Irreducibility of the Picard-Fuchs equation related to the Lotka-Volterra polynomial \( x^2 y^2 (1 - x - y) \)

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Abstract

We prove that the Zarisky closure of the monodromy group of the polynomial
\( x^2 y^2 (1 - x - y) \) is the symplectic group \( Sp(4, \mathbb{C}) \). This shows that some previous results about this monodromy representation are wrong.

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1 Introduction.

The study of a germ of a vector fields in $\mathbb{R}^4$ with two pairs of non-resonant imaginary eigenvalues reduces to the study of the special perturbations

$$xy(1 - x - y)d \log H + \varepsilon_1 y dx + \varepsilon_2 y x^2 dx = 0$$

of the integrable quadratic foliation

$$xy(1 - x - y)d \log H = 0$$

with first integral $H = x^p y^q(1 - x - y)$, e.g. [1, chapter 1, section 4.6]. Equivalently, we may consider the following generalized Lotka-Volterra system associated to (2)

$$\begin{align*}
x' &= x[q(1 - x - y) - y] \\
y' &= y[x - p(1 - x - y)]
\end{align*}$$

and the perturbed foliation (1) is associated then to

$$\begin{align*}
x' &= x[q(1 - x - y) - y] \\
y' &= y[x - p(1 - x - y) + \varepsilon_1 + \varepsilon_2 x^2].
\end{align*}$$

The limit cycles of the perturbed system correspond to the zeros of the displacement map

$$\Delta(h) = \frac{1}{h} \int_{H=h} x^{p-1} y^q(\varepsilon_1 + \varepsilon_2 x^2)dx + O(\varepsilon_1^2 + \varepsilon_2^2).$$

where $\Gamma_h = \{H = h\}$ is a continuous family of ovals (closed orbits) of the non-perturbed system (3). The unicity of the limit cycle of (1), (4) was shown first by [10, Zoladek, 1986], who proved the monotonicity of the function

$$F(h) = \frac{1}{h} \int_{H=h} x^{p-1} y^q dx / \int_{H=h} x^{p+1} y^q dx$$

on the maximal interval $(0, h_1)$ where the ovals of $\{(x, y) : H(x, y) = h\}$ exist. Note that the system (4) has $\{x = 0\}$ and $\{y = 0\}$ as invariant lines. In this relation, recall that a plane quadratic vector field with an invariant line has a unique limit cycle (if any), as explained by [5, Coppel, 1989], but see also [9, Zegeling and Kooij, 1994].

The case of a more general quadratic perturbations of (3) was studied also by [11, Zoladek, 1994] and revised recently in [12]. The common point of the above mentioned papers is, that they use ad hoc methods based on apriori estimates. The essential reason why all these estimates hold remains hidden.

In 1985 Van Gils and Horozov [8] gave an overview of the various methods, which have been used at this time to prove the uniqueness of the limit cycles for the perturbations of the generalized Lotka-Volterra system (3). The central result of their paper is that in the particular case, in which $p = q$ is an integer, the functions

$$I(h) = \alpha \int_{H=h} x^{p+1} y^p dx + \beta \int_{H=h} x^{p-1} y^p dx, \alpha, \beta \in \mathbb{R}, h \in (0, h_1)$$
satisfy a Picard-Fuchs equation of second order, whose coefficients are rational in $h^{1/p}$. The authors used then topological arguments as the Rolle’s theorem, to bound the zeros of $I(h)$ in terms of the degrees of the coefficients of the Picard-Fuchs equation, from which the result of Zoladek [10] follows.

More precisely, if $h_1 > 0$ and $h_2 = 0$ are the critical values of the Lotka-Volterra integral $H = x^p y^p (1 - x - y)$, then the Abelian integral $I(h)$ [5] allows an analytic continuation from $(0, h_1)$ to a small neighbourhood of $h_1$. This follows from the Picard-Lefschetz formula and the fact, that the oval $\{H = h\}$ represents a cycle vanishing at $h_1$. At the other end of the interval, at $h = 0$, the function is not analytic, but has a logarithmic type of singularity, as it follows from a generalised Picard-Lefschetz formula [8]. Namely

$$I(h) = J(z) \ln(z) + K(z), \ z = h^{1/p}$$

for suitable functions $J(z), K(z)$, which are analytic in a neighbourhood of $z = 0$. Note that $I(h) + 2\pi \sqrt{-1}J(z)$ is an analytic continuation of $I(h)$ and hence it is an Abelian integral too. It has therefore a similar logarithmic type singularity at $h = h_1$.

Following [8], denote by $W$ the Wronskian

$$W = \det \begin{pmatrix} \frac{d}{dz} I(z) & \frac{d}{dz} J(z) \\ \frac{d^2}{dz^2} I(z) & \frac{d^2}{dz^2} J(z) \end{pmatrix}$$

The Picard-Lefschetz formula then implies, that $W = W(h)$ as a function in $h$ allows an analytic continuation from the interval $(0, h_1)$ to a neighborhood of $h_1$, and also that $W(h)$ allows an analytic continuation from the interval $(0, h_1)$ to a covering of a punctured neighbourhood of $h = 0$, in which $W$ is analytic in $z = h^{1/p}$. From this, the authors concluded that $W$ is in fact a rational function in $z = h^{1/p}$, and even computed it explicitly [8, Lemma 1].

