The generalization of the addition property for soliton type processes

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Abstract
A generalization of the addition relation for the Riemann theta functions and its limiting version for exponential functions appearing in soliton type equations are reported. The presented form seems to be particularly useful when processes in \( N + 1 \), \( (N > 1) \) space-time are analyzed. The commonly applied bilinear and trilinear approaches, restricted to the pure soliton processes, represent particular cases of the reported formalism. As an example, the dispersion equation for either quasiperiodic or soliton processes following 2+1 Calogero-Bogoyavlenskij-Schiff equation is derived.

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Running title: Generalization of addition property...

1 INTRODUCTION
As it is well known, solitons represent the localized excitations and appear in the numerous branches of physics when nonlinear description is essential, as in superconductivity, plasma physics, fiber optics, domain wall or fluid dynamics, protein chains etc. Unfortunately, the well developed soliton theory practically deals with the one-dimensional dynamical systems. The same conclusion relates to the quasi-periodic processes. Because of the physical reasons, many research groups make efforts in order to find the methods for multidimensional solitons analysis. The results obtained hitherto have however only a contributory character. Therefore below we present, it seems, a slightly more general approach to the truly multidimensional solitons and quasi periodic-processes.

In order to illustrate the problem we start with a simple example of the famous Kadomtsev - Petviashvili equation

\[ 3u_{yy} = [4u_t - 6uu_x + u_{xxx}]_x, \]

repeating some arguments of Dubrovin [1]. Symbols \((,)_x\) or \(u_x\) denote relevant partial derivatives. Looking for the solution in form

\[ u = u(x, y, t) = -2 \ln \tau (z_1, ..., z_g), \quad z_i = k_i x + l_i y + w_i t + z_0, i = 1, ..., g, \]

a substitution to the equation (1) leads to

\[ \frac{\tau_{xxxx} \tau - 4 \tau_{xxx} \tau_x + 3 (\tau_{xx})^2}{\tau^2} + 4 \frac{\tau_{xt} \tau - \tau_x \tau_t}{\tau^2} - 3 \frac{\tau_{yy} \tau - (\tau_y)^2}{\tau^2} = -8d, \]

if the first constant of integration is zero, and the second one is equal to \(-8d\). According to the Hirota approach, there is introduced the bilinear differential operator \(D_x\) which if applied to the ordered pair of functions \(f(x, y, t)\) and \(g(x, y, t)\) is defined as

\[ D_x^N (f \circ g) := (\partial_w)^N [f(x + w, y, t) g(x - w, y, t)]_{w=0}. \]

Then equation (3) can be written as

\[ [D_x^4 + 4D_x D_t - 3(D_y)^2] (\tau \circ \tau) + 8d \tau^2 = 0. \]
Next, for \( d = 0 \), making use the standard Hirota procedure one can find multisoliton solutions of (1).

It is worth notice that the case \( d \neq 0 \) is not considered in frame of the bilinear technique since it leads to quasiperiodic solutions and periods of the periodic subprocesses tend to infinity as \( d \to 0 \).

There is an opinion that a huge success of the bilinear formalism in soliton theory [2], [3] can be linked with the addition property for \( \tau \)-functions.

\[
\tau (z + w) \tau (z - w) = \sum_{\varepsilon} W(w; \varepsilon) Z(z; \varepsilon),
\]

where \( z, w \in \mathbb{C}^g, \tau : \mathbb{C}^g \to \mathbb{C}, W, Z : \mathbb{C}^g \times \mathbb{Z}^g \to \mathbb{C} \), the sum is over a finite set and \( Z(z; \varepsilon) \) enumerated by \( \varepsilon \in \mathbb{Z}^g_2 \) form a set of linearly independent functions. Exponential functions appearing in multisoliton solutions have this property as well as the Riemann theta functions leading in turn to the quasiperiodic solutions.

The essential feature of the equation (6) is that the product of shifted \( \tau \)-functions admits factorization.

Relation (6) should be considered only as the \( \tau \)-function property. This means that putting either \( w = 0 \) or \( z = 0 \) we have \( \tau^2(z) = \sum_{\varepsilon} W(0; \varepsilon) Z(z; \varepsilon) \) or \( \tau(w) \tau(-w) = \sum_{\varepsilon} W(w; \varepsilon) Z(0; \varepsilon) \), respectively. If the proper determinants do not vanish, functions \( W(w; \varepsilon) \) and \( Z(z; \varepsilon) \) can be expressed by \( \tau \)-functions.

(Nota bene, if \( \tau \)-function is identified with the Riemann theta functions \( \theta(z) \) - see below, for \( \varepsilon, \varepsilon' \in \mathbb{Z}^g_2 \), we have \( \theta(z + \varepsilon'/2) = \theta(z - \varepsilon'/2) \), then \( \theta(\varepsilon'/2; \varepsilon) = \delta_{\varepsilon, \varepsilon'} \) - Kronecker symbol and \( Z(z; \varepsilon) = \theta^2(z + \varepsilon/2) \).)

