COHOMOLOGY OF INVARIANT DRINFELD TWISTS
ON GROUP ALGEBRAS

PIERRE GUILLOT AND CHRISTIAN KASSEL

ABSTRACT. We show how to compute a certain group \( H^2_\ell(G) \) of equivalence classes of invariant Drinfeld twists on the algebra of a finite group \( G \) over a field \( k \) of characteristic zero. This group is naturally isomorphic to the second lazy cohomology group \( H^2_\ell(O_k(G)) \) of the Hopf algebra \( O_k(G) \) of \( k \)-valued functions on \( G \). When \( k \) is algebraically closed, the answer involves the group of outer automorphisms of \( G \) induced by conjugation in the group algebra as well as the set of all pairs \((A, b)\), where \( A \) is an abelian normal subgroup of \( G \) and \( b : \hat{A} \times \hat{A} \to k^\times \) is a \( k^\times \)-valued \( G \)-invariant non-degenerate alternating bilinear form on the dual \( \hat{A} \). When the ground field \( k \) is not algebraically closed, we use algebraic group techniques to reduce the computation of \( H^2_\ell(G) \) to a computation over the algebraic closure. As an application of our results, we compute \( H^2_\ell(G) \) for a number of groups.

Introduction

The aim of this paper is to show how to compute a certain group \( H^2_\ell(G) \) of equivalence classes of invariant Drinfeld twists on the group algebra of a finite group \( G \). Drinfeld twists were introduced by Drinfeld in his work [11] on quasi-Hopf algebras. They were used by Etingof and Gelaki [14] to classify the triangular semisimple cosemisimple Hopf algebras. There is an abundant literature on the subject, starting with Movshev’s article [23] and including several papers by Davydov, Etingof, Gelaki and others; see [1, 7, 8, 9, 12, 13, 14, 15, 17].

There are mainly two reasons why we restrict to invariant Drinfeld twists:
(a) firstly, invariant twists form a group under multiplication whereas general twists do not;
(b) secondly, the group \( H^2_\ell(G) \) we consider identifies naturally with the second lazy cohomology group \( H^2_\ell(O_k(G)) \) of the Hopf algebra \( O_k(G) \) of \( k \)-valued functions on \( G \):

\[
H^2_\ell(G) \cong H^2_\ell(O_k(G)).
\]

The lazy cohomology group \( H^2_\ell(H) \) was defined by Schauenburg [27] for any Hopf algebra \( H \); it was systematically investigated by Bichon and Carnovale in [3]. The group \( H^2_\ell(H) \) has been used to compute the Brauer group of a Hopf algebra and to equip the category of projective representations of \( H \) with the structure of a crossed \( \pi \)-category in the sense of Turaev [31].

When \( H \) is a cocommutative Hopf algebra, then \( H^2_\ell(H) \) coincides with the Hopf algebra cohomology group \( H^2(H, k) \) introduced by Sweedler in [29]. In the special case where \( H = k[G] \) is a group algebra, \( H^2_\ell(H) \) is isomorphic to the cohomology group \( H^2(G, k^\times) \) of the group \( G \) acting trivially on \( k^\times \). There are very few examples of non-cocommutative Hopf algebras for which the lazy cohomology has been determined. So the computation of the lazy cohomology of the function algebras \( O_k(G) \), which are in general not cocommutative, was our prime motivation.

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We also felt that such a computation might lead to a better understanding of the corresponding “versal torsors” constructed in [2].

Let us summarize our results. Given a group $G$, let $\text{Int}_k(G)$ be the group of automorphisms of $G$ induced by conjugation in the group algebra $k[G]$; this group contains the group $\text{Inn}(G)$ of inner automorphisms of $G$. We first show that the quotient group $\text{Int}_k(G)/\text{Inn}(G)$ embeds into $H^2_G(G)$. As a consequence, there are finite groups $G$ for which $H^2_G(G)$ is non-abelian. This settles an open question: there are indeed (finite-dimensional) Hopf algebras $H$ such that the lazy cohomology group $H^2_H$ is not abelian.

Next, let $B(G)$ be the set of pairs $(A, b)$, where $A$ is an abelian normal subgroup of $G$ and $b : A \times \hat{A} \to k^\times$ is a $k^\times$-valued $G$-invariant non-degenerate alternating bilinear form on the Pontryagin dual $\hat{A}$ of $A$. When the ground field $k$ is algebraically closed and of characteristic zero, we construct a map

$$\Theta : H^2_B \to B(G)$$

whose fibres are the left cosets of $\text{Int}_k(G)/\text{Inn}(G)$; we show that this map is surjective if in addition the order of $G$ is odd. In case $B(G)$ is trivial, we have

$$H^2_B \cong \text{Int}_k(G)/\text{Inn}(G).$$

We also identify the set $B(G)$ with a colimit (in the category of sets) of $H^2(\hat{A}, k^\times)^G$ over all abelian normal subgroups $A$ of $G$. As an application, if $G$ is a group of odd order with $\text{Int}_k(G) = \text{Inn}(G)$ and with a unique maximal abelian normal subgroup $A$, then $H^2_B(G)$ has the following simple expression:

$$H^2_B(G) \cong H^2(\hat{A}, k^\times)^G.$$

Note that although these results can be expressed purely in group-theoretical terms, their proofs rely on techniques originating in the theory of quantum groups.

We use the previous results to compute $H^2_B(G)$ in a number of examples. We first give a list of groups for which $H^2_B(G)$ is trivial: this includes all simple groups and all symmetric groups. We also exhibit groups for which $H^2_B(G)$ is non-trivial: if $G$ is the wreath product $\mathbb{Z}/p!\mathbb{Z}/p$ with $p$ an odd prime, then $H^2_B(G)$ is an elementary abelian $p$-group of rank $(p - 1)/2$. For the alternating group $A_4$, we have $H^2_B(A_4) \cong \mathbb{Z}/2$ and we give explicitly an invariant twist generating this group. We also consider a group of order 32 with non-trivial $\text{Int}_k(G)/\text{Inn}(G)$; we write down an invariant twist generating the subgroup of $H^2_B(G)$ coming from $\text{Int}_k(G)/\text{Inn}(G)$.

We also present a general sufficient condition for $H^2_B(G)$ to be abelian, namely it is so if the tensor product of any two irreducible representations of $G$ is a multiplicity-free direct sum of irreducible representations.

When the ground field $k$ is not algebraically closed, we use Lie algebra techniques to compare the group $H^2_B(G/k) = H^2_B(G)$ to the group $H^2_B(G/k)$ of gauge equivalence classes of invariant Drinfeld twists with coefficients in the algebraic closure $\bar{k}$ of $k$. When $k$ is large enough, we obtain an exact sequence of the form

$$1 \to H^1(k, \mathbb{Z}(G)) \to H^2_B(G/k) \to H^2_G(\bar{k}) \to 1,$$

where $H^1(k, \mathbb{Z}(G))$ is the first Galois cohomology of $k$ with coefficients in the centre of $G$. This reduces the computation of $H^2_B(G/k)$ over an arbitrary field to a computation over its algebraic closure.

The paper is organized as follows. In Section 1 we recall the definition of Drinfeld twists and we define what we call the twist cohomology of a Hopf algebra $H$; when $H$ is finite-dimensional we show that the twist cohomology of $H$ is isomorphic to the lazy cohomology of the dual Hopf algebra.
In Section 2 we apply the content of the previous section to the case when $H$ is the algebra of a finite group $G$. We prove that $\text{Int}_k(G)/\text{Inn}(G)$ embeds into $H^2_G(G)$, from which we deduce that $H^2_G(G)$ is not always abelian.

In Section 3 we deal with the case where $G$ is a finite abelian group and compute $H^2_G(G)$ in terms of alternating bilinear forms. We also show that any abelian group with a non-degenerate alternating bilinear form is of the form $A \times A$.

In Section 4 we present our main results involving the map $\Theta$. We give its main properties and identify $B(G)$ with a colimit.

In Section 5 we exploit the semi-simplicity of $k[G]$ to deduce the above-mentioned condition that ensures that $H^2_G(G)$ is abelian.

In Section 6 we remark that the lazy cohomology of $H^2_G(G)$ can be computed in terms of algebraic groups. We determine the corresponding Lie algebras and establish the exact sequence above.

In Section 7 we use our main theorem to compute $H^2_G(G)$ for a number of specific groups.

**Convention.** Throughout the paper, we fix a field $k$ over which all our constructions are defined. In particular, all linear maps are supposed to be $k$-linear and unadorned tensor products mean tensor products over $k$. For simplicity, we assume that the ground field $k$ is of characteristic zero. The reader can check that all our results hold, except possibly those of Section 6, under the more general hypothesis that the characteristic of $k$ does not divide the orders of the groups we consider.

For any algebra $A$ we denote the group of invertible elements of $A$ by $A^\times$.

1. **Invariant Drinfeld twists**

In this section we first recall what Drinfeld twists on a Hopf algebra are. Next we use the Drinfeld twists that are invariant to attach to each Hopf algebra $H$ two “cohomology groups” $H^1_{\text{tw}}(H)$ and $H^2_{\text{tw}}(H)$ in such a way that, when $H$ is finite-dimensional, these groups are isomorphic to the lazy cohomology groups of the dual Hopf algebra.

1.1. **Twists.** Let $H$ be a Hopf algebra. A Drinfeld twist on $H$, or simply a twist on $H$, is an invertible element $F$ of $H \otimes H$ satisfying the condition

\[(F \otimes 1) (\Delta \otimes \text{id}_H)(F) = (1 \otimes F) (\text{id}_H \otimes \Delta)(F),\]

where $\Delta : H \to H \otimes H$ denotes systematically the coproduct of $H$. Twists were first defined by Drinfeld [11].

The main feature of a twist $F$ lies in the fact that it endows $H$ with a new Hopf algebra structure: this new Hopf algebra, which we denote by $H^F$, is equal to $H$ as an algebra, has the same counit as $H$, but has a new coproduct $\Delta^F$ given for all $a \in H$ by

\[\Delta^F(a) = F \Delta(a) F^{-1}.\]

The group $H^\times$ of invertible elements of $H$ acts on the set of twists by

\[a \cdot F = (a \otimes a) F \Delta(a^{-1}),\]

where $a \in H^\times$ and $F$ is a twist. Using standard terminology, we say that twists belonging to the same orbit under $H^\times$ are gauge equivalent. Gauge equivalent twists $F, F'$ give rise to isomorphic Hopf algebras $H^F$ and $H^{F'}$; more precisely, if $F' = a \cdot F$, then the map $x \mapsto axa^{-1}$ induces an isomorphism $H^F \to H^{F'}$ of Hopf algebras. For details, see [11, 23, 14, 1].
In general the product in \((H \otimes H)^\times\) of two twists is not a twist. To turn the set of twists into a group, we restrict to a special class of twists. A twist \(F\) is called invariant if
\begin{equation}
\Delta(a) F = F \Delta(a)
\end{equation}
for all \(a \in H\). We will show below that invariant twists form a group. Let us observe beforehand that if \(\varepsilon\) is an invariant twist, then \(\Delta^F = \Delta\), where \(\Delta^F\) is the twisted coproduct defined by (1.2). Hence \(H^F = H\) for such a twist (nevertheless an invariant twist may change the possibly existing quasi-triangular structure of \(H\), as we shall see in Section 4.1). Note also that if \(a \in H\) is a group-like element, i.e., such that \(\Delta(a) = a \otimes a\), and if \(F\) is an invariant twist, then \(a \cdot F = F\).

In the literature twists are sometimes assumed to satisfy the additional condition
\begin{equation}
(\varepsilon \otimes \text{id})(F) = 1 = (\text{id} \otimes \varepsilon)(F),
\end{equation}
where \(\varepsilon : H \to k\) is the counit of \(H\). We say that a twist is normalized if Equations (1.5) hold.

**Lemma 1.1** – A twist \(F\) is normalized if and only if it satisfies the condition
\begin{equation}
(\varepsilon \otimes \varepsilon)(F) = 1.
\end{equation}

**Proof.** Let \(a = (\varepsilon \otimes \text{id})(F)\) and \(b = (\text{id} \otimes \varepsilon)(F)\); these are invertible elements of \(H\). Applying \(\text{id} \otimes \varepsilon \otimes \text{id}\) to both sides of (1.1), we obtain \((b \otimes 1) F = (1 \otimes a) F\). Multiplying by \(F^{-1}\) and applying \(\varepsilon \otimes \text{id}\), we obtain \(a = \varepsilon(b) 1\). Similarly, applying \(\text{id} \otimes \varepsilon\) yields \(b = \varepsilon(a) 1\). It follows that \(a, b\) are equal scalar multiples of the unit 1 and we have
\[a = b = (\varepsilon \otimes \varepsilon)(F)\.
\]
From this the desired equivalence can be deduced easily. \(\square\)

1.2. **Twist cohomology.** We now show how to construct two cohomology groups out of invariant twists. To this end, we consider two groups \(A^1(H)\) and \(A^2(H)\) defined as follows: \(A^1(H)\) is the centre of \(H^\times\) and \(A^2(H)\) consists of all invertible elements of \(H \otimes H\) satisfying (1.4).

For \(a \in A^1(H)\), set
\begin{equation}
\delta^1(a) = (a \otimes a) \Delta(a^{-1}) \in (H \otimes H)^\times.
\end{equation}

For \(F \in A^2(H)\), set
\begin{equation}
\delta^2_L(F) = (F \otimes 1) (\Delta \otimes \text{id})(F) \in (H \otimes H \otimes H)^\times
\end{equation}
and
\begin{equation}
\delta^2_R(F) = (1 \otimes F) (\text{id} \otimes \Delta)(F) \in (H \otimes H \otimes H)^\times.
\end{equation}

It follows from the definitions that the equalizer
\[
Z^2(H) = \{F \in A^2(H) \mid \delta^2_L(F) = \delta^2_R(F)\}
\]
is the set of invariant twists on \(H\).

Let \(A^3(H)\) be the group of invertible elements \(\omega \in H \otimes H \otimes H\) satisfying
\begin{equation}
\Delta^\omega(a) \omega = \omega \Delta^\omega(a)
\end{equation}
for all \(a \in H\). Here \(\Delta^\omega = (\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta\) stands for the iterated coproduct.

