Quantum Phase Reduction for Continuous Measurement

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(Dated: August 26, 2022)

We develop a general framework of phase reduction theory for quantum nonlinear oscillators. Employing quantum trajectory theory, we define the limit cycle trajectory and the phase space according to a stochastic Schrödinger equation. Unlike classical dynamics, because a perturbation, which is applied to quantum states, is represented by a unitary transformation, phase response curves are calculated with respect to generators of Lie algebra. In the classical limit, we show that the proposed approach reproduces the conventional phase reduction theory. The derived phase equation numerically demonstrates that quantum noise desynchronizes the quantum van der Pol oscillators.

Introduction.—Synchronization, i.e., the adjustment of intrinsic rhythm to an external periodic perturbation, is a ubiquitous phenomenon in a wide range of science among biology, chemistry and physics. Extensive efforts in fundamental science and in technical applications have been made to study synchronization [1–4]. In the last decade, interest in synchronization has been extended from classical dynamics to quantum regime. Various researches are reported to study on synchronization of nonlinear oscillators that show quantum effects, such as spins [5, 6], optomechanical systems [7, 8], cold atoms [9, 10], superconducting circuits [11], quantum heat engines [12–15], and (discrete or continuous) time crystals [16–18]. In particular, to explore quantum synchronization, theoretical models of limit cycle, i.e. self-sustained oscillator adaptable to weak perturbation, are proposed in open quantum system, such as quantum van der Pol oscillators [19–23], spin oscillators [24], and qubit models [25]. Furthermore, several experiments have demonstrated quantum synchronization of limit cycles in a laboratory setting [6, 18, 26–28]. However, in a quantum framework with no classical analogue, defining the measure of quantum synchronization is challenging, and there are several approaches for measuring quantum synchronization. In classical dynamics, synchronization is measured directly through trajectories of nonlinear oscillators in the phase space. In contrast, in quantum dynamics, synchronization is quantified mainly by two approaches, global synchronization error [29, 30] and correlation between local observables [31, 32]. Furthermore, some indicators have been suggested as witness of quantum synchronization such as, mutual information [33], quantum discord [31, 34], and entanglement [35]. However, several studies have reported that there is no one-to-one correspondence between quantum synchronization and the above-mentioned indicators [5, 36–38]. Furthermore, these measures are often inappropriate for evaluating synchronization processes [8].

Therefore, to describe quantum limit cycles in phase dynamics as in classical dynamics, we propose a quantum phase reduction theory for continuous measurement. The phase reduction theory [1, 2] is a powerful tool for analyzing limit cycle dynamics, reducing the multidimensional dynamics of a weakly perturbed limit cycle to one-dimensional dynamics in the phase space. By continuously monitoring the environment to which oscillatory systems are coupled, quantum trajectories of the system come to obey a stochastic Schrödinger equation (SSE) [39–41]. When the effect of quantum noise is sufficiently weak, these trajectories fluctuate around a deterministic trajectory, and we identify the deterministic trajectory uniquely corresponding to the continuous measurement. Because a perturbation in the quantum limit cycle dynamics, unlike that in classical dynamics, is represented by a unitary transformation, we calculate a phase response to a perturbation based on Lie algebra. Therefore, we derive a quantum phase equation from a Lindblad master equation that describes a weakly perturbed dissipative system. Furthermore, in the classical limit, the proposed approach reproduces the conventional phase reduction theory. Through the quantum van der Pol oscillators, we show the proposal approach reproduces the definitions of the limit cycle trajectory, the phase, the perturbation, and the phase response curves of the conventional phase reduction theory. Our approach differs from the semi-classical phase reduction approach [42] in that, by continuous measurement, we do not resort to a semi-classical approximation where the stochastic trajectories fluctuate around the classical trajectory. Through a simulation of the quantum van der Pol limit cycle oscillators, we numerically demonstrate how quantum limit cycle oscillators are desynchronized due to quantum noise.

