WEAKLY MIXING POLYGONAL BILLIARDS

JON CHAIKA AND GIOVANNI FORNI

Abstract. We prove that there exists a $G_δ$ dense set of (non-rational) polygons such the billiard flow is weakly mixing with respect to the Liouville measure (on the unit tangent bundle to the billiard). This follows, via a Baire category argument, from showing that for any translation surface the product of the flows in almost every pair of directions is ergodic with respect to Lebesgue measure. This in turn is proven by showing that for every translation surface the flows in almost every pair of directions do not share non-trivial common eigenvalues.

1. Introduction

A basic question in ergodic theory is the dynamical properties of the billiard flow in various (planar) domains (which we will call “tables”) [Kat05, MT, Tab, CheMar, Gut]. Frequently, the table is assumed to have piecewise $C^1$ boundary and the billiard is assumed to be a massless point traveling without friction on the interior of the table which experiences elastic collision when it hits the boundary of the table. We will be concerned with such a system where the table is a polygon. The main result of this paper is,

Theorem 1.1. There exists a weakly mixing billiard flow in a polygon.

In fact, we answer a conjecture stated by E. Gutkin and A. Katok (see [GutKa, §1] that the set of polygonal tables with a weakly mixing billiard flow is a dense $G_δ$ subset of the appropriate space of all polygonal tables with a fixed number of vertices.

Let $P$ denote a polygon. The billiard flow is a flow $F^t : (P \times S^1) / \sim \to (P \times S^1) / \sim$, where $(p,θ) \sim (q,ψ)$ if $p = q \in \partial P$ and the angle between $θ$ and the side of $P$ at $p$ is $\pi$ minus the angle between $ψ$ and the side of $P$ at $p$. This flow is defined for any orbit that does not orbit into the vertices of $P$. This flow preserves the Lebesgue measure on

2010 Mathematics Subject Classification. 37A25, 37E35, 30F60, 32G15.

Key words and phrases. Billiards in Polygons, Rational Polygonal Billiards, Weak Mixing Flows, Teichmüller flow, Moduli space of Abelian differentials.
$P \times S^1$, defined as the product of the Lebesgue measure on $P$ times the Lebesgue measure on $S^1$ (pushed forward by $\sim$).

A natural dynamical consequence of this result is that there exists a polygon $P$ of area 1 so that for any rectangles $R, R' \subset P$ and for any intervals $I, I' \subset S^1$, for almost every $p, p' \in P$ and $\theta, \psi \in S^1$ we have

$$\frac{|\{0 \leq t \leq T : F^t(p, \theta) \in R \times I \text{ and } F^t(p', \psi) \in R' \times I'\}|}{T} \to \frac{|R|}{|I|} \cdot \frac{|R'|}{|I'|}.$$

S. Kerckhoff, H. Masur and J. Smillie proved that there were ergodic billiard flows in polygons [KMS]. The significant general results about billiard flows in polygons are C. Boldrighini, M. Keane and F. Marchetti’s result that they have at most a countable set of directions containing periodic trajectories [BKM] and A. Katok’s result that the billiard flow has zero entropy [Kat87, Section 3]. By a result of A. Katok [Kat87, Section 3] the zero entropy property implies a subexponential bound on the growth of complexity, in particular for the counting function of generalized diagonals and periodic orbits. Recently, D. Scheglov [Sch13], [Sch19+] improved this bound in the case of almost all triangles to “weakly exponential”. We recall that H. Masur [Mas90] showed the counting function of generalized diagonals grows quadratically for rational tables, and it has been conjectured that polynomial bounds should hold for general typical polygons ([Kat87, §4]). The most significant recent results on the ergodic properties of billiards in polygons, all of which are in different contexts, are the results of A. Avila and V. Delecroix [AvDel] and D. Aulicino, A. Avila and V. Delecroix [AuAvDel], which prove weak mixing in almost all directions in certain rational polygons and the proof by A. Málaga and S. Troubetzkoy [MaTr] of the weak mixing property in almost all directions for rational billiards in generic polygons with vertical - horizontal sides. In these cases the unit tangent bundle splits into invariant surfaces, and the weak mixing property is proved with respect to the two dimensional measure on the generic invariant surface, as opposed to our setting in which we prove weak mixing with respect to the (3-dimensional) Liouville measure on the unit tangent bundle. Another recent result is the proof by J. Bobok and S. Troubetzkoy [BoTr] of topological weak mixing for the billiard map (that is, for the first return map to the sides of the polygon, which is a $\mathbb{Z}$-action, as opposed to the billiard flows we consider) in the generic polygon.

Similar to the proof of existence of ergodic billiards [KMS], and of the directional weak mixing results of [GutKa], our result is obtained from
a result about every translation surface via a Baire category argument. A translation surface is a pair \((X, \omega)\) where \(X\) is a Riemann surface and \(\omega\) is an Abelian differential. From \(\omega\) we obtain an \(S^1\) family of vector fields on \(X\), which in turn give flows \(F^t_{r\omega}\) on \(X\) and a Lebesgue measure on \(X\). We denote this normalized Lebesgue measure \(\lambda^2\), regardless of the surface, which is preserved by these flows.

**Theorem 1.2.** For a.e. \((\theta, \phi) \in S^1 \times S^1\) the flow \(F^t_{r\omega} \times F^t_{r\phi\omega}\) is \(\lambda^2 \times \lambda^2\) ergodic.

P. Hubert and the first named author [ChHu] previously showed that for almost every surface (with respect to any \(SL(2, \mathbb{R})\)-invariant measure) the product of the flow in almost every direction is uniquely ergodic. Our methods for establishing Theorem 1.2 are spectral, as we show the following, which because the straight line flow on any translation surface is \(\lambda^2\) ergodic in almost every direction [KMS], is well known to imply Theorem 1.2 (see for instance [KT], Prop. 4.2):

**Theorem 1.3.** For every \(\alpha \neq 0\) and every translation surface \(\omega\) we have \(|\{\theta \in S^1 : F^t_{r\omega} \text{ has eigenvalue } e^{2\pi i \alpha}\}| = 0\).

A. Avila and the second named author [AvFo] showed that for almost every translation surface (in genus at least 2) the flow in almost every direction is weakly mixing, which implies the above result for almost every surface. In fact, by an announcement of D. Aulicino, A. Avila and V. Delecroix [AuAvDel], Theorem 1.3 holds for almost every surface with respect to every \(SL(2, \mathbb{R})\) invariant measure. (They show that for any \(SL(2, \mathbb{R})\) ergodic measure, which is not supported on branched covers of tori, the vertical flow on almost every surface is weakly mixing. This gives Theorem 1.2 for these measures. By an argument based on rigidity sequences, one can show that Theorem 1.3 holds for any branched cover of a torus. In fact, one can prove, see for instance [FH19], that for any \(\alpha \in \mathbb{R} \setminus \{0\}\) and for the flow in almost every direction on the branched cover of the torus, there exists a rigidity sequence \(t_j\) so that \(e^{2\pi i \alpha t_j}\) does not converge to 1).

Note, it is well known that single cylinder surfaces give many examples of surfaces (even with full orbit closure) where every direction has a non-constant eigenfunction.

### 1.1. Organization of the paper

In Section 1.2 below we gather some fundamental open questions in the ergodic theory of billiards in polygons. In Section 2 we recall some basic material about the Teichmüller flow and the renormalization cocycle for translation flows,
the Kontsevich–Zorich cocycle over the Teichmüller flow on the Hodge bundle. We conclude the section, in Section 2.1, with an outline of our argument. In Section 3 we derive several consequences of the work of [EskMir], [EskMirMo] and [ChEs], including results on the growth of vectors under the action of the Kontsevich–Zorich cocycle and on averages along horocycle arcs of the pushforward of “height” functions (functions as in Theorem 3.8) for the moduli space by the geodesic flow. Section 4 contains, in Section 4.1, preparatory results on the growth of curves in the Hodge bundle, which are derived from results of the previous section, and, in Section 4.2, some standard large deviation results, which are included for convenience of the reader. Section 5 contains the core of the argument, that is, the key proposition on the controlled growth of curves under the Kontsevich-Zorich cocycle (Prop. 5.1). In Section 6 after recalling Veech’s criterion for weak mixing, we prove our main result, Theorem 1.3. Finally, in Section 7 we derive our result on the existence of weakly mixing polygonal billiards.

1.2. Open questions.

Question 1.4. Does every translation surface $\omega$ where the flow is not weakly mixing in almost every direction have the property that the line $\mathbb{R}[\Im(r_{\theta}\omega)]$ spanned by the cohomology class $[\Im(r_{\theta}\omega)]$ of the imaginary part of its rotation $r_{\theta}\omega$ intersects an integer translate of the stable subspace of the $SL(2, \mathbb{R})$ subbundle in a set of directions $\theta \in S^1$ of positive measure?

Question 1.5. For every translation surface is the flow in almost every pair of directions uniquely ergodic? Spectrally singular (modulo constants)?

There are also many natural questions about the flow of billiards in irrational polygons.

Question 1.6. Is there a polygon with a mixing billiard flow? Topologically mixing? Minimal? Is the billiard flow in every irrational polygon ergodic? Weak mixing? Mixing?

Note that numerical experiments [CasPros] suggest that every billiard in an irrational triangle is mixing.

1.3. Acknowledgments. The first named author thanks DMS-1452762, a Poincaré chair, a Sloan fellowship and a Warnock chair for support. The first named author also thanks the University of Maryland for its hospitality. The second named author was supported by
the NSF grant DMS 1600687 and by a Research Chair of the Fondation Sciences Mathématiques de Paris (FSMP). He would also like to thank the University of Utah for its hospitality. The authors would like to thank CIRM, IHP, University of Zurich, ETH Zurich and Oberwolfach for their hospitality during work on this project.

2. Background

As stated earlier, a translation surface is a pair \((X, \omega)\) where \(X\) is a finite type Riemann surface and \(\omega\) is an Abelian differential. The space of translation surfaces is stratified by the order and number of the zeros of the Abelian differential. Let \(\mathcal{H}(\alpha)\) be the moduli space of translation surfaces \((X, \omega)\) such that \(\omega\) has \(k\) zeros of orders \(\alpha := (\alpha_1, \ldots, \alpha_k)\), which we call a stratum. The (real) Hodge bundle of the stratum is a bundle over each stratum whose fiber at \((X, \omega)\) is \(H^1(X; \mathbb{R})\). Given a translation surface \((X, \omega)\), integrating \(\omega\) provides charts for \(X \setminus \Sigma\) where \(\Sigma\) is the set of zeros of \(\omega\). From its action on these charts, the group \(SL(2, \mathbb{R})\) of real matrices with determinant one acts on each translation surface, preserving the stratum it’s in.

The group \(SL(2, \mathbb{R})\) also acts on the Hodge bundle since it acts on the base \(\mathcal{H}(\alpha)\) of the bundle and this action can be lifted to the bundle by parallel transport of cohomology classes with respect to a natural flat connection.

The Kontsevich-Zorich cocycle is given by the action of \(SL(2, \mathbb{R})\) on the Hodge bundle. It is a cocycle over the action of the group \(SL(2, \mathbb{R})\) on the stratum \(\mathcal{H}(\alpha)\). See the lecture notes of the second named author and Carlos Matheus [FoMa] for a detailed description of the above material.

Let

\[
    h_s = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \hat{h}_s = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}, \quad g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix},
\]

denote, respectively, the Teichmüller horocycle flows and the Teichmüller geodesic flow. Also let

\[
    r_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}
\]

denote the group of rotations.

Let \(I \subset \mathbb{R}\) be any interval. Given a horocycle arc \(\{h_s \omega \mid s \in I\}\) at \(\omega \in \mathcal{H}(\alpha)\), a horocycle section at \(\omega\) is a map \(\phi : I \to H^1(M, \mathbb{R})\) such that \(\phi(s) \in H^1_{h_s \omega}(M, \mathbb{R})\). Here \(H^1(M, \mathbb{R})\) denotes the Hodge bundle
over the moduli space of Abelian differentials and $H^1_ω(M, \mathbb{R})$ denotes the fiber at $ω ∈ \mathcal{H}(α)$. 

For every $s ∈ I$, let $π_ω(s) : H^1_{h_sω}(M, \mathbb{R}) → H^1_ω(M, \mathbb{R})$ denote the linear map given by the parallel transport along the horocycle arc joining $h_sω$ to $ω$, that is, $\{h_s−σω|σ ∈ [0, s]\}$. Given a horocycle section $φ : I → H^1(M, \mathbb{R})$, the curve $φ_ω : I → H^1_ω(M, \mathbb{R})$ is defined as the composition

$$φ_ω(s) = π_ω(s) ∘ φ(s), \text{ for all } s ∈ I.$$

Since the Kontsevich–Zorich cocycle is defined by parallel transport, the following commutation relation holds: for all $v ∈ H^1_{h_sω}(M, \mathbb{R})$,

$$π_{gω} e^{2t} s v \circ KZ(g_t, h_sω) v = KZ(g_t, ω) \circ π_ω(s) v.$$

It follows that for any horocycle section $φ : I → H^1(M, \mathbb{R})$ at $ω$ we have the identity

$$KZ(g_t, h_sω) φ(s) = π_{gω}^{-1} e^{2t} s KZ(g_t, ω) φ_ω(s).$$

In other terms, since $g_t h_sω = h_{e^{2t} s} g_t ω$ the section

$$KZ(g_t, h_{e^{−2t} s} ω) φ(e^{−2t} s) = π_{gω}^{-1} s KZ(g_t, ω) φ_ω e^{−2t} s$$

is a horocycle section at $g_t ω$, hence in order to compute the evolution of horocycle sections under the Teichmüller flow it is enough to compute the evolution of curves $φ_ω : I → H^1_ω(M, \mathbb{R})$ under the maps

$$φ_ω(s) → KZ(g_t, ω) φ_ω e^{−2t} s.$$

Let $| \cdot |$ denote the Hodge norm. We recall that the Hodge norm of a cohomology class $γ ∈ H^1(X, \mathbb{C})$ is given by

$$|γ| = \sqrt{\int_Xγ ∧ * γ},$$

where $*$ denotes the Hodge star operator. This norm depends on the Riemann surface (but not the Abelian differential), though we usually suppress this dependance. See the survey [FoMaZo] for a detailed description. For every $g ∈ SL(2, \mathbb{R})$ let $\|KZ(g, ω)\|$ denote the operator norm with respect to the Hodge norms on $H^1_ω(M, \mathbb{R})$ and $H^1_{gω}(M, \mathbb{R})$. If $F ⊂ H^1(M, \mathbb{R})$ is a subbundle of the (real) Hodge bundle, let $\|KZ(g, ω)\|_F$ denote the operator norm with respect to the Hodge norm restricted to $F$. 

**Lemma 2.1.** Let $\text{dist}$ denote the hyperbolic distance on $SL(2, \mathbb{R})/SO(2)$. For any Abelian differential $ω ∈ \mathcal{H}(α)$ and for all $g ∈ SL(2, \mathbb{R})$, we have $\|KZ(g, ω)\| ≤ e^{\text{dist}(g, Id)}$ and moreover $|KZ(g, ω)v| ≥ e^{−\text{dist}(g, Id)}|v|$. 


Proof. For $g = g_t$ a diagonal element of $SL(2, \mathbb{R})$, the upper bound holds for the Hodge norm by the first variation formulas [Fo, §2] or [FoMa, §3.5]. The lower bounds also follows since $KZ(g_t, \omega)$ is invertible with inverse $K(g_t^{-1}, g_t \omega)$, hence for $t \geq 0$

$$|v| = |KZ(g_t^{-1}, g_t \omega)KZ(g_t, \omega)v| \leq e^t|KZ(g_t, \omega)v|.$$  

For a general $g \in SL(2, \mathbb{R})$, the result follows from the KAK decomposition of $SL(2, \mathbb{R})$, since the action on the Hodge bundle of the group $K = SO(2) \subset SL(2, \mathbb{R})$ of rotations is isometric. □

Lemma 2.2. For every $c > 0$ there is a compact set $K$ and $t_0 > 0$ so that if $\omega \in K$ then $|\{-1 \leq s \leq 1 : g_t h_s \omega \in K\}| > 2 - c$ for all $t > t_0$.

