HOW TO QUANTIZE THE ANTIBRACKET

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Abstract. The uniqueness of (the class of) deformation of Poisson Lie algebra \(\mathfrak{po}(2n)\) has long been a completely accepted folklore. Actually this is wrong as stated, because its validity depends on the class of functions that generate \(\mathfrak{po}(2n)\) (e.g., it is true for polynomials but false for Laurent polynomials).

We show that, unlike \(\mathfrak{po}(2n|m)\), its quotient modulo center, the Lie superalgebra \(\mathfrak{h}(2n|m)\) of Hamiltonian vector fields with polynomial coefficients, has exceptional extra deformations for \((2n|m) = (2|2)\) and only in this superdimension. We relate this result to the complete description of deformations of the antibracket (also called the Schouten or Buttin bracket).

We show that, whereas the representation of the deform (the result of deformation aka quantization) of the Poisson algebra in the Fock space coincides with the simplest space on which the Lie algebra of commutation relations acts, this coincidence is not necessary for Lie superalgebras.

§1. Introduction

This is an edited version of the paper preprinted in Erwin Schrödinger International Institute for Mathematical Physics (875; www.esi.ac.at) and published in Theor. Math. Phys. To save space, and having in mind most general target audience, mainly interested in answers, we have omitted boring calculations (including an exposition of important but inaccessible paper [Ko2]). The omitted material whose documentation took far too long time will be published together with the details of the proof of our classification of simple vectorial Lie superalgebras: an expounding of [LS2]. As usual, when one deletes something “obvious” one should be extra careful and we are sorry to say that we did throw away several cocycles (fortunately, on isomorphic algebras). We also modify the final text by disclaiming Shmelev’s interpretation of \(\mathfrak{h}(2|2)\) which we used to trustfully rewrite from paper to paper.

We compensate the proofs omitted by extensive background: several vital, not just important, notions (for example, that of Lie superalgebra) are not as well known as is the general belief.

1.1. General setting of our problem. The problem we consider is usually breezily formulated. In order not to get confused and derive our main result, we do our best to formulate it extra carefully.

In 1977, M. Marinov asked one of us: “How to quantize this “new mechanic ([L1])” of yours? Will the Planck’s constant be odd?!?” In 1987, S. Sternberg repeated the question in connection with his studies with Kostant [KS]. For a preliminary answer, see [L3], where the importance of odd parameters and the “queer” analog of \(\mathfrak{gl}\) was indicated. Here we concentrate on other issues but again, as in [L3], tirelessly emphasize the importance of the “point functor” approach to Lie superalgebras. In particular, if we deal with their deformations one needs odd parameters. For the convenience of the reader, all necessary background is collected in §4 (Background).
The main questions, before we start counting how many quantizations of the Poisson algebra are possible, are what is quantization and what is the Poisson algebra?

There are many interpretations of the notion “quantization”. We consider quantization as a deformation, and it is vital to start with a lucid description of the class in which we deform our object, to say nothing of the lucid description of the object itself. For example, in the simplest case, when the supermanifold \( M \) is \( \mathbb{C}^{2n|m} \) (or \( \mathbb{R}^{2n|m} \)) equipped with a symplectic structure, we consider the superspace \( \mathcal{F} \) of functions on \( M \). There are two natural structures on \( \mathcal{F} \): that of an associative (and supercommutative) superalgebra and that of a Lie superalgebra, called the Poisson superalgebra and denoted by \( \mathfrak{po}(2n|m) \). Therefore we must first select one of the two problems: describe either

1) deformation of the associative superalgebra \( \mathcal{F} \) (usually, one sacrifices commutativity) or

2) deformation of the Lie superalgebra \( \mathfrak{po}(2n|m) \).

Both problems can be solved by computing a certain cohomology (Hochschild one for deformations of the associative algebra structure, Lie one for deformations of the Lie algebra structure; passage to superalgebras only brings in some extra signs). Problem (1) was considered from various angles by Flato et. al. [Bea], Neroslavsky and Vlassov [NV], De Wilde and Lecomte, Drinfeld, Fedosov [Fe, D1], Kontsevich [Kon1] to name a few.

It was always clear that Problems (1) and (2) are related; here we intend not to replace one with the other but manifestly separate them and concentrate on Problem (2). Dirac was, perhaps, the first to consider it: indeed, quantization in [Dir] is understood as follows. Let \((M, \omega)\) be a symplectic manifold (locally: domain), \(\{\cdot, \cdot\}\) the corresponding Poisson bracket. Assume that all functions (“observables”) depend on a parameter \(t\) (time). Then quantization is a passage from the classical equations of motion with Hamiltonian \(H\)

(1) \[ \dot{f} = \{f, H\} \]

to quantum ones

(2) \[ \dot{\hat{f}} = [\hat{f}, \hat{H}], \]

where \(\hat{f}\) and \(\hat{H}\) are operators (acting in a space to be specified) and \([\cdot, \cdot]\) is the commutator.

Let us express the elements of the Poisson algebra (the product in which is \(\{\cdot, \cdot\}\)) as contact fields \(K_f\) with generating function \(f\) (see (60)), so that eq. (1) becomes

(3) \[ \dot{K_f} = [K_f, K_H]. \]

In this formulation, it becomes manifest that the structure of associative and commutative algebra on the space of functions which label the classical operators \(K_f\) is beside the point; whereas quantization is, equally manifestly, a deformation of the Lie algebra structure inside the variety of Lie algebras.

For a long time Vey’s paper [V] was the only one where the deformation of Lie structure was studied (cf. [Hc]); in [L3] and here we follow this approach.

Observe a totally different from anyone’s approach to quantization due to Berezin [B], where the dimensions of the algebras deformed (in Berezin’s sense) can vary under deformation and where the convergence of the series expansion of the formal parametric family of multiplications is investigated.

Concerning Problem (1), Shereshevskii [SI] was the first, as far as we know, to show that the space of deformations of the associative structure on \(\mathcal{F}(M)\) is “not less”, in a sense, than the space of affine connections on \(M\) and is, therefore, too huge to be of interest (is undescribable). Kontsevich [Kon1] understood that this space should be considered modulo
certain gauge transformations and showed that this makes the quotient space describable (one dimensional). Superization of this result is a routine job performed in [Bo].

In pre-Kontsevich era, to diminish the number of deformations of the associative structure, people usually assumed that the following “correspondence principle” holds

$$\lim_{\hbar \to 0} \frac{f \ast_{\hbar} g - g \ast_{\hbar} f}{\hbar} = \{f, g\}.$$  

This, actually, amounts to replacement of Problem (2) by Problem (1).

**Remark.** Observe that Kontsevich even considered the Lie bracket constructed from not necessarily non-degenerate odd bivector field. The bracket thus obtained is sometimes also called Poisson bracket augmenting the already considerable confusion.

### 1.2. Cohomology depend on the type of functions.

The number of nonequivalent deformations of the Lie (super)algebra $g$ may also depend on the type of functions involved in the description of $g$ (smooth, analytic, polynomial, etc.): compare the parametric family $\text{svect}(1|n)$ of divergence free vector fields with Laurent polynomials as coefficients [GLS], with the rigid Lie superalgebra $\text{svect}(1|n)$ of divergence free vector fields with polynomial coefficients.

The uniqueness of deformation of $\mathfrak{po}(2n)$ was a folklore since long ago. When Batalin and Tyutin [BT1] actually proved the statement for Poisson Lie superalgebras $\mathfrak{po}(2n|m)$ (generated by functions of a certain class), they found it difficult to publish the result because it was dubbed as “known” (although no proof was ever published even in the purely even case, cf. [V], review [Bea] and more recent [HG], the object under study being the existence, not uniqueness). However, one should be very careful here: for arbitrary generating functions, the statement is wrong, because its validity depends on the class of functions: For example, for polynomials this is true, but false for Laurent polynomials, cf. [Dzh], see also [KT].

Another example is a multiparameter quantization of functions on the orbits of simple Lie groups in the coadjoint representation, cf. [DGS], [Kon2], [GL2].

### 1.3. What a Lie superalgebra is.

Lie superalgebras had appeared in topology in 1930’s or earlier. So when somebody offers a “better than usual” definition of a notion which seemed to have been established about 70 year ago this might look strange, to say the least. Nevertheless, the answer to the question “what is a Lie superalgebra?” is still not a common knowledge. Indeed, the naive definition (“apply the Sign Rule to the definition of the Lie algebra”) is manifestly inadequate for considering the (singular) supervarieties of deformations and applying representation theory to mathematical physics, for example, in the study of the coadjoint representation of the Lie supergroup which can act on a supermanifold but never on a superspace (an object from another category). So, to deform Lie superalgebras, apply group-theoretical methods in “super” setting, etc., we must be able to recover a supermanifold from a superspace, and vice versa.

A proper definition of Lie superalgebras is as follows, cf. [L3]. The *Lie superalgebra* in the category of supermanifolds corresponding to the “naive” Lie superalgebra $L = L_0 \oplus L_1$ is a linear supermanifold $\mathcal{L} = (L_0, \mathcal{O})$, where the sheaf of functions $\mathcal{O}$ consists of functions on $L_0$ with values in the Grassmann superalgebra on $L_1^*$. This supermanifold should be such that for “any” (say, finitely generated, or from some other appropriate category) supercommutative superalgebra $C$, the space $\mathcal{L}(C) = \text{Hom}(\text{Spec}C, \mathcal{L})$, called the space of $C$-points of $\mathcal{L}$, is a Lie algebra and the correspondence $C \longrightarrow \mathcal{L}(C)$ is a functor in $C$. (A. Weil introduced this approach in algebraic geometry in 1953; in super setting it is called *the language of points or families*, see [L].) This definition might look terribly complicated, but fortunately one can
show that the correspondence \( L \leftrightarrow L \) is one-to-one and the Lie algebra \( L(C) \), also denoted \( L(C) \), admits a very simple description: \( L(C) = (L \otimes C)_0 \).

A Lie superalgebra homomorphism \( \rho : L_1 \rightarrow L_2 \) in these terms is a functor morphism, i.e., a collection of Lie algebra homomorphisms \( \rho_C : L_1(C) \rightarrow L_2(C) \) compatible with morphisms of supercommutative superalgebras \( C \rightarrow C' \). In particular, a representation of a Lie superalgebra \( L \) in a superspace \( V \) is a homomorphism \( \rho : L \rightarrow \mathfrak{gl}(V) \), i.e., a collection of Lie algebra homomorphisms \( \rho_C : L(C) \rightarrow (\mathfrak{gl}(V) \otimes C)_0 \).

**Example.** Consider a representation \( \rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V) \). The tangent space of the moduli superspace of deformations of \( \rho \) is isomorphic to \( H^1(\mathfrak{g}; V \otimes V^*) \). For example, if \( \mathfrak{g} \) is the \( 0|n \)-dimensional (i.e., purely odd) Lie superalgebra (with the only bracket possible: identically equal to zero), its only irreducible representations are the trivial one, \( 1 \), and \( \Pi(1) \). Clearly, \( 1 \otimes 1^* \simeq \Pi(1) \otimes \Pi(1)^* \simeq 1 \), and because the superalgebra is commutative, the differential in the cochain complex is trivial. Therefore \( H^1(\mathfrak{g}; 1) = E^1(\mathfrak{g}^*) \simeq \mathfrak{g}^* \), so there are \( \dim \mathfrak{g} \) odd parameters of deformations of the trivial representation. If we consider \( \mathfrak{g} \) “naively” all of the odd parameters will be lost.

Which of these infinitesimal deformations can be extended to a global one is a separate much tougher question, usually solved ad hoc, see [F].

In this paper we deal with a similar problem: we deform the Lie superalgebra structure, i.e., the superbracket. Deformations of any superstructure can, of course, have odd parameters. Yu. Manin writes that this is obvious [Man] but he is overoptimistic: even live classics still sometimes deliberately ignore odd parameters, see, e.g., [CK]. Physicists easier accept odd (and other infinitesimal) parameters; odd parameters are the cornerstone of supersymmetry ([WZ]); in several famous papers Witten clearly illustrated the importance of odd parameters, see, for example, [W]. Witten’s papers triggered an avalanche of elaborations among which we would like to point out [CK, Man, IR, Sm].

### 1.4. Quantization, as we understand it.

Quantization of \( \mathfrak{po}(2n|m) \) consists of two steps:

1. Deformation of the Lie superalgebra \( \mathfrak{po}(2n|m) \)
2. Realization of the deform by operators in some space (the Fock space).

There is also a step somewhat aside, 0-th step:

0. Prequantization, i.e., realization of \( \mathfrak{po}(2n|m) \) by operators in some “classical version” of the Fock space.

Execution of Steps (1) and (2) seems to be routine; their superization has only two novel features: realization of \( \mathfrak{po}(2n|m) \) for \( m \) odd not by all differential operators but by a part similar to the “queer” analog of the general Lie algebra, the one which preserves a complex structure given by an odd operator. For details, see [L3]. Another novel feature is described in sec. 1.5.

For a complete description of prequantizations, see [BSS, Sm1, Ko6].

Related with the prequantization is description of representations of (anti)commutation relations (RCR). (For an approach distinct from ours, see [B1].) It turns out that the representation of the deform (after quantization) of \( \mathfrak{po}(2n|0) \) in the Fock space coincides with the simplest space on which RCR act. Observe that this coincidence is not necessary for Lie superalgebras.

Here we consider the odd analogs of the Poisson bracket, namely, the antibracket (Schouten or Buttin bracket) and the deformations of the antibracket, other than quantizations. For each of these Lie superalgebras with these brackets, i.e., for \( \mathfrak{po}(2m|n) \), \( \mathfrak{b}(n) \) and each member \( b\lambda(n) \) of the the one-parameter set of deformations of \( \mathfrak{b}(n) \), we will investigate its quantization in the above sense.
Speaking about antibracket, recall that the Buttin superalgebra $b(n)$ is the superspace of functions with reversed parity on the $n|n$-dimensional superspace endowed with the Lie superalgebra structure given by the antibracket; $b(n)$ can also be realized as the superspace of multivector fields (with reversed parity) on $p|q$-dimensional superspace for any $p, q$ such that $p + q = n$, $p, q \geq 0$ and with the Lie superalgebra structure given by the Schouten bracket.

