Symmetry-Protected Topological Entanglement

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We propose an order parameter for the Symmetry-Protected Topological (SPT) phases which are protected by Abelian on-site symmetries. This order parameter, called the SPT entanglement, is defined as the entanglement between $A$ and $B$, two distant regions of the system, given that the total charge (associated with the symmetry) in a third region $C$ is measured and known, where $C$ is a connected region surrounded by $A$ and $B$ and the boundaries of the system. In the case of 1-dimensional systems we prove that at the limit where $A$ and $B$ are large and far from each other compared to the correlation length, the SPT entanglement remains constant throughout a SPT phase, and furthermore, it is zero for the trivial phase while it is nonzero for all the non-trivial phases. Moreover, we show that the SPT entanglement is invariant under the low-depth quantum circuits which respect the symmetry, and hence it remains constant throughout a SPT phase in the higher dimensions as well. Finally, we show that the concept of SPT entanglement leads us to a new interpretation of the string order parameters and based on this interpretation we propose an algorithm for extracting the relevant information about the SPT phase of the system from the string order parameters.

Symmetry-Protected Topological (SPT) order is a new kind of order at zero temperature which cannot be described by the traditional Landau paradigm of symmetry breaking \[1^4\]. Different SPT orders remain distinct from each other in the presence of an appropriate symmetry. However, there is no local order parameter to distinguish SPT ordered phases from the trivial phase, and furthermore SPT ordered phases do not exhibit long-range order between distant sites. First known example of a SPT phase is the Haldane phase of the antiferromagnetic spin chains with odd integer spins, which is protected by the $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry corresponding to $\pi$ rotations around an orthogonal axis \[4^6\].

It is often claimed that (symmetry-protected) topological order of a ground state is fundamentally related to its entanglement structure. So, it is desirable to have order parameters which directly detect this entanglement structure. In fact, in the case of topological order, topological entanglement entropy can reveal the presence of topological order based on the entanglement properties of the ground state \[7^9\]. In this paper, we propose a new quantity, called SPT entanglement (SPT-Ent), which detects SPT ordered phases based on their entanglement properties. We show that for Abelian symmetries and in the appropriate limits, SPT-Ent is a universal quantity, i.e. in any arbitrary dimension it remains constant throughout a SPT phase, similar to the fact that topological entanglement entropy remains constant throughout a topological phase \[8^9\]. Furthermore, in the case of 1-dimensional systems we calculate SPT-Ent in the Matrix Product State (MPS) framework and we show that it always successfully detects the presence of SPT order. We also show that the concept of SPT-Ent leads us to a new interpretation of the string order parameters and a systematic method for extracting the relevant information about the phase from them.

Although the main results of this paper are focused on the field of SPT order, it also contributes by showing that the resource theory point of view to entanglement can be useful in the study of many-body systems (See e.g. \[10\] for a recent review). In this point of view, entanglement of a given state is characterized by the equivalence class of all states that can be reversibly transformed to the state via Local Operations and Classical Communications (LOCC). All states in the same equivalence class have the same amount of entanglement relative to all measures of entanglement. This point of view to entanglement is essential for deriving the results of this paper. This suggests that the resource theory point of view to entanglement might also be useful in the study of other many-body problems.

Another interesting aspect of this work is that it clearly shows how in a many-body problem, the distinction between entanglement and correlations becomes important. Although in the recent years tools of entanglement theory, and in particular entanglement measures, have been extensively used in the study of many-body systems, it is not always clear that why they are relevant in this context, and why they cannot be replaced by other measures of correlation, such as mutual information. The defining property of entanglement measures, which distinguishes them from other measures of correlation, is that they are non-increasing under classical communication, while they can increase under quantum communication. Interestingly, this distinction plays a clear role in the arguments that lead to the properties of SPT-Ent.

Definition: Consider a finite-dimensional lattice system. Let $G$ be an Abelian symmetry with on-site linear unitary representations on the sites of this system. Let $A$ and $B$ be two non-overlapping regions of the system, and $C$ be a connected region surrounded by $A$ and $B$ and the boundaries of the system (see Fig 1a). Suppose we
measure the total charge associated with the symmetry $G$ in the region $C$ and obtain charge $\kappa$. Here, each $\kappa$ corresponds to a distinct irreducible representation (irrep) of $G$ as $g \rightarrow e^{i\kappa(g)}$. Let $\Pi^{(C)}_\kappa$ be the projector to the subspace of states with charge $\kappa$ in region $C$. By measuring the total charge in $C$ on state $\rho$, the charge $\kappa$ is obtained with the probability $p_\kappa = \text{tr}(\rho \Pi^{(C)}_\kappa)$, and given this outcome the reduced state of $AB$ after the measurement will be $\rho^{(AB)}_\kappa \equiv \frac{\text{tr}_{AB}(\Pi^{(C)}_\kappa \rho)}{p_\kappa}$, where the trace is over all the sites in the system except those in $A$ and $B$.

**Summary of results:** In this paper we report three major results about SPT-Ent. We also find an interesting connection between SPT-Ent and the string order parameters. In the following, we provide a summary of these results:

1. We prove that in the limit where two regions $A$ and $B$ are far from each other compared to the correlation length, the SPT-Ent is always zero in the trivial phase (See Theorem 2 and the definition below it). It follows that

**Corollary 1** A non-vanishing SPT-Ent between two distant regions, in the limit where the distance between them is much larger than the correlation length, indicates the presence of SPT order.

This result provides a method for detecting SPT order in arbitrary dimension. As we show later, this method can detect all SPT ordered phases in 1-d systems. Note that, unlike most other proposed approaches for detecting SPT order, this method works even if the translational symmetry is broken.

Theorem 2 gives a criterion that should be satisfied by all states which can be prepared by low-depth symmetric circuits starting from a product state. For instance, using this result and by proving that SPT-Ent does not vanish in the ground state of Majorana chain \[13\] for certain values of parameters, one can show that these states cannot be generated from a product state using a low-depth symmetric circuit \[14\]. This result is recently proven in \[15, 28\] using totally different approaches.

2. Theorem 2 implies that SPT-Ent remains constant throughout the trivial phase. Making an assumption about the decay of correlations, we can extend this result to all SPT phases, and show that at the limit where regions $A$ and $B$ are large and far from each other compared to the correlation length, SPT-Ent is a universal quantity, i.e. it remains constant throughout the phase (See Theorems 2). This result also holds in arbitrary dimension. The proof is analogous to the argument that shows topological entanglement entropy remains constant throughout a phase \[8, 9\]. It is based on a reasonable, but unproven, assumption that in the limit where $A$ and $B$ are large and far from each other, small deformations of the boundaries of these regions do not affect the SPT-Ent. Since SPT-Ent is monotonically increasing with the sizes of $A$ and $B$ (See Eq. (5)), given that the correlations are short range and the system is sufficiently homogenous (e.g., translational invariant), this assumption looks reasonable.

3. We calculate SPT-Ent of 1-d systems in the MPS framework, and prove that, at the limit where $A, B$ and $C$ are large compared to the correlation length, SPT-Ent is: 1) independent of the sizes of these regions, 2) constant throughout all SPT phases, and 3) zero for the trivial phase while it is nonzero for all the non-trivial phases. Indeed, we show that in this limit the SPT-Ent is equal
to the entanglement of a maximally entangled state of a pair of $d_{\omega,j}$-dimensional systems, where $d_{\omega,j}$ is the dimension of the projective irreps of the group $G$ in the cohomology class $[\omega]$ which characterizes the SPT phase of the system $[1,4]$. Note that, in general, irreps whose factor system belong to the same cohomology class have different dimensions. But, interestingly it turns out that for Abelian groups all such irreps have the same dimension. (See lemma 11 in the Supplementary Material [12].) According to the classifications of the SPT phases of 1-d systems, the equivalence class $[\omega]$ uniquely determines the SPT phase $[1,4]$. Therefore, in the case of 1-d systems direct calculation of SPT-Ent confirms our general result that SPT-Ent is constant throughout SPT phases. It turns out the parameter $d_{\omega}$ has a natural interpretation as the edge mode degeneracy associated with each edge of an open chain $[13,22]$. Note that the parameter $d_{\omega}$ is only defined for 1-d systems.

