Bounded solutions for a class of Hamiltonian systems

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Abstract

We obtain bounded for all \( t \) solutions of ordinary differential equations as limits of the solutions of the corresponding Dirichlet problems on \((-L, L)\), with \( L \to \infty \). We derive a priori estimates for the Dirichlet problems, allowing passage to the limit, via a diagonal sequence. This approach carries over to the PDE case.

Key words: Bounded for all \( t \) solutions, a priori estimates.

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1 Introduction

For \(-\infty < t < \infty\), we consider the equation

\[
(1.1) \quad u'' - a(t)u^3 = f(t),
\]

with continuous functions \( a(t) > 0 \) and \( f(t) \). Clearly, “most” solutions of (1.1) blow up in finite time, for both increasing and decreasing \( t \). By using two-dimensional shooting, S.P. Hastings and J.B. McLeod [3] showed that the equation (1.1) has a uniformly bounded on \((-\infty, \infty)\) solution, in case of constant \( a(t) \) and uniformly bounded \( f(t) \). Their proof used some non-trivial topological property of a plane. We use a continuation method and passage to the limit as in P. Korman and A.C. Lazer [4] to obtain the existence of a uniformly bounded on \((-\infty, \infty)\) solution for (1.1), and for similar systems. We produce a bounded solution as a limit of the solutions of the corresponding Dirichlet problems

\[
(1.2) \quad u'' - a(t)u^3 = f(t) \quad \text{for } t \in (-L, L), \quad u(-L) = u(L) = 0,
\]

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as $L \to \infty$. If $f(t)$ is bounded, it follows by the maximum principle that the solution of (1.2) satisfies a uniform in $L$ a priori estimate, which allows passage to the limit.

Then we use a variational approach motivated by P. Korman and A.C. Lazer [4] (see also P. Korman, A.C. Lazer and Y. Li [5]), to get a similar result for a class of Hamiltonian systems. Again, we consider the corresponding Dirichlet problem on $(-L, L)$, which we solve by the minimization of the corresponding functional, obtaining in the process a uniform in $L$ a priori estimate, which allows passage to the limit as $L \to \infty$.

We used a similar approach to obtain uniformly bounded solutions for a class of PDE systems of Hamiltonian type. The challenge was to adapt the elliptic estimates in case only the $L^\infty$ bound is known for the right hand side.

2 A model equation

**Theorem 2.1.** Consider the equation (for $u = u(t)$)

\[
(2.1) \quad u'' - a(t)u^3 = f(t),
\]

where the given functions $a(t) \in C(\mathbb{R})$ and $f(t) \in C(\mathbb{R})$ are assumed to satisfy

\[
|f(t)| \leq M, \quad \text{for all } t \in \mathbb{R}, \text{ and some constant } M > 0,
\]

and

\[
a_0 \leq a(t) \leq a_1, \quad \text{for all } t \in \mathbb{R}, \text{ and some constants } a_1 \geq a_0 > 0.
\]

Then the problem (2.1) has a classical solution uniformly bounded for all $t \in \mathbb{R}$, i.e., $|u(t)| \leq K$ for all $t \in \mathbb{R}$, and some $K > 0$. Such a solution is unique.

**Proof.** We shall obtain a bounded solution as a limit of solutions to the corresponding Dirichlet problems

\[
(2.2) \quad u'' - a(t)u^3 = f(t) \quad \text{for } t \in (-L, L), \quad u(-L) = u(L) = 0,
\]

as $L \to \infty$. To prove the existence of solutions, we embed (2.2) into a family of problems

\[
(2.3) \quad u'' - \lambda a(t)u^3 = f(t) \quad \text{for } t \in (-L, L), \quad u(-L) = u(L) = 0,
\]

with $0 \leq \lambda \leq 1$. The solution at $\lambda = 0$, and other $\lambda$, can be locally continued in $\lambda$ by the implicit function theorem, since the corresponding linearized problem

\[
u''(t) - 3\lambda a(t)u^2(t)w(t) = 0 \quad \text{for } t \in (-L, L), \quad w(-L) = w(L) = 0
\]

has only the trivial solution $w(t) \equiv 0$, as follows by the maximum principle. Multiplying (2.3) by $u$ and integrating, we get a uniform in $\lambda$ bound on $H^1$ norm of the solution, which implies the bound in $C^2$ (using Sobolev’s embedding and
the equation (2.3); this bound depends on $L$. It follows that the continuation can be performed for all $0 \leq \lambda \leq 1$. At $\lambda = 1$, we get the desired solution of (2.2).

