Special Hermitian structures obtained from hyperbolic suspensions

Anna Fino, Gueo Grantcharov, Misha Verbitsky

Abstract. Motivated by the construction based on topological suspension of a family of compact non-Kähler complex manifolds with trivial canonical bundle given by L. Qin and B. Wang in [QW], we study toric suspensions of balanced manifolds by holomorphic automorphisms. In particular, we show that toric suspensions of Calabi-Yau manifolds are balanced. We also prove that suspensions associated with hyperbolic automorphisms of hyperkähler manifolds do not admit any pluriclosed, astheno-Kähler or p-pluriclosed Hermitian metric. Moreover, we consider natural extensions for hypercomplex manifolds, providing some explicit examples of compact holomorphic symplectic and hypercomplex non-Kähler manifolds. We also show that a modified suspension construction provides examples with pluriclosed metrics.

Contents

1 Introduction  2
2 Hyperkähler manifolds and their automorphisms  3
  2.1 Hyperkähler manifolds and the BBF form  3
  2.2 Classification of automorphisms of hyperkähler manifolds  4
3 Toric suspensions  5
  3.1 Toric suspensions: definition and basic properties  5
  3.2 Hyperbolic suspensions  6
4 Balanced metrics on Calabi-Yau suspensions  7
5 Hyperbolic holomorphically symplectic suspensions  8
  5.1 Hyperbolic holomorphically symplectic suspensions  8
  5.2 Balanced, pluriclosed and LCK Hermitian metrics  9
  5.3 Strongly positive and weakly positive (p,p)-currents  9
  5.4 Hyperbolic automorphisms and Cantat-Ding-Sibony currents  11
6 Examples of suspensions of hyperkähler manifolds  12

1 Anna Fino is partially supported by by Project PRIN 2017 “Real and complex manifolds: Topology, Geometry and Holomorphic Dynamics” and by GNSAGA (Indam), Gueo Grantcharov is supported by a grant from the Simons Foundation (#853269). Misha Verbitsky is partially supported by CNPq - Process 310952/2021-2 and FAPERJ E-26/202.912/2018.
1 Introduction

Finding examples of special non-Kähler metrics on compact complex manifolds has become a question of increasing interest in recent years. It is partly due to developments in Physics related to Hull-Strominger system ([St], [Hu]) and generalized geometry ([Gu1], [Gu2], [Hi]). In [QW] a new class of examples of non-Kähler manifolds with trivial canonical bundle and nice topological properties have been introduced. It is based on the topological suspension construction.

Given in general a smooth manifold \( M \) and a diffeomorphism \( f \) of \( M \), the mapping torus (or suspension) of \( f \) is defined to be the quotient \( M_f \) of the product \( M \times \mathbb{R} \) by the \( \mathbb{Z} \)-action defined by

\[
(p, t) \rightarrow (f^n(p), t + n).
\]

As a consequence \( dt \) defines a nonsingular closed 1-form on \( M_f \) tangent to the fibration

\[
M_f \rightarrow S^1 = \mathbb{R}/\mathbb{Z}.
\]

Moreover, the vector field \( \frac{\partial}{\partial t} \) on \( M \times \mathbb{R} \) defines a vector field on \( M_f \), the suspension of the diffeomorphism \( f \). There is a natural correspondence between the orbits of \( f \) and the trajectories of the vector field. Mapping tori have been used in [Li] to construct examples of co-symplectic and co-Kähler manifolds.

The suspension construction can be extended to complex manifolds in the following way. Given a complex manifold \( M \), a set of commuting holomorphic automorphisms \( f_j, j = 1, \ldots, 2k \), of \( M \) and a lattice \( \Lambda \subset \mathbb{C}^k \) of rank \( 2k \), generated by \( \xi_1, \ldots, \xi_{2k} \), one can define an action of \( \mathbb{Z}^{2k} = \langle \xi_1, \ldots, \xi_{2k} \rangle \) on \( M \times \mathbb{C}^k \) via \( \varphi_j(m, z) = (f_j(m), z + \xi_j) \). The quotient of \( M \times \mathbb{C}^k \) by the action of \( \mathbb{Z}^{2k} \) is called the toric suspension of \( (M, f_1, \ldots, f_{2k}) \). In particular, if \( f \) is an automorphism of a complex manifold \( M \) and \( T^2 = \mathbb{C}/\mathbb{Z}^2 \) an elliptic curve, one can construct the complex suspension of \( f \) as the toric suspension \( S(f) \) of \( M \) associated with the pair \( (f, \text{Id}_M) \). In the present paper we study the metric properties of the constructed manifolds, like the existence of balanced metrics, that is, Hermitian metrics with co-closed fundamental
form. We also extend the construction to produce hypercomplex manifolds with special metric properties.

Using a different construction related to automorphisms of 3-dimensional Sasakian manifolds, we construct suspensions admitting pluriclosed metrics, that is, the Hermitian metrics with $\partial\bar{\partial}$-closed fundamental forms.

In Section 2 and 3 we present the necessary information on hyperkähler manifolds and toric suspension construction. In Section 4 we prove that the complex toric suspension of a balanced manifold $M$ by two commuting holomorphic diffeomorphisms preserving a volume form is balanced. As a corollary we show that if $M$ is a Calabi-Yau manifold and $f$ is an automorphism of $M$ preserving the holomorphic volume form, then the complex suspension $S(f)$ has trivial canonical bundle and admits a balanced metric.

In Section 5 we show that the balanced manifolds constructed using any hyperbolic automorphism of hyperkähler manifolds do not admit any $p$-pluriclosed and locally conformally Kähler (LCK) metric. In Section 6 we recover the construction in [QW] as toric suspension of a Kummer surface and we generalize it to suspension of the Hilbert scheme of points on Kummer surfaces. In Section 7 we discuss the natural extensions of toric suspensions on hypercomplex manifolds and their metric structures. As an application we construct explicit examples of compact holomorphic symplectic and hypercomplex non-Kähler manifolds. The examples are in fact pseudo-hyperkähler and admit quaternionic balanced metric, but no HKT metric.

Finally in Section 8 we show how using automorphisms of Sasakian and Kähler manifolds it is also possible to construct suspensions admitting pluriclosed metrics recovering a recent example constructed in [FP2], as a compact 3-step solvmanifold.

## 2 Hyperkähler manifolds and their automorphisms

Here we introduce the necessary background materials on hyperkähler geometry. We follow [AV], [Bea], [Bes], [Ca], [Ka].

### 2.1 Hyperkähler manifolds and the BBF form

**Definition 2.1:** A hyperkähler manifold is a compact, Kähler, holomorphically symplectic manifold.

**Definition 2.2:** A hyperkähler manifold $M$ is called to be of maximal holonomy (also: simple, or IHS) if $\pi_1(M) = 0$ and $H^{2,0}(M) = \mathbb{C}$.

**Theorem 2.3:** (Bogomolov’s decomposition [Bo])
Any hyperkähler manifold admits a finite covering which is a product of a torus and several hyperkähler manifolds of maximal holonomy.

**Remark 2.4:** From now on all hyperkähler manifolds are assumed to be of maximal holonomy.

**Theorem 2.5:** (Fujiki [Fu])
Let \( \eta \in H^2(M, \mathbb{Z}) \), and \( \dim M = 2n \), where \( M \) is hyperkähler. Then \( \int_M \eta^{2n} = cq(\eta, \eta)^n \), for some primitive integer quadratic form \( q \) on \( H^2(M, \mathbb{Z}) \), and \( c > 0 \) a rational number.

**Definition 2.6:** This primitive integral quadratic form \( q \) on \( H^2(M, \mathbb{Z}) \) is called Bogomolov-Beauville-Fujiki form, or BBF form. It is defined by the Fujiki’s relation uniquely, up to a sign. The sign is determined from the following formula (Bogomolov, Beauville, see [Bea])

\[
\lambda q(\eta, \eta) = \int_M \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} - \frac{n-1}{n} \left( \int_M \eta \wedge \Omega^{n-1} \wedge \overline{\Omega} \right) \left( \int_M \eta \wedge \Omega^n \wedge \overline{\Omega}^{n-1} \right),
\]

where \( \Omega \) is the holomorphic symplectic form on \( M \) and \( \lambda > 0 \).

**Remark 2.7:** The BBF form \( q \) has signature \((3, b_2 - 3)\) when extended on \( H^2(M, \mathbb{R}) \). It is negative definite on primitive forms, and positive definite on \( \langle \Omega, \overline{\Omega}, \omega \rangle \), where \( \omega \) is a Kähler form. On \((1,1)\)-forms \( \eta \) it can be written as \( q(\eta, \eta) = \text{const} \int_M \eta \wedge \eta \wedge \Omega^{n-1} \wedge \overline{\Omega}^{n-1} \).

