}^2 < \infty.
\]
This implies \(D_K (\mathcal{H}_K^1) \supset \mathcal{H}_K\). Thus we have \(D_K (\mathcal{H}_K^1) = \mathcal{H}_K\). Therefore \(D_K\) is a self-adjoint operator of
\[
\text{dom}(D_K) = \left\{ \bigoplus_{s \in S^n} \xi_s \in \mathcal{H}_K ; \sum_{s \in S^n} 1 \sum_{i=0}^{2^n-1} |\xi_s(i)|^2 < \infty \right\}.
\]

Note that we have \(\rho_K(\mathcal{A}_K)(\mathcal{H}_K^1) \subset \mathcal{H}_K^1\) and \(F_K = D_K |D_K|^{-1}\). We now prove some regularity of \(F_K\) and \(D_K\).

Lemma 2.2. We have the followings:

(1) \([F_K, a] \in \mathcal{K}(\mathcal{H}_K)\) for any \(a \in C(V_K)\).
(2) \([D_K, a] \in \mathcal{B}(\mathcal{H}_K)\) for any \(a \in \mathcal{A}_K\).
(3) \(|D_K|^{-1} \in \mathcal{K}(\mathcal{H}_K)\).
(4) \((D_K^2 + 1)^{-1/2} \in \mathcal{K}(\mathcal{H}_K)\).
(5) \(|D_K|^{-p} \in \mathcal{L}^1(\mathcal{H}_K) \iff p > \text{dim}_S(K)\), where \(\mathcal{L}^1(\mathcal{H}_K)\) is the set of trace class operators on \(\mathcal{H}_K\).

(6) \((D_K^2 + 1)^{-p/2} \in \mathcal{L}^1(\mathcal{H}_K) \iff p > \text{dim}_S(K)\).

**Proof.**

(1) First, we take \(a \in \mathcal{A}_K\). For any \(s \in S^{\times j}\), we have

\[
[F_K, a]|_{\mathcal{H}_S} = \frac{1}{\sqrt{n}} \left[ \Delta_n a \circ G_n \right]^{\text{inv}}(\Delta_n a \circ G_n).
\]

Therefore, the operator norm \(||[F_K, a]|_{\mathcal{H}_S}||\) is less than

\[
\text{Lip}(a) \cdot \varepsilon_s = \text{Lip}(a) \cdot \prod_{k=1}^{j} r_{s_k}.
\]

Thus \([F_K, a]\) is compact for \(a \in \mathcal{A}_K\) since we have \(\prod_{k=1}^{j} r_{s_k} \leq \max_{s \in S} r_{s}^{j} \to 0\) as \(j \to \infty\). The case for any continuous function is proved by the denseness of \(\mathcal{A}_K\) in \(C(V_K)\).

(2) For any \(s \in S^{\times j}\), we have

\[
[D_K, a]|_{\mathcal{H}_S} = \frac{1}{\sqrt{n}} \left( \prod_{k=1}^{j} r_{s_k} \right)^{-1} \left[ \Delta_n a \circ G_n \right]^{\text{inv}}(\Delta_n a \circ G_n).
\]

So the operator norm \(||[D_K, a]|_{\mathcal{H}_S}||\) is less than \(\text{Lip}(a)\), which is independent of \(j\). Therefore \([D_K, a]\) is bounded on \(\mathcal{H}_K\).

(3) Because of \(|D_K| = \bigoplus_{s \in S^{\infty}} \frac{1}{e_s} E_{2^n}\), we have \(|D_K|^{-1} = \bigoplus_{j=0}^{\infty} \bigoplus_{s \in S^{\times j}} \left( \prod_{k=1}^{j} r_{s_k} \right) E_{2^n}\).

Thus \(|D_K|^{-1}\) is compact since we have \(\prod_{k=1}^{j} r_{s_k} \to 0\) as \(j \to \infty\).

(4) Because of \(D_K^2 + 1 = \bigoplus_{s \in S^{\infty}} \left( \frac{1}{e_s^2} + 1 \right) E_{2^n}\), we have

\[
(D_K^2 + 1)^{-1/2} = \bigoplus_{j=0}^{\infty} \bigoplus_{s \in S^{\times j}} \left( \prod_{k=1}^{j} r_{s_k}^{-2} + 1 \right)^{-1/2} E_{2^n}.
\]

Thus \((D_K^2 + 1)^{-1/2}\) is a compact operator.
(5) Because of $|D_K|^{-p} = \bigoplus_{j=0}^{\infty} \bigoplus_{s \in S^j} \left( \prod_{k=1}^{j} r_{s_k}^{-p} \right) E_{2n}$, we have

$$\text{Tr}(|D_K|^{-p}) = \sum_{j=0}^{\infty} \sum_{s \in S^j} 2^n \prod_{k=1}^{j} r_{s_k}^{-p} = 2^n \sum_{j=0}^{\infty} \left( \sum_{s=1}^{N} r_{s_k}^{-p} \right)^j.$$ 

Thus we have

$$|D_K|^{-p} \in L^1(\mathcal{H}_K) \iff \sum_{s=1}^{N} r_{s_k}^{-p} < 1.$$ 

This implies $|D_K|^{-p} \in L^1(\mathcal{H}_K) \iff p > \dim_S(K)$.

