Stratified flows over a complex relief

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Abstract. A non-linear problem on steady non-homogeneous flows over an uneven bottom is considered. A semi-analytical model deals with asymptotic solutions of the Dubreil-Jacotin—Long equation of uniformly stratified fluid. Approximate solutions are constructed by the perturbation procedure combined with the Fourier method of modal expansion. Stationary wave patterns forced above the rough terrain of finite horizontal extension are calculated and analyzed.

1. Introduction
In this paper, a mathematical model of non-homogeneous fluid flows over complex topography is considered. Such stratified flows are of interest to meteorology in the context of study air currents above the mountain ranges [1, 2, 3]. It is well known that stationary wave patterns can occur in the lee of the terrain obstacles for appropriate upwind conditions [4]. In the theory of lee waves, exact analytic solutions are known only for simplest topographies such as single bell-shaped or semi-circular obstacle [5, 6]. Interference of lee waves over several obstacles has been studied experimentally in the papers [7, 8], and semi-analytically in the papers [9, 10, 11, 12, 13]. We develop an analytical approach involving von Mises transformation of the flow domain into the fixed one which provides efficient numerical calculation of the flow patterns over complex terrain.

2. Basic Equations
We consider a steady two-dimensional flow of a heavy inviscid incompressible fluid in the horizontal layer of finite depth affected by the gravity. A basic model involves the non-linear Euler equations

\begin{align*}
\rho (uu_x + vu_y) + p_x &= 0, & \rho (uv_x + vv_y) + p_y &= -\rho g, \\
u \rho_x + v \rho_y &= 0, & u_x + v_y &= 0,
\end{align*}

where \( \rho \) is the fluid density, \((u, v)\) is the fluid velocity, \( p \) is the pressure and \( g \) is the gravity acceleration. It is supposed that the flow is bounded from above by a rigid lid \( y = H \) where \( H \) is identified with the troposphere depth. The topography has the form \( y = h(x) \), so the boundary conditions at the bottom and the top lid are

\begin{align*}
v - uh_x &= 0|_{y = h(x)}, & v &= 0|_{y = H}.\end{align*}
It is supposed that the upstream flow is presented by the uniform current having constant speed $U$ and the known density profile $\rho_\infty(y)$, so we have the condition

$$(u, v) \to (U, 0), \quad \rho \to \rho_\infty \quad (x \to -\infty)$$  \hspace{1cm} (4)$$

Introducing the stream function $\psi$ by means of $u = \psi_y, v = -\psi_x$, we obtain the dependence $\rho = \rho(\psi)$ which is specified by the upstream condition (4) as $\rho(\psi) = \rho_\infty(\psi/U)$. It is assumed that the buoyancy frequency $N$ defined by the formula $N^2(y) = -g\rho'_\infty(y)/\rho_\infty(y)$ is constant in the upstream flow. In this case, the density $\rho_\infty$ depends exponentially on the height, $\rho_\infty(y) = \rho_0 \exp(-N^2 y/g)$ where $\rho_0$ is the reference density at $y = 0$. For this stratification, the basic dimensionless parameters are the Boussinesq parameter $\sigma$ and the squared inverse densimetric Froude number $\lambda$ defined by the formulae

$$\sigma = \frac{N^2 H}{g}, \quad \lambda = \frac{\sigma g H}{U^2}.$$  \hspace{1cm} (5)

The parameter $\sigma$ determines the slope of the density profile at upstream flow, and parameter $\lambda$ qualifies the sub- or super-criticality of this flow. In addition, we use dimensionless height of topography $\alpha = a/H$ as small parameter by applying the perturbation method. Selecting the height $H$ as the length scale and upwind speed $U$ as the velocity scale, dimensionless variables are introduced as follows:

$$(x, y, h) = H(\bar{x}, \bar{y}, \bar{h}), \quad \psi = UH \bar{\psi}.$$  \hspace{1cm} (6)

In these variables, the shape of the bottom topography rewrites as $\bar{y} = \alpha \bar{h}(\bar{x})$, and upstream density profile takes the dimensionless form $\rho_\infty/\rho_0 = \exp(-\sigma \bar{y})$.

