Homogeneous Killing spinor space-times

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Abstract. A classification of Petrov type D Killing spinor space-times admitting a homogeneous conformal representant is presented. For each class a canonical line-element is constructed and a physical interpretation of its conformal members is discussed.

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1. Introduction

In the area of exact solutions of the Einstein field equations space-times admitting Killing spinors occupy a special position. Many known and important exact solutions admit Killing spinors: the Friedmann-Lemaître-Robertson-Walker, Kerr, Kantowski-Sachs, Schwarzschild interior and Wahlquist’s rotating perfect fluid metrics are just a few examples crossing one’s mind. More generally every conformally flat space-time admits Killing spinors, as well as every locally rotationally symmetric perfect fluid, every Petrov type D vacuum solution and, with the exception of the Plebanski-Hacyan metrics, every Petrov type D (doubly aligned) Einstein-Maxwell solution. As the existence of a Killing spinor is a conformally invariant property of a space-time, this suggests the construction of new, physically relevant, exact solutions (for example rotating perfect fluids or non-aligned Petrov type D Einstein-Maxwell solutions), by finding suitable conformal representants of Killing spinor space-times. Because of the particular relevance of the Petrov type D situation, a chief concern —but an undertaking which has not yet been completed— is the construction of all so-called KS space-times. These were defined in [11] as non-conformally flat space-times admitting a non-null valence two Killing spinor $X_{AB}$. Alternatively they can be characterised as the class of Petrov type D conformal Killing-Yano space-times: their repeated principal Weyl spinors are aligned with the principal spinors of $X_{AB}$ [14],

$$X_{AB} = X_{0(A' B')} ,$$

and define geodesic shear-free null congruences, while the square $P_{ab} = D_{a}{}^{c} D_{cb}$ of the conformal Killing-Yano two-form $D_{ab} = X_{AB} \epsilon_{A'B'} + \bar{X}_{A'B'} \epsilon_{AB}$ is a conformal Killing tensor of Segre type $[(11)(11)]$. There is always a conformal representant in which $P_{ab}$ is a
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Killing tensor, namely when the modulus $|X|$ of the Killing spinor is constant (by means of a global rescaling this constant can be taken $= 1$): this conformal representant will be called the unitary representant. To facilitate the discussion a Killing tensor of Segre type $[(11)(11)]$ with two non-constant (double) eigenvalues will be called regular, in contrast to the ones with one or two constant eigenvalues, which will be called semi-regular or exceptional respectively. The Killing tensor existing in the unitary representant belongs to the latter family and vice-versa: if a representant admits an exceptional Killing tensor, then that representant is –modulo a constant rescaling– the unitary one. Generically a KS space-time admits conformal representants with a regular Killing tensor: these are the space-times considered by Jeffryes [8] and which play an important role in for example Einstein-Maxwell theory (for example all fully aligned Petrov type D Einstein-Maxwell solutions [5], with the exception of the Plebanski-Hacyan metrics [14], belong to this category). KS space-times which do not possess a conformal representant with a regular Killing tensor necessarily belong to Jeffryes’ [8] classes $I$ or $I_N$: while the semi-regular case was dealt with in [11] for class $I_N$ and in [17] for class $I$, the existence of KS space-times of Jeffryes’ class $I$ admitting neither regular nor semi-regular Killing tensors, was demonstrated only recently [2]. These space-times were obtained by imposing some algebraic restrictions on the curvature and turned out to be homogeneous, having a 4-dimensional maximal isometry group. These results also indicated that a set of space-times might have been overlooked in [11]: the form of the homogeneous class I metrics suggests the existence of a homogeneous limit of class $I_N$, contradicting the property that all $I_N$ space-times of [11] admit a 3-dimensional maximal isometry group. Indeed, it turned out that in [11] the possibility was overlooked that all Cartan scalars could be constants, thereby giving rise to a homogeneous space-time. Together with the fact that it was not at all obvious from [8] which of the KS space-times of classes II, III, III$_N$, IV did admit an isometry group of dimension $\geq 4$, this seemed to justify a systematic investigation of all KS space-times admitting a homogeneous conformal representant.