This conclusion, that $W$ is a rational function in $h^{1/p}$ is, however, wrong. Indeed, there exist functions of moderate growth, analytic on the universal covering of $\mathbb{C} \setminus \{0, h_1\}$, with a branch on $(0, h_1)$ which are analytic at $h_1$, analytic in $h^{1/p}$ at $h = 0$, but still not algebraic in $h$. To construct an example, consider a second order Fuchs equation with singular points at $0, h_1, \infty$ and Riemann scheme

$$\begin{pmatrix} 0 & h_1 & \infty \\ 0 & 0 & \alpha \\ \frac{1}{p} & 0 & \beta \end{pmatrix}$$

where $\alpha + \beta + 1/p = 1$. Let $I(h)$ be a (branch of a) solution on $(0, h_1)$, analytic in a small neighbourhood of $h = h_1$. For generic values of the characteristic exponent $\alpha$ (or $\beta$) our equation has no algebraic solutions. Therefore $I(h)$ can not be analytic in a neighbourhood of $h = 0$ too. It is concluded that $I(h)$ is analytic in $h^{1/p}$ in a neighbourhood of $h = 0$. Clearly, this argument points out a gap, but does not disproof the result of [8].
The purpose of the present note is to study in more detail the monodromy representation of the Lotka-Volterra polynomial $H = x^p y^p (1 - x - y)$, in the case when $p$ is an integer. The knowledge of this monodromy representation allows, according to [3], to compute the minimal degree of the differential equation satisfied by $I(h)$. We shall show in this way, that in the first non trivial case $p = 2$, the Abelian integral $I(h)$ satisfies a linear differential equation of minimal degree four, even if the coefficients are supposed to be algebraic functions. Thus, the result of [3] Lemma 1 is definitely wrong. Note also, that the the computation of [3, section 3.3], and in particular Corollary 4 there, are also wrong.

We prove in fact a more general result about the attached Lie group $G$, which is the Zarisky closure of the monodromy group of the polynomial $H(x, y)$. Namely, we show that in the case $p = 2$ the group $G$ is isomorphic to the symplectic group $Sp(4, \mathbb{C})$, see Theorem 2. As the standard representation of $Sp(4, \mathbb{C})$ is irreducible, then according to Corollary 6 we obtain

*The Abelian integral* (3) *satisfies a Fuchs type equation of minimal degree four, even if its coefficients are supposed to be algebraic functions in* $h$.

The paper is organized as follows. In the next short section we give some background, concerning the reduction of the degree of Picard-Fuchs operators. In section 2 we determine explicitly the monodromy operators, related to the two singular critical values of $H$. This (long) computation is contained in principal in [3], in the case of arbitrary integers $p, q$ and $H = x^p y^q (1 - x - y)$. In the particular case $p = q$, part of these computations simplify, and for this reason we give here an independent treatment.

It is a straightforward observation, that the monodromy representation of $H = x^p y^p (1 - x - y)$ is reducible. We have in fact a two-dimensional plane $V_2$ of zero-cycles on which the monodromy acts as identity, as well a complementary $2p$-dimensional plane $V_{2p}$, invariant under the action of the monodromy group. The nature of this sub-representation $V_{2p}$ is studied in the last section 3 in the simplest non-trivial case $p = 2$. By taking the Zarisky closure of the monodromy group, we find that the sub-representation $V_4$ coincides with the standard representation of the symplectic group $Sp(4, \mathbb{C})$. This, combined with section 2 implies the claims about the degree of the Picard-Fuchs equation.

### 2 Reduction of the degree of Picard-Fuchs equations.

In this section we summarize, following [3], the necessary facts about the reduction of the degree of Picard-Fuchs differential operators.

To a non-constant polynomial $f \in \mathbb{C}[x, y]$ we associate its monodromy representation

$$
\pi_1(\mathbb{C} \setminus S, b) \rightarrow Aut(H_1(f_b, \mathbb{Z}))
$$

where $f_t = f^{-1}(t) \subset \mathbb{C}^2$ are the fibers of the fibration $f : \mathbb{C}^2 \rightarrow \mathbb{C}$. The image of the fundamental group $\pi_1(\mathbb{C} \setminus S, b)$ is the monodromy group $\mathcal{M}$. The Zarisky closure $G = \overline{\mathcal{M}}$ of $\mathcal{M}$ is a linear algebraic group, embedded in $GL_d(\mathbb{C})$, $d = \dim H_1(f_b, \mathbb{Z})$. It is nothing but the differential Galois group of a generic Picard-Fuchs system, related
to the fibration defined by $f$. In the sequel, an important role is played by the connected component $G^0$ of $G$, containing the identity transformation.

