BV SOLUTIONS OF A CONVEX SWEEPING PROCESS WITH A COMPOSED PERTURBATION

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Abstract. A measurable sweeping process with a composed perturbation is considered in a separable Hilbert space. The values of the moving set generating the sweeping process are closed, convex sets. The retraction of the sweeping process is bounded by a positive Radon measure. The perturbation is the sum of two multivalued mappings. The values of the first one are closed, bounded, not necessarily convex sets. It is measurable in the time variable, is Lipschitz continuous in the phase variable, and satisfies a conventional growth condition. The values of the second one are closed, convex, not necessarily bounded sets. We assume that this mapping has a closed with respect to the phase variable graph.

The remaining assumptions concern the intersection of the second mapping and the multivalued mapping defined by the growth conditions. We suppose that this intersection has a measurable selector and has certain compactness properties.

We prove the existence of solutions for our inclusion. The proof is based on the author’s theorem on continuous with respect to a parameter selectors passing through fixed points of contraction multivalued maps with closed, non-convex, decomposable values depending on the parameter, and the classical Ky Fan fixed point theorem. The results which we obtain are new.

1. Introduction. Let $H$ be a separable Hilbert space with the norm $\| \cdot \|$, metric $d(\cdot, \cdot)$, inner product $\langle \cdot, \cdot \rangle$ and null element $\Theta$. Introduce the following notation:

$\mathbb{R}^+ = [0, +\infty)$, $\mathbb{R}^{-} = [0, +\infty]$, $T = [0, a] \subset \mathbb{R}^+$, $a > 0$, $C : T \to H$ is a multivalued mapping with convex, closed values, $U, V : T \times H \times H \to H$ are multivalued mappings with closed values. By $\text{exc} \{C(s), C(t)\}$, $s \leq t$, $s, t \in T$ we denote the excess of the set $C(s)$ over the set $C(t)$

$$\text{exc} \{C(s), C(t)\} = \sup \{d(x, C(t)) : x \in C(s)\}.$$ 

In what follows, we always assume that the following inequality holds:

$$\text{exc} \{C(s), C(t)\} \leq \mu([s, t]), \quad s \leq t, \quad s, t \in T,$$

where $\mu$ is a positive Radon measure on $T$. We denote by $\lambda$ the Lebesgue measure and by $L^1(T, H)$ the space of Lebesgue integrable functions from $T$ to $H$.

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Consider a measurable sweeping process

\[-dx \in \mathcal{N}(C(t); x(t)) + F(t, x(t)) \, d\lambda,\]
\[F(t, x) = U(t, x) + V(t, x),\]
\[x(0) \in C(0),\]

where \(\mathcal{N}(C(t); \cdot)\) is the normal cone of the set \(C(t)\) in the sense of convex analysis, and \(U, V : T \times H \to H\) are multivalued mappings with closed values.

**Definition 1.1.** By a solution of inclusion (1.2) we mean a triple \((x; u; v)(\cdot), u(\cdot), v(\cdot)\) such that

1) \(x(u; v)(\cdot)\) is a right-continuous function of bounded variation from \(T\) to \(H\), \(x(u; v)(0) = x_0, x(u; v)(t) \in C(t), t \in T\) and \(u(\cdot), v(\cdot) \in L^1(T, H)\);

2) there exists a positive Radon measure \(\nu\) absolutely continuously equivalent to the measure \(\mu + \lambda\) such that the differential measure \(dx(u; v)\) generated by the function \(x(u; v)(\cdot)\) is absolutely continuous with respect to the measure \(\nu\), and the density \(\frac{dx(u; v)}{d\nu}(t)\) of the measure \(dx(u; v)\) with respect to the measure \(\nu\) and the functions \(u(\cdot), v(\cdot)\) satisfy the inclusions

\[\frac{dx(u; v)}{d\nu}(t) + (u(t) + v(t)) \frac{d\lambda}{d\nu}(t) \in -\mathcal{N}(C(t); x(u; v)(t)) \, \nu \text{ a.e.},\]
\[u(t) \in U(t, x(u; v)(t)) \, \lambda \text{ a.e.},\]
\[v(t) \in V(t, x(u; v)(t)) \, \lambda \text{ a.e.}.\]

Positive Radon measures are absolutely continuously equivalent if each of them is absolutely continuous with respect to the other.

Our definition of a solution to measurable sweeping process follows [29] and others.

Let \(\overline{B}\) be the unit closed ball in \(H\) centered at the point \(\Theta\). For a bounded set \(D \subset H\) we denote
\[\|D\| = \{\sup \|x\|; x \in D\}.\]

We denote by \(\text{haus}(\cdot, \cdot)\) the Hausdorff metric on the space of all nonempty, closed, bounded sets.

We make the following assumptions.

**Hypotheses \(H(U)\).** The multivalued mapping \(U : T \times H \to H\) with closed, not necessarily convex values has the following properties:

1) the mapping \(t \to U(t, x)\) is measurable;
2) the inequalities

\[\text{haus}(U(t, x), U(t, y)) \leq k(t) \|x - y\| \, \lambda \text{ a.e.},\]
\[x, y \in H, \ k(\cdot) \in L^1(T, \mathbb{R}^+), \]
\[\|U(t, x)\| = \sup\{\|u\|; u \in U(t, x)\} \leq m_1(t) + n_1(t) \|x\| \, \lambda \text{ a.e.},\]
\[x \in H, \ m_1(\cdot), n_1(\cdot) \in L^1(T, \mathbb{R}^+)\] hold.

**Hypotheses \(H(V)\).** The multivalued mapping \(V : T \times H \to H\) with closed convex values has the following properties:

1) the inequality

\[d(\Theta, V(t, x)) < m_2(t) + n_2(t) \|x\| \, \lambda \text{ a.e.},\]
\[x \in H, \ m_2(\cdot), n_2(\cdot) \in L^1(T, \mathbb{R}^+)\] holds;
2) the mapping
\[ t \rightarrow V(t, x) \cap (m_2(t) + n_2(t)\|x\|)B, \quad x \in H \]
has a \( \lambda \)-measurable selector, and the mapping \( x \rightarrow V(t, x) \) has a closed graph for \( \lambda \) almost every \( t \in T \);

3) for every bounded set \( D \subset H \) the set
\[ V(t, D) \cap (m_2(t) + n_2(t)\|D\|)B, \quad x \in H \]
is relatively compact for \( \lambda \) almost every \( t \in T \), where \( V(t, D) = \{ \cup V(t, x); \quad x \in D \} \).

Since inequality (1.9) is strict, for \( \lambda \) almost every \( t \) the set \( V(t, x) \cap (m_2(t) + n_2(t)\|x\|)B, \quad x \in H \) is not empty. Hence, Hypothesis \( H(V) \) 2) is meaningful.

**Main result**

**Theorem 1.1.** Let inequality (1.1) and Hypotheses \( H(U), \ H(V) \) hold. Then, the measurable sweeping process (1.2), (1.3) has a solution \((x(u; v)(\cdot), u(\cdot), v(\cdot))\) and
\[ \|x(u; v)(t) - x(u; v)(t - 0)\| \leq \mu(\{t\}), \quad t \in T. \]

If the measure \( \nu \) is absolutely continuous with respect to the measure \( \lambda \), then the measures \( \nu \) and \( \lambda \) are absolutely continuously equivalent. This will be the case, for example, when the measure \( \mu \) is absolutely continuous with respect to the measure \( \lambda \). In this case, the sets of \( \nu \) measure zero and \( \lambda \) measure zero coincide. Hence, from (1.4) it follows that
\[ \frac{dx(u; v)}{d\lambda}(t) + u(t) + v(t) \in \mathcal{N}(C(t); x(u; v)(t)) \]
\( \lambda \) almost everywhere.

If the differential measure \( dx(u; v) \) of a right continuous function of bounded variation \( x(u; v)(\cdot) \) is absolutely continuous with respect to the Lebesgue measure \( \lambda \), then the function \( t \rightarrow x(u; v)(\cdot) \) is absolutely continuous and \( \lambda \) almost everywhere on \( T \) we have
\[ \frac{dx(u; v)}{d\lambda}(t) = \frac{dx(u; v)(t)}{dt}. \]

From this equality and (1.10) we see that when Hypotheses \( H(U), \ H(V) \) hold the differential inclusion
\[ \frac{dx}{dt} \in -\mathcal{N}(C(t), x(t)) + F(t, x(t)), \quad (1.11) \]
\[ F(t, x) = U(t, x) + V(t, x) \quad (1.12) \]
has an absolutely continuous solution.