We claim that there is a uniform in $L$ bound in $C^2[-L,L]$ for any solution of (2.2), i.e., there is a constant $K > 0$, so that for all $t \in [-L,L]$, and all $L > 0$,

$$|u(t)| \leq K, \quad |u'(t)| \leq K, \quad \text{and} \quad |u''(t)| \leq K.$$

Indeed, if $t_0$ is a point of positive maximum of $u(t)$, then from the equation (2.3) we get

$$-a_0u^3(t_0) \geq f(t_0) \geq -M,$$

which gives us an upper bound on $u(t_0)$. Arguing similarly at a point of negative minimum of $u(t)$, we get a lower bound on $u(t)$, and then conclude the first inequality in (2.4). From the equation (2.3) we get a uniform bound on $|u''(t)|$. Note that for all $t \in \mathbb{R}$, we can write

$$u(t + 1) = u(t) + u'(t) + \int_{t}^{t+1} (t + 1 - \xi)u''(\xi)\,d\xi,$$

from which we immediately deduce a uniform bound on $|u'(t)|$.

We now take a sequence $L_j \to \infty$, and denote by $u_j(t) \in H^1_0(-\infty,\infty)$ the bounded solution of the problem (2.2) on the interval $(-L_j,L_j)$, extended as zero to the outside of the interval $(-L_j,L_j)$. For all $t_1 < t_2$, writing

$$|u_j(t_2) - u_j(t_1)| = \left|\int_{t_1}^{t_2} u'_j \,dt\right| \leq \sqrt{t_2 - t_1} \left(\int_{t_1}^{t_2} (u'_j)^2 \,dt\right)^{1/2} \leq K(t_2 - t_1),$$

in view of (2.6), we conclude that the sequence $\{u_j(t)\}$ is equicontinuous and uniformly bounded on every interval $[-L_p,L_p]$. By the Arzela-Ascoli theorem, it has a uniformly convergent subsequence on every $[-L_p,L_p]$. So let $\{u_{j_k}(t)\}$ be a subsequence of $\{u_j(t)\}$ that converges uniformly on $[-L_1,L_1]$. Consider this subsequence on $[-L_2,L_2]$ and select a further subsequence $\{u_{j_k}^{(2)}\}$ of $\{u_{j_k}(t)\}$ that converges uniformly on $[-L_2,L_2]$. We repeat this procedure for all $m$, and then take the diagonal sequence $\{u_{j_k}^{(m)}\}$. It follows that it converges uniformly on any bounded interval to a function $u(t)$.

Expressing $(u_{j_k}^{(m)})''$ from the equation (2.2), we conclude that the sequence $\{(u_{j_k}^{(m)})''\}$, and then also $\{(u_{j_k}^{(m)})'\}$ (in view of (2.6)), converge uniformly on bounded intervals. Denote $v(t) := \lim_{k \to \infty} (u_{j_k}^{(m)})''(t)$. For $t$ belonging to any bounded interval $(a,b)$, similarly to (2.7), we write

$$u_{j_k}^{(m)}(t) = u_{j_k}^{(m)}(a) + (t - a) (u_{j_k}^{(m)})'(a) + \int_{a}^{t} (t - \xi) (u_{j_k}^{(m)})''(\xi)\,d\xi,$$

and conclude that $u(t) \in C^2(-\infty,\infty)$, and $u''(t) = v(t)$. Hence, we can pass to the limit in the equation (2.2), and conclude that $u(t)$ solves this equation on
We have $|u(t)| \leq K$ on $(-\infty, \infty)$, proving the existence of a uniformly bounded solution.

Turning to the uniqueness, the difference $w(t)$ of any two bounded solutions $u(t)$ and $\bar{u}(t)$ of (2.1) would be a bounded for all $t$ solution of the linear equation

\begin{equation}
\tag{2.7}
w'' - b(t)w = 0,
\end{equation}

with $b(t) = a(t)(u^2 + u\bar{u} + \bar{u}^2) > 0$. It follows that $w(t)$ is convex when it is positive. If at some $t_0$, $w(t_0) > 0$ and $w'(t_0) > 0$ ($w'(t_0) < 0$), then $w(t)$ is unbounded as $t \to \infty$ ($t \to -\infty$), a contradiction. A similar contradiction occurs if $w(t_0) < 0$ for some $t_0$. Therefore, $w \equiv 0$.

**Remark 1.** To prove the existence of solutions of (2.2), we could alternatively consider the corresponding variational functional $J(u) : H^1_0(-L, L) \to \mathbb{R}$, defined by

\[
J(u) = \int_{-L}^{L} \left[ \frac{(u')^2}{2} + a(t)\frac{u^4}{4} + f(t)u \right] dt.
\]

Since for any $\epsilon > 0$

\[
\left| \int_{-L}^{L} f(t)u dt \right| \leq \epsilon \int_{-L}^{L} u^2 dt + c(\epsilon) \int_{-L}^{L} f^2 dt
\]

\[
\leq \epsilon \int_{-L}^{L} u^2 dt + c_1, \text{ with } c_1 = c_1(L, \epsilon),
\]

and

\[
\int_{-L}^{L} u^2 dt \leq c_2(L) \int_{-L}^{L} (u')^2 dt,
\]

we see (noting $a(t)u^4 \geq 0$) that

\[
J(u) \geq c_3 \int_{-L}^{L} (u')^2 dt - c_4
\]

for some $c_3, c_4 > 0$, so that $J(u)$ is bounded from below, coercive and convex in $u'$. Hence $J(u)$ has a minimizer in $H^1_0(-L, L)$, which gives us a classical solution of (2.2), see e.g., L. Evans [1]. However, to get a uniform in $L$ estimate of $\int_{-L}^{L} (u')^2 dt$ (needed to conclude the equicontinuity in (2.6)), one would have to assume that $\int_{-\infty}^{\infty} f^2(t) dt < \infty$, giving a weaker result than above.

