Quantum Measurements and Contextuality

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Abstract

In quantum physics the term ‘contextual’ can be used in more than one way. One usage, here called ‘Bell contextual’ since the idea goes back to Bell, is that if $A$, $B$, and $C$ are three quantum observables, with $A$ compatible (i.e., commuting) with $B$ and also with $C$, whereas $B$ and $C$ are incompatible, a measurement of $A$ might yield a different result (indicating that quantum mechanics is contextual) depending upon whether $A$ is measured along with $B$ (the $\{A,B\}$ context) or with $C$ (the $\{A,C\}$ context). An analysis of what projective quantum measurements measure shows that quantum theory is Bell noncontextual: the outcome of a particular $A$ measurement when $A$ is measured along with $B$ would have been exactly the same if $A$ had, instead, been measured along with $C$.

A different definition found in [1], here called ‘globally (non)contextual’ to distinguish it from the Bell variety, refers to whether there is (‘noncontextual’), or is not (‘contextual’), an ‘empirical model’ that simultaneously assigns probabilities in a consistent manner to the outcomes of measurements of a certain collection of observables, not all of which are compatible. A simple example shows that an empirical model can exist even in a situation where the (supposed) measurement probabilities cannot refer to properties of a quantum system, and hence lack physical significance, even though mathematically well-defined. It is noted that the quantum sample space required for interpreting measurements of incompatible properties in separate runs of an experiment has a tensor product structure, a fact sometimes overlooked.

Contents

1 Introduction 2
2 Bell Contextual 3
3 Quantum Measurements 4

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1 Introduction

The terms ‘context’ and ‘contextual’ are used with a variety of different meanings. In this article they refer to the concepts as they appear in quantum physics, in particular quantum foundations; there is no attempt to relate them to usages which occur in other fields, such as psychology. Unfortunately, even in the restricted domain of quantum foundations one finds more than one usage and, what is equally unfortunate, carelessness on the part of some authors whose definitions are not clearly stated, and sometimes hard to extract from the context (to use a different sense of that word) in which their discussions appear. All of which adds to the general confusion which has given quantum foundations a bad reputation among physicists working in other disciplines. In this paper two quite distinct usages of ‘(non)contextual’ will be discussed. One is due to Bell, and appears in older work on quantum foundations; for this I will use the term ‘Bell (non)contextual’. It is discussed in Secs. 2 to 4. The other is found in Abramsky et al. [1]; I refer to it as ‘global (non)contextual’, and discuss it in Sec. 5. Undoubtedly other definitions are to be found in the literature; I have not attempted a survey.

As the title suggests, my aim is to relate ‘(non)contextual’ to the topic of quantum measurements. Here there is enormous confusion, as exemplified by the well-known measurement problem. Alas, ‘measurement’ is another term often used in quantum foundations without a clear meaning, so of necessity any discussion of how it relates to contextuality must be based on some understanding of, if not solution to, the measurement problem. In fact there are (at least) two measurement problems. The first, widely discussed, problem is how to think about the macroscopic outcome of a measurement, since unitary time evolution often results in a quantum superposition (Schrödinger cat) of possible measurement outcomes. The second, less discussed but equally important, is how to infer from the macroscopic outcome the earlier microscopic property that the apparatus was designed to measure. Both problems are discussed in Sec. 3. The second is the one most intimately connected to notions of quantum contextuality.

The approach employed in the remainder of this paper is, without apology, based upon the consistent histories (CH), or decoherent histories interpretation of quantum mechanics. It is, so far as I know, the only interpretation presently available which can provide a complete solution to both measurement problems. There is no room in a short article to reproduce its essential features; the reader will find an accessible overview in [2], and a helpful introduction to the way it deals with the aforementioned measurement problem(s) in [3–5]. The conclusion in Sec. 4 is that quantum theory is Bell noncontextual, in agreement with earlier work in [6].
and in Sec. V E of [5]. The ‘global contextuality’ (my terminology) defined in [1] is considered in Sec. 5. There is an overall summary in Sec. 6.

