On generalized median triangles and tracing orbits

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Abstract. We study generalization of median triangles on the plane with two complex parameters. By specialization of the parameters, we produce periodical motion of a triangle whose vertices trace each other on a common closed orbit.

Contents

0. Introduction 1
1. Generalized median operators 2
2. Reduction of 18-fold ways of $M_{wx/yz}$ 4
3. Tracing orbits of triangles 7
References 11

0. Introduction

Given a triangle $\Delta = \Delta ABC$ on a plane, one forms its medial (or midpoint) triangle $S(\Delta) = \Delta A'B'C'$ which, by definition, is a triangle obtained by joining the midpoints $A', B', C'$ of the sides $BC, CA, AB$ respectively. The median triangle $M(\Delta) = \Delta A''B''C''$ of $\Delta = \Delta ABC$ is a triangle whose three sides are parallel to the three medians $AA', BB', CC'$ of $\Delta$. To position $M(\Delta)$, it is convenient to impose extra condition that $M(\Delta)$ shares its centroid with $\Delta$ as well as with $S(\Delta)$. To fix labels of vertices of $M(\Delta)$, one can set, for example, $\overrightarrow{AA'} = \overrightarrow{A''B'}$, $\overrightarrow{BB'} = \overrightarrow{B''C'}$, $\overrightarrow{CC'} = \overrightarrow{C''A'}$.

Arithmetic interest on median triangles can be traced back to Euler who found a smallest triangle made of three integer sides and three integer medians: there exists $\Delta ABC$ with $AB = 136$, $BC = 174$, $CA = 170$, $AA' = 127$, $BB' = 131$ and $CC' = 158$ (cf. [2]). In recent years, geometrical constructions of nested triangles in more general senses call attentions of researchers (e.g., [1], [6]). In particular, M.Hajja [3] studied a generalization of the above constructions $S(\Delta)$ and $M(\Delta)$ by introducing a real parameter $s \in \mathbb{R}$ to replace the midpoints of the sides by more general $(s : 1 - s)$-division points. Recently in [5], the former construction for $S(\Delta)$ was generalized so as to have two complex parameters $\Delta \mapsto S_{p,q}(\Delta)$ ($p, q \in \mathbb{C}$, $pq \neq 1$).

The aim of the first part of this paper is, following the line of [5], to extend the procedure for $M(\Delta)$ to a collection of operations of the forms $\Delta \mapsto M_{wx/yz}(\Delta)$ so that the sides of $M_{wx/yz}(\Delta)$ are given by vectors joining vertices of $\Delta$ and of $S_{p,q}(\Delta)$ in 18-fold ways of label correspondences (See Definition 1.5 below). After studying mutual relations of the 18-fold ways, we will find that only three ways among them are essential. Then, applying the finite Fourier transforms of triangles, we obtain operators $S[\eta, \eta']$ and $M_{wx/yz}[\eta, \eta']$ which behave smoothly with the parameter $(\eta, \eta')$ running over the full space $\mathbb{C}^2$ (the former was already closely studied in [5]).

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In the second part of the present paper, we will study ‘dancing’ of triangles $S[\eta, \eta'](\Delta)$ and $M^{yx/yz}[\eta', \eta'](\Delta)$ along with periodical parameters $(\eta(t), \eta'(t)) \in \mathbb{C}^2$ ($t \in \mathbb{R}/\mathbb{Z}$). In particular, we search conditions under which the three vertices of a triangle trace one after the other in motion along a single common orbit. Basic examples including “choreographic three bodies dancing on a figure eight” will also be illustrated.

1. Generalized median operators

Throughout this paper, we use the notations: $i := e^{2\pi i/4}$, $\rho := e^{2\pi i/6}$, $\omega := e^{2\pi i/3}$.

We consider any triangle lies on the complex plane $\mathbb{C}$ and identify it with the multiset of vertices $\{a_0, a_1, a_2\}$ on $\mathbb{C}$. It is often useful to say that a vector $\Delta = (a_0, a_1, a_2)$ on $\mathbb{C}$ is a triangle triple representing the triangle $\{a_0, a_1, a_2\}$. For any choice of $w, x, y, z \in \mathbb{Z}/3\mathbb{Z}$, we shall write $a_w = a_i$, the $i$-th coordinate of $\Delta$ such that $i \mod 3 = x$.

