Incomplete Fuzzy Soft Sets and Their Application to Decision-Making

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Abstract: The research of incomplete fuzzy soft sets is of paramount importance in fuzzy soft sets, where the combination of incomplete fuzzy soft set and decision-making problem is of great significance. Incomplete information in fuzzy soft sets leads to more uncertainty and ambiguity in decision-making. The focus of this paper is to propose an algorithm of fuzzy soft set based decision-making problems under incomplete information. On the basis of the weighted function, we introduce the notions of weighted incomplete soft sets and weighted incomplete fuzzy soft sets, and show an approach to weighted incomplete fuzzy soft sets for dealing with decision-making. Considering the missing weight function, the concept of incomplete weighted fuzzy soft sets is presented. Meanwhile, we apply the incomplete weighted fuzzy soft sets to solve the decision-making problem. As modal-style operators for fuzzy soft sets have a precise description of attributes possessed by objects, we apply modal-style operator for incomplete fuzzy soft set to deal with decision-making and propose a new algorithm to make it more accurate and simple.

Keywords: incomplete fuzzy soft set; weighted incomplete fuzzy soft set; incomplete weighted fuzzy soft set; modal-style operator; decision-making

1. Introduction

Classical mathematical tools are not always successful in dealing with complex problems involved in uncertainty, imprecision and vagueness. In 1999, Molodtsov [1] initiated the theory of soft sets as a mathematical tool for dealing with uncertainties, which is not affected by the difficulties of existing methods. The soft set theory is different from traditional tools for dealing with uncertainties, such as fuzzy set theory [2] and rough set theory [3]; it is free from the inadequacy of the parametrization tools of these theories. Subsequently, the generalized models of soft sets (hybrid soft sets) appeared rapidly, and people became more and more interested in the practical applications of hybrid soft set theories [4–6], especially with regard to their applications in decision-making [4,5,7]. Decision making is considered a cognitive-based human activity for selecting the best alternative. The decision-making in a soft environment often requires decision makers to provide evaluation information about the criteria and the alternatives with a hybrid soft set. The combinations of soft sets with generalized fuzzy sets are typical models of hybrid soft sets. In this direction, Maji et al. [8,9] extended the theory of soft sets to the fuzzy soft sets. In terms of fuzzy soft set based decision-making methods, Roy and Maji [10] provided a comparison score based method for decision-making in a fuzzy soft environment. In Reference [11], a computational tool called D-score table is introduced, which improves the traditional decision-making process based on a fuzzy soft set and proves the convenience when attributes change across the decision process. Feng et al. [12] proposed an adjustable approach to a fuzzy soft set on the basis of decision-making by level soft sets. In addition, a novel adjustable approach based on decision rules is presented. Recently, we introduced the notion of modal-style
operator of fuzzy soft sets and put forward a new approach to deal with decision-making based on a modal-style operator of fuzzy soft sets [13].

However, there may be unknown, missing or inexistent data in the process of collecting data. Therefore, standard soft sets (fuzzy soft sets) with incomplete information must be taken into account, which requires the inspection of incomplete soft sets (incomplete fuzzy soft sets). Yan and Zhi [14] studied data analysis methods on the basis of incomplete soft sets environment. The data filling way for incomplete soft sets is presented by Qin et al. [15]. However, the real value of missing data often exists perfect uncertainty in some cases. It seems that we need to avoid estimating the decision-making procedure. With this in mind, Jose Carlos R. A. [16] introduced a new decision standard based soft set under incomplete information. Based on Laplace’s principle of indifference, they supposed that the $*$ of all tables can be replaced with 0 or 1 with equal probability. They browsed all completed forms that are generated from the initial incomplete table. Every case is evaluated by their respective selection values, and we rank the alternatives according to the proportion of the table for which they have obtained the highest selection value. Under the environment of incomplete fuzzy soft sets, Deng and Wang [17] presented an object-parameter way for forecasting the unknown entries of fuzzy soft sets. Based on the similarity measures of fuzzy sets, Liu [18] put forward an existing way for predicting unknown data in an incomplete fuzzy soft set that has a few limitations and introduced a novel adjustable object-parameter method to forecast unknown data in incomplete fuzzy soft sets. So far, most researchers just predict and estimate an incomplete data set in the incomplete fuzzy soft set, and do not combine with the practical decision-making problem. Meanwhile, considering the perfect uncertainty of the real value of missing data, we extend a decision-making problem of the incomplete soft set to incomplete fuzzy soft sets by learning from Jose’s idea. Since the level soft set is the bridge connecting the soft set and fuzzy soft set, we convert the incomplete fuzzy soft set into the incomplete soft set through the level soft set and use Jose’s idea to deal with the decision problem with incomplete information. The weight function of a weighted fuzzy soft set decides the importance of parameters. We combine the weight function and incomplete fuzzy soft set for presenting a new concept called a weighted incomplete fuzzy soft set. Meanwhile, we put forward the algorithm for handing decision-making based on the weighted incomplete fuzzy soft set. Based on the above, we have incomplete fuzzy soft set when we miss data in the information table of fuzzy soft set. What should we do when we miss weighted value of a certain parameter in the information table of weighted incomplete fuzzy soft sets? Considering this point, we put forward a novel notion of incomplete weighted fuzzy soft sets and apply the novel notion to solve decision-making problems. We divide the missing weighted values into intervals, and evaluate them by combining the selected values of each feasible filling table. For the extended decision algorithm for incomplete fuzzy soft sets, we know it is reasonable and good, but the computational process is very complex and tedious. Modal-style operators for fuzzy soft sets have a precise description of attributes possessed by objects. In order to avoid huge computational complexity and improve accuracy of decision-making, we propose a new algorithm based on incomplete fuzzy soft sets. In this algorithm, we apply a modal-style operator for a fuzzy soft set for incomplete information system.

It is worth noting that our target is not to estimate or complete missing data. Instead, we give suggestions on how to make suitable choices when data or weight function are missed in fuzzy soft sets. The content of this paper is organized as follows: Section 2 reviews some fundamental notions of fuzzy sets, soft sets and incomplete fuzzy soft sets. Section 3 presents an algorithm of an incomplete fuzzy soft set for solving decision-making problems. In Section 4, we introduce the definitions of weighted incomplete fuzzy soft set (and weighted incomplete soft set) and show the algorithm of weighted incomplete fuzzy soft set based decision-making. In Section 5, we introduce a novel notion of incomplete weighted fuzzy soft set and apply them to decision-making problems. Section 6 shows a decision-making algorithm for incomplete fuzzy soft set associated with modal-style operator. Finally, our discussion is included in Section 7 and conclusions are included in Section 8.
2. Preliminaries

This section reviewed some fundamental concepts of fuzzy sets, soft sets and incomplete fuzzy soft sets. See especially [1,2,8,19] for further details and background.