2 Bell Contextual

So far as I know, the term ‘contextual’ first appears in relation to quantum foundations in an early paper by Bell [7]. It is not altogether easy to follow Bell’s presentation, but I believe the general idea is the following; see, e.g., the discussion in Sec. VII of [8]. Let $A$, $B$, and $C$ be three quantum observables—for present purposes an ‘observable’ is any Hermitian operator on a (finite-dimensional) Hilbert space—and assume that $A$ commutes with $B$ and with $C$, but $B$ does not commute with $C$. Hence according to a principle of standard (textbook) quantum mechanics, it is possible to measure $A$ along with $B$, or $A$ along with $C$, but one cannot measure all three, $A$, $B$, and $C$ at the same time, i.e., in a single experimental run. Bell noted that for this reason a measurement of $A$ along with $B$ requires a different apparatus than a measurement of $A$ along with $C$, and, with good reason, asked the question: would the value obtained for $A$ be the same when measured with $B$ as when measured with $C$? If the answer is ‘Yes’ then quantum mechanics (or whatever theory one is using to analyze this situation) is noncontextual, whereas if the answer is ‘No’, or at least ‘No’ in certain cases, quantum mechanics (or the theory in question) is contextual in the sense that the measurement outcome for $A$ depends upon whether it is measured along with $B$, the $\{A, B\}$ context or together with $C$, the $\{A, C\}$ context.

This distinction seems clear enough until one gives it careful thought and finds that there are conceptual traps hidden inside what looks at the outset like a straightforward definition. One of these is found in the term ‘measured’: what is a quantum measurement? Does one mean a macroscopic measurement outcome, a pointer position if we use the picturesque, albeit archaic, terminology of quantum foundations? Or is it the prior microscopic property inferred from the pointer position? Let us assume the latter: the apparatus pointer indicates that observable $A$ had the value $A = a_1$ in a particular run of the experiment, one in which $B$ was measured at the same time as $A$ using what we might call the $AB$ apparatus. If, instead, in this particular run, $A$ had been measured along with $C$ using the (necessarily different) $AC$ apparatus, would the $A$ pointer have ended up once again indicating $A = a_1$?

Note that this is not at all the same question as asking whether the probability distribution for $A$ outcomes during a set of repeated $AB$ measurements, all starting with the same initial state for the particle, would be the same as that of the $A$ outcomes during a similar set of $AC$ measurements. Standard textbook quantum mechanics tells us that if we start with a particular initial state the marginal distribution of $A$ outcomes computed from the joint distribution of two commuting observables $A$ and $B$ will be the same as that computed from the joint distribution of $A$ and $C$. The question raised by Bell was not the identity of distributions, but the identity of outcomes in the same run. Since $A$, $B$, and $C$ cannot be measured simultaneously, this is a counterfactual question: $A$ was measured with $B$; what would have been the $A$ outcome had it (contrary to fact) been measured along with $C$?

Thus the question of whether quantum mechanics is or is not Bell contextual has embedded in it some issues in quantum foundations concerning which it is safe to say there is no general agreement. My position regarding them will emerge in the next section; all I can ask
of the sceptical reader is to pay attention to the arguments and try and assess their validity
with an open mind.

3 Quantum Measurements

Consider a projective measurement on a system, hereafter thought of as a particle, of an observable \( A = A^\dagger \) with spectral form

\[
A = \sum_j a_j P_j, \quad \sum_j P_j = I_p \tag{1}
\]

