ON THE EXTENSION OF THE ERDŐS–MORDELL TYPE INEQUALITIES

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Abstract. We discuss the extension of inequality $R_A \geq \frac{1}{2} r_a + \frac{1}{2} r_c$ to the plane of triangle $\triangle ABC$. Based on the obtained extension, in regard to all three vertices of the triangle, we get the extension of Erdős-Mordell inequality, and some inequalities of Erdős-Mordell type.

1. Introduction

Let triangle $\triangle ABC$ be given in Euclidean plane. Denote by $R_A, R_B$ and $R_C$ the distances from the arbitrary point $M$ in the interior of $\triangle ABC$ to the vertices $A, B$ and $C$ respectively, and denote by $r_a, r_b$ and $r_c$ the distances from the point $M$ to the sides $BC, CA$ and $AB$ respectively (Figure 1).

![Figure 1: Erdős-Mordell inequality](image)

Then Erdős-Mordell inequality is true:

$$R_A + R_B + R_C \geq 2(r_a + r_b + r_c)$$

(1)

whereat equality holds if and only if triangle $ABC$ is equilateral and $M$ is its center. This inequality was conjectured by P. Erdős as Amer. Math. Monthly Problem 3740 in 1935. [9], after his experimental conjecture in 1932. [13]. It was proved by L.J. Mordell in 1935, (in Hungarian, according to [13]), and as the solution of the Problem 3740 in 1937. [22].
Considering the Erdös-Mordell inequality (1) the goal of this research is to determine areas in the plane of the triangle, where the following three inequalities are valid:

\[ R_A \geq \frac{c}{a} r_b + \frac{b}{a} r_c \]  
\[ R_B \geq \frac{c}{b} r_a + \frac{a}{b} r_c \]  
\[ R_C \geq \frac{b}{c} r_a + \frac{a}{c} r_b \]

where \( a = |BC|, \ b = |CA|, \ c = |AB| \).

In this paper we determine a set of points \( E \) for which

\[ R_A + R_B + R_C \geq \left( \frac{c}{b} + \frac{b}{c} \right) r_a + \left( \frac{c}{a} + \frac{a}{c} \right) r_b + \left( \frac{a}{b} + \frac{b}{a} \right) r_c \]

is valid. It is known that the triangular area of \( \triangle ABC \) is contained in the set \( E \) \( \{3\}, \{4\}, \{11\}, \{13\}, \{14\}, \{26\} \). Here we show that the set \( E \) is greater than the triangle \( \triangle ABC \), and we give a geometric interpretation of the set \( E \).

The proofs of Erdös-Mordell inequality are often based on different proofs of inequality (2), as given in \( \{4\}, \{6\}, \{7\}, \{11\}, \{12\}, \{23\}, \{26\} \). N. Derigades in \( \{8\} \) proved the inequality (5) valid in the whole plane of the triangle, where \( r_a, r_b \) and \( r_c \), are signed distances. A similar result was given by B. Malešević \( \{20\}, \{21\} \).

Note that V. Pambuccian \( \{24\} \) recently proved that the Erdös-Mordell inequality is equivalent to non-positive curvature. Overview of recent results on Erdös-Mordell inequalities and related inequalities is given in \( \{1\} - \{3\}, \{5\}, \{8\}, \{10\}, \{13\} - \{21\}, \{24\}, \{25\}, \{27\} - \{30\} \).

2. The Main Results

In this section we analyze only the inequality (2). Let \( \triangle ABC \) be a triangle with vertices \( A(0,r), B(p,0), C(q,0) \), \( p \neq q, r \neq 0 \). Without diminishing generality, let \( p < q \). We denote by \( M(x,y) \) an arbitrary point in the plane of the triangle \( \triangle ABC \). The distance from the point \( M \) to the point \( A \), and the distance from the point \( M \) to the straight lines \( b \) and \( c \) are given by functions:

\[ R_A = \sqrt{x^2 + (y - r)^2} \]  
\[ r_b = \frac{|-qy - rx + qr|}{\sqrt{r^2 + q^2}} \]  
\[ r_c = \frac{|py + rx - pr|}{\sqrt{r^2 + p^2}} \]

respectively. Consider the inequality (2) related to the vertex \( A \). The analytical notation of this inequality is:

\[ \sqrt{x^2 + (y - r)^2} \geq \frac{\sqrt{r^2 + p^2}}{|q-p|} \cdot \frac{|-qy - rx + qr|}{\sqrt{r^2 + q^2}} + \frac{\sqrt{r^2 + q^2}}{|q-p|} \cdot \frac{|py + rx - pr|}{\sqrt{r^2 + p^2}}, \]
\[ |q - p| \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \sqrt{x^2 + (y - r)^2} \geq (r^2 + p^2) |qy - rx + qr| + (r^2 + q^2) |py + rx - pr|. \] (10)

