How to preserve symmetries with cut-off regularized integrals?

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Abstract

We present a prescription to calculate the quadratic and logarithmic divergent parts of several integrals employing a cutoff in a coherent way, i.e. in total agreement with symmetry requirements. As examples we consider one-loop Ward identities for QED and a phenomenological chiral model.

1 Introduction

In quantum field theory, ultraviolet (UV) divergences always emerge from momentum integration in the loops. In order to identify the physical content of the theory under consideration, one has to subtract these divergences with a definite regularization procedure. Among them, the dimensional regularization is probably the most popular one because it is easy to use and respects all the symmetry properties of the initial lagrangian $[1, 2, 3]$. The latter aspect is of fundamental importance and could justify by itself the choice of dimensional regularization.

Nevertheless, the introduction of an explicit cut-off is sometimes advantageous, for example as soon as one considers renormalization group equations in the Wilsonian approach $[3, 4]$ or effective theories $[5]$. In that context, the problem concerning the non-respect of symmetries arises. A three-momentum cut-off has been proved to be useful, e.g., in the case of the Nambu-Jona-Lasinio model (see, e.g., Ref. $[6]$) but it does not respect Lorentz invariance. This can be avoided by using a four-momentum cut-off, but the latter suffers from the fact that it violates gauge invariance, too. This can be seen for instance from the fact that the one-loop photon

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polarization tensor in QED is no longer transverse. The problem has been solved by Schwinger within his proper-time approach \cite{7}. However, his formalism is very different from the usual Feynman diagram technique. In the literature there exists a modified version of the proper-time regularization which can directly be applied to Feynman diagrams \cite{8}, but as shown later in this paper, the results are ambiguous since they depend on the way the propagators are written. On purely phenomenological grounds, the problem is sometimes “solved” by simply imposing gauge invariance at hand, i.e., if we keep the example of the photon polarization tensor in QED, by projecting it onto the transverse part. This method is of course not very satisfactory.

Finally, another method which respects gauge invariance is that by Pauli and Villars \cite{9}. The masses of the fictitious particles which are introduced by this method can be interpreted in the sense of a cut-off \cite{10}. However, as soon as the theory allows for particles having different masses propagating in the same loop, this method becomes impracticable. In conclusion, a regularization scheme with an explicit cut-off dependence and preserving symmetries would be very desirable. In this paper, we will not give the ultimate solution of this long-standing problem. However, within the context of renormalization group equations, it is often sufficient to look at the divergent parts of the integrals. A prescription to calculate these divergent parts within a cutoff scheme preserving symmetries is thus on its own an interesting problem. We emphasize in particular the importance of a consistent treatment of quadratic divergences, which in the case of QED always cancel at the end of the calculation, but which survive in other theories and can give rise to a very strong running of the model parameters within the renormalization group \cite{4,11,12}.

We will show through several examples how we can conciliate the explicit dependence on a cut-off and the requirements of symmetry. In Sect. 2, we will start by giving an example where the proper-time method fails. In Sect. 3, we give a prescription which allows to conciliate all constraints. Then, in the remaining sections, we will give some illustrations through several examples (QED in Sect. 4, a phenomenological chiral model in Sect. 5).

2 Proper-time approach

As already mentioned in the introduction, there are essentially three regularization methods implying a cut-off: the naive (three or four dimensional) momentum cut-off, the method by Pauli-Villars and the proper-time one. For the reasons detailed in the introduction we are concentrating on a modified version of the latter, which can immediately be applied to the calculation of Feynman diagrams. However, as we are going to show, ambiguities may arise with this method.

To be more explicit, let us start by giving two expressions which can appear in a one loop
calculation:

\[
A = \int \frac{d^4k}{i(2\pi)^4} \frac{k^2}{(k^2 - m^2)^2},
\]

\[
B = \int \frac{d^4k}{i(2\pi)^4} \frac{1}{(k^2 - m^2)} + \int \frac{d^4k}{i(2\pi)^4} \frac{m^2}{(k^2 - m^2)^2}.
\]

Although both expressions are divergent, we expect a useful regularization procedure to guarantee the equality \(A = B\). Once the Wick rotation is done, the proper-time regularization method as stated in the literature \([8]\) consists in rewriting the integrals and introducing a cut-off \(\Lambda\) via:

\[
\frac{\Gamma(n)}{(k^2 + m^2)^n} = \int_0^\infty d\tau \tau^{n-1} e^{-\tau(k^2 + m^2)} \to \int_{1/\Lambda^2}^\infty d\tau \tau^{n-1} e^{-\tau(k^2 + m^2)},
\]

where \(\Gamma\) is the usual Euler function.

In this paper we are only interested in the divergent part of the integrals, i.e. the parts which stay finite in the limit \(\Lambda \to \infty\). Applying the proper-time method to (1) and (2), one obtains for the divergent contributions:

\[
A_{\text{div}} = -\frac{2}{(4\pi)^2} \left( \Lambda^2 - m^2 \ln \Lambda^2 \right),
\]

\[
B_{\text{div}} = -\frac{1}{(4\pi)^2} (\Lambda^2 - 2m^2 \ln \Lambda^2),
\]

i.e., the proper-time regularization procedure breaks the formal equality \(A = B\). The logarithmic term is the same but the coefficient for the quadratic divergence depends on the way the integrals are written. Of course, one might argue that rewriting (1) in two parts is not correct, since it is divergent, and decide to forbid such a manipulation for the computation of divergences. Such a rule would be arbitrary but acceptable if one could be sure that the results are in agreement with all symmetry requirements. Actually, we will see in the following sections that this is not the case.

More generally, this example exhibits the fact that some ambiguities can arise, so that a special care has to be taken in the computation of divergences. Therefore, we have to proceed differently in order to obtain a consistent dependence on \(\Lambda\). Such a method is developed in the next section.

### 3 Consistent Approach

When dealing with a quantum field theory containing scalar and vector fields, we are confronted with integrals of the typical form

\[
\int \frac{d^4k}{i(2\pi)^d} \frac{k^a}{(k^2 - m^2)^b}.
\]

\(^1\)in this paper, we will only consider UV divergences
Here it is supposed that Feynman parameters have already been introduced, such that the “mass” $m$ might depend on several parameters $x_1, ..., x_n$. The integrals over the Feynman parameters are omitted since they are irrelevant for our discussion. Tensor integrals with $k^n k^n \cdots$ in the numerator are transformed using

$$
\int d^d k k^\mu k^\nu f(k^2) = \frac{g^{\mu\nu}}{d} \int d^d k k^2 f(k^2) \quad (7)
$$

$$
\int d^d k k^n k^\rho k^\eta f(k^2) = \frac{g^{\mu\nu}g^{\rho\eta} + g^{\mu\rho}g^{\nu\eta} + g^{\mu\eta}g^{\nu\rho}}{d(d+2)} \int d^d k k^4 f(k^2) \quad (8)
$$

and analogous rules in the case of more than four indices. In the case of fermion fields, the same integrals appear after the traces over gamma matrices have been evaluated. For our purposes, $b$ and $a$ can take the values $b = 1, \ldots, 4$ and $a = 0, \ldots, 2b - 2$, such that the integral (6) is at most quadratically divergent. Despite we want to calculate our results in $d = 4$, we leave $d$ unevaluated for reasons which will be explained below.

