Computations over Local Rings in Macaulay2

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Abstract. Local rings are ubiquitous in algebraic geometry. Not only are they naturally meaningful in a geometric sense, but also they are extremely useful as many problems can be attacked by first reducing to the local case and taking advantage of their nice properties. Any localization of a ring $R$, for instance, is flat over $R$. Similarly, when studying finitely generated modules over local rings, projectivity, flatness, and freeness are all equivalent.

We introduce the packages PruneComplex, Localization and LocalRings for Macaulay2. The first package consists of methods for pruning chain complexes over polynomial rings and their localization at prime ideals. The second package contains the implementation of such local rings. Lastly, the third package implements various computations for local rings, including syzygies, minimal free resolutions, length, minimal generators and presentation, and the Hilbert–Samuel function.

The main tools and procedures in this paper involve homological methods. In particular, many results depend on computing the minimal free resolution of modules over local rings.

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I. Introduction

Local rings were first defined by Wolfgang Krull as a Noetherian ring with only one maximal ideal. The name, originating from German "Stellenring," points to the fact that such rings often carry the geometric information of a variety in the neighborhood of a point, hence local.

The procedures presented in this paper are described as pseudocodes and implemented in Macaulay2, a computer algebra software specializing in algebraic geometry and commutative algebra. In many situations, the power of Macaulay2 lies in its algorithms for computing Gröbner bases using monomial orderings. In general, however, there are no known monomial orders for local rings. The only exception to this is for localization with respect to a maximal ideal, for which Mora’s tangent cone algorithm provides a minimal basis suitable for computations. The main motivation for this work is studying local properties of algebraic varieties near irreducible components of higher dimension, such as the intersection multiplicity of higher dimensional varieties.

I.1. Definitions. We begin with basic definitions. A ring shall always mean a commutative ring with a unit.

Definition I.1. A ring $R$ is Noetherian if any non-empty set of ideals of $R$ has maximal members.

An $R$-module $M$ is Noetherian if any non-empty set of submodules of $M$ has maximal members.

Definition I.2. A ring $R$ is called a quasi-local ring if it has only one maximal ideal. A Noetherian quasi-local ring $R$ is called a local ring.

There are many examples of local rings, as defined above, in the nature. For example, the ring $k[[x_1, x_2, \ldots, x_n]]$ of formal power series with $n$ indeterminates is a (complete) local ring where the maximal ideal is the ideal of all ring elements without a constant term.

Definition I.3. Let $R$ be a ring, $M$ an $R$-module, and $S \subset R$ a multiplicative closed set.

The localization of $R$ at $S$ is the ring $R[S^{-1}] = S^{-1}R := \{(r,s) : r \in R, s \in S\}/\sim$, where $(r,s) \sim (r',s')$ if and only if the is $t \in S$ such that $t(s'r - sr') = 0$. The equivalence class of $(r,s)$ is denoted $\frac{r}{s}$. Moreover, there is a canonical homomorphism $\varphi : R \to R[S^{-1}]$ given by $r \mapsto \frac{r}{1}$. This homomorphism induces a bijection between the set of prime ideals of $R[S^{-1}]$ and the set of prime ideals of $R$ that do not intersect $S$.

The localization of $M$ at $S$ is the $R[S^{-1}]$-module $S^{-1}M$ defined similarly. In particular, we have $S^{-1}M = R[S^{-1}] \otimes_R M$ (see [3, Lemma 2.4]).

In general, one can obtain quasi-local rings from any ring $R$ through localization at a prime ideal:

Definition I.4. Let $R$ be a ring and $p \subset R$ be any prime ideal.

The localization of $R$ at $p$ is defined as $R_p = R[S^{-1}]$ where $S = R \setminus p$. For any $R$-module $M$, denote $M_p := S^{-1}M = R_p \otimes_R M$.

Proposition I.5. Let $R$ and $p$ be as defined above, then $(R_p, pR_p)$ is a quasi-local ring.

\[\text{In particular, the inclusion } \text{Spec}R[S^{-1}] \to \text{Spec}R \text{ is a flat morphism.}\]
Proof Recall that $\varphi : R \to R_p$ induces a bijection between the set of prime ideals of $R_p$ and the set of prime ideals of $R$ that do not intersect $S = R \setminus p$, i.e., prime ideals of $R$ that are contained in $p$. In particular, if $q \subset R_p$ is a prime, then $\varphi^{-1}(q) \subset p$, hence $q \subset pR_p = m$. Therefore $m$ contains every prime ideal of $R_p$ and is the unique maximal ideal.

This procedure is canonical in the sense that it satisfies the following universal property:

**Proposition I.6** (Universal Property of Localization). Let $\varphi : R \to R[S^{-1}]$ be given by $r \mapsto r/1$. Suppose there is a ring $R'$ along with a homomorphism $\psi : R \to R'$ such that for any $s \in S$, $\psi(s)$ is a unit in $R'$, then there is a unique homomorphism $\sigma : R[S^{-1}] \to R'$ such that $\psi = \sigma \varphi$ (see [3, pp. 60]).

Another useful property of localization is flatness:

**Definition I.7.** An $R$-module $F$ is flat if $- \otimes_R F$ is an exact functor. That is, if

$$0 \to M'' \to M \to M' \to 0$$

is an exact sequence of $R$-modules, then

$$0 \to M'' \otimes_R F \to M \otimes_R F \to M' \otimes_R F \to 0$$

is again an exact sequence.

**Lemma I.8** ([3, Proposition 2.5]). Let $R$ be a ring and $S \subset R$ be a multiplicative closed set. The localization $R[S^{-1}]$ is a flat $R$-module.

Later, we will see conditions under which this functor is faithfully flat. For now, this lemma has various interesting applications:

**Corollary I.9.** Let $R$ and $S$ be as above and let $p$ be a prime.

1. For any ideal $I \subset R$, $(R/I)_p = R_p/I_p$.
2. Recall that for any $R$-modules $M$ and $N$, $\text{Hom}_R(M, N)$ is the abelian group of $R$-module homomorphisms from $M$ to $N$. In particular, $\text{Hom}_R(M, N)$ is itself an $R$-module, and if $M$ is finitely presented, we have an isomorphism $\text{Hom}_R(S^{-1})((S^{-1}M, S^{-1}N) \cong S^{-1}\text{Hom}_R(M, N)$.
3. For ideals $I, J \subset R$, recall that $J : I = \{r \in R : rI \subset J\}$ is the quotient ideal. More generally, we can write $J : I \cong \text{Hom}_R(R/I, R/J)$. In particular, $J_p : I_p = (J : I)_p$.