(6) Because of

$$(D_K^2 + 1)^{-p/2} = \bigoplus_{j=0}^{\infty} \bigoplus_{s \in S^j} \left( \prod_{k=1}^{j} r_{s_k}^{-2} + 1 \right)^{-p/2} E_{2n},$$

we have

$$\text{Tr}((D_K^2 + 1)^{-p/2}) = \sum_{j=0}^{\infty} \sum_{s \in S^j} 2^n \left( \prod_{k=1}^{j} r_{s_k}^{-2} + 1 \right)^{-p/2}.$$ 

Thus we have

$$\sum_{j=0}^{\infty} \sum_{s \in S^j} 2^n \prod_{k=1}^{j} r_{s_k}^{-p} \leq \text{Tr}((D_K^2 + 1)^{-p/2}) \leq \sum_{j=0}^{\infty} \sum_{s \in S^j} 2^n \prod_{k=1}^{j} r_{s_k}^{-p},$$

that is we have

$$2^{n-p/2} \sum_{j=0}^{\infty} \left( \sum_{s=1}^{N} r_{s_k}^{-p} \right)^j \leq \text{Tr}((D_K^2 + 1)^{-p/2}) \leq 2^n \sum_{j=0}^{\infty} \left( \sum_{s=1}^{N} r_{s_k}^{-p} \right)^j.$$ 

This implies

$$(D_K^2 + 1)^{-p/2} \in L^1(\mathcal{H}_K) \iff \sum_{s=1}^{N} r_{s_k}^{-p} < 1 \iff p > \dim_S(K).$$

\[\square\]

**Theorem 2.3.** The pair $(\mathcal{H}_K, F_K)$ is an even Fredholm module over $C(V_K)$ with the $\mathbb{Z}_2$-grading $\epsilon_K = \bigoplus_{S \in S^{\infty}} \epsilon$. The pair $(\mathcal{H}_K, F_K)$ is a $(\lceil \dim_S(K) \rceil + 1)$-summable even Fredholm module over $\mathcal{A}_K$. In particular, if we have $\dim_S(K) < n$, an operator

$$[F_K, a^1][F_K, a^2] \cdots [F_K, a^n]$$

is of trace class for any $a^1, a^2, \ldots, a^n \in \mathcal{A}_K$.  

Proof. By the definition of $F_K$, we have $F_K^2 = 1$, $F_K^* = F_K$, and $F_K \varepsilon_K + \varepsilon_K F_K = 0$. $[F_K, a]$ is also a compact operator by Lemma 2.2. Therefore, $(H_K, F_K)$ is an even Fredholm module over $C(V_K)$.

Next we prove summability of the Fredholm module $(H_K, F_K)$ over $A_K$. Since $[D_K, a]$ is a bounded operator for $a \in A_K$ and $|D_K|^{-([\text{dim}_S(K)] + 1)}$ is of trace class, we have

$$
[F_K, a^1][F_K, a^2] \cdots [F_K, a^{[\text{dim}_S(K)] + 1}]
= [D_K, a^1]|D_K|^{-1}[D_K, a^2]|D_K|^{-1} \cdots [D_K, a^{[\text{dim}_S(K)] + 1}]|D_K|^{-1}
= [D_K, a^1][D_K, a^2] \cdots [D_K, a^{[\text{dim}_S(K)] + 1}]|D_K|^{-([\text{dim}_S(K)] + 1)} \in \mathcal{L}^1(H_K)
$$

for $a^1, a^2, \ldots, a^{[\text{dim}_S(K)] + 1} \in A_K$. Here, we have $|[D_K]|^{-1}, T = 0$ if $T \in \mathcal{B}(H_K)$ is a direct sum of operators on each $n$-cubes $f_\gamma(s)$. Therefore, $(H_K, F_K)$ is a $([\text{dim}_S(K)] + 1)$-summable even Fredholm module.

\[ \square \]

**Theorem 2.4.** The triple $(A_K, H_K, D_K)$ is an even $QC^\infty$-spectral triple of spectral dimension $\text{dim}_S(K)$.

Proof. By the definition of $D_K$ and Lemma 2.2, $(A_K, H_K, D_K)$ is an even spectral triple of spectral dimension $\text{dim}_S(K)$. $(A_K, H_K, D_K)$ is also of $QC^\infty$-class since we have $|[D_K], T = 0$ for an operator $T \in \mathcal{B}(H_K)$ of the direct sum of operators on $n$-cubes $f_\gamma(s)$.

We next prove a nonvanishing property of the $K^0$-class of the Fredholm module $(H_K, F_K)$.

**Theorem 2.5.** Denote by $X_1, \ldots, X_k$ the all connected components of $V \cup \bigcup_{s \in S} f_s(\gamma_n)$. Then, if there is $X_i$ such that

$$
\#(V_0 \cap X_i) \neq \#(V_1 \cap X_i),
$$

the Connes-Chern character $\text{Ch}_s(H_K, F_K) \in H^\text{even}_A(A_K)$ induces a non-zero additive map $K_0(C(V_K)) \cong K_0(A_K) \to \mathbb{C}$ by the Connes pairing. In particular, $[H_K, F_K] \in K^0(C(V_K))$ is not trivial.

Proof. Set

$$
d_0 = \#(V_0 \cap X_i), \quad d_1 = \#(V_1 \cap X_i)
$$

and

$$
p(x) = \begin{cases} 
1 & x \in X_i \\
0 & \text{otherwise}
\end{cases}
$$
for \( x \in V_K \). Then, \( p \) is a continuous function and we have

\[
\text{index}(pF_K^+p : p\mathcal{H}_K^+ \to p\mathcal{H}_K) = \text{index}(pU_np : p\ell^2(V_0) \to p\ell^2(V_1))
= d_0 - d_1 \neq 0.
\]

Therefore, we have \( \text{Ch}_*(\mathcal{H}_K, F_K) \neq 0 \) on \( K_0(C(V_K)) \).

**Remark 2.6.** The assumption in Theorem 2.5 does not hold for some examples such as the Sierpinski carpet (see subsection 4.3) and the \( n \)-cube \( \gamma_n \). In these cases, the Connes-Chern character induces the \( 0 \)-map on \( K_0(\mathcal{A}_K) \).

**Remark 2.7.** As remarked in Remark 1.9, we can define a Fredholm module on \( C(V) \) by using any unitary matrix \( U \) instead of \( U_n \). All properties in subsection 2.1 hold without changing proofs in such a situation.