**Lemma 1.2** – (a) The map \(\delta^1 : A^1(H) \to (H \otimes H)^\times\) is a group homomorphism whose image \(\text{Im}(\delta^1)\) is a central subgroup of \(A^2(H)\).

(b) The maps \(\delta^2_L, \delta^2_R : A^2(H) \to (H \otimes H \otimes H)^\times\) are group homomorphisms whose images are subgroups of \(A^3(H)\).

(c) The group \(Z^2(H)\) is a subgroup of \(A^2(H)\) containing \(\text{Im}(\delta^1)\) as a central subgroup.
Proof. This is straightforward and left to the reader. \hfill \Box

In view of Lemma 1.2, we may define the following groups.

**Definition 1.3** – The twist cohomology groups of a Hopf algebra $H$ are given by
\begin{align}
H^1_{tw}(H) &= \text{Ker}(\delta^1 : A^1(H) \to A^2(H)) \\
H^2_{tw}(H) &= Z^2(H)/\text{Im}(\delta^1).
\end{align}

A priori there is no reason why $H^2_{tw}(H)$ should be abelian, and indeed in Section 2.2 we shall prove the following.

**Proposition 1.4** – There are finite-dimensional Hopf algebras $H$ for which the group $H^2_{tw}(H)$ is not abelian.

By contrast, $H^1_{tw}(H)$ is abelian since $A^1(H)$ is. The following result is an immediate consequence of the definitions.

**Proposition 1.5** – The group $H^1_{tw}(H)$ is the multiplicative group of central group-like elements of $H$.

**Remark 1.6** – Twist cohomology can also be defined using normalized invariant twists. Indeed, first observe that the normalized invariant twists form a subgroup $\bar{Z}^2(H)$ of $Z^2(H)$. In view of Lemma 1.1, we have the group decomposition
\[ Z^2(H) = k^\times (1 \otimes 1) \times \bar{Z}^2(H). \]

On the other hand, we can decompose $A^1(H)$ as
\[ A^1(H) = k^\times 1 \times \bar{A}^1(H), \]
where $\bar{A}^1(H)$ consists of the elements $a \in A^1(H)$ such that $\varepsilon(a) = 1$. It is easy to check that the homomorphism sends $\bar{A}^1(H)$ into $\bar{Z}^2(H)$ and maps the trivial factor $k^\times 1$ isomorphically onto $k^\times (1 \otimes 1)$. Therefore, $H^1_{tw}(H)$ is equal to the kernel of $\delta^1 : \bar{A}^1(H) \to \bar{Z}^2(H)$ and
\[ H^2_{tw}(H) \cong \bar{Z}^2(H)/\delta^1(\bar{A}^1(H)). \]

### 1.3. Gauge equivalence for invariant twists

Recall from Section 1.1 that the group $H^\times$ acts on the set of twists by (1.3) and that twists belonging to the same orbit are called gauge equivalent.

We may wonder which invertible elements of $H$ preserve the set of invariant twists. To answer this question, we introduce the group $N(H)$ of elements $a \in H^\times$ such that
\begin{equation}
\Delta(axa^{-1}) = (a \otimes a) \Delta(x) (a^{-1} \otimes a^{-1})
\end{equation}
for all $x \in H$. For each $a \in H$ consider the algebra automorphism $\text{ad}(a)$ of $H$ sending $x$ to $axa^{-1}$. It is easy to check that an element $a \in H^\times$ satisfies Condition (1.13) if and only if $\text{ad}(a)$ is an automorphism of Hopf algebras.

It is easily verified that the group $N(H)$ contains the centre $A^1(H)$ of $H^\times$, the group $G(H)$ of group-like elements of $H$, and their product $A^1(H)G(H)$ as normal subgroups.

**Proposition 1.7** – (a) For any invariant twist $F$, the twist $a \cdot F$ is invariant if and only if $a \in N(H)$.

(b) Let $F$ be an invariant twist. If $F' = a \cdot F$ for some $a \in N(H)$, then $F$ and $F'$ differ by an element of $\text{Im}(\delta^1) = \delta^1(A^1(H))$ if and only if $a \in A^1(H)G(H)$.

(c) The map $a \mapsto a \cdot (1 \otimes 1)$ from $N(H)$ to $Z^2(H)$ induces an injective group homomorphism
\[ N(H)/A^1(H)G(H) \hookrightarrow H^2_{tw}(H). \]
Proof. (a) By definition, the twist $a \cdot F$ is invariant if and only if
\[
\Delta(x)(a \otimes a) F \Delta(a^{-1}) = (a \otimes a) F \Delta(a^{-1}) \Delta(x)
\]
for all $x \in H$. In view of the invariance and the invertibility of $F$, this is equivalent to
\[
\Delta(x)(a \otimes a) \Delta(a^{-1}) = (a \otimes a) \Delta(a^{-1}) \Delta(x),
\]
which in turn is equivalent to Condition (1.13), in which $a$ has been replaced by $a^{-1}$; equivalently, $a^{-1}$, hence $a$, belongs to $N(H)$.

(b) There is $z \in A^1(H)$ such that $F' = \delta^1(z) F$ if and only if
\[
(a \otimes a) \Delta(a^{-1}) = (z \otimes z) \Delta(z^{-1}),
\]
which is equivalent to $\Delta(z^{-1}a) = z^{-1}a \otimes z^{-1}a$, hence to $z^{-1}a \in G(H)$. The conclusion follows.

(c) In view of (a) and (b) it is enough to check that the map $a \mapsto a \cdot (1 \otimes 1)$ preserves products. We note the following equality for any $a, b \in N(H)$ and any invariant twist $F$:
\[
(a \cdot (1 \otimes 1))(b \cdot F) = (ab) \cdot F.
\]
This equality follows from an easy computation using the invariance of $b \cdot F$. To conclude, apply (1.14) to $F = 1 \otimes 1$. \hfill \Box

Let us consider the following subgroups of the group of Hopf algebra automorphisms of $H$:
\[
\text{Int}(H) = \text{ad}(N(H)) \quad \text{and} \quad \text{Inn}(H) = \text{ad}(G(H)).
\]
The group $\text{Inn}(H)$ is a normal subgroup of $\text{Int}(H)$. Since the kernel of $\text{ad}$ is $A^1(H)$, it follows that $\text{ad}(a)$, where $a \in N(H)$, belongs to $\text{Inn}(H)$ if and only if $a$ belongs to $A^1(H) G(H)$. Therefore, we have a natural isomorphism of groups
\[
N(H)/A^1(H) G(H) \cong \text{Int}(H)/\text{Inn}(H).
\]
In view of this isomorphism, Proposition 1.7 can be summarized as follows.

**Corollary 1.8.** (a) The group $\text{Int}(H)/\text{Inn}(H)$ acts freely on $H^2_{\text{tw}}(H)$. Two elements of $H^2_{\text{tw}}(H)$ belong to the same orbit under this action if and only if they can be represented by gauge equivalent invariant twists.

(b) Gauge equivalence and equivalence modulo $\text{Inn}(\delta^1)$ define the same relation on invariant twists if and only if $\text{Int}(H)/\text{Inn}(H) = 1$.

(c) The group $H^2_{\text{tw}}(H)$ has a subgroup isomorphic to $\text{Int}(H)/\text{Inn}(H)$.

### 1.4. Lazy cohomology.

The definition of twist cohomology is motivated by lazy cohomology. Indeed, it is well known that the twists on a finite-dimensional Hopf algebra are in bijection with the two-cocycles on the dual Hopf algebra. As we will see in Section 1.5, invariant twists correspond to so-called lazy two-cocycles. Using the latter, Schauenburg [27] defined the lazy cohomology of a Hopf algebra.

Let us first recall the definition of lazy cohomology; we follow [3] On the dual vector space $H^* = \text{Hom}(H, k)$ there is an associative product, called the convolution product. It is defined for $\lambda, \mu \in \text{Hom}(H, k)$ by
\[
(\lambda \mu)(x) = \sum (x) \lambda(x_1) \mu(x_2)
\]
for all $x \in H$. Here and in the sequel we make use of the Heyneman-Sweedler sigma-notation for the image
\[
\Delta(x) = \sum (x) x_1 \otimes x_2
\]
of an element $x \in H$ under the coproduct.
Under the convolution product, \( \text{Hom}(H, k) \) becomes a monoid whose unit is the counit \( \varepsilon \) of \( H \). The group of invertible elements of this monoid is denoted by \( \text{Reg}(H) \).

An element \( \lambda \in \text{Reg}(H) \) is called lazy if
\[
\sum_{(x)} \lambda(x_1) x_2 = \sum_{(x)} \lambda(x_2) x_1
\]
for all \( x \in H \). The set \( \text{Reg}^1(H) \) of lazy elements of \( \text{Reg}(H) \) is a central subgroup of the latter.

Since \( H \otimes H \) is a Hopf algebra, we may consider the group \( \text{Reg}(H \otimes H) \) of invertible elements \( \sigma \in \text{Hom}(H \otimes H, k) \). We define \( \text{Reg}^2(H) \) as the subgroup of \( \text{Reg}(H \otimes H) \) consisting of all those \( \sigma \in \text{Reg}(H \otimes H) \) such that
\[
\sum_{(x), (y)} \sigma(x_1 \otimes y_1) x_2 y_2 = \sum_{(x), (y)} \sigma(x_2 \otimes y_2) x_1 y_1 \in H
\]
for all \( x, y \in H \). The elements of \( \text{Reg}^2(H) \) are called lazy.

A lazy left 2-cocycle of \( H \) is an element \( \sigma \in \text{Reg}^2(H) \) satisfying the equations
\[
\sum_{(x), (y)} \sigma(x_1 \otimes y_1) \sigma(x_2 y_2 \otimes z) = \sum_{(y), (z)} \sigma(y_1 \otimes z_1) \sigma(x \otimes y_2 z_2)
\]
for all \( x, y, z \in H \). We denote by \( Z^2(H) \) the set of lazy 2-cocycles of \( H \). Chen [6] was the first to observe that this set is a group under the convolution product.

For \( \lambda \in \text{Reg}(H) \), define \( \partial(\lambda) \in \text{Reg}(H \otimes H) \) for all \( x, y \in H \) by
\[
\partial(\lambda)(x \otimes y) = \sum_{(x), (y)} \lambda(x_1) \lambda(y_1) \lambda^{-1}(x_2 y_2),
\]
where \( \lambda^{-1} \) is the convolution inverse of \( \lambda \). When restricted to \( \text{Reg}^1(H) \), the map \( \lambda \mapsto \partial(\lambda) \) becomes a group homomorphism
\[
\partial : \text{Reg}^1(H) \to \text{Reg}^2(H),
\]
whose image \( B^2(H) \) is a central subgroup of \( Z^2(H) \).

**Definition 1.9** – The lazy cohomology groups \( H^1_l(H) \) and \( H^2_l(H) \) are given by
\[
H^1_l(H) = \text{Ker}(\partial : \text{Reg}^1(H) \to \text{Reg}^2(H))
\]
and
\[
H^2_l(H) = Z^2_l(H)/B^2_l(H).
\]

The group \( H^1_l(H) \) is abelian since \( \text{Reg}^1_l(H) \) is. The group \( H^2_l(H) \) classifies the bicleft biGalois objects of \( H \) up to isomorphism, as was shown in [3, Th. 3.8].

When the Hopf algebra \( H \) is cocommutative, then the lazy cohomology groups \( H^i_l(H) \) (\( i = 1, 2 \)) coincide with the Hopf algebra cohomology groups \( H^i(H, k) \) constructed by Sweedler in [29]. In the special case where \( H = k[G] \) is the algebra of a group \( G \), it follows from [29, Th. 3.1] that for \( i = 1, 2 \),
\[
H^i_l(k[G]) \cong H^i(G, k^\times),
\]
where \( H^i(G, k^\times) \) is the cohomology of the group \( G \) acting trivially on \( k^\times \). Note that when the group \( k^\times \) is divisible (e.g., if \( k \) is algebraically closed), then
\[
H^2_l(k[G]) \cong H^2(G, k^\times) \cong \text{Hom}(H_2(G, Z), k^\times),
\]
where \( H_2(G, Z) \) is the second integral homology group of \( G \).
1.5. Relating twist cohomology to lazy cohomology. We have already observed that for any Hopf algebra $H$ the dual vector space $H^*$ carries an algebra structure. If in addition $H$ is finite-dimensional, then $H^*$ is a Hopf algebra with coproduct given by
\[ \Delta(\lambda)(x \otimes y) = \lambda(xy) \]
for $\lambda \in H^*$ and $x, y \in H$. (Here we identify Hom$(H \otimes H, k)$ with $H^* \otimes H^*$.) The counit of $H^*$ is given by $\varepsilon(\lambda) = \lambda(1)$.

The following result relates twist cohomology to lazy cohomology.

**Theorem 1.10** – For each finite-dimensional Hopf algebra $H$ we have
\[ H^2_{tw}(H) \cong H^1_{tw}(H^*) \quad \text{and} \quad H^2_{tw}(H) \cong H^2_{tw}(H^*). \]

As a consequence of the theorem and of Proposition 1.4, we obtain the following.

**Corollary 1.11** – There are finite-dimensional Hopf algebras $H$ for which the group $H^2_{tw}(H)$ is not abelian.

**Proof of Theorem 1.10.** Identify an element $a \in H$ with $\lambda_a : H^* \to k$ given by $\lambda_a(\alpha) = \alpha(a)$ for all $\alpha \in H^*$. Similarly identify an element $F \in H \otimes H$ with $\sigma_F : H^* \otimes H^* \to k$ given by $\sigma_F(\alpha \otimes \beta) = (\alpha \otimes \beta)(F)$ for all $\alpha, \beta \in H^*$. It is easy to check that $a \in H$ is invertible if and only if $\lambda_a \in \text{Reg}(H^*)$ and that $F \in H \otimes H$ is invertible if and only if $\sigma_F \in \text{Reg}(H^* \otimes H^*)$. It is also well known that $F \in H \otimes H$ is a twist if and only if $\sigma_F$ satisfies (1.19). We conclude the proof with the help of the following lemma whose proof is straightforward.