Let $\delta(t) \subset H_1(f_t, \mathbb{Z})$ be a continuous family of cycles, and $\omega$ a polynomial one-form. The Abelian integral

$$I(t) = \int_{\delta(t)} \omega$$

satisfies a Picard-Fuchs equation of minimal degree $d$

$$I^{(d)} + a_1 I^{(d-1)} + \ldots + a_d I = 0$$

whose coefficients are rational functions in $t$, $a_i \in \mathbb{C}(t)$. Let $V \subset H_1(f_b, \mathbb{C})$ be a vector plane, invariant under $G$. Then obviously $d \leq \dim V$. Let $\gamma \in V$ and $\gamma(t)$ the corresponding continuous family of cycles (a locally constant section of the homology bundle). The set of such $\gamma$ with the property $\int_{\gamma(t)} \omega \equiv 0$ is an invariant sub-plane of $V$ which we denote by $V_1$. We conclude that $d = \dim V - \dim V_1$.

Consider now the following reduction problem:

Find a differential equation

$$I^{(d_0)} + b_1 I^{(d_0-1)} + \ldots + b_{d_0} I = 0$$

(6)

of minimal degree $d_0 \leq d$, such that $b_i = b_i(t)$ are algebraic functions in $t$.

The computation of the degree $d_0$ goes along the same lines as above, except that the Lie group $G$ is replaced by $G^0$. Namely, let $V^0 \subset H_1(f_b, \mathbb{C})$ be a plane, invariant under $G^0$, that is to say a sub-representation of $G^0$. Such a plane was called virtually invariant in [3]. It follows that $d_0 \leq \dim V^0$. Let $\gamma \in V^0$ and $\gamma(t)$ the corresponding continuous family of cycles as above. The set of such $\gamma$ with the property $\int_{\gamma(t)} \omega \equiv 0$ is a virtually invariant sub-plane of $V^0$ which we denote by $V^0_1$. We conclude that

**Theorem 1.** $d_0 = \dim V^0 - \dim V^0_1$.

**Corollary 1.** If the representation of $G^0$ on $V^0$ is irreducible, then the minimal degree of the differential operator (6) equals $\dim V^0$.

See [3] for proofs.

### 3 Topology of the fibration, defined by the polynomial $x^p y^p(1 - x - y)$

Denote

$$f = x^p y^p(1 - x - y)$$

for some $p \in \mathbb{N}^*$. The polynomial $f$ has two critical values $t_1 = \frac{x^{2p}}{(2p+1)^2 + 1}$ and $t_2 = 0$ and non-isolated critical points along the lines $x = 0$ and $y = 0$. For $t \neq t_{1,2}$ the algebraic curve $\Gamma_t = \{(x, y) \in \mathbb{C}^2 : f(x, y) = t\}$ is a smooth Riemann surface and its fundamental group has $2p + 2$ generators, see Fig.5. As $\Gamma_t$ has three punctures (at infinity), then it is a genus $p$ algebraic curve. We wish to describe the "continuous
variation of $\Gamma_t$ when $t$ varies along closed circuits in $\mathbb{C} \setminus \{t_1, t_2\}$, or equivalently, the topology of the fibration

$$f : \mathbb{C}^2 \to \mathbb{C} \setminus \{t_1, t_2\}$$

$$(x, y) \mapsto f(x, y) = x^p y^p (1 - x - y)$$

The calculation of this geometric monodromy is in general a difficult task, see the survey of Siersma [7].

To begin with, we first localize $f$ at the singular points $(0,1)$ and $(1,0)$ and obtain a germ of analytic function with non-isolated critical points. We study first their geometric monodromy, which is straightforward.

3.1 The germ $(x + \ldots)^p(y + \ldots)$.

Let $f : \mathbb{C}^2, 0 \to \mathbb{C}, 0$ be a germ of analytic function, such that

$$f_0 = \{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$$

defines a germ of a divisor with simple normal crossing and multiplicities 1 and $p$. In appropriate coordinates in a suitable neighborhood of the origin we have $f(x, y) = x^p y$, which will be assumed until the end of this section.

The marked fibration associated to $f$ is by definition the usual fibration

$$f : \mathbb{C}^2 \to \mathbb{C} \setminus 0$$

$$(x, y) \mapsto f(x, y) = x^p y$$

whose fibers are the Riemann surfaces (topological cylinders)

$$f_t = \{(x, y) : x^p y = t\}$$

with $p + 1$ marked points corresponding to the intersection of $f_t$ with the two fixed lines $\{x = 1\}$ and $\{y = 1\}$

$$S_t = \{(1, t), (t^{1/p} e^{2k\pi i/p}, 1), k = 0, 1 \ldots, p - 1\}.$$ 

Continuous deformation of $t$ induces an isotopy of the marked fibers $f_t$. A continuous variation of $t$ along a closed circuit about the origin induces therefore a diffeomorphism (geometric monodromy)

$$M : f_t \to f_t, t \neq 0$$

defined up to an isotopy, which permutes the marked points. We wish to describe $M$ along the same lines, as in the case of an isolated singularity of Morse type, $p = 1$, e.g. [2].

