Reflexivity of a Banach Space with a Countable Vector Space Basis

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Abstract

All most all the function spaces over real or complex domains and spaces of sequences, that arise in practice as examples of normed complete linear spaces (Banach spaces), are reflexive. These Banach spaces are dual to their respective spaces of continuous linear functionals over the corresponding Banach spaces. For each of these Banach spaces, a countable vector space basis exists, which is responsible for their reflexivity. In this paper, a specific criterion for reflexivity of a Banach space with a countable vector space basis is presented.

Keywords: Normed Linear Spaces; Banach Spaces; Vector Space Basis; Weak Topology.

1 Introduction

All most all the function spaces over real or complex domains and spaces of sequences, that arise in practice as examples of normed complete linear spaces (Banach spaces), are reflexive. The topological property of being reflexive is that these Banach spaces are dual to their respective spaces of continuous linear functionals over the corresponding Banach spaces. For a reflexive Banach space, the double dual space is isometrically isomorphic to the Banach space, and they are essentially the same spaces. For each of these Banach spaces, a countable vector space basis exists, which is responsible for their reflexivity. A vector space basis for a Banach space can be easily formulated, taking the analogy from a Riesz basis for a Hilbert space. However, these Banach spaces are not plainly \( L^p \)-function or \( \ell^p \)-sequence spaces. For a Banach space with a countable vector space basis, the projection maps onto finite dimensional component subspaces are continuous and surjective, and hence open maps. The coefficient sequence of any vector in the Banach space, with respect to a basis, for which the dual basis functionals are normalized, can be shown to be bounded with respect to sup-norm, and the linear transformation mapping a vector to its coefficient sequence becomes continuous. The dual basis linear functionals become continuous and form a basis for the dual space. A specific criterion for reflexivity of a Banach space with a countable vector space basis is presented.
2 Main Results

Let \( \mathbb{N} \) be the set of positive integers, and let \( \mathbb{R} \) and \( \mathbb{C} \) be the fields of real and complex numbers, respectively, equipped with absolute value norm, denoted by \( | \cdot | \). Let \( \mathbb{F} \) be \( \mathbb{R} \) or \( \mathbb{C} \).

A a topologically complete normed linear (vector) space over \( \mathbb{F} \) is called a Banach space. Let \( \ell^p (\mathbb{N}, \mathbb{F}) \) be the Banach space of \( \mathbb{F} \)-valued sequences, equipped with the \( p \)-norm, for \( 1 \leq p \leq \infty \). Let \( e_i \) be the standard Euclidean vector, with \( j \)-th component of \( e_i \) equal to the Kronecker \( \delta_{i,j} \), for \( i, j \in \mathbb{N} \). The set \( \{ e_i : i \in \mathbb{N} \} \) forms a basis for \( \ell^p (\mathbb{N}, \mathbb{F}) \), for \( 1 \leq p \leq \infty \), and the basis is countable.

2.1 Countable Basis Banach Spaces

Let \( \mathcal{B} \) be a Banach space, with norm \( \| \cdot \|_B \), and let \( \{ \xi_i : i \in \mathbb{N} \} \subset \mathcal{B} \). If every element \( \mathbf{x} \in \mathcal{B} \) can be expressed as a linear combination \( \mathbf{x} = \sum_{i \in \mathbb{N}} c_i \xi_i \), for some scalars \( c_i \in \mathbb{F} \), where \( i \in \mathbb{N} \), then the Banach space is said to admit a countable basis, which is \( \{ \xi_i : i \in \mathbb{N} \} \). By the uniqueness of the linear combination, it is assumed that if \( \sum_{i \in \mathbb{N}} d_i \xi_i = 0 \), for some scalars \( d_i \in \mathbb{F} \), then \( d_i = 0 \), for every \( i \in \mathbb{N} \). Componentwise addition and subtraction is required to hold. The space of continuous linear functionals, denoted by \( \mathcal{B}^* \), is called the dual of \( \mathcal{B} \), with its dual norm \( \| \cdot \|_{B^*} \). The double dual, or bidual, is denoted by \( \hat{\mathcal{B}} \), with its bidual norm \( \| \cdot \|_{\hat{B}^*} \).

