The Prime Spectrum and Representation Theory of the 2 × 2 Reflection Equation Algebra*

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Abstract

The theory of generalized Weyl algebras is used to study the 2 × 2 reflection equation algebra \( A = A_q(M_2) \) in the case that \( q \) is not a root of unity, where the \( R \)-matrix used to define \( A \) is the standard one of type \( A \). Simple finite dimensional \( A \)-modules are classified, finite dimensional weight modules are shown to be semisimple, \( \text{Aut}(A) \) is computed, and the prime spectrum of \( A \) is computed along with its Zariski topology. Finally, it is shown that \( A \) satisfies the Dixmier-Moeglin equivalence.

1 Introduction

Throughout, \( k \) is a field and \( q \in k^\times \) is not a root of unity.

Let \( n \) be a positive integer. Consider the action by right conjugation of the algebraic group \( \text{GL}_n(k) \) of invertible \( n \times n \) matrices on the space \( M_n(k) \) of all \( n \times n \) matrices:

\[
M \xrightarrow{g \in \text{GL}_n(k)} g^{-1}Mg.
\]

At the level of coordinate rings, the action map becomes an algebra homomorphism

\[
\mathcal{O}(M_n) \to \mathcal{O}(M_n) \otimes \mathcal{O}(\text{GL}_n),
\]

where we have dropped mention of the base field \( k \) to simplify notation. This gives \( \mathcal{O}(M_n) \) the structure of a comodule-algebra over the Hopf algebra \( \mathcal{O}(\text{GL}_n) \). We shall consider what happens when this picture is carried into a quantum algebra setting. The construction of [21] yields a noncommutative deformation \( \mathcal{O}_q(M_n) \) of \( \mathcal{O}(M_n) \), using the the \( R \)-matrix

\[
R_{ij}^{lk} = \begin{cases} 
q & \text{if } i = j = k = l \\
1 & \text{if } i = j, k = l, \text{ and } i \neq k \\
q^{-1} & \text{if } i > j, i = l, \text{ and } j = k \\
0 & \text{otherwise.}
\end{cases}
\]

More precisely, the \( k \)-algebra \( \mathcal{O}_q(M_n) \) has a presentation with \( n^2 \) generators \( \{ t_{ij} \mid 0 \leq i, j \leq n \} \) and the relations

\[
R_{ab}^{ik} t_{ij}^a t_{ij}^b = R_{ij}^{ab} k_{ij}^a k_{ij}^b.
\]

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where repeated indices are summed over. Further, \( \mathcal{O}_q(M_n) \) is a bialgebra in a way that matches the comultiplication on \( \mathcal{O}(M_n) \) induced by matrix multiplication in \( M_n \):

\[
\Delta(t^i_j) = t^i_k \otimes t^k_j \\
\epsilon(t^i_j) = \delta^i_j.
\]

Inverting a suitable central determinant-like element in \( \mathcal{O}_q(M_n) \) yields a noncommutative deformation \( \mathcal{O}_q(\mathfrak{gl}_n) \) of \( \mathcal{O}(\mathfrak{gl}_n) \); see [2] I.2.4] for example.

One may attempt to mimic the map of (1) for \( \mathcal{O}_q(M_n) \) with the hope of making this algebra a comodule-algebra over \( \mathcal{O}_q(\mathfrak{gl}_n) \),

\[
\mathcal{O}_q(M_n) \rightarrow \mathcal{O}_q(M_n) \otimes \mathcal{O}_q(\mathfrak{gl}_n) \\
t^i_j \mapsto t^i_k \otimes S(t^k_j),
\]

but such a prescription yields only a coaction map and not an algebra homomorphism. The remedy is to replace the \( \mathcal{O}_q(\mathfrak{gl}_n) \)-comodule \( \mathcal{O}_q(M_n) \) by a different noncommutative deformation of \( \mathcal{O}(M_n) \). The needed construction is provided by the transmutation theory of Majid, presented in [19]; it is a \( k \)-algebra \( A_q(M_n) \) with \( n^2 \) generators \( \{ u^i_j | 0 \leq i, j \leq n \} \) and the relations

\[
R^k_{im}R^m_{qr} u^k_i u^r_p = R^k_{im}R^m_{qp} u^p_i u^i_r,
\]

where the \( R \)-matrix is still (2), the same one used to build \( \mathcal{O}_q(M_n) \). Replacing (3) with

\[
A_q(M_n) \rightarrow A_q(M_n) \otimes \mathcal{O}_q(\mathfrak{gl}_n) \\
u^i_j \mapsto u^i_k \otimes S(t^k_j)
\]

does give an algebra homomorphism, making \( A_q(M_n) \) a comodule-algebra over \( \mathcal{O}_q(\mathfrak{gl}_n) \) and providing a more suitable "quantization" of (1). The algebra \( A_q(M_n) \) is referred to as a braided matrix algebra by Majid, and as a reflection equation algebra elsewhere in the literature.

We shall focus on the case \( n = 2 \), the \( 2 \times 2 \) reflection equation algebra, denoted throughout by \( A := A_q(M_2) \). It is generated by \( u_{ij} \) for \( i, j \in \{1, 2\} \) with the relations given in (1), which simplify to:

\[
\begin{align*}
u_{11}u_{22} &= u_{22}u_{11} \\
u_{11}u_{12} &= u_{12}(u_{11} + (q^{-2} - 1)u_{22}) \\
u_{22}u_{12} &= q^2u_{12}u_{22} \\
u_{21}u_{12} - u_{12}u_{21} &= (q^{-2} - 1)u_{22}(u_{22} - u_{11}).
\end{align*}
\]

Observe that \( u_{12} \) and \( u_{22} \) normalize the subalgebra generated by \( u_{11} \) and \( u_{22} \), and they do so via inverse automorphisms of that subalgebra. This suggests that \( A \) is a generalized Weyl algebra, a fact this paper is devoted to exploiting.

**Brief History**

The "reflection equation" (4) was first introduced by Cherednik in his study [3] of factorizable scattering on a half-line, and reflection equation algebras later emerged from Majid’s transmutation theory in [18]. In [16], Kulish and Sklyanin prove several things about \( A = A_q(M_2) \). They show that \( A \) has a \( k \)-basis consisting of monomials in the generators \( u_{ij} \). They compute the center of \( A \). They find a determinant-like element of \( A \) and they show that inverting \( u_{22} \) and setting the determinant-like element equal to 1 yields \( U_q(\mathfrak{s}\mathfrak{l}_2) \), and they note that this can be used to pull back representations of \( U_q(\mathfrak{s}\mathfrak{l}_2) \) to representations of \( A \). (We shall see in this paper that all irreducible representations that are not annihilated by \( u_{22} \) arise in this way.) Domokos and Lenagan address \( A_q(M_n) \) for general \( n \) in [9]. They show that \( A_q(M_n) \) is a noetherian domain, and that it has a \( k \)-basis consisting of monomials in the generators \( u_{ij} \).
Paper Outline  Section 2 builds up the needed background and notation regarding generalized Weyl algebras (GWAs), most of which is a collection of results from [2], [3], [4], and [10]. A description of homogeneous ideals of GWAs is given in section 2.2 and localization is explored in section 2.3. Section 2.4 addresses GK dimension by transporting the arguments of [17] for skew Laurent rings into the GWA setting. Section 2.5 explores a certain aspect of the finite dimensional representation theory of GWAs, focusing on the setting that will apply to the $2 \times 2$ reflection equation algebra when $q$ is not a root of unity.

Section 3 applies GWA theory to our $2 \times 2$ reflection equation algebra $A$. Normal elements are identified and then used to compute the automorphism group. Sections 3.1 and 3.2 contain a classification of finite dimensional simple $A$-modules and an identification of a large class of semisimple $A$-modules. Finally, the prime spectrum of $A$ is fully worked out in section 3.3 and some consequences are explored.

Notation  All rings are rings with 1, and they are not necessarily commutative. Given a ring $R$ and an automorphism $\sigma$ of $R$, we use $R[x;\sigma]$ to denote the skew polynomial ring and $R[x^{\pm};\sigma]$ to denote the skew Laurent ring. Our convention for the twisting is such that $xr = \sigma(r)x$ for $r \in R$. If there is further twisting by a $\sigma$-derivation $\delta$, then the notation becomes $R((x^{\pm};\sigma))$ to denote the skew Laurent series ring. Given a subset $G$ of a ring $R$, we indicate by $\langle G \rangle$ the two-sided ideal of $R$. When there is some ambiguity as to the ring in which ideal generation takes place, we resolve it by using a subscript $\langle G \rangle_R$ or by writing $\langle G \rangle \circ R$.

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2  Generalized Weyl Algebras

Generalized Weyl algebras, henceforth known as GWAs, were introduced by Bavula in [1]. Examples include the ordinary Weyl algebra and the classical and quantized universal enveloping algebras of $\mathfrak{sl}_2$.

We shall define GWAs by presenting them as rings over a given base ring. A ring $S$ over a ring $R$, also known as an $R$-ring, is simply a ring homomorphism $R \to S$. A morphism $S \to S'$ of rings over $R$ is a ring homomorphism such that

$$\begin{array}{ccc}
R & \longrightarrow & S' \\
S & \longrightarrow & \downarrow \\
& \downarrow & \\
& R & \leftarrow \\
& \downarrow & \\
& \downarrow & \\
& S & \leftarrow
\end{array}$$

commutes. Given any set $\mathcal{X}$, one can show that a free $R$-ring on $\mathcal{X}$ exists. This provides meaning to the notion of a presentation of a ring over $R$; it can be thought of as a ring over $R$ satisfying a universal property described in terms of the relations.

Definition 1: Let $R$ be a ring, $\sigma$ an automorphism of $R$, and $z$ an element of the center of $R$. The GWA based on this data is the ring over $R$ generated by $x$ and $y$ subject to the relations

$$\begin{align*}
yx & = z \\
xr & = \sigma(r)x \\
xy & = \sigma(z) \\
yr & = \sigma^{-1}(r)y
\end{align*} \quad \forall \ r \in R. \tag{7}$$

We denote this construction by $R[x, y; \sigma, z]$.
and we adapt some useful notation from [2] as follows. Define
\[ v_n = \begin{cases} x^n & n \geq 0 \\ y^{-n} & n \leq 0 \end{cases} \]
for \( n \in \mathbb{Z}_{\geq 0} \), and define
\[ \sigma^{[j,k]}(z) = \prod_{l=j}^{k} \sigma^l(z) \]
for integers \( j \leq k \). We take a product over an empty index set to be 1. Define the following special elements of \( \mathbb{Z}(\mathbb{R}) \):
\[ [\, [n,m] \,] = \begin{cases} \sigma^{[n+m+1,n]}(z) & n > 0, m < 0, |n| \geq |m| \\ \sigma^{[1,n]}(z) & n > 0, m < 0, |n| \leq |m| \\ \sigma^{[n+1,m+1]}(z) & n < 0, m > 0, |n| \geq |m| \\ \sigma^{[n+1,0]}(z) & n < 0, m > 0, |n| \leq |m| \\ 1 & \text{for other } n, m \in \mathbb{Z}. \end{cases} \]
(8)
Now we have \( v_nv_m = [n,m]v_{n+m} \) for \( n, m \in \mathbb{Z} \).

2.1 Basic Properties

This section lays down some basic ring-theoretic properties of GWAs, ones we will need to reference in later sections. The following two propositions are easy observations:

**Proposition 2:** There is an \( \mathbb{R} \)-ring homomorphism \( \phi : \mathbb{R}[x,y;\sigma,z] \rightarrow \mathbb{R}[x^\pm;\sigma] \) sending \( x \) to \( x \) and \( y \) to \( xz - 1 \). There is also an \( \mathbb{R} \)-ring homomorphism \( \phi' : \mathbb{R}[x,y;\sigma,z] \rightarrow \mathbb{R}[x^\pm;\sigma] \) sending \( x \) to \( xz \) and \( y \) to \( x - 1 \).

**Proposition 3:** \( \mathbb{R}[x,y;\sigma,z] \) has the alternative expression \( \mathbb{R}[y,x;\sigma^{-1},\sigma(z)] \), and \( \mathbb{R}[x,y;\sigma,z]^\text{op} \) can be expressed as \( \mathbb{R}^\text{op}[y,x;\sigma^{-1},\sigma(z)] \).

Proposition 3 roughly means that if we prove something (ring-theoretic) about \( x \), then we get a \( y \) version of the result by swapping \( x \) and \( y \), replacing \( \sigma \) with \( \sigma^{-1} \), and replacing \( z \) with \( \sigma(z) \). And if we prove something left-handed, then we get a right-handed version of the result by replacing \( \sigma \) with \( \sigma^{-1} \) and replacing \( z \) with \( \sigma(z) \).

**Proposition 4:** Consider a GWA \( \mathbb{R}[x,y;\sigma,z] \).

1. It is a free left (right) \( \mathbb{R} \)-module on \( \{ v_i \mid i \in \mathbb{Z} \} \).
2. It has a \( \mathbb{Z} \)-grading with homogeneous components \( Rv_n = v_nR \):
   \[ R[x,y;\sigma,z] = \bigoplus_{n \in \mathbb{Z}} Rv_n \]
3. It contains a copy of the ring \( \mathbb{R} \) as the subring of degree zero elements. The subring generated by \( R \) and \( x \) is a skew polynomial ring \( \mathbb{R}[x;\sigma] \), and the subring generated by \( R \) and \( y \) is \( \mathbb{R}[y;\sigma^{-1}] \).
4. It is left (right) noetherian if \( \mathbb{R} \) is left (right) noetherian.

**Proof:** See [22, Lemma II.3.1.6] for a proof of assertion 1. Assertions 2 and 3 are then easily shown. Assertion 4 was proven in [2, Proposition 1.3].

The following results are now routine.
Proposition 5: Let $W = R[x, y; z, \sigma]$ be a GWA. The homomorphisms of Proposition 5 are injective if and only if $z \in R$ is regular, and they are isomorphisms if and only if $z \in R$ is a unit.

Corollary 6: A GWA $W = R[x, y; \sigma, z]$ is a domain if and only if $R$ is a domain and $z \neq 0$.

Proposition 7: Let $W = R[x, y; \sigma, z]$ be a GWA. Then $x, y \in W$ are regular if and only if $z \in R$ is regular.

The center of a GWA is often easily described when its coefficient ring is a domain:

Proposition 8: Let $R$ be a domain, and let $\sigma$ be an automorphism of $R$ such that $\sigma|_{Z(R)} : Z(R) \to Z(R)$ has infinite order. Then $Z(R[x, y; \sigma, z])$ is $Z(R)^o$, the subring of $Z(R)$ fixed by $\sigma$.

Proof: If $a \in Z(R)^o$, then a commutes with $R$, $x$, and $y$ and is therefore central. Suppose for the converse that $a = \sum_{m \in \mathbb{Z}} a_m v_m$ is central. Then $xa = ax$ and $ya = ay$ require that $\sigma(a_m) = a_m$ for all $m \in \mathbb{Z}$. Given any nonzero $m \in \mathbb{Z}$, our hypothesis ensures that there is some $r \in Z(R)$ such that $\sigma^m(r) \neq r$. Now $ra = ar$ requires $ra_m = a_m \sigma^m(r)$, so $a_m = 0$. Thus $a = a_0 \in R^o$. Finally, $a$ commutes with $R$, so $a \in Z(R)^o$.

There are similar and easily verified facts about skew Laurent polynomials and skew Laurent series:

Proposition 9: Let $R$ be a domain, and let $\sigma$ be an automorphism of $R$ such that $\sigma|_{Z(R)} : Z(R) \to Z(R)$ has infinite order. Then $Z(R[x^\pm; \sigma]) = Z(R)^o$, the subring of $Z(R)$ fixed by $\sigma$. Similarly, $Z(R[[x^\pm; \sigma]]) = Z(R)^o$.

Under some stronger conditions, one can also characterize the normal elements of a GWA:

Proposition 10: Let $R$ be a domain, and let $\sigma$ be an automorphism of $R$ such that there is an $r \in Z(R)$ which is not fixed by any nonzero power of $\sigma$. Then the normal elements of $W = R[x, y; \sigma, z]$ are homogeneous.

Proof: Suppose that $a = \sum a_m v_m \in W$ is a nonzero normal element. Then $ra = ab$ for some $b \in W$. Looking at the highest degree and lowest degree terms of $ra$, and considering that $R$ is a domain, $b$ must have degree 0 in order for $ab$ to have the same highest and lowest degree terms. Thus $b \in R$. Now $ra = ab$ becomes $ra_m = a_m \sigma^m(b)$ for all $m \in \mathbb{Z}$. Since $r$ is central, we may cancel the $a_m$ whenever it is nonzero. If $a_m$ is nonzero for multiple $m \in \mathbb{Z}$, then $r = \sigma^m(b) = \sigma^{m+n}(b)$ for some $m, n \in \mathbb{Z}$ with $n \neq 0$. But $r = \sigma^n(r)$ would contradict our assumption on $r$, so $a$ must be homogeneous.

Proposition 11: Let $R$ be a commutative domain, $\sigma$ an automorphism, and $z \in R$ such that $\sigma^m(z)$ is never a unit multiple of $z$ for nonzero $m \in \mathbb{Z}$. Then the normal elements of $W = R[x, y; \sigma, z]$ are the $r \in R$ such that $\sigma(r)$ is a unit multiple of $r$.

Proof: Suppose that $r \in R$ and $\sigma(r) = ur$, where $u \in R^\times$. Then $rR = Rr$ because $R$ is commutative, $xr = r(ux)$, $ry = r(yu^{-1})$, and $ry = (yu)r$. Thus $r$ is normal in $W$. Now assume for the converse that $a \in W$ is normal and nonzero. By Proposition 10 using the fact that $z$ is not fixed by any nonzero powers of $\sigma$, $a$ is homogeneous. Write it as $a = a_m v_m$.

Suppose that $m \geq 0$, so that $a = a_m x^m$. For some $b \in W$, $ar = ba$. Clearly $b$ must have the form $b_1 x$ for some $b_1 \in R$, so we have $a_m = b_1 \sigma(a_m)$. Thus $a_m R \subseteq \sigma(a_m) R$. For some $c \in W$, $xa = ac$. Then $c$ must have the form $c = c_1 x$ for some $c_1 \in R$, so we have $\sigma(a_m) = a_m \sigma^m(c_1)$. Thus $\sigma(a_m) R \subseteq a_m R$. We
Thus, cancelling the $a_m$ cancels the $m < 1$. If $z \in (a)$ for some $\sigma \in \text{Aut}(R)$, then we may use writing out cosets.

**Definition 13:** Whenever $I$ is a subset of a GWA $R[x, y; \sigma, z]$ and $m \in \mathbb{Z}$, $I_m$ shall denote the subset $I_m := \{r \in R \mid rv_m \in I\}$ of $R$ and $I_m^{\text{op}}$ shall denote $I_m^{\text{op}} := \{r \in R \mid v_mr \in I\}$.

**Remark 14:** $I_m^{\text{op}}$ is a notational device for working with the symmetry $R[x, y; \sigma, z]^{\text{op}} = R^{\text{op}}[x, y; \sigma^{-1}, \sigma(z)]$. It transfers the definition of $I_m$ to the GWA structure on the opposite ring. Note that the relation is that $I_m^{\text{op}} = \sigma^{-m}(I_m)$ for all $m \in \mathbb{Z}$.

Propositions 15 to 18 were essentially observed in [4].

**Proposition 15:** Let $I$ be a right $R[x; \sigma]$-submodule of $R[x, y; \sigma, z]$. The $I_n$ are right ideals of $R$, and they satisfy

$$I_{-1(n+1)} \sigma^{-n}(z) \subseteq I_{-n} \quad I_n \subseteq I_{n+1}$$

### 2.2 Ideals

We will establish in this section a notation for discussing the homogeneous ideals of a GWA. We will also explore a portion of the prime spectrum of a GWA. First, note that quotients by ideals in the coefficient ring work as they ought to:

**Proposition 12:** Let $W = R[x, y; \sigma, z]$ be a GWA, with $J \triangleleft R$ an ideal such that $\sigma(J) = J$. Let $I \triangleleft W$ be generated by $J$. Then there is a canonical isomorphism

$$W/I \cong (R/J)[x, y; \hat{\sigma}, z + J],$$

where $\hat{\sigma}$ is the automorphism of $R/J$ induced by $\sigma$.

We will generally abuse notation and reuse the labels “$\sigma$” and “$z$” instead of putting hats on things or writing out cosets.

**Definition 13:** Whenever $I$ is a subset of a GWA $R[x, y; \sigma, z]$ and $m \in \mathbb{Z}$, $I_m$ shall denote the subset $I_m := \{r \in R \mid rv_m \in I\}$ of $R$ and $I_m^{\text{op}}$ shall denote $I_m^{\text{op}} := \{r \in R \mid v_mr \in I\}$.

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Propositions 15 to 18 were essentially observed in [4].

**Proposition 15:** Let $I$ be a right $R[x; \sigma]$-submodule of $R[x, y; \sigma, z]$. The $I_n$ are right ideals of $R$, and they satisfy

$$I_{-1(n+1)} \sigma^{-n}(z) \subseteq I_{-n} \quad I_n \subseteq I_{n+1}$$

(10)
for all \( n \in \mathbb{Z}_{\geq 0} \). Thus, a homogeneous right \( R[x; \sigma]\)-submodule \( I \) of \( R[x, y; \sigma, z] \) has the form \( \bigoplus_{n \in \mathbb{Z}} I_n v_n \) for a family \((I_n)_{n \in \mathbb{Z}}\) of right ideals of \( R \) satisfying (10). Further, any such family \((I_n)_{n \in \mathbb{Z}}\) defines a right \( R[x; \sigma]\)-submodule of \( R[x, y; \sigma, z] \) in this way.