### 2.2 Classification of automorphisms of hyperkähler manifolds

**Remark 2.8:** The indefinite orthogonal group \( O(m, n), m, n > 0 \), is the Lie group of all linear transformations of an \( l \)-dimensional real vector space that leave invariant a nondegenerate, symmetric bilinear form \( q \) of signature \((m, n)\), where \( l = m + n \). \( O(m, n) \) has 4 connected components. We denote the connected component of 1 by \( SO^+(m, n) \). We call a vector \( v \) positive if its square is positive, i.e. if \( q(v, v) > 0 \).

**Definition 2.9:** Let \( V \) be a real vector space of dimension \( n + 1 \) with a quadratic form \( q \) of signature \((1, n)\), \( \text{Pos}(V) = \{ x \in V \mid q(x, x) > 0 \} \) its positive cone, and \( \mathbb{P}^+V \) projectivization of \( \text{Pos}(V) \). Denote by \( g \) any \( SO(V) \)-invariant Riemannian structure on \( \mathbb{P}^+V \) (it is easy to see that \( g \) is unique up
to a constant multiplier). Then \((\mathbb{P}^1 V, g)\) is called **hyperbolic space**, and the group \(SO^+(V)\) the **group of oriented hyperbolic isometries**.

**Theorem 2.10:** Let \(n > 0, \) and \(\alpha \in SO^+(1, n)\) is an isometry acting on \(V\). Then one and only one of the following three cases occurs

(i) \(\alpha\) has an eigenvector \(x\) with \(q(x, x) > 0\) (\(\alpha\) is "**elliptic isometry**");

(ii) \(\alpha\) has an eigenvector \(x\) with \(q(x, x) = 0\) and a real eigenvalue \(\lambda_x\) satisfying \(|\lambda_x| > 1\) (\(\alpha\) is "**hyperbolic isometry**", or **loxodromic isometry**);

(iii) \(\alpha\) has a unique eigenvector \(x\) with \(q(x, x) = 0\) with eigenvalue 1, and no fixed points on \(\mathbb{P}^1 V\) (\(\alpha\) is "**parabolic isometry**").

For a proof see for instance \([K]\).

**Remark 2.11:** All eigenvalues of elliptic and parabolic isometries have absolute value 1. Hyperbolic and elliptic isometries are semisimple (that is, diagonalizable over \(\mathbb{C}\)).

**Definition 2.12:** Notice that any complex automorphism of a hyperkähler manifold acts by isometry on the space \(H^{1,1}(M, \mathbb{R})\) with the BBF metric which has signature \((1, b_2 - 2)\). A complex automorphism \(f\) of a hyperkähler manifold \(M\) is called **elliptic (parabolic, hyperbolic)** if the induced action \(f^*\) of \(f\) is elliptic (parabolic, hyperbolic) on \(H^{1,1}(M, \mathbb{R})\).

Further on we shall need the following lemma.

**Lemma 2.13:** Let \(M\) be a hyperkähler manifold, \(f : M \rightarrow M\) a hyperbolic automorphism, and \(\eta \in H^{1,1}(M, \mathbb{R})\) a non-zero \(f^*\)-invariant class. Then \(q(\eta, \eta) < 0\).

**Proof:** Let \(v_+, v_-\) be eigenvectors of \(f^*\) with the real eigenvalues \(\lambda > 1\) and \(\lambda^{-1}\). Then any invariant vector of \(f^*\) belongs to \(\langle v_+, v_-\rangle\). However, \(q\) is negative definite on the space spanned by the other eigenvectors because signature of \(q\) on \(H^{1,1}(M, \mathbb{R})\) is \((1, b_2 - 2)\).  

### 3 Toric suspensions

#### 3.1 Toric suspensions: definition and basic properties

**Definition 3.1:** Let \(M\) be a complex manifold, and \(f_1, ..., f_2k \in \text{Aut}(M)\) a set of commuting holomorphic automorphisms of \(M\). Let \(\Lambda \subset \mathbb{C}^k\) be a lattice
of rank $2k$, generated by $\xi_1, \ldots, \xi_{2k}$. Define an action of $\mathbb{Z}^{2k} = \langle \varphi_1, \ldots, \varphi_{2k} \rangle$ on $M \times \mathbb{C}^k$ via $\varphi_j(m, z) = (f_j(m), z + \xi_j)$. In other words, $\mathbb{Z}^{2k}$ acts on $\mathbb{C}^k$ as a shift by the corresponding element of $\Lambda$ and on $M$ as an automorphism obtained as an appropriate product of $f_i$. The quotient $(M \times \mathbb{C}^k)/\mathbb{Z}^{2k}$ is called the toric suspension of $(M, f_1, \ldots, f_{2k})$.

**Remark 3.2:** The toric suspension is clearly complex analytic, holomorphically fibered over the torus $\mathbb{C}^k/\Lambda$, but not necessarily Kähler.

**Theorem 3.3:** Let $S(M, f_1, \ldots, f_{2k})$ be a toric suspension, with $M$ a compact Kähler manifold. Then $S(M, f_1, \ldots, f_{2k})$ is Kähler if and only if there is a Kähler class $[\omega] \in H^{1,1}(M)$ such that $f^*_i([\omega]) = [\omega]$.

**Proof:** See the proof of Theorem 3.4.1 in the paper [Ma].

### 3.2 Hyperbolic suspensions

The following definition is motivated by the classification of the automorphism groups of hyperbolic manifolds, such as a K3 surface.

**Definition 3.4:** Let $f : M \rightarrow M$ be an automorphism of a compact complex manifold of Kähler type. We say that $f$ is a **hyperbolic automorphism** if the action of $f$ on $H^{1,1}(M, \mathbb{R})$ has a unique eigenvector $\eta$ with an eigenvalue $f^*\eta = \lambda \eta$ such that $\lambda > 1$.

We list some immediate properties of hyperbolic automorphisms.

**Proposition 3.5:** Let $f : M \rightarrow M$ be a hyperbolic automorphism of a compact complex manifold of Kähler type, and $\eta \in H^{1,1}(M, \mathbb{R})$ an eigenvector with an eigenvalue $f^*\eta = \lambda \eta$ such that $\lambda > 1$. Denote by $\text{Kah}(M) \subset H^{1,1}(M, \mathbb{R})$ the Kähler cone of $M$. Then

(i) $\eta$ belongs to the closure of the Kähler cone.

(ii) $\int_M \eta^n = 0$, where $n = \dim_{\mathbb{C}} M$. In particular, $\eta \notin \text{Kah}(M)$.

(iii) The action of $f$ on $\text{Kah}(M)$ has no fixed points.

**Proof:** Let $S \subset H^{1,1}(M, \mathbb{R})$ be the sum of all eigenspaces of $f$ on $H^{1,1}(M, \mathbb{R})$ not containing $\lambda$. Since $\lambda$ is the biggest eigenvalue, for any $v \in H^{1,1}(M, \mathbb{R}) \setminus S$, one has $\lim_i \frac{(f^*)_i(v)}{\lambda^i} = \eta$. Since $\text{Kah}(M)$ is open, this is also true for general Kähler class $\omega$. We obtained $\eta$ as a limit of Kähler forms. This proves (i).

To see (i), we notice that $\int_M \eta^n = \int_M f^*(\eta)^n = \lambda^n \int_M \eta^n$. 

---

_A. Fino, G. Grantcharov, M. Verbitsky_  
Special Hermitian structures
To obtain (iii), assume that $f$ fixes a Kähler class $\omega$ on $M$. Using compactness of the space of currents, we may represent a limit of Kähler classes $\eta$ by a positive, closed current $\Theta$. Then

$$\int_M \Theta \wedge \omega^{n-1} = \int_M f^* \Theta \wedge (f^* \omega)^{n-1} = \lambda \int_M \Theta \wedge \omega^{n-1},$$

giving a contradiction. ■

**Remark 3.6:** Since an automorphism of a hyperkähler manifold preserves its Kähler cone, and the eigenvector $x$ with $|\lambda_x| > 1$ sits on the boundary of the Kähler cone (Proposition 3.5), the number $\lambda_x$ is positive.

