q-Fourier Transform and its inversion-problem

M. C. Rocca$^{1,2}$ and A. Plastino$^1$

$^1$ Instituto de Física (IFLP-CCT-Conicet)
Universidad Nacional de La Plata (UNLP)
C.C. 67 (1900) La Plata, Argentina

$^2$ Departamento de Física
Fac. de Ciencias Exactas, UNLP

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Abstract

Tsallis’ q-Fourier transform is not generally one-to-one. It is shown here that, if we eliminate the requirement that $q$ be fixed, and let it instead “float”, a simple extension of the $F_q$—definition, this procedure
restores the one-to-one character.

KEYWORDS: q-Fourier transform, generalization, one to one character, statistical mechanics, nonextensive statistical mechanics.
1 Introduction

Nonextensive statistical mechanics (NEXT) [1, 2, 3], a current generalization of the Boltzmann-Gibbs (BG) one, is actively studied in diverse areas of Science. NEXT is based on a nonadditive (though extensive [4]) entropic information measure characterized by the real index $q$ (with $q = 1$ recovering the standard BG entropy). It has been applied to variegated systems such as cold atoms in dissipative optical lattices [5], dusty plasmas [6], trapped ions [7], spin glasses [8], turbulence in the heliosheath [9], self-organized criticality [10], high-energy experiments at LHC/CMS/CERN [11] and RHIC/PHENIX/Brookhaven [12], low-dimensional dissipative maps [13], finance [14], galaxies [15], Fokker-Planck equation’s applications [16], etc.

NEXT can be advantageously expressed via $q$-generalizations of standard mathematical concepts (the logarithm and exponential functions, addition and multiplication, Fourier transform (FT) and the Central Limit Theorem (CLT) [17, 22, 25]). The $q$-Fourier transform $F_q$ exhibits the nice property of transforming $q$-Gaussians into $q$-Gaussians [17]. Recently, plane waves, and the representation of the Dirac delta into plane waves have been also generalized [18, 19, 21, 22].

A serious problem afflicts $F_q$. It is not generally one-to-one. A detailed
example is discussed below. In this work we show that by recourse to a rather simple but efficient stratagem that consists in

- eliminating the requirement that $q$ be fixed and instead
- let it “float”,

one restores the one-to-one character.

2 Generalizing the q-Fourier transform

We define, following [17], a q-Fourier transform of $f(x) \in L^1(\mathbb{R})$, $f(x) \geq 0$ as

$$F(k, q) = [H(q - 1) - H(q - 2)] \times 
\int_{-\infty}^{\infty} f(x)\{1 + i(1 - q)kx[f(x)]^{(q-1)}\} \frac{1}{1-q} \, dx$$

(2.1)

where $H(x)$ is the Heaviside step function.

The only difference between this definition and that given in [17] is that $q$ is not fixed and varies within the interval $[1, 2)$. Herein lies the hard-core of our presentation. This simple change of perspective makes it is easy to find the inversion-formula for (2.1) by recourse to the inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \lim_{\epsilon \to 0^+} \int_{1}^{2} \frac{1}{F(k, q)\delta(q - 1 - \epsilon) \, dq} \right] e^{-ikx} \, dk.$$  (2.2)
As a consequence, we see that this q-Fourier transform is one-to-one, unlike what happens in [23], [24]. In the next section we give an illustrative example.

3 Example

As an illustration we discuss the example given by Hilhorst in Ref. ([22]).

Let $f(x)$ be

$$
f(x) = \begin{cases} 
(\frac{1}{x})^\beta; & x \in [a, b]; \ 0 < a < b; \ \lambda > 0 \\
0; & x \text{ outside } [a, b].
\end{cases}
$$

(3.1)

The corresponding q-Fourier transform is

$$
F(k, q) = \lambda^\beta \int_a^b x^{-\beta} \left\{1 + i(1 - q)k\lambda^{\beta(q-1)}x^{1-\beta(q-1)}\right\} \frac{1}{1-q} \, dx.
$$

(3.2)

Effecting the change of variables

$$
y = x^{1-\beta(q-1)},
$$

we have for (3.2)

$$
F(k, q) = [H(q - 1) - H(q - 2)] \times \\
\frac{\lambda^\beta}{1 - \beta(q - 1)} \int_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-q}} \left\{1 + i(1 - q)k\lambda^{\beta(q-1)}y\right\} \frac{1}{1-q} \, dy.
$$