Proposition 2.1. Let \( \mathcal{B} \) be a Banach space, with norm \( \| \cdot \|_B \), and let \( \{ \xi_i : i \in \mathbb{N} \} \subset \mathcal{B} \) be a basis for \( \mathcal{B} \). Let \( \mathbb{I} \subset \mathbb{N} \) and \( \mathbb{J} = \mathbb{N} \setminus \mathbb{I} \), and let \( \mathcal{L}(\mathbb{I}) \) and \( \mathcal{L}(\mathbb{J}) \) be the subspaces spanned by \( \{ \xi_i : i \in \mathbb{I} \} \) and \( \{ \xi_j : j \in \mathbb{J} \} \), respectively. The following statements hold:

1. Both \( \mathcal{L}(\mathbb{I}) \) and \( \mathcal{L}(\mathbb{J}) \) are closed linear subspaces of \( \mathcal{B} \), with respect to \( \| \cdot \|_B \).
2. If at least one of the index sets \( \mathbb{I} \) and \( \mathbb{J} \) is finite, then \( \mathcal{B} = \mathcal{L}(\mathbb{I}) \oplus \mathcal{L}(\mathbb{J}) \), i.e., every vector \( \mathbf{x} \in \mathcal{B} \) can be expressed as the sum of vectors \( \mathbf{x} = \mathbf{y} + \mathbf{z} \), for some \( \mathbf{y} \in \mathcal{L}(\mathbb{I}) \) and \( \mathbf{z} \in \mathcal{L}(\mathbb{J}) \), uniquely.
3. If at least one of the index sets \( \mathbb{I} \) and \( \mathbb{J} \) is finite, the projection operators \( P_I \) and \( P_J \), defined by \( P_I(\mathbf{x}) = \mathbf{y} \) and \( P_J(\mathbf{x}) = \mathbf{z} \), for \( \mathbf{x} \in \mathcal{B} \), with \( \mathbf{y} \in \mathcal{L}(\mathbb{I}) \) and \( \mathbf{z} \in \mathcal{L}(\mathbb{J}) \), such that \( \mathbf{x} = \mathbf{y} + \mathbf{z} \), are both well defined and continuous.

Proof. If \( \mathbb{I} = \emptyset \) or \( \mathbb{J} = \emptyset \), the statements vacuously hold.

For a singleton set \( \{ i \} \), for some \( i \in \mathbb{N} \), \( \mathcal{L}(\{ i \}) = \{ c \xi_i : c \in \mathbb{F} \} \), and \( \| c \xi_i \|_B = |c| \cdot \| \xi_i \|_B \), whence \( \mathcal{L}(\{ i \}) \) is topologically closed in \( \mathcal{B} \) with respect to \( \| \cdot \|_B \). If \( \mathbf{x} = \sum_{j \in \mathbb{N}} c_j \xi_j \in \mathcal{B} \), for some scalars \( c_j \in \mathbb{F} \), then \( \mathbf{y} = c_i \xi_i \in \mathcal{L}(\{ i \}) \), \( \mathbf{z} = \sum_{j \in \mathbb{N} \setminus \{ i \}} c_j \xi_j \in \mathcal{L}(\mathbb{N} \setminus \{ i \}) \), and the expression \( \mathbf{x} = \mathbf{y} + \mathbf{z} \) is uniquely determined. Therefore, \( \mathcal{L}(\mathbb{N} \setminus \{ i \}) \) is a closed linear subspace of \( \mathcal{B} \), with respect to \( \| \cdot \|_B \), as well.

Now \( \mathcal{L}(\mathbb{I}) = \cap_{j \in \mathbb{J}} \mathcal{L}(\mathbb{N} \setminus \{ j \}) \) and \( \mathcal{L}(\mathbb{J}) = \cap_{i \in \mathbb{I}} \mathcal{L}(\mathbb{N} \setminus \{ i \}) \), hence both are closed. The remaining part is obvious.

