Generic rigidity of frameworks with orientation-preserving crystallographic symmetry

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Abstract

We extend our generic rigidity theory for periodic frameworks in the plane to frameworks with a broader class of crystallographic symmetry. Along the way we introduce a new class of combinatorial matroids and associated linear representation results that may be interesting in their own right. The same techniques immediately yield a Maxwell-Laman-type combinatorial characterization for frameworks embedded in 2-dimensional cones that arise as quotients of the plane by a finite order rotation.
1. Introduction

A crystallographic framework is an infinite planar structure, symmetric with respect to a crystallographic group, made of fixed-length bars connected by universal joints with full rotational freedom. The allowed continuous motions preserve the lengths and connectivity of the bars (as in the finite framework case) and (this is the new addition) symmetry with respect to the group $\Gamma$. However, the representation of $\Gamma$ is not fixed and may change. Figures [1] and [2] show examples. A crystallographic framework is rigid when the only allowed motions (that, additionally, must act on the representation of $\Gamma$) are Euclidean isometries and flexible otherwise.

The topic of this paper is the following question: Which crystallographic frameworks are rigid and which are flexible? In its most general form, this question doesn't seem
computationally tractable: even for finite frameworks, the best known algorithms rely on exponential-time Gröbner basis computations. However, generically—and almost all crystallographic frameworks are generic—we can say more with Theorem 1 (stated below in Section 1.3): generic rigidity and flexibility depend on the combinatorial type of the framework, given by a colored graph, which is a finite, directed graph with elements of a group on the edges. Moreover, Theorem 1 is a “good characterization” in that a polynomial time combinatorial algorithm can decide whether a colored graph corresponds to generically rigid crystallographic frameworks.

![Figure 2: A $\Gamma_4$-crystallographic framework: (a) A piece of an infinite crystallographic framework with $\Gamma_4$ symmetry. The group $\Gamma_4$ is generated by an order 4 rotation and translations. The fundamental domain of the $\Gamma_4$-action on $\mathbb{R}^2$ is shown as a dashed box. (b) The associated colored graph capturing the underlying combinatorics. The color coding conventions are as in Figure 1.](image)

Thus, Theorem 1 is a true analog of the landmark Maxwell-Laman Theorem [7, 12] from rigidity theory, which characterizes generic rigidity and flexibility of finite frameworks in the plane. We stress that the genericity hypotheses made by Theorem 1 are on the geometry of the framework only, which is the same as genericity assumptions from the theory of finite frameworks.

1.1. Algebraic definition of rigidity and flexibility A $\Gamma$-crystallographic framework is given by the data $(\tilde{G}, \varphi, \tilde{\ell})$, where $\tilde{G} = (\tilde{V}, \tilde{E})$ is an infinite graph, $\Gamma$ is a crystallographic group, $\varphi$ is a free $\Gamma$-action with finite quotient on $\tilde{G}$, and $\tilde{\ell}$ is an assignment of positive
lengths to each edge $ij \in \tilde{E}$. To keep the terminology in this framework manageable, we will refer simply to frameworks when the context is clear, with the understanding that the frameworks appearing in the paper are crystallographic.

A realization $G(p, \Phi)$ of the abstract framework $(\tilde{G}, \varphi, \tilde{\ell})$ is defined to be an assignment $p = (p_i)_{i \in \tilde{V}}$ of points to the vertices of $\tilde{G}$ and a representation $\Phi$ of $\Gamma \hookrightarrow \text{Euc}(2)$ by Euclidean isometries acting discretely and co-compactly, such that

$$\|p_i - p_j\| = \tilde{\ell}_{ij} \quad \text{for all edges } ij \in \tilde{E}$$

$$\Phi(\gamma) \cdot p_i = p_{\gamma(i)} \quad \text{for all group elements } \gamma \in \Gamma \text{ and vertices } i \in \tilde{V}$$

Equation (1) says that a realization respects the given edge lengths, which appears in the theory of finite frameworks. Equation (2) says that, if we hold $\Phi$ fixed, regarded as a map $p : \tilde{V} \to \mathbb{R}^2$, $p$ is equivariant. However, $\Phi$ is, in general, not fixed. This is a very important feature of the model: the motions available to the framework include those that deform the representation $\Phi$ of $\Gamma$, provided this happens in a way compatible with the abstract $\Gamma$-action $\varphi$.

1.2. Rigidity via realization and configuration spaces The realization space $\mathcal{R}(\tilde{G}, \varphi, \tilde{\ell})$ (shortly $\mathcal{R}$) of an abstract framework is defined as the set of its realizations. Motions of the framework are, then, continuous paths in the realization space. To factor out trivial motions, we define the configuration space $\mathcal{C}$ to be $\mathcal{C} = \mathcal{R}/\text{Euc}(2)$. With this definition, we can formally define rigidity: a realization $G(p, \Phi)$ is rigid if it is isolated in $\mathcal{C}$; otherwise the realization of the framework is flexible, and there is a continuous path in $\mathcal{C}$ through $(p, \Phi)$ giving a motion of the framework. (See Section 25 for a detailed treatment of these spaces.)

We remark that the definition makes it clear that we are interested in what is sometimes called “local rigidity” in the literature: the configuration space may have multiple connected components, each with a different dimension. We are not concerned with the stronger notion of “global rigidity”, which requires that $\mathcal{C}$ be a single point.

1.3. Main result: Crystallographic Maxwell-Laman Our main result is the following “Maxwell-Laman-type” theorem for crystallographic frameworks where the symmetry group is generated by translations and a finite order rotation. The “$\Gamma$-colored-Laman graphs” appearing in the statement are defined in Section 13; genericity is defined in detail in Section 26 but the term is used in the standard sense of algebraic geometry: generic frameworks are the (open, dense) complement of a proper algebraic subset of the configuration space.

**Theorem 1.** Let $\Gamma$ be a crystallographic group generated by translations and rotations. A generic $\Gamma$-crystallographic framework $(\tilde{G}, \varphi, \tilde{\ell})$ is minimally rigid if and only if its colored quotient graph is $\Gamma$-colored-Laman.
1.4. The Main Theorem for orbifolds An alternative interpretation of Theorem 1 is that it characterizes rigidity of finite frameworks in Euclidean orbifolds with geodesic bars. The orbifold is obtained by taking the quotient $\mathbb{R}^2/\Gamma$, where $\Gamma$ is generated by translations and rotations. This is what is meant elsewhere in the literature when “torus” [17, 18] or “cone” [25] frameworks are discussed. Since we don’t work in this formalism, we leave the issue of an intrinsic Theorem 1 aside.

1.5. Cone frameworks A particularly interesting simplification—that we will see as a “warm up” for Theorem 1—is when the symmetry is given by a rotation around the origin through angle $2\pi/k$. In this case, the quotient is a flat cone with opening angle $2\pi/k$, so we call such frameworks cone frameworks. For the purposes of cone frameworks, we will identify $\mathbb{Z}/k\mathbb{Z}$ with this subgroup of $SO(2)$.

The formalism is very similar to that for crystallographic frameworks, except everything is finite. A cone framework is given by $(\tilde{G}, \varphi, \tilde{\ell})$, where $\tilde{G} = (\tilde{V}, \tilde{E})$ is a finite graph, $\varphi$ is a free $\mathbb{Z}/k\mathbb{Z}$-action, and $\tilde{\ell}$ is an assignment of positive lengths to each edge $ij \in \tilde{E}$. Realizations $\tilde{G}(p)$ of the abstract framework $(\tilde{G}, \varphi, k, \tilde{\ell})$ are point sets $p = (p_i)_{i \in \tilde{V}}$ satisfying

$$||p_j - p_i|| = \tilde{\ell}_{ij} \quad \text{for all edges } ij \in \tilde{E}$$

$$\gamma \cdot p_i = p_{\gamma(i)} \quad \text{for all group elements } \gamma \in \mathbb{Z}/k\mathbb{Z} \text{ and vertices } i \in \tilde{V}$$

and the definitions of the realization and configurations spaces, and well as rigidity and flexibility are similar to the crystallographic case.

We prove the following theorem in Section 28: cone-Laman graphs are defined in Section 15.

**Theorem 2.** A generic cone framework is minimally rigid if and only if the associated colored graph $(G, \gamma)$ is cone-Laman.

1.6. Crystallographic direction networks In order to prove the rigidity Theorems 1 and 2, we will use crystallographic direction networks. A $\Gamma$-crystallographic direction network $(\tilde{G}, \varphi, \tilde{d})$ consists of an infinite graph $\tilde{G}$ with a free $\Gamma$-action $\varphi$ on the edges and vertices, and an assignment of a direction $\tilde{d}_{ij}$ to each edge $ij \in \tilde{E}$.

We define a realization $G(p, \Phi)$ of $(\tilde{G}, \varphi, \tilde{d})$ to be a mapping of $\tilde{V}$ to a point set $p$ and a representation $\Phi$ of $\Gamma$ by Euclidean isometries such that

$$\langle p_i - p_j, \tilde{d}_{ij} \rangle = 0 \quad \text{for all edges } ij \in \tilde{E}$$

$$\Phi(\gamma) \cdot p_i = p_{\gamma(i)} \quad \text{for all group elements } \gamma \in \Gamma \text{ and vertices } i \in \tilde{V}$$

Since setting all the $p_i$ equal and $\Phi$ to be trivial produces a realization, the realization space is never empty. For our purpose, though, such realizations are degenerate. We

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1The proof tells us more, namely that the same theorems about cone frameworks are true for any order $k$ rotation, but for simplicity, we restrict ourselves to the case arising as part of the crystallographic setting.
define a realization of a crystallographic direction network to be \textit{faithful} if none of the edges of \( G \) are realized with coincident endpoints.

1.7. Crystallographic Direction Network Theorem Our second main result is an exact characterization of when a generic direction network admits a faithful realization, in the spirit of Whiteley’s Parallel Redrawing Theorem \[26, \text{Section 4}\].

\textbf{Theorem 3.} Let \( \Gamma \) be a crystallographic group generated by translations and rotations. A generic realization of a \( \Gamma \)-crystallographic direction network \((\tilde{G}, \varphi, \tilde{d})\) has a faithful realization if and only if its associated colored graph is \( \Gamma \)-colored-Laman. This realization is unique up to translation and scaling.

1.8. Proof strategy for Theorem 1 The deduction of the rigidity Theorem 1 from Theorem 3 uses the natural extension of our \textit{periodic direction network method} from \[10\]. Briefly, the steps are:

- We reduce the problem of rigidity, as is standard in the field, to a linearization called \textit{infinitesimal rigidity}. (This is defined in Section 26)
- We then show that minimal infinitesimal rigidity of a colored graph \((G, \gamma)\) coincides with generic direction networks on \((G, \gamma)\) having a faithful realization up to translation and scaling. (This is done in Section 27)
- Theorem 1 is then immediate from Theorem 3.

Although the steps in Sections 25–27 are, in light of \[10, 24\] somewhat routine, we remark at this point that the translation between infinitesimal rigidity and faithful direction network realizability does \textit{not} go through when the symmetry group contains reflections. Thus, this additional hypothesis is forced by our proof method. While, with some additional effort, we might be able to extend the Direction Network Theorem 3 to all two-dimensional crystallographic groups, this improvement would not, by itself, give a more general rigidity theorem.

1.9. Roadmap Most of the work in this paper is in the proof of Theorem 3 which proceeds in three parts:

- Part I studies the crystallographic groups \( \Gamma_k \) for \( k = 2, 3, 4, 6 \), giving convenient coordinates to their representation spaces (Sections 3–5) and developing a matroid on the \( \Gamma_k \) (Proposition 8.2).
- Part II contains the combinatorial part of the proof of Theorem 3, developing \( \Gamma \)-graded sparse graphs (definitions are given in Sections 12 and 13) in terms of matroidal (Proposition 12.3) and decomposition (Proposition 12.4) properties.
Part III then develops the theory of direction networks and links the combinatorics of colored graphs defined by sparsity conditions to the geometry of direction networks. The main result of Part III is Theorem 3 which is deduced from Proposition 21.1.

Readers familiar with [10] will notice that the broad strokes of the proof plan is similar, but that there is no “natural representation” step, in which dependence and independence in colored graph matroids are related to determinantal formulas. The reason for this is that, in the crystallographic case, the variables arising from direction network realization problems do not separate out as cleanly. Thus, an alternative viewpoint of Part III is that it introduces new techniques for proving linear representability of sparsity matroids.

1.10. Related work The results of this paper are a direct extension of the theory we introduced in [10], and they stand on a similar foundation. Our paper [10] contains a detailed discussion from several historical perspectives.

The general area of rigidity with symmetry has been somewhat active in the past few years, but the results here are independent of much of it. For completeness, we review some work along similar lines. A specialization of our [10, Theorem A] is due to Ross [17, 18]. Schulze [20, 21] and Schulze and Whiteley [22] studied the question of when “incidental” symmetry induces non-generic behaviors in finite frameworks, which is a different setting than the “forced” symmetry we consider here and in [10]. Ross, Schulze, and Whiteley [19] have studied the present problems, but they do not give any combinatorial characterizations. Borcea and Streinu [4] have proposed a kind of “doubly generic” periodic rigidity, where the combinatorial model does not include the colors on the quotient graph.

1.11. Acknowledgements We thank Igor Rivin for encouraging us to take on this project and many productive discussions on the topic. This work is part of a larger effort to understand the rigidity and flexibility of hypothetical zeolites, which is supported by CDI-I grant DMR 0835586 to Rivin and M. M. J. Treacy. LT’s final preparation of this paper was funded by the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no 247029.
I. Groups

2. Crystallographic group preliminaries

In this section, we review some basic facts about crystallographic groups generated by translations and rotations.

2.1. Facts about the Euclidean group

The Euclidean isometry group $Euc(d)$ in any dimension $d$ admits the following short exact sequence:

$$1 \to \mathbb{R}^d \to Euc(d) \to O(d) \to 1$$

where $O(d)$ is the orthogonal group. The subgroup $\mathbb{R}^d < Euc(d)$ is the subgroup of translations and $Euc(d) \to O(d)$ is the map that associates to an isometry $\psi$ its derivative at the origin $D\psi_0$. This short exact sequence splits, since $O(d)$ is naturally isomorphic to the subgroup of $Euc(d)$ consisting of isometries fixing the origin.

Consequently, $Euc(d)$ is isomorphic to the semidirect product $\mathbb{R}^d \rtimes O(d)$ with group operation:

$$(v, r) \cdot (v', r') = (v + r \cdot v', r r')$$

Since our setting is 2-dimensional, from now on, we are interested in $Euc(2)$. In the two dimensional case, we have the following simple lemma, which we state without proof.

**Lemma 2.1.** Any nontrivial orientation-preserving isometry of the Euclidean plane is either a rotation around a point or a translation.

Thus, when we refer to orientation-preserving elements of $Euc(2)$ we call them simply “rotations” or “translations”. We denote the counterclockwise rotation around the origin through angle $2\pi/k$ by $R_k$.

2.2. Crystallographic groups

A 2-dimensional crystallographic group $\Gamma$ is a group admitting a discrete cocompact faithful representation $\Gamma \to Euc(2)$. We will denote by $\Phi$ discrete faithful representations of $\Gamma$. In this paper we are interested in the case where all the group elements are represented by rotations and translations (i.e., we disallow reflections and glides).

Bieberbach’s Theorems [2, 3] classify all crystallographic groups, and there are precisely five 2-dimensional crystallographic groups containing only translations and rotations. The first group which we denote by $\Gamma_1$ is $\mathbb{Z}^2$. The rest are all semidirect products.
of $\mathbb{Z}^2$ with a cyclic group. Namely, for $k = 2, 3, 4, 6$, we have $\Gamma_k = \mathbb{Z}^2 \rtimes \mathbb{Z}/k\mathbb{Z}$. The action on $\mathbb{Z}^2$ by the generator of $\mathbb{Z}/k\mathbb{Z}$ is given by the following table.

| $k$ | 2         | 3         | 4         | 6         |
|-----|-----------|-----------|-----------|-----------|
| matrix | $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ | $\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$ |

We define the $\mathbb{Z}^2$ subgroup of $\Gamma_k$ to be the translation subgroup of $\Gamma_k$ and denote it by $\Lambda(\Gamma_k)$. We denote $\gamma \in \Gamma_k$, $k = 2, 3, 4, 6$ as $\gamma = (t, r)$ with $t \in \mathbb{Z}^2$ and $r \in \mathbb{Z}/k\mathbb{Z}$.

2.3. Remark on groups considered Since we are only interested in crystallographic groups of this form, the rest of the paper will consider $\Gamma_k$ only (and not more general crystallographic groups). Moreover, since the main objective of this paper is [10, Theorem A] when $k = 1$, we will treat only $k = 2, 3, 4, 6$ in what follows. However, the theory presented here specializes to $\Gamma_1$.

2.4. Finitely generated subgroups If $\gamma_1, \ldots, \gamma_t$ are element of $\Gamma_k$, we denote the subgroup generated by the $\gamma_i$ as $\langle \gamma_1, \ldots, \gamma_t \rangle$. If $\Gamma^1, \ldots, \Gamma^t$ are a sequence of finitely generated subgroups then $\langle \Gamma^1, \Gamma^2, \ldots, \Gamma^t \rangle$ denotes the subgroup generated by the union of some choice of generators for each $\Gamma^i$.

3. Representation space

$\Gamma$-crystallographic frameworks and direction networks are required to be symmetric with respect to the group $\Gamma$. However, the representation is allowed to flex. In this section, we formalize this flexing.

3.1. The representation space Let $\Gamma$ be a crystallographic group. We define the representation space $\text{Rep}(\Gamma)$ of $\Gamma$ to be

$$\text{Rep}(\Gamma) = \{ \Phi : \Gamma \to \mathbb{R}^2 \rtimes O(2) \mid \Phi \text{ is a discrete faithful representation} \}$$

3.2. Motions in representation space For our purposes a 1-parameter family of representations is a continuous motion if it is pointwise continuous. More precisely, identify $\text{Euc}(2) \cong \mathbb{R}^2 \times O(2)$ as topological spaces. Suppose $\Phi_t : \Gamma \to \text{Euc}(2)$ is a family of representations defined for $t \in (-\epsilon, \epsilon)$ for some $\epsilon > 0$. Then, $\Phi_t$ is a continuous motion through $\Phi_0$ if $\Phi_t(\gamma)$ is a continuous path in $\text{Euc}(2)$ for all $\gamma \in \Gamma$.

3.3. Coordinates for representations We now show how to give convenient coordinates for the representation space for each $\Gamma_k$ for $k = 2, 3, 4, 6$; by the classification of 2-dimensional crystallographic groups, these are the only cases we need to check. This next lemma follows readily from Bieberbach’s Theorems, but we give a proof in in Section 3.6 for completeness.
Lemma 3.1. The representation spaces of each of the $\Gamma_k$ can be given coordinates as follows:

- $\text{Rep}(\Gamma_2) \cong \{v_1, v_2, w \in \mathbb{R}^2 : v_1 \text{ and } v_2 \text{ are linearly independent}\}$
- $\text{Rep}(\Gamma_k) \cong \{v_1, w, \varepsilon \mid v_1 \neq 0, \varepsilon = \pm 1, v_1, w \in \mathbb{R}^2\}$ for $k = 3, 4, 6$

The vectors specify the “$\mathbb{R}^2$-part” of the image of a generator in $\text{Euc}(2) \cong \mathbb{R}^2 \rtimes O(2)$. The $v_i$ will be the $\mathbb{R}^2$-part of translational generators, and $w$ the $\mathbb{R}^2$-part of a rotational generator. The vector $w$ determines the rotation center, but is not the rotation center itself.

3.4. Coordinates for finite-order rotations

The following lemma describes the coordinates of an order $k$ rotation in $\text{Euc}(2)$, and it makes the meaning of the vector $w$ appearing in the statement of Lemma 3.1 precise: it determines how an order $k$ rotation acts on the origin.

Lemma 3.2. Let $\psi$ be an orientation-preserving element of $\text{Euc}(2)$. Then $\psi$ has order $k = 2, 3, 4, 6$ if and only if it is of the form $(w, R_k \varepsilon)$, where $R_k$ is the order $k$ counterclockwise rotation through angle $2\pi/k$.

Proof. If $\psi$ has the required form, then $\psi^k$ is $(w + R_k \cdot w + \cdots + R_k^{+k-1} \cdot w, R_k^{\pm k})$. The first coordinate corresponds to walking along the boundary of a regular $k$-gon, so it is the identity, and the second evidently is as well. On the other hand, if $\psi$ has order $k$ then an arbitrary point is either fixed or its iterated images under $\psi$ are the vertices of a regular polygon, but not necessarily visited in cyclic order. More specifically, a rotation though angle $j \frac{2\pi}{k}$ has order $k$ if and only if $j$ has order $k$ in $\mathbb{Z}/k\mathbb{Z}$. For $k = 2, 3, 4, 6$, however, $1$ and $-1$ are the only such $j$. \hspace{1cm} \Box

3.5. Generators for $\Gamma_k$

We also need a description of the generating sets for each of the $\Gamma_k$, which follows from their descriptions as semi-direct products of $\mathbb{Z}^2 \rtimes \mathbb{Z}/k\mathbb{Z}$.

Lemma 3.3. The following are generating sets for each of the $\Gamma_k$:

- $\Gamma_2$ is generated by the set $\{((1, 0), 0), ((0, 1), 0), ((0, 0), 1)\}$.
- $\Gamma_k$ is generated by the set $\{((1, 0), 0), ((0, 0), 1)\}$ for $k = 3, 4, 6$.

For convenience, we set the notation $r_k = ((0, 0), 1)$, $t_1 = ((1, 0), 0)$, and $t_2 = ((0, 1), 0)$. We now have the pieces in place to prove Lemma 3.1.

3.6. Proof of Lemma 3.1

We let $\Phi \in \text{Rep}(\Gamma_k)$ be a discrete, faithful representation. Thus $\Phi$ is determined by the images of the generators, so Lemma 3.3 tells us we need only to check $t_1$, $t_2$, and $r_k$.

The generators $t_i$ must always be mapped to translations: since they are infinite order and $\Phi$ is faithful, the only other possibility is an infinite order rotation. This would contradict $\Phi$ being discrete. Thus:
• For $k = 2$, $t_1$ and $t_2$ are mapped to translations $(v_1, \text{Id})$ and $(v_2, \text{Id})$.

• For $k = 3, 4, 6$, $t_1$ is mapped to a translation $(v_1, \text{Id})$.

Moreover, faithfulness and discreteness force:

• All the images $v_i$ to be non-zero.

• The images $v_1$ and $v_2$ to be linearly independent for $k = 1, 2$.

By Lemma 3.2 we must have $\Phi(r_k) = (w, R^\varepsilon_k)$ for some $w \in \mathbb{R}^2$ and $\varepsilon \in \{-1, 1\}$. Since $R_2$ is order 2, we have $\Phi(r_2) = (w, R_2)$ and $\varepsilon$ is unnecessary for $\Gamma_2$.

In the other direction, given the data described in the statement of the lemma, we simply define $\Phi(t_i)$ and $\Phi(r_k)$ as above. When $k = 3, 4, 6$, we set $\Phi(t_2) = (R_2^\varepsilon v_1, \text{Id})$. For arbitrary elements of $\Gamma$, we define $\Phi((m_1, m_2), m_3) = \Phi(t_1)^{m_1}\Phi(t_2)^{m_2}\Phi(r_k)^{m_3}$. It is straightforward to check $\Phi$ as defined is a homomorphism and is discrete and faithful.

