Quantum adiabatic search with decoherence in the instantaneous energy eigenbasis

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In Phys. Rev. A 71, 060312(R) (2005) the robustness of the local adiabatic quantum search to decoherence in the instantaneous eigenbasis of the search Hamiltonian was examined. We expand this analysis to include the case of the global adiabatic quantum search. As in the case of the local search the asymptotic time complexity for the global search is the same as for the ideal closed case, as long as the Hamiltonian dynamics is present. In the case of pure decoherence, where the environment monitors the search Hamiltonian, we find that the time complexity of the global quantum adiabatic search scales like $N^{3/2}$, where $N$ is the list length. We moreover extend the analysis to include success probabilities $p < 1$ and prove bounds on the run time with the same scaling as in the conditions for the $p \to 1$ limit. We supplement the analytical results by numerical simulations of the global and local search.

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I. INTRODUCTION

Quantum adiabatic algorithms, first conceived in Ref. [1], have been developed as an alternative to the traditional circuit model of quantum computation. An important aspect of the adiabatic approach is its expected resilience to various kinds of open system effects, which makes it a promising candidate for robust quantum computation. The basic reason for this appealing feature is that the adiabatic quantum computer operates near the instantaneous ground state of a time-dependent Hamiltonian and can therefore be expected to be insensitive to relaxation and open system effects among the excited states. The robustness of adiabatic quantum computation has been discussed in the literature for different kinds of perturbations, such as relaxation [2], noise [3], decoherence [4-6], and unitary control errors [7].

In this paper, we elaborate on the robustness of the quantum adiabatic search [1, 5, 8] in the presence of decoherence in the instantaneous eigenbasis of the search Hamiltonian. In particular, we extend the analysis of the local search [5] in Ref. [1] to global adiabatic search [1]. We further address the dependence of the asymptotic time complexity of the search procedure on the success probability for finding the searched item.

To model adiabatic search in the presence of decoherence we consider master equations of the form ($\hbar = 1$ from now on)

$$\dot{\rho} = -iA[H, \rho] - B[W, [W, \rho]],$$

where $\rho$ is the density operator, and $A, B$ are constants such that $B \geq 0$. For convenience, we also assume that $A \geq 0$. The operator $H$ is taken to be the time-dependent Hamiltonian of the adiabatic search [1, 7], and $W$ is a time-dependent Hermitian operator such that $[H, W] = 0$, which gives decoherence in the instantaneous eigenbasis of $H$.

Decoherence with respect to the energy eigenbasis of a system has been shown to occur in certain regimes of system-environment interaction [8]. Apart from this general regime of weak interaction and dominating self-Hamiltonians one can find other settings where this type of decoherence occurs. One example is control errors in terms of fluctuations in the energy levels of the system [10]. Note also that decoherence in the energy eigenbasis does not necessarily imply the particular form of Eq. (1). However, this choice provides a sufficiently simple model to be treated analytically, and yet results in non-trivial behavior. Moreover, the form of Eq. (1) quite generally guarantees the solution to be a proper density operator at all times.

Our results show that for $A \neq 0$ and list length $N$, it is sufficient with a run time of the order $N$ in the global case and $\sqrt{N}$ in the local case, as is the case for the ideal closed evolution. These results suggest that the adiabatic quantum computer is protected against this type of decoherence, which is promising for physical realizations.

In the wide-open case where $A = 0$ [11], with the special choice $W = H$, the scaling changes to $N^{3/2}$ and $N$, for the global and local search, respectively.

Although the concept of adiabaticity has a well-defined meaning for closed systems, there is no obviously unique generalization to the case of open systems. One may consider several different generalizations focusing on various aspects of the standard adiabaticity. For example, in Refs. [12, 13, 14] the concept of adiabaticity is based upon Jordan block decompositions of the superoperator that describe the evolution of the open system. A different approach, designed for systems that are weakly open, has been put forward in Ref. [15]. This latter approach is based upon taking decoupling of the energy subspaces of the ideal Hamiltonian as the starting point. Here, we use a third method based upon a feature of the adiabatic quantum computer, namely that the evolution takes place near an eigenstate of the ideal Hamiltonian.
This property is in contrast with, e.g., implementations of holonomic quantum gates \( R \), where it is essential that the gate can operate on arbitrary superpositions, without too large errors. For the functioning of the adiabatic quantum computer, however, it is sufficient that the state of the system remains close to the correct eigenstate. In the present interpretation of the word “adiabaticity” we only require that the probabilities of finding the system in the instantaneous eigenstates of \( H(s) \) are conserved, but we allow superpositions of the instantaneous eigenstates to decay into mixtures.

The structure of the article is as follows. In Sec. II we derive an integral equation from the original master equation. Section III gives a brief overview of the adiabatic search algorithm, and we calculate some quantities which are used later. In Sec. IV we prove a sufficient condition for the success probability of the adiabatic search to approach unity in the semi-open case. We consider both global and local search. Furthermore, a condition on the decoherence term of the master equation, which is used in the proofs in Sec. IV is examined. In Sec. V we demonstrate that the wide-open case is essentially different from the semi-open case. We prove sufficient conditions for the success probability to approach unity in both the global, as well as the local case. We further prove that the sufficient conditions are also necessary. In Sec. VI bounds on the run time for fixed success probabilities less than unity, are given. We end with the conclusions.

II. THE MASTER EQUATION

In this section we develop the basic equation used in our analysis. Since it is our intention to use adiabaticity we consider a fixed family of Hamiltonians \( H(s) \), parametrized, in a sufficiently smooth way, by the parameter \( s \). We assume that the parameter \( s \) is proportional to the time \( t \), such that \( s = 0 \) at time \( t = 0 \) and \( s = 1 \) at time \( t = T \). The time \( T \) is referred to as the “run time”. We also assume a family of Hermitian operators \( W(s) \), to be used in the master equation

\[
\frac{d}{dt} \rho(t) = -i[A[H(t/T), \rho(t)] - B[W(t/T), [W(t/T), \rho(t)]]].
\]

(2)

We assume that both \( H(t/T) \) and \( W(t/T) \) are Hermitian and non-degenerate at each \( t \). Moreover, we assume \( [H(t/T), W(t/T)] = 0 \) for all \( t \). By a change of variables \( s = t/T \) in Eq. 2 one obtains

\[
\frac{d}{ds} \rho(s) = -iTA[H(s), \rho(s)] - TBBW(s), [W(s), \rho(s)]]
\]

(3)

where \( \rho(s) = \rho(sT) \). Let \( \{ |E_n(s)\rangle \}_n \) be instantaneous orthonormal eigenvectors of \( H(s) \), i.e., \( H(s)|E_n(s)\rangle = E_n(s)|E_n(s)\rangle \). Since \( H(s) \) is assumed to be non-degenerate the arbitrariness in the choice of eigenvectors is limited to phase factors. Let \( w_n(s) \) denote the eigenvalues of \( W(s) \). If we insert \( \rho(s) = \sum_n \rho_{nm}(s)|E_n(s)\rangle \langle E_m(s)| \) into Eq. 3, the equations for the components \( \rho_{nm}(s) \) become

\[
\frac{d}{ds} \rho_{nm}(s) = -iTA[E_n(s) - E_m(s)]\rho_{nm}(s) - TBB\{w_n(s) - w_m(s)\}^2\rho_{nm}(s) - i[Z(s), \rho(s)]_{nm},
\]

(4)

where \( Z(s) \) is an Hermitian matrix with elements

\[
Z_{nm}(s) = i\langle \dot{E}_n(s)|E_m(s)\rangle
\]

(5)

and \( \rho(s) \) is the matrix with elements \( \rho_{nm}(s) \).

Consider an ordinary differential equation

\[
\frac{d}{ds} x(s) = L_1(s)x(s) + L_2(s)x(s),
\]

(6)

where \( L_1(s) \) and \( L_2(s) \) are s-dependent linear operators acting on the vector \( x \). If the space is finite-dimensional, \( L_1(s) \) and \( L_2(s) \) are sufficiently well behaved as functions of \( s \), and if \( L_1(s) \) fulfills \( [L_1(s), L_1(s')] = 0 \) for all \( s, s' \), then Eq. 6 can be rewritten as the following integral equation

\[
x(s) = e^{\int_0^s L_1(s')ds'} x(0)
+ e^{\int_0^s L_1(s')ds'} \int_0^s e^{-\int_0^{s'} L_1(s'')ds''} L_2(s')x(s')ds'.
\]

(7)

In the present case we have \( x(s) = \rho(s) \) and

\[
(L_1(s)\rho(s))_{nm} = -iTA\{E_n(s) - E_m(s)\}\rho_{nm}(s)
- TBB\{w_n(s) - w_m(s)\}^2\rho_{nm}(s),
\]

(8)

\[
L_2(s)\rho(s) = - [Z(s), \rho(s)].
\]

(9)

Hence, Eq. 6 can be rewritten as

\[
\rho_{nm}(s) = e^{-i\int_0^s TA(s')Q_{nm}(s')ds'}\rho_{nm}(0)
- i\int_0^s e^{-i\int_0^{s'} TA(s'')Q_{nm}(s'')} [Z(s'), \rho(s')]_{nm}ds',
\]

(10)

where

\[
Q_{nm}(s) = \int_0^s [w_n(s') - w_m(s')]^2ds',
\]

(11)

\[
R_{nm}(s) = \int_0^s [E_n(s') - E_m(s')]ds'.
\]

(12)

III. THE SEARCH PROBLEM

Our task is to find a single marked item in a collection of \( N \) items. The \( N \)-element search problem is associated with an \( N \)-dimensional Hilbert space with orthonormal basis \( \{|k\rangle \}_{k=1}^N \), where the marked item corresponds
yielding the orthonormal basis 

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |k\rangle. \quad (14)$$

The ground state $|\psi\rangle$ of the initial Hamiltonian $H(0)$ is easy to prepare, while the ground state of $H(1)$ gives the solution of the search problem. Hence, if we assume the evolution is adiabatic, this family of Hamiltonians takes us from the initial state $|\psi\rangle$ to the marked state $|\mu\rangle$, and thus solves the search problem. A crucial observation is that the problem is essentially two-dimensional, since the space $H_R$ spanned by $|\psi\rangle$ and $|\mu\rangle$ is the only relevant subspace. We may Gram-Schmidt orthogonalize $\{|\psi\rangle, |\mu\rangle\}$, yielding the orthonormal basis

$$\{|\psi\rangle, |\bar{\psi}\rangle\}, \quad |\bar{\psi}\rangle = \frac{\sqrt{N}|\mu\rangle - |\psi\rangle}{\sqrt{N-1}}. \quad (15)$$

