Code for prompt numerical computation of the leading order GPD evolution

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Abstract

This paper describes the design and work of a set of computer routines capable for numerical computation of generalized parton distributions (GPDs) evolution at the leading order. The main intention of this work is to present a fast-working computer code making possible fitting of GPDs parameters to the data on hard electron-nucleon scattering.

1 Introduction

Generalized parton distributions [1, 2, 3, 4, 5] parameterize matrix elements of quark and gluon twist-2 operators between nucleon states with non-equal momenta. They can be represented as functions of fractions $x$ and $\xi$ of longitudinal momentum of the proton carried by the parton and of $t = (p_1 - p_2)^2$, the square of the 4-momentum transfer between initial and final protons$^1$ (see, Fig. 1).

![Partonic picture for GPDs](image)

Figure 1: Partonic picture for GPDs. The value of $\xi$ is given by $\xi = x_B/(2 - x_B)$, where $x_B$ is Bjorken variable.

GPDs are the basis for description of hard elastic lepton-nucleon scattering in terms of QCD, just like the ordinary PDFs are the basis of QCD description of the DIS. Besides the description of hard elastic reactions, GPDs give access to the orbital momentum of partons in the nucleon [6] and 3-dimensional nucleon structure [7, 8, 9, 10]. The importance of GPDs for understanding of the nucleon structure raised a problem of their extraction from hard elastic

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$^1$The variable $t$ does not enter the evolution equations and is therefore omitted in this paper.
scattering data, first of all from deeply virtual Compton scattering (DVCS). The activity of measuring DVCS observables at lepton-nucleon facilities is quite intense [11, 12, 13, 14, 15].

Similar to PDFs, GPDs are subject of perturbative QCD evolution which should be taken care of in extraction of GPDs from the data on lepton-nucleon scattering. The LO kernels for GPDs evolution are well-known [1, 2, 3, 4, 5, 16, 17]. The NLO results have been published [18]. At present, there exists a code [19, 20] which can be used for leading and next-to-leading order evolution of GPDs. However, on a 1.7 GHz Pentium-4 machine this code takes about 15 seconds for LO evolution and about 40 seconds for the NLO which is not fast enough for performing fitting procedure requiring very large number of evolution routine calls. Evolution equations for GPDs at the LO can be also solved by a decomposition of GPDs in series of special functions [21, 22]. The advantage of this method is that it does not require introduction of a difference scheme for $Q^2$. However, the decomposition is not trivial for numerical implementation and the number of terms in the series required to achieve a reliable precision can be large (depending on the input GPDs).

In the present paper another code is introduced\footnote{The code is available at http://www.ifh.de/~vinnikov/gpd_evol.tar.gz}, which does the LO GPD evolution with a reasonable precision for $\sim 10^{-3}$ seconds, making possible its usage for fitting. The fast work was achieved by optimization of the computational procedure. It uses 4-th order Runge-Kutta method for solving the evolution equations. Since in the broad DGLAP region ($|x| > \xi$) the variation of GPDs is not sizable, and in the narrow ERBL region ($|x| < \xi$) is quite significant, the convolution integrals are performed on a logarithmic grid so that the number of points in ERBL and DGLAP regions is the same. The code itself is also optimized. For example, the operation of division is avoided as much as possible. This was done since division on the machines commonly used (such as Pentium-IV) is performed 6 times slower than multiplication. So, if an expression to be computed reads $a/b/c$, its optimized version is $a/(b \ast c)$. Since the NLO expressions are considerably larger then the LO ones, their implementation is not done at the moment.

## 2 Evolution kernels

The LO kernels for GPDs evolution are well-known. In the code those published in [5, 16, 17] are used\footnote{Literally, the expressions from Ref. [17] were taken. Due to the symmetry relations of singlet GPDs, these kernels are equivalent to those published in Ref. [5], (see Eq. 5.21 of Ref. [5]).}. The numerical computation of the evolution requires that singularities in the kernel to be cancelled analytically. The kernels with regularized singularities in the form they take in the code are given in the Appendix.

The kernels assume the definition of the GPDs given in [23]. In the forward limit they obey

$$
\begin{align*}
H_{NS}(x, \xi = 0, t = 0) &= u(x) - d(x), \\
H_Q(x, \xi = 0, t = 0) &= u(x) + d(x) + s(x), \\
H_G(x, \xi = 0, t = 0) &= x g(x), \\
\tilde{H}_Q(x, \xi = 0, t = 0) &= \Delta u(x) + \Delta d(x) + \Delta s(x), \\
\tilde{H}_G(x, \xi = 0, t = 0) &= x \Delta g(x),
\end{align*}
$$

with $q(x) = q_V(x) + \bar{q}(x)$, $\Delta q(x) = \Delta q_V(x) + \Delta \bar{q}(x)$, $\bar{q}(-x) = -\bar{q}(x)$, $g(-x) = -g(x)$, $\Delta \bar{q}(-x) = \Delta \bar{q}(x)$, $\Delta g(-x) = \Delta g(x)$. Note that quark singlet and non-singlet densities are not divided by $n_f$. This is important in the singlet sector, where non-diagonal parts of the kernel depend on the
normalization of the quark singlet distribution. For non-singlet channel, isovector combination was chosen to represent the work of the code. It is clear that any other non-singlet combination of GPDs can be taken as well.