**Lemma 1.12** – (a) Under the previous identifications, the map
\[ \partial : \text{Reg}(H^*) \to \text{Reg}(H^* \otimes H^*) \]
of (1.20) coincides with the map $\delta^1 : H^x \to (H \otimes H)^x$.
(b) The linear map $\lambda_a \in \text{Reg}(H^*)$ is lazy if and only if $a \in H^x$ is central.
(c) The linear map $\sigma_F \in \text{Reg}(H^* \otimes H^*)$ is lazy if and only if $F$ is invariant.

**Remark 1.13** – Bichon and Carnovale [3, Sect. 1] associated to each Hopf algebra $H$ certain groups $\text{CoInn}(H)$ and $\text{CoInt}(H)$ of Hopf algebra automorphisms of $H$. If $H$ is finite-dimensional and $H^*$ is the dual Hopf algebra, then $\text{CoInt}(H) \cong \text{Int}(H^*)$ and $\text{CoInn}(H) \cong \text{Inn}(H^*)$ (the groups of Hopf algebra automorphisms of $H$ and $H^*$ are naturally isomorphic). There is a version of Corollary 1.8 in this context; see [3, Lemmas 1.11–1.13]. In [3, Ex. 7.5] a 32-dimensional (non-commutative, non-cocommutative) Hopf algebra with non-trivial $\text{CoInt}(H)/\text{CoInn}(H)$ was constructed.

## 2. Hopf Algebras of Groups

In this section we apply the content of Section 1 to the main case of interest to us, namely when $H$ is the algebra of a group.

### 2.1. The Hopf algebra $k[G]$.
Let $G$ be a group and $k[G]$ the corresponding group algebra over the ground field $k$. It is well known that $H = k[G]$ carries the structure of a Hopf algebra with coproduct $\Delta$, counit $\varepsilon$, and antipode $S$ given by
\[ \Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1} \]
for all $g \in G$. The only group-like elements of $H$ are the elements of $G$.

In this case $A^3(H)$ is equal to the centre of $k[G]^x$ and $A^2(H)$ is the group of $G$-invariant elements of the algebra $(k[G] \otimes k[G])^G$ of elements of $k[G] \otimes k[G]$ fixed under the diagonal conjugation action, i.e., $A^2(H)$ consists of the invertible $G$-invariant elements of $k[G] \otimes k[G]$. Similarly, $A^3(H)$ is the group of invertible $G$-invariant elements of $k[G] \otimes k[G] \otimes k[G]$. 
Notation 2.1 – We shall henceforth denote $H_{tw}^i(k[G])$ $(i = 1, 2)$ of Definition 1.3 by $H_i^G(G)$, or by $H_i^G(G/k)$ if we need to stress the ground field $k$, as will be the case in Section 6.

In view of Proposition 1.5, $H_1^G(G)$ coincides with the centre of $G$; in particular, it is independent of the ground field $k$. So the main question is to determine the group $H_2^G(G)$.

Example 2.2 – Let us consider a situation where $H$ is a group $H$.

We now claim that if $k[G]$ has trivial units, then

$$H_2^G(G) = 1.$$ Indeed, since $k[G] \otimes k[G] = k[G \times G]$ has trivial units, any element $F \in A^2(k[G])$ is of the form $F = \lambda g \otimes h$, where $\lambda \in k^\times$ and $g, h \in G$. One checks that

$$\delta_2^G(F) = \lambda^2 g^2 \otimes hg \otimes h \quad \text{and} \quad \delta_2^G(F) = \lambda^2 g \otimes gh \otimes h^2.$$ It follows that any element of $\mathbb{Z}^2(k[G])$ is of the form $\lambda 1 \otimes 1$, which belongs to the image of $\delta^3$ since it is equal to $\delta^3(\lambda 1)$.

Observe that, if $k[G]$ has trivial units, then $G$ is necessarily torsion free, hence infinite.

If $G$ is a finite group, then the Hopf algebra $k[G]$ is finite-dimensional and we may consider its dual Hopf algebra. The latter is the algebra $O_k(G)$ of $k$-valued functions on $G$. For each $g \in G$ let $e_g$ be the characteristic function of the singleton $\{g\}$. The set $(e_g)_{g \in G}$ is a basis of $O_k(G)$, dual to the basis $(g)_{g \in G}$ of $k[G]$; it is also an orthogonal family of primitive idempotents. The coproduct $\Delta$, counit $\varepsilon$, and antipode $S$ of $O_k(G)$ are given by

$$\Delta(e_g) = \sum_{h \in G} e_h \otimes e_{h^{-1}g}, \quad \varepsilon(e_g) = \delta_{g, 1}, \quad S(e_g) = e_{g^{-1}}$$ for all $g \in G$.

By Theorem 1.10 we have

$$H_i^G(G) = H_i^{tw}(k[G]) \cong H_i^G(O_k(G)) \quad (i = 1, 2).$$

The constructions of Section 1.2 applied to the group algebra $k[G]$ thus give a direct definition of the lazy cohomology of the function Hopf algebra $O_k(G)$; we shall use this direct definition to compute $H_2^G(O_k(G))$ in Section 4.

2.2. Automorphism groups. We now apply the content of Section 1.3 to the case $H = k[G]$. If $a \in N(k[G])$, then for all $g \in G$,

$$\Delta(aga^{-1}) = (a \otimes a) \Delta(g)(a^{-1} \otimes a^{-1}) = (a \otimes a)(g \otimes g)(a^{-1} \otimes a^{-1}) = (aga^{-1} \otimes aga^{-1}).$$

Hence, $aga^{-1}$ belongs to $G$ for all $g$. In other words, $a$ belongs to the normalizer $N_k(G)$ of $G$ in $k[G]$. The converse holds and we have

$$N(k[G]) = N_k(G).$$
Since \( G(k[G]) = G \), the group \( \operatorname{Inn}(k[G]) \) coincides with the group \( \operatorname{Inn}(G) \) of inner automorphisms of \( G \). We set \( \operatorname{Int}_k(G) = \operatorname{Int}(k[G]) \); this is the group of automorphisms of \( G \) induced by conjugation by invertible elements of \( k[G] \). As a special case of (1.15), we have the isomorphism
\[
(2.4) \quad N_k(G)/A^1(k[G]) G \cong \operatorname{Int}_k(G)/\operatorname{Inn}(G).
\]

In the present context Corollary 1.8 takes the following form.

**Corollary 2.3** – (a) The group \( \operatorname{Int}_k(G)/\operatorname{Inn}(G) \) acts freely on \( H^2_k(G) \). Two elements of \( H^2_k(G) \) belong to the same orbit under this action if and only if they can be represented by gauge equivalent invariant twists.

(b) Gauge equivalence and equivalence modulo \( \operatorname{Im}(\delta^1) \) define the same relation on invariant twists if and only if \( \operatorname{Int}_k(G)/\operatorname{Inn}(G) = 1 \).

(c) The group \( H^2_k(G) \) has a subgroup isomorphic to \( \operatorname{Int}_k(G)/\operatorname{Inn}(G) \).

In Section 7.1 we shall check that \( \operatorname{Int}_k(G)/\operatorname{Inn}(G) = 1 \) for a number of classical groups. Actually, finding finite groups with non-trivial \( \operatorname{Int}_k(G)/\operatorname{Inn}(G) \) is not so straightforward. The easiest way to obtain such a group is via the group \( \operatorname{Aut}_c(G) \) of class-preserving automorphisms of \( G \), i.e., the ones that preserve each conjugacy class. Let us explain how \( \operatorname{Int}_k(G) \) and \( \operatorname{Aut}_c(G) \) are related when \( G \) is a finite group. First, observe that if \( k \subseteq \bar{k} \) is an extension of fields, then \( \operatorname{Int}_{\bar{k}}(G) \) is a subgroup of \( \operatorname{Int}_k(G) \). Now, by [18, Prop. 2.5], if \( k \) is big enough (for instance, if it is algebraically closed) and its characteristic is prime to the order of \( G \), then
\[
(2.5) \quad \operatorname{Int}_{\bar{k}}(G) = \operatorname{Aut}_c(G).
\]
The inclusion \( \operatorname{Int}_k(G) \subset \operatorname{Aut}_c(G) \) follows immediately for any field \( k \) whose characteristic is prime to the order of \( G \). This inclusion can also be deduced from the following simple argument: by definition, for any \( \varphi \in \operatorname{Int}_k(G) \) and any \( g \in G \), the image \( \rho(\varphi(g)) \) under a representation \( \rho \) of the group is conjugate to \( \rho(g) \); hence, \( \chi(\varphi(g)) = \chi(g) \) for any character \( \chi \) of \( G \) and any \( g \in G \); it follows that \( \varphi \) preserves each conjugacy class.

Burnside [5] was the first to exhibit examples of finite groups with non-trivial \( \operatorname{Aut}_c(G)/\operatorname{Inn}(G) \), hence such that \( \operatorname{Int}_k(G)/\operatorname{Inn}(G) \neq 1 \) for large enough \( k \). His smallest example was of order 729; Wall [32] much later found one of order 32 (we shall discuss this example in Section 7.6). Note that the triviality of \( \operatorname{Int}_k(G)/\operatorname{Inn}(G) \) is related to the isomorphism problem for group algebras (see [21, 25]).

We are now ready to prove Proposition 1.4.

**Proof of Proposition 1.4.** It is enough to find a finite group such that \( H^2_k(G) \) is non-abelian. Now, C.-H. Sah [26] constructed for each prime \( p \) and each integer \( m \geq 3 \) a \( p \)-group \( G \) of order \( p^{5m} \) for which \( \operatorname{Aut}_c(G)/\operatorname{Inn}(G) \) is non-abelian. It follows from (2.5) and from Corollary 2.3 (c) that for such a group \( G \) (and an appropriate ground field \( k \)) the group \( H^2_k(G) \) is non-abelian.

Note that the smallest example exhibited by Sah is of order \( 2^{15} \), that is roughly a thousand times bigger than the smallest group for which \( \operatorname{Aut}_c(G)/\operatorname{Inn}(G) \neq 1 \).

3. THE ABELIAN CASE

In this section we further assume that the ground field \( k \) is algebraically closed. Under this assumption we describe \( H^2_k(A) \) for any finite abelian group \( A \). It is worth indulging in the details for we shall see later that, in the case of a general finite group, its abelian subgroups control the invariant twists.

All twists for \( A \) are obviously invariant, so what we describe is simply the group of twists modulo gauge equivalence. Observe also that \( \operatorname{Int}_k(A) = \operatorname{Inn}(A) = \{ \text{id} \} \).
in this case; it follows from Corollary 2.3 that gauge equivalence coincides with equivalence modulo \( \text{Im}(\delta^1) \).

### 3.1. General picture

Let \( A \) be a finite abelian group and \( \hat{A} = \text{Hom}(A, k^\times) \) be its group of characters. Recall that the discrete Fourier transform is the map

\[
    k[A] \to \mathcal{O}_k(\hat{A}) : g \mapsto \hat{g}
\]

defined for all \( g \in A \) and \( \chi \in \hat{A} \) by \( \hat{g}(\chi) = \chi(g^{-1}) \). It is easy to check that it is a Hopf algebra isomorphism. We now compute \( H^2(\hat{A}) \).

**Proposition 3.1** — We have \( H^2_t(A) \cong H^2(\hat{A}, k^\times) \).

**Proof.** We have the sequence of isomorphisms

\[
    H^2(\hat{A}) \cong H^2(\mathcal{O}_k(A)) \cong H^2(k[\hat{A}]) \cong H^2(\hat{A}, k^\times).
\]

The first isomorphism follows from (2.2), the second one from the discrete Fourier transform, and the last one from (1.21). \( \square \)

### 3.2. Explicit formulas

The easily-remembered isomorphism of Proposition 3.1 can be made explicit. For the proof we used an isomorphism between \( A \) and \( \hat{A} \). Following [23, Prop. 3], we exhibit a direct isomorphism between \( H^2_t(A) \) and \( H^2(\hat{A}, k^\times) \) as follows.

For \( \rho \in \hat{A} \), let \( e_\rho \) be the corresponding idempotent in \( k[A] \). To a \( k^\times \)-valued two-cocycle \( c \) on \( \hat{A} \) we associate the element

\[
    F = \sum_{\rho, \sigma} c(\rho, \sigma) e_\rho \otimes e_\sigma \in k[A] \otimes k[A].
\]

Then \( F \) is a twist for \( k[A] \). Conversely, given a twist \( F \), define \( c \) by

\[
    c(\rho, \sigma) = (\rho \otimes \sigma)(F).
\]

Then \( c \) is a \( k^\times \)-valued two-cocycle on \( \hat{A} \).

It is easily checked that gauge equivalence of twists corresponds to cohomological equivalence of two-cocycles. Moreover, the group of two-cocycles modulo coboundaries is equal to \( H^2(\hat{A}, k^\times) \). Thus we have the desired explicit isomorphism.

Next, we note that since \( k^\times \) is divisible, the universal coefficient theorem and a well-known computation of the second cohomology of an abelian group imply that

\[
    H^2(\hat{A}, k^\times) \cong \text{Hom}(H_2(\hat{A}, \mathbb{Z}), k^\times) \cong \text{Hom}(\Lambda^2_{\mathbb{Z}} \hat{A}, k^\times).
\]

As a result, with our assumption on \( k \), the second cohomology group turns out to be the group of \( k^\times \)-valued alternating bilinear forms on \( \hat{A} \). This can be made explicit as follows: the alternating form corresponding to a two-cocycle \( c \) is

\[
    b(\rho, \sigma) = \frac{c(\sigma, \rho)}{c(\rho, \sigma)},
\]

where \( \rho, \sigma \in \hat{A} \). Thus \( b \) measures the symmetry default of \( c \).

Just as \( c \) defines a twist \( F \in k[A] \otimes k[A] \), the bilinear form \( b \) allows us to introduce

\[
    R(A, b) = \sum_{\sigma, \tau \in \hat{A}} b(\sigma, \tau) e_\sigma \otimes e_\tau.
\]

The element \( R(A, b) \in k[A] \otimes k[A] \) is invertible and we have \( R(A, b) = F_{21} F^{-1} \).

The two-tensor \( R(A, b) \) determines \( F \) up to gauge equivalence as follows.

