CUT-OFF FOR LAMPLIGHTER CHAINS ON TORI: DIMENSION INTERPOLATION AND PHASE TRANSITION

AMIR DEMBO*, JIAN DING†, JASON MILLER‡, AND YUVAL PERES§

ABSTRACT. Suppose that $G$ is a finite, connected graph and $X$ is a lazy random walk on $G$. The lamplighter chain $X^\circ$ associated with $X$ is the lazy random walk on the wreath product $G^\circ = Z_2 \wr \mathcal{G}$, the graph whose vertices consist of pairs $(f, x)$ where $f$ is a $\{0, 1\}$-labeling of the vertices of $G$ and $x$ is a vertex in $G$. In each step, $X^\circ$ moves from a configuration $(f, x)$ by updating $x$ to $y$ using the transition rule of $X$, and if $x \neq y$, replacing $f_x$ and $f_y$ by two independent uniform random bits. The mixing time of the lamplighter chain on the discrete torus $Z^\circ_n$ is known to have a cutoff at a time asymptotic to the cover time of $Z_n^d$, if $d = 2$, and to half the cover time if $d \geq 3$. We show that the mixing time of the lamplighter chain on $G_n(a) = Z_n^d \times Z_{\log n}$ has a cutoff at $\psi(a)$ times the cover time of $G_n(a)$ as $n \to \infty$, where $\psi(a)$ is a weakly decreasing map from $[0, \infty)$ onto $[1/2, 1)$. In particular, as $a > 0$ varies, the threshold continuously interpolates between the known thresholds for $Z^2_n$ and $Z^3_n$. Perhaps surprisingly, we find a phase transition (non-smoothness of $\psi$) at the point $a_* = \pi r_3 (1 + \sqrt{2})$, where $r_3$ is the effective resistance from 0 to $\infty$ in $Z^3$.

1. INTRODUCTION

1.1. Setup. Suppose that $G$ is a finite, connected graph with vertices $V(G)$ and edges $E(G)$, respectively. Let $X(G) = \{f : V(G) \to \{0, 1\}\}$ be the set of markings of $V(G)$ by elements of $\{0, 1\}$. The wreath product $Z^2_2 \wr G$ is the graph whose vertices consist of pairs $(f, x)$ where $f \in X(G)$ and $x \in V(G)$. There is an edge between $(f, x)$ and $(g, y)$ if and only if $\{x, y\} \in E(G)$ and $f_z = g_z$ for all $z \notin \{x, y\}$. Suppose that $P(x,y)$ is the transition kernel for a lazy random walk $X$ on $G$. That is, $P$ is given by

\[ P(x,y) := P_{x}[X_{1} = y] = \begin{cases} 1/2 & \text{if } x = y, \\ 1/2d(x) & \text{if } \{x, y\} \in E(G), \end{cases} \]

where $d(x)$ is the degree of $x \in V(G)$ and $P_x$ denotes the law under which $X_0 = x$. The lamplighter walk $X^\circ$ is the lazy random walk on $G^\circ$. Explicitly, it moves from a configuration $(f, x)$ by

1. picking $y$ adjacent to $x$ in $G$ according to $P$, then
2. if $y \neq x$, updating each of the values of $f_x$ and $f_y$ independently according to the uniform measure on $\{0, 1\}$.

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The value of \( f_z \) for all other \( z \in V(G) \) remains fixed. We refer to the \( f_x \) as the state of the lamp at \( x \). If \( f_x = 1 \) (resp. \( f_x = 0 \)) we say that the lamp at \( x \) is on (resp. off); this is the source of the name “lamplighter.” Note that the projection of \( X^\infty \) to \( G \) evolves as a lazy random walk on \( G \). It is easy to see that the unique stationary distribution of \( X^\infty \) is given by the product measure
\[
\pi^\infty((f, x)) = \pi(x) 2^{-|G|}
\]
where \( \pi \) is the (unique) stationary distribution of \( P \) and \(|G| = |V(G)| \).

The purpose of this work is to determine the asymptotics of the total variation mixing time of the lamplighter walk on a particular sequence of graphs. In order to state our main results precisely and put them into context, we will first review some basic terminology from the theory of Markov chains. Suppose that \( \mu, \nu \) are measures on a finite probability space. The total variation distance between \( \mu, \nu \) is given by
\[
\|\mu - \nu\|_{TV} = \max_A |\mu(A) - \nu(A)| = \frac{1}{2} \sum_x |\mu(x) - \nu(x)|.
\]

The \( \epsilon \)-total variation mixing time of a transition kernel \( P \) on a graph \( G \) with stationary distribution \( \pi \) is given by
\[
t_{\text{mix}}(G, \epsilon) = \min \left\{ t \geq 0 : \max_{x \in V(G)} \|P^t(x, \cdot) - \pi\|_{TV} \leq \epsilon \right\}.
\]

Throughout, we let \( t_{\text{mix}}(G) = t_{\text{mix}}(G, \frac{1}{2}) \). Lazy random walk on a family of graphs \((G_n)\) is said to exhibit cutoff if
\[
\lim_{n \to \infty} \frac{t_{\text{mix}}(G_n, \epsilon)}{t_{\text{mix}}(G_n, 1 - \epsilon)} = 1 \quad \text{for all} \quad \epsilon > 0.
\]

For each \( x \in V(G) \) let \( \tau(x) = \min\{k \geq 0 : X_k = x\} \) be the hitting time of \( x \). With \( E_x \) the expectation associated with \( P_x \), the maximal hitting time of \( G \) is given by
\[
t_{\text{hit}} = t_{\text{hit}}(G) = \max_{x,y \in V(G)} E_y[\tau(x)]
\]
and the cover time of \( G \) is
\[
t_{\text{cov}} = t_{\text{cov}}(G) = \max_{y \in V(G)} E_y \left[ \max_{x \in V(G)} \tau(x) \right].
\]

The mixing time of \( G^\infty \) was first studied by Häggström and Jonasson in [HJ97] in the case of the complete graph \( K_n \) and the one-dimensional cycle \( Z_n \). Their work implies a total variation cutoff with threshold \( \frac{1}{2} t_{\text{cov}}(K_n) \) in the former case and that there is no cutoff in the latter. The connection between \( t_{\text{mix}}(G^\infty) \) and \( t_{\text{cov}}(G) \) is explored further in [PR04] (see also the account given in [LPW09, Chapter 19]), in addition to developing the relationship between \( t_{\text{hit}}(G) \) and the relaxation time (i.e. inverse spectral gap) of \( G^\infty \), and the relationship between exponential moments of the size of the uncovered set \( U(t) \) of \( G \) at time \( t \) and the uniform, i.e. \( \ell_\infty \)-mixing time of \( G^\infty \). In particular, it is shown in [PR04, Theorem 1.3] that if \( (G_n) \) is a sequence of graphs with \(|V(G_n)| \to \infty \) and \( t_{\text{hit}}(G_n) = o(t_{\text{cov}}(G_n)) \) then
\[
\frac{1}{2}(1 + o(1))t_{\text{cov}}(G_n) \leq t_{\text{mix}}(G_n^\infty) \leq (1 + o(1))t_{\text{cov}}(G_n) \quad \text{as} \quad n \to \infty.
\]
By combining the results of [DPRZ06] and [Ald91], it is observed in [PR04] that $t_{\text{mix}}((\mathbb{Z}_2^n)\cong)$ has a threshold at $t_{\text{cov}}(\mathbb{Z}_2^n)$. Thus, (1.5) gives the best universal bounds, since $K_n$ attains the lower bound and $\mathbb{Z}_2^n$ attains the upper bound. In [MPT12], it is shown that $t_{\text{mix}}((\mathbb{Z}_d^n)\cong) \sim \frac{1}{2}t_{\text{cov}}(\mathbb{Z}_d^n)$ when $d \geq 3$ and more generally that $t_{\text{mix}}(G_n) \sim \frac{1}{2}t_{\text{cov}}(G_n)$ whenever $(G_n)$ is a sequence of graphs with $|V(G_n)| \to \infty$ satisfying certain uniform local transience assumptions. This prompted the question [MPT12, Section 7] of whether for each $\alpha \in (\frac{1}{2}, 1)$ there exists a (natural) family of graphs $(G_n)$ such that $t_{\text{mix}}(G_n) \sim \alpha t_{\text{cov}}(G_n)$ as $n \to \infty$. In this work we give an affirmative answer to this question by analyzing the lamplighter walk on a thin 3D torus. Finally, bounds for the relaxation and uniform mixing times are respectively obtained in [KP12] and [KMP11].

**Figure 1.** A typical configuration of the lamplighter on a $5 \times 5$ planar grid. The colors indicate the state of the lamps and the circle gives the position of the underlying random walker.

### 1.2. Main results.
Fix $a > 0$. We consider the mixing time for the SRW $X^\circ_k$, $k \in \mathbb{N}$, on the lamplighter group based on the 3D thin tori $G_n(a) = (V_n, E_n) = \mathbb{Z}_n^2 \times \mathbb{Z}_h$ of size $n \times n \times h$, where $h = [a \log n]$. From the main result of [DPRZ04] we know that the cover time of the 2D projection of SRW on $G_n(a)$ to $\mathbb{Z}_n^2$ is given by

$$t_{\text{cov}} := \frac{3}{2}t_{\text{cov}}(\mathbb{Z}_n^2)$$

where $t_{\text{cov}}(\mathbb{Z}_n^2) := \frac{4}{\pi}n^2(\log n)^2(1 + o(1))$ (where the factor $\frac{3}{2}$ is due to the lazy steps of walk in the $h$-direction, which occur with probability $\frac{1}{3}$). Let $r_3$ denote the resistance $0 \leftrightarrow \infty$ for the SRW in $\mathbb{Z}_3^3$. That is,

$$r_3 = \frac{1}{6q}, \quad \text{where} \quad q = P_0[T_0 = \infty],$$

and $T_0$ denotes the return time to zero by SRW in $\mathbb{Z}_3^3$ (see [LPW09] Proposition 9.5) for the relation $r_3 = \frac{1}{6q}$ and an explicit formula for $q$). Setting hereafter

$$\phi := \pi r_3 a$$

we prove the following theorem in Section 2.
Theorem 1.1. The cover time \( t_{\text{cov}}(a, n) \) of \( G_n(a) \) by SRW is given by
\[
t_{\text{cov}}(a, n) = (1 + o(1)) C(a, n), \quad \text{as } n \to \infty
\]
where
\[
C(a, n) := (1 + 2\phi) t_{\text{cov}}^{\square}
\]
and \( \phi \) is as in (1.6).

Figure 2. The function \( \Psi \) from (1.8) which, by Theorem 1.2, gives the asymptotic ratio of \( t_{\text{mix}}/t_{\text{cov}}^{\square} \). Also shown are the corresponding Peres-Revelle upper and lower bounds of \( 2\phi + 1 \) and \( \phi + \frac{1}{2} \), respectively, for \( t_{\text{mix}}/t_{\text{cov}}^{\square} \); recall (1.5). The threshold \( 1 + \sqrt{2} \) is where \( t_{\text{mix}}/t_{\text{cov}}^{\square} \) obtains the Peres-Revelle lower bound.

Our main result establishes cutoff for SRW \( \{X_k^a\} \) on the lamplighter group based on \( G_n(a) \) and determines its location as a function of the height parameter \( a \).

Theorem 1.2. Total-variation cut-off occurs for \( \{X_k^a\} \) on \( G_n(a) \) at \( \Psi(\phi) t_{\text{cov}}^{\square} \), where
\[
\Psi(\phi) := \begin{cases} 
(1 + (1 - \frac{1}{\sqrt{2}})\phi)^2, & \text{if } \phi \leq \sqrt{2} + 1, \\
\frac{1 + 2\phi}{2}, & \text{if } \phi > \sqrt{2} + 1.
\end{cases}
\]
In particular, \( t_{\text{mix}} = (\Psi(\phi) + o(1)) t_{\text{cov}}^{\square} \).

Comparing Theorems 1.1 and 1.2 we see that the ratio between the mixing time of \( \{X_k^a\} \) and the cover time \( C(a, n) \) of the base graph by the SRW \( \{X_k\} \), monotonically interpolates between the fraction of the cover time necessary to mix in two dimensions (ratio 1) [DPRZ04, PR04] and fraction in three dimensions (ratio 1/2) [MP12]. This gives an affirmative answer to the first question posed in [MP12, Section 7].
We note in passing that for all $\phi > 0$ the value of $t_{\text{mix}}/t_{\text{cov}}^2 \to \Psi(\phi)$ is bounded away of its trivial bound 1. The latter corresponds to the mixing time for lamplighter group on the 2D torus of side length $n$ that corresponds to the base sub-graph $(x_1, x_2, 1)$ of $G_n(a)$ (which as shown in [PR04] coincides with the cover time $t_{\text{cov}}^2(1 + o(1))$ for the corresponding (lazy) 2D projected SRW). However, when $\phi \geq \sqrt{2} + 1$ asymptotically $t_{\text{mix}}$ matches the elementary bound $t_{\text{mix}} \geq \frac{(1+o(1))C(a, n)}{2}$ (see (1.7), and [LPW09, Lemmas 19.3 and 19.4]), which applies for the lamplighter group on any base graph having maximal hitting time which is significantly smaller than the corresponding cover time.

The mixing time of $X^\circ = (\mathcal{E}, X)$ is dominated by the first coordinate $\mathcal{E}$ since the amount of time it takes for $X$ to mix is negligible compared to that required by $X^\circ$. We can sample from $\mathcal{E}(t)$ by:

1. sampling the range $\mathcal{C}(t)$ of the lazy random walk run for time $t$, then
2. marking the vertices of $\mathcal{C}(t)$ with either 0 or 1 by i.i.d. fair coin flips.

Determining the mixing time of $X^\circ$ is thus typically equivalent to computing the threshold $t$ where the corresponding marking becomes indistinguishable from a uniform marking of $V(G)$ by i.i.d. fair coin flips. This in turn can be viewed as a statistical test for the uniformity of the uncovered set $\mathcal{U}(t)$ of $X$. That is, $X^\circ(t)$ is not mixed for as long as $\mathcal{U}(t)$ exhibits some non-trivial systematic geometric structure, thereby our work is connected to the literature on the geometric structure of the last visited points by the random walk [Bel13, BH91, DPRZ04, DPRZ06, MP12, MS13].

We next provide a variational formula for $\Psi(\phi)$ of (1.8), whose interpretation in the sequel shall explain the asymptotic structure of $\mathcal{U}(t_{\text{cov}}^2)$ by the SRW on $G_n(a)$ and shed light on the relevance of $\Psi(\cdot)$ to the mixing time.

