Critical Exponents of the Chiral Potts Model
from Conformal Field Theory

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The \(Z_N\)-invariant chiral Potts model is considered as a perturbation of a \(Z_N\) conformal field theory. In the self-dual case the renormalization group equations become simple, and yield critical exponents and anisotropic scaling which agree with exact results for the super-integrable lattice models. The continuum theory is shown to possess an infinite number of conserved charges on the self-dual line, which remain conserved when the theory is perturbed by the energy operator.

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The $Z_N$ chiral Potts models, first introduced as models of systems exhibiting commensurate-incommensurate and melting transitions [1,2], have more recently become of interest as exactly solvable lattice models whose Boltzmann weights satisfy the star-triangle relation, yet do not enjoy the so-called ‘difference property’ possessed by previously studied such models [3]. The chiral Potts models are also unusual in that they are intrinsically anisotropic in their critical properties [4,5], unlike most other such lattice critical systems in which rotational symmetry is recovered in the scaling limit, so that they are described by a conformal field theory.

A special case of the chiral Potts model, the so-called superintegrable case, has proved to be especially tractable [2,5,6,7]. It becomes critical at the self-dual point, and the critical exponents describing the specific heat singularity and the anisotropic scaling of the correlation lengths have been obtained exactly: $\alpha = 1 - 2/N$, $\nu_1 = 1$, and $\nu_2 = 2/N$ [4,5]. In addition, the presumably exact conjectures $\beta_j = j(N - j)/2N^2$ have been made [2,4,8] for the exponents governing the order parameters $\langle e^{ij}\theta \rangle \sim (-t)^{\beta_j}$. Interestingly enough, the values found for the ‘thermodynamic’ exponents $\alpha$ and $\beta_j$ agree with those of the isotropic $Z_N$ models [9] obtained when the chirality parameter is set to zero, whose continuum limit at criticality is believed to correspond to the $Z_N$ conformal field theories first studied by Fateev and Zamolodchikov [10]. From the renormalization group point of view, this result is puzzling, since, as will be discussed below, the chiral perturbation of the isotropic model appears to be relevant, and therefore should be expected to modify the critical exponents in an essential way.

In this paper, we give a simple explanation of this apparent paradox, showing why the thermodynamic exponents are not changed. In addition, we give a simple derivation of the anisotropic scaling exponents $\nu_1$ and $\nu_2$. Our methods are strictly
valid only for weak chirality, close to the isotropic point. However, they suggest that
the exponents are in fact independent of the magnitude of the chirality parameter,
to all orders in perturbation theory. The main non-universal feature, which appears
to vary continuously along the critical line, is the orientation of the principal axes
with respect to which anisotropic scaling occurs. In addition, we show that there
is a two-dimensional parameter space in which these models are integrable in the
continuum sense, that is, they possess an infinite number of conserved charges.

In the simplest case, \( N = 3 \), the reduced lattice Hamiltonian for this model on
a square lattice with sites labelled by pairs of integers \((x, y)\) is

\[
S = -\sum_{x,y} (K_x \cos(\Delta_x \theta_{x,y} - \delta_x) + K_y \cos(\Delta_y \theta_{x,y} - \delta_y)) ,
\]

(1)

where the \( \theta_{x,y} \) are angles which are integer multiples of \( 2\pi/3 \), and \( \Delta_\mu \ (\mu = x, y) \) is
the lattice difference operator. The original version of this model, due to Ostlund
[1], corresponds to \( \delta_y = 0 \). The self-dual case, of interest here, occurs for \( \delta_y = \pm i\delta_x \).
The superintegrable point then corresponds to \( \delta_x = \pi/6 \). When \( \delta_x = \delta_y = 0 \), this
model is the ordinary 3-state Potts model, whose continuum limit at the critical
point is described by the \( Z_3 \)-invariant conformal field theory with central charge \( c = \frac{4}{5} \). When the chiral perturbation is turned on, to which operators in the continuum
theory does it couple? On the lattice, \( \delta_\mu \) couples to \( \sin \Delta_\mu \theta \). This transforms under
90° lattice rotations like a vector, and so should, in the continuum limit where full
rotational invariance is recovered, transform as a combination of operators with
spins \( \pm (4n + 1) \), where \( n = 0, 1, \ldots \). The most relevant such operators of the \( Z_3 \)
conformal field theory, which are not total derivatives, have spin \( \pm 1 \), and (labelled
by their scaling dimensions \((h, \tilde{h})\)) are \( \Phi_{(\frac{7}{6}, \frac{7}{6})} \) and \( \Phi_{(\frac{5}{6}, \frac{5}{6})} \). Note that they have
overall scaling dimension \( x = \frac{9}{5} < 2 \). This means that they are relevant in the
renormalization group sense. Considerations of rotational symmetry then imply
that $\delta = \delta_x + i\delta_y$ couples to $\Phi(\frac{7}{5}, \frac{2}{5})$ and $\bar{\delta} = \delta_x - i\delta_y$ to $\Phi(\frac{2}{5}, \frac{7}{5})$.