where the \( a_j \) are the distinct eigenvalues of \( A \), thus \( a_j \neq a_k \) if \( j \neq k \), and the \( P_j \) are projectors, \( P_j = P_j^\dagger = P_j^2 \), which are mutually orthogonal: for \( j \neq k \), \( P_j P_k = 0 \). The collection \( \{P_j\} \) of orthogonal projectors which sum to the identity \( I_p \) on the Hilbert space \( \mathcal{H}_p \) of the particle is a projective decomposition of the identity or PDI. The measuring process is assumed to be such that if the particle is initially in some state \( |\psi\rangle \) with the (microscopic) property \( P_j \), i.e., \( P_j|\psi\rangle = |\psi\rangle \), then its interaction with the measurement apparatus will, by unitary time evolution, lead to a state \( |\Psi\rangle \) which lies in a subspace with projector \( M_j \) of the apparatus Hilbert space \( \mathcal{H}_m \), i.e., \( M_j|\Psi\rangle = |\Psi\rangle \). The subspace \( M_j \) corresponds to the (macroscopic) apparatus pointer being in the position \( j \). Since the different pointer positions are macroscopically distinct, we can safely assume that \( M_j M_k = 0 \) for \( j \neq k \), and by adding an additional projector \( M_0 \) to cover all other possibilities (e.g., the apparatus has broken down), we can assume that \( \{M_j\} \) is a projective decomposition of the identity \( I_m \) of the apparatus Hilbert space. Note that we are thinking of the \( \{M_j\} \) as referring to a later time when the measurement is over and the particle has disappeared, or else has become a very small part of what we call the apparatus—this gets around any need to ‘collapse’ the particle wavefunction at the end of the measurement.

Next we need to dispose of the two measurement problems. The first arises when the initial particle state is some superposition of states corresponding to different eigenvalues of \( A \), say

\[
|\psi\rangle = \sum_j r_j |\phi_j\rangle, \quad P_k|\psi_j\rangle = \delta_{jk} |\phi_j\rangle \tag{2}
\]

with at least two of the \( r_j \) unequal to zero. Then unitary time evolution will, as is well-known, lead to a later apparatus state \( |\Psi\rangle \) which no longer falls in just one of the pointer subspaces \( \{M_j\} \), so it is a macroscopic quantum superposition, or in popular terminology a Schrödinger cat. What shall we do with it? Let us follow the advice of Born and use the Schrödinger wave \( |\Psi\rangle \) to assign a probability

\[
p_j = \langle \Psi | M_j | \Psi \rangle \tag{3}
\]

to each measurement outcome. When used in this manner I refer to \( |\Psi\rangle \) as a pre-probability, i.e., something used to generate a probability distribution, as in (3), to be carefully distinguished from the projector \( [\Psi] = |\Psi\rangle \langle \Psi| \) that represents a hard-to-interpret quantum property, the weird ‘cat state’.

The CH interpretation treats quantum mechanics, following Born, as a stochastic theory in which time development must be understood using probabilities. But as in classical
physics, probabilities require a sample space of mutually exclusive possibilities, one and only one of which is thought to take place in a particular run of the experiment. In quantum theory a sample space is always a PDI. The orthogonality of the projectors ensures that only one, and the fact that they sum to the identity means at least one, of these possibilities will occur in any particular situation. In classical physics the choice of a sample space is usually quite straightforward, and if two or more spaces are of interest in some situation it causes no difficulty, as one can always combine them. But in quantum mechanics one needs to make different choices depending on what aspect of a situation one is interested in. Thus let \{P_j\} and \{Q_k\} be two PDIs for the same system. If and only if they are compatible,

\[ P_j Q_k = Q_k P_j \quad \text{for all } j \text{ and } k \]  

(4)
can there be a common refinement, consisting of all the nonzero \(P_j Q_k\); otherwise they are incompatible and cannot be combined. The CH approach resolves (or evades or tames) the standard quantum paradoxes by means of the single framework rule, which prohibits as meaningless those arguments which reach some conclusion by (often implicitly) combining properties belonging to incompatible PDIs or frameworks; the latter term can include PDIs involving events at different times.

