Asymptotic Functions of Entire Functions

Aimo Hinkkanen1 · Joseph Miles1 · John Rossi2

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Abstract

If \( f \) is an entire function and \( a \) is a complex number, \( a \) is said to be an asymptotic value of \( f \) if there exists a path \( \gamma \) from 0 to infinity such that \( f(z) - a \) tends to 0 as \( z \) tends to infinity along \( \gamma \). The Denjoy–Carleman–Ahlfors Theorem asserts that if \( f \) has \( n \) distinct asymptotic values, then the rate of growth of \( f \) is at least order \( n/2 \), mean type. A long-standing problem asks whether this conclusion holds for entire functions having \( n \) distinct asymptotic (entire) functions, each of growth at most order \( 1/2 \), minimal type. In this paper conditions on the function \( f \) and associated asymptotic paths are obtained that are sufficient to guarantee that \( f \) satisfies the conclusion of the Denjoy–Carleman–Ahlfors Theorem. In addition, for each positive integer \( n \), an example is given of an entire function of order \( n \) having \( n \) distinct, prescribed asymptotic functions, each of order less than \( 1/2 \).

Keywords  Entire function · Asymptotic function · Asymptotic value

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Dedicated to the memory of Walter K. Hayman, FRS.

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✉ John Rossi
rossij@vt.edu
Aimo Hinkkanen
aimo@illinois.edu
Joseph Miles
joe@math.uiuc.edu

1 Department of Mathematics, University of Illinois at Urbana-Champaign, 1409 W. Green St., Urbana, IL 61801, USA
2 Department of Mathematics, Virginia Tech, 225 Stanger St., Blacksburg, VA 24061-1026, USA
1 Introduction

Suppose that \( f(z) \) is an entire function in the complex plane \( \mathbb{C} \). An entire function \( a(z) \) is said to be an asymptotic function for \( f \) if there exists a path \( \gamma \) in \( \mathbb{C} \) from 0 to infinity such that \( f(z) - a(z) \) tends to 0 as \( z \) tends to infinity along \( \gamma \); we then say that \( f \) is asymptotic to \( a \) on \( \gamma \). In the case that \( a(z) \) is a constant function, it is said to be an asymptotic value of \( f \). The Denjoy–Carleman–Ahlfors Theorem (\cite{2}; in a very general form for subharmonic functions this can be found in \cite[Thm. 8.9, p. 562]{17}) states that if \( f \) has \( n \) distinct asymptotic values then the rate of growth of \( f \) is at least order \( n/2 \), mean type. This bound is known to be sharp \cite[p. 1]{1}. For consider

\[
f(z) = \int_0^z (w^{-n/2} \sin w^{n/2}) \, dw.
\]

Then \( f \) has order \( n/2 \), and if \( 0 \leq \nu \leq n-1 \) and \( z = r e^{2\nu i \pi/n} \), where \( r > 0 \), we have

\[
f(z) \to e^{2\nu i \pi/n} \int_0^\infty x^{-n/2} \sin(x^{n/2}) \, dx
\]
as \( r = |z| \to \infty \).

It is an open question \cite[Prob. 2.3]{16} whether the analogue of the Denjoy–Carleman–Ahlfors Theorem holds for asymptotic functions \( a(z) \) with growth at most order \( 1/2 \), minimal type. For such functions it is known that the minimum modulus is unbounded on a sequence of circles with radii tending to infinity. Absent this minimum modulus condition on the asymptotic functions \( a(z) \), there are easy examples of entire functions of finite order with infinitely many asymptotic functions. Such an example is \( f(z) = \sin(\sqrt{z})/\sqrt{z} \) or \( f(z) = e^{-z} \), and \( a_j(z) = j f(z) \) for each positive integer \( j \). Note that both \( f(z) \) and \( a_j(z) \) tend to 0 as \( z \to \infty \) along the positive real axis.

Throughout this paper we let \( B(0, r) \) denote the disk \( \{ z \in \mathbb{C} : |z| < r \} \) and let \( S(0, r) \) be its boundary circle \( \{ z \in \mathbb{C} : |z| = r \} \), where \( r > 0 \). If \( f \) is entire and \( D \) is a domain in \( \mathbb{C} \), we write

\[
M(r, D, f) = \sup\{|f(z)| : z \in D \cap S(0, r)\}.
\]

We denote the usual maximum modulus \( M(r, \mathbb{C}, f) \) of \( f \) by \( M(r, f) \). A path \( \gamma \) from 0 to infinity is said to be segmental if it is a Jordan arc consisting of line segments whose vertices accumulate only at infinity.

