A CHARACTERIZATION OF FINITE QUOTIENTS OF ABELIAN VARIETIES

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ABSTRACT. We provide a characterization of quotients of Abelian varieties by actions of finite groups that are free in codimension-one via vanishing conditions on the orbifold Chern classes. The characterization is given among a class of varieties with singularities that are more general than quotient singularities, namely among the class of klt varieties. Furthermore, for a semistable (respectively stable) reflexive sheaf $\mathcal{E}$ with vanishing first and second orbifold Chern classes over a projective klt variety $X$, we show that $\mathcal{E}|_{X_{\text{reg}}}$ is a locally-free and flat sheaf given by a linear (irreducible unitary) representation of $\pi_1(X_{\text{reg}})$ and that it extends over a finite Galois cover $\tilde{X}$ of $X$ étale over $X_{\text{reg}}$ to a locally-free and flat sheaf given by an equivariant linear (irreducible unitary) representation of $\pi_1(\tilde{X})$.

1. INTRODUCTION

In the current paper, we provide sufficient conditions for a (semi)stable reflexive sheaf over a class of projective varieties with mild singularities, specifically klt singularities, to be locally-free up to a finite cover. A characterization of quotients of Abelian varieties by finite groups acting freely in codimension-one follows. This is achieved by tracing a correspondence between polystable (respectively stable) reflexive sheaves with zero Chern classes and (irreducible) unitary representations of the fundamental group (see (3.1.6) and Theorem 1.1) that goes back to the celebrated results of Narasimhan-Seshadri [NS65], Donaldson [Don87], and Uhlenbeck-Yau [UY86] on stable holomorphic vector bundles:

1.0.1. On a compact Kähler manifold $X$ of dimension $n$ with a Kähler form $\omega$, a vector bundle $\mathcal{E}$ is stable with vanishing first and second Chern classes, that is

$$c_1(\mathcal{E})_\mathbb{R} = 0 \quad \text{and} \quad \int_X c_2(\mathcal{E}) \wedge [\omega]^{n-2} = 0,$$

if and only if it comes from an irreducible unitary representation of $\pi_1(X)$.

We recall that the notion of stability (in the sense of Mumford-Takemato) requires the notion of the slope $\mu_\omega$ of a coherent sheaf $\mathcal{F}$ provided by:

$$\mu_\omega(\mathcal{F}) := \int_X \frac{c_1(\mathcal{F}) \wedge [\omega]^{n-1}}{\text{rank}(\mathcal{F})}.$$

We say that a coherent sheaf $\mathcal{E}$ is (semi)stable with respect to $[\omega]$ if the inequality

$$\mu_\omega(\mathcal{F}) < \mu_\omega(\mathcal{E}) \quad \text{(respectively $\mu_\omega(\mathcal{F}) \leq \mu_\omega(\mathcal{E})$)}$$

holds for every coherent subsheaf $\mathcal{F}$ of $\mathcal{E}$ with $0 < \text{rank}(\mathcal{F}) < \text{rank}(\mathcal{E})$. The notion of (semi)stability generalizes in the projective category to the case when $X$ is an $n$-dimensional normal projective variety with $n - 1$ ample divisors $H_1, \ldots, H_{n-1}$. In

2010 Mathematics Subject Classification. 14B05, 14J17, 14J32, 14E20, 14L30, 32J26, 32J27, 32Q30, 32Q26.

Key words and phrases. Classification Theory, Uniformization, Torus Quotients, Minimal Model Program, KLT Singularities.
this case, a coherent sheaf $E$ is said to be (semi)stable with respect to the polarization $h := (H_1, \ldots, H_{n-1})$ if (1.0.2) holds with the slope of a subsheaf $F$ above replaced by

$$\mu_h(F) := c_1(F) \cdot H_1 \cdot \ldots \cdot H_{n-1}/\text{rank}(F).$$

Note that this is well defined since $c_1(F) = c_1(\det F)$ and $\det F$ is invertible outside the singular locus of $X$, which is of codimension two or more. We say that a reflexive (or torsion-free) coherent sheaf $E$ on $X$ is generically semi-positive if all its torsion-free quotients $F$ have semipositive slopes $\mu_h(F)$ with respect to every polarization $h := (H_1, \ldots, H_{n-1})$ with all $H_i$ ample. We note that if $\det E$ is numerically trivial, then this condition is equivalent to the semistability of $E$ (or of the dual of $E$) with respect to all such polarizations, i.e., to $E$ being generically semi-negative (equivalently generically semi-positive). Also, in the case $E$ is generically semi-positive or generically semi-negative and $\det E$ is $\mathbb{Q}$-Cartier, then $\det E$ (or equivalently $c_1(E)$) is numerically trivial if and only if $c_1(E) \cdot H_1 \cdot \ldots \cdot H_{n-1} = 0$ for some $(n-1)$-tuple of ample divisors $(H_1 \cdot \ldots \cdot H_{n-1})$, see Lemma 2.4. An important example of a generically semi-positive sheaf is the cotangent sheaf of a non-uniruled normal projective variety [Miy85].

Throughout, we work with a normal complex projective variety $X$ whose singularities are mild enough so that we have a meaningful notion of intersection numbers between the first two $\mathbb{Q}$-Chern classes (or orbifold Chern classes) of a reflexive coherent sheaf $F$ over $X$ and general complete intersection hyperplane sections. More precisely, we ask $X$ to have an orbifold structure (i.e., having only quotient singularities) in codimension-2 (see Section 2.B). This means that, if we cut down $X$ by $(n-2)$ very ample divisors, the general resulting surface has only isolated quotient singularities, and hence inherits an orbifold structure (or $Q$-structure). Hence, there is a well-defined intersection pairing between $\mathbb{Q}$-Chern classes $\hat{c}_2(F), \hat{c}_1(F)$ and $(n-2)$-tuples of ample divisors on $X$ and similarly for the second $\mathbb{Q}$-Chern character $\hat{c}_2(F) := (\hat{c}_1(F) - 2\hat{c}_2)/2$, see section 2. The same holds for the intersection pairing between the first $\mathbb{Q}$-Chern class (or $\mathbb{Q}$-Chern character) $\hat{c}_1(F) (= \hat{c}_1(F))$ and $(n-1)$-tuples of ample divisors. In this context, we have the following analog of (1.0.1) for projective $X$ (see Section 3 for its proof).

**Theorem 1.1.** Let $X$ be an $n$-dimensional normal projective variety, $X_{\text{reg}}$ its nonsingular locus and $F$ a reflexive coherent sheaf on $X$. Assume that $X$ has only quotient singularities in codimension-2. Then (the analytification of) $F|_{X_{\text{reg}}}$ comes from an irreducible unitary representation (respectively, an unitary representation) of $\pi_1(X_{\text{reg}})$ if and only if, for some (and in fact for all) ample divisors $H_1, \ldots, H_{n-1}$ on $X$, we have:

1. The reflexive sheaf $F$ is stable (respectively, polystable) with respect to the polarization $h := (H_1, \ldots, H_{n-1})$.
2. The first and second $\mathbb{Q}$-Chern characters of $F$ verify the vanishing conditions:

$$\hat{c}_1(F) \cdot H_1 \cdot \ldots \cdot H_{n-1} = 0,$$
$$\hat{c}_2(F) \cdot H_1 \cdot \ldots \cdot H_{n-2} = 0.$$

In particular, $F$ is locally free on $X_{\text{reg}}$ and is generically semi-positive in this case.

This will then lead us to our main theorem below, which gives a characterization for finite quotients of Abelian varieties that are étale in codimension-1.

We recall that a klt space is a normal $\mathbb{Q}$-Gorenstein space $X$ with at worst klt singularities ([KM98]), i.e. $X$ admits a desingularization $\pi : \tilde{X} \to X$ that satisfies $a_i > -1$ for
1.2 and sets the stage for this paper. We give there, such a generalization of the classical uniformization theorem of S.T. Yau (Theorem 3) and the second via Miyaoka’s theorem on the generic semipositivity of the cotangent sheaf of a non-uniruled variety by working out the following orbifold generalization of Simpson’s correspondence between coherent sheaves with flat connections and (semi)stable bundles with zero first and second Q-Chern characters. Both of these proofs generalize to the case of other classical quotients, such as orbifold quotients of the ball, to be treated elsewhere.

Theorem 1.2 (Characterization of Finite Quotients of Abelian Varieties). Let $X$ be an $n$-dimensional normal projective variety. Then $X$ is a quotient of an Abelian variety by a finite group acting freely in codimension-1 if and only if $X$ has at most klt singularities and we have

\begin{align}
(1.2.1) & \quad K_X \equiv 0, \\
(1.2.2) & \quad \text{The second } \mathbb{Q}\text{-Chern class of } \mathcal{T}_X := (\Omega^1_X)^* \text{ respects the vanishing condition} \\
& \quad \hat{c}_2(\mathcal{T}_X) \cdot A_1 \cdot \ldots \cdot A_{n-2} = 0,
\end{align}

for some $(n-2)$-tuple of ample divisors $(A_1, \ldots, A_{n-2})$.

Remark 1.3. We remark that the two conditions (1.2.1) and (1.2.2) in the theorem may be replaced by the following equivalent set of assumptions for some and hence for all polarization $h = (H_1, \ldots, H_{n-1})$, with the $H_i$ ample (see Explanation 4.1):

\begin{align}
(1.3.1) & \quad \text{The tangent sheaf } \mathcal{T}_X \text{ is semistable with respect to } h. \\
(1.3.2) & \quad \text{The first and second } \mathbb{Q}\text{-Chern characters of } \mathcal{T}_X \text{ verify the vanishing conditions} \\
& \quad \hat{c}_1(\mathcal{T}_X) \cdot H_1 \cdot \ldots \cdot H_{n-1} = 0, \\
& \quad \hat{c}_2(\mathcal{T}_X) \cdot H_1 \cdot \ldots \cdot H_{n-2} = 0.
\end{align}