Provided that the space-time is of Petrov type $D$ (because of the resulting alignment of the Killing spinor with the Weyl spinor, see equation (1)), the modulus $|X|$ of the Killing spinor is then a geometric invariant and is therefore constant: the homogeneous members of this family are then precisely the unitary representant and its constant rescalings.

In §2 I present the main equations describing KS space-times with a homogeneous unitary representant. In §3 I give a complete classification of these space-times, explicitly listing the corresponding line-elements. In §4 a possible physical interpretation of the conformal representants of the different classes is discussed.

2. Main equations

Following the notations and conventions of [2], the tetrad basis vectors are taken as $k, \ell, m, m'$ with $-k^a \ell_a = 1 = m^a m'_a$. The correspondence with the Newman-Penrose operators [13] and the basis one-forms is given by $(m^a, m'^a, \ell^a, k^a) \leftrightarrow (\delta, \delta, \Delta, D)$ and
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\((\bar{m}_a, m_a, -k_a, -l_a) \leftrightarrow (\omega^1, \omega^2, \omega^3, \omega^4)\). The description of the problem becomes most compact when using the Geroch-Held-Penrose formalism‡: writing a symmetric (non-null) spinor \(X_{AB}\) as in (11) and using \(\omega, \iota\) as the basis spinors, the property that \(X_{AB}\) is a Killing spinor implies (see [8] for details)

\[ \kappa = \sigma = 0 \] (2)

and

\[ \mathcal{P} X = -\rho X, \]
\[ \mathcal{\bar{\sigma}} X = -\tau X, \] (3)

together with their ‘primed versions’ and \(X' = X\), namely \(\kappa' = \sigma' = 0\) and \(\mathcal{P}'X = -\rho'X, \mathcal{\bar{\sigma}}'X = -\tau'X\). It follows that the Weyl tensor is of Petrov type D (or O) and that \(k, \ell, m, \bar{m}\) are its principal null directions.

In the unitary representant \(|X|\) is constant and hence

\[ \rho + \bar{\rho} = \tau + \bar{\tau}' = 0. \] (5)

The main equations become then

a) the integrability conditions expressing the existence of the Killing spinor:

\[ \mathcal{P}'\rho - \mathcal{P}\rho' = 0, \quad \mathcal{\bar{\sigma}}'\tau - \mathcal{\bar{\sigma}}\tau' = 0, \quad \mathcal{P}\tau' - \mathcal{\bar{\sigma}}'\rho = 0, \] (6)

b) the GHP equations:

\[ \mathcal{P} \rho = 0, \] (7)
\[ \mathcal{\bar{\sigma}} \rho = 2\rho \tau + \Phi_{01}, \] (8)
\[ \mathcal{P} \tau = 2\rho \tau + \Phi_{01}, \] (9)
\[ \mathcal{\bar{\sigma}} \tau = 0, \] (10)
\[ \mathcal{P} \rho' - \mathcal{\bar{\sigma}} \tau' = -\rho \rho' + \tau \bar{\tau}' - \Psi_2 - \frac{1}{12} R \] (11)
\[ \Phi_{00} = -\rho^2, \quad \Phi_{02} = -\tau^2, \] (12)
\[ E = -\frac{R}{12} - \rho \rho' + \tau \tau', \] (13)

where \(E\) is the real part of \(\Psi_2 = E + iH\),

c) the Bianchi equations (in which \(\Psi_2, \Phi_{11}\) and \(R\) are constants as a consequence of the homogeneity assumption):

\[ \mathcal{P} \Phi_{01} = -\rho (4 \Phi_{01} + 5 \tau \rho), \] (14)
\[ \mathcal{P}' \Phi_{01} = \rho' \Phi_{01} - \rho \Phi_{12} + \tau (3 \Psi_2 + \tau \bar{\tau} - 2 \Phi_{11}), \] (15)
\[ \mathcal{\bar{\sigma}} \Phi_{01} = -\tau (4 \Phi_{01} + 5 \tau \rho), \] (16)

