ON THE STEADY STATE BIFURCATION OF THE CAHN-HILLIARD/ALLEN-CAHN SYSTEM

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ABSTRACT. In this paper, the steady state bifurcation of the Cahn-Hilliard/Allen-Cahn system is investigated. By using the Lyapunov-Schmidt method, combining with the implicit function theorem, we prove that this system bifurcates from the trivial solution to the nontrivial solution branch as parameter crosses certain critical value. The expression of bifurcated solution is also obtained.

1. Introduction. In order to model simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point, Cahn and Novick-Cohen [2] derived the Cahn-Hilliard/Allen-Cahn system in the following form, denoted as (CH/AC):

\[
\begin{align*}
\frac{\partial u}{\partial t} &= h^2 \Delta (g(u + v) + g(u - v) - h^2 \Delta u), \quad x \in \Omega, \\
\frac{\partial v}{\partial t} &= -g(u + v) + g(u - v) - \alpha v + h^2 \Delta v, \quad x \in \Omega, \\
\frac{\partial u}{\partial n} |_{\partial \Omega} &= 0, \quad \frac{\partial \Delta u}{\partial n} |_{\partial \Omega} = 0, \quad \frac{\partial v}{\partial n} |_{\partial \Omega} = 0, \\
u(x, 0) &= u_0(x), \quad v(x, 0) = v_0(x), \quad x \in \Omega,
\end{align*}
\]

(1)

where the unknown function \(u\) denotes the average concentration of one of the components and is a conserved quantity, and \(v\) is an order parameter. Furthermore, \(h\) is a positive parameter which represents the lattice spacing and the parameter \(\alpha\) reflects the location of the system within the phase diagram. \(\Delta\) is the Laplace operator, \(\Omega \subset \mathbb{R}^n\) (\(1 \leq n \leq 3\)) is a \(C^\infty\) bounded domain and \(g(s) = -\frac{1}{2} s + s^3\). It is well known that the (1) can be considered as a system encompassing both the Cahn-Hillard and the Allen-Cahn equations. In particular, when \(u = \frac{1}{2}\), (1) reduces to the Allen-Cahn equation, and when \(v = 0\), (1) reduces to the Cahn-Hillard equation. Allen-Cahn equation and Cahn-Hillard equation have been intensively studied (see for instance [8, 4, 3, 9, 13, 7]). Long time asymptotics for the CH/AC system are developed in Novick-Cohen [12].

Up to now, we find several mathematical results on diffuse interface model for simultaneous order-disorder and phase separation. In [1] Brochet Hilhorst and Novick-Cohen proved the existence of maximal attractor for the CH/AC system. In [5] Gokieli and Ito studied the CH/AC system with constraints, they obtained the

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existence of global attractor for the CH/AC system, and proved that any element of the $\omega$-limit set of the initial data is a solution of the steady-state problem associate with the CH/AC system. In [6], Gokieli and Marcinkowski presented a numerical method for solving the CH/AC system. The simulation results give a first intuition about the stationary state form and stability.

Inspired by the above work, we concerned with the steady state bifurcation of the CH/AC system, to wit the nontrivial solution of the following boundary value problem:

$$
\begin{align*}
- h^2 \Delta^2 u - \lambda \Delta u + \Delta (2u^3 + 6uv^2) &= 0, \quad x \in \Omega, \\
h^2 \Delta v + (\lambda - \alpha)v - 2v^3 - 6u^2v &= 0, \quad x \in \Omega, \\
\frac{\partial u}{\partial n} |_{\partial \Omega} = 0, \quad \frac{\partial \Delta u}{\partial n} |_{\partial \Omega} = 0, \quad \frac{\partial v}{\partial n} |_{\partial \Omega} = 0,
\end{align*}
$$

(2)

where $u, v, h, \alpha, \lambda$ as in (1). $\triangle$ is the Laplace operator, $\Omega \subset \mathbb{R}^n$ ($1 \leq n \leq 3$) is a $C^\infty$ bounded domain.

Due to the Landau average field theory (also see [5, 6]), we have

$$h, \alpha, \lambda > 0. \quad (3)$$

By the work of Ma and Wang [10], we know

$$\lambda \propto \frac{T_c - T}{T}, \quad (4)$$

where $T$ represents the temperature, $T_c$ describes the critical temperature. We shall use the Lyapunov-Schmidt method combining with the implicit function theorem to reduce the equation (2) to a finite dimensional algebraic equation, then consider the bifurcation of the reduced equation. We prove that (2) bifurcates from the trivial solution to the nontrivial solution branch as parameter $\lambda$ crosses a certain critical value. The expression of bifurcated solution is also obtained. Our method of proof strongly rely on Ma and Wang’s book [10, 11].

The rest of the paper is arranged as follows. In Section 2, we will convert the steady state equation into abstract form. In Section 3, we will state and prove our main result. The discussion and conclusion are carried out in Section 4.