**Proposition 3.2** — If \( F, F' \) are twists for \( k[A] \) such that \( F'_{21} F'^{-1} = F_{21} F^{-1} \), then \( F \) and \( F' \) are gauge equivalent.

In Section 4.2 we shall give an analogous result for an arbitrary group.
Proof. The result can be deduced from the proof of [17, Lemma 5.7]; here is a direct argument. Consider the twist $f = F^{-1}F'$. It is symmetric in the sense that $f_{21} = f$. The cocycle corresponding to $f$ has an associated alternating form $b_0$ such that $R(A, b_0) = f_{21}f^{-1}$. Since $f$ is symmetric, we have $R(A, b_0) = 1 \otimes 1$, which implies that $b_0(\sigma, \tau) = 1$ for all $\sigma, \tau \in \hat{A}$. Therefore, under the isomorphism of Proposition 3.1, the twist $f$ represents the trivial element of $H^2_\Omega(A)$. Consequently, $F$ and $F'$ represent the same element of $H^2_\Omega(A)$, which by Proposition 1.7 and the equality $\text{Int}_k(A) = \text{Im}(A)$ yields the conclusion.

We now end these computational remarks with a comment on the inverse bijection, taking a $k^\times$-valued alternating bilinear form $b$ on $\hat{A}$ to the class of a two-cocycle $c$ modulo coboundaries. One observes that $b$ can itself be viewed as a two-cocycle on $\hat{A}$ whose corresponding bilinear form is $\hat{b}$, so, in order to find $c$, we look for a “square root” of $b$. This is easily achieved when $A$, hence $\hat{A}$, has odd order: in this case $\rho \mapsto \rho/2$ is an automorphism of $\hat{A}$ and we may set

$$c(\rho, \sigma) = b(\rho/2, \sigma) = b(\rho, \sigma/2) = b(\rho, \sigma)^{1/2}.$$ 

for $\rho, \sigma \in \hat{A}$. The square root in the previous formula is uniquely defined since $b$ takes its values in a group of roots of unity of odd order. It is trivial to check that the bilinear form corresponding to this cocycle $c$ is indeed $b$.

3.3. Non-degenerate bilinear forms & cocycles. Let $b$ be a $k^\times$-valued alternating bilinear form on $\hat{A}$. By (3.1) it defines an element $R(A, b) \in k[A] \otimes k[A]$. We shall say that $b$ is minimal if there are no proper subgroups $B \subset A$ such that $R(A, b) \in k[B] \otimes k[B]$.

**Proposition 3.3** – The alternating bilinear form $b$ is minimal if and only if it is non-degenerate.

One way to see this is by considering the map $f : \mathcal{O}_k(A) \rightarrow k[A]$ defined by $R(A, b)$, that is by $f(x) = (x \otimes \text{id})(R(A, b))$. It is easy to see that $R(A, b)$ is minimal if and only if $f$ is surjective, and that $b$ is non-degenerate if and only if $f$ is injective. For reasons of dimension, these conditions are equivalent.

Alternatively, Proposition 3.3 can deduced from the following lemma, which we will need later anyway. It asserts that the constructions of the previous paragraph are in a sense natural in $A$.

**Lemma 3.4** – Let $B$ be any subgroup of $A$, and let $p : \hat{A} \rightarrow \hat{B}$ be the restriction map. Given an alternating bilinear form $b'$ on $\hat{B}$, define $b(\sigma, \tau) = b'(p(\sigma), p(\tau))$ on $\hat{A}$. Then $R(B, b') = R(A, b)$ in $k[A] \otimes k[A]$.

Conversely, if $b$ is any alternating bilinear form on $\hat{A}$ such that $R(A, b)$ belongs to $k[B] \otimes k[B]$, then $b(\sigma, \tau) = b'(p(\sigma), p(\tau))$ for some form $b'$ on $\hat{B}$.

**Proof.** We have

$$R(A, b) = \sum_{\sigma, \tau \in \hat{A}} b(\sigma, \tau) e_\sigma \otimes e_\tau \quad \text{and} \quad R(B, b') = \sum_{\kappa, \lambda \in \hat{B}} b'(\kappa, \lambda) e_\kappa \otimes e_\lambda.$$ 

The equality $R(B, b') = R(A, b)$ follows from

$$e_\kappa = \sum_{\sigma \in p^{-1}(\kappa)} e_\sigma.$$ 

This last equality is obtained by an elementary computation with characters. It also makes the converse clear, for it is now apparent that $R(A, b)$ lies in $k[B] \otimes k[B]$ if and only if $b$ is constant on the fibres of $p$. $\square$
It is customary to talk about non-degenerate two-cocycles when the associated bilinear form \( b \) is non-degenerate. We shall also speak of minimal two-cocycles in the same sense. It turns out that the existence of a non-degenerate two-cocycle imposes severe restrictions on \( A \), as is shown in the following proposition, which is well-known (see [19, Thm. 4.8] or [9, Lemma 5.2]).

**Proposition 3.5** – Any finite abelian group \( A \) with a non-degenerate alternating bilinear form \( b : A \times A \to k^\times \) is of symmetric type, i.e., of the form \( A \cong A' \times A' \).

4. The main theorem

In this section we present our main results. We again assume that the ground field \( k \) is algebraically closed.

The material in Sections 4.1, 4.2, 4.3 is known to the experts (see for example the unpublished manuscript [7]). We include it for convenience.

4.1. Quasi-triangular Hopf algebras. Let \( H \) be a Hopf algebra with coproduct \( \Delta \). Denote the opposite coproduct of \( H \) by \( \Delta^\text{op} \); by definition \( \Delta^\text{op}(a) = \Delta(a)_{21} \) for all \( a \in H \).

Let \( H \) be a Hopf algebra. A universal \( R \)-matrix for \( H \) is an invertible element \( R \in H \otimes H \) such that

\[
(4.1) \quad \Delta^\text{op}(a) = R \Delta(a) R^{-1}
\]

for all \( a \in H \), and

\[
(4.2) \quad (\Delta \otimes \text{id}_H)(R) = R_{13} R_{23} \quad \text{and} \quad (\text{id}_H \otimes \Delta)(R) = R_{13} R_{12}.
\]

A quasi-triangular Hopf algebra is a Hopf algebra equipped with a universal \( R \)-matrix. For more, see [20, Chap. VIII].

Given a universal \( R \)-matrix \( R \) for \( H \), it is possible to construct another one with the help of a twist on \( H \). More precisely, for any twist \( F \) on \( H \), set

\[
R_F = F_{21} R F^{-1}.
\]

Then \( R_F \) is a universal \( R \)-matrix for the Hopf algebra \( H_F \) obtained from \( H \) by twisting the coproduct following (1.2). If the twist \( F \) is invariant, then as observed in Section 1.1, we have \( \Delta^F = \Delta \) and therefore \( H_F = H \). In this case \( R_F \) is another universal \( R \)-matrix for \( H \).

If \( F, F' \) are gauge equivalent invariant twists on \( H \), then the corresponding universal \( R \)-matrices \( R_F, R_{F'} \) are conjugate. Indeed, if \( F' = a \cdot F \) for some invertible element \( a \in H \), then

\[
(4.3) \quad R_{F'} = (a \otimes a) R_F (a \otimes a)^{-1}.
\]

It follows that, if \( F \) and \( F' \) represent the same element of \( H^2_{\text{tw}}(H) \), i.e., if \( F' = a \cdot F \) for some invertible central element \( a \), then \( R_{F'} = R_F \).

In the sequel we shall also make use of Radford’s work on minimal quasi-triangular Hopf algebras: in [24] it was proved that any quasi-triangular Hopf algebra \( H \) with universal \( R \)-matrix \( R = \sum s_i \otimes t_i \) contains a unique smallest (finite-dimensional) quasi-triangular Hopf subalgebra \( H(R) \) with the same universal \( R \)-matrix \( R \) (it is the smallest Hopf subalgebra of \( H \) containing the elements \( s_i \) and \( t_i \)).

He also proved that if \( H \) is either commutative or cocommutative, then \( H(R) \) is bicommutative, i.e., both commutative and cocommutative; see [24, Prop. 13].
4.2. Quasi-triangular structures on group algebras. Consider the case where $H = k[G]$ is the algebra of a finite group $G$. Since $k[G]$ is cocommutative, we can equip it with the trivial universal $R$-matrix $1 \otimes 1$. It follows from the discussion in Section 4.1 that if $F$ is a $G$-invariant twist on $k[G]$, then
\[ R_F = F_{21} (1 \otimes 1) F^{-1} = F_{21} F^{-1} \]
is a universal $R$-matrix for $k[G]$. We have observed above that if $F$ and $F'$ represent the same element of $H^2_{tw}(H) = H^2_{tw}(G)$, then $R_{F'} = R_F$. Conversely, the universal $R$-matrix $R_F = F_{21} F^{-1}$ determines $F^{-1}$ up to gauge equivalence.

**Proposition 4.1** – When $F$ and $F'$ are invariant twists, then $R_{F'} = R_F$ if and only if $F^{-1}$ and $F'^{-1}$ are gauge equivalent.

*Proof.* We start with the “only if” part. Since $F, F'$ are invariant twists, so is $f = F^{-1} F'$. The equality $R_{F'} = R_F$ implies that $f$ is a symmetric twist, i.e., $f_{21} = f$. It follows from [14, Cor. 3] that $f$ is gauge equivalent to $1 \otimes 1$. In other words, there is an invertible element $a \in H$ such that $f = (a \otimes a) \Delta(a^{-1})$. Therefore,
\[ F^{-1} = f F'^{-1} = (a \otimes a) \Delta(a^{-1}) F'^{-1} = (a \otimes a) F'^{-1} \Delta(a^{-1}). \]

It follows that $F^{-1}$ and $F'^{-1}$ are gauge equivalent.

The converse is trivial: if $F^{-1} = a \cdot F'^{-1}$ as above, then
\[ R_F = \Delta(a) R_{F'} \Delta(a^{-1}) = R_{F'}, \]
by invariance of $R_{F'}$. \hfill \Box

For any quasi-triangular Hopf algebra $H$ with universal $R$-matrix $R = \sum s_i \otimes t_i$, the *Drinfeld element* $u_R$ is defined to be
\[ u_R = \sum S(t_i) s_i, \]
where $S$ is the antipode of $H$ (see [10, Sect. 2], [20, Sect. VIII.4]). We need the following lemma, which has been observed in the proof of [15, Prop. 3.4]. We give a proof for the sake of completeness.

**Lemma 4.2** – Let $G$ be a finite group and $F$ an invariant twist on $k[G]$. If $R_F = F_{21} F^{-1}$, then $u_{R_F} = 1$.

*Proof.* Any Drinfeld element $u_R$ associated to a universal $R$-matrix $R$ satisfies the relations
\[ S^2(a) = u_R a u_R^{-1} \quad \text{and} \quad \Delta(u_R) = (R_{21} R)^{-1} (u_R \otimes u_R) \]
for all $a \in H$; see [10, Prop. 2.1 and 3.2] or [20, Prop. VIII.4.1 and VIII.4.5]. The antipode of any cocommutative Hopf algebra being involutive, it follows from the first equation in (4.6) that $u_{R_F}$ is a central element of $k[G]$. Moreover, for $R = R_F$, we have
\[ R_{21} R = F_{21} F^{-1} F_{21} F^{-1} = 1 \otimes 1. \]
It then follows from the second equation in (4.6) that $\Delta(u_{R_F}) = u_{R_F} \otimes u_{R_F}$. In other words, $u_{R_F}$ is a central group-like element of $k[G]$, hence a central element of the group. To prove that $u_{R_F} = 1$, we apply [1, Th. 3.4] to the case where $H = k[G]$ and $J = F^{-1}$. Since $k[G]$ is unimodular (and cocommutative), this theorem states that if $\lambda \in k[G]^*$ is a non-zero integral, then so is $u_{R_F} \rightarrow \lambda = \sum \lambda \lambda_1 \lambda_2 (u_{R_F})$. Now by definition of an integral in $k[G]^*$, we have $g \lambda(g) = \lambda(g)1$ for all $g \in G$. It follows that $\lambda = g e_1$, where $e_1$ is the characteristic function of the unit in $G$ and $g$ is some scalar. We take $e_1$ as a non-zero integral in $k[G]^*$. Now
\[ (u_{R_F} \rightarrow e_1)(g) = \sum (e_1)(g) (e_1)(u_{R_F}) = e_1(g u_{R_F}) \]
for all $g \in G$, which means that $u_{R_F} \to e_1$ is the characteristic function of the singleton $\{u_{R_F}^{-1}\}$. It is then clear that the latter can be an integral in $k[G]^*$ only if $u_{R_F} = 1$. 

For any universal $R$-matrix for $k[G]$, we may consider the minimal quasi-triangular Hopf subalgebra $k[G](R)$ introduced in Section 4.1. Since $k[G]$ is commutative, $k[G](R)$ is necessarily bicommutative. Hence, $k[G](R)$ must be the group algebra of an abelian subgroup $A$ of $G$:

$$k[G](R) = k[A].$$

Now, $k[A]$ is isomorphic to the algebra $O_k(\hat{A})$ of functions on $\hat{A} = \text{Hom}(A, k^\times)$, the braidings of which have a particularly simple form. We proceed to explain this next.

### 4.3. Quasi-triangular structures on function algebras

We shall consider the Hopf algebra $H$ of functions (resp. smooth functions, resp. regular functions, . . .) on a group $G$ (resp. Lie group, resp. algebraic group, . . .). In this case a universal $R$-matrix is none other than a function $b$ on $G \times G$ and the equations (4.1) and (4.2) take a particularly simple form. Equation (4.1) does not involve $b$ since $H$ is commutative; it actually forces $H$ to be cocommutative so that the group $G$ must be abelian. Equations (4.2) become the functional equations

$$b(xy, z) = b(x, z)b(y, z) \quad \text{and} \quad b(x, yz) = b(x, y)b(x, z)$$

for all $x, y, z \in G$. Thus $b$, which is assumed to be invertible, is just a $k^\times$-valued bilinear form on the abelian group $G$. In view of the material in Section 3.2, we may ask when this form is alternating. We give a condition involving the Drinfeld element $u_R$ defined by (4.5). It follows from this definition that $u_R$ becomes the function

$$u_R(x) = b(x^{-1}, x) = b(x, x)^{-1}$$

for all $x \in G$. Therefore, $u_R = 1$ implies that $b$ is alternating, and conversely.