Theorem 1.2 is a generalization of the classical uniformization theorem of S.T. Yau which states that a compact Kähler manifold $X$ of dimension $n$ with Kähler class $[w]$ that satisfies the equalities $c_1(X)_R = 0$ and $\int_X c_2(X) \wedge [w]^{n-2} = 0$ is uniformized by $\mathbb{C}^n$. Yau established this result, as a consequence of his solution to Calabi’s conjecture [Yau78], by proving that a compact Kähler manifold with vanishing real first Chern class is Ricci-flat. The problem of extending this result to the setting of canonical singularities in general was proposed in the remarkable paper of Shepherd-Barron and Wilson [SBW94]. There, they show that threefolds with at most canonical singularities with numerically trivial first and second $\mathbb{Q}$-Chern classes are finite quotient of Abelian threefolds (unramified in codimension-1). Our basic setup and strategies closely follow those of [SBW94]. Such theorems for terminal varieties ($a_i > 0$ in the equality (1.1.1)) and more generally for klt varieties that are smooth in codimension-2 have recently been established by the same route by Greb, Kebekus and Peternell [GKP14a] and sets the stage for this paper. We give two different proofs of the above theorem, the first via a result on the polystability of the tangent sheaf ([GKP12, Theorem 3]) and the second via Miyaoka’s theorem on the generic semipositivity of the cotangent sheaf of a non-uniruled variety by working out the following orbifold generalization of Simpson’s correspondence between coherent sheaves with flat connections and (semi)stable bundles with zero first and second $\mathbb{Q}$-Chern characters. Both of these proofs generalize to the case of other classical quotients, such as orbifold quotients of the ball, to be treated elsewhere.
Theorem 1.4 (Desingularization of Semistable Reflexive Sheaves Up to a Finite Cover). Let \( X \) be a normal projective variety with at most quotient singularities in codimension two, \( \mathfrak{h} := (H_1, \ldots, H_{n-1}) \) a polarization on \( X \) and \( \mathcal{E} \) a coherent reflexive sheaf on the analytic variety \( X^\text{an} \). Then \( \left. \mathcal{E} \right|_{X_{\text{reg}}} \) is locally free and flat, i.e. given by a representation of \( \pi_1(X_{\text{reg}}) \), if and only if \( \mathcal{E} \) is \( \mathfrak{h} \)-semistable and verifies
\[
\widehat{c}_1(\mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-1} = 0,
\]
\[
\widehat{c}_2(\mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-2} = 0.
\]
In this case, if \( X \) is furthermore klt, then there exists a finite, Galois morphism \( f : Y \rightarrow X \) étale over \( X_{\text{reg}} \), independent of \( \mathcal{E} \), such that \( (f^*\mathcal{E})^{**} \) is locally-free, equivariantly flat and with numerically trivial determinant over \( Y \).

It goes without saying that Yau’s resolution of the Calabi conjecture and the Donaldson-Uhlenbeck-Yau theorem are basic ingredients in our proof of Theorem 1.4. Another is the orbifold (or log) Bogomolov inequality obtained by Kawamata in [Kaw92] via a well-known argument of Miyaoka. Beside these classical results among others, key to our current proof also include the recent resolution of the Lipman-Zariski conjecture for klt orbifolds by Kawamata and the elusive \( \eta \)-semistable sheaves of an orbifold (or log) (K.)Bogomolov inequality obtained by Kawamata in [Kaw92] via a well-known argument of Miyaoka. Beside these classical results among others, key to our current proof also include the recent resolution of the Lipman-Zariski conjecture for klt orbifolds by Kawamata and the elusive \( \eta \)-semistable sheaves of an orbifold (or log) (K.)Bogomolov inequality obtained by Kawamata in [Kaw92] via a well-known argument of Miyaoka.

1.A. Acknowledgements. The authors owe a debt of gratitude to Stefan Kebekus for fruitful conversations leading to a strengthening of results in the first draft of this paper and for his kind invitation for the first author’s short visit to Freiburg where these took place. We also thank Chenyang Xu for a quick answer to a pertinent question of the paper.

2. Preliminaries on \( \mathbb{Q} \)-structures and \( \mathbb{Q} \)-Chern classes

In this section, we first give a very brief overview of the theory of \( \mathbb{Q} \)-vector bundles on an orbifold \( X \) (or Satake’s \( V \)-bundles on a \( V \)-manifold [Sat56]) and their \( \mathbb{Q} \)-Chern classes. Then we provide, via the Bogomolov inequality for the \( \mathbb{Q} \)-Chern classes of semistable \( \mathbb{Q} \)-bundles, a numerical criterion for \( \mathbb{Q} \)-Chern classes of generically semi-positive reflexive sheaves to vanish (Section 2.B.5). Finally we collect some basic facts on the behaviour of reflexive sheaves under a natural class of finite surjective maps between normal varieties.

2.A. Reflexive operations. In this paper, all objects are defined over \( \mathbb{C} \), all coherent sheaf on an algebraic (or analytic) variety \( X \) are \( \mathcal{O}_X \)-modules and all torsion-free or reflexive sheaves on \( X \) are coherent. We follow the convention of denoting the reflexive hull of a coherent sheaf \( F \) of rank \( r \) by \( F^{**} \). Similarly we define the reflexive exterior power by \( \wedge^{[i]} F := (\wedge^i F)^{**} \). In particular, \( \det F \) is the reflexivization \( (\wedge^r F)^{**} \) of \( \wedge^r F \). For a morphism \( f : Y \rightarrow X \), the reflexive pull-back of a coherent sheaf \( F \) on \( X \) is denoted by \( f'^* F := (f^* F)^{**} \). A useful fact about reflexive sheaves on a normal variety \( X \) is that such a sheaf \( \mathcal{E} \) is locally free on an open subsets \( X_0 \) of \( X_{\text{reg}} \) with codimension \( \geq 2 \) complement in \( X \) and that, for any such open subset, \( \mathcal{E} = i_* \left( \left. \mathcal{E} \right|_{X_0} \right) \) where \( i : X_0 \hookrightarrow X \) is the inclusion. In particular, reflexive pullbacks behave well under composition of finite morphisms between normal varieties. For an in-depth discussion of reflexive sheaves and reflexive operations we invite the reader to consult Hartshorne [Har80] and [GKKP11].
2.B. Local constructions. For a reflexive sheaf over a complex analytic variety $X$ with at most quotient singularities in codimension-2, we first define the $Q$-Chern classes via metric Chern-forms analytically locally. We then define, in the case $X$ is algebraic, the $Q$-Chern classes for an algebraic reflexive sheaf as those of its analyticization. We also discuss conditions that guarantee their vanishing considered as multilinear forms on the Néron-Severi space when $X$ is projective.

2.B.1. $Q$-vector bundles and $Q$-Chern classes. Let $X$ be a complex analytic variety with at most quotient singularities. Let $\{U_\alpha\}$ be a finite cover of $X$ with local uniformizations, that is, for each $\alpha$ there exists a complex manifold $X_\alpha$ and a finite, proper, holomorphic map $p_\alpha : X_\alpha \to U_\alpha$ such that $U_\alpha = X_\alpha / G_\alpha$, where $G_\alpha = \text{Gal}(X_\alpha / U_\alpha)$. Following what is now standard terminology, see the appendix of [GKKP11] (or [Mum83]), we call $G_\alpha$-sheaves the $G_\alpha$-equivariant coherent sheaves on $X_\alpha$ and their $G_\alpha$-equivariant subsheaves the $G_\alpha$-subsheaves. We call a coherent sheaf $\mathcal{E}$ on $X$ a $Q$-vector bundle, if for each $\alpha$, there exists a $G_\alpha$-locally-free sheaf $\mathcal{E}_\alpha$ on $X_\alpha$ such that its $Q_\alpha$-invariants $\mathcal{E}_\alpha^{Q_\alpha}$ descend to $\mathcal{E}|_{U_\alpha}$, i.e. $\mathcal{E}_\alpha^{Q_\alpha} = \mathcal{E}|_{U_\alpha}$.

Now, let $h_\alpha$ be a collection of hermitian $G_\alpha$-invariant metrics on $\mathcal{E}_\alpha$ verifying the natural compatibility conditions on overlaps. Such a collection exists by a partition of unity argument (subordinate to $\{U_\alpha\}$) since the $X_\alpha$'s are locally isomorphic over the $U_\alpha$ overlaps. Denote the $i$-th Chern form of $h_\alpha$ by $\Theta_{\alpha,i}(\mathcal{E}, h_\alpha)$. These forms are $G_\alpha$-invariant by construction and naturally give rise to $Q$-Chern forms $\Theta_i$ (over $X_{\text{reg}}$) defined by the local data $p^{G}_{\alpha} : (\Theta_i)|_{U_\alpha} = \Theta_{\alpha,i}(\mathcal{E}, h_\alpha)$. These define natural cohomology classes $\tilde{\Theta}_i(\mathcal{E}) := [\Theta_i(\mathcal{E}, h_\alpha)] \in H^{2i}(X, \mathbb{Q})$ independent of the choice of the metrics $h_\alpha$ and are called the $Q$-Chern classes (or orbifold-Chern classes) of the $Q$-bundle $\mathcal{E}$. See for example [Kaw92] and note that $V$-manifolds satisfy Poincaré duality with coefficients $\mathbb{Q}$.

2.B.2. Reflexive sheaves as $Q$-vector bundles. A reflexive sheaf $\mathcal{E}$ on a complex analytic normal surface with at worst quotient singularities has a natural $Q$-vector bundle structure defined by the locally free sheaves $\mathcal{E}_\alpha := p^{G}_{\alpha}([\mathcal{E}|_{U_\alpha}])$. More generally, let $X$ be any normal complex analytic variety with at most quotient singularities in codimension-2, that is, the maximal subvariety $X_1$ of $X$ with an orbifold structure, defined by removing the non-orbifold locus from $X$ has $\text{codim}_{X}(X \setminus X_1) \geq 3$. Then any reflexive sheaf on $X$ has a $Q$-vector bundle structure on an open subset $X_2 \hookrightarrow X_1$ with codimension-3 complement. In particular, we may define $Q$-Chern classes of a reflexive sheaf by restricting to $X_2$.

Assume further that $X$ is projective. We define the $Q$-Chern classes of an algebraic reflexive sheaf $\mathcal{E}$ restricted to $X_2$ to be those of the analytification $\mathcal{E}^{\text{anit}}$ of $\mathcal{E}$. Since $\text{codim}_{X}(X \setminus X_2) \geq 3$, there is are well defined intersection pairings between $\tilde{\Theta}_1(\mathcal{E})$ and any $(n-1)$-tuples of ample divisors $(H_1, \ldots, H_{n-1})$ and between any linear combination $\Delta := a \cdot \tilde{\Theta}_2(\mathcal{E}) + b \cdot \tilde{\Theta}_2(\mathcal{E})$ and the $(n-2)$-tuples $(H_1, \ldots, H_{n-2})$, the latter pairing via

\begin{equation}
\Delta \cdot H_1 \cdot \ldots \cdot H_{n-2} := \Delta|_{X_2} \cdot H_1 \cdot \ldots \cdot H_{n-2}.
\end{equation}

Remark 2.1. We recall that klt spaces have only quotient singularities in codimension-2 since a generic surface cut out by hyperplanes is klt and a klt surface has only quotientsingularities from rudiments of classification theory [KM98], see for example [GKKP11, Prop. 9.4]. Therefore our discussion above is valid for any reflexive sheaf over a projective klt variety. Also varieties with only quotient singularities are klt from the fact that a finite morphism $f : Y \to X$ étale in codimension-1 between normal varieties preserve the klt condition by [KM98, Prop. 5.20]. In particular, if $X$ as given in this previous sentence has only quotient singularities in codimension-1, then so does $Y$ (since $X_1$ is klt in this case).
Remark 2.2. [Evaluating Q-Chern Classes on Complete Intersection Surfaces] Let $X$ and $\mathcal{E}$ be as before. Define $S := D_1 \cap \ldots \cap D_{n-2}$ to be the orbifold projective surface in $X_2$ cut out by general members $D_i$ of basepoint-free linear system $|m \cdot H_i|$ for $m$ being a sufficiently large positive integer. It is an easy and well-known fact that $\mathcal{E}|_S$ is reflexive. In particular $\mathcal{E}|_S$ has a natural Q-vector bundle structure (one can also see this by simply restricting the Q-vector bundle structure that is already enjoyed by $\mathcal{E}|_X$ to the general surface $S \hookrightarrow X_2$). We may interpret the intersection number in (2.0.1) as the rational number $(1/m^{n-1}) \cdot \Delta(\mathcal{E}|_S)$ (we note that this number is obtained by integrating over the fundamental class of $S$ and is independent of the choice of $S$ for fixed $H_1, \ldots, H_{n-2}$ and $m$.

