Gauge model at finite temperature with massive quarks and at finite density on anisotropic lattice

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Critical properties of QCD and the chiral condensate at finite density are analytically studied on an anisotropic lattice in the approximation \( SU(N) \approx Z(N) \). Asymptotic behavior of the partition function and its continuum limit are discussed.

In the previous paper \[1\] we considered the pure gluon sector of QCD at finite temperature on lattice with the anisotropy \( \xi \equiv a_\sigma/a_\tau \neq 1 \) [where \( a_\sigma(a_\tau) \) is the spatial (temporal) spacing]. At present the use of anisotropic lattices \( \xi \gg 1 \) becomes a very popular technique \[2\]. In this paper we include matter fields into the consideration. The approximation \( SU(N) \approx Z(N) \) used here does not cover all features of the SU(N) gauge theory, but since the SU(N) and Z(N) theories belong to the same universality class \[3\], it is commonly believed that the Z(N) degrees of freedom are responsible for many important aspects of the SU(N) phase structure \[4\].

1. Z(N) model on anisotropic lattice

Here we consider lattice QCD in the Hamiltonian limit where all the terms proportional to \( 1/\xi \) are neglected \[4\]; so the partition function can be written as

\[ Z = \sum_{\{\sigma\},\{\bar{\psi}\},\{\psi\}} \exp\{-S^G - S^F\} \tag{1} \]

with the gluon action

\[ -S^G = \kappa_\tau \sum_{\vec{x},\tau;n} \sigma_{\vec{x},\tau;0} \sigma_{\vec{x},\tau+0;n} \sigma_{\vec{x}+n,\tau;0} \sigma_{\vec{x},\tau;n} + c.c., \]

where \( \sigma_{\vec{x},\tau;\nu} \in Z(N) \), and \( \kappa_\tau \equiv \frac{2N_c}{g_\tau^2} \xi \). For the fermion part of the action, we choose the form \[6\] in the approximation \( \xi \gg 1 \)

\[ -S^F = n_f a_\sigma^3 \sum_x \bar{\psi}_x D_{\vec{x},x}^0 \psi_x, \tag{2} \]

\[
D_{\vec{x},x}^0 = \frac{r - \gamma_0}{2} \sigma_0(x) \delta_{\vec{x}}^0 - \frac{r + \gamma_0}{2} \sigma_0^*(x) \delta_{\vec{x}}^0 - (r + ma_\tau)\delta_{\vec{x}}^0,
\]

where \( r \) is the Wilson parameter \((0 < r \leq 1)\), \( n_f \) is the number of flavors.

For gluon fields we use the periodic boundary conditions \( \sigma_{\vec{x},\tau;\mu} = \sigma_{\vec{x}+N_{\tau},\tau;\mu} \) and fix the Hamiltonian gauge \( \sigma_{\vec{x},\tau;0} = 1 + \delta_\tau(\Omega_{\vec{x}} - 1) \); \( \Omega_{\vec{x}} = \prod_{\tau=0}^{N_{\tau}-1} \sigma_{\vec{x},\tau;0} \) is the Polyakov loop. Fermion fields obey standard antiperiodic boundary conditions.

Summing over the spatial variables \( \sigma_{\vec{x};n} \) \[6\], one may easily get

\[ -S^G_{\epsilon f} = \kappa_\tau (N_c) \sum_{\vec{x},n} \Omega_{\vec{x}} \Omega_{\vec{x}+n}^* + c.c., \tag{4} \]

where the effective coupling \( \kappa_\tau \) is defined as

\[ \frac{1 - e^{-3\kappa_\tau}}{1 + 2e^{-3\kappa_\tau}} = \left\{ \frac{1 - e^{-3\kappa_\tau}}{1 + 2e^{-3\kappa_\tau}} \right\}^{N_c}. \]

The effective action \[6\] coincides with the action obtained in \[8\] in a strong coupling approximation where the magnetic part of the action is suppressed by factor \( g^{-2} \). On an extremely anisotropic lattice \((\xi \to \infty)\), the similar suppression appears due to factor \( \xi^{-1} \), the model may be studied beyond the strong coupling area \[8\] and the limit \( a_\tau \ll a_\sigma \to 0 \) may be monitored.

In the weak coupling region, the effective coupling \( \kappa_\tau \)

\[
\kappa_\tau \approx -\frac{1}{3} \ln \left\{ \frac{1 - e^{-3N_c\kappa_\tau}}{1 + 2e^{-3N_c\kappa_\tau}} \right\}.
\]
\[ \lim_{N_r \to \infty} \kappa_r = \text{const} \] leads to the reasonable renormalization condition on the bare constant \( \kappa_r \sim g^{-2} \sim \ln \frac{1}{\Lambda^2} \).

We consider the special case \( r = 1 \) for the fermionic action. Taking into account that 
\[
(\delta_{x-x',x'})^\dagger = \delta_{x+x',x'},
\] the action \( S \) can be rewritten as
\[
-S^F = n_f \delta^3 \sum_x \left( \bar{\psi}_x^\dagger \Delta_{x,x}^\dagger \psi_x^\dagger + \bar{\psi}_x^\dagger \Delta_{x,x} \psi_x \right)
\]
with \( \psi(x) = \frac{1 + m}{2} \) and
\[
\Delta_{x,x} = \delta_{x,x} \left( \sigma_0(x) \delta^r_{i,j} - (1 + m \cdot m) \delta^r_{i,j} \right).
\]