We represent the Hamiltonian family in Eq. (13), restricted to the space $H_R$, as matrices in the basis defined in Eq. (13). After the substitution $s = t/T$, we get

$$M(s) = \begin{bmatrix} \frac{sN-1}{s\sqrt{N}} - s\frac{\sqrt{N-1}}{N} \\ -s\frac{\sqrt{N-1}}{N} \end{bmatrix}, \quad (16)$$

which has the eigenvalues

$$E_0(s) = -\frac{1}{2} - \frac{1}{2} \Delta(s), \quad E_1(s) = -\frac{1}{2} + \frac{1}{2} \Delta(s) \quad (17)$$

and orthonormal eigenvectors

$$e^{(0)} = \frac{\sqrt{2}}{\sqrt{\Delta^2 - \Delta D}} \begin{bmatrix} \sqrt{N} \Delta - D \\ s \frac{\sqrt{N-1}}{N} \end{bmatrix} = \begin{bmatrix} e^{(0)}_1(s) \\ e^{(0)}_2(s) \end{bmatrix}, \quad (18)$$

$$e^{(1)} = \frac{\sqrt{2}}{\sqrt{\Delta^2 + \Delta D}} \begin{bmatrix} \sqrt{N} \Delta + D \\ -s \frac{\sqrt{N-1}}{N} \end{bmatrix} = \begin{bmatrix} e^{(1)}_1(s) \\ e^{(1)}_2(s) \end{bmatrix},$$

where

$$\Delta(s) = E_1(s) - E_0(s) = \sqrt{\frac{1 + (N-1)(2s - 1)^2}{N}}, \quad (19)$$

$$D(s) = -1 + 2s \frac{N-1}{N}. \quad (20)$$

The two relevant eigenvectors of $H(s)$ are

$$|E_0(s)\rangle = e^{(0)}_1(s)|\psi\rangle + e^{(0)}_2(s)|\bar{\psi}\rangle,$$

$$|E_1(s)\rangle = e^{(1)}_1(s)|\psi\rangle + e^{(1)}_2(s)|\bar{\psi}\rangle, \quad (21)$$

where $|E_0(s)\rangle$ is the ground state.

We next calculate the matrix $Z(s)$, defined in Eq. (15). In order to save some efforts we first notice that $Z_{00}(s)$ as well as $Z_{11}(s)$ are identically zero, since $e^{(0)}_0$, $e^{(1)}_1$, and $e^{(1)}_1$ are real valued. Secondly $Z(s)$ has to be an Hermitian matrix, which follows from differentiation of the relation $\langle E_n(s)|E_m(s)\rangle = \delta_{nm}$.

By combining Eq. (15) with Eqs. (18), (19), and (20), one obtains the following result

$$Z_{01}(s) = \frac{-i\sqrt{N-1}}{1 + (N-1)(2s - 1)^2}. \quad (22)$$

Define

$$Z(s) = |Z_{01}(s)| = \frac{\sqrt{N-1}}{1 + (N-1)(2s - 1)^2}. \quad (23)$$

Note that the maximum $\sqrt{N-1}$ of $Z(s)$ is obtained at $s = 1/2$. A useful property of $Z$ is

$$\int_0^1 Z(s)ds \leq \frac{\pi}{2}, \quad (24)$$

for all $N \geq 1$.

Concerning the reduction to the two-dimensional problem, one should note that, due to the assumption $|H(s), W(s)\rangle = 0$, the operator $W(s)$ block-diagonalizes with respect to the eigenspaces of $H(s)$. This implies that the solution of Eq. (22) remains in the subspace $H_R$ once started there. Thus, we only need to consider the master equation in Eq. (22) on this particular subspace.

Define

$$Q(s) = Q_{10}(s), \quad R(s) = R_{10}(s), \quad (25)$$

$$Y(s) = \rho_{00}(s) - \rho_{11}(s) = 2\rho_{00}(s) - 1. \quad (26)$$

From Eq. (10) one can derive the following equation for $Y(s)$,
\[
Y(s) = Y(0) - 4 \int_0^s e^{-TBQ(s')} Z(s') \text{Re} \left[ e^{-iT A R(s')} \rho_{10}(0) \right] ds' \\
- 4 \int_0^s e^{-TBQ(s')} \int_0^{s'} e^{TBQ(s'')} \text{Re} \left[ e^{-iT A R(s'')} e^{iT A R(s'')} \right] Z(s') Z(s'') Y(s'') ds'' ds'.
\]

(27)

As seen from Eq. (26), the value of \( Y(s) \) determines to what degree the system is in the ground state, i.e., \( Y = 1 \) when the system is in the ground state, and \( Y = -1 \) when the system is in the first excited state.

IV. SEMI-OPEN SYSTEM

A. Global search

We begin with a comment on our terminology regarding the “openness” of quantum systems. With a “wide-open” \( 11 \) system we intend a system governed by Eq. (1), with \( A = 0 \). As a contrast to the wide-open case we define “semi-open” as \( A > 0 \). Finally, a “closed” system denotes the case \( B = 0 \).

We define

\[
\Gamma(s) = w_1(s) - w_0(s),
\]

(28)

\[
R(s) = \int_0^s \Delta(s') ds',
\]

(29)

\[
Q(s) = \int_0^s \Gamma^2(s') ds',
\]

(30)

\[
\delta = \min_{s \in [0,1]} \Delta(s) = \frac{1}{\sqrt{N}},
\]

(31)

\[
\gamma = \min_{s \in [0,1]} \Gamma(s).
\]

(32)

In the proofs we use the following restriction on the decoherence term

\[
\int_0^1 Z(s) \left| \frac{d}{ds} \Gamma^2(s) \right| ds \leq K,
\]

(33)

where \( K \) is a constant independent of \( N \). This assumption essentially means that the fluctuations of \( \Gamma \) in \( s \) are not allowed to grow with the list-length \( N \). The restriction to an \( N \)-independent \( K \) can be relaxed. We discuss this issue, as well as the condition in general, in Sec. IV C.

From Eq. (27) it follows that

\[
|Y(0) - Y(s)| \leq 4 |J(s)| + 4 |I(s)|,
\]

(34)

where

\[
I(s) = \frac{1}{2} I_+(s) + \frac{1}{2} I_-(s),
\]

(35)

\[
I_\pm(s) = \int_s^1 e^{-T \Delta(s') \mp i A R(s')} Z(s') u_\pm(s') ds',
\]

(36)

\[
u_\pm(s') = \int_0^{s'} e^{T \Delta(s'') \mp i A R(s'')} Z(s'') Y(s'') ds'',
\]

(37)

\[
J(s) = \int_0^1 e^{-TBQ(s')} Z(s') \text{Re} \left[ e^{-iT A R(s')} \rho_{10}(0) \right] ds'.
\]

(38)

For the sake of simplicity we separate the treatment of the terms \( I \) and \( J \).

1. The term \( I \)

To begin with we define the following integral and perform a partial integration
\[
G_{\pm}[g](s) = \int_0^s e^{-T[BQ(s') \pm iAR(s')]} Z(s') g(s') ds' \\
= \int_0^s e^{-T[BQ(s') \pm iAR(s')]} \left[ B \frac{dQ}{ds'}(s') \pm iA \frac{dR}{ds'}(s') \right] \frac{Z(s') g(s')}{B \frac{dQ}{ds'}(s') \pm iA \frac{dR}{ds'}(s')} ds' \\
= -\frac{1}{T} e^{-T[BQ(s) \pm iAR(s)]} \frac{Z(s) g(s)}{B \frac{dQ}{ds}(s) \pm iA \frac{dR}{ds}(s)} \\
+ \frac{1}{T} \sqrt{N-1} \frac{g(s)}{B \frac{dQ}{ds}(s) \pm iA \frac{dR}{ds}(s)} \bigg|_{s=0} \\
+ \frac{1}{T} \int_0^s e^{-T[BQ(s) \pm iAR(s)]} \frac{d}{ds'} \left( \frac{Z(s')}{B \frac{dQ}{ds'}(s') \pm iA \frac{dR}{ds'}(s')} \right) g(s') ds' \\
+ \frac{1}{T} \int_0^s e^{-T[BQ(s) \pm iAR(s)]} \frac{Z(s')}{B \frac{dQ}{ds'}(s') \pm iA \frac{dR}{ds'}(s')} \frac{d}{ds'} g(s') ds'.
\]  

(39)

If we put \( g(s) = u_{\pm}(s) \) into Eq. (39), the result is

\[
I_{\pm}(s) = G_{\pm}[u_{\pm}](s) \\
= -\frac{1}{T} e^{-T[BQ(s) \pm iAR(s)]} \frac{Z(s) g(s)}{B \frac{dQ}{ds}(s) \pm iA \frac{dR}{ds}(s)} \int_0^s e^{T[BQ(s') \pm iAR(s')]} Z(s') Y(s') ds' \\
+ \frac{1}{T} \int_0^s e^{-T[BQ(s) \pm iAR(s)]} \frac{d}{ds'} \left( \frac{Z(s')}{B \frac{dQ}{ds'}(s') \pm iA \frac{dR}{ds'}(s')} \right) \int_0^{s'} e^{T[BQ(s'') \pm iAR(s'')]} Z(s'') Y(s'') ds'' ds' \\
+ \frac{1}{T} \int_0^s \frac{Z(s')}{B \frac{dQ}{ds'}(s') \pm iA \frac{dR}{ds'}(s')} Z(s') Y(s') ds'.
\]  

(40)

In the next step we calculate an upper bound to \( |I_{\pm}(s)| \). To do this we use that \( |Y(s)| \leq 1 \). Moreover, since \( s' \geq s'' \) it follows that \( Q(s') \geq Q(s'') \), which implies \( \exp\{-TB[Q(s') - Q(s'')]\} \leq 1 \). We also use that \( dQ/ds = \Gamma^2(s), dR/ds = \Delta(s) \), and Eq. (24) to obtain

\[
|I_{\pm}(s)| \leq \frac{1}{2} \int_0^s \frac{d}{ds'} \left( \frac{Z(s')}{B \Gamma^2(s') \pm iA \Delta(s')} \right) ds' \\
+ \frac{1}{T} \int_0^s \frac{Z(s')}{B \Gamma^2(s') \pm iA \Delta(s')} ds' \\
+ \frac{1}{2} \int_0^s \frac{Z(s)}{B \Gamma^2(s) \pm iA \Delta(s)} ds'.
\]  

(41)

Now we use the fact that \( Z(s) \leq \sqrt{N-1} \), as well as Eqs. (31) and (32), which result in

\[
|I_{\pm}(s)| \leq \frac{1}{2} \int_0^s \frac{d}{ds'} \left( \frac{Z(s')}{B \Gamma^2(s') \pm iA \Delta(s')} \right) ds' \\
+ \frac{\sqrt{N-1}}{T} \sqrt{B^2 \gamma^4 + A^2 \Delta^2}.
\]  

(42)

