Thermodynamics of Born–Infeld black holes

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Received 21 July 2008, in final form 28 August 2008
Published 31 October 2008
Online at stacks.iop.org/CQG/25/225009

Abstract
We discuss the horizon structure for Born–Infeld black holes in the context of Einstein–Born–Infeld gravity. We show that the entropy function formalism agrees with a direct calculation of the entropy. With the entropy function formalism we also obtain the entropy when an axion–dilaton system as well as gravitational derivative corrections are included.

PACS numbers: 04.65.+e, 04.70.−s

1. Introduction
The work of Wald [1] has led to great progress in the understanding of the entropy of black holes. This approach is valid in theories with invariance under general coordinate invariance, and has been applied in particular to actions with gravitational higher-derivative corrections (see [2] for an overview). For extremal black holes the entropy formalism ([3], see also [4]) can be applied. This method depends only on the action of the theory being considered, in particular, it does not require the explicit black-hole solution.

In this paper we emphasize higher-derivative terms in the matter contributions to the action. We are especially interested in the coupling of gravity and Born–Infeld electromagnetism. This higher-derivative version of Maxwell’s theory leads to a generalization of the Reissner–Nordstrom black hole. The corresponding explicit black-hole solutions have been known for a long time, see, e.g., [5–10]. Born–Infeld theory is interesting for many, not unrelated, reasons: it satisfies electric–magnetic duality invariance, a property which also holds in the presence of an axion–dilaton system [22]. Dirac–Born–Infeld theories, which include additional scalars, correspond to the low-energy limit of D-branes [11, 12] as well as heterotic and type I string theories. Also there is an interest for cosmological applications, see [13] and references therein. Different aspects of the Born–Infeld black holes have also been studied in, e.g., [14–21].
We focus on the black-hole solutions of Einstein–Born–Infeld (EBI) theory, and the extension that includes an axion and dilaton field in an $SL(2, \mathbb{R})$ invariant way (EBIDA) [22]. In EBI theory, where the exact black-hole solution is available, we can compare the result for the entropy by a direct calculation using the solution, to the entropy function calculation for the extremal case. In EBIDA theory there is to the best of our knowledge no exact solution available. In this case we start from the black-hole solution of Maxwell electromagnetism with axion and dilaton [26]. In the case of two or more vector fields there is an extremal limit and the entropy formalism can again be compared to the results from the solution. In the EBIDA case the entropy can only be obtained by the entropy function formalism. For completeness we extend these results to include also gravitational higher-derivative terms.

This paper is organized as follows. In section 2 we discuss the black-hole solution in EBI electrodynamics and its horizon structure. We obtain thermodynamic quantities directly using the solution, and also from the near-horizon limit and the entropy function formalism. The results of the EBIDA case can be found in section 3, and their extension with higher derivative $R^2$ terms are in section 4. Our conclusions are in section 5. In an appendix we include the equations of motion for the EBIDA case.

2. Born–Infeld black holes

It has been known for a long time that the Reissner–Nordstrom solution to the Einstein–Maxwell system can be extended to the Einstein–Born–Infeld case [5–7, 10]. The Einstein–Born–Infeld Lagrangian is of the form

$$\mathcal{L}_{EBI} = \sqrt{-\det g} R + \frac{4}{b^2} (\sqrt{-\det g} - \sqrt{-\det (g + b F)}).$$

The spherically symmetric static solution with electric charge $q$ and magnetic charge $p$ is of the form

$$ds^2 = -G(r) \, dt^2 + \frac{dr^2}{G(r)} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2),$$

$$F_{rt} = \frac{q}{\sqrt{r^2 + a^2}}, \quad F_{\theta\phi} = p \sin \theta,$$

$$a^2 = b \sqrt{q^2 + p^2},$$

with

$$G(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} \left( \frac{4r g(r)}{3} + \frac{2r^4}{3a^4} \left( 1 - \sqrt{1 + \frac{a^2}{r^4}} \right) \right),$$

$$g(r) = \frac{1}{2a} \left( -F \left( \varphi, \frac{1}{2} \right) + 2K \left( \frac{1}{2} \right) \right), \quad (0 < \varphi \leq \pi/2)$$

$$= \frac{1}{2a} F \left( \pi - \varphi, \frac{1}{2} \right), \quad (\pi/2 \leq \varphi < \pi),$$

$$r = a \tan(\varphi/2),$$

where $F$ and $K$ are incomplete and complete elliptic integrals of the first kind.