In [5], for $p, q \in \mathbb{C}$ with $pq \neq 1$, we introduced an operation $S_{p,q}$ on the triangle triples defined by

$$S_{p,q}(a_0, a_1, a_2) = (a_0', a_1', a_2') : \begin{cases} a_0' = \alpha_{p,q} a_0 + \beta_{p,q} a_1 + \gamma_{p,q} a_2; \\ a_1' = \alpha_{p,q} a_1 + \beta_{p,q} a_2 + \gamma_{p,q} a_0; \\ a_2' = \alpha_{p,q} a_2 + \beta_{p,q} a_0 + \gamma_{p,q} a_1, \end{cases}$$

where,

$$\alpha_{p,q} = \frac{p(1-q)}{1-pq}, \beta_{p,q} = \frac{q(1-p)}{1-pq}, \gamma_{p,q} = \frac{(1-p)(1-q)}{1-pq}.$$ 

When $p, q$ are real numbers, $S_{p,q}(\Delta)$ can be obtained from intersection points of certain two cevian triples of $\Delta$ as introduced in [4]. For convenience, we shall call $S_{p,q}$ a generalized cevian operator on triangles also for complex parameters $p, q$. Since $\alpha_{p,q} + \beta_{p,q} + \gamma_{p,q} = 1$, it is easy to see that the centroids of $\Delta = (a_0, a_1, a_2)$ and of $\Delta' := S_{p,q}(\Delta) = (a_0', a_1', a_2')$ coincide and that

$$\sum_{i \in \mathbb{Z}/3\mathbb{Z}} a_{y+i}a_{x+i}' = 0$$

for any choice of $w, x \in \mathbb{Z}/3\mathbb{Z}$. This determines, for each $(y, z) \in (\mathbb{Z}/3\mathbb{Z})^2$ with $y \neq z$, a unique triangle triple $\Delta'' = (a_0'', a_1'', a_2'')$ by the conditions:

$$\Delta'' \text{ share the centroid with } \Delta, \Delta', \text{ in other words, } \Delta, \Delta' \text{ and } \Delta'' \text{ are centroid;}$$

$$a_{y+i}a_{x+i}' = a_{y+i}''a_{x+i}'' \quad (i \in \mathbb{Z}/3\mathbb{Z}).$$

Definition 1.5 ((p, q)-median triangle). Let $p, q \in \mathbb{C}$ with $pq \neq 1$, and $w, x, y, z \in \mathbb{Z}/3\mathbb{Z}$ with $y \neq z$. Given a triangle triple $\Delta = (a_0, a_1, a_2)$ with $\Delta' = S_{p,q}(\Delta) = (a_0', a_1', a_2')$, we define the triangle triple

$$M^{yx/yz}_{p,q}(\Delta) := \Delta'',$$

where $\Delta'' = (a_0'', a_1'', a_2'')$ is determined by the condition (1.3)-(1.4). We shall call $M^{yx/yz}_{p,q}$ a generalized median operator on triangles.

Example 1.6 (Prototype). Let $\Delta ABC$ be a triangle represented by a triple $\Delta = (a, b, c) \in \mathbb{C}^3$. Let us illustrate the classical case in Introduction in our terminology: As noted in [5] Example 1.2, the midpoint triangle $S(\Delta) = \Delta A'B'C'$ is given by $S_{0,1/2}(\Delta)$. The median triangle $M(\Delta) = \Delta A''B''C''$ labeled by the condition $\overrightarrow{AA'} = A''\overrightarrow{B'}, \overrightarrow{BB'} = B''\overrightarrow{C'}, \overrightarrow{CC'} = C''\overrightarrow{A'}$ is then given by $M^{10/01}_{0,1/2}(\Delta)$.
Example 1.7. In [3], M.Hajja discusses three types of triangles called the s-medial, the s-Routh, and the s-median triangles with a real parameter $s \in \mathbb{R}$. The $(p, q)$-median triangle introduced above generalizes Hajja’s s-median triangle. Start with a triangle $\Delta ABC$ represented by a positive triangle triple $\Delta = (a, b, c)$ satisfying $\text{Im}(\frac{a}{c}) > 0$. Form first $\Delta' = (a', b', c')$ to be $\mathcal{S}_{0,1-s}(\Delta)$ (called the s-median of $\Delta$), the triangle whose vertices are $(s : 1-s)$-division points of the edges of $\Delta$. The s-median triangle of $\Delta$, written $\mathcal{H}_s(\Delta)$, is, by definition, a triangle $\{a'', b'', c''\}$ such that $\frac{a}{c} = \frac{b''}{c''}$, $\frac{b}{c} = \frac{c''}{a''}$, and $\frac{c}{a} = \frac{a''}{b''}$. Without loss of generality, we may assume $\mathcal{H}_s(\Delta)$ and $\Delta$ are concentroid, i.e. $a + b + c = a'' + b'' + c''$ so that $\mathcal{H}_s(\Delta)$ is uniquely determined from $\Delta$. In our above definition, we find $\mathcal{H}_s(\Delta)$ to be $\mathcal{M}_{0,1/2-s}(\Delta)$.

