ELLIPTIC K3 SURFACES WITH $p^n$-TORSION SECTIONS

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ABSTRACT. We classify elliptic K3 surfaces in characteristic $p$ with $p^n$-torsion section. For $p^n \geq 3$ we verify conjectures of Artin and Shioda, compute the heights of their formal Brauer groups, as well as Artin invariants and Mordell–Weil groups in the supersingular cases.

INTRODUCTION

The geometry and arithmetic of K3 surfaces is a fascinating subject of algebraic geometry. Moreover, this class of surfaces provides a rich source of conjectures that are difficult to come by.

In this paper, we consider K3 surfaces in positive characteristic $p$ that are elliptically fibered. Moreover, we assume that the fibration possesses a torsion section of order $p^n$. Such surfaces have already been studied by Schweizer [Schw05]. Recall, e.g., from [K-M85, Chapter 12], that the Igusa moduli functor, which classifies ordinary elliptic curves with $p^n$-torsion sections, is representable by a smooth affine curve, the so-called Igusa curve $\text{Ig}(p^n)_{\text{ord}}$ if $p^n \geq 3$. Using Igusa’s results [Ig68], we first strengthen results of [Schw05] and [D-K09]:

**Theorem.** Elliptic K3 surfaces with $p^n$-torsion section in characteristic $p$ exist for $p^n \leq 8$ only. If the fibration has constant $j$-invariant then $p^n = 2$.

Using the universal elliptic curves over the Igusa curves and the results [L-S08] on their Néron models over their cusps and the supersingular locus, we explicitly classify elliptic K3 surfaces with $p^n$-torsion sections for $p^n \geq 3$.

Next, translation by a $p$-torsion section of an elliptic fibration induces a $\mathbb{Z}/p\mathbb{Z}$-action, i.e., a wild $p$-cyclic automorphism. Such wild automorphisms on K3 surfaces have been studied in general by Dolgachev and Keum [D-K01]. Using their results, we illustrate and strengthen these results in case the wild automorphism arises from translation by a $p$-torsion section. For example, we determine the fixed point set of translation by a $p$-torsion section in bad fibers of the elliptic fibration, which extends work of Miranda and Persson [M-P89] from the prime-to-$p$ case.

Before stating one of our main results, let us state a couple of conjectures on the arithmetic of elliptic K3 surfaces. First, let us recall that a surface is called Shioda-supersingular if the rank of its Néron–Severi group is equal to its second

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Betti number. In [Sh74b], Shioda has shown that unirational surfaces are Shioda-supersingular and conjectured the converse in [Sh77a]. On the other hand, a surface is called Artin-supersingular if its formal Brauer group has infinite height. Artin [Ar74] has shown that unirational K3 surfaces are Artin-supersingular and conjectured the converse. Moreover, he proved in loc. cit. that Shioda-supersingular surfaces are Artin-supersingular and conjectured the converse. Thus

**Conjecture.** For K3 surfaces,

1. (Shioda) Shioda-supersingularity implies unirationality,
2. (Artin) Artin-supersingularity implies unirationality,
3. (Artin) Artin-supersingularity implies Shioda-supersingularity.

For elliptic K3 surfaces these two notions of supersingularity coincide [Ar74].

In characteristic 2, there is another conjecture by Artin [Ar74], which does not only imply the above conjectures but also gives a geometric explanation of the above conjectures:

**Conjecture** (Artin). In characteristic 2, an elliptic fibration on a supersingular K3 surface arises via Frobenius pullback from a rational elliptic surface.

Unfortunately, such a conjecture cannot be true in general in characteristic $p \geq 3$, see Section 3 for discussion. However, for elliptic K3 surfaces with $p^n$-torsion sections a beautiful picture emerges:

**Theorem.** Let $X \to \mathbb{P}^1$ be an elliptic K3 surface in positive characteristic $p$ with $p^n$-torsion sections and $p^n \geq 3$. Then the following are equivalent:

1. The elliptic fibration arises as Frobenius pullback from a rational elliptic fibration.
2. $X$ is unirational.
3. $X$ is supersingular.
4. The fibration has precisely one additive fiber.

**Corollary.** The conjectures of Artin and Shioda hold for elliptic K3 surfaces with $p^n$-torsion sections if $p^n \geq 3$.

Let us recall that the moduli space of K3 surfaces is stratified by the height $h$ of the formal Brauer group, which takes every value $1 \leq h \leq 10$ or $h = \infty$. Furthermore, the moduli space of surfaces with $h = \infty$, i.e., the Artin-supersingular surfaces, is stratified by the Artin invariant $\sigma_0$, which takes every value $1 \leq \sigma_0 \leq 10$. For our surfaces we prove the following alternative

**Proposition.** For an elliptic K3 surface with $p^n$-torsion section in characteristic $p$ and $p^n \geq 3$ there are two possibilities:

1. either the elliptic fibration has precisely one additive fiber and the surface is supersingular ($h = \infty$),
2. or the elliptic fibration has precisely two additive fibers and the surface is ordinary ($h = 1$)
In characteristic 2, a connection between the height of the formal Brauer group and the singular fibers of an elliptic fibration has already been observed by Artin [Ar74]. For \( p^n \geq 7 \) there is only one elliptic K3 surface with \( p^n \)-torsion section and it is supersingular. On the other hand, the generic elliptic K3 surface with \( p^n \)-torsion section with \( p^n \leq 5 \) is ordinary.

Concerning the Artin invariants of the supersingular surfaces we obtain the following characterization:

**Theorem.** The Artin invariant \( \sigma_0 \) of a supersingular and elliptic K3 surface with \( p^n \)-torsion in characteristic \( p \) satisfies \( \sigma_0 \leq \sigma_0(p^n) \) where

\[
\begin{array}{c|cccc}
  p^n & 8 & 7 & 5 & 4 & 3 \\
  \sigma_0(p^n) & 1 & 1 & 2 & 3 & 6
\end{array}
\]

Conversely, a supersingular K3 surface in characteristic \( p \) with \( \sigma_0 \leq \sigma_0(p^n) \) possesses an elliptic fibration with \( p^n \)-torsion section.

We also determine the Mordell–Weil groups and find explicit Weierstraß equations of these fibrations. In particular, we obtain explicit and complete families of supersingular K3 surfaces with \( \sigma_0 \leq \sigma_0(p^n) \) in characteristic \( p \). To obtain these results in characteristic \( p \leq 3 \) we use semi-universal deformations of the \( E_8^2 \)-singularity (\( p = 3 \)) and the \( E_8^2 \)-singularity (\( p = 2 \)).

On the other hand, elliptic K3 surfaces with 2-torsion section in characteristic 2 are much harder to come by. This has to do with the fact that there is no Igusa curve to "tame" the situation. It turns out that there are extra classes. For example, fibrations with constant \( j \)-invariant have to be considered and there are classes where the formal Brauer group has height 2, i.e., the above alternative does no longer hold. We refer to Theorem 5.2 for the precise structure result.

The article is organized as follows: In Section 1 we recall a couple of general facts about the Igusa moduli problem and show that elliptic K3 surfaces with \( p^n \)-torsion sections can exist for \( p^n \leq 8 \) only. In Section 2 we analyze the fixed locus of translation by a \( p \)-torsion section in an elliptic fibration. In Section 3 we compute the height of the formal Brauer group in terms of the additive fibers of an elliptic fibration. This already yields some of our main theorems for \( p \neq 2 \). In Section 4 we give an explicit classification for \( p \neq 2 \) and compute the Artin invariants in the supersingular cases. The rest of the article takes place in characteristic 2 only: in Section 5 we prove the general structure result, and classify the new, "exotic" classes in Section 6. Finally, in Section 7 we deal with 4- and 8-torsion sections and use again the corresponding Igusa curves.

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1. Igusa Curves

In this section we first recall the Igusa moduli problem and the Igusa curves and use these results to show that elliptic K3 surfaces with $p^n$-torsion section can exist only if $p^n \leq 8$.

Let us recall, e.g. from [K-M85, Chapter 12.3], that the Igusa moduli functor $[\text{Ig}(p^n)_{\text{ord}}]$ associates to every scheme $S$ over $\mathbb{F}_p$ the set of ordinary elliptic curves $E$ over $S$ such that the $n$-fold Frobenius pullback $E^{(p^n)} = (F^n)^*(E)$ possesses a $p^n$-torsion section. If $p^n \geq 3$ then this functor is representable by a smooth and affine curve over $\mathbb{F}_p$, the Igusa curve $\text{Ig}(p^n)_{\text{ord}}$. We denote by $E \to \text{Ig}(p^n)_{\text{ord}}$ the universal family. Thus, if $X \to B$ is an elliptic fibration in characteristic $p$ with $p^n$-torsion section, and if $U \subseteq B$ denotes the open set over which the fibres are ordinary elliptic curves, then there exists a classifying morphism $\varphi : U \to \text{Ig}(p^n)_{\text{ord}}$ such that the restriction $X|_U \to U$ is isomorphic to $(F^n)^*(E) \to U$.

The geometry of the normal compactification $\text{Ig}(p^n)_{\text{ord}}$ of $\text{Ig}(p^n)_{\text{ord}}$ has been studied in [Ig68]. For example, if $n = 1$ and $p \geq 3$, which is the case that we will be needing most in the sequel, then the $j$-invariant induces a Galois morphism $\text{Ig}(p^n)_{\text{ord}} \to \mathbb{P}^1$, whose Galois group is cyclic of order $(p - 1)/2$. This morphism is totally ramified over the supersingular $j$-values and totally split over $j = \infty$, i.e., there are $(p - 1)/2$ points lying above infinity, the so-called cusps. The degenerating behavior of the universal family $E \to \text{Ig}(p^n)_{\text{ord}}$ over the supersingular points and the cusps has been determined in [L-S08].

**Theorem 1.1.** An elliptic K3 surface $X \to \mathbb{P}^1$ with $p^n$-torsion sections in positive characteristic $p$ satisfies the inequality $p^n \leq 8$. Moreover, if the fibration has constant $j$-invariant then $p = 2$ and $n = 1$.

**Proof.** We first deal with the case of constant $j$-invariant. Since the $p^n$-torsion section is different from the zero section, the generic fiber is ordinary and so the ordinary locus $U \subseteq \mathbb{P}^1$ is open and dense. Moreover, if $p^n \geq 3$ then the Igusa moduli problem is representable and constant $j$-invariant implies that the classifying morphism $\varphi : U \to \text{Ig}(p^n)_{\text{ord}}$ is constant. Thus, $X|_U \to U$ is a product family (the Igusa curve is a fine moduli space), and not birational to a K3 surface. Hence in this case we have $p^n = 2$.

We may thus assume that the fibration has non-constant $j$-invariant, and again, the ordinary locus $U \subseteq \mathbb{P}^1$ is open and dense. Also, we may assume $p^n \geq 3$, i.e., that the Igusa moduli problem is representable. Then the classifying morphism $\varphi : U \to \text{Ig}(p^n)_{\text{ord}}$ is dominant, which implies that $\text{Ig}(p^n)_{\text{ord}}$ is a rational curve. The genera of the Igusa curves have been determined in [Ig68] and a straightforward computation shows that these curves are rational if and only if $p^n \leq 11$.

Let us first exclude $p = 11$. In this case $\text{Ig}(11)_{\text{ord}}$ has 5 cusps. Hence our fibration has at least 5 fibres with potentially multiplicative reduction. By [L-S08, Theorem 4.3] we have in fact multiplicative reduction. Thus, our family has at least 5 fibres with multiplicative reduction, necessarily of type $\text{I}_n$, where 11 divides all
elliptic curve at the cusps and additive reduction at the supersingular points.

Proof. These curves are rational if and only if \[\rho(X) > 50\], i.e., \(X\) is not a K3 surface.

The remaining case \(p^n = 9\) is excluded similarly and we leave it to the reader. \(\square\)

**Remark 1.2.** Non-existence of elliptic K3 surfaces with \(p\)-torsion sections for \(p \geq 11\) has been shown in [D-K09 Theorem 2.13]. Under the assumption that the fibration does not have constant \(j\)-invariant, Theorem [1.1] has been shown in the remark after [Schw05 Theorem 2.3], using methods closely related to ours. Nevertheless, we decided to give a proof in our setup, i.e., by analyzing the classifying morphisms to the Igusa curves and their universal families.

The proof shows that Igusa curves that are rational are crucial for the description of elliptic K3 surfaces with \(p\)-torsion sections. Igusa’s results [Ig68] show that these curves are rational if and only if \(p^n \leq 11\). For our explicit classification later on, and in order to obtain equations when needed, we determine Weierstraß equations in these cases.