The key point is not to try to regularize all these integrals separately, but to deduce them from a single one. We introduce

$$
I(\alpha, \beta) = \int \frac{d^d k}{i(2\pi)^d} \frac{1}{(\alpha k^2 - \beta m^2)},
$$

(9)

$\alpha$ and $\beta$ being some arbitrary parameters (which will be taken equal to 1 at the end of the calculation). It is easy to see that all the integrals of the form (9) can be rewritten as linear combinations of $I(\alpha, \beta)$ and its derivatives. For instance, the expressions $A$ and $B$ defined by Eqs. (10) and (11) can be written as

$$
A = \frac{\partial}{\partial \alpha} I(\alpha, \beta) \bigg|_{\alpha=1, d=4}.
$$

(10)

and

$$
B = I(\alpha, \beta) \big|_{\alpha=1, d=4} + \frac{\partial}{\partial \beta} I(\alpha, \beta) \bigg|_{\alpha=1, d=4}.
$$

(11)

It is straightforward to show the formal relation

$$
I(\alpha, \beta) = \alpha^{-d/2} \int \frac{d^d k}{i(2\pi)^d} \frac{1}{(k^2 - \beta m^2)} = \alpha^{-d/2} I(1, \beta).
$$

(12)

and the remaining integral $I(1, \beta)$ can be evaluated in $d = 4$ with the help of the proper-time method [8]

$$
I(\alpha = 1, \beta)_{d=4} = -\frac{1}{(4\pi)^2} \left( \Lambda^2 - \beta m^2 \ln \Lambda^2 \right) + \ldots
$$

(13)

(the dots represent terms which stay finite for $\Lambda \to \infty$), and hence

$$
I_{div}(\alpha, \beta) = -\frac{\alpha^{-d/2}}{(4\pi)^2} \left( \Lambda^2 - \beta m^2 \ln \Lambda^2 \right).
$$

(14)
For reasons which will become clear later, we have not yet set \( d = 4 \) in the prefactor \( \alpha^{-d/2} \). One could hope that, if one derived all needed integrals from this single one by taking derivatives with respect to \( \alpha \) and \( \beta \), everything would be consistent. However, there still remains a subtlety as we will see now.

From Eq. (10) we obtain

\[
A_{\text{div}} = -\frac{1}{(4\pi)^2} \left( \frac{d}{2} \Lambda^2 - \frac{d}{2} m^2 \ln \Lambda^2 \right).
\]  

(15)

and from Eq. (11)

\[
B_{\text{div}} = -\frac{1}{(4\pi)^2} (\Lambda^2 - 2m^2 \ln \Lambda^2).
\]  

(16)

If we set \( d = 4 \), we see that the formal equality \( A = B \) is violated. However, we can correct this by keeping \( d = 4 \) only in the term \( \propto \ln \Lambda^2 \), but setting \( d = 2 \) in the coefficient of the term \( \propto \Lambda^2 \). If we do so, the expressions for \( A_{\text{div}} \) and \( B_{\text{div}} \) coincide. This prescription can be generalized to all other integrals under consideration. Although this prescription seems to be rather arbitrary, it allows us to obtain a consistent set of regularized elementary integrals such that the final result is independent of the way the non-regularized integrals are written. These integrals are listed in the appendix.

To generalize the preceding result to loops containing derivative couplings, let us now consider a more complicated example:

\[
C^{\mu\nu} = \int \frac{d^d k}{i(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - m^2)^3}.
\]  

(17)

With the help of Eqs. (7) and (9), this integral can be written as

\[
C^{\mu\nu} = \frac{g^{\mu\nu}}{d} \int \frac{d^d k}{i(2\pi)^d} \frac{k^4}{(k^2 - m^2)^3} = \frac{g^{\mu\nu}}{2d} \frac{\partial^2}{\partial \alpha^2} I(\alpha, \beta) \bigg|_{\alpha = \beta = 1}.
\]  

(18)

Note that we leave \( d \) undetermined until the end of the calculation. Using Eq. (14), we obtain

\[
C^{\mu\nu}_{\text{div}} = -\frac{(d + 2)g^{\mu\nu}}{8(4\pi)^2} (\Lambda^2 - \beta m^2 \ln \Lambda^2),
\]  

(19)

and then, following the prescription given above, we finally set \( d = 2 \) in the term \( \propto \Lambda^2 \) and \( d = 4 \) in the term \( \propto \ln \Lambda^2 \):

\[
C^{\mu\nu}_{\text{div}} = -\frac{g^{\mu\nu}}{4(4\pi)^2} (2\Lambda^2 - 3m^2 \ln \Lambda^2).
\]  

(20)

This method seems very unusual concerning the way to treat the dimensionality of the divergences. However, Veltman [13] already noticed that quadratic divergences are associated with \( d = 2 \) in the context of dimensional regularization, whereas the logarithmic part has to be
treated in dimension \( d = 4 \). More recently, Harada et al. used a similar trick \([4, 14, 15, 16]\) in order to derive the renormalisation group equations for different models of hadronic effective theories. They show [4] that one cannot obtain correct results if one simply works in dimension \( d = 4 \) during the calculation. In fact, the first integrals listed in our appendix have been obtained in Ref. [4] by calculating them in dimensional regularization, expanding the results around \( d = 2 \) and \( d = 4 \), and identifying the resulting poles according to the rules

\[
\frac{1}{2 - d} \rightarrow \frac{\Lambda^2}{8\pi}, \quad (21)
\]

\[
\frac{1}{4 - d} \rightarrow \frac{\ln \Lambda^2}{2}. \quad (22)
\]

We mention that with the help of Eqs. (7) and (8) we can reproduce in this way all the other integrals in our appendix, too. Within this dimensional regularization method, the remaining finite parts of the integrals can in principle be calculated, but they are independent of \( \Lambda \) and therefore irrelevant for the renormalization group equations.

Although we are not giving a formal proof, the examples in the next sections show that our method satisfies symmetry requirements automatically. Let us now come to the first example, the QED lagrangian.

## 4 QED Vacuum Polarization at One Loop

In order to test the procedure previously developed, it is important to consider a simple gauge theory. This is why we start with the regularization of the QED lagrangian in this part. With standard notations, the lagrangian reads:

\[
\mathcal{L} = \bar{\psi}(i\partial \! \! \! / \! - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA^\mu \bar{\psi}\gamma_\mu\psi - \frac{\lambda}{2}(\partial_\mu A^\mu)^2. \quad (23)
\]

When one writes explicitly the gauge invariance of the generating functional, one can obtain some general relations and constraints for \( n \)-points Green functions, called the Ward identities.