Recall that the order \( \rho = \rho(f) \) of an entire function \( f \) is defined by

\[
\rho(f) = \limsup_{r \to \infty} \frac{\log \log M(r, f)}{\log r}.
\]

If \( 0 < \rho < +\infty \), we say that \( f \) is of minimal type if \( \log M(r, f) = o(r^\rho) \) as \( r \to \infty \), and we say that \( f \) is of mean type if

\[
0 < \limsup_{r \to \infty} \frac{\log M(r, f)}{r^\rho} < +\infty.
\]
The strongest result on [16, Prob. 2.3] is due to Fenton [7], who proved the following theorem.

**Theorem A** If \( f \) is entire with \( n \) distinct asymptotic functions of order less than \( 1/4 \), then the growth of \( f \) is at least order \( n/2 \), mean type.

In [17, Thm. 8.13, p. 577] Hayman showed that Fenton’s methods can be used to obtain this result for asymptotic functions with growth no faster than order \( 1/4 \), minimal type. For an earlier result of Somorjai with asymptotic functions of order < \( 1/30 \), see [21].

Stronger results are known if the asymptotic paths are rays. In this case Denjoy [4,5] showed that if \( f \) has order \( \mu \) then the number of distinct asymptotic functions with order less than \( 1/(2 + \mu^{-1}) \) is at most \( 2\mu \).

Dudley Ward and Fenton [6] obtained the following result.

**Theorem B** Suppose that \( f \) is analytic in the sector \( D = \{ z \in \mathbb{C} : |\arg z| < \eta \} \) for some \( \eta \in (0, \pi) \) and is continuous on \( \partial D \). Suppose that \( a(z) \) and \( b(z) \) are entire, each with order less than \( 1/(2 + 2\eta \pi^{-1}) \), are not identically zero and satisfy

\[
f(te^{i\eta}) - a(te^{i\eta}) \to 0 \quad \text{as} \ t \to +\infty \quad (1)
\]

and

\[
f(te^{-i\eta}) - b(te^{-i\eta}) \to 0 \quad \text{as} \ t \to +\infty. \quad (2)
\]

Then

\[
\liminf_{r \to \infty} \frac{\log M(r, D, f)}{r^{\pi/(2\eta)}} > 0.
\]

Elementary arguments lead from Theorem B to the following corollary.

**Corollary A** Suppose that \( f \) is entire and that \( \gamma_1, \gamma_2, \ldots, \gamma_n \) are distinct rays emanating from 0. Suppose for \( 1 \leq j \leq n \) that \( a_j(z) \) are distinct entire functions of order less than \( 1/(2 + 2\eta^{-1}) \) and that \( f(z) \) is asymptotic to \( a_j(z) \) on \( \gamma_j \). Then

\[
\liminf_{r \to \infty} \frac{\log M(r, f)}{r^{\pi/2}} > 0.
\]

Hinkkanen and Rossi [18] obtained the conclusion of Theorem B under the relaxed assumption that the open set \( D \) is bounded by paths \( \gamma_1 \) and \( \gamma_2 \), not necessarily disjoint away from 0, with (1) holding on \( \gamma_1 \) and (2) holding on \( \gamma_2 \) and that the angular measure \( m(D \cap S(0, r)) \) of \( D \cap S(0, r) \) satisfies

\[
m(D \cap S(0, r)) \leq 2\eta
\]

for all large \( r \).

For further results in this direction, see [8–12].

The principal result of this paper is the following theorem.
Theorem 1.1 Suppose that $\gamma_1, \gamma_2, \ldots, \gamma_n$ are $n$ simple segmental paths from 0 to infinity, disjoint except at the origin, arranged in counterclockwise fashion. Let $D_j$ be the Jordan domain between $\gamma_j$ and $\gamma_{j+1}$, where $\gamma_{n+1} = \gamma_1$. Suppose that $f$ is entire and that $a_j(z)$, for $1 \leq j \leq n$, are distinct entire functions with order $\rho(a_j) < 1/2$ such that $f$ is asymptotic to $a_j$ on $\gamma_j$. Suppose that there exists a number $\kappa > \rho := \max\{\rho(a_j) : 1 \leq j \leq n\}$ such that

$$\limsup_{z \to \infty, z \in D_j} \frac{\log |f(z)|}{|z|^\kappa} > 0$$

for each $j$. Then

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{r^{n/2}} > 0.$$

We note in particular that if (3) holds with $\kappa = 1/2$, then (4) holds.