**Proposition 1.3.** The universal elliptic curves over \(\mathrm{Ig}(p^n)_{\text{ord}}\) for \(p^n \leq 11\) are given by the following equations over \(\mathbb{F}_p[t]\):

| \(p^n\) | \(\mathcal{E}\) \(\in\) \(\mathcal{E}^{(p)}\) \(\mathcal{E}^{(p^2)}\) \(\mathcal{E}^{(p^3)}\) \(\mathcal{E}^{(p)}\) \(\mathcal{E}^{(p^2)}\) \(\mathcal{E}^{(p^3)}\) | \(\text{singular fibres}\) |
|---|---|---|---|---|---|---|---|
| 11 | \(\mathcal{E}\) : \(y^2 = x^3 + (t - 1)^{-1}tx + 5t^{-1}(t - 1)^{11}\) | \(5 \times I_1, \ III^*, \ III^*\) |
| \(\mathcal{E}^{(p)}\) : \(y^2 = x^3 + (t - 1)^{-11}t^{11}x + 5t^{-11}(t - 1)^{11}\) | \(5 \times I_1, \ II, \ III\) |
| \(\mathcal{E}^{(p^2)}\) : \(y^2 + txy = x^3 - t^3(t^2 - 1)\) | \(3 \times I_1, \ IV_1^*\) |
| \(\mathcal{E}^{(p^3)}\) : \(y^2 + txy = x^3 + tx + (t^2 - t)\) | \(3 \times I_1, \ II\) |
| \(8\) | \(\mathcal{E}\) : \(y^2 + xy = x^3 + (t + 1)\) | \(2 \times I_1, \ III_1\) |
| \(\mathcal{E}^{(p)}\) : \(y^2 + xy = x^3 + t^2(1 + t^2)\) | \(2 \times I_2, \ I_1^*\) |
| \(\mathcal{E}^{(p^2)}\) : \(y^2 + xy = x^3 + t^4(1 + t^4)\) | \(2 \times I_4, \ I_1\) |
| \(\mathcal{E}^{(p^3)}\) : \(y^2 + xy = x^3 + t^8(1 + t^8)\) | \(2 \times I_8, \ I^*_1\) |
| \(7\) | \(\mathcal{E}\) : \(y^2 = x^3 + t^3x + 5t^6\) | \(3 \times I_1, \ III^*\) |
| \(\mathcal{E}^{(p)}\) : \(y^2 = x^3 + tx + 5t^{12}\) | \(3 \times I_7, \ III\) |
| \(5\) | \(\mathcal{E}\) : \(y^2 = x^3 + 3t^4x + t^5\) | \(2 \times I_1, \ II^*\) |
| \(\mathcal{E}^{(p)}\) : \(y^2 = x^3 + 3t^4x + t\) | \(2 \times I_5, \ II\) |
| \(4\) | \(\mathcal{E}\) : \(y^2 + xy = x^3 + t\) | \(I_1, \ II_1^*\) |
| \(\mathcal{E}^{(p)}\) : \(y^2 + xy = x^3 + t^2\) | \(I_2, \ III_1^*\) |
| \(\mathcal{E}^{(p^2)}\) : \(y^2 + xy = x^3 + t^4\) | \(I_4, \ I^*_1\) |
| \(3\) | \(\mathcal{E}\) : \(y^2 + txy = x^3 - t^3\) | \(I_1, \ II_1^*\) |
| \(\mathcal{E}^{(p)}\) : \(y^2 + txy + t^2y = x^3\) | \(I_3, \ IV_1^*\) |

All places of bad reduction are defined over \(\mathbb{F}_p\) with split multiplicative reduction at the cusps and additive reduction at the supersingular points.

**Proof.** As an example we do the case \(p = 7\) and leave the others to the reader: The elliptic curve \(\mathcal{E}\) for \(p = 7\) given in the table has Hasse invariant \([1] \in \mathbb{F}_p/\mathbb{F}_p^{\times (p-1)}\),
which implies that \( E^{(p)} \) has a \( \mathbb{F}_p[t] \)-rational \( p \)-division point. Thus, there exists a morphism \( \varphi : \text{Spec} \mathbb{F}_p[t] \to \mathbb{Ig}(p)^{\text{ord}} \) such that \( E \) is the pullback of the universal elliptic curve over \( \mathbb{Ig}(p)^{\text{ord}} \) via \( \varphi \). Since the \( j \)-invariant of \( E \) is not constant, it follows that \( \varphi \) is a finite morphism. The curve \( \mathbb{Ig}(p)^{\text{ord}} \) has \( (p - 1)/2 = 3 \) cusps over which the universal family degenerates into \( I_1 \)-fibers [L-S08, Theorem 10.3]. Since the same is true for \( E \), we get \( \deg \varphi = 1 \), i.e., \( \varphi \) is an isomorphism. \( \square \)

2. Wild \( p \)-cyclic actions

Since we are dealing with elliptic fibrations with \( p \)-torsion sections in positive characteristic \( p \), translation by such a torsion section gives rise to a wild automorphism, and we may apply the results of [D-K01]. For K3 surfaces, we will see that there are at most two additive fibers and if there are two such fibers then the elliptic fibration arises as Frobenius pullback from an elliptic K3 surface. To fix notation, let \( X \to B \) be an elliptic surface with zero section \( \sigma_0 \) and \( p \)-torsion section \( \sigma_p \).

We denote by \( G \) the cyclic group of order \( p \) generated by translations by \( \sigma_p \) and set \( Y := X/G \). Note that the elliptic fibration \( X \to B \) induces an elliptic fibration \( Y \to B \) and we get a diagram of elliptic fibrations over \( B \)

\[
Y \to X \to X/G \cong Y,
\]

where the first map is purely inseparable (relative Frobenius over \( B \)) and the second is an Artin-Schreier morphism.

We now analyze the action of \( G \) induced on the fibers. In characteristic zero and for multiplicative reduction, this has been worked out in [M-P89, Section 2]. If \( X_0 \) denotes a special fiber of the fibration we will denote by \( (\sigma_0 \cdot \sigma_p)_0 \) the intersection number of \( \sigma_0 \) and \( \sigma_p \) in the fiber \( X_0 \). Finally, we denote by \( F_0 \) the reduced fixed point scheme of the \( G \)-action on \( X_0 \), see also the discussion in [D-K01, Remark 2.7].

**Proposition 2.1.** Let \( X \to B \) be an elliptic fibration in characteristic \( p \) with \( p \)-torsion section \( \sigma_p \). Let \( X_0 \) be a special fiber and let \( F_0 \) be the reduced fixed point scheme of the \( \sigma_p \)-translation on \( X_0 \).

If \( X_0 \) has semi-stable reduction and more precisely, if the reduction is

1. good and ordinary then \( (\sigma_0 \cdot \sigma_p)_0 = 0 \) and \( F_0 = \emptyset \),
2. good and supersingular then \( (\sigma_0 \cdot \sigma_p)_0 \geq 1 \) and \( F_0 = X_0 \),
3. bad multiplicative then \( (\sigma_0 \cdot \sigma_p)_0 = 0 \) and \( F_0 = \emptyset \).

If \( X_0 \) has additive reduction and \( (\sigma_0 \cdot \sigma_p)_0 \geq 1 \) then \( F_0 = X_0 \).

If \( X_0 \) has additive reduction, \( (\sigma_0 \cdot \sigma_p)_0 = 0 \) and the reduction type is

1. II, III, IV then \( F_0 \) equals the unique point that is not smooth over the base of the fibration,
2. \( \Gamma_n^* \) \( (p \neq 2) \), \( IV^* \) \( (p \neq 3) \), \( III^* \) \( (p \neq 2) \), \( II^* \) then \( F_0 \) is a curve, equal to the union of all multiplicity \( \geq 2 \)-components of \( X_0 \).

In characteristic \( p \leq 3 \) the situation is the same if \( \sigma_p \) does not specialize into the component group of \( X_0 \). If it does and if the reduction type is
(1) IV* then $p = 3$ and $F_0$ is one point, which lies on the component of multiplicity 3.
(2) III* then $p = 2$ and $F_0$ is one point, namely the intersection of the component of multiplicity 4 and the one of multiplicity 2,
(3) I* then $p = 2$ and $F_0$ depends on the component into which $\sigma_p$ specializes:

| reduction type | specialization into $F_0$ |
|---------------|--------------------------|
| $I_0$ or $I_1^*$ | 1 point |
| $I_n^*$, $n \geq 3$, $n$ odd | necessarily $\Theta_1$ | a curve |
| $I_n^*$, $n \geq 2$, $n$ even | $\Theta_1$ | a curve |
| $\Theta_2, \Theta_3$ | 1 point, |

where the $\Theta_i$'s are those irreducible components of multiplicity 1 that do not intersect with $\sigma_0$. Furthermore, $\Theta_2$ and $\Theta_3$ pass through the same component of multiplicity 2.

**Proof.** The generic fiber of the fibration is an ordinary elliptic curve, and $\sigma_p$ generates a subgroup scheme isomorphic to $\mathbb{Z}/p\mathbb{Z}$. By [T-O70], this group scheme can either specialize to $\alpha_p$ or $\mathbb{Z}/p\mathbb{Z}$ in $X_0$. Now, if $X_0$ is good and ordinary then the $p$-torsion subgroup scheme $X_0[p]$ of $X_0$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z}) \times \mu_p$, which implies that $\sigma_p$ cannot meet $\sigma_0$ and $F_0 = \emptyset$. Similarly, if $X_0$ has multiplicative reduction then $\sigma_p$ has to specialize into the component group and again $F_0 = \emptyset$. In case of good and supersingular reduction $X_0[p]$ is infinitesimal, which implies $(\sigma_0 \cdot \sigma_p)_0 \geq 1$ and $F_0 = X_0$. In case of additive reduction and $(\sigma_0 \cdot \sigma_p)_0 \geq 1$ then $\sigma_p$ induces an $\alpha_p$-action on $X_0$, thus $F_0 = X_0$.

We may thus assume that $X_0$ has additive reduction and $(\sigma_0 \cdot \sigma_p)_0 = 0$. In particular, we obtain a non-trivial $\mathbb{Z}/p\mathbb{Z}$-action on $X_0$. Also, unless $X_0$ is of type II, this fiber is a union of $\mathbb{P}^1$'s. Moreover, $F_0$ is connected by [D-K01], and is thus one point or a connected curve. The next thing to note is that a $\mathbb{Z}/p\mathbb{Z}$-action on $\mathbb{P}^1$ in characteristic $p$ has either precisely one fixed point or the action is trivial. Also, components of $X_0$ get mapped to components and a point of $F_0$ where two components meet has to be mapped to another such point under the $\mathbb{Z}/p\mathbb{Z}$-action. From these facts one can easily work out $F_0$, which we leave to the reader. $\square$

Let us recall from [D-K01] that the fixed locus of a $\mathbb{Z}/p\mathbb{Z}$-action on a K3 surface is either a finite set of at most two points or a connected curve. Combining these results with Proposition 2.1 we obtain our first structural result:

**Theorem 2.2.** Let $X \to \mathbb{P}^1$ be an elliptic K3 surface with $p$-torsion section in positive characteristic $p$. Then the fibration has at least one and at most two fibers that are neither multiplicative nor ordinary.

Moreover, if there are two such fibers then $p \leq 5$, these fibers have additive reduction, translation by $\sigma_p$ has precisely two fixed points and the elliptic fibration arises as Frobenius pullback from an elliptic K3 surface.

**Proof.** If the fibration has neither additive nor good supersingular fibers then translation by $\sigma_p$ acts without fixed points by Proposition 2.1. By [D-K01] Theorem 2.4] this implies $p = 2$ and that $Y = X/G$ is an Enriques surface, which is absurd.
since genus one-fibrations on Enriques have multiple fibers and thus can never be elliptic, i.e., with zero section.

The fixed locus of the $\sigma_p$-translation consists either of at most two points or is a connected curve by [D-K01]. On the other hand, every fiber that is neither multiplicative nor good ordinary has a non-trivial contribution to the fixed locus by Proposition 2.1. This implies that there can be at most two fibers that are good supersingular or additive.

Moreover, if there are two such fibers then the fixed locus consists of two points. By Proposition 2.1 these two fibers have additive reduction and [D-K01, Theorem 2.4] implies $p \leq 5$ and that $Y \to \mathbb{P}^1$ is an elliptic K3 surface (it cannot be Enriques by the reasons given above).