For a Banach space \( \mathcal{B} \), with a countable basis \( \{ \xi_i : i \in \mathbb{N} \} \subset \mathcal{B} \), the algebraic dual basis is the set of linear functionals \( \{ \hat{\xi}_i : i \in \mathbb{N} \} \), defined by their action on the basis vectors by the condition that \( \hat{\xi}_i(\xi_j) = \delta_{i,j} \), for \( i, j \in \mathbb{N} \). Proposition 1 (item 3) shows that \( \hat{\xi}_i \) is continuous, hence bounded, i.e., \( 1 \leq \| \hat{\xi}_i \|_{B^*} < \infty \), for every \( i \in \mathbb{N} \). Let \( \eta_i = \| \hat{\xi}_i \|_{B^*} \xi_i \) and \( \hat{\eta}_i = \| \xi_i \|_B \hat{\xi}_i \), for \( i \in \mathbb{N} \). Then, \( \| \hat{\eta}_i \|_{B^*} = 1 \) and \( \hat{\eta}_i(\eta_j) = \delta_{i,j} \), for \( i, j \in \mathbb{N} \). Moreover, \( \{ \hat{\eta}_i : i \in \mathbb{N} \} \)
forms a basis for $\hat{B}$: if $\hat{f} \in \hat{B}$, then, by the algebraic action of $\hat{f}$ on the vector space basis $\{\eta_i : i \in \mathbb{N}\}$ for $B$, it holds that $\hat{f} = \sum_{i \in \mathbb{N}} \hat{f}(\eta_i) \hat{\eta}_i$, and the uniqueness of the algebraic expression is obvious.

**Proposition 2.2.** Let $B$ be a Banach space, with a countable basis, $\{\eta_i : i \in \mathbb{N}\}$, such that $\{\hat{\eta}_i : i \in \mathbb{N}\}$ is a normalized basis for $\hat{B}$, i.e., $\|\hat{\eta}_i\|_B = 1$ and $\{\hat{\eta}_i : i \in \mathbb{N}\}$ forms a basis for $\hat{B}$, and let $x = \sum_{i \in \mathbb{N}} c_i \eta_i$, for some $c_i \in \mathbb{F}$, where $i \in \mathbb{N}$. Then, $\sup_{i \in \mathbb{N}} |c_i| \leq \|x\|_B$.

**Proof.** Let $\hat{T}$ be the formal linear mapping defined by $\hat{T}(d) = \sum_{i \in \mathbb{N}} d_i \hat{\xi}_i$, for $d = (d_1, d_2, d_3, \ldots) \in \ell^1(\mathbb{N}, \mathbb{F})$. For $m, n \in \mathbb{N}$, and $g_n = \sum_{i=1}^n d_i \hat{\xi}_i$, the estimate $\|g_{m+n} - g_n\|_B = \left\| \sum_{i=n+1}^{m+n} d_i \hat{\eta}_i \right\|_B \leq \sup_{n+1 \leq i \leq m+n} \sum_{i=n+1}^{m+n} |d_i|$, shows that $\{g_i : i \in \mathbb{N}\}$ is a Cauchy sequence in $\hat{B}$, and $\hat{T}(d) \in \hat{B}$, with the norm of $\hat{T}$, as a linear transformation from $\ell^1(\mathbb{N}, \mathbb{F})$ into $\hat{B}$, being at most 1. Let $\hat{T} : \hat{B} \rightarrow \ell^\infty(\mathbb{N}, \mathbb{F})$ be the adjoint linear transformation of $\hat{T}$. The linear transformation norm of $\hat{T}$ coincides with that of $\hat{T}$, and $\hat{T}$ becomes continuous. Let $\hat{\eta}_i$ be the natural embedding of $\eta_i$ in $\hat{B}$. For $x = \sum_{i \in \mathbb{N}} c_i \eta_i$, for some scalars $c_i \in \mathbb{F}$, $i \in \mathbb{N}$, the natural embedding of $x$ in $\hat{B}$ is $\hat{x} = \sum_{i \in \mathbb{N}} c_i \hat{\eta}_i$, which is an isometric isomorphism, by Hahn-Banach theorem, i.e., $\|x\|_B = \|\hat{x}\|_\hat{B}$. If $\|\hat{x}\|_\hat{B} \leq 1$, then $|\hat{x}(\hat{f}(e_i))| = |\hat{x}(\hat{\eta}_i)| = |c_i| \leq 1$, for every $i \in \mathbb{N}$. By the linearity of $\hat{T}$, the embedding, the content follows.