3.7. Degenerate representations  When we are dealing with “collapsed realizations” of direction networks in Part III we will need to work with certain degenerate representations of $\Gamma_k$. The space

$$\overline{\text{Rep}(\Gamma_k)}$$

is defined to be representations of $\Gamma_k$ where we allow the $v_i$ to be any vectors. Topologically this is the closure of $\text{Rep}(\Gamma_k)$ in the space of all (not necessarily discrete or faithful) representations $\Gamma_k \to \text{Euc}(2)$.

3.8. Rotations and translations in crystallographic groups  As we have defined them, 2-dimensional crystallographic groups are abstract groups admitting a discrete faithful representation to $\text{Euc}(2)$. However, as we saw in the proof of Lemma 3.1 all group elements in $\Lambda(\Gamma_k)$ must be mapped to translations, and all group elements outside $\Lambda(\Gamma_k)$ must be mapped to rotations. Consequently, we will henceforth call elements of $\Lambda(\Gamma)$ “translations” and elements outside of $\Lambda(\Gamma_k)$ “rotations” (even though technically they are elements of an abstract group).

4. Subgroup structure

This short section contains some useful structural lemmas about subgroups of $\Gamma_k$.

4.1. The translation subgroup  For a subgroup $\Gamma' < \Gamma_k$, we define its translation subgroup $\Lambda(\Gamma')$ to be $\Gamma' \cap \Lambda(\Gamma_k)$. (Recall that $\Lambda(\Gamma_k)$ is the subgroup $\mathbb{Z}^2$ coming from the semidirect product decomposition of $\Gamma_k$.)
4.2. Facts about subgroups  With all the definitions in place, we state several lemmas about subgroups of $\Gamma_k$ that we need later.

**Lemma 4.1.** Let $\Gamma' < \Gamma_k$ be a subgroup of $\Gamma_k$, and suppose $\Gamma' \neq \Lambda(\Gamma')$. Then $\Gamma'$ is generated by one rotation and $\Lambda(\Gamma')$.

*Proof.* We need only observe that $\Gamma_k/\Lambda(\Gamma_k)$ is finite cyclic and contains $\Gamma'_k/\Lambda(\Gamma'_k)$ as a subgroup. ☐

This next lemma is straightforward, but useful. We omit the proof.

**Lemma 4.2.** Let $r_1, r_2 \in \Gamma_k$ be rotations. Then $\langle r_1, r_2 \rangle$ is a finite cyclic subgroup consisting of rotations if and only if some nontrivial powers $r_1^p$ and $r_2^q$ commute.

**Lemma 4.3.** Let $r' \in \Gamma_2$ be a rotation and $\Gamma' < \Lambda(\Gamma_2)$ a subgroup of the translation subgroup of $\Gamma_2$. Then $\Lambda(\langle r', \Gamma' \rangle) = \Gamma'$; i.e., after adding the rotation $r'$, the translation subgroup of the group generated by $r'$ and $\Gamma'$ is again $\Gamma'$.

*Proof.* All translation subgroups of $\Gamma_2$ are normal, and so the set $\{ gh \mid g = r', Id \ h \in \Gamma' \}$ is a subgroup and is equal to $\langle r', \Gamma' \rangle$. Clearly, the only translations are those elements of $\Gamma'$. ☐

5. The restricted representation space and its dimension

To define our degree of freedom heuristics in Part II, we need to understand how representations of $\Gamma_k$ restrict to subgroups $\Gamma' < \Gamma_k$, or equivalently, which representations of $\Gamma'$ extend to $\Gamma_k$. For $\Gamma' < \Gamma_k$, the restricted representation space of $\Gamma'$ is the image of the restriction map from $\Gamma_k$ to $\Gamma$, i.e.,

$$\text{Rep}_{\Gamma_k}(\Gamma') = \{ \Phi : \Gamma' \to \text{Euc}(2) \mid \Phi \text{ extends to a discrete faithful representation of } \Gamma_k \}$$

We define the notation $\text{rep}_{\Gamma_k}(\Gamma') := \dim \text{Rep}_{\Gamma_k}(\Gamma')$, since the dimension of $\text{Rep}_{\Gamma_k}(\Gamma')$ is an important quantity in what follows. Since it will be useful later, we also define:

$$T(\Gamma') := \begin{cases} 0 & \text{if } \Gamma' \text{ has a rotation} \\ 2 & \text{if } \Gamma' \text{ has no rotations} \end{cases}$$

Equivalently, we may define $T(\Gamma')$ as the dimension of the space of translations commuting with $\Gamma'$. In Section 23, we will show that $T(\Gamma')$ is the dimension of the space of collapsed solutions of a direction network for a connected graph $G'$ satisfying $\rho(\pi_1(G')) = \Gamma'$.

The dimension $\text{rep}_{\Gamma_k}(\Gamma')$ of the restricted representation space $\text{Rep}_{\Gamma_k}(\Gamma')$ is an important quantity for counting the degrees of freedom in a direction network. We now develop some properties of $\text{rep}_{\Gamma_k}(\cdot)$ and how it changes as new generators are added to a finitely generated subgroup.
5.1. Translation subgroups  For translation subgroups \( \Gamma' < \Gamma \), we are interested in the dimension of \( \text{Rep}_\Gamma(\Gamma') \). The following lemma gives a characterization for translation subgroups in terms of the rank of \( \Gamma' \).

**Lemma 5.1.** Let \( \Gamma' < \Gamma_k \) be a nontrivial subgroup of translations.

- If \( k = 3, 4, 6 \), then \( \text{rep}_{\Gamma_k}(\Gamma') = 2 \).
- If \( k = 1, 2 \), then \( \text{rep}_{\Gamma_k}(\Gamma') = 2r \), where \( r \) is the minimal number of generators of \( \Gamma' \).

In particular, \( \text{rep}_{\Gamma_k}(\Gamma') \) is even.

**Proof.** Suppose \( k = 3, 4, \) or 6. By Lemma 3.1, the space of representations of \( \Gamma_k \) is 4-dimensional and is uniquely determined by the parameters \( v_1, w \) and the sign \( \epsilon \). The group \( \Lambda(\Gamma_k) \cong \mathbb{Z}^2 \) is generated by \( t_1 \) and \( r_k t_1 r_k^{-1} \), and so any \( \gamma \in \Lambda(\Gamma_k) \) can be written uniquely as \( t_1^{m_1} r_k t_2^{m_2} r_k^{-1} \) for integers \( m_1, m_2 \). Thus, since \( \Phi(\gamma) \) is a translation,

\[
\Phi(\gamma) = (\Phi(t_1))^{m_1} \Phi(r_k) (\Phi(t_2))^{m_2} \Phi(r_k)^{-1} = (m_1 v_1, \text{Id}) (w, R_k^\epsilon) (m_2 v_1, \text{Id}) (w, R_k^{-\epsilon}) = (m_1 v_1 + m_2 R_k^\epsilon v_1, \text{Id})
\]

Hence, regardless of \( w \), any representation with the same \( v_1, \epsilon \) parameters restricts to the same representation on \( \Lambda(\Gamma_k) \) and thus also on \( \Gamma' \).

Suppose \( k = 1, 2 \). In this case by the proof of Lemma 3.1 any discrete faithful representation \( \Lambda(\Gamma_k) \rightarrow \text{Euc}(2) \) extends to a discrete faithful representation of \( \Gamma_k \). Since \( \Lambda(\Gamma_k) \cong \mathbb{Z}^2 \), any discrete faithful representation of its subgroups to \( \mathbb{R}^n \) extends to \( \Lambda(\Gamma_k) \) and hence \( \Gamma_k \). Hence \( \text{rep}_{\Gamma_k}(\Gamma') \) is equal to the dimension of representations \( \Gamma' \rightarrow \mathbb{R}^2 \) which is twice the size of a minimal generating set of \( \Gamma' \), since it is a free abelian group.

\[\square\]

5.2. The radical of a subgroup  In Section 3, we will introduce a matroid on the elements of a crystallographic group. To prove the required properties, we need to know how the translation subgroup \( \Lambda(\cdot) \) changes as generators are added to a subgroup of \( \Gamma_k \).

The radical of a subgroup \( \Gamma' < \Gamma \), which we now define and develop, is the key tool for doing this.

We define the radical, \( \text{Rad}(\Gamma') \), of \( \Gamma' \) to be the largest subgroup containing \( \Gamma' \) such that

\[
\text{rep}_\Gamma(\Lambda(\Gamma')) = \text{rep}_\Gamma(\Lambda(\text{Rad}(\Gamma'))) \quad \text{and} \quad T(\Gamma') = T(\text{Rad}(\Gamma'))
\]

(7)

It is called the radical since it contains at least all the roots of nontrivial elements of \( \Gamma' \), by Lemma 5.6 below.
5.3. Properties of the radical  The following sequence of lemmas enumerates the properties of the radical that we will use in the sequel.

**Lemma 5.2.** Let $\Gamma' < \Gamma_k$ be a subgroup of $\Gamma_k$. Then the radical $\text{Rad}(\Gamma')$ is well-defined.

**Proof.** First let $k = 2$. There are two cases. If $\Gamma'$ contains only translations, we set

$$\text{Rad}(\Gamma') = \{ t \in \Lambda(\Gamma_2) : t^i \in \Gamma' \text{ for some power } i \text{ of } t \}$$

Any subgroup $\Gamma'' < \Gamma_2$ containing $\Gamma'$ with $T(\Gamma') = T(\Gamma'')$ and $\text{rep}_T(\Gamma') = \text{rep}(\Gamma'')$ must be a translation group of the same rank as $\Gamma'$ and by definition of $\text{Rad}(\Gamma')$ is the largest such subgroup. Also, note that $\text{Rad}(\Gamma')$ and $\Gamma'$ necessarily have the same rank.

Otherwise $\Gamma'$ contains a rotation $r'$. In this case, we set

$$\text{Rad}(\Gamma') = \langle r', \text{Rad}(\Lambda(\Gamma')) \rangle$$

By Lemma 4.3, for $\text{Rad}(\Gamma')$ defined this way, the translation subgroup $\Lambda(\text{Rad}(\Gamma'))$ is just $\text{Rad}(\Lambda(\Gamma'))$ which by the previous paragraph is the largest translation subgroup containing $\Lambda(\Gamma')$ and having the same rank. Any subgroup $\Gamma'' < \Gamma_2$ containing $\Gamma$ must be of the form $\Gamma'' = \langle r', \Lambda(\Gamma'') \rangle$ with $\Lambda(\Gamma') < \Lambda(\Gamma'')$. If additionally $\text{rep}_T(\Lambda(\Gamma')) = \text{rep}_T(\Lambda(\Gamma''))$, then $\Lambda(\Gamma'') < \text{Rad}(\Lambda(\Gamma'))$ and $\Gamma'' < \text{Rad}(\Gamma')$.

Now we suppose that $k = 3, 4, 6$. There are four possibilities for $\Gamma'$:

- If $\Gamma'$ is trivial, then we define $\text{Rad}(\Gamma')$ to be trivial, and this choice is clearly canonical.
- If $\Gamma'$ is a cyclic group of rotations, then Lemma 4.2 guarantees that there is a unique largest cyclic subgroup containing it, and we define this to be $\text{Rad}(\Gamma')$.
- If $\Gamma'$ has only translations, then we define $\text{Rad}(\Gamma') = \Lambda(\Gamma_k)$.
- If $\Gamma'$ has translations and rotations, then some power of both standard generators for $\Gamma_k$ from Lemma 3.3 lies in $\Gamma'$. It follows that that defining $\text{Rad}(\Gamma') = \Gamma_k$ is the canonical choice.

The construction used to prove Lemma 5.2 gives us the following structural description of the radical.

**Proposition 5.3.** Let $\Gamma' < \Gamma_k$ be a subgroup of $\Gamma_k$ for $k = 2, 3, 4, 6$. Then if $k = 2$,

- If $\Gamma'$ is a translation subgroup, then $\text{Rad}(\Gamma')$ is the subgroup of translations with a non-trivial power in $\Gamma'$.
- If $\Gamma'$ has translations and rotations, then $\text{Rad}(\Gamma') = \langle r', \text{Rad}(\Lambda(\Gamma')) \rangle$.

If $k = 2, 3, 4, 6$, then there are four possibilities for the radical:
• If $\Gamma'$ is trivial, the radical is trivial.

• If $\Gamma'$ is cyclic, the radical is a cyclic subgroup of order $k$.

• If $\Gamma'$ is a translation subgroup, the radical is the translation subgroup of $\Gamma_k$.

• If $\Gamma'$ has translations and rotations, the radical is all of $\Gamma_k$.

Another immediate corollary of Lemma 5.2 is that we may “pass to radicals” if we are interested in $\text{rep}_{\Gamma_k}(\cdot)$ and $T(\cdot)$.

**Proposition 5.4.** Let $\Gamma'$ be a subgroup of $\Gamma_k$. Then

\[
\text{rep}_{\Gamma_k}(\Gamma') = \text{rep}_{\Gamma_k}(\text{Rad}(\Gamma'))
\]

\[
T(\Gamma') = T(\text{Rad}(\Gamma'))
\]

The radical also has a monotonicity property.

**Lemma 5.5.** Let $\Gamma' < \Gamma_k$ be a finitely-generated subgroup of $\Gamma_k$, and let $\Gamma'' < \Gamma'$ be a subgroup of $\Gamma'$. Then $\text{Rad}(\Gamma'') < \text{Rad}(\Gamma')$.

**Proof.** Pick a generating set of $\Gamma''$ that extends to a generating set of $\Gamma'$. Analyzing the cases in Proposition 5.3 shows that the radical cannot become smaller after adding generators. \(\square\)

As mentioned above, this next lemma provides some justification for the terminology “radical”.

**Lemma 5.6.** Let $\Gamma' < \Gamma_k$ be a subgroup of $\Gamma_k$. If some power $\gamma^i$ of $\gamma$ is not the identity and $\gamma^i \in \Gamma'$, then $\gamma \in \text{Rad}(\Gamma')$.

**Proof.** If $\gamma$ is a translation, this is clear by Proposition 5.3. Now let $\gamma$ be a rotation with $\text{Id} \neq \gamma^\ell \in \Gamma'$ and $\ell \neq 1$. Together these hypotheses imply that $k$ is 3, 4 or 6, and so we see that $\text{Rad}(\Gamma')$ is either all of $\Gamma_k$ or finite and cyclic of order $k$. In the first case, we are clearly done, and the second follows from Lemma 4.2 and the fact that $\Gamma'$ itself is finite and cyclic. \(\square\)

**Lemma 5.7.** Let $\Gamma' < \Gamma_k$ be a translation subgroup of $\Gamma_k$, and let $\gamma \in \Gamma_k$. Then $\text{Rad}(\gamma \Gamma' \gamma^{-1}) = \text{Rad}(\Gamma')$; i.e., the radical of translation subgroups is fixed under conjugation.

**Proof.** For $k = 2$ this follows from the fact that all translation subgroups are normal. For $k = 3, 4, 6$ it is immediate from the definitions. \(\square\)

**Lemma 5.8.** Let $\Gamma' < \Gamma_k$ be a subgroup of $\Gamma_k$, and let $\Gamma'' < \Lambda(\Gamma_k)$ be a translation subgroup of $\Gamma_k$. Then $\text{Rad}(\langle \Lambda(\Gamma'), \Gamma'' \rangle) = \text{Rad}(\Lambda(\langle \Gamma', \Gamma'' \rangle))$. 

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Proof. The proof is in cases based on $k$. For $k = 3, 4, 6$, either $\Gamma''$ is trivial or both sides of the desired equation are $\Lambda(\Gamma_k)$, by Proposition 5.3. Either way, the lemma follows at once.

Now suppose that $k = 2$. If $\Gamma'$ is a translation subgroup, then the lemma follows immediately. Otherwise, we know that $\Gamma'$ is generated by a rotation $r'$ and the translation subgroup $\Lambda(\Gamma')$. Applying Lemma 4.3 we see that

$$\Lambda(\langle \Gamma', \Gamma'' \rangle) = \Lambda(\langle r', \Lambda(\Gamma'), \Gamma'' \rangle) = \langle \Lambda(\Gamma'), \Gamma'' \rangle$$

from which the lemma follows. 

5.4. The quantity $\text{rep}_{\Gamma_k}(\Gamma') - T(\Gamma')$ The following statement plays a key role in the matroidal construction of Section 8.

Proposition 5.9. Let $\Gamma' < \Gamma_k$ be a subgroup of $\Gamma_k$, and let $\gamma \in \Gamma_k$ be an element of $\Gamma_k$. Then,

$$\text{rep}_{\Gamma_k}(\Lambda(\langle \Gamma', \gamma \rangle)) - T(\langle \Gamma', \gamma \rangle) - (\text{rep}_{\Gamma_k}(\Lambda(\Gamma')) - T(\Gamma')) = \begin{cases} 2 & \text{if } \gamma \notin \text{Rad}(\Gamma') \\ 0 & \text{otherwise} \end{cases}$$

i.e., the quantity $\text{rep}_{\Gamma_k}(\cdot) - T(\cdot)$ increases by two after adding $\gamma$ to $\Gamma'$ if and only if $\gamma \notin \text{Rad}(\Gamma')$ and otherwise the increase is zero.

Proof. If $\gamma \in \text{Rad}(\Gamma')$, this follows at once from the definition, since the quantity $\text{rep}_{\Gamma}(\Gamma') - T(\Gamma')$ depends only on the radical.

Now suppose that $\gamma \notin \text{Rad}(\Gamma')$. Since the radical is defined in terms of $\text{rep}_{\Gamma_k}(\cdot)$ and $T(\cdot)$, Lemma 5.5 implies that at least one of $\text{rep}_{\Gamma_k}(\cdot)$ or $-T(\cdot)$ increases, it is easy to see from the definition that either type of increase is by at least 2. We will show that the increase is at most 2, from which the lemma follows. The rest of the proof is in three cases, depending on $k$.

Now we let $k = 3, 4, 6$. The only way for the increase to be larger than 2 is for $\Gamma'$ to be trivial and $\text{Rad}(\langle \gamma \rangle) = \Gamma_k$. This is clearly impossible given the description from the proof of Lemma 5.2.

To finish, we address the case $k = 2$. Suppose $\gamma$ is a translation. Then $T(\langle \gamma, \Gamma' \rangle) = T(\langle \Gamma' \rangle)$, since adding $\gamma$ as a generator doesn’t give us a new rotation if one wasn’t already present in $\Gamma'$. Lemmas 4.1 and 4.3 imply that $\Lambda(\langle \gamma, \Gamma' \rangle) = \langle \gamma, \Lambda(\Gamma') \rangle$. Hence, the rank of the translation subgroup increases by at most 1, and so, by Lemma 5.1, $\text{rep}_{\Gamma_k}(\cdot)$ by at most 2.

Now suppose that $\gamma$ is a rotation. If $\Gamma'$ has no rotations, then Lemma 4.3 implies $\Lambda(\langle \gamma, \Gamma' \rangle) = \Gamma'$, and so $T(\cdot)$ decreases and $\text{rep}_{\Gamma_k}(\cdot)$ is unchanged. If $\Gamma'$ has rotations, then $\Gamma' = \langle r', \Lambda(\Gamma') \rangle$ for some rotation $r' \in \Gamma'$. Since $k = 2$, $r' \gamma$ is a translation and so

$$\Lambda(\langle \gamma, \Gamma' \rangle) = \Lambda(\langle \gamma, r', \Lambda(\Gamma') \rangle) = \Lambda(\langle r', \gamma, \Lambda(\Gamma') \rangle) = \langle r' \gamma, \Lambda(\Gamma') \rangle$$

Thus, in this case, the number of generators of the translation subgroup increases by at most one and $T(\cdot)$ is unchanged. By Lemma 5.1, the proof is complete. 


6. Teichmüller space and the centralizer

The representation spaces defined in the previous two sections are closely related to the degrees of freedom in the crystallographic direction networks we study in the sequel. In this section, we develop the Teichmüller space and centralizer, which play the same role for frameworks.

6.1. Teichmüller space  The Teichmüller space of $\Gamma_k$ is defined to be the space of discrete faithful representations modulo conjugation by $\text{Euc}(2)$, i.e. $\text{Teich}(\Gamma_k) = \text{Rep}(\Gamma_k)/\text{Euc}(2)$. For a subgroup $\Gamma' < \Gamma$, we define its restricted Teichmüller space to

$$\text{Teich}_{\Gamma_k}(\Gamma') = \text{Rep}_{\Gamma_k}(\Gamma')/\text{Euc}(2).$$

Correspondingly, we define $\text{teich}_{\Gamma_k}(\Gamma') = \dim(\text{Teich}_{\Gamma_k}(\Gamma'))$.

6.2. The centralizer  For a subgroup $\Gamma' \leq \Gamma_k$ and a discrete faithful representation $\Phi : \Gamma \to \text{Euc}(2)$, the centralizer of $\Phi(\Gamma')$ which we denote $\text{Cent}_{\text{Euc}(2)}(\Phi(\Gamma'))$ is the set of elements commuting with all elements in $\Phi(\Gamma')$. We define $\text{cent}_{\Gamma_k}(\Gamma')$ to be the dimension of the centralizer $\text{Cent}_{\text{Euc}(2)}(\Phi(\Gamma'))$. The quantity $\text{cent}_{\Gamma_k}(\Gamma')$ is independent of $\Phi$, and we can compute it. Since we don’t depend on Lemma 6.1 or Proposition 6.2 for any of our main results, we skip the proofs in the interest of space.

**Lemma 6.1.** Let notation be as above. The dimension $\text{cent}_{\Gamma_k}(\Gamma')$ of $\text{Cent}_{\text{Euc}(2)}(\Phi(\Gamma'))$ is independent of the representation $\Phi$. Furthermore, $\text{cent}(\Gamma') \geq T(\Gamma')$, and in particular,

$$\text{cent}(\Gamma') = \begin{cases} 
0 & \text{if } \Gamma' \text{ contains rotations and translations} \\
1 & \text{if } \Gamma' \text{ contains only rotations} \\
2 & \text{if } \Gamma' \text{ contains only translations} \\
3 & \text{if } \Gamma' \text{ is trivial}
\end{cases}$$

As a corollary, we get the following proposition relating $\text{rep}_{\Gamma_k}(\cdot)$ and $T(\cdot)$ to $\text{teich}_{\Gamma_k}(\cdot)$ and $\text{cent}(\cdot)$.

**Proposition 6.2.** Let $\Gamma' < \Gamma_k$. Then:

(A) If $\Gamma'$ contains a translation, then $T(\Gamma') = \text{cent}_{\Gamma_k}(\Gamma')$. Otherwise, $T(\Gamma') = \text{cent}_{\Gamma_k}(\Gamma') - 1$.

(B) If $\Gamma'$ is a non-trivial translation subgroup, then $\text{teich}_{\Gamma_k}(\Gamma') = \text{rep}_{\Gamma_k}(\Gamma') - 1$.

(C) If $\Gamma'$ is trivial, then $\text{teich}_{\Gamma}(\Gamma') = \text{rep}_{\Gamma}(\Gamma') = 0$.

(D) For any $\Gamma' < \Gamma_k$, $\text{rep}_{\Gamma_k}(\Lambda(\Gamma')) - T(\Gamma') = \text{teich}_{\Gamma_k}(\Lambda(\Gamma')) - \text{cent}_{\Gamma_k}(\Gamma')$. 

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7. Matroid preliminaries

The concepts of matroids and their linear representability play a key role in the results of this paper. In Section 8, we will define a matroidal structure on, essentially, $\Gamma_k$. In this section, we review the parts of matroid theory we will need in the sequel.