We now discuss the dynamical significance of the bounded solution, established in Theorem 2.1, let us call it $u_0(t)$. The difference of any two solutions of (2.1) satisfies (2.7). We see from (2.7) that any two solutions of (2.1) intersect at most once. Also from (2.7), we can expect $u_0(t)$ to have one-dimensional stable manifold as $t \to \pm \infty$. It follows that $u_0(t)$ provides the only possible asymptotic form of the solutions that are bounded as $t \to \infty$ (or $t \to -\infty$), while all other solutions become unbounded.
Next we show that the conditions of this theorem cannot be completely removed. If \( a(t) \equiv 0 \), then for \( f(t) = 1 \), all solutions of (2.1) are unbounded as \( t \to \pm \infty \). The same situation may occur in case \( a(t) > 0 \), if \( f(t) \) is unbounded. Indeed, the equation

\[
(2.8) \quad u'' - u^3 = 2 \cos t - t \sin t - t^3 \sin^3 t
\]

has a solution \( u(t) = t \sin t \). Let \( \tilde{u}(t) \) be any other solution of (2.8). Then \( w(t) = u(t) - \tilde{u}(t) \) satisfies (2.7), with \( b(t) = u^2 + u \tilde{u} + \tilde{u}^2 > 0 \). Clearly, \( w(t) \) cannot have points of positive local maximum, or negative local minimum. But then \( \tilde{u}(t) \) cannot remain bounded as \( t \to \pm \infty \), since in such a case the function \( w(t) \) would be unbounded with points of positive local maximum and negative local minimum. It follows that all solutions of (2.8) are unbounded as \( t \to \pm \infty \).

The approach of Theorem 2.1 is applicable to more general equations and systems. For example, we have the following theorem.

**Theorem 2.2.** Consider the system (for \( u = u(t) \) and \( v = v(t) \))

\[
(2.9) \quad \begin{cases}
    u'' - a_1(t)f(u, v) = h_1(t), \\
    v'' - a_2(t)g(u, v) = h_2(t).
\end{cases}
\]

Assume that the functions \( a_i(t) \in C(\mathbb{R}) \) satisfy \( a_0 \leq a_i(t) \leq a_1 \) for all \( t \in \mathbb{R} \) and some constants \( 0 < a_0 \leq a_1 \), while \( h_i(t) \in C(\mathbb{R}) \) are uniformly bounded, \( i = 1, 2 \). Assume that the functions \( f(x, y) \) and \( g(x, y) \) are continuous on \( \mathbb{R}^2 \), and

\[
(2.10) \quad f(x, y) \to \infty (-\infty) \text{ as } x \to \infty (-\infty), \text{ uniformly in } y,
\]

and

\[
(2.11) \quad g(x, y) \to \infty (-\infty) \text{ as } y \to \infty (-\infty), \text{ uniformly in } x.
\]

Assume that

\[
(2.12) \quad xf(x, y) \geq \alpha, \quad \text{and } yg(x, y) \geq \alpha,
\]

for some \( \alpha \in \mathbb{R} \), and all \( (x, y) \in \mathbb{R}^2 \). Assume finally that the quadratic form in \((w, z)\)

\[
(2.13) \quad a_1(t)f_z(x, y)w^2 + (a_1(t)f_y(x, y) + a_2(t)g_z(x, y)) wz + a_2(t)g_y(x, y)z^2
\]

is positive semi-definite for all \( t, x \) and \( y \). Then the problem (2.9) has a classical solution uniformly bounded for all \( t \in (-\infty, \infty) \).

**Proof.** To prove the existence of solutions for the corresponding Dirichlet problem on \((-L, L)\),

\[
(2.14) \quad \begin{cases}
    u'' - a_1(t)f(u, v) = h_1(t) \quad \text{for } t \in (-L, L), \quad u(-L) = u(L) = 0, \\
    v'' - a_2(t)g(u, v) = h_2(t) \quad \text{for } t \in (-L, L), \quad v(-L) = v(L) = 0,
\end{cases}
\]
we embed it into a family of problems

\begin{align}
(2.15) \quad \begin{cases}
  u'' - \lambda a_1(t) f(u, v) = h_1(t) & \text{for } t \in (-L, L), \quad u(-L) = u(L) = 0, \\
  v'' - \lambda a_2(t) g(u, v) = h_2(t) & \text{for } t \in (-L, L), \quad v(-L) = v(L) = 0,
\end{cases}
\end{align}