Derivation.—In open quantum dynamics, quantum limit cycle oscillators are usually described by a Lindblad equation [43, 44]. Let \( \rho(t) \) be a density operator at time \( t \) whose time evolution is governed by

\[
\frac{d\rho}{dt} = -i[H, \rho] + \sum_{k=1}^{M} \gamma_k D[L_k] \rho, \tag{1}
\]

where \( D[O] \) is the dissipator defined by \( D[O] \rho \equiv O \rho O^\dagger - }
sents the quantum van der Pol oscillator model [19] and prescribed by the following diffusive SSE in the Stratonovich the homodyne detection scheme, the evolution can be de-
tained by continuously monitoring the environment. In
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cle model [24], where
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measurable quantum state. To describe the dynamics, we
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\[ 1/2 (O^\dagger O \rho + \rho O^\dagger O), \] and \( \gamma_k \) is the jump rate of quantum jump \( L_k \). Because we want to obtain a general phase reduction approach that can be applied to quantum limit cycle models, we do not specify the jump operators \( L_k \). For example, \( \{L_k\} = \{a, a^\dagger\} \) represents the quantum van der Pol oscillator model [19] and \( \{L_k\} = \{S_+S_z, S_-S_z\} \) represents the spin-1 limit cycle model [24], where \( a, a^\dagger \) are the annihilation and creation operators, respectively, and \( \{S_k\} \) are Pauli matrices. Here, we have to emphasize that a Lindblad equation describes a density operator, not the dynamics of the measurable quantum state. To describe the dynamics, we employ quantum trajectory theory [45]. Quantum trajectory theory describes the stochastic evolution of a pure state of the system \(|\psi\rangle\), conditioned on measurements obtained by continuously monitoring the environment. In the homodyne detection scheme, the evolution can be described by the following diffusive SSE in the Stratonovich form [39–41] (see Appendix A and B for details).

\[
d|\psi\rangle = \left[ -iH_{\text{eff}} + \sum_{k=1}^{M} \frac{1}{2} (\langle L_k^\dagger L_k \rangle - L_k^\dagger L_k) + \langle X_k \rangle \left( L_k \mp \frac{\langle X_k \rangle}{2} \right) \right] |\psi\rangle \, dt + \sum_{k=1}^{M} \left( L_k \mp \frac{\langle X_k \rangle}{2} \right) |\psi\rangle \circ dW_k(t),
\]

where \( \circ \) denotes Stratonovich calculus, \( H_{\text{eff}} = H - i/2 \sum_{k=1}^{M} L_k^\dagger L_k \) is a non-Hermitian operator (an effective Hamiltonian), and \( X_k \equiv L_k + L_k^\dagger \) is a quadrature of the system. Here, \( \langle O \rangle \) denotes the expectation value of \( O \) with respect to state \(|\psi\rangle\), \( \langle \psi | O | \psi \rangle \). Random variables \( dW_k \) are Wiener increments that satisfy \( \mathbb{E}[dW_k] = 0, \mathbb{E}[dW_k^2] = dt \), where \( \mathbb{E}[\cdot] \) denotes the average over all possible trajectories. The associated stochastic homodyne currents \( J_k(t) \) are

\[
J_k(t) \equiv \langle X_k \rangle + \xi_k(t),
\]

where \( \xi_k(t) \equiv dW_k/dt \). In general, a limit cycle trajectory in quantum dynamics and the phase space along it are not well defined. In the classical stochastic dynamics of limit cycles, a stochastic differential equation is represented by adding noise terms to a given deterministic differential equation [46–49]. In contrast, quantum dynamics are stochastic in nature and the deterministic equation is not given. To then realize a quantum phase reduction, we have to define the deterministic limit cycle and the phase along it. In classical dynamics, the deterministic limit cycle dynamics corresponds to an equation obtained by removing noise terms in a stochastic differential equation in the Stratonovich form. As an analogue of classical cases, we remove noise terms from an SSE in the Stratonovich form and define the resulting equation as the deterministic limit cycle dynamics

\[
d|\psi\rangle = \left[ -iH + \sum_{k=1}^{M} \frac{1}{2} \langle (L_k^\dagger L_k) \rangle - L_k^\dagger L_k + \langle X_k \rangle \left( L_k \mp \frac{\langle X_k \rangle}{2} \right) \right] |\psi\rangle \, dt + \sum_{k=1}^{M} \left( L_k \mp \frac{\langle X_k \rangle}{2} \right) |\psi\rangle \circ dW_k(t),
\]