Proof. By [Ath, Section 2.3], there exists $V$ a continuous, proper function on $H$, and constants $c, \gamma, a, b, C$ and $\tau$ so that, for all $t \geq \tau$,

$$\int_0^{2\pi} V(g_t r_\theta \omega) d\theta \leq ce^{-\gamma t} V(\omega) + b$$

and moreover, $a^{-1} V(\omega) < V(g \omega) < a V(\omega)$ for all $g \in SL(2, \mathbb{R})$ with $|g| < C$. By choosing $N$ large enough and using that $r_\theta = \hat{h}_{\tan(\theta)} h_{\log(\cos(\theta))} h_{-\tan(\theta)}$ we have the lemma for $K = V^{-1}[0, N]$. □

Let $\nu$ be an $SL(2, \mathbb{R})$-invariant measure on a stratum of translation surfaces. We say $F$ is a $\nu$-almost everywhere invariant subbundle if for $\nu$-almost every $\omega$ and any $g \in SL(2, \mathbb{R})$ we have that $g$ sends $F_\omega$ to $F_{g\omega}$, that is, $KZ(g, \omega)F_\omega = F_{g\omega}$.

Following [ChEs, Def. 1.3] we say that the KZ cocycle has a $\nu$-measurable almost invariant splitting $F$ if there exists a finite set of proper subbundles $F_1, \ldots, F_n \subset F$ such that $F_i \cap F_j = \{0\}$ $\nu$-almost everywhere, for all $1 \leq i, j \leq n$, and for $\nu$-almost all $\omega$ and almost all $g$, the linear map $KZ(g, \omega)$ sends the set $\{F_{1, \omega}, \ldots, F_{n, \omega}\}$ to the set $\{F_{1, g\omega}, \ldots, F_{n, g\omega}\}$. Following [ChEs, Def. 1.4], we say that the cocycle acts strongly irreducibly on $F$ with respect to the measure $\nu$ if it does not admit any measurable almost invariant splitting.

The span of $\Re(\omega), \Im(\omega)$ (respectively the real and imaginary part of the Abelian differential $\omega \in \mathcal{H}(\alpha)$) defines a smooth invariant, symplectic subbundle of the Hodge bundle, which is then $\nu$-invariant for any $SL(2, \mathbb{R})$ invariant measure $\nu$ on $\mathcal{H}(\alpha)$. We call this the $SL(2, \mathbb{R})$ subbundle. Let $\hat{F}$ denote its symplectic complement, which is also a $\nu$-almost everywhere invariant subbundle.
2.1. Outline of proof. To prove Theorem 1.3 most of our work is to rule out the ‘weak stable space’ (in the terminology of [AvFo]). This is different from the approach of [AvDel] and our understanding of the approach of [AuAvDel], where they rule it out for ‘structural’ reasons. It also differs from the approach in [AvFo], because it is centered more on moduli space. As in the previous approaches we apply the Veech criterion [Ve84] (Lemma 6.1 of this paper) which morally says that it suffices to show that for a.e. \( \alpha \) there exists \( c > 0 \) and a compact set \( K \) so that for arbitrarily large \( t \) we have

\[
\|KZ(g_t, r_{\theta}\omega)3(r_{\theta}\omega)\|_Z > c
\]

and \( g_tr_{\theta}\omega \in K \) (we actually need two additional appropriate times in \( K \)). We show that for every \( \omega \) this one parameter family of classes are transverse to any integer translate of the stable \( SL(2, \mathbb{R}) \)-bundle. This allows us to apply Lemma 4.6 which makes the assignment

\[
\theta \rightarrow \frac{KZ(g_t, r_{\theta}\omega)3(r_{\theta}\omega)}{|KZ(g_t, r_{\theta}\omega)3(r_{\theta}\omega)|},
\]

after rescaling the segment by the geodesic flow which exponentially expands it, closer to a constant curve (for typical \( \theta \)). This lets us apply Proposition 3.1 (a modification of [ChEs]) to have that \( \|KZ(g_t, r_{\theta}\omega)3(r_{\theta}\omega)\| \) typically grows in \( t \). The key Proposition 5.1 shows that, under appropriate assumptions, there exists a \( 0 < \rho < 1 \) so that for an appropriately chosen segment of angles \( J \) we have that for most \( \theta \in J \) there is an \( s > 0 \) so that

\[
\|KZ(g_{t+s}, r_{\theta}\omega)3(r_{\theta}\omega)\|_Z > (\|KZ(g_t, r_{\theta}\omega)3(r_{\theta}\omega)\|_Z)^\rho
\]

and \( g_{t+s}r_{\theta}\omega \in K \). A key step in the proof (Lemma 5.3) is a new large deviations estimate for the measure of the set of directions, on any translation surface, where the Kontsevich-Zorich cocycle grows slower than expected. This complements [AAEKMU], Theorem 1.5, which proves a similar result for when the growth of the cocycle is larger than expected. Iterating Proposition 5.1 we can avoid the issues of ‘descendants’ (contrary to [AvFo], §3 especially (3.6), although this terminology is not used there). Another difference with [AvFo]’s approach is we treat the “weak stable” and stable subbundles at the same time. We also need to treat the unstable-\( SL(2, \mathbb{R}) \) subbundle and sub-bundles where the Kontsevich-Zorich cocycle acts isometrically, which we do by straightforward or standard arguments.

Throughout this proof we treat horocycles instead of circles because they behave better under the geodesic flow, and relate horocycles to...
Section 6.2 proves Theorem 1.1 via the previously mentioned, and now standard, Baire Category argument. In this section we follow the approach of [Vo97].

3. Making vectors grow

This section proves Propositions 3.1 and 3.2 and then develops, in a straightforward way, some machinery from [EskMirMo].

Let \( \hat{F} \) denote the complementary subbundle, that is, the symplectic orthogonal, of the \( \text{SL}(2, \mathbb{R}) \) subbundle and let \( \nu \) be any \( \text{SL}(2, \mathbb{R}) \)-invariant ergodic probability measure.

**Proposition 3.1.** (Chaika-Eskin, [ChEs]) If \( F \subset \hat{F} \) is an equivariant subbundle, where the Kontsevich-Zorich cocycle acts strongly irreducibly and with a positive exponent, then there exists \( \lambda := \lambda_F > 0 \) such that, for all \( \delta, \epsilon > 0 \), and for all \( L \) sufficiently large, there exist \( \hat{\epsilon} > 0 \) and an open set \( \hat{U} := \hat{U}(\delta, L) \) so that \( \nu(\hat{U}) > 1 - \delta \), and for all \( \omega \in \hat{U} \) and for all parallel horocycle sections \( v \) at \( \omega \), that is, for all the maps \( v : [-1, 1] \to F \) such that \( v(s) \in F_{h_s \omega} \) and \( \pi(\omega) v(s) = v(0) \), we have

\[
\text{Leb}\{ -1 \leq s \leq 1 : |KZ(g_L, h_s \omega) u| \geq e^{AL(1-\epsilon)} |u|, \text{ for all } u \in F_{h_s \omega} \text{ with } \angle(u, v(s)) < \hat{\epsilon} \} > 2 - \delta.
\]

Proposition 3.1 is proved below.

For any compact subset \( K \), for any \( \omega \) and \( t > 0 \), let

\[
v_K(\omega, t) = |\{ \tau \in [0, t] : g_\tau(\omega) \in K \}|.
\]

**Proposition 3.2.** Let \( F \subset \hat{F} \) be an equivariant subbundle of \( \hat{F} \). For any compact subset \( K \) there exists \( \lambda := \lambda_{F, K} \in [0, 1) \) such that

- \( \sup_{v \in F_\omega} \frac{|KZ(g_\tau(\omega)) v|}{|v|} \leq \exp \left( t - (1 - \lambda) v_K(\omega, t) \right) ; \)
- \( \inf_{v \in F_\omega} \frac{|KZ(g_\tau(\omega)) v|}{|v|} \geq \exp \left( -t + (1 - \lambda) v_K(\omega, t) \right). \)

**Proof.** Both estimates follow from the first variational formulas for the Hodge norm [Fo, §2] or [FoMa, §3.5]. \( \square \)

We now prove Proposition 3.1.

3.1. **Proof of Proposition 3.1.** Let \( \lambda > 0 \) denote the largest element of the Lyapunov spectrum restricted to \( F \).

Let \( \mu \) be an \( \text{SO}(2) \) invariant, compactly supported probability measure on \( \text{SL}(2, \mathbb{R}) \), whose support generates a dense subgroup of \( \text{SL}(2, \mathbb{R}) \). Let \( M \) be any \( \text{SL}(2, \mathbb{R}) \)-invariant suborbifold. We recall
that Eskin and M. Mirzakhani [EskMir] and A. Eskin, M. Mirzakhani and A. Mohammadi [EskMirMo] proved that all orbit closures of the $SL(2, \mathbb{R})$ action on the moduli space are “affine” suborbifolds supporting a unique “affine” probability invariant measure.

Let $E_{\text{good}}(L, \epsilon)$ be the set of all surfaces $\omega \in M$ so that for every $v \in F_\omega$ there exists a subset $H(v) \subset SL(2, \mathbb{R})^L$

$$\mu^L(H(v)) > 1 - \epsilon,$$

and for all $(h_1, \ldots, h_L) \in H(v)$, we have

$$e^{(\lambda - \epsilon)L}|v| < |KZ(h_L \cdots h_1, \omega)v| < e^{(\lambda + \epsilon)L}|v|$$

**Lemma 3.3.** [Chaika-Eskin [ChEs], Lemma 2.11] For all $\epsilon > 0$ we have $\lim_{L \to \infty} \nu(E_{\text{good}}(L, \epsilon)) = 1$.

Note that by assumption the cocycle acts strongly irreducibly on $F$ justifying the application of [ChEs]. The continuity of the Hodge norm implies that $E_{\text{good}}(L, \epsilon)$ is open.

To prove the proposition we need to relate random walks to Teichmüller geodesics via standard techniques:

**Lemma 3.4** (Sublinear Tracking). There exists $\lambda > 0$ (depending only on $\mu$), and for $\mu^N$-almost all $\bar{g} = (g_1, \ldots, g_n, \ldots) \in SL(2, \mathbb{R})^N$ there exists $\bar{\theta} = \bar{\theta}(\bar{g}) \in \mathbb{R}$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \| (g_{\lambda n} r_\theta) (g_n \cdots g_1)^{-1} \| = 0.$$

Furthermore, the distribution of $\bar{\theta}$ is uniform, i.e

$$\mu^N \left( \{ \bar{g} \in SL(2, \mathbb{R})^N : \bar{\theta}(\bar{g}) \in [\theta_1, \theta_2] \} \right) = \frac{\theta_2 - \theta_1}{2\pi}.$$

Note that (3) follows from the fact that $\mu$ is $SO(2)$ invariant.

Because, when $\theta \notin \{ \pm \frac{\pi}{2} \}$,

$$r_\theta = \begin{pmatrix} 1 & 0 \\ \tan(\theta) & 1 \end{pmatrix} \begin{pmatrix} \cos(\theta) & 0 \\ 0 & \sec(\theta) \end{pmatrix} \begin{pmatrix} 1 & -\tan(\theta) \\ 0 & 1 \end{pmatrix}$$

the previous lemma implies:

**Lemma 3.5** (Sublinear Tracking). There exists $\lambda > 0$ (depending only on $\mu$), and $\mu^N$-almost all $\bar{g} = (g_1, \ldots, g_n, \ldots) \in SL(2, \mathbb{R})^N$ there exists $\bar{s} = \bar{s}(\bar{g}) \in (-\infty, +\infty)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \log \| (g_{\lambda n} h_{\bar{s}}) (g_n \cdots g_1)^{-1} \| = 0.$$

Furthermore, the distribution of $\bar{s}$ is in the measure class of Lebesgue.
Now let \( \hat{E}_{\text{good}}(L, \tilde{\epsilon}) \) be the set of all surfaces \( \omega \in \mathcal{M} \) so that, for every \( v \in \mathcal{F}_\omega \), we have

\[
\text{Leb}\{s \in [-1, 1] : e^{(\lambda-\tilde{\epsilon})L} < \frac{|KZ(g_L, h_s\omega)v'(s)|}{|v'(s)|} < e^{(\lambda+\tilde{\epsilon})L}\} > 2 - \delta.
\]

Lemmas 3.5 and 3.3 imply:

**Lemma 3.6.** \( \lim_{L \to \infty} \nu(\hat{E}_{\text{good}}(L, \tilde{\epsilon})) = 1. \)

We conclude the proof of Proposition 3.1. Let \( \delta > 0 \) and \( 0 < \tilde{\epsilon} < \frac{1}{2} \epsilon. \) By Lemma 3.6, for \( L \) large enough we have that \( \nu(\hat{E}_{\text{good}}(L, \tilde{\epsilon})) > 1 - \delta \) and \( e^{\frac{1}{2}L} > 8. \)

We then choose \( \hat{U} := \hat{E}_{\text{good}}(L, \tilde{\epsilon}) \) which is an open set by continuity of the Hodge norm.

By construction, for all \( \omega \in \hat{U} \) and for all parallel horocycle sections \( v \) at \( \omega \), that is for all the maps \( v : [-1, 1] \to \mathcal{F} \) such that \( v(s) \in \mathcal{F}_{h_s\omega} \) and \( \pi_\omega(s)v(s) = v(0) \), we have

\[
\text{Leb}\{s \in [-1, 1] : e^{(\lambda-\tilde{\epsilon})L} < \frac{|KZ(g_L, h_s\omega)v(s)|}{|v(s)|} < e^{(\lambda+\tilde{\epsilon})L}\} > 2 - \delta.
\]

Let now \( u \in V_{h_s\omega} \) be a vector such that \( \angle(u, v(s)) < \hat{\epsilon}. \) We can write \( u = av(s) + bv^\perp(s) \) with \( v^\perp(s) \perp v(s) \) (with respect to the Hodge inner product) and by assumption we have

\[
|a| \geq \frac{1 - \hat{\epsilon}}{2} \frac{|u|}{|v(s)|} \quad \text{and} \quad |b| \leq 2\hat{\epsilon} \frac{|u|}{|v^\perp(s)|}.
\]

It follows that, if \( \hat{\epsilon} < e^{-L}e^{(\lambda-\epsilon)L}/8, \) by applying Lemma 2.1, we have

\[
|KZ(g_L, h_s\omega)u| \geq e^{(\lambda-\epsilon)L} \frac{1 - \hat{\epsilon}}{2} \frac{1}{2} |u| - 2\hat{\epsilon}e^L|u| \geq \frac{e^{(\lambda-\epsilon)L}L}{8}|u| > e^{(\lambda-\epsilon)L}|u|.
\]

The argument is completed.

### 3.2. Developing.

**Theorem 3.7.** *(Eskin-Mirzakhani-Mohammadi)* Let \( U \) be an open set, \( \nu \) be an \( SL(2, \mathbb{R}) \) invariant and ergodic measure. For any \( \epsilon > 0 \) there exists a finite set of invariant manifolds, \( \mathcal{Z}_1, \ldots, \mathcal{Z}_n \) so that for any \( C \subset \text{supp}(\nu) \setminus \bigcup_{i=1}^n \mathcal{Z}_i \), compact, there exists \( T_1 > 0 \) so that for all \( \omega \in C \) and for all \( T \geq T_1 \),

\[
\frac{1}{2T} \int_{-1}^{1} \int_{0}^{T} \chi_U(g_t h_s \omega) \, dt \, ds > \nu(U) - \epsilon.
\]
This follows from [EskMirMo, Theorem 2.7] by choosing \( \phi \in C_c(H) \), with \( \text{supp}(\phi) \subset U, 0 \leq \phi \leq 1 \) with \( \| \phi \|_1 > \nu(U) - \frac{\epsilon}{2} \).