with $0 \leq \lambda \leq 1$. The implicit function theorem applies, since the corresponding linearized problem

\begin{align}
\begin{cases}
  u'' - \lambda a_1(t) f(u, v) w + f_y(u, v) z = 0 & \text{for } t \in (-L, L), \\
  v'' - \lambda a_2(t) g(u, v) w + g_y(u, v) z = 0 & \text{for } t \in (-L, L), \\
  w(-L) = w(L) = z(-L) = z(L) = 0
\end{cases}
\end{align}

has only the trivial solution $w = z = 0$. This follows by multiplying the first equation by $w$, the second one by $z$, integrating, adding the results, and using the condition (2.13). Using (2.12), we obtain a uniform in $\lambda$ bound on the $H^1$ norm of the solution of (2.15), so that the continuation can be performed for all $0 \leq \lambda \leq 1$. At $\lambda = 1$, we obtain a solution of (2.15).

From the first equation in (2.14) and the assumption (2.10) we conclude the bound (2.14) on $u(t)$, and a similar bound on $v(t)$ follows from the second equation in (2.14) and the assumption (2.11), the same way as we did for a single equation. Using the equations in (2.14), we obtain uniform bounds on $u''$ and $v''$, and the uniform bounds on $u'$ and $v'$ follow from (2.15). Hence, we have the estimates (2.14) for $u$ and $v$. We then let $L \to \infty$, and pass to the limit along the diagonal sequence, as in the proof of Theorem 2.1, to conclude the proof of Theorem 2.2.

**Example 1.** Theorem 2.2 applies in case $f(x, y) = x + x^{2n+1} + r(y)$, $g(x, y) = y + y^{2m+1} + s(x)$, with positive integers $n$ and $m$, assuming that the functions $r(y)$ and $s(x)$ are bounded and have small enough derivatives for all $x$ and $y$, and the functions $a_i(t)$ and $b_i(t)$, $i = 1, 2$, satisfy the assumptions of the theorem.

### 3 Bounded solutions of Hamiltonian systems

We use variational approach to get a similar result for a class of Hamiltonian systems. We shall be looking for uniformly bounded solutions $u \in H^1(\mathbb{R}; \mathbb{R}^m)$ of the system

\begin{align}
(3.1) \quad u_{ii} - a(t) V_{zz_i}(u_1, u_2, \ldots, u_m) = f_i(t), \quad i = 1, \ldots, m.
\end{align}

Here $u_i(t)$ are the unknown functions, $a(t)$ and $f_i(t)$ are given functions on $\mathbb{R}$, $i = 1, \ldots, m$, and $V(z)$ is a given function on $\mathbb{R}^m$.

**Theorem 3.1.** Assume that $a(t) \in C(\mathbb{R})$ satisfies $a_0 \leq a(t) \leq a_1$ for all $t$, and some constants $0 < a_0, a_1$. Assume that $f_i(t) \in C(\mathbb{R})$, with $|f_i(t)| \leq M$ for some $M > 0$ and all $i$ and $t \in \mathbb{R}$. Also assume that $V(z) \in C^1(\mathbb{R}^m)$ satisfies

\begin{align}
(3.2) \quad \lim_{z_i \to +\infty} V_{z_i} = +\infty, \quad \lim_{z_i \to -\infty} V_{z_i} = -\infty, \quad \text{uniformly in all } z_j \neq z_i,
\end{align}

6
and

\begin{equation}
(3.3) \quad a(t)V(z) + \sum_{i=1}^{m} z_i f_i(t) \geq -f_0(t), \text{ for all } t \in R, \text{ and } z_i \in R,
\end{equation}

with some \( f_0(t) > 0 \) satisfying \( \int_{-\infty}^{\infty} f_0(t) \, dt < \infty \). Then the system (3.1) has a uniformly bounded solution \( u_i(t) \in H^1(\mathbb{R}) \), \( i = 1, \ldots, m \) (i.e., for some constant \( K > 0 \), \( |u_i(t)| < K \) for all \( t \in \mathbb{R} \), and all \( i \)).

**Proof.** As in the previous section, we approximate solution of (3.1) by solutions of the corresponding Dirichlet problems \( i = 1, \ldots, m \)

\begin{equation}
(3.4) \quad u''_i - a(t)V_{z_i}(u) = f_i(t), \quad \text{for } t \in (\mathbb{R} \setminus \mathbb{R}) \text{, } \quad \text{as } L \to \infty.
\end{equation}

Solutions of (3.4) can be obtained as critical points of the corresponding variational functional \( J(u) : [H^1_0(\mathbb{R} \setminus \mathbb{R})]^m \to \mathbb{R} \) defined as

\[ J(u) := \int_{-L}^{L} \left[ \sum_{i=1}^{m} \left( \frac{1}{2} u''_i(t) + u_i(t)f_i(t) \right) + a(t)V(u(t)) \right] \, dt. \]

By (3.3), \( J(u) \geq c_1(L) \sum_{i=1}^{m} \|u_i\|_{H^1(\mathbb{R} \setminus \mathbb{R})} - c_2 \), for some positive constants \( c_1 \) and \( c_2 \), so that \( J(u) \) is bounded from below, coercive and convex in \( u' \). Hence, \( J(u) \) has a minimizer in \( [H^1_0(\mathbb{R} \setminus \mathbb{R})]^m \), giving us a classical solution of (3.4), see e.g., L. Evans [1].

