Analytical and numerical solution of coupled KdV-MKdV system

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March 28, 2022

Abstract

The matrix 2x2 spectral differential equation of the second order is considered on x in \((-\infty, +\infty)\). We establish elementary Darboux transformations covariance of the problem and analyze its combinations. We select a second covariant equation to form Lax pair of a coupled KdV-MKdV system. The sequence of the elementary Darboux transformations of the zero-potential seed produce two-parameter solution for the coupled KdV-MKdV system with reductions. We show effects of parameters on the resulting solutions (reality, singularity). A numerical method for general coupled KdV-MKdV system is introduced. The method is based on a difference scheme for Cauchy problems for arbitrary number of equations with constants coefficients. We analyze stability and prove the convergence of the scheme which is also tested by numerical simulation of the explicit solutions.

1 Introduction

There are two complementary approaches to integrable systems: analytical and numerical ones to be developed. Even most profound analytical IST method cannot give explicit solution of general Cauchy problem while numeric can, but is rather compulsory in use: calculations could need powerful computers. May be most transparent of analytical methods are based on algebraic structures associated with a problem. To such structures belongs Darboux transformations covariance of Lax representation of nonlinear equations that yields a powerful tool for explicit solutions production. We investigate applications of special kind of such discrete symmetry - to be called elementary ones \[10\]. Its elementarity simply means that a product of such transformations generate the standard one \[10, 13\]. Here we study combinations of such transforms that do not coincide with binary ones \[12\] and hence are not so known.

The main ideas of numerical integration of such integrable systems go up to the famous properties of the equations as the Lax pair and infinite series of conservation laws existence \[7\]. From a point of view of general theory of such systems some hopes are concerned with a development of the finite-difference or other approximations of the systems. Namely if one could prove a convergence and stability theorems for such difference systems (existence of solutions is implied), a way to existence and uniqueness of solutions is opened \[6\].

The coupled KdV-MKdV system arises in many problems of mathematical physics. Some integrable systems are associated with a polynomial spectral problem and have Virasoro symmetry algebras are considered \[17\]. A dispersive system describing a vector multiplet interacting with the KdV field is a member of a bi-Hamiltonian integrable hierarchy \[8\]. Recently a multisymplectic numerical twelve points scheme was produced. This scheme is equivalent to the multisymplectic Preissmann scheme and is applied.
to solitary waves over long time interval \[14\]. The coupled KdV-MKdV system is also connected to other physical applications \[11\].

The general system we consider in this work have the following form

\[
\theta_t^n + \sum_{m,k} \left( g_{m,k}^{n,1} \theta^m \theta_x^k + g_{m,k}^{n,2} (\theta^m)^2 \theta_x^k + g_{m,k}^{n,3} \theta^m \theta_x^k + g_{m,k}^{n,4} \theta^m \theta_{xx}^k + g_{m,k}^{n,5} (\theta^m)_{xx}^k \right) + d_n \theta_{xxx}^n = 0, \tag{1}
\]

where \( n, m, k = 1, 2, \ldots, N \) are the dependent variables numbers. Nonlinear coefficients are \( g_{m,k}^{n,l} \), \( l = 1, 2, \ldots, 5 \) and \( d_n \) are dispersion coefficients.

In particular, for the system under consideration \((N = 3)\) the variables \( \theta^1, \theta^2, \theta^3 \) are denoted by \( f, u, v \) to have the integrable system \[10\]

\[
\begin{align*}
f_t + \frac{1}{2} f_{xxx} + \frac{3}{2} (uf)_x - \frac{3}{4} f_x f^2 &= 0, \\
u_t - \frac{1}{4} u_{xxx} - \frac{3}{2} u_x u + 3v v_x + \frac{3}{4} u_x f^2 - \frac{3}{2} (f_x v)_x &= 0, \\
v_t + \frac{1}{2} v_{xxx} + \frac{3}{4} v_x u - \frac{3}{4} (vf^2)_x + \frac{3}{2} u_{xx} f + \frac{3}{2} u_x f_x &= 0 \tag{2}
\end{align*}
\]

The Lax pair is given in \[10\]. The system exhibits two integrable reductions having explicit solutions, Hirota-Satsuma \[4, 1\] and a two components KdV-MKdV system \[10, 12\]. Krishnan \[5\] showed that a generalized KdV-MKdV system have solitary wave solutions and investigate the effects of increasing the nonlinearity of one variable on the existence of solitary waves. Some form of KdV-MKdV system have explicit solutions in terms of Jacobi elliptic functions \[2\].

In this work we present explicit solutions for a system of three equations \((N = 3)\) that have not been specified in \[10, 12\]. We study this two-parameter explicit solutions and show effects of choosing these parameters on the solutions. We demonstrate the use of two arbitrary elementary DTs \[10\] and its special choice that holds a hereditary of the reduction to built explicit solutions to the KdV-MKdV system \(2\).