One of these identities concerns the transversality of the photon correlation function represented in Fig. 1a. Using standard Feynman rules, it reads:

\[
-i \Pi_{\mu\nu}(p) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} Tr \left[ \gamma_\mu \frac{i}{k^2 - p^2 - m} \gamma_\nu \frac{i}{k^2 - m} \right]. \quad (24)
\]

After introducing the Feynman parameters, we can rewrite (24) as follows:

\[
\Pi_{\mu\nu}(p) = 4e^2 \int \frac{d^4k}{i(2\pi)^4} \int_0^1 dx \frac{2k_\mu k_\nu + [m^2 + x(1 - x)p^2 - k^2]g_{\mu\nu} - 2x(1 - x)p_\mu p_\nu}{(k^2 - \Delta^2)^2}, \quad (25)
\]

where \( \Delta^2 = m^2 - x(1 - x)p^2 \). Using the integrals listed in the appendix, we finally obtain

\[
\Pi_{\mu\nu} = \frac{e^2}{12\pi^2} \left( p^2 g_{\mu\nu} - p_\mu p_\nu \right) \ln \Lambda^2 + \ldots \quad (26)
\]
Figure 1: One-loop Feynman diagrams for vacuum polarization $\Pi^{\mu\nu}(q)$, vertex correction $\Gamma^\mu(p,p)$, and electron self-energy $\Sigma(p)$ in QED.

(again, the dots represent terms which stay finite for $\Lambda \to \infty$), so that we recover immediately the transversality of the photon, as required by the $U(1)$ gauge symmetry. A direct computation with a naive cut-off or the proper-time method does not give this result. It should also be noticed that during the calculation quadratic divergences appear, but in the final result they cancel, as it should be.

Another quite simple and remarkable consequence of gauge invariance is the relation between the vertex function $\Gamma^\mu(p,p)$ and the electron self-energy $\Sigma(p)$, depicted in Figs. 1b and 1c, respectively. The corresponding Ward identity reads:

$$\Gamma^\sigma(p,p) = -\frac{\partial}{\partial p^\sigma} \Sigma(p), \quad (27)$$

where

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{i} \left[ \frac{g_{\mu\nu}}{k^2} + \left( \frac{1}{\lambda} - 1 \right) \frac{k_\mu k_\nu}{k^4} \right] \gamma^\mu \frac{i}{\not{p} - \not{k} - m} \gamma^\nu, \quad (28)$$

and

$$-ie\Gamma^\sigma(p,p) = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{1}{i} \left[ \frac{g_{\mu\nu}}{k^2} + \left( \frac{1}{\lambda} - 1 \right) \frac{k_\mu k_\nu}{k^4} \right] \gamma^\mu \frac{i}{\not{p} - \not{k} - m} \gamma^\sigma \frac{i}{\not{p} - \not{k} - m} \gamma^\nu. \quad (29)$$

After some manipulations on gamma matrices and using the formulas listed in the appendix, one obtains

$$-i\Sigma(p) = \frac{i(-ie)^2}{16\pi^2} \ln \Lambda^2 \left( \left( \frac{1}{\lambda} + 3 \right)m - \frac{1}{\lambda} \not{p} \right) + \ldots \quad (30)$$

and

$$-ie\Gamma^\sigma(p,p) = \frac{-(-ie)^3}{16\pi^2\lambda} \ln \Lambda^2 \gamma^\sigma + \ldots, \quad (31)$$

so that the Ward identity connecting three-point and two-point Green functions is satisfied.

There are of course other Ward identities to be checked, even at one loop level, but the examples given above are considered as significant tests for the consistency of our procedure. We will therefore go on to our second example, which is a phenomenological chiral model.
5 Phenomenological chiral model

As a last illustration, we consider in this section a model which reproduces very well the phenomenology of the hadronic vector and axial-vector correlators in the low-energy region (see [17]). This model contains $\pi$, $\sigma$, $\rho$, and $a_1$ mesons as elementary fields. Using the notations defined in [17], we write the lagrangian in the following form

$$
\mathcal{L} = \frac{1}{2} \partial_{\mu} \Phi \cdot \partial^{\mu} \Phi - \frac{\mu^2}{2} \Phi \cdot \Phi - i g Y_{\mu} \Phi \cdot \partial^{\mu} \Phi - \frac{h_1}{2} Y_{\mu} \Phi \cdot Y^{\mu} \Phi + \frac{h_2}{4} \Phi \cdot \Phi Y_{\mu} Y^{\mu} - \frac{\lambda^2}{4} (\Phi \cdot \Phi)^2
$$

$$
+ c \sigma - \frac{1}{8} \text{tr} (\partial_{\mu} Y_{\nu} - \partial_{\nu} Y_{\mu}) (\partial^{\mu} Y^{\nu} - \partial^{\nu} Y^{\mu}) + \frac{m_0^2}{4} \text{tr} Y_{\mu} Y^{\mu} - \frac{\xi}{4} \text{tr} (\partial_{\mu} Y^{\mu})^2 ,
$$

(32)

where $\Phi = \begin{pmatrix} \sigma \\ \vec{\pi} \end{pmatrix}$ are the scalar and pseudoscalar fields, and $Y_{\mu} = \begin{pmatrix} \vec{\rho}_{\mu} \\ \vec{a}_{1\mu} \end{pmatrix} \cdot \vec{T}_5$ are the vector and axial vector fields. $\vec{T}$ and $\vec{T}_5$ are $O(4)$ matrices related to the global $SU(2)_L \times SU(2)_R$ chiral symmetry, fulfilling the commutation relations

$$
[T_i , T_j] = i \varepsilon_{ijk} T_k , \quad [T_i , T_5^j] = i \varepsilon_{ijk} T_5^k , \quad [T_5^i , T_5^j] = i \varepsilon_{ijk} T_k .
$$

(33)

Experimental information about $\rho$ and $a_1$ mesons is obtained from electromagnetic processes and $\tau$ decay data. As a consequence, it is necessary to introduce the photon and the $W$ fields and thus replace the ordinary derivatives by covariant ones, following

$$
D_{\mu} \Phi = \left( \partial_{\mu} - i e A_{\mu} T_3 - \frac{i e \cos \theta_C}{\sin \theta_W} (W_{1\mu} T_1^L + W_{2\mu} T_2^L) \right) \Phi ,
$$

$$
D_{\mu} Y_{\nu} = \partial_{\mu} Y_{\nu} - i e A_{\mu} [T_3 , Y_{\nu}] - \frac{i e \cos \theta_C}{\sin \theta_W} (W_{1\mu} [T_1^L , Y_{\nu}] + W_{2\mu} [T_2^L , Y_{\nu}]) .
$$

(34)

In order to determine the photon self-energy, one has to evaluate the different contributions depicted in Ref. [17] figures 10 and 11. Thanks to the method described in the preceding section, it is possible to show after some tedious calculations that the total $\gamma$ self-energy is simply proportional to $q^2 g_{\mu\nu} - q_{\mu} q_{\nu}$, as it should be. Furthermore, we have explicitly checked that this result cannot be obtained by a direct use of the proper-time method.