Next, the additive fibers tend to be potentially supersingular, which is important for the computation of the formal Brauer group in Section 3. The following extends results from [L-S08].

**Proposition 2.3.** Let $X \to B$ be an elliptic fibration with $p^n$-torsion sections and $p^n \geq 3$. Then every additive fiber has potentially supersingular reduction.

**Proof.** For $p \geq 5$ this is [L-S08, Theorem 4.3] and [L-S08, Remark 4.4].

For $p^n = 3$ and $p^n = 4$ there is a universal elliptic curve over $\text{Ig}(p^n)_{\text{ord}}$, which degenerates into multiplicative fibers at places of potentially multiplicative reduction, see [L-S08, Section 12] for $\text{Ig}(3)_{\text{ord}}$ and Proposition [L-S08, Proposition 1.3] for $\text{Ig}(4)_{\text{ord}}$. Since every elliptic fibration with $p^n$-torsion section pulls back from these, we conclude that the only additive fibers can come from potentially supersingular places. □

**Proposition 2.4.** Let $X \to \mathbb{P}^1$ be an elliptic K3 surface with $p$-torsion section $\sigma_p$ in positive characteristic $p$. Then either

1. $\sigma_0 \cdot \sigma_p = 0$ and there are no fibers with good supersingular reduction, i.e., every potentially supersingular fiber has additive reduction, or
2. $\sigma_0 \cdot \sigma_p = 1$, the characteristic is $p = 2$, the fibration is semi-stable, and there is precisely one fiber with good supersingular reduction.

**Proof.** Suppose that $\sigma_0 \cdot \sigma_p \geq 1$. Then the fixed locus is a connected curve by [D-K01, Corollary 3.6] and from Proposition 2.1 and Proposition 2.2, we infer that there is only one fiber whose reduction is neither good ordinary nor multiplicative. Moreover, the intersection of $\sigma_0$ and $\sigma_p$ takes place in this fiber. In particular, there is at most one additive fiber and if there is one, then $\sigma_p$ does not specialize into the component group of that fiber. We denote by $I_{pn_v}$ with $v = 1, \ldots$ the multiplicative fibers and applying [Sh90, Theorem 8.6], we get

$$6 \leq 4 + 2(\sigma_0 \cdot \sigma_p) = \sum_v n_v \frac{k_v(p - k_v)}{p},$$

where the $k_v$’s are integers $1 \leq k_v \leq p - 1$ that encode which component of $I_{pn_v}$ is hit by $\sigma_p$. Basic calculus tells us

$$\frac{k_v(p - k_v)}{p} \leq \frac{p}{4},$$
where the inequality is strict if \( p \neq 2 \). On the other hand, we know
\[
24 = c_2(X) = \sum_v p n_v + a
\]
where \( a = 0 \) if and only if there are no additive fibers. We conclude
\[
6 \leq 4 + 2(\sigma_0 \cdot \sigma_p) \leq \sum_v n_v \frac{p}{4} \leq \frac{c_2(X)}{4} = 6,
\]
i.e., we have equality everywhere. Thus, \( \sigma_0 \cdot \sigma_p = 1 \), the characteristic equals \( p = 2 \) (else the second inequality could not be an equality) and there are no additive fibers (else the third inequality could not be an equality).

\[\square\]

**Remark 2.5.** The second alternative does exist and a complete classification is given in Proposition 5.2

### 3. Unirationality and the Formal Brauer Group

In this section we relate potentially supersingular fibres of an elliptic K3 surface with \( p \)-torsion sections to its formal Brauer group. For \( p \geq 3 \) this implies that these surfaces are either unirational or ordinary. It also implies conjectures of Artin and Artin–Shioda in this case.

We start by recalling the following fundamental result of [A-M77]: if \( X \) is a smooth surface over \( k = \overline{k} \) with smooth Picard scheme, e.g., a K3 surface, then the functor on the category of finite local \( k \)-algebras \( A \) with residue field \( k \)
\[
\widehat{Br} : A \mapsto \ker \left( H^2_{\text{ét}}(X \times A, \mathbb{G}_m) \to H^2_{\text{ét}}(X, \mathbb{G}_m) \right)
\]
is pro-represented by a smooth formal group of dimension \( h^2(X, \mathcal{O}_X) \), the formal Brauer group \( \widehat{Br}(X) \) of \( X \).

For a K3 surface, the height \( h \) of the formal Brauer group is \( \infty \) or an integer \( 1 \leq h \leq 10 \) and all values are taken [Ar74, Corollary 7.7]. Moreover, \( h \) determines the Newton polygon on second crystalline cohomology [Ill79, Section II.7.2]. In particular, the extreme cases are as follows:
- \( h = 1 \) if and only if Newton- and Hodge- polygon coincide, i.e., the K3 surface is ordinary, and
- \( h = \infty \) if and only if the Newton polygon is a straight line, i.e., the K3 surface is supersingular.

To be more precise about the notion of supersingularity, we recall

**Definition 3.1.** A K3 surface is called *supersingular in the sense of Artin* if its formal Brauer group has infinite height. A surface is called *supersingular in the sense of Shioda* if it satisfies \( \rho = b_2 \).

A K3 surface that is Shioda-supersingular is also Artin-supersingular [Ar74, Theorem 0.1]. The Artin–Mazur conjecture states that also the converse holds [Ill79, Remarque II.5.13]. Since this conjecture is known to be true for elliptic K3 surfaces [Ar74, Theorem 1.7], we do not have to distinguish between these two notions of supersingularity.
Unirational K3 surfaces are Shioda-supersingular \cite{Sh74b}, as well as Artin-supersingular \cite{Ar74}. For both notions, the converse is conjectured, see \cite[Question II]{Sh77a} and \cite{Ar74}. Thus, we summarize

**Conjecture 3.2.** For elliptic K3 surfaces,

1. (Shioda) Shioda-supersingularity implies unirationality,
2. (Artin) Artin-supersingularity implies unirationality,

This conjecture is known to hold for Fermat quartics \cite{Sh74b}. Kummer surfaces in \( p > 2 \) \cite{Sh77b}, and thus for supersingular K3 surfaces with Artin invariant \( \sigma_0 \leq 2 \) \cite{Og78}. Also it holds in characteristic 2 \cite{R-S79}, and for supersingular K3 surfaces with Artin invariant \( \sigma_0 \leq 3 \) in characteristic 5 \cite{P-S06}.

In characteristic 2, there is another conjecture by Artin \cite[p.552]{Ar74}, which, if true, would imply the previous conjectures and gives a geometric explanation for them – note that this conjecture is supported by a dimension count \cite[p.552]{Ar74}:

**Conjecture 3.3 (Artin).** In characteristic 2, an elliptic fibration on a supersingular K3 surface arises via Frobenius pullback from a rational elliptic surface.

Unfortunately, such a conjecture cannot be true in characteristic \( p \geq 3 \). Here is a counter-example:

**Example 3.4.** Let \( S_4 \) be the Fermat quartic in \( \mathbb{P}^3 \), which has been shown in \cite{Sh74b} to be supersingular in all characteristics \( p \) for which there exists a \( \nu \) s.t. \( p^\nu \equiv -1 \mod 4 \), e.g. in \( p = 3 \). This surface possesses a genus one fibration with six fibers of type \( I_4 \), see \cite[Section IV.2]{B-H85}. The associated Jacobian fibration \( X \to \mathbb{P}^1 \) is a supersingular elliptic K3 surface, again with six fibers of type \( I_4 \). If it were the Frobenius pullback of some other elliptic surface then the elliptic fibration of \( X^{(1/p)} \) would have six fibers of type \( I_n \) such that \( pn = 4 \), giving \( p = 2 \) as only possibility. Thus, \( X \to \mathbb{P}^1 \) is a supersingular K3 surface whose elliptic fibration is not a Frobenius pullback from another elliptic fibration.

The following result links the height of the formal Brauer group to the number of potentially supersingular fibers of the elliptic fibration:

**Theorem 3.5.** Let \( X \to \mathbb{P}^1 \) be an elliptic K3 surface with \( p \)-torsion section in characteristic \( p \), whose fibration does not have constant \( j \)-invariant. Then the fibration has at least one and at most two fibres with potentially supersingular reduction. Moreover,

1. if there is one fiber with potentially supersingular reduction then the formal Brauer group has height \( h \geq 2 \).
2. if there are two fibers with potentially supersingular reduction then the formal Brauer group has height \( h = 1 \).

**Proof.** We know \( p \leq 7 \) by Theorem 1.1. Since we assumed the fibration not to have constant \( j \)-invariant, the map from the base to the \( j \)-line is dominant, whence surjective and there is at least one fiber with potentially supersingular reduction.
Being a K3 surface, we may assume that the elliptic fibration is given by a Weierstrass equation
\begin{equation}
y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)
\end{equation}
where the \(a_i(t)\)'s are polynomials of degree \(\leq 2i\), i.e., \(a_i(t) = \sum_{j=0}^{2i} a_{ij} t^j\).

Assume \(p = 2\). Then the formal Brauer group of \(X\) has height \(h = 1\) if and only if \(a_{11} \neq 0\) by [Ar74, Theorem (2.12)] (the extra assumptions of this theorem are not needed for this statement). A fiber with potentially supersingular reduction is given by the vanishing of \(j(t) = a_1(t)^{12}/\Delta(t)\). Since \(\deg a_1(t) \leq 2\), the fibration has at most two such fibers. Moreover, if the fibration has two such fibers then \(a_{11} \neq 0\), which implies \(h = 1\). On the other hand, if there is only one such fiber then \(a_{11} = 0\), which implies \(h \geq 2\).

Now, assume that \(p = 3\). A straight forward, but tedious calculation shows that \(h = 1\) is equivalent to \(a_{11}^2 + a_{22} \neq 0\) in this case. After a suitable change of coordinates, we may assume \(a_1(t) = 0\). In this case, the Hasse invariant of the generic fiber is given by the class of \(-a_2(t)\) in \(k(t)^\times/k(t)^\times\). Moreover, since the fibration has a 3-torsion section, the Hasse invariant is trivial, i.e., \(-a_2(t)\) is a square. On the other hand, fibers with potentially supersingular reduction fulfill \(0 = c_4(t) = b_2(t)^2 = a_2(t)^2\) in this case. From \(\deg a_2(t) \leq 4\) we conclude that there are at least two such fibers. Moreover, the fibration has two such fibers if and only if \(a_{22} \neq 0\), i.e., if and only if \(h = 1\).

Next, assume \(p = 5\). Then we may assume \(a_1(t) = a_2(t) = a_3(t) = 0\). Computing the Hasse invariant, we see that then \(2a_4(t)\) has to be a fourth power in order for the fibration to possess 5-torsion sections. The vanishing of \(c_4(t) = 2a_4(t)\) is necessary for a fiber to have potentially supersingular reduction. From \(\deg a_4(t) \leq 8\) and the fact that \(2a_4(t)\) is a fourth power we conclude that there are at most two such fibers. A tedious calculation shows that \(h = 1\) is equivalent to \(2a_{44} \neq 0\) under our assumptions. As in the previous cases, having two fibers with potentially supersingular reduction is equivalent to \(2a_{44} \neq 0\), and thus equivalent to \(h = 1\).

We leave \(p = 7\) to the reader. Alternatively, one can use Theorem [4.1] below, by which there is only one such surface. It has one fiber with potentially supersingular reduction. The elliptic fibration arises as Frobenius pullback from a rational elliptic surface, i.e., this unique surface is unirational, whence fulfills \(h = \infty\). □

**Remark 3.6.** In characteristic 2, this connection between potentially supersingular fibers and the height of the formal Brauer group has already been observed by Artin [Ar74, p.552].

Thus, in order to obtain supersingular elliptic K3 surfaces with \(p\)-torsion sections we have to look at fibrations with constant \(j\)-invariant, which can exist for \(p = 2\) only, or at fibrations that have precisely one potentially supersingular fiber.