We now consider the duality properties of these operators. The duality transformation involves three steps: first the Boltzmann weights on each link are Fourier transformed with respect to $\Delta_\mu \theta$, introducing new integer-valued conjugate link variables $n_\mu$, then the sums over the original variables $\theta_{x,y}$ are carried out, which give constraints of the form $\Delta_\mu n_\mu = 0 \mod 3$ at each lattice site. These are then solved by writing $n_\mu \propto \epsilon_{\mu\nu} \tilde{\theta}_\nu$, where the $\tilde{\theta}_\nu$ are angles defined on the dual lattice, and $\epsilon_{\mu\nu}$ is the totally antisymmetric symbol. Since, in the Fourier transform, $n_\mu$ couples linearly to $i\delta_\mu$, it follows that $\tilde{\theta}_\nu$ will couple to $i\epsilon_{\nu\mu} \delta_\mu$. Keeping track of the factors more carefully, we find that under duality,

$$\delta_\mu \to i\epsilon_{\mu\nu} \delta_\nu,$$  \hspace{1cm} (2)

so that $\delta \to \delta$ and $\bar{\delta} \to -\bar{\delta}$. The model with $\bar{\delta} = 0$, $\delta \neq 0$ is thus self-dual. Equivalently we may say that, under duality, $\Phi(\frac{7}{5}, \frac{2}{5}) \to \Phi(\frac{2}{5}, \frac{7}{5})$; while* $\Phi(\frac{2}{5}, \frac{7}{5}) \to -\Phi(\frac{2}{5}, \frac{7}{5})$. This is consistent with the fusion rules of these operators, $\Phi(\frac{2}{5}, \frac{7}{5}) \cdot \Phi(\frac{2}{5}, \frac{7}{5}) \sim \Phi(\frac{2}{5}, \frac{2}{5}) + \cdots$, since the leading thermal operator $\Phi(\frac{2}{5}, \frac{7}{5})$ is odd under duality.

Thus we are led to consider the field theory described by the action

$$S = S_0 + \delta \int \Phi(\frac{7}{5}, \frac{2}{5}) d^2x + \bar{\delta} \int \Phi(\frac{2}{5}, \frac{7}{5}) d^2x + \tau \int \Phi(\frac{2}{5}, \frac{2}{5}) d^2x,$$  \hspace{1cm} (3)

where $S_0$ is the action of the $Z_3$ conformal field theory. The self-dual subspace, with which we shall be mostly concerned, then corresponds to $\bar{\delta} = \tau = 0$. We are therefore considering a conformal field theory perturbed by an irreducible spin one operator. This will turn out to have greatly simplifying consequences.

* This apparent lack of symmetry between $\Phi(\frac{2}{5}, \frac{7}{5})$ and $\Phi(\frac{2}{5}, \frac{7}{5})$ can be traced to a particular choice for $\epsilon_{\mu\nu}$. In fact, there are two possible duality transformations corresponding to each choice of sign. Under the other kind, it is $\Phi(\frac{2}{5}, \frac{7}{5})$ which is even.
Let us consider the renormalization group equation for the dimensionless coupling $g = \delta a^{1/5}$, reflecting the flow of this coupling under changes in the microscopic lattice cut-off $a$. It is conventional to work within a scheme where the lattice cut-off is replaced by one which respects the rotational symmetry of the theory. In particular, we may consider an expansion of the partition function in terms of the connected correlation functions of $\Phi(z, z)$, with an ultraviolet regulator which prohibits any two of the integration arguments approaching more closely than $a$.

