Gravitating cosmic superconducting tubes
in the Einstein gauged non-linear $\sigma$-model in (3+1)-dimensions

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Abstract

In this paper we construct the first analytic examples of (3 + 1)-dimensional self-gravitating regular cosmic tube solutions which are superconducting, free of curvature singularities and with non trivial topological charge in the Einstein-$SU(2)$ non-linear $\sigma$-model. These gravitating topological solitons at large distance from the axis look like a (boosted) cosmic string with an angular defect given by the parameters of the theory, and near the axis the parameters of the solutions can be chosen so that the metric is singularity free and without angular defect. The curvature is concentrated on a tube around the axis. These solutions are similar to global strings but regular everywhere, and the non-linear $\sigma$-model regularizes the global string in a similar way as a non-Abelian field regularizes the Dirac monopole. Also, these solutions can be promoted to self consistent solutions of the fully coupled Einstein Maxwell non-linear $\sigma$-model in which the non-linear $\sigma$-model is minimally coupled both to the $U(1)$ gauge field and to General Relativity. The analysis shows that these solutions behave as superconductors as they carry a persistent current even when the $U(1)$ field vanishes. Such persistent current cannot be continuously deformed to zero as it is tied to the topological charge of the solutions themselves. The peculiar features of the gravitational lensing of these gravitating solitons are shortly discussed.
1 Introduction

Topological defects are formed in phase transitions when a system goes from a state of higher symmetry to a state of lower symmetry. They can be classified as local or global depending on the fact if a local or global symmetry is broken. Topological defects occur in very different areas of physics like, for example, condensed matter physics, high energy physics and cosmology. In condensed matter physics perhaps the most famous and intuitive example is the formation of domain walls in ferromagnetic materials, which separate domains with different magnetization. Another famous example are the formation of vortex lines in superfluid helium and line defects (dislocations) in crystals. A topological defect can sometimes be completely regular in which case it is known as a topological soliton. They play a fundamental role in quantum field theory, nuclear physics and high energy physics [1, 2].

The first example of stable topological soliton in three space dimensions was proposed by Skyrme and it is known as Skyrmion [3]. It has the remarkable property of possessing Fermionic excitations despite the fact that the dynamical field is an $SU(2)$-valued scalar field. At leading order in the ’t Hooft large $N$ expansion [4, 5, 6], the Skyrme model represents a (phenomenologically successful) low energy description of QCD. The Skyrme model arose from a clever modification of the non-linear $\sigma$-model (NLSM henceforth), which is the low energy description of the dynamics of Pions (for nice reviews see e.g. [7], [8]). The Skyrme term was added to
the NLSM in order to avoid Derrick’s no-go scaling argument \[9\] preventing the existence of static soliton solutions of finite energy. On the other hand, it should be kept in mind that the elegant arguments in \[10\] (see also \[11\], \[12\], \[13\], \[14\], and references therein) to show that Skyrmions represent Fermions (at least semi-classically) are only based on the existence of stable solitons with non-trivial third homotopy class while they do not use directly the Skyrme term in itself. Thus, it is extremely interesting to search for alternative ways to avoid Derrick’s scaling argument in order to achieve “\textit{Fermions out of Bosons}” with the simplest possible ingredients. The main approaches to avoid the no-go scaling argument in \[9\] are, first of all, to minimally couple the NLSM to gravity and/or to Maxwell theory. Indeed, using the techniques developed in \[15\], \[16\], \[17\], \[18\], \[19\], \[20\], \[21\], \[22\], \[23\], \[24\], \[25\], \[26\], \[27\], \[28\], \[29\], \[30\], exact self-gravitating NLSM solutions with non-trivial topological charge have been found in \[31\], \[32\] and \[33\]. Secondly, it is very helpful to construct time-dependent ansatz for the SU(2)-valued matter field with the property that the energy-momentum tensor is time-independent (this idea is the SU(2) generalization of the Bosons star ansatz for a U(1)-charged scalar field: see \[34\], \[35\] and references therein). Such a generalization has been achieved in \[20\], \[25\], \[29\] and \[30\].

Further relevant topological solitons which play an important role in high energy as well as condensed matter physics, are the Abrikosov-Nielsen-Olesen vortex line \[36\], \[37\] and the ‘t Hooft-Polyakov monopole in the SU(2) Yang-Mills-Higgs system \[38\], \[39\]. It is worth to mention that the latter at large distances looks like a Dirac monopole, however the Dirac monopole, which describes a point-like magnetic charge, is singular at the origin whereas the ‘t Hooft-Polyakov monopole is regular at the origin. Thus, the non-Abelian internal symmetry group in the ‘t Hooft-Polyakov monopole “regularizes” the Dirac Monopole’s singularity. In the present paper we will see a similar effect in the case of gravitating hadronic tubes as we will explain later.

The formation of topological defects are very important in grand unified theories as the actual symmetry group of the standard model is supposed to be a result of a series of spontaneous symmetry breaking of a larger symmetry group. This means that topological defects play a fundamental role from microscopic scale to extremely large scale namely in cosmology due to the fact that the universe in its evolution expanded and cooled down, and therefore went through several phase transitions were topological defects have formed. As in cosmology the most relevant interaction is gravity, the topological defects must be studied in the context of field theories coupled to gravity.

In cosmology the topological defect which attracted the most attention of the scientific community are the cosmic strings. The simplest exact cosmic string solution is given by an energy-momentum tensor concentrated in a line (for example the \(z\) axis) \[40\]

\[
T_{\alpha\beta} = \mu \delta(x) \delta(y) \text{diag}(1, 0, 0, 1) .
\]

The exact solution of the Einstein field equations associated to this energy-momentum tensor, written in cylindrical coordinates, is locally but not globally flat

\[
ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2 ,
\]

as the range of the angular coordinate \(\theta\) is not as usual \(2\pi\) but

\[
0 \leq \theta \leq 2\pi(1 - 4G\mu) ,
\]

where \(G\) and \(\mu\) are the gravitational constant and the mass of the cosmic string per unit length, respectively. Therefore this locally flat space-time has a conical defect, an angular deficit \(\Delta = 8G\mu\) and has a curvature singularity on the \(z\) axis. A possible way to smooth out the singularity is to smear the energy-momentum tensor on a cylinder of finite radius \(\delta\) and it is possible to find exact solutions \[41\], \[42\], \[43\]. The principal problem of this procedure is that there
is a sharp boundary whose radius is arbitrary and moreover the energy-momentum tensor is not derived from some fundamental action principle. In order to find a cosmic string solution from a fundamental action principle usually the Einstein-Yang-Mills-Higgs action is used, but unfortunately no exact solutions are known. An explicit global cosmic string metric (where the fundamental field is a Goldstone boson instead of the Higgs and Yang-Mills fields) has the peculiarity that the matter distribution has no sharp boundary and the angular defect varies with the distance from the symmetry axis \[44\]. This metric has a curvature singularity at a finite distance from the axis. It was shown that non-singular global strings can exist if there is an explicit time dependence in the metric \[45, 46\].

The existence of cosmic strings has many important cosmological and astrophysical implications. For example, cosmic strings have been proposed to have a role in the galaxy formation as a source of density perturbations \[47\]. Cosmic strings have also observable effects through gravitational lensing being the most known effect the formation of double images \[40\], \[48\], \[49\]. It is worth to point out that the space-time generated by a thin string (with a Dirac delta matter source) does not exert force on a test particle being locally flat, but the existence of a conical defect still generates double images. Perhaps one of the most fascinating aspects of cosmic strings is that, under certain conditions, they become superconducting as it was shown in the pioneering article of Witten \[50\]. This can have observable effects as such superconducting strings would act as sources of synchrotron radiation or high energy cosmic rays \[51\]. Moreover it has also been proposed that superconducting strings moving in a magnetized plasma can be a mechanism for the production of gamma ray bursts \[52\].

Due to the many important cosmological and astrophysical implications of cosmic strings it is of great interest to find analytic non-singular solutions which can be derived from some fundamental action principle which leaves no arbitrariness in the choice of fields and their potentials. Indeed, as it has been explained before, in the case of the Einstein-Yang-Mills-Higgs action no such exact solutions are known. The known exact solutions are the thin string with Dirac delta matter source and the global string, both of them possess curvature singularities which would go against the cosmic censorship conjecture. An important point is how to choose a fundamental matter field. Here we will consider the (gauged) NLSM as it is an effective low energy description of the dynamics of Pions (as well as of their electromagnetic properties).