**Lemma 1.3.** For $s > 0$ and $\rho, z \in [0, 1]$, let

\begin{equation}
(1.9) \quad b_\rho(z) = 1 - \rho - \frac{s(1 - z)^2}{1 - \rho}, \quad \alpha_\rho(z) = \frac{s z^2}{2} + \phi,
\end{equation}

with $b(z) = b_0(z)$, $\alpha(z) = \alpha_0(z)$ and the convention that $b_1(z) = -\infty 1_{\{z \neq 1\}}$. Then, $\Psi(\cdot)$ of (1.8) emerges from the following variational problem:

\begin{equation}
(1.10) \quad \Psi(\phi) = \inf \{ s \geq 1 : \forall \rho, z \in [0, 1], b_\rho(z) \geq 0 \implies \alpha_\rho(z) \geq \rho \}
\end{equation}

\begin{equation}
(1.11) \quad = \sup \{ s \geq 1 : \exists \rho, z \in [0, 1], \text{ such that } b_\rho(z) \geq 0 \text{ and } \alpha_\rho(z) \leq \rho \}.
\end{equation}

**Proof.** First, set $h(\rho) := \sqrt{\rho(\rho + \rho/2)}$, $t := \sqrt{5}$ and

\begin{equation}
(1.12) \quad t_\star = \sup_{\rho \in [0, 1]} \{ h(\rho) + 1 - \rho \}.
\end{equation}

The conditions $b_\rho(z) \geq 0$ and $\alpha_\rho(z) \geq \rho$ are then re-expressed as $tz \geq t - (1 - \rho)$ and $tze \geq h(\rho)$, respectively. So, with the optimal choice being $z = z_\star := 1 - (1 - \rho)/t$, it follows that (1.10) holds if and only if $t \geq t_\star$. That is, $\Psi(\phi) = t_\star^2$. Further, considering at $t = t_\star$ the optimal $z_\star = h(\rho)/h(\rho) + 1 - \rho$, yields the identity (1.11). Finally, in (1.12) the optimal choice is $\rho = \rho_\star = (\sqrt{2} - 1)\phi$, but in case $\phi \geq 1/(\sqrt{2} - 1)$ it is out of range and one needs to settle instead for $\rho = 1$. One easily checks that $h(\rho_\star) = \phi + \sqrt{2}$, while $h(1) = \sqrt{\phi + \sqrt{2}}$, hence with $t_\star$ monotone increasing in $\phi$ it is easy to confirm from the preceding that $t_\star^2 = \Psi(\phi)$ is given by the explicit formula (1.8), as claimed.

□
The functions $\alpha_\rho(z)$ and $b_\rho(z)$ control the structure of set of points in $G_n(a)$ that are not visited by $\{X_k, k \leq st^{\square}_{\text{cov}}\}$. Indeed, for any $\rho \in [0,1]$ we associate with each $x \in G_n(a)$ a type $z \in [0,1]$ according to the number of excursions of the srw, by time $st^{\square}_{\text{cov}}$, across the 2D cylindrical annulus of radii $Mhn^\rho$ and $M^2hn^\rho$, centered at the 2D projection of $x$. Our parameters are such that for $n \to \infty$ followed by $M \to \infty$, with high probability about $n^{2b_\rho(z)+o(1)}$ of the $n^{2(1-\rho)+o(1)}$ such annuli are of $z$-type and points $x \in G_n(a)$ well within each such $z$-type annulus, are unvisited by the srw with probability $n^{-\alpha_\rho(z)+o(1)}$ (c.f. Remark 5.13 detailing this in case of $\rho = 0$).

When upper bounding $t_{\text{mix}}$ in Section 3 it thus suffices to consider only admissible types, namely, to assume $b_\rho(z) \geq 0$ for all $\rho$ (see (3.5) for the way we do it), and by (1.10) the discrepancy of about $n^{-\alpha_\rho(z)}$ between the fraction of lamps which are off and those which are on within each such annulus, is then buried under the inherent noise level of $n^{-\rho}$, as long as $s > \Psi(\phi)$. However, upper bounding $t_{\text{mix}}$ is technically challenging since it requires sharp control on exponential moments of the size of the unvisited set corresponding to each $z$-type. Reducing this first to somewhat sparse sub-lattice $A$ of $G_n(a)$ allows us to further prove that the approximate independence of the corresponding Bernoulli($n^{-\alpha_\rho(z)}$) variables applies for $\rho = 0$, even in terms of tail probabilities (see Remark 5.13), which by careful extension of the concept of $z$-types to the annuli profiles $z \in [0,1]^L$ corresponding to $\rho_k = k/L$, $k = 0,1,\ldots,L$, allows us to control aforementioned exponential moments (even though at any $\rho > 0$ the corresponding Bernoulli($n^{-\alpha_\rho(z)}$) variables are no longer asymptotically independent).

The lower bound on $t_{\text{mix}}$ is proved in Section 4 based on the dual representation (1.11) of $\Psi(\phi)$. Specifically, it tells us that for any $s < \Psi(\phi)$ there exist admissible types $\rho, z' \in (0,1)$ (such that $b_\rho(z') \geq 0$), for which $\alpha_\rho(z') < \rho$. Approximating such $\rho$ by $\rho_k := [\rho L]/L$, the maximum discrepancy at time $st^{\square}_{\text{cov}}$ between “off-lamps” and “on-lamps” over large enough (and spatially well separated) collection $A_{2D,k}$ of 2D disjoint cylinders of radii $hn^{\rho_k}$, far exceeds its value under the invariant (uniform) law for the srw $\{X^o\}$. Consequently it serves as a suitable statistics for distinguishing between the law of the lamplighter group at that time and its invariant law.

Remark 1.4. When proving the lower bound on $t_{\text{mix}}$ in Section 4 we note that the most likely way to have type $z(0) = z$ at the $O(h)$ size 2D annulus corresponding to $\rho = 0$, is via the profile $z(\rho) = 1 - (1-\rho)(1-z)$ (which linearly interpolates between $z(0) = z$ and $z(1) = 1$). For each $\rho, z \in [0,1]$ there are then about $n^{2(1-\rho)b_0(z)+o(1)}$ such disjoint annuli of radii $O(hn^\rho)$ and $z(\rho)$-type. However, we find there that such profiles are highly unlikely for the set of unvisited points of $G_n(a)$ by time $st^{\square}_{\text{cov}}$ (which is why for upper bounding $t_{\text{mix}}$ one needs to control large deviations of all possible $z(\cdot)$-type profiles).

2. Cover time for the thin torus: proof of Theorem 1.1

The Gaussian Free Field (in short gff), on finite, connected graph $G = (V,E)$, with respect to some fixed $v_0 \in V$, is the stochastic process $\{\eta_u\}_{u \in V}$ with $\eta_{v_0} = 0$,
whose density with respect to Lebesgue measure on \( V \setminus \{ v_0 \} \) is proportional to
\begin{equation}
\exp \left( -\frac{1}{4} \sum_{u,v} |\eta_u - \eta_v|^2 \right),
\end{equation}
where we used \( u \sim v \) to denote \( \{u, v\} \in E \). An important connection between GFF and the SRW on \( \mathcal{G} \) is the following identity (see for example, \[\text{[Lyo09, Chapter 2]}\]):
\begin{equation}
E[(\eta_u - \eta_v)^2] = R_{\text{eff}}(u, v).
\end{equation}
Here \( R_{\text{eff}}(u, v) \) is the effective resistance between \( u \in V \) and \( v \in V \) in the electrical network associated with \( \mathcal{G} \) by placing a unit resistor on each edge \( \{u, v\} \in E(\mathcal{G}) \) (and we sometimes use \( R_{\text{eff}}^G(u, v) \) to emphasize the underlying graph \( \mathcal{G} \), in case of possible ambiguity). Our proof of Theorem 1.1 relies on the following relation between the cover time \( t_{\text{cov}}(\mathcal{G}) \) of \( \mathcal{G} \) by SRW and the maximum of the corresponding GFF.

**Theorem 2.1** ([Din11]). Consider a sequence of graphs \( \mathcal{G}_n = (V_n, E_n) \) of uniformly bounded maximal degrees, such that \( t_{\text{hit}}(\mathcal{G}_n) = o(t_{\text{cov}}(\mathcal{G}_n)) \) as \( n \to \infty \). For each \( n \), let \( \{\eta_v\}_{v \in V_n} \) denote a GFF on \( \mathcal{G}_n \) with \( \eta_{v_0}^n = 0 \) for certain \( v_0^n \in V_n \). Then, as \( n \to \infty \),
\begin{equation}
t_{\text{cov}}(\mathcal{G}_n) = (1 + o(1))|E_n|\left( E\{ \sup_{v \in V_n} \eta_v \} \right)^2.
\end{equation}

In light of the preceding theorem, the key to the proof of Theorem 1.1 is an estimate on the expected supremum for the associated GFF. To this end, we start with few estimates of effective resistances assuming familiarity with the connection between random walks and electric flows (see for example [Lyo09, Chapter 2]).

**Lemma 2.2.** Let \( \{X_n\} \) denote the SRW on the graph \( \mathcal{G} = (V, E) \) started at some \( o \in V \), independent of the Geometric random variable \( T \). Then, there exists a current flow \( \theta = \{\theta_{u,v} : \{u, v\} \in E\} \) with unit current source at \( o \), current \( p_v := P[X_T = v] \) reaching each \( v \in V \), and the Dirichlet energy bound
\[ D(\theta) := \sum_{(u,v) \in E} \theta_{u,v}^2 \leq \frac{1}{d_o} \sum_{n=0}^T E\{1_{\{X_n = o\}}\} \].

**Proof.** Let \( t = P[T \geq 1] \in (0, 1) \). Set \( \ell_v := \frac{1}{d_v} E\{\sum_{n=0}^T 1_{\{X_n = v\}}\} \) and \( N(u, v) := \sum_{n=0}^{T-1} 1_{\{X_n = u, X_{n+1} = v\}} \), for each \( u, v \in V \). Then, due to the memoryless property of Geometric random variables, clearly
\[ p_v = 1_{v = o} + \sum_{u : u \sim v} (EN(u, v) - EN(v, u)) = 1_{v = o} + t \sum_{u : u \sim v} (\ell_u - \ell_v) \].
Thus, the current flow \( \theta_{u,v}^o := t(\ell_u - \ell_v) \) on \( \{u, v\} \in E \), together with external unit current into \( o \), results with current \( p_v \) reaching each \( v \in V \). Furthermore,
\[ \sum_{u \sim v} (\theta_{u,v}^o)^2 = \frac{t^2}{T} \sum_{u \sim v} (\ell_u - \ell_v)^2 \leq t \ell_o \sum_{u \sim v} (\ell_u - \ell_v) \]
\[ \leq t \ell_o \sum_{v : v \sim o} (\ell_o - \ell_v) \leq \ell_o \],
since \( t \sum_{v : v \sim o} (\ell_u - \ell_v) = -p_u \leq 0 \) for all \( u \neq o \), and is at most one at \( u = o \). \( \square \)

We will also need the following claim.
Lemma 2.3. For any graph $G = (V, E)$, let $R$ be the diameter for the effective resistance. Consider a collection of numbers $\{\rho_v : v \in V\}$ such that $\sum_{v \in V} \rho_v = 0$ and $\frac{1}{2} \sum_{v \in V} |\rho_v| = 1$, and let $\Theta$ denote the collection of all flows on $G$ such that at any vertex $v$ the difference between out-going and in-coming flow is $\rho_v$. Then,

$$\min_{\theta \in \Theta} \{D(\theta)\} \leq R.$$  

Proof. Let $V^+ = \{v \in V : \rho_v \geq 0\}$ and $V^- = V \setminus V^+$. By assumption on $\rho$, there exists a function $w : V^+ \times V^- \mapsto [0, \infty)$ such that

$$\sum_{u \in V^-} w(v, u) = \rho_v \text{ for all } v \in V^+ \text{ and } \sum_{u \in V^+} w(u, v) = -\rho_v \text{ for all } v \in V^-.$$  

So in particular we have $\sum_{v \in V^+, u \in V^-} w(v, u) = 1$. For $(v, u) \in V^+ \times V^-$, let $\theta^{v,u}$ be an electric current which sends unit amount of flow from $v$ to $u$ (so in particular $D(\theta^{v,u}) \leq R_{\text{eff}}(v, u)$). Denoting $\theta := \sum_{v \in V^+, u \in V^-} w(v, u) \theta^{v,u}$, by our construction of $w(\cdot, \cdot)$ we see that $\theta \in \Theta$. It remains to bound the Dirichlet energy of $\theta$. By Cauchy-Schwartz inequality, we get that

$$D(\theta) = \sum_{e \in E} \theta_e^2 = \sum_{e \in E} \left( \sum_{v \in V^+, u \in V^-} w(v, u) \theta_e^{v,u} \right)^2 \leq \sum_{e \in E} \sum_{v \in V^+, u \in V^-} w(v, u) (\theta_e^{v,u})^2 \leq \sum_{v \in V^+, u \in V^-} w(v, u) D(\theta^{v,u}) \leq R,$$

completing the proof of the lemma.

\[\square\]

Lemma 2.4. With $R_{\text{eff}}(\cdot, \cdot)$ denoting effective resistances on $G_n(a) = (V_n, E_n)$, we have that for all $x, x' \in G_n(a)$,

(2.4) $$R_{\text{eff}}(x, x') \leq 2r_3 + \frac{1}{a\pi} + o(1).$$

Furthermore, for $x = (y, 0)$ and $x' = (y', 0)$ where $y, y' \in \mathbb{Z}^2$ and $\|y - y\|_{\mathbb{Z}_n^2} \geq 2a \log n$, we have

(2.5) $$R_{\text{eff}}(x, x') = 2r_3 + \frac{1}{a \log n} \log \|y - y\|_{\mathbb{Z}_n^2} + o(1).$$

Proof. Fixing arbitrary $x, x' \in G_n(a)$ we establish (2.4) upon constructing a flow of $1 + o(1)$ current from $x$ to $x'$ whose Dirichlet energy is at most $2r_3 + 1/(a\pi) + o(1)$. To this end, for $\{X_n\}$ a SRW on $G_n(a)$ and an independent Geometric random variable $T$ of mean $(\log n)^4$, let $p_v = P_x[X_T = v]$ for $v \in G_n(a)$, and $p_{(i)} := \sum_{v \in \mathbb{Z}_n^2 \times (i)} p_v$ (namely, the probability that the “vertical” coordinate of $X_T$ is at $i \in \mathbb{Z}_n^2$). We claim that

(2.6) $$\frac{1}{6} E_x \left[ \sum_{t=0}^{T} I_{\{X_t = x\}} \right] = r_3 + o(1).$$

In order to see the lower bound in (2.6), we note that the random walk is the same as a random walk in $\mathbb{Z}^3$ in the first $h = \lfloor a \log n \rfloor$ steps, during which period the expected number of visits accumulated at $x$ is already $6(r_3 + o(1))$. Setting $N = (\log n)^5$, since $E(T1_{T \geq N}) \to 0$, we get the matching upper bound upon showing that

(2.7) $$E_x \left[ \sum_{t=h}^{N} I_{\{X_t = x\}} \right] = o(1).$$
To this end, with \( A \) denoting the event that simultaneously for all \( h \leq t \leq N \), the number of vertical steps made by the SRW up to time \( t \) is in the range \((t/10, t/2)\), we clearly have that \( P[A^c] \leq (\log n)^{-r} \) for any \( r \) finite and all \( n \) large enough. Therefore

\[
E_x \left[ \sum_{t=h}^{N} 1_{\{X_t = x\}} \right] \leq NP[A^c] + E_x \left[ \sum_{t=h}^{N} 1_{\{X_t = x, A\}} \right] = o(1) + \sum_{t=h}^{N} \frac{O(1)}{\sqrt{\log n}} \frac{O(1)}{t} = o(1),
\]

with the term \( \frac{O(1)}{\sqrt{\log n}} \) upper bounding the probability of the SRW returning at time \( t \) to its starting height (referring to its vertical coordinate), and \( O(1/t) \) bounding the probability of its 2D projection returning to the starting point, respectively (we obtain their independence upon conditioning on the number of vertical steps the SRW made up to time \( t \)). Combined with (2.7), this completes the verification of (2.6).