‡ for details the reader is referred to [9], but to ease comparison with the Newman-Penrose formalism, remember that the GHP weighted operators \(\mathcal{P}, \mathcal{P}', \mathcal{\bar{\sigma}}, \mathcal{\bar{\sigma}}'\) generalise the NP operators \(D, \Delta, \delta, \bar{\delta}\), while the GHP variables \(\kappa', \sigma', \rho', \tau'\) replace the NP variables \(-\nu, -\lambda, -\mu, -\pi\)
In \([8]\) KS space-times were classified, according to the behaviour of their GHP spin coefficients. The following cases were distinguished: class \(I\) \((\rho \tau \neq 0, \rho' = 0)\), class \(II\) \((\tau \tau' \neq 0, \rho = \rho' = 0)\), class \(III\) \((\rho \rho' \neq 0, \tau = \tau' = 0)\), class \(III_N\) \((\rho 
eq 0, \rho' = \tau = \tau' = 0)\) and class \(IV\) \((\rho = \rho' = \tau = \tau' = 0)\). In classes \(II, III, III_N\) and \(IV\) conformal representatives with regular Killing tensors always exist. In principle one can use the results of \([8]\) in order to obtain information about any possible homogeneous members in these classes. This is however not obvious and therefore the analysis is presented from scratch in the unitary representant.

All 0-weighted combinations of GHP quantities, such as \(\rho \tau \) and \(\tau \tau'\), are geometric invariants and therefore are constants in a homogeneous space-time: for each class the consequences of this elementary observation will be analysed separately.

### 3.1. Class \(I\)

Defining the variables \(\phi, \phi'\) by

\[
\tau' \Phi_{01} = -3 \rho \tau \tau' - 2 \phi, \quad \tau \Phi_{21} = -3 \rho' \tau \tau' - 2 \phi' \phi',
\]

homogeneity implies \(\mathcal{D}(\tau \tau') = 0\) and hence, using \([9]\), \(\phi - \bar{\phi} = 0\). If a class \(I\) conformal representant would admit a regular Killing tensor, then \([17]\) also \(\phi + \bar{\phi}\) would be 0 and hence \(\phi = 0\). Then however Bianchi equation \([14]\), which in terms of the functions \(\phi, \phi'\) reads

\[
\mathcal{D} \phi = \frac{2 \rho}{|\tau|^2} (|\tau|^4 - |\phi|^2),
\]

would lead to an inconsistency. If, on the other hand, a class \(I\) conformal representant would admit a semi-regular Killing tensor, then it would belong to the space-times discussed in \([17]\). As the latter’s unitary representant possesses at most a 1-dimensional isometry group, all homogeneous members of class \(I\) must be given by the exceptional Killing tensor space-times discussed in \([2]\). These were obtained by fixing the null tetrad \((\omega^a)\) such that \(\rho = i Q r, \rho' = -i r\) and \(\tau = m\), after which the curvature components are given by \([12]\) and

\[
R, \Psi_2, \Phi_{11}, \Phi_{01} = 20(m^2 + Q r^2), -\frac{8}{3} (m^2 + Q r^2), -\frac{7}{2} (m^2 - Q r^2), -5 \tau \rho,
\]
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(Q = ±1 and r, m are real parameters). With respect to suitably chosen tetrads (Ω^a) or (σ^a) the line-elements obtained were:

\( Q = 1: \)

\[
ds^2 = 2(\Omega^{12} + \Omega^{22} - Q\Omega^{32} + Q\Omega^{42}),
\]

\[
\Omega^1 = \frac{1}{2(m^2 - r^2)}(2m^2 dt - rdv) - \frac{r}{K(x, y)}(ydx - xdy),
\]

\[
\Omega^2 = \frac{1}{K(x, y)}(\sin vdx + \cos vdy),
\]

\[
\Omega^3 = \frac{1}{K(x, y)}(\cos vdx - \sin vdy),
\]

\[
\Omega^4 = \frac{m}{2(m^2 - r^2)}(-2rdt + dv) + \frac{m}{K(x, y)}(ydx - xdy),
\]

with

\( K(x, y) = 1 - (m^2 - r^2)(x^2 + y^2). \)

