LOGARITHMIC SOBOLEV AND SHANNON’S INEQUALITIES
AND AN APPLICATION TO THE UNCERTAINTY PRINCIPLE

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

TAKAYOSHI OGAWA
Mathematical Institute, Tohoku University
Sendai, 980-8578, Japan

KENTO SERAKU
Tamachi Branch, Risona Bank Co., Ltd.
Tokyo 108-014, Japan

Abstract. The uncertainty principle of Heisenberg type can be generalized via the Boltzmann entropy functional. After reviewing the $L^p$ generalization of the logarithmic Sobolev inequality by Del Pino-Dolbeault [6], we introduce a generalized version of Shannon’s inequality for the Boltzmann entropy functional which may be regarded as a counter part of the logarithmic Sobolev inequality. Obtaining best possible constants of both inequalities, we connect both the inequalities to show a generalization of uncertainty principle of the Heisenberg type.

1. Logarithmic Sobolev inequality. We consider the relation between the sharp logarithmic Sobolev inequality with the generalized Shannon inequality and the uncertainty principle of Heisenberg type.

The logarithmic Sobolev inequality is a version of the Sobolev inequality in the Sobolev spaces. Let $W^{1,p}(\mathbb{R}^n)$ be the Sobolev space defined by

$$W^{1,p}(\mathbb{R}^n) = \{ f \in L^p(\mathbb{R}^n); \nabla f \in L^p(\mathbb{R}^n) \},$$

where $1 \leq p \leq \infty$. When $p = 2$, we use the abbreviated notation $H^1 = H^1(\mathbb{R}^n) = W^{1,2}(\mathbb{R}^n)$. Stam [17] first obtained the logarithmic Sobolev inequality in $H^1(\mathbb{R}^n)$ and later on Gross [7] reconsidered the inequality with the Gaussian measure and showed a relation with the hypercontractivity of the semi-group in the probability theory. In particular it is considered as the infinite dimensional version of the Sobolev inequality for higher dimensions $n \geq 3$:

$$\|f\|_{W^{1,p}(\mathbb{R}^n)} \leq S_b \|\nabla f\|_2,$$  \hspace{1cm} (1.1)

where

$$\|f\|_p \equiv \left( \int_{\mathbb{R}^n} |f(x)|^p dx \right)^{\frac{1}{p}}$$

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and

\[ S_2^2 = \frac{1}{n(n-2)} \left( \frac{\Gamma(n)}{\Gamma(\frac{n}{2})} \right)^{\frac{2}{n}}. \] (1.2)

Here \( \Gamma(\cdot) \) denotes the Gamma function (cf. Rosen [13], Talenti [18]).

We first show the \( L^2 \) based inequality of the logarithmic Sobolev inequality due to Stam and Gross. The inequality is presented as the following form (see also Weisssler [19], Lieb-Loss [10] and Ledoux [9]).

**Proposition 1.1** (The logarithmic Sobolev inequality). Let \( n \geq 2 \). For any \( a > 0 \) and \( f \in H^1(\mathbb{R}^n) \), the following inequality holds

\[
\int_{\mathbb{R}^n} |f(x)|^2 \log \left( \frac{|f(x)|^2}{\|f\|_2^2} \right) dx + n(1 + \log a)\|f\|_2^2 \leq \frac{a^2}{\pi} \|\nabla f\|_2^2. \] (1.3)

The equality is attained by

\[ f(x) = \exp \left( -\frac{\pi |x|^2}{2a^2} \right). \]

One can choose \( a \geq \frac{1}{e} \) as

\[
\int_{\mathbb{R}^n} |f(x)|^2 \log \left( \frac{|f(x)|^2}{\|f\|_2^2} \right) dx \leq \frac{1}{\pi e^2} \|\nabla f\|_2^2
\]

and then the inequality does not depend on the dimension and it is also valid for infinite dimensional case.

The inequality (1.3) and its proof is deeply related with various important inequalities of Young and Hardy-Littlewood-Sobolev type as well as the hypercontractivity of the heat and other semi-group with the sharp constants (cf. Weisssler [19]).

One can optimize the parameter \( a > 0 \) appearing in Proposition 1.1 and then the equivalent form of (1.3) is obtained as follows which is obtained by Weisssler [19]:

**Corollary 1.2** (Sharp logarithmic Sobolev inequality). Let \( n \geq 2 \) and \( f \in H^1(\mathbb{R}^n) \). Then the equivalent form of the inequality (1.3) is

\[
\int_{\mathbb{R}^n} \frac{|f(x)|^2}{\|f\|_2^2} \log \left( \frac{|f(x)|^2}{\|f\|_2^2} \right) dx \leq \frac{n}{2} \log \left( \frac{2}{n\pi e} \|\nabla f\|_2^2 \right). \] (1.4)

The constant in the right hand side is the best possible and it is attained by \( G_\mu(x) \equiv \frac{1}{4\pi \mu} e^{-\frac{|x-x_0|^2}{4\mu}} \) for any \( \mu > 0 \) and \( x_0 \in \mathbb{R}^n \).

**Proof of Corollary 1.2.** To derive (1.4) from (1.3), we set

\[
\int_{\mathbb{R}^n} |f(x)|^2 \log \left( \frac{|f(x)|^2}{\|f\|_2^2} \right) dx \leq \frac{a^2}{\pi} \|\nabla f\|_2^2 - n \log(a)\|f\|_2^2 \equiv F(a)
\]

and optimize the right hand side with the parameter \( a > 0 \). Since

\[
F'(a) = \frac{2}{\pi} \|\nabla f\|_2^2 a - \frac{n\|f\|_2^2}{a} = 0
\]

and the minimum of \( F(a) \) is realized by \( a = a_0 \) with

\[
a_0^2 = \frac{n\pi \|f\|_2^2}{2 \|\nabla f\|_2^2},
\]
the minimum of the right hand side is
\[ \int_{\mathbb{R}^n} |f(x)|^2 \log \left( \frac{|f(x)|^2}{\|f\|_2^2} \right) dx \leq \frac{n}{2} \|f\|_2^2 \log \left( \frac{2}{n\pi e} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} \right). \]

To see the optimality, by substituting \(|f(x)|^2 = G_\mu(x) = \frac{1}{(4\pi\mu)^{n/2}} \exp \left(-\frac{|x|^2}{4\mu}\right)\) with \(\|f\|_2 = \|G_\mu\|_1 = 1\), it follows by noting
\[ \int_{\mathbb{R}^n} e^{-|x|^b}|x|^b dx = \frac{n}{b} \pi^{\frac{n}{2}} \Gamma \left(\frac{n}{b} + 1\right) \Gamma \left(\frac{n}{2} + 1\right)^{-1} \]
that
\[ \frac{2}{n} \int_{\mathbb{R}^n} |f(x)|^2 \log(|f(x)|^2) dx = \frac{2}{n} \int_{\mathbb{R}^n} G_\mu(x) \left(-\frac{n}{2} \log(4\pi\mu) - \frac{|x|^2}{4\mu}\right) dx \]
\[ = - \log(4\pi\mu) - 1. \]

On the other hand, the right hand side of (1.5) is
\[ \log \left( \frac{2}{n\pi e} \int_{\mathbb{R}^n} |\nabla G_\mu|^2 dx \right) = \log \left( \frac{1}{4\pi e} \right). \]
Hence for any \(\mu > 0\), \(G_\mu(x)\) attains the best possible constant.

On the other hand, to derive the original inequality, we divide the both side of (1.4) by \(\|f\|_2\) to see
\[ \int_{\mathbb{R}^n} \frac{|f(x)|^2}{\|f\|_2} \log \left( \frac{|f(x)|^2}{\|f\|_2^2} \right) dx \leq \frac{n}{2} \log \left( \frac{2}{n\pi e} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} \right). \]

Then introducing a parameter \(a > 0\), we have
\[ \frac{n}{2} \log \left( \frac{2}{n\pi e} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} \right) \]
\[ = \frac{n}{2} \log \left( \frac{a^2}{\pi} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} \right) + \frac{n}{2} \log \left( \frac{2}{nae^2} \right) \]
\[ \leq \frac{n}{2} \log \frac{n}{2} + \frac{n}{2} + \exp \left( \log \left( \frac{a^2}{\pi} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} \right) \right) + \frac{n}{2} \log \left( \frac{2}{nae^2} \right) \]
\[ = \frac{n}{2} \log \frac{n}{2e} + \frac{a^2}{\pi} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} + \frac{n}{2} \log \left( \frac{2}{nae^2} \right) \]
\[ = \frac{a^2}{\pi} \frac{\|\nabla f\|_2^2}{\|f\|_2^2} - n \left( 1 + \log a \right). \]

**Remark 1.3.** The Sobolev inequality (1.1) with the best constant (1.2) formally implies the logarithmic Sobolev inequality (1.4) as a limit of \(n \to \infty\) (see Appendix below).