In the singlet sector, the symmetry properties $H_Q(-x, \xi, t) = -H_Q(x, \xi, t)$, $H_G(-x, \xi, t) = H_G(x, \xi, t)$, $\tilde{H}_Q(-x, \xi, t) = \tilde{H}_Q(x, \xi, t)$ and $\tilde{H}_G(-x, \xi, t) = -\tilde{H}_G(x, \xi, t)$ are used. This allowed to reduce the computational time by a factor of 4. Thus, the singlet kernels are given only for $x \geq 0$.

For all parts of the kernel, convolution at $x = \pm \xi$ was done separately. The kernels at $x = \pm \xi$ were taken as $x \to \xi$ limits of the corresponding DGLAP or (equivalently for $x \to \pm \xi$) ERBL expressions.

The running strong coupling constant is taken as a solution of the NLO RG equation

$$\frac{d \alpha_s(\mu)}{d \ln(\mu^2)} = -b_0 \alpha_s^2(\mu) - b_1 \alpha_s^3(\mu) - b_2 \alpha_s^4(\mu)$$

(2)

where the coefficients $b_0$ and $b_1$ are given by

$$b_0 = \frac{1}{12\pi} (33 - 2N_f),$$

$$b_1 = \frac{1}{24\pi^2} (153 - 19N_f),$$

$$b_2 = \frac{1}{3456\pi^3} \left(77139 - 15099N_f + 325N_f^2\right).$$

(3)

Here $N_F$ is the number of active quark flavors. The initial condition for Eq. 2 is taken as the value of $\alpha_s(M_Z)$. Then the equation is solved back to smaller values of $\mu$. Down to $\mu = m(b) = 4.3$ GeV the number of active flavors is taken to be equal 5. Between $\mu = m(c) = 1.3$ GeV and $\mu = m(b)$, $N_f = 4$ is used. Below $\mu = m(c)$ $N_f$ is 3. Note that the number of active flavors $N_f$ entering into the $\alpha_s(\mu)$ RG equation is different from the number of flavors in the proton $n_f$ which enters into the quark-gluon evolution kernel. In the code, $n_f$ is always equal to 3.

3 Computation of the convolution integrals

The integrals are computed on a logarithmic grid. The required parameterization for non-singlet case reads:

$$x_i = -\delta e^{-(i-2n)\gamma} + \delta \quad (-1 \leq x \leq 0, 0 \leq i \leq 2n),$$

$$x_i = \delta e^{-(i-2n)\gamma} - \delta \quad (0 < x \leq 1, 2n < i \leq 4n),$$

(4)

(5)

where $\gamma$ and $\delta$ are given by

$$\gamma = \frac{1}{n} \ln\left(\frac{1}{2} + \frac{1}{2\zeta} \sqrt{9\xi^2 - 12\xi + 4}\right), \quad \delta = \frac{\xi^2}{1 - 2\zeta}.$$  

(6)

This grid has the following useful properties:

$$x_0 = -1, \ x_n = -\xi, \ x_{2n} = 0, \ x_{3n} = \xi, \ x_{4n} = 1$$

(7)

i.e. it provides the same level of detailization for ERBL and DGLAP regions.

For the singlet case the grid size is reduced so that to cover only $0 \leq x \leq 1$ interval:

$$x_i = \delta e^{i\gamma} - \delta \quad (0 \leq i \leq 2n),$$

(8)
so that

\[ x_0 = 0, \quad x_n = \xi, \quad x_{2n} = 1. \]  

(9)

The parameters \( \gamma \) and \( \delta \) are the same as in the non-singlet case (Eq. 6).