In this paper we construct the first examples of analytic and singularity free cosmic tube solutions for the self-gravitating SU(2)-NLSM. At large distance the metric behave in a similar way to the one of a cosmic string boosted in the axis and has an angular defect related to the parameter of the theory, however near the axis the parameter of the solution can be chosen in such way that the solution is free of singularities and without angular defect. This means that these solutions are free of singularities everywhere and the angular defect depends on the distance from the axis. It is also worth point out that the matter field do not have a sharp boundary and the curvature reaches its maximum on a tube around the axis rather than on the axis itself. All these features make the new solutions similar to a global string with the big difference that they are regular instead of having a singularity at finite distance from the axis. The cosmic tubes found here are related with global strings in the same way as non-Abelian monopole are related with Dirac monopoles. In other words, the non-linear $\sigma$-model regularizes the global string keeping a similar behavior at large distances. These solution also possess non-trivial topological charge (the third homotopy class) and can be promoted to full solutions of the Einstein Maxwell NLSM in which the NLSM field is minimally coupled both to the $U(1)$ field as well as to gravity. These gauged solutions carry a persistent current even when the $U(1)$ gauge field is zero and therefore becomes superconducting in the sense of \[50\]. In particular, the superconducting currents are tied to the topological charge so that they cannot be deformed continuously to zero (that is why they are persistent). It is worth to emphasize that the gravitating solitons constructed in the present paper only involve degrees of free-
dom arising from low energy QCD minimally coupled with General Relativity without the need of additional potentials.

The structure of the paper is the following: in Sections II and III, we present the model, give our ansatz and the corresponding field equations. In section IV we construct the exact regular cosmic tube solutions of the Einstein NLSM and show that they possess non trivial topological charge. Also we study the geodesic equations and its physical properties. In Section V it will be shown how to promote the found solutions to be solutions of the gauged Einstein NLSM system and how these are superconducting configurations. In the last section our conclusions are detailed.

2 The Einstein $SU(2)$-NLSM

The Einstein-NLSM theory is described by the action
\[
I[g,U] = \int d^4x \sqrt{-g} \left( \frac{\mathcal{R}}{2\kappa} + \frac{K}{4} \text{Tr}[L^\mu L_\mu] \right),
\]
where $\mathcal{R}$ is the Ricci scalar and $L_\mu$ are the Maurer-Cartan form components $L_\mu = U^{-1} \nabla_\mu U$ for $U \in SU(2)$, being $\nabla_\mu$ the covariant derivative. Here $\kappa$ is the gravitational constant and the positive coupling $K$ is fixed by experimental data. In our convention $c = \hbar = 1$ and Greek indices run over the four dimensional space-time with mostly plus signature.

In order to produce a correct physical interpretation of the topological solitons here constructed and compare with the cosmic string solutions already existing in the literature, in this work we will refer to configurations with vanishing cosmological constant, however the techniques used here are also effective in presence of a cosmological constant. We hope to come back on this interesting issue in a future publication.

The complete Einstein-NLSM equations read
\[
\nabla^\mu L_\mu = 0, \quad G_{\mu\nu} = \kappa T_{\mu\nu},
\]
where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ the energy-momentum tensor of the NLSM given by
\[
T_{\mu\nu} = -\frac{K}{2} \text{Tr} \left[ L_\mu L_\nu - \frac{1}{2} g_{\mu\nu} L^\alpha L_\alpha \right].
\]
The winding number of the configurations reads
\[
w_B = \frac{1}{24\pi^2} \int \rho_B, \quad \rho_B = \text{Tr}[\epsilon^{ijk} L_i L_j L_k].
\]
When the topological density $\rho_B$ is integrated on a space-like surface $\omega_B$ represents the baryon number. We will only consider configurations in which $\rho_B \neq 0$. A necessary (but, in general, not sufficient) condition in order to have non-vanishing topological charge is
\[
d\alpha \wedge d\Theta \wedge d\Phi \neq 0
\]
where $\alpha$, $\Theta$ and $\Phi$ are the three scalar degrees of freedom appearing in the standard parametrization of the $SU(2)$-valued scalar field defined in Eq. [8].
3 Ansatz and Field Equations

To construct analytical solutions in this theory we will use the generalized hedgehog ansatz [15], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], which is defined as

\[
U^{\pm}(x^{\mu}) = \cos(\alpha) \mathbf{1}_2 \pm \sin(\alpha) \ n^i t_i, \quad n^i n_i = 1 ,
\]

\[
n^1 = \sin \Theta \cos \Phi , \quad n^2 = \sin \Theta \sin \Phi , \quad n^3 = \cos \Theta ,
\]

where \( t_i \equiv i \sigma_i \) with \( \sigma \) are the Pauli matrices. This ansatz is defined over the Weyl-Lewis-Papapetrou metric

\[
\begin{align*}
\text{ds}^2 &= -B_0^2 \left( \frac{2e^{f_0}}{B_0} - \omega_s G \right) \text{d}t^2 - 2B_0^2 \left( \frac{e^{f_0}}{B_0} - \omega_s G \right) \text{d}t \text{d}z + B_0^2 e^{f_0} G \text{d}z^2 + e^{-2R} \left( \text{d}r^2 + \text{d}\theta^2 \right) ,
\end{align*}
\]

(9)

where in general \( G = G(r, \theta) , \ R = R(r, \theta) \) while \( B_0, \omega_s, f_0 \) are arbitrary constants. Since the metric determinant of the section spanned by \( t \) and \( z \) is negative definite (and \( e^{-2R} \) is definite positive as we will show in the next sections),

\[
\begin{vmatrix}
g_{tt} & g_{tz} \\
g_{zt} & g_{zz}
\end{vmatrix} = -B_0^2 < 0 ,
\]

(10)

the space-time with this metric is always Lorentzian regardless of the metric components in the \( t - z \). As it will be discussed in the next sections, in order to have an integer value of the topological charge, one must allow the whole range of real number as the domain of the coordinate \( r \):\n
\[-\infty < r < \infty .
\]

(11)

and \( \theta \) is an angular coordinate with range \( \theta \in [0, 2\pi] \).

According to [29], [30] we will consider a matter field in the form

\[
\alpha = \alpha(r) , \quad \Theta = q\theta , \quad \Phi = \omega_s t + z ,
\]

(12)

where \( \omega_s \) is a constant. Note that the ansatz defined here above satisfies the necessary condition in Eq. (7) in order to possess a non-trivial topological charge. The sufficient conditions will be discussed in the following sections.

This ansatz is very useful for (at least) three reasons. The first one is because Eq. (12) implies the relations

\[
\nabla_{\mu} \Phi \nabla^{\mu} \Phi = 0 , \quad \nabla_{\mu} \Theta \nabla^{\mu} \Phi = 0 ,
\]

which simplify greatly the NLSM equations. The second reason is that Eq. (12) allows to avoid the Derrick’s scale argument [2] as it is a time dependent ansatz which, however is compatible with a stationary metric. Thirdly, the three coupled field equations for the NLSM in the metric defined in Eq. (9) reduce to the single second order ODE for \( \alpha(r) \):

\[
\alpha'' - \frac{q^2}{2} \sin(2\alpha) = 0 .
\]

(13)

It is important to note that this second order equation for \( \alpha(r) \) can be reduced to the following first order equation

\[
(\alpha')^2 - q^2 \sin^2 \alpha = E_0 ,
\]

(14)
where $E_0$ is an integration constant. However, the compatibility with the Einstein equation requires that $E_0 = 0$. This situation is different from what happens in flat space-time [29], [30], where the integration constant $E_0$ can be non-zero.

Quite remarkably, the Einstein equations with the energy-momentum tensor corresponding to the NLSM configuration defined in Eq. (12) are reduced to only two solvable equations:

$$R'' - K\kappa q^2 \sin^2 \alpha = 0 , \quad (15)$$

$$\left(\partial_r^2 + \partial_\theta^2\right)G + 2C_0 \sin^2(q\theta)e^{-2R} \sin^2 \alpha = 0 , \quad (16)$$

where $C_0 = K\kappa e^{f_0}/B_0^2$. Note that once $R$ and $\alpha$ are solved, $G$ can be obtained directly. In fact, Eq. (16) is nothing but a flat linear Poisson equation in two dimensions in which the source term is known explicitly (as $\alpha(r)$ and $R(r)$ have been determined in Eqs. (13) and (15)). Hence, Eq. (16) can be solved, for instance, using the method of Green’s function. In the next sections we will construct the solution using a direct method. At this point, it is worth to emphasize that the function $G$ depends explicitly on $\theta$. As the coordinate $\theta$ plays the role of an angular coordinate going around the hadronic tube, the fact that $G$ depends on $\theta$ implies that the present family of gravitating solitons is not axi-symmetric (as we will clarify in the next sections). The physical role of the function $G$ will be discussed in the analysis of the curvature invariants and of the geodesics.