7.1. Matroids given by bases

A matroid $M$ is a combinatorial structure that captures some essential features of linear dependence and independence over a ground set $E$. Matroids have many equivalent definitions (see, e.g., the monograph [13]), but a convenient one for graph-theoretic matroids is by the bases $\mathcal{B}(M) \subseteq 2^E$, which must satisfy the following axioms:

- **Non-triviality** $\mathcal{B}(M) \neq \emptyset$.
- **Equal size** If $A$ and $B$ are in $\mathcal{B}(M)$, then $|A| = |B|$.
- **Base exchange** If $A$ and $B$ are in $\mathcal{B}(M)$, then there are elements $a \in A \setminus B$ and $b \in B \setminus A$ such that $A + b - a \in \mathcal{B}(M)$.

The size of bases is defined to be the rank of the matroid.

For readers new to matroids, we note that basis exchange corresponds to Steinitz exchange between bases of a finite-dimensional vector space. The canonical example of a matroid has the ground set the edges of the complete graph $K_n$ on $n$ vertices and the bases the spanning trees; this is usually called the graphic matroid in the literature. All the axioms are readily verified in this case.

7.2. Matroids given by rank functions

Let $E$ be a set and $f$ a non-negative, integer-valued function defined on subsets of $E$. We define $f$ to be monotone, if for all $A \subseteq B \subseteq E$, $f(A) \leq f(B)$. We define $f$ to be submodular, if for any subsets $A$ and $B$ of $E$:

$$f(A \cup B) + f(A \cap B) \leq f(A) + f(B)$$

which is called the submodular inequality. Submodular functions are an important class in optimization theory, since they capture a kind of “combinatorial concavity”. A more “local” characterization of submodularity, which will be easier for us to work with is along these lines. Let $A \subseteq B \subseteq E$, and let $e \in E \setminus B$. Then, $f$ is submodular if and only if for all such $A$, $B$ and $x$

$$f(A \cup \{x\}) - f(A) \geq f(B \cup \{x\}) - f(B)$$

An alternative characterization of a matroid $M$ on a ground set $E$ is by its rank function, which we denote $f_M$. An integer-valued set function $f_M$ on $E$ is the rank function of a matroid $M$ if:

- **Non-negativity** $f_M$ is non-negative and zero on $\emptyset$
**Monotonicity** $f_M$ is monotone

**Submodularity** $f_M$ is submodular

**Normalization** For all $A \subset E$ and $e \in E \setminus A$, $f_M$ increases by zero or one when $e$ is added to $A$.

For example, the rank function of the graphic matroid is given as follows: the rank of a subset $E'$ of edges of $K_n$ spanning $n'$ vertices and $c'$ connected components is simply $n' - c'$.

7.3. **Connection between rank functions and bases** The conversion between the characterization by rank function and bases is as follows. Given the rank function $f_M$ of a matroid $M$, the bases are

$$\mathcal{B}(M) = \{A \subset E : f_M(A) = |A| \text{ and } f_M(A) = f_M(E)\}$$

Given the bases $\mathcal{B}(M)$, the rank function is given by

$$f_M(A) = \max_{B \in \mathcal{B}(M)} |A \cap B|$$

7.4. **Infinite ground sets** Readers familiar with matroids will notice that we have not required the ground set to be finite. This is intentional: since we will be working with ground sets involving $\Gamma_k$, which is infinite, our ground set will be as well. Because all the matroids we deal with are finite rank (depending on $n$ and $\Gamma$), all the required theory goes through.

7.5. **Matroids from submodular functions** A fundamental theorem of matroid theory, due to Edmonds and Rota [5], and extended to the case where $E$ may be infinite by Pym and Perfect [14], in matroid theory gives a recipe for moving from submodular functions to matroids.

**Theorem 4** ([5]). Let $E$ be a set and $f$ be a non-negative, monotone, finite, integer-valued function on subsets of $E$. Then the collection of subsets

$$\{A \subset E : f(A) = |A| \text{ and for all subsets } A' \subset A, |A'| \leq f(A')\}$$

gives the bases of a matroid.

We define the matroid arising from Theorem 4 to be $M_f$.

7.6. **Matroid union** We will use as an essential tool, the following construction.

**Theorem 5** ([5]). Let $M_1$ and $M_2$ be matroids on a common ground set $E$, and let $f_1$ and $f_2$ be submodular functions such that $M_i$ is $M_{f_i}$ as in Theorem 4. Then the matroid $M_{f_1+f_2}$, obtained from Theorem 4 has, as its bases, the subsets of $E$

$$\{A \subset E : A = A_1 \cup A_2, \text{ with } A_1 \cap A_2 = \emptyset \text{ and } A_i \in \mathcal{B}(M_i)\}$$
8. A matroid on crystallographic groups

We now define and study a matroid $M_{\Gamma_k,n}$ for $k = 2, 3, 4, 6$.

8.1. Preview of $\Gamma-(1, 1)$ graphs and $M_{\Gamma_k,n}$ In Section 14, we will relate $M_{\Gamma_k,n}$ to "$\Gamma-(1, 1)$ graphs", which are defined in Section 12.5. The results here, roughly speaking, are the group theoretic part of the proof of Proposition 12.3 in Section 14.

To briefly motivate the definitions given next, $\Gamma-(1, 1)$ graphs need not be connected, and each connected component is associated with a finitely generated subgroup of $\Gamma_k$. The ground set of $M_{\Gamma_k,n}$ and the $A_i$ defined below capture this situation. The operations of conjugating and fusing, defined here in Sections 8.9 and 8.10 will be interpreted graph theoretically in Section 14.

8.2. The ground set

For the definition of the ground set, we fix $\Gamma$ and a natural number $n \geq 1$. The ground set $E_{\Gamma_k,n}$ is defined to be:

$$E_{\Gamma_k,n} = \{(\gamma, i) : 1 \leq i \leq n\}$$

In other words the ground set is $n$ labeled copies of $\Gamma_k$.

Let $A \subset E_{\Gamma_k,n}$. We define some notations:

- $A_i = \{\gamma : (\gamma, i) \in A\}$; i.e., $A_i$ is the group elements from copy $i$ of $\Gamma_k$ in $A$. Some of the $A_i$ may be empty and $A_i$ can be a multi-set. $A$ may equivalently defined by the $A_i$.

- $\Gamma_{A_i} = \langle \gamma : \gamma \in A_i \rangle$; i.e., the subgroup generated by the elements in $A_i$.

- $\Lambda(A) = \langle \Lambda(\Gamma_{A_1}), \Lambda(\Gamma_{A_2}), \ldots, \Lambda(\Gamma_{A_n}) \rangle$; the translation subgroup generated by the translations in each of the $\Gamma_{A_i}$.

- $c(A)$ is the number of $A_i$ that are not empty.

8.3. The rank function

We now define the function $g_1(A)$ for $A \subset E_{\Gamma_k,n}$ to be

$$g_1(A) = n + \frac{1}{2} \text{rep}_{\Gamma_k}(\Lambda(A)) - \frac{1}{2} \sum_{i=1}^{n} T(\Gamma_{A_i})$$

The meaning of the terms in $g_1(A)$ are as follows:

- The second term is a global adjustment for the representation space of the group generated by the translations in each of the $\Gamma_{A_i}$. We note that this is not the same as the translation group $\Lambda(\bigcup_{i=1}^{n} A_i)$, which includes translations arising as products of rotations in different $A_i$.

- The quantity $n - \frac{1}{2} \sum_{i=1}^{n} T(\Gamma_{A_i}) = \sum_{i=1}^{n} (1 - \frac{1}{2} T(\Gamma_{A_i}))$ is a local adjustment based on whether $\Gamma_{A_i}$ contains a rotation: each term in the latter sum is one if $\Gamma_{A_i}$ contains a rotation and otherwise it contributes nothing.
8.4. An analogy to uniform linear matroids To give some intuition about why the construction above might be matroidal, we observe that Proposition 5.9, interpreted in matroidal language gives us:

**Proposition 8.1.** Let $A$ be a finite subset of $\Gamma_k$ generating a subgroup $\Gamma_A$. Then the function

$$r(A) = \frac{1}{2} \left( \text{rep}_{\Gamma_k}(\Gamma_A) - T(\Gamma_A) \right)$$

is the rank function of a matroid on the ground set $\Gamma_k$.

The matroid in the conclusion of Proposition 8.1 is a kind of uniform linear matroid, with $\Gamma_k$ playing the role of a vector space and $r$ the role of dimension of the linear span. Since the function $g_1$, defined above, builds on $r$, one might expect that it inherit a matroidal structure. We verify this next.

8.5. $M_{\Gamma_k,n}$ is a matroid The following proposition is the main result of Part I.

**Proposition 8.2.** The function $g_1$ is the rank function of a matroid $M_{\Gamma_k,n}$.

The proof depends on Lemmas 8.3 and 8.4 below, so we defer it for the moment. The strategy is based on the observation that when $n = 1$, the ground set is essentially $\Gamma_k$. In this case, submodularity and normalization of $g_1$ (the most difficult properties to establish) follow immediately from Proposition 5.9. The motivation of Lemmas 8.3 and 8.4 is to reduce, as much as possible, the proof of the general case to $n = 1$.

**Lemma 8.3.** Let $A \subset E_{\Gamma_k,n}$, and set $\Gamma_{A,\ell} = \langle \Gamma_A, \ell(A) \rangle$. Then, for all $1 \leq \ell \leq n$,

(A) $\text{Rad}(\Lambda(A)) = \text{Rad}(\Lambda(\Gamma_{A,\ell}))$

(B) $T(\Gamma_A, \ell) = T(\Gamma_{A,\ell})$

**Proof.** The statement (A) is immediate from Lemma 5.8. (B) follows from the fact that $\Lambda(A)$ is a translation subgroup of $\Gamma_k$, so $\Gamma_{A,\ell}$ has a rotation if and only if $\Gamma_A, \ell$ does. □

**Lemma 8.4.** Let $A \subset E_{\Gamma_k,n}$, and set $\Gamma_{A,\ell} = \langle \Gamma_A, \ell(A) \rangle$. If $B = A + (\gamma, \ell)$ and $\Gamma_{B,\ell} = \langle \Gamma_B, \ell(A) \rangle$, then,

$$\Gamma_{B,\ell} = \langle \gamma, \Gamma_{A,\ell} \rangle$$

**Proof.** First we observe that

$$\Gamma_{B,\ell} = \langle \gamma, \Gamma_A, \ell(A) \rangle \geq \langle \gamma, \Gamma_A, \ell(A) \rangle$$

so to finish the proof we just have to show that

$$\Lambda(B) \leq \langle \gamma, \Gamma_A, \ell(A) \rangle$$

Since $\Gamma_{B,i} = \Gamma_{A,i}$ for all $i \neq \ell$, it follows that

$$\Lambda(B) = \langle \Lambda(A), \Lambda(\langle \gamma, \Gamma_A, \ell(A) \rangle) \rangle \leq \langle \gamma, \Gamma_A, \ell(A) \rangle$$

□
8.6. Proof of Proposition 8.2 We check the rank function axioms (from Section 7.2).

Non-negativity: This follows from the fact that $\text{rep}_{k}(\cdot)$ is non-negative, and the sum of the $\frac{1}{2}T(\cdot)$ terms cannot exceed $n$.

Monotonicity: Immediate from Lemma 5.5.

Normalization: To prove that $g_{1}$ is normalized, let $A \subset E_{k,n}$ and $B = A + (\gamma, \ell)$. Since all the $T(\gamma, i)$ terms cancel except for the ones with $i = \ell$, the increase is given by

$$g_{1}(B) - g_{1}(A) = \frac{1}{2} \left( \text{rep}_{k}(\Lambda(B)) - \text{rep}_{k}(\Lambda(A)) - T(\Gamma_{B, \ell}) + T(\Gamma_{A, \ell}) \right)$$

Because the r.h.s. is an invariant of the radical by Proposition 5.4, we pass to radicals and apply Lemma 8.3 to see that the r.h.s. is equal to

$$\frac{1}{2} \left( \text{rep}_{k}(\Lambda(\Gamma'_{B, \ell})) - \text{rep}_{k}(\Lambda(\Gamma'_{A, \ell})) - T(\Gamma'_{B, \ell}) + T(\Gamma'_{A, \ell}) \right)$$

Using Lemma 8.4 then tells us that this can be simplified further to

$$\frac{1}{2} \left( \text{rep}_{k}(\Lambda(\langle \gamma, \Gamma'_{A, \ell} \rangle)) - \text{rep}_{k}(\Lambda(\Gamma'_{A, \ell})) - T(\langle \gamma, \Gamma'_{A, \ell} \rangle) + T(\Gamma'_{A, \ell}) \right)$$

at which point Proposition 5.9 applies, and we conclude that the increase is either zero or one.

Submodularity: Inspecting the argument for normalization and using Lemma 5.5 one more time gives submodularity, since, if $A' \subset A$ and $\gamma \notin \text{Rad}(\Gamma_{A, \ell})$, then $\gamma \notin \text{Rad}(\Gamma'_{A, \ell})$.

This gives us the submodular inequality (8).

8.7. The bases and independent sets With the rank function of $M_{\Gamma, n}$ determined, we can give a structural characterization of its bases and independent sets. Let $A \subset E_{k,n}$.

We define $A$ to be independent if

$$|A| = g_{1}(A)$$

If $A$ is independent and, in addition

$$|A| = c(A) + \frac{1}{2} \text{rep}_{k}(\Lambda(\Gamma_{k}))$$

we define $A$ to be tight. A (not-necessarily independent) set $A$ with $c(A)$ parts that contains a tight subset on $c(A)$ is defined to be spanning.

We define the classes

$$\mathcal{B}(M_{\Gamma, n}) = \left\{ B \subset E_{\Gamma, n} : B \text{ is independent and } |B| = n + \text{rep}(\Gamma) \right\}$$

$$\mathcal{I}(M_{\Gamma, n}) = \left\{ B \subset E_{\Gamma, n} : B \text{ is independent} \right\}$$

It is now immediate from Proposition 8.2 that

Lemma 8.5. The classes $\mathcal{I}(M_{\Gamma, n})$ and $\mathcal{B}(M_{\Gamma, n})$ are the independent sets and bases of the matroid $M_{\Gamma, n}$. 

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8.8. Structure of tight sets  We also have a structural characterization of the tight independent sets in \(M_{\Gamma, n}\).

**Lemma 8.6.** An independent set \(A \in I(M_{\Gamma, n})\) is tight if and only if it is one of two types:

(A) Each of the non-empty \(A_i\) contains a rotation. One exceptional non-empty \(A_i\) contains \(\frac{1}{2} \text{rep}_{\Gamma_k}(\Lambda(\Gamma_k))\) additional elements, and \(\text{rep}_{\Gamma_k}(\Lambda(\Gamma_{A_i})) = \text{rep}_{\Gamma_k}(\Lambda(\Gamma_k))\), and all the rest of the \(A_i\) contain a single rotation only.

(B) Each of the \(c(A)\) contains a rotation. Two exceptional non-empty \(A_i\) (w.l.o.g., \(A_1\) and \(A_2\)) contain, between them, \(\frac{1}{2} \text{rep}(\Lambda(\Gamma_k))\) additional elements and \(\text{rep}_\Gamma(\langle \Lambda(\Gamma_{A_1}), \Lambda(\Gamma_{A_2}) \rangle) = \text{rep}_\Gamma(\Lambda(\Gamma))\).

Type (B) is only possible when \(\Gamma_k = \Gamma_2\).

**Proof.** One direction is straightforward: A set \(A \subset E_{\Gamma, n}\) of either type (A) or (B) satisfies, by hypothesis, \(|A| = c(A) + \frac{1}{2} \text{rep}(\Lambda(\Gamma))\); by construction \(T(\Gamma_{A_i})\) is zero for all the non-empty \(A_i\) and \(\text{rep}_\Gamma(\Lambda(A)) = \text{rep}_\Gamma(\Lambda(\Gamma))\).

On the otherhand, assuming that \(A\) is tight, we see that each non-empty part has to contain a rotation, and, since \(A\) is independent there are only one (for \(k = 3, 4, 6\)) or two (\(k = 2\)) additional elements in \(A\). Thus, the \(A_i\) containing these extra elements need to generate the translation subgroup of \(\Gamma_k\). \(\square\)

8.9. Conjugation of independent sets  Let \(A \in I(M_{\Gamma_k})\) be an independent set, and suppose, w.l.o.g., that \(A_1, A_2, \ldots, A_{c(A)}\) are the non-empty parts of \(A\). Let \(\gamma_1, \gamma_2, \ldots, \gamma_{c(A)}\) be elements of \(\Gamma_k\). The conjugation of \(A\) by \(\gamma_1, \gamma_2, \ldots, \gamma_{c(A)}\) is defined to be

\[
\{(\gamma_i^{-1} A_i \gamma_i, i) : 1 \leq i \leq c(A)\}
\]

Conjugation preserves independence in \(M_{\Gamma_k, n}\).

**Lemma 8.7.** Let \(A \in I(M_{\Gamma_k})\) be an independent set. Then the conjugation of \(A\) by \(c(A)\) elements \(\gamma_1, \ldots, \gamma_{c(A)}\) is also independent.

**Proof.** Lemma [5.7] implies that the radical of translation subgroups is preserved under conjugation, and whether or not \(A_i\) contains a rotation is as well. Since the rank function \(g_1\) is determined by these two properties of the \(A_i\), we are done. \(\square\)

8.10. Separating and fusing independent sets  Let \(A \in I(M_{\Gamma_k})\) be an independent set. A **separation** of \(A\) is defined to be the following operation:

- Select \(i\) and \(j\) such that \(A_j\) is empty.
- Select a (potentially empty) subset \(A'_i \subset A_i\) of \(A_i\).
- Replace elements \((\gamma, i) \in A'_i\) with \((\gamma, j)\).
Separation preserves independence in $M_{\Gamma_k,n}$.

**Lemma 8.8.** Let $A \in \mathcal{I}(M_{\Gamma_k})$ be an independent set. Then any separation of $A$ is also an independent set.

**Proof.** Let $B$ be a separation of $A$. If the subset $A_i'$ in the definition of a separation is empty, then $B$ is the same as $A$, and there is nothing to prove.

An independent set is either tight or a subset of a tight set. (Bases in particular are tight.) Consequently, by Lemma 8.6, either $B_i$ or $B_j$ consists of a single element. Assume w.l.o.g., it is $B_j$. Define $C \subset E_{\Gamma,n}$ as $C_k = B_k$ for $k \neq j$ and $C_j$ empty; i.e. $C$ is $B$ with the single element in $B_j$ dropped. Then $C$ is a subset of $A$ and hence independent. If $B_j$ consists of a rotation, then adding it to $C$ clearly preserves independence. If $B_j$ consists of a translation $\gamma$, then since $A$ is independent we must have $\gamma \notin \text{Rad}(\Lambda(C))$. Consequently $B = C + (\gamma, j)$ is independent since $\text{Rad}(\Lambda(B)) > \text{Rad}(\Lambda(C))$ and hence $\text{rep}_{\Gamma_k}(\Lambda(B)) > \text{rep}_{\Gamma_k}(\Lambda(C))$. \qed

The reverse of separation is fusing a set $A$ on $A_i$ and $A_j$. This operation replaces $A_i$ with $A_i \cup A_j$ and makes $A_j$ empty. Fusing does not, in general, preserve independence, but it takes tight sets to spanning ones.

**Lemma 8.9.** Let $A$ be a tight independent set, and suppose that $A_i$ and $A_j$ are non-empty. Then, after fusing $A$ on $A_i$ and $A_j$, the result is a spanning set (with one less part).

**Proof.** Let $B$ be the set resulting from fusing $A$ on $A_i$ and $A_j$. By hypothesis, all the non-empty $A_\ell$ contain a rotation, so this is true of the non-empty $B_\ell$ as well. The lemma then follows by noting that $\Lambda(A) \leq \Lambda(B)$, so the same is true of the radicals by Lemma 5.5. Thus, $g_1(B) = c(B) + \text{rep}(\Lambda(\Gamma_k))$, and this implies $B$ is spanning. \qed
II. Sparse graphs

9. Colored graphs and the map \( \rho \)

We will use colored graphs, which are also known as “gain graphs” (e.g., [18]) or “voltage graphs” [28] as the combinatorial model for crystallographic frameworks and direction networks. In this section we give the definitions and explain the relationship between colored graphs and graphs with a free \( \Gamma_k \)-action.

9.1. Colored graphs

Let \( G = (V,E) \) be a finite, directed graph, with \( n \) vertices and \( m \) edges. We allow multiple edges and self-loops, which are treated the same as other edges. A \( \Gamma_k \)-colored-graph (shortly, colored graph) \( (G, \gamma) \) is a finite, directed multigraph \( G \) and an assignment \( \gamma = (\gamma_{ij})_{ij \in E(G)} \) of a group element \( \gamma_{ij} \in \Gamma_k \) (the “color”) to each edge \( ij \in E(G) \).

9.2. The covering map

Although we work with colored graphs because they are technically easier, crystallographic frameworks were defined in terms of infinite graphs \( \tilde{G} \) with a free \( \Gamma \)-action \( \varphi \) with finite quotient. In fact, the formalisms are equivalent. The following is a straightforward specialization of covering space theory (see, e.g., [6, Section 1.3]), but we provide the dictionary for the convenience of the reader.

Let \( (G, \gamma) \) be a colored graph, we define its lift \( \tilde{G} = (\tilde{V}, \tilde{E}) \) by the following construction:

- For each vertex \( i \in V(G) \), there is a subset of vertices \( \{i_{\gamma}\}_{\gamma \in \Gamma} \subset V(\tilde{G}) \) (the fiber over \( i \)).
- For each (directed) edge \( ij \in E(G) \) with color \( \gamma_{ij} \), and for each \( \gamma \in \Gamma_k \), there is an edge \( i_{\gamma} j_{\gamma' \gamma_{ij}} \) in \( E(\tilde{G}) \) (the fiber over \( ij \)).
- The \( \Gamma \)-action on vertices is \( \gamma \cdot i_{\gamma'} = i_{\gamma \gamma'}. \) The action on edges is that induced by the vertex action.

Now let \( (\tilde{G}, \varphi) \) be an infinite graph with a free \( \Gamma_k \)-action that has finite quotient. We associate a colored graph \( (G, \gamma) \) to \( (\tilde{G}, \varphi) \) by the following construction, which we define to be the colored quotient:

\[\text{This terminology comes from Igor Rivin [16], and is consistent with [10].}\]
Let $\tilde{G} = \tilde{G} / \Gamma$ be the quotient of $\tilde{G}$ by $\Gamma$, and fix an (arbitrary) orientation of the edges of $\tilde{G}$ to make it a directed graph. By hypothesis, the vertices of $\tilde{G}$ correspond to the vertex orbits in $\tilde{G}$ and the edges to the edge orbits in $\tilde{G}$.

For each vertex orbit under $\Gamma$ in $\tilde{G}$, select a representative $\tilde{i}$.

For each edge orbit $\tilde{ij}$ in $\tilde{G}$ there is a unique edge that has the representative $\tilde{i}$ as its tail. There is also a unique element $\gamma_{ij} \in \Gamma$ such that the head of $\tilde{ij}$ is $\gamma_{ij}(\tilde{j})$. We define this $\gamma_{ij}$ to be the color on the edge $ij \in G$.

The projection map from $(\tilde{G}, \varphi)$ to its colored quotient is the function that sends a vertex $\tilde{i} \in V(\tilde{G})$ its representative $i \in V(G)$. Figures 1 and 2 both show examples; the color coding of the vertices in the infinite developments indicated the fibers over vertices in the colored quotient.