In the case where $A$ is an abelian group and $G = \hat{A}$, we can use the Fourier transform, as introduced in Section 3.1 and made explicit in Section 3.2: it provides an isomorphism between the Hopf algebras $O_k(G)$ and $k[A]$. A bilinear form $b$ on $G$, viewed as a universal $R$-matrix for $O_k(G)$, thus gives rise to a universal $R$-matrix for $k[A]$; the latter is none other than $R(A, b)$, as defined by (3.1).

When $b$ is alternating, we have proved in Section 3.2 that it corresponds to a two-cocycle $c$, and in turn $c$ defines a twist $F$ on $k[A]$ such that $R_F = R(A, b)$. From Lemma 4.2 we have the following.

**Lemma 4.3** – When $b$ is an alternating bilinear form on $\hat{A}$, the Drinfeld element of the universal $R$-matrix $R(A, b)$ is 1.

### 4.4. The map $\Theta$

For any finite group $G$, let $B(G)$ denote the set of pairs $(A, b)$, where $A$ is an abelian normal subgroup of $G$ and $b : \hat{A} \times \hat{A} \to k^\times$ is a $k^\times$-valued $G$-invariant non-degenerate alternating bilinear form on $\hat{A}$. The letter $B$ is meant to suggest either “bilinear” or “braiding”. We see $B(G)$ as a pointed (in particular, non-empty) set, the distinguished element being $((1), 1)$, the trivial bilinear form on the trivial subgroup. We write this element simply 1.

Let us define a map of sets

$$\Theta : H^2_k(G) \to B(G)$$

as follows. An invariant twist $F$ on $k[G]$ determines by Formula (4.4) a universal $R$-matrix $R_F = F_{21}F^{-1}$. As observed in Section 4.2, there is a unique abelian subgroup $A$ of $G$ such that

$$k[G](R_F) = k[A].$$
Moreover, \( R_F \) corresponds to a bilinear form \( b \) on \( \widehat{A} \), and \( R_F = R(A,b) \). We set \( \Theta(F) = (A,b) \) and we call \( A \) the socle of \( F \). Note that if \( A \) is the socle of \( F \), then \( R_F \) belongs to \( k[A] \otimes k[A] \); if moreover \( F \in k[A] \otimes k[A] \), then we say that \( F \) is supported by \( A \). (More on this in the next section).

**Lemma 4.4** – (a) Let \( R \) be a universal \( R \)-matrix for \( k[G] \) such that \((g \otimes g) R = R(g \otimes g) \) for all \( g \in G \). If \( A \) is the abelian subgroup of \( G \) such that \( k[G](R) = k[A] \), then \( A \) is normal.

(b) The map \( \Theta \) is well-defined.

Proof. (a) The conjugation by \( g \in G \) induces an automorphism of the Hopf algebra \( k[G] \) that sends the \( G \)-invariant tensor \( R \) to itself. Therefore it sends \( k[G](R) = k[A] \) to itself, which implies that \( A \) is a normal subgroup.

(b) We have observed above that \( R_F \), and thus \((A,b)\), depends only on the class of \( F \) in \( H^2_F(G) \). It follows from Part (a) that \( A \) is normal. By Lemma 4.2, the Drinfeld element \( u_{R_F} \) is equal to 1, and therefore the bilinear form \( b \) is alternating. It is minimal, and so also non-degenerate by Proposition 3.3. It is obviously \( G \)-invariant.

We now state our main theorem.

**Theorem 4.5** – Assume that the ground field \( k \) is algebraically closed and of characteristic zero. For any finite group \( G \) the map \( \Theta : H^2_F(G) \to B(G) \) enjoys the following properties:

(a) The fibres of \( \Theta : H^2_F(G) \to B(G) \) are the left cosets of \( \text{Int}_k(G)/\text{Inn}(G) \).

(b) The subset \( \Theta^{-1}(1) \) is a subgroup of \( H^2_F(G) \) isomorphic to \( \text{Int}_k(G)/\text{Inn}(G) \).

(c) If the order of \( G \) is odd, then any element \( (A,b) \) in \( B(G) \) is of the form \( \Theta(F) \) for an \( F \) which is supported by \( A \). In particular, \( \Theta \) is surjective in this case.

Proof. (a) Assume that \( \Theta(F) = \Theta(F') = (A,b) \). It follows that \( R_F = R_{F'} = R(A,b) \), and by Proposition 4.1, the twists \( F^{-1} \) and \( F'^{-1} \) are gauge equivalent:

\[
F^{-1} = (a \otimes a) F'^{-1} \Delta(a^{-1}) = (a \otimes a) \Delta(a^{-1}) F'^{-1}.
\]

Since \( F^{-1} \) and \( F'^{-1} \) are invariant, we deduce from Proposition 1.7 (a) that \( a \) belongs to \( N_G(k) = N(k[G]) \). From (1.14) we obtain \((a \cdot (1 \otimes 1))^{-1} = a^{-1} \cdot (1 \otimes 1) \), which can be rewritten as \([a \otimes a] \Delta(a^{-1})]^{-1} = (a^{-1} \otimes a^{-1}) \Delta(a) \). As a result,

\[
F = F' (a^{-1} \otimes a^{-1}) \Delta(a) .
\]

Thus \( F \) and \( F' \) belong to the same left coset of \( \text{Int}_k(G)/\text{Inn}(G) \) (see Proposition 1.7 (c) together with the isomorphisms (1.15) and (2.4)). The converse is clear.

(b) This follows from Part (a). Note that \( \Theta^{-1}(1) \) is the subgroup isomorphic to \( \text{Int}_k(G)/\text{Inn}(G) \) mentioned in Corollary 2.3 (c).

(c) Assume that \( G \) has odd order, and let \((A,b) \in B(G) \). As observed in Section 3.2, the fact that \( \widehat{A} \) has odd order allows us to pick the map \( c : (x,y) \mapsto b(x/2,y) \) as a two-cocycle whose associated bilinear form is \( b \). Since \( b \) is \( G \)-invariant, so is \( c \). Therefore the twist \( F \in k[A] \otimes k[A] \) determined by \( c \) according to the recipe of Section 3.2 is also invariant. We have \( R_F = R(A,b) \), which implies that \( \Theta(F) = (A,b) \).

Since the sets \( \text{Int}_k(G)/\text{Inn}(G) \) and \( B(G) \) are finite, we deduce the following corollary (we shall give an alternating proof in Section 6).

**Corollary 4.6** – The group \( H^2_G(G) \) is finite.

Theorem 4.5 has the following obvious consequences.

**Corollary 4.7** – If \( B(G) = \{1\} \), then \( H^2_F(G) \cong \text{Int}_k(G)/\text{Inn}(G) \).
Corollary 4.8 – When \( \text{Int}_k(G) = \text{Inn}(G) \), then \( \Theta : H^2_k(G) \rightarrow \mathcal{B}(G) \) is injective. If in addition \( G \) has odd order, then \( \Theta \) is a bijection of sets.

4.5. The set \( \mathcal{B}(G) \) as a colimit. If \( A \subset B \) is an inclusion of abelian subgroups of \( G \), then the natural map

\[
H^2(\hat{A}, k^\times)^G \rightarrow H^2(\hat{B}, k^\times)^G
\]

is injective. Indeed, it suffices to check that \( H^2(\hat{A}, k^\times) \rightarrow H^2(\hat{B}, k^\times) \) is injective. Since \( H^2(\hat{A}, k^\times) \cong \text{Hom}(\Lambda^2 \hat{A}, k^\times) \) for all abelian groups \( A \), this is equivalent to the surjectivity of \( \Lambda^2 \hat{B} \rightarrow \Lambda^2 \hat{A} \), which follows from the surjectivity of the natural homomorphism \( \hat{B} \rightarrow \hat{A} \) between the Pontryagin duals.

Now let \( C \) be the category whose objects are the abelian normal subgroups of \( G \) and whose arrows are the inclusions. It follows that the colimit of \( A \mapsto H^2(\hat{A}, k^\times)^G \) over \( C \) is essentially a union, which we denote by \( \text{colim} H^2(\hat{A}, k^\times)^G \).

As a matter of notation, given \( b \in H^2(\hat{A}, k^\times)^G \), where \( A \) is an abelian normal subgroup of \( G \), we shall write \([A, b]\) for the corresponding element of \( \text{colim} H^2(\hat{A}, k^\times)^G \). Of course several choices for \( A \) and \( b \) can lead to the same element in the colimit.

Lemma 4.9 – An element \( \beta \in \text{colim} H^2(\hat{A}, k^\times)^G \) defines a universal \( R \)-matrix \( R(\beta) \in k[G] \otimes k[G] \) with Drinfeld element \( 1 \) by the formula \( R(\beta) = R(A, b) \), where \( A \) and \( b \) are such that \( \beta = [A, b] \).

Proof. We need first to show that \( R(A, b) = R(B, b') \) whenever \([A, b] = [B, b']\) in the colimit. By the very construction of this colimit, it is enough to check this when \( A, B, b \) and \( b' \) are as in Lemma 3.4. The same lemma yields the result, and \( R(\beta) \) is well-defined. We use Lemma 4.3 to conclude that the Drinfeld element of \( R(\beta) \) is 1.

Proposition 4.10 – The map \( \mathcal{B}(G) \rightarrow \text{colim} H^2(\hat{A}, k^\times)^G \) that sends \((A, b)\) to \([A, b]\) is a bijection of sets.

Proof. We construct the inverse map. Given \( \beta \in \text{colim} H^2(\hat{A}, k^\times)^G \), let \( R(\beta) \) be as in the previous lemma, and let \( A \) be defined by \( k[G](R(\beta)) = k[A] \). It is clear that \( R(\beta) \) is \( G \)-invariant; so we see from Lemma 4.4(a) that \( A \) is abelian and normal.

The \( R \)-matrix \( R(\beta) \) yields a bilinear form \( b \) on \( \hat{A} \). The form \( b \) is alternating since the Drinfeld element of \( R(\beta) \) is 1, it is non-degenerate by Proposition 3.3, and it is \( G \)-invariant. Therefore, \((A, b) \in \mathcal{B}(G) \). We also observe from Lemma 3.4 that \( \beta = [A, b] \). It is now clear that \( \beta \mapsto (A, b) \) is the inverse of \((A, b) \mapsto [A, b] \).

In a sense the elements of \( \mathcal{B}(G) \) provide canonical representatives for the equivalence classes in the colimit. In the sequel we will sometimes use the notation \((A, b)\) in order to speak of the element \([A, b]\) in the colimit; this notation carries the extra information that \( b \) is non-degenerate in accordance with Section 4.4.

The reader should beware that the notation \( \text{colim} H^2(\hat{A}, k^\times)^G \) does not suggest any group law, for the colimit is taken in the category of sets. Nevertheless, we can at least construct a partially defined multiplication on it.

Suppose given \((A, b)\) and \((B, b')\), and assume that \( A \) and \( B \) are both contained in a larger abelian normal subgroup \( C \) of \( G \). Then \( b \) and \( b' \) can be viewed as elements of \( H^2(\hat{C}, k^\times) \) and composed in this group; their product is still \( G \)-invariant and thus defines the element \([C, bb']\). We put \([A, b] \cdot [B, b'] = [C, bb']\) whenever \( C \) exists with the aforementioned properties. Note that this definition does not depend on the choice of \( C \): indeed if \( C' \) is another choice, then \( C \cap C' \) is yet another abelian normal subgroup of \( G \) containing both \( A \) and \( B \), and it has inclusion maps to \( C \) and \( C' \).
In terms of \( \mathcal{B}(G) \) the above product can be, somewhat less conveniently, described as \( (D, bb') \), where \( D \) is determined by \( k[G](R(C, bb')) = k[D] \); here \( D \) is a subgroup of \( C \) and we have written \( bb' \) for the unique element in \( H^2(D, k^\times) \) that maps to the product of \( b \) and \( b' \) in \( H^2(\widehat{C}, k^\times) \).

The following lemma gives a sufficient condition for the map \( \Theta \) to be compatible with this structure.

**Lemma 4.11** — Let \( F \) and \( F' \) be invariant twists for \( G \) and suppose that \( F \) commutes with \( R_{F'} \). Under this assumption,

(a) we have \( R_{FF'} = R_F R_{F'} \);

(b) if \( \Theta(F) \) and \( \Theta(F') \) can be composed in \( \mathcal{B}(G) \), then \( \Theta(F) \Theta(F') = \Theta(FF') \).

**Proof.** (a) We simply compute:

\[
R_{FF'} = (FF')_{21}(FF')^{-1} = F_{21}F'_{21}(F')^{-1}F^{-1} = F_{21}R_{F'}F^{-1} = R_F R_{F'}.
\]

(b) Let \( \Theta(F) = (A, b) = \alpha \) and \( \Theta(F') = (B, b') = \beta \). By assumption there exists an abelian normal subgroup \( C \) of \( G \) containing \( A \) and \( B \). Let \( \gamma = \alpha \beta = [C, bb'] \). The \( R \)-matrices \( R(\alpha) \), \( R(\beta) \), and \( R(\gamma) \) can all be viewed as tensors in \( k[C] \otimes k[C] \). Correspondingly, we view \( b, b' \), and \( bb' \) as bilinear forms on \( \widetilde{C} \). It is clear from (3.1) that \( R(C, b) R(C, b') = R(C, bb') \). However, \( R(C, b) = R(\alpha) \) by definition, and similarly, \( R(C, b') = R(\beta) \) and \( R(C, bb') = R(\gamma) \).

Note that for any invariant twist \( J \) we always have \( R(\Theta(J)) = R_J \) as a trivial consequence of the definitions. This remark shows, on the one hand, that we have \( R_F = R(\alpha) \), \( R_{F'} = R(\beta) \), and \( R_{FF'} = R(\Theta(FF')) \); on the other hand, combining (a) with the equality \( R(\alpha) R(\beta) = R(\gamma) \), which we have just established, we deduce \( R_{FF'} = R(\gamma) \). The identity \( R(\Theta(FF')) = R(\gamma) \) clearly implies \( \Theta(FF') = \gamma \).