An immediate application to the present discussion arises from the fact that the \{M_j\}, the possible measurement outcomes, form a PDI. But there is another PDI with just two projectors, \(|\Psi\rangle\text{ and } I_m - |\Psi\rangle\text{}, with \(|\Psi\rangle = |\Psi\rangle\langle\Psi|\) the weird ‘cat’ property introduced above, and \(I_m\) the identity on the apparatus Hilbert space \(\mathcal{H}_m\). This PDI, \{|\Psi\rangle, I_m - |\Psi\rangle\}, is incompatible, with the \{M_j\} PDI when, as we are assuming, two or more of the \(r_j\) are nonzero. Both PDIs are valid quantum descriptions, but they cannot be combined. One cannot say, at least while being careful in the use of language, that \(|\Psi\rangle\) corresponds to the pointer being simultaneously in several locations, since talk of \(|\Psi\rangle\) renders talk of pointer positions represented by the \(M_j\) meaningless.\footnote{In the quantum logic of Birkhoff and von Neumann \cite{9}, the combination of \(|\Psi\rangle\) and some \(M_j\) is meaningful, but false. This, alas, is not very helpful; see the discussion in Sec. 4.6 of \cite{10}.} In summary, the CH approach resolves the first measurement problem, a Schrödinger cat state \(|\Psi\rangle\) of the output, by employing a framework or PDI \{\(M_k\)\} as the sample space to which \(|\Psi\rangle\) assigns probabilities according to the Born rule, rather than using the PDI \{|\Psi\rangle, I_m - |\Psi\rangle\}, in which talk of the pointer having a position is meaningless.

To justify the belief of the experimental physicist that the apparatus he has carefully constructed to carry out some sort of measurement of a microscopic property does what it was designed to do, we need to solve the second quantum measurement problem. In the case of the projective measurement of \(A\), this means being able to infer from the outcome, the pointer position \(M_j\), in a particular run of the experiment, that in this run the particle actually had the property \(P_j\), corresponding to the eigenvalue \(a_j\) of \(A\), immediately before the measurement took place. The theoretical analysis that justifies this requires the use of quantum histories, and in the interests of brevity we refer the reader to other work \cite{3,5} for details. Experimenters will generally want to calibrate a piece of apparatus to check that it is working properly before using it to gather data. The simplest sort of calibration is to send into the apparatus, on several successive runs, particles which are known to have a specific property \(P_j\), and checking that the pointer always ends up in the corresponding position \(M_j\). After doing this for all the different \(j\) values, the experimenter will feel justified in assuming
that when a particle in an unknown state arrives that causes the pointer to end up at \( M_j \), in this case the particle just before the measurement had the property \( P_j \).

This sounds sensible, but what if the particle was initially prepared not in an eigenstate of \( A \), but instead in some superposition, as in (2), with several nonzero \( r_j \)? Granted, there is no reason, if we employ the framework \( \{ M_j \} \) for analyzing the macroscopic outcome, to suppose the pointer will end up in a weird superposition; instead it will arrive at a specific position, which will vary from run to run. But before the measurement took place, did not the particle have the property \( |\psi\rangle \)? That assumes a PDI \( \{ |\psi\rangle, I_p - |\psi\rangle \} \), but there is an alternative, namely the PDI \( \{ P_j \} \). These two PDIs are incompatible and cannot be combined, and it is the PDI \( \{ P_j \} \) which is useful in answering the question as to whether a given measurement outcome \( M_j \) revealed the prior property \( P_j \). The CH analysis justifies treating \( |\psi\rangle \) in this situation as a pre-probability which can be used to assign a probability

\[
p_j = \langle \psi | P_j | \psi \rangle
\]

to the property \( P_j \), which will later lead (with certainty) to the measurement outcome \( M_j \). Both (3) and (5) are marginals of the joint distribution

\[
\Pr(P_j, M_k) = \delta_{jk} p_j,
\]

which results from a CH analysis employing histories and an extended Born rule. Combining (3) and (6) yields the conditional probability

\[
\Pr(P_k | M_j) = \delta_{jk},
\]

which says that if the pointer ends up at \( M_j \) the particle earlier had the property \( P_j \).