The following lemma is straightforward:

**Lemma 9.1.** Let $(G, \gamma)$ be a $\Gamma_k$-colored graph. Then its lift is well defined, and is an infinite graph with a free $\Gamma_k$-action. If $(\tilde{G}, \varphi)$ is an infinite graph with a free $\Gamma_k$-action, then it is the lift of its colored quotient, and the projection map is well-defined and a covering map.

### 9.3. The map $\rho$

Let $(G, \gamma)$ be a colored graph, and let $P = \{e_1, e_2, \ldots, e_t\}$ be any closed path in $G$; i.e., $P$ is a not necessarily simple walk in $G$ that starts and ends at the same vertex crossing the edges $e_i$ in order. If we select a vertex $b$ as a base point, then the closed paths are elements of the fundamental group $\pi_1(G, b)$.

We define the map $\rho$ as:

$$\rho(P) = \gamma_{e_1}^{e_1} \cdots \gamma_{e_t}^{e_t}$$

where $e_i$ is 1 if $P$ crosses $e_i$ in the forward direction (from tail to head) and $-1$ otherwise. For a connected graph $G$ and choice of base vertex $i$, the map $\rho$ induces a well-defined homomorphism $\rho : \pi_1(G, i) \to \Gamma$.

### 9.4. Cyclic groups

The preceding development of colored graphs is in terms of a crystallographic group $\Gamma$, but the construction is quite general, and it also works for any group such as e.g. $\mathbb{Z}/k\mathbb{Z}$. Since $\mathbb{Z}/k\mathbb{Z}$ is abelian, it is easy to check that $\rho$ depends only on its image on cycles in $G$ only, which makes the theory simpler. The following is Lemma 9.1 adapted for $\mathbb{Z}/k\mathbb{Z}$-colored graphs.

**Lemma 9.2.** Let $(G, \gamma)$ be a $\mathbb{Z}/k\mathbb{Z}$-colored graph. Then its lift is well defined, and is a finite graph with a free $\mathbb{Z}/k\mathbb{Z}$-action. If $(\tilde{G}, \varphi)$ is a finite graph with a free $\mathbb{Z}/k\mathbb{Z}$-action, then it is the lift of its colored quotient, and the projection map is well-defined and a covering map.

### 10. The subgroup of a $\Gamma_k$-colored graph

The map $\rho$, defined in the previous section, is fundamental to the results of this paper. In this section, we develop properties of the $\rho$-image of a colored graph $(G, \gamma)$ and connect it with the matroid $M_{\Gamma_k,n}$ which was defined in Section 8.
10.1. Colored graphs with base vertices

Let $(G, \gamma)$ be a colored graph with $n$ vertices and $c$ connected components $G_1, G_2, \ldots, G_c$. We select a base vertex $b_i$ in each connected component $G_i$, and denote the set of base vertices by $B$. The triple $(G, \gamma, B)$ is then defined to be a marked colored graph.

If $(G, \gamma, B)$ is a marked colored graph then $\rho$ induces a homomorphism from $\pi_1(G_i, b_i)$ to $\Gamma_k$. In the rest of this section, we show how to use these homomorphisms to define a map from $(G, \gamma)$ to $E_{\Gamma_k, n}$, the ground set of the matroid $M_{\Gamma_k, n}$.

10.2. Fundamental closed paths generate the $\rho$-image

Let $(G, \gamma, B)$ be a marked colored graph with $n$ vertices and $c$ connected components. Select and fix a maximal forest $F$ of $G$, with connected components $T_1, T_2, \ldots, T_c$. The $T_i$ are spanning trees of the connected components $G_i$ of $G$, with the convention that when a connected component $G_i$ has no edges.

With this data, we define, for each edge $i j \in E(G) - E(F)$ the fundamental closed path of $i j$ to be the path that:

- Starts at the base vertex $b_\ell$ in the same connected component $G_\ell$ as $i$ and $j$.
- Travels the unique path in $T_\ell$ to $i$.
- Crosses $i j$.
- Travels the unique path in $T_\ell$ back to $v_\ell$.

Fundamental closed paths with respect to $F$ in $G_i$ generate $\pi_1(G_i, b_i)$ [6, Proposition 1A.2].

10.3. From colored graphs to sets in $E_{\Gamma_k, n}$

We now let $(G, \gamma, B)$ be a marked colored graph and fix a choice of spanning forest $F$. We associate with $(G, \gamma, B, F)$ a subset $A(G, B, F)$ of $E_{\Gamma_k, n}$ (defined in Section 8) as follows:

- For each edge $i j \in E(G) - E(T_\ell)$, let $P_{ij}$ be the fundamental closed path with respect to $T_\ell$ and $b_\ell$ of $i j$.
- Add an element $(\rho(P_{ij}), \ell)$ to $A(G, B, F)$.

The following is immediate from the previous discussion.

**Lemma 10.1.** Adopting the notation from Section 8, $\Gamma_{A(G, B, F), \ell} = \rho(\pi_1(G_\ell, v_\ell))$.

Since we will show, in Section 12, that the invariants we need are independent of $B$ and $F$, we frequently suppress them from the notation when the context is clear.

11. Map-graph preliminaries

The families of colored graphs we define in the next sections have, as their underlying (uncolored, undirected) multi-graphs, a map-graph structure. In this short section, we define map graphs and review the properties we need.
11.1. Map-graphs and sparsity A map-graph is a graph in which every connected component has exactly one cycle. In this definition, self-loops correspond to cycles. A 2-map-graph is a graph that is the edge-disjoint union of two spanning map-graphs. See Figure 3 for an example; observe that map-graphs do not need to be connected.

Figure 3: A 2-map-graph with its certifying decomposition into map-graphs indicated by edge color.

11.2. The overlap graph Let $G$ be a 2-map-graph and fix a decomposition into two spanning map-graphs $X$ and $Y$. Let $X_i$ and $Y_i$ be the connected components of $X$ and $Y$, respectively. Also select a base vertex $x_i$ and $y_i$ for each connected component of $X$ and $Y$, with all base vertices on the cycle of their component. Denote the collection of base vertices by $B$.

We define the overlap graph of $(G, X, Y, B)$ to be the directed graph with:

- Vertex set $B$.
- A directed edge from $x_i$ to $y_i$ if $y_i$ is a vertex in $X_i$.
- A directed edge from $y_i$ to $x_i$ if $x_i$ is a vertex in $Y_i$.

Figure 4 gives an example. The property of the overlap graph we need is:

**Proposition 11.1.** Let $G$ be a 2-map-graph with fixed decomposition and choice of base vertices. The overlap graph of $(G, Y, R, B)$ has a directed cycle in each connected component.

**Proof.** Every vertex has exactly one incoming edge, since each vertex is in exactly one connected component of each of $X$ and $Y$. Thus, as an undirected graph, the overlap graph is a map-graph (see, e.g., [23]).

12. $\Gamma$-$(2,2)$ graphs

In this section we define $\Gamma$-$(2,2)$ graphs which are the first of two key families of colored graphs introduced in this paper (the second is $\Gamma$-colored-Laman graphs, defined in Section 13). We also state the main combinatorial results on $\Gamma$-$(2,2)$ graphs, but defer the proof of a key technical result, Proposition 12.3, to Section 14.
12.1. The translation subgroup of a colored graph

Let \((G, \gamma, B)\) be a marked colored graph, as in Section 10, with connected components \(G_1, G_2, \ldots, G_c\) and base vertices \(b_1, b_2, \ldots, b_c\). Recall from Section 9.3 that, with this data, there is a homomorphism

\[ \rho : \pi_1(G_i, b_i) \to \Gamma_k \]

We define \(\Lambda(G, B)\) to be

\[ \Lambda(G, B) = \langle \Lambda(\rho(G_i, b_i)) : i = 1, 2, \ldots, c \rangle \]

We define \(\text{rep}_{\Gamma_k}(G) = \text{rep}_{\Gamma_k}(\Lambda(G, B))\). As the notation suggests, \(\text{rep}_{\Gamma_k}(G)\) is independent of the choice of base vertices \(B\).

**Lemma 12.1.** Let \((G, \gamma, B)\) be a marked colored graph. The quantity \(\text{rep}_{\Gamma_k}(G)\) is independent of the choice of base vertices, and so is a property of the underlying colored graph \((G, \gamma)\).

*Proof.* Changing base vertices corresponds to conjugation. Lemma 5.7 implies that the radical of \(\Lambda(G, B)\) is preserved under conjugation. Since \(\text{rep}_{\Gamma_k}(\cdot)\) depends only on the radical, the lemma follows.

12.2. The quantity \(T\) for a colored graph

Let \((G, \gamma, B)\) be a marked colored graph, with \(G\) connected (and so a single base vertex \(b\)). We define \(T(G)\) to be \(T(\rho(\pi_1(G, b)))\). The proof of the following lemma is entirely similar to that of Lemma 12.1.

**Lemma 12.2.** Let \((G, \gamma, B)\) be a marked colored graph. The quantity \(T(G)\) is independent of the choice of base vertices, and so is a property of the underlying colored graph \((G, \gamma)\).
Figure 5: An example of a $\Gamma$-(2, 2) graph when $\Gamma = \Gamma_3$.

12.3. $\Gamma$-(2, 2) graphs We are now ready to define $\Gamma$-(2, 2) graphs. Let $(G, \gamma)$ be a colored graph with $n$ vertices and $c$ connected components $G_i$. We define the function $f$ to be

$$f(G) = 2n + \text{rep}_{\Gamma_k}(G) - \sum_{i=1}^{c} T(G_i)$$

A colored graph $(G, \gamma)$ on $n$ vertices and $m$ edges is defined to be a $\Gamma$-(2, 2) graph if:

- The number of edges $m$ is $2n + \text{rep}(\Lambda(\Gamma_k))$ (i.e., it is the maximum possible value for $f$).
- For every subgraph $G'$ of $G$, with $m'$ edges, $m' \leq f(G')$.

We note that it is essential that the definition is made over all subgraphs, and not just vertex-induced or connected ones. Figure 5 shows an example of a $\Gamma$-(2, 2) graph.

12.4. Direction network derivation Before continuing with the development of the combinatorial theory, we quickly motivate the definition of $\Gamma$-(2, 2) graphs. Readers who are not familiar with rigidity and direction networks may want to either skip to Section 12.5 and revisit this, purely informative, section after reading the definitions in Part III.
Proposition 21.1, in Section 21 below, says that a generic direction network on a \( \Gamma \)-colored graph \((G, \gamma)\) has only collapsed realizations (with all the points on top of each other and a trivial representation for the \( \Lambda(\Gamma) \)), if and only if \((G, \gamma)\) is \( \Gamma\text{-}(2,2) \).

The definition of the function \( f \) comes from analyzing the degree of freedom of collapsed realizations. For any realization \( G(p, \Phi) \), we can translate it (which preserves directions), so that \( \Phi(r_k) \) has the origin as its rotation center. Then, restricted to a subgraph \( G' \) of \( G \):

- The total number of variables involved in the equations giving the edge directions is \( 2n' + \text{rep}_{\Gamma_k}(G') \). Since we fix \( \Phi(r_k) \) to rotate at the origin (see Section 21 for an explanation why we can do this), the only variability left in \( \Phi \) is \( \Phi(\Lambda(\Gamma_k)) \). Since \( \text{rep}_{\Gamma_k}(G') \) measures how much of \( \Lambda(\Gamma) \) is “seen” by \( G' \), this is the term we add.

- Each connected component \( G_i \) has a \( T(G'_i) \)-dimensional space of collapsed realizations. If \( G'_i \) has a rotation, then a collapsed realization of the lift \( \tilde{G}'_i \) must lie on the corresponding rotation center since a solution must be rotationally symmetric. When \( G'_i \) has no rotation, no such restriction exists, and there are 2-dimensions worth of places to put the collapsed \( \tilde{G}'_i \). Each collapsed connected component is independent of the others, so this term is additive over connected components.

The heuristic above coincides with the definition of the function \( f \).

12.5. \( \Gamma\text{-}(1,1) \) graphs We will characterize \( \Gamma\text{-}(2,2) \) graphs in terms of decompositions into simpler \( \Gamma\text{-}(1,1) \) graphs \(^2\) which we now define.

Let \((G, \gamma)\) be a colored graph and select a base vertex \( b_i \) for each connected component \( G_i \) of \( G \). We define \((G, \gamma)\) to be a \( \Gamma\text{-}(1,1) \) graph if:

- \( G \) is a map-graph plus \( \frac{1}{2} \text{rep}_{\Gamma_k}(\Lambda(\Gamma_k)) \) additional edges.
- For each connected component \( G_i \) of \( G \), \( \rho(\pi_1(G_i, b_i)) \) contains a rotation.
- We have \( \text{rep}_{\Gamma_k}(G) = \text{rep}_{\Lambda(\Gamma_k)}(\Gamma_k) \), i.e., \( \text{Rad}(\Lambda(G,B)) = \Lambda(\Gamma_k) \).

Although we do not define \( \Gamma\text{-}(1,1) \) graphs via sparsity counts, there is an alternative characterization in these terms. We define the function \( g(G) \) to be

\[
g(G) = n + \frac{1}{2} \text{rep}_{\Gamma_k}(G) - \frac{1}{2} \sum_{i=1}^{c} T(G_i)
\]

where \((G, \gamma)\) is a colored graph and \( n \) and \( c \) are the number of vertices and connected components. Notice that \( g = \frac{1}{2} f \). In Section 14 we will show:

\(^2\)The terminology of “(2,2)” and “(1,1)” comes from the fact that spanning trees of finite graphs are “(1,1)-tight” in the sense of \([3]\). The \( \Gamma\text{-}(1,1) \) graphs defined here are, in a sense made more precise in \([10] \) Section 5.2], analogous to spanning trees. We don’t go into details here in the interest of space, since the analogy isn’t necessary for any of the proofs.
**Proposition 12.3.** The family of $\Gamma$-$(1,1)$ graphs gives the bases of a matroid, and the rank of the $\Gamma$-$(1,1)$ matroid is given by the function:

$$g(G) = n + \frac{1}{2} \text{rep}_{\Gamma_1}(G) - \frac{1}{2} \sum_{i=1}^c T(G_i)$$

In particular, this implies that $g$ is non-negative, submodular, and monotone.

### 12.6. Decomposition characterization of $\Gamma$-$(2,2)$ graphs

The key combinatorial result about $\Gamma$-$(2,2)$ graphs, that is used in an essential way to prove the “collapsing lemma” Proposition 21.1 is the following.

**Proposition 12.4.** Let $(G, \gamma)$ be a colored graph. Then $(G, \gamma)$ is a $\Gamma$-$(2,2)$ graph if and only if it is the edge-disjoint union of two spanning $\Gamma$-$(1,1)$ graphs.

**Proof.** Since $f = 2g$, and Proposition 12.3 implies that $g$ meets the hypothesis of Theorem 4 we conclude that the $\Gamma$-$(2,2)$ graphs are also the bases of a matroid. Theorem 5 then says that the $\Gamma$-$(2,2)$ matroid must coincide with the class of colored graphs defined by the desired decomposition. \[\square\]

### 13. $\Gamma$-colored Laman graphs

We are now ready to define $\Gamma$-colored-Laman graphs, which are the colored graphs characterizing minimally rigid generic frameworks in Theorem 1. Just as for $\Gamma$-$(2,2)$ graphs, we define them via sparsity counts.

![Figure 6: Examples of $\Gamma$-colored-Laman graphs: (a) a $\Gamma_2$-colored-Laman graph; (b) a $\Gamma_3$-colored-Laman graph](image-url)
13.1. Definition of $\Gamma$-colored-Laman graphs  Let $(G, \gamma)$ be a colored graph, and let $f$ be the sparsity function defined in Section 12. The most direct definition of the sparsity function $h$ for $\Gamma$-colored-Laman graphs is:

$$h(G) = f(G) - 1$$

A colored graph $(G, \gamma)$ is defined to be $\Gamma$-colored-Laman if:

- $G$ has $n$ vertices and $m = 2n + \text{rep}_{\Gamma_k}(\Lambda(\Gamma_k)) - T(\Gamma_k) - 1$ edges.
- For all subgraphs $G'$ spanning $m'$ edges, $m' \leq h(G')$

Figure 6 shows some examples of $\Gamma$-colored-Laman graphs. If a colored graph is a subgraph of a $\Gamma$-colored-Laman graph, then it is defined to be $\Gamma$-colored-Laman sparse. Equivalently, $(G, \gamma)$ is $\Gamma$-colored-Laman sparse if only the “$m' \leq h(G')$” condition above holds.

13.2. Alternate formulation of $\Gamma$-colored-Laman graphs  While the definition of $h$ is all that is needed to prove Theorem 1, it does not give any motivation in terms of a degree-of-freedom count. We now give an alternate formulation of $\Gamma$-colored-Laman via the Teichmüller space and the centralizer, which were defined in Section 6, that will let us do this.

Let $(G, \gamma, B)$ be a marked colored graph with connected components $G_1, \ldots, G_c$ and $n$ vertices, and let $\Lambda(G, B)$ be its translation subgroup as defined in Section 12.1. We define

$$\text{teich}_{\Gamma_k}(G) = \text{teich}_{\Gamma_k}(\Lambda(G, B))$$

which, by a proof nearly identical to that of Lemma 12.1, is well-defined and independent of the choice of base vertices.

For a component $G_\ell$ with base vertex $i_\ell$, we set $\text{cent}_{\Gamma_k}(G_\ell) = \text{cent}_{\Gamma_k}(\rho(\pi_1(G_\ell, i_\ell)))$. For similar reasons, $\text{cent}_{\Gamma_k}(G_\ell)$ is also independent of the base vertex.

We can now define a “more natural” sparsity function

$$h'(G) = 2n + \text{teich}_{\Gamma_k}(G) - \left( \sum_{i=1}^{c} \text{cent}_{\Gamma_k}(G_i) \right)$$

The class of colored graphs defined by $h'$ is the same as that arising from $h$, giving a second definition of $\Gamma$-colored-Laman graphs. Since Lemma 13.1 is not used to prove any further results, we omit the proof.

**Lemma 13.1.** A colored graph $(G, \gamma)$ is $\Gamma$-colored-Laman if and only if:

- $G$ has $n$ vertices and $m = 2n + \text{teich}(\Gamma) - \text{cent}(\Gamma)$ edges.
- For all subgraphs $G'$ spanning $m'$ edges, $m' \leq h'(G')$
13.3. Degree of freedom heuristic  The function $h'$ is amenable to an interpretation that allows us, by Lemma 13.1, to give a rigidity-theoretic “degree of freedom” derivation of $\Gamma$-colored-Laman graphs. This section is expository, and readers unfamiliar with rigidity theory may skip to Section 13.4 and return here after reading Part IV.

Given a framework with underlying colored graph $(G, \gamma)$, with the graph $G$ having $n$ vertices and $c$ connected components $G_1, G_2, \ldots, G_c$, we find that:

- We have $2n$ degrees of freedom from the points. From the representation $\Phi : \Gamma_k \rightarrow \text{Euc}(2)$, there are $\text{rep}(\Gamma_k)$ degrees of freedom, but if we mod out by trivial motions from $\text{Euc}(2)$, we have $\text{teich}(\Gamma_k)$ degrees of freedom left. However, we have only $\text{teich}(\Gamma_k)(G)$ degrees of freedom that apply to $G$.

- Each connected component has $\text{cent}(\Gamma_k)(G_i)$ trivial degrees of freedom. Since elements in the centralizer for $G_i$ commute with those in $\rho(\pi_1(G_i))$, we may “push the vertices of $G_i$ around” with the centralizer elements while preserving symmetry. Since these motions always exist, they are trivial.

This heuristic corresponds to the function $h'$.

13.4. Edge-doubling characterization of $\Gamma$-colored-Laman graphs  The main combinatorial fact about $\Gamma$-colored-Laman graphs we need is the following simple characterization by edge-doubling (cf. [9, 15]).

Proposition 13.2. Let $\Gamma = \Gamma_k$ for $k = 2, 3, 4, 6$ be a crystallographic group and let $(G, \gamma)$ be a $\Gamma$-colored graph. Then $(G, \gamma)$ is $\Gamma$-colored-Laman if and only if for any edge $i j \in E(G)$, the colored graph $(G', \gamma')$ obtained by adding a copy of $i j$ to $G$ with the same color results in a $\Gamma$-(2, 2) graph.

Proof. This is straightforward to check once we notice that $(G, \gamma)$ is $\Gamma$-colored-Laman if and only if no subgraph $G'$ with $m'$ edges has $m' = f(G')$. \qed

13.5. $\Gamma$-colored-Laman circuits  Let $(G, \gamma)$ be a colored graph. We define $(G, \gamma)$ to be a $\Gamma$-colored-Laman circuit if it is edge-wise minimal with the property of being not $\Gamma$-colored-Laman sparse. More formally, $(G, \gamma)$ is a $\Gamma$-colored-Laman circuit if:

- $(G, \gamma)$ is not $\Gamma$-colored-Laman sparse

- For all colored edges $i j \in E(G)$, $(G - i j, \gamma)$ is $\Gamma$-colored-Laman sparse

As the terminology suggests, $\Gamma$-colored-Laman circuits are the circuits of the matroid that has, as its bases, $\Gamma$-colored-Laman graphs. The following lemmas are immediate from the definition.

Lemma 13.3. Let $(G, \gamma)$ be a colored graph. If $(G, \gamma)$ is not $\Gamma$-colored-Laman sparse, then it contains a $\Gamma$-colored-Laman circuit as a subgraph.
Lemma 13.4. Let \((G, \gamma)\) be a colored graph with \(n\) vertices and \(m\) edges. Then \((G, \gamma)\) is a \(\Gamma\)-colored-Laman circuit if and only if:

- The number of edges \(m = f(G)\)
- For all subgraphs \(G'\) of \(G\), on \(m'\) edges, \(m' < f(G')\)

Here \(f\) is the colored-(2, 2) sparsity function defined in Section 12.

14. \(\Gamma\)-(1, 1) graphs: proof of Proposition 12.3

With the definitions and main properties of \(\Gamma\)-(2, 2) and \(\Gamma\)-colored-Laman graphs developed, we prove:

Proposition 12.3. The family of \(\Gamma\)-(1, 1) graphs gives the bases of a matroid, and the rank of the \(\Gamma\)-(1, 1) matroid is given by the function:

\[
g(G) = n + \frac{1}{2} \text{rep}_{\Gamma_k}(G) - \frac{1}{2} \sum_{i=1}^{c} T(G_i)
\]

In particular, this implies that \(g\) is non-negative, submodular, and monotone.

With this, the proof of Proposition 12.4 is also complete. The rest of this section is organized as follows: first we prove that the \(\Gamma\)-(1, 1) graphs give the bases of a matroid and then we argue that the rank function of this matroid is, in fact, the function \(g\), defined in Section 12.

We recall from Section 10 that, for a marked colored graph \((G, \gamma, B)\) with a fixed spanning forest \(F\), the map \(\rho\), defined in Section 9, induced a map from \((G, \gamma, B, F)\) to \(E_{\Gamma_k,n}\), the ground set of the matroid \(M_{\Gamma_k,n}\) from Section 8. We adopt the notation of Section 10 and denote the image of this map by \(A(G, B, F)\).

We start by studying \(A(G, B, F)\) in more detail.

14.1. Rank of \(A(G, B, F)\) As defined, the set \(A(G, B, F)\) depends on a choice of base vertices for each connected component and a spanning forest \(F\) of \(G\). Since we are interested in constructing a matroid on colored graphs without additional data, the first structural lemma is that the rank of \(A(G, B, F)\) in \(M_{\Gamma_k,n}\) is independent of the choices for \(B\) and \(F\).