Thus it seems that relation (6) is really very powerful and there are a few practical reasons for such statement which we shall discuss below.

First of all, without any additional assumptions for function having addition property one can easy calculate even derivatives of the equation (6), obtaining some combinations of derivatives of \( \tau \)-function logarithm, (6)

\[
2L_{ij} = \sum_{\varepsilon} W_{w_1w_j}(0; \varepsilon) \left[ Z(z; \varepsilon) / \tau^2(z) \right],
\]

\[
L_{ijkl} + 2(3 \times L_{ij} L_{kl}) = \sum_{\varepsilon} W_{w_iw_jw_kw_l}(0; \varepsilon) \left[ Z(z; \varepsilon) / \tau^2(z) \right],
\]

\[
L_{ijklmn} + 2(15 \times L_{ij} L_{klmn}) + 4(15 \times L_{ij} L_{kl} L_{mn}) = \sum_{\varepsilon} W_{w_iw_jw_kw_lw_mw_n}(0; \varepsilon) \left[ Z(z; \varepsilon) / \tau^2(z) \right],
\]

etc., where \( L := \ln \tau(z) \), \( L_i := \partial_{z_i} \ln \tau(z) \) and we use a shorthand notation \( (3 \times L_{ij} L_{kl}) := L_{ij} L_{kl} + L_{ik} L_{jl} + L_{il} L_{kj} \) i.e. including all permutations.

Note that the "basis" functions \( Z(z; \varepsilon) / \tau^2(z) \) for all operators are the same. Therefore, the differentiation rules (6), (8), (9), etc., automatically reconstruct the Korteweg - de Vries - Kotera - Sawada hierarchy giving a tool for a derivation of relevant dispersion equations [1], [2].

In order to elucidate problem of dispersion equations, let us observe that (6) can be written as

\[
L_{xxxx} + 6(L_{xx})^2 + 4L_{xt} - 3L_{yy} = -8d,
\]

and, due to the linear dependence of arguments of \( L = \ln \tau(z_1, ..., z_g) \) on space and time coordinates, as

\[
\sum_{p,q,r,s=1}^{g} k_p k_q k_r k_s \left[ L_{z_p z_q z_r z_s} + 2(3 \times L_{z_p z_q} L_{z_r z_s}) \right] + \sum_{p,q=1}^{g} (4k_p w_q - 3l_p l_q) L_{z_p z_q} = -8d.
\]

The shorthand notation \( 3 \times \ldots \) was explained before. Now, if \( \tau \)-function has the addition property, we can apply rules (7) and (8) obtaining

\[
\sum_{\varepsilon} \left[ \sum_{p,q,r,s=1}^{g} k_p k_q k_r k_s W_{w_p w_q w_r w_s}(0; \varepsilon) + \sum_{p,q=1}^{g} (4k_p w_q - 3l_p l_q) W_{w_p w_q}(0; \varepsilon) + 8d W(0; \varepsilon) \right] Z(z; \varepsilon) = 0.
\]
since \( \tau^2 (z) = \sum_\varepsilon W (0; \varepsilon) Z (z; \varepsilon) \). Finally, the Kadomtsev-Petviashvili equation (1) has a solution if in a relevant class of \( \tau \)-functions (exponential or Riemann theta function), for any \( \varepsilon \in \mathbb{Z}_2^g \) the system of \( 2^g \) algebraic dispersion equations
\[
\sum_{p,q,r,s=1}^g k_pk_qk_rk_sW_{wpqwrs} (0; \varepsilon) + \sum_{p,q=1}^g (4kpw_p - 3lpql) W_{wpqw} (0; \varepsilon) + 8dW (0; \varepsilon) = 0
\]
(13) has a nontrivial solution. The broader discussion, particularly concerning quasiperiodic solutions, one can find in [1]: [2] [3] [4]. Here, we want to underline only a formal similarity of (5) and (13) equations and to underline that equation (13) remains valid also for \( j \neq 0 \), i.e., also for quasiperiodic solutions.

The second argument relates to another class of equations which are solved by means of so-called trilinear operator. The trilinear operator \( T \) and the complex conjugate trilinear operator \( T^* \) are defined as [5]: [6]
\[
T (f \circ g \circ h) = (\partial_{z_1} + j\partial_{z_2} + j^2\partial_{z_3}) \ f (z_1) g (z_2) h (z_3) \big|_{z_1=z_2=z_3=z},
\]
(14)
\[
T^* (f \circ g \circ h) = (\partial_{z_1} + j^2\partial_{z_2} + j\partial_{z_3}) \ f (z_1) g (z_2) h (z_3) \big|_{z_1=z_2=z_3=z},
\]
(15)
where \( j = \exp (i2\pi/3) \).