If
\[ \text{exc} (C(s), C(t)) \leq |\omega(t) - \omega(s)|, \quad s \leq t, \]
where \( \omega : T \rightarrow \mathbb{R} \) is an absolutely continuous function, then as the measure \( \mu \) appearing in one can take the measure
\[ \mu(A) = \int_A |\dot{\omega}(\tau)| \, d\tau, \quad A \in \mathcal{B}, \]
where \( \mathcal{B} \) is the \( \sigma \)-algebra of Borel sets. Therefore, if inequality (1.13) and Hypotheses \( H(U), \ H(V) \) hold, then inclusion (1.11), (1.12) has an absolutely continuous solution.
The proof of Theorem 1.1 is based on the theorem on selectors continuous with respect to a parameter and passing through the fixed points of multivalued contraction mappings with closed, nonconvex, decomposable values depending on the parameter [29] and the classical fixed point theorem by Ky Fan [18]. The application of the fixed point theorem for upper semicontinuous multivalued mappings with closed convex values in a Banach space to prove the existence of solutions to multivalued evolution equations in a Hilbert space was apparently originated in the work [6]. The same theorem was used in [7] to prove the existence of solutions for an ordinary differential equation with upper semicontinuous convex-valued right-hand side.

In the seventies J.J. Moreau introduced and thoroughly investigated the so-called sweeping processes, which have a particular form of differential inclusion (1.11) with \( F \equiv \Theta \). The original motivation of his work was to model quasistatic evolution in elastoplasticity, friction dynamics, granular material, contact dynamics [21]–[23].

Since then, many other applications have been given, such as application to switched electrical circuits [1], nonsmooth mechanics [2, 10], crowd motion [19], hysteresis in elasto-plastic models [16] among others.

Moreover, due to the development of new techniques to deal with differential inclusions involving normal cones, new variants of the sweeping processes have been introduced. We can mention the state dependent sweeping process, the second order sweeping process, and some other variants (see [17] and references therein).

We can write differential inclusion (1.11) with the perturbation (1.12) as the control system

\[
\frac{dx}{dt} \in -\mathcal{N}(C(t), x(t)) + u(t) + v(t),
\]

with a control \( u(t) \) and an obstacle \( v(t) \) with the constraints

\[
u(t) \in U(t, x(t)), \]

\[
v(t) \in V(t, x(t)).
\]

From (1.15), (1.16) it follows that control system (1.14) can be considered as a Lur’e dynamical system with a set-valued feedback \( U(t, x) \) and a set-valued obstacle \( V(t, x(t)) \) in the feedback loop [3].

The nonconvexity of values of the mapping \( U(t, x) \) arises in dynamical systems with control switches. In such systems the constraint (1.15) usually has the form

\[
u(t) \in f(t, x(t)) \cdot \text{bd} \, \bar{B},
\]

where \( f : T \times H \to R^+ \), and \( \text{bd} \, \bar{B} \) is the boundary of the unit ball \( \bar{B} \).

Recently, multivalued Lur’e dynamical systems have attracted a lot of attention from researchers (see, e.g., [11, 12] and others).

In the survey paper [12], the authors consider a wide class of multivalued Lur’e systems and establish its connection with sweeping processes.

We mention the works which are closely related to our results, i.e. the papers where one addresses the existence of right continuous BV solutions to sweeping processes. Among those are the references [27]–[25].

In [14], one considers a scalar convex-valued upper semicontinuous in the phase variable perturbation with closed values.

Composed perturbations were considered in [24]. In this work one studied the existence of BV solutions of a measurable sweeping process with the values of the multivalued mapping \( C(t) \) being r-prox-regular, closed sets and with a perturbation...
being the sum of a single-valued and a multivalued mappings. The single-valued mapping satisfies the Lipschitz condition and the multivalued mapping is scalarly upper semicontinuous with convex, compact values satisfying linear growth conditions with respect to a compact set. Even though the properties of the mappings in the perturbation are stated for the variables \((t, x)\) which belong to the graph of the mapping \(t \to C(t)\), Theorem 4.1 in [24] is nevertheless a particular case of our Theorem 1.1 when the values of the mapping \(C(t)\) are convex, closed sets. The same is true for the absolutely continuous version of Theorem 3.1 in [8]. This point will be addressed in more details in the last section of the present paper.

If \(U(t, x) \equiv \Theta, t \in T, x \in H\), then our Theorem 1.1 generalizes Theorem 1.1 in [30], where one assumes that the values of the mapping \(V(t, x)\) are convex, compact sets.

If \(V(t, x) \equiv \Theta, t \in T, x \in H\), then from Theorem 1.1 one obtains a quite general result on existence of a solution for a measurable sweeping process with a convex-valued perturbation with bounded values. However, a still more general result in this respect is contained in the statement 1) of Theorem 1.1 [28], where the values of the mapping \(U(t, x)\) can be unbounded sets.

The present article consists of Introduction and four sections.

In Introduction, we formulate our problem, state the main result, provide possible applications and compare our results with the existing ones.

In the second section, we introduce notation and give some definitions and auxiliary results necessary to prove our main result.

In the third section, we introduce a sweeping process with a single-valued perturbation depending only on time and we study properties of its solution set with respect to the set of perturbations.

In the forth paragraph, we consider the multivalued Nemytskii operator with closed, nonconvex, decomposable values.

In the fifth section, we prove the main result and give some comments.

2. Main notations, definitions and preliminary data. Let \(Y\) be a metric space, \(cY\) be the collection of all nonempty closed sets from \(Y\), \(cbY\) be the family of all bounded sets from \(cY\) with the Hausdorff metrics \(haus\(Y, \cdot, \cdot\)\) and \(compY\) be the family of all nonempty, compact sets from \(Y\). For a topological vector space \(Z\) by \(\omega-Z\) we denote the space \(Z\) endowed with the weak topology. If \(D \subset Z\), then \(\omega-D\) means that \(D\) is endowed with the topology induced by that of the space \(\omega-Z\).

By \(\overline{co}D\) we mean the closed, convex hull of a set \(D \subset Z\).

Let \(W\) be a topological space. A multivalued mapping \(F: W \rightarrow Y\) is called lower semicontinuous if for any open set \(E \subset Y\) the set \(F^{-1}(E) = \{w \in W; F(w) \cap E \neq \emptyset\}\) is open.

We note that if \(W\) is a metric space, the definition of lower semicontinuity is equivalent to the following one: for any \(w \in W\), \(y \in F(w)\) and any sequence \(w_n \in W, n \geq 1, w_n \rightarrow w\) there exists a sequence \(y_n \in F(w_n), n \geq 1\), converging to \(y\).

A multivalued mapping \(F: W \rightarrow Y\) is called upper semicontinuous if for any open set \(E \subset Y\) the set \(F^{+}(E) = \{w \in W; F(w) \subset E\}\) is open.

If \(Y\) is a compact metric space, \(W\) is a metric space and \(F: W \rightarrow Y\) is a multivalued mapping with closed values, then the upper semicontinuity is equivalent to the closedness of the graph of the mapping.
A multivalued mapping \( F : T \to cH \) is called measurable [15], if for any closed set \( E \subset H \) the set \( F^{-1}(E) = \{ t \in T; F(t) \cap E \neq \emptyset \} \) is an element of the \( \sigma \)-algebra \( \Sigma \) of Lebesgue measurable sets from \( T \).

A set \( K \) of measurable mappings \( u : T \to H \) is called decomposable if for any \( u, v \in K \), \( \Lambda \in \Sigma \) the element \( \chi(\Lambda)u + \chi(T \setminus \Lambda)v \) belongs to the set \( K \), where \( \chi(\Lambda) \) is the characteristic function of the set \( \Lambda \).

The space of all integrable with respect to the Lebesgue measure \( \lambda \) mappings from \( T \) to \( H \) we denote by \( L^1(T,H) \).

A measurable multivalued mapping \( \Gamma : T \to cb \) is called integrally bounded if there exists a function \( m(\cdot) \in L^1(T,\mathbb{R}^+) \) such that

\[
\| \Gamma(t) \| = \sup\{\|u\|; u \in \Gamma(t)\} \leq m(t) \text{ a.e.}
\]

The space of all measurable, integrally bounded mappings \( \Gamma : T \to cb \) we denote by \( L^1(T,cb) \), and by \( dcbl^1(T,H) \) we mean the collection of all closed, bounded, decomposable sets from \( L^1(T,H) \).

If \( \Gamma(\cdot) \in L^1(T,cb \), then \( S_\Gamma \) is the collection of all integrable with respect to the measure \( \lambda \) selectors of the mapping \( t \to \Gamma(t) \), which, as it is well known, is an element of the space \( dcbl^1(T,H) \).

Denote by \( V_+(T,H) \) the space of all right continuous functions \( x : T \to H \) of bounded variation with the topology of uniform convergence on \( T \). The space \( V_+(T,H) \) is normalizable with the norm

\[
\| x(\cdot) \|_{V_+} = \sup\{\|x(t)\|; t \in T\}.
\]

It is well known that a function \( x(\cdot) \in V_+(T,H) \) has the left limit \( x(t-0) \) at every point \( t \in [0,a[ \).

Let \( U \subset V_+(T,H) \). The set \( U \) is called right equicontinuous in a point \( s \in [0,a[ \), if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \|x(s) - x(t)\| \leq \varepsilon \) for all \( x(\cdot) \in U \) and \( t \in [s,s+\delta[ \).

