CHAIN STRUCTURE IN SYMPLECTIC ANALYSIS OF A CONSTRAINED SYSTEM

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Abstract

We show that the constraint structure in the chain by chain method can be investigated within the symplectic analysis of Faddeev-Jackiw formalism.
1 Introduction

In our previous paper [1] we showed that the traditional constraint structure of Dirac formalism [2, 3] can also be obtained from symplectic analysis [5, 6, 7] of Faddeev-Jackiw formalism [4]. In the traditional Dirac method, at each level of consistency, constraints divide to first- and second-class. Hence, the consistency of second class constraints at that level determines a number of Lagrange multipliers, while the consistency of first-class ones leads to constraints of the next level. For this reason this method is called level by level.

An alternative method, called chain by chain, has been recently introduced [8], in which constraints are collected as first- and second-class chains. In this method the consistency of each individual constraint in a definite chain gives the next element of that chain, assuming that it is not the terminating element. Such a chain structure possesses suitable properties in constructing the gauge generating function [9, 10] as well as the process of gauge fixing [11].

In this paper we show that the chain structure can also be derived from the symplectic analysis. In other words, following the singularity properties of symplectic two-form, one can find suitable null-eigenvectors for it such that the resulting constraints emerge in a chain structure. This would be done in section (2). In section (3) we give more technical details about terminating elements of the chains, together with discussing the main properties of the chains (when they are first class and when they are second class). Some examples are given in section (4) and our concluding remarks are given in section (5).

2 Chain Structure

Consider a phase space with coordinates $y^i (i = 1, \ldots, 2N)$ specified by the first order Lagrangian

$$L = a_i(y)\dot{y}^i - H(y)$$

where $H(y)$ is the canonical Hamiltonian of the system. The equations of motion read

$$f_{ij}\dot{y}^j = \partial_i H$$

(2)
where
\[ f_{ij} \equiv \partial_i a_j(y) - \partial_j a_i(y) \]  
(3)

It is called presymplectic tensor. We denote it in matrix notation as \( f \).

Suppose it is non-singular. Let \( f^{ij} \) be the components of the inverse, \( f^{-1} \).

Then from (2) we have
\[ \dot{y}^i = \{y^i, H\}, \]  
(4)

where the Poisson bracket \( \{ , \} \) is defined as
\[ \{F(y), G(y)\} = \partial_i F \partial_j G f^{ij}. \]  
(5)

Suppose we want to impose the set of primary constraints \( \Phi^{(1)} \) to the system. To do this, as stated in [1], one should add the consistency term \( \eta^\mu \Phi_0^{(1)} \) to the canonical Hamiltonian and extend the phase space to include the Lagrange multipliers \( \eta^\mu \). This gives the next order Lagrangian
\[ L^{(1)} = (a_i - \eta^\mu A_{\mu i}) \dot{y}^i - H(y) \]  
(6)

where
\[ A_{\mu i} = \partial_i \Phi_\mu^{(1)}. \]  
(7)

Considering \( Y \equiv (y^i, \eta^\mu) \) as coordinates, the symplectic tensor \( F \) reads
\[ F = \begin{pmatrix} \frac{f}{A} & A \\ -A & 0 \end{pmatrix}. \]  
(8)

The equations of motion in the matrix notation are
\[ F \dot{Y} = \partial H \]  
(9)

Using operations that keep the determinant invariant, it is easy to show that
\[ \det F = \det \begin{pmatrix} \frac{f}{A} & A \\ 0 & A f^{-1} A \end{pmatrix} = (\det f)(\det A f^{-1} A). \]  
(10)

Assuming \( \det f \neq 0 \), \( F \) would be singular if \( C \equiv A f^{-1} A \) is singular. Using (5) and (7) we have
\[ C_{\mu \nu} = \{ \Phi_\mu^{(1)}, \Phi_\nu^{(1)} \}. \]  
(11)
Now we want to give a different approach compared to our previous work in [1]. In that paper we investigated all null eigenvectors of $F$. Here, considering the matrix $C_{\mu\nu}$ in more detail, we concentrate only on the first row (and first column). Suppose $\Phi^{(1)}_1$ has vanishing Poisson bracket with all primary constraints, i.e. $\{\Phi^{(1)}_1, \Phi^{(1)}_\mu\} \approx 0$. Then it is clear that $F$ has the null eigenvector
\begin{equation}
\left( \partial_i \Phi^{(1)}_1 f^{ij}, 1, 0, \ldots, 0 \right) .
\end{equation}