**Proposition 3.7.** Let \(X \to \mathbb{P}^1\) be an elliptic K3 surface with \(p\)-torsion section in characteristic \(p \geq 3\) that has precisely one potentially supersingular fiber. Then the elliptic fibration arises as Frobenius pullback from a rational elliptic surface. In particular, \(X\) is unirational and supersingular (\(h = \infty\)).
**Proof.** Let $I_{p,n_v}, v = 1, \ldots, n$ be the multiplicative fibers. Since $p \geq 3$, the fibration does not have constant $j$-invariant and thus there exist places of potentially multiplicative reduction which are multiplicative by Proposition 2.3. Now, by Proposition 2.4 the potentially supersingular fiber is additive, say with $m$ components and Swan conductor $\delta$ and we obtain

$$24 = c_2(X) = \sum_v pn_v + (2 + \delta + (m - 1))$$

We also know that $X \to \mathbb{P}^1$ arises as Frobenius pullback from some elliptic fibration $Y \to \mathbb{P}^1$, which has multiplicative fibers $I_{n_v}, v = 1, \ldots$. This fibration has one additive fiber also with Swan conductor $\delta$ and with, say, $m'$ components. Using (3) we obtain

$$c_2(Y) = \sum_v n_v + (2 + \delta + (m' - 1)) \leq \frac{22 - \delta}{p} + (2 + \delta + (m' - 1))$$

Since $p \neq 2$, reduction of type $I^{*}_n$ with $n \geq 1$ is potentially multiplicative and thus cannot occur as the additive fiber of $Y \to \mathbb{P}^1$. Inspecting the list of additive fibers we obtain $m' \leq 9$.

On the other hand, $Y$ is either rational or K3, i.e, $c_2(Y) = 12$ or $c_2(Y) = 24$. If $p \geq 5$ then $\delta = 0$ and (4) implies $c_2(Y) < 24$, which implies that $Y$ is rational. If $p = 3$ then $c_2(Y) = 24$ could only be achieved if $\delta \geq 20$. However, this contradicts (3), since $\sum_v pn_v \geq p = 3$. Thus, $Y$ is a rational surface also for $p = 3$. $\square$

**Remark 3.8.** We will see in Section 5 that the statement is wrong for $p = 2$.

We now come to one of the main results of this article, which relates the geometry of the elliptic fibration to supersingularity and unirationality.

**Theorem 3.9.** Let $X \to \mathbb{P}^1$ be an elliptic K3 surface with $p$-torsion sections in characteristic $p \geq 3$. Let $\varphi : \mathbb{P}^1 \to \mathcal{Ig}(p)^{\text{ord}}$ be the compactified classifying morphism. Then the following are equivalent:

1. $X$ arises as Frobenius pullback from a rational elliptic surface
2. $X$ is a Zariski surface
3. $X$ is unirational
4. $X$ is supersingular
5. the fibration has precisely one fiber with additive reduction
6. $\varphi$ is totally ramified over the supersingular point of $\mathcal{Ig}(p)^{\text{ord}}$

In particular, the conjectures of Artin–Shioda (Conjecture 3.2) and Artin (Conjecture 3.3) hold for this class of surfaces.

**PROOF.** Since $p \geq 3$ the fibration does not have constant $j$-invariant by Theorem 1.1 and so $\varphi$ is surjective. Also $p \leq 7$ by loc. cit., which implies that $\mathcal{Ig}(p)^{\text{ord}}$ has precisely one supersingular point. By Proposition 2.3 all additive fibers are potentially supersingular, which gives (5) $\Rightarrow$ (6). By Proposition 2.4 potentially supersingular fibers are additive and we get (6) $\Rightarrow$ (5).

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold in general. The implication (4) $\Rightarrow$ (5) follows from Theorem 3.5. Finally, (5) $\Rightarrow$ (1) follows from Proposition 3.7. $\square$
Corollary 3.10. The Artin–Shioda conjecture holds for elliptic K3 surfaces with $p$-torsion sections.

Proof. For $p \geq 3$ this is Theorem 3.9 and for $p = 2$ it follows from [R-S79]. □

Remark 3.11. We will see in Section 5 that Theorem 3.9 also holds for elliptic K3 surfaces with 4-torsion sections in characteristic 2.

Let us finally reformulate Theorem 3.9 in terms of the “other” surfaces:

Theorem 3.12. Let $X \to \mathbb{P}^1$ be an elliptic K3 surface with $p$-torsion sections in characteristic $p \geq 3$. Then the following are equivalent:

1. $X$ is ordinary
2. $X$ is not unirational
3. $X$ arises as Frobenius pullback from a K3 surface
4. the fibration has precisely two fibers with additive reduction

Moreover, such surfaces can exist in characteristic $p \leq 5$ only.

Proof. By Theorem 2.2 case (4) can happen in characteristic $p \leq 5$ only.

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold in general. The implication (3) $\Rightarrow$ (4) follows from Theorem 3.9 and Theorem 2.2. Finally, the implication (4) $\Rightarrow$ (1) follows from Theorem 3.5. □

Remark 3.13. As we shall see in Section 5 there do exist elliptic K3 surfaces with 2-torsion section that are neither unirational nor ordinary.

4. THE EXPLICIT CLASSIFICATION

Having established the general picture in the previous sections, we now give a detailed classification of elliptic K3 surfaces with $p$-torsion section in characteristic $p \geq 3$. This is achieved by studying the classifying morphism to the Igusa curve and the Néron model of the universal family over $\operatorname{Ig} (p)_{\text{ord}}$. We pay special attention to the arising supersingular surfaces.

Let us recall the following from [Sh90]: the Néron–Severi group $\operatorname{NS}(X)$ of an elliptic surface together with its intersection pairing is made up of two natural subgroups: the trivial lattice $T$, which is associated to the singular fibers, and the Mordell–Weil group $\operatorname{MW}(X)$, which arises from sections of the elliptic fibration and the Néron–Tate height pairing. Inside this group sits the narrow Mordell–Weil group $\operatorname{MW}^0(X)$ consisting of those sections that lie fiberwise on the same component as the zero-section. For rational elliptic surfaces these groups have been worked out explicitly in [O-S91].

For the singular fibers we use Kodaira’s notation. For example, $I_n$ denotes the multiplicative reduction where a singular fiber consists of $n$ smooth rational curves forming a cycle. In case of additive reduction and in characteristic $p \leq 3$ there is a further invariant, namely the Swan conductor $\delta$ of a singular fiber, which we add as index. Thus, $I_n^*\delta$ stands for additive reduction of type $I_n^*$ with Swan conductor $\delta$. We refer to [Sil94, Chapter IV] for definitions and details.

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Let us also recall that the discriminant of $\operatorname{NS}(X)$ for a supersingular K3 surface $X$ is of the form $p^{2\sigma_0}$ for some integer $1 \leq \sigma_0 \leq 10$, called the Artin invariant.
All values actually occur and surfaces with Artin invariant $\leq \sigma_0$ form a $(\sigma_0 - 1)$-dimensional subspace inside the moduli space of all supersingular surfaces [Og78]. Finally, there is only surface with $\sigma_0 = 1$ [Og78] and in $p \geq 3$ surfaces with $\sigma_0 \leq 2$ are Kummer surfaces by [Og78] and [Sh79].

**Characteristic 7.**

**Theorem 4.1.** There exists only one elliptic K3 surface $X \to \mathbb{P}^1$ with 7-torsion section in characteristic 7 up to isomorphism. It has the following invariants:

| singular fibers | $\sigma_0$ | $\text{MW}^\circ(X)$ | $\text{MW}(X)$ |
|-----------------|-----------|----------------|---------------|
| III, $3 \times I_7$ | 1         | $A_1(7)$       | $A_1^s(7) \oplus (\mathbb{Z}/7\mathbb{Z})$ |

The Weierstraß equation is given by the following:

$$y^2 = x^3 + tx + t^{12}.$$  

In particular, it is the unique supersingular K3 surface with Artin invariant $\sigma_0 = 1$.

**Proof.** As in Section 1 we denote by $\varphi$ the classifying morphism to $\text{Ig}(7)^{\text{ord}}$ and by $\mathcal{E} \to \text{Ig}(7)^{\text{ord}}$ the universal curve. An analysis of the multiplicative fibers as in the proof of Theorem 1.1 shows that $\deg \varphi \geq 2$ is impossible. Hence $\varphi$ is an isomorphism, proving uniqueness. Since $\mathcal{E}(7)$ corresponds in fact a K3 surface, we get existence. The singular fibres are listed in Proposition 1.3.

Denote by $Y \to \mathbb{P}^1$ the elliptic fibration corresponding to $\mathcal{E}$. Then $Y$ is rational, which implies that $X$ is a Zariski surface and thus unirational. The singular fibres are given in Proposition 1.3 and thus the root lattice of $Y$ is $E_7$. From the tables in [O-S91] we see that the (narrow) Mordell-Weil lattice is $\text{MW}(Y) \cong A_1^s$ and $\text{MW}^\circ(Y) \cong A_1$, respectively.

Now, Frobenius induces an inclusions of lattices

$$\text{MW}(Y)_{\text{free}}(p) \subseteq \text{MW}(X)_{\text{free}},$$

which is of some finite index $\mu$. Taking determinants, we obtain

$$\mu^2 = \frac{\det \text{MW}(Y)_{\text{free}}(p)}{\det \text{MW}(X)_{\text{free}}}.$$  

After plugging in Lemma 4.3 below, we obtain

$$\mu^2 = \frac{\det A_1^s(7)}{\det \text{NS}(X) |\text{MW}(X)_{\text{tor}}|^2} \det(U \oplus A_6^{\oplus 3} \oplus A_1) = \frac{1}{7} \cdot 7 \cdot 7^2 \cdot 7^3 \cdot 2,$$

which yields $\mu = 1$. Thus, $\sigma_0 = 1$ and $\text{MW}(X) \cong A_1^s(7) \oplus (\mathbb{Z}/7\mathbb{Z})$.

**Remark 4.2.** Existence and uniqueness of this surface have already been shown in [Schw05, Examples 2.4].

**Lemma 4.3 ([Sh90, Theorem 8.7]).** Let $X$ be an elliptic surface whose $j$-invariant is not constant. Then

$$\det \text{NS}(X) = \frac{\det \text{MW}(X)_{\text{free}} \cdot \det T}{|\text{MW}(X)_{\text{tor}}|^2},$$

where $T$ denotes the trivial lattice.
Characteristic 5.

**Theorem 4.4.** In characteristic 5, the classifying morphism $\varphi$ of an elliptic K3 surface with $5$-torsion section is finite of degree 2. Conversely, if $\varphi : \mathbb{P}^1 \to \text{Ig}(5)_{\text{ord}}$ is a morphism of degree 2 then the associated elliptic fibration with $5$-torsion section is a K3 surface.

More precisely, the surfaces have the following invariants:

| singular fibers | dim | $\sigma_0$ | $\text{MW}^\circ(X)$ | $\text{MW}(X)$ |
|-----------------|-----|------------|----------------------|----------------|
| $2 \times \Pi, 4 \times I_5$ | 2   |            |                      |                |
| $2 \times \Pi, I_{10}, 2 \times I_5$ | 1   |            |                      |                |
| $2 \times \Pi, 2 \times I_{10}$ | 0   |            |                      |                |
| $IV, 4 \times I_5$ | 1   | 2          | $A_2(5)$             | $A_2^2(5) \oplus \mathbb{Z}/5\mathbb{Z}$ |
| $IV, I_{10}, 2 \times I_5$ | 0   | 1          | (30)                 | $\langle \frac{5}{2} \rangle \oplus \mathbb{Z}/5\mathbb{Z}$ |

Here, $\text{dim}$ denotes the dimension of the family. For the supersingular surfaces, this list also gives Artin invariants $\sigma_0$ and their (narrow) Mordell–Weil lattices.

**Remark 4.5.** The surfaces with two $\Pi$-fibers arise as Frobenius pullbacks from Shioda’s sandwich surfaces [Sh06]. From this fact one obtains another proof of their non-supersingularity.

**Proof.** The proof is analogous to the proof of Theorem 4.1. We leave it to the reader to show that the classifying morphism $\varphi$ is of degree 2. Then we obtain the complete classification of these surfaces in terms of the branch points of the classifying morphism: To do so, let $\mathcal{E} \to \text{Ig}(5)_{\text{ord}}$ be the universal elliptic curve over the Igusa curve. By Proposition 1.3, its Weierstraß equation is given by

$$y^2 = x^3 + 3t^4x + t^5,$$

which has a singular fiber of type $\Pi^*$ over $t = 0$ and fibers of type $I_1$ over $t = \pm 1$. Note that this surface is a rational extremal elliptic surface.