We will discuss some general properties of the solution (2.2)–(2.8). It is then convenient to set $p = 0$. Magnetic charges can always be reinstated by using the duality property of Born–Infeld electrodynamics.
Figure 1. The horizon structure of Born–Infeld black holes as a function of the parameters $b$ and $q$. To the left of curve I there is one horizon, in the area between curves I and II there are two horizons, that coalesce on II. To the right of II there are no horizons.

For $b = 0$ we recover the Reissner–Nordstrom solution. If in addition $q = 0$ the function $G(r)$ has one zero and we find the Schwarzschild solution with horizon at $r_H = 2m$. For $0 < q < m$, $G(r)$ has two zeros at $m \pm \sqrt{m^2 - q^2}$. For $q = m$ these two horizons coalesce at $r_H = m$, for $q > m$ there is no black hole.

For arbitrary $b > 0$ the following properties are obtained (see figure 1). The analogue of the Schwarzschild solution is found for

$$b > \frac{4q^3 K^2(1/2)}{9m^2}. \tag{2.9}$$

In the region where (2.9) is satisfied $G(r) \to -\infty$ for $r \to 0$, it has one zero, and increases to the asymptotic value $G(r) = 1$ at $r \to \infty$. On the boundary I determined by (2.9) $G'(0) = 0$, and $G(0) = 1 - 2q/b$. In contrast, for

$$b < \frac{4q^3 K^2(1/2)}{9m^2} \tag{2.10}$$

$G(r) \to +\infty$ for $r \to 0$, and there is a region where $G(r)$ has two zeros. In the $(q, b)$ plane this region is bounded above by (2.10), and bounded to the right by the curve II that runs from $(m, 0)$ (extremal Reissner–Nordstrom) to the point

$$P_e = (qe/m, 2qe/m), \quad \frac{qe}{m} = \frac{3}{\sqrt{2K(1/2)}} = 1.144 \ldots, \tag{2.11}$$

where the curves I and II intersect on the curve II $G(r)$ and $G'(r)$ vanish at the point

$$r_H = \sqrt{q^2 - \frac{1}{4}b^2}, \tag{2.12}$$

which corresponds to the horizon of the extremal Born–Infeld black hole. So there is a finite range of extremal black holes. The horizon (2.12) shrinks from $r_H = q$ at $b = 0$ to $r_H = 0$ at $b = 2q(P_e)$. To the right of both curves I and II there are no real zeros of $G(r)$ and there is no black hole. This implies that for a large enough values of $b$ ($b > 2.288 \ldots$) there is no analogue of the Reissner–Nordstrom black hole for any value of $q$. 

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For a metric of the form (2.2) the requirement that \( G(r_H) = 0 \) and \( G'(r_H) = 0 \) (which corresponds to an extremal solution with horizon \( r_H \)) gives a near-horizon metric of the form (we now normalize to \( m = 1 \))

\[
\begin{align*}
\text{d} s^2 &= \frac{2}{G''(r_H)} \left( -r^2 \text{d}t^2 + \frac{1}{r^2} \text{d}r^2 + r^2_H (\text{d}\theta^2 + \sin^2\theta \text{d}\phi^2) \right), \\
&\quad \text{where } r = r_H + \lambda r', \ t' = \frac{1}{2} \lambda t G''(r_H), \text{in the limit } \lambda \to 0.
\end{align*}
\]

For the Born–Infeld case we find

\[
G''(r_H) = \frac{2}{q^2 + \frac{1}{4} b^2},
\]

with \( r_H \) given in (2.12). The entropy of the extremal Born–Infeld black hole is the area of the horizon,

\[
S = 16\pi^2 \left( q^2 - \frac{1}{4} b^2 \right),
\]

where we have set the gravitation coupling equal to \( G_N = 1/16\pi \). Thus the entropy for the Born–Infeld black hole is smaller than for a Reissner–Nordstrom black hole of the same mass and charge.