From Theorem 1.1 we obtain the following corollary.

Corollary 1.2 Suppose for $1 \leq j \leq n$ that $a_j(z)$ are distinct entire functions with order $\rho(a_j) < 1/2$. For $1 \leq j \leq n$, let $\gamma_j$ be the ray with $\arg z = 2\pi j/n$. Suppose that $f$ is entire and that $f$ is asymptotic to $a_j$ on $\gamma_j$. Then

$$\liminf_{r \to \infty} \frac{\log M(r, f)}{r^{n/2}} > 0.$$

In comparing Corollary 1.2 with Corollary A, we note that Corollary 1.2 treats asymptotic functions $a(z)$ of all orders less than $1/2$, but requires that the rays be equally spaced.

We also show that Fenton’s result in the special case that the $a_j(z)$ are polynomials follows quite easily from Theorem 1.1.

Finally, in Sect. 3 we give an example of an entire function of order $n$ having $n$ distinct, prescribed asymptotic functions of order $< 1/2$. Our method does not seem capable of constructing an entire $f$ of order $n$ with $2n$ such prescribed asymptotic functions.

2 Proofs

2.1 A Lemma

Fundamental to our approach is the following lemma.

Lemma 2.1 Suppose that $\gamma_1$ and $\gamma_2$ are simple segmental paths from 0 to infinity, disjoint except for the origin. Let $D$ be a Jordan domain with $\partial D = \gamma_1 \cup \gamma_2$. For $t > 0$, let $\Phi(t)$ be the angular measure of $D \cap S(0, t)$. Suppose that $a_1$ and $a_2$ are distinct entire functions with orders $\rho(a_1)$ and $\rho(a_2)$ satisfying $\rho = \max\{\rho(a_1), \rho(a_2)\} < 1/2$, such that $f$ is asymptotic to $a_j$ on $\gamma_j$ for $j = 1, 2$. Suppose that there exists a number $\kappa > \rho$ such that
\[
\limsup_{z \to \infty} \frac{\log |f(z)|}{|z|^\kappa} > 0.
\]

Then there exists \( R_1 > 0 \) such that for all \( R > R_1 \), we have

\[
\log M(R, D, f) \geq \frac{\pi}{8} \exp \left\{ \pi \int_{R_1}^{R} \frac{dt}{\Phi(t)} \right\} \geq \frac{\pi}{8} \left( \frac{R}{R_1} \right)^{1/2}.
\]

**Proof** Without loss of generality, we may assume that \( \kappa < 1/2 \). Select numbers \( \kappa_1 \) and \( \kappa_2 \) such that \( \rho < \kappa_1 < \kappa_2 < \kappa \). There exists a number \( C \geq 1 \) such that for all \( z \in \partial D \) we have

\[
\max\{\log |a_1(z)|, \log |a_2(z)|\} < C + |z|^\kappa_1.
\]

There exists \( M' \geq 1 \) such that for \( j = 1, 2 \),

\[
\log |f(z) - a_j(z)| \leq M'
\]

for all \( z \in \gamma_j \) and hence

\[
\log |f(z)| \leq \log^+ |a_j(z)| + M' + \log 2 < C + |z|^\kappa_1 + M' + \log 2.
\]

With \( M = C + M' + \log 2 \), we have

\[
\log |f(z)| < M + |z|^\kappa_1
\]

for all \( z \in \partial D \).

Set

\[
A_0 = \frac{20}{\left( \frac{1}{2} - \kappa_1 \right) 4^{1/2 - \kappa_1}} > 1.
\]

We choose \( R_0 \) so large that for all \( r \geq R_0 \) we have

\[
1 + M + A_0 r^\kappa_1 < r^\kappa_2.
\]

The following lemma is from [17, Lem. 8.13, p. 583].