4 Quantum Mechanics is Bell Noncontextual

With a proper quantum-mechanical understanding of how to interpret measurements in a way that makes sense and connects with laboratory practice, we are now in a position to analyze the \( ABC \) situation introduced in Sec. 2. Figure 1 is a schematic diagram of a measurement apparatus. In Fig. 1(a) the particle to be measured enters from the left, the apparatus measures \( A \), and the outcome is indicated by the location of a pointer on the right, which can be in one of several possible positions: the solid arrow represents a particular outcome and the dashed arrows are alternative possibilities. The modified apparatus in Fig. 1(b) has a handle which can be set in one of two positions, \( B \) or \( C \). When in the \( B \) position the apparatus will measure \( B \) at the same time as \( A \), with the value of \( B \) shown by a second pointer; while if the handle is in position \( C \) it measures \( A \) together with \( C \), and the second pointer indicates the measured value of \( C \). Moving the handle changes what happens inside the apparatus, and it can be set at the very last moment, just before the particle enters on the left side.

Needless to say, the careful experimenter will want to make separate sets of calibration runs, one set with the handle in position \( B \), and another set with it in position \( C \). In the first set the particles with eigenvalues \( (a_j, b_k) \) for \( A \) and \( B \) are repeatedly sent into the apparatus
Figure 1: (a) Apparatus to measure $A$ for particle entering from the left, with outcome shown by the pointer on the right, which can be in one of several possible positions. (b) Apparatus which will measure $A$ together with $B$ or with $C$, depending on the setting of the handle, which is shown in position $B$, but can be moved to the $C$ position (dashed). The second pointer indicates the $B$ or the $C$ outcome.

to verify that the $A$ and the $B$ pointers arrive at the correct positions, while the second set uses particles with eigenvalues $(a_j, c_l)$.

With the calibrations completed, consider a run in which the apparatus handle is in position $B$ and at the end of the measurement the first pointer indicates that the particle possessed the property $P_2$ corresponding to the eigenvalue $a_2$, while the second pointer indicates that the particle also had, say, the property $B = b_1$. Now suppose the handle had been in position $C$ rather than $B$ during this run; would the $A$ pointer nonetheless have indicated a prior property $P_2$ corresponding to $a_2$? This is at least plausible, because the handle could have been shifted, from $B$ to $C$, just before the arrival of the particle, and since the particle already had the property $P_2$ before arriving at the apparatus, see the discussion in Sec. 3, it is hard to see how a later shift of the handle could have altered this. It may be worth noting that the counterfactual analysis used in the foregoing discussion is consistent with a proposal made some years ago, [11] and Ch. 19 of [10], and which has stood the test of time; see, for example, a discussion with Stapp concerning nonlocality in [12, 13].

But what about retrocausal influences? Could a later change of the position of the handle have altered an earlier property of the particle? So far as I know, there is not the slightest experimental evidence for such retrocausal influences. Proposals for retrocausality sometimes arise out of discussions of quantum correlations that violate Bell inequalities, along with a desire to avoid attributing them to nonlocal influences; see, e.g., [14]. However, since such correlations have a fairly straightforward explanation in terms of ordinary causes once one has cleared away the fog associated with quantum measurement problems, see [3, 15], there seems at present no reason to take such retrocausation proposals seriously.

Until someone points out a flaw in the argument given above, or in its predecessors in [5, 6], I will continue to maintain that quantum mechanics is Bell noncontextual. Authors who claim that quantum mechanics is contextual and cite Bell’s work to support their claim either have not understood what is at issue, or have not known how to analyze quantum measurements in a consistent and fully quantum mechanical fashion.
5 An Alternative Definition of ‘Contextual’

5.1 Global Contextuality

The term ‘contextual’ has been used in recent years in ways different from Bell’s proposal. In some cases this is simply carelessness, but in others there is a careful definition related to, but distinctly different from, Bell contextuality as exemplified in the ABC setup of Sec. 2. One instance of a different definition is found in the work of Abramsky et al. [1], to which I here attach the name global contextuality in order to distinguish it from Bell’s version.