Let us now apply the entropy function formalism [3] to obtain the entropy in this case. We start from the near-horizon solution parametrized as

\[
\begin{align*}
\text{d}s^2_{NH} &= v_1 \left( -r^2 \text{d}t^2 + \frac{\text{d}r^2}{r^2} \right) + v_2 (\text{d}\theta^2 + \sin^2\theta \text{d}\phi^2), \\
F_{rt} &= e, \quad F_{\theta\phi} = p \sin\theta.
\end{align*}
\]

The entropy function is given by

\[
E = 2\pi (16\pi q e - f(e, p, v_1, v_2)),
\]

where \( f \) is the Lagrangian, evaluated in the near-horizon limit (2.16), and integrated over the angles,

\[
f(e, p, v_1, v_2) = \int \text{d}\theta \text{d}\phi \mathcal{L}_{NH}.
\]

In the following sections, we will extend this result to include an axion–dilaton system as well as gravitational higher-derivative corrections. Also in this case the entropy will be proportional to the area of the horizon. However, in the cases where no analytic black-hole solution is available, we have to depend on the entropy formalism to obtain the entropy in terms of the charges and the Born–Infeld parameter \( b \).
3. Axion–dilaton

In this section, we discuss the effect of adding an axion–dilaton system to the results of section 2. In this case there is, to our knowledge, no explicit solution available for the Born–Infeld case. We start with an explicit solution for the Maxwell case, and will obtain for that case the entropy both by the explicit calculation using the solution and from the entropy formalism.

The Maxwell case was developed in a number of papers \cite{23–26}, we will use the form given in the last reference. The action for \(N\) vector fields is given by

\[
\mathcal{L} = \sqrt{-g} \left( R + 2\partial\Phi \right)^2 + \frac{1}{2} e^{2\Phi} \left( \partial a \right)^2 - e^{-2\Phi} \sum_i F_i F_i + a \sum_i F_i + F_i.
\]  

The field \(a\) represents the axion, \(\Phi\) corresponds to the dilaton.

The solution is

\[
\begin{align*}
\text{ds}^2 &= -G(r) \text{d}t^2 + G(r)^{-1} \text{d}r^2 + R^2(r) (\text{d}\theta^2 + \sin^2 \theta \text{d}\phi^2), \\
G(r) &= (r - m - r_0)(r - m + r_0)/R^2(r), \\
F_{tr} &= e^{\Phi_0} q_i ((r + \Sigma)^2 + \Delta^2) + 2p_i \text{d}r, \\
F_{\theta\phi} &= -e^{\Phi_0} p_i, \\
e^{2\Phi} &= e^{2\Phi_0} \frac{(r + \Sigma)^2 + \Delta^2}{r^2 - \Delta^2 - \Sigma^2}, \\
a(r) &= a_0 - e^{2\Phi_0} 2\Delta r \frac{(r + \Sigma)^2 + \Delta^2}{(r + \Sigma)^2 + \Delta^2}.
\end{align*}
\]  

The mass \(m\), dilaton and axion charges \(\Sigma\), \(\Delta\), and the electric and magnetic charges \(q_i\) and \(p_i\) are subject to the following conditions:

\[
\begin{align*}
r_0^2 &= m^2 + \Delta^2 + \Sigma^2 - (P^2 + Q^2) \geq 0, \\
P \cdot Q + m \Delta &= 0, \\
-P^2 + Q^2 + 2m \Sigma &= 0,
\end{align*}
\]  

where

\[
\begin{align*}
P^2 &= \sum_i p_i^2, \\
Q^2 &= \sum_i q_i^2, \\
P \cdot Q &= \sum_i p_i q_i.
\end{align*}
\]  

Before turning to the extremal case with \(N\) vector fields we briefly consider this system of equations for \(N = 1\), i.e., one vector. Then the constraints imply

\[
\Delta^2 + \Sigma^2 = \left( \frac{p_i^2 + q_i^2}{2m} \right)^2, \\
r_0^2 = \left( m - \frac{p_i^2 + q_i^2}{2m} \right)^2.
\]  