Also we modify a numerical method \[7, 3\] for solution of system \(1\). It is a difference scheme for Cauchy problems for arbitrary number of equations with constants coefficients. The scheme preserves two conservation laws for the KdV type equations and the order of error of the difference formulas is improved \[7, 15\]. The convergence is proved and stability is analyzed giving the conditions taken in account in choosing time and space step sizes \[16, 9\].

The present work is organized as follows. Section 2 introduces the matrix spectral equation of the second order with 2 x 2 matrix coefficients and two elementary DTs. We select the second equation of the Lax pair and derive the compatibility conditions. The product of these two transformations yields the standard DT \[10, 13\]. Section 3 illustrates how the, first, elementary DT is used to produce solution to the KdV equation as well as the general evolution equation generated by the compatibility conditions of the Lax pair. Explicit solutions are introduced for the case of zero initial potentials of the matrix problem. In section 4 we consider a reduction constraints on the potential of the matrix spectral equation. This reduction gives an automorphism that relates two pairs of solution of the spectral equation for two spectral parameters. We use this results in the compound elementary DTs to produce an explicit solution to a coupled KdV-MKdV system that results from the compatibility conditions of Lax pair under this reduction. The effects of these parameters on the solution (reality, singularity) is analyzed. Section 5 introduces a numerical method for solving coupled KdV-MKdV system \(1\). We produce a difference scheme for a Cauchy problem with initial condition rapidly decreasing at both infinities. The main steps of the scheme convergence and stability analysis is shown while the details are explained in appendix A and B. The scheme is tested by applying it to integrable coupled KdV-MKdV system and the numerical results are compared with explicit formulas obtained in section 4.
2 Lax pair spectral equations and the elementary DTs

Consider a matrix spectral equation of the second order with spectral parameter \( \lambda \) and \( 2 \times 2 \) matrix coefficients.

\[
\Psi_{xx} + F \Psi_x + U \Psi = \lambda \sigma_3 \Psi
\]  

(3)

where the vector \( \Psi = (\Psi_1, \Psi_2)^T \) and the matrix potentials are \( U = \{u_{ij}\}, F = \{f_{ij}, f_{ii} = 0\}, i = 1, 2 \) while \( \sigma_3 = diag(1, -1) \) is the Pauli matrix.

For equation (3) we perform two elementary Darboux transforms [10]. The first one is

\[
\tilde{\Psi}_1 = \Psi_{1x} + \epsilon_{11} \Psi_1 + \epsilon_{12} \Psi_2, \quad \epsilon_{11} = - \left( \varphi_{1x} + \frac{1}{2} f_{12} \varphi_2 \right) / \varphi_1, \quad \epsilon_{12} = f_{12} / 2,
\]

\[
\tilde{\Psi}_2 = \Psi_2 + \epsilon_{21} \Psi_1, \quad \epsilon_{21} = - \varphi_2 / \varphi_1,
\]

\[
\tilde{\varphi}_1 = (\vartheta_x + \epsilon_{11}) \varphi_3 + \epsilon_{12} \varphi_4, \quad \tilde{\varphi}_2 = \varphi_4 + \epsilon_{21} \varphi_3.
\]

(4)

where \( (\varphi_1, \varphi_2)^T \) and \( (\varphi_3, \varphi_4)^T \) are two solutions of (3) corresponding to different spectral parameters.

Substituting the above expressions for \( \tilde{\Psi}_1, \tilde{\Psi}_2 \) into (3) and collecting the coefficients of \( \Psi_1, \Psi_2 \) and their derivatives we obtain the expressions for new potentials as

\[
\tilde{f}_{12} = u_{12} + f_{12} \epsilon_{11},
\]

\[
\tilde{f}_{21} = -2 \epsilon_{21},
\]

\[
\tilde{u}_{11} = u_{11} - 2 \epsilon_{11} \varphi_x - f_{12} \epsilon_{21} - f_{21} \epsilon_{12},
\]

\[
\tilde{u}_{12} = u_{12} - \epsilon_{12} \varphi_x + \epsilon_{11} u_{12} - \epsilon_{12} \left( u_{11} + u_{22} \right),
\]

\[
\tilde{u}_{21} = f_{21} - 2 \epsilon_{21} \varphi_x - \tilde{f}_{21} \epsilon_{11},
\]

\[
\tilde{u}_{22} = u_{22} - \epsilon_{21} u_{12} - \tilde{u}_{21} \epsilon_{12} - \tilde{f}_{21} \epsilon_{12}.
\]

(5)

