Approximation by Normal Distribution for a Sample Sum in Sampling Without Replacement from a Finite Population

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Abstract
A sum of observations derived by a simple random sampling design from a population of independent random variables is studied. A procedure finding a general term of Edgeworth asymptotic expansion is presented. The Lindeberg condition of asymptotic normality, Berry-Esseen bound, Edgeworth asymptotic expansions under weakened conditions and Cramer type large deviation results are derived.

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1 Introduction
Let \( \pi_N = (Y_{1N}, ..., Y_{NN}) \) be a population of independent random variables (r.v.) and \( r = (r_1, ..., r_N) \) be a random vector (r.vec.) independent of \( Y_{1N}, ..., Y_{NN} \) and such that \( P \{r_1 = k_1, ..., r_N = k_N\} = (N!)^{-1} \), for any permutation \( k = (k_1, ..., k_N) \) of the numbers 1, ..., \( N \). Let \( S_{nN} = Y_{r_1N} + ... + Y_{r_nN}, \ 1 \leq n \leq N, \) that is a sum of \( n \) r.v.s chosen at random without replacement from the population \( \pi_N \). We are interested in approximation of the distribution of \( S_{nN} \) by a normal distribution.

The sample sum \( S_{nN} \) is of interest in the context of its statistical applications. Under hypothesis of homogeneity of two samples, the two-sample linear rank statistics can be reduced to that of sample sum \( S_{nN} \) when the elements of the population are non-random real numbers; in this case, the elements of the population become the scores of the rank statistic, see Bickel and von Zwet (1978). Situation where \( Y_{kN} \) are random variables arises, for
instance, in the problems associated with two-stage samples, in particular in stratified random sampling, in a linear estimation for the mean total of a stratified population, see Cochran (1963), Mirakhmedov et al. (2015). Note also if \( n = N \), then \( S_{nN} \) simply is a sum of independent r.v.s. Many authors have studied the sample sum. In the situation when \( Y_{kN} \) are non-random a sufficient and necessary condition, similar to the well-known Lindeberg condition, for asymptotic normality of \( S_{nN} \) have been presented by Erdös and Renyi (1959) and Hajek (1960); we should refer also to Wald and Wolfowitz (1944) who for the first time have proved asymptotic normality of \( S_{nN} \) under certain conditions. The rate in the central limit theorem (CLT) was obtained by Bikelis (1969) and Höglund (1978). The second-order approximation results were derived by Robinson (1978), Bickel and von Zwart (1978), Babu and Singh (1985), Babu and Bai (1996) and Bloznelis (2000). Also Cramer’s large deviation result has been proved by Robinson (1977) and Hu et al. (2007b). The case when the elements of the population \( \pi_N \) are r.v.s has been extensively studied since the paper of von Bahr (1972), who have established for the first time a bound for the remainder term in CLT; later non-uniform variant of von Bahr’s result was obtained by Mirakhmedov and Nabiev (1976). Von Bahr’s bound has been improved in Corollary of Mirakhmedov (1985) and in Theorem 1 of Zhao et al. (2004). Two terms Edgeworth asymptotic expansion result has been established by Mirakhmedov and Nabiev (1979) and Hu et al. (2007a), whereas expansion with arbitrary number of terms was obtained by Mirakhmedov (1983). Large deviation results follows from Theorems 11 and 12 of Mirakhmedov (1996).

In this paper, we consider the general situation when the elements of the population \( \pi_N \) are r.v.s, which may be degenerate (i.e., non-random). The aim of the paper is threefold. The first, in Section 2, we present a procedure finding a general term of Edgeworth asymptotic expansion for the sample sum; our rule based on slightly generalized formula of Erdös and Renyi (1959), and used fact that the integrant of that formula is the characteristic function (ch.f.) of an independent two-dimensional r.vec.s. This rule is considerably simpler than that of Mirakhmedov (1983), who have used formula of von Bahr (1972) and Bloznelis (2000), who gives just two terms of asymptotic expansion for the case when the elements of a population are non-random numbers. The second, we obtain Berry-Esseen bound and Edgeworth expansion result under weakened moment conditions, namely we assume that \( E|Y_{mN}|^{k+\delta} < \infty \), \( m = 1, ..., N \), where \( 0 < \delta \leq 1 \), \( k \) is the number of terms in asymptotic expansion (notice that the normal distribution is the first term). A general Theorems 3.1 and 3.2 of Section 3 give the bound for the remainder term in a unusually form; this together with
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above mentioned fact on the integrand of Erdos and Reny formula allows to write estimations of the remainder terms in term of Lyapunov ratios of a associated r.v.s \( Z_m \), see (2.2) and (2.5). By turn, this fact allows us to improve concerning theorems of Zhao et al. (2004) and Hu et al. (2007a) by weakening the moment conditions, bringing some correction and giving one more term of expansion, see Remarks 3.4 and 3.5. Thirdly, we note that the sample sum is a special case of so namely decomposable statistics, considered for instance in Mirakhmedov et al. (2014). Using this fact, we derive for the first time the Lindeberg type condition of asymptotic normality and Cramer’s type large deviation result. These results are extension of concerning theorems of Hajek (1960) and Robinson (1977), respectively. Thus, in the paper, complete spectrum of results on approximation by normal distribution is collected. Main assertions are presented in Section 3, whereas the proofs are given in Section 4.

In what follows, \( Bi(p) \) is used to denote Bernoulli distribution with probability of success \( p \); \( \mathcal{L}(\xi) \) stands for the distribution of the r.v. (r.vec.) \( \xi \); \( c, C \) with or without an index are positive universal constants whose value may differ at each occurrence. In the sequel notations, we shall suppress the dependence on \( N \) whenever it is convenient.

2 Procedure of Finding a General Term of Asymptotical Expansion of the ch.f. of \( S_{nN} \)

Write \( S_{nN} = Y_{1N}\eta_1 + ... + Y_{NN}\eta_N \), where \( \eta_m = 1 \) if \( Y_{mN} \) appears in the sample, else \( \eta_m = 0 \), i.e. \( \mathcal{L}(\eta_m) = Bi(p) \) with \( p = n/N \) and \( \eta_1 + ... + \eta_N = n \). The joint distribution of \( (\eta_1, ..., \eta_N) \) can be viewed as conditional joint distribution of the r.vec. \( (\xi_1, ..., \xi_N) \) under \( \zeta_N := \xi_1 + ... + \xi_N = n \), where \( \xi_1, ..., \xi_N \) are independent r.v.s and \( \mathcal{L}(\xi_m) = Bi(p) \). Hence, \( E \left( e^{it\widehat{S}_{nN}/\zeta_N} = n \right) = E e^{itS_{nN}} \), where \( \widehat{S}_{nN} = Y_{1N}\xi_1 + ... + Y_{NN}\xi_N \). This together with equality

\[
E \left( e^{it\widehat{S}_{nN} + i\tau(\zeta_N - n)} \right) = E \left( e^{i\tau(\zeta_N - n)} E \left( e^{it\widehat{S}_{nN}/\zeta_N} \right) \right) = \sum_{k=0}^{\infty} e^{i\tau(k-n)} P\{\zeta_N = k\} E \left( e^{it\widehat{S}_{nN}/\zeta_N} \right),
\]

and fact that \( \widehat{S}_{nN} \) and \( \zeta_N \) are a sums of independent r.v.s, implies by Fourier inversion

\[
E \exp\{itS_{nN}\} = \frac{1}{\sqrt{2\pi d_n(p)}} \int_{-\infty}^{\infty} \prod_{m=1}^{N} E \exp\{itY_{mN}\xi_m + i\tau(\xi_m - p)/\sqrt{nq]\} d\tau, \quad (2.1)
\]

where \( q = 1 - p, d_n(p) := \sqrt{2\pi nq} P\{\zeta_N = n\} = \sqrt{2\pi nq C_n^m p^n q^{N-n}}. \)
If \( Y_{mN} = a_{mN}, m = 1, \ldots, N \) are non-random real numbers then (2.1) can be written in the form

\[
E \exp \{itS_{nN}\} = \frac{1}{\sqrt{2\pi} d_n(p)} \int_{-\pi\sqrt{nq}}^{\pi\sqrt{nq}} e^{-i\tau/\sqrt{nq}} \prod_{m=1}^{N} (q + p \exp \{ita_{mN} + i\tau/\sqrt{nq}\}) d\tau.
\]

In fact this is the formula of Erdős and Renyi (1959) which had been used to prove the first satisfactory CLT for the sample sum. The formula had motivated for a number of related studies; for instance Hajek (1960), Bikelis (1969), Babu and Singh (1985) and Bloznelis (2000). Set

\[
\gamma = N^{-1} \sum_{m=1}^{N} EY_{mN}, \sigma^2 = \frac{1}{N} \sum_{m=1}^{N} \left[ E(Y_{mN} - \gamma)^2 - p (E(Y_{mN} - \gamma))^2 \right],
\]

\[
Z_{mN}^x = (Y_{mN} - \gamma) x - p E(Y_{mN} - \gamma).
\]

Note that \( ES_{nN} = n\gamma \) and \( S_{nN} - n\gamma = Z_{1N}^n + \ldots + Z_{NN}^n. \) From formula (2.1), we obtain

\[
\varphi_n(t) := E \exp \left\{ it \frac{S_{nN} - n\gamma}{\sigma \sqrt{n}} \right\} = \frac{\Theta_N(t)}{d_n(p)} = \frac{\Theta_N(t)}{\Theta_N(0)},
\]

where

\[
\Theta_N(t) = \frac{1}{\sqrt{2\pi}} \int_{-\pi\sqrt{nq}}^{\pi\sqrt{nq}} \prod_{m=1}^{N} \psi_m(t, \tau) d\tau,
\]

\[
\psi_m(t, \tau) = E \exp \left\{ \frac{itZ_{mN}^{\xi_m}}{\sigma \sqrt{n}} + \frac{i\tau (\xi_m - p)}{\sqrt{nq}} \right\},
\]

because \( d_n(p) = \Theta_N(0), \) due to inversion formula for the local probability. In what follows just to keep notation simple, we put

\[
Z_m = Z_{mN}^{\xi_m}.
\]

Note that \( n\sigma^2 = VarZ_1 + \ldots + VarZ_N, \) and

\[
\sum_{m=1}^{N} \text{cov} (Z_m, \xi_m) = 0.
\]

In regard to formula (2.3), we remark that the integrand in \( \Theta_N(t) \) is the ch.f. of a sum of independent two dimensional r.vec.s. This fact is crucial in our next considerations, in particular, in constructing the terms of asymptotic expansion of the ch.f. of the sample sum. Indeed, let \( E \not|Z_m|^k < \infty, \)
\( k \geq 3 \), and \( P_{mN}(t, \tau) \), \( m = 1, 2, \ldots \), be the well-known polynomials of \( \tau \) and \( t \) from the theory of asymptotic expansion of the ch.f. of a sum of independent r.vec.s, see Bhattacharya and Ranga Rao (1976) p.52 (henceforth to be referred to as BR). In our case, the concerning sum is \((Z, \xi) = (\tilde{Z}_1, \tilde{\xi}_1) + \ldots + (\tilde{Z}_N, \tilde{\xi}_N)\), where

\[
\tilde{Z}_m = Z_m / \sigma \sqrt{n} \quad \text{and} \quad \tilde{\xi}_m = (\xi_m - p) / \sqrt{nq}.
\]

It is essential that this sum has zero expectation, uncorrelated components and a unit covariance matrix, due to (2.6). Hence, the integrand in \( \Theta_N(t) \), being the ch.f. of the sum \((Z, \xi)\), can be approximated by a power-series in \( N^{-1/2} \) whose coefficients are polynomials of \( t \) and \( \tau \) containing the common factor \( \exp\{- (t^2 + \tau^2)/2\} \), hence, the series can be integrated wrt \( \tau \) over the interval \((-\infty, \infty)\). The degree of \( P_{mN}(t, \tau) \) is \( 3m \) and minimal degree is \( m + 2 \); the coefficients of \( P_{mN}(t, \tau) \) only involve the cumulants of the random vectors \((Z, \xi)\) of order less than or equal to \( m + 2 \), in particular:

\[
\frac{1}{\sqrt{N}} P_{1N}(t, \tau) = \frac{i^3}{6} \sum_{m=1}^{N} E \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^3,
\]

\[
\frac{1}{N} P_{2N}(t, \tau) = \frac{i^4}{24} \sum_{m=1}^{N} \left( E \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^4 - 3 \left( E \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^2 \right)^2 \right) + \frac{1}{2N} P_{1N}^2(t, \tau).
\]