4 Gravitating tubes

4.1 Solving the system

Eqs. (13) and (15) can be solved analytically, and the expressions for $\alpha(r)$ and $R(r)$ are given by

$$\alpha(r) = 2 \arctan \exp (qr + C_1) , \quad (17)$$

$$R(r) = K\kappa \ln \left( \cosh(qr + C_1) \right) + C_2r + C_3 , \quad (18)$$

where $C_1$, $C_2$ and $C_3$ are integration constants. Replacing the above in Eq. (16) we obtain the final equation for $G$,

$$\partial_r^2 G + \partial_\theta^2 G + \frac{2K\kappa}{B_0^2} e^{-2(C_2r+C_3)+f_0} \cosh^{-2(K\kappa+1)}(qr + C_1) \sin^2(q\theta) = 0 . \quad (19)$$

At this point it is important to emphasize that considering only Eqs. (8), (9), (12), (17) and (18) the complete Einstein-NLSM system has been reduced to a single equation for the metric function $G$ given in Eq. (19).

The energy density (measured by a co-moving observer) of these configurations is given by

$$T_{\hat{0}\hat{0}} = K \left[ q^2 e^{2C_2r+C_3} \text{sech}^{2(1-K\kappa)}(qr + C_1) + \frac{\omega_s e^{f_0} \sin^2(q\theta) \text{sech}^2(qr + C_1)}{B_0(2e^{f_0} - B_0^2\omega_s G)} \right]$$

where $T_{\hat{0}\hat{0}} = \mathbf{T}(\hat{e}_0, \hat{e}_0)$ for $\hat{e}_0 = \frac{1}{\sqrt{-g_{\hat{0}\hat{0}}}} \partial_\hat{t}$.

Note that we should impose the constraint on the integration constants

$$C_2 < (1 - K\kappa) |q| , \quad (21)$$

to avoid divergence of the energy density at $r \to \pm\infty$. 


4.2 An analytical solution

Using that \( \sin^2(x) = \frac{1 - \cos(2x)}{2} \), Eq. (16) becomes

\[
(\partial_r^2 + \partial_\theta^2)G + C_0 e^{-2R(r)} \sin^2 \alpha(r) - C_0 \cos(2q\theta) \times e^{-2R(r)} \sin^2 \alpha(r) = 0 .
\]  (22)

The function \( G \) can be expressed by the sum of two functions \( G_1(r) \) and \( G_2(r, \theta) \)

\[
G(r, \theta) = G_1(r) + G_2(r, \theta) ,
\]  (23)

which satisfy

\[
\frac{d^2G_1}{dr^2} = -C_0 e^{-2R(r)} \sin^2 \alpha(r) ,
\]
\[
(\partial_r^2 + \partial_\theta^2)G_2 = C_0 \cos(2q\theta) \times e^{-2R(r)} \sin^2 \alpha(r) .
\]  (24)

The first equation is trivially solved by a double integral with respect to \( r \),

\[
G_1(r) = -C_0 \int_{-\infty}^r dr_1 \int_{-\infty}^{r_1} dr_2 e^{-2R(r_2)} \sin^2 \alpha(r_2) .
\]  (25)

We can solve the equation for \( G_2 \) by separation of variables, obtaining

\[
G_2(r, \theta) = y \psi_1(r) \cos(2q\theta) ,
\]  (26)

for some real constant \( y \), and \( \psi_1(r) \) satisfying

\[
\psi_1'' - 4q^2 \psi_1 = \frac{C_0}{y} \times e^{-2R(r)} \sin^2 \alpha(r) .
\]  (27)

Eq. (27) can be solve in terms of elliptic functions using the method of variation of parameters. The solution of the homogeneous equation is

\[
\psi_h = a\psi^{(1)}_h + b\psi^{(2)}_h = ae^{2qr} + be^{-2qr} ,
\]

with \( a, b \) integration constants. On the other hand, a particular solution can be found through

\[
\psi_p = A(r)\psi^{(1)}_h + B(r)\psi^{(2)}_h ,
\]

where

\[
A(r) = \int \frac{-\psi^{(2)}_h P(r)}{W(\psi^{(1)}_h, \psi^{(2)}_h)} dr ,
\]
\[
B(r) = \int \frac{\psi^{(1)}_h P(r)}{W(\psi^{(1)}_h, \psi^{(2)}_h)} dr ,
\]

here \( W(\psi^{(1)}_h, \psi^{(2)}_h) = -4q \) is the Wronskian, and

\[
P(r) = \frac{C_0}{y} e^{-2R} \sin^2 \alpha .
\]
Therefore, the solution of Eq. (27) is given by

\[
\psi_1 = \psi_h + \psi_p = ae^{2qr} + be^{-2qr} + e^{-2(C_1 + qr)} \cosh^{-2K\kappa}(C_1 + qr) \left( C_0 + C_0(e^{2(C_1 + qr)} - 1) \text{Hypergeometric2F1}[1, -1 - K\kappa, 1 + K\kappa, -e^{2(C_1 + qr)}] \right).
\]

4.3 Regularity

Here we will discuss the regularity of the metric. Taking into account Eqs. (17) and (18), the Ricci scalar \(S\), the Kretschmann scalar, the square of the Ricci tensor and of the Weyl tensor for the metric (9) read

\[
S = 2Kq^2 e^{2(C_2 + C_3)} \cosh^{2(K\kappa - 1)}(qr + C_1), \quad R^\mu\nu\rho\sigma R_{\mu\nu\rho\sigma} = S^2, \quad R^\mu\nu R_{\mu\nu} = \frac{1}{2}S^2,
\]

\[
C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} = \frac{4K^2q^2}{3} \left[ g e^{2C_2 + C_3} \cosh^{K\kappa - 1}(qr + C_1) \right]^4,
\]

and are all regular everywhere (in particular, \(S\) is regular at \(r \to \pm\infty\)) if we impose the following condition on the integration constants:

\[
|C_2| < (1 - K\kappa) |q|.
\]

One may wonder whether the above condition in Eq. (29) is enough to ensure the regularity of the metric. Indeed, it is possible to compute explicitly the main fourteen curvature invariants that are usually considered in the literature to analyze, in four dimensions, the issue of regularity \(53, 54\). Such invariants are

\[
I_1 = S = R^\mu_\mu = 2e^{2R} R^\mu_\mu, \quad I_2 = R^\mu_\nu R^\nu_\mu = \frac{1}{2}I_1^2, \quad I_3 = R^\mu_\nu R^\nu_\rho R^\rho_\mu = \frac{1}{4}I_1^3, \quad I_4 = R^\mu_\nu R^\nu_\rho R^\rho_\sigma R^\sigma_\mu = \frac{1}{8}I_1^4,
\]

\[
J_1 = A_{\mu\nu\rho\sigma}g^{\mu\rho}g^{\nu\sigma} = \frac{1}{3}I_1^2, \quad J_2 = B_{\mu\nu\rho\sigma}g^{\mu\rho}g^{\nu\sigma} = \frac{1}{18}I_1^3, \quad J_3 = E_{\mu\nu\rho\sigma}g^{\mu\rho}g^{\nu\sigma} = 0, \quad J_4 = F_{\mu\nu\rho\sigma}g^{\mu\rho}g^{\nu\sigma} = 0,
\]

\[
K_1 = C_{\mu\nu\rho\sigma}R^{\mu\nu}R^{\rho\sigma} = \frac{1}{12}I_1^3, \quad K_2 = A_{\mu\nu\rho\sigma}R^{\mu\nu}R^{\rho\sigma} = \frac{1}{36}I_1^4, \quad K_3 = C_{\mu\nu\rho\sigma}Q^{\mu\nu}Q^{\rho\sigma} = \frac{1}{48}I_1^5,
\]

\[
K_4 = A_{\mu\nu\rho\sigma}Q^{\mu\nu}Q^{\rho\sigma} = \frac{1}{144}I_1^6, \quad K_5 = D_{\mu\nu\rho\sigma}Q^{\mu\nu}Q^{\rho\sigma} = 0.
\]

with

\[
A_{\mu\nu\rho\sigma} = C_{\mu\nu\alpha\beta}C_{\gamma\delta\rho\sigma}g^{\alpha\gamma}g^{\beta\delta}, \quad B_{\mu\nu\rho\sigma} = C_{\mu\nu\alpha\beta}A_{\gamma\delta\rho\sigma}g^{\alpha\gamma}g^{\beta\delta},
\]

\[
D_{\mu\nu\rho\sigma} = B_{\mu\nu\rho\sigma} - \frac{J_2}{12}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) - \frac{1}{4}J_1C_{\mu\nu\rho\sigma},
\]

\[
D_{\mu\nu\rho\sigma} = \frac{1}{\sqrt{J_3}}D_{\mu\nu\rho\sigma}, \quad E_{\mu\nu\rho\sigma} = C_{\mu\nu\alpha\beta}D_{\gamma\delta\rho\sigma}g^{\alpha\gamma}g^{\beta\delta},
\]

\[
F_{\mu\nu\rho\sigma} = C_{\mu\nu\alpha\beta}E_{\gamma\delta\rho\sigma}g^{\alpha\gamma}g^{\beta\delta}, \quad Q_\mu^\nu = R^\mu_\rho R^\rho_\nu.
\]