We then find

\[
\begin{align*}
G(r) &= \frac{r - 2m + \frac{p_i^2 + q_i^2}{2m}}{r + \frac{p_i^2 + q_i^2}{2m}}, \\
R^2(r) &= r^2 - \left( \frac{p_i^2 + q_i^2}{2m} \right)^2.
\end{align*}
\]  

This solution with a single horizon, when the axion field is absent, is the one found in \cite{24}. The curvature singularity is at \(r_S = (p_i^2 + q_i^2)/2m\). The solution has a horizon at \(r_H \equiv 2m - r_S\), if \(r_H > r_S\), which corresponds to \(p_i^2 + q_i^2 < 2m^2\). If \(p_i^2 + q_i^2 = 2m^2\) the horizon and curvature singularity coincide. This charged axion–dilaton black hole with a single vector has no extremal limit.
Now we go back to (3.6) and the \( N \)-vector case. Here the constraints imply
\[
    r_0^2 = \frac{1}{4m^2}(2m^2 - P^2 - Q^2)^2 - 4S^2, \quad (3.10)
\]
where
\[
    S = \sqrt{P^2Q^2 - (P \cdot Q)^2}. \quad (3.11)
\]
The curvature singularity is at \( r_S^2 = \Delta^2 + \Sigma^2 \). The solution becomes extremal for \( r_0 = 0 \), which leads to
\[
    2m^2 = P^2 + Q^2 \pm 2S. \quad (3.12)
\]
The condition that the horizon for the extremal case is outside the curvature singularity leads to
\[
    (r_H^2) = m^2 = \frac{1}{2}(P^2 + Q^2 + 2S), \quad (3.13)
\]
The near-horizon limit of the metric is
\[
    ds^2 = (m^2 - \Delta^2 - \Sigma^2)\left(-\rho^2 d\tau^2 + \frac{d\rho^2}{\rho}\right) + (m^2 - \Delta^2 - \Sigma^2)(d\theta^2 + \sin^2\theta d\phi^2), \quad (3.14)
\]
with, for the case where \( m^2 \) is given by (3.13),
\[
    m^2 = \Delta^2 + \Sigma^2 = 2S. \quad (3.15)
\]
The metric exhibits the usual \( AdS_2 \times S_2 \) symmetry. The difference with (2.13) is due to the fact that the curvature singularity is not at \( r = 0 \) in this case. The entropy is given by
\[
    S = 16\pi^2((r_H^2) - (r_S^2)) = 32\pi^2 S. \quad (3.16)
\]
Now we will treat the same case by the entropy formalism. The entropy function is
\[
    \mathcal{E} = 2\pi \left(16\pi Q \cdot E - 4\pi \left(2(v_1 - v_2) - 4aP \cdot E - 2e^{-2\Phi} \frac{P^2v_1^2 - E^2v_2^2}{v_1v_2}\right)\right). \quad (3.17)
\]
Here \( E = (e_1, \ldots, e_N) \) is the vector of electric fields \( F_{\mu\nu} \). We require that \( \mathcal{E} \) is extremal under variations of \( e_i, v_1, v_2, a \) and \( e^{-2\Phi} \). This gives the following equations:
\[
    q_i + ap_i - e^{-2\Phi} e_i v_2/v_1 = 0, \quad (3.18)
\]
\[
    P \cdot E = 0, \quad (3.19)
\]
\[
    2e^{-2\Phi}(P^2 v_1/v_2 - E^2v_2/v_1) = 0, \quad (3.20)
\]
\[
    1 - e^{-2\Phi}(P^2/v_1 + E^2v_2/v_1) = 0, \quad (3.21)
\]
\[
    1 - e^{-2\Phi}(P^2 v_1/v_2 + E^2/v_1) = 0. \quad (3.22)
\]
The solution is
\[
    v_1 = v_2 = 2S, \quad (3.23)
\]
\[
    a = -P \cdot Q/P^2, \quad (3.24)
\]
\[
    e^{-2\Phi} = S/P^2, \quad (3.25)
\]
\[
    e_i = (P^2 q_i - P \cdot Q p_i)/S. \quad (3.26)
\]
If we substitute this back into the entropy function we find
\[
    \mathcal{E} = 32\pi^2 S, \quad (3.27)
\]
in agreement with (3.16).
To obtain information on the entropy for the Born–Infeld case we have to use the entropy function formalism, since there is no explicit solution available. The action reads

$$L = \sqrt{-\det g} \left( R + 2(\partial \Phi)^2 + \frac{1}{2} e^{4\Phi} (\partial a)^2 \right) + a \sum F_i^* F_i + \sum B_i, \quad (3.28)$$

with

$$B_i = \frac{4}{b^2} (\sqrt{-\det g} - \sqrt{-\det (g + be^{-\Phi} F_i)}). \quad (3.29)$$