Lemma 14.1. Let \((G, \gamma, B)\) be a marked colored graph with connected components \(G_1, G_2, \ldots, G_c\) and fix a spanning forest \(F\) of \(G\). Then the rank of \(A(G, B, F)\) in the matroid \(M_{\Gamma_k,n}\) is invariant under changing the base vertices and spanning forest.

Proof. For convenience, shorten the notation \(A(G, B, F)\) to \(A\). By Lemma 10.1, \(\rho(\pi_1(G_\ell, v_\ell)) = \Gamma_{A_\ell}\). Changing the spanning forest \(F\) just picks out a different set of generators for
π₁(G, vℓ), and so does not change Γₐ, and thus the rank in MΓ, which does not depend on the generating set, is unchanged.

To complete the proof, we show that changing the base vertices corresponds, in EΓ, to applying the conjugation operation defined in Section 8 to A. Suppose that G is connected and fix a spanning tree F and a base vertex b. If P is a closed path starting and ending at b, for any other vertex b′ there is a path P′ that: starts at b′, goes to b along a path Pbb′, follows P, and then returns from b to b′ along Pbb′ in the other direction. We have ρ(P′) = ρ(Pbb′)ρ(P)ρ(Pbb′)⁻¹, so P and P′ have conjugate images. Thus changing base vertices corresponds to conjugation, and by Lemma 8.7 we are done after considering connected components one at a time.

In light of Lemma 14.1, when we are interested only in the rank of A(G, B, F), we can freely change B and F. Thus, we define the notation A(G) to suppress the dependence on B and F.

14.2. Adding or Deleting a Colored Edge and A(G) In the proof of the basis exchange property, we will need to start with a Γ-(1, 1) graph, and add a colored edge to it. There are two possibilities: the edge ij is in the span of some connected component Gℓ of G or it is not. Each of these has an interpretation in terms of how A(G + ij) is different from A(G).

Lemma 14.2. Let (G, γ) be a colored graph and let ij be a colored edge. Then:

(A) If the edge ij is in the span of a connected component, Gℓ of G, then A(G + ij) is A(G) + (γ, ℓ), where γ is the image of the fundamental closed path of ij with respect to some spanning tree and base vertex of Gℓ.

(B) If the edge ij connects two connected components Gℓ and Gr of G, then A(G + ij) is a fusing operation (defined in Section 8) on A(G) after a conjugation. In particular, in the notation of Section 8, A(Gℓ) and A(Gr) are fused. Conversely, A(G) is a conjugation of a separation of A(G + ij).

Proof. Statement (A) follows from the fact that if we pick a base vertex and spanning tree of Gℓ, then adding the colored edge ij to Gℓ induces exactly one new fundamental closed path.

For statement (B), w.l.o.g., assume that G has two connected components and ij connects them. Since ij is in any spanning tree of G + ij, it follows that every fundamental closed path in G + ij has ρ-image conjugate to a closed path in G, so A(G + ij) consists of group elements conjugate to elements in A(Gℓ) and A(Gr), as required. The converse is clear since the inverse of a conjugation is a conjugation, and the inverse of fusing is separating. □
14.3. $\Gamma$-(1, 1) graphs and tight independent sets in $M_{\Gamma,k,n}$  $\Gamma$-(1, 1) graphs $(G, \gamma)$ have a simple characterization in terms of $A(G)$: they correspond exactly to the situations in which $A(G)$ is tight and independent.

**Lemma 14.3.** Let $(G, \gamma)$ be a colored graph. Then $(G, \gamma)$ is $\Gamma$-(1, 1) if and only if $A(G)$ is tight and independent in $M_{\Gamma,k,n}$.

Proof. We recall that Lemma 8.6 gave a structural characterization of tight independent sets in $M_{\Gamma,k,n}$. The proof follows by translating the definitions from Section 8.7 into graph theoretic terms. In this proof we adopt the notation of Section 8.7 and we remind the reader that a subset $A \subset E_{\Gamma,k,n}$ is tight if it is independent in $M_{\Gamma,k,n}$ and has:

$$|A| = c(A) + \frac{1}{2} \text{rep}(\Lambda(\Gamma_k))$$

elements.

We first suppose that $A(G)$ is tight, and show that $(G, \gamma)$ is a $\Gamma$-(1, 1) graph. By construction $A(G)$ has an element $(\gamma, \ell)$ if and only if there is some edge $ij$ in the connected component $G_\ell$ not in the spanning forest $F$ used to compute $A(G)$. It then follows that, if $A(G)$ is tight, each connected component of $G_\ell$ of $G$ has at least one more edge than $G \cap F$. This implies that $G$ contains a spanning map graph. Because $|A(G)| = c(A) + \frac{1}{2} \text{rep}(\Lambda(\Gamma_k))$ it then follows that $G$ is a map-graph plus $\frac{1}{2} \text{rep}(\Lambda(\Gamma_k))$ additional edges, which are the combinatorial hypotheses for being a $\Gamma$-(1, 1) graph.

Now we use the fact that $A(G)$ is independent in $M_{\Gamma,k,n}$. Independence implies that, if non-empty, $A(G)$ contains a rotation, from which it follows that, for each connected component $G_\ell$ of $G$, $\rho(\pi_\ell(G_i, b_i))$ does as well. Similarly, independence implies that $\text{rep}_{\ell}(\Lambda(A(G))) = \text{rep}_{\ell}(\Lambda(\Gamma_k))$, so the same is true for $\text{rep}_{\ell}(G)$. We have now shown that $(G, \gamma)$ is a $\Gamma$-(1, 1) graph.

The other direction is straightforward to check. $\square$

14.4. $\Gamma$-(1, 1) graphs form a matroid  We now have the tools to prove that the $\Gamma$-(1, 1) graphs form the bases of a matroid. We take as the ground set the graph $K_{\Gamma,k,n}$ on $n$ vertices that has one copy of each possible directed edge $ij$ or self-loop $ij$ with color $\gamma \in \Gamma_k$.

**Lemma 14.4.** The set of $\Gamma$-(1, 1) graphs on $n$ vertices form the bases of a matroid on $K_{\Gamma,k,n}$.

Proof. We check the basis axioms (defined in Section 7).

- **Non-triviality:** There is some $\Gamma$-(1, 1) graph on $n$ vertices. An uncolored tree plus $\frac{1}{2} \text{rep}(\Gamma) + 1$ edges, each of which is colored by a standard generator for $\Gamma$ is clearly $\Gamma$-(1, 1). Thus the set of bases is not empty.
- **Equal size:** By definition, all $\Gamma$-(1, 1) graphs have the same number of edges.
- **Base exchange:** The more difficult step is checking basis exchange. To do this we let $G$ be a $\Gamma$-(1, 1) graph and $ij$ a colored edge of some other $\Gamma$-(1, 1) graph which is not in $G$.  

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It is sufficient to check that there is some colored edge \(i'j' \in E(G)\) such that \(G + ij - i'j'\) is also a \(\Gamma\)-(1, 1) graph. Let \((G', \gamma')\) be the colored graph \((G + ij, \gamma)\).

Pick base vertices \(B\) and a spanning forest \(F\) of \(G'\) that contains the new edge \(ij\). By Lemma 14.1, forcing \(ij\) to be in \(F\) does not change the rank of \(A(G', B, F)\) in \(M_{\Gamma_k,n}\). Lemma 14.2 then implies that \(A(G', B, F)\) is spanning, but not independent, in \(M_{\Gamma_k,n}\). Thus there is an element of \(A(G', B, F)\) that can be removed to leave a tight, independent set. Since \(ij\) is in \(F\), this element does not correspond to \(ij\). The basis exchange axiom then follows from the characterization of \(\Gamma\)-(1, 1) graphs in Lemma 14.3. \(\square\)

14.5. The rank function of the \(\Gamma\)-(1, 1) matroid  
Now we compute the rank function of the \(\Gamma\)-(1, 1) matroid. The following lemma is immediate from the definitions.

**Lemma 14.5.** Let \((G, \gamma)\) be a colored graph with \(n\) vertices and \(c\) connected components. Then

\[
g(G) = n - c + g_1(A(G))
\]

where \(g_1\) is the rank function of the matroid \(M_{\Gamma_k,n}\).

We can use this to show:

**Lemma 14.6.** Let \((G, \gamma)\) be a colored graph that is independent in the \(\Gamma\)-(1, 1) matroid with \(m\) edges. Then \(m = g(G)\).

**Proof.** By definition \((G, \gamma)\) is a subgraph of some \(\Gamma\)-(1, 1) graph \((G', \gamma')\). By Lemma 14.3, \(m' = g(G')\), where \(m'\) is the number of edges of \(G'\). It suffices to show that deleting an edge preserves this equality and independence of \(A(G')\). By Lemma 14.2, deleting an edge is equivalent to either removing an element from \(A(G')\) or separating and conjugating \(A(G')\) and these both preserve independence of \(A(G')\). In the first case, \(g_1(A(\cdot))\) drops by 1 while \(n'\) and \(c'\) remain constant, and in the second case \(n'\) and \(g_1(A(\cdot))\) remain constant while \(c'\) increases by 1. \(\square\)

We can now compute the rank function of the \(\Gamma\)-(1, 1) matroid.

**Lemma 14.7.** The function \(g\) is the rank function of the \(\Gamma\)-(1, 1) matroid.

**Proof.** Let \((G, \gamma)\) be an arbitrary colored graph with \(n\) vertices and \(c\) connected components. As discussed in Section 7, the rank of \((G, \gamma)\) in the \(\Gamma\)-(1, 1) matroid is the maximum size of the intersection of \(G\) with a \(\Gamma\)-(1, 1) graph. Lemma 14.6 implies that what we need to show is that a maximal independent subgraph \((G', \gamma)\) of \((G, \gamma)\) has \(g(G)\) edges.

We construct \(G'\) as follows. First pick a base vertex for every connected component of \(G\) and a spanning forest \(F\) of \(G\). Initially set \(G'\) to be \(F\). Then add edges one at a time to \(G'\) from \(G - F\) so that \(A(G')\) remains independent in \(M_{\Gamma_k,n}\) until the rank of \(A(G')\) is equal to that of \(A(G)\). This is possible by the matroidal property of \(M_{\Gamma_k,n}\) and Lemma 14.1 which says the rank of \(A(G')\) is invariant under the choices of spanning forest and base vertices.
When the process stops, \( A(G') \) is independent in \( M_{\Gamma_k,n} \), so \( G' \) is in the \( \Gamma'\)-\((1, 1) \) matroid by Lemma \[14.3\] By construction \( G' \) has

\[
m' = n - c + g_1(A(G))
\]

edges, which is \( g(G) \) by Lemma \[14.5\]

14.6. Proof of Proposition \[12.3\] The proposition is immediate from Lemmas \[14.4\] and \[14.7\]

15. Cone-\((2, 2)\) and cone-Laman graphs

We now develop the combinatorial language for cone frameworks and direction networks. Since it runs parallel to that for crystallographic direction networks, but is simpler, we will be somewhat brief. Figure \[7\] shows some examples of colored graphs defined in this section.

Figure 7: Examples of \( \mathbb{Z}/3\mathbb{Z} \) colored graphs: (a) a cone-\((1, 1)\) graph; (b) a cone-\((2, 2)\) graph; (c) a cone-Laman graph. Edges without directions and colors have the identity group element coloring them.

15.1. Cone-\((2, 2)\) graphs  Let \( (G, \gamma) \) be a \( \mathbb{Z}/k\mathbb{Z} \) colored graph with \( n \) vertices. We define \( (G, \gamma) \) to be a cone-\((2, 2)\) graph if:

- \( G \) has \( m = 2n \) edges.
- For all subgraphs with \( m' \) edges, \( n' \) vertices, and connected components \( G_1, G_2, \ldots, G_c \),

\[
m' \leq 2n' - \sum_{i=1}^{c} T(G_i)
\]
The quantity $T$ is the same one defined in Section 12, since all elements of $\mathbb{Z}/k\mathbb{Z}$ are represented by rotations. If only the second condition holds, then $(G, \gamma)$ is defined to be cone-(2, 2) sparse.

15.2. Cone-(1, 1) graphs We define $(G, \gamma)$ to be a cone-(1, 1) graph if $G$ is a map-graph and the cycle in each connected component has non-trivial $\rho$-image.

The sparsity characterization of cone-(1, 1) graphs is:

**Lemma 15.1.** The cone-(1, 1) graphs on $n$ vertices are the bases of a matroid that has as its rank function

$$r(G') = n' - \frac{1}{2} \sum_{i=1}^{c} T(G_i)$$

where $n'$ and $c'$ are the number of vertices and connected components in $G'$.

Lemma 15.1 follows from a simplification of the arguments in Section 14, but can also be obtained via [27, ”Matroid Theorem”].

15.3. Characterization of cone-(2, 2) graphs From Theorem 5 and the matroidal Lemma 15.1 we get a decomposition characterization of cone-(2, 2) graphs.

**Lemma 15.2.** Let $(G, \gamma)$ be a colored graph. The $(G, \gamma)$ is cone-(2, 2) if and only if it is the edge-disjoint union of two cone-(1, 1) graphs.

A corollary, by the Matroid Union Theorem 5 is:

**Lemma 15.3.** The family of cone-(2, 2) graphs on $n$ vertices gives the bases of a matroid.

15.4. Cone-Laman graphs Let $(G, \gamma)$ be a $\mathbb{Z}/k\mathbb{Z}$ colored graph with $n$ vertices. We define $(G, \gamma)$ to be a cone-Laman graph if:

- $G$ has $m = 2n - 1$ edges.
- For all subgraphs with $m'$ edges, $n'$ vertices, and connected components $G_1, G_2, \ldots, G_c$,

$$m' \leq 2n' - 1 - \sum_{i=1}^{c} T(G_i)$$

If only the second condition holds, we define $(G, \gamma)$ to be cone-Laman-sparse.

The relationship between cone-Laman and cone-(2, 2) graphs is similar to that from the crystallographic case, and has the same proof.

**Lemma 15.4.** Let $(G, \gamma)$ be a $\mathbb{Z}/k\mathbb{Z}$-colored graph. Then, $(G, \gamma)$ is a cone-Laman graph if and only if $G$ becomes a cone-(2, 2) graph after doubling any edge.
15.5. Cone-Laman graphs are connected Although cone-(2, 2) graphs need not be connected, cone-Laman graphs are.

**Lemma 15.5.** Let \((G, \gamma)\) be a cone-Laman graph. Then \(G\) is connected.

**Proof.** Let \(G\) have \(n\) vertices. By hypothesis, \(G\) has \(2n - 1\) edges, and any subgraph on \(n'\) vertices and \(m'\) edges satisfies \(m' \leq 2n' - 1\). The lemma then follows from \([8, Lemma 4]\). \(\Box\)

15.6. Cone-Laman circuits Let \((G, \gamma)\) be a \(\mathbb{Z}/k\mathbb{Z}\) colored graph with \(n\) vertices. We define \((G, \gamma)\) to be a cone-Laman circuit if:

- \((G, \gamma)\) is not a cone-Laman-sparse.
- \((G - i j, \gamma)\) is cone-Laman-sparse for any colored edge \(i j \in E(G)\).

A fact we need is that cone-Laman circuits are always connected.

**Lemma 15.6.** Let \((G, \gamma)\) be a cone-Laman circuit. Then \(G\) is connected, and is either:

- A connected cone-(2, 2) graph, if \(T(G) = 0\).
- A graph on \(n\) vertices with \(m' \leq 2n' - 2\) for all subgraphs, on \(n'\) vertices and \(m'\) edges, if \(T(G) = 2\).

**Proof.** Let \(G\) have \(n\) vertices, \(m\) edges, and \(c\) connected components \(G_i\) with \(n_i\) vertices and \(m_i\) edges. Because \((G, \gamma)\) becomes cone-Laman sparse after the removal of any edge, we must have

\[m_i = 2n_i - T(G_i)\]

for every connected component, since otherwise one of the \(G_i\) would not be cone-Laman sparse after removing one edge. This then implies that none of the \(G_i\) is cone-Laman sparse, so there must be only one of them.

The structural statement then comes from noting that the cone-(2, 2) sparsity function bounds the number of edges in any subgraph. \(\Box\)

16. Generalized cone-(2, 2) graphs

As a technical tool in the proof of Theorem \([3]\), we will use generalized cone-(2, 2) graphs. These are \(\Gamma_k\)-colored graphs, which we will define in terms of a decomposition property.
16.1. Generalized cone-(1, 1) graphs Let $(G, \gamma)$ be a $\Gamma_k$-colored graph. We define $(G, \gamma)$ to be a generalized cone-(1, 1) graph if, after considering the $\rho$-image modulo the translation subgroup, the result is a cone-(1, 1) graph. Equivalently, $(G, \gamma)$ is a generalized cone-(1, 1) graph if:

- $G$ is a map graph
- The $\rho$-image of the cycle in each connected component of $G$ is a rotation

The difference between cone-(1, 1) graphs and generalized cone-(1, 1) graphs is that the rotations need not be around the same center. Nonetheless, the proof of the following lemma is nearly the same as that of Lemma 15.1.

**Lemma 16.1.** The generalized cone-(1, 1) graphs on $n$ vertices are the bases of a matroid that has as its rank function

$$r(G') = n' - \frac{1}{2} \sum_{i=1}^{c} T(G_i)$$

where $n'$ and $c'$ are the number of vertices and connected components in $G'$.

16.2. Relation to $\Gamma$-(1, 1) graphs Generalized cone-(1, 1) graphs are related to $\Gamma$-(1, 1) graphs by this next sequence of lemmas.

**Lemma 16.2.** Let $(G, \gamma)$ be a $\Gamma$-(1, 1) graph. Then $(G, \gamma)$ contains a generalized cone-(1, 1) graph as a spanning subgraph.

**Proof.** This follows from the definition, since each connected component $G_i$ of $G$ has $T(G_i) = 0$. It follows that $G_i$ has a spanning subgraph that is a connected map-graph with its cycle having a rotation as its $\rho$-image. \hfill $\square$

Let $(G, \gamma)$ be a $\Gamma$-(1, 1) graph, and let $(G', \gamma)$ be a spanning generalized cone-(1, 1) subgraph. One exists by Lemma 16.2. We define $(G', \gamma)$ to be a g.c.-basis of $(G, \gamma)$.

**Lemma 16.3.** Let $(G, \gamma)$ be a $\Gamma_k$-colored $\Gamma$-(1, 1) graph for $k = 3, 4, 6$. Let $(G', \gamma)$ be a g.c.-basis of $(G, \gamma)$, and let $ij$ be the (unique) edge in $E(G) - E(G')$. Then either:

- The colored edge $ij$ is a self-loop and the color $\gamma_{ij}$ is a translation.
- There is a unique minimal subgraph $G''$ of $G$, such that the $\rho$-image of $(G'', \gamma)$ includes a translation, $ij$ is an edge of $G''$, and if $vw \in E(G'')$, then $(G' + ij - vw, \gamma)$ is also a g.c.-basis of $(G, \gamma)$.

**Proof.** If $ij$ is a self-loop colored by a translation, then it is a circuit in the matroid of generalized cone-(1, 1) graphs on the ground set $(G, \gamma)$. Otherwise, the subgraph $G''$ the lemma requires is just the fundamental generalized-cone-(1, 1) circuit of $ij$ in $(G', \gamma)$.

\hfill $\square$
16.3. Generalized cone-(2, 2) graphs  Let \((G, \gamma)\) be a \(\Gamma_k\)-colored graph. We define \((G, \gamma)\) to be a generalized cone-(2, 2) graph if it is the union of two generalized cone-(1, 1) graphs. Theorem 5 implies that:

**Lemma 16.4.** The generalized cone-(2, 2) graphs on \(n\) vertices give the bases of a matroid.

The other fact about generalized cone-(2, 2) graphs is their relationship to \(\Gamma\)-(2, 2) graphs.

**Lemma 16.5.** Let \((G, \gamma)\) be a \(\Gamma\)-(2, 2) graph. Then \((G, \gamma)\) contains a generalized cone-(2, 2) graph as a spanning subgraph.

**Proof.** This is immediate from Lemma 15.2 and Lemma 16.2
III. Direction networks

17. Cone direction networks

As a warm up for crystallographic direction networks, we will study cone direction networks. Our main result on cone direction networks is the natural adaptation of Theorem 3. Full definitions are given in Section 17.1 below. Genericity means that Theorem 6 is true for all but a proper algebraic subset of the space of edge-direction assignments, and will be made precise in Section 20.

**Theorem 6.** A generic realization of a cone direction network \((\tilde{G}, \varphi, \tilde{d})\) has a faithful realization if and only if its associated colored graph is cone-Laman. This realization is unique up to scaling.

In the rest of this section we give the required definitions and indicate the proof strategy. The proof is then carried out in Sections 18–20.

17.1. Cone and colored direction networks A cone direction network \((\tilde{G}, \varphi, \tilde{d})\), is given by a finite graph \(\tilde{G}\), a free \(\mathbb{Z}/k\mathbb{Z}\)-action \(\varphi\) on \(\tilde{G}\), and an assignment \(\tilde{d} = (\tilde{d}_{ij})_{ij \in E(\tilde{G})}\) of a direction to each edge, such that:

\[
\tilde{d}_{\gamma \cdot i j} = R_k^\gamma \cdot \tilde{d}_{i j} \quad \text{for all } \gamma \in \mathbb{Z}/k\mathbb{Z}
\]

Recall from Section 2 that we let \(\mathbb{Z}/k\mathbb{Z}\) act on \(\mathbb{R}^2\) by mapping the generator to \(R_k\), the counter-clockwise rotation through angle \(2\pi/k\) around the origin; when the context is clear, we will simply write \(\gamma \cdot p_i\) for this action. Note that \(\tilde{d}\) is completely defined by assigning a direction to one edge in each \(\mathbb{Z}/k\mathbb{Z}\)-orbit of edges in \(\tilde{G}\).

By Lemma 9.2, the combinatorial data of a cone direction network is contained in its colored quotient graph. We define a colored direction network \((G, \gamma, d)\) to be a \(\mathbb{Z}/k\mathbb{Z}\)-colored graph \((G, \gamma)\) along with an assignment of a direction to every edge.

17.2. The realization problem The realization problem for a cone direction network is to find a point set \(p_i\) for each \(i \in V(\tilde{G})\) so that each edge \(ij \in E(\tilde{G})\) is in the direction \(\tilde{d}_{ij}\). The realization space of a cone direction network is defined to be:

\[
\left\langle p_j - p_i, \tilde{d}_{ij} \right\rangle = 0 \quad \text{for all edges } ij \in E(\tilde{G})
\]

\[
\gamma \cdot p_i = p_{\gamma \cdot i} \quad \text{for all vertices } i \in V(\tilde{G})
\]
The unknowns are the points \( p_i \) and the given data are the directions \( \tilde{d}_{ij} \). We denote points in the realization space by \( \tilde{G}(p) \).

Because the directions \( \tilde{d}_{ij} \) respect the \( \mathbb{Z}/k\mathbb{Z} \)-action \( \varphi \) on \( \tilde{G} \), the realization space is identified with the following system (denoted \( (G, \gamma, d) \)) defined on the quotient graph \( (G, \gamma) \):

\[
\left\langle \gamma_{ij}^{-1}d_{ij}, p_i \right\rangle + \left\langle d_{ij}, -p_i \right\rangle = 0 \tag{9}
\]

The points \( p_i \) for each \( i \in V(G) \) are the unknowns and the directions \( d_{ij} \) are the given data. We denote points in the realization space by \( G(p) \).