The definitions in [1] are given in terms of measurements, but if we assume these are projective measurements the definitions can be translated into the language of quantum projectors and subspaces as follows. One starts with a collection $X$ of observables, Hermitian operators all acting on the same Hilbert space, some of which may be incompatible (i.e., fail to commute) with others. A measurement context is a subset of of commuting observables from $X$, and associated with it is a quantum sample space consisting of the common refinement of the PDIs associated with these observables, i.e., all nonzero products of the corresponding projectors; see the discussion connected with (4). One is interested in a particular collection, denoted by $M$ in [1], of measurement contexts, and an empirical model assigns separate probability distributions to each measurement context in $M$, using its associated sample space as defined above. These distributions must satisfy a condition called the compatibility of marginals: any set of commuting observables that lies in the intersection $C_1 \cap C_2$ of two measurement contexts $C_1$ and $C_2$ must be assigned the same probabilities by the two distributions. An empirical model is said to be noncontextual if there is a single global probability distribution for the entire collection of observables which on every measurement context agrees with the probabilities assigned by the empirical model, while if such a global distribution does not exist the empirical model is said to be contextual. The meaning of these terms will (I hope) be made clearer by the specific example considered next.

5.2 An Example

Let us apply the above definition to the $ABC$ case considered earlier in Sec. 2 where $\{A, B\}$ is one measurement context and $\{A, C\}$ a different measurement context. Since $A$ and $B$ commute, the corresponding PDIs, let them be $\{P_j\}$ and $\{Q_k\}$, will have a common refinement whose sample space consists of all the nonzero products $P_jQ_k = Q_kP_j$, and the empirical model assigns a probability distribution, which is to say a nonnegative number $p_{jk}$ to each of these so that they sum to 1. Similarly for the $\{A, C\}$ context: if $\{R_l\}$ is the PDI corresponding to observable $C$, the empirical model will assign probabilities to the nonzero $P_jR_l = R_lP_j$. The two distributions must be mutually consistent (compatibility of the marginals) in that they assign the same probability to each projector $P_j$ that lies in the overlap, which is to say

$$\Pr(P_j) = \sum_k \Pr_1(P_j, Q_k) = \sum_l \Pr_2(P_j, R_l), \quad (8)$$

where $\Pr_1()$ and $\Pr_2()$ are the two probability distributions that together constitute the empirical model; $\Pr_1(P_j, Q_k)$ is the probability of $P_jQ_k$, and $\Pr_2(P_j, R_l)$ the probability
assigned to $P_j R_l$. The empirical model is (in my notation) \textit{globally noncontextual} if and only if there is a global joint probability distribution $\Pr_g(P_j, Q_k, R_l)$ such that

$$\Pr_1(P_j, Q_k) = \sum_l \Pr_g(P_j, Q_k, R_l), \quad \Pr_2(P_j, R_l) = \sum_k \Pr_g(P_j, Q_k, R_l),$$

(9)

This clearly differs from Bell contextuality in that \(8\) refers to a probability distribution of $A$, not the specific value revealed on a particular run of the experiment. But another very important difference lies in the ‘compatibility of marginals’ conditions in \(9\). While $\Pr_1(P_j, Q_k)$ is a well-defined quantum probability that assigns a number to each $P_j Q_k$ of the PDI, and likewise $\Pr_2(P_j, R_l)$ is well-defined, the same in \textit{not} true of $\Pr_g(P_j, Q_k, R_l)$. Since $Q_k$ and $R_l$ will in at least some cases fail to commute, as implied by the assumption $BC \neq CB$, there will be instances in which the product $P_j Q_k R_l$ is not a projector, so we are not dealing with well-defined quantum probabilities.