The second elementary DT is performed after the first one and can be obtained by reversing the indices \( 1 \to 2 \) and \( 2 \to 1 \) to get, for example, the following potentials

\[
\tilde{f}_{21} = \tilde{u}_{21} + \tilde{f}_{21} \epsilon_{22}, \quad \tilde{f}_{21} = - \left( \varphi_{2x} + \frac{1}{2} f_{12} \varphi_1 \right) / \varphi_2,
\]

\[
\tilde{u}_{22} = \tilde{u}_{22} - 2 \tilde{f}_{22} \varphi_x - \tilde{f}_{21} \tilde{\varphi}_1 - \tilde{f}_{12} \tilde{\varphi}_2, \quad \tilde{\varphi}_1 = \tilde{\varphi}_1 / \tilde{\varphi}_2, \quad \tilde{\varphi}_2 = \tilde{\varphi}_2 / \tilde{\varphi}_1.
\]

(6)

The spectral equation (3) is considered as the first equation of the Lax pair, take the second as

\[
\Psi_t = \Psi_{xxx} + B \Psi_x + C \Psi
\]

(7)

where \( B = \frac{3}{2} diag(U) + \frac{3}{2} F_x + \frac{3}{4} F^2 \)

and \( C = \frac{3}{2} U_x - \frac{3}{4} diag(U) x - \frac{3}{4} (f_{12} u_{21} + f_{21} u_{12}) I + \frac{3}{8} (f_{12, x} f_{21} - f_{12} f_{21, x}) \sigma_3 + \frac{3}{4} (u_{11} - u_{22}) \sigma_3 F. \)

Equation (7) is also covariant under transformations (4), (5). The compatibility conditions have the following form

\[
F_t - F_{3x} + B_{2x} - 3 U_{2x} + 2 C_x + F B_x - \sigma_3 B \sigma_3 F_x + U B
\]

\[- \sigma_3 B \sigma_3 U + F C - \sigma_3 C \sigma_3 F = 0, \]

\[
U_t - U_{3x} + C_{2x} + U C - \sigma_3 C \sigma_3 U + F C_x - \sigma_3 B \sigma_3 U_x = 0.
\]

(8)

and the transformations (4), (5) determine a discrete symmetry of (2).
3 Solution of two coupled KdV-MKdV equations and KdV equation via the first elementary DT

For a spectral parameter $\lambda$ and a seed potential $F, U$ we obtain the solutions $\varphi_1, \varphi_2$ to the pair (3), (7). Then performing the first elementary DT to obtain the new potentials $\tilde{F}, \tilde{U}$ which are solutions to the system (2). For the case of zero seed potential the solutions, $\varphi_1$ and $\varphi_2$ of the system (3), (7) have the form

$$\varphi_1 = c_1 e^{ax + \lambda t} + c_2 e^{-(ax + \lambda t)},$$
$$\varphi_2 = d_1 e^{iax} + d_2 e^{-(iax + \lambda t)}.$$

(9)

where $c_1, c_2, d_1, d_2$ are arbitrary constants, $a = \sqrt{\lambda}$ and $i$ is the imaginary unit. System (2) reduced (for the only nonzero elements) to the following

$$f_{21t} + \frac{1}{2} f_{21xxx} + \frac{3}{4} f_{21u_{11x}} = -\frac{3}{2} u_{11} u_{21},$$
$$u_{11t} - \frac{1}{4} u_{11xxx} - \frac{3}{2} u_{11} u_{11x} = 0,$$
$$u_{21t} + \frac{1}{2} u_{21xxx} + \frac{3}{4} u_{21} u_{11x} + \frac{3}{2} u_{21x} u_{11} = \frac{3}{4} u_{11} f_{21xx} + \frac{3}{4} u_{11}^2 f_{21}.$$

(10)

where $f_{12} = 0, u_{12} = 0, u_{22} = 0$ and tildes are omitted for simplicity. This system with explicit solution obtained from (4), (3) as

$$f_{21} = \left(2 e^{(1-i)\lambda(a^2 t + x)} \left(d_2 e^{-2ia t + d_1 e^{2ia x}}\right) / (c_2 + c_1 e^{2a(a^2 t + x)})\right),$$
$$u_{11} = \left(8 a^2 c_1 c_2 e^{2a(a^2 t + x)} / (c_2 + c_1 e^{2a(a^2 t + x)})\right)^2,$$
$$u_{21} = \left(-2 ia e^{(1-i)\lambda(a^2 t + x)} \left(d_2 e^{-2ia t + d_1 e^{2ia x}}\right) / (c_2 + c_1 e^{2a(a^2 t + x)})\right).$$

(11)

where $c_1, c_2, d_1, d_2$ are arbitrary constants and $a = \sqrt{\lambda}$.