Its application we illustrate by example of the Satsuma equation [7]
\[
F_{xx}F_{yy}F - F_{xx} (F_y)^2 - F_{yy} (F_x)^2 + 2F_{xy}F_xF_y - (F_{xy})^2 F = 0,
\]
(16) which then can be written then as
\[
(T_xT_{xx}T_yT_{yy} - T_{xx}^2T_{yy}^2) (F \circ F \circ F) = 0.
\]
(17)

Observe now that the addition property for \( F \)-functions in version (1) is not applicable, since (16) consists triads of \( F \) derivatives in contrast to (1) where we have dealt with the pairs only. Therefore the left hand side of a relevant version of addition property (1), if exists, ought to contain the product of three shifted functions, (or even more).

The next argument relates to discrete equations and their reduction to dispersion equations. For example, considering the KdV-type completely integrable difference-difference equation
\[
u (x, t + d) - u (x, t - d) = u (x, t + d) \ u (x, t - d) \ u (x + d, t) - u (x - d, t),
\]
(18)
\[
u (x, t) := \tau (x + d, t) \tau (x - d, t) / \tau (x, t + d) \tau (x, t - d) - 1,
\]
(19)
it is seen that equation (1) enables to write the products \( \tau (x + d, t) \tau (x - d, t) \tau (x, t + d) \tau (x, t - d) \) in a compact form leading to the dispersion equation (1). The shift \( w = d \) plays a role of a step in a difference equation. On the other hand, if the multidimensional version of difference-difference equation is considered, it requires an introduction of a few and independent steps with respect to each coordinate. This in a natural way leads to the form of addition property, where on the left had side the product of a few shifted functions appear.

Now, one can ask about the class of functions having the property (1). The oldest and the most known functions having this property are the Riemann theta functions [8] [9] [10].
\[
\theta (z|B) = \sum_{n \in \mathbb{Z}^g} \exp [i\pi (2 \langle z, n \rangle + \langle n, Bn \rangle)]
\]
(20)
where \( z \in \mathbb{C}^g \), \( B \in \mathbb{C}^{g \times g} \) is the Riemann matrix, (i.e. symmetric with positively defined imaginary part), \( \langle z, n \rangle := \sum_{j=1}^g z_jn_j \). The equivalent of (1) takes then the form
\[
\theta (z + w|B) \theta (z - w|B) = \sum_{\varepsilon \in \mathbb{Z}_2^g} \exp [i\pi (2 \langle z, \varepsilon \rangle + \langle \varepsilon, B\varepsilon \rangle)] \theta (z + B\varepsilon|2B) \exp (i2\pi < 2w, \varepsilon >) \theta (w + B\varepsilon|2B)
\]
(21)
where symbol $\varepsilon \in \mathbb{Z}_g^2$ denote g-fold sum over $\varepsilon_j = 0, 1; i = 1, ..., g$. This relation can be also written down by so called $\theta$ - functions with characteristics.

The second class is represented by exponential functions

$$E \left( z | \tilde{B} \right) = \sum_{n \in \mathbb{Z}_g^2} \exp \left[ i \pi \left( 2 \langle z, n \rangle + \langle n, \tilde{B} n \rangle \right) \right]$$

(22)

which appear in solutions of standard soliton equations. Matrix $\tilde{B} \in \mathbb{C}^{g \times g}$, (although sometimes it is convenient to assume that diagonal elements of $\tilde{B}$ are real). In our opinion this is just a source of the bilinear operator applicability. The combinations of theta and exponential functions leading to processes when solitons propagate on a quasiperiodic background form the next class of functions having addition property.

The fourth class, which to our knowledge has no physical application, is generated by integral of product of Gaussian function and Riemann theta functions with characteristics, where integration takes place with respect to the second characteristics.

Concluding our motivation, the interpretation of the shift $w$ in (21) as a step suggests that for solitons in $N + 1$ space-time a few independent steps $w_1, ..., w_N$ should be introduced. This leads to the generalization of the relation (21) or more precisely of equation (22) and such generalization is just the aim of this note. We believe that the reported below relations can be useful for some multidimensional soliton type problems.

2 THE GENERALIZED ADDITION PROPERTY

In order to repeat the procedure as for (7-9) in case of a few independent variables it would be sufficient to have relation

$$\tau \left( z + u^{(0)} \right) \tau \left( z + u^{(1)} \right) ... \tau \left( z + u^{(J-1)} \right) = \sum_{\varepsilon} W \left( w^{(1)}, ..., w^{(J-1)}; \varepsilon \right) Z \left( z; \varepsilon \right)$$

(23)

for some class of $\tau$ -functions, where $u^{(j)} = u^{(j)} \left( w^{(1)}, ..., w^{(J)} \right) \in \mathbb{C}^g$, $j = 0, ..., J - 1$.