Let \( y = kx + r, k \in \mathbb{R}, \) then the inequality (10) reads as follows:

\[ |x| |q - p| \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \sqrt{1 + k^2} \geq |x| \left( (r^2 + p^2) |qk - r| + (r^2 + q^2) |pk + r| \right) \] (11)

For \( x = 0, \) the previous inequality is reduced to an equality which solution is the point \( A(0, r). \) For \( x \neq 0 \) we obtain inequality by a single variable \( k: \)

\[ |q - p| \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \sqrt{1 + k^2} \geq (r^2 + p^2) |qk - r| + (r^2 + q^2) |pk + r|. \] (12)

Solution of the inequality (12) reduces to four cases per parameter \( k: \)

\begin{align*}
(\alpha_1) : & \begin{cases} 
  pk + r \geq 0 \\
  -qk - r \geq 0,
\end{cases} \\
(\alpha_2) : & \begin{cases} 
  pk + r < 0 \\
  -qk - r \geq 0,
\end{cases} \\
(\alpha_3) : & \begin{cases} 
  pk + r \geq 0 \\
  -qk - r < 0,
\end{cases} \\
(\alpha_4) : & \begin{cases} 
  pk + r < 0 \\
  -qk - r < 0.
\end{cases}
\end{align*} (13-16)

Note that the value \( k \) corresponds to the points \( (x, y) \in \mathbb{R}^2 \) located on the straight line \( y = kx + r. \) With its values, the mentioned parameter of the line \( y = kx + r \) decomposes \( \mathbb{R}^2 \) on four corner areas. Inquiring the existence of parameter \( k \) (i.e. the pencil of lines \( y = kx + r \) through the vertex \( A) \) depending on the signs of parameters \( p, q \) and \( r, \) we provide the following table of existing corner areas (\( \alpha_1 \) - (\( \alpha_4 \)):

|   | \( p \) | \( q \) | \( r \) | \( (\alpha_1) \) | \( (\alpha_2) \) | \( (\alpha_3) \) | \( (\alpha_4) \) |
|---|---|---|---|---|---|---|---|
| 1. | >0 | >0 | >0 | + | + | + | - |
| 2. | <0 | >0 | >0 | + | - | + | + |
| 3. | <0 | <0 | >0 | - | + | + | + |
| 4. | >0 | >0 | <0 | - | + | + | + |
| 5. | <0 | >0 | <0 | + | + | - | + |
| 6. | <0 | <0 | <0 | + | + | + | - |
| 7. | =0 | >0 | >0 | + | - | + | - |
| 8. | =0 | >0 | <0 | - | + | - | + |
| 9. | <0 | =0 | >0 | - | - | + | + |
| 10. | <0 | =0 | <0 | + | + | - | - |

**Table 1:** The existence of the corner area depending on the parameters \( p, q \) and \( r \)
The corner areas ($\alpha_1$) and ($\alpha_4$) are always in the interior of $\angle BAC$ and its cross angle, while the areas ($\alpha_2$) and ($\alpha_3$) are in the interior of its supplementary angle (Figure 2).

Let us consider the equation:

\[(q - p) \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \sqrt{1+k^2} = \pm (qk + r) \mp (r^2 + p^2) \pm (r^2 + q^2) \pm (r^2 + q^2) |pk + r|. \quad (17)\]

I) Let $k$ fulfill ($\alpha_1$) or ($\alpha_4$). Then the previous equation can be rewritten in a way that follows, with positive sign (+) in the case of area ($\alpha_1$) and negative sign (−) in the case of area ($\alpha_4$)

\[(q - p) \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \sqrt{1+k^2} = \pm ((qk - r)(r^2 + p^2) + (pk + r)(r^2 + q^2)) \quad (18)\]

i.e.

