Irreducible symplectic 4-folds and Eisenbud-Popescu-Walter sextics

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Abstract

Eisenbud Popescu and Walter have constructed certain special sextic hypersurfaces in \( \mathbb{P}^5 \) as Lagrangian degeneracy loci. We prove that the natural double cover of a generic EPW-sextic is a deformation of the Hilbert square of a \( K3 \) surface \( (K3)^2 \) and that the family of such varieties is locally complete for deformations that keep the hyperplane class of type \((1,1)\) - thus we get an example similar to that (discovered by Beauville and Donagi) of the Fano variety of lines on a cubic 4-fold. Conversely suppose that \( X \) is a numerical \( (K3)^2 \), that \( H \) is an ample divisor on \( X \) of square 2 for Beauville’s quadratic form and that the map \( X \to H^\vee \) is the composition of the quotient \( X \to Y \) for an anti-symplectic involution on \( X \) followed by an immersion \( Y \hookrightarrow H^\vee \); then \( Y \) is an EPW-sextic and \( X \to Y \) is the natural double cover. If a conjecture on the behaviour of certain linear systems holds this result together with previous results of ours implies that every numerical \( (K3)^2 \) is a deformation of \( (K3)^2 \).

1 Introduction

A compact Kähler manifold is irreducible symplectic if it is simply connected and it carries a holomorphic symplectic form spanning \( H^{2,0} \) (see [1, 7]). An irreducible symplectic surface is nothing else but a \( K3 \) surface. Higher-dimensional irreducible symplectic manifolds behave like \( K3 \) surfaces in many respects [7, 8] however their classification up to deformation of complex structure is out of reach at the moment. Let \( S \) be a \( K3 \); the Hilbert square \( S^{[2]} \) i.e. the blow-up of the diagonal in the symmetric square \( S^{(2)} \) is the simplest example of an irreducible symplectic 4-fold. An irreducible symplectic 4-fold \( M \) is a numerical \( (K3)^2 \) if there exists an isomorphism of abelian groups

\[
\psi : H^2(M; \mathbb{Z}) \xrightarrow{\sim} H^2(S^{[2]}; \mathbb{Z}) \tag{1.1}
\]

such that

\[
\int_M \alpha^4 = \int_{S^{[2]}} \psi(\alpha)^4 \tag{1.2}
\]

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for all $\alpha \in H^2(M; \mathbb{Z})$\textsuperscript{1}. In \textsuperscript{12} we studied numerical $(K3)^{[2]}$’s with the goal of classifying them up to deformation of complex structure. We proved that any numerical $(K3)^{[2]}$ is deformation equivalent to an $X$ carrying an ample divisor $H$ such that

$$\int_X c_1(H)^4 = 12, \quad \dim |H| = 5$$

(1.3)

(the first equation is equivalent to $c_1(H)$ being of square 2 for the Beauville form), and such that the rational map

$$X \dashrightarrow |H|^\vee$$

(1.4)

satisfies one of the following two conditions:

(a) There exist an anti-symplectic involution $\phi: X \rightarrow X$ with quotient $Y$ and an embedding $Y \hookrightarrow |H|^\vee$ such that (1.4) is the composition of the quotient map $f: X \rightarrow Y$ and the embedding $Y \hookrightarrow |H|^\vee$.

(b) Map (1.4) is birational onto a hypersurface of degree between 6 and 12.

In this paper we describe explicitly all the $X$ occurring in Item (a) above. Notice that $Y$ is singular because smooth hypersurfaces in $\mathbb{P}^5$ are simply connected. Moreover the singular locus is a surface because $\phi$ is anti-symplectic. Thus $Y$ is far from being a generic sextic hypersurface; we will show that it belongs to a family of sextics constructed by Eisenbud, Popescu and Walter, see Example (9.3) of \textsuperscript{4}. We will prove that conversely a generic EPW-sextic has a natural double cover which is a deformation of $(K3)^{[2]}$. Since EPW-sextics form an irreducible family we get that the $X$’s satisfying (a) above are deformation equivalent. Actually if $(X_i, f_i^* O_Y(1))$ are two polarized couples where $f_i: X_i \rightarrow Y_i$ satisfy (a) above for $i = 1, 2$ then we may deform $(X_1, f_1^* O_Y(1))$ to $(X_2, f_2^* O_Y(1))$ through polarized deformations. In particular all the explicit examples of $f: X \rightarrow Y$ satisfying (a) above that were constructed in \textsuperscript{11} are equivalent through polarized deformations - this answers positively a question raised in Section (6) of \textsuperscript{11}. We recall that no examples are known of $X$ satisfying Item (b) above; in \textsuperscript{12} we conjectured that they do not exist - one result in favour of the conjecture is that if $X$ satisfies (a) above then all small deformations of $X$ keeping $c_1(f^* O_Y(1))$ of type $(1, 1)$ also satisfy (a) above see the proposition at the end of Section (4) of \textsuperscript{12}. If our conjecture is true then the results of this paper together with the quoted results of \textsuperscript{12} give that numerical $(K3)^{[2]}$’s are deformation equivalent to the Hilbert square of a $K3$. Before stating precisely our main results we recall the construction of EPW-sextics. Let $V$ be a 6-dimensional vector space and $\mathbb{P}(V)$ be the projective space of 1-dimensional sub vector spaces $\ell \subset V$. Choose an isomorphism $\text{vol}: \wedge^6 V \xrightarrow{\sim} \mathbb{C}$ and let $\sigma$ be the symplectic form on $\wedge^3 V$ defined by wedge product, i.e. $\sigma(\alpha, \beta) := \text{vol}(\alpha \wedge \beta)$; thus $\wedge^3 V \otimes O_{\mathbb{P}(V)}$ has the structure of a symplectic vector-bundle of rank 20. Let $F$ be the sub-vector-bundle of $\wedge^3 V \otimes O_{\mathbb{P}(V)}$ with fiber over $\ell \in \mathbb{P}(V)$ equal to

$$F_\ell := \text{Im} \left( \ell \otimes \wedge^2 (V/\ell) \hookrightarrow \wedge^3 V \right).$$

(1.5)\footnote{For the experts: the Beauville quadratic form and the Fujiki constant of $M$ are the same as those of $S^{[3]}$.}
Thus we have an exact sequence
\[ 0 \to F \xrightarrow{\rho} \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \]  
(1.6)
We have \( rk(F) = 10 \) and \( \sigma|_{F_1} = 0 \); thus \( F \) is a Lagrangian sub-bundle of \( \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \). Let \( LG(\wedge^3 V) \) be the symplectic Grassmannian parametrizing \( \sigma \)-Lagrangian subspaces of \( \wedge^3 V \). For \( A \in LG(\wedge^3 V) \) we let
\[ \lambda_A : F \to (\wedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)} \]  
(1.7)
be Inclusion \((1.6)\) followed by the projection \((\wedge^3 V) \otimes \mathcal{O}_{\mathbb{P}(V)} \to (\wedge^3 V/A) \otimes \mathcal{O}_{\mathbb{P}(V)} \). Since the vector-bundles appearing in \((1.7)\) have equal rank we have \( \det(\lambda_A) \in H^0((\det F)^{-1}) \). We let \( Y_A \subset \mathbb{P}(V) \) be the zero-scheme of \( \det(\lambda_A) \). Let \( \omega := c_1(\mathcal{O}_{\mathbb{P}(V)}(1)) \); a straightforward computation gives that
\[ c(F) = 1 - 6\omega + 18\omega^2 - 34\omega^3 + \ldots \]  
(1.8)
In particular \( \det F \cong \mathcal{O}_{\mathbb{P}(V)}(-6) \). Thus \( Y \) is always non-empty and if \( Y \neq \mathbb{P}(V) \) then \( Y \) is a sextic hypersurface. An \( EPW \)-\textit{sextic} is a hypersurface in \( \mathbb{P}(V) \) which is equal to \( Y_A \) for some \( A \in LG(\wedge^3 V) \). In Section \( 4 \) we describe explicitly the non-empty Zariski-open \( LG(\wedge^3 V)^0 \subset LG(\wedge^3 V) \) parametrizing \( A \) such that the following hold: \( Y_A \) is a sextic hypersurface smooth at all points where the map \( \lambda_A \) of \((1.7)\) has corank 1, the analytic germ \( (Y_A, \ell) \) at a point \( \ell \) where \( \lambda_A \) has corank 2 is isomorphic to the product of a smooth 2-dimensional germ times an \( A_1 \)-singularity and furthermore \( \lambda_A \) has corank at most 2 at all points of \( \mathbb{P}(V) \).

Let \( A \in LG(\wedge^3 V)^0 \); then \( Y_A \) supports a quadratic sheaf as defined by Casnati and Catanese \( 3 \) and hence there is a natural double cover \( X_A \to Y_A \) with \( X_A \) smooth - see Section \( 4 \). In Section \( 5 \) we will prove the following result.

**Theorem 1.1.** Keep notation as above. Then the following hold.

(1) Suppose that \( X, H \) are a numerical \( (K3)^2 \) and an ample divisor on \( X \) such that both \((1.3)\) and Item (a) above hold. Then there exists \( A \in LG(\wedge^3 V)^0 \) such that \( f : X \to Y \) is identified with the natural double cover \( X_A \to Y_A \).

(2) For \( A \in LG(\wedge^3 V)^0 \) let \( X_A \to Y_A \) be the natural double cover defined in Section \( 4 \) and let \( H_A \) be the pull-back to \( X_A \) of \( \mathcal{O}_{Y_A}(1) \). Then \( X_A \) is an irreducible symplectic variety deformation equivalent to \( (K3)^2 \) and both \((1.3)\) and Item (a) above hold with \( X = X_A \) and \( H = H_A \).

We recall that Beauville-Donagi \( 2 \) proved the following result: if \( Z \subset \mathbb{P}^5 \) is a smooth cubic hypersurface the Fano variety \( F(Z) \) parametrizing lines on \( Z \) is a deformation of \( (K3)^2 \). They also proved that the family of \( F(Z) \)'s polarized by the Plücker line-bundle is locally complete. Similarly the family of \( X_A \)'s that one gets by letting \( A \) vary in \( LG(\wedge^3 V)^0 \) is also a locally complete family of polarized varieties by the proposition at the end of Section (4) of \( 12 \) that we have already quoted. This is confirmed by the following computation. The tangent space to \( LG(\wedge^3 V) \) at a point \( A \) is isomorphic to \( Sym_2 A^\vee \) and hence \( \dim LG(\wedge^3 V) = 55 \). Since \( \dim \mathbb{P}GL(V) = 35 \) we get that \( \dim(LG(\wedge^3 V)/\mathbb{P}GL(V)) = 20 \) which is the number of moduli of a polarized deformation of \( (K3)^2 \). We remark that the Beauville-Donagi family and the family of \( X_A \)'s are the only explicit examples of a locally complete family of higher dimensional polarized irreducible symplectic
varieties. Notice that our conjecture amounts to the statement that the family of $X_A$'s is globally complete once we take into account the limiting $X_A$'s one gets for $A \in (\mathbb{L}G(\wedge^3 V) \setminus \mathbb{L}G(\wedge^3 V)^0)$. An interesting feature of EPW-sextics is that they are preserved by the duality map i.e. the dual of an EPW-sextic $Y_A$ is an EPW-sextic, see Section 3. Thus duality defines a regular involution on an open dense subset of the moduli space of numerical $(K3)^2$'s polarized by a divisor $H$ satisfying $\mathcal{L} \approx H^2$ and Item (a) above, see Section 6. It would be interesting to know explicitly which $A \in \mathbb{L}G(\wedge^3 V)$ correspond to special 4-folds e.g. Hilbert squares of a $K3$; we discuss this problem in Section 7.

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### 2 EPW-sextics

We will explicitly describe those EPW-sextics whose only singularities are the expected ones - the main result is Proposition 2.8. We start by recalling (see [1] [5]) how one defines natural subschemes $D_i(A,F) \subset \mathbb{P}(V)$ such that

$$\text{supp} D_i(A,F) = \{ \ell \in \mathbb{P}(V) \mid \dim(F_\ell \cap A) \geq i \}.$$ (2.1)

**Definition 2.1.** Let $U \subset \mathbb{P}(V)$ be an open subset. A symplectic trivialization of $\wedge^3 V \otimes O_U$ consists of a couple $(\mathcal{L}, \mathcal{H})$ of trivial transversal Lagrangian sub-vector-bundles $\mathcal{L}, \mathcal{H} \subset \wedge^3 V \otimes O_U$.

Let $(\mathcal{L}, \mathcal{H})$ be a symplectic trivialization of $\wedge^3 V \otimes O_U$. The symplectic form $\sigma$ defines an isomorphism

$$\mathcal{H} \xrightarrow{\sim} \mathcal{L}^\vee, \quad \alpha \mapsto \sigma(\alpha, \cdot)$$ (2.2)

and hence we get a direct sum decomposition

$$\mathcal{L} \oplus \mathcal{L}^\vee = \wedge^3 V \otimes O_U.$$ (2.3)

Conversely a direct sum decomposition (2.3) with $\mathcal{L}, \mathcal{L}^\vee$ trivial Lagrangian sub-vector-bundles such that $\sigma$ induces the tautological isomorphism $\mathcal{L}^\vee \xrightarrow{\sim} \mathcal{L}^\vee$ gives a symplectic trivialization of $\wedge^3 V \otimes O_U$; this is how we usually present a symplectic trivialization of $\wedge^3 V \otimes O_U$.

**Claim 2.2.** Let $A \in \mathbb{L}G(\wedge^3 V)$ and $\ell_0 \in \mathbb{P}(V)$. There exists an open affine $U \subset \mathbb{P}(V)$ containing $\ell_0$ and a symplectic trivialization (2.3) of $\wedge^3 V \otimes O_U$ such that for every $\ell \in U$ both $A$ and $F_\ell$ are transversal to $\mathcal{L}_\ell^\vee$.