**Lemma 2.2** Let \( \phi \) be a non-negative continuous convex function of \( \log t \) for \( 0 \leq t < \infty \) with \( \phi(0) = 0 \), and suppose that for some \( \delta > 0 \), we have

\[
\int_{\delta}^{\infty} \frac{\phi(t)}{t^{3/2}} dt < \infty.
\]

Let \( D \) be a domain in the plane such that every boundary point of \( D \) is regular for Dirichlet’s problem and such that for all \( r \in (0, \infty) \), the circle \( S(0, r) \) intersects the
complement of $D$. Then there is a function $u$, continuous and non-negative in $\overline{D}$ and harmonic in $D$, such that $u(z) = \phi(|z|)$ for all $z \in \partial D$ and such that for all $z \in D, we have$

$$
\phi(|z|) \leq u(z) \leq 20|z|^{1/2} \int_{4|z|}^{\infty} \frac{\phi(t)}{t^{3/2}} \, dt. \tag{6}
$$

Hayman used an estimate for harmonic measure that is not as strong as is known. Hence the number 20 on the right hand side of (6) can be reduced but we do not try to do this as it is not important for us.

We use $\phi(t) = t^{\kappa_1}$ for our earlier choice of $\kappa_1$ and calculate

$$
20|z|^{1/2} \int_{4|z|}^{\infty} \frac{\phi(t)}{t^{3/2}} \, dt = A_0|z|^{\kappa_1}.
$$

We apply Lemma 2.2 to obtain a harmonic function $u$ on $D$ such that for all $z \in D$ we have

$$
|z|^{\kappa_1} \leq u(z) \leq A_0|z|^{\kappa_1}.
$$

By our assumptions, there exists $z_1 \in D$ with $|z_1| > R_0$ such that

$$
\log |f(z_1)| > |z_1|^{\kappa_2} > A_0|z_1|^{\kappa_1} + M + 1. \tag{7}
$$

We set $R_1 = |z_1|$.

Suppose that $R > R_1$. Let $D(R)$ be the component of $D \cap B(0, R)$ containing $z_1$. We note that $\partial D(R)$ lies in $\partial D$ except for a non-empty union of open arcs in $S(0, R)$. Let $U = U_R$ be the harmonic function in $D(R)$ that satisfies $U(z) = 0$ if $z \in \partial D(R)$ lies in the interior of an arc in $S(0, R)$ lying in $\partial D(R)$, while $U(z) = |z|^{\kappa_1}$ if $z \in \partial D(R)$ and $|z| < R$ (and thus $z \in \partial D$). Let $u$ be the harmonic function obtained above from Lemma 2.2 with the choice $\phi(t) = t^{\kappa_1}$. Evidently $U - u$ is harmonic on $D(R)$ and $U(z) - u(z) \leq 0$ for all $z \in \partial D(R)$. Thus $U(z) - u(z) \leq 0$ for all $z \in D(R)$.

Let

$$
M'(R, D, f) = \max\{|f(z)| : |z| = R, \ z \in \partial D(R)| \leq M(R, D, f).
$$

Write $H(R) = S(0, R) \cap \partial D(R)$. Let $\omega(R, z)$ denote the harmonic measure of $H(R)$ at $z \in D(R)$. The function

$$
w(z) = U(z) + M + \omega(R, z) \log M'(R, D, f)
$$

is harmonic on $D(R)$ and

$$
\log |f(z)| \leq w(z)
$$

for all $z \in \partial D(R)$ by (5). Then for all $z \in D(R)$

$$
\log |f(z)| \leq w(z) \leq u(z) + M + \omega(R, z) \log M(R, D, f). \tag{8}
$$
Setting $z = z_1$ we obtain from (7)

$$A_0|z_1|^{\kappa_1} + M + 1 \leq \log |f(z_1)| \leq A_0|z_1|^{\kappa_1} + M + \omega(R, z_1) \log M(R, D, f),$$

implying that

$$1 \leq \omega(R, z_1) \log M(R, D, f).$$

If we now let $\Phi(t)$ be the angular measure of $D(R) \cap S(0, t)$, we have $\Phi(t) \leq \Phi_1(t)$, and by [14, Thm. 6.2, (6.4), p. 149 and p. 158] we get

$$\omega(R, z_1) \leq \frac{8}{\pi} \exp \left\{ -\pi \int_{R_1}^{R} \frac{dt}{t^{\Phi_1}(t)} \right\} \leq \frac{8}{\pi} \exp \left\{ -\pi \int_{R_1}^{R} \frac{dt}{t^{\Phi_1}(t)} \right\}.$$ (9)

We rearrange to obtain

$$\log M(R, D, f) \geq \frac{\pi}{8} \exp \left\{ \pi \int_{R_1}^{R} \frac{dt}{t^{\Phi_1}(t)} \right\}.$$ (10)

The second inequality in our conclusion follows from the fact that $\Phi(t) \leq 2\pi$.

