On Physical Layer Security over Fox’s H-Function Wiretap Fading Channels

Long Kong, Student Member, IEEE, Georges Kaddoum, Member, IEEE, Hatim Chergui, Member, IEEE

Abstract—Most of the well-known fading distributions, if not all of them, could be encompassed by Fox’s H-function fading. Consequently, we investigate the exact and asymptotic behavior of physical layer security (PLS) over Fox’s H-function fading wiretap channels. In particular, closed-form expressions are derived for secrecy outage probability (SOP), probability of non-zero secrecy capacity (PNZ), and average secrecy capacity (ASC). These expressions are given in terms of either univariate or bivariate Fox’s H-function. In order to show the comprehensive effectiveness of our derivations, three metrics are respectively listed over the following frequently used fading channels, including Rayleigh, Weibull, Nakagami-m, α − µ, Fisher-Snedecor (F-S) F, and extended generalized-K (EGK). Our tractable results are more straightforward and general, besides that, they are feasible and applicable, especially the SOP, which was mostly limited to the lower bound in literature due to the difficulty of achieving closed-form expressions. In order to validate the accuracy of our analytical results, Monte-Carlo simulations are subsequently performed for the special case α − µ fading channels. One can observe perfect agreements between the exact analytical and simulation results, and highly accurate approximations between the exact and asymptotic analytical results.

Index Terms—Physical layer security, Fox’s H-function wiretap fading channels, Mellin transform, secrecy outage probability, probability of non-zero secrecy capacity, average secrecy capacity.

I. INTRODUCTION

SECRECY analysis of wireless system always varies from one statistical model to another, which is mainly due to the limitation of each fading model. For example, the gamma-gamma distribution was introduced to model the free space optical (FSO) communication link [1], [2], and Fisher-Snedecor (F-S) F to model the device-to-device communication [3], [4]. As such, many endeavors have been drawn to investigate the mathematical characteristics of secure transmission for different communication scenarios.

Dating back to the fundamental works of physical layer security (PLS) from information theoretical perspective, Shannon and Wyner are undoubtedly called the pioneers in this field [5], [6]. They had established the mathematical background of perfect secrecy and wiretap channel model. Later on, Wyner’s classic wiretap model was investigated over additive white Gaussian noise channel (AWGN) and Rayleigh fading channels [7], [8]. Over the past decades, plenty of research efforts have been pursued on the investigation of PLS over various fading channels, such as Rayleigh [8], Rician [9], Nakagami-m, Weibull [10], Lognormal [11], generalized-K [12]–[15], and α − µ (or, equivalently, generalized gamma) [16]–[20], etc. Secrecy outage probability (SOP), the probability of non-zero secrecy capacity (PNZ), and average secrecy capacity (ASC) are the three typical and frequently studied secrecy metrics.

As more new communication topologies appear, e.g., device-to-device (D2D) communications, FSO communications, intervehicle communication, millimeterwave (mmWave) communications, wireless body area networks (WBAN), and cognitive radios, the existing models become obsolete. As such, more advanced and better suited fading models were subsequently proposed and analyzed, such as α − µ [21], κ − µ/η − µ [22], F-S F [3], [4], the extended generalized-K (EGK) [23], and cascaded α − µ fading [24], among many other fading channels.

With the emergence of various fading models, a unified and generic fading model is required to subsume most, if not all, of these fading distributions. Fox’s H-function distribution, reported in [25]–[27], is one possible model to accommodate various fading models with high flexibility. It was first introduced in [28] and [29] as a pure mathematical finding, and can be generalized to Gamma, exponential, Chi-square, Weibull, Rayleigh, and Half-Normal distribution, etc. Other examples, including generalized-K, α − µ, F-S F, and EGK, were recently explored by Alhennawi et al. [25] and Rahama et al. [30], respectively. These findings were achieved by transforming these probability density distributions (PDFs) of received signal-to-noise ratios (SNRs) in the manner of Fox’s H-function.

The feasibility and applicability of Fox’s H-function distribution as a general fading model for wireless communication is not new. In [23], a variation of Fox’s H-function fading model was proposed as a general model for most well-known distribution. Jeong et al. found that Fox’s H-function distribution offers a better fading model of vehicle-to-vehicle (V2V) communication than other ordinary fading distributions [31]. More recently, Alhennawi et al. in [25] derived the symbol error rate (SER) and channel capacity of single- and multiple-branch diversity receivers when communicating over Fox’s H-function fading channels. As a consequence, the advantages of Fox’s H-function fading are threefold:

• The unity and the genericity of its form for most distribution, e.g., Rayleigh, Nakagami-m, Weibull, and α − µ, etc;
• the simplicity and the generality of it to derive to derive the key performance metrics of wireless communications

L. Kong and G. Kaddoum are with the LaCIME lab, Department of Electrical Engineering, Ecole de technologie supérieure (ETS), Université du Québec, Montréal (Québec), Montreal, Canada, H3C 1K3, e-mails: (long.kong.1@ens.etsmtl.ca, georges.kaddoum@etsmtl.ca).
H. Chergui is with the National Institute of Telecommunications (INPT), Rabat, Morocco. e-mail: chergui@ieee.org.
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systems, e.g., outage probability, SER, and channel capacity [25].

- The possibility of using its distribution to study the PLS analysis over $\alpha - \mu$, F-S $\mathcal{F}$ fading channels [18], [20], [24], [32].

To the best of the authors’ knowledge, apart from the investigation of PLS over the aforementioned fading channels [8]–[20], including generalized-$K$, $\alpha - \mu$, $\kappa - \mu$ [33]–[35], F-S $\mathcal{F}$ [24], no works has ever been found to analyze the PLS over the general Fox’s $H$-function fading channels.