**Proposition 16**: Let \( I \) be a right ideal of \( R[x, y; \sigma, z] \). The \( I_n \) are right ideals of \( R \), and they satisfy

\[
\begin{align*}
I_{-(n+1)} & \supseteq I_n \\
I_{-(n+1)} \sigma^{-n}(z) & \subseteq I_n \\
I_n & \subseteq I_{n+1} \\
I_n \sigma^{-n+1}(z) & \supseteq I_{n+1}
\end{align*}
\]  

(11)

for all \( n \in \mathbb{Z}_{\geq 0} \). Thus, a homogeneous right ideal \( I \) of \( R[x, y; \sigma, z] \) has the form \( \bigoplus_{n \in \mathbb{Z}} I_n v_n \) for a family \((I_n)_{n \in \mathbb{Z}}\) of right ideals of \( R \) satisfying (11). Further, any such family \((I_n)_{n \in \mathbb{Z}}\) defines a right ideal of \( R[x, y; \sigma, z] \) in this way.

**Proposition 17**: Let \( I \) be a left ideal of \( R[x, y; \sigma, z] \). The \( I_n \) are left ideals of \( R \), and they satisfy

\[
\begin{align*}
\sigma^{n+1}(I_{-(n+1)}) & \supseteq \sigma^n(I_n) \\
\sigma^{n+1}(I_{-(n+1)}) \sigma^{-n+1}(z) & \subseteq \sigma^n(I_n) \\
\sigma^{-n}(I_n) & \subseteq \sigma^{-(n+1)}(I_{n+1}) \\
\sigma^{-n}(I_n) \sigma^{-n+1}(z) & \supseteq \sigma^{-(n+1)}(I_{n+1})
\end{align*}
\]  

(12)

for all \( n \in \mathbb{Z}_{\geq 0} \). Thus, a homogeneous left ideal \( I \) of \( R[x, y; \sigma, z] \) has the form \( \bigoplus_{n \in \mathbb{Z}} I_n v_n \) for a family \((I_n)_{n \in \mathbb{Z}}\) of left ideals of \( R \) satisfying (12). Further, any such family \((I_n)_{n \in \mathbb{Z}}\) defines a left ideal of \( R[x, y; \sigma, z] \) in this way.

**Proposition 18**: Let \( I \) be an ideal of \( R[x, y; \sigma, z] \). The \( I_n \) are ideals of \( R \), and they satisfy (11) and (12) for all \( n \in \mathbb{Z}_{\geq 0} \). Thus, a homogeneous ideal \( I \) of \( R[x, y; \sigma, z] \) has the form \( \bigoplus_{n \in \mathbb{Z}} I_n v_n \) for a family \((I_n)_{n \in \mathbb{Z}}\) of ideals of \( R \) satisfying (11) and (12). Further, any such family \((I_n)_{n \in \mathbb{Z}}\) defines an ideal of \( R[x, y; \sigma, z] \) in this way.

We may depict (11) and (12) by the following diagrams:

\[
\begin{array}{ccccc}
\cdots & \supseteq & I_{-2} & \supseteq & I_{-1} \\
& & \sigma^{-1}(z) & & \\
\cdots & \supseteq & \sigma^2(I_{-2}) & \supseteq & \sigma(I_{-1}) \\
& & \sigma^2(z) & & \\
& & I_0 & \subseteq & I_1 \\
& & \sigma(z) & & \\
& & \sigma^{-1}(I_0) & \subseteq & \sigma^{-2}(I_1) \\
& & \sigma^{-1}(z) & & \\
& & \cdots & \subseteq & \cdots
\end{array}
\]

We may also depict an alternative way of stating (12).

\[
\begin{align*}
I_{-(n+1)} & \supseteq \sigma^{-1}(I_n) \\
\sigma(I_{-(n+1)}) \sigma(z) & \subseteq I_n \\
\sigma^{-1}(I_n) & \subseteq \sigma^{-1}(I_{n+1}) z,
\end{align*}
\]  

(13)

by the following diagram:

\[
\begin{array}{ccccccc}
\cdots & \sigma^{-1} & \sigma^{-1} & \sigma & \cdots \\
I_{-2} & \sigma^{-1} & I_{-1} & \sigma & I_0 & \sigma & I_1 & \sigma & \cdots
\end{array}
\]

The following lemma will be useful for working out the prime spectrum of certain GWAs.

**Lemma 19**: Let \( W = R[x, y; \sigma, z] \) be a GWA such that \( R^e \subseteq R \) has the following property:

\[
\forall I \triangleleft R^e, \ RIR \cap R^e = I.
\]  

(14)

Then there are mutually inverse inclusion-preserving bijections

\[
\begin{align*}
\{ I \mid I \triangleleft R^e \} & \leftrightarrow \{ WIW \mid I \triangleleft R^e \} \\
I & \leftrightarrow WIW \\
I \cap R^e & \leftrightarrow I
\end{align*}
\]  

(15)
Now let $S = \{WPW \mid p \in \text{spec}(R^p)\}$ and assume that $S \subseteq \text{spec}(W)$. Assume also that extension of ideals to $R$ preserves intersections in the following sense: for any family $(I_\alpha)_{\alpha \in A}$ of ideals of $R^p$,
\[
\bigcap_{\alpha \in A} RI_\alpha R = R\left(\bigcap_{\alpha \in A} I_\alpha\right)R. \tag{16}
\]
Then (16) restricts to a homeomorphism
\[
\text{spec}(R^p) \approx S.
\]

**Proof:** Given any $I \triangleleft R^p$,
\[
WJW = \bigoplus_{m \in \mathbb{Z}} RIRv_m, \tag{17}
\]
because the right hand side satisfies the conditions of Proposition 18 needed to make it an ideal of $W$. Given $I, J \triangleleft R^p$ with $WJW \subseteq WJW$, we have $RIR \subseteq RJR$ from looking at the degree zero component. From (14) we can then deduce that $I \subseteq J$. The converse of this is clear: $I \subseteq J \Rightarrow WJW \subseteq WJW$. Putting this information together, we have the inclusion-preserving correspondence (15).

Now assume that $S \subseteq \text{spec}(W)$ and that (10) holds. Let $\phi : \text{spec}(R^p) \to S$ be the restriction of (15). We show that the bijection $\phi$ is a homeomorphism.

$\phi$ is a closed map: Given any $I \triangleleft R^p$, one has that $p \supseteq I$ if and only if $WpW \supseteq WIW$, for all $p \in \text{spec}(R^p)$. That is, the collection of $p \in \text{spec}(R^p)$ that contain $I$ is mapped by $\phi$ onto the collection of $P \subseteq S$ that contain $WIW$.

$\phi$ is continuous: Let $K \triangleleft W$. Define $\mathcal{J} := \{J \triangleleft R^p \mid K \subseteq WJW\}$ and $I := \bigcap \mathcal{J}$ (with an intersection of the empty set being $R^p$). For $p \in \text{spec}(R^p)$, if $WpW \supseteq K$, then $p \in \mathcal{J}$, so $p \supseteq I$. And if $I \subseteq p$, then
\[
K \subseteq \bigcap_{j \in \mathcal{J}} WJW = W\left(\bigcap_{j \in \mathcal{J}} RJR\right)W = WIW \subseteq WpW,
\]
where the first equality is an application of Proposition 20 to $R \subseteq W$, and the second equality is due to the assumption (10). We have therefore shown that the collection of $P \subseteq S$ that contain $K$ pulls back via $\phi$ to the collection of $p \in \text{spec}(R^p)$ that contain $I$.

We identify in the following proposition one situation in which the condition (16) holds for a given family of ideals.

**Proposition 20:** Let $A \subseteq B$ be rings such that $B$ is a free left $A$-module with a basis $(b_j)_{j \in \mathcal{J}}$ for which $Ab_j = b_j A$ for all $j \in \mathcal{J}$. Let $(I_\alpha)_{\alpha \in A}$ be a family of ideals of $A$ satisfying $b_j I_\alpha \subseteq I_\alpha b_j$ for all $j$ and $\alpha$. Then
\[
\bigcap_{\alpha \in A} BI_\alpha B = B\left(\bigcap_{\alpha \in A} I_\alpha\right)B. \tag{18}
\]

**Proof:** We begin by showing that $b_j \left(\bigcap_{\alpha \in A} I_\alpha\right) \subseteq \left(\bigcap_{\alpha \in A} I_\alpha\right)b_j$ for all $j$. Consider any $j \in \mathcal{J}$ and any $r \in \bigcap_{\alpha \in A} I_\alpha$. There is, for each $\alpha \in A$, an $r'_\alpha \in I_\alpha$ such that $b_j r = r'_\alpha b_j$. Since $b_j$ came from a basis for $AB$, all the $r'_\alpha$ are equal, and so we’ve shown that $b_j \left(\bigcap_{\alpha \in A} I_\alpha\right) \subseteq \left(\bigcap_{\alpha \in A} I_\alpha\right)b_j$ for all $j$.

Let $I$ be any ideal of $A$ satisfying $b_j I \subseteq Ib_j$ for all $j$. Observe that $\bigoplus_{j \in \mathcal{J}} Ib_j$ is then an ideal of $B$, and hence it is the extension of $I$ to an ideal of $B$. Applying this principle to $I = I_\alpha$ for $\alpha \in A$, and also applying it to $I = \bigcap_{\alpha \in A} I_\alpha$, (18) follows from the fact that
\[
\bigcap_{\alpha \in A} \left(\bigoplus_{j \in \mathcal{J}} I_\alpha b_j\right) = \bigoplus_{j \in \mathcal{J}} \left(\bigcap_{\alpha \in A} I_\alpha\right) b_j.
\]

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2.3 Localizations

Proposition 21: Let \( W = R[x, y; r, \sigma] \) be a GWA with \( z \) regular. Then \( S := \{1, x, x^2, \ldots \} \) is an Ore set of regular elements, and the corresponding ring of fractions is given by the homomorphism \( \phi : W \to R[x^\pm; \sigma] \) of Proposition 3.

Proof: That the elements of \( S \) are regular comes from Proposition 3. If we can show that \( \phi \) is the localization homomorphism for a right ring of fractions of \( W \) with respect to \( S \), then by [12, Theorem 6.2] we will have that \( S \) is a right Ore set. Then it will also be a left Ore set due to Proposition 3 of course with the same ring of fractions, by [12, Proposition 6.5].

So we have only two things to verify: that \( \phi(S) \) is a collection of units and that elements of \( R[x^\pm; \sigma] \) have the form \( \phi(w)\phi(s)^{-1} \) with \( w \in W \) and \( s \in S \). The former statement is obvious. For the latter, consider an arbitrary \( p \in R[x^\pm; \sigma] \). There is some \( n \in \mathbb{Z}_{\geq 0} \) such that \( px^n \in R[x; \sigma] \). Observe that, by Proposition 5 \( \phi \) maps the \( R \)-subring \( R[x; \sigma] \) of \( W \) generated by \( x \) isomorphically to the \( R \)-subring \( R[x; \sigma] \) of \( R[x^\pm; \sigma] \). So

\[
p = \phi(\phi^{-1}(px^n))\phi(x^n)^{-1},
\]

proving that \( \phi \) gives a right ring of fractions. That \( \phi \) also works as a left ring of fractions is obtained for free using Proposition 3.

Proposition 22: Let \( W = R[x, y; r, \sigma, z] \) be a GWA. Let \( S \subseteq R \) be a right denominator set, and assume that \( \sigma(S) = S \). Then \( S \) is a right denominator set of \( W \), and the associated localization map has the following description: Let \( \phi_0 : R \to RS^{-1} \) be the localization map for the right ring of fractions of \( R \). Let \( \hat{\sigma} \) be the automorphism of \( RS^{-1} \) induced by \( \sigma \), and let \( \hat{z} = \phi_0(z) \). Let \( \phi : W \to RS^{-1}[x, y; \hat{\sigma}, \hat{z}] \) be the extension of \( R \to RS^{-1}[x, y; \hat{\sigma}, \hat{z}] \) to \( W \) that sends \( x \) to \( x \) and \( y \) to \( y \). This is the desired localization map. In short,

\[
WS^{-1} = (RS^{-1})[x, y; \hat{\sigma}, \hat{z}].
\]

An analogous statement holds for left denominator sets.

Proof: Note that \( \hat{\sigma} \) exists due to our hypothesis \( \sigma(S) = S \). And the extension \( \phi_0 \) of \( \phi_0 \) exists because GWA relations hold where needed. If we can show that \( \phi \) really does define a right ring of fractions of \( R[x, y; r, \sigma, z] \) with respect to \( S \), then it will follow that \( S \) is a right denominator set in \( R[x, y; r, \sigma, z] \) (by [12, Theorem 10.3] for example). Thus, three things need to be verified: that \( \phi(S) \) is a collection of units, that elements of \( RS^{-1}[x, y; \hat{\sigma}, \hat{z}] \) have the form \( \phi(w)\phi(s)^{-1} \) with \( w \in W \) and \( s \in S \), and that the kernel of \( \phi \) is \( \{w \in W \mid ws = 0 \text{ for some } s \in S\} \). That \( \phi(S) \) is a collection of units is obvious.

Let \( \sum_{i \in \mathbb{Z}} a_i v_i \) be an arbitrary element of \( RS^{-1}[x, y; \hat{\sigma}, \hat{z}] \). Get a “common right denominator” \( s \in S \) and elements \( r_i \) of \( R \) so that \( a_i = \phi_0(r_i)\phi_0(s)^{-1} \) for all \( i \in \mathbb{Z} \) (see [12, Lemma 10.2a], noting that all but finitely many of the \( a_i \) vanish). Then

\[
\sum a_i v_i = \sum \phi_0(r_i)\phi_0(s)^{-1}v_i = \sum \phi_0(r_i)v_i\hat{\sigma}^{-1}(\phi_0(s)^{-1}) = \sum \phi(r_i,v_i)\phi_0(\sigma^{-1}(s))^{-1} \]

After a further choice of common denominator, we see that \( \sum_{i \in \mathbb{Z}} a_i v_i \) has the needed form. It remains to examine the kernel of \( \phi \). Let \( w = \sum r_i v_i \) be an arbitrary element of \( W \). If \( ws = 0 \) with \( s \in S \), then \( \phi(w) \) must vanish because \( \phi(s) \) is a unit. Assume for the converse that \( 0 = \phi(w) = \sum \phi_0(r_i)v_i \). Then \( r_i \in \ker(\phi_0) \) for all \( i \), so there are \( s_i \in S \) such that \( r_i s_i = 0 \) for all \( i \). By [12, Lemma 4.21], there are \( b_i \in R \) such that the products \( \sigma^{-1}(s_i)b_i \) are all equal to a single \( s \in S \). Then

\[
ws = \sum r_i s_i v_i \sigma^{-1}(s_i)b_i = \sum r_i s_i v_i b_i = 0.
\]

Thus \( \ker(\phi) = \{w \in W \mid ws = 0 \text{ for some } s \in S\} \), and this completes the proof of the right-handed version of the theorem. The left-handed version can be obtained for free from Proposition 3.
We will generally abuse notation and reuse the labels “$\sigma$” and “$z$” instead of putting hats on things.

**Corollary 23:** The localization of $W = R[x, y; \sigma, z]$ at the multiplicative set $S$ generated by $\{\sigma^n(z) \mid n \geq 0\}$ is a skew Laurent ring $(RS^{-1})[x^\pm; \sigma]$, where the localization map extends the one $R \to RS^{-1}$ by sending $x$ to $x$ and $y$ to $yz^{-1}$.

**Proof:** Use Proposition 22 to describe the localization. Then observe that it is isomorphic to a skew Laurent ring by Proposition 5 since $z$ has become a unit.

### 2.4 GK Dimension

Throughout this section, $R$ denotes an algebra over a field $k$, $z$ a central element, $\sigma : R \to R$ an algebra automorphism, and $W$ the GWA $R[x, y; \sigma, z]$.

**Proposition 24:** $\text{GK}(W) \geq \text{GK}(R) + 1$

**Proof:** Since $W$ contains a copy of the skew polynomial ring $R[x; \sigma]$, the problem reduces to showing that $\text{GK}(R[x; \sigma]) \geq \text{GK}(R) + 1$. The proof is standard (c.f. [15, Lemma 3.4]).

Under what conditions can Proposition 24 be upgraded to an equality? We look to the skew Laurent case, i.e. the case in which $z$ is a unit, for some guidance.

**Definition 25:** An algebra automorphism $\sigma : R \to R$ is locally algebraic iff for each $r \in R$, $\{\sigma^n(r) \mid n \geq 0\}$ spans a finite dimensional subspace of $R$. Equivalently, $\sigma$ is locally algebraic if and only if every finite dimensional subspace of $R$ is contained in some $\sigma$-stable finite dimensional subspace of $R$.

It was shown in [17] Prop. 1] that if $\sigma$ is locally algebraic, then $\text{GK}(R[x^\pm; \sigma]) = \text{GK}(R) + 1$. The locally algebraic assumption was also shown to be partly necessary in [23], for example when $R$ is a commutative domain with finitely generated fraction field. So we should at least adopt the locally algebraic assumption. Unfortunately, it is difficult to apply the result of [17] to a general GWA; the process of inverting $z$, as in Corollary 24 does not make it simple to carry along GK dimension information. For one thing, $z$ is typically not central or even normal in $W$. Also, a locally algebraic $\sigma$ can fail to induce a locally algebraic automorphism of the localized algebra. So we instead proceed with a direct calculation:

**Theorem 26:** Assume that the automorphism $\sigma : R \to R$ is locally algebraic. Then

$$\text{GK}(W) = \text{GK}(R) + 1.$$  

**Proof:** Given Proposition 24, it remains to show that $\text{GK}(W) \leq \text{GK}(R) + 1$. Let $Z$ denote the linear span of $\{\sigma^n(z) \mid n \geq 0\}$, and $\{1\}$. Consider any affine subalgebra of $W$; let $X$ be a finite dimensional generating subspace for it. We first enlarge $X$ to a subspace $\bar{X}$ of the form

$$\bar{X} := \bigoplus_{|m| \leq m_0} U v_m,$$

where $U$ is a finite dimensional $\sigma$-stable subspace of $R$ with $Z \subseteq U$. Here is a procedure for doing this: for $m \in \mathbb{Z}$, let $\pi_m : W \to R$ denote the $m$th projection map coming from the left $R$-basis $(v_m)_{m \in \mathbb{Z}}$ of $W$. Let $m_0 = \max\{|m| \mid \pi_m(X) \neq 0\}$. Now $\sum_{|m| \leq m_0} \pi_m(X)$ is a finite dimensional subspace of $R$, so it is contained in a $\sigma$-stable subspace $U$ of $R$. It is harmless to include $Z$ in $U$ (note that $Z$ is finite dimensional because $\sigma$, and hence also $\sigma^{-1}$, is locally algebraic). This gives us $\bar{X}$ defined by (19), with $X \subseteq \bar{X}$.  

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Next, we show that
\[ X^n \subseteq \bigoplus_{|m| \leq nm_0} U^{n+(n-1)m_0} v_m \]  
for \( n \geq 1 \). It holds by definition when \( n = 1 \), so assume that \( n > 1 \) and that (20) holds for \( \hat{X}^{n-1} \). Then the induction goes through:
\[
\begin{align*}
\hat{X}^n &= \hat{X}^{n-1} \hat{X} \subseteq \left( \bigoplus_{|m| \leq (n-1)m_0} U^{n-1+(n-2)m_0} v_m \right) \left( \bigoplus_{|m| \leq m_0} U v_m \right) \\
&= \bigoplus_{|m| \leq nm_0} \sum_{m_1 + m_2 = m} U^{n-1+(n-2)m_0} v_{m_1} U v_{m_2} \\
&\subseteq \bigoplus_{|m| \leq nm_0} \sum_{m_1 + m_2 = m} U^{n+(n-1)m_0} v_{m_1 + m_2} = \bigoplus_{|m| \leq nm_0} U^{n+(n-1)m_0} v_m.
\end{align*}
\]
For the inclusion in the final line we used the fact, evident from (8), that \( [m_1, m_2] \in Z_{\text{min}}(m_1, m_2) \subseteq U_{\text{min}}([m_1], [m_2]) \). With (20) established, we have
\[
\dim(\hat{X}^n) \leq (2nm_0 + 1) \dim(U^{n+(n-1)m_0})
\]
for all \( n \geq 1 \). The theorem follows:
\[
\text{GK}(k(X)) \leq \text{GK}(k(\hat{X})) \leq 1 + \text{GK}(R).
\]

2.5 Representation Theory

Modules over GWAs have been explored and classified under various hypotheses by several authors. A classification of simple \( R[x, y; \sigma, z] \)-modules is obtained in [3] for \( R \) a Dedekind domain with restricted minimum condition and with a condition placed on \( \sigma \): that maximal ideals of \( R \) are never fixed by any nonzero power of \( \sigma \). These results are expanded in [5] and further in [10], where indecomposable weight modules with finite length as \( R \)-modules are classified for \( R \) commutative. In the latter work, the authors introduce \emph{chain} and \emph{circle} categories to handle maximal ideals of \( R \) that have infinite and finite \( \sigma \)-orbit respectively. Another expansion of the work of [5] was carried out in [20], where the simple \( R \)-torsion modules were classified relaxing all assumptions on \( R \) (even commutativity), but with the assumption that \( \sigma \) acts freely on the set of maximal left ideals of \( R \). In order to establish notation and put the spotlight on a particular setting that will be of use to us, we proceed with our own development.