The regularity condition at \(r \to \pm\infty\) is the same as (29). Thus, all the curvature invariants which can be built from the metric in Eqs. (9) and (18) are regular everywhere (and, moreover, they do not depend on the function \(G(r, \theta)\)) if the condition in Eq. (29) on the integration constants holds.

One can also verify that the space-time is of Petrov type II. One more constraint on the integration constant \(C_2\) will arise from the analysis of the geodesic in the following sections.
4.4 Periodicity of $U$ and the topological charge

In this subsection we will discuss a simple but deep property of the $SU(2)$ valued scalar field $U$ which has a very important consequence. First of all, one can notice that when

$$q = \frac{1}{2} + n,$$

with $n$ an integer, the topological charge is non-zero while, when $q$ is an integer, the topological charge vanishes. This can be seen as follows. The topological density corresponding to the NLSM configuration in Eq. (12) reads

$$\rho_B = (12q \sin q \sin^2 \alpha) \alpha'.$$

Thus, as the range of $\theta$ is $[0, 2\pi]$, the integral of the above density

$$B = \int W r d\theta \wedge dz,$$

is non-zero if and only if Eq. (30) holds. It is also worth to note here that the $z$ coordinate goes along the axis of the tube and along the topological density. Thus, in a sense, the quantity

$$\frac{B}{L_z} = \int W dr \wedge d\theta$$

can be interpreted as the topological charge per unit of length of the tube, with $L_z = \int dz$.

Here it is worth to emphasize that, when the condition in Eq. (30) holds, $\theta$ is a proper angular coordinate with range $[0, 2\pi]$. First of all, the metric itself is periodic with period $2\pi$ as it depends on $\theta$ only through the function $G(r, \theta)$. The function $G(r, \theta)$ depends only through the factor $\cos (2q \theta)$ in Eq. (16). Secondly, the energy-momentum tensor of the $SU(2)$-valued scalar field is also periodic in $\theta$ with the same period $2\pi$ when the condition in Eq. (30) holds.

However, the matrix $U$ itself is not periodic in $\theta$ with the same period $2\pi$ as it reads

$$U = \begin{pmatrix} \cos \alpha + i \sin \alpha \cos (q \theta) & i e^{-i(z + \omega t)} \sin \alpha \sin (q \theta) \\ i e^{i(z + \omega t)} \sin \alpha \sin (q \theta) & \cos \alpha - i \sin \alpha \cos (q \theta) \end{pmatrix}.$$

The “lack of periodicity” can be compensated by an internal Isospin rotation (and this explains why the energy-momentum tensor corresponding to the above NLSM configuration is periodic with period $2\pi$). This can be seen as follows: let’s consider an Isospin transformation to the $U$ matrix

$$U \rightarrow \hat{U} = M_I U M_I^{-1},$$

with $M_I \in SU(2)$ a constant matrix given by

$$M_I = i \begin{pmatrix} 0 & e^{-c} \\ e^c & 0 \end{pmatrix},$$

1As it will be discussed in the next subsection, it is possible to choose the integration constants of the solution in such a way to eliminate the deficit angle close to the origin: see Eq. (42) and the discussion below.
with \( c \) a constant. It is direct to check that

\[
U(\theta = 0) = \hat{U}(\theta = 2\pi) .
\]

This means that \( U \) is periodic up to an Isospin transformation and, consequently, \( \theta \) is a proper angular coordinate with range \([0, 2\pi]\) when \( q \) is half-integer.

Thus, as it happens with the spin-from-Isospin effect, the internal symmetry group plays a fundamental role. In that case, the spin-from-Isospin effect is generated by the possibility to require “spherical symmetry up to an internal rotation”. In the present case, the condition that \( U \) satisfies periodic boundary conditions up to an Isospin rotation is enough to ensure that \( T_{\mu\nu} \) is periodic and \( B \neq 0 \).

### 4.5 Coordinate transformation and the asymptotic behavior

To analyze the nature of the metric in Eq. (9), it is convenient to make the following coordinate transformation:

\[
X(r) = \int_{-\infty}^{r} e^{-R(y)} dy = \int_{-\infty}^{r} \frac{e^{-c_2 y - c_3}}{(\cosh(qy + C_1))^{K\kappa}} dy . \tag{31}
\]

It is obvious that \( X(r) \) increases monotonically as \( r \) increases, since

\[
\frac{dX}{dr} = \frac{e^{-c_2 r - c_3}}{(\cosh(qr + C_1))^{K\kappa}} > 0 . \tag{32}
\]

Moreover, to make \( X(r) \) well-defined, it is necessary to impose that

\[
C_2 < K\kappa |q| . \tag{33}
\]

This assumption gives us the range of \( X \) given by

\[-\infty < r < \infty \implies 0 < X < \infty . \tag{34}\]

With this coordinate transformation, the metric becomes,

\[
\text{ds}^2 = -\frac{B_0^2 \omega_s}{e^{f_0}} \left( \frac{2e^{f_0}}{B_0} - \omega_s G \right) dt^2 - \frac{2B_0^2}{e^{f_0}} \left( \frac{e^{f_0}}{B_0} - \omega_s G \right) dz^2 + \frac{B_0^2}{e^{f_0}} G d\theta^2 + \frac{e^{2\tilde{R}(X)}}{b^{2\tilde{R}(X)}} d\theta^2 ,
\]

where \( \tilde{R}(X) = (R \circ r)(X) \).

As we will show now, the function \( e^{-2\tilde{R}(X)} \) of the new cylindrical radial coordinate \( X \) both for \( X \) close to zero and for \( X \to \infty \) is proportional to \( X^2 \):

\[
e^{-2\tilde{R}(X)} \xrightarrow{X \to 0} \Delta_0 X^2 , \quad e^{-2\tilde{R}(X)} \xrightarrow{X \to \infty} \Delta_\infty X^2 .
\]

This implies that \( \theta \) is an angular coordinate. Consequently, it is very important to determine the coefficients \( \Delta_0 \) and \( \Delta_\infty \) which determine the effective deficit angles seen from observers “very close to” and “very far from” the axis of the tube, respectively.

Note that we have the following two limits of the function \( R \):

\[
r \to \pm \infty \implies R \to (C_2 \pm K\kappa |q|) r , \tag{35}
\]
so that we also have
\[ r \to \pm \infty \implies e^{-R} \to 2K\kappa e^{\mp K\kappa C_1 - C_3} e^{-(C_2 \pm K\kappa|q|)r}. \] (36)

In the limit of \( r \gg r_+ \) for a sufficiently large \( r_+ \gg 0 \), we find that
\[
X(r) = X_1 + \frac{2^{K\kappa} \times e^{K\kappa C_1 + C_3}}{-(C_2 + K\kappa|q|)} e^{-(C_2 + K\kappa|q|)r} - e^{-(C_2 + K\kappa|q|)r_+}
\approx \frac{2^{K\kappa} \times e^{-(K\kappa C_1 + C_3)}}{-(C_2 + K\kappa|q|)} e^{-(C_2 + K\kappa|q|)r}. \] (37)

Here, we have defined a finite constant
\[
X_1 \equiv \int_{-\infty}^{r_+} e^{-C_2y - C_3} \frac{e}{\cosh(|q|y + C_1)}^{K\kappa} dy,
\] (38)
and in the last line, we used the assumption that \( C_2 + K\kappa|q| < 0 \). Thus, we obtain
\[
e^{-2R} \approx (K\kappa|q| + C_2)^2 X^2, \quad \text{at} \quad r \gg 1. \] (39)