It follows from these results that by quantifying SPT-Ent we can construct order parameters to detect SPT order. Consider the negativity, $N(\sigma) \equiv (\| \sigma^{T_A} \|_1 - 1)/2$, as an example of continuous entanglement measures [17], where $\sigma^T_A$ denotes the partial transpose and $\| \cdot \|_1$ is the trace norm. Then, the above result implies that at the limit where $A$, $B$, and $C$ are much larger than the correlation length,

$$\lim \sum_\kappa \left\| \left[ \text{tr}_{\mathbb{C}}(\Pi_\kappa^C) \rho \right]^{T_A} \right\|_1 - 1 = d_{\omega} - 1. \quad (3)$$

The system is in a non-trivial phase iff $[\omega]$ is non-trivial $[1,4]$, or equivalently if $d_{\omega} > 1$. So, the left-hand side of Eq. (3) can serve as an order parameter whose value for the large blocks of $A$, $B$, and $C$ only detects the presence of SPT order, but also reveals the dimension of the irreps in the equivalence class $[\omega]$ which characterizes the phase. For example, in the case of Haldane phase where $d_{\omega} = 2$, the left-hand side of Eq. (3) converges to one, while it converges to zero in the trivial phase.

New perspective on string order parameters: There exists a striking connection between SPT-Ent and the string type order parameters, which are the traditional tool for detecting the SPT order $[18,21,24]$. Consider the string operators in the form of $O_{ij}(g) \equiv X_i^{(A)} \otimes_{l \in C} u(l)(g) \otimes Y_j^{(B)}$, where $\{X_i^{(A)}\}$ and $\{Y_j^{(B)}\}$ are basis for the space of operators acting on $A$ and $B$ respectively, and $g \rightarrow u(l)(g)$ is the representation of $G$ on site $l$. The relation between the string order parameters and SPT-Ent can be established using the Fourier transform

$$\sum_{g \in G} e^{-i\kappa(g)} \text{tr}(\rho \, O_{ij}(g)) = \text{tr} \left( \left[ |\kappa\rangle \langle \kappa|^C \otimes X_i^{(A)} \otimes Y_j^{(B)} \right] \Omega^{(AB)(C)}(\rho) \right). \quad (4)$$

These equations for different charges $\kappa$ and different $X_i^{(A)}$ and $Y_j^{(B)}$, uniquely determine $\Omega^{(AB)(C)}(\rho)$, and hence the SPT-Ent of $\rho$. This observation suggests each string order parameter $\text{tr}(\rho \, O_{ij}(g))$ alone does not have any information about the SPT phase of the system, but together they provide enough information to find the SPT-Ent of state, and hence to detect the presence of SPT order. Therefore, it follows from our results on SPT-Ent that, in contrast to what has been suggested before, in the case of Abelian symmetries string order parameters can be used to detect the presence of SPT order.

This interpretation of string order parameters leads us to an algorithm for extracting the relevant information about SPT order of the system: One first uses the Fourier transform of the string order parameters to find the state $\Omega^{(AB)(C)}(\rho)$ via Eqs. (4) and then checks if this state is entangled or not. In fact, using this algorithm one can even find the dimension $d_{\omega}$ corresponding to the phase of system.

Generalizations of SPT-Ent: In the above, the concept of SPT-Ent is defined based on the notion of the charge associated with the symmetry group $G$, which protects the SPT phase. But, in general, instead of the charge associated with $G$, one can consider the charge associated with any subgroup $G'$ of $G$, and define a generalized notion of SPT-Ent based on such generalized charges. Using the fact these generalized charges are conserved under unitary dynamics with symmetry $G$, it is straightforward to show that these generalized versions of SPT-Ent are also universal quantities which remain constant throughout the phase. Indeed, even if the group $G$ in Non-Abelian, as long as the subgroup $G'$ whose charge is measured, is Abelian this result still holds. We explore these generalizations of SPT-Ent in [13].

In the rest of the paper we give a formal presentation of our main results. The proofs are presented in the Supplementary Materials (SM) [12].

SPT-entanglement in arbitrary dimension: Different SPT phases can be classified based on the equivalence classes of states induced by the symmetric low-depth quantum circuits $[24]$. According to this classification, two states are in the same SPT phase if one can be approximately transformed to the other by a low-depth circuit $V = \prod_{i=1}^n U_i$ where each $U_i$ is a product of a set of unitaries which i) act locally on non-overlapping regions of the system, and ii) are invariant under the symmetry. The circuit should be low-depth in the sense that the depth $l$ times the maximum range of each unitary in the circuit is bounded by some range $R$ which in the thermodynamics limit is negligible compared to the system size.

Let $A$ and $B$ be two non-overlapping regions with arbitrary sizes, and $C$ be a connected region surrounded by $A$, $B$ and the boundaries of the system.

**Theorem 2** Suppose there exists a symmetric circuit with range bounded by $R$, which transforms state $\rho$ to a product state. Then, the SPT-Ent of state $\rho$ between
any two regions $A$ and $B$ with distance more than $4 R$ is zero, i.e. for any measure of entanglement $E$, it holds that $\text{dist}(A, B) > 4 R$ implies $E(\Omega^{(A'B'C)}(\rho)) = 0$.

A state is said to be in the trivial phase if it can be transformed to a product state using a low-depth quantum circuit. Therefore, this theorem implies that for all states in the trivial phase, the SPT-Ent between two regions with distance much larger than the correlation length, is zero. This proves corollary 1.

Theorem 2 implies that in the trivial phase SPT-Ent remains constant throughout the phase. Making an assumption about the homogeneity of state, we can extend this result to all SPT phases, and show that in general SPT-Ent is a universal quantity.

To show this we first note that, according to lemma 1 in SM [12] by making regions $A$ and $B$ larger we will monotonically increase SPT-Ent, that is for any non-overlapping regions $A'$ and $B'$ it holds that

$$A \subseteq A', B \subseteq B' \Rightarrow \Omega^{(A'B'C')}(\rho) \xrightarrow{\text{LOCC}} \Omega^{(A'B'C)}(\rho)$$

where $C' = C \setminus (A' \cup B')$ is the subregion of $C$ surrounded by $A'$, $B'$ and the boundaries of the system. The arrow means that the transformation from the first state to the second is possible via Local Operations and Classical Communication, where locality is defined relative to partition $A'|K_C|B'$, or equivalently relative to $A'|K_C|B'$. The above relation holds for arbitrary state $\rho$ and arbitrary regions $A$ and $B$. Generally, the SPT-Ent between the larger regions $A'$ and $B'$ is greater than the SPT-Ent between the smaller regions $A$ and $B$, and hence this transformation is not reversible via LOCC. However, assuming that the system is sufficiently homogenous and regions $A$ and $B$ are large and far from each other compared to the correlation length, one expects that a small increase in the sizes of $A$ and $B$ cannot increase SPT-Ent anymore. In other words, it seems natural to assume that at this limit the inverse transformation should also be possible via LOCC, i.e $\Omega^{(A'B'C)}(\rho) \xrightarrow{\text{LOCC}} \Omega^{(A'B'C')}(\rho)$. Indeed, our results prove the validity of this assumption in the case of 1-d systems.

According to the following theorem, this assumption implies that in all SPT phases SPT-Ent is a universal quantity. Let $A_l$ and $B_l$ be the balls of radius $l \text{ about } A$ and $B$ respectively. Let $C_l \equiv C \setminus (A_l \cup B_l)$ be the subregion of $C$ surrounded by $A_l$, $B_l$ and the boundaries of the system. Finally, let region $D$ be the complement of regions $A$, $B$ and $C$. We prove in SM [12] that

**Theorem 3** Suppose there exists a symmetric circuit with range bounded by $R$, which maps state $\rho_1$ to $\rho_2$. Assume $\rho_1$ and $\rho_2$ satisfy the homogeneity conditions $\Omega^{(A'B'C)}(\rho_{1,2}) \xrightarrow{\text{LOCC}} \Omega^{(A_l'B_l'C_l)}(\rho_{1,2})$, for some $l > 2R$. Furthermore, in the case where $D$ is non-empty, assume the distance between $C$ and $D$ is more than $2R$. Then, the SPT-Ent of $\rho_1$ and $\rho_2$ between $A$ and $B$ relative to $C$ are equal, i.e. $\Omega^{(A'B'C)}(\rho_1) \equiv \Omega^{(A'B'C)}(\rho_2)$.

