On the Use of Integrals to Evaluate Series of Rational Terms

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Introduction  In the January 2006 issue of The College Mathematics Journal, M. Andreoli posed the following problem [1]:

**Problem 819.** For \( n \geq 1 \), evaluate

\[
\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)\cdots(k+n)} .
\]

This article is written with the hope to draw attention to a method that has been proposed independently by the author in [2] and Wheelon in [3] and that allows one to find exact values for a large class of convergent series of rational terms. The method has been extended to many series in the papers of Lesko and Smith [4] and Efthimiou [5].

We will outline both the author’s and Wheelon’s variations of the method in order to compare the similarities and differences, and then use them to compute the series (II). Finally, we apply the methods to additional series that generalize (II). The results thus obtained seem to be new.

**The Core Idea of the Method**  Let a series of the form \( \sum_{k \in I} u_k v_k \), where \( I \) is a subset of \( \mathbb{Z} \), be given. Then it is convenient to write only one of the factors, say \( v_k \), as an integral transform

\[
v_k = \int_a^b \tilde{v}_k(t) f(t) \, dt ,
\]

and thus

\[
\sum_{k \in I} u_k v_k = \sum_{k \in I} u_k \int_a^b \tilde{v}_k(t) f(t) \, dt .
\]
Assuming that the order of the operations of summation and integration can be exchanged

$$\sum_{k \in I} u_k v_k = \int_a^b \left( \sum_{k \in I} u_k \tilde{v}_k(t) \right) f(t) \, dt.$$  

In this article we shall always exchange the order of the two operations assuming that the reader knows how to reason for its validity. Details on this may be found in the original papers [2] and [4] where the Laplace transform is used. If one can find an explicit function $h(t) = \sum_{k \in I} u_k \tilde{v}_k(t)$, then he has succeeded in writing the initial series in a simple integral representation:

$$\sum_{k \in I} u_k v_k = \int_a^b h(t) f(t) \, dt.$$  

If, furthermore, the integration can be performed, then analytic answers for the initial series are obtained.

**Series of rational terms**  Consider a series for which $I = \{1, 2, \ldots \}$, $u_k = 1$, and $v_k = Q(k)/P(k)$, where $Q(k)$ and $P(k)$ are two polynomials in $k$:

$$S = \sum_{k=1}^{\infty} \frac{Q(k)}{P(k)}.$$  \hspace{1cm} (2)

To ensure convergence, we may assume that $\deg Q + 2 \leq \deg P$.

Problem 819 of *The College Mathematics Journal* is clearly a special case of (2) for $Q(k) = 1$ and $P(k) = k(k+1)(k+2) \cdots (k+n)$.

**The author’s variation**  In the method presented in [2], the expansion of the general rational term of the series to partial fractions plays a central role. We shall then assume that the sum $S$ can be written in terms of partial fractions:

$$S = \sum_{k=1}^{\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{A_{ij}}{(k+a_i)^j},$$

where the constants $A_{ij}$ are uniquely determined by the partial fraction decomposition of each summand. Details for this decomposition can be found in [2].

Using the identity

$$\frac{1}{A^L} = \frac{1}{(L-1)!} \int_0^{+\infty} x^{L-1} e^{-Ax} \, dx,$$  \hspace{1cm} (3)
we write the series in integral form (which is valid only if \(a_i > -1\), for all \(i\)):

\[
S = \sum_{k=1}^{\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m_i} \frac{A_{ij}}{(j-1)!} \int_0^{+\infty} x^{j-1} e^{-(k+a_i)x} \, dx .
\] (4)

By interchanging the integration over \(x\) with the addition over \(k\), we find

\[
S = \sum_{i=1}^{\ell} \sum_{j=2}^{m_i} \frac{A_{ij}}{(j-1)!} \int_0^{+\infty} x^{j-1} \frac{e^{-(a_i+1)x}}{1-e^{-x}} \, dx \\
+ \sum_{i=1}^{\ell} A_{i1} \int_0^{+\infty} \frac{e^{-(a_i+1)x} - 1}{1-e^{-x}} \, dx .
\]

To further simplify this sum we need to know the values of the constants \(A_{ij}\). The integrals may be expressed in terms of the polygamma functions [2].