### 2.2 Reflexivity of a Countable Basis Banach Space

Let $B$ be a Banach space with a countable basis $\{\xi_i : i \in \mathbb{N}\}$, and $\hat{B}$ be the dual space of with $B$ generated by the dual basis $\{\hat{\xi}_i : i \in \mathbb{N}\}$. In this subsection, three specific assumptions regarding structure of $B$ and $\hat{B}$ are assumed to hold good, for the theorem of this section.

**Assumption 1.** For every $x \in B$, with $x = \sum_{i \in \mathbb{N}} c_i \xi_i$, for some scalars $c_i \in \mathbb{F}$, if $y_n = \sum_{i=1}^n c_i \xi_i$, then $\|y_n\|_B \leq \|x\|_B$, for every $n \in \mathbb{N}$.

The assumption just stated can hold, for a very large collection of Banach spaces. It is particularly a nondecreasing norm, with addition of more terms: $\|y_n\|_B \leq \|y_{n+1}\|_B$, for each $n \in \mathbb{N}$. In general, for arbitrary scalars $c_i \in \mathbb{F}$, with $|c_i|$ bounded and $y_n = \sum_{j=1}^n c_j \xi_j$, where $i, n \in \mathbb{N}$, it may not be true that $\{\|y_n\|_B : n \in \mathbb{N}\}$ is a bounded set of nonnegative real numbers, even though $\|y_n\|_B$ is nondecreasing, for $n \in \mathbb{N}$.

**Assumption 2.** For any scalars $c_i \in \mathbb{F}$, where $i \in \mathbb{N}$, with $y_n = \sum_{i=1}^n c_i \xi_i$, if $\|y_n\|_B \leq b$, for some fixed $b > 0$ and every $n \in \mathbb{N}$, then $\sum_{i \in \mathbb{N}} c_i \xi_i \in B$, and if $x = \sum_{i \in \mathbb{N}} c_i \xi_i \in B$, then $\|x\|_B \leq b$.

The assumption just stated can hold, for a very large collection of Banach spaces. By the embedding of $y_n = \sum_{i=1}^n c_i \xi_i \in B$ into the double dual $\hat{y}_n = \sum_{i=1}^n c_i \hat{\xi}_i \in \hat{B}$, followed by an appeal to the compactness of the closed convex unit sphere of $\hat{B}$, centered at the origin, with respect to weak* topology of $\hat{B}$, if $\|y_n\|_B \leq b$, then the sequence $\hat{y}_n = \sum_{i=1}^n c_i \hat{\xi}_i$ converges to $\hat{z} = \sum_{i \in \mathbb{N}} c_i \hat{\xi}_i$ with $\|\hat{z}\|_\hat{B} \leq b$. The assumption implies that there exists
$x \in \mathcal{B}$, such that the natural embedding of $x$ into $\hat{\mathcal{B}}$ is $\hat{z}$.

