Counting statistics for arbitrary cycles in quantum pumps

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Statistics of charge transport in an adiabatic pump are determined by the dynamics of the scattering matrix $S(t)$. We show that, up to an integer offset, the statistics depend only on the corresponding path $\mathbf{N}(t) = S^\dagger \sigma_3 S$ in the coset space (the sphere for a single channel). For a general loop $S(t)$ we solve for the noise-minimizing pumping strategy. The average current is given by the area enclosed by $\mathbf{N}(t)$ in the coset space; its minimal noise by the area of a minimal surface (soap film) spanned by $\mathbf{N}(t)$ in the space of all matrices. We formulate conditions for quantization of the pumped charge.

This result implies that the statistics are determined, up to an integer offset, by the path $\mathbf{N}(t) = S^\dagger \sigma_3 S$ in the (matrix realization of the) coset space $U(2M)/U(M) \times U(M)$. In the single-channel case $\mathbf{N}(t) = \mathbf{n} \sigma$ reduces to a contour $\mathcal{C} = \{\mathbf{n}(t)\}$ on the unit sphere in 3D. We find that at low temperature the average pumped charge is given by the area enclosed by $\mathcal{C}$ on the sphere (cf. Ref. $\text{[10]}$),

$$\langle Q \rangle = \frac{1}{4\pi} A_{\text{sphere}}. \quad (3)$$

This area is defined only modulo 1 since a surface with the edge $\mathcal{C}$ can cover any of two complementary pieces of the sphere or even cover the sphere several times. A way of fixing this integer is discussed below. When the contour $\mathbf{N}(t)$ [but not necessarily $S(t)$] is small, the pumped charge is quantized. Indeed, in Refs. $\text{[3,11]}$ quantization was found under these conditions.

Let us begin by sketching our main results. Our consideration is valid also in the presence of a voltage bias, since the latter can be gauged away at the expense of a phase of the scattering matrix. We will show that the statistics of current are invariant under a local symmetry phase of the scattering matrix. We will show that the latter can be gauged away at the expense of a local integral involving $\mathbf{N}(t)$. We transform it to a local integral over the time disk defined as follows: For the analysis of driving at frequency $\omega$ we can replace the time axis by a unit circle $C_t$: $w = e^{i\omega t}$. There is a unique harmonic ($\Delta \mathbf{n} = 0$) extension of the mapping $\mathbf{n}(w)$ from $C_t$ into the disk $D_t$. The noise is given by the Dirichlet functional,

$$\langle \langle Q^2 \rangle \rangle = \frac{1}{8\pi} \int_{D_t} (\partial_t \mathbf{n})^2 d^2 x, \quad (4)$$

where $w = x_1 + ix_2$. For a given contour $\mathcal{C}$ the details of its traversal in time depend on the pulse shape.
of the driving fields. Unlike the average charge $Q$, the noise value is sensitive to the pulse shape, i.e., to the time parametrization. Optimal pulse shapes minimizing the noise were found in several cases $[14]$. Here we solve the problem of noise optimization for an arbitrary cycle. Specifically, we show that the pumping cycle is optimal when the mapping $\omega(u)$ is conformal, $(\partial_u \omega)^2 = 0$. The minimal noise value is the area of the minimal surface (soap film) spanned by $C$,

$$\langle \langle Q^2 \rangle \rangle_{\min} = \frac{1}{4\pi} A_{\min}. \quad (5)$$

Similar results hold in the many-channel case.

**Invariance.** Physically, multiplication of the scattering matrix $U(t)$ by $U(t)$ just redistributes the scattered particles, without affecting the correlations at the scattering center. The outgoing states acquire an extra time-dependent phase which changes the time these particles need to reach the reservoirs. As a result, the extra charge $W_i \equiv \oint U(t_i) dU_i/4\pi i$ is transferred to the lead $i = \text{L or R}$, and we get $[14]$. For periodic $U(t)$ these numbers are integer. (Note that no net charge accumulation near the scatterer implies $W_L = -W_R$).