### 2.2. Quantized differential form on self-similar sets

Note that all similitudes on \( \gamma_n \) form \( f_s(x) = r_sT_sx + b_s \) for an orthogonal matrix \( T_s \in O(n) \) and \( b_s \in \mathbb{R}^n \). It is easy to calculate the quantum differential form \([F_K, x^\alpha]\) in the case for \( \gamma_n = [0, 1]^n \) and \( T_s = E_n \) (for any \( s \in S \)), which is the direct sum of the matrix \( d_n x^\alpha \); see Proposition 1.6. We can also express \([F_K, x^\alpha]\) explicitly for the general case and show that they satisfy “a variation” of the Clifford relation.

**Proposition 2.8.** We have

\[
[F_K, x^\alpha][F_K, x^\beta] = \begin{cases} 
- [F_K, x^\beta][F_K, x^\alpha] & \alpha \neq \beta \\
- \bigoplus_{s \in S^{\infty}} \frac{e_s^2}{n} E_{2^n} & \alpha = \beta
\end{cases}.
\]

**Proof.** Take an orthogonal matrix \( T_S = [t_{ij}]_{i,j} \in O(n) \) and a vector \( b_S \in \mathbb{R}^n \) such that the image of the affine transformation \( g_S(x) = e_S T_S x + b_S \) of \( [0, 1]^n \) equals \( f_S(\gamma_n) \) and \( g_S(x) \) preserves the numbering the vertices of \( [0, 1]^n \) and \( f_S(\gamma_n) \). If we assume \( \gamma_n = [0, 1]^n \), we have \( f_S = g_S \). Note that we have

\[
[F_K, x^\alpha]|_{\mathcal{H}_S} = \frac{1}{\sqrt{n}} \left[ \Delta_n x^\alpha \circ G_n \right]^{-t} \Delta_n x^\alpha \circ G_n \right].
\]

Recall that \( v_{2j} - v_{2i-1} = \pm e_S T_S e_k \) when \( g_S^{-1}(v_{2j}) \) is connecting \( g_S^{-1}(v_{2i-1}) \) by an edge of the \( n \)-cube \( [0, 1]^n \) parallel with \( x^k \)-direction and \( T_S e_k = \sum_{\alpha=1}^n t_{\alpha k} e_\alpha \), and we have

\[
[F_K, x^\alpha]|_{\mathcal{H}_S} = \frac{e_S}{\sqrt{n}} \sum_{j=1}^n t_{\alpha j} e_\alpha^{(n)}.
\]
Thus,

\[
[F_K, x^\alpha][F_K, x^\beta]|_{\mathcal{H}_s} = \frac{e_s^2}{n} \left( \sum_{j=1}^n t_\alpha j e^{(n)}_j \right) \left( \sum_{j=1}^n t_\beta k e^{(n)}_k \right) = \frac{e_s^2}{n} \sum_{j,k} t_\alpha j t_\beta k e^{(n)}_j e^{(n)}_k
\]

\[
= \frac{e_s^2}{n} \sum_{j \neq k} t_\alpha j t_\beta k e^{(n)}_j e^{(n)}_k - \frac{e_s^2}{n} \sum_{j=1}^n t_\alpha j t_\beta j
\]

\[
= \left\{ \begin{array}{ll} 
\frac{e_s^2}{n} \sum_{j \neq k} t_\alpha j t_\beta k e^{(n)}_j e^{(n)}_k & (\alpha \neq \beta) \\
-e_s^2 E_{2n} & (\alpha = \beta) 
\end{array} \right.
\]

Therefore, we have

\[
[F_K, x^\alpha][F_K, x^\beta] = \left\{ \begin{array}{ll} 
-[F_K, x^\beta][F_K, x^\alpha] & (\alpha \neq \beta) \\
-\sum_{s \in S_{\infty}} e_s^2 E_{2n} & (\alpha = \beta) 
\end{array} \right.
\]

By Proposition 2.8, we get an explicit formula for \([[[F_K, x^1] \cdots [F_K, x^n]]\).

**Proposition 2.9.** We have

\[
[[F_K, x^1] \cdots [F_K, x^n]] = \sum_{s \in S_{\infty}} \frac{e_s^n}{n^{n/2}} E_{2n}.
\]

**Proof.** Similar to the proof of Proposition 1.10.

**Remark 2.10.** Set \(e^K_\alpha = \sum_{s \in S_{\infty}} e^{(n)}_\alpha\), and we have the following Clifford relation

\[
e^K_\alpha e^K_\beta = \left\{ \begin{array}{ll}
-e^K_\beta e^K_\alpha & (\alpha \neq \beta) \\
-id_{H_K} & (\alpha \neq \beta)
\end{array} \right.
\]

Thus we can regard \(e^K_\alpha\) as a 0-\(Q\)-form in the sense of [8].

### 3. Dixmier traces

In this section, we calculate the Dixmier trace of two operators. In general, the value for the second operator changes if the Fredholm operator \(F_n\) changes to a different Fredholm operator.
3.1. Dixmier trace of $|D_K|^{-p}$

In this subsection, we calculate the Dixmier trace of $|D_K|^{-p}$. This is given by the residue at the pole of the zeta function $\zeta_{D_K}(s) = \text{Tr}(|D_K|^{-s})$.

**Theorem 3.1.** For any $p \geq \dim_S(K)$, we have $|D_K|^{-p} \in \mathcal{L}^{(1,\infty)}(\mathcal{H}_K)$ and

$$\text{Tr}_\omega(|D_K|^{-p}) = \begin{cases} -2^n \left( \dim_S(K) \sum_{s=1}^N r_s^{\dim_S(K)} \log r_s \right)^{-1} & (p = \dim_S(K)) \\ 0 & (p > \dim_S(K)) \end{cases}.$$ 

Thus we have

$$\text{Tr}_\omega(f|D_K|^{-\dim_S(K)}) = -2^n \left( \dim_S(K) \sum_{s=1}^N r_s^{\dim_S(K)} \log r_s \right)^{-1} \int_K f|K| d\Lambda$$

for any $f \in C(V_K)$ by the Riesz-Markov-Kakutani representation theorem. Here, $\Lambda$ is the $\dim_S(K)$-dimensional Hausdorff probability measure of $K$.