\[(q - p) \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \sqrt{1+k^2} = \pm (q - p)(r(q+p) + k(pq - r^2)) \quad (19)\]

abbreviated written as
\[
\lambda \sqrt{1+k^2} = \pm \beta k \pm \gamma = \begin{cases} 
\beta k + \gamma, & k \in (\alpha_1) \\
-\beta k - \gamma, & k \in (\alpha_4)
\end{cases}
\] (20)

where at:
\[
\lambda = (q-p) \sqrt{r^2+p^2} \sqrt{r^2+q^2} \quad \text{and} \quad \lambda > 0
\] (21)
\[
\beta = (pq-r^2)(q-p)
\] (22)
\[
\gamma = r(q^2-p^2).
\] (23)

As \( p < q \), the equation (19) can be divided by \( q-p \neq 0 \) and then squared:
\[
(r^2+p^2)(r^2+q^2)(1+k^2) = (r(q+p)+k(pq-r^2))^2
\] (24)

which transforms into
\[
(r(p+q)k-(pq-r^2))^2 = 0.
\] (25)

Based on the above equation, we conclude that there exists the unique solution:
\[
k_1 = \frac{pq-r^2}{r(p+q)}
\] (26)

only if, for \( k = k_1 \):
\[
\pm \beta k \pm \gamma \geq 0
\] (27)
is valid.

Hence, the straight line \( y = k_1x + r \) is in the interior of \( \angle BAC \) and its cross angle, or it doesn’t exist. The cases where values \( k_1 \) from the formula (26) does not meet the condition (27) are presented in the Table 1 with:

- in the case 1: \( k_1 > r/q \iff p(q^2+r^2) > 0 \);
- in the case 3: \( k_1 > r/p \iff (-q)(p^2+r^2) > 0 \);
- in the case 4: \( k_1 < -r/p \iff p(q^2+r^2) > 0 \);
- in the case 6: \( k_1 < -r/p \iff (-q)(p^2+r^2) > 0 \).

**Lemma 1.** For \( k \in (\alpha_1) \cup (\alpha_4) \) inequality (12) is valid, where equality holds for \( k = k_1 \) if (27) is fulfilled.

**Proof.** (12) \( \iff (r(p+q)k-(pq-r^2))^2 \geq 0 \). \( \square \)

**Corollary 1.** Inequality (12) is valid for lines \( b \) and \( c \).

**II** Let \( k \) fulfill \( (\alpha_2) \) or \( (\alpha_3) \). Then equation (17) can be rewritten in a way that follows, with negative sign \((-\)\) in the case of area \( (\alpha_2) \) and positive sign \((+)\) in the case of area \( (\alpha_3) \)
\[
(q-p) \sqrt{r^2+p^2} \sqrt{r^2+q^2} \sqrt{1+k^2} = \pm((qk+r)(r^2+p^2)+(pk+r)(r^2+q^2))
\] (28)
or abbreviated written as

\[ \lambda \sqrt{1 + k^2} = \pm \delta k \pm \varepsilon = \begin{cases} \delta k + \varepsilon, & k \in (\alpha_3) \\ -\delta k - \varepsilon, & k \in (\alpha_2) \end{cases} \]  

(29)

with parameters:

\[ \lambda = (q - p) \sqrt{r^2 + p^2} \sqrt{r^2 + q^2} \text{ and } \lambda > 0 \]

\[ \delta = (r^2 + pq)(q + p) \]

\[ \varepsilon = r(2r^2 + q^2 + p^2) \]  

The equation (29) is considered under the following condition:

\[ \pm \delta k \pm \varepsilon \geq 0. \]  

(32)

By squaring the equation (29) we obtain

\[ P(k) = \lambda^2 (1 + k^2) - (\pm \delta k \pm \varepsilon)^2 = (\lambda^2 - \delta^2) k^2 - 2\delta \varepsilon k + (\lambda^2 - \varepsilon^2) = 0. \]  

(33)

For the square trinomial

\[ P(k) = \hat{A}k^2 + \hat{B}k + \hat{C} \]

coefficients \( \hat{A}, \hat{B}, \hat{C} \) are determined by:

\[ \hat{A} = \lambda^2 - \delta^2 = (q - p)^2(r^2 + p^2)(r^2 + q^2) - (r^2 + pq)^2(q + p)^2 \]

(35)

\[ \hat{B} = -2\delta \varepsilon = -2r (r^2 + pq)(q + p)(2r^2 + q^2 + p^2) \]

(36)

\[ \hat{C} = \lambda^2 - \varepsilon^2 = (r^2 + pq)((pq - r^2)(q - p)^2 - 2r(2r^2 + q^2 + p^2)) \].