The following is immediate from the definitions and Lemma 9.2.

**Lemma 17.1.** Given a colored direction network \( (G, \gamma, d) \), its lift to a cone direction network \( (\tilde{G}, \varphi, \tilde{d}) \) is well-defined, and the realization spaces of \( (G, \gamma, d) \) and \( (\tilde{G}, \varphi, \tilde{d}) \) are canonically isomorphic and hence of the same dimension.

In light of Lemma 17.1, we can move back and forth between the two settings freely. In our proofs, we will start with a colored direction network and study the dimension of its realization space via geometric arguments involving the lift. This next lemma is a corollary of Lemma 17.1 but an explicit proof is instructive.

**Lemma 17.2.** Let \( (G, \gamma) \) be a colored graph and \( (\tilde{G}, \varphi) \) its lift. Assigning a direction to one representative of each edge orbit under \( \varphi \) in \( \tilde{G} \) gives a well-defined colored direction network \( (G, \gamma, d) \).

**Proof.** Adopt the indexing scheme for the vertices and edges of \( \tilde{G} \) from Section 9. If the direction \( d \) is assigned to an edge \( i, j \in E(\tilde{G}) \), we assign the direction \( \gamma_{ij}^{-1} \cdot d \) to \( ij \in E(G) \). Since we assign a direction to only one edge in the fiber over \( ij \), this procedure gives a well-defined assignment of directions to the edges of \( G \), and it is easy to check that lifting these directions agrees with the assignments made to \( \tilde{G} \).

**17.3. Collapsed and faithful realizations** The realization space of a cone direction network is never empty: it is always possible to put all the points \( p_i \) at the origin, in which case the realization equations are trivially satisfied. We define such realizations to be collapsed. Similarly, if \( i, j \in E(\tilde{G}) \) is an edge and a realization sets \( p_i = p_j \), we define the edge \( i, j \) to be collapsed in that realization.

For the purposes of rigidity theory, collapsed realizations are degenerate. We define a realization to be faithful if it has no collapsed edges. Thus, the content of Theorem 6 is that cone-Laman graphs are, generically, the maximal colored graphs underlying direction networks with faithful realizations.

**17.4. Proof strategy for Theorem 6** We deduce Theorem 6 from the following “collapsing lemma”.

**Proposition 17.3.** A generic cone direction network that has as its colored quotient graph a cone-(2, 2) graph has only collapsed realizations.
Given Proposition 17.3, the proof of Theorem 6 uses an “edge doubling trick” employed to prove the analogous [10] Theorem B:

- We start with a generic cone direction network with an underlying cone-Laman graph. This has a one-dimensional realization space.
- We then observe that if there is a collapsed edge, the realization space is equivalent to that coming from a generic direction network on the same graph with a doubled edge, which is cone-(2, 2).
- Proposition 17.3 then says the realization space is, in fact, zero dimensional, which is a contradiction.

Although these steps, which are carried out in Section 20, require some technical care, they are straightforward. Most of the work is involved in proving Proposition 17.3. Since the variables in the realization system for a colored direction network with \( \mathbb{Z}/k\mathbb{Z} \) symmetry, do not separate for \( k = 3, 4, 6 \), as in the finite [24] or periodic [10] cases, we make a geometric argument as opposed to using the Laplace expansion as is done in [10, 24]. The approach is as follows:

- We start with a cone-(2, 2) graph, and decompose it into two edge-disjoint cone-(1, 1) graphs \( X \) and \( Y \), which is allowed by the combinatorial Lemma 15.2 and select base vertices.
- We then assign a direction to each connected component of \( X \) and \( Y \) that forces any realization to have a specific structure that is only possible in collapsed realizations.

These steps are carried out in Sections 18 and 19.

18. Generic linear projections

For the proof of Proposition 17.3 in the next section, we will need several geometric lemmas.

18.1. Affine lines Given a unit vector \( \mathbf{v} \in \mathbb{R}^2 \) and a scalar \( s \in \mathbb{R} \), we denote by \( \ell(\mathbf{v}, s) \) the affine line

\[
\langle \mathbf{p}, \mathbf{v}^\perp \rangle = s
\]

18.2. An important linear equation The following is a key lemma which will determine where certain points must lie when solving a cone direction network.

**Lemma 18.1.** Suppose \( R \) is a non-trivial rotation about the origin, \( \mathbf{v}^* \) is a unit vector and \( \mathbf{p} \) satisfies

\[
(R - I)\mathbf{p} = \lambda \mathbf{v}^*
\]
for some $\lambda \in \mathbb{R}$. Then, for some $C \in \mathbb{R}$, we have $p = Cv$ where $v = R_{\pi/2}R^{-1/2}v^*$, $R^{-1/2}$ is some square root of $R^{-1}$, and $R_{\pi/2}$ is the counter-clockwise rotation through angle $\pi/2$.

Proof. A computation shows that $(R - I)R^{-1/2} = R^{1/2} - R^{-1/2}$ is a multiple of $R_{\pi/2}$, from which the Lemma follows. \[\square\]

18.3. The linear projection $T(v, w, R)$ Let $k \in \mathbb{N}$ be at least three, $v$ and $w$ be unit vectors in $\mathbb{R}^2$, and $R$ some nontrivial rotation. Denote by $v^*$ the vector $(R^{1/2} \cdot v) \perp$ for some choice of square root of $R$.

We define $T(v, w, R)$ to be the linear projection from $\ell(v, 0)$ to $\ell(w, 0)$ in the direction $v^*$. The following properties of $T(v, w, R)$ are straightforward.

Lemma 18.2. Let $v$ and $w$ be unit vectors, and $R$ a nontrivial rotation. Then, the linear map $T(v, w, R)$:

- Is defined if $v^*$ is not in the same direction as $w$.
- Is identically zero if $v^*$ and $v$ are collinear.
- Is otherwise never zero.

18.4. The scale factor of $T(v, w, R)$ The image $T(v, w, R) \cdot v$ is equal to $\lambda w$, for some scalar $\lambda$. We define the scale factor $\lambda(v, w, R)$ to be this $\lambda$.

We then need two elementary facts about the scaling factor of $T(v, w, R)$. First, it is either identically zero or depends rationally on $v$ and $w$.

Lemma 18.3. Let $v$ and $w$ be unit vectors such that $v^*$ and $w$ are linearly independent. Then the scaling factor $\lambda(v, w, R)$ of the linear map $T(v, w, R)$ is given by

$$\frac{\langle v, (v^*)^\perp \rangle}{\langle w, (v^*)^\perp \rangle}$$

Proof. The map $T(v, w, R)$ is equivalent to the composition of:

- perpendicular projection from $\ell(v, 0)$ to $\ell((v^*)^\perp, 0)$, followed by
- the inverse of perpendicular projection $\ell(w, 0) \rightarrow \ell((v^*)^\perp, 0)$.

The first map scales the length of vectors by $\langle v, (v^*)^\perp \rangle$ and the second by $\langle w, (v^*)^\perp \rangle$. \[\square\]

From Lemma [18.3] it is immediate that

Lemma 18.4. The scaling factor $\lambda(v, w, R)$ is identically 0 precisely when $R$ is an order two rotation. If $R$ is not an order 2 rotation, then $\lambda(v, w, R)$ approaches infinity as $v^*$ approaches $\pm w$. 

\[46\]
18.5. Generic sequences of the map \( T(v, w, R) \) Let \( v_1, v_2, \ldots, v_n \) be unit vectors, and \( S_1, S_2, \ldots, S_n \) be rotations of the form \( R_k^i \) where \( R_k \) is a rotation of order \( k \). We define the linear map \( T(v_1, S_1, v_2, S_2, \ldots, v_n, S_n) \) to be

\[
T(v_1, S_1, v_2, S_2, \ldots, v_n, S_n) = T(v_n, v_1, S_n) \circ T(v_{n-1}, v_n, S_{n-1}) \circ \cdots \circ T(v_1, v_2, S_1)
\]

This next proposition plays a key role in the next section, where it is interpreted as providing a genericity condition for cone direction networks.

**Proposition 18.5.** Let \( v_1, v_2, \ldots, v_n \) be pairwise linearly independent unit vectors, and \( S_1, S_2, \ldots, S_n \) be rotations of the form \( R_k^i \). Then if the \( v_i \) avoid a proper algebraic subset of \( (\mathbb{S}^1)^n \) (that depends on the \( S_j \)), the map \( T(v_1, S_1, v_2, S_2, \ldots, v_n, S_n) \) scales the length of vectors by a factor of \( \lambda \neq 1 \).

**Proof.** If any of the \( T(v_i, v_{i+1}, S_i) \) are identically zero, we are done, so we assume none of them are. The map \( T(v_1, S_1, v_2, S_2, \ldots, v_n, S_n) \) then scales vectors by a factor of:

\[
\lambda(v_1, v_2, S_1) \cdot \lambda(v_2, v_3, S_2) \cdots \lambda(v_{n-1}, v_n, S_{n-1}) \cdot \lambda(v_n, v_1, S_n)
\]

which we denote \( \lambda \). That \( \lambda \) is constantly one is a polynomial statement in the \( v_i \) by Lemma 18.3, and so it is either always true or holds only on a proper algebraic subset of \( (\mathbb{S}^1)^n \). This means it suffices to prove that there is one selection for the \( v_i \) where \( \lambda \neq 1 \). We will show that \( |\lambda| \) can be made arbitrarily large, which implies that, in particular, it is not a constant.

Select the \( v_i \) so that the projection \( T(v_i, v_j, S_j) \) is defined for all \( i \) and \( j \). We hold \( v_2, \ldots, v_n \) fixed and consider what happens as we change \( v_1 \). As we change \( v_1 \), the contributions to \( \lambda \) from all the terms except \( \lambda(v_1, v_2, S_1) \) and \( \lambda(v_n, v_1, S_n) \) are fixed, so their contribution to \( \lambda \) is a constant as \( v_1 \) changes.

Here is the key observation: in a small neighborhood of the direction that makes \( v_1^* = v_2 \), \( \lambda(v_n, v_1) \) is uniformly bounded, since \( v_n^* \) is bounded away from \( v_1 \) by our initial choice of the \( v_i \). On the other hand, by Lemma 18.4 \( \lambda(v_1, v_2) \) is unbounded on the same neighborhood, and thus, \( |\lambda| \) can be made arbitrarily large. \( \square \)

19. Direction networks on cone-(2, 2) graphs collapse

In this section, we prove:

**Proposition 17.3.** A generic cone direction network that has as its colored quotient graph a cone-(2, 2) graph has only collapsed realizations.

The organization follows the outline given in Section 17.4. We separate the proof into two major cases: order \( k \geq 3 \) rotations (Section 19.4) and order two rotations (Section 19.3). Both make use of the results from Section 19.2 which relate the combinatorics of cone-(1, 1) graphs to the geometric lemmas of Section 18.
19.1. Genericity The meaning of generic in the statement of Proposition 17.3 is the standard one from algebraic geometry: the set of direction assignments for which the proposition fails to hold is a proper algebraic subset of the space of direction assignments.

Because the system (9) is square and homogeneous, the only solutions are all zero if and only if (9) has full rank, which is a polynomial statement in the given directions $d_{ij}$. It then follows that if we can construct one set of directions for which all realizations are collapsed, the same statement is true generically. Moreover, in this case, it is easy to describe the non-generic set: it is the set of directions for which the formal determinant of (9) vanishes.

The rest of this section, then, is occupied with showing such directions exist.

19.2. Assigning directions for map-graphs Let $(G, \gamma)$ be a $\mathbb{Z}/k\mathbb{Z}$-colored graph that is a connected cone-(1, 1) graph. Recall from Section 15 that this means that $G$ is a tree plus one edge and that the unique cycle in $G$ has non-trivial image under the map $\rho$. We also select and fix a base vertex $b \in V(G)$ that is on the cycle.

The next lemma shows that we can assign directions to the edges of $G$ so that the realization of the resulting direction network all have the structure similar to that shown in Figure 8. This will be the main “gadget” that we use in the proof of Proposition 17.3 below.

**Figure 8:** The structure of the realization of a cone-(1, 1) graph provided by Lemma 19.1 when $k = 4$ and the order of the rotation carried by the cycle is 4 ($\gamma = 1$ in the notation of Lemma 19.1). Every vertex lies on one of the dashed lines, which are determined by the order 4 rotational symmetry and the vector $v$. The fiber over the base vertex (black) lies at the crossings.

**Lemma 19.1.** Let $k = 3, 4, 6$, let $(G, \gamma)$ be a $\mathbb{Z}/k\mathbb{Z}$-colored graph that is a connected cone-(1, 1) graph with a base vertex $b$ on the unique cycle in $G$. Let $\gamma \in \mathbb{Z}/k\mathbb{Z}$ be the $\rho$-image of the cycle in $G$, and let $v$ be a unit vector. We can assign directions $d$ to the edges of $G$ so that, in all realizations of the resulting cone direction network that is the lift of $(G, \gamma, d)$:
• The directions from the origin to the points realizing the fiber over the base vertex \( b \) are in directions \( R_k \cdot v \)

• The rest of the points all lie on the lines between \( p_{i:b} \) and \( p_{(i+\gamma):b} \) as \( i \) ranges over \( \mathbb{Z}/k\mathbb{Z} \).

Proof. By Lemma 17.1 and Lemma 17.2, we can just assign directions in the lift \( \tilde{G} \) of \( G \).

We start by selecting an edge \( bi \in E(G) \) that is:

• Incident on the base vertex \( b \)

• On the cycle in \( G \)

Such an edge exists because \( G \) is a map-graph and \( b \) is on the cycle. \( G - bi \) is a spanning tree \( T \) of \( G \).

Since \( T \) is contractible, it lifts to \( k \) disjoint copies of itself in \( \tilde{G} \). Select one of these copies and denote it \( \tilde{T} \). Note that \( \tilde{T} \) hits the fiber over every edge in \( G \) except for \( bi \) exactly one time and the fiber over every vertex exactly one time.

Assign every edge in \( \tilde{T} \) the direction \( v^* = (R_k^{\gamma/2} \cdot v)^\perp \). By Lemma 17.2 this assigns a direction to every edge in \( G \) except for \( bi \). From the observations above, it now follows by the connectivity of \( T \) that in any realization of the cone direction network induced on \( \tilde{G} - \pi^{-1}(ij) \) any point lies on the \( R_k \)-orbit of a single affine line in the direction of \( v^* \).

Now select the edge in the fiber over \( bi \) incident on the copy of \( i \) in \( \tilde{T} \). Assign this edge the direction \( v^* \) as well. Let the set of directions induced on \( G \) be \( \tilde{d} \).

Denote by \( p_b \) the realization of the copy of \( b \) in \( \tilde{T} \) in a realization of \( (\tilde{G}, \varphi, \tilde{d}) \). As we have noted above, the realization of every vertex of \( \tilde{G} \) is on one of the lines \( \ell(R_k \cdot v^*, s) \) where \( s \) is determined by \( p_b \). The selection of direction for the edge \( bi \) further forces that if \( p_b \) is in the fiber over \( b \), that \( R \cdot p_b - p_b \) is in the direction \( v \).

It now follows from Lemma 18.1, applied to a rotation of the same order as \( R^\gamma \), that \( p_b \) is in the direction \( v \), which is what we wanted.

19.3. Proof of Proposition 17.3 for order 2 rotations Decompose the cone-(2,2) graph \( (G, \gamma) \) into two edge-disjoint cone-(1,1) graphs \( X \) and \( Y \). The order of the \( \rho \)-image of any cycle in either \( X \) or \( Y \) is always 2, so the construction of Lemma 19.1 implies that by assigning the same direction \( v \) to every edge in \( X \) every vertex in any realization lies on a single line through the origin in the direction of \( v \). Similarly for edges in \( Y \) in a direction \( w \) different than \( v \).

Since every vertex is at the intersection of two skew lines through the origin, the proposition is proved.

19.4. Proof of Proposition 17.3 for rotations of order \( k \geq 3 \) In what follows we let \( (G, \gamma) \) be a cone-(2,2) graph on \( n \) vertices. We fix a decomposition of \( (G, \gamma) \) into two cone-(1,1) graphs. This is possible by Lemma 15.2.
Let \( G_i \) be the connected components of the two cone-(1, 1) graphs. Which part of the decomposition \( G_i \) comes from is not important in what follows, so we suppress it in the notation. All the information in the decomposition we need comes from the overlap graph, defined in Section 11. Select a base vertex \( b_i \) on the cycle of each of the \( G_i \). Let \( D \) be the overlap graph of the decomposition, and index the vertex set of \( D \) by \( B \), the collection of \( b_i \).

**Assigning directions** For each \( G_i \), select a unit vector \( v_i \) such that:

- The \( v_i \) are all different.
- Any subset of the \( v_i \) are generic in the sense of Proposition 18.5.

This is possible, since Proposition 18.5 rules out only a measure zero subset of directions for each sub-collection.

Now, for each \( G_i \) we assign, in the notation of Lemma 19.1, the direction \( v_i^* \) to the edges in \( G_i \) as prescribed by Lemma 19.1. This is well-defined, since the \( G_i \) partition the edges of \( G \). (They clearly overlap on the vertices—we will exploit this fact below—but it does not prevent us from assigning edge directions independently.)

We define the resulting colored direction network to be \((G, \gamma, d)\) and the lifted cone direction network \((\tilde{G}, \phi, \tilde{d})\). We also define, as a convenience, the rotation \( S_i \) as the rotation such that \( v_i = (S_i^{1/2} v_i^*) \perp \) to be \( S_i \).

**Local structure of realizations** Let \( G_i \) and \( G_j \) be distinct connected cone-(1, 1) components and suppose that there is a directed edge \( b_i b_j \) in the overlap graph \( D \). We have the following relationship between \( p_{b_i} \) and \( p_{b_j} \) in realizations of \((\tilde{G}, \varphi, \tilde{d})\).

**Lemma 19.2.** Let \( \tilde{G}(p) \) be a realization of the cone direction network \((\tilde{G}, \varphi, \tilde{d})\) defined above. Let vertices \( b_i \) and \( b_j \) in \( V(G) \) be the base vertices of \( G_i \) and \( G_j \), and suppose that \( b_i b_j \) is a directed edge in the overlap graph \( D \). Let \( p_{\gamma' \cdot \tilde{b}_j} \) be the realization of any vertex in the fiber over \( b_i \) in \( V(\tilde{G}) \). Then there is a vertex \( \gamma' \cdot \tilde{b}_j \) in the fiber over \( b_j \) such that \( p_{\gamma' \cdot \tilde{b}_j} = T(\gamma, \varphi, \tilde{d}) \cdot p_{\gamma' \cdot \tilde{b}_j} \).

The proof is illustrated in Figure 9.

**Proof.** By Lemma 19.1, the fiber over every vertex in \( G_i \) lies on a line \( \ell(R^*_k \cdot v_i^*, s) \) for some scalar \( s_i \). The scalar \( s_i \) is determined by location of any \( p_{b_i} \) representing a vertex in the fiber over \( b_i \), since the \( p_{\cdot b_i} \) are all equal to some multiple of \( R^*_k \cdot v_i \).

In particular, the vertices in the fiber over \( b_j \) are on these lines. Additionally, Lemma 19.1 applied to \( G_j \), says that the vertices in the fiber over \( b_j \) are all equal to some scalar multiple of \( R^*_k \cdot v_j \). This is exactly the situation captured by the map \( T(\gamma, \varphi, \tilde{d}) \). \( \square \)
Figure 9: Example of the local structure of the proof of Proposition 17.3: the directions we assign imply that if $b_1b_2$ is an edge in the overlap graph, then the base vertex of $G_1$ can be obtained from the base vertex of $G_2$ via the linear projection $T(v_1,v_2,S_1)$.

**Base vertices on cycles in $D$ must be at the origin** Let $b_i$ be the base vertex in $G_i$ that is also on a directed cycle in $D$. The next step in the proof is to show that all representatives in $b_i$ must be mapped to the origin in any realization of $(\tilde{G}, \varphi, \tilde{d})$.

**Lemma 19.3.** Let $\tilde{G}(p)$ be a realization of the cone direction network $(\tilde{G}, \varphi, \tilde{d})$ defined above, and let $b_i \in V(G)$ be a base vertex that is also on a directed cycle in $D$ (one exists by Proposition 11.1). Then all vertices in the fiber over $b_i$ must be mapped to the origin.

**Proof.** Iterated application of Lemma 19.2 along the cycle in $D$ that $b_i$ is on tells us that any vertex in the fiber over $b_i$ is related to another vertex in the same fiber by a linear map meeting the hypothesis of Proposition 18.5. This implies that if any vertex in the fiber over $b_i$ is mapped to a point not the origin, some other vertex in the same fiber is mapped to a point at a different distance to the origin. This is a contradiction, since all realizations $\tilde{G}(p)$ are symmetric with respect to $R_k$, so, in fact the fiber over $b_i$ was mapped to the origin. \qed

**All base vertices must be at the origin** So far we have shown that every base vertex $b_i$ that is on a directed cycle in the overlap graph $D$ is mapped to the origin in any realization $\tilde{G}(p)$ of $(\tilde{G}, \varphi, \tilde{d})$. However, since every base vertex is connected to the cycle in its connected component by a directed path in $D$, we can show all the base vertices are at the origin.

**Lemma 19.4.** Let $\tilde{G}(p)$ be a realization of the cone direction network $(\tilde{G}, \varphi, \tilde{d})$ defined above. Then all vertices in the fiber over $b_i$ must be mapped to the origin.

**Proof.** The statement is already proved for base vertices on a directed cycle in Lemma 19.3. Any base vertex not on a directed cycle, say $b_i$, is at the end of a directed path which starts at a vertex on the directed cycle. Thus $p_{\gamma\cdot b_i}$ is the image of 0 under some linear map and hence is at the origin. \qed
All vertices must be at the origin  The proof of Proposition 17.3 then follows from the observation that, if all the base vertices $b_i$ must be mapped to the origin in $\tilde{G}(\mathbf{p})$, then Lemma 19.1 implies that every vertex in the lift of $G_i$ lies on a family of $k$ lines intersecting at the origin. (This is the degenerate form of Figure 8 where the base vertex is at the origin.)

Since every vertex is in the span of two of the $G_i$, and these families of lines intersect only at the origin, we are done: $\tilde{G}(\mathbf{p})$ must put all the points at the origin. □

20. Proof of Theorem 6

This section gives the proof of:

**Theorem 6.** A generic realization of a cone direction network $(\tilde{G}, \varphi, \hat{\mathbf{d}})$ has a faithful realization if and only if its associated colored graph is cone-Laman. This realization is unique up to scaling.

20.1. Generic rank of the colored realization system  Proposition 17.3 is a geometric statement, but it has an algebraic interpretation. In matroidal language, this next lemma says that, in matrix form, the system (9) is a generic linear representation for the cone-(2, 2) matroid.

**Lemma 20.1.** Let $(G, \gamma)$ be a $\mathbb{Z}/k\mathbb{Z}$-colored graph with $n$ vertices and $m$ edges. Then, if $(G, \gamma)$ is cone-(2, 2) sparse, the generic rank of the system (9) is $m$.

**Proof.** If $(G, \gamma)$ is cone-(2, 2) sparse, the matroid property of cone-(2, 2) graphs (Lemma 15.3) implies that it can be extended to a cone-(2, 2) graph $(G', \gamma')$. Form a generic direction network on $(G', \gamma')$. By Proposition 17.3 the colored realization system for this extended direction network has rank $2n$, so it follows that all $m$ of the equations in its restriction to $(G, \gamma)$ are independent. □

In particular, since cone-Laman graphs are cone-(2, 2) sparse, we get:

**Lemma 20.2.** Let $(G, \gamma)$ be a cone-Laman graph with $\mathbb{Z}/k\mathbb{Z}$ colors. Then the generic rank of the system (9) is $2n - 1$.