In particular, if all similarity ratios $r_s$ are equal, we have

$$\text{Tr}_\omega(|D_K|^{-\dim_S(K)}) = \frac{2^n}{\log N}.$$ 

**Proof.** By the proof of Lemma 2.2, we have

$$\text{Tr}(|D_K|^{-p}) = 2^n \sum_{j=0}^\infty \left( \sum_{s=1}^N r_s^p \right)^j = 2^n \left( 1 - \sum_{s=1}^N r_s^p \right)^{-1}.$$ 

Thus we have

$$(z - 1)\text{Tr}(|D_K|^{-zp}) = 2^n \frac{z - 1}{1 - \sum_{s=1}^N r_s^zp} = 2^n \frac{z - 1}{\sum_{s=1}^N \left( r_s^{\dim_S(K)} - r_s^zp \right)} = 2^n \left( \sum_{s=1}^N \frac{r_s^{\dim_S(K)} - r_s^zp}{z - 1} \right)^{-1}.$$ 

and the following value

$$\text{Tr}_\omega(|D_K|^{-p}) = \lim_{z \to +1} (z - 1)\text{Tr}(|D_K|^{-zp}) = 2^n \left( \sum_{s=1}^N \lim_{z \to +1} \frac{r_s^{\dim_S(K)} - r_s^zp}{z - 1} \right)^{-1}.$$
converges for $p \geq \dim_S(K)$. Finally, we get
\[
\text{Tr}_\omega(|D_K|^{-p}) = -2^n \left( \sum_{s=1}^{N} \frac{d}{dz} \bigg|_{z=1} \frac{r_s^{\dim(K)}}{r_s^{\dim(K)} \log r_s} \right)^{-1}
\]
for $p = \dim_S(K)$ and
\[
\text{Tr}_\omega(|D_K|^{-p}) = 0
\]
for $p > \dim_S(K)$.

### 3.2. Dixmier trace of $[[F_K, x^1] \cdots [F_K, x^n]]^p$

In this subsection, we calculate the Dixmier trace of $[[F_K, x^1] \cdots [F_K, x^n]]^p$ by using Proposition 2.9.

**Theorem 3.2.** We have $[[F_K, x^1] [F_K, x^2] \cdots [F_K, x^n]]^p \in \mathcal{L}^{1, \infty}(\mathcal{H}_K)$ for any $p \geq \frac{1}{n} \dim_S(K)$. Moreover, we have
\[
\text{Tr}_\omega([[[F_K, x^1] [F_K, x^2] \cdots [F_K, x^n]]^p) = \frac{1}{n^{p/2}} \text{Tr}_\omega(|D_K|^{-p})
\]
for $p = \frac{1}{n} \dim_S(K)$.

Thus we have
\[
\text{Tr}_\omega(f[[F_K, x^1] [F_K, x^2] \cdots [F_K, x^n]]^{\frac{1}{n} \dim(K)})
\]
\[
= -2^n \left( \frac{d^{\dim(S(K))}}{dz} \sum_{s=1}^{N} \frac{r_s^{\dim(K)}}{r_s^{\dim(K) \log r_s}} \right)^{-1} \int_K |f| d\Lambda
\]
for any $f \in C(V_K)$ by the Riesz-Markov-Kakutani representation theorem. Here, $\Lambda$ is the $\dim_H(K)$-dimensional Hausdorff probability measure of $K$. 
Proof. By Proposition 2.9, we have
\[
[[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]]^p = \bigoplus_{S \in \mathcal{S}^n} \frac{e_S^{np}}{n^{np/2}} E_{2n}.
\]
Therefore, we get
\[
\text{Tr}( [[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]]^p ) = 2^n \sum_{j=0}^{\infty} \sum_{(s_1, \ldots, s_j) \in \mathcal{S}^j} \frac{1}{n^{np/2}} \prod_{k=1}^{j} r_{s_k}^{np}
\]
and the following condition
\[
\text{Tr}( [[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]]^p ) < \infty \iff p > \frac{1}{n} \dim_S(K).
\]
If \( p \) satisfies the above condition, the LHS can be written as
\[
\text{Tr}( [[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]]^p ) = \frac{2^n}{n^{np/2}} \left( 1 - \sum_{s=1}^{N} r_s^{np} \right)^{-1}.
\]
Therefore, the similar proof of Theorem 3.1 implies
\[
[[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]]^p \in \mathcal{L}^{(1, \infty)}(\mathcal{H}_K)
\]
for \( p \geq \frac{1}{n} \dim_S(K) \). Moreover, we get
\[
\text{Tr}_\omega([[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]]^p)
\]
\[
= \lim_{z \to +1} (z - 1) \text{Tr}( [[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]]^p )
\]
\[
= \frac{2^n}{n^{\dim_S(K)/2}} \left( \sum_{s=1}^{N} \lim_{z \to +1} \frac{r_s^{\dim_S(K)}}{z - 1} \right)^{-1}
\]
\[
= -\frac{2^n}{n^{\dim_S(K)/2}} \left( \dim_S(K) \sum_{s=1}^{N} r_s^{\dim_S(K)} \log r_s \right)^{-1}
\]
for \( p = \frac{1}{n} \dim_S(K) \) and
\[
\text{Tr}_\omega([[F_K, x^1][F_K, x^2] \cdots [F_K, x^n]]^p) = 0
\]
for \( p > \frac{1}{n} \dim_S(K) \).
4. Examples

In this section, we apply arguments in Section 2 and 3 to some examples.