To treat the integrand in Eq. (42), we define

\[
X(s) = \left| \frac{d}{ds} \left( \frac{Z(s)}{B \Gamma^2(s) \pm iA \Delta(s)} \right) \right|,
\]

which can be estimated as

\[
X(s) \leq \frac{1}{\sqrt{B^2 \gamma^4 + A^2 \delta^2}} \left| \frac{d}{ds} Z(s) \right| \\
+ \frac{B}{B^2 \gamma^4 + A^2 \delta^2} \frac{Z(s)}{ds} \left| \frac{d}{ds} \Gamma^2(s) \right| \\
+ \frac{A}{B^2 \gamma^4 + A^2 \delta^2} Z(s) \left| \frac{d}{ds} \Delta(s) \right|.
\]  

(43)

The function \( Z(s) \) is symmetric around \( s = 1/2 \) and increasing on the interval \([0, 1/2]\). It follows that the function \( |dZ/ds| \) is also symmetric around \( s = 1/2 \), and that \( |dZ/ds| = dZ/ds \) on \([0, 1/2]\). Thus

\[
\int_0^s \left| \frac{d}{ds} Z(s') \right| ds' \leq \int_0^1 \left| \frac{d}{ds'} Z(s') \right| ds' \leq 2 \sqrt{N-1}.
\]  

(44)

Since both \( Z(s) \) and \( \Delta(s) \) are symmetric around \( s = 1/2 \), \( Z(s)|d\Delta/ds| \) has the same symmetry. Moreover, \( \Delta(s) \) is increasing on the interval \([1/2, 1]\). Hence, \( Z(s)|d\Delta/ds| = Z(s) \Delta/ds \) on \([1/2, 1]\), which leads to

\[
\int_0^s Z(s) \left| \frac{d}{ds} \Delta(s) \right| ds \leq \int_0^1 Z(s) \left| \frac{d}{ds} \Delta(s) \right| ds \leq 2 \sqrt{N-1} \frac{N}{N} \leq 2.
\]  

(45)
By combining Eq. (33) with Eqs. (13), (14), and (15), one obtains

\[ \int_0^s X(s')ds' \leq \frac{2\sqrt{N-1}}{\sqrt{B^2\gamma^4 + A^2}} + \frac{BK + 2A}{B^2\gamma^4 + A^2}. \]  \hfill (46)

By combining Eqs. (31), (35), (42), and (40) it follows that

\[ |I(s)| \leq \frac{\pi N}{2T} \left( \frac{4}{\sqrt{B^2\gamma^4 N + A^2}} + \frac{BK + 2A}{B^2\gamma^4 N + A^2} \right). \]  \hfill (47)

2. The term \( J \)

Equation (38) can be rewritten as

\[ J(s) = \text{Re} \left( \rho_{10}(0) \int_0^s e^{-T[BJ(s')+iAR(s')]}Z(s')ds' \right). \]  \hfill (48)

We write \( |J(s)| \leq |\rho_{10}(0)| |G_+(1)(s)| \), with \( G \) as in Eq. (39) with \( g(s) \equiv 1 \). By a line of reasoning very similar to the one in the previous section one obtains

\[ |G_+(1)(s)| \leq \frac{1}{T} \int_0^s \left| \frac{d}{ds'} \left( \frac{Z(s')}{BT^2(s') + iA\Delta(s')} \right) \right| ds'. \]

\[ + \frac{1}{T} \frac{1}{\sqrt{B^2\gamma^4(0) + A^2}} \left( \frac{Z(s)}{\sqrt{B^2\gamma^4(s) + A^2}} \right), \]  \hfill (49)

which results in

\[ |J(s)| \leq |\rho_{10}(0)| \left( \frac{1}{T\sqrt{N}} \frac{1}{\sqrt{B^2\gamma^4(0) + A^2}} \right)^3 \]

\[ + |\rho_{10}(0)| N \frac{1}{T} \left( \frac{1}{\sqrt{B^2\gamma^4 N + A^2}} \right) \]

\[ + |\rho_{10}(0)| N \frac{1}{T} \left( \frac{BK + 2A}{B^2\gamma^4 N + A^2} \right). \]  \hfill (50)

3. Collecting the results

By combining Eqs. (34), (47), and (50), one obtains

\[ |Y(0) - Y(s)| \leq 4|\rho_{10}(0)| \left( \frac{1}{T\sqrt{N}} \frac{1}{\sqrt{B^2\gamma^4(0) + A^2}} \right)^3 \]

\[ + 4|\rho_{10}(0)| N \frac{1}{T} \left( \frac{1}{\sqrt{B^2\gamma^4 N + A^2}} \right) \]

\[ + 4|\rho_{10}(0)| N \frac{1}{T} \left( \frac{BK + 2A}{B^2\gamma^4 N + A^2} \right) \]

\[ + 8N \frac{1}{T} \left( \frac{1}{\sqrt{B^2\gamma^4 N + A^2}} \right) \]

\[ + 2N \frac{2}{T} \frac{BK + 2A}{B^2\gamma^4 N + A^2}. \]  \hfill (51)

Note that

\[ B^2\gamma^4 N + A^2 \geq A^2, \]  \hfill (52)

\[ B^2T^4(0) + A^2 \geq A^2, \]  \hfill (53)

which, as long as \( A > 0 \), implies that

\[ |\rho_{00}(0) - \rho_{00}(s)| \leq \frac{1}{T^{\sqrt{N}}} \frac{1}{A} \left( \frac{1}{A} N \frac{BK + 5A}{A^2} + \frac{N BK + 5A}{A^2} \right), \]  \hfill (54)

where we have used Eq. (20). The leading term in Eq. (51) is proportional to \( N/T \). In the \( N/T \to \infty \) limit the probability to find the system in the groundstate is conserved. Thus, it is a sufficient condition for adiabaticity that \( T \gg N \).

B. Local search

As is well known [1], the standard global adiabatic search algorithm performs as the classical search, but the local adiabatic search [2, 8] outperforms the classical algorithms. Here we further elaborate and extend the analysis of Ref. [2] concerning the effect of decoherence on the local adiabatic search. The idea behind the local search is to adjust the speed of parameter change in such a way that more time is spent near the minimum energy gap. This is obtained by a reparametrization of the functions \( H(s) \) and \( W(s) \) to \( H(f(s)) \) and \( W(f(s)) \), where \( f \) is a strictly increasing, sufficiently smooth function, such that \( f(0) = 0 \) and \( f(1) = 1 \). The new master equation becomes

\[ \frac{d}{ds}\rho(s) = -iT[H(f(s)), \rho(s)] \]

\[ -BT[W(f(s)), [W(f(s)), \rho(s)]]. \]  \hfill (55)

By a change of variables \( r = f(s) \) and by defining \( \tilde{\rho}(r) = \rho(f^{-1}(r)) \), Eq. (55) becomes

\[ \frac{d}{dr}\tilde{\rho}(r) = -iT\frac{d}{dr}\rho(f^{-1}(r))[H(r), \tilde{\rho}(r)] \]

\[ -BT\frac{d}{dr}\rho(f^{-1}(r))[W(r), [W(r), \rho(r)]]. \]  \hfill (56)

Note that \( \tilde{\rho}(0) = \rho(0) \) and \( \tilde{\rho}(1) = \rho(1) \).

The equation for \( \tilde{Y}(r) = 2\rho_{00}(r) - 1 \) is fully analogous to Eq. (27) with \( Q \) and \( R \) replaced by \( \tilde{Q} \) and \( \tilde{R} \), where

\[ \tilde{Q}(r) = \int_0^r r^2(r') \frac{d}{dr}\rho f^{-1}(r') dr', \]  \hfill (57)

\[ \tilde{R}(r) = \int_0^r r^2(r') \frac{d}{dr}\rho f^{-1}(r') dr'. \]  \hfill (58)
The choice
\[ f^{-1}(r) = \frac{1}{T} \int_0^r \frac{1}{\Delta^2(r')} dr', \quad (59) \]
\[ L = \int_0^1 \frac{1}{\Delta^2(r')} dr' \quad (60) \]
gives the optimal efficiency for the quantum search problem \[\Gamma \leq \frac{1}{2} \frac{\pi}{\sqrt{N-1}}\]. One obtains
\[ \tilde{Q}(r) = \frac{1}{L} \int_0^r \frac{\Gamma^2(r')}{\Delta^2(r')} dr', \quad (61) \]
\[ \tilde{R}(r) = \frac{1}{L} \int_0^r \frac{1}{\Delta(r')} dr', \quad (62) \]
\[ L = \frac{N}{\sqrt{N-1}} \arctan(\sqrt{N-1}) \leq \frac{\pi}{2} \frac{N}{\sqrt{N-1}} \quad (63) \]
Define
\[ \zeta = \min_{s \in [0,1]} \frac{\Gamma^2(s)}{\Delta(s)}. \quad (64) \]
As in the global case we let \(|\tilde{Y}(0) - \tilde{\tilde{Y}}(r)| \leq 4|\tilde{I}(r)| + 4|\tilde{\tilde{I}}(r)|, \) and as before we separate the treatment of the terms \(\tilde{I}\) and \(\tilde{\tilde{I}}\). The terms \(\tilde{I}(r)\) and \(\tilde{\tilde{I}}(r)\) are defined as in Eqs. (35) and (37), but with \(Q\) and \(R\) replaced by \(\tilde{Q}\) and \(\tilde{R}\), respectively.

