Microscopic Spectral Density of the Dirac Operator in Quenched QCD

P.H. Damgaard\textsuperscript{a}, U.M. Heller\textsuperscript{b} and A. Krasnitz\textsuperscript{c}

\textsuperscript{a}) The Niels Bohr Institute
Blegdamsvej 17
DK-2100 Copenhagen
Denmark

\textsuperscript{b}) SCRI
Florida State University
Tallahassee, FL 32306-4130
USA

\textsuperscript{c}) Unidade de Ci\^encias Exactas e Humanas
Universidade do Algarve
Campus de Gambelas, P-8000 Faro
Portugal

March 25, 2022

Abstract

Measurements of the lowest-lying eigenvalues of the staggered fermion Dirac operator are made on ensembles of equilibrium gauge field configurations in quenched SU(3) lattice gauge theory. The results are compared with exact analytical predictions in the microscopic finite-volume scaling regime.
The study of the eigenvalue spectrum of the Dirac operator in QCD and related gauge theories has a long history, most notably due to the relation between zero modes, near-zero modes, topology, and chiral symmetry breaking. More recently the subject has undergone a strong revival. This was originally sparked by the observation made by Leutwyler and Smilga [1] that in sectors of fixed topological charge \( \nu \) one can derive exact analytical results for the distribution of small eigenvalues of the Dirac operator. The idea is that there exists a finite-volume scaling regime for the euclidean gauge theory where the partition function can be computed exactly. One crucial input is the assumption that chiral symmetry is spontaneously broken according to the conventional scenario. The finite-size scaling regime is unphysical in the sense that it restricts the associated pseudo-Goldstone bosons to be very light, so light that only their zero-momentum modes contribute significantly to the euclidean path integral. The conditions for this simplification are two-fold. First, the euclidean finite-volume partition function of QCD must be accurately described by the effective hadronic excitations, rather than the underlying quark and gluon degrees of freedom. This imposes the large-volume condition \( \frac{1}{\Lambda_{QCD}} \ll L \), where \( L \) is the linear extent of the four-volume. Second, this linear extent \( L \) should always be insignificant compared with the Compton wavelength of the pseudo-Goldstone bosons: \( L \ll \frac{1}{m_{\pi}} \). While this latter condition is inappropriate for actual, physical, no-so-light pions, it is of course just the right limit for studying spontaneous chiral symmetry breaking. More importantly in this context: it is ideally suited for finite-volume studies of lattice gauge theory.

A major breakthrough in the understanding of the Dirac operator spectrum in the above finite-volume region has been the suggestion by Verbaarschot and co-workers that random matrix theory can give exact analytical predictions for the spectrum in a suitably defined microscopic regime [2, 3]. Gauge theories based on various gauge groups and with fermions in either fundamental or adjoint representations fall into essentially just three different universality classes, which beautifully fit into corresponding classifications of random matrix theory ensembles [4]. In this paper we shall restrict ourselves to SU(3) gauge theory with quarks in the fundamental representation of the gauge group, where universality of all microscopic correlation functions, including the microscopic spectral density itself, has been proven [3]. Moreover, it has recently been shown that all these exact results can also be derived directly from the finite-volume partition function of QCD [8, 9], without recourse to random matrix theory. Also the universal smallest-eigenvalue distribution can be computed in this way [10].

So far the analytical predictions for the microscopic Dirac operator spectrum have been tested in 4-dimensional SU(2) lattice gauge theory with staggered fermions [11], and in 3-dimensional SU(3) lattice gauge theory with staggered fermions [2]. There have also been preliminary studies of the microscopic spectrum in 2-dimensional U(1) lattice gauge theory with a fixed-point Dirac operator [13]. However, none of these cases test the most important universality class of all, namely that of (in the language of random matrix theory) the chiral Unitary Ensemble. This is the universality class of 4-dimensional SU(\( N_c \)) gauge theories with \( N_c \geq 3 \), and quarks in the fundamental representation.

In this paper we shall present the first results for a lattice gauge theory determination of the microscopic Dirac operator spectrum of SU(3) gauge theory with a staggered fermion Dirac operator. The biggest advantage of using a staggered Dirac operator is that the microscopic spectrum at any value of the gauge coupling should coincide with the continuum predictions, since the universality class of this lattice-regularized Dirac operator in this case coincides with the universality class of the continuum Dirac operator (for fermions in the same color representation). The main disadvantage is related to the fact that all comparisons with analytical results should be done in sectors of fixed topological charge \( \nu \).