An SSE is usually represented and calculated in the Ito form for computational and statistical convenience. It is emphasized that we should remove noise terms from an SSE in the Stratonovich interpretation, not from that in the Ito interpretation. There are two reason for this. One is the chain rule of differentiation calculation. The phase reduction requires coordinate transformation between a state vector and a phase coordinate. The transformation is performed via the chain rule of the differentiation, which holds only in the Stratonovich form but not in the Ito form. The other reason is norm preservation. Note that the norm of Eq. (4) is preserved, \( d||\psi|| = 0 \), where
the norm is defined as \( \| \psi \| \equiv \sqrt{\langle \psi | \psi \rangle} \). Therefore, Eq. (4) stands on its own as pure-state dynamics. This is not the case for the Ito interpretation. It is a nontrivial property that an SSE with noise terms removed also stands as pure-state dynamics because, unlike the case for classical dynamics, the deterministic dynamics Eq. (4) is not given. When Eq. (4) satisfies the following equation for a period \( T \), \( |\psi\rangle \) has a limit cycle solution \( |\psi_0\rangle \) to which \( |\psi\rangle \) converges:

\[
\lim_{t \to \infty} \| \langle \psi(t) | \psi(t + T) \rangle \| = 1. \tag{5}
\]

Since \( U(1) \) has no physical effect on the state \( |\psi\rangle \) [50], Eq. (5) defines the same phase about \( U(1) \) transformation. Figure 1 shows a quantum trajectory generated by Eq. (2) that fluctuates around deterministic limit cycle solution \( |\psi_0\rangle \) of Eq. (4) in the quantum van der Pol oscillator model. We define the phase on a quantum limit cycle using the deterministic trajectory \( |\psi_0\rangle \). There are several schemes for the phase reduction in classical stochastic systems [47, 48, 51]. Because we want to emphasize simplicity of representation, we derive the phase equation by following the procedure of [47]. The phase \( \theta \) is defined along the limit cycle solution \( |\psi_0\rangle \). Under Eq. (4), we define the phase \( \theta \) so that the phase changes at constant frequency \( \omega = 2\pi/T \). Furthermore, the phase \( \theta \) outside the limit cycle solution \( |\psi_0\rangle \) is defined by an isochron under Eq. (4) as follows:

\[
\Theta(|\psi(t)\rangle) = \Theta \left( \lim_{n \to \infty} |\psi(t + nT)\rangle \right), \tag{6}
\]

where the phase function \( \Theta(|\psi\rangle) \) represents the phase at state \( |\psi\rangle \). By virtue of the convergence to the limit cycle solution \( |\psi_0\rangle \), the isochron defines the phase outside the limit cycle solution. We here assume that the perturbation is sufficiently weak that state \( |\psi\rangle \) is in the vicinity of the limit cycle solution \( |\psi_0\rangle \).

It should be mentioned that our definition of the phase response curve differs from that of the classical counterpart. While for the classical limit cycle the phase is defined in Euclidean space, for the quantum limit cycle the state is defined by the unitary group in Hilbert space. Therefore, the bases are not basis vectors in Euclidean space but rather the generators of unitary group \( U(N) \) [52]. The generator of \( U(N) \) can be decomposed into the generators of \( U(1) \) and those of the special unitary group \( SU(N) \). \( U(1) \) represents scalar multiplication, while \( SU(N) \) is a unitary group with determinant \( \det(U) = 1 \). For example, generators of \( SU(2) \) correspond to Pauli matrices. By the phase definition of Eq. (5), \( U(1) \) has no effect on the phase, so only \( SU(N) \) needs to be considered in the phase space. In quantum limit cycles, the perturbation is represented by an infinitesimal unitary transformation and the phase response curve is calculated for the infinitesimal unitary transformation. Based on Lie algebra, an arbitrary infinitesimal unitary transformation is represented by Taylor expansion as \( U |\psi\rangle = \exp \left( \sum_{l=1}^{N^2-1} i g_l E_l + i g_0 I \right) |\psi\rangle \approx |\psi\rangle + \sum_{l=1}^{N^2-1} i g_l E_l |\psi\rangle + i g_0 |\psi\rangle \), where \( E_l \) are generators of \( SU(N) \), \( I \) is the identity matrix, and real coefficients \( g_l \) satisfy \( |g_l| \ll 1 \). Figure 2 shows \( SU(2) \) and the generators of \( SU(2) \) on the Bloch sphere. The phase response curves for the generators \( E_l \) are represented as