**SPT-Ent in 1-d systems**: In the following we calculate SPT-Ent in the case of 1-d systems using MPS framework. We start by a short review of the results on the classification of SPT phases in MPS framework (We follow the presentation of [25]).

For a 1-d system with short range correlations by blocking the sites in the large blocks a translationally invariant ground state will converge to a fixed point in the form of

$$|\Psi\rangle = S^{\otimes N} |\lambda\rangle^{\otimes N}$$

where $|\lambda\rangle = \sum_k \lambda_k |k\rangle |k\rangle$ is a virtual entangled state between adjacent virtual sites with Schmidt coefficients $\{\lambda_k\}$, and $S$ is an isometry which maps two virtual sites $t_L$ and $t_R$ to a physical site $i$. This isometry can indeed be thought as a Renormalization Group (RG) transformation on a block [26]. It is shown that for any gapped 1-d system with a unique ground state, by blocking the physical sites the ground state can be approximated by a state in the form of $|\Psi\rangle$ with an accuracy which is exponential in the block sizes [25].

Consider a symmetry group $G$ with on-site linear unitary representation $g \rightarrow u(g)$. After blocking $L$ sites together this symmetry will be represented by $g \rightarrow U(g)$ on each block, where $U(g) = u(g)^{\otimes L}$. Assume $|\Psi\rangle$ is invariant under this symmetry, i.e. $U(g)^{\otimes N} |\Psi\rangle = |\Psi\rangle$. Then, it turns out that there always exists a projective representation $g \rightarrow V(g)$ of group $G$ such that $U_i(g) S = S [V_{iL}(g) \otimes V_{iR}^*(g)]$ and $[V_{iL}^*(g) \otimes V_{(i+1)L}(g)] |\lambda\rangle = |\lambda\rangle$ where $V^*(g)$ is the complex conjugate of $V(g)$ [13]. Note that for any phase $e^{i\theta(g)}$ the representation $g \rightarrow e^{i\theta(g)} V(g)$ will also satisfy the above equations and so $V(g)$ is defined only up to a phase. Let $\omega$ be the factor system of the representation $g \rightarrow V(g)$, i.e. $V(g) V(h) = \omega(g,h) V(gh)$. Then, the gauge transformation $g \rightarrow e^{i\theta(g)} V(g)$ induces an equivalence relation on the space of the factor systems: $\omega$ and $\omega'$ are equivalent if $\omega(g,h) = \omega'(g,h) e^{i(\theta(gh) - \theta(g) - \theta(h))}$ for some phase $e^{i\theta(\cdot)}$. We denote the equivalence class of the factor system $\omega$ by $[\omega]$. In lemma 11 in SM [12] we prove that for Abelian groups all irreps whose factor systems are in a given equivalence class $[\omega]$ have the same dimension, denoted by $d_{[\omega]}$. Note that $d_{[\omega]} = d_{[\omega']}$.

Let $A$ and $B$ be two non-neighbour blocks of a 1-d system and $C$ be a connected region between $A$ and $B$. Note that these regions are not necessarily large. We prove in SM [12] that

**Theorem 4** For any state $|\Psi\rangle$ in the form of Eq. (6), that is for any fixed point of the RG, the SPT-Ent between $A$ and $B$ relative to $C$, i.e. the bipartite entanglement of state $\Omega^{(A'B'C)}(\Psi)$, is equal to the entanglement of a maximally entangled state of a pair of $d_{[\omega]}$-dimensional systems.
Since the SPT phase of state $|\Psi\rangle$ is uniquely determined by the equivalence class $[\omega]$, this theorem implies that for states in the form of Eq. (6), the SPT-Ent depends only on the SPT phase of the state. Furthermore, since by blocking the sites, a general MPS ground state converges to a fixed point in the form of Eq (6), in the limit where $A,B$ and $C$ are large, this result applies to any general MPS with SPT order. It is worth pointing out that other approaches have been proposed recently for detecting SPT order based on the entanglement of the fixed point of the RG [28-30].

Conclusion: In this paper we introduced the concept of SPT-entanglement and proved that it remains constant throughout a SPT phase in arbitrary dimension. Also we calculated SPT entanglement in 1-d systems and showed that it can always detect the presence of SPT order. In the future works we show that this property of SPT ordered phases is closely related to the fact that certain SPT phases can be used as quantum wires in the measurement based quantum computation [22-30]. We will also show that the results on SPT Entanglement, can be used to find a lower bound on localizable entanglement [31].

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Symmetry-Protected Topological Entanglement: Supplementary Material

Basic properties of SPT-Entanglement (arbitrary dimensional systems)

In the following $\Pi^{(X)}_\alpha$ denotes the projector to the subspace with charge $\alpha$ in region $X$.

Monotonicity of SPT-Ent (Equation 5 in the paper)

Let $A$ and $B$ be two disjoint regions of the system and $C$ be a connected region surrounded by $A$, $B$ and the boundaries of the system.

Lemma 5 For any non-overlapping regions $A'$ and $B'$ it holds that

$$A \subseteq A', B \subseteq B' \Rightarrow \Omega^{(A'B'|C')} (\rho) \xrightarrow{\text{LOCC}} \Omega^{(AB|C)} (\rho),$$

where $C' \equiv C \setminus (A' \cup B')$ is the subregion of $C$ surrounded by $A'$, $B'$ and the boundaries of the system.

![Region partition](image)

FIG. 2: Region $A'$ is partitioned to region $A$ and region $\Delta A$. Similarly, region $B'$ is partitioned to region $B$ and region $\Delta B$. Region $C' \equiv C \setminus (A' \cup B')$ is the subregion of $C$ surrounded by $A'$, $B'$ and the boundaries of the system.

Proof. Recall that

$$\Omega^{(A'B'|C')} (\rho) = \sum_\tau q_\tau |\tau \rangle \langle \tau |^{(K_{C'})} \otimes \rho^{(A'B')}_\tau,$$

where

$$q_\tau \rho^{(A'B')}_\tau \equiv \text{tr}_{A'B'} (\Pi^{(C')}_\tau \rho),$$

where the trace is over all sites except the sites in regions $A'$ and $B'$. Similarly

$$\Omega^{(AB|C)} (\rho) = \sum_\kappa p_\kappa |\kappa \rangle \langle \kappa |^{(K_C)} \otimes \rho^{(AB)}_\kappa,$$

where

$$p_\kappa \rho^{(AB)}_\kappa \equiv \text{tr}_{ABC} (\Pi^{(C)}_\kappa \rho).$$

The key point here is that Abelian charges are additive, that is to measure the total charge of a region we can measure the charges of different parts of that regions and then add them up. In other word, measuring the total charge can be implemented via local measurements and classical communication, which is required to communicate the outcomes of these measurements and find the total charge.

Region $C$ can be partitioned as the union of the pairwise disjoint regions $C'$, $\Delta A$ and $\Delta B$, where

$$\Delta A \equiv C \cap (A' \setminus A), \quad \Delta B \equiv C \cap (B' \setminus B).$$
So, the total charge in region $C$ can be written as the sum of the charges in $C'$, $\Delta A$ and $\Delta B$. In other words,

$$\Pi^{(C)}_\kappa = \sum_{\tau, \tau_A, \tau_B} \Pi^{(C')}_{\tau} \otimes \Pi^{(\Delta A)}_{\tau_A} \otimes \Pi^{(\Delta B)}_{\tau_B}$$

(11)

where the summation is over all the charges $\tau, \tau_A$ and $\tau_B$ that adds up to $\kappa$. It follows that

$$p_\kappa^{(AB)} = \text{tr}_{AB}(\Pi^{(C)}_\kappa \rho) = \sum_{\tau, \tau_A, \tau_B} q_\tau \text{tr}_{AB}\left(\left[\Pi^{(\Delta A)}_{\tau_A} \otimes \Pi^{(\Delta B)}_{\tau_B}\right] \rho^{(A'B')}_{\tau}\right)$$

(12)

where the trace is over all the system except the sites in $A$ and $B$.

Based on this observation we can now explicitly define a LOCC which transforms the state $\Omega^{(A'B')(C')} (\rho)$ to the state $\Omega^{(AB)(C)} (\rho)$. Suppose two local parties acting on $A'$ and $B'$ measure the charges in regions $\Delta A$ and $\Delta B$ and obtain charges $\tau_A$ and $\tau_B$ respectively. Then, they implement the following transformation on the classical register:

$$|\tau\rangle \quad \rightarrow \quad |\tau_A + \tau_B + \tau\rangle.$$

Note that to implement this transformation they need to communicate via a classical channel.