The collection of operators $\mathcal{S} := \{\mathcal{S}_{p,q} \mid (p, q) \in \mathbb{C}^2, pq \neq 1\}$ is incomplete in the sense that the composition $\mathcal{S}_{p,q} \mathcal{S}_{p',q'}$ may not always be of the form of an $\mathcal{S}_{p,q} \in \mathcal{S}$. The lesson found in our previous work [5] to remedy this defect is to introduce new parameters $(\eta, \eta') \in \mathbb{C}^2$ by

$$\eta := \frac{p - q}{1 - pq} + \frac{(p - 1)(2q - 1)}{1 - pq} \omega, \quad \eta' := \frac{p - q}{1 - pq} + \frac{(p - 1)(2q - 1)}{1 - pq} \omega^2.$$

It turned out that we can introduce $\mathcal{S}[\eta, \eta']$ well defined for all pairs $(\eta, \eta') \in \mathbb{C}^2$ (that coincide with $\mathcal{S}_{p,q}$ when $pq \neq 1$) so that the composition law $\mathcal{S}[\eta_1, \eta'_1] \mathcal{S}[\eta_2, \eta'_2] = \mathcal{S}[\eta_1 \eta_2, \eta'_1 \eta'_2]$ holds to provide a natural multiplicative monoid structure on the collection

$$\mathcal{S} := \{\mathcal{S}[\eta, \eta'] \mid (\eta, \eta') \in \mathbb{C}^2\}.$$

Now, regarding triangle triples as column vectors in $\mathbb{C}^3$, we easily see that the operations $\mathcal{S}_{p,q}$ and $\mathcal{S}[\eta, \eta']$ naturally determine linear transformations (3 by 3 matrices in $M_3(\mathbb{C})$) acting on $\mathbb{C}^3$ on the left. Below, we shall identify those operators as their matrix representatives in $M_3(\mathbb{C})$. Let

$$I := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad W := \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix}.$$

The following proposition summarizes basic properties for $\mathcal{S}[\eta, \eta'] \in \mathcal{S}$:

**Proposition 1.10 ([5]).** Notations being as above, we have:

(i) $\mathcal{S} = \{\alpha I + \beta J + \gamma J^2 \mid \alpha + \beta + \gamma = 1\} \subset M_3(\mathbb{C})$.

(ii) $\mathcal{S}[\eta, \eta'] = W \cdot \text{diag}(\omega^\eta, \omega^\eta') \cdot W^{-1}$

$$= \frac{1}{3}(1 + \eta + \eta')I + \frac{1}{3}(1 + \eta \omega + \eta' \omega^2)J + \frac{1}{3}(1 + \eta \omega^2 + \eta' \omega)J^2 \quad (\eta, \eta' \in \mathbb{C}).$$

Let us turn to generalized median operators. We first extend $\mathcal{M}^{wx/yz}_{p,q}$ ($pq \neq 1$) to the new parameters $(\eta, \eta') \in \mathbb{C}^2$. Below, we understand the number $\omega^x$ and the matrix $J^y$ in the obvious sense for each $x \in \mathbb{Z}/3\mathbb{Z}$.

**Definition 1.11 ((\eta, \eta')-median triangles).** Let $\eta, \eta' \in \mathbb{C}$, and let $w, x, y, z \in \mathbb{Z}/3\mathbb{Z}$ with $y \neq z$. Given a triangle triple $\Delta = (a_0, a_1, a_2)$ with $\Delta' = \mathcal{S}[\eta, \eta'](\Delta) = (a'_0, a'_1, a'_2)$, we define the triangle triple

$$\mathcal{M}^{wx/yz}[\eta, \eta'](\Delta) := \Delta''$$

where $\Delta'' = (a''_0, a''_1, a''_2)$ is determined by the condition (1.3)-1.4.

It is not difficult to see that $\mathcal{M}^{wx/yz}[\eta, \eta'] \in \mathcal{S}$. In fact, we have the following explicit formula:

**Proposition 1.12.** Given $\eta, \eta' \in \mathbb{C}$ and $w, x, y, z \in \mathbb{Z}/3\mathbb{Z}$ with $y \neq z$, we have

$$(J^y - J^y)\mathcal{M}^{wx/yz}[\eta, \eta'] = J^x \mathcal{S}[\eta, \eta'] - J^y.$$
Proof. Let $\Delta = (a_0, a_1, a_2)$ be a triangle triple, and write $\Delta' = S[\eta, \eta'](\Delta) = (a_0', a_1', a_2')$ and $\Delta'' = M^{xz/yz}[\eta, \eta'](\Delta) = (a_0'', a_1'', a_2'')$. The assertion essentially amounts to seeing the identity

$$(J^x - J^y)(\Delta'') = J^x(\Delta') - J^y(\Delta).$$

Observe that the 1st component of $J^x(\Delta') - J^y(\Delta)$ is $a_0 a_2^2$, and that the 1st component of $J^x(\Delta'') - J^y(\Delta'')$ is $a_0'' a_2''$. They coincide with each other by definition. Similarly, one can see the coincidence of their 2nd and 3rd components, as they are the 1st components of the above after $\Delta$ replaced by $J\Delta$, $J^2\Delta$. One can extend the identity also for degenerate triangle triples by easy argument of continuity, and hence conclude the matrix identity as asserted. \hfill \Box

Although the factor $(J^x - J^y)$ in LHS of the above Proposition 1.12 is not an invertible matrix, the centroid condition (1.3) determines $M^{xz/yz}[\eta, \eta']$ in $S$ as seen in the following corollary.

In fact, the generalized median operator $M^{xz/yz}[\eta, \eta']$ turns out to be reduced to a generalized cevian operator $S[\eta_0, \eta_1]$ after a simple change of parameters:

**Corollary 1.13.** Notations being as in Proposition 1.12, we have

$$M^{xz/yz}[\eta, \eta'] = S[\eta_0, \eta_1]$$

where

$$\eta_0 = \frac{\eta_0^x - \omega^y}{\omega^x - \omega^y}, \quad \eta_1 = \frac{\eta_1^x - \omega^y}{\omega^x - \omega^y}.$$

Proof. Let $N = \frac{1}{3}(1 + J + J^2)$ (i.e., the matrix with all entries $\frac{1}{3}$). The fact that $M = M^{xz/yz}[\eta, \eta']$ preserves the centroids of triangles implies that $NM = N$. Hence it follows from Proposition 1.12 that $(*) : (J^x - J^y + N)M = J^xS[\eta, \eta'] - J^y + N$. Since the matrix $(J^x - J^y + N) \in S$ is invertible, the identity $(*)$ determines $M$ which itself lies in $S$ by Proposition 1.10 (ii) and gives rise to

$$\begin{bmatrix} 0 & \omega^x - \omega^y \\ \omega^x - \omega^y & \omega^x - \omega^y \end{bmatrix} \begin{bmatrix} 1 \\ \eta_1 \\ \eta_0 \end{bmatrix} = \begin{bmatrix} 1 \\ \omega^x - \omega^y \end{bmatrix} \begin{bmatrix} 1 \\ \eta_1 \\ \eta_0 \end{bmatrix} - \begin{bmatrix} 1 \\ \omega^x - \omega^y \end{bmatrix} \begin{bmatrix} 1 \\ \eta_1 \\ \eta_0 \end{bmatrix},$$

after conjugation by $W$. This settles the asserted formula on $(\eta_0, \eta_1)$. \hfill \Box

2. **Reduction of 18-fold ways of $M^{xz/yz}$**

The upper label $wx/yz$ for a generalized median operator $M^{xz/yz}[\eta, \eta']$ is to be given from the collection of $(w, x, y, z) \in (\mathbb{Z}/3\mathbb{Z})^4$ with $y \neq z$.

Since the condition (1.4) is stable under simultaneous shifts of labels in $wx/yz$, we have the identity $M^{xz/yz}[\eta, \eta'] = M^{x+1,z+1}[\eta, \eta']$ which will be listed below in (2.2). As a consequence, there are 18 different ways of labels up to the shifts in $\mathbb{Z}/3\mathbb{Z}$. However, there are many other identities in-between generalized cevian and median operators as shown in the following list (2.1)-(2.7). We omit their proofs, for these identities in $S$ for $(\eta, \eta') \in \mathbb{C}^2$ and for labels $wx/yz$ can be easily verified (for example, by simple calculations of matrices using Propositions 1.10 and 1.12), once they are discovered.
we shall translate the above results for
samples chosen from 18 types of labels, where
the last three triangles are also dependent by a linear relation (2.6).

\[
\Delta = (\eta, \eta') = (\omega, \omega')
\]

with \( \eta, \eta' \). We translate Corollary 1.13 in the form

\[
S[\eta, \eta'] = S[\eta_0, \eta_0'] \cdot J = J \cdot S[\eta_0, \eta_0']
\]

and vice versa. The following table shows some samples chosen from 18 types of labels, where

\[
\Delta'' = (a_0'', a_1'', a_2'') = M_{p,q}^{ux/yz}(\Delta) = S_{p,q}(\Delta)
\]

for \( \Delta = (a_0, a_1, a_2) \) and \( \Delta' = (a_0', a_1', a_2') = S_{p,q}(\Delta) \).
Example 2.11. In Example 1.7, we identified Hajja’s s-median operator $H_s$ with $M_{s^{-1}}^{00/12}$ for $s \in \mathbb{R}$. The above formula (2.10) (cf. Table I) translates it as

\begin{equation}
H_s = M_{s^{-1}}^{00/12} = S_{\frac{2s}{s^2 + 1}}.
\end{equation}