2.5.1 Simple Modules of Finite Dimension

Let \( R \) be a commutative \( k \)-algebra and let \( W = R[x, y; \sigma, z] \) be a GWA. Let \( W \cdot V \) be a finite dimensional simple left \( W \)-module. It contains some simple left \( R \)-module \( V_0 \), which has an annihilator \( m := \text{ann}_R V_0 \in \text{max spec } R \). The automorphism \( \sigma \) acts on \text{max spec } R, and the behavior of \( V \) depends largely on whether \( m \) sits in a finite or an infinite orbit. We’d like to deal with the infinite orbit case, so assume that \( \sigma^i(m) = \sigma^j(m) \Rightarrow i = j \) for \( i, j \in \mathbb{Z} \).

Let \( e_0 \) be a nonzero element of \( V_0 \), so we have \( m = \text{ann}_R e_0 \). For \( i \in \mathbb{Z} \), let \( e_i = v_i e_0 \). Notice that for \( i \in \mathbb{Z} \) and \( r \in m \), we have
\[
\sigma^i(r) e_i = \sigma^i(r) v_i e_0 = v_i r e_0 = 0,
\]
as an $W$-module. Let $e_i$ denote $1 \in R/\sigma^i(m)$ as an element of $\bigoplus_{i=0}^{n-1} R/\sigma^i(m)$ for $0 \leq i \leq n-1$, and let $e_{n-1} = e_n = 0$. Then $\mathcal{A} V$ is isomorphic to $\bigoplus_{i=0}^{n-1} R/\sigma^i(m)$ if $\bigoplus_{i=0}^{n-1} R/\sigma^i(m)$ is given the following $W$-action:

$$x(re_i) = \sigma(r)e_{i+1}$$

$$y(re_i) = \sigma^{-1}(r)ze_{i-1}.$$

One could check explicitly that forming the $\infty$-orbit. Then

$\sigma_i(m) \subseteq \text{ann}_R e_i$. So whenever $e_i \neq 0$, $\sigma_i(m) = \text{ann}_R e_i$. We use this to argue that the subspaces $Re_i$ are independent: Consider a vanishing combination

$$\sum_{i \in \mathcal{I}} r_i e_i = 0$$

where $\mathcal{I} \subseteq \mathbb{Z}$ is finite and $e_i \neq 0$ for $i \in \mathcal{I}$. For any $j \in \mathcal{I}$, choose a $c_j \in \left(\prod_{i \in \mathcal{I} \setminus \{j\}} \sigma^i(m)\right) \setminus \sigma^j(m)$, and apply it to (21). The result is $c_j r_j e_j = 0$, which implies that $c_j r_j \in \sigma^j(m)$, so $r_j \in \sigma^j(m)$ and $r_j e_j = 0$.

Since we assumed $V$ to be finite dimensional, only finitely many of the $e_i$ may be nonzero. In particular, there is some $e_{i_0} \neq 0$ such that $e_{i_0-1} = 0$ (a “lowest weight vector”). We may as well shift our original indexing so that this $e_{i_0}$ is $e_0$. (After all, $e_0$ was only assumed to be a nonzero element of some simple $R$-submodule of $V$ with annihilator having infinite $\sigma$-orbit, and $e_{i_0}$ would have fit the bill just as well.) Similarly, on the other end, there is some $n \geq 0$ so that $e_{n-1} \neq 0$ and $e_n = 0$. Note that these definitions imply that $e_i = x^i e_0$ is nonzero for $0 \leq i \leq n-1$.

It is now clear that $\bigoplus_{i=0}^{n-1} Re_i$ is a $W$-submodule of $V$:

$$x(re_i) = \sigma(r)x e_i$$

$$y(re_i) = \sigma^{-1}(r)y e_i = \sigma^{-1}(r)z e_{i-1}. \quad (22)$$

So, since $\mathcal{A} V$ is simple, $\bigoplus_{i=0}^{n-1} Re_i = V$. Each $Re_i$ for $0 \leq i \leq n - 1$ is isomorphic as an $R$-module to $R/\sigma^i(m)$. Knowing this and knowing that the $W$-action is described by (22), we have pinned down $\mathcal{A} V$ up to isomorphism. Let us also pin down $e_0$ and $m$.

Applying $xy$ and $yx$ to the extreme “edges” of the module shows that $\sigma(z), \sigma^{-n+1}(z) \in m$:

$$\sigma(z) e_0 = x(y e_0) = 0 \Rightarrow \sigma(z) \in m$$

$$z e_{n-1} = y(x e_{n-1}) = 0 \Rightarrow z \in \sigma^{n-1}(m).$$

Further, $n > 0$ is minimal with respect to this property: if we had $0 < i < n$ with $\sigma^{-i+1}(z) \in m$, then $y e_i = 0$, so $Re_i + \cdots + Re_n$ would be a proper nontrivial submodule of $V$.

This allows us to characterize $Re_0$ as $\text{ann}_V(y)$, as follows. The inclusion $Re_0 \subseteq \text{ann}_V(y)$ is obvious since $y$ normalizes $R$. Suppose that $y.\left(\sum_{i=0}^{n-1} r_i e_i\right) = 0$, where $r_i \in R$. Then $0 = \sum_{i=1}^{n-1} \sigma^{-1}(r_i)z e_{i-1}$, so for each $1 \leq i \leq n-1$ we have $\sigma^{-1}(r_i)z \in \sigma^{i-1}(m)$. The minimality of $n$ discussed above implies that $z \notin \sigma^{i-1}(m)$, so we have $\sigma^{-1}(r_i) \in \sigma^{i-1}(m)$, and hence $r_i \in \sigma^i(m) = \text{ann}_R(e_i)$, for $1 \leq i \leq n-1$. So $\sum_{i=0}^{n-1} r_i e_i = r_0 e_0 \in Re_0$, proving that $\text{ann}_V(y) = Re_0$. We have also gained a nice internal description for $m$: it is $\text{ann}_R(\text{ann}_V(y))$. Let us record what has been established so far:

**Lemma 27:** Let $\mathcal{A} V$ be a finite dimensional simple left $W$-module, where $W = R[x, y; \sigma, z]$ and $R$ is a commutative $k$-algebra. Assume that $V$ contains some simple $R$-submodule with annihilator having infinite $\sigma$-orbit. Then $\text{ann}_V(y)$ is just such an $R$-submodule. Let $m = \text{ann}_R(\text{ann}_V(y))$. Then $\sigma(z) \in m$, there is a minimal $n > 0$ such that $\sigma^{-n+1}(z) \in m$, and $V$ is isomorphic to

$$\bigoplus_{i=0}^{n-1} R/\sigma^i(m) \quad (23)$$

as an $R$-module. Let $e_i$ denote $1 \in R/\sigma^i(m)$ as an element of (24) for $0 \leq i \leq n-1$, and let $e_{n-1} = e_n = 0$. Then $\mathcal{A} V$ is isomorphic to (24) if (24) is given the following $W$-action:

$$x(re_i) = \sigma(r)e_{i+1}$$

$$y(re_i) = \sigma^{-1}(r)z e_{i-1}.$$
about $W$ by instead realizing these modules as quotients by certain left ideals. We will run into a family of infinite dimensional simple modules along the way; the construction mimics the Verma modules typical to the treatment of representations of $\mathfrak{sl}_2$ \cite{13} II.7 and $U_q(\mathfrak{sl}_2)$ \cite{6} I.4.

**Definition 28:** Let $R$ be a commutative ring, $W = R[x, y; \sigma, z]$, and $m$ a maximal ideal of $R$ with infinite $\sigma$-orbit. Define $I_m := Wm$ to be the left ideal of $W$ that $m$ generates, and define $M_m$ to be the $\mathbb{Z}$-graded left $W$-module $M_m := W/I_m$. Define $e_i$ to be the image of $v_i$ in $M_m$ for $i \in \mathbb{Z}$.

Note that

$$I_m = \bigoplus_{i \in \mathbb{Z}} \sigma^i(m)v_i;$$

the inclusion $\supseteq$ is due to the fact that $v_i m = \sigma^i(m)v_i$, and $\subseteq$ holds because the right hand side is a left ideal of $W$ (condition \cite{12} is satisfied).

**Lemma 29:** Let $R$ be a commutative $k$-algebra, $W = R[x, y; \sigma, z]$, and $m$ a maximal ideal of $R$ with infinite $\sigma$-orbit. The submodules of $M_m$ are of the following types:

1. $0$ or $M_m$
2. $\bigoplus_{i \geq j} R e_i$ for some $j > 0$ with $\sigma^{-j+1}(z) \in m$
3. $\bigoplus_{i \leq -j'} R e_i$ for some $j' > 0$ with $\sigma^{j'}(z) \in m$
4. a sum of a submodule of type 2 and one of type 3

**Proof:** Let $S$ be a proper nontrivial submodule of $M_m$. We first show that $S$ is homogeneous, so that if $\sum a_i e_i \in S$ with a certain $a_i e_i \neq 0$, then $e_j \in S$.

**Claim:** $S$ is homogeneous.

**Proof:** Suppose that $a \in S$, say $a = \sum_{i \in I} a_i e_i$ with $I \subseteq \mathbb{Z}$ finite and $a_i \in R \setminus \sigma^i(m)$ for $i \in I$. Let $j \in I$, and choose an element $c$ of $\left( \prod_{i \in I \setminus \{j\}} \sigma^i(m) \right) \setminus \sigma^j(m)$. Then $ca = ca_j e_j \in S$. Since $c, a_j \in R \setminus \sigma^j(m)$, $ca_j$ is a unit mod $\sigma^j(m)$. Hence $e_j \in S$.

Define vector subspaces $M^+ := \bigoplus_{i \geq 0} R e_i$ and $M^- := \bigoplus_{i < 0} R e_i$ of $M_m$. Since $S$ is proper and homogeneous, $S = (S \cap M^+) \oplus (S \cap M^-)$.

To show that $S$ is of type 2 or 3 then, it suffices to show that $S \cap M^+$ is a type 2 submodule when it is nonzero, and that $S \cap M^-$ is a type 3 submodule when it is nonzero.

Assume that $S \cap M^+ \neq 0$. Then $e_j \in S$ for some $j > 0$; let $j > 0$ be minimal such that this happens. By applying powers of $x$, we see that $S \cap M^+ = \bigoplus_{i \geq j} R e_i$. Since $e_{j-1} \notin S$, $ye_j = xe_{j-1}$ must vanish. This happens if and only if $z \in \sigma^{j-1}(m)$, i.e. if and only if

$$\sigma^{-j+1}(z) \in m.$$  \hspace{1cm} (24)

Now assume that $S \cap M^- \neq 0$. Let $j' > 0$ be minimal such that $e_{-j'} \in S$. By applying powers of $y$, we see that $S \cap M^- = \bigoplus_{i \leq -j'} R e_i$. Since $e_{-j'+1} \notin S$, $xe_{-j'} = \sigma(z)e_{-j'+1}$ must vanish. This happens if and only if $\sigma(z) \in \sigma^{-j'+1}(m)$, i.e. if and only if

$$\sigma^{j'}(z) \in m.$$  \hspace{1cm} (25)

Finally, it is routine to check that \cite{14} are actually submodules of $M_m$, considering the equivalences mentioned in \cite{24} and \cite{25}.

\[ \blacksquare \]
This shows that $M_m$ has a unique largest proper submodule, $N_m$, given by
\[ N_m := \bigoplus_{i \leq -n'} R e_i \oplus \bigoplus_{i \geq n} R e_i \quad (\text{with } n, n' \text{ possibly } \infty) \] (26)
where $n > 0$ is chosen to be minimal such that $\sigma^{-n+1}(z) \in m$ (or $\infty$ if this never occurs) and $n' > 0$ is chosen to be minimal such that $\sigma^{n'}(z) \in m$ (or $\infty$ if this never occurs). For example, if $m$ is disjoint from $\{\sigma^i(z) \mid i \in \mathbb{Z}\}$, then $N_m = 0$ and $M_m$ is simple.

**Theorem 30:** Let $R$ be a commutative $k$-algebra and $W = R[x, y; \sigma, z]$.

1. Let $m$ be a maximal ideal of $R$ with infinite $\sigma$-orbit. Assume that $R$ is affine. The simple module $V_m := M_m/N_m$ is finite dimensional if and only if there are $n, n' > 0$ such that
\[ \sigma^{-n+1}(z), \sigma^{n'}(z) \in m. \]

2. Let
\[ \mathcal{M} = \{ m \in \text{max spec } R \mid \text{m has infinite } \sigma \text{-orbit, } \sigma(z) \in m, \text{ and } \sigma^{-n+1}(z) \in m \text{ for some } n > 0 \}. \]

Any finite dimensional simple left $W$-module $V$ that contains a simple $R$-submodule with annihilator having infinite $\sigma$-orbit is isomorphic to $V_m$ for exactly one $m \in \mathcal{M}$, namely $m = \text{ann}_{R}(\text{ann}_{W}(y))$.

3. If $m \in \mathcal{M}$ and $n > 0$ is minimal such that $\sigma^{-n+1}(z) \in m$, then $V_m \cong W/(Wm + Wy + Wx^n)$.

**Proof:** Assertion 1 follows from Lemma 24, the definition of $N_m$, and the fact (due to the Nullstellensatz) that each $R/\sigma'(m)$ is finite dimensional when $R$ is affine. For assertion 2, suppose that $\mathcal{W}V$ is simple, finite dimensional, and contains a simple $R$-submodule with annihilator having infinite $\sigma$-orbit. Lemma 27 pins $V$ down as isomorphic to the left $W$-module in (23). This construction is in turn isomorphic to $V_m$, where $m = \text{ann}_{R}(\text{ann}_{W}(y))$, and the lemma guarantees that $\sigma(z), \sigma^{-n+1}(z) \in m$ for some $n > 0$. Hence $m \in \mathcal{M}$ and $V \cong V_m$. Since $m = \text{ann}_{R}(\text{ann}_{W}(y))$, no two $V_m$ for $m \in \mathcal{M}$ can be isomorphic. Assertion 3 amounts to the fact that, under the given hypotheses, $N_m$ is the submodule of $M_m$ generated by the cosets $y + I_m$ and $x^n + I_m$. \[ \blacksquare \]

### 2.5.2 Weight Modules of Finite Dimension

In further pursuit of finite dimensional modules, we now explore a class of modules that includes the semisimple ones. We continue with the notation $W = R[x, y; \sigma, z]$ and the assumption that $R$ is a commutative $k$-algebra. Let $\mathcal{W}X$ be finite dimensional and semisimple. Consider the $R$-submodule spanned by annihilators of maximal ideals,
\[ S := \sum_{m \in \text{max spec } R} \text{ann}_{X} m. \]
It is in fact a $W$-submodule of $X$, since $x$ and $y$ map $\text{ann}_{X} m$ into $\text{ann}_{X} \sigma(m)$ and $\text{ann}_{X} \sigma^{-1}(m)$ respectively. Since we assumed $X$ to be semisimple, $S$ has a direct sum complement $S'$ in $\mathcal{W}X$. If $S'$ were nonzero, then it contains some simple $R$-submodule which is then annihilated by some maximal ideal of $R$, contradicting $S' \cap S = 0$. Thus our assumption that $\mathcal{W}X$ is semisimple requires $X = S$. We now wonder when this condition is sufficient for semisimplicity.

**Definition 31:** Let $R$ be a commutative $k$-algebra. A $W$-module where $W = R[x, y; \sigma, z]$ is a weight module if and only if it is semisimple as an $R$-module. Note that this is equivalent to saying that $X$ is spanned by annihilators of maximal ideals of $R$. The support $\text{supp } X$ of an $R$-module $X$ is the collection of maximal ideals $m$ of $R$ such that $\text{ann}_{X} m$ is nonzero.
Let us collect some elementary facts about weight modules for use in the coming semisimplicity theorem.

**Proposition 32:** Let $R$ be a commutative $k$-algebra, $X$ a semisimple $R$-module, and $\# Y \leq \# X$. Then

$$X = \bigoplus_{m \in \max \spec R} \ann_X m$$

and the canonical map $X \to X/Y$ induces an isomorphism of $R$-modules

$$\(\ann_X m)/(\ann_Y m) \cong \ann_{X/Y} m$$

for each $m \in \max \spec R$.

**Proof:** By assumption, $X$ is a direct sum of simple $R$-submodules. Each simple $R$-submodule is isomorphic to $R/m$ for some $m \in \max \spec R$. Thus $X$ is a direct sum of the $\ann_X m$; each $\ann_X m$ is actually just the $(R/m)$-homogeneous component of $X$. Since $Y$ is a submodule of $X$, it is semisimple and has its own decomposition

$$Y = \bigoplus_{m \in \max \spec R} \ann_Y m = \bigoplus_{m \in \max \spec R} Y \cap \ann_X m. \tag{28}$$

Fix an $m \in \max \spec R$. It is clear that the canonical map $X \to X/Y$ restricts to an $R$-homomorphism $\ann_X m \to \ann_{X/Y} m$ with kernel $Y \cap \ann_X m = \ann_Y m$. To see that it is surjective, consider any $x + Y \in \ann_{X/Y} m$. Write $x$ as $\sum_{n \in \max \spec R} x_n$, where $x_n \in \ann_X n$. Since $mx \subseteq Y$, the decomposition (28) gives $mx_n \subseteq Y$ for all $n$. When $n \neq m$, this implies that $x_n \in Y$ since $m$ contains a unit mod $n$. Thus $x + Y$ is the image of $x_m$ under $\ann_X m \to \ann_{X/Y} m$. $lacksquare$

Since we only focused on simple finite dimensional $W$-modules of a certain type, we will only attempt to get at the weight modules whose composition factors are of that type. Adapting the “chain” and “circle” terminology from [10].

**Definition 33:** Let $R$ be a commutative $k$-algebra and let $W = R[x, y; \sigma, z]$. A finite dimensional module $wX$ is of chain-type if and only if every $m \in \supp X$ has infinite $\sigma$-orbit.

**Proposition 34:** Let $R$ be a commutative $k$-algebra, $W = R[x, y; \sigma, z]$, and $wX$ a chain-type finite dimensional weight module. Let $\mathcal{M}$ be as in (27). Then each composition factor of $X$ has the form $V_m$ for some $m \in \mathcal{M}$, and

$$\supp X \cap \mathcal{M} = \{m \in \mathcal{M} \mid V_m \text{ is a composition factor of } wX\}. \tag{29}$$

**Proof:** Choose a $W$-module composition series $0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X$. It can be refined into a composition series for $\# X$, so since $\# X$ is semisimple we have:

$$\# X \cong \bigoplus_{i=1}^r \bigoplus \{(R/m)^{(k)} \mid R/m \text{ is a composition factor of } X_i/X_{i-1} \text{ with multiplicity } k\}. \tag{30}$$

In particular, each $X_i/X_{i-1}$ contains some simple $R$-submodule whose annihilator comes from $\supp X$ and therefore has infinite $\sigma$-orbit. Theorem 30 applies: for $1 \leq i \leq r$, $X_i/X_{i-1} \cong V_{m_i}$ for a unique $m_i \in \mathcal{M}$. The right hand side of (29) is then $\{m_1, \ldots, m_r\}$. For each $1 \leq i \leq r$, let $n_i > 0$ be minimal such that $\sigma^{-n_i+1}(z) \in m_i$.

Knowing that $X_i/X_{i-1} \cong V_{m_i}$, we can read off the support of $X$ from (30):

$$\supp X = \{\sigma^\ell(m_i) \mid 1 \leq i \leq r \text{ and } 0 \leq \ell \leq n_i - 1\}.$$
Suppose that $\sigma^j(m_i) \in \mathcal{M}$, with $1 \leq i \leq r$ and $0 \leq j \leq n_i - 1$. Then $\sigma(z) \in \sigma^j(m_i)$, so $\sigma^{-\ell + 1}(z) \in m_i$. The minimality of $n_i$ forces $\ell = 0$. This proves that $\text{supp} X \cap \mathcal{M} = \{m_1, \ldots, m_r\}$, and the latter is the right hand side of (29). □

Next, we identify a condition on $\text{supp} X \cap \mathcal{M}$ that we will show guarantees semisimplicity for $wX$.

**Definition 35:** Let $R$ be a commutative $k$-algebra, $\sigma$ an automorphism, and $z \in R$. Let $\mathcal{M}$ be as in (27). A subset $S \subset \mathcal{M}$ has separated chains if and only if the following holds: whenever $m \in S$ and $n > 0$ is minimal such that $\sigma^{-n+1}(z) \in m$, it follows that $\sigma^n(m) \not\in S$.