Remark. 1) What we call the “Buttin bracket” here was discovered in the pre-super era by Schouten; Buttin first proved that this bracket establishes a Lie superalgebra structure. The interpretations of the Buttin superalgebra $b(n)$ similar to that of the Poisson algebra $po(2n|m)$ and of the elements of $le(n) = b(n)/\text{center}$ as analogs of Hamiltonian vector fields was given in [L1]. The Buttin bracket and “odd mechanics” introduced in [L1] was rediscovered by Batalin and Vilkovisky (and, even earlier, by Zinn-Justin, but his papers went mainly unnoticed, as observed in [FLSf]); it gained a great deal of currency under the name antibracket; in several papers Batalin and Vilkovisky demonstrated its importance, see reviews [GPS], [BT2].

Not every deformation qualifies to be considered as quantization. Roughly speaking, having started with a Lie (super)algebra of vector fields on a superspace of certain dimension, we should, after quantization, obtain an algebra which possesses a representation in the space of halved functional dimension. The deforms (results of the deformation) of $b(n)$ given by (10) are denoted by $b_\lambda(n)$, see Background. NONE of the deformations of $b(n)$ is quantizations in the above sense (there are no representations of halved dimension). The one we call quantization just looks similar to the only quantization of Poisson algebra.

For the odd versions of prequantizations, i.e., representations of $b_\lambda(n)$, and related with them description of the representations of $le(n)$, see [L2], [Ko6].

Step (1) was performed by Kochetkov in [Ko1]–[Ko5] (except for a case missed; we will also show that this omission is inessential) for the Buttin superalgebras $b_\lambda(n)$, and for $h(2n|m)$ except for $nm \neq 0$. Here we correct Kochetkov’s result and complete Step (2) started in [L3].

1.5. Numerous Fock spaces. There are two major types of the new Fock spaces:

1) Fix a realization (grading) of $po$ or $b$. Then $po_-$ and $b_-$ can be considered as the analog of the Heisenberg Lie superalgebra of anti-commutation relations. The relatively new message is that while for $po_-$ there is only one (up to a character and parity change) irreducible representation, there are several non-isomorphic irreducible representations for the (anti)commutation relations represented by $b_-$. Throughout the paper we insist on considering $po$ and its “odd” analog, $b$, as well as the deform of the latter, $b_\lambda$, as Lie superalgebras. These Lie superalgebras, and especially their quantum analogs, are, in a sense, analogs of $gl(V)$. The Lie algebra $gl(V)$ has many irreducible representations (realized in tensors constructed on $V$ and $V^*$, in modules with vacuum vector, etc.), and so do all the above mentioned Lie superalgebras. Contrariwise, the associative algebra $\text{Mat}(V)$ of endomorphisms of $V$ or its matrix version, $\text{Mat}(\text{dim } V)$, though isomorphic to $gl(V)$ as a vector space, has only one irreducible module, $V$ (and so do the super versions of $\text{Mat}(V)$, even the queer versions, $Q(V)$). This module $V$ is exactly what is called the Fock space. So the Fock space is the analog of the standard or identity representation for $gl(V)$. Therefore, considering representations of the Lie (super)algebra, we should take the “smallest” representation, the one which plays the role of the identity one for $gl(V)$ for the role of an analog of the Fock space, especially in the “classical” case, that of $po$ or $b$. 

2) Every “nonstandard realization” of \( \mathfrak{po} \) and \( \mathfrak{b} \) has its own Fock space; several ones in case of \( \mathfrak{b} \). Here we draw attention of the reader to the following phenomenon. Even for the “conventional” Poisson superalgebra \( \mathfrak{po}(2n|m) \), there are several analogs of the Fock space representations corresponding to several nonstandard realizations \( \mathfrak{po}(2n|m; r) \) of \( \mathfrak{po}(2n|m) \). For example, these realizations for \( m = 2k \) are given by the following gradings of the generating functions \( p, q, \xi, \eta \):

\[
\begin{align*}
\deg p_i &= \deg q_i = \deg \xi_j = \deg \eta_j = 1 \text{ for any } i \text{ and } j > r; \\
\deg \xi_j &= 0, \quad \deg \eta_j = 2 \text{ for } 0 \leq j \leq r.
\end{align*}
\]

For the complete list of nonstandard realizations — one of the main results of classification of simple vectorial Lie superalgebras, see [Sch], [LS1]. We only need some of them, see sec. A.7. Here we describe in detail only one of these realizations, the standard one (Theorem 3.2).

Observe that though for distinct nonstandard realization the analogs of \( \mathfrak{hei} \) and \( \mathfrak{ab} \) are of different dimensions, the adjoint representations of \( \mathfrak{po} \) and \( \mathfrak{b} \) are the “smallest” ones and, in contradistinction with numerous analogs of Fock spaces for representations of (anti)commutation relations, are unique.

1.6. Main results. 1) We observe that the Lie superalgebras \( \mathfrak{h}(2|2) \) of Hamiltonian vector fields can be included into a parametric family \( \mathfrak{h}_\lambda(2|2) \). Theorem 2.1 shows how deformations of the Lie superalgebra structures given by the Poisson bracket and antibracket are interrelated, namely, we show that \( \mathfrak{h}_\lambda(2|2) \) is isomorphic to the regrading \( \mathfrak{b}_\lambda(2; 2) \) of \( \mathfrak{b}_\lambda(2) \).

From here we deduce new exceptional quantizations of the Lie superalgebras \( \mathfrak{h}(2|2) \) of Hamiltonian vector fields and the Buttin superalgebra \( \mathfrak{b}(2) \).

2) As a corollary of the above we deduce that the Lie superalgebra of Hamiltonian vector fields may have more quantizations than the corresponding Poisson one. It seemed natural to expect that the Lie superalgebra \( \mathfrak{h}(2n|m) \) of Hamiltonian vector fields — the quotient of \( \mathfrak{po}(2n|m) \) modulo center — has exactly one quantization, as many as \( \mathfrak{po}(2n|m) \). These great expectations are justified almost always, except for \( (2n|m) = (2|2) \), when the parameters of deformation belong to a singular supervariety almost completely described by Kochetkov [Ko1], [Ko5]. His omission should have been obvious in view of our earlier result [ALS, Sh] but everybody overlooked it; perhaps, because it does not actually matter. Here we conclude the description of the deformations of \( \mathfrak{h}(2n|m) \) and relate them with the complete description of quantizations (deformations) of the antibracket and its quotient modulo center, \( \mathfrak{le}(n) \).

3) Unlike the quantized Poisson algebra \( \mathfrak{po}(2n|0) \) and \( \mathfrak{hei}(2n|0) \) which have exactly ONE realization by means of creation and annihilation operators in the Fock space, the quantized Lie superalgebra \( \mathfrak{ab}(n) \), the “odd” version of \( \mathfrak{hei}(2n|m) \), has \( n+1 \) distinct Fock spaces, one of which is of finite dimension. An important feature here: odd parameters of representations are a must.

Remarks. 1) Kochetkov proved [Ko2] that the subalgebra \( \mathfrak{sb}(n) \) of divergence-free multivector fields (superfields harmonic with respect to the odd Laplacian) is not rigid and described the corresponding cocycles in [Ko2], see also more accessible [L3]. The main deformation of \( \mathfrak{b}(n) \) described below preserves \( \mathfrak{sb}(n) \); this means that deformations of \( \mathfrak{sb}(n) \) are of a different nature. Still, the restriction of the quantization onto \( \mathfrak{sb}(n) \) is nontrivial; note that Kochetkov showed that there are also other deformations of \( \mathfrak{sb}(n) \).

2) Our second main result shows that the Lie superalgebra of Hamiltonian vector fields may have more quantizations than the corresponding Poisson one just once: in dimension \( (2|2) \). Contrariwise, \( \mathfrak{le}(n) \), the Lie superalgebra analogous to \( \mathfrak{h}(2n|m) \), is always more rigid.
than $b(n)$: namely, the quantization does induce a deformation of $\mathfrak{le}(n)$, but that is all: $\mathfrak{le}(n)$ has no other deformations.

3) The passage to real forms is always possible whereas exposition and study are easier over $\mathbb{C}$. So in what follows we work over $\mathbb{C}$. (Passage from $\mathbb{C}$ to $\mathbb{R}$ should be performed with caution: compare “Theorem” 9 of \cite{K} with correct results of M. Parker and Serganova \cite{S} and with \cite{LS2}.)

Note that using theorems from \cite{F} the volume of calculations can be reduced to a negligible amount in the contact case as well (sec. 3.3). These simplifications are applicable to vectorial Lie superalgebras with polynomial coefficients. The case of Laurent coefficients, especially for centrally extended algebras, is quite different technically (or at least so it looks to us at the moment); for partial results, see \cite{Ko4}.

1.7. On two confusions. 1) The tendency to mix the elements of the Lie superalgebra $\mathfrak{po}(2n|m)$ labelled by functions with the functions themselves (that generate an associative and supercommutative superalgebra with respect to the dot product) introduces a mess and hinders the study of quantization in our sense, i.e., deformation of $\mathfrak{po}(2n|m)$ as a Lie superalgebra.

To emphasize the distinction, we will denote the associative (super)algebras by Latin characters (say, $A$); the same space considered as a Lie (super)algebra with the (super)bracket $[x, y]$ instead of the dot product $xy$ will be denoted by the corresponding Gothic letter ($\mathfrak{a}$) or subscript $L$ for Lie ($A_L$).

2) The situation is further worsened by the “common knowledge” of the following “fact” (see, e.g., Remark on p. 66 in Kac’s paper \cite{K}):

(6) \textit{there exists an associative superalgebra $A$ such that $A_L \simeq \mathfrak{po}(2n|m)$.}

This statement is wrong. We will explain why and eventually give a correct formulation, but first consider an example which illuminates the problem.

In textbooks and papers (see, e.g., \cite{Pe}) the following description of $\mathfrak{po}(2n)$ can be encountered:

$\mathfrak{po}(2n)$ \textit{is generated} (presumably, as an associative algebra, whereas in fact it is a Lie algebra; for its presentation as a Lie algebra in terms of generators and relations, see \cite{LP}) by the $p_i$ and $q_i$ for $i = 1, \ldots, n$ subject to relations

(7) \[ \{p_i, q_j\} = i\hbar \delta_{ij} \]

and the bracket should satisfy the Leibniz rule:

(8) \[ \{f, gh\} = \{f, g\}h + g\{f, h\}. \]

Obviously, Eq. \cite{8} is not part of the definition of $\mathfrak{po}(2n)$ but one of its properties, a particular case of the Lie derivative along the vector field generated by $f$, see \cite{BSS}. (Eq. \cite{8} is, however, a part of a definition of a generalized Poisson structure, the one determined by a degenerate bivector, as, e.g., in \cite{Kon1}.)

Eqs. \cite{7} are identities that determine the Heisenberg Lie algebra $\mathfrak{hei}(2n)$ whose space is a $(2n + 1)$-dimensional space $W \oplus \mathbb{C}z$, where $W$, spanned by $p, q$, is endowed with the non-degenerate skew-symmetric form $B$, and $z$ lies in the center and the Lie bracket is given by \cite{7} with the right hand side multiplied by $z$.

Our nihilistic stand towards associative algebras is justified (we hope) by our results and some clarification of the general picture. But in other problems one \textit{has} to consider both structures together. In his studies of integrable systems Drinfeld even introduced the notion
of Poisson–Lie algebra (with both an associative and Lie multiplications related by Leibniz rule). Here we do not consider this notion.

1.7.1. From \( \mathfrak{hei}(2n) \) we construct \( \mathfrak{po}(2n) \) in two steps.

Step 1. We consider the associative algebra \( \text{Weyl}(2n) = U(\mathfrak{hei}(2n)) \). Because we are interested, mostly or only, in irreducible representations of \( \mathfrak{hei}(2n) \), we recall Schur’s lemma and fix the central charge, rather than consider it a parameter, i.e., identify \( z \) with \( \hbar \).

The associative algebra \( \text{diff}(n) \) of differential operators with polynomial coefficients on an \( n \)-dimensional space can be viewed as \( U(\mathfrak{hei}(2n))/(z - \hbar) \). Both \( \text{Weyl}(2n) \), and its quotient \( \text{diff}(n) \) are often called the Weyl algebra, from the context one can usually guess which of the two is meant.

Step 2. The Poisson algebra is not isomorphic to \( \text{diff}(2n) = \text{diff}(2n)_L \) but is obtained from \( \text{diff}(2n) \) by contraction, i.e., the passage to the quasi-classical limit as \( \hbar \to 0 \) after we set \( p = i\hbar \partial _\theta \) in (0.3).

Alternatively, one can define the Poisson algebra as isomorphic to \( \text{gr}(\text{diff}(2n)) \), the graded Lie algebra associated with filtration of \( \text{diff}(2n) \) induced by the natural filtration of the enveloping algebra \( U(\mathfrak{hei}(2n)) \).

1.8. Related problems. 1) Having established the uniqueness of the quantization, it is desirable to have a regular procedure for it. On the flat space, there are several ways to pass from the function (i.e., the symbol, generating an element of the Poisson algebra) to the corresponding operator; these procedures are Weyl, Wick, \( pq \), etc., quantizations. Description of quantizations on the spaces locally equivalent (in terms of \( G \)-structures) to classical domains had been started only recently, see \[ \text{DLQ}] \, \[ \text{LO}] \.

2) For presentation (i.e., description of generators and defining relations) of \( \mathfrak{po}(2n) \) (problem discussed in sec. 1.7) as of a Lie algebra, see \[ \text{LP} \]: for superizations and open problems, see \[ \text{GLP} \].

3) For \( q \)-quantization of the finite dimensional Poisson Lie superalgebras, see \[ \text{LSA} \]; one can also try to derive the \( q \)-quantum version of defining relations of the Poisson Lie superalgebras given in \[ \text{GLP} \].

§2. DEFORMATIONS OF THE BUTTIN SUPERALGEBRA AND ITS SUBALGEBRAS

For preliminaries and definitions, see Appendix: background.