Eliminating the pressure $p$ from the momentum equations (1) reduces the system (1)–(3) to the equivalent boundary value problem for the non-linear Dubreil – Jacotin — Long equation [1] which has the dimensionless formulation

$$\Delta \psi + \lambda(\psi - y) = \frac{1}{2} \sigma \left( |\nabla \psi|^2 - 1 \right) \quad (\alpha h(x) < y < 1) \hspace{1cm} (5)$$

$$\psi = 0 |_{y=\alpha h(x)}, \quad \psi = 1 |_{y=1} $$  \hspace{1cm} (6)

Here $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator, and the bar is dropped in notation of variables. The upstream condition takes the form

$$\psi(x, y) \to y \quad (x \to -\infty).$$  \hspace{1cm} (7)

3. Constructing analytic solution

The uneven bottom topography essentially complicates solving the problem (5)–(7) even in the case of the simplest bottom shape $h(x)$. Therefore we involve the von Mises independent variables $(x, \psi)$ in order to transform original flow domain to the unit strip in the $(x, \psi)$–plane. Now we seek the streamlines in the form $y = Y(x, \psi)$ so the equation (5) takes a more complicated form

$$-\frac{\partial}{\partial x} \frac{Y_x}{Y_\psi} + \frac{\partial}{\partial \psi} \frac{1 + Y_x^2}{Y_\psi^2} = \lambda(Y - \psi) + \frac{1}{2} \sigma \left( \frac{1 + Y_x^2}{Y_\psi^2} - 1 \right) \quad (0 < \psi < 1) \hspace{1cm} (8)$$

but the boundary conditions (6) simplify as

$$Y(x, 0) = \alpha h(x), \quad Y(x, 1) = 1.$$  \hspace{1cm} (9)
The upstream condition (7) reduces to

\[ Y(x, \psi) \to \psi \quad (x \to -\infty) \]  \hspace{1cm} (10)

The advantage of the problem (8)–(10) is a formulation of equations in rectangular domain with the exact topographic condition at the bottom line \( \psi = 0 \). However, the limitation appears while the von Mises transformation suggests all the streamlines to be not overhanging. Thus, the nonlinear model (8)–(10) may serve for simulating the topographic flows over a gently shaped relief having relatively small height of peaks. Thus, we are looking for the solution \( Y \) by the perturbation method with a small parameter \( \alpha \)

\[ Y(x, \psi) = \psi + \alpha w_0(x, \psi) + \alpha^2 w_1(x, \psi) + \ldots \]

Substituting this power series into (8)–(9) leads to the recursive set of equations

\[ L(\sigma, \lambda) w_n = f_n \quad (0 < \psi < 1; \quad n = 0, 1, 2, \ldots), \]  \hspace{1cm} (11)

\[ w_0(x, 0) = h(x), \quad w_0(x, 1) = 0; \quad w_n(x, 0) = w_n(x, 1) = 0 \quad (n = 1, 2, \ldots) \]  \hspace{1cm} (12)

Here \( L(\sigma, \lambda) \) is the linear second-order differential operator acting by the formula

\[ L(\sigma, \lambda) w = w_{xx} + w_{yy} - \sigma w_{\psi} + \lambda w, \]

and the non-linearity realizes by the recursive operators \( f_n = f_n(w_0, \ldots, w_{n-1}) \). Particulary, we have

\[ f_0 = 0, \quad f_1(w_0) = (w_0 x w_0^\psi)_x + \frac{1}{2} (w_0^2_x + 3w_0^2_\psi)_\psi - \frac{1}{2} \sigma (w_0^2 + 3w_0^2_\psi). \]

Further, we apply a version of the Fourier method using the eigenfunctions

\[ \varphi_n(x, \psi) = e^{ikx} e^{\sigma \psi/2} \sin \pi n \psi \quad (n = 1, 2, 3, \ldots) \]

which satisfy the equation \( L(\sigma, \lambda_n) \varphi_n = 0 \) with the eigenvalues \( \lambda_n = \pi^2 n^2 + k^2 + \sigma^2/4 \). The leading-order solution \( w_0 \) has the form

\[ w_0(x, \psi) = e^{\sigma \psi/2} \left\{ y(\psi) h(x) + \sum_{n=1}^{\infty} w_0^{(n)}(x) \sin \pi n \psi \right\} \]  \hspace{1cm} (13)

where the function

\[ y(\psi) = \frac{\sin k_0 (1 - \psi)}{\sin k_0}, \quad k_0 = \sqrt{\lambda - \sigma^2/4} \]  \hspace{1cm} (14)

presents the hydrostatic mode, and the infinite series in (13) involves normal modes of the waves generated by topography. More precisely, for upstream flow with a given value of the Froude number \( \lambda \) taken from the interval