It is well-known that although a remnant of chiral symmetry is exactly preserved in the lattice theory with staggered fermions, the interplay between gauge-field topology and fermions is a subtle issue in this formulation. In particular, the existence of \( \nu \) fermion zero modes in a gauge field configuration with topological charge \( \nu \) is jeopardized. Earlier studies of the microscopic Dirac operator spectrum
in SU(2) lattice gauge theory with staggered fermions [11] have suggested that to a large degree the effects of non-trivial gauge field topology are insignificant at feasible values of the gauge field coupling, and hence can be ignored at this stage. While we feel that this issue still deserves further clarification, we shall here take the same stand as in ref. [11], and simply sum over all gauge field configurations, irrespective of topology. Comparison should, in the same spirit, be made only with analytical \( \nu = 0 \) predictions.

Before considering the numerical algorithm that computes the Dirac operator eigenvalues in a given gauge field configuration, it is instructive to estimate the number of small eigenvalues it is meaningful to include in our study. The crucial observation here is that in the microscopic scaling region near \( \lambda \sim 0 \), the Dirac operator spectrum will have an average level spacing of order unity. Specifically, we shall be concerned with the rescaled, microscopic, spectral density \[2\]

\[
\rho_s(\zeta) = \frac{1}{V} \rho \left( \frac{\zeta}{V \Sigma} \right),
\]

where \( \Sigma = \langle \bar{\psi} \psi \rangle = \pi \rho(0) \), and \( \rho(\lambda) \) denotes the usual (macroscopic) spectral density. We choose a convention where the chiral condensate is defined, with a summation over the number of colors \( N_c \), by (for one flavor of mass \( m \))

\[
\langle \bar{\psi} \psi \rangle = \frac{1}{V} \frac{\partial}{\partial m} Z(m).
\]

Here \( Z(m) \) is the finite-volume partition function. The macroscopic spectral density is normalized so that \( \rho(\lambda) d\lambda \) is the mean number of eigenvalues per unit volume, as in ref. [1]. The exact, universal, analytical prediction from both random matrix theory [3, 5] and the finite-size partition functions [8, 9] reads

\[
\rho_s(\zeta) = \pi \rho(0) \left[ \frac{\zeta}{2} \left( J_{N_f+\nu}(\zeta) - J_{N_f+\nu+1}(\zeta) \right) J_{N_f+\nu-1}(\zeta) \right],
\]

with \( \zeta \equiv \lambda V \Sigma = \pi \rho(0) V \lambda \). This analytical expression has the following qualitative features: It vanishes at \( \zeta = 0 \) (consistent with the fact that at any finite volume there is no spontaneous chiral symmetry breaking), rises sharply, and then undergoes exponentially damped oscillations around the constant value \( \rho(0) \). The oscillations take place on a scale of order \( \Delta \zeta \sim 1 \), and, in fact, correspond directly to fluctuations over distinct sharp peaks of the first few individual eigenvalues. While the analytical prediction is exact (can be achieved to any degree of accuracy) in the infinite-volume limit with \( L \ll 1/m_\pi \), it is clear that at any finite volume the prediction will fail eventually. In any case, since the distinct feature of the prediction, the oscillations, die out exponentially, we are in fact only interested in averaging over the first very few eigenvalues. It would thus be considerable overkill to compute, for each gauge field configuration, more than a few lowest-lying eigenvalues. We use the Ritz functional method [14] to compute the lowest 10 eigenvalues (and eigenvectors) of \(-\mathcal{D}^2\), with \( \mathcal{D} \) the staggered Dirac operator

\[
\mathcal{D}_{x,y} = \frac{1}{2} \sum_{\mu} \eta_\mu(x) \left( U_\mu(x) \delta_{x+\mu,y} - U_{\mu}^+(y) \delta_{x,y+\mu} \right) = \mathcal{D}_{\epsilon,o} + \mathcal{D}_{o,e}.
\]

Here \( \eta_\mu(x) = (-1)^{\sum_{\nu < \mu} x_{\nu}} \) are the usual staggered phase factors. Introducing \( \epsilon(x) = (-1)^{\sum_{\nu} x_{\nu}} \), we have indicated that \( \mathcal{D} \) connects even, \( \epsilon(x) = 1 \), with odd, \( \epsilon(x) = -1 \), sites and vice versa.

For clarity, we recall some well known properties of the staggered Dirac operator here, see e.g. ref. [15]. The staggered Dirac operator is antihermitian and therefore \(-\mathcal{D}^2\) is hermitian and positive (semi-) definite. \( \epsilon(x) \) plays the rôle that \( \gamma_5 \) plays for the continuum Dirac operator, e.g. \( \{ \mathcal{D}, \epsilon \} = 0 \), and it is easy to see that the eigenvalues of \( \mathcal{D} \) come in pairs \( \pm i\lambda \). If \( \psi(x) \) is the eigenvector with eigenvalue