The above observation implies that the sharp form of the logarithmic inequality can be extended into the dimension free version of the inequality (1.3).

The logarithmic Sobolev inequality also holds not only on the Euclidian spaces with the Lebesgue measure but with the Gauss measure. The both exposition is equivalent (see for instance, Lieb-Loss [10]). One of the generalization connecting to the version with the Gaussian measure is given by the \(L^1\) form of the inequality as follows: Substituting \(f^{\frac{1}{2}}\) into \(f\) in (1.4), one can obtain the \(L^1\) based version of the logarithmic inequality.
Corollary 1.4. Let $n \geq 2$. For any non-negative $f \in L^1(\mathbb{R}^n)$ with $f^{1/2} \in H^1(\mathbb{R}^n)$, it holds that
\[
\int_{\mathbb{R}^n} f(x) \log(f(x)) \, dx \leq \frac{n}{2} \|f\|_1 \log \left( \frac{1}{2n\pi e} \int_{\mathbb{R}^n} f(x) \left| \nabla \log \left( f(x) \right) \right|^2 \, dx \right).
\]
The constant in the right hand side is the best possible and it attains by $f(x) = G_{\mu}(x - x_0)$, with $x_0 \in \mathbb{R}^n$.

The resulting inequality can be seen by the following normalized form:
\[
\int_{\mathbb{R}^n} f(x) \log(f(x)) \, dx \leq \frac{n}{2} \log \left( \frac{1}{2n\pi e} \int_{\mathbb{R}^n} f(x) \left| \nabla \log \left( f(x) \right) \right|^2 \, dx \right)
\]
for any non-negative $f$ with $f^{1/2} \in H^1(\mathbb{R}^n)$ and $\|f\|_1 = 1$.

2. Logarithmic Sobolev inequality in $L^p$. Weissler [19] modified the Stam-Gross logarithmic inequality into $L^p$ based spaces with using the heat evolution semi-group. Del Pino-Dolbeault [6] obtained the inequality and the best possible constant (Ledoux [8] for the case $p = 1$).

Proposition 2.1 ([6], [8]). Let $1 \leq p < n$ and $1/p + 1/p' = 1$. Suppose $f \in W^{1,p}(\mathbb{R}^n)$ with $f \geq 0$ and $\|f\|_p = 1$, then it holds that
\[
\int_{\mathbb{R}^n} f(x)^p \log(f(x))^p \, dx \leq \frac{n}{2} \log \left( B_{n,p} \int_{\mathbb{R}^n} \left| \nabla f \right|^p \, dx \right),
\]
where
\[
B_{n,p} = \frac{p}{n} \left( p - 1 \right)^{p-1} \pi^{-\frac{p}{2}} \left( \frac{\Gamma \left( \frac{n}{2} + 1 \right)}{\Gamma \left( \frac{n}{2} + \frac{p}{2} + 1 \right)} \right)^{\frac{p}{2}}
\]
is the best possible. Besides the best constant is attained by $e^{-\tilde{c}_{n,p'}|x|^{p'}}$ with
\[
\tilde{c}_{n,p'} = \left( \int_{\mathbb{R}^n} e^{-|x|^{p'}} \, dx \right)^{\frac{1}{p'}} = \left( \frac{\pi^{\frac{p}{2}} \Gamma \left( \frac{n}{2} + \frac{p}{2} + 1 \right)}{\Gamma \left( \frac{n}{2} + 1 \right)} \right)^{\frac{1}{p'}}
\]
up to translation and dilation.

Let $\dot{W}^{1,p}(\mathbb{R}^n)$ be the homogeneous Sobolev space defined by
\[
\dot{W}^{1,p}(\mathbb{R}^n) = \{ f \text{ measurable on } \mathbb{R}^n; \nabla f \in L^p(\mathbb{R}^n) \}.
\]
Del Pino-Dolbeault [6] utilize the best constant for the Gagliardo-Nirenberg inequality:

Proposition 2.2 (Sharp constant for Gagliardo-Nirenberg’s inequality [5]). Let $r \equiv p \frac{\frac{1}{p} - \frac{1}{q}}{\frac{1}{p} - \frac{1}{q}}$. For any $f \in L^q \cap \dot{W}^{1,p}$, it holds that
\[
\|f\|_r \leq C_{GN} \|\nabla f\|_p^\theta \|f\|_q^{1-\theta},
\]
where
\[
\frac{1}{r} = \frac{1-\theta}{q} + \theta \left( \frac{1}{p} - \frac{1}{n} \right)
\]
\[ C_{GN} = \left( \frac{q - p}{p \sqrt{\pi}} \right)^{\frac{q}{p}} \left( \frac{pq}{n(q - p)} \right)^{\frac{p}{q}} \times \left( \frac{np - (n - p)q}{pq} \right)^{\frac{1}{q}} \left( \frac{\Gamma \left( \frac{q - 1}{q - p} \right) \Gamma \left( \frac{n}{p} + 1 \right)}{\Gamma \left( \frac{p - 1}{p} \frac{np - (n - p)q}{q - p} \right) \Gamma \left( \frac{n}{p} + 1 \right)} \right)^{\frac{p}{q}} \]

is the best possible, where \( \Gamma(\cdot) \) denotes the Gamma function. The best constant is attained by

\[ w_q(x) = \left( 1 + \frac{q - p}{p - 1} |x|^{\frac{p}{p - 1}} \right)^{-\frac{q - 1}{q - p}} \]

up to dilation and translation.

For the completeness, we give the main part of the proof of Proposition 2.1. **Proof of Proposition 2.1.** We only show the case when \( 1 < p < n \). The extremal case \( p = 1 \) can be treated in a different way (cf. [8]). Taking the logarithm of both sides of the inequality (2.4), and the limit of passing \( q \to p \), we have

\[ \lim_{q \to p} \frac{1}{\theta} \log \left( \frac{|f|^r}{\|f\|_q} \right) - \lim_{q \to p} \frac{1}{\theta} \log C_{GN} \leq \lim_{q \to p} \log \left( \frac{\|\nabla f\|_p}{\|f\|_q} \right) = \log \left( \frac{\|\nabla f\|_p}{\|f\|_q} \right). \] \hspace{1cm} (2.6)

Noting

\[ \theta = (q - p) \left( \frac{n}{(p - 1)p^2} + o(1) \right) \]

the first term of the left hand side is identified as

\[ \frac{p}{n} \int_{\mathbb{R}^n} \frac{|f(x)|^p}{\|f\|_p^p} \log \left( \frac{|f(x)|}{\|f\|_p} \right) dx. \] \hspace{1cm} (2.7)

Indeed, by setting \( g(\lambda) = \log \|f\|_{p \frac{p - 1}{p - 1}}, h(\lambda) = \log \|f\|_\lambda \), the first term of the left hand side of (2.6) is

\[ I = \frac{(p - 1)p^2}{n} \lim_{q \to p} \frac{1}{q - p} \log \left( \frac{\|f\|_r}{\|f\|_q} \right) \]

\[ = \frac{(p - 1)p^2}{n} \lim_{q \to p} \left( \frac{1}{q - p} (\log \|f\|_{p \frac{p - 1}{p - 1}} - \log \|f\|_p) - \frac{1}{q - p} (\log \|f\|_q - \log \|f\|_p) \right) \]

\[ = \frac{(p - 1)p^2}{n} \left( g'(p) - h'(p) \right), \]

where

\[ h'(p) = \frac{1}{p\|f\|_p} \int_{\mathbb{R}^n} |f(x)|^p \log \left( \frac{|f(x)|}{\|f\|_p} \right) dx. \]