This satisfies the requirements of electric–magnetic duality [22]. In this case the entropy function is

$$E_{BI} = 2\pi \left( 16\pi Q \cdot E - 4\pi \left( 2(v_1 - v_2) - 4a P \cdot E + \sum B_{NI} \right) \right), \quad (3.30)$$

where the Born–Infeld contribution is now expressed in terms of fields in the near-horizon limit,

$$B_{NIi} = \frac{4}{b^2} \left( v_1 v_2 - \sqrt{(v_1^2 - b^2 e^2 \Phi)(v_2^2 + b^2 p_i^2 e^{-2\Phi})} \right). \quad (3.31)$$

We obtain the following equations from the variation of $E_{BI}$:

$$0 = q_i + a p_i - \frac{\partial B_{NIi}}{4 \partial e_i}, \quad (3.32)$$

$$0 = P \cdot E, \quad (3.33)$$

$$0 = \sum \frac{\partial B_{NIi}}{\partial e_i}, \quad (3.34)$$

$$0 = 2 + \sum \frac{\partial B_{NIi}}{\partial v_1}, \quad (3.35)$$

$$0 = -2 + \sum \frac{\partial B_{NIi}}{\partial v_2}. \quad (3.36)$$

In the general case we have not found an explicit solution of these equations. In an expansion in $b^2$ it is possible to find a solution for the variables $e_i, v_1, v_2, a$ and $e^{-2\Phi}$ to order $b^3$. The corresponding entropy is

$$E_{BI} = 16\pi^2 \left( 2S - \frac{b^2}{16(p_i^2)^2} \sum (e_i^2 + p_i^2)^2 + O(b^4) \right), \quad (3.37)$$

where $e_i$ takes the value (3.26).

For the special case $N = 2$, where $S = |p_1 q_2 - p_2 q_1|$ we find

$$v_1 = 2|p_2 q_1 - p_1 q_2| + \frac{b^2}{8}, \quad v_2 = 2|p_2 q_1 - p_1 q_2| - \frac{b^2}{8}, \quad (3.38)$$

$$a = -\frac{p_1 q_1 + p_2 q_2}{p_1^2 + p_2^2}, \quad (3.39)$$

$$e^{-2\Phi} = \frac{|p_2 q_1 - p_1 q_2|}{p_1^2 + p_2^2}, \quad (3.40)$$

$$e_1 = \frac{p_2 (p_2 q_1 - p_1 q_2)}{|p_2 q_1 - p_1 q_2|}. \quad (3.41)$$
\[ e_2 = - \frac{p_1(p_2 q_1 - p_1 q_2)}{|p_2 q_1 - p_1 q_2|}. \]  

(3.42)

Except for the \( b^2 \)-contribution in \( v_1 \) and \( v_2 \) this is the same solution as for the Maxwell case, see (3.23)–(3.26). See also the Born–Infeld solution without axion–dilaton, where there is a similar structure (2.20). The entropy is

\[ E_{BI} = 16\pi^2 \left( 2|p_2 q_1 - p_1 q_2| - \frac{b^2}{8} \right). \]  

(3.43)

This simple solution, and the similarity with the results of section 2, might be helpful in finding an explicit solution for the Born–Infeld case with axion and dilaton for \( N = 2 \).