Thus in this case, we may and will understand $\hat{c}_2(\mathcal{E})$ (and similarly $\hat{c}_2^2(\mathcal{E})$), following [SBW94], as multilinear forms on the Néron-Severi space $\text{NS}(X)_Q$ (see below).

2.B.3. Numerical triviality of Q-Chern classes on the Picard group.

Definition 2.3 (Numerical Triviality of Q-Chern Classes of Reflexive Sheaves). Let $X$ be a normal projective variety with only quotient singularities in codimension-2. For a reflexive sheaf $\mathcal{E}$ on $X$, we say that the first and second Q-Chern classes of $\mathcal{E}$ are numerically trivial on the Picard group (or simply trivial on $X$), and we write $\hat{c}_1(\mathcal{E}) \equiv_{ns} 0, i = 1, 2$, if $\hat{c}_i(\mathcal{E})$ defines a vanishing multilinear form on $\text{NS}(X)_Q$. Since the $\mathbb{R}$-span of the ample classes is open in $\text{NS}(X)_\mathbb{R}$, we have

$$\hat{c}_1(\mathcal{E}) \equiv_{ns} 0 \iff \hat{c}_1(\mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-1} = 0 \forall (H_1, \ldots, H_{n-1}),$$
$$\hat{c}_2(\mathcal{E}) \equiv_{ns} 0 \iff \hat{c}_2(\mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-2} = 0 \forall (H_1, \ldots, H_{n-2}),$$

where $H_1, \ldots, H_{n-1}$ are ample divisors on $X$.

In the case $\det \mathcal{E}$ is $\mathbb{Q}$-Cartier, it is an elementary exercise to show that $\hat{c}_1(\mathcal{E}) \equiv_{ns} 0$ if and only if $\hat{c}_1(\mathcal{E}) \equiv 0$. One can also appeal directly to the numerical triviality criterion of Kleiman given in [Kle66] to see this. When it is not $\mathbb{Q}$-Cartier however, it makes little sense to talk about numerical triviality in the usual sense since intersection with arbitrary curves has no sense and, even if it does, this notion would be different from the numerical triviality on the Picard group in general.

2.B.4. Bogomolov inequality for Q-bundles. With the setup as above and $\text{rank}(\mathcal{E}) = r$, a natural combination of Q-Chern classes as defined in (2.0.1) is

$$\Delta_B(\mathcal{E}) := 2r \cdot \hat{c}_2(\mathcal{E}) - (r - 1) \cdot \hat{c}_1^2(\mathcal{E}).$$

The combination appears for example in the Bogomolov inequality for semistable sheaves. According to [Kaw92, Lem. 2.5] any semistable reflexive sheaf $\mathcal{F}$ on a projective normal surface $S$ with only quotient singularities verifies the **Bogomolov inequality**

\begin{equation}
\Delta_B(\mathcal{F}) \geq 0.
\end{equation}

Now assume that $\mathcal{E}$ is semistable with respect to a polarization $(H_1, \ldots, H_{n-1})$. Then, according to the classical result of Mehta-Ramanathan [MR82], the restriction $\mathcal{E}|_S$ is also semistable, where $S := D_1 \cap \ldots \cap \hat{D}_i \cap \ldots \cap D_{n-1}$ is the complete intersection surface cut out by general members $D_i \in |m \cdot H_i|, m \gg 0$ and $i \in \{1, \ldots, n - 1\}$, after removing an ample divisor $H_i$ from the $(n - 1)$-tuple $(H_1, \ldots, H_{n-1})$. Therefore, thanks to the Bogomolov inequality (2.3.1), we have

\begin{equation}
\Delta_B(\mathcal{E}) \cdot H_1 \cdot \ldots \cdot \hat{H}_i \cdot \ldots \cdot H_{n-1} \geq 0 \forall i,
\end{equation}

where $(H_1, \ldots, \hat{H}_i, \ldots, H_{n-1})$ is the $(n - 2)$-tuple of ample divisors defined by removing the ample divisor $H_i$ from $(H_1, \ldots, H_{n-1})$. 
2.B.5. Numerical triviality criterion for $Q$-Chern classes. Generically semi-positive reflexive sheaves (over normal varieties with only quotient singularities in codimension-2) are central objects of this paper. In the next lemma we show that the $Q$-Chern classes of such sheaves verify a natural numerical triviality criterion (on the Picard group).

**Lemma 2.4** (A Criterion for the Numerical Triviality of $\mathcal{C}_1$ and $\mathcal{C}_2$ on the Picard group). Let $X$ be a normal projective variety $X$ with only quotient singularities in codimension-2 and $\mathcal{E}$ a generically semi-positive reflexive sheaf on $X$. Assume that

\[
(2.4.1) \quad \mathcal{C}_1(\mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-1} = 0
\]

holds for some $(n - 1)$-tuple of ample divisors $(H_1, \ldots, H_{n-1})$, then $\mathcal{C}_1(\mathcal{E}) \equiv_{ns} 0$.

If we assume furthermore that

\[
(2.4.2) \quad \mathcal{C}_2(\mathcal{E}) \cdot H_1' \cdot \ldots \cdot H_{n-2}' = 0
\]

for an $(n - 2)$-tuple of ample divisors $(H_1', \ldots, H_{n-2}')$, then $\mathcal{C}_2(\mathcal{E}) \equiv_{ns} 0$.

**Proof.** Aiming for a contradiction, suppose there exists a polarization $(A_1, \ldots, A_{n-1})$ such that $\mathcal{C}_1(\mathcal{E}) \cdot A_1 \cdot \ldots \cdot A_{n-1} \neq 0$. Then, by the generic semi-positivity assumption, we have

\[
(2.4.3) \quad \mathcal{C}_1(\mathcal{E}) \cdot A_1 \cdot \ldots \cdot A_{n-1} > 0.
\]

Now let $m \in \mathbb{N}^+$ be a sufficiently large integer such that $(mH_i - A_i)$ is ample for all $i \in \{1, \ldots, n-1\}$ and consider the equality

\[
m^{n-1} \cdot \mathcal{C}_1(\mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-1} = \mathcal{C}_1(\mathcal{E}) \cdot ((mH_1 - A_1) + A_1) \cdot \ldots \cdot ((mH_{n-1} - A_{n-1}) + A_{n-1}).
\]

The right-hand side of (2.4.3) is strictly positive by (2.4.2) (and by the generic semi-positivity of $\mathcal{E}$). But the left hand-side is equal to zero by the assumption, a contradiction.

To prove the numerical triviality of $\mathcal{C}_2(\mathcal{E})$, we argue similarly by using the Bogomolov inequality (2.3.2): First we observe that the generic semi-positivity of $\mathcal{E}$ together with $\mathcal{C}_1(\mathcal{E}) \equiv_{ns} 0$ implies that $\mathcal{E}$ is semistable independent of the choice of a polarization. Therefore the second $Q$-Chern class of $\mathcal{E}$ is "pseudo-effective" in the sense that, thanks to Bogomolov (2.3.2), the inequality

\[
(2.4.4) \quad \mathcal{C}_2(\mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq 0
\]

holds for all $(n - 2)$-tuples of ample divisors $(H_1, \ldots, H_{n-2})$. Suppose to the contrary that there exists ample divisors $A'_1, \ldots, A'_{n-2}$ such that $\mathcal{C}_2(\mathcal{E}) \cdot A'_1 \cdot \ldots \cdot A'_{n-2} \neq 0$, i.e. 

\[
(2.4.5) \quad \mathcal{C}_2(\mathcal{E}) \cdot A'_1 \cdot \ldots \cdot A'_{n-2} > 0.
\]

Set $m' \in \mathbb{N}^+$ to be a sufficiently large positive integer such that $(m'H_i - A'_i)$ is ample for all $i \in \{1, \ldots, n - 2\}$. Now from the equality

\[
m'^{n-2} \cdot \mathcal{C}_2(\mathcal{E}) \cdot H_1' \cdot \ldots \cdot H'_{n-2} = \mathcal{C}_2(\mathcal{E}) \cdot ((m'H_1' - A'_1) + A'_1) \cdot \ldots \cdot ((m'H_{n-2}' - A'_{n-2}) + A'_{n-2})
\]

we can extract a contradiction by observing that, although the left-hand side is equal to zero, the right-hand side is strictly positive by the Bogomolov inequality 2.3.2 and the inequality 2.4.5. □
Remark 2.5. Lemma 2.4 (see also the discussion after Definition 2.3) in particular shows that for generically semipositive reflexive sheaves $\mathcal{E}$ whose determinant ($\det \mathcal{E}$) is a Q-Cartier divisor, the two sets of vanishing conditions $[\det(\mathcal{E}) \equiv 0, \mathcal{E}_2(\mathcal{E}) \equiv_\alpha 0]$ and $[\mathcal{E}_1(\mathcal{E}) \cdot A_1 \cdot \ldots \cdot A_{n-1} = 0, \mathcal{E}_2(\mathcal{E}) \cdot H_1 \cdot \ldots \cdot H_{n-2} = 0]$, where $(A_1, \ldots, A_{n-1})$ and $(H_1, \ldots, H_{n-2})$ are any $(n-1)$ and $(n-2)$-tuples of ample divisors in $X$, are equivalent. So for example in the case of a non-uniruled Q-Gorenstein variety $X$, the vanishing conditions in (1.1.2) when $\mathcal{F} = \mathcal{F}_X$ is the same as $K_X \equiv 0$ and $\mathcal{E}_2(\mathcal{F}_X) \equiv_\alpha 0$.

2.C. Q-sheaves and global constructions. The theory of Q-sheaves was introduced by Mumford [Mum83] as an algebraic generalization of Q-vector bundles to a much larger class of coherent sheaves for which a meaningful notion of Chern classes can be defined. In this section we briefly recall some elementary facts that are needed for our results and we refer to Mumford [Mum83] (see also [Meg92]) for a detailed account of this theory.