Within such a scheme, the coefficients of the renormalization group $\beta$-function to any finite order in perturbation theory are given by integrals over correlation functions evaluated in the unperturbed theory, which has rotational symmetry. This has the consequence that, to all orders in perturbation theory, the renormalization group equation for $g$ is given by its first term only:

$$\dot{g} = \frac{1}{5}g,$$  \hspace{1cm} (4)

because any higher order terms $g^n$ with $n > 1$ would not transform correctly under rotations. Explicit calculations at low orders confirm this result. In the same way, if we consider any scalar perturbation of the action $S$, for example the thermal operator, then its renormalization group equation can contain no dependence on $g$, so that

$$\dot{\tau} = \frac{6}{5}\tau + O(\tau^3) \hspace{1cm} (5)$$

The same is true for all other such scalar quantities which could enter the free energy.

The renormalization group equations, within such a scheme, are therefore trivial. There are only two possible fixed points, $g = 0$ (the ordinary 3-state Potts model) and $g \to \infty$, to which all theories with $g \neq 0$ flow. Note that this fixed point lies at infinity in a rotationally invariant regularization scheme, but in others,
for example a lattice scheme where terms $O(g^{4n+1})$ may appear on the right hand side of Eq. (4), it may move into a finite value. The second equation Eq. (5) implies that the renormalization group eigenvalues of all scalar operators are the same at $g = 0$ and $g \to \infty$. This has the consequence that the exponents governing the singular behaviour of the free energy as a function of temperature, magnetic field, and so on are the same. It is consistent with the observation that they agree also at the superintegrable point.

Next, we discuss the anisotropic scaling of correlation functions. This is a consequence of the appearance of a novel kind of space-time symmetry, which is most easily seen through the properties of the stress tensor. The argument is a simple generalization of that given by Zamolodchikov [11] for perturbations of a conformal field theory by a scalar operator. At the conformal point $\delta = 0$ there are two non-zero components $T = T_{zz}$ and $\bar{T} = T_{\bar{z}\bar{z}}$, where $\partial_{\bar{z}} T = \partial_{z} \bar{T} = 0$. The other components $T_{z \bar{z}}$ and $T_{\bar{z}z}$ vanish. When $\delta \neq 0$, this is no longer true. A correlation function $\langle T(z) \ldots \rangle$ is modified to first order in $\delta$ by a term

$$\delta \int \langle T(z) \Phi(z_1, \bar{z}_1) \ldots \rangle d^2z_1 .$$

Using the operator product expansion

$$T(z) \cdot \Phi(z_1, \bar{z}_1) = \frac{7}{32} \frac{(z - z_1)^2}{\Phi(z, \bar{z})} + \frac{1}{z - z_1} \partial_z \Phi(z_1, \bar{z}_1) + \cdots ,$$

and the fact that $\partial z^{-1} = \frac{\pi}{z} \delta^{(2)}(z - z_1)$, it follows that

$$\partial_{\bar{z}} T = \frac{\pi}{2} (1 - \frac{7}{5}) \delta \Phi + O(\delta^2) .$$

Now we should expect to be able to express all the terms on the right hand side in terms of the renormalized coupling constant and renormalized operators of the conformal field theory. But Eq. (4) implies that $\delta$ needs no renormalization (as may be checked, there are no new ultraviolet divergences in correlation functions),
and a simple dimensional argument shows that the operator multiplying the term of order $\delta^n$ would have scaling dimensions $(2 + 2n/5, 1 - 3n/5)$. Since these must all be non-negative, only the $n = 1$ term is in fact present. It then follows from the conservation equation $\partial_z T + \partial_{\bar{z}} T_{\bar{z}z} = 0$ that

$$T_{\bar{z}z} = -\frac{2}{3}(1 - \frac{7}{5}) \delta \Phi_{(\bar{z}, \bar{z})} ,$$

(9)

and similarly that

$$T_{zz} = -\frac{2}{3}(1 - \frac{2}{5}) \delta \Phi_{(z, \bar{z})} .$$

(10)

The fact that these are not equal implies that rotational symmetry is broken, as expected. Instead we see that

$$T_{\bar{z}z} = -\frac{2}{3} T_{zz} .$$

(11)