**Proof.** The set of Lagrangian subspaces of $(\wedge^3 V)$ which are transversal to a given Lagrangian subspace is an open dense subset of $\mathbb{L}G(\wedge^3 V)$. Thus there exists $C \in \mathbb{L}G(\wedge^3 V)$ which is transversal both to $A$ and to $F_{\ell_0}$. Since the condition of being transversal is open there exists an open affine $U \subset \mathbb{P}(V)$ containing $\ell_0$ such that $F_\ell$ is transversal to $C$ for all $\ell \in U$. Let $B \in \mathbb{L}G(\wedge^3 V)$ be transversal to $C$; thus

$$\wedge^3 V = B \oplus C$$ (2.4)

\[\text{The second equation of (1.3) follows from the first equation - see [12].}\]
and hence \((B \otimes O_U, C \otimes O_U)\) is a symplectic trivialization of \(\wedge^3 V \otimes O_U\). Letting \(\mathcal{L} := B \otimes O_U\) we have an isomorphism \(\mathcal{L}' \cong C \otimes O_U\) induced by \(\sigma\) and we write the chosen symplectic trivialization as \((2.4)\); by construction both \(A\) and \(F_i\) are transversal to \(\mathcal{L}'_j\) for every \(\ell \in U\).

Choose \(A \in \mathbb{L}G(\wedge^3 V)\). Let \(\ell_0 \in \mathbb{P}(V)\). Assume that we have \(U\) and a symplectic trivialization of \(\wedge^3 V \otimes O_U\) as in the claim above. We will define a closed degeneracy subscheme \(D_i(A, F, U, \mathcal{L}, \mathcal{L}') \subset U\) - after doing this we will define the subcheme \(D_i(A, F) \subset \mathbb{P}(V)\) by gluing together the local degeneracy loci. Via \((2.3)\) we may identify both \(A \otimes O_U\) and \(F_i|_U\) with the graphs of maps

\[
q_A, q_F : \mathcal{L} \to \mathcal{L}'
\]

because for every \(\ell \in U\) both \(A\) and \(F_i\) are transversal to \(\mathcal{L}'_j\). Since both \(A \otimes O_U\) and \(F_i|_U\) are Lagrangian sub-vector-bundles the maps \(q_A\) and \(q_F\) are symmetric. Choosing a trivialization of \(\mathcal{L}\) we view \(q_A, q_F\) as symmetric 10 \(\times\) 10 matrices with entries in \(\mathbb{C}[U]\).

**Definition 2.3.** Keep notation as above. We let \(D_i(A, F, U, \mathcal{L}, \mathcal{L}') \subset U\) be the closed subscheme defined by the vanishing of determinants of \((11-i)\times(11-i)\)-minors of \((q_A - q_F)\).

Notice that the definition above makes sense because if we change the trivialization of \(\mathcal{L}\) the relevant determinants are multiplied by units of \(\mathbb{C}[U]\).

**Lemma 2.4.** Let \(U_1, U_2 \subset \mathbb{P}(V)\) be open affine and \(\mathcal{L}_1 \oplus \mathcal{L}'_1 = \wedge^3 V \otimes O_{U_1}\) be symplectic trivializations. Then

\[
D_i(A, F, U_1, \mathcal{L}_1, \mathcal{L}'_1) \cap U_1 \cap U_2 = D_i(A, F, U_2, \mathcal{L}_2, \mathcal{L}'_2) \cap U_1 \cap U_2.
\]

**Proof.** It suffices to prove the lemma for \(U_1 = U_2 = U\). The constructions above can be carried out more generally for a trivialization \(\mathcal{V} \oplus \mathcal{W} \cong \wedge^3 V \otimes O_U\) where \(\mathcal{V}, \mathcal{W} \subset \wedge^3 V \otimes O_U\) are trivial rank-10 sub-vector-bundles (not necessarily Lagrangian) such that for every \(\ell \in U\) both \(A\) and \(F_i\) are transversal to \(\mathcal{W}_i\). We identify \(A \otimes O_U\) and \(F_i|_U\) with the graphs of maps \(q_A : \mathcal{V} \to \mathcal{W}\) and \(q_F : \mathcal{V} \to \mathcal{W}\) respectively - notice that in general it does not make sense to ask whether \(q_A, q_F\) are symmetric! Trivializing \(\mathcal{V}\) and \(\mathcal{W}\) we view \(q_A, q_F\) as 10 \(\times\) 10 matrices with entries in \(\mathbb{C}[U]\). We let \(D_i(A, F, U, \mathcal{V}, \mathcal{W}) \subset U\) be the subscheme defined by the vanishing of determinants of \((11-i)\times(11-i)\)-minors of \((q_A - q_F)\). One checks easily that if we change \(\mathcal{V}\) (leaving \(\mathcal{W}\) fixed) or if we change \(\mathcal{W}\) (leaving \(\mathcal{V}\) fixed) the scheme \(D_i(A, F, U, \mathcal{V}, \mathcal{W})\) remains the same. The lemma follows immediately.

Now we define \(D_i(A, F)\). Consider the collection of symplectic trivializations \(\mathcal{L}_j \oplus \mathcal{L}'_j = \wedge^3 V \otimes O_U\), with \(U_j \subset \mathbb{P}(V)\) open affine and the corresponding closed subschemes \(D_i(A, F, U_j, \mathcal{L}_j, \mathcal{L}'_j)\). By Claim \((2.2)\) the subsets \(U_j\) cover \(\mathbb{P}(V)\) and by Lemma \((2.4)\) the \(D_i(A, F, U_j, \mathcal{L}_j, \mathcal{L}'_j)\) for different \(j\)'s match on overlaps; thus they glue together and they define a closed subscheme \(D_i(A, F) \subset \mathbb{P}(V)\). Clearly \((2.1)\) holds and furthermore \(D_{i+1}(A, F)\) is a subscheme of \(D_i(A, F)\). We claim that

\[
Y_A := D_1(A, F).
\]

It is clear from \((2.1)\) that \(supp(Y_A) = supp(D_1(A, F))\) and hence we need only check that the scheme structures coincide in a neighborhood of any point \(\ell_0 \in \mathbb{P}(V)\).
There exists $B \in LG(\wedge^3 V)$ which is transversal both to $F_{t_0}$ and $A$. There is an open neighborhood $U$ of $t_0$ such that $B$ is transversal to $F_{\ell}$ for $\ell \in U$. The symplectic form $\sigma$ defines an isomorphism $B \cong A^\vee$. Consider the symplectic trivialization of $\wedge^3 V \otimes O_U$ given by $L := A \otimes O_U$ and $L^\vee := B \otimes O_U$; since $q_A = 0$ we have $D_1(A, F, U, L, L^\vee) = Y_A \cap U$ and we are done. To simplify notation we let

$$W_A := D_2(A, F).$$

As shown in Section 11 - see 11.4 - $Y_A$ is never empty. We claim that $W_A$ is never empty as well. In fact Formula (6.7) of [5] and Equation (1.8) give that if $W_A$ has the expected dimension i.e. 2 (see Equations (2.13)-(2.14)) or is empty then the cohomology class of the cycle $[W_A]$ is

$$cl([W_A]) = 2c_3(F) - c_1(F)c_2(F) = 40w^3.\quad (2.9)$$

Since the right-hand side of the above equation is non-zero it follows that necessarily $W_A \neq \emptyset$.

**Definition 2.5.** Let $LG(\wedge^3 V)^\times \subset LG(\wedge^3 V)$ be the set of $A$ such that for all $\ell \in \mathbb{P}(V)$ we have

$$\dim A \cap F_{\ell} \leq 2,\quad (2.10)$$

i.e. $D_3(A, F) = \emptyset$. Let $LG(\wedge^3 V)^0 \subset LG(\wedge^3 V)^\times$ be the set of $A$ which do not contain a non-zero completely decomposable element $v_0 \wedge v_1 \wedge v_2$.

A straightforward dimension count gives the following result.

**Claim 2.6.** Both $LG(\wedge^3 V)^\times$ and $LG(\wedge^3 V)^0$ are open dense subsets of $LG(\wedge^3 V)$.

We will show that $LG(\wedge^3 V)^0$ is the open subset of $LG(\wedge^3 V)$ parametrizing $A$ such that $Y_A$ and $W_A$ are as nice as possible. First we describe $\Theta_{t_0} D_3(A, F)$ at a point $t_0 \in D_3(A, F)$. Proceeding as in the proof of Claim (2.2) we consider a symplectic trivialization

$$\wedge^3 V \otimes O_U = A \otimes O_U \oplus A^\vee \otimes O_U\quad (2.11)$$

where $U \subset \mathbb{P}(V)$ is a suitable open affine subset containing $t_0$ and $A^\vee \in LG(\wedge^3 V)$ is transversal to $A$ and to $F_{\ell}$ for every $\ell \in U$. We view $F \otimes O_U$ as the graph of a symmetric map $q_F: A \otimes O_U \to A^\vee \otimes O_U$. Let

$$U \xrightarrow{\psi} Sym_2 A^\vee \quad \ell \mapsto \psi(\ell) := q_F(\ell)\quad (2.12)$$

and let $\Sigma_i \subset Sym_2 A^\vee$ be the closed subscheme parametrizing quadratic forms of corank at least $i$. Since the map $q_A: A \otimes O_U \to A^\vee \otimes O_U$ whose graph is $A \otimes O_U$ is zero we have

$$D_i(A, F) \cap U = \psi^* \Sigma_i.\quad (2.13)$$

We recall that $\Sigma_i$ is an irreducible local complete intersection with

$$cod(\Sigma_i, Sym_2 A^\vee) = i(i + 1)/2.\quad (2.14)$$


Let $\overline{r} \in (\Sigma_i \setminus \Sigma_{i+1})$. Then $\Sigma_i$ is smooth at $\overline{r}$ and the tangent space $\Theta_{\overline{r}}\Sigma_i$ is described as follows. Identify $Sym_2 A^\vee$ with its tangent space at $\overline{r}$, then

$$\Theta_{\overline{r}}\Sigma_i = \{ q \in Sym_2 A^\vee \mid q|_{\ker(\overline{r})} = 0 \}.$$  \hfill (2.15)

Thus we also get a natural identification

$$(N_{\Sigma_i/ Sym_2 A^\vee})_{\overline{r}} = Sym_2 \ker(\overline{r}).$$  \hfill (2.16)

**Lemma 2.7.** Keep notation as above. Suppose that $A \in \mathcal{LG}(\wedge^3 V)^\times$. Let $\ell_0 \in D(A,F)$ and let

$$A \cap F_{\ell_0} = \ell_0 \otimes W$$  \hfill (2.17)

where $W \subset \wedge^2(V/\ell_0)$. The composition

$$\Theta_{\ell_0}\mathbb{P}(V) \xrightarrow{d\psi(\ell_0)} Sym_2 A^\vee \rightarrow Sym_2 \ker(\ell_0)^\vee$$  \hfill (2.18)

is surjective if and only if $W$ contains no non-zero decomposable element.

**Proof.** Let $\ell_0 = \mathbb{C}t_0$. Choose a codimension 1 subspace $V_0 \subset V$ transversal to $\ell_0$. Thus we have

$$V/\ell_0 \cong \oplus V_0, \quad W \subset \wedge^2 V_0.$$  \hfill (2.19)

The map

$$
\begin{array}{cccc}
V_0 & \rightarrow & \mathbb{P}(V) \\
\alpha & \mapsto & [v_0 + \alpha]
\end{array}
$$  \hfill (2.20)

gives an isomorphism between $V_0$ and an open affine subspace of $\mathbb{P}(V)$ containing $\ell_0$ - with $0 \in V_0$ corresponding to $\ell_0$. Shrinking $U$ if necessary we may assume that $U \subset V_0$ is an open subset containing $0$. For $u \in U$ the map $\psi(u) : A \rightarrow A^\vee$ is characterized by the equation

$$\alpha + \psi(u)(\alpha) = (v_0 + u) \wedge \gamma(u,\alpha), \quad \gamma(u,\alpha) \in \wedge^2 V_0$$  \hfill (2.21)

where $\alpha \in A$. Thus when we view $\psi(u)$ as a symmetric bilinear form we have the formula

$$\psi(u)(\alpha,\beta) = \text{vol}((v_0 + u) \wedge \gamma(u,\alpha) \wedge \beta)$$  \hfill (2.22)

for $\alpha, \beta \in A$. Now assume that $\alpha, \beta \in \ker(\ell_0)$ and hence

$$\alpha = v_0 \wedge \alpha_0, \quad \beta = v_0 \wedge \beta_0, \quad \alpha_0, \beta_0 \in \wedge^2 V_0.$$  \hfill (2.23)

Let

$$\tau \in \Theta_{\ell_0} U \cong V_0$$  \hfill (2.24)

and let $u(t)$ be a “parametrized curve” in $U$ with $u(0) = 0$ and $\dot{u}(0) = \tau$. Then

$$d\psi(\tau)(v_0 \wedge \alpha_0, v_0 \wedge \beta_0) = \frac{d}{dt}|_{t=0} \text{vol}((v_0 + u(t)) \wedge \gamma(u(t),v_0 \wedge \alpha_0) \wedge v_0 \wedge \beta_0).$$  \hfill (2.25)

Differentiating and observing that $\gamma(0,v_0 \wedge \alpha_0) = \alpha_0$ we get that

$$d\psi(\tau)(v_0 \wedge \alpha_0,v_0 \wedge \beta_0) = -\text{vol}(v_0 \wedge \tau \wedge \alpha_0 \wedge \beta_0).$$  \hfill (2.26)

Let’s prove the proposition. If $i = 0$ there is nothing to prove. If $i = 1$ let $\ker(\ell_0) = \mathbb{C}v_0 \wedge \alpha_0$. We apply the above formula with $\beta_0 = \alpha_0$; since $\tau \in V_0$ is
exists a decomposable non-zero $\gamma$ and hence for all $\gamma$ is surjective. Given Formula (2.26) this implies surjectivity of Composition (2.18).