### 4. Gravitational higher-derivative corrections

In this section we will consider gravitational higher-derivative terms of the form \( R^2 \). We will add to the Lagrangians (2.1), (3.28) terms of the form

\[ L_\alpha = \alpha \Phi^2 (x R_{\mu}\nu\rho R_{\mu}\nu\rho - 4y R_{\mu}\mu, R_{\mu}\nu + z R^2). \]  

(4.1)

Here again it is difficult, if not impossible, to obtain exact solutions. What we can do is to work to order \( \alpha^3 \). In this case the result depends only on the solution at \( \alpha^0 \), corrections to this solution due to the higher-derivative terms contribute only to terms of order \( \alpha^2 \) and higher.

In the calculation of the entropy function we use

\[ R_{\mu}\nu\rho R_{\mu}\nu\rho = \frac{4(v_1^2 + v_2^2)}{v_1^2 v_2^2}, \quad R_{\mu}\nu = \frac{2(v_1^2 + v_2^2)}{v_1^2 v_2^2}, \quad R^2 = \frac{4(v_1^2 - v_2^2)^2}{v_1^2 v_2^2}. \]  

(4.2)

Working to order \( \alpha \), and including axion and dilaton, we find for the case \( N = 2 \),

\[ E = 16\pi^2 \left( 2|p_2 q_1 - p_1 q_2| - \frac{b^2}{8} \right) \]

\[ - \frac{64\pi^2 \alpha e^{-\Phi_0}}{256|p_2 q_1 - p_1 q_2|^2 (x - 2y) + b^4 (x - 2y + 2z)}. \]  

(4.3)

Here \( e^{-\Phi_0} \) corresponds to the solution (3.40). The Gauss–Bonnet combination \( x = y = z = 1 \) depends on the charges only through \( e^{-\Phi_0} \), and is independent of the Born–Infeld parameter \( b \).

### 5. Conclusions

The main conclusion is that the entropy function formalism works well in all cases considered in this paper: Einstein–Born–Infeld black holes, and the various extensions thereof, including in particular the axion–dilaton case. In this last case we cannot compare with the result of an exact solution, but the result of the entropy function agrees, in various limits, with known results.

The simplicity of the EBIDA entropy for \( N = 2 \) seems to suggest that an explicit black-hole solution might be obtained, however, we have not succeeded in this respect. An extension of our work would be to include not only a general \( R^2 \) gravitational higher-derivative term, but a complete order \( \alpha' \) correction indicated by string theory. In specific cases, such as the heterotic string, this can be pushed to higher orders in \( \alpha' \). It would be interesting to see how the entropy function formalism copes with such an extension.

3 Of course, in string theory \( b \) is proportional to \( \alpha \), so one should also truncate the Born–Infeld system. However, there is no problem in keeping the complete \( b \)-dependence, which we therefore do.
Acknowledgments

We are grateful to Tomas Ortín, Sjoerd de Haan and Ashoke Sen for useful discussions. SP thanks the Centre for Theoretical Physics in Groningen for their hospitality. WC and MdR are supported by the European Commission FP6 program MRTN-CT-2004-005104 in which WC and MdR are associated to Utrecht University.

Appendix. Equations of motion

In this appendix we present the equations of motion for the Maxwell and Born–Infeld cases, corresponding to the action given in section 3. The metric is always of the form
\[
d s^2 = -G(r) \, dt^2 + G(r)^{-1} \, dr^2 + R^2(r) \, (d\theta^2 + \sin^2 \theta \, d\phi^2).
\]  
(A.1)

We give the equations for a single vector field. The electric and magnetic fields appear as
\[
F_{rt} = E(r), \quad F_{\theta\phi} = M(r) \sin \theta.
\]  
(A.2)

The equation for the vector field is of the form
\[
0 = (M(r) a(r) + E(r) e^{-2\Phi(r)} X(r))',
\]  
(A.3)

where the prime indicates differentiation with respect to \(r\), and
\[
X(r) = R^2(r) + \frac{b^2}{2} e^{-2\Phi(r)} \left( \frac{M^2(r)}{R^2(r)} + E^2(r) R^2(r) \right), \quad \text{Maxwell case, } b \to 0,
\]  
(A.4)

\[
X(r) = \sqrt{R^4(r) e^{2\Phi(r)} + b^2 M^2(r)} \left( e^{2\Phi(r)} - b^2 E^2(r) \right), \quad \text{Born–Infeld case, } b \neq 0.
\]  
(A.4)