1. The term \(\tilde{I}\)

The analogue of Eq. (40), is obtained by replacing \(Q\) and \(R\) by \(\tilde{Q}\) and \(\tilde{R}\), respectively. We use \(d\tilde{Q}/dr = \Gamma^2(\tilde{r})/[L\Delta^2(\tilde{r})]\), \(d\tilde{R}/dr = 1/[L\Delta(\tilde{r})]\), \(\exp[-TB(\tilde{Q}(r) - \tilde{Q}(r'))] \leq 1\) if \(\tilde{r} \geq \tilde{r}'\), as well as Eq. (24), which result in
\[ |\tilde{I}_\pm(r)| \leq \frac{L \pi}{T} \int_0^r d\tilde{r}' \left| \frac{Z(\tilde{r}')}{B \frac{\Gamma^2(\tilde{r}')}{\Delta(\tilde{r}')} \mp iA \frac{1}{\Delta(\tilde{r}')}} \right| d\tilde{r}' + \frac{1}{T} \int_0^r \left| \frac{Z(\tilde{r}')}{\sqrt{B^2 \frac{\Gamma^2(\tilde{r}')}{\Delta(\tilde{r}')} + A^2}} \right| d\tilde{r}' \quad (65) \]
By using Eq. (64) it follows that
\[ |\tilde{I}_\pm(r)| \leq \frac{L \pi}{T} \int_0^r d\tilde{r}' \left| \frac{Z(\tilde{r}')}{B \frac{\Gamma^2(\tilde{r}')}{\Delta(\tilde{r}')} \mp iA} \right| d\tilde{r}' + \frac{1}{\sqrt{B^2 \zeta^2 + A^2}} \int_0^r Z^2(\tilde{r}') \Delta(\tilde{r}') d\tilde{r}' \]
\[ + \frac{L \pi}{T} \int_0^r \left| \frac{Z(\tilde{r}')}{B \frac{\Gamma^2(\tilde{r}')}{\Delta(\tilde{r}')} \mp iA} \right| d\tilde{r}' + \frac{L \pi}{T} \int_0^r \frac{Z(\tilde{r}') \Delta(\tilde{r}')}{\sqrt{B^2 \frac{\Gamma^2(\tilde{r}')}{\Delta(\tilde{r}')} + A^2}} d\tilde{r}' \quad (66) \]
One may use \(Z(\tilde{r}) \Delta(\tilde{r}) \leq \sqrt{(N-1)/N}\) and Eq. (24) to obtain
\[ |\tilde{I}_\pm(r)| \leq \frac{L \pi}{T} \int_0^r d\tilde{r}' \left| \frac{Z(\tilde{r}')}{B \frac{\Gamma^2(\tilde{r}')}{\Delta(\tilde{r}')} \mp iA} \right| d\tilde{r}' + \frac{\pi L}{T} \frac{1}{\sqrt{N-1}} \frac{1}{\sqrt{B^2 \zeta^2 + A^2}} \quad (67) \]
The derivative contained in the integrand on the right-hand side of the above equation is performed by first multiplying both the numerator and the denominator by \(\Delta(\tilde{r}')\) and then using \(Z(\tilde{r}) \Delta^2(\tilde{r}) = \sqrt{(N-1)/N}\). Moreover, we use Eq. (64). The result is
\[ \int_0^r \left| \frac{d}{d\tilde{r}'} \left( \frac{Z(\tilde{r}')}{B \frac{\Gamma^2(\tilde{r}')}{\Delta(\tilde{r}')} \pm iA} \right) \right| d\tilde{r}' \leq \frac{B}{B^2 \zeta^2 + A^2} \int_0^1 Z(\tilde{r}) \left| \frac{d}{d\tilde{r}'} \frac{1}{\Delta^2(\tilde{r})} \right| d\tilde{r}' \]
\[ + \frac{A}{B^2 \zeta^2 + A^2} \int_0^1 Z(\tilde{r}) \left| \frac{d}{d\tilde{r}'} \Delta(\tilde{r}) \right| d\tilde{r}'. \quad (68) \]
Note that in the inequality above we have extended the integration interval from \([0, r]\) to \([0, 1]\). By combining Eqs. (35), (44), (65), and (67) one obtains
\[ |\tilde{I}(r)| \leq \frac{\pi^2 \sqrt{N}}{T} \frac{1}{\sqrt{B^2 \zeta^2 + A^2}} + \frac{A}{B^2 \zeta^2 + A^2} \quad (69) \]
2. The term \(\tilde{\tilde{I}}\)

As in the case of global search one obtains \(|\tilde{\tilde{I}}(r)| \leq |\tilde{\tilde{\tilde{I}}}(0)| \left| \frac{G + 1}{1} \right|(r)\), with \(G\) as in Eq. (38) with \(g(r) \equiv 1\), and \(Q\) and \(R\) replaced by \(\tilde{Q}\) and \(\tilde{R}\). Note that \(\tilde{\tilde{\tilde{I}}}(0) = \rho(\tilde{f}^{-1}(0)) = \rho(0)\) and thus \(\tilde{\tilde{\tilde{I}}}(0) = \tilde{\tilde{I}}(0)\). Using similar
arguments as in the previous sections, one obtains
\[ |\mathcal{G}_+[1](r)| \leq \frac{L}{T} \int_0^r \frac{d}{dr'} \left( \frac{Z(r') \Delta(r')}{B^2 \Delta^2(r') + iA} \right) dr' + \frac{L}{T} \frac{\sqrt{N-1}}{\sqrt{N}} \frac{1}{\sqrt{B^2 T^2(0) + A^2}} + \frac{L}{T} \frac{Z(r) \Delta(r)}{\sqrt{B^2 \xi^2 + A^2}} \] (70)

Next we use \( Z(r) \Delta(r) \leq \sqrt{(N-1)/N} \) together with Eqs. (33), (45), (63), and (68), yielding
\[ |\mathcal{J}(r)| \leq \frac{\pi}{2} |\bar{\rho}_{00}(0)| \frac{\sqrt{N}}{T} \frac{1}{\sqrt{B^2 \xi^2 + A^2}} + \frac{\pi}{2} |\bar{\rho}_{00}(0)| \frac{1}{T} \frac{1}{\sqrt{B^2 T^2(0) + A^2}} + \frac{\pi}{2} |\bar{\rho}_{00}(0)| \frac{\sqrt{N}}{T} \frac{BK}{\sqrt{B^2 T^2(0) + A^2}} \frac{N}{N-1} + \frac{\pi}{2} |\bar{\rho}_{00}(0)| \frac{2A}{\sqrt{B^2 \xi^2 + A^2}} \] (71)

3. Collecting the results

We combine Eqs. (33) and (71) together with \( \bar{Y}(0) - \bar{Y}(r) \leq 4|\mathcal{J}(r)| + 4|\mathcal{J}(r)| \), \( A > 0 \), \( B^2 \xi^2 + A^2 \geq A^2 \), and \( B^2 T^2(0) + A^2 \geq A^2 \). Furthermore, we assume \( N \geq 2 \), which gives \( \sqrt{N/(N-1)} \leq \sqrt{2} \), and results in \( \bar{Y} \)
\[ |\bar{\rho}_{00}(0) - \bar{\rho}_{00}(r)| \leq \left( |\bar{\rho}_{00}(0)| + \frac{\pi}{2} \right) \sqrt{2\pi BK \frac{\sqrt{N}}{A^2 T}} + (3|\bar{\rho}_{00}(0)| + 2\pi) \frac{\sqrt{N}}{A T} + |\bar{\rho}_{00}(0)| \frac{1}{A T} \] (72)

The leading term of this equation is proportional to \( \sqrt{N/T} \). Hence, a sufficient condition for adiabaticity is \( T \gg \sqrt{N} \). Equation (72) can be expressed in terms of the solution \( \rho(s) \) of Eq. (65) by the substitution \( r = f(s) \). However, this change of variables does not affect the condition for adiabaticity.

C. The condition

Both for the global and local search we have restricted the decoherence term of the master equation to fulfill the condition in Eq. (68). This condition puts a limit on the fluctuations of \( \Gamma \) with respect to \( s \), as a function of the list-length \( N \). However, one may note that the condition can be relaxed, in the sense that we may allow \( K \) to be dependent on \( N \). By inserting \( K = cN^a \), where \( c \geq 0 \) and \( a \geq 0 \) are constants, into Eqs. (54) and (73), one finds that the new scaling becomes \( N^{1+a} \) and \( N^{1/2+a} \) for the global and local search, respectively. Thus, the scalings get worse, but one finds that there is a “window” \( 0 < a < 1/2 \) where the local search still outperforms the classical search, although not with the optimal efficiency of the Grover search [17].

In the following we attempt to obtain an indication of to what extent the condition in Eq. (68) is restrictive or not. For the sake of simplicity we do so for the more restrictive case where \( K \) is independent of \( N \). We assume \( \Gamma(s) = \eta(\Delta(s)) \), where \( \eta : (0, \infty) \to (0, \infty) \), is an increasing and sufficiently smooth function. Now, a larger value of \( B^2 T^2(0) \) implies stronger decoherence. Thus, we have chosen to consider a decoherence model for which the strength of the decoherence increases with the energy difference between the two instantaneous eigenstates, but we do not specify any details on how it grows. We intend to derive a sufficient condition on \( \eta \) for \( \Gamma \) to fulfill the condition in Eq. (68).

By the assumptions on \( \eta \) and the form of \( \Delta \), it follows that \( \Gamma \) is decreasing on the interval \([0, 1/2]\) and increasing on \([1/2, 1]\). Partial integration implies
\[ \int_0^1 Z(s) \frac{d}{ds} \Gamma^2(s) \, ds = 2 \int_{1/2}^1 Z(s) \frac{d}{ds} \eta^2(\Delta(s)) \, ds \]
\[ = -2\sqrt{N-1} \eta^2(N^{-1/2}) -2 \int_{1/2}^1 \eta^2(\Delta(s)) \frac{d}{ds} Z(s) \, ds \]
\[ + 2\sqrt{N-1} \eta^2(1). \] (73)

We use \( \eta^2(N^{-1/2}) \geq 0 \) and \( 2\eta^2(1) \sqrt{N-1} = 2\eta^2(1) \). Moreover, we use \( Z(s) \Delta^3(s) = Z(N-1)/N \), which leads to
\[ \int_0^1 Z(s) \frac{d}{ds} \Gamma^2(s) \, ds \leq 4 \frac{\sqrt{N-1}}{N} \int_{1/2}^1 \eta^2(\Delta(s)) \frac{d}{ds} \Delta^3(s) \, ds \]
\[ + 2\eta^2(1). \] (74)

Only the integral on the right-hand side of the above expression may potentially grow when \( N \) increases. If we treat this term separately we find
\[ \frac{\sqrt{N-1}}{N} \int_{1/2}^1 \eta^2(\Delta(s)) \frac{d}{ds} \Delta^3(s) \, ds \leq \frac{1}{\sqrt{N}} \int_{1/\sqrt{N}}^1 \eta^2(\Delta) \Delta^3 \, d\Delta. \]

This implies that a sufficient condition for the inequality in Eq. (68) to hold, is that the expression on the right-hand side of the above equation, is bounded. From the assumptions on \( \eta \) it follows that the limiting behavior of this integral is determined by the asymptotic behavior of the function \( \eta \) as \( \Delta \to 0 \). Assume that \( \eta(x) \leq C x^\sigma \) for some constants \( C \) and \( \sigma \geq 0 \). Then it follows that
\[ \frac{1}{\sqrt{N}} \int_{1/\sqrt{N}}^1 \eta^2(x) x^3 \, dx \leq \begin{cases} \frac{C^2 N^a}{2} \frac{N^{1+a}}{N^{3/2}}, & \sigma = 1, \\ \frac{C^2 N^{a/2}}{2(\sigma-1)} \left( N^{1+a} - N^{a/2} \right), & \sigma \neq 1. \end{cases} \] (75)
In the case where $\sigma = 1$, the function $\Gamma$ satisfies the inequality in Eq. (33), since $(\ln N)/\sqrt{N}$ is a bounded function on the nonzero natural numbers. In the case where $\sigma \neq 1$, one can see that the right-hand side of Eq. (15) contains two terms. None of these terms increase in magnitude, with increasing $N$, if $\sigma \geq 1/2$. Hence, we may conclude that for all $\sigma \geq 1/2$ the condition in Eq. (33) is fulfilled. In other words, for a sufficiently smooth, positive, and increasing function $\eta$, the only condition we have required is $\eta(x) \leq Cx^\sigma$ with $\sigma \geq 1/2$.