The second equation in (3) is the KdV equation while the remaining are a two components coupled KdV-MKdV system that was solved by elementary DT.

4 Solution of three coupled KdV-MKdV equations via the compound elementary DTs

Existence of different kinds of automorphism causes special constraints (3). Multiplying (3) by $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to have

$$\sigma_1 \Psi_{xx} + \sigma_1 F \Psi_x + \sigma_1 U \Psi = \lambda \sigma_1 \sigma_3 \Psi$$

(12)

but $\sigma_1 \sigma_3 = -\sigma_3 \sigma_1$ and consider the conditions $\sigma_1 F = F \sigma_1$ and $\sigma_1 U = U \sigma_1$ that means

$$f_{12} = f_{21} = f, \quad u_{11} = u_{22} = u, \quad u_{12} = u_{21} = v.$$  

(13)

So (12) becomes

$$(\sigma_1 \Psi)_{xx} + F (\sigma_1 \Psi)_x + U (\sigma_1 \Psi) = -\lambda \sigma_3 (\sigma_1 \Psi)$$
The above automorphism $\Psi (\lambda) \leftarrow \sigma_1 \Psi (-\lambda)$ relates two pair of solutions $(\varphi_1, \varphi_2)$ and $(\varphi_3, \varphi_4)$ of (3) corresponding to different values of spectral parameter $\lambda, -\lambda$ as

$$
\begin{pmatrix}
\varphi_3(-\lambda) \\
\varphi_4(-\lambda)
\end{pmatrix} = \sigma_1
\begin{pmatrix}
\varphi_1(\lambda) \\
\varphi_2(\lambda)
\end{pmatrix} = \begin{pmatrix}
\varphi_2(\lambda) \\
\varphi_1(\lambda)
\end{pmatrix}.
$$

Using this result in the elementary DTs (4), (5) and (6) to obtain the expressions for the new potentials $f, u, v$. In the case of zero initial potentials these new potentials have the following forms

$$
\begin{align*}
f &= 2\frac{\varphi_1(\varphi_2)_x - \varphi_2(\varphi_1)_x}{(\varphi_1)^2 - (\varphi_2)^2}, \\
u &= \left(\frac{(\varphi_1^2)_x - (\varphi_2^2)_x}{(\varphi_1)^2 - (\varphi_2)^2}\right) + 2\left(\frac{\varphi_1(\varphi_2)_x - \varphi_2(\varphi_1)_x}{(\varphi_1)^2 - (\varphi_2)^2}\right)^2, \\
v &= 2\left(\frac{\varphi_1(\varphi_2)_x - \varphi_2(\varphi_1)_x}{(\varphi_1)^2 - (\varphi_2)^2}\right)_x + \frac{(\varphi_1(\varphi_2)_x - \varphi_2(\varphi_1)_x)((\varphi_1^2)_x - (\varphi_2^2)_x)}{(\varphi_1^2 - (\varphi_2)^2)^2}. \tag{14}
\end{align*}
$$

where $\varphi_1, \varphi_2$ are as in (3) with $c_1, c_2, d_1, d_2$ are arbitrary constants and $a = \sqrt{\lambda}$.

The above expressions are solutions of system (2) that reduced under the reduction conditions (3) to system (4).

The choice of the arbitrary constants $(c_1, c_2, d_1, d_2)$ affects on the behavior of the solution in formula (3). For example choosing equal constants $c_1 = c_2 = d_1 = d_2 = 0.5$ (we choose the value to be 0.5 to simplify the resulting formula but the idea valid for any value) the solutions have the form

$$
\begin{align*}
f &= 2a(\sin \eta_1 \cosh \eta_2 - \cos \eta_1 \sinh \eta_2)/(\cosh^2 \eta_2 - \cos^2 \eta_1), \\
u &= 2a^2(\sin \eta_1 \cosh \eta_2 + \cos \eta_1 \sinh \eta_2)^2/(\cosh^2 \eta_2 - \cos^2 \eta_1)^2, \\
v &= 2a^2(\cos 3\eta_1 \cosh \eta_2 - 2\sin \eta_1 \sinh \eta_2)(\cos 2\eta_1 + \cosh 2\eta_2 + 2) - \cos \eta_1 \cosh 3\eta_2)/(\cosh^2 \eta_2 - \cos^2 \eta_1)^2. \tag{15}
\end{align*}
$$

where $\eta_1 = a^3 t - ax$, $\eta_2 = a^3 t + ax$, $a = \sqrt{\lambda}$ is real.

We see that the above expression (15) is singular at $\eta_2 = 0, \eta_1 = n\pi, n = 0, 1, 2, ...$. Hence we have singularity at $(x = \frac{n\pi}{2a}, t = \frac{n\pi}{2a^3})$.