**Assumption 3.** For every $\hat{f} \in \hat{\mathcal{B}}$, with $\hat{f} = \sum_{i \in \mathbb{N}} d_i \hat{\xi}_i$, for some scalars $d_i \in \mathbb{F}$, it holds that

$$
\lim_{n \to \infty} \left\| \sum_{i = n+1}^{\infty} d_i \hat{\xi}_i \right\|_\mathcal{B} = 0
$$

Let $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \hat{\mathcal{L}}((1, \ldots, n))$, where $\hat{\mathcal{L}}((1, \ldots, n))$ is the closed linear subspace generated by $\{\hat{\xi}_i : 1 \leq i \leq n\}$, for $n \in \mathbb{N}$. If the assumption just stated holds, then $\mathcal{B}$ is the topological closure of $\mathcal{M}$, with respect to $\| \cdot \|_\mathcal{B}$. The assumption is satisfied, if the coefficient sequence is in $\ell^p(\mathbb{N}, \mathbb{F})$, for some finite $p \geq 1$, possibly depending on $\hat{f}$, i.e., $1 \leq p < \infty$. The following is the main result.

**Theorem 2.3. (Rabin)** Let $\mathcal{B}$ be a Banach space with a countable basis $\{\xi_i : i \in \mathbb{N}\}$. If the three assumptions stated above hold, then $\mathcal{B}$ is reflexive, i.e., the double dual $\hat{\hat{\mathcal{B}}}$ is isometrically isomorphic to $\mathcal{B}$.

**Proof.** Let $S(r) = \{x \in \mathcal{B} : \|x\|_\mathcal{B} < r\}$, for $r > 0$, be the open subset of $\mathcal{B}$, of radius $r$, centered at the origin, and $\overline{S(r)} = \{x \in \mathcal{B} : \|x\|_\mathcal{B} \leq r\}$, for $r > 0$, be the closed subset of $\mathcal{B}$, of radius $r$, centered at the origin.

Let $V_i = \{d_i \in \mathcal{F} : |d_i| < \|\hat{\xi}_i\|_\mathcal{B}\}$ be the open set of $\mathcal{F}$, of radius $\|\hat{\xi}_i\|_\mathcal{B}$, centered at the origin, and $\overline{V_i} = \{d_i \in \mathcal{F} : |d_i| \leq \|\hat{\xi}_i\|_\mathcal{B}\}$ be the closed set of $\mathcal{F}$, of radius $\|\hat{\xi}_i\|_\mathcal{B}$, centered at the origin.

The product topology on $\overline{S(1)}$ is generated by the basic open sets

$$\left\{ \sum_{i \in \mathbb{N}} c_i \xi_i \in \overline{S(1)} : c_i \in U_i, \ i \in \mathbb{N} \right\}
$$

where $U_i$ is an open subset of $\overline{V_i}$, such that $U_i = \overline{V_i}$, for all but possibly finitely many indexes $i \in \mathbb{N}$.

Let $T : \overline{S(1)} \to \prod_{i \in \mathbb{N}} V_i$ be the on-to-one continuous function mapping $x = \sum_{i \in \mathbb{N}} c_i \xi_i$ to the coefficient sequence $(c_1, c_2, \ldots) \in \prod_{i \in \mathbb{N}} V_i$. Then, the basic open set of $\overline{S(1)}$ as just defined is the set $T^{-1}(\prod_{i \in \mathbb{N}} U_i)$, where $\prod_{i \in \mathbb{N}} U_i$ is an open subset of the countable infinite product $\prod_{i \in \mathbb{N}} V_i$.

There are two parts in the proof: in the first part, $T(\overline{S(1)})$ is shown to be closed in $\prod_{i \in \mathbb{N}} V_i$ (hence $\overline{S(1)}$ becomes compact with respect to the product topology), and in the second part, the weak topology of $\overline{S(1)}$ is shown to coincide with the product topology of $\overline{S(1)}$.

**Part 1.** Since the projections onto finite dimensional subspaces are continuous and also open maps, the set

$$\Gamma_n = \left\{ (c_1, \ldots, c_n) \in \mathbb{F}^n : \left\| \sum_{i \in \mathbb{N}} c_i \xi_i \right\|_\mathcal{B} \leq 1, \text{ for some scalars } c_i \in \mathcal{F} \text{ and } i \geq n+1 \right\}
$$