**Definition 3.7:** Let $f : M \to M$ be an automorphism of a compact complex manifold of Kähler type, and $T^2 = \mathbb{C}/\mathbb{Z}^2$ an elliptic curve. Consider a toric suspension $S(f)$ of $M$ associated with the pair $(f, \text{Id}_M)$. This manifold is called a **complex suspension of $f$**. We call $S(f)$ a **hyperbolic suspension** if $f$ is hyperbolic.

**Remark 3.8:** The toric suspension $S(f)$ of $M$ associated with the pair $(f, \text{Id}_M)$ can be viewed as the product manifold $M_f \times S^1$, where $M_f$ is the mapping torus of $M$ by $f$ obtained as the quotient of $M \times \mathbb{R}$ by the $\mathbb{Z}$-action $(p, t) \to (f^{-n}(p), t + n)$.

If $(t, s)$ are local coordinates on $\mathbb{R} \times S^1$, then $\frac{\partial}{\partial t}$ on $M \times \mathbb{R}$ defines a vector field $X_f$ on $S$ called the **suspension vector field** of $f$ (see [Gol]). Note that the vector field $X_f - i \frac{\partial}{\partial s}$ on $S$ is holomorphic. Moreover, the vector fields $X_f, \frac{\partial}{\partial s}$ provide a natural splitting $TS(f) = T_{vert} S \oplus \pi^* TE$, which defines a flat Ehresmann connection on $S(f)$, which we call the **standard connection**. We will denote by $\theta$ the associated connection 1-form such that $\theta + \sqrt{-1} ds$ is a $(1,0)$-form with respect to the complex structure on $S(f)$.

**Remark 3.9:** By Proposition 3.5 (iii) and Theorem 3.3, a hyperbolic suspension is never Kähler.

### 4 Balanced metrics on Calabi-Yau suspensions

Balanced metrics were introduced in [Mic]. For further properties and examples see e.g. [AB], [AB2] and [FLY].

**Definition 4.1:** Let $(M, I, h)$ be a complex Hermitian manifold, $\dim_{\mathbb{C}} M = n$, and $\omega$ the fundamental $(1,1)$-form associated to $h$. We say that $h$ is **balanced** if $\omega^{n-1}$ is closed.
The main result of the present Section is the following theorem.

**Theorem 4.2:** Let $M$ be a balanced compact manifold of complex dimension $n$ and $f_1, f_2 \in \text{Aut}(M)$ two commuting holomorphic automorphisms preserving a volume form $V$. Denote by $\pi : S \to E$ the corresponding suspension over an elliptic curve $E$. Assume that $M$ is balanced. Then $S$ is also balanced.

**Proof.** Let $\omega_E$ be a Kähler form on $E$. Recall that a smooth fibration $\pi : S \to E$ over an elliptic curve is called essential ([Mic]) if $\pi^*(\omega_E)$ is not Aeppli exact, i.e. $\pi^*(\omega_E)$ cannot be equal to $\overline{\partial} \alpha + \partial \alpha$, for every $(1,0)$-form $\alpha$. Michelsohn ([Mic]) proves that the total space $S$ of an essential fibration with balanced fibers over a complex curve is balanced. To prove Theorem 4.2 it remains only to show that $\pi^*(\omega_E)$ is not Aeppli exact. By contradiction assume that $\pi^*(\omega_E)$ is Aeppli exact.

Since $V$ is $f_j$-invariant, $j = 1, 2$, we may extend $V$ to a form $V_h$ on $S$ vanishing on horizontal vector fields of this Ehresmann connection. Then the form $V_h$ is of type $(n, n)$, positive and closed. Since $V_h$ vanishes on any horizontal vector, the form $\pi^*(\omega_E) \wedge V_h$ is of maximal degree and positive, so $\int_S \pi^*(\omega_E) \wedge V_h > 0$. But by the assumption that $\pi^*(\omega_E)$ is Aeppli exact and Stokes Theorem we have $\int_S \pi^*(\omega_E) \wedge V_h = 0$, a contradiction. ■

5 Hyperbolic holomorphically symplectic suspensions

5.1 Hyperbolic holomorphically symplectic suspensions

**Definition 5.1:** Let $M$ be a hyperkähler manifold and $f : M \to M$ a hyperbolic automorphism preserving the holomorphic symplectic form. Denote by $S$ the corresponding hyperbolic suspension, fibered over $T^2$ with fiber $M$. Then $S$ is called a hyperbolic holomorphically symplectic suspension.

**Proposition 5.2:** Let $S$ be a hyperbolic holomorphically symplectic suspension or Calabi-Yau hyperbolic suspension. Then $S$ is balanced and non-Kähler Calabi-Yau.

**Proof:** In both cases there exists an invariant non-vanishing holomorphic section $\Theta$ of the canonical bundle of $M$. Therefore, $V := \Theta \wedge \overline{\Theta}$ is a $f$-invariant volume on $M$. By Theorem 4.2 $S$ is balanced. By Remark 3.9
S is non-Kähler. Moreover, the form \((\theta + \sqrt{-1} ds) \wedge \Theta\) (Remark 3.8) is a non-vanishing holomorphic section of the canonical bundle of S.

### 5.2 Balanced, pluriclosed and LCK Hermitian metrics

The study of special Hermitian metrics posed also the question of compatibility between different structures of non-Kähler type. We recall the conjecture in [FV] according to which a compact complex manifold admitting both a pluriclosed, i.e. whose Hermitian form \(\omega\) satisfies \(dd^c \omega = 0\), and a balanced metric is Kähler. This has been already proven for specific cases in the papers [Ve14, Ch, FV, FLY, O, FP, FS]. A similar question was posed in [STW] (see also [Fe]) for a compact complex manifold of complex dimension \(n\) admitting a balanced metric and an astheno-Kähler metric, i.e. whose Hermitian form satisfies \(dd^c \omega^{n-2} = 0\). A negative answer to this question was given in [FGV, LU]. For conjectures related to the existence of locally conformally Kähler metrics - the ones that satisfies \(d\omega = \theta \wedge \omega\), see the book [OV].

Based on the previous discussion one can formulate the following general conjecture:

**Conjecture 5.3:** Let \(X\) be a compact complex manifold, \(n := \dim \mathbb{C} X > 2\). Assume that two of the following assumptions occur.

(i) \(X\) admits a Hermitian form \(\omega\) which is locally conformally Kähler, that is, satisfies \(d\omega = \theta \wedge \omega\).

(ii) \(X\) admits a Hermitian form \(\omega\) which is balanced

(iii) \(X\) admits a Hermitian form \(\omega\) which is \(p\)-pluriclosed, that is, satisfies \(dd^c(\omega^p) = 0\), for \(p = 1, 2, \ldots, n-3\) if \(n > 3\) or for \(p = 1\) if \(n = 3\).

Then \(X\) admits a Kähler structure.

In this section, we prove this conjecture when \(X\) is a suspension of a hyperkähler manifold \(M\) associated with a hyperbolic automorphism of \(M\). The non-existence of locally conformally Kähler metrics on these examples follows from Proposition 37.8 in [OV].

### 5.3 Strongly positive and weakly positive \((p, p)\)-currents

Here we recall that a \((p, p)\)-current on a complex manifold \(X\) is an element of the Frechet space dual to the space of \((n-p, n-p)\) complex forms
Λ^{n-p,n-p}(X). In the compact case, the space of \((p,p)\)-currents can be identified with the space of \((p,p)\)-forms with distribution coefficients and the duality is given by integration. So for any \((p,p)\)-current \(T\) and a form \(\alpha\) of type \((n-p,n-p)\) we have

\[
\langle T, \alpha \rangle = \int_X T \wedge \alpha.
\]

The operators \(d\) and \(d^c\) can be extended to \((p,p)\)-currents by using the duality induced by the integration, i.e., \(dT\) and \(d^cT\) are respectively defined via the relations

\[
\langle dT, \beta \rangle = -\int_X T \wedge d\beta, \quad \langle d^cT, \beta \rangle = -\int_X T \wedge d^c\beta.
\]

We recall now the definition of a positive \((p,p)\)-form (see e.g. [Dem]).