2.1. The main deformation. (After \[ \text{ALS} \].) As is clear from the definition of the Buttin bracket, see \[ \text{SU} \], there is a regrading (namely, \( b(n; n) \), given by deg \( \xi_i = 0 \), deg \( q_i = 1 \) for all \( i \)) under which \( b(n) \), initially of depth 2, takes the form \( g = \bigoplus _{i \geq -1} g_i \) with \( g_0 \simeq \text{vect}(0|n) \) and \( g_{-1} \cong \Pi(\mathbb{C}[\xi]) \). Now, let us replace the \( \text{vect}(0|n) \)-module \( g_{-1} \) of functions (with inverted parity) with the module of \( \lambda \)-densities, i.e., set \( g_{-1} \cong \Pi(\text{Vol}(0|n)^{\lambda}) \), where the action of \( g_0 = \text{vect}(0|n) \) is given for any \( D \in g_0 \) and \( f \in \mathbb{C}[\xi] \) by the formulas

\[
L_D(f \text{vol}^{\lambda}_\xi) = \Big(D(f) + (-1)^{p(D)p(f)}\lambda f \text{div} D\Big) \cdot \text{vol}^{\lambda}_\xi \quad \text{and} \quad \text{p}(\text{vol}^{\lambda}_\xi) = \bar{\text{I}}.
\]

Define \( b_\lambda(n; n) \), a deform of \( b(n; n) \), as the Cartan prolong

\[
b_\lambda(n; n) := (g_{-1}, g_0)_* = (\Pi(\text{Vol}(0|n)^{\lambda}), \text{vect}(0|n))_*.
\]

These \( b_\lambda(n; n) \) for all \( \lambda \)’s constitute the main deformation. (Though main, this deformation is not the quantization of the Buttin bracket, cf. sec. 2.2.)
The deform $b_\lambda(n)$ of $b(n)$ is a regrading of $b_\lambda(n; n)$ described as follows. Let $\lambda = \frac{2a}{n(a-b)}$; set

$$b_{a,b}(n) = \{ M_f \in m(n) \mid a \\text{ div} M_f = (-1)^{p(f)}2(a-bn) \frac{\partial f}{\partial \tau}\}.$$  

(11)  

Taking into account the explicit form (18) of the divergence of $M_f$ we see that

$$b_{a,b}(n) = \{ M_f \in m(n) \mid (bn - aE) \frac{\partial f}{\partial \tau} = a \Delta f \} =$$

$$\{ D \in \text{vect}(n|n+1) \mid L_D(v\delta_q^a_0, \xi, \alpha_0^a-bn) = 0 \}.$$  

(12)  

It is subject to a direct check that $b_{a,b}(n)$ is another notation for $b_\lambda(n)$, where $\lambda = \frac{2a}{n(a-b)}$. This shows that $\lambda$ actually runs over the projective line $\mathbb{C}P^1$, not $\mathbb{C}$.

Observe the following isomorphisms:

$$b_{n+,b}(n) \cong sm(n); \quad b_{n-,a}(2; 2) \cong b_{1/2}(2; 2) \cong h(2|2), \quad \text{and } b_{-a,-b}(n) \cong b_{a,b}(n).$$  

(13)  

Moreover, $b_{1,2}(2; 2) \cong b(2|2)$, where $b(2|2)$, the deform of $h(2|2)$, is described in sec. 3.2.

The Lie superalgebra $b(n) = b_0(n)$ is not simple: it has an $\varepsilon$-dimensional, i.e., $(0|1)$-dimensional, center. At $\lambda = 1$ and $\infty$ the Lie superalgebra $b_\lambda(n)$ is not simple either: it has a simple ideal of codimension $\varepsilon^n$ and $\varepsilon^{n+1}$, respectively, cf. [LS]. The corresponding exact sequences are

$$0 \rightarrow \mathbb{C} \cdot M_1 \rightarrow b(n) \rightarrow \text{le}(n) \rightarrow 0,$$

$$0 \rightarrow b_1(n) \rightarrow \mathbb{C} \cdot M_{\xi_1 \ldots \xi_n} \rightarrow 0,$$

$$0 \rightarrow b_\infty(n) \rightarrow \mathbb{C} \cdot M_{\tau \xi_1 \ldots \xi_n} \rightarrow 0.$$  

(14)  

Clearly, at the exceptional values of $\lambda$, i.e., $0$, $1$, and $\infty$, the deformations of $b_\lambda(n)$ should be investigated extra carefully. As we will see immediately, it pays: in each of exceptional points we find extra deformations.

The Lie superalgebras $b_\lambda(n)$ are simple for $n > 1$ and $\lambda \neq 0, 1, \infty$. It is also clear that the $b_\lambda(n)$ are non-isomorphic for distinct $\lambda$’s for $n > 2$.

**Grozman’s twist of the Schouten bracket.** 1) The Schouten bracket was originally defined on the superspace of multivector fields on a manifold, i.e., on the superspace of sections of the exterior algebra (over the algebra $F$ of functions) of the tangent bundle, $\Gamma(\Lambda^* (T(M))) \cong \Lambda^*_{\mathbb{F}}(\text{ Vect}(M))$. The explicit formula of the Schouten bracket (in which the hatted slot should be ignored, as usual) is

$$[X_1 \wedge \cdots \wedge X_k, Y_1 \wedge \cdots \wedge Y_l] =$$

$$\sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k \wedge Y_1 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_l.$$  

(15)  

With the help of Sign Rule we easily superize formula (15) for the case when $M$ is replaced with a supermanifold $\mathcal{M}$. The relation of the supersersion of (15) thus obtained with (10) is as follows. Let $x$ and $\xi$ be the even and odd coordinates on $\mathcal{M}$. Setting $\theta_i = \Pi(\frac{\partial}{\partial x_i}) = \hat{x}_i$, $q_j = \Pi(\frac{\partial}{\partial \xi_j}) = \hat{\xi}_j$, we get an identification of the Schouten bracket of multivector fields on $\mathcal{M}$ with the Buttin bracket of functions on the supermanifold $\hat{\mathcal{M}}$ whose coordinates are $x, \xi$ and $\hat{x}, \hat{\xi}$; any transformation of $x, \xi$ induces that of the checked coordinates.

2) In [G], Grozman classified all bilinear invariant differential operators acting in the spaces of sections of tensor fields on any manifold. In this remarkable paper, he also introduced
a one-parameter deformation of the Schouten bracket related with the one we call “main deformation”. Namely, he introduced the operator
\[ X \text{vol}^\mu, Y \text{vol}^\nu \mapsto ((\nu - 1)(\mu + \nu - 1)\text{div}X \cdot Y +
(\nu - 1)^p(X)(\mu - 1)(\mu + \nu - 1)X\text{div}Y -
(\mu - 1)(\nu - 1)\text{div}(XY) \text{vol}^{\mu + \nu}, \]
where the divergence of a polyvector field is best described in local coordinates \((x, \hat{x})\) on the supermanifold \(\hat{M}\) associated with any supermanifold \(M\), see formula (80).

Grozman’s Lie superalgebra on twisted polyvector fields on \(M\) given by formula (22) can be realized as a subalgebra of the Lie superalgebra of divergence-free polyvector fields \(sb^{n+1}\) of on \(M \times \mathbb{R}_+\), or the Lie subsuperalgebra of functions (with respect to the Buttin bracket) on the associated supermanifold with checked coordinates. The exact formula:
\[ X \text{vol}^\lambda \mapsto t^{-\lambda}X + \frac{1}{\lambda - 1}t^{-\lambda+1}\frac{\partial}{\partial t} \wedge \text{div}(X); \]
in terms of sec. A.4 the right hand side of (17) is \(t^{-\lambda}f(x, \xi) + \frac{1}{\lambda - 1}t^{-\lambda+1}\frac{\partial}{\partial t} \Delta(f)\).

The case \(n = 2\). Let \(\tilde{n}\) denote the grading
\[ \deg q_i = 0, \deg \xi_i = 0 \text{ for } i = 1, \ldots, n. \]
Then we have an analog of representation (10):
\[ \mathfrak{g}_i = \left( \Pi(\Lambda^{i+1}(\text{vect}(n|0))) \right) \otimes \text{Vol}^{-i\lambda} \text{ for } i = -1, 0, \ldots, n - 1. \]
Since
\[ \Lambda^n(\text{vect}(n|0)) \simeq \text{Vol}^{-1}(n|0), \]
we see that if \(\lambda = -\frac{1}{2}\), then
\[ \mathfrak{g}_{-1} \simeq \mathfrak{g}_1 \simeq \text{Vol}^{-1/2}. \]
Generally,
\[ \mathfrak{b}_\lambda(2; 2) \simeq \mathfrak{b}_{-1-\lambda}(2; 2) \text{ or, which is the same, } \mathfrak{h}_\lambda(2|2; 2) \simeq \mathfrak{h}_{-1-\lambda}(2|2; 2). \]
In particular, we have an additional outer automorphism \(T_\pm : \mathfrak{g}_{-1} \leftrightarrow \mathfrak{g}_1\) of \(\mathfrak{g} = \mathfrak{b}_{-1/2}(2; 2)\).

Now recall (see Background) that the natural symmetric paring
\[ (f\sqrt{\text{vol}}, g\sqrt{\text{vol}}) = \int fg\text{vol} \]
(well defined on functions with compact support and formally extended to formal semi-densities) becomes skew symmetric on the purely odd space, and hence determines a central extension. This is the well-known extension that determines the Poisson superalgebra; this cocycle is of degree \(-2\).

Now observe that (compare with (19))
\[ \Lambda^n(\text{vect}(0|n)) \simeq \text{Vol}(0|n) \]
which leads to the isomorphism
\[ \mathfrak{b}_\lambda(2; 2) \simeq \mathfrak{b}_{1-\lambda}(2; 2), \text{ or, which is the same, } \mathfrak{h}_\lambda(2|2) \simeq \mathfrak{h}_{1-\lambda}(2|2). \]
In particular, (this is the trifle Kochetkov, and all of us, missed):
\[ \mathfrak{b}_{1/2}(2; 2) \simeq \mathfrak{b}_{-3/2}(2; 2). \]
2.2. Quantizations: retelling [Ko1], [Ko2], [L3] and more. The deformation $b_\lambda(n)$ of $b(n)$ that connects $b(n)$ with $\mathfrak{sm}(n)$ will be referred to as the main one. The other deformations, called singular ones, are no less interesting. Of particular interest are the ones corresponding to $\lambda = 0$ and (for $n = 2$) to $\lambda = 1/2$ and $\lambda = -3/2$: they are quantizations.

For $\mathfrak{g} = b_\lambda(n)$, set $H = H^2(\mathfrak{g}; \mathfrak{g})$. (Recall (see [F]) that the superspace $H$ is usually identified with the tangent space to the singular supervariety of parameters of deformations of $\mathfrak{g}$ at the point corresponding to $\mathfrak{g}$.)

**Theorem.** 1) $\dim H = (1|0)$ for $\mathfrak{g} = b_\lambda(n)$ unless $\lambda = 0, -1, 1, \infty$ for $n > 2$. For $n = 2$, in addition to the above $\dim H \neq (1|0)$ at $\lambda = 1/2$.

2) At exceptional values of $\lambda$ listed in 1) we have

- $\dim H = (2|0)$ at $\lambda = \pm 1$ and $\lambda$ odd, or $\lambda = \infty$ and $\lambda$ even, or $\lambda = 1/2$ (or $\lambda = -3/2$) and $n = 2$.
- $\dim H = (1|1)$ at $\lambda = 0$, or $\lambda = \infty$ and $\lambda$ odd, or $\lambda = \pm 1$ and $\lambda$ even.

The corresponding cocycles $C$ are given by the following nonzero values in terms of the generating functions $f$ and $g$, where $d_1(f)$ is the degree of $f$ with respect to odd indeterminates only (here $k = (k_1, \ldots, k_n)$; we set $q^k = q_1^{k_1} \cdots q_n^{k_n}$ and $|k| = \sum k_i$): (21)

| $b_\lambda(n)$ | $p(C)$ | $C(f, g)$ |
|----------------|--------|-----------|
| $b_0(n)$ | odd | $(-1)^{p(f)}(d_1(f) - 1)(d_1(g) - 1)f g$ |
| $b_{-1}(n)$ | $n + 1 \pmod{2}$ | $f = q^k, \quad g = q^l \mapsto (4 - |k| - |l|)|q^{k+l}|\xi_1 \cdots \xi_n + \tau\Delta(q^{k+l}|\xi_1 \cdots \xi_n)$ |
| $b_{1}(n)$ | $n + 1 \pmod{2}$ | $f = \xi_1 \cdots \xi_n, \quad g \mapsto \begin{cases} (d_1(g) - 1)g & \text{if } g \neq af, a \in \mathbb{C} \\ (2n - 1)f & \text{if } g = f \text{ and } n \text{ is even} \end{cases}$ |
| $b_{\infty}(n)$ | $n \pmod{2}$ | $f = \tau\xi_1 \cdots \xi_n, \quad g \mapsto \begin{cases} (d_1(g) - 1)g & \text{if } g \neq af, a \in \mathbb{C} \\ 2f & \text{if } g = f \text{ and } n \text{ is odd} \end{cases}$ |

On $b_2(2) \simeq \mathfrak{h}(2|2)$ (the latter being a regrading of $\mathfrak{h}(2|2)$, see sec. A.7) the cocycle is the one induced on $b_2(2|2) = \mathfrak{po}(2|2)/\text{center}$ by the usual quantization of $\mathfrak{po}(2|2)$: we first quantize $\mathfrak{po}$ and then take the quotient modulo center (generated by constants).

3) The space $H$ is diagonalizable with respect to the Cartan subalgebra of $\mathfrak{det} \mathfrak{g}$; the cocycle $M$ corresponding to the main deformation is one of the eigenvectors. Let $C$ be another eigenvector in $H$, it determines a singular deformation. The only cocycles $kM + lC$ that can be extended to a global deformation are those for $kl = 0$, i.e., either $M$ or $C$.

All the singular deformations of the bracket $\{\cdot, \cdot\}_{\text{odd}}$ in $b_\lambda(n)$ (except the one for $\lambda = 1/2$ and $n = 2$) have a very simple form even for the even $\hbar$:

$$\{f, g\}^{\text{sing}}_{\hbar} = \{f, g\}_{\text{odd}} + \hbar \cdot C(f, g)$$ for any $f, g \in b_\lambda(n)$.

**Remark.** C. Roger observed that the singular deformation (quantization) of $b_0(n) = b(n)$ is, up to sign, the wedge product of two 1-cocycles, the derivations $f \mapsto (d_1(f) - 1)f$. He also advises to note that the cocycle on $b_2(2) \simeq \mathfrak{h}(2|2)$ induced by the quantization of $\mathfrak{po}(2|2)$ is a straightforward superization of the well-known Vey’s cocycle [GS].

Since the elements of $b_\lambda(n)$ are encoded by functions (for us: polynomials) in $\tau$, $q$ and $\xi$ subject to one relation with an odd left hand side in which $\tau$ enters, it seems plausible that the bracket in $b_\lambda(n)$ can be, at least for generic values of parameter $\lambda$, expressed solely in terms of $q$ and $\xi$. Indeed, here is the explicit formula (in which $\{f, g\}_{B,B}$ is the usual
antibracket):  

\[(23) \quad \{f_1, f_2\}_\lambda^{\text{main}} = \{f_1, f_2\}_{B.B.} + \lambda (c_\lambda(f_1, f_2) f_1 \Delta f_2 + (-1)^{p(f_1)} c_\lambda(f_2, f_1) (\Delta f_1) f_2),\]

where

\[(24) \quad c_\lambda(f_1, f_2) = \frac{\deg f_1 - 2}{2 + \lambda (\deg f_2 - n)}\]

and \(\deg\) is computed with respect to the standard grading \(\deg q_i = \deg \xi_i = 1\).