### A. Our Work and Contributions

The contributions of this paper are multifold, which are listed as follows:

1) Novel exact and closed-form expressions are initially derived for the secrecy metrics, including the SOP, PNZ, and ASC. Our formulations, in terms of univariate or bivariate Fox’s $H$-function, are given in simple and tractable mathematical fashion.

2) The difficulty of deriving closed-form expressions for SOP explicitly lies in tractable integrals. Consequently, many works can be found on the development of lower bound of SOP ($\mathcal{P}_{\text{out}} = \text{Pr}(C_s \geq R_u)$). Since the lower bound of SOP is actually the complementary of probability of non-zero secrecy capacity, i.e., $\mathcal{P}_{\text{nz}} = \text{Pr}(C_s > 0)$, it is much easier to obtain the lower bound of SOP and PNZ, which can be found in [21]. Strictly speaking, our work fills this gap of lacking exact closed-form SOP expression over those fading channels.

3) The obtained general and unified secrecy metrics’ expressions are found identical with the existing works when being compared with Monte-Carlo simulation results. On the other hand, the obtained secrecy expressions can be straightforward applied to other transformable but not listed herein wiretap fading channels.

4) The asymptotic behavior of these secrecy metrics is also obtained for the sake of providing simple but highly accurate approximations of secrecy metrics at high average signal-to-noise (SNR) regime.

Resultantly, the obtained analytical expressions are especially beneficial since the analytical expressions themselves (i) provide a unified approach to analyze the PLS over the generalized fading model; (ii) serve as an efficient and convenient tool to validate and compare the special cases of Fox’s $H$-function fading channels; and (iii) enable researchers and wireless communication engineers to quickly evaluate secrecy performance when encountering security risks.

### B. Structure and Notations

The rest of this paper is structured as follows: Section II simply illustrates Fox’s $H$-function fading and its Mellin transform. In Section III system model and problem formulation are presented. Secrecy analysis is conducted in Sections IV and V together with several examples. Afterwards, in Section VI numerical results and discussions are presented. Finally, Section VII concludes the paper.

### II. PRELIMINARY

#### A. Fox’s $H$-function Fading

Consider a wireless communication link over a fading channel, where the instantaneous SNR at user $k$, $\gamma_k$, follows Fox’s $H$-function PDF, given by [28]

$$f_k(\gamma_k) = \kappa H_{p,q}^{m,n}(\lambda \gamma_k), \quad \lambda > 0, \quad \kappa > 0,$$

where $\lambda$ and $\kappa$ are constants such that $\int_0^\infty f_k(\gamma_k) d\gamma_k = 1$. $\gamma_k$ is a shorthand for $(x_1, y_1), \cdots, (x_l, y_l)$. Step (a) is developed by expressing Fox’s $H$-function in terms of its definition [36] eq. (1.2)]. $A_i > 0$ for all $i = 1, \cdots, p$, and $B_l > 0$ for all $l = 1, \cdots, q$. $0 \leq m \leq q$, $0 \leq n \leq p$, $L$ is a suitable contour separating the poles of the gamma functions $\Gamma(b_l + B_l s)$ from the poles of the gamma functions $\Gamma(1 - a_i - A_i s)$,

$$\Theta_k(s) = \frac{\prod_{i=1}^{m} \Gamma(b_l + B_l s) \prod_{i=1}^{n} \Gamma(1 - a_l - A_l s)}{\prod_{i=m+1}^{q} \Gamma(1 - b_l - B_l s) \prod_{i=n+1}^{p} \Gamma(a_l + A_l s)}.$$ (2)

The cumulative distribution function (CDF) of the received SNR at user $k$, i.e., $\gamma_k$ is given by [28] eqs. (3.9) and (3.7)]

$$F_k(\gamma_k) = \frac{\kappa}{\lambda} H_{p+l,q+1}^{m+n+1}(\lambda \gamma_k) \begin{bmatrix} (1,1),(a_i+A_i)_{p} \end{bmatrix}_{(b_l+B_l)_{q},(0,1)}, \quad (3a)$$

or

$$F_k(\gamma_k) = 1 - \frac{\kappa}{\lambda} H_{p+l,q+1}^{m+n+1}(\lambda \gamma_k) \begin{bmatrix} (a_i+A_i)_{p},(1,1) \end{bmatrix}_{(b_l+B_l)_{q},(0,1)}, \quad (3b)$$

where $F_k(\gamma)$ is the complementary CDF (CCDF). For the convenience of notations, $\Theta_k^f$ and $\Theta_k^m$ are used thereafter to denote that for the PDF and CDF from Fox’s $H$-function, respectively. The Mellin transform of $f_k(\gamma)$ is defined and given as [25] eq. (5)] [36] eq. (2.8)].