Fuzzy set theory initiated by Zadeh [2] offers a suitable framework for representing and processing fuzzy concepts according to permit partial memberships. Let $U$ be a non-empty set, known as universe. A fuzzy set $\theta$ on $U$ is termed to be a membership function $\theta : U \to [0,1]$. For $h \in U$, the membership value $\theta(h)$ actually specifies the degree where $h$ pertains to the fuzzy set $\theta$. In the following, $P(U)$ and $F(U)$ define the family of all subsets of $U$ and the family of all fuzzy sets of $U$, respectively.

In 1999, Molodtsov [1] initiated the definition of soft sets. Assume that $U$ is the universe set and $E$ is the set of parameters associated with $U$ like, attributes, properties, or characteristics of objects in $U$. $(U, E)$ will be known as a soft space. The definition of soft sets is presented as follows:

**Definition 1** ([1]). A pair $(f, S)$ is termed to be a soft set on $U$ if $S \subseteq E$ and $f : S \to P(U)$.

Videlicet, a soft set in Definition 1, is a parameterized family of subsets of universe set. For $s \in S$, $f(s)$ may be regarded as the set of $s$-approximate elements of the soft set $(f, S)$.

Molodtsov [1] noted that Zadeh’s fuzzy set may be regarded as a special case of the soft set. Maji et al. [8] researched hybrid structures with respect to both fuzzy sets and soft sets. The definition of fuzzy soft sets was presented as a fuzzy generalization of soft sets.

**Definition 2** ([8]). Assume that $(U, E)$ is a soft space. A pair $(g, S)$ is termed to be a fuzzy soft set on $U$ if $S \subseteq E$ and $g : S \to F(U)$.

In Definition 2, fuzzy sets on the universe $U$ replace the crisp subsets of $U$. Thus, each soft set may be regarded as a fuzzy soft set.

**Definition 3** ([20]). A pair $(g^{-1}, E)$ is said to be a pseudo fuzzy soft set on $U$, if $g^{-1} : U \to F(E)$ is a mapping from $U$ into the set of all fuzzy subsets of $E$.

Intuitively, a pseudo fuzzy soft set on $U$ is a family of fuzzy subsets of the parameters set $E$. For any $o \in U$, $g^{-1}(o)$ may be regarded as the set of $o$-approximate parameters. $g^{-1}(o)(e)$ is the degree with which $o$ possesses the attribute $e$. Mathematically speaking, the concepts of pseudo fuzzy soft set and fuzzy soft set are equivalent. In fact, if $(g, E)$ is a fuzzy soft set on $U$, $(g^{-1}, E)$ is a pseudo fuzzy soft set on $U$, where $g^{-1} : U \to F(E)$ is given by $g^{-1}(o)(e) = g(e)(o)$ for each $o \in U$ and $e \in E$. Conversely, suppose that $(g^{-1}, E)$ is a pseudo fuzzy soft set on $U$, $(g^{-1}, E)$ determines a fuzzy soft set $(g, E)$ on $U$ by $g(e)(o) = g^{-1}(o)(e)$ for each $o \in U$ and $e \in E$.

Furthermore, Feng [12] presented the notion of level soft sets associated with fuzzy soft sets as follows:

**Definition 4** ([12]). Assume that $(f, S)$ is a fuzzy soft set on the universe $U$, where $s \in S \subseteq E$ and $E$ is the parameter set. For $\kappa \in [0,1]$, the $\kappa$–level soft set of the fuzzy soft set $(f, S)$ is a crisp soft set $(f_\kappa, S)$ defined by

$$f_\kappa(s) = \{ u \in U : f(s)(u) \geq \kappa \}. \quad (1)$$

In Definition 4, the level (or threshold) assigned to every parameter is always the constant value $\kappa \in [0,1]$. However, in dealing with practical problems, the threshold is changed with the actual situation. Hence, Feng preferred to use a function rather than a constant number as the threshold [12] for membership values.
Definition 5 ([12]). Assume that \((f, S)\) is a fuzzy soft set on the universe \(U, s \in S \subseteq E,\) and \(E\) is the parameter set. Suppose that \(\lambda : S \rightarrow [0, 1]\) is a fuzzy set in \(S\) which is said to be a threshold fuzzy set. The level soft set of the fuzzy soft set \((f, S)\) associated with the fuzzy set \(\lambda\) is a crisp soft set \((f_\lambda, S)\) defined by

\[
f_\lambda(s) = \{u \in U : f(s)(u) \geq \lambda(s)\}.
\]

A soft set or a fuzzy soft set can be regarded as an information system or an information table. For the soft set, each entry in this table is 1 or 0 decided by whether an object pertains to the range of a parameter or not. For the fuzzy soft set, every entry pertains to the interval \([0, 1]\) and is determined by the membership degree of an object on a parameter. Assume that the domain of every soft set or fuzzy soft set is \(U = \{v_1, v_2, \ldots, v_m\}\) and the parameters set is \(S = \{s_1, s_2, \ldots, s_n\}\). If there exists incomplete data in the information table of a soft set or fuzzy soft set, respectively, the soft set or fuzzy soft set is termed to be an incomplete soft set or incomplete fuzzy soft set, where the unknown data is represented by “*”. For instance, Tables 1 and 2 [18] represent an incomplete soft set \((g, S)\) and an incomplete fuzzy soft set \((f, S)\), respectively. In Table 1, we know all membership values of objects on parameters except \(v_2\) on \(s_1\) and \(v_3\) on \(s_2\). The unknown data are expressed by “*” in the information table, namely, \(g(s_1)(v_2) = *\) and \(g(s_2)(v_3) = *\). Similarly, unknown data of fuzzy soft set are \(v_2\) on \(s_2\) and \(v_3\) on \(s_1\) in Table 2, which is \(f(s_2)(v_2) = *\) and \(f(s_2)(v_3) = *\).

**Table 1.** Incomplete soft set \((g, S)\).

|   | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \(s_5\) | \(s_6\) |
|---|---------|---------|---------|---------|---------|---------|
| \(v_1\) | 1       | 1       | 0       | 0       | 0       | 0       |
| \(v_2\) | *       | 0       | 1       | 0       | 1       | 0       |
| \(v_3\) | 1       | *       | 1       | 0       | 0       | 1       |
| \(v_4\) | 0       | 0       | 0       | 1       | 0       | 1       |

**Table 2.** Incomplete fuzzy soft set \((f, S)\).

|   | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \(s_5\) | \(s_6\) |
|---|---------|---------|---------|---------|---------|---------|
| \(v_1\) | 0.9     | 0.4     | 0.5     | 0.4     | 0.8     | 0.8     |
| \(v_2\) | 0.8     | *       | 0.5     | 0.7     | 0.6     | 0.3     |
| \(v_3\) | 0.4     | *       | 0.9     | 0.9     | 0.5     | 0.9     |
| \(v_4\) | 0.9     | 0.8     | 0.9     | 0.4     | 0.7     | 0.5     |

3. Incomplete Fuzzy Soft Set Based Decision-Making

Fuzzy soft set based on complete information has been widely applied to solving a decision-making problem. However, incomplete data sets are widely used in the real world. A small error during the measurement process, limitation on collecting data, errors in data understanding and other factors may directly lead to missing data. Recently, many scholars presented some algorithms to predicting unknown data in an incomplete soft set. If we apply the predicted results to deal with the decision-making problem, decision results must have certain errors. In dealing with decision-making problem, Alcantud [19] presented an algorithm that give advice about where choices should be made when data are missing in the context of soft sets instead of filling any missing data. Similarly, we find that scholars only predict any missing data in the incomplete fuzzy soft set. However, there exist certain errors in the predicted data when we solve practice problems. Thus, we should consider giving advice on which choice should be made when data are missing in the context of fuzzy soft sets. Thus, we extend this idea to incomplete fuzzy soft sets based decision-making and present an algorithm in an incomplete fuzzy soft set. Assume that \(U\) is an initial universe set and \(E\) is the set of parameters associated with \(U\).