\( Q = -1: \)

\[
ds^2 = 2(\sigma^1 \sigma^2 - \sigma^3 \sigma^4),
\]

\[
\sigma^1 = \frac{1}{2\sqrt{2}}[(\mathcal{E}z^2 - \frac{i}{m^2 + r^2}\mathcal{E}^{-1})dx - \mathcal{E}dz],
\]

\[
\sigma^3 = \frac{1}{2(m^2 + r^2)}[mdt + (m + r)zdx + rdy],
\]

\[
\sigma^4 = \frac{1}{2(m^2 + r^2)}[mdt + (m - r)zdx - rdy],
\]

with

\( \mathcal{E} = \exp(\frac{r^2y + m^2t}{r^2 + m^2}). \)

Note that the \( m = r \) limit of (26) is conformally flat, while for \( r \rightarrow m \) the metric (23) has a singular limit of Petrov type D, in which the tetrad can be rewritten as

\[
\Omega^1 = dv - u + 2mx dy,
\]

\[
\Omega^2 = \sin 2mv dx + \cos 2mv dy,
\]

\[
\Omega^3 = \cos 2mv dx - \sin 2mv dy,
\]

\[
\Omega^4 = du - 2mx dy.
\]

In all cases the dimension of the isometry group is 4 (there is no isotropy as \( \rho \) and \( \tau \) are \( \neq 0 \)).

3.2. Class \( I_N \)

When \( \rho' = 0 \) \( \Phi_{12} = 0 \). There are then always \( \Omega^a \) conformal representants admitting a regular or semi-regular Killing tensor. As in the previous paragraph homogeneity implies that \( \phi \) is real and the same argument as before shows that the
regular case leads to an inconsistency. The semi-regular case was treated in \[11\]: there however only the hypersurface homogeneous situation was considered and the possibility was overlooked (see equations (2.15)) that all the Cartan invariants could be space-time constants. The Bianchi equations, together with the assumption that \( \phi, \Psi, U \) and \( V \) are constant, imply then \( \phi = \pm |\tau|^2, U = -|\tau|^2/2, V = 2\phi - |\tau|^2 \) and
\[
\Psi_2 = \frac{4}{3}|\tau|^{-2}(\phi + 2|\tau|^2)(\phi - |\tau|^2).
\]
Writing \(|\tau| = m \) (constant) the non-conformally flat cases are characterised by
\[
R, \Psi_2, \Phi_{11}, \Phi_{01} = 20m^2, -\frac{8}{3}m^2, -\frac{7}{2}m^2, -5im^2,
\]
which, when compared with \[22\], shows that the corresponding metrics should be obtainable by taking the \( Q \to \infty \) limit of class I (after replacing \( r \) by \( r/Q \)). As the class I metrics \[23,26,29\] were constructed assuming explicitly \( Q = \pm 1 \), this limit is not easy to recognise and therefore these metrics are constructed below from scratch. First the tetrad is fixed by means of a boost and a rotation such that \( \rho = im \) and \( \tau = \pi = m \). The Cartan equations become then
\[
\begin{align*}
d\omega^1 &= -2m\omega^4 \wedge (i\omega^1 + \omega^3), \\
d\omega^3 &= 2im\omega^2 \wedge \omega^1 - 2m\omega^3 \wedge (\omega^1 + \omega^2), \\
d\omega^4 &= 2m\omega^4 \wedge (\omega^1 + \omega^2).
\end{align*}
\]
It follows that \( i(\omega^1 - \omega^2) - \omega^4 \) is closed and that (the dual vector field of) \( \omega^4 \) is hypersurface-orthogonal. This allows one to write \( \omega^4 = xdu \ (dx \wedge du \neq 0) \), after which \[32\] implies
\[
\omega^1 + \omega^2 = -\frac{1}{2mx}dx + vdu \ (dx \wedge du \wedge dv \neq 0).
\]
Writing \( i(\omega^1 - \omega^2) = \omega^4 + 2dy, \ [31] \) shows that
\[
\omega^3 = \frac{1}{4mx}dv - dy + fdu,
\]
with \( f = -(x^2 + v^2)/(4x) + F \) and \( F \) an arbitrary function of \( u \). As the resulting spin coefficients and curvature components depend on the parameter \( m \) only, it follows\[9\] that a coordinate transformation \textit{must} exist making \( F = 0 \). The null tetrad becomes herewith
\[
\begin{align*}
\omega^3 &= \frac{1}{4mx}dv - dy - \frac{1}{4x}(x^2 + v^2)du, \ 
\omega^4 &= 2xdy \\
\omega^1 &= -\frac{1}{4mx}dx + \frac{1}{2}(v - ix)du - idy.
\end{align*}
\]
After a coordinate transformation \( v \to v/(mx) \), the line-element can then be written as
\[
ds^2 = \frac{1}{8m^2x^2}(dx^2 + 8(v^2 - m^2x^2)du^2) + 2(dv + xdu)^2 - \frac{1}{2m^2x}dudv. \ [35]
\]
Again the maximal dimension of the isometry group is 4.

