Stability of $\Phi^4$ oscillatons

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Abstract. We investigate the stability of oscillatons with a quartic self-interaction. Oscillatons are spherically symmetric solutions of the coupled Einstein-Klein-Gordon for the case of a real scalar field. It is shown that there exist equilibrium configurations which are stables under small and strong (radial) perturbations. Some basic numerical tools are implemented that shall help in the study of arbitrary equilibrium configurations.

1. Introduction
Seidel and Suen were the first that found non-singular and asymptotically flat solutions of the coupled Einstein-Klein-Gordon (EKG) equations, in which both the (real) scalar field and the metric are time-dependent [1, 2]. These solutions are so-called oscillatons that is the short name for oscillating soliton stars. Time-dependent of oscillatons arise because we working with real scalar field, and this case is different from its complex counterpart, boson stars, for which the spacetime geometry is static[3].

In an exhaustive analysis presented in[4], it was shown that oscillatons are classified into stable (S-branch) and unstable (U-branch) configurations. S-oscillatons are stable configurations under small radial perturbations, and they typically migrate to other S-profiles if strongly perturbed. On the other hand, U-oscillatons are intrinsically unstable: they migrate to the S-branch if their mass is moderate, but they may collapse into black holes if their mass is large enough.

As we mention in [5], oscillatons may have some importance in both Astrophysics and Cosmology, because scalar fields have been proposed as candidates to be dark matter in the Universe [6, 7, 8, 9]. Scalar field dark matter would form cosmological structure with properties similar to those of oscillatons. More recently, it has been suggested that multistate bosonic objects could also provide an alternative explanation for the flat rotation curves in galaxies [10, 11, 12, 13].

In this work, we study the evolution of oscillatons with a quartic self-interaction in the scalar field potential, and analyze the stability in some selected cases following closely the numerical tools presented in Ref.[4].

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2. Mathematical background

We solve the system of the Einstein-Klein-Gordon (EKG) equations for a spherically-symmetric metric of the form

\[ ds^2 = -\alpha^2(t, r)dt^2 + a^2(t, r)dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right), \]

where \( a^2(t, r) \) is the radial metric function, and \( \alpha^2(t, r) \) is the lapse function. The action that describes our self-gravitating system is

\[ \int d^4x \sqrt{-g} \left( \frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu} \Phi_{,\mu} \Phi_{,\nu} - V(\Phi) \right), \]

where \( \Phi \) is a real scalar field endowed with a scalar field potential

\[ V(\Phi) = \frac{1}{2} m^2 \Phi^2 + \frac{1}{4} \lambda \Phi^4. \]

The parameter \( m_\Phi \) is the mass of the scalar field, and \( \lambda \) is the quartic interaction parameter.

The evolution of metric functions \( a(t, r) \) and \( \alpha(t, r) \) is governed by Einstein equations:

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} \]

The scalar field dynamics is described by the Klein-Gordon (KG) equation, which is obtained from the conservation equations for the scalar field energy-momentum tensor

\[ T_{\mu\nu}^{\text{\tiny KG}} = \Phi_{,\mu} \Phi_{,\nu} - \frac{1}{2} g_{\mu\nu} \left( \Phi_{,\gamma} \Phi_{,\gamma} + m_\Phi^2 \Phi^2 + \frac{\lambda}{2} \Phi^4 \right) \]

The KG equation is equivalent to the following set of first order differential equations

\[ \Phi_{,t} = a \Pi \]

\[ \Pi_{,t} = \frac{1}{x^2} \left( \frac{x^2 a \Psi}{a} \right)_{,x} - a \alpha (\Phi + \Lambda \Phi^3) \]

\[ \Psi_{,t} = \left( \frac{\alpha \Pi}{a} \right)_{,x} \]

Equations (6-11) are the total set of evolution equations that we solve numerically for different values of the \( \Lambda \) parameter.
3. Boundary conditions

Note that the set of equations is singular at \( x = 0 \). We take a spatial grid of the form \( x_i = (i - 1/2)\Delta x \), to avoid the singularity at the origin. From the evolution equation of \( \Phi \), see Eq. (9), we can see that it not necessary to apply boundary condition for \( \Phi \), because the evolution equation can be integrated all the way from the boundary point \( x_0 = -\Delta/2 \) to the outer boundary point.

As in Ref.[4], we use the fictitious point \( x_0 = -\Delta x/2 \) to impose appropriate parity conditions: \( \Pi \) is even and \( \Psi \) is odd. At the outer boundary, we assume that \( \Pi \) behaves as an outgoing wave pulse of the form:

\[
\Pi = u(x - t)/x
\]

where \( u \) is an arbitrary function. In differential form this becomes

\[
\partial_x \Pi + \partial_t \Pi + \Pi/x = 0
\]

By using finite difference in (13), we can solve it to find the unknown boundary value at the new time level. Because \( \Pi \) behaves as an outgoing wave at the boundary, so does \( \Phi \). Then, the outgoing wave boundary condition applied to \( \Phi \) can imply that at the outer boundary

\[
\Psi = -\Pi - \Phi/x.
\]

We used this expression to obtain boundary values for \( \Psi \) once those of \( \Phi \) and \( \Pi \) are known. Boundary condition for the metric functions are the following. Local flatness at the origin implies that \( a(x = 0) = 1 \) and \( \partial_x a(x = 0) = 0 \), and this two conditions imply that \( a(x_1) = 1 + \mathcal{O}(x^3) \). The Hamiltonian constraint (6) is then integrated outwards to obtain \( a(x) \).