Multiplying both sides of (9) with (12) gives the secondary constraint
\begin{equation}
\Phi^{(2)}_1 = \{ \Phi^{(1)}_1, H \} .
\end{equation}

Next, we consider the consistency of $\Phi^{(1)}_1$ and add the term $\eta^{(2)}_1 \dot{\Phi}^{(2)}_1$ to the Hamiltonian. We array the phase space coordinates as
\[ Y \equiv (y^i; \eta^{(1)}_1, \eta^{(2)}_1; \eta^{(1)}_2, \ldots, \eta^{(1)}_m). \]

Then the matrix $A$ at this stage reads
\begin{equation}
A \equiv \left( \partial_i \Phi^{(1)}_1, \partial_i \Phi^{(2)}_1; \partial_i \Phi^{(2)}_2, \ldots, \partial_i \Phi^{(1)}_m \right) .
\end{equation}

The matrix $C$ should also be improved to
\begin{equation}
C = \begin{pmatrix}
C_{11} & C_{1\nu} \\
C_{\mu 1} & C_{\mu\nu}
\end{pmatrix}
\end{equation}

where
\begin{equation}
C_{\mu\nu}^{(nm)} = \{ \Phi^{(n)}_\mu, \Phi^{(m)}_\nu \}
\end{equation}

(So far, we have $m = 1, 2$ just for $\mu = 1$. However, for $\mu > 1$ only $m = 1$ is present.)

Suppose again that $\Phi^{(2)}_1$ has vanishing Poisson brackets with all primary constraints. Then a new null eigenvector would emerge for $F$ as
\begin{equation}
\left( \partial_i \Phi^{(2)}_1 f^{ij}, 0, 1; 0, \ldots, 0 \right) .
\end{equation}

Multiplying both sides of equations of motion (9) with (17) gives the third level constraint
\begin{equation}
\Phi^{(3)}_1 = \{ \Phi^{(2)}_1, H \} .
\end{equation}

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The process will be continued by adding the term $\eta_{(3)}^1 \dot{\Phi}_{(3)}^1$ to the Hamiltonian and extend the set of coordinates to include $\eta_{(3)}^1$ as well. Proceeding in this way one produces the constraint chain that begins with primary constraint $\Phi_{(1)}^1$ (while keeping the primary constraints $\Phi_{(1)}^2, \Phi_{(1)}^3, \ldots, \Phi_{(1)}^m$ as they are). The first chain terminates if no new constraint emerge at the terminating point, or if the singularity of $F$ due to the constraints of first chain disappears. We postpone the discussion on "how the chain would terminate" to the next section.

The first chain terminated, one should proceed to the next chain beginning with $\Phi_{(1)}^2$. At this stage the (rectangular) matrix $A$ in (8) is as follows

$$A \equiv \left( \partial \Phi_{(1)}^1, \ldots, \partial \Phi_{(N_1)}^1; \partial \Phi_{(1)}^2, \ldots, \partial \Phi_{(1)}^m \right).$$

(19)

Suppose $\Phi_{(1)}^2$ has vanishing Poisson brackets with the constraints of the first chain and with $\Phi_{(1)}^\mu, \mu > 2$ as well. It is easy to find that the symplectic matrix $F$ has the following null eigenvector

$$\begin{pmatrix}
\partial \Phi_{(1)}^2 f^{ij}; \overbrace{0, \ldots, 0}^{N_1}; 1, 0, \ldots, 0
\end{pmatrix}.$$  

(20)

Multiplying with equation of motion gives the next constraint $\Phi_{(1)}^2$ of the second chain. Then the second chain (beginning with $\Phi_{(1)}^2$) can be knitted in the same way as the first one. The second chain terminated, one can produce the constraints of the third chain, and so on. In this way one can construct the whole system of constraints (within the symplectic analysis) as a collection of constraint chains.