We claim that $\rho$ is injective. In fact assume that we have $[a] \neq [b]$ and $[a \wedge \alpha] = [b \wedge \beta]$. Then $\text{span}(a) = \text{span}(\beta) = S$ where $S \subset V_0$ is a subspace of dimension 4. Since $\alpha, \beta$ span $W$ we get that $\text{span}(\gamma) = S$ for all $\gamma \in W$. Thus

$$\gamma \wedge \gamma \in \text{Im}(\wedge^2 S \to \wedge^2 V_0)$$

(2.28)

for all $\gamma \in W$. Since $G_r(2,S) \subset \mathbb{P}(\wedge^2 S)$ is a hypersurface we get that there exists a decomposable non-zero $\gamma \in W$, contradiction. Thus $\rho$ is injective; since $\rho$ is defined by quadratic polynomials we get that $\text{Im}(\rho)$ is a conic in $\mathbb{P}(\wedge^2 V_0)$ and hence

$$\wedge^4 V_0^\vee \to H^0(\mathcal{O}_{\mathbb{P}(W)}(2))$$

(2.29)

is surjective. Given Formula (2.26) this implies surjectivity of Composition (2.18). \qed

**Proposition 2.8.** Keep notation as above. Suppose that $A \in \mathbb{L}G(\wedge^3 V)^\times$. The following statements are equivalent.

1. $A \in \mathbb{L}G(\wedge^3 V)^0$.
2. $(D_i(A,F) \setminus D_{i+1}(A,F))$ is smooth of codimension equal to the expected codimension $i(i + 1)/2$ for all $i$.
3. $(Y_A \setminus W_A)$ is smooth and for every $\ell_0 \in W_A$ the following holds. There exist $U \subset \mathbb{P}(V)$ open in the analytic topology containing $\ell_0$ and $x,y,z$ holomorphic functions on $U$ vanishing at $\ell_0$ with $dx(\ell_0), dy(\ell_0), dz(\ell_0)$ linearly independent such that $Y_A \cap U = V(xz - y^2)$.

**Proof.** We prove equivalence of (1) and (2) - the proof of equivalence of (1) and (3) is similar, we leave it to the reader. Let $\ell_0 \in (D_i(A,F) \setminus D_{i+1}(A,F))$. Since we have Equation (2.14) and since $(\Sigma_i \setminus \Sigma_{i+1})$ is smooth of codimension $i(i + 1)/2$ (see Equation (2.14)) $D_i(A,F)$ is smooth of codimension $i(i + 1)/2$ at $\ell_0$ if and only if Composition (2.18) is surjective. Let $\ell_0 = C_{v_0}$. By Lemma (2.7) Composition (2.18) is surjective if and only if $A$ does not contain a non-zero completely decomposable element divisible by $v_0$, i.e. of the form $v_0 \wedge v_1 \wedge v_2$.

The proposition follows immediately. \qed

**Remark 2.9.** Let $A \in \mathbb{L}G(\wedge^3 V)^0$.

1. By Proposition (2.8) $Y_A$ has canonical singularities; since $Y_A$ is a sextic adjunction gives that $Y_A$ has Kodaira dimension 0.
2. The family of $Y_A$’s (for $A \in \mathbb{L}G(\wedge^3 V)^0$) is locally trivial.
3 The dual of an EPW-sextic

Let \( A \in LG(\wedge^3 V) \) be such that \( Y_A \) is a reduced hypersurface. For \( \ell \in Y_A^{nm} \) a smooth point of \( Y_A \) let \( T_\ell Y_A \subset \mathbb{P}(V^\vee) \) be the projective tangent space to \( Y_A \) at \( \ell \); the dual \( Y_A^\vee \subset \mathbb{P}(V^\vee) \) is (as usual) the closure of \( \bigcup_{\ell \in Y_A^{nm}} T_\ell Y_A \). We will show that if \( A \) is generic then \( Y_A^\vee \) is isomorphic to an EPW-sextic. The trivialization \( \nu: \wedge^6 V \overset{\sim}{\longrightarrow} \mathbb{C} \) defines a trivialization \( \nu\circ: \wedge^6 V^\vee \overset{\sim}{\longrightarrow} \mathbb{C} \) and hence a symplectic form \( \sigma^\vee \) on \( \wedge^3 V^\vee \); let \( LG(\wedge^3 V^\vee) \) be the symplectic Grassmannian parametrizing \( \sigma^\vee \)-Lagrangian subspaces of \( \wedge^3 V^\vee \). For \( A \in LG(\wedge^3 V) \) we let

\[
A^\perp := \{ \phi \in \wedge^3 V^\vee | \langle \phi, A \rangle = 0 \} \subset \wedge^3 V^\vee.
\] (3.1)

As is easily checked \( A^\perp \in LG(\wedge^3 V^\vee) \). Thus we have an isomorphism

\[
\delta_V: LG(\wedge^3 V) \overset{\sim}{\longrightarrow} LG(\wedge^3 V^\vee) \quad A \quad \mapsto \quad A^\perp.
\] (3.2)

**Proposition 3.1.** Keep notation as above and assume that

\[
A \in LG(\wedge^3 V)^0 \cap \delta_V^{-1}LG(\wedge^3 V^\vee)^0.
\] (3.3)

Then \( Y_A^\vee = Y_A^\perp \).

**Proof.** We claim that it suffices to prove that

\[
Y_A^\vee \subset Y_A^\perp.
\] (3.4)

In fact by Item (1) of Remark (2.9) we know that \( Y_A \) is not covered by positive-dimensional linear spaces and hence \( Y_A^\vee \) is 4-dimensional; since \( A^\perp \in LG(\wedge^3 V^\vee)^0 \) we know that \( Y_A^\perp \) is irreducible (see Proposition (2.8)) and hence (3.4) implies that \( Y_A^\vee = Y_A^\perp \). Let’s prove (3.4). Let \( \psi \) be as in (2.12) and let’s adopt the notation introduced in the proof of Lemma (2.7). Let \( \ell_0 = \mathbb{C} v_0 \in Y_A^{nm} \). By Proposition (2.8) we know that \( \ell_0 \notin W_A \) and hence ker \( \psi(\ell_0) = \mathbb{C} v_0 \cap A_0 \) with \( A_0 \in \wedge^2 V_0 \) an indecomposable element; let \( J_0 \subset V_0 \) be the span of \( A_0 \), thus \( \dim J_0 = 4 \) because \( A_0 \) is indecomposable. Let \( E_0 \subset V \) be the codimension-1 subspace spanned by \( v_0 \) and \( J_0 \). It follows immediately from (2.20) with \( \beta_0 = A_0 \) that

\[
T_{\ell_0} Y_A = \mathbb{P}(E_0).
\] (3.5)

Now notice that \( v_0 \cap A_0 \in \wedge^3 E_0 \cap A \) and hence

\[
\{0\} \neq \left( \wedge^3 V / \wedge^3 E_0 + A \right)^\vee = \left( \wedge^3 E_0 \right)^\perp \cap A^\perp.
\] (3.6)

Let \( E_0^\perp = \mathbb{C} \phi_0 \); then

\[
(\wedge^3 E_0)^\perp = \mathbb{C} \phi_0 \otimes \wedge^2 (V^\vee / \mathbb{C} \phi_0) = F_{\mathbb{C} \phi_0}.
\] (3.7)

By (3.5) and (3.6) - (3.7) we get that \( T_{\ell_0} Y_A \in Y_A^\perp \); this proves (3.4). \( \square \)

By the above proposition duality defines a rational involution on the set of projective equivalence classes of EPW-sextics. We will show later on - see Section (7) - that a generic EPW-sextic is not self-dual, i.e. the rational involution defined by duality is not the identity.
4 Double covers of EPW-sextics

We give the details of the following observation: for $A \in \mathbb{LG}(\wedge^3 V)^0$ the variety $Y_A$ supports a quadratic sheaf (see Definition (0.2) of [3]) and if $X_A \to Y_A$ is the associated double cover then $X_A$ is smooth. Let $A \in \mathbb{LG}(\wedge^3 V)$ and let $A^\vee \subset \wedge^3 V$ be a Lagrangian subspace transversal to $A$ - see Section 2. Thus we have

$$\wedge^3 V = A \oplus A^\vee.$$ (4.1)

Let

$$\tilde{\lambda}_A: \wedge^3 V \otimes \mathcal{O}_P(V) \to A^\vee \otimes \mathcal{O}_P(V),$$ (4.2)

be the projection corresponding to Decomposition (4.1). Let $\nu$ and $\lambda_A$ be given by (1.6) and (1.7) respectively; then $\lambda_A = \tilde{\lambda}_A \circ \nu$. We will study the sheaf $\text{coker}(\lambda_A)$ fitting into the exact sequence

$$0 \to F \xrightarrow{\lambda_A} A^\vee \otimes \mathcal{O}_P(V) \to \text{coker}(\lambda_A) \to 0.$$ (4.3)

**Proposition 4.1.** Keep notation as above and assume that $A \in \mathbb{LG}(\wedge^3 V)^0$. Let $\ell_0 \in \mathbb{P}(V)$.

(1) If $\ell_0 \not\in Y_A$ then $\text{coker}(\lambda_A)$ is zero in a neighborhood of $\ell_0$.

(2) If $\ell_0 \in (Y_A \setminus W_A)$ there exist an open affine $U \subset \mathbb{P}(V)$ containing $\ell_0$ and trivializations $F|_U \cong \mathcal{O}_U^9 \oplus \mathcal{O}_U^9$ and $A \otimes \mathcal{O}_U \cong \mathcal{O}_U^9 \oplus \mathcal{O}_U$ such that (4.3) restricted to $U$ reads

$$0 \to \mathcal{O}_U^9 \oplus \mathcal{O}_U \xrightarrow{(1d,x)} \mathcal{O}_U^9 \oplus \mathcal{O}_U \to \text{coker}(\lambda_A) \otimes \mathcal{O}_U \to 0$$ (4.4)

where $x$ is a local generator of $I_{Y_A \cap U}$.

(3) If $\ell_0 \in W_A$ there exist an open affine $U \subset \mathbb{P}(V)$ containing $\ell_0$ and trivializations $F|_U \cong \mathcal{O}_U^8 \oplus \mathcal{O}_U^2$ and $A \otimes \mathcal{O}_U \cong \mathcal{O}_U^8 \oplus \mathcal{O}_U^2$ such that (4.3) restricted to $U$ reads

$$0 \to \mathcal{O}_U^8 \oplus \mathcal{O}_U^2 \xrightarrow{(1d,M)} \mathcal{O}_U^8 \oplus \mathcal{O}_U^2 \to \text{coker}(\lambda_A) \otimes \mathcal{O}_U \to 0$$ (4.5)

where

$$M = \begin{pmatrix} x & y \\ y & z \end{pmatrix},$$ (4.6)

with $x, y, z$ generators of $I_{W_A \cap U}$.

**Proof.** This is a straightforward consequence of the proof of Lemma 4.1; we leave the details to the reader.

We will need a few results on the sheaf $\text{coker}(\lambda_A)$ and sheaves which locally look like $\text{coker}(\lambda_A)$.

**Definition 4.2.** A coherent sheaf $F$ on a smooth projective variety $Z$ is a Casnati-Catanese sheaf if for every $p \in Z$ there exists $U \subset Z$ open in the classical topology containing $p$ such that one of the following holds:

(1) $F|_U = 0$. 

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(2) There exist \( x \in \text{Hol}(U) \) with \( x(p) = 0 \), \( dx(p) \neq 0 \) and an exact sequence
\[
0 \to \mathcal{O}_U \xrightarrow{x} \mathcal{O}_U \to \mathcal{F} \to 0. \tag{4.7}
\]

(3) There exist \( x, y, z \in \text{Hol}(U) \) vanishing at \( p \) with \( dx(p), dy(p), dz(p) \) linearly independent and an exact sequence
\[
0 \to \mathcal{O}_U^2 \xrightarrow{M} \mathcal{O}_U^2 \to \mathcal{F} \to 0 \tag{4.8}
\]
where \( M \) is the map defined by Matrix (4.6).

Thus \( \text{coker}(\lambda_A) \) is a typical example of a Casnati-Catanese sheaf. If \( \mathcal{F} \) is a Casnati-Catanese sheaf the schematic support of \( \mathcal{F} \) is a divisor \( D \) on \( Z \); thus letting \( i: D \hookrightarrow Z \) be the inclusion we have
\[
\mathcal{F} = i_* \mathcal{G} \tag{4.9}
\]
for a coherent sheaf \( \mathcal{G} \) on \( D \). In particular \( \text{coker}(\lambda_A) = i_* \zeta_A \) for \( i: Y_A \hookrightarrow \mathbb{P}(V) \) the inclusion map and \( \zeta_A \) a certain coherent sheaf on \( Y_A \).

**Proposition 4.3.** Let \( \mathcal{F} \) be a Casnati-Catanese sheaf on \( Z \). Let \( D \) be the schematic support of \( \mathcal{F} \) and hence \( \mathcal{F} = i_* \mathcal{G} \) where \( i: D \hookrightarrow Z \) is the inclusion, see (4.9).