Letting \( u(\lambda) = \frac{\theta(\lambda - 1)}{p - 1} \) and \( g(\lambda) = h(u(\lambda)) \),

\[ g'(p) = h'(u(p)) u'(p) = \frac{1}{p - 1} \frac{1}{\|f\|_p^p} \int_{\mathbb{R}^n} |f(x)|^p \log \left( \frac{|f(x)|}{\|f\|_p} \right) dx. \]

Hence we obtain (2.7). Next we consider the second term of the left hand side of (2.6):

\[ II = \lim_{q \to p} \frac{1}{\theta} \log C_{GN}. \]
Since the function in (2.5), i.e.,
\[ w_q(x) = \left(1 + \frac{q-p}{p-1} |x|^\frac{p}{p-1} \right)^{-\frac{p-1}{q-p}} \]
attains the best possible constant of (2.4) (see [14]) and
\[ w_q(x) \to w(x) = e^{-|x|^{p'}} \]
as \( q \to p \), it follows that
\[ II = \lim_{q \to p} \frac{1}{q-p} \log C_{GN} \]
\[ = - \log \left( \frac{\|\nabla w\|_p}{\|w\|_p} \right) + \frac{(p-1)p^2}{n} \lim_{q \to p} \frac{1}{q-p} \log \left( \frac{\|w_q\|_r}{\|w_q\|_q} \right). \] (2.8)

Recall that \( p' \) is the Hölder conjugate exponent of \( p \). The second term of the right hand side of (2.8) is
\[ \frac{1}{n} \int_{\mathbb{R}^n} \frac{w^p(x)}{\|w\|_p^p} \log \left( \frac{w^p(x)}{\|w\|_p^p} \right) \, dx \]
\[ + \frac{(p-1)p^2}{n} \left( \lim_{q \to p} \frac{1}{q-p} \log \left( \frac{\|w_q\|_r}{\|w_q\|_q} \right) - \lim_{q \to p} \frac{1}{q-p} \log \left( \frac{\|w\|_r}{\|w\|_q} \right) \right) \] (2.9)

We split the second term of the right hand side of (2.9) as
\[ \frac{(p-1)p^2}{n} (III - IV) \]

with
\[
III = \lim_{q \to p} \frac{1}{q-p} \log \left( \frac{\|w_q\|_r}{\|w_q\|_q} \right)
\]
\[ = \lim_{q \to p} \left( \frac{1}{r(q-p)} \log \|w_q\|_r - \frac{1}{q(q-p)} \log \|w_q\|_q \right)
\]
\[ = \frac{1}{p} \lim_{q \to p} \frac{1}{q-p} \left( \log \|w_q\|_r - \log \|w_q\|_q \right) - \frac{1}{p^2(p-1)} \log(\|w\|_p^p). \]

Analogously to before,
\[ \lim_{q \to p} \frac{1}{q-p} \log \left( \frac{\|w_q\|_r}{\|w_q\|_q} \right)
\]
\[ = \lim_{q \to p} \left( \frac{\log \|w_q\|_r - \log \|w_q\|_q}{q-p} \right)
\]
\[ = \lim_{q \to p} \left( \frac{\log \|w_q\|_r - \log \|w_q\|_p}{q-p} - \frac{\log \|w_q\|_q - \log \|w_q\|_p}{q-p} \right)
\]
\[ = \left( \frac{p}{p-1} - 1 \right) \frac{d}{dp} \left( \log(\|w\|_p^p) \right)
\]
\[ = \frac{1}{p-1} \frac{1}{\|w\|_p^p} \int_{\mathbb{R}^n} w(x)^p \log(w(x)) \, dx. \]
Since \( w_q(x) \to w(x) \) as \( q \to p \) pointwisely and accordingly in \( L^r \) and \( L^s \) by a standard procedure, we see that \( III = IV \), and hence
\[
\lim_{q \to p} \frac{1}{q - p} \log C_{\text{GN}} = -\log \left( \frac{\|\nabla w\|_p}{\|w\|_p} \right) + \frac{1}{q - p} \left( \log \left( \frac{\|w\|_r}{\|w\|_q} \right) \right)
\]
\[
= -\log \left( \frac{\|\nabla w\|_p}{\|w\|_p} \right) + \frac{1}{n} \int_{\mathbb{R}^n} w^p(x) \log \left( \frac{w^p(x)}{\|w\|_p^p} \right) \, dx
\]
\[
= -\log \left( \frac{\|\nabla w\|_p}{\|w\|_p} \right) + \frac{1}{n} \int_{\mathbb{R}^n} w^p(x) \log (w^p(x)) \, dx - \frac{1}{n} \log(\|w\|_p^p).
\]

Since \( w \) is explicitly given, we see the exact values of those norms as follows:
\[
\|w\|_p^p = p^{-\hat{p}} \left( \frac{p - 1}{p} \right) 2\pi^{\frac{n}{2}} \Gamma \left( \frac{n}{p} \right) \Gamma \left( \frac{n}{2} \right)^{-1}
\]
\[
\|\nabla w\|_p^p = \left( \frac{p}{p - 1} \right)^{p - 2} \hat{p} \pi^{-\frac{n}{2}} \Gamma \left( \frac{n}{p} \right) \Gamma \left( \frac{n}{2} \right)^{-1}
\]
by using (1.5). Thus
\[
\frac{1}{n} \int_{\mathbb{R}^n} w^p(x) \log (w^p(x)) \, dx = -\frac{1}{p} (p - 1).
\]
Hence we conclude that
\[
\lim_{q \to p} \frac{1}{q - p} \log C_{\text{GN}} = -\log \left( \frac{\|\nabla w\|_p}{\|w\|_p} \right) + \frac{1}{n} \int_{\mathbb{R}^n} w^p(x) \log (w^p(x)) \, dx - \frac{1}{n} \log(\|w\|_p^p)
\]
\[
= -\log \left( \frac{\|\nabla w\|_p}{\|w\|_p} \right) - \frac{1}{p} \left( p - 1 \right) + \frac{1}{n} \log \left( p^{-\hat{p}} \left( \frac{p - 1}{p} \right) 2\pi^{\frac{n}{2}} \Gamma \left( \frac{n}{p} \right) \Gamma \left( \frac{n}{2} \right)^{-1} \right)
\]
\[
= \frac{1}{p} \log \left( \frac{p}{n} \left( \frac{p - 1}{e} \right)^{p - 1} \pi^{-\frac{n}{2}} \Gamma \left( \frac{n}{p} + 1 \right) \Gamma \left( \frac{n}{2} + 1 \right) \right)
\]
\[
= \frac{1}{p} \log B_{n,p}.
\]

The non-normalized version of (2.1)-(2.2) is given by
\[
\int_{\mathbb{R}^n} f(x)^p \log \left( \frac{f(x)^p}{\|f\|_p^p} \right) \, dx \leq \frac{n}{p} \|f\|_p^p \log \left( B_{n,p} \|f\|_p^p \int_{\mathbb{R}^n} |\nabla f|^p \, dx \right) .
\]  \hspace{1cm} (2.10)

The \( L^1 \) version of the inequality (2.10) now reads as follows:

**Corollary 2.3.** Let \( 1 < p < n \). For any \( f^{1/p} \in W^{1,p}(\mathbb{R}^n) \) with \( f \geq 0 \),
\[
\int_{\mathbb{R}^n} f(x) \log \left( \frac{f(x)}{\|f\|_1} \right) \, dx \leq \frac{n}{p} \|f\|_1 \log \left( b_{n,p} \|f\|_1 \int_{\mathbb{R}^n} f(x) |\nabla \log f(x)|^p \, dx \right) .
\]  \hspace{1cm} (2.11)
Equivalently,
\[ \int_{\mathbb{R}^n} f(x) \log f(x) dx \leq \frac{n}{p} \|f\|_1 \log \left( \frac{b_{n,p}}{\|f\|_1^{1-p}} \int_{\mathbb{R}^n} f(x)|\nabla \log f(x)|^p dx \right), \quad (2.12) \]
where
\[ b_{n,p} = \frac{B_{n,p}}{p^p} = \frac{1}{ne^{p-1}} \left[ \left( \frac{1}{p} \right)^{\frac{n}{2}} \pi^{-\frac{n}{2}} \left( \frac{\Gamma\left(\frac{n}{2} + 1\right)}{\Gamma\left(\frac{n}{p} + 1\right)} \right) \right]^{\frac{p}{n}} \quad (2.13) \]
is the best possible constant.