\[
Z_l(\theta) = \frac{\partial \Theta}{\partial g_l} \left. \right|_{g_l=0} = \lim_{g_l \to 0} \frac{\Theta(\exp(i g_l E_l |\psi_0(\theta)\rangle) - \Theta(|\psi_0(\theta)\rangle)}{g_l}, \tag{7}
\]

where \( |\psi_0(\theta)\rangle \) represents state \( |\psi\rangle \) on limit cycle solution \( |\psi_0\rangle \) with phase \( \theta \) and the partial derivative with respect to \( g_l \) means the partial derivative with respect to a unitary transformation by generator \( E_l \). This formulation meets the definition of the quantum phase response curve. Although Eq. (4) is not represented by an Hermitian dynamics, an arbitrary infinitesimal change of a pure state can be represented as an unitary transformation. Therefore, Eqs. (2) and (4) should be represented as unitary transformations since any norm-preserving transformation can be expressed by a unitary transformation. By the phase definition, the deterministic terms in Eq. (2) correspond to a constant \( \omega \) in the phase space, we focus on the stochastic terms. The stochastic terms can be decomposed into the components parallel and orthogonal.
where Hermitian operators $H_k$ are defined by
\[ H_k \equiv i(L_k - \langle L_k \rangle) |\psi\rangle \langle \psi | + \text{H.c.} \tag{9} \]

In Eq. (8), from the first to the second line, we decompose the stochastic terms into the parallel and the orthogonal components to state $|\psi\rangle$. The parallel components correspond to $U(1)$ transformation. From the second to the third line, the orthogonal components come to be represented by traceless Hermitian operators $H_k$, which are defined in Eq. (9). Traceless Hermitian operators $H_k$ perform rotation in the plane spanned by $|\psi\rangle$ and $(L_k - \langle L_k \rangle) |\psi\rangle$. Thanks to the trace-orthogonal property of Lie algebra, traceless Hermitian operators $H_k$ can be decomposed into a linear combination of $SU(N)$ generators as follows:
\[ H_k = \sum_{i=1}^{N^2-1} g_{k,i} E_i, \tag{10} \]
\[ g_{k,i} \equiv \text{Tr}[H_k E_i]. \tag{11} \]

In the same way, the deterministic terms in Eq. (2) are represented by a traceless Hermitian operator and decomposed into Lie algebra generators with coefficients $f_l$. By the phase definition of Eq. (5), $U(1)$ has no effect on the phase and the phase response to $U(1)$ is trivially zero. As a result, we consider only $SU(N)$ components for the phase space. Therefore, a quantum phase equation is derived by the chain rule as follows:
\[ \frac{d\theta}{dt} = \sum_{l=1}^{N^2-1} \frac{\partial \theta}{\partial y_l} \frac{dg_l}{dt} = \frac{1}{2} \sum_{i=1}^{N^2-1} Z_i(\theta) f_i(\theta) + \sum_{k=1}^{M} \sum_{l=1}^{N^2-1} Z_l(\theta) g_{k,l}(\theta) \circ \xi_k(t) \]
\[ = \omega + \sum_{k=1}^{M} \sum_{l=1}^{N^2-1} Z_l(\theta) g_{k,l}(\theta) \circ \xi_k(t). \tag{12} \]