\[\text{The microscopic spectral density can be viewed as a blow-up of the point } \rho(0) \text{ due to the finite volume. We should thus have } \lim_{\zeta \to \infty} \rho_s(\zeta) = \rho(0), \text{ a matching condition which indeed is satisfied.}\]
\[ i \lambda \text{ then } \epsilon(x) \psi(x) \text{ is the eigenvector with eigenvalue } -i \lambda. \] It is well known that \(-\mathcal{D}^2\) only couples even with even and odd with odd sites. Furthermore, if \(\psi_e\) is a normalized eigenvector of \(-\mathcal{D}^2\) with eigenvalue \(\lambda^2\), non-zero only on even sites, then \(\psi_o = \frac{1}{\sqrt{2}} \mathcal{D}_{o,e} \psi_e\) is also a normalized eigenvector of \(-\mathcal{D}^2\) with eigenvalue \(\lambda^2\), non-zero only on odd sites. The eigenvectors of \(\mathcal{D}\) with eigenvalues \(\pm i \lambda\) are then \(\psi_{\pm}(x) = \frac{1}{\sqrt{2}} (\psi_e(x) \mp i \psi_o(x))\).

In any case, it is clear that to obtain the lowest 10 positive (imaginary) eigenvalues of \(\mathcal{D}\) it is sufficient to compute the lowest 10 eigenvalues of \(-\mathcal{D}_{e,e}^2\), working only on the even sublattice, and then taking the square root. This is the approach we have taken here, for which the Ritz functional method is very well suited.

We have generated configurations that are essentially statistically independent by employing a pure gauge algorithm of typically 3 to 4 microcanonical overrelaxation sweeps, followed by one heatbath update sweep, and repeating this process 10 times between eigenvalue computations. To convert our eigenvalue measurements, or actually a histogram of all measured eigenvalues, into a microscopic spectral density, see eq. (1), we need an estimate of \(\Sigma\), or of \(\rho(0)\), in the infinite volume limit. On a finite lattice \(\rho(\lambda)\), obtained by normalizing a histogram of the lowest eigenvalues appropriately, will oscillate about the infinite volume \(\rho(0)\). We thus obtain \(\rho(0)\) just by averaging the finite volume \(\rho(\lambda)\) over the first few (leaving out the very first) oscillations. Our results, together with information about the number of measurements and the lattice size, is given in Table 1.

Our results here are restricted to the quenched approximation, which, in the analytical context can be achieved simply by taking the \(N_f \to 0\) limit of the formula (3). While the quenched approximation obviously saves an enormous amount of computer time, it also has another unique advantage: The massless limit of the Dirac operator is taken automatically. This means that the “valence Goldstone boson” in principle, were there no lattice artefacts, would be strictly massless. One boundary of the scaling regime, \(L \ll 1/m_\pi\) is thus trivially satisfied from the very beginning. On the lattice, the other boundary, \(1/\Lambda_{QCD} \ll L\), is effectively replaced by the condition that the volume should be large in physical units. This means that the lattice volume in lattice units must increase steeply with the gauge coupling \(\beta = 6/g^2\) if we wish to go towards weak coupling, and continuum scaling. Another way of saying that is as follows: to obtain reasonable estimates of \(\rho(0)\) – i.e. for \(\rho(\lambda)\) to be approximately constant (apart from the predicted oscillations), we need sufficiently large lattices in physical units. Empirically we find that \(L \gtrsim 2N_{\tau,c}(\beta)\) is needed, with \(N_{\tau,c}(\beta)\) denoting the approximate temporal extension of a system to be at the deconfinement transition point at given gauge coupling \(\beta\). For example, we found a \(6^4\) lattice at \(\beta = 5.7\) (\(N_{\tau,c}(5.7) \approx 4\)) and an \(8^4\) lattice at \(\beta = 5.85\) (\(N_{\tau,c}(5.85) \approx 6\)) not to be sufficiently

\begin{table}[h]
\centering
\begin{tabular}{|l|l|l|l|}
\hline
\(\beta\) & \(L\) & \text{meas} & \(\Delta \lambda\) & \(\rho(0)\) \\
\hline
4.0 & 4 & 5000 & 0.0007 & 0.483(3) \\
4.5 & 4 & 4000 & 0.0008 & 0.448(4) \\
5.1 & 4 & 4000 & 0.0010 & 0.368(3) \\
5.1 & 6 & 4000 & 0.00018 & 0.368(3) \\
5.1 & 8 & 1500 & 0.00006 & 0.363(5) \\
5.55 & 6 & 4000 & 0.0003 & 0.192(2) \\
\hline
\end{tabular}
\caption{Our estimates of \(\rho(0)\) for our various gauge field ensembles. The third column gives the number of configurations analyzed, and the forth column the bin size in the histograms.}
\end{table}
large.