V. THE WIDE-OPEN CASE WITH $\Gamma(s) = \Delta^\sigma(s)$

The wide-open case is obtained if we let $A = 0$ (and $B = 1$ for convenience) which in the global search case yields the master equation

$$\frac{d}{ds} \rho(s) = -T [W(s), [W(s), \rho(s)]]$$

(76)

and in the local search case yields

$$\frac{d}{dr} \bar{\rho}(r) = -T \frac{df^{-1}}{dr} [W(r), [W(r), \bar{\rho}(r)]]$$

(77)

where we have changed variables to $r = f(s)$ as described in Sec. [IVB].

Even with the assumption that $W(s)$ should be simultaneously diagonalizable with $H(s)$, we still have a rather large freedom in the choice of $W$, in terms of how $\Gamma$ depends on the parameter $s$ and the list-length $N$. In the previous sections it was only necessary to require that $\Gamma$ should fulfill the condition in Eq. (33). In this case, however, we need to specify more about $W$.

Our main object of study in this section is the case $W(s) = H(s)$, which corresponds to $\Gamma(s) = \Delta(s)$. However, in order to prove that in the wide-open case the asymptotic behavior is dependent on the choice of $W$, we consider the generalization $\Gamma(s) = \Delta^\sigma(s)$. Note in particular that if $\sigma = n$, where $n$ is an odd natural number, then $\Gamma(s) = \Delta^n(s)$ can be obtained with $W(s) = 2^{n-1}(H(s) + (1/2)I)^n$.

Here we prove that in the wide-open case, with $\Gamma(s) = \Delta^\sigma(s)$, the global and local conditions for adiabaticity are $T \gg N^{\sigma+1/2}$ and $T \gg N^\sigma$, respectively. We first prove that these are sufficient conditions. Thereafter follows a proof of necessity.

A. Global search: Sufficiency

Let $A = 0$ and $B = 1$ in Eq. (24) and assume that $\Gamma(s) = \Delta^\sigma(s)$, which gives

$$Q(s) = \int_0^s \Delta^{2\sigma}(s')ds'.$$

(78)

Assume $\sigma \geq 0$, and use $\Delta(s) \geq \delta = 1/\sqrt{N}$. It follows that $Q(s) \geq N^{-\sigma} s$, for all $s \geq 0$. Moreover, since $s' \geq s''$, it follows that $Q(s'') - Q(s') \geq N^{-\sigma}(s' - s'')$. This can be used, together with $|Y(s)| \leq 1$, to show that Eq. (27) leads to

$$|Y(0) - Y(s)| \leq 4 \int_0^s \int_0^s e^{-T N^{-\sigma}(s'' - s')} Z(s') Z(s'') ds'' ds' + 4|\rho_{10}(0)| \int_0^s e^{-T N^{-\sigma} s'} Z(s') ds'.$$

(79)

One may use $Z(s) \leq \sqrt{N-1}$, integrate, and use Eq. (24) to obtain

$$|\rho_{00}(0) - \rho_{00}(s)| \leq (2|\rho_{10}(0)| + \pi) N^{\sigma+1/2} T.$$

(80)

Thus we have proved that a sufficient condition for $\rho_{00}(s)$ to approach $\rho_{00}(0)$ is $T \gg N^{\sigma+1/2}$, if $\sigma \geq 0$.

B. Local search: Sufficiency

Here we require $\sigma$ to fulfill $\sigma \geq 1$. As before we let $A = 0$ and $B = 1$ in Eq. (27), but we replace $Q$ with $\bar{Q}$ defined as

$$\bar{Q}(r) = \int_0^r \Delta^{2\sigma}(r') \frac{df^{-1}}{dr'} dr'$$

$$\geq \frac{1}{L \int_0^r \Delta^{2\sigma-2}(r') dr'}$$

$$\geq \frac{1}{LN^{1-\sigma} r},$$

(81)

with $L$ as in Eq. (83), and where we in the inequality have used $\Delta(s) \geq \delta = 1/\sqrt{N}$ and the assumption $\sigma \geq 1$. Inserting Eq. (81) into Eq. (24) yields

$$|\bar{Y}(0) - \bar{Y}(r)|$$

$$\leq 4 \int_0^r \int_0^r e^{-(r' - r'')T/(LN^{\sigma-1})} Z(r') Z(r'') dr'' dr' + 4|\bar{\rho}_{10}(0)| \int_0^r e^{-r'T/(LN^{\sigma-1})} Z(r') dr'.$$

(82)

By a reasoning very similar to the derivation leading from Eq. (79) to Eq. (80) it follows that

$$|\bar{Y}(0) - \bar{Y}(r)| \leq 2 \pi \frac{LN^{\sigma-1} \sqrt{N - 1}}{T} + 4|\bar{\rho}_{10}(0)| \frac{LN^{\sigma-1} \sqrt{N - 1}}{T},$$

(83)

which, together with Eq. (83), gives

$$|\bar{\rho}_{00}(0) - \bar{\rho}_{00}(r)| \leq (2|\bar{\rho}_{10}(0)| + \pi) \frac{\pi N^\sigma}{2 T}.$$  

(84)

Thus, a sufficient condition for adiabaticity is $T \gg N^\sigma$. 

C. Necessity

In Secs. [V A] and [V B] we proved sufficient conditions for conservation of the probability to find the system in the ground state, throughout the evolution. Here we prove that these sufficient conditions are also necessary, but under the simplifying assumption that we start in the ground state of the system. Moreover, we focus on the end point \( s = 1 \), i.e., we consider the success probability \( p = \rho_0(1) \), and necessary conditions for \( p \) to approach unity.

Due to the effectively two-dimensional Hilbert space of the problem we may use a Bloch vector representation in terms of which the state of the system and the Hermitian operator \( W \) read

\[
\begin{align*}
\rho(s) &= \frac{1}{2} [\hat{1} + \mathbf{v}(s) \cdot \mathbf{\sigma}], \\
W(s) &= \frac{1}{2} [(w_0(s) + w_1(s))\hat{1} + \mathbf{u}(s) \cdot \mathbf{\sigma}],
\end{align*}
\]

(85)

(86)

where \( \mathbf{v} \) and \( \mathbf{u} \) are elements in \( \mathbb{R}^3 \), \( ||\mathbf{v}|| \leq 1 \), and \( \mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) are the Pauli spin operators. It follows that \( \mathbf{u}(s) = -\Gamma(s)\mathbf{q}(s) \), where \( \mathbf{q}(s) \) is the Bloch unit-vector representing the ground state of \( H(s) \).

If we use the assumption \( \Gamma(s) = \Delta^\sigma(s) \), then Eq. (91) can be rewritten in the Bloch representation as

\[
\dot{\mathbf{v}} = -T\xi(s)\{\mathbf{v} - \mathbf{q}(s)\mathbf{q}(s) \cdot \mathbf{v}\},
\]

(87)

where

\[
\xi(s) = \Delta^{2\sigma}(s), \quad \sigma \geq 0.
\]

(88)

In the local case Eq. (97) again becomes Eq. (87), but with the parameter \( s \) replaced by \( r \) and

\[
\xi(r) = \Delta^{2\sigma}(r) \frac{df^{-1}}{dr} = \frac{\Delta^{2\sigma - 2}(r)}{L}, \quad \sigma \geq 1.
\]

(89)

Most of the material presented in this subsection is independent of whether we consider the global or the local search. The difference is exclusively carried by the function \( \xi \). Due to this we will in the rest of this subsection use the variable \( s \), although we should, to be correct, use \( r \) in the local search case. In Sec. [V C 2] we properly use the variable \( r \) again.

Let \( |\psi\rangle \) and \( |\varphi\rangle \) defined in Eq. (13) correspond to the north and south pole of the Bloch sphere, respectively. The Bloch vector \( \mathbf{q} \), corresponding to the instantaneous ground state, can then be expressed as

\[
\mathbf{q}(s) = \frac{1}{\Delta(s)} \left( 2s \frac{\sqrt{N - 1}}{N}, 0, 1 - 2s \frac{N - 1}{N} \right). 
\]

(90)

Note that \( \mathbf{q} \) moves in the \( x-z \) plane only, and that \( \mathbf{q}(0) = (0, 0, 1) \). Let \( \theta(s) \) be the angle between \( \mathbf{q}(s) \) and \( \mathbf{q}(0) \). This angle can be expressed as

\[
\cos(\theta(s)) = \mathbf{q}(s) \cdot \mathbf{q}(0) = \frac{1 - (N - 1)(2s - 1)}{\sqrt{N(\sqrt{1 + (N - 1)(2s - 1)})}}.
\]

(91)

For \( N \geq 2 \) the right-hand side of the above equation is strictly decreasing on the interval \( 0 \leq s \leq 1 \), beginning at the value 1 and ending at the value \( 2/N - 1 \). It follows that we can choose \( \theta(s) \) as an increasing function with range in \([0, \pi]\).

Since the system is started in the ground state \( \rho(0) = |E_0(0)\rangle\langle E_0(0)| \), the initial condition is \( \mathbf{v}(0) = \mathbf{q}(0) \). Since both \( \mathbf{q}(s) \) and \( \mathbf{v}(0) \) are fixed to the \( x-z \) plane it follows, from the form of Eq. (97), that the motion of \( \mathbf{v} \) is fixed to the \( x-z \) plane. Thus, we may restrict our analysis to this plane only.

**Lemma 1** Let \( \mathbf{q}_0, \mathbf{q}', \) and \( \mathbf{q}'' \) be three unit vectors in \( \mathbb{R}^2 \), and let \( \mathbf{v} \) be such that \( ||\mathbf{v}|| \leq 1 \). Suppose that the angle between \( \mathbf{q}_0 \) and \( \mathbf{q}' \) is greater than or equal to the angle between \( \mathbf{q}_0 \) and \( \mathbf{q}'' \). Moreover, let both these angles be in the interval \([0, \pi]\). Then

\[
\mathbf{v} \cdot \mathbf{q}' \geq \mathbf{q}_0 \cdot \mathbf{q}'
\]

implies

\[
\mathbf{v} \cdot \mathbf{q}'' \geq \mathbf{q}_0 \cdot \mathbf{q}''.
\]

(92)

(93)

**Proof.** Consider a coordinate system such that \( \mathbf{q}_0 \) corresponds to \((1, 0)\), \( \mathbf{q}' \) to \((\cos \theta_1, \sin \theta_1)\), and \( \mathbf{q}'' \) to \((\cos \theta_2, \sin \theta_2)\), with \( 0 \leq \theta_1 \leq \theta_2 \leq \pi \). In the cases when \( \theta_1, \theta_2 \) assumes possible combinations of \( 0, \pi \), the lemma is trivially true. Hence, we assume \( 0 < \theta_1 \leq \theta_2 < \pi \). If we let \( \mathbf{v} = (x, y) \), then Eq. (92) becomes \( x \cos \theta_1 + y \sin \theta_1 \geq \cos \theta_1 \). By using \( \sin \theta_1 > 0 \), this can be rewritten as

\[
y \geq (1 - x) \cot \theta_1.
\]

(94)

Since cotangent is a decreasing function on the interval \((0, \pi)\), and since \( 1 - x \geq 0 \), it follows that the right-hand side of Eq. (92) is larger than or equal to \((1 - x) \cot \theta_2\). The above procedure can be reversed to obtain Eq. (93).

