Probability of failure model in mechanical component because of fatigue

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Abstract. Fatigue is one of the most common modes in the failure of a mechanical component. Fatigue experiment shows that the coefficient of variation of fatigue life data is ranging from 30% to 40%. High deviation of the data means the deterministic prediction of fatigue life may not be valid anymore, hence probability-based method to calculate the probability of failure due to fatigue damaged is conducted. The probability of failure prediction methodology is based on damage fraction concept. As the usage cycles increase, the damage fraction increases until it reaches critical damage point where a component fails. Damage fraction is a function of ultimate strength and fatigue limit, in which both are modelled as random variables with normal distribution. The validation is done by variating standard deviation of each random variables. The validation result shows that increasing standard deviation of both ultimate strength and fatigue limit will increase the probability of failure.

1. Introduction
Fatigue is a failure mode due to high number of cycle load that cause crack to initiate and propagate, the crack then continues to propagate until fracture occurs. Fatigue loading is recognized as one of the most common failure modes in mechanical components. Fatigue failure analysis is an important issue for reliability analysis and structure design in many industries including power generation industry, automotive industry and aerospace industry [1, 2]. Fatigue failure of material is depend on the interaction of large stress with critical flow. In essence, fatigue is controlled by the weakest link of material with the probability of weak link increasing with the material volume [3, 4].

Earlier models of fatigue damage in literature focus on the deterministic nature of the process. One of the most popular deterministic approach of fatigue modelling is S-N curve, which is based on data of fatigue test. This curve shows the relation between fatigue life, N and cyclic stress amplitude, Sₐ and assumes that fatigue damage accumulation is linear. In a typical fatigue test, a specimen is subjected to a constant amplitude stress and the number of cycles to failure for a specific stress level is recorded. The fatigue lives corresponding to a specific stress level can be known from S-N curve. As the applied stress level decreases, the number of cycles to failure increases [5]. Some materials, such as steel and titanium have a fatigue limit, which is the highest stress that a material can withstand for an infinite number of cycle without breaking. If these materials are applied with stress level below or equal to their fatigue
limit, they will have infinite life. Fatigue limit is affected by several factors that are size, surface finish, surface treatment, temperature, environment, and type of loading.

Deterministic fatigue damage models tend to be conservative and produce undefinable levels of risk in components. In any fatigue experiment, there are always some amount of scatter in fatigue data [6]. This scatter is generally the effect of variety of random factors. Some of the factors include inconsistencies of surface finish, deviations in specimen alignment, differences in applied loading, and inconsistent individual stress. Those sources of scatter are generally mitigated through careful specimen preparation and handling, calibration of laboratory equipment, replicable experimental procedures, and use of identical specimens made from similar material from the same supplier. But even those steps are taken, scatter in fatigue data is still observed due to slight differences in the microstructure of each specimen, which produces different condition for crack initiation and propagation in the specimen.

The statistical scatter in fatigue experiment data is very large, with cycles to failure data having coefficient of variation typically ranging from 30% until 40% and sometimes as high as 150% [7]. This has led some researchers for finding new methods to predict the reliability and remaining life of components. The advantage of the new method will enhance the confidence in final product because it can accommodate the uncertainty and cost saving. The uncertainty in fatigue loading may result from couple of things, such as manufacturing process, geometry of the component, and loading process. In manufacturing process, different manufactures will have different tolerances in producing material, hence the same material will have slightly different mechanical properties. Different environment also has influence, a controlled laboratory environment will give more precise experimental result compared to laboratory where many distractions are present. Because of these, even under the same experimental condition such as constant amplitude loading, fatigue test may give random number of fatigue life with specific distribution.

This paper provide methodology of modelling probabilistic damage to predict the probability of failure of a component. The methodology then need to be tested with some condition to prove its validity.

2. Material and methods
Failure condition occurs when the load is more than or equal to the resistance. In this model, failure condition of mechanical component because of fatigue \( Z(n) \) is defined when the damage accumulation \( D_n \) greater than or equal to the critical damage \( D_{cr} \). Damage accumulation \( D_n \) is a function of two random variables that are ultimate strength \( S_u \) and fatigue limit \( S_e \), while critical damage \( D_{cr} \) is assumed to be constant with value one. Failure condition according to definition above can be written

\[
Z(n) = D_{cr} - D_n \leq 0
\]

(1)

Figure 1 gives illustration about the interaction between load and resistance. One of the most widely used damage accumulation model is the linear damage accumulation, also known as Palmgren-Miner rule [8]

\[
D_n = \frac{n}{N_f}
\]

(2)