1. If \( q \geq 2 \) then \( \text{Tor}_q(\mathcal{F}, \mathcal{E}) = 0 \) for any (abelian) sheaf \( \mathcal{E} \) on \( Z \).
2. The sheaf \( \mathcal{G} \) is locally isomorphic to \( \mathcal{G}^\vee := \text{Hom}(\mathcal{G}, \mathcal{O}_D) \). In particular \( \mathcal{G} \) is pure i.e. there does not exist a non-zero subsheaf of \( \mathcal{G} \) supported on a subscheme of \( D \) of dimension strictly smaller than \( \dim D \).
3. The map of sheaves \( \mathcal{O}_D \to \text{Hom}(\mathcal{G}, \mathcal{G}) \) which associates to \( f \in \mathcal{O}_D, p \) multiplication by \( f \) is an isomorphism.
4. There is an isomorphism
\[
\text{Ext}^1(\mathcal{F}, \mathcal{O}_Z) \cong i_* \left( \mathcal{G}^\vee \otimes N_{D/Z} \right). \tag{4.10}
\]

**Proof.** (1) (2) and (3) follow immediately from the given local resolutions of \( \mathcal{F} \).

(4): Let \( \mathcal{E}_0 \to \mathcal{F} \) be a surjection with \( \mathcal{E}_0 \) locally-free and let \( \mathcal{E}_1 \) be the kernel of the surjection. Thus we have an exact sequence
\[
0 \to \mathcal{E}_1 \xrightarrow{h} \mathcal{E}_0 \to \mathcal{F} \to 0. \tag{4.11}
\]
By Item (1) the sheaf \( \mathcal{E}_1 \) is locally-free. The dual of (4.11) is the exact sequence
\[
0 \to \mathcal{E}_0^\vee \xrightarrow{h^\vee} \mathcal{E}_1^\vee \xrightarrow{\partial} \text{Ext}^1(\mathcal{F}, \mathcal{O}_Z) \to 0. \tag{4.12}
\]
Multiplication defines an inclusion
\[
\mathcal{F} \otimes \mathcal{O}_Z(-D) = (\mathcal{E}_0/\mathcal{E}_1) \otimes \mathcal{O}_Z(-D) \to \mathcal{E}_0(-D)/\mathcal{E}_1(-D). \tag{4.13}
\]
Since \( \mathcal{F} \) is supported on \( D \) we have an inclusion \( \mathcal{E}_0(-D) \hookrightarrow \mathcal{E}_1 \) and hence an inclusion
\[
\mathcal{E}_0(-D)/\mathcal{E}_1(-D) \hookrightarrow \mathcal{E}_1/\mathcal{E}_1(-D) = \mathcal{E}_1 \otimes \mathcal{O}_D. \tag{4.14}
\]
Composing Map (4.13) and Map (4.14) we get an inclusion
\[ \mathcal{G} \otimes O_D(-D) \hookrightarrow \mathcal{E}_1|_D \]  
whose dual is a surjection
\[ \mathcal{E}_1^\vee|_D \rightarrow \mathcal{G} \otimes N_{D/Z}. \]  
Since \( \mathcal{F} \) is supported on \( D \) the connecting homomorphism map of (4.12) annihilates \( \mathcal{E}_1^\vee(-D) \) and hence it may be identified with a quotient map of \( \mathcal{E}_1^\vee|_D \): a local computation shows that the quotient map is (4.15). This proves Item (4).

We set
\[ \xi_A := \zeta_A \otimes O_{Y_A}(-3). \]  

### Proposition 4.4

Keep notation as above and assume that \( A \in LG(\wedge^3 V)^0 \).

1. There exists a symmetric isomorphism
\[ \alpha_A: \xi_A \sim \xi_A^\vee \]  
   defining a commutative multiplication map
\[ \bar{\alpha}_A: \xi_A \otimes \xi_A \rightarrow O_{Y_A}. \]  

2. Multiplication (4.19) is an isomorphism away from \( W_A \) and near \( \ell_0 \in W_A \) is described as follows. There exist an open affine \( U \subset Y_A \) containing \( \ell_0 \), global generators \( \{e_1, e_2\} \) of \( \xi_A \otimes O_U \) and \( x, y, z \in C[U] \) generating the ideal of \( W_A \cap U \) such that
\[ \bar{\alpha}_A(e_1 \otimes e_1) = x, \quad \bar{\alpha}_A(e_1 \otimes e_2) = \bar{\alpha}_A(e_2 \otimes e_1) = y, \quad \bar{\alpha}_A(e_2 \otimes e_2) = z. \]  

3. Any map \( \gamma: \xi_A \rightarrow \xi_A^\vee \) is a constant multiple of \( \alpha_A \).

**Proof.** Let
\[ \bar{\mu}_A: \wedge^3 V \otimes O_{\mathbb{P}(V)} \rightarrow A \otimes O_{\mathbb{P}(V)} \]  
be the projection given by Decomposition (4.11) and let \( \mu_A: F \rightarrow A \otimes O_{\mathbb{P}(V)} \) be defined by \( \mu_A := \bar{\mu}_A \circ \nu \) where \( \nu \) is as in (1.6). The diagram
\[ \begin{array}{ccc}
F & \xrightarrow{\lambda_A} & A^\vee \otimes O_{\mathbb{P}(V)} \\
\downarrow{\mu_A} & & \downarrow{\mu_A^\vee} \\
A \otimes O_{\mathbb{P}(V)} & \xrightarrow{\lambda_A^\vee} & F^\vee
\end{array} \]  
is commutative because \( F \xrightarrow{(\mu_A, \lambda_A)} (A \oplus A^\vee) \otimes O_{\mathbb{P}(V)} \) is a Lagrangian embedding. The map \( \lambda_A \) is an injection of sheaves because \( Y_A \neq \mathbb{P}(V) \) and hence also \( \lambda_A^\vee \) is an injection of sheaves. Thus there is a unique \( \beta_A: i_\ast \zeta_A \rightarrow Ext^1(i_\ast \zeta_A, O_{\mathbb{P}(V)}) \) making the following diagram commutative with exact horizontal sequences:
\[ \begin{array}{ccc}
0 & \rightarrow & F & \xrightarrow{\lambda_A} & A^\vee \otimes O_{\mathbb{P}(V)} & \rightarrow & i_\ast \zeta_A & \rightarrow & 0 \\
\downarrow{\mu_A} & & \downarrow{\mu_A^\vee} & & \downarrow{\alpha_A^\vee} & & \downarrow{\beta_A} & & \\
0 & \rightarrow & A \otimes O_{\mathbb{P}(V)} & \xrightarrow{\lambda_A^\vee} & F^\vee & \rightarrow & Ext^1(i_\ast \zeta_A, O_{\mathbb{P}(V)}) & \rightarrow & 0
\end{array} \]  

By Isomorphism (4.10) we get that \( \beta_A \) may be viewed as a map \( \beta_A: \zeta_A \sim \zeta_A^\vee(6) \).
Claim 4.5. Let $Z$ be a smooth projective variety and $\mathcal{F}$ be a Casnati-Catanese sheaf on $Z$. Suppose that there exist vector-bundles $\mathcal{E}_0, \mathcal{E}_1$ on $Z$ and an exact sequence

$$
0 \rightarrow \mathcal{E}_1 \xrightarrow{\lambda} \mathcal{E}_0 \xrightarrow{\mu} \mathcal{F} \xrightarrow{\beta} 0
$$

(4.24)

Then $\beta$ is an isomorphism if and only if the map

$$
\mathcal{E}_1 \xrightarrow{(\lambda, \mu)} \mathcal{E}_0 \oplus \mathcal{E}_0^\vee
$$

(4.25)

is an injection of vector-bundles i.e. it is injective on fibers.

Proof. Let $\mathcal{F} = i_* \mathcal{G}$ where $i : D \hookrightarrow Z$ is the inclusion. First we notice that $\beta$ is an isomorphism if and only if it is surjective; in fact by Items (4) and (2) of Proposition 4.3, we have local identifications of the sheaves $\text{Hom}(\mathcal{F}, \text{Ext}^1(\mathcal{F}, \mathcal{O}_Z))$ with $\text{Hom}(\mathcal{G}, \mathcal{G})$ and it follows from Item (3) of the same proposition that a map of stalks $\mathcal{G}_p \rightarrow \mathcal{G}_p$ is an isomorphism if and only if it is surjective. By Nakayama’s Lemma (or by a direct computation) we get that $\beta$ is an isomorphism if and only if for every $p \in Z$ the map from the fiber of $\mathcal{F}$ at $p$ to the fiber of $\text{Ext}^1(\mathcal{F}, \mathcal{O}_Z)$ at $p$ is surjective. As is easily checked there exist $U \subset Z$ open in the classical topology containing $p$, trivial vector-bundles $\mathcal{A}_0, \mathcal{B}_0$ on $U$ for $i = 0, 1$ and isomorphisms $\mathcal{E}_i|_U \cong \mathcal{A}_i \oplus \mathcal{B}_i$ such that the restriction of (4.24) to $U$ reads

$$
0 \rightarrow \mathcal{A}_1 \oplus \mathcal{B}_1 \xrightarrow{(\phi, \psi)} \mathcal{A}_0 \oplus \mathcal{B}_0 \xrightarrow{\mu} \mathcal{F}|_U \xrightarrow{\beta} 0
$$

(4.26)

where $\phi : \mathcal{A}_1 \rightarrow \mathcal{A}_0$ is an isomorphism and $\psi : \mathcal{B}_1 \rightarrow \mathcal{B}_0$ is a standard resolution of $\mathcal{F}|_U$ as given in Definition 4.2, i.e. $\mathcal{B}_1 = \mathcal{B}_0 = 0$ if (1) of Definition 4.2 holds, $\mathcal{B}_1 \cong \mathcal{B}_0 \cong \mathcal{O}_U$ and $\psi$ is the map of (4.24) if (2) of Definition 4.2 holds and finally if (3) of Definition 4.2 holds then $\mathcal{B}_1 \cong \mathcal{B}_0 \cong \mathcal{O}_U^\vee$ and $\psi$ is the map of (4.24). The map $\beta$ is a surjection from the fiber of $\mathcal{F}$ at $p$ to the fiber of $\text{Ext}^1(\mathcal{F}, \mathcal{O}_U)$ if and only if the composition

$$
B_0 \xrightarrow{\pi^\vee} \mathcal{A}_0 \oplus \mathcal{B}_0 \xrightarrow{\mu^\vee} \mathcal{A}_1 \oplus \mathcal{B}_1^\vee \xrightarrow{\psi} \mathcal{B}_1^\vee
$$

(4.27)

is surjective at $p$. On the other hand since $\psi$ is zero at $p$ the map $((\phi, \psi), \mu)$ is injective at $p$ if and only if the composition

$$
\mathcal{B}_1 \xrightarrow{\pi^\vee} \mathcal{A}_1 \oplus \mathcal{B}_1 \xrightarrow{\mu} \mathcal{A}_0^\vee \oplus \mathcal{B}_0^\vee
$$

(4.28)

is injective at $p$. Since $\psi$ vanishes at $p$ while $\phi^\vee$ is an isomorphism at $p$ the equality $(\phi^\vee, \psi^\vee) \circ \mu = \mu^\vee \circ (\phi, \psi)$ gives that the composition

$$
\mathcal{B}_1 \xrightarrow{\pi^\vee} \mathcal{A}_1 \oplus \mathcal{B}_1 \xrightarrow{\mu} \mathcal{A}_0^\vee \oplus \mathcal{B}_0^\vee
$$

(4.29)
vanishes at $p$. Thus (4.28) is injective at $p$ if and only if the composition

$$B_1 \overset{\pi}{\longrightarrow} A_1 \oplus B_1 \xrightarrow{\mu} A_0' \oplus B_0' \overset{\nu}{\longrightarrow} B_0'$$

(4.30)

is injective at $p$. Since Composition (4.30) is the transpose of Composition (4.27) this proves that $\beta$ is a surjection at $p$ if and only if $((\phi, \psi), \mu)$ is injective at $p$. \qed

The above result together with Exact Sequence (4.23) gives that $\beta_A$ is an isomorphism. We define the map $\alpha_A$ of (4.18) to be the tensor product of $\beta_A$ times the identity map on $O_{Y_A}(-3)$. Since $\beta_A$ is an isomorphism we get that $\alpha_A$ is an isomorphism. The map $\alpha_A^* - \alpha_A$ is zero on $(Y_A \setminus W_A)$ because $\xi_A$ is locally-free of rank 1 on $(Y_A \setminus W_A)$. By Item (2) of Proposition (4.3) $\xi_A$ is pure and hence $\xi_A$ is pure; thus $\alpha_A^* - \alpha_A$ is zero on $Y_A$ i.e. $\alpha_A$ is symmetric. This proves Item (1). Let’s prove Item (2). Since $\xi_A$ is locally-free of rank 1 away from $W_A$ it is clear that (4.13) is an isomorphism away from $W_A$. Equation (4.20) defines a symmetric isomorphism $\mu: \xi_A \otimes O_U \rightarrow \xi_A \otimes O_U$. By Item (3) of Proposition (4.3) any other symmetric isomorphism $\xi_A \otimes O_U \rightarrow \xi_A \otimes O_U$ is equal to $f \cdot \mu$ where $f \in \mathbb{C}[U]^\times$ is invertible; Item (2) of the proposition follows at once from this. Lastly we prove Item (3). The composition $\alpha_A^{-1} \circ \gamma$ is a global section of $Hom(\xi_A, \xi_A)$ and by Item (3) of Proposition (4.3) we get that $\alpha_A^{-1} \circ \gamma$ is multiplication by a constant $c$; thus $\gamma = c \alpha_A$. \qed

Now we are ready to define the double cover of $X_A$ for $A \in \mathbb{L}G(\wedge^3 V)^0$. Map (4.19) gives $O_{Y_A} + \xi_A$ the structure of a commutative finite $O_{Y_A}$-algebra. Let

$$X_A := Spec(O_{Y_A} + \xi_A)$$

(4.31)

and $f: X_A \rightarrow Y_A$ be the structure map. Thus $f$ is a finite map of degree 2; let $\phi: X_A \rightarrow X_A$ be the covering involution - thus $\phi$ corresponds to the map on $O_{Y_A} + \xi_A$ which is the identity on $O_{Y_A}$ and multiplication by $(-1)$ on $\xi_A$. Equivalently

$$f_* O_{X_A} = O_{Y_A} + \xi_A$$

(4.32)

with $O_{Y_A}$ and $\xi_A$ the $+1$ and $(-1)$-eigensheaf respectively. It follows at once from our lemma that $f$ is ramified exactly over $W_A$ and that $X_A$ is smooth. Let $A$ vary in $\mathbb{L}G(\wedge^3 V)^0$. By Item (2) of Remark (2.9) the family of $Y_A$’s is locally trivial; since $\mathbb{L}G(\wedge^3 V)^0$ is irreducible we get the following result.