**Proof** We present the proofs as they exemplify the usefulness of flatness.

1. Consider the exact sequence of $R$-modules $0 \to I \to R \to R/I \to 0$. Since $- \otimes_R R_p$ is flat, we have another exact sequence $0 \to I_p \to R_p \to (R/I)_p \to 0$. Exactness of this sequence gives $(R/I)_p = \text{coker}(I_p \to R_p) = R_p/I_p$.
2. See [3, Proposition 2.10].
3. Using (1) and (2) we get:

$$J_p : I_p = \text{Hom}_{R_p}(R_p/I_p, R_p/J_p) = \text{Hom}_{R_p}((R/I)_p, (R/J)_p) \cong \text{Hom}_R(R/I, R/J)_p = (J : I)_p.$$

Similar techniques will be used in future sections to prove correctness of procedures.
I.1.1. Geometric Intuition. At this point, it is appropriate to elaborate on the geometric picture behind localization: Let \( A = \mathbb{k}[x_1, \ldots, x_r] \) be the polynomial ring with \( r \) indeterminates and consider an affine variety \( X \) in \( \mathbb{A}_k^r = \text{Spec} A \), the affine \( r \)-space. Suppose \( R \) is the affine coordinate ring of \( X \) and take any point \( P \in X \) corresponding to a prime ideal \( p \subset R \) of functions vanishing at \( P \). Now, in order to study the behavior of \( X \) “near” \( P \), we can invert all functions in \( R \) that do not vanish at \( P \), i.e., adjoin inverses for all functions in \( R \setminus p \). This process yields exactly the local ring \( R_p \), the coordinate ring of the neighborhood of \( P \) in \( X \).

Clearly the local ring of \( R \) at a prime \( p \) is, in general, not finitely generated over \( R \). This is the main obstacle in developing computational methods (such as Gröbner bases) for local rings. However, certain computations remain tractable when we consider finitely generated modules over Noetherian rings.

I.2. Preliminaries. In the next few definitions we describe the main tools used in our computations.

Definition I.10. Let \( R \) be a ring and \( M \) an \( R \)-module.

A projective resolution of \( M \) is an acyclic complex of \( R \)-modules

\[
P_\bullet : \cdots \to P_2 \to P_1 \to P_0 \to M \to 0
\]

such that each \( P_i \) is a projective \( R \)-module. In particular, every \( R \)-module has a projective resolution.

Resolutions are used widely in homological algebra to construct invariants of objects. In particular, using the following proposition we know that every finitely generated module over a local ring has a free resolution.

Proposition I.11. For finitely generated modules over local rings, flatness, projectivity and freeness are all equivalent.

Proof See [4, Theorem 7.10].

An extraordinarily useful tool in the theory of local rings is the following lemma:

Lemma I.12 (T. Nakayama, cf. [3, Corollary 4.8]). Let \( (R, \mathfrak{m}) \) be a quasi-local ring and let \( M \) be a finite \( R \)-module. Then a subset \( u_1, \ldots, u_n \) of \( M \) is a generating set for \( M \) if and only if the set of residue classes \( \{u'_i\} \) is a generating set for \( M/\mathfrak{m}M \) over the field \( R/\mathfrak{m}^2 \).

Corollary I.13. Let \( (R, \mathfrak{m}) \) and \( M \) be as above.

1. If \( M/\mathfrak{m}M = 0 \), then \( M = 0 \).
2. If \( N \) is a submodule of \( M \) such that \( M = \mathfrak{m}M + N \), then \( M = N \).
3. Any generating set for \( M \) contains a minimal generating set for \( M \) as a subset; if \( u_1, \ldots, u_m \) and \( v_1, \ldots, v_n \) are two minimal generating sets of \( M \), then \( m = n \) and there is an invertible \( n \times n \) matrix \( T \) over \( R \) such that \( v = uT \) (see [2, Corollary 5.3]).

Proof See [4, Section 2].

This corollary implies that minimal free resolutions are well defined over local rings, but first we need to make a new definition:

\(^{\text{2}}\)Nagata, following Nakayama, credits this lemma to W. Krull and G. Azumaya, whose generalization of this lemma holds for any ring \( R \) and its Jacobson radical \( \mathfrak{m} \), defined as the intersection of all maximal ideals of \( R \). Note that in local rings the Jacobson radical is the same as the unique maximal ideal
Definition I.14. Let \((R, \mathfrak{m})\) and \(M\) be as above.

Let \(m_1, \ldots, m_n\) be a generating set for \(M\). The relation module of the elements \(m_i\) is the set \(N\) of elements \((r_1, \ldots, r_n) \in R^n\) such that \(\sum r_i m_i = 0\). Clearly \(N\) has an \(R\)-module structure. In particular, there is an exact sequence:

\[
0 \to N \to R^n \to M \to 0.
\]

When \(m_i\) form a minimal generating set for \(M\), \(N\) is called the relations module of \(M\).

The \(i\)-th Syzygy module of \(M\), denoted as \(\text{Syz}_R^i(M)\), is defined as the relation module of the \((i-1)\)-th Syzygy module of \(M\) when \(i > 0\) and \(M\) when \(i = 0\).

Note that the definition above can be extended to other rings, but the resulting module depends on the choice of the generating set. In particular, the corollary above implies that over local rings the syzygies of finite modules are unique up to isomorphism:

Theorem I.15 (\cite{2}, Theorem 26.1). Let \((R, \mathfrak{m})\) be a local ring and \(M\) be a finite \(R\)-module. The \(i\)-th Syzygy module of \(M\) is unique up to isomorphism. In particular, if \(N\) is the relation module of an arbitrary generating set \(u_i\) for \(M\), then \(N = \text{Syz}_R^1(M) \oplus R^k\) for some \(k\).

Proof Follows from I.13 (2).

Remark I.16. Suppose \(\{m_i\}\) is a generating set for \(M\) with \(n_0\) elements and the relation module \(\text{Syz}_R^1(\{m_i\})\) has a generating set with \(n_1\) elements. Then, using the exact sequence in the definition above we can find an exact sequence:

\[
R^{n_1} \to R^{n_0} \to M \to 0.
\]

Repeating this process inductively for \(\text{Syz}_R^i(\{m_i\})\), we can construct a free resolution for \(M\).

Definition I.17. Let \((R, \mathfrak{m})\) be a local ring and let \(M\) be an \(R\)-module.

A minimal free resolution \(F\) of \(M\) is a resolution:

\[
F_\bullet : \cdots \xrightarrow{\partial} F_2 \xrightarrow{\partial} F_1 \xrightarrow{\partial} F_0 \to M \to 0
\]

such that for every differential, \(R/\mathfrak{m} \otimes_R \partial = 0\). That is to say, there are no units in the differentials.

Remark I.18. Theorem I.15 implies that any resolution of \(M\) contains the minimal free resolution as a summand and the minimal free resolution is unique up to changes of basis.

Minimal free resolutions, when they exist, capture many structural invariants of modules, which is why many results are limited to the local and graded cases where Nakayama’s Lemma holds. In particular, everything that follows also applies to the graded case by setting \(\mathfrak{m}\) to be the irrelevant ideal. That is, when \(R = \oplus_{i \geq 0} R_i\) is a finitely generated graded algebra over the field \(k = R_0\) and \(M\) is a finitely generated graded \(R\)-module.