Define functions \( G_{mN}(t) \) such that \( G_{0N}(t) = 1 \) and

\[
G_{mN}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} P_{m,N}(t, \tau) \exp \left\{- \frac{\tau^2}{2} \right\} d\tau, m = 1, 2, \ldots,
\]

and write

\[
Q_{kN}(t) = e^{-\frac{t^2}{2}} \sum_{m=0}^{k} N^{-m/2} G_{m,N}(t).
\]

Due to Theorem 9.11 of BR (1976), this series is \((k + 1)\)-term asymptotic expansion of the integral \( \Theta_N(t) \). Finally, we get the \((k + 1)\)-term asymptotic expansion of \( \varphi_n(t) \) by dividing \( Q_{kN}(t) \) with \( Q_{kN}(0) \). Also in order to estimate remainder terms in this approximation one can use directly Theorem 9.11, Lemma 9.5 and Lemma 14.3 of BR in an evidently simplified form.
In particular, three term asymptotic expansion of the \( \varphi_n(t) \) is

\[
W_{3N}(t) = e^{-\frac{t^2}{2}} \left( 1 + N^{-1/2}G_{1N}(t) + N^{-1}(G_{2N}(t) - G_{2N}(0)) \right)
\]

\[
= e^{-t^2/2} \left\{ 1 + \frac{(it)^3}{6} \sum_{m=1}^{N} E\tilde{Z}_m^3 + \frac{(it)^6}{72} \left( \sum_{m=1}^{N} E\tilde{Z}_m^3 \right)^2 \right\}
\]

\[
+ \frac{(it)^4}{24} \left( \sum_{m=1}^{N} E\tilde{Z}_m^4 - 3 \left( \sum_{m=1}^{N} (E\tilde{Z}_m^2)^2 + \left( \sum_{m=1}^{N} E\tilde{Z}_m^2 \tilde{\xi}_m \right)^2 \right) \right) + (it)^2 \frac{pq\alpha_{20}}{2n\sigma^2}
\]

\[
= e^{-t^2/2} \left\{ 1 + \frac{(it)^3}{6\sqrt{n}\sigma^3} \Lambda_1 + \frac{(it)^6}{72n\sigma^6} \Lambda_2^2 + \frac{(it)^4}{24n\sigma^4} \Lambda_2 + (it)^2 \frac{pq\alpha_{20}}{2n\sigma^2} \right\}, \quad (2.11)
\]

where

\[
\alpha_{kl} = \alpha_{kl}^{(1)}, \quad \alpha_{kl}^{(i)} = \frac{1}{N} \sum_{m=1}^{N} \left( E(Y_{mN} - \gamma)^k \right)^i \left( E(Y_{mN} - \gamma) \right)^l. \quad (2.12)
\]

\[
\Lambda_1 = \alpha_{30} - 3p\alpha_{21} + 2p^2\alpha_{23} + 12p^2\alpha_{22} - 6p^3\alpha_{40} - 3p\alpha_{20}^{(2)} - 3q(\alpha_{20} - 2p\alpha_{20})^2.
\]

The terms of asymptotic expansion of the distribution function of \( S_{nN}/\sigma\sqrt{n} \) can be obtained by standard way: formally substitute

\[
(-1)^\nu \frac{d^\nu}{du^\nu} \Phi(u) = -e^{-u^2/2} H_{\nu-1}(u)/\sqrt{2\pi}, \quad \text{where } \Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2} dt,
\]

instead of \((it)^\nu e^{-t^2/2}\) for each \( \nu \) in the expression for \( W_{3N}(t)e^{-t^2/2} \), see Lemma 7.2 of BR, p. 53, where \( H_{\nu}(x) \) is the \( \nu \)-th order Hermit-Chebishev polynomial. Denoting \( W_{3N}(u) \) for three term asymptotic expansion, we have

\[
W_{3N}(u) = \Phi(u) - \frac{e^{-u^2/2}}{\sqrt{2\pi}} \left\{ \frac{H_2(u)}{6\sigma^3\sqrt{n}} \Lambda_1 \right. \\
+ \frac{1}{72n\sigma^6} \left[ H_5(u)\Lambda_1^2 + 3H_3(u)\sigma^2\Lambda_2 + 36H_1(u)p\sigma^4\alpha_{20} \right], \quad (2.13)
\]

with \( H_1(u) = u, \ H_2(u) = u^2 - 1, \ H_3(x) = u^3 - 3u, \ H_5(x) = u^5 - 10u^3 + 15u. \)
Remark 2.1. Let the elements of the population are non-random real numbers, viz. \( Y_{mN} = a_{mN} \). Put
\[
\bar{a} = N^{-1} (a_{1N} + ... + a_{NN}), \quad b^2 = N^{-1} ( (a_{1N} - \bar{a})^2 + ... + (a_{NN} - \bar{a})^2 ),
\]
\[
\tilde{a}_{mN} = (a_{mN} - \bar{a}) / b, \quad A_k = N^{-1} (\tilde{a}^k_{1N} + ... + \tilde{a}^k_{NN}).
\] (2.14)

Then
\[
W_{3N}(u) = \Phi(u) - e^{-u^2/2} \left\{ \frac{H_2(u)(1 - 2p)}{6\sqrt{nq}} A_3 + \frac{H_5(u)(1 - 4pq)}{72nq} A_3^2 \right. \\
+ \left. \frac{H_3(u)}{24nq} \left( (1 - 6pq)A_4 - 3(1 - 4pq) \right) \right\}.
\]

Remark 2.2. If \( \gamma = 0 \) and \( \alpha_{20} = 1 \),
\[
\psi_m(t, \tau) = \exp \left\{ -i 2\sqrt{n} \tilde{a}_m Y_{mN} - \frac{i \tau \sqrt{n}}{\sqrt{nq}} \right\} + p \exp \left\{ i t Y_{mN} - p E Y_{mN} \right\} \frac{1}{\sqrt{n}} + i \tau q \frac{1}{\sqrt{nq}}.
\] (2.16)

Formula (2.3) with \( \psi_m(t, \tau) \) written in the form (2.16) has been used (without proof) by Zhao et al. (2004) and Hu et al. (2007a). They subsequently proven few lemmas on the integrant of \( \Theta_N(t) \) by working with (2.16) in order to get bounds for the remainder terms in the first- and second-order asymptotic of the ch.f. \( \varphi_n(t) \). In contrast, based on the formula (2.3) and idea that the integrand of \( \Theta_N(t) \) is the ch.f. of a sum of \( N \) independent two-dimensional r.vec.s. (the fact which was not remarked by the referred authors), the algorithm presented above is more stream and general in constructing asymptotic expansion of any “length”. It is also much simpler than that of Mirakhmedov (1983) and Bloznelis (2000).

The formula (2.3) assumes that the r.v.s \( \xi_m \) are asymptotically not degenerate, hence, the case \( n = N \), i.e., the case of sum of independent r.v.s, is excluded. In Zhao et al. (2004) and Hu et al. (2007a), this limitation was covered using the approach suggested by von Bahr (1972). Let
\[
\phi_m(u) = E e^{iuY_{mN}}, \quad b_m(t) = e^{i2\alpha_{20}/2n\sigma^2} E e^{itY_{mN}/\sigma} - 1,
\] (2.17)
\[
B_j(t) = \frac{(-1)^{j+1}}{j} \sum_{m=1}^{N} b^j_m(t), \quad j = 1, ..., n;
\]
von Bahr (1972) have proved the following formula

\[ e^{t^2\alpha_2/2\sigma^2} \varphi_n(t) = \sum_{i_j \geq 0, 1 \leq j \leq n} \prod_{j=1}^{n} \frac{(pB_j(t))^{i_j}}{i_j!} C \left( \sum_{j=1}^{n} j i_j, n, n \right). \] (2.18)

In respect of formula (2.18), we make the following comments. Let us write 1 instead of the \( C(n,N,\cdot) \), then the right-hand side has form

\[ \exp \{ pB_1(t) + \ldots + p^n B_n(t) \} \]

where \( i \geq j \) and \( \ell_{i,n} \) are some magnitudes depending on moments of the r.v.s \( Y_{m,N} \). Hence, when we wish to approximate \( \varphi_n(t) \) by ch.f. of a normal distribution, we should approximate \( C(n,N,\cdot) \) by 1, separate \( \ell_{i,n} t^j / n^{(j-2)/2} \), where \( i \geq j \) and \( \ell_{i,n} \) are some magnitudes depending on moments of the r.v.s \( Y_{m,N} \). For the remainder term in CLT; later Mirakhmedov and Nabiev (1976) had proved the non-uniform variant of von Bahr’s result: \( c \max_{1 \leq m \leq N} E |Y_{m,N} - E Y_{m,N}|^3 / \sqrt{n} \sigma^3 \); while Zhao et al. (2004) obtained Berry-Esseen bound, which further improve the result of von Bahr, see below Remark 3.5.

When we wish to get \( s \)-terms Edgeworth asymptotic expansion, in addition to the above procedure, we must separate the terms \( \ell_{i,n} t^j / n^{(j-2)/2} \), \( j = 2, \ldots, s \) from \( pB(t) + \ldots + p^n B_n(t) \), estimate properly the sum \( p^{s+1} B_{s+1}(t) + \ldots + p^n B_n(t) \), get the asymptotic expansions for \( C(n,N,\cdot) \) and \( \exp \{ pB_1(t) + \ldots + p^n B_n(t) \} \) using Stirling’s formula and Taylor expansion idea respectively. Finally we obtain the asymptotic expansion of \( \varphi_n(t) \) by multiplying the asymptotic expansions. This algorithm has been developed by Mirakhmedov (1983) giving the exact formula for the first four terms; the algorithm use more complicated algebra compared to that above presented based on formula (2.3). Recently, two terms Edgeworth expansion with improved bound for a remainder term are given by Hu et al. (2007a).

In this paper, we combine both approaches in developing the methods of aforementioned papers and assuming weaker moment condition.

## 3 Main Results

We still use the notations of Sections 1 and 2; see (2.2), (2.5), (2.7), (2.12) and (2.14). Additionally, \( I\{A\} \) stands for the indicator function of the set \( A \).
and

$$\Delta_{jN} = \sup_u \left| P \{ S_{nN} < u \sigma \sqrt{n} + n\gamma \} - W_{jN}(u) \right|, j = 1, 2, 3;$$

where $W_{1N}(u) = \Phi(u)$, $W_{3N}(u)$ is given in (2.13) and

$$W_{2N}(u) = \Phi(u) + \frac{(1 - u^2)e^{-u^2/2}}{6\sigma^3\sqrt{2\pi n}} \Lambda_1.$$
where

\[ \beta_{k,N} = \sum_{m=1}^{N} E \left| \tilde{Z}_m \right|^k = \beta^{(1)}_{k,N} + qp^{k-1} \beta^{(2)}_{k,N}, \]

\[ \beta^{(1)}_{k,N} = \frac{N^{-1} \sum_{m=1}^{N} E \left| Y_{mN} - pE Y_{mN} - q \gamma \right|^k}{n^{(k-2)/2} \sigma^k}, \]

\[ \beta^{(2)}_{k,N} = \frac{N^{-1} \sum_{m=1}^{N} \left| E (Y_{mN} - \gamma) \right|^k}{n^{(k-2)/2} \sigma^k}, \text{ } k \geq 2. \]

Under the condition (2.15) \( \sigma^2 = 1 - p \alpha_{20} \geq q \), since \( \alpha_{20} \leq \alpha_{02} \), and hence

\[ \beta_{k,N} \leq \tilde{\beta}_{k,N}. \] (3.4)