To obtain continuous solutions we can choose $c_1 = c_2, d_1 = d_2 = r, c_1$, $r$ is real constant. We again choose $c_1 = 0.5$ following the previous concept. So (4) have the form

$$
\begin{align*}
f &= 2ar (\cosh \eta_2 \sin \eta_1 - \cos \eta_1 \sinh \eta_2)/(\cosh^2 \eta_2 - r^2 \cos^2 \eta_1), \\
u &= a^2 (1 - r^4 - r^4 \cos 2\eta_1 + \cosh 2\eta_2 + r^2 \sin 2\eta_1 \sinh 2\eta_2)/(\cosh^2 \eta_2 - r^2 \cos^2 \eta_1)^2, \\
v &= 2a^2 r(((-7 + 6r^2 + 2r^2 \cos 2\eta_1) \cos \eta_1 \cosh \eta_2 - \cos \eta_1 \cosh 3\eta_2) - 2(1 + r^2 + r^2 \cos 2\eta_1 + \cosh 2\eta_2) \sin \eta_1 \sinh \eta_2))/(1 + r^2 + r^2 \cos 2\eta_1 - \cosh 2\eta_2)^2. \tag{16}
\end{align*}
$$

where $a, \eta_1, \eta_2$ as in (3)

Choosing this parameter $(r)$ to be $r < 1$ gives real nonsingular solutions. The above formula (4) is built from elliptic and periodic functions so it does not preserve its symmetry but its localized as shown in figures (1.a,b) below.
Fig.(1.a) Non-singular solutions, f, u and v (r=0.5), a=2, t=0.

Fig.(1.b) Propagation of solutions, f, u and v (r=0.5), a=2, t=1.

Fig.(1) The solutions in (4) does not preserve its symmetry but its localized.

Choosing the parameter r to be $r > 1$ in formula (4) gives singular solutions as shown figure (2) below.

Fig.(2) Singular solutions, f, u and v (r=2), a=2, t=0.

Moreover the choice of these arbitrary constants ($c_1$, $c_2$, $d_1$, $d_2$) as well as the spectral parameter $\lambda$ affects the reality of the resulting solution. As example for $\lambda = -2i m^2$, $m$ is real and choosing $c_1 = c_2 = d_1 = d_2 = 0.5$, we get real solution

\[
f = m(\cos 2\zeta_1 \sinh \zeta_2 - \sinh \zeta_2 - \sin \zeta_1 \cosh 2\zeta_2 + \sin \zeta_1)/(0.25 \cosh 2\zeta_2 - 0.25)(1 - \cos 2\zeta_1)
\]

where $\zeta_1 = 2mx + 4m^2t$, $\zeta_2 = 2mx - 4m^2t$, while choosing $c_1 = c_2 = 1, d_1 = d_2 = 2$ give the following complex solution

\[
f = m^*(-8(-5 \sinh \zeta_2 \cos 2\zeta_1 + 5 \sinh \zeta_2 + 5 \sin \zeta_1 \cosh 2\zeta_2 - 5 \sin \zeta_1) - 8i(6 \sinh \zeta_2 + 3 \sin 2\zeta_1 \sinh 2\zeta_2 + 6 \sin \zeta_1))/(17 \cosh 2\zeta_2 + 10 + 36 \cos \zeta_1 \cosh \zeta_2 - 8 \cos 2\zeta_1 \cosh 2\zeta_2 + 17 \cos 2\zeta_1)
\]

5 The numerical method

5.1 The difference scheme

For the coupled KdV-MKdV system (4) we introduce a numerical (finite difference) method of solution (6, 15). This scheme is valid for arbitrary number of equations with constant coefficients and of the form

\[
\frac{\theta_{i,j+1}^{n} - \theta_{i,j}^{n}}{\tau} + \sum_{m,k}(g_{m,k}^{1} \theta_{i,j}^{m,k} \theta_{i+1,j+1}^{k} - \theta_{i-1,j+1}^{k})/2h + g_{m,k}^{2} \theta_{i,j}^{m,j} \theta_{i+1,j+1}^{k} - \theta_{i-1,j+1}^{k})/2h + g_{m,k}^{3} \theta_{i,j}^{m,j} \theta_{i+1,j+1}^{k} - \theta_{i-1,j+1}^{k})/2h + g_{m,k}^{4} \theta_{i,j}^{m,j} \theta_{i+1,j+1}^{k} - \theta_{i-1,j+1}^{k})/2h = 0
\]

where $i$ and $j$ are the discrete space and time respectively. The time step is denoted by $\tau$ while $h$ denotes spatial step.
5.2 Stability analysis of the scheme