More formally, the transformation rule $[14]$ for $\langle Q \rangle$ follows from the Brouwer formula $[14]$. For the higher cumulants one can use the result $[14]$ for the generating function $\chi(\lambda) = \sum Q P(Q)e^{i\lambda Q}$,

$$\chi(\lambda) = \det \left[ 1 + n_F(t', t)(S_{\lambda}^\dagger(t)S_{\lambda}(t) - 1) \right]. \quad (6)$$

Here $S_{\lambda}(t) \equiv e^{-i\lambda/4} S(t)e^{i\lambda/4}$ and $n_F(t', t) = i/[2\pi(t' - t + i0)]$ is the Fourier transform of the Fermi distribution. In Ref. $[14]$ by separating phases and amplitudes of $S(t)$, this result was presented in a form, which immediately implies the transformation rule $[14]$. Indeed, the determinant in Eq. $\langle 5 \rangle$ of Ref. $[14]$ is invariant under $[14]$ and the quantity $\tilde{N}$ in the prefactor is shifted by $W$.

To express $\chi(\lambda)$ via $N$ we notice that $S_{\lambda} = e^{i\lambda S/4} S(t)e^{-i\lambda S/4}$ and $n_F(t', t) = i/[2\pi(t' - t + i0)]$ is the Fourier transform of the Fermi distribution. In Ref. $[14]$ by separating phases and amplitudes of $S(t)$, this result was presented in a form, which immediately implies the transformation rule $[14]$. Indeed, $\chi(\lambda) = \det \left[ 1 + n_F(t', t)(e^{i\lambda/2} - 1)S_{\lambda}(t) - 3 \right]$. (7)

At $T = 0$, multiplying by $1 + n_F(t', t)(e^{i\lambda} - 1)S_{\lambda}(t) - 3 \right]$. (8)

Note that the result $[14]$ is explicitly invariant under global rotations, $N(t) \rightarrow V(N(t))V$ [corresponding to transformations $S(t) \rightarrow S(t)V$].

Eqs. $[14]$, $[15]$, $[16]$ involve $N$, but not $S$, and hence can define $P(\chi)$ only up to an integer offset. Indeed, the infinite product of the eigenvalues of these integral operators can be regularized in many ways $[14]$. Notice that for the operators $[14]$, due to strong degeneracy, one can choose eigenstates that span a narrow frequency range, of order $\omega$. For those far above the Fermi level ($n_F = 0$) the eigenvalues are 1. Deep in the Fermi sea ($n_F = 1$) the eigenvalues appear in pairs $e^{\pm i\lambda/2}$ with the product 1. Although regularization procedures can pair them in different ways, $\chi(\lambda)$ can be changed only by an even power of $e^{\pm i\lambda/2}$, which gives an integer shift of $Q$.

**Pumped charge and the area.** The expression for $\langle Q \rangle = \partial_\lambda \chi(\lambda = 0)$ contains a singularity whose regularization requires the knowledge of the full $S(t)$ and gives an integral over the period $[14]$.\]

$$\langle Q \rangle = \frac{1}{4\pi i} \oint_C \text{Tr}(\sigma_3 dSS') \quad (9)$$

The loop $C = \{S(t)\}$ can be contracted to a point, uniquely up to continuous deformations. In the process the loop spans a surface $D$. (For a two-parametric pump there is a natural choice of $D$ corresponding to the interior of the contour in the parameter plane.) Using Stokes’ theorem, we rewrite $[14]$ as a surface integral, which further reduces to the “area” of the corresponding surface $D$ in the coset space,

$$\langle Q \rangle = \int_D \frac{\text{Tr}(\sigma_3 dS \land dS')}{4\pi i} = \int_D \frac{\text{Tr}(N dN \land dN)}{16\pi i}. \quad (10)$$

Note that the integrand is the curvature of the fiber bundle $S \rightarrow N = S^3 \sigma_3 S$. In the single-channel case $N = n\sigma_3$, and we obtain $[14]$: \[ [14] \]

$$\langle Q \rangle = \frac{1}{8\pi} \int_D \epsilon_{ijk} n_i dn_j \land dn_k. \quad (11)$$

One can try to define the “integral part” $Q_{\text{int}}$ of $\langle Q \rangle$ as follows: Let us parametrize scattering matrices as $S = US^0[N]$ with a matrix $S^0$, defined for any $N$, and a matrix $U$ as in Eq. $[13]$, and assign to each cycle $S(t)$ the winding number of the corresponding $U(t)$. This attempt fails, since there is no continuous global map $S^0[N]$. In fact, any two loops in $U(2M)$ can be deformed into each other, i.e., any $Q_{\text{int}}$ is discontinuous under certain contour deformations. However, continuous maps $S^0[N]$ do exist for contractible regions, and one can introduce $Q_{\text{int}}$ for contours $S(t)$, for which $N(t)$ does not leave such a region. Examples are regions of matrices $S$ without perfectly transmitting (or reflecting) channels. In particular, the integer $\tilde{N}$ introduced in Ref. $[14]$ changes abruptly when the pumping cycle contains a scattering matrix $S(t)$ with a perfectly transmitting channel.