2. Operator equation. In this section, we consider the steady state equation of the CH/AC system and convert the steady state equation into abstract form. Let

$$H = \left\{ u \in L^2(\Omega) \mid \int_\Omega u dx = 0 \right\}, \quad X = \left\{ (u, v) \in H \times L^2(\Omega) \right\},$$

$$X_1 = \left\{ (u, v) \in [H^4(\Omega) \times H^2(\Omega)] \cap X \mid \frac{\partial u}{\partial n} |_{\partial \Omega} = 0, \frac{\partial \Delta u}{\partial n} |_{\partial \Omega} = 0, \frac{\partial v}{\partial n} |_{\partial \Omega} = 0 \right\}.$$

We define the operators $L_\lambda = -A + B_\lambda$ and $G : X_1 \rightarrow X$ by

$$-A(U) = \begin{pmatrix} -h^2 \Delta^2 u \\ h^2 \Delta v \end{pmatrix}, \quad B_\lambda(U) = \begin{pmatrix} -\lambda \Delta u \\ (\lambda - \alpha)v \end{pmatrix}, \quad (5)$$

$$G(U) = \begin{pmatrix} \Delta (2u^3 + 6uv^2) \\ -(2v^3 + 6u^2v) \end{pmatrix}, \quad (6)$$

where $U = (u, v)$. Then the steady state equation (2) is equivalent to the following operator equation:

$$L_\lambda U + G(U) = 0. \quad (7)$$
3. Steady state bifurcation of the CH/AC system.

3.1. Eigenvalues and eigenfunctions. The linearized eigenvalue equations of (2) are given by

\[
\begin{align*}
-h^2 \Delta^2 u_1 - \lambda \Delta u_1 &= \mu u_1, \\
h^2 \Delta u_2 + (\lambda - \alpha)u_2 &= \mu u_2, \\
\frac{\partial u_1}{\partial n} |_{\partial \Omega} &= 0, \\
\frac{\partial u_2}{\partial n} |_{\partial \Omega} &= 0, \\
\int_{\Omega} u_1 dx &= 0.
\end{align*}
\]

(8)

Let \( \rho_k \) and \( e_k \) be the eigenvalues and eigenfunctions of the following eigenvalue problem

\[
\begin{align*}
-\triangle e_k &= \rho_k e_k, \quad x \in \Omega, \\
\frac{\partial e_k}{\partial n} |_{\partial \Omega} &= 0, \\
\int_{\Omega} e_k dx &= 0.
\end{align*}
\]

(9)

It is known that the eigenvalues of (9) satisfy

\[
0 < \rho_1 \leq \rho_2 \leq \cdots,
\]

\[
\rho_k \to \infty \quad (k \to \infty),
\]

and the eigenfunctions \( \{e_k\} \) of (9) constitute an orthogonal basis of \( H \). Especially, when \( \Omega = [0, L]^n \), then the eigenvalues and eigenfunctions of (9) satisfy

\[
\rho_k = \frac{|K|^{2} \pi^2}{L^2}, \quad e_k = \cos \frac{k_1 \pi x_1}{L} \cos \frac{k_2 \pi x_2}{L} \cdots \cos \frac{k_n \pi x_n}{L},
\]

(10)

where

\[
|K|^2 = k_1^2 + k_2^2 + \cdots + k_n^2, \quad |K|^2 \neq 0, \quad k_i = 0, 1, 2, \ldots (i = 1, 2, \ldots, n).
\]

It is easy to see that \( \rho_1 = \frac{\pi^2}{L^2} \) is the first eigenvalue of (9) and

\[
e_1 = \cos \frac{k_1 \pi x_1}{L} \cos \frac{k_2 \pi x_2}{L} \cdots \cos \frac{k_n \pi x_n}{L}, \quad k_1^2 + k_2^2 + \cdots + k_n^2 = 1
\]

is the eigenfunction corresponding to \( \rho_1 \).

**Lemma 3.1.** Let \( \beta_j^k(\lambda)(j = 1, 2, k = 1, 2, \ldots) \) be eigenvalues solving (8). Then \( \beta_j^k(\lambda)(j = 1, 2, k = 1, 2, \ldots) \) are real and

\[
\beta_1^1(\lambda) = \rho_1(\lambda - h^2 \rho_1) \begin{cases} > 0, & \lambda > \lambda_0, \\ = 0, & \lambda = \lambda_0, \\ < 0, & \lambda < \lambda_0, \end{cases}
\]

\[
\beta_j^k(\lambda_0) = h^2 \rho_k - \rho_k < 0, \quad k = 2, 3, \ldots,
\]

\[
\beta_j^2(\lambda_0) = -h^2 \rho_k + h^2 \rho_1 - \alpha < 0, \quad k = 1, 2, \ldots.
\]

**Proof.** Let \( M_k(\lambda) \) be the matrix given by

\[
M_k(\lambda) = \begin{pmatrix} -h^2 \rho_k^2 + \lambda \rho_k & 0 \\ 0 & -h^2 \rho_k + \lambda - \alpha \end{pmatrix}.
\]