Now, by (2.6) and Lemma 2.2 there exists a unit current flow \( \theta^{(x)} \) out of \( x \), with current inflow of \( p_v \) into each \( v \in G_n(a) \) and

\[
\mathcal{D}(\theta^{(x)}) = \sum_{(u,v) \in E_n} (\theta^{(x)}_{u,v})^2 \leq r_3 + o(1).
\]

Setting \( p'_v := P_y[X_T = v] \) and \( p'_{[i]} := \sum_{v \in Z_n^2 \times \{i\}} p'_v \), we have by the same reasoning a unit current flow \( \theta^{(x')} \) out of \( x' \), with current inflow \( p'_v \) into each \( v \in G_n(a) \) and

\[
\mathcal{D}(\theta^{(x')}) \leq r_3 + o(1).
\]

Furthermore, by time \( T \) which is concentrated around \( ET \gg h^3 \), the vertical component of \( \{X_n\} \) is so nearly uniformly distributed that

\[
\max_i |hp_{[i]} - 1| = o(1) = \max_i |hp'_{[i]} - 1|.
\]

Next, fixing \( i \in Z_h \) set \( \rho_i, \rho'_i \in [0, 1] \) such that

\[
\rho_ip_{[i]} = \rho'_ip'_{[i]} = \min\{p_{[i]}, p'_{[i]}\}
\]

so there exist zero-net current flows on the sub-graph \( Z_n^2 \times \{i\} \) of \( G_n(a) \), with outflow \( \rho_ip_v \) and inflow \( \rho'_ip'_v \) at each \( v \in Z_n^2 \times \{i\} \). Let \( \theta^i \) denote the flow of minimal Dirichlet energy among all such current flows and \( |\theta^i| = \frac{1}{2} \sum_{v \in Z_n^2 \times \{i\}} |\rho_ip_v - \rho'_ip'_v| \) its total flow. Then, by Lemma 2.3 we have that

\[
\mathcal{D}(\theta^i) \leq |\theta^i|^2 \text{Diam}_{R_{\text{eff}}}(Z_n^2),
\]

where \( \text{Diam}_{R_{\text{eff}}}(Z_n^2) \) is the diameter for the resistance metric in the torus \( Z_n^2 \). Recalling (2.10), we obtain that

\[
\sum_i \mathcal{D}(\theta^i) \leq \frac{1 + o(1)}{h} \text{Diam}_{R_{\text{eff}}}(Z_n^2).
\]

Combined with the standard estimate

\[
\text{Diam}_{R_{\text{eff}}}(Z_n^2) \leq \frac{1 + o(1)}{\pi} \log n
\]

(see, e.g., [Dim12, Lemma 3.4]), we arrive at

\[
\sum_i \mathcal{D}(\theta^i) \leq \frac{1 + o(1)}{h} \text{Diam}_{R_{\text{eff}}}(Z_n^2) \leq \frac{1}{a\pi}(1 + o(1)).
\]
Consider now the current flow \( \theta^* \) from \( x \) to \( x' \) obtained by combining \( \theta^{(x)} \) with the union of all flows \( \{ \theta^i, i \in \mathbb{Z}_b \} \) and the current flow \( -\theta^{(x')} \). The net amount of current reaching sub-graph \( Z_n^2 \times \{ i \} \) is then \( p[i] - p'[i] \), so by (2.10), the flow from \( x \) to \( x' \) via \( \theta^* \) is \( 1 + o(1) \), whereas by (2.8), (2.9) and (2.11), its Dirichlet energy is at most

\[
D(\theta^{(x)}) + \sum_i D(\theta^i) + D(\theta^{(x')}) \leq 2r_3 + \frac{1}{\alpha n} + o(1),
\]

completing the proof of the upper bound (2.4).

For the lower bound, we let \( C_x \) and \( C_{x'} \) be cubes of side-length \( \log \log n \) centered around \( x \) and \( x' \), respectively. Let \( G_{a,n} \) be the graph obtained by identifying \( \partial C_x \) (also \( \partial C_{x'} \)) as a single vertex, as well as identifying \( \{(z,i): 1 \leq i \leq h\} \) as a single vertex for each \( z \in \mathbb{Z}_n^2 \). By Rayleigh monotonicity principle, we see that

\[
R_{\text{eff}}(x, x') \geq R_{\text{eff}}(x, \partial C_x) + R_{\text{eff}}(x', \partial C_{x'}) + R_{\text{eff}}^C(\partial C_x, \partial C_{x'}). \]

It is clear that \( R_{\text{eff}}(x, \partial C_x) = R_{\text{eff}}(x', \partial C_{x'}) = r_3 + o(1) \). In addition, by triangle inequality we see that

\[
R_{\text{eff}}^C(\partial C_x, \partial C_{x'}) \geq R_{\text{eff}}^{(x,x')} - R_{\text{eff}}^{(x,x')} - R_{\text{eff}}^{(x',x')} = \frac{1}{h}(R_{\text{eff}}(y,y') - 2R_{\text{eff}}(o, \partial C_o)) = \frac{1}{\pi a \log n} \log \|y - y'^2\| + o(1),
\]

where \( C_o \) is a 2D box of side-length \( \log \log n \) centered around \( o \), and the last equality follows for example from [Din12 Lemma 3.4]. Altogether, this gives the desired lower bound on the effective resistance. \( \square \)

The following lemma is useful in comparing the maxima of two Gaussian processes (see for example [Fer75 Corollary 2.1.3]).

**Lemma 2.5 (Sudakov-Fernique).** Let \( A \) be an arbitrary finite index set and let \( \{X_a\}_{a \in A} \) and \( \{Y_a\}_{a \in A} \) be two centered Gaussian processes such that

\[(2.12) \quad \mathbb{E}(X_a - X_b)^2 \geq \mathbb{E}(Y_a - Y_b)^2, \text{ for all } a, b \in A.\]

Then \( \mathbb{E}[\max_{a \in A} X_a] \geq \mathbb{E}[\max_{a \in A} Y_a] \).

We are now ready to estimate the maximum of the GFF on the thin torus.

**Lemma 2.6.** Let \( \{\eta_v: v \in G_n(a)\} \) be a GFF on \( G_n(a) \) with \( \eta_v = 0 \). Then,

\[
\mathbb{E}[\max_{v \in V_n} \eta_v] = 2\sqrt{r_3 + \frac{1}{\alpha n} + o(1)}\sqrt{\log n}.
\]

**Proof.** By (2.2) and Lemma 2.4, we get that \( \text{Var}(\eta_v) \leq 2r_3 + \frac{1}{\alpha n} + o(1) \) for all \( u, v \in G_n(a) \). Hence, the union bound for centered Gaussian variables \( \{\eta_u - \eta_v: u \in G_n(a)\} \) exceeding \( r \) followed by integration over \( r \geq 0 \), yields the stated upper bound. For the lower bound, we employ a comparison argument. Let \( A \) be a 2D box of side-length \( n/(8h) \), and let \( \{\xi_v: v \in A\} \) be a GFF on \( A \) with Dirichlet boundary condition (i.e., \( \xi|_{\partial A} = 0 \)). Now define mapping \( g: A \to G_n(a) \) by \( g(v) = (2h v, 0) \). It is well known that (see, e.g., [LL10 Theorem 4.4.4 and Proposition 4.6.2])

\[
R_{\text{eff}}^A(u, v) = \frac{1}{\pi} \log \|u - v\|_2 + O(1).
\]
Combined with Lemma 2.4, it yields that for all \( u, v \in A \)

\[ R_{\text{eff}}^G(u, v) = (2ar3\pi + 1 + o(1))h^{-1}R_{\text{eff}}^A(u, v). \]

Applying (2.2) and Lemma 2.5, we obtain that

\[ E[\max_{v \in V_\kappa \eta_v}] \geq \sqrt{2ar3\pi + 1 + o(1)}h^{-1/2}E[\max_{u \in A \xi_u}]. \]

Combined with [BDG01, Theorem 2] which states that \( E[\max_{u \in A \xi_u}] = (\sqrt{2/\pi} + o(1)) \log n, \) this yields the desired lower bound on \( E[\max_{v \in V_\kappa \eta_v}]. \)

As \( |E_n| = 3an^2 \log n(1 + o(1)) \), upon combining Theorem 2.1 and Lemma 2.6, we immediately obtain Theorem 1.1.

3. Upper Bound on Mixing Time

Letting \( s' = s + \epsilon \) we start the lamplighter chain with all lamps off and initial position uniformly chosen in \( \mathcal{G}_n(a) \), run the SRW on lamplighter group for \( s't_{\text{cov}} \) steps, and show that for any \( s > \Psi(\phi) \) and \( \epsilon > 0 \) the total-variation distance between its law and the uniform law goes to zero as \( n \to \infty \). To this end, let \( \tau_x \) denote the first hitting time of a point \( x \in \mathcal{G}_n(a) \) by the SRW on the base graph \( \mathcal{G}_n(a) \), with \( \Gamma_s := \{ x \in \mathcal{G}_n(a) : \tau_x > s't_{\text{cov}} \} \) denoting the subset of \( \mathcal{G}_n(a) \) unvisited by the SRW up to time \( s't_{\text{cov}} \). Then, by [MP12, Lemma 3.1 and Proposition 3.2], it suffices to find an event \( \mathcal{G} \) measurable on the path of SRW on the base graph \( \mathcal{G}_n(a) \) up to time \( s't_{\text{cov}} \), such that as \( n \to \infty \)

\[ P[\mathcal{G}] \to 1 \quad \text{and} \quad E[2^{\Gamma_s \cap \Gamma'_s} 1_{\mathcal{G}_s}] \to 1, \]

where \( \Gamma'_s \) and \( \mathcal{G}' \) correspond to a second, independent copy of the SRW on \( \mathcal{G}_n(a) \).

To this end, we set a \( 2r \)-sized 3D sub-lattice in \( \mathcal{G}_n(a) \) and for \( k = 0, 1, \ldots, L \), consider additional \( 2R_k \)-sized 2D sub-lattices of the base of \( \mathcal{G}_n(a) \), denoted \( \mathcal{A}^{s,*,k}_{2D} \).

We assume that \( R''_0 \leq R_0 \leq R''_1 \leq \cdots \leq R_L \leq n \) are of integer ratios (when relevant). For each such collection of sub-lattices, let \( \mathcal{A} \) denote the collection of 3D sub-lattice points whose 2D-projections fall for each \( k = 0, 1, \ldots, L \) in the \( 2R''_k \)-sized sub-squares around some base point \( y_k \in \mathcal{A}^{s,*,k}_{2D} \), with the relevant sequence of base points for each \( x \in \mathcal{A} \) denoted by \( y = (y_0, \ldots, y_L) \) with \( y_k = y_k(x) \). For each \( 0 \leq k \leq L \) we further denote by \( \mathcal{A}^{s,*,k}_{2D} \) the set \( \{ y \in \mathcal{A}^{s,*,k}_{2D} : y = y_k(x) \} \) for some \( x \in \mathcal{A} \). Note that \( \mathcal{G}_n(a) \) is covered by union of \( \kappa^\prime := (2r)^3 \prod_{k=0}^{L} (R_k/R''_k)^2 \) such collections \( \mathcal{A}^{s,*,k}_{2D} \), where \( i = 1, \ldots, (2r)^3 \) indicates a translate of our 3D-lattice and \( j = (j_0, j_1, \ldots, j_L) \) with \( j_k = 1, \ldots, (R_k/R''_k)^2 \) a \( 2R''_k \)-translate of the 2D-lattice \( \mathcal{A}^{s,*,k}_{2D} \) (with respect to the sparser 2D-lattice \( \mathcal{A}^{s,*,k}_{2D} \)). Thus, taking \( \mathcal{G} = \cap_{i,j} \mathcal{G}_{i,j} \) for \( \mathcal{G}_{i,j} \) which are translates of some event \( \mathcal{G} \) associated with one collection \( \mathcal{A} \), by translation invariance of the law of path of SRW on torus \( \mathcal{G}_n(a) \) and Hölder’s inequality, we see that (3.1) holds provided that for \( n \to \infty \), and uniformly with respect to the relative shifts between the 2D sub-lattices that correspond to each choice of \( j \).

\[ \kappa^\prime P[\mathcal{G}] \to 0 \quad \text{and} \quad E[2^{\kappa^\prime |\mathcal{A} \cap \Gamma_s \cap \Gamma'_s|} 1_{\mathcal{G}_s}] \to 1. \]

Proceeding with the proof of (3.2), we use the short notation \( \mathcal{A}^{2D}, R, R', R'' \) and \( y(x) \) for \( \mathcal{A}^{2D,0}, R_0, R'_0, R''_0 \) and \( y_0(x) \), respectively, setting \( R''_k = M R''_k, R_k = M R''_k, r' = M \) and \( r = Mr' \) for large constant \( M > 1 \) (so \( \kappa^\prime = \kappa'(M, L) \) is independent of...
$n$, where $R'' = h$, then $R''_k = n^{k/L} R''$ for $1 \leq k \leq L - 1$, while $n = RM^3$ for $R := R_L$.

We further consider the disjoint $3D$-annuli between pairs of concentric balls with outer radius $r$ and inner radius $r'$, centered at each $x \in A$ and for $k = 0, \ldots, L$, the disjoint cylindrical annuli in $G_n(a)$, of height $h$ and based on concentric $2D$ disks of inner and outer radii $R'_k$ and $R_k$, respectively, centered at each $y_k \in A_{2D,k}$.

For any $k$, each cylindrical annulus decomposes the path of the srw on $G_n(a)$ into $R_k$-excursions, each starting at outer cylinder boundary and run till hitting the inner cylinder boundary (which we call the excursion’s external part), then going back till exiting the outer cylinder (called the excursion’s internal part). Note that for each $k$, conditional on their starting and ending points, the internal parts of various $R_k$-excursions of our collection of cylindrical annuli are mutually independent of each other. Further, for all $n$ large enough and any $k \geq 1$, each $R_{k-1}$-sized cylindrical annulus is strictly inside one of the $R'_k$-sized sub-cylinders we consider, so its $R_{k-1}$-excursions decompose the internal parts of each of the corresponding $R_k$-excursions. Similarly, for all $n$ large enough, our $r$-balls are strictly inside one $R'_k$-sub-cylinder and each of them further decomposes the internal parts of each of the corresponding $R_0$-excursions into what we call $r$-excursions, i.e. start at the ball boundary, run till hitting its $r'$-centered sub-ball (external parts) and then going back till exiting the $r$-ball. Here again, conditional on their starting and ending points the internal part of the various $r$-excursions associated with the collection $A$ are independent of each other.

Let
\begin{equation}
\overline{NC}'(s) := 2s \frac{(\log n)^2}{\log (R/R')} ,
\overline{NB}'(s) := \frac{4s r'}{a} \log n ,
\end{equation}
which, as shown in the sequel, are the typical excursion counts by time $s t_{\text{cov}}^{(\cdot)}$. The next definition summarizes the type of deviations from typical values which are of concern in evaluating $t_{\text{mix}}$.

**Definition 3.1.** Fixing small $\eta > 0$ such that $1/\eta$ is integer, for $\underline{z} = (z_0, \ldots, z_L)$ with $z_k \leq z_L = 1$, $k = 0, \ldots, L - 1$ integer multiples of $\eta$, we say that $y = (y_0, \ldots, y_L)$, or equivalently, that $y_0 \in A_{2D,0}$, is of $\underline{z}$-type if the first $(z_k - 2\eta)^2 \overline{NC}'(s)$ relevant $R_k$-excursions (for the cylindrical annulus centered at $y_k$), are completed within the first $\overline{NC}'(s)$ $R_L$-excursions for cylindrical annulus centered at $y_L$, and in case $z_k < 1$ require also that the first $(z_k - \eta)^2 \overline{NC}'(s)$ are not completed during these $R_L$-excursions. We similarly say that $x \in A$ is of $\underline{z}$-type (for $z$ an integer multiple of $\eta$), if the first $(z - 3\eta)^2 \overline{NB}'(s)$ relevant $r$-excursions are completed within the first $\overline{NC}'(s)$ $R_L$-excursions for cylindrical annulus centered at $y_L(x)$, and in case $z < 1$ require also that the first $(z - 2\eta)^2 \overline{NB}'(s)$ of those $r$-excursions are not completed during said $R_L$-excursions.