**Andreoli’s problem** In problem 819 of *The College Mathematics Journal* the rational function \(1/P(k)\) can be easily expanded in partial fractions

\[
\frac{1}{P(k)} = \sum_{i=0}^{n} \frac{A_i}{k+i} ,
\] (5)

with

\[
A_i = \frac{(-1)^i}{n!} \binom{n}{i} .
\]

This can be proved easily by multiplying equation (5) by \(k + j\) and then taking the limit \(k \to -j\). We thus have

\[
\sum_{k=1}^{\infty} \frac{1}{P(k)} = \frac{1}{n!} \int_0^{+\infty} \frac{e^{-x}}{1-e^{-x}} \sum_{i=0}^{n} \binom{n}{i} (-1)^i (e^{-x})^i \, dx \\
= \frac{1}{n!} \int_0^{+\infty} \frac{e^{-x}}{1-e^{-x}} (1-e^{-x})^n \, dx \\
= \frac{1}{n!} \int_0^{+\infty} e^{-x} (1-e^{-x})^{n-1} \, dx .
\] (6)

By a change of variables \(u = e^{-x}\),

\[
\sum_{k=1}^{\infty} \frac{1}{P(k)} = \frac{1}{n!} \int_0^1 (1-u)^{n-1} du = \frac{1}{n \cdot n!} .
\] (7)
A generalization  One can easily generalize Andreoli’s proposal: For \( n \geq 1 \), we can evaluate

\[
S(a, b) = \sum_{k=1}^{\infty} \frac{1}{[a + kb][a + (k + 1)b][a + (k + 2)b] \cdots [a + (k + n)b]}.
\]  \( \text{(8)} \)

Following the method described above, we find

\[
S(a, b) = \frac{1}{b^n n!} \int_{0}^{+\infty} e^{-(a+b)x} (1 - e^{-bx})^{n-1} dx.
\]

The change of variables \( u = e^{-x} \) gives the integral the form

\[
S(a, b) = \frac{1}{b^n n!} \int_{0}^{1} u^{a+b-1} (1 - u^b)^{n-1} du.
\]

The last integral is a B-function integral:

\[
\int_{0}^{1} (1 - u^\ell)^{n-1} u^{m-1} du = \frac{1}{\ell} B\left(\frac{m}{\ell}, n\right), \quad m, n, \ell > 0.
\]  \( \text{(9)} \)

The sum \( S(a, b) \) is thus equal to \( B(a/b + 1, n) \) which value can be easily computed through the \( \Gamma \)-function. Finally, we get

\[
S(a, b) = \frac{1}{nb} \prod_{i=1}^{n} \frac{1}{a + ib}.
\]

Wheelon’s variation  The method presented by Wheelon [3] for the computation of the series \( \text{(2)} \) assumes that \( Q(k) = 1 \) and \( P(k) \) can be factorized as a product of first order monomials. In this method, the convolution theorem as used by Feynman [6] plays a central role. In particular, starting from

\[
\frac{1}{ab} = \int_{0}^{1} \frac{dx}{[ax + b(1 - x)]^2},
\]

one can show the general formula

\[
\frac{1}{\prod_{i=1}^{\ell} \alpha_i^{m_i}} \frac{1}{\prod_{i=1}^{\ell} \Gamma(m_i)} \int_{0}^{\ell-1} \int_{0}^{\ell-1} \int_{0}^{\ell-1} \cdots \int_{0}^{\ell-1} \frac{1}{\prod_{i=1}^{\ell} x^{m_i-1} (1 - \sum_{i=1}^{\ell-1} x_i)^{m_i-1}} dx_1 dx_2 \cdots dx_{\ell-1}.
\]

In physics literature, this formula is known as Feynman’s integral and the \( x \)'s are called Feynman’s parameters.
If $Q(k) = 1$ and $P(k)$ given by

$$P(k) = (k + a_1)^{m_1}(k + a_2)^{m_2} \cdots (k + a_\ell)^{m_\ell},$$

(10)

where the $a_i$, $i = 1, 2, \ldots, \ell$ are distinct real numbers, none of them a negative integer and all $m_i$ are positive integers, then the series (2) is written

$$S = \frac{\Gamma(\sum_{i=1}^{\ell} m_i)}{\prod_{i=1}^{\ell} \Gamma(m_i)} \sum_{k=1}^{\infty} \int \cdots \int_{0 \leq \sum_{i=1}^{\ell-1} x_i \leq 1} \frac{(\prod_{i=1}^{\ell-1} x_i^{m_i-1})(1 - \sum_{i=1}^{\ell-1} x_i)^{m_{\ell-1}-1}}{\sum_{i=1}^{\ell-1} x_i^{m_i}} dx_1 dx_2 \cdots dx_{\ell-1}

= \frac{1}{\prod_{i=1}^{\ell} \Gamma(m_i)} \sum_{k=1}^{\infty} \int \cdots \int_{0 \leq \sum_{i=1}^{\ell-1} x_i \leq 1} \frac{(\prod_{i=1}^{\ell-1} x_i^{m_i-1})(1 - \sum_{i=1}^{\ell-1} x_i)^{m_{\ell-1}-1}}{\sum_{i=1}^{\ell-1} x_i^{m_i}} \times

\times \int_0^{+\infty} \frac{(\sum_{i=1}^{\ell-1} x_i^{m_i-1}) e^{-y[k+a_\ell+\sum_{i=1}^{\ell-1} x_i(a_i-a_\ell)]}}{1 - e^{-y}} dy.