One can obtain the constant independent of the dimension as follows:

**Corollary 2.4.** For \( f \in W^{1,p}(\mathbb{R}^n) \) with \( 2 \leq p < n \),
\[ \int_{\mathbb{R}^n} |f(x)|^p \log \left( \frac{|f(x)|^p}{\|f\|_p^p} \right) dx \leq D_p \frac{\|
abla f\|_p^p}{\|f\|_p^p}, \]
where
\[ D_p = e^{-p}(p - 1)^{p-1} \pi^{-\frac{p}{2}} \]
is a constant independent of the dimension.

**Lemma 2.5 (Young’s inequality).** For any \( \alpha > 0 \) and \( \beta > 0 \),
\[ \alpha \beta \leq e^\alpha + \beta \log \beta - \beta. \quad (2.14) \]

**Proof of Lemma 2.5.** Let \( t = \phi(s) \) and \( s = \psi(t) \equiv \phi^{-1}(t) \) be a pair of the (modified) Young function, then Young’s inequality (see Fig. 1.1 below) gives
\[ \alpha \beta \leq \int_0^\alpha \phi(s)ds + \int_1^\beta \psi(t)dt. \]
Choosing \( \phi(s) = e^s \) and \( \psi(t) = \log t \), it follows
\[ \alpha \beta \leq \int_0^\alpha e^s \, ds + \int_1^\beta \log t \, dt = e^\alpha + \beta \log \beta - \beta. \]
Proof of Corollary 2.4. From (2.1),
\[
\int_{\mathbb{R}^n} |f(x)|^p \log \left( \frac{|f(x)|^p}{\|f\|_p^p} \right) \, dx \\
\leq \frac{n}{p} \log \left( \frac{p}{n} \left( \frac{p-1}{e} \right)^{p-1} \pi^{-\frac{1}{2}} \left( \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{p}{p} + 1)} \right)^\frac{p}{n} \|\nabla f\|_p^p \right).
\]
Applying the Young inequality (2.14) in Lemma 2.5, then by free parameters \(a > 0\) and \(b > 0\),
\[
\frac{n}{p} \log \left( \frac{p}{n} \left( \frac{p-1}{e} \right)^{p-1} \pi^{-\frac{1}{2}} \left( \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{p}{p} + 1)} \right)^\frac{p}{n} \|\nabla f\|_p^p \right) \\
\leq \frac{a^p}{b} \frac{\|\nabla f\|_p^p}{\|f\|_p^p} + \frac{n}{p} \log \left( \frac{n}{p} - \frac{n}{p} + \frac{n}{p} \log \left( \frac{b}{a^p n} \left( \frac{p-1}{e} \right)^{(p-1)} \pi^{-\frac{1}{2}} \left( \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{p}{p} + 1)} \right)^\frac{p}{n} \right) \right) \\
= \frac{a^p}{b} \frac{\|\nabla f\|_p^p}{\|f\|_p^p} - n(1 + \log a) + \frac{n}{p} \log \left( b(p-1)^{p-1} \pi^{-\frac{1}{2}} \right) + \log \left( \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{p}{p} + 1)} \right).
\]
Choosing \(b^{-1} = (p-1)^{p-1} \pi^{-\frac{1}{2}}\), we obtain
\[
\int_{\mathbb{R}^n} |f(x)|^p \log \left( \frac{|f(x)|^p}{\|f\|_p^p} \right) \, dx + n(1 + \log a) \\
\leq a^p (p-1)^{p-1} \pi^{-\frac{1}{2}} \frac{\|\nabla f\|_p^p}{\|f\|_p^p} + \log \left( \frac{\Gamma(\frac{n}{2} + 1)}{\Gamma(\frac{p}{p} + 1)} \right). \tag{2.15}
\]
From (2.15),
\[
\int_{\mathbb{R}^n} |f(x)|^p \log \left( \frac{|f(x)|^p}{\|f\|_p^p} \right) \, dx + n \left( \log ea - \log \left( \frac{p' \Gamma(\frac{n}{2})}{2 \Gamma(\frac{n}{p})} \right)^\frac{1}{p} \right) \\
\leq a^p (p-1)^{p-1} \pi^{-\frac{1}{2}} \frac{\|\nabla f\|_p^p}{\|f\|_p^p}
\]
and choosing \(a\) such that
\[
a \geq a_c \equiv e^{-1} \left( \frac{p' \Gamma(\frac{n}{2})}{2 \Gamma(\frac{n}{p})} \right)^\frac{1}{p}, \tag{2.16}
\]
the resulting inequality is
\[
\int_{\mathbb{R}^n} |f(x)|^p \log \left( \frac{|f(x)|^p}{\|f\|_p^p} \right) \, dx \leq a_c^p (p-1)^{p-1} \pi^{-\frac{1}{2}} \frac{\|\nabla f\|_p^p}{\|f\|_p^p}. \tag{2.17}
\]
One can observe that if \(2 \leq p\) then \(p' \leq 2\) and from (2.16),
\[
a_c = e^{-1} \left( \frac{p' \Gamma(\frac{n}{2})}{2 \Gamma(\frac{n}{p})} \right)^\frac{1}{p} \leq e^{-1}
\]
and the choice of \(a\) in (2.16) is independent of the spatial dimension \(n\) if \(a \geq e^{-1}\). Hence the inequality (2.17) is independent of spatial dimension if \(a_c = e^{-1}\) under \(2 \leq p\). \(\square\)
3. Shannon’s inequality. We introduce the weighted Lebesgue space $L^p_b(\mathbb{R}^n)$ as
$$L^p_b(\mathbb{R}^n) = \{ f \in L^p_{\text{loc}}(\mathbb{R}^n); \langle x \rangle^b f \in L^p(\mathbb{R}^n) \}.$$ 
Shannon [15] (cf. [16]) obtained an inequality between the informatics entropy and the second moment of information variables. The continuous analogue is well known in the case $L^2$ space (cf. Beckner-Pearson [2]). We show the $L^1_b$ generalization of the Shannon inequality obtained in Ogawa-Wakui [12] (cf. Bercher [3] and [4]).

**Theorem 3.1** (Shannon). Let $n \geq 1$ and $b > 0$. For any non-negative function $f \in L^1_b(\mathbb{R}^n)$, it holds that
\[
\int_{\mathbb{R}^n} f(x) \log (f(x))^{-1} dx \leq \frac{n}{b} \| f \|_1 \log \left( \frac{c_{n,b} \int_{\mathbb{R}^n} |x - \bar{x}|^b f(x) dx}{\| f \|_1^{1+\frac{b}{n}}} \right), 
\] (3.1)
or equivalently,
\[
\int_{\mathbb{R}^n} f(x) \log \left( \frac{f(x)}{\| f \|_1} \right)^{-1} dx \leq \frac{n}{b} \| f \|_1 \log \left( \frac{c_{n,b} \int_{\mathbb{R}^n} |x - \bar{x}|^b f(x) dx}{\| f \|_1} \right), 
\] (3.2)
where $\bar{x}$ is the $b$-th moment center of $f$, i.e.,
\[
\int_{\mathbb{R}^n} |x - \bar{x}|^b f(x) dx = \inf_{y \in \mathbb{R}^n} \int_{\mathbb{R}^n} |x - y|^b f(x) dx
\] (3.3)
and
\[
c_{n,b} = \frac{b e}{n} \left( \int_{\mathbb{R}^n} e^{-|x|^b} dx \right)^{b/n} = \frac{b e}{n} \left( e^{\frac{n}{b}} \Gamma \left( \frac{b}{n} + 1 \right) \right)^{\frac{1}{b}}
\] (3.4)
is the best possible. Besides the best constant is attained by $e^{-\tilde{c}_{n,b}|x|^b}$ up to translation and dilation, where $\tilde{c}_{n,b}$ is given by (2.3).