The transformation from the second line to the third line is performed by the phase definition. The phase equation (12) in the Stratonovich form can be converted to an equivalent equation in the Ito form [53]
\[ \frac{d\theta}{dt} = \omega + \frac{1}{2} \sum_{k=1}^{M} \frac{dY_k(\theta)}{d\theta} Y_k(\theta) + \sum_{k=1}^{M} Y_k(\theta) \xi_k(t), \tag{13} \]
where $Y_k(\theta) = \sum_{i=1}^{N^2-1} Z_i(\theta) g_{k,i}(\theta)$. Let us elaborate on the difference between our approach and that of Ref. [42], which is the extant phase reduction approach for quantum systems. Ref. [42] employs a quasi-probability distribution, such as the P, Q, or Wigner representation, to derive a phase equation from the semi-classical Fokker-Planck equation. While such research resorts to the semi-classical approximation, our approach defines the phase reduction theory within a quantum framework. To explain the difference in detail, let us examine the quantum van der Pol oscillator defined by
\[ \frac{d\rho}{dt} = -i[a^\dagger a, \rho] + \gamma_1 g D[a^\dagger] \rho + \gamma_1 d D[a] \rho + \gamma_2 D[a^2] \rho. \tag{14} \]

The quantum van der Pol model describes limit cycle dynamics at a quantum scale. In quantum systems, measurement outcomes are stochastic in nature. Then, position $x = 1/\sqrt{2}(a + a^\dagger)$ and momentum $p = -i/\sqrt{2}(a - a^\dagger)$ are evaluated through their expectation values as $\langle x \rangle_\rho$ and $\langle p \rangle_\rho$, respectively, where $\langle O \rangle_\rho = \text{Tr}[O \rho]$. In the classical limit $\langle a^\dagger a \rangle_\rho \gg 1$ (i.e., the system is at a macroscopic scale), Eq. (14) gives the equation of momentum as [20]
\[ \langle \dot{x} \rangle_\rho + \langle \dot{\bar{x}} \rangle_\rho = \frac{1}{\epsilon} \left( 1 - \frac{\langle x \rangle_\rho^2 + \langle \bar{x} \rangle_\rho^2}{A_c^2} \right) + O(\epsilon^2), \tag{15} \]
where $\epsilon \equiv \gamma_{1g} - \gamma_{1d}$, the difference between two rates of one-particle gain and loss, and $A_c = \sqrt{1/\gamma_{2d}}$. Equation (15) recovers the classical van der Pol oscillator model up to $O(\epsilon^2)$. For the semi-classical approximation, the previous research [42] assumes that the deterministic limit cycle trajectory, around which stochastic trajectories fluctuate, is defined in the classical limit $\gamma_{1g} \gg \gamma_{2d}$. Then, the previous approach can be applied only to systems near the classical limit where the conventional phase reduction theory is applied. Since our approach reduces a quantum state $|\psi\rangle$ to the phase based on the quantum trajectory theory, our approach can be applied in the quantum regime $\gamma_{2d} \gg \gamma_{1g}$ as will be demonstrated in the Example section.

Thus far, we have been concerned with regimes ranging from semi-classical to quantum. Historically, the phase reduction theory was demonstrated in the context of classical deterministic dynamics. We next show that our quantum phase reduction theory is connected with the conventional phase reduction theory in the classical limit. By a phase transformation (see Appendix D for details), $|\psi\rangle \rightarrow e^{i\phi(t)} |\psi\rangle$, the diffusive SSE [Eq. (2)] is rewritten as [39, 55]
\[ d |\psi\rangle = dt \left[ -iH + \sum_{k=1}^{M} \left( -\frac{1}{2} L_k L_k + \langle L_k^\dagger \rangle L_k - \frac{1}{2} \langle L_k \rangle^2 \right) \right] |\psi\rangle + \sum_{k=1}^{M} dW_k (L_k - \langle L_k \rangle) |\psi\rangle. \tag{16} \]
In the classical limit, the state $|\psi\rangle$ is considered coherent and satisfies $a|\psi\rangle = a|\psi\rangle$, $a^\dagger|\psi\rangle = \alpha^*|\psi\rangle + |x\rangle$, and $|\alpha| \gg 1$, where $|x\rangle = a^\dagger|\psi\rangle - \alpha^*|\psi\rangle$. In Eq. (16), the stochastic terms are so weak compared to the deterministic terms that the dynamics can be considered as deterministic. As discussed above, in the classical limit, the quantum van der Pol oscillator model recovers the classical model in Euclidean space. Since the expectation is calculated along the limit cycle in Hilbert space, our definition of limit cycle is equivalent to the conventional one in the classical limit. This equivalence applies also to the perturbation and phase response. Perturbation in Euclidean space is reproduced by momentum operator $p$ in Hilbert space as follows. By the unitary perturbation $d|\psi\rangle/dt = -i p|\psi\rangle$, the derivative of the expectation value of the position is unity $d\langle x \rangle/dt = -i \langle [x,p]\rangle = 1$. An arbitrary real perturbation in Euclidean space $d|\psi\rangle/dt = f(t)$ is then obtained by unitary perturbation in Hilbert space $d|\psi\rangle/dt = -if(t)p|\psi\rangle$. Because of this fact, the phase response can be calculated in the same way for both the conventional approach and our approach. As shown above, our approach is connected with the conventional approach in the classical limit.