Definition I.19. Let \((R, \mathfrak{m})\), \(M\), \(F_\bullet\) be as above and suppose \(F_i \cong R^{\oplus r_i}\).

The \(i\)-th Betti number of \(M\), \(\beta_i^R(M)\) is defined as the rank of the \(i\)-th module in the minimal free resolution of \(M\):

\[
\beta_i^R(M) = \text{rk}_RF_i = r_i.
\]

Equivalently, this is the minimal number of generators of the \(i\)-th Syzygy module of \(M\):

\[
\text{Syz}_R^i(M) = \ker \partial_{i-1} = \text{Im} \partial_i \cong \text{coker} \partial_{i+1}.
\]
I.3. Artinian Local Rings. Besides dimension and projective dimension, a very useful tool in building invariants for local rings and modules over them is the length. Another useful construction in commutative ring theory is completion. Localization at a prime ideal followed by completion at the maximal ideal a common step in many proofs. Many problems, can be attacked by first reducing to the local case and then to the complete case. In particular, Artinian local rings are automatically complete and therefore share the nice properties of complete local rings. As an example, when \((R, \mathfrak{m})\) is local, \(\hat{R}\) is local and faithfully flat over \(R\).

In this section we review important results leading up to the definition of the Hilbert-Samuel function.

**Definition I.20.** A ring \(R\) is **Artinian** if any descending chain of ideals is finite.

**Definition I.21.** A **composition series** of an \(R\)-module \(M\) is a chain of inclusions:

\[ M = M_0 \supset M_1 \supset \cdots \supset M_n, \]

where the inclusions are strict and each \(M_i/M_{i+1}\) is a simple module.

The **length of** \(M\) is defined as the least length of a composition series for \(M\), or \(\infty\) if \(M\) has no finite composition series.

**Theorem I.22** ([3, Theorem 2.14]). Let \(R\) be a ring \(R\). The following conditions are equivalent:

- \(R\) is Artinian.
- \(R\) has finite length as an \(R\)-module.
- \(R\) is Noetherian and all the prime ideals in \(R\) are maximal.

Let \((R, \mathfrak{m})\) be an Artinian local ring. This theorem implies that \(\mathfrak{m}\) is its only prime ideal.

**Theorem I.23** ([3, Theorem 2.13]). Let \(R\) be any ring and \(M\) be as above. \(M\) has a finite composition series if and only if \(M\) is Artinian. Moreover, every composition series for \(M\) has the same length.

In particular, over local rings we have \(M_i/M_{i+1} \cong R/\mathfrak{m}\) for all \(i\). Note that over an Artinian local ring, the chain \(\mathfrak{m}^i M \supset \mathfrak{m}^{i+1} M \supset \cdots\) eventually terminates. That is to say, \(\mathfrak{m}^n M = \mathfrak{m}^{n+1} M\), hence by Lemma I.12, \(\mathfrak{m}^n M/\mathfrak{m}^{n+1} M = 0\) implies that \(\mathfrak{m}^n M = 0\). Therefore, every composition series is equivalent to a refinement of the chain:

\[ M \supset \mathfrak{m}M \supset \cdots \supset \mathfrak{m}^n M = 0. \]

Therefore, we can compute length by computing the sum of lengths of each term in the sequence above. Recall that by Lemma I.12, \(\mathfrak{m}^i M\) and \(R/\mathfrak{m}\)-vector space \(\mathfrak{m}^i M/\mathfrak{m}^{i+1} M\) have the same length and number of generators. Moreover, the length and number of basis elements of a vector space are equal, so the length of \(\mathfrak{m}^i M\) is the size of its minimal generating set.

**Corollary I.24** ([3, Corollary 2.17]). Let \(R\) be a Noetherian ring and \(M\) be a finite \(R\)-module. The following conditions are equivalent:

- \(M\) has finite length.
- All the primes that contain the annihilator of \(M\) are maximal.
- \(R/\text{Ann}(M)\) is Artinian.
Lastly:

**Corollary I.25** ([3 Corollary 2.18]). Let $R$ and $M$ be as above. Suppose $I = \text{Ann}(M)$ and $p$ is a prime of $R$ containing $I$. Then $M_p = M \otimes_R R_p$ has finite length if and only if $p$ is minimal among primes containing $I$.

In particular, when $M = R/I$, we conclude that $M_p = R_p/I_p$ is Artinian if and only if $p$ is the minimal prime over $I$.

**Definition I.26.** Let $(R, m)$ be a local ring and let $M$ be a finitely generated $R$-module.

Ideal $q \subset m$ is a **parameter ideal for** $M$ if $M/qM$ has finite length. Equivalently, $q$ is a parameter ideal for $M$ if: $\text{rad}(\text{ann}(M/qM)) = m$.

A **system of parameters** is a sequence $(x_1, \ldots, x_d) \subset m$ such that $m^n \subset (x_1, \ldots, x_d)$ for $n >> 0$.

Geometrically, for local ring of a variety $X$ at $P$, a system of parameters is a local coordinate system for $X$ around $P$.

**Definition I.27.** Let $R$ and $M$ be as above and let $q$ be a parameter ideal for $M$.

The **Hilbert-Samuel function of $M$ with respect to parameter ideal $q$** is:

$$H_{q,M}(n) := \text{Length}(q^n M/q^{n+1}M).$$

Note that $q^n M/q^{n+1}M$ is a module over the Artinian ring $R/(q + \text{ann}(M))$, hence the length is finite.

**Corollary I.28.** Let $R$ and $M$ be as above, then $m$ is a parameter ideal for $M$. In particular:

$$\text{Length}_R(M) = \sum_{i=0}^{n} \text{Length}_R(m^i M/m^{i+1}M) = \sum_{i=0}^{n} H_{m,M}(i).$$

II. Elementary Computations

The procedures in this section are implemented in Macaulay2 language and are available across three packages: Localization, LocalRings, and PruneComplex [10]. In order to run the examples, it suffices to load LocalRings as follows:

Macaulay2, version 1.9.2
i1 : needsPackage "LocalRings"

The first step in performing computations over local rings is defining a proper data structure that permits defining the usual objects of study, such as ideals, modules, and complexes over local rings. We have implemented localization of polynomial rings with respect to prime ideals as a limited type of the field of fractions in Localization:

```plaintext
i2 : R = ZZ/32003[x,y,z];
i3 : P = ideal"x,y,z";
i4 : RP = localRing(R, P)
o4 : LocalRing
```

Let $M_p$ be a module over the local ring $R_p$. Our first computation is to find a free resolution of $M_p$. 