We write the classifying morphism $\varphi = \varphi_{\alpha,\beta} : \mathbb{P}^1 \to \text{Ig}(5)_{\text{ord}}$ as

$$t = \frac{\alpha s^2 + \beta}{s^2 + 1},$$

whose branch points are $t = \alpha$ and $t = \beta$, where $t$ (resp. $s$) is a local parameter of $\text{Ig}(5)_{\text{ord}}$ (resp. $\mathbb{P}^1$). Then our surfaces arise as pull-backs along Frobenius $F$ and $\varphi_{\alpha,\beta}$:

$$
\begin{array}{ccc}
X & \xrightarrow{Y} & \mathcal{E} \\
\downarrow & & \downarrow \\
\mathbb{P}^1 & \xrightarrow{F} & \mathbb{P}^1 & \xrightarrow{\varphi_{\alpha,\beta}} & \text{Ig}(5)_{\text{ord}}
\end{array}
$$

The elliptic surface $Y$ is given by the Weierstraß equation

$$y^2 = x^3 + 3(\alpha s^2 + \beta)^4 x + (\alpha s^2 + \beta)^5 (s^2 + 1),$$
and depending on $\alpha$ and $\beta$ we obtain the following list

| $\{\alpha, \beta\} \cap \{0, \pm 1\}$ | singular fibers of $X$ | singular fibers of $Y$ | $Y$ |
|--------------------------------------|------------------------|------------------------|-----|
| $\emptyset$                          | $2 \times \Pi, 4 \times I_5$ | $2 \times \Pi^*, 4 \times I_1$ | $K3$ |
| $\{1\}, \{-1\}$                     | $2 \times \Pi, I_{10}, 2 \times I_5$ | $2 \times \Pi^*, I_2, 2 \times I_1$ | $K3$ |
| $\{1, -1\}$                         | $2 \times \Pi, 2 \times I_{10}$ | $2 \times \Pi^*, 2 \times I_2$ | $K3$ |
| $\{0\}$                             | $IV, 4 \times I_5$ | $IV^*, 4 \times I_1$ | rational |
| $\{0, 1\}, \{0, -1\}$              | $IV, I_{10}, 2 \times I_5$ | $IV^*, I_2, 2 \times I_1$ | rational |


giving the explicit classification of our surfaces.

By Theorem 3.9 the supersingular surfaces are precisely those that arise as Frobenius pullbacks from rational elliptic surfaces. It remains to determine the Mordell–Weil groups and Artin invariants.

For the $(IV, I_{10}, 2 \times I_5)$-surface this can be done as in the proof of Theorem 4.1 and we leave it to the reader.

Let $X \to \mathbb{P}^1$ be a $(IV, 4 \times I_5)$-surface. Using [O-S91], we see that it arises via Frobenius pullback from rational elliptic surface $Y \to \mathbb{P}^1$ with $MW^0(Y) \cong A_2$. From (5) we get an inclusion of Mordell–Weil lattices and once we have shown equality our assertion follows. Now, $MW^0(Y)$ is generated by two sections $P_1, P_2$ with $\langle P_1, P_2 \rangle = 2$, which implies that both neither meet the zero-section nor specialize into the component groups of the singular fibers. By Lemma 4.6 below, these two sections cannot lie in the image of $V : MW(X) \to MW(Y)$.

Now, denote by $K$ the function field of $\mathbb{P}^1$ and let $E$ and $E^{(p)}$ be the generic fibers of $Y$ and $X$ over $\text{Spec} \ K$. Multiplication by $p$ induces an exact sequence

\[
0 \to \ker(V) \to E^{(p)}(K)/F(E(K)) \xrightarrow{V} E(K)/pE(K) \to 0,
\]

where $V$ denotes Verschiebung. Knowing that $P_1$ and $P_2$ do not lie in the image of $V$, this implies $\nabla V = 0$ in the sequence above and we obtain the desired equality of Mordell–Weil lattices.

**Lemma 4.6.** Let $R$ be complete DVR with field of fractions $K$ of characteristic $p \geq 5$ and perfect residue field $k$. Let $E$ be an elliptic curve over $K$ and assume that $E^{(p)}$ has a $K$-rational $p$-division point. Assume moreover, that $E$ has additive reduction that is not of type $\Pi^*$ if $p = 5$. If $P \in E^{(p)}(K)$ then $V(P)$, where $V$ denotes Verschiebung, specializes into the component group or to zero in the Néron model $\mathcal{E}$ of $E$.

**Proof.** Let $\pi \in R$ be a uniformizer. Set $L := K(\pi^{1/12})$ and denote by $S$ the integral closure of $R$ in $L$. Then $L/K$ is totally ramified, $\varpi := \pi^{1/12}$ is a uniformizer on $S$. Denote by $\nu_\varpi$ and $\nu_\varpi^p$ normalized valuations, i.e., $\nu_\varpi(\pi) = \nu_\varpi^p(\varpi) = 1$ and $\nu_\varpi^p(x) = 12 \nu_\varpi(x)$ for all $x \in R$.

Since $p \geq 5$, the curve $E$ acquires semi-stable reduction over $L$, which is good and supersingular [L-S08 Theorem 4.3]. Let us denote by $\mathcal{E}$ minimal Weierstraß equations and assume that the singularity (in case of bad reduction) lies in $(0, 0)$.

For a section $P = (x_0, K, y_0, K)$ we set $t_{0,K} := y_0/K, x_{0,K}$ and note that $\nu_\varpi(t_{0,K}) < 0$ if and only if $P$ specializes to zero in the Néron model, as well as $\nu_\varpi(t_{0,K}) > 0$.
if and only if $P$ specializes non-trivially into the component group. Now, we run Tate’s algorithm and suppose we have to reduce $r_1$-times to get from $\mathcal{E}_K^{(p)} \times_K L$ to $\mathcal{E}_L^{(p)}$. By our assumptions on $p$ and $L/K$ we have $r_1 = \nu_{\pi}(\Delta_{\text{min}})$, where $\Delta_{\text{min}}$ denotes the minimal discriminant of $\mathcal{E}_K^{(p)}$. Then $P$, considered as a section of $\mathcal{E}_L^{(p)}$, fulfills $\nu_{\pi}(t_{0,L}) = 12\nu_{\pi}(t_{0,K}) - r_1$.

Next, $V$ induces a map $\mathcal{E}_L^{(p)} \to \mathcal{E}_L$. Both elliptic curves have good supersingular reduction and on the level of tangent spaces, this map is multiplication by the Hasse invariant [K-M85, Chapter 12.4]. Then, for appropriate local parameters $e$, $e^{(p)}$ around zero, $V$ is given by $e^{(p)} \mapsto H \cdot e + ...$ for some lift of the Hasse invariant to $S$. As this lift we may choose the “naive” Hasse invariant in the sense of raising a homogeneous Weierstraß equation to the $p$-th power and taking the coefficient of $(xyz)^{p-1}$. If we set $h := \nu_{\infty}(H)$ then, $h > 0$ since we have supersingular reduction and $h$ is divisible by $p - 1$ as there is an $L$-rational $p$-division point on $E_L$. Thus, if $V(P) = (x'_{0,L}, y'_{0,L})$ in $\mathcal{E}_L$, we set $t'_{0,L} = y'_{0,L}/x'_{0,L}$ and get $\nu_{\pi}(t'_{0,L}) = (12\nu_{\pi}(t_{0,K}) - r_1)(h + 1)$.

Suppose we have to reduce $r_2$-times in the Tate algorithm to get from $\mathcal{E}_K \times L$ to $\mathcal{E}_L$. Then we finally obtain

$$\nu_{\pi}(t'_{0,K}) = \nu_{\pi}(t_{0,K}) + \frac{r_2 - r_1}{12} + \left(\nu_{\pi}(t_{0,K}) - \frac{r_1}{12}\right) \cdot h.$$

Let us first assume that $P \in E_K(K)$ does not specialize into the component group, which means $\nu_{\pi}(t_{0,K}) \leq 0$. Recall that $h > 0$ and that $p - 1$ divides $h$. Moreover, from the tables of minimal discriminants we get $r_2 - r_1 \leq 8$ (note that reduction of type $I_n^*, n \geq 2$ is impossible by [L-S08, Corollary 4.5]). Thus, if $p \geq 7$ or if $r_2 - r_1 < 8$ we get $\nu_{\pi}(t''_{0}) < 0$, i.e., $V(P)$ specializes to zero in the Néron model of $E_K$. The only case where this may fail is $p = 5$ and $r_2 - r_1 = 8$, i.e., $\mathcal{E}_K^{(p)}$ has reduction of type $\Pi^*(r_1 = 2)$ and $\mathcal{E}_K$ has reduction of type $\Pi^*(r_2 = 10)$.

Finally, assume that $P$ specializes into the component group of $\mathcal{E}_K^{(p)}$. Then there exists an integer $m$, prime to $p$, such that $mP$ does not specialize into the component group any more. By the previous discussion $V(mP)$ specializes to zero in the Néron model of $\mathcal{E}_K$. Now, as a group scheme, the special fiber of $\mathcal{E}_K$ is $G_\pi \times \Phi$, where $\Phi$ is the component group of $\mathcal{E}_K$. Since $G_\pi$ does not have $m$-torsion, it follows that $V(P)$ specializes to zero or into the component group of $\mathcal{E}_K$. \qed

The following result makes sure that we find in fact complete families of supersingular K3 surfaces.

**Proposition 4.7.** Let $X$ be an elliptic K3 surface with $p^n$-torsion section in characteristic $p$. Assume that $X$ is supersingular with Artin-invariant $\sigma_0$. Then, every (Shioda-)supersingular K3 surface with Artin invariant $\sigma_0$ in characteristic $p$ possesses an elliptic fibration with $p^n$-torsion section.

**Proof.** To give a (quasi-)elliptic fibration on $X$ is equivalent to giving an isometric embedding of a hyperbolic lattice $U$ of rank 2 into $\text{NS}(X)$. 


Then, the trivial lattice $T$ is the sub-lattice of $\text{NS}(X)$ generated by $U$ and all $x \in U^\perp$ with $x^2 = -2$, see [Sh90]. By [Sh90, Theorem 1.3] the torsion sections of the fibration correspond to the torsion of $\text{NS}(X)/T$.

The Néron–Severi group of a (Shioda-)supersingular K3 surface is uniquely determined by $p$ and $\sigma_0$ by [R-S79, Theorem 2']. Thus, by the previous discussion, if one of these surfaces possesses a (quasi-)elliptic fibration with $p^n$-torsion section then so do all of them.

However, we have to rule out the possibility that the isometric embedding of $U$ into $\text{NS}(X)$ corresponding to the elliptic fibration on $X$ gives rise to a quasi-elliptic fibration on another K3 surface $Y$ with the same $p$ and $\sigma_0$: if $p \geq 5$ or if $\text{rank}(T) < 22$ then the fibration on $Y$ is automatically elliptic and the quasi-elliptic case cannot occur at all. And finally, if $p \leq 3$ and $\text{rank}(T) = 22$ then the elliptic fibration on $X$ is extremal and these K3 surfaces have been explicitly classified in [Ito02]. It turns out that these surfaces have Artin invariant $\sigma_0 = 1$, i.e., $X$ is isomorphic to $Y$. □

Together with Theorem 4.4 we immediately conclude

**Corollary 4.8.** Every (Shioda-)supersingular K3 surface with $\sigma_0 \leq 2$ in characteristic 5 possesses an elliptic fibration with 5-torsion section. □

**Characteristic 3.** We denote by $O \in \overline{\text{Ig}(3)_{\text{ord}}}$ the unique supersingular point.

**Theorem 4.9.** In characteristic 3, the classifying morphism $\varphi$ for an elliptic K3 surface with 3-torsion section is finite of degree fulfills $2 \leq \deg \varphi \leq 6$. More precisely,

1. $\deg \varphi = 2$ and $\varphi^{-1}(O)$ consists of two points.
2. $\deg \varphi = 3$, $\varphi$ is separable and $\varphi^{-1}(O)$ consists of two points.
3. $\deg \varphi = 4$ and $\varphi^{-1}(O)$ consists of one or two points.
4. $\deg \varphi = 5$ and $\varphi^{-1}(O)$ consists of one point or two points with ramification index $e = 2$ and $e = 3$.
5. $\deg \varphi = 6$ and $\varphi^{-1}(O)$ consists of one point or two points with ramification index $e = 3$.

Conversely, if $\varphi$ is as above then the associated elliptic fibration with 3-torsion section is a K3 surface.

**Proof.** The proof is analogous to the proof of Theorem 4.1 (but lengthier and with more subcases) and we leave it to the reader. □

From this description it is easy to obtain a complete list of these surfaces as before. However, since this list is rather long, we have decided not to include it here. Instead, we only determine the supersingular K3 surfaces with 3-torsion sections. By Theorem 4.9, these are precisely the surfaces, where the classifying morphism is totally ramified over $O \in \overline{\text{Ig}(3)_{\text{ord}}}$. As before, $\varphi$ denotes the classifying morphism.