**Theorem 3.2.** There exists a constant \( C_j > 0 \) such that for any \( T > 0 \) and each \( j = 1, 2, 3 \)

\[ \Delta_{jN} \leq C_j \left( \max \left( \beta_{1+j+\delta,N}, T^{-1} \right) + (nq)^{-(j-1+\delta)/2} + \chi_N \left( T_jN/\sigma \sqrt{n}, T/\sigma \sqrt{n} \right) \right), \]

where

\[ \chi_N(d_0, d_1) = I\{d_0 < d_1\} \int_{d_0 \leq |t| \leq d_1} \left| \frac{1}{t} \right| dt \]

\[ \leq C \min \left( \chi_{1N}(d_0, d_1), \chi_{2N}(d_0, d_1) \right) I\{d_0 < d_1\}, \] (3.5)

\[ \chi_{1N}(d_0, d_1) = \sqrt{nq} \ln d_1 \exp \left\{ -n \left( 1 - \sup_{d_0 \leq |t| \leq d_1} \frac{1}{n} \sum_{m=1}^{N} \left| E e^{itY_{mN}} \right| \right) \right\}, \]

and

\[ \chi_{2N}(d_0, d_1) = \sqrt{nq} \ln d_1 \exp \left\{ -nq \left( 1 - \sup_{d_0 \leq |t| \leq d_1} \frac{1}{n} \sum_{m=1}^{N} \left| E e^{itY_{mN}} \right| \right) \right\}. \]

**Remark 3.2.** The \( \chi_{1N}(d_0, d_1) \) is better than \( \chi_{2N}(d_0, d_1) \) under certain conditions such as \( q \) is close to zero but many enough of elements of the population are not degenerate r.v.s. In contrast, \( \chi_{2N}(a,b) \) is applicable for the case when all elements of a population are degenerate, i.e., non-random, viz. \( Y_{mN} = a_{mN}, m = 1, ..., N, \) but \( q \) is not too close to zero. In this case, \( \chi_{2N}(d_0, d_1) \) is exponentially small, for instance, if for a given \( d_0 > 0, \) there exist \( d_1 > 0, \varepsilon > 0 \) and \( \delta > 0 \) not depending of \( n \) such that

\[ \# \left\{ k : \left| \frac{t}{a_{kN} - \bar{a}} / b \sqrt{n} - x - 2\pi \nu \right| > \varepsilon \right\} \geq \delta n \]

for any fixed \( x, \) all \( n \geq 1, \) \( t \in (d_0, d_1) \) and integer \( \nu, \) see Bickel and von Zwet (1978), Robinson (1978) and Mirakhmedov (1983).
Choosing $T = 0.01 \left( \beta_{2+\delta,N} + (nq)^{-1/2} \right)^{-1}$, $T = \beta_{3+\delta,N}^{-1}$, and $T = \beta_{4+\delta,N}^{-1}$, respectively, we obtain from Theorem 2 the following:

**Corollary 3.1.** For any $\delta \in (0,1)$, there exists a positive constants $C_l$ such that

(a) $\Delta_{1N} \leq C_1 \left( \beta_{2+\delta,N} + (nq)^{-\delta/2} \right)$

(b) $\Delta_{2N} \leq C_2 \left( \beta_{3+\delta,N} + (nq)^{-(1+\delta)/2} + \chi_N \left( 0.001 \sigma^2 / V_{3,N}, (\beta_{3+\delta,N} \sigma \sqrt{n})^{-1} \right) \right)$

(c) $\Delta_{3N} \leq C_3 \left( \beta_{4+\delta,N} + (nq)^{-(2+\delta)/2} + \chi_N \left( 0.001 \sigma^2 / V_{3,N}, (\beta_{4+\delta,N} \sigma \sqrt{n})^{-1} \right) \right)$

where $V_{k,N} = N^{-1} \sum_{m=1}^{N} E \left| Y_{m,N} - \gamma \right|$. Taking $T = \beta_{3+\delta,N}$ and $T = \beta_{4+\delta,N}$, see notation (3.3), we obtain

**Corollary 3.2.** Let condition (2.15) is fulfilled. Then for any $\delta \in (0,1]$ there exists a positive constants $C_l$ such that

(a) $\Delta_{2N} \leq C_2 \left( \tilde{\beta}_{3+\delta,N} + (nq)^{-(1+\delta)/2} + \chi_N \left( 0.001 \sigma^2 / V_{3,N}, 2^{3+\delta} \sqrt{n^\delta \sigma^{1+\delta}} \right) \right)$

(b) $\Delta_{3N} \leq C_3 \left( \tilde{\beta}_{4+\delta,N} + (nq)^{-(2+\delta)/2} + \chi_N \left( 0.001 \sigma^2 / V_{3,N}, 2^{4+\delta} \sqrt{n^{1+\delta} \sigma^{2+\delta}} \right) \right)$

Set

$$
\mu_{k,N} = \frac{p}{(\sigma \sqrt{n})^k} \sum_{m=1}^{N} E \left| Y_{m,N} - \gamma \right|^k = V_{k,N} / \sigma^k n^{(k-2)/2}.
$$

**Theorem 3.3.** There exists a constant $C_j > 0$ such that for any $\delta \in (0,1]$

$$
\Delta_{jN} \leq C_j \left( \max \left( \mu_{1+j+\delta,N}, T^{-1} \right) + \chi_N \left( \tilde{T}_j / \sigma \sqrt{n}, T / \sigma \sqrt{n} \right) \right),
$$

for each $j = 1, 2$ and $3$, and any $T \geq 0.115 \mu_{\min(3,1+j+\delta),N}^{-1} \overset{def}{=} \tilde{T}_j$.

Choosing $T = 0.115 \mu_{2+\delta,N}^{-1}$, $T = \mu_{3+\delta,N}^{-1}$ and $T = \mu_{4+\delta,N}^{-1}$, respectively, we obtain from Theorem 3.3 the following

**Corollary 3.3.** For any $\delta \in (0,1)$, there exists a positive constants $C_l$ such that

(a) $\Delta_{1N} \leq C_1 \mu_{2+\delta,N}$

(b) $\Delta_{2N} \leq C_2 \left( \mu_{3+\delta,N} + \chi_N \left( 0.001 \sigma^2 V_{3,N}^{-1}, n^{\delta/2} \sigma^{2+\delta} V_{3+\delta,N}^{-1} \right) \right)$,
(c) \( \Delta_{3N} \leq C_3 \left( \mu_{4+\delta,N} + \chi_N \left( 0.001\sigma^2/V_{3,N}, n^{(1+\delta)/2}\sigma^{3+\delta}V_{4+\delta,N}^{-1} \right) \right) \).

We note that under condition (2.15)

\[
\sigma^{2+\delta}V_{3+\delta,N}^{-1} \leq 2^{2(3+\delta)}\sqrt{\sigma^{1+\delta}} \quad \text{and} \quad n^{(1+\delta)/2}\sigma^{3+\delta}V_{4+\delta,N}^{-1} \leq 2^{2(4+\delta)}\sqrt{n^{1+\delta}\sigma^{2+\delta}}.
\]

**Remark 3.3.** We restricted ourselves to just two and three term asymptotic expansions, only to keep the level of complexity of the expressions low. The results can be further extended for \( s \)-term asymptotic expansion, \( s \geq 3 \), but at the expense of added complexity in the proof.

**Remark 3.4.** It is easy to see that \( \beta_{k,N} \leq 2^{k-1}(1 + p^{k-1})\mu_{k,N} \). Nevertheless, Theorem 3.3 and Corollary 3.3 give better bounds for remainder term wrt Theorem 3.2 and Corollary 3.1 in the case when \( q \) close to zero; in particular if \( q = 0 \) then the concerning known results on sum of independent r.v.s. follows from Corollary 3.3.

**Remark 3.5.** The main results of Zhao et al. (2004) and Hu et al. (2007a) respectively state that: if condition (2.15) is fulfilled then

\[
\Delta_{1N} \leq C \min \left( \beta_{3,N} + (nq)^{-1/2}, \mu_{3,N} \right),
\]

\[
\Delta_{2N} \leq C \min \left( \beta_{4,N} + (nq)^{-1}, \mu_{4,N} \right)
\]

\[+ \min \left( \chi_{1N} \left( 0.01\sigma^2/V_{3,N}, 16\sigma\sqrt{n} \right), \chi_{2N} \left( 0.01\sigma^2/V_{3,N}, 16\sigma\sqrt{n} \right) \right). \]

In contrast, Parts (a) and (b) of Corollaries 3.1 and 3.3 gives more general bounds using the moments of lower order \( 2 + \delta \) and \( 3 + \delta \), \( 0 < \delta \leq 1 \), respectively. Moreover, since inequality (3.4), the case \( \delta = 1 \) of Corollaries 3.1 and 3.3 improves of their theorems, using Lyapunov’s ratios \( \beta_{k,N} \) instead of \( \beta_{k,N} \). Also Part (c) gives one more term of expansion. If \( Y_{m,N} = a_{mN} \), \( j = 3 \) are real non-random numbers then the main results of Robinson (1978) and Bloznelis (2000) follows from Corollary 3.2.

Further, we note that the condition (2.15) is essential in theorems of Zhao et al. (2004) and Hu et al. (2007a); failure to comply of (2.15) leads to a distortion of the remainder term; see Corollary 1 of Zhao et al. (2004). In contrast, our results, except Corollary 3.2, are invariant with respect to the condition (2.15). Therefore, without loss of generality and just in order to simplify the algebra in course of proofs we shall assume condition (2.15).

**Remark 3.6.** It can be readily checked that

\[
\frac{\Lambda_1}{\sigma^3\sqrt{n}} = \sum_{m=1}^{N} E\tilde{Z}_m^3.
\]

(3.6)
Hence, the Lyapunov’s ratio of the r.v.s $Z_m$, i.e., $\beta_{k,N}$, is natural characteristic in estimate of the remainder terms in approximation by normal distribution, in contrast of $\hat{\beta}_{k,N}$. Moreover, due to part (b) of Corollaries 3.1 and (3.6), we conjecture that there exist a constant $C > 0$ such that $\Delta_{1N} \leq C\beta_{3,N}$.

Let $P_N(x) = P\{S_{nN} < x\sigma \sqrt{n} + n\gamma\}$.

**THEOREM 3.4.** Let

$$\lim_{n \to \infty} pq > 0 \text{ and } \lim_{n \to \infty} N^{-1} \sum_{m=1}^{N} E \exp \{ H |Y_{mN}| \} < \infty, \exists H > 0. \quad (3.7)$$

Then for all $x \geq 0$, $x = o(\sqrt{N})$

$$\frac{1 - P_N(x)}{1 - \Phi(x)} = \exp \left\{ \frac{x^3}{\sqrt{N}} L_N \left( \frac{x}{\sqrt{N}} \right) \right\} \left( 1 + O \left( \frac{x + 1}{\sqrt{N}} \right) \right), \quad (3.8)$$

$$\frac{P_N(-x)}{\Phi(-x)} = \exp \left\{ - \frac{x^3}{\sqrt{N}} L_N \left( - \frac{x}{\sqrt{N}} \right) \right\} \left( 1 + O \left( \frac{x + 1}{\sqrt{N}} \right) \right), \quad (3.9)$$

where $L_N(v) = \ell_{0N} + \ell_{1N}v + ...$ is a power series that for all sufficiently large $N$ is majorized by a power series with coefficients not depending on $N$, and is convergent in some disc, so that $L_N(v)$ converges uniformly in $N$ for sufficiently small values of $|v|$. Particularly

$$\ell_{0N} = \frac{1}{6} \sum_{m=1}^{N} E\tilde{Z}_m^3, \ell_{1N} = \frac{1}{8} \left( \frac{1}{3} \sum_{m=1}^{N} E\tilde{Z}_m^4 - \left( \sum_{m=1}^{N} E\tilde{Z}_m^2 \xi_m \right)^2 - \sum_{m=1}^{N} (E\tilde{Z}_m^3)^2 \right).$$

**COROLLARY 3.4.** Let the conditions (3.7) is satisfied. Then for all $x \geq 0$, $x = o(N^{1/6})$

$$\frac{1 - P_N(x)}{1 - \Phi(x)} = 1 + O \left( \frac{x^3 + 1}{\sqrt{N}} \right) \quad \text{and} \quad \frac{P_N(-x)}{\Phi(-x)} = 1 + O \left( \frac{x^3 + 1}{\sqrt{N}} \right). \quad (3.10)$$