The last expression for $s = -3, \frac{3}{2}$ appears to be singular as $S_{\infty}, S_{\frac{3}{2}}$ respectively, but these singularities can be removed in the language of $(\eta, \eta')$-parameters: Indeed, by (1.8) we can interpret $S_{s^{-1}} = S[\omega + (1 - s) \omega^2, \omega^2 + (1 - s) \omega]$, hence from Definition 1.11 we obtain $M_{s^{-1}}^{00/12} = M^{00/12}[\omega + (1 - s) \omega^2, \omega^2 + (1 - s) \omega]$. Corollary 1.13 then allows us to compute

\begin{equation}
H_s = M^{00/12}[\omega + (1 - s) \omega^2, \omega^2 + (1 - s) \omega]
\end{equation}

which makes senses on all $s \in \mathbb{C}$. Finally, formulas (2.11)-(2.15) transform $H_s$ into various expressions of generalized medians. For example, for generic complex parameter $s$, one has:

\begin{equation}
H_s = M_{\frac{s^2}{s^2 + 1}, \frac{1}{s^2 + 1}} = M_{\frac{2s}{s^2 + 1}} = M_{\frac{1}{s^2 + 1}}^{02/01}.
\end{equation}

Example 2.15 (Parameters for $M_{p,q}^{wx/yz} = S_{p,q}$). Let $wx/yz$ be a given label with $w, x, y, z \in \mathbb{Z}/3\mathbb{Z}$, $y \neq z$. By Proposition 1.12, we find that $M_{p,q}^{wx/yz}[\eta, \eta'] = S[\eta, \eta']$ has a unique solution in the form

\[ S[\eta, \eta'] = (J^\omega + J^\gamma - J^\alpha)^{-1} J^\omega = \alpha + \beta J + \gamma J^2 \]

summarized in the following table:

| Label | $\alpha$ , $\beta$ , $\gamma$ | Label | $\alpha$ , $\beta$ , $\gamma$ | Label | $\alpha$ , $\beta$ , $\gamma$ |
|-------|-----------------------------|-------|-----------------------------|-------|-----------------------------|
| 00/01 | 4/7 , 2/7 , 1/7             | 01/01 | 1 , 0 , 0                   | 02/01 | 1/2 , 1/2 , 0               |
| 00/10 | 0 , 0 , 1                   | 01/10 | 1/7 , 2/7 , 4/7             | 02/10 | 0 , 1/2 , 1/2               |
| 00/02 | 4/7 , 1/7 , 2/7             | 01/02 | 1/2 , 0 , 1/2               | 02/02 | 1 , 0 , 0                   |
| 00/20 | 0 , 1 , 0                   | 01/20 | 0 , 1/2 , 1/2               | 02/20 | 1/7 , 4/7 , 2/7             |
| 00/12 | 1/2 , 0 , 1/2               | 01/12 | 2/7 , 1/7 , 4/7             | 02/12 | 0 , 0 , 1                   |
| 00/21 | 1/2 , 1/2 , 0               | 01/21 | 0 , 1 , 0                   | 02/21 | 2/7 , 4/7 , 1/7             |
Recall from Proposition 1.10 (ii) that the corresponding parameter \((\eta, \eta')\) for each case is given by
\[
\begin{align*}
\eta &= \alpha + \beta \omega^2 + \gamma \omega, \\
\eta' &= \alpha + \beta \omega + \gamma \omega^2.
\end{align*}
\]
Next we search parameters \((p, q)\) with \(pq \neq 1\) satisfying \(M_{p,q}^{wx/yz} = S_{p,q}\) from the above table. They are classified into the following three kinds:

(i) \(S_{p,q} \in \{J\} (= \{J, J^2\})\), \hspace{1cm} [01/01, 00/10, 02/02, 00/20, 02/12, 01/21];

(ii) \(S_{p,q} = \frac{1}{2} (J + J^j) (i \neq j)\), \hspace{1cm} [02/01, 02/10, 01/02, 01/20, 00/12, 00/21];

(iii) \(S_{p,q} = \alpha J + \beta J + \gamma J^2 \{\{\alpha, \beta, \gamma\} = \{\frac{1}{7}, \frac{2}{7}, \frac{4}{7}\}\}\), \hspace{1cm} [00/01, 01/10, 00/02, 02/20, 01/12, 02/21].