It is convenient to consider the fundamental groupoid $\pi_1(f_t, S_t)$ of the pair $(f_t, S_t)$ [4], which replaces the common fundamental group $\pi_1(f_t, \ast)$ in the case $p = 1$. Recall that this groupoid is the set of homotopy classes of loops $\gamma : [0,1] \to f_t$ such that $\gamma(0), \gamma(1) \in S_t$. Two loops $\gamma_1, \gamma_2$ are composable, if $\gamma_2(1) = \gamma_1(0)$ and in this case
the homotopy class of $\gamma_1 \circ \gamma_2 \in \pi_1(f_t, S)$ is well defined. In the case when $S_t = *$ is a single point $\pi_1(f_t, S_t) = \pi_1(f_t, *)$ is the usual fundamental group of the fiber $f_t$. The fundamental groupoid $\pi_1(f_t, S)$ is freely generated by $p + 1$ loops, as shown on fig.1 in the particular case $p = 3$.

The geometric monodromy $M$ defines a homomorphisms

$$m_* : \pi_1(f_t, S_t) \to \pi_1(f_t, S_t).$$

Clearly, $M$ permutes cyclically the points $(t^{1/p}, 1)$ and fixes $(t, 1)$. Define a loop $\gamma_0$ connecting $(t, 1)$ to a point in the set $\{(t^{1/p}, 1)\}$ and then inductively

$$\gamma_k = m_*^k \gamma_0, k = 0, 1, ..., p.$$

**Proposition 1.** The $p + 1$ loops $\gamma_0, \gamma_1, ..., \gamma_p$ generate the groupoid $\pi_1(f_t, S_t)$, the closed loop $\alpha = \gamma_0^{-1} \circ \gamma_p$ generates the fundamental group $\pi_1(f_t, *) = \mathbb{Z}$. (see fig. 1).

**Proof.** The projection $(x, y) \to x$ maps $f_t$ isomorphically to $\mathbb{C}^*$, and the images of the marked points are

$$1, t^{1/p} e^{2k\pi i/p}, k = 0, 1, ..., p - 1.$$

Alternatively, in the ball $B_R = \{(x, y) \in \mathbb{C}^2 : |x|^2 + |y|^2 \leq R^2\}$, for sufficiently small $|t|$ the surface $f_t \cap B_R$ is projected under $(x, y) \to x$ to the annulus shown on fig.1. When $t$ makes one turn around the origin, in a clockwise direction, the marked points $t^{1/p} e^{2k\pi i/p}$ permute cyclically, and the corresponding paths $\gamma_k$ connecting 1 to $t^{1/p} e^{2k\pi i/p}$ are as on fig. 1. Note that in the case $p = 1$ we get the usual Picard-Lefschetz formula. \Halmos

For a further use, note the following

**Corollary 2.** It is always possible to choose the initial loop $\gamma_0$ in such a way, that $\gamma_i, 0 \leq i \leq p,$ are non-intersecting, and non self-intersecting loops. In this case the union $\bigcup^p_{i=0} \gamma_i \subset f_t$ is an embedded planar graph, which is a deformation retract of the marked cylinder $f_t$.

Consider finally the relative homology group $H_1(f_t, S_t) = H_1(f_t, S_t, \mathbb{Z})$ which is isomorphic as a $\mathbb{Z}$-module to $\mathbb{Z}^{p+1}$. The generators of $H_1(f_t, S_t)$ are represented by the paths $\gamma_0, \gamma_1, ..., \gamma_p$ defined in Proposition 1. We conclude that the corresponding equivalence classes of paths

$$[\gamma_0], [\gamma_1], ..., [\gamma_{p-1}], [\gamma_p] = [\alpha] + [\gamma_0] \in H_1(f_t, S_t).$$

define a basis of the relative homology group.

The diffeomorphism $M = f_t \to f_t$ induces a homomorphism (Picard-Lefschetz monodromy operator)

$$M_* : H_1(f_t, S_t) \to H_1(f_t, S_t).$$

Note that $M_*[\alpha] = [\alpha]$ and in the basis

$$[\gamma_0], [\gamma_1], ..., [\gamma_{p-1}], [\alpha]$$
Figure 1: The fundamental groupoid $\pi_1(f_t, S)$ of the cylinder $f_t = \{x^3y = t\}$ with four marked points.

$M_*$ is represented by the matrix

\[
\begin{pmatrix}
0 & \cdots & 0 & 1 & 0 \\
1 & \cdots & 0 & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots \\
0 & \cdots & 1 & 0 & 0 \\
0 & \cdots & 0 & 1 & 1
\end{pmatrix}
\]

with characteristic polynomial

\[(-1)^{p+1}(\lambda - 1)(\lambda^p - 1).\]

3.2 The topology of the fiber $f_t = \{x^py^p(1 - x - y) = t\}$ when $t$ is close to $t_2 = 0$.

Our aim in this section is to construct explicit generators for the fundamental group of $f_t$ which allows a simple description of the action of the monodromy transformations of the fibration, defined by $f$. The generators will be presented in the form of an embedded graph, which is a deformation retract of $f_t$ (see Corollary 2).