In the above algorithm, we provide suggestions which should be selected when data are missing in fuzzy soft sets instead of filling missing data with the predicted data. We evaluate every feasible
plan by the choice value, and then give alternative options according to the proportion of the table that has the highest choice value. Due to a level soft set bridge, the gap between fuzzy soft sets and crisp soft sets, we convert the fuzzy soft set into a crisp soft set by using the level soft set, that is to say, we do not use a fuzzy soft set directly to dealing with a decision-making problem but to use a crisp soft set derived from them after selecting certain thresholds. Based on a certain threshold, every membership value of objects on parameters convert 0 or 1, where missing membership value of objects on parameters turn into “*”, which are replaced with either 0 or 1. In addition, 0 and 1 own the same probability. We should calculate the choice value of each object in each case for the level soft set. Then, we get the object that has the highest percentage of cases with complete information. For illustrating the basic ideal of Algorithm 1, we provide an illustrate example, where an incomplete fuzzy soft set is obtained by deleting some data of fuzzy soft set in [11].

Algorithm 1 for incomplete fuzzy soft set based decision making:

Step 1: Consider the incomplete fuzzy soft set \( (f, E) \), where \( u \in U, e \in E \).

Step 2: Give a threshold fuzzy set \( E : S \rightarrow [0, 1] \) and compute the level soft set \( (\lambda, E) \) with respect to the threshold fuzzy set \( \lambda \).

Step 3: Present the level soft set \( (\lambda, E) \) in tabular form, where missing data are still denoted by the sign “*”.

Step 4: Compute the choice value \( c_i \) of \( u_i \) according to \( (\lambda, E) \), where \( c_i = \sum_j f_{\lambda}(u_i)(e_j) \).

Step 5: In the modified \( s \times t \) matrix, list the cells with value \( * \) as \( ((u_{i1}, e_{j1}), \ldots, (u_{iw}, e_{jw})) \).

Step 6: By every vector \( \alpha = (\alpha_1, \ldots, \alpha_w) \in \{0, 1\}^w \), we construct a \( s \times t \) matrix \( P_\alpha = (p_{ui, ej})_{s \times t} \) where

1. \( p_{ui, ej} = f_{\lambda}^{-1}(u_i)(e_j) \) if \( (u_i, e_j) \) is not listed in \( ((u_{i1}, e_{j1}), \ldots, (u_{iw}, e_{jw})) \), where \( f_{\lambda}^{-1}(u_i)(e_j) = \lambda(e_j) \).
2. \( p_{ui, ej} = \alpha_v \) if \( (u_i, e_j) = (u_{iv}, e_{vj}), v \in \{1, 2, \ldots, w\} \).

Step 7: For every \( u_i \), let \( n_{ui} \) regarded as the number of vectors \( \alpha = (\alpha_1, \ldots, \alpha_w) \in \{0, 1\}^w \) for which object \( u_i \) maximizes the choice value at \( P_\alpha \). Let \( o_{ui} = n_{ui}/2^w \). Define \( o_{ui} = 0 \) for dominated alternatives.

Step 8: The optimal decision is to choose \( u_i \) satisfying \( o_{ui} = \max_{i=1}^{n} o_{ui} \).

Step 9: If \( l \) has multiple values, then any one may be selected.

Example 1. Consider the incomplete fuzzy soft set \( (f, S) \) given in Table 3, where \( U = \{v_1, v_2, v_3, v_4, v_5, v_6\} \) is a universe of houses and \( S = \{s_1, s_2, s_3, s_4, s_5\} \) is the set of parameters. Mr. X (say) has interest in buying a house. \( s_1, s_2, s_3, s_4 \) and \( s_5 \) stand for “expensive”, “beautiful”, “wooden”, “cheap” and “in the surroundings”, respectively. Namely, Mr. X wants to buy houses that extremely conform to the attributes in \( S \).

In the incomplete fuzzy soft set \( (f, S) \), we miss some data, namely, \( f(s_1)(v_2) = *, f(s_2)(v_3) = * \) and \( f(s_4)(v_8) = * \). For simplicity, we regard 0.5-level soft set as the threshold fuzzy set and get the 0.5-level soft set \( (f_{0.5}, S) \) in Table 4 where unknown data of the 0.5-level soft set still are denoted by “*”. By Steps 4 and 5, we remove 1, 4 row and get a new \( 4 \times 5 \) matrix in Table 5. It is clear that we get \( w = 3 \) and list the cells with value \( * \) as \( ((v_{21}, s_{11}), (v_{32}, s_{22}), (v_{51}, s_{41})) \). For every \( \alpha \in \{0, 1\}^w \), we construct a \( 8 \times 5 \) matrices \( P_\alpha \) in Tables 6–13, that is \( P_{a_1}, P_{a_2}, P_{a_3}, P_{a_4}, P_{a_5}, P_{a_6}, P_{a_7}, P_{a_8} \). Meanwhile, we also get the choice values of the houses in the each table of Tables 6–13. According to the proportion of the table in which have the highest choice value, we observe that \( v_3 \) owns the highest choice value in all eight tables, that is, \( n_{v_3} = 8 \). \( v_3 \) has the highest choice value at \( P_{a_5} \) and \( P_{a_7} \), that is, \( n_{v_3} = 2 \). \( v_5 \) has the highest choice value at \( P_{a_4} \) and \( P_{a_7} \), that is, \( n_{v_5} = 2 \). \( v_6 \) has the highest choice value at \( P_{a_1}, P_{a_3}, P_{a_4} \) and \( P_{a_7} \), that is, \( n_{v_6} = 4 \). Hence, we have \( o_{v_2} = 8/2^3 = 1, o_{v_3} = 2/2^3 = 0.25, o_{v_5} = 2/2^3 = 0.25 \) and \( o_{v_6} = 4/2^3 = 0.5 \). The dominated objects are denoted by \( o_{v_1} = 0 \). It is obvious that the maximum value is \( o_{v_2} \) and so the optimal decision is to choose \( v_2 \).
Table 3. Incomplete fuzzy soft set \((f, S)\).