A set \( U \subset V_+(T,H) \) is called left equicontinuous at a point \( s \in ]0,a[ \), if for any \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that \( \|x(s) - x(t)\| \leq \varepsilon \) for all \( x(\cdot) \in U \) and \( t \in ]s-\delta,s[ \).

A set \( U \subset V_+(T,H) \) is called unilaterally equicontinuous if it is simultaneously left and right equicontinuous at every point \( s \in ]0,a[ \), right equicontinuous at the point 0 and left equicontinuous at the point \( a \).

The variation of a function \( x(\cdot) \in V_+(T,H) \) we denote by \( \text{var} x(\cdot) \). A set \( U \subset V_+(T,H) \) is called uniformly bounded in norm and variation if there exists a constant \( M > 0 \) such that

\[
\|x(t)\| \leq M, \quad t \in T, \quad x(\cdot) \in U,
\]

\[
\text{var} x(\cdot) \leq M, \quad x(\cdot) \in U.
\]

Let \( B \) be the \( \sigma \)-algebra of Borel sets from \( T \). By a positive Radon measure we mean a scalar positive measure defined on the \( \sigma \)-algebra \( B \). The variation of a measure \( m : B \to H \) we denote by \( |m|(\cdot) \). If \( |m|(T) < \infty \), then the measure \( m \) is a measure with bounded variation. In this case, the variation \( |m|(\cdot) \) of the measure \( m(\cdot) \) is a positive Radon measure.

A positive Radon measure \( \nu \) is absolutely continuous with respect to a measure \( \mu \), if from \( \mu(A), \ A \in B \) it follows that \( \nu(A) = 0 \). If the converse is also true, then the measures \( \mu \) and \( \nu \) are called absolutely continuously equivalent.

A measure \( m : B \to H \) is absolutely continuous with respect to a positive Radon measure \( \mu(\cdot) \), if the measure \( |m|(\cdot) \) is absolutely continuous with respect to the
measure $\mu$. For a positive Radon measure $\nu$ on $T$ by $L^1_\nu(T,H)$ we denote the space of equivalence classes of all $\nu$ measurable mappings $g : T \to H$ such that $\|g(\cdot)\|$ is an element of the space $L^1_\nu(T,\mathbb{R}^+)$. If a measure of bounded variation $m : B \to H$ is absolutely continuous with respect to a positive Radon measure $\mu$, then according to the Radon-Nikodym theorem there exists a function $\hat{m} \in L^1_\mu(T,H)$ such that

$$m(A) = \int_A \hat{m}(\tau) \, d\mu(\tau), \quad A \in B.$$ 

The function $t \to \hat{m}(t)$ is called the density of the measure $m$ with respect to the measure $\mu$ and is denoted by $\frac{dm}{d\mu}(\cdot)$. 

Lemma 2.1. Let $U \subset V_+(T,H)$ and the following conditions are fulfilled:

1) the set $U$ is unilaterally equicontinuous;
2) the set

$$U(t) = \{x(t); \ x(\cdot) \in U\}, \quad t \in T$$

is relatively compact;
3) the set $U$ is uniformly bounded in norm and variation.

Then, the set $U$ is a relatively compact subset of the space $V_+(T,H)$, i.e. from any sequence $x_n(\cdot) \in U$, $n \geq 1$, one can extract a subsequence $x_{nk}(\cdot) \in U$, $k \geq 1$, uniformly converging to some function $x(\cdot) \in V_+(T,H)$.

Lemma 2.1 follows from Lemma 2.1 [30].

Corollary 2.1. On every unilaterally equicontinuous and bounded in variation set $U \subset V_+(T,H)$ the topologies of uniform and pointwise convergences coincide.

Corollary 2.1 is a restatement of Corollary 2.3 [30].

Lemma 2.2. Let Hypotheses $H(V)$ hold. Then, for any function $x(\cdot) \in V_+(T,H)$ the mapping

$$t \to V(t,x(t)) \cap (m_2(t) + n_2(t)\|x(t)\|)B$$

has an $\lambda$ integrable selector.

Lemma 2.3. If a function of bounded variation $x : T \to H$ is right continuous, then there exists a unique measure $m : B \to H$ such that for any $0 \leq c \leq d \leq a$ we have

$$m([c,d]) = x(d) - x(c). \quad (2.1)$$

The statement of Lemma 2.3 follows from Theorem 1 [13, p. 358].

The measure $m(\cdot)$ is usually called the differential measure generated by the function $x(\cdot)$ and is denoted by $dx$. If $\dot{x}(\cdot) \in L^1_\nu(T,H)$ and $x(t) = x(0) + \int_{[0,t]} \dot{x}(\tau) \, d\nu(\tau)$, $t \in T$, then the function $x(t)$ is a right continuous function of bounded variation, the differential measure $dx$ generated by the function $x(\cdot)$ is absolutely continuous with respect to the measure $\nu$ and $\dot{x}(\cdot)$ is the density of the measure $dx$ with respect to the measure $\nu$, i.e.

$$\frac{dx}{d\nu}(t) = \dot{x}(t) \ \nu \text{ a.e.} \quad (2.2)$$

It is known [9, Chapter V, p. 43, Theorem 1] that if the Lebesgue measure $\lambda$ is absolutely continuous with respect to a positive measure Radon $\nu$, then the function
\( \hat{x} : T \to H \) is integrable with respect to the measure \( \lambda \) if and only if the function \( t \to \hat{x}(t) \frac{d\lambda}{d\nu}(t) \) is \( \nu \) integrable. In this case, it holds that

\[
\int_{[0,t]} \hat{x}(t) \, d\lambda(t) = \int_{[0,t]} \hat{x}(t) \frac{d\lambda}{d\nu}(t) \, d\nu(t), \quad t \in T. \tag{2.3}
\]

**Proposition 2.1.** Let \( \nu \) be a positive Radon measure, \( x : T \to H \) be a right continuous function of bounded variation with the differential measure \( dx \) being absolutely continuous with respect to the measure \( \nu \). Then, the function \( t \to \|x(t)\|^2 \) is a right continuous function of bounded variation and

\[
\frac{\|x(t)\|^2}{2} \leq \frac{\|x(0)\|^2}{2} + \int_{[0,t]} \langle x(\tau), \frac{dx}{d\nu}(\tau) \rangle \, d\nu(\tau), \quad t \in T. \tag{2.4}
\]

Proposition 2.1 follows from Proposition 3.3 in [27].

**Proposition 2.2.** Let \( r(t), t \in T \) be the solution of the differential equation

\[
\dot{r}(t) = m(t) + n(t) r(t), \quad r(0) = M, \quad t \in T, \tag{2.5}
\]

\( m(\cdot), n(\cdot) \in L^1(T, \mathbb{R}^+), \ M \geq 0 \). If a function \( x(\cdot) \in V_+(T, H) \) satisfies the inequality

\[
\|x(t)\| \leq M + \int_{[0,t]} (m(s) + n(s)\|x(s)\|) \, d\lambda(s), \quad t \in T \tag{2.6}
\]

then

\[
\|x(t)\| \leq r(t), \quad t \in T. \tag{2.7}
\]

This proposition follows from Lemma 5.1 in [30].

3. **Auxiliary results.** In this section, we state several results which are crucial in proving the main results of the paper.

In what follows, we assume that inequality (1.1) holds, and instead of \( \lambda \) a.e. we write a.e.

Consider the differential inclusion

\[
-dx \in \mathcal{N}(C(t); x(t)) + f(t) \, d\lambda,
\]

\( x(0) = x_0 \in C(0), \quad f(\cdot) \in L^1(T, \mathbb{R}^+). \)

Let \( \hat{r}(\cdot) \in L^1(T, \mathbb{R}^+) \) and

\[
r(t) = r_0 + \int_{[0,t]} \hat{r}(\tau) \, d\lambda(\tau), \quad t \in T, \tag{3.2}
\]

\( r_0 \geq \|x_0\| \) and \( dr \) be the differential measure generated by the function \( r(t) \). Then, \( dr \) is a positive Radon measure. Consider the measure \( \nu \) defined by

\[
\nu = \mu + dr + \lambda, \tag{3.3}
\]

which is absolutely continuously equivalent to the measure \( \mu + \lambda \).

**Theorem 3.1** (Theorem 4.1 [30]). For any \( f(\cdot) \in L^1(T, H) \) satisfying the inequality

\[
\|f(t)\| \leq \hat{r}(t) \text{ a.e.} \tag{3.4}
\]

there exists a unique solution \( x(f)(t) \) of inclusion (3.1), which satisfies the inequality

\[
\|x(f)(t) - x(f)(s)\| \leq \nu([s,t]) + dr([s,t]), \quad s \leq t, \tag{3.5}
\]
and for the density \( \frac{dx}{d\nu}(t) \) of the differential measure \( dx(f) \) generated by the function \( x(f) \) the following inclusion
\[
- \frac{dx(f)}{d\nu}(t) - f(t) \frac{d\lambda}{d\nu} \in \mathcal{N}(C(t); x(f)(t)) \quad \text{a.e.}
\] (3.6)
holds.