**Proof of Theorem 3.1.** A proof for the case $b = 2$ can be seen in [11]. To show (3.1), it suffices to show that for $f \in L^1_b(\mathbb{R}^n)$ with $\bar{x} = 0$ and $\| f \|_1 = 1$,
\[
\int_{\mathbb{R}^n} f(x) \log f(x)^{-1} dx \leq \frac{n}{b} \log \left( c_{n,b} \int_{\mathbb{R}^n} |x|^b f(x) dx \right)
\] (3.5)
since by scaling $x \to \lambda^{-1}x$ it is equivalent with
$$\lambda^{-n} \int_{\mathbb{R}^n} f(x) \log f(x)^{-1} dx \leq \frac{n}{b} \log \left( c_{n,b} \lambda^{-(n+b)} \int_{\mathbb{R}^n} |x|^b f(x) dx \right)$$
and setting $\lambda^n = \| f \|_1$ yields the above inequality (3.1).

Let $B_b(x) = e^{-\tilde{c}_{n,b}|x|^b}$ with the constant $\tilde{c}_{n,b}$ chosen as $\| B_b \|_1 = 1$. By Jensen’s inequality,
\[
\exp \left( \int_{\mathbb{R}^n} f(x) \log (f(x))^{-1} dx - \int_{\mathbb{R}^n} f(x) \log (B_b(x))^{-1} dx \right) \leq \int_{\mathbb{R}^n} \left( \frac{f(x)}{B_b(x)} \right)^{-1} f(x) dx = \int_{\mathbb{R}^n} e^{-\tilde{c}_{n,b}|x|^b} f(x) dx = 1.
\]
Hence
\[
\int_{\mathbb{R}^n} f(x) \log (f(x))^{-1} dx \leq \tilde{c}_{n,b} \int_{\mathbb{R}^n} |x|^b f(x) dx.
\] (3.6)
Applying the $L^1$-invariant scaling $f(x) \rightarrow f_\lambda(x) = \lambda^n f(\lambda x)$ to (3.6),
\[
\frac{b}{n} \int_{\mathbb{R}^n} f(x) \log (f(x))^{-1} \, dx \leq \frac{b}{n} c_{n,b} \lambda - b \int_{\mathbb{R}^n} |x|^b f(x) \, dx + \frac{b}{n} \log \lambda \int_{\mathbb{R}^n} f(x) \, dx
= \frac{b c_{n,b}}{n} \lambda - b \int_{\mathbb{R}^n} |x|^b f(x) \, dx + \log \lambda.
\]
(3.7)

Optimizing (3.7) by setting $\lambda^b = \frac{b c_{n,b}}{n} \int_{\mathbb{R}^n} |x|^b f(x) \, dx$, we obtain
\[
\frac{b}{n} \int_{\mathbb{R}^n} f(x) \log (f(x))^{-1} \, dx \leq 1 + \log \left( \frac{b c_{n,b}}{n} \int_{\mathbb{R}^n} |x|^b f(x) \, dx \right)
= \log \left( \frac{b c_{n,b}}{n} \int_{\mathbb{R}^n} |x|^b f(x) \, dx \right).
\]

Finally we notice that $B_b(x) = \exp\left(-\tilde{c}_{n,b} |x|^b\right)$ attains the best possible constant of the inequality (3.5). To see this, it suffices to show that $B_b(x) = e^{-\tilde{c}_{n,b} |x|^b}$ gives the equality of (3.6). Indeed,
\[
\frac{b}{n} \int_{\mathbb{R}^n} B_b(x) \log B_b^{-1}(x) \, dx = \frac{b c_{n,b}}{n} \int_{\mathbb{R}^n} |x|^b B_b(x) \, dx.
\]

Then letting $B_{b,\mu}(x) = \mu^n e^{-\tilde{c}_{n,b} |x|^b}$ with $\mu^b = \frac{b c_{n,b}}{n} \int_{\mathbb{R}^n} |x|^b B_b(x) \, dx$, we see that $B_{b,\mu}(x)$ attains the corresponding equality to (3.5). For the $b$-th moment of $B_b(x)$, by setting $t = 1$ in
\[
-\frac{n}{b} t^{-\frac{n+b}{b}} = \frac{d}{dt} \int_{\mathbb{R}^n} e^{-\tilde{c}_{n,b} t |x|^b} \, dx = -\tilde{c}_{n,b} \int_{\mathbb{R}^n} |x|^b e^{-\tilde{c}_{n,b} t |x|^b} \, dx,
\]
we obtain
\[
\int_{\mathbb{R}^n} |x|^b e^{-\tilde{c}_{n,b} t |x|^b} \, dx = \int_{\mathbb{R}^n} |x|^b B_b(x) \, dx = \frac{n}{b} \tilde{c}_{n,b}^{-1}
\]
and this yields that $\mu^b = 1$. This proves that the inequality (3.5) is sharp. \hfill \square

Remark 3.2. The original version of the Shannon inequality is given for the discrete version as $\sum_{i=1}^{n} p_i \log p_i^{-1} \leq \log n$.

As a corollary of Theorem 3.1, we obtain the following.

Corollary 3.3. Let $n \geq 2$, $b > 0$ and $c_{n,b}$ be defined in (3.4). Then for any non-negative function $f \in L^1_1(\mathbb{R}^n)$,
\[
\int_{\{x \in \mathbb{R}^n : f(x) \leq 1\}} f(x) \log \left( \frac{f(x)}{\|f\|_{L^1_1}} \right)^{-1} \, dx
\leq \frac{n}{b} \|f\|_{L^1_1} \log \left( \frac{c_{n,b} \left( \int_{\{x \in \mathbb{R}^n : f(x) \leq 1\}} |x - \bar{x}|^b f(x) \, dx \right)}{\|f\|_{L^1_1}} \right).
\]
(3.8)

where $\|f\|_{L^1_1} \equiv \int_{\{x \in \mathbb{R}^n : f(x) \leq 1\}} f(x) \, dx$. The constant $c_{n,b}$ on the right hand side is the best possible and given by (3.4).

Remark 3.4. The $b$-th moment center appears in (3.8) is the one that $f$ is not restricted in $f \leq 1$. 
4. Uncertainty inequality.

4.1. **Heisenberg’s uncertainty principle.** Heisenberg’s uncertainty principle is identified by

\[
\|f\|_2^2 \leq \frac{2}{n} \left( \int_{\mathbb{R}^n} |\nabla f|^2 dx \int_{\mathbb{R}^n} |x|^2 |f(x)|^2 dx \right)^{1/2}.
\] (4.1)

See, for more relevant generalization, Beckner [1]. On the other hand, the sharp logarithmic Sobolev inequality combined with the generalized version of the Shannon inequality yield a generalization of the uncertainty principle involving the Boltzmann-Shannon entropy functional. We show the following generalization:

**Theorem 4.1.** Let $1 < p < n$ and $1/p + 1/p' = 1$. For any non-negative function $f \in L_{p'}^1$ with $f^{1/p} \in W^{1,p}$. Then it holds that

\[
- \frac{n}{p'} \log \left( \frac{c_{n,p'} \left( \int_{\mathbb{R}^n} |x - \bar{x}|^{p'} f(x) dx \right)}{\|f\|_1} \right) \leq \int_{\mathbb{R}^n} \frac{f(x)}{\|f\|_1} \log \left( \frac{f(x)}{\|f\|_1} \right) dx
\]

\[
\leq \frac{n}{p} \log \left( \frac{b_{n,p} \left( \int_{\mathbb{R}^n} |\nabla \log f(x)|^{p} f(x) dx \right)}{\|f\|_1} \right),
\]

where $\bar{x}$ is the $p'$-th moment center of $f$ given by (3.3) and $b_{n,p}$ and $c_{n,p'}$ are given in (2.13) and (3.4), respectively. In particular,

\[
\|f\|_1 \leq \frac{1}{n} \left( \int_{\mathbb{R}^n} |x - \bar{x}|^{p'} f(x) dx \right)^{1/p'} \left( \int_{\mathbb{R}^n} |\nabla \log f(x)|^{p} f(x) dx \right)^{1/p},
\] (4.2)

where $n^{-1}$ is the best possible constant and it is attained by $f(x) = e^{-\tilde{c}_{n,p'} |x|^{p'}}$ with

\[
\tilde{c}_{n,p'} = \pi^{p'} \left( \frac{\Gamma \left( \frac{n}{p'} + 1 \right)}{\Gamma \left( \frac{n}{p' + 1} \right)} \right)^{\frac{n}{p'}}.
\]

The inequality (4.1) is a direct consequence of (4.2) by setting $p = p' = 2$, $|f|^2$ into $f$ and noting $\bar{x}$ gives the minimum for the moment of $f$.