There are only $\kappa_o = \eta^{-(L+1)}$ possible $\underline{z}$-types. Hence, setting $\kappa = \kappa_o^2 \kappa_o^2$, $\mathcal{G} = \bigcap_{\underline{z}} \mathcal{G}_{\underline{z}}$ and
\[ \Gamma_{\underline{z},\underline{z}} := \{ x \in A : y(x) \text{ of } \underline{z} \text{- type} \} , \]
we further get from Hölder’s inequality that (3.2) holds provided that for any two types $\underline{z}$ and $\underline{z}'$, as $n \to \infty$,
\begin{equation}
\kappa \mathbb{P} [\mathcal{G}_{\underline{z}}] \to 0 \quad \text{and} \quad \mathbb{E} [2^{\kappa |\Gamma_{\underline{z},\underline{z}}|} 1_{\mathcal{G}_{\underline{z}}^c} 1_{\mathcal{G}_{\underline{z}'}^c}] \to 0 .
\end{equation}
Turning hereafter to prove (3.4), we start by defining the relevant truncation events.

**Definition 3.2.** For each \( \eta > 0 \) and type \( z \), define the event \( \mathcal{G}_z = \mathcal{G}_z(\eta) \) consisting of:

(a) By time \( s t_{\text{cov}} \) the srw on \( \mathcal{G}_n(a) \) completes for each \( R_L \)-sized cylindrical annulus centered at \( y_L \in \mathcal{A}_{2D,L} \) the corresponding first \( \mathcal{N}^\eta(s) \) excursions.

(b) For any \( \rho_k := k/L, k = 0, \ldots, L - 1 \), there are at most \( n^{2\rho_k(z_k)+\eta} \) points \( y_k \in \mathcal{A}_{2D,k} \) to which corresponds some \( y_0 \in \mathcal{A}_{2D,0} \) of \( z \)-type.

(c) If \( x \in \mathcal{A} \) is such that \( y_0(x) \) is of \( z \)-type (cylindrical annuli), then \( x \) is also \( z \)-type (in terms of \( r \)-excursions), for some \( z \geq z_0 \).

**Remark 3.3.** In particular, by requirement (a) of Definition 3.2 we can and shall modify \( \Gamma_{s,z} \) to denote the subset of those \( x \in \mathcal{A} \) with \( y(x) \) of \( z \)-type, which are not visited by the srw during its first \( \mathcal{N}^\eta(s) \) excursions for cylindrical annulus centered at \( y_L(x) \). Further, by requirement (b) of Definition 3.2, for all \( \eta > 0 \) small enough this set is empty whenever \( \rho_k(z_k) < 0 \) for some \( 0 \leq k \leq L \). Hence, in (3.4) only exponential moments of admissible \( z \)-types are to be considered. That is, we may and shall assume when computing these that

\[
\sqrt{s} \leq \min_{k=0}^{L-1} \left\{ \frac{1 - \rho_k}{1 - z_k} \right\}.
\]

We proceed with analysis lemma that is key to the success of our scheme for bounding exponential moments as in (3.4) for admissible \( z \)-types and all \( s > \Psi(\phi) \).

**Lemma 3.4.** Let \( \Psi_{L,\eta}(\phi) \) denote, per given \( L \) and \( \eta \), the minimal value of \( s \geq 1 \), such that if type \( z \) is admissible, then for any \( m = 0, \ldots, L \),

\[
\gamma_{m,\eta}(z) := \alpha(z_0 - 4\eta) - m\eta - \frac{1}{L} \sum_{k=1}^{m} \left[ \frac{1}{L} - 2sL(z_k - z_{k-1} - 2\eta) \right] \geq \eta.
\]

Then, with \( \Psi(\cdot) \) given by the variational problem (1.10), we have that

\[
\Psi(\phi) = \lim_{L \to \infty} \lim_{\eta \to 0} \lim \{ \Psi_{L,\eta}(\phi) \}.
\]

**Proof.** Recall that \( z_L = 1 \) and note that the limit

\[
\Psi_L(\phi) := \lim_{\eta \to 0} \{ \Psi_{L,\eta}(\phi) \},
\]

exists and corresponds to the requirement that \( \gamma_m(z,0) \geq 0 \) for \( m = 0, \ldots, L \) and admissible \( z \). Further, setting \( \Delta_k := tL(z_k - z_{k-1}) \), for \( k = 1, \ldots, L \) and \( t := \sqrt{s} \) we have that

\[
\phi\alpha(z_0) = (tz_0)^2 = (t - \frac{1}{L} \sum_{k=1}^{L} \Delta_k)^2,
\]

yielding that \( \sqrt{\Psi_L(\phi)} \) is merely the infimum over all \( t \geq 1 \) such that for \( m = 0, \ldots, L \),

\[
(t - \frac{1}{L} \sum_{k=1}^{L} \Delta_k)^2 \geq \phi \left[ \frac{m+1}{L} - \frac{2}{L} \sum_{k=1}^{m} (\Delta_k) \right],
\]
whenever $z \in [0,1]^{L+1}$ satisfies $\Box$. That is, denoting by $D$ the collection of all $\Delta := (\Delta_1, \ldots, \Delta_L) \in \mathbb{R}^L$ such that
\begin{equation}
\delta_r := \frac{1}{L-r} \sum_{k=r+1}^{L} \Delta_k \in [0,1] \quad \forall 0 \leq r < L,
\end{equation}
we have that
\[\sqrt{\Psi_L(\phi)} = \max_{m=0}^{\infty} \max_{\Delta \in D} \{t_m(\Delta)\},\]
with $t_m(\Delta)$ the smallest $t \geq 1$ for which (3.8) holds, per given $m$ and $\Delta$. The value of $t_m(\Delta)$ depends only on $t_m$ and $(\Delta_1, \ldots, \Delta_m)$, and further, given $t_m$ and $\sum_{k=1}^{m} \Delta_k$, by Cauchy-Schwartz the maximal value of $t_m(\Delta)$ is attained for $\Delta_k = \Delta$ constant over $1 \leq k \leq m$. Thus, setting $\delta = t_m$, we deduce that $\sqrt{\Psi_L(\phi)}$ is bounded above by the minimal $t \geq 1$ such that
\begin{equation}
(t - (1 - \rho)\delta - \rho \Delta)^2 \geq \phi \rho [1 - 2(\Delta)^2] + \phi \frac{1}{L},
\end{equation}
for any $\delta \in [0,1]$, $\Delta \in \mathbb{R}$ and $\rho \in [0,1]$ for which $\rho L = m$ is integer valued. Note that (3.10) trivially holds whenever $\Delta > 1$ and $\rho > 0$ (whereas for $\rho = 0$ the value of $\Delta$ is irrelevant). Further, since $t \geq 1 \geq \rho$, $\delta \geq 0$, if (3.10) holds for $\Delta = 0$, it also holds for any $\Delta < 0$. Consequently, suffices to consider (3.10) only for $\Delta, \delta \in [0,1]$. Each choice of $(\Delta, \delta)$ in the latter range corresponds to $\underline{\Delta} = (\Delta_1, \ldots, \Delta, \delta, \ldots, \delta)$ in $D$, hence we conclude that the right-side of (3.7) equals the minimal $s = t^2 \geq 1$ satisfying (3.10) for all $\delta \in [0,1]$, $\rho \in (0,1]$ and $\Delta \geq 0$. To match this with (1.10) we equivalently set $(1 - \rho)\delta = t(1 - w)$ and $\rho \Delta = t(w - z)$ with $1 \geq w \geq z$ such that $b_\rho(w) \geq 0$ for $s = t^2$ (corresponding to $\delta \leq 1$). This transforms (3.10), in terms of $z$ and $w$, to the inequality
\[\alpha(z) + \frac{2s(w - z)^2}{\rho} \geq \rho.
\]
Now, by elementary calculus we find that
\begin{equation}
\alpha_\rho(w) = \inf_{z \leq w} \left\{ \alpha(z) + \frac{2s(w - z)^2}{\rho} \right\}
\end{equation}
(with infimum attained at $z_* := (2/\rho)w/(2/\rho + 1/\phi)$). Comparing the preceding with (1.10) we thus conclude that (3.7) holds, as claimed. \hfill \Box

In view of Lemma 3.4, our strategy for proving the stated upper bound on mixing time is thus to set $s = \Psi_{L,\eta}(\phi) + \epsilon$ and consider first $n \to \infty$ followed by $M \to \infty$, then $\eta \downarrow 0$, $L \to \infty$ and finally $\epsilon \to 0$. In doing so, let us denote by $N_{C_{y_k,k,j,w}}$, for $k < j \leq L$ and $w \in [0,1]$ (with $N_{C_y} := N_{C_{y,0,L,1}}$, the number of $R_k$-excursions for $y_k \in A\cup{k,k,k}$, completed during the first $w^2N\mathcal{C}^*(s)$ $R_j$-excursions for the corresponding $y_j \in A\cup{j,j,j,j}$, and let $N_{C_{y_k,L}}$ stand for the number of latter $R_L$-excursions completed by time $s't_{1/\epsilon}^\uparrow$. Similarly, for $x \in A$ and $z \geq \eta$, we denote by $H_{x,z}$ the event of not hitting $x$ during the first $z^2N_{\mathcal{B}}^*(s)$ of the corresponding $r$-excursions and by $N_{B_x,z}$ the number of those $r$-excursions during the first $z^2N_{\mathcal{B}}^*(s)$ excursions of the $R_0$-cylindrical annulus centered at $y(x)$. With these notations in place, we establish (3.4) as a consequence of the following two lemmas, whose proofs are provided in Sections 5 and 6 respectively.
Lemma 3.5. For any fixed $s > 1$, $1 \geq z > \eta > 0$, and $M \geq M_0(\eta, z)$ large enough, as $n \to \infty$ we have that both,
\begin{align}
(3.12) \quad & n^{\alpha(z-\eta)}P[H_{0,z}] \to 0 \quad \text{and} \quad n^{\alpha(z+\eta)}P[H_{0,z}] \to \infty \\
(3.13) \quad & n^2(\log n)P\left[\overline{NB}^*(s)^{-1}NB_{0,z} \notin [(z-\eta)^2, (z+\eta)^2]\right] \to 0
\end{align}

Lemma 3.6. For any fixed $s > 1$, any positive integer $L$, $w, z \geq \eta \geq 0$ and $L \geq j > k \geq 0$, we have for all $M \geq M_1(\eta, z, w, j, k)$ large enough, as $n \to \infty$, that both
\begin{align}
(3.14) \quad & n^M P[|NC_{0,L}-NC^*(s')| \geq \eta NC^*(s')] \to 0, \\
(3.15) \quad & \limsup_{n \to \infty} \frac{\log P[NC_{0,k,j,w}(s) \leq \eta NC^*(s)] + 2s(w-z)^2}{\rho_j - \rho_k} \leq \eta.
\end{align}

Remark 3.7. As will be clear from the proof of Lemma 3.5, the bound (3.12) applies uniformly when we condition on the start and end point for each of the relevant $r$-excursions. Similarly, (3.13) holds uniformly with respect to position of $0$ within the $2R^n$-sized square centered at $y(0)$ and start/end points of the relevant $R$-excursions. Likewise, (3.15) holds uniformly with respect to start/end points of relevant $R_j$-excursions as well as relative position of $0$ within the $2R^n_j$-sized square centered at $y_j(0)$ (with same applying in case of (3.14), just with $L$ instead of $j$).

Indeed, considering (3.14) and taking the union over the at most $(M^3/2)^2$ possible values of $y_L$, clearly requirement (a) in Definition 3.2 is satisfied with probability going to one in $n$ (see, Remark 3.7). Next, for $z_k < 1$, $k < L$, by Definition 3.1 having $y_0$ of $\zeta$-type inside the $R_k$-sized annulus centered at $y_k \in A_{2D,k}$ requires that $NC_{y_k,k,L,1} \leq (z_k - \eta)^2 NC^*(s)$. With $|A^*_{2D,k}| = (n/2R_k)^2 \leq n^{2-2\rho_k}$ it follows from (3.15) that the expected number of such points $y_k$ is $o(n^{2\rho_k(z_k)+\eta})$, hence by Markov’s inequality and union over $0 < k < L$, we deduce that also requirement (b) of Definition 3.2 holds with probability going to one in $n$ (the case $z_k = 1$ trivially holds by the bound on $|A^*_{2D,k}|$). Similarly, if $y_0(x)$ is of $\zeta$-type and $x$ is not of $\zeta$-type with some $z \geq z_0$, then necessarily $NB_{x,z_0-2\eta} < (z_0 - 3\eta)^2 NC^*(s)$. Thus, by union over at most $n^2 \log n$ points of $A$, we deduce from (3.13) that requirement (c) of Definition 3.2 also holds with probability going to one in $n$ (where by requirement (b) we know that for all $\eta$ small enough it suffices to consider here only $z_0 \geq 5\eta$). With $\kappa$ independent of $n$, this establishes the first/left part of (3.4).

As we mentioned before, in dealing with the second/right part of (3.4) it suffices to consider only pairs of admissible types, with $z_0 \geq 5\eta$ and further by requirement (c) of Definition 3.2 assume that the number of $r$-excursions of each point $x$ considered is at least $(z_0 - 3\eta)^2 NB^*(s)$. Given the position of their starting and ending points, the $r$-excursions for different choices of $x \in A$ are mutually independent and further independent of the random subset $\Gamma_x(0)$ of $A_{2D,0}$. Recall our modification of $\Gamma_x$ per Remark 3.3, hence $x \in A$ for which the $R$-sized annulus around $y(x)$ is of $z_0$-type for one walk and $z'_0$-type for the other, is by (3.12) in both $\Gamma_x$ and $\Gamma'_{x,z}$ with probability much smaller than $\bar{p} := n^{-\alpha(z_0-4\eta)}\bar{\alpha}(z'_0-4\eta)$ (utilizing here the independence of the two copies of the SRW on $G_{0}(a)$ considered in (3.4). Next, fixing $s > \Psi_{L,\eta}(\phi)$ and
admissible types \( \bar{z}, \bar{z}' \), let
\[
\Gamma_{\bar{z}}(k) := \{ y_k \in A_{2D,k} \text{ for some } y \text{ of } \bar{z} \text{ - type} \},
\]
for \( k = 0, \ldots, L \), with \( \Gamma'_{\bar{z}}(k) \) the same sets for an independent srw on \( G_n(a) \). In particular,
\[
\Gamma_{\bar{z}}(0) := \{ y_0 \in A_{2D,0} \text{ of } \bar{z} \text{ - type} \},
\]
and in view of the uniformity of (3.12) per Remark 3.7, we deduce from the preceding discussion that \( |\Gamma_{s,z} \cap \Gamma'_{s,z}'| \) is stochastically dominated by the sum of \( J_0 := |\Gamma_{\bar{z}}(0) \cap \Gamma'_{\bar{z}}(0)| \) i.i.d. Binomial\((m, \bar{p})\) random variables, say \( \{\xi_{\ell}\} \), which are further independent of \( J_0 \), where
\[
m := |\{ x \in A : y(x) = y \}| \leq (2R')^2 h.
\]
The right-side of (3.4) is thus an immediate consequence of
(3.16)
\[
\mathbf{E} \left[ (1 + h^4\bar{p})^{J_0} \right] \to 1,
\]
and the fact that for all \( n \) large enough,
(3.17)
\[
\mathbf{E}[2^{n\xi}] = [1 + (2^k - 1)\bar{p}]^n \leq 1 + eu m, \text{ which is applicable whenever } um \in [0, 1].
\]
Turning to prove (3.16), let \( J_k := |\Gamma_{\bar{z}}(k) \cap \Gamma'_{\bar{z}}(k)| \), for \( k = 0, \ldots, L - 1 \), and note that for any \( k = 0, \ldots, L - 1 \), given their starting and ending points, the inner parts of the \( R_{k+1} \)-cylinders for different choices of \( y_{k+1} \in A_{2D,k+1} \) are independent of each other, and of the random subset \( \Gamma_{\bar{z}}(k + 1) \). Thus, as in the preceding derivation, the contributions \( \{\xi_{\ell}, \ell = 1, \ldots, J_{k+1}\} \) to \( J_k \) that correspond to the possible \( y_{k+1} \in A_{2D,k+1} \cap \Gamma'_{\bar{z}}(k + 1) \), are stochastically dominated by mutually independent random variables \( \{\tilde{\xi}_{\ell}\} \), each having maximal size \( m_k \) and mean \( m_k \bar{p}_k \), which are further independent of \( J_{k+1} \). Here, \( m_k := n^2(\rho_{k+1} - \rho_k) = n^2/L \) bounds the maximal number of points \( y_k \in A_{2D,k} \) inside a \( R_{k+1} \)-cylinder around \( y_{k+1} \in A_{2D,k+1} \), and considering the definition of \( \bar{z}, \bar{z}' \), and the uniform upper bound of (3.15) in case \( j = k + 1 \), \( w = z_j - 2\eta \) and \( z = z_k \) (or \( w = z_\ell' - 2\eta \) and \( z = z_k' \), we deduce by the independence of the two copies of the srw on \( G_n(a) \) that
\[
\bar{p}_k := n^2 - 2sL(z_{k+1} - 2\eta - z_k) + 2sL(z_{k+1} - 2\eta - z_k) \leq 1.
\]
Thus, though each \( \xi_{\ell} \) is no longer Binomial (there are dependencies within each \( R_{k+1} \)-cylinder), starting with \( k = 0 \), \( u_0 = h^4\bar{p} \) and following same argument as in derivation of (3.17), we have for \( k = 0, 1, \ldots, L - 1 \) and \( u_{k+1} = eu_k \bar{p}_k m_k \), that
(3.18)
\[
\mathbf{E} \left[ (1 + u_k)^{J_k} \right] \leq \mathbf{E} \left[ (1 + u_k)^{\xi_{\ell}} \right]^{J_{k+1}} \leq \mathbf{E} \left[ (1 + u_k)^{J_{k+1}} \right],
\]
provided \( u_k m_k \leq 1 \) for \( k = 0, \ldots, L - 1 \). The latter inequality holds since for \( s > \Psi_{L,\eta}(\phi) \)
\[
u_km_k = e^{kh^4n - \gamma_{L,\eta}(z) - \gamma_{L,\eta}(z')} \to 0,
\]
by the definition of \( \Psi_{L,\eta}(\phi) \) and \( \gamma_{L,\eta}(z) \) (c.f. (3.6)). Iterating (3.18) over \( 0 \leq k \leq L - 1 \), we thus find that
\[
\mathbf{E} \left[ (1 + u_0)^{J_0} \right] \leq \mathbf{E} \left[ (1 + u_L)^{J_L} \right] \to 1.
\]
since \( J_L \leq |A_{2D,L}| \) is uniformly bounded (in \( n \)), whereas by the preceding, \( u_L \to 0 \) as \( n \to \infty \).