Namely, consider the real triangle, with vertices the singular points $(0, 0), (1, 0), (0, 1)$ in $\mathbb{R}^2$. Let $U \subset \mathbb{C}^2$ be a suitable tubular neighborhood of this triangle. We suppose that $\partial U$ is transversal to the complex lines

\[
\{x = 0, y = 0, x + y + 1 = 0\} \subset \mathbb{C}^2
\]}
and moreover $f_0 \cap U$ is a deformation retract of $f_0$. It follows that for sufficiently small $|t|$ the border $\partial U$ is transversal also to $f_t$, and that $f_t \cap U$ is a deformation retract of $f_t$. This allows to localize our description of the fiber $f_t$ near the triangle with vertices $(0,0), (1,0), (0,1)$, see fig. 2.

Consider three cross-sections (complex discs) transversal to the sides, dividing the triangle into three pieces, with corresponding tubular neighborhoods $U_{13}, U_{23}, U_{12}$, where $U = U_{13} \cup U_{23} \cup U_{12}$. The cross sections intersect the fiber $f_t \subset \mathbb{C}^2$ for sufficiently small non-zero $|t|$ in exactly $p, p$ and 1 points respectively, which will be the marked points from the section 3.1. Let $U_{12}$ be the tubular neighborhood, containing $(0,0)$. Then $f_t \cap U_{12}$ has $p$ connected components (topological cylinders) which coincide with the fibers of a Morse polynomial, and it is retracted to $p$ disjoint segments. The fibers $f_t \cap U_{13}$ and $f_t \cap U_{23}$ are described as in the section 3.1 Corollary 2.

We construct now a graph, embedded in $f_t \cap U$ which is a deformation retract of $f_t$. For this purpose we take together the corresponding graphs near the singular point $(0,0), (1,0), (0,1)$, constructed in section 3.1. The assembling of these graphs is shown on fig. 3 in the case $p = 3$, the general case of arbitrary $p$ being analogous. It is easy to check (by making use of a partition of the unity) that the resulting graph is embedded in $f_t$ and is a deformation retract of $f_t$.

### 3.3 The monodromy of the fibers $f_t$ along closed paths near the singular value $t = 0$.

The next step is to describe the action of the monodromy transformation $M : f_t \to f_t$, corresponding to the singular value $t = 0$, which once again easily follows from the preceding constructions. More specifically, we shall describe the linear operator $M \in \text{Aut}(H_1(f_t, \mathbb{Z}))$. We begin by choosing a suitable basis of $H_1(f_t, \mathbb{Z})$. In a suitable neighborhood of $(x,y) = (0,0)$ the fiber $f_t = \{x^py^p(1-x-y) = t\}$ has $p$ connected components homeomorphic to cylinders, and each component carries a vanishing cycle.
Figure 3: Assembling of a deformation retract of the fiber $f_t = \{x^3y^3(1-x-y) = t\}$.

denoted
$$\delta^k_{12}, \, k = 0, 1, \ldots, p - 1.$$ 
At $(0, 1)$ and $(1, 0)$ we have vanishing cycles denoted respectively $\delta_{13}$ and $\delta_{23}$. We have therefore $p + 2$ vanishing cycles, additional $p$ cycles are constructed as follows
$$\delta_k = M^k \delta_0, \, k = 0, 1, \ldots, p - 1$$
where $\delta_0 = \delta(t)$ is represented by the oval of $f_t$ for $t_1 < t < t_2$. The $2p + 2$ cycles which we described are independent in $H_1(f_t, \mathbb{Z})$.

**Proposition 2.** The monodromy operator $M_*$ acts as follows
$$\delta^0_{12} \rightarrow \delta^1_{12} \rightarrow \cdots \rightarrow \delta^{p-1}_{12} \rightarrow \delta^0_{12}, \; \delta_{13} \rightarrow \delta_{13}, \; \delta_{23} \rightarrow \delta_{23}$$
and
$$\delta_0 \rightarrow \delta_1 \rightarrow \cdots \rightarrow \delta_{p-1}, \; \delta_{p-1} \rightarrow \delta_0 - \delta^0_{12} - \delta_{13} - \delta_{23}.$$

**Proof.** Having described the monodromy of the fiber $f_t$ localized around the singular points of $f$, everything is obvious, except $M_0 \delta_{p-1} = M_0^p \delta_0$. To compute $M_0 \delta_{p-1} = M_0^p \delta_0$ we use the action of $M^p$ described in the relative homology of the local fibers. As noted in [8] (or see fig. 1) $M^p$ is a Picard-Lefschetz operator, and
$$M_0^p \delta_0 = \delta_0 - \delta^0_{12} - \delta_{13} - \delta_{23}.$$ 

It is straightforward to check that the characteristic and the minimal polynomials of $M_*$ are equal respectively to
$$(\lambda - 1)^2(\lambda^p - 1)^2, \, (\lambda - 1)(\lambda^p - 1)^2$$

**Corollary 3.** The orbit of the cycle $\delta_0$ under the action of $M_*$ spans the $2p$-dimensional vector space generated by
$$\delta_i, \delta^i_{12} + \delta_{13} + \delta_{23}, \, i = 0, 1, \ldots, p - 1.$$
3.4 The monodromy of the fibration $f : \mathbb{C}^2 \rightarrow \mathbb{C} \setminus \{t_1, t_2\}$.