In a similar way, the limit of \( r \ll r_- \) for a sufficiently small \( r_- \ll 0 \) is found to be
\[
X(r) = X_2 - \frac{2^{K\kappa} \times e^{K\kappa C_1 - C_3}}{-(C_2 - K\kappa|q|)} e^{-(C_2 - K\kappa|q|)r} - e^{-(C_2 - K\kappa|q|)r_-}
\approx \frac{2^{K\kappa} \times e^{K\kappa C_1 - C_3}}{-(C_2 - K\kappa|q|)} e^{-(C_2 - K\kappa|q|)r}. \] (40)

Here, we also have defined a finite constant
\[
X_2 \equiv \int_{-\infty}^{r_-} e^{-C_2y - C_3} \frac{e}{\cosh(|q|y + C_1)}^{K\kappa} dy,
\] (41)
and in the last line we have used the fact that \( r \ll r_- \ll 0 \). Therefore, we get
\[
e^{-2R} \approx (K\kappa|q| - C_2)^2 X^2 e^{-2C_3}, \quad \text{at} \quad r \ll 0. \] (42)

Now, we see that we can choose \( C_3 \) such that the angular deficit is 1 near the axis defined by \( X(r) = 0 \) so that \( \theta \) becomes a proper angular coordinate. The ratio of the values of \( g_{\theta\theta} \) at two infinities becomes,
\[
\frac{g_{\theta\theta}(r = \infty)}{g_{\theta\theta}(r = -\infty)} = \frac{g_{\theta\theta}(X = \infty)}{g_{\theta\theta}(X = 0)} = \left( \frac{K\kappa|q| + C_2}{K\kappa|q| - C_2} \right)^2 < 1. \] (43)
4.6 Geodesics

Let \( \lambda \) be an affine parameter of geodesics in our space-time. The geodesic equations are found to be

\[
\ddot{t} - B_0 e^{-f_0} (\omega_s \dot{t} + \dot{z}) \frac{dG}{d\lambda} = 0 ,
\]

\[
\ddot{z} + \omega_s B_0 e^{-f_0} (\omega_s \dot{t} + \dot{z}) \frac{dG}{d\lambda} = 0 ,
\]

\[
\ddot{r} - (\dot{t}^2 - \dot{\theta}^2) (C_2 + K \kappa q \tanh(q r + C_1)) - \frac{1}{2} B_0^2 (\omega_s \dot{t} + \dot{z})^2 e^{2C_2 r + 2C_3 - f_0} \cosh^{2K\kappa} (q r + C_1) \partial_r G = 0 ,
\]

\[
\ddot{\theta} - 2\dot{r} \dot{\theta} (C_2 + K \kappa q \tanh(q r + C_1)) - \frac{1}{2} B_0^2 (\omega_s \dot{t} + \dot{z})^2 e^{2C_2 r + 2C_3 - f_0} \cosh^{2K\kappa} (q r + C_1) \partial_\theta G = 0 ,
\]

where

\[
\frac{dG}{d\lambda} = \dot{r} \partial_r G + \dot{\theta} \partial_\theta G ,
\]

and the dot denotes the derivative with respect to \( \lambda \).

From the first two equations, we obtain

\[
\omega_s \dot{t} + \dot{z} = 0 \quad \Rightarrow \quad \omega_s t + z = a \lambda + b ,
\]

\[
\ddot{z} - \omega_s \dot{t} + 2 B_0 e^{-f_0} \omega_s (\omega_s \dot{t} + \dot{z}) \frac{dG}{d\lambda} = 0 .
\]

for some constants \( a \) and \( b \).

Then, the Eqs. (45), (46), and (47) become

\[
\ddot{z} + a B_0 e^{-f_0} \omega \frac{dG}{d\lambda} = 0 ,
\]

\[
\ddot{r} - (\dot{t}^2 - \dot{\theta}^2) (C_2 + K \kappa q \tanh(q r + C_1)) - \frac{1}{2} B_0^2 a^2 e^{2C_2 r + 2C_3 - f_0} \cosh^{2K\kappa} (q r + C_1) \partial_r G = 0 ,
\]

\[
\ddot{\theta} - 2 \dot{r} \dot{\theta} (C_2 + K \kappa q \tanh(q r + C_1)) - \frac{1}{2} B_0^2 a^2 e^{2C_2 r + 2C_3 - f_0} \cosh^{2K\kappa} (q r + C_1) \partial_\theta G = 0 .
\]

4.6.1 Geodesics of constant \( r \) and \( \theta \)

Let us consider a geodesic of constant \( r \) and \( \theta \) with the trajectory given by \( (t(\lambda), r_0, \theta_0, z(\lambda)) \) for some constants \( r_0 \) and \( \theta_0 \). Then, \( G(\lambda) = G(r_0, \theta_0) \) is constant so that

\[
\left. \frac{dG}{d\lambda} \right|_{(r_0, \theta_0)} = 0 .
\]

Thus, the Eqs. (44) and (45) allow a trivial solution with

\[
t(\lambda) = t_1 \lambda + t_0 , \quad z(\lambda) = (a - t_1) \lambda + (b - t_0) ,
\]

for some constants \( t_0 \) and \( t_1 \).

4.6.2 Geodesics on surfaces of constant \( z \)

On the hypersurface of constant \( z \), the Eq. (45) becomes

\[
\omega_s B_0 e^{-f_0} \omega_s \dot{t} \frac{dG}{d\lambda} = 0 ,
\]
from which we obtain
\[
\frac{dG}{d\lambda} = \dot{r} \partial_r G + \dot{\theta} \partial_\theta G = 0 .
\] (57)

Then, the Eq. (44) becomes
\[
\ddot{t} = 0 \quad \implies \quad t = \frac{a}{\omega_s} \lambda + b .
\] (58)

For convenience, let’s put
\[
a = \omega_s, \quad b = 0 ,
\] so that
\[
t = \lambda .
\] (59)

Then, the remaining geodesic equations become
\[
\ddot{r} - (\dot{r}^2 - \dot{\theta}^2) (C_2 + K \kappa \tanh(q r + C_1)) - \frac{1}{2} B_0^2 \omega_s^2 e^{2C_2 r + 2C_3 - f_0} \cosh^{2K \kappa} (q r + C_1) \partial_r G = 0 ,
\] (60)
\[
\ddot{\theta} - 2 \dot{r} \dot{\theta} (C_2 + K \kappa \tanh(q r + C_1)) - \frac{1}{2} B_0^2 \omega_s^2 e^{2C_2 r + 2C_3 - f_0} \cosh^{2K \kappa} (q r + C_1) \partial_\theta G = 0 .
\] (61)

Using Eq. (57) in a linear combination \((\dot{r} \times (52) + \dot{\theta} \times (53))\) yields
\[
\frac{1}{2} \frac{d}{d\lambda} \left( \dot{r}^2 + \dot{\theta}^2 \right) - \dot{r} \left( \dot{r}^2 + \dot{\theta}^2 \right) (C_2 + K \kappa \tanh(q r + C_1)) = 0 ,
\]
or equivalently,
\[
\frac{d}{d\lambda} \left[ \frac{1}{2} \ln \left( \dot{r}^2 + \dot{\theta}^2 \right) - C_2 r - K \kappa \ln \left\{ \cosh(q r + C_1) \right\} \right] = 0 .
\] (62)

Thus, for some constant \(C_4\), we have
\[
\dot{r}^2 + \dot{\theta}^2 = e^{2(C_2 r + C_4)} \cosh^{2K \kappa} (q r + C_1) .
\] (63)

We should assume that
\[
C_2 < - K \kappa |q| ,
\] (64)
to avoid infinity velocities at large \(r\). The line element for the geodesic with constant \(z\) is
\[
ds^2 = - \frac{B_0^2 \omega_s}{e^{f_0}} \left( \frac{2e^{f_0}}{B_0} - \omega_s G \right) dt^2 + e^{-2R} (dr^2 + d\theta^2) .
\]

Divided by the affine parameter \(t\), it can be written as
\[
B_0 \omega_s \left( B_0 \omega_s e^{-f_0} G - 2 \right) + e^{-2R} (\dot{r}^2 + \dot{\theta}^2) = \frac{ds^2}{dt^2} \equiv \epsilon .
\]