**Proposition 36:** Let $R$ be a commutative $k$-algebra, $\sigma$ an automorphism, and $z \in R$. Let $\mathcal{M}$ be as in (27), and suppose that $S \subset \mathcal{M}$ has separated chains. Then given $m, m' \in S$ and $n, n' > 0$ minimal such that $\sigma^{-n+1}(z) \in m$ and $\sigma^{-n'+1}(z) \in m'$,

1. $m' \in \{m, \sigma(m), \ldots, \sigma^{n-1}(m)\}$ only if $m' = m$
2. $\sigma^{-1}(m'), \sigma^{-n'}(m') \not\in \{m, \sigma(m), \ldots, \sigma^{n-1}(m)\}$.

**Proof:** Suppose that $m' = \sigma^j(m)$, where $0 \leq j \leq n - 1$. Then $\sigma(z) \in \sigma^j(m)$, so $\sigma^{-\ell + 1}(z) \in m$. The minimality of $n$ then forces $\ell = 0$, whence $m' = m$.

Suppose that $\sigma^{-1}(m') = \sigma^j(m)$, where $0 \leq j \leq n - 1$. Then $\sigma(z) \in m' = \sigma^{j+1}(m)$, so $\sigma^{-\ell + (j+1)+1}(z) \in m$. The minimality of $n$ then forces $\ell = n$, which gives $\sigma^n(m) = m' \in S$. This contradicts the assumption that $S$ has separated chains.

Suppose that $\sigma^{-n'}(m') = \sigma^j(m)$, where $0 \leq j \leq n - 1$. Then we have $\sigma^{-\ell + 1}(z) \in m$, since $\sigma^{-n'+1}(z) \in m'$. The minimality of $n$ then forces $\ell = 0$, which gives $\sigma^{-n'}(m') = m \in S$, contradicting the assumption that $S$ has separated chains. □

**Theorem 37:** Let $R$ be a commutative $k$-algebra, let $W = R[x, y; \sigma, z]$, and let $\mathcal{M}$ be as in (27). Let $X$ be a chain-type finite dimensional weight left $W$-module. If $\text{supp} X \cap \mathcal{M}$ has separated chains, then $X$ is semisimple.

**Proof:** Assume the hypotheses. Choose a composition series for $wX$:

$$0 = X_0 \subseteq X_1 \subseteq \cdots \subseteq X_r = X.$$  

For $1 \leq i \leq r$, $X_i/X_{i-1} \cong V_{m_i}$ for a unique $m_i \in \mathcal{M}$, and $\{m_1, \ldots, m_r\}$ has separated chains (Proposition 37). Let $n_1, \ldots, n_r$ be the distinct items among $m_1, \ldots, m_r$, with respective multiplicities $t_1, \ldots, t_r$. For each $1 \leq j \leq s$, let $n_j > 0$ be minimal such that $\sigma^{-n_j+1}(z) \in n_j$.

For any $a \in \text{max spec } R$, we iteratively apply Proposition 31 to obtain:

$$\dim_{R/a} \text{ann}_X a = \dim_{R/a} \text{ann}_{X_r/X_{r-1}} a + \dim_{R/a} \text{ann}_{X_{r-1}/X_{r-2}} a + \cdots = \sum_{i=1}^r \dim_{R/a} \text{ann}_{V_{m_i}} a$$

$$= \sum_{i=1}^r \dim_{R/a} \text{ann}_{V_{m_i}} a \quad (31)$$

Fix $a$ with $1 \leq j \leq s$. Apply 31 to the case $a = n_j$ and use Proposition 31 to obtain

$$\dim_{R/n_j} \text{ann}_X n_j = t_j.$$  

Apply 31 to the cases $a = \sigma^{-1}(n_j)$ and $a = \sigma^{n_j}(n_j)$ and use Proposition 31 to obtain

$$\text{ann}_X (\sigma^{-1}(n_j)) = \text{ann}_X (\sigma^{n_j}(n_j)) = 0.$$  


Let $b_1', \ldots, b_{t_j}'$ be an $(R/n_j)$-basis for $\text{ann}_X n_j$. Each $Wb_i'$ is a nonzero homomorphic image of $wW$ in which $n_i, y, z$ and $x^{n_j}$ are killed; $n_j$ is killed because $b_i'$ came from $\text{ann}_X n_j$, and $y$ and $x^{n_j}$ are killed because they map $\text{ann}_X n_j$ into $\text{ann}_X (\sigma^{-1}(n_j))$ and $\text{ann}_X (\sigma^{n_j}(n_j))$ respectively. By Theorem 29 it follows that each $Wb_i'$ is isomorphic to $M_{n_j}/N_{n_j} =: V_{n_j}$.

Do this for all $j$. Let

$$S = \sum_1^t \sum_1^{t_j} Wb_i',$$

a semisimple $W$-module of $X$. Since $\text{ann}_X n_j$ is $R$-spanned by $b_1', \ldots, b_{t_j}'$, we have (by Proposition 32)

$$\text{ann}_X n_j \cong (\text{ann}_X n_j)/(\text{ann}_S n_j) = 0$$

for all $j$. By the Jordan-Hölder theorem, any simple $W$-module of $X/S$ is isomorphic to $V_{n_j}$ for some $j$. Therefore $X/S$ must be 0. That is, $X = S$ is semisimple. □

If $\mathcal{M}$ as a whole has separated chains, then we conclude from this theorem that all chain-type finite dimensional weight modules are semisimple. There is a converse:

**Proposition 38:** Let $R$ be an affine commutative $k$-algebra, let $W = R[x, y; \sigma, z]$, and let $\mathcal{M}$ be as in (27). If $\mathcal{M}$ does not have separated chains, then there is a chain-type finite dimensional weight left $W$-module that is not semisimple.

**Proof:** If $\mathcal{M}$ does not have separated chains, there is some $m \in \mathcal{M}$ such that $\sigma^n(m) \in \mathcal{M}$, where $n > 0$ is minimal such that $\sigma^{-n+1}(z) \in m$. Let $n' > 0$ be minimal such that $\sigma^{-n'+1}(z) \in \sigma^n(m)$. Then $m$ contains $\sigma(z), \sigma^{-n+1}(z)$, and $\sigma^{-(n+n')}(z)$. Hence

$$S := \bigoplus_{i \leq -1} R_{e_i} \oplus \bigoplus_{i \geq n+n'} R_{e_i}$$

is a submodule of $M_m$, by Lemma 29. Let $wX = M_m/S$. This is isomorphic to $\bigoplus_{0 \leq \leq n+n'}/R/\sigma'(m)$ as an $R$-module, so $wX$ is a chain-type finite dimensional weight left $W$-module. Since $M_m$ contains a unique largest proper submodule

$$N_m = \bigoplus_{i \leq -1} R_{e_i} \oplus \bigoplus_{i \geq n} R_{e_i}$$

and $N_m$ properly contains $S$, $X$ contains a unique largest proper nontrivial submodule $N_m/S$. Therefore $X$ cannot be semisimple. □

**Theorem 39:** Let $R$ be an affine commutative $k$-algebra, let $W = R[x, y; \sigma, z]$, and let $\mathcal{M}$ be as in (27). The following are equivalent:

1. All chain-type finite dimensional weight left $W$-modules are semisimple.
2. $\mathcal{M}$ has separated chains.
3. For any maximal ideal $m$ of $R$ with infinite $\sigma$-orbit, there are no more than two integers $i$ such that $\sigma^i(z) \in m$.
4. For any $m \in \mathcal{M}$, there is exactly one $n > 0$ such that $\sigma^{-n+1}(z) \in m$.

**Proof:** The equivalence 1–2 is due to Theorem 27 and Proposition 38.

2 $\implies$ 3 Assume that 3 fails. Let $m$ be in $\mathcal{M}$ and let $i < j$ be positive integers such that $\sigma^{-i+1}(z), \sigma^{-j+1}(z) \in m$. We may assume that $i > 0$ is minimal such that $\sigma^{-i+1}(z) \in m$. Observe that $\sigma(z), \sigma^{-(j-i)+1}(z) \in \sigma^i(m)$. This implies that $\sigma^i(m) \in \mathcal{M}$, so $\mathcal{M}$ does not have separated chains.
Assume that \( \mathfrak{M} \) fails; let \( m \) be a maximal ideal of \( R \) with infinite \( \sigma \)-orbit and with \( \sigma^i(z), \sigma^j(z), \sigma^k(z) \in m \), where \( i < j < k \) are integers. We may assume that \( j > i \) is minimal such that \( \sigma^i(z) \in m \) and that \( k > j \) is minimal such that \( \sigma^k(z) \in m \). Let \( n := \sigma^{-k+1}(m) \). Observe that \( n \in \mathfrak{M} \) since \( \sigma(z) \in n \) and \( \sigma^{-(k-j)+1}(z) \in n \). Since \( \sigma^{-(k-i)+1}(z) \in n \) as well, with \( k - i \neq k - j \), we see that \( \mathfrak{M} \) fails.

Suppose that \( \mathfrak{M} \) does not have separated chains. Then there is some \( m \in \mathfrak{M} \) such that \( \sigma^n(m) \in \mathfrak{M} \), where \( n > 0 \) is minimal such that \( \sigma^{n+1}(z) \in m \). Let \( n' > 0 \) be minimal such that \( \sigma^{-n'+1}(z) \in \sigma^n(m) \). Since \( \sigma(z), \sigma^{-n+1}(z), \sigma^{-(n+n')}(z) \in m \), \( \mathfrak{M} \) fails to hold.

\section{The 2 \times 2 Reflection Equation Algebra}

We now shift our focus to a specific GWA, the algebra \( A \) defined in (10). Define an automorphism \( \sigma \) of the polynomial ring \( k[u_{22}, u_{11}, z] \) by

\[
\begin{align*}
\sigma(u_{22}) &= q^2 u_{22} \\
\sigma(u_{11}) &= u_{11} + (q^2 - 1)u_{22} \\
\sigma(z) &= z + (q^2 - 1)u_{22}u_{11} - u_{11}.
\end{align*}
\]

The algebra \( A \) is a GWA over the above polynomial algebra, with \( x \) being \( u_{21} \) and \( y \) being \( u_{12} \):

\[ A \cong k[u_{22}, u_{11}, z][x, y; \sigma, z]. \]

This can be verified by defining mutually inverse homomorphisms in both directions using universal properties. One checks that the reflection equation relations \( (6) \) hold in the GWA, and that the GWA relations \( (7) \) hold in \( A \).

\textbf{Proposition 40:} \( A \) is a noetherian domain of GK dimension 4.

\textbf{Proof:} In [9] Proposition 3.1, polynomial sequences and Gröbner basis techniques are used to show that \( A_n(M_n) \) is a noetherian domain for all \( n \). Theorem 11 and Corollary 10 give an alternative way to see this for \( A = A_n(M_2) \).

It is also observed in [9] that the Hilbert series of \( A_n(M_n) \) can be determined using [10] (7.37)]. One may deduce from the Hilbert series that the GK dimension of \( A_n(M_n) \) is \( n^2 \). Theorem 26 gives an alternative way to see this for \( A = A_n(M_2) \), since \( \sigma \) is locally algebraic.

By a change of variables in \( k[u_{22}, u_{11}, z] \) we can greatly simplify the expression of \( A \) as a GWA. Consider the change of variables:

\[ u = u_{22}, \quad t = u_{11} + q^{-2}u_{22}, \quad d = z - q^{-2}u_{11}u_{22} \quad (32) \]

Now we have

\[ A \cong k[u, t, d][x, y; \sigma, z], \]

where \( z = d + q^{-2}tu - q^{-4}u^2 \) and

\[ \begin{align*}
\sigma(u) &= q^2 u \\
\sigma(t) &= t \\
\sigma(d) &= d.
\end{align*} \]

The special elements \( t \) and \( d \) of \( A \) are, up to a scalar multiple, the quantum trace and quantum determinant explored in [18].

Since \( q \) is not a root of unity, \( \sigma \) has infinite order. We may therefore apply Proposition 5 to determine the center of \( A \):
**Proposition 41:** $Z(A) = k[t, d]$.

This was also computed in [16], and a complete description of the center of $A_q(M_n)$ for arbitrary $n$ is given in [17].

Using the fact that $q$ is not a root of unity, the elements
\[
\sigma^m(z) = d + q^{2m-2}tu - q^{4m-4}u^2
\]
of $k[u, t, d]$, for $m \in \mathbb{Z}$, are pairwise coprime. This allows us to get at the normal elements of $A$, which gives us a handle on its automorphism group:

**Theorem 42:** The automorphism group of $A$ is isomorphic to $(k^\times)^2$, with $(\alpha, \gamma) \in (k^\times)^2$ corresponding to the automorphism given by
\[
\begin{pmatrix}
u_{11} & \nu_{12} \\ \nu_{21} & \nu_{22}\end{pmatrix} \mapsto \begin{pmatrix} \alpha \nu_{11} & \alpha \nu_{12} \\ \alpha \nu_{21} & \alpha \nu_{22} \end{pmatrix}
\]

**Proof:** Let $\psi : A \to A$ be an automorphism. By Proposition [11] the nonzero normal elements of $A$ are the $\sigma$-eigenvectors in $k[u, t, d]$. That is, they all have the form $u'f(t, d)$ for some polynomial $f(t, d)$ and some $i \in \mathbb{Z}_{\geq 0}$. Since $k[u, t, d]$ is the linear span of such elements, it is preserved by $\psi$. Since $u$ is normal, $\psi(u) = u'f(t, d)$ for some $i$ and $f$, and similarly $\psi^{-1}(u) = u'g(t, d)$ for some $j$ and $g$. Note that $k[t, d]$, being the center of $A$, is also preserved by $\psi$. Therefore $u = \psi(\psi^{-1}(u)) = u'fi\psi(g)$ implies that $i = 1$ and $f$ is a unit. So $\psi(u) = au$ for some $a \in k^\times$.

Observe that $\psi(xu) = a^{-1}\psi(xu) = a^{-1}q^2\psi(xu) = q^2\psi(xu)$. Any $a \in A$ with the property that $au = q^2ua$ is a sum of homogeneous such $a$’s, and a homogeneous such $a$ is $bu_m$ for some $b \in k[u, t, d]$ and some $m \in \mathbb{Z}$ such that
\[
q^2bu_m = bu_mu = q^{2m}bu_m.
\]
This equation requires that either $b = 0$ or $m = 1$. Therefore $\psi(x) = bx$ for some nonzero $b \in k[u, t, d]$. The same argument applies to $\psi^{-1}$, and it is easy to deduce from this that $b$ must be a unit, i.e. $b \in k^\times$.

Similarly, using the fact that $\psi(y)u = q^{-2}\psi(y)$, we get that $\psi(y) = cy$ for some $c \in k^\times$.

For any $m > 0$, we have
\[
b^{m-1}\psi(\sigma^m(z)) x^{m-1} = \psi(\sigma^m(z)) x^{m-1} = \psi(x^m) = b^m x^m = b^m \sigma^m(z) x^{m-1}.
\]
It follows that $\psi(\sigma^m(z)) = b\sigma^m(z)$ for all $m > 0$. Considering that $\sigma^m(z) = d + q^{2m-2}tu - q^{4m-4}u^2$, the linear span of $\{\sigma(z), \sigma^2(z), \sigma^3(z)\}$, for instance, contains $\{d, tu, u^2\}$. So $b\sigma(u^2) = \psi(u^2) = \alpha^2 u^2$, i.e. $bc = \alpha^2$. And $bctu = \psi(tu) = \alpha \psi(t)u$, so $\psi(t) = \alpha t$. And $bctd = \psi(d)$, so $\psi(d) = \alpha^2 d$. Letting $\gamma = ba^{-1}$, so that $\psi(x) = (\alpha \gamma)x$ and $\psi(y) = (\alpha \gamma)^{-1}y$, we see that $\psi$ is the automorphism corresponding to $(\alpha, \gamma)$ in the theorem statement. One easily checks that there is such an automorphism for every $(\alpha, \gamma) \in (k^\times)^2$, and that composition of automorphisms corresponds to multiplication in $(k^\times)^2$.

**3.1 Finite Dimensional Simple Modules**

The finite dimensional simple modules over $A$ come in two types: the ones annihilated by $u_{22}$ and the ones on which $u_{22}$ acts invertibly. This observation follows from the fact that since $u_{22}$ is normal, its
annihilator in any A-module is a submodule. The former are modules over \( A/(u_{22}) \), a three-variable polynomial ring. The latter are addressed by Theorem 30 given the GWA structure \( [33] \); we proceed to apply the theorem and state a classification.

Assume that \( k \) is algebraically closed. Let \( R \) denote the coefficient ring \( k[u, t, d] \) of \( A \) as a GWA. Maximal ideals of \( R \) take the form \( \mathfrak{m}(u_0, t_0, d_0) := (u - u_0, t - t_0, d - d_0) \) for some scalars \( u_0, t_0, d_0 \in k \). They get moved by \( \sigma^n \) to \( \mathfrak{m}(q^{-2n}u_0, t_0, d_0) \) for \( n \in \mathbb{Z} \), so \( \mathfrak{m}(u_0, t_0, d_0) \) has infinite \( \sigma \)-orbit if and only if \( u_0 \neq 0 \). Therefore a finite dimensional simple left \( A \)-module contains a simple \( R \)-submodule with annihilator having infinite \( \sigma \)-orbit if and only if \( u = u_{22} \) acts nontrivially. Theorem 30 requires us to consider the condition \( \sigma^{-n+1}(z), \sigma^{n'}(z) \in \mathfrak{m}(u_0, t_0, d_0) \) if and only if

\[
\sigma^{-n+1}(z) = d + q^{-2n}tu - q^{-4n}u^2 \\
\sigma^{n'}(z) = d + q^{2n'-2}tu - q^{4n'-4}u^2,
\]

a straightforward calculation shows that, as long as \( u_0 \neq 0 \), one has \( \sigma^{-n+1}(z), \sigma^{n'}(z) \in \mathfrak{m}(u_0, t_0, d_0) \) if and only if

\[
d_0 = -q^{2n'-n-1}u_0^2 \\
t_0 = (q^{-2n} + q^{2(n'-1)})u_0.
\]

Define for \( u_0 \in k^\times \) and \( t_0, d_0 \in k \) the left \( A \)-module \( M(u_0, t_0, d_0) := A/(\mathfrak{m}(u_0, t_0, d_0)) \), and let \( e_i \) denote the image of \( v_i \) in it for all \( i \in \mathbb{Z} \). Let \( N(u_0, t_0, d_0) \) be the submodule \( \bigoplus_{n \leq i \leq n'} R e_i \oplus \bigoplus_{n \leq i < n'} R e_i \), where \( n > 0 \) is chosen to be minimal such that \( d_0 + q^{-2n}tu_0 - q^{-4n}u_0^2 = 0 \) (or \( \infty \) if this does not occur), and \( n' > 0 \) is chosen to be minimal such that \( d_0 + q^{2n'-2}tu_0 - q^{4n'-4}u_0^2 = 0 \) (or \( \infty \) if this does not occur). We observed in the general setting \( [20] \) that this is the unique largest proper submodule of \( M(u_0, t_0, d_0) \). Define \( V(u_0, t_0, d_0) \) to be the simple left \( A \)-module \( M(u_0, t_0, d_0)/N(u_0, t_0, d_0) \). As an \( R \)-module, this is isomorphic to

\[
\bigoplus_{-n' < i < n} R/\sigma^i(\mathfrak{m}(u_0, t_0, d_0)),
\]

so it has dimension \( n + n' - 1 \) when \( n \) and \( n' \) are finite. Putting together our observations and applying Theorem 30 we have:

**Theorem 43:** Assume that \( k \) is algebraically closed.

1. Let \( u_0 \in k^\times \) and let \( t_0, d_0 \in k \). The simple left \( A \)-module \( V(u_0, t_0, d_0) \) is finite dimensional if and only if there are \( n, n' > 0 \) such that \( [31] \) holds.

2. Let \( n > 0 \). Any \( n \)-dimensional simple left \( A \)-module \( V \) that is not annihilated by \( u = u_{22} \) is isomorphic to

\[
V_n(u_0) := V(u_0, t_0 = (q^{-2n} + 1)u_0, d_0 = -q^{-2n}u_0^2)
\]

for a unique \( u_0 \in k^\times \), namely the eigenvalue of \( u_{22} \) on \( \text{ann}_V(u_{12}) \).

These simple modules are all pullbacks of simple \( U_q(sl_2) \)-modules along homomorphisms. Define, for each \( \alpha \in k^\times \), an algebra homomorphism \( \psi_\alpha : A \to U_q(sl_2) \):

\[
u_{11} \quad u_{12} \quad \mapsto \quad q^{-1}(q^{-1})^2\alpha EF + \alpha K^{-1} \\
u_{21} \quad u_{22} \quad \mapsto \quad q^{-1}(q^{-1})^2\alpha KF \\
\alpha K.
\]

Such homomorphisms can be shown to exist by checking that the relations \( [9] \) hold inside \( U_q(sl_2) \) for the desired images of the \( u_{ij} \). The definition we use for \( U_q(sl_2) \) is given in \( [3] \text{ I.3} \). For \( n > 0 \), consider the \( n \)-dimensional simple left \( U_q(sl_2) \)-module \( V(n-1, +) \) defined in \( [3] \text{ I.4} \). By using \( x \) and \( y \) as “raising” and “lowering” operators in the usual way, one can easily verify that the pullback \( \overline{V}(n-1, +) \) of \( V(n-1, +) \) along \( \psi_\alpha \) is a simple \( A \)-module. Identifying an \( \alpha \)-eigenvalue of \( \alpha q^{-n-1} \), we conclude that \( \overline{V}(n-1, +) \cong \overline{V}_n(\alpha q^{-n-1}) \). This gives:

20
Theorem 44: Assume that $k$ is algebraically closed. Every finite dimensional simple left $A$-module that is not annihilated by $u = u_{22}$ is the pullback of some simple left $U_q(\mathfrak{sl}_2)$-module along $\psi_\alpha$ for some $\alpha \in k^\times$.