We prove stability with respect to small perturbations of initial conditions \([16, 9]\). It is the boundness of the discrete solution with respect to small perturbation of the initial data. We give here the main steps while the details are presented in Appendix A. We can write

\[
d\theta_j^{n,j+1} = T_{i,r}^{n,j+1} d\theta_r^{n,j} = T_{i,r}^{n,j+1} T_{i,r}^{n,j} d\theta_{i,r}^{n,j-1} = \Pi_r (T_{i,r}^n)^r d\theta_{r}^{n,o}
\]

where \(d\theta_j^{n,j+1}\) is perturbations of the discrete solution, \(d\theta_r^{n,o}\) small perturbation of the initial data and \(T_{i,r}^{n,j+1}\) is a differentiable operator. Stability required the boundedness of \(\Pi_r (T_{i,r}^n)^r\), i.e. \(\|T^r\|\) is bounded.

We found that

\[
\tau \leq \text{(constant)} \cdot h^6
\]

5.3 Convergence proof for the scheme

We prove that the solution of \([17]\) converges to the solution of \([1]\) if the exact solution is continuously differentiable one \([16, 9]\). We introduce here the main points for the scheme convergence and give the details in Appendix B.

\(\theta_i^j\) is the difference solution of \([17]\), \(u_i^j\) is the exact solution. Hence the error \(v_i^j\) is given by \(v_i^j = \theta_i^j - u_i^j\).

Introducing \(L_2\) norm defined by \(\|v_i^j\| = \left( \sum \sum (v_i^j)^2 \right)^{1/2}\)

The scheme converges when the norm of that error \(\|v_i^j\| \to 0\) as \((\tau, h \to 0)\)

We found that \(\|v_i^j+1\| \leq P(M) (\tau + h^2)\), where \(P(M)\) is a polynomial in the bounded constant \(M = \tau / e^{a(r-1)}\) and \(a\) as in \([18]\). Hence the convergence proved.

5.4 Numerical calculations and test

The coupled KdV-MKdV system \([2]\) is solved numerically using scheme \([17]\) with initial condition from \([1]\) at \(t = 0\) and the results are compared with the explicit formulas \([1]\). The percentage errors are shown in the following plots.

Fig.3 percentage errors of the numerical solutions relative to the explicit solutions.
The results of the test confirms the validity of the numerical scheme we propose. It also illustrates the errors of evaluation that could be estimated by the resulting inequalities of the scheme convergence proof.

6 Conclusion

Darboux transformations covariance of Lax representation of nonlinear equations is a powerful tool for explicit solutions production. Here we investigate applications of special kind of such discrete symmetry - to be called elementary ones. We use these elementary DT to produce explicit solutions for coupled KdV-MKdV system. The iteration of DT can be formulated in form of determinants representations \[10, 12\]. A numerical method for general coupled KdV-MKdV system is introduced. It is a difference scheme for Cauchy problems for arbitrary number of equations with constants coefficients. We analyze stability and prove the convergence of the scheme. The scheme keeps two conservation laws chosen in analogy with KdV type equations. Analyzing stability and proving the convergence beside comparing the numerical results with explicit formulas allow us to use the numerical scheme to systems with arbitrary coefficients that is presumably non-integrable. Obviously the coupled KdV systems are successfully treated by our scheme \[3\].

Acknowledgment We thank S. B. Kshevetskii for useful discussions about numerical scheme for the problem under consideration.

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Stability analysis of the scheme

We prove stability with respect to small perturbations (because we consider nonlinear equations) of initial conditions. Strictly speaking it is the boundness of the discrete solution in terms of small perturbation of the initial data. Consider the differential

\[ T_{i,r}^{n,j+1} = \{ \partial \theta_{i,j}^{n+1} / \partial \theta_{r,j}^{n} \}, \quad d \theta_{r,j}^{n} = \{ \theta_{i+1,j}^{n}, \theta_{i,j}^{n}, \theta_{i-1,j}^{n}, \theta_{i,j+1}^{n}, \theta_{i,j+2}^{n} \}^t \]

and define the norm \( \| d \theta^j \| = \left( \sum_{r} \left( \sum_{n} (d \theta_{r,j}^{n})^2 \right)^{1/2} \right) \)