The first two cases are uninteresting: (i) occurs when \((p, q) = (0, 0), (1, \ast), (\ast, 1)\) (where \(\ast \neq 1\)) so that \(S_{p,q}\) simply represents a permutation of vertex labels (cf. [5], (2.2)); (ii) occurs when \(S_{p,q}(\Delta)\) represents the midpoint triangle, while the sides of \(M_{p,q}^{wx/yz}(\Delta)\) consists of the half sides of \(S_{p,q}(\Delta)\) when \((p, q) = (0, \frac{1}{2}), (\frac{1}{2}, 0)\). (Note: \(S_{p,q} = \frac{1}{2} (1 + J)\) never occurs). However, (iii) yields geometrically nontrivial cases when \((p, q) = (\frac{4}{7}, \frac{2}{7}), (\frac{2}{7}, \frac{4}{7}), (\frac{4}{7}, \frac{2}{7}), (\frac{2}{7}, \frac{4}{7}), (\frac{1}{2}, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})\). These are operations for Routh’s triangles discussed in [5], Example 4.3.

![Figure 1. Type (iii) for \(M_{p,q}^{wx/yz}(\Delta) = S_{p,q}(\Delta)\) with \(\Delta = \Delta(0) = (0, 1, 0.7 + 0.5i)\)](image)

3. Tracing orbits of triangles

Since our operators \(S[\eta, \eta']\) are realized as linear actions on \(\mathbb{C}^3\) fixing \((1, 1, 1)\), they commute with every complex affine transformation of triangles, in other words, \(S[\eta, \eta']\) commutes with any mapping of the form \((a, b, c) \mapsto (f(a), f(b), f(c))\) where \(f: z \mapsto \lambda z + \nu (\lambda, \nu \in \mathbb{C})\). This is not always the case for real affine transformations. We first begin with the following simple

**Lemma 3.1.** Let \((\eta, \eta') \in \mathbb{C}^2\). The operation \(S[\eta, \eta']\) commutes with the real affine transformations of triangles if and only if \(\eta = \eta'\), i.e., \(\eta\) and \(\eta'\) are complex conjugate to each other.

**Proof.** Recall that any real affine transformation of the complex plane \(\mathbb{C}\) can be written as \(f_{\lambda, \mu, \nu}(z) = \lambda z + \mu \bar{z} + \nu\) with \(\lambda, \mu, \nu \in \mathbb{C}\). Given a triangle triple \(\Delta = (a, b, c)\) and \(f = f_{\lambda, \mu, \nu}\), write \(f(\Delta) := (f(a), f(b), f(c))\) for the image of \(\Delta\) by \(f\). Then, one computes
\[
S[\eta, \eta'] \left( f_{\lambda, \mu, \nu} \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \right) = f_{\lambda, \mu, \nu} \left( S[\eta, \eta'] \left( \begin{array}{c} a \\ b \\ c \end{array} \right) \right) + \mu \cdot \left( S[\eta, \eta'] - S[\bar{\eta}, \bar{\eta}'] \right) \left( \begin{array}{c} b \\ c \end{array} \right).
\]
The commutativity of $S[\eta, \eta']$ and $f_{\lambda, \mu, \nu}$ holds if and only if $\mu = 0$ (i.e., $f_{\lambda, \mu, \nu}$ is complex affine) or $S[\eta, \eta'] = \frac{1}{3}(1 + \eta + \eta')I + \frac{1}{3}(1 + \eta \omega + \eta' \omega^2)J + \frac{1}{3}(1 + \eta \omega^2 + \eta' \omega)J^2$ is in $M_3(\mathbb{R})$. The latter condition is easily seen to be equivalent to $\tilde{\eta} = \eta'$.

In [5], we called $S[\eta, \eta'] \in \mathbb{S}$ an area-preserving operator if the associated parameters $\eta, \eta' \in \mathbb{C}$ satisfy $|\eta| = |\eta'| = 1$. The set of area-preserving operators forms a compact multiplicative torus in $GL_3(\mathbb{C})$. Since $S[\eta, \eta']^k = S[\eta^k, \eta'^k]$ ($k \in \mathbb{Z}$), iteration of area-preserving operators can be interpolated by one-parameter family of the form

$$\{S[e^{2\pi int}, e^{2\pi int}]\}_{t \in \mathbb{R}}.$$ 

We are particularly interested in the case where three vertices move along a single closed orbit cyclically replacing positions of each other after $t \mapsto t + \pi / 3$ so that the total motion is left invariant after $t \mapsto t + \pi$. Note that, in this situation, we may assume $\pi = 1$ and $n, m$ are coprime integers without loss of generality. Taking this into accounts, we are led to start with a more general setup: Suppose we are given $\Delta \in \mathbb{C}^3$ and two continuous functions $\eta, \eta' : \mathbb{R} \to (\mathbb{R} / \mathbb{Z} \to \mathbb{C})$ (with period 1). We shall consider the periodic maps $\mathbb{R} \to \mathbb{R} / \mathbb{Z} \to \mathbb{C}^3$ in the form

$$\Delta(t) = S[\eta(t), \eta'(t)](\Delta) \text{ or } \mathcal{M}^{wx/yz}[\eta(t), \eta'(t)](\Delta).$$