(3.3)
Now, (3.3) can be rewritten in the useful form

\[
F(k, q) = [H(q - 1) - H(q - 2)] \times \\
\left\{ \left\{ H(q - 1) - H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] \right\} \times \\
\frac{\chi^{\beta}}{1 - \beta(q - 1)} \int_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{\frac{\beta(2-q)}{1-\beta(q-1)}} \left\{ 1 + i(1 - q)k\lambda^{\beta(q-1)}y \right\}^{\frac{1}{1-q}} dy + \\
\left\{ H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] - H(q - 2) \right\} \times \\
\frac{\chi^{\beta}}{\beta(q - 1) - 1} \int_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}} \left\{ 1 + i(1 - q)k\lambda^{\beta(q-1)}y \right\}^{\frac{1}{1-q}} dy \right\}.
\]

Taking into account that the involved integrals are defined in a finite interval, we can cast (3.4) as

\[
F(k, q) = [H(q - 1) - H(q - 2)] \times \\
\left\{ \left\{ H(q - 1) - H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] \right\} \times \\
\frac{\chi^{\beta}}{1 - \beta(q - 1)} \lim_{\epsilon \to 0^+} \int_{a^{1-\beta(q-1)}}^{b^{1-\beta(q-1)}} y^{\frac{\beta(2-q)}{1-\beta(q-1)}} \left\{ 1 + i(1 - q)(k + i\epsilon)\lambda^{\beta(q-1)}y \right\}^{\frac{1}{1-q}} dy + \\
\left\{ H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] - H(q - 2) \right\} \times \\
\frac{\chi^{\beta}}{\beta(q - 1) - 1} \lim_{\epsilon \to 0^+} \int_{b^{1-\beta(q-1)}}^{a^{1-\beta(q-1)}} y^{\frac{\beta(q-2)}{1-\beta(q-1)}} \left\{ 1 + i(1 - q)(k + i\epsilon)\lambda^{\beta(q-1)}y \right\}^{\frac{1}{1-q}} dy \right\}.
\]

(3.5)
We now use results of the Integral's table [26] to evaluate (3.5) and get

\[
\lim_{\epsilon \to 0^+} \int_{\alpha^{1-\beta(q-1)}}^{\infty} y^{-\frac{\beta(2-q)}{1-\beta(q-1)}} \left\{ 1 + i(1 - q)(k + i\epsilon)\lambda^{\beta(q-1)}y \right\}^{\frac{1}{1-\beta}} dy =
\]

\[
\frac{(q - 1)[1 - \beta(q - 1)]\alpha^{\frac{2-q}{q-1}}}{(2-q)[(1-q)i(k+i0)\lambda^{\beta}]^{\frac{1}{q-1}}} \times
\]

\[
F \left( \frac{1}{q - 1}, \frac{2-q}{(q - 1)[1 - \beta(q - 1)]}, \frac{1}{q - 1} + \frac{\beta(2-q)}{1 - \beta(q - 1)} \right) - \frac{1}{(1-q)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}}
\]

\[
F \left( \frac{1}{q - 1}, \frac{\beta - 1}{\beta(q - 1) - 1}, \frac{\beta q - 2}{\beta(q - 1) - 1} \right) - (q - 1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}
\]

and

\[
\lim_{\epsilon \to 0^+} \int_{0}^{\infty} y^{\frac{\beta(2-q)}{q-1-1}} \left\{ 1 + i(1 - q)(k + i\epsilon)\lambda^{\beta(q-1)}y \right\}^{\frac{1}{1-\beta}} dy =
\]

\[
\frac{[\beta(q - 1) - 1]a^{1-\beta}}{\beta - 1} \times
\]

\[
F \left( \frac{1}{q - 1}, \frac{\beta - 1}{\beta(q - 1) - 1}, \frac{\beta q - 2}{\beta(q - 1) - 1} \right) - (q - 1)i(k+i0)\lambda^{\beta(q-1)}a^{1-\beta(q-1)}
\]

where \( F(a, b, c; z) \) is the hypergeometric function. Thus we obtain for \( F(k, q) \)

\[
F(k, q) = [H(q - 1) - H(q - 2)] \times
\]

\[
\left\{ \left\{ H(q - 1) - H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] \right\} \right\} \times
\]

\[
\frac{(q - 1)\lambda^{\beta}}{(2-q)[(1-q)i(k+i0)\lambda^{\beta}]^{\frac{1}{q-1}}} \times
\]

\[
\left\{ a^{\frac{2-q}{q-1}} F \left( \frac{1}{q - 1}, \frac{2-q}{(q - 1)[1 - \beta(q - 1)]}, 1 + \frac{\beta(2-q)}{1 - \beta(q - 1)} \right) \right\}
\]