### 2.2 Proof of Theorem 1.1

We now turn to the proof of Theorem 1.1. For each $j$, let $\Phi_j(t)$ be the angular measure of $D_j \cap S(0, t)$. We apply Lemma 2.1 on each $D_j$ to conclude that there exists $R_1(j) > 0$ such that for all $R > R_1(j)$,

$$\log M(R, D_j, f) \geq \frac{\pi}{8} \exp \left\{ \pi \int_{R_1(j)}^{R} \frac{dt}{t^{\Phi_j(t)}} \right\}.$$ (11)

Let $R_1 = \max\{R_1(j) : 1 \leq j \leq n\}$. Then for all $R > R_1$ we have

$$\log M(R, f) \geq \frac{\pi}{8} \exp \left\{ \pi \int_{R_1(j)}^{R} \frac{dt}{t^{\Phi_j(t)}} \right\}$$

for each $j$ with $1 \leq j \leq n$.

We have

$$n^2 \leq \left( \sum_{j=1}^{n} \frac{1}{\Phi_j(t)} \right) \left( \sum_{j=1}^{n} \Phi_j(t) \right) \leq 2\pi \sum_{j=1}^{n} \frac{1}{\Phi_j(t)}.$$

implying that

$$\frac{n^2}{2} \int_{R_1}^{R} \frac{dt}{t} \leq \pi \sum_{j=1}^{n} \int_{R_1}^{R} \frac{dt}{t^{\Phi_j(t)}}.$$
Thus for all $R > R_1$, there exists $j = j(R)$ such that

$$\frac{n}{2} \int_{R_1}^{R} \frac{dt}{t} \leq \pi \int_{R_1}^{R} \frac{dt}{t \Phi_j(t)}.$$ 

So for $R > R_1$, with $j = j(R)$ we have

$$\log M(R, f) \geq \frac{\pi}{8} \exp \left\{ \pi \int_{R_1}^{R} \frac{dt}{t \Phi_j(t)} \right\} \geq \frac{\pi}{8} \left( \frac{R}{R_1} \right)^{n/2},$$

establishing (4).

### 2.3 Proof of Corollary 1.2

We now prove Corollary 1.2. Let $D_j$ be the sector between $\gamma_j$ and $\gamma_{j+1}$ (with $\gamma_{n+1} = \gamma_1$). From elementary considerations we have

$$\limsup_{r \to \infty} \frac{\log M(r, f)}{r^{1/2}} > 0.$$ 

Trivially for each $R$ there exists $j = j(R)$ such that $\log M(R, f) = \log M(R, D_j, f)$. Thus there exists $j$ such that

$$\limsup_{z \to \infty} \frac{\log |f(z)|}{|z|^{1/2}} > 0.$$ 

We may thus apply Lemma 2.1 on this $D_j$ to conclude that there exists $R_1 = R_1(j) > 0$ such that for all $R > R_1$

$$\log M(R, f) \geq \log M(R, D_j, f) \geq \frac{\pi}{8} \exp \left\{ \pi \int_{R_1(j)}^{R} \frac{dt}{2\pi \ln^{-1}} \right\} = \frac{\pi}{8} \left( \frac{R}{R_1} \right)^{n/2}. $$

Remark

Our proofs in fact yield somewhat stronger versions of Theorem 1.1 and Corollary 1.2. The assumption that all of the asymptotic functions are distinct can be weakened. Both Theorem 1.1 and Corollary 1.2 depend on applications of Lemma 2.1, each application requiring only that the asymptotic functions associated with adjacent paths are distinct. Thus both results hold even if $f$ has fewer than $n$ distinct asymptotic functions on the $n$ disjoint paths, provided only that asymptotic functions associated with adjacent paths are distinct.

Similar considerations apply to the Denjoy–Carleman–Ahlfors Theorem and Fenton’s theorem, as an examination of the arguments in [2,7] shows. In fact, more can
be said in the case of asymptotic values. A path to infinity on which \( f \) tends to a constant \( a \), finite or infinite, corresponds to a singularity of \( f^{-1} \) lying over \( a \). More than one singularity can lie over a given \( a \). Certain singularities are classified as direct singularities. If \( f \) is meromorphic of order \( \rho \), Ahlfors [3] showed that \( f^{-1} \) has no more than \( \max\{2\rho, 1\} \) direct singularities. If \( f \) is entire, all singularities lying over infinity are direct. From this it follows that if \( f \) is entire of order \( \rho \), \( f \) has no more than \( 2\rho \) singularities lying over finite points, implying the Denjoy–Carleman–Ahlfors Theorem. For details, including a necessary modification of Ahlfors’s arguments, see [19, pp. 309–313] or [20, pp. 303–307].