The dilaton and axion equations of motion are, respectively,
\[
0 = -(2G(r) R^2(r) \Phi(r)')' + e^{4\Phi(r)} G(r) (a(r)')^2 - 2e^{-2\Phi(r)} \left( \frac{M^2(r)}{X(r)} - E^2(r) X(r) \right),
\]  
(A.5)

\[
0 = (e^{4\Phi(r)} G(r) R^2(r) a(r)')' + 4 E(r) M(r).
\]  
(A.6)

There are three independent components of the Einstein equations. Taking convenient linear combinations of these gives the following three equations of motion:
\[
0 = e^{4\Phi(r)} (a(r)')^2 + 4 (\Phi(r)')^2 - \frac{1}{R^2(r)} (\left( R^2(r) \right)')^2 - 2R^2(r) (R^2(r))''),
\]  
(A.7)

\[
0 = -2 + (G(r) R^2(r)')' - \frac{4R^2(r)}{b^2} \left( 1 - \frac{X(r)}{R^2(r)} \right),
\]  
(A.8)

\[
0 = -2 + (G(r) R^2(r)')'' - \frac{4R^2(r)}{b^2} \left( 2 - \frac{X(r)}{R^2(r)} - \frac{R^2(r)}{X(r)} \right).
\]  
(A.9)

In (A.9) we can write
\[
(G(r) R^2(r))'' = (G(r)' R^2(r))' + (G(r) R^2(r))',
\]  
(A.10)

which we can use to combine with (A.8), giving instead of (A.9),
\[
0 = (G(r)' R^2(r))' - \frac{4R^2(r)}{b^2} \left( 1 - \frac{R^2(r)}{X(r)} \right).
\]  
(A.11)
A.1. The Einstein–Born–Infeld black hole

Here we obtain the solution discussed in section 2. Thus we have Born–Infeld electromagnetism, Einstein gravity, but no dilaton and axion. Solve equation (A.7) by

\[ R^2(r) = r^2. \]

(A.12)

Then we can solve (A.3), which gives

\[ E(r) R^2(r) = Q \sqrt{1 - b^2 E(r)^2} \rightarrow E(r) = \frac{Q}{\sqrt{r^4 + b^2 Q^2}} = \frac{Q}{X(r)}. \]

(A.13)

There are two remaining equations (A.8) and (A.9),

\[ -2 + 2 (r G(r))' = \frac{4 r^2}{b^2} \left( 1 - \frac{Q}{r^2 E(r)} \right), \]

(A.14)

\[ -2 + 2 (r G(r))' + r (r G(r))'' = \frac{4 r^2}{b^2} \left( 2 - \frac{Q}{r^2 E(r)} - \frac{r^2 E(r)}{Q} \right). \]

(A.15)

Note that (A.14) and (A.15) combine to

\[ (r G(r))^" = \frac{4 r^2}{Q} \left( 1 - \frac{r^2 E(r)}{Q} \right). \]

(A.16)

The solution for the vector field (A.13) can be written as

\[ E(r) = -Q g(r)', \quad g(r)' = -\frac{1}{\sqrt{r^4 + b^2 Q^2}}. \]

(A.17)

From (A.15) we then obtain

\[ (r G(r))^" = \frac{4 r^2}{b^2} + \frac{4 r^2 g(r)'}{b^2}, \]

(A.18)

which we also obtain by differentiating (A.14). So (A.14) is the integrated form of (A.16). We can then integrate (A.14) to

\[ r G(r) = -2m + r + \frac{2 r^3}{3 b^2} \left( 1 - \frac{\sqrt{r^4 + b^2 Q^2}}{r^2} \right) + \frac{4 Q^2 g(r)}{3}. \]

(A.19)

which is the solution (2.8).