\[ \pi^2 m^2 + \frac{1}{4} \sigma^2 < \lambda < \pi^2 (m + 1)^2 + \frac{1}{4} \sigma^2 \]

we obtain a \( m \)-modal lee wave solution having the coefficients

\[ w_0^{(n)}(x) = -\frac{c_n}{k_n} \int_{-\infty}^{x} \sin k_n(x - s) h''(\xi) d\xi \quad (n = 1, \ldots, m) \]
with the oscillation wave-number \( k_n = \sqrt{\lambda - \sigma^2/4 - \pi^2 n^2} \), and

\[
w^{(n)}_0(x) = \frac{c_n}{2\kappa_n} \int_{-\infty}^{+\infty} e^{-\kappa_n |x-s|} h''(\xi) \, d\xi \quad (n = m + 1, m + 2, \ldots)
\]

with the attenuation wave-number \( \kappa_n = \sqrt{\pi^2 n^2 + \sigma^2/4 - \lambda} \). Here \( c_n \) are the known Fourier coefficients of the hydrostatic modal function \( y(\psi) \) from (14) calculated with respect to the harmonic basis \( \{ \sin \pi n \psi \}_{n=1}^{\infty} \). The constructed solution \( w_0 \) satisfies automatically no wave upstream condition \( w_0 \to 0 \) as \( x \to -\infty \), and it has the lee-wave downstream behavior

\[
w_0(x, \psi) \sim e^{\sigma \psi/2} \sum_{n=1}^{m} \left( a_j^{(0)} \sin k_n x + b_j^{(0)} \cos k_n x \right) \sin \pi n \psi
\]
as \( x \to +\infty \). The constant wave amplitudes \( a_j^{(0)} \) and \( b_j^{(0)} \) depend here on the upstream flow parameters as well as on the topography shape. The non-linear correction to the leading-order solution \( w_0 \) can be found by constructing the function \( w_1 \) via a similar analytical procedure which involves the right-hand nonlinearity term \( f_1(w_0) \) calculated by the first step of this iterative method.

4. Numerical calculations

Wolfram Mathematica [14] was used by semi-analytic calculations of the flows for a wide class of the bottom obstacles. Symbolic computer algebra was used to represent a truncated solution (13), which provides fast convergence. Numerical simulations used a series of ten basic harmonics for the series presenting the functions \( w_0 \) and \( w_1 \). Computations involved the discretization with 50 discrete points in the horizontal direction \( x \) and 10 points in \( \psi \). The calculation of the non-linear second-order solution used 6-order interpolation splines for the leading-order discrete linear solution. Multiple series of calculations were carried out on the computer cluster at Novosibirsk State University.

Preliminary simulations used the first-order linear solution [11] and the second-order non-linear solution [12]. Wave solutions calculated for the topography with a finite number of peaks showed that the linear theory overestimates the magnitude of the waves in the interference zone. Figure 1 shows the 1-mode stratified flow over the double-bumped topography with the fixed parameters \( \alpha = 0.05 \), \( \lambda = 15 \), \( \sigma = 02 \), but the distance between the bumps can vary. It is clear, that the wave patterns are very sensitive to this variation. However, the nonlinear correction smooths essentially the wave crests over the second obstacle only. This overestimating phenomenon of the linear theory has been pointed in the laboratory experiments [7] with two bell-shaped obstacles.

The calculations also found the fragmentation effect which occurs for near-field wave patterns forced by multi-bumped topography of finite extension. Experimental setup was considered in the paper [8] with sinusoidal topography of finite extension. The existence of sharp fronts separating the flow domains having different wave scales was discussed in our paper [13]. Namely, using the Fourier expansions allow to evaluate analytically the impact of modal decomposition on the fragmentation observed in numerical experiments. Figure 2 illustrates a 1-mode stratified flow over irregular-shaped obstacle with the parameters \( \alpha = 0.16 \), \( \lambda = 15 \), \( \sigma = 0.1 \). These dimensionless values of parameters correspond to an upstream wind speed of 10 m/s in the flow over the terrain having maximum peak height of 1600 m by the depth of troposphere 10 km. The sheared wind profile in Fig. 2 demonstrates the air jet over the peak with the maximum velocity about 50 m/s.
Figure 1. Stratified flow over two-bumped topography. Thin lines correspond to the linear model.

Figure 2. The streamlines (left) and the fluid velocity profile (right) in the stratified flow over irregular topography.
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