We can write \( d \theta_{r,j}^{n+1} = T_{i,r}^{n,j+1} d \theta_{r,j}^{n} = T_{i,r}^{n,j+1} T_{i,r}^{n,j-1} = \Pi (T_{i,r}^{n})^T d \theta_{r,o}^{n} \)

where \( d \theta_{r,j}^{n+1} \) is perturbations of the discrete solution, \( d \theta_{r,o}^{n} \) small perturbation of the initial data. Stability required the boundedness of \( \Pi (T_{i,r}^{n})^T \) i.e \( \| T^r \| \) is bounded. We calculate \( T \) from [17] as follow

\[ T_{i,r}^{n,j+1} = \delta_{i,r} - \tau \sum_{m,k} \left( \frac{g_{m,k}}{2h} \left[ \theta_{i,j}^{m,j} (\delta_{i+1,r} - \delta_{i-1,r}) + \delta_{i,r} (\theta_{i+1,j}^{k,j} - \theta_{i-1,j}^{k,j}) \right] \right. \\
\left. + \frac{g_{m,k}^2}{2h} \left[ (\theta_{i,j}^{m,j})^2 (\delta_{i+1,r} - \delta_{i-1,r}) + 2 \theta_{i,j}^{m,j} \delta_{i,r} (\theta_{i+1,j}^{k,j} - \theta_{i-1,j}^{k,j}) \right] \right. \\
\left. + \frac{g_{m,k}^3}{2h} \left[ (\theta_{i,j}^{m,j} - \theta_{i,j}^{m,j}) (\delta_{i+1,r} - \delta_{i-1,r}) + (\delta_{i+1,r} - \delta_{i-1,r}) (\theta_{i+1,j}^{k,j} - \theta_{i-1,j}^{k,j}) \right] \right. \\
\left. + \frac{g_{m,k}^4}{2h} \left[ \theta_{i,j}^{m,j} (\delta_{i+1,r} - 2 \delta_{i,r} + \delta_{i-1,r}) + \delta_{i,r} (\theta_{i+1,j}^{k,j} - 2 \theta_{i,j}^{k,j} + \theta_{i-1,j}^{k,j}) \right] \right. \\
\left. + \frac{g_{m,k}^5}{2h} \left[ \theta_{i,j}^{m,j} \theta_{i,j}^{k,j} (\delta_{i+1,r} - \delta_{i-1,r}) + \theta_{i,j}^{m,j} (\theta_{i+1,j}^{k,j} - \theta_{i-1,j}^{k,j}) \delta_{i,r} + \theta_{i,j}^{k,j} (\theta_{i+1,j}^{k,j} - \theta_{i-1,j}^{k,j}) \delta_{i,r} \right] \right) \\
\left. - \frac{\tau d_{i,n}}{2h^3} [\delta_{i+2,r} - 2 \delta_{i+1,r} + 2 \delta_{i-1,r} - 2 \delta_{i-2,r}] \right) \]

Rewriting (19) in terms of identity (E), symmetric (S) and anti-symmetric (A) matrices

\[ \| S^{j+1} \| \leq \tau \max_{l,n,m,k} \left[ g_{m,k}^{n,l} \right] \max_{i,m,k} \left( | \theta_{x,i}^{m,j} | + | \theta_{x,i}^{k,j} | \right), \]

\[ \| A^{j+1} \| \leq \frac{1}{h} \max_{l,n,m,k} \left[ g_{m,k}^{n,l} \right] \max_{i,m,k} \left( | \theta_{x,i}^{m,j} | + \frac{3}{h} \max_n |d_n| \right), \]

where \( \theta_{x,i}^{j} = \frac{\theta_{i+1,j} - \theta_{i,j}}{2h} \), \( n, m, k = 1, 2, \ldots, N, \quad l = 1, 2, \ldots, 5 \).
\[ \|T^{j+1}\|^2 = \|(T^{j+1})^* T^{j+1}\| = \|(E - A^{j+1} + S^{j+1})(E + A^{j+1} + S^{j+1})\| \]
\[ \leq 1 + 2 \|S^{j+1}\| + (\|A^{j+1}\| + \|S^{j+1}\|)^2 \]
\[ \leq e^{a(\tau,h)\tau}, \]

\[ a(\tau, h) = 2 \max_{i, n, m, k} \left| g_{m, k}^{n, l} \right| \max_{i, m, k} \left( |\theta_i^{m, j}| |\theta_{x, i}^{k, j}| + \tau \max_{i, m, k} \left| g_{m, k}^{n, l} \right| \max_{i, m, k} \left( |\theta_i^{m, j}| |\theta_{x, i}^{k, j}| \right) + \frac{1}{h} \max_{i, n, m, k} \left| g_{m, k}^{n, l} \right| \max_{i, m, k} \left( |\theta_i^{m, j}| |\theta_{x, i}^{k, j}| \right) \right| + \frac{3}{h^3} \max \left| d_n \right| \right|^2 \]

We have here a conditional stability. That is we require that \( \tau \to 0 \) more faster than \( h \to 0 \). Namely we need
\[ \tau \leq (\tan^{-1} t) \cdot h^6 \]

8 Appendix B
The scheme convergence

We prove the convergence by proving that the norm of the error (between the difference solution and the exact solution) vanishes as the mesh is refined. Let \( \theta^i \) the difference solution of (17), \( u^i \) the exact solution. The error \( v^i \) is given by \( v^i = \theta^i - u^i \).