### 3. The main deformation of \(\mathfrak{h}(2|2)\)

Comparison of the non-positive terms of the \(\mathbb{Z}\)-gradings shows that \(\mathfrak{b}_\lambda(2; 2) \cong \mathfrak{h}_\lambda(2|2)\).

In Eq. (12) we have interpreted \(\mathfrak{b}_\lambda(n)\) as preserving a complicated tensor \(\omega_{a, c, e}^{\lambda - bn}\).

#### 3.1. Theorem

Set \(h(\lambda) = \frac{2\lambda - 1}{\lambda}\). Then \(D \in \mathfrak{vect}(2|2)\) belongs to \(\mathfrak{b}_\lambda(2; 2)\) if and only if

\[(25) \quad D = D_f = H_f + h(\lambda) W_f, \quad \text{where} \quad W_f = \left( \int_0^p \frac{\partial f}{\partial \eta} \frac{\partial dp}{\partial \xi} + (-1)^{p(f)} \frac{\partial f}{\partial \xi} \frac{\partial dp}{\partial \eta} \right) \partial_p + (-1)^{p(f)} \frac{\partial f}{\partial \xi} \partial_\eta \]

for some \(f \in \mathbb{C}[p, q, \xi, \eta]\). Then, for \(f, g \in \mathbb{C}[p, q, \xi, \eta]\), we have

\[(26) \quad [D_f, D_g] = D_{(f, g)_{B.B.}} + h(\lambda) D_{c(f, g)},\]

where

\[(27) \quad \begin{aligned}
c(f, g) &= -\frac{\partial f}{\partial p} \int_0^p \frac{\partial g}{\partial \eta} \frac{\partial dp}{\partial \xi} + \frac{\partial g}{\partial p} \int_0^p \frac{\partial f}{\partial \xi} \frac{\partial dp}{\partial \eta} \\
\frac{\partial}{\partial \eta} \left( \int_{(0, q)}^{(p, q)} \left( (-1)^{p(f)} \frac{\partial f}{\partial \eta} \frac{\partial dp}{\partial \xi} - \frac{\partial f}{\partial \xi} \frac{\partial dp}{\partial \eta} \right) p \right) &+ \\
\int_{(0, 0)}^{(0, q)} \left( (-1)^{p(f)} \frac{\partial f}{\partial \eta} \frac{\partial dp}{\partial \xi} + \frac{\partial f}{\partial \xi} \frac{\partial dp}{\partial \eta} \right) p &+ \frac{\partial f}{\partial \eta} \frac{\partial dp}{\partial \xi} |_{p=0, q=0}.
\end{aligned}\]

Observe that the formula

\[(28) \quad [H_f, H_g]_{\text{new}} = H_{(f, g)_{B.B.}} + h(\lambda) \cdot H_{c(f, g)}\]

determines a deformation of \(\mathfrak{h}(2|2)\) (which is the main deformation of \(\mathfrak{b}_{1/2}(2)\)) but (and this agrees with \([BT]\)) the formula

\[(29) \quad \{f, g\}_{\text{new}} = \{f, g\}_{B.B.} + h(\lambda) \cdot c(f, g)\]

does not determine a deformation of \(\mathfrak{po}(2|2)\) because \((29)\) does not satisfy the Jacobi identity.

#### 3.2. Deformations of \(\mathfrak{g} = \mathfrak{b}_{1/2}(n; n)\)

Clearly, \(\mathfrak{g}_{-1}\) is isomorphic to \(\Pi(\sqrt{V}0\ell)\). Therefore there is an embedding

\[(30) \quad \mathfrak{b}_{1/2}(n; n) \subset \begin{cases} 
\mathfrak{h}(2^{n-1}|2^{n-1}) & \text{for } n \text{ even} \\
\mathfrak{le}(2^{n-1}) & \text{for } n \text{ odd}.
\end{cases}\]

It is tempting to determine quantizations of \(\mathfrak{g}\) in addition to those considered by Kochetkov, as the composition of embedding \([33]\) and the subsequent quantization.

For \(n = 2\), when \([33]\) is not just an embedding but an isomorphism, this certainly works and we get the following extra quantization of the antibracket described in Theorem 1.2: we first deform the antibracket to the point \(\lambda = \frac{1}{2}\) along the main deformation, and then quantize it as the quotient of the Poisson superalgebra. This scheme fails to give new algebras for \(n = 2k > 2\):
Theorem. For \( n = 2k > 2 \), the image of \( \mathfrak{b}_{1/2}(n; n) \) under embedding (30) is rigid under the quantization of the ambient.

Proof: direct verification.

3.3. General algebras are rigid. The rigidity of contact and pericontact series was earlier established by painstaking calculations due to Shmelev [Sm] for the series \( \mathfrak{f} \) and Kochetkov [Ko2] for the series \( \mathfrak{m} \). It is, however, an example of the general statements on cohomology of coinduced modules (see [F]) and immediately follows from the later computations of cohomologies of \( \mathfrak{gl}(m|n) \), \( \mathfrak{osp}(m|2n) \), and \( \mathfrak{pe}(n) \), see [FL2] and [F], and the following observation: as modules over themselves, the algebras \( \mathfrak{g} = \mathfrak{vect} \), \( \mathfrak{f} \) and \( \mathfrak{m} \) are expressed as modules of generalized tensor fields (see sec. A.6) as follows:

\[
\text{vect}(m|n) = T(\text{id}_{\mathfrak{g}_0(m|n)}); \quad \mathfrak{f}(2m + 1|n) = T(\mathbb{C}[-2]_{\mathfrak{g}_0}); \quad \mathfrak{m}(n) = \Pi(T(\mathbb{C}[-2]_{\mathfrak{g}_0})),
\]

where \( \mathbb{C}[k] \) is the representation of \( \mathfrak{g}_0 \) (in the standard grading of \( \mathfrak{g} \)) trivial on the simple part and such that the center \( z \) of \( \mathfrak{g}_0 \) acts as multiplication by \( k \in \mathbb{C} \), where the central element \( z \) is selected to act on \( \mathfrak{g}_0 \), as multiplication by \( i \in \mathbb{Z} \).

Thus, the adjoint modules are coinduced, and therefore we have:

**Theorem.** \( H^2(\mathfrak{g}; \mathfrak{g}) \simeq H^2(\mathfrak{g}_0; \text{id}_{\mathfrak{g}_0}) = 0 \) for \( \mathfrak{g} = \text{vect}(m|n) \), and \( H^2(\mathfrak{g}; \mathfrak{g}) \simeq H^2(\mathfrak{g}_0; \mathbb{C}[-2]_{\mathfrak{g}_0}) = 0 \) for \( \mathfrak{f}(2m + 1|n) \) and \( \mathfrak{m}(n) \).

### §4. Representations of \( \mathfrak{hei}(2n|m) \) and \( \mathfrak{ab}(n) \)

We begin with observation that only for the standard realization (see sec. A.7) the relation between the commutation/anticommutation relations (represented by the elements of negative degree from the Poisson or Buttin Lie superalgebra) and the Poisson or Buttin Lie superalgebra itself is the same as for Lie algebra \( \mathfrak{po}(2n) \). (To see the difference most graphically, consider the finite dimensional case, say, \( \mathfrak{po}(0|2n) \) with the grading \( \deg \xi_i = 0 \), \( \deg \eta_i = 1 \) for all \( i \).)

4.1. Lemma. (Sg1) 1) Let \( V \) be a vector superspace, and \( P_V(C) = (V \otimes C)_0 \) be the set of its \( C \)-points. Then \( V \simeq W \) if and only if \( P_V(C) \simeq P_W(C) \) for all supercommutative superalgebras \( C \).

2) Let \( \mathfrak{g} \) and \( \mathfrak{h} \) be Lie superalgebras. Then \( \mathfrak{g} \simeq \mathfrak{h} \) if and only if \( P_{\mathfrak{g}}(C) \simeq P_{\mathfrak{h}}(C) \) as Lie algebras for all supercommutative superalgebras \( C \).

3) Let \( V \) and \( W \) be two modules over \( \mathfrak{g} \). The modules are isomorphic if and only if \( P_V(C) \simeq P_W(C) \) as modules over \( P_{\mathfrak{g}}(C) \) for all supercommutative superalgebras \( C \).

iv) It suffices to verify the above conditions for \( C = \Lambda(N) \) with \( N \) “sufficiently large”.

Remark. One should not replace \( N \) with \( \infty \), as Berezin did: though we only have to verify one condition instead of infinitely many ones, we acquire infinite topological difficulties, cf. [D2].

4.2. Irreducible representations of \( \mathfrak{hei}(2n|m) \) and its analogs. The following statement and its analog, heading 1) of Theorem 4.3, are particular case of a result of Sergeev [Sg2] (that corrects “Theorem” 7 of [K]):

**Theorem.** Let us represent the superspace of \( \mathfrak{hei}(2n|m) \) as \( W \oplus \mathbb{C}z \), where \( W \) is endowed with the form \( B \), see §2 and §3 and represent \( W \) as \( V \oplus V^* \) if \( m = 2k \) or \( W = V \oplus V^* \oplus U \) if \( m = 2k + 1 \), where \( V \) and \( V^* \) are isotropic with respect to the form \( B \) and each of dimension \( n|k \). Then over \( \mathbb{C} \), the only irreducible representations of \( \mathfrak{hei}(2n|m) \) are isomorphic to the following Fock superspaces: \( \mathcal{F}_h \simeq \mathbb{C}[V] \) if \( m = 2k \) and this is a \( G \)-type representation; or
Let $W$. Theorem. 1) irreducible representations: the $q$-ab $\text{ab}(n) = W \oplus \mathbb{C}z$. Let $W$ be spanned by the even elements $q_1, \ldots, q_n$ and odd elements $\theta_1, \ldots, \theta_n$.

**Theorem.** 1) Over any commutative algebra with the zero odd part, $\text{ab}(n)$ has only two irreducible representations: the 1|0-dimensional trivial module $1$ and $\Pi(1)$.

2) Let $C$ be a supercommutative superalgebra with $C_1 \neq 0$ and $\xi \in C_1$. There are $n + 1$ distinct irreducible $\text{ab}(n; C)$-modules $F_i$, $0 \leq i \leq n$, corresponding to odd parameters describing the tangent space to the trivial representation $1$. Namely, for a nonzero vector $v$, set $zv = \xi v$ and

\[ q_i v = \cdots = q_n v = \theta_{i+1} v = \cdots = \theta_n v = 0 \text{ for } i = 0, 1, \ldots, n \]

and define

\[ F_i = \text{ind}_{\text{Span}(q_1, \ldots, q_i, \theta_{i+1}, \ldots, \theta_n; z)}(C^2 v) \simeq C[\lambda_1, \ldots, \lambda_n; x_{i+1}, \ldots, x_n]. \]

The explicit realization of the operators is:

\[ q_i \mapsto \xi \frac{\partial}{\partial x_i}, \ldots, q_n \mapsto \xi \frac{\partial}{\partial x_n}, q_{i+1} \mapsto x_{i+1}, \ldots, q_n \mapsto x_n, \theta_1 \mapsto \lambda_1, \ldots, \theta_i \mapsto \lambda_i; \theta_{i+1} \mapsto -\xi \frac{\partial}{\partial x_{i+1}}, \ldots, \theta_n \mapsto -\xi \frac{\partial}{\partial x_n}. \]

Thus, as superspaces,

\[ \text{po}(2n|2m) \simeq C[q, \frac{\partial}{\partial q}, \xi, \frac{\partial}{\partial \xi}], \text{dim } \text{U}(\text{hei}(2n|2m))/(z - h); \]

\[ \text{Pi}(\text{b}(n) \otimes C[\xi]) \simeq (\text{U}(\text{ab}(n)) \otimes C[\xi])/(z - \xi), \]

where for the antibracket we have to consider everything over $C$ to account for the odd parameters.

---

§5. Appendix: Background

**A.1. Linear algebra in superspaces. Generalities.** A superspace is a $\mathbb{Z}/2$-graded space; for any superspace $V = V_0 \oplus V_1$, denote by $\Pi(V)$ another copy of the same superspace: with the shifted parity, i.e., $(\Pi(V))_i = V_{i+1}$. The superdimension of $V$ is $\text{dim } V = p + q \epsilon$, where $\epsilon^2 = 1$ and $p = \text{dim } V_0$, $q = \text{dim } V_1$. Usually, $\text{dim } V$ is expressed as a pair $(p, q)$; in this notation the useful formula $\text{dim } V \otimes W = \text{dim } V \cdot \text{dim } W$ looks mysterious whereas with $\epsilon$ this is clear.

A superspace structure in $V$ induces the superspace structure in the space $\text{End}(V)$. A basis of a superspace is always a basis consisting of homogeneous vectors. Let $p_i$ denote the parity of $i$th basis vector, then the matrix unit $E_{ij}$ is supposed to be of parity $p_i + p_j$ and the bracket of supermatrices is defined via Sign Rule:

- if something of parity $p$ moves past something of parity $q$ the sign $(-1)^{pq}$ accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity.

More examples of application of Sign Rule: setting $[X, Y] = XY - (-1)^{p(X)p(Y)}YX$ we get the notion of the supercommutator and the ensuing notions of the supercommutative superalgebra and the Lie superalgebra (that in addition to superskew-commutativity satisfies the super Jacobi identity, i.e., the Jacobi identity amended with the Sign Rule). The superderivation of a superalgebra $A$ is a linear map $D : A \rightarrow A$ that satisfies the super Leibniz rule

\[ D(ab) = D(a)b + (-1)^{p(D)p(a)}aD(b), \]
In particular, let \( A = \mathbb{C}[x] \) be the free supercommutative polynomial superalgebra in \( x = (x_1, \ldots, x_n) \), where the superstructure is determined by the parities of the indeterminates: \( p(x_i) = p_i \). Partial derivatives are defined (with the help of super Leibniz Rule) by the formulas \( \frac{\partial x}{\partial x_j} = \delta_{i,j} \). Clearly, the collection \( \text{Der} A \) of all superderivations of \( A \) is a Lie superalgebra whose elements are of the form \( \sum f_i(x) \frac{\partial}{\partial x_i} \).