2.4 Fenton’s theorem for polynomials \( a_j \)

We next turn to a proof, based on Theorem 1.1, of Fenton’s theorem in the case that all \( a_j \) are polynomials. We treat the situation where not all \( a_j \) are assumed to be distinct.

Suppose that \( f \) is entire and that \( F = \{\gamma_1, \gamma_2, \ldots, \gamma_m\} \) is a collection of paths from 0 to infinity, each with an associated polynomial \( a_j \) such that \( f \) is asymptotic to \( a_j \) on \( \gamma_j \). Suppose that two of these paths, say \( \gamma_j \) and \( \gamma_k \), intersect on a sequence of points tending to infinity. Then \( a_j - a_k \) tends to 0 on this sequence, implying that \( a_j \equiv a_k \); in this circumstance, we delete one of \( \gamma_j \) and \( \gamma_k \), obtaining a subcollection of \( F \). If two paths in this subcollection intersect on a sequence tending to infinity, we again delete one of them. We continue until we arrive at a subcollection \( F' \) with the property that there is a disk \( D \) centered at the origin containing all points of intersection of any two members of \( F' \). After renumbering, we write \( F' = \{\gamma_1, \gamma_2, \ldots, \gamma_q\} \). After an obvious modification of the paths on \( D \), we may suppose that the paths in \( F' \) are simple, segmental, disjoint except at the origin, and numbered in counterclockwise order. If any two adjacent paths in \( F' \), say \( \gamma_j \) and \( \gamma_{j+1} \) (with \( \gamma_{q+1} = \gamma_1 \)), are associated with the same asymptotic polynomial, we delete one of these paths. We continue this process until we obtain a subcollection \( F'' = \{\gamma_1, \gamma_2, \ldots, \gamma_n\} \) of disjoint simple segmental paths numbered in counterclockwise order such that the polynomials associated with any two adjacent paths are distinct.

Every collection \( F \) has such a subcollection \( F'' \). Our task is to show that \( f \) has growth at least order \( n/2 \), mean type, where \( n \) is the number of paths in such a subcollection \( F'' \). We note that if all the asymptotic polynomials associated with the paths in \( F \) are distinct, then \( F = F'' \), after obvious modifications of the paths on a disk \( D \). Similarly, \( F = F'' \) (after obvious modifications) if all the paths in \( F \) are disjoint outside some disk and the asymptotic polynomials associated with adjacent paths are distinct.

We adopt the notation of Theorem 1.1. We consider such a collection \( F'' \) and for \( 1 \leq j \leq n \) write

\[
a_j(z) = \sum_{p=0}^{d_j} b_{pj} z^p
\]

where \( d_j \) is the degree of \( a_j \).
Fix $j$ and assume, without loss of generality, that $d_j \geq d_{j+1}$. Replacing $f$ by $f - a_{j+1}$ and $a_j$ by $a_j - a_{j+1}$ without changing notation, we may assume $a_{j+1} \equiv 0$ and $a_j \not\equiv 0$. Our goal is to show that in $D_j$, the modified function $\log |f(z)|$ grows at least like $|z|^{1/2}$, clearly implying that our original $\log |f(z)|$ does as well.

If $a_j$ is constant, it follows from familiar arguments in the case of asymptotic values that $f$ is unbounded in $D_j$. On $\partial D_j$, we have $\log |f(z)| \leq M$ for some $M > 1$. For every $z \in \partial D_j$ with $|z| < R$, we have (8) with $u(z) = 0$. We note that

$$\omega(R, z) = O\left(\frac{|z|}{R}\right)^{1/2}$$

from (9). Letting $M_j(R) = M(R, D_j, f)$, we find from (8) that if there is a sequence of $R \to \infty$ on which $\log M_j(R) = o(R^{1/2})$, then $f$ is bounded in $D_j$, which is a contradiction. Thus we must have

$$\liminf_{R \to \infty} \frac{\log M_j(R)}{R^{1/2}} > 0.$$  (10)