Another important test of our method is the Goldstone theorem. If there is no explicit symmetry breaking term ($c = 0$ in our case), but the global symmetry is spontaneously broken in the vacuum, then there has to be a massless particle which corresponds to the pion in our case. Moreover, we have to take into account the mixing between $\pi$ and $a_1$. This calculation is highly non trivial; one has to evaluate all the diagrams depicted in Ref. [17] (see figure 6). Writing the self-energy for the pion in this model, one finally obtains in the chiral limit ($c = 0$)

$$
\Sigma_{\pi\pi}(k^2) = \frac{3g^2}{16\pi^2 \xi} (1 - 3\xi) k^2 \ln \Lambda^2 + \ldots ,
$$

(35)

i.e., the pion self-energy is simply proportional to $k^2$ and therefore does not destroy the Goldstone character of the pion. It is also interesting to note that, as in the case of the photon, all
quadratic divergences have canceled in the final result. The above expression for $\Sigma_{\pi\pi}$ does not include the resummation of the $\pi - a_1$ mixing term. When including this mixing we also recover the Goldstone theorem ($\Sigma_{\pi\pi}(k^2 = 0) = 0$), but the expression for the total pion self-energy is far more complicated, such that we refrain from giving the full expression here. Again, this result is non trivial and we should note that, up to now, it has only been obtained in a dimensional regularization approach (see [17]).

In all these examples, we have seen that the way of computing divergences described in the third section respects symmetry requirements.

6 Conclusions

We have seen in this paper how to handle logarithmic and quadratic divergences in a cut-off regularization scheme in a consistent way, i.e., in a way satisfying constraints from symmetry requirements. For example, we have shown that our approach was preserving the transversality of the photon polarization tensor in QED as well as the transversality of the photon polarization tensor and the Goldstone theorem in the case of a phenomenological chiral model including $\pi$, $\sigma$, $\rho$, and $a_1$ mesons. In all these examples, the quadratic divergences cancel at the end of the calculation due to the symmetry properties of the model. Recently our method has also been applied to QED with one extra dimension [12], where the quadratic divergences survive and give rise to a power-law in the running of the effective four-dimensional gauge coupling. We have to stress, however, that the present method is only designed to compute the divergent parts of the integrals.

Our aim is now to use this regularization scheme in order to derive renormalization group equations for the phenomenological chiral model described in the last section. In fact, it is interesting to incorporate the scalar degree of freedom to extend the results of Harada et al. [14] [15] [16] concerning the chiral symmetry restoration.

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APPENDIX: TABLE OF INTEGRALS

We present in this appendix a detailed and complete list of regularized integrals necessary for the calculation of all integrals described in the main text and in [17]. As already mentioned, only the divergent parts are given.

\begin{align*}
\int \frac{d^4k}{i(2\pi)^4} \frac{1}{(k^2 - m^2)^2} & = -\frac{1}{(4\pi)^2} [\Lambda^2 - m^2 \ln \Lambda^2] + \ldots & (36) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^2}{(k^2 - m^2)^2} & = \frac{1}{(4\pi)^2} \ln \Lambda^2 + \ldots & (37) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^2}{(k^2 - m^2)^2} & = -\frac{1}{(4\pi)^2} [\Lambda^2 - 2m^2 \ln \Lambda^2] + \ldots & (38) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 - m^2)^2} & = -\frac{g^{\mu\nu}}{2(4\pi)^2} [\Lambda^2 - m^2 \ln \Lambda^2] + \ldots & (39) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^2}{(k^2 - m^2)^3} & = \frac{1}{(4\pi)^2} \ln \Lambda^2 + \ldots & (40) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^\mu k^\nu}{(k^2 - m^2)^3} & = \frac{g^{\mu\nu}}{4(4\pi)^2} \ln \Lambda^2 + \ldots & (41) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^4}{(k^2 - m^2)^3} & = -\frac{1}{(4\pi)^2} [\Lambda^2 - 3m^2 \ln \Lambda^2] + \ldots & (42) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^2 k^\mu k^\nu}{(k^2 - m^2)^3} & = -\frac{1}{4(4\pi)^2} [2\Lambda^2 - 3m^2 \ln \Lambda^2] + \ldots & (43) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^\mu k^\nu k^\rho k^\eta}{(k^2 - m^2)^3} & = -\frac{1}{8(4\pi)^2} \left[ g^{\mu\nu} g^{\rho\eta} + g^{\mu\rho} g^{\nu\eta} + g^{\mu\eta} g^{\nu\rho} \right] [\Lambda^2 - m^2 \ln \Lambda^2] + \ldots & (44) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^4}{(k^2 - m^2)^4} & = \frac{1}{(4\pi)^2} \ln \Lambda^2 + \ldots & (45) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^2 k^\mu k^\nu}{(k^2 - m^2)^4} & = \frac{g^{\mu\nu}}{4(4\pi)^2} \ln \Lambda^2 + \ldots & (46) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^\mu k^\nu k^\rho k^\eta}{(k^2 - m^2)^4} & = \frac{1}{24(4\pi)^2} \left[ g^{\mu\nu} g^{\rho\eta} + g^{\mu\rho} g^{\nu\eta} + g^{\mu\eta} g^{\nu\rho} \right] \ln \Lambda^2 + \ldots & (47) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^6}{(k^2 - m^2)^4} & = -\frac{1}{(4\pi)^2} [\Lambda^2 - 4m^2 \ln \Lambda^2] + \ldots & (48) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^4 k^\mu k^\nu}{(k^2 - m^2)^4} & = -\frac{g^{\mu\nu}}{2(4\pi)^2} [\Lambda^2 - 2m^2 \ln \Lambda^2] + \ldots & (49) \\
\int \frac{d^4k}{i(2\pi)^4} \frac{k^2 k^\mu k^\nu k^\rho k^\eta}{(k^2 - m^2)^4} & = -\frac{1}{24(4\pi)^2} \left[ g^{\mu\nu} g^{\rho\eta} + g^{\mu\rho} g^{\nu\eta} + g^{\mu\eta} g^{\nu\rho} \right] [3\Lambda^2 - 4m^2 \ln \Lambda^2] + \ldots & (50)
\end{align*}
References

[1] G. ’t Hooft and M.J.G. Veltman, Nucl. Phys. B 44 (1972) 189.

[2] S. Weinberg, The Quantum Theory of Fields, Vol. 1: Foundations (Cambridge University Press, 1995).

[3] S. Weinberg, The Quantum Theory of Fields, Vol. 2: Modern applications (Cambridge University Press, 1996).

[4] M. Harada and K. Yamawaki, Phys. Rep. 381, 1 (2003).

[5] H. Georgi, Ann. Rev. Nucl. Part. Sci. 43 (1993) 209.

[6] S. P. Klevansky, Rev. Mod. Phys. 64 (1992) 649.

[7] J. Schwinger, Phys. Rev. 74 (1948) 1439.

[8] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Oxford Science Publications, 1993).

[9] W. Pauli and F. Villars, Rev. Mod. Phys. 21 (1949) 434.

[10] C. Itzykson and J.B. Zuber, Quantum Field Theory (Mc Graw-Hill, 1980).

[11] K.R. Dienes, E. Dudas, and T. Gherghetta, Nucl. Phys. B 537 (1999) 47.