Let $X$ be a normal projective variety with only quotient singularities. Then according to Mumford ([Mum83, Chapt. 2]) there exists a Q-structure given by the collection of charts $(U_\alpha, p_\alpha : X_\alpha \to U_\alpha)$, where $U_\alpha$ are quasi-projective, $p_\alpha$ are étale in codimension-1 and $X_\alpha$ are smooth. Let $G_\alpha := \text{Gal}(X_\alpha/U_\alpha)$. We call a coherent sheaf $\mathcal{E}$ on $X$ a Q-sheaf, if there exists coherent $G_\alpha$-sheaves $\mathcal{E}_\alpha$ on $X_\alpha$ such that $\mathcal{E}_\alpha^{G_\alpha} = \mathcal{E}|_{U_\alpha}$.

Now, let $K$ be a Galois extension of the function field $k(X)$ containing all the function fields $k(X_\alpha)$ and let $\hat{X}$ be the normalization of $X$ in $K$. Let $G$ be the Galois group. By construction, the corresponding finite morphism $p : \hat{X} \to X$ factors though each $p_\alpha : X_\alpha \to U_\alpha$, i.e. there exists a collection of finite morphisms $q_\alpha : \hat{X}_\alpha \to X_\alpha$ giving a commutative diagram

$$
\begin{array}{ccc}
\hat{X}_\alpha & \xrightarrow{q_\alpha} & X_\alpha \\
\downarrow p|_{\hat{X}_\alpha} & & \downarrow p_\alpha \\
U_\alpha & & 
\end{array}
$$

For a Q-sheaf (or a Q-vector bundle) $\mathcal{E}$ on $X$, we can define a coherent sheaf on $\hat{X}$ by gluing together the local data given by $(\hat{X}_\alpha, \hat{\mathcal{E}}_\alpha := q_\alpha^*(\mathcal{E}_\alpha))$. A result of Mumford ([Mum83, Prop. 2.1]) shows that when the global cover $\hat{X}$ is Cohen-Macaulay, any sheaf $\hat{\mathcal{E}}$ on $\hat{X}$ arising from a Q-sheaf on $X$ admits finite resolution by locally-free sheaves and hence admits well-defined Chern classes. In this situation we define the i-th Q-Chern class of $\mathcal{E}$ as a $(G$-invariant) cohomology class on $\hat{X}$ by $\hat{c}_i(\mathcal{E}) := (1/|G|) c_i(\hat{\mathcal{E}})$. Following [Mum83], which identifies the G-invariant cohomology class on $\hat{X}$ with homology classes of complementary dimension on $X$ for the intersection theory on $X$, we define the intersection of $\hat{c}_1(\mathcal{E})$ with cycles on $X$ by its intersection with the pullback cycle on $\hat{X}$ and we see that this definition agrees with the analytic one in Section 2.B via the projection formula. We refer the reader to [Mum83] for the intersection theory of Q-sheaves and note in particular that, as any normal surface is Cohen-Macaulay, we can always define Q-Chern classes of Q-sheaves on a normal irreducible surface $S$ with only quotient singularities and that the algebraic Hodge-Index theorem holds for $\hat{c}_1$ by considering it on a desingularization of $\hat{S}$.

Remark 2.6 (An Equivalent Definition for $\hat{c}_1 \equiv 0$). Let $S$ be a normal surface with only quotient singularities. Let $p : \hat{S} \to S$ be the global cover that was constructed above and $\hat{\mathcal{E}}$ the $G$-locally-free sheaf on $\hat{S}$. It is not difficult to see that

$$
(2.6.1) \quad \hat{c}_1(\mathcal{E}) \equiv 0 \iff c_1(\mathcal{E}) \equiv 0.
$$

The reason is that the Q-factoriality of $S$ (recall that any normal variety with quotient singularity is Q-factorial [KM98, Prop. 5.15]), together with $\hat{c}_1(\mathcal{E}) \equiv 0$, implies that
det(\mathcal{E}) is numerically trivial as a Q-Cartier divisor. Notice that by construction we have \det(\mathcal{E}) = p^*[\alpha](\mathcal{E}). Therefore, the projection formula for Chern classes of pull-back bundles implies that \( c_1(\mathcal{E}) \cdot A = 0 \), for every ample divisor \( A \) in \( \tilde{\mathcal{S}} \). The equivalence (2.6.1) now follows from Kleiman’s numerical triviality criterion for Q-Cartier divisors [Kle66, Prop. 3] which gives the elementary (linear algebra) fact that over a normal, irreducible projective variety a Q-Cartier divisor is numerically trivial if it has zero intersection with all ample polarizations.

2.D. **Behaviour of reflexive sheaves under finite quasi-étale morphisms.** We briefly review some elementary facts about the behaviour of reflexive sheaves under finite morphisms that are étale in codimension-1, i.e. finite quasi-étale morphisms, ending with a simple observation regarding the stability of the tangent sheaf of klt varieties. We follow the standard convention that Galois morphisms are finite.

**Lemma 2.7** *(Q-Chern Classes and Reflexive Pull-Backs)*. Let \( X \) be a normal projective variety with at most quotient singularities in codimension-2. Let \( f : Y \rightarrow X \) be a finite Galois morphism that is étale in codimension-1 (with \( Y \) normal). Then \( Y \) has at most quotient singularities in codimension-2 and, for \( \Delta \) a linear combination of \( \alpha_2 \) and \( \alpha_2 \), a reflexive sheaf \( \mathcal{E} \) on \( X \) satisfies \( \Delta(\mathcal{E}) \cdot H_1 \cdots H_{n-2} = 0 \) for some \( (n-2) \)-tuple of ample divisors \( (H_1, \ldots, H_{n-2}) \) if and only if

\[
\Delta(f^*[\alpha](\mathcal{E})) \cdot f^*H_1 \cdots f^*H_{n-2} = 0.
\]

Moreover, \( \Delta(\mathcal{E}) \equiv_{ns} 0 \) if and only if \( \Delta(f^*[\alpha](\mathcal{E})) \equiv_{ns} 0 \) and, in the case \( \det\mathcal{E} \) is Q-Cartier, \( \det\mathcal{E} \) is numerically trivial if and only if \( f^*[\det\mathcal{E}] = \det(f^*[\alpha]\mathcal{E}) \).

**Proof.** The first claim follows from Remark 2.1. Now, define \( \mathcal{G} := f^*[\alpha](\mathcal{E}) \) and let \((U_a, p_a : X_a \rightarrow U_a; U_a = X_a/G_a)\) be the local Q-structure (see Section 2.B) for \( X_1 \), where \( X_1 \) is equal to \( X \) minus its non-orbifold locus. Let \( \{\mathcal{E}_a\} \) be the collection of \( G_a \)-sheaves on \( X_a \). Define \( V_a := f^{-1}U_a \) and let \( Y_a := X_a \times_{U_a} V_a \) be the fibre product given by the base change \( p_a : X_a \rightarrow U_a \) with the corresponding commutative diagram

\[
\begin{array}{ccc}
Y_a & \xrightarrow{r_a} & V_a \\
\downarrow & & \downarrow \\
X_a & \xrightarrow{f_a := f|_{V_a}} & U_a.
\end{array}
\]

Since \( X_a \) is smooth and \( f_a : V_a \rightarrow U_a \) is étale in codimension-1, so is \( g_a \). From the purity of the branch locus, we find that \( g_a \) is étale. In particular \( Y_a \) is smooth. From the commutativity of the diagram, we have \( g_a = g_a^*(\mathcal{E}_a) \), \( g_a \) being the locally-free sheaf on \( Y_a \) invariant under the action of \( H_a := \text{Gal}(Y_a/U_a) \) such that \( g_a^{H_a} = \mathcal{G}|_{V_a} \). In particular given a collection of \( G_a \)-invariant metrics \( h_a \) on \( X_a \) and the corresponding Chern forms \( \Theta_{a,i}(\mathcal{E}_a, h_a) \) (see Section 2.B), we have induced \( H_a \)-invariant Chern forms \( \hat{\Theta}_{a,i}(\mathcal{G}, g_a^*h_a) \) on \( Y_a \) given by \( g_a^*(\Theta_{a,i}) \). Thus \( \hat{\xi}(\mathcal{G}|_{Y_1}) = (f|_{X_1})^*\hat{\xi}(\mathcal{E}|_{X_1}) \in H^{2i}(Y_1, \mathbb{Q}) \), where \( Y_1 := f^{-1}(X_1) \). The lemma now follows from the projection formula.

The last statement follows from summing the polarization divisors over the action of the Galois group of \( f \), since \( \det\mathcal{E} \) and \( \Delta(f^*[\alpha](\mathcal{E})) \) are invariant under this group. Note that if \( \det\mathcal{E} \) is Q-Cartier (meaning that \( \mathcal{L} := \mathcal{O}(m\det\mathcal{E}) \) is invertible for some \( m \in \mathbb{Z} \)) then \( f^*[\det\mathcal{E}] \) is also (since \( f^*\mathcal{L} = f^*[m\det\mathcal{E}] \) by the normality of \( Y \) and the equality on the smooth locus of \( Y \)).
Lemma 2.8. Let $f : Y \to X$ be a finite Galois morphism between normal varieties that is étale in codimension-1, $\mathcal{E}$ a reflexive sheaf on $X$ and $h := (H_1, \ldots, H_{n-1})$ a fixed polarization on $X$ with $H_i$ ample. Then $\mathcal{E}$ is semistable with respect to $h$ if and only if $f^*\mathcal{E}$ is semistable with respect to $f^*h := (f^*H_1, \ldots, f^*H_{n-1})$.

Proof. The proof follows from the fact that the subsheaf of $f^*\mathcal{E}$ with maximal $(f^*h)$-slope is unique and thus invariant under the action of the group $\text{Gal}(Y/X)$.

Let $X$ be any non-uniruled normal projective variety. The generic semi-positivity result of Miyaoka [Miy85] says that the tangent sheaf of $X$, Thm. 1.4], obtained from careful analysis of Remark 1.13 and/or Remark 2.10, while Proposition 2.8 and go back at least to [SBW94]. We note that the proofs of Theorem 2.10, Sect. 5] uses only Proposition 2.8.

Proposition 2.9 (Semipositivity of the Tangent Sheaf of Klt Varieties). Let $X$ be a klt projective variety. If $K_X \equiv 0$, then $\mathcal{T}_X$ is generically semi-positive.

Proof. Corollary 3.2 (or Theorem 1.4) implies that there exists a finite Galois cover $f : Z \to X$ étale over $X_{\text{reg}}$ which pulls back numerically trivial $\mathbb{Q}$-Cartier divisors on $X$ to Cartier ones on $Z$, i.e. a simultaneous index one cover for such divisors. This implies that $K_Z \equiv_{\mathbb{Q}} f^*K_X$ is Cartier. But $Z$ has at worst klt singularities by Remark 2.1. Hence $Z$ has only canonical singularities. From our discussion above, $Z$ is not uniruled. Thus $\mathcal{T}_Z = f^*\mathcal{T}_X$ is generically semi-positive, and therefore by Lemma 2.8 so is $\mathcal{T}_X$.