(37)

Let us consider the equation:

\[ \hat{A} = -4pq r^4 + (p^4 + q^4 - 4pq^3 - 4p^3 q - 2p^2 q^2)r^2 - 4p^3 q^3 = 0. \]

(38)

It has real solutions for \( r \) in the following form:

\[ \begin{cases} r_{1,2} = \frac{1}{4\sqrt{pq}} \left( (q - p)^2 \pm \sqrt{(q - p)^4 - 16p^2 q^2} \right) > 0 \\ r_{3,4} = -\frac{1}{4\sqrt{pq}} \left( (q - p)^2 \pm \sqrt{(q - p)^4 - 16p^2 q^2} \right) < 0 \end{cases} \]

(39)

iff

\[ \left( p \geq 0 \land q \geq (3 + 2\sqrt{2})p \right) \lor \left( p < 0 \land q \leq (3 - 2\sqrt{2})p \right). \]

(40)

**Remark 1.** When \( p < 0 \) and \( q > 0 \) then \( \hat{A} = 4|p|qr^4 + (q^2 - p^2)^2 r^2 + 4|p|q \left( p^2 + q^2 \right)^2 r^2 + 4|p|^3 q^3 > 0 \) is valid. Note that the equation \( \hat{A} = 0 \) is not considered for \( p = 0 \) or \( q = 0 \) (because we obtain the contradictions: \( p = 0, q \neq 0: \hat{A} = r^2 q^4 = 0 \implies r = 0 \); i.e. \( p \neq 0, q = 0: \hat{A} = r^2 p^4 = 0 \implies r = 0 \)).
We distinguish the cases:

**a)** Let \( r = r_j \) for some \( j = 1, 2, 3, 4 \), then \( \hat{A} = 0 \). In this case, \( \hat{B} \neq 0 \), because \( r^2 + pq \neq 0 \) and \( q + p \neq 0 \) (in the case of equilateral triangle, there will be valid \( q + p = 0 \) and then \( r = \pm pi, i = \sqrt{-1} \)). Therefore, by solving the linear equation \( \hat{B} k + \hat{C} = 0 \) we find that:

\[
k_2 = \frac{-\hat{C}}{\hat{B}} = \frac{\lambda^2 - \varepsilon^2}{2\delta \varepsilon} = \frac{(q-p)^2 (r^2+p^2) (r^2+q^2) - r^2 (2r^2+q^2+p^2)^2}{2r(q+p)(2r^2+q^2+p^2)}.
\]

For \( p < 0 \) and \( q > 0 \) the case \( a) \) is not considered (because \( \hat{A} > 0 \)). Let us examine when the value \( k_2 \) meet the condition (32). It is valid that:

\[
\pm \delta k_2 + \varepsilon \geq 0 \iff \pm (\delta k_2 + \varepsilon) = \pm \left( \frac{\lambda^2 - \varepsilon^2}{2\delta \varepsilon} + \varepsilon \right) = \pm \left( \frac{\lambda^2 + \varepsilon^2}{2\varepsilon} \right) \\
\geq 0.
\]

Based on \( \varepsilon = r(2r^2+q^2+p^2) \) we conclude:

- if \( r > 0 \) then \( \delta k_2 + \varepsilon \geq 0 \) is fulfilled, whereby \( k_2 \) fulfills condition (32) and \( k_2 \in (\alpha_3) \);
- if \( r < 0 \) then \( -\delta k_2 - \varepsilon \geq 0 \) is fulfilled, whereby \( k_2 \) fulfills condition (32) and \( k_2 \in (\alpha_2) \).

In this case, the line \( y = k_2 x + r \) is in the exterior of \( \angle BAC \) and its cross angle.

**b)** Let \( r \neq r_j \) for each \( j = 1, 2, 3, 4 \), then \( \hat{A} \neq 0 \) and in this case, by solving the quadratic equation (33), we find the values:

\[
k_{2,3} = -\frac{\delta \varepsilon \pm \sqrt{\hat{\lambda}^2 (\delta^2 + \varepsilon^2 - \lambda^2)}}{\delta^2 - \hat{\lambda}^2} = \frac{r(p + q)(r^2+pq)(q^2+p^2+2r^2) \pm 2 (r^2+p^2) (r^2+q^2) (q-p) \sqrt{r^2+pq}}{(q-p)^2 (r^2+p^2) (r^2+q^2) - (r+pq)^2 (q+p)^2}.
\]

If \( r^2+pq \geq 0 \) then exists \( k_{2,3} \in \mathbb{R} \). Incidence of \( k_{2,3} \in \mathbb{R} \) to the area \( (\alpha_3) \), as to the area \( (\alpha_2) \) is determined by the inequality (32). The expression \( \delta k_{2,3} + \varepsilon \) exists for \( \delta \neq \pm \lambda \), whereby the expression \( \delta k_{2,3} + \varepsilon \) is either positive or negative (because \( \delta k_{2,3} + \varepsilon = 0 \iff \delta = \pm \lambda \)).