**Definition 5.4**: A weakly positive (strictly weakly positive) \((p,p)\)-form on a complex manifold \(X\) is a real \((p,p)\)-form \(\eta\) such that for any complex subspace \(V \subset T^1 \cdot 0 \cdot X\), \(\dim_C V = p\), the restriction \(\eta|_V\) is a non-negative volume form (positive volume form). Weakly positive condition is equivalent to

\[
\partial^p \eta(v_1, \overline{v}_1, v_2, \overline{v}_2, \ldots, v_p, \overline{v}_p) \geq 0,
\]

for every tangent vectors \(v_1, \ldots, v_p \in T_x^{1,0} X\). A real \((p,p)\)-form \(\eta\) is called strongly positive (strictly strongly positive) if it can be locally expressed as a sum

\[
\eta = \partial^p \sum_{j_1, \ldots, j_p} \eta_{j_1 \ldots j_p} \xi_{j_1} \wedge \overline{\xi}_{j_1} \wedge \ldots \wedge \xi_{j_p} \wedge \overline{\xi}_{j_p},
\]

running over the set of \(p\)-tuples \(\xi_{j_1}, \xi_{j_2}, \ldots, \xi_{j_p}\) of \((1,0)\)-forms, with \(\eta_{j_1 \ldots j_p} \geq 0\) \((\eta_{j_1 \ldots j_p} > 0)\).

All strongly positive forms are also weakly positive. The strongly positive and the weakly positive forms form closed, convex cones in the space of real \((p,p)\)-forms, see for instance [Dem]. These two cones are dual with respect to the Poincare pairing

\[
\Lambda^{p,p}(X, \mathbb{R}) \times \Lambda^{n-p,n-p}(X, \mathbb{R}) \to \Lambda^{n,n}(X, \mathbb{R}).
\]

For \((1,1)\)-forms and \((n-1,n-1)\)-forms, the strong positivity is equivalent to weak positivity. Finally, a product of a weakly positive form and a strongly positive one is always weakly positive (however, a product of two weakly positive forms may be not weakly positive). A product of strongly positive forms is still strongly positive.
A strongly/weakly positive \((p,p)\)-current is a current taking non-negative values on weakly/strongly positive compactly supported \((n-p,n-p)\)-forms.

**Definition 5.5:** A \((p,p)\)-current \(T\) is called **weakly positive** if

\[
i^{n-p} \int_X T \wedge \alpha_1 \wedge \overline{\alpha_1} \wedge ... \alpha_{n-p} \wedge \overline{\alpha_{n-p}} \geq 0,
\]

for every \((1,0)\)-forms \(\alpha_1, ... \alpha_{n-p}\) with inequality being strict for at least one choice of \(\alpha_i\)'s. The current \(T\) is called **strongly positive** if the inequality is strict for every non-zero \(\alpha_1 \wedge \alpha_1 \wedge ... \alpha_{n-p} \wedge \alpha_{n-p}\).

**Definition 5.6:** A \((p,p)\)-current \(T\) is called **strictly strongly positive** (resp. **strictly weakly positive**) if

\[
T > \varepsilon \omega
\]

for a strictly strongly positive (resp. strictly weakly positive) \((p,p)\)-form \(\omega\) and a positive number \(\varepsilon\).

**Claim 5.7:** The cone of strongly positive \((p,p)\)-currents is dual to the cone of strictly weakly positive \((p,p)\)-forms, the cone of weakly positive \((p,p)\)-currents is dual to the cone of strictly strongly positive \((p,p)\)-forms,

\[
\square
\]

The main result of this section is the following

**Theorem 5.8:** Let \(f \in \text{Aut}(M)\) be a hyperbolic automorphism of a hyperkähler manifold, and denote by \(\pi : S \to E\) the suspension \(S(f)\) of \((M,f)\). Then \(S\) admits a \(dd^c\)-exact, strongly positive \((p,p)\)-current \(\beta\) for any \(p = 2, 3, ..., n-1\), where \(n := \dim_{\mathbb{C}} M\).

We prove this theorem in Subsection 5.4. Theorem 5.8 immediately implies the following.

**Corollary 5.9:** Let \(S\) be a hyperbolic suspension over a hyperkähler manifold \(M\), as in Theorem 5.8. Then \(S\) does not admit a \(dd^c\)-closed strictly weakly positive \((n-p+1,n-p+1)\)-form \(U\). In particular, \(S\) is not \(k\)-pluriclosed for any \(k = 1, 2, ..., n-1\).

**Proof:** Let \(\beta = dd^c \alpha\) be a current introduced in Theorem 5.8. If \(U\) is \(dd^c\)-closed strictly weakly positive \((n-p+1,n-p+1)\)-form \(U\), we have

\[
0 < \int_M U \wedge \beta = \int_M dd^c U \wedge \alpha = 0,
\]

which is impossible. \(\square\)

### 5.4 Hyperbolic automorphisms and Cantat-Dinh-Sibony currents

Let \(f\) be a hyperbolic automorphism of a hyperkähler manifold \(M\), \(\dim_{\mathbb{C}} M = n\), and \(p = 1, 2, ..., n-1\), and denote by \(\lambda\) its eigenvalue which satisfies \(|\lambda| > 1\).
(it is unique by [BKLV]). Since $f$ preserves the Kähler cone, it preserves the positive cone of $M$, hence $\lambda > 1$. Recall that the mass of a positive $(p,p)$-current $v$ on a Kähler manifold $M$ is $\int_M v \wedge \omega^{n-p}$.

The action of $f^*$ on $H^{2p}(M)$ has $\lambda^p$ as the maximal eigenvalue ([BKLV]), hence the mass of $\frac{1}{\lambda^p}(f^*)^k \omega^p$ is bounded. The set of positive $(p,p)$-currents of bounded mass is compact ([Dem]). Therefore the sequence $\{\frac{1}{\lambda^p}(f^*)^k \omega^p\}_{k=1,\ldots,\infty}$ has a limit point. The eigenspace corresponding to $\lambda^p$ in $H^{p,p}(M)$ has multiplicity 1, as shown in [BKLV]. By [DS, Theorem 4.3.1], the limit of a subsequence $\lim_k \frac{1}{\lambda^p}(f^*)^k \omega^p$ is a unique positive $(p,p)$-current $\sigma$ which satisfies $f^* \sigma = \lambda \sigma$. We call it the Cantat-Dingh-Sibony current (Cantat prove this result for $(1,1)$-currents on a K3 surface [Ca], and Dingh-Sibony for all dimensions).

Using the decomposition $TS = T_{\text{vert}} S \oplus \pi^* TE$ indiced by the flat Ehresmann connection on $S$, we can consider the bundle $\mathbb{D} := D^{p,p}_\pi(S)$ of fiberwise currents as a local system on $E$; the monodromy of this local system is given by the map $v \mapsto f^* v$. Identifying local systems and flat bundles, we can consider $\mathbb{D}$ as a bundle with flat connection.

Consider a real line bundle $L \subset \mathbb{D}$ spanned by the Cantat-Dingh-Sibony current $\sigma$. This line bundle is preserved by the natural flat connection on $\mathbb{D}$, and its monodromy map is multiplication by $\lambda$. Choose a trivialization of $L$ such that the corresponding connection 1-form $\theta$ satisfies $d\theta = 0$ and $d(I\theta) = 0$. Using the decomposition $TS = T_{\text{vert}} S \oplus \pi^* TE$, we can embed the sections of $\mathbb{D}$ into the space of $(p,p)$-currents on $S$. Let $\alpha$ be the current on $S$ associated with the section of $L \subset \mathbb{D}$ constructed above. Since $\sigma$ is a limit of closed currents, $d\sigma = 0$ and we have $d\alpha = \alpha \wedge \theta$ and $\beta := dd^c \alpha = \alpha \wedge \theta \wedge I\theta$. The current $\beta$ is $dd^c$-exact. Since $\beta$ is a limit of the wedge power of strongly positive $(1,1)$-forms, it is strongly positive. This proves Theorem 5.8.

6 Examples of suspensions of hyperkähler manifolds

We briefly recall the examples of suspensions of Kummer K3 surfaces from [QW] first.

Take the complex 2-torus $\mathbb{T}$ given by the quotient of $\mathbb{C}^2$ by the standard lattice generated by the unit vectors $(1,0), (i,0), (0,1), (0,i)$. Consider the involution of $\mathbb{C}^2$ given by multiplication by $-1$, i.e. $(z,w) \rightarrow (-z,-w)$. The involution descends to an involution $\sigma$ of the torus $\mathbb{T}$ with 16 fixed points $p_1, \ldots, p_{16}$. The quotient space $\mathbb{T}/ <1, \sigma>$ has 16 double points. The singularities can be resolved by blowing the singularities up, yieldings a smooth compact surface containing 16 mutually disjoint smooth rational curves $C_j$. This is the Kummer surface $Km$ associated to $\mathbb{T}$. There is an
alternative description of the Kummer surface. Let \( X \) denote the surface obtained by blowing up \( T \) at each of the points \( p_1, \ldots, p_{16} \). Let \( E_j \cong \mathbb{P}^1 \) be the exceptional divisor over \( p_j \). The involution \( \sigma \) of \( T \) lifts to an involution \( \tau \) of \( X \) with the fixed set \( E = E_1 \cup \ldots \cup E_{16} \). The eigenvalues of the differential of \( \tau \) at every point of \( E \) are \( \pm 1 \). So the quotient \( X/\langle 1, \tau \rangle \) is smooth and contains 16 rational \((-2)\)-curves \( C_j \cong \mathbb{P}^1 \), the images of the rational \((-1)\)-curves \( E_j \) in \( X \). The quotient is a Kummer surface \( \text{Km} \).