\[\S \text{ the explicit construction of this transformation is not obvious}\]
3.3. Class II

Imposing $\rho = \rho' = 0$ (and hence $\Phi_{00} = \Phi_{22} = 0$), the Bianchi equations, together with the constancy of $R, \Phi_{11}, \Psi_2$ and $|\tau| = m$, lead to $\Phi_{01} = \Phi_{12} = 0$ and

$$R, \Psi_2, \Phi_{11} = -4m^2 - 8V, \frac{2}{3}(V - m^2), V - \frac{1}{2}m^2.$$  

The null tetrad can now be partially fixed by rotating such that $\tau = m$. Switching to the Newman-Penrose formalism, it furthermore follows from the Bianchi equations that the spin coefficients $\epsilon$ and $\gamma$ are real and that $\beta = \alpha$. The Cartan equations show then that $\omega^1 - \omega^2$ is closed and that $(\text{the dual vector fields of}) \omega^3$ and $\omega^4$ are hypersurface-orthogonal. This allows one to partially fix a boost such that $\omega^3$ is closed too, implying $\epsilon = 0$ and $\alpha = \tau/2$. The Newman-Penrose equations reduce then to

$$D\gamma = 2m^2 + 2V, \quad \delta\gamma = \delta\gamma = 0,$$  

(36)

while the Cartan equations become

$$d\omega^1 = 2\tau\omega^3 \wedge \omega^4,$$  

(37)

$$d\omega^3 = 0,$$  

(38)

$$d\omega^4 = 2\gamma\omega^4 \wedge \omega^3.$$  

(39)

It follows that $\omega^3 = du$ and $\omega^4 = e^p dv$, with $p$ a function of $u$ and $v$: by (36) $2\omega^3 \omega^4$ is then the metric of a two-space of constant curvature. One should now distinguish the flat and non-flat cases:

a) when $V + m^2 \neq 0$ the coordinate $v$ can be re-defined such that $\omega^4 = kv^{-2}(dv - du)$, with, by (36), $2k = -1/(m^2 + V)$. One has then $\gamma = 1/v$ and, putting $\omega^1 - \omega^2 = idy$ (37) can be integrated to give

$$\omega^1 + \omega^2 = dx + \ell(v)(dv - du), \quad \omega^1 - \omega^2 = idy,$$

$$\omega^3 = du, \quad \omega^4 = kv^{-2}(dv - du),$$

(40)

with $\ell = 2m/(m^2 + V)$. Note that the case $V = m^2$ is conformally flat.

b) when $V + m^2 = 0$ the integration is straightforward and one obtains

$$\omega^1 = 2m(dx + idy + udv), \quad \omega^3 = du, \quad \omega^4 = dv,$$

(41)

yielding the line element

$$ds^2 = 4m^2((dx + udv)^2 + dy^2) - dudv.$$  

(42)

As there is a residual 1-dimensional group of boost isotropies, both families (a) and (b) admit a 5 dimensional isometry group.
3.4. Class III