Then all eigenvalues \( \mu = \beta_j^k(\lambda) \) of (8) satisfy

\[
M_k x_k^j = \beta_j^k(\lambda) x_k^j, \quad j = 1, 2, \quad k = 1, 2, \cdots,
\]

where

\[
\beta_j^k(\lambda) = -h^2 \rho_k^2 + \lambda \rho_k, \quad \beta_j^2(\lambda) = -h^2 \rho_k + \lambda - \alpha,
\]

\[
x_k^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad x_k^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad k = 1, 2, \cdots.
\]
Also, the eigenvectors $e_k^j(x)(j = 1, 2, k = 1, 2, \cdots)$ of (8) corresponding to $\lambda_k^j(\lambda)(j = 1, 2, k = 1, 2, \cdots)$ are
\[
e^j_k(x) = \begin{pmatrix} e_k(x) \\ 0 \end{pmatrix}, \quad e^2_k(x) = \begin{pmatrix} 0 \\ e_k(x) \end{pmatrix}, \quad k = 1, 2, \cdots,
\]
where $e_k$ is as in (10). Thus the proof is complete. \hfill \Box

**Remark 1.** By Lemma 3.1 we know that $\lambda_k^j(\lambda)(j = 1, 2, k = 1, 2, \cdots)$ are real eigenvalues of $L_\lambda$, and
\[
\beta_1^j(\lambda) = \rho_1(\lambda - h^2\rho_1) \begin{cases} > 0, & \lambda > \lambda_0, \\
= 0, & \lambda = \lambda_0, \\
< 0, & \lambda < \lambda_0, \end{cases}
\]
\[
\beta_k^j(\lambda_0) = h^2\rho_k(\rho_1 - \rho_k) < 0, \quad k = 2, 3, \cdots,
\]
\[
\beta_k^j(\lambda) = -h^2\rho_k + h^2\rho_1 - \alpha < 0, \quad k = 1, 2, \cdots.
\]
It is easy to verify that $\beta_k^j(\lambda)(j = 1, 2, k = 1, 2, \cdots)$ are real eigenvalues of the conjugate operator $L_\lambda^*$, and $e^j_k(x)(j = 1, 2, k = 1, 2, \cdots)$ are the eigenvectors of $L_\lambda^*$ corresponding to $\beta_k^j(\lambda)(j = 1, 2, k = 1, 2, \cdots)$.

**Remark 2.** When $\Omega = [0, \pi]^n$ $(1 \leq n \leq 3)$, one can straightly verify that the algebraic multiplicity $m$ of the first eigenvalue $\rho_1 = 1$ of $L_\lambda$ is completely determined by the dimension of domain $\Omega$, i.e., $m = n$.

### 3.2. Steady state bifurcation for general bounded domain case
In this subsection we consider the case that the domain $\Omega$ is a general bounded domain. For steady state equation (2) we have the following steady state bifurcation theorem.

**Theorem 3.2.** Assume the algebraic multiplicity of the eigenvalue $\rho_1$ is one. Then the following assertions hold true:

(i) (2) has no bifurcated branch on $\lambda < h^2\rho_1$ and has exactly two branches on $\lambda > h^2\rho_1$.

(ii) The bifurcated solutions $U_\lambda$ can be expressed as
\[
U_\lambda = \pm \left[2\rho_1 \int_\Omega (e_1(x))^4 \, d\Omega \right]^{-\frac{1}{2}} \beta_1^1(\lambda)^{-\frac{1}{2}} e_1^1 + o(\beta_1^1(\lambda)^{-\frac{1}{2}}).
\]

**Proof.** By the spectral theorem for general linear completely continuous fields [11], we obtain that the space $X_1$ and $X$ can be decomposed into
\[
\begin{cases}
X_1 = E_1 \oplus E_2, \\
X = E_1 \oplus \overline{E_2}, \\
\overline{E_2} = \text{closure of } E_2 \text{ in } X,
\end{cases}
\]
for $\lambda$ near $\lambda_0$, where
\[
E_1 = \text{span}\{e^1_1\}, \\
E_2 = \text{span}\{e^2_2, e^3_3, \cdots, e^k_k, \cdots, e^1_1, e^2_2, \cdots, e^2_2, \cdots\}.
\]
By (12), for any $U \in X_1$, we have
\[
U = \bar{\eta} + \eta,
\]
where
\[
\bar{\eta} = xe^1_1 \in E_1, \quad \eta = \sum_{k=2}^{\infty} y^k_1 e^1_1 + \sum_{k=1}^{\infty} y^2_k e^2_k \in E_2.
\]
Inserting (13) into (7), then the operator equation (7) can be written in the following form

\[
\beta_1^1(\lambda)xe_1^1 + \sum_{k=2}^{\infty} \beta_1^k(\lambda)y_k^1e_k^1 + \sum_{k=1}^{\infty} \beta_2^k(\lambda)y_k^2e_k^2 + G(\bar{x} + \bar{y}) = 0. \tag{14}
\]

From section 3.1, we can assume that

\[
\langle e_i^m, e_j^n \rangle = \begin{cases} 
0 & \text{if } i \neq j \text{ or } m \neq n, \\
1 & \text{if } i = j, \ m = n.
\end{cases}
\tag{15}
\]