Let \( \hat{C}^2 \) be the surface obtained by blowing up \( C^2 \) at every point of the discrete set \( \pi^{-1}(\{p_1, \ldots, p_{16}\}) \), where \( \pi : C^2 \to T \) is the quotient map, we have the following diagram

\[
\begin{array}{ccc}
\hat{C}^2 & \longrightarrow & X \\
\downarrow \pi & & \downarrow \pi \\
C^2 & \longrightarrow & T \\
& \longrightarrow & T/\langle 1, \sigma \rangle \\
\end{array}
\]

By the Lefschetz Theorem on \((1,1)\)-forms we have that the Picard group of \( \text{Km} \) is isomorphic to \( H^2(\text{Km}, \mathbb{Z}) \cap H^{1,1}(\text{Km}) \), so the rank of the Picard group of \( \text{Km} \) is 20. Moreover, the Picard group of \( \text{Km} \) is generated by the 16 exceptional divisors \( E_i \) and by the pull-back by \( \pi \) of divisors on \( T/\langle 1, \sigma \rangle \).

The canonical \((2,0)\)-form \( dz_1 \wedge dz_2 \) on \( C^2 \) induces a nowhere vanishing \((2,0)\)-form on \( T \). Therefore, the pullback of this form on \( X \) induces a holomorphic \((2,0)\)-form on the Kummer surface.

Let \( A \in SL(2, \mathbb{Z} + \sqrt{-1} \mathbb{Z}) \) be a matrix with \( |\text{tr}(A)| > 2 \), so that it is diagonalizable with eigenvalues \( \lambda, \lambda^{-1} \). Let \( dv_1, dv_2 \) be respectively the associated eigenvectors of the induced map on \( H^1(T, \mathbb{C}) \cong \Lambda^1(\mathbb{R}^4) \). Denote by \( A \) also the induced map on \( \Lambda^k(\mathbb{R}^4) \). Then \( A \) preserves the holomorphic \((2,0)\)-form \( dv_1 \wedge dv_2 \) on \( T \) and the divisor \( D = \sum_{i=1}^{16} E_i \). So it defines a holomorphic transformation \( \varphi_A \) on \( \text{Km} \) preserving the induced holomorphic \((2,0)\)-form. In particular the \( \mathbb{Z} \)-action on \( T \times \mathbb{R} \times S^1 \) generated by

\[
f: (p, x, y) \to (A(p), x + 1, y)
\]

extends to an action on \( \text{Km} \times \mathbb{R} \times S^1 \), generated by the hyperbolic automorphism \( f \). The quotient is a compact complex manifold \( A(\text{Km}) \) with trivial canonical bundle and satisfying the hard Lefschetz property, such that its real homotopy type is formal as shown in [QW].

In a similar way we can construct hyperbolic automorphisms preserving the holomorphic symplectic form on higher-dimensional hyperkähler manifolds arising as Hilbert scheme of points on \( \text{Km} \). More precisely, \( f : \text{Km} \to \text{Km} \) extends to \( f^{[n]} : \text{Km}^{[n]} \to \text{Km}^{[n]} \) on the Hilbert scheme of order \( n \) of
Km in a natural way: to a zero-dimensional subscheme $Z \subset \text{Km}$ we assign $f(Z)$. According to Beauville (and PhD thesis by P. Beri for example [Be]), $f^{[n]}$ preserves the holomorphic symplectic form if and only if $f$ does. Now we can construct the suspension $\text{A}(\text{Km}^{[n]})$ using $f^{[n]}$ and obtain:

**Theorem 6.1:** The space $\text{A}(\text{Km}^{[n]})$ for $n \geq 1$ is a non-Kähler compact complex manifold with trivial canonical bundle which admits a balanced metric and it is not $k$-pluriclosed for any $k = 1, 2, \ldots, 2n - 1$.

**Proof:** The fact that it is balanced follows from Proposition 5.2 and Corollary 5.9.

The metric in the examples above is not explicit. But if we consider the suspension over the real 4-torus $\mathbb{T}^4$ we can define such metric explicitly. For a matrix $A$ as above, $A(dv_1 \wedge dv_1) = |\lambda|^2 dv_1 \wedge dv_1$ and $A(dv_2 \wedge dv_2) = |\lambda|^2 dv_2 \wedge dv_2$. Consider the differential forms on $\mathbb{R} \times S^1$ given by $\alpha_1 = |\lambda|^{-2x} dx \wedge dy$ and $\alpha_2 = |\lambda|^{-2x} dx \wedge dy$. Then they are closed and invariant under the action in (6.1). Moreover the form $\alpha_1 + \alpha_2$ descends to a positive definite $(2, 2)$-form on the suspension $\text{A}(\mathbb{T}^4)$ of the 4-torus defined by this action. In particular it defines a balanced metric by the observation of Michelson [Mic]. We can directly check that

$$\omega = |\lambda|^{-2x} dv_1 \wedge dv_1 + |\lambda|^{2x} dv_2 \wedge dv_2 + dx \wedge dy$$

is invariant and satisfies $d\omega^2 = 0$. Hence it defines a balanced metric.

**Remark 6.2:** We restrict ourselves here to the more explicitly described examples, but many of the known compact hyperkähler manifolds admit hyperbolic automorphisms. We expect that the topological properties of $\text{A}(\text{Km})$ from [QW] are also valid for $\text{A}(\text{Km}^{[n]})$. Note that the manifold $\text{A}(\mathbb{T}^4)$ can be also described as the almost abelian solvmanifold $M^0(c)$ in [AFLM] (see also Section 3 in [FMS]). By Theorem 4.1 in [FP] the associated almost abelian Lie algebra, which is isomorphic to $b_6$ in the notation of [AFLM], admits a balanced metric.

### 7 Holomorphically symplectic and hypercomplex structures on toric suspensions with 4-dimensional base

In this section we show how the toric suspensions could be used to construct examples of compact holomorphic symplectic and hypercomplex non-Kähler manifolds. The examples are in fact pseudo-hyperkähler. We also discuss their metric structure.
7.1 General construction and example

We consider a toric suspension of $M$, where $M$ is a real 8-torus, over a real 4-torus base. More precisely we consider $S = S(T^8, f, id, id, id)$, where $f$ is a diffeomorphism of $T^8$ defined by a matrix $A \in SL(8, \mathbb{R})$. We choose $A$ as in the following Lemma:

**Lemma 7.1:** Consider on $\mathbb{R}^8$ the hypercomplex structure $(I, J, K)$ defined, in terms of the standard basis $(e_1, \ldots, e_8)$, by

$Ie_1 = e_3, \quad Ie_2 = e_4, \quad Ie_5 = -e_7, \quad Ie_6 = -e_8,$
$Je_1 = -e_5, \quad Je_2 = -e_6, \quad Je_3 = -e_7, \quad Je_4 = -e_8,$

and the pseudo hyperhermitian metric $h = (e^1)^2 - (e^2)^2 + (e^3)^2 - (e^4)^2 + (e^5)^2 - (e^6)^2 + (e^7)^2 - (e^8)^2$.

Then there exists an integer matrix

$$A = \begin{pmatrix}
1 & 0 & 1 & 0 & -1 & -1 & 0 & 1 \\
0 & -1 & 0 & -1 & -1 & 0 & 1 & 1 \\
-1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & -1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\
1 & 0 & -1 & -1 & 0 & -1 & 0 & -1 \\
0 & -1 & -1 & -1 & -1 & 0 & 1 & 0 \\
-1 & -1 & -1 & 0 & 0 & 1 & 0 & -1
\end{pmatrix} \in SL(8, \mathbb{Z})$$

preserving the pseudo hyperhermitian structure $(I, J, K, h)$ and such $A^k \neq Id$, for every $k > 1$.

**Proof:** Since $A$ commutes with the matrices associated to $I$ and $J$ with respect to the standard basis $(e_1, \ldots, e_8)$ and $A^t HA = H$, where $H$ is the matrix associated to $h$, we have that $A$ preserves the pseudo hyperhermitian structure $(I, J, K, h)$. Moreover, $A^k \neq Id$, for every $k > 1$, since $A$ has eigenvalues $\pm i(1 + \sqrt{2})$ and $\pm i(\sqrt{2} - 1)$ of multiplicity two.