20.2. Proof of Theorem 6  We prove each direction of the statement in turn. Since it is technically easier, we prove the equivalent statement on colored direction networks. The theorem then follows by Lemma 17.1.

**Cone-laman graphs generically have faithful realizations**  Let $(G, \gamma)$ be a cone-Laman graph with $n$ vertices. Lemma 15.4 implies that doubling any edge $ij$ of $(G, \gamma)$ results in a cone-(2, 2) graph $(G + ij, \gamma)$. Select edge directions for $G$ such that, for every $(G + ij, \gamma)$ obtained from $G$ by doubling an edge, these directions on the edges of $G$, plus some direction on the added copy of the edge $ij$ yield directions on $(G + ij, \gamma)$ that
are generic in the sense of Proposition 17.3. This is possible, since the desired directions lie in the intersection of a finite number of full measure subsets of the space of direction assignments.

Define \((G, \gamma, d)\) to be the colored direction network with these directions. By Lemma 20.1, the realization space is 1-dimensional, since the system (9) has rank \(2n - 1\). There must be some realization of \(G\) that is not entirely collapsed: \(G\) is connected by Lemma 15.5 and, since the realization space is 1-dimensional, it allows non-trivial scalings. Together these facts imply that some edge is not realized with coincident endpoints.

Now we suppose, for a contradiction, that some edge \(ij\) is collapsed in a non-collapsed realization \(G(p)\) of \((G, \gamma, d)\). Because the realization space is one-dimensional, all other realizations are scalings of \(G(p)\), which implies that \(ij\) is collapsed in all realizations of \((G, \gamma, d)\). Adding a second copy of the colored edge \(ij\) and giving it a different direction forces \(ij\) to be collapsed in all realizations, and so we see that the realization space of \((G, \gamma, d)\) is exactly the same as that of \((G + ij, \gamma, d)\).

We are now at a contradiction. The directions \(d\) are chosen such that \((G + ij, \gamma, d)\) is generic in the sense of Proposition 17.3, and this implies that \((G + ij, \gamma, d)\), and so \((G, \gamma, d)\) has a zero-dimensional realization space. Since \((G, \gamma, d)\) always has at least a one-dimensional realization space, we are done.

**Cone-Laman circuits have collapsed edges** For the other direction, we suppose that \((G, \gamma)\) has \(n\) vertices and is not cone-Laman. If the number of edges \(m\) is less than \(2n - 1\), then the realization space of any direction network is at least 2-dimensional, so it contains more than just rescalings. Thus, we assume that \(m \geq 2n - 1\). In this case, \(G\) is not cone-Laman sparse, so it contains a cone-Laman circuit \((G', \gamma)\). Thus, we are reduced to showing that any cone-Laman circuit has, generically, only realizations with collapsed edges, since these then force collapsed edges in any realization of a generic colored direction network on \((G, \gamma)\).

We recall that Lemma 15.6 says there are two types of cone-Laman circuits \((G', \gamma)\):

- \((G', \gamma)\) is a cone-(2, 2) graph.
- \((G', \gamma)\) has \(T(G') = 2\), \(n'\) vertices, \(m' = 2n' - 2\) edges, and is cone-(2, 2) sparse.

If \((G', \gamma)\) is a cone-(2, 2) graph, then Proposition 17.3 applies to it, and we are done. For the other type of cone-(2, 2) circuit, Lemma 20.1 implies that, for generic directions, a direction network has rank \(2n' - 2\), so the realization space is two-dimensional. Because \(T(G') = 2\), the \(\rho\)-image of \((G', \gamma)\) is trivial, so \(G'\) lifts to \(k\) disjoint copies of itself. We can construct realizations of the lifted direction network by picking one of these copies of \(G'\) in the lift as representatives and putting the vertices on top of each other at an arbitrary point in the plane. Since this is a 2-dimension family of realizations with all edges collapsed, this family is, in fact the entire realization space, completing the proof. 

\(\square\)
21. Crystallographic direction networks

Let \((\tilde{G}, \varphi)\) be a graph with a \(\Gamma_k\)-action \(\varphi\). A crystallographic direction network \((\tilde{G}, \varphi, \tilde{d})\) is given by, \((\tilde{G}, \varphi)\) and an assignment of a direction \(\tilde{d}_{ij}\) to each edge \(ij \in E(\tilde{G})\).

21.1. The realization problem

A realization of a crystallographic direction network is given by a point set \(p = (p_i)_{i \in V(\tilde{G})}\) and a representation \(\Phi\) of \(\Gamma_k\) such that:

\[
\begin{align*}
\langle p_j - p_i, \tilde{d}_{ij}^\perp \rangle &= 0 \quad \text{for all edges } ij \in E(\tilde{G}) \quad (10) \\
p_{\gamma \cdot i} &= \Phi(\gamma) \cdot p_i \quad \text{for all vertices } i \in V(\tilde{G}) \quad (11)
\end{align*}
\]

We observe that for a crystallographic direction networks to be realizable, the \(\Gamma_k\)-orbit of any edge needs to be given the \(\phi\)-equivariant directions; i.e. if \(i'j' = \gamma \cdot ij\), then \(d_{i'j'}\) is \(d_{ij}\) rotated by the rotational part of \(\gamma\). From now on we require \(\phi\)-equivariance of directions. We denote realizations by \(\tilde{G}(p, \Phi)\), to indicate the dependence on \(\Phi\).

The definition of collapsed and faithful realizations is similar to that for cone direction networks. An edge \(ij\) is collapsed in a realization \(\tilde{G}(p, \Phi)\) if \(p_i = p_j\). A realization is collapsed when all the edges are collapsed and \(\Phi\) is trivial. A representation is trivial if it maps the translation generators of \(\Gamma_k\) to the zero vector. A realization is faithful if no edge is collapsed and \(\Phi\) is not trivial.

Although our techniques don’t require it in this section, for convenience, we will consider only realizations that map the rotational generator \(r_k\) of \(\Gamma_k\) to the rotation around the origin \(R_k\) through angle \(2\pi/k\). The dimension of the resulting realization space is two less than the dimension of the realization space where the rotation center of \(\Phi(r_k)\) is not pinned down.

21.2. Direction Network Theorem

As in the case of cone direction networks, all the information about a crystallographic direction network is captured by its colored quotient graph. We make this precise in Section 21.4 below. Our main theorem on crystallographic direction networks is

**Theorem 3.** Let \(\Gamma\) be a crystallographic group generated by translations and rotations. A generic realization of a \(\Gamma\)-crystallographic direction network \((\tilde{G}, \varphi, \tilde{d})\) has a faithful realization if and only if its associated colored graph is \(\Gamma\)-colored-Laman. This realization is unique up to translation and scaling.

21.3. Proof of Theorem 3

The key proposition, which is proved in Section 23 is:

**Proposition 21.1.** A generic crystallographic direction network that has as its colored quotient graph a \(\Gamma\)-(2, 2) graph has only collapsed realizations.

It then follows that:
Proposition 21.2. A generic crystallographic direction network that has, as its colored quotient graph, a $\Gamma$-colored-Laman circuit has only realizations with collapsed edges.

Proposition 21.2 is proved in Section 24. The proof of Theorem 6 then goes through with appropriate modifications.

21.4. Colored direction networks As we did with cone direction networks, we will make use of colored crystallographic direction networks to study crystallographic direction networks. Since there is no chance of confusion, we simply call these “colored direction networks” in the next several sections.

A colored direction network $(G, \gamma, d)$ is given by a $\Gamma_k$-colored graph $(G, \gamma)$ and an assignment of a direction $d_{ij}$ to every edge $ij$. The realization system for $(G, \gamma, d)$ is given by

$$\left( \Phi(\gamma_{ij}) \cdot p_j - p_i, d_{ij} \right) = 0$$

(12)

The unknowns are the representation $\Phi$ of $\Gamma_k$ and the points $p_i$. We denote points in the realization space by $G(p, \Phi)$.

The following two lemmas linking crystallographic direction networks and colored direction networks have the same proofs as Lemmas 17.1 and 17.2

Lemma 21.3. Given a colored direction network $(G, \gamma, d)$, its lift to a crystallographic direction network $(\tilde{G}, \varphi, \tilde{d})$ is well-defined and the realization spaces of $(G, \gamma, d)$ and $(\tilde{G}, \varphi, \tilde{d})$ are isomorphic. In particular, they have the same dimension.

Lemma 21.4. Let $(G, \gamma)$ be a $\Gamma_k$-colored graph and $(\tilde{G}, \varphi)$ its lift. Assigning a direction to one representative of each edge orbit under $\varphi$ in $\tilde{G}$ gives a well defined colored direction network $(G, \gamma, d)$.

This next lemma, which is also immediate from the definitions, describes collapsed edges in terms of colored direction networks.

Lemma 21.5. Let $(G, \gamma, d)$ be a colored direction network and let $G(p, \Phi)$ be a realization of $(G, \gamma, d)$. Let $(\tilde{G}, \varphi, \tilde{d})$ be the lift of $(G, \gamma, d)$ and $\tilde{G}(p, \Phi)$ be the associated lift of $G(p, \Phi)$. Then a colored edge $ij \in E(G)$ lifts to an orbit of collapsed edges in $\tilde{G}(p, \Phi)$ if and only if

$$p_i = \Phi(\gamma_{ij}) \cdot p_j$$

in $G(p, \Phi)$.

In light of Lemmas 21.3–21.5, we may switch freely between the formalisms, and we do so in subsequent sections.
21.5. Proof strategy for Proposition 21.1  The main difference between Proposition 21.1 and Proposition 17.3 is that we need to account for the flexibility of $\Phi$. To do this, we bootstrap the proof using generalized cone-$(2,2)$ graphs (from Section 16). The steps are:

- We show that, for fixed $\Phi$, a generic direction network on a generalized cone-$(2,2)$ graph has a unique solution (Proposition 22.1).

- Then we allow $\Phi$ to flex. We show that by adding $\text{rep}(\Lambda(\Gamma_k))$ edges that extend a generalized cone-Laman graph to a $\Gamma$-$(2,2)$ graph, realizations of a generic direction network are forced to collapse.

This is done in Sections 22 and 23.

22. Direction networks on generalized cone-$(2,2)$ graphs

Let $(G, \gamma)$ be a generalized cone-$(2,2)$ graph. In light of Proposition 17.3, it is intuitive that the realization system (12) should have generic rank $2n$ for a colored direction network on $(G, \gamma)$, since cone direction networks are a “special case”. In this section we verify that intuition by giving the reduction to.

**Proposition 22.1.** Fix a faithful representation $\Phi$ of $\Gamma_k$. Holding $\Phi$ fixed, a generic crystallographic direction network that has a generalized cone-$(2,2)$ graph as its colored quotient has a unique solution.

Proposition 22.1 is immediate from the following statement and Lemma 21.3.

**Proposition 22.2.** Let $(G, \gamma)$ be a generalized cone-$(2,2)$ graph with $n$ vertices. Then the generic rank of the realization system (12) is $2n$.

22.1. Proof of Proposition 22.2 Expanding (12) we get

$$\left\langle \Phi(\gamma_{ij}) \cdot \mathbf{p}_j - \mathbf{p}_i, \mathbf{d}_{ij}^\perp \right\rangle = \left\langle \Phi(\gamma_{ij}) \cdot \mathbf{p}_j, \mathbf{d}_{ij}^\perp \right\rangle - \left\langle \mathbf{p}_i, \mathbf{d}_{ij}^\perp \right\rangle$$

We define $\Phi(\gamma_{ij}), \in \text{SO}(2)$ to be the rotational part of $\Phi(\gamma_{ij})$ and $\Phi(\gamma_{ij}), \in \mathbb{R}^2$ to be the translational part, so that $\Phi(\gamma_{ij}) \cdot \mathbf{p} = \Phi(\gamma_{ij}), \cdot \mathbf{p} + \Phi(\gamma_{ij}),$. In this notation, (13) becomes

$$\left\langle \Phi(\gamma_{ij}), \cdot \mathbf{p}_j, \mathbf{d}_{ij}^\perp \right\rangle + \left\langle \Phi(\gamma_{ij}), \mathbf{d}_{ij}^\perp \right\rangle - \left\langle \mathbf{p}_i, \mathbf{d}_{ij}^\perp \right\rangle$$

Finally, since the rotational part $\Phi(\gamma_{ij}),$ preserves the inner product, we see that (12) is equivalent to the inhomogeneous system

$$\left\langle \mathbf{p}_j, \Phi(\gamma_{ij})^{-1}, \cdot \mathbf{d}_{ij}^\perp \right\rangle - \left\langle \mathbf{p}_i, \mathbf{d}_{ij}^\perp \right\rangle = -\left\langle \Phi(\gamma_{ij}), \mathbf{d}_{ij}^\perp \right\rangle$$

The l.h.s. of (15) is the same as (9), and thus the generic rank of (15) is at least as large as that of (9). The proposition then follows from Proposition 17.3.
23. Proof of Proposition 21.1

We now have the tools in place to prove:

**Proposition 21.1.** A generic crystallographic direction network that has as its colored quotient graph a $\Gamma$-(2, 2) graph has only collapsed realizations.

The proof is split into two cases, $\Gamma_2$ and $\Gamma_k$ for $k = 3, 4, 6$.

23.1. Proof for rotations of order 3, 4, and 6 Let $(G, \gamma)$ be a $\Gamma$-(2, 2) graph. We construct a direction network on $(G, \gamma)$ that has only collapsed solutions, from which the desired generic statement follows.

**Assigning directions** We select directions $d$ for each edge in $G$ such that they are generic in the sense of Proposition 22.1 for every g.c.-basis of $(G, \gamma)$. Define the resulting colored direction network to be $(G, \gamma, d)$.

**The realization space of any spanning g.c.-basis** With these direction assignments, we can compute the dimensions of the realization space for the direction network induced on any spanning g.c.-basis of $(G, \gamma)$. One exists by Lemma 16.5.

**Lemma 23.1.** Let $(G', \gamma)$ be a spanning g.c.-basis of $(G, \gamma)$. Then the realization space of the induced direction network $(G', \gamma, d)$ is 2-dimensional, and linearly depends on the representation $\Phi$.

**Proof.** The dimension comes from Proposition 22.1 and comparing the number of variables to the number of equations in the realization system (12). Moving the variables associated with $\Phi$ to the right completes the proof.  

**A g.c.-basis with non-collapsed complement** Let $(G', \gamma)$ be a g.c.-basis of $(G, \gamma)$. By edge counts, there are exactly two edges $ij$ and $vw$ in the complement of $(G', \gamma)$.

**Lemma 23.2.** There is a g.c.-basis $(G', \gamma)$ of $(G, \gamma)$ such that the edges $ij$ and $vw$ in its complement are non-collapsed in some realization of $(G', \gamma, d)$.

**Proof.** By Proposition 12.4, we can decompose $(G, \gamma)$ into two spanning $\Gamma$-(1, 1) graphs $X$ and $Y$. Since $\Gamma$-(1, 1) graphs are all generalized cone-(1, 1) graphs plus an edge for $k = 3, 4, 6$, we can assume, w.l.o.g., that $X - ij$ and $Y - vw$ are generalized cone-(1, 1). Define $X'$ to be $X - ij$ and $Y'$ to be $Y - vw$. It follows that $X' \cup Y'$ is a g.c.-basis of $(G, \gamma)$.

Let $X''$ be the fundamental g.c.-$(1, 1)$ circuit of $ij$ in $X'$. By Lemma 16.3, the $\rho$-image of $X''$ contains a translation. If every edge in $X''$ is collapsed in every realization of the direction network $(X' \cup Y', \gamma, d)$, this implies that $\Phi$ must always be trivial in any realization. Proposition 22.1 would then imply that the realization space is 0-dimensional, which is a contradiction to Lemma 23.1.
Here is the key step of the argument: since some edge $i'j'$ in $X''$ is not collapsed, we can do a basis exchange (on the g.c.-$(1,1)$ matroid) to find a g.c.-basis with $i'j'$ and $vw$ in its complement, and $i'j'$ not collapsed in some realization.

We then repeat the argument on $Y'$ and $vw$. Since this will not affect $i'j'$, we are done.

Now we select a g.c.-basis $(G',\gamma)$ with the property of Lemma 23.2, and let $G(p,\Phi)$ be a realization in which the edges $ij$ and $vw$ are both non-collapsed. The rest of the proof will be to show that, adding back $ij$ and $vw$ forces all realizations of $(G,\gamma,d)$ to collapse.

The realization space of $(G'+ij,\gamma,d)$ We first consider adding back $ij$.

Lemma 23.3. The realization space of $(G'+ij,\gamma,d)$ is 1-dimensional.

Proof. We know that $ij$ is not collapsed, so the Lemma will follow provided that the direction of $v = p_j - p_i$ is non-constant in realizations of $(G',\gamma,d)$ as a function of $\Phi$. In this case, simply setting $d_{ij} = v$ would impose a new linear constraint, decreasing the dimension of the realization space by one.

To see that the direction of $v$ is not constant as $\Phi$ varies, observe that assigning a direction $d_{ij}$ other than $v$ to $ij$ would then force $ij$ to collapse in any realization of $(G'+ij,\gamma,d)$. In turn, using the edge swapping argument from Lemma 23.2 the entire g.c.-$(1,1)$ circuit of $ij$ in $X'$ collapses, resulting in a zero-dimensional realization space. This contradicts Lemma 23.1 in that it implies the realization space of $(G',\gamma,d)$ was 1-dimensional.

In light of Lemma 23.3, we set the direction $d_{ij}$ to be $v$. This is allowed, since it preserve the realization $G(p,\Phi)$ we obtained from Lemma 23.2 and $ij$ is, by definition, outside of the g.c.-basis $(G',\gamma)$.

The representation $\Phi$ must be trivial Now we consider adding the edge $vw$ to $(G'+ij,\gamma,d)$.

Lemma 23.4. The representation $\Phi$ is trivial in any realization of $(G,\gamma,d)$.

Proof. The realization space of $(G'+ij,\gamma,d)$ is 1-dimensional by Lemma 23.3 and so it consists only of rescalings of the realization $G(p,\Phi)$ in which $p_v$ and $p_w$ are distinct. Setting the direction $d_{vw}$ to any direction other than that of $p_w - p_v$ then gives the Lemma: the new constraint then forces the edge $vw$ to collapse, and with it, using the argument used to show Lemma 23.2 its g.c.-$(1,1)$ circuit in $Y'$, and consequently $\Phi$.

All realizations are collapsed The existence of a g.c.-basis of $(G,\gamma)$ and Proposition 22.1 guarantee a unique realization of $(G,\gamma,d)$ depending on $\Phi$. When $\Phi$ is trivial, this is the completely collapsed solution.
23.2. Proof for rotations of order 2 Let \((G, \gamma)\) be a \(\Gamma-(2, 2)\) graph. Again, we will assign directions so that the resulting direction network \((G, \gamma, d)\) has only collapsed solutions. The proof has a slightly different structure from the \(k = 3, 4, 6\) case. The main geometric lemma is the following.

**Lemma 23.5.** Let \((X, \gamma)\) be a \(\Gamma-(1, 1)\) graph with \(\Gamma_2\) colors. Then any realization \(X(p, \Phi)\) of a colored direction network \((X, \gamma, d)\) that assigns the same direction \(v\) to every edge lifts to a realization \(\tilde{X}(p, \Phi)\) such that every vertex lies on a single line in the direction of \(v\).

**Proposition 21.1 for \(\Gamma_2\) from Lemma 23.5** With Lemma 23.5, the Proposition follows readily: the combinatorial Proposition 12.4 says we may decompose \((G, \gamma)\) into two spanning \(\Gamma-(1, 1)\) graphs, which we define to be \(X\) and \(Y\). We assign the edges of \(X\) a direction \(v_X\) and the edges of \(Y\) a linearly independent direction \(v_Y\). Applying Lemma 23.5 to \(X\) and \(Y\) separately shows that every vertex of a lifted realization \(\tilde{G}(p, \Phi)\) must lie in two skew lines. This is possible only when they are all at the intersection of these lines, implying only collapsed realizations.

**Proof of Lemma 23.5** Let \((X, \gamma)\) be a \(\Gamma-(1, 1)\) graph with \(\Gamma_2\) colors, and let \((X, \gamma, d)\) be a direction network that assigns all the edges the same direction. Let \((X', \gamma)\) be a g.c.-\((1, 1)\) basis of \((X, \gamma)\); one exists by Lemma 16.2.

First we consider one connected component \(X''\) of \(X'\).

**Lemma 23.6.** Let \((X'', \gamma, d)\) be a connected g.c.-\((1, 1)\) graph, and let \(d\) assign the same direction \(v\) to every edge. Then, in any realization of the lifted crystallographic direction network \((\tilde{X}, \varphi, d)\), every vertex and every edge lies on a line in the direction \(v\) through a rotation center.

**Proof.** We reason similarly to the way we did in Section 19.3. Because the \(\rho\)-image of \(X''\) contains an order 2 rotation \(r\), for some vertex \(i \in V(X'')\), there is a vertex \(\tilde{i}\) in the fiber over \(i\) such that \(p_{\tilde{i}} - p_{r \cdot \tilde{i}} = p_{\tilde{i}} - \Phi(r) \cdot p_{\tilde{i}}\) is in the direction \(v\). Because \(\Phi(r)\) is a rotation through angle \(\pi\), this means that \(p_{\tilde{i}}\) and \(p_{r \cdot \tilde{i}}\) lie on a line through the rotation center of \(r\) in the direction \(v\). Because \(X''\) is connected, and edge directions are fixed under an order 2 rotation, the same is then true for every vertex in the connected component \(\tilde{X}_0''\) of the lifted realization \(\tilde{X}(p, \Phi)\) that contains \(p_{\tilde{i}}\).

The lemma then follows by considering translates of \(\tilde{X}_0''\).

Considering the connected components one at a time, Lemma 23.6 readily implies

**Lemma 23.7.** Let \((X', \gamma, d)\) be a g.c.-\((1, 1)\) graph, and let \(d\) assign the same direction \(v\) to every edge. Then, in any realization of the lifted crystallographic direction network \((\tilde{X}, \varphi, d)\), every vertex and every edge lies on a line in the direction \(v\) through a rotation center.
To complete the proof, we recall that the $\rho$-image of $(X, \gamma)$ contains two linearly independent translations $t$ and $t'$. If $\Phi(t)$ or $\Phi(t')$ is not in the direction $v$, by Lemma 23.7 there is some edge in the lifted realization $\tilde{X}(p, \Phi)$ that has one endpoint on one line in the direction $v$ and the other endpoint on a translation of this line. This is incompatible with all edge edges of $X$ being assigned the direction $v$, so we conclude that $\Phi(t)$ and $\Phi(t')$ are both in the direction $v$, from which the Lemma follows.

24. Proof of Proposition 21.2

We now prove the “Maxwell direction” of Theorem 3:

Proposition 21.2. A generic crystallographic direction network that has, as its colored quotient graph, a $\Gamma$-colored-Laman circuit has only realizations with collapsed edges.