2.E. Extending flat sheaves. We collect here some key ingredients in our proofs. The first two are found in [GKP14a], while Proposition 2.12 gives necessary generalizations.

Theorem 2.10 (Removing the Singularity Contribution to the Algebraic Fundamental Group by a Finite Map, [GKP14a, Thm. 1.4]). Let $X$ be a normal quasi-projective klt variety. Then, there is a Galois morphism $f : Y \to X$ étale in codimension-1 with $Y$ normal such that the inclusion $Y_{\text{reg}} \to Y$ induces an isomorphism $\hat{\pi}_1(Y_{\text{reg}}) \cong \hat{\pi}_1(Y)$.

Proposition 2.11 (Extending Flat Sheaves, see [GKP14a, Thm. 1.13] and/or Remark 5.2). Let $Y$ be a normal analytic variety satisfying the conclusion of Theorem 2.10. For every locally-free analytic sheaf $\mathcal{F}$ on $Y_{\text{reg}}$ given by a representation of $G = \hat{\pi}_1(Y)$, there exists a locally-free analytic sheaf $\mathcal{F}$ on $Y$ given by a representation of $G$ such that $\mathcal{F} \mid Y_{\text{reg}} \cong \mathcal{F}$.

Proposition 2.12 (Enriques-Severi Lemma on Identifying Flat Sheaves via Restriction, compare [GKP14a, Sect. 5]). Let $X$ be a normal projective variety with at most quotient singularities in codimension-2, $H$ an ample line bundle and $\mathcal{V}$ a bounded family of reflexive sheaves on $X$. Let $D$ be a general member of a high enough multiple of $H$. Then a reflexive sheaf $\mathcal{F}$ over $X$ belongs to $\mathcal{V}$ if and only if $\mathcal{F} \mid D$ is isomorphic to an element of $\mathcal{V}$ restricted to $D$. It is flat and locally free over $X$ (respectively over its smooth locus $X_{\text{reg}}$) if and only if its restriction to $D$ (respectively to $D_{\text{reg}}$) is.

These results are, at least in similar forms for the case of canonical threefolds, strategic in [SBW94] and go back at least to [Miy87]. For the convenience of the reader, a proof of the first proposition can be found in Remark 5.2 and the second in the appendix. Nevertheless, Theorem 2.10 is a principal theorem in [GKP14a] obtained from careful analysis via a result of Chenyang Xu on the finiteness of the algebraic local fundamental groups for klt varieties [Xu12]. We note that the proofs of Theorem 1.1 and of the first part of Theorem 1.4 uses only Proposition 2.12.
3. Stable Reflexive Sheaves with Vanishing Q-Chern Classes

We recall that a hermitian metric $h$ on a holomorphic vector bundle $\mathcal{E}$ over a compact Kähler manifold with Kähler form $\omega$ is said to satisfy the Einstein condition if

$$i \Lambda \omega F = \lambda \text{id}_{\mathcal{E}}$$

for some $\lambda \in \mathbb{R}$ where $F$ is the curvature of the unitary connection compatible with the holomorphic structure. From the classical result of Donaldson, Uhlenbeck and Yau, we know that given a compact Kähler manifold $X$ of dimension $n$ and a Kähler class $[\omega]$, every $[\omega]$-stable holomorphic vector bundle $\mathcal{E}$ admits a hermitian metric $h$ whose associated unitary connections is Hermitian-Einstein. Moreover if $c_1(\mathcal{E}) = 0$ and

$$\int_X c_2(\mathcal{E}) \wedge [\omega]^{n-2} = 0,$$

then $(\mathcal{E}, h)$ is flat. Our aim in this section is to prove a generalization of this result to the case of reflexive sheaves over normal projective varieties with only quotient singularities in codimension-2, namely Theorem 1.1. For this, we first examine how stability of a $Q$-vector bundle $\mathcal{E}$ over a complex projective surface $S$ with at most quotient singularities behaves under blowing-ups: Let $(U_a, p_a: X_a \to U_a)$ be the local $Q$-structure of $S$ (Section 2.B). Let $p: \tilde{S} \to S$ be the global finite cover with Galois group $G$ and let $\mathcal{E}$ be the locally-free sheaf on $\tilde{S}$ such that $\mathcal{E}^G = \mathcal{E}$ (Section 2.C). We study the stability of $\pi^*(\mathcal{E})$ on a $G$-equivariant desingularization $\pi: \tilde{S} \to \tilde{S}$ (whose existence is guaranteed classically or by [BM95] for example). Note that the actions of $G$ on $\tilde{S}$ and on $\mathcal{E}$ lift uniquely to actions of $G$ on $S$ and on $\pi^*(\mathcal{E})$ respectively. For expediency, we now allow all of our polarization divisors to be $Q$-Cartier.

**Proposition 3.1** (Lifting Stability to $G$-Equivariant Desingularizations). Let $\mathcal{E}$ be a reflexive coherent sheaf on a normal projective surface $S$ with only quotient singularities. Fix an ample divisor $H$ on $S$. With the setup as above, if $\mathcal{E}$ is stable with respect to $H$, then there exists a $G$-invariant polarization $\tilde{H}$ on $\tilde{S}$ such that $\mu_{\tilde{H}}(\tilde{\mathcal{E}}) = \mu_H(\mathcal{E})$, where $\tilde{\mathcal{E}} := \pi^* \mathcal{E}$, and that $\tilde{\mathcal{E}}$ is $G$-stable with respect to $\tilde{H}$, that is, for every $G$-equivariant subsheaf $\tilde{\mathcal{G}} \subset \tilde{\mathcal{E}}$, we have the strict inequality

$$\mu_{\tilde{H}}(\tilde{\mathcal{G}}) < \mu_{\tilde{H}}(\tilde{\mathcal{E}}).$$

**Proof.** First note that $\tilde{\mathcal{E}}$ is semistable with respect to $\tilde{H} := p^*(H)$. For otherwise the destabilizing subsheaf of $\tilde{\mathcal{E}}$, as it is $G$-equivariant (saturated and thus given by a pull-back bundle outside codimension-2), would descend to a proper $H$-destablizing subsheaf of $\mathcal{E}$. It is also $G$-stable; otherwise there would exist a saturated $G$-equivariant semistable subsheaf $\tilde{\mathcal{G}} \subset \tilde{\mathcal{E}}$ of strictly smaller rank with $\mu_{\tilde{H}}(\tilde{\mathcal{G}}) = \mu_{\tilde{H}}(\tilde{\mathcal{E}})$, which would descend to a saturated semistable subsheaf $\mathcal{G} \subset \mathcal{E}$ with $\mu_H(\mathcal{G}) = \mu_H(\mathcal{E})$ and contradict the stability of $\mathcal{E}$. Note also that one can arrange $(\pi^* (\tilde{H}) - E')$ to be ample by a suitable choice of an effective and $\pi$-exceptional $Q$-divisor $E'$ and, by averaging over $G$ if necessary, $G$-invariant. Set $\tilde{H}^0 = (\pi^* (\tilde{H}) - E')$ for this choice. Since $\tilde{\mathcal{E}}$ is locally free, $\tilde{\mathcal{E}} = \pi^* \mathcal{E}$ is trivial along the (reduced) exceptional divisor $E$ of $\pi$ and hence $\mu_{\tilde{H}^0}(\tilde{\mathcal{E}}) = \mu_{\tilde{H}}(\tilde{\mathcal{E}})$.

Now, let $\tilde{\mathcal{F}}$ be any $G$-subsheaf of $\tilde{\mathcal{E}}$. Let $U$ be a Zariski-open subset of $\tilde{S}$ with $\text{codim}_S(\tilde{S} \setminus U) = 2$ such that $\pi|_{\tilde{S} \setminus U} : \pi^{-1}(U) \to U$ is an isomorphism. Let $\mathcal{F}^0 := (\pi^{-1}|_{\pi^{-1}(U)})^* \tilde{\mathcal{F}}$ be the $G$-subsheaf of $\mathcal{E}|_U$ induced by $\tilde{\mathcal{F}}$. Define $\mathcal{F} := \mathcal{F}^0 \mid U$ to be the coherent extension of $\mathcal{F}^0$ across $\tilde{S} \setminus U$ and notice that $\mathcal{F}$ defines a $G$-equivariant...
coherent subsheaf of $\mathcal{E}$. Now from the $G$-stability of $\mathcal{E}$ with respect to $\hat{H}$, we infer that the inequality
\begin{equation}
(3.1.2) \quad \mu_{\pi^*\hat{H}}(\mathcal{F}) < \mu_{\pi^*\hat{H}}(\mathcal{E}) \quad \text{(in fact $\mu_{\pi^*\hat{H}}(\mathcal{F}) \leq \mu_{\pi^*\hat{H}}(\mathcal{E}) - \frac{1}{\text{rank}(\mathcal{E})^2}$)}
\end{equation}
holds. On the other hand it is easy to see that the set
\begin{equation}
(3.1.3) \quad \{ \mu_{\hat{H}}(\mathcal{F}) : \mathcal{F} \text{ coherent subsheaf of } \mathcal{E} \}
\end{equation}
admits an upper-bound. Therefore for all sufficiently small $\delta \in \mathbb{Q}^+$, we can always exploit the $G$-stability of $\mathcal{E}$ (3.1.2) to ensure that the inequality (3.1.1) holds by choosing $\hat{H} := \pi^*\hat{H} + \delta H^0$, i.e. $\mathcal{E}$ is $G$-stable with respect to $\hat{H}$.

It is well known from the celebrated theorems of Donaldson and Uhlenbeck-Yau and from their proofs, see for example [Pra93, Thm. 5] and the remarks given therein, that given a compact Kähler manifold equipped with the holomorphic action of a compact Lie group $G$ and a $G$-invariant Kähler form $w$, any holomorphic vector bundle that is $G$-stable with respect to $w$ admits a $G$-invariant Hermitian-Einstein connection. Therefore, as our choice of polarization $\hat{H}$ in the above proposition is $G$-invariant, the locally-free sheaf $\mathcal{E} = \pi^*\tilde{\mathcal{E}}$ carries a Hermitian-Einstein connection $\tilde{D} : \tilde{\mathcal{E}} \to \tilde{\mathcal{E}} \otimes \Omega^1_S$ that is $G$-invariant.