|   | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \(s_5\) |
|---|---------|---------|---------|---------|---------|
| \(v_1\) | 0.1     | 0.5     | 0.3     | 0.4     | 0.3     |
| \(v_2\) | *       | 0.5     | 0.2     | 0.3     | 0.6     |
| \(v_3\) | 0.1     | *       | 0.4     | 0.5     | 0.1     |
| \(v_4\) | 0.7     | 0.2     | 0.2     | 0.2     | 0.3     |
| \(v_5\) | 0.2     | 0.6     | 0.3     | *       | 0.3     |
| \(v_6\) | 0.9     | 0.2     | 0.1     | 0.1     | 0.8     |

Table 4. The 0.5-level soft set \((f_{0.5}, S)\).

|   | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \(s_5\) |
|---|---------|---------|---------|---------|---------|
| \(v_1\) | 0       | 1       | 0       | 0       | 0       |
| \(v_2\) | *       | 1       | 0       | 0       | 1       |
| \(v_3\) | 0       | *       | 0       | 1       | 0       |
| \(v_4\) | 1       | 0       | 0       | 0       | 0       |
| \(v_5\) | 0       | 1       | 0       | *       | 0       |
| \(v_6\) | 1       | 0       | 0       | 0       | 1       |

Table 5. Incomplete soft set \((f'_{0.5}, S)\).

|   | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \(s_5\) |
|---|---------|---------|---------|---------|---------|
| \(v_2\) | *       | 1       | 0       | 0       | 1       |
| \(v_3\) | 0       | *       | 0       | 1       | 0       |
| \(v_5\) | 0       | 1       | 0       | *       | 0       |
| \(v_6\) | 1       | 0       | 0       | 0       | 1       |

Table 6. \(P_{a_1}\) matrix.

|   | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \(s_5\) | \(c_i\) |
|---|---------|---------|---------|---------|---------|---------|
| \(v_2\) | 0       | 1       | 0       | 0       | 1       | 2       |
| \(v_3\) | 0       | 0       | 0       | 1       | 0       | 1       |
| \(v_5\) | 0       | 1       | 0       | 0       | 0       | 1       |
| \(v_6\) | 1       | 0       | 0       | 0       | 1       | 2       |

Table 7. \(P_{a_2}\) matrix.

|   | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \(s_5\) | \(c_i\) |
|---|---------|---------|---------|---------|---------|---------|
| \(v_2\) | 1       | 1       | 0       | 0       | 1       | 3       |
| \(v_3\) | 0       | 0       | 0       | 1       | 0       | 1       |
| \(v_5\) | 0       | 1       | 0       | 0       | 0       | 1       |
| \(v_6\) | 1       | 0       | 0       | 0       | 1       | 2       |

Table 8. \(P_{a_3}\) matrix.

|   | \(s_1\) | \(s_2\) | \(s_3\) | \(s_4\) | \(s_5\) | \(c_i\) |
|---|---------|---------|---------|---------|---------|---------|
| \(v_2\) | 0       | 1       | 0       | 0       | 1       | 2       |
| \(v_3\) | 0       | 1       | 0       | 1       | 0       | 2       |
| \(v_5\) | 0       | 1       | 0       | 0       | 0       | 1       |
| \(v_6\) | 1       | 0       | 0       | 0       | 1       | 2       |
4. Weighted Incomplete Fuzzy Soft Sets Based Decision Making

In 1996, Lin [21] defined a new theory of mathematical analysis, denoted by the theory of weighted soft sets. Maji et al. [22] presented the table of a weighted soft set by Lin’s idea. Afterwards, Feng [12] introduced the definition of weighted fuzzy soft sets and applied weighted fuzzy soft set to solve decision-making problems. The concept of weighted fuzzy soft sets gives a mathematical frame for modeling and analyzing decision-making problems, in which all the selected parameters may be unequally important.

**Definition 6 ([12]).** Let \((U, E)\) be a soft space, \(S \subseteq E\). If \((f, S)\) is a fuzzy soft set on \(U\) and \(W : S \rightarrow [0, 1]\) is a weight function, where the weight is \(W_i = W(s_j)\) for every \(s_j \in S\), the fuzzy soft set is termed to be weighted fuzzy soft set, which is expressed by \((f, S, W)\).

Note that the weight function in the weighted fuzzy soft set can be used as a threshold fuzzy set, that is, the level soft set can be considered.
Now, we combine incomplete fuzzy soft set with a weighted function to generate a new definition called a weighted incomplete fuzzy soft set as follows:

**Definition 7.** If there is incomplete data in the information table of a weighted fuzzy soft set (weighted function is not missed), the weighted fuzzy soft set is said to be a weighted incomplete fuzzy soft set, and the unknown data are marked as the sign “∗”.

If we replace weighted fuzzy soft set with weighted soft set in Definition 7, we get the definition of a weighted incomplete soft set. The weighted soft set is said to be a weighted incomplete soft set if there is incomplete data in the information table of a weighted soft set (weighted function is not missed), where the unknown data are marked as the sign “∗”. From a mathematical point of view, the level soft set with unknown data is equivalent to the weighted incomplete soft set. The weight function of a weighted incomplete fuzzy soft set decides the importance of parameters. By revising Algorithm 1, we get Algorithm 2 of weighted incomplete fuzzy soft set based on decision-making problems. In Algorithm 2, we compute the weighted choice value by the weights of parameters where the weight function can be considered as a threshold fuzzy set. In real life, one has different standards for different things. Thus, Algorithm 2 can better deal with decision-making problem.

**Algorithm 2** for weighted incomplete fuzzy soft set based decision-making:

**Step 1:** Consider the weighted incomplete fuzzy soft set (\(f, E, W\)), where \(u \in U, e \in E\).

**Step 2:** Give a threshold fuzzy set \(\lambda : E \to [0, 1]\) for decision-making and compute the level soft set \((f_\lambda, E, W)\) with respect to the threshold fuzzy set \(\lambda\).

**Step 3:** Present the level soft set \((f_\lambda, E, W)\) in tabular form, where missing data are still denoted by the sign “∗”.

**Step 4:** Compute the weighted choice value \(c_i\) of \(u_i\) according to \((f_\lambda, E, W)\), where \(c_i = \sum (f_\lambda(u_i)(e_j) \times W_j)\).

Remove each row with the smallest weighted choice value \(c_i\) (whether ∗ takes 0 or 1, the choice value \(c_i\) is the smallest). The number of remaining objects is denoted by \(s\), and the number of attributes from \(E\) is denoted by \(t\). The reduced weighted incomplete soft set is denoted by \((f'_\lambda, E, W)\).

**Step 5:** In the modified \(s \times t\) matrix, list the cells with value ∗ as \(((u_{i1}, e_{j1}), \ldots, (u_{iw}, e_{jw}))\).