Note that generally $\Delta(0)$ may not be the same as the initial $\Delta$ and that $\Delta(t)$ may degenerate at some $t$ even if $\Delta$ is given as a non-degenerate triangle. The family $\{\Delta(t)\}_t$ will be called collision-free if, for every $t \in \mathbb{R}$, $\Delta(t)$ is a (degenerate or non-degenerate) triangle with three distinct vertices. We sometimes regard the time parameter $t \in \mathbb{R}$ also as $t \in \mathbb{R} / \mathbb{Z}$ when no confusion could occur.

**Definition 3.2.** Notations being as above, we say the family $\{\Delta(t)\}_{t \in \mathbb{R} / \mathbb{Z}}$ to have a single tracing orbit in ascending (resp. descending) order, if $J \Delta(t) = \Delta(t + \frac{1}{3})$ (resp. $= \Delta(t - \frac{1}{3})$).

If $\{\Delta(t)\}_t$ has a single tracing orbit in ascending order, and $\Delta(t)$ is written as $(a_0(t), a_1(t), a_2(t))$, then, $a_0(t) = a_2(t + \frac{1}{3}) = a_1(t + \frac{2}{3}) = a_0(t + 1)$ for all $t \in \mathbb{R}$. We may interpret a collision-free family with this property as a motion of three particles $a_0, a_1, a_2$ moving along a single closed orbit so that they trace each other chronologically with $a_0 \to a_1 \to a_2 \to a_0$.

**Proposition 3.3.** Let $\Delta \in \mathbb{C}^3$ and $\eta, \eta' : \mathbb{R} / \mathbb{Z} \to \mathbb{C}$ be continuous functions with period 1.

(i) $\{S[\eta(t), \eta'(t)](\Delta)\}_t$ has a single tracing orbit in ascending (resp. descending) order if and only if

$$\eta(t + \frac{1}{3}) = \eta(t) \omega^{-1}, \eta'(t + \frac{1}{3}) = \eta'(t) \omega \quad (t \in \mathbb{R})$$

$$\left( \text{resp. } \eta(t - \frac{1}{3}) = \eta(t) \omega^{-1}, \eta'(t - \frac{1}{3}) = \eta'(t) \omega \quad (t \in \mathbb{R}) \right)$$

holds.

(ii) Let $wx/yz$ be a label for generalized median operators. Then, $\{\mathcal{M}^{wx/yz}[\eta(t), \eta'(t)](\Delta)\}_t$ has a single tracing orbit in ascending (resp. descending) order if and only if $\tilde{\eta}(t) := \eta(t) - \omega^{x-y}, \tilde{\eta}'(t) := \eta'(t) - \omega^{y-x}$ satisfy

$$\tilde{\eta}(t + \frac{1}{3}) = \tilde{\eta}(t) \omega^{-1}, \tilde{\eta}'(t + \frac{1}{3}) = \tilde{\eta}'(t) \omega \quad (t \in \mathbb{R}).$$

$$\left( \text{resp. } \tilde{\eta}(t - \frac{1}{3}) = \tilde{\eta}(t) \omega^{-1}, \tilde{\eta}'(t - \frac{1}{3}) = \tilde{\eta}'(t) \omega \quad (t \in \mathbb{R}) \right).$$

**Proof.** (i) follows immediately from (2.1). To prove (ii), we use Corollary 1.13 to express $\mathcal{M}^{wx/yz}[\eta(t), \eta'(t)](\Delta)$ as $S[\eta_0(t), \eta_1(t)]$. Then, apply (i) for the latter form. \qed
Now, let us turn back to the area-preserving parameters \( \eta(t) = e^{2\pi int}, \eta'(t) = e^{2\pi int} \) with coprime integers \( m, n \in \mathbb{Z} \) and examine some typical cases.

**Example 3.4.** Let \( \Delta \) be a triangle triple and \( m, n \) coprime integers. By Proposition 3.3(i), the family
\[
\{ \Delta(t) = \mathcal{S}[e^{2\pi int}, e^{2\pi int}](\Delta) \}_{t \in \mathbb{R}}
\]
has a single tracing orbit if \( m + n \equiv 0 \pmod{3} \). The vertices move in ascending (resp. descending) order if \( m \equiv 2 \pmod{3} \) (resp. \( m \equiv 1 \pmod{3} \)).