3.2 Finite Dimensional Weight Modules

Keep the notation and assumptions of the previous section. The weight $A$-modules are the ones that decompose into simultaneous eigenspaces for the actions of $u$, $t$, and $d$; this is what it means to be semisimple over $R = k[u, t, d]$ when $k$ is algebraically closed. In this section, we simply apply Theorem 39 to $A$.

We observed in the previous section that the only maximal ideals $m(u_0, t_0, d_0)$ of $R$ with finite $\sigma$-orbit are ones with $u_0 = 0$. Hence the chain-type finite-dimensional weight $A$-modules are exactly the ones on which $u$ acts as a unit.

In the previous section we identified the set $\mathcal{M}$ defined in (27) as

$$\mathcal{M} = \{ m \in \text{max spec } R \mid m \text{ has infinite } \sigma\text{-orbit, } \sigma(z) \in m, \text{ and } \sigma^{-n+1}(z) \in m \text{ for some } n > 0 \}$$

We will show that statement 4 of Theorem 39 holds for $A$. Let $m = m(u_0, (q^{-2n} + 1)u_0, d_0 = -q^{-2n}u_0^2)$ be an element of $\mathcal{M}$. Suppose that $\sigma^{-n'}(z) \in m$, where $n' > 0$. Then, using (34), we have:

\begin{align*}
(q^{-2n} + 1)u_0 &= (q^{-2n'} + 1)u_0' \quad (35) \\
q^{-2n}u_0^2 &= q^{-2n'}u_0'^2. \quad (36)
\end{align*}

Using (35) to eliminate $u_0'$ from (36), we obtain

$$q^{-2n} = q^{-2n'} \left( \frac{q^{-2n} + 1}{q^{-2n'} + 1} \right)^2,$$

which simplifies to

$$(q^{2n} - q^{2n'}) = q^{2n-2n'} (q^{2n} - q^{2n'}).$$

This requires that $n = n'$. Therefore Theorem 39 applies to $A$ and gives:

Theorem 45: Finite-dimensional weight left $A$-modules on which $u = u_{22}$ acts as a unit are semisimple.

3.3 Prime Spectrum

We rely on the expression of $A$ as a GWA in (33):

$$k[u, t, d][x, y; \sigma, z]$$

$$\sigma : u \mapsto q^2u, t \mapsto t, d \mapsto d$$

$$z = d + q^{-2}tu - q^{-4}u^2.$$

We can get at all the prime ideals of $A$ by considering various quotients and localizations. Let us begin by laying out notation for the algebras to be considered:

- $A/\langle u \rangle$ is simply a polynomial ring,
  $$A/\langle u \rangle \cong k[u_{11}, u_{12}, u_{21}].$$

A glance at the reflection equation relations (6) is enough to see this.
Proposition 46: Let $A$ denote the localization of $A$ at the set of powers of $u$, a denominator set because $u$ is normal. By Proposition [22] this is $k[u^+, t, d][x, y; \sigma, z]$. By Proposition [12] $A_u/(t, d) = k[u^+][x, y; \sigma]$. In this quotient, $z$ is a unit: $z = -q^{-1}u^2$. Hence, by Proposition [5]

$$A_u/(t, d) = k[u^+][x^+; \sigma].$$

Let $A_{ud}$ denote the localization of $A_u$ at the set of powers of $d$, a denominator set because $d$ is central. By Proposition [22] this is $k[u^+, t, d^+][x, y; \sigma, z]$. By Proposition [12]

$$A_{ud}/(t) = k[u^+, d^+][x, y; \sigma, z].$$

Let $A_{ut}$ denote the localization of $A_u$ at the set of powers of $t$, a denominator set because $t$ is central. By Proposition [22] this is $k[u^+, t^+, d][x, y; \sigma, z]$. Let $A_{utx}$ denote the localization of this at the set of powers of $x$; by Proposition [21] this is indeed a denominator set, and we obtain $A_{utx} = k[u^+, t^+, d][x^+; \sigma]$. Then

$$A_{utx}/(d) = k[u^+, t^+][x^+; \sigma].$$

Let $A_{utxd}$ denote the localization of $A_{utx}$ at the set of powers of $d$:

$$A_{utxd} = k[u^+, t^+, d^+][x^+; \sigma].$$

What will turn out to be missing from this list is an algebra that gives us access to those prime ideals of $A_{ut}$ that contain some power of $x$. We cover this in the next section.

### 3.3.1 Primes of $A_{ut}$ That Contain a Power of $x$

We write $A_{ut}$ as

$$A_{ut} = R[x, y; \sigma, z],$$

where $R = k[u^+, t^+, d]$. A reminder about our notation: a subscript on a subset of a GWA indicates a certain subset of its base ring, seen in Definition [13]. Define

$$r_n = (q^{2n} + 1)^2d + q^{2n}t^2$$

for $n \in \mathbb{Z}$; these elements of $R$ will help us to understand the ideal of $A_{ut}$ generated by a power of $x$:

**Proposition 46:** Let $n \in \mathbb{Z}_{>0}$. Then

$$\prod_{j=n-i+1}^{n} r_n \in (x^n)_{n-i}$$

for all $0 \leq i \leq n$.

**Proof:** The induction will rely on the following observations:

1. For $n \geq 1$, $r_n \in (\sigma^n(z), z)_R$.
2. Let $I$ be an ideal of a GWA $R[x, y; \sigma, z]$. For $n \geq 1$, $R^n \cap I_n = (\sigma^n(z), z)_R \subseteq I_{n-1}$.

Direct calculation verifies observation [1]:

$$r_n = \frac{q^{2n+2} - 1}{q^{2n} - 1}(u^{-1}(\sigma^n(z) - z)) + \frac{q^{2n+1} - 1}{q^{2n} - 1}(q^{2n}z - \sigma^n(z)),$$

and observation [2] follows from Proposition [18]. The $i = 0$ case, $1 \in (x^n)_n$, is trivial. Assume that $0 \leq i < n$ and that [23] holds for $i$. Then $a := r_{n-i} \in (x^n)_{n-i}$ has $a(\sigma^{n-i}(z), z)_R \subseteq (x^n)_{n-(i+1)}$. Hence, by observation [1] $a r_{n-i} \in (x^n)_{n-(i+1)}$, proving [23] for $i + 1$. 


Proposition 47: Assume that \( n \geq 1 \) and \( P \in \text{spec}(A_{ut}) \). If \( x^n \in P \) and \( x^{n-1} \notin P \), then \( r_n \in P \).

Proof: From the case \( i = n \) of Proposition 46,

\[
 r_1 r_2 \cdots r_n \in P.
\]

Since this is a product of central elements in \( A_{ut} \), we conclude that \( r_n' \in P \) for some \( n' \). In particular, \( r_n' \in P_{n-1} \). Applying Proposition 46 with \( i = 1 \), we also have \( r_n' \in P_{n-1} \). Since \( t \) is a unit, and since \( q \) is not a root of unity, it is clear from (37) that \( 1 \in \langle r_n, r_n' \rangle_R \) if \( n \neq n' \). We assumed that \( x^{n-1} \notin P \), so \( n' = n \). ■

So when considering homogeneous prime ideals \( P \) of \( A_{ut} \) that contain a power of \( x \), we can eliminate a variable by factoring out the ideal generated by one of the \( r_i \). Namely, we may factor out \( \langle r_n \rangle \) if \( n \geq 1 \) is taken to be minimal such that \( x^n \in P \), and we may then consider \( P \) as a prime ideal of \( A(n) := A_{ut}/\langle r_n \rangle \). Using Proposition 12, this algebra is isomorphic to \( k[u^\pm, t^\pm][x, y; \sigma, z_n] \), where

\[
 z_n = \frac{-q^{2n}}{(q^{2n} + 1)^2} + q^{-2}u - q^{-4}u^2. \tag{39}
\]

Let \( R(n) \) denote \( k[u^\pm, t^\pm] \), thought of as \( R/\langle r_n \rangle_R \). The ideal generated by \( x^n \) can be pinned down completely in \( A(n) \). We again start by defining some special elements of the base ring that will help us break things down. Make the following definitions for \( n \in \mathbb{Z} \):

\[
 s_n^j = u - \frac{q^{2j}}{q^{2n} + 1} t \quad \text{for } j \in \mathbb{Z}.
\]

\[
 J_n^m = \{ j \in \mathbb{Z} \mid 1 \leq j \leq n - m \} \quad \text{for } m \geq 0 \tag{40}
\]

\[
 J_m^m = \{ j \in \mathbb{Z} \mid m + 1 \leq j \leq n \}
\]

\[
 \pi_m^n = \prod_{j \in J_m^m} s_n^j \quad \text{for } m \in \mathbb{Z}.
\]

Here is a way to visually organize these definitions for the example \( n = 3 \):

\[
 1 = \pi_3^3, \quad s_1^3 = \pi_2^2, \quad s_2^3 = \pi_1^3, \quad s_3^3 = \pi_0^3
\]

\[
 s_1^2 \cdot s_2^2 \cdot s_3^2 = \pi_{-1}^2, \quad s_1^3 \cdot s_2^3 \cdot s_3^3 = \pi_{-2}^3, \quad 1 = \pi_{-3}^3
\]

Observe that \( \sigma(z_n) = -s_n^n s_0^0 \) and that

\[
 \sigma^{-1}(s_n^j) = q^{-2}s_{n+1}^j, \tag{41}
\]

so that

\[
 \sigma^i(z_n) = -q^{4i-4}s_{n-i+1}^n s_{i-1}^n \tag{42}
\]

for \( n, i \in \mathbb{Z} \). Finally, observe that the \( s_n^j \) are pairwise coprime over various \( j \), since \( q \) is not a root of unity.

For the next results, we abstract this situation.
**Proposition 48**: Fix $n \geq 1$. Consider an arbitrary GWA $A = R[x,y;\sigma,z]$ and sequence $(s_j)_{j \in \mathbb{Z}}$ of elements of $R$ such that

1. $R$ is commutative,
2. $z$ is a unit multiple of $s_1 s_{n+1}$,
3. $\sigma^{-1}(s_j)$ is a unit multiple of $s_{j+1}$,
4. and $(s_i, s_j)_R = R$ for all $i \neq j$.

Define $J_m$ as $J_m^n$ is defined in (40) and let $\pi_m = \prod_{j \in J_m} s_j$. Then we have

$$\langle x^n \rangle_m = (\pi_m)_R$$

for $m \in \mathbb{Z}$.

**Proof**: The sequence of ideals $\langle \pi_m \rangle_R$ satisfies the conditions needed in Proposition 18 in order for $\bigoplus_{m \in \mathbb{Z}} \langle \pi_m \rangle_R = R$ to define an ideal of $A$, as can be checked using our assumptions 2 and 3. Since $\pi_n = 1$, the latter ideal contains $x^n$. This gives the inclusion $\langle x^n \rangle_m \subseteq \langle \pi_m \rangle_R$ for $m \in \mathbb{Z}$. To get equality we must show that

$$\pi_m \in \langle x^n \rangle_m$$

for all $m \in \mathbb{Z}$.

For $m \geq n$, (43) holds trivially. Assume that (43) holds for a given $m$, with $1 \leq m \leq n$. Then:

$$\langle x^n \rangle_{m-1} \supseteq \langle \pi_m \sigma^n(z), \sigma^{-1}(\pi_m)z \rangle_R$$

$$= \left( \prod_{j=1}^{n-m} s_j \right) \left( s_{n-(m-1)} s_{1-m-1} \right) \left( \prod_{j=2}^{n-(m-1)} s_j \right) \left( s_{n+1} s_1 \right) \left( x^n \right)_R$$

$$= \pi_{m-1} \langle s_{1-m}, s_{n+1} \rangle_R = \pi_{m-1} R.$$  

Line (44) is due to the induction hypothesis and Proposition 18. Line (45) uses assumptions 2 and 3. And line (46) uses assumption 4. Hence, by induction, (43) holds for $m \geq 0$.

Now assume that $1 - n \leq m \leq 0$ and that (43) holds for $m$. We can apply a similar strategy to what was done for (44)-(46):

$$\langle x^n \rangle_{m-1} \supseteq \langle \pi_m, \sigma^{-1}(\pi_m) \rangle_R = \left( \prod_{j=-m+1}^{n} s_j , \prod_{j=-m+2}^{n+1} s_j \right)_R$$

$$= \pi_{m-1} \langle s_{-m+1}, s_{n+1} \rangle_R = \pi_{m-1} R.$$  

Hence, by induction, (43) holds for all $m \geq -n$. In particular (the case $m = -n$), $y^n \in \langle x^n \rangle$. Thus (43) holds trivially for $m < -n$.

**Corollary 49**: In the setup of Proposition 48 $\langle x^n \rangle = \langle y^n \rangle$.

**Proof**: We shall make use of Proposition 18 to exploit symmetries in the hypotheses of Proposition 48. Let us use hats to denote our new batch of input data to Proposition 48. Consider $A$ as a GWA $R[\tilde{x}, \tilde{y}; \tilde{\sigma}, \tilde{z}]$, with elements $(s_j)_{j \in \mathbb{Z}}$ of $R$, where

$$\tilde{x} = y \quad \tilde{y} = x \quad \tilde{\sigma} = \sigma^{-1} \quad \tilde{s}_j = s_{n+1-j}$$

This satisfies the hypotheses of Proposition 48. Following along the notations needed to state the conclusion, define

$$\tilde{\pi}_m = \prod_{j \in \tilde{J}_m} \tilde{s}_j$$

$$\text{for} \ m \in \mathbb{Z}.$$
and also define
\[ \hat{l}_m = I_{-m} \] (which is \( \{ r \in R \mid rv_{-m} \in I \} \)) \hspace{1cm} (49)
whenever \( I \subseteq A \), to match Definition [3] with the new GWA structure. Observe that
\[ \{ n + 1 - j \mid j \in J_m \} = J_{-m} \] \hspace{1cm} (50)
for all \( m \in \mathbb{Z} \), so that \( \hat{\pi}_m = \pi_{-m} \). The conclusion of Proposition [8] for the items with hats is then that
\[ \langle \hat{x}^n \rangle_m = \langle \hat{\pi}_m \rangle_R \]
for all \( m \in \mathbb{Z} \).

In order to get at the homogeneous primes of \( A_{(\alpha)} \) that contain \( x^n \), we now seek to describe all the homogeneous ideals of \( A_{(\alpha)} \) that contain \( x^n \). Statements of the next few results remain in a general GWA setting, in order to continue taking advantage of the symmetry of GWA expressions.

**Proposition 50:** Assume the setup of Proposition [8]. Fix arbitrary integers \( \ell_1 \leq \ell_2 \). There is an element \( e_0 \) of \( R \) such that, setting \( e_j = \sigma^{-j}(e_0) \) for \( j \in \mathbb{Z} \), the family \((e_j)_{j \in \mathbb{Z}}\) satisfies:

1. \( e_j \equiv 1 \mod s_j \) for \( j \in \mathbb{Z} \).
2. \( e_j \equiv 0 \mod s_i \) for distinct \( i, j \in \{ \ell_1, \ldots, \ell_2 \} \).
3. \( e_{\ell_1}, \ldots, e_{\ell_2} \) is a collection of orthogonal idempotents that sum to 1, where bars denote cosets with respect to \( \langle \prod_{i=\ell_1}^{\ell_2} s_i \rangle \).

**Proof:** The \( s_j \), for \( j \in \mathbb{Z} \), are pairwise coprime as elements of \( R \). The Chinese Remainder Theorem (CRT) provides an \( e_0 \) in \( R \) which is congruent to 1 mod \( s_0 \) and congruent to 0 mod \( s_i \) for all nonzero \( i \in \{ \ell_1 - \ell_2, \ldots, \ell_2 - \ell_1 \} \). Then for \( j \in \mathbb{Z} \) we have that \( \sigma^{-j}(e_0) \) is congruent to 1 mod \( s_j \) and congruent to 0 mod \( s_i \) for all \( i \in \{ \ell_1 - \ell_2 + j, \ldots, \ell_2 - \ell_1 + j \} \) with \( i \neq j \). Setting \( e_j = \sigma^{-j}(e_0) \) gives us [1] and [2].

Part of the CRT says that
\[ \left\langle \prod_{i=\ell_1}^{\ell_2} s_i \right\rangle = \bigcap_{i=\ell_1}^{\ell_2} \langle s_i \rangle, \]
and [3] easily follows from this using [1] and [2]. \( \blacksquare \)

**Proposition 51:** Assume the setup of Proposition [8]. Let \((e_j)_{j \in \mathbb{Z}}\) be as in Proposition [50] with \( \ell_1 = 1 \) and \( \ell_2 = n \). There are mutually inverse inclusion-preserving bijections

\[
\begin{align*}
\{ \text{homogeneous right } R[x; \sigma] \text{-submodules } I \} \\
\{ \text{of } A \text{ containing } x^n \}
\end{align*}
\]

\[
\begin{align*}
\leftrightarrow \\
\{ \text{families } (I_{m_j} \mid m \in \mathbb{Z}, j \in J_m) \text{ of ideals of } R \text{ satisfying [52]} \} \\
\text{for all } m, j
\end{align*}
\]

\[ I \quad \leftrightarrow \quad (I_m + \langle s_j \rangle \mid m \in \mathbb{Z}, j \in J_m) \]

\[ \bigoplus_{m \in \mathbb{Z}} \left( \langle \pi_m \rangle + \sum_{j \in J_m} I_{m,j} e_j \right) v_m \quad \leftrightarrow \quad (I_{m_j} \mid m \in \mathbb{Z}, j \in J_m), \]

where the condition [52] is that
\[ I_{-(m+1),j} \subseteq I_{-m,j} \forall j \in J_{-(m+1)} \quad \text{and} \quad I_{m,j} \subseteq I_{m+1,j} \forall j \in J_{m+1} \]

for all \( m \in \mathbb{Z}_{\geq 0} \).
Proof: Combining Propositions 15 and 48 we obtain the following correspondence:

\[
\begin{align*}
\{ \text{homogeneous right } R[x; \sigma]-\text{submodules} \} & \leftrightarrow \{ \text{sequences } (I_m)_{m \in \mathbb{Z}} \text{ of ideals of } R \text{ satisfying the conditions } (10) \text{ of Proposition } 15 \text{ with } \pi_m \in I_m \text{ for all } m 
\end{align*}
\]

\[
\begin{align*}
I & \leftrightarrow (I_m)_{m \in \mathbb{Z}} \\
\bigoplus_{m \in \mathbb{Z}} I_m v_m & \leftrightarrow (I_m)_{m \in \mathbb{Z}}.
\end{align*}
\]

The \(s_j\), for \(j \in \mathbb{Z}\), are pairwise coprime as elements of \(R\). So an ideal \(I_m\) of \(R\) containing \(\pi_m\) corresponds, via the CRT, to a collection of ideals \((I_{mj})_{j \in J_m}\) such that \(s_j \in I_{mj}\) for \(j \in J_m\). Explicitly, the correspondence is:

\[
\begin{align*}
\{ \text{sequences } (I_m)_{m \in \mathbb{Z}} \text{ of ideals of } R \text{ with } \pi_m \in I_m \text{ for all } m \} & \leftrightarrow \{ \text{families } (I_{mj} = I_m + (s_j) ) \mid m \in \mathbb{Z}, j \in J_m \} \\
(I_m)_{m \in \mathbb{Z}} & \mapsto (I_{mj} = I_m + (s_j) )_{m \in \mathbb{Z}} \\
\left( (I_m) + \sum_{j \in J_m} I_{mj} e_j \right)_{m \in \mathbb{Z}} & \leftrightarrow (I_m)_{m \in \mathbb{Z}}.
\end{align*}
\]

In order to make use of this with (53), we need to express the condition (10) of Proposition 15 in terms of the \(I_{mj}\). Let \((I_m)_{m \in \mathbb{Z}}\) be a sequence of ideals of \(R\) with \(\pi_m \in I_m\) for all \(m\), and let \((I_m)_{m \in \mathbb{Z}, j \in J_m}\) be the family of ideals it corresponds to in (54). For \(m \in \mathbb{Z}_{\geq 0}\),

\[
I_m \subseteq I_{m+1} \iff (\pi_m) + \sum_{j=1}^{n_m} I_{mj} e_j \subseteq (\pi_{m+1}) + \sum_{j=1}^{n_m} I_{m+1,j} e_j 
\]

\[
\Rightarrow (s_i) + I_{mi} \subseteq (s_i) + I_{m+1,i} \quad \forall i \in J_{m+1}
\]

\[
\Rightarrow I_{mi} \subseteq I_{m+1,i} \quad \forall i \in J_{m+1}
\]

\[
\Rightarrow I_m \subseteq I_{m+1}.
\]

Line (56) is obtained by adding \((s_i)\) to both sides of the inclusion in line (55), and using the properties of the \(e_j\) from Proposition 51. Line (58) is due to the fact that \(s_i \in I_{m+1,i}\). Line (58) can be seen by looking at (55) and noting that \(e_{n_m} \in (\pi_{m+1})\) because \(e_{n_m}\) vanishes mod \(s_j\) for \(j \in J_{m+1}\). For similar reasons we also have, for \(m \in \mathbb{Z}_{\geq 0}\),

\[
I_{-(m+1)} \sigma^{-m}(z) \subseteq I_{-m} \iff \left( (\pi_{-(m+1)}) + \sum_{j=m+2}^{n} I_{-(m+1),j} e_j \right) s_{n+m+1} s_{m+1} \subseteq (\pi_{-m}) + \sum_{j=m+1}^{n} I_{-m,j} e_j 
\]

\[
\Rightarrow s_{n+m+1}(\pi_{-m}) + \sum_{j=m+2}^{n} I_{-(m+1),j} s_{n+m+1} s_{m+1} e_j \subseteq (\pi_{-m}) + \sum_{j=m+1}^{n} I_{-m,j} e_j 
\]

\[
\Rightarrow (s_i) + I_{-(m+1),i} s_{n+m+1} s_{m+1} \subseteq I_{-m,i} \quad \forall i \in J_{-(m+1)}
\]

\[
\Rightarrow I_{-(m+1),i} \subseteq I_{-m,i} \quad \forall i \in J_{-(m+1)}
\]

\[
\Rightarrow I_{-(m+1)} \sigma^{-m}(z) \subseteq I_{-m}.
\]

The only subtlety this time is that line (62) relies on the fact that \(s_{n+m+1}\) and \(s_{m+1}\) are units modulo \(s_i\) for all \(i \in J_{-(m+1)}\). We conclude that the condition (10) of Proposition 15 holds for \((I_m)_{m \in \mathbb{Z}}\) if and only if the condition (59) holds for \((I_{mj})_{m \in \mathbb{Z}, j \in J_m}\). Combining this fact with the correspondences (53) and (54) yields the desired correspondence (51). \(\blacksquare\)
Corollary 52: Assume the setup of Propositions 48 and 51. There are mutually inverse inclusion-preserving bijections

\[
\{ \text{homogeneous right } R[y; \sigma^{-1}]-\text{submodules } I \} \quad \leftrightarrow \quad \{ \text{families } (I_{m,j} \mid m \in \mathbb{Z}, j \in J_m) \text{ of ideals of } R \text{ satisfying (65)} \}
\]

\[
I \quad \leftrightarrow \quad (I_m + \langle s_j \rangle \mid m \in \mathbb{Z}, j \in J_m)
\]

\[
\bigoplus_{m \in \mathbb{Z}} \left( \langle \pi_m \rangle + \sum_{j \in J_m} I_{m,j} e_j \right) v_m \quad \leftrightarrow \quad (I_{m,j} \mid m \in \mathbb{Z}, j \in J_m),
\]

where the condition (65) is that
\[
I_{-(m+1),j} \supseteq I_{-m,j} \forall j \in J_{-(m+1)} \quad \text{and} \quad I_{m,j} \supseteq I_{m+1,j} \forall j \in J_{m+1}
\]

for all \( m \in \mathbb{Z}_{\geq 0} \).