The reason is following: the derivatives of (23) with respect to $w^{(j)}$ will relate only to $W$-function, leaving $Z \left( z; \varepsilon \right)$ unchanged. On the other hand the derivatives of l.h.s.of (23) with respect to $w^{(j)}$ one can change into derivatives of $\tau$ - functions with respect $z_i$ which is necessary for an application in the soliton theory since usually $z$ is linear in space and time variables, ($z := k_x x + k_y y + ... + k_z z + \omega t \in \mathbb{C}^g$). Since as a $\tau$ -function, either exponential or Riemann theta functions are usually chosen and since exponential functions can be considered as a particular case of the second ones, we shall start from the Riemann theta functions.

There are numerous transformations for $\theta$ - functions, but to our knowledge the J-th order addition relation, as here, was considered only by Koizumi and cited in [8]. But his factorization of the r.h.s. was completely different than required in (23) and thus it is rather useless for our purposes. Nevertheless for the standard $\theta$ - functions according to (20) one can prove the identity which coincides with our demand (23), [8].

Theorem 1.

If $z, u^{(k)} \in \mathbb{C}^g$, $k = 0, ..., J - 1$, such that $\sum_{k=0}^{J-1} u^{(k)} = 0$, then

$$\theta \left( z + u^{(0)} | B \right) \theta \left( z + u^{(1)} | B \right) ... \theta \left( z + u^{(J-1)} | B \right) =$$

$$= \sum_{\varepsilon \in \mathbb{Z}_g^2} \exp \left[ i \pi \left( 2 \langle z, \varepsilon \rangle + \langle \varepsilon, B \varepsilon \rangle \right) \right] \theta \left( J z + B \varepsilon | J B \right) \times$$

$$\times \exp \left( i 2 \pi < u^{(0)}, \varepsilon > \right) \theta \left( \begin{pmatrix} u^{(0)} - u^{(1)} + B \varepsilon \\ u^{(0)} - u^{(2)} + B \varepsilon \\ ... \\ u^{(0)} - u^{(J-1)} + B \varepsilon \end{pmatrix} \right) \begin{bmatrix} 2B & B & ... & B \\ B & 2B & ... & B \\ ... & ... & ... & ... \\ B & B & ... & 2B \end{bmatrix}$$

(24)
where all $\theta$-functions are of order $g$, with exception of the last one on the r.h.s. which is of order $(J-1)g$.

The proof we report in Appendix 1.

Observe that the $g$-fold sum now is over $\varepsilon_j = 0, 1, \ldots, J-1$, (it contains $J^g$ elements). If instead of $J$ vectorial parameters $u^{(k)}$, among which only $J-1$ are independent, the new $J-1$ vectorial parameters $w^{(k)}$, $(k = 1, \ldots, J-1)$ are introduced by

\[
\begin{array}{l}
\ u^{(k)} = \left\{ \begin{array}{l}
\sum_{m=1}^{J-1} j^{m} w^{(m+k-J)} + \sum_{m=1}^{J-1} j^{m} w^{(m+k-1)}, \text{ for } k = 1, \ldots, J-1, \\
\sum_{k=1}^{J-1} w^{(k)}, \text{ for } k = 0,
\end{array} \right.
\end{array}
\]

(25)

where $j = \exp(i2\pi/J)$, the relation $\sum_{k=0}^{J-1} u^{(k)} = 0$ is satisfied automatically since $\sum_{k=1}^{J-1} u^{(k)} = -\sum_{k=1}^{J-1} w^{(k)}$.

Thus equation (24) with parameters $u_k$ given by (25) is not unique. This one adopted here however, gives a correspondence with trilinear operators introduced earlier in soliton theory, [7].

For fixed $J$, equations (25) can be inverted, leading to

\[
w^{(k)} = \frac{1}{j-1} \left( u^{(k)} - j u^{(k+1)} (1 - \delta_{k,J-1}) - j u^{(1)} \delta_{k,J-1} \right), \text{ for } k = 1, \ldots, J-1,
\]

(26)

where $\delta_{k,J-1}$ is the standard Kronecker symbol. Note that the choice of $w$ parameters as follows from (25) or (26) is not unique. This one adopted here however, gives a correspondence with trilinear operators introduced earlier in soliton theory, [7].

Applying a procedure denoted as the soliton limit [7] to the identity (24) we are able present rewrite relations (25) for exponential functions (24).

Theorem 2.

\[
E \left( z + u^{(0)} | B \right) E \left( z + u^{(1)} | B \right) \ldots E \left( z + u^{(J-1)} | B \right) = \sum_{\varepsilon \in \mathbb{Z}_J} \left\{ \exp \left[ i\pi 2 \left( \langle z + u^{(0)} | \varepsilon \rangle + \langle z + u^{(1)} | \varepsilon \rangle \right) \right] \right.
\]

\[
\times \left[ \sum_{m \in \mathbb{Z}_J} c(\varepsilon, m) \exp \left[ i\pi 2 \left( J z + \varepsilon B \right) \right] \right] \}
\]

\[
\times \sum_{n^{(1)}, \ldots, n^{(J-1)} \in \mathbb{Z}_J} \left\{ C(J, \varepsilon, n) \exp \left[ i\pi 2 \left( \sum_{k=1}^{J-1} \langle n^{(k)} + B \varepsilon \rangle, n^{(k)} \right) \right] \right\}
\]

where "cut-off" functions $c(\varepsilon, m)$ and $C(J, \varepsilon, n)$ are given by

\[
c(\varepsilon, m) = \prod_{j=0}^{g} \left( \delta_{m_j,0} + \delta_{m_j,1} \delta_{\varepsilon_j,0} \right);
\]