3 Terminal elements

In this section we want to see how the constraint chains may terminate. We will also discuss that whether the chains are first or second class. For this reason we begin with a one-chain system. In this case the "chain by chain method" coincides exactly with level by level method investigated in the framework of symplectic analysis in [1]. Suppose the chain, beginning with $\Phi_{(1)}^1$, terminates after $N_1$ steps. Then the rectangular matrix $A$ in (8) is

$$A \equiv \left( \partial \Phi_{(1)}^1, \ldots, \partial \Phi_{(N_1)}^1 \right).$$

(21)
and the matrix elements of $C \equiv \hat{A} f^{-1} A$ would be

$$C^{(nm)} = \{\Phi^{(n)}, \Phi^{(m)}\} \quad n, m = 1, \ldots, N_1. \quad (22)$$

Using the Jacobi identity it is possible to show that

$$C = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & C^{(N_1 - 1)} \\
\vdots & \vdots & \ddots & \vdots \\
0 & C^{(N_1 - 1)2} & \cdots & C^{(N_1 - 1)(N_1 - 1)} & C^{(N_1 - 1)N_1}
\end{pmatrix} \quad (23)$$

and

$$\det C = (-1)^{N_1} \det \left( C^{1N_1} \right)^{N_1}. \quad (24)$$

If $C^{1N_1} \equiv \{\Phi^{(1)}, \Phi^{(N_1)}\} \approx 0$ then from (10) and (24) we have $\det F \approx 0$. In order that $\Phi^{(N_1)}$ be the terminal element, multiplying the null eigenvector

$$\begin{pmatrix}
\partial_i \Phi^{(N_1)} f^{ij} ; 0, \ldots, 1
\end{pmatrix} \quad (25)$$

with the equations of motion (9) should give no new constraint. This would be so if $\{\Phi^{N_1}, H\} \approx 0$.

If, on the other hand, $C^{1N_1} \neq 0$ we have $\det C \neq 0$. This means that all constraints of the chain are second class. Moreover, $\det F \neq 0$ and the symplectic two-form would be invertible. As shown in [7] the inverse of $F$ can be written as

$$F^{-1} = \begin{pmatrix}
\frac{f^{-1} - f^{-1} AC^{-1} \hat{A} f^{-1}}{C^{-1} A f^{-1}} & -f^{-1} AC^{-1} \\
C^{-1} A f^{-1} & C^{-1}
\end{pmatrix}. \quad (26)$$

where $A$ is defined in (21). Then $F^{-1}$ defines a new bracket between functions of the original phase space that is the same as the Dirac bracket[1]. Using (26), the equations of motion (9) can be solved for $\hat{q}^{(n)}$'s. Imposing the results to the Lagrangian and adding a total derivative is equivalent to redefining the original Hamiltonian as

$$H \rightarrow H - \{H, \Phi^{(n)}\} C_{nm} \Phi^{(m)} \quad (27)$$
where $C_{nm}$’s are elements of $C^{-1}$.

The whole procedure is in full agreement with Dirac approach in the framework of chain by chain method[8]. That is, a one chain system is completely first or second class. In the latter case the system is called \textit{self-conjugate}, since the matrix $C$ of the Poisson brackets of constraints of the chain, i.e. (23), is non-singular.

Now let consider a system with two chains. This is the case when we are given two primary constraints, say $\Phi^{(1)}$ and $\Psi^{(1)}$.

Considering the consistency of constraints in the framework of chain by chain method, the authors of [8] have shown that a double chain system may be of four categories. both first class, one first class and one self conjugate second class, both self conjugate second class, and finally two \textit{cross conjugate} second class. In the following we show that the same things may emerge as the results of the symplectic analysis, provided that one follows similar steps of chain by chain method.

\textbf{i) two first class chains}

In the chain by chain method, this happens when the terminal element of first chain, say $\Phi^{(N_1)}$, has vanishing Poisson brackets with primary constraints $\Phi^{(1)}$ and $\Psi^{(1)}$ as well as with Hamiltonian and then knitting the second chain, the same thing is true for the terminal element $\Psi^{(N_2)}$.