In this case, we reduce the equation (14) to the following equation

\[
\begin{cases}
\beta_1^1(\lambda)x + \langle G(\bar{x} + \bar{y}), e_1^1 \rangle = 0, \\
\beta_1^k(\lambda)y_k + \langle G(\bar{x} + \bar{y}), e_k^1 \rangle = 0, \ k = 2, \ldots, \\
\beta_2^k(\lambda)y_k^2 + \langle G(\bar{x} + \bar{y}), e_k^2 \rangle = 0, \ k = 1, \ldots.
\end{cases}
\tag{16}
\]

Since \((x, y_1^2, \ldots, y_k^1, \ldots, y_{k+1}^2, \ldots) = (0, 0, \ldots, 0, \ldots, 0, \ldots)\) satisfies the following equation

\[
\begin{cases}
\beta_1^k(\lambda)y_k^1 + \langle G(\bar{x} + \bar{y}), e_k^1 \rangle = 0, \ k = 2, \ldots, \\
\beta_2^k(\lambda)y_k^2 + \langle G(\bar{x} + \bar{y}), e_k^2 \rangle = 0, \ k = 1, \ldots,
\end{cases}
\tag{17}
\]

and near \(\lambda = \lambda_0\) by Remark 1, we have

\[
\beta_1^k(\lambda) < 0, \ k = 2, \ldots,
\]

\[
\beta_2^k(\lambda) < 0, \ k = 1, \ldots.
\]

Then, by the implicit function theorem, we obtain from (17) a solution

\[
\begin{cases}
y_k^1 = \Phi_1^k(x, \lambda) = o(|x|), \ k = 2, \ldots, \\
y_k^2 = \Phi_2^k(x, \lambda) = o(|x|), \ k = 1, \ldots.
\end{cases}
\tag{18}
\]

Inserting (18) into (16), we can obtain the following bifurcation equation

\[
\beta_1^1(\lambda)x + \langle H(x), e_1^1 \rangle = 0, \tag{19}
\]

where

\[
H(x) = G(xe_1^1 + \sum_{k=2}^{\infty} \Phi_1^k(x, \lambda)e_k^1 + \sum_{k=1}^{\infty} \Phi_2^k(x, \lambda)e_k^2).
\]

It is easy to see that

\[
x e_1^1 + \sum_{k=2}^{\infty} \Phi_1^k(x, \lambda)e_k^1 + \sum_{k=1}^{\infty} \Phi_2^k(x, \lambda)e_k^2
\]

\[= x \begin{pmatrix} e_1^1 \\ 0 \end{pmatrix} + \sum_{k=2}^{\infty} \Phi_1^k(x, \lambda) e_k^1 + \sum_{k=1}^{\infty} \Phi_2^k(x, \lambda) e_k^2 =: \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},
\]

where

\[
u_1 = xe_1 + \sum_{k=2}^{\infty} \Phi_1^k(x, \lambda)e_k, \ u_2 = \sum_{k=1}^{\infty} \Phi_2^k(x, \lambda)e_k.
\]

Thus, one can obtain that
\[ \langle H(x), e_1 \rangle_X = \int_{\Omega} \Delta (2u_1^3 + 6u_1 u_2^2) e_1(\tilde{x}) d\tilde{x} \]
\[ = \int_{\Omega} \rho_1 \left( -2u_1^3 - 6u_1 u_2^2 \right) e_1(\tilde{x}) d\tilde{x} \]
\[ = \int_{\Omega} \rho_1 \left[ -2(xe_1(\tilde{x}) + \sum_{k=2}^{\infty} \Phi_k(x, \lambda)e_k(\tilde{x})) \right] \]
\[ - 6(xe_1(\tilde{x}) + \sum_{k=2}^{\infty} \Phi_k(x, \lambda)e_k(\tilde{x})) \]
\[ \times \left( \sum_{k=1}^{\infty} \Phi_k^2(x, \lambda)e_k(\tilde{x}) \right) e_1(\tilde{x}) d\tilde{x} \]
\[ = -2\rho_1 x^3 \int_{\Omega} \frac{\left[ e_1(\tilde{x}) \right]^4}{4} d\tilde{x} + o(|x|^3). \]

Then we infer from above equality and \((19)\) that

\[ \beta_1^1(\lambda)x - 2\rho_1 x^3 \int_{\Omega} \frac{\left[ e_1(\tilde{x}) \right]^4}{4} d\tilde{x} + o(|x|^3) = 0. \] \((20)\)

Therefore, it follows from \((20)\) that

\[ x = \pm \left[ 2\rho_1 \int_{\Omega} \frac{\left[ e_1(\tilde{x}) \right]^4}{4} d\tilde{x} \right]^{-\frac{1}{2}} \left[ \beta_1^1(\lambda) \right]^{\frac{1}{2}} + o\left( \left[ \beta_1^1(\lambda) \right]^{\frac{1}{2}} \right), \]

which implies that when \( \lambda > \lambda_0 \) the steady-state equation \((2)\) has bifurcated solution

\[ U_\lambda = \pm \left[ 2\rho_1 \int_{\Omega} \frac{\left[ e_1(\tilde{x}) \right]^4}{4} d\tilde{x} \right]^{-\frac{1}{2}} \left[ \beta_1^1(\lambda) \right]^{\frac{1}{2}} e_1^1 + o\left( \left[ \beta_1^1(\lambda) \right]^{\frac{1}{2}} \right). \]

Thus the proof is complete. \( \square \)

**Remark 3.** Theorem 3.2 and \((4)\) yield the following physical results:

1. If the temperature is high, then concentration of one of the components will keep a uniform spatial distribution.
2. If the temperature becomes low enough, then the spatial distribution of the concentration of one of the components will not keep uniform but change with space.