![Figure 1. Probability of failure illustration using interaction between load and resistance](image-url)
where \( n \) is the number of applied cycles and \( N_f \) is the number of cycles to failure. To obtain the value of \( N_f \), Basquin [9] proposed mathematical equation that is

\[
N_f = \left( \frac{S_{nf}}{a} \right)^{-\frac{1}{b}}
\]

(3)

\[
a = \left( \frac{(f \times S_u)^2}{S_e} \right)
\]

\[
b = -\frac{1}{3} \log_{10} \left( \frac{f \times S_u}{S_e} \right)
\]

where \( S_{nf} \) is corrected alternating stress, \( S_u \) is ultimate strength, \( S_e \) is fatigue limit and \( f \) is fatigue strength fraction. Corrected alternating stress \( S_{nf} \) is required because in the engineering practice fully reversed loading conditions that is maximum stress and minimum stress completely reversed or it has zero mean stress is very rare. Goodman [10] propose mathematics equation to calculate \( S_{nf} \) base on Goodman diagram

\[
\frac{S_a}{S_{nf}} + \frac{S_m}{S_u} = 1
\]

where \( S_a \) is the alternating stress and \( S_m \) is the mean stress. Substitute equation (2) and (3) to (1) to obtain

\[
Z(n) = 1 - \frac{n}{\left( \frac{S_{nf}}{a} \right)^{-\frac{1}{b}}}
\]

(4)

Failure function \( Z(n) \) is a function of some variables such as ultimate strength \( S_u \), fatigue limit \( S_e \), fatigue strength fraction \( f \), alternating stress \( S_a \), and mean stress \( S_m \). In this model, two variables that are \( S_u \) and \( S_e \) are defined as random variables with normal distribution, \( S_u \sim N(\mu_1, \sigma_1) \) and \( S_e \sim N(\mu_2, \sigma_2) \) [11,12,13] and the other variables are constant. Furthermore, we can define \( Z(n) \) as a function of two random variables that are \( S_u \) and \( S_e \) or \( Z(n) = g(S_u, S_e) \). The probability of failure can be defined as

\[
P(Z(n) \leq 0) = P(g(S_u, S_e) \leq 0) = \iint_{g(S_u, S_e) \leq 0} f(S_u, S_e) \, dS_u \, dS_e
\]

(5)

where \( f(S_u, S_e) \) is joint probability function of random variable \( S_u \) and \( S_e \). The probability of failure illustration can be seen in Figure 2.

![Figure 2](image-url)
To make it easier to calculate, random variables $S_u$ and $S_e$ are normalized and it becomes $U_1 = \frac{S_u - \mu_1}{\sigma_1}$ and $U_2 = \frac{S_e - \mu_2}{\sigma_2}$. This process is valid as long as $S_u$ and $S_e$ are assumed to be independent. Beside that, we also linearize $g(U_1, U_2)$ to become

$$g(U_1, U_2) \approx g(u_1^*, u_2^*) + \nabla g(u_1^*, u_2^*)^T \left( \frac{u_1 - u_1^*}{u_2 - u_2^*} \right)$$

(6)

where $u_1^*$ and $u_2^*$ are the coordinate which make the distance from $g(U_1, U_2)$ to the origin become smallest. Equation (5) can be written as

$$P(Z(n) \leq 0) \approx P \left( g(u_1^*, u_2^*) + \nabla g(u_1^*, u_2^*)^T \left( \frac{u_1 - u_1^*}{u_2 - u_2^*} \right) \leq 0 \right)$$

(7)

Figure 3 give an illustration about the result of normalization and linearization process.

**Figure 3.** Normalization of random variables $S_u, S_e$ and linearization of failure function $g(U_1, U_2)$.

We use Lagrange method to find the value of $u_1^*$ and $u_2^*$. Our problem become minimize

$$f(U_1, U_2) = U_1^2 + U_2^2$$

with constraint

$$g(U_1, U_2) = 1 - \frac{n}{\left( \frac{S_{nf} \times (U_2 \sigma_2 + \mu_2)}{f \times (U \sigma_1 + \mu_1)} \right)^{\frac{1}{2}}} = 0$$

The probability of failure in Equation (7) can be calculated as follows

$$P \left( g(u_1^*, u_2^*) + \nabla g(u_1^*, u_2^*)^T \left( \frac{u_1 - u_1^*}{u_2 - u_2^*} \right) \leq 0 \right) = P(\nabla g(u_1^*, u_2^*)^T \left( \frac{u_1 - u_1^*}{u_2 - u_2^*} \right) \leq 0)$$

(8)

Supposed that $\nabla g(u_1^*, u_2^*)^T = (v_1^*, v_2^*)$ and $c = -v_1^* u_1^* - v_2^* u_2^*$

$$P \left( g(u_1^*, u_2^*) + \nabla g(u_1^*, u_2^*)^T \left( \frac{u_1 - u_1^*}{u_2 - u_2^*} \right) \leq 0 \right) = P(v_1^* U_1 + v_2^* U_2 \leq -c)$$