4. Lower bound on mixing time

Fixing \( s < \Psi(\phi) \), in view of (3.11) and the variational formulation (1.11) of \( \Psi(\phi) \), there exist \( \rho, \delta \) and \( w > z > (1 + w/\rho)\delta \), all in \( (0,1] \), such that

\[
(4.1) \quad b_\rho(w - z) \geq 2\delta \quad \text{and} \quad \alpha(z + 3\delta) + \frac{2s(w - z + \delta)^2}{\rho - \delta} \leq \rho - 5\delta.
\]

Next, fixing \( \epsilon > 0 \) let \( s' = (1 - \epsilon)s \) and \( Q_{s'} \) denote the law of the SRW on the lamplighter group at time \( s't_{\text{cov}}^3 \), starting from all lamps off, with \( Q_\infty \) the uniform law over the set of \( 2^V \) possible lamp configurations. To take advantage of results of [DPRZ06] for the 2D projection of the SRW on \( G_n(a) \), we consider throughout this section \( R_k = R_{k+1} = (k!)^3 \) for \( k = 1, \ldots, m \) with \( m \) an integer such that \( n = K_m := m^3R_m \) for some \( \gamma \in [b + 12, b + 16] \) (and \( b \geq 10 \) a universal constant from [DPRZ06] Lemma 4.2). Let \( Z_m \) denote a maximal set of \( 4R_{m+4} \)-separated points on 2D base of \( G_n(a) \) excluding those within distance \( R_m \) of the starting position 0 of the 2D projected SRW, such that \( (0, 2R_m) \in Z_m \) (as in [DPRZ06] Equation (10.3)) for \( \beta = \rho \) and \( K_m = n \). Let now

\[
Z'_m := Z_m \bigcap \bigcup_{v_i} C(v_i, R_{m-2}),
\]

where the collection \( \{v_i\} \) forms a maximal \( 4R_m \)-separated set on the 2D base of \( G_n(a) \).

Clearly, \( t_{\text{mix}} \geq s't_{\text{cov}}^3 \) if \( \|Q_{s'} - Q_\infty\|_{\text{TV}} \to 1 \), with cut-off holding (in view of Section 3), should this apply for any \( s < \Psi(\phi) \) and \( \epsilon > 0 \). To this end, for any \( v \in Z'_m \), let \( \kappa_n(v) \) denote the number of points of \( G_n(a) \) in the \( R_{m-2} \)-sized cylinder around \( v \), and \( G_v \) denote the difference of “off-lamps” minus “on-lamps” among these points. Considering

\[
W_n = \max_{v \in Z'_m} \{G_v\},
\]

it then suffices to show that as \( n \to \infty \),

\[
(4.2) \quad Q_\infty[W_n \geq n^{\rho + \delta}] \to 0 \quad \text{and} \quad Q_{s'}[W_n < n^{\rho + \delta}] \to 0.
\]

Turning to prove (4.2), note first that under \( Q_\infty \) the variables \( \{G_v, v \in Z'_m\} \) are mutually independent, with \( G_v \) having the law of the sum of \( \kappa_n(v) \) i.i.d. symmetric \( \{\pm 1\} \)-valued variables \( \{I_{v,j}\} \). Further, \( \sup_v \kappa_n(v) \leq Cn^{2\rho} \) and \( |Z'_m| \leq Cn^{2(1 - \rho)} \) for some \( C \) finite and all \( n \). Recall that \( \mathbb{E}[e^{\zeta I_{v,j}}] \leq e^{\zeta^2/2} \) for all \( \zeta \), hence by the union bound over at most \( Cn^{2} \) values of \( v \in Z'_m \) and the uniform tail bound

\[
(4.3) \quad \sup_{r \leq Cn^{2\rho}} \mathbb{P} \left[ \sum_{j=1}^r I_{v,j} \geq n^{\rho + \delta} \right] \leq e^{-n^{\delta + C\zeta^2/2}},
\]

we have that the left-side of (4.2) holds for any \( \delta > 0 \). Turning to deal with the right-side of (4.2), note that \( \log n = (3 + o(1))m \log m \) and for \( R' = R_m/R_m' = m^3 \) the value of \( N^\beta(s) \) of (3.3) is within \( 1 + o(1) \) (as \( n \to \infty \)), of \( n_m(2s) := 6sm^2 \log m \) of [DPRZ06]. Thus, the probability of event \( A_s \) where \( n_m(2s) \) exceeds the maximal number of \( R_m \)-excursions over all \( n^2 \) cylindrical annuli centered at some \( x \) on the
2D base of \( \mathcal{G}_n(a) \), which are made by the SRW on \( \mathcal{G}_n(a) \) up to time \( s't^2_{\text{cov}} \), approach one as \( n \to \infty \) (this follows from [DPRZ06, Equation (3.18)], the union bound and standard exponential tail bounds on the number of actual steps taken by a lazy SRW). Next, with \( \tilde{U}_v \) denoting the number of unvisited points within the relevant \( R_{pm-2} \)-sized cylinder on \( \mathcal{G}_n(a) \), during the first \( n_m(2s) \) excursions by the SRW of the corresponding \( R_m \)-sized cylinder, we claim that the right-side of (4.2) holds as soon as

\[
(4.4) \quad \mathbb{P}[\max_{v \in \mathcal{Z}_m} \{\tilde{U}_v\} \geq 2n^{\rho+\delta}] \to 1.
\]

Indeed, \( Q_s[W_n < n^{\rho+\delta}] \) is bounded above by \( \mathbb{P}[\mathbf{A}_v^\star] \) plus the probability of the complement of the event in (4.4) and

\[
(4.5) \quad \sum_{v \in \mathcal{Z}_m} Q_s\left[ \sum_{j \notin \tilde{U}_v} I_{v,j} \leq -n^{\rho+\delta} \right].
\]

Since conditional on the path of the SRW on \( \mathcal{G}_n(a) \) the variables \( \{I_{v,j}, j \notin \tilde{U}_v\} \) retain under \( Q_s \) their symmetric i.i.d. ±1-valued law, the sum in (4.5) is small by the uniform tail bound of (4.3), so it suffices to prove that (4.4) holds. To this end, let \( N_{m,k}^v \) denote the number of \( R_k \)-excursions during the first \( n_m(2s) \) excursions by the SRW of the corresponding \( R_m \)-cylinder centered at \( v \). Then, considering [DPRZ06, Equation (10.3)] for \( \beta = \rho \) and \( \gamma \beta = w - \delta \) (which results with \( (1 - \beta)(2 - a^*) = 2b_\gamma(w - \delta) \geq 4\delta \)), we call \( v \in \mathcal{Z}_m \) an \((m, \beta)\)-pre-qualified point if \( N_{m,k}^v \in [\hat{n}_k - \delta, \hat{n}_k + \delta] \) for \( \hat{n}_k \) of [DPRZ06, Equation (10.2)] and all \( \beta m \leq k \leq m - 1 \).

Note that we have increased by \( m^{12} \) the value of \( K_m = n \) in order to compensate for such factor in \( |\mathcal{Z}_m'|/|\mathcal{Z}_m| \), as needed en route to [DPRZ06, Equation (10.7)]. Further, while for \((m, \beta)\)-qualified points the derivation of [DPRZ06, Equation (10.3)] applies only for \( 2s < 2 \), we have no such restriction here, on account of removing the event \( \mathbf{A}_v^\star \) when defining pre-qualified points (hence no longer needing [DPRZ06, Equation (10.8)]). Now, from [DPRZ06, Equation (10.3)] we have that with probability approaching one as \( n \to \infty \), there exists an \((m, \beta)\)-pre-qualified \( v \in \mathcal{Z}_m' \) which in particular has less than \( w^2 n_m(2s) \) excursions of the \( R_m \)-sized cylindrical annulus around it, completed during the first \( n_m(2s) \) of the corresponding \( R_m \)-excursions. Thus, ordering the points of \( \mathcal{Z}_m' \) in some non-random fashion, we select the first \( v^* \in \mathcal{Z}_m' \) with less than \( w^2 n_m(2s) \) excursions of \( R_m \)-sized cylindrical annulus around it within first \( n_m(2s) \) excursions of corresponding \( R_m \)-sized annulus. Now, since by definition, points in \( \mathcal{Z}_m' \) are \( 4R_{pm-4} \)-separated and the \( R_m \)-sized cylindrical annulus around each is of distance \( R_{m-1} \geq R_{pm} \) from any (other) point of \( \mathcal{Z}_m' \), our choice of \( v^* \) is measurable on the \( \sigma \)-algebra generated by the exterior parts of \( R_{pm} \)-excursions of the SRW for \( v^* \).

Setting

\[
(4.6) \quad \lambda := 2s \frac{(w - z + \delta)^2}{(\rho - \delta)^2},
\]

we proceed to show that conditional on \( v = v^* \), the event

\[
\mathbf{A} := \{\bar{U}_v \geq \eta (2-\lambda) (\rho - \delta) - \delta\}
\]

occurs with probability \( 1 - o_n(1) \), where \( \bar{U}_v \) counts the points \( y \) in the maximal set \( Z_{\delta m}(v) \) of \( 4R_{\delta m} \)-separated points in the disc \( C(v, R_{pm-2}) \) on the 2D base of \( \mathcal{G}_n(a) \), for
which the 2D (projected) SRW made no more than \(z^2 n_m(2s)\) of the corresponding \(R_{\delta m}\)-excursions within its first \(w^2 n_m(2s)\) completed excursions of the \(R_{\rho m}\)-sized cylindrical annulus centered at \(v\). To this end, we first apply [DPRZ06, Lemma 2.4] for \(r = R_{\rho m-2}\), \(R = R_{\rho m-1}\), \(R' = R_{\rho m}\) and the event \(A\), which is measurable on the interior parts of first \(w^2 n_m(2s)\) excursions for \(R_{\rho m}\)-sized cylindrical annulus around \(v\), to deduce that suffices to show that the unconditional probability (for non-random \(v\), is \(P[A] = 1 - o_n(1)\). Then, setting \(\hat{R} = R_{\rho m} + R_{\rho m-2}\) and \(\hat{\rho} = R_{\rho m-1} - R_{\rho m-2}\), as in [DPR07, proof of Lemma 6.1], note that for the choice of positive parameters \(\gamma := \rho, \beta := \omega, \eta := \delta\) and \(A := ((z - \delta) \rho - w \delta)/(w - z + \delta)\), each \((m, \rho)\)-pre-sluggish point (see definition there), of \(Z_{\delta m}(v)\) contributes to \(\mathcal{U}_v\). In the course of proving [DPR07, Lemma 6.2] it is shown that the number of \((m, \rho)\)-pre-sluggish points concentrates with probability \(1 - o_n(1)\) around its mean value, which turns to be \(R_m^{(2-\lambda)(\gamma - \eta) - o_n(1)}\) (see [DPR07, Equations (6.6) and (6.7)]), out of which it follows that \(P[A] \to 1\) as \(n \to \infty\). We note in passing that the values of \(\lambda, \beta, \gamma\) considered here are outside the range of validity of [DPR07, Lemmas 6.1 and 6.2]. However, this restriction in [DPR07] is only due to the requirement [DPR07, Equation (6.10)] that any \((m, \gamma)\)-pre-sluggish point should be with very high probability also \((m, \gamma)\)-sluggish, a requirement we completely abandon here. Once again, note that we can determine which \(y \in Z_{\delta m}(v)\) are \((m, \gamma)\)-pre-sluggish, based only on the exterior parts of all relevant \(R_{\delta m}\)-excursions, so in particular, with probability \(1 - o_n(1)\) the variable \(\hat{U}_v\), stochastically dominates the sum of \(n^{(2-\lambda)(\rho - \delta) - \delta}\) i.i.d. Bernoulli \(q_n\) variables, where \(q_n\) is the minimal probability (over the location of the excursions end points), that \(x = (y, 0)\) is not visited by the SRW on \(Z_n(a)\) during its first \(z^2 n_m(2s)\) excursions of the \(R_{\delta m}\)-sized cylinder centered at \(x\).

Recall that \(n_m(2s)\) is within factor \(1 + o(1)\) of the value of \(\overline{NC}^*(s)\) for \(R_{\delta m} = (\delta m)^3 R_{\delta m}\). Hence, by (3.15) for some \(M\) large enough, as \(n \to \infty\) the probability of having at least \((z + \delta)^2 \overline{NC}^*(s)\) of the \(R_{\rho}\)-excursions during the first \(z^2 n_m(2s)\) of the corresponding \(R_{\delta m}\)-excursions, is bounded away from one. In view of Lemma 3.5 this implies that \(n^{\alpha(z + 3\delta)} q_n \to \infty\).

Finally, it follows from (4.1) and (4.6) that

\[
(2 - \lambda)(\rho - \delta) - \delta - \alpha(z + 3\delta) \geq \rho + 2\delta
\]

hence by the CLT (for Binomial random variables), we conclude from the preceding discussion that (4.4) holds.

5. Proof of Lemma 3.5 3D-like tail probabilities

5.1. Evaluation of typical values. Recall the typical excursion counts

\[
\overline{NC}^*(s) := 2s \left(\frac{\log n}{\log(R/R')}\right)^2 \quad \text{and} \quad \overline{NB}^*(s) := \frac{4s\rho'}{a} \log n,
\]

by time \(s_{\text{conv}}^L\), as in (3.3), setting \(R = MR', R' = MR''\) and \(R'' = h\) integer valued, unless explicitly specified otherwise.