Let $f \in C_0^\infty(\mathbb{R}^n)$, $f(x) \geq 0$. The inequality (4.2) is known to obtain by the direct estimate for all $1 < p < \infty$. Since $\text{div} (x - \bar{x}) = n$,

\[
\int_{\mathbb{R}^n} f(x) dx = \frac{1}{n} \int_{\mathbb{R}^n} (\text{div} (x - \bar{x})) f(x) dx = -\frac{1}{n} \int_{\mathbb{R}^n} (x - \bar{x}) \cdot \nabla f(x) dx
\]

\[
\leq \frac{1}{n} \int_{\mathbb{R}^n} |x - \bar{x}|^{1/p'} f(x) (x - \bar{x}) f(x) \text{div} (\nabla f(x)) dx
\]

\[
\leq \frac{1}{n} \left( \int_{\mathbb{R}^n} |x - \bar{x}|^{p'} f(x) dx \right)^{1/p'} \left( \int_{\mathbb{R}^n} |\nabla \log f(x)|^{p} f(x) dx \right)^{1/p}.
\]

We illustrate a different method using the logarithmic Sobolev inequality and the generalized version of the Shannon inequality in $L^p$. 
Proof of Theorem 4.1. From Theorem 3.1 with $b = p'$,

$$
\int_{\mathbb{R}^n} f(x) \log (f(x))^{-1} dx \leq \frac{n}{p'} \log \left( \frac{c_{n,p'} (\int_{\mathbb{R}^n} |x-x'|^{p'} f(x) dx)}{\|f\|_1^{1+\frac{2n}{p'}}} \right)
$$

$$
= \log \left( \frac{c_{n,p'} (\int_{\mathbb{R}^n} |x-x'|^{p'} f(x) dx)}{\|f\|_1^{\frac{n}{p'}+1}} \right),
$$

where

$$
c_{n,p'} = \left( \frac{p' e}{n} \right)^{\frac{n}{2}} \left( \frac{2\pi \frac{n}{2}}{p'} \Gamma \left( \frac{n}{2} \right) \right)^{-1} = \left( \frac{p' e}{n} \right)^{\frac{n}{2}} \|e^{-|x|^{p'}}\|_1.
$$

From the sharp version of the logarithmic Sobolev inequality (2.11) with the best constant (2.13),

$$
\int_{\mathbb{R}^n} \frac{f(x)}{\|f\|_1} \log (f(x)) dx \leq \frac{n}{p} \log \left( \frac{b_{n,p} (\int_{\mathbb{R}^n} \nabla \log f(x)^{p'} dx)}{\|f\|_1^{\frac{n}{2}}} \right)
$$

$$
= \log \left( \frac{b_{n,p} (\int_{\mathbb{R}^n} \nabla \log f(x)^{p'} dx)}{\|f\|_1^{\frac{n}{2}}} \right),
$$

where

$$
b_{n,p} = \left( \frac{1}{nep-1} \right)^{\frac{n}{2}} \left( \frac{1}{p'} \right)^{\frac{n}{2}} \left( \frac{p'}{2\pi} \Gamma \left( \frac{n}{2} \right) \right)^{-1} \left( \frac{n}{2} \right)^{-1} = \left( \frac{1}{nep-1} \right)^{\frac{n}{2}} \left( \frac{1}{p'} \right)^{\frac{n}{2}} \|e^{-|x|^{p'}}\|_1^{-1}.
$$

Combining (4.3) with (4.4) and (4.5) with (4.6), we immediately obtain that for any $1 \leq p < n$,

$$
- \log \left( \frac{c_{n,p'} (\int_{\mathbb{R}^n} |x-x'|^{p'} f(x) dx)}{\|f\|_1^{\frac{n}{p}}} \right) \leq \int_{\mathbb{R}^n} \frac{f(x)}{\|f\|_1} \log \left( \frac{f(x)}{\|f\|_1} \right) dx
$$

$$
\leq \log \left( \frac{b_{n,p} (\int_{\mathbb{R}^n} \nabla \log f(x)^{p'} dx)}{\|f\|_1^{\frac{n}{2}}} \right). \tag{4.7}
$$

Namely

$$
0 \leq \|f\|_1 \log \left( \frac{c_{n,p'} b_{n,p}}{\|f\|_1^{\frac{n}{p'}+\frac{2n}{p'}}} \left( \int_{\mathbb{R}^n} |x-x'|^{p'} f(x) dx \right)^{\frac{n}{2}} \left( \int_{\mathbb{R}^n} \nabla \log f(x)^{p'} dx \right)^{\frac{n}{2}} \right)
$$

or equivalently

$$
\|f\|_1^n \leq c_{n,p'} b_{n,p} \left( \int_{\mathbb{R}^n} |x-x'|^{p'} f(x) dx \right)^{\frac{n}{2}} \left( \int_{\mathbb{R}^n} \nabla \log f(x)^{p'} dx \right)^{\frac{n}{2}},
$$
where the constant is identified by
\[
    c_{n,p}^{\frac{n}{p}} b_{n,p}^{\frac{n}{p}} = \left( \frac{p' e}{n} \right)^{\frac{n}{p}} \| e^{-|x|^p} \|_{1} \times \left( \frac{1}{n e^{p-1}} \right)^{\frac{n}{p}} \left( \frac{1}{p'} \right)^{\frac{n}{p}} \| e^{-|x|^{p'}} \|_{1}^{-1}
\]
\[
    = \left( \frac{p' e}{n} \right)^{\frac{n}{p}} \left( \frac{1}{n e^{p-1}} \right)^{\frac{n}{p}} \left( \frac{1}{p'} \right)^{\frac{n}{p}}
\]
\[
    = \left( \frac{c}{n} \right)^{\frac{n}{p}} \left( \frac{1}{n e^{p-1}} \right)^{\frac{n}{p}} = n^{-n}. \tag{4.8}
\]

The inequality is identified with the uncertainty principle (4.2) of Heisenberg type in $L^p$ with the best constant. The best constant is apparently attained by the common extremizer function $e^{-|x|^p}$.

In the above proof, combining the generalized Shannon inequality and the sharp logarithmic Sobolev inequality imply a version of the uncertainty principle via the Boltzmann-Shannon entropy functional for $f$, in particular the estimate (4.7).

In view of Theorem 4.1, one may regard that the logarithmic Sobolev inequality also holds for $p \geq n$. The result by Del Pino-Dolbeault [6] does not cover these cases because of the restriction on the sharp constant of the Gagliardo-Nirenberg inequality (2.4). Since the best possible constant coincides in (4.8), we conjecture that the best possible constant for the logarithmic Sobolev inequality for $n \geq p$ still given by $b_{n,p}$ in (4.6).