**Proposition 4.6.** The varieties $X_A$ for $A$ varying in $\mathbb{L}G(\wedge^3 V)^0$ are all deformation equivalent.

## 5 Proof of Theorem (1.1)

Throughout this section $X$ is a numerical $(K3)^2$ with an ample divisor $H$ such that both (1.3) and Item (a) of Section (??) hold. Thus we have an anti-symplectic involution $\phi: X \rightarrow X$ with quotient map $f: X \rightarrow X/\langle \phi \rangle =: Y$ and an embedding

$$j: Y \hookrightarrow |H|', \quad \deg Y = 6$$

(5.1)

such that $j \circ f$ is the tautological map $X \rightarrow |H|'$. Let $X^\phi$ be the fixed locus of $\phi$; then $sing Y \cong X^\phi$. Since $\phi$ is anti-symplectic $X^\phi$ is a smooth Lagrangian...
surface - not empty because a smooth hypersurface in $\mathbb{P}^5$ is simply connected. Thus
\[
\dim(\text{sing} Y) = 2
\]
and $f$ is unramified over $(Y \setminus \text{sing} Y)$. We have a decomposition
\[
f_* \mathcal{O}_X = \mathcal{O}_Y \oplus \eta,
\]
where $\eta$ is the $(-1)$-eigensheaf of the involution $\phi^* : f_* \mathcal{O}_X \to f_* \mathcal{O}_X$.

**Claim 5.1.** Keep notation as above. The sheaf $j_* \eta$ is a Casnati-Catanese sheaf on $|H|^\vee$.

**Proof.** Let $p \in (|H|^\vee \setminus Y)$; then $j_* \eta$ is zero in a neighborhood of $p$. Let $p \in (Y \setminus \text{sing} Y)$; then $\eta$ is locally-free of rank 1 in a neighborhood of $p$ (in $Y$) and hence we see that (2) of Definition 4.2 holds. Let $p \in \text{sing} Y$. Let $f^{-1}(p) = \{\tilde{p}\}$. Since $\dim X^\phi = 2$ there exist a $\tilde{U} \subset X$ open in the classical topology containing $\tilde{p}$ with analytic coordinates $\{u_1, u_2, v_1, v_2\}$ centered at $\tilde{p}$ and an open $U \subset |H|^\vee$ containing $p$ with analytic coordinates $\{x_1, \ldots, x_5\}$ centered at $p$ such that
\[
\phi^* (u_1, u_2, v_1, v_2) = (-u_1, -u_2, v_1, v_2)
\]

(5.4)
\[
f^* (x_1, x_2, x_3, x_4, x_5) = (u_1^2, u_2^2, -u_1 u_2, v_1, v_2).
\]

(5.5)
It follows from (5.4) that
\[
j_* \eta|_U \text{ is generated by } (u_1, u_2).
\]

(5.6)
A straightforward computation gives the free presentation
\[
0 \to \mathcal{O}_U^2 \frac{x_2}{x_3} \xrightarrow{x_1} \mathcal{O}_U^{(u_2, u_1)} (j_* \eta)|_U \to 0.
\]

(5.7)
Thus (3) of Definition 4.2 holds. 

For future use we notice the following: keeping notation as in the above proof we may assume by shrinking $U$ that $f^{-1}(U) = \tilde{U}$ and then
\[
Y \cap U = V(x_1 x_2 - x_3^3).
\]

(5.8)
Multiplication on $f_* \mathcal{O}_X$ defines a symmetric map
\[
\overline{\alpha} : \eta \otimes \eta \to \mathcal{O}_Y
\]

(5.9)
making $\eta$ a quadratic sheaf in the sense of Casnati-Catanese [3]. A straightforward computation shows that $\overline{\alpha}$ defines a symmetric isomorphism
\[
\alpha : \eta \xrightarrow{\sim} \eta' := \text{Hom}(\eta, \mathcal{O}_Y).
\]

(5.10)
The symmetric map $\overline{\alpha}$ makes $\mathcal{O}_Y \oplus \eta$ a commutative $\mathcal{O}_Y$-algebra and we have the tautological isomorphism
\[
X \cong \text{Spec}(\mathcal{O}_Y \oplus \eta).
\]

(5.11)
The main result of this section is the following.
Theorem 5.2. Keep notation and hypotheses as above. There exists \( A \in \mathbb{L}G(\wedge^3 V)^0 \) such that \( Y = Y_A \) and \( \eta \cong \xi_A \).

The proof of Theorem 5.2 will be given at the end of this section. Let’s show that Theorem (1.1) follows from Theorem (5.2). Let’s prove (1) of Theorem (1.1) i.e. that \( f : X \to Y \) is identified with the natural double cover \( X_A \to Y_A \). According to Theorem (5.2) we may identify \( \eta \) and \( \xi_A \). We claim that with this identification the map \( \alpha \) of (5.10) gets identified with a non-zero constant multiple of \( \alpha_A \); in fact \( \alpha \) is non-zero and hence the claim follows from Item (3) of Proposition (4.4). Thus multiplying the isomorphism \( \eta \simeq \xi_A \) by a suitable constant we may assume that the map \( \alpha \) gets identified with \( \alpha_A \); by (5.11) and (4.31) we get that \( f : X \to Y \) is identified with \( X_A \to Y_A \). Let’s prove (2) of Theorem (5.2) i.e. that if \( A \in \mathbb{L}G(\wedge^3 V)^0 \) then \( X_A \to Y_A \) is a deformation of \( (K^3)^2 \). By Proposition (4.6) the \( X_A \) for \( A \in \mathbb{L}G(\wedge^3 V)^0 \) are all deformation equivalent. Since every Kähler deformation of an irreducible symplectic manifold is an irreducible symplectic manifold it suffices to prove that there exists one \( A \in \mathbb{L}G(\wedge^3 V)^0 \) such that \( X_A \) which is a symplectic irreducible variety deformation equivalent of \( (K^3)^2 \). By Item (1) of Theorem (1.1) it suffices to exhibit \( X, H \), where \( X \) is a deformation of \( (K^3)^2 \) and \( H \) is an ample divisor on \( X \) such that both (1.3) and Item (a) of Section (1) hold. Such an example was given by Mukai (Ex.(5.17) of [10]), details are in Subsection (5.4) of [11]. We briefly describe the example. Let \( F \subset \mathbb{P}^6 \) be “the” Fano 3-fold of index 2 and degree 5 i.e. the transversal intersection of \( Gr(2, \mathbb{C}^5) \subset \mathbb{P}(\wedge^2 \mathbb{C}^5) \) and a 6-dimensional linear subspace of \( \mathbb{P}(\wedge^2 \mathbb{C}^5) \). Let \( Q \subset \mathbb{P}^6 \) be a quadric hypersurface intersecting transversely \( F \) and let

\[
S := F \cap Q.
\]

Thus \( (S, \mathcal{O}_S(1)) \) is a generic polarized \( K3 \) surface of degree 10. We assume that

\[
E \cdot c_1(\mathcal{O}_S(1)) \equiv 0 \pmod{10}, \quad E \text{ divisor on } S,
\]

this is Hypothesis (4.8) of [11] and it holds for \( Q \) belonging to an open dense subset of the space of all quadrics. We have \( \dim |I_F(2)| = 4 \) and \( \dim |I_S(2)| = 5 \). Let \( \Sigma \) be the degree-7 divisor on \( |I_S(2)| \) parametrizing singular quadrics. Every \( Q \in |I_F(2)| \) is singular and hence

\[
\Sigma = |I_S(2)| + Y.
\]

Since \( \deg \Sigma = 7 \) we have \( \deg Y = 6 \). The generic \( Q \in Y \) has corank 1 and hence it has two rulings by 3-dimensional linear spaces. Thus there exists a natural double cover of an open dense subset of \( Y \). It turns out (see Ex.(5.17) of [10]) and Subsection (5.4) of [11]) that the double cover extends to a double cover \( f : X \to Y \), that

\[
X = \{ F \text{ stable sheaf on } S, \ rk(F) = 2, \ c_1(F) = c_1(\mathcal{O}_S(1)), \ c_2(F) = 5 \}/\text{isomorphism}
\]

and that \( X \) is a deformation of \( S^{[2]} \). Let \( H := f^*\mathcal{O}_Y(1) \). As shown in Subsection (5.4) of [11] both (5.13) and Item (a) of Section (1) hold. This proves that (2) of Theorem (1.1) holds.
5.1 A locally free resolution of $j_*(\eta \otimes \mathcal{O}_Y(3))$

**Proposition 5.3.** Let $X$ be a numerical $(K3)^{[2]}$ with an ample divisor $H$ such that both Proposition 5.2 and Item (a) of Section 5 hold. If $k \geq 3$ then $\mathcal{O}_X(kH)$ is very ample.

**Proof.** By hypothesis $nH$ is very ample for some $n \gg 0$ and hence it suffices to prove that the multiplication map

$$H^0(\mathcal{O}_X(kH)) \otimes H^0(\mathcal{O}_X(H)) \to H^0(\mathcal{O}_X((k+1)H)) \quad (5.16)$$

is surjective for $k \geq 3$. By Proposition 5.2 there exists a plane $\Lambda \subset |H|^\vee$ such that $\Lambda \cap (\text{sing} Y) = \emptyset$ and $\Lambda$ is transversal to $Y$. Let $C := \Lambda \cap Y$ and $\tilde{C} := f^{-1}C$. Then $C$ is a smooth plane sextic by (5.1) and $\pi := f|_C: \tilde{C} \to C$ is an unramified double cover. Let’s prove that

$$H^0(\mathcal{O}_C(kH)) \otimes H^0(\mathcal{O}_C(H)) \to H^0(\mathcal{O}_C((k+1)H)) \quad (5.17)$$

is surjective for $k \geq 3$. We have

$$\pi_* \mathcal{O}_C = \mathcal{O}_C \oplus \lambda \quad (5.18)$$

where $\lambda$ is a non-trivial square root of $\mathcal{O}_C$. Thus

$$H^0(\mathcal{O}_C(kH)) = H^0(\pi_* \mathcal{O}_C) = H^0(\mathcal{O}_C(k)) \oplus H^0(\lambda(k)). \quad (5.19)$$

Since the multiplication map $H^0(\mathcal{O}_C(k)) \otimes H^0(\mathcal{O}_C(1)) \to H^0(\mathcal{O}_C(k+1))$ is surjective it suffices to prove surjectivity of

$$H^0(\lambda(k)) \otimes H^0(\mathcal{O}_C(1)) \to H^0(\lambda(k+1)) \quad (5.20)$$

for $k \geq 3$. Since $C$ is a smooth plane sextic adjunction gives $K_C \cong \mathcal{O}_C(3)$; since $\lambda \not\cong \mathcal{O}_C$ we get that $h^1(\lambda(k)) = 0$ for $k \geq 3$. Thus

$$\chi(\lambda(k)) = 6k - 9, \quad k \geq 3. \quad (5.21)$$

Let $U \subset H^0(\mathcal{O}_C(1))$ be a 2-dimensional subspace spanned by sections $\epsilon_0, \epsilon_1$ with no common zeroes. Consider the multiplication map

$$H^0(\lambda(k)) \otimes U \to H^0(\lambda(k+1)). \quad (5.22)$$

By the base-point-free pencil trick we have

$$\ker(\mu) = \{(\sigma \epsilon_0) \otimes \epsilon_1 - (\sigma \epsilon_1) \otimes \epsilon_0 \mid \sigma \in H^0(\lambda(k-1))\}. \quad (5.23)$$

Using Formula (5.21) one gets that $\dim \text{Im}(\mu) = h^0(\lambda(k+1))$ and hence $\mu$ is surjective: thus Map (5.20) is surjective and hence also Map (5.17) is surjective.

Now we prove that Map (5.20) is surjective. Let $X = X_4 \supset X_3 \supset X_2 \supset X_1 = C$ be a chain of smooth linear sections of $X$, i.e. $X_3 \in |H|$ and $X_2 = D \cap D'$ where $D, D' \in |H|$ intersect transversely. We claim that the restriction map

$$H^0(\mathcal{O}_X(sH)) \to H^0(\mathcal{O}_{X_{s-1}}(sH)) \quad (5.24)$$

is surjective for $s = 1$ and $s \geq 3$. It suffices to show that

$$h^1(\mathcal{O}_X((s-1)H)), \quad s = 1, \quad s \geq 3. \quad (5.25)$$

This follows from the Lefschetz Hyperplane Section Theorem and Kodaira Vanishing. Surjectivity of Map (5.24) for $s = 1$ and $s \geq 3$ together with surjectivity of (5.17) gives surjectivity of (5.10) by an easy well-known argument. \qed
Let
\[ \theta := \eta \otimes O_Y(3). \]  
(5.26)

**Corollary 5.4.** Keep notation and hypotheses as above. Then \( \theta \) is globally generated.

**Proof.** Let \( H^0(O_X(3H))^{-} \subset H^0(O_X(3H)) \) be the \((-1)\)-eigenspace for the action of \( \phi^* \). Then
\[
H^0(O_X(3H)) = H^0(O_Y(3H)) \oplus H^0(\theta), \quad H^0(\theta) = H^0(O_X(3H))^{-}. \tag{5.27}
\]
Let \( p \in Y \) and let \( \theta_p \) be the fiber of \( \theta \) at \( p \); we must show that
\[
H^0(O_X(3H))^{-} \longrightarrow \theta_p \tag{5.28}
\]
is surjective. Assume that \( p \in (Y \setminus \text{sing} Y) \). Let \( f^{-1}(p) = \{p_1, p_2\} \) and let \( L_{p_i} \) be the fiber of \( O_X(3H) \) at \( p_i \). Since \( O_X(3H) \) is very ample the evaluation map \( H^0(O_X(3H)) \to (L_{p_1} \oplus L_{p_2}) \) is surjective. Since \( \theta_p \) is identified with the \((-1)\)-eigenspace for the action of \( \phi^* \) on \( (L_{p_1} \oplus L_{p_2}) \) we get that \( (5.28) \) is surjective. Finally assume that \( p \in \text{sing} Y \) and let \( f^{-1}(p) = \{\tilde{p}\} \). Then \( (5.26) \) identifies \( \theta_p \) with \( \Omega_{X,\tilde{p}}^3 \) and Map \( (5.25) \) with differentiation at \( \tilde{p} \). Since \( O_X(3H) \) is very ample the differential at \( \tilde{p} \) of the map \( X \to |3H|^\vee \) is injective; it follows that \( (5.28) \) is a surjection. \( \square \)