At this point we can formulate sufficient conditions for the quantization of the pumped charge: The fractional part of $\langle Q \rangle$ vanishes for small contours $N(t)$ in the coset space, which remain close to their initial point over the pumping cycle. An example: the minimal and maximal conductance, $g = 0$ and $g = M$, is achieved at the points $N = \pm \sigma_3$. Hence keeping $g$ close to one of these values throughout the cycle guarantees the quantization. We evaluate the integral $[14]$ for such cycles and estimate the accuracy of quantization as

$$\delta Q \lesssim g \quad \text{for } g \approx 0, \quad \delta Q \lesssim M - g \quad \text{for } g \approx M. \quad (12)$$
For a single channel the fractional part is the area \( \{ \} \) within the small contour \( n(t) \).

For a single channel, \( M = 1 \), the scattering matrix can be parametrized by the conductance \( g \) and three phases:

\[
S(g, \alpha, \beta, \phi) = e^{i\phi/2} \left( \begin{array}{cc}
\sqrt{1-g} e^{i\alpha} & i\sqrt{g} e^{-i\beta} \\
i\sqrt{g} e^{i\beta} & \sqrt{1-g} e^{-i\alpha}
\end{array} \right). \tag{13}
\]

The components of the unit vector \( n \) then are

\[ n_z = 1 - 2g; \quad n_x + i n_y = -2i \sqrt{g(1-g)} e^{i(\alpha-\beta)}. \tag{14} \]

Using these expressions we can explain the charge quantization found, for instance, in Refs. \([3,4]\). For the pumping cycles studied the system encircled the resonance point \( g = 1 \) in the parameter plane at a sufficient distance from it so that \( g \approx 0 \) throughout the cycle. The corresponding loop \( n(t) \) encircled the north pole \( (g = 0; \text{Fig. } 5) \). Since the interior of the loop in the parameter plane contained the resonance point, the surface \( D \) in \( (1) \) covered the lower part of the sphere, i.e., almost the whole sphere, \( Q \approx 1 \).

**Noise optimization.** For the noise, given by the \( \chi^2 \) term in the Taylor series of \( \ln \chi(\lambda) \), we obtain a double integral over the unit circles \( w = e^{i\omega t}, w' = e^{i\omega t'} \):

\[
\langle \langle Q^2 \rangle \rangle = \int_{|w|=1} \int_{|w'|=1} \frac{dw \, dw'}{(w-w')^2} \text{Tr} [1 - N(t)N(t')] \tag{15}
\]

Let \( N(t) = \sum_{k \geq 0} N_k \exp(i k \omega t) + \text{h.c.} \). Then the (unique) harmonic extension of the mapping \( w \to N(w) \) from the circle into the disk \( |w| < 1 \) is given by

\[
N(w) = N^+(w) + N^-(w) = \sum_{k \geq 0} N_k w^k + \text{h.c.}. \tag{16}
\]

Expanding the integrand \( \langle \langle Q^2 \rangle \rangle \) near \( t' = t \) one sees that we are justified in replacing the integration over \( w \) by

\[
\int_{|w|=1} \rightarrow \frac{1}{2} \int_{|w|=1-\varepsilon} + \frac{1}{2} \int_{|w|=1+\varepsilon}, \quad \varepsilon \to 0. \tag{17}
\]

Eqs. \( (14), (17) \) and the invariance under time reversal \( w, w' \to w, w' \) allow us to use complex analysis to do the integration over \( w \) in \( (16) \):

\[
\langle \langle Q^2 \rangle \rangle = \text{Tr} \int \frac{N(w)}{16\pi i} \left[ dw \, \partial_{w} N^+(w) - \partial_{w} \partial_{w} N^-(w) \right]. \tag{18}
\]

Finally, Stokes’ theorem gives the integral over the interior \( D_t = \{ w = x_1 + i x_2 : |w| \leq 1 \} \) of the time circle,

\[
\langle \langle Q^2 \rangle \rangle = \frac{1}{16\pi} \int_{D_t} d^2 x \text{Tr} [\partial_t (\partial_t N)^2]. \tag{19}
\]

which reduces to \( \langle \langle Q^2 \rangle \rangle \) in the single-channel case. Note that the functional \( \langle \langle Q^2 \rangle \rangle \) is well-defined for any surface \( N(w) \). The harmonic surface \( \{ \} \) provides the minimal value to \( \langle \langle Q^2 \rangle \rangle \) among the maps \( N(w) \) with the fixed value at the boundary, \( N(t) \).