**Example 3.5** (Steiner ellipse). The special case \( m = -1, n = 1 \) of Example 3.4 is
\[
\Delta(t) = \mathcal{S}[e^{-2\pi it}, e^{2\pi it}](\Delta).
\]
In this case, starting from \( \Delta(0) = \Delta \), the vertices of a triangle move on an ellipse with sides tangent to an interior ellipse. For an easy proof for the case \( \Delta \) is non-degenerate, one can apply Lemma 3.1 to deform \( \Delta \) to the equilateral triangle \((1, \omega, \omega^2)\) in real affine geometry. If \( \Delta = (0, 1, u+v\sqrt{-1}) \), then the circumscribed ellipse has the following equation in XY-coordinates of \( \mathbb{C} \).
\[
v^2 \left( X - \frac{1 + u}{3} \right)^2 + (v - 2uv) \left( X - \frac{1 + u}{3} \right) \left( Y - \frac{v}{3} \right) + (1 - u + u^2) \left( Y - \frac{v}{3} \right)^2 = \frac{v^2}{3}.
\]

**Example 3.6.** The following three collections of figures illustrate the family \( \mathcal{S}[e^{2\pi it}, e^{2\pi int}](\Delta) \) for \( n \equiv 2 \pmod{3} \), \( \mathcal{S}[e^{2\pi int}, e^{2\pi it}](\Delta) \) for \( m \equiv 2 \pmod{3} \) and some other types from Example 3.4 respectively. We start from \( \Delta = \Delta(0) = (0, 1, 0.7 + 0.5i) \).
Example 3.7 (Median orbits). By Proposition 3.3 (ii), the median triangle family

\[ \{ \mathcal{M}^{0x/01}[\eta(t), \eta'(t)](\Delta) \}_{t \in \mathbb{R}} \]

along with \( \eta(t) = e^{2\pi int} + \omega^x \), \( \eta'(t) = e^{2\pi int} + \omega^{-x} \) (\( m + n \equiv 0 \pmod{3} \), \( x \in \mathbb{Z}/3\mathbb{Z} \)) has a single tracing orbit. The following figure starts from \( \Delta = (0, 1, 0.7 + 0.5i) \), \( \Delta(0) = (\frac{4}{5} + \frac{1}{5}i, \frac{7}{30} + \frac{1}{6}i, \frac{2}{5}) \). According to (2.8), the orbit is independent of the choice of \( x \in \mathbb{Z}/3\mathbb{Z} \). We also observe that it is similar to the orbit \( \{ \mathcal{S}[e^{-10\pi it}, e^{4\pi it}] (\Delta) \}_{n \in \mathbb{R}} \) illustrated in the previous example.

It is not necessary for us to persist in area-preserving parameters in Proposition 3.3. Simple linear sums of \( e^{2\pi int} \) with \( m \equiv \pm 1 \pmod{3} \) (\( \pm \) depends on \( \eta, \eta' \) individually) already provide us with a number of remarkable examples. In this paper, we content ourselves with showing the following few cases among them.
Example 3.8 (Figure eight cevian orbit). Let $\Delta = (0, i, -i)$ be a degenerate triangle, and set $
(t) = -2e^{2\pi it} + e^{-4\pi it}$, $
'(t) = 2e^{-2\pi it} + e^{4\pi it}$. Then, the vertices of $\Delta(t) = S[\n(t), \n'(t)](\Delta)$ moves on a single figure 8 curve $X^2 = \frac{3}{16}X^4 + Y^2$ in XY-coordinates of $\mathbb{C}$: The first vertex of $\Delta(t)$ moves along $\frac{2}{3}\sqrt{3}\cos(t) + i\frac{2}{3}\sqrt{3}\sin(2t)$ ($t \in \mathbb{R}$) and the other two vertices chase it on the same orbit.

$\xleftarrow{\rightarrow}$

![Figure 7](image)

Example 3.9 (Figure eight median orbit). We provide another example for Proposition 3.3 (ii). Starting from $\Delta = (0, 4, 3+i)$, the median triangle family $\{M^{01/01}_{\n(t), \n'(t)}(\Delta)\}_{t \in \mathbb{R}}$ along with $\n(t) = -2e^{2\pi it} + e^{-4\pi it} + \omega$, $\n'(t) = 2e^{-2\pi it} + e^{4\pi it} + \omega^2$ gives a figure eight orbit.

$\xleftarrow{\rightarrow}$

![Figure 8](image)

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