for $n \in \mathbb{N}$, is a closed subset of $\prod_{i = 1}^{n} V_i$. Now, the set $A_n = \Gamma_n \times \prod_{i = n+1}^{\infty} V_i$ is a closed subset of $\prod_{i \in \mathbb{N}} V_i$, with respect to the product topology, for every $n \in \mathbb{N}$. If $\left\| \sum_{i \in \mathbb{N}} c_i \xi_i \right\|_\mathcal{B} \leq 1$, then $\left\| \sum_{i = 1}^{n} c_i \xi_i \right\|_\mathcal{B} \leq 1$, by Assumption 1, and therefore, $T(\overline{S(1)}) \subseteq A_n$, for every $n \in \mathbb{N}$. Conversely, if $(c_1, \ldots, c_n, c_{n+1}, \ldots) \in A_n$, for every $n \in \mathbb{N}$, then $\left\| \sum_{i = 1}^{n} c_i \xi_i \right\|_\mathcal{B} \leq 1$, for every $n \in \mathbb{N}$, by Assumption 1, and $\sum_{i \in \mathbb{N}} c_i \xi_i \in \mathcal{B}$, with $\left\| \sum_{i \in \mathbb{N}} c_i \xi_i \right\|_\mathcal{B} \leq 1$, by Assumption 2. Thus,

$$\bigcap_{n \in \mathbb{N}} A_n = T(\overline{S(1)})
$$

and $T(\overline{S(1)})$ is a closed subset of $\prod_{i \in \mathbb{N}} V_i$, hence compact with respect to the product topology.

**Part 2.** The weak topology on $\mathcal{B}$ is generated by subbasic open sets $\mathcal{V}(\hat{f}, z, \rho)$, where $\hat{f} \in \hat{\mathcal{B}}$, $z \in \mathcal{B}$ and $\rho > 0$.
some appropriately chosen basis. Let $g = \sum_{i=1}^n d_i \xi_i$, for some fixed $n \geq N(\hat{f}, \delta)$. If $|\hat{g}(y) - \hat{g}(x)| < \frac{\delta}{4}$, then $|\hat{f}(y) - \hat{f}(x)| \\
\leq |\hat{f}(y) - \hat{g}(y)| + |\hat{g}(y) - \hat{g}(x)| + |\hat{g}(x) - \hat{f}(x)| < \frac{\delta}{4}(\|y\|_\mathcal{N} + \|y\|_\mathcal{M}) + \frac{\delta}{4} < \delta$, whenever $x, y \in S(1)$. Thus, $V(g, x, \frac{\delta}{4}) \cap \overline{S(1)} \subseteq V(\hat{f}, z, \rho) \cap \overline{S(1)}$, for $x \in V(\hat{f}, z, \rho) \cap \overline{S(1)}$, with $\delta = \delta(\hat{f}, z, \rho) = \rho - |\hat{f}(z) - \hat{f}(x)|$, and some appropriately chosen $g \in \mathcal{M}$.

In the product topology, all the functionals in $\mathcal{M}$ remain continuous, and the product topology on $\overline{S(1)}$ coincides with the weak topology on $\overline{S(1)}$. Thus, $\overline{S(1)}$ is compact in the weak topology. □

**Observation** The compactness of the set $\prod_{i \in \mathbb{N}} V_i$ can be proved without using the axiom of choice. It is also possible to take $B$ to be the dual of another Banach space $\mathcal{N}$, and apply the criterion to infer the reflexivity of $\mathcal{N}$.

### 3 Miscellany and Applications

A vector space basis is very important for a vector space. For an infinite dimensional topological vector space, a precise formulation of a vector space basis can be expressed, extending the definition of linear independence for linear combinations of finitely many vectors to series. For a Banach space with a countable vector space basis, some more interesting and useful facts can be proved.

**Theorem 3.1. (Weak Denseness of Finite Linear Combinations)** Let $B$ be a Banach space with a countable basis $\{\xi_i : i \in \mathbb{N}\}$, and $\mathcal{M} = \bigcup_{n \in \mathbb{N}} \mathcal{L}(\{1, \ldots, n\})$, where $\mathcal{L}(\{1, \ldots, n\})$ is the closed linear subspace generated by $\{\xi_i : 1 \leq i \leq n\}$, for $n \in \mathbb{N}$. Then, $\mathcal{M}$ is weakly dense in $B$.