We can gain some information about the torsion in \( H^2_1(G) \) from Lemma 4.11.

**Corollary 4.12** — If \( F \) commutes with \( R_F \), then the powers of \( \Theta(F) \) are well-defined. Moreover, if \( \Theta(F) \) has order \( n \), then the order of \( F \) in \( H^2_1(G) \) is \( nd \), where \( d \) divides \( \text{card}(	ext{Int}_k(G)/\text{Inn}(G)) \).

**Proof.** The lemma and an immediate induction imply that \( \Theta(F^k) = \Theta(F)^k \) for any non-negative integer \( k \). Now apply Theorem 4.5. \( \square \)

**Remark 4.13** — The condition that \( F \) commute with \( R_F \) is trivially satisfied if \( F \) is supported by its socle (recall that this means \( F \in k[A] \otimes k[A] \)) for \( F \) and \( R_F \) in this case both belong to the commutative algebra \((k[A] \otimes k[A])\). Theorem 4.5 guarantees that any twist \( F \) is supported by its socle in the particular case when \( G \) has odd order and \( \text{Int}_k(G) = \text{Inn}(G) \). The next corollary goes further with a very special class of examples.

**Corollary 4.14** — Assume that \( G \) is a group of odd order such that \( \text{Int}_k(G) = \text{Inn}(G) \). If \( G \) has a unique maximal abelian normal subgroup \( A \), then

\[
H^2_1(G) \cong H^2(\widehat{A}, k^\times)^G.
\]

**Proof.** It follows from Theorem 4.5 and Proposition 4.10 that \( H^2_1(G) \) is via the map \( \Theta \) in bijection with \( H^2(\widehat{A}, k^\times)^G \) as a set. Let us check that \( \Theta \) is a homomorphism in this case.

Now, the same theorem shows that any element of \( H^2_1(G) \) can be represented by a twist \( F \) that is supported by its socle \( A_F \). Suppose that \( F \) and \( F' \) are chosen in this way. By hypothesis, \( A_F \subset A \) and \( A_{F'} \subset A \), and so \( F \) and \( F' \) are both
in $k[A] \otimes k[A]$. Clearly then $F$ commutes with $R_{F'}$, since $A$ is abelian. Lemma 4.11 then implies that $\Theta(F) \Theta(F') = \Theta(FF')$, as desired. \hfill \qed

5. SOME CONSEQUENCES OF SEMI-SIMPURITY

The aim of this short section is to obtain a sufficient condition on a finite group $G$ for the group $H^2(G)$ to be abelian and to prepare the ground for the next section. We keep assuming that $k$ is algebraically closed.

In this situation, $\hat{k}[G]$ is semi-simple and there is an algebra isomorphism

\begin{equation}
\hat{k}[G] \cong \prod_{\rho \in \hat{G}} \text{End}_k(V_\rho),
\end{equation}

where $\rho$ runs over the set $\hat{G}$ of irreducible representations of $G$. Consequently,

\begin{equation}
\hat{k}[G]^G \cong \prod_{\rho \in \hat{G}} \text{End}_k(V_\rho),
\end{equation}

Since the isomorphism (5.1) sends any element $a \in \hat{k}[G]$ to the left multiplication by $a$ on each representation space $V_\rho$, it sends the conjugation action of $G$ on $\hat{k}[G]$ to the action of $G$ on $\text{End}_k(V_\rho)$ given by $(gf) : v \mapsto gf(g^{-1}v)$ for all $g \in G$, $f \in \text{End}_k(V_\rho)$. It follows from this and from Schur’s lemma that the subalgebra $\hat{k}[G]^G$ of $G$-invariants elements is given by

\begin{equation}
\hat{k}[G]^G \cong \prod_{\rho \in \hat{G}} \text{End}_G(V_\rho) \cong \mathcal{O}_k(\hat{G}).
\end{equation}

From the above we also deduce the $G$-invariant algebra isomorphisms

\begin{equation}
\hat{k}[G] \otimes \hat{k}[G] \cong \prod_{\rho,\sigma \in \hat{G}} (\text{End}_k(V_\rho) \otimes \text{End}_k(V_\sigma)) \cong \prod_{\rho,\sigma \in \hat{G}} \text{End}_k(V_\rho \otimes V_\sigma).
\end{equation}

Decomposing the tensor products $V_\rho \otimes V_\sigma$ into simple modules

\begin{equation}
V_\rho \otimes V_\sigma \cong \bigoplus_{\tau \in \hat{G}} V^{\ell}_{\tau,\rho,\sigma},
\end{equation}

we obtain the $G$-invariant algebra isomorphism

\begin{equation}
\hat{k}[G] \otimes \hat{k}[G] \cong \prod_{\rho,\sigma \in \hat{G}} \text{End}_k\left(\bigoplus_{\tau \in \hat{G}} V^{\ell}_{\tau,\rho,\sigma}\right).
\end{equation}

We claim that the subalgebra $(\hat{k}[G] \otimes \hat{k}[G])^G$ of $G$-invariant elements of $\hat{k}[G] \otimes \hat{k}[G]$ can be decomposed into a product of matrix algebras as follows:

\begin{equation}
(\hat{k}[G] \otimes \hat{k}[G])^G \cong \prod_{\rho,\sigma \in \hat{G}} \prod_{\tau \in \hat{G}} M_{d^{\ell}_{\rho,\sigma}}(k).
\end{equation}

Indeed, taking $G$-invariants of both sides of (5.3) and using Schur’s lemma twice, we obtain

\begin{equation}
(\hat{k}[G] \otimes \hat{k}[G])^G \cong \prod_{\rho,\sigma \in \hat{G}} \text{End}_G\left(\bigoplus_{\tau \in \hat{G}} V^{d^{\ell}_{\rho,\sigma}}_{\tau,\rho,\sigma}\right)
\end{equation}

\begin{equation}
\cong \prod_{\rho,\sigma \in \hat{G}} \prod_{\tau \in \hat{G}} \text{End}_G(V^{d^{\ell}_{\rho,\sigma}}_{\tau,\rho,\sigma})
\end{equation}

\begin{equation}
\cong \prod_{\rho,\sigma \in \hat{G}} \prod_{\tau \in \hat{G}} M_{d^{\ell}_{\rho,\sigma}}(\text{End}_G(V_\tau))
\end{equation}

\begin{equation}
\cong \prod_{\rho,\sigma \in \hat{G}} \prod_{\tau \in \hat{G}} M_{d^{\ell}_{\rho,\sigma}}(k).
\end{equation}
We thus see that the algebra $(k[G] \otimes k[G])^G$ is semisimple; moreover, it is commutative if and only if all multiplicities $d^\tau_{\rho,\sigma}$ are equal to 1 or 0. In this case we shall say that $G$ has no multiplicities.

Observe that an abelian group has no multiplicities. Note also that a quotient of a group without multiplicities has no multiplicities.

The following result provides a sufficient condition for $\text{H}^2_{\ell}(G)$ to be abelian.

**Proposition 5.1** – If $G$ has no multiplicities, then $A^2(k[G])$ and $\text{H}^2_{\ell}(G)$ are abelian groups.

**Proof.** Since $A^2(k[G]) = (\{k[G] \otimes k[G]\}^G)^\times$, it follows from (5.4) that

$$A^2(k[G]) \cong \prod_{\rho,\sigma \in G} \prod_{\tau \in \widehat{G}} GL_{d^\tau_{\rho,\sigma}}(k).$$

Therefore, if $d^\tau_{\rho,\sigma} \leq 1$ for all $\rho, \sigma, \tau \in \widehat{G}$, the group $A^2(k[G])$ is a $k$-split torus, which is abelian. Since $\text{H}^2_{\ell}(G)$ is a subquotient of $A^2(k[G])$, it is abelian as well. \(\square\)

**Remark 5.2** – A consequence of Proposition 5.1 is that, when $G$ has no multiplicities, any invariant twist $F$ commutes with $R_F$: indeed $F$ and $R_F$ both belong to $A^2(k[G])$. In this case we may appeal to Corollary 4.12 in order to study the torsion in $\text{H}^2_{\ell}(G)$.

### 6. Rationality Questions

In this section we no longer assume that the ground field $k$ is algebraically closed. Let $\bar{k}$ be the algebraic closure of $k$. We fix a finite group $G$.

**6.1. The point of view of algebraic groups.** For any $k$-algebra $K$, and for $i = 1, 2, 3$, we define $A^i(K)$ to be the group of $G$-invariant invertible elements in $K[G]^\otimes_i$, where $G$ acts diagonally by conjugation. We can see $A^i$ as a group scheme over $k$. By Section 5 this group scheme is smooth since $A^i(K)$ is a product of general linear groups $GL_n(K)$ whenever $K$ contains $\bar{k}$. When $K = k$, we have $A^i(k) = A^i(k[G])$ in the notation of Section 1.2.

We define $\delta^i : A^1 \to A^2$ and $\delta^2_{\ell}, \delta^2_R : A^2 \to A^3$ by the same formulas as in Section 1.2; these are morphisms of group schemes. The equalizer $\Gamma$ of $\delta^2_{\ell}$ and $\delta^2_R$ is a smooth group scheme. Moreover,

$$\Gamma(k) = Z^2(k[G])$$

by Section 1.2. Using Notation 2.1, we obtain

$$\Gamma(k)/\delta^i(A^1(k)) = \text{H}^2_{\ell}(G/k).$$

**6.2. Tangent Lie algebras.** Let $\Gamma^0$ be the identity component of $\Gamma$. We have the following.

**Proposition 6.1** – The image of $\delta^1 : A^1(\bar{k}) \to A^2(\bar{k})$ is $\Gamma^0(\bar{k})$.

**Proof.** Since we work in characteristic 0, we can use the machinery of Lie algebras. For a group scheme $S$, we write Lie $S$ for the Lie algebra tangent to $S$. For instance, $\text{Lie} A^i(K) = (K[G]^\otimes_i)^G$ for $i = 1, 2, 3$.

The tangent maps $\text{Lie} \delta^1$, $\text{Lie} \delta^2_{\ell}$, $\text{Lie} \delta^2_R$ are given by the following formulas:

$$\text{Lie} \delta^1(Y) = Y \otimes 1 + 1 \otimes Y - \Delta(Y),$$

$$\text{Lie} \delta^2_{\ell}(X) = X \otimes 1 + (\Delta \otimes \text{id})(X),$$

$$\text{Lie} \delta^2_R(X) = 1 \otimes X + (\text{id} \otimes \Delta)(X),$$

where $Y \in \bar{k}[G]^G$ and $X \in (\bar{k}[G] \otimes \bar{k}(G))^G$. The Lie algebra $\text{Lie} \Gamma(\bar{k})$ is the subspace of those tangent vectors $X \in \text{Lie} A^2(\bar{k})$ such that $\text{Lie} \delta^2_{\ell}(X) = \text{Lie} \delta^2_R(X)$. Moreover,
the proposition will be proved once we show that such an $X$ must be of the form $\text{Lie}\,\delta^1(Y)$ for some $Y$ tangent to $A^1$. This basic translation from algebraic groups to Lie algebras, over an algebraically closed field of characteristic 0, is classical and follows for example from the material in [4, Chap. II, §6 6–7].

As it happens, the formulas above for the tangent maps make sense for any $Y \in \bar{k}[G]$ and $X \in \bar{k}[G] \otimes \bar{k}[G]$, not necessarily $G$-invariant. So if we prove that the following is an exact sequence

\[ (6.2) \quad 0 \longrightarrow \bar{k}[G] \overset{\text{Lie}\,\delta^1}{\longrightarrow} \bar{k}[G] \otimes^2 \bar{k}[G] \overset{\text{Lie}\,\delta^2 \otimes \text{Lie}\,\delta_R^2}{\longrightarrow} \bar{k}[G] \otimes^3 \bar{k}[G], \]

then the proposition will follow by taking $G$-fixed points. It is easy to check that the composite map $(\text{Lie}\,\delta^2_R - \text{Lie}\,\delta^2_L) \circ \text{Lie}\,\delta^1 = 0$, so that (6.2) is a complex of vector spaces. One also verifies that the map $\text{Lie}\,\delta^2_R : \bar{k}[G] \rightarrow \bar{k}[G] \otimes^2 \bar{k}[G]$ is injective. So in order to prove the exactness of (6.2), it suffices to check it at $\bar{k}[G] \otimes^2 \bar{k}[G]$. To achieve this, we count dimensions. The subspace $V$ of $\bar{k}[G] \otimes^2 \bar{k}[G]$ spanned by the elements $g \otimes h$ with $g \neq h$ has dimension $n^2 - n$, where $n$ is the order of $G$. A simple computation shows that the map $\text{Lie}\,\delta^2_R - \text{Lie}\,\delta^2_L$ is injective on $V$. Therefore, its rank is at least $n^2 - n$. We deduce that the kernel of $\text{Lie}\,\delta^2_R - \text{Lie}\,\delta^2_L$ has dimension at most $n^2 - (n^2 - n) = n$, which is the dimension of the image of $\text{Lie}\,\delta^1$. This shows that the kernel of $\text{Lie}\,\delta^2_R - \text{Lie}\,\delta^2_L$ is equal to the image of $\text{Lie}\,\delta^1$. \hfill \Box

As a consequence, we obtain another proof of Corollary 4.6.

**Corollary 6.2** – The group $H^2_L(G/\bar{k})$ is finite.

**Proof.** By Proposition 6.1,

\[ (6.3) \quad H^2_L(G/\bar{k}) = \Gamma(\bar{k})/\delta^1(\bar{A}^1(\bar{k})) = \Gamma(\bar{k})/\Gamma^0(\bar{k}) = \pi_0(\Gamma), \]

which is finite. \hfill \Box

6.3. **The rationality exact sequence.** We are now able to compare $H^2_L(G/k)$ and $H^2_L(G/\bar{k})$.

**Theorem 6.3** – Assume that all $\bar{k}$-representations of $G$ can be realized over $k$. Then there is an exact sequence

\[ 1 \longrightarrow H^1(k, Z(G)) \longrightarrow H^2_L(G/k) \longrightarrow H^2_L(G/\bar{k}) \longrightarrow 1, \]

where $H^1(k, Z(G))$ is the first Galois cohomology group of $k$ with coefficients in the centre of $G$.