Example.—Using the derived phase equation (12), we now demonstrate the effect of quantum noise on quantum synchronization. As an example, we consider the quantum van der Pol oscillator in a rotating frame [20]:

$$\frac{dp}{dt} = -i[H,p] + \gamma_1 gD[a^\dagger]p + \gamma_1 dD[a]p + \gamma_2 dD[a^2]p,$$

where the Hamiltonian $H = i\Omega(a^\dagger - a)$ and $\Omega$ is the strength of a resonant drive. Equation (17) is the rotating frame representation of Eq. (14), subjected to a resonant drive. The Hamiltonian in laboratory frame is $H_0 = \omega_0 a^\dagger a + \Omega i e^{-i\omega t}a^\dagger + H.c.$, where the system has a natural frequency $\omega_0$ and rotates with an external resonant drive. For brevity of expression, we here approximate the quantum van der Pol oscillator by limiting the bosonic Fock state to three levels [22]. This approximation is valid for $\gamma_1 g, \gamma_1 d \ll \gamma_2 d$ where two-particle loss is dominant and dynamics are essentially restricted to the lowest three levels, we consider this case in the following. Figure 3 shows the results of numerical simulations using $N = 100$ oscillators, where the parameters set in $\Omega \ll \gamma_2 d$ the quantum response regime are fixed at $\Omega = 0.1$, $\gamma_1 g = 0.004$, $\gamma_1 d = 0$, and $\gamma_2 d = 2$. We set all the oscillators fully synchronized in the initial state. Figure 3 shows that quantum noise desynchronizes the oscillators in the phase space. Figure 4 shows the phase response curve $Z_l(\theta)$. Although the phase response curves are calculated for all eight generators, five of the phase response curves vanish. The only phase response curves that do not vanish are those of the three generators. These are the generators of special orthogonal unitary group $SO(3)$ and represent rotations about the orthogonal axes, which are parallel to their zero-eigenvectors. That is, the phase is shifted only by rotation about the axes.

Conclusion.—In this Letter, we proposed a quantum phase reduction of a Lindblad equation by continuous measurement. By continuous measurement, the quantum limit cycle is identified as a pure state trajectory, and the phase response to a unitary transformation such as a weak perturbation is defined based on Lie algebra. Since the dynamics of a quantum limit cycle can be described by the phase equation as in the classical phase reduction, analysis techniques used in the classical phase reduction approach may be applied to our approach as well. Furthermore, through simulations of the quantum van der Pol oscillator, we calculate the phase shift of the quantum limit cycle due to quantum noise. The present study can be used for unveiling quantum limit cycles. For instance, it is possible to obtain a quantum limit cycle through optimization of the phase response curve, as was done previously for classical systems [56].

ACKNOWLEDGMENTS

This work was supported by JSPS KAKENHI Grant Number JP22H03659.