**COROLLARY 3.5.** Let the condition (3.7) is satisfied. Then for all $x \geq 0$, $x = o(N^{1/6})$

$$|P_N(x) - \Phi(x)| = O \left( \frac{(x^2 + 1) \exp \{-x^2/2\}}{\sqrt{N}} \right). \quad (3.11)$$
Corollary 3.6. Let $Y_{mN} = a_{mN}$, $m = 1, ..., N$; that is the elements of the population are non-random real numbers. If $\lim_{n \to \infty} pq > 0$ then for all $x \geq 0$, $x = o(\sqrt{N})$ the relations (3.8) and (3.9) hold, and for all $x \geq 0$, $x = o(N^{1/6})$ the relations (3.10) and (3.11) hold. Also, in this case, in notation (2.14)

$$\ell_{0N} = \frac{1 - 2p}{6\sqrt{pq}} A_3, \ell_{1N} = \frac{1 - 6pq}{24pq} A_4 - \frac{(1 - 2p)^2}{8pq} - \frac{1}{8pq} A_3^2.$$

Remark 3.7. For the case when $Y_{mN} = a_{mN}$, Theorem 1 of Robinson (1977) shows that the relation (3.8) is true for all $x \geq 0$, $x = o\left(\sqrt{N}/\max_\sim a_{mN}\right)$. Also we refer to the recently paper of Hu et al. (2007b) where it was shown that for $0 \leq x \leq C \min\left(\sqrt{nq}/\max_\sim \tilde{a}_{mN}, (nq)^{1/6}/B_3^{1/3}\right)$

$$1 - P_N(x) = (1 - \Phi(x)) \left(1 + O\left((x + 1)^3B_3/\sqrt{nq}\right)\right),$$

where in addition to the notation (2.14) we denote

$$B_l = N^{-1}\left(|\tilde{a}_{1N}|^l + ... + |\tilde{a}_{NN}|^l\right). \quad (3.12)$$

Remark 3.8. Notice that the sample sum is a special case of the statistics of form $\sum_{m=1}^N f_m(\eta_m)$, so namely decomposable statistics (DS), in a simple random sample scheme without replacement, where $f_1, ..., f_N$ are functions defined on the set of non-negative integers, $\eta_m$ is a frequency of $m$-th element of the population in a sample of size $n$. Therefore, from general theorems on DS, one can derive some results for the sample sum. In particular, Theorems 3.1 and 3.4 are derived from concerning theorems of Mirakhmedov (1996) and Mirakhmedov et al. (2014). For details of DS see the referred papers and references within.

4 Lemmas and Proofs

Before getting to proof of the assertions of Section 3, we shall prove several auxiliary lemmas. Actually, Lemma 4.1 and 4.2 below are valid for any sequence of independent r.vec.s with zero expectations and uncorrelated sums of components; moreover, if the r.vec.s has a finite moments of integer order, say $k > 2$, then Lemma 4.1 is a corollary of Theorem 9.11 of BR (1976); authors could not find in literature concerning assertions when the $k$ may be non-integer as we require.

In what follows, we still use the notations of Sections 2 and 3, and assume $E|Z_{m,N}|^k < \infty$, where $k > 2$ is not necessary to be integer. Additionally just to keep the notation simple, we put $\kappa_{l,N} = \sum_{m=1}^N E|\tilde{\xi}_m|^l =$
\((q^{l-1} + p^{l-1})(nq)^{-(l-2)/2}\). We shall use the following relation, see formula (10) of Mirakhmedov (1992): for any complex \(z\), integer \(l \geq 0\) and \(\delta \in (0, 1]\)

\[
e^z = \sum_{j=0}^{l} \frac{z^j}{j!} + \theta \frac{2^{1-\delta}z^{l+\delta}}{(1+\delta) \cdot \ldots \cdot (l+\delta)}e^{|\text{Re} z|},
\]  

(4.1)

here and everywhere in what follows \(\delta \in (0, 1]\), and we use the same symbol \(\theta\) for a value such that \(|\theta| \leq 1\), although they may not be the same at a different occurrence; also we use well-known inequalities between moments and Lyapunov’s ratio, see for instance BR (1976, Lemma 6.2) and Petrov (1975, Ch. IV, Lemma 2):

\[
E |\xi|^i \leq \left( E |\xi|^i \right)^{j/i}, 1 \leq j \leq i; \beta_{1N}^{1/(l-2)} \leq \beta_{kN}^{1/(k-2)}, 3 \leq l \leq k.
\]  

(4.2)

In what follows, without loss of generality, we shall assume that condition (2.15) is fulfilled. The general case can be reduced to that of satisfying (2.15) by considering \((Y_{mN} - \gamma) / \sqrt{\alpha_{20}}\) instead of \(Y_{mN}\). So, throughout this section \(Z_m = Y_{mN} \xi_m - pEY_{mN}\) and \(\sigma^2 = 1 - p\alpha_{02}\).

Set

\[
\tilde{P}_{1N}(t, \tau) = 1, \tilde{P}_{2N}(t, \tau) = 1 + \frac{1}{\sqrt{N}} P_{1N}(t, \tau), \tilde{P}_{3N}(t, \tau) = 1 + \frac{1}{\sqrt{N}} P_{1N}(t, \tau) + \frac{1}{N} P_{2N}(t, \tau),
\]

\[
\Psi_N(t, \tau) = \prod_{m=1}^{N} \psi_m(t, \tau) \quad \text{and} \quad \Im_{j, N} = c_1 \min \left( \beta_{1+j+\delta, N}^{-1/(1+j+\delta)}, \kappa_{1+j+\delta, N}^{-1/(1+j+\delta)} \right).
\]

**Lemma 4.1.** If

\[
\beta_{1+j+\delta, N} \leq 1 \quad \text{and} \quad \max(|t|, |\tau|) \leq \Im_{j, N},
\]  

(4.3)

a positive \(c_1 \leq 0.1\), then there exist \(C_1 > 0\) such that for each \(k = 0, 1\) and \(j = 1, 2, 3\) one has

\[
\left| \frac{\partial^k}{\partial t^k} \left( \Psi_N(t, \tau) - e^{-t^2 + \tau^2 / 2} \tilde{P}_{jN}(t, \tau) \right) \right| \leq C_1 \left( \beta_{1+j+\delta, N} + \frac{1}{(nq)^{(j-1+\delta)/2}} \right) \left( 1 + |t|^{1+j+\delta} + |\tau|^{1+j+\delta} \right) e^{-t^2 + \tau^2 / 4}.
\]

**Proof.** The case \(j = 1\) follows from Part (1) of Lemma A of Mirakhmedov (2005). We restrict ourselves by proof of the case \(j = 3\); the case \(j = 2\) can
be proved in a very similar manner with less algebra. The proof uses an idea of the proof of Theorem 8.6 of BR p.64. We have

\[
|\psi_m(t, \tau) - 1| \leq \frac{1}{2} E \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^2 \\
\leq \left( t^2 \left( E \left| \tilde{Z}_m \right|^{4+\delta} \right)^{2/(4+\delta)} + \tau^2 \left( E \left| \tilde{\xi}_m \right|^{4+\delta} \right)^{2/(4+\delta)} \right) \\
\leq t^2 \beta_{4+\delta,N}^{2/(4+\delta)} + \tau^2 \kappa_{4+\delta,N}^{2/(4+\delta)} < 2c_1, \quad (4.4)
\]

Hence, we can write

\[
\Psi_N(t, \tau) = \exp \left\{ \sum_{m=1}^N \ln \psi_m(t, \tau) \right\} = \exp \left\{ - \sum_{m=1}^N \sum_{r=1}^{\infty} \frac{1}{r} (1 - \psi_m(t, \tau))^r \right\}.
\]

Note that (see (2.8) and (2.9))

\[
\frac{1}{\sqrt{N}} \frac{\partial}{\partial t} P_{1N}(t, \tau) = \frac{i^3}{2} \sum_{m=1}^N E\tilde{Z}_m \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^2,
\]

\[
\frac{1}{N} \frac{\partial}{\partial t} P_{2N}(t, \tau) = \frac{i^4}{6} \sum_{m=1}^N E\tilde{Z}_m \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^3 \\
- \frac{i^4}{2} \sum_{m=1}^N E \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^2 E\tilde{Z}_m \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right) \\
+ \frac{i^6}{12} \sum_{m=1}^N E \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^3 \sum_{m=1}^N E\tilde{Z}_m \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^2.
\]

To keep the notations simple, we set

\[
\nabla_N(t, \tau) = \sum_{m=1}^N \left( (1 - \psi_m(t, \tau)) + 2^{-1} (1 - \psi_m(t, \tau))^2 \right) - \frac{t^2 + \tau^2}{2}.
\]

Now,

\[
\Psi_N(t, \tau) - \exp \left\{ - \frac{t^2 + \tau^2}{2} \right\} \left( 1 + \frac{1}{\sqrt{N}} P_{1N}(t, \tau) + \frac{1}{N} P_{2N}(t, \tau) \right) \\
= \exp \left\{ - \frac{t^2 + \tau^2}{2} \right\} \left[ \exp \left\{ - \nabla_N(t, \tau) \right\} - 1 - \frac{1}{\sqrt{N}} P_{1N}(t, \tau) - \frac{1}{N} P_{2N}(t, \tau) \right]
\]
\[ + \exp \left\{ -\nabla N(t, \tau) - \frac{t^2 + \tau^2}{2} \right\} \left( \exp \left\{ - \sum_{m=1}^{N} \sum_{j=3}^{\infty} j^{-1} (1 - \psi_m(t, \tau))^j \right\} - 1 \right) \]

\[ = A_1(t, \tau) + A_2(t, \tau). \quad (4.5) \]

Also

\[ \frac{\partial}{\partial t} A_1(t, \tau) = -tA_1(t, \tau) + A_{11}(t, \tau), \quad (4.6) \]

where

\[ A_{11}(t, \tau) = \exp \left\{ -\nabla N(t, \tau) \right\} \left( \sum_{m=1}^{N} (2 - \psi_m(t, \tau)) \frac{\partial}{\partial t} \psi_m(t, \tau) \right. \]

\[ + \frac{i^3}{2} \sum_{m=1}^{N} E\tilde{Z}_m (t\tilde{Z}_m + \tau\tilde{\xi}_m)^2 - \frac{i^4}{6} \sum_{m=1}^{N} E\tilde{Z}_m (t\tilde{Z}_m + \tau\tilde{\xi}_m)^3 \]

\[ + \frac{i^2}{2} \sum_{m=1}^{N} E (t\tilde{Z}_m + \tau\tilde{\xi}_m)^2 E\tilde{Z}_m (t\tilde{Z}_m + \tau\tilde{\xi}_m) \]

\[ + \frac{i^3}{2} \left( \exp \left\{ -\nabla N(t, \tau) \right\} - 1 - \frac{i^3}{6} \sum_{m=1}^{N} E (t\tilde{Z}_m + \tau\tilde{\xi}_m)^3 \right) \]

\[ \sum_{m=1}^{N} E\tilde{Z}_m (t\tilde{Z}_m + \tau\tilde{\xi}_m)^2 \]

\[ + \frac{i^4}{2} \left( \exp \left\{ -\nabla N(t, \tau) \right\} - 1 \right) \left( \frac{1}{3} \sum_{m=1}^{N} E\tilde{Z}_m (t\tilde{Z}_m + \tau\tilde{\xi}_m)^3 \right. \]

\[ - \sum_{m=1}^{N} E (t\tilde{Z}_m + \tau\tilde{\xi}_m)^2 E\tilde{Z}_m (t\tilde{Z}_m + \tau\tilde{\xi}_m) \right) \]

\[ \equiv A_{11}^{(1)}(t, \tau) + A_{11}^{(2)}(t, \tau) + A_{11}^{(3)}(t, \tau). \quad (4.7) \]

Next,

\[ \frac{\partial}{\partial t} A_2(t, \tau) = A_2(t, \tau) \sum_{m=1}^{N} (2 - \psi_m(t, \tau)) \frac{\partial}{\partial t} \psi_m(t, \tau) \]

\[ + \exp \left\{ -\nabla N(t, \tau) - \frac{t^2 + \tau^2}{2} \right\} \]

\[ \exp \left\{ - \sum_{m=1}^{N} \sum_{j=3}^{\infty} j^{-1} (1 - \psi_m(t, \tau))^j \right\} \]

\[ \sum_{m=1}^{N} \sum_{j=3}^{\infty} (1 - \psi_m(t, \tau))^{j-1} \frac{\partial}{\partial t} \psi_m(t, \tau) \]
Approximation by normal distribution for a sample sum

\[ \equiv A_2^{(1)}(t, \tau) + A_2^{(2)}(t, \tau). \] (4.8)