If we further assume that $\hat{\text{ch}}_2(\mathcal{E}) = 0$ and $\mu_{\hat{H}}(\mathcal{E}) = 0$, then $\text{ch}_2(\mathcal{E}) = 0$ and, with a Kähler form $w$ representing $c_1(\hat{H})$,
\begin{equation}
(3.1.4) \quad \frac{\sqrt{-1}}{2\pi} \int_{\mathcal{S}} \text{tr}(\Lambda_w F_{\tilde{D}}) d\text{vol}(w) = \int_{\mathcal{S}} \text{Ric}(\tilde{D}) \wedge [w] = c_1(\mathcal{E}) \cdot \hat{H} = 0.
\end{equation}
In particular the unitary connection $\tilde{D}$ is flat, i.e. $\tilde{D}^2 = 0$. The flatness of $\tilde{D}$ follows from the fact that $\tilde{D}$ is Hermitian-Einstein (3.0.1), so that the vanishing condition (3.1.4) implies the vanishing of $\Lambda_w F_{\tilde{D}}$, and from the well-known Riemann bilinear identity, c.f. page 16 of [Sim92] or equations 4.2 and 4.3 in Chapter IV of [Kob87], which takes the form
\begin{equation}
(3.1.5) \quad \int_{\mathcal{S}} \text{ch}_2(\mathcal{E}) = C(||\Lambda_w F_{\tilde{D}}||^2_{L^2} - ||F_{\tilde{D}}||_{L^2}^2)
\end{equation}
for some positive constant $C$.

Now, since $\tilde{D}$ is $G$-invariant, it induces a flat connection on the analytic locally-free sheaf $(\mathcal{E}|_{S_{\text{reg}}}^\text{an})$ restricted to the smooth locus $S_{\text{reg}}$ of $\mathcal{S}$ with a finite number of (smooth) points removed. Since removing smooth points from $S_{\text{reg}}$ does not change its fundamental group, we find that $(\mathcal{E}|_{S_{\text{reg}}}^\text{an})$ is given by a unitary representation $\rho_S : \pi_1(S_{\text{reg}}) \to U(r, \mathbb{C}) \subset GL(r, \mathbb{C})$ with $r$ the rank of $\mathcal{E}$. This representation is irreducible for otherwise $(\mathcal{E}|_{S_{\text{reg}}}^\text{an})$ is a holomorphic direct sum of proper subbundles $E_0, E_1, \ldots$ with zero slope and we reach a contradiction to the stability of $\mathcal{E}$ by the existence of the zero slope reflexive subsheaf $j_* \mathcal{O}_{S_{\text{reg}}}^\text{an}(E_0)$ in $j_* (\mathcal{E}|_{S_{\text{reg}}}^\text{an})^\text{an} = \mathcal{E}^\text{an}$. For future references, we summarize this discussion as follows.

3.1.6. Given a normal irreducible surface $S$ with only quotient singularities, let $\mathcal{E}$ be a stable (respectively polystable) reflexive sheaf over $S$. If $\hat{\text{c}}_1(\mathcal{E}) \cdot A = 0$ for some ample divisor $A$ on $S$ and $\hat{\text{ch}}_2(\mathcal{E}) = 0$, then the analytification of $\mathcal{E}|_{S_{\text{reg}}}^\text{an}$ is given by a unitary, irreducible (respectively possibly reducible) representation $\rho_S : \pi_1(S_{\text{reg}}) \to GL(r, \mathbb{C})$. 
3.A. Proof of Theorem 1.1. Observe that we only need to treat the stable case. This is because if \( \mathcal{F} = \oplus_i \mathcal{F}_i \) is polystable, then, for each \( i \), as \( \mu_\mathcal{F}_i = 0 \), the Hodge-index theorem implies that
\[
\hat{c}_2(\mathcal{F}_i) \cdot H_1 \cdot \ldots \cdot H_{n-2} \leq 0 \text{ and so the Bogomolov inequality (2.3.2)}
\]
\[
\hat{c}_2(\mathcal{F}_i) \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq \frac{1}{r_i} \hat{c}_2(\mathcal{F}_i) \cdot H_1 \cdot \ldots \cdot H_{n-2} \quad (r_i := \text{rank}(\mathcal{F}_i))
\]
gives \( \hat{c}_2(\mathcal{F}_i) \cdot H_1 \cdot \ldots \cdot H_{n-2} \geq 0 \) for all \( i \). Hence, the additivity of the second Q-Chern character implies that every \( h \)-stable component \( \mathcal{F}_i \) verifies \( \hat{c}_2(\mathcal{F}_i) \cdot H_1 \cdot \ldots \cdot H_{n-2} = 0 \).

We first assume that \( (\mathcal{F}|_{X_{\text{reg}}})^{\text{an}} \) is given by an irreducible unitary representation of \( \pi_1(X_{\text{reg}}) \). Then, for every smooth irreducible curve \( C \) cut out by general hyperplane sections corresponding to high enough multiples of a polarization \( h \)—where the Theorem of Mehta-Ramanathan [MR82] holds—the restriction \( (\mathcal{F}|_C)^{\text{an}} \) also comes from an irreducible unitary representation of \( \pi_1(C) \) via the surjectivity (by the Lefschetz theorem) of the push-forward of the fundamental group induced by the inclusion \( C \hookrightarrow X_{\text{reg}} \). Now, according to the classical result of Narasimhan and Seshadri, the bundle \( (\mathcal{F}|_C)^{\text{an}} \) is stable with degree zero. Therefore \( \mathcal{F} \) is stable of degree zero with respect to any polarization \( h \) (and hence is generically semi-positive). This gives one direction of the theorem.

To prove the reverse direction, we argue as follows: Let \( \mathcal{V} \) be the family of reflexive sheaves \( \mathcal{G} \) on \( X \) of rank \( r \) whose restriction \( (\mathcal{G}|_{X_{\text{reg}}})^{\text{an}} \) is defined by an irreducible unitary representation of \( \pi_1(X_{\text{reg}}) \). Recall that \( \mathcal{V} \) is a bounded family: The well-known fact that \( \pi_1(X_{\text{reg}}) \) is finitely presented implies that the collection of irreducible unitary representations of \( \pi_1(X_{\text{reg}}) \) in \( \text{GL}(r, \mathbb{C}) \) is parametrized by a subvariety of \( \text{GL}(r, \mathbb{C}) \times \ldots \times \text{GL}(r, \mathbb{C}) \) (rank(\( \pi_1(X_{\text{reg}}) \))-times). Therefore, the family of locally-free analytic sheaves on \( X_{\text{an}}^{\text{reg}} \) coming from irreducible unitary representation of \( \pi_1(X_{\text{reg}}) \) is bounded. See also 6.0.4 for another proof of this boundedness. Now, let \( S := D_1 \cap \ldots \cap D_{n-2} \) be the complete intersection surface cut out by general members \( D_i \in |m \cdot H_i|, \; m \gg 0 \), which has only quotient singularities and a natural \( \mathbb{Q} \)-structure \( (U_a, \mathcal{V}_a : X_a \rightarrow \mathbb{U}_a) \). Let us denote the restriction \( \mathcal{F}|_S \) by \( \mathcal{F}_S \). We know, by the stable restriction theorem of Mehta-Ramanathan [MR84] generalized to the singular case (an easy consequence of [Lan04, Theorem 5.2] applied to a desingularization of \( X \) or of more classical arguments [Fle84] adapted to our situation) that \( \mathcal{F}_S \) is stable with respect to \( H_{n-1} \) (restricted to \( S \)). It is also a basic fact that \( \mathcal{F}_S \) is reflexive; For given an exact sequence
\[
0 \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow \mathcal{Q} \rightarrow 0
\]
provided by [Har80], where \( \mathcal{G} \) is locally free and \( \mathcal{Q} \) torsion free (which is a characterization of the reflexivity of \( \mathcal{F} \)), the right exact restriction of this sequence to \( D = D_1 \) is necessarily exact by the fact that such a restriction preserves torsion-freeness by [HL10, Lemma 1.1.12] and preserves exactness over the open part of \( D \) where all three sheaves are locally free. Hence by (3.1.6), we find that \( \mathcal{F}^\text{an}|_{S_{\text{reg}}} \) is defined by an irreducible unitary representation \( \rho_S : \pi_1(S_{\text{reg}}) \rightarrow \text{GL}(r, \mathbb{C}) \). On the other hand, from the Lefschetz hyperplane section theorem for quasi-projective varieties [HL85], the inclusion \( S_{\text{reg}} \hookrightarrow X_{\text{reg}} \) induces a group isomorphism
\[
\pi_1(S_{\text{reg}}) \cong \pi_1(X_{\text{reg}}).
\]
Thus \( \rho_S \) gives rise to an irreducible unitary representation \( \rho : \pi_1(X_{\text{reg}}) \rightarrow \text{GL}(r, \mathbb{C}) \). That is, there exists a locally-free, flat, analytic sheaf \( \mathcal{G}^\circ \) on \( X_{\text{reg}} \) (coming from an irreducible
unitary representation) whose restriction to $S$ is isomorphic to $\mathcal{F}^\text{an}_S$:

\[ (3.1.8) \quad \mathcal{F}^\text{an}_S|_{S_{\text{reg}}} \cong \mathcal{F}|_{S_{\text{reg}}} \]

Theorem 1.1 now follows from Proposition 2.12. \hfill \Box

The next corollary is now an immediate consequence of Theorem 1.1 and the result on extending flat sheaves (Theorem 2.11) across the singular locus of a klt variety after going to a suitable cover (where the contributions of the singularities to the algebraic fundamental group of the smooth locus disappear), Theorem 2.10.

**Corollary 3.2** (Desingularization of (Poly)Stable Reflexive Sheaves with Vanishing $\mathbf{Q}$-Chern Classes Up to a Finite Quasi-étale Cover). Let $X$ be a klt projective variety. There exists a finite Galois morphism $f : Y \to X$ étale over $X_{\text{reg}}$ with Galois group $G$ such that $f^*\mathcal{F}$ is a locally-free sheaf given by a $G$-equivariantly irreducible unitary representation of $\pi_1(Y)$ (respectively, a direct sum of such sheaves) for every reflexive sheaf $\mathcal{F}$ on $X$ verifying the conditions (1.1.1) and (1.1.2) in Theorem 1.1.

This holds in particular for rank-one reflexive sheaves associated to $\mathbf{Q}$-Cartier divisors that are numerically equivalent to zero.

**3.B. A proof of Theorem 1.2 via polystability.** Let $g : Z \to X$ be the global index-1 cover provided in the last part of Corollary 3.2 (see also the proof of Proposition 2.9). Then $K_Z = g^*[X]$ is a numerically trivial Cartier divisor and $Z$ has only canonical singularities. According to the main result of [GKP12], there exists a quasi-étale cover $h : \hat{Z} \to Z$ where $\mathcal{F}_\hat{Z}$ is polystable with respect to the polarization $h'^* = (h'^* (H_1), \ldots, h'^* (H_{n-1}))$ with $h' := h \circ \hat{g}$. Since both $g$ and $h$ are unramified in codimension-1, we have the sheaf isomorphism $\det \mathcal{F}_\hat{Z} \cong \det (h'^* \mathcal{F}_X)$ by the ramification formula. As a result, the natural inclusion of reflexive sheaves $\mathcal{F}_\hat{Z} \to h'^* \mathcal{F}_X$ is an isomorphism. On the other hand, as the $\mathbf{Q}$-Chern classes behave well under quasi-étale morphisms (see Lemma 2.7), the assumption $c_2(\mathcal{F}_X) \cdot H_1 \cdots H_{n-2} = 0$ implies that

\[ c_2(\mathcal{F}_\hat{Z}) \cdot h'^* (H_1) \cdots h'^* (H_{n-2}) = 0. \]

Therefore by Corollary 3.2 we have a morphism $f : Y \to \hat{Z}$ that is étale in codimension-1 such that $\mathcal{F}_Y = f^* \mathcal{F}_\hat{Z}$ is locally-free (and flat). According to the resolution of Lipman-Zariski conjecture for klt spaces [GKKP11, Thm. 6.1], this implies that $Y$ is smooth. In particular $X$ has only quotient singularities. But again, as $f$ is étale in codimension-1, we have $K_Y \equiv 0$ and $c_2(\mathcal{F}_Y) \cdot f^*(h'^* (H_1)) \cdots f^*(h'^* (H_{n-2})) = 0$.