In symplectic analysis, the procedure of knitting the first chain is described more or less in section (2), leading to matrix $A$, given in (19). One should notice that the singularity of $F$ given in (8) is not removed at this step. In fact, there are $N_1$ null eigenvectors corresponding to $N_1$ first class constraints of the first chain. However, to begin knitting the second chain, we keep these null eigenvectors as they are and just search for \textit{new null eigenvectors} corresponding to second chain. In other words, to find the $(n+1)$th element of the second chain we use the following null eigenvector:

$$\begin{pmatrix}
\partial_i \Psi^{(n)} f^{ij}; \\
0, \cdots, 0; \\
0, \cdots, 0; \\
1
\end{pmatrix}$$

(28)

Multiplying this with equations of motion (9), as before, gives the constraint

$$\Psi^{(n+1)} = \{\Psi^{(n)}, H\}.$$  

(29)
Knitting the $\Psi$-chain in this manner, we reach finally the terminal element $\Psi^{(N_2)}$ which commute with $H$. We remember that the final symplectic two-form possesses $N_1 + N_2$ null eigenvectors.

**ii) One first and one second class chain**

By now, it may have been clear to the reader that in order to knit the constraint chains similar steps should be followed in symplectic analysis and Dirac formalism. In symplectic analysis we search for appropriate null eigenvectors while in Dirac formalism we investigate directly the consistency conditions.

For the case under consideration suppose the $\Phi$-chain is first class and the $\Psi$-chain is second class. In this case the matrix $F$ has the following from

\[
F = \begin{pmatrix}
    f & A_1 & A_2 \\
    -A_1 & 0 & 0 \\
    -A_2 & 0 & 0
\end{pmatrix}
\]  

(30)

where

\[
A_1^{(n)} = \partial_i \Phi^{(n)} \quad n = 1, \ldots, N_1 \\
A_2^{(n')} = \partial_i \Psi^{(n')} \quad n' = 1, \ldots, N_2
\]  

(31)

Concerning the part $A_1$ (and $-\tilde{A}_1$), the matrix $F$ has $N_1$ null eigenvectors as

\[
\left( \partial_i \Phi^{(n)} f^{ij} ; \underbrace{0, \ldots, 1, \ldots, 0}_{N_1} ; \underbrace{0, \ldots, 0}_{N_2} \right).
\]

However, since the constraints in $\Psi$-chain are second class the singularity in part $A_2$ has been removed. If one omits the columns and and rows of part $A_1$, the remaining matrix

\[
F_{\text{inv}} = \begin{pmatrix}
    f & A_2 \\
    -A_2 & 0
\end{pmatrix}
\]  

(32)

would be invertible, where its inverse is something similar to (26).

If conversely $\Phi$- chain were second class and the $\Psi$- chain were first class, then $A_1$-part would be invertible and $A_2$-part would have $N_2$ null eigenvectors.
iii) two self conjugate second class chains

In Dirac formalism this is the case when both chains are second class and the terminating element of each chain has non-vanishing Poisson bracket with the top element of the same chain. As shown in [8] it is possible to redefine the Hamiltonian and constraints in such a way that constraints of one chain commute with the constraints of the other chain.

Following the steps of chain by chain method in symplectic analysis, one finally reaches to the symplectic two-form shown in (30) and (31); but this time it is invertible. Considering the algebra of Poisson brackets one can show that the inverse can be written as:

\[
F^{-1} = \begin{pmatrix}
    \frac{1}{\det(C)} & -\frac{1}{\det(C)} A_i C_i^{-1} A_i f^{-1} & -f^{-1} A_1 C_1^{-1} & -f^{-1} A_2 C_2^{-1} \\
    C_1^{-1} A_1 f^{-1} & C_1^{-1} & 0 & 0 \\
    C_2^{-1} A_2 f^{-1} & 0 & C_2^{-1} & 0 \\
\end{pmatrix}, \quad (33)
\]

where

\[
C_{nm} = \{ \Phi^{(n)} , \Phi^{(m)} \}
\]

\[
C_{n'm'} = \{ \Psi^{(n')}, \Psi^{(m')} \} \quad (34)
\]

iv) two cross-conjugate second class chains

In this case the chains have the same length and the terminal element of each chain has non-vanishing Poisson bracket with the top element of the other chain [8]. Each constraint in \( \Phi \)-chain finds its conjugate in the \( \Psi \)-chain and vice-versa.