$$M[f_k(\gamma_k), s] = \int_0^\infty f_k(\gamma_k) \gamma^{s-1} d\gamma_k = \kappa \lambda^{-s} \Theta_k(s). \quad (4)$$
TABLE I: Exact expressions of $f_k(\gamma_k)$ for different special cases of Fox’s $H$-function distribution

| In. SNR | $f_k(\gamma_k)$ |
|---------|----------------|
| $\alpha - \mu$ | $f_k(\gamma_k) = \kappa H_{0.1}^{1,1} \left( \text{ayk} \left( \mu - \frac{1}{\mu}, \frac{1}{\mu} \right) \right)$, where $\kappa = \frac{\mu}{\gamma_k \epsilon}, \lambda = \frac{\mu}{\gamma_k \epsilon}, \beta = \frac{\left(\mu + \frac{1}{\mu}\right)}{\epsilon}$, $\text{F-S } F$ |
| $\text{EGK}$ | $f_k(\gamma_k) = \kappa H_{0.2}^{1,1} \left( \text{ayk} \left( m_k - \frac{1}{\gamma_k}, 1 \right) \right)$, where $\kappa = \frac{\gamma_k m_k}{\gamma_k m_k - 1}, \lambda = \frac{\gamma_k m_k}{\gamma_k m_k - 1}, \beta = \frac{\lambda}{\gamma_k m_k}$, $\text{EGK}$ |

B. Special cases

As mentioned before, Fox’s $H$-function distribution provides enough flexibility to accommodate most fading distributions. As a result, the objective herein is to list some well-known examples, such as $\alpha - \mu$, F-S $F$, and EGK, as shown in Table. I, where $\gamma_k$ is the average received SNR at user $k$.

III. SYSTEM MODEL AND PROBLEM FORMULATION

A. System Model

The Alice-Bob-Eve classic wiretap model is used here to illustrate a legitimate transmission link (Alice $\rightarrow$ Bob) in the presence of a malicious eavesdropper. In such a wiretap channel model, the transmitter Alice (A) wishes to send secret messages to the intended receiver Bob (B) in the presence of an eavesdropper Eve (E); the link between A and B is called the main channel, whereas the one between A and E is named as the wiretap channel. It is assumed that (i) all users are equipped with a single antenna; (ii) both links are independent and subjected to Fox’s $H$-function fading; (iii) a perfect channel state information (CSI) is available at all users.

As a result, the received SNRs at B and E are denoted as $\gamma_k, k \in \{B, E\}$, which follow Fox’s $H$-function PDF, and are respectively given by

$$f_B(\gamma_B) = \kappa B H_{1,1}^{m_k, m_k} \left( \text{ayB} \left( a_k, a_k \right) \right), \quad \gamma_B > 0, \quad (5a)$$

$$f_E(\gamma_E) = \kappa E H_{1,1}^{m_k, m_k} \left( \text{ayE} \left( a_k, a_k \right) \right), \quad \gamma_E > 0. \quad (5b)$$

B. Problem Formulation

According to [3], the secrecy capacity over fading wiretap channels is defined as the difference between the main channel capacity $C_M = \log_2(1 + \gamma_B)$ and the wiretap channel capacity $C_W = \log_2(1 + \gamma_E)$ as the following form,

$$C_S = \begin{cases} C_M - C_W, & \gamma_B > \gamma_E \\ 0, & \text{otherwise}. \end{cases} \quad (6)$$

In other words, a positive secrecy capacity can be assured if and only if the received SNR at Bob has a superior quality than that at Eve’s.

1) Secrecy Outage Probability: The outage probability of the secrecy capacity is defined as the probability that the secrecy capacity $C_S$ falls below the target secrecy rate $R_t$, i.e.,

$$\mathcal{P}_{out}(R_t) = Pr(C_S < R_t). \quad (7)$$

Technically speaking, SOP can be conceptually explained as two cases: (i) $C_S < R_t$ whilst positive secrecy capacity is surely guaranteed; (ii) secrecy outage definitely happens when $C_S$ is non-positive. To this end, (7) can be rewritten as follows [12], [38],

$$\mathcal{P}_{out}(R_t) = Pr(\gamma_B \leq R_t \gamma_E + R_s - 1) = \int_0^{R_t} F_B(\gamma_B) f_E(\gamma_E) d\gamma_E, \quad (8)$$

where $R_s = 2^{R_t}$, $\gamma_0 = R_t \gamma_E + \mathcal{W}$, and $\mathcal{W} = R_s - 1$.

2) Probability of Non-Zero Secrecy Capacity: The PNZ refers to the event that the positive secrecy capacity can be surely achieved, namely $Pr(C_S > 0)$, thus respecting its definition, (6) can be further rewritten as follows,

$$\mathcal{P}_{nz} = Pr(\gamma_B > \gamma_E) = \int_0^{\infty} \int_0^{R_s} f_B(\gamma_B) f_E(\gamma_E) d\gamma_E d\gamma_B = \int_0^{\infty} f_B(\gamma_B) F_E(\gamma_E) d\gamma_E. \quad (9)$$

3) Average Secrecy Capacity: By using some simple mathematical manipulations, the ASC can be further re-expressed as the sum of three terms, which are given by [13]

$$\bar{C}_S = \int_0^{\infty} \log_2(1 + \gamma_B) f_B(\gamma_B) F_E(\gamma_E) d\gamma_B$$

$$\quad + \int_0^{\infty} \log_2(1 + \gamma_E) f_E(\gamma_E) F_B(\gamma_B) d\gamma_E$$

$$\quad - \int_0^{\infty} \log_2(1 + \gamma_B) f_E(\gamma_E) d\gamma_E. \quad (10)$$

For the brevity of following derivations, let $g_k(\gamma_k) = \ln(1 + \gamma_k) f_B(\gamma_k)$.