After introduction of the logarithmic grid the convolution integrals can be computed using the standard equidistant Simpson’s formula taking into account the Jacobian for the grid:

\[
\int_{-1}^{1} f(x) dx = \int_{0}^{2n} \frac{1}{\gamma} (\delta - x_i) f(x_i) di + \int_{2n}^{4n} \frac{1}{\gamma} (\delta + x_i) f(x_i) di.
\]

(10)

The uncertainty in the integrals can be resolved with the L'Hopital’s rule. Since for the numerical integration the Simpson’s \( O(\Delta x^4) \) method is used, the numerical computation of the derivative requires \( O(\Delta x^3) \) precision. Depending on the position of the singularity (availability of other grid points on the left and on the right), one of the following formulas is be used:

\[
f'(x_i) = \frac{1}{6\gamma(\delta + x)} (-11f(x_i) + 18f(x_{i+1}) - 9f(x_{i+2}) + 2f(x_{i+3})),
\]

(11)

\[
f'(x_i) = \frac{1}{6\gamma(\delta + x)} (-2f(x_{i-1}) - 3f(x_i) + 6f(x_{i+1}) - f(x_{i+2})),
\]

(12)

\[
f'(x_i) = \frac{1}{6\gamma(\delta + x)} (f(x_{i-2}) - 6f(x_{i-1}) + 3f(x_i) + 2f(x_{i+1})),
\]

(13)

\[
f'(x_i) = \frac{1}{6\gamma(\delta + x)} (-2f(x_{i-3}) + 9f(x_{i-2}) - 18f(x_{i-1}) + 11f(x_i)).
\]

(14)

For \( x < 0 \) the denominator should be replaced by \( 6\gamma(\delta - x) \).

To verify the convolution code, a computation was done for a simple functional form GPDs. For the non-singlet, spin-independent quark-gluon and gluon-gluon parts and spin-dependent quark-quark and gluon-quark parts

\[ H_{\text{test}}(x, \xi, Q^2) = 1 - x^2 \]

(15)

was taken, and for singlet spin-independent quark-quark and gluon-quark parts and spin-dependent quark-gluon and gluon-gluon parts

\[ H_{\text{test}}(x, \xi, Q^2) = x - x^3. \]

(16)

It was found that taking \( n = 50 \) provides 4-6 digits precision for all the integrals. It should be noted that the test functions are not convenient for the Simpson method calculation since in the logarithmic grid they fall steeply when \( x \to 1 \). However, they made possible analytical computation of the integrals and therefore they were used in the test. The real GPDs which fall rapidly as \( x \to \pm 1 \) should be computed with use of Simpson’s method more precisely on such a grid.

\section{The evolution procedure}

Basing on the integrals obtained in the previous section, the code for GPD evolution was written. It is based on the 4-th order Runge-Kutta method. Indeed, after introduction of the grid \( 5 \) the convolution integrals are transformed into sums. Therefore, the integro-differential
evolution equation is transformed into a set of $4n + 1$ ordinary differential equations of the form
\[
\frac{dH_i}{d\ln Q^2} = F_i(\ln Q^2, H_0, H_1, \ldots H_{4n}),
\] (17)
where $H_i = H(x_i, \xi, t, \ln Q)$. The 4-th order Runge-Kutta scheme on the $\ln Q^2$-grid with $m$ nodes
\[
\ln Q_j^2 = \ln Q_0^2 + jd \ln Q^2, \quad 0 \leq j \leq m - 1
\] (18)
therefore reads:
\[
H_i^j = H_i^{j-1} + \frac{d\ln Q^2}{6} \left( k_i^{(1)} + 2k_i^{(2)} + 2k_i^{(3)} + k_i^{(4)} \right),
\] (19)
where
\[
k_i^{(1)} = F_i(\ln Q_j^2, H_0, H_1, \ldots H_{4n}),
\] (20)
\[
k_i^{(2)} = F_i(\ln Q_j^2 + \frac{d\ln Q^2}{2}, H_0 + \frac{d\ln Q^2}{2}k_0^{(1)}, H_1 + \frac{d\ln Q^2}{2}k_1^{(1)}, \ldots H_{4n} + \frac{d\ln Q^2}{2}k_{4n}^{(1)}),
\] (21)
\[
k_i^{(3)} = F_i(\ln Q_j^2 + \frac{d\ln Q^2}{2}, H_0 + \frac{d\ln Q^2}{2}k_0^{(2)}, H_1 + \frac{d\ln Q^2}{2}k_1^{(2)}, \ldots H_{4n} + \frac{d\ln Q^2}{2}k_{4n}^{(2)}),
\] (22)
\[
k_i^{(4)} = F_i(\ln Q_j^2 + d\ln Q^2, H_0 + d\ln Q^2 k_0^{(3)}, H_1 + d\ln Q^2 k_1^{(3)}, \ldots H_{4n} + d\ln Q^2 k_{4n}^{(3)}).
\] (23)

The results of the computation are presented in Figs. 2-7 for typical kinematics of HERMES and HERA. The code works both in forward (from smaller to larger $Q^2$) and backward (from larger to smaller $Q^2$) directions. As it can be seen, choice of $n = 20$ and $m = 1$ already gives satisfactory results. Distinctions from the careful solution with $n = 500$ and $m = 500$ can be noticed around $Q^2 = 20$, $\xi = 0.5\%$ accuracy at HERA kinematics. For Pentium-IV 1.7 GHz $\alpha \approx 1.4 \cdot 10^{-6}$ sec in the non-singlet case and $\approx 2 \cdot 10^{-6}$ in the singlet case. This means that computation of evolution with $n=20, m=1$ takes about $10^{-3}$ sec.