In particular, the Lemma claims that $A$ preserves the pseudo-hyperk"ahler structure on $\mathbb{R}^8$ and the lattice generated by $e_1, \ldots, e_8$. Such a matrix defines a hyperbolic diffeomorphism $f_A$ of $T^8$ which preserves the pseudo-hyperkahler structure and in particular both the hypercomplex structure and the holomorphic symplectic form. We note that any such matrix which preserves a positive-definite metric, has all eigenvalues on the unit circle so cannot be hyperbolic. Now we have:

**Theorem 7.2:** The hyperbolic toric suspension $S = S(T^8, f_A, id, id, id)$, where $T^8$ is the 8-dimensional real torus obtained by factoring $\mathbb{R}^8$ by the
standard lattice and $A_f$ is the diffeomorphism of $T^8$ defined by the matrix $A$ from Lemma 7.1 carries a pseudo-hyperkähler structure. In particular for a fixed complex structure $I$, it carries both hypercomplex and holomorphic symplectic structures, but no Kähler metrics. Moreover, the Obata connection of the hypercomplex structure is flat.

**Proof:** Since $f_A$ preserves both the hypercomplex and the holomorphic symplectic structures on $\mathbb{R}^8$ and descends to $T^8$, then the action $(p, t) \rightarrow (f_A^n(p), t + n)$ of $\mathbb{Z}$ on $T^8 \times \mathbb{R}^4$ preserves the induced natural structures obtained as product of the ones on $T^8$ and the canonical hyperkähler structure on $\mathbb{R}^4$. Also the hypercomplex structure is clearly with flat Obata connection. Then all structures descends to the quotient $S$. □

**Remark 7.3:** Note that pseudo-hyperkähler structures on 12-dimensional compact solvmanifolds are constructed in [Ya] and are associated to the almost abelian Lie algebras

$$\Psi_I(\Psi_J(g)) = \text{span}_\mathbb{R}\{U_1^1, U_1^2, U_1^3, U_1^4, V_1^1, V_1^2, V_1^3, V_1^4, V_2^1, V_2^2, V_2^3, V_2^4\}$$

with Lie bracket

$$[U_1^j, V_2^h] = c_{1j}^1 V_1^h + c_{1j}^2 V_2^h, \quad j = 1, 2, \quad h = 1, 2, 3, 4,$$

and hypercomplex structure $(I, J, K)$ defined by

$$IU_1^1 = U_2^1, \quad IV_1^1 = V_2^1, \quad IV_1^2 = V_2^2, \quad IU_1^3 = U_2^3, \quad IU_1^4 = U_2^4, \quad IV_1^3 = V_2^3, \quad IV_1^4 = V_2^4,$$

$$JU_1^1 = U_2^1, \quad JV_1^1 = V_2^1, \quad JV_1^2 = V_2^2, \quad JU_1^3 = U_2^3, \quad JU_1^4 = U_2^4, \quad JV_1^3 = V_2^3, \quad JV_1^4 = V_2^4.$$

So in particular the associated solvable Lie group is a semidirect product of the form $(\mathbb{R} \times \mathbb{R}^8) \rtimes \mathbb{R}^3$. In the notation of [Ya] $g = a \times b$, with $a = \text{span}_\mathbb{R}\{U_1^1\}$ and $b = \text{span}_\mathbb{R}\{V_1^1, V_1^3\}$ and by Theorem 6.6 in [Ya] if $b$ has a non-degenerate 2-form which is closed on $g$, then $(\Psi_I(\Psi_J(g)), I, J, K)$ admits a compatible pseudo-hyperkähler structure. The previous condition is satisfied if $c_{11}^1 = -c_{12}^2$. The hyperbolic toric suspension $S = S(T^8, f_A, id, id, id)$ corresponds to the compact solvmanifold constructed as a quotient of the solvable Lie group $H$ whose Lie algebra is $\mathfrak{h} := \Psi_I(\Psi_J(g))$ with $ad_{U_1^1} = \text{diag}(1, -1, 1, -1, 1, -1, 1, -1)$. If we consider the basis

$$f_1 = V_1^1, \quad f_2 = V_2^1, \quad f_3 = V_1^2, \quad f_4 = V_2^2, \quad f_5 = V_1^3, \quad f_6 = V_2^3,$$

$$f_7 = V_1^4, \quad f_8 = V_2^4, \quad f_9 = U_1^1, \quad f_{10} = U_2^1, \quad f_{11} = U_1^3, \quad f_{12} = U_2^3,$$

the structure equations of $\mathfrak{h}$ are

$$df^i = f^j \wedge f^0, \quad i = 1, 3, 5, 7, \quad df^j = -f^i \wedge f^0, \quad j = 2, 4, 6, 7,$$

$$df^k = 0, \quad k = 9, \ldots, 12. \quad (7.1)$$
The pseudo-hyperkähler structure on \( \mathfrak{h} \) is given by \((I, J, K, \omega_I, \omega_J, \omega_K)\), where

\[
I f_1 = f_3, \quad I f_2 = f_4, \quad I f_5 = -f_7, \quad I f_6 = -f_8, \quad I f_9 = f_{10}, \quad I f_{11} = -f_{12}, \\
J f_1 = f_5, \quad J f_2 = f_6, \quad J f_3 = f_7, \quad J f_4 = f_8, \quad J f_9 = f_{11}, \quad J f_{10} = f_{12},
\]

\(IJ = -JI = K\) and

\[
\omega_I = 2(-f^1 \wedge f^2 - f^3 \wedge f^4 + f^5 \wedge f^6 + f^7 \wedge f^8 + f^9 \wedge f^{10} - f^{11} \wedge f^{12}), \\
\omega_J = 2(f^1 \wedge f^8 + f^4 \wedge f^5 - f^2 \wedge f^7 - f^3 \wedge f^6 + f^9 \wedge f^{11} + f^{10} \wedge f^{12}), \\
\omega_K = 2(f^1 \wedge f^6 - f^4 \wedge f^7 - f^2 \wedge f^5 + f^3 \wedge f^8 - f^9 \wedge f^{12} + f^{10} \wedge f^{11}).
\]

With respect to the basis of \((1,0)\)-forms with respect to \(I\)

\[
\eta_1 = f^1 + if^3, \quad \eta_2 = f^2 + if^4, \quad \eta_3 = f^5 - if^7, \\
\eta_4 = f^6 - if^8, \quad \eta_5 = f^9 + if^{10}, \quad \eta_6 = f^{11} - f^{12}
\]

(7.2)

we have

\[
J \eta_1 = \overline{\eta}_3, \quad J \eta_2 = \overline{\eta}_4, \quad J \eta_3 = -\overline{\eta}_1, \quad J \eta_4 = -\overline{\eta}_2, \quad J \eta_5 = \overline{\eta}_6, \quad J \eta_6 = -\overline{\eta}_5
\]

(7.3)

and the associated \((2,0)\)-form \(\omega_J + i\omega_K\) is given by

\[
\omega_J + i\omega_K = 2(\eta_5 \wedge \eta_6 + i\eta_1 \wedge \eta_4 - i\eta_2 \wedge \eta_3).
\]

Note that

\[
J(\omega_J + i\omega_K) = 2(\overline{\eta}_5 \wedge \eta_6 - i\overline{\eta}_1 \wedge \eta_4 + i\overline{\eta}_2 \wedge \eta_3)
\]

and that the two \((4,0)\)-forms \(\eta_1 \wedge \eta_3 \wedge \eta_5 \wedge \eta_6\) and \(\eta_2 \wedge \eta_4 \wedge \eta_5 \wedge \eta_6\) are both \(\bar{\partial}\)-exact.