Proposition 5.1. Fixing \(y = (y_0, \ldots, y_L)\) with \(y_k \in A_{2D,k}\) for \(0 \leq k \leq L\) let \(\overline{NC}_{2,k}(s)\) denote the expectation of \(\overline{NC}_{2,k}(s) := \overline{NC}_{y_k,k,L,1}\) in case \(k < L\). That is, the number of \(R_k\)-excursions for such \(y\), completed during the first \(\overline{NC}^*(s)\) of the \(R_L\)-excursions for the corresponding \(y_L \in A_{2D,L}\), with \(\overline{NC}_{y,L}(s)\) denoting the
number of latter \( R_L \)-excursions completed by time \( s t_{\text{cov}}^{1/2} \). Then, for any \( \delta > 0 \), some \( C = C(\delta) > 0 \), all \( M \geq M_0(\delta) \), \( n \geq n_0(\delta, M) \) and \( 0 \leq k \leq L \), we have that

\[
(1 - \delta) \mathcal{N} C^* (s) \leq \mathcal{N} C_{y,k} (s) \leq (1 + \delta) \mathcal{N} C^* (s),
\]

\[(5.1)\]

\[
P \left[ \left| \mathcal{N} C_{y,L} (s) - \mathcal{N} C_{y,L} (s) \right| \geq \delta \mathcal{N} C_{y,L} (s) \right] \leq \exp (-C s (\log n)^2)
\]

\[(5.2)\]

\[
P \left[ \left| \mathcal{N} C_{y,k} (s) - \mathcal{N} C_{y,k} (s) \right| \geq \delta \mathcal{N} C_{y,k} (s) \right] \leq n^{-C s \delta^2}
\]

\[(5.3)\]

**Proof of Proposition 5.1.** Note that \( \mathcal{N} C_{y,L} (s) \) counts the number of excursions between concentric 2D-disks of radii \( R_L \) and \( R_L \) by the projected srw on \( \mathbb{Z}^2 \) during its first \( \frac{4}{7} n^2 (\log n)^2 (1 + o(1)) \) steps [DPRZ04]. (The factor 2/3 is due to elimination of all vertical steps of the original srw on \( \mathcal{G}_n (a) \).) Our first assertion, namely (5.1) in case \( k = L \), thus follows from [DPRZ06] Lemma 3.2. That is, \( \mathcal{N} C_{y,L} (s) \) is up to leading order given by \( \mathcal{N} C^* (s) \). Since \( R_L / R' \) is independent of \( n \), (5.2) likewise follows from [DPRZ06] Lemma 3.2. Note that \( |A_{2D,L}^* | \) does not grow with \( n \). Consequently, this concentration result further implies that \( \mathcal{N} C_{y,k} (s) = \mathcal{N} C_{y,L} (s) (1 + o(1)) \) for \( 0 \leq k < L \), completing the proof of (5.1). The same argument also gives (5.3). \( \square \)

We consider hereafter \( R = M R' = M^2 R'' \), with both \( M \) and \( R'' \geq h \) large enough integers.

**Proposition 5.2.** Suppose that \( x, x' \in \mathcal{G}_n (a) \) with \( |x - x'| \leq R'' \). Let \( \zeta_{t,s}^{x'} \) be the time that the first \( z^2 \mathcal{N} C^* (s) \) excursions from \( \partial \mathcal{C}(x', R') \) to \( \partial \mathcal{C}(x', R) \) have been completed. Then, for any \( \eta > 0 \), all \( M \geq M_0 (\eta) \) and \( n \geq n_0 (\eta, M) \), we have that

\[
(1 - \eta) z^2 \mathcal{N} \mathcal{B}^* (s) \leq \mathbb{E} [\mathcal{N} \mathcal{B}(\zeta_{t,s}^{x'} / t_{\text{cov}})] \leq (1 + \eta) z^2 \mathcal{N} \mathcal{B}^* (s).
\]

The main steps in the proof of Proposition 5.2 are contained in the following sequence of lemmas.

**Lemma 5.3.** Fix \( R \in \mathbb{N} \) and suppose that \( f \) is a positive harmonic function on \( \mathcal{B}(0, R) \) in \( \mathbb{Z}^3 \). For every \( M \geq 2 \), we have that

\[
\max_{u, u' \in \mathcal{B}(0, R')} \frac{f(u)}{f(u')} = 1 + O(M^{-1}).
\]

\[(5.4)\]

Likewise, if \( x \in \mathcal{G}_n (a) \), \( R \leq n/2 \), and \( f \) is a positive harmonic function on \( \mathcal{C}(0, R) \) in \( \mathcal{G}_n (a) \) then for every \( M \geq 2 \) we have that

\[
\max_{u, u' \in \mathcal{B}(0, R')} \frac{f(u)}{f(u')} = 1 + O(M^{-1}).
\]

\[(5.5)\]

**Proof.** We will first give the proof of (5.4). The Harnack inequality [Law91] Theorem 1.7.2] implies that there exists a constant \( C_0 > 0 \) such that

\[
\max_{u, u' \in \mathcal{B}(0, R/2)} \frac{f(u)}{f(u')} \leq C_0.
\]

\[(5.6)\]

It thus follows from [Law91] Theorem 1.7.1] that there exists a constant \( C_1 > 0 \) such that any \( u, u' \in \mathcal{B}(0, R') \) we have

\[
|f(u) - f(u')| \leq R' \frac{C_1}{R} \max_{v \in \mathcal{B}(0, R/2)} f(v).
\]

\[(5.7)\]
Combining (5.6) with (5.7) gives (5.4). Observe that (5.5) follows from (5.4) because any function which is harmonic on $\overline{C(x, R)}$ may be lifted to a harmonic function on a cylinder in $\mathbb{Z}^3$ with radius $R$ and periodic boundary conditions.

We let hereafter $\sigma_W$ denote the first exit time of the SRW $\{X_k\}$ from a given $W \subset \mathcal{G}_n(a)$, and $\mathcal{L}_{s,t}(W) := \sum_{k=s+1}^{t} 1_{\{X_k \in W\}}$ for the SRW local time of $W$ between (possibly random) times $s \leq t$, using also $\mathcal{L}_t(W)$ for $\mathcal{L}_{0,t}(W)$.

**Lemma 5.4.** Fix $x \in \mathcal{G}_n(a)$, $M \geq 2$, $r \leq h$, and let $B' = \mathcal{B}(x, r')$. Suppose that $Z \geq 0$ is a random variable which depends only on $X_{[0,\sigma_{r'}]}$. Fix $W \subseteq \mathcal{G}_n(a)$ which contains $\mathcal{B}(x, \frac{r}{2})$. Then we have that

\[
\max_{w,w' \in \partial W} \sum_{u \in B'} \frac{\mathbb{E}_u[Z | X_{\sigma_w} = w]}{\mathbb{E}_u[Z | X_{\sigma_w} = w']} = 1 + O(M^{-1}).
\]

In particular,

\[
\max_{w \in \partial W} \sum_{u \in B'} \frac{\mathbb{E}_u[Z]}{\mathbb{E}_u[Z | X_{\sigma_w} = w]} = 1 + O(M^{-1}).
\]

**Proof.** Fix $u \in B'$ and $w \in \partial W$. Then we have that

\[
\mathbb{E}_u[Z | X_{\sigma_w} = w] = \sum_{v \in \partial B'} \mathbb{E}_u[Z | X_{\sigma_{w'}} = v] \mathbb{P}_u[X_{\sigma_{w'}} = v | X_{\sigma_w} = w].
\]

By Bayes' rule, we can write

\[
\mathbb{P}_u[X_{\sigma_{w'}} = v | X_{\sigma_w} = w] = \frac{\mathbb{P}_u[X_{\sigma_w} = w | X_{\sigma_{w'}} = v] \mathbb{P}_u[X_{\sigma_{w'}} = v]}{\mathbb{P}_u[X_{\sigma_w} = w]}.
\]

By the strong Markov property, the ratio on the right hand side of (5.9) is bounded from above by

\[
\max_{v,v' \in \partial B'} \frac{\mathbb{P}_v[X_{\sigma_w} = w]}{\mathbb{P}_{v'}[X_{\sigma_w} = w]}.
\]

As $v \rightarrow \mathbb{P}_v[X_{\sigma_w} = w]$ for $v \in \mathcal{B}(x, \frac{r}{2})$ is harmonic, by Lemma 5.3 the expression in (5.10) is in turn equal to $1 + O(M^{-1})$. Combining this with (5.8) and using that $Z \geq 0$ implies the desired result.

**Lemma 5.5.** Suppose that $x, x', y, z \in \mathcal{G}_n(a)$ with $|x - x'| \leq R''$, $y \in \partial B'$ with $B' = \mathcal{B}(x, r')$, and $w \in \partial C$ with $C = C(x', R)$. Let $Z$ be the number of excursions that $X_{[0,\sigma_c]}$ makes from $\partial B'$ to $\partial B$ with $B = \mathcal{B}(x, r)$. Fix $w \in \partial C$. There exists a constant $c_1 > 0$ depending on $r > 0$ but neither on $M$ nor $w$, such that the law of $Z$ conditional on $\{X_{\sigma_c} = w\}$ is stochastically dominated by $1 + Y$, for $Y$ a Geometric random variable with parameter $c_1/M$. In particular, we have that

\[
\mathbb{E}_y[Z | X_{\sigma_c} = w] \rightarrow 1 \quad \text{as} \quad n \rightarrow \infty \quad \text{and} \quad M \rightarrow \infty
\]

uniformly in $y$ and $w$. Moreover, for each $z \in \partial \mathcal{B}(x, h)$ we have that

\[
\mathbb{E}_z[\mathcal{L}_{\sigma_c}(B')] | X_{\sigma_c} = w \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty \quad \text{and} \quad M \rightarrow \infty
\]

uniformly in $z$ and $w$. 

Proof. Suppose that \( u \in \partial \mathcal{B} \). For each \( j \), we let \( \sigma_j^x = \sigma_{\mathcal{B}(x,j)} \). Let \( A_M = \mathcal{B}(x, \frac{h}{M}) \setminus C(x', \frac{h}{M}) \). We first claim that there exists a constant \( C_1 > 0 \) such that
\[
\mathbb{P}_u[X_{\sigma_j^x} \notin A_M \mid X_{\sigma_C} = w] \leq \frac{C_1}{M}.
\]
Indeed, Lemma 5.4 implies that it suffices to prove that there exists a constant \( C_1 > 0 \) such that \( \mathbb{P}_u[X_{\sigma_j^x} \notin A_M] \leq \frac{C_1}{M} \) in place of the bound (5.13). The claim thus follows from [Law91, Lemma 1.7.4]. Lemma 5.4 combined with [Law91, Theorem 1.5.4] implies that there exists a constant \( C_2(r) > 0 \) such that
\[
\mathbb{P}_u[\tau_{\partial \mathcal{B}} < \sigma_{h/4}^x \mid X_{\sigma_C} = w] \leq \frac{C_2(r)}{M}.
\]
We next claim that there exists a constant \( C_3(M) > 0 \) that for all \( u \in A_M \) we have that
\[
\mathbb{P}_u[\tau_{\partial \mathcal{B}} < \sigma_C \mid X_{\sigma_C} = w] \leq \frac{C_3(M)}{\log \log n}.
\]
Indeed, by Bayes’ rule we can rewrite the above as
\[
\frac{\mathbb{P}_u[X_{\sigma_C} = w \mid \tau_{\partial \mathcal{B}} < \sigma_C] \mathbb{P}_u[\tau_{\partial \mathcal{B}} < \sigma_C]}{\mathbb{P}_u[X_{\sigma_C} = w]}
\]
By [Law91 Exercise 1.6.8], the final factor is of order \( C_3(M)/\log \log n \), so to complete the proof of (5.13) it suffices to show that the ratio above is bounded. Applying the strong Markov property for the first time that \( X \) hits \( \partial \mathcal{B}(x, \frac{h}{M}) \), it in turn suffices to show that
\[
\max_{u, u' \in \partial \mathcal{B}(x, \frac{h}{M})} \frac{\mathbb{P}_u[X_{\sigma_C} = w]}{\mathbb{P}_{u'}[X_{\sigma_C} = w]}
\]
is bounded. This follows from [Law91 Theorem 1.7.2] since \( u \mapsto \mathbb{P}_u[X_{\sigma_C} = w] \) is harmonic. Combining (5.13), (5.14), and (5.15) yields the claim regarding the law of \( Z \). This, in turn, implies (5.12). \( \square \)

Lemma 5.6. Suppose \( x, x' \in \mathcal{G}_n(a) \) with \( |x - x'| \leq R'' \). Let \( \mathcal{C} = \mathcal{C}(x', R), \mathcal{C}' = \mathcal{C}(x', R') \) and \( G^w(y, x) \) denoting the Green function for \( X \) stopped upon hitting \( \partial \mathcal{C} \) conditioned on exiting \( \mathcal{C} \) at a given \( w \in \partial \mathcal{C} \). Then, for any \( y \in \mathcal{C} \setminus \mathcal{C}' \) and \( \alpha < 2 \)
\[
G^w(y, x) = \frac{3 + O(M^{-1})}{\pi h} \left( \log R - \log |y - x| + o(|y - x|^{-\alpha}) + O(n^{-1}) \right).
\]

Proof. Let \( \tau_x \) be the first time that \( X \) hits \( x \), and let \( \tau_x^+ \) be the first positive hitting time. By the strong Markov property of \( X \) at time \( \tau_x^+ \), we have
\[
\mathbb{P}_x[X_{\sigma_C} = w \mid \tau_x^+ \leq \sigma_C] = \mathbb{P}_x[X_{\sigma_C} = w],
\]
i.e., the events \( \{X_{\sigma_C} = w\} \) and \( \{\tau_x^+ \leq \sigma_C\} \) are independent. Thus
\[
\mathbb{P}_x[\tau_x^+ > \sigma_C \mid X_{\sigma_C} = w] = \mathbb{P}_x[\tau_x^+ > \sigma_C];
\]
taking reciprocals, \( G^w(x, x) = G(x, x) \), where \( G \) is the (unconditioned) Green kernel for \( X \) stopped upon hitting \( \partial \mathcal{C} \).