Using Eqs. (12), it is straightforward to check that after this LOCC the joint state of regions $A$, $B$ and the classical register is described by $\Omega^{(AB)(C)} (\rho)$. This completes the proof. ■

**Effect of symmetric low-depth circuits on the SPT-Entanglement**

As we defined in the paper, a low-depth circuit is a circuit in which the depth $l$ times the maximum range of each unitary in the circuit is bounded by some range $R$ which in the thermodynamics limit is negligible compared to the system size. An important feature of the low-depth circuits is that under this type of transformations localized quantum information cannot propagate outside of a region with diameter of order $R$. Equivalently, in the Heisenberg picture, under these circuits a local observable evolves to an observable whose support is restricted to a region with the diameter of order $R$. This implies that

**Proposition 6** Consider two circuits $T$ and $T'$ whose ranges are bounded by $R$. If in the ball of radius $l > R$ around a region $A$ the local unitaries which form these two circuits are identical, then the two circuits act identically on region $A$, i.e. for any states $\rho$ of the system

$$\text{tr}_{A}\left(T\rho T^\dagger\right) = \text{tr}_{A}\left(T'\rho T'^\dagger\right).$$

(13)

where the trace is over all the sites in the system except the sites in region $A$.

Let $A$ and $B$ be two disjoint regions of the system, $C$ be a connected region surrounded by $A$, $B$ and the boundaries of the system, and let region $D$ be the rest of the system, i.e. all the sites not covered by $A$, $B$ and $C$.

Define $A_l$ and $B_l$ to be the balls of radius $l < \frac{1}{2}\text{dist}(A, B)$ about regions $A$ and $B$ respectively, where $\text{dist}(A, B)$ is the distance of $A$ and $B$. Let $C_l \equiv C \setminus (A_l \cup B_l)$ be the subregion of $C$ surrounded by $A_l$, $B_l$ and the boundaries of the system.

**Theorem 7** Let $V$ be a symmetric low-depth circuit with range bounded by $R$. Assume, in the case where $D$ is non-empty, $\text{dist}(D, C) > 2R$. Then, for any $l$ in the interval $2R < l < \text{dist}(A, B)/2$ it holds that

$$\Omega^{(A_lB_l|C_l)} (V \rho V^\dagger) \xrightarrow{LOCC} \Omega^{(AB)(C)} (\rho).$$

(14)

**Proof.** Consider the circuit $V = V_1 \cdots V_N$ where all $V_i : 1 \leq i \leq N$ are local, symmetric unitaries. By assumption the depth of this circuit times the maximum range of each unitary in the circuit is bounded by $R$. Consider the inverse circuit $V^\dagger = V_N^\dagger \cdots V_1^\dagger$. This is also a symmetric circuit whose range is bounded by $R$. Now based on this circuit we define the symmetric local circuits $V_{\text{inv}}^{(A)}$ and $V_{\text{inv}}^{(B)}$ using the following recipe: The circuit $V_{\text{inv}}^{(A)} (V_{\text{inv}}^{(B)})$ is obtained by replacing all the local unitaries in the circuit $V_N^\dagger \cdots V_1^\dagger$ whose support are not restricted to $A_l$ ($B_l$) by the identity operator. So, $V_{\text{inv}}^{(A)}$ and $V_{\text{inv}}^{(B)}$ are local symmetric circuits whose support are restricted to $A_l$ and $B_l$ respectively.
Next, we note that the circuit \((V_{\text{inv}}^{(A)}) \otimes V_{\text{inv}}^{(B)})V\) is a symmetric circuit with range \(2R\). It is straightforward to show that \((V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)})V\) has the following decomposition

\[
(V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)})V = S^{(C)} \otimes S^{(D)}
\]

where \(S^{(C)}\) is a symmetric unitary acting on \(C\) and \(S^{(D)}\) is a symmetric unitary acting on \(D\). This simply follows by applying the proposition 6 for the regions \(A\), \(B\), \(C\) and separately. For instance, in the case of region \(A\), proposition 6 implies that if \(l\) is larger than the range of \((V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)})V\) then the effect of \((V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)})V\) and \(V^\dagger V = I\), on region \(A\) are indistinguishable. So, for \(l > 2R\), \((V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)})V\) acts as identity on \(A\). A similar argument implies that \((V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)})V\) acts as identity on \(B\) too. Finally, note that the distance between \(C\) and \(D\) by assumption is larger than \(2R\), and since the range of \((V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)})V\) is bounded by \(2R\), it cannot generate any correlation between \(C\) and \(D\). This proves Eq. (15).

The important consequence of Eq. (15) is that, under \((V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)})V\) region \(C\) evolves unitarily, and since this unitary is symmetric, it follows that under \((V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)})V\) the charge in region \(C\) is conserved.

Now we can define the LOCC which transforms state \(\Omega^{(A)}\) over regions \(A\) and \(B\) respectively.

1. Two local parties apply local unitaries \(V_{\text{inv}}^{(A)}\) and \(V_{\text{inv}}^{(B)}\) on regions \(A_l\) and \(B_l\) respectively.
2. Then, they measure the total charges in regions \(\Delta A \equiv C \cap (A_l \setminus A)\), and \(\Delta B \equiv C \cap (B_l \setminus B)\).
3. Finally, they apply the following transformation on the classical register

\[
|\tau\rangle \mapsto |\tau + \tau + \tau\rangle
\]

where \(\tau_A\) and \(\tau_B\) are charges of regions \(\Delta A\) and \(\Delta B\) respectively.

In the following we show that after the above LOCC transformation, the joint state of regions \(A\) and \(B\) and the classical register is described by \(\Omega^{(AB|C)}(\rho)\).

Recall that

\[
\Omega^{(A_l B_l | C)}(V \rho V^\dagger) = \sum_{\tau} q_{\tau} |\tau\rangle \langle \tau| (K_{\tau}) \otimes \sigma_{\tau}(A_l B_l),
\]

where

\[
q_{\tau} \sigma_{\tau}(A_l B_l) = \text{tr}_{A_l B_l} \left( \Pi_{\tau}^{(C)} V \rho V^\dagger \right),
\]

and the trace is over all sites in the system except the sites in regions \(A_l\) and \(B_l\). Similarly

\[
\Omega^{(AB|C)}(\rho) = \sum_{\kappa} p_{\kappa} |\kappa\rangle \langle \kappa| (K_{\kappa}) \otimes \rho^{(AB)}_{\kappa},
\]

where

\[
p_{\kappa} \rho^{(AB)}_{\kappa} = \text{tr}_{\overline{ABC}} (\Pi_{\kappa}^{(C)} \rho).
\]

It is straightforward to see that after the above LOCC the joint state of regions \(A\) and \(B\) and the classical register will be described by

\[
\sum_{\tau, \tau_A, \tau_B} q_{\tau} |\tau + \tau + \tau\rangle \langle \tau + \tau + \tau| (K_{\tau}) \otimes \text{tr}_{\overline{AB}} \left( \Pi_{\tau}^{(A)} \otimes \Pi_{\tau}^{(B)} \right) |V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}\rangle \sigma_{\tau}(A_l B_l) |V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}\rangle^\dagger
\]