Denote by \( S(\hat{r}) \) the set of all \( f(\cdot) \in L^1(T, H) \) satisfying inequality (3.4).

Let \( \mathcal{R}_S(x_0) \) be the set of all solutions \( x(f)(\cdot) \) of inclusion (3.1) for \( f(\cdot) \in S(\hat{r}) \).

**Theorem 3.2 (Theorem 4.2 [30]).** The following statements are true:

a) the set \( \mathcal{R}_S(x_0) \) is not empty;

b) the following inequality holds
\[
\|x(f)(t) - x(f)(s)\| \leq \nu([s, t]) + dr([s, t]),
\] (3.7)
s \leq t, s, t \in T, \ f(\cdot) \in S(\hat{r});

c) for any \( f(\cdot) \in S(\hat{r}) \) the solution \( x(f)(\cdot) \) is unique and
\[
\|x(f_1)(t) - x(f_2)(t)\| \leq \int_{[0, t]} \|f_1(\tau) - f_2(\tau)\| d\lambda(\tau),
\] (3.8)
t \in T, f_i(\cdot) \in S(\hat{r}), i = 1, 2;

d) the set \( \mathcal{R}_S(x_0) \) is uniformly bounded in norm and variation;

e) the set \( \mathcal{R}_S(x_0) \) is unilaterally equicontinuous and for any \( x(f)(\cdot) \in \mathcal{R}_S(x_0) \) and \( t \in [0, a] \) the following inequality holds
\[
\|x(f)(t) - x(f)(t - 0)\| \leq \mu([t]).
\]

**Theorem 3.3.** Let a sequence \( f_n(\cdot) \in L^1(T, H) \), \( n \geq 1 \), converges in the space \( L^1(T, H) \) to a function \( f(\cdot) \). Then, there exists a subsequence \( f_{n_k}(\cdot) \), \( k \geq 1 \), of the sequence \( f_n(\cdot) \), \( n \geq 1 \), and a function \( q(\cdot) \in L^1(T, \mathbb{R}^+) \) such that
\[
f_{n_k}(t) \to f(t) \quad \text{a.e. as} \ k \to \infty,
\] (3.9)
\[
\|f_{n_k}(t)\| \leq q(t) \quad \text{a.e.}, \quad k \geq 1.
\] (3.10)

**Proof.** Since \( f_n(\cdot) \), \( n \geq 1 \), is a Cauchy sequence, we have
\[
\lim_{n,m \to \infty} \int_T \|f_n(t) - f_m(t)\| d\lambda(t) = 0.
\]

Hence, there exists a sequence \( n_k \), \( k \geq 1 \), \( n_1 < n_2 < \ldots n_k < \ldots \), such that
\[
\int_T \|f_n(t) - f_{n_k}(t)\| d\lambda(t) \leq 2^{-k} \quad \text{for} \ n \geq n_k, \quad k \geq 1.
\]

Let
\[
g_1(t) = f_{n_1}(t), \ g_k(t) = f_{n_k}(t) + f_{n_1}(t), \quad k \geq 2,
\] (3.11)
\[
\varphi_k(t) = g_{k+1}(t) - g_k(t), \quad k \geq 1.
\] (3.12)

Then,
\[
\int_T \|g_{k+1}(t) - g_k(t)\| d\lambda(t) \leq 2^{-k}, \quad k \geq 2.
\] (3.13)
From Theorem 3.7 [26, Ch. 4, §4] and (3.12) it follows that the series \( \sum_{k=1}^{\infty} \| \varphi_k(t) \| \) converges almost everywhere and its sum is a Lebesgue integrable function. Consequently, the series \( \sum_{k=1}^{\infty} \varphi_k(t) \) converges almost everywhere, its sum \( S(t) \) is a Lebesgue integrable function and [26, Theorem 3.7]

\[
\int_T S(t) \, d\lambda(t) = \sum_{k=1}^{\infty} \int_T \varphi_k(t) \, d\lambda(t),
\]

\[
\lim_{k \to \infty} \int_T \| R_k(t) \| \, d\lambda(t) = 0, \tag{3.14}
\]

where \( R_k(t) = \sum_{m \geq k+1} \varphi_m(t) \) is the remainder of the series. Since

\[
S_k(t) = \sum_{m=1}^{k} \varphi_m(t) = f_{n_{k+1}}(t),
\]

from this equality we infer that the sequence \( f_{n_k}(t), \ k \geq 1 \), converges almost everywhere to \( S(t) \). Using (3.14) we obtain

\[
\lim_{k \to \infty} \int_T \| f_{n_k}(t) - S(t) \| \, d\lambda(t) = 0.
\]

From this equality it follows that

\[
\lim_{k \to \infty} f_{n_k}(t) = f(t) = S(t) \text{ a.e.}
\]

Therefore, (3.9) is proved.

Let

\[
\beta_k(t) = \| f_{n_{k+1}}(t) \| - \| f_{n_k}(t) \|, \quad k \geq 1.
\]

Since

\[
\beta_k(t) \leq \| f_{n_{k+1}}(t) - f_{n_k}(t) \| = \| \varphi_k(t) \|
\]

and the series \( \sum_{k=1}^{\infty} \| \varphi_k(t) \| \) converges almost everywhere to a Lebesgue integrable function \( S(t) \), the series \( \sum_{k=1}^{\infty} \beta_k(t) \) converges almost everywhere and its sum \( \beta(t) \) is a Lebesgue integrable function.

Let

\[
q_k(t) = \sum_{m \geq k} \beta_m(t) = \sum_{m \geq k} \| f_{n_{m+1}}(t) \| - \| f_{n_m}(t) \|.
\]

Then,

\[
q_k(t) = q_{k+1}(t) + \| f_{n_{k+1}}(t) \| - \| f_{n_k}(t) \|.
\]

Therefore,

\[
q_{k+1}(t) - q_k(t) = -\| f_{n_k}(t) \| + \| f_{n_{k+1}}(t) \| - \| f_{n_k}(t) \| \leq \| f_{n_k}(t) \| - \| f_{n_{k+1}}(t) \|, \quad k \geq 1.
\]

From this inequality we infer that

\[
q_{k+1}(t) + \| f_{n_{k+1}}(t) \| \leq q_k(t) + \| f_{n_k}(t) \|, \quad k \geq 1.
\]

Consequently,

\[
\| f_{n_{k+1}}(t) \| \leq q_1(t) + \| f_{n_1}(t) \| = \beta(t) + \| f_{n_1}(t) \|.
\]
Setting $q(t) = \beta(t) + \|f_{n_k}(t)\|$, from the last inequality we obtain
\[
\|f_{n_k}(t)\| \leq q(t) \text{ a.e., } \quad k \geq 1.
\]
Since $\beta(\cdot) \in L^1(T, \mathbb{R}^+)$, inequality (3.10) and, thus, the theorem is proved.

Theorem 3.3 is a generalization of the classical Lebesgue’s dominated convergence theorem (see, e.g., Theorem 3.5 [26]).

Consider the differential inclusion
\[
-dx \in \mathcal{N}(C(t); x(t)) + (u(t) + v(t)) \, d\lambda,
\]
\[
u(\cdot), v(\cdot) \in L^1(T, H), \quad x(0) = x_0 \in C(0).
\]

According to (3.18) the set
\[
\{\cup u_n(t); n \geq 0\} \subset H
\]
is relatively compact for almost all $t \in T$, then the sequence $x(u_n; v_n)(\cdot), n \geq 1$ of solutions of inclusion (3.15) converges in the space $V_+(T, H)$ to the solution $x(u_0; v_0)(\cdot)$ of inclusion (3.15).

**Proof.** Using Theorem 3.3 we find a subsequence $u_{n_k}(\cdot), k \geq 1$, and a function $q(\cdot) \in L^1(T, \mathbb{R}^+)$ such that
\[
u_{n_k}(t) \to u_0(t) \text{ a.e., } \quad k \to \infty,
\]
\[
\|u_{n_k}(t)\| \leq q(t) \text{ a.e., } \quad k \geq 0.
\]
According to (3.18) the set
\[
\{\cup u_{n_k}(t); k \geq 1\} \subset H
\]
is relatively compact for almost all $t \in T$.

Let
\[
f_k(t) = u_{n_k}(t) + v_{n_k}(t), \quad k \geq 0,
\]
where $u_{n_0}(t) = u_0(t), v_{n_0}(t) = v_0(t), 
\hat{r}(t) = q(t) + \alpha(t)$.

Then, from (3.16)–(3.20) it follows that
\[
\|f_k(t)\| \leq \hat{r}(t) \text{ a.e., } \quad k \geq 0
\]
and the set
\[
\{\cup f_k(t); k \geq 0\} \subset H
\]
is relatively compact for almost all $t \in T$.