Appendix A: Stochastic calculus

Typically, a stochastic differential equation is represented by two different forms, the Ito and Stratonovich forms, which are respectively given by

$$f(t)dW(t) \equiv f(t)(W(t+dt) - W(t)),$$

$$(A1)$$

$$f(t)\circ dW(t) \equiv f\left(t + \frac{dt}{2}\right)(W(t+dt) - W(t)).$$

$$(A2)$$

In the Stratonovich interpretation, a calculus is performed at the midpoint of the interval $[t, t+dt]$. The two forms can be converted to each other via

$$f(t)\circ dW(t) = f(t)dW(t) + \frac{1}{2}df(t)dW(t).$$

$$(A3)$$

The conversion of Eq. (A3) is performed according to the Ito rule:

$$dW(t)^2 = dt,$$

$$(A4)$$

$$dW(t)dt = 0.$$  

$$(A5)$$

Appendix B: Ito and Stratonovich conversion of stochastic Schrödinger equation

A diffusive stochastic SSE is represented by the Ito form [39, 40]

$$d|\psi\rangle = \frac{dt}{iH_{eff}} + \sum_{k=1}^{M} \frac{X_k}{2} \left( L_k - \frac{X_k}{4} \right) |\psi\rangle + \sum_{k=1}^{M} dW_k(t) \left( L_k - \frac{X_k}{2} \right) |\psi\rangle.$$  

$$(B1)$$
According to Appendix A, Eq. (B1) can be converted to the equivalent Stratonovich form as follows:

\[ d\psi = \left[-iH_{\text{eff}} + \sum_{k=1}^{M} + \frac{1}{2} (L_k^1 L_k^1 + (X_k) \left(L_k - \frac{(X_k)}{2}\right) + \frac{1}{4} \left(-2L_k^2 + (L_k^2) + (L_k^2)\right) \right] |\psi\rangle \, dt \\
+ \sum_{k=1}^{M} \left(L_k - \frac{(X_k)}{2}\right) |\psi\rangle \, dW_k(t). \quad (B2) \]

The conversion from Eq. (B1) to Eq. (B2) is performed according to the multivariate Ito rule [53]:

\[ dW_m(t) dW_n(t) = \delta_{m,n} dt, \quad (B3) \]
\[ dW_m(t) dt = 0, \quad (B4) \]

where \( \delta_{m,n} \) is Kronecker’s delta.

**Appendix C: Generators of SU(3)**

The generators of SU(3) correspond to Gell-Mann matrices \( \{\lambda_l\} \).

\[
\begin{aligned}
\lambda_1 &= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \lambda_2 &= \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, & \lambda_3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\
\lambda_4 &= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, & \lambda_5 &= \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{bmatrix}, & \lambda_6 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \\
\lambda_7 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{bmatrix}, & \lambda_8 &= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}.
\end{aligned}
\]

Gell-Mann matrices are orthogonal with respect to the Hilbert-Schmidt inner product:

\[ \text{Tr}[\lambda_m \lambda_n] = 2\delta_{mn}. \quad (C2) \]

The generators \( E_l \) of SU\((N)\) are defined so that they are normalized with respect to the trace norm:

\[ E_l \equiv \frac{1}{\sqrt{2}} \lambda_l. \]

**Appendix D: Phase transformation**

Since \( U(1) \) has no physical effect on the state \( |\psi\rangle \), by phase transformation \( |\psi'\rangle \equiv e^{i\phi(t)} |\psi\rangle \), we can obtain a equation physically equivalent to an SSE (2):

\[ d|\psi'\rangle = \left(i\frac{d\phi}{2} - \frac{1}{2}(d\phi)^2\right) |\psi'\rangle + e^{i\phi}(1 + i\phi) d|\psi\rangle. \quad (D1) \]

In particular, Eq. (16) is obtained by the phase transformation \( |\psi'\rangle \equiv e^{i\phi(t)} |\psi\rangle \), which satisfies the differential equation as follows:

\[ d\phi = \frac{i}{2} \sum_{k=1}^{M} \langle X_k \rangle (L_k - I_k^1) dt + \frac{i}{2} \sum_{k=1}^{M} \langle X_k \rangle (L_k - I_k^1) dW_k. \quad (D2) \]

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