**Lemma 2** If \( \mathbf{v}(s) \) is the solution of Eq. (97) with initial condition \( \mathbf{v}(0) = \mathbf{q}(0) \), and with \( \mathbf{q}(s) \) as in Eq. (90), then

\[
\mathbf{v}(s) \cdot \mathbf{q}(s) \geq \mathbf{q}(0) \cdot \mathbf{q}(s), \quad \forall s \in [0, 1].
\]

(95)

**Proof.** The strategy of this proof is to make a Cauchy-Euler polygon approximation of Eq. (99), and show an inequality similar to Eq. (95) for every step size. Since the solution \( \mathbf{v}(s) \) is obtained in the limit of small step sizes, the solution will satisfy the inequality in Eq. (95).

Let \( (s_k)_{k=0}^M \) be a partition of the interval \([0, 1]\) such that \( s_0 = 0 \). The partition has the step size \( \Delta s = 1/M \). Let \( \mathbf{q}_k = \mathbf{q}(s_k) \) and let

\[
\mathbf{v}_0 = \mathbf{q}_0, \\
\mathbf{v}_{k+1} = \mathbf{v}_k - \Delta s T \xi(s_k)[\mathbf{v}_k - \mathbf{q}_k(\mathbf{q}_k \cdot \mathbf{v}_k)].
\]

(96)
Let $v^{(Dk)}(s)$ be the linear interpolation of $v_0, v_1, \ldots, v_k$, meaning that on the interval $[s_k, s_{k+1}]$ the function $v^{(Dk)}(s)$ is defined by
\[ v^{(Dk)}(s) = v_{k+1} \frac{s - s_k}{s_{k+1} - s_k} + v_k \frac{s_{k+1} - s}{s_{k+1} - s_k}. \] (97)

Now we prove that the approximation $v^{(Dk)}(s)$ converges uniformly on the interval $[0, 1]$ to the solution $v(s)$, as $Dk \to 0$. Define
\[ f(s, v) = -T \xi(s) \{ v - q(s) \} |q(s) \cdot v\}. \] (98)

In both the global and the local case the function $f$ can be shown to be continuous on $\mathbb{R} \times \mathbb{R}^3$, and to have a continuous partial derivative with respect to $s$. If one uses the assumption $\sigma \geq 0$ in the global case, and $\sigma \geq 1$ as well as the additional assumption $N \geq 2$ in the local case, one finds
\[ 0 \leq \xi(s) \leq 1. \] (99)

It follows that $f$ fulfills the Lipschitz condition
\[ ||f(s, x) - f(s, y)|| \leq T ||x - y||, \quad (s, x), (s, y) \in [0, 1] \times \mathbb{R}^3. \] (100)

If we assume that $Dk \leq 1/T$, then Eq. (99) implies
\[ ||1 - Dk T \xi(s)|| \leq 1, \quad \forall s \in [0, 1]. \] (101)

By induction using Eqs. (99) and (101) one can show that $||v_k|| \leq 1$. This in turn implies that the linear interpolation $v^{(Dk)}(s)$ fulfills
\[ ||v^{(Dk)}(s)|| \leq 1, \quad \forall s \in [0, 1]. \] (102)

The inequality $||v_k|| \leq 1$, together with Eq. (98), can be used to show that
\[ ||v_{k+1} - v_k|| \leq Dk T \xi(s_k) ||v_k|| \leq (s_{k+1} - s_k) T. \] (103)

This gives
\[ ||v^{(Dk)}(s'') - v^{(Dk)}(s')|| \leq T ||s'' - s'||, \quad s', s'' \in [0, 1]. \] (104)

The fact that the partial derivative of $f$, with respect to $s$, is continuous on $\mathbb{R} \times \mathbb{R}^3$, implies that there exists a constant $C$ such that
\[ ||f(s'', v) - f(s', v)|| \leq C ||s'' - s'||, \] (105)

for all $s', s'' \in [0, 1]$, and all $v \in \mathbb{R}^3 : ||v|| \leq 1$. By combining Eqs. (100)-(105) one obtains
\[ ||f(s'', v^{(Dk)}(s'')) - f(s', v^{(Dk)}(s'))|| \leq (T^2 + C) ||s'' - s'||, \] (106)

for all $s', s'' \in [0, 1]$. This can be used to show that
\[ ||v^{(Dk)}(s'') - v^{(Dk)}(s') - \int_{s'}^{s''} f(s, v^{(Dk)}(s)) ds|| \leq C ||s'' - s'||, \] (107)

for all $s', s'' \in [0, 1]$, and where $C = Dk (T^2 + C)$. Since both $f$ and $v^{(Dk)}$ are continuous, Eq. (107) implies that $v^{(Dk)}$ is an $\epsilon$-approximative solution to Eq. (57).

Since $f$ is continuous and fulfills the Lipschitz condition in Eq. (100) it follows from Eq. (107) that
\[ ||v^{(Dk)}(s) - v(s)|| \leq ||v^{(Dk)}(0) - v(0)|| e^{Ts} + Dk (T^2 + C) (e^{Ts} - 1), \] (108)

for all $s \in [0, 1]$. Since $v(0) = v^{(Dk)}(0) = q(0)$ it follows from Eq. (108) that $v^{(Dk)}(s)$ converges uniformly on $[0, 1]$ to the solution $v(s)$, as $Dk \to 0$. (One may note that the solution $v(s)$ exists and is unique, since $f$ is continuous and Lipschitz on $[0, 1] \times \mathbb{R}^3$.) The solution $v(s)$ is continuous as it is the limit function of the continuous and uniformly converging functions $v^{(Dk)}(s)$ on the subset $[0, 1]$ of $\mathbb{R}$. (See Theorem 7.12 in Ref. [21].)

In the following we focus on the proof of the inequality in Eq. (95). We use the fact that the vectors $v_k, q_k$ are restricted to the $x$-$z$ plane of the Bloch sphere. By induction we prove that $v_{k+1} \cdot q_k \geq v_k \cdot q_k$.

By using Eq. (100) one finds
\[ v_{k+1} \cdot q_k = v_k \cdot q_k. \] (109)

If this is combined with the induction hypothesis $v_k \cdot q_k \geq q_0 \cdot q_k$ we obtain
\[ v_{k+1} \cdot q_k \geq q_0 \cdot q_k. \] (110)

The angle between $q_{k+1}$ and $q_0$ is larger than the angle between $q_k$ and $q_0$, and these angles are in the range $[0, \pi]$. According to Lemma 1 with $q' = q_k$ and $q'' = q_{k+1}$, it follows from Eq. (110) that
\[ v_{k+1} \cdot q_{k+1} \geq q_0 \cdot q_{k+1}. \] (111)

Since the induction hypothesis is true for $k = 0$, we obtain
\[ v_k \cdot q_k \geq q_0 \cdot q_k, \] (112)

for all $0 \leq k \leq M$. This is true regardless of the choice of step size $Dk$.

Suppose there exists a point $s$ for which the inequality $v(s) \cdot q(s) \geq q(0) \cdot q(s)$ does not hold. Then there exists a number $a > 0$, such that $||v(s) - q(0)|| \cdot q(s) = -a < 0$. By continuity of $v$ and $q$, there must exist an interval $s_1 < s < s_2$ such that
\[ ||v(s) - q(0)|| \cdot q(s) \leq -a/2, \quad \forall s \in [s_1, s_2]. \] (113)

Now consider approximations $v^{(Dk)}(s)$ in this interval.
\[ ||v^{(Dk)}(s) - q(0)|| \cdot q(s) \leq ||v^{(Dk)}(s) - v(s)|| \cdot q(s) - a/2 \]
\[ \leq \sup_{s \in [s_1, s_2]} ||v^{(Dk)}(s) - v(s)|| \cdot q(s) - a/2. \] (114)
Since \( v^{(Ds)}(s) \) converges uniformly to \( v(s) \), as \( Ds \to 0 \), there exists a sufficiently small \( Ds \) such that \( \sup \| v^{(Ds)}(s) - v(s) \| < a/2 \). Thus, for a sufficiently small \( Ds \) it follows that \( \| v^{(Ds)}(s) - \bar{q}(0) \| \cdot \bar{q}(s) < 0 \), for all \( s \in [s_1, s_2] \). This contradicts Eq. (112), since we may choose \( s \) to be some element of the partition \( \{(s_k)_{k=0}^M \} \) inside the interval \([s_1, s_2] \). (Such an element exists if \( Ds \) is sufficiently small.) Hence, by contradiction we have proved the statement of the lemma. \( \Box \)

1. Global search

In the following we let \( Y_T \) denote the solution of Eq. (120) for a given run time \( T \), with \( A = 0 \) and \( B = 1 \). Similarly we let \( \rho_T \) denote the solution of Eq. (60), for the run time \( T \).

Lemma 3

\[ Y_T(s) \geq Y_0(s), \quad T \geq 0, \quad 0 \leq s \leq 1. \]  

Proof. If \( \rho_T(0) = |E_0(0)\rangle \langle E_0(0)| \), then, by construction, we have

\[ Y_T(s) = 2|E_0(s)\rangle \langle E_0(s)| \rho_T(s) - 1 = v_T(s) \cdot q(s), \]  

where \( v_T \) is the Bloch vector representing \( \rho_T \). Similarly \( Y_0(s) = v_0(s) \cdot q(s) \), where \( v_0(s) \) is the Bloch vector corresponding to \( \rho_0(s) \) which is the solution of Eq. (60) with \( T = 0 \), and initial condition \( \rho_0(0) = |E_0(0)\rangle \langle E_0(0)| \). The solution is \( \rho_0(s) = |E_0(0)\rangle \langle E_0(0)| \). We may conclude that \( v_0(s) = q(0) \). Hence, \( Y_T(s) \geq Y_0(s) \) is only Eq. (95) in disguise. \( \Box \)

If we combine Eq. (27) with Eq. (115) and let \( A = 0 \), \( B = 1 \), we find the following inequality

\[ 1 - Y_T(1) \geq I_0, \]  

where

\[ I_0 = 4 \int_0^1 \int_0^{y(1)} e^{-T(Q(s')-Q(s''))} Z(s')Z(s'')Y_0(s'')ds'ds'' \]  

and where we have used that \( Y_T(0) = 1 \).