Denote
\[
x(f_k)(\cdot) = x(u_{n_k}; v_{n_k})(\cdot), \quad k \geq 0.
\]
Using (3.24), (3.22) and Theorem 3.2 we see that the set $\{\cup x(f_k)(\cdot); k \geq 0\}$ is uniformly bounded in norm and variation, and it is unilaterally equicontinuous. Then, according to Theorem 2.1 [20, Chapter 0] there exists a subsequence $x(f_{k_m})(\cdot), m \geq 1$, of the sequence $x(f_k)(\cdot), k \geq 1$, converging pointwise in the
weak topology of the space $H$ to some function of bounded variation $y(\cdot)$. From Proposition 2.1, (3.3), (3.22), (3.6) we infer that
\[
\frac{1}{2}\|x(f_{k_m})(t) - x(f_0)(t)\|^2 \leq \int_{[0,t]} \langle x(f_{k_m}) - x(f_0), f_0(t) - f_{k_m}(t) \rangle \, d\lambda(t) \cdot d\nu(t), \quad t \in T.
\]
Since the functions $t \rightarrow \langle x(f_{k_m})(t) - x(f_0)(t), f_0(t) - f_{k_m}(t) \rangle$, $m \geq 1$, are Lebesgue integrable, similarly to (2.3) we obtain
\[
\frac{1}{2}\|x(f_{k_m})(t) - x(f_0)(t)\|^2 \leq \int_{[0,t]} \langle x(f_{k_m}) - x(f_0), f_0(t) - f_{k_m}(t) \rangle \, d\lambda(t).
\]
Since the function $y(\cdot)$ is Lebesgue integrable, we rewrite the last inequality in the form
\[
\frac{1}{2}\|x(f_{k_m})(t) - x(f_0)(t)\|^2 \leq \int_{[0,t]} \langle x(f_{k_m}) - y, f_0(t) - f_{k_m}(t) \rangle \, d\lambda(t) + \int_{[0,t]} \langle y - x(f_0), f_0(t) - f_{k_m}(t) \rangle \, d\lambda(t).
\]
From (3.23) it follows that the set $\{\cup(f_0(t) - f_{k_m}(t)); m \geq 1\} \subset H$ is relatively compact for almost all $t \in T$. Since the sequence $x(f_{k_m})(\tau) - y(\tau)$, $k \geq 1$, $\tau \in T$ is bounded, the sequence of functions $h \rightarrow \langle x(f_{k_m})(\tau) - y(\tau), h \rangle$, $m \geq 1$, $h \in H$ is equicontinuous. It is well known that on every equicontinuous set the topology of pointwise convergence coincides with the topology of uniform convergence on compact sets. Hence,
\[
\lim_{m \to \infty} \langle x(f_{k_m})(\tau) - y(\tau), f_0(\tau) - f_{k_m}(\tau) \rangle = 0 \text{ a.e.}
\]
Now, from (3.22), the uniform boundedness in norm of the sequence $x(f_{k_m})(t)$, $m \geq 1$, and Lebesgue’s dominated convergence theorem it follows that
\[
\lim_{m \to \infty} \int_{[0,t]} \langle x(f_{k_m}) - y(\tau), f_0(t) - f_{k_m}(t) \rangle = 0, \quad t \in T.
\]
The second integral in (3.25) converges to zero, since the sequence $f_{k_m}(-)$, $m \geq 1$, converges to $f_0(-)$ in the topology of the space $\omega-L^1(T,H)$. Therefore, from (3.25) we see that the sequence $x(f_{k_m})(t)$, $m \geq 1$, for every $t \in T$ converges to $x(f_0)(t)$ in the space $H$. Then, Corollary 2.1 implies that the sequence $x(f_{k_m})(\cdot)$, $m \geq 1$, converges to $x(f_0)(\cdot)$ in the space $V_+(T,H)$. According to (3.21) we have proved that if the assumptions of Theorem 3.4 are fulfilled, then there exists a subsequence $x(u_{m_n}; v_{m_n})(\cdot)$, $m \geq 1$, of the sequence $x(u_n; v_n)(\cdot)$, $n \geq 1$, converging to $x(u_0; v_0)(\cdot)$ in the space $V_+(T,H)$.

Suppose that the sequence $x(u_n; v_n)(\cdot)$, $n \geq 1$, itself does not converge to $x(u_0; v_0)(\cdot)$ in the space $V_+(T,H)$. Then, there exists a subsequence $x(u_{m_k}; v_{m_k})(\cdot)$, $k \geq 1$, of the sequence $x(u_n, v_n)(\cdot)$, $n \geq 1$, such that any subsequence of the sequence $x(u_{m_k}; v_{m_k})(\cdot)$, $k \geq 1$, does not converge to $x(u_0; v_0)(\cdot)$. Repeating the reasoning above to the sequences $u_{m_k}(-)$ and $v_{m_k}(-)$, $k \geq 1$, and taking into account the fact that for $v_0(-)$, $u_0(-)$ inclusion (3.15) has a unique solution $x(u_0; v_0)(\cdot)$ we
arrive at a contradiction. Therefore, the sequence \(x(u_n; v_n)(\cdot), n \geq 1\), converges in the space \(V_+(T, H)\) to the solution \(x(u_0; v_0)\). The theorem is proved.

4. Nemytskii multivalued operator and its properties. In what follows, we assume that inequality (1.1) and Hypotheses \(H(U), H(V)\) hold. Since \(\lambda\) is the Lebesgue measure, instead of \(d\lambda\) we write \(d\tau\).

Let \(x(\Theta; \Theta)(\cdot)\) be the solution of inclusion (3.15) for \(u(t) \equiv \Theta, v(t) \equiv \Theta, t \in T\) and

\[
M = \sup\{\|x(\Theta; \Theta)\|, t \in T\}. \tag{4.1}
\]

Denote by \(M\) the solution of the differential equation (2.5) with

\[
m(t) = m_1(t) + m_2(t), \quad n(t) = n_1(t) + n_2(t), \quad t \in T, \tag{4.2}
\]

where \(m_i(\cdot), n_i(\cdot), i = 1, 2\) are the functions from inequalities (1.8), (1.9).

Consider the sets

\[
S_U = \{u(\cdot) \in L^1(T, H); \|u(t)\| \leq m_1(t) + n_1(t) r(t) \text{ a.e.}\}, \tag{4.3}
\]

\[
S_V = \{v(\cdot) \in L^1(T, H); \|v(t)\| \leq m_2(t) + n_2(t) r(t) \text{ a.e.}\}. \tag{4.4}
\]

Setting \(\hat{r}(t) = \hat{r}(t)\) and using Theorems 3.1, 3.2 we see that for any \(u(\cdot) \in S_U, v(\cdot) \in S_V\) inclusion (3.15) has a unique solution \(x(u; v)(\cdot)\) and for the measure \(\nu\) defined by (3.3) we have

\[
- \frac{dx(u; v)}{dv}(t) - (u(t) + v(t)) \frac{d\lambda}{dv}(t) \in \mathcal{N}(C(t); x(u; v)(t)) \nu \text{ a.e.} \tag{4.5}
\]

and for any \(t \in T\)

\[
\|x(u; v)(t) - x(u; v)(t - 0)\| \leq \mu(t), \quad \text{a.e. (4.6)}
\]

In view of (4.1), (4.2) and (3.8), (3.15) we obtain

\[
\|x(u; v)(t)\| \leq M + \int_{[0, t]} \|u(\tau) + v(\tau)\| d\tau \leq
\]

\[
\leq M + \int_{[0, t]} (m(\tau) + n(\tau)r(\tau)) d\tau = r(t), \quad t \in T \tag{4.7}
\]

for \(u(\cdot) \in S_U, v(\cdot) \in S_V\).

Let \(T(u; v)\) be the operator which with every \(u(\cdot), v(\cdot) \in L^1(T, H)\) associates the unique solution \(x(u; v)(\cdot)\) of inclusion (3.15), i.e.

\[
x(u; v)(\cdot) = T(u; v)(\cdot). \tag{4.8}
\]

We note that the existence and uniqueness of a solution of equation (3.15) for any \(u(\cdot), v(\cdot) \in L^1(T, H)\) follow from Theorem 3.1 when we take \(\hat{r}(t) = \|u(t)\| + \|v(t)\|\).