Consider a specific case of three operators

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

(10)
on a 3-dimensional Hilbert space, where the lower right $2 \times 2$ blocks of $A$, $B$, and $C$ are, respectively, the identity, the $\sigma_x$, and the $\sigma_z$ Pauli matrices. Thus it is obvious that $A$ commutes with both $B$ and $C$, whereas $BC \neq CB$. Each observable has two eigenvalues equal to $+1$ and one equal to $-1$. If probabilities are assigned using the density operator

$$\rho = \begin{pmatrix} p & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{pmatrix}; \quad p \geq 0, r \geq 0; \quad p + 2r = 1;$$

(11)

the joint probabilities of $A$ and $B$, and of $A$ and $C$, are those in the first two boxes in the following table, where $a$, $b$, and $c$ denote eigenvalues of $A$, $B$, and $C$.

$$\begin{array}{|c|c|} \hline a & b = 1 \\ \hline r & -1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline a & c = 1 \\ \hline r & -1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|c|} \hline a & b = c & 1 & -1 \\ \hline r & b \neq c & 0 & 0 \\ \hline \end{array}$$

(12)

The values in the third box represent a global probability distribution chosen to produce the $A, B$ and $A, C$ marginals in the first two boxes. It does not correspond to anything in the quantum Hilbert space since, for example, it assigns to the triple $a = 1$, $b = -1$, $c = -1$ a probability $r$. If it existed the corresponding Hilbert subspace would be one in which, thinking of the lower right $2 \times 2$ blocks as representing a spin-half particle, one would have $S_z = \sigma_z/2 = -1/2$ \textit{AND} $S_x = \sigma_x/2 = -1/2$. There is no such subspace apart from the 0 element, to which one cannot, of course, assign a positive probability. This is consistent with what students learn in an introductory course: the $S_x$ and $S_z$ components of angular momentum of a spin-half particle cannot be measured simultaneously. In other words, this particular empirical model, which is globally noncontextual, does not correspond to anything in Hilbert space quantum physics.
5.3 Probabilities of Measurements?

To be sure, in [1] the probabilities that define the empirical model do not relate in a direct way to microscopic properties; instead they are the probabilities of measurements. But, one may ask, measurements of what? They are clearly not the sorts of measurements discussed in Sec. 3 nor does the treatment in [1] indicate how they might be realized with some form of quantum apparatus (the only kind found in modern laboratories). One response might be that if incompatible observables cannot be measured simultaneously, they could be measured separately in separate runs of the experiment. For example, in some runs the handle in Fig. 1(b) is at B, while in others it is at the C position, with the incoming particle always prepared in the same initial state. Since the different runs are independent, they need not be carried out in succession. One could just as well imagine two pieces of apparatus, the first with the handle always at the B location, the second always at C, that simultaneously carry out measurements on two identically-prepared particles. This joint measurement can be analyzed using an obvious extension of the approach in Sec. 4 and the two pointer readings related to a PDI, a sample space, representing properties before measurement. But now the sample space is the tensor product of those for the individual particles 1 and 2, and if we choose it in such a way that the apparatuses reveal prior properties, it consists of commuting projectors of the form

\[ P_{1j}Q_{1k} \otimes P_{2j'}R_{2l} \]  

in a fairly obvious notation. This, of course, is very different from the nonexistent (from a quantum perspective) sample space of triples that form the arguments of \( \text{Pr}_s() \) in (9).

Granted, the study of probabilities not directly connected to quantum physical reality may nonetheless yield some useful insights as to what goes on in the quantum world, and given the widely acknowledged conceptual difficulties of quantum theory, new sources of insight are welcome. But to avoid adding further confusion to the confused state of quantum foundations, it would be valuable if discussions of this sort were to clearly distinguish ideas and interpretations that apply directly to quantum mechanics from those in which one is, in effect, employing classical models (in the present instance ‘empirical models’) in place of quantum physics.