**Theorem 3.8.** ([EskMirMo, Proposition 2.13]) Let \( M \subset H \) be an affine invariant submanifold. (In this proposition \( M = \emptyset \) is allowed.) Then there exists an \( SO(2) \)-invariant function \( f_M : H \rightarrow [1, \infty] \) with the following properties:

1. \( f_M(\omega) = \infty \) if and only if \( \omega \in M \), and \( f_M \) is bounded on compact subsets of \( H \setminus M \). For any \( \rho > 0 \), the set \( \{ \omega : f_M(\omega) \leq \rho \} \) is a compact subset of \( H \setminus M \).
2. There exists \( b > 0 \) (depending on \( M \)) and for every \( 0 < c < 1 \) there exists \( t_0 > 0 \) (depending on \( M \) and \( c \)) such that for all \( \omega \in H \setminus M \) and all \( t > t_0 \),
   \[
   \frac{1}{2\pi} \int_0^{2\pi} f_M(g_t r_\theta \omega) d\theta \leq cf_M(\omega) + b.
   \]
3. There exists \( \sigma > 1 \) and \( V \subset SL(2, \mathbb{R}) \) a neighborhood of the identity so that for all \( g \in V \) and all \( \omega \in H \),
   \[
   \sigma^{-1}f_M(\omega) \leq f_M(g\omega) \leq \sigma f_M(\omega).
   \]

This implies a similar result for horocycles:

**Lemma 3.9.** ([AAEKMU, Lemma 3.5]) Let \( f_M \) be as in Theorem 3.8. Then there exists a constant \( b' > 0 \) so that for all \( 0 < a < 1 \) there exists \( \bar{t}_0 = \bar{t}_0(a) \) such that for all \( t > \bar{t}_0 \) and for all \( \omega \in H \setminus M \) we have
   \[
   \int_{-1}^{1} f_M(g_t h_s \omega) ds < af_M(\omega) + b'.
   \]

For each \( \rho \in \mathbb{R}^+ \), let \( \mathcal{C}_\rho = \{ \omega : f_M(\omega) \leq \rho \} \) and, for all \( N \in \mathbb{N} \), let
   \[
   Z_N = \left\{ s \in [-1, 1] : g_{jN} h_s \omega \notin \mathcal{C}_\rho \text{ for all } j \in \{1, 2, \ldots, N\} \right\}.
   \]

**Proposition 3.10.** ([AAEKMU, Proposition 3.7]) Let \( f_M \) be as in Theorem 3.8 and let \( b' > 0 \) and \( \bar{t}_0 = \bar{t}_0(a) \) be as in Lemma 3.9. There exist \( C_1 > 1 \) (independent of \( \omega \) and \( a \)) such that for all \( a \in (0, 1) \), all \( \rho > C_1 b'/a \), all \( t \geq \bar{t}_0 \) such that \( e^t \in \mathbb{N} \) and all \( N \in \mathbb{N} \),
   \[
   \int_{Z_{N-1}} f(g_{Nt} h_s \omega) ds < (2a)^N f_M(\omega) + (2a)^{N-1} b'.
   \]

By choosing \( a < \min\{\frac{1}{2}, \frac{1}{2b'}\} \) we obtain:

**Corollary 3.11.** ([cf Ath, Theorem 1.1 (2)]) There exists \( \rho \in \mathbb{R}, \zeta < 1 \) so that for all \( S, T \) large enough if \( \omega \in \mathcal{C}_\rho \)
   \[
   \text{Leb}(\{ s \in [-1, 1] : \cup_{t \in [S, S+T]} \{ g_{t} h_s \omega \} \cap \mathcal{C}_\rho = \emptyset \}) < \zeta^T.
   \]
Proposition 3.12. [AAEKMU, Proposition 3.9] (cf. KKLM, Theorem 1.5) Let $\mathcal{C}_\rho$ be as above for a function $f_M$ as in Theorem 3.8. Let us assume that $\omega \notin \mathcal{M}$. For any $\delta, a \in (0, 1)$ there exist $\rho_0 > 1$ and $t_0 > 1$, depending only on $a$, such that for all $\rho \geq \rho_0$, all $t \geq t_0$ such that $e^t \in \mathbb{N}$, and all $N \in \mathbb{N}$, the set
\[
\{ s \in [-1, 1] : \frac{1}{N} \sum_{j=1}^{N} \chi_{\mathcal{C}_\rho}(g_j h_s \omega) < 1 - \delta \}
\]
can be covered by at most $2^N C_1^N (2a)^{\delta N} e^{2tN} C(\omega)$ intervals of radius $e^{-2tN}$, where $C(\omega) = \max\{1, \alpha(\omega)/\rho\}$.

Note that, if $a < \frac{1}{2(2C_1)}$, the measure of the set in the statement of Prop. 3.12 decays exponentially in $N$.

Corollary 3.13. Let $U$ be an open set whose boundary has measure zero. For all $\epsilon > 0$ there exist numbers $\xi \in (0, 1)$, $t_0 > 0$, $N_0 \in \mathbb{N}$ and a set $V_\epsilon := V_\epsilon(U) \subset \text{supp}(\nu)$, open in $\text{supp}(\nu)$, and $T_2 \in \mathbb{R}^+$ so that for any $T > T_2$:

(i) $\nu(V_\epsilon) > 1 - \epsilon$;
(ii) for $\nu$ almost every $\omega$ we have $\liminf_{S \to \infty} \frac{\{0 \leq t \leq S : g_t \omega \in V_\epsilon\}}{S} > 1 - \epsilon$;
(iii) if $\omega \in V_\epsilon$ we have $\left| \frac{1}{2T} \int_{-1}^{1} \int_0^T \chi_U(g_t h_s \omega)dt\, ds - \nu(U) \right| < \epsilon$;
(iv) for all $t \geq t_0$, $N \geq N_0$ and $\omega \in V_\epsilon$ we have
\[
\left| \{ s \in [-1, 1] : \frac{1}{N} \sum_{j=1}^{N} \chi_{V_\epsilon}(g_j h_s \omega) < 1 - \epsilon \} \right| < \xi^N.
\]

This uses the following standard definition and result:

Definition 3.14. Let $\omega \in \mathcal{H}$ and $\nu$ be the unique, $\text{SL}(2, \mathbb{R})$-invariant and ergodic probability measure with $\text{supp}(\nu) = \overline{\text{SL}(2, \mathbb{R})}\omega$. We say $\omega$ is Birkhoff generic if
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \phi(g_t \omega)dt = \int_\mathcal{H} \phi d\nu
\]
for all $\phi \in C_c(\mathcal{H})$.

Lemma 3.15. Let $S$ be a set so that $\nu(\partial S) = 0$. If $\omega$ is Birkhoff generic then
\[
\lim_{T \to \infty} \frac{1}{T} \int_0^T \chi_S(g_t \omega)dt = \nu(S).
\]
If $U$ is an open set and $\omega$ is Birkhoff generic

$$\liminf_{T \to \infty} \frac{1}{T} \int_0^T \chi_U(g_t(\omega)) dt \geq \nu(U).$$

**Proof of Corollary 3.13.** First, (ii) follows free of charge by the fact that $V_\epsilon$ is open and Lemma 3.15. Given an open set $U$ and $\epsilon > 0$, we then construct an open set $V_\epsilon$ with properties (i), (iii) and (iv).

For the given $U$ and $\epsilon > 0$, we apply Theorem 3.7 to obtain a finite number of manifolds $Z_1, \ldots, Z_n$. For each of these manifolds we build a function $f_i$ as in Theorem 3.8 and let $C(s) = \{\omega : f_i(\omega) \leq s\}$.

Applying Proposition 3.12 to each $f_i$ with $\delta < \frac{\epsilon}{n}$ and $a < \frac{1}{2(2C_1)^{1/2}}$ (where $C_1$ is as in the proposition), there exists $\rho_i, t_i$ as in the proposition, and so that $\nu(\cap C_i) > 1 - \epsilon$. Following the sentence after the proposition, for any $\sigma > 1$ there exist constants $\zeta_1, \ldots, \zeta_n < 1$ and natural numbers $N_1, \ldots, N_n$ so that for each $i \in \{1, \ldots, n\}$, for all $t \geq t_i$, for all $\omega \in C_i$ and $N \geq N_i$,

$$\left| \{s \in [-1, 1] : \frac{1}{N} \sum_{j=1}^N \chi_{\cap C_i} (g_{jt}(\omega)) < 1 - \epsilon \} \right| < \zeta_i^N.$$

Now, letting $\xi > \max\{\zeta_i\}$, there exist $N_0, t_0$ so that, if $\omega \in \cap C_i$ for all $i \in \{1, \ldots, n\}$, then for all $N \geq N_0$ and $t \geq t_0$ we have

$$\left| \{s \in [-1, 1] : \frac{1}{N} \sum_{j=1}^N \chi_{\cap C_i} (g_{jt}(\omega)) < 1 - \epsilon \} \right| < \xi^N.$$

We define the set $V_\epsilon$ to be the interior in $\text{supp}(\nu)$ of the compact set $C = \text{supp}(\nu) \cap \cap C_i$, so that property (iv) holds by the above argument. By applying Theorem 3.7 to the compact set

$$C := \text{supp}(\nu) \cap \bigcap_{i=1}^n C_i,$$

and the given $\epsilon > 0$, we conclude that there exists $T_2 > 0$ such that property (iii) holds for all $T > T_2$. Finally, it follows from the definitions that $\nu(V_\epsilon) \geq \nu(\cap C_i) > 1 - \epsilon$, hence property (i) is also proved. 

We will of course be interested in $V_\epsilon$ chosen for $U = \hat{U}$ and this is what $\hat{V}_\epsilon$ denotes for the remainder of the paper. We further assume that $\hat{V}_\epsilon$ is contained in a fixed compact set.
Some preparatory results

4.1. Making curves grow. The main result of the next section is Lemma 4.8, which applies Proposition 3.1 to say that the geodesic flow image of (projectively) Lipschitz horocycle sections with small enough (projective) Lipschitz constants in a subbundle as in Proposition 3.1 typically grow. It also collects some results which says that the geodesic flow image of horocycle sections have their Lipschitz (and usually their projective Lipschitz) constants improve.

Definition 4.1. A horocycle section \( \phi : [a, b] \to H^1(M, \mathbb{R}) \) at \( \omega \) (that is, a map \( \phi : [a, b] \to H^1(M, \mathbb{R}) \) such that \( \phi(s) \in H_{h, \omega}(M, \mathbb{R}) \)) is \( K \)-Lipschitz at \( \omega \), if the curve \( \phi_\omega : [a, b] \to H^1_\omega(M, \mathbb{R}) \), obtained by parallel transport along the horocycle, (that is, such that \( \phi_\omega(s) = \pi_\omega(s) \circ \phi(s) \) for all \( s \in [a, b] \)) is \( K \)-Lipschitz with respect to the Hodge norm on \( H^1_\omega(M, \mathbb{R}) \).

Lemma 4.2. If \( \phi : [a, b] \to H^1(M, \mathbb{R}) \) is a \( K \)-Lipschitz horocycle section at \( \omega \), then \( KZ(g_t)(\phi) : [e^{2t}a, e^{2t}b] \to H^1(M, \mathbb{R}) \) given by
\[
KZ(g_t)(\phi)(s) = KZ(g_t, h_{se^{-2t}})\phi(se^{-2t})
\]
is a \( Ke^{-t} \)-Lipschitz horocycle section at \( g_t \omega \).

Proof. By definition \( \phi : [a, b] \to H^1(M, \mathbb{R}) \) is a \( K \)-Lipschitz horocycle section at \( \omega \) if the parallel transport \( \phi_\omega : I \to H^1_\omega(M, \mathbb{R}) \) is a \( K \)-Lipschitz map with respect to the Hodge norm. We recall that the map \( \phi_\omega \) is defined, for all \( s \in I \), as a composition \( \phi_\omega(s) = \pi_\omega(s) \circ \phi(s) \) with the parallel transport \( \pi_\omega(s) : H^1_{h, \omega}(M, \mathbb{R}) \to H^1_\omega(M, \mathbb{R}) \).

By the commutativity of parallel transport and the KZ cocycle, we have
\[
\pi_{g_t \omega}(s)KZ(g_t, h_{se^{-2t}})\phi(se^{-2t}) = KZ(g_t, \omega)\phi_\omega(se^{-2t}),
\]
hence by Lemma 2.1 the map defined as \( KZ(g_t, \phi)_{g_t \omega}(s) = KZ(g_t, \omega)\phi_\omega(se^{-2t}) \) is \( Ke^{-t} \)-Lipschitz.

Definition 4.3. A horocycle section \( \phi : [a, b] \to H^1(M, \mathbb{R}) \) at \( \omega \) is projectively \( \kappa \)-Lipschitz at \( \omega \), for any \( s \in [a, b] \), if the curve \( \phi_\omega : [a, b] \to H^1_\omega(M, \mathbb{R}) \) obtained by parallel transport along the horocycle is \( K \)-Lipschitz with respect to the Hodge norm on \( H^1_\omega(M, \mathbb{R}) \) and the Hodge norm \( \|\phi_\omega(s)\|_\omega \) is bounded below by \( K/\kappa > 0 \).

Remark 4.4. For any bounded interval \([a, b] \subset \mathbb{R}\), there exists a constant \( C_{a,b} > 0 \) such that the following holds. For any section...
Remark 4.5. Every $K$-Lipschitz section $\phi : [a, b] \to H^1(M, \mathbb{R})$ (projectively) $\kappa$-Lipschitz at $\omega$, and for any $s \in [a, b]$, the section $\phi$ is also (projectively) $C_{s, b} \kappa$-Lipschitz at $h_s \omega$.

Lemma 4.6. If $\phi : [a, b] \to H^1(M, \mathbb{R})$ is a projectively $\kappa$-Lipschitz horocycle section at $\omega$, then $KZ(g_t)(\phi) : [e^{2t}a, e^{2t}b] \to H^1(M, \mathbb{R})$ given by $KZ(g_t)(\phi)(s) = KZ(g_t, h_{se^{-2t}})\phi(se^{-2t})$ is a projectively $\kappa$-Lipschitz horocycle section at $g_t\omega$. If in addition $\phi : [a, b] \to \mathcal{F} \subset \hat{\mathcal{F}}$, then for any compact set $K$ there exists $\lambda_{\mathcal{F}, K} \in [0, 1)$ such that the following holds. For any $\omega$ and for any $t > 0$, let

$$v_K(\omega, t) := |\{ t' \in [0, t] : g_{t'}\omega \in K \}|.$$

Then for all $t > 0$ the horocycle section $KZ(g_t)(\phi) : [e^{2t}a, e^{2t}b] \to \hat{\mathcal{F}}$ is projectively $\kappa_t$-Lipschitz at $g_t\omega$, where

$$\kappa_t \leq \kappa \exp(-(1 - \lambda_{\mathcal{F}, K})v_K(\omega, t)).$$

Proof. By Lemma 4.2, if the horocycle section $\phi$ is $K$-Lipschitz at $\omega$, then $KZ(g_t)(\phi)_{g\omega}$ is $Ke^{-t}$-Lipschitz, and by Lemma 2.1 it follows that $\|KZ(g_t)(\phi)_{g\omega}(s)\| \geq e^{-t}\|\pi(\phi)(s)\|$, for all $s \in [a, b]$. It follows immediately that if $\|\pi(\phi)(s)\|_{\omega} \geq K/\kappa$, then

$$\|KZ(g_t)(\phi)_{g\omega}(s)\| \geq e^{-t}\|\phi(\omega)(s)\| \geq Ke^{-t}/\kappa,$$

hence $KZ(g_t)(\phi)$ is projectively $\kappa$-Lipschitz at $g_t\omega$, establishing the first claim.