For the lapse function we impose \( \alpha = 1/a \) as an outer boundary condition, this is because in a vacuum our slicing condition implies that we are in Schwarzschild coordinates. Then, we are assuming that our boundary conditions are sufficiently far away as to be always in vacuum. Thus, the slicing condition is integrated inwards to obtain \( \alpha(x) \).

4. Numerical Results

For our numerical experiments, we use an equilibrium configuration found in[5] as the initial condition for \( \Phi(0, x) \), and from this we obtain \( \Psi(0, x) = \Phi'(0, t) \) and \( \Pi(0, x) = 0 \) through Eqs.(8).

We apply a transformation to the evolution equation corresponding to \( \Pi \), as Eq. (10) could have some problems if discretized without care, because it is not second order accurate due to the presence of the factor \( 1/x^2 \) in the principal part. The transformation that guarantees the accuracy of our simulations at the origin is

\[
\Pi_\ell = 3 \frac{\partial}{\partial x^3} \left( \frac{x^2 a \Psi}{a} \right) - a \alpha \Phi
\]

and this is the equation which we discretized. In Fig. 1, we show the values obtained for the masses and fundamental frequencies of equilibrium configurations as functions of the central value of the first scalar field coefficient \( \phi_1(0) \) for the equilibrium configuration with \( \Lambda = 1[2, 4, 14, 5] \). The critical mass of the configurations, given by the maximum value of the mass curve, is \( M_{\Phi C} = 0.694(m_p^2/m_\Phi) \), which is reached at \( \phi_{1c}(0) = 0.48 \); the corresponding fundamental frequency is \( \Omega_{\Phi} = 0.859 \). Before the critical mass value (left side) there are the values of masses that represent the S-branch, and after critical mass value (right side) there are the values masses that represent the U-branch.

Also in Fig. 1, we show the function \( x^2 \rho \) at \( x = 100 \) for three different times. We can observe that the different evolution coincide very well up to \( x = 20 \), but they differ significantly for
large radii where $x^2 \rho < 10^{-5}$. This means that part of the scalar field has to be reflected from the boundary, because our boundary conditions correspond, strictly, to the massless case. In any case, the error is not significant and we were able to keep it under control in the numerical evolutions.

![Figure 1](image1.png)

**Figure 1.** Plots of $M_\Phi$ as functions of the central value $\phi_1(0)$ for $\Lambda = 1$ (left). And function $x^2 \rho$ at $x = 100$ for three different times and $\Lambda = 1$

On the other hand, in Fig. 2 we show the maximum value of the radial metric $a(t, x)$ as read from the numerical evolution up to $t = 5000$. We see that it maintain the same oscillatory pattern in all time, and we take this as evidence of the stability of this equilibrium configuration. To verify the convergence of our numerical method, we use the momentum constrain equation

$$\beta := a_{,t} - \frac{1}{2} x a \phi_1 \Pi = 0$$  \hspace{1cm} (16)

Its evolution is also shown in Fig. 2, in which we plot the $L^2$ norm of the value of $\beta$ across the grid as a function of the time for three different resolutions. This shows that our numerical code is second order convergent.

![Figure 2](image2.png)

**Figure 2.** (Left) Maximum values as function of the time of the radial metric function $g_{rr} = a^2(t, r)$ for a stable equilibrium configuration. The numerical outer boundary is located at $x = 100$, the resolution is $\Delta x = 0.01$, and the quartic parameter is $\Lambda = 1$. (Right) The $L^2$ norm of the momentum constrain, see Eq. (16), for three different spatial resolutions: $\Delta x = 0.01$ (fine), $\Delta x = 0.02$ (medium) and $\Delta x = 0.04$ (coarse). The numerical code is then second order convergent.
We present some evolutions of different perturbed S-oscillatons in Fig. 3. We can see that when the perturbation is strong, the oscillaton migrate back to the line of equilibrium configurations if the initial mass is less than the critical value, \( M_c = 0.694 \), see Fig. 1.

Figure 3. Evolution of different strongly perturbed oscillatons. The perturbated oscillatons migrate and settles down onto another equilibrium configuration (represented by de solid line) when the initial mass is less than the critical one, \( M_c = 0.694 \).

5. Final Comments

Results obtained for oscillatons with quartic self-interaction indicate that their behavior is similar tho those obtained in Refs.[15, 4] for the case of a quadratic potential only.

It is known that massive oscillatons may be classified into stable and unstable configurations, and that this classification can be read off from the mass function similar to that shown in Fig. 1. We have only shown the existence of stable quartic oscillatons under small and strong (radial) perturbations.

More work is needed to have a complete classification of the stability and, in particular, to determine the existence of unstable configurations. This is research under progress that we expect to report elsewhere.

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