In the proof, we will use the following statement (cf. [10, Lemma 14.2] for the case when the $\rho$-image is a translation subgroup)

Lemma 24.1. Let $(G, \gamma, d)$ be a colored direction network on a colored graph $(G, \gamma)$ with connected components $G_1, G_2, \ldots, G_c$. Then $(G, \gamma, d)$ has at least

$$\text{rep}_{\Gamma_k}(\Lambda(\Gamma_k)) - \text{rep}_{\Gamma_k}(G) + \sum_{i=1}^c T(G_i)$$

dimensions of solutions with all edges collapsed and the origin as a rotation center.

We defer the proof of Lemma 24.1 to Section 24.2 and first show how Lemma 24.1 implies Proposition 21.2.

24.1. Proof of Proposition 21.2

Let $(G, \gamma)$ be a $\Gamma$-colored-Laman circuit with $n$ vertices, $m$ edges, and $c$ connected components $G_1, G_2, \ldots, G_c$. By Lemma 13.4, we have

$$m = 2n + \text{rep}_{\Gamma_k}(G) - \sum_{i=1}^c T(G_i)$$

It follows from Proposition 21.1 that for generic directions, a colored direction network $(G, \gamma, d)$ has a

$$2n + \text{rep}_{\Gamma_k}(\Lambda(\Gamma_k)) - m = \text{rep}_{\Gamma_k}(\Lambda(\Gamma_k)) - \text{rep}_{\Gamma_k}(G) + \sum_{i=1}^c T(G_i)$$

dimensional space of realizations with the origin as a rotation center. Applying Lemma 24.1 shows that in all of them every edge is collapsed.

24.2. Proof of Lemma 24.1

For now, assume that the colored graph $(G, \gamma)$ is connected. Select a base vertex $b$. 

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Representations that are trivial on $\Lambda(G, b)$ Let $\Phi \in \overline{\text{Rep}}_{\Gamma_k}(\Lambda(\Gamma_k))$ be such that

$$\Phi(t) = ((0, 0), \text{Id})$$

for all translations $t \in \Lambda(G, b)$. These representations form a $(\text{rep}_{\Gamma_k}(\Lambda(\Gamma_k)) - \text{rep}_{\Gamma_k}(G))$-dimensional space.

Collapsed realizations for a fixed representation Now we show that there are $T(G)$ dimensions of realizations with all edges collapsed. We do this with an explicit construction. There are two cases.

Case 1: $T(G) = 2$. In this case, we know that the subgroup generated by $\rho(\pi_1(G, b))$ is a translation subgroup. Fix a spanning tree $T$ of $G$ and a point $p_b \in \mathbb{R}^2$. We will construct a realization with vertex $b$ mapped to $p_b$ and all edges collapsed.

For any pair of vertices $i$ and $j$, define $Q_{ij}$ to be the path in $T$ from $i$ to $j$ and define $\eta_{ij}$ to be $\rho(Q_{ij})$. We then set $p_i = \Phi(\eta_{bi}^{-1}) \cdot p_b$ for all vertices $i \in V(G)$ other than $b$. Thus all vertex locations are determined by $p_b$, giving a 2-dimensional space of realizations for this $\Phi$. We need to check that all edges are collapsed.

If $ij$ is an edge of $T$ with color $\gamma_{ij}$, then we have

$$\gamma_{ij}^{-1} = \eta_{bi}^{-1} \cdot \eta_{bj}$$

Using this relation, we see that

$$p_j = \Phi(\eta_{bj}^{-1}) \cdot p_b = \Phi(\eta_{ij}^{-1} \cdot \eta_{bi}^{-1}) \cdot p_b = \Phi(\gamma_{ij}^{-1}) \cdot p_i$$

so the edge $ij$ is collapsed. If $ij$ is not an edge in $T$, then the fundamental closed path $P_{ij}$ of $ij$ relative to $T$ and $b$ follows $Q_{bi}$, crosses $ij$, and returns to $b$ along $Q_{jb}$. This gives us the relation

$$\gamma_{ij} = \eta_{bi}^{-1} \cdot \rho(P_{ij}) \cdot \eta_{bj}$$

We then compute

$$\Phi(\gamma_{ij}) \cdot p_j = (\Phi(\eta_{bi}^{-1}) \cdot \Phi(\rho(P_{ij})) \cdot \Phi(\eta_{bj})) \cdot p_j$$

Since $\Phi$ is trivial on the $\rho$-images of fundamental closed paths, the r.h.s. simplifies to

$$\Phi(\eta_{bi}^{-1}) \cdot \Phi(\eta_{bj}) \cdot p_j = \Phi(\eta_{bi}^{-1}) \cdot p_b = p_i$$

and we have shown that all edges are collapsed.

Case 2: $T(G) = 0$. We adopt the notation from Case 1. As before, we fix a spanning tree $T$ and a representation $\Phi$ that is trivial on the translation subgroup $\Lambda(G, b)$. By Lemma 4.1, $\rho(\pi_1(G, b))$ is generated by a translation subgroup $\Gamma' < \Lambda(G, b)$ and a rotation $r \in \Gamma_k$. We set $p_b$ to be on the rotation center of $\Phi(r)$ and define the rest of the $p_i$ as before: $p_i = \Phi(\eta_{bi}^{-1}) \cdot p_b$. Observe that $\Phi(r)$ then fixes $p_b$. 

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For edges $ij$ in the tree $T$, the argument that $ij$ is collapsed from Case 1 applies verbatim. For non-tree edges $ij$, a similar argument relating the fundamental closed path $P_{ij}$ to $Q_{bi}$ and $Q_{bj}$ yields the relation

$$
\gamma_{ij} = \eta_{bi}^{-1} \cdot \rho(P_{ij}) \cdot \eta_{bj}
$$

Since $\Phi$ is trivial on translations $t \in \Gamma'$, we see that

$$
\Phi(\gamma_{ij}) = \Phi(\eta_{bi}^{-1}) \cdot \Phi(r) \cdot \Phi(\eta_{bj})
$$

We then compute

$$
\Phi(\gamma_{ij})p_j = \Phi(\eta_{bi}^{-1}) \cdot \Phi(r) \cdot \Phi(\eta_{bj}) \cdot p_j = \Phi(\eta_{bi}^{-1}) \cdot \Phi(r) \cdot p_b
$$

Because $\Phi(r) \cdot p_b = p_b$, the r.h.s. simplifies to $p_i$, and so the edge $ij$ is collapsed.

**Multiple connected components** The proof of the lemma is completed by considering connected components one at a time to remove the assumption that $G$ is connected.  
\[\Box\]
IV. Rigidity

25. Crystallographic and colored frameworks

We now return to the setting of crystallographic frameworks, leading to the proof of Theorem [1] in Section [27]. The overall structure is very similar to [10, Sections 16–18], but we give sufficient detail for completeness. Here is the roadmap to the rest of the paper:

- In this section we give the continuous rigidity theory for crystallographic frameworks and the related colored crystallographic frameworks.
- Section [26] introduces infinitesimal rigidity and defines genericity for crystallographic frameworks.
- The proof of Theorem [1] is then in Section [27].
- We conclude with a discussion of cone frameworks and the proof of Theorem [2] in Section [28].

25.1. Crystallographic frameworks

We recall the following definition from the introduction: a crystallographic framework $(\tilde{G}, \varphi, \tilde{\ell})$ is given by:

- An infinite graph $\tilde{G}$
- A free action $\varphi$ on $\tilde{G}$ by a crystallographic group $\Gamma$ with finite quotient
- An assignment of a length $\ell_{ij}$ to each edge $ij \in \tilde{E}$

In what follows, $\Gamma$ will always be one of the groups $\Gamma_2$, $\Gamma_3$, $\Gamma_4$, or $\Gamma_6$.

25.2. The realization space

A realization $\tilde{G}(p, \Phi)$ of a crystallographic framework $(\tilde{G}, \varphi, \tilde{\ell})$ is given by an assignment $p = (p_i)_{i \in \tilde{V}}$ of points to the vertices of $\tilde{G}$ and a representation $\Phi$ of $\Gamma \hookrightarrow \text{Euc}(2)$ by Euclidean isometries acting discretely and co-compactly, such that

\[ ||p_i - p_j|| = \tilde{\ell}_{ij} \quad \text{for all edges } ij \in \tilde{E} \quad (16) \]
\[ \Phi(\gamma) \cdot p_i = p_{\gamma(i)} \quad \text{for all group elements } \gamma \in \Gamma \text{ and vertices } i \in \tilde{V} \quad (17) \]
We see that (17) implies that, to be realizable at all, the framework \((\tilde{G}, \varphi, \tilde{\ell})\) must assign the same length to each edge in every \(\Gamma\)-orbit of the action \(\varphi\). The condition \((16)\) is the standard one from rigidity theory that says the distances between endpoints of each edge realize the specified lengths.

We define the realization space \(R(\tilde{G}, \varphi, \tilde{\ell})\) (shortly \(R\)) of a crystallographic framework to be the set of all realizations
\[
R(\tilde{G}, \varphi, \tilde{\ell}) = \{ (p, \Phi) : \tilde{G}(p, \Phi) \text{ is a realization of } (\tilde{G}, \varphi, \tilde{\ell}) \}
\]

25.3. The configuration space The group \(\text{Euc}(2)\) of Euclidean isometries acts naturally on the realization space. Let \(\psi \in \text{Euc}(2)\) be an isometry. For any point \((p, \Phi) \in R\),
\[
(\psi \circ p, \Phi^\psi)
\]
is a point in \(R\) as well where \(\Phi^\psi\) is the representation defined by
\[
\Phi^\psi(\gamma) = \psi \Phi(\gamma) \psi^{-1}.
\]

We define the configuration space \(C(\tilde{G}, \varphi, \tilde{\ell})\) (shortly \(C\)) of a crystallographic framework to be the quotient \(R/\text{Euc}(2)\) of the realization space by Euclidean isometries.

Since the spaces \(R\) and \(C\) are subsets of an infinite-dimensional space, there are some technical details to check that we omit in the interest of brevity. Interested readers can find a development for the periodic setting in \([11\text{, Appendix A}]\). The present crystallographic case proceeds along the same lines.

25.4. Rigidity and flexibility A realization \(\tilde{G}(p, \Phi)\) is defined to be (continuously) rigid if it is isolated in the configuration space \(C\). Otherwise it is flexible. As the definition makes clear, rigidity is a local property that depends on a realization.

A framework that is rigid, but ceases to be so if any orbit of bars is removed is defined to be minimally rigid.

25.5. Colored crystallographic frameworks In principle, the realization and configuration spaces \(R(\tilde{G}, \varphi, \tilde{\ell})\) and \(C(\tilde{G}, \varphi, \tilde{\ell})\) of crystallographic frameworks could be complicated infinite dimensional objects. In this section, we will show that they are, in fact, equivalent to the finite-dimensional configuration spaces of colored crystallographic frameworks, which will be technically simpler to work with.

A colored crystallographic framework (shortly a colored framework) is a triple \((G, \gamma, \ell)\), where \((G, \gamma)\) is a \(\Gamma_k\)-colored graph and \(\ell = (\ell_{ij})_{ij \in E(G)}\) is an assignment of a length to each edge.

The relationship between crystallographic and colored frameworks is similar to that between their direction network counterparts: using the arguments for Lemmas \([17.1]\) and \([17.2]\) we see that each colored framework has a well-defined lift to a crystallographic framework and each crystallographic framework has, as its quotient, a colored framework.

\[\text{The reference } [11] \text{ is an earlier version of } [10].\]
25.6. The colored realization and configuration spaces  A realization $G(p, \Phi)$ of a colored framework is an assignment of points $p = (p_i)_{i \in V(G)}$ and a representation $\Phi$ of $\Gamma_k$ by Euclidean isometries acting discretely and cocompactly such that

$$||\Phi(\gamma_{ij}) \cdot p_j - p_i||^2 = \ell_{ij}^2$$

for all edges $ij \in E(G)$. The realization space $\mathcal{R}(G, \gamma, \ell)$ is then defined to be

$$\mathcal{R}(G, \gamma, \ell) = \{(p, \Phi) : G(p, \Phi) \text{ is a realization of } (G, \gamma, \ell)\}$$

The Euclidean group $\text{Euc}(2)$ acts naturally on $\mathcal{R}(G, \gamma, \ell)$ by

$$\psi \cdot (p, \Phi) = (\psi \cdot p, \Phi \psi)$$

where $\psi$ is a Euclidean isometry. Thus we define the configuration space $\mathcal{C}(G, \gamma, \ell)$ to be the quotient $\mathcal{R}(G, \gamma, \ell)/\text{Euc}(2)$ of the realization space by the Euclidean group.

25.7. The modified configuration space  Because it is technically simpler, we will consider the modified realization space $\mathcal{R}'(G, \gamma, \ell)$, which we define to be:

$$\mathcal{R}'(G, \gamma, \ell) = \{(p, \Phi) : G(p, \Phi) \text{ is a realization of } (G, \gamma, \ell) \text{ with } \Phi(r_k) \text{ fixing the origin}\}$$

Recall that $r_k$ is the rotation of order $k$ that is one of the generators of $\Gamma_k$. The modified configuration space $\mathcal{C}'(G, \gamma, \ell)$ is then defined to be the quotient $\mathcal{R}'(G, \gamma, \ell)/O(2)$ of the modified realization space by the orthogonal group $O(2)$. Since every representation $\Phi \in \text{Rep}(\Gamma_k)$ is conjugate to a representation $\Phi'$ that has the origin as a rotation center by a Euclidean translation, this next lemma follows immediately.

**Lemma 25.1.** Let $(G, \gamma, \ell)$ be a colored framework. Then the configuration space $\mathcal{C}(G, \gamma, \ell)$ is homeomorphic to the modified configuration space $\mathcal{C}'(G, \gamma, \ell)$.

From the definition and Lemma 3.1 we see that the modified configuration space is an algebraic subset of $\mathbb{R}^{2n} \times \mathbb{R}^4$, for $\Gamma_2$ and of $\mathbb{R}^{2n} \times \mathbb{R}^2$ for $\Gamma_k$ with $k = 3, 4, 6$.

25.8. Colored rigidity and flexibility  We now can define rigidity and flexibility in terms of colored frameworks. A realization $G(p, \Phi)$ of a colored framework is rigid if it is isolated in the configuration space and otherwise flexible. Lemma 25.1 implies that a realization is rigid if and only if it is isolated in the modified configuration space.

25.9. Equivalence of crystallographic and colored rigidity  The connection between the rigidity of crystallographic and colored frameworks is captured in the following proposition, which says that we can switch between the two models.

**Proposition 25.2.** Let $(\tilde{G}, \varphi, \tilde{\ell})$ be a crystallographic framework and let $(G, \gamma, \ell)$ be an associated colored framework quotient. Then the configuration spaces $\mathcal{C}(\tilde{G}, \varphi, \tilde{\ell})$ and $\mathcal{C}'(G, \gamma, \ell)$ are homeomorphic.

**Proof.** This follows from the definitions and a straightforward computation. □
26. Infinitesimal and generic rigidity

As discussed above, the modified realization space $\mathcal{R}'(G, \gamma, \ell)$ of a colored framework is an algebraic subset of $\mathbb{R}^{2n+2r}$, where $r$ is the rank of the translation subgroup $\Lambda(\Gamma_k)$. The coordinates are given as follows:

- The first $2n$ coordinates are the coordinates of the points $p_1, p_2, \ldots, p_n$
- The final $2r$ coordinates are the vectors $v_i$ specifying the representation of the translation subgroup $\Lambda(\Gamma_k)$. (Since we have “pinned” a rotation center to the origin, the vector $w$ from Lemma 3.1 is also fixed.)

26.1. Infinitesimal rigidity

As is typical in the derivation of Laman-type theorems, we relax the condition of rigidity, we linearize the problem by considering the tangent space of $\mathcal{R}'(G, \gamma, \ell)$ near a realization $G(p, \Phi)$.

The vectors in the tangent space are infinitesimal motions of the framework, and they can be characterized as follows. Let $(q, u_1, u_2) \in \mathbb{R}^{2n+4}$ for $k = 2$ or $(q, u_1) \in \mathbb{R}^{2n+2}$ for $k = 3, 4, 6$. To this vector there is an associated representation $\Phi'$ defined by $\Phi'(r_k) = (0, R_k)$ and $\Phi'(t_i) = (u_i, \text{Id})$. Then differentiation of the length equations yield this linear system ranging over all edges $ij \in E(G)$:

$$\left\langle \Phi(\gamma_{ij}) \cdot p_j - p_i, \Phi'(\gamma_{ij}) \cdot q_j - q_i \right\rangle$$

The given data are the $p_i$ and $\Phi$, and then unknowns are the $q_i$ and $\Phi'$. A realization $G(p, \Phi)$ of a colored framework is defined to be infinitesimally rigid if the system (18) has a 1-dimensional solution space. A realization that is infinitesimally rigid but ceases to be so when any colored edge is removed is minimally infinitesimally rigid.

26.2. Infinitesimal rigidity implies rigidity

A standard kind of result relating infinitesimal rigidity and rigidity for generic frameworks holds in our setting. Since our realization space is finite, adapting standard arguments (see e.g. [1]) to our situation is not hard, so we omit a proof.

Lemma 26.1. If a realization $G(p, \Phi)$ of a colored framework is infinitesimally rigid, then it is rigid.

26.3. Generic rigidity

The converse of Lemma 26.1 does not hold in general, but it does for nearly all realizations. Let $(G, \gamma, \ell)$ be a colored framework. A realization $G(p, \Phi)$ is defined to be regular for $(G, \gamma, \ell)$ if the rank of the system (18) is maximal over all realizations.

Whether a realization is regular depends on both the colored graph $(G, \gamma)$ and the given lengths $\ell$. Let $G(p, \Phi)$ be a regular realization of a colored framework. If, in addition, the rank of (18) at $G(p, \Phi)$ is maximal over all realizations of colored frameworks with the same colored graph $(G, \gamma)$, we define $G(p, \Phi)$ to be generic.
We define the rank of (18) at a generic realization to be its \textit{generic rank}. Since it depends on formal minors of the matrix underlying (18) only, it is a property of the colored graph \((G, \gamma)\).

If \((G, \gamma, \ell)\) is a framework with generic realizations, it is immediate that the set of non-generic realizations is a proper algebraic subset of the realization space. Alternatively, if we consider frameworks as being induced by realizations, the set of non-generic realizations is a proper algebraic subset of \(\mathbb{R}^{2n+2r}\), where \(r = 1\) for \(\Gamma_3, \Gamma_4,\) and \(\Gamma_6,\) and \(r = 2\) for \(\Gamma_2\).

For generic realizations, a standard argument (again, along the lines of [11]) shows that rigidity and infinitesimal rigidity coincide.

\textbf{Proposition 26.2.} \textit{A generic realization of a colored framework \((G, \gamma, \ell)\) is rigid if and only if it is infinitesimally rigid.}

\section*{27. Proof of Theorem 1}

All the tools are in place to prove our main theorem:

\textbf{Theorem 1.} \textit{Let \(\Gamma\) be a crystallographic group generated by translations and rotations. A generic \(\Gamma\)-crystallographic framework \((\tilde{G}, \varphi, \tilde{\ell})\) is minimally rigid if and only if its colored quotient graph is \(\Gamma\)-colored-Laman.}

The proof occupies the rest of this section.

\subsection*{27.1. Reduction to colored frameworks}

By Proposition 25.2, it is sufficient to prove the statement of Theorem 1 for colored frameworks. Proposition 26.2 then implies that the Theorem will follow from a characterization of generic infinitesimal rigidity for colored frameworks.

Thus, to prove the theorem, we show that, for a colored graph \((G, \gamma)\) with \(n\) vertices and \(m = 2n + \text{rep}_{\Gamma_k} (\Lambda(\Gamma_k)) - 1\) edges, the generic rank of the system (18) is \(m\) if and only if \((G, \gamma)\) is a \(\Gamma\)-colored-Laman graph.

\subsection*{27.2. Necessity: the “Maxwell direction”}

We recall the definition of the sparsity function \(h(G)\) from Section 13, which defines \(\Gamma\)-colored-Laman graphs. We have, for a colored graph \((G, \gamma)\) with \(n\) vertices and \(c\) connected components \(G_1, G_2, \ldots, G_c,\)

\[ h(G) = 2n + \text{rep}_{\Gamma_k} (\Lambda(\Gamma_k)) - 1 - \sum_{i=1}^{c} T(G_i) \]

That colored-Laman-sparsity is necessary for the system (18) to have independent equations is captured in the following proposition.

\textbf{Proposition 27.1.} \textit{Let \((G, \gamma)\) be a colored graph. Then the generic rank of the system (18) is at most \(h(G)\).}
Proof. Let \( G(p, \Phi) \) be any realization of a colored framework on a colored graph \((G, \gamma)\) with no collapsed edges. That is select a representation \( \Phi \) of \( \Gamma_k \) and points \( p_i \), such that, \( \Phi(\gamma_{ij}) \cdot p_j \neq p_i \) for all edges \( ij \in E(G) \).

We now define the direction \( d_{ij} \) to be \( (\Phi(\gamma_{ij}) \cdot p_j - p_i) \perp \) for each edge \( ij \in E(G) \). These directions define a colored direction network \((G, \gamma, d)\) with the property that any solution to this direction network corresponds to an infinitesimal motion of the colored framework realization \( G(p, \Phi) \).

Lemma 24.1 implies that there are dimensions of realizations with every edge collapsed. By construction, there is a non-collapsed realization of this direction network as well: it is simply \((p, \Phi)\) rotated by \( \pi/2 \). Since this is not obtained by taking linear combinations of realizations where every edge is collapsed, the dimension of the space of infinitesimal motions is always at least

\[
\text{rep}_{\Gamma_k}(\Lambda(\Gamma_k)) - \text{rep}_{\Gamma_k}(G) + \sum_{i=1}^{c} T(G_i) + 1
\]

The proposition follows by subtracting from \( 2n + \text{rep}_{\Gamma_k}(\Lambda(\Gamma_k)) \) and comparing to \( h(G) \).

27.3. Sufficiency: the “Laman direction”  The other direction of the proof of Theorem 1 is this next proposition

Proposition 27.2. Let \((G, \gamma)\) be a \( \Gamma \)-colored-Laman graph. Then the generic rank of the system 18 is \( h(G) \).

Proof. It is sufficient to construct a single example at which this rank is attained, since the generic rank is always at least the rank for any specific realization. We will do this using direction networks.

Let \((G, \gamma)\) be a \( \Gamma \)-colored-Laman graph, and select a direction \( d_{ij} \) for each edge \( ij \in E(G) \), such that both \( d \) and \( d^\perp = (d_{ij}^\perp) \) are generic in the sense of Theorem 3. By Theorem 3 the colored direction network \((G, \gamma, d)\) has a unique, faithful solution \((p, \Phi)\), which implies that, for all edges \( ij \in E(G) \)

\[
\Phi(\gamma_{ij}) \cdot p_j - p_i = \alpha_{ij} d_{ij}
\]

for some non-zero scalar \( \alpha_{ij} \in \mathbb{R} \). It follows that, by replacing \( d_{ij} \) with \( \Phi(\gamma_{ij}) \cdot p_j - p_i \) in the direction realization system (12) we obtain (18). Since \( d^\perp \) is also generic for Theorem 3, we conclude that (18) has full rank as desired. \( \square \)
28. Cone frameworks

For the group $\mathbb{Z}/k\mathbb{Z}$, the counterpart of Theorem 1 is

**Theorem 2.** A generic cone framework is minimally rigid if and only if the associated colored graph $(G, \gamma)$ is cone-Laman.

The theory for cone frameworks follows the same lines as that for $\Gamma_k$-crystallographic frameworks. Since all the steps from Sections 25-27 go through with appropriate modifications (which are simplifications) we omit the details in the interest of space.
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