Based on the Corollary 1, the straight lines \( y = k_s x + r, (s = 2, 3) \) are in the exterior of \( \angle BAC \) and its cross angle (Figure 3).

Consider the limiting case for \( k_{2,3} \) when \( r \to r_j \). Note that \( \hat{A} = \lambda^2 - \delta^2 \to 0 \) is valid, whereat from

\[
k_{2,3} = \frac{-\varepsilon}{(\delta - \lambda) (\delta + \lambda)} \cdot \left( \delta \pm \frac{|\lambda| \sqrt{1 + \frac{\delta^2 - \lambda^2}{\varepsilon^2}}}{\sqrt{1 + \frac{\delta^2 - \lambda^2}{\varepsilon^2}}} \right)
\]

follows

\[
\lim_{r \to r_j} k_2 = \frac{-\varepsilon}{(\delta + \lambda)} \quad \land \quad \lim_{r \to r_j} k_3 = \infty.
\]
Figure 3: The existence of the lines $y = k_{s}x + r$, $(s = 2, 3)$ depending on the parameter $\hat{A}$

Related to the $\angle BAC$ we distinguish the cases:

1. $\angle BAC < \pi/2 \iff r^2 + pq > 0$ and if $\hat{A} \neq 0$ then there are two real and different values of $k_2$ and $k_3$. In this case, the following lemma is valid:

**Lemma 2.** For $\angle BAC < \pi/2$, $k \in (\alpha_2) \cup (\alpha_3)$ the inequality (12) is valid, just in the cases:

1. $\hat{A} > 0 \land k \in [-\infty, k_2] \cup [k_3, +\infty] \setminus ((\alpha_1) \cup (\alpha_4))$;
2. $\hat{A} = 0 \land k \in [-\infty, k_2] \setminus ((\alpha_1) \cup (\alpha_4))$;
3. $\hat{A} < 0 \land k \in [k_2, k_3] \setminus ((\alpha_1) \cup (\alpha_4))$;

where the equality holds for $k = k_2$ or $k = k_3$.

2. If $\angle BAC = \pi/2 \iff r^2 + pq = 0$ then $\hat{A} = -qp(q - p)^4$, $\hat{B} = 0$ and $\hat{C} = 0$, according to the equation (42) that $k_{2,3} = 0$. Hence is valid:

**Lemma 3.** For $\angle BAC = \pi/2$ and $k \in (\alpha_2) \cup (\alpha_3)$ the inequality (12) is valid. The equality is valid only for $k = 0$.

Proof. (12) $\iff \hat{A}k^2 + \hat{B}k + \hat{C} \geq 0 \iff -qp(q - p)^4 k^2 \geq 0$. $\blacksquare$
3. \(<\angle BAC > \pi/2 \iff r^2 + pq < 0\). In this case, for: \(r^2 < -pq\) and for the coefficient \(\hat{A}\):
\[
\hat{A} > 4r^6 + (p^4 + q^4) r^2 + 4 (p^2 + q^2) r^4 - 2r^6 + 4p^2q^2 r^2
\]
\[
= 2r^6 + 4 (p^2 + q^2) r^4 + (p^4 + q^4 + 4p^2q^2) r^2 > 0
\]
is valid. Since \(k_{2,3} \in \mathbb{C}\) and \(\hat{A} > 0\) the inequality (12) is valid, which proves the claim:

**Lemma 4.** For \(<\angle BAC > \pi/2\) and \(k \in (\alpha_2) \cup (\alpha_3)\) the inequality (12) is valid in the strict form.

Based on the previous considerations in I) and II), follows:

**Statement 1.** The inequality (12) holds in following cases:
\(k \in (\alpha_1) \cup (\alpha_4)\)

or
\(k \in (\alpha_2) \cup (\alpha_3)\) for \(<\angle BAC \geq \pi/2\)

i.e.
\[
k \in [-\infty, k_2] \cup [k_3, +\infty] \setminus ((\alpha_1) \cup (\alpha_4)) \land \hat{A} > 0
\]
\[
k \in [-\infty, k_2] \setminus ((\alpha_1) \cup (\alpha_4)) \land \hat{A} = 0
\]
\[
k \in [k_2, k_3] \setminus ((\alpha_1) \cup (\alpha_4)) \land \hat{A} < 0,
\]

for \(<\angle BAC < \pi/2\).