We now take a sequence \( L_j \to \infty \), and denote by \( u_j(t) \in H^1(\mathbb{R}; \mathbb{R}^m) \) a vector solution of the problem (3.4) on the interval \( (-L_j, L_j) \), extended as zero vector to the outside of the interval \( (-L_j, L_j) \). By our condition (3.2), we conclude a component-wise bound of \( |u_j(t)| \), uniformly in \( j \) and \( t \). The crucial observation (originated from [1]) is that the variational method provides a uniform in \( j \) bound on \( \|u'_j(t)\|_{L^2(\mathbb{R} \setminus \mathbb{R}, \mathbb{R}^m)} \). Indeed, we have \( H^1_0(\mathbb{R} \setminus \mathbb{R}) \subset H^1_0(\mathbb{R} \setminus \mathbb{R}) \) for \( L > L_j \). If we now denote by \( M_L \) the minimum value of \( J(u) \) on \( H^1_0(\mathbb{R} \setminus \mathbb{R}) \), then \( M_L \) is non-increasing in \( L \) (there are more competing functions for larger \( L \)) and in particular \( J(u_j) \leq M_L \) if \( L_j > 1 \). In view of the condition (3.2), this provides us with a uniform in \( j \) bound on \( \int_{-L_j}^{L_j} \sum_{i=1}^{m} (u'_{j,i}(t))^2 \, dt \), from which we conclude that the sequence \( \{u_j(t)\} \) is equicontinuous on every bounded interval (as in (2.10) above). With the sequence \( \{u_j(t)\} \) equicontinuous and uniformly bounded on every interval \( [-L_p, L_p] \), it converges uniformly to some \( u \in C(\mathbb{R}; \mathbb{R}^m) \) on \( [-L_p, L_p] \). From the equation (3.4), we have uniform convergence of \( \{u'_j\} \), and hence uniform convergence of \( \{u_j\} \) follows from (2.10). We complete the proof as in the proof of Theorem 2.1.

**Example 2.** Consider the case \( m = 2 \), \( V(z) = z_1^4 + z_2^2 + h(z_1, z_2) \), with \( h(z_1, z_2) > 0 \) and \( h(z_1, z_2) \) bounded on \( \mathbb{R}^2 \). We consider the system

\[
\begin{cases}
    u''_1 - a(t) (4u_1^3 + h(z_1, u)) = f_1(t), \\
    u''_2 - a(t) (2u_2 + h(z_2, u)) = f_2(t),
\end{cases}
\]
where the functions \(a(t), f_1(t), f_2(t)\) satisfy the assumptions of Theorem 3.1. Applying Young’s inequality, we obtain

\[
|u_1(t)f_1(t)| \leq c u_1^4(t) + c_1 f_1^{4/3}(t),
\]

and

\[
|u_2(t)f_2(t)| \leq c u_2^2(t) + c_2 f_2^{2}(t).
\]

Therefore, we get for some \(c_3 > 0\)

\[
a(t) (u_1^4 + u_2^2 + h(u_1, u_2)) + u_1(t)f_1(t) + u_2(t)f_2(t) \geq -c_3 \left( f_1^{4/3}(t) + f_2^{2}(t) \right).
\]

Hence, Theorem 3.1 applies provided that \(\int_{-\infty}^{\infty} \left( f_1^{4/3}(t) + f_2^{2}(t) \right) dt < \infty\).

**4 Bounded solutions of Hamiltonian PDE systems**

In this section, we use a combination of the variational approach and elliptic estimates to show that similar results can be obtained for Hamiltonian PDE systems. We shall be looking for uniformly bounded solutions \(u = (u_1, \ldots, u_m) \in H^1(\mathbb{R}^n; \mathbb{R}^m)\), for \(n > 1\), of the system

\[
\Delta u_i - a(x)V_{z_i}(u) = f_i(x), \quad i = 1, \ldots, m.
\]

Here \(u_i(x)\) are the unknown functions, \(a(x)\) and \(f_i(x)\) are given functions on \(\mathbb{R}^n\), \(i = 1, \ldots, m\), and \(V(z)\) is a given function on \(\mathbb{R}^m\). We shall denote the gradient of \(a(x)\) by \(Da(x)\).