[12] T. Varin, J. Welzel, A. Deandrea, and D. Davesne, arXiv:hep-ph/0610130 (2006).

[13] M.J.G. Veltman, Acta Phys. Polon. B 12 (1981) 437.

[14] M. Harada and K. Yamawaki, Phys. Rev. D 64 (2001) 014023.

[15] M. Harada and K. Yamawaki, Phys. Rev. Lett. 87 (2001) 152001.

[16] Y. Hidaka, O. Morimatsu and M. Ohtani, Phys. Rev. D 73 (2006) 036004.

[17] M. Urban, M. Buballa and J. Wambach, Nucl. Phys. A 697 (2002) 338.
How to preserve symmetries with cut-off regularized integrals?

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Abstract

We present a prescription to calculate the quadratic and logarithmic divergent parts of several integrals employing a cutoff in a coherent way, i.e., in total agreement with symmetry requirements. As examples we consider one-loop Ward identities for QED and a phenomenological chiral model.

1 Introduction

In quantum field theory, ultraviolet (UV) divergences always emerge from momentum integration in the loops. In order to identify the physical content of the theory under consideration, one has to subtract these divergences with a definite regularization procedure. Among them, the dimensional regularization is probably the most popular one because it is easy to use and respects all the symmetry properties of the initial lagrangian [1, 2, 3]. The latter aspect is of fundamental importance and could justify by itself the choice of dimensional regularization.

Nevertheless, the introduction of an explicit cut-off is sometimes advantageous, for example as soon as one considers renormalization group equations in the Wilsonian approach [3, 4] or effective theories [5]. In that context, the problem concerning the non-respect of symmetries arises. A three-momentum cut-off has been proved to be useful, e.g., in the case of the Nambu-Jona-Lasinio model (see, e.g., Ref. [6]) but it does not respect Lorentz invariance. This can be avoided by using a four-momentum cut-off, but the latter suffers from the fact that it violates gauge invariance, too. This can be seen for instance from the fact that the one-loop photon
polarization tensor in QED is no longer transverse. The problem has been solved by Schwinger within his proper-time approach [7]. However, his formalism is very different from the usual Feynman diagram technique. In the literature there exists a modified version of the proper-time regularization which can directly be applied to Feynman diagrams [8], but as shown later in this paper, the results are ambiguous since they depend on the way the propagators are written. On purely phenomenological grounds, the problem is sometimes “solved” by simply imposing gauge invariance at hand, i.e., if we keep the example of the photon polarization tensor in QED, by projecting it onto the transverse part. This method is of course not very satisfactory.

Finally, another method which respects gauge invariance is that by Pauli and Villars [9]. The masses of the fictitious particles which are introduced by this method can be interpreted in the sense of a cut-off [10]. However, as soon as the theory allows for particles having different masses propagating in the same loop, this method becomes impracticable. In conclusion, a regularization scheme with an explicit cut-off dependence and preserving symmetries would be very desirable. In this paper, we will not give the ultimate solution of this long-standing problem. However, within the context of renormalization group equations, it is often sufficient to look at the divergent parts of the integrals. A prescription to calculate these divergent parts within a cutoff scheme preserving symmetries is thus on its own an interesting problem. We emphasize in particular the importance of a consistent treatment of quadratic divergences, which in the case of QED always cancel at the end of the calculation, but which survive in other theories and can give rise to a very strong running of the model parameters within the renormalization group [4, 11, 12].

We will show through several examples how we can conciliate the explicit dependence on a cut-off and the requirements of symmetry. In Sect. 2, we will start by giving an example where the proper-time method fails. In Sect. 3, we give a prescription which allows to conciliate all constraints. Then, in the remaining sections, we will give some illustrations through several examples (QED in Sect. 4, a phenomenological chiral model in Sect. 5).

2 PROPER-TIME APPROACH

As already mentioned in the introduction, there are essentially three regularization methods implying a cut-off: the naive (three or four dimensional) momentum cut-off, the method by Pauli-Villars and the proper-time one. For the reasons detailed in the introduction we are concentrating on a modified version of the latter, which can immediately be applied to the calculation of Feynman diagrams. However, as we are going to show, ambiguities may arise with this method.

To be more explicit, let us start by giving two expressions which can appear in a one loop
calculation:

\[ A = \int \frac{d^4k}{i(2\pi)^4} \frac{k^2}{(k^2 - m^2)^2}, \]  
\[ B = \int \frac{d^4k}{i(2\pi)^4} \frac{1}{(k^2 - m^2)} + \int \frac{d^4k}{i(2\pi)^4} \frac{m^2}{(k^2 - m^2)^2}. \]

Although both expressions are divergent, we expect a useful regularization procedure to guarantee the equality \( A = B \). Once the Wick rotation is done, the proper-time regularization method as stated in the literature \cite{8} consists in rewriting the integrals and introducing a cut-off \( \Lambda \) via:

\[ \Gamma(n) \frac{1}{(k^2 + m^2)^n} = \int_0^\infty d\tau \tau^{n-1} e^{-\tau(k^2 + m^2)} \rightarrow \int_0^\infty d\tau \tau^{n-1} e^{-\tau(k^2 + m^2)}, \]

where \( \Gamma \) is the usual Euler function.

In this paper we are only interested in the divergent part of the integrals, i.e. the parts which stay finite in the limit \( \Lambda \to \infty \). Applying the proper-time method to (1) and (2), one obtains for the divergent contributions:

\[ A_{div} = -\frac{2}{(4\pi)^2} (\Lambda^2 - m^2 \ln \Lambda^2), \]
\[ B_{div} = -\frac{1}{(4\pi)^2} (\Lambda^2 - 2m^2 \ln \Lambda^2), \]

i.e., the proper-time regularization procedure breaks the formal equality \( A = B \). The logarithmic term is the same but the coefficient for the quadratic divergence depends on the way the integrals are written. Of course, one might argue that rewriting (1) in two parts is not correct, since it is divergent, and decide to forbid such a manipulation for the computation of divergences. Such a rule would be arbitrary but acceptable if one could be sure that the results are in agreement with all symmetry requirements. Actually, we will see in the following sections that this is not the case.

More generally, this example exhibits the fact that some ambiguities can arise, so that a special care has to be taken in the computation of divergences. Therefore, we have to proceed differently in order to obtain a consistent dependence on \( \Lambda \). Such a method is developed in the next section.

3 CONSISTENT APPROACH

When dealing with a quantum field theory containing scalar and vector fields, we are confronted with integrals of the typical form

\[ \int \frac{d^dk}{i(2\pi)^d} \frac{k^a}{(k^2 - m^2)^b}, \]

\[ ^1 \text{in this paper, we will only consider UV divergences} \]
Here it is supposed that Feynman parameters have already been introduced, such that the “mass” \( m \) might depend on several parameters \( x_1, \ldots, x_n \). The integrals over the Feynman parameters are omitted since they are irrelevant for our discussion. Tensor integrals with \( k^\mu k^\nu \) \( \cdots \) in the numerator are transformed using

\[
\int d^d k^\mu k^\nu f(k^2) = \frac{g^{\mu\nu}}{d} \int d^d k^2 f(k^2)
\]

and analogous rules in the case of more than four indices. In the case of fermion fields, the same integrals appear after the traces over gamma matrices have been evaluated. For our purposes, \( b \) and \( a \) can take the values \( b = 1, \ldots, 4 \) and \( a = 0, \ldots, 2b - 2 \), such that the integral (6) is at most quadratically divergent. Despite we want to calculate our results in \( d = 4 \), we leave \( d \) unevaluated for reasons which will be explained below.