4.1. Cantor dust

The Cantor dust is a generalization of the middle third Cantor set to a higher dimension. Let \( CD_n \) be the Cantor dust defined on \( \gamma_n = [0, 1]^n \) and the similitudes be

\[
f_s(x) = \frac{1}{3}x + \frac{2}{3} \sum_{\alpha=1}^{n} a_{\alpha} e_{\alpha} \quad (x \in \gamma_n, s = 0, 1, 2, \ldots, 2^n - 1).
\]

Here, we write \( a_n a_{n-1} \cdots a_2 a_1 \) as a number \( s \) in binary and \( e_{\alpha} \) is the standard basis of \( \mathbb{R}^n \). Since \( CD_n \) satisfies the open set condition, we have \( \dim_H(CD_n) = \dim_S(CD_n) = n \log_3 2 \). We also have \( V_{CD_n} = CD_n \) since \( V \subset \bigcup_{s=0}^{2^n-1} f_s(V) \). Then, we get

\[
\mathcal{A}_{CD_n} = \text{Lip}(CD_n) \text{ and } C(V_{CD_n}) = C(CD_n).
\]

![Figure 5. The first 3 steps of construction of CD2.](image)

Since all \( f_s(\gamma_n) \) are disconnected each other and also \( \#(V_0 \cap f_1(\gamma_n)) = 1 \) and \( \#(V_1 \cap f_1(\gamma_n)) = 0 \), the \( K^0 \)-class of \( (H_{CD_n}, F_{CD_n}) \) in \( K^0(C(CD_n)) \) does not vanish by Theorem 2.5.

**Theorem 4.1.** The Connes-Chern character

\[
\text{Ch}_s(H_{CD_n}, F_{CD_n}) \in H^\text{even}_1(\text{Lip}(CD_n))
\]

induces a non-zero additive map \( K_0(C(CD_n)) \to \mathbb{C} \). In particular, \([H_{CD_n}, F_{CD_n}]\) is not trivial in \( K^0(C(CD_n)) \).

Since \( \dim_S(CD_n) = n \log_3 2 \), we also get the following results.
Corollary 4.2. \( (1) \) \( (\mathcal{H}_{CD_n}, F_{CD_n}) \) is a \([n \log_3 2] + 1\)-summable even Fredholm module over \( \text{Lip}(CD_n) \).

\( (2) \) \( (\text{Lip}(CD_n), \mathcal{H}_{CD_n}, D_{CD_n}) \) is a \( QC^\infty \)-spectral triple of spectral dimension \( n \log_3 2 \).

Corollary 4.3. We have the following.

\( (1) \) \( \text{Tr}(|D_{CD_n}|^{-p}) = \frac{2^n \cdot 3^p}{3^p - 2^n} \) for any \( p > n \log_3 2 \).

\( (2) \) \( \text{Tr}_\omega(|D_{CD_n}|^{-n \log_3 2}) = \frac{2^n}{n \log 2} \).

\( (3) \) \( \text{Tr}_\omega(f|D_{CD_n}|^{-n \log_3 2}) = \frac{2^n}{n \log 2} \int_{CD_n} f \, d\Lambda \) for any \( f \in C(CD_n) \). Here, \( \Lambda \) is the \( (n \log_3 2) \)-dimensional Hausdorff probability measure of \( CD_n \).

Corollary 4.4. An operator \( |[F_{CD_n}, x^1][F_{CD_n}, x^2] \cdots [F_{CD_n}, x^n]|^{\log_3 2} \) is of \( L^{(1,\infty)} \)-class and we have

\[
\text{Tr}_\omega(|[F_{CD_n}, x^1][F_{CD_n}, x^2] \cdots [F_{CD_n}, x^n]|^{\log_3 2}) = \frac{2^n}{n(2n \log_3 2)/2 \log 2}.
\]

Thus we have

\[
\text{Tr}_\omega(f|[F_{CD_n}, x^1][F_{CD_n}, x^2] \cdots [F_{CD_n}, x^n]|^{\log_3 2}) = \frac{2^n}{n(2n \log_3 2)/2 \log 2} \int_{CD_n} f \, d\Lambda
\]

for any \( f \in C(CD_n) \). Here, \( \Lambda \) is the \( (n \log_3 2) \)-dimensional Hausdorff probability measure of \( CD_n \).

4.2. Middle third Cantor set, revisited

In this subsection, we focus on the middle third Cantor set \( CS = CD_1 \).

First, we see a relationship between our Fredholm module and Connes’ Fredholm module defined in [3, Chapter IV. 3. \( \varepsilon \)]. We recall Connes’ Fredholm module \( H \) and \( F \) on \( C(CS) \). Let \( I_{i,j} = (a_{i,j}, b_{i,j}) \) \( (i \in \mathbb{N}, j = 1, 2, \ldots, 2^i) \) be open intervals in \( [0, 1] \) which are defined as

\[
I_{1,1} = \left( \frac{1}{3}, \frac{2}{3} \right) \text{ and } I_{i+1,j} = \left( \frac{2b_{i,j-1} + a_{i,j}}{3}, \frac{b_{i,j-1} + 2a_{i,j}}{3} \right).
\]

Here, we set \( b_{i,0} = 0 \) and \( a_{i,i+1} = 1 \). The middle third Cantor set satisfies \( CS = [0, 1] \setminus \bigcup_{i,j} I_{i,j} \). Connes defined

\[
H = \bigoplus_{i,j} \ell^2(\{a_{i,j}, b_{i,j}\}) \text{ and } F = \bigoplus_{i,j} F_1.
\]

Note that \( H \oplus \ell^2(\{0, 1\}) \equiv \mathcal{H}_{CS} \) as Hilbert spaces.
Lemma 4.5. Let $a < b < c$ be real numbers. We assume

$$[\ell^2((a, b)), F_1], [\ell^2((b, c)), F_1], [\ell^2((a, c)), F_1] \in K^0(C([a, b, c]))$$

under homomorphisms $K^0(C([a, b])) \to K^0(C([a, b, c])), K^0(C([b, c])) \to K^0(C([a, b, c]))$ and $K^0(C([a, c])) \to K^0(C([a, b, c]))$ defined by inclusions $\{a, b\} \to \{a, b, c\}$, $\{b, c\} \to \{a, b, c\}$ and $\{a, c\} \to \{a, b, c\}$, respectively. Then we have

$$[\ell^2((a, b)), F_1] + [\ell^2((b, c)), F_1] = [\ell^2((a, c)), F_1] \text{ in } K^0(C([a, b, c])).$$