Now we turn to optimization of pumping: The cyclic evolution of \( S(t) \) is achieved by periodic changes in external parameters that control the scattering or the bias. For a fixed trajectory in the parameter space the rate of motion can be varied. These changes do not affect the average pumped charge \( \langle \langle Q \rangle \rangle \) but do influence the noise \( \langle \langle Q^2 \rangle \rangle \). Notice that \( \frac{1}{2} \text{Tr} [\partial_t (\partial_t N)^2] \geq \text{Tr} [\partial_t (\partial_t N)^2] \cdot \text{Tr} [\partial_t (\partial_t N)^2] \geq \text{area} \) spanned by \( \partial_t N \) and \( \partial_t N \) in the matrix space. Hence the noise value always exceeds the area [defined by the scalar product \( \langle A, B \rangle = \text{Tr}(A^1 B) / 2 \)]

\[
\langle \langle Q^2 \rangle \rangle \geq \frac{1}{4\pi} A[N(w)]. \tag{20}
\]

The equality is achieved only if \( \partial_t N \) are orthogonal, \( \text{Tr} [\partial_t (\partial_t N) \partial_t N] = 0 \), and have the same length, \( \text{Tr} [\partial_t (\partial_t N)^2] = \text{Tr} [\partial_t (\partial_t N)^2] \). These two conditions together can be written as \( \text{Tr} [\partial_t (\partial_t N)^2] = 0 \) and characterize conformal mappings \( N(w) \).

Since any regular surface can be parametrized conformally, the minimal values of both sides of Eq. \( (19) \) coincide. The minimal noise value is thus given by the minimal area \( \{ \} \) of a surface spanned by the loop \( N(t) \). The optimal pumping with this minimal noise value for a given loop is achieved when the corresponding harmonic mapping \( N(w) \) is conformal. Notice that the \( SL_2(\mathbb{R}) \) time reparametrization symmetry, \( \langle \langle Q^2 \rangle \rangle \) \( w \to (w + a)/(1 + \bar{a}w) \), preserves the classes of harmonic and conformal maps. For \( N \) cycles \( N_N(w) \equiv N_1(w^N) \) and \( N_2 \left( \prod_{n=1}^N (w + a_n)/(1 + \bar{a}_n w) \right) \) give the same statistics.

**Applications.** Our findings give a new perspective on the analysis of pumping cycles discussed in the literature. Consider the cycles

\[
S_\beta(t) = e^{-i\phi(t)} s_{\beta/2} S(0) e^{i\phi(t)} s_{\beta/2}, \quad \Delta \phi = 2\pi N, \tag{21}
\]