We now prove that this state is equal to \(\Omega^{(AB|C)}(\rho)\), which completes the proof the theorem. To prove this we need to show that

\[
p_{\kappa} \rho_{\kappa} = \sum_{\tau, \tau_A + \tau_B = \infty} q_{\tau} \times \text{tr}_{\overline{ABC}} \left( \Pi_{\tau}^{(A)} \otimes \Pi_{\tau}^{(B)} \right) |V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}\rangle \sigma_{\tau}(A_l B_l) |V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}\rangle^\dagger
\]
Here, we simplify the right-hand side
\[
\sum_{\tau, \tau_A, \tau_B} q_{\tau} \cdot \text{tr}_{\tau_A \tau_B}\left(\Pi_{\tau_A}^{(\Delta A)} \otimes \Pi_{\tau_B}^{(\Delta B)} \left[ [V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}] \sigma_{\tau}^{(A,B)} [V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}]^\dagger \right] \right)
\]
\[
= \sum_{\tau, \tau_A, \tau_B} \text{tr}_{\tau_A \tau_B}\left(\Pi_{\tau_A}^{(\Delta A)} \otimes \Pi_{\tau_B}^{(\Delta B)} \left[ [V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}] \Pi_{\tau}^{(\hat{C})} V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}]^\dagger \right] \right)
\]
\[
= \sum_{\tau, \tau_A, \tau_B} \text{tr}_{\tau_A \tau_B}\left(\Pi_{\tau_A}^{(\Delta A)} \otimes \Pi_{\tau_B}^{(\Delta B)} \left[ [V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}] [V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}]^\dagger \right] \right)
\]
\[
= \text{tr}_{\tau_A \tau_B}\left(\Pi_{\tau}^{(\hat{C})} [V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}] [V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}]^\dagger \right)
\]
\[
= \text{tr}_{\tau_A \tau_B}\left(\Pi_{\tau}^{(\hat{C})} [S(C) \otimes S(D)] [S(C) \otimes S(D)]^\dagger \right)
\]
\[
= \text{tr}_{\tau_A \tau_B}(\Pi_{\tau}^{(\hat{C})} \rho) = p_c \rho_c
\]
where to get the third line we have used the fact that the support of $\Pi_{\tau}^{(\hat{C})}$ is restricted to $\hat{C}$ and so it commutes with $V_{\text{inv}}^{(A)} \otimes V_{\text{inv}}^{(B)}$ (whose support is restricted to $\Delta A \cup \Delta B$). To get the fourth line we have used the additivity of Abelian charges, i.e.
\[
\Pi_{\tau}^{(\hat{C})} = \sum_{\tau, \tau_A, \tau_B} \Pi_{\tau}^{(\hat{C})} \otimes \Pi_{\tau_A}^{(\Delta A)} \otimes \Pi_{\tau_B}^{(\Delta B)}
\]  
(21)
To get the fifth line we have used Eq. (15), and finally to get the sixth line we have used the fact that $S(D)$ acts trivially on $C$ and the fact that since $S(C)$ is a symmetric unitary acting on region $C$, it preserves the charge in this region, i.e. for all charges $\tau$ it holds that
\[
[\Pi_{\tau}^{(C)}, S(C)] = 0.
\]  
(22)
This completes the proof. ■

**SPT-Entanglement vanishes in the trivial phase (Proof of theorem 2)**

In this section, we first repeat the statement of theorem 2 and then prove it.

**Theorem 2** Suppose there exists a symmetric circuit with range bounded by $R$, which transforms state $\rho$ to a product state. Then, the SPT-Ent of state $\rho$ between any two regions $A$ and $B$ with distance more than $4R$ is zero, i.e. for any measure of entanglement $E$, it holds that $\text{dist}(A,B) > 4R$ implies $E(\Omega^{(AB)}(\rho)) = 0$.

**Proof.** For simplicity we assume $A$, $B$, and $C$ cover the entire system, and so $D$ is empty. Otherwise, if this is not the case, we can extend region $D$ to $A$ and $B$. Assuming the result holds in the case where $D$ is empty, SPT-Entanglement is zero between these extended regions. But, since tracing over the added region cannot increase entanglement, it follows that SPT-Entanglement is also zero between the original regions $A$ and $B$.

Suppose $V$ is the local symmetric circuit with range bounded by $R$ which maps state $\rho$ to a product state denoted by $\sigma$, i.e. $V\rho V^\dagger = \sigma$, or equivalently $\sigma = V^\dagger V\rho$. Note that $V^\dagger$ is also a symmetric circuit with range bounded by $R$.

Then from theorem 1 we find that for all $l$ in the interval $2R < l < \text{dist}(A,B)/2$ it holds that
\[
\Omega^{(A_l B_l |\hat{C}_l)}(\sigma) \xrightarrow{\text{LOCC}} \Omega^{(AB)}(\rho).
\]  
(23)
where $A_l$ and $B_l$ are the balls of radius $l$ about regions $A$ and $B$ respectively, and $\hat{C}_l \equiv C \setminus (A_l \cup B_l)$ is the subregion of $C$ surrounded by $A_l$, $B_l$ and the boundaries of the system. But, since $\sigma$ is unentangled, it follows that $\Omega^{(A_l B_l |\hat{C}_l)}(\sigma)$ is also unentangled. Since $\Omega^{(AB)}(\rho)$ can be obtained from $\Omega^{(A_l B_l |\hat{C}_l)}(\sigma)$ via LOCC, it follows that $\Omega^{(AB)}(\rho)$ is also unentangled. This completes the proof. ■
SPT-Entanglement is a universal quantity in all SPT phases (Proof of theorem 3)

In this section, we first repeat the statement of theorem 3 and then prove it.

Let \( A_l \) and \( B_l \) be the balls of radius \( l \) about regions \( A \) and \( B \) respectively. Let \( \tilde{C}_l \equiv C \setminus (A_l \cup B_l) \) be the subregion of \( C \) surrounded by \( A_l, B_l \) and the boundaries of the system. Finally, let region \( D \) be the complement of regions \( A, B \) and \( C \). Then

**Theorem 3** Suppose there exists a symmetric circuit with range \( R \) which maps state \( \rho_1 \) to \( \rho_2 \). Assume \( \rho_1 \) and \( \rho_2 \) satisfy the homogeneity conditions \( \Omega^{(AB|C)}(\rho_{1,2}) \xrightarrow{\text{LOCC}} \Omega^{(AB|\tilde{C}_l)}(\rho_{1,2}) \), for some \( l > 2R \). Furthermore, in the case where \( D \) is non-empty, assume the distance between \( C \) and \( D \) is more than \( 2R \). Then, the SPT-Ent of \( \rho_1 \) and \( \rho_2 \) between \( A \) and \( B \) relative to \( C \) are equal, i.e., \( \Omega^{(AB|C)}(\rho_1) \xrightarrow{\text{LOCC}} \Omega^{(AB|C)}(\rho_2) \).

**Proof.** Let \( V \) be the unitary with depth \( R \) which transforms \( \rho_1 \) to \( \rho_2 \), i.e. \( V\rho_1V^\dagger = \rho_2 \). Then according to theorem 7 for any \( l \) in the interval \( 2R < l < \text{dist}(A, B)/2 \) there exists LOCC which realizes the transformation

\[
\Omega^{(A_lB_l|\tilde{C}_l)}(\rho_2) \xrightarrow{\text{LOCC}} \Omega^{(AB|C)}(\rho_1) \tag{24}
\]

But according to the assumptions, for some \( l > 2R \) there exists LOCC which realizes the transformation

\[
\Omega^{(AB|C)}(\rho_2) \xrightarrow{\text{LOCC}} \Omega^{(A_lB_l|\tilde{C}_l)}(\rho_2) \tag{25}
\]

The above two equations together imply that there exists LOCC which realizes the transformation

\[
\Omega^{(AB|C)}(\rho_2) \xrightarrow{\text{LOCC}} \Omega^{(AB|C)}(\rho_1) \tag{26}
\]

On the other hand, \( \rho_1 = V^\dagger \rho_2 V \) and since \( V \) is a symmetric unitary with range bounded by \( R \), so is \( V^\dagger \). Therefore, we can repeat the above argument and show that there exists LOCC which realizes

\[
\Omega^{(AB|C)}(\rho_2) \xrightarrow{\text{LOCC}} \Omega^{(AB|C)}(\rho_1) \tag{27}
\]

This completes the proof. □

**One-dimensional systems**

In this section of supplementary materials we prove theorem 4 in the paper. This theorem determines the SPT-entanglement of states in the form of

\[
|\Psi\rangle = S^\otimes N|\lambda\rangle^\otimes N \tag{28}
\]

where \(|\lambda\rangle = \sum_k \lambda_k|k\rangle|k\rangle\) is a “virtual” entangled state between adjacent virtual sites with Schmidt coefficients \(\{\lambda_k\}\), and \(S\) is an isometry which maps two virtual sites \(i_L\) and \(i_R\) to the physical site \(i\) (See Fig. 3). To find the

![FIG. 3: The isometry S maps two virtual sites \(i_L\) and \(i_R\) to the physical site \(i\).](image)

SPT-Entanglement of state \(\Psi\) between \(A\) and \(B\) relative to \(C\), we need to calculate the bipartite entanglement of state

\[
\Omega^{(AB|C)}(\Psi) \equiv \sum_{\kappa} p_\kappa |\kappa\rangle \langle \kappa|^{(K_C)} \otimes \rho_\kappa^{(AB)} \tag{29}
\]
relative to $AK_C|B$ partition, or equivalently, relative to $A|K_CB$ partition, where

$$\rho^{(AB)}_{\kappa} = \frac{1}{p_\kappa} \text{tr}_{\mathcal{T}}(\Pi^{(C)}_\kappa |\Psi\rangle\langle\Psi|\Pi^{(C)}_\kappa).$$

(30)

In the following, we show that the entanglement of this state is equal to the entanglement of a maximally entangled state of a pair of $d^\omega$-dimensional systems. Recall that $d^\omega$ is the dimension of the irreps of $G$ whose factor systems belong to $[\omega]$, the equivalence class that the factor system of the representation $g \to V(g)$ belongs to. Also, recall that $d^\omega = d^{\omega^*}$ (See lemma 11).