\[4\text{In singlet case, the function index runs from 0 to } 2n \text{ instead of 0 to } 4n \text{ in the non-singlet sector.}
\]
\[5\text{R.h.s. of these equations depend explicitly (i.e. not via GPDs) on } Q^2 \text{ through } \alpha_s(Q^2).\]
Figure 2: Evolution of the quark non-singlet GPD $H_{NS}$ for $\xi = 0.1$ from $Q^2 = 4 \text{ GeV}^2$ to $1 \text{ GeV}^2$ (left panel) and for $\xi = 10^{-4}$ from $Q^2 = 4 \text{ GeV}^2$ to $40 \text{ GeV}^2$ (right panel). The initial GPD is obtained using the double-distribution ansatz [24, 25] with $b = 1$ and CTEQ6M [26] PDF at the scale of $4 \text{ GeV}^2$. For the evolved distribution, $n$ and $m$ represent parameters of the numerical evolution procedure described in Eqs. (5) and (18) correspondingly.

Figure 3: Evolution of the quark singlet GPD $H_Q$ for $\xi = 0.1$ from $Q^2 = 4 \text{ GeV}^2$ to $1 \text{ GeV}^2$ (left panel) and for $\xi = 10^{-4}$ from $Q^2 = 4 \text{ GeV}^2$ to $40 \text{ GeV}^2$ (right panel). See caption of Fig. 2.

Figure 4: Evolution of the gluon GPD $H_G$ for $\xi = 0.1$ from $Q^2 = 4 \text{ GeV}^2$ to $1 \text{ GeV}^2$ (left panel) and for $\xi = 10^{-4}$ from $Q^2 = 4 \text{ GeV}^2$ to $40 \text{ GeV}^2$ (right panel). See caption of Fig. 2 (the input profile parameter $b = 2$ was taken).
Figure 5: Evolution of the spin-dependent non-singlet GPD $\tilde{H}_{NS}$ for $\xi = 0.1$ from $Q^2 = 4 \text{ GeV}^2$ to 1 GeV$^2$ (left panel) and for $\xi = 10^{-4}$ from $Q^2 = 4 \text{ GeV}^2$ to 40 GeV$^2$ (right panel). The initial GPD is obtained using the double-distribution ansatz [24, 25] with $b = 1$ and Blümlein-Böttcher [27] PDF at the scale of 4 GeV$^2$. For the evolved distribution, $n$ and $m$ represent parameters of the numerical evolution procedure described in Eqs. 5 and 18 correspondingly.

Figure 6: Evolution of the spin-dependent singlet quark GPD $\tilde{H}_Q$ for $\xi = 0.1$ from $Q^2 = 4 \text{ GeV}^2$ to 1 GeV$^2$ (left panel) and for $\xi = 10^{-4}$ from $Q^2 = 4 \text{ GeV}^2$ to 40 GeV$^2$ (right panel). See caption of Fig. 5.

Figure 7: Evolution of the spin-dependent gluon GPD $\tilde{H}_G$ for $\xi = 0.1$ from $Q^2 = 4 \text{ GeV}^2$ to 1 GeV$^2$ (left panel) and for $\xi = 10^{-4}$ from $Q^2 = 4 \text{ GeV}^2$ to 40 GeV$^2$ (right panel). See caption of Fig. 5 (the input profile parameter $b = 2$ was taken).
5 Existing numerical instabilities

Even though the code is quite effective, instabilities stemming from the complicate structure of the kernels still arise in the output. So far, only one instability was found. It is related to the fact that due to the “+” regularization procedure the convolution integral at a given value of $x$ depends to a large extent on the value of derivative of the GPD at this point. Therefore, if an input GPD has sudden spikes (not related to a possible semi-singular behavior at $x = \xi$ which the code can handle), the code can amplify them. This type of instability is illustrated in Fig. 8. The problem occurs only in the gluon-gluon part of the kernel and only for evolution in the backward direction, i.e. from larger $Q^2$ to smaller one. It should be noted that the smaller is the step in $x$, the bigger is the value of the derivative the sudden spikes produce. Therefore, when for some reason the task is to perform evolution with very high precision, the stability of the input GPDs must be taken care of.

6 Summary

A code for prompt numerical computation of leading order GPDs evolution is presented. Its fundamentals are computation of the convolution integrals on a logarithmic grid and 4-th order Runge-Kutta method for integration of the differential equations. The code makes possible fast computation of the GPD evolution. On a 1.7 GHz Pentium-4 machine computational time required to perform evolution in the $Q^2$ range from 4 to 1 GeV$^2$ or from 4 to 40 GeV$^2$ is of order of $10^{-3}$ seconds. Such code makes possible fitting of GPDs parameters from data on hard elastic lepton-nucleon reactions.