Following all the steps needed to knit the chains, finally the symplectic two-form \( F \), is as written in (30) and (31), noticing that \( N_1 = N_2 \). Suppose \( A \equiv (A_1, A_2) \) is a rectangular matrix with \( 2N_1 \) columns. Then the inverse of \( F \) would be as in (26). It should be noted that the matrix \( C \) of Poisson brackets in this case is as follows:

\[
C = \begin{pmatrix}
    0 & X \\
    -X & 0 \\
\end{pmatrix} \quad (35)
\]

where

\[
X_{nm} = \{ \Phi^{(n)} , \Psi^{(m)} \} \quad (36)
\]
Analyzing the general case, i.e. the multi-chain system, is more or less similar to the two-chain system. The chains are collected as first class, self-conjugate second class, and couples of cross-conjugate second class chains. The essential points to reach such a system of constraints can be understood from the discussions given above, however, the details are complicated and does not lead to any new point.

4 Example

As an example, consider the Lagrangian

$$L = \frac{1}{2} (\dot{x} - ay)^2 - byz - \frac{1}{2} cy^2 - \frac{1}{2} dz^2$$

(37)

where $x$, $y$ and $z$ are variables and $a$, $b$, $c$ and $d$ are parameters. The primary constraints are $\Phi_1^{(1)} = P_y$ and $\Phi_1^{(2)} = P_z$. The canonical Hamiltonian is

$$H_c = \frac{1}{2} P_x^2 + a P_x y + byz + \frac{1}{2} cy^2 + \frac{1}{2} dz^2.$$  

(38)

The secondary constraints are $\Phi_2^{(1)} = a P_x + bz + cy$ and $\Phi_2^{(2)} = by + dz$.

Different types of a two-chain system can be obtained by suitable choices of parameters. For $a = d = 1$ and $b = c = 0$ the chains are

$$\begin{align*}
\Phi_1^{(1)} &= P_y \\
\Phi_1^{(2)} &= P_z \\
\Phi_2^{(1)} &= P_x \\
\Phi_2^{(2)} &= z .
\end{align*}$$  

(39)

The first chain is first-class and the next one is second-chain. For $a = b = 0$ and $c = d = 1$ we have two self-conjugate chain as follows

$$\begin{align*}
\Phi_1^{(1)} &= P_y \\
\Phi_1^{(2)} &= P_z \\
\Phi_2^{(1)} &= y \\
\Phi_2^{(2)} &= z .
\end{align*}$$  

(40)

Finally for $a = c = d = 0$ and $b = 1$ there are two cross-conjugate chains as

$$\begin{align*}
\Phi_1^{(1)} &= P_y \\
\Phi_1^{(2)} &= P_z \\
\Phi_2^{(1)} &= z \\
\Phi_2^{(2)} &= y .
\end{align*}$$  

(41)
Now let discuss the above system in symplectic analysis. The first order Lagrangian is

\[ L = P_x \dot{x} + P_y \dot{y} + P_z \dot{z} - \left( \frac{1}{2} P_x^2 + aP_x y + byz + \frac{1}{2} cy^2 + \frac{1}{2} dz^2 \right) \]  

(42)

with the primary constraints \( P_y \) and \( P_z \). Suppose \((y^1, \ldots, y^6)\) stand for \((x, y, z, P_x, P_y, P_z)\). Adding the consistency term \( \eta^1 \dot{P}_y, \eta^2 \dot{P}_z \) to the Lagrangian (see Eq. 6) the symplectic two-form \( F \) for coordinates

\[ Y \equiv (y^1, \ldots, y^6, \eta^1, \eta^2) \]

is similar to Eq. 8 in which \( f \) is a \( 6 \times 6 \) symplectic matrix and

\[ A = \begin{pmatrix} 
0 & 0 \\
0 & 0 \\
0 & 0 \\
0 & 0 \\
1 & 0 \\
0 & 1 
\end{pmatrix}. \]  