The first of them, studied extensively by Levitov et al. \([1,4]\), describes conductors under the voltage bias \(-i\phi(t)/\epsilon\), as one can see by applying a gauge transformation. The cycle \( S_\alpha \) was discussed in Ref. \([4]\). These cycles differ only by a transformation \( U(\phi(t)) = e^{i\phi(t)} s_{\beta/2} \), hence the statistics coincide up to a shift by the winding number: \( P_\beta(Q - N) = P_\alpha(Q) \). Indeed, the same pulse shape \( \phi(t) = \omega t \) and others, generated by \( SL_2(\mathbb{R}) \) was found optimal for both cycles. The statistics for this optimal cycle \( S_\alpha(t) \) are related to the well-known binomial distribution for a conductor under a constant positive bias by \( P^{\text{opt}}_\alpha(Q - N) = P^{\text{opt}}_\alpha(Q) \), in agreement with Ref. \([4]\). For a single channel the vector \( n(t) \) follows a line of constant latitude for both \( S_\alpha \) and \( S_\beta \): \( \beta(t) - \beta(0) = \phi(t) \) or \( \alpha(t) - \alpha(0) = \phi(t) \). The pumped charge \( \langle Q \rangle \) is given by the area above this line, \( g \), for \( \langle \langle Q \rangle \rangle \), and below this line, \( 1 - g \), for \( \langle \langle Q \rangle \rangle \). The minimal noise value \( \{ \} \) is the area \( g(1 - g) \) of the sphere’s cross-section. The optimal pumping corresponds to the trivial homotopy of the time disk onto this cross-section, this map being obviously harmonic and conformal.
The rotational invariance of $\chi(\lambda)$ implies that the counting statistics for any circular path $n(t)$ is the same as for biased conductors. In particular, the optimal pumping uniformly traverses the circle and gives rise to a binomial distribution. As an example consider the cycle
\begin{equation}
S_\beta(t) = \left( \frac{\cos \eta(t)}{\sin \eta(t)} \sin \eta(t) - \cos \eta(t) \right), \quad \Delta \eta = 2\pi, \quad (22)
\end{equation}
during which the conductance $g = \sin^2 \eta(t)$ oscillates. We find that $\langle Q \rangle = 0$ and $n(t)$ traverses twice the meridian in Fig. 2. Thus $P(Q)$ coincides with the distribution for the equator, $S_\beta$ at $g = 1/2$, and the pulse $\eta(t) = \omega t$ from Ref. 14 is optimal. For this pulse $P(Q)$ is the shifted binomial distribution for $N = 2$ cycles, $P_{opt}^n(Q) = P_{opt}^n(Q + 1) \left|_{N=2} = \frac{1}{4} \left( \frac{2}{Q+1} \right) \right.$, in agreement with Ref. 14.

Our geometric approach allows us to obtain relations between current and noise for broad classes of pumping cycles. For small loops $n(t)$ the minimal surface lies within the sphere and the analysis simplifies. If the loop $S(t)$ is also small, the system is in the weak pumping regime with $\langle Q \rangle, \langle Q^2 \rangle \ll 1$. In this case we have for a general (possibly self-intersecting) loop $\langle Q \rangle = (A_+ - A_-)/4\pi \leq \langle Q^2 \rangle_{\min} = (A_+ + A_-)/4\pi$, where $A_{\pm}$ are the contributions to the enclosed area with positive (resp. negative) orientations. The weak-pumping regime was studied very recently by Levitov, who found that the transport is described by two uncorrelated Poisson processes (transporting charge to the right and to the left), reducing in some cases to a single process. Our inequality $\langle Q^2 \rangle \geq \langle Q \rangle$ is in agreement with Levitov's findings, with $A_{\pm}/4\pi$ being the rates of the two Poisson processes for an optimal cycle. The equality, the criterion for the reduction to a single Poisson process in the weak-pumping regime, is thus reached only for optimally traversed contours enclosing the area of a constant orientation (in particular, for non-self-intersecting loops). The simplest example of such a cycle considered in Ref. 17 corresponds in our terms to $n(t)$ traversing uniformly a small circle. Generally, for weak harmonic driving $n(t)$ encircles an ellipse. The optimal pulse shape, given by the conformal map of $D_I$ onto this ellipse, involves elliptic integrals. Further, for a general small polygon the optimal pumping is given by the Schwarz-Christoffel formula, describing a map of $D_I$ onto its interior (also reducing to elliptic integrals for a rectangle). The results concerning the weak-pumping regime can be generalized to the many-channel case. Using local complex coordinates in the coset space, we found that the ratio $\langle Q^2 \rangle/\langle Q \rangle \geq 1$, reaching the minimal value unity (corresponding to a single Poisson process) only for optimal cycles with a complex analytic (or antianalytic) minimal surface $N(w)$.

For the strong-pumping regime our description also gives new results. For instance, in the interesting case of a single channel and a contour $n(t)$ without self-intersection, we find that $\langle Q^2 \rangle_{\min} \leq \delta \langle Q \rangle$ to closest integer.

In conclusion, we have shown that the counting statistics of a sample subject to a periodic pumping (and possibly to a voltage bias) are determined by a path in the coset space and given by Eq. (8). The average pumped charge and its minimal variance for an arbitrary pumping cycle are given by the area of encircled by this path in the coset space and the area of the minimal surface spanned by this path, respectively. The optimal pumping strategy can be found as a harmonic conformal map of the time disk onto this minimal surface. Our results represent a unifying framework for analysis of transport statistics in various realizations of pumping.

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