By an appropriate rescaling \(\epsilon\) becomes \(-1\) or \(0\) for time-like or null geodesic, respectively. Using Eq. (63), we have
\[
e^{-2R} (\dot{r}^2 + \dot{\theta}^2) = e^{-2C_2 r - 2C_3} \sech^{2K \kappa} (q r + C_1) \times e^{2(C_2 r + C_4)} \cosh^{2K \kappa} (q r + C_1) = e^{-2C_3 + 2C_4} .
\]

Thus, the line element of the geodesic with constant \(z\) gives a relation between constants as follows
\[
B_0 \omega_s \left( B_0 \omega_s e^{-f_0} G - 2 \right) + e^{-2C_3 + 2C_4} = \epsilon = 0 .
\]
Using Eq. (63) in the geodesic equation of \( r \), from Eq. (60) one finds that

\[
\ddot{r} - 2\dot{r}^2 \left( C_2 + Kq \tanh(qr + C_1) \right) - e^{2C_2r} \cosh^{2K} (qr + C_1) \times \left\{ e^{2C_3} \left( C_2 + Kq \tanh(qr + C_1) \right) + \frac{1}{2} B_0^2 \omega_s^2 e^{2C_3 - f_0} \partial_r G \right\} = 0 .
\]  

After solving this equation, one should plug the solution to the equation of motion for \( \theta \) in Eq. (61) to find \( \theta(t) \). This completes the solving process of the geodesic equations with constant \( z \). In principle, this problem can be reduced to an effective one-dimensional Newtonian problem observing that, along the geodesics with constant \( z \), one has

\[
\dot{r} \partial_r G + \dot{\theta} \partial_{\theta} G = 0 \Rightarrow \frac{d\theta}{dr} = -\frac{\partial_r G}{\partial_{\theta} G} ,
\]  

where \( G(r, \theta) \) is defined in Eqs. (23), (25) and (28). The reason is that from Eq. (66) one can determine \( \theta = \theta(r) \) along the geodesics with constant \( z \). Once \( \theta = \theta(r) \) has been determined, one can insert it into Eq. (63) obtaining a first order Newtonian-like equation of the form

\[
\dot{r}^2 = V_{\text{eff}}(r) , \quad V_{\text{eff}}(r) = \frac{e^{2(C_2r + C_3)} \cosh^{2K} (qr + C_1)}{1 + \left( \frac{d\theta}{dr} \right)^2} .
\]

However, Eq. (66) is a quite complicated first order non-autonomous differential equation for \( \theta(r) \) due to the explicit form of \( G(r, \theta) \) in Eqs. (23), (25) and (28). The relevant issue of the behavior of geodesics in this family of gravitating solitons deserve a more detailed analysis on which we hope to come back in a future publication.

In the usual case of cosmic strings, the angle defect of gravitational lensing is independent of the initial distance of the particle from the source. But in our case, the distribution of source is smoothly spread out to \( r = \infty \), so that the angle defect depends on \( r_0 \), the initial location of geodesic motion. This would be one of the most distinguished properties of our solution.

### 4.7 Constraint on integration constant

Combining the regularity conditions on the integration constants in Eqs. (21), (29), (33) and (64) we obtain the following single inequality

\[
-(1 - K\kappa) |q| < C_2 < -K\kappa |q| .
\]

Thus, when \( C_2 \) satisfies the above condition and \( C_3 \) is chosen as in Eq. (42) all the curvature invariants of the metric are regular and the geodesics behave in a reasonable way. Note that the experimental value of the Pions coupling constant is such that \( 0 < K\kappa \ll 1 \). On the other hand, the integration constant \( C_1 \) is a free parameter which fixes the location of the point about which the profile function \( \alpha(r) \) is symmetric.
5 Gauged gravitating tubes

5.1 The Einstein-NLSM-Maxwell theory

The action of the $U(1)$ gauged NLSM is

$$I[g, U, A] = \int d^4x \sqrt{-g} \left[ \frac{R}{2\kappa} + \frac{K}{4} \text{Tr}(L^\mu L_\mu) - \frac{1}{4} F_{\mu\nu}F^{\mu\nu} \right],$$

and the field equations read

$$D_\mu L^\mu = 0, \quad G_{\mu\nu} = \kappa(T_{\mu\nu} + \bar{T}_{\mu\nu}),$$

where the current $J^\mu$ is given by

$$J^\mu = \frac{K}{2} \text{Tr} \left[ \hat{O} L^\mu \right], \quad \hat{O} = U^{-1} t_3 U - t_3,$$

being the electromagnetic energy-momentum tensor. Note that in the action in Eq. (68) there is a quadratic term in $A_\mu$.

The topological charge in this case is given by

$$w_B = \frac{1}{24\pi^2} \int_\Sigma \rho_B,$$

where

$$\rho_B = \epsilon^{ijk} \text{Tr} \left[ (U^{-1} \partial_i U) (U^{-1} \partial_j U) (U^{-1} \partial_k U) - \partial_i [3A_j t_3 (U^{-1} \partial_k U + (\partial_k U) U^{-1})] \right].$$

5.2 Gravitating tubes coupled to the electromagnetic field

Let’s consider the following Maxwell potential

$$A_\mu = \left( u, 0, 0, \frac{1}{\omega_s} u \right), \quad u = u(r, \theta),$$

together with the metric defined in Eq. (9) and the matter field in Eq. (12). This is a very convenient ansatz [29, 30], a direct computation reveals that the three field equations for the gauged NLSM reduce (once again) to

$$\alpha'' - \frac{q^2}{2} \sin 2\alpha = 0,$$

so that

$$\alpha(r) = 2 \arctan(e^{qr+C_1}).$$

On the other hand, the four Maxwell equations reduce to just one linear equation:

$$\Delta u - 2K e^{-2R}(\omega_s - 2u) \sin^2 \alpha \sin^2 q\theta = 0.$$
Note that this linear equation for $\Psi = \omega_s - 2u$ can be easily solved, at least numerically, since $\alpha(r)$ is explicitly known and $R(r)$ can be also determined explicitly. In fact, there are two non-trivial Einstein equations that can be combined to obtain an uncoupled equation for $R$

$$R'' - \frac{1}{2}K\kappa(\alpha'^2 + q^2 \sin^2 \alpha) = 0,$$

so that, as in the case without Maxwell,

$$R(r) = K\kappa \log(\cosh(qr + C_1)) + C_2 r + C_3,$$  \hspace{1cm} (79)

and the following equation for $G$

$$\Delta G + \frac{2\kappa}{B_0^2 \omega_s^2} e^{f_0 - 2R} \left( K \sin^2 \alpha \sin^2 q\theta (\omega_s - 2u)^2 + e^{2R(\nabla u)^2} \right) = 0.$$ \hspace{1cm} (80)

Hence, once again, the function $G$ satisfy a flat two-dimensional Poisson equation in which the source is explicitly known. Therefore, once the Maxwell equation in Eq. (78) has been solved, the function $G$ can be determined explicitly using several methods (such as the Green function). Resuming, with the ansatz for $U$ and $A_\mu$ in Eqs. (12), (76) and for the metric in Eq. (9), the Einstein-Maxwell-NLSM field equations reduce to Eqs. (78) and (80), where $\alpha(r)$ and $R(r)$ are in Eqs. (77) and (79).

The energy density measured in an orthonormal frame is found to be

$$T^0_0 = \frac{K e^{f_0}}{2\omega_s B_0(2e^{f_0} - B_0 \omega_s G)} \left[ 2\sin^2(q\theta) \sech^2(qr + C_1)(2u - \omega_s)^2 + e^{2C_2 r + C_3} \cosh^{2K\kappa}(qr + C_1) \left\{ (\partial_r u)^2 + (\partial_\theta u)^2 \right\} \right]$$

$$+ K q^2 e^{2C_2 r + C_3} \sech^{2(1-K\kappa)}(qr + C_1),$$ \hspace{1cm} (81)

where we used the tetrad given by

$$\hat{e}_0 = \frac{1}{\sqrt{-g_{00}}} \hat{\partial}_t.$$

The topological charge density including the Callan-Witten term is

$$\rho_B = \partial_r \left( 6q(\alpha - \sin \alpha \cos \alpha) \sin(q\theta) + \frac{12q}{\omega_s} \sin \alpha \cos \alpha \sin(q\theta) \cdot u \right) + \partial_\theta \left( \frac{12}{\omega_s} \cos(q\theta) \cdot u \partial_r \alpha \right).$$

The non-vanishing components of the current are given by

$$J_\mu = 2K \sin^2 \alpha \sin^2 q\theta (\partial_\mu \Phi - 2A_\mu),$$ \hspace{1cm} (82)

and the electric and magnetic field read

$$E_r = -\partial_r u, \hspace{1cm} E_\theta = -\partial_\theta u, \hspace{1cm} B_r = -\frac{1}{B_0} e^{2R} \partial_\theta u, \hspace{1cm} B_\theta = \frac{1}{B_0} e^{2R} \partial_r u.$$

The plots here below as well as the review of the Witten construction clearly show why these solutions of the Einstein-Maxwell-NLSM represent regular topologically non-trivial gravitating superconducting tubes.
In order to plot the relevant physical functions we have set our parameters as

\[ C_1 = 0, \quad C_2 = -\frac{1}{50}, \quad C_3 = 0, \quad K = \frac{1}{10}, \quad \kappa = \frac{1}{4}, \quad q = \frac{1}{2}, \quad \omega = 1, \quad f_0 = 0, \quad B_0 = \frac{1}{2}. \]  \hspace{1cm} (83)

![Figure 1](image1)

Figure 1: From left to right, the \( \alpha \) profile and the Ricci scalar \( S \) as functions of the \( r \) coordinate and metric function \( e^{-2R(X)} \) as a function of the \( X \) coordinate.