Proposition II.1. Let $R_p$ and $M_p$ be as above. Suppose we have a module $N$ over the parent ring $R$ such that $N \otimes_R R_p = M_p$ and suppose $C_\bullet$ is a free resolution of $N$. Then $C_\bullet \otimes_R R_p$ is a free resolution for $M_p$.

Proof Recall from Lemma I.8 that $R_p$ is a flat module over $R$, hence the the sequence $C_\bullet \otimes_R R_p$ remains exact and is a resolution for $N \otimes_R R_p = M_p$. Moreover, if $F$ is a free $R$-module then $F \otimes_R R_p$ is a free $R_p$-module. Therefore $C_\bullet \otimes_R R_p$ is a free resolution for $M_p$. 

Fortunately, efficient procedures for finding free resolutions for homogeneous and non-homogeneous modules over polynomial rings using Gröbner basis methods already exist. Therefore, one way to find a free resolution of $M_p$ is to begin with finding a suitable module $N$, then find a free resolution $C_\bullet$ for $N$ and tensor it with $R_p$ to get a free resolution for $M_p$. 

Remark II.2. Finitely generated modules can be defined in Macaulay2 in four different ways:

- Free module: $F = R^n$.
- Submodule of a free module; such modules arise as the image of a map of free modules:
  \[
  R^m \begin{bmatrix} g_1 & g_2 & \cdots & g_m \end{bmatrix} \rightarrow G \rightarrow 0.
  \]
  The matrix of generators $[g_1 \ g_2 \ \cdots \ g_m]$ captures the data in $G$.
- Quotient of a free module; such modules arise as the cokernel of a map of free modules:
  \[
  R^n \begin{bmatrix} h_1 & h_2 & \cdots & h_m \end{bmatrix} \rightarrow R^n \rightarrow R^n/H \rightarrow 0.
  \]
  A matrix of relations $[h_1 \ h_2 \ \cdots \ h_m]$ captures the data in $R^n/H$.
- Subquotient modules; given two maps of free modules $g : R^m \rightarrow R^{m+n}$ with $G = \text{Im}(g)$ and $h : R^n \rightarrow R^{m+n}$ with $H = \text{Im}(h)$, the subquotient module with generators of $g$ and relations of $h$ is given by $(H + G)/H$. That is to say:
  \[
  R^m \begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix} \rightarrow R^{m+n} \begin{bmatrix} h_1 & h_2 & \cdots & h_m & g_1 & g_2 & \cdots & g_n \end{bmatrix} \rightarrow (H + G)/H \rightarrow 0.
  \]
  Two matrices, one for relations in $H$ and one for generators of $G$ are needed to store $(G + H)/H$.

Note that the class of subquotient modules contains free modules and is closed under the operations of taking submodules and quotients. In particular, submodules are subquotient modules with relations module $H = 0$ and quotients modules are subquotient modules with generator module $G = R^n$.

Proposition II.3. Using the notation above, consider a $R_p$-module $M_p$ with generator module $G_p$ and relation module $H_p$, i.e. $M_p = (G_p + H_p)/H_p$, such that $\frac{a_1}{u_1}, \ldots, \frac{a_m}{u_m}$ generate $G_p \subset R_p$ and $\frac{b_1}{v_1}, \ldots, \frac{b_m}{v_m}$ generate $H_p \subset R_p$ where $h_i, g_i \in R_p$ and $u_i, v_i \in R_p = R \setminus p$. Let $G$ and $H$ be the $R$-modules generated by the $g_i$ and $h_i$, respectively, and let $N = (G + H)/H$ be the $R$-module with generator module $G$ and relation module $H$. Then $N \otimes_R R_p = M_p$. 

\[\text{Note that this methods would also apply to localization with respect to a multiplicative closed set.}\]
**Proof** Consider the exact sequence of $R$-modules:

$$0 \to H \to H + G \to \frac{H + G}{H} \to 0.$$ 

By Lemma I.8 $R_p$ is a flat $R$-module, so we have an exact sequence of $R_p$-modules:

$$0 \to H \otimes_R R_p \to (H + G) \otimes_R R_p \to \left( \frac{H + G}{H} \right) \otimes_R R_p \to 0.$$ 

Hence:

$$N \otimes_R R_p = \left( \frac{H + G}{H} \right) \otimes_R R_p = \left( \frac{H + G}{H} \right) \otimes_R R_p = \frac{H \otimes_R R_p + G \otimes_R R_p}{H \otimes_R R_p}.$$ 

Note that both $G_p$ and $H_p$ are submodules of free module $R^r_p$, therefore it suffices to show $G \otimes_R R_p = G_p$ for any submodule of a free module. Observe that without loss of generality $G_p$ can be generated by $g_i$, as $u_i \in R^*_p$. Note that $mG_p + G \otimes_R R_p = \sum m g_i + \sum R_p g_i = \sum R_p g_i = G_p$. Therefore $G_p = G \otimes_R R_p$ and $H_p = H \otimes_R R_p$ by Corollary I.13. $\square$

**Procedure II.4 (liftUp).** Given a matrix $M_p$ over local ring $R_p$, returns a matrix $M$ over $R$ such that $M_p = R_p \otimes_R M$. This is done by multiplying each column of $M_p$ with the least common denominator of elements in that column, then formally lifting up the matrix to $R$.

**Input:** matrix $MP$ over $RP$  
**Output:** matrix $M$ over $R$ 

**Begin**  
for i from 1 to number of columns of $MP$ do  
    col <- i-th column of $MP$  
    d <- LCM of denominators of elements of col  
    $MP$ <- $MP$ with the i-th column multiplied by $d$  
    $M$ <- formally lift $MP$ from $RP$ to $R$  
RETURN $M$  

End

As shown above, this is enough to lift any module. Given a matrix $M_p$ over local ring $R_p$, we can find maps $g : R^m_p \to R^r_p$ and $h : R^n_p \to R^r_p$ such that $G_p = \text{Im}(g)$ is the relation module of $M_p$ and $H_p = \text{Im}(h)$ is the generator module of $M_p$. The matrix representing the maps is found using Macaulay2 commands **generators** and **relations**. Then the matrices $g$ and $h$ can be lift up, which then give $G$ and $H$ as their images. After that we have the lift up as $M = (G + H)/H$. In Macaulay2, the command **subquotient** returns the subquotient module given the matrices $g$ and $h$.

**Input:** module $MP$ over $RP$  
**Output:** module $M$ over $R$ 

**Begin**  
g <- **generators** $MP$  
g <- liftUp g  
h <- **relations** $MP$  
h <- liftUp h  
M <- **subquotient**($g$, $h$)
Example II.5. Computing free resolution of a Gorenstein ideal of projective dimension 2 with 5 generators [Example 5, pp. 127]:

```
i5 : use RP;
i6 : IP = ideal(x^3 + y^3, x^3 + z^3, x*y/(z+1), x*z/(y+1), y*z/(x+1))
o6 = ideal (x + y , x + z , -----, -----, -----)
      z + 1 y + 1 x + 1

o6 : Ideal of RP
i7 : I = liftUp IP
o7 = ideal (x + y , x + z , x*y, x*z, y*z)
o7 : Ideal of R
i8 : C = res I
1 5 5 1
o8 = R <-- R <-- R <-- R <-- 0
0 1 2 3 4
i9 : C ** RP
1 5 5 1
o9 = RP <-- RP <-- RP <-- RP <-- 0
0 1 2 3 4
```

Caution II.6. A priori there is no reason to believe that this resolution is minimal.