(28)

\[
C(J, \varepsilon, n) = \prod_{j=1}^{g} \left( \sum_{k=0}^{J-1} \delta_{\varepsilon_j,0} \right) \left( \sum_{k=0}^{J-1} \delta_{N_j(J),k-1} \right)
\]

(29)

\[
N_j(J) = \sum_{k=1}^{J-1} n_j^{(k)}; \quad s^{(k)} = u^{(0)} - u^{(k)}, \quad k = 1, \ldots, J-1,
\]

(30)

$u^{(k)}$ are defined by (25) and $\tilde{B}$ matrix is such that $\text{Diag \ Im} \ B \equiv 0$, which however does not infringe a generality.

The proof follows from the observation that exponential functions $E \left( z | B \right)$ can be obtained from theta functions $\theta(\varepsilon | B)$ as a limiting relation.
3 PARTICULAR CASES

For $J = 2$, since $j = -1$, the identity (24) reduces to the commonly known (21) formula, and similarly for exponential functions. For $J = 3$, since then $j = \exp(i2\pi/3)$, the identity (24) reduces to

$$\sum_{\varepsilon \in \mathbb{Z}_3} W\left(\mathbf{w}^{(1)}, \mathbf{w}^{(2)}; \varepsilon\right) Z(\mathbf{z}; \varepsilon),$$

where $\mathbf{z}, \mathbf{w}^{(1)}, \mathbf{w}^{(2)} \in C^g$ and are arbitrary. This relations rewritten for exponential functions shows a close relation with trilinear operator analyzed in papers of [7]

4 AN EXAMPLE OF APPLICATION

Let us consider the Calogero-Bogoyavlenskij-Schiff (CBS) equation ($\eta = 0$), which follows also from a reduction of self-dual Yang-Mills equation and its modification by Grammaticos-Ramani-Hietarinta ($\eta = 1$), of which soliton solutions were reported in [8]. After substitution $\Phi = 2L_x$, where $L = \ln \tau$ this equation reduces to

$$\eta L_{gyy} + 4L_{xxt} + L_{xxyy} + 8L_{xx}L_{xyy} + 4L_{xy}L_{xxx} = C.$$ (35)

We assume that the integration constant $C$ is independent of $y$ and $t$ as well. Moreover, if $\tau = \tau(\mathbf{z}) = \tau(z_1, ..., z_g)$, $z_k = \kappa_k x + \nu_k y + \omega_k t$, $k = 1, ..., g$, the equation (35) becomes ($L_{ijk} := \partial_z \partial_{z_j} \partial_{z_k} \ln \tau(\mathbf{z})$)

$$\sum_{ijk} (\eta \nu_k \nu_j \nu_k + 4 \kappa_i \kappa_j \kappa_k) L_{ijk} +$$

$$= \frac{1}{12} \sum_{ijklm} [(8 \kappa_i \kappa_j \kappa_k \nu_m + 4 \kappa_i \nu_j \kappa_k \kappa_l \kappa_m) (L_{ijklm} + 12L_{ij} L_{klm})] = C.$$
As the $\tau$-function we can choose either $\theta$-functions or exponential $E$-functions, (or even some combinations of both in spirit of those reported in [3]).

On the other hand for $J = 3$, substituting $w = w^{(1)}, v = w^{(2)}$ in (31) for typographic reasons and denoting

$$S(z, w, v) = \tau(z + w + v) \tau(z + jw + j^2v) \tau(z + j^2w + jv),$$

(37)

simple but a little tedious calculations give the necessary relations

$$\partial_{w_i} S(z, w, v)|_{w = v = 0} = S(0, 0, 0) = 3L_{ijk}$$

$$\partial_{w_i, w_j, w_k} S(z, w, v)|_{w = v = 0} = 3[L_{ijkm} + 12L_{ij} L_{klm}].$$

(38)

(39)

Derivatives of (31) with respect to $w$ and $v$ deal with the $W$ function and since $S(z, 0, 0) = [\theta(z)|B]^3$, we have (see Appendix 2)

$$L_{ijk} = \frac{1}{3} \sum_{\varepsilon \in \mathbb{Z}_3^3} [W_{w_i, w_j, w_k}(w, v; \varepsilon)|_{w = v = 0} Z(z; \varepsilon)] / \theta^3(z|B)$$

(40)

$$L_{ijkm} + 12L_{ij} L_{klm} = \frac{1}{3} \sum_{\varepsilon \in \mathbb{Z}_3^3} [W_{w_i, w_j, w_k, w_m}(w, v; \varepsilon)|_{w = v = 0} Z(z; \varepsilon)] / \theta^3(z|B).$$

(41)

Coefficients $W(w, v; \varepsilon)$ and functions $Z(z; \varepsilon)$ are given now either by (32) or (33), respectively, if quasiperiodic solutions are considered or by their soliton limit in spirit of (27).