This class is the Sachs’ transform [6] of class II. All calculations are therefore similar to those of the previous paragraph, but now a distinction has to be made between the cases \( \omega^3 \pm \omega^4 \) being closed: this is responsible for the appearance of the extra parameter \( Q = \pm 1 \) below. Starting with \( \tau = \tau' = 0 \) and the constancy of \( R, \Phi_{11}, \Psi_2 \) and \( \rho \rho' \), the only other non-0 curvature components turn out to be \( \Phi_{00} = -\rho^2 \) and \( \Phi_{22} = -\rho^2 \). Fixing a boost such that \( \rho = iQr, \mu = ir \) \((r \in \mathbb{R} \text{ constant})\), the Newman-Penrose and Bianchi equations imply

\[
R, \Phi_2, \Phi_{11} = 8(U - \frac{1}{2}Qr^2), -\frac{2}{3}(U + Qr^2), U + \frac{1}{2}Qr^2,
\]

with the spin coefficients satisfying \( \tau = -\epsilon, \bar{\tau} = -\gamma, \beta = -\bar{\alpha} \). The Cartan equations become

\[
d\omega^1 = \omega^1 \wedge (2\alpha \omega^2 + (2\gamma - ir)\omega^3 + (2\epsilon - iQr)\omega^4),
\]

\[
d\omega^3 = 2iQr\omega^2 \wedge \omega^1,
\]

\[
d\omega^4 = 2i\omega^2 \wedge \omega^1.
\]

As (the dual vector field of) \( \omega^1 \) is hypersurface-orthogonal, one can partially fix a spatial rotation such that \( \omega^1 = P^{-1}d\zeta \) with \( P \) real and \( \zeta \) complex. As \( \omega^4 - Q\omega^3 \) is closed, a coordinate \( u \) is defined locally by \( 2du = \omega^4 - Q\omega^3 \). Introducing a fourth coordinate \( w \) such that \( \omega^3 \) is spanned by \( d\zeta, \bar{d}\zeta, dw \), equation (43) and its complex conjugate imply that \( P = P(\zeta, \bar{\zeta}), \epsilon = iQr/2, \gamma = ivr/2, \beta = -\bar{\alpha} \) and \( \alpha = \alpha(\zeta, \bar{\zeta}) \). Herewith the remaining Newman-Penrose equations are, in analogy with (36), given by

\[
\delta\alpha + \bar{\delta}\alpha - 4\alpha\bar{\alpha} = 2U - 2Qr^2, \quad D\alpha = \Delta\alpha = 0,
\]

expressing that \( 2\omega^1\omega^2 = P^{-2}d\zeta d\bar{\zeta} \) is the metric of a two-space of constant curvature. Defining standard coordinates in this two-space by

\[
\omega^1 = (1 + \frac{k}{4}x^2)^{-1}(dx + i\,dy)
\]

and writing \( \omega^3 = f\,dx + g\,dy + h\,dw \), equation (44) shows that \( f, g, h \) are functions of \( x, y \) and \( w \), allowing one to put \( w = 1 \) and \( g = 0 \). Integration of (44) gives then

\[
f = 4Qr \frac{xy}{(1 + \frac{k}{4}x^2)^2},
\]

after which the line-element becomes \((w = v - Qu)\)

\[
ds^2 = 2(1 + \frac{k}{4}x^2)^{-2}(dx^2 + x^2dy^2) - 2Q[(f\,dx + dv)^2 - du^2].
\]

The constant \( k \) is related to \( U \) and \( r \) by

\[
U = Qr^2 + \frac{k}{8}.
\]

There is an obvious residual isotropy group of spatial rotations and hence these space-times admit a 5-dimensional isometry group. The Gödel metric is obtained as the special case \( U = 0, Q = 1 \). Note that the case \( k + 16Qr^2 = 0 \) is excluded, as it gives rise to a conformally flat metric.
3.5. Class \( \text{III}_N \)

This is the special case \( \rho' = 0 \) of class \( \text{III} \): fixing the boost such that \( \rho = ir \) (\( r \in \mathbb{R} \) constant) and proceeding in exactly the same way as in the previous paragraph, one recovers the expressions for \( \omega^1, \omega^3 \) (but with \( Q \) replaced by 1), while \( \omega^4 \) becomes exact. The line-element reads then

\[
\text{d}s^2 = 2(1 + \frac{k}{4}x^2)^{-2}(\text{d}x^2 + x^2\text{d}y^2) - 2\text{d}udv - 2f\text{d}x\text{d}u,
\]

with

\[
f = 4Qr\frac{xy}{(1 + \frac{k}{4}x^2)^2}
\]

and \( U = \frac{k}{8} \) (now the flat case \( k = 0 \) is excluded). As before there is a 5-dimensional isometry group.