Suppose now that $d_j \geq 1$. Let $\alpha$ be a value that $f$ takes infinitely often, say (at least) at distinct values $c_p$ for $1 \leq p \leq d_j$. Set

$$P(z) = \prod_{p=1}^{d_j} (z - c_p)$$

and

$$g(z) = \frac{f(z) - \alpha}{P(z)}.$$

Then $g$ is entire, and $g(z) \to 0$ as $z \to \infty$ along $\gamma_{j+1}$ (since $f(z) \to 0$). As $z \to \infty$ along $\gamma_j$, we have $f(z) - a_j(z) \to 0$, so that

$$g(z) = o(1) + \frac{a_j(z) - \alpha}{P(z)} = o(1) + \frac{a_j(z)}{P(z)} = o(1) + b_{d_j}$$

since $\deg a_j = \deg P = d_j$. Recall that $b_{d_j} \neq 0$. Applying the previous arguments to $g$ instead of $f$, we see that (10) holds if $M_j(R)$ refers to $g$, and hence also holds if $M_j(R)$ refers to $f$. The above works in all domains $D_j$, so that we may apply Theorem 1.1 to conclude that

$$\liminf_{R \to \infty} \frac{\log M(R, f)}{R^{n/2}} > 0.$$
3 An Example

Theorem 3.1 Let \( n \) be a positive integer. For \( 1 \leq j \leq n \), suppose that \( a_j \) is an entire function of growth no faster than order \( n \), minimal type. There exists an entire function \( f \) of order \( n \), mean type, such that each \( a_j \), for \( 1 \leq j \leq n \), is an asymptotic function of \( f \).

Our proof is an adaptation of a technique that Fuchs and Hayman ([13], see also [15, pp. 80–83]) used to assign deficiencies of an entire function arbitrarily subject only to the condition that the sum of the deficiencies does not exceed 1. We prove the following lemma.

Lemma 3.2 Let \( n \) be a positive integer. Let

\[ c_n = \int_0^\infty e^{-tn} \, dt. \]

Let the path \( \Gamma \) be the boundary of a sector of opening \( 2\pi/n \) given by

\[ \Gamma(t) = \begin{cases} -te^{-i\pi/n}, & -\infty < t < 0, \\ te^{i\pi/n}, & 0 \leq t < \infty. \end{cases} \]

Let \( \Omega_1 = \{ re^{i\theta} : r > 0 \text{ and } |\theta| < \pi/n \} \) be the inside of \( \Gamma \) and \( \Omega_2 = \{ re^{i\theta} : r > 0 \text{ and } \pi/n < \theta < 2\pi - \pi/n \} \) be the outside. Then there exists an entire function \( \varphi \) such that

\[ \varphi(z) = \begin{cases} e^{-z} - \left( \frac{c_n}{\pi} \sin \frac{\pi}{n} \right) z^{-1} + O(|z|^{-2}), & z \in \Omega_1, \\ \left( \frac{c_n}{\pi} \sin \frac{\pi}{n} \right) z^{-1} + O(|z|^{-2}), & z \in \Omega_2. \end{cases} \] (11)

uniformly as \( z \to \infty \).

Proof of Lemma 3.2. For \( z \in \mathbb{C}\setminus\Gamma \), define

\[ E(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{e^{w^n}}{w - z} \, dw. \]

We note that if \( w \in \Gamma \), then \( w^n = -|w|^n \) and thus \( E \) is analytic for \( z \in \mathbb{C}\setminus\Gamma \). Let

\[ I = \frac{1}{2\pi i} \int_{\Gamma} e^{w^n} \, dw = \frac{1}{2\pi i} \int_0^\infty e^{-t^n} \left( e^{i\pi/n} - e^{-i\pi/n} \right) \, dt = \frac{1}{\pi} c_n \sin \frac{\pi}{n}. \]

Note that

\[ \frac{1}{w - z} + \frac{1}{z} = \frac{w}{z(w - z)}. \]

Thus

\[ L := \frac{1}{2\pi i} \int_{\Gamma} \frac{we^{w^n}}{z(w - z)} \, dw = E(z) + \left( \frac{c_n}{\pi} \sin \frac{\pi}{n} \right) \frac{1}{z}. \] (12)

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Consider $z \notin \Gamma$ with $|z| = R$ for large $R > 0$. Let $\Gamma_1$ be the portion of $\Gamma$ with modulus at most $R/2$, $\Gamma_2$ the portion of $\Gamma$ with modulus between $R/2$ and $2R$, and $\Gamma_3$ the portion of $\Gamma$ with modulus at least $2R$. Let

$$d_n = \int_0^\infty t e^{-t^n} \, dt.$$ 

We have

$$\left| \frac{1}{2\pi i} \int_{\Gamma_1} \frac{w}{z(w-z)} e^{wn} \, dw \right| + \left| \frac{1}{2\pi i} \int_{\Gamma_3} \frac{w}{z(w-z)} e^{wn} \, dw \right| < \left( \frac{2}{\pi R^2} + \frac{1}{\pi R^2} \right) d_n. \quad (13)$$