**Proof.** The weak topology on $B$ is generated by subbasic open sets $V(\hat{f}, x, \epsilon)$, where $\hat{f} \in \mathcal{B}$, $x \in B$ and $\epsilon > 0$, where $V(\hat{f}, x, \epsilon) = \{y \in B : |\hat{f}(y) - \hat{f}(x)| < \epsilon\}$.

Let $x = \sum_{i \in \mathbb{N}} c_i \xi_i \in B$, for some scalars $c_i \in \mathbb{F}$, for $i \in \mathbb{N}$. Let $y_n = \sum_{i=1}^n c_i \xi_i$, so that $y_n \in \mathcal{L}(\{1, \ldots, n\})$, for $n \in \mathbb{N}$. For every $f = \sum_{i \in \mathbb{N}} d_i \hat{\xi}_i$, for any scalars $d_i \in \mathbb{F}$, for $i \in \mathbb{N}$, and $\epsilon > 0$, there exists $N(f, x, \epsilon) \in \mathbb{N}$, such that $|f(y_n) - f(x)| < \epsilon$, for every $n \geq N(f, x, \epsilon)$. Now, for a basic open subset $V(\hat{f}, x, \epsilon) \cap \ldots \cap V(\hat{f}_m, x, \epsilon_m)$, for some $m \in \mathbb{N}$, let $\delta = \min\{\epsilon_i : 1 \leq i \leq m\}$ and $\nu = \max\{N(\hat{f}_i, x, \delta) : 1 \leq i \leq m\}$. Then, $|\hat{f}_i(y_n) - \hat{f}_i(x)| < \delta$, for $1 \leq i \leq m$, for every $n \geq \nu$. Thus, every weak basic open set centered at $x$ intersects $\mathcal{M}$, and $x$ is a limit point of $\mathcal{M}$ with respect to the weak topology. □
Theorem 3.2. (Weak Denseness of a Countable Set) Let $\mathcal{B}$ be a Banach space with a countable basis $\{\xi_i : i \in \mathbb{N}\}$, and $\mathcal{M}_Q = \bigcup_{n \in \mathbb{N}} \mathcal{L}_Q(\{1, ..., n\})$, where $\mathcal{L}_Q(\{1, ..., n\})$ is the countable subset obtained by collecting linear combinations of vectors in $\{\xi_i : 1 \leq i \leq n\}$, with rational number coordinates, for $n \in \mathbb{N}$. Then, $\mathcal{M}_Q$ is weakly dense in $\mathcal{B}$.

Proof. $\mathcal{M}_Q$ is strongly dense in $\mathcal{M}$, where $\mathcal{M}$ is as defined in Theorem 3.1 and weakly dense in $\mathcal{B}$. \qed

Corollary 3.2.1. Let $\mathcal{B}$, $\mathcal{M}$ and $\mathcal{M}_Q$ be as in Theorems 3.1 and 3.2. For any continuous linear functional $\hat{g}$ defined on $\mathcal{M}_Q$, such that $|\hat{g}(x)| \leq \|x\|_B$, for $x \in \mathcal{M}_Q$, there is a unique continuous extension $\hat{h}$ of $\hat{g}$ to $\mathcal{M}$, and $\hat{h}$ satisfies $|\hat{h}(x)| \leq \|x\|_B$, for every $x \in \mathcal{M}$.

Proof. $\mathcal{M}_Q$ is strongly dense in $\mathcal{M}$. \qed

For a property that depends continuously with respect to the weak topology, in order to show that the property holds for all of $\mathcal{B}$, it suffices to show that the same holds for $\mathcal{M}$.

4 Conclusions

The projection maps and dual basis linear functionals a Banach space with a countable vector space basis are shown to be continuous open maps. The dual basis linear functionals form a basis for the dual space, and the double dual basis linear functionals form a basis for the double dual space. A specific criterion for reflexivity is presented.

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