**Step 6:** By every vector \(\alpha = (\alpha_1, \ldots, \alpha_w) \in \{0, 1\}^w\), we construct a \(s \times t\) weighted matrix \(P_\alpha = (p_{u_i e_j})_{s \times t}\) where

\[
\begin{align*}
(1) \quad p_{u_i e_j} &= f_\lambda^{-1}(u_i)(e_j) \text{ if } (u_i, e_j) \text{ is not listed in } ((u_{i1}, e_{j1}), \ldots, (u_{iw}, e_{jw})), \quad f_\lambda^{-1}(u_i)(e_j) = f_\lambda(e_j)(u_i). \\
(2) \quad p_{u_i e_j} &= \alpha_v \text{ if } (u_i, e_j) = (u_{iv}, e_{jv}), v \in \{1, 2, \ldots, w\}.
\end{align*}
\]

**Step 7:** For every \(u_i\), let \(n_{u_i}\) be regarded as the number of vectors \(\alpha = (\alpha_1, \ldots, \alpha_w) \in \{0, 1\}^w\) for which object \(u_i\) maximizes the weighted choice value at \(P_\alpha\). Let \(o_{u_i} = n_{u_i}/2^w\). Define \(o_{u_i} = 0\) for dominated alternatives.

**Step 8:** The optimal decision is to choose \(u_i\) if \(o_{u_i} = \max_{i=1,\ldots,p} o_{u_i}\).

**Step 9:** If \(I\) has multiple values, then any one may be selected.

To illustrate the above idea, we reconsider Example 1 presented in Section 3. Suppose that Mr. X imposes the following weights on the parameters \(S\): \(W_1 = 0.9\) for \(s_1\), \(W_2 = 0.6\) for \(s_2\), \(W_3 = 0.6\) for \(s_3\), \(W_4 = 0.7\) for \(s_4\) and \(W_5 = 0.5\) for \(s_5\). We have Table 14 of weighted incomplete fuzzy soft set. For simplicity, we still regard 0.5-level soft set as the threshold fuzzy set and get the 0.5-level soft set \((f_{0.5}, S, W)\) in Table 15 where unknown data of the 0.5-level soft set are still denoted by “∗”.

Similar to Algorithm 3, we remove 1 row and get a new \(5 \times 5\) reduction matrix in Table 16. Meanwhile, we construct eight \(5 \times 5\) matrices \(P_i\) in Tables 17–24, that is, \(P_{a_1}, P_{a_2}, P_{a_3}, P_{a_4}, P_{a_5}, P_{a_6}, P_{a_7}\). We also get the weighted choice values of the houses in each table of Tables 17–24. We observe that \(v_2\) owns the highest choice value at \(P_{a_2}, P_{a_5}, P_{a_6}\) and \(P_{a_7}\), that is, \(n_{v_2} = 4\). \(h_6\) has the highest choice value at \(P_{a_1}, P_{a_5}, P_{a_4}\) and \(P_{a_7}\), that is, \(n_{h_6} = 4\). \(v_3, v_4\) and \(v_5\) do not have the highest choice value, that is,
$n_{v2} = n_{v4} = n_{v5} = 0$. Hence, we have $o_{v2} = o_{v6} = 4/2^3 = 0.5$, $o_{v3} = o_{v4} = o_{v5} = 0$. The dominated objects are denoted by $o_{v1} = 0$. It is obvious that maximum values are $o_{v2}$ and $o_{v6}$, and so $v_2$ and $v_6$ are selected as the optimal decision.

Table 14. Weighted incomplete fuzzy soft set $(f, S, W)$.

| $s_1$, $W_1 = 0.9$ | $s_2$, $W_2 = 0.6$ | $s_3$, $W_3 = 0.6$ | $s_4$, $W_4 = 0.7$ | $s_5$, $W_5 = 0.5$ |
|---------------------|---------------------|---------------------|---------------------|---------------------|
| $v_1$               | 0.1                 | 0.5                 | 0.3                 | 0.4                 | 0.3                 |
| $v_2$               | $*$                 | 0.5                 | 0.2                 | 0.3                 | 0.6                 |
| $v_3$               | 0.1                 | $*$                 | 0.4                 | 0.5                 | 0.1                 |
| $v_4$               | 0.7                 | 0.2                 | 0.2                 | 0.2                 | 0.3                 |
| $v_5$               | 0.2                 | 0.6                 | 0.3                 | $*$                 | 0.3                 |
| $v_6$               | 0.9                 | 0.2                 | 0.1                 | 0.1                 | 0.8                 |

Table 15. The 0.5-level soft set $(f_{0.5}, S, w)$.

| $s_1$, $W_1 = 0.9$ | $s_2$, $W_2 = 0.6$ | $s_3$, $W_3 = 0.6$ | $s_4$, $W_4 = 0.7$ | $s_5$, $W_5 = 0.5$ |
|---------------------|---------------------|---------------------|---------------------|---------------------|
| $v_1$               | 0                   | 1                   | 0                   | 0                   |
| $v_2$               | $*$                 | 1                   | 0                   | 0                   | 1                   |
| $v_3$               | 0                   | $*$                 | 0                   | 1                   | 0                   |
| $v_4$               | 1                   | 0                   | 0                   | 0                   | 0                   |
| $v_5$               | 0                   | 1                   | 0                   | $*$                 | 0                   |
| $v_6$               | 1                   | 0                   | 0                   | 0                   | 1                   |

Table 16. Weighted incomplete reduction matrix.

| $s_1$, $W_1 = 0.9$ | $s_2$, $W_2 = 0.6$ | $s_3$, $W_3 = 0.6$ | $s_4$, $W_4 = 0.7$ | $s_5$, $W_5 = 0.5$ |
|---------------------|---------------------|---------------------|---------------------|---------------------|
| $v_2$               | $*$                 | 1                   | 0                   | 0                   | 1                   |
| $v_3$               | 0                   | $*$                 | 0                   | 1                   | 0                   |
| $v_4$               | 1                   | 0                   | 0                   | 0                   | 0                   |
| $v_5$               | 0                   | 1                   | 0                   | $*$                 | 0                   |
| $v_6$               | 1                   | 0                   | 0                   | 0                   | 1                   |

Table 17. $P_{a_1}$ matrix.

| $s_1$, $W_1 = 0.9$ | $s_2$, $W_2 = 0.6$ | $s_3$, $W_3 = 0.6$ | $s_4$, $W_4 = 0.7$ | $s_5$, $W_5 = 0.5$ | $c_i$ |
|---------------------|---------------------|---------------------|---------------------|---------------------|-------|
| $v_2$               | 0                   | 1                   | 0                   | 0                   | 1     | 1.1   |
| $v_3$               | 0                   | 0                   | 0                   | 1                   | 0     | 0.7   |
| $v_4$               | 1                   | 0                   | 0                   | 0                   | 0     | 0.9   |
| $v_5$               | 0                   | 1                   | 0                   | 0                   | 0     | 0.6   |
| $v_6$               | 1                   | 0                   | 0                   | 0                   | 1     | 1.4   |

Table 18. $P_{a_2}$ matrix.