**Theorem 4.1.** Assume that \(a(x), f_i(x) \in C^\infty(\mathbb{R}^n)\) and \(V(z) \in C^\infty(\mathbb{R}^m)\). In addition, assume that there exist constants \(0 < a_0 \leq a_1\) and \(M > 0\) such that \(a_0 \leq a(x) \leq a_1\) and \(|f_i(x)|, |Da(x)|, |Df_i(x)| \leq M\) for all \(x \in \mathbb{R}^n\) and \(i = 1, \ldots, m\). Assume also that

\[
\lim_{z_j \to \infty} V_{z_j} = \infty, \lim_{z_i \to -\infty} V_{z_i} = -\infty, \quad \text{uniformly in all } z_j \neq z_i,
\]

and

\[
a(x)V(z) + \sum_{i=1}^{m} z_i f_i(x) \geq -f_0(x),
\]

for all \(x \in \mathbb{R}^n\), \(z \in \mathbb{R}^m\) and some function \(f_0(x) > 0\) satisfying \(\int_{\mathbb{R}^n} f_0(x) dx < \infty\). Then the system (4.1) has a uniformly bounded classical solution \(u(x)\), with \(u_i(x) \in C^2(\mathbb{R}^n)\), \(i = 1, \ldots, m\).

As in the proof of Theorem 3.1, we approximate solutions of the system (4.1) by solutions of the following system

\[
\begin{align*}
\Delta u_i(x) - a(x)V_{z_i}(u(x)) = f_i(x) & \quad \text{for } x \in B_L(0), \\
u_i(x) = 0 & \quad \text{for } x \in \partial B_L(0),
\end{align*}
\]

where \(B_L(0) = \{x \in \mathbb{R}^n : |x| < L\}\).
Lemma 4.1. Assume that $a(x), f_i(x) \in C^\infty(\mathbb{R}^n)$ and $V(z) \in C^\infty(\mathbb{R}^m)$. In addition, assume that the condition (4.3) is satisfied. Then the system (4.4) has a classical solution $u_L = (u_{L,1}, \ldots, u_{L,m}) \in C^2(B_L(0); \mathbb{R}^m)$.

Proof. We consider the following variational approach: the functional

$$J(u) := \int_{B_L(0)} \left[ \frac{1}{2} |\nabla u_i|^2 + u_i(x)f_i(x) \right] + a(x)V(u(x)) \right] \, dx$$

is minimized over $H_0^1(B_L(0); \mathbb{R}^m)$. From the condition (4.3), we have

$$J(u) \geq c_1(L)\|u\|^2_{H^1(B_L(0))} - c_2$$

for some positive constants $c_1, c_2$. Therefore, $J$ is bounded below, coercive and convex in $\nabla u$. Hence, it has a minimizer $u_L \in H_0^1(B_L(0); \mathbb{R}^m)$ that satisfies the system (4.4). (See Theorem 2 in Section 8.2.2 of [1].) Now $u_L$ solves the following elliptic system

$$\begin{cases}
\Delta u_{L,i} = a(x)V_{z_i}(u_L) + f_i(x) \quad &\text{in } B_L(0), \\
u_{L,i} = 0 \quad &\text{on } \partial B_L(0).
\end{cases}$$

For any $i$, since $a, f_i$ and $V$ are all smooth and $u_L \in H_0^1$, it follows from standard elliptic estimates that $u_{L,i} \in H^3(B_L(0))$, and therefore $u_L \in H^3(B_L(0); \mathbb{R}^m)$. (See Theorem 8.13 in [2].) By a bootstrapping argument and the Sobolev embedding theorem, one has $u_{L,i} \in C^2(B_L(0))$ for all $i$ and hence $u_L$ is a classical solution to (4.4).

In the next lemma, we apply interior estimates for classical solutions of the Poisson equation to the function $u_L$ found in Lemma 4.1. We introduce some notations from [2]. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and $u \in C^{2,\alpha}(\Omega)$ for some $0 < \alpha < 1$. We set

$$|D^k u|_{\alpha;\Omega} := \sup_{|\beta|=k} \sup_{\Omega} |D^\beta u|, \quad k = 0, 1, 2,$$

and

$$[D^k u]_{\alpha;\Omega} := \sup_{|\beta|=k} \sup_{x,y \in \Omega, x \neq y} \frac{|D^\beta u(x) - D^\beta u(y)|}{|x-y|^\alpha}.$$

Lemma 4.2. Given $L > 2$ and $0 < \alpha < 1$, under the assumptions of Theorem 4.1, there exists a constant $K$ independent of $L$ such that the function $u_L$ found in Lemma 4.1 satisfies

$$\|u_L\|_{0,B_L(0)}, |D u_L|_{0,B_L(0)}, |D^2 u_L|_{0,B_L(0)}, |D^2 u_L|_{\alpha,B_L(0)}, |D^3 u_L|_{\alpha,B_L(0)} \leq K,$$

where $L' = L - 1$ and $L'' = L - 2$. 