If $\phi : [a, b] \to \hat{\mathcal{F}}$, it follows from Proposition 3.2 that there exists $\lambda := \lambda_{\mathcal{F}, K} \in [0, 1)$ such that

$$\|KZ(g_t)(\phi)_{g\omega}(s)\| = \|KZ(g_t, \omega)(\phi_{\omega}(s))\| \geq e^{-t(1-\lambda)v_K(\omega, t)}\|\phi_{\omega}(s)\|.$$

Since $\phi$ is projectively $\kappa$-Lipschitz at $\omega$, it satisfies the lower bound $\|\phi_{\omega}(s)\| \geq K/\kappa$, for all $s \in [a, b]$ with $K > 0$ its Lipschitz constant. It follows that

$$\|KZ(g_t)(\phi)_{g\omega}(s)\| \geq (K/\kappa)e^{-t(1-\lambda)v_K(\omega, t)} = Ke^{-t/(\kappa e^{-t(1-\lambda)v_K(\omega, t)}},$$

which implies that the section $KZ(g_t)(\phi)$ is projectively $\kappa_t$-Lipschitz at $g_t\omega$ with $\kappa_t \leq Ke^{-t(1-\lambda)v_K(\omega, t)}$, as stated.
Lemma 4.7. Let $\mathcal{F} \subset \hat{F}$ be a subbundle where $KZ$ acts strongly irreducibly and with a positive exponent. There exists $\lambda > 0$ such that the following holds. For any $\delta, \epsilon > 0$ and for all sufficiently large $L > 0$, there exist $\kappa_0 := \kappa_0(a, b, \delta, L)$ and an open set $\bar{U} := \bar{U}(\delta, L)$ (as in Proposition 3.1) with $\nu(\bar{U}) > 1 - \delta$, such that for all $\kappa \in (0, \kappa_0)$, for all $t \geq 0$, for all $s_0 \in [a + e^{-2t}, b - e^{-2t}]$ and for all horocycle sections $\phi : [a, b] \to \mathcal{F} \subset \hat{F}$ projectively $\kappa$-Lipschitz at $h_{s_0}\omega$, whenever $g_t h_{s_0}\omega \in \bar{U}$, we have that

$$(8) \quad \text{Leb}(\{ s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : |KZ(g_{t+L}, h_s \omega)(\phi(s))| < e^{\lambda L(1-\epsilon)}|KZ(g_t, h_s \omega)(\phi(s))| \}) < \delta e^{-2t}.$$ 

Proof. The lemma follows by Proposition 3.1 applied to the horocycle $\{ h_{s_0}g_t(h_{s_0}\omega)|s \in [-1, 1] \}$ at $g_t h_{s_0} \omega$. Let us consider the image $KZ(g_t)(\phi)$ of the section $\phi$ for $s \in [s_0 - e^{-2t}, s_0 + e^{-2t}]$. By definition this is the restriction of a horocycle section at $g_t \omega$, and after reparametrization can be regarded as a horocycle section at $g_t h_{s_0} \omega = h_{e^{2t}s_0}g_t \omega$, defined as $KZ(g_t, h_{s_0+e^{-2t}s})\phi(s_0 + e^{-2t} s)$ for $s \in [-1, 1]$. By Lemma 1.6 since $\phi$ is projectively $\kappa$-Lipschitz at $h_{s_0}\omega$, then $KZ(g_t)(\phi)$, and its reparametrization, are still projectively $\kappa$-Lipschitz at $g_t h_{s_0} \omega$. It follows that the ratio $K/m$ between the Lipschitz constant $K$ and the minimum Hodge norm $m$ of the curve

$$KZ(g_t)(\phi)_{g_t h_{s_0} \omega}(s) = \pi_{g_t h_{s_0} \omega}(KZ(g_t)(\phi)(s)), \quad \text{for } s \in [-1, 1],$$

is at most $\kappa$, hence, for all $s \in [-1, 1]$ we have that

$$\angle\left(KZ(g_t)(\phi)_{g_t h_{s_0} \omega}(s), KZ(g_t)(\phi)_{g_t h_{s_0} \omega}(s_0)\right) < \kappa$$

therefore there exists $\kappa_0 > 0$ such that, for $\kappa \in (0, \kappa_0)$, the horocycle section

$$KZ(g_t)(\phi)(e^{2t}s_0 + s) = KZ(g_t, h_{s_0+e^{-2t}s})\phi(s_0 + e^{-2t}s)$$

makes an angle at most $\hat{\epsilon}$ (with respect to the Hodge norm on $H^1_{h_{s_0} g_t h_{s_0} \omega}(M, \mathbb{R})$) with a parallel section, so that since

$$KZ(g_{t+L}, h_{s_0+e^{-2t}s})\phi(s_0 + e^{-2t}s) = KZ(g_L, h_{s_0} g_t h_{s_0}\omega) KZ(g_t)(\phi)(e^{2t}s_0 + s)$$

by Proposition 3.1 we have

$$\text{Leb}(\{-1 \leq s \leq 1 : |KZ(g_{t+L}, h_{s_0+e^{-2t}s})\phi(s_0 + e^{-2t}s)| \geq e^{\lambda L(1-\epsilon)}|KZ(g_t, h_{s_0+e^{-2t}s})\phi(s_0 + e^{-2t}s)| \}) > 2 - \delta,$$

which implies the statement by change of variables.

$\Box$
The main result of this section is the following lemma. Let $\delta > 0$ and let $L > 0$ be sufficiently large.

Let $\hat{U} := \hat{U}(\delta, L)$ be the open set given by Proposition 3.1 and Lemma 4.7 and, for all small $\epsilon > 0$, let $\hat{V}_\epsilon := V_\epsilon(\hat{U}(\delta, L)) \subset \text{supp}(\nu)$ denote the open subset (which depends on $\hat{U}(\delta, L)$) given by Corollary 3.13).

**Notation:** For all $\epsilon, \delta > 0$, let

$$
\eta(\epsilon, \delta) := \sqrt{3(\epsilon + \delta)} + 2\sqrt{\delta}, \quad \text{and} \quad 
\mu(\epsilon, \delta) := \lambda(1 - \epsilon) - (1 + \lambda(1 - \epsilon))\eta(\epsilon, \delta).
$$

**Lemma 4.8.** For any bounded interval $[a, b] \subset \mathbb{R}$, for all $\delta > 0$, for all $L > 0$ sufficiently large, there exists $\kappa_0 := \kappa_0(a, b, \delta, L)$ such that the following holds. For all $\kappa \in (0, \kappa_0)$, there exists $T_3 > 0$ and $t_0 > 0$ such that, for all $s_0 \in [a + e^{-2t}, b - e^{-2t}]$, whenever

- $g_{t_0} h_{s_0} \omega \in \hat{V}_\epsilon(\hat{U}(\delta, L))$,
- $\phi : [a, b] \to H^1(M, \mathbb{R})$ is a projectively $\kappa$-Lipschitz horocycle section at $\omega$,
- $t > t_0$ and $T > T_3$

we have

$$
\text{Leb}\left(\{s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : |KZ(g_{t+T}, h_s \omega)\phi(s)| < e^{\mu(\epsilon, \delta)T} |KZ(g_t, h_s \omega)\phi(s)|\} \right) < \eta(\epsilon, \delta)e^{-2t}.
$$

**Proof.** Let $\delta > 0$ and $L > 0$ sufficiently large be fixed. By Corollary 3.13 (iii) (and the fact that $\nu(\hat{U}) > 1 - \delta$), since $g_{t_0} h_{s_0} \omega \in \hat{V}_\epsilon$, we have that, for all $T > T_2$,

$$
\int_0^T \int_{-1}^1 \chi_{\hat{U}}(g_{t_0} h_s g_{t+T} h_{s_0} \omega)dsd\ell \\
= e^{2t} \int_0^T \int_{-e^{-2t}}^{e^{-2t}} \chi_{\hat{U}}(g_{t+T} h_{s_0 + s} \omega)dsd\ell > 2T(1 - \delta - \epsilon).
$$

That is,

$$
\sum_{j=0}^{\lfloor T \rangle - 1} \int_0^L e^{2t} \int_{-e^{-2t}}^{e^{-2t}} \chi_{\hat{U}}(g_{t+T} h_{s_0 + s} \omega)dsd\ell > 2T - 2T(\epsilon + \delta) - L.
$$
From this we have that there exists \( a \in [0, L] \) so that
\[
\sum_{j=0}^{\lfloor T \rfloor-1} e^{2t} \int_{e^{-2t}}^{e^{2t}} \chi_\mathcal{U}(gt_{t+a+jL}h_{s_0+s\omega})ds > \frac{2T - 2T(\epsilon + \delta) - L}{L}.
\]

It follows that except for a set of \( s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] \) with measure at most \( e^{-2t} \sqrt{2(\epsilon + \delta) + \frac{L}{T}} \) we have
\[
\sum_{j=0}^{\lfloor T \rfloor-1} \chi_\mathcal{U}(gt_{t+a+jL}h_{s\omega}) > \left(1 - \sqrt{2(\epsilon + \delta) + \frac{L}{T}}\right) \frac{T}{L}.
\]

For \( t \geq t_0 \) and for all \( s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] \), let
\[
J(s) := \# \left\{ j \in \{0, \ldots, \lfloor T/L \rfloor - 1 \} : gt_{t+a+jL}h_{s\omega} \in \hat{U} \right\}
\]
but \(|KZ(g_{t+a+(j+1)L}, h_{s\omega})\phi(s)| < e^{\lambda L(1-\epsilon)}|KZ(g_{t+a+jL}, h_{s\omega})\phi(s)|\). We claim that
\[
(11) \quad \text{Leb}\left( \left\{ s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : J(s) > 2\sqrt{\delta} \frac{T}{L} \right\} \right) < 2\sqrt{\delta} e^{-2t}.
\]
In fact, for each \( j \), let \( \chi_j \) denote the characteristic function of the set
\[
(12) \quad \{ s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : gt_{t+a+jL}h_{s\omega} \in \hat{U} \}
\]
but \(|KZ(g_{t+a+(j+1)L}, h_{s\omega})\phi(s)| < e^{\lambda L(1-\epsilon)}|KZ(g_{t+a+jL}, h_{s\omega})\phi(s)|\). Since the interval \([a, b]\) is bounded, by Remark 4.1 there exists a constant \( C_{a,b} > 0 \) such that if the section \( \phi : [a, b] \to H^1(M, \mathbb{R}) \) is projectively \( \kappa \)-Lipschitz at \( \omega \), then \( \phi \) is projectively \( C_{a,b,\kappa} \)-Lipschitz at \( h_{s\omega} \), for all \( s \in [a, b] \). It then follows from Lemma 4.6 that, for all \( j \in \mathbb{N} \), the section \( KZ(g_{t+a+jL})\phi : [e^{2t}a, e^{2t}b] \to H^1(M, \mathbb{R}) \), is still projectively \( C_{a,b,\kappa} \)-Lipschitz at \( g_{t+a+jL}h_{s\omega} \). Thus, by Lemma 4.7, there exists \( \kappa_0 := \kappa_0(a, b, \delta, L) \) such that, for all \( \kappa \in (0, \kappa_0) \) if \( t \geq t_0 \), and \( g_{t+a+jL}h_{s\omega} \in \hat{U} \), then
\[
\text{Leb}\left( \left\{ s' \in [s-e^{-2(t+a+jL)}, s+e^{-2(t+a+jL)}] : |KZ(g_{t+a+(j+1)L}, h_{s\omega})\phi(s)| < e^{\lambda L(1-\epsilon)}|KZ(g_{t+a+jL}, h_{s\omega})\phi(s)| \right\} \right) < \delta e^{-2(t+a+jL)}.
\]
Thus,
\[
\text{Leb}\left( \left\{ s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : g_{t+a+jL}h_{s\omega} \in \hat{U} \right\} \right) \text{ but } |KZ(g_{t+a+(j+1)L}, h_{s\omega})\phi(s)| < e^{\lambda L(1-\epsilon)}|KZ(g_{t+a+jL}, h_{s\omega})\phi(s)| \left( 4\delta e^{-2t} \right).
\]
It then follows that
\[ \int_{e^{-2t}}^{e^{-2t}} \sum_{j=0}^{\lfloor T/L \rfloor - 1} \chi_j(s_0 + s) ds \leq \sum_{j=0}^{\lfloor T/L \rfloor - 1} \int_{e^{-2t}}^{e^{-2t}} \chi_j(s_0 + s) ds \leq 4 \lfloor T/L \rfloor \delta e^{-2t}, \]
which finally implies that
\[ \text{Leb} \{ s \in [s_0 - e^{-2t}, s_0 + e^{-2t}] : \sum_{j=0}^{\lfloor T/L \rfloor - 1} \chi_j(s_0 + s) > 2\sqrt{\delta} \lfloor T/L \rfloor \} \leq 2\sqrt{\delta} e^{-2t}, \]
as claimed.

Choosing \( T_3 > 0 \) so large that \( \frac{T_3}{L} < \epsilon + \delta \) gives a set of measure at least \( [1 - (\sqrt{3(\epsilon + \delta)} + 2\sqrt{\delta})] e^{-2t} \), where we have that, whenever \( T > T_3 \), for at least \( \frac{T}{L}(1 - 3(\epsilon + \delta) - 2\sqrt{\delta}) \) of our indices the cocycle grows by at least \( e^{\lambda L(1-\epsilon)} \). By Lemma 2.1, the cocycle reduces the Hodge norm by multiplication times a factor larger than \( e^{-L} \) on the remaining indices, thereby giving the bound. \( \square \)

### 4.2. Some probabilistic results.

We collect below some well-known probabilistic results for the convenience of the reader.

**Lemma 4.9.** Let \((\Omega, \mu)\) be a probability space and \( F_j : (\Omega, \mu) \to \{0, 1\} \) be a sequence of random variables such that there exists \( 0 < \rho < 1 \) so that for any \( j \), the conditional probability that \( F_j \) is 1 given \( F_1, \ldots, F_{j-1} \) is at least \( \rho \). Let \( G_i : (\Omega, \mu) \to \{0, 1\} \) be independent and so that \( \mu(G_i^{-1}(1)) = \rho \). Then for all \( \ell \) and \( r \),
\[
\mu(\{ \omega : \sum_{i=1}^{\ell} F_i(\omega) \leq r \}) \leq \mu(\{ \omega : \sum_{i=1}^{\ell} G_i(\omega) \leq r \}).
\]

By standard large deviations results we obtain:

**Corollary 4.10.** Let \((\Omega, \mu)\) be a probability space and \( F_j : (\Omega, \mu) \to \{0, 1\} \) be a sequence of random variables such that there exists \( 0 < \rho < 1 \) so that for any \( j \), the conditional probability that \( F_j \) is 1 given \( F_1, \ldots, F_{j-1} \) is at least \( \rho \). For all \( \epsilon > 0 \) there exists \( C_1, C_2 > 0 \) so that
\[
\mu(\{ \omega : \sum_{j=1}^{\ell} F_j(\omega) \leq (\rho - \epsilon)\ell \}) \leq C_1 e^{-C_2 \ell}.
\]

**Corollary 4.11.** Let \((\Omega, \mu)\) be a probability space, \( k \in \mathbb{N} \) and \( F_j : (\Omega, \mu) \to \{0, 1\} \) be a sequence of random variables such that there exists \( 0 < \rho < 1 \) so that for any \( j \), the conditional probability that \( F_j \)
is 1 given $F_1, \ldots, F_{j-k}$ is at least $\rho$. For all $\epsilon > 0$ there exists $C_3, C_4 > 0$ so that
\[
\mu(\{\omega : \sum_{j=1}^{\ell} F_j(\omega) \leq (\rho - \epsilon)\ell\}) \leq C_3 e^{-C_4 \ell}.
\]

Proof. Let us consider the $k$ sequences of random variables: $(F_{1+jk})_{j \in \mathbb{N}}, (F_{2+jk})_{j \in \mathbb{N}}, \ldots, (F_{k+jk})_{j \in \mathbb{N}}$. For each $i \in \{1, \ldots, k\}$ the sequence $(F_{i+jk})_{j \in \mathbb{N}}$ satisfies the hypothesis of Corollary 4.10. In fact, by assumption the conditional probability that $F_{i+jk}$ is 1 given $F_1, \ldots, F_{i+jk-k}$ is at least $\rho$. Since $\{F_i, \ldots, F_{i+(j-1)k}\} \subset \{F_1, \ldots, F_{i+jk-k}\}$, it follows that the conditional probability that $F_{i+jk}$ is 1 given $F_{i+1}, \ldots, F_{i+(j-1)k}$ is also at least $\rho$. By Corollary 4.10 we therefore have that for each $i \in \{1, \ldots, k\}$
\[
\mu(\{\omega : \sum_{j=1}^{\ell} F_{i+jk}(\omega) \leq (\rho - \epsilon)\ell\}) \leq C_1(\epsilon)e^{-C_2(\epsilon)\ell}.
\]