| $s_1$, $W_1 = 0.9$ | $s_2$, $W_2 = 0.6$ | $s_3$, $W_3 = 0.6$ | $s_4$, $W_4 = 0.7$ | $s_5$, $W_5 = 0.5$ | $c_i$ |
|---------------------|---------------------|---------------------|---------------------|---------------------|-------|
| $v_2$               | 1                   | 1                   | 0                   | 0                   | 1     | 2     |
| $v_3$               | 0                   | 0                   | 0                   | 1                   | 0     | 0.7   |
| $v_4$               | 1                   | 0                   | 0                   | 0                   | 0     | 0.9   |
| $v_5$               | 0                   | 1                   | 0                   | 0                   | 0     | 0.6   |
| $v_6$               | 1                   | 0                   | 0                   | 0                   | 1     | 1.4   |
Table 19. $P_{a_5}$ matrix.

| $c_i$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|-------|-------|-------|-------|-------|-------|
|       | 0     | 0     | 1     | 0     | 1     |
|       | 1     | 1     | 0     | 0     | 0     |
|       | 0     | 0     | 1     | 0     | 0     |
|       | 0     | 0     | 0     | 0     | 1     |
|       | 1     | 0     | 0     | 0     | 1     |

Table 20. $P_{a_6}$ matrix.

| $c_i$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|-------|-------|-------|-------|-------|-------|
|       | 0     | 0     | 1     | 0     | 1     |
|       | 1     | 1     | 0     | 0     | 0     |
|       | 0     | 0     | 1     | 0     | 0     |
|       | 0     | 0     | 0     | 0     | 1     |
|       | 1     | 0     | 0     | 0     | 1     |

Table 21. $P_{a_7}$ matrix.

| $c_i$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|-------|-------|-------|-------|-------|-------|
|       | 1     | 1     | 0     | 0     | 0     |
|       | 1     | 0     | 0     | 0     | 0     |
|       | 0     | 1     | 0     | 0     | 0     |
|       | 1     | 0     | 0     | 0     | 1     |
|       | 0     | 0     | 0     | 0     | 1     |

Table 22. $P_{a_8}$ matrix.

| $c_i$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|-------|-------|-------|-------|-------|-------|
|       | 0     | 0     | 1     | 0     | 0     |
|       | 1     | 0     | 0     | 0     | 0     |
|       | 0     | 1     | 0     | 0     | 0     |
|       | 0     | 0     | 0     | 0     | 1     |
|       | 1     | 0     | 0     | 0     | 1     |

Table 23. $P_{a_9}$ matrix.

| $c_i$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|-------|-------|-------|-------|-------|-------|
|       | 0     | 0     | 1     | 0     | 0     |
|       | 0     | 1     | 0     | 0     | 0     |
|       | 1     | 0     | 0     | 0     | 0     |
|       | 1     | 0     | 0     | 0     | 1     |
|       | 0     | 0     | 0     | 0     | 1     |

Table 24. $P_{a_{10}}$ matrix.

| $c_i$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ |
|-------|-------|-------|-------|-------|-------|
|       | 0     | 0     | 1     | 0     | 0     |
|       | 0     | 1     | 0     | 0     | 0     |
|       | 1     | 0     | 0     | 0     | 0     |
|       | 0     | 0     | 0     | 0     | 1     |
|       | 0     | 0     | 0     | 0     | 1     |
5. Incomplete Weighted Fuzzy Soft Sets Based Decision Making

In the above sections, we discuss the notions of incomplete fuzzy soft sets and weighted incomplete fuzzy soft sets, and apply them to solve decision-making problems. In the incomplete fuzzy soft set, there exist unknown data in the information table of fuzzy soft set. Weighted incomplete fuzzy soft sets are obtained by introducing weighting functions into incomplete fuzzy soft sets. It is normal that we miss some data for some reason in real life. When we miss data in the information table of a fuzzy soft set, we have an incomplete fuzzy soft set. What should we do when we miss a weighted value of a certain parameter in the information table of weighted incomplete fuzzy soft sets? For resolving the problem, we will introduce a new concept called incomplete weighted fuzzy soft sets.

Definition 8. If there exist incomplete data in the weight function for a weighted incomplete fuzzy soft set, the incomplete fuzzy soft set is said to be an incomplete weighted fuzzy soft set, and unknown weighted values are marked as the sign “⋆”.

Similarly, the incomplete soft set is said to be an incomplete weighted soft set if incomplete data exist in the weight function for weighted incomplete soft set, where unknown weighted value are marked as the sign “⋆”. In order to understand Definition 8, we consider incomplete fuzzy soft sets in Table 14 and attach incomplete weight functions on the incomplete fuzzy soft set. All weight values on parameters are known except those of ˜α. Similarly, the incomplete soft set is said to be an incomplete weighted soft set if incomplete data exist in the weight function for incomplete soft set. All weight values on parameters are known except those of ˜α. Thus, we get five matrices $W^1$, $W^2$, $W^3$, $W^4$, $W^5$ by using five intervals.