3.6. Class \( \text{IV} \)

Here \( \rho = \rho' = \tau = \tau' = 0 \) and the only non-vanishing curvature components are the constants

\[ R, \Psi_2, \Phi_{11} = 8(U - V), \frac{2}{3}(V - U), U + V. \]

The four basis vector fields are hypersurface-orthogonal and one can partially fix a boost and a spatial rotation such that

\[
\omega^1 = P^{-1}\text{d}\zeta, \quad \omega^3 = Q^{-1}\text{d}u, \quad \omega^4 = Q^{-1}\text{d}v,
\]

with \( P \) and \( Q \) depending on \( \zeta, \bar{\zeta} \) and \( u, v \) respectively. The surviving Newman-Penrose equations are, with real \( \epsilon, \gamma \):

\[
D\gamma - \Delta\epsilon + 4\epsilon\gamma = 2V, \quad \delta\gamma = \delta\epsilon = 0,
\]

\[
\delta\alpha + \bar{\alpha}\delta - 4\alpha\bar{\alpha} = 2U, \quad D\alpha = \Delta\alpha = 0,
\]

expressing that these space-time are products of two constant-curvature two-spaces, of signature 0 and 2 respectively. Alternatively they can be characterised as the conformally symmetric[11,12] space-times of Petrov type D. With \( Q = e^p \) and \( P = e^p \) one obtains

\[
p_{\zeta\bar{\zeta}} = 2Ue^{-2p}, \]

\[
q_{uv} = 2Ve^{-2q},
\]

i.e. the Ricci scalars of the two-spaces \( (u = \text{const}, v = \text{const}) \) and \( (\zeta = \text{const}) \) are respectively given by \( 8U \) and \( 8V \). The line-element can be written as

\[
\text{d}s^2 = 2(1 + 2U\zeta\bar{\zeta})^{-2}\text{d}\zeta\text{d}\bar{\zeta} - 2(1 + 2Vuv)^{-2}\text{d}udv,
\]

with \( U \neq V \) as otherwise the space-time is conformally flat. Because of the residual freedom of boosts and spatial rotations, (the non-conformally flat members of) these space-times admit a 6-dimensional isometry group.
4. Interpretation

With the exception of (48), which for $U = 0, Q = 1$ reduces to the Gödel metric and of (57), which for $U + V > 0$ is the $\Lambda \neq 0$ generalisation of the Bertotti-Robinson metric $[3, 15]$ (the unique conformally flat non-null Einstein-Maxwell solution when $\Lambda = 0$), none of the above Killing spinor space-times has an immediate physical interpretation. Some of their conformal representatives however might be interpreted as Einstein-Maxwell, perfect fluid or pure radiation space-times. Addressing this question directly in coordinates, leads to insuperable problems. A better strategy is to stay within the Newman-Penrose formalism and to express the spin coefficients and curvature components of the conformally transformed metric in terms of those of the unitary representant and of the directional derivatives $\partial_a \Omega$ of the conformal factor $\Omega$. This still leads to a complicated algebraic and differential consistency analysis for the unknowns $D\Omega, \Delta\Omega, \delta\Omega, \nabla\Omega$ and $n_1, n_2, n_3, n_4$ (in the case of a perfect fluid $n_a = \sqrt{w + pu_a}$ with $u^a$ the 4-velocity), or (in the case of a Maxwell field) the 6 components $\Phi_0, \Phi_1, \Phi_2$ of the Maxwell spinor. In the following paragraphs this procedure (for details of which the reader is referred to $[2]$) is applied to the different classes of homogeneous KS space-times discussed above.