If the distance from $z$ to $\Gamma_2$ is at least 1, we set $\Gamma_2^* = \Gamma_2$. If the distance from $z$ to $\Gamma_2$ is less than 1, we indent away from $z$ the segment of $\Gamma_2$ with distance less than 1 from $z$ to the arc of circle with distance at least 1 from $z$ to obtain a modified $\Gamma_2^*$. By Cauchy’s Theorem we have

$$\int_{\Gamma_2} \frac{w e^{wn}}{z(w-z)} \, dw = \int_{\Gamma_2^*} \frac{w e^{wn}}{z(w-z)} \, dw.$$ 

We conclude that

$$\left| \frac{1}{2\pi i} \int_{\Gamma_2} \frac{w e^{wn}}{z(w-z)} \, dw \right| = O \left( R e^{-(R/2)^n} \right) = O \left( \frac{1}{R^2} \right). \quad (14)$$

Combining (12), (13), and (14), we obtain

$$\left| E(z) + \left( \frac{c_n}{\pi} \sin \frac{\pi}{n} \right) \frac{1}{z} \right| = O \left( \frac{1}{R^2} \right) \quad (15)$$

for $z \notin \Gamma$.

Let $E_2(z)$ be $E(z)$ for $z \in \Omega_2$ and let $E_1(z)$ be $E(z)$ in $\Omega_1$. By replacing a short segment of $\Gamma$ by a circular arc lying in $\Omega_1$, we see from Cauchy’s Theorem that $E_2$ can be continued analytically across $\Gamma$ onto $\Omega_1$ and, by the Cauchy Integral Formula, that $E_2$ is entire and

$$E_2(z) = E_1(z) + e^{zn} \quad (16)$$

for all $z \in \Omega_1$. We set $\varphi(z) = E_2(z)$ and conclude from (15) and (16) that (11) holds. This proves Lemma 3.2.

We now turn to the proof of Theorem 3.1. Letting $\varphi$ be as in Lemma 3.2 for $1 \leq j \leq n$ we set

$$\varphi_j(z) = \varphi \left( e^{-2\pi i j/n} z \right).$$
Define
\[ f(z) = \sum_{j=1}^{n} \frac{\varphi_j(z) a_j(z)}{e^{zn}}. \] (17)

Clearly \( f \) is entire of order \( n \) mean type.

Let \( \gamma_j \) be the ray \( \{re^{2\pi ij/n}: r > 0\} \). Suppose that \( z \in \gamma_{j_0} \). Then \( e^{-2\pi ij_0/n}z > 0 \) and so lies in \( \Omega_1 \). Thus
\[ \varphi_{j_0}(z) = \varphi(e^{-2\pi ij_0/n}z) = e^{zn} + O(|z|^{-1}) = e^{|z|^n} + O(|z|^{-1}). \]

Hence
\[ \frac{\varphi_{j_0}(z) a_{j_0}(z)}{e^{zn}} - a_{j_0}(z) = \left( \frac{e^{zn} + O(|z|^{-1})}{e^{zn}} - 1 \right) a_{j_0}(z) = O(|z|^{-1})a_{j_0}(z) = e^{|z|^n}O(|z|^{-1}). \]

implying that
\[ \lim_{z \to \infty} \left( \frac{\varphi_{j_0}(z) a_{j_0}(z)}{e^{zn}} - a_{j_0}(z) \right) = 0. \] (18)

We now consider \( z \in \gamma_{j_0} \) and \( j \neq j_0 \). We note that \( e^{-2\pi ij/n}z \in \Omega_2 \). By (11),
\[ |\varphi_j(z)| = |\varphi(e^{-2\pi ij/n}z)| = O(|z|^{-1}). \]

Consequently, for \( z \in \gamma_{j_0} \),
\[ \left| \sum_{\substack{j=1 \atop j \neq j_0}}^{n} \varphi_j(z) a_j(z) e^{zn} \right| \leq \left( \sum_{j=1}^{n} |a_j(z)| \right) O(|z|^{-1}) e^{|z|^n} = o(1). \] (19)

The combination of (17), (18), and (19) completes the proof of Theorem 3.1.

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