Recall from Theorem I.15 that a free resolution $F_\bullet$ is minimal when there are no units in any of the differentials $\partial : F_i \rightarrow F_{i-1}$. Therefore we can minimize resolutions by iteratively removing all units from the differentials while preserving the complex.

Proposition II.7. Using the notation as before, suppose we have a free resolution for $M$:

$$
\cdots \rightarrow F_{i+1} \xrightarrow{\partial_{i+1}} F_i \xrightarrow{\partial_i} F_{i-1} \xrightarrow{\partial_{i-1}} F_{i-2} \cdots \rightarrow F_0 \rightarrow M \rightarrow 0.
$$

If the matrix of $\partial_i$ contains a unit, we can construct a free resolution $\tilde{F}_\bullet$ for $M$ such that $F_i \cong \tilde{F}_i \oplus R$ and $F_{i-1} \cong \tilde{F}_{i-1} \oplus R$.

Proof

Suppose $F_i \cong R^{\oplus r_i}$ and that for each $i$ we can represent $\partial_i$ as a $r_{i-1} \times r_i$ matrix with entries in $R$. Then $M(\partial_i)$ can be written as:

$$
M(\partial_i) = \begin{bmatrix}
  a_{1,1} & \cdots & a_{1,r_i} \\
  \vdots & & \vdots \\
  a_{r_{i-1},1} & \cdots & a_{r_{i-1},r_i}
\end{bmatrix}.
$$

such that the element $u = a_{m,n}$ is a unit. Note that if $S$ and $T$ are change of coordinate matrices for $F_i$ and $F_{i-1}$ respectively, then without loss of generality we can consider the free resolution:
In particular, we can find invertible matrices $S$ and $T$ such that $S\partial T$ has $u$ in the top left corner of the matrix. This can be done by swapping the $n$-th and first columns of $M(\partial_i)$ while swapping $n$-th and first rows of $M(\partial_{i+1})$, and similarly swapping the $m$-th and first columns of $M(\partial_i)$ while swapping $m$-th and first rows of $M(\partial_{i-1})$.

Now that $u = a_{1,1}$, let $c_j$ denote the $j$-th column. For each $1 < j \leq r_i$, we can use changes of coordinate to add $-a_{1,j}v_1/u$ to the column $c_j$ without losing exactness. As a result, the top row of the matrix is now all zero except for $u$ in the top right corner. Similarly, denoting the $j$-th row by $v_j$, we can zero out the first column (with the exception of $u$) by adding $-a_{j,1}v_1/u$ to the $j$-th row for $1 < j \leq r_i-1$. The result is:

$$M(\partial_i') = \begin{bmatrix}
    u & 0 & 0 & \ldots & 0 \\
    0 & a_{1,1} & a_{1,2} & \ldots & a_{1,r_i} \\
    0 & a_{2,1} & a_{2,2} & \ldots & a_{2,r_i} \\
    \vdots & \vdots & \vdots & \ddots & \vdots \\
    0 & a_{r_i-1,1} & a_{r_i-1,2} & \ldots & a_{r_i-1,r_i}
\end{bmatrix}.$$

That is to say, $\tilde{F}_*$ can be decomposed as direct sum of two exact sequences:

$$\cdots \rightarrow F_{i+1} \xrightarrow{\partial_{i+1}'} R^{\oplus r_{i-1}} \xrightarrow{\varphi} R^{\oplus r_{i-1}} \xrightarrow{\partial_{i-1}'} F_{i-2} \rightarrow \cdots \rightarrow F_0 \rightarrow M \rightarrow 0$$

$$\cdots \rightarrow 0 \xrightarrow{u} R \xrightarrow{u} R \xrightarrow{0} \cdots$$

Therefore, without loss of generality, we can remove the first row of $M(\partial_{i+1})$ and first column of $M(\partial_i)$ as well as first column of $M(\partial_{i-1})$ and first row of $M(\partial_i)$ to get the maps for $\tilde{F}_*$. □

Note that we can also perform the same operation on resolutions of homogeneous or inhomogeneous modules, but in the inhomogeneous case minimal resolutions are not well-defined. This procedure is implemented in `PruneComplex`:

**Procedure II.8 (pruneUnit).** Given a chain complex as a list of compatible matrices $C$ and index $i$ of one of the differentials such that the element in the last row and column is a unit, this procedure prunes the resolution as described by the proposition above.

**Input:** chain complex $C$ and integer $i$

**Output:** chain complex $C$

**Begin**

1. $M \leftarrow C_i$
2. $m \leftarrow$ number of rows of $M$
3. $n \leftarrow$ number of columns of $M$
4. $u \leftarrow M_{(m,n)}$
5. if $C_(i-1)$ exists then
   1. $C_(i-1) \leftarrow C_(i-1)$ with first column deleted
6. for $c$ from 2 to $n$ do
   1. $C_i \leftarrow C_i$ with first column times $(M_{(1,c)}/u)$ subtracted from column $c$
7. $C_i \leftarrow C_i$ with first row deleted
8. $C_i \leftarrow C_i$ with first column deleted
9. if $C_(i+1)$ exists then
   1. $C_(i+1) \leftarrow C_(i+1)$ with first row deleted
One direction for improving this procedure is removing multiple units simultaneously by collecting them in a square matrix concentrated in the lower right corner. In general, before using this procedure, we need to find a suitable unit and move it to the bottom right corner of the matrix. By suitable unit we mean a unit that has the lowest number of terms in its row and column. This distinction is important in order to keep the procedures efficient.

**Procedure II.9** (**pruneDiff**). Given a chain complex as a list of compatible matrices $C$ and index $i$ of one of the differentials, completely prunes the differential by iteratively finding a suitable unit, moving it to the end, then calling **pruneUnit** to remove the unit.

**Input:** chain complex $C$, integer $i$

**Output:** chain complex $C$

**Begin**

```
while there are units in $C_i$ do
    (m,n) <- the coordinates of the unit with the sparsest row and column in $C_i$
    $C_{i-1}$ <- $C_{i-1}$ with column $m$ and first column swapped
    $C_i$ <- $C_i$ with row $m$ and first row swapped
    $C_i$ <- $C_i$ with column $n$ and first column swapped
    $C_{i+1}$ <- $C_{i+1}$ with row $n$ and first row swapped
    pruneUnit($C$, i)
RETURN $C$
```

**End**

Note that the resulting differential does not contain any units, however, it is not necessarily a differential of the minimal complex, as there may be units in the adjacent differentials.