4.1. Classes I, I$_N$, II and III$_N$

For class I, determined by (23-24), (26-27), (29), it was shown $[2]$ that there are no perfect fluid (neither aligned nor non-aligned), pure radiation nor Einstein-Maxwell (neither singly nor doubly aligned) interpretations. The same conclusions hold for classes I$_N$ and III$_N$, determined by (33) and (50) respectively: the analysis is quasi-identical, although some care is necessary to exclude the singly aligned Einstein-Maxwell case, as the vanishing of the Maxwell spinor components $\Phi_0$ or $\Phi_2$ is now not equivalent).

For class II, determined by (40) and (41), the Einstein-Maxwell case demands some more work than in I or I$_N$, as $D\Omega = 0$ cannot be excluded a priori on the basis of equations (69a) or (69f) of $[2]$. As all results—for perfect fluids, pure radiation and Maxwell fields—are again negative (and identical to those for class I) and as the calculations are quite long and little illuminating, they will not be repeated here.

4.2. Class III

The class III metrics, determined by (48), clearly exhibit local rotational symmetry (LRS)$[16]$: they belong to the LRS I family when $Q = 1$ and to the LRS III family when $Q = -1$. A search for a perfect fluid interpretation along the lines of $[2]$ reveals that there is no such interpretation when $Q = -1$: this is in agreement with the fact that the class III metrics all have a purely electric Weyl tensor ($\Psi_2 = -(k + 16Qr^2)/12$) and that no LRS III purely electric perfect fluids exist$[10]$. On the other hand, when $Q = 1$ a perfect fluid interpretation does exist and is necessarily aligned: multiplying the metrics (48) with a scale factor $\Omega = 1/\sinh((k/2 + 4r^2)^{1/2}u)$ transforms (48) into
a purely electric “stiff fluid” metric \((dp/dw = 1)\). Purely electric LRS I metrics are generalisations of the Gödel-metric: they are stiff fluids, the metric of which can be written in the standard form\(^{10}\),

\[
ds^2 = \frac{1}{j(x)^2} \left[ \frac{dx^2}{j(x)^2} - (dt + qr^2 e^\psi d\phi)^2 + e^{2\psi}(dr^2 + r^2 d\phi^2) \right]
\]

(58)

with \(e^{-\psi} = 1 + \frac{k}{4} r^2\) and \(j^2 = \left( \frac{k}{2} + q^2 \right)x^2 + c_1 x + c_2\). In fact, dropping the factor \(j^{-2}\) from (58), a coordinate transformation brings these metrics in the form (48). This shows that for LRS I perfect fluids the condition of being purely electric is equivalent with being conformally homogeneous. Again, as for class I there are no (doubly nor singly) aligned Einstein-Maxwell nor pure radiation interpretations.

### 4.3. Class IV

As remarked at the end of the previous section the class IV metric (52) itself represents an aligned non-null Einstein-Maxwell field (provided \(U + V > 0\)). Again one can show that there is no pure radiation interpretation (aligned nor non-aligned), but both aligned and non-aligned perfect fluid interpretations do exist: surprisingly the general solution for the non-aligned case can be given in explicit form. For a detailed discussion of the ensuing solutions, see [18]

### 5. Discussion

Homogeneous KS space-times, being of Petrov type D, admit either a 4, 5, or 6-dimensional group \(G_n\) of isometries. The following list of space-times exhausts these three families:

- \(G_4\): class I, determined by (23-24), (26-27), (29) and class \(I_N\), determined by (33)
- \(G_5\): class II, determined by (40) and (41), class III, determined by (48) and class \(III_N\), determined by (50)
- \(G_6\): class IV, determined by (57).

None of these metrics admits (a conformal representant with) a pure radiation interpretation. A perfect fluid interpretation only exists for the class III metric (48) with \(Q = 1\) and the class IV metrics. The \(Q = -\) class III metrics are precisely the purely electric LRS I stiff fluid metrics, generalising the Gödel-metric, while the class IV metrics are either aligned and form a sub-class of the LRS II perfect fluids, or they are non-aligned and then can be given in explicit form. The resulting space-times, discussed in [18], in general admit no symmetries.

A different problem is related to the non-null and non-aligned Petrov type D Einstein-Maxwell solutions. Very few of these being explicitly known (Griffiths’ 1986
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solution being the only one known to the author), it remains a tantalising question whether any such solution might be constructable by conformally transforming any of the KS space-times discovered to date.

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