Finally we have
\[
\{\omega : \sum_{j=1}^{\ell} F_j(\omega) \leq (\rho - \epsilon)\ell\} \subset \bigcup_{i=1}^{k} \{\omega : \sum_{j=1}^{\ell/k} F_{i+jk}(\omega) \leq (\rho - \epsilon)\ell/k\},
\]
hence there exists $\ell_0 := \ell_0(k, \epsilon)$ such that for $\ell \geq \ell_0$ we have
\[
\begin{align*}
\mu(\{\omega : \sum_{j=1}^{\ell} F_j(\omega) \leq (\rho - \epsilon)\ell\}) &\leq \sum_{i=1}^{k} \mu(\{\omega : \sum_{j=1}^{\ell/k} F_{i+jk}(\omega) \leq (\rho - \epsilon/2)\ell/k\}) \\
&\leq kC_1(\epsilon/2)e^{-C_2(\epsilon/2)(\ell/k-2)}.
\end{align*}
\]
Thus the estimate in the statement holds (for $\ell$ sufficiently large) with $C_3(\epsilon) = kC_1(\epsilon/2)e^{2C_2(\epsilon/2)}$ and $C_4(\epsilon) = C_2(\epsilon/2)$. \qed

5. The key proposition

Proposition 5.1. Let $\kappa_0$ be as in Lemma 4.8. There exist constants $\sigma > 0, \tau > 0$ so that for all large enough compact sets, $\mathcal{K}$ there exists $\gamma_\mathcal{K} > 0$ such that all horocycle sections $\phi : [-1, 1] \to \mathcal{F} \subset \hat{\mathcal{F}}$ at $\omega$, under the conditions that, for some $s_0 \in [-1, 1]$ we have

(a) $e^t$ large enough and $\gamma \in (0, \gamma_\mathcal{K})$ ;
(b) $g_t h_{s_0} \omega \in \mathcal{K}$,
(c) the section $\phi_{t,s_0} : [-1, 1] \to H^1(M, \mathbb{R})$ defined by

$$\phi_{t,s_0}(s) = KZ(g_t, h_{s_0 + se^{-2t}})\phi(s_0 + se^{-2t})$$

is $\gamma K$-Lipschitz at $h_{s_0} \omega$, with $\kappa \in (0, \min\{1, \kappa_0\})$,

(d) $\max_{s\in[-1,1]} \|\phi_{t,s_0}(s)\|_Z = \gamma$,

there exists a set $S_{t,s_0} \subset [s_0 - e^{-2t}, s_0 + e^{-2t}]$ of Lebesgue measure

$$\text{Leb}(S_{t,s_0}) > 2(1 - \gamma^\sigma)e^{-2t}$$

such that, for each $s \in S_{t,s_0}$, there exists $\ell := \ell(s)$ such that

(A) $\frac{1}{2}\log|\gamma| \leq \ell \leq \frac{3}{2}\log|\gamma|$;

(B) $g_t + \ell h_{s} \omega \in \mathcal{K}$;

(C) $\max_{s'\in[-1,1]} \|\phi_{t+\ell,s}(s')\|_Z \geq \|\phi_{t+\ell,s}(0)\|_Z > e^{\ell}(1 - \kappa)\gamma$;

Note that by our assumption on $\ell(s_0)$, Conclusion [C] implies that there exists $0 < \rho < 1$ so that : for all $s \in S_{t,s_0}$ and for $\ell = \ell(s)$, we have

$$\|KZ(g_{\ell}, g_t h_{s} \omega)\phi(s)\|_Z > (1 - \kappa)\frac{\gamma^\rho}{\gamma} \|KZ(g_t, h_{s} \omega)\phi(s)\|_Z \geq (1 - \kappa)^2 \|KZ(g_t, h_{s} \omega)\phi(s)\|_Z^\rho.$$ 

Before the beginning the proof of Proposition 5.1 we reduce lower estimates on the distance to the integer lattice to lower bounds on the norm of vectors in the cohomology bundle over a fixed compact set.

**Lemma 5.2.** For any compact subset $\mathcal{K}$ of the moduli space there exists a constant $\gamma_\mathcal{K} > 0$ such that for $0 < \gamma < \gamma_\mathcal{K}$ the following holds. Let $\phi : [-1, 1] \to H^1(M, \mathbb{R})$ be a $\gamma$-Lipschitz horocycle section at $\omega$. If $g_\omega \in \mathcal{K}$ and $\max_{s\in[-1,1]} \|\phi(s)\|_Z \leq \gamma$, then there exists a parallel section $z(s) \in H^1_{h_{\omega}}(M, \mathbb{Z})$ such that, for all $\ell \leq 3|\log|\gamma||/4$ and for all $s \in [-1, 1]$ we have

$$\|KZ(g_{\ell}, h_{s} \omega)\phi(s)\|_Z \geq |KZ(g_{\ell}, h_{s} \omega)(\phi(s) - z(s))|.$$ 

**Proof.** We remark that over any compact set $\mathcal{K}$ the Hodge length of the shortest vector of the integer lattice $H^1(M, \mathbb{Z})$ has a positive minimum $\delta_\mathcal{K} > 0$. Since, by hypothesis, $\max \|\phi(s)\|_Z \leq \gamma$ it follows that there exists $z \in H^1_{\omega}(M, \mathbb{Z})$ such that

$$|\phi(0) - z| = \max_{s\in[-1,1]} \|\phi(s)\|_Z \leq \gamma.$$ 

Let $z(s) \in H^1_{h_{\omega}}(M, \mathbb{Z})$ denote the section given by the parallel transport of $z \in H^1_{h_{\omega}}(M, \mathbb{Z})$, that is, the section such that $\pi_\omega(s)z(s) = z$.

Note that these are the parameters that [C] refers to.
for all $s \in [-1, 1]$. Similarly, let $\tilde{\phi}(s) \in H^1_{h,\omega}(M, \mathbb{R})$ denote the parallel transport of the vector $\phi(0)$, that is, the section $\tilde{\phi}(s)$ such that $\pi_s(\tilde{\phi}(s)) = \phi(0)$, for all $s \in [-1, 1]$.

Since for all $s \in [-1, 1]$ the vector $\phi(s) - z(s)$ is obtained from the vector $\phi(s_0) - z$ by parallel transport along a horocycle of length at most 2, for all $s \in [-1, 1]$ we have

$$|\tilde{\phi}(s) - z(s)| \leq 10\gamma,$$

and since by hypothesis $\phi$ is a $\gamma$-Lipschitz section it follows that

$$|\phi(s) - \tilde{\phi}(s)| \leq 10\gamma.$$  

We remark that since $\ell \leq |\log \gamma|/2$ we have that $\|KZ(g_{\ell}, \omega)\| \leq \gamma^{-3/4}$. Hence

$$|KZ(g_{\ell}, h_{s}\omega)(\phi(s) - z(s))| \leq 20\gamma^{1/4}.$$  

Thus it suffices to choose $\gamma'_{K}$ so that $(\gamma'_{K})^{1/4} < \delta_{K}/40$.  

As a consequence, under the hypotheses of the above lemma, a lower bound on $|KZ(g_{\ell}, h_{s}\omega)(\phi(s) - z(s))|$ is equivalent to a lower bound on $\|KZ(g_{\ell}, h_{s}\omega)\psi(s)\|_{Z}$ and up to replacing $\phi(s)$ with $\phi(s) - z(s)$ we can estimate $|KZ(g_{\ell}, h_{s}\omega)\phi(s)|$ from below.

Proposition 5.1 follows from the next lemma, whose proof we defer until after the proposition’s proof.

For any $\epsilon, \delta > 0$, let $\eta := \eta(\epsilon, \delta) > 0$ and $\mu := \mu(\epsilon, \delta) > 0$ be the constants defined in formula (3).

**Lemma 5.3.** Let $T_0 > 0$ be sufficiently large and so that $e^{T_0} \in \mathbb{N}$ and let $T > T_0$. Let $\kappa \in (0, \kappa_0)$ and let $\psi : [-1, 1] \to H^1(M, \mathbb{R})$ be a horocycle section at $\omega \in \mathcal{K}$ such that $\psi$ is projectively $\kappa$-Lipschitz. For all $j \in \{0, \ldots, \lfloor T/T_0 \rfloor - 1\}$, for $s \in [-1, 1]$, let

$$\Phi_{jT_0}(s) := KZ(g_{jT_0}, h_{s}\omega)\psi(s).$$

If $\epsilon, \delta > 0$ are small enough there exists $\sigma' > 0$ such that

$$|\Phi_{(j+1)T_0}(s)| \geq e^{(\mu-\epsilon)T_0} |\Phi_{jT_0}(s)|$$

$$> (1 - 16\eta) \left(\frac{T}{T_0}\right)^{\sigma'} > 2(1 - e^{-\sigma'T}).$$

**Remark 5.4.** Let $\mathcal{F} \subset \hat{\mathcal{F}}$ be an equivariant subbundle, where the Kontsevich-Zorich cocycle acts strongly irreducibly with a positive exponent, and let $\lambda$ be the largest such exponent. Let $v : [-1, 1] \to \mathcal{F}$ be a parallel horocycle section at $\omega$. By the above lemma and Lemma 2.1
one has that for any $\epsilon' > 0$ there exists $c > 0$ and $T_1 \in \mathbb{R}$ so that for all $T > T_1$ we have

$$\lambda(\{s \in [-1, 1] : |KZ(g_T, h_s\omega)v(s)| < e^{(\lambda - \epsilon')T} \}) < e^{-cT}.$$ 

More generally, one can prove such a large deviation result for any Lipschitz horocycle section $\psi : [-1, 1] \to \mathcal{F}$. Indeed by Proposition 3.12 (applying it to a function $f_0$) there exists a compact set $K$ and a constant $c$ so that the measure of the set of $s \in [-1, 1]$ so that $|\{0 \leq t \leq T : g_t h_s\omega \notin K\}| < cT$ decays exponentially with $T$. (Indeed one can choose $\rho$ so big that if $g_t h_s\omega \in C_{\rho_0}$ then $g_{t+\tau}\omega \in C_{\rho} = K$ for all $0 \leq \tau \leq t_0$.) By Lemma 4.6, the transported horocycle section $KZ(g_T)(\psi)$ at such an $s \in [-1, 1]$ is $\kappa_T$-projectively Lipschitz with a constant $\kappa_T$ which decays exponentially with $T$.

Proof of Proposition 5.1 assuming Lemma 5.3. For all $c, \gamma > 0$, let $B := B(c, \gamma)$ denote the set of $s \in [-1, 1]$ such that for all $\ell \in [\frac{1}{2}|\log \gamma|, (\frac{1}{2} + c)|\log \gamma|]$ we have $g_\ell h_s g_t h_s \omega \notin K$. We first prove that for any $c > 0$, and for $\gamma > 0$ sufficiently small, the Lebesgue measure of $B$ is polynomially small in $\gamma > 0$.

To prove an upper bound on the measure of $B$ we apply Corollary 3.11 with $S = \frac{1}{2}|\log \gamma|$ and $T := c|\log \gamma|$ sufficiently large to derive that there exists $\zeta \in (0, 1)$ such that

$$\text{Leb}(B) \leq \zeta^{|\log \gamma|} = \gamma^{c|\log \zeta|}.$$ 

Let $\psi = \phi_{t,s_0}$. By the hypotheses of the Proposition the horocycle section $\phi_{t,s_0}$ at $g_t h_s \omega$, defined in formula (14), satisfies the hypotheses of Lemma 5.3. Indeed, by the hypotheses (c) and (d) of the Proposition, the section $\phi_{t,s_0}$ is $\kappa$-projectively Lipschitz at $h_s \omega$ with $\kappa \in (0, \kappa_0)$.

Let $T_0 > 0$ be sufficiently large and so that $e^{T_0} \in \mathbb{N}$ and let $\gamma > 0$ such that $T := \frac{1}{2}|\log \gamma| > T_0$. As in Lemma 5.3 we partition the interval $[0, \frac{1}{2}|\log \gamma|]$ into $R := \left[\frac{|\log \gamma|}{2}, \frac{1}{2}\right]$ equal intervals of length $T_0 > 0$ and an additional interval of length at most $T_0$. Let $G$ be the set of formula (17) in the statement of the Lemma 5.3 applied to the horocycle section $\phi_{t,s_0}$, and let $S_{t,s_0}$ be defined as

$$S_{t,s_0} := \{s_0 + s'e^{-2t} : s' \in G \setminus B\}.$$ 

Lemma 5.3 and the previous paragraph establishes the measure lower bound of formula (15), for any $\sigma < \min(\sigma', c|\log \zeta|)$. 

Lemma 2.1 there exists \( \mu \) to verify Condition (C). Since cycle section \( \phi \) and because the Hodge norm can change by a factor of at most 
\[
|g|_{c} > \text{for } s \text{ by Lemma 5.2, there exists a parallel section } z \text{ by formula (18), we have the lower bound }
\]
\[
\ell_{s} \in \left[ \frac{1}{2} \right] \log \gamma, \left( \frac{1}{2} + c \right) \log \gamma \right]
\]
so that \( g_{t+\ell_{s}}h_{s}\omega = g_{\ell_{s}}h_{e^{2t}(s-s_{0})}g_{t}h_{s_{0}}\omega \in \mathcal{K} \).

Clearly conditions \([A]\) and \([B]\) are satisfied for \( \ell := \ell_{s} \). It remains to verify Condition \([C]\). Since \( e^{2t}(s-s_{0}) \in \mathcal{G} \), by formula (17) and Lemma 5.2 there exists a parallel section \( z \) such that
\[
\|z\| \leq \left( \frac{1}{2} + c \right) \log \gamma \text{, for } c > 0 \text{ sufficiently small there exists } \gamma \text{ such that, for all } s \in [-1, 1],
\]
\[
|KZ(g_{RT_{0}}, g_{t}h_{s}\omega)\phi(s)| = |KZ(g_{RT_{0}}, h_{e^{2t}(s-s_{0})}g_{t}h_{s_{0}}\omega)\phi_{t_{0},s_{0}}(e^{2t}(s-s_{0}))| = |\phi_{RT_{0}}(e^{2t}(s-s_{0}))| \geq e^{\mu'RT_{0}}|\phi_{t_{0},s_{0}}(e^{2t}(s-s_{0}))| = e^{\mu'RT_{0}}|\phi(s)|. 
\]
and because the Hodge norm can change by a factor of at most \( e^{\pm(c)\log \gamma + T_{0}} \) from time \( RT_{0} = \left( \frac{1}{2} \frac{\log \gamma}{T_{0}} \right) T_{0} \) to time \( \ell_{s} \leq \left( \frac{1}{2} + c \right) \log \gamma \),

Since, by our assumption on \( s \), \( g_{t}h_{e^{2t}(s-s_{0})}g_{t}h_{s_{0}}\omega \in \mathcal{K} \), by \((c)\) the horo-cycle section \( \phi_{t,s_{0}} \) is \( \gamma \)-Lipschitz at \( h_{e^{2t}(s-s_{0})}g_{t}h_{s_{0}}\omega \) and by \((d)\)
\[
(18) \max_{s' \in [-1, 1]} \|\phi_{t,s_{0}}(s')\|_{Z} = \gamma,
\]
by Lemma 5.2 there exists a parallel section \( z_{t,s_{0}} : [-1, 1] \to H^{1}(M, \mathbb{Z}) \) such that \( z(t,s_{0}) = H^{1}(M, \mathbb{Z}) \) with
\[
(19) \|KZ(g_{t}, h_{s}g_{t}h_{s_{0}}\omega)\phi_{t_{0},s_{0}}(s')\|_{Z} = \|KZ(g_{t}, h_{s}g_{t}h_{s_{0}}\omega)(\phi_{t,s_{0}}(s') - z_{t,s_{0}}(s'))\|_{Z}.
\]
In fact, we can apply Lemma 5.2 to the section \( \phi_{t,s_{0}} \) to get a parallel integer section \( z_{t,s_{0}} : [-1, 1] \to H^{1}(M, \mathbb{Z}) \) at \( g_{t}h_{s_{0}}\omega \) such that the above identity holds. By applying the above argument to the curve \( \phi_{t,s_{0}} - z_{t,s_{0}} \) we therefore conclude that, for \( s \in S_{t,s_{0}} \),
\[
\|\phi_{t+s}^\ell(0)\|_{Z} = \|KZ(g_{t}, g_{t}h_{s}\omega)\phi_{t,s_{0}}(s)(s)\|_{Z} \geq e^{\ell_{t}}\|\phi_{t,s_{0}}(s)\|_{Z}.
\]
Finally since by hypothesis the section \( \phi_{t,s_{0}} \) is \( \kappa \gamma \)-Lipschitz and by formula (18), we have the lower bound
\[
\|\phi_{t+s}^\ell(0)\|_{Z} \geq \min_{s' \in [-1, 1]} \|\phi_{t,s_{0}}(s')\|_{Z} \geq (1 - \kappa)\gamma,
\]
which completes the argument. \( \square \)
Proof of Lemma 5.3 Let us recall that by definition, for \( s \in [-1, 1] \),

\[
\Phi_{jT_0}(s) := KZ(g_{jT_0}, h_\omega) \phi(s).
\]

We say that the pair \((j, s)\) is good if \( g_{jT_0} h_\omega \in V \). Observe that if \((j, s)\) is good then, since by hypothesis the horocycle section \( \phi \) is projectively \( \kappa \)-Lipschitz, by Lemma 4.8 for sufficiently large \( T_0 > 0 \), we have

\[
Leb \left( \left\{ s' \in [s - e^{-2jT_0}, s + e^{-2jT_0}] : |\Phi_{(j+1)T_0, s_0}(s')| < e^{\eta T_0} |\Phi_{jT_0, s_0}(s')| \right\} \right) < \eta e^{-2jT_0}.
\]

We now wish to use this estimate and Corollary 4.11 to complete the proof of the lemma. To satisfy the assumptions of Corollary 4.11 we define a sequence of nested partitions of the interval \([-1, 1]\):

\[
P^{(j)} = \{ [ie^{-2jT_0}, (i + 1)e^{-2jT_0}] \}_{i = -e^{-2jT_0}}^{-1}.
\]

Let \( P^{(j)}(s) \) be the unique element of \( P^{(j)} \) such that \( s \in P^{(j)}(s) \) and \( P^{(j)}_{s} = [ie^{-2jT_0}, (i + 1)e^{-2jT_0}] \).