**Proof.** Taking $\text{Gal}(\bar{k}/k)$-fixed points in the exact sequence

\[ 1 \longrightarrow \Gamma^0(\bar{k}) \longrightarrow \Gamma(\bar{k}) \longrightarrow \pi_0(\Gamma) \rightarrow 1 \]

yields the exact sequence

\[ 1 \longrightarrow \Gamma^0(k) \longrightarrow \Gamma(k) \longrightarrow \pi_0(\Gamma) \rightarrow H^1(k, \Gamma^0). \]

Dividing out the first two groups by the central subgroup $\delta^1(A^1(k))$, and identifying terms using (6.1) and (6.3), we obtain the exact sequence

\[ (6.4) \quad 1 \longrightarrow \Gamma^0(k)/\delta^1(A^1(k)) \longrightarrow H^2_L(G/k) \longrightarrow H^2_L(G/\bar{k}) \longrightarrow H^1(k, \Gamma^0) \rightarrow 1. \]

The hypothesis on $k$ implies in particular that $A^1$ is a $k$-split torus. By Proposition 6.1, $\Gamma^0$ is also a $k$-split torus, so that $H^1(k, \Gamma^0) = 1$ by Hilbert’s Theorem 90.

Now take $\text{Gal}(k/\bar{k})$-fixed points again, this time in the exact sequence

\[ 1 \longrightarrow Z(G) \longrightarrow A^1(\bar{k}) \overset{\delta^1}{\longrightarrow} \Gamma^0(\bar{k}) \rightarrow 1. \]

We obtain the exact sequence

\[ (6.5) \quad 1 \longrightarrow Z(G) \longrightarrow A^1(k) \overset{\delta^1}{\longrightarrow} \Gamma^0(k) \longrightarrow H^1(k, Z(G)) \longrightarrow H^1(k, A^1). \]
By Hilbert’s Theorem 90 again, \( H^1(k, A^1) = 1 \), so that
\[
\Gamma^0(k) / \delta^1(A^1(k)) \cong H^1(k, Z(G)).
\]
This shows that the exact sequence of the theorem is (6.4).

\[\Box\]

**Remark 6.4** – From the sequences (6.4) and (6.5) we deduce without any assumptions on \( k \) that the natural map \( H^2_k(G/k) \to H^2_k(G/\bar{k}) \) is injective whenever the centre of \( G \) is trivial. If moreover \( k \) happens to satisfy the hypothesis of Theorem 6.3, then this map is surjective, too.

**Remark 6.5** – It follows from the proof of Theorem 6.3 that any element of \( H^1(k, Z(G)) \) can be represented by a map \( \gamma_a : \Gal(\bar{k}/k) \to Z(G) \) of the form \( \gamma_a(\sigma) = \sigma_1a \sigma^{-1} \) for some \( a \in A^1(\bar{k}) \). The image of \( \gamma_a \) in \( H^2_k(G/k) \) can be represented by the invariant twist \( \delta^1(a) = (a \otimes a) \Delta(a^{-1}) \).

### 7. Examples and computations

In this section we again assume that the ground field is algebraically closed and we compute \( H^2_k(G) \) for some finite groups, as an application of Theorem 4.5 and its corollaries.

#### 7.1. Examples of groups with trivial \( H^2_k(G) \).

**Theorem 7.1** – Let \( k \) be an algebraically closed field. Then \( H^2_k(G) = 1 \) if \( G \) belongs to the following list of finite groups:

(i) the simple groups,

(ii) the symmetric groups \( S_n \),

(iii) the groups \( \SL_n(\mathbb{F}_q) \),

(iv) the groups \( \GL_n(\mathbb{F}_q) \) when \( n \) is coprime to \( q - 1 \).

The simple groups include:

(a) all alternating groups \( A_n \), except \( A_4 \) (the latter contains Klein’s Vierergruppe as a normal subgroup and will be treated in Section 7.5),

(b) all projective linear groups \( \PSL_n(\mathbb{F}_q) \), except \( \PSL_2(\mathbb{F}_2) \) and \( \PSL_2(\mathbb{F}_3) \); we have \( \PSL_2(\mathbb{F}_2) \cong S_3 \) and \( \PSL_2(\mathbb{F}_3) \cong A_4 \), which are not simple.

**Proof.** By Corollary 4.7 it suffices to check that \( \mathcal{B}(G) \) is trivial and \( \text{Int}_k(G) = \text{Inn}(G) \) for each of these groups.

Let us start with \( \mathcal{B}(G) \). Recall that it consists of all pairs \((A, b)\), where \( A \) is an abelian normal subgroup of \( G \) and \( b \) is a \( G \)-invariant nondegenerate alternating bilinear form on \( A \). By Proposition 3.5, any abelian group having a non-degenerate alternating bilinear form is necessarily of symmetric type. Any group \( G \) contains an abelian normal subgroup of symmetric type, namely the trivial group. If it does not contain any other abelian normal subgroup of symmetric type, then the set \( \mathcal{B}(G) \) is trivial.

So to prove that \( \mathcal{B}(G) \) is trivial for the groups listed in the theorem, it is enough to check that they do not contain any non-trivial abelian normal subgroups of symmetric type. This holds for

(i) the finite simple groups for obvious reasons;

(ii) the symmetric groups because the only proper normal subgroups they contain are the alternating groups;

(iii) the groups \( \GL_n(\mathbb{F}_q) \) and \( \SL_n(\mathbb{F}_q) \) because their only abelian normal subgroups are groups of roots of unity, which are cyclic, hence not of symmetric type.

Let us now check that \( \text{Int}_k(G) = \text{Inn}(G) \) for the above groups. Since \( k \) is algebraically closed, it is enough by (2.5) to check that \( \text{Aut}_c(G) = \text{Inn}(G) \).

Now, for any simple group \( G \), Feit and Seitz [16] proved that \( \text{Aut}_c(G) = \text{Inn}(G) \) using the classification of finite simple groups.
For the symmetric groups $S_n$, the argument goes as follows. Each class-preserving automorphism necessarily maps any transposition to a transposition. Now it is well known (and a simple exercise) that any automorphism of $S_n$ that preserves transpositions is necessarily inner. For $n \neq 6$ one can also use the fact that all automorphisms of $S_6$ are inner.

Let us now deal with $SL_n(\mathbb{F}_q)$. If $(n, q) = (2, 2)$, then $SL_n(\mathbb{F}_q) = PSL_n(\mathbb{F}_q) = S_3$ and we conclude as above. Now assume that $(n, q) \neq (2, 2)$. A class-preserving automorphism $\varphi$ of $SL_n(\mathbb{F}_q)$ necessarily fixes each element of the centre $Z$ of the group, hence induces a class-preserving automorphism of $PSL_n(\mathbb{F}_q) = SL_n(\mathbb{F}_q)/Z$.

If $(n, q) \neq (2, 3)$, then the latter is simple and it follows from the above-mentioned result by Feit and Seitz that the automorphism induced by $\varphi$ on $PSL_n(\mathbb{F}_q)$ is inner.

If $(n, q) = (2, 3)$, then $PSL_n(\mathbb{F}_q) \cong A_4$, whose automorphisms are all inner. We are thus reduced to the case when $\varphi$ induces the identity on $PSL_n(\mathbb{F}_q)$. It follows that there is a group homomorphism $z : SL_n(\mathbb{F}_q) \to Z$ such that for all $g \in SL_n(\mathbb{F}_q)$ we have $\varphi(g) = z(g)g$. Since $\varphi$ fixes the elements of $Z$, we have $z(g) = 1$ for all $g \in Z$. Therefore, $z$ factors through $PSL_n(\mathbb{F}_q)$. The range of $z$ being abelian, it must factor through the abelianization of $PSL_n(\mathbb{F}_q)$, which is trivial since $PSL_n(\mathbb{F}_q)$ is simple and not abelian. Hence, $z(g) = 1$ for all $g \in SL_n(\mathbb{F}_q)$. It follows that $\varphi$ is the identity.

Since $n$ is coprime to the order $q - 1$ of the cyclic group $\mathbb{F}_q^\times$, it follows that $\mathbb{F}_q^\times$ is uniquely $n$-divisible and that any element $g \in GL_n(\mathbb{F}_q)$ can be written (uniquely) as $g = \lambda u$, where $\lambda$ is the unique $n$-th root of the determinant of $g$ and $u \in SL_n(\mathbb{F}_q)$. Let $\varphi$ be a class-preserving automorphism of $GL_n(\mathbb{F}_q)$. We necessarily have $\det(\varphi(g)) = \det(g)$. Therefore $\varphi$ preserves the subgroup $SL_n(\mathbb{F}_q)$.

From the decomposition $g = \lambda u$, we easily deduce that the restriction of $\varphi$ to $SL_n(\mathbb{F}_q)$ is class-preserving. It follows from the above discussion that this restriction is an inner automorphism. We can therefore assume that the restriction of $\varphi$ to $SL_n(\mathbb{F}_q)$ is the identity. Since it is also the identity on the central matrices, it is the identity on $GL_n(\mathbb{F}_q)$. We have thus proved that any class-preserving automorphism of $GL_n(\mathbb{F}_q)$ is the conjugation by some element of $SL_n(\mathbb{F}_q)$. \(\square\)

7.2. The wreath product $\mathbb{Z}/p \wr \mathbb{Z}/p$ for $p$ odd. It provides a first simple example of a group with non-trivial $H^2_f(G)$.

**Proposition 7.2** – The group $H^2_f(\mathbb{Z}/p \wr \mathbb{Z}/p)$ is an elementary abelian $p$-group of rank $(p - 1)/2$.

**Proof.** For simplicity, we work over the field $\mathbb{C}$ of complex numbers. If $G$ is the wreath product $\mathbb{Z}/p \wr \mathbb{Z}/p$, then we can write $G$ as a semi-direct product $G = V \rtimes C$, where $C$ is a cyclic group of order $p$, generated by an element $\sigma$, and $V = \mathbb{F}_p[C]$, on which $C$ acts by left multiplication.

The abelian normal subgroups of $G$ are exactly the submodules of $V$. It will follow from Corollary 4.14 that

$$H^2_f(G) = H^2(V, k^\times)^G,$$

if we can only prove that $\text{Aut}_G(V) = \text{Inn}(G)$.

So let $\varphi$ be a class-preserving automorphism of $G$; we wish to prove that $\varphi$ is inner. By composing $\varphi$ with a suitable inner automorphism, we may assume that $\varphi(\sigma) = \sigma$, and with this assumption we see that $\varphi$ induces an automorphism of the $\mathbb{F}_p[C]$-module $V$. Now, $\sigma$ has only one eigenvector $\varepsilon_1$ (up to a scalar), with eigenvalue 1. Therefore we can find a basis $\{\varepsilon_1, \ldots, \varepsilon_p\}$ of $V$ such that the matrix of $\sigma$ in this basis is given by a single Jordan block. Moreover, the commutant of a
The conjugates of \( \varepsilon_i \) are of the form \( \varepsilon_i + b\varepsilon_{i-1} \) for \( b \in \mathbb{F}_p \). As a result, if a linear map of \( V \) given by a matrix as above is to send each \( \varepsilon_i \) to one of its conjugates, then it must act like conjugation by \( \sigma^b \) (that is, all the constants in the matrix are 0 except for \( a \) and \( b \), and \( a = 1 \)). This proves that \( \varphi \) is inner.

Let us now compute \( H^2(\hat{V},\mathbb{C}^\times)^G \). First we use the exponential exact sequence
\[
0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{C}^{\exp} \longrightarrow \mathbb{C}^\times \longrightarrow 0,
\]
which yields an isomorphism \( H^2(\hat{V},\mathbb{C}^\times) \cong H^3(\hat{V},\mathbb{Z}) \) (and likewise for any finite group). Since \( V \) is elementary abelian, it is well known that \( H^*(\hat{V},\mathbb{Z}) \) is the subspace of \( H^*(\hat{V},\mathbb{F}_p) \) of those elements killed by the Bockstein \( \beta : H^*(\hat{V},\mathbb{F}_p) \rightarrow H^{*+1}(\hat{V},\mathbb{F}_p) \). Recall that in low degrees we have the following identifications:
\[
(7.1) \quad H^2(\hat{V},\mathbb{Z}) = H^1(\hat{V},\mathbb{C}^\times) = \text{Hom}(\hat{V},\mathbb{C}^\times) = \hat{V} = V.
\]
(The first identification comes from the exponential exact sequence again.) Also we have
\[
(7.2) \quad H^1(\hat{V},\mathbb{F}_p) = \text{Hom}(\hat{V},\mathbb{F}_p) = \text{Hom}(\hat{V},\mathbb{C}^\times) = \hat{V} = V.
\]
Let us pick a basis \( (e_1,\ldots,e_p) \) of \( V \) such that \( \sigma(e_i) = e_{i+1} \) (taking indices modulo \( p \)). Each element \( e_i \) gives rise, via (7.1), to an element of \( H^2(\hat{V},\mathbb{F}_p) \) which we still denote by \( e_i \). Via (7.2) it can also be identified with an element of \( H^1(\hat{V},\mathbb{F}_p) \), which we denote by \( u_i \) in the sequel. The full mod \( p \) cohomology ring is then
\[
H^*(\hat{V},\mathbb{F}_p) = \mathbb{F}_p[e_1,\ldots,e_p] \otimes \Lambda(u_1,\ldots,u_p).
\]
Therefore \( H^3(\hat{V},\mathbb{F}_p) \) is isomorphic to \( V \otimes V \otimes \Lambda^3 V \) as an \( \mathbb{F}_p[\mathbb{C}] \)-module, or equivalently as an \( \mathbb{F}_p[\mathbb{G}] \)-module. We wish to determine the kernel of the Bockstein restricted to the subspace of \( G \)-invariant elements in this module: this will be \( H^3(\hat{V},\mathbb{Z})^G = H^2(\hat{V},\mathbb{C}^\times)^G \). Since the Bockstein is visibly injective on the \( \Lambda^3 V \) summand (see details on the action below), we focus on \( V \otimes V \). Let
\[
v_k = \sum_{i=1}^{p} u_i e_{i+k},
\]
for \( 0 \leq k \leq p-1 \) (again the indices are to be understood modulo \( p \)). It is easy to see that the elements \( v_k \) span the subspace of fixed points.