IV. SECRECY METRICS CHARACTERIZATION

To begin the characterization of secrecy performance over Fox’s $H$-function fading channels, one useful and unified theorem is first provided. This theorem is essentially beneficial to the acquisition of the final closed-form expressions for the aforementioned three secrecy metrics.
Theorem 1. Consider a general fading channel where the received SNR’s PDF is \( f(y) \) and another function \( u(y) \). Suppose their Mellin transform are \( M[f(y), s] \) and \( M[u(y), s] \), respectively. If the Mellin transform of \( u(y) \) exists, then by using Parseval’s formula for Mellin transform [39, eq. (8.3.23)], we have
\[
\int_0^\infty f(y)u(y)dy = \frac{1}{2\pi j} \int_L M[f(y), s]M[u(y), 1-s]ds,
\]
where \( L \) is the integration path from \( -\infty \) to \( +\infty \), and \( \nu \) is a constant.

The aforementioned Theorem is recalled to make a basis for the following derivations. To this end, we have the following remark.

Remark 1. The SOP, PNZ, and ASC over Fox’s H-function fading wiretap channels are respectively given by
\[
\begin{align*}
\mathcal{P}_{out} &= \frac{1}{2\pi j} \int_L M[F_B(\gamma_B), 1-s]M[f_E(\gamma_E), s]ds, \\
\mathcal{P}_{nz} &= \frac{1}{2\pi j} \int_L M[F_E(\gamma_B), 1-s]M[f_B(\gamma_B), s]ds, \\
\mathcal{C}_s &= \frac{1}{2\pi j} \int_L M[g_B(\gamma_E), 1-s]M[F_E(\gamma_B), s]ds \\
&\quad + \frac{1}{2\pi j} \int_L M[g_E(\gamma_E), 1-s]M[F_B(\gamma_B), s]ds \\
&\quad - \frac{1}{2\pi j} \int_L M[f_E(\gamma_E), 1-s]M[\ln(1+\gamma_E), s]ds
\end{align*}
\]
\[\text{Proof.} \text{ Recalling (8), (9), and (10), and then using Theorem 1 the proofs for (12a), (12b), and (12c) are directly accomplished.}\]

A. SOP Characterization

Theorem 2. The SOP over Fox’s H-function fading wiretap channels is given by [13], shown at the top of next page.

Proof. See Appendix A.

1) Lower Bound of SOP: As \( \gamma_B \) and \( \gamma_E \) tend to \( \infty \), we have
\[
\mathcal{P}_{out} = Pr \left( \log_2 \left( \frac{1 + \gamma_B}{1 + \gamma_E} \right) < R_0 \right) \\
= Pr \left( \log_2 \left( \frac{\gamma_B}{\gamma_E} \right) < R_0 \right) \\
= \int_0^\infty F_B(R_0, y)f_E(y)dy.
\]

Proposition 1. As \( \gamma_B \) and \( \gamma_E \) tend to \( \infty \), the lower bound of SOP over Fox’s H-function fading channels is given by [15].

Proof. By applying the Mellin transform of the product of two Fox’s H-function [40, eq. (2.25.1.1)], the proof is accomplished.

B. PNZ Characterization

Theorem 3. The PNZ over Fox’s H-function wiretap fading channels is given by [16], which is shown at the top of next page.

Proof. According to (12b), \( M[F_E(\gamma_B), 1-s] \) and \( M[f_B(\gamma_B), s] \) are separately given by
\[
\begin{align*}
M[F_E(\gamma_B), 1-s] &= \frac{k_B}{\lambda_E^{s-1}} F_B(1-s), \\
M[f_B(\gamma_B), s] &= \frac{k_B}{\lambda_E^s} F_B(s).
\end{align*}
\]

Next, substituting (17a) and (17b) into (12b), yields the following result
\[
\mathcal{P}_{nz} = \frac{k_Bk_E}{2\lambda_E^{1+\nu}} \int_L \Theta_B(s)F_B(1-s) \left( \frac{\lambda_B}{\lambda_E} \right)^{1-s} ds,
\]
\[\text{Subsequently, directly applying the definition of univariate Fox’s H-function, the proof is achieved.}\]

Alternatively, we provide another method to prove (16). Revisiting (9) and directly replacing \( f_B(\gamma_B) \) and \( F_E(\gamma_B) \) with their expressions, we have
\[
\begin{align*}
\mathcal{P}_{nz} &= \frac{k_Bk_E}{\lambda_E^2} \int_0^\infty H_{p_1,q_1}^{m_1,n_1} \left[ \frac{\lambda_B\gamma_B}{(a_i, A_i)=1; p_1} \right] \\
&\times H_{p_2+1,q_2+1}^{m_2,n_2,1} \left[ \frac{\lambda_E\gamma_B}{(b_i, B_i)=1; q_2} \right] d\gamma_B,
\end{align*}
\]
\[\text{where the last step is readily derived by using the Mellin transform of the product of two Fox’s H-function [40, eq. (2.25.1.1)].}\]

C. ASC Characterization

Theorem 4. The ASC over Fox’s H-function wiretap fading channels is given by
\[
\mathcal{C}_s = \frac{1}{\ln(2)} (I_1 + I_2 - I_3),
\]
where \( I_1 \) and \( I_2 \) are respectively given by [21a] and [21b], shown at the top of next page.

\[\text{Proof.} \text{ See Appendix B.}\]