We consider the exterior differential as a superderivation of the superalgebra of exterior differential forms, so \( dx \) is even for any odd \( x \) and we can consider not only polynomials in \( dx \). Smooth or analytic functions in \( dx \) are called \emph{pseudodifferential forms} on the supermanifold with coordinates \( x \), see \[\text{BL}\]. We needed them to interpret \( \mathfrak{h}_\lambda(2|2) \).

**A.1.1. General linear superalgebras: two types.** The general linear Lie superalgebra of all supermatrices of given format \( \text{Par} \) (an ordered collection of parities of basis vectors, or just a superdimension) is denoted by \( \mathfrak{gl}(\text{Par}) \). Any matrix from \( \mathfrak{gl}(\text{Par}) \) can be expressed as the sum of its even and odd parts; in the standard (simplest) format this is the following block expression:

\[
(A \ B) = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} + \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}, \quad p \left( \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \right) = 0, \quad p \left( \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \right) = 0.
\]

The \emph{supertrace} is the map \( \mathfrak{gl}(\text{Par}) \to \mathbb{C}, (A_{ij}) \mapsto \sum(-1)^{p_i}A_{ii} \). Since \( \text{str}[x, y] = 0 \), the subsuperspace of supertraceless matrices constitutes the \emph{special linear} Lie subsuperalgebra \( \mathfrak{sl}(\text{Par}) \).

Another super versions of \( \mathfrak{gl}(n) \) is called the \emph{queer} Lie superalgebra and is defined as the Lie superalgebra that preserves the complex structure given by an \emph{odd} operator \( J \), i.e., is the centralizer \( C(J) \) of \( J \):

\[
q(n) = C(J) = \{ X \in \mathfrak{gl}(n|n) \mid [X, J] = 0 \}, \quad \text{where} \quad J^2 = -\text{id}.
\]

It is clear (over \( \mathbb{C} \)) that by a change of basis we can reduce \( J \) to the form \( J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1 & 0 \end{pmatrix} \).

In the standard format we have

\[
q(n) = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \right\}.
\]

On \( q(n) \), the \emph{queertrace} is defined: \( \text{qtr} : \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mapsto \text{tr}B \). Denote by \( \mathfrak{sq}(n) \) the Lie superalgebra of \emph{queertraceless} matrices.

Observe that the identity representations of \( q \) and \( \mathfrak{sq} \) in \( V \), though irreducible in the non-graded sense, are not irreducible in the non-graded sense: take homogeneous (with respect to parity) and linearly independent vectors \( v_1, \ldots, v_n \) from \( V \); then \( \text{Span}(v_1 + J(v_1), \ldots, v_n + J(v_n)) \) is an invariant subspace of \( V \) which is not a subsuperspace. On such inhomogeneous irreducible representations, see \[\text{Lan}\].

A representation is \emph{irreducible of general type} or just of \emph{G-type} if there is no nontrivial invariant subspace. An irreducible representation is called \emph{irreducible of Q-type} (\( Q \) is after the general queer Lie superalgebra); if it has no invariant subsuperspace but \emph{has} a nontrivial invariant subspace.

**A.1.2. Lie superalgebras that preserve bilinear forms: two types.** Given a linear map \( F \) of superspaces, there exists a corresponding dual map \( F^* \) between the dual superspaces; if \( A \) is the supermatrix corresponding to \( F \) in a basis of format \( \text{Par} \), then the \emph{supertransposed} matrix \( A^{st} \) corresponds to \( F^* \):

\[
(A^{st})_{ij} = (-1)^{(p_i + p_j)(p_i + p(A))} A_{ji}.
\]
The supermatrices $X \in \mathfrak{gl}(\text{Par})$ such that
\begin{equation}
X^t B + (-1)^{p(X)p(B)} BX = 0 \quad \text{for an homogeneous matrix } B \in \mathfrak{gl}(\text{Par})
\end{equation}
constitute the Lie superalgebra $\text{aut}(B)$ that preserves the bilinear form on $V$ with matrix $B$.

Recall that the supersymmetry of the homogeneous form $\omega$ means that its matrix $B$ satisfies the condition $B^u = B$, where
\begin{equation}
B^u = \begin{pmatrix}
R^t & (-1)^{p(B)}T^t \\
(-1)^{p(B)}S^t & -U^t
\end{pmatrix}
\end{equation}
for the matrix $B = \begin{pmatrix} R & S \\ T & U \end{pmatrix}$.

Similarly, skew-supersymmetry of $B$ means that $B^u = -B$. Thus, we see that the upsetting of bilinear forms $u : \text{Bil}(V, W) \to \text{Bil}(W, V)$, which, for the spaces $V = W$, is expressed on matrices in terms of the transposition, becomes a new operation on supermatrices.

The most popular canonical forms of the nondegenerate supersymmetric form are the ones whose supermatrices in the standard format are the following canonical ones, $B_{ev}$ or $B'_{ev}$:
\begin{equation}
B_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},
\end{equation}
or
\begin{equation}
B'_{ev}(m|2n) = \begin{pmatrix} \text{antidiag}(1, \ldots, 1) & 0 \\ 0 & J_{2n} \end{pmatrix}.
\end{equation}

The usual notation for $\text{aut}(B_{ev}(m|2n))$ is $\mathfrak{osp}(m|2n)$ or, more precisely, $\mathfrak{osp}^{sy}(m|2n)$. Observe that the passage from $V$ to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric forms, preserved by the “symplectico-orthogonal” Lie superalgebra, $\mathfrak{sp}'o(2n|m)$ or, better say, $\mathfrak{osp}^{sk}(m|2n)$, which is isomorphic to $\mathfrak{osp}^{sy}(m|2n)$ but has a different matrix realization. We never use notation $\mathfrak{spo}(2n|m)$ in order to prevent confusion with the special Poisson superalgebra.

In the standard format the matrix realizations of these algebras are:
\begin{equation}
\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & X^t \\ X & A & B \\ -Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^{sk}(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y & -X^t & E \end{pmatrix} \right\},
\end{equation}
where $\begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n), \quad E \in \mathfrak{o}(m)$ and $^t$ is the usual transposition.

A non-degenerate supersymmetric odd bilinear form $B_{odd}(n|n)$ can be reduced to a canonical form whose matrix in the standard format is $J_{2n}$. A canonical form of the superskew odd non-degenerate form in the standard format is $\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. Observe that we did not make a mistake here: with a minus for the symmetric form and with a plus for the skew form!

The usual notation for $\text{aut}(B_{odd}(\text{Par}))$ is $\mathfrak{pe}(\text{Par})$. The passage from $V$ to $\Pi(V)$ establishes an isomorphism $\mathfrak{pe}^{sy}(\text{Par}) \cong \mathfrak{pe}^{sk}(\text{Par})$. This Lie superalgebra is called, as A. Weil suggested, periplectic. The matrix realizations in the standard format of these superalgebras are:
\begin{equation}
\mathfrak{pe}^{sy}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \quad \text{where } B = -B^t, \quad C = C^t \right\}; \quad \mathfrak{pe}^{sk}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \quad \text{where } B = B^t, \quad C = -C^t \right\}.
\end{equation}

\footnote{An “odd [analog of] symplectic” (the orthogonal group preserves lines, as its name reflects ($\circ \rho \theta \dot{\varsigma}$ = straight, direct (the opposite of crooked is $\varepsilon \dot{i} \theta \dot{\varsigma}$)), symplectic ($\sigma \nu \mu \pi \lambda \kappa \varepsilon \iota \nu$) = intertwine or interweave, and $\pi \varepsilon \mu \sigma \sigma \dot{\varsigma}$ means odd, as opposed to even.)}
We note that although $\mathfrak{osp}(m|2n) \simeq \mathfrak{osp}(2n|m)$, as well as $\mathfrak{pe}^{sp}(n) \simeq \mathfrak{pe}^{sk}(n)$, the difference between these isomorphic Lie superalgebras is sometimes crucial, see [LS1] and Remark A.5.2.

The *special periplectic* superalgebra is $\mathfrak{sp}(n) = \{ X \in \mathfrak{pe}(n) \mid \text{str} X = 0 \}$. Of particular interest will be also $\mathfrak{sp}(n)_{a,b} = \mathfrak{sp}(n) \oplus \mathbb{C}(az + bd)$, where $z = 1_{2n}$, $d = \text{diag}(1_n, -1_n)$. Indeed, it is the linear part of $\mathfrak{b}_{a,b}(n)$.

**A.2. Vectorial Lie superalgebras. The standard realization.** The elements of the Lie algebra $\mathcal{L} = \mathfrak{der} \mathbb{C}[u]$ are considered as vector fields. The Lie algebra $\mathcal{L}$ has only one maximal subalgebra $\mathcal{L}_0$ of finite codimension (consisting of the fields that vanish at the origin). The subalgebra $\mathcal{L}_0$ determines a filtration of $\mathcal{L}$: set

\[(47) \quad \mathcal{L}_{-1} = \mathcal{L} \quad \text{and} \quad \mathcal{L}_i = \{ D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}] \subseteq \mathcal{L}_{i-1} \} \quad \text{for} \ i \geq 1 .
\]

The associated graded Lie algebra $L = \bigoplus_{i \geq -1} L_i$, where $L_i = \mathcal{L}_i / \mathcal{L}_{i+1}$, consists of the vector fields with *polynomial* coefficients.

**A.2.1. Superization.** For $\mathcal{L} = \mathfrak{der} \mathbb{C}[u, \xi]$ suppose $\mathcal{L}_0 \subset \mathcal{L}$ is a maximal subalgebra of finite codimension and containing no ideals of $\mathcal{L}$. Let $\mathcal{L}_{-1}$ be a minimal subspace of $\mathcal{L}$ containing $\mathcal{L}_0$, different from $\mathcal{L}_0$ and $\mathcal{L}_0$-invariant. A *Weisfeiler filtration* of $\mathcal{L}$ is determined by setting for $i \geq 1$:

\[(48) \quad \mathcal{L}_{-i-1} = [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} \quad \text{and} \quad \mathcal{L}_i = \{ D \in \mathcal{L}_{i-1} \mid [D, \mathcal{L}_{-1}] \subseteq \mathcal{L}_{i-1} \} .
\]

Since the codimension of $\mathcal{L}_0$ is finite, the filtration takes the form

\[(49) \quad \mathcal{L} = \mathcal{L}_{-d} \supset \cdots \supset \mathcal{L}_0 \supset \cdots
\]

for some *depth* $d$. Considering the subspaces $[\mathcal{L}, \mathcal{L}]$ as the basis of a topology, we can complete the graded or filtered Lie superalgebras $L$ or $\mathcal{L}$; the elements of the completion are the vector fields with formal power series as coefficients. Although the structure of the graded algebras is easier to describe, in applications the completed Lie superalgebras are usually needed.

Unlike Lie algebras, simple vectorial superalgebras possess *several* non-isomorphic maximal subalgebras of finite codimension, see sec. A.7.

1) *General algebras.* Let $x = (u_1, \ldots, u_n, \theta_1, \ldots, \theta_m)$, where the $u_i$ are even indeterminates and the $\theta_j$ are odd ones. Set $\mathfrak{vect}(n|m) = \mathfrak{der} \mathbb{C}[x]$; it is called the *general vectorial Lie superalgebra*.

On vectorial superalgebras, there are two types of trace. The divergences (depending on a fixed volume element) belong to one of them, various linear functionals that vanish on the brackets (traces) belong to the other type. Accordingly, the *special (divergence free)* subalgebra of a vectorial algebra $\mathfrak{g}$ is denoted by $\mathfrak{sg}$, e.g., $\mathfrak{vect}(n|m)$ and $\mathfrak{s vect}(n|m)$, and the traceless subalgebra of $\mathfrak{g}$ is denoted $\mathfrak{g}^t$.

2) *Special algebras.* The *divergence* of the field $D = \sum_i f_i \frac{\partial}{\partial u_i} + \sum_j g_j \frac{\partial}{\partial \theta_j}$ is the function (in our case: a polynomial, or a series)

\[(50) \quad \text{div} D = \sum_i \frac{\partial f_i}{\partial u_i} + \sum_j (-1)^{p(g_j)} \frac{\partial g_j}{\partial \theta_j} .
\]

- The Lie superalgebra $\mathfrak{svect}(n|m) = \{ D \in \mathfrak{vect}(n|m) \mid \text{div} D = 0 \}$ is called the *special or divergence-free vectorial superalgebra*.

It is clear that it is also possible to describe $\mathfrak{svect}(n|m)$ as $\{ D \in \mathfrak{vect}(n|m) \mid L_D \text{vol}_x = 0 \}$, where $\text{vol}_x$ is the volume form with constant coefficients in coordinates $x$ (see sec. A.6) and $L_D$ the Lie derivative with respect to $D$. 
The Lie superalgebra \( \mathfrak{svect}_\lambda(0|m) = \{ D \in \mathfrak{vect}(0|m) \mid \text{div} (1 + \lambda \theta_1 \cdots \theta_m) D = 0 \} \), where \( p(\lambda) \equiv m \pmod{2} \), — the deform of \( \mathfrak{svect}(0|m) \) — is called the deformed special or divergence-free vectorial superalgebra. Clearly, \( \mathfrak{svect}_\lambda(0|m) \cong \mathfrak{svect}_\mu(0|m) \) for \( \lambda \mu \neq 0 \). So we briefly denote these deforms by \( \mathfrak{svect}(0|m) \).

Observe that, for \( m \) odd, the parameter of deformation \( \lambda \) is odd.