To pass from the first expression to the second, we used formula (3). Wheelon does not present explicitly the above formula; he only describes his method in a simple case and then he lists exact results for series for which $P(k)$ is a product of up to four distinct monomials. The result for Andreoli’s problem (which will be computed in the next section) does not appear in [3].

Andreoli’s problem again When $m_i = 1$, for all $i$, Feynman’s integral can be simplified considerably. In particular, one can write

$$\frac{1}{\alpha_1 \alpha_2 \cdots \alpha_\ell} = (\ell - 1)! \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{\ell-2}} dx_{\ell-1} \times

\frac{1}{[\alpha_1 + x_1(\alpha_2 - \alpha_1) + \cdots + x_{\ell-2}(\alpha_{\ell-1} - \alpha_{\ell-2}) + x_{\ell-1}(\alpha_\ell - \alpha_{\ell-1})]^{\ell}},$$
If \( \ell = n + 1 \) and \( \alpha_i = k + i - 1, i = 1, 2, \ldots, n + 1 \), then
\[
\frac{1}{k(k+1)(k+2)\cdots(k+n)} = n! \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{n-1}} dx_n \frac{1}{[k + \sum_{i=1}^n x_i]^{n+1}}
\]
\[
= \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \frac{1}{[k + \sum_{i=1}^n x_i]^{n+1}}
\]
\[
= \frac{1}{n!} \int_0^1 dx_1 \int_0^1 dx_2 \cdots \int_0^1 dx_n \int_0^{\infty} y^n e^{-y[k + \sum_{i=1}^n x_i]} dy
\]
\[
= \frac{1}{n!} \int_0^{\infty} e^{-yk} y^n \left( \int_0^1 e^{-yx} dx \right)^n dy
\]
\[
= \frac{1}{n!} \int_0^{\infty} e^{-yk}(1 - e^{-y})^n dy ,
\]
(11)

and
\[
\sum_{k=1}^{\infty} \frac{1}{k(k+1)(k+2)\cdots(k+n)} = \frac{1}{n!} \int_0^{\infty} (1 - e^{-y})^n \sum_{k=1}^{\infty} e^{-yk} dy
\]
\[
= \frac{1}{n!} \int_0^{\infty} (1 - e^{-y})^{n-1} e^{-y} dy ,
\]
which is identical to equation (6) found by the partial fractions method.

**A Related Result**  Equation (11) is valid for any positive number \( k \) (we shall rename it \( x \)), not necessarily an integer. If instead we sum over \( n \)
\[
\sum_{n=0}^{\infty} \frac{1}{x(x+1)(x+2)\cdots(x+n)} = \int_0^1 u^{x-1} \sum_{n=0}^{\infty} \frac{(1-u)^n}{n!} du
\]
\[
= e \int_0^1 u^{x-1} e^{-u} du ,
\]
(12)

where we have made a change of variable \( u = e^{-x} \). In this expression we can now expand \( e^{-u} \) in a Taylor series and perform the elementary integrals to find
\[
\sum_{n=0}^{\infty} \frac{1}{x(x+1)(x+2)\cdots(x+n)} = e \left[ \frac{1}{x} - \frac{1}{1!} \frac{1}{x+1} + \frac{1}{2!} \frac{1}{x+2} - \cdots \right] .
\]
(13)

**Additional Results**  The series proposed by Andreoli and its generalization (8) are well-known: see page 234 of [7]. Equations (13) and (12) are also found in [7] (page 317). Equation (13) — although not equation (12) — is presented as an exercise (ex. 126, page 272) in Knopp’s book [8]. In this section we extend the results
presented above to other series. The results obtained seem to be new; we have not found them in [7] or [8].