The monodromy group is generated by two operators, $M_1$ and $M_2 = M$ related to the singular values $t_1, t_2 = 0$. The value $t_1$ corresponds to a Morse critical point and the operator $M_1$ is given by the Picard-Lefschetz formula. Its description amounts to compute the intersection form on $H_1(f, \mathbb{Z})$.

**Proposition 3.** The intersection numbers of the $2p + 2$ cycles

$\delta_{13}, \delta_{23}, \delta_i, \delta^i_{12}, 0 \leq i, j \leq p - 1$

are as follows

$$(\delta^i_{12} \cdot \delta^j_{12}) = 0, (\delta^i_{12} \cdot \delta_{13}) = 0, (\delta^i_{12} \cdot \delta_{23}) = 0, (\delta_{13} \cdot \delta_{23}) = 0$$

$$(\delta_i \cdot \delta_{13}) = (\delta_i \cdot \delta_{23}) = 1$$

$$(\delta_i \cdot \delta^i_{12}) = 1; (\delta_i \cdot \delta^j_{12}) = 0, i \neq j$$

$$(\delta_i \cdot \delta_j) = -1, i < j.$$

The first and the third lines are obvious. For the second, we choose an orientation on $\delta_{13}, \delta_{23}$ in such a way, that

$$(\delta_0 \cdot \delta_{13}) = (\delta_0 \cdot \delta_{23}) = 1$$

and then use the invariance of the intersection number under the action of the monodromy. The only non-trivial fact is the fourth line. We note that according to fig. 3, the intersection number of the relative cycles $\delta_i \cap U_{12}, \delta_j \cap U_{12}$ are well defined and equal 0. The fibration defined by $f$ in $U_{23} \cup U_{13}$ can be further continuously deformed in a way, which does not change the topology of the fibers and their monodromy. Namely, by such a deformation we may replace $f$ by

$$\tilde{f} = (1 - x - y)x^p(1 - x))$$

and consider the linear projection

$$\pi : U_{23} \cup U_{13} \rightarrow U_{23} \cup U_{13} \cap \{1 - x - y = 0\}$$
parallel to the lines $x = \text{const.}$ As in the section 3.1 the projection

$$
\pi : U_{23} \cup U_{13} \cap f_t \rightarrow U_{23} \cup U_{13} \cap \{1 - x - y = 0\}
$$

is an injective local biholomorphism, and its image is shown on fig. 4. When $t$ makes one turn around the origin, the two marked points corresponding to the ends of the relative cycle $\delta_0 \cap U_{23} \cup U_{13}$ turn in the same direction. The result is the relative cycle $\delta_1 \cap U_{23} \cup U_{13}$ shown on fig. 4. This already proves that

$$(\delta_0 \cdot \delta_i) = (\delta_0 \cdot \delta_j) = \pm 1, 1 \leq i, j \leq p - 1$$

and by invariance of the intersection form the numbers

$$(\delta_i \cdot \delta_j), 1 \leq i < j \leq p - 1$$

are all equal to the either +1 or to −1. We have finally

$$(\delta_0 \cdot \delta_1) = (M_{p-1}^1 \delta_0 \cdot M_{p-1}^1 \delta_1) = (\delta_{p-1} \cdot M_p^1 \delta_0)$$
$$= (\delta_{p-1} \cdot (\delta_0 - \delta_0^{12} - \delta_{13} - \delta_{23})) = -(\delta_0 \cdot \delta_{p-1}) - 2$$
$$= - (\delta_0 \cdot \delta_1) - 2$$

and hence $$(\delta_0 \cdot \delta_1) = -1.$$  \hfill \Box

4 A case study : the polynomial $x^2y^2(1 - x - y)$.

In this section we consider in detail the first non-trivial case $p = 2$, in which the fibers $f_t$ are genus two Riemann surfaces with three punctures. For definiteness, denote $M_1, M_2 = M \in \text{Aut}(H_1( f_t, \mathbb{Z}))$ the monodromy operators associated to simple closed loops around $t_1$ or respectively $t_2 = 0$. The monodromy group $\mathbf{M}$ of the polynomial $x^2y^2(1 - x - y)$ is then the subgroup of $\text{Aut}(H_1( f_t, \mathbb{Z}))$ generated by $M_1, M_2$. The smallest algebraic variety containing $\mathbf{M}$ is an algebraic group, denoted $G$. It is the Zarisky closure of $\mathbf{M}$.

We note that $(H_1(f_t, \mathbb{Z}))$ carries a (degenerate) intersection form $\omega$ of rank $p = 2$ invariant under the action of $M_1, M_2$. It is easily verified, that $\mathbf{M}$ and hence $G$ is isomorphic to a subgroup of the symplectic group $Sp(4, \mathbb{C})$.

We shall prove the following

\textbf{Theorem 2.} The Zarisky closure of the monodromy group of the polynomial $x^2y^2(1 - x - y)$ is isomorphic to the symplectic group $Sp(4, \mathbb{C})$.