By the strong Markov property for \( X \) conditioned on \( \{X_{\sigma_C} = w\} \), we have that
\[
G^w(y, x) = \mathbb{P}_y[\tau_x \leq \sigma_C \mid X_{\sigma_C} = w] G^w(x, x).
\]
By Bayes' rule, we can write the first factor on the right side as
\[ P_y[\tau_x \leq \sigma_C \mid X_{\sigma_C} = w] = \frac{P_y[X_{\sigma_C} = w \mid \tau_x \leq \sigma_C]}{P_y[X_{\sigma_C} = w]} P_y[\tau_x \leq \sigma_C] \]
\[ = \frac{P_x[X_{\sigma_C} = w]}{P_y[X_{\sigma_C} = w]} P_y[\tau_x \leq \sigma_C]. \]

Since \( G(y, x) = P_y[\tau_x \leq \sigma_C]G(x, x) \), combining the above we see that
\[ (5.16) \quad G^{w}(y, x) = \frac{P_y[X_{\sigma_C} = w]}{P_y[X_{\sigma_C} = w]} G(y, x). \]

Since \( v \mapsto P_v[X_{\sigma_C} = w] \) is harmonic, by Lemma 5.3 we arrive at
\[ (5.17) \quad G^{w}(y, x) = (1 + O(M^{-1}))G(y, x). \]

It thus remains only to estimate \( G(y, x) \). To this end, let \( G_{\mathbb{Z}_2^n} \) denote the Green function associated with the projected (unconditioned) random walk in \( \mathbb{Z}_2^n \) stopped upon exiting the disk of radius \( R \) centered at (the projection of) \( x \). Note that the projected random walk has a 1/3 holding probability since this is the probability that the (unprojected) walk moves in the vertical direction. Let \( W_x \) be those points in \( G_n(a) \) whose 2D projection is equal to \( x \). Then
\[ (5.18) \quad G_{\mathbb{Z}_2^n}(y, x) = \sum_{z \in W_x} G(y, z). \]

Since \( z \mapsto G(y, z) \) (for \( y \) fixed) is harmonic for \( z \neq y \), Lemma 5.3 implies that
\[ (5.19) \quad \frac{G(y, z)}{G(y, z')} = 1 + O(M^{-1}) \quad \text{for all} \quad z, z' \in W_x. \]

Moreover, [Law91] Proposition 1.6.7] gives us that for every \( \alpha < 2 \) we have
\[ G_{\mathbb{Z}_2^n}(y, x) = \frac{3}{\pi} (\log(n) - \log |y - x|) + o(|y - x|^{-\alpha}) + O(n^{-1}) \]
(recall the 1/3 laziness). Combining this with (5.18) and (5.19) tells us that for every \( \alpha < 2 \) we have
\[ G(y, x) = (1 + O(M^{-1})) \left( \frac{3}{\pi} (\log(n) - \log |y - x|) + o(|y - x|^{-\alpha}) + O(n^{-1}) \right). \]

Combining this with (5.17) gives the result. \( \square \)

**Lemma 5.7.** Fix \( x, x' \in G_n(a) \) with \( |x - x'| \leq R' \), let \( C' = C(x', R') \), \( C = C(x', R) \), and \( B' = B(x, r') \). For each \( z \in \partial B' \) and \( w \in \partial C' \), we let
\[ \mathcal{E}_{B', z}^w = E_z[\mathcal{L}_{\sigma_C}(B') \mid X_{\sigma_C} = w]. \]

Then,
\[ (5.20) \quad \frac{\mathcal{E}_{B', z}^w}{2(r')^2} \to 1 \quad \text{as} \quad n \to \infty \quad \text{then} \quad r' \to \infty \quad \text{and then} \quad M \to \infty. \]

**Proof.** We will first reduce (5.20) to a computation which involves srw in \( \mathbb{Z}^3 \). Let \( \sigma_{h/M}^x \) be the first time that \( X \) leaves \( B(x, \frac{h}{2M}) \). Then we have that
\[ \mathcal{E}_{B', z}^w = E_z[\mathcal{L}_{\sigma_{h/M}^x}(B') \mid X_{\sigma_C} = w] + E_z[\mathcal{L}_{\sigma_{h/M}^x}(B') \mid X_{\sigma_C} = w]. \]
The latter term is \( o(1) \) as \( n \to \infty \) and then \( r' \to \infty \) by \([5.12]\) of Lemma 5.5. By Lemma 5.4, we know that the former term is equal to \( (1 + O(1/M))E_z[\mathcal{L}_{\sigma_{h/M}}(\mathcal{B}')] \).

Let \( \tilde{X} \) be a random walk in \( \mathbb{Z}^3 \) and let \( \tilde{Z} \) be the amount of time that \( \tilde{X} \) spends in \( \mathcal{B}' \). Using \([\text{Law91}], \text{Theorem 1.5.4}\) and that \( h = \Theta(\log n) \) we in turn have that

\[
E_z[\mathcal{L}_{\sigma_{h/M}}(\mathcal{B}')] = E_z[\tilde{Z}] + O(1). \tag{5.21}
\]

By \([\text{Law91}], \text{Theorem 1.5.4}\), we have

\[
\frac{1}{(r')^2}E_z[\tilde{Z}] \to \int_{B(0,1)} \frac{c_3}{|u - 1|} du \quad \text{as} \quad r' \to \infty,
\]

where \( du \) denotes Lebesgue measure on \( \mathbb{R}^3 \), \( c_3 := 2/(3\pi) \) is given explicitly in \([\text{LL10}], \text{Theorem 4.3.1, top of page 82}\), and \( B(0,1) = \{ v \in \mathbb{R}^3 : |v| < 1 \} \) is the unit ball in \( \mathbb{R}^3 \); we note that an additional factor of \( r' \) appears in the normalization from spatially rescaling. The convergence, moreover, is uniform in the starting point \( z \) of \( \tilde{X} \). In particular, this implies that the asymptotic behavior of \( E_z[\tilde{Z}] \) does not depend on the specific choice of \( z \in \partial \mathcal{B}' \). Thus to compute the leading order of \( E_z[\tilde{Z}] \) for \( z \in \partial \mathcal{B}' \) and \( r' \) large, it suffices to compute the leading order of \( E_z[\tilde{Z}] \) as we first take a limit as \( |z| \to \infty \) and then \( r' \to \infty \). Applying \([\text{Law91}], \text{Theorem 1.5.4}\) once more along with \([\text{Law91}], \text{Proposition 1.5.10}\), we see that

\[
E_z[\tilde{Z}] \sim \frac{4\pi(r')^3}{3} c_3 \quad \text{and} \quad P_z[\tilde{Z} > 0] \sim \frac{r'}{|z|} \quad \text{as} \quad |z| \to \infty \quad \text{and then} \quad r' \to \infty.
\]

Combining gives \((5.20)\). □

**Lemma 5.8.** Fix \( x, x' \in \mathbb{G}_n(a) \) with \( |x - x'| \leq R' \), let \( C' = \mathcal{C}(x', R') \), \( C = \mathcal{C}(x', R) \), and \( \mathcal{B}' = \mathcal{B}(x, r') \). Let

\[
F_{C,B} := \frac{\nabla\mathcal{B}'(s)}{\nabla\mathcal{C}'(s)} = \frac{2r'}{h} \log(R/R'). \tag{5.23}
\]

Fix \( z \in \partial C' \) and \( w \in \partial C \). Then

\[
Q_{C,B,z}^w := P_z[\tau_{\mathcal{B}'} < \sigma_C \mid X_{\sigma_C} = w] = F_{C,B}(1 + o(1)).
\]

**Proof.** Recall that if \( Z \) is a non-negative random variable with \( P[Z > 0] > 0 \) then \( P[Z > 0] = E[Z]/E[Z \mid Z > 0] \). For \( z \in \partial C' \), we let

\[
\mathcal{E}_{\mathcal{B}',z}^w = E_z[\mathcal{L}_{\sigma_C}(\mathcal{B}') \mid X_{\sigma_C} = w] \quad \text{and} \quad \mathcal{H}_{\mathcal{B}',z}^w = E_z[\mathcal{L}_{\sigma_C}(\mathcal{B}') \mid X_{\sigma_C} = w, \tau_{\mathcal{B}'} < \sigma_C].
\]

Then, we have that

\[
Q_{C,B,z}^w = \frac{\mathcal{E}_{\mathcal{B}',z}^w}{\mathcal{H}_{\mathcal{B}',z}^w}.
\]

We thus arrive at the stated formula for \( Q_{C,B,z}^w \) by showing that as \( n \to \infty \), then \( r \to \infty \) and finally \( M \to \infty \),

\[
\mathcal{E}_{\mathcal{B}',z}^w \sim \frac{4(r')^3}{h} \log(R/R') \quad \text{and} \quad \mathcal{H}_{\mathcal{B}',z}^w \sim 2(r')^2.
\]
Proof of Proposition 5.2. By the strong law of large numbers, suffices to show that

\[ \mathcal{E}_{C(x,r')}^{w}(z) := \mathbb{E}_{z}[\mathcal{L}_{\sigma_{C}}(C(x,r')) \mid X_{\sigma_{C}} = w] \sim 3(r')^{2} \log(R/R'). \]

Applying Lemma 5.6 a second time implies that

\[ \mathbb{E}_{\partial} \text{the expected number of excursions from} \]

Note that by Lemma 5.6 the Green’s function \( G^{w}(z, \bar{x}) \) is approximately constant on points \( \bar{x} \in C(x, r') \) when \( X \) starts at \( z \in C' \). Consequently,

\[ \frac{1}{3} \pi(r')^{3} \frac{\pi(r')^{2} h}{3 h} = \frac{4 r'}{\pi(r')^{2} h} \]

leading to the stated formula \((5.24)\) for \( \mathcal{E}_{B'}^{w} \). Finally, note that

\[ \mathcal{H}_{B',z}^{w} = \mathbb{E}_{z}[\mathcal{E}_{B',X_{\tau'}^{w}}^{w} \mid X_{\sigma_{C}} = w, \tau_{B'} < \tau_{C}] , \]

hence \((5.25)\) follows from Lemma 5.7. \( \square \)

Proof of Proposition 5.2. By the strong Markov property and Lemma 5.3, we see that the ratio is equal to

\[ \mathbb{P}_{y}[\tau_{x} < \tau_{B} \mid X_{\tau_{B}} = z] = (1 + O(M^{-1}))\Delta. \]

Proof. Applying Bayes’ rule, we first note that

\[ \frac{\mathbb{P}_{x}[X_{\tau_{B}} = z]}{\mathbb{P}_{y}[X_{\tau_{B}} = z]} = 1 + O(M^{-1}). \]

Suppose that \( \bar{X} \) is a srw on \( \mathbb{Z}^{3} \) and let \( \tau_{x}, \tau_{B} \) be the corresponding stopping times. Then

\[ \mathbb{P}_{y}[\tau_{x} < \tau_{B}] = 1 - \frac{\mathbb{P}_{y}[\bar{\tau}_{x} = \infty]}{\mathbb{P}_{y}[\bar{\tau}_{x} = \infty \mid \bar{\tau}_{x} \geq \tau_{B}]} . \]

By [Law91, Proposition 6.5.1] (the constant \( c_{3} \) from this proposition is the same as \( c_{3} \) from [Law91, Theorem 1.5.4]) and using the strong Markov property in the second equality, we have

\[ \mathbb{P}_{y}[\bar{\tau}_{x} = \infty] \sim 1 - \frac{c_{3} q}{r'} \quad \text{and} \quad \mathbb{P}_{y}[\bar{\tau}_{x} = \infty \mid \bar{\tau}_{x} \geq \tau_{B}] \sim 1 - \frac{c_{3} q}{r}. \]

Combining this with \((5.27)\) yields the stated value of \( \Delta \) up to leading order. \( \square \)
5.2. Tail probabilities for 3D type events. In this section, we will establish tail probabilities for 3D type events, which imply both (3.12) and (3.13). Both estimates will be established in the strong sense of Remark 3.7.

**Proposition 5.10.** Fix \( x \in G_0(a) \) and let \( \mathcal{F}_x \) be the \( \sigma \)-algebra generated by the entrance and exit points of all the excursions of \( X \) from \( \partial B(x, r') \) to \( \partial B(x, r) \). Let \( \xi_{x,s}^z \) be the first time that \( X \) has completed \( z^2N\mathcal{B}^* (s) \) excursions from \( \partial B(x, r') \) to \( B(x, r) \). For every \( \eta > 0 \) such there exists \( M_0 = M_0(\eta) \) such that for every \( M \geq M_0 \) there exists \( n_0 = n_0(\eta, M) \) such that \( n \geq n_0 \) implies

\[
n^{-\alpha(z+\eta)} \leq P_y [\tau_x > \xi_{x,s}^z | \mathcal{F}_x] \leq n^{-\alpha(z-\eta)}
\]

uniformly in \( y \in G_0(a) \setminus B(x, r') \).

**Proof.** The inner parts of these excursions are independent of each other given \( \mathcal{F}_x \). Thus, the conditional probability considered in (5.28) is the product of \( z^2N\mathcal{B}^* (s) \) probabilities. Proposition 5.9 implies that there exists \( \delta = \delta(M) \downarrow 0 \) as \( M \to \infty \) such that each of these probabilities is bounded from above by \( (1 - \Delta + \delta) \) uniformly in the initial and terminal points of the excursion. Note that

\[
(1 - \Delta)z^2N\mathcal{B}^* (s) \leq \exp(-\Delta z^2N\mathcal{B}^* (s)) = n^{-\alpha(z)}.
\]

The upper bound asserted in the statement of the proposition (and hence the estimate stated in (3.12)) follows since \( \alpha(z-\eta) < \alpha(z) \). The lower bound is proved similarly. \( \square \)

We now turn to establish (3.13). Recall the definition of \( \xi_{x,s}^z \) as given in Proposition 5.2.

**Proposition 5.11.** For every \( \eta > 0 \) there exists \( \gamma > 0 \) such that for all \( n, r' \in \mathbb{N} \) we have that

\[
P \left[ \frac{\mathcal{N}\mathcal{B}(\xi_{x,s}^z/\overline{t}_{\text{cov}})}{\mathcal{N}\mathcal{B}^* (s)} \notin [(z-\eta)^2, (z+\eta)^2] \right] \leq n^{-\gamma r'}.
\]

**Proof.** Fix \( \eta > 0 \). We are first going to show that there exists \( \gamma = \gamma(\eta) > 0 \) such that for all \( n, r' \in \mathbb{N} \)

\[
P \left[ \mathcal{N}\mathcal{B}(\xi_{x,s}^z/\overline{t}_{\text{cov}}) < (z-\eta)^2\mathcal{N}\mathcal{B}^* (s) \right] \leq n^{-\gamma r'}.
\]

Lemma 5.5 and Lemma 5.8 together imply that there exists \( \delta = \delta(M) \downarrow 0 \) as \( M \to \infty \) and \( n_0 = n_0(M) \) such that for all \( n \geq n_0 \) the following is true. The number of excursions from \( \partial B(x, r') \) to \( \partial B(x, r) \) within one excursion from \( \partial \mathcal{C}(x, R) \) to \( \partial \mathcal{C}(x, R') \) is stochastically bounded below by a Bernoulli random variable \( J \) of mean \( p = F_{\mathcal{C}, \mathcal{B}}(1 - \delta) \) uniformly in the initial and terminal points of the excursion. Thus, the probability considered in (5.30) is bounded above by the probability \( P \) that a Binomial\( (N, p) \) variable, \( N = z^2N\mathcal{C}^* (s) \), takes at most the value \( (z-\eta)^2\mathcal{N}\mathcal{B}^* (s) \).

Let \( q = 1 - p \). An application of the Chernoff bound thus implies that, for each \( \theta > 0 \), we have

\[
\frac{1}{N} \log(P) \leq -\theta + \log(1 - q + qe^\theta) = -\theta + \log(1 + q\theta + O(q\theta^2)) = \theta(q - 1) + O(q\theta^2) = -p\theta + O(q\theta^2).
\]
Fix $\theta > 0$ sufficiently small so that
\[-p\theta + O(q\theta^2) \leq -\frac{p\theta}{2}.
\]
Then we have
\[
\mathbb{P}[J \leq (z - \eta)^2 N \mathcal{B}^r(s)] \leq e^{-\theta_p N/2} = e^{-\frac{z^2 \theta N}{2}} = n^{-\gamma r'}
\]
where $\gamma = \theta s z^2 / a$. In particular, the value of $\gamma > 0$ is independent of $r'$.

The same argument also gives that, by possibly decreasing $\gamma = \gamma(\eta) > 0$ we have that
\[
\mathbb{P}\left[ N \mathcal{B}(\zeta_{z,s}^x/t_{\text{cov}}^x) > (z + \eta)^2 N \mathcal{B}^r(s) \right] \leq n^{-\gamma r'}
\]
Indeed, the only difference being that we need to replace the i.i.d. copies of Bernoulli $J$ by i.i.d. copies of the product of Bernoulli $\tilde{J}$ of mean $F_{\tilde{C}, \tilde{B}}(1 - \delta)$ and $1 + Y$ for the Geometric random variable $Y$ of success probability $c_1 / M$ as established in Lemma 5.5.

By combining Proposition 5.10 with Proposition 5.11 we obtain the following.