7
\[ b^{\frac{q-2}{q-1}} F \left( \frac{1}{q-1}, \frac{2 - q}{(q - 1)(1 - \beta(q-1))}, \frac{1}{q - 1} \right) \frac{\beta(2 - q)}{1 - \beta(q-1)} \] 

\[ + \left\{ H \left[ q - \left( 1 + \frac{1}{\beta} \right) \right] - H(q - 2) \right\} \frac{\lambda^\beta}{\beta - 1} \times \] 

\[ \left\{ a^{1 - \beta} F \left( \frac{1}{q - 1}, \frac{\beta - 1}{\beta(q - 1) - 1}, \frac{\beta q - 2}{\beta(q - 1) - 1} \right) \right\} \] 

\[ (q - 1)i(k + i0)\lambda^{\beta(q-1)}a^{1 - \beta(q-1)} - \] 

\[ b^{1 - \beta} F \left( \frac{1}{q - 1}, \frac{\beta - 1}{\beta(q - 1) - 1}, \frac{\beta q - 2}{\beta(q - 1) - 1} \right) \] 

\[ (q - 1)i(k + i0)\lambda^{\beta(q-1)}b^{1 - \beta(q-1)} \} \right\} \). \tag{3.8} \]

As we can see from (3.8), \( F(k, q) \) is dependent of \( a \) and \( b \), and, as consequence, one-to-one as has been shown in Section 2.

However, and this is the crucial issue, if we fix \( q \) and select \( \beta = 1/(q - 1) \), (3.8) simplifies and adopts the appearance

\[ F(k, q) = \lambda^{\frac{q}{q-1}} \frac{q - 1}{2 - q} [H(q - 1) - H(q - 2)] \times \] 

\[ \left[ a^{\frac{q-2}{q-1}} F \left( \frac{1}{q - 1}, \frac{2 - q}{q - 1}, \frac{2 - q}{q - 1}; (q - 1)i(k + i0)\lambda \right) \right. - \] 

\[ b^{\frac{q-2}{q-1}} F \left( \frac{1}{q - 1}, \frac{2 - q}{q - 1}, \frac{2 - q}{q - 1}; (q - 1)i(k + i0)\lambda \right) \] 

\[ \left. (q - 1)i(k + i0)\lambda^{\beta(q-1)}a^{1 - \beta(q-1)} \right\}. \tag{3.9} \]
With the help of the result given in [27] for

\[ F(-a, b, b, -z) = (1 + z)^a, \]

we obtain for (3.9):

\[ F(k, q) = \lambda^{q^{-1}} q^{-1} [H(q - 1) - H(q - 2)] \]
\[ \left( a^{q^{-1}} - b^{q^{-1}} \right) \left[ 1 + (1 - q)ik\lambda \right]^{1/q}. \]  

(3.10)

Using now the expression for \( \lambda \) of [22], i.e.,

\[ \lambda = \left[ \left( \frac{q - 1}{2 - q} \right) \left( a^{q^{-1}} - b^{q^{-1}} \right) \right]^{1-q}, \]

we have, finally,

\[ F(k, q) = [H(q - 1) - H(q - 2)] \left[ 1 + (1 - q)ik\lambda \right]^{1/q}, \]  

(3.11)

which is the result given by Hilthorst in [22], demonstrating that \( F(k, q) \) is not one-to-one. As a conclusion we can say that for fixed \( q \) the q-Fourier transform is NOT one-to-one. On the contrary, as we have shown in section 2, when \( q \) is NOT fixed, the q-Fourier transform is indeed one-to-one.

**Conclusions**

In the present communication we have discussed the NOT one-to-one nature of the q-Fourier transform \( F_q \). We have shown that, if we eliminate the
requirement that $q$ be fixed and let it “float” instead, such simple extension of the $F_q$—definition restores the desired one-to-one character.

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