The scheme converges when the norm \( \|V^i\| \to 0 \) as \( (\tau, h \to 0) \) where the norm is defined as \( \|V^i\| = \sqrt{\sum_{i} \left( v_i^{n, j} \right)^2 h} \)

substitute in (17) by \( \theta^i = v^i + u^i \) keeping in mind that for \( \theta^i \) equation (17) is \( O(\tau + h^2) \) and using the operator \( T \) defined by

\[ v_i^{n, j} = \tau \sum_{m, k} \left( g_{m, k}^{n, 1} \left( u_i^{m, j} \frac{v_{i+1}^{k, j} - v_{i-1}^{k, j}}{2h} + u_i^{m, j} \frac{v_{i+1}^{k, j} - v_{i-1}^{k, j}}{2h} \right) + \frac{1}{h} \sum_{m, k} \left( u_i^{m, j} \frac{v_{i+1}^{k, j} - v_{i-1}^{k, j}}{2h} + u_i^{m, j} \frac{v_{i+1}^{k, j} - v_{i-1}^{k, j}}{2h} \right) \right) \]

So we obtain

\[ v_i^{n, j+1} = \tau \sum_r T_{i r}^{j+1} v_r^{n, j} + \tau f_{m, k, i}^{n, j} \]

where \( f_{m, k, i}^{n, j} = \sum m, k \left( g_{m, k}^{n, 1} \left( u_i^{m, j} \frac{v_{i+1}^{k, j} - v_{i-1}^{k, j}}{2h} + u_i^{m, j} \frac{v_{i+1}^{k, j} - v_{i-1}^{k, j}}{2h} \right) + \frac{1}{h} \sum_{m, k} \left( u_i^{m, j} \frac{v_{i+1}^{k, j} - v_{i-1}^{k, j}}{2h} + u_i^{m, j} \frac{v_{i+1}^{k, j} - v_{i-1}^{k, j}}{2h} \right) \right) \]

\[ + \frac{n, 4}{h} \sum m, k \left( u_i^{m, j} \frac{v_{i+1}^{k, j} - v_{i-1}^{k, j}}{2h} + u_i^{m, j} \frac{v_{i+1}^{k, j} - v_{i-1}^{k, j}}{2h} \right) \]

\[ + O(\tau + h^2) \]
\[ \|f\|^2 = \left( \sum_i (f_{m,k,i}^n)^2 \right)^{1/2} \]
\[
\leq \frac{|g_{m,k}^1|_{\text{max}}}{h^{3/2}} \|Vj\|^2 + \frac{|g_{m,k}^2|_{\text{max}}}{h^2} \|Vj\|^3 + \frac{|g_{m,k}^3|_{\text{max}}}{h^{5/2}} \|Vj\|^2 + \frac{|g_{m,k}^4|_{\text{max}}}{h^{5/2}} \|Vj\|^2 \\
+ \frac{|g_{m,k}^5|_{\text{max}}}{h^2} \|Vj\|^3 + O(\tau + h^2) \\
\leq \frac{|g_{m,k}^1|_{\text{max}}}{h^{3/2}} \|Vj\|^3 + O(\tau + h^2), \quad |g_{m,k}^l|_{\text{max}} = \max_{n,m,k,l} g_{m,k}^l 
\]

Using Schwartz inequality so (20) becomes
\[
\|V_j + 1\| \leq \|T_j + 1\| \|V_j\| + \tau \|f_j\| \\
\leq \|T_j + 1\| \|V_j\| \|V_{j-1}\| + \tau \left( (T_j + 1) \|f_{j-1}\| + \|f_j\| \right) \\
\leq e^{\alpha \tau j} \|V_0\| + \tau \left( e^{\alpha \tau (j-1)} \|f_0\| + e^{\alpha \tau (j-2)} \|f_1\| + \ldots + \|f_j\| \right) \\
\leq e^{\alpha \tau j} \|V_0\| + M \left( |g_{m,k}^l|_{\text{max}} \|V_{j+1}\|^3 + M O(\tau + h^2) \right), \quad M = \tau \frac{e^{\alpha \tau j-1}}{e^{\alpha \tau - 1}} 
\]

Using \(\|V_0\| = 0\), the above inequality has the solution
\[
\|V_{j+1}\| \leq P(M) O(\tau + h^2), \quad P(M) \text{ is a polynomial in } M. 
\]
Since \(M\) is bounded then \(\|V_{j+1}\| \to 0\) as \(\tau, h \to 0\) and the convergence proved.