Using (4.7) we infer that for the set

\[
T(S_U; S_V)(t) = \{x(u; v)(t); u(\cdot) \in S_U, v(\cdot) \in S_V\}, \tag{4.9}
\]

\(t \in T\), the following inequality holds

\[
\|T(S_U; S_V)(t)\| \leq r(t), \quad t \in T. \tag{4.10}
\]

According to Hypothesis \(H(V) 3)\) the set

\[
V(t, r(a)B) \cap (m_2(t) + n_2(t) r(a))B, \quad t \in T = [0, a]
\]
is relatively compact for almost every $t \in T$. Since the solution $r(t)$, $r(0) = M$ of equation (2.5) is a nondecreasing function, according to (4.10)
\[ T(S_U; v)(t) \subset r(a)B, \quad t \in T. \]
Hence, the set
\[ V(t, T(S_U; S_V)(t)) \cap (m_2(t) + n_2(t)\|T(S_U; S_V)(t)\|)B \]
is relatively compact for almost all $t \in T$.
Let
\[ W(t) = V(t, T(S_U; S_V)(t)) \cap (m_2(t) + n_2(t)\|T(S_U; S_V)(t)\|)B, \quad t \in T. \quad (4.11) \]
Then, the values of the multivalued mapping $t \to \bar{\omega}W(t)$ are convex, compact sets for almost every $t \in T$. According to (4.10), (4.11) we have
\[ \|\bar{\omega}W(t)\| \leq m_2(t) + n_2(t)r(t). \quad (4.12) \]
Let
\[ S_{\omega\omega}W = \{v(\cdot) \in L^1(T, H); v(t) \in \bar{\omega}W(t) \text{ a.e.}\}. \quad (4.13) \]
From Lemma 2.2 it follows that the multivalued mapping
\[ t \to V(t, T(u; v)(t)) \cap (m_2(t) + n_2(t)\|T(u; v)(t)\|)B \]
has integrable selectors. Hence, the set $S_{\omega\omega}W$ is not empty and it has the following properties:
\begin{itemize}
  \item[a)] $S_{\omega\omega}W$ is a nonempty, convex, compact subset of the space $\omega\cdot L^1(T, H);
  \item[b)] for any $v \in S_{\omega\omega}W$ the inequality
    \[ \|v(t)\| \leq m_2(t) + n_2(t)r(t) \quad \text{a.e.} \]
    holds;
  \item[c)] the set
    \[ S_{\omega\omega}W(t) = \{v(t); v(\cdot) \in S_{\omega\omega}W\} \subset H \]
is compact for almost all $t \in T$.
\end{itemize}

**Lemma 4.2.** The operator $T(u; v)$ defined by (4.8) is continuous from $L^1(T, H) \times \omega\cdot S_{\omega\omega}W$ to $V_u(T, H)$.

The lemma follows from the properties a)–c) of the set $S_{\omega\omega}W$ and Theorem 3.4.

Let $u(\cdot), v(\cdot) \in L^1(T, H)$. Consider the multivalued mapping $t \to U(t, T(u; v)(t))$. Since the function $t \to T(u; v)(t)$ is a right continuous function of bounded variation, this mapping is Lebesgue measurable. Then, from Hypotheses $H(U)$ we infer that the mapping $t \to U(t, T(u; v)(t))$ is an element of the space $L^1(T, \text{cb} H)$. Hence, the set
\[ \Phi(u; v) = \{f(\cdot) \in L^1(T, H); f(t) \in U(t, T(u; v)(t)) \text{ a.e.}\} \quad (4.14) \]
is an element of the space $\text{dcb} L^1(T, H)$. Thus, we can define the multivalued mapping $\Phi : L^1(T, H) \times L^1(T, H) \to \text{dcb} L^1(T, H)$ which is called the Nemytskii multivalued operator.

On the space $L^1(T, H)$ consider the function
\[ P(x) = \int_T \rho(t, x(t)) dt, \quad (4.15) \]
where
\[ \rho(t, x(t)) = (\exp(-2 \int_0^t k(\tau) \, d\tau)) \| x(t) \|, \quad t \in T, \]  
and \( k(\cdot) \) is the function appearing in inequality (1.7). It is clear that the function \( P(x) \) is a norm equivalent to the norm \( \| x \|_{L^1} \) on the space \( L^1(T, X) \).

The Hausdorff distance between elements from \( \text{cb} \, L^1(T, H) \) when the space \( L^1(T, H) \) is endowed with the norm \( P(x) \) we denote by \( \text{haus}_P(\cdot, \cdot) \).

**Theorem 4.1.** The Nemytskii operator \( \Phi(u; v) \) has the properties:

1) the operator \( \Phi(u; v) \) is continuous from \( L^1(T, H) \times \omega-S_{co}^{W} \) to the space \( \text{cb} \, L^1(T, H) \) endowed with the Hausdorff metric \( \text{haus}_{L^1}(\cdot, \cdot) \);

2) the inequality
\[ \text{haus}_{L^1}(\Phi(u_1; v), \Phi(u_2; v)) \leq L \| u_1 - u_2 \|_{L^1}, \]  
where \( u_i(\cdot) \in L^1(T, H), \ i = 1, 2, \ v(\cdot) \in \omega-S_{co}^{W} \) holds, where
\[ L = \int_T k(t) \, dt; \]  
\[ \text{haus}_P(\Phi(u_1; v), \Phi(u_2; v)) \leq \frac{1}{2} P(u_1 - u_2), \]  
where \( v(\cdot) \in \omega-S_{co}^{W}, \ u_i \in L^1(T, H), \ i = 1, 2. \)

**Proof.** Using Proposition 2.4 in [31] we obtain
\[ \text{haus}_{L^1}(\Phi(u_1; v_1), \Phi(u_2; v_2)) \leq \]  
\[ \int_T \text{haus}(U(t, \mathcal{T}(u_1; v_1)(t)), U(t, \mathcal{T}(u_2; v_2)(t))) \, dt. \]  
Hence, according to (1.7) and (4.18) we have
\[ (\Phi(u_1; v_1), \Phi(u_2; v_2)) \leq L \| \mathcal{T}(u_1; v_1)(\cdot) - \mathcal{T}(u_2; v_2)(\cdot) \|_{V^+}. \]  
The statement 1) of Theorem 4.1 follows from the last inequality and Lemma 4.2. Inequality (4.17) follows from inequality (4.20) and the inequality
\[ \| \mathcal{T}(u_1; v)(t) - \mathcal{T}(u_2; v)(t) \| \leq \int_{[0, t]} \| u_1(\tau) - u_2(\tau) \| \, d\tau. \]  
Now, we prove inequality (4.19). From (1.7) and (4.21) we obtain
\[ \text{haus}(U(t, \mathcal{T}(u_1; v)(t)), U(t, \mathcal{T}(u_2; v)(t))) \leq k(t) \int_0^t \| u_1(\tau) - u_2(\tau) \| \, d\tau. \]  
Using this inequality and (4.14)–(4.16) we arrive at the inequality
\[ \text{haus}_P(\Phi(u_1; v), \Phi(u_2; v)) \leq \]  
\[ \leq \int_T (\exp(-2 \int_0^t k(\tau) \, d\tau)) \times k(t) (\int_0^t \| u_1(\tau) - u_2(\tau) \| \, d\tau) \, dt. \]  
Integrating the right-hand side of this inequality by parts we obtain
\[ \text{haus}_P(\Phi(u_1; v), \Phi(u_2; v)) \leq \frac{1}{2} \left( \int_T (\exp(-2 \int_0^t k(\tau) \, d\tau)) \| u_1(t) - u_2(t) \| \, dt \right). \]
Now, inequality (4.19) follows from (4.22), (4.15), (4.16). The theorem is proved.

For a fixed $v(\cdot) \in \omega-S_{\mathfrak{c}W}$ denote by $(\text{Fix } \Phi)(v)$ the set of fixed points of the Nemytskii operator $\Phi(u; v)$.

**Theorem 4.2.** The following statements are true:

a) for any $v(\cdot) \in \omega-S_{\mathfrak{c}W}$ the set $(\text{Fix } \Phi)(v)$ is not empty;

b) there exists a continuous function $u : \omega-S_{\mathfrak{c}W} \to L^1(T, H)$ such that $u(v) \in (\text{Fix } \Phi)(v)$, i.e.

$$
(\text{Fix } \Phi)(v) \ni u(v) \in (\text{Fix } \Phi)(v), \quad v \in \omega-S_{\mathfrak{c}W}.
$$

**Proof.** From Theorem 4.1 it follows that for a fixed $u(\cdot) \in L^1(T, H)$ the mapping $v \to \Phi(u; v)$ is lower semicontinuous from the compact metric space $\omega-S_{\mathfrak{c}W}$ to $L^1(T, H)$ and it has closed, bounded, decomposable values. Now the statement of the theorem follows from inequality (4.19) and Theorem 1.1 [29], if we consider $v(\cdot)$ as a parameter.

5. **Main result and comments.** In this section, we give the proof of Theorem 1.1 and compare our results with those of the works [24, 8].