4.2. Some generalization. If the two exponents $p$ and $q$ are not necessarily the Hölder conjugate exponents, one may obtain a generalized version of the uncertainty principle:

**Proposition 4.2.** Let $n \geq 2$ and $1 < p, q < \infty$ with $p < n$. Then for any non-negative $f \in \dot{W}^{1,p}(\mathbb{R}^n)$ with $f \in L^1_q(\mathbb{R}^n)$, it holds
\[
    \|f\|_{1}^{\frac{1}{p} + \frac{1}{q}} \leq b_{n,p}^{\frac{n}{p}} c_{n,q}^{\frac{n}{q}} \left( \int_{\mathbb{R}^n} |x - \bar{x}|^q f(x) \, dx \right)^{\frac{1}{q}} \left( \int_{\mathbb{R}^n} |\nabla \log f(x)|^p f(x) \, dx \right)^{\frac{1}{p}},
\]
where
\[
    b_{n,p}^{\frac{n}{p}} c_{n,q}^{\frac{n}{q}} = \left( \frac{1}{n e^{p-1}} \right)^{\frac{n}{p}} \left( \frac{1}{p'} \right)^{\frac{n}{p}} \| e^{-|x|^p} \|_{1}^{-1} \times \left( \frac{g e}{n} \right)^{\frac{n}{p}} \| e^{-|x|^{p'}} \|_{1}
\]
\[
    = \left( \frac{1}{n \frac{n}{p} + \frac{n}{q} e^{p-1} - \frac{n}{q}} \right)^{\frac{n}{p}} \left( \frac{p' \Gamma \left( \frac{n}{q} \right)}{\Gamma \left( \frac{n}{p} \right)} \right).
\]

5. Generalization. Since the extremal function for the inequalities (3.1) and (3.2) satisfies $e^{-|x|^p} \leq 1$, denoting $\|f\|_{L^1((f \leq 1))} = \|f\|_{1,*}$, we arrange the resulting inequality into $f(x) \rightarrow f(x) \chi_{(f \leq 1)}(x)$. Let $1 < p < n$. Noting $\lim_{r \to 0} r \cdot \log r \to 0$,
\[
    \int_{\{f \leq 1\}} f(x) \log \left( \frac{f(x)}{f(x)} \right) \, dx \leq \frac{n}{p} \|f\|_{1,*} \log \left( \frac{c_{n,p}^{\frac{n}{p}} \left( \int_{\{f \leq 1\}} |x - \bar{x}|^p f \, dx \right)}{\|f\|_{1,*}^{\frac{n}{p} + \frac{n}{q}}} \right) \tag{5.1}
\]
\[
    = \|f\|_{1,*} \log \left( \frac{c_{n,p}^{\frac{n}{p}} \left( \int_{\{f \leq 1\}} |x - \bar{x}|^p f \, dx \right)^{\frac{n}{p}}}{\|f\|_{1,*}^{\frac{n}{p} + \frac{n}{q}}} \right).
\]
Similarly substituting \( f(x) \to f(x)\chi_{\{|f(x)| \geq 1\}}(x) \) in (2.12) and denoting \( \|f\|_{L^1(|f| \geq 1)} \equiv \|f\|_{1^*}, \)

\[
\int_{\{f \geq 1\}} f(x) \log \left( f(x) \right) dx \\
\leq \frac{n}{p} \|f\|_{1^*} \log \left( \frac{b_{n,p}}{\|f\|_{1^*}} \int_{\{f \geq 1\}} f(x) \nabla \log \chi_{f \geq 1} f(x) \right) .
\]

Namely,

\[
\int_{\{f \geq 1\}} f(x) \log \left( \frac{f(x)}{\|f\|_{1^*}} \right) dx \\
\leq \|f\|_{1^*} \log \left( \frac{b_{n,p}}{\|f\|_{1^*}} \left( \int_{\{f \geq 1\}} f(x) \nabla \log \chi_{f \geq 1} f(x) \right) \right)^{\frac{1}{p}} . \tag{5.2}
\]

Combining the inequalities (5.1) and (5.2),

\[
\int_{\{f \leq 1\}} f(x) \log(f(x))^{-1} dx + \int_{\{f \geq 1\}} f(x) \log f(x) dx \\
\leq \|f\|_{1^*} \log \left( \frac{c_{n,p} \int_{\{f \leq 1\}} |x - \bar{x}|^p f dx}{\|f\|_{1^*}^{1+\frac{p}{n}}} \right)^{\frac{1}{p}} \\
+ \|f\|_{1^*} \log \left( \frac{b_{n,p} \int_{\{f \geq 1\}} f(x) \nabla \log \chi_{f \geq 1} f(x) \right) \right)^{\frac{1}{p}} . \tag{5.3}
\]

Since \( \|f\|_{1^*} \leq \|f\|_1 \), it follows from (5.3) that

\[
\int_{\mathbb{R}^n} \frac{f(x)}{\|f\|_1} \log\left| \frac{f(x)}{\|f\|_1} \right| dx \\
\leq \log \left( \frac{c_{n,p}^{\frac{n}{p}} b_{n,p}^{\frac{n}{p}}}{\|f\|_{1^*}^{\frac{n}{p}} \|f\|_{1^*}^{\frac{n}{p}}} \left( \int_{\{f \leq 1\}} |x - \bar{x}|^p f(x) dx \right)^{\frac{1}{p}} \right. \\
\times \left. \left( \int_{\{f \geq 1\}} \nabla \log \chi_{f \geq 1} f(x) \right) \right)^{\frac{1}{p}} .
\]

and thus we have

\[
\|f\|_{1^*}^{\frac{n}{p} + 1} \|f\|_{1^*}^{\frac{n}{p} - 1} \exp \left( \int_{\mathbb{R}^n} \frac{f(x)}{\|f\|_1} \log f(x) dx \right) \\
\leq c_{n,p}^{\frac{n}{p}} b_{n,p}^{\frac{n}{p}} \left( \int_{\{f \leq 1\}} |x - \bar{x}|^p f(x) dx \right)^{\frac{1}{p}} \\
\times \left( \int_{\{f \geq 1\}} \nabla \log \chi_{f \geq 1} f(x) \right) \right)^{\frac{1}{p}} . \tag{5.4}
\]

Denoting the constant in (5.4) explicitly, we obtain the generalized version of uncertainty principle as follows:
Theorem 5.1 (Generalized uncertainty principle). Let \( n \geq 2 \) and \( 1 < p < n \). For any non-negative function \( f \in L^p_\mu(\mathbb{R}^n) \) with \( f^{1/p} \in W^{1,p}(\mathbb{R}^n) \), it holds

\[
\|f\|_{L^p_\mu}^{\frac{1}{p} + \frac{1}{p'} + \frac{1}{2}} \|f\|_{L^1_\mu}^{\frac{1}{p} - \frac{1}{2}} \exp\left(\frac{1}{n} \int_{\mathbb{R}^n} f(x) \log f(x) \, dx\right) \\
\leq \frac{1}{n} \left( \int_{\mathbb{R}^n} |x - \bar{x}|^p f(x) \, dx \right)^{\frac{1}{p}} \left( \int_{\mathbb{R}^n} |\nabla \log f(x)|^p f(x) \, dx \right)^{\frac{1}{p}},
\]

where \( \|f\|_{L^p_\mu} \) and \( \|f\|_{L^1_\mu} \) denote \( \|f\|_{L^p((f \leq 1))} \) and \( \|f\|_{L^1((f \geq 1))} \), respectively.

6. Appendix.

6.1. A version on the Gaussian measure. The logarithmic Sobolev inequality is known to hold on the space with the Gaussian measure and it is equivalent to the version in the Euclidean space with the Lebesgue measure. Connecting with this, we see from the sharp inequality (1.6) that the Gaussian weight version holds: Substituting \( g(x)G_\mu(x) \) into \( f(x) \) with \( \|gG_\mu\|_1 = 1 \) in (1.6), and by

\[
\log G_\mu(x) = -\frac{|x|^2}{4\mu} - \frac{n}{2} \log(4\pi\mu)
\]

one can see that

\[
\int_{\mathbb{R}^n} g(x) \log (g(x)) G_\mu(x) \, dx - \frac{1}{4\mu} \int_{\mathbb{R}^n} |x|^2 g(x) G_\mu(x) \, dx - \frac{n}{2} \log(4\pi\mu) \\
\leq \frac{n}{2} \log \left( \frac{1}{2n\pi e} \int_{\mathbb{R}^n} g(x) \left|\nabla \log (g(x)) - \frac{x}{2\mu} \right|^2 G_\mu(x) \, dx \right) \\
= \frac{n}{2} \log \left( \frac{1}{2n\pi e} \int_{\mathbb{R}^n} g(x) \left( \nabla \log (g(x)) \right)^2 - \frac{x}{\mu} \cdot \nabla \log (g(x)) + \frac{|x|^2}{4\mu^2} \right) G_\mu(x) \, dx \right). \tag{6.1}
\]