D. Special Cases

Accommodating the closed-form expressions for secrecy performance metrics in the corresponding entries in Table 1 directly yields the results, as displayed in Table 2. After some simple algebraic manipulations, one can observe the obtained results herein are identically consistent with the existing works [16, 18, 20, 32].
\[ P_{\text{out}} = 1 - \frac{\kappa B E W}{\lambda_B R_s} H^{1}_{0,1;m_1+1,m_1,1} \left[ R_s \frac{1}{\lambda_E W'} \lambda_B W \begin{array}{c} 2, (1, 1) \\ 1, (1) \end{array} \right] \begin{array}{c} (1-d_l, D_l)_{i=1:q_1} \\ (1-a_i, A_i)_{i=1:p_1} \\ (0, 1) \end{array}, \] (13)

\[ P_{\text{out}}^L = 1 - \frac{\kappa B E}{\lambda_B A E} H^{m_1+2,1,m_1+2}_{1,1;1,2,1,2} \left[ \frac{\lambda_B R_s}{\lambda_E} \right] \begin{array}{c} (a_i + A_i, A_i)_{i=1:p_1} \\ (0, 1) \end{array}, \] (15)

\[ P_{\text{nz}} = \frac{\kappa B E}{\lambda_B A E} H^{m_1+2,1,m_1+2}_{1,1;1,2,1,2} \left[ \frac{\lambda_E}{\lambda_B} \right] \begin{array}{c} (1, 1) \\ (d_l + D_l)_{i=1:q_1} \end{array}, \] (16)

\[ I_1 = \frac{\kappa B E}{\lambda_B A E} H^{m_1+2,1,m_1+2}_{1,1;1,2,1,2} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \begin{array}{c} (1, 1) \\ (1, 1) \end{array}, \] (21a)

\[ I_2 = \frac{\kappa B E}{\lambda_B A E} H^{m_1+2,1,m_1+2}_{1,1;1,2,1,2} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \begin{array}{c} (1, 1) \\ (1, 1) \end{array}, \] (21b)

### Table II: Exact expressions of \( P_{\text{out}}, P_{\text{nz}} \) and \( \tilde{C}_s \) for different special cases of Fox’s \( H \)-function distribution

| \( P_{\text{out}} = 1 - \frac{\kappa B E W}{\lambda_B R_s} H^{1}_{0,1;1,1,1} \left[ R_s \frac{1}{\lambda_E W'} \lambda_B W \right] \) | \( P_{\text{out}}^L = 1 - \frac{\kappa B E}{\lambda_B A E} H^{1,1,1,1}_{1,1,1} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \) | \( P_{\text{nz}} = \frac{\kappa B E}{\lambda_B A E} H^{1,1,1,1}_{1,1,1} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \) | \( \alpha - \mu \) |
|---|---|---|---|
| \( \tilde{C}_s = \frac{\kappa B E}{\lambda_B A E} H^{1,1,1,1}_{1,2,1,1,2} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \) | \( \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \) | \( \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \) | \( \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \) | (1, 1) |

### F-S T

| \( P_{\text{out}} = 1 - \frac{\kappa B E W}{\lambda_B R_s} H^{1}_{0,1,2,1} \left[ R_s \frac{1}{\lambda_E W'} \lambda_B W \right] \) | \( P_{\text{out}}^L = 1 - \frac{\kappa B E}{\lambda_B A E} H^{1,1,2,1,2,2}_{1,1,2,1,2} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \) | \( P_{\text{nz}} = \frac{\kappa B E}{\lambda_B A E} H^{1,1,2,1,2,2}_{1,1,2,1,2} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \) |
|---|---|---|
| \( \tilde{C}_s = \frac{\kappa B E}{\lambda_B A E} H^{1,1,2,1,2,2}_{1,1,2,1,2} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \) | \( \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \) | \( \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \) | \( \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \) | (1, 1) |

### EGK

| \( P_{\text{out}} = 1 - \frac{\kappa B E W}{\lambda_B R_s} H^{1}_{0,1,1,2} \left[ R_s \frac{1}{\lambda_E W'} \lambda_B W \right] \) | \( P_{\text{out}}^L = 1 - \frac{\kappa B E}{\lambda_B A E} H^{1,1,1,2,2,2}_{1,1,1,2,2,2} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \) | \( P_{\text{nz}} = \frac{\kappa B E}{\lambda_B A E} H^{1,1,1,2,2,2}_{1,1,1,2,2,2} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \) |
|---|---|---|
| \( \tilde{C}_s = \frac{\kappa B E}{\lambda_B A E} H^{1,1,1,2,2,2}_{1,1,1,2,2,2} \left[ \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \right] \) | \( \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \) | \( \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \) | \( \frac{1}{\lambda_E} \frac{\lambda_B}{\lambda_E} \) | (1, 1) |
V. ASYMPTOTIC SECURİTY METRICS CHARACTERIZATION

The obtained secrecy expressions are given in terms of either univariate or bivariate Fox’s $H$-function. In order to provide more insights at high or low $\tilde{\gamma}_B$ regime, the asymptotic behavior of the three aforementioned secrecy metrics are developed in this section.

According to the expansions of the univariate and bivariate Fox’s $H$-functions can be derived by evaluating the residue of the corresponding integrands at the closest poles to the contour, namely, the minimum pole on the right for large Fox’s $H$-function arguments and the maximum pole on the left for small ones.

A. Asymptotic SOP

The lower bound of SOP is still expressed in terms of Fox’s $H$-function, in order to study the asymptotic behavior of SOP, the lower bound of SOP is further simplified by expanding the univariate Fox’s $H$-function. Consequently, at high $\tilde{\gamma}_B$ regime, we have $\frac{1}{\lambda_B} \rightarrow \infty$. By using the expanding rule, the asymptotic SOP is given by (23), shown at the top of next page.

For the sake of high accuracy, the asymptotic SOP at high $\tilde{\gamma}_B$ regime is evaluated at $\tau = 0$ and $\tau = -\alpha_E \mu_B$, and is given by (25).