This implies that if \( Y_T(1) \) is to approach 1, then \( I_0 \) has to go to zero, or be negative, since \( Y_T \leq 1 \). However, \( I_0 \geq 0 \), as is shown below.

Equation (91) yields

\[ Y_0(s) = q(s) \cdot q(0) \geq \frac{-(N-1)(2s-1)}{\sqrt{N} \sqrt{1+(N-1)(2s-1)^2}}. \]  

By inserting Eq. (119) into Eq. (118) and by making the change of variables \( x = \sqrt{N-1}(2s'-1) \) and \( y = \sqrt{N-1}(2s''-1) \) one obtains

\[ I_0 \geq \int_{-\sqrt{N-1}}^{\sqrt{N-1}} \int_{-\sqrt{N-1}}^{\sqrt{N-1}} \frac{-ye^{-T[Q(x)-Q(y)]}}{(1+x^2)(1+y^2)^{3/2}} \, dx \, dy. \]  

In the above expression we have introduced the function \( \kappa(x) = 1/2 + x/(2\sqrt{N-1}) \). Furthermore,

\[ TQ(\kappa(x)) = T \int_0^{\kappa(x)} \frac{1}{N^\sigma} \left[ 1 + (N-1)(2s'-1)^2 \right] ds' \]

\[ = \alpha(\Phi(x) - \Phi(-\sqrt{N-1})), \]  

where

\[ \alpha = \frac{T}{2N^\sigma \sqrt{N-1}} \]

and

\[ \Phi(x) = \int_0^{x} (1 + x^2)^{\sigma} \, dx. \]  

Note that \( \Phi(x) = -\Phi(x) \). By using Eq. (104) one can write

\[ TQ(\kappa(x)) - TQ(\kappa(y)) = \alpha [\Phi(x) - \Phi(y)]. \]

Define

\[ I(\alpha, \beta) = \int_{-\beta}^{\beta} f_\alpha(x) \int_{-\beta}^{x} g_\alpha(y) \, dy \, dx, \]

\[ f_\alpha(x) = e^{-\alpha \Phi(x)} \frac{1}{1+x^2}, \quad g_\alpha(y) = e^{\alpha \Phi(y)} \frac{y}{(1+y^2)^{3/2}}. \]

From Eq. (120) it follows that \( I_0 \) fulfills the inequality

\[ I_0 \geq \sqrt{\frac{N-1}{N}} I(\alpha, \sqrt{N-1}). \]  

The definition in Eq. (125) yields

\[ \frac{d}{d\beta} I(\alpha, \beta) = f_\alpha(\beta) \int_{-\beta}^{\beta} y \sinh(\alpha \Phi(y)) \, dy + 2e^{-\alpha \Phi(\beta)} \int_{-\beta}^{\beta} \cosh(\alpha \Phi(x)) \, dx. \]

The second equality in the expression above is obtained by separating the integrals \( \int_{-\beta}^{\beta} \) into \( \int_{-0}^{\beta} \) and make the change of variables \( x \rightarrow -x \) (and \( y \rightarrow -y \)) in the \( 0^\beta \) integrals.

If one uses the inequality

\[ \frac{y}{(1+y^2)^{3/2}} \leq \frac{\beta}{\sqrt{1+\beta^2}} \frac{1}{1+y^2}, \quad \forall y \in [0, \beta]. \]  

in Eq. (127), one obtains

\[ \frac{d}{d\beta} I(\alpha, \beta) \geq 2 \frac{\beta e^{-\alpha \Phi(\beta)}}{(1+\beta^2)^{3/2}} \int_{0}^{\beta} e^{-\alpha \Phi(x)} \sqrt{1+x^2} \, dx = F(\alpha, \beta). \]  

(129)
Since the integrand of the above integral is strictly positive, one can conclude that
\[ \frac{d}{d\beta} I(\alpha, \beta) > 0, \quad \beta > 0. \] (130)

Hence, for a fixed \( \alpha \) the function \( I(\alpha, \beta) \) is strictly increasing. Similarly, \( \sqrt{(N-1)/N} \) increases with \( N \). If we assume \( N \geq 2 \) it follows that
\[ I_0 \geq \frac{1}{\sqrt{2}} I(\alpha, \sqrt{(N-1)/N}) \geq \frac{1}{\sqrt{2}} I(\alpha, 1) \]
\[ \geq \frac{1}{\sqrt{2}} \int_0^1 F(\alpha, \beta') d\beta' > 0. \] (131)

We can conclude that if \( I_0 \to 0 \), then \( I(\alpha, 1) \to 0 \), necessarily. This, in turn, implies \( \int_0^1 F(\alpha, \beta') d\beta' \to 0 \). We now investigate how the last expression depends on \( \alpha \).

\[ \frac{d}{d\alpha} \int_0^1 F(\alpha, \beta') d\beta' = \int_0^1 \frac{d}{d\alpha} F(\alpha, \beta') d\beta', \] (132)
\[ \frac{dF}{d\alpha} = -2 \beta \Phi(\beta) e^{-\alpha \Phi(\beta)} \int_0^\beta \frac{e^{-\alpha \Phi(x)}}{1 + x^2} dx \]
\[ -2 \beta e^{-\alpha \Phi(\beta)} \int_0^\beta \frac{\Phi(x) e^{-\alpha \Phi(x)}}{1 + x^2} dx. \] (133)

Both terms on the right-hand side in the above expression are strictly less than zero, for \( \beta > 0 \). It follows that \( \int_0^1 F(\alpha, \beta') d\beta' \) is a strictly decreasing function of \( \alpha \).

Thus, a necessary condition for the integral to go to zero is that \( \alpha \) goes to infinity. Due to the form of \( \alpha \) in Eq. (122), it follows, for large \( N \), that \( T N^{-(\sigma+1/2)} \to \infty \) is a necessary condition for \( Y_T(1) \to 1 \). Since \( Y_T(1) \to 1 \) is equivalent to \( p \to 1 \), we have shown that the sufficient condition \( T \gg N^{\sigma+1/2} \), proved in Sec. V A, is also necessary.

2. Local search

The proof of the necessary condition in the local case resembles rather closely the proof of the global search case. However, we assume that \( \sigma \geq 1 \). We let \( \tilde{Y}_T \) denote the solution of Eq. (27) for a given run time \( T \), with \( A = 0 \) and \( B = 1 \), and with \( Q \) replaced by \( \tilde{Q} \) defined in Eq. (51). Similarly we let \( \tilde{\rho}_T \) denote the solution of Eq. (77), for the given run time \( T \).

It is straightforward to prove the counterpart of Lemma 3
\[ \tilde{Y}_T(r) \geq \tilde{Y}_0(r), \quad T \geq 0, \quad 0 \leq r \leq 1. \] (134)

Similarly we can obtain \( 1 - \tilde{Y}_T(1) \geq \tilde{I}_0 \) with \( \tilde{I}_0 \) as in Eq. (115), but with \( Q \) replaced by \( \tilde{Q} \). Moreover,
\[ \tilde{Y}_0(r) = q(r) \cdot q(0) \geq \frac{-(N-1)(2r-1)}{\sqrt{N} \sqrt{1+(N-1)(2r-1)^2}}. \] (135)

Due to Eq. (133), Eq. (120) remains true if we replace \( I_0 \) and \( Q(\kappa(x)) \) with \( \tilde{Q}(\kappa(x)) \). We also obtain
\[ T \tilde{Q}(\kappa(x)) = \tilde{\alpha} [\Phi(x) - \Phi(-\sqrt{N-1})], \] (136)
where
\[ \tilde{\alpha} = \frac{T}{2LN^{-\sigma / 2} \sqrt{N-1}} = \frac{T}{2N^\sigma \arctan(\sqrt{N-1})}. \] (137)

and
\[ \tilde{\Phi}(x) = \int_0^x (1 + x^2)^{\sigma/2} dx'. \] (138)

Now we define \( \tilde{I}(\tilde{\alpha}, \beta) \) as in Eq. (126) but with \( \alpha \) replaced by \( \tilde{\alpha} \), and \( f_\alpha, g_\alpha \) replaced by
\[ \tilde{f}_\alpha(x) = e^{-\tilde{\alpha} \Phi(x)} \frac{1}{1 + x^2}, \quad \tilde{g}_\alpha(y) = e^{\tilde{\alpha} \Phi(y)} \frac{-y}{(1 + y^2)^{3/2}}. \] (139)

The line of reasoning from Eq. (120) to Eq. (133) is unchanged up to a replacement of \( s, \alpha, \Phi, I_0, \) and \( I(\alpha, \beta) \) with \( r, \tilde{\alpha}, \tilde{I}_0, \) and \( \tilde{I}(\tilde{\alpha}, \beta) \). We define \( \tilde{F}(\tilde{\alpha}, \beta) \) in analogy with \( F(\alpha, \beta) \) in Eq. (129).

Analogously to the global search case we find that a necessary condition for the limit \( Y_T(1) \to 1 \) is that \( \tilde{\alpha} \to \infty \), since \( \int_0^1 \tilde{F}(\tilde{\alpha}, \beta') d\beta' \) is a strictly decreasing function in \( \tilde{\alpha} \). For large \( N \), it follows, due to the form of \( \tilde{\alpha} \) in Eq. (137), that \( T N^{-(\sigma+1/2)} \to \infty \) is a necessary condition for \( p \to 1 \). Thus, we have proved that the sufficient condition for adiabaticity, proved in Sec. V A, is also necessary.

VI. SUCCESS PROBABILITIES AND THE \( p \to 1 \) LIMIT

We have so far investigated the conditions for the success probability \( p \) to approach unity, which echoes the formulation of the standard adiabatic theorem. However, for an actual implementation of the adiabatic quantum computer we have to be satisfied with a success probability less than unity in order to have a finite run time. It thus seems reasonable to ask how much of the results obtained for the \( p \to 1 \) limit survives when \( p < 1 \).

Consider the set of pairs of list lengths and run times \( (N, T) \) that gives precisely the success probability \( p \). One may note that \( T \) does not necessarily have to be a function of \( N \), as there may accidentally be several values of \( T \), for a fixed \( N \), which give the same \( p \). We show that the run time \( T \) is bounded by curves with the same asymptotic behavior as found in the \( p \to 1 \) cases.

We assume that the system initially is in the ground state, i.e., \( \rho_{00}(0) = 1 \) and \( \rho_{01}(0) = 0 \). We denote \( p = \rho_{00}(1) \). The strategy is to obtain expressions of the form \( 1 - p \leq U(T/N^\nu) \) or \( 1 - p \geq V(T/N^\nu) \). If \( U \) and \( V \) are continuous and strictly decreasing functions, we may invert them, on suitable domains, and obtain bounds \( N^\nu V^{-1}(1-p) \leq T \leq N^\nu U^{-1}(1-p) \). We derive
both upper and lower bounds in the wide-open case. In the semi-open case we obtain upper bounds only.