Considering functions $Z(z; \varepsilon)$ as independent, (36) reduces to the system of 3$^g$ algebraic dispersion equations for any $\varepsilon \in \mathbb{Z}_3^3$

$$\sum_{ijk} (\eta \nu_i \nu_j \nu_k + 4\kappa_i \kappa_j \kappa_k) W_{w_i, w_j, w_k}(w, v; \varepsilon)|_{w = v = 0} +$$

$$\sum_{ijklm} \kappa_i \kappa_j \kappa_k \kappa_l \nu_m W_{w_i, w_j, w_k, w_l, w_m}(w, v; \varepsilon)|_{w = v = 0} = 3CW(w, v; \varepsilon)|_{w = v = 0}$$

(42)

which determines relations between $\kappa_i, \nu_i, \omega_i$ ($i = 1...g$) and also $C$, if $\eta = 1$. Its nontrivial solution, if exists, gives the quasiperiodic solutions of 2+1 CBS equations in form of $\Phi = 2[\ln \theta(z|B)]_x$ with $z_i = \kappa_i x + \nu_i y + \omega_i t$. For higher genus $g$, it gives also the additional conditions on matrix $B$, since system (27) can be overdetermined. Then.

Multisoliton solutions of this equation, reported in paper [8], were found by means of the trilinear operator formalism, where the third roots $\sqrt[3]{T}$ appear as above, but they can be also derived using equation (27) taken for $J = 3$. There is an interesting similarity between trilinear soliton version of CBS equation reported in [8] and the left hand side of equation (42) which when $C \neq 0$ is valid also for quasi-periodic processes. It follows from the fact that trilinear operators $T$ and $T^*$ used in papers [3] and [8] coincide with used here operators $\partial/\partial w$ and $\partial/\partial v$, respectively.

In conclusion, it seems that the presented here generalized addition formula can be useful for other $N + 1$ soliton type equations.

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6 APPENDIX 1

The proof of Theorem 1.

Lemma 1

If $B$ is Riemannian then also matrix $B' = \begin{bmatrix} 2B & \cdots & B \\ \vdots & \ddots & \vdots \\ B & \cdots & 2B \end{bmatrix}$ (with diagonal blocks $2B$ and otherwise $B$) is also Riemannian. Indeed, both matrices $B$ and $B'$ are symmetric. Moreover, if $\langle \mathbf{n}, B \mathbf{n} \rangle \geq 0$, for any $\mathbf{n} \in \mathbb{Z}^g$ (with equality for $\mathbf{n} = \mathbf{0}$), then

$$\left\langle \left( \mathbf{n}_1, \ldots, \mathbf{n}_{j-1} \right), B' \left( \mathbf{n}_1, \ldots, \mathbf{n}_{j-1} \right)^T \right\rangle = \sum_{k=1}^{J-1} \langle \mathbf{n}_k, B \mathbf{n}_k \rangle + \left\langle \left( \sum_{k=1}^{J-1} \mathbf{n}_k \right), B \left( \sum_{k=1}^{J-1} \mathbf{n}_k \right) \right\rangle \geq 0,$$

(with equality for all $\mathbf{n}_k = \mathbf{0}$).

By the definition of $\theta$-function [20], the l.h.s of (44) takes form

$$\sum \sum \exp \left[ \cdots \exp \left[ \cdots \exp \left[ \cdots \exp \left[ \sum \sum \exp \left[ i \pi \left( 2 \sum_{j=0}^{J-1} \left( \left\langle \mathbf{z} + \mathbf{u}^{(j)} \right \rangle, \mathbf{n}^{(j)} \right\rangle + \sum_{j=0}^{J-1} \left\langle \mathbf{n}^{(j)} B \mathbf{n}^{(j)} \right\rangle \right) \right) \right) \right) \right) \right) \right) = (43)$$

Substituting $\mathbf{n} = \mathbf{n}^{(0)} - \sum_{j=1}^{J-1} \mathbf{n}^{(j)}$, we have

$$\sum \sum \exp \left[ i \pi \left( 2 \sum_{j=0}^{J-1} \left( \langle \mathbf{z} + \mathbf{u}^{(0)}, \mathbf{n} - \sum_{i=1}^{J-1} \mathbf{n}^{(i)} \rangle \right) + \sum_{j=1}^{J-1} \langle \mathbf{n}^{(j)} B \mathbf{n}^{(j)} \rangle \right) \right) \right) = (44)$$