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A Explicit expressions for the kernels

A.1 Non-singlet part

After cancellation of the singularities thanks to the “+” operation the kernels are ready for practical usage. At \( x < -\xi \) the non-singlet part of the evolution equation reads:

\[
\frac{dH_{NS}(x, \xi, Q^2)}{d\ln Q^2} = \frac{2\alpha_s(Q^2)}{3\pi} \left[ \frac{x}{\xi} \int_{\xi}^{\frac{3}{2}} dy \left( \frac{2\xi^2 - x^2 - y^2}{(y - x)(y^2 - \xi^2)} \right) \left( H_{NS}(y, \xi, Q^2) - H_{NS}(x, \xi, Q^2) \right) \right. \\
+ \left. H_{NS}(x, \xi, Q^2) \left( \frac{3}{2} + 2\ln(1 + x) + \frac{x - \xi}{2\xi} \ln((\xi - x)(1 - \xi)) - \frac{x + \xi}{2\xi} \ln((-\xi - x)(1 + \xi)) \right) \right],
\]

At \(-\xi < x < \xi\):

\[
\frac{dH_{NS}(x, \xi, Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{3\pi} \left[ \frac{x}{\xi} \int_{\xi}^{1} dy \left( \frac{x + \xi(y - x + 2\xi)}{\xi(y + \xi)(y - x)} \right) \left( H_{NS}(y, \xi, Q^2) - H_{NS}(x, \xi, Q^2) \right) \right.
\\
+ \left. \int_{-1}^{x} dy \left( \frac{x - \xi(y - x - 2\xi)}{\xi(y - \xi)(y - x)} \right) \left( H_{NS}(y, \xi, Q^2) - H_{NS}(x, \xi, Q^2) \right) \right. \\
+ \left. H_{NS}(x, \xi, Q^2) \left( 3 + 2\ln \frac{1 - x^2}{1 + \xi} + \frac{x - \xi}{\xi} \ln(x - \xi) - \frac{x + \xi}{\xi} \ln(x + \xi) \right) \right],
\]

\( x > \xi \):

\[
\frac{dH_{NS}(x, \xi, Q^2)}{d\ln Q^2} = \frac{2\alpha_s(Q^2)}{3\pi} \left[ \frac{x}{\xi} \int_{\xi}^{1} dy \frac{x^2 + y^2 - 2\xi^2}{(y - x)(y^2 - \xi^2)} \left( H_{NS}(y, \xi, Q^2) - H_{NS}(x, \xi, Q^2) \right) \right.
\\
+ \left. H_{NS}(x, \xi, Q^2) \left( \frac{3}{2} + 2\ln(1 - x) + \frac{x - \xi}{2\xi} \ln((x - \xi)(1 + \xi)) - \frac{x + \xi}{2\xi} \ln((x + \xi)(1 - \xi)) \right) \right].
\]

A.2 Singlet quark-quark part

In the DGLAP region singlet quark-quark evolution is governed by the same kernel as the non-singlet (Eq [29]). At \( 0 < x < \xi \)

\[
\frac{dQH_Q(x, \xi, Q^2)}{d\ln Q^2} = \frac{\alpha_s(Q^2)}{3\pi} \left[ \frac{x}{\xi} \int_{\xi}^{1} \frac{H_Q(y, \xi, Q^2)}{\xi + y} dy + 4x \int_{\xi}^{1} \frac{H_Q(y, \xi, Q^2) - H_Q(x, \xi, Q^2)}{y^2 - x^2} dy \\
+ 2\frac{\xi - x}{\xi} \int_{0}^{\frac{\pi}{2}} H_Q(y, \xi, Q^2) \frac{ydy}{y^2} - 4(\xi^2 - x^2) \int_{0}^{\xi} \frac{H_Q(y, \xi, Q^2) - H_Q(x, \xi, Q^2)}{(\xi^2 - y^2)(y^2 - x^2)} ydy \right. \\
+ \left. 2H_Q(x, \xi, Q^2) \left( \ln \frac{1 - x}{1 + x} + 2\ln \frac{x + 3}{2} \right) \right].
\]
A.3 Quark-gluon part

In this part and also in the gluon-gluon part of the kernel a pole at \( y = \xi \) must be regularized. For this, subtraction of the \( H_G(\xi, \xi, Q^2) \) is done. If \( H_G(x, \xi, Q^2) \) is subtracted instead, this lead to larger of inaccuracy of the Simpson’s method since the function

\[
\left( H_G(y, \xi, Q^2) - H_G(x, \xi, Q^2) \right) / (y - \xi)
\]

varies very rapidly around \( y = x \).