Finally, repeating the previous procedure iteratively for each differential, we can remove all units from all differentials to minimize the free resolution. This task is accomplished in the **pruneComplex** procedure. Note that in general the order of pruning differentials is arbitrary, but in special cases there may be a right choice.

**II.1. Is the Smooth Rational Quartic a Cohen-Macaulay Curve?**

**Example II.10.** In this example we test whether the smooth rational quartic curve is locally a Cohen-Macaulay. Define the rational quartic curve in $\mathbb{P}^3$:

```
i2 : R = ZZ/32003[a..d];
i3 : I = monomialCurveIdeal(R,{1,3,4})
o3 = ideal (b*c - a*d, c - b*d, a*c - b*d, b - a*c)
o3 : Ideal of R
```

Compute a free resolution for $I$:

```
i4 : C = res I
    1 | 4 | 4 | 1
```
Localize the resolution at the origin:

```
      o5 : Ideal of R

    i6 : RM = localRing(R, M);
    i7 : D = C ** RM;
    i7 : E = pruneComplex D

1  4  4  1
---  ---  ---  ---
0  1  2  3
```

That is to say, the rational quartic curve is not Cohen-Macaulay at the origin \( m \). Therefore the curve is not Cohen-Macaulay in general. Now we localize with respect to a prime ideal:

```
      o9 : Ideal of R

    i10 : RP = localRing(R, P);
    i11 : D' = C ** RP;
    i12 : E' = pruneComplex D'

1  2  1
---  ---  ---
0  1  2  3
```

However, the curve is Cohen-Macaulay at the prime ideal \( p \) (and in fact any other prime ideal).

III. Other Computations

Using the procedures described in the previous section many other procedures already implemented for polynomial rings can be extended to work over local rings. In particular, this works well when the result of the procedure can be described in free resolution. The common trick here is to use \texttt{liftUp} to lift the object to a polynomial ring, perform the procedure, tensor the resolution describing the result with \( R_p \), then prune the resolution.

III.1. Computing Syzygy Modules. Perhaps the easiest example of this trick is in finding a minimal resolution for a module, as demonstrated in Section II.1. The steps involved in that example used to implement the following procedure in \texttt{LocalRings}:

\textbf{Procedure III.1 (resolution).} Given a module \( M \) over local ring \( R_p \) returns a minimal free resolution for \( M \). The first step is finding a map of free modules \( f : R_p^n \rightarrow R_p^m \) such that \( M \cong \coker(f) \). This is accomplished using the \texttt{Macaulay2} command \texttt{presentation}, which returns the matrix of the map \( f \).
Input: module M  
Output: chain complex C  

Begin
   RP <- ring M  
f <- presentation M  
f' <- liftUp f  
M' <- coker f'  
C <- resolution M'  
CP <- C ** RP  
CP <- pruneComplex CP  
RETURN CP  
End

To further demonstrate the trick mentioned above, we computing the first syzygy module over a local ring in the next example, then give a short procedure for computing the syzygy.

Example III.2. Define a the coordinate ring of $A^6$:

```
i2 : R = ZZ/32003[vars(0..5)];
```

Define the local ring at the origin:

```
i6 : M = ideal"a,b,c,d,e,f";
i7 : RM = localRing(R, M);
```

Consider the cokernel module of an arbitrary matrix $f$ over $R_m$:

$$
R_m^3 \xrightarrow{f} R_m^3 \rightarrow N \rightarrow 0
$$

```
i9 : f
i9 = | -abc+def 0 -b3+acd |
     | 0 abc-def ab2-cd2-c |
     | ab2-cd2-c -b3+acd 0 |
     3 3
```

Lift the matrix to $R$ and compute its first two syzygies:

```
i10 : f' = liftUp f;
i10 : Matrix R <---- R
i11 : g' = syz f';
i11 : Matrix R <---- R
i12 : h' = syz g';
i12 : Matrix R <---- 0
```

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That is to say, the lift of our module has a free resolution:

$$0 \xrightarrow{h'} R^1 \xrightarrow{g'} R^3 \xrightarrow{f'} R^3 \rightarrow N' \rightarrow 0.$$ 

Now we tensor the differentials with $R_m$:

```plaintext
i13 : g = g' ** RM;
```

```plaintext
o13 : Matrix RM <--- RM
```

```plaintext
i14 : h = h' ** RM;
```

```plaintext
o14 : Matrix RM <--- 0
```

```plaintext
i15 : C = {mutableMatrix g, mutableMatrix h};
```

So our module has a free resolution:

$$0 \xrightarrow{h} R^1_m \xrightarrow{g} R^3_m \xrightarrow{f} R^3_m \rightarrow N \rightarrow 0.$$ 

Now we prune the resolution:

```plaintext
i16 : pruneDiff(C, 1)
```

```plaintext
o16 = {| b3-acd |, 0}
```

```plaintext
| ab2-cd2-c |
```

```plaintext
| -abc+def |
```

That is to say, we have a resolution:

$$0 \xrightarrow{h} R^1_m \xrightarrow{g} R^3_m \xrightarrow{f} R^3_m \rightarrow N \rightarrow 0.$$ 

We can test that our result is a syzygy matrix:

```plaintext
i17 : GM = matrix C#0;
```

```plaintext
i18 : FM * GM == 0
```

```plaintext
o18 = true
```

Recall from Definition [1.19] that $\text{Syz}_{R_p}(N) = \text{Im} \partial_i$ is the syzygy module of $M$. So the first syzygy module of $N$ is given by:

```plaintext
i26 : image GM
```

```plaintext
o26 = image | b3-acd |
```

```plaintext
| ab2-cd2-c |
```

```plaintext
| -abc+def |
```

```plaintext
3
```

Corollary III.3. Let $R_p$ and $M$ be as above. Suppose $N$ is the $R$-module described in II.3. Then $\text{Syz}_{R_p}^1(M) = \text{Syz}_{R}^1(N) \otimes_R R_p$. 

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Proof Recall that the syzygy module of $M$ is the minimal module such that we have a short exact sequence $0 \rightarrow \text{Syz}_R^1(N) \rightarrow R^n \rightarrow N \rightarrow 0$. By Lemma I.8 we have a short exact sequence $0 \rightarrow \text{Syz}_R^1(N) \otimes_R R_p \rightarrow R^n_p \rightarrow N \otimes_R R_p = M \rightarrow 0$. Which proves the corollary. □

The steps above are implemented in the \texttt{syz} procedure in \texttt{LocalRings}:

Procedure III.4 (syz). Given a matrix $M$ over local ring $R_p$ returns the first syzygy matrix of $M$.