To the end of the section we give the proof of this remarkable fact. A basis of the first homology group $H_1(f_t, \mathbb{Z})$ will be chosen as in the preceding section

$$\delta_0, \delta_1, \delta_0^{12}, \delta_{12}, \delta_{13}, \delta_{23}$$  \hfill (8)

where $\delta_0 = \delta_0(t)$ is a cycle vanishing at the unique Morse critical point when $t$ tends to $t_1$, $\delta_{13}, \delta_{23}$ are vanishing cycles at the singular points $(1,0), (0,1)$, and $\delta_0^{12}, \delta_{12}$ are
cycles vanishing at \((0,0)\). The cycle \(\delta_1\) is the image of \(\delta_0\) under the action of the monodromy operator about the singular value \(t = 0\), \(\delta_1 = M_* \delta_0\). The cycles \((8)\) are represented by closed loops on the Riemann surface \(f_t\), and by abuse of notation we denote these loops by the same letter. The closed loops can be chosen in a way that their union is a deformation retract of \(f_t\), see fig.5.

To compute \(M_1\) we note that according to Proposition 3 the sign of intersection indexes of the cycles of \(H_1(f_t, \mathbb{Z})\) can be chosen as follows

|     | \(\delta_0\) | \(\delta_1\) | \(\delta_{12}^0\) | \(\delta_{12}^1\) | \(\delta_{13}\) | \(\delta_{23}\) |
|-----|--------------|--------------|------------------|------------------|--------------|--------------|
| \(\delta_0\) | 0            | -1           | 1                | 0                | 1            | 1            |
| \(\delta_1\) | 1            | 0            | 0                | 1                | 1            | 1            |
| \(\delta_{12}^0\) | -1           | 0            | 0                | 0                | 0            | 0            |
| \(\delta_{12}^1\) | 0            | -1           | 0                | 0                | 0            | 0            |
| \(\delta_{13}\) | -1           | -1           | 0                | 0                | 0            | 0            |
| \(\delta_{23}\) | -1           | -1           | 0                | 0                | 0            | 0            |

The monodromy operators \(M_1\) and \(M_2^2\) in this basis \((8)\) are represented by the fol-
lowing matrices (denoted by the same letter).

\[
M_1 = \begin{pmatrix}
1 & -1 & 1 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
M_2 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

The homology group \(H_1(f_t, \mathbb{Z})\) splits into two invariant subspaces under \(M\)

\[H_1(\Gamma_h, \mathbb{C}) = V_1 \oplus V_2\]

where

\[V_1 = \text{Span}\{\delta_0, \delta_1, \delta_{12} + \delta_{13} + \delta_{23}, \delta_{13} + \delta_{12} + \delta_{23}\},
V_2 = \text{Span}\{\delta_{13} - \delta_{23}, \delta_{13} + \delta_{23} - 2\delta_{12} - 2\delta_{12}\}\]

The monodromy group \(M\) acts on \(V_2\) as the identity transformation, and on \(V_1\) in the basis \(\delta_0, \delta_1, \delta_{12} + \delta_{13} + \delta_{23}, \delta_{13} + \delta_{12} + \delta_{23}\) the monodromy operators are represented by the following matrices (which we denote by the same letters)

\[
M_1 = \begin{pmatrix}
1 & -1 & 3 & 2 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
M_2 = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}.
\]

Let \(\mathfrak{g}\) be the Lie algebra of \(G\), that is to say the tangent space \(T_I G\) of the variety \(G\) at the identity matrix \(I\). Clearly \(\mathfrak{g}\) is isomorphic to a sub-algebra of \(sp(4, \mathbb{C})\) and to prove Theorem 2 it will be enough to check that \(\mathfrak{g}\) is isomorphic to \(sp(4, \mathbb{C})\). For this let us note first that

\[
M_1^k = \begin{pmatrix}
1 & -k & 3k & 2k \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \in M, \forall k \in \mathbb{Z}
\]

which implies

\[
M_1^z = \begin{pmatrix}
1 & -z & 3z & 2z \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \in G, \forall z \in \mathbb{C}
\]
and hence
\[
a = \begin{pmatrix}
0 & -1 & 3 & 2 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \in \mathfrak{g}
\]

Similarly
\[
M_2^2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & -1 & 0 & 1
\end{pmatrix},
M_2M_1M_2^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 1 & 2 & 3 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

and taking powers of these matrices we conclude that
\[
c = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
b = \begin{pmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 2 & 3 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

belong to \( \mathfrak{g} \). We shall check that in fact \( \mathfrak{g} \) is generated as a Lie algebra by \( a, b, c \).

**Proposition 4.** The Lie algebra \( \mathfrak{g} \) generated by the matrices \( a, b, c \) is isomorphic to the symplectic algebra \( \mathfrak{sp}(4, \mathbb{C}) \).