The key point is not to try to regularize all these integrals separately, but to deduce them from a single one. We introduce

\[
I(\alpha, \beta) = \int \frac{d^d k}{i(2\pi)^d} \frac{1}{(\alpha k^2 - \beta m^2)}
\]

\( \alpha \) and \( \beta \) being some arbitrary parameters (which will be taken equal to 1 at the end of the calculation). It is easy to see that all the integrals of the form (6) can be rewritten as linear combinations of \( I(\alpha, \beta) \) and its derivatives. For instance, the expressions \( A \) and \( B \) defined by Eqs. (1) and (2) can be written as

\[
A = -\frac{\partial}{\partial \alpha} I(\alpha, \beta) \bigg|_{\alpha=\beta=1,d=4}.
\]

and

\[
B = I(\alpha, \beta)\big|_{\alpha=\beta=1,d=4} + \frac{\partial}{\partial \beta} I(\alpha, \beta) \bigg|_{\alpha=\beta=1,d=4}.
\]

It is straightforward to show the formal relation

\[
I(\alpha, \beta) = \alpha^{d/2} \int \frac{d^d k}{i(2\pi)^d} \frac{1}{(k^2 - \beta m^2)} = \alpha^{d/2} I(1, \beta).
\]

and the remaining integral \( I(1, \beta) \) can be evaluated in \( d = 4 \) with the help of the proper-time method [8]

\[
I(1, \beta)_{d=4} = -\frac{1}{(4\pi)^2} \left( \Lambda^2 - \beta m^2 \ln \Lambda^2 \right) + \ldots
\]

(the dots represent terms which stay finite for \( \Lambda \to \infty \)), and hence

\[
I_{div}(\alpha, \beta) = -\frac{\alpha^{-d/2}}{(4\pi)^2} \left( \Lambda^2 - \beta m^2 \ln \Lambda^2 \right).
\]
For reasons which will become clear later, we have not yet set $d = 4$ in the prefactor $\alpha^{-d/2}$. One could hope that, if one derived all needed integrals from this single one by taking derivatives with respect to $\alpha$ and $\beta$, everything would be consistent. However, there still remains a subtlety as we will see now.

From Eq. (10) we obtain

$$A_{\text{div}} = -\frac{1}{(4\pi)^2} \left( \frac{d}{2} \Lambda^2 - \frac{d}{2} m^2 \ln \Lambda^2 \right).$$

(15)

and from Eq. (11)

$$B_{\text{div}} = -\frac{1}{(4\pi)^2} (\Lambda^2 - 2m^2 \ln \Lambda^2).$$

(16)

If we set $d = 4$, we see that the formal equality $A = B$ is violated. However, we can correct this by keeping $d = 4$ only in the term $\propto \ln \Lambda^2$, but setting $d = 2$ in the coefficient of the term $\propto \Lambda^2$. If we do so, the expressions for $A_{\text{div}}$ and $B_{\text{div}}$ coincide. This prescription can be generalized to all other integrals under consideration. Although this prescription seems to be rather arbitrary, it allows us to obtain a consistent set of regularized elementary integrals such that the final result is independent of the way the non-regularized integrals are written. These integrals are listed in the appendix.

To generalize the preceding result to loops containing derivative couplings, let us now consider a more complicated example:

$$C_{\mu\nu} = \int \frac{d^d k}{i(2\pi)^d} \frac{k^\mu k^\nu}{(k^2 - m^2)^3}. $$

(17)

With the help of Eqs. (7) and (9), this integral can be written as

$$C_{\mu\nu} = \frac{g_{\mu\nu}}{d} \int \frac{d^d k}{i(2\pi)^d} \frac{k^4}{(k^2 - m^2)^3} = \frac{g_{\mu\nu}}{d} \frac{\partial^2}{\partial \alpha^2} I(\alpha, \beta) \bigg|_{\alpha = \beta = 1}. $$

(18)

Note that we leave $d$ undetermined until the end of the calculation. Using Eq. (14), we obtain

$$C_{\text{div}}^{\mu\nu} = -\frac{(d+2)g_{\mu\nu}}{8(4\pi)^2} (\Lambda^2 - \beta m^2 \ln \Lambda^2),$$

(19)

and then, following the prescription given above, we finally set $d = 2$ in the term $\propto \Lambda^2$ and $d = 4$ in the term $\propto \ln \Lambda^2$:

$$C_{\text{div}}^{\mu\nu} = -\frac{g_{\mu\nu}}{4(4\pi)^2} (2\Lambda^2 - 3m^2 \ln \Lambda^2).$$

(20)

This method seems very unusual concerning the way to treat the dimensionality of the divergences. However, Veltman [13] already noticed that quadratic divergences are associated with $d = 2$ in the context of dimensional regularization, whereas the logarithmic part has to be
treated in dimension $d = 4$. More recently, Harada et al. used a similar trick [4, 14, 15, 16] in order to derive the renormalisation group equations for different models of hadronic effective theories. They show [4] that one cannot obtain correct results if one simply works in dimension $d = 4$ during the calculation. In fact, the first integrals listed in our appendix have been obtained in Ref. [4] by calculating them in dimensional regularization, expanding the results around $d = 2$ and $d = 4$, and identifying the resulting poles according to the rules

$$\frac{1}{2 - d} \rightarrow \frac{\Lambda^2}{8\pi},$$

$$\frac{1}{4 - d} \rightarrow \frac{\ln \Lambda^2}{2}. \quad (21)$$

We mention that with the help of Eqs. (7) and (8) we can reproduce in this way all the other integrals in our appendix, too. Within this dimensional regularization method, the remaining finite parts of the integrals can in principle be calculated, but they are independent of $\Lambda$ and therefore irrelevant for the renormalization group equations.

Although we are not giving a formal proof, the examples in the next sections show that our method satisfies symmetry requirements automatically. Let us now come to the first example, the QED lagrangian.

## 4 QED Vacuum Polarization at One Loop

In order to test the procedure previously developed, it is important to consider a simple gauge theory. This is why we start with the regularization of the QED lagrangian in this part. With standard notations, the lagrangian reads:

$$\mathcal{L} = \bar{\psi}(i\beta - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eA^\mu \bar{\psi}\gamma^\mu\psi - \frac{\lambda}{2}(\partial_\mu A^\mu)^2. \quad (23)$$

When one writes explicitly the gauge invariance of the generating functional, one can obtain some general relations and constraints for $n$-points Green functions, called the Ward identities.