References

[1] Wald R M 1993 Black hole entropy is the Noether charge Phys. Rev. D 48 3427 (arXiv:gr-qc/9307038)
[2] Mohaupt T 2007 Supersymmetric black holes in string theory Fortsch. Phys. 55 519 (arXiv:hep-th/0703035)
[3] Sen A 2005 Black hole entropy function and the attractor mechanism in higher derivative gravity J. High Energy Phys. JHEP09(2005)038 (arXiv:hep-th/0506177)
[4] Sen A 2007 Black hole entropy function, attractors and precision counting of microstates arXiv:0708.1270
[5] Demianski M 1986 Static electromagnetic geon Found. Phys. 16 187
[6] Wiltshire D L 1988 Black holes in string generated gravity models Phys. Rev. D 38 2445
[7] Gibbons G W and Rasheed D A 1995 Electric–magnetic duality rotations in nonlinear electrodynamics Nucl. Phys. B 454 185 (arXiv:hep-th/9506035)
[8] Clement G and Gal’stsov D 2000 Solitons and black holes in Einstein–Born–Infeld–dilaton theory Phys. Rev. D 62 124013 (arXiv:hep-th/0007228)
[9] Breton N 2002 Horizon structure of Born–Infeld black hole Prepared for Conf. on Topics in Mathematical Physics, General Relativity, and Cosmology on the Occasion of the 75th Birthday of Jerzy F Plebański (Mexico City, Mexico, 17–20 Sep, 2002)
[10] Breton N and Garcia-Salcedo R 2007 Nonlinear electrodynamics and black holes arXiv:hep-th/0702008
[11] Andreev O D and Tseytlin A A 1988 Partition function representation for the open superstring effective
action: cancellation of Mobius infinities and derivative corrections to Born–Infeld Lagrangian Nucl. Phys. B 311 205
[12] Leigh R G 1989 Dirac–Born–Infeld action from Dirichlet sigma model Mod. Phys. Lett. A 4 2767
[13] M Banados A 2008 Born–Infeld action for dark energy and dark matter arXiv:0801.4103
[14] Rasheed D A 1997 Non-linear electrodynamics: zeroth and first laws of black hole mechanics
arXiv:hep-th/9702087
[15] Tamaki T and Torii T 2000 Gravitating Blon and Blon black hole with dilaton Phys. Rev. D 62 061501
(arXiv:gr-qc/0004071)
[16] Tamaki T and Torii T 2001 Dyonic Blon black hole in string inspired model Phys. Rev. D 64 024027
(arXiv:gr-qc/0101083)
[17] Tamaki T 2004 Black hole solutions coupled to Born–Infeld electrodynamics with derivative corrections
J. Cosmol. Astropart. Phys. JCAP05(2004)004 (arXiv:gr-qc/0310099)
[18] Yazadjiev S S, Fiziev P P, Boyadjiev T L and Todorov M D 2001 Electrically charged Einstein–Born–Infeld
black holes with massive dilaton Mod. Phys. Lett. A 16 2143 (arXiv:hep-th/0105165)
[19] Chandrasekhar B, Yavartanoo H and Yun S 2008 Non-supersymmetric attractors in BI black holes Phys. Lett.
B 660 392 (arXiv:hep-th/0611240)
[20] Chandrasekhar B 2007 Born–Infeld corrections to the entropy function of heterotic black holes Braz. J. Phys.
37 349 (arXiv:hep-th/0604028)
[21] Stefanov I Z, Yazadjiev S S and Todorov M D 2007 Scalar–tensor black holes coupled to Born–Infeld nonlinear
electrodynamics Phys. Rev. D 75 084036 (arXiv:0704.3784)
[22] Gibbons G W and Rasheed D A 1996 SL(2, R) Invariance of non-linear electrodynamics coupled to an axion
and a dilaton Phys. Lett. B 365 46 (arXiv:hep-th/9509141)
[23] Shapere A D, Trivedi S and Wilczek F 1991 Dual dilaton dyons Mod. Phys. Lett. A 6 2677
[24] Garfinkle D, Horowitz G T and Strominger A 1991 Charged black holes in string theory Phys. Rev. D 43 3140
Garfinkle D, Horowitz G T and Strominger A 1992 Charged black holes in string theory Phys. Rev. D 45 3888
(erratum)
[25] Ortin T 1993 Electric–magnetic duality and supersymmetry in stringy black holes Phys. Rev. D 47 3136
(arXiv:hep-th/9208078)
[26] Kallosh R and Ortin T 1993 Charge quantization of axion-dilaton black holes Phys. Rev. D 48 742
(arXiv:hep-th/9302109)