This implies the existence of an unusual symmetry, under transformations which are mixtures of rotations and dilatations

$$z \rightarrow e^{3\omega} z , \quad \bar{z} \rightarrow e^{2\omega} \bar{z} ,$$

(12)

where $\omega$ is arbitrary. However, we see immediately that these transformations do not preserve the reality of $z + \bar{z}$ and $i(z - \bar{z})$, and therefore do not appear to be physical. This requires careful discussion. Since $\delta y$ is pure imaginary, the Boltzmann weights are not necessarily real and positive. This means that there is no straightforward interpretation as a statistical model. Several of the usual properties of such systems, such as insensitivity of bulk thermodynamic quantities to boundary conditions, can fail in this case, and it appears that the usual arguments supporting the existence of a continuum limit may not go through either. However, as pointed out by several authors, although the transfer matrix corresponding to Eq. (1) is not real, it is hermitian, as is the quantum hamiltonian which arises
in the limiting case. Therefore such a transfer matrix, or quantum hamiltonian, should define a sensible quantum theory in $1 + 1$ dimensions with the possibility of a continuum limit as a quantum field theory.* Within this picture the symmetry Eq. (12) is now physical, since in real time $z$ and $\bar{z}$ get replaced by light-cone co-ordinates $x^\pm = t \pm x$, while Eq. (11) becomes $T_{-+} = \frac{2}{3} T_{+-}$. The generator of this symmetry is $\frac{5}{2} D + \frac{1}{2} M$, where $D$ is the generator of dilatations, and $M$ that of Lorentz boosts.

This has important consequences for the spectrum. If the hamiltonian and momentum operators are denoted by $H$ and $P$, and we introduce their light-cone combinations $P^\pm = H \pm P$, then it follows from the commutation relations of $H$, $P$, $D$ and $M$ that if $|p^+, p^-\rangle$ is an eigenstate, so is $|e^{-2\omega p^+}, e^{-3\omega p^-}\rangle$. The density of states is constant along curves $p^- = \kappa \delta^{-5/2} (p^+)^{3/2}$, where $\kappa$ is constant. Within perturbation theory in $\delta$, only states with positive $p^+$ and $p^-$, and therefore positive $\kappa$, may arise, since the unperturbed correlation functions have support only inside the light-cone. However, this does not rule out the possibility of non-perturbative states with $\kappa < 0$. For the spectrum of $H$ to be bounded from below, there must be a minimum value of $\kappa$, say $\kappa_{\text{min}}$. If $\kappa_{\text{min}} < 0$, the ground state will then lie on this curve, and will have non-zero eigenvalue of $P$, implying a spontaneous breaking of translation symmetry. In fact, exact calculations at the superintegrable point [12], and numerical calculations for more general values of the chiral parameter [13], indicate that this in fact happens. As the critical line is approached, a whole sequence of first-order transitions takes place in which states with non-zero momentum cross the lowest energy state in the zero-momentum

* These observations do not invalidate our previous assertions about the structure of the renormalization group equations, which were made within the context of perturbation theory, which is well-defined in Euclidean space. However, it should also be possible to work entirely in real time, and require a perturbative regularization procedure which respects Lorentz invariance.
sector. The results for the critical exponents, however, refer to quantities evaluated in the perturbative sector, whose lowest energy state we denote by $|\Psi_0\rangle$.

A two-point correlation function of an arbitrary scalar operator $\phi$ can be written as an integral over a spectral density

$$
\langle \Psi_0 | T[\phi(x^+, x^-)\phi(0, 0)] | \Psi_0 \rangle = \int \rho(p^+, p^-) e^{ip\cdot x} dp^+ dp^- ,
$$

where

$$
\rho(p^+, p^-) = \sum_n |\langle \Psi_0 | \phi | n \rangle|^2 \delta(p_n^+ - p^+) \delta(p_n^- - p^-)
$$

which will have the scaling form

$$
\rho(p^+, p^-) = (p^+ p^-)^{-1} f \left( \frac{\delta(p^+)^{2/5}}{(p^-)^{3/5}} \right) ,
$$