### 7.2 Compatible metric structures

On a hypercomplex manifold \((M, I, J, K)\) we always have a **hyper-Hermitian** (positive-definite) metric, that is a metric compatible with the complex structures \(I, J, K\). When the fundamental forms \(\omega_I, \omega_J, \omega_K\) are closed the metric is hyperkähler, but in the example constructed in Theorem 7.2 such metric doesn’t exist. A generalization of hyperkähler condition is the condition \(\partial \Omega = \partial(\omega_J + i\omega_K) = 0\) in which case the metric is called **hyperkähler with torsion** (shortly HKT). In fact one can characterize the HKT condition in terms of \(\Omega\): if there is a \((2,0)\) form w.r.t. \(I \Omega\), such that \(\partial \Omega = 0\), \(\Omega(JX, JY) = -\Omega(X, Y)\), and \(\Omega(X, JX) > 0\) for non-zero \((1,0)\) \(X\), then the metric \(g(X, Y) = Re \Omega(X, JY)\) is HKT. The HKT metric is a good candidate for a quaternionic analog of Kähler metrics in complex geometry - it arises from a local quaternionic-subharmonic potential and gives rise to a
Hodge theory (see [GP] and [Ve02]). The existence or non-existence of HKT, in 8-dimensional case depends on purely holomorphic data (see [GLV]). In hypercomplex geometry the analog of the balanced condition is called quaternionic balanced (see [LW]) and such metric satisfies $\partial(\Omega^{n-1}) = 0$, where $2n$ is the complex dimension of the manifold. We have the following:

**Theorem 7.4:** The hypercomplex manifold $S = S(T^8, f_A, id, id, id)$ from Theorem 7.2 admits a quaternionic balanced metric, but admits no HKT metrics.

**Proof:** We consider the solvmanifold model of $S$ from Remark 6.3. We use the same $(1,0)$-forms $\eta_i, 1 \leq i \leq 6$, and complex structure $J$ as in (7.2) and (7.3). Note that the hypercomplex structure has Obata holonomy in $SL(n, \mathbb{H})$, since it has a closed and real $(6,0)$-form. From the structure equations (7.1) we see that the $(2,0)$-form $\Omega = \eta_1 \wedge \eta_3 + \eta_2 \wedge \eta_4 + \eta_5 \wedge \eta_6$ satisfies the condition

$$\partial \Omega^2 = \partial(\eta_1 \wedge \eta_3 \wedge \eta_5 \wedge \eta_6 + \eta_2 \wedge \eta_4 \wedge \eta_5 \wedge \eta_6 + \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \eta_4) = 0,$$

but $\partial \Omega \neq 0$, so $\Omega$ defines a quaternionic-balanced metric. On the other side, by averaging argument (see [FG]), if there is any HKT metric, then there exists an invariant one. Working by contradiction, we assume that there is a $(2,0)$-form $\tilde{\Omega}$ with $\partial(\tilde{\Omega}) = 0$ which is $J$-anti-invariant and positive, so defines an HKT metric. Then $\tilde{\Omega}$ has the form

$$\tilde{\Omega} = \sum_{\alpha, \beta} a_{\alpha \beta} \eta_\alpha \wedge J\eta_\beta,$$

where $a_{\alpha \beta}$ is a Hermitian and positive definite matrix. Now we can adapt the Harvey-Lawson property for $SL(n, \mathbb{H})$ manifolds from (see [GLV]) and use it explicitly. Since the $(4,0)$-form $\alpha = \eta_1 \wedge \eta_3 \wedge \eta_5 \wedge \eta_6 + \eta_2 \wedge \eta_4 \wedge \eta_5 \wedge \eta_6$ is $\partial$-exact and the $(6,0)$-form $\beta = \eta_1 \wedge \eta_2 \wedge \eta_3 \wedge \eta_4 \wedge \eta_5 \wedge \eta_6$ is closed, then we have

$$\int_M \tilde{\Omega} \wedge \alpha \wedge \overline{\beta} = 0,$$

by integration by parts. On the other side $vol = \beta \wedge \overline{\beta}$ is a volume form and $\tilde{\Omega} \wedge \alpha \wedge \overline{\beta} = a_{11} + a_{22} > 0$, so

$$\int_M \tilde{\Omega} \wedge \omega^{(4,0)} \wedge \overline{\beta} > 0,$$

and we get a contradiction.

**8 Pluriclosed metrics from suspensions**

We recall the following
Definition 8.1: Let \((M, I)\) be a complex manifold. We say that a \(I\)-Hermitian metric \(h\) is pluriclosed (or SKT) if its fundamental form is \(\partial\bar{\partial}\)-closed.

Using automorphisms of Kähler manifolds it is also possible to construct suspensions admitting pluriclosed metrics in the following way.

Let \((X^3, \xi, \eta, \varphi, \Phi)\) be a 3-dimensional Sasakian manifold and \((Y^{2n}, I, g, \omega)\) be a Kähler manifold of complex dimension \(n\).

On the product \(X^3 \times Y^{2n} \times \mathbb{R}\) we can define a complex structure \(\tilde{I}\) such that \(\tilde{I}(\xi) = \frac{\partial}{\partial t}\), where \(t\) is the coordinate on \(\mathbb{R}\) and \(\tilde{I} = I\) on \(Y^{2n}\). The 2-form \(\tilde{\omega} = \eta \wedge dt + \omega\) is then a positive \((1,1)\)-form on \((X^3 \times Y^{2n} \times \mathbb{R}, \tilde{I})\). Since \(d\eta = \Phi\), by a direct computation we obtain

\[d\tilde{\omega} = \Phi \wedge dt\]

and

\[dT^B = d(\tilde{I}d\tilde{\omega}) = -d(\Phi \wedge \eta) = 0,\]

where \(T^B = \tilde{I}d\tilde{I}\omega\) is the so-called Bismut torsion form. Therefore we have the following

Theorem 8.2: Let \((X^3, \xi, \eta, \varphi, \Phi)\) be a Sasakian 3-dimensional manifold, \((Y^{2n}, I, g, \omega)\) a Kähler manifold and \(f = (f_1, f_2)\) a diffeomorphism of \(X^3 \times Y^{2n}\) such that \(f_1\) is a diffeomorphism of \(X^3\) preserving the Sasakian structure \((\xi, \eta, \varphi, \Phi)\) and \(f_2\) is a diffeomorphism of \(Y^{2n}\) preserving the Kähler structure \((I, g, \omega)\). Then the suspension of \(X^3 \times Y^{2n}\) by \(f\) has a pluriclosed (non-Kähler) metric.

Example 8.3: An application of the previous construction gives the example of compact solvmanifold constructed in [FP2]. More precisely, let \(G\) be the simply connected 3-step solvable Lie group with structure equations

\[
\begin{align*}
    de^1 &= e^2 \wedge e^3, \\
    de^2 &= -e^2 \wedge e^8, \\
    de^3 &= e^3 \wedge e^8, \\
    de^4 &= be^5 \wedge e^8, \\
    de^5 &= -be^4 \wedge e^8, \\
    de^6 &= be^7 \wedge e^8, \\
    de^7 &= -be^6 \wedge e^8, \\
    de^8 &= 0,
\end{align*}
\]

with \(b = \frac{2\pi}{\log(2 + \sqrt{3})}\). By [FP2] \(G\) has the left-invariant complex structure

\[Ie_1 = -e_2, Ie_3 = e_8, Ie_4 = e_5, Ie_6 = e_7,\]

\[-19-

and admits a compact quotient by a lattice $\Gamma$. The $J$-Hermitian metric $g = \sum_{i=1}^{8} (e_i)^2$ is pluriclosed since the Bismut torsion 3-form $T^B = Id\omega$ is the closed 3-form $-e_1 \wedge e_2 \wedge e_3$. The compact solvmanifold $\Gamma \backslash G$ can be obtained as a suspension of the product of the 3-Sasakian manifold given by the compact quotient of the real 3-dimensional Heisenberg group by a lattice and the standard torus $T^4$. Moreover, the compact solvmanifold can be viewed also as the total space of a bundle over a circle with fibre a circle bundle over a 6-torus.

References

[AFLM] de Andrés, L.C., Fernández, M., de León, M. Mencá, J.J., *Some six dimensional compact symplectic and complex solvmanifolds*, Rendiconti di Mat. Roma 12 (1992), 59–67. (Cited on page 14.)

[AB] Alessandrini, L., Bassanelli, G., *Transforms of currents by modifications and 1-convex manifolds*, Osaka J. Math. 40 (2003), 717–740. (Cited on page 7.)

[AB2] Alessandrini, L., Bassanelli, G., *Wedge product of positive product of positive currents and balanced manifolds*, Tohoku Math. J. 60 (2008), 123–134. (Cited on page 7.)

[AV] Amerik, E., Verbitsky, M., *Construction of automorphisms of hyperkähler manifolds*, Compos. Math. 153 (2017), 1610–1621. (Cited on page 3.)

[Bea] Beauville, A., *Variétés Kähleriennes dont la première classe de Chern est nulle*. J. Diff. Geom. 18 (1983), 755–782. (Cited on pages 3 and 4.)

[Be] Beri, P., On birational transformations and automorphisms of some hyperkähler manifolds, PhD-thesis, Université de Poitiers, 2020. (Cited on page 14.)