The “if” direction of Theorem 1.2 now follows from the classical uniformization result due to the fundamental work of Yau in [Yau78] on the existence of a Ricci-flat metric in this case. See for example [Kob87, IV.Cor. 4.15] or argue directly that the Ricci flat metric is actually flat using the Riemann bilinear relations (3.1.5) so that the fundamental group of $Y$ must act by isometry on the flat universal cover $\mathbb{C}^n$, which then must be an extension of a lattice in $\mathbb{C}^n$ by a group of isometries of the lattice fixing the origin, this latter group easily seen to be finite.

To prove the “only if” direction of Theorem 1.2, notice that if $X$ is a finite quotient of an Abelian variety by a finite group acting freely in codimension-1, then it follows from the definition that $\hat{c}_1(\mathcal{F}_X) = \hat{c}_2(\mathcal{F}_X) = 0$. We observe easily that $X$ is normal in this case and thus according to [KM98, Prop. 5.20] that $X$ has at most klt singularities. \hfill \Box
4. SEMISTABLE $\mathbb{Q}$-SHEAVES AND THEIR CORRESPONDENCE WITH FLAT SHEAVES

We give a proof of Theorem 1.4 following the classical approach via the Jordan-Hölder filtration of a semistable sheaf, which also explicit a natural and necessary part of Simpson’s proof of his celebrated correspondence. Theorem 1.2 is then a corollary of the local freeness result (up to a finite cover) of Theorem 1.4 and the generic semipositivity theorem of the cotangent sheaf due to Miyaoka [Miy85, Miy87]. We remark that our orbifold generalizations in Theorem 1.4 of this important result of Simpson seem not to be previously known even in dimension 2, when $X$ is an orbifold (for which the orbifold fundamental group is given by $\pi_1(X_{\text{reg}})$).

4.A. Proof of Theorem 1.4. If $E$ is $h$-stable, then the result follows from Theorem 1.1 and Corollary 3.2.

In general, consider first the case when $X = S$ is a surface. Let $f: \hat{S} \to S$ be the Galois, étale in codimension-1 cover given in Theorem 2.10, with Galois group $G$, over which locally-free flat sheaves on $\hat{S}_{\text{reg}}$ extend (Proposition 5.2). Denote the reflexive $G$-sheaf $f^{[*]} E$ by $\hat{E}$. It is semistable (and hence, by its $G$-equivariance, $G$-semistable) with respect to $\hat{h} := f^* h$ by the uniqueness of the maximal destabilizing subsheaf of $\hat{E}$, so that this subsheaf is a $G$-subsheaf, and the semistability of $E$. Let $0 = \hat{E}_0 \subset \hat{E}_1 \subset \hat{E}_2 \subset \ldots \subset \hat{E}_{\text{r}_i - 1} \subset \hat{E}_i = E$

be a $(G)$-semistable filtration of $\hat{E}$, by which we mean that each $\hat{E}_i$ is a saturated zero-slope $\hat{h}$-semistable $(G)$-subsheaf of $\hat{E}$. Then $\hat{E}_i := \hat{E}_i / \hat{E}_{i-1}$ is a zero-slope $(G)$-semistable torsion-free sheaf. Any such filtration can be completed on general grounds to a (possibly longer) filtration of the same type but with $\hat{E}_i (G)$-stable, known as a $(G)$-Jordan-Hölder filtration of $\hat{E}$. In particular, these latter filtrations exist for $\hat{E}$ (but in general non-unique) and any $G$-filtration can be completed to a Jordan-Hölder filtration. Note that the torsion-freeness of $\hat{E}_i$ implies that it differs from $\hat{E}_i := (\hat{E}_i)^{\ast \ast}$ at most on a codimension two subset of $\hat{S}$ so that $\hat{E}_i$ inherits the stability conditions of $\hat{E}_i$.

We analyze the following two special cases of (4.0.1):

Case 1, when (4.0.1) is the Jordan-Hölder filtration of $\hat{E}$:

By [Meg92, Lem. 10.9], we have $\hat{c}_2(\hat{E}_i) \leq \hat{c}_2(\hat{E}_i)$ with equality if and only if $\hat{E}_i = \hat{E}_i$. As $\hat{c}_1(\hat{E}_i) = \hat{c}_1(\hat{E}_i)$, this is the same as $\hat{c}_2(\hat{E}_i) \leq \hat{c}_2(\hat{E}_i)$. Since $\hat{c}_1(\hat{E}_i) : H_1 = 0$, the Hodge-index theorem implies that

(4.0.2) $\hat{c}_1^2(\hat{E}_i) \leq 0$.

But as $\hat{E}_i$ is $H_1$-semistable, we have, thanks to the Bogomolov inequality (2.3.2), that

$$-\hat{c}_2(\hat{E}_i) = 2\hat{c}_2(\hat{E}_i) - \hat{c}_1^2(\hat{E}_i) \geq \frac{-1}{r_i} \hat{c}_1^2(\hat{E}_i) \quad (r_i = \text{rank}(\hat{E}_i))$$

$$\geq 0, \quad \text{by the inequality (4.0.2).}$$

That is, $\hat{c}_2(\hat{E}_i) \leq 0$ for all $i$. Together, by the additivity of the second $\mathbb{Q}$-Chern character for direct sums, we have

(4.0.3) $0 = \hat{c}_2(\hat{E}) = \sum_i \hat{c}_2(\hat{E}_i) \leq \sum_i \hat{c}_2(\hat{E}_i) \leq 0$,

and thus $\hat{c}_2(\hat{E}_i) = \hat{c}_2(\hat{E}_i) = 0$. It follows that $\hat{E}_i = \hat{E}_i$, which is hence, by (3.1.6) and Proposition 2.11, locally-free and flat for each $i$. This in turn implies that each $\hat{E}_i$ is an
iterated extension of flat and locally free sheaves and is thus locally free.

Case 2, when (4.0.1) is the $G$-Jordan-Holder filtration of $\mathcal{E}$:

Since such a filtration can be completed to a Jordan-Holder filtration of $\mathcal{E}$, Case 1 implies that each $G$-stable $\mathcal{E}_i$ is locally free. Let $\pi: \hat{S} \to S$ be a $G$-equivariant resolution of $S$. The local freeness of the terms in the $G$-Jordan-Holder filtration of $\mathcal{E}$ as well as in its grading and the $G$-equivariance implies that the filtration lifts to a locally free filtration of $\pi^* (\mathcal{E})$ with terms $\pi^* (\mathcal{E}_i)$. By Proposition 3.1, the locally free sheaves $\pi^* (\mathcal{E}_i)$ are all $G$-stable with respect to a fixed $G$-invariant polarization. As before, every $G$-stable grading $\pi^* (\mathcal{E}_i)$ of the filtration of $\pi^* (\mathcal{E})$ induced by that of $\mathcal{E}$ admits a $G$-invariant unitarily flat connection. Hence, $\pi^* (\mathcal{E})$ is an iterated extension by unitarily flat and locally free sheaves which is therefore, by Simpson’s correspondence [Sim92, Cor. 3.10] (and the remarks immediately after that corollary), endowed with a unique flat connection. Since the filtration is $G$-equivariant, the uniqueness of this flat connection implies that it is $G$-invariant. This yields a $G$-invariant flat connection on $\mathcal{E} |_{S_{\text{reg}}}$ and therefore one on $\mathcal{E}$ by its reflexivity and Proposition 2.11. As a result $\mathcal{E} |_{S_{\text{reg}}}$ is locally free and flat and therefore comes from a representation of $\pi_1 (S_{\text{reg}})$.

We have thus established the theorem in the case $X$ is a surface.

For higher dimensional $X$, exactly the same argument as that of Section 3, via induction on $\dim X$ using Proposition 2.12, gives the semistable analog of Theorem 1.1 and Corollary 3.2. The remaining (very last) part of Theorem 1.4 now follows from the last part of Lemma 2.7. \hfill $\Box$

4.B. A proof of Theorem 1.2 via semistability. According to Proposition 2.9 the tangent sheaf $\mathcal{T}_X$ is generically semi-positive, that is, as $K_X \equiv 0$, $\mathcal{T}_X$ is semistable independent of the choice of polarization. Therefore, Theorem 1.4 says that there exists a finite quasi-étale cover $f: Y \to X$ such that $f^{[*]} \mathcal{T}_X$ is locally-free. But since $f$ is unramified in codimension-1, we have the sheaf isomorphism $\det \mathcal{T}_Y \cong \det (f^{[*]} \mathcal{T}_X)$ by the ramification formula. As a result, the natural inclusion of reflexive sheaves $\mathcal{T}_Y \to f^{[*]} \mathcal{T}_X$ is an isomorphism and $\mathcal{T}_Y$ is locally-free. The rest of the proof is identical to that of the polystable case. \hfill $\Box$

We now briefly explain the equivalence of the two sets of conditions $\{(1.2.1), (1.2.2)\}$ and $\{(1.3.1), (1.3.2)\}$.