We have

\[ \sum_{m=1}^{N} (1 - \psi_m(t, \tau)) = \frac{t^2 + \tau^2}{2} - \frac{i^3}{6} \sum_{m=1}^{N} E \left( t \tilde{Z}_m + \tau \tilde{\xi}_m \right)^3 + r(t, \tau), \] (4.9)

with \( |r(t, \tau)| \leq 3^{-1} \left( t^4 \beta_{4,N} + \tau^4 \kappa_{4,N} \right), \) and

\[ \sum_{m=1}^{N} |1 - \psi_m(t, \tau)|^2 \leq 2 \left( t^4 \beta_{4,N} + \tau^4 \kappa_{4,N} \right). \] (4.10)

Using the last and first inequalities of (4.4), we obtain

\[ \left| \sum_{m=1}^{N} \sum_{r=3}^{\infty} r^{-1} (1 - \psi_m(t, \tau))^r \right| \leq \frac{1}{3(1 - 2c_1^2)} \sum_{m=1}^{N} |1 - \psi_m(t, \tau)|^3 \]

\[ \leq \frac{4c_1^{2-\delta}}{3(1 - 2c_1^2)} \left( |t|^{4+\delta} \beta_{4+\delta,N} + |\tau|^{4+\delta} \kappa_{4+\delta,N} \right) \leq \frac{8c_1^{\delta}}{3(1 - 2c_1^2)}. \] (4.11)

Now, use Taylor expansion idea, (2.6), inequalities (4.2), condition (4.3) and (4.9), (4.10) to get (see notation (2.8), (2.9))

\[ -\nabla N(t, \tau) = \frac{i^3}{6} \sum_{m=1}^{N} E \left( t \tilde{Z}_m + \tau \tilde{\xi}_m \right)^3 \]

\[ + \frac{i^4}{24} \sum_{m=1}^{N} E \left( t \tilde{Z}_m + \tau \tilde{\xi}_m \right)^4 - \frac{i^4}{8} \sum_{m=1}^{N} \left( E \left( t \tilde{Z}_m + \tau \tilde{\xi}_m \right)^2 \right)^2 + r_1(t, \tau) \]

\[ \equiv \frac{1}{\sqrt{N}} P_{1,N}(t, \tau) + \frac{1}{N} \left( P_{2,N}(t, \tau) - \frac{1}{2} P_{1,N}^2(t, \tau) \right) + r_1(t, \tau) \] (4.12)

\[ = \frac{i^3}{6} \sum_{m=1}^{N} E \left( t \tilde{Z}_m + \tau \tilde{\xi}_m \right)^3 + r_2(t, \tau) \] (4.13)

\[ = r_3(t, \tau), \] (4.14)

\[ \nabla_N^2(t, \tau) = \frac{1}{N} P_{1,N}^2(t, \tau) + r_4(t, \tau), \] (4.15)

\[ |\nabla N(t, \tau)|^3 \leq r_5(t, \tau), \] (4.16)
where
\[ |r_l(t, \tau)| \leq C(c_1, \delta) \left( |t|^{4+\delta} \beta_{4+\delta,N} + |\tau|^{4+\delta} \kappa_{4+\delta,N} \right), \quad l = 1, 4, 5; \] (4.17)

\[ |r_2(t, \tau)| \leq 4 \left( t^4 \beta_{4+\delta,N} + \tau^4 \kappa_{4+\delta,N} \right), \] (4.18)

\[ |r_3(t, \tau)| \leq \frac{2}{3} \left( |t|^{\beta_{4+\delta,N}^{1/(2+\delta)}} + |\tau|^{\kappa_{4+\delta,N}^{1/(2+\delta)}} \right) \leq \frac{2c_1}{3} \left( t^2 + \tau^2 \right) < \frac{1}{15} \left( t^2 + \tau^2 \right); \] (4.19)

and
\[
\sum_{m=1}^{N} \left( 2 - \psi_m(t, \tau) \right) \frac{\partial}{\partial t} \psi_m(t, \tau) = -t + \frac{i^3}{2} \sum_{m=1}^{N} E\tilde{Z}_m \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^2 \\
+ \frac{i^4}{6} \sum_{m=1}^{N} E\tilde{Z}_m \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^3 \\
- \frac{i^4}{2} \sum_{m=1}^{N} E \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right)^2 \\
E\tilde{Z}_m \left( t\tilde{Z}_m + \tau\tilde{\xi}_m \right) + r_6(t, \tau), \] (4.20)

where
\[ |r_6(t, \tau)| \leq 4 \left( |t|^{4+\delta} + |\tau|^{4+\delta} \right) \beta_{4+\delta,N} + |\tau|^{4+\delta} \kappa_{4+\delta,N} \right). \]

Also
\[ \left| \sum_{m=1}^{N} \left( 2 - \psi_m(t, \tau) \right) \frac{\partial}{\partial t} \psi_m(t, \tau) \right| \leq 3 \left( |t| + |\tau| \right). \] (4.21)

At last, using (4.4) and that
\[ \left| \frac{\partial}{\partial t} \psi_m(t, \tau) \right| \leq |t| E\tilde{Z}_m^2 + |\tau| E \left| \tilde{Z}_m \xi_m \right| \]
we have
\[
\left| \sum_{m=1}^{N} \sum_{j=3}^{\infty} (1 - \psi_m(t, \tau))^{j-1} \frac{\partial}{\partial t} \psi_m(t, \tau) \right| \leq \frac{1}{1 - 2c_1^2} \sum_{m=1}^{N} |1 - \psi_m(t, \tau)|^2 \left| \frac{\partial}{\partial t} \psi_m(t, \tau) \right| \\
\leq C(c_1, \delta) \left( |t|^{4+\delta} + |\tau|^{4+\delta} \right) \left( \beta_{4+\delta,N} + \kappa_{4+\delta,N} \right). \] (4.22)

To prove Lemma 4.1, it is sufficient to get appropriate upper bound for the module of \( A_1(t, \tau), A_2(t, \tau) \) and their derivative. To get a bound
(i) for the $|A_1(t, \tau)|$: use (4.1) with $\delta = 1$, $l = 2$ and (4.12)–(4.16) and the last inequality of (4.19);

(ii) for the $|A_2(t, \tau)|$: use (4.1) with $\delta = 1$, $l = 0$ and both inequalities of (4.10), equality (4.13), the last inequality of (4.19);

(iii) for the $|A_{11}(t, \tau)|$: successively to estimate $|A^{(1)}_{11}(t, \tau)|$ use (4.20), (4.14) and (4.16), to estimate $|A^{(2)}_{11}(t, \tau)|$ use (4.1) with $\delta = 1$, $l = 1$ and (4.15), (4.13), (4.18), (4.14), (4.18), to estimate $|A^{(3)}_{11}(t, \tau)|$ use (4.1) with $\delta = 1$, $l = 0$ and (4.14) with the first inequality of (4.19);

(iv) for the $|\partial_t A_2(t, \tau)|$: use to estimate $A^{(1)}_2(t, \tau)$ term the bound already obtained for the $|A_2(t, \tau)|$ and (4.21); next for the $A^{(2)}_2(t, \tau)$ term use (4.22), (4.11), (4.14) and the last inequality of (4.19).

Applying the results of (i)–(iv) in the (4.5), (4.6), (4.7) and (4.8), also using inequalities (4.2), condition (4.3) after some algebra one can complete the proof of Lemma 4.1; the details are omitted.

**Lemma 4.2.** Let $\max(\beta_{2+\delta,N}+(nq)^{-1/2}) \leq 0.01$. If $|t| \leq 0.315\beta_{2+\delta,N}^{-1}$ and $|\tau| \leq 0.315\sqrt{nq}$ then for each $k = 0, 1$ one has

$$\left| \frac{\partial^k}{\partial t^k} \Psi_N(t, \tau) \right| \leq \exp \left\{ -\frac{t^2 + \tau^2}{10} \right\}.$$ 

**Proof of Lemma 4.2.** Follows from Part (2) of Lemma A of Mirakhmedov (2005).

**Lemma 4.3.** There exists a $c_1 > 0$ such that if $|t| \leq c_1 (\beta_{2+\delta,N}+(nq)^{-1/2})^{-1}$ and $0.0625\sqrt{nq} \leq |\tau| \leq \pi\sqrt{nq}$ then for each $k = 0, 1$ one has

$$\left| \frac{\partial^k}{\partial t^k} \Psi_N(t, \tau) \right| \leq \exp \left\{ -0.09nq \right\}.$$ 

**Proof.** For a given r.v. $\varsigma$ say, let $\varsigma* = \varsigma - \varsigma^*$, where $\varsigma^*$ is an independent copy of $\varsigma$. We have

$$|\psi_m(t, \tau)|^2 = |\psi_m(0, \tau)|^2 + E \left[ (e^{it\tilde{Z}_m^*} - 1) (e^{it\xi_m^*} - 1) \right] + E \left( e^{it\tilde{Z}_m^*} - 1 \right)$$

$$\leq |\psi_m(0, \tau)|^2 + |t| |\tau| E \left| \tilde{Z}_m^* \xi_m^* \right| + t^2 E \tilde{Z}_m^{*2}.$$
Using this and the inequalities $x < \exp\{x - 1\}$,

$$\left| \frac{\partial}{\partial t} \prod_{m=1}^{N} \psi_m(t, \tau) \right| \leq \sum_{m=1}^{N} \left| \frac{\partial}{\partial t} \psi_m(t, \tau) \right| \prod_{l \neq m} |\psi_l(t, \tau)|, \quad \left| \sum_{m=1}^{N} \frac{\partial}{\partial t} \psi_m(t, \tau) \right| \leq |t| + 2|\tau|,$$

we find for $k = 0, 1$

$$\left| \frac{\partial^k}{\partial t^k} \Psi_N(t, \tau) \right| \leq \sqrt{e} (|t| + |\tau|)^k \exp \left\{ \pi |t| + 2t^2 - \frac{1}{4} \sum_{m=1}^{N} \left( 1 - |\psi_m(0, \tau)|^2 \right) \right\}.$$

(4.23)

On the other hand, $|E \exp\{i\tau \xi_m\}|^2 \leq \exp \left\{ -4pq \sin^2 \tau/2 \right\}$,

$$\sin^2 \frac{\tau}{2} \geq \frac{\tau^2}{\pi^2}, |\tau| \leq \pi, 1 - e^{-u} \geq \frac{1 - e^{-c}}{c} u, 0 \leq u \leq c,$$

Hence, for $0.0625 \sqrt{nq} \leq |\tau| \leq \pi \sqrt{nq}$, we have $\sum_{m=1}^{N} \left( 1 - |\psi_m(0, \tau)|^2 \right) \geq (1 - e^{-1})nq/4$. This inequality together with (4.23) yield

$$\left| \frac{\partial^k}{\partial t^k} \Psi_N(t, \tau) \right| \leq \sqrt{e} (|t| + |\tau|)^k \exp \left\{ -nq \left( (1 - e^{-1}) 4^{-1} - (\pi + 2c_1) c_1 \right) \right\} \leq \exp \left\{ -0.09nq \right\},$$

since $|t| \leq c_1 \left( \beta_{2+\delta,N} + (nq)^{-1/2} \right)^{-1} \leq c_1 \sqrt{nq}$, and here we choose $c_1 \leq (1 - e^{-1})/16(\pi + 1) \approx 0.0126$. Lemma 4.3 follows.

**Lemma 4.4.** There exists a $c_1 > 0$ such that if $|t| \leq c_1 \beta_{2+\delta,N}^{-1}$, see (3.3), and $0.0625 \sqrt{nq} \leq |\tau| \leq \pi \sqrt{nq}$ then for each $k = 0, 1$ one has

$$\left| \frac{\partial^k}{\partial t^k} \Psi_N(t, \tau) \right| \leq 3 (|t| + |\tau|)^k \exp \left\{ -cnq \right\}.$$

**Proof.** The proof follows by using the Newton-Leibniz rule for derivative of a product, reasons very similar to that of the proof of Lemma 2 of Zhao et al. (2004) and inequality

(4.24)

Here $c_1 > 0$ so small that $8c_1^2 + c_1 \leq (1 - \cos(1/16))/24$, hence $0 < c_1 \leq 0.0329$. 