**Theorem 4.10.** Every (Shioda-)supersingular K3 surface with Artin invariant $\sigma_0 \leq 6$ in characteristic 3 possesses an elliptic fibration with 3-torsion section.
The complete list of these surfaces is given by the following table:

### deg $\varphi = 6$ (separable)

| singular fibers | dim | $\sigma_0$ | $\text{MW}^0(X)$ | $\text{MW}(X)$ |
|-----------------|-----|------------|-------------------|----------------|
| $\Pi_4, 6 \times I_3$ | 5   | 6          | $E_8(3)$          | $E_8(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\Pi_4, I_6, I_3 \times 4$ | 4   | 5          | $E_7(3)$          | $E_7(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\Pi_4, I_9, I_3 \times 3$ | 3   | 4          | $E_6(3)$          | $E_6^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\Pi_4, I_6 \times 2, I_3 \times 2$ | 3   | 4          | $D_6(3)$          | $D_6^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\Pi_4, I_{12}, I_3 \times 2$ | 2   | 3          | $D_5(3)$          | $D_5^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\Pi_4, I_6 \times 3$ | 2   | 3          | $D_4(3) \oplus A_1(3)$ | $D_4^*(3) \oplus A_1^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\Pi_4, I_9, I_6, I_3$ | 2   | 3          | $A_5(3)$          | $A_5^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\Pi_4, I_{15}, I_3$ | 1   | 2          | $A_4(3)$          | $A_4^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\Pi_4, I_{12}, I_6$ | 1   | 2          | $A_3(3) \oplus A_1(3)$ | $A_3^*(3) \oplus A_1^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_2, 6 \times I_3$ | 4   | 5          | $E_6(3)$          | $E_6^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_2, I_6, I_3 \times 4$ | 3   | 4          | $A_5(3)$          | $A_5^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_2, I_9, I_3 \times 3$ | 2   | 3          | $A_2(3) \oplus D_2^*(3)$ | $A_2^*(3) \oplus Z/3\mathbb{Z}$ |
| $\text{IV}_2, I_6 \times 2, I_3 \times 2$ | 2   | 3          | $L_4(3)$          | $L_4^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_2, I_{12, 3} \times 2$ | 1   | 2          | $L_3(3)$          | $L_3^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_2, I_6 \times 3$ | 1   | 2          | $A_1(3) \oplus L_2(3)$ | $A_1^*(3) \oplus L_2^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $I_{0,0}, I_3 \times 6$ | 3   | 4          | $D_4(3)$          | $D_4^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $I_{0,0}^*, I_6, 4 \times I_3$ | 2   | 3          | $A_1(3) \oplus D_2^*(3)$ | $A_1^*(3) \oplus D_2^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $I_{0,0}^*, I_6 \times 2, I_3 \times 2$ | 1   | 2          | $A_1(3) \oplus D_2^*(3)$ | $A_1^*(3) \oplus D_2^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $I_{0,0}^*, I_9, I_3 \times 3$ | 1   | 2          | $L_2(3)$          | $L_2^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $I_{0,0}^*, I_6 \times 3$ | 0   | 1          | $A_1(3)$          | $A_1^*(3) \oplus \mathbb{Z}/6\mathbb{Z}$ |
| $I_{0,0}^*, I_{12, 3} \times 2$ | 0   | 1          | $A_1(3)$          | $A_1^*(3) \oplus \mathbb{Z}/6\mathbb{Z}$ |

### deg $\varphi = 6$ (inseparable)

| singular fibers | dim | $\sigma_0$ | $\text{MW}^0(X)$ | $\text{MW}(X)$ |
|-----------------|-----|------------|-------------------|----------------|
| $\text{IV}_2, 2 \times I_9$ | 1   | 2          | $A_2(3)$          | $A_2^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_2, I_{18}$ | 0   | 1          | $\langle 18 \rangle$ | $\langle 18 \rangle \oplus \mathbb{Z}/3\mathbb{Z}$ |

### deg $\varphi = 5$

| singular fibers | dim | $\sigma_0$ | $\text{MW}^0(X)$ | $\text{MW}(X)$ |
|-----------------|-----|------------|-------------------|----------------|
| $\text{IV}_5, 5 \times I_3$ | 4   | 5          | $E_8(3)$          | $E_8(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_5, I_6, 3 \times I_3$ | 3   | 4          | $E_7(3)$          | $E_7^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_5, 2 \times I_6, I_3$ | 2   | 3          | $D_6(3)$          | $D_6^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_5, I_9, 2 \times I_3$ | 2   | 3          | $E_6(3)$          | $E_6^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_5, I_6, I_6$ | 1   | 2          | $A_5(3)$          | $A_5^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_5, I_{12, 3}$ | 1   | 2          | $D_4(3)$          | $D_4^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
| $\text{IV}_5, I_{15}$ | 0   | 1          | $A_4(3)$          | $A_4^*(3) \oplus \mathbb{Z}/3\mathbb{Z}$ |
matrices are given by $f$ proceed as in the proof of Theorem 4.4 in order to obtain explicit equations: let for certain characteristic.

Then we substitute supersingular point $y$ of index $2$.

Here, $L_2$, $L_3$, and $L_4$ are lattices of rank $2$, $3$, and $4$, all of determinant $12$, whose matrices are given by

$$L_2 = \begin{pmatrix} 4 & -2 \\ -2 & 4 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & -1 \\ -1 & -1 & 4 \end{pmatrix}, \quad L_4 = \begin{pmatrix} 4 & -1 & 0 & 1 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ 1 & 0 & -1 & 2 \end{pmatrix}.$$

Also, the notation $3.L$ for a lattice $L$ stands for a lattice that has $L$ as a sublattice of index $3$.

PROOF. By Theorem 3.9, the classifying morphism $\varphi$ is totally ramified over the supersingular point $O \in \operatorname{Ig}(3)^{\mathrm{ord}}$. This gives $4 \leq \deg \varphi \leq 6$ by Theorem 4.9. We proceed as in the proof of Theorem 4.9 in order to obtain explicit equations: let $f_3(s)$, $f_4(s)$ and $f_5(s)$ be polynomials of degree $3$, $4$ and $5$ with no zero in $s = 0$. Then we substitute

$$t = \frac{s^6}{f_5(s)}, \quad t = \frac{s^5}{f_4(s)} \quad \text{and} \quad t = \frac{s^4}{f_3(s)}$$

into the Weierstraß equation $y^2 + txy = x^3 - t^5$ of the universal family over $\operatorname{Ig}(3)^{\mathrm{ord}}$, see Proposition 1.3. In all cases this leads to a Weierstraß equation

$$y^2 = x^3 + s^2 x^2 + s^5 + r_4 s^4 + r_3 s^3 + r_2 s^2 + r_1 s + r_0$$

for certain $(r_4, r_3, r_2, r_1, r_0) \in \mathbb{A}^5_k$. Depending on the degree of $\varphi$ these coefficients satisfy the following conditions:

$$\begin{align*}
\deg \varphi = 6 : & \quad r_1 r_0 \neq 0 \\
\deg \varphi = 5 : & \quad r_1 \neq 0, \quad r_0 = 0 \\
\deg \varphi = 4 : & \quad r_2 \neq 0, \quad r_1 = r_0 = 0
\end{align*}$$

Note that the generic surfaces of each degree correspond to the extremal rational surfaces of the cases $1C$, $1D$ and $3C$ of [La94 §3].

It is remarkable that these rational elliptic surfaces appear in the family of elliptic surfaces related to the semi-universal deformation of the $E_8^2$-singularity in characteristic $3$, which is given by

$$y^2 = x^3 + (t^2 + s)x^2 + (q_1 t + q_0)x + t^5 + r_4 t^4 + r_3 t^3 + r_2 t^2 + r_1 t + r_0.$$

To obtain elliptic K3 surfaces with $3$-torsion section we have to take the Frobenius pullback of these surfaces. Then the non-trivial $3$-torsion sections of the fibration are explicitly given by

$$(-r^5 + r^4 t + r^3 t^2 + r^2 t^3 + r t^4 + t^5, \pm r^3 (r^5 + r^4 t + r^3 t^2 + r^2 t^3 + r t^4 + t^5)_{\frac{1}{r}} + \frac{1}{r^3} t^4)$$
(For deg \( \varphi = 4 \) one needs to modify slightly because of the minimality of the equation.)

By Lemma 4.3 and the preceding argument, the index of \( \text{MW}(Y)_{\text{free}}(3) \) inside \( \text{MW}(X)_{\text{free}} \) is related to the Artin invariant of \( X \) for each case in the table. From this observation we obtain an upper bound for the Artin invariant. On the other hand, since all the surfaces in the table can be realized inside the families corresponding to the semi-universal deformation of the \( E_8 \)-singularity as noted above, the dimension of the surface having the given type of singular fibers inside the moduli space is bounded from below. This gives the Artin invariants for the cases deg \( \varphi = 4 \) and deg \( \varphi = 6 \).

For the case deg \( \varphi = 5 \) we need a more precise analysis. Let \( X \) be an elliptic K3 surface with 3-torsion sections whose singular fibers are of type IV \( 5 \times I_3 \). Then we have \( \mu^2 = 3^{12 - 2\sigma_0(X)} \), where \( \mu \) is the index of \( \text{MW}(Y)_{\text{free}}(3) \) inside \( \text{MW}(X)_{\text{free}} \). This implies \( \sigma_0(X) \leq 6 \). On the other hand, these surfaces are realized inside the semi-universal deformation of the \( E_8 \)-singularity, which yields \( \sigma_0(X) \geq 5 \). Thus, we have to decide whether \( \mu = 1 \) or \( \mu = 3 \) holds true. Assume \( \mu = 1 \). From \( \text{MW}(Y)_{\text{free}} = \text{MW}^0(Y) = E_8 \) we get \( \text{MW}(X)_{\text{free}} = \text{MW}^0(X) = E_8(3) \). However, the 3-torsion sections of this surface do not lie in \( \text{MW}^0(X) \), which produces many free sections in \( \text{MW}(X) \) that do not lie in \( \text{MW}^0(X) \), a contradiction. Thus, \( \mu = 3 \) and we obtain \( \sigma_0(X) = 5 \). The other cases can be treated similarly using Lemma 4.11.

Since we have found examples for all Artin invariants \( \sigma_0 \leq 6 \), Proposition 4.7 tells us that every (Shioda-)supersingular K3 surface with \( \sigma_0 \leq 6 \) possesses an elliptic fibration with 3-torsion section.

**Lemma 4.11.** With the notations as before, the index of \( \text{MW}^0(Y) \) inside \( \text{MW}(Y)_{\text{free}} \) divides the index of \( \text{MW}^0(X) \) inside \( \text{MW}(X)_{\text{free}} \).

5. Characteristic 2

In this section we deal with elliptic K3 surfaces with 2-torsion section in characteristic 2. The classification in this case has much more subcases as for \( p \geq 3 \) since the fibration may have constant \( j \)-invariant, additive fibers may not be potentially supersingular and potentially supersingular may have good reduction.

We start with a useful result, which directly follows from [D-K01]:

**Proposition 5.1.** Let \( X \rightarrow \mathbb{P}^1 \) be an elliptic K3 surface with 2-torsion section in characteristic 2. Then \( X = Y^{(2)} \) for some elliptic fibration \( Y \rightarrow \mathbb{P}^1 \). Moreover, denote by \( G \) the group of order 2 that acts on \( X \) via translating by the 2-torsion point. Then \( Y = X/G \) and there are two cases

1. \( G \) has one or two fixed points and \( Y \) is a K3 surface
2. The fixed locus of \( G \) is a connected curve and \( Y \) is a rational surface. In particular, \( X \) is unirational in this case.

**Proof.** Let us recall that multiplication by 2 on generic fibers of the fibration factors as \( Y \rightarrow Y^{(2)} = X \rightarrow X/G = Y \), cf. (1).

If $G$ has a finite number of fixed points then there are at most two of them by \cite[Theorem 2.4]{D-K01}. If $G$ acted without fixed points, then $Y$ would be an Enriques surface, which is absurd, cf. the proof of Theorem\cite{2.2}. If $G$ has one fixed point then $X/G$ is a K3 surface by \cite[Theorem 2.4]{D-K01} and \cite[Remark 2.6]{D-K01}. And if $G$ has two fixed points then $X/G$ is also a K3 surface by \cite[Theorem 2.4]{D-K01}.

If $G$ has non-isolated fixed points then the fixed locus is a connected curve by \cite[Corollary 3.6]{D-K01} and the quotient $X/G$ is rational \cite[Theorem 3.7]{D-K01}.