One may note that the upper bounds can be seen as sufficient conditions on the run time to achieve a success probability which is at least $p$.

### A. The wide-open case, global and local

An upper bound for the global wide-open search can be obtained directly from Eq. (80). The result is

$$T \leq N^{\sigma+1/2} \frac{\pi}{1 - p}, \quad \sigma \geq 0. \quad (140)$$

We obtain a lower bound by using results from Sec. VC.1. By combining Eqs. (117) and (131) one obtains

$$1 - p \geq \frac{1}{2\sqrt{2}} \int_{0}^{1} F(\alpha, \beta) d\beta \equiv C(\alpha), \quad (141)$$

where $F$ is defined in Eq. (120), and $\alpha$ in Eq. (122). We know from Sec. VC.1 that $C$ is strictly decreasing in $\alpha$. Moreover, $C$ is continuous. Hence, $C$ is invertible and

$$T \geq 2N^\sigma \sqrt{N - 1} C^{-1}(1 - p), \quad N \geq 2. \quad (142)$$

The bounds in Eqs. (146) and (147) are sufficient conditions on the run time to obtain at least the success probability $p$, for a given list length $N$.

### C. Numerical analysis

![FIG. 1: (Color online) Global search with $W(s) = H(s)$. Each curve shows log$_2$ $T$ vs log$_2$ $N$, where $T$ is the run time needed to obtain a given success probability $p$ at $s = 1$, and where $N$ is the list length. The dimensionless run time $T$ is the quotient between the physical run time and $T_0$ defined in Eq. (148). The dotted curves correspond to the wide-open case, defined by the degree of openness $\omega = 1$. The dashed curves correspond to the semi-open case $\omega = 0.9$, and the solid correspond to $\omega = 0.5$. Within each group, each curve corresponds to a success probability, which counted from below is $p = 0.4, 0.5, 0.6, 0.7, 0.8$. As seen, the asymptotic slope appears to tend to $\nu = 3/2, \nu = 1, \nu = 0$, independent of $p$. For the semi-open cases the asymptotic slope appears to tend to $\nu = 1, \nu = 0$, independent of $p$.](image)

We have numerically calculated the actual run times of the wide-open global and local search. Let us choose $A = \cos(\omega \pi / 2)$ and $B = \sin(\omega \pi / 2)$. This choice of $A$ and $B$ is merely a convenient way to obtain an interpolation between the closed case $\omega = 0$ and the wide-open case $\omega = 1$, without effectively changing $T$. Moreover we have chosen $W(s) = H(s)$. Thus, we consider an environment that monitors the energy of the system. The initial condition at $s = 0$ is $\rho(0) = |\psi\rangle\langle\psi|$ with $|\psi\rangle$ as in Eq. (13). In other words, we begin at the initial ground state of the Hamiltonian in Eq. (10).

We should also comment upon the units. As seen in Sec. II we have put the energy gap between the ground
state and the degenerate first excited state to 1 in the initial Hamiltonian $H(0)$. Depending on the actual system used to realize the search procedure, the energy gap is actually $\Delta \mathcal{E}$ in some appropriate energy unit. To put the energy gap to 1 corresponds to measuring energy in units of $\Delta \mathcal{E}$. Furthermore, as we have chosen $\hbar = 1$, it follows that the run time $T$, used throughout this paper, can be seen as the quotient $T = T/\hat{T}$, of the physical run time $\hat{T}$ and $\hat{T}$ defined by

$$T_0 = \Delta \mathcal{E}^{-1}. \quad (148)$$

In Fig. 1 the dotted curves correspond to the wide-open global search with success probability ranging from $p = 0.4$ to $p = 0.8$. The curves seem to follow closely to a linear asymptote with slope $\nu = 3/2$, independent of the choice of $p < 1$. The dotted curves in Fig. 1 similarly correspond to the wide-open local search case. The asymptotic slope of these curves seems to be $\nu = 1$ for every choice of $p < 1$.

We have also calculated the actual run times of the semi-open global and local search numerically. Again we use $W(s) = H(s)$, $A = \cos(\omega \pi/2)$, and $B = \sin(\omega \pi/2)$. The results for the global case are shown in Fig. 1 in form of the dashed curves corresponding to $\omega = 0.9$, and the solid curves corresponding to $\omega = 0.5$. All these curves seem to tend to the asymptotic slope $\nu = 1$, which is in line with our analytical results. In Fig. 2 we similarly have created plots for the semi-open local search case. These again correspond to the dashed curves with $\omega = 0.9$, and the solid curves with $\omega = 0.5$. The results are qualitatively similar to the semi-open global search case, but with asymptotic slope $\nu = 1/2$.

Figures 1 and 2 show the global and local search case for a fixed success probability $p = 1/2$ but for different degrees of openness $\omega$. Figure 1 suggests that in the semi-open cases the asymptotic slope is $\nu = 1$, while in the wide-open case the asymptotic slope is $\nu = 3/2$, which is consistent with our analytical results. In Fig. 2 the curves in the local search case appears to have a qualitatively similar behavior to the global search case, except that the asymptotic slope is $\nu = 1$ in the wide-open case, and $\nu = 1/2$ in the semi-open case.

From Figs. 1 and 2 one may obtain an intuitive understanding of the “discontinuity” of the scaling of the run times, when going from the semi-open to the wide-open case. The closer we get to the wide-open case (i.e., the closer $\omega$ gets to 1), the higher up in list-lengths the curve “follows” the curve of the wide-open case, before it turns to its asymptotic slope. In this context one should also note that for every fixed list-length $N$, the system is continuous with respect to $\omega$. In other words, the discontinuity only appears for the asymptotes in the $N \to \infty$ limit, not in any system corresponding to a fixed list. The list-length $N$ corresponds to the dimension of the total Hilbert space, but in the effective two-dimensional Hilbert space the list-length only enters as a parameter in the Hamiltonian. If we are to understand the discontinuity of the asymptotic scaling as a function of the degree of openness, in the case $W(s) = H(s)$, we may consider how the parameter $N$ enters the terms $A[H, \rho]$ and $B[H, [H, \rho]]$ in the master equation. The minimum energy gap of the Hamiltonian is proportional to $1/\sqrt{N}$. Thus, at the minimum gap the list length $N$ enters as $A/\sqrt{N}$ in the Hamiltonian part of the master equation, but as $B/N$ in the decohering part. No matter how small $A$ is, as long as it is nonzero, $A/\sqrt{N}$ always dominates $B/N$ in the limit of large $N$. Although this, strictly speaking, only holds at $s = 1/2$, it seems intuitively reasonable that the Hamiltonian term dominates the evolution, for large $N$, as long as $A \neq 0$. Intuitively it also seems reasonable that if the Hamiltonian dominates the evolution, then the Hamiltonian determines the asymptotic scaling of the run time, and thus the scaling becomes the one of the ideal closed case. However, if $A = 0$ then the decohering term determines the scaling.

VIII. CONCLUSIONS

We investigate the effect of decoherence on the adiabatic search of a marked element in a disordered list of $N$ elements. More specifically we consider deco-
we prove that these sufficient conditions are $T \gtrsim N$, where $T$ is the run time needed to obtain the success probability $p = 1/2$ at $s = 1$, and where $N$ is the list length. The dimensionless run time $T$ is the quotient between the physical run time and $T_0$ defined in Eq. (148). Each line shows the result for a given value of the degree of openness $\omega$. Counted from below, the lines correspond to $\omega = 0, 0.1, \ldots, 0.9, 1$. Hence, the uppermost line corresponds to the wide-open case $\omega = 1$ and the lowermost line corresponds to the closed case $\omega = 0$. All the lines seem to tend to the slope $\nu = 1$, except the uppermost which seems to tend to $\nu = 3/2$.

We prove sufficient conditions for the $p \to 1$ limit of the success probability $p$. In the semi-open case ($A \neq 0$) we prove that these sufficient conditions are $T \gtrsim N$ and $T \gg \sqrt{N}$ in the global and local search case, respectively, where $T$ is the run time of the adiabatic search. In other words, the results for the ideal adiabatic search ($B = 0$) remains valid even in the presence of decoherence with respect to the instantaneous eigenbasis.

In the wide-open case ($A = 0$), sufficient and necessary conditions for the $p \to 1$ limit, are deduced. In the case where $W = H$ we find that the conditions for $p \to 1$ are $T \gg N^{3/2}$ and $T \gg N$ for the global and local search, respectively.

We finally investigate how the run time $T$ depends on $N$ for a fixed success probability $p$ less than unity. We find that it is sufficient with run times that scales like $N$ and $\sqrt{N}$, in the semi-open case, as for the ideal global and local adiabatic searches. In the wide-open case we instead find the scalings $N^{3/2}$ and $N$, for the global and local search, respectively.

In view of these results the following picture emerges. In the semi-open cases the Hamiltonian dynamics dominates the behavior of the system for sufficiently large list lengths. To be more precise, an increased degree of eigenbasis decoherence does increase the run time of the adiabatic search, but only through the constant $C$ of the asymptote $CN^\nu$. In the wide-open case, however, the protective effect of the Hamiltonian dynamics is not present. Consequently, the asymptotic behavior depends directly on the choice of $W$. As a consequence the asymptotic behavior of the wide-open case may be distinctly different from that of the semi-open case, although we have the same choice of $W$. One may note that the abrupt change in asymptotic behavior with respect to the transition from semi-open to wide-open system, raises a warning sign concerning the use of the wide-open evolution as an approximation to an “almost” wide-open evolution. It should be noted, however, that for any fixed list length, the system is continuous with respect to the degree of openness, and that the discontinuity rather concerns the asymptotic scaling of the run time in the limit of large list-lengths.

The decoherence model provided by Eq. (11) implies that we disregard the memory effects in the environment and the corresponding fluctuations, which seems justifiable if the relevant time scales of the evolution is much
larger than the dominating time scales of the memory of the environment. The modeling of memory effects generally leads to integro-differential equations \cite{23, 24}, or to time-local non-Markovian equations \cite{25}. To include such effects into the analysis of the influence of the environment on the efficiency of the adiabatic quantum computer would be interesting but challenging.

Finally we note that, in spite of the apparent difference between the adiabatic quantum computing scheme and the traditional circuit model, it has been shown that these two models are, in a certain sense, equivalent \cite{26, 27}. However, this equivalence does not concern the robustness to noise, relaxation, or decoherence. Previous findings suggest that the adiabatic quantum computer to a certain extent should be resilient against various kinds of open system effects. The results of this paper provide further evidence for this.

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