For the sake of high accuracy, the asymptotic SOP at high $\tilde{\gamma}_B$ regime is evaluated at $\tau = 0$ and $\tau = -\alpha_E \mu_B$, and is given by (25).

B. Asymptotic PNZ

The asymptotic PNZ at high or low $\tilde{\gamma}_B$ regime, is computed by evaluating the residues of analytical PNZ, given in (16). According to , Fox’s $H$-function can be further simplified by choosing the dominate term of the Mellin-Barnes type integral. As such, we can evaluate the residue of the PNZ at low $\tilde{\gamma}_B$ regime, at the point

\[ \tau = \min_{l=1, m_1, l=1, m_1} \left( \frac{d_l + D_l}{D_l}, \frac{a_i + A_i - 1}{A_i} \right). \]

Assuming the case of a simple pole, the asymptotic PNZ is thereafter given in (27).

Take the case $\alpha = \mu$ as an example, applying the obtained result, the asymptotic PNZ at low $\tilde{\gamma}_B$ regime is evaluated at $s = -\alpha_E \mu E$ and thereafter given by (28).

C. Asymptotic ASC

By applying the expansion rule, in the case of high $\tilde{\gamma}_B$, the asymptotic ASC is given by (29), which is obtained by individually expanding $I_1$ and $I_2$, respectively. The detailed proof for (29) is referenced to Appendix C.

Similarly, take the case $\alpha = \mu$ as an example, we arrive at the asymptotic ASC at high $\tilde{\gamma}_B$ regime by

\[ I_1 \approx \frac{k_B k_E}{\lambda_B \lambda_E} \Gamma(\mu_B) \Gamma(\mu_E) \left[ \frac{\Psi_0(\mu_B)}{\alpha_B} - \ln(\lambda_B) \right], \]

\[ I_2 \approx \frac{k_B k_E}{\mu_B \lambda_B \lambda_E} H_{2,3}^{3,1} \left( \frac{\lambda_E}{\lambda_B} \right) \left( (0, 1), (1, 1), (\mu_B + \frac{\alpha_E \mu_B}{\alpha_E}, 0, 1), (0, 1) \right). \]

VI. NUMERICAL RESULTS AND DISCUSSIONS

In this section, Monte-Carlo simulations are used to validate the correctness of our analytical derivations demonstrated in Sections IV and V, particularly, over one special case of Fox’s $H$-function wiretap fading channel, i.e., $\alpha = \mu$ wiretap fading channels. For the simplicity of notations, the curves with markers are Monte-Carlo simulation results, whereas the lines represent the analytical results.

In order to validate the analytical accuracy of our derivations, Monte-Carlo simulation outcomes together with analytical results are presented in Figs. [1-3] with regard to the aforementioned three secrecy performance metrics over $\alpha = \mu$ fading channels. Apparently, these figures show that our mathematical representations are in perfect agreements with simulation results.

In Fig. 1, the SOP against $\tilde{\gamma}_B$ is plotted for several fading scenarios, such as Rayleigh, Weibull, Nakagami-$m$, and $\alpha = \mu$. As observed from the figure, specifically, the Nakagami-$m$ ($\alpha = 2, \mu = m$) against Rayleigh ($\alpha = 2, \mu = 1$) and Weibull ($\alpha$ is the fading parameter, $\mu = 1$), respectively. One can conclude that larger $\alpha$ and $\mu$ values result in lower SOP. This is mainly because lower $\alpha$ and $\mu$ values represent serious non-linearity and sparse clustering, i.e., worse channel conditions. This phenomenon also remains true for the PNZ, as shown in Fig. 2. In addition, the lower bound of SOP and the asymptotic SOP are also plotted.

As depicted in Fig. 2 both the exact and asymptotic behavior of $P_{nz}$ are plotted against $\tilde{\gamma}_B$ for Rayleigh, Weibull, Nakagami-$m$, and $\alpha = \mu$. Compared with the exact result, one can conclude that our asymptotic PNZ behaves well at low $\tilde{\gamma}_B$ regime.

The ASC against the ratio of $\tilde{\gamma}_B$ and $\bar{\gamma}_E$ is presented in Fig. 3 and as expected, there is a perfect match between our analytical and simulated results. Also, one can obtain two insights from this graph: on one hand, lower $\alpha$ values lead to higher ASC, no matter whoever experiences severe fading environment. The insight obtained from this figure just vividly demonstrates how information-theoretic security exploits the fading property of wireless transmission medium to ensure secure transmission. On the other hand, a potential malicious eavesdropper can also benefit from poor channel conditions, since worse fading channels reversely enable them to possess better capability to successfully access and wiretap the main channel to a certain extent.

\[ \text{It is worthy to mention that (i) the } \alpha = \mu \text{ fading channel is implemented by using the WAFO toolbox, (ii) the numerical evaluation of univariate and bivariate Fox’s } H \text{-function for MATLAB implementations are based on the method proposed in Table II and Appendix A, respectively.} \]
\[ \mathcal{P}_{\text{out}} L = 1 - \lim_{s \to -\tau} \left( \frac{\sum_{i=1}^{n} \Gamma(b_i + B_i s) \prod_{i=1}^{m} \Gamma(1 - c_i - C_i) \prod_{i=1}^{n} \Gamma(1 - A_i + A_i s)}{\prod_{i=1}^{m} \Gamma(1 + b_i + B_i s) \prod_{i=1}^{n} \Gamma(1 - b_i - B_i s) \prod_{i=1}^{n} \Gamma(a_i + A_i + A_i s) \prod_{i=1}^{n} \Gamma(c_i + C_i - C_i s)} \right) \left( \frac{\lambda_E}{\lambda B R s} \right)^s, \] where \( \tau = \max \left( \frac{-b_i + B_i}{B_1} \right) \).