Proof. Set $b = b_1 = b_2$, $\{a, b\} = \{a, b_1\}$ and $\{b, c\} = \{b_2, c\}$. We have

$$[\ell^2((a, b_1)), F_1] + [\ell^2((b_2, c)), F_1] = [\ell^2((a, b_1)) \oplus \ell^2((b_2, c)), F_1 \oplus F_1]$$

$$= \begin{bmatrix} \ell^2((a, c)) \oplus \ell^2((b_1, b_2)) & E_2 \\ E_2 & E_2 \end{bmatrix}.$$ 

Here the $\mathbb{Z}_2$-grading operator of the last Fredholm module is defined by $\bar{\epsilon} = \epsilon \oplus (-\epsilon)$. Set

$$T_t = \begin{bmatrix} F_1 \cos t & \sin t \\ \sin t & -F_1 \cos t \end{bmatrix}$$

on $\ell^2((a, c)) \oplus \ell^2((b_1, b_2))$. Then we have $T_t \bar{\epsilon} + \epsilon T_t = 0$, $T_0 = F_1 \oplus (-F_1)$ and $T_{\pi/2} = \begin{bmatrix} E_2 \\ E_2 \end{bmatrix}$. Thus we have

$$[\ell^2((a, b_1)), F_1] + [\ell^2((b_2, c)), F_1] = \begin{bmatrix} \ell^2((a, c)) \oplus \ell^2((b_1, b_2)), F_1 \oplus (-F_1) \\ \ell^2((a, c)), F_1 \\ \ell^2((b_1, b_2)), F_1 \end{bmatrix}$$

$$= [\ell^2((a, c)), F_1] - [\ell^2((b_1, b_2)), F_1]$$

$$= [\ell^2((a, c)), F_1].$$

Here, the last equality is given by $b = b_1 = b_2$. \hfill \Box

By Lemma 4.5, we have

$$[H, F] + [\mathcal{H}_{CS}, F_{CS}] = [H_{CS}, F_{CS}] + [\ell^2([0, 1]), F_1].$$

Therefore we have $[H, F] = [\ell^2([0, 1]), F_1]$ in $K^0(C(CS))$. On the other hand, if set

$$p_k(x) = \begin{cases} 1 & x \in [0, 1/3^k] \cap CS \\ 0 & \text{otherwise} \end{cases}$$

for $x \in CS$, then we get $\langle [H_{CS}, F_{CS}], [p_k] \rangle = k$ and $\langle [\ell^2([0, 1]), F_1], [p_k] \rangle = 1$ by the index pairing between $K$-homology and $K$-theory. Thus a pair $([H_{CS}, F_{CS}], [H, F])$ is linearly independent on $\mathbb{Z}$ in $K^0(C(CS))$. 
Second, we set similitudes

\[ f_1(x) = \frac{1}{3}x, \quad f_2(x) = \frac{1}{3}x + \frac{2}{3}e_1 \]

for \( x \in \gamma_2 \) and denote by \( K \) the self-similar set defined by the IFS \((\gamma_2, \{f_1, f_2\})\). Then we get \( K = CS \times \{0\} \) as sets. So the Fredholm module \((H_K, F_K)\) is a novel Fredholm module of the middle third Cantor set. Note that we have \( V_K \neq K \) and \( (\bigcup_{S \in S_n} V_S) \cap K \neq 0 \) in this case.

\[ \text{Figure 6. The first 3 steps of construction of } K. \]

4.3. Sierpinski carpet and its higher dimensional analogue

The Sierpinski carpet is another generalization of the middle third Cantor set to a “2-dimensional space”. The Menger sponge is also an analogue of the Sierpinski carpet but in a “3-dimensional space”. In this subsection, we delve into such self-similar sets in \( n \)-dimensional space \((n \geq 2)\). Let \( S_n \subset \mathbb{N} \cup \{0\} \) be the index set defined by

\[ S_n = \{s \in \mathbb{N} \cup \{0\} : 0 \leq s \leq 3^n - 1 \text{ and at most one} \]

of digits equals 1 in ternary expression of \( s \} \).

For example, for \( n = 2, 3 \), we have \( S_2 = \{0, 1, 2, 3, 5, 6, 7, 8\} \) and

\[ S_3 = S_2 \cup \{9, 11, 15, 17, 18, 19, 20, 21, 23, 24, 25, 26\}. \]

Define similitudes \( f_s : \gamma_n \to \gamma_n \) for \( s \in S_n \) by

\[ f_s(x) = \frac{1}{3}x + \frac{1}{3} \sum_{\alpha=1}^{n} a_\alpha e_\alpha. \]

Here, we use a number \( s \) to express \( a_na_{n-1} \cdots a_2a_1 \) in ternary. We write \( SC_n \) as the self-similar set on the IFS \((\gamma_n, S_n, \{f_s\}_{s \in S_n})\). When \( n = 2 \) and \( 3 \), \( SC_2 \) is the Sierpinski
carpet and $SC_3$ is the Menger sponge. Since $CD_n$ satisfies the open set condition, we have $\dim_H(CD_n) = \dim_S(CD_n) = \log_3(\#S_n) = \log_3(2^{n-1}(n + 2))$. We have $V_{SC_n} = SC_n$ since $V \subset \bigcup_{s \in S_n} f_s(V)$. Then, we get

$$\mathcal{A}_{SC_n} = \text{Lip}(SC_n), \quad C(V_{SC_n}) = C(SC_n).$$

Since $X = V \cup \bigcup_{s \in S_n} f_s(\gamma_n)$ is connected, we have $\#(V_0 \cap X) = \#(V_1 \cap X)$; the assumption in Theorem 2.5 does not hold.