The classification of elliptic K3 surfaces with 2-torsion in characteristic 2 is now as follows, where $h$ denotes the height of the formal Brauer group as discussed in Section\cite{3}.

**Theorem 5.2.** Let $X \to \mathbb{P}^1$ be an elliptic K3 surface with 2-torsion section in characteristic 2.

If the fibration has constant $j$-invariant then the singular fibers are either

1. one additive fiber of type $I^*_{12,6}$, and then $h \geq 2$, or
2. two additive fibers, both of type $I^*_{4,2}$, and then $h = 1$.

If the fibration does not have constant $j$-invariant, then we have the following cases:

1. the fibration has precisely one additive fiber, which is potentially supersingular. In this case $h \geq 2$ holds true.
2. the fibration is semi-stable and there is precisely one fiber with good and supersingular reduction. Moreover, $X$ is unirational and $h = \infty$.
3. the fibration has precisely two fibers with additive reduction, both of which are potentially supersingular. In this case $h = 1$ holds true.
4. the fibration has precisely two fibers with additive reduction, one of which is potentially supersingular and the other one is potentially ordinary of type $I^*_{14,2}$. In this case $h = 1$ holds true.

**Proof.** Let

$$y^2 + a_1(t)xy + a_3(t)y = x^3 + a_2(t)x^2 + a_4(t)x + a_6(t)$$

be a global Weierstraß equation of the K3 surface, where the $a_i(t)$’s are polynomials of degree $\leq 2i$. We denote by $\sigma_2$ the 2-torsion section and let $G$ be the group of order 2 generated by translation by $\sigma_2$.

In order to have additive reduction at $t_0$ it is necessary that $a_1(t_0) = 0$. As $\deg a_1(t) \leq 2$, it follows that there are at most two places of additive reduction. Moreover, from $j = a_1^2/\Delta$ we infer that for a place $t_0$ to have potentially supersingular reduction, again, $a_1(t_0) = 0$ is necessary.

**Case 1:** assume that $a_1(t)$ has a double zero. Then we get $h \geq 2$ from \cite[Theorem 2.12]{Ar74}.

If the fibration does not have constant $j$-invariant then there has to be at least one place of potentially supersingular reduction, which corresponds to the double zero of $a_1(t)$. Then this fiber has either additive reduction and we are in case (1)
or else this fiber has good supersingular reduction and we are in case (2). In this latter case \((\sigma_2 \cdot \sigma_0) \geq 1\), translation by \(\sigma_2\) fixes the whole supersingular fiber and the quotient \(X/G\) is rational by Proposition 5.1. In particular, \(X\) is unirational and thus \(h = \infty\).

If the fibration has constant \(j\)-invariant then the only singular fibers can be of type \(I^*_4+8d\) for some \(d \geq 0\), which have minimal discriminant \(12d + 12\) and Swan conductor \(2 + 4d\) by \[L-S08\] Proposition 15.1. Since the minimal discriminants add up to \(c_2(X) = 24\) and there is at most one additive fiber it has to be of type \(I^*_4,2\).

**CASE 2:** assume that \(a_1(t)\) has two distinct zeros. From \[Ar74\] Theorem 2.12 we obtain \(h = 1\). In particular, \(X/G\) is a K3 surface and the \(G\)-action has one or two fixed points by Proposition 5.1.

In the case where \(j(t)\) is constant it has to be a unit at both simple zeros of \(a_1(t)\), i.e., \(\Delta\) has a zero of order 12 at both places. Depending on whether the discriminant is minimal, the reduction at such a place is either good or of type \(I^*_4\) by \[L-S08\] Proposition 15.1. Since the sum of the minimal discriminants is equal to \(c_2(X) = 24\) we must have two fibers of type \(I^*_4,2\).

We may thus assume that the fibration does not have constant \(j\)-invariant.

First, assume that both places are potentially supersingular. Then both places have additive reduction since the \(G\)-action has two fixed points and would fix a supersingular fiber completely by Proposition 2.1. This is case (3).

Now, assume that one of the zeros of \(a_1(t)\) corresponds to a place with potentially ordinary or potentially multiplicative reduction. Not both zeros can belong to places of good or multiplicative reduction since there is at least one potentially supersingular fiber. Let \(t_0\) be the place with potentially good or ordinary reduction.

By \[L-S08\] Section 15] the minimal discriminant at this place equals

\[v(\Delta) = 12 + 12d - 2v(t_0)(j) \geq 12\]

and the reduction is of type \(I^*_4+8d-2v(j)\). As explained in loc. cit. such an additive and not potentially supersingular fiber arises a quadratic twist from an elliptic fibration \(X' \to E^1\) that has semi-stable reduction at the place corresponding to the \(I^*_4+8d-2v(j)\)-fiber. This quadratic twist may be arranged in such a way that the other fibers are not affected, which implies \(c_2(X') < c_2(X)\). Since \(j(X) = j(X')\) the fibration still has non-constant \(j\)-invariant after twisting and thus \(c_2(X') \neq 0\). In particular \(c_2(X') = 12\), i.e., \(X'\) is a rational surface. This implies that the minimal discriminant of the fiber of type \(I^*_4+8d-2v(j)\) equals 12, i.e., \(d = 0\) and \(v(j) = 0\) and we get a fiber of type \(I^*_4,2\) with potentially ordinary reduction. Also the potentially supersingular fiber must have additive reduction or else the \(G\)-action would fix a supersingular fiber, but we already now that \(G\) fixes only two points.

\[\square\]

**Remark 5.3.** Compared to characteristic \(p \geq 3\) the new, “exotic” classes are fibrations with constant \(j\)-invariant, as well as classes (2) and (4) in the case of non-constant \(j\)-invariant. We will classify them completely in Section 6. There, it will turn out that they are supersingular if only if the elliptic fibration arises as Frobenius pullback from a rational surface, as predicted by Artin’s Conjecture 3.3.
In characteristic \( p \geq 3 \), an elliptic K3 surface with \( p \)-torsion section that has precisely one fiber with potentially supersingular reduction is supersingular, unirational and its elliptic fibration arises as Frobenius pullback from a rational elliptic surface.

The following examples have one additive and potentially supersingular fiber, i.e., the height \( h \) of the formal Brauer group is at least 2 by Theorem 3.5. However, these surfaces are not supersingular and their elliptic fibrations arise as Frobenius pullback from K3 surfaces, i.e., the alternative of Theorem 3.12 does not hold in characteristic 2.

**Proposition 5.4.** Let \( X \to \mathbb{P}^1 \) be the elliptic K3 surface given by the Weierstraß equation

\[
y^2 + t^2xy + t^2y = x^3 + (1 + t)x^2 + t.
\]

The elliptic fibration is a 4-fold Frobenius pullback. More precisely,

| \( j \) - invariant | singular fibers | type | height of \( \hat{Br}_X \) |
|---------------------|----------------|------|--------------------------|
| \( X^{(1/2)} \)    | \( t^{16} \)   | \( I_{16}, I_{16} \) | K3 | 2                        |
| \( X^{(1/4)} \)    | \( t^8 \)      | \( I_{4,6}, I_{8} \) | K3 | 2                        |
| \( X^{(1/8)} \)    | \( t^4 \)      | \( I_{8,6}, I_{4} \) | K3 | 2                        |
| \( X^{(1/16)} \)   | \( t^2 \)      | \( I_{10,6}, I_{2} \) | K3 | 2                        |

The elliptic fibrations of \( X \), \( X^{(1/2)} \), \( X^{(1/4)} \) and \( X^{(1/8)} \) possess 2-torsion sections and arise as Frobenius pullbacks from K3 surfaces.

**Proof.** The computation of the singular fibers is straight forward and left to the reader. Moreover, all surfaces are K3 surfaces and since they are related by Frobenius pullbacks the heights of their formal Brauer groups coincide. Thus, it suffices to compute the formal Brauer group of one surface and we take the one of the statement of the proposition. Making a coordinate change to achieve \( a_2 = 0 \) in the Weierstraß equation we can apply [Ar74, Theorem (2.12)] and obtain \( h = 2 \). □

These surfaces belong to class (1) with non-constant \( j \)-invariant of Theorem 5.2. We shall see further examples with \( h = 2 \) and iso-trivial fibrations in the next section.

### 6. THE EXOTIC CLASSES IN CHARACTERISTIC 2

This sections deals with the classes of Theorem 5.2 that do no exist for \( p \geq 3 \).

**Fibrations with constant \( j \)-invariant.** This class coincides with the Kummer surfaces studied by Shioda in [Sh74a]:

**Proposition 6.1.** Every elliptically fibered K3 surface with constant \( j \)-invariant and 2-torsion section in characteristic 2 arises as minimal desingularization of

\[
(E_1 \times E_2)/G \to E_2/G \cong \mathbb{P}^1,
\]

where \( E_1 \) is an ordinary and \( E_2 \) is an arbitrary elliptic curve, and \( G \cong \mathbb{Z}/2\mathbb{Z} \) acts via the sign involution on each factor.
Conversely, for any two elliptic curves \( E_1, E_2 \), where \( E_1 \) is ordinary, a minimal desingularization of (7) yields an elliptic K3 surface with constant \( j \)-invariant and 2-torsion section. More precisely,

| \( E_2 \)   | singular fibers | \( \rho \) | \( h \) |
|------------|-----------------|----------|--------|
| ordinary   | \( 2 \times I_{1,2}^* \) | \( 18 \leq \rho \leq 20 \) | 1      |
| supersingular | \( I_{12,6}^* \) | 18      | 2      |

In particular, these surfaces cannot be supersingular, and \( h = 2 \) is possible.

**Proof.** Since the generic fiber is ordinary, such a surface is a quadratic twist of a trivial fibration. Thus, \( X \) arises via \( (E_1 \times C)/G = C/G \), where \( \varphi : C \rightarrow \mathbb{P}^1 \) is an Artin–Schreier morphism of degree 2. The group \( G = \mathbb{Z}/2\mathbb{Z} \) acts via the sign involution on \( E_1 \) and via the Galois action on \( C \). From Theorem 5.2 we know that the fibration \( X \rightarrow \mathbb{P}^1 \) has either one fiber of type \( I_{12,6}^* \) or two fibers of type \( I_{1,2}^* \). From [L-S08, Section 15] it then follows that \( \varphi \) is ramified in one point with four non-trivial higher ramification groups (the \( I_{12,6}^* \)-case) or in two points with two non-trivial higher ramification groups (the \( 2 \times I_{1,2}^* \)-case). In both cases \( C \) is an elliptic curve, and the Galois action coincides with the sign involution. In case, \( \varphi \) is ramified in one point, its \( p \)-rank is trivial [Cr84, Corollary 1.8], and thus \( C \) is supersingular. Similarly, if \( \varphi \) is ramified in two points then \( C \) is ordinary.

Conversely, it is easy to see that this construction yields elliptic K3 surfaces with 2-torsion section.

The rank \( \rho \) of the Néron–Severi group has been determined in [Sh74a]. We set \( A = E_1 \times E_2 \), where \( E_1 \) is an ordinary elliptic curve. Then the height of \( \hat{\text{Br}}(A) \) is 1 or 2 depending on whether \( E_2 \) is ordinary or supersingular [G-K03, Lemma 6.2]. Since \( A/G \) has only rational singularities [Sh74a] we can conclude as in the proof of [G-K03, Theorem 6.1] that the formal Brauer groups of \( A/G \) and \( X \) are isomorphic. Since \( A \rightarrow A/G \) is an Artin-Schreier covering of degree 2, there is a non-trivial trace map, and as in the proof of [G-K03, Theorem 6.1] we conclude that the formal Brauer groups of \( A \) and \( A/G \) are isomorphic. \( \Box \)

**Semi-stable fibrations.** Class (2) with non-constant \( j \)-invariant in Theorem 5.2 is closely related to rational elliptic surfaces. These surfaces are unirational and supersingular.

**Proposition 6.2.** Let \( X \rightarrow \mathbb{P}^1 \) be an elliptic K3 surface with 2-torsion section in characteristic 2 whose fibration is semi-stable. Then \( X \rightarrow \mathbb{P}^1 \) arises as Frobenius pullback from a rational elliptic surface \( Y \rightarrow \mathbb{P}^1 \) with semistable fibration.

Conversely, if \( Y \rightarrow \mathbb{P}^1 \) is a rational elliptic surface with semistable fibration, then its Frobenius pullback yields an elliptic K3 surface with 2-torsion section.