**Proposition 5.12.** For every $\eta > 0$ there exists $M_0 = M_0(\eta)$ such that for all $M \geq M_0$, there exist $n_0 = n_0(\eta, M)$ such that for all $n \geq n_0$, we have
\[-\alpha(z + \eta) \leq \frac{\log \mathbb{P}[\tau_x > \zeta_{z,s}^x]}{\log n} \leq -\alpha(z - \eta) \quad \text{for all } n \quad \text{large enough.}
\]

**Remark 5.13.** Equipped with these facts, we have the approximate picture that at time $s t_{\text{cov}}^\square$, the unvisited subset of SRW on $G_n(a)$ within the part of $A$ corresponding to each fixed $y \in \mathcal{A}_{2D}^z$ of $z$-type, is well modeled, even in terms of tail probabilities, as collection of i.i.d. Bernoulli $(n^{-\alpha(z)})$ variables, which given types of all points in $\mathcal{A}_{2D}$, are further mutually independent subsets (across the choice of $y$).

6. **Proof of Lemma 3.6: 2D excursion counts at various radii**

This section is devoted to the proof of (3.15). To this end, recall our notations of $R'' = h$, $R = M^2 h$ and for any fixed $L \in \mathbb{N}$ and $k \in \{0, \ldots, L\}$, having $\rho_k = k / L$ and $R_k = n^{\rho_k} R$. Fixing $w, z$ and $j \in \{k + 1, \ldots, L\}$ we let $\mathcal{N} C_{y_k, k, j, w}(s)$ count the number of $R_k$-excursions for $y_k \in \mathcal{A}_{2D,k}$ completed during the $w^2 \mathcal{N} C^*(s)$ first $R_j$-excursions for the corresponding $y_j \in \mathcal{A}_{2D,j}$, with (3.15) stating that for each $\eta \in (0, w \wedge z)$ there exists $M_0 = M_0(\eta)$ such that for all $M \geq M_0$ and $n \geq n_0(\eta, M)$
\[
(6.1) \quad \left| \frac{\log \mathbb{P}[\mathcal{N} C_{0,k,j,w}(s) \leq (z - \eta)^2 \mathcal{N} C^*(s)]}{\log n} + \frac{2s(w - z)^2}{\rho_j - \rho_k} \right| \leq \eta.
\]

Using [DPRZ06] Lemma 2.3 we stochastically dominate $\mathcal{N} C_{0,k,j,w}(s)$ from above and below by comparable variables of a much simpler form and thereby establish (6.1) upon studying the tail behavior of the latter variables. Specifically, fixing $0 \leq k < j \leq L$, set for each $n \in \mathbb{N},$
\[
p_{k \to j}(n) := \frac{\log R_k - \log R'_k}{\log R_j - \log R'_k} \quad \text{and} \quad p_{j \to k}(n) := \frac{\log R_j - \log R'_j}{\log R_j - \log R'_k}.
\]
Note that \( p_{k \to j}(n) \) (resp. \( p_{j \to k}(n) \)) approximates the probability that the srw \( X \) starting from a point in \( \partial \mathcal{C}(0, R_k) \) (resp. \( \partial \mathcal{C}(0, R_j) \)) hits \( \partial \mathcal{C}(0, R_j) \) (resp. hits \( \partial \mathcal{C}(0, R_k) \)) before hitting \( \partial \mathcal{C}(0, R_j) \) [Law91]. Moreover,

\[
\lim_{M \to \infty} \lim_{n \to \infty} \frac{p_{k \to j}(n) \mathcal{N}(s)}{\log n} = \lim_{M \to \infty} \lim_{n \to \infty} \frac{p_{j \to k}(n) \mathcal{N}(s)}{\log n} = \frac{2s}{\rho_j - \rho_k}.
\]

We next show that \( \mathcal{N}_{0,k,j,w}(s) \) is stochastically comparable to

\[
Z_{w,s}(p, p') := \sum_{i=1}^{\mathcal{N}(s)} J_i (1 + Y_i),
\]

where \((J_i)\) and \((Y_i)\) are two independent sequences of i.i.d. random variables, such that each \( J_i \) is Bernoulli\((p)\), each \( Y_i \) is Geometric\((p')\), and the parameters \( p \in (0,1) \) and \( p' \in (0,1) \) are comparable to \( p_{j \to k}(n) \) and \( p_{k \to j}(n) \), respectively.

**Lemma 6.1.** For every \( c > 1 \), \( w > 0 \) and \( L \geq j > k \geq 0 \), all \( M \geq M_0(c, L) \) and \( n \geq n_0(c, L, M) \), if \( p > cp_{j \to k}(n) \) and \( p' < p_{k \to j}(n)/c \), then the law of \( \mathcal{N}_{0,k,j,w}(s) \) is stochastically dominated from above by \( Z_{w,s}(p, p') \). Likewise, if \( p < p_{j \to k}(n)/c \) and \( p' > cp_{k \to j}(n) \) then the law of \( \mathcal{N}_{0,k,j,w}(s) \) is stochastically dominated from below by \( Z_{w,s}(p, p') \).

**Proof.** For each \( i \), let \( \tilde{J}_i \) denote the indicator of the event that the \( i \)th excursion \( E_i \) of the srw \( X \) from \( \partial \mathcal{C}(0, R_j') \) to \( \partial \mathcal{C}(0, R_j) \) hits \( \partial \mathcal{C}(0, R_k) \). We also let \( \tilde{Y}_i \) denote the number of returns that the srw \( X \) makes to \( \mathcal{C}(0, R_k) \) from \( \partial \mathcal{C}(0, R_k) \) before exiting \( \mathcal{C}(0, R_k) \) during \( E_i \). Then,

\[
\mathcal{N}_{0,k,j,w}(s) = \sum_{i=1}^{\mathcal{N}(s)} \tilde{J}_i (1 + \tilde{Y}_i).
\]

Let \( \mathcal{F} \) denote the \( \sigma \)-algebra generated by the entrance and exit points of all excursions \( \{E_i\} \) and \( \mathcal{H} \) denote the \( \sigma \)-algebra generated by \( \mathcal{F} \) as well as all entrance and exit points of the excursions of \( X \) from \( \partial \mathcal{C}(0, R'_k) \) to \( \partial \mathcal{C}(0, R_k) \). By DPRZ06 Lemma 2.3 and Law91, we have that,

\[
\lim_{M \to \infty} \liminf_{n \to \infty} \inf_i \left\{ \frac{P[\tilde{J}_i = 1 | \mathcal{F}]}{p_{j \to k}(n)} \right\} = \lim_{M \to \infty} \limsup_{n \to \infty} \sup_i \left\{ \frac{P[\tilde{J}_i = 1 | \mathcal{F}]}{p_{j \to k}(n)} \right\} = 1,
\]

\[
\lim_{M \to \infty} \liminf_{n \to \infty} \inf_{i, \ell} \left\{ \frac{P[\tilde{Y}_i = \ell | \mathcal{H}, \tilde{Y}_i \geq \ell]}{p_{k \to j}(n)} \right\} = \lim_{M \to \infty} \limsup_{n \to \infty} \sup_{i, \ell} \left\{ \frac{P[\tilde{Y}_i = \ell | \mathcal{H}, \tilde{Y}_i \geq \ell]}{p_{k \to j}(n)} \right\} = 1.
\]

Combining (6.4) and (6.5) yields the desired result because the excursions \( \{E_i\} \) are conditionally independent given \( \mathcal{F} \). \( \square \)

By Lemma 6.1 it suffices to prove the bounds of (6.1) for \( Z_{w,s}(p, p') \) (with \( p/p_{j \to k}(n) \to 1 \) and \( p'/p_{k \to j}(n) \to 1 \)) in place of \( \mathcal{N}_{0,k,j,w}(s) \). Turning to determine the asymptotics of the corresponding probabilities, set

\[
\Lambda_{p,p'}(\theta) := \log \mathbb{E}[e^{-\theta \tilde{J}_i (1+\tilde{Y}_i)}] = \log \left( 1 - p + \frac{pp'}{e^\theta - 1 + p'} \right) \quad \text{for} \quad \theta \geq 0,
\]
and for each \(0 \leq z \leq w \leq 1\), let
\[
I_{p,p'}(z, w) := \frac{1}{p} \inf_{\theta \geq 0} \left\{ z^2 \theta + w^2 \Lambda_{p,p'}(\theta) \right\}.
\]

**Lemma 6.2.** Fix \(\kappa \in (0, \infty)\). Then, we have that for \(w \geq \sqrt{\kappa}z > 0\),
\[
I_\kappa(z, w) := \lim_{p \to 0} I_{p,\kappa}(z, w) = \inf_{v \geq 0} \left( vz^2 - \frac{vw^2}{\kappa + v} \right) = -(w - \sqrt{\kappa}z)^2.
\]

Let \(\theta_p \in [0, \infty)\) be the unique value so that \(\Lambda'_{p,\kappa}(\theta_p) = -(z/w)^2\). Then,
\[
\begin{align*}
\lim_{p \to 0} \frac{\theta_p}{p} & = \sqrt{\kappa} \frac{w}{z} - \kappa := v_* \geq 0, \\
\lim_{p \to 0} p^2 \Lambda''_{p,\kappa}(\theta_p) & = 0.
\end{align*}
\]

**Proof.** We begin by making the substitution \(\theta := \log(1 + pv)\) for \(v \geq 0\), and setting \(f_p(v) := p^{-1} \log(1 + pv)\) rewrite \(I_{p,\kappa}(z, w)\) as
\[
I_{p,\kappa}(z, w) = \inf_{v \geq 0} \left\{ z^2 f_p(v) + w^2 f_p\left(\frac{-v}{\kappa + v}\right) \right\}.
\]

Since \(f_p(v) \uparrow \infty\) as \(v \to \infty\), the infimum in (6.9) is attained at some finite \(v_p\). Further, with \(p \mapsto f_p(v)\) non-increasing, there exists a universal finite constant \(V\) such that \(v_p\) takes its values in \([0, V]\) as \(p \to 0\) and \(\kappa\) fixed. This allows us to change the order of the limit in \(p\) and the infimum over \(v\), yielding
\[
I_\kappa(z, w) = \inf_{v \geq 0} \lim_{p \to 0} \left\{ z^2 f_p(v) + w^2 f_p\left(\frac{-v}{\kappa + v}\right) \right\}.
\]

Since \(f_p(v) \to v\) for \(p \to 0\), the first assertion of the lemma follows upon verifying that the infimum in (6.6) is achieved at \(v_* \geq 0\).

As for confirming (6.7) and (6.8), let \(F_p(v) := f_p(F_0(v))\) for \(F_0(v) = -v/(\kappa + v)\), so \(\Lambda_{p,\kappa}(\theta) = pF_p(v)\), under the substitution \(\theta = \log(1 + pv)\). Differentiating both sides of this identity twice and rearranging, we find that
\[
p^2 \Lambda''_{p,\kappa}(\theta) = p(1 + pv)(F''_p(v)(1 + pv) + pF'_p(v)).
\]

Since the infimum in the definition of \(I_{p,\kappa}(z, w)\) is attained at \(\theta_p\), necessarily \(\theta_p = pF_p(v_p)\). Thus, as \(p \to 0\) we have that \(p^{-1}(e^\theta_p - 1) = v_p \to v_*\), from which (6.7) follows. Further, \(F'_p(v_p) \to F'_0(v_*)\) and \(F''_p(v_p) \to F''_0(v_*)\), yielding (6.8) in view of (6.10). \[\square\]

As explained before, (6.1) follows upon combining Lemma (6.1) with the following lemma.

**Lemma 6.3.** Fix \(s \geq 1\), \(\kappa \in (0, \infty)\) and \(w \geq \sqrt{\kappa}z > 0\). If \(p_n \log n\) are uniformly bounded above and uniformly bounded away from zero, then
\[
\lim_{n \to \infty} \frac{1}{p_n \mathcal{NC}^*(s)} \log P[Z_{w,s}(p_n, \kappa p_n) \leq z^2 \mathcal{NC}^*(s)] = -(w - \sqrt{\kappa}z)^2.
\]

**Proof.** Fix \(s \geq 1\), \(\kappa \in (0, \infty)\) and \(w \geq \sqrt{\kappa}z > 0\). Now, for any \(p \in (0, 1)\) we get by applying Chernoff’s bound, then optimizing over \(\theta \geq 0\) that
\[
\frac{1}{p \mathcal{NC}^*(s)} \log P[Z_{w,s}(p, \kappa p) \leq z^2 \mathcal{NC}^*(s)] \leq I_{p,\kappa}(z, w).
\]
Thus, in view of (6.6), considering \( p = p_n \to 0 \) yields the upper bound in (6.11).

For the lower bound we use a change of measure analogous to the proof of the lower bound in Cramer’s theorem (see [DZ10, Theorem 2.2.3]). Specifically, fixing \( p \in (0, 1) \) and \( \delta > 0 \) small (we eventually send \( \delta \to 0 \)), we set \( \theta = \theta_p \geq 0 \) be the unique value such that \( \Lambda^\prime_p(\theta_p) = -(z - \delta)^2/w^2 \) and probability measure \( P_\theta \) given by

\[
\frac{dP_\theta}{dP} = \exp \left(- \theta \, Z_{w,s}(p, \kappa p) - w^2 \eta \, \mathcal{NC}^*(s) \Lambda_p(\theta) \right).
\]

Considering event \( A_{p,\kappa p} = \{(\mathcal{NC}^*(s))^{-1} Z_{w,s}(p, \kappa p) \in [(z - 2\delta)^2, z^2]\} \), we clearly have then

\[
(6.13) \quad P[A_{p,\kappa p}] \geq P_\theta[A_{p,\kappa p}] \exp \left(w^2 \eta \, \mathcal{NC}^*(s) \Lambda_p(\theta) + \theta(z - 2\delta)^2 \mathcal{NC}^*(s) \right).
\]

Adding and subtracting \( \theta(z - \delta)^2 \mathcal{NC}^*(s) \) in the exponent on RHS of (6.13), then setting there \( \theta = \theta_p \), we see that \( P[A_{p,\kappa p}] \) is further bounded below by

\[
P_{\theta_p}[A_{p,\kappa p}] \exp \left(p \mathcal{NC}^*(s) I_{\kappa p}(z - \delta, w) - w \mathcal{NC}^*(s) \theta_p \right),
\]

where \( \eta := (z - \delta)^2 - (z - 2\delta)^2 \). We now complete the proof by taking \( p = p_n \) (we will suppress the subscript \( n \)). Indeed, note that under \( P_\theta \) the variables \( J_i(1 + Y_i) \) are i.i.d. each having mean \((z - \delta)^2/w^2\) and variance \( \Lambda''_{p,\kappa p}(\theta) \). Further, \( p^2 \eta \mathcal{NC}^*(s) \) is bounded away from zero, so by (6.8) we see that \( \text{Var}_{\theta_p} \left( \mathcal{NC}^*(s)^{-1} Z_{w,s}(p, \kappa p) \right) \to 0 \) as \( n \to \infty \), while \( E_{\theta_p} \left( \mathcal{NC}^*(s)^{-1} Z_{w,s}(p, \kappa p) \right) = -w^2 \Lambda_{p,\kappa p}(\theta_p) = (z - \delta)^2 \). Consequently,

\[
\lim_{n \to \infty} \frac{1}{p \mathcal{NC}^*(s)} \log P_{\theta_p}[A_{p,\kappa p}] = 0.
\]

Hence, by (6.6) and (6.7) we have that

\[
\liminf_{n \to \infty} \frac{1}{p \mathcal{NC}^*(s)} \log P[Z_{w,s}(p, \kappa p) \leq z^2 \mathcal{NC}^*(s)] \geq -(w - \sqrt{\kappa(z - \delta)})^2 - 2\eta.
\]

The stated lower bound follows by considering \( \delta \to 0 \) (so \( \eta \to 0 \) as well).

\[\Box\]

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