At \( 0 < x < \xi \):

\[
\frac{d_G H_G(x, \xi, Q^2)}{d \ln Q^2} = n_f \frac{\alpha_s(Q^2)}{2\pi \xi^2} \left[ \int_0^x \frac{x^4 + \xi^3 x - 2\xi^2 x^2 - (\xi^2 - \xi x)y^2}{(\xi^2 - y^2)^2} \left( H_G(y, \xi, Q^2) - H_G(\xi, \xi, Q^2) \right) dy \right. \\
\left. + \xi x \int_x^1 \frac{H_G(y, \xi, Q^2)}{(\xi + y)^2} dy + H_G(\xi, \xi, Q^2) \left( \frac{x^2}{\xi + x} + \frac{\xi^2 - x^2}{2\xi} \ln \frac{\xi + x}{\xi - x} \right) \right] 
\]

\[(31)\]

\( x > \xi \):

\[
\frac{d_G H_G(x, \xi, Q^2)}{d \ln Q^2} = n_f \frac{\alpha_s(Q^2)}{2\pi} \left[ \int_x^1 \frac{y^2 - 2xy + 2x^2 - \xi^2}{(\xi^2 - y^2)^2} \left( H_G(y, \xi, Q^2) - H_G(\xi, \xi, Q^2) \right) dy \right. \\
\left. - H_G(\xi, \xi, Q^2) \left( \frac{\xi^2 - x^2}{2\xi^3} \ln \frac{(1 + \xi)(x - \xi)}{(\xi + x)(1 - \xi)} + \frac{x - x^2}{\xi^2(\xi^2 - 1)} \right) \right] 
\]

\[(32)\]

A.4 Gluon-quark part

\( 0 < x < \xi \):

\[
\frac{d_Q H_G(x, \xi, Q^2)}{d \ln Q^2} = \frac{2\alpha_s(Q^2)}{3\pi} \left[ \frac{(\xi - x)^2}{\xi} \int_0^x \frac{H_Q(y, \xi, Q^2) - H_Q(\xi, \xi, Q^2)}{\xi^2 - y^2} dy \right. \\
\left. + \int_x^1 \frac{\xi^2 + 2\xi y - x^2}{\xi(\xi + y)} H_Q(y, \xi, Q^2) dy + H_Q(\xi, \xi, Q^2) \left( \frac{\xi - x^2}{2\xi} \ln \frac{\xi^2}{\xi^2 - x^2} \right) \right],
\]

\[(33)\]

\( x > \xi \):

\[
\frac{d_Q H_G(x, \xi, Q^2)}{d \ln Q^2} = \frac{2\alpha_s(Q^2)}{3\pi} \int_x^1 \frac{\xi^2 - x^2 - 2y^2 + 2xy}{\xi^2 - y^2} H_Q(y, \xi, Q^2) dy,
\]

\[(34)\]

A.5 Gluon-gluon part

For the pure gluonic part it was found that simple subtraction of the GPD value at \( y = x \) does not provide high enough precision. To overcome the difficulty, a partial fractioning was introduced so that integrand in each of the integrals had only one pole (at \( y = x \) or \( y = \xi \)). This brought formal complication into the evolution kernel. However, practical numerical calculations became more stable and precise.
0 < x < \xi:

\[
\frac{d_G H_G(x, \xi, Q^2)}{d \ln Q^2} = \frac{3\alpha_s(Q^2)}{4\pi} \left[ \int_x^1 \frac{6\xi^2 + 4\xi y - 2x^2}{\xi(y^2 - x^2)} H_G(y, \xi, Q^2) dy \\
+ \frac{1}{4} \int_x^1 \frac{\xi^2 y + 2\xi x^2 + x^2 y}{(\xi + y)^2(y - x)} \left( H_G(y, \xi, Q^2) - H_G(x, \xi, Q^2) \right) dy \\
- \frac{1}{\xi} \int_0^x \frac{6\xi^2 + 4\xi x - 2y^2}{(\xi - y)^2} \left( H_G(y, \xi, Q^2) - H_G(x, \xi, Q^2) \right) dy \\
+ \frac{1}{(\xi - x)^2} \int_0^x \frac{6\xi^2 + 4\xi x^2 + x^2 y}{(\xi - y)^2} \left( H_G(y, \xi, Q^2) - H_G(x, \xi, Q^2) \right) dy \\
- \frac{1}{(\xi + x)^2} \int_0^x \frac{\xi^2 x + 2\xi y^2 + xy^2}{\xi^2 - y^2} \left( H_G(y, \xi, Q^2) - H_G(x, \xi, Q^2) \right) dy \\
- \frac{1}{(\xi + x)^2} \left( H_G(\xi, \xi, Q^2) - H_G(x, \xi, Q^2) \right) \left( \frac{x}{\xi - x} - \frac{1}{2} \ln \frac{\xi + x}{\xi - x} \right) \\
- \frac{1}{(\xi + x)^2} \left( H_G(\xi, \xi, Q^2) - H_G(x, \xi, Q^2) \right) \left( \xi(\xi + x) \ln \frac{\xi + x}{\xi - x} - x(2\xi + x) \right) \\
+ \frac{1}{(\xi + x)^2} \left( 2(\xi - x) \frac{x}{\xi^2} + \frac{1}{\xi^2} (2\xi + x)(\xi - x) \ln \frac{\xi + x}{\xi - x} \right) \\
+ 2H_G(x, \xi, Q^2) \left( \ln \frac{1 - x}{1 + \xi} - \frac{2}{1 + \xi} + \frac{11}{6} - \frac{n_f}{9} \right) \right],
\]