[Bes] Besse, A., *Einstein Manifolds*, Springer-Verlag, New York (1987). (Cited on page 3.)

[Bo] Bogomolov, F. A., *On the decomposition of Kähler manifolds with trivial canonical class*, Math. USSR-Sb. 22 (1974), 580–583. (Cited on page 3.)

[BKLV] Bogomolov, F., Kamenova, L., Lu, S., Verbitsky, M., *On the Kobayashi pseudometric, complex automorphisms and hyperkähler manifolds*, Geometry over nonclosed fields, 1-17, Simons Symp., Springer, Cham, 2017. (Cited on page 12.)

[Ca] Cantat, S., *Dynamique des automorphismes des surfaces K3*, Acta Math. 187 (2001), no. 1, 1–57. (Cited on pages 3 and 12.)

[Ch] Chiose, I., *Obstructions to the existence of Kähler structures on compact complex manifolds*, Proc. Amer. Math. Soc. 142, No. 10 (2014), 3561–3568. (Cited on page 9.)
A. Fino, G. Grantcharov, M. Verbitsky

Special Hermitian structures

[Dem] Demailly, J.-P., Complex analytic and differential geometry, 2012 (http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf). (Cited on pages 10 and 12.)

[DS] Dinh, T.-C., Sibony, N., Super-potentials for currents on compact Kähler manifolds and dynamics of automorphisms, J. Algebraic Geom. 19 (2010), no. 3, 473–529. (Cited on page 12.)

[Fe] Fei, T., Construction of Non-Kähler Calabi-Yau Manifolds and new solutions to the Strominger System, Adv. Math. 302 (2016) 529–550. (Cited on page 9.)

[FMS] Fernández, M., Munóz, V., Santisteban, J. A., Cohomologically Kähler manifolds with no Kähler metrics, Int. J. Math. Math. Sci. 52 (2003), 3315–3325. (Cited on page 14.)

[FG] Fino, A., Grantcharov, G., Properties of manifolds with skew-symmetric torsion and special holonomy, Adv. Math. 189 (2004), no. 2, 439–450. (Cited on page 18.)

[FGV] Fino, A., Grantcharov, G., Vezzoni, L., Astheno-Kähler and balanced structures on fibrations, Int. Math. Res. Not. IMRN 2019, no. 22, 7093–7117. (Cited on page 9.)

[FP] Fino, A., Paradiso, F., Balanced Hermitian structures on almost abelian Lie algebras, J. Pure Applied Algebra 227 (2023), no. 2, Paper No. 107186. (Cited on pages 9 and 14.)

[FP2] Fino, A., Paradiso, F., Hermitian structures on a class of almost nilpotent solvmanifolds, J. Algebra 609 (2022), 861–925. (Cited on pages 3 and 19.)

[FS] Freibert, M., Swann, A., Compatibility of balanced and SKT metrics on two-step solvable Lie groups, arXiv:2203.16638. (Cited on page 9.)

[Fu] Fujiki, A., On the de Rham Cohomology Group of a Compact Kähler Symplectic Manifold, Adv. Stud. Pure Math. 10 (1987), 105–165. (Cited on page 4.)

[FV] Fino, A., Vezzoni, L., On the existence of balanced and SKT metrics on nilmanifolds, Proc. Amer. Math. Soc., 144(6), (2016), 2455–2459. (Cited on page 9.)

[FLY] Fu, J., Li , J., Yau, S.-T., Balanced metrics on non-Kähler Calabi-Yau threefolds, J. Diff. Geom. 90 (2012), 81–129. (Cited on pages 7 and 9.)

[GoI] Goldman, W., Geometric structures on manifolds, 2021 (http://www.math.umd.edu/~wmg/gstom.pdf). (Cited on page 7.)

[GLV] Grantcharov, G., Lejmi, M., Verbitsky, M., Existence of HKT metrics on hypercomplex manifolds of real dimension 8, Adv. Math. 320 (2017), 1135–1157. (Cited on page 18.)

[GP] Grantcharov, G., Poon, Y. S., Geometry of hyperKähler connections with torsion, Commun. Math. Phys. 213 (2000), 19–37. (Cited on page 18.)
[Gu1] Gualtieri, M., Generalized complex geometry, PhD-thesis, University of Oxford, 2003, arXiv:math/0401221. (Cited on page 2.)

[Gu2] Gualtieri, M., Generalized Kähler geometry, Comm. Math. Phys. 331 (2014), no. 1, 297–331. (Cited on page 2.)

[Hi] Hitchin, H., Generalized Calabi-Yau Manifolds, Q. J. Math. 54 (2003), no. 3, 281–308. (Cited on page 2.)

[Hu] Hull, C., Superstring compactifications with torsion and space-time supersymmetry, In Turin 1985 Proceedings “Superunification and Extra Dimensions” (1986), 347–375. (Cited on page 2.)

[Ka] Kapovich, M., Kleinian groups in higher dimensions. In “Geometry and Dynamics of Groups and Spaces. In memory of Alexander Reznikov”, M.Kapranov et al (eds). Birkhauser, Progress in Mathematics, Vol. 265, 2007, p. 485-562. (Cited on pages 3 and 5.)

[LU] Latorre, A., Ugarte, L., On non-Kähler compact complex manifolds with balanced and astheno-Kähler metrics, C. R. Math. Acad. Sci. Paris 355, no. 1 (2017), 90–93. (Cited on page 9.)

[LW] Lejmi, M., Weber, P., Quaternionic Bott-Chern cohomology and existence of HKT metrics, Q. J. Math. 68 (2017), no. 3, 705–728. (Cited on page 18.)

[Li] Li, H., Topology of co-symplectic/co-Kähler manifolds, Asian J. Math. 12 (2008), 527–544. (Cited on page 2.)

[Ma] Manjarín, M., Normal almost contact structures and non-Kähler compact complex manifolds, Indiana Univ. Math. J. 57 (2008), no. 1, 481–507. (Cited on page 6.)

[Mic] Michelsohn, M. L., On the existence of special metrics in complex geometry, Acta Math. 143 (1983), 261–295. (Cited on pages 7, 8, and 14.)

[OV] Ornea, L., Verbitsky, M., Principles of Locally Conformally Kähler Geometry, arXiv:2208.07188. (Cited on page 9.)

[O] Otiman, A., Special Hermitian metrics on Oeljeklaus-Toma manifolds, Bull. Lond. Math. Soc. 54 (2022), 655–667. (Cited on page 9.)

[St] Strominger, A. E., Superstrings with torsion, Nuclear Phys. B 274(2) (1986), 253–284. (Cited on page 2.)

[QW] Qin, L., Wang, B., A family of compact complex and symplectic Calabi-Yau manifolds that are non-Kähler, Geom. Topol. 22 (2018), 2115–2144. (Cited on pages 1, 2, 3, 12, 13, and 14.)

[STW] Székelyhidi, G., Tosatti, V., Weinkove, B., Gauduchon metrics with prescribed volume form, Acta Math. 219 (2017), no. 1, 181–211. (Cited on page 9.)

[Ve02] Verbitsky, M., HyperKähler manifolds with torsion, supersymmetry and Hodge theory, Asian J. Math. 6 (2002), no. 4, 679–712. (Cited on page 18.)
[Ve14] Verbitsky, M., *Rational curves and special metrics on twistor spaces*, Geom. Topol. **18** (2014), no. 2, 897–909. (Cited on page 9.)

[Ya] Yamada, T., *A construction of compact pseudo-Kähler solvmanifolds with no Kähler structures*, Tsukuba J. Math. **29** (2005), 79-109. (Cited on page 16.)

Anna Fino  
Dipartimento di Matematica G. Peano  
Universitá di Torino  
via Carlo Alberto 10, 10123 Torino, Italy  
Also:  
Department of Mathematics and Statistics  
Florida International University  
Miami Florida, 33199, USA  
annamaria.fino@unito.it, afino@fiu.edu

Gueo Grantcharov  
Department of Mathematics and Statistics  
Florida International University  
Miami Florida, 33199, USA  
granchg@fiu.edu

Misha Verbitsky  
Instituto Nacional de Matemática Pura e Aplicada (IMPA)  
Estrada Dona Castorina, 110  
Jardim Botânico, CEP 22460-320  
Rio de Janeiro, RJ - Brasil  
Also:  
Laboratory of Algebraic Geometry,  
National Research University Higher School of Economics,  
Department of Mathematics, 6 Usacheva street, Moscow, Russia.  
verbit@impa.br