The Bockstein is a derivation that vanishes on each \( e_i \) and \( \beta(u_i) = e_i \). Because the elements \( e_i \) commute with one another, we see that \( \beta(v_k) = \beta(v_{-k}) \). On the other hand the elements \( \beta(v_k) \) for \( 0 \leq k \leq (p-1)/2 \) are linearly independent. Finally, the kernel of \( \beta \), restricted to the module of invariant elements, has dimension \( (p-1)/2 \).

\[\square\]

7.3. The group \( \mathbb{Z}/2 \wr \mathbb{Z}/2 \). We turn our attention to the case \( p = 2 \). This will complete the series of wreath products and will illustrate our method for groups of even order. Note that \( G = \mathbb{Z}/2 \wr \mathbb{Z}/2 \) is the dihedral group of order 8.

**Proposition 7.3** – The group \( H^2_1(\mathbb{Z}/2 \wr \mathbb{Z}/2) \) is trivial.
Proof. We keep the notation of the previous section and we observe that $\text{Aut}_e(G) = \text{Inn}(G)$ as before. It then follows from (2.5) and Theorem 4.5 that $H^2_e(G)$ injects into $B(G)$.

The first difference with the odd $p$ case appears when we look for normal abelian subgroups: there are exactly two maximal such subgroups, namely $E_1 = V = \langle e_1, e_2 \rangle$ and $E_2 = \langle \sigma, e_1 e_2 \rangle$ (the analogue of $E_2$ for odd $p$ is not normal). Both $E_1$ and $E_2$ are elementary abelian 2-groups of rank 2.

Let $A = \mathbb{Z}/2 \times \mathbb{Z}/2$ and let $(\tilde{e}_1, \tilde{e}_2)$ be a basis of $\hat{A}$. We shall write the group law on $\hat{A}$ multiplicatively. One can see directly that there is only one non-trivial non-degenerate alternating bilinear form $b$ on this group with values in $\mathbb{C}^\times$, namely the one determined by

$$b(\tilde{e}_1, \tilde{e}_1) = b(\tilde{e}_2, \tilde{e}_2) = 1 \quad \text{and} \quad b(\tilde{e}_1, \tilde{e}_2) = b(\tilde{e}_2, \tilde{e}_1) = -1.$$ 

(In other words, this form is the determinant when the subgroup $\{1, -1\}$ of $\mathbb{C}^\times$ is identified with $\mathbb{F}_2$.) The form $b$ is evidently invariant under all automorphisms of $\mathbb{Z}/2 \times \mathbb{Z}/2$. It follows that $B(G)$ has exactly three elements. Theorem 4.5 then implies that $H^2_e(G)$ has at most three elements.

Now the reader can check that $G$ has a well-defined automorphism fixing $e_1 e_2$ and exchanging $e_2$ and $e_1$; it also exchanges $E_1$ and $E_2$. From this, it follows that if the image of $H^2_e(G)$ in $B(G)$ had more than one element, then it would have all three elements. Let us show that this is impossible.

One checks easily that $G$ has no multiplicities in the sense of Section 5. According to Remark 5.2, we can apply Corollary 4.12, from which it follows that the elements of $H^2_e(G)$ have order dividing 2. This is impossible in a group of order 3, so that finally $H^2_e(G)$ is trivial. \hfill \Box

This is an example where the map $\Theta$ is not surjective. It seems instructive to spell out the reason why the argument used in the proof of Theorem 4.5 (c) does not work here.

Lemma 7.4 – Let $\mathbb{Z}/2 = \langle \sigma \rangle$ act on $\hat{A} = \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle \tilde{e}_1, \tilde{e}_2 \rangle$ by $\sigma(\tilde{e}_1) = \tilde{e}_2$. Then there is no $\sigma$-invariant normalized two-cocycle $c$ on $A$ whose associated form is $b$.

Proof. Since $b(\tilde{e}_1, \tilde{e}_2) = -1$, we would have $c(\tilde{e}_1, \tilde{e}_2) = -c(\tilde{e}_2, \tilde{e}_1)$; by invariance of $c$, we would also have $c(\tilde{e}_1, \tilde{e}_2) = c(\tilde{e}_2, \tilde{e}_1)$, which yields a contradiction. \hfill \Box

The reader will see that the lemma applies to $\hat{E}_1$ and $\hat{E}_2$ under an appropriate choice of generators. It shows directly that the non-trivial elements in $B(G)$ do not correspond to invariant twists on the corresponding abelian groups. The argument above is the only one we have in order to prove that these elements of $B(G)$ are not of the form $\Theta(F)$.

7.4. A group of order 27. We turn to the example of $G = (\mathbb{Z}/3 \times \mathbb{Z}/3) \times \mathbb{Z}/3$, where $C = \mathbb{Z}/3$ acts on $V = (\mathbb{Z}/3)^2$ via a single Jordan block with eigenvalue 1. This group is somewhat intermediate between $\mathbb{Z}/3 \wr \mathbb{Z}/3$ and $\mathbb{Z}/2 \wr \mathbb{Z}/2$; it provides an example of a group $G$ whose abelian normal subgroups are not contained in a maximal one and at the same time has a non-trivial $H^2_e(G)$.

Proposition 7.5 – We have $H^2_e(G) = \mathbb{Z}/3 \times \mathbb{Z}/3$.

Proof. Write $c$ for the generator of $C$ and $(e_1, e_2)$ for a basis of $V$ such that the action of $c$ is given by a Jordan block. The element $e_1$ is central, and from $c e_2 c^{-1} = e_1 e_2$ we deduce that any map from $G$ to an abelian group has $e_1$ in its kernel. In particular, $e_1$ belongs to all the normal subgroups of $G$. Thus we can read off these normal subgroups of $G$ from the subgroups of $G/\langle e_1 \rangle$, which is an elementary abelian 3-group of order 9.
In this way we find exactly four abelian normal subgroups of $G$ of square order, say $E_i$, $1 \leq i \leq 4$: they are all elementary abelian 3-groups of order 9, a basis being given in each case respectively by $(e_1, e_2)$, $(e_1, c)$, $(e_1, e_2c)$, and $(e_1, e_2c^2)$. In each case, we can view $E_i$ as a $\mathbb{F}_3$-vector space of dimension 2, and $e_1$ is an eigenvector for the action of $G$ with eigenvalue 1. One checks in all four cases that there are no other eigenvectors, or equivalently that the action of $G$ factors through a copy of $\mathbb{Z}/3$ acting via a single Jordan block with eigenvalue 1. In particular, the action preserves the determinant. It is easy to see that the same can be said of the action of $G$ on the Pontryagin duals $\hat{E}_i$.

As it turns out, the only alternating bilinear forms on $\mathbb{F}_3^2$ are proportional to the determinant (here we identify the group of third roots of unity in $\mathbb{C}^\times$ with $\mathbb{F}_3$). Therefore, on each $\hat{E}_i$ we have precisely two $G$-invariant non-degenerate alternating bilinear forms, namely the determinant and its opposite. In total, $\mathcal{B}(G)$ has nine elements, taking the trivial one into account.

It is easy to check that $\text{Int}_k(G) = \text{Im}(G)$. It then follows from Theorem 4.5 that $H^2_\theta(G)$ has order 9. Moreover, Corollary 4.12 also applies (see Remark 4.13) and we deduce that the elements of $H^2_\theta(G)$ have order dividing 3. The result follows. \hfill $\square$

7.5. The alternating group $A_4$. This example is similar to $\mathbb{Z}/2 \triangleright \mathbb{Z}/2$, as the group $A_4$ is a semi-direct product $V \rtimes C$, where as above $V$ is a copy of $\mathbb{Z}/2 \times \mathbb{Z}/2$ generated by the permutations $e_1 = (1,2)(3,4)$ and $e_2 = (1,3)(2,4)$, while $C$ is a copy of $\mathbb{Z}/3$ generated by the permutation $\sigma = (1,2,3)$. However, we now have the following result (where $b$ is as in Section 7.3).

**Lemma 7.6** – Let $\mathbb{Z}/3 = \langle \sigma \rangle$ act on $\hat{V} = \mathbb{Z}/2 \times \mathbb{Z}/2 = \langle \hat{e}_1, \hat{e}_2 \rangle$ by $\sigma(\hat{e}_1) = \hat{e}_2$ and $\sigma(\hat{e}_2) = \hat{e}_1 \hat{e}_2$. Then there does exist a $\sigma$-invariant normalized two-cocycle $c$ on $A$ whose associated form is $b$.

**Proof.** We only need to set the value of $\lambda = c(\hat{e}_1, \hat{e}_1)$, and $\mu = c(\hat{e}_1, \hat{e}_2)$. All other values of $c$ are dictated by the requirement that it be normalized, $\sigma$-invariant, and subject to the condition $c(y, x) = b(x, y)c(x, y)$. We proceed by inspection and set $\lambda = -1$, $\mu = 1$. We checked (with the help of a computer) that $c$ is indeed a two-cocycle. \hfill $\square$

**Proposition 7.7** – The group $H^2_\theta(A_4)$ has order two.

**Proof.** It is an elementary exercise to prove that all automorphisms of $A_4$ are inner, so that $\Theta$ is injective. The subgroup $V$ is the only abelian normal subgroup, and again we see that $\mathcal{B}(G)$ has precisely two elements, the non-trivial one being $(V, b)$ with $b$ as above. Thus $H^2_\theta(G)$ has at most two elements.

A non-trivial twist $F$ is afforded by Lemma 7.6. Indeed, applying the recipe of Section 3.2, assuming that $\hat{e}_i$ is taken to be the character of $V$ whose kernel is generated by $e_i$, and writing $e_3 = e_1 e_2$, the cocycle $c$ exhibited in the proof of the lemma yields

$$4F = 1 \otimes 1 - (e_1 \otimes e_1 + e_2 \otimes e_2 + e_3 \otimes e_3)$$
$$+ (1 \otimes e_1 + e_2 \otimes e_1 + e_3 \otimes e_1) + (1 \otimes e_2 + e_2 \otimes e_1 + e_3 \otimes e_1)$$
$$+ (e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_3 - e_3 \otimes e_2 + e_3 \otimes e_1 - e_1 \otimes e_3).$$

By construction $F$ is $A_4$-invariant and maps to $(V, b)$ under the map $\Theta$. \hfill $\square$

7.6. A group considered by G. E. Wall. Our final example will be the group $G$ generated by three generators $s$, $t$, $u$ subject to the relations

$$s^2 = t^2 = u^8 = 1, \quad st = ts, \quad sus^{-1} = u^3, \quad tut^{-1} = u^5.$$  

It is isomorphic to the semi-direct product $\mathbb{Z}/8 \rtimes \text{Aut}(\mathbb{Z}/8)$. 
This group was considered by Wall [32], who proved that Aut_\(\ell\)(G)/\text{Inn}(G) = \mathbb{Z}/2, the non-trivial element being represented by the automorphism \(\alpha\) given by 
\[\alpha(s) = u^4s, \quad \alpha(t) = u^4t, \quad \text{and} \quad \alpha(u) = u.\]

By (2.5) the automorphism \(\alpha\) is the conjugation by an element \(a\) of the normalizer \(N_\ell(G)\) of \(G\) in \(k[G]^\times\). A computer search for \(a\) yielded the element
\[a = \frac{1}{2} (1 + u^4) + \frac{\sqrt{2}}{4} u (1 - u^2 - u^4 + u^6).\]

One checks directly that \(a^2 = 1\) (so \(a\) is invertible) and
\[as = u^4sa, \quad at = u^4ta, \quad au = ua,\]
so that conjugation by \(a\) does indeed induce the automorphism \(\alpha\).

It follows from Proposition 1.7 that \(H_2^\ell(G)\) has a subgroup of order 2, generated by \(F = (a \otimes a) \Delta(a^{-1})\). The symmetric invariant twist \(F\) is determined by
\[8F = 2 (u_{00} + u_{44}) + (u_{11} + u_{33} + u_{55} + u_{77})
+ u_{01} + u_{03} + u_{05} + u_{07} + u_{12} + u_{16} + u_{25} + u_{23} + u_{35} + u_{36} + u_{67}
+ u_{10} + u_{13} + u_{30} + u_{32} + u_{50} + u_{52} + u_{70} + u_{71} + u_{73} + u_{53} + u_{63} + u_{76}
- (u_{13} + u_{14} + u_{15} + u_{16} + u_{23} + u_{27} + u_{34} + u_{37} + u_{45} + u_{47} + u_{56} + u_{57})
- (u_{31} + u_{41} + u_{51} + u_{61} + u_{32} + u_{72} + u_{43} + u_{73} + u_{54} + u_{74} + u_{65} + u_{75}),\]
where we set \(u_{ij} = u^i \otimes u^j\) for \(i, j \in \{0, 1, \ldots, 7\}\).

Let us now describe \(B(G)\). The group \(G\) has only one abelian normal subgroup \(\hat{A}\) of square order, namely the copy of \(\mathbb{Z}/2 \times \mathbb{Z}/2\) generated by \(t\) and \(u^4\). On \(\hat{A}\) there is only one non-degenerate alternating bilinear form \(b\), and it is invariant under all automorphisms of \(\hat{A}\), as already observed a couple of times. As a result, \(B(G)\) has exactly two elements (counting the trivial one).

It follows from Theorem 4.5 and from Int_{\ell}(G)/\text{Inn}(G) = \mathbb{Z}/2 that \(H_2^\ell(G)\) has order 4 or 2 according as \(\Theta\) is surjective or not. In this instance we were not able to conclude. We can at least observe that the action of \(G\) on \(\hat{A}\) is such that Lemma 7.4 applies, from which we deduce that there is no invariant twist \(F \in k[\hat{A}] \otimes k[\hat{A}]\) such that \(\Theta(F) = (A, b)\). However, we do not know whether there is an \(F \in k[G] \otimes k[G]\) with this property.

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**Univerrité de Strasbourg & CNRS, Institut de Recherche Mathématique Avancée,**
7 Rue René Descartes, 67084 Strasbourg, France

*E-mail address*: guillot@math.u-strasbg.fr, kassel@math.u-strasbg.fr

*URL*: www-irma.u-strasbg.fr/~guillot/, www-irma.u-strasbg.fr/~kassel/