Using the Feynman integral and equation (3), we can show that

$$\frac{1}{k(k+2)(k+4)\cdots(k+2n)} = \frac{1}{n!} \int_0^1 u^{k-1} (1-u^2)^{n-1} du.$$  \hfill (14)

By renaming $k$ as $x$, summing over $n$, and following the steps that led to equation (13), we can find

$$\sum_{n=0}^{\infty} \frac{1}{x(x+2)(x+4)\cdots(x+2n)} = e^{1/2} \int_0^1 u^{x-1} e^{-u^2/2} du$$

$$= e^{1/2} \left[ \frac{1}{x} - \frac{1}{2!} \frac{1}{x+2} + \frac{1}{2^2 2!} \frac{1}{x+4} - \cdots \right].$$  \hfill (15)

On the other had, summing in (14) over $k$ gives:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+4)\cdots(k+2n)} = \frac{1}{2^n n!} \int_0^1 (1-u^2)^n \frac{1}{1-u} du.$$  \hfill (16)

We could have arrived at the same integral representation if we had used the partial fraction decomposition. The integral of the right hand side may be rewritten as

$$\int_0^1 (1-u^2)^{n-1} (1+u) du = \int_0^1 (1-u^2)^{n-1} du + \int_0^1 (1-u^2)^{n-1} u du.$$  

These integrals in turn can be computed\(^1\) by the use of the B-function (9). Finally, we find

$$\sum_{k=1}^{\infty} \frac{1}{k(k+2)(k+4)\cdots(k+2n)} = \frac{1}{2n} \left( \frac{1}{(2n-1)!!} + \frac{1}{(2n)!!} \right).$$  \hfill (16)

Empowered with the results that have been presented so far, one can ask if we can prove similar results for the series

$$F(x) = \sum_{n=0}^{\infty} \frac{1}{x(x+\ell)(x+2\ell)\cdots(x+n\ell)},$$

$$S(\ell) = \sum_{k=1}^{\infty} \frac{1}{k(k+\ell)(k+2\ell)\cdots(k+n\ell)}.$$  

Indeed, it is straightforward to show that

$$F(x) = e^{1/\ell} \int_0^1 u^{x-1} e^{-u/\ell} du$$

$$= e^{1/\ell} \left[ \frac{1}{x} - \frac{1}{\ell} \frac{1}{x+\ell} + \frac{1}{\ell^2 2!} \frac{1}{x+2\ell} - \cdots \right].$$  \hfill (17)

\(^1\)Actually, the second integral is basic.
and that
\[ S(\ell) = \frac{1}{\ell^n n!} \int_0^1 (1 - u^\ell)^n \frac{1}{1 - u} \, du. \]

The integral of the right hand side may be rewritten as
\[ \int_0^1 (1 - u^\ell)^{n-1} (1 + u + u^2 + \cdots + u^{\ell-1}) \, du, \]
that is, a sum of integrals of the form (9). We thus find
\[ S(\ell) = \frac{1}{\ell n} \left( \frac{1}{(\ell n)\ell} + \frac{1}{(\ell n - 1)\ell} + \frac{1}{(\ell n - 2)\ell} + \cdots + \frac{1}{(\ell n - (\ell - 1))\ell} \right), \tag{18} \]
where the symbol \( l^\ell \) is an extension of the double factorial and stands for the product of all numbers between 1 and \( \ell n - m \) that, when divided by \( \ell \), give residue \( \ell - m \). For example, \((3n)!! = (3n)!^3 = 3 \cdot 6 \cdot 9 \cdot \cdots (3n), (3n - 1)!! = (3n - 1)!^3 = 2 \cdot 5 \cdot 8 \cdot \cdots (3n - 1), \) and so on.

**Conclusion** Taking a problem proposed by Andreoli as the origin of this article, we have attempted to draw the attention of the reader to the use of integral transforms in the computation of exact values for series and make him appreciate the ease and transparency of the method. In the process, we have derived some results (17) and (18) that seem to be new or, at least, do not appear in the commonly used references [7] and [8].

The method can be applied to a wide variety of series as explained in the articles of Efthimiou [2, 5] and Lesko and Smith [4] and the book of Wheelon [3]. The reader is encouraged to look at these references for additional information. Hopefully, he will use the method to search for new results. He may begin with the series:

\[ F(x; a, b) = \sum_{n=0}^{\infty} \frac{1}{[a + xb][a + (x + \ell)b][a + (x + 2\ell)b] \cdots [a + (x + n\ell)b]}, \]
\[ S(\ell; a, b) = \sum_{k=1}^{\infty} \frac{1}{[a + kb][a + (k + \ell)b][a + (k + 2\ell)b] \cdots [a + (k + n\ell)b]} . \]

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