(9)

$U_1$ and $U_2$ are random variable with standard normal distribution so its mean is 0 and its variance is 1. Since $U_1$ and $U_2$ are obtained consecutively from $S_u$ and $S_e$ then it is obvious to assume independency. Define new random variable $X = v_1^* U_1 + v_2^* U_2$

$$P(Z(n) \leq 0) \approx P \left( g(u_1^*, u_2^*) + \nabla g(u_1^*, u_2^*)^T \left( \frac{u_1 - u_1^*}{u_2 - u_2^*} \right) \leq 0 \right) = P(X \leq -c)$$

(10)
where $X$ has normal distribution with mean equal to 0 and variance equal to $v_1^2 + v_2^2$.

3. Case study and analysis
Mathematics model for predicting the probability of failure of a component due to fatigue loading has been developed. We apply this model to calculate the probability of failure from 6061-T6 Aluminum and make some variation about its critical damage, standard deviation of ultimate strength, and standard deviation of fatigue limit to ensure the accuracy of the model. Table 1 summarize the data which is used for the calculation.

Table 1. Data for calculating the probability of failure 6061-T6 Aluminum.

| Random Variable | Constant Variable |
|-----------------|-------------------|
| Variable        | Distribution      | Mean     | Standard Deviation       | Variable   | Value |
| Ultimate Strength $S_u$ | Normal         | 310 Mpa  | 15 Mpa                  | Critical damage $D_{cr}$ | 1       |
| Fatigue limit $S_e$ | Normal          | 95 MPa   | 5 Mpa                   | Alternating stress $S_a$ | 220 MPa  |
|                  |                  |          |                         | Mean stress $S_m$     | 0       |
|                  |                  |          |                         | Fatigue strength fraction $f$ | 0.9     |

The mean value of ultimate strength $S_u$, and fatigue limit $S_e$ are obtained from ASM Metals Handbook [14] and the standard deviation value is chosen 5% from its mean. Alternating stress $S_a$ and mean stress $S_m$ value depend on researcher, while the value of fatigue strength fraction $f$ is found to be 0.9 for material with ultimate strength less than 482 MPa.

Figure 4 shows the relationship between the probability of failure and the number of usage cycles of the component. The probability of failure remains lower (almost constant) for early period and it starts increasing as usage cycles increase. The earlier part of the graph that shows almost constant probability of failure can be represented as the crack initiation period while increasing probability of failure phase is an indicative of crack propagation period. From the figure above, the mathematics model that has been developed can be concluded valid, because the probability of failure increase with increase in usage cycles.

Figure 4. Probability of failure plot for alternating stress level 220 MPa.
Another validation is needed in order to ensure that the mathematics model has given accurate result.

Equation 4 stated that failure condition will happen when the damage due to usage cycles reaches the critical damage. In most cases, the value of critical damage is assumed to be 1, but it has been observed that the values of critical damage can range from 0.5 to 2.2 [15]. In order to validate the mathematics model, three different value of critical damage is tested.

Figure 5 shows probability of failure for three different values of $D_{cr}$. These probabilities of failure plot reveal the trend of increasing probability with decrease in the value of $D_{cr}$. The critical damage $D_{cr}$ is the value of threshold damage before fatigue failure happens, when the value of $D_{cr}$ is decreased, then the probability of fatigue failure will increase. From Figure 5, it can be concluded that by variating the value of $D_{cr}$ the mathematics model gives accurate result.

Ultimate strength $S_u$ and fatigue limit $S_e$ are random variables with normal distribution. In the two previous calculation, the standard deviation of $S_u$ is 15 MPa and the standard deviation of $S_e$ is 5 MPa. Different value of standard deviation will have an effect in the probability of failure. Standard deviation will affect the shape of normal distribution and the bigger standard deviation means that the data more spread out from the mean.

We know from Equation 4 that the probability of failure $Z(n)$ is a function of $S_u$ and $S_e$, because of that by variating the standard deviation of $S_u$ and $S_e$ it will affect the value of $Z(n)$. Figure 6a and 6b show the bigger standard deviation implies the higher probability of failure. This is because increasing the standard deviation will make the failure area also increase.
4. Conclusion
A probability approach based on cumulative damage to predict the probability of failure of a component subjected to failure loading has been developed. The methodology uses a load-resistance interference model and assumes ultimate strength and fatigue limit as random variables with normal distribution. Critical damage, standard deviation of ultimate strength and standard deviation of fatigue limit are three factors that affect the probability of failure of a component. Decreasing the value of critical damage will increase the probability of failure, while decreasing the standard deviation of ultimate strength and fatigue limit will decrease it.

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