6 Summary and Conclusion

The careful analysis in Sec. 3 of what it is that quantum measurements actually measure is needed to address the question of whether quantum mechanics is or is not contextual in the sense originally introduced by Bell, Sec. 2. Given that \( A \) was measured together with a compatible (AB=BA) observable \( B \), the \( \{A,B\} \) context for \( A \), would the outcome for \( A \) in this particular run of the experiment have been the same if, instead, \( A \) had been measured together with a compatible (AC = CA) observable \( C \), the \( \{A,C\} \) context for \( A \)? The question is of interest when \( B \) and \( C \) are incompatible observables, \( BC \neq CB \), so \( A \), \( B \), and \( C \) cannot be measured together in a single experiment. When an analysis of the situation is carried out, Sec. 3 using the consistent histories interpretation of quantum mechanics, which is capable of describing the entire measurement process in fully quantum-mechanical terms, the conclusion, Sec. 4 is clear: If the apparatus, properly constructed and calibrated,
indicates that \( A \) had a particular value, say \( a_1 \), prior to the measurement of \( A \) together with \( B \), then in this particular run the same result for \( A \) would have been obtained had \( A \) been measured along with \( C \), or with any other observable compatible with \( A \). Thus quantum mechanics is noncontextual, or, to be more precise, ‘Bell noncontextual’, if one uses Bell’s definition. It is worth noting that were quantum theory Bell contextual it would cast grave doubt on the results of experiments, since equipment is typically designed to measure some specific property without concern about what other compatible quantities might happen to be measured at the same time.

However, there are other definitions of contextuality, and a version found in [1] and discussed in Sec. 5 to which I have given the name global contextuality, differs from Bell contextuality in two important respects. First, rather than referring to a particular outcome of the \( A \) measurement, the focus is on the probability distribution of these outcomes. Second, ‘noncontextual’ is defined in terms of the existence (and ‘contextual’ in terms of the nonexistence) of an ‘empirical model’, understood as a single global probability distribution which assigns (marginal) probability distributions to all of the observables under consideration, whether or not they commute. The example in Sec. 5.2 of such an empirical model for three observables in a 3-dimensional Hilbert space does not make quantum-mechanical sense in that there is no quantum sample space (projective decomposition of the identity) to which the probabilities of the empirical model can be assigned. If, on the other hand, one supposes that various incompatible measurements are carried out in different runs of an experiment, the corresponding quantum sample space, using tensor products of quantum properties, Sec. 5.3 does not support the definition of an ‘empirical model’ given in [1]. Thus while global contextuality is a well-defined mathematical concept it is not clear that it represents anything of physical significance in the quantum world. It may nonetheless be a source of useful physical insights, but there is also the danger that it might lead, as have various other classical concepts applied in the quantum domain, to yet more paradoxes and confusion.

What Bell and global (non)contextuality have in common is the notion that a PDI associated with an observable can lie in the intersection of two or more incompatible PDIs, or ‘contexts’ (or ‘frameworks’ in the language of CH), and that one should pay attention to this crucial respect in which quantum theory differs from classical physics. What has been lacking in many previous discussions, and which I have attempted to supply, is a description of the simplest form of quantum measurement, a projective measurement\(^2\) in order to understand what is actually measured; i.e., what the macroscopic outcome of the measurement reveals about the microscopic state of the measured system before the measurement took place. A large number of paradoxes in quantum foundations arise from a failure to analyze quantum measurements in fully quantum-mechanical terms: i.e., Hilbert space rather than, say, the (classical) hidden variables employed by Bell and his followers. For example, claims, based on violations of Bell inequalities, that the quantum world is nonlocal evaporate when these classical variables are replaced by a proper quantum analysis [3,15,16]. Similarly, discussions of contextuality, at least to the extent that they relate to measurements, need to be based on a clear understanding of the quantum measurement process.

Finally, to repeat what was stated in Sec. 1 the analysis presented here is concerned

\(^2\)An extension to POVMs, ‘generalized measurements’, will be found in [5].
only with the use of ‘(non)contextual’ in quantum physics. Whether or not, and if so how, these ideas carry over to the use of (non)contextuality in psychology is something I must leave to those far more knowledgeable about that subject than I.

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