**Remark 4.6.** The Sierpinski carpet $SC_2$ is a compact set in $\mathbb{R}^2$. We have $K_0(C(SC_2)) = \mathbb{Z}$ which is generated by (matrix valued) constant functions on $SC_2$, and the index pairing between $K$-theory and $K$-homology induces the 0-map $K_0(C(SC_2)) \to \mathbb{Z}$. Therefore we get $[\mathcal{H}_{SC_2}, F_{SC_2}] = 0$ in $K^0(C(SC_2))$ by [6, Theorem 7.5.5].

On the other hand, we can construct a non-trivial Fredholm module corresponding to the Sierpinski carpet in a manner similar to the construction shown in subsection 4.2. Define $z : \gamma_1 \to \gamma_1$ by $z(t) = \frac{1}{3}t$ and $f_s^z = (f_s, z) : \gamma_3 \to \gamma_3$ for $s \in S_2$. Then we get a new IFS $(\gamma_3, S_2, \{f_s^z\}_{s \in S_2})$. Denote by $\widetilde{SC_2}$ the self-similar set on the new IFS, and we get $\widetilde{SC_2} = SC_2 \times \{0\}$. The corresponding Fredholm module $(\mathcal{H}_{\widetilde{SC_2}}, F_{\widetilde{SC_2}})$ represents a non-trivial element in $K^0(C(V_{\widetilde{SC_2}}))$.

**Remark 4.7.** The construction of IFS in Remark 4.6 can be extended to general cases: let $(\gamma_n, S, \{f_s\}_{s \in S})$ be an IFS and $K$ its self-similar set. Then $(\gamma_{n+1}, S, \{(f_s, z)\}_{s \in S})$ is a new IFS and the corresponding self-similar set denote by $\widetilde{K}$ satisfies $\widetilde{K} = K \times \{0\}$ and $[\mathcal{H}_{\widetilde{K}}, F_{\widetilde{K}}] \neq 0$ in $K^0(C(V_{\widetilde{K}}))$.

Since $\dim_S(SC_n) = \log_3(2^{n-1}(n + 2))$, we get the following results.

**Corollary 4.8.** (1) $(\mathcal{H}_{SC_n}, F_{SC_n})$ is a $([\log_3(2^{n-1}(n + 2))] + 1)$-summable even Fredholm module over $\text{Lip}(SC_n)$. 
Corollary 4.9. We have the following.

1. \( \text{Tr}(|D_{SC_n}|^{-p}) = \frac{2^n \cdot 3^p}{3^p - 2^{n-1}(n+2)} \) for any \( p > \log_3(2^{n-1}(n+2)) \).

2. \( \text{Tr}_\omega(|D_{SC_n}|^{-\log_3(2^{n-1}(n+2))}) = \frac{2^{2n}}{\log(2^{n-1}(n+2))} \).

3. \( \text{Tr}_\omega(f|D_{SC_n}|^{-\log_3(2^{n-1}(n+2))}) = \frac{2^n}{\log(2^{n-1}(n+2))} \int_{SC_n} f \, d\Lambda \) for any \( f \in C(SC_n) \).

Here, \( \Lambda \) is the \((\log_3(2^{n-1}(n+2)))\)-dimensional Hausdorff probability measure of \( SC_n \).

Corollary 4.10. For \( d = \frac{1}{n} \log_3(2^{n-1}(n+2)) \), we have

\[
||[F_{SC_n},x^1][F_{SC_n},x^2] \cdots [F_{SC_n},x^n]|^d \in L^{(1,\infty)}(H_{SC_n})
\]

and

\[
\text{Tr}_\omega(||[F_{SC_n},x^1][F_{SC_n},x^2] \cdots [F_{SC_n},x^n]|^d) = \frac{2^n}{n^{nd/2} \log(2^{n-1}(n+2))}.
\]

Thus we have

\[
\text{Tr}_\omega(f|[F_{CD_n},x^1][F_{CD_n},x^2] \cdots [F_{CD_n},x^n]|^d) = \frac{2^n}{n^{nd/2} \log(2^{n-1}(n+2))} \int_{SC_n} f \, d\Lambda
\]

for any \( f \in C(SC_n) \). Here, \( \Lambda \) is the \((\log_3(2^{n-1}(n+2)))\)-dimensional Hausdorff probability measure of \( SC_n \).

4.4. With rotations

Let \( R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \) be a rotation matrix. Let also \( f_1, f_2, f_3, f_4 \) be four similarities defined by

\[
f_s(x) = \frac{1}{2\sqrt{2}} R \left( x - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) + b_s.
\]

Here, we set

\[
b_1 = \frac{1}{4} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad b_2 = \frac{1}{4} \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \quad b_3 = \frac{1}{4} \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \quad b_4 = \frac{1}{4} \begin{bmatrix} 3 \\ 3 \end{bmatrix}.
\]

The IFS \((\gamma_2, \{f_1, f_2, f_3, f_4\})\) is defined by using a rotation of angle \( \theta \). We get the self-similar set \( K \) on the IFS \((\gamma_2, \{f_1, f_2, f_3, f_4\})\) that satisfies the open set condition.
Then we have $V_K \neq K$ and $(\bigcup_{S \in \{1,2,3,4\}} V_S) \cap K = \emptyset$. Since $\{(0,0)\}$ is a connected component of $V \cup \bigcup_{S \in \{1,2,3,4\}} f_S(\gamma_2)$, the Fredholm module $(\mathcal{H}_K, F_K)$ defines a non-trivial element in $K^0(C(V_K))$.

**Corollary 4.11.**

1. $(\mathcal{H}_K, F_K)$ is a 2-summable even Fredholm module over $\mathcal{A}_K$.
2. $(\mathcal{A}_K, \mathcal{H}_K, D_K)$ is a $QC^\infty$-spectral triple of spectral dimension $\frac{4}{3}$.

**Corollary 4.12.** We have the following.

1. $\text{Tr}(|D_K|^{-p}) = \frac{4}{2^{p/2} - 4}$ for any $p > \frac{4}{3}$.
2. $\text{Tr}_\omega(|D_K|^{-4/3}) = \frac{2}{\log 2}$.
3. $\text{Tr}_\omega(f|D_K|^{-4/3}) = \frac{2}{\log 2} \int_K f \text{d}\Lambda$ for any $f \in C(V_K)$. Here, $\Lambda$ is the $4/3$-dimensional Hausdorff probability measure of $K$.