One of these identities concerns the transversality of the photon correlation function represented in Fig. 1a. Using standard Feynman rules, it reads:

$$-i\Pi_{\mu\nu}(p) = -(-ie)^2 \int \frac{d^4k}{(2\pi)^4} Tr \left[ \gamma_\mu \frac{i}{k - \beta - m} \gamma_\nu \frac{i}{k - m} \right]. \quad (24)$$

After introducing the Feynman parameters, we can rewrite (24) as follows:

$$\Pi_{\mu\nu}(p) = 4e^2 \int \frac{d^4k}{i(2\pi)^4} \int_0^1 dx \frac{2k_\mu k_\nu + [m^2 + x(1 - x)p^2 - k^2]g_{\mu\nu} - 2x(1 - x)p_\mu p_\nu}{(k^2 - \Delta^2)^2}, \quad (25)$$

where $\Delta^2 = m^2 - x(1 - x)p^2$. Using the integrals listed in the appendix, we finally obtain

$$\Pi_{\mu\nu} = \frac{e^2}{12\pi^2} (p^2 g_{\mu\nu} - p_\mu p_\nu) \ln \Lambda^2 + \ldots \quad (26)$$
Figure 1: One-loop Feynman diagrams for vacuum polarization $\Pi^{\mu\nu}(q)$, vertex correction $\Gamma_\mu(p, p)$, and electron self-energy $\Sigma(p)$ in QED.

(again, the dots represent terms which stay finite for $\Lambda \to \infty$), so that we recover immediately the transversality of the photon, as required by the $U(1)$ gauge symmetry. A direct computation with a naive cut-off or the proper-time method does not give this result. It should also be noticed that during the calculation quadratic divergences appear, but in the final result they cancel, as it should be.

Another quite simple and remarkable consequence of gauge invariance is the relation between the vertex function $\Gamma_\mu(p, p)$ and the electron self-energy $\Sigma(p)$, depicted in Figs. 1b and 1c, respectively. The corresponding Ward identity reads:

$$\Gamma_\sigma(p, p) = -\frac{\partial}{\partial p^\rho} \Sigma(p),$$  \hfill (27)

where

$$-i\Sigma(p) = (-ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{i} \left[ \frac{g_{\mu\nu}}{k^2} + \left( \frac{1}{\lambda} - 1 \right) \frac{k_\mu k_\nu}{k^4} \right] \gamma^\mu \frac{i}{\slashed{p} - \slashed{k} - m} \gamma^\nu $$ \hfill (28)

and

$$-ie\Gamma_\sigma(p, p) = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{1}{i} \left[ \frac{g_{\mu\nu}}{k^2} + \left( \frac{1}{\lambda} - 1 \right) \frac{k_\mu k_\nu}{k^4} \right] \gamma^\mu \frac{i}{\slashed{p} - \slashed{k} - m} \gamma_\sigma \frac{i}{\slashed{p} - \slashed{k} - m} \gamma^\nu.$$ \hfill (29)

After some manipulations on gamma matrices and using the formulas listed in the appendix, one obtains

$$-i\Sigma(p) = \frac{i(-ie)^2}{16\pi^2} \ln \Lambda^2 \left( \left( \frac{1}{\lambda} + 3 \right) m - \frac{1}{\Lambda} \rho \right) + \ldots$$ \hfill (30)

and

$$-ie\Gamma_\sigma(p, p) = \frac{-(-ie)^3}{16\pi^2\lambda} \ln \Lambda^2 \gamma_\sigma + \ldots,$$ \hfill (31)

so that the Ward identity connecting three-point and two-point Green functions is satisfied.

There are of course other Ward identities to be checked, even at one loop level, but the examples given above are considered as significant tests for the consistency of our procedure. We will therefore go on to our second example, which is a phenomenological chiral model.
5 Phenomenological chiral model

As a last illustration, we consider in this section a model which reproduces very well the phenomenology of the hadronic vector and axial-vector correlators in the low-energy region (see [17]). This model contains $\pi$, $\sigma$, $\rho$, and $a_1$ mesons as elementary fields. Using the notations defined in [17], we write the lagrangian in the following form

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \cdot \partial^\mu \Phi - \frac{\mu^2}{2} \Phi \cdot \Phi - ig Y_\mu \Phi \cdot \partial^\mu \Phi - \frac{h_1}{2} Y_\mu \Phi \cdot Y^\mu \Phi + \frac{h_2}{4} \Phi \cdot \Phi \text{tr} Y_\mu Y^\mu - \frac{\lambda^2}{4} (\Phi \cdot \Phi)^2$$

$$+ c \sigma - \frac{1}{8} \text{tr} (\partial_\mu Y_\nu - \partial_\nu Y_\mu) (\partial^\mu Y^\nu - \partial^\nu Y^\mu) + \frac{m_0^2}{4} \text{tr} Y_\mu Y^\mu - \frac{\xi}{4} \text{tr} (\partial_\mu Y^\mu)^2,$$

(32)

where $\Phi = \begin{pmatrix} \sigma \\
\vec{\pi} \end{pmatrix}$ are the scalar and pseudoscalar fields, and $Y_\mu = \vec{\rho}_\mu \cdot \vec{T} + \vec{a}_1 \mu \cdot \vec{T}^5$ are the vector and axial vector fields. $\vec{T}$ and $\vec{T}^5$ are $O(4)$ matrices related to the global $SU(2)_L \times SU(2)_R$ chiral symmetry, fulfilling the commutation relations

$$[T_i, T_j] = i \epsilon_{ijk} T_k, \quad [T_i, T^5_j] = i \epsilon_{ijk} T^5_k, \quad [T^5_i, T^5_j] = i \epsilon_{ijk} T^5_k.$$

(33)

Experimental information about $\rho$ and $a_1$ mesons is obtained from electromagnetic processes and $\tau$ decay data. As a consequence, it is necessary to introduce the photon and the $W$ fields and thus replace the ordinary derivatives by covariant ones, following

$$D_\mu \Phi = \left( \partial_\mu - ie A_\mu T_3 - \frac{i e \cos \theta_C}{\sin \theta_W} (W_1 T^L_1 + W_2 T^L_2) \right) \Phi,$$

$$D_\mu Y_\nu = \partial_\mu Y_\nu - ie A_\mu [T_3, Y_\nu] - \frac{i e \cos \theta_C}{\sin \theta_W} (W_1 [T^L_1, Y_\nu] + W_2 [T^L_2, Y_\nu]).$$

(34)

In order to determine the photon self-energy, one has to evaluate the different contributions depicted in Ref. [17] figures 10 and 11. Thanks to the method described in the preceding section, it is possible to show after some tedious calculations that the total $\gamma$ self-energy is simply proportional to $q^2 g_{\mu\nu} - q_\mu q_\nu$, as it should be. Furthermore, we have explicitly checked that this result cannot be obtained by a direct use of the proper-time method.