### 3.3. Steady state bifurcation for \(n\)-dimensional box case.

Since the algebraic multiplicity of the eigenvalue \(\rho_1\) is not one may lead to much richer bifurcated behavior, in this subsection we consider the case that the domain \(\Omega\) is an \(n\)-dimensional box, i.e., \(\Omega = [0, \pi]^n\) (\(1 \leq n \leq 3\)). For equation \((2)\) we have the following steady state bifurcation theorem.

**Theorem 3.3.** For domain \(\Omega = [0, \pi]^n\) (\(1 \leq n \leq 3\)) the following assertions hold true:

1. If \(n = 1\), then the problem \((2)\) bifurcates from \((U, \lambda) = (0, h^2)\) on \(\lambda > h^2\) to two branches.
2. If \(n = 2\), then the problem \((2)\) bifurcates from \((U, \lambda) = (0, h^2)\) on \(\lambda > h^2\) to eight branches.
3. If \(n = 3\), then the problem \((2)\) bifurcates from \((U, \lambda) = (0, h^2)\) on \(\lambda > h^2\) to twenty six branches.
4. Each branch \(\Gamma(\lambda)\) bifurcates from \((0, h^2)\) for \((2)\) is regular.
Proof. For $\Omega = [0, \pi]^n$ ($1 \leq n \leq 3$), the proof is divided into a few steps in the following.

**Step 1. Reduction Equation.**

Let

$$U = \sum_{|K| \geq 1} y_K e_K^1 + \sum_{|K| \geq 1} z_K e_K^2.$$  

Then

$$U = \left( \begin{array}{c} \sum_{|K| \geq 1} y_K e_K \\ \sum_{|K| \geq 1} z_K e_K \end{array} \right) = \left( \begin{array}{c} u \\ v \end{array} \right),$$

(21)

where $e_K$ as in (10). The steady equation (2) can be expressed as

$$\beta_1^1(\lambda)y_i + \frac{2}{\pi^n} \int_{\Omega} \Delta(2u^3 + 6uv^2) \cos x_i \, dx = 0, \quad i = 1, 2, \ldots, n,$$

(22)

$$\beta_1^2(\lambda)z_i + \frac{2}{\pi^n} \int_{\Omega} (-2v^3 - 6u^2 v) \cos x_i \, dx = 0, \quad i = 1, 2, \ldots, n,$$

(23)

$$\beta_1^k(\lambda)y_K + \frac{1}{||e_K||^2} \int_{\Omega} \Delta(2u^3 + 6uv^2)e_K \, dx = 0, \quad |K| > 1,$$

(24)

$$\beta_2^k(\lambda)z_K + \frac{1}{||e_K||^2} \int_{\Omega} (-2v^3 - 6u^2 v)e_K \, dx = 0, \quad |K| > 1.$$  

(25)

We find that

$$\int_{\Omega} \Delta(2u^3 + 6uv^2)e_K \, dx = -|K|^2 \int_{\Omega} (2u^3 + 6uv^2)e_K \, dx.$$  

Then (22) and (23) can be rewritten as

$$\beta_1^1(\lambda)y_i - \frac{2}{\pi^n} \int_{\Omega} (2u^3 + 6uv^2) \cos x_i \, dx = 0, \quad i = 1, 2, \ldots, n,$$

(26)

$$\beta_1^2(\lambda)z_i - \frac{2}{\pi^n} \int_{\Omega} (2u^3 + 6u^2 v) \cos x_i \, dx = 0, \quad i = 1, 2, \ldots, n.$$  

(27)

For $K = (k_1, \ldots, k_n)$, we denote

$$\beta_{2j}^1 = \beta_{1K}^1, \quad y_{2j} = y_K \text{ if } k_j = 2 \text{ and } k_i = 0 \text{ for } i \neq j,$$

$$\beta_{2j}^2 = \beta_{1K}^2, \quad z_{2j} = z_K \text{ if } k_j = 2 \text{ and } k_i = 0 \text{ for } i \neq j,$$

$$\beta_{1j}^1 = \beta_{1K}^1, \quad y_{1j} = y_K \text{ if } k_i = k_j = 1 \text{ and } k_l = 0 \text{ for } l \neq i, j,$$

$$\beta_{1j}^2 = \beta_{1K}^2, \quad z_{1j} = z_K \text{ if } k_i = k_j = 1 \text{ and } k_l = 0 \text{ for } l \neq i, j.$$  