9
Proof. We fix an arbitrary index \( i \in \{1, ..., m \} \), and omit the subscript \( L \). Therefore, we denote \( u = u_L \) and \( u_i = u_{L,i} \). Suppose \( x_0 \in B_L(0) \) is such that \( u_i(x_0) \) is a positive maximum of \( u_i \). Then since \( \Delta u_i(x_0) \leq 0 \), it follows from (4.4) that

\[
a(x_0)V_{z_i}(u(x_0)) + f_i(x_0) \leq 0
\]

and hence

\[
V_{z_i}(u(x_0)) \leq \frac{M}{a_0}.
\]

The assumption (4.2) and (4.4) then guarantee that \( u_i(x_0) \) is bounded from above independent of \( L \). Similarly, we have the minimum of \( u_i \) is bounded from below independent of \( L \). Since this holds for all \( i \), we deduce

\[
|u|_{0,B_L(0)} \leq K_0
\]

for some \( K_0 \) independent of \( L \).

We denote \( F_i(u, x) := a(x)V_{z_i}(u) + f_i(x) \). It follows from Lemma 4.1 and (4.7) that \( F_i \in C^2(B_L(0)) \) and \( |F_i(u, x)|_{0,B_L(0)} \) is bounded independent of \( L \). Let \( \bar{x} \in B_L(0) \) and \( w \) be the Newtonian potential of \( F_i \) on \( B_1(\bar{x}) \), then it is clear that \( u_i = \omega + v \) for some harmonic function \( v \) on \( B_1(\bar{x}) \). For all \( x \in B_1(\bar{x}) \) we have

\[
w(x) = \int_{B_1(\bar{x})} \Gamma(x - y)F_i(u(y), y)dy
\]

and

\[
Dw(x) = \int_{B_1(\bar{x})} D\Gamma(x - y)F_i(u(y), y)dy,
\]

where \( \Gamma \) is the fundamental solution of the Laplacian in \( \mathbb{R}^n \) (see [2] Lemma 4.1). Using properties of \( \Gamma \) and uniform boundedness of \( F_i \), it is easy to check that

\[
|w|_{0,B_1(\bar{x})} \leq C|F_i|_{0,B_1(\bar{x})} \quad \text{and} \quad |Dw|_{0,B_1(\bar{x})} \leq C|F_i|_{0,B_1(\bar{x})}
\]

for some constant \( C \) depending only on \( n \). Therefore we have

\[
|v|_{0,B_1(\bar{x})} \leq |u_i|_{0,B_1(\bar{x})} + |w|_{0,B_1(\bar{x})} \leq C \left( |u_i|_{0,B_1(\bar{x})} + |F_i|_{0,B_1(\bar{x})} \right).
\]

Using interior estimates for harmonic functions (see [2] Theorem 2.10), we have

\[
|Dv|_{0,B_{\frac 12}(\bar{x})} \leq C|v|_{0,B_1(\bar{x})}
\]

for some constant \( C \) depending only on \( n \), since for any \( x \in B_{\frac 12}(\bar{x}) \), we have \( \text{dist}(x, \partial B_1(\bar{x})) \geq \frac 12 \). Now combining (4.9)-(4.10) we obtain

\[
|Dv|_{0,B_{\frac 12}(\bar{x})} \leq C \left( |u_i|_{0,B_1(\bar{x})} + |F_i|_{0,B_1(\bar{x})} \right)
\]

for some constant \( C \) depending only on \( n \). This along with (4.8) yields

\[
|Du_i|_{0,B_{\frac 12}(\bar{x})} \leq C \left( |u_i|_{0,B_1(\bar{x})} + |F_i|_{0,B_1(\bar{x})} \right) \leq C \left( |u_i|_{0,B_L(0)} + |F_i|_{0,B_L(0)} \right)
\]
for some constant $C$ depending only on $n$. Now since $\bar{x}$ is arbitrary in $B_L(0)$, it follows that

$$|Du|_{0;B_L(0)} \leq C \left( |u_1|_{0;B_L(0)} + |F_i|_{0;B_L(0)} \right).$$

In particular, since $|u_1|_{0;B_L(0)}$ and $|F_i|_{0;B_L(0)}$ are bounded independent of $L$, we obtain a uniform bound on $|Du|_{0;B_L(0)}$ independent of $L$. Hence we have

$$(4.11) \quad |Du|_{0;B_L(0)} \leq K_1$$

for some $K_1$ independent of $L$.

By assumption, both $|Da|_{a;\mathbb{R}^n}$ and $|Df_i|_{0;\mathbb{R}^n}$ are bounded. Since $V$ is smooth, and both $|u_1|_{0;B_L(0)}$ and $|Du|_{0;B_L(0)}$ are bounded independent of $L$, it is clear that $|DF_i|_{0;B_L(0)}$ is bounded independent of $L$. It follows that $[F_i]_{a;B_L(0)}$ is bounded independent of $L$. For all $\bar{x} \in B_{L'}(0)$ we deduce from [2] Theorem 4.6 that

$$\left( \frac{1}{3} \right)^2 |D^2u|_{0;B\bar{x}(\bar{x})} + \left( \frac{1}{3} \right)^{2+\alpha} |D^2u|_{a;B\bar{x}(\bar{x})}$$