Let $S_V$ and $S_Y$ be the sets defined by equalities (4.3) and (4.4). According to Theorem 4.2 and (4.23) there exists a continuous function $u : \omega-S_{\mathfrak{c}W} \to L^1(T, H)$ such that $u(v) \in (\text{Fix } \Phi)(v)$. Using (4.14) we obtain

$$
u(v)(t) \in U(t, T(u(v); v))(t) \text{ a.e.} 
$$

(5.1)

Introduce the notation

$$
u(S_{\mathfrak{c}W}) = \{u(v); v(\cdot) \in S_{\mathfrak{c}W}\}. 
$$

**Lemma 5.1.** The following inclusions hold

$$
S_{\mathfrak{c}W} \subset S_V, \quad u(S_{\mathfrak{c}W}) \subset S_Y.
$$

(5.2)

**Proof.** The first inclusion follows from (4.12), (4.13) and (4.4).

Let $v(\cdot) \in S_{\mathfrak{c}W}$. Then from (1.9), (3.15), (3.8), (4.1), (5.1) similarly to (4.7) we obtain

$$
\|x(u(\hat{v}); \hat{v})(t)\| \leq M + \int_{0, t} (m_1(\tau) + n_1(\tau)\|x(u(\hat{v}); \hat{v})(\tau)\|) d\tau + \\
+ \int_{0, t} (m_2(\tau) + n_2(\tau)r(\tau)) d\tau.
$$

(5.3)

Let $r_1(t), r_1(0) = M$ be the solution of the differential equation

$$
\dot{r}_1(t) = (m_1(t) + m_2(t) + n_2(t)r(t)) + n_1(t)r_1(t).
$$

(5.4)

Using (2.7), (4.2) and (5.4) we obtain

$$
\dot{\hat{v}}(t) - \dot{r}_1(t) = n_1(t)(r(t) - r_1(t)).
$$

From this equality we infer that $r(t) = r_1(t), t \in T$. Then, (5.3), (5.4) and Proposition 2.2 imply that

$$
\|x(u(\hat{v}); \hat{v})(t)\| \leq r_1(t) = r(t), \quad t \in T, \quad \hat{v}(\cdot) \in S_{\mathfrak{c}W}.
$$

(5.5)

From this inequality, (5.1) and (1.8) it follows that

$$
\|u(\hat{v})(t)\| \leq m_1(t) + n_1(t)r(t), \quad \hat{v}(\cdot) \in S_{\mathfrak{c}W}.
$$

Therefore, in view of (4.3) the second inclusion in (5.2) holds. The lemma is proved.
Proof of Theorem 1.1. Consider the operator $\mathcal{T}(u(v); v)$ from $\omega-S_{co}W$ to $V_+(T, H)$. From the statement 1) in Theorem 4.1, the statement c) in Theorem 4.2 and Lemma 4.2 we infer that the mapping $v \to \mathcal{T}(u(v); v)$ is continuous from $\omega-S_{co}W$ to $V_+(T, H)$.

Let

$$F(t, x) = V(t, x) \cap (m_2(t) + n_2(t)\|x\|)B$$

(5.6)

and

$$S_F(v) = \{f(\cdot) \in L^1(T, H); f(t) \in F(t, \mathcal{T}(u(v); v)(t)) \text{ a.e.}, \ v \in S_{co}W\}. \tag{5.7}$$

From Lemma 2.2 and Hypotheses $H(V)$ it follows that $S_F(v)$ is nonempty, convex, compact subset of the space $\omega-L^1(T, H)$. From (5.2), (4.11) and (4.13) we infer that

$$S_F(v) \subset S_{co}W, \quad v(\cdot) \in S_{co}W. \tag{5.8}$$

Hence, we can define the multivalued mapping $v \to S_F(v)$ with nonempty, convex, weakly compact values from the convex, compact, metrizable set $\omega-S_{co}W$ to $\omega-S_{co}W$.

Let

$$K = \{\mathcal{T}(u(v); v); \ v(\cdot) \in \omega-S_{co}W\}.$$ 

Then the set $K$ is a compact subset of the space $V_+(T, H)$. From Corollary 2.1 in [30] it follows that there exists a compact set $D \subset H$ such that

$$\mathcal{T}(u(v); v)(t) \in D, \quad t \in T, \ v(\cdot) \in S_{co}W. \tag{5.9}$$

Using Hypothesis $H(V)$ 3 we see that the values of the mapping $t \to V(t, D) \cap (m_2(t) + n_2(t)\|D\|)B$ are relatively compact sets for almost every $t \in T$.

Denote

$$V^*(t) = V(t, D) \cap (m_2(t) + n_2(t)\|D\|)B,$$

Then, the values of the mapping $t \to \omega V^*(t)$ are convex, compact sets for almost every $t \in T$. Since $F(t, x) \subset \omega V^*(t)$, $x \in D$, from Hypothesis $H(V)$ 2 it follows that the mapping $x \to F(t, x)$, $x \in D$ is upper semicontinuous with convex, compact values for almost every $t \in T$.

Let a sequence $v_n \in \omega-S_{co}W$, $n \geq 1$, weakly converge to $v(\cdot) \in \omega-S_{co}W$. Then the sequence $\mathcal{T}(u(v_n); v_n)(t)$, $n \geq 1$, converges to $\mathcal{T}(u(v); v)$ in the space $V_+(T, H)$. Since $\mathcal{T}(u(v_n); v_n)(t) \in D$, $t \in T$, $n \geq 1$, and the mapping $x \to F(t, x)$ is upper semicontinuous for almost every $t \in T$, we have

$$\bigcap_{n=1}^{\infty} \bigcup_{k \geq n} F(t, \mathcal{T}(u(v_n); v_n)(t)) \subset F(t, \mathcal{T}(u(v); v)(t)) \text{ a.e.} \tag{5.10}$$

If a sequence $f_n(\cdot) \in S_F(v_n)$, $n \geq 1$, weakly converges to $f$, then from (5.10), (5.7) and Mazur’s lemma for weakly converging sequences we infer that $f(t) \in F(t, \mathcal{T}(u(v); v)(t))$ a.e. Hence, according to (5.7) we have $f(\cdot) \in S_F(v)$. Consequently, the mapping $v \to S_F(v)$, $v \in \omega-S_{co}W$ has a weakly closed graph. Then, from the inclusion (5.8) we deduce that the mapping $v \to S_F(v)$, $v \in \omega-S_{co}W$ is upper semicontinuous from $\omega-S_{co}W$ to $\omega-S_{co}W$ with convex, compact values. According to the Ky Fan Theorem [18] there exists a fixed point $v_*$ of the mapping $v \to S_F(v)$, i.e.

$$v_* \in S_F(v_*). \tag{5.11}$$

Let $u_*(\cdot) = u(v_*(\cdot))$, $x(u_*(v_*)\cdot)(t) = \mathcal{T}(u(v_*(v_*))(t)$. Then, from (5.11), (5.7), (5.6), (5.1), (4.5) and (4.6) we obtain

$$-\dot{x}(u_*(v_*))(t) \in \mathcal{N}(C(t); x(u_*(v_*))(t)) + (u_*(t) + v_*(t))d\lambda,$$
\[-\frac{dx(u_\ast;v_\ast)}{dv}(t) - (u_\ast(t) + v_\ast(t)) \frac{d\lambda}{dv}(t) \in \mathcal{N}(C(t);x(u_\ast;v_\ast)(t)) \text{ } \nu \text{ a.e.} \]
\[u_\ast(t) \in U(t,x(u_\ast;v_\ast)(t)) \text{ a.e.,} \]
\[v_\ast(t) \in V(t,x(u_\ast;v_\ast)(t)) \text{ a.e.,} \]
\[\|x(u_\ast;v_\ast)(t) - x(u_\ast;v_\ast)(t - 0)\| \leq \mu(\{t\}), \quad t \in T. \]

Therefore, according to (1.4)–(1.6) the triple \((x(u_\ast;v_\ast)(\cdot), u_\ast(\cdot), v_\ast(\cdot))\) is a solution of inclusion (1.2). Thus, Theorem 1.1 is proved.

**Remark 5.1.** From (5.11) and (5.5) it follows that
\[\|x(u_\ast;v_\ast)(t)\| \leq r(t), \quad t \in T.\]

In the work [24] the following theorem was proved.

**Theorem 5.1 (Theorem 4.1 [24]).** Let \(C : T \to H\) be an \(r\)-prox-regular multivalued mapping with \(r \in [0, +\infty)\) and there exists a finite positive Radon measure \(\mu\) on \(T\) with \(\sup_{s \in [0,a]} \mu(\{s\}) < r/2\). For all \(y \in H, s,t \in T, s < t\) the following inequality holds
\[|d(y,C(t)) - d(y,C(s))| \leq \mu([s,t]). \tag{5.12}\]

Let \(F : T \times H \to H\) be a multivalued mapping with nonempty, convex, compact values such that
(i) \(F(\cdot,\cdot)\) is scalarly upper semicontinuous;
(ii) there exists a compact set \(K \subset \overline{B}\) and a function \(\beta : T \to \mathbb{R}^+\) with \(\beta(\cdot) \in L^1(T,\mathbb{R}^+)\), such that
\[F(t,x) \subset \beta(t)(1 + \|x\|)K, \quad t \in T, x \in \bigcup_{s \in T} C(s). \tag{5.13}\]

Let \(f : T \times H \to H\) be a single-valued mapping such that
(iii) for every \(\rho > 0\) there exists a function \(L_\rho : T \to \mathbb{R}^+, L_\rho(\cdot) \in L^1(T,\mathbb{R})\) such that
\[\|f(t,x) - f(t,y)\| \leq L_\rho(t)\|x - y\|, \quad t \in T, x,y \in \rho\overline{B}. \tag{5.14}\]
(iv) \(f(\cdot, x)\) is Lebesgue measurable for all \(x \in H\) and there exists a function \(\alpha : T \to \mathbb{R}^+\) \(\alpha(\cdot) \in L^1(T,\mathbb{R})\) such that
\[\|f(t,x)\| \leq \alpha(t)(1 + \|x\|), \quad t \in T, x \in \bigcup_{s \in T} C(s). \tag{5.15}\]

Then, for every \(u_0 \in C(0)\) the perturbed sweeping process
\[-du \in \mathcal{N}(C(t);u(t)) + (F(t,u(t)) + f(t,u(t))) d\lambda, \tag{5.16}\]
\(u(0) = u_0\) has a solution satisfying the inequality \(\|u(t) - u(t-0)\| \leq 2\mu(\{t\}), t \in [0,a]\).