Since

\[
\log \left( -\frac{2\mu}{ne} \int_{\mathbb{R}^n} g(x) \cdot \nabla \log (g(x)) G_\mu(x) \, dx \right) \\
= \log \left( -\frac{2}{ne} \frac{1}{2\mu} \int_{\mathbb{R}^n} |x|^2 G_\mu(x) g(x) \, dx + \frac{2}{e} \right) \tag{6.2}
\]

and

\[
\frac{1}{4\mu} \int_{\mathbb{R}^n} |x|^2 g(x) G_\mu(x) \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^n} x \cdot \nabla \log (g(x)) g(x) G_\mu(x) \, dx + \frac{n}{2} \\
\leq \frac{\mu}{2} \int_{\mathbb{R}^n} \left|\nabla \log (g(x))\right|^2 g(x) G_\mu(x) \, dx + \frac{1}{8\mu} \int_{\mathbb{R}^n} |x|^2 g(x) G_\mu(x) \, dx + \frac{n}{2}
\]

namely

\[
\frac{1}{4\mu} \int_{\mathbb{R}^n} |x|^2 g(x) G_\mu(x) \, dx \leq \mu \int_{\mathbb{R}^n} \left|\nabla \log (g(x))\right|^2 g(x) G_\mu(x) \, dx + n, \tag{6.3}
\]
we have from (6.1)-(6.3) that
\[
\int_{\mathbb{R}^n} g(x) \log (g(x)) G_\mu(x) dx \\
\leq \frac{n}{2} \log \left( \frac{2\mu}{ne} \int_{\mathbb{R}^n} g(x) |\nabla \log (g(x))|^2 G_\mu(x) dx \right) \\
+ \frac{2\mu}{ne} \left( -\frac{1}{2\mu^2} + \frac{1}{4\mu^2} \right) \int_{\mathbb{R}^n} |x|^2 g(x) G_\mu(x) dx + \frac{2}{e}
\]
\[
+ \mu \int_{\mathbb{R}^n} |\nabla \log (g(x))|^2 g(x) G_\mu(x) dx + n
\]
\[
\leq \frac{n}{2} \log \left( \frac{2\mu e}{n} \int_{\mathbb{R}^n} g(x) |\nabla \log (g(x))|^2 G_\mu(x) dx \right) \\
+ \frac{1}{2\mu e} \int_{\mathbb{R}^n} |x|^2 g(x) G_\mu(x) dx + \frac{2}{e}
\]
\[
+ \mu \int_{\mathbb{R}^n} |\nabla \log (g(x))|^2 g(x) G_\mu(x) dx + n.
\]
Letting \( d\gamma = G_\mu(x) dx \) and one can see that
\[
\int_{\mathbb{R}^n} g(x) \log (g(x)) d\gamma \leq \frac{n}{2} \log \left( \frac{2\mu e}{n} \int_{\mathbb{R}^n} |\nabla \log (g(x))|^2 g(x) d\gamma \right) \\
+ \mu \int_{\mathbb{R}^n} |\nabla \log (g(x))|^2 g(x) d\gamma.
\]
After applying the \( L^1 \)-invariant scaling: \( g_\lambda(x) \equiv \lambda^n g(\lambda x) \),
\[
\int_{\mathbb{R}^n} g(x) \log (g(x)) d\gamma + n \log \lambda \cdot \int_{\mathbb{R}^n} g(x) d\gamma \\
\leq \frac{n}{2} \log \left( \frac{2\mu e}{n} \int_{\mathbb{R}^n} |\nabla \log (g(x))|^2 g(x) d\gamma \right) \\
+ \mu \int_{\mathbb{R}^n} |\nabla \log (g(x))|^2 g(x) d\gamma.
\]
Since \( \|G_\mu\|_1 = 1 \), choosing
\[
\lambda^2 = \frac{2\mu e}{n} \int_{\mathbb{R}^n} |\nabla \log (g(x))|^2 g(x) d\gamma,
\]
we obtain the logarithmic Sobolev inequality for the Gaussian weight (Gauss measure) as follows:
\[
\int_{\mathbb{R}^n} g(x) \log (g(x)) d\gamma \leq \mu \int_{\mathbb{R}^n} |\nabla \log (g(x))|^2 g(x) d\gamma.
\]

6.2. Connection from the Sobolev inequality to the logarithmic Sobolev inequality. The following argument is informed by one of the referee. The analogous observation is given by Beckner-Pearson [2]. The sharp Sobolev inequality implies the sharp logarithmic Sobolev inequality (1.4) as follows. Let \( f \in H^1 \) and for simplicity we assume that \( \|f\|_2 = 1 \). Then by (1.1) and Jensen’s inequality,
\[
\frac{2}{n-2} \int_{\mathbb{R}^n} |f|^2 \log |f|^2 dx \leq \log \left( \int_{\mathbb{R}^n} |f|^\frac{2n}{n-2} dx \right) \leq -\frac{n}{n-2} \log \left( S_b^2 \int_{\mathbb{R}^n} |\nabla f|^2 dx \right),
\]
where \( S_b \) is the best possible constant for the Sobolev inequality given by (1.2). Plugging \( g(\hat{x}) = f(x_1) f(x_2) \cdots f(x_k) \) for \( \hat{x} \in \mathbb{R}^{kn} \) with \( \hat{x} = (x_1, x_2, \ldots, x_k) \) and \( x_i \in \mathbb{R} 

\( \mathbb{R}^n \) into (6.4), we see by noticing 
\( \| g \|_{L^2(\mathbb{R}^n)}^2 = \| f \|_{L^2(\mathbb{R}^n)}^{2k} = 1 \) and 
\( \| \nabla g \|_{L^2(\mathbb{R}^n)}^2 = k \| \nabla f \|_{L^2(\mathbb{R}^n)}^2 \) that

\[
k \int_{\mathbb{R}^n} |f|^2 \log |f|^2 \, dx = \int_{\mathbb{R}^n} |g|^2 \log |g|^2 \, d\tilde{x}
\leq \frac{kn}{2} \log \left( \frac{1}{\pi kn(kn-2)} \frac{\Gamma(kn)}{\Gamma\left(\frac{kn}{2}\right)} \right)^{\frac{2}{kn}} \int_{\mathbb{R}^n} |\nabla g|^2 \, d\tilde{x}
\leq \frac{kn}{2} \log \left( \frac{1}{\pi kn(kn-2)} \frac{\Gamma(kn)}{\Gamma\left(\frac{kn}{2}\right)} \right)^{\frac{2}{kn}} k \int_{\mathbb{R}^n} |\nabla f|^2 \, dx.
\]

Namely we obtain with using the Stirling formula \( n! \simeq \sqrt{2\pi e} n^{-\frac{1}{2}} e^n \) that

\[
\int_{\mathbb{R}^n} |f|^2 \log |f|^2 \, dx
\leq \frac{n}{2} \log \left( \frac{1}{\pi n^2 (kn-2)^2} \frac{\Gamma(kn+1)}{\Gamma\left(\frac{kn}{2}+1\right)} \right)^{\frac{2}{kn}} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx
\leq \frac{n}{2} \log \left( \frac{1}{\pi n^2 (kn-2)^2} \frac{\sqrt{2\pi kn(e/kn)^{kn}}} {\sqrt{2\pi kn/2(kn/2e)^{kn/2}}} \right)^{\frac{2}{kn}} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx
\leq \frac{n}{2} \log \left( \frac{1}{\pi n^2 (kn-2)^2} \frac{\sqrt{2\pi kn(e/kn)^{kn}}}{(2\pi kn/2)^{kn/2}} \right)^{\frac{2}{kn}} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx
\leq \frac{n}{2} \log \left( \frac{2n}{e\pi n(n-\frac{k}{2})} \right)^{\frac{2}{kn}} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx.
\]

Observing that \( k \to \infty \) we obtain the sharp inequality (1.4). The above observation is based on the fact that the logarithmic Sobolev inequality is invariant for the independent event for the probabilistic object.

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*E-mail address:* ogawa@math.tohoku.ac.jp