We claim that by the remark preceding the statement of Lemma 5.3 if \( g_{jT_0} h_\omega \in K \) then, for any \( s' \in P^{(j+1)}(s) \),

\[
(1 + 2e^{-2T_0})^{-1} \leq \left| \frac{\Phi_{jT_0}(s)}{\Phi_{jT_0}(s')} \right| \leq (1 + 2e^{-2T_0}).
\]

In fact, since the Teichmüller distance between \( g_{jT_0} h_\omega \) and \( g_{jT_0} h'_\omega \) is at most \( e^{-2T_0} \), there exists \( g \in SL(2, \mathbb{R}) \) at a hyperbolic distance at most \( e^{-2T_0} \) from the identity in \( SL(2, \mathbb{R})/SO(2) \) such that, for any cohomology class \( v \in \mathcal{H}^1_{h_\omega}(M, \mathbb{R}) \), we have

\[
KZ(g_{jT_0}, h_\omega)(\pi_{h_\omega}(v)) = KZ(g, g_{jT_0} h_\omega)KZ(g_{jT_0}, h'_\omega)(v).
\]

By Lemma 2.1 it follows that

\[
|KZ(g_{jT_0}, h_\omega)\pi_{h_\omega}(v)| \leq \exp(e^{-T_0}) |KZ(g_{jT_0}, h'_\omega)(v)|,
\]

so that for \( v = \phi(s') \) we derive the estimate

\[
\frac{|KZ(g_{jT_0}, h_\omega)\pi_{h_\omega}(\phi(s'))|}{|KZ(g_{jT_0}, h'_\omega)\phi(s')|} \leq \exp(e^{-2T_0}).
\]

Let \( K > 0 \) denote the Lipschitz constant of \( \phi \) at \( \omega \). Since the section \( \phi \) is \( \kappa \)-projectively Lipschitz at \( \omega \), there exists a constant \( C > 0 \) such that it
is $CK$-Lipschitz and $C\kappa$-projectively Lipschitz at $h_s\omega$. By Lemma 2.1 we then have

$$\|KZ(g_{jT_0}, h_s\omega)\phi(s')\| \geq e^{-jT_0}\|\phi(s')\| \geq e^{-jT_0}(\|\phi(s)\| - CK) \geq e^{-jT_0}K(\kappa^{-1} - C).$$

Since $\|KZ(g_{jT_0}, h_s\omega)\| \leq e^{jT_0}$ and $|s - s'| \leq e^{-2(j+1)T_0}$ for $s' \in P^{(j+1)}(s)$, we also have

$$|KZ(g_{jT_0}, h_s\omega)(\phi(s) - \pi_{h_s\omega}(\phi(s')))| \leq CK e^{-2(j+1)T_0},$$

so that we have derived the estimate

$$\frac{|KZ(g_{jT_0}, h_s\omega)(\phi(s) - \pi_{h_s\omega}(\phi(s')))|}{|KZ(g_{jT_0}, h_s\omega)\phi(s')|} \leq \frac{CK e^{-2T_0}}{1 - C\kappa}.$$

The upper bound in formula (21) then follows from the above estimates, for $\kappa \in (0, \kappa_0)$ and for $T_0 > 0$ sufficiently large. The lower bound follows by symmetry, hence the claim is proved.

Now assume $T_0$ is large enough so that $e^{-cT_0} < (1 + e^{-2T_0})^{-1}$ and let

$$B_j = \{s : \exists s' \in P^{(j)}(s) \text{ with } \frac{|\Phi_{(j+1)T_0}(s')|}{|\Phi_{jT_0}(s')|} \leq e^{(\mu - c)T_0} \leq e^{\mu T_0(1 + 2e^{-2T_0})^{-1}}\},$$

let $\mathcal{I}_j(s) = \{k < j - 1 : s \in B_k\}$ and

$$G_j = \{s : \exists s' \in P^{(j+1)}(s) \text{ with } g_{jT_0}h_s\omega \in V_\varepsilon\}.$$  

Observe that (20) and (21) give

$$\text{Leb}\{s \notin B_j : s \in G_j \text{ and } \mathcal{I}_j(s) = \nu\} > (1 - 2\eta)\text{Leb}\{s \in G_j : \mathcal{I}_j(s) = \nu\}.$$  

Indeed, by definition $s, s' \in P^{(j)}$ for some $k$ implies $\mathcal{I}_j(s) = \mathcal{I}_j(s')$ and (20) and (21) gives the measure estimate conditioned to $s \in G_j$ (which is equivalent to $s' \in G_j$).

This lower bound on the conditional probability provides the large deviations estimate via Corollary 4.11. Indeed, by applying the corollary to the sequence of random variables $F_j$ equal, for all $j \in \mathbb{N}$, to the characteristic functions of the sets of $s \notin B_j \cap G_j$, we have that in the complement of a set of measure exponentially small in $\frac{T}{T_0}$,

$$\#\{0 \leq j \leq \left[\frac{T}{T_0}\right] : s \notin B_j \text{ or } s \notin G_j\} > (1 - 8\eta)\left[\frac{T}{T_0}\right].$$
(for all large enough $T > T_0$). This is because by formula (23) the sequence of random variables $\{F_j\}$ defined above satisfies the assumptions of Corollary 4.11. Consider the set of $s \in [-1, 1]$ such that

$$\{|\{j \in \{0, \ldots, \lfloor \frac{T}{T_0} \rfloor\}: g_j T_0 h_s \omega \in V_t\}| > (1 - 4\eta)[\frac{T}{T_0}] \cdot$$

Note that by Corollary 3.13 (iv) (which we may apply if $T_0$ is at least $t_0$) the complement of this set has that its measure decays exponentially with $[\frac{T}{T_0}]$. The intersection of the sets given by (24) and (25) satisfies the desired conditions, hence the proof of the Proposition is complete. □

6. Proof of Theorem 1.3

6.1. The Veech criterion. To prove Theorem 1.3 we need a condition to rule out the flow $F^t_{r\omega}$ having $e^{2\pi i \alpha}$ as an eigenvalue. This is Lemma 6.1 below, which is essentially due to Veech. Following Veech [Ve84, §7] we have the following criterion for a translation flow $F^t_{r\omega}$ to not have $e^{2\pi i \alpha}$ as an eigenvalue. There is a $c > 0$ and a sequence of transversals $\{J_i\}$ so that

- $|J_i| \to 0$.
- $F^s_{r\omega}(J_i)$ are disjoint intervals for all $0 \leq s \leq \frac{c}{\ell(J_i)}$, where $\ell(J_i)$ denotes the length of the projection of $J_i$ in the direction $\theta$.
- If $T$ is the IET given by the first return to $J_i$ and $I_1, \ldots, I_d$ are the intervals that define $T$ then $|I_a| > c|I_b|$ for all $a, b$.

Let $r_i$ be the vector of return times of $F^t_{r\omega}$ to $J_i$. If $\lim \sup_{i \to \infty} \|r_i \alpha\|_Z \neq 0$ then $e^{2\pi i \alpha}$ is not an eigenvalue of $F^t_{r\omega}$. We have the following trivial consequence of this:

**Lemma 6.1.** (Veech) Let $e^{2\pi i \alpha}$ be a nontrivial eigenvalue of $F^t_{\omega}$. For every compact set $K$ there exists $\ell_K$ so that, if there exist $\ell \geq \ell_K$ and a diverging sequence $(t_i) \subset \mathbb{R}^+$ with the property that $g_{t_i} \omega \in K$, $g_{t_i - \ell} \omega \in K$ and $g_{t_i + \ell} \omega \in K$, then

$$\lim_{i \to \infty} \|KZ(g_{t_i}, \omega)(\alpha \Xi(\omega))\|_Z = 0.$$  

**Proof.** For each compact set $K$ there exist constants $a_K, b_K$ and $d_K > 0$ so that if $\omega' \in K$ then we can write $\omega'$ as a zippered rectangles over an interval $I$ of length at most $a_K$, in the horizontal direction and one endpoint a singularity (that is, as a suspension with piece-wise constant roof function). Moreover the heights of the rectangles are at most $b_K$. Lastly $d_K$ is the length of the shortest saddle connection on all $\omega' \in K$. 

Because there are singularities on each endpoint of the rectangles, if \( u \) is the width of a rectangle, there is a saddle connection on the surfaces with holonomy \((u, r)\) with \( r < b_K \). So if \( \ell > \log \left( \frac{br}{2d_K} \right) \) and \( g_\ell \omega' \in K \) then \( u \geq \frac{1}{2}d_K e^{-\ell} \). In fact, the holonomy of the same saddle connection on \( g_\ell \omega \) is \((e^\ell u, e^{-\ell}r)\) and for \( \ell > \log \left( \frac{2b_K}{d_K} \right) \) we have

\[
e^{-\ell} |r| \leq \frac{d_K}{2b_K} b_K = d_K/2.\]

Since \( g_\ell \omega' \in K \) its shortest saddle connection has length at least \( d_K \), hence we have \( e^\ell |u| \geq d_K/2 \), as claimed.

Likewise, because one endpoint of \( I \) is a singularity, if \( F^s_{\omega'} \) is discontinuous on \( I \) then there is a saddle connection on \( \omega' \) with holonomy \((a, v)\) with \(|v| \leq |s|\) and \(|a| < a_K \). Thus if \( \ell > \log \left( \frac{a_K}{2d_K} \right) \) and \( g_{-\ell} \omega' \in K \), then \( F^s_{\omega'} \) acts continuously on \( I \) for all \(-\frac{1}{2}e^{-\ell}d_K < s < \frac{1}{2}e^{-\ell}d_K \). In fact, the saddle connection on \( g_{-\ell} \omega' \) has holonomy \((e^{-\ell}a, e^\ell v)\). For \( \ell > \log \left( \frac{2a_K}{d_K} \right) \) we have

\[
e^{-\ell} |a| \leq \frac{d_K}{2a_K} a_K = d_K/2,\]

hence \( e^\ell |v| \geq d_K/2 \) and so \(|s| \geq d_K/2e^{-\ell} \), as claimed.

We choose \( \ell_K = \max \{ \log \left( \frac{a_K}{2d_K} \right), \log \left( \frac{b_K}{2d_K} \right) \} \).

Now, if \( \omega' = g_t \omega \), by \( g_{-t} \) we can transport \( I \) back to \( \omega \) and obtain an interval \( I' \) so that \(|I'| = \ell'(I') \leq e^{-t}a_K \). Observe that the second and third bullet points of the Veech criterion still hold since they state properties which are equivariant with respect to the action of the Teichmüller flow \( g_t \). We now need to understand the return times to \( I' \). Note that they are the imaginary parts of a basis for (absolute) homology of the surface and are all of bounded size when transported to \( g_t \omega \) (because \( g_t \omega = \omega' \in K \)). Thus the return time vector is given by \( KZ(g_t, \omega) \mathfrak{Z}(\omega) \) and the lemma follows.  

\[\square\]

### 6.2. Eliminating the stable subbundle of the \( SL(2, \mathbb{R}) \)-bundle and isometric subbundles.

Recall the decomposition of \( r_\theta \) given by formula \((4)\). Since the horocycle \( h_t \) fixes the real part of holomorphic differentials, that is, it fixes the leaves of the stable foliation of the Teichmüller flow, we have that \( g_{\log \cos \theta} h_{-\tan \theta \omega} \) belongs to the same stable manifold as \( r_\theta \omega \), hence \( h_{-\tan \theta \omega} \) and \( r_\theta \omega \) have the same vertical foliation. The vertical flow of \( h_{-\tan \theta \omega} \) equals the vertical flow of \( r_\theta \omega \) after a linear reparametrization by multiplication times \( \cos \theta \). It follows that if \( \alpha \) is a (fixed) eigenvalue for the vertical flow of \( r_\theta \omega \), then \( \alpha \sec(\theta) \) is an eigenvalue for the vertical flow of \( h_{-\tan \theta \omega} \), thus by the
Veech criterion (Lemma 6.1) we have that if \( K \) is a compact set, \( \ell \geq \ell_K \), \( g_t h_s \omega, g_{t+\ell} h_s \omega \) and \( g_{t+\ell} h_s \omega \in K \) with \( t_i \to \infty \) then

\[
\lim_{i \to \infty} \| KZ(g_t, h_s) \alpha \sec(\theta) [\Re(h - \tan \theta \omega)] \| \to 0.
\]

We conclude that if \( \alpha \) is a fixed eigenvalue for the vertical flow of \( r_{\theta} \omega \) for a positive measure set of \( \theta \in \mathbb{T} \), then \( \alpha \sec(\arctan s) \) is an eigenvalue for the vertical flow of \( h_s \omega \), for a positive measure set of \( s \in \mathbb{R} \), which implies that for a positive measure set of \( s \in \mathbb{R} \) we have that if \( g_t h_s \omega, g_{t+\ell} h_s \omega \) and \( g_{t+\ell} h_s \omega \in K \) with \( t_i \to \infty \) then

\[
\lim_{i \to \infty} \| KZ(g_t, h_s \omega) \psi(s) \| \to 0
\]

where \( \psi(s) \) is the horocycle section

\[
\psi(s) := \alpha \sec(\arctan s)) [\Re(h_s \omega)) = \alpha \sec(\arctan s)] [\Im(\omega)).
\]

**Lemma 6.2.** The section \( \psi \) has range in \( SL(2, \mathbb{R}) \) subspace at \( \omega \) and it is transverse to any integer translate of the stable subbundle of the \( SL(2, \mathbb{R}) \) subbundle.