Let \( \epsilon: H^0(\theta) \otimes O_{|H|^\vee} \to j_* \theta \) be the evaluation map. By the above corollary \( \epsilon \) is surjective; let \( G \) be the kernel of \( \epsilon \). Thus we have an exact sequence
\[
0 \to G \longrightarrow H^0(\theta) \otimes O_{|H|^\vee} \xrightarrow{\epsilon} j_* \theta \to 0. \tag{5.29}
\]

**Proposition 5.5.** Keep notation and assumptions as above. Then
\[
G \cong \Omega^3_{|H|^\vee}(3). \tag{5.30}
\]

**Proof.** First we prove that \( G \) is locally-free. By Claim \( 5.1 \) the sheaf \( j_* \eta \) is Casnati-Catanese and hence so is \( j_* \theta \). By (1) of Proposition \( 4.3 \) we get that \( G \) is locally-free. Since \( G \) is locally-free Beilinson’s spectral sequence with
\[
E^1_{p,q} = H^q(G(p)) \otimes \Omega_{|H|^\vee}^{3-p}(-p) \tag{5.31}
\]
converges in degree 0 to the graded sheaf associated to a filtration on \( G \), see \[13\] p. 240. Thus it suffices to prove that
\[
h^q(G(p)) = \begin{cases} 0 & \text{if } -5 \leq p \leq 0 \text{ and } (p,q) \neq (-3,3), \\ 1 & \text{if } (p,q) = (-3,3). \end{cases} \tag{5.32}
\]
This follows from a straightforward computation which goes as follows. Tensorizing \( (5.26) \) by \( O_{|H|^\vee}(p) \) and taking the associated cohomology exact sequence we get
\[
\cdots \longrightarrow H^0(\theta) \otimes H^{q-1}(O_{|H|^\vee}(p)) \longrightarrow H^{q-1}(\theta(p)) \longrightarrow H^q(G(p)) \longrightarrow H^0(\theta) \otimes H^q(O_{|H|^\vee}(p)) \longrightarrow \cdots \tag{5.33}
\]
(We let $\theta(p) := \theta \otimes \mathcal{O}_Y(p)$.) From this one easily gets that
\begin{equation}
 h^1(G) = 0, \quad h^0(G(p)), \quad -5 \leq p \leq 0,
\end{equation}
and that
\begin{equation}
 h^{q-1}(\theta(p)) = h^q(G(p)), \quad -5 \leq p \leq 0, \quad (p, q) \neq (0, 1).
\end{equation}

We compute the left-hand side. The map $f$ is finite and we have (5.3) - (5.26); thus
\begin{equation}
 h^{q-1}(\mathcal{O}_X((3 + p)H)) = h^{q-1}(f_*\mathcal{O}_X((3 + p)H)) = h^{q-1}(\mathcal{O}_Y(3 + p)) \oplus h^{q-1}(\theta(p)).
\end{equation}

In order to compute $h^{q-1}(\theta(p))$ we first compute $h^{q-1}(\mathcal{O}_X((3 + p)H))$. Kodaira Vanishing gives that $h^{q-1}(\mathcal{O}_X((3 + p)H)) = 0$ for $2 \leq q \leq 5$ and $p \neq -3$ and hence by (5.36) we get that
\begin{equation}
 h^{q-1}(\theta(p)) = 0, \quad 2 \leq q \leq 5, \quad p \neq -3.
\end{equation}

Now consider $q = 1$. We claim that
\begin{equation}
 h^0(\mathcal{O}_X((3 + p)H)) = h^0(\mathcal{O}_Y((3 + p)H)), \quad -5 \leq p \leq -1.
\end{equation}

For $-5 \leq p \leq -3$ the equation is trivial and for $p = -2$ it holds by hypothesis. To check equality for $p = -1$ we apply Formula (4.0.4) of [12]:
\begin{equation}
 \chi(\mathcal{O}_X(nH)) = \frac{1}{2}n^4 + \frac{5}{2}n^2 + 3.
\end{equation}

Since $H$ is ample Kodaira Vanishing gives that $h^0(\mathcal{O}_X(2H)) = \chi(\mathcal{O}_X(2H))$ and by the above formula we get $h^0(\mathcal{O}_X(2H)) = 21$. On the other hand a straightforward computation gives that $h^0(\mathcal{O}_Y(2)) = 21$ and by (5.38) this finishes the proof of (5.38). Thus we have proved that
\begin{equation}
 h^0(\theta(p)) = 0, \quad -5 \leq p \leq -1.
\end{equation}

Finally consider $h^{q-1}(\mathcal{O}_X)$: it vanishes for $q = 2, 4$ and
\begin{equation}
 h^{q-1}(\mathcal{O}_X) = 1 = h^{q-1}(\mathcal{O}_Y), \quad q = 1, 5.
\end{equation}

On the other hand $h^2(\mathcal{O}_X) = 1$ and $h^2(\mathcal{O}_Y) = 0$. This proves that
\begin{equation}
 h^{q-1}(\theta(-3)) = \begin{cases} 0 & \text{if } q \neq 3, \\ 1 & \text{if } q = 3. \end{cases}
\end{equation}

Equation (5.32) follows from Formulae (5.34), (5.35), (5.37), (5.40) and (5.42). \hfill \Box

By the above proposition and by (5.29) we have an exact sequence
\begin{equation}
 0 \to \Omega^3_{\mathcal{H}^\vee}(3) \xrightarrow{\kappa} H^0(\theta) \otimes \mathcal{O}_{\mathcal{H}^\vee} \xrightarrow{j_* \theta} 0.
\end{equation}
5.2 Proof of Theorem (5.2)

Claim 5.6. Keep notation as above. There exists an isomorphism
\[
\beta: j_* \theta \cong \text{Ext}^1(j_* \theta, \mathcal{O}|_H) .
\] (5.44)

Proof. Since \( j_* \theta \) is a Casnati-Catanese sheaf we have an isomorphism
\[
\text{Ext}^1(j_* \theta, \mathcal{O}|_H) \cong j_* (\theta \otimes N_{Y/H})
\] (5.45)
because of Item (4) of Proposition (4.3). By (5.28) and Isomorphism (5.10) we get
\[
j_* (\theta \otimes N_{Y/H}) = j_* (\eta \otimes \mathcal{O}_Y(-3) \otimes N_{Y/H}) \cong j_* (\eta \otimes \mathcal{O}_Y(-3) \otimes N_{Y/H}) .
\] (5.46)
By (5.41) we have \( N_{Y/H} \cong \mathcal{O}_Y(6) \) and hence we get that
\[
j_* (\eta \otimes \mathcal{O}_Y(-3) \otimes N_{Y/H}) \cong j_* (\eta \otimes \mathcal{O}_Y(3)) = j_* \theta .
\] (5.47)
The above equations prove (5.44). \( \square \)

Let \( \kappa, \epsilon \) be as in (5.43) and \( \beta \) be as in (5.44). We claim that there exists a map
\[
s: H^0(\mathcal{O}|_H) \to \left( \Omega^3_{H|\nu}((3)) \right) = \text{Ext}^3_{H|\nu}(-3)
\] (5.48)
such that the following diagram is commutative:
\[
\begin{array}{cccccc}
0 & \longrightarrow & \Omega^3_{H|\nu}(3) & \xrightarrow{\kappa} & H^0(\theta) \otimes \mathcal{O}|_H & \xrightarrow{\epsilon} & j_* \theta & \longrightarrow & 0 \\
& & \downarrow s & & \downarrow s & & \downarrow \beta \\
0 & \longrightarrow & H^0(\theta) \otimes \mathcal{O}|_H & \xrightarrow{\kappa} & \Theta^3_{H|\nu}(-3) & \xrightarrow{\partial} & \text{Ext}^1(j_* \theta, \mathcal{O}|_H) & \longrightarrow & 0
\end{array}
\] (5.49)
In fact this follows from the results of Casnati-Catanese [3] or of Eisenbud-Popescu-Walter [4]; by the proof of Claim (2.1) of [3] the obstruction to existence of \( s \) lies in \( H^1(Sym_2 (H^0(\theta) \otimes \mathcal{O}|_H)^\nu) \) which is zero and hence \( s \) exists.

Remark 5.7. Proposition (1.6) of [3] does not hold with \( F = j_* \theta \) because \( \chi(j_* \theta(-3)) \) is not even, see Theorem (9.1) of [4] - in fact \( \chi(j_* \theta(-3)) = 1 \). Thus unlike the surfaces considered by Casnati-Catanese the 4-fold \( Y \) cannot be presented as the degeneracy locus of a symmetric map of vector-bundles.

Claim 5.8. Keep notation and assumptions as above. Then
\[
\Omega^3_{H|\nu}(3) \xrightarrow{(\kappa, s^\nu)} (H^0(\theta) \otimes H^0(\theta)^\nu) \otimes \mathcal{O}|_H
\] (5.50)
is an injection of vector-bundles. The image of \( (\kappa, s^\nu) \) is Lagrangian for the tautological symplectic form on \( H^0(\theta) \otimes H^0(\theta)^\nu \) given by
\[
\lambda((\alpha, \psi), (\alpha', \psi')) := \psi(\alpha) - \psi'(\alpha')
\] (5.51)
Proof. The sheaf \( j_* \theta \) is a Casnati-Catanese sheaf on \( |H| \); since \( \beta \) is an isomorphism we get by Claim (5.6) that \( (\kappa, s^\nu) \) is an injection of vector-bundles. The tautological symplectic form vanishes on \( Im(\kappa, s^\nu) \) by commutativity of Diagram (5.49). Since \( \Omega^3_{H|\nu}(3) \) has rank 10 it follows that \( Im(\kappa, s^\nu) \) is Lagrangian. \( \square \)

20
We will show that Diagram (5.49) can be identified with Diagram (4.23) for a suitable \( A \). Let \( V \) be a 6-dimensional complex vector-space and \( F \hookrightarrow \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \) be the sub-vector-bundle defined by (1.5).

**Proposition 5.9.** Keep notation as above. Then

\[
F \cong \Omega^3_{\mathbb{P}(V)}(3).
\]  

**Proof.** Let \( Q := \Theta_{\mathbb{P}(V)}(-1) \). Thus we have the Euler sequence

\[
0 \to \mathcal{O}_{\mathbb{P}(V)}(-1) \to V \otimes \mathcal{O}_{\mathbb{P}(V)} \to Q \to 0
\]  

and by definition \( F \cong (\wedge^2 Q)(-1) \). The perfect pairing \( \wedge^2 Q \times \wedge^3 Q \to \wedge^5 Q \cong \mathcal{O}_{\mathbb{P}(V)}(1) \) gives an isomorphism \( \wedge^2 Q \cong (\wedge^3 Q^\vee)(1) \) and hence

\[
F \cong \wedge^3 Q^\vee \cong \Omega^3_{\mathbb{P}(V)}(3).
\]  

In order to identify (5.49) with (4.23) we will need a few properties of the vector-bundle \( F \cong \Omega^3_{\mathbb{P}(V)}(3) \).

**Proposition 5.10.** Keep notation as above. The dual of Exact Sequence (1.6) defines an isomorphism

\[
\wedge^3 V^\vee \cong H^0(F^\vee).
\]  

**Proof.** Exact Sequence (5.53) gives an exact sequence

\[
0 \to \wedge^2 V \otimes \mathcal{O}_{\mathbb{P}(V)}(-1) \to \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \to \wedge^3 Q \to 0.
\]  

This induces an isomorphism

\[
\wedge^3 V \cong H^0(\wedge^3 Q).
\]  

The symplectic form on \( \wedge^3 V \) gives an identification \( \wedge^3 V^\vee \cong \wedge^3 V \) and we have \( F^\vee \cong \wedge^3 Q \) by (5.54). With these identifications the map of (5.55) is identified with the map of (5.57). Thus (5.55) is an isomorphism.

**Proposition 5.11.** Keep notation as above. Assume that \( \mathcal{W} \) is a symplectic vector-bundle and that \( \mu : F \to \mathcal{W} \) is an injection of vector-bundles such that \( \mu(F) \) is a Lagrangian sub-vector-bundle. Then \( \mu^\vee : \mathcal{W}^\vee \to F^\vee \) induces an isomorphism

\[
H^0(\mathcal{W}^\vee) \cong H^0(F^\vee).
\]  

**Proof.** Since \( \mu(F) \) is Lagrangian the symplectic form on \( \mathcal{W} \) induces an isomorphism \( \mathcal{W}/\mu(F) \cong F^\vee \). Thus we have an exact sequence

\[
0 \to F \xrightarrow{-\mu} \mathcal{W} \to F^\vee \to 0.
\]  

and its dual

\[
0 \to F \xrightarrow{\mu^\vee} \mathcal{W}^\vee \to F^\vee \to 0.
\]  

The above exact sequence with \( \mathcal{W} = \wedge^3 V \otimes \mathcal{O}_{\mathbb{P}(V)} \) gives

\[
0 = h^0(F) = h^1(F)
\]  

because of Proposition (5.10). Now consider (5.60) in general: we get that \( H^0(\mathcal{W}) \to H^0(F^\vee) \) is an isomorphism because of Equation (5.61).
By Proposition 5.9 we have $\Omega^3|_{\mathbb{H}^3}(3) \cong F$. By Claim 5.8 and Proposition 5.11 we have a sequence of isomorphisms

$$\wedge^3 V^\vee \xrightarrow{\sim} H^0(F^\vee) \xrightarrow{\sim} H^0(\wedge^3 \Theta|_{\mathbb{H}^3}(\neg 3)) \xrightarrow{\sim} H^0(\theta)^\vee \oplus H^0(\theta).$$

(5.62)

Let

$$\rho: H^0(\theta) \oplus H^0(\theta)^\vee \xrightarrow{\sim} \wedge^3 V$$

be the transpose of the composition of the maps in (5.62). Then (abusing notation)

$$\rho(\Omega^3|_{\mathbb{H}^3}(3)) = F.$$

(5.64)

Thus (5.49) starts looking like (5.23). One missing link: we have not proved that $\rho(H^0(\theta)^\vee)$ is Lagrangian for the symplectic form $\sigma$ on $\wedge^3 V$ defined in Section 4.