Input: matrix M
Output: first syzygy matrix of M

Begin
\begin{align*}
    & \text{RP} \leftarrow \text{local ring of } M \\
    & f' \leftarrow \text{liftUp } M \\
    & g' \leftarrow \text{syz } f' \\
    & h' \leftarrow \text{syz } g' \\
    & g \leftarrow g' \ast \ast \text{ RP} \\
    & h \leftarrow h' \ast \ast \text{ RP} \\
    & C \leftarrow \{g, h\} \\
    & C \leftarrow \text{pruneDiff}(C, 1) \\
    & \text{RETURN } C \#0 \\
\end{align*}
End

III.2. Computing Minimal Generators and Minimal Presentation.

Procedure III.5 (mingens). Given a module $M$ over local ring $R_p$, returns the matrix of minimal generators of $M$. This is accomplished by lifting up the module, finding a resolution of length one (i.e., computing the syzygy once), then localizing the resolution and pruning it to ensure we have minimal generators.

Input: module M
Output: matrix of minimal generators of M

Begin
\begin{align*}
    & \text{RP} \leftarrow \text{ring of } M \\
    & \text{-- Free Module:} \\
    & \quad \text{if } M \text{ is a free module then} \\
    & \quad \text{RETURN } \text{generators } M \\
    & \text{-- Module defined by generators:} \\
    & \quad \text{if } M \text{ is submodule of a free module then} \\
    & \quad \quad f \leftarrow \text{generators } M \\
    & \quad \quad f' \leftarrow \text{liftUp } f \\
    & \quad \quad g' \leftarrow \text{syz } f' \\
    & \quad \quad g \leftarrow g' \ast \ast \text{ RP} \\
    & \quad \quad C \leftarrow \{f, g\} \\
    & \quad \quad C \leftarrow \text{pruneDiff}(C, 1) \\
    & \quad \text{RETURN } C \#0 \\
    & \text{-- Module defined by relations:} \\
    & \quad \text{if } M \text{ is quotient of a free module then} \\
    & \quad \quad f \leftarrow \text{relations } M
\end{align*}
\[
\begin{align*}
f' & \leftarrow \text{liftUp } f \\
g' & \leftarrow \text{syz } f' \\
g & \leftarrow g' \times \text{RP} \\
C & \leftarrow \{f, g\} \\
C & \leftarrow \text{pruneDiff}(C, 1) \\
\text{RETURN } C#0
\end{align*}
\]

-- Module defined by generators and relations:
  if M is a subquotient then
    \[
    \begin{align*}
f & \leftarrow \text{generators } M \\
g & \leftarrow \text{relations } M \\
f' & \leftarrow \text{liftUp } f \\
g' & \leftarrow \text{liftUp } g \\
h' & \leftarrow \text{modulo}(f, g) \\
h & \leftarrow h' \times \text{RP} \\
C & \leftarrow \{h\} \\
C & \leftarrow \text{pruneComplex } C \\
\text{RETURN } C#0
\end{align*}
\]
End

The command \text{modulo} returns a matrix \( h \) whose image is the pre-image of the image of \( g \) under \( f \), i.e., \( \text{Im}(h) = f^{-1}(\text{Im}(g)) \).

A very similar procedure can be used to compute minimal presentation of modules.

\textbf{Procedure III.6 (minimalPresentation).} Given a module \( M \) over local ring \( R_p \), returns the module \( N \cong M \) with minimal generators and relations. This is accomplished by lifting up the module, finding a resolution of length two (i.e., computing the syzygy twice), then localizing the resolution and pruning it to ensure we have minimal generators and relations.

\textbf{Input:} module \( M \)
\textbf{Output:} module \( N \) with minimal generators and relations

\textbf{Begin}
\[
\begin{align*}
\text{RP} & \leftarrow \text{ring of } M \\
-- \text{Free Module:} \\
\text{if } M \text{ is a free module then} \\
N & \leftarrow M \\
-- \text{Module defined by generators:} \\
\text{if } M \text{ is submodule of a free module then} \\
f & \leftarrow \text{generators } M \\
f' & \leftarrow \text{liftUp } f \\
g' & \leftarrow \text{syz } f' \\
h' & \leftarrow \text{syz } g' \\
g & \leftarrow g' \times \text{RP} \\
h & \leftarrow h' \times \text{RP} \\
C & \leftarrow \{g, h\} \\
C & \leftarrow \text{pruneComplex } C \\
N & \leftarrow \text{coker } C#0
\end{align*}
\]
\textbf{End}
The command \texttt{modulo} returns a matrix $h$ whose image is the pre-image of the image of $g$ under $f$, i.e., $\text{Im}(h) = f^{-1}(\text{Im}(g))$.

\textit{Remark III.7.} In the implemented package, the isomorphism $N \simarrow M$ is stored in $N$.\texttt{cache.pruningMap}.
This is done by keeping track of every coordinate change in \texttt{pruneUnit}.

\textbf{III.3. Computing Length and the Hilbert-Samuel Function.} Recall the definition of the Hilbert-Samuel function from Definition I.27. The first step in computing this function is computing length of modules over local rings.

\textbf{Procedure III.8 (length).} Given a module $M$ over local ring $R_p$ returns the length of $M$.

\textbf{Input:} module $M$

\textbf{Output:} integer $n$

\textbf{Begin}

\texttt{(RP, m) <- ring of $M$ and its maximal ideal}

$s \leftarrow 0$

\texttt{REPEAT}

\texttt{N <- minimalPresentation $M$}

$n \leftarrow \text{number of generators of } N$

$M \leftarrow m \ast M$

$s \leftarrow s + n$

\texttt{UNTIL } $n = 0$

\textbf{End}
Now we can compute the Hilbert-Samuel function of a finitely generated module \( M \) with respect to a parameter ideal \( q \) as follows:

**Procedure III.9 (hilbertSamuelFunction).** Given a parameter ideal \( q \) for module \( M \) over local ring \( R_p \) and an integer \( n \), returns the value of the Hilbert-Samuel function at \( n \). Recall that when the parameter ideal is the maximal ideal of \( R_p \), we have \( \text{Length}(m^i N/m^{i+1} N) = \text{Length}(m^i N) \), which is easier to compute.