Proof: We shall apply Proposition 51 while viewing \( A \) as a GWA with the alternative GWA structure \( R[y, x; \sigma^{-1}, \sigma(z)] \). Make the definitions (47)-(49), and also define

\[ \hat{\epsilon}_j = e_{n+1-j}. \]

This data satisfies the hypotheses of Proposition 51 and allows us to conclude that there is a correspondence

\[
\{ \text{homogeneous right } R[y; \sigma^{-1}]-\text{submodules } I \} \quad \leftrightarrow \quad \{ \text{families } (\hat{I}_{m,j} \mid m \in \mathbb{Z}, j \in J_m) \text{ of ideals of } R \text{ satisfying (68)} \}
\]

\[
I \quad \leftrightarrow \quad (\hat{I}_m + \langle \hat{s}_j \rangle \mid m \in \mathbb{Z}, j \in J_m)
\]

\[
\bigoplus_{m \in \mathbb{Z}} \left( \langle \pi_m \rangle + \sum_{j \in J_m} \hat{I}_{m,j} \hat{e}_j \right) v_m \quad \leftrightarrow \quad (\hat{I}_{m,j} \mid m \in \mathbb{Z}, j \in J_m),
\]

where the condition (68) is that
\[
\hat{I}_{-(m+1),j} \subseteq \hat{I}_{-m,j} \forall j \in J_{-(m+1)} \quad \text{and} \quad \hat{I}_{m,j} \subseteq \hat{I}_{m+1,j} \forall j \in J_{m+1}
\]

for all \( m \in \mathbb{Z}_{\geq 0} \). Using the observation (60) and reindexing by \((m, j) \mapsto (-m, n+1-j)\), this becomes the correspondence (69). \( \blacksquare \)

Proposition 53: Assume the setup of Proposition 48. All ideals of \( A \) containing \( x^n \) are homogeneous.

Proof: Let \((e_j)_{j \in \mathbb{Z}}\) be as in Proposition 48 with \( \ell_1 = -n + 2 \) and \( \ell_2 = 2n - 1 \). Let \( I \) be any ideal of \( A \) containing \( x^n \), and let \( \sum_{m \in \mathbb{Z}} a_m v_m \in I \) be an arbitrary element. Then since \( \langle x^n \rangle = \langle y^n \rangle \), from Corollary 49 we have \( v_m \in I \) for \( m \geq n \) and for \( m \leq -n \), and the problem is reduced to considering

\[
\sum_{m=-n+1}^{n-1} a_m v_m \in I
\]

and needing to show that \( a_m v_m \in I \) for \( m \in \{-n + 1, \ldots, n - 1\} \). Consider any \( j, j' \in \{1, \ldots, n\} \). Multiplying on the left by \( e_j \) and on the right by \( e_{j'} \) yields

\[
I \ni \sum_{m=-n+1}^{n-1} e_j a_m v_m e_{j'} = \sum_{m=-n+1}^{n-1} a_m e_j \sigma^m(e_{j'}) v_m = \sum_{m=-n+1}^{n-1} a_m e_j e_{j'-m} v_m.
\]
When \(-n + 1 \leq m \leq n - 1\) and \(1 \leq j' \leq n\), we have \(-n + 2 \leq j' - m \leq 2n - 1\). So, since \(\pi_0 \in I\) (due to Proposition 48), the product \(e_j e_{j'-m}\) that appears above vanishes mod \(I\) unless \(j' - m = j\), in which case it is congruent to \(e_j\) mod \(I\). Thus we have
\[
a_{j' - j} e_j e_{j'-j} \in I
\]
for all \(j, j' \in \{1, \ldots, n\}\). When \(j \in J_m\), we have \(j, m + j \in \{1, \ldots, n\}\), so this shows that \(a_m e_j v_m \in I\), for all \(m \in \{-n + 1, \ldots, n - 1\}\) and \(j \in J_m\), and in particular that
\[
\sum_{j \in J_m} a_m e_j v_m \in I.
\]

Fix an \(m \in \{-n + 1, \ldots, n - 1\}\). Since \(\pi_m \in I_m\) (due to Proposition 48), \(\sum_{j \in J_m} e_j \equiv 1\) mod \(I_m\). Hence \(a_m v_m \in I\).

**Corollary 54:** Assume the setup of Propositions 48 and 51. There are mutually inverse inclusion-preserving bijections

\[
\{ \text{ideals } I \text{ of } A \text{ containing } x^n \} \leftrightarrow \begin{cases} 
\text{families } (I_m) \ | \ m \in \mathbb{Z}, \ j \in J_m \text{ of} \\
\text{ideals of } R \text{ satisfying (70) with } s_j \in I_m \\
\text{for all } m, j
\end{cases}
\]

\[
I \leftrightarrow (I_m + (s_j) \ | \ m \in \mathbb{Z}, \ j \in J_m)
\]

\[
\bigoplus_{m \in \mathbb{Z}} \left( \langle \pi_m \rangle + \sum_{j \in J_m} I_m e_j \right) v_m \leftrightarrow (I_m \ | \ m \in \mathbb{Z}, \ j \in J_m),
\]

where the condition (70) is that

\[
\begin{align*}
I_{-(m+1),j} & = I_{-m,j} \forall j \in J_{-(m+1)}, \\
\sigma(I_{-(m+1),j}) & = I_{-m,j-1} \forall j \in J_{-(m+1)}, \quad \text{and} \quad I_{m,j} = I_{m+1,j} \forall j \in J_{m+1},
\end{align*}
\]

for all \(m \in \mathbb{Z}_{\geq 0}\).

**Proof:** We shall deduce left-handed versions of 51 and 54 by viewing \(A^{op}\) as a GWA \(R[x; y; \sigma^{-1}, \sigma(z)]\). Recall the notation \(I_{m}^{op}\) from Definition 13 and Remark 14. Let us use hats to keep track of things in terms of this GWA structure: \(A^{op} = R[\hat{x}, \hat{y}; \hat{\sigma}, \hat{\sigma}(z)]\); define

\[
\hat{x} = x, \quad \hat{y} = y, \quad \hat{\sigma} = \sigma^{-1}, \quad \hat{\sigma}(z) = \sigma(z), \quad \hat{\sigma}^{-1}(s_{n+1-j}) = s_{n+1-j}, \quad \hat{\sigma}^{-1}(e_{n+1-j}) = e_{n+1-j}, \quad \hat{\pi}_m = \prod_{j \in J_m} \hat{s}_j.
\]

This data satisfies the hypotheses of Propositions 51 and 54 so we obtain correspondences

\[
\{ \text{homogeneous left } R[x; \sigma]-\text{submodules} \} \leftrightarrow \begin{cases} 
\text{families } (\hat{I}_m) \ | \ m \in \mathbb{Z}, \ j \in J_m \text{ of} \\
\text{ideals of } R \text{ satisfying (73) with } \hat{s}_j \in \hat{I}_m \\
\text{for all } m, j
\end{cases}
\]

\[
\bigoplus_{m \in \mathbb{Z}} \sigma^m \left( \langle \hat{\pi}_m \rangle + \sum_{j \in J_m} \hat{I}_m e_j \right) v_m \leftrightarrow (\hat{I}_m^{op} + (\hat{s}_j) \ | \ m \in \mathbb{Z}, \ j \in J_m)
\]

where the specified conditions are that

\[
\hat{I}_{-(m+1),j} \subseteq \hat{I}_{-m,j} \forall j \in J_{-(m+1)}, \quad \hat{I}_{m,j} \subseteq \hat{I}_{m+1,j} \forall j \in J_{m+1}.
\]
\[ \hat{I}_{(m+1),j} \supseteq \hat{I}_{m,j} \quad \forall j \in \mathcal{J}_{-(m+1)}, \quad \text{and} \quad \hat{I}_{m,j} \supseteq \hat{I}_{m+1,j} \quad \forall j \in \mathcal{J}_{m+1} \]  

(74)

for all \( m \in \mathbb{Z}_{\geq 0} \). To make this useful, we transform the expression for the families of ideals in the right hand side of (72) as follows:

\[ I_{mj} = \sigma^m(\hat{I}_{m,n+1-(j+m)}). \]

The index sets \( \mathcal{J}_m \) have symmetries that can be used to reindex sums and products after applying this transformation: \( \mathcal{J}_m = \{ n + 1 - j \mid j \in \mathcal{J}_m \} = \{ j - m \mid j \in \mathcal{J}_m \} \). A consequence is that \( \sigma^m(\hat{x}_m) \) is a unit multiple of \( \pi_m \). One now makes the routine substitutions and reindexings in (72)-(74) to obtain correspondences

\[
\begin{align*}
\{ \text{homogeneous left } R[x;\sigma]-\text{submodules} \} & \leftrightarrow \{ \text{families } (I_{mj} \mid m \in \mathbb{Z}, j \in \mathcal{J}_m) \text{ of ideals of } R \text{ satisfying } (76) \text{ with } s_j \in I_{mj} \} \\
& \text{for all } m,j.
\end{align*}
\]

\[
\begin{align*}
\{ \text{homogeneous left } R[y;\sigma^{-1}]-\text{submodules} \} & \leftrightarrow \{ \text{families } (I_{mj} \mid m \in \mathbb{Z}, j \in \mathcal{J}_m) \text{ of ideals of } R \text{ satisfying } (77) \text{ with } s_j \in I_{mj} \} \\
& \text{for all } m,j.
\end{align*}
\]

(75)

where the specified conditions are now that

\[
\begin{align*}
\sigma(I_{-(m+1),j}) & \subseteq I_{-(m+1),j} \quad \forall j \in \mathcal{J}_{-(m+1)}, \\
\sigma(I_{m+1,j}) & \subseteq I_{m+1,j} \quad \forall j \in \mathcal{J}_{m+1}, \\
\sigma(I_{-(m+1),j}) & \supseteq I_{-(m+1),j-1} \quad \forall j \in \mathcal{J}_{-(m+1)}, \quad \text{and} \\
\sigma(I_{m+1,j}) & \supseteq I_{m+1,j} \quad \forall j \in \mathcal{J}_{m+1}.
\end{align*}
\]

(76) \hspace{1cm} (77)

for all \( m \in \mathbb{Z}_{\geq 0} \).

Note that, by Corollary 49, an ideal of \( A \) contains \( x^n \) if and only if it contains \( y^n \). And note that, by Proposition 50, all ideals of \( A \) are homogeneous. Hence we may combine (71), (74), and (77) to obtain the correspondence (69), and the condition in (70) is just the conjunction of conditions (72), (75), (76), and (77).

We now specialize back to the algebra \( A_{(n)} = A_{n!}/(r_n) \). Corollary 54 applies to \( A_{(n)} \) with the elements of \( R_{(n)} \) defined in (40).

**Proposition 55:** For \( n \geq 1 \), there are mutually inverse inclusion-preserving bijections

\[
\{ \text{ideals } I/(x^n) \text{ of } A_{(n)}/(x^n) \} \leftrightarrow \{ \text{ideals } \hat{I}_n \text{ of } k[t^{\pm}] \}
\]

\[
\begin{align*}
I/(x^n) & \mapsto (I_0 + (s^n_1)_{R_{(n)})} \cap k[t^{\pm}]
\end{align*}
\]

\[
\begin{align*}
\bigoplus_{m \in \mathbb{Z}} \left( \langle \pi_m^n \rangle + \langle \hat{I}_n \rangle \right) v_m \quad /\langle x^n \rangle & \leftrightarrow \quad \hat{I}_n.
\end{align*}
\]

(78)

**Proof:** Let \( e_i^n \) for \( j \in \mathbb{Z} \) be as in Proposition 50 with \( \ell_1 = 1 \) and \( \ell_2 = n \). In particular they are elements of \( R_{(n)} \) such that \( e_i^n \) is congruent to \( 1 \mod s_i^n \) for \( j \in \mathbb{Z} \) and congruent to \( 0 \mod s_i^n \) for all distinct \( i, j \in \{ 1, \ldots, n \} \), and \( \sigma^{-1}(e_i^n) = e_j^{n+1} \) for all \( j \in \mathbb{Z} \). For \( j \in \mathbb{Z} \), the algebra \( R_{(n)}/\langle s^n_j \rangle \) is isomorphic to \( k[t^{\pm}] \), and the isomorphism is the composite

\[ k[t^{\pm}] \hookrightarrow R_{(n)} \rightarrow R_{(n)}/\langle s^n_j \rangle. \]

We obtain from this a correspondence of ideals for each \( j \):

\[
\begin{align*}
\{ \text{ideals of } R_{(n)} \text{ containing } s^n_j \} & \leftrightarrow \{ \text{ideals of } k[t^{\pm}] \} \\
J & \mapsto \quad \hat{J} = J \cap k[t^{\pm}]
\end{align*}
\]

(79)
This allows us to restate the correspondence that we obtain from Corollary \[54\] as
\[
\{ \text{ideals } I \text{ of } A(n) \text{ containing } x^n \} \leftrightarrow \{ \text{families } (\tilde{I}_{mj} \mid m \in \mathbb{Z}, j \in J_m^n) \text{ of } \}
\]
\[
I \mapsto \left( \left( I_m + (s_j^n) \right) \cap k[t^\pm] \mid m \in \mathbb{Z}, j \in J_m^n \right)
\]
(80)
where the condition \[31\] is that
\[
\tilde{I}_{-(m+1),j} = \tilde{I}_{-m,j} \forall j \in J_{m+1}^n, \quad \text{and } \quad \tilde{I}_{-(m+1),j} = \tilde{I}_{-m,j-1} \forall j \in J_{m+1}^n
\]
(81)
for all \( m \in \mathbb{Z}_{\geq 0} \). The \( \sigma \) has disappeared from the condition \[40\] because \( \sigma \) fixes \( k[t^\pm] \). Notice that \[31\] simply says that all the ideals in the family \( (\tilde{I}_{mj} \mid m \in \mathbb{Z}, j \in J_m^n) \) are equal. So we may as well give them all one name, \( \tilde{I}_m \). We may also simplify the expression of the left hand side of \[80\]: for \( m \in \mathbb{Z} \), we have
\[
\langle \pi_m^n + \sum_{j \in J_m^n} (\tilde{I}_m + (s_j^n))e_j^n \rangle = \langle \pi_m^n + (s_j^n)e_j^n \mid j \in J_m^n \rangle + \sum_{j \in J_m^n} \langle \tilde{I}_m \rangle e_j^n
\]
\[
= \langle \pi_m^n + (s_j^n)e_j^n \mid j \in J_m^n \rangle + \langle \tilde{I}_m \rangle
\]
\[
= \langle \pi_m^n \rangle + \langle \tilde{I}_m \rangle.
\]
(82)
(83)
Line \[82\] is due to the fact that \( \sum_{j \in J_m^n} e_j^n \) is congruent to 1 mod \( \pi_m^n \). Line \[83\] is due to the fact that \( s_j^n e_j^n \) is congruent to 0 mod \( \pi_m^n \) for all \( j \in J_m^n \). Thus we obtain \[78\].

**Proposition 56: Products of ideals are preserved by the correspondence \[78\].**

**Proof:** Let \( a, b \) be ideals of \( k[t^\pm] \), and let \( I/(x^n), J/(x^n) \) be the respective corresponding ideals of \( A(n)/(x^n) \) via \[78\]. We must show that the product \( ab \) corresponds via \[78\] to \( (I/(x^n))(J/(x^n)) = (IJ + (x^n))/(x^n) \). That is, we must show that \((IJ + (x^n))a + (s_1^n) \cap k[t^\pm] = ab \). Using the fact that all of the ideals on the right hand side of \[80\] are equal,
\[
(I_m + (s_j^n)) \cap k[t^\pm] = a
\]
\[
(J_m + (s_j^n)) \cap k[t^\pm] = b
\]
(84)
for all \( m \in \mathbb{Z}, j \in J_m^n \). The contraction \((IJ)_0\) of the product \(IJ\) consists of sums of products of homogeneous terms of opposite degree; i.e. terms of the form
\[
(a\pi_m^n) \cdot (b\pi_{-m}) = a\pi_{m}^n(b)\pi_{-m}
\]
for \( m \in \mathbb{Z} \). Hence \((IJ + (x^n))_0 + (s_1^n)\) can be written as
\[
(IJ + (x^n))_0 + (s_1^n) = (IJ)_0 + (\pi_0^n + (s_1^n) = (IJ)_0 + (s_1^n) = \sum_{m \in \mathbb{Z}} [m, -m]I_m\pi_{m}^n(J_{-m} + (s_1^n)).
\]
Observe the following:

**Claim:** If \( m \in \{0, \ldots, n-1\} \), then \([m, -m] \) is a unit mod \( (s_1^n)_{R(n)} \). Otherwise, it is in \( (s_1^n)_{R(n)} \).

**Proof:** If \( m < 0 \), then \([m, -m] = \sigma^{0, m+1}[z_m] \) is divisible by \( z_m \), which is divisible by \( s_1^n \). If \( m > n - 1 \), then \([m, -m] = \sigma^{1, m}[z_m] \) is divisible by \( \sigma^n(z_m) \), which is also divisible by \( s_1^n \). If \( m = 0 \), then \([m, -m] = 1 \) is a unit mod \( s_1^n \). Finally, assume that \( m \in \{1, \ldots, n-1\} \). Then \([m, -m] = \sigma^{1, n}[z_m] \) is a unit multiple of the product
\[
\prod_{i=1}^{m} s_{n-i+1} \prod_{i=1}^{m} s_{1-i}.
\]
Observe that the assumption $1 \leq m \leq n-1$ precludes $s_i^n$ from being a factor in the product above. Since the $s_i^n$ are pairwise coprime, it follows that $\llbracket m, -m \rrbracket$ is a unit mod $s_i^n$.