**Proof.** We have seen in the proof of Theorem 5.2 that \( Y \rightarrow \mathbb{P}^1 \) is rational. Moreover, the elliptic fibration on \( Y \) must be semi-stable because the one on \( X \) is. We leave the converse to the reader. \( \Box \)

**Remark 6.3.** In [Ito09, Section 4], an 8-dimensional family of semistable rational elliptic surfaces related to the deformation of an \( E_8^4 \)-singularity is constructed. Via
Frobenius pullback we obtain an 8-dimensional family of semistable elliptic K3 surfaces with Artin invariants $1 \leq \sigma_0 \leq 9$, see [Ito09, Theorem 5.2].

**Additive and potentially ordinary fibers.** Also, Class (4) with non-constant $j$-invariant in Theorem 5.2 is closely related to rational elliptic surfaces. However, being ordinary, these surfaces are neither unirational nor supersingular.

In order to state the result, let us introduce the following notation: For a point $Q \in \mathbb{P}^1$ denote by $\psi_Q : \mathbb{P}^1 \to \mathbb{P}^1$ the Artin–Schreier morphism of degree 2 that is branched over $Q$.

**Proposition 6.4.** Let $X \to \mathbb{P}^1$ be an elliptic K3 surface with non-constant $j$-invariant and 2-torsion section in characteristic 2 that possesses a potentially ordinary fiber of type $I_{2,4}^*$, say, at $Q \in \mathbb{P}^1$. Then there exists a rational elliptic surface $X' \to \mathbb{P}^1$ with 2-torsion section and good ordinary reduction at $Q$ such that $X$ arises as quadratic twist from $X'$ via $\psi_Q$.

Conversely, if $X' \to \mathbb{P}^1$ is a rational elliptic surface with 2-torsion section and with good ordinary reduction at $Q \in \mathbb{P}^1$ then the quadratic twist of $X'$ with respect to $\psi_Q$ yields an elliptic K3 surface with 2-torsion section and a potentially ordinary fiber of type $I_{2,4}^*$ above $Q$.

**Proof.** From [L-S08, Section 15] we see that the $I_{2,4}^*$-fiber arises from an elliptic fibration $X' \to \mathbb{P}^1$ as quadratic twist $\psi : C \to \mathbb{P}^1$, which is totally ramified at $Q$. If $Q$ is the only branch point, which we can and will assume, then $X'$ has the same singular fibers as $X$ but has good reduction at $Q$. In particular, $c_2(X') < c_2(X)$, and so $X' \to \mathbb{P}^1$ is a rational elliptic surface. Since the reduction type of $X \to \mathbb{P}^1$ at $Q$ has Swan conductor $\delta = 2$, we conclude that $\psi$ has two non-trivial higher ramification groups, i.e., $\psi = \psi_Q$. We leave the converse to the reader. $\square$

**Remark 6.5.** Note that $X$ and $X'$ have the same numbers and types of singular fibers (including Swan conductors) except for the $I_{2,4}^*$-fiber at $Q$ which is induced on $X$ by the quadratic twist.

**Example 6.6.** To illustrate this case with an example, consider the universal elliptic curve $E \to \operatorname{Ig}(4)_{\text{ord}}$. Then $E^{(2)} \to \operatorname{Ig}(4)_{\text{ord}}$ corresponds to a rational elliptic surface with 2-torsion section. Twisting with respect to $\psi_Q : \mathbb{P}^1 \to \mathbb{P}^1$, where $Q \in \operatorname{Ig}(4)_{\text{ord}} \subset \mathbb{P}^1$ corresponds to the ordinary $j$-value $j = 1$, we obtain

$$y^2 + xy = x^3 + \frac{1}{t+1}x^2 + t^2.$$  

This is an elliptic K3 surface with 2-torsion section, having singular fibers of type $I_2, III^*_1$ (potentially supersingular) and $I_{2,4}^*$ (potentially ordinary).

7. **SECTIONS OF ORDER 4 AND 8**

In this section we classify elliptic K3 surfaces with 8- and 4-torsion sections in characteristic 2, using again the Igusa curves. They turn out to belong to classes (1) and (3) with non-constant $j$-invariant of Theorem 5.2. It also turns out that Theorem 3.9 and Theorem 3.12 hold for them. Thus, these surfaces behave like the ones in characteristic $p \geq 3$. 
**8-torsion sections.** The following result is proved as Theorem 4.1, which is why we leave it to the reader:

**Theorem 7.1.** There exists only one elliptic K3 surface $X \to \mathbb{P}^1$ with 8-torsion section in characteristic 2 up to isomorphism. It has the following invariants:

| singular fibers | $\sigma_0$ | $MW^\circ(X)$ | $MW(X)$ |
|-----------------|------------|---------------|----------|
| $I_{1,1}^*, 2 \times I_8$ | 1 | $A_1(2)$ | $A_1^*(2) \oplus (\mathbb{Z}/8\mathbb{Z})$ |

The Weierstraß equation is given by the following:

$$y^2 + t^2xy = x^3 + x + t^4.$$ 

In particular, it is the unique supersingular K3 surface with Artin invariant $\sigma_0 = 1$.

**Remark 7.2.** Having Artin invariant $\sigma_0 = 1$, it is a generalized Kummer surface [Schr07]. An explicit Weierstraß equation is given in Proposition 1.3, but we note that uniqueness and an equation have already been obtained in [Schw05, Examples 2.4].

**4-torsion sections.** As before, we denote by $O \in Ig(4)_{\text{ord}}$ the unique supersingular point. Since our proof works as for $p = 3$ or $p = 7$, we leave it to the reader and only state the result:

**Theorem 7.3.** In characteristic 2, the classifying morphism $\varphi$ for an elliptic K3 surface with 4-torsion section is finite of degree $2 \leq \deg \varphi \leq 4$. More precisely,

1. $\deg \varphi = 2$, $\varphi$ is separable and $\varphi^{-1}(O)$ consists of two points, or
2. $\deg \varphi = 3$ and $\varphi^{-1}(O)$ consists of one point or two points, or
3. $\deg \varphi = 4$ and $\varphi^{-1}(O)$ consists of one point or two points with ramification index $e = 2$ (wildly ramified).

Conversely, if $\varphi$ is as above then the associated elliptic fibration with 4-torsion section is a K3 surface.
More precisely, depending on the branch points we obtain the following table, where \( X = Y^{(4)} \) and the type of \( Y \) is tabled in the last column.

| \( \deg \varphi \) | singular fibers | \( \dim \) | \( h \) | \( Y \) |
|---------------------|-----------------|---------|-------|--------|
| 2                   | \( \varphi \) separable: |         |       |        |
|                     | \( 2 \times I^{*}_{1,1} \) & \( 2 \times I_{4} \) | 2       | 1     | K3     |
|                     | \( 2 \times I^{*}_{1,1} \) & \( I_{8} \)     | 1       | 1     | K3     |
| 3                   | \( I^{*}_{1,1,III_{1}} \) & \( 3 \times I_{4} \) | 3       | 1     | K3     |
|                     | \( I^{*}_{1,1,III_{1}} \) & \( I_{8}, I_{4} \) | 2       | 1     | K3     |
|                     | \( I^{*}_{1,1,III_{1}} \) & \( I_{12} \)     | 1       | 1     | K3     |
|                     | \( I^{*}_{3,3} \) & \( 3 \times I_{4} \) | 2       | \( \infty \) | rational |
|                     | \( I^{*}_{3,3} \) & \( I_{8}, I_{4} \) | 1       | \( \infty \) | rational |
|                     | \( I^{*}_{3,3} \) & \( I_{12} \)     | 0       | \( \infty \) | rational |
| 4                   | \( \varphi \) separable: |         |       |        |
|                     | \( 2 \times III_{1} \) & \( 4 \times I_{4} \) | 4       | 1     | K3     |
|                     | \( 2 \times III_{1} \) & \( I_{8}, 2 \times I_{4} \) | 3       | 1     | K3     |
|                     | \( 2 \times III_{1} \) & \( I_{12}, I_{4} \) | 2       | 1     | K3     |
|                     | \( I^{*}_{0,2} \) & \( 4 \times I_{4} \) | 3       | \( \infty \) | rational |
|                     | \( I^{*}_{0,2} \) & \( I_{8}, 2 \times I_{4} \) | 2       | \( \infty \) | rational |
|                     | \( I^{*}_{0,2} \) & \( I_{12}, I_{4} \) | 1       | \( \infty \) | rational |
| \( \varphi \) inseparable but not purely inseparable: |         |       |       |        |
|                     | \( 2 \times III_{1} \) & \( 2 \times I_{8} \) | 2       | 1     | K3     |
|                     | \( 2 \times III_{1} \) & \( I_{16} \)     | 1       | 1     | K3     |
|                     | \( I^{*}_{1,1} \) & \( 2 \times I_{8} \) | 0       | \( \infty \) | rational |
| \( \varphi \) purely inseparable: |         |       |       |        |
|                     | \( I^{*}_{1,1} \) & \( I_{16} \)     | 0       | \( \infty \) | rational |

**Remark 7.4.** There are two unique surfaces in this list:

- If \( \deg \varphi = 4 \), \( \varphi \) is inseparable but not purely inseparable and \( \varphi \) totally ramified over \( O \), we obtain the unique surface with \( 8 \)-torsion section.
- If \( \deg \varphi = 4 \) and \( \varphi \) is purely inseparable, the resulting elliptic K3 surface is extremal. Such surfaces in characteristic \( p = 2, 3 \) have been studied and classified in [Ito02]. In fact, our surface appears in Table 1 of loc. cit.

Similar to characteristic 3, the generic supersingular surfaces can be related to deformations of singularities. Namely,

\[
y^2 + txy = x^3 + t^5 + r_4 t^4 + r_3 t^3 + r_2 t^2
\]

defines a 3-dimensional family of rational elliptic surfaces. This family arises as subfamily of the semi-universal deformation of a \( E_8^{(4)} \)-singularity. Then all \( Y^{(4)}_{\lambda} \rightarrow \mathbb{P}^1 \) are elliptic K3 surfaces with \( 4 \)-torsion sections. We leave the following result, whose proof is analogous to the one of Theorem 4.10 to the reader.

**Theorem 7.5.** Every (Shioda-)supersingular K3 surface with Artin invariant \( \sigma_0 \leq 4 \) in characteristic 2 possesses an elliptic fibration with \( 4 \)-torsion section.
The complete list of these surfaces is given by the following table.

| deg $\varphi$ | singular fibers | dim | $\sigma_0$ | $\text{MW}^0(X)$ | $\text{MW}(X)$ |
|---------------|-----------------|-----|------------|------------------|-----------------|
| $3$           | $I_3^*, I_4^*$  | $2$ | $3$        | $D_4(2)$         | $D_4^*(2) \oplus \mathbb{Z}/4\mathbb{Z}$ |
| $3$           | $I_8, I_4^*$    | $1$ | $2$        | $A_3$           | $A_3^* \oplus \mathbb{Z}/4\mathbb{Z}$ |
| $3$           | $I_4^* I_{12}$  | $0$ | $1$        | $A_2$           | $A_2^* \oplus \mathbb{Z}/4\mathbb{Z}$ |
| $4$ $\varphi$ separable: |                       |     |            |                  |                 |
| $I_{0,2}^*$   | $4 \times I_4$  | $3$ | $4$        | $D_4(2)$         | $D_4^*(2) \oplus \mathbb{Z}/4\mathbb{Z}$ |
| $I_{0,2}^*$   | $I_8, 2 \times I_4$ | $2$ | $3$        | $A_3(2)$        | $A_3^*(2) \oplus \mathbb{Z}/4\mathbb{Z}$ |
| $I_{0,2}^*$   | $I_{12}, I_{4}$ | $1$ | $2$        | $A_2(2)$        | $A_2^*(2) \oplus \mathbb{Z}/4\mathbb{Z}$ |
| $\varphi$ inseparable but not purely inseparable: |                       |     |            |                  |                 |
| $I_{1,1}^*$   | $2 \times I_8$  | $0$ | $1$        | $A_1(2)$        | $A_1^*(2) \oplus \mathbb{Z}/8\mathbb{Z}$ |
| $\varphi$ purely inseparable: |                       |     |            |                  | $\{0\}$, $\mathbb{Z}/4\mathbb{Z}$ |

Moreover, from the table above we see that the implications $(4) \Rightarrow (5) \Rightarrow (1)$ of Theorem $3.9$ hold for these surfaces. Also, we see that these surfaces can only be ordinary or supersingular. Thus,

Theorem 7.6. Theorem $3.9$ and Theorem $3.12$ hold for elliptic $K3$ surfaces with $4$-torsion section in characteristic $2$.

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