x > \xi:

\[
\frac{d_G H_G(x, \xi, Q^2)}{d \ln Q^2} = \frac{3\alpha_s(Q^2)}{\pi} \left[ \int_x^1 \frac{y \xi + x \xi + xy - x^2}{(\xi - y)(\xi + y)^2} \left( H_G(y, \xi, Q^2) - H_G(x, \xi, Q^2) \right) dy \\
+ \int_x^1 \frac{H_G(y, \xi, Q^2) - H_G(x, \xi, Q^2)}{y - x} dy \\
- \frac{1}{x} \int_x^1 \frac{x^2 + y^2}{(\xi^2 - y^2)^2} \left( H_G(y, \xi, Q^2) - H_G(x, \xi, Q^2) \right) dy \\
+ \left( H_G(\xi, \xi, Q^2) - H_G(x, \xi, Q^2) \right) \frac{2\xi^2 x - x^3}{4\xi^3} \ln \frac{(\xi - x)(\xi + 1)}{(\xi + x)(\xi - 1)} \\
- \frac{x^2 + \xi^2}{2\xi^2} \left( H_G(\xi, \xi, Q^2) - H_G(x, \xi, Q^2) \right) \left( \frac{\xi(1 - x)}{(\xi + x)(\xi + 1)} + \frac{x - \xi}{\xi^2 - 1} + \frac{x}{\xi + x} \right) \\
+ H_G(x, \xi, Q^2) \left( \frac{x^2 + \xi^2(1 - x)}{2\xi^2(\xi^2 - 1)} + \frac{1}{2\xi^2} - \frac{x}{4\xi^3} \right) \ln \frac{1 + \xi}{x + \xi} \\
- \frac{1}{2} \ln \frac{1 - \xi^2}{(1 - x)^2} + \left( \frac{1}{2\xi^2} + \frac{x}{4\xi^3} \right) (\xi - x)^2 \ln \frac{1 - \xi}{x - \xi} + \frac{11}{12} - \frac{n_f}{18} \right] \right],
\]
A.6 Spin-dependent singlet quark-quark part

\[ 0 < x < \xi: \]
\[
\frac{dQ}{dlnQ^2} \tilde{H}_Q(x, \xi, Q^2) = \frac{2\alpha_s(Q^2)}{3\pi} \left[ \int_0^x \frac{2\xi y + x^2 + y^2}{(\xi + y)(y^2 - x^2)} \left( \tilde{H}_Q(y, \xi, Q^2) - \tilde{H}_Q(x, \xi, Q^2) \right) dy \right. \\
\left. - (\xi - x) \int_0^x \frac{2\xi x + y^2 + x^2}{(\xi^2 - y^2)(y^2 - x^2)} \left( \tilde{H}_Q(y, \xi, Q^2) - \tilde{H}_Q(x, \xi, Q^2) \right) dy \right. \\
\left. + \tilde{H}_Q(x, \xi, Q^2) \left( \ln \frac{1 - x^2}{(\xi + x)(\xi + 1)} + \frac{\xi - x}{2\xi} \ln \frac{\xi + x}{\xi - x} + \frac{3}{2} \right) \right]. \tag{37}
\]

For \( x > \xi \) the evolution kernel is given by the same expression as the non-singlet spin-independent one (Eq. 29).

A.7 Spin-dependent quark-gluon part

\[ 0 < x < \xi: \]
\[
\frac{dG}{dlnQ^2} \tilde{H}_Q(x, \xi, Q^2) = \frac{n_f \alpha_s(Q^2)}{2\pi} \left[ \int_0^x \frac{\tilde{H}_G(y, \xi, Q^2)}{(\xi + y)^2} dy \right. \\
\left. - 2(\xi - x) \int_0^x \frac{\tilde{H}_G(y, \xi, Q^2) - \tilde{H}_G(\xi, \xi, Q^2)}{(\xi^2 - y^2)^2} y dy \right. \\
\left. - \frac{x^2}{\xi^2(\xi + x)} \tilde{H}_G(\xi, \xi, Q^2) \right]. \tag{38}
\]

\[ x > \xi: \]
\[
\frac{dG}{dlnQ^2} \tilde{H}_Q(x, \xi, Q^2) = -n_f \alpha_s(Q^2) \left[ \int_0^x \frac{\tilde{H}_G(y, \xi, Q^2)}{(\xi^2 - y^2)^2} dy \right. \\
\left. + \frac{1+x}{\xi^2 - 1} \tilde{H}_G(\xi, \xi, Q^2) \right]. \tag{39}
\]