**Proof.** The stable stable subspace of the \( SL(2, \mathbb{R}) \) subspace at \( h_s \omega \) is generated by the cohomology class \([\Re(h_s \omega)] \in H^1_{h_s \omega}(M, \mathbb{R})\). For any \( z \in H^1_{\omega}(M, \mathbb{Z}) \) we consider the equation (with \( a \in \mathbb{R} \))

\[
\psi(s) - z = a \Re(h_s \omega) = a(\Re(\omega) + s \Im(\omega))
\]

By definition of \( \psi(s) \) we derive

\[
z \wedge \Im(\omega) = -a \Re(\omega) \wedge \Im(\omega)
\]

\[
z \wedge \Re(\omega) = (as - \alpha \sec(\arctan s)) \Re(\omega) \wedge \Im(\omega)
\]

Thus the coefficient \( a \in \mathbb{R} \) is uniquely determined (given \( z \in H^1_{\omega}(M, \mathbb{Z}) \)) by the first equation, hence the second equation has a unique solution.

\[\square\]

**Proposition 6.3.** For every compact set \( K \) there exists a constant \( \gamma''_K > 0 \) such that the following holds. If there exist \( t_0 > 0 \) and an integer class \( z \in H^1_{g_{t_0} \omega}(M, \mathbb{Z}) \) such that

- \( g_{t_0} \omega \in K \) and \( \{ t > 0 | g_t \omega \in K \} \) has upper density at least \( \frac{2}{4} \);
- \( |KZ(g_{t_0}, \omega)(\alpha [\Im(\omega)]) - z| \in (0, \gamma''_K) \);
- \( KZ(g_{t_0}, \omega)(\alpha [\Im(\omega)]) - z \) belongs to an isometric subbundle \( E_0 \) for the Kontsevich–Zorich cocycle,

then \( e^{2\pi i \alpha} \) is not an eigenvalue for the vertical flow \( F^t_{\omega} \) of \( \omega \).

\[\footnote{Note that this condition cannot hold if \( \alpha = 0 \).}\]
Proof of Proposition 6.3. Let $\gamma''_{K}$ be so small that the minimum Hodge distance between points of the lattice $H^1_\omega(M, \mathbb{Z})$ for $\omega \in K$ is at least $10\gamma''_{K}$. By the isometry property of the Kontsevich–Zorich cocycle on $E_0$, we have that for all $t \geq t_0$,

\begin{align}
\parallel KZ(g_{t_0}, \omega)(\alpha[\Im(\omega)])\parallel_Z &= \parallel KZ(g_{t_0}, \omega)(\alpha[\Im(\omega)]) - z\parallel_Z = \parallel KZ(g_{t-t_0}, g_{t_0}\omega)(z)\parallel_Z \leq \gamma''_{K},
\end{align}

hence, whenever $g_t\omega \in K$, since $KZ(g_{t-t_0}, g_{t_0}\omega)(z) \in H^1_{g_t\omega}(M, \mathbb{Z})$, by the choice of the constant $\gamma_K > 0$, we have

\[ \parallel KZ(g_t, \omega)(\alpha[\Im(\omega)])\parallel_Z = \parallel KZ(g_{t_0}, \omega)(\alpha[\Im(\omega)])\parallel_Z \neq 0. \]

Since by hypothesis that the set $\{ t > 0 : g_t\omega \in K \}$ has upper density $\frac{3}{4}$, it can be proved (by an elementary argument left to the reader) that for any $\ell_0 > 0$ there exists $\ell > \ell_0$ and a diverging sequence $(t_i)$ such that $g_{t_i}\omega, g_{t_i-\ell}\omega$ and $g_{t_i+\ell}\omega \in K$ for all $i \in \mathbb{N}$. Hence, we may apply the Veech criterion (Lemma 6.1) and $\alpha$ is not an eigenvalue, as stated. □

6.3. Proof of Theorem 1.3. Before completing the proof, we recall that if Kontsevich-Zorich cocycle does not act isometrically on a subbundle then Proposition 5.1 can be applied to it.

Theorem 6.4. (c.f. [EskMir], [FoMaZo]) Let $\phi$ be a horocycle section. For almost every $s \in \mathbb{R}$, one of the following two (not mutually exclusive) possibilities holds

1. $\phi(s)$ has a non-trivial projection onto a continuous $SL(2, \mathbb{R})$-equivariant subbundle where the Kontsevich-Zorich cocycle acts isometrically.
2. $\phi(s)$ has a non-trivial projection onto a continuous $SL(2, \mathbb{R})$-equivariant subbundle where the Kontsevich-Zorich cocycle has a positive exponent and acts strongly irreducibly.

Proof. By [EskMir] Theorem A.6] the splitting is semi-simple and each irreducible block is strongly irreducible (in the sense that it does not admit a measurable almost invariant splitting) and it is either isotropic or symplectic. By [EskMir] Theorem A.5] if the subbundle is isotropic then all of the exponents are zero. By [EskMir] Theorem A.4], if all the exponents are zero then the cocycle acts isometrically. So, we assume that we are considering an irreducible symplectic block where the cocycle has a non-zero exponent. Since the cocycle is symplectic, it has both a positive and a negative exponent (which are opposite of each other). □
Proof of Theorem 1.3. We proceed with a proof by contradiction. Let 
B ⊂ [−1, 1] denote the set of points for which the conclusion of the 
theorem fails, and assume that the measure of B is ξ > 0. Let \( K \) be a 
compact set so that Lemma 2.2 and Proposition 5.1 hold with \( c = .999 \) 
and \( \mu(\text{Int}(K)) > .999 \). Let \( a = \max \{ t_0, \ell_K \} \) where \( t_0 \) is as in Lemma 2.2 and \( \ell_K \) is as in Lemma 6.1. We choose \( \gamma_0 > 0 \) small enough so 
that Proposition 5.1 holds with the compact set \( K \), \( \gamma_0 \leq \gamma''_K \) where \( \gamma''_K \) is as in Proposition 6.3, and 
one additional condition below (so that the RHS in the estimate of 
formula (33) below is at most \( \frac{1}{2} \)). By Lemma 6.1 there exists \( t_0 > 0 \) so 
that

(27) \[
\text{Leb}\{ s \in [-1, 1] : \text{for all } t \geq t_0 \text{ either } \| KZ(g_t, h_s \omega)\psi(s)\|_Z < e^{-a-1}\gamma_0 \\
\text{or } g_t h_s \omega \notin K, \text{ or } g_{t-a} h_s \omega \notin K, \text{ or } g_{t+a} h_s \omega \notin K \} > .99\xi.
\]

Let \( B' \) denote the set in the LHS of formula (27). By the Lebesgue 
density theorem, there exists \( r_0 > 0 \) so that

(28) \[
\text{Leb}\{ s \in B' : \text{Leb}(B(s, r) \cap B') > 1.99 r \text{ for all } r \leq r_0 \} > .98 \xi.
\]

Let \( B'' \) be the set in the LHS of formula (28). We now choose \( t \geq t_0 \) 
and \( s \in B'' \) such that \( h_s \omega \) is Birkhoff generic, which is a full measure 
condition by [ChEs, Theorem 1.1], so that

- \( e^{-2t} < r_0 \)
- \( g_t h_s \omega \in K \)
- \( \| KZ(g_t, h_s \omega)\psi(s)\|_Z = \gamma < \gamma_0 \).

The first assumption is trivial. The second and third assumption can be 
simultaneously satisfied by Lemma 6.1 and Birkhoff genericity. (Indeed 
\( \mu(K) > .999 > \frac{3}{4} \) and so, for any \( \ell \), there exist arbitrarily large \( t \) so 
that \( g_{t-\ell} h_s \omega, g_v h_s \omega, g_{t+\ell} h_s \omega \in K \).)

By the third, and last, of the above assumptions there exists an 
integer class \( z \in H^1_{g_t h_s \omega}(M, \mathbb{Z}) \) such that

\[
| KZ(g_t, h_s \omega)(\psi(s) - z) | = \gamma < \gamma_0.
\]

This smallness assumption shows that, if \( \psi(s) - z \) is contained in an 
isometric subbundle, then for almost every \( \theta \in S^1 \), the flow in direction 
\( \theta \) does not have \( \alpha \) as an eigenvalue. Indeed, we are assuming all but 
the first assumption of Proposition 6.3 and that assumption holds for 
almost every \( s \in [-1, 1] \), by Birkhoff genericity.

Thus, up to the translation of \( \psi \) by the integer vector \( z \), that is, 
by considering the section \( \psi - z \) instead of \( \psi \), we can assume that \( \psi \)
has a non-zero projection on a strongly irreducible subbundle with a non-zero top Lyapunov exponent.

By Lemma \[6.2\] the section \(\psi\) is transverse to any integer translate of the stable subbundle of the \(SL(2, \mathbb{R})\) subbundle, so clearly the projection to the stable subspace of the \(SL(2, \mathbb{R})\) subbundle can be ignored.

To complete the proof we now separately consider the projection of \(\phi\) to the unstable part of the \(SL(2, \mathbb{R})\) subbundle and to the symplectic orthogonal \(\hat{F}\) of the \(SL(2, \mathbb{R})\) subbundle.

We show that for each of these individually, at the time \(t' > t\) when the norm of the section has grown to size \(\gamma_0\) under the action of the cocycle, the subset of \(B(s, e^{-2t})\) of points which are not in the set in formula \[27\] has at least half the measure of \(B(s, e^{-2t})\), a contradiction which will conclude our argument. First, we consider the projection onto the unstable part of the \(SL(2, \mathbb{R})\)-subbundle. By Lemma \[2.2\], for any \(t' > t_0 + a\) we have that

\[
\text{Leb}\{s' \in B(s, e^{-2t}) : g_{t'} h_s \omega, g_{t'-a} h_s \omega \text{ and } g_{t'+a} h_s \omega \in K}\) > 1.994 e^{-2t}.
\]

Let \(\delta > 0\) be the norm of the projection of \(KZ(g_t, h_s \omega) \psi(s)\) onto the unstable part of the \(SL(2, \mathbb{R})\) subbundle. Let \(\ell = \log(\frac{20}{\delta}) + 1\). Because the unstable part of the \(SL(2, \mathbb{R})\) subbundle grows by a constant factor \(e^{t'}\) at time \(t' > 0\), we have our claim by applying \[29\] at \(t' = t + \ell\).

It now suffices to prove the analogous result for \(\hat{F}\). Let \(\phi\) be the projection of \(\psi\) on \(\hat{F}\). By choosing a possibly larger \(t > 0\), we can further assume that

- \(g_t h_s \omega \in K\),
- the horocycle section \(\phi_0 : [-1, 1] \to H^1(M, \mathbb{R})\) defined by
  \[
  \phi_0(s') = KZ(g_t, h_s e^{-2t} \omega) \phi(s + e^{-2t} s')
  \]
  is \(\kappa \gamma'\)-Lipschitz at \(h_s \omega\), where \(\kappa < \kappa_0\) as in Proposition \[5.1\].
- \(\max_{s' \in [-1, 1]} \|\phi_0(s')\|_Z = \gamma' < \gamma_0\).

Indeed by Lemma \[4.6\] for every compact set \(K'\), there exists \(v > 0\) so that to establish the second bullet point it suffices to show that \(\|\{0 < t'' < t : g_{t''} h_s \omega \in K'\}\| > v > 0\) for all \(s' \in [s - e^{-2t}, s + e^{-2t}]\). We choose \(K'\) to be the closure of a small neighborhood of \(K\). By the Birkhoff genericity of \(h_s \omega\) with respect to the Teichmüller flow, there exists \(t_1\) so that \(\|\{0 < t'' < t_1 : g_{t''} h_s \omega \in K\}\| > v > 0\). There exists \(U\), a small neighborhood of \(s\), so that for every \(s' \in U\) and \(0 \leq t'' \leq t_1\),

\[
g_{t''} h_s \omega \in K' \text{ whenever } g_{t''} h_s \omega \in K.
\]
The second bullet point follows by choosing \( t \geq t_1 \) so that \( s' \in [s-e^{-2t}, s+e^{-2t}] \subset U \).

We now iteratively apply Proposition 5.1. For every \( j \in \mathbb{N} \), let \( S_j \subset [s-e^{-2t}, s+e^{2t}] \) denote the set of points such that Proposition 5.1 can be applied \( j \)-times. Inductively, by applying Proposition 5.1 for \( j-1 \) times, for a point \( s_0 \in S_j \) we obtain real numbers \( \ell_1(s_0), \ldots, \ell_j(s_0) \). Let \( L_j(s_0) := \ell_1(s_0) + \cdots + \ell_j(s_0) \) and, for all \( s' \in [-1,1] \), let

\[
\phi_j(s') := KZ(g_{t+L_j(s_0)}, h_{s'}\omega)\phi(s'),
\]

which restricts on the interval \([s_0 - e^{-2(t+L_j(s_0))}, s_0 + e^{-2(t+L_j(s_0))}]\) to a reparametrization of the section \( \phi_{t+L_0(s_0)} \) introduced in Proposition 5.1. Since the section \( \phi_{t+L_j(s_0),s_0} \) is \( \gamma' \)-Lipschitz and has maximum at least \( \gamma \) it follows that

\[
\min \phi_{t+L_j(s_0),s_0} \geq (1-\kappa) \max \phi_{t+L_j(s_0),s_0}.
\]

Let then \( (x_j) \) denote the sequence defined by recursion as follows: let \( x_0 = \min \phi_0 \) and let

\[
x_{j+1} = (1-\kappa)^2 x_j^\rho.
\]

Notice that, for all \( j \in \mathbb{N} \),

\[
x_j = (1-\kappa)^{2+\cdots+2\rho} x_0^\rho \geq (1-\kappa)^{2/(1-\rho)} x_0^\rho.
\]

Observe that, if

\[
(30) \quad \beta := \log_{x_0}(\gamma_0) - 2(1-\rho)^{-1} \log_{x_0}(1-\kappa) > 0.
\]

and \( v = \lceil \log_{x_0}(\beta) \rceil \) then

\[
(31) \quad \gamma_0^\rho > x_v \geq \gamma_0.
\]

Now, by induction, if \( s' \in S_{j+1} \) and \( \|KZ(g_{t+L_i(s')}, h_{s'}\omega)\phi(s')\|_Z < \gamma_K \) for \( i \leq j \) then

\[
(32) \quad \|KZ(g_{t+L_{j+1}(s')}, h_{s'}\omega)\phi(s')\|_Z \geq x_{j+1}.
\]

Indeed, inductively \( x_i \leq \|KZ(g_{t+L_i(s')}, h_{s'}\omega)\phi(s')\|_Z \) and so it follows by formula (16). By (15) we have that the the measure of the set of

\[\text{Note that if } s' \in S_j \text{ then we can apply Proposition 5.1 at } h_{s'}\omega \text{ iff } \|KZ(g_{t+L_j(s')}, h_{s'}\omega)\phi(s')\| \text{ is small enough. Indeed, } e^{t+L_j(s')} > e^t, g_{t+L_j(s')} h_{s'}\omega \in K \text{ by the definition of } S_j. \text{ The Lipschitz assumption follows because Lemma 4.2 gives that the Lipschitz constant gets smaller, while at the same time } \|KZ(g_t, h_{s'}\omega)\phi(s')\| \geq \|KZ(g_t, h_{s'}\omega)\phi(s')\| \text{ and so the requirement on the Lipschitz constant gets laxer. So the second bullet point implies (6).} \]
\( s' \in [s-e^{-2t}, s+e^{-2t}] \) so that \( \| KZ(g_{t+\min(j',s' \notin S_j)}(s'), h_{s'} \omega) \phi(s') \|_Z < \gamma_0 \)
is at most (since \( v = \lfloor \log_\rho(\beta) \rfloor \) and by formula \( (30) \))
\[
\sum_{j=0}^{v} (x_j) | a | \leq \sum_{j=0}^{v} (x_0 | a |) | a | = \sum_{i \geq 0} \left( \frac{\gamma_0}{(1-\kappa)^{2/(1-\rho)}} \right)^{i | a | a} .
\]

Clearly this bound goes to 0 with \( \gamma_0 \), and so if \( \gamma_0 \) is small enough we
are left with at least a subset of conditional measure greater than \( \frac{1}{2} \). That is, for a set of \( s' \in [s-e^{-2t}, s+e^{-2t}] \) of conditional measure at least \( \frac{1}{2} \) we have that for each \( s' \) in this set:
- there exists \( \tau_{s'} \geq t \) so that \( \| KZ(g_{\tau_{s'}}, h_{s'} \omega) \phi(s') \|_Z \geq \gamma_0 \),
- \( g_{\tau_{s'}} h_{s'} \omega \in K \).