We infer from (24) and (25) that

$$y_{2j} = \frac{8}{\pi^n \beta_{2j}^1(\lambda)} \int_{\Omega} (2u^3 + 6uv^2) \cos 2x_j \, dx$$

$$= \frac{8}{\pi^n \beta_{2j}^1(\lambda)} \int_{\Omega} \left[ 2 \left( \sum_{|K| \geq 1} y_K e_K \right)^3 + 6 \left( \sum_{|K| \geq 1} y_K e_K \right) \times \left( \sum_{|K| \geq 1} z_K e_K \right)^2 \right] \cos 2x_j \, dx$$

$$= o(|q|^2),$$
\[
\begin{align*}
  y_{ij} &= \frac{8}{\pi^n \beta_{ij}^1} \int_\Omega \left( 2u^3 + 6uv^2 \right) \cos x_i \cos x_j \, dx \\
  &= \frac{8}{\pi^n \beta_{ij}^1} \int_\Omega \left[ 2 \left( \sum_{|K| \geq 1} y_K e_K \right)^3 \\
  &\hspace{1cm} + 6 \left( \sum_{|K| \geq 1} y_K e_K \right) \times \left( \sum_{|K| \geq 1} z_K e_K \right)^2 \right] \cos x_i \cos x_j \, dx \\
  &= o(|q|^2),
\end{align*}
\]

\[
\begin{align*}
  z_{ij} &= \frac{2}{\pi^n \beta_{ij}^2} \int_\Omega \left( 2u^3 + 6uv^2 \right) \cos 2x_j \, dx \\
  &= \frac{2}{\pi^n \beta_{ij}^2} \int_\Omega \left[ 2 \left( \sum_{|K| \geq 1} z_K e_K \right)^3 \\
  &\hspace{1cm} + 6 \left( \sum_{|K| \geq 1} y_K e_K \right)^2 \left( \sum_{|K| \geq 1} z_K e_K \right) \right] \cos 2x_j \, dx \\
  &= o(|q|^2),
\end{align*}
\]

\[
\begin{align*}
  z_{ij} &= \frac{4}{\pi^n \beta_{ij}^2} \int_\Omega \left( 2u^3 + 6uv^2 \right) \cos x_i \cos x_j \, dx \\
  &= \frac{4}{\pi^n \beta_{ij}^2} \int_\Omega \left[ 2 \left( \sum_{|K| \geq 1} z_K e_K \right)^3 \\
  &\hspace{1cm} + 6 \left( \sum_{|K| \geq 1} y_K e_K \right)^2 \left( \sum_{|K| \geq 1} z_K e_K \right) \right] \cos x_i \cos x_j \, dx \\
  &= o(|q|^2),
\end{align*}
\]

where \( q = (y_1, \ldots, y_n, z_1, \ldots, z_n) \in \mathbb{R}^{2n} \). In a similar way, one can obtain

\[
\begin{align*}
  \{ y_K = o(|q|^2), \forall |K| > 1, \\
  z_K = o(|q|^2), \forall |K| > 1. \}
\end{align*}
\]

(28)

Based on (21) and (28) we have

\[
\begin{align*}
  \int_\Omega u^3 \cos x_i \, dx \\
  &= \int_\Omega \left( \sum_{j=1}^n y_j \cos x_j \right)^3 \cos x_i \, dx + o(|q|^3) \\
  &= y_i^3 \int_\Omega \cos^3 x_i \, dx + \sum_{j \neq i} 3y_i y_j^2 \int_\Omega \cos^2 x_i \cos x_j \, dx + o(|q|^3) \\
  &= \frac{3\pi^n}{4} \left( \frac{1}{2} y_i^3 + \sum_{j \neq i} y_i y_j^2 \right) + o(|q|^3),
\end{align*}
\]

\[
\begin{align*}
  \int_\Omega uv^2 \cos x_i \, dx \\
  &= \int_\Omega \left( \sum_{j=1}^n y_j \cos x_j \right) \left( \sum_{j=1}^n z_j \cos x_j \right)^2 \cos x_i \, dx + o(|q|^3)
\end{align*}
\]
Thus, the reduction equation (26) and (27) can be expressed

\[
= \sum_{j=1}^{n} y_i z_j^2 \int_{\Omega} \cos^2 x_i \cos^2 x_j \, dx
\]

\[
+ \sum_{j \neq i} 2 y_j z_j z_i \int_{\Omega} \cos^2 x_i \cos^2 x_j \, dx + o(|q|^3)
\]

\[
= \frac{3\pi^n}{8} y_i z_i^2 + \frac{\pi^n}{4} \sum_{j \neq i} (y_i z_j^2 + 2 y_j z_j z_i) + o(|q|^3),
\]

\[
\int_{\Omega} \omega^3 \cos x_i \, dx
\]

\[
= \int_{\Omega} \left( \sum_{j=1}^{n} z_j \cos x_j \right)^3 \cos x_i \, dx + o(|q|^3)
\]

\[
= \frac{3\pi^n}{4} \left( \frac{1}{2} z_i^3 + \sum_{j \neq i} z_j z_j^2 \right) + o(|q|^3),
\]

\[
\int_{\Omega} u^2 v \cos x_i \, dx
\]

\[
= \int_{\Omega} \left( \sum_{j=1}^{n} y_j \cos x_j \right)^2 \left( \sum_{j=1}^{n} z_j \cos x_j \right) \cos x_i \, dx + o(|q|^3)
\]

\[
= \sum_{j=1}^{n} z_j y_j^2 \int_{\Omega} \cos^2 x_j \cos^2 x_i \, dx
\]

\[
+ \sum_{j \neq i} 2 z_j y_j y_i \int_{\Omega} \cos^2 x_i \cos^2 x_j \, dx + o(|q|^3)
\]

\[
= \frac{3\pi^n}{8} z_i y_i^2 + \frac{\pi^n}{4} \sum_{j \neq i} (z_i y_j^2 + 2 z_j y_j y_i) + o(|q|^3).
\]