To illustrate the basic thought of Algorithm 3, we use Algorithm 2’s example where all weight values on parameters are known except those of ˜W2 in Table 25. Due to the similarity of Algorithm 2, we directly get the incomplete weighted reduction matrix Table 26 by step 2 to step 5 and compute the incomplete weighted choice value of each $P_k$ by Tables 17–24 where ˜W2 is unknown, that is, for $P_{a_1}$, $c_{a_1} = \{0.5 + *, 0.7, 0.9, *, 1.4\}$, for $P_{a_2}$, $c_{a_2} = \{1.4 + *, 0.7, 0.9, *, 1.4\}$, for $P_{a_3}$, $c_{a_3} = \{0.5 + *, 0.7 + *, 0.9, *, 1.4\}$, for $P_{a_4}$, $c_{a_4} = \{0.5 + *, 0.7, 0.9, 0.7 + *, 1.4\}$, for $P_{a_5}$, $c_{a_5} = \{1.4 + *, 0.7 + *, 0.9, *, 1.4\}$, for $P_{a_6}$, $c_{a_6} = \{1.4 + *, 0.7, 0.9, 0.7 + *, 1.4\}$, $P_{a_7}$, $c_{a_7} = \{0.5 + *, 0.7 + *, 0.9, 0.7 + *, 1.4\}$, $P_{a_8}$, $c_{a_8} = \{1.4 + *, 0.7 + *, 0.9, 0.7 + *, 1.4\}$. Thus, 5 × 8 matrix $P$ is represented as Table 27. For a column $c_{a_1}$ in Table 27, the maximum choice value $C^{'a_1}$ without “⋆” is 1.4 and the maximum choice value $C_{a_1}^{'a_1}$ with “⋆” is 0.5 + *, and then $C^{'a_1} = 0.9$. Similarly, we get $C^{'a_2} = 0.7$. Furthermore, it is divided into five intervals: $0 \leq * \leq 0.7$, $0.7 < * < 0.9$, $0.9 < * \leq 1$, $* = 0.7$, $* = 0.9$. For $0 \leq * \leq 0.7$, $n_{a_1}^1 = n_{a_2}^1 = n_{a_3}^1 = 4$, $n_{a_4}^1 = n_{a_5}^1 = n_{a_6}^1 = 0$. For $0.7 < * < 0.9$, $n_{a_2}^2 = 4$, $n_{a_3}^2 = n_{a_4}^2 = 2$, and $n_{a_5}^2 = n_{a_6}^2 = 0$. For $0.9 < * \leq 1$, $n_{a_3}^3 = 5$, $n_{a_4}^3 = n_{a_5}^3 = n_{a_6}^3 = 2$, and $n_{a_2}^4 = n_{a_6}^4 = 0$. For $* = 0.7$, $n_{a_2}^4 = n_{a_6}^4 = 4$, $n_{a_3}^4 = n_{a_5}^4 = 2$, and $n_{a_4}^4 = 0$. For $* = 0.9$, $n_{a_2}^5 = 5$, $n_{a_3}^5 = n_{a_4}^5 = n_{a_5}^5 = 2$, $n_{a_6}^5 = 1$, and $n_{a_7}^5 = 0$. In order to facilitate the observation, the results are represented in Table 28. It is clear that we select $v_2$ and $v_6$ in $0 \leq * < 0.7$. In $0.7 < * < 0.9$, $0.9 < * \leq 1$ and $* = 0.9$, 0.7 < * < 0.9, 0.9 < * \leq 1, * = 0.7, * = 0.9. For 0 \leq * \leq 0.7, n_{a_1}^1 = n_{a_2}^1 = n_{a_3}^1 = 4, n_{a_4}^1 = n_{a_5}^1 = n_{a_6}^1 = 0. For 0.7 < * < 0.9, n_{a_2}^2 = 4, n_{a_3}^2 = n_{a_4}^2 = 2, and n_{a_5}^2 = n_{a_6}^2 = 0. For 0.9 < * \leq 1, n_{a_3}^3 = 5, n_{a_4}^3 = n_{a_5}^3 = n_{a_6}^3 = 2, and n_{a_2}^4 = n_{a_6}^4 = 0. For * = 0.7, n_{a_2}^4 = n_{a_6}^4 = 4, n_{a_3}^4 = n_{a_5}^4 = 2, and n_{a_4}^4 = 0. For * = 0.9, n_{a_2}^5 = 5, n_{a_3}^5 = n_{a_4}^5 = n_{a_5}^5 = 2, n_{a_6}^5 = 1, and n_{a_7}^5 = 0. In order to facilitate the observation, the results are represented in Table 28.
we select \( v_2 \). In \( * = 0.7 \), we select \( v_2 \) and \( v_6 \). By summarizing and merging, we choose \( v_2 \) and \( v_6 \) in \( 0 \leq * \leq 0.7 \) and choose \( v_2 \) in \( 0.7 < * \leq 1 \). Thus, the corresponding optimal decision is to select \( v_2 \) since \( v_2 \) is a common object in two intervals.

Algorithm 3 for incomplete weighted fuzzy soft set based decision-making:

**Step 1:** Considering incomplete weighted fuzzy soft set \((f, E, \overline{W})\), where \( u \in U, e \in E \).

**Step 2:** Give a threshold fuzzy set \( \lambda : E \rightarrow [0, 1] \) for decision-making and compute the level soft set \((f_\lambda, E, \overline{W})\) with respect to the threshold fuzzy set \( \lambda \).

**Step 3:** Present the level soft set \((f_\lambda, E)\) in tabular form, where missing data \((f_\lambda(u_i)(e_j))\) are still denoted by the sign \( \ast \) and missing weighted values are still marked as the sign \( \ast \).

**Step 4:** Compute the incomplete weighted choice value \( c_i \) of \( u_i \) according to \((f_\lambda, E, \overline{W})\), where \( c_i = \sum_j (f_\lambda(u_i)(e_j)) \times W_j \).

Remove each row with the smallest incomplete weighted choice value \( c_i \). (Whether \( \ast \) and \( \ast \) takes 0 or 1, the choice value \( c_i \) is the smallest). The number of remaining objects is denoted by \( s \), and the number of attributes from \( E \) is denoted by \( t \). The reduced incomplete weighted soft set is denoted by \((f_\lambda, E, \overline{W})\).

**Step 5:** In the modified \( s \times t \) matrix, list the cells with value \( \ast \) as \((u_{i_1}, e_{j_1}, W_1), \ldots, (u_{i_w}, e_{j_w}, W_w)\).

**Step 6:** By every vector \( \alpha = (\alpha_1, \ldots, \alpha_w) \in \{0, 1\}^w \), we construct a \( s \times t \) incomplete weighted matrix \( \overline{P}_\alpha = (p_{u_i,e_j})_{s \times t} \) where

1. \( p_{u_i,e_j} = f^{-1}_\lambda(u_i)(e_j) \) if \( (u_i, e_j) \) is not listed in \((u_{i_1}, e_{j_1}), \ldots, (u_{i_w}, e_{j_w})\), where \( f^{-1}_\lambda(u_i)(e_j) = f_\lambda(e_j)(u_i) \).
2. \( p_{u_i,e_j} = \alpha_v \) if \( (u_i, e_j) \) = \( (u_{i_v}, e_{j_v}) \), \( v \in \{1, 2, \ldots, w\} \).

**Step 7:** By computing the incomplete weighted choice value \( c_{\alpha_i} \) of each \( \overline{P}_\alpha = (p_{u_i,e_j})_{s \times t} \), we construct a new \( s \times a \) matrix \( \overline{P} = (p_{u_i,c_{\alpha_i}}) \) where each column of the matrix \( \overline{P} \) represents the incomplete weighted choice value of each \( \overline{P}_\alpha \) for every object. Missing weighted values are still marked as the sign \( \ast \). (The weighted choice value containing unknown data is regarded as an incomplete weighted choice value.)

**Step 8:** Get the maximum choice value \( C_{\alpha} \) without \( \ast \) and the maximum choice value \( C_{\alpha'} \) with \( \ast \) in each column of the matrix. Compute the critical values \( \bar{C}_q = |C_{\alpha} - C_{\alpha'}| \) in each column of the matrix where \( \ast \) is regarded as 0. Let \( \bar{C}_q \) remove duplicate values and get \( \bar{C}_q \) where \( q \) is the number of \( \bar{C}_q \).

**Step 9:** Sorting and classifying each critical value \( \bar{C}_q \) such that we get \( 2q + 1 \) matrices \( R^q \) in different intervals where \( k = \{1, \ldots, 2q + 1\} \).

**Step 10:** For every \( u_i \), let \( n_{u_i} \) be regarded as the number whose object \( u_i \) maximizes the weighted choice value at \( R^q \). Select the object corresponding to the maximum number \( n_{u_i} \) in each interval. Summarize and merge the interval values belonging to the same object.

**Step 11:** In the object corresponding to the modified interval value, the object with the most common attributes is chosen as the optimal decision \( u_i \).