![Figure 2](image2)

Figure 2: The metric function \( G \) as a function of \( r \) and \( \theta \).

From Fig. 1 and Fig. 2 we can see that even if the metric is not axi-symmetric due to the explicit \( \theta \)-dependence of \( G(r, \theta) \) in Eq. (80), all the curvature invariants are axi-symmetric as they only depend on \( r \). Thus, the plots of all the curvature invariants are very similar and they all show a smooth peak at finite distance from the origin (remember that, in the coordinate \( r \), the origin is at \( r \rightarrow -\infty \)). As we expected, \( e^{-2R(X)} \) goes as \( X^2 \) when \( X \rightarrow 0 \) as well as when \( X \rightarrow \infty \).

![Figure 3](image3)

Figure 3: The Energy density \( T_{00} \) and topological density \( \rho_B \) as functions of \( r \) and \( \theta \).
In Fig. 3 the energy density associated to a comoving observer has two parts. The first one only depends on “r” and, as the curvature invariants, has its smooth maximum at the same finite distance from the axis. The second part depends both on “r” and on “θ” and the corresponding peak is at the same distance from the axis as the peak of the first term and, in θ is localized around θ ∼ π. The peaks of the energy density and the peaks of the topological density coincide, as expected.

From the plots in Fig. 4 one can see that \( J_\mu \neq 0 \) only at the position of the two peaks and tends to zero out. Here we have imposed the following boundary conditions,

\[
u(r, 2\pi) - u(r, 0) = 0, \quad G(r, 2\pi) - G(r, 0) = 0.
\]

### 5.3 Why the tubes are superconductors?

#### 5.3.1 Review of the Witten argument

Before discussing the superconducting nature of the present solutions, we will shortly review the results in [50] (which have been considerably generalized in many subsequent works: see for instance [55], [56], [57], [58], [59], [60], and references therein).

The main motivation to introduce such topological objects is related, of course, to the spectacular observable effects that such objects could have (were they to exist: see the original reference [50] as well as [61], [62], [63], [64], [65], [66], [67], [68], [69], [70], [71], [72], [73], [74], [75], [76], [77], [78] and references therein). The second motivation is related to the fact that such remarkable objects can be constructed using quite reasonable ingredients. Many of the examples available in the literature do not use exclusively building blocks within the standard model\(^2\). For instance, extra U(1) gauge potential as well as Higgs-like scalar fields are often important ingredients while in [55], [56], [57], [58], [59], [60], supersymmetry plays a fundamental role.

The starting point of [50] is the following Lagrangian\(^3\)

\[
L_{\text{kin}} = -\frac{1}{4} (F^2 + B^2) + |D\sigma|^2 + |D\psi|^2,
\]

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad B_{\mu\nu} = \partial_\mu S_\nu - \partial_\nu S_\mu,
\]

\[
D_\mu \sigma = (\partial_\mu + ieA_\mu) \sigma, \quad D_\mu \psi = (\partial_\mu + ieS_\mu) \psi.
\]

\(^2\)A very interesting exception is the superconducting strings constructed in [80] in the electroweak sector. On the other hand, their stability properties have not been fully understood yet.

\(^3\)We will change the notation of [50] slightly in order to avoid confusions at later stages.
The above theory has an $U(1) \times U(1)$ gauge symmetry (the first corresponding to $A_{\mu}$ and the second to $S_{\mu}$). In order for the above theory to support superconducting strings, it is necessary to include an interaction potential between the two Higgs fields $\sigma$ and $\psi$. The choice of [50] was

$$V(\sigma, \psi) = \frac{\lambda}{8} \left( |\psi|^2 - \mu^2 \right)^2 + \frac{\lambda}{4} |\sigma|^4 + f |\sigma|^2 |\psi|^2 - m^2 |\sigma|^2 \right), \quad (85)$$

$$S_{\text{tot}} = \int d^4x \sqrt{g} (L_{\text{kin}} + V(\sigma, \psi)) \right). \quad (86)$$

The reasons behind this choice are the following: The first necessary ingredient is the breakdown of the gauge symmetry corresponding to $S_{\mu}$ in order to ensure the existence of vortices. The Higgs field $\psi$ in the core of the vortex field is usually assumed to only depend on the two spatial coordinates (say, $r$ and $\theta$) transverse to the vortex axis (which is along the $z$-axis). Then, one must require that, in the vacuum, $\langle \sigma \rangle = 0$. At this point, with a clever choice of the range of the parameters of the Higgs potential, one can achieve the following situation. Despite the fact that the (minimization of the) kinetic energy tends to suppress $\langle \sigma \rangle \neq 0$ within the core, if one chooses $m^2$ to be positive, the potential energy will favor $\langle \sigma \rangle \neq 0$ within the core. As it was shown in [50], this can indeed happen. In other words, there is an open region in the parameter space in which $\langle \sigma \rangle = 0$ asymptotically but $\langle \sigma \rangle \neq 0$ within the core of the vortex associated to $\psi$. This is a fundamental technical step since the superconducting currents (to be described in a moment) are sustained by the region in which $\langle \sigma \rangle \neq 0$. If $\sigma_0(r, \theta)$ minimizes the energy of the string, then the superconducting current is associated with the (slowly varying) phase $\Theta$ of $\sigma_0(r, \theta)$. One can achieve this by introducing the dependence on $z$ and $t$ in $\sigma$ as follows:

$$\sigma(r, \theta, z, t) = \sigma_0(r, \theta) \exp[i\Theta(z, t)] \right). \quad (87)$$

The expression of the current is

$$\mathcal{J} \approx 2e\sigma \left( \partial \Theta + e A \right) \right), \quad (88)$$

which is made of two factors. The first factor $\left( \partial \Theta + e A \right)$ is responsible for the dynamics of the zero modes along the strings (associated to the phase $\Theta$). However, such a factor by itself would be “useless” as it needs to “rely on” something. This “something” is the factor $\sigma$. Thus, first of all, the first factor needs $\sigma$ to be different from zero somewhere. For superconducting strings, as it has been already emphasized, the spatial region where $\sigma \neq 0$ is tube-shaped. However, this is not enough: the configurations in which $\sigma \neq 0$ within a tube-shaped region must be stable, otherwise the current would decay.[4] In the settings of [50], [55], [56], [57], [58], [59], [60], the linear stability of the configurations where $\sigma \neq 0$ within the string (as well as $\sigma$ approaching to zero outside) were established by direct methods (such as linear perturbation theory). Note also that the current defined above cannot have arbitrarily large values since $\sigma$ has a maximum value determined by the Higgs potential. Once the stability of such tube-shaped regions (which are going to host the superconducting currents) has been established, one can ask:

*Under which circumstances the above current is superconducting?*

One needs a mechanism which keeps such current perpetually alive even in the absence of an external gauge potential. At this point, topology comes into play. Since $\Theta$ is only defined modulo $2\pi$, the integral over a close loop of the above current will not vanish in general even when one turns off the electromagnetic field. Indeed, one can build a topological invariant associated to (the integral of) $\Theta$. Thus, such a current cannot relax when the topological invariant associated to $\Theta$ is non-zero.