We begin our comparison between theory and Monte Carlo results with the distribution of the smallest eigenvalue. Here the analytical prediction, which again has been proven to be universal in the random matrix theory context, and derivable directly from the finite-volume partition functions too \[10\], reads

\[
P_{\text{min}}(\zeta) = \pi \rho(0) \zeta \exp \left[ -\frac{\zeta^2}{4} \right].
\] (5)

Results for different lattice sizes of volume \(L^4\) are shown in the histograms of Fig. 1, which are compared with the analytical prediction (5). For each \(\beta\)-value we have first determined \(\rho(0)\) as described earlier, which means that these plots are all parameter-free. The agreement is seen to be excellent indeed.

Next, we have extracted the spectral density \(\rho(\lambda)\) near \(\lambda \sim 0\) by binning eigenvalues in small intervals (see Table 1), and normalizing the resulting density as described previously. By rescaling the small eigenvalues according to \(\zeta = \pi \rho(0)V\lambda\), we obtain the microscopic spectral density as shown in Fig. 2. Again we find absolutely excellent agreement with the exact analytical prediction (5), which is shown as the full line. We have the poorest statistics for \(\beta = 5.1, L = 8\) (only 1500 independent configurations), and indeed this is the plot for which we get the least impressive agreement. However, taken together, over several different \(\beta\)-values and several different volume sizes, the precise prediction is very well reproduced.

We conclude by noting that the exact analytical predictions for the microscopic Dirac operator spectrum, and the smallest Dirac eigenvalue distribution, now have been successfully confirmed by direct numerical computation in SU(3) lattice gauge theory with staggered fermions. The relevant universality class is, in the language of random matrix theory, that of the chiral Unitary Ensemble. For gauge group \(SU(N_c)\) with \(N_c \geq 3\) and fermions in the fundamental representation, it coincides with the universality class of the continuum Dirac operator.

ACKNOWLEDGEMENT: P.H.D. and U.M.H. acknowledge the support of NATO Science Collaborative Research Grant No. CRG 971487. The work of P.H.D has also been partially supported by EU TMR grant no. ERBFMRXCT97-0122, and the work of U.M.H. has been supported in part by DOE contracts DE-FG05-85ER250000 and DE-FG05-96ER40979. A.K. acknowledges the funding by the Portuguese Fundação para a Ciência e a Tecnologia, grants CERN/S/FAE/1111/96 and CERN/P/FAE/1177/97.
Figure 1: The distribution of the lowest eigenvalue for our different lattice ensembles of Table 1.
Figure 2: The microscopic spectral density for our different lattice ensembles of Table 8.
References

[1] H. Leutwyler and A. Smilga, Phys. Rev. D46 (1992) 5607.

[2] E.V. Shuryak and J.J.M. Verbaarschot, Nucl. Phys. A560 (1993) 306.

[3] J.J.M. Verbaarschot and I. Zahed, Phys. Rev. Lett. 70 (1993) 3852.
   J.J.M. Verbaarschot, Phys. Lett. B329 (1994) 350; Nucl. Phys. B426 (1994) 559.

[4] J.J.M. Verbaarschot, Phys. Rev. Lett. 72 (1994) 2531.

[5] G. Akemann, P.H. Damgaard, U. Magnea and S.M. Nishigaki, Nucl. Phys. B487 (1997) 721.
   P.H. Damgaard and S. Nishigaki, Nucl. Phys. B518 (1998) 495.

[6] J. Jurkiewicz, M.A. Nowak and I. Zahed, Nucl. Phys. B478 (1996) 605; E: B513 (1998) 759.

[7] T. Wilke, T. Guhr and T. Wettig, Phys. Rev. D57 (1998) 6486.

[8] P.H. Damgaard, Phys. Lett. B424 (1998) 322.
   G. Akemann and P.H. Damgaard, Nucl. Phys. B528 (1998) 411; Phys. Lett. B432 (1998) 390.

[9] J. Osborn, D. Toublan and J.J.M. Verbaarschot, [hep-th/9806110].

[10] S.M. Nishigaki, P.H. Damgaard and T. Wettig, [hep-th/9803007].

[11] M.E. Berbenni-Bitsch, S. Meyer, A. Schäfer, J.J.M. Verbaarschot and T. Wettig, Phys. Rev. Lett. 80 (1998) 1146.
   M.E. Berbenni-Bitsch, S. Meyer and T. Wettig, [hep-lat/9804030].

[12] P.H. Damgaard, U.M. Heller, A. Krasnitz and T. Madsen, [hep-lat/9803012], to appear in Phys. Lett. B.

[13] F. Farchioni, I. Hip, C.B. Lang and M. Wohlgenannt, [hep-lat/9809049].

[14] B. Bunk, K. Jansen, M. Lüsch er and H. Simma, DESY Report (Sept. 1994);
   T. Kalkreuter and H. Simma, Comp. Phys. Comm. 93 (1996) 33.

[15] J. Smit and J.C. Vink, Nucl. Phys B286 (1987) 485;
   T. Kalkreuter, Phys. Rev. D48 (1993) 1926.