Thus, the reduction equation (26) and (27) can be expressed

\[
\beta_1^1(\lambda) y_i - 3 \left( \frac{1}{2} y_i^3 + \sum_{j \neq i} y_j y_j^2 \right) - 6 \left[ \frac{3}{4} y_i z_i^2 + \sum_{j \neq i} \left( \frac{1}{2} y_i z_j^2 + y_j z_j z_i \right) \right] + o(|q|^3) = 0, \quad i = 1, \ldots, n,
\]

\[
\beta_2^2(\lambda) z_i - 3 \left( \frac{1}{2} z_i^3 + \sum_{j \neq i} z_j z_j^2 \right) - \left[ \frac{3}{4} z_i y_i^2 + \sum_{j \neq i} \left( \frac{1}{2} z_i y_j^2 + z_j y_j y_i \right) \right] + o(|q|^3) = 0, \quad i = 1, \ldots, n.
\]

**Step 2. Approximate Equation**

Since \((z_1, \ldots, z_n, y_1, \ldots, y_n) = (0, \ldots, 0, 0 \cdots, 0)\) satisfies (30), and when \(\lambda = \lambda_0\) one can deduce from Remark 1 that \(\beta_1^1(\lambda_0) < 0\). Then, by the implicit function theorem, we obtain from (30)

\[
z_i = \Phi_i(y) = o(|y|), \quad i = 1, \ldots, n,
\]
where \( y = (y_1, \cdots, y_n) \). Thus, the reduced equation (29) and (30) are equivalent to the following equation
\[
\beta_1^1(\lambda) y_i - 3 \left( \frac{1}{2} y_i^3 + \sum_{j \neq i} y_i y_j^2 \right) + o(|Q|^3) = 0, \quad i = 1, \cdots, n.
\]
(32)

Consider the following approximate equations of (32)
\[
\beta_1^1(\lambda) y_i - 3 \left( \frac{1}{2} y_i^3 + \sum_{j \neq i} y_i y_j^2 \right) = 0, \quad i = 1, \cdots, n.
\]
(33)

It is clear that if all solutions of (33) are regular, then the number of solutions of (33) and (32) are the same.

**Step 3. Bifurcated Branch**

When \( n = 1 \), (33) reduces to the following equation
\[
\beta_1^1(\lambda) y_1 - 3 \left( \frac{1}{2} y_1^3 \right) = 0.
\]
(34)

It is easy to see that (34) has the following solutions
\[
Y_1 = 0, Y_{2,3} = \pm \frac{\sqrt{6} \beta_1^1(\lambda)}{3}.
\]
The Jacobian matrix of the vector field in (34) are given by
\[
M_1(y_1) = \beta_1^1(\lambda) - \frac{3y_1^2}{2}.
\]
(35)

Then we derive from (35) that
\[
M_1(Y_1) = \beta_1^1(\lambda) > 0,
M_1(Y_2) = -2\beta_1^1(\lambda) < 0,
M_1(Y_3) = -2\beta_1^1(\lambda) < 0,
\]
which implies that bifurcated branches are regular. Therefore, we derive Assertion (i).

When \( n = 2 \), (33) reduces to the following equations
\[
\begin{align*}
\beta_1^1(\lambda) y_1 - 3 \left( \frac{1}{2} y_1^3 + y_1 y_2^2 \right) &= 0, \\
\beta_1^1(\lambda) y_2 - 3 \left( \frac{1}{2} y_2^3 + y_2 y_1^2 \right) &= 0.
\end{align*}
\]
(39)

It is easy to see that (39) has the following eight nontrivial solutions
\[
Y_{1,2} = \left(0, \pm \frac{\sqrt{6} \beta_1^1(\lambda)}{3}\right), \quad Y_{3,4} = \left(\pm \frac{\sqrt{2} \beta_1^1(\lambda)}{3}, 0\right),
\]
\[
Y_{5,6,7,8} = \left(\pm \frac{\sqrt{2} \beta_1^1(\lambda)}{3}, \pm \frac{2 \beta_1^1(\lambda)}{3}\right).
\]
The Jacobian matrix of the vector field in (39) are given by
\[
M_2(y_1, y_2) = \begin{pmatrix}
\beta_1^1(\lambda) - 3 \left( \frac{3}{2} y_1^2 + y_2^2 \right) & -6y_1 y_2 \\
-6y_1 y_2 & \beta_1^1(\lambda) - 3 \left( \frac{3}{2} y_2^2 + y_1^2 \right)
\end{pmatrix}.
\]
(40)