3) The algebras that preserve Pfaff equations and differential 2-forms. Having denoted \( u = (t, p_1, \ldots, p_n, q_1, \ldots, q_n) \) set

\[
\tilde{\alpha}_1 = dt + \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq m} \theta_j d\theta_j \quad \text{and} \quad \omega_0 = d\alpha_1.
\]

The form \( \tilde{\alpha}_1 \) is called contact, the form \( \tilde{\omega}_0 \) is called symplectic. Sometimes it is more convenient to redenote the \( \theta \)'s and set

\[
\xi_j = \frac{1}{\sqrt{2}}(\theta_j - i\theta_{r+j}); \quad \eta_j = \frac{1}{\sqrt{2}}(\theta_j + i\theta_{r+j}) \quad \text{for} \quad j \leq r = \lfloor m/2 \rfloor \quad \text{(here} \quad i^2 = -1), \quad \theta = \theta_{2r+1}
\]

and in place of \( \tilde{\omega}_0 \) or \( \tilde{\alpha}_1 \) take \( \alpha_1 \) and \( \omega_0 = d\alpha_1 \), respectively, where

\[
\alpha_1 = dt + \sum_{1 \leq i \leq n} (p_i dq_i - q_i dp_i) + \sum_{1 \leq j \leq r} (\xi_j d\eta_j + \eta_j d\xi_j) \begin{cases} +\theta d\theta & \text{if} \quad m = 2r \\ +\lambda d\lambda & \text{if} \quad m = 2r + 1. \end{cases}
\]

The Lie superalgebra that preserves the Pfaff equation \( \alpha_1(X) = 0 \) for \( x \in \mathfrak{vect}(2n+1|m) \), i.e., the superalgebra

\[
\mathfrak{e}(2n+1|m) = \{ D \in \mathfrak{vect}(2n+1|m) \mid L_D \alpha_1 = f_D \alpha_1 \text{ for some } f_D \in \mathbb{C}[t, p, q, \theta] \},
\]

is called the contact superalgebra. The Lie superalgebra

\[
\mathfrak{po}(2n|m) = \{ D \in \mathfrak{e}(2n+1|m) \mid L_D \alpha_1 = 0 \}
\]

is called the Poisson superalgebra. (A geometric interpretation of the Poisson superalgebra: it is the Lie superalgebra that preserves the connection with form \( \alpha \) in the line bundle over a symplectic supermanifold with the symplectic form \( d\alpha \).)

Similarly, set \( u = q = (q_1, \ldots, q_n) \), let \( \theta = (\xi_1, \ldots, \xi_n; \tau) \) be odd. Set

\[
\alpha_0 = d\tau + \sum_{i} (\xi_i dq_i + q_i d\xi_i), \quad \omega_1 = d\alpha_0
\]

and call these forms the pericontact and periplectic, respectively. Observe that this pericontact form is even.

The Lie superalgebra that preserves the Pfaff equation \( \alpha_0(X) = 0 \) for \( x \in \mathfrak{vect}(n|n+1) \), i.e., the superalgebra

\[
\mathfrak{m}(n) = \{ D \in \mathfrak{vect}(n|n+1) \mid L_D \alpha_0 = f_D \cdot \alpha_0 \text{ for some } f_D \in \mathbb{C}[q, \xi, \tau] \}
\]

is called the pericontact superalgebra.

The Lie superalgebra

\[
\mathfrak{b}(n) = \{ D \in \mathfrak{m}(n) \mid L_D \alpha_0 = 0 \}
\]

is referred to as the Buttin superalgebra. (A geometric interpretation of the Buttin superalgebra: it is the Lie superalgebra that preserves the connection with form \( \alpha_1 \) in the line bundle of rank \( \varepsilon \) over a periplectic supermanifold, i.e., a supermanifold with the periplectic form \( d\alpha_0 \).)

The Lie superalgebras

\[
\mathfrak{sm}(n) = \{ D \in \mathfrak{m}(n) \mid \text{div} D = 0 \}, \quad \mathfrak{sb}(n) = \{ D \in \mathfrak{b}(n) \mid \text{div} D = 0 \}
\]
are called the divergence-free (or special) pericontact and special Buttin superalgebras, respectively.

Remark. A relation with finite dimensional geometry is as follows. Clearly, \( \ker \alpha_1 = \ker \tilde{\alpha}_1 \). The restriction of \( \tilde{\omega}_0 \) to \( \ker \alpha_1 \) is the orthosymplectic form \( B_{ev}(m|2n) \); the restriction of \( \omega_0 \) to \( \ker \tilde{\alpha}_1 \) is \( B'_{ev}(m|2n) \). Similarly, the restriction of \( \omega_1 \) to \( \ker \alpha_0 \) is \( B_{odd}(n|n) \).

### A.3. Generating functions.

A laconic way to describe that preserves \( f \in M \) respectively.

\[
K_f = (2 - E)(f) \frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t} E,
\]

where \( E = \sum_i y_i \frac{\partial}{\partial y_i} \) (here the \( y_i \) are all the coordinates except \( t \)) is the Euler operator (which counts the degree with respect to the \( y_i \)), and \( H_f \) is the hamiltonian field with Hamiltonian \( f \) that preserves \( d\tilde{\alpha}_1 \):

\[
H_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) - (-1)^{p(f)} \sum_{j \leq m} \left( \frac{\partial f}{\partial \eta_j} \frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right).
\]

The choice of the form \( \alpha_1 \) instead of \( \tilde{\alpha}_1 \) only affects the shape of \( H_f \) that we give for \( m = 2k + 1 \):

\[
H_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial}{\partial p_i} \right) - (-1)^{p(f)} \sum_{j \leq k} \left( \frac{\partial f}{\partial \xi_j} \frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j} \frac{\partial}{\partial \xi_j} + \frac{\partial f}{\partial \theta} \frac{\partial}{\partial \theta} \right).
\]

\* Even form \( \alpha_0 \). For \( f \in \mathbb{C}[q, \xi, \tau] \), we set:

\[
M_f = (2 - E)(f) \frac{\partial}{\partial \tau} - Le_f - (-1)^{p(f)} \frac{\partial f}{\partial \tau} E,
\]

where \( E = \sum_i y_i \frac{\partial}{\partial y_i} \) (here the \( y_i \) are all the coordinates except \( \tau \)), and

\[
Le_f = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial q_i} \right).
\]

Since

\[
L_{K_f}(\alpha_1) = 2 \frac{\partial f}{\partial \tau} \alpha_1 = K_1(f) \alpha_1,
\]

\[
L_{M_f}(\alpha_0) = -(-1)^{p(f)} 2 \frac{\partial f}{\partial \tau} \alpha_0 = -(-1)^{p(f)} M_1(f) \alpha_0,
\]

it follows that \( K_f \in \mathfrak{k}(2n + 1|m) \) and \( M_f \in \mathfrak{m}(n) \). Observe that

\[
p(Le_f) = p(M_f) = p(f) + 1.
\]

\* To the (super)commutators \([K_f, K_g] \) or \([M_f, M_g] \) there correspond contact brackets of the generating functions:

\[
[K_f, K_g] = K_{(f, g)} \text{, b.}; \quad [M_f, M_g] = M_{(f, g)} \text{, m. b.}
\]

The explicit formulas for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on \( t \) (resp. \( \tau \)).

The Poisson bracket \( \{\cdot, \cdot\}_{pb} \) (in the realization with the form \( \omega_0 \)) is given by the formula

\[
\{f, g\}_{pb} = \sum_{i \leq n} \left( \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} \right) - (-1)^{p(f)} \sum_{j \leq m} \frac{\partial f}{\partial \theta_j} \frac{\partial g}{\partial \theta_j} \text{ for } f, g \in \mathbb{C}[p, q, \theta].
\]
and in the realization with the form $\omega_0$ for $m = 2k + 1$ it is given by the formula

$$
\{f, g\}_{P.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} \right) - (1)^{\rho(f)} \left( \sum_{i \leq m} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} + \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} \right) + \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} \right)
$$

for $f, g, \in \mathbb{C}[p, q, \xi, \eta, \theta]$.

The Buttin bracket $\{\cdot, \cdot\}_{B.b.}$ is given by the formula

$$
\{f, g\}_{B.b.} = \sum_{i \leq n} \left( \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} + (1)^{\rho(f)} \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial q_i} \right)
$$

for $f, g, \in \mathbb{C}[q, \xi]$.

In terms of the Poisson and Buttin brackets, respectively, the contact brackets are

$$
\{f, g\}_{k.b.} = (2 - E)(f) \frac{\partial g}{\partial t} - \frac{\partial f}{\partial t} (2 - E)(g) - \{f, g\}_{P.b.}
$$

and

$$
\{f, g\}_{m.b.} = (2 - E)(f) \frac{\partial g}{\partial t} + (1)^{\rho(f)} \frac{\partial f}{\partial t} (2 - E)(g) - \{f, g\}_{B.b.}
$$

The Lie superalgebras of Hamiltonian fields (or Hamiltonian superalgebra) and its special subalgebra (defined only if $n = 0$) are

$$
\mathfrak{h}(2n|m) = \{D \in \mathfrak{vect}(2n|m) \mid L_D \omega_0 = 0 \} \text{ and } \mathfrak{s}\mathfrak{h}(m) = \{H_f \in \mathfrak{h}(0|m) \mid \int f vol = 0 \}.
$$

The “odd” analogs of the Lie superalgebra of Hamiltonian fields are the Lie superalgebra of vector fields $\mathcal{L}f$ introduced in [L1] and its special subalgebra:

$$
\mathfrak{l}(n) = \{D \in \mathfrak{vect}(n|n) \mid L_D \omega_1 = 0 \} \text{ and } \mathfrak{sl}(n) = \{D \in \mathfrak{l}(n) \mid \text{div}D = 0 \}.
$$

It is not difficult to prove the following isomorphisms (as superspaces):

$$
\mathfrak{e}(2n + 1|m) \cong \text{Span}(K_f \mid f \in \mathbb{C}[t, p, q, \xi]); \quad \mathfrak{l}(n) \cong \text{Span}(\mathcal{L}f \mid f \in \mathbb{C}[q, \xi]);
$$

$$
\mathfrak{m}(n) \cong \text{Span}(M_f \mid f \in \mathbb{C}[t, q, \xi]); \quad \mathfrak{h}(2n|m) \cong \text{Span}(H_f \mid f \in \mathbb{C}[p, q, \xi]).
$$

A.4. Divergence-free subalgebras. Since

$$
divK_f = (2n + 2 - m)K_1(f),
$$

it follows that the divergence-free subalgebra of the contact Lie superalgebra either coincides with it (for $m = 2n + 2$) or is isomorphic to the Poisson superalgebra. For the pericontact series, the situation is more interesting: the divergence free subalgebra is simple.

Since

$$
divM_f = (-1)^{\rho(f)2} \left( (1 - E) \frac{\partial f}{\partial \tau} - \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i} \right),
$$

it follows that

$$
\mathfrak{s}\mathfrak{m}(n) = \text{Span} \left( M_f \in \mathfrak{m}(n) \mid (1 - E) \frac{\partial f}{\partial \tau} = \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i} \right).
$$

In particular,

$$
div\mathcal{L}f = (-1)^{\rho(f)2} \sum_{i \leq n} \frac{\partial^2 f}{\partial q_i \partial \xi_i}.
$$
The odd analog of the Laplacian, namely, the operator

$$\Delta = \sum_{i \leq n} \frac{\partial^2}{\partial q_i \partial \xi_i}$$

on a periplectic supermanifold appeared in physics under the name of BRST operator, cf. [GPS]. The vector fields from $\mathfrak{sl}(n)$ are generated by harmonic functions, i.e., such that $\Delta(f) = 0$.

**A.5. The Cartan prolongs.** To define $b_\lambda(n)$, one of our main characters, and several related algebras we need the notion of the Cartan prolong. So let us recall the definition and generalize it somewhat. Let $\mathfrak{g}$ be a Lie algebra, $V$ a $\mathfrak{g}$-module, $S^i$ the operator of the $i$th symmetric power. Set $\mathfrak{g}_{-1} = V$, $\mathfrak{g}_0 = \mathfrak{g}$ and, for $i > 0$, define the $i$th Cartan prolong (the result of Cartan’s prolongation) of the pair $(\mathfrak{g}_{-1}, \mathfrak{g}_0)$ as

$$\mathfrak{g}_i = \{X \in \text{Hom}(\mathfrak{g}_{-1}, \mathfrak{g}_{-1}) \mid X(v_0)(v_1, \ldots, v_i) = X(v_1)(v_0, \ldots, v_i) \text{ for any } v_0, v_1, \ldots, v_i \in \mathfrak{g}_{-1}\} = (S^i(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_0) \cap (S^{i+1}(\mathfrak{g}_{-1}^*) \otimes \mathfrak{g}_{-1}).$$

(Here we consider $\mathfrak{g}_0$ as a subspace in $\mathfrak{g}_{-1}^* \otimes \mathfrak{g}_{-1}$, so the intersection is well-defined.)

The Cartan prolong of the pair $(V, \mathfrak{g})$ is $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* = \bigoplus_{i \geq -1} \mathfrak{g}_i$.

Suppose that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful. Then, clearly,

$$\begin{align*}
(\mathfrak{g}_{-1}, \mathfrak{g}_0)_* & \subset \text{vect}(n) = \text{det} \mathbb{C}[x_1, \ldots, x_n], \text{ where } n = \dim \mathfrak{g}_{-1} \text{ and } \\
\mathfrak{g}_i & = \{D \in \text{vect}(n) \mid \deg D = i, [D, X] \in \mathfrak{g}_{i-1} \text{ for any } X \in \mathfrak{g}_{-1}\}.
\end{align*}$$

It can be easily verified that the Lie algebra structure on $\text{vect}(n)$ induces same on $(\mathfrak{g}_{-1}, \mathfrak{g}_0)_*$.

Of the four simple vectorial Lie algebras, three are Cartan prolongs: $\text{vect}(n) = (\text{id}, \mathfrak{gl}(n))_*$, $\mathfrak{s vect}(n) = (\text{id}, \mathfrak{sl}(n))_*$ and $\mathfrak{h}(2n) = (\text{id}, \mathfrak{sp}(n))_*$. The fourth one — $\mathfrak{f}(2n+1)$ — is the result of a trifle more general construction described as follows.

**A.5.1. A generalization of the Cartan prolong.** Let $\mathfrak{g}_- = \bigoplus_{-d \leq i \leq -1} \mathfrak{g}_i$ be a nilpotent $\mathbb{Z}$-graded Lie algebra and $\mathfrak{g}_0 \subset \text{det}_0 \mathfrak{g}$ a Lie subalgebra of the $\mathbb{Z}$-grading-preserving derivations. For $i > 0$, define the $i$-th prolong of the pair $(\mathfrak{g}_-, \mathfrak{g}_0)$ as

$$\mathfrak{g}_i = ((S^i(\mathfrak{g}_-^*) \otimes \mathfrak{g}_0) \cap (S^{i+1}(\mathfrak{g}_-^*) \otimes \mathfrak{g}_{-1}))_i,$$

where the subscript $i$ in the right hand side singles out the component of degree $i$.

Define $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ to be $\bigoplus_{i \geq -d} \mathfrak{g}_i$; then, as is easy to verify, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie algebra.

What is the Lie algebra of contact vector fields in these terms? Denote by $\mathfrak{hei}(2n)$ the Heisenberg Lie algebra: its space is $W \oplus \mathbb{C} \cdot z$, where $W$ is a $2n$-dimensional space endowed with a non-degenerate skew-symmetric bilinear form $B$ and the bracket in $\mathfrak{hei}(2n)$ is given by the following relations:

$$\begin{align*}
z \text{ is even and lies in the center and } [v, w] = B(v, w) \cdot z \text{ for any } v, w \in W.
\end{align*}$$

Clearly, $\mathfrak{f}(2n+1) \cong (\mathfrak{hei}(2n), \mathfrak{csp}(2n))_*$. 