In the following we first show that using LOCC the state $|\text{ME}_{d^\omega}\rangle$, a maximally entangled state of a pair of $d^\omega$-dimensional systems, can be transformed to the state $\Omega^{(AB)(C)}(\Psi)$. We denote this transformation by

$$|\text{ME}_{d^\omega}\rangle \xrightarrow{\text{LOCC}} \Omega^{(AB)(C)}(\Psi).$$

(31)

To prove this we use the following lemma

**Lemma 8** Consider projective unitary representations of an Abelian symmetry $G$ on two local systems $R$ and $\bar{R}$ and assume the factor systems of these representations respectively belong to the equivalence classes $[\omega]$ and $[\omega^*]$. Then, the total charge of $R$ and $\bar{R}$ can be measured via LOCC by consuming a maximally entangled state of a pair of $d^\omega$-dimensional systems.

The lemma is proven at the end of the supplementary materials.

Then, we prove the reverse, i.e. we show that using LOCC the state $\Omega^{(AB)(C)}(\Psi)$ can be transformed to the state $|\text{ME}_{d^\omega}\rangle$. We denote this transformation by

$$\Omega^{(AB)(C)}(\Psi) \xrightarrow{\text{LOCC}} |\text{ME}_{d^\omega}\rangle.$$  

(32)

To prove this we use the following lemma proven at the end of the supplementary materials

**Lemma 9** Let $g \to T_R(g)$ and $g \to T_{\bar{R}}(g)$ be the projective unitary representations of an Abelian symmetry $G$ on the local systems $R$ and $\bar{R}$ respectively. Assume the factor systems of these representations belong to the equivalence classes $[\omega]$ and $[\omega^*]$ respectively. Let $\rho$ be an arbitrary (pure or mixed) state of the joint system of $R$ and $\bar{R}$ with a definite charge, i.e. $\forall g \in G$

$$[T_R(g) \otimes T_{\bar{R}}(g)] \rho = \rho [T_R(g) \otimes T_{\bar{R}}(g)] = e^{i\kappa(g)} \rho$$

for some 1-d representation $g \to e^{i\kappa(g)}$. Then, there exists an LOCC which transforms the state $\rho$ to a maximally entangled state of a pair of $d^\omega$-dimensional systems.

Equations (31) and (32) together prove theorem 1. We proceed by proving these equations and then we prove lemmas 8 and 9.

**Proof of**  $|\text{ME}_{d^\omega}\rangle \xrightarrow{\text{LOCC}} \Omega^{(AB)(C)}(\Psi)$

Suppose two local parties, Alice and Bob, each has a pair of systems $(R_A, \bar{R}_A)$ and $(R_B, \bar{R}_B)$ where the Hilbert spaces of all of these four systems are isomorphic to the Hilbert space of the virtual systems which describe the state $\Psi$. Assume the symmetry $G$ is represented by $g \to V(g)$ on $R_A/B$ and by $g \to V^*(g)$ on $\bar{R}_A/B$. Suppose Alice and Bob each prepares their local systems $\bar{R}_{A/B} \otimes R_{A/B}$ in the state $|\lambda\rangle$ (See Fig. 4). Note that $|\lambda\rangle$ satisfies $\forall g \in G : |V^*(g) \otimes V(g)|\lambda\rangle = |\lambda\rangle$. Then, Alice and Bob measure the total charge in the joint system formed by $R_A$ and $\bar{R}_B$ (See Fig. 4).

In general, when the representation $g \to V(g)$ is non-Abelian, this measurement is non-local, i.e. it cannot be realized via LOCC alone. However, according to the lemma 8 by consuming a maximally entangled state of a pair of $d^\omega$-dimensional systems this measurement can be realized via LOCC. Then, after this measurement Alice and Bob discard $R_A$ and $\bar{R}_B$ but they keep the record of the outcome of the charge measurement. Let $\Pi^{(R_A\pi_B)}_\kappa$ be the projector to the subspace in which the total system formed by $R_A$ and $\bar{R}_B$ has charge $\kappa$. Then, the probability of
observing charge $\kappa$ in this measurement is $q_\kappa = \langle \lambda | R_\kappa | \psi \rangle^2$. Furthermore, given the charge $\kappa$ is obtained in this measurement the state of $R_A R_B$ after the measurement will be

$$\sigma^{(AB)}_\kappa = \frac{1}{q_\kappa} \text{tr}_{R_A R_B} \left( \Pi^{(R_A R_B)}_\kappa \langle \lambda | \lambda \rangle \right)$$

We will prove that for all charges $\kappa$ it holds that $q_\kappa = p_\kappa$ and furthermore

$$\rho^{(AB)}_\kappa = \mathcal{E}_A \otimes \mathcal{E}_B (\sigma^{(AB)}_\kappa)$$

where $\mathcal{E}_A$ and $\mathcal{E}_B$ are local operations acting on Alice’s and Bob’s systems respectively. This implies that $\mathcal{E}_A \otimes \mathcal{E}_B$ transforms $\sum_\kappa q_\kappa | \kappa \rangle \langle \kappa | \otimes \sigma^{(AB)}_\kappa$ to $\Omega^{(AB)}(\Psi)$. This in turn implies that there exists a LOCC which consumes a maximally entangled state of a pair of $d_{ijkl}$-dimensional systems and create $\Omega^{(AB)}(\Psi)$ which proves Eq.(31).

So, to complete the proof we need to show Eq.(34) holds. The fact that $q_\kappa = p_\kappa$ can also be shown similarly.). Let $\Pi^{(C)}_\kappa$ be the projector to the subspace of states in which the region C has the total charge $\kappa$. This means that

$$\Pi^{(C)}_\kappa \left( \bigotimes_{i \in C} U_i(g) \right) = \left( \bigotimes_{i \in C} U_i(g) \right) \Pi^{(C)}_\kappa = e^{i \kappa(g)} \Pi^{(C)}_\kappa$$

where $g \rightarrow e^{i \kappa(g)}$ is the 1-dimensional representation of the symmetry group $G$ corresponding to the charge $\kappa$ and $g \rightarrow U_i(g)$ is the representation of the symmetry on the physical site $i$. Similarly, let $\Pi^{(C)}_\kappa$ be the projector to the subspace corresponding to the charge $\kappa$ of the representation $g \rightarrow \bigotimes_{i \in C} V_{i_L}(g) \otimes V_{i_R}^*(g)$ where $g \rightarrow V_{i_L}(g)$ and $g \rightarrow V_{i_R}^*(g)$ are the representation of symmetry $G$ on the virtual sites $i_L$ and $i_R$ respectively. This means that

$$e^{i \kappa(g)} \tilde{\Pi}^{(C)}_\kappa = \left( \bigotimes_{i \in C} V_{i_L}(g) \otimes V_{i_R}^*(g) \right) \tilde{\Pi}^{(C)}_\kappa$$

$$= \tilde{\Pi}^{(C)}_\kappa \left( \bigotimes_{i \in C} V_{i_L}(g) \otimes V_{i_R}^*(g) \right)$$

As we have seen before $g \rightarrow V(g)$ is defined such that

$$U_i(g) S = S \left[ V_{i_L}(g) \otimes V_{i_R}^*(g) \right]$$
This implies that
\[ \Pi_k^{(C)} S^\otimes N = S^\otimes N \tilde{\Pi}_k^{(C)} \] (38)

Since \(|\Psi\rangle = S^\otimes N |\lambda\rangle^\otimes N\) we conclude that
\[ \Pi_k^{(C)} |\Psi\rangle = S^\otimes N \tilde{\Pi}_k^{(C)} |\lambda\rangle^\otimes N \] (39)

Comparing the above equation with the definition of \(\rho_k^{(AB)}\) given by Eq. (30) and noting that \(S\) is an isometry we can conclude that
\[ \rho_k^{(AB)} \cong \frac{1}{p_k} \text{tr}_{AB} \left( \tilde{\Pi}_k^{(C)} |\lambda\rangle \langle \lambda|^\otimes N \tilde{\Pi}_k^{(C)} \right) \] (40)

where the equality holds up to local isometries.