\[ \mathcal{P}_{\text{nz}} \approx \lim_{s \to -\tau} \left( \frac{\left( \frac{\lambda_E}{\lambda B} \right)^s \sum_{i=1}^{n} \Gamma(b_i + B_i s) \prod_{i=1}^{m} \Gamma(d_i + D_i - D_i s) \prod_{i=1}^{n} \Gamma(1 - a_i + A_i + A_i s) \prod_{i=1}^{n} \Gamma(1 - c_i - C_i) \prod_{i=1}^{n} \Gamma(b_i + B_i - B_i s)}{\prod_{i=1}^{m} \Gamma(1 - b_i - D_i - D_i s) \prod_{i=1}^{n} \Gamma(a_i + A_i + A_i s) \prod_{i=1}^{n} \Gamma(c_i + C_i - C_i s) \prod_{i=1}^{n} \Gamma(1 - b_i - B_i + B_i s)} \right), \] \[ I_1 \approx \frac{\kappa B E}{\lambda B A E} \left[ \ln \left( \frac{1}{\lambda B} \right) + \sum_{i=1}^{m} B_i \Psi_0(b_i + B_i s) - \sum_{i=m}^{m} B_i \Psi_0(b_i + B_i s) - \sum_{i=m}^{m} A_i \Psi_0(a_i + A_i + A_i s) \right] \lim_{s \to -u} \left( 0, \frac{b_i + B_i}{B_1} \right)_{i=1, \ldots, m_1}, \] where \( u = \max \left( \left\{ \left( \frac{c_i + C_i - 1}{c_i} \right)_{i=1, \ldots, m_2} \right\} \right) \).

\[ I_2 \approx \lim_{s \to -u} \left( s - u \right) \prod_{i=1}^{m} \Gamma(b_i + B_i + B_i s) \left( \frac{\lambda E}{\lambda B} \right)^s \left[ \sum_{i=1}^{n} \frac{\kappa B E}{\lambda B A E} \Gamma(s) \prod_{i=1}^{m} \Gamma(1 - a_i - A_i + A_i s) \prod_{i=1}^{m} \Gamma(1 - c_i - C_i + C_i s) \prod_{i=1}^{m} \Gamma(b_i + B_i + B_i s) \prod_{i=1}^{m} \Gamma(1 - b_i - B_i - B_i s) \prod_{i=1}^{m} \Gamma(a_i + A_i + A_i s) \prod_{i=1}^{m} \Gamma(c_i + C_i - C_i s) \prod_{i=1}^{m} \Gamma(1 - b_i - D_i - D_i s) \prod_{i=1}^{m} \Gamma(a_i + A_i + A_i s) \prod_{i=1}^{m} \Gamma(c_i + C_i - C_i s) \prod_{i=1}^{m} \Gamma(1 - b_i - D_i - D_i s) \right), \] where \( u = \min \left( \left\{ \frac{b_i + B_i}{B_1} \right\}_{i=1, \ldots, m_1} \right) \).

To obtain a fair comparison, the asymptotic ASC is also depicted in Fig. 4. Again, it can be seen that the asymptotic ASC presents a highly accurate approximation to the exact ASC, especially at high \( \tilde{\gamma}_B \) regime.

VII. CONCLUSION

Since Fox’s \( H \)-function fading channel can subsume most of the fading models, this paper comprehensively investigated the secrecy metrics, including the SOP, PNZ, and ASC, over Fox’s \( H \)-function wiretap fading channels. Closed-form expressions respecting the three metrics were derived in a general and unified manner, which are given in terms of the univariate or bivariate Fox’s \( H \)-function. In addition, those closed-form expressions were further used to boost the acquire of asymptotic behavior of the secrecy metrics. The asymptotic ones were much simpler and highly accurate for practical usage.

In addition, for the sake of providing more insights on some well-known fading models, several special cases of Fox’s \( H \)-function distribution were particularly explored, including \( \alpha - \mu \), F-S, \( \mathcal{F} \), and EGK. Those examples were further elaborated with the general form, and their accuracy was also compared with Monte-Carlo simulation results. As observed and discussed, the advantages of those general mathematical representations are listed as follows: (i) they are consistent with the existing works; (ii) it provides a unified generic approach to other fading models which can be expanded in terms of Fox’s \( H \)-function fading distribution.

APPENDIX A

Proof of the Theorem

At the very beginning, revisiting (12a)

\[ \mathcal{P}_{\text{out}} = \int_{0}^{\infty} \tilde{F}_B(\gamma_0)\tilde{f}_E(\gamma_E) d\gamma_E \]

\[ = 1 - \int_{0}^{\infty} \tilde{F}_B(\gamma_0)\tilde{f}_E(\gamma_E) d\gamma_E \]

\[ = 1 - \frac{1}{2\pi} \int_{\gamma_1} M(\tilde{\gamma}_B(\gamma_0), 1 - s)M(\tilde{\gamma}_E, s) ds, \]
and using the definition of Mellin transform and Fox’s $H$-function, we arrive at $\mathcal{M}[F_B(s)]$

\[
\begin{align*}
    \mathcal{M}[F_B(\gamma_0), 1 - s] \\
    = \int_0^{\infty} \gamma^{-s}_0 F_B(\gamma_0) d\gamma \\
    = \frac{k_B}{2\lambda_B \pi j} \int_{L_1} \Theta_f(\xi) \lambda_B^{-\xi} \int_0^{\infty} \gamma^{-s}_E \gamma_0^{-\xi} d\gamma d\xi, \quad (32)
\end{align*}
\]