Explanation 4.1. Assume that $X$ is a klt projective variety verifying condition (1.2.1), i.e. $K_X \equiv 0$. Then by Lemma 2.9 we know that there exists a finite quasi-étale morphism $f: Y \to X$ such that $\mathcal{T}_Y$ is generically semipositive. But $f$ being étale in codimension-1 implies that $\mathcal{T}_Y = f^{[*]} \mathcal{T}_X$ and thus $\mathcal{T}_X$ is also generically semipositive by Lemma 2.8, establishing condition (1.3.1). As $c_1(\mathcal{T}_X)$ agrees with $c_1(\mathcal{T}_Y) = c_1(K_X)$ in codimension-1 by construction, we have $\mathcal{T}_1(\mathcal{T}_X)|_S \equiv 0$ for a general surface $S$ cut out by $n - 2$ very ample divisors and thus $c_2^\mathcal{T}(\mathcal{T}_X): S = 0$. Hence (1.3.2) follows from (1.2.2). Conversely, assume that the conditions (1.3.1) and (1.3.2) hold. Then, by Theorem 1.4 we know that there exists a finite quasi-étale morphism $f: Y \to X$ such that $\mathcal{T}_Y = f^{[*]}(\mathcal{T}_X)$ is flat and locally free so that it has vanishing (orbifold) Chern classes and, in particular, $K_Y$ is numerically trivial. The ramification formula $K_Y = f^{[*]}(K_X)$ (together with the projection formula) implies that $K_X \equiv 0$ and condition (1.2.2) follows by Lemma 2.7.
5. Remarks on the Projectively-flat Case

Remark 5.1 (The Projectively-Flat Case). The arguments of Section 3 lead to a slight generalization of Theorem 1.1 of the correspondence between stability of reflexive sheaves and representations of fundamental groups. Indeed, over a projective variety $X$ with only quotient singularity in codimension-2, any $h$-polystable reflexive sheaf $\mathcal{E}$ of rank $r$, where $h := (H_1, \ldots, H_{n-1})$, $H_i$ being ample, verifying the Bogomolov equality

\[
(5.1.1) \quad (2r \cdot c_2(\mathcal{E}) - (r - 1) \cdot c_1^2(\mathcal{E})) \cdot H_1 \cdot \ldots \cdot H_{n-2} = 0
\]

restricts to a unitary, projectively-flat and locally-free sheaf on $X_{\text{reg}}$, i.e. $\mathcal{E}|_{X_{\text{reg}}}$ is locally free and comes from a representation $\rho^\circ : \pi_1(X_{\text{reg}}) \to \text{PU}(r, \mathbb{C})$. When $X$ is smooth (or compact Kähler) and $\mathcal{E}$ is locally-free, the fact that $\mathcal{E}$ is projectively-flat is well-known and follows from the Bogomolov equality and equation (3.1.5):

\[
(5.1.2) \quad \frac{1}{r} \int_X c_1^2(\mathcal{E}) \wedge [w]^{n-2} = \int_X \text{ch}_2(\mathcal{E}) = B(||A_wF_D||^2 - ||F_D||^2) \wedge [w]^n,
\]

where $B$ is a constant, and the existence of a unique Hermitian-Einstein connection $D$ on $\mathcal{E}$ ([UY86] and [Don87]). Hence, exactly the same argument as in Section 3 applies.

Remark 5.2 (Projectively-Flat Sheaves over Klt Varieties). In the case $X$ is a klt variety, Proposition 2.11 also easily generalizes to give the existence of a normal projective variety $Y$ and a finite morphism $f : Y \to X$ that is etale in codimension-1 such that $f^*(\mathcal{E})$ is a projectively-flat and locally-free sheaf on $Y$, for every polystable reflexive sheaf $\mathcal{E}$ that satisfies the Bogomolov equality. For this purpose we only need to show (as in [GKP14a, Sect. 11]) that every representation $\rho^\circ : \pi_1(Y_{\text{an}}) \to G \subset \text{PGL}(r, \mathbb{C})$ factors through a representation $\rho : \pi_1(Y_{\text{an}}) \to G$:

\[
(5.2.1) \quad \pi_1(Y_{\text{an}}) \xrightarrow{i_*} \pi_1(Y_{\text{an}}) \xrightarrow{\rho} G,
\]

where $i_*$ is induced by the inclusion map $i : Y_{\text{reg}} \hookrightarrow Y$. Notice that the functoriality of profinite completions implies that we have a commutative diagram

\[
\begin{array}{ccc}
\hat{G} & \xrightarrow{\rho^\circ} & \hat{\pi}_1(Y_{\text{an}}) \\
\downarrow \hat{i}_* \cong & & \downarrow \hat{i}_* \\
G & \xrightarrow{\rho^\circ} & \pi_1(Y_{\text{an}})
\end{array}
\]

whose vertical arrows are the profinite completions of the corresponding groups. Now by [FL81], the induced group homomorphism $i_*$ is surjective. Furthermore as $\text{PGL}(r, \mathbb{C})$ is a finite quotient of $\text{SL}(r, \mathbb{C})$ and as every finitely-generated linear group is residually finite by the classical theorem of Malcev, we find that $G$ is residually finite so that the profinite completion of $G$, given by $G \to \hat{G}$, is an injection. This finishes the claim.

6. Appendix: A Proof of Proposition 2.12 (The Enriques-Severi Lemma)

Let $\mathcal{F}$ be a reflexive sheaf on $X$ and $\mathcal{V}$ be the family of algebraic reflexive sheaves $\mathcal{G}$ on $X$ of the same rank such that $(\mathcal{G}|_{X_{\text{reg}}})_{\text{an}}$ is defined by a representation of $\pi_1(X_{\text{reg}})$. 
Claim 6.0.2. There exists a sufficiently large positive integers $d$ such that for every general member $D$ of $|dH|$ we have

\begin{equation}
H^1(D, \mathcal{H}om(\mathcal{G}, \mathcal{F}) \otimes \mathcal{O}_X(-dH)) = 0, \quad \forall \mathcal{G} \in \mathcal{V}.
\end{equation}

In particular, the natural map $H^0(X, \mathcal{H}om(\mathcal{G}, \mathcal{F})) \to H^0(D, \mathcal{H}om(\mathcal{G}, \mathcal{F})|_D)$ is onto.

Proof of Claim 6.0.2. The existence of $d$ is guaranteed by the following two facts:

6.0.4. Boundedness of the Family $\mathcal{V}$: The boundedness argument given in the proof of Theorem 1.1 also applies here, but we offer another approach as follows. By repeatedly taking hyperplane sections and by applying the Lefschetz theorem of Hamm and Lê [HL85], we assume without loss of generality for the proof that $X$ is a surface. We go now to the Galois klt cover $\tilde{Y}$ with $\tilde{\pi}_1(\mathcal{Y}_{\text{reg}}) = \pi_1(\mathcal{Y})$ provided by Theorem 2.10 and notice that, over $\mathcal{X}_{\text{reg}}, \mathcal{V}$ can be identified with the family of locally free flat sheaves on $\mathcal{Y}$ that are equivariant with respect to the Galois group of the cover which one can further identify with the same on an equivariant resolution of singularity $h: \tilde{Y} \to Y$ of $Y$. Since the latter family has vanishing Chern classes by construction, Hirzebruch-Riemann-Roch implies that the Hilbert polynomial of a member of this family with respect to $h^*H$ for an ample polarization $H$ is a constant polynomial over this family. As $Y$ is has only rational singularities, the Leray spectral sequence and the projection formula show that the same holds for the family on $\mathcal{Y}$, which is therefore bounded by [HL10, Corollary 3.3.7]. Note that the argument gives the vanishing of the $h$-slope of every member of $\mathcal{V}$ for all $h$.

6.0.5. Vanishing: The normality of $X$ together with the assumption that $\mathcal{F}$ is reflexive imply that $\mathcal{H}om(\mathcal{G}, \mathcal{F})$ is reflexive (since its double dual is then $\mathcal{H}om(\mathcal{G}^{**}, \mathcal{F})$) the germs (of sections) of which corresponds naturally and bijectively with that of $\mathcal{H}om(\mathcal{G}, \mathcal{F})$ and so, thanks to [Har80, Prop. 1.3], that depth($\mathcal{H}om(\mathcal{G}, \mathcal{F}))_x \geq 2$, for all $x \in X$. Therefore, according to [SGA2, §XII, Corollary 1.4], we can find a positive integer $d$ for which the equality (6.0.3) holds for any fixed member $\mathcal{G}$ of the family $\mathcal{V}$. The claim follows as $\mathcal{V}$ is bounded.

Now suppose $\mathcal{F}|_{D_0}$ is locally free and flat, where $D_0 := D \cap \mathcal{X}_{\text{reg}} \subset D_{\text{reg}}$. Then it comes from a representation of $\pi_1(D_0)$. We suppose further, by taking $d$ sufficiently large, that $\mathcal{F}|_{D}$ is reflexive over $D$ and, by the Lefschetz hyperplane section theorem for quasi-projective varieties [HL85], that the group homomorphism $\pi_1(D_0) \to \pi_1(X_{\text{reg}})$ induced by the closed embedding $D_0 \to X_{\text{reg}}$ is an isomorphism. This means that $\rho|_{D}$ gives rise to a representation $\rho: \pi_1(X_{\text{reg}}) \to GL(r,\mathbb{C})$ and thus that there exists a locally-free, flat, analytic sheaf $\mathcal{G}$ on $X_{\text{reg}}$ whose restriction to $D_0$ is isomorphic to $\mathcal{F}|_{D_0}$. Define the reflexive sheaf $\mathcal{G}$ on $X$ by $\mathcal{G} := i_*(\mathcal{G})$, where $i: X_{\text{reg}} \hookrightarrow X$ is the natural inclusion. Then the isomorphism (in fact, identification) of the locally-free sheaves $\mathcal{G}|_{D_0}$ and $\mathcal{F}|_{D_0}$ over $D_0$ yields a section $i_0$ of $\mathcal{H}om(\mathcal{G}|_{D_0}, \mathcal{F}|_{D_0}) = \mathcal{H}om(\mathcal{G}|_{D}, \mathcal{F}|_{D})|_{D_0}$. As $\mathcal{H}om(\mathcal{G}|_{D}, \mathcal{F}|_{D})$ is reflexive over $D$ (since $\mathcal{F}|_{D}$ is) and as $D$ is normal, $i_0$ extends to a section $i_0$ of $\mathcal{H}om(\mathcal{G}, \mathcal{F})|_{D}$, and hence a section of $\mathcal{H}om(\mathcal{G}, \mathcal{F})|_{D}$ which is therefore induced by a section $i$ of $\mathcal{H}om(\mathcal{G}, \mathcal{F})|_{D}$. By Claim 6.0.2, $i$ extends to a section $i$ of $\mathcal{H}om(\mathcal{G}, \mathcal{F})$ which we will identify with its induced morphism $i: \mathcal{G} \to \mathcal{F}$.

The reflexivity of the two sheaves $\mathcal{G}$ and $\mathcal{F}$ implies that they are isomorphic over an open set intersecting $D_0$; the restriction functor from an open set $U_0 \hookrightarrow X_{\text{reg}}$ where the torsion free sheaves in the exact sequence $0 \to \ker i \to \mathcal{G} \to \text{image} i \to 0$ are locally free


(hence having complement of codimension $\geq 2$ in $X$) to the nonempty closed subvariety $D \cap U_0$ is exact and the rank of a morphism is locally constant where-ever maximal. In particular, the morphism $\bar{\iota}$ is injective as $\ker \bar{\iota}$ is torsion free. Also, since $\mathcal{F}$ and $\mathcal{G}$ are locally free in codimension-2 on $X_{\text{reg}}$, their slopes with respect to the polarization $h = (h_D, H)$ are determined by the slopes of their restriction to $D$ with respect to a polarization $h_D$ on $D$. The last statement of Proposition 2.12 now follows as an injective morphism between reflexive sheaves of the same rank and slope and having maximal rank on an open is necessarily an isomorphism in codimension one, and hence an isomorphism.

The argument of the last paragraph works verbatim to give the first part of Proposition 2.12 for a bounded family $\mathcal{V}$ of reflexive sheaves.

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