Let \(\tilde{\rho}_k^{(a_R b_L)}\) be the reduced state of the virtual sites \(a_R\) and \(b_L\) when the total system is in the state \(\tilde{\Pi}_k^{(C)} |\lambda\rangle / \sqrt{p_k}\)
\[ \tilde{\rho}_k^{(a_R b_L)} \cong \frac{1}{p_k} \text{tr}_{a_R b_L} \left( \tilde{\Pi}_k^{(C)} |\lambda\rangle \langle \lambda|^\otimes N \tilde{\Pi}_k^{(C)} \right) \] (41)

where the trace is over all the virtual sites except \(a_R\) and \(b_L\). Then, one can easily show that there exists local operations \(E_A\) and \(E_B\) such that
\[ \tilde{\rho}_k^{(AB)} = E_A \otimes E_B (\tilde{\rho}_k^{(a_R b_L)}) \] (42)

On the other hand, using the fact that \(|\lambda\rangle\) has the trivial charge, it turns out that
\[ \tilde{\rho}_k^{(a_R b_L)} = \sigma_k^{(AB)} \] (43)

These two equations together prove Eq. (34) and this completes the proof of \(|\text{ME}_{d(w)}\rangle \xrightarrow{\text{LOCC}} \Omega^{(AB)(C)}(\Psi)\).

Proof of \(\Omega^{(AB)(C)}(\Psi) \xrightarrow{\text{LOCC}} |\text{ME}_{d(w)}\rangle\)

To show that \(\Omega^{(AB)(C)}(\Psi) \xrightarrow{\text{LOCC}} |\text{ME}_{d(w)}\rangle\) it is sufficient to show that for all \(\kappa\) with nonzero \(p_k \), the state \(\rho_k^{(AB)}\) can be transformed to \(|\text{ME}_{d(w)}\rangle\) under LOCC. But as we have seen in Eq. (40), \(\rho_k^{(AB)}\) is equal to \(\frac{1}{p_k} \text{tr}_{AB} \left( \tilde{\Pi}_k^{(C)} |\lambda\rangle \langle \lambda|^\otimes N \tilde{\Pi}_k^{(C)} \right)\) up to local isometries. So to prove \(\Omega^{(AB)(C)}(\Psi) \xrightarrow{\text{LOCC}} |\text{ME}_{d(w)}\rangle\) it is sufficient to prove that
\[ \frac{1}{p_k} \text{tr}_{AB} \left( \tilde{\Pi}_k^{(C)} |\lambda\rangle \langle \lambda|^\otimes N \tilde{\Pi}_k^{(C)} \right) \xrightarrow{\text{LOCC}} |\text{ME}_{d(w)}\rangle \] (44)

To prove this we show that
\[ \left[ V_{a_R}^* (g) \otimes V_{b_L}^* (g) \right] \tilde{\Pi}_k^{(C)} |\lambda\rangle^\otimes N = e^{-ik(g)} \tilde{\Pi}_k^{(C)} |\lambda\rangle^\otimes N \] (45)

This equation together with lemma 9 imply Eq. (44).

So, to complete the proof in the following we show Eq. (45) holds. We first note that Eq. (36) implies
\[ \left[ \bigotimes_{l \in C} V_{i_l}^* (g) \otimes V_{i_R}^* (g) \right] \tilde{\Pi}_k^{(C)} |\lambda\rangle^\otimes N = e^{ik(g)} \tilde{\Pi}_k^{(C)} |\lambda\rangle^\otimes N \] (46)

On the other hand,
\[ \left[ \bigotimes_{l \in C} V_{i_l} (g) \otimes V_{i_R}^* (g) \right] \tilde{\Pi}_k^{(C)} |\lambda\rangle^\otimes N \]
\[ = \tilde{\Pi}_k^{(C)} \left[ \bigotimes_{l \in C} V_{i_l} (g) \otimes V_{i_R}^* (g) \right] |\lambda\rangle^\otimes N \]
\[ = \tilde{\Pi}_k^{(C)} \left[ V_{a_R}^* (g^{-1}) \otimes V_{b_L} (g^{-1}) \right] |\lambda\rangle^\otimes N \]
\[ = [V_{a_R}^* (g^{-1}) \otimes V_{b_L} (g^{-1})] \tilde{\Pi}_k^{(C)} |\lambda\rangle^\otimes N \] (47)
where in the second line we have used Eq. (36), in the third line we have used the fact that
\[ \forall g \in G : \ (V_r^* (g) \otimes V_{(i+1,L)} (g)) \ |\lambda \rangle = |\lambda \rangle \] (48)
and in the fourth line we have used the fact that \( \Pi_{\kappa}^{(C)} \) acts trivially on the virtual sites \( a_R \) and \( b_L \). Equations (46) and (47) together imply Eq. (45) and this completes the proof.

Proof of lemma 8

Proof. Consider systems \( r \) and \( \tau \) with \( d_{[\omega]} \)-dimensional Hilbert spaces, \( \mathcal{H}_r \) and \( \mathcal{H}_{\tau} \). Assume the representation of the symmetry \( G \) on the systems \( r \) and \( \tau \) are respectively \( g \rightarrow t(g) \) and \( g \rightarrow t'(g) \), where \( g \rightarrow t(g) \) is an irrep of \( G \) whose factor system belongs to the equivalence class \([\omega]\). Let \( |\Theta \rangle \in \mathcal{H}_\tau \otimes \mathcal{H}_r \) be the state with the trivial charge, i.e. the state which satisfies

\[ \forall g \in G : \ [t^* (g) \otimes t(g)] |\Theta \rangle = |\Theta \rangle \] (49)

Since \( g \rightarrow t(g) \) is an irrep it follows that \( |\Theta \rangle \) is uniquely specified by the above equation up to a phase and, furthermore, it is a maximally entangled state.

Now assume \( r \) and \( \tau \) are in state \( |\Theta \rangle \). Since the total charge of \( r \) and \( \tau \) is trivial, the total charge of the joint systems \( R\mathcal{R}_{\tau}\tau \) is equal to the total charge of the joint systems \( R\mathcal{R}_r \). Now consider the representation of the symmetry \( G \) on \( R \) and \( \tau \) as a joint system on one side, and also its representation on \( \mathcal{R}_r \) and \( r \) as another joint system on the other side. One can easily see that the factor systems of both of these representations belong to the trivial equivalence class. Since the group is Abelian this means that both of these representations are Abelian. It follows that to measure \( \kappa \), the total charge of \( R\mathcal{R}_{\tau}\tau \), one can measure \( \kappa_A \), the total charge of \( \mathcal{R}_{\tau} \), and \( \kappa_B \), the total charge of \( \mathcal{R}_r \), locally and then adds them up together to obtain \( \kappa = \kappa_A + \kappa_B \) (See Fig. 4). This means that the total charge of \( R\mathcal{R}_r \) can be measured via LOCC and by consuming the entangled state \( |\Theta \rangle \).

Proof of lemma 9

We prove this using the following lemma.

Lemma 10 Any projective unitary representation \( g \rightarrow T(g) \) of an Abelian group \( G \) is equivalent to the tensor product of two representations as \( T(g) = W [t(g) \otimes t'(g)] W^\dagger \) where \( W \) is a unitary, \( g \rightarrow t(g) \) is a fixed irrep whose factor system is in the equivalence class that the factor system of the original representation \( g \rightarrow T(g) \) belongs to it, and \( g \rightarrow t'(g) \) is a linear (Abelian) unitary representation of \( G \).

This lemma is proven at the end of the supplementary materials.