This simplifies the expression above:

$$
\sum_{m \in \mathbb{Z}} [m, -m] I_m \sigma^m(J_{-m}) + \langle s_i^n \rangle = \sum_{m=0}^{n-1} I_m \sigma^m(J_{-m}) + \langle s_i^n \rangle.
$$

Now we calculate what is needed:

$$
\left( (IJ + \langle x^n \rangle)_0 + \langle s_i^n \rangle \right) \cap k[t^\pm] = \left( \sum_{m=0}^{n-1} I_m \sigma^m(J_{-m}) + \langle s_i^n \rangle \right) \cap k[t^\pm]
$$

$$
= \left( \sum_{m=0}^{n-1} (I_m + \langle s_i^n \rangle) \sigma^m(J_{-m} + \langle s_m^n \rangle) + \langle s_i^n \rangle \right) \cap k[t^\pm]
$$

$$
= \left( \sum_{m=0}^{n-1} (a + \langle s_i^n \rangle) \sigma^m(b + \langle s_m^n \rangle) + \langle s_i^n \rangle \right) \cap k[t^\pm]
$$

$$
= \left( \sum_{m=0}^{n-1} ab + \langle s_i^n \rangle \right) \cap k[t^\pm]
$$

$$
= ab. \quad (86)
$$

Line (85) uses (84), and lines (85) and (86) both make use of the correspondence (79). \hfill \blacksquare

**Corollary 57:** For $n \geq 1$, there is a homeomorphism

$$
\text{spec}(A_{(n)}/\langle x^n \rangle) \approx \text{spec}(k[t^\pm])
$$

given by

$$
P/(x^n) \mapsto (P_0 + \langle s_i^n \rangle_{R_{(n)}}) \cap k[t^\pm]
$$

$$
\bigoplus_{m \in \mathbb{Z}} (\langle \sigma_m^n \rangle + \langle p \rangle) v_m / \langle x^n \rangle \leftrightarrow p \quad (87)
$$

**Proof:** Propositions 55 and 56 \hfill \blacksquare

### 3.3.2 The Prime Spectrum of $A$

Express the algebra $A$ as a GWA according to (33). Let $X$ denote the set of positive powers of $x$. Define $r_n \in A$ for $n \geq 1$ as in (37). Also define $s_i^n$, $J_n^m$, and $\sigma_m^n$ for $n \geq 1$ and $j, m \in \mathbb{Z}$ as in (40), but with everything taking place in $A$. Define the following subsets of spec($A$):

$$
T_1 = \{ P \in \text{spec}(A) \mid u \in P \},
$$

$$
T_2 = \{ P \in \text{spec}(A) \mid P = (P \cap k[t, d]) \}, \text{ and}
$$

$$
T_{3n} = \{ P \in \text{spec}(A) \mid u, t \notin P, P \cap X = \{ x^n, x^{n+1}, \ldots \} \} \text{ for } n \geq 1.
$$

**Theorem 58:** The prime spectrum of $A$ is, as a set, the disjoint union of $T_1$, $T_2$, and $T_{3n}$ for $n \geq 1$. Each of these subsets is homeomorphic to the prime spectrum of a commutative algebra as follows:

- $\text{spec}(k[u_{11}, u_{12}, u_{21}]) \approx T_1$ via $p \mapsto \langle u \rangle + \langle p \rangle$.
- $\text{spec}(k[t, d]) \approx T_2$ via $p \mapsto \langle p \rangle$.
• \( \text{spec}(k[t^\pm]) \approx T_{3n} \) via \( p \mapsto \langle \pi_n^m v_m \mid -n \leq m \leq n \rangle + \langle p \cap k[t] \rangle \), for all \( n \geq 1 \).

Our proof will make use of the localizations and quotients of \( A \) that were described in the introduction to section 3.3. Many of them are quantum tori, so it will help that the prime spectrum of a quantum torus is known.

**Definition 59:** A quantum torus over a field \( k \) is an iterated skew Laurent algebra

\[ k[x_1^\pm][x_2^\pm; \tau_2] \cdots [x_n^\pm; \tau_n] \]

for some \( n \in \mathbb{Z}_{\geq 0} \) and some automorphisms \( \tau_2, \ldots, \tau_n \) such that \( \tau_i(x_j) \) is a nonzero scalar multiple of \( x_j \) for all \( i \in \{2, \ldots, n\} \) and \( j \in \{1, \ldots, i-1\} \).

**Lemma 60:** [11, Corollary 1.5b] Contraction and extension provide mutually inverse homeomorphisms between the prime spectrum of a quantum torus and the prime spectrum of its center.

**Proof of Theorem 58:** Consider the partition of \( \text{spec}(A) \) into subsets \( S_1, \ldots, S_6 \) given by the following tree, in which branches represent mutually exclusive possibilities:

\[
\begin{align*}
P & \in \text{spec}(A) \\
u & \in P & u & \notin P \\
 & \in S_1 & \\
t, d & \in P & d & \notin P, t \in P \\
P & \in S_2 & P & \in S_3 \\
t & \notin P \\
X & \cap P = \emptyset & X & \cap P \neq \emptyset \\
P & \in S_4 & P & \in S_5 \\
d & \in P & d & \notin P \\
P & \in S_6 \\
\end{align*}
\]

It is easy to verify that

\[
S_1 = T_1, \\
S_6 = \bigsqcup_{n \geq 1} T_{3n}.
\]

To establish that \( \{T_1, T_2\} \cup \{T_{3n} \mid n \geq 1\} \) is a partition of \( \text{spec}(A) \), we will show that

\[
S_2 \cup S_3 \cup S_4 \cup S_5 = T_2.
\] (88)

Let \( P \in T_2 \) and let \( p = P \cap k[t, d] \). Then, using the same reasoning as in (17), \( P_m = p k[u, t, d] \) for all \( m \in \mathbb{Z} \). In particular, \( u \notin P \), so \( P \notin S_4 \), and \( P_n = P_0 \) for all \( n \geq 1 \), so \( P \notin S_6 \). This establishes the inclusion \( \supseteq \) of (88). We now address the reverse inclusion.

\( S_2 \subseteq T_2 \): Since \( u \) is normal, a prime ideal of \( A \) that excludes \( u \) also excludes any power of \( u \). So \( S_2 \approx \text{spec}(A_u/(t, d)) \). Since \( A_u/(t, d) = k[u^\pm][x^\pm; \sigma] \) is a quantum torus, Lemma 60 and Proposition

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Talgebras. Lemma 60 and Proposition 9 give that $S$ back to $S_2 = \{(t,d)\} \subseteq T_2$.

$S_3 \subseteq T_2$: Since $d$ is central, a prime ideal of $A$ that excludes $d$ also excludes any power of $d$. So $S_3 \approx \text{spec}(A_{ud}/\langle t \rangle)$.

Claim: In the algebra $A_{ud}/\langle t \rangle = k[u^d, d^d][x,y;\sigma,z]$, one has $\langle x^n \rangle = \langle 1 \rangle$ for all $n \in Z^{\geq 0}$.
Proof: Let $n \geq 1$. Multiplying $x^n$ by $y$ on either side shows that $\langle x^n \rangle_{n-1}$ contains $z$ and $\sigma(z)$.

Thus all prime ideals of $k[u^d, d^d][x,y;\sigma,z]$ are disjoint from the set of powers of $x$. Therefore by localization, using Proposition 21 and Theorem 67, $\text{spec}(k[u^d, d^d][x,y;\sigma,z]) \approx \text{spec}(k[u^d, d^d][x^d;\sigma])$. Lemma 60 and Proposition 9 give that $S_3 \approx \text{spec}(k[d^d])$. Let’s start with a $p \in \text{spec}(k[d^d])$ and follow it back to $S_4$: $p$ corresponds to its extension $\langle p \rangle \in A_{u^d,d^d}$ and let us follow $p$ back to $S_3$: $p$ corresponds to its extension $\langle p \rangle \in A_{u^d,d^d}$, which in turn corresponds to $\langle d \rangle + \langle p \rangle = \langle d \rangle + \langle p \cap k[d] \rangle \in A_{u^d,d^d}$. Now $(d) + (p \cap k[d]) \in A$ is prime because the quotient by it is $(k[u,d]/(p \cap k[d]))[x,y;\sigma,z]$, a GWA over a domain with $z \neq 0$. Hence by Lemma 68, $\langle p \rangle$ corresponds to $(d) + (p \cap k[d]) \in A$. So

$S_4 \subseteq T_2$: Since $t$ is central, $S_4 \approx \text{spec}(A_{u^d,d^d}/\langle d \rangle)$. Since $A_{u^d,d^d}/\langle d \rangle = k[u^d, d^d][x^d;\sigma]$ is a quantum torus, Lemma 69 and Proposition 9 give that $S_4 \approx \text{spec}(k[t^d])$. Let $p \in \text{spec}(k[t^d])$, and let us follow $p$ back to $S_3$: $p$ corresponds to its extension $\langle p \rangle \in A_{u^d,d^d}$, which in turn corresponds to $\langle d \rangle + \langle p \rangle = \langle d \rangle + \langle p \cap k[t] \rangle \in A_{u^d,d^d}$. Now $(d) + (p \cap k[t]) \in A$ is prime because the quotient by it is $(k[u,d]/(p \cap k[t]))[x,y;\sigma,z]$, a GWA over a domain with $z \neq 0$. Hence by Lemma 68, $\langle p \rangle$ corresponds to $(d) + (p \cap k[t]) \in A$. So

$S_5 \subseteq T_2$: We have $S_5 \approx \text{spec}(A_{u^d,d^d})$. Since $A_{u^d,d^d} = k[u^d, t^d, d^d][x^d;\sigma]$ is a quantum torus, Lemma 69 and Proposition 9 give $S_5 \approx k[t^d,d^d]$. Let $p \in \text{spec}(k[t^d,d^d])$, and let us follow it back to $S_5$: $p$ corresponds to its extension $\langle p \rangle \in A_{u^d,d^d}$, which in turn corresponds to $\langle d \rangle + \langle p \rangle = \langle d \rangle + \langle p \cap k[t] \rangle \in A_{u^d,d^d}$. Now $(d) + (p \cap k[t]) \in A$ is prime because the quotient by it is $(k[u,d]/(p \cap k[t]))[x,y;\sigma,z]$, a GWA over a domain with $z \neq 0$. Hence by Lemma 68, $\langle p \rangle$ corresponds to $(d) + (p \cap k[t]) \in A$. So

We have established $\langle u \rangle \subseteq T_1 \cup T_2 \cup \bigcup_{n \geq 1} T_{3n}$.

The remainder of the proof establishes homeomorphisms of $T_1$, $T_2$, and the $T_{3n}$ to spectra of commutative algebras.

$T_1$: Clearly, $T_1$ is homeomorphic to the prime spectrum of $A/\langle u \rangle \cong k[u_{11}, u_{12}, u_{21}]$ via $p \mapsto \langle u \rangle + \langle p \rangle$. 

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\( T_2 \): Note that the ring extension \( k[u, t, d]^p = k[t, d] \subseteq k[u, t, d] \) satisfies the condition \( (14) \). It also satisfies the condition \( (15) \), due to Proposition 20. We may therefore apply Lemma 19 to conclude that \( T_2 \approx \text{spec}(k[t, d]) \), with \( p \in \text{spec}(k[t, d]) \) corresponding to \( (p) \triangleleft A \).

\( T_3n \): Let \( n \geq 1 \). We have \( T_{3n} \approx \text{spec}(A_{(n)}/(x^{n})) \). By Corollary 57, we in turn have \( \text{spec}(A_{(n)}/(x^{n})) \approx \text{spec}(k[t^{\infty}]) \). Let \( p \in \text{spec}(k[t^{\infty}]) \), and let us follow it back to \( T_{3n} \). In Corollary 57 \( p \) corresponds to

\[
\bigoplus_{m \in \mathbb{Z}} (\langle \pi_m^n \rangle + \langle p \rangle)_{vm},
\]

which is

\[
\langle \pi_m^n_{vm} \mid -n \leq m \leq n \rangle + \langle p \rangle \triangleleft A_{(n)} = A_{ut}/(r_{n}).
\]

(89) Applying Lemma 68 is not as trivial in this situation, so we will check the needed hypotheses carefully. Write \( p \cap k[t] \) as \( (p) \in \text{spec}(k[t]) \), where \( p \) is either zero or it is some irreducible polynomial in \( t \) that is not divisible by \( t \). Let \( A = A/(r_{n}) \) and let \( R = k[u, t] \). Then \( A \) is a GWA \( R[x, y, \sigma, z_n] \), with the \( z_n \) given in (24). Note that the extension of the ideal \( (r_{n}) \triangleleft A \) to \( A_{ut} \) is \( (r_{n}) \triangleleft A_{ut} \). So by Proposition 60 \( A_{ut}/(r_{n}) \) is the localization of \( A \) at \( S := \{u^it^j \mid i, j \in \mathbb{Z} \geq 0\} \subseteq A \). Define

\[
\mathcal{G} := \{\pi_m^n_{vm} \mid -n \leq m \leq n\} \cup \{p\} \subseteq A.
\]

Let \( P = \bigoplus_{m \in \mathbb{Z}} (\pi_m^n, p)_{vm} \triangleleft A \). Check that this is an ideal of \( A \) by using (41) and (42) to verify that the conditions of Proposition 18 are met. \( P \) is generated by \( \mathcal{G} \) as a right ideal of \( A \); this takes care of hypothesis 1 of Lemma 68 Hypothesis 2 requires some work to verify. First we need:

Claim: For \( m \in \mathbb{Z} \),

\[
(\pi_m^n, p)_R = \bigcap_{j \in J_m^n} \langle s_j^n, p \rangle_{R'}.
\]

(90) Proof: Assume that \(-n < m < n\), otherwise there is nothing to prove (take an empty intersection to be \( R \)). If \( p = 0 \), then (90) follows from the fact that \( R \) is a UFD and \( \pi_m^n \) is a product of the non-associate irreducibles \( s_j^n \in R \). Assume that \( p \neq 0 \), so \( p \) is an irreducible polynomial in \( t \) that is not divisible by \( t \). For convenience of notation, let \( s_1, \ldots, s_r \) be the elements of \( \{s_j^n \mid j \in J_m^n\} \). We will show that

\[
\langle s_1s_2 \cdots s_r, p \rangle = \langle s_1, p \rangle \cap \langle s_2 \cdots s_r, p \rangle
\]

(91) and then (90) will follow by repeating the same principle with induction.

The inclusion \( \subseteq \) of (91) is obvious. For \( \supseteq \), suppose that \( \alpha s_1 + \gamma p = \beta s_2 \cdots s_r + \delta p \), where \( \alpha, \beta, \gamma, \delta \in R \). Then \( (\gamma - \delta)p \in \langle s_1, s_2 \cdots s_r \rangle \). We can see that \( p \) is regular mod \( \langle s_1, s_2 \cdots s_r \rangle \) by using an isomorphism \( R/(s_1) \cong k[t] \) that fixes \( t \): the image of \( p \) under \( R \to R/(s_1) \cong k[t] \) is itself, and the image of \( \langle s_2 \cdots s_r \rangle \) is \( \langle t^{r-1} \rangle \) (since \( q \) is not a root of unity). Hence we have \( (\gamma - \delta) \in \langle s_1, s_2 \cdots s_r \rangle \).

Write it as \( (\gamma - \delta) = \varepsilon s_1 + \zeta s_2 \cdots s_r \), for some \( \varepsilon, \zeta \in R \). Then

\[
(\alpha + \varepsilon p)s_1 = (\alpha s_1) + (\varepsilon s_1)p = (\beta s_2 \cdots s_r + \delta p - \gamma p) + (\gamma - \delta - \zeta s_2 \cdots s_r)p
\]

\[
= (\beta - \zeta p)s_2 \cdots s_r.
\]

Since \( s_1, s_2, \ldots, s_r \) are non-associate irreducibles, it follows that \( (\beta - \zeta p) = \eta s_1 \) for some \( \eta \in R \). Finally,

\[
\alpha s_1 + \gamma p = \beta s_2 \cdots s_r + \delta p = (\eta s_1 + \zeta p)s_2 \cdots s_r + \delta p
\]

\[
= \eta s_1s_2 \cdots s_r + (\zeta s_2 \cdots s_r + \delta)p,
\]

proving (91).

Now we can verify hypothesis 2 the ideal (89) is already known to be prime and:

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Corollary 61: The algebra \( A \) is \( S \)-torsionfree.

**Proof:** It suffices to check that \( (R/\langle \pi_m^n, p \rangle)_R \) is \( S \)-torsionfree for all \( m \in \mathbb{Z} \), for if
\[
\left( \sum_{m \in \mathbb{Z}} a_m v_m \right) u^t \in P,
\]
then \( q^{2m} a_m v_m u^t \in \langle \pi_m^n, p \rangle \) for each \( m \in \mathbb{Z} \). By (90), the problem further reduces to checking that \( (R/\langle s_j^n, p \rangle)_R \) is \( S \)-torsionfree for each \( m \in \mathbb{Z} \) and \( j \in J_m^n \). \( R/\langle s_j^n, p \rangle \) is isomorphic to \( k[t] \) by an isomorphism that fixes \( t \), so \( R/\langle s_j^n, p \rangle \cong k[t]/(p) \) is a domain and in particular \( S \)-torsionfree.

For hypothesis 9 the nontrivial case to check is \( g = \pi_m^n v_m \) and \( s = u^t \). In this case we have
\[
gs = s(q^{2m} g) \in gS \cap sP.
\]
Therefore \( P \circ A/(r_n) \) is the contraction of \( \mathcal{S} \). Pulling back to \( A \), we conclude that \( p \in \text{spec}(k[t^\pm]) \) corresponds to
\[
\langle \pi_m^n v_m \mid -n \leq m \leq n \rangle + (r_n) + (p \cap k[t]) \triangleleft A.
\]

Corollary 61: The algebra \( A \) is a noetherian UFD. (See [7] for the definition of noetherian UFD).

**Proof:** Having just listed all the prime ideals of \( A \), we simply check off the needed conditions:

- \( A \) is a noetherian domain.
- Every nonzero prime ideal of \( A \) contains a nonzero principal prime ideal. (Here a principal ideal is one generated by a single normal element). **Proof:** For \( T_1 \) and \( T_2 \) this is obvious. For \( P \in T_{3n} \), \( n \geq 1 \), note that \( P \) contains \( (r_n) \) for \( T_2 \).
- Height one primes of \( A \) are completely prime. **Proof:** Since \( (r_n) \) is properly contained in any \( P \in T_{3n} \) for \( n \geq 1 \), the primes in \( T_{3n} \) are not height one. We check that all the other primes are completely prime. Suppose \( P \in T_2 \). Then \( P \) is generated in the commutative coefficient ring \( k[u, t, d] \) of the GWA \( A = k[u, t, d][x, y; \sigma, z] \) and it does not contain \( z \), so Proposition 12 shows that \( A/P \) is a GWA over a domain, and hence a domain. For \( P = \langle u \rangle \) \( (\langle p \rangle) \in T_1 \), \( A/P \) is \( k[u_{11}, u_{12}, u_{21}]/p \), which is a domain.

Since \( A \) is noetherian, every closed subset of \( \text{spec}(A) \) is a finite union of irreducible closed subsets. The topology of \( \text{spec}(A) \) is therefore known if all inclusions of prime ideals are known. We address in the following proposition those inclusions that are not already expressed in Theorem 3.

Proposition 62: The inclusions among the prime ideals of \( A \) are as follows:

1. Inclusions coming from the homeomorphisms \( T_1 \approx \text{spec}(k[u_{11}, u_{12}, u_{21}]) \), \( T_2 \approx \text{spec}(k[t, d]) \), and \( T_{3n} \approx \text{spec}(k[t^\pm]) \) for \( n \geq 1 \).
2. Let \( P \in T_1 \). No prime in \( T_2 \) contains \( P \), and no prime in \( T_{3n} \) contains \( P \) for any \( n \).
3. Let \( P \in T_2 \), say \( P = \langle p \rangle \) with \( p \in \text{spec}(k[t, d]) \).
   a) The set of \( Q \in T_1 \) that contain \( P \) is
   \[
   \{ \langle u \rangle \mid q \in \text{spec}(k[u_{11}, u_{12}, u_{21}]) \text{ and } p \subseteq \phi^{-1}(q) \},
   \]
   where \( \phi \) is the homomorphism \( \phi: k[t, d] \rightarrow k[u_{11}, u_{12}, u_{21}] \) that sends \( t \) to \( u_{11} \) and \( d \) to \( u_{12}u_{21} \).
   b) Let \( n \geq 1 \). The set of \( Q \in T_{3n} \) that contain \( P \) is
   \[
   \{ \langle \pi_m^n v_m \mid -n \leq m \leq n \rangle + (r_n) + (q \cap k[t]) \mid q \in \text{spec}(k[t^\pm]) \text{ and } p \subseteq \eta_n^{-1}(q) \},
   \]
   where \( \eta_n \) is the homomorphism \( \eta_n: k[t^\pm, d] \rightarrow k[t^\pm] \) that sends \( t \) to \( t \) and \( d \) to \( -\frac{\phi_n^2}{(\phi_n^2 + 1)}t^2 \).
4. Let $n \geq 1$ and let $P \in T_{3n}$, say

$$P = \langle \pi_m^n v_m \mid -n \leq m \leq n \rangle + (r_n) + \langle p \cap k[t] \rangle$$

with $p \in \text{spec}(k[t^{\pm}])$. If $p = 0$, then the only $Q \in T_1$ containing $P$ is

$$\langle u_{11}, u_{22}, u_{21}, u_{12} \rangle.$$  

If $p \neq 0$, then no prime in $T_1$ or $T_2$ contains $P$, and no prime in $T_{3n'}$ contains $P$ for any $n' \neq n$.

**Proof:** The inclusions of assertion 1 are addressed by the homeomorphisms in Theorem 5.

2. If $P \in T_1$, then $u \in P$. If $Q \in T_2$ then $Q_0$ (using the notation of Definition 13) is generated in $k[u, t, d]$ by elements of $k[t, d]$, so $Q$ cannot contain $u$ and therefore cannot contain $P$. If $Q \in T_{3n}$, then by definition $Q$ cannot contain $u$ and therefore cannot contain $P$.