A.8 Spin-dependent gluon-quark part

\[ 0 < x < \xi: \]
\[
\frac{dQ}{dlnQ^2} \tilde{H}_G(x, \xi, Q^2) = \frac{2\alpha_s(Q^2)}{3\pi} \left[ \int_0^x \frac{\tilde{H}_Q(y, \xi, Q^2)}{\xi + y} dy \right. \\
\left. - \frac{(\xi - x)^2}{\xi} \int_0^x \frac{\tilde{H}_Q(y, \xi, Q^2)}{\xi^2 - y^2} y dy \right]. \tag{40}
\]

\[ x > \xi: \]
\[
\frac{dQ}{dlnQ^2} \tilde{H}_G(x, \xi, Q^2) = \frac{2\alpha_s(Q^2)}{3\pi} \int_0^x \frac{\xi^2 + x^2 - 2xy}{\xi^2 - y^2} \tilde{H}_Q(y, \xi, Q^2) dy. \tag{41}
\]

A.9 Spin-dependent gluon-gluon part

As it was in the case of the spin-independent gluon-gluon kernel, the spin-dependent gluon-gluon kernel requires partial fractioning of the denominator so that each of the integrals would have only one pole, which can be easily handled by subtraction.
\[
0 < x < \xi:
\]
\[
\frac{d_G \tilde{H}_G(x, \xi, Q^2)}{d \ln Q^2} = \frac{3\alpha_s(Q^2)}{2\pi} \left[ 2x \int_x^1 \frac{\xi^2 + 2\xi y + 2y^2 - x^2}{(\xi + y)^2(y^2 - x^2)} \left( \tilde{H}_G(y, \xi, Q^2) - \tilde{H}_G(x, \xi, Q^2) \right) dy 
+ 2 \frac{\xi - x}{\xi + x} \int_0^x \frac{y^2 - \xi^2 - 2x^2 - 2\xi x}{(\xi^2 - y^2)^2} \left( \tilde{H}_G(y, \xi, Q^2) - \tilde{H}_G(\xi, \xi, Q^2) \right) y dy 
+ \frac{2}{(\xi + x)^2} \int_0^x \frac{y^2 - \xi^2 - 2x^2 - 2\xi x}{y^2 - x^2} \left( \tilde{H}_G(y, \xi, Q^2) - \tilde{H}_G(x, \xi, Q^2) \right) y dy \right) (42)
\]
\[
- \left( \tilde{H}_G(\xi, \xi, Q^2) - \tilde{H}_G(x, \xi, Q^2) \right) \left( \frac{2x^3}{\xi^2(\xi + x)} + \frac{x^2}{(\xi + x)^2} + \ln \frac{\xi^2}{\xi^2 - x^2} \right) 
+ \tilde{H}_G(x, \xi, Q^2) \left( \frac{2x(1 - x)}{\xi^2(\xi + x)} - \frac{2x^3}{\xi^2(\xi + x)} + \ln \frac{x^2(1 - x)}{\xi^2(1 + x)} \right) \right].
\]
\[
 x > \xi:
\]
\[
\frac{d_G \tilde{H}_G(x, \xi, Q^2)}{d \ln Q^2} = \frac{3\alpha_s(Q^2)}{2\pi} \left[ 2x \int_x^1 \frac{\tilde{H}_G(y, \xi, Q^2) - \tilde{H}_G(x, \xi, Q^2)}{y - x} dy 
+ 2 \int_x^1 \frac{x(\xi^2 + y^2) - y(x^2 + y^2 - \xi^2)}{(\xi^2 - y^2)^2} \left( \tilde{H}_G(y, \xi, Q^2) - \tilde{H}_G(x, \xi, Q^2) \right) dy \right) (43)
\]
\[
+ \tilde{H}_G(x, \xi, Q^2) \left( \frac{2x - 2}{1 - \xi^2} + \ln \frac{(1 - x)^2}{1 - \xi^2} \right) + 2 \tilde{H}_G(\xi, \xi, Q^2) \left( \frac{2x - \xi^2 - x^2}{\xi^2 - 1} + 1 \right) 
+ \left( \tilde{H}_G(y, \xi, Q^2) - \tilde{H}_G(\xi, \xi, Q^2) \right) \left( \frac{2\xi^2 - x}{\xi^2 - 1} - 2 + \ln \frac{\xi^2 - x^2}{\xi^2 - 1} \right) \right].
\]
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