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[4] As it will be explained in the next subsection, the best option would be a setting in which a suitable non-vanishing topological charge enforces $\sigma$, at the same time, to be different from zero in some spatial region and to approach to zero outside. In this way, topology would ensure the stability of the superconducting current.
Basic ingredients A very nice pedagogical construction has been achieved in \[79\] where, assuming that symmetry breaking is always generated by suitable Higgs or scalar potentials, the author reduced the basic ingredients to the skeleton. The author required that bulk theory should have an unbroken global non-Abelian symmetry (let us assume, just for concreteness, that such a group is \(SU(2)\)) allowing the existence of string-like configurations. Moreover, on such string-like configurations, one has to break \(SU(2)\) down to a subgroup \(U(1)\). As it has been already emphasized, such requirements are usually taken care of by introducing suitable potentials for the scalars. In fact, the Einstein-Maxwell-NLSM has all the above ingredients already “built-in” and there is no need to introduce any potential: the typical interactions among the Maurer-Cartan forms associated to the Isospin degrees of freedom do the job.

Secondly, it would be very nice to use topology not only to ensure the persistent character of the current but also to guarantee the appearance of regions where \(\sigma \neq 0\) (such that \(\sigma\) approaches zero outside). Higher topological charges can be useful, as we will discuss in the following subsection.

5.3.2 The superconductivity of the tubes

The main features of the above expressions in Eq. \([82]\) for the current of these topologically non-trivial gauged crystals are the following.

1) The current does not vanish even when the electromagnetic potential vanishes \((u = 0)\).

2) Such a “current left over” \(J_{(0)\mu}\):

\[
J_{(0)\mu} = 2K \sin^2(\alpha) \sin^2(q\theta) \partial_{\mu} \Phi,
\]

(where \(\Phi\) has been defined in Eq. \([12]\)) which survives even when the Maxwell field is turned off, is maximal where the energy density is maximal (namely, where \(\sin^2(\alpha) \sin^2(q\theta) = 1\) which defines the positions of the peaks in the energy density as well as in the topological density) and vanishes rapidly far from the peaks.

3) Such residual current \(J_{(0)\mu}\) cannot be turned off continuously. This can be seen as follows. There are three ways to “kill” \(J_{(0)\mu}\). The first way is to deform \(\alpha\) to an integer multiple of \(\pi\) (but this is impossible as such a deformation would change the topological charge). The second way is to deform \(q\theta\) to an integer multiple of \(\pi\) (but also this deformation is impossible due to the conservation of the topological charge). The third way is to deform \(\Phi\) to a constant (but also this deformation cannot be achieved). Note also that \(\Phi\) is defined modulo \(2\pi\) (as the \(SU(2)\) valued field \(U\) depends on \(\cos \Phi\) and \(\sin \Phi\) rather than on \(\Phi\) itself). This implies that the line integral of \(\partial_{\mu} \Phi\) along a closed contour does not necessarily vanish (as it happens in the original Witten argument).

The above characteristics show that the above residual current is a persistent current which cannot vanish as it is topologically protected. This, by definition, implies that \(J_{(0)\mu}\) defined in Eq. \([89]\) is a superconducting current supported by the present gauged tubes.

5.4 Comments on the peculiar features of gravitational lensing in these gravitating gauged tubes

It is a good place to comment the peculiar features of the gravitational lensing of these gravitating gauged superconducting tubes. As it is well known, one of the main aim of gravitational lensing (a classic detailed textbook is \([81]\)) is the analysis of light rays in curved space-times of physical interest. The physical argument which provides gravitational lensing with sound basis is the geometrical optics approximation of the Maxwell equations in curved space-times. Namely, in many situations of high interest in astrophysics and cosmology, the analysis of light-like geodesics is already enough to get relevant information (avoiding, in this way, the analysis of the full Maxwell equations in the space-times of interest which is considerably more difficult).
Thus, roughly speaking, in the usual cases one analyzes the Maxwell equations

$$\nabla^\mu F_{\mu\nu} = 0,$$  \hspace{1cm} (90)

within the eikonal approximation. The relevance of null geodesics arises from the fact that the photon is massless. However, in the present case, the Maxwell field does not couple only to gravity but also, obviously, to the NLSM through the covariant derivative and the current defined in Eqs. (69), (71) and (72). Therefore, the Maxwell equations within the background defined by gravitating solitons in Eqs. (9), (17), (18), (23) and (28) (or in its gauged version discussed here above) are not the ones in Eq. (90) but the ones obtained by the (variation with respect to $A_\mu$ of the) action in Eqs. (68) and (69). Such equations read

$$\nabla^\mu f_{\mu\nu} = -4K \sin^2(q \theta) \sin^2(\alpha(r)) \left( a_\nu + \partial_\nu \Omega \right),$$ \hspace{1cm} (91)

$$0 = \nabla^\nu \left[ \sin^2(q \theta) \sin^2(\alpha(r)) \left( a_\nu + \partial_\nu \Omega \right) \right],$$ \hspace{1cm} (92)

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu,$$ \hspace{1cm} (93)

where $\nabla^\mu$ is the Levi-Civita covariant derivative of the metric defined in Eqs. (9), (18), (23) and (28), $a_\nu$ is the electromagnetic perturbation and $\alpha(r)$ defined in Eq. (17). The need of the gauge transformation $\Omega$ (determined by the condition in Eq. (92)) arises from the fact that we are considering electromagnetic perturbations keeping fixed the $SU(2)$ valued field $\bar{U}$ (which, as it has been already emphasized, plays the role of the background). As it happens in the usual Ginzburg-Landau description of superconductors, the mass-like term in Eq. (91) for the electromagnetic perturbations arises from the terms quadratic in $A_\mu$ in the action in Eqs. (68) and (69). Consequently, in order to analyze the propagation of light-rays in these gravitating tubes it is mandatory to develop the geometrical optics approximation corresponding to the system in Eqs. (91), (92) and (93). This is a very interesting but rather difficult topic on which we hope to come back in a future publication. Here we only want to emphasize that the physical effects of the mass-like term in Eq. (91) are very small far from the peaks in the energy density and topological density of the gravitating soliton which are defined by the conditions

$$\sin^2(q \theta) \sin^2(\alpha(r)) = 1.$$

Thus, the light rays which propagate very far from the position of the tube do not feel the presence of the gravitating soliton itself and follow light-like geodesics. However, close to the peaks the light rays will deviate considerably from light-like geodesics.

### 6 Conclusions

The first example of analytic and curvature singularity free cosmic tube solutions for Einstein $SU(2)$ NLSM have been found. The metric at large distance from the axis looks similar to a boosted cosmic string. The matter distribution has no sharp boundary and the curvature is concentrated at a finite distance from the axis. The angular defect of the solutions depends on the distance from the axis and the parameters of the solutions can be chosen in such a way that it vanishes near the axis but not at large distance. These properties make the solution similar to a global string but with the fundamental difference that while the global string has a curvature singularity at a finite distance from the axis whereas the new solutions are singularity free. Due to the non-Abelian nature of $SU(2)$ the singularity has been

\[5\] If we would consider perturbations both of the Maxwell field and of the NLSM then the $U(1)$ gauge invariance would be manifest again. However, being the present $SU(2)$ background a soliton, it is a reasonable approximation to keep it fixed as it is much heavier than the photon.
smeared out in a similar way as the Yang-Mills field in the ’t Hooft-Polyakov monopole smears out the singularity of the Dirac monopole.

Due to the non-Abelian symmetry group the most natural way to impose a periodicity condition on the $SU(2)$ field is up to an inner space rotation. This very natural boundary condition allows the solution to carry non-trivial topological charge.

One of the most remarkable aspects of these solutions is that they can also be promoted to solutions of the gauged Einstein NLSM, i.e. the $SU(2)$ field is minimally coupled to both the $U(1)$ field and gravity and without neglecting the corresponding Maxwell equations “sourced” by the currents arising from the NLSM. The gauged solutions are characterized by the fact that they can carry a persistent current even when the Maxwell field is zero, which means that they are superconducting. Moreover, the superconducting current is also topologically protected.

It is worth to point out that in cosmology one of the most important observational consequences of the existence of cosmic strings is gravitational lensing. The overwhelming majority of papers dealing with gravitational lensing assume axi-symmetry of the cosmic string. The analytic solution found here however is not axi-symmetric and therefore the geodesic equation becomes non-trivial. An interesting feature of the solution found here is that it is highly repulsive in the core of the tube, the matter distribution has no sharp boundary and therefore spreads to infinity, the deflection angle depends from the initial distance from the source. It is reasonable to suppose that these non trivial features should have interesting observational consequences and will be object of study in further investigations.

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