**Proof.** As \( \mathfrak{g} \subset \mathfrak{sp}(4, \mathbb{C}) \) it is enough to compute the Cartan decomposition of \( \mathfrak{sp}(4, \mathbb{C}) \) with respect to the intersection form on \( V_1 \subset H_1(f_t, \mathbb{Z}) \), and verify that the basis of the decomposition belongs to \( \mathfrak{g} \). Note first that
\[
[[a, b], a] = -2a, [[a, b], b] = 2b
\]

and hence the matrices \( a, b, [a, b] \) generate \( \mathfrak{sl}_2(\mathbb{C}) \). This suggests that the matrix \( H_1 = [a, b] \) belongs to the Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \). To find a second element \( H_2 \) of \( \mathfrak{h} \) we compute (some) eigenvectors of \( \text{ad}_{H_1} : \mathfrak{g} \to \mathfrak{g} \), where \( \text{ad}_{H_1}(X) = H_1X - XH_1 \), until finding an appropriate candidate for \( H_2 \), after what we display the various root spaces. Namely, let
\[
X_{21} = -3[a, b] + [a, c] - 4a, X_{12} = 3[a, b] + [b, c] + 4b, H_2 = [X_{21}, X_{12}].
\]

Then \( \mathfrak{h} = \langle H_1, H_2 \rangle \) is the Cartan subalgebra and let
\[
\lambda_1, \lambda_2 \in \mathfrak{h}^*, \lambda_1(H_1) = 1, \lambda_1(H_2) = -5, \lambda_2(H_1) = 0, \lambda(H_2) = 5.
\]

be a basis of the dual space \( \mathfrak{h}^* \). Then it is straightforward to check that \( \pm \lambda_1 \pm \lambda_2 \) are roots with corresponding one-dimensional roots spaces \( \mathfrak{g}_{\pm \lambda_1 \pm \lambda_2} \) spanned by the vector on the second line of Table 1 where
\[
Y_{12} = [X_{21}, b], Z_{12} = [X_{12}, a], U_1 = b, V_1 = a, U_2 = [[X_{21}, b], X_{21}], V_2 = [X_{12}, [X_{12}, a]].
\]

\[\square\]
Table 1: Root spaces of $\mathfrak{g}$

| $\mathfrak{g}_{\lambda_1 - \lambda_2}$ | $\mathfrak{g}_{-\lambda_1 + \lambda_2}$ | $\mathfrak{g}_{\lambda_1 + \lambda_2}$ | $\mathfrak{g}_{-\lambda_1 - \lambda_2}$ | $\mathfrak{g}_{2\lambda_1}$ | $\mathfrak{g}_{-2\lambda_1}$ | $\mathfrak{g}_{2\lambda_2}$ | $\mathfrak{g}_{-2\lambda_2}$ |
|--------------------------------------|--------------------------------------|--------------------------------------|--------------------------------------|-----------------------------|-----------------------------|-----------------------------|-----------------------------|
| $X_{12}$                             | $X_{21}$                             | $Y_{12}$                             | $Z_{12}$                             | $U_1$                        | $V_1$                        | $U_2$                        | $V_2$                        |

5 Concluding remarks

Let $f = f(x,y)$ be an arbitrary non-constant polynomial. The set $\mathcal{A}$ of its non-regular values is finite and therefore we can consider the monodromy representation of the fundamental group $\pi_1(\mathbb{C} \setminus \mathcal{A}, \ast)$ on $H_1(f^{-1}(t), \mathbb{Z})$. Clearly the representation preserves the intersection form of the first homology group $H_1(f^{-1}(t), \mathbb{Z})$.

The subplane $V_0 \subset H_1(f^{-1}(t), \mathbb{Z})$ of zero-cycles (the kernel of the intersection form) is invariant, and $\pi_1(\mathbb{C} \setminus \mathcal{A}, \ast)$ acts on it trivially. Therefore the reduced representation of the fundamental group on $V = H_1(f^{-1}(t), \mathbb{Z})/V_0$ is well defined too, and $V$ carries an invariant non-degenerate intersection form. The reduced monodromy group is thus a subgroup of $Sp(2p, \mathbb{C})$ and denote by $G$ its Zarisky closure. Here $2p = \dim V$ and $p$ is the genus of the Riemann surface of $f^{-1}(t)$, $t \notin \mathcal{A}$.

It is well known that for generic $f$ (e.g. Morse plus polynomials) we have $G = Sp(2p, \mathbb{C})$. According to Theorem 2 this holds true also in the special Lotka-Volterra case $f = x^p y^p (1 - x - y)$, $p = 2$. We conjecture that $G = Sp(2p, \mathbb{C})$ for every integer $p \geq 1$.

On the other hand, if $f$ is a composite polynomial, $f = g \circ h$, where $h : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a polynomial mapping, we can not expect that $G = Sp(2p, \mathbb{C})$. A simple example is $f = y^2 + P(x^2)$ for $P$ a polynomial of degree at least three. The natural involution $x \rightarrow -x$ induces a decomposition $H_1(f^{-1}(t), \mathbb{Z})/V_0 = V_+ \oplus V_-$ where $V_\pm$ are invariant under $G$ so $G \neq Sp(2p, \mathbb{C})$. We note that by the Ritt theorem 3 a univariate polynomial $f \in \mathbb{C}[x]$ is composite if and only if its monodromy group is imprimitive. Are there examples of non-composite bivariate polynomials $f = f(x,y)$, such that $G \neq Sp(2p, \mathbb{C})$?

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