3. Conclusion

For the vertex \(A\), let us define
\[
E_A = \left\{(x, y) \mid R_A \geq \frac{c}{a} r_b + \frac{b}{a} r_c\right\},
\]
and for the vertices \(B\) and \(C\), let us define
\[
E_B = \left\{(x, y) \mid R_B \geq \frac{c}{b} r_a + \frac{a}{b} r_c\right\},
\]
\[
E_C = \left\{(x, y) \mid R_C \geq \frac{b}{c} r_a + \frac{a}{c} r_b\right\},
\]
respectively. Based on the analysis of the inequalities (2), (3) and (4), the inequality (5) is valid in the intersection of the areas:
\[
E = E_A \cap E_B \cap E_C.
\]

(43)

Therefore follows

**Statement 2.** Erdős-Mordell inequality is valid in the area \(E\).

Let us define the set \(M\) by the intersection of the corner areas formed from \(E_A\), \(E_B\) and \(E_C\), containing the initial triangle. Then the set of points \(M\) is quadrilateral or hexagonal shape, and is contained the area \(E\) (Figure 4).
Figure 4: Extension of the triangle ABC to the area $M \subset E$

Let us define Erdős-Mordell curve in the plane of triangle, by the following equation:

$$R_A + R_B + R_C = 2(r_a + r_b + r_c), \quad (44)$$

where

$$R_A = \sqrt{x^2 + (y - r)^2}, \quad R_B = \sqrt{(x - p)^2 + y^2}, \quad R_C = \sqrt{(x - q)^2 + y^2},$$

$$r_a = \frac{|y(q - p)|}{\sqrt{(q - p)^2}} = |y|, \quad r_b = \frac{|-q(y - r) - rx|}{\sqrt{r^2 + q^2}}, \quad r_c = \frac{|-p(y - r) - rx|}{\sqrt{r^2 + p^2}}.$$

The curve (44) is a union of parts of algebraic curves of order eight (Figure 5).

Figure 5: Erdős-Mordell curve and the area $E$
Let us denote by $E'$ the part of the plane $\mathbb{R}^2$ bounded by the Erdős-Mordell’s curve and consisting the triangle $\triangle ABC$. Thus, according to the fact that inequality (5) is valid in the area of the triangle $\triangle ABC$, and based on continuity, it follows that inequality (5) is valid in the area $E'$. Remark that the area $E'$ allows us to precise when, except for the inequality (5), some of the inequalities (2), (3) and/or (4) are true. For example, in the area $(E' \setminus E_A) \cap E_B \cap E_C$ the inequalities (5), (4), (3) are true and (2) is not true. At end of this section let us emphasize that the following statement is true.

**STATEMENT 3.** All geometric inequalities based on the inequalities (2), (3) and (4) can be extended from the triangle interior to the area $E'$.

**EXAMPLE 1.** In the area $E$, the inequality of Child [7] is valid:

$$R_A \cdot R_B \cdot R_C \geq 8 \cdot r_a \cdot r_b \cdot r_c$$  \hspace{1cm} (45)

because, based on inequality between arithmetic and geometric mean, follows:

$$a \cdot R_A \geq b \cdot r_c + c \cdot r_b \geq 2\sqrt{b \cdot c \cdot r_b \cdot r_c}$$  \hspace{1cm} (46)

$$b \cdot R_B \geq c \cdot r_a + a \cdot r_c \geq 2\sqrt{c \cdot a \cdot r_c \cdot r_a}$$  \hspace{1cm} (47)

$$c \cdot R_C \geq a \cdot r_b + b \cdot r_a \geq 2\sqrt{a \cdot b \cdot r_a \cdot r_b}.$$  \hspace{1cm} (48)

Hence, by multiplying the left and right sides of inequalities (46) - (48), we get the inequality (45) in the area $E$.  \hfill $\blacksquare$

At the end of this paper, let us set up an open problem (proposed by anonymous reviewer): prove or disprove that there exist a positive number $\varepsilon$ such that the area of $E'$ is bigger than $1+\varepsilon$ times the area of the triangle for every triangle. Thus, we set a conjecture: for the finite area of $E'$ the value $\varepsilon$ is determined in the case of equilateral triangle.

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