Substituting $\mathbf{n} = \mathbf{n}^{(0)} - \sum_{j=1}^{J-1} \mathbf{n}^{(j)}$, we have

$$\sum \sum \exp \left[ i \pi \left( 2 \sum_{j=0}^{J-1} \left( \langle \mathbf{z} + \mathbf{u}^{(0)}, \mathbf{n} - \sum_{i=1}^{J-1} \mathbf{n}^{(i)} \rangle \right) + \sum_{j=1}^{J-1} \langle \mathbf{n}^{(j)} B \mathbf{n}^{(j)} \rangle \right) \right) \right) \right) = (45)$$

Substituting $\mathbf{n} = \mathbf{n}^{(0)} - \sum_{j=1}^{J-1} \mathbf{n}^{(j)}$, we have

$$\sum \sum \exp \left[ i \pi \left( 2 \sum_{j=0}^{J-1} \left( \langle \mathbf{z} + \mathbf{u}^{(0)}, \mathbf{n} - \sum_{i=1}^{J-1} \mathbf{n}^{(i)} \rangle \right) + \sum_{j=1}^{J-1} \langle \mathbf{n}^{(j)} B \mathbf{n}^{(j)} \rangle \right) \right) \right) \right) = (46)$$

Substituting $\mathbf{n} = \mathbf{n}^{(0)} - \sum_{j=1}^{J-1} \mathbf{n}^{(j)}$, we have

$$\sum \sum \exp \left[ i \pi \left( 2 \sum_{j=0}^{J-1} \left( \langle \mathbf{z} + \mathbf{u}^{(0)}, \mathbf{n} - \sum_{i=1}^{J-1} \mathbf{n}^{(i)} \rangle \right) + \sum_{j=1}^{J-1} \langle \mathbf{n}^{(j)} B \mathbf{n}^{(j)} \rangle \right) \right) \right) \right) = (47)$$

Substituting $\mathbf{n} = \mathbf{n}^{(0)} - \sum_{j=1}^{J-1} \mathbf{n}^{(j)}$, we have

$$\sum \sum \exp \left[ i \pi \left( 2 \sum_{j=0}^{J-1} \left( \langle \mathbf{z} + \mathbf{u}^{(0)}, \mathbf{n} - \sum_{i=1}^{J-1} \mathbf{n}^{(i)} \rangle \right) + \sum_{j=1}^{J-1} \langle \mathbf{n}^{(j)} B \mathbf{n}^{(j)} \rangle \right) \right) \right) \right) = (48)$$

Substituting $\mathbf{n} = \mathbf{n}^{(0)} - \sum_{j=1}^{J-1} \mathbf{n}^{(j)}$, we have

$$\sum \sum \exp \left[ i \pi \left( 2 \sum_{j=0}^{J-1} \left( \langle \mathbf{z} + \mathbf{u}^{(0)}, \mathbf{n} - \sum_{i=1}^{J-1} \mathbf{n}^{(i)} \rangle \right) + \sum_{j=1}^{J-1} \langle \mathbf{n}^{(j)} B \mathbf{n}^{(j)} \rangle \right) \right) \right) \right) = (49)$$
The next substitution is \( n = Jm + \varepsilon \), with \( n \in \mathbb{Z}^q \), \( \varepsilon \in \mathbb{Z}^q_j \) and thus \( \sum_{n \in \mathbb{Z}^q} = \sum_{m \in \mathbb{Z}^q} \sum_{\varepsilon \in \mathbb{Z}^q_j} \), due to standard properties of theta functions \([1],[6]\) that

\[
\theta (z + Am|A) = \exp \left\{ -i\pi \left[ 2 \langle z, m \rangle + \langle m, Am \rangle \right] \right\} \theta (z|A), \text{ if } z \in \mathbb{C}^n, A \in \mathbb{C}^{n \times n} \text{ and } m \in \mathbb{Z}^n,
\]