**A.5.2. Lie superalgebras of vector fields as Cartan prolongs.** The superization of the constructions from sec. A.5 are straightforward: via Sign Rule. We thus obtain:

$$\begin{align*}
\text{vect}(m|n) = (\text{id}, \mathfrak{gl}(m|n))_*; \quad \mathfrak{s vect}(m|n) = (\text{id}, \mathfrak{sl}(m|n))_*; \\
\mathfrak{b}(2m|n) = (\text{id}, \mathfrak{osp}^{sk}(m|2n))_*; \\
\mathfrak{le}(n) = (\text{id}, \mathfrak{pe}^{sk}(n))_*; \quad \mathfrak{sl}(n) = (\text{id}, \mathfrak{sp}^{sk}(n))_*.
\end{align*}$$
Remark. Observe that the Cartan prolongs \((\text{id}, \mathfrak{osp}^{sk}(m|2n))_s\) and \((\text{id}, \mathfrak{pe}^{sk}(n))_s\) are finite dimensional.

The generalization of Cartan prolongations described in sec. A.5.1 has, after superization, two analogs associated with the contact series \(\mathfrak{f}\) and \(\mathfrak{m}\), respectively.

- Let \(\mathfrak{he}(2n|m)\) or \(\mathfrak{he}(W)\) on the direct sum of a \((2n, m)\)-dimensional superspace \(W\) endowed with a non-degenerate skew-symmetric bilinear form \(B\) and the \((1, 0)\)-dimensional space spanned by \(z\).

Clearly, we have \(\mathfrak{f}(2n+1|m) = (\mathfrak{he}(2n|m), \mathfrak{cosp}^{sk}(m|2n))_s\). More generally, given \(\mathfrak{he}(2n|m)\) and a subalgebra \(\mathfrak{g}\) of \(\mathfrak{cosp}^{sk}(m|2n)\), we call \((\mathfrak{he}(2n|m), \mathfrak{g})_s\) the \(k\)-prolong of \((W, \mathfrak{g})\), where \(W\) is the identity \(\mathfrak{osp}^{sk}(m|2n)\)-module.

- The “odd” analog of \(\mathfrak{f}\) is associated with the following “odd” analog of \(\mathfrak{he}(2n|m)\). Denote by \(\mathfrak{ab}(n)\) or \(\mathfrak{ab}(W)\) the antibracket Lie superalgebra: its space is \(W \oplus \mathbb{C} \cdot z\), where \(W\) is an \(n|n\)-dimensional superspace endowed with a non-degenerate skew-symmetric odd bilinear form \(B\); the bracket in \(\mathfrak{ab}(n)\) is given by the following relations:

\[
(\mathfrak{ab}(n), \mathfrak{g}) = (\mathfrak{he}(2n|m), \mathfrak{g})_s + \mathfrak{g} \cdot B + B \cdot \mathfrak{g}\]

Clearly, \(\mathfrak{m}(n) = (\mathfrak{ab}(n), \mathfrak{pc}^{sk}(n))_s\). More generally, given \(\mathfrak{ab}(n)\) and a subalgebra \(\mathfrak{g}\) of \(\mathfrak{pc}^{sk}(n)\), we call \(\mathfrak{ab}(n), \mathfrak{g}\) the \(m\)-prolong of \((W, \mathfrak{g})\), where \(W\) is the identity \(\mathfrak{pc}^{sk}(n)\)-module.

### A.6. The modules of tensor fields.

To advance further, we have to recall the definition of the modules of tensor fields over \(\mathfrak{vect}(m|n)\) and its subalgebras, see [29], [31]. For any other \(\mathbb{Z}\)-graded vectorial Lie superalgebra, the construction is identical.

Let \(\mathfrak{g} = \mathfrak{vect}(m|n)\) and \(\mathfrak{g}_{\geq} = \bigoplus_{i \geq 0} \mathfrak{g}_i\). Clearly, \(\mathfrak{vect}_0(m|n) \cong \mathfrak{gl}(m|n)\). Let \(V\) be the \(\mathfrak{gl}(m|n)\)-module with the lowest weight \(\lambda = \text{lwt}(V)\). Make \(V\) into a \(\mathfrak{g}_{\geq}\)-module setting \(\mathfrak{g}_+ \cdot V = 0\) for \(\mathfrak{g}_+ = \bigoplus_{i \geq 0} \mathfrak{g}_i\). Let us realize \(\mathfrak{g}\) by vector fields on the \(m|n\)-dimensional linear supermanifold \(\mathcal{C}^{m|n}\) with coordinates \(x = (u, \xi)\). The superspace \(T(V) = \text{Hom}_U(\mathfrak{g}_2)(U(\mathfrak{g}), V)\) is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to \(\mathbb{C}[[x]] \otimes V\). Its elements have a natural interpretation as formal tensor fields of type \(V\). When \(\lambda = (a, \ldots, a)\) we will simply write \(T(a)\) instead of \(T(\lambda)\). We will usually consider \(\mathfrak{g}\)-modules induced from irreducible \(\mathfrak{g}_0\)-modules.

Examples: \(\mathfrak{vect}(m|n)\) as \(\mathfrak{vect}(m|n)\)- and \(\mathfrak{svect}(m|n)\)-modules is \(T(\text{id})\). Further examples: \(T(\mathfrak{h})\) is the superspace of functions; \(\text{Vol}(m|n) = T(1, \ldots, 1; -1, \ldots, -1)\) (the semicolon separates the first \(m\) “even” coordinates of the weight with respect to the matrix units \(E_{ij}\) of \(\mathfrak{gl}(m|n)\) from the “odd” coordinates) is the superspace of densities or volume forms. We denote the generator of \(\text{Vol}(m|n)\) corresponding to the ordered set of coordinates \(x\) by \(\text{vol}(x)\). The space of \(\lambda\)-densities is \(\text{Vol}^\lambda(m|n) = T(\lambda, \ldots, \lambda; -\lambda, \ldots, -\lambda)\); we denote its generator by \(\text{vol}^\lambda(x)\). In particular, \(\text{Vol}^\lambda(0|n) = T(\bar{\lambda})\) but \(\text{Vol}^\lambda(0|n) = T(\bar{\lambda})\).

Remark. To view the volume element as \(\text{"d}m\text{d}u\text{d}^n\xi\) is totally wrong: the Berezinian (superdeterminant) can never appear as a factor under the changes of variables. One can try to use the usual notations of differentials provided all the differentials anticommute. Then the linear transformations that do not intermix the even \(u\)'s with the odd \(\xi\)'s multiply the volume element \(\text{vol}(x)\), viewed as the fraction \(\text{vol}(x) = \frac{\partial x_1 \cdots \partial x_m}{\partial \xi_1 \cdots \partial \xi_n}\), by the Berezinian of the transformation. But how could we justify this? Let \(x = (u, \xi)\). If we consider the usual, exterior, differential forms, then the \(dx_i\)’s super anti-commute, hence, the \(d\xi_i\) commute; whereas if we consider the symmetric product of the differentials, as in the metrics, then the \(dx_i\)’s supercommute, hence, the \(du_i\) commute. However, the \(\frac{\partial}{\partial \xi}\) anticommute and, from transformations’ point of view, \(\frac{\partial}{\partial \xi} = \frac{1}{\frac{\partial}{\partial x}}\). The notation, \(du_1 \cdots du_m \cdot \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}\), suggested by V. Ogievetsky, is, nevertheless, still wrong: almost any transformation \(A : (u, \xi) \mapsto (v, \eta)\) sends \(du_1 \cdots du_m \cdot \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}\) to the correct element, \(\text{ber}(A)(du^m \cdot \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n})\), plus extra terms. Indeed, the fraction \(du_1 \cdots du_m \cdot \frac{\partial}{\partial x_1} \cdots \frac{\partial}{\partial x_n}\) is the highest weight vector of an indecomposable
$\mathfrak{gl}(m/n)$-module and $\text{vol}(x)$ is the notation of the image of this vector in the 1-dimensional quotient module modulo the invariant submodule that consists precisely of all the extra terms.

### A.7. Nonstandard realizations.

| Lie superalgebra | its $\mathbb{Z}$-grading |
|------------------|--------------------------|
| $\mathfrak{vect}(n|m; r)$, $0 \leq r \leq m$ | $\deg u_i = \deg \xi_j = 1$ for any $i, j$ | $(\ast)$ |
| $\mathfrak{m}(n; r)$, $0 \leq r \leq n$, $r \neq n - 1$ | $\deg \tau = 2$, $\deg q_i = \deg \xi_j = 1$ for any $i$ | $(\ast)$ |
| $\mathfrak{t}(2n + 1|m; r)$, $0 \leq r \leq [m/2]$, $r \neq k - 1$ for $m = 2k$ and $n = 0$ | $\deg p_i = \deg q_i = \deg \xi_j = \deg \eta_j = \deg \theta_k = 1$ for any $i, j, k$ | $(\ast)$ |
| $\mathfrak{t}(1|2m; m)$ | $\deg t = 2$ |

For the reasons why $r$ cannot take value $n - 1$ for $\mathfrak{m}(n)$ and $k - 1$ for $\mathfrak{t}(1|2k)$, irrelevant in this paper but vital in other problems, we refer the reader to [LS1].

**Comments:** The gradings in the series $\mathfrak{vect}$ induce the gradings in the series $\mathfrak{svect}$; the gradings in $\mathfrak{m}$ induce the gradings in $\mathfrak{b}_\lambda$, $\mathfrak{le}$, $\mathfrak{slc}$, $\mathfrak{b}$, $\mathfrak{sb}$; the gradings in $\mathfrak{t}$ induce the gradings in $\mathfrak{po}$, $\mathfrak{h}$.

In (87) we consider $\mathfrak{t}(2n + 1|m)$ as preserving the Pfaff equation $\alpha(X) = 0$ for $X \in \mathfrak{vect}(2n + 1|m)$, where

$$\alpha = dt + \sum_{i \leq n} (p_i dq_i - q_i dp_i) + \sum_{j \leq r} (\xi_j d\eta_j + \eta_j d\xi_j) + \sum_{k \geq m - 2r} \theta_k d\theta_k.$$  

The standard realizations correspond to $r = 0$, they are marked by $(\ast)$. Observe that the codimension of $\mathcal{L}_0$ attains its minimum in the standard realization.

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**References**

[ALSh] Alekseevsky D., Leites D., Shchepochkina I., New examples of simple Lie superalgebras of vector fields. C.r. Acad. Bulg. Sci., v. 34, N 9, 1980, 1187–1190 (in Russian)

[Bea] Bayen F., Flato M., Fronsdal C., Lichnerowicz A., Sternheimer D. Deformation theory and quantization. I. Deformations of symplectic structures. Ann. Physics 111 (1978), no. 1, 61–151

[BT1] Batalin I., Tyutin I., General local solution to the cyclic Jacobi equation, manus ca 1990, unpublished

[BT2] Batalin I., Tyutin I., Generalized Field-Antifield formalism. In: Dobrushin R. et. al. (eds.) Topics in Statistical and theoretical Physics (F.A.Berezin memorial volume), Transactions of AMS, series 2, v. 177, 1996, 23–43

[B] Berezin, F. A., General concept of quantization. Comm. Math. Phys. 40 (1975), 153–174; id., Quantization. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 38 (1974), 1116–1175; id., Quantization in complex symmetric spaces. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 39 (1975), no. 2, 363–402, 472

[B1] Berezin, F. A., Some remarks on the representations of commutation relations. (Russian) Uspehi Mat. Nauk 24 1969 no. 4 (148), 65–88
[BSS] Bernstein J., The Lie superalgebra $osp(1|2)$, connections over symplectic manifolds and representations of Poisson algebras. In: [L4], 9/1987-13, 1-60;
Shapovalov, A. V. Invariant differential operators and irreducible representations of finite dimensional Hamiltonian and Poisson Lie superalgebras. Serdica, 7 (1981), no. 4, 337–342 (in Russian);
Shmelev G. S. Irreducible representations of infinite dimensional Hamiltonian and Poisson Lie superalgebras, and invariant differential operators. Serdica, 8 (1982), no. 4, 408–417 (in Russian); id., Irreducible representations of Poisson Lie superalgebras and invariant differential operators. Funktsional. Anal. i Prilozhen. 17 (1983), no. 1, 91–92 (in Russian)

[BL] Bernstein J., Leites D., Invariant differential operators and irreducible representations of Lie superalgebras of vector fields. Sel. Math. Sov., v. 1, N 2, 1981, 143–160

[Bo] Bordemann M., The deformation quantization of certain super-Poisson brackets and BRST cohomology. Conférence Moshe Flato 1999, Vol. II (Dijon), 45–68, Math. Phys. Stud., 22, Kluwer Acad. Publ., Dordrecht, 2000; math.QA/0003218

[CK] Cheng S., Kac V., Generalized Spencer cohomology and filtered deformations of $Z$-graded Lie superalgebras. Adv. Theor. Math. Phys. 2 (1998), no. 5, 1141–1182; math.RT/9805039

[D1] Deligne P., Déformations de l’algèbre des fonctions d’une variété symplectique: comparaison entre Fedosov et De Wilde, Lecomte. (French) [Deformations of the algebra of functions of a symplectic manifold: comparison of the Fedosov and De Wilde-Lecomte methods] Selecta Math. (N.S.) 1 (1995), no. 4, 667–697

[D2] Deligne P. (et. al., eds.) Quantum fields and strings: a course for mathematicians. Vol. 1, 2. Material from the Special Year on Quantum Field Theory held at the Institute for Advanced Study, Princeton, NJ, 1996–1997. AMS, Providence, RI; Institute for Advanced Study (IAS), Princeton, NJ, 1999. Vol. 1: xxii+723 pp.; Vol. 2: pp. i–xxiv and 727–1501

[DWL] De Wilde, M. and Lecomte, P., Existence of star product and of formal deformation of the Poisson Lie algebra of arbitrary symplectic manifold, Lett. Math. Phys. 7 (6) (1983), 487–496.

[Dir] Dirac, P.A.M. Principles of Quantum Mechanics, 4th ed. Oxford, England: Oxford University Press, 1982. (also in: Dirac, P. A. M. The collected works of P. A. M. Dirac: 1924–1948. Edited and with a preface by R. H. Dalitz. Cambridge University Press, Cambridge, 1995. xxiv+i-310 pp.)