**Step 12:** If \( i \) has multiple values, then any one may be selected.

| \( s_1, W_1 = 0.9 \) | \( s_2, W_2 = \ast \) | \( s_3, W_3 = 0.6 \) | \( s_4, W_4 = 0.7 \) | \( s_5, W_5 = 0.5 \) |
|---------------------|-----------------|-----------------|-----------------|-----------------|
| \( v_2 \) | \( * \) | 1 | 0 | 0 | 1 |
| \( v_3 \) | 0 | \( * \) | 0 | 1 | 0 |
| \( v_4 \) | 1 | 0 | 0 | 0 | 0 |
| \( v_5 \) | 0 | 1 | 0 | \( * \) | 0 |
| \( v_6 \) | 1 | 0 | 0 | 0 | 1 |
we get

where → is a fuzzy implication operator and ⊗ is a t-norm on [0, 1]. In a fuzzy formal context, \( R(x,a) \) represents the degree to which \( x \) owning the attribute \( a \), where the value \( R(x,a) \) is the number in \([0,1]\). ↓ and ⋄ of modal-style operators represent the two extreme cases in describing a set of attributes based on the related objects. Suppose that unknown data of incomplete fuzzy soft sets can be replaced with 0 or 1. We have a precise description of attributes possessed by objects in the four extremely cases when we use modal-style operators to handing decision-making problem. That is, we have two extreme cases if unknown data are replaced with 0. Similarly, we also have two extreme cases if unknown data are replaced with 1. Now, we propose the following algorithm for dealing with incomplete fuzzy soft set based decision-making problem by using modal-style operators.

Now, we apply the above algorithm to the incomplete fuzzy soft set \((f,S)\) described by Table 3. Let \(\rightarrow_G\) (Godel) be fuzzy implication operator. By step 2, we have optimum decision \(\theta^0_0 = 0.9/s_1 + 0.6/s_2 + 0.4/s_3 + 0.5/s_4 + 0.8/s_5 \) and \(\theta_1 = 1/s_1 + 1/s_2 + 0.4/s_3 + 1/s_4 + 0.8/s_5\). By simple calculation, we get

\[
\begin{align*}
\theta^0_1 &= 0.1/v_1 + 0/v_2 + 0/v_3 + 0.2/v_4 + 0/v_5 + 0.1/v_6; \\
\theta^0_2 &= 0.5/v_1 + 0.6/v_2 + 0.5/v_3 + 0.7/v_4 + 0.6/v_5 + 0.9/v_6; \\
\theta^1_1 &= 0.1/v_1 + 0.2/v_2 + 0.1/v_3 + 0.2/v_4 + 0.2/v_5 + 0.1/v_6; \\
\theta^1_2 &= 0.5/v_1 + 1/v_2 + 1/v_3 + 0.7/v_4 + 1/v_5 + 0.9/v_6.
\end{align*}
\]

Afterwards, we compute the related choice values \(\sigma = 0.6/v_1 + 0.9/v_2 + 0.6/v_3 + 0.9/v_4 + 0.9/v_5 + 1.0/v_6\). It is obvious that the maximum choice value is 1.0, and \(v_5\) is chosen as the optimal alternative.

Since the incomplete fuzzy soft set \((f,E)\) in Table 3 is deleted from the fuzzy soft set of [11], we know the missing data are \(f(v_2)(s_1) = 0.3, f(v_3)(s_2) = 0.7 \) and \(f(v_3)(s_4) = 0.2\).
In Reference [11], Kong et al. thought that $v_6$ should be the best alternative because it has the maximum choice value. Thus, improved Algorithm 4 is more accurate than Algorithm 1 and reduces the computational complexity.

**Algorithm 4** for incomplete fuzzy soft set based decision-making by using modal-style operators:

**Step 1:** Considering the incomplete fuzzy soft set $(f, E)$, where $u \in U$, $e \in E$.

**Step 2:** All unknown data are replaced with 0 and input the optimum decision $\theta_0 \in f(E)$ given by

$$\theta_0(e_i) = \max \{ f(u_i)(e_j) | u_i \in U, e_j \in E \}.$$

**Step 3:** Calculate $\theta^\downarrow_0$ and $\theta^\diamond_0$.

**Step 4:** All unknown data are replaced with 1 and input the optimum decision $\theta_1 \in f(E)$ given by

$$\theta_1(e_i) = \max \{ f(u_i)(e_j) | u_i \in U, e_j \in E \}.$$

**Step 5:** Calculate $\theta^\downarrow_1$ and $\theta^\diamond_1$.

**Step 6:** Calculate the choice value $\sigma(u)$ for each $u \in U$, where

$$\sigma(u) = \frac{\theta^\downarrow_0(u) + \theta^\downarrow_1(u) + \theta^\diamond_0(u) + \theta^\diamond_1(u)}{2}.$$

**Step 7:** The optimal decision is to choose $l \in U$ satisfying $\sigma(l) = \max_{u \in U} \sigma(u)$.

**Step 8:** If $l$ has multiple values, then any one may be selected.

7. **Discussion**

Research on fuzzy soft sets under complete information has been very active and many important results have been achieved in the decision-making aspect. As information of most practical applications is not fully available, incomplete or partially known information needs to be processed. However, most researchers just predict and estimate an incomplete data set in the incomplete fuzzy soft set, and do not combine with the practical decision-making problem. Thus, we focus on the application of fuzzy soft set based decision-making with incomplete information in this paper.

Firstly, we give an algorithm of an incomplete fuzzy soft set based decision-making from the perspective of probability theory. Due to the importance of weight function, we put forward the concepts of weighted incomplete fuzzy soft set and weighted incomplete soft set and develop a decision-making method based on weighted incomplete fuzzy soft sets. Considering the missing weight function, a novel notion of incomplete weighted fuzzy soft sets is proposed and applied to handle the decision-making problem. While the presented decision algorithm of an incomplete fuzzy soft set is reasonable and good, the process raises the problem of computational complexity in the case of a large number of missing values. As modal-style operators for fuzzy soft sets have a precise description of attributes possessed by objects, we apply a modal-style operator for an incomplete fuzzy soft set to deal with decision-making and propose a new algorithm to make it more accurate and simple.

Similarly, it leads to a huge amount of computation if there exist multiple missing weight values. This can be an even more crucial problem. Thus, we should improve the application of incomplete weighted fuzzy soft sets in decision-making in the future to make it more simple and convenient. Meanwhile, one can consider to apply incomplete fuzzy soft set to a multi-criteria group decision-making problem.

8. **Conclusions**

In this study, some basic notions and properties are reviewed. Furthermore, an algorithm is given to deal with incomplete fuzzy soft set based decision making problem. Based on the weighted function, we put forward the concepts of weighted incomplete fuzzy soft set and weighted incomplete soft set and develop a decision making method based on weighted incomplete fuzzy soft sets. Considering the missing weight function, a novel notion of incomplete weighted fuzzy soft sets is proposed and applied to handle the decision making problem. Finally, we improve the previous algorithm of incomplete fuzzy soft set based decision making and propose a new algorithm to make it more accurate and simple. In further research, one can consider to apply incomplete fuzzy soft set to multi-criteria group decision making problem.
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