Let
\[ W_{1N}(t) = \exp\left\{-t^2/2\right\}, W_{2N}(t) = e^{-t^2/2} \left(1 + \frac{(it)^3}{6\sqrt{n}\sigma^3} \Lambda_1\right), \quad (4.25) \]
and \( W_{3N}(t) \) be defined in (2.11). Recall the notation (3.2), (3.3) and note that
\[ T_{1,N} = 0.01 \max \left( -\beta_{2+\delta,N}, \left( \beta_{2+\delta,N} + (nq)^{-1/2}\right)^{-(1)} \right), \]
\[ T_{2,N} = T_{3,N} = 0.01 \max \left( -\beta_{3,N}, \left( \beta_{3,N} + (nq)^{-1/2}\right)^{-(1)} \right). \]

**Lemma 4.5.** For each \( j = 1, 2, 3 \) and \( k = 0, 1 \) one has: if \( \beta_{1+j+\delta,N} \leq 1, \max \left( \beta_{2+\delta,N}, (nq)^{-1/2}\right) \leq 0.01 \) and \( |t| \leq T_{j,N} \) then there exist a constant \( C > 0 \) such that
\[
\left| \frac{\partial^k}{\partial t^k} (\varphi_n(t) - W_j N(t)) \right| \leq C \left( \left( 1 + |t|^{2+j}\right) e^{-t^2/12} (\beta_{2+j+\delta,N} + \kappa_{2+j+\delta,N}) + e^{-cnq} \right). \]

**Proof.** We assume that \( \Im j,N \leq T_{j,N} \), see notation of Lemma 4.1; the case \( \Im j,N > T_{j,N} \) is simpler needing considerably less algebra.

Let \( |t| \leq \Im j,N \). We have
\[
\frac{\partial^k}{\partial t^k} (\Theta_N(t) - Q_i N(t)) = \frac{1}{\sqrt{2\pi}} \int_{|\tau| \leq \Im j,N} \frac{\partial^k}{\partial t^k} \left( \Psi_N(t, \tau) - e^{-t^2/2} (1 + \tilde{P}_j N(t, \tau)) \right) d\tau
\]
\[ + \frac{1}{\sqrt{2\pi}} \int_{\Im j,N \leq |\tau| \leq 0.315/\sqrt{m}} \frac{\partial^k}{\partial t^k} \Psi_N(t, \tau) d\tau + \frac{1}{\sqrt{2\pi}} \int_{0.315/\sqrt{m} \leq |\tau| \leq \pi/\sqrt{m}} \frac{\partial^k}{\partial t^k} \Psi_N(t, \tau) d\tau
\]
\[ + \frac{1}{\sqrt{2\pi}} \int_{|\tau| \geq \Im j,N} e^{-\tau^2/2} \frac{\partial^k}{\partial t^k} e^{-t^2/2} (1 + \tilde{P}_j N(t, \tau)) d\tau = D_1 + D_2 + D_3 + D_4. \]

Applying Lemmas 4.1, 4.2 and 4.3 to estimate \( D_1, D_2, \) and \( D_3 \), respectively, after simple calculations, we obtain
\[
|D_1 + D_2 + D_3| \leq C \left( \left( 1 + |t|^{2+i}\right) e^{-t^2/12} (\beta_{2+j+\delta,N} + \kappa_{2+j+\delta,N}) + e^{-cnq} \right). \]

It is readily seen that
\[
|D_4| \leq C e^{-t^2/2} e^{-\pi^2 nq/4}. \]
Now let \( \mathcal{S}_{j,N} \leq |t| \leq T_{j,N} \). Write
\[
\frac{\partial^k}{\partial t^k} (\Theta_N(t) - Q_{jN}(t)) = \frac{1}{\sqrt{2\pi}} \int_{|\tau| \leq 0.315\sqrt{nq}} \frac{\partial^k}{\partial t^k} \Psi_N(t, \tau) d\tau + D_3 - \frac{\partial^k}{\partial t^k} Q_{jN}(t)
\]
\[= \tilde{D}_2 + D_3 - \frac{\partial^k}{\partial t^k} Q_{jN}(t). \tag{4.29}\]
Due to definition of \( Q_{iN}(t) \), it easy to see that: for \( \mathcal{S}_{jN} \leq |t| \leq T_{jN} \)
\[
\left| \frac{\partial^k}{\partial t^k} Q_{jN}(t) \right| \leq C \left( 1 + |t|^{2+j} \right) e^{-t^2/4} (\beta_{2+j+\delta,N} + \kappa_{2+j+\delta,N}). \tag{4.30}\]

Note that \( T_{1N} \leq 0.01\beta_{2+\delta,N}^{-1}, \ T_{2N} = T_{3N} \leq 0.01\beta_{3,N}^{-1} \), hence, we can use Lemma 4.2 to estimate \( \tilde{D}_2 \); also to estimate \( D_3 \) we apply Lemmas 4.3 and 4.4; finally, we obtain: if \( \mathcal{S}_{jN} \leq |t| \leq T_{jN} \) then
\[
\left| \tilde{D}_2 + D_3 \right| \leq C \left( (\beta_{2+j+\delta,N} + \kappa_{2+j+\delta,N}) e^{-t^2/12} + e^{-cnq} \right). \tag{4.31}\]
Thus, from (4.26)–(4.31), we have: if \( |t| \leq T_{jN} \) then for \( k = 0 \) and \( k = 1 \)
\[
\left| \frac{\partial^k}{\partial t^k} (\Theta_N(t) - Q_{jN}(t)) \right| \leq C \left( (1 + |t|^{2+j}) e^{-t^2/12} (\beta_{2+j+\delta,N} + \kappa_{2+j+\delta,N}) + e^{-cnq} \right). \tag{4.32}\]

On the other hand using Stirling’s formula, it is easy to show that
\[
\Theta_N(0) = 1 + \frac{1}{N} G_{2,N}(0) + \frac{2\theta}{(nq)^2}. \tag{4.33}\]
Applying (4.32) and (4.33) in the formula (2.3) and using inequality between Lyapunov’s ratios with quite evident calculations, we complete the proof of Lemma 4.5.

Remark 4.1. It may remarked that Lemma 4.5 can be further extended for \( s \)-term asymptotic expansion and any \( k = 0, 1, \ldots, s, \ s \geq 3 \), but at the expense of added complexity in the proof. Such an extension of Lemma 4.5 can be used to derive asymptotic expansion of the moments of the sample sum. In particular, one can show that, see notation (2.12)
\[
E(S_{nN} - ES_{nN})^3 = n\Lambda_1 + Cn^{3/2}\sigma^3 (\beta_{3+\delta,N} + (nq)^{-1+\delta/2}),
\]
\[
E(S_{nN} - ES_{nN})^4 = n^2\sigma^4 \left( 3 + \frac{\Lambda_2}{n\sigma^4} + 4\frac{pq\alpha_2}{n\sigma^2} + C(\beta_{4+\delta,N} + (nq)^{-2+\delta/2}) \right).
\]
In the case, \( Y_{mN} = a_{mN} \) are non-random real numbers these formulas have the following form, see notations (2.14) and (3.12),

\[
E(S_{nN} - E S_{nN})^3 = n(1 - 2p) A_3 + C \sqrt{nq}((1 - 3pq) A_4 + 1),
\]

\[
E(S_{nN} - ES_{nN})^4 = (nq)^2 \left( 3 + \frac{1}{nq}((1 - 6pq) A_4 - 3) + \frac{16p}{n} + C \frac{1}{(nq)^{3/2}}((p^3 + q^3)(1 + B_5)) \right).
\]

An upper bound for \( ES^4_{nN} \) has been presented in Lemma 2.6 of Rosen (1967).

**Lemma 4.6.** Let

\[
\mu_{2+\delta,N} \leq \min \left( 2^{-1}, 2^{-(2+\delta)/2}(nq)^{-\delta/2} \right).
\]

Then for \(|t| \leq c_1(\delta)\mu_{2+\delta,N}^{-1}, c_1(\delta) := ((1 + \delta)(2 + \delta)/16)^{1/\delta} \), one has

\[
\prod_{m=1}^{N} \left| \psi_m(t, \tau) \right| \leq \exp \left\{ -p^2 \frac{t^2}{4} \right\}.
\]

**Proof.** Set

\[
v_{s,m} = E |Y_{m,N}|^s, \bar{v}_{s,m} = \max(1, v_{s,m}), \bar{\vartheta}_m \leq \text{Var} Y_{m,N}, V_{s,N} = N^{-1} \sum_{m=1}^{N} v_{s,m}, \bar{\vartheta}^2 = N^{-1} \sum_{m=1}^{N} \vartheta_m^2 = 1 - \alpha_{02}.
\]

Since \( \mu_{2+\delta,N} = n^{-\delta/2} \sigma^{-(2+\delta)} V_{2+\delta,N} \) and (4.34), we have \( 2q^{\delta/2+\delta} V_{2+\delta} \leq \sigma^2 \). Therefore, \( 2q\alpha_{02} \leq 2q V_{2+\delta} \leq 2q^{\delta/2+\delta} V_{2+\delta} \leq \sigma^2 = \vartheta^2 + q\alpha_{02}, \) hence

\[
\sigma^2 \geq \vartheta^2 = \sigma^2 - q\alpha_{02} \geq \sigma^2 - \sigma^2/2 \geq \sigma^2/2. \quad (4.35)
\]

Apply (4.1) with \( l = 2 \) to get

\[
\left| E e^{i t Y_{mN}/\sigma \sqrt{n}} \right|^2 \leq 1 - \frac{t^2 \vartheta_m^2}{n\sigma^2} + c(\delta) \left| \frac{t}{\sigma \sqrt{n}} \frac{v_{m,2+\delta}}{(\sqrt{n\sigma})^{2+\delta}} \right|^2, c(\delta) = 4/(1 + \delta)(2 + \delta).
\]

Using this inequality and that \( q^2 + 2pq \left| E e^{i t Y_{mN}/\sigma \sqrt{n}} \right| \leq 1 - p^2 \), we obtain

\[
\left| \psi_m(t, \tau) \right|^2 \leq E \exp \left\{ i \tau \frac{Y_{mN} \xi_m}{\sigma \sqrt{n}} + i \frac{\xi_m}{\sqrt{npq}} \right\}^2 = \left( q + p \left| E e^{i t Y_{mN}/\sigma \sqrt{n}} \right|^2 \right)^2 \leq 1 - p^2 \left( 1 - \left| E e^{i t Y_{mN}/\sigma \sqrt{n}} \right|^2 \right) \leq 1 - p^2 \frac{t^2 \vartheta_m^2}{n\sigma^2} + c(\delta) p^2 \frac{\left| t/\sigma \sqrt{n} \right|^{2+\delta}}{(\sigma \sqrt{n})^{2+\delta}}.
\]
Hence, for $|t| \leq c_1(\delta)\mu_{2+\delta,N}^{-1}$, we have
\[
\prod_{m=1}^{N} |\psi_m(t, \tau)|^2 \leq \exp\left\{-p^2t^2\frac{\sigma^2}{\sigma^2} + c(\delta)p^2 |t|^{2+\delta} \mu_{2+\delta,N}\right\}
\leq \exp\left\{-p^2t^2\left(\frac{\sigma^2}{\sigma^2} - c(\delta)c_1(\delta) |t|^{\delta} \mu_{2+\delta,N}\right)\right\} \leq \exp\left\{-p^2t^2\right\},
\]
since (4.35) and that $\mu_{2+\delta,N} \leq 1$. Lemma 4.6 is proved.

**Lemma 4.7.** Let $p \geq 1/2$ and $\mu_{1+j+\delta,N} \leq 1$, $j = 1, 2, 3$. Then for $|t| \leq 0.1\mu_{1+j+\delta,N}^{-1/(1+j+\delta)}$ one has
\[
|\varphi_n(t) - W_{jN}(t)| \leq c_3 |t|^j \left(1 + |t|^{1+\delta}\right) \mu_{1+j+\delta,N} e^{-t^2/4}.
\]
where $W_{jN}(t)$ are defined in (2.11) and (4.25).