[DGS] Donin J., Gurevich D., Shnider S., Quantization of function algebras on semisimple orbits in $g^{*}$, q-alg/9607008

[DLO] Duval C., Lecomte P., Ovsienko V., Conformally equivariant quantization: existence and uniqueness. Ann. Inst. Fourier (Grenoble) 49 (1999), no. 6, 1999–2029; math.DG/9902032

[Dzh] Dzhumadildaev A., Virasoro type Lie algebras and deformations. Z. Phys. C 72 (1996), no. 3, 509–517

[Fe] Fedosov B., A simple geometrical construction of deformation quantization, J. Differential Geom. 40 (1994), no. 2, 213–238; Fedosov B., Deformation quantization and index theory. Akademie Verlag, Berlin, 1996;
Gelfand I., Retakh V., Shubin M., Fedosov manifolds. Adv. Math. 136 (1998), no. 1, 104–140

[F] Fuks (Fuchs) D. B., Cohomology of Infinite Dimensional Lie Algebras. Translated from the Russian by A. A. B. Bovinsky. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1986. xii+339 pp.

[FL] Fuchs D. B., Leites D., Cohomology of Lie superalgebras. C. r. Acad. Bulg. Sci., v. 37, 12, 1984, 1595–1596

[FLSf] Fulp R., Lada T., Stasheff J., Noether’s variational theorem II and the BV formalism. Proceedings of the 22nd Winter School “Geometry and Physics” (Srni’, 2002). Rend. Circ. Mat. Palermo (2) Suppl. No. 71 (2003), 115–126; math.QA/0204079

[GK] Gendenshtein L. E.; Krive I. V. Supersymmetry in quantum mechanics. Soviet Phys. Uspekhi 28 (1985), no. 8, 645–666 (1986); translated from Uspekhi Fiz. Nauk 146 (1985), no. 4, 553–590 (Russian)

[GPS] Gomis J., Paris J., Samuel S., Antibracket, antifields and gauge-theory quantization, Phys. Rept. 259 (1995), n.1–2, 1–191; hep-th/9412228

[G] Grozman P., Classification of bilinear invariant operators on tensor fields. (Russian) Funktsional. Anal. i Prilozhen. 14 (1980), no. 2, 58–59; for details, see math.RT/0509562
HOW TO QUANTIZE THE ANTI-BRACKET

[GL2] Grozman P., Leites D., Lie superalgebras of supermatrices of complex size. Their generalizations and related integrable systems. In: by E. Ramírez de Arellano, M. V. Shapiro, L. M. Tovar and N. L. Vasilevski (eds.) Proc. International Sympos. Complex Analysis and related topics, Mexico, 1996, Birkhauser Verlag, 1999, 73–105; math.RT/0202177

[GLP] Grozman P., Leites D., Poletaeva E., Defining relations for simple Lie superalgebras of polynomial vector fields. In: Ivanov E. et. al. (eds.) Supersymmetries and Quantum Symmetries (SQS’99, 27-31 July, 1999), Dubna, JINR, 387–396; math.RT/0202152

[GLS] Grozman P., Leites D., Shchepochkina I., Lie superalgebras of string theories, Acta Mathematica Vietnamica, v. 26, 2001, no. 1, 27–63; hep-th/9702120

[GLS3] Grozman P., Leites D., Shchepochkina I., Invariant differential operators on supermanifolds and The Standard Model. In: Olshanetsky M., Vainshtein A., (eds.) M. Marinov memorial volume, World Sci., 2002, 508–555; math.RT/0202193; ESI preprint 1111 (2001) (http://www.esi.ac.at)

[GS] Guillemin V., Sternberg Sh., Symplectic techniques in physics. Second edition. Cambridge University Press, Cambridge, 1990. xii+468 pp.

[HG] Hazewinkel M., Gerstenhaber M. (eds.) Deformation theory of algebras and applications, Kluwer, Dordrecht, 1988, viii+1030 pp.

[K] Kac V., Lie superalgebras, Adv. Math. v. 26, 1977, 8–96

[Ko1] Kotchetkov Yu. Déformations de superalgèbres de Buttin et quantification. C.R. Acad. Sci. Paris, ser. I, 299:14, 1984, 643–645

[Ko2] Kotchetkov Yu. Deformations of Lie superalgebras. VINITI Depositions, Moscow 1985, # 384–85 (in Russian)

[Ko3] Kochetkov, Yu. Yu. Relations and deformations of odd Hamiltonian superalgebras. (Russian) Mat. Zametki 63 (1998), no. 3, 391–401; translation in Math. Notes 63 (1998), no. 3-4, 342–351; van den Hijligenberg, N.; Kotchetkov, Y.; Post, G. Deformations of $S(0,n)$ and $H(0,n)$. Internat. J. Algebra Comput. 3 (1993), no. 1, 57–77

[Ko4] van den Hijligenberg, N. W.; Kotchetkov, Yu. Yu. The absolute rigidity of the Neveu-Schwarz and Ramond superalgebras. J. Math. Phys. 37 (1996), no. 11, 5858–5868

[Ko5] van den Hijligenberg N., Kotchetkov Y.; Post G., Deformations of vector fields and Hamiltonian vector fields on the plane. Math. Comp. 64 (1995), no. 211, 1215–1226; Kochetkov, Yu. Yu. Deformations of the Hamiltonian Lie algebra $H(2)$. (Russian) Funktsional. Anal. i Prilozhen. 28 (1994), no. 3, 77–79; translation in Funct. Anal. Appl. 28 (1994), no. 3, 211–213

[Ko6] Kochetkov, Yu. Yu., Induced irreducible representations of Leites superalgebras. In: Onishchik A. et al (eds.) Voprosy teorii grupp i gomologicheskoy algebry Problems in group theory and in homological algebra (Russian), 120–123, 139, Yaroslav. Gos. Univ., Yaroslavl, 1983; id., Discrete modules for Lie superalgebras of the series Le., ibid., 1989, 142–148

[Ko7] Kochetkov, Yu. Yu. Singular vectors, invariant operators and discrete modules for the Lie algebra of Hamiltonian vector fields $H(n)$. (Russian) Uspekhi Mat. Nauk, 44 (1989), no. 5(269),167–168; translation in Russian Math. Surveys, 44 (1989), no. 5, 201–202

[KT] Konstein S. E., Tyutin I., Cohomologies of the Poisson superalgebra on (2,n)-superdimensional spaces; hep-th/0411235

[Kon2] Kontsevich M., Deformation quantization of algebraic varieties. EuroConférence Moshé Flato 2000, Part III (Dijon). Lett. Math. Phys. 56 (2001), no. 3, 271–294;

[Kon1] Kontsevich, M., Deformation quantization of Poisson manifolds, Lett. Math. Phys. 66 (2003), no. 3, 157–216; q-alg/9709040

[KS] Kostant B.; Sternberg S., Symplectic reduction, BRS cohomology, and infinite-dimensional Clifford algebras. Ann. Physics, 176 (1987), no. 1, 49–113

[Lan] Landweber G., Representation rings of Lie superalgebras; math.RT/0403203

[LO] Lecomte P.B.A., Ovsienko V.Yu., Projectively equivariant symbol calculus, Lett. Math. Phys. 49 (1999), no. 3, 173–196; math.DG/9809061

[L] Leites D., Introduction to the supermanifold theory. Russian Math. Surveys, v. 35, n.1, 1980, 3–53; id., Supermanifold theory, Karelia brach of the USSR Acad. Sci., Petrozavodsk, 1983, 200 pp. (Russian)(expanded in [L4])

[L1] Leites D., New Lie superalgebras and mechanics. Soviet Math. Doklady, v. 18, n. 5, 1977, 1277–1280

[L2] Leites D., Lie superalgebras. In: Modern Problems of Mathematics. Recent developments, v. 25, VINITI, Moscow, 1984, 3–49 (in Russian; English translation in: JOSMAR (J. Soviet Math.) v. 30 (6), 1985, 2481–2512)
[L3] Leites D., Clifford algebras as superalgebras, and quantization. (Russian) Teoret. Mat. Fiz. 58 (1984), no. 2, 229–232; id, Quantization. Supplement 3. In: F. Berezin, M. Shubin. Schrödinger equation, Kluwer, Dordrecht, 1991, 483–522

[L4] Leites D. (ed.) Seminar on Supermanifolds, Reports of Stockholm University, 1987–1990, nn. 1–35, 2000 pp.

[LP] Leites D., Poletaeva E., Defining relations for classical Lie algebras of polynomial vector fields. Math. Scand. 81 (1997), no. 1, 5–19; [math.RT/0510019]

[LSa] Leites D., Shapovalov A., Manin-Olshansky triples for Lie superalgebras, J. Nonlinear Math. Phys., 2000, v. 7, no. 2, 120–125

[LS1] Leites D., Shchepochkina I., Classification of simple Lie superalgebras of vector fields, preprint MPIM-2003-28 (www.mpim-bonn.mpg.de)

[LS2] Leites D., Shchepochkina I., Classification of simple real Lie superalgebras of vector fields,

[Man] Manin Y. I., Gauge field theory and complex geometry, second edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 289. Springer-Verlag, Berlin, 1997. xii+346 pp

[NV] Neroslavsky, O.; Vlassov, A. Sur les déformations de l’algèbre des fonctions d’une variété symplectique. (French) C. R. Acad. Sci. Paris Sér. I Math. 292 (1981), no. 1, 71–73

[Pe] Perelomov A., Integrable systems of classical mechanics and Lie algebras. Vol. I. Birkhäuser Verlag, Basel, 1990. x+307 pp.

[R] Raina, A., An algebraic geometry view of currents in a model quantum field theory on a curve. C. R. Acad. Sci. Paris Sér. I Math. 318 (1994), no. 9, 851–856

[S] Serganova V., Classification of simple real Lie superalgebras and symmetric superspaces. Funktsional. Anal. i Prilozhen. 17 (1983), no. 3, 46–54; id., Automorphisms and real forms of Lie superalgebras of string theories. Funktsional. Anal. i Prilozhen. 19 (1985), no. 3, 75–76. (A detailed exposition in: [L4], v. 22.);

[Sch] Shchepochkina I., The five exceptional simple Lie superalgebras of vector fields. [hep-th 9702120] The five exceptional simple Lie superalgebras of vector fields and their fourteen regradings, Representation Theory (electronic), v. 3, 1999, 435–443; [math.RT/9904079]

[Sg1] Sergeev A., An analog of the classical invariant theory for Lie superalgebras. I, II. Michigan Math. J. 49 (2001), no. 1, 113–146, 147–168; [math.RT/9810113] [math.RT/9904079]

[Sg2] Sergeev A., Irreducible representations of solvable Lie superalgebras, Represent. Theory (electronic), v. 3, 1999, 435–443; [math.RT/9810109]

[Sm] Shmelev, G. S. Contact Lie superalgebras are rigid. (Russian) C. R. Acad. Bulgare Sci. 36 (1983), no. 5, 569–570

[Sm1] Shmelev G. S. Differential $H(2n, m)$-invariant operators and indecomposable osp(2, 2n)-representations. (Russian) Funktsional. Anal. i Prilozhen. 17 (1983), no. 4, 94–95; id., Irreducible representations of Poisson Lie superalgebras and invariant differential operators. (Russian) Funktsional. Anal. i Prilozhen. 17 (1983), no. 1, 91–92; id., Irreducible representations of infinite-dimensional Hamiltonian and Poisson Lie superalgebras, and invariant differential operators. (Russian) Serdica 8 (1982), no. 4, 408–417 (1983); id., Invariant operators on a symplectic supermanifold. (Russian) Mat. Sb. (N.S.) 120(162) (1983), no. 4, 528–539

[Sm2] Shmelev G. S., Differential operators that are invariant with respect to the Lie superalgebra $H(2, 2; \lambda)$ and its irreducible representations. (Russian) Funktsional. Anal. i Prilozhen. 17 (1983), no. 4, 94–95; id., Irreducible representations of Poisson Lie superalgebras and invariant differential operators. (Russian) Funktsional. Anal. i Prilozhen. 17 (1983), no. 1, 91–92; id., Irreducible representations of infinite-dimensional Hamiltonian and Poisson Lie superalgebras, and invariant differential operators. (Russian) Serdica 8 (1982), no. 4, 408–417 (1983); id., Invariant operators on a symplectic supermanifold. (Russian) Mat. Sb. (N.S.) 120(162) (1983), no. 4, 528–539

[Shu] Shubin, M. A., Semiclassical asymptotics on covering manifolds and Morse inequalities. Geom. Func. Anal. 6 (1996), no. 2, 370–409; Shubin M., Novikov inequalities for vector fields. The Gelfand Mathematical Seminars, 1993–1995, Gelfand Math. Sem., Birkhäuser Boston, Boston, MA, 1996, 243–274

[V] Vey J., Déformations du crochet de Poisson d’une variété symplectique. Comm. Math. Helv., 50, 1975, 421–454

[WZ] Wess, J.; Zumino, B., Supergauge transformations in four dimensions. Nuclear Phys. B70 (1974), 39–50;
HOW TO QUANTIZE THE ANTIBRACKET

Wess J., Supersymmetry-supergravity. In: J. A. de Azcárraga (ed.) *Topics in quantum field theory and gauge theories* (Proc. VIII Internat. GIFT Sem. Theoret. Phys., Salamanca, 1977), pp. 81–125, Lecture Notes in Phys., 77, Springer, Berlin-New York, 1978;

Wess, J., Supersymmetry/supergravity. Concepts and trends in particle physics (Schladming, 1986), Springer, Berlin, 1987, 29–58;

Wess J., Zumino B., Superspace formulation of supergravity. Phys. Lett. B 66 (1977), no. 4, 361–364;

Wess J., Supersymmetry/supergravity. In: H. Latal and H. Mitter (eds.) *Concepts and trends in particle physics* (Schladming, 1986), 29–58, Springer, Berlin, 1987; Wess J., Introduction to supersymmetric theories. In: Dž. Šijački, N. Bilić, B. Dragović and D. Popović. *Frontiers in particle physics '83* (Dubrovnik, 1983), 104–131, World Sci. Publishing, Singapore, 1984;

Wess J., Bagger J., *Supersymmetry and supergravity*. Princeton Series in Physics. Princeton University Press, Princeton, N.J., 1983. i+180 pp

[W] Witten E., Phys. Lett. B 77 (1978), no. 4-5, 394–400; id., Grand unification with and without supersymmetry. In: O. Castanos, A. Frank and L. Urrutia (eds.) *Introduction to supersymmetry in particle and nuclear physics* (Mexico City, 1981), 53–76, Plenum, New York, 1984

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