the expression \([49]\) becomes

\[
\begin{align*}
\sum_{m \in \mathbb{Z}^q} \sum_{\varepsilon \in \mathbb{Z}^q_j} & \exp \left\{ i\pi \left[ 2 \langle z + u^{(0)} , (Jm + \varepsilon) \rangle + \langle (Jm + \varepsilon) , B(Jm + \varepsilon) \rangle \right] \right\} \times \\
& \times \theta \left( \begin{bmatrix} u^{(0)} - u^{(1)} + B \varepsilon \\ u^{(0)} - u^{(J-1)} + B \varepsilon \end{bmatrix} + \begin{bmatrix} 2B & \ldots & B \\ \ldots & \ldots & \ldots \\ 2B & \ldots & 2B \end{bmatrix} \begin{bmatrix} m \\ m \end{bmatrix} \begin{bmatrix} 2B & \ldots & B \\ \ldots & \ldots & \ldots \\ 2B & \ldots & 2B \end{bmatrix} \right) \\
& = \sum_{\varepsilon \in \mathbb{Z}^q_j} \exp \left\{ i\pi \left[ 2 \langle z + u^{(0)} , (Jm + \varepsilon) \rangle + \langle (Jm + \varepsilon) , (Jm + \varepsilon) \rangle \right] \right\} \times \\
& \times \exp \left\{ -i\pi \left[ 2 \left( \sum_{j=1}^{J-1} \left( u^{(0)} - u^{(j)} + B \varepsilon \right) , m \right) + \langle m, (J - 1)Bm \rangle \right] \right\} \times \\
& \times \theta \left( \begin{bmatrix} u^{(0)} - u^{(1)} + B \varepsilon \\ u^{(0)} - u^{(J-1)} + B \varepsilon \end{bmatrix} + \begin{bmatrix} 2B & \ldots & B \\ \ldots & \ldots & \ldots \\ 2B & \ldots & 2B \end{bmatrix} \begin{bmatrix} m \\ m \end{bmatrix} \begin{bmatrix} 2B & \ldots & B \\ \ldots & \ldots & \ldots \\ 2B & \ldots & 2B \end{bmatrix} \right) \\
& = \sum_{\varepsilon \in \mathbb{Z}^q_j} \exp \left\{ i\pi \left[ 2 \langle (z + u_0) , \varepsilon \rangle + \langle \varepsilon , B \varepsilon \rangle \right] \right\} \theta \left( Jz + \sum_{j=0}^{J-1} u^{(j)} + B \varepsilon |JB \right) \times \\
& \times \theta \left( \begin{bmatrix} u^{(0)} - u^{(1)} + B \varepsilon \\ u^{(0)} - u^{(J-1)} + B \varepsilon \end{bmatrix} + \begin{bmatrix} 2B & \ldots & B \\ \ldots & \ldots & \ldots \\ 2B & \ldots & 2B \end{bmatrix} \begin{bmatrix} m \\ m \end{bmatrix} \begin{bmatrix} 2B & \ldots & B \\ \ldots & \ldots & \ldots \\ 2B & \ldots & 2B \end{bmatrix} \right)
\end{align*}
\]
i.e. we have obtained the right hand-side of \([24]\) under condition that \( \sum_{j=0}^{J-1} u^{(j)} = 0 \).
7 APPENDIX 2

Algorithm for derivatives of W-function.

The W-function derivatives and thus the combination of derivatives of \( \tau \)-function leading to some hierarchy of pde in \( N+1 \) space-time can be derived algorithmically using e.g. the Mathematica program.

Starting from the more useful form of (23)

\[
\exp \sum_{j=1}^{J} \left[ \ln \tau (z + u^{(j)}) - \ln \tau (z) \right] = \sum_{\varepsilon} W \left( w^{(1)}, \ldots, w^{(J-1)}; \varepsilon \right) \frac{Z(z; \varepsilon)}{[\tau (z)]^j} \tag{54}
\]

with \( u^{(k)} = u^{(k)} (w^{(1)}, \ldots, w^{(J-1)}) \) and differentiating with respect to the components of \( w^{(k)} \) vectors we obtain the expressions for combinations of derivatives of \( \tau \)-function logarithms. A simple algorithm for this calculation we illustrate here giving two examples.

\[
J = 3; \quad j = \exp (i2\pi/J) = \exp (i2\pi/3); \quad w^{(k)} \in C^g, \quad k = 1, 2
\]

\[
\frac{\partial^p+q}{(\partial w^{(1)})^p (\partial w^{(2)})^q} W \left( w^{(1)}, w^{(2)} \right) = \frac{\partial^{p+q}}{(\partial w^{(1)})^p (\partial w^{(2)})^q} \exp \left[ L \left( w^{(1)} + w^{(2)} \right) + L \left( jw^{(1)} + j^2w^{(2)} \right) + L \left( j^2w^{(1)} + jw^{(2)} \right) - 3L (0) \right], \tag{55}
\]

\[
J = 5; \quad j = \exp (i2\pi/J) = \exp (i2\pi/5); \quad w^{(k)} \in C^g, \quad k = 1, \ldots, 4
\]

\[
\frac{\partial^p+q}{(\partial w^{(1)})^p (\partial w^{(2)})^q} W \left( w^{(1)}, \ldots, w^{(4)} \right) = \frac{\partial^{p+q}}{(\partial w^{(1)})^p (\partial w^{(2)})^q} \exp \left[ L \left( w^{(1)} + w^{(2)} + w^{(3)} + w^{(4)} \right) + L \left( jw^{(1)} + j^2w^{(2)} + j^3w^{(3)} + j^4w^{(4)} \right) + L \left( j^2w^{(1)} + j^3w^{(2)} + j^4w^{(3)} + jw^{(4)} \right) + L \left( j^3w^{(1)} + j^4w^{(2)} + j^2w^{(3)} + j^2w^{(4)} \right) + L \left( j^4w^{(1)} + j^3w^{(2)} + j^4w^{(3)} + j^2w^{(4)} \right) - 5L (0) \right], \tag{56}
\]

where have used a shorthand notation \( L (w) := \ln \tau (z + w) \).
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