In the work [8] instead of inequality (5.12) one considered the inequality
\[|d(y,C(t)) - d(y,C(s))| \leq |v(t) - v(s)|, \quad y \in H, s,t \in T, \quad v : T \to \mathbb{R}\]
where \(v : T \to \mathbb{R}\) is an absolutely continuous function. Assuming that \(F(\cdot,\cdot)\) and \(f(\cdot,\cdot)\) possess the same properties as in Theorem 5.1 the existence of an absolutely continuous solution was proved (Theorem 3.1 [8]). Since a closed, convex set is r-prox-regular, in the case when the values of the mapping \(C(t)\) are closed, convex sets in the works [24, 8], Theorem 4.1 in [24] is a particular case of our Theorem 1.1, and Theorem 3.1 [8] follows from Theorem 1.1.

To prove this, let \(pr(C(t), x)\) be the projection of a point \(x \in H\) to the set \(C(t)\). The function \(t,x \to pr(C(t); x)\) enjoys the following properties [28]:
1) the function \(t \to pr(C(t); x)\) is Lebesgue measurable for any \(x \in H\);
2) the following inequality holds
\[
\|pr(C(t); x) - pr(C(t); y)\| \leq \|x - y\|, \quad x, y \in H, \; t \in T, \quad (5.17)
\]
\[
\|pr(C(t); x)\| \leq d(\Theta, C(0)) + \mu(T) + 2\|x\|, \quad t \in T. \quad (5.18)
\]
Let
\[
M(r) = d(\Theta, C(0)) + \mu(T) + 2r, \quad r \geq 0,
\]
\[
K_r(t) = L_{M(r)}(t) + 1, \quad (5.20)
\]
\(K_r(t) > 0, \; t \in T, \; K_r(\cdot) \in L^1(T, \mathbb{R}^+)\), where \(L_{M(r)}(t)\) is the function appearing in (5.14). Denote
\[
\hat{f}(t, x) = f(t, pr(C(t); x)), \quad x \in H, \; t \in T. \quad (5.21)
\]
Then, from (5.15), (5.18) it follows that
\[
\|\hat{f}(t, x)\| < m_1(t)(1 + \|x\|), \quad x \in H, \; t \in T, \quad (5.22)
\]
where
\[
m_1(t) = 2(\alpha(t) + 1)(1 + d(\Theta, C(0)) + \mu(T)), \quad m_1(t) > 0, \; m_1(\cdot) \in L^1(T, \mathbb{R}^+).
\]
Using (5.14), (5.17), (5.18), (5.20), (5.21) we infer that the function \(t \rightarrow \hat{f}(t, x), \; x \in H\) is measurable and
\[
\|\hat{f}(t, x) - \hat{f}(t, y)\| \leq K_r(t)\|x - y\|, \quad x, y \in r \mathcal{B}. \quad (5.23)
\]
Let
\[
\hat{F}(t, x) = F(t, pr(C(t); x)), \quad x \in H, \; t \in T. \quad (5.24)
\]
Without loss of generality, we can assume that the compact set \(K\) in (5.13) is convex and balanced.

Denote
\[
m_2(t) = 2(\beta(t) + 1)(1 + d(\Theta, C(0)) + \mu(T)).
\]
Then, from (5.13) it follows that
\[
\|\hat{F}(t, x)\| < m_2(t)(1 + \|x\|), \quad x \in H, \; t \in T, \quad (5.25)
\]
\[
\hat{F}(t, x) \subset m_2(t)(1 + \|x\|)K, \quad x \in H, \; t \in T, \; m_2(\cdot) \in L^1(T, \mathbb{R}^+). \quad (5.26)
\]
From (5.25), (5.26) we infer that
\[
\hat{F}(t, x) \cap (m_2(t) + m_2(t)\|x\|) \mathcal{B} \neq \emptyset, \quad x \in H, \; t \in T
\]
and for any bounded set \(D \subset H\) the set
\[
\hat{F}(t, D) \cap (m_2(t) + m_2(t)\|D\|) \mathcal{B}
\]
is relatively compact for any \(t \in T\). From the assumption (ii) we see that the mapping \((t, x) \rightarrow \hat{F}(t, x)\) has a closed graph in \(T \times H \times H\). Since the function \((t, x) \rightarrow pr(C(t); x)\) is measurable in \(t\) and continuous in \(x\), there exists an increasing by inclusion sequence \(T_1 \subset T_2 \subset \ldots \subset T_n \subset \ldots \subset T\) of closed sets such that \(\lambda(T \setminus \bigcup_{n=1}^{\infty} T_n) = 0\) and the restriction of the function \(pr(C(t); x)\) to \(T_n \times H, \; n \geq 1\), is continuous. Then, the restriction of the mapping \(\hat{F}(t, x)\) to \(T_n \times H, \; n \geq 1\), has a closed graph in \(T_n \times H \times H\). Hence, for almost every \(t \in T\) the mapping \(x \rightarrow \hat{F}(t, x)\) has a closed graph in \(H \times H\). Using the statement (iii) in Theorem 3.5 [15] we see that the mapping \(t \rightarrow \hat{F}(t, x), \; x \in H\) is measurable. Since the mapping
\[
t \rightarrow (m_2(t) + m_2(t)\|x\|) \mathcal{B}
\]
is measurable, and, thus, it has a measurable selector [15].
Let 
\[ m(t) = m_1(t) + m_2(t), \quad n(t) = n_1(t) + n_2(t), \quad t \in T \]
and \( r(t), \quad r(0) = M \) be the solution of the differential equation (2.5). Let \( \gamma = \max \{ r(t); \quad t \in T \} \) and
\[ \Phi(t, x) = \hat{f}(t, pr(\gamma B; x)). \quad (5.27) \]
Then from (5.22), (5.23), (5.27) and the measurability of the function \( t \rightarrow f(t, x) \)
it follows that the function \( \Phi(t, x) \) possesses the following properties:

1) the function \( t \rightarrow \Phi(t, x) \) is Lebesgue measurable;
2) \( \| \Phi(t, x) - \Phi(t, y) \| \leq K \gamma \| x - y \|, \quad x, y \in H \);
3) \( \| \Phi(t, x) \| < m_1(t)(1 + \| x \|), \quad x \in H, \quad t \in T. \)

Consider the differential inclusion
\[ -du \in \mathcal{N}(C(t); u(t)) + (\Phi(t, u(t)) + \hat{F}(t, u(t))) d\lambda, \quad (5.28) \]
\[ u(0) = u_0 \in C(0). \]

From the properties of the function \( \Phi(t, x) \) and the multivalued mapping \( \hat{F}(t, x) \)
and Theorem 1.1 it follows that inclusion (5.28) has a solution \( u(t), \quad u(0) = u_0 \in C(0) \) which satisfies, in view of Remark 5.1, the inequality
\[ \| u(t) \| \leq r(t), \quad t \in T. \quad (5.29) \]

Since
\[ u(t) \in C(t), \quad (5.30) \]
from (5.29), (5.30), (5.21), (5.24), (5.27) we infer that
\[ F(t, u(t)) = \hat{F}(t, u(t)), \quad f(t, u(t)) = \Phi(t, u(t)), \quad t \in T. \]

Therefore, the function \( u(t), \quad u(0) = u_0 \) is a solution of inclusion (5.16).

Despite the seeming generality expressed in Hypotheses (i)–(iv), when the values
of the mapping \( t \rightarrow C(t) \) are closed, convex sets, Theorem 5.1 (Theorem 4.1 [24]) follows from our Theorem 1.1. Similarly, if inequality (1.13) holds, then Theorem 3.1 [8] is a corollary of our Theorem 1.1. In conclusion, we note that inequality (1.1)
follows from inequality (5.12), and the values of the mapping \( V(t, x) \) in Theorem 1.1 are closed, convex sets, whereas in [24, 8] the values of the mapping \( F(t, x) \) are convex, compact sets.

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