By Lemma 2.2 if \( \gamma_0 \) is sufficiently small (so that in Proposition 5.1 \( \ell(s) \geq | \log \gamma_0 | / 2 > t_0 \), hence \( \tau_{s'} - t \geq t_0 \)) the subset of \( s' \in [s-e^{-2t}, s+e^{-2t}] \) such that \( g_{\tau_{s'}+a} h_{s'} \omega \) or \( g_{\tau_{s'}+a} h_{s'} \omega \) are not in \( K \) has conditional measure at most \( 1/9 \). By Lemma 2.1 we have that
\[
\| KZ(g_{\tau_{s'}}, h_{s'} \omega) \phi(s') \|_Z \geq \gamma_0 \implies \| KZ(g_{\tau_{s'}+a}, h_{s'} \omega) \phi(s') \|_Z \geq e^{-a} \gamma_0 .
\]

By the above lower bound, together with the condition that \( g_{\tau_{s'}}, h_{s'} \omega \in K \), \( g_{\tau_{s'}+a} h_{s'} \omega \in K \) and \( g_{\tau_{s'}+a} h_{s'} \omega \in K \), we conclude that \( s' \notin B' \).
Since the conditional measure of such \( s' \in [s-e^{-2t}, s+e^{-2t}] \) is at least \( 1/2 - 1/9 > 1/4 \), this contradicts that \( s \in B'' \) (because \( e^{-2t} < r_0 \)).

7. Proof of Theorem 1.1

7.1. Setup. Let \( P \) be a polygon and \( T_1(P) \) be the phase space for its
billiard flow. Let \( X(P) = T_1(P) \times T_1(P) \).

The phase space \( T_1(P) \) is endowed with the Liouville measure for the
billiard flow, hence the space \( X(P) \) can be endowed with the product
measure. All integrals below will be taken with respect to the square
of the Liouville measure on \( X(P) \).

We also need to define \( Lip_\epsilon(X(P)) \) the space of \( c \)-Lipschitz functions
on \( X(P) \). For instance, we view \( T_1(P) \) as a quotient of \( P \times S^1 \) (which
is a metric space) and use the path metric to define a metric on \( T_1(P) \).

Lemma 7.1. For every \( \epsilon > 0 \) and \( T > 0 \) we have
\[
\{ P : \int_{X(P)} | \frac{1}{T} \int_0^T f(F^t(x,\theta),F^t(y,\psi))dt - \int_{X(P)} f | < \epsilon 
\text{ for all } f \in Lip_\epsilon(X(P)) \}
\]
is open.
By Theorem 1.2 we have

**Corollary 7.2.** For every $\epsilon > 0$ there exists $N_\epsilon$ such that the following holds. If $P$ is a rational polygon with the property that the group of reflections about its sides (translated to the origin) contains a rotation by $\frac{2\pi}{M}$ with $M \geq N_\epsilon$, then there exists $T_0$ such that for all $T \geq T_0$ and $f \in Lip_1(X(P))$ we have

$$\int_{X(P)} \left| \frac{1}{T} \int_0^T f(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} f \right| < \epsilon.$$ 

**Proof.** Consider the flat surface $M_P$ obtained by unfolding $P$. By Theorem 1.2 we have that for almost every pair of directions the product flow is ergodic on $M_P \times M_P$. Considering these flows on $T_1(P)$, by our assumption on the group of reflections, they equidistribute in the product of the table cross $M \geq N_\epsilon$ evenly spaced copies of a discrete set in $S^1$. The corollary follows. \qed

**Proof of Theorem 1.1 assuming Lemma 7.1.** Let $n$ be given and, for all $k \in \mathbb{N}$, let $U_k$ be the set of polygons $P$ with $n \geq 3$ sides so that there exists $T > 0$ such that, for all $f \in Lip_1(X(P))$,

$$\int_{X(P)} \left| \frac{1}{T} \int_0^T f(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} f \right| < \frac{1}{k}.$$ 

By Lemma 7.1, $U_k$ is open for all $k \in \mathbb{N}$ as union (over $T > 0$) of a family of open sets. By Corollary 7.2 it is dense for all $k \in \mathbb{N}$. So by the Baire Category theorem there exists $P \in \cap_k U_k$. Because 1-Lipschitz functions have dense span in the set of continuous functions, the product of the billiard flow with itself on $X(P)$ is ergodic. In fact, if $P \in \cap_k U_k$ we have that, for every $k \in \mathbb{N}$, there exists $T_k > 0$ such that for all $f \in Lip_1(X(P))$,

$$\int_{X(P)} \left| \frac{1}{T_k} \int_0^{T_k} f(F^t(x, \theta), F^t(y, \psi)) dt - \int_{X(P)} f \right| < \frac{1}{k}.$$ 

Ergodicity then follows (for instance) from the ergodic theorem. The ergodicity of the product billiard flow implies that the billiard flow is weakly mixing. \qed

**Definition 7.3.** (Y. Vorobets [Vo97], Definition 2.1) We say a polygon $Q$ is a $\delta$-perturbation of $P$ if there exists a homeomorphism $\phi_{P,Q}: P \to Q$ that establishes a one-to one correspondence between the vertices of the two polygons and so that the distance between the corresponding vertices is at most $\delta$. 


For every polygon $P$ there exists $\delta_P > 0$ such that for $\delta < \delta_P$, any $\delta$-perturbation $Q$ of $P$ has a triangulation whose triangles are in a bijective correspondence with triangles of a triangulation of $P$. For instance, one can triangulate $P$ by diagonals in an arbitrary manner. Then for $\delta < d(P)/2$, with $d(P)$ the minimum of the smallest non-zero distance between vertices, for each two vertices of $P$ which can be joined by a diagonal inside $P$, the corresponding vertices of $Q$ can also be joined by a diagonal inside $P$. Then there exists a triangulation of $Q$ corresponding to the above triangulation of $P$ by diagonals.

In the next lemma we assume the homeomorphism $\phi_{P,Q}$ is as above, affine on the triangles in a (fixed) triangulation of $P$ and takes them to the corresponding triangles of $Q$.

**Lemma 7.4.** For every table $P$ and for every $\epsilon > 0$, there exists $\delta > 0$ such that if $Q$ is a $\delta$-perturbation of $P$, then

$$\text{Lip}_1(X(Q)) \subset \phi^{-1}_{P,Q}(\text{Lip}_1(X(P))).$$

This is straightforward.

**Lemma 7.5.** If $P$ is in the set described by (34) then there exists $\epsilon' > 0$ so that

$$\int_{X(P)} |\frac{1}{T} \int_0^T f(F^t(x,\theta), F^t(y,\psi))dt - \int_{X(P)} f| < \epsilon$$

for all $f \in \text{Lip}_{1+\epsilon'}(X(P))$.

**Proof.** Indeed $\text{Lip}_1(X(P))$ is compact in $\| \cdot \|_{\text{sup}}$ and so there exists $\epsilon''$ so that

$$\int_{X(P)} |\frac{1}{T} \int_0^T f(F^t(x,\theta), F^t(y,\psi))dt - \int_{X(P)} f| < \epsilon - \epsilon''$$

for all $f \in \text{Lip}_1(X(P))$.

Let $\epsilon' = \frac{1}{9}(\frac{1}{9(1-\epsilon''}) - 1)$.

For any polygon $P$, let $\pi^{(1)} : P \times S^1 \to P$ and $\pi^{(2)} : P \times S^1 \to S^1$ denote the canonical projections.

**Proposition 7.6.** (Vorobets \cite{Vo97, Proposition 2.3}) Let $Q$ be a $\delta$-perturbation of $P$. For each $t > 0$ there exists $B \subset P \times S^1$ dependent on the polygons $P, Q$, the map $\phi := \phi_{P,Q}$ and $t$ and of measure at most $C_3(C_1t + C_2)^3\delta$ such that for each $(x,\theta) \in P \times S^1 \setminus B$ and for each $\tau$, $0 \leq \tau \leq t$, at least one of the following two possibilities hold:
The distance between $\pi_{P}^{(1)} F_{P}^{t}(x, v)$ and $\phi^{-1}(\pi_{Q}^{(1)} F_{Q}^{t}(\phi(x), v)$ is at most $C_4(C_1 t + C_2)^2\delta$ and the angle between the directions of $\pi_{P}^{(2)} F_{P}^{t}(x, v)$ and $\pi_{Q}^{(2)} F_{Q}^{t}(\phi(x), v)$ is at most $C_5(C_1 t + C_2)^2\delta$.

(2) The points $\pi_{P}^{(1)} F_{P}^{t}(x, v)$ and $\pi_{Q}^{(1)} F_{Q}^{t}(\phi(x), v)$ lie at a distance at most $C_6(C_1 t + C_2)^2\delta$ from the boundaries of $P$ and $Q$ respectively. The constants $C_1, \ldots, C_6$ are positive and depend only on $P$.

From this proposition, we derive:

**Corollary 7.7.** For any $f \in \text{Lip}_c(X(P))$, $T$ and $\epsilon > 0$ there exists $\delta > 0$ so that for any $Q$, a $\delta$-perturbation of $P$ we have

$$
\left| \int_{X(Q)} f \circ \phi_{P,Q}^{-1}(F_{P}(x, \theta), F_{Q}(y, \psi)) dt - \int_{X(P)} f(F_{P}(x, \theta), F_{Q}(y, \psi)) dt \right| < \epsilon.
$$

Note for fixed $c$ and $T$, $\delta$ can be chosen uniformly over $\text{Lip}_c(X(P))$.

**Proof of Lemma 7.1.** Let $\epsilon < \frac{1}{4}$. It suffices to show that for any $P$ which belongs to the set in formula (34) there exists $\delta_P > 0$ so that any $\delta_P$-perturbation also belongs to the set (34). Because $P$ belongs to the set (34), it satisfies the inequality in (34) for $\epsilon - \epsilon''$ for some $\epsilon'' > 0$. We choose $\delta > 0$ so that Lemma 7.4 implies that, if $Q$ is a $\delta$-perturbation of $P$, then $\text{Lip}_c(X(Q)) \subset \text{Lip}_c^{1+\epsilon''}(X(P))$, formula (35) is satisfied with $\epsilon = \epsilon''$ and

$$
\left| \int_{X(Q)} f \circ \pi_{P,Q}^{-1} - \int_{X(P)} f \right| < \frac{\epsilon''}{9}.
$$

It follows that any $\delta$-perturbation of $P$ satisfies (34). \qed

**References**

[AAEKMU] Al-Saqban, H; Apisa, P; Erchenko, A; Khalil, O; Mirzadeh, S; Uyanik, C. Exceptional Directions for the Teichmüller Geodesic Flow and Hausdorff Dimension, preprint, arXiv:1711.10542v2 (to appear in Journal of Eur. Math. Soc.).

[Ath] Athreya, J. Quantitative recurrence and large deviations for Teichmuller geodesic flow. Geom. Dedicata 119 (2006), 121–140.

[AuAvDel] Aulicino, D; Avila, A; Delecroix, V. Announcement

[AvDel] Avila, A; Delecroix, V. Weak mixing directions in non-arithmetic Veech surfaces J. Amer. Math. Soc. 29, no. 4 (2016), 1167–1208.
Weak mixing for interval exchange transformations and translation flows. Ann. of Math. **165** (2007), no. 2, 637–664.

Bobok, J; Troubetzkoy, S. Topologically weakly mixing polygonal billiards. Proceedings AMS. DOI: https://doi.org/10.1090/proc/14135

Boldrighini, C; Keane, M; Marchetti, F. Billiards in polygons. Ann. Probab. **6** (1978), no. 4, 532–540.

Casati, G; Prosen, T. Mixing property of triangular billiards. Phys. Rev. Lett. **83** (1999), 4729–4732.

Chaika, J; Eskin, A. Every flat surface is Birkhoff and Oseledets generic in almost every direction. J. Mod. Dyn. **9** (2015), 1–23.

Chaika, J; Hubert, P. Circle averages and disjointness in typical flat surfaces on every Teichmüller disc, Bull. London Math. Soc. **49**, no. 5 (2017), 755–769.

Chernov, N; Markarian, R. Chaotic billiards. Mathematical Surveys and Monographs **127**. American Mathematical Society, Providence, RI, 2006. xii+316 pp.

Eskin, A; Mirzakhani, M. Invariant and stationary measures for the SL(2, R) action on moduli space. Publ. Math. Inst. Hautes Études Sci. **127** (2018), 95–324.

Eskin, A; Mirzakhani, M; Mohammadi, A. Isolation, equidistribution, and orbit closures for the SL(2, R) action on moduli space. Ann. of Math. **182** (2015), no. 2, 673–721.

Ferenczi, S; Hubert, P. Rigidity of square-tiled interval exchange transformations. J. Mod. Dynam. **14** (2019), 153-177.

Filip, S. Zero Lyapunov exponents and monodromy of the Kontsevich–Zorich cocycle. Duke Math. J. **166** (2017) (4), 657–706.

Forni, G. Deviation of ergodic averages for area-preserving flows on surfaces of higher genus, Ann. of Math. **155**, no. 1, (2002), 1–103.

Forni, G; Matheus, C. Introduction to Teichmüller theory and its applications to dynamics of interval exchange transformations, flows on surfaces and billiards, in Lectures from the Bedlewo Summer School 2011, edited by F. Rodriguez Hertz. J. Mod. Dynam. **8** (3-4), 2014, 271–436.

Forni, G; Matheus, C; Zorich, A. Lyapunov spectrum of invariant subbundles of the Hodge bundle Ergod. th. & Dynam. Sys. **34** (2014) 353-408.

Gutkin, E. Billiard dynamics: an updated survey with the emphasis on open problems. Chaos **22**, No. 2, 026116 (2012).

Gutkin, E; Katok, A. Weakly mixing billiards, Springer Lecture Notes in Math. **1345** (1988), 163–176.

Kadyrov, S; Kleinbock, D; Lindenstrauss, E; Margulis, G. Singular systems of linear forms and non-escape of mass in the space of lattices. Journal d’Analyse Mathématique **133**, (1) (2017) 253–277.

Katok, A. B. The growth rate for the number of singular and periodic orbits for a polygonal billiard, Commun. Math. Phys. **111** (1987), 151–160.

Katok, A. B. Billiard table as a playground for a mathematician. Surveys in modern mathematics, 216–242, London Math. Soc. Lecture Note Ser. **321**, Cambridge Univ. Press, Cambridge, 2005.
[KT] Katok, A. B; Thouvenot, J.-P. Spectral properties and combinatorial constructions in ergodic theory, Handbook of dynamical systems. Vol. 1B, Elsevier B. V., Amsterdam, 2006, 649–743.

[KMS] Kerckhoff, S; Masur, H; Smillie, J. Ergodicity of Billiard Flows and Quadratic Differentials, Ann. of Math. 124, no. 2, (1986), 293–311.

[MaTr] Málaiga Sabogal, A; Troubetzkoy, S. It Weakly mixing polygonal billiards. Bull. Lond. Math. Soc. 49 (2017), no. 1, 141–147.

[Mas82] Masur, H. Interval Exchange Transformations and Measured Foliations, Ann. of Math. 115, No. 1 (1982), 169–200.

[Mas90] Masur, H. The growth rate of trajectories of a quadratic differential, Ergod. Th. & Dynam. Sys. 10 (1990), 151–176.

[MT] Masur, H; Tabachnikov, S. Rational billiards and flat structures. Handbook in Dynamical Systems Vol. 1A, Elsevier, 2002, 1015–1089.

[Sch13] Scheglov, D. Growth of periodic orbits and generalized diagonals for typical triangle billiards, J. Mod. Dynam. 7 (2013), 31–44.

[Sch19+] Scheglov, D. Complexity growth of a typical triangular billiard is weakly exponential, preprint, arXiv:1208.4679v1 (to appear in Journal d’Analyse Mathématique).

[Tab] Tabachnikov, S. Geometry and billiards. Student Mathematical Library 30. American Mathematical Society, Providence, RI; Mathematics Advanced Study Semesters, University Park, PA, 2005. xii+176 pp.

[Ve84] Veech, W. The metric theory of interval exchange transformations. I. Generic spectral properties. Amer. J. Math. 106 (1984), no. 6, 1331–1359.

[Vo97] Vorobets, Y. Ergodicity of billiards in polygons. Mat. Sb. 188, no. 3 (1997), 65–112; Sb. Math. 188, no. 3 (1997), 389–434

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH, SALT LAKE CITY, UT USA

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD USA

E-mail address: chaika@math.utah.edu
E-mail address: gforni@umd.edu