where we have used the facts that $\phi$ transforms under dilatations according to its scaling dimension $x_\phi$, and is a Lorentz scalar, and that $|\Psi_0\rangle$ is annihilated by $P^+$ and $P^-$. All this is supposed to be valid on the self-dual line. When the thermal perturbation is added, the scaling function $\Phi$ will also depend on the additional scaling variable $\tau (p^+ p^-)^{-3/5}$ (where the exponent $\frac{3}{5}$ reflects the value of the thermal eigenvalue.) We may write this two-variable scaling form alternatively as

$$
\rho(p^+, p^-) = (p^+ p^-)^{x_\phi - 1} \tilde{f} \left( p^+ \xi^-, p^- \xi^+ \right) ,
$$

where

$$
\xi^+ \sim \delta^{-1} \tau ,
\xi^- \sim \delta^{-1} \tau^{-2/3} .
$$

We therefore identify the anisotropic correlation length exponents $\nu_1 = 1$ and $\nu_2 = 2/3$.

It will be noticed immediately that the principal axes with respect to which this anisotropic scaling takes place are $x^\pm$, not $t$ and $x$ as is found in studies of
the superintegrable case [2,4]. We now offer an explanation of this. The lattice
theory differs from the continuum action Eq. (3) by terms which break rotational,
or Lorentz, symmetry. These terms are irrelevant in the isotropic theory at the
critical point, since they correspond to operators of spin 4 or higher. When such an
isotropic critical theory is perturbed by a scalar operator, it is generally true that
such operators remain irrelevant, at least in the scaling region. This is consistent
with the observation that no new terms in their renormalization arise from the
scalar perturbation. When a theory is perturbed by a non-scalar operator, however,
this is no longer valid. Indeed, in the present case the coupling constant $g_s$ of an
operator of spin $s$ can be renormalized at order $g^s$ by the interaction, so that its
renormalization group equation has the form

$$\dot{g}_s = y_s g_s + \text{const.} g^s + \cdots .$$

(18)

A particular case of interest is the $(zz)$ component of the stress tensor $T$ whose
coupling $g_2$ will get a contribution at order $g^2$. In fact, this may be computed from
the operator product expansion $\Phi(\frac{7}{5}, \frac{2}{5}) \cdot \Phi(\frac{7}{5}, \frac{2}{5}) \sim \frac{7}{2} T + \cdots$ to give

$$\dot{g}_2 = \frac{7\pi}{2} g^2 + \cdots .$$

(19)

The stress tensor is an example of a redundant operator. When added to the action,
it does not modify the critical exponents, but instead its effect may be removed
by an appropriate co-ordinate transformation. In our case, this is accomplished by
sending*

$$x^+ \rightarrow x^+ + 7\pi^2 g^2 x^- .$$

(20)

There are, in addition, higher order effects in $a$ which come from terms in Eq. (19)
like $g_4 g^6$, and so on. All of these will be non-universal, and cannot be calculated

* The operator product expansion coefficient $\frac{7}{2}$ comes from the scaling dimension $\frac{7}{5}$ normal-
ized by $\frac{2}{5} = \frac{2}{5}$. The extra factor of $2\pi$ in Eq. (20) arises from that in the conventional
definition of the stress tensor.
within our continuum approach. Note that all these additional operators do not affect the simple renormalization group equation Eq. (4) for \( g \), since they are all have even spin. Although this shows that non-universal lattice effects may rotate the principal scaling axes away from the light-like directions \( x^\pm \), it does not explain why the axes should be precisely parallel to \( t \) and \( x \) at the superintegrable point. Perhaps there is a hidden symmetry at this point which requires such an alignment.

Although the arguments above have been presented for \( N = 3 \) for clarity, they generalize straightforwardly to arbitrary \( N \). In that case, in the isotropic \( Z_N \) model, the leading thermal operator has scaling dimensions \( (2/(N+2), 2/(N+2)) \), and the leading spin 1 operator \( ((N+4)/(N+2), 2/(N+2)) \). We take this as coupling to the chiral parameter \( \delta \). All the previous arguments then go through, and in particular we find that \( T_{-+} = \frac{2}{N}T_{+-} \). This leads to anisotropic correlation length exponents \( \nu_1 = 1, \nu_2 = 2/N \), in agreement with the exact results found at the superintegrable point [4,5].