**Proposition 5.12.** Let $V$ be a 6-dimensional complex vector-space. Suppose that $\tau: \wedge^3 V \times \wedge^3 V \rightarrow \mathbb{C}$ is a symplectic form such that $\tau|_{F_i} = 0$ for every $\ell \in \mathbb{P}(V)$. Then $\tau = c\sigma$ for a certain $c \in \mathbb{C}^*$.

**Proof.** Let $\{e_0, \ldots, e_5\}$ be a basis of $V$. Let $S \subset \mathcal{P}\{\{0, \ldots, 5\}\}$ be the family of $I \subset \{0, \ldots, 5\}$ of cardinality 3. For $\{i, j, k\} \in S$ with $i < j < k$ we let $e_I := e_i \wedge e_j \wedge e_k$. Choose an ordering of $S$; then $\{\ldots, e_1, \ldots\}_{I \in S}$ is basis of $\wedge^3 V$. Let $\alpha, \beta \in \wedge^2 V$. For $i \in \{0, \ldots, 5\}$ we have $e_i \wedge \alpha, e_i \wedge \beta \in F_{e_i}$; by our hypothesis we get that $\tau(e_i \wedge \alpha, e_i \wedge \beta) = 0$ and hence

$$\tau(e_I, e_J) = 0 \text{ if } I \cap J \neq \emptyset.$$ 

(5.65)

Let $j \in \{0, \ldots, 5\}$; then

$$0 = \tau((e_i + e_j) \wedge \alpha, (e_i + e_j) \wedge \beta) = \tau(e_i \wedge \alpha, e_j \wedge \beta) + \tau(e_j \wedge \alpha, e_i \wedge \beta).$$

(5.66)

This implies that

$$\text{sign}(I, I^c)\tau(e_I, e_{I^c}) = \text{sign}(J, J^c)\tau(e_J, e_{J^c}), \quad I, J \in S$$

(5.67)

where $I^c, J^c$ are the complements of $I, J$ in $\{0, \ldots, 5\}$ respectively. The proposition is an immediate consequence of (5.65) and (5.67). □

**Corollary 5.13.** Keep notation and hypotheses as above. Then

$$\rho^* \sigma = c\lambda$$

(5.68)

where $\rho$ is the isomorphism of (5.64), $\sigma$ is the symplectic form defined in Section 4, $c$ is a non-zero constant and $\lambda$ is the tautological symplectic form defined in (5.57).

**Proof.** The corollary is equivalent to the equality

$$(\rho^{-1})^* \lambda = c^{-1} \sigma, \quad c \in \mathbb{C}^*.$$ 

(5.69)

By Claim 5.8, $\rho(\Omega^3|_{\mathbb{H}^3}(3))$ (yes, we abuse notation again) is a Lagrangian sub-vector-bundle of $\wedge^3 V \otimes O_{\mathbb{P}(V)}$ equipped with symplectic form $(\rho^{-1})^* \lambda$. By Equality (5.64) we get that for any $\ell \in \mathbb{P}(V)$ the restriction of $(\rho^{-1})^* \lambda$ to $F_{\ell}$ is zero; by Proposition 5.2 we get that (5.69) holds. □
Let $A := \rho(H^0(\theta)^\vee)$. Since $H^0(\theta)^\vee$ is a Lagrangian subspace of $H^0(\theta) \oplus H^0(\theta)^\vee$ equipped with the symplectic form $\lambda$ the above corollary gives that $A \in LG(\Lambda^3 V)$. It is clear from Diagram (5.40) that we have equality of reduced $Y = (Y_A)_{red}$ where $(Y_A)_{red}$ is the reduced $Y_A$. Since $\deg Y = 6 = \deg(Y_A)$ and $Y, Y_A$ are both Cartier divisors we get that

$$Y = Y_A.$$  \hspace{1cm} (5.70)

**Claim 5.14.** Keep notation and assumptions as above. Then $A \in LG(\Lambda^3 V)^0$.

**Proof.** First notice that $A \in LG(\Lambda^3 V)^\times$ simply because $j_4 \theta$ is a Casnati-Catanese sheaf. We notice also that

$$D_1(A, F) \setminus D_2(A, F) = Y \setminus singY.$$ \hspace{1cm} (5.71)

Thus we may apply Proposition (2.8) in order to prove the claim. Item (3) of the proposition is satisfied by (5.71) and (5.5) and hence $A \in LG(\Lambda^3 V)^0$. Alternatively one can check that Item (3) of Proposition (2.8) holds. The only unproved fact is that $W_A$ is smooth. By (2.11), (5.4) and (5.5) we know that $W_A$ is a local complete intersection; since $singY = (W_A)_{red}$ and $singY$ is smooth we get that if $W_A$ is not smooth then it is not reduced. Thus by Formula (5.4) we get that it suffices to show that $\deg(singY) = 40$; this follows at once form the formulae in Item (1) of Theorem (1.1) of [12].

Comparing Diagrams (4.23) and (5.49) we see that $\theta = \zeta_A$; by (1.17) and (6.1) we get that $\eta = \xi_A$. This completes the proof of Theorem (5.2).

### 6 An involution on a moduli space

Let $K^0_3$ be the set of isomorphism classes of couples $(X, H)$ where $X$ is a numerical $(K3)^{[2]}$ and $H$ an ample divisor on $X$ such that (1.3) and (a) of Section 11 both hold; couples $(X_i, H_i)$ for $i = 1, 2$ are isomorphic if there exists an isomorphism $\psi: X_1 \rightarrow X_2$ such that $\psi^*H_2 \sim H_1$. By Theorem 1.1 we have an identification

$$K^0_3 = LG(\Lambda^3 V)^0//\text{PGL}(V)$$ \hspace{1cm} (6.1)

where $V$ is a 6-dimensional complex vector-space. We remark that the second equality of (1.3) follows from the first one. Furthermore the first equality of (1.3) should be thought of as the analogue of self-intersection 2 for an ample divisor on a $K3$ surface, see Footnote (1). We recall also that Condition (a) is an open condition. Thus $K^0_3$ is an open subset of the moduli space of couples $(X, H)$ where $X$ is a numerical $(K3)^{[2]}$ and $H$ an ample divisor on $X$ of square 2 for Beauville’s quadratic form - actually the larger moduli space of couples $(X, H)$ with $H$ big and nef is a better setting for what follows. We represent a point of $K^0_3$ by $[X_A]$ where $A \in LG(\Lambda^3 V)^0$. Let $\delta_V: LG(\Lambda^3 V) \xrightarrow{\sim} LG(\Lambda^3 V^\vee)$ be the isomorphism of (3.2). The subset of $LG(\Lambda^3 V)^0$ appearing in (3.3) is open dense and $\text{PGL}(V)$-invariant hence

$$U := (LG(\Lambda^3 V)^0 \cap \delta_V^{-1}LG(\Lambda^3 V^\vee)^0) //\text{PGL}(V).$$ \hspace{1cm} (6.2)

is an open and dense subset of $K^0_3$. If $[X_A] \in U$ then we have the natural double cover $X_{A^+} \rightarrow Y_{A^+}$ and furthermore $[X_{A^+}] \in K^0_2$ by Theorem 1.1.
Hence duality defines an involution

\[
\begin{array}{ccc}
\mathcal{U} & \delta & \mathcal{U} \\
[X_A] & \rightarrow & [X_A] \\
\end{array}
\] (6.3)

Let’s prove that \(\delta\) is not the identity. We consider the example of Mukai that we presented in the proof that Theorem (5.2) implies Theorem (1.1). Thus \(S\) is a \(K3\) given by (5.12), satisfying (5.13), and \(Y, X\) are given by (5.14) and (5.15) respectively. Then \(Y \subset |I_S(2)|\) corresponds to a certain \(M \in LG(\wedge^3 H^0(I_S(2)))^0\) i.e. \(Y = Y_M\). According to Subsubsection (5.4.2) of [11] the dual \(Y_M^\vee \subset |I_S(2)|\) is described as follows. By (5.13) \(S\) is cut out by quadrics and contains no lines hence we have a well-defined regular map

\[
\begin{array}{ccc}
S[2] & \xrightarrow{g} & |I_S(2)|^\vee \\
[Z] & \mapsto & \{Q \mid Q \supset \langle Z \rangle \}
\end{array}
\] (6.4)

where \(\langle Z \rangle \subset \mathbb{P}^6\) is the unique line containing \(Z\). Then \(Y_M^\vee = Im(g)\). One has \(|I_F(2)| \in Im(g)\) - in fact \(|I_F(2)|\) is the image by \(g\) of

\[
B_S := \{[Z] \in S[2] \mid \langle Z \rangle \subset F\}.\] (6.5)

Iskovskih (Cor. (6.6) of [9]) proved that \(B_S \cong \mathbb{P}^2\). We proved that \(g\) has degree 2 onto its image and that \(deg Y_M^\vee = 6\). (6.6)

For \([Z] \in (S[2] \setminus B_S)\) there exists a unique conic \(C \subset F\) containing \(Z\). Then \(C \cap S\) is a scheme of length 4 containing \(Z\) and there is a well-defined residual scheme \(Z'\) of \(Z\), see Subsection (4.3) of [11]. Let \(\phi([Z]) := [Z']\); then

\[
g^{-1}(g([Z])) = \{[Z], \phi([Z])\}.\] (6.7)

For \([Z]\) generic \(Z'\) can be characterized as the unique \(Z' \subset S\) of length 2 such that

\[
Z' \cap Z = \emptyset, \quad \langle Z' \rangle \cap \langle Z \rangle \neq \emptyset\] (6.8)

**Proposition 6.1.** Keep notation and assumptions as above. Then \(|I_F(2)| \in Y_M^\vee\) is a point of multiplicity 3.

**Proof.** Let \(\Lambda \subset |I_F(2)|\) be a generic linear subspace with \(\dim \Lambda = 3\). Thus

\[
\Lambda^\perp := \{\Omega \in |I_S(2)|^\vee \mid \Lambda \subset \Omega\}
\] (6.9)

is a generic line in \(|I_S(2)|^\vee\) containing \(|I_F(2)|\). By (6.6) we must prove that

\[
\sharp (\Lambda^\perp \cap Y_M^\vee \setminus \{|I_F(2)|\}) = 3.\] (6.10)

Let \([Z] \in S[2];\) then \(g([Z]) \in (\Lambda^\perp \cap Y_M^\vee \setminus \{|I_F(2)|\})\) if and only if

\[
\langle Z \rangle \subset \bigcap_{t \in \Lambda} Q_t \quad \langle Z \rangle \not\subset F.\] (6.11)

(Here \(Q_t \subset \mathbb{P}^6\) is the quadric corresponding to \(t\).) In the proof of Item (1) of Lemma (4.20) of [11] we showed that

\[
\bigcap_{t \in \Lambda} Q_t = F \cup \langle C \rangle\] (6.12)
where $C \subset F$ is a certain conic with span $\langle C \rangle$ such that $F \cap \langle C \rangle = C$. Let $\mathcal{Q} \subset \mathbb{P}^6$ be a quadric such that $S = F \cap \mathcal{Q}$, see (5.12). By (5.13) the surface $S$ does not contain conics and hence $\mathcal{Q} \cap C$ is a finite set of length 4. Since $\Lambda$ is chosen generically $\mathcal{Q} \cap C$ consists of 4 distinct points $p_1, \ldots, p_4$ and hence

\[ \langle C \rangle \cap S = \{p_1, \ldots, p_4\}. \quad (6.13) \]

Thus $[Z] \in S^{[2]}$ satisfies (6.11) if and only if $Z$ is a subset of $\{p_1, \ldots, p_4\}$; since there are 6 such $Z$'s and since $g$ has degree 2 onto its image (see (6.7)) we get that (6.10) holds.

The above proposition shows that the involution $\delta$ is not the identity. In fact $M \in LG(\wedge^3 H^0(I_S(2)))$ and hence every singular point of $Y_M$ has multiplicity 2. Since $Y_M^\vee$ has a point of multiplicity 3 we get that $Y_M^\vee$ is not projectively isomorphic to $Y_M$. The proposition also shows that if we want the codomain of $\delta$ to be $K_2$ then $\delta$ is not defined at $Y_M$.

Let $\mathcal{K}_2$ be the set of isomorphism classes of couples $(X, H)$ where $X$ is a numerical $(K3)^{[2]}$, $H$ is a big and nef divisor on $X$ such that (1.3) holds and furthermore $(X, H)$ is a “limit” of $(X', H')$ parametrized by $\mathcal{K}_2^0$; is it true that $\delta$ extends to a regular involution defined on all of $\mathcal{K}_2^0$?

### 7 Particular examples of EPW-sextics

The question we briefly address is the following. Given $(X, H)$ satisfying (1.3) and Item (a) of Section (1) how do we describe an $A \in LG(\wedge^3 H^0(\mathcal{O}_X(H)))$ such that $X_A \cong X$? More generally we may ask the same question for an $(X', H')$ which is a limit of $(X, H)$ as above. A similar question for the Fano variety of lines on a cubic 4-fold is studied by Hassett in [6].