By Proposition 1.5, the quantized differential forms $[F_K, x^\alpha] (\alpha = 1, 2)$ are given as

$$[F_K, x^1] = \bigoplus_{j=0}^{\infty} \bigoplus_{S \in S^\times} \frac{e_S}{\sqrt{2}} \left[ \begin{array}{cccc} 0 & 0 & \cos j\theta & -\sin j\theta \\ 0 & 0 & -\sin j\theta & -\cos j\theta \\ -\cos j\theta & \sin j\theta & 0 & 0 \\ \sin j\theta & \cos j\theta & 0 & 0 \end{array} \right],$$

$$[F_K, x^2] = \bigoplus_{j=0}^{\infty} \bigoplus_{S \in S^\times} \frac{e_S}{\sqrt{2}} \left[ \begin{array}{cccc} 0 & 0 & \sin j\theta & \cos j\theta \\ 0 & 0 & \cos j\theta & -\sin j\theta \\ -\sin j\theta & -\cos j\theta & 0 & 0 \\ -\cos j\theta & \sin j\theta & 0 & 0 \end{array} \right].$$
Thus, we have

$$[[F_K, x^1][F_K, x^2]] = \bigoplus_{s \in S^\infty} \frac{e_s^2}{2} E_4.$$ 

This implies

**Corollary 4.13.** An operator $$[[F_K, x^1][F_K, x^2]]^{2/3}$$ is of $$L^{(1, \infty)}$$-class and we have

$$\text{Tr}_\omega([[F_K, x^1][F_K, x^2]]^{2/3}) = \frac{\sqrt{2}}{\log 2}.$$ 

Thus we have

$$\text{Tr}_\omega(f[[F_K, x^1][F_K, x^2]]^{2/3}) = \frac{\sqrt{2}}{\log 2} \int_K f|_K \, d\Lambda$$

for any $$f \in C(V_K)$$. Here, $$\Lambda$$ is the $$\frac{4}{3}$$-dimensional Hausdorff probability measure of $$K$$.

**4.5. Without the open set condition**

In this subsection, we present an example of a self-similar set that does not satisfy the open set condition. In this case, we can detect the similarity dimension by using our Fredholm module but not detect the Hausdorff dimension explicitly.

Let $$(\gamma, S = \{1, 2, 3, 4, 5\}, \{f_s\}_{s \in S})$$ be an IFS defined to be

$$f_1(x) = \frac{1}{3}x, \quad f_2(x) = \frac{1}{3}x + \frac{2}{3}e_1, \quad f_3(x) = \frac{1}{3}x + \frac{2}{3}e_2,$$

$$f_4(x) = \frac{1}{3}x + \frac{2}{3}e_1 + \frac{2}{3}e_2, \quad f_5(x) = \frac{2}{3}x + \frac{1}{6}e_1 + \frac{1}{6}e_2.$$ 

Note that the IFS does not satisfy the open set condition. Let $$K$$ be the self-similar set on the IFS. Since we have $$V \subset \bigcup_{s=1}^5 f_s(V)$$, we have $$V_K = K$$. The similarity dimension $$s = \text{dim}_S(K)$$ of $$K$$ is given by the following identity

$$4 \cdot \left(\frac{1}{3}\right)^s + \left(\frac{2}{3}\right)^s = 1.$$ 

We can easily check $$1 < s < 2$$.

**Corollary 4.14.**

1. $$(\mathcal{H}_K, F_K)$$ is a 2-summable even Fredholm module over $$\text{Lip}(K)$$. 
2. $$(\text{Lip}(K), \mathcal{H}_K, D_K)$$ is a $$QC^{\infty}$$-spectral triple of spectral dimension $$s$$.

**Corollary 4.15.** We have the following.
\begin{enumerate}
\item \( \text{Tr}(|D_K|^{-p}) = \frac{4 \cdot 3^p}{3^p - 2^p - 4} \) for any \( p > s \).
\item \( \text{Tr}_\omega(|D_K|^{-\dim_S(K)}) = \frac{4 \cdot 3^s}{3^s s \log 3 - 2^s s \log 2} \)
\item \( \text{Tr}_\omega(f|D_K|^{-\dim_S(K)}) = \frac{4 \cdot 3^s}{3^s s \log 3 - 2^s s \log 2} \int_K f \, d\Lambda \) for any \( f \in C(K) \). Here, \( \Lambda \) is the \( \dim_H(K) \)-dimensional Hausdorff probability measure of \( K \).
\end{enumerate}

**Corollary 4.16.** An operator \( \|[F_K, x^1][F_K, x^2]\|^{s/2} \) is of \( L^{(1, \infty)} \)-class and we have
\[
\text{Tr}_\omega(\|[F_K, x^1][F_K, x^2]\|^d) = \frac{2^{2-s/2} \cdot 3^s}{3^s s \log 3 - 2^s s \log 2}.
\]
Thus we have
\[
\text{Tr}_\omega(f|F_K, x^1][F_K, x^2|)^d) = \frac{2^{2-s/2} \cdot 3^s}{3^s s \log 3 - 2^s s \log 2} \int_K f \, d\Lambda
\]
for any \( f \in C(K) \). Here, \( \Lambda \) is the \( \dim_H(K) \)-dimensional Hausdorff probability measure of \( K \).

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Takashi Maruyama
Takashi Maruyama, System Platform Research Laboratories, NEC Corporation, 1753 Shimonumabe, Nakahara-ku, Kawasaki, Kanagawa, Japan; 49takashi@nec.com; 49takashi@gmail.com

Tatsuki Seto
General Education and Research Center, Meiji Pharmaceutical University, 2-522-1 Noshio, Kiyose-shi, Tokyo, Japan; tatsukis@my-pharm.ac.jp