$$\leq C \left[ |u_1|_{0;B\bar{x}(\bar{x})} + \left( \frac{1}{3} \right)^2 |F_i|_{0;B\bar{x}(\bar{x})} + \left( \frac{2}{3} \right)^\alpha |F_i|_{a;B\bar{x}(\bar{x})} \right]$$

$$\leq C \left( |u_1|_{0;B_L(0)} + |F_i|_{0;B_L(0)} + |F_i|_{a;B_L(0)} \right)$$

for some constant $C$ depending only on $n$ and $\alpha$. Since $\bar{x} \in B_{L'}(0)$ is arbitrary and the above right hand side is bounded independent of $L$, we conclude that

$$(4.12) \quad |D^2u|_{0;B_{L'}(0)}, \quad |D^2u|_{a;B_{L'}(0)} \leq K_2$$

for some $K_2$ independent of $L$. Putting $(4.7)$, $(4.11)$, $(4.12)$ together and setting $K := \max\{K_1, K_2, K_3\}$, we obtain $(4.5)$. \hfill \square

**Proof of Theorem 4.1** We take an increasing sequence $\{L_j\}$, with $L_1 > 2$ and $\lim_{j \to \infty} L_j = \infty$, and denote by $u_j = u_{L_j}$ the function found in Lemma 4.1. We extend $u_j$ to be zero outside $B_{L_j}(0)$. Note that $u_j \in C^{2,\alpha}(B_{L_j}(0); \mathbb{R}^m)$ but does not need to be smooth on $\mathbb{R}^n$. On each $B_{L_j'}(0)$, it follows from Lemma 4.2 that the sequences $\{u_j\}_{j \geq p}$, $\{Du_j\}_{j \geq p}$ and $\{D^2u_j\}_{j \geq p}$ are all uniformly bounded and equicontinuous. Using the diagonal arguments as in the proof of Theorem 2.4 one can find a subsequence $\{u_{j_k}\}$ such that $\{u_{j_k}\}$, $\{Du_{j_k}\}$ and $\{D^2u_{j_k}\}$ are all uniformly convergent on all $B_{L_j'}(0)$. In particular, there exists $u \in C(\mathbb{R}^n; \mathbb{R}^m)$ such that

$$(4.13) \quad u_{j_k} \to u \quad \text{uniformly on all bounded domains in } \mathbb{R}^n.$$ 

It is clear from Lemma 4.2 that $u$ is bounded on $\mathbb{R}^n$. It remains to show that the vector valued function $u$ satisfies the system $(4.1)$. Let $\Omega \subset \mathbb{R}^n$
be any bounded convex domain and \( i \in \{1, \ldots, m\} \) be any index. Note that \( u_{jk,i} \in C^2(\Omega) \) for all \( k \) sufficiently large, and there exist \( v \in C(\Omega; \mathbb{R}^n) \) and \( w \in C(\Omega; \mathbb{R}^{n \times n}) \) such that

\[
\nabla u_{jk,i} \to v \quad \text{and} \quad \nabla^2 u_{jk,i} \to w \quad \text{uniformly on} \ \overline{\Omega},
\]

where \( \nabla^2 u_{jk,i} \) is the Hessian matrix of \( u_{jk,i} \). Fix \( x_0 \in \Omega \). For any \( x \in \Omega \), we have

\[
u_{jk,i}(x) = u_{jk,i}(x_0) + \int_{l_{x_0}^x} \nabla u_{jk,i}(s) \cdot \tau ds,
\]

where \( l_{x_0}^x \) is the line segment joining \( x_0 \) and \( x \) and \( \tau \) is the unit tangent vector of \( l_{x_0}^x \). Using (4.13) and (4.14), we obtain

\[
u_i(x) = u_i(x_0) + \int_{l_{x_0}^x} v(s) \cdot \tau ds,
\]

and therefore \( u_i \in C^1(\Omega) \) and \( \nabla u_i = v \). Using similar arguments and (4.14), we obtain that \( v \in C^1(\Omega) \) and \( \nabla v = w \), and hence \( u_i \in C^2(\Omega) \) and \( \nabla^2 u_i = w \) in \( \Omega \). For \( k \) sufficiently large, we know \( u_{jk,i} \) solves

\[
\Delta u_{jk,i} - a(x)V_{z_i}(u_{jk}) = f_i(x), \quad \text{for} \ x \in \Omega.
\]

Passing to the limit as \( k \to \infty \), we have

\[
\Delta u_i - a(x)V_{z_i}(u) = f_i(x), \quad \text{for} \ x \in \Omega.
\]

Since this holds for all bounded convex domains \( \Omega \in \mathbb{R}^n \), we conclude that \( u \in C^2(\mathbb{R}^n; \mathbb{R}^m) \) is a bounded solution of the system (4.1).

**Remark 2.** We can apply Theorem 4.1 to the system given in Example 2, but with smooth \( h \) and the functions \( a(x), f_1(x), f_2(x) \) satisfying the additional assumptions in Theorem 4.1.

**References**

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