#### 7.1 Another description of $Y_A$

Let $V$ be a 6-dimensional complex vector-space. Let $LG(\wedge^3 V)^\dagger \subset LG(\wedge^3 V)$ be the open subset of $A$ such that there exists $W \subset V$ of codimension 1 such that

\[ \wedge^3 W \cap A = \emptyset. \quad (7.1) \]

For $A \in LG(\wedge^3 V)^\dagger$ we will describe $Y_A$ as a set of degenerate quadrics in $\mathbb{P}(\wedge^3 W)$. Using this description we will propose an answer to the question asked above for those $Y$ that were described by Mukai in Ex.(5.17) of [10] and that we have used in Sections (5)-(6). Let $\ell_0 \in \mathbb{P}(V)$ be such that

\[ V = \ell_0 \oplus W. \quad (7.2) \]

Thus we have a decomposition

\[ \wedge^3 V = (\ell_0 \otimes \wedge^2 W) \oplus \wedge^3 W. \quad (7.3) \]

Both addends are Lagrangian subspaces of $\wedge^3 V$ and hence the symplectic form $\sigma$ induces an isomorphism

\[ \wedge^3 W \overset{\sim}{\rightarrow} (\ell_0 \otimes \wedge^2 W)^\vee. \quad (7.4) \]
We choose a non-zero $v_0 \in \ell_0$. Multiplication by $v_0$ defines an isomorphism $\wedge^2 W \to \ell_0 \otimes \wedge^2 W$ and hence \(\text{(4.4)}\) becomes an isomorphism
\[
\wedge^3 W \xrightarrow{\sim} \wedge^2 W^\vee. \tag{7.5}
\]
Let $\text{vol}_W : \wedge^5 W \xrightarrow{\sim} \mathbb{C}$ be the trivialization defined by setting $\text{vol}_W(\tau) = \text{vol}(v_0 \wedge \tau)$; then Isomorphism \(\text{(4.3)}\) is given by the perfect pairing
\[
\wedge^2 W \times \wedge^3 W \longrightarrow \mathbb{C}
\alpha,\beta \longmapsto \text{vol}_W(\alpha \wedge \beta) \tag{7.6}
\]
Let
\[U := \mathbb{P}(V) \setminus \mathbb{P}(W). \tag{7.7}\]
Tensorizing \(\text{(4.3)}\) by $\mathcal{O}_U$ we get a symplectic trivialization of $\wedge^3 V \otimes \mathcal{O}_U$ with $\mathcal{L} := \wedge^2 W \otimes \mathcal{O}_U$ and $\mathcal{L}^\vee := \wedge^3 W \otimes \mathcal{O}_U$. Notice that $F_\ell$ is transversal to $\wedge^3 W$ for every $\ell \in U$. Thus $D_1(A,F,U,\mathcal{L},\mathcal{L}^\vee)$ is well-defined. Let $q_A \in \text{Sym}^2(\wedge^2 W^\vee)$ be as in \(\text{(4.3)}\). For $w \in W$ let $q_w \in \text{Sym}^2(\wedge^2 W^\vee)$ be the quadratic form defined by
\[q_w(\alpha) := w \wedge \alpha \wedge \alpha. \tag{7.8}\]
Let $\ell = [v_0 + w]$; as is easily checked $q_w$ is the quadratic form defined by the symmetric map $\phi_{F_\ell}$. The map
\[
W \quad \longrightarrow \quad U \\
|w| \quad \mapsto \quad [v_0 + w] \tag{7.9}
\]
is an isomorphism; from now on we identify $U$ with $W$ via the above map. Thus
\[
D_1(A,F,U,\mathcal{L},\mathcal{L}^\vee) = Y_A \setminus \mathbb{P}(W) = \{w \in W | \det(q_A - q_w) = 0\}. \tag{7.10}
\]
Now notice that the $q_w$’s are the Plücker quadratic forms whose zero-locus is $\text{Gr}(2,W) \subset \mathbb{P}(\wedge^2 W)$. Let $Q_A \subset \mathbb{P}(\wedge^2 W)$ be $Q_A = V(q_A)$ and let $Z_A \subset \mathbb{P}(\wedge^2 W)$ be
\[Z_A := Q_A \cap \text{Gr}(2,W). \tag{7.11}\]
Thus $|I_{Z_A}(2)|$ is the span of $Q_A$ and $|I_{\text{Gr}(2,W)}(2)|$, in particular $|I_{Z_A}(2)| \cong \mathbb{P}^5$. Let $\Sigma_A$ be the degree-10 divisor on $|I_{Z_A}(2)|$ parametrizing singular quadrics. Each quadric $V(q_w) \in |I_{\text{Gr}(2,W)}(2)|$ is singular with $\dim(\text{sing}V(q_w)) = 3$ and hence we have
\[
\Sigma_A = 4|I_{\text{Gr}(2,W)}(2)| + \Sigma'_A. \tag{7.12}
\]
Thus $\Sigma'_A$ is a degree-6 divisor. The above discussion proves the following result.

**Proposition 7.1.** Keep notation and assumptions as above. Then
\[Y_A \cong \Sigma'_A. \tag{7.13}\]

Now we can propose an explicit description of those $M \in LG(\wedge^3 V)^0$ such that $Y_M$ is one of Mukai’s examples. Actually we propose a description of those $A \in LG(\wedge^3 V)$ for which $Y_A \cong Y_M$; if the description is correct then one simply sets $M = A^\perp$. By Proposition \(\text{(5.1)}\) we know that $Y_M$ has a point of multiplicity $3$ hence we choose $A$ such that
\[
\dim(\text{sing}Q_A) = 2 \tag{7.14}
\]
because $\Sigma_A$ will have a point of multiplicity at least 3 at $Q_A$ - of multiplicity equal to 3 if $A$ is generic. The question is: is it true that if (7.14) holds then
\[ Y_A \cong g(S[2]) \] (7.15)
where $S \subset \mathbb{P}^6$ is a certain $K3$ surface of genus 6 and $g$ is as in Section [13]. Our observation is that one can associate to such an $A$ a $K3$ surface $S \subset \mathbb{P}^6$. In fact consider the duals $Q_A^\vee, \text{Gr}(2, W)^\vee \subset \mathbb{P}(\wedge^2 W)^\vee$. By (7.14) $Q_A^\vee$ is a smooth quadric hypersurface in $\text{sing}(Q_A)^\perp \cong \mathbb{P}^6$ and $\text{Gr}(2, W)^\vee = \text{Gr}(2, W)^\vee$. Thus
\[ S := Q_A^\vee \cap \text{Gr}(2, W)^\vee \] (7.16)
is indeed a $K3$ surface of genus 6; our guess is that (7.15) holds with the above $S$.

7.2 Non-reduced $Y_A$‘s

We will consider examples of $(X_0, H_0)$ a limit of $(X, H)$ with $X$ a numerical $(K3)[2]$ and $H$ an ample divisor on $X$ satisfying both (7.20) and Item (a) of Section [13] and such that $|H_0|$ is base-point free with
\[ X_0 \rightarrow |H_0|^\vee \] (7.17)
of degree higher than 2, namely 4 or 6. In the examples there is an anti-symplectic involution $\phi : X \rightarrow X$ with quotient map $f : X \rightarrow Y$. There is a map $j_0 : Y \rightarrow |H_0|^\vee$ such that Map (7.17) is the composition
\[ X_0 \xrightarrow{f_0} Y_0 \xrightarrow{j_0} |H_0|^\vee. \] (7.18)
Of course $j_0$ is not an embedding, it is finite onto its image. We consider the decomposition $f_{0*}\mathcal{O}_{X_0} = \mathcal{O}_{Y_0} \oplus \eta$ where $\eta$ is the $(-1)$-eigenspace for the action of $\phi^*_0$. Let $\theta := \eta \otimes j_{0*}\mathcal{O}_{|H_0|^\vee}(3)$. Assuming that $\theta$ is globally generated we may consider the exact sequence
\[ 0 \rightarrow G \rightarrow H^0(\theta) \otimes \mathcal{O}_{|H_0|^\vee} \xrightarrow{\epsilon, j_{0*}\theta} 0 \] (7.19)
where $\epsilon$ is the evaluation map. Then $G \cong \Omega^3(3)|H_0|^\vee$. One can construct a commutative diagram as in (7.19) and proceed as in Section [6] to get a decomposition into Lagrangian subspaces
\[ \wedge^3 H^0(\mathcal{O}_{X_0}(H_0))^\vee = H^0(\theta) \oplus H^0(\theta)^\vee. \] (7.20)
Then $H^0(\theta)^\vee \in \text{LG}(\wedge^3 H^0(\mathcal{O}_{X_0}(H_0))^\vee)$ is a point corresponding to $X_0$. The question of course is to describe Decomposition (7.20). Our first example is $X_0 = S[2]$ where $\pi : S \rightarrow \mathbb{P}^2$ is a double cover branched over a smooth sextic. Thus $S$ is a $K3$ surface and $H_S := \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$ is an ample divisor on $S$ with $H_S \cdot H_S = 2$. Let $L \subset \mathbb{P}^2$ be a line and let
\[ D_L := \{ p_1 + p_2 \in S[2] | \{ p_1, p_2 \} \cap L \neq \emptyset \}. \] (7.21)
Then $D_L$ is an ample Cartier divisor on $X_0$, call it $H_0$ (thus the pull-back of $H_0$ by the desingularization $S[2] \rightarrow S[2]$ is big and nef). There exist smoothings $X$ of $X_0$ for which $H_0$ deforms to an ample divisor $H$ on $X$. Then $X$ is a
deformation of $(K3)^2$ and $H$ satisfies \ref{13} and Item (a) of Section \ref{1}. The involution $\phi_0: X_0 \to X_0$ is given by
\[
\phi_0(p_1 + p_2) = \iota(p_1) + \iota(p_2)
\]  
(7.22)
where $\iota: S \to S$ is the covering involution of $\pi$. Let $q: S^2 \to S^{(2)}$ be the quotient map; then $q^*\mathcal{O}_{X_0}(H_0) \cong \mathcal{O}_S(H_S) \boxtimes \mathcal{O}_S(H_S)$ and hence
\[
q^*H^0(\mathcal{O}_{X_0}(kH_0)) = \text{Sym}^2 H^0(\mathcal{O}_S(kH_S)).
\]  
(7.23)
Let $U := H^0(\mathcal{O}_S(H_S)) = H^0(\mathcal{O}_{P^2}(1))$. From \ref{1223} we get that $H^0(\mathcal{O}_{X_0}(H_0)) \cong \text{Sym}^2 U$. Furthermore the image of $j_0: Y_0 \to \mathbb{P}(\text{Sym}^2 U^\vee)$ is the chordal variety of the Veronese surface, i.e. the discriminant cubic, and $j_0$ is of degree 2 onto its image. Let $\sigma \in H^0(\mathcal{O}_S(3H_S))$ be a generator of the $(−1)$-eigenspace for the action of $\iota^*$; thus the divisor of $\sigma$ is the ramification divisor of $\pi$. One easily checks that we have an isomorphism
\[
\text{Sym}^3 U \underbrace{\xrightarrow{\sim}}_{\alpha} \quad q^*H^0(\theta)
\]
(Here $\pi^*: \text{Sym}^3 U = H^0(\mathcal{O}_{P^2}(3)) \hookrightarrow H^0(\mathcal{O}_S(3H_S))$. On the other hand we have the decomposition of $GL(U)$-modules
\[
\wedge^3 H^0(\mathcal{O}_{X_0}(H_0))^\vee = \wedge^3(\text{Sym}^2 U^\vee) = \text{Sym}^3 U \oplus \text{Sym}^3 U^\vee.
\]  
(7.25)
One can check that given Isomorphism \ref{13} this is Decomposition \ref{12} in the present case. The last example is similar but we must state that we have not checked the details. Let $S \subset P^3$ be a smooth quartic containing no lines and let $X_0 := S^{[2]}$. Let $U := H^0(\mathcal{O}_S(1))$. We have a map
\[
X_0 = S^{[2]} \xrightarrow{\sigma} \text{Gr}(2, U^\vee)
\]
(7.26)
Let $p: \text{Gr}(2, U^\vee) \hookrightarrow \mathbb{P}(\wedge^2 U^\vee) = \mathbb{P}^5$ be the Plücker embedding and let $H_0 := (pg)^*\mathcal{O}_{P^5}(1)$ be the Plücker class. Then $(X_0, H_0)$ is the limit of $(X, H)$ where $H$ satisfies \ref{13} and (a) of Section \ref{1}. The involution $\phi_0: X_0 \to X_0$ associates to $[Z]$ the residual of $Z$ in $(Z) \cap S$. Then $j_0: Y_0 \to \text{Gr}(2, U^\vee)$ is finite of degree 3. Since $H^0(\mathcal{O}_{X_0}(H_0)) = \wedge^2 U$ it is natural to guess that in the present case Decomposition \ref{12} is the decomposition of $GL(U)$-modules
\[
\wedge^3(\wedge^2 U^\vee) = \text{Sym}^2 U \oplus \text{Sym}^2 U^\vee.
\]  
(7.27)

References

[1] A. Beauville, *Variétés Kähleriennes dont la première classe de Chern est nulle*, J. Differential geometry 18, 1983, pp. 755-782.

[2] A. Beauville - R. Donagi, *La variétés des droites d’une hypersurface cubique de dimension* 4, C. R. Acad. Sci. Paris Sér. I Math. 301, 1985, pp. 703-706.

[3] G. Casnati - F. Catanese, *Even sets of nodes are bundle symmetric*, J. Diff. Geom. 47, 1997, pp. 237-256.
[4] D. Eisenbud - S. Popescu - C. Walter, *Lagrangian subbundles and codimension 3 subcanonical subschemes*, Duke Math. J. 107, 2001, pp. 427-467.

[5] W. Fulton - P. Pragacz, *Schubert Varieties and Degeneracy Loci*, LNM 1689.

[6] B. Hassett, *Special cubic fourfolds*, Compositio Math. 120, 2000, pp. 1-23.

[7] D. Huybrechts, *Compact hyper-Kähler manifolds: basic results*, Invent. Math. 135, 1999, pp. 63-113.

[8] D. Huybrechts, *Erratum: “Compact hyper-Kähler manifolds: basic results”* [Invent. Math. 135 (1999), no. 1, 63–113], Invent. Math. 152, 2003, pp. 209-212.

[9] V. A. Iskovskih, *Fano 3-folds, I*, Math. USSR Izvestija, Vol. 11, 1977, pp. 485-527.

[10] S. Mukai, *Moduli of vector bundles on K3 surfaces and symplectic manifolds*, Sugaku Expos. 1, 1988, pp. 139-174.

[11] K. G. O’Grady, *Involutions and linear systems on holomorphic symplectic manifolds*, math.AG/0403519 to appear on GAFA.

[12] K. G. O’Grady, *Irreducible symplectic 4-folds numerically equivalent to (K3)[2]*, math.AG/0504434.

[13] C. Okonek - M. Schneider - H. Spindler, *Vector bundles on Projective spaces*, Progress in Mathematics 3, Birkhäuser (1980).

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