Note that the decomposition \( T(g) = W [t(g) \otimes t'(g)] W^\dagger \) is not unique. In particular, by choosing a different unitary \( \tilde{W} \) one can obtain a decomposition as \( T(g) = \tilde{W} [t(g) \otimes r(g)] \tilde{W}^\dagger \) where the representations \( g \rightarrow r(g) \) and \( g \rightarrow t'(g) \) are not necessarily equivalent. (One can show that for a particular cohomology class, called the maximally non-commuting class, by a proper choice of the unitary \( W \) the representation \( g \rightarrow t'(g) \) can always be chosen to be trivial. However, in general the representation \( g \rightarrow t'(g) \) cannot be chosen to be trivial.)

This lemma implies that the representation \( g \rightarrow T(g) \) induces a decomposition of the Hilbert space as \( \mathcal{H} \cong \mathcal{M} \otimes \mathcal{N} \) where \( g \rightarrow t(g) \) acts irreducibly on \( \mathcal{M} \) and \( g \rightarrow t'(g) \) acts on \( \mathcal{N} \). Note that for a given representation \( g \rightarrow T(g) \) the decomposition of \( \mathcal{H} \) as \( \mathcal{H} \cong \mathcal{M} \otimes \mathcal{N} \) is uniquely specified. On the other hand, by decomposing the representation \( g \rightarrow t'(g) \) to 1-dimensional irreps of \( G \), one can decompose \( \mathcal{N} \) as \( \mathcal{N} \cong (\bigoplus_{\lambda} \mathcal{N}_\lambda) \) where the subspace \( \mathcal{N}_\lambda \) is the subspace on which the representation \( g \rightarrow t'(g) \) acts as 1-d irrep \( g \rightarrow e^{i\lambda(g)} \) (We assume each \( \lambda \) in the summation corresponds to a distinct 1-d irrep \( g \rightarrow e^{i\lambda(g)} \)). Considering this decomposition of \( \mathcal{H} \), an arbitrary vector \( |\phi \rangle \in \mathcal{H} \) can be expanded as

\[ |\phi \rangle = \sum_{i=1}^{d_{[\omega]}} \sum_{\lambda} \langle i | \otimes |\lambda \rangle \otimes |\phi(i,\lambda)\rangle \] (50)

where \( \{|i\rangle : 1 \leq i \leq d_{[\omega]}\} \) is a basis for \( \mathcal{M} \), \( \{|\lambda \rangle\} \) labels different inequivalent 1-d irrep of \( G \) corresponding to different sectors of \( \mathcal{N} \cong (\bigoplus_{\lambda} \mathcal{N}_\lambda) \) and \( |\phi(i,\lambda)\rangle \) is a vector in \( \mathcal{N}_\lambda \).
where the mixed states via a standard purification argument. These charges. This completes the proof of lemma 9 for pure states. This proof can be easily extended to the case of

\( \kappa \)

know

\( g \)

induce decompositions of the representations

\( \rho \)

\( \kappa \)

\( \theta \)

\( |\theta(\eta)\rangle \in M^{(R)} \otimes M^{(\overline{R})} \) to be the state which satisfies

\[ |\theta(\eta)\rangle = e^{i\eta(g)}|\theta(\eta)\rangle \]

where \( g \to e^{i\eta(g)} \) is an arbitrary 1-d representation of the Abelian group \( G \) which shows up in the representation \( g \to |\theta(\eta)\rangle \otimes t^*(g) \). Note that this state is maximally entangled and is uniquely defined by the above equation up to a phase.

Consider an arbitrary state \( |\phi\rangle \in H^{(R)} \otimes \overline{H}^{(R)} \) which has a definite charge \( \kappa \), i.e. it satisfies

\[ |T_R(g) \otimes T_{\overline{R}}(g)||\phi\rangle = e^{i\kappa(g)}|\phi\rangle \]

Then, one can easily show that \( |\phi\rangle \) has a decomposition in the following form

\[ |\phi\rangle = \sum_{\lambda_a, \lambda_b} |\Theta(\kappa - \lambda_a - \lambda_b)||\lambda_a, \lambda_b\rangle|\phi(\kappa, \lambda_a, \lambda_b)\rangle \]

where \( |\phi(\kappa, \lambda_a, \lambda_b)\rangle \in N^{(R)}_{\lambda_a} \otimes N^{(\overline{R})}_{\lambda_b} \) and we have used the notation introduced in Eq.(50).

According to this decomposition we can conclude that by performing local measurements \{\{\lambda_a\}\} and \{\{\lambda_b\}\} we project the state \( |\phi\rangle \) to \( |\Theta(\kappa - \lambda_a - \lambda_b)||\lambda_a, \lambda_b\rangle|\phi(\kappa, \lambda_a, \lambda_b)\rangle \). Therefore, after this measurement the subsystem \( M^{(R)} \otimes M^{(\overline{R})} \) is in the state \( |\Theta(\kappa - \lambda_a - \lambda_b)\rangle \) which is a maximally entangled. Since after this measurement two parties know \( \kappa, \lambda_a \) and \( \lambda_b \) they can transform this state via LOCC to a standard maximally entangled state independent of these charges. This completes the proof of lemma 9 for pure states. This proof can be easily extended to the case of mixed states via a standard purification argument.

**Proof of lemma 10**

The lemma 10 is an immediate consequence of the following lemma

**Lemma 11** Let \( g \to u^{(\beta)}(g) \) and \( g \to u^{(\gamma)}(g) \) be two irreps of an Abelian group \( G \) whose factor systems are in the same cohomology class. Then, their dimension is the same, and furthermore there exists a unitary \( W \) and a 1-d representation of group \( g \to e^{i\kappa(g)} \) such that

\[ \forall g \in G : u^{(\beta)}(g) = e^{i\kappa(g)}Wu^{(\gamma)}(g)W^\dagger. \]

**Proof.** Let \( u^{(\gamma^*)}(g) \) be the complex conjugate of \( u^{(\gamma)}(g) \). Then, one can easily see that \( g \to u^{(\beta)}(g) \otimes u^{(\gamma^*)}(g) \) is a representation of \( G \) whose factor system belongs to the trivial cohomology class. Since the group is Abelian all of its irreps in the trivial cohomology class are 1-dimensional and so this representation can be decomposed to 1-d irreps of \( G \). Let the normalized vector \( |\Theta(\kappa)\rangle \) be a 1-dimensional subspace on which \( g \to u^{(\beta)}(g) \otimes u^{(\gamma^*)}(g) \) acts irreducibly, i.e.

\[ \forall g \in G : \left[ u^{(\beta)}(g) \otimes u^{(\gamma^*)}(g) \right]|\Theta(\kappa)\rangle = e^{i\kappa(g)}|\Theta(\kappa)\rangle \]

where \( g \to e^{i\kappa(g)} \) is the corresponding 1-d representation of \( G \). But, since the representations \( \beta \) and \( \gamma \) are irreducible using the Schur’s lemma we can conclude that \( |\Theta(\kappa)\rangle \) should be a maximally entangled vector. This means that
the dimension of the irrep $\beta$ and $\gamma$ are the same. We denote this dimension by $d$. Since $|\Theta(\kappa)\rangle$ is a maximally entangled normalized vector it follows that there exists a unitary $W$ such that $|\Theta(\kappa)\rangle = (W \otimes I)|\psi(+)\rangle$ where $|\psi(+)\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle$ and $I$ is the identity operator acting on the $d$-dimensional space. Then Eq. (56) implies

$$\left[(W^\dagger u(\beta)(g)W) \otimes u(\gamma)^* (g)\right] |\psi(+)\rangle = e^{i\kappa(g)} |\psi(+)\rangle$$

(57)

But, since $(A \otimes B)|\psi(+)\rangle = (AB^T \otimes I)|\psi(+)\rangle$, where $B^T$ is the transpose of $B$ we find that

$$\left[\left(W^\dagger u(\beta)(g)W u(\gamma)^\dagger (g)\right) \otimes I\right] |\psi(+)\rangle = e^{i\kappa(g)} |\psi(+)\rangle$$

(58)

This implies that $W^\dagger u(\beta)(g)W u(\gamma)^\dagger (g) = e^{i\kappa(g)} I$ and so

$$\forall g \in G : \ u(\beta)(g) = e^{i\kappa(g)} W u(\gamma)(g)W^\dagger$$

(59)

which completes the proof. ■