Another important test of our method is the Goldstone theorem. If there is no explicit symmetry breaking term ($c = 0$ in our case), but the global symmetry is spontaneously broken in the vacuum, then there has to be a massless particle which corresponds to the pion in our case. Moreover, we have to take into account the mixing between $\pi$ and $a_1$. This calculation is highly non trivial; one has to evaluate all the diagrams depicted in Ref. [17] (see figure 6). Writing the self-energy for the pion in this model, one finally obtains in the chiral limit ($c = 0$)

$$\Sigma_{\pi\pi}(k^2) = \frac{3g^2}{16\pi^2 \xi} (1 - 3\xi) k^2 \ln \Lambda^2 + \ldots,$$

(35)

i.e., the pion self-energy is simply proportional to $k^2$ and therefore does not destroy the Goldstone character of the pion. It is also interesting to note that, as in the case of the photon, all
quadratic divergences have canceled in the final result. The above expression for $\Sigma_{\pi\pi}$ does not include the resummation of the $\pi - a_1$ mixing term. When including this mixing we also recover the Goldstone theorem ($\Sigma_{\pi\pi}(k^2 = 0) = 0$), but the expression for the total pion self-energy is far more complicated, such that we refrain from giving the full expression here. Again, this result is non trivial and we should note that, up to now, it has only been obtained in a dimensional regularization approach (see [17]).

In all these examples, we have seen that the way of computing divergences described in the third section respects symmetry requirements.

6 CONCLUSIONS

We have seen in this paper how to handle logarithmic and quadratic divergences in a cut-off regularization scheme in a consistent way, i.e., in a way satisfying constraints from symmetry requirements. For example, we have shown that our approach was preserving the transversality of the photon polarization tensor in QED as well as the transversality of the photon polarization tensor and the Goldstone theorem in the case of a phenomenological chiral model including $\pi$, $\sigma$, $\rho$, and $a_1$ mesons. In all these examples, the quadratic divergences cancel at the end of the calculation due to the symmetry properties of the model. Recently our method has also been applied to QED with one extra dimension [12], where the quadratic divergences survive and give rise to a power-law in the running of the effective four-dimensional gauge coupling. We have to stress, however, that the present method is only designed to compute the divergent parts of the integrals.

Our aim is now to use this regularization scheme in order to derive renormalization group equations for the phenomenological chiral model described in the last section. In fact, it is interesting to incorporate the scalar degree of freedom to extend the results of Harada et al. [14, 15, 16] concerning the chiral symmetry restoration.

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APPENDIX: TABLE OF INTEGRALS

We present in this appendix a detailed and complete list of regularized integrals necessary for the calculation of all integrals described in the main text and in [17]. As already mentioned, only the divergent parts are given.

\[
\begin{align*}
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{1}{k^2 - m^2} \right) &= -\frac{1}{(4\pi)^2} \left[ \Lambda^2 - m^2 \ln \Lambda^2 \right] + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{1}{k^2} \right) &= -\frac{1}{(4\pi)^2} \ln \Lambda^2 + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^2}{k^2 - m^2} \right) &= -\frac{1}{(4\pi)^2} \left[ \Lambda^2 - 2m^2 \ln \Lambda^2 \right] + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^{\mu}k^{\nu}}{k^2 - m^2} \right) &= -\frac{g^{\mu\nu}}{2(4\pi)^2} \left[ \Lambda^2 - m^2 \ln \Lambda^2 \right] + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^2}{k^2 - m^2} \right) &= \frac{1}{(4\pi)^2} \ln \Lambda^2 + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^2}{k^2 - m^2} \right) &= \frac{g^{\mu\nu}}{4(4\pi)^2} \ln \Lambda^2 + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^4}{k^2 - m^2} \right) &= -\frac{1}{(4\pi)^2} \left[ \Lambda^2 - 3m^2 \ln \Lambda^2 \right] + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^{\mu}k^{\nu}k^{\rho}}{k^2 - m^2} \right) &= -\frac{1}{4(4\pi)^2} \left[ 2\Lambda^2 - 3m^2 \ln \Lambda^2 \right] + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^{\mu}k^{\nu}k^{\rho}k^{\sigma}}{k^2 - m^2} \right) &= -\frac{1}{8(4\pi)^2} \left[ g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} \right] \left[ \Lambda^2 - m^2 \ln \Lambda^2 \right] + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^4}{k^2 - m^2} \right) &= \frac{1}{(4\pi)^2} \ln \Lambda^2 + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^{\mu}k^{\nu}k^{\rho}}{k^2 - m^2} \right) &= \frac{g^{\mu\nu}}{4(4\pi)^2} \ln \Lambda^2 + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^{\mu}k^{\nu}k^{\rho}k^{\sigma}}{k^2 - m^2} \right) &= -\frac{1}{24(4\pi)^2} \left[ g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} \right] \ln \Lambda^2 + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^6}{k^2 - m^2} \right) &= -\frac{1}{(4\pi)^2} \left[ \Lambda^2 - 4m^2 \ln \Lambda^2 \right] + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^4k^{\mu}k^{\nu}}{k^2 - m^2} \right) &= -\frac{g^{\mu\nu}}{2(4\pi)^2} \left[ \Lambda^2 - 2m^2 \ln \Lambda^2 \right] + \ldots \\
\int \frac{d^4k}{i(2\pi)^4} \left( \frac{k^2k^{\mu}k^{\nu}k^{\rho}}{k^2 - m^2} \right) &= -\frac{1}{24(4\pi)^2} \left[ g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho} \right] \left[ 3\Lambda^2 - 4m^2 \ln \Lambda^2 \right] + \ldots
\end{align*}
\]
References

[1] G. ’t Hooft and M.J.G. Veltman, Nucl. Phys. B 44 (1972) 189.

[2] S. Weinberg, The Quantum Theory of Fields, Vol. 1: Foundations (Cambridge University Press, 1995).

[3] S. Weinberg, The Quantum Theory of Fields, Vol. 2: Modern applications (Cambridge University Press, 1996).

[4] M. Harada and K. Yamawaki, Phys. Rep. 381, 1 (2003).

[5] H. Georgi, Ann. Rev. Nucl. Part. Sci. 43 (1993) 209.

[6] S. P. Klevansky, Rev. Mod. Phys. 64 (1992) 649.

[7] J. Schwinger, Phys. Rev. 74 (1948) 1439.

[8] J. Zinn-Justin, Quantum Field Theory and Critical Phenomena (Oxford Science Publications, 1993).

[9] W. Pauli and F. Villars, Rev. Mod. Phys. 21 (1949) 434.

[10] C. Itzykson and J.B. Zuber, Quantum Field Theory (Mc Graw-Hill, 1980).

[11] K.R. Dienes, E. Dudas, and T. Gherghetta, Nucl. Phys. B 537 (1999) 47.

[12] T. Varin, J. Welzel, A. Deandrea, and D. Davesne, arXiv:hep-ph/0610130 (2006).

[13] M.J.G. Veltman, Acta Phys. Polon. B 12 (1981) 437.

[14] M. Harada and K. Yamawaki, Phys. Rev. D 64 (2001) 014023.

[15] M. Harada and K. Yamawaki, Phys. Rev. Lett. 87 (2001) 152001.

[16] Y. Hidaka, O. Morimatsu and M. Ohtani, Phys. Rev. D 73 (2006) 036004.

[17] M. Urban, M. Buballa and J. Wambach, Nucl. Phys. A 697 (2002) 338.