**Proof.** Proof based on the formula (2.18) of Bahr, hence, we use denotes (2.17). Also we shall develop some ideas of Mirakhmedov (1983), Zhao et al. (2004) and Hu et al. (2007a).

Let $c_0(k, \delta) = 2^1-\delta/(1+\delta)...(k+\delta)$ and $c_2(k, \delta) = (10c_0(k, \delta)(k+1!)^{-1/\delta} \leq 0.1$, with $k \geq 2$. Due to (2.15) and inequalities (4.2), we observe that
\[
V_{k+\delta,N} \geq 1. \quad (4.36)
\]
Hence
\[
\mu_{k+\delta,N}^{-1} = n^{(k-2+\delta)/2}\sigma^{k+\delta}/V_{k+\delta,N} \leq n^{(k-2+\delta)/2}\sigma^{k+\delta}. \quad (4.37)
\]
If $|t| \leq c_2(k, \delta)\mu_{k+\delta,N}^{-1/(k+\delta)}$, then also
\[
|t| \leq c_2(k, \delta)\sigma n^{(k-2+\delta)/2(k+\delta)} \quad (4.39)
\]
and
\[
t^2/2n\sigma^2 \leq c_2^2(k, \delta)/2n^{2/(k+\delta)} \leq 0.005. \quad (4.40)
\]
Set $d_{ms} = EY_{mN}^s$. Applying (4.1) and (4.36)-(4.39) with appropriate $l$ and $k$, respectively, we obtain
\[
b_m(t) = \frac{itd_{m1}}{\sigma\sqrt{n}} + \frac{t^2}{2n\sigma^2}(\alpha_{20} - d_{m2}) + \frac{(it)^3}{6(\sigma\sqrt{n})^3}(d_{m3} - 3d_{m1}\alpha_{20})
\]
\[
+ \frac{(it)^4}{24(\sigma\sqrt{n})^4} \left(d_{m4} - 6d_{m2}\alpha_{20} + 3\alpha_{20}^2\right) + \theta \frac{t^{4+\delta}v_{m,4+\delta}}{6(\sigma\sqrt{n})^{4+\delta}}, \quad (4.41)
\]
Approximation by normal distribution for a sample sum

\[ b_m(t) = \frac{itd_{m1}}{\sigma\sqrt{n}} + \frac{t^2}{2n\sigma^2}(\alpha_{20} - d_{m2}) + \frac{(it)^3}{6(\sigma\sqrt{n})^3}(d_{m3} - 3d_{m1}\alpha_{20}) + \frac{t^{3+\delta}v_{m,4+\delta}/(4+\delta)}{2(\sigma\sqrt{n})^{3+\delta}}, \]

(4.42)

\[ b_m(t) = \frac{itd_{m1}}{\sigma\sqrt{n}} + \frac{t^2}{2n\sigma^2}(\alpha_{20} - d_{m2}) + \theta \frac{t^3\bar{v}_{m,2+\delta}}{(\sigma\sqrt{n})^{2+\delta}}, \]

(4.43)

\[ b_m(t) = \frac{itd_{m1}}{\sigma\sqrt{n}} + \theta \frac{t^2\bar{v}_{m,2}}{n\sigma^2}, \]

(4.44)

\[ b_m(t) = 1.10\frac{t\bar{v}_{m,1}}{\sigma\sqrt{n}}. \]

(4.45)

Let us prove the case \( j = 3 \). Note that

\[ |t| \frac{\bar{v}_{m,4+\delta}^{1/(4+\delta)}}{\sigma\sqrt{n}} \leq 0.2. \]

(4.46)

Using (4.37), (4.38), (4.46), inequality \( \bar{v}_{m,l} \leq \bar{v}_{m,k+\delta} \leq \bar{v}_{m,k+\delta}, 1 \leq l \leq 4+\delta, \) and successively (4.41)–(4.45), we obtain

\[ pB_1(p) = \frac{(it)^3}{6\sigma^3\sqrt{n}}\alpha_{30} + \frac{(it)^4}{24n\sigma^4}(\alpha_{40} - 3\alpha_{20}^2) + \frac{t^{4+\delta}}{6}\mu_{4+\delta,N}, \]

(4.47)

\[ p^2B_2(p) = \frac{t^2}{2\sigma^2}p\alpha_{02} - \frac{(it)^3}{2\sigma^3\sqrt{n}}p\alpha_{21} \]

(4.48)

\[ p^3B_3(p) = \frac{(it)^3}{3\sigma^3\sqrt{n}}p^2\alpha_{03} - \frac{(it)^4}{2n\sigma^4}p^2(\alpha_{20}\alpha_{02} - \alpha_{22}) + \theta t^{4+\delta}\mu_{4+\delta,N}, \]

(4.49)

\[ p^4B_4(p) = -\frac{(it)^4}{4n\sigma^4}p^3\alpha_{04} + \theta t^{4+\delta}\mu_{4+\delta,N}, \]

(4.50)

\[ \left| p^l B_l(t) \right| \leq \left( 1.01 |t| \mu_{4+\delta,N}^{1/(4+\delta)} \right)^l \leq 1.05 |t|^{4+\delta} \mu_{4+\delta,N}(0.101)^{l-5}, \text{ for } 5 \leq l \leq n. \]

(4.51)

For \( |t| \leq 0.1\mu_{4+\delta,N}^{-1/(4+\delta)} \) from (4.47)–(4.51) we have

\[ \sum_{l=5}^n \left| p^l B_l(t) \right| \leq 1.113 |t|^{4+\delta} \mu_{4+\delta,N}, \]

(4.52)
and

\[ \sum_{l=1}^{4} p^l B_l(p) = \frac{t^2}{2\sigma^2} p\alpha_0^2 + \frac{(it)^3}{6\sigma^3\sqrt{n}} \Lambda_1 + \frac{(it)^4}{24n\sigma^4} \Lambda_2 + \theta t^{4+\delta} \mu_{4+\delta,N} \]  

(4.53)

\[ = \frac{t^2}{2\sigma^2} p\alpha_0^2 + \theta \frac{t^2}{8}, \]  

(4.54)

since \( \mu_{4+\delta,N} \leq 1 \) and hence \( \mu_{4+\delta,N}^{-1/(4+\delta)} \leq \mu_{4+\delta,N}^{-1/(2+\delta)} \leq \mu_{4,N}^{-1/2} \leq \mu_{3,N}^{-1} \). Thus, (4.52) and (4.54) gives

\[ \sum_{l=1}^{n} \left| p^l B_l(p) \right| = \frac{t^2}{2\sigma^2} p\alpha_0^2 + \theta \frac{t^2}{4}. \]  

(4.55)

Next, we shall use arguments similar as in the relations (3.11)–(3.22) of Mirakhmedov (1983) with application above (4.46)–(4.55). Use the Stirling’s formula to get for \( 0 < r \leq n \)

\[ 0 \leq C(n, N, r) - 1 + \frac{1-p}{2n} r(r-1) \leq 3 \frac{1-p^2}{n^2} r^4. \]  

(4.56)

Now rewrite (2.16) in the form

\[ e^{t^2\alpha_2/2\sigma^2} \varphi_n(t) = I_1 + I_2 + I_3, \]  

(4.57)

where

\[ I_1 = \sum \prod_{l=1}^{n} \frac{(pB_l(t))^{i_l}}{i_l!} C \left( n, N, \sum_{l=1}^{n} li_l \right), \]

here the summation is over all \( i_l \geq 0, l = 1, 2, 3, 4 \) and \( i_l > 0 \) for at least one \( l = 5, \ldots, n; \)

\[ I_2 = \sum_{i_l \geq 0, \atop 1 \leq l \leq 4} \prod_{l=1}^{4} \frac{(pB_l(t))^{i_l}}{i_l!} \left( C \left( n, N, \sum_{l=1}^{4} li_l \right) - 1 + \frac{1-p}{2n} \sum_{l=1}^{4} li_l \left( \sum_{l=1}^{4} li_l - 1 \right) \right), \]

\[ I_3 = \sum_{i_l \geq 0, \atop 1 \leq l \leq 4} \prod_{l=1}^{4} \frac{(pB_l(t))^{i_l}}{i_l!} \left( 1 - \frac{1-p}{2n} \sum_{l=1}^{4} li_l \left( \sum_{l=1}^{4} li_l - 1 \right) \right). \]
Because \( C(n, N, r) \leq 1 \) it follows from (4.52) and (4.55)

\[
|I_1| \leq \exp\left\{ \sum_{l=1}^{4} |p^l B_l(t)| \right\} \left( \exp\left\{ \sum_{l=5}^{n} |p^l B_l(t)| \right\} - 1 \right)
\leq c |t|^{4+\delta} \mu_{4+\delta,N} \exp\left\{ \frac{t^2}{2\sigma^2} pq_{02} + \theta \frac{t^2}{4} \right\}. \tag{4.58}
\]

Using (4.56), we obtain

\[
|I_2| \leq \frac{1}{n^2} \sum_{l=1}^{4} \prod_{i=1}^{l} \frac{|pB_l(t)|^{i_l}}{i_l!} \left( \sum_{l=1}^{4} l_{i_l} \right)^4
\leq \frac{c}{n^2} \exp\left\{ \sum_{l=1}^{4} |p^l B_l(t)| \right\} \sum_{l=1}^{4} \left( |p^l B_l(t)| + |p^l B_l(t)|^4 \right)
\leq \frac{c}{n^2} t^2 (1 + t^6) \exp\left\{ \frac{t^2}{2\sigma^2} pq_{02} + \theta \frac{t^2}{4} \right\}. \tag{4.59}
\]

As in Mirakhmedov (1983, equality (3.17)), we have

\[
I_3 = \sum_{i_l \geq 0, \ 1 \leq l \leq 4} \prod_{l=1}^{4} \frac{(p^l B_l(t))^{i_l}}{i_l!} - q \frac{1}{2n} \sum_{i_l \geq 0, \ 1 \leq l \leq 4} \prod_{l=1}^{4} \frac{(p^2 B_2(t))^{i_l}}{i_l!} \left( 4i_2(i_2-1) + 2i_2 \right)
\leq \sum_{i_l \geq 0, \ 1 \leq l \leq 4} \prod_{l=1}^{4} \frac{(p^l B_l(t))^{i_l}}{i_l!} \left( \sum_{l=1}^{4} l_{i_l} \left( \sum_{l=1}^{4} l_{i_l} - 1 \right) \right) - \left( 4i_2(i_2-1) + 2i_2 \right)
\leq \exp\left\{ \sum_{l=1}^{4} p^l B_l(t) \right\} \left( 1 - \frac{q}{n} p^2 B_2(t) \left( 2p^2 B_2(t) + 1 \right) - \frac{q}{2n} \sum_{l,j=1}^{4} c_{l,j} (p^l B_l(t))^{k_l} (p^j B_j(t))^{k_j} \right)
\]

where \( k_l + k_j = 2, \ k_l \geq 0, \ k_2 = 1 \) and \( c_{i,j} \) are some constants (we do not need to have \( c_{i,j} \) in explicit form, although it can be found easily as a long formula). Use (4.47)–(4.50), (4.52), (4.54), inequalities \( q_02 \leq q_020 \leq \sigma^2 \) (because \( p \geq 1/2 \)) after some algebra we obtain for \(|t| \leq 0.1 \mu_{4+\delta,N}^{-1/(4+\delta)}\)
\[ I_3 = \exp \left\{ \frac{t^2}{2\sigma^2} p\alpha_0 + \theta \frac{t^2}{8} \right\} \left( 1 + \frac{(it)^3}{6\sigma^3\sqrt{n}} \Lambda_1 + \frac{(it)^6}{t2n\sigma^6} \Lambda_1^2 \right) + \frac{(it)^4}{24n\sigma^4} \left( \Lambda_2 + \frac{12qp^2\alpha_0^2}{2n\sigma^2} \right) + \frac{p\theta^2\alpha_0^2}{2n\sigma^2} + c\theta \left( t^3 + t^{4+\delta} \right) \mu_{4+\delta,N} \right). \quad (4.60) \]

The case \( j = 3 \) of Lemma 4.7 follows from (4.57)–(4.60).

Proof of the cases \( j = 1 \) and \( j = 2 \) is a very similar to proof of the case \( j = 3 \) with quite evident algebra using (4.37)–(4.40) with \( k = 2, k = 3 \) respectively and inequality \( 0 \leq C(n, N, r) - 1 \leq n^{-1}r^2 \) instead of (4.56). The details are omitted.