Finally, we give a simple argument that these continuum models should be integrable. In the conformal field theory, there is an infinite number of conserved charges corresponding to holomorphic and antiholomorphic currents \( (T(s)(z), 0) \) and \( (0, \overline{T(s)}(\overline{z})) \) of arbitrarily high spin \( s \). When the theory is perturbed by the thermal operator \( \Phi_{2/N+2, 2/N+2} \), it is well known that an infinite subset of these remain conserved, with spins \( s \) following the exponents of of \( A_{N-1} \) modulo the Coxeter number [14]. This may be traced to the property that the residue at \( z_1 = z \) in the operator product expansion of \( T(s)(z) \) and \( \Phi_{2/N+2, 2/N+2}(z_1, \overline{z_1}) \) is a total derivative with respect to \( z \) [11]. Similarly, an infinite set of antiholomorphic currents remain conserved for an analogous reason. Now all of the \( Z_N \) conformal field theories possess an extended \( W \)-symmetry, generated by conserved currents \( W^{(n)}(z) \) and \( \overline{W}^{(n)}(\overline{z}) \) [15]. With respect to this extended symmetry, the chiral fields
of dimension \(((N + 4)/(N + 2), 2/(N + 2))\) are not primary, but \(W\)-descendents of the form \(W^{(n)}_{-1}(n)\Phi_{2/N+2,2/N+2}\) for some \(n\). Since the holomorphic current \(W^{(n)}\) commutes with all the antiholomorphic ones, it follows that all the antiholomorphic currents which remain conserved under a purely thermal perturbation remain conserved when the chiral perturbation is also present. In general, however, the holomorphic currents are no longer conserved. It is unclear at this point what consequences may be drawn from this. The theory with both types of perturbation is expected to be massive, and therefore to possess a sensible \(S\)-matrix. However, in the absence of Lorentz invariance (which holds only when \(\delta = 0\)), or the mixed symmetry discussed earlier (which holds only when \(\tau = 0\)), there is no simple way of determining how these conserved charges act on the asymptotic states.

The manifolds of integrability are the planes \(\delta = 0\) and \(\delta = 0\) in the space parametrized by \((\tau, \delta, \bar{\delta})\). These appear to differ from the lattice case, where, close to the isotropic point, the exact solution manifold, expressed in our coordinates, has the form \(\tau \sim \delta \bar{\delta}\). It is possible that irrelevant couplings are responsible for this difference, or that continuum and lattice integrability are rather different properties.

In passing, we comment on the more physical case of the uniaxial model with \(\delta_x \neq 0, \delta_y = 0\). In this case symmetry arguments are are far less powerful. The renormalization group equation for \(\delta_x\) is no longer trivial, and gets corrections at order \(\delta_x^5\). These are very difficult to compute, involving as they do an integral over a 6-point function. However, they do suggest the possibility of a non-trivial fixed point for \(\delta_x \neq 0\). If this were to occur, it would correspond to a Lifshitz point, whose existence is still controversial in numerical studies of the \(N = 3\) model [16].

In summary, we have given simple symmetry arguments for the values of the critical exponents of the \(Z_N\) chiral Potts models on the self-dual line. These argu-
ments are valid in the limit of weak chirality, but the results are independent of the its magnitude, and are consistent with those found at the superintegrable point. We have argued that the continuum theory makes sense only in Minkowski space, and found that it exhibits an unusual space-time symmetry, which allows for the observed spontaneous breaking of translational symmetry in the true ground state. Our analysis indicates that, while the anisotropic correlation length exponents are universal, the principal axes to which they refer are not, and that these should rotate away from the light-like directions as the chirality parameter $\delta$ is increased. This may explain the inconclusive results obtained for mass-gap scaling in numerical work for small values of $\delta$ [2]. Crossover theory suggests that the asymptotic regime should be reached only for reduced temperatures of the order of $\delta^{2N/(N-2)}$, which will be a very narrow region, especially for $N = 3$. However, even then, the exponent $\nu_1 = 1$ should be observed only if the gap to states with the correct non-zero value of the momentum is considered. In all other cases, the other exponent $\nu_2 = 2/N$ should be observed. It would be interesting to see whether the predicted rotation of the scaling axes occurs in the vicinity of the superintegrable point, where numerical work should be more conclusive.

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