Then we derive from (40) that the eigenvalues of \( M_2(Y_i) (i = 1, 2, 3, 4) \) are all negative, and \( M_2(Y_i) (i = 5, 6, 7, 8) \) have one positive eigenvalue one negative eigenvalue, which implies that bifurcated branches are regular. Hence, Assertion (ii) is derived.
When \( n = 3 \), (33) reduces to the following equation
\[
\begin{align*}
\beta_1'(\lambda)y_1 - 3\left(\frac{1}{2}y_1^3 + y_1y_2^2 + y_1y_3^2\right) &= 0, \\
\beta_1'(\lambda)y_2 - 3\left(\frac{1}{2}y_2^3 + y_2y_1^2 + y_2y_3^2\right) &= 0, \\
\beta_1'(\lambda)y_3 - 3\left(\frac{1}{2}y_3^3 + y_3y_1^2 + y_3y_2^2\right) &= 0.
\end{align*}
\]

(41)

It is easy to see that (41) has the following 26 nontrivial solutions
\[
\begin{align*}
Y_{1,2} &= (0, 0, \pm \sqrt{\frac{6\beta_1'(\lambda)}{3}}, 0), \\
Y_{3,4} &= (0, 0, \pm \sqrt{\frac{6\beta_1'(\lambda)}{3}}, 0), \\
Y_{5,6} &= \left(\pm \sqrt{\frac{6\beta_1'(\lambda)}{3}}, 0, 0\right), \\
Y_{7,8,9,10} &= \left(0, \pm \sqrt{\frac{2\beta_1'(\lambda)}{3}}, 0, 0\right), \\
Y_{11,12,13,14} &= \left(\pm \sqrt{\frac{2\beta_1'(\lambda)}{3}}, 0, \pm \sqrt{\frac{2\beta_1'(\lambda)}{3}}\right), \\
Y_{15,16,17,18} &= \left(\pm \sqrt{\frac{2\beta_1'(\lambda)}{3}}, \pm \sqrt{\frac{2\beta_1'(\lambda)}{3}}, 0\right), \\
Y_{19,20,21,22,23,24,25,26} &= \left(\pm \sqrt{\frac{30\beta_1'(\lambda)}{15}}, \pm \sqrt{\frac{30\beta_1'(\lambda)}{15}}, \pm \sqrt{\frac{30\beta_1'(\lambda)}{15}}\right).
\end{align*}
\]

The Jacobian matrix of the vector field in (41) are given by
\[
M_3(y_1, y_2, y_3) = \begin{pmatrix}
\beta_1'(\lambda) - l_1(Y) & -6y_1y_2 & -6y_1y_3 \\
-6y_1y_2 & \beta_1'(\lambda) - l_2(Y) & -6y_2y_3 \\
-6y_1y_3 & -6y_2y_3 & \beta_1'(\lambda) - l_3(Y)
\end{pmatrix},
\]

(42)

where
\[
\begin{align*}
l_1(Y) &= 3\left(\frac{3}{2}y_1^2 + y_2^2 + y_3^2\right), \\
l_2(Y) &= 3\left(\frac{3}{2}y_2^2 + y_1^2 + y_3^2\right), \\
l_3(Y) &= 3\left(\frac{3}{2}y_3^2 + y_1^2 + y_2^2\right).
\end{align*}
\]

Then we derive from (42) that the eigenvalues of \( M_3(Y)(i = 1, \ldots, 6) \) are all negative, \( M_3(Y)(i = 7, \ldots, 18) \) have one positive eigenvalue two negative eigenvalues and \( M_3(Y)(i = 19, \ldots, 26) \) have two positive eigenvalues one negative eigenvalue, which implies that bifurcated branches are regular. Therefore, we derive Assertion (iii).

The proof of the theorem is complete.

\[4.\textbf{Discussion and conclusion.}\] It follows from Lemma 3.1 that the critical control parameter \( \lambda_0 \) is obtained as
\[
\lambda_0 = h^2\rho_1
\]

(43)
in which \( h \) is a positive parameter which represents the lattice spacing, and \( \rho_1 \) is the first eigenvalue of Laplace operator \(-\Delta\) in bounded domain \( \Omega \). It is known that \( \rho_1 \) is inversely proportional to diameter of \( \Omega \). Thus, (43) indicates that the diameter of domain \( \Omega \) is larger, the steady state bifurcation from trivial solution more easily happens. Similarly, the value of \( h \) is also a key factor for steady state bifurcation from the trivial solution. The larger \( h \), the system (2) is harder to undergo a steady state bifurcation.

Theorem 3.2, 3.3 indicate that after undergoing a steady state bifurcation, the new state of the system (2) is not unique. The number of new steady-states is only
determined by the domain and its dimension. In the special case of the algebraic multiplicity of the eigenvalue $\rho_1$ is one, the new states are two steady-states. When the algebraic multiplicity of the eigenvalue $\rho_1$ is two, the new states are eight steady-states. When the algebraic multiplicity of the eigenvalue $\rho_1$ is three, the new states are twenty six steady-states. Although we only consider the steady state bifurcation of (2) under one type of physical boundary condition, our theoretical analysis in present work can be applied to consider other boundary condition, for instance, periodic boundary condition.

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