Proof of Theorem 3.1. Follows (see Remark 3.7) in immediate manner from Theorem 4.2 of Mirakhmedov et al. (2014) by putting \( \omega_m = 1 \), \( f_{m,N}(x) = Y_{mN} \cdot x \), \( m = 1, ..., N \), and \( \mathcal{L}(\xi_m) = Bi(p) \).

Proof of Theorem 3.2. First of all, we note that by using inequalities (4.2) and definition of \( W_{j,N}(u) \) one can observe that \( \Delta_{j,N} \leq c, \) for some \( c > 0 \). Hence, if there is a constant \( c_0 > 0 \) such that \( \max(\beta_{1+j+\delta,N}, T^{-1}) \geq c_0 \) then Theorem 3.2 is hold true with \( C_j = c/c_0 \). Therefore, we shall assume that \( \max(\beta_{1+j+\delta,N}, T^{-1}) \leq c_0 < 0.01 \). Remark that this condition implies, due to inequalities (4.2), that \( \beta_{2+\delta,N} \leq 0.01 \) also. Below \( W_{j,N}(t) \) are from (2.11) and (4.25) and \( T_{j,N} \) from Lemma 4.5. By Esseen’s smoothing lemma, see for instance Feller (1966), and the fact that

\[
|\varphi_n(t) - W_{j,N}(t)| \leq |t| \max_{|u| \leq |t|} \left| \frac{\partial}{\partial u} (\varphi_n(u) - W_{j,N}(u)) \right|,
\]

we have: for arbitrary \( T > 0 \)

\[
\Delta_{j,N} \leq \frac{1}{\pi} \int_{|t| \leq T} \left| \frac{\varphi_n(t) - W_{j,N}(t)}{t} \right| \, dt + \frac{24}{T\sqrt{2\pi}}, \quad (4.61)
\]

\[
\leq \frac{1}{\pi} \max_{|u| \leq 1} \left| \frac{\partial}{\partial u} (\varphi_n(u) - W_{j,N}(u)) \right| + \frac{1}{\pi} \int_{1 \leq |t| \leq T} \left| \frac{\varphi_n(t) - W_{j,N}(t)}{t} \right| \, dt + \frac{24}{T\sqrt{2\pi}}.
\]

If \( T \leq T_{j,N} \) then the proof of Theorem 3.2 is concluded by applying last inequality and Lemma 4.5.

Let now \( T \geq T_{j,N} \), then

\[
\Delta_{j,N} \leq \frac{1}{\pi} \max_{|u| \leq 1} \left| \frac{\partial}{\partial u} (\varphi_n(u) - W_{j,N}(u)) \right| + \frac{1}{\pi} \int_{1 \leq |t| \leq T_{j,N}} \left| \frac{\varphi_n(t) - W_{j,N}(t)}{t} \right| \, dt + \frac{1}{\pi} \int_{T_{j,N} \leq |t| \leq T} \left| \frac{W_{j,N}(t)}{t} \right| \, dt
\]

\[
+ \frac{1}{\pi} \int_{T_{j,N} \leq |t| \leq T} \left| \frac{\varphi_n(t)}{t} \right| \, dt + \frac{24}{T\sqrt{2\pi}} = J_1 + J_2 + J_3 + J_4 + \frac{24}{T\sqrt{2\pi}}, \quad (4.62)
\]
Using Lemma 4.5 and taking into account the exponential factor of \( W_jN(t) \), we obtain

\[
J_1 + J_2 + J_3 \leq C \left( \beta_{2+j+\delta,N} + \frac{1}{(nq)\beta_{\frac{1}{2}}} \right). \tag{4.63}
\]

Theorem 3.2 follows from (4.62) and (4.63) because \( J_4 = \chi_N(T_{jN}/\sigma\sqrt{n}, \pi) \).

In view of equality \( E e^{itY_{mN}\xi_m + i\tau}\xi_m} = 1 + p \left( E e^{i(\tau + tY_{mN})} - 1 \right) \), we have

\[
\left| E e^{itY_{mN}\xi_m + i\tau}\xi_m} \right| \leq 1 + p \left( 1 - \left| E e^{itY_{mN}} \right| \right) \quad \text{and} \quad \left| E e^{itY_{mN}\xi_m + i\tau}\xi_m} \right|^2 = 1 - 2pq \left( 1 - E \cos(tY_{mN} + \tau) \right).
\]

Therefore,

\[
\prod_{m=1}^{N} \left| E e^{itY_{mN}\xi_m + i\tau}\xi_m} \right| \leq \exp \left\{ -n \left( 1 - \frac{1}{N} \sum_{m=1}^{N} \left| E e^{itY_{mN}} \right| \right) \right\}, \tag{4.64}
\]

and

\[
\prod_{m=1}^{N} \left| E e^{itY_{mN}\xi_m + i\tau}\xi_m} \right| \leq \exp \left\{ -pq \sum_{m=1}^{N} \left( 1 - E \cos(\tau + tY_{mN}) \right) \right\}
\leq \exp \left\{ -pq \left( 1 - \frac{1}{N} \sum_{m=1}^{N} \left| E e^{i\tau + tY_{mN}} \right| \right) \right\}. \tag{4.65}
\]

On the other hand, by formula (2.3)

\[
\chi_N(T_{jN}/\sigma\sqrt{n}, \pi) = \frac{\sqrt{nq}}{\Theta_N(0)} \int_{T_{jN}/\sigma\sqrt{n} \leq |\tau| \leq T/\sigma\sqrt{n}} \frac{1}{|\tau|} \int_{|\tau| \leq \pi} \prod_{m=1}^{N} \left| E e^{itY_{mN}\xi_m + i\tau}\xi_m} \right| d\tau dt. \tag{4.66}
\]

The inequality (3.5) follows from (4.33), (4.64), (4.65) and (4.66).

**Proof of Theorem 3.3.** Let \( p \leq 1/2 \), then \( q \geq 1/2 \), and taking into account that \( \sigma^2 \leq 1 \) and (4.36), we have

\[
(nq)^{-\frac{k-2}{2}} \leq (2/n)^{\frac{k-2}{2}} \leq V_{k,N} = (2/n\sigma^2)^{\frac{k-2}{2}} \sigma^{k-2} V_{k,N} \leq 2\mu_{k,N}, \quad \text{therefore}, \quad \beta_{k,N} + (nq)^{-\frac{k-2}{2}} \leq (2^{k-1} + 1)\mu_{k,N} + (2/n)^{(k-2)/2} \leq (2^{k-1} + 3)\mu_{k,N}, \quad \text{since} \quad \beta_{k,N} \leq 2^{k-1}(1 + p^{k-1})\mu_{k,N}. \tag{4.67}
\]

Thus, if \( p \leq 1/2 \) or \( (nq)^{-\frac{k-2}{2}} \leq 2^{k-2}\mu_{k,N} \), then Theorem 3.3 follows from Theorem 3.2.
Let now $p > 1/2$ and $(nq)^{-(k-2)/2} > 2^{k/2} \mu_{k,N}$. In view of (4.61), we have

$$
\Delta_N \leq \frac{1}{\pi} \left( \int_{|t| \leq \bar{T}_2} \left| \frac{\varphi_n(t) - W_{j,N}(t)}{t} \right| dt + \int_{\bar{T}_2 \leq |t| \leq T} \left| W_{j,N}(t) \right| dt \right. + \int_{T \leq |t| \leq T} \left| \frac{\varphi_n(t)}{t} \right| dt \\
+ \left. \int_{T \leq |t| \leq T} \left| \frac{\varphi_n(t)}{t} \right| dt \right) + \frac{24}{T \sqrt{2 \pi}},
$$

where $\bar{T}_2 = 0.1\mu^{-1}_{1+i+\delta,N}$ and $T_2 = c_1(\delta)\mu^{-1}_{2+\delta,N}$, $c_1(\delta)$ is a constant defined in Lemma 4.5. Applying here Lemmas 4.6, 4.7 and definitions of $W_{j,N}(t)$ after simple calculations, we complete the proof of Theorem 3.3.

**Proof of Corollary 3.1.** Note that

$$
0.01 \left( \beta_{3+\delta,N} + (nq)^{-1/2} \right)^{-1} \leq T_1 \text{ and } T_{2N} = T_{3N} \geq 0.01 \beta_{3,N}^{-1} \geq 0.001 \sigma \sqrt{n}/V_{3,N}.
$$

(4.68)

because $\beta_{3,N}^{-1} \leq \beta_{3,N}^{(1)} + \beta_{3,N}^{(2)} \leq 9V_{3,N}/\sigma^3 \sqrt{n}$.

Therefore, Corollary 3.1 follows immediately from Theorem 3.2 by putting $T = 0.01 \left( \beta_{3+\delta,N} + (nq)^{-1/2} \right)^{-1}$, $T = \beta_{3+\delta,N}^{-1}$ and $T = \beta_{4+\delta,N}^{-1}$ for the cases (i), (ii), and (iii), respectively.

**Proof of Corollary 3.2.** Let $\alpha_{02} \leq 1/4$. Then $(1 - 2p\alpha_{02})^2 = (1 - p\alpha_{02})^2 - 2p\alpha_{02}(1 - 2p\alpha_{02}) + p^2 \alpha_{02}^2 \geq (1 - p\alpha_{02})(1 - 3p\alpha_{02}) \geq (1 - p\alpha_{02})(1 - 3p/4) \geq \sigma^2/4$. Use this fact and inequality (4.2) to get: for $k \geq 3$

$$
\beta_{k,N} \geq \beta_{1,k,N}^{(1)} \geq \frac{n^{-(k-2)/2} \sigma^{-k}}{k!} \left( \sum_{m=1}^{N} E(Y_{m,N} - pEY_{m,N})^2 \right)^{k/2} \geq \frac{n^{-(k-2)/2}}{\sigma^2/4} \geq n^{-(k-2)/2}(\sigma^2)^{-k/2}.
$$

Hence,

$$
\left( \beta_{1,k,N}^{(1)} \sigma \sqrt{n} \right)^{-1} \leq 2^{k/2} n^{(k-3)/2} \sigma^{-1/2} n^{-(k-2)/2}, k \geq 3.
$$

(4.69)

Let $\alpha_{02} > 1/4$. Then $\sigma^2 \beta_{k,N}^{(2)} \geq \frac{n^{-(k-2)/2} \sigma^{-1/2}}{\alpha_{02}^{1/2}}$, and hence noting that $\sigma \leq 1$, we obtain

$$
\left( \sigma^2 \beta_{k,N}^{(2)} \right)^{-1} / \sigma \sqrt{n} \leq 2^{k/2} n^{(k-3)/2} \sigma^{-1/2} n^{-(k-2)/2}, k \geq 3.
$$

(4.70)
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Put now in Theorem 3.2 \( T = \beta_{k,N}^{-1} \leq \min \left\{ \left( \beta_{k,N}^{(1)} \right)^{-1}, \left( \sigma^2 \beta_{k,N}^{(2)} \right)^{-1} \right\} \), use (3.4), (4.68) and (4.69), (4.70) with \( k = 3 + \delta \) and \( k = 4 + \delta \) respectively to complete the proof of Corollary 3.2.

**Proof of Corollary 3.3.** Follows immediately from Theorem 3.3 and (4.68), by choosing \( t \) as it is indicated just before Corollary 3.3.

**Proof of Theorem 3.4.** Follows (see Remark 3.7) from Theorem 12 of Mirakhmedov (1996) by putting \( f_{m,N}(x) = Y_{mN} \cdot x, m = 1, ..., N, L(\xi_m) = Bi(p) \), and noting that \( \sigma^2 \geq q \) is bounded away from zero.

**Proofs of Corollaries 3.4 and 3.5.** Follow from Theorem 3.4 immediately because the right hand side of (3.8) and (3.9) is \( 1 + O(x^3/\sqrt{N}) \) for \( x = o(N^{1/6}) \) and \( 1 - \Phi(x) \leq (x\sqrt{2\pi})^{-1} \exp\{-x^2/2\}, x > 0 \).

**Proof of Corollary 3.6.** Follows from Theorem 3.4 because we can assume without losing of generality that \( \max |a_{m,N}| \leq c \), and hence the condition (3.7) is fulfilled.

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