where step (a) is developed by interchanging the order of two integrals. The inner integral in (32) can be further expressed as

\[
\begin{align*}
    \int_0^{\infty} \gamma^{-s}_E \gamma_0^{-\xi} d\gamma &= \mathcal{W}^{-\xi} \int_0^{\infty} \gamma^{-s}_E \left( 1 + \frac{R_s}{\mathcal{W} \gamma} \right)^{-\xi} d\gamma \\
    &= \int_0^{\infty} \mathcal{W}^{-\xi} \left( \frac{R_s}{\mathcal{W} \gamma} \right)^{-1-\xi} d\gamma \\
    &= \frac{\mathcal{W}^{1-s}}{1-s}. \quad (33)
\end{align*}
\]

Next, deploying the definition of the bivariate Fox’s $H$-function [37], the proof is achieved.

**APPENDIX B**

**PROOF FOR THEOREM**

Since the logarithm function can be alternatively re-expressed in terms of Fox’s $H$-function with the help from [40], eq. (8.4.5.4) and [40], eq. (8.3.21.21),

\[
\ln(1 + x) = H^{2,2}_{1,2} \left[ x \begin{array}{c}(1, 1), (1, 1) \\ (1, 1), (0, 1) \end{array} \right], \quad (36)
\]

For the ease of proof, we take the proof for $I_1$ as an example.

\[
I_1 = \frac{1}{2\pi j} \int_{L_1} \mathcal{M}[F_E(\gamma_B), s] \mathcal{M}[G(\gamma_B), 1 - s] ds \quad (37)
\]

\[
\mathcal{M}[F_E(\gamma_B), s] = \frac{k_E}{\lambda_E s} \Theta_E(s), \quad (38)
\]
\[
M[F_B(\gamma_0), 1 - s] = \frac{\kappa_B}{2\pi j} \left( \frac{R_s}{\mathcal{W}} \right)^{s-1} \Gamma(1-s) \int_{L_1} \frac{\Gamma(\xi + s - 1)\Theta^f_\mathcal{E}(\xi)}{\Gamma(\xi)} (\lambda_B \mathcal{W})^{-\xi} d\xi
\]

where \(\Theta^f_\mathcal{E}(s)\) is shown in (39) at the top of next page.

\(M[g_k(\gamma_k), 1 - s]\) can be regarded as the Mellin transform of the product of two Fox’s-\(H\)-function (40) eq. (2.25.1.1), which is given by (40) and shown at the top of next page.

Next, substituting (38) and (40) into (37), yields the following result

\[
I_1 = -\frac{\kappa_B^2 k_E}{4\pi^2 \lambda_B \lambda_E} \int_{L_1} \int_{L_2} \frac{\Theta(s, \xi) \Theta(\xi)}{\lambda_E^2} (s - 1) d\xi d\xi, \quad (41)
\]

where \(\Theta(s, \xi)\) and \(\Theta(\xi)\) are given by (42), shown at the top of next page.

Next, replacing \(\xi = -\eta, s = -\tau, I_1\) can be expressed as (21a) in terms of the bivariate Fox’s-\(H\)-function. In particular, when \(n_1 = 0, I_1\) is further simplified in terms of the extended generalized bivariate Fox’s-\(H\)-function.

Following the same methodology, \(I_2\) can be obtained. \(I_3\) can be finally achieved from (28 eq. (18)).

\[
I_3 = \frac{k_E}{2\pi j} \int_{L_1} M[\ln(1 + \gamma_\mathcal{E}), s] \lambda_E^{1+s} \Theta^f_\mathcal{E}(1-s) ds
\]

and subsequently when \(\frac{s_E}{\lambda_E} \to \infty\), we evaluate the residue at \(s\), where

\[
s = \max \left[0, \left( -\frac{b_1 + B_l}{B_l} \right) \right]_{i=1, \ldots, m_1} \left( c_i + \frac{c_i - 1}{c_i} \right)_{i=1, \ldots, m_2}. \quad (46)
\]

Considering all poles are simple, we arrive at the derived asymptotic \(I_1\).

Similarly, at high \(\tilde{\gamma}_B\) regime, \(I_2\) can be obtained at the point \(u = \min \left( \frac{b_i + B_l}{B_l} \right)_{i=1, \ldots, m_2}\), we complete the proof.

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\[ M[\gamma_B(\gamma_B), 1 - s] = \int_0^\infty \gamma_B^x \ln(1 + \gamma_B)f_B(\gamma_B) d\gamma_B = \kappa_B \int_0^\infty \gamma_B^{x-2} H_{2,2}^{m_1, m_1+2} \left[ \frac{1}{\lambda_B} (1, (1, 1), (1, 1), (0, 1)) \right] \left( 1, (1, 1), (1, 1) - \alpha_j (1 - s) \right) A_j \right]_{j=1}^{q_1} d\gamma_B \\
\]

\[ \Theta(s, \xi) = \prod_{i=1}^{n_1} \frac{\Gamma(1 - A_i + A_i s + A_j \xi)}{\Gamma(b_i + B_i - B_j + B_j s - B_j \xi)} \prod_{l=1}^{m_1} \frac{\Gamma(1 - b_l - B_l + B_l s + B_l \xi)}{\Gamma(1 - B_l)} \]

\[ \Theta(\xi) = \frac{\Gamma(1 + \xi) \Gamma(-\xi) \Gamma(-\xi)}{\Gamma(1 - \xi)}. \]