(43)

It has two null-eigenvectors as follows

\[ v_1 = (0, 1, 0, 0, 0, 0; 1, 0) \]
\[ v_2 = (0, 0, 1, 0, 0, 0; 0, 1). \]  

(44)

According to the procedure given in this paper, we should consider them one by one. Multiplying \( v_1 \) form left by the equations of motion (9) gives the second level constraint of the first chain as

\[ \Phi^{(2)}_1 = aP_x + bz + cy. \]  

(45)

Now let consider different choice of parameters:

**i)** suppose \( a = d = 1 \) and \( b = c = 0 \). Then \( \Phi^{(2)}_1 = P_x \) which commute with primary constraints and Hamiltonian. So the singularity of symplectic two-form due to first chain remains in the system, and the first chain terminates at this step. Then multiplying the null eigenvector \( v_2 \) (see Eq. 44) by equations of motion (9) gives the constraint \( \Phi^{(2)}_2 = z \) which is conjugate to \( \Phi^{(1)}_2 = P_z \). In this way the system of constraints (39) are reproduced (one first-class and one second-class).
ii) Suppose \( a = b = 0 \) and \( c = d = 1 \). Then \( \Phi_1^{(2)} = y \) is conjugate to \( \Phi_1^{(1)} = P_y \). Adding the consistency term \( \eta_1^{(2)} \dot{y} \) to the Lagrangian, the singularity of symplectic two-form due to first chain disappears. So the first chain is second class and terminates at \( \Phi_1^{(2)} \). Again from null eigenvector \( v_2 \) the second level constraint \( \Phi_2^{(2)} = z \) emerges \( v_2 \) which is conjugate to \( \Phi_2^{(1)} = P_z \). The singularity due to second chain also disappears by adding \( \eta_2^{(2)} \dot{z} \) to the Lagrangian. As observed, the system of two self-conjugate chain given in (40) is derived from the symplectic analysis.

iii) Suppose \( a = c = d = 0 \) and \( b = 1 \). Then \( \Phi_1^{(2)} = z \) is conjugate to \( \Phi_2^{(1)} = P_z \). According to the algorithm of chain by chain method [8] in such a situation (when the last element of a chain a chain does not commute with some other primary constraint) one should begin to knit the next chain and then investigate the consistency condition of both chains simultaneously. In symplectic analysis, this should be done by considering the next null eigenvector, i.e. \( v_2 \) (see Eq. 44). Multiplying the equations of motion (9) by \( v_2 \) gives the constraint \( \Phi_2^{(2)} = y \). In this way the two cross-conjugate chains (41) would be reproduced.

Adding the consistency term

\[
\eta_1^{(1)} \dot{P}_y + \eta_1^{(2)} \dot{P}_z + \eta_2^{(1)} \dot{z} + \eta_2^{(2)} \dot{y}
\]

to the Lagrangian the \( 6 \times 4 \) matrix \( A \) in the symplectic two-form (8) takes the form

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

It is obvious that singularity of symplectic two-form is removed and using \( F^{-1} \) the brackets induced in phase space (see eq. 5) is the same as Dirac brackets due to second class constraints \( P_y, P_z, z, y \).

5 Conclusion

In this work we showed that the symplectic analysis is able to construct the constraints in a chain structure. In fact, at each stage one can play suitably
with null-eigenvectors of the symplectic tensor to produce any set of desired constraints. To construct the constraint chains one should act with null-eigenvectors one by one, such that at each stage only one chain gains a new constraint.

We think that this work shows once again the essential equivalence between symplectic analysis and the Dirac method. In fact, there are some hidden calculation in Faddeev-Jackiw formalism and symplectic analysis which is more or less equivalent to what done in traditional Dirac method. We tried to show some of theses detailed calculations.

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