3a. Assume the setup of assertion 3a. Suppose that $Q \in T_1$, and write it as $\langle u \rangle + \langle q \rangle$ with $q \in \text{spec}(k[u_{11}, u_{12}, u_{21}])$. Then $P \subseteq Q$ if and only if $\langle u \rangle + \langle p \rangle \subseteq Q$, which holds if and only if $(\langle u \rangle + \langle p \rangle)/\langle u \rangle \subseteq Q/\langle u \rangle$ holds in $A/\langle u \rangle$. The following composite is the homomorphism $\phi$ that we defined:

$$k[t, d] \twoheadrightarrow A \twoheadrightarrow A/\langle u \rangle \cong k[u_{11}, u_{12}, u_{21}]$$

$$p \quad Q \quad \leftrightarrow \quad Q/\langle u \rangle \leftrightarrow q.$$ 

We see that $P \subseteq Q$ if and only if $\phi(p) \subseteq q$. This holds if and only if $p \subseteq \phi^{-1}(q)$, so assertion 3a is proven.

3b. Assume the setup of assertion 3b. Suppose that $Q \in T_{3n}$, and write it as

$$\langle \pi_m^n v_m \mid -n \leq m \leq n \rangle + (r_n) + \langle q \cap k[t] \rangle$$

with $q \in \text{spec}(k[t^{\pm}])$. Then

$$Q = \bigoplus_{m \in \mathbb{Z}} (\langle \pi_m^n v_m \rangle + \langle q \cap k[t] \rangle) v_m;$$

the inclusion $\supseteq$ is clear and the inclusion $\subseteq$ follows from the fact that the right hand side is an ideal of $A$, which can be verified by using 11 and 12 to check that the conditions of Proposition 13 are met.

In particular, $Q_0 = \langle \pi_0^n v_0 \rangle + \langle q \cap k[t] \rangle$. Now assertion 3b is proven as follows:

$$P \subseteq Q \iff p \subseteq Q_0 = \langle \pi_0^n v_0 \rangle + \langle q \cap k[t] \rangle$$

$$\iff p \subseteq (r_n)_{k[t, d]} + \langle q \cap k[t] \rangle$$

$$\iff p \subseteq (r_n)_{k[t, d]} + \langle q \rangle$$

$$\iff (r_n)_{k[t, d]} + \langle q \rangle \cap k[t, d] \subseteq (r_n)_{k[t, d]} + \langle q \rangle$$

$$\iff \eta_n(p) \subseteq \eta_n(q).$$

Line 92 is due to the fact that $Q_0 \cap k[t, d] = (r_n)_{k[t, d]} + \langle q \cap k[t] \rangle$. Line 93 is due to the fact that $k[t, d]$ mod the ideal $(r_n)_{k[t, d]} + \langle q \cap k[t] \rangle$ is $t$-torsionfree. Line 94 is due to the fact that $\eta_n$ is the following composite:

$$k[t^{\pm}, d] \hookrightarrow k[t^{\pm}, d]/(r_n) \cong k[t^{\pm}]/\langle q \rangle + (r_n) \leftrightarrow q.$$  

4. Assume the setup of assertion 4. Let $Q \in T_1$ such that $P \subseteq Q$, say $Q = \langle u_{22} \rangle + \langle q \rangle$ with $q \in \text{spec}(k[u_{11}, u_{12}, u_{21}])$. Then $Q$ contains a power of $x$ and a power of $y$, so $q$ contains $u_{21}$ and $u_{12}$. $Q$ also contains $r_n$, which is equivalent to $q^n u_{11}$ modulo $(u_{22}, u_{12}, u_{21})$. $Q$ contains, and therefore equals, the maximal ideal $\langle u_{11}, u_{22}, u_{21}, u_{12} \rangle$. The containment $P \subseteq \langle u_{11}, u_{22}, u_{21}, u_{12} \rangle$ clearly holds if $p = 0$. But if $p$ is nonzero, then it contains some polynomial in $t$ with nonzero constant term, which is not in $\langle u_{11}, u_{22}, u_{21}, u_{12} \rangle$. Thus, nothing in $T_1$ contains $P$ when $p \neq 0$. 
If $Q \in T_2$, then $Q = \bigoplus_{m \in \mathbb{Z}} Q_0 v_m$ does not contain any power of $x$. So nothing in $T_2$ contains $P$.

Now suppose that $Q \in T_{3n'}$ with $n' \neq n$, and suppose for the sake of contradiction that $P \subseteq Q$. Then $Q$ contains $r_n$ and $r_{n'}$. Since $n \neq n'$, it follows that $t, d \in Q$. Write $Q$ as

$$Q = \langle \pi_m v_m \mid -n' \leq m \leq n' \rangle + \langle r_{n'} \rangle + \langle q \cap k[t] \rangle$$

with $q \in \text{spec}(k[t^\pm])$. Since $t \in Q$, we must have $q = 0$. We have a contradiction:

$$d \in Q_0 = \langle \pi_0', r_{n'} \rangle \trianglelefteq k[u, t, d].$$

Proposition 63: The algebra $A$ does not have normal separation (see [12, Ch 12] for the definition).

Proof: Let $P = \langle \pi_0, x, y, r_1 \rangle$ and let $Q = \langle r_1 \rangle$, both prime ideals of $A$. We will show that no element of $P \setminus Q$ is normal modulo $Q$. Note that $k[u, t, d]/\langle r_1 \rangle \cong k[u, t]$, and let $R = k[u, t]$. Using Proposition 12, $A/Q$ is isomorphic to $W := R[x, y; \sigma, z = -q^{-4} s_1 s_2]$, and $P/Q$ becomes

$$\mathcal{P} := \langle \pi_0, x, y \rangle = \bigoplus_{m > 0} R y^m \oplus \langle s_1 \rangle_R \oplus \bigoplus_{m > 0} R x^m.$$

By Proposition 11 the nonzero normal elements of $W$ are the $\sigma$-eigenvectors in $R$. Thus, they are all of the form $u^i f(t)$ for some polynomial $f(t)$ and some $i \in \mathbb{Z}_{\geq 0}$. But $\mathcal{P}$ cannot contain such elements, since $\mathcal{P}_0 = \langle s_1 \rangle_R$. ■

Proposition 64: The algebra $A$ is not catenary.

Proof: Let $n \geq 1$. The information in Proposition 63 implies that the following two chains of primes are saturated:

One reason to compute the prime spectrum of an algebra is to make progress towards the lofty goal of knowing its complete representation theory. The idea is to make progress by trying to know the algebra’s primitive ideals, those ideals that arise as annihilators of irreducible representations. Since primitive ideals are prime, one approach is to determine the prime spectrum of the algebra and then attempt to locate the primitives living in it. The Dixmier-Moeglin equivalence, when it holds, provides a topological criterion for picking out primitives from the spectrum; see [6, II.7-II.8] for definitions.

Theorem 65: The algebra $A$ satisfies the Dixmier-Moeglin equivalence, and its primitive ideals are as follows:
• The primitive ideals in $T_1$ are $\langle u \rangle + (p)$ for $p \in \text{max spec } k[u_{11}, u_{12}, u_{21}]$.
• The primitive ideals in $T_2$ are $(p)$ for $p \in \text{max spec } k[t, d]$.
• The primitive ideals in $T_{3n}$ are $\langle \pi_m^nu_m \mid -n \leq m \leq n \rangle + (p \cap k[t])$ for $n \geq 1$ and $p \in \text{max spec } k[t^\pm]$.

**Proof:** We first observe that $\mathcal{A}$ satisfies the Nullstellensatz over $k$. For this we can use [5, II.7.17], which applies because $\mathcal{A}$ is an iterated skew polynomial algebra

$$\mathcal{A} \cong k[u_{11}][u_{22}][u_{12}; \tau][u_{21}; \tau', \delta']$$

for a suitable choice of $\tau, \tau', \delta'$. It then follows from [6, II.7.15] that the following implications hold for all prime ideals of $\mathcal{A}$:

$$\text{locally closed } \implies \text{primitive } \implies \text{rational}.$$  

To establish the Dixmier-Moeglin equivalence for $\mathcal{A}$, it remains to close the loop and show that rational primes are locally closed. We shall deal separately with the three different types of primes identified in Theorem 52.

$T_1$: Suppose that $P \in T_1$, say $P = \langle u \rangle + (p)$ with $p \in \text{spec}(k[u_{11}, u_{12}, u_{21}])$. Then $\mathcal{A}/P \cong k[u_{11}, u_{12}, u_{21}]/p$. It follows that $P$ is rational if and only if $p$ is a maximal ideal of $k[u_{11}, u_{12}, u_{21}]$. In this case $P$ will be maximal and therefore locally closed. Thus, rational primes in $T_1$ are locally closed.

$T_2$: Suppose that $P \in T_2$, say $P = \langle p \rangle$ with $p \in \text{spec}(k[t, d])$. Then, using Proposition 49, $\mathcal{A}/P$ is a GWA $R[x, y; \sigma, z]$, where $R := k[u_{11}, u_{12}, u_{21}]/(p)$. Since $z = d + q^{-1}tu - q^{-1}u^2$ is regular in $R$, Proposition 61 tells us that $R[x, y; \sigma, z]$ embeds into the skew Laurent polynomial algebra $R[x^\pm; \sigma]$. Let $K$ denote the fraction field of $R$. The skew Laurent polynomial algebra $R[x^\pm; \sigma]$ embeds into the skew Laurent series algebra $K((x^\pm; \sigma))$. (We are abusing notation and writing $\sigma$ for the induced automorphism of $K$.) Since the skew Laurent series algebra is a division ring, we obtain an induced embedding of the Goldie quotient ring $\text{Fract}(\mathcal{A}/P)$ into it:

$$\text{Fract}(\mathcal{A}/P) \hookrightarrow K((x^\pm; \sigma)).$$

For something to be in the center of $\text{Fract}(\mathcal{A}/P) \cong \text{Fract}(R[x, y; \sigma, z])$, it must at least commute with $R$ and $x$. This is sufficient to place it in the center of $K((x^\pm; \sigma))$, so

$$Z(\text{Fract}(\mathcal{A}/P)) \cong Z(K((x^\pm; \sigma))) \cap \text{Fract}(\mathcal{A}/P). \quad (95)$$

According to Proposition 63 the center of $K((x^\pm; \sigma))$ is the fixed subfield $K^\sigma$. Since $K$ is wholly contained in $Z(\text{Fract}(\mathcal{A}/P)) \cong Z(\text{Fract}(R[x, y; \sigma, z]))$, (95) becomes

$$Z(\text{Fract}(\mathcal{A}/P)) \cong K^\sigma.$$

Now to compute $K^\sigma$. Since

$$R = k[u_{11}, u_{12}, u_{21}]/(p) \cong (k[t, d]/p)[u],$$

$K$ is the rational function field $L(u)$, where $L$ is the fraction field of $k[t, d]/p$.

**Claim:** $K^\sigma = L$.

**Proof:** Observe that $\sigma$ fixes $L$ and sends $u$ to $q^2u$. Consider any nonzero $f/g \in K^\sigma = L(u)^\sigma$, where $f, g \in L[u]$ are coprime. We have $\sigma(f)g = f\sigma(g)$. Since $f$ and $g$ are coprime, it follows that $f \mid \sigma(f)$. Similarly, since $\sigma(f)$ and $\sigma(g)$ are coprime, $\sigma(f) \mid f$. It follows that $\sigma(f) = \alpha f$ for some $\alpha \in L$. From $\sigma(f)g = f\sigma(g)$ it follows that also $\sigma(g) = \alpha g$. We have an eigenspace decomposition for the action of $\sigma$ as an $L$-linear operator on $L[u]$; it is $\bigoplus_{i \geq 0} Lu^i$, with distinct eigenvalues since $q$ is not a root of unity. Since $f$ and $g$ are $\sigma$-eigenvectors with the same eigenvalue $\alpha$, there is some $i \geq 0$ such that $f = f_0u^i$ and $g = g_0u^i$, where $f_0, g_0 \in L$. Thus, $f/g = f_0/g_0 \in L$. 

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We have found that

\[ Z(\text{Fract}(A/P)) \cong L. \]

The fraction field \( L \) of \( k[t, d]/p \) is algebraic over \( k \) if and only if \( p \triangleleft k[t, d] \) is maximal. Thus, \( P \) is rational if and only if \( p \) is maximal.

Now assume that \( P \) is rational and hence that \( p \) is maximal. For any \( q \in \text{spec}(k[t, d]) \) and any \( n \geq 1 \), define

\[ Q_{q, n} := \langle t^n v_m \mid -n \leq m \leq n \rangle + \langle r_n \rangle + \langle q \cap k[t] \rangle \in T_{3n}. \]

If no \( Q_{q, n} \) contains \( P \), then by using Proposition 62 we can see that \( \{P\} = V(P) \cap (\text{spec}(A) \setminus V(u)) \), so \( P \) is locally closed. Suppose, on the other hand, that \( Q_{q, n} \) contains \( P \) for some \( q \in \text{spec}(k[t, d]) \) and \( n \geq 1 \). We claim that this can occur for at most one \( n \).

**Claim:** If \( Q_{q, n} \) and \( Q_{q', n'} \) both contain \( P \), then \( n = n' \).

**Proof:** According to assertion 3b of Proposition 62, we have

\[ p \subseteq \eta_n^{-1}(q) \quad \text{and} \quad p \subseteq \eta_{n'}^{-1}(q'), \]

where \( \eta_n, \eta_{n'} \) are the homomorphisms defined there. Since \( p \) generates a maximal ideal of \( k[t^\pm, d] \), this forces

\[ \eta_n^{-1}(q) = \eta_{n'}^{-1}(q'). \]

We have

\[ d + \frac{q^{2n}}{(q^{2n} + 1)^2} t^2, \quad d + \frac{q^{2n'}}{(q^{2n'} + 1)^2} t^2 \in \eta_n^{-1}(q) = \eta_{n'}^{-1}(q'), \]

so

\[ \left( \frac{q^{2n}}{(q^{2n} + 1)^2} - \frac{q^{2n'}}{(q^{2n'} + 1)^2} \right) t^2 \in \eta_n^{-1}(q) = \eta_{n'}^{-1}(q'). \]

Since we cannot have \( \eta_n^{-1}(q) = k[t^\pm, d] \), the quantity in parentheses must vanish. This leads to the equation

\[ 0 = q^{2n}(q^{2n'} + 1)^2 - q^{2n'}(q^{2n} + 1)^2 = (q^{n'} - q^n)(q^{n'} + q^n)(q^{n+n'} - 1)(q^{n+n'} + 1). \]

Since \( q \) is not a root of unity, it follows that \( n = n' \).

Hence \( \{P\} = V(P) \cap (\text{spec}(A) \setminus (V(u) \cup V(x^n))) \), and again we see that \( P \) is locally closed. Thus we have shown that all rational primes in \( T_2 \) are locally closed.

**Claim:** Suppose that \( n \geq 1 \) and \( P \in T_{3n} \), say \( P = \langle t^n v_m \mid -n \leq m \leq n \rangle + \langle r_n \rangle + \langle p \cap k[t] \rangle \) with \( p \in \text{spec}(k[t^\pm]) \). We will show that if \( P \) is rational, then \( p \) must be maximal. Assume that \( P \) is rational, but \( p \) is not maximal (i.e. \( p = 0 \)). Then \( t \in Z(\text{Fract}(A/P)) \) is algebraic over \( k \), so for some nonzero polynomial \( f(T) \in k[T] \) we must have \( f(t) = 0 \in Z(\text{Fract}(A/P)) \). For the element \( t \) of \( A \), this means that \( f(t) \in P \). We can describe \( P \) explicitly in terms of its homogeneous components:

\[ P = \bigoplus_{m \in \mathbb{Z}} \langle t^n, r_n \rangle v_m. \]

The inclusion \( \supseteq \) is obvious, and the equality can be verified by checking that the conditions of Proposition 62 are met by the right hand side and it indeed defines a two-sided ideal. So the fact that \( f(t) \in P \) can be refined to \( f(t) \in \langle t^n, r_n \rangle k[u, t, d] \). Pushing this fact into \( k[u, t, d]/\langle r_n \rangle \cong k[u, t] \) gives

\[ f(t) \in \langle r^n_0 \rangle_{k[u, t]} \]

which is clearly false.

Thus we have shown that when \( P \) is rational, \( p \triangleleft k[t^\pm] \) must be maximal. Using Proposition 62 we see that in this case \( \{P\} = V(P) \). So all rational primes in \( T_{3n} \) are locally closed.
We have now shown that all rational prime ideals of $\mathcal{A}$ are locally closed, and we conclude that $\mathcal{A}$ satisfies the Dixmier-Moeglin equivalence. Further, we have pinpointed which primes are rational in $T_1$ and $T_2$. As for $T_{3n}$, we have found for $P = \langle \pi^m \nu_n \mid -n \leq m \leq n \rangle + \langle \nu_n \rangle + \langle \mathfrak{p} \cap k[t] \rangle$ that

$$P \text{ rational } \implies \mathfrak{p} \text{ maximal } \implies P \text{ locally closed.}$$

Putting this information together and applying the Dixmier-Moeglin equivalence, we conclude that the primitive ideals of $\mathcal{A}$ are as stated in the theorem.

4 Appendix

There are a few aspects of noncommutative localization that make an appearance throughout this work and that rely on the noetherian hypothesis. For the reader’s convenience, we lay them out here. Proposition 66 says that localization “commutes” with factoring out an ideal, and it is a standard fact. Theorem 67 says that the usual correspondence of prime ideals along a localization is a homeomorphism, also a standard fact. Finally, Lemma 68 provides a way to describe the pullback of a prime ideal along a localization by using a “nice” generating set.

Proposition 66: Let $S$ be a right denominator set in a right noetherian ring $R$. Let $I$ be an ideal of $R$, with extension $I^e$ to $RS^{-1}$. Then:

- $I^e$ is an ideal of $RS^{-1}$.
- $\bar{S} := \{ s + I \mid s \in S \}$ is a denominator set of $R/I$.
- The canonical homomorphism $\phi : R/I \to (RS^{-1})/I^e$ gives a right ring of fractions for $R/I$ with respect to $\bar{S}$. That is, there is an isomorphism $\bar{\phi} : (R/I)\bar{S}^{-1} \cong (RS^{-1})/I^e$ making the following diagram commute:

$$\begin{array}{ccc}
R & \xrightarrow{\text{loc}} & RS^{-1} \\
\downarrow & & \downarrow \text{loc} \\
R/I & \xrightarrow{\bar{\phi}} & (R/I)\bar{S}^{-1}
\end{array}$$

Theorem 67: Let $S$ be a right denominator set in a right noetherian ring $R$. Then contraction and extension of prime ideals are inverse homeomorphisms:

$$\text{spec}(RS^{-1}) \approx \{ Q \in \text{spec}(R) \mid Q \cap S = \emptyset \}. \quad (96)$$

Lemma 68: Let $R$ be a right noetherian ring, $S \subseteq R$ a right denominator set, and $\phi : R \to RS^{-1}$ the localization map. Let $\mathcal{G} \subseteq R$ and assume the following:

1. The right ideal $P$ generated by $\mathcal{G}$ is a two-sided ideal of $R$.
2. Either $P$ is a prime ideal of $R$ disjoint from $S$, or $\langle \phi(\mathcal{G}) \rangle$ is a prime ideal of $RS^{-1}$ and $(R/P)_R$ is $S$-torsionfree.
3. For all $g \in \mathcal{G}$ and $s \in S$,

$$gs \cap sP \neq \emptyset.$$ 

Then

$$P = \phi^{-1}(\langle \phi(\mathcal{G}) \rangle).$$

That is, the ideal of $RS^{-1}$ generated by $\phi(\mathcal{G})$ contracts to the ideal of $R$ generated by $\mathcal{G}$. 

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Proof: Assumption 3 guarantees that the right ideal of $RS^{-1}$ generated by $\phi(G)$ is a two-sided ideal. Let superscripts “e” and “c” denote extension and contraction of ideals along $\phi$. Observe that

$$\langle \phi(G) \rangle = \{ \sum_{i=1}^{n} \phi(g_i)\phi(r_i)\phi(s_i)^{-1} \mid n \in \mathbb{Z}_{\geq 0}, r_i \in R, s_i \in S, g_i \in G \text{ for } 1 \leq i \leq n \}$$

$$= \{ \sum_{i=1}^{n} \phi(g_i r_i)\phi(s_i)^{-1} \mid s \in S, n \in \mathbb{Z}_{\geq 0}, r_i \in R, g_i \in G \text{ for } 1 \leq i \leq n \}$$

(97)

$$= \{ \phi(a)\phi(s)^{-1} \mid s \in S, a \in \langle G \rangle \} = P^e.$$

In line (97), we used the fact that it is possible to get a “common right denominator” for a finite list of right fractions; see [12, Lemma 10.2]. Now assumption 2 implies that $P$ is a prime ideal of $R$ disjoint from $S$, either trivially or by [12, Theorem 10.18b]. To finish, we use the correspondence between prime ideals disjoint from $S$ and prime ideals of $RS^{-1}$:

$$P = P^{ec} = (\phi(G))^c = \phi^{-1}(\langle \phi(G) \rangle).$$

Note that assumption 3 of Lemma 68 holds trivially whenever $G$ or $S$ is central.

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