**Input:** parameter ideal \( q \), module \( M \), integer \( n \)  
**Output:** integer \( r \)

**Begin**

\[
(RP, m) \leftarrow \text{ring of } M \text{ and its maximal ideal} \\
N \leftarrow q^n \ast M \\
\text{if } q = m \text{ then} \\
\quad \text{RETURN length } N \\
\text{else} \\
\quad \text{RETURN length}(N/(q \ast N))
\]

**End**

**Example III.10.** Consider the twisted cubic curve and an embedded curve defined by ideal \( I \):

\[
i2 : R = \ZZ/32003[x,y,z,w];
i3 : P = \text{ideal } "\text{yw-z}\text{2, }xw-yz, \text{zx-y}\text{2}\text{"} \\
\]

\[
o3 = \text{ideal } (-z + y\ast w, -y\ast z + x\ast w, -y + x\ast z) \\
i4 : I = \text{ideal } "z(yw-z)\text{2-w(xw-yz)}, xz-y\text{2}\text{"} \\
o4 = \text{ideal } (-z + 2y\ast z\ast w - x\ast w, -y + x\ast z)
\]

One can check that the radical of \( I \) is \( p \), hence \( I \) is a \( p \)-primary ideal.

\[
i5 : \text{codim } I = \text{ codim } P \\
o5 = \text{true} \\
i6 : \text{radical } I = \text{ P} \\
o6 = \text{true}
\]

In particular, \( p \) is the minimal prime above \( I \), hence by Corollary I.25, \( R_p/IR_p \) is Artinian. Finally, we compute the length and Hilbert-Samuel function of \( R_p/IR_p \):

\[
i7 : RP = \text{localRing}(R, P); \\
i8 : N = RP^1/promote(I, RP) \\
o8 = \text{cokernel } \mid -z3+2yzw-xw2 -y2+zx \mid \\
o8 : RP-module, quotient of RP
\]
IV. EXAMPLES AND APPLICATIONS IN INTERSECTION THEORY

An important application of computation of length over local rings is in intersection theory. In the following examples we compute various functions using the \texttt{LocalRings} package.

\textbf{Example IV.1.} Computing the geometric multiplicity of intersections.

Bézout’s theorem tells us that the number of points where two curves meet is at most the product of their degrees. Consider the parabola $y = x^2$ and lines $y = x$ and $y = 0$:

\begin{verbatim}
i2 : R = ZZ/32003[x,y];
i3 : C = ideal"y-x^2"; -- parabola y=x^2
i4 : D = ideal"y-x"; -- line y=x
i5 : E = ideal"y"; -- line y=0
\end{verbatim}

The (naive) geometric multiplicity of the intersection of the curve $C$ with the curves $D$ and $E$ at the point $(1, 1)$ is given by the length of the Artinian ring $R_{(x-1,y-1)}/(C+D)R_{(x-1,y-1)}$ [7 pp. 9]:

\begin{verbatim}
i6 : use R;
i7 : P = ideal"y-1,x-1";
i8 : RP = localRing(R, P);
i9 : length (RP^1/promote(C+D, RP))
o9 = 1
i10 : length (RP^1/promote(C+E, RP))
o10 = 0
\end{verbatim}

Similarly, we can find the geometric multiplicity of intersections at the origin:

\begin{verbatim}
i11 : use R;
i12 : P = ideal"x,y"; -- origin
i13 : RP = localRing(R, P);
i14 : length(RP^1/promote(C+D, RP))
o14 = 1
i15 : length(RP^1/promote(C+E, RP))
o15 = 2
\end{verbatim}

Now consider the curves $y = x^2$ and $y = x^3$:

\begin{verbatim}
i2 : R = ZZ/32003[x,y];
i3 : C = ideal"y-x^3";
i4 : D = ideal"y-x^2";
i5 : E = ideal"y";
\end{verbatim}
Once again, we can compute the geometric multiplicity of intersections at the origin:

```
i6 : use R;
i7 : P = ideal"x,y";
i8 : RP = localRing(R, P);
i9 : length(RP^1/promote(C+D, RP))
o9 = 2
i10 : length(RP^1/promote(C+E, RP))
o10 = 3
```

And at the point (1, 1):

```
i11 : use R;
i12 : P = ideal"x-1,y-1";
i13 : RP = localRing(R, P);
i14 : length(RP^1/promote(C+D, RP))
o14 = 1
i15 : length(RP^1/promote(C+E, RP))
o15 = 0
```

**Example IV.2.** Computing the Hilbert-Samuel series.

Consider the ring \( \mathbb{Z}/32003 \mathbb{Z}[x, y] \) as a module over itself:

```
i2 : R = ZZ/32003[x,y];
i3 : RP = localRing(R, ideal gens R);
i4 : N = RP^1;
i5 : q = ideal"x2,y3"
2 3
o5 = ideal (x , y )
```

First we compute the Hilbert-Samuel series with respect to the maximal ideal:

```
i6 : for i from 0 to 5 list hilbertSamuelFunction(N, i) -- n+1
o6 = {1, 2, 3, 4, 5, 6}
```

In particular, \( R_p \) is not of finite length. Moreover, we compute the Hilbert-Samuel series with respect to the parameter ideal \((x^2, y^3)\):

```
i7 : for i from 0 to 5 list hilbertSamuelFunction(q, N, i) -- 6(n+1)
o7 = {6, 12, 18, 24, 30, 36}
```

**Example IV.3.** Computing multiplicity of a zero dimensional variety at a the origin [9] Teaching the Geometry of Schemes, §5].

Consider the variety defined by the ideal \((x^5 + y^3 + z^3, x^3 + y^5 + z^3, x^3 + y^3 + z^5)\):

```
i2 : R = QQ[x,y,z];
i3 : RP = localRing(R, ideal gens R);
i4 : I = ideal"x5+y3+z3,x3+y5+z3,x3+y3+z5"
5 3 3 5 3 5 3 3
o4 = ideal (x , y , z )
```

```
\( o4 = \text{ideal}(x + y + z, y + x + z, z + x + y) \)

Now we compute the length:
\[
i5 : M = \text{RP}^1/I
\]
\[
o5 = \text{cokernel} \begin{vmatrix} x_5 + y_3 + z_3 & y_5 + x_3 + z_3 & z_5 + x_3 + y_3 \end{vmatrix}
\]
\[
o5 : \text{RP-module, quotient of RP}
\]
\[
i6 : \text{length}(\text{RP}^1/I)
\]
\[
o6 = 27
\]

Note that the sum of the terms in the Hilbert-Samuel function gives the same number:
\[
i7 : \text{for } i \text{ from } 0 \text{ to } 6 \text{ list hilbertSamuelFunction}(M, i)
\]
\[
o7 = \{1, 3, 6, 7, 6, 3, 1\}
\]

V. Other Examples from Literature

Various other examples of computations involving local rings (as well as updated versions of this thesis) are available online on author’s webpage at: [https://ocf.berkeley.edu/~mahrud/thesis/examples.m2](https://ocf.berkeley.edu/~mahrud/thesis/examples.m2)

VI. Conclusion and Future Directions

Over the course of this thesis, a framework for performing various computations over localization of polynomials with respect to a prime ideal has been established. Therefore the next logical steps are extending the framework to support localization with respect to arbitrary multiplicative closed sets and improving the efficiency of the algorithms. This can be accomplished by developing new theoretical techniques, such as a local monomial order, or by transferring the computations to the Macaulay2 engine.

The author is in particular interested in applying the machinery developed here to study resolutions of singularities and conjectures regarding minimal free resolutions.

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