The chemical potential may be introduced in accordance with \( \Theta = e^{i\phi} \) and, after the integration over \( \psi(x) \) fields and taking into account that \( \lim_{a \to 0} (1 + a \cdot m) = e^{m/T} \), we get for the Z(3) gauge group
\[
-S^F_{eff} = \eta - \frac{\Omega}{2} + \frac{\Omega^*}{2} - \frac{2 m f}{\sqrt{3}} \phi + \text{const}
\]
with \( \eta = \frac{1}{\sqrt{3}} \).

\[
\frac{2}{3} \ln \left( \frac{m}{T} + \frac{\mu}{T} \right) - \frac{1}{3} \ln \left( \frac{m^3}{T} + \frac{\mu^3}{T} \right) = \frac{3}{4} \phi(m, \mu)
\]

\[
\phi(m, \mu) = \frac{1}{2i} \ln \left( \frac{m}{T} + \frac{i \mu}{T} + \frac{i \Delta_\mu}{T} \right); \quad -\pi < \phi < \pi.
\]

In the asymptotic area \( m >> T; \mu >> T \) the real part of fermion contribution remains essential only when \( m \simeq \mu \) but outside this region \( \eta \) rapidly disappears. It is also easy to show that the imaginary part of fermion contribution disappears both for \( \frac{\mu}{T} \to 0 \) and (if \( \frac{m}{T} >> 1 \)) for \( \frac{m}{T} \to 0 \) and
\[
\phi = \{ \pm \frac{\pi}{3}; \frac{\mu}{T} >> \frac{m}{T} > 1; \frac{\mu}{T} \to \pm \infty, \frac{\mu}{T} \to \pm \infty \}.
\]

The effective action \( S_{eff} = S_{eff}^G + S_{eff}^F \) corresponds to the extended Potts (\( N=3 \)) or Ising (\( N=2 \)) model which has been studied in detail in \( [11] \). Our consideration essentially differs from that of \( [11] \) by the specific form of 'external sources' \( \eta \) and \( \phi \). In particular, the fermion part of the action depends on \( m, \mu, T \) only through the combinations of \( \frac{\mu}{T} \) and \( \frac{m}{T} \). In addition, in our approach the parameter \( m \) may be arbitrary small. All parameters are expressed in physical units, so the fermion part of the effective action does not change in \( \alpha \to 0; \quad N_r \to \infty \) limit.

2. Mean spin approximation

Here we apply the mean spin approach \( [1] \) to compute analytically the average value of Polyakov loop \( \langle \Omega \rangle \), which allows us to estimate \( \langle \bar{\psi} \psi \rangle \) at finite density. Let us introduce \( \Omega = re^{i\theta} = \frac{1}{V} \sum_x \Omega_x \) (mean spin), where \( V \) is the volume in lattice units. Adjusting the definition of quasiaverages \( [12] \) to the considered case

\[
\langle q \rangle = \sum_{(\sigma)} q \delta \left( 2V r \cos \theta - 2 \sum_x \text{Re}\Omega_x \right) + \left( \frac{1}{\sqrt{3}} \sum_x \text{Im}\Omega_x \right),
\]

we may write for the gluon part of the effective action

\[
\exp \{-S_{eff}^G\} \equiv \frac{\exp \{-S_{eff}^G\} \times \langle \bar{\psi} \psi \rangle}{\langle \bar{\psi} \psi \rangle} \quad (8)
\]

with

\[
-\frac{1}{V} \ln \langle \bar{\psi} \psi \rangle = \frac{L}{2} \sum_{k=0}^{2} l_k \ln l_k;
\]

\[
l_k = \frac{1 + \frac{r}{2} \cos \left( \frac{2\pi k}{3} \right)}{3}.
\]

The gluon part of the effective action up to \( \gamma^3 \) terms can then be written as \( [1] \)

\[
-S_{eff}^G = \gamma^2 + \frac{5}{3} \gamma^2 r^2 \left( 1 + 2r \cos \frac{2\pi k}{3} - 2r^2 \right).
\]

\footnote{\ It seems worth noting that the fermion part of the action enjoys symmetries \( \eta(m, \mu) = \eta(m, -\mu); \quad \phi(\mu) = -\phi(-\mu) \) and \( \eta(m, \mu) = \eta(\mu, m) \).}

\footnote{\ It is easy to show that the 'mean spin' method in the lowest order in \( \gamma \) coincides with the simple version of the 'mean field' method.}
with \( \gamma = 3 \frac{2}{1+2e^{-\frac{m}{T}}} \). Therefore we may finally write

\[-\frac{S_{eff}}{V} = \Phi = \eta r \cos \theta + \frac{2inF\phi r}{\sqrt{3}} \sin \theta - \mathcal{L} - S_{eff}^G \frac{G}{V}.\]

To calculate the partition function we shall find the saddle point \( \Omega_0 = r_0 e^{i\theta_0} \) where both \( \text{Re}S_{eff}(\theta, r_0) = \min \text{Re}S_{eff} \) and \( |\text{Im}S_{eff}(\theta, r_0)| = \min |\text{Im}S_{eff}| \). Although at any \( \theta \neq \pi n \) \( \text{Im}S_{eff}(\theta, r_0) \neq 0 \), it oscillates in thermodynamical limit with a very high frequency \( \sim V \sin \theta \) and \( \int_{V - \pi}^{V + \pi} \text{Im}S_{eff} (\theta, r_0, V') dV' \) becomes negligible at any large \( V \). On the other hand, \( \text{Re}S_{eff}(\theta, r_0) \) gains the minimum at \( \eta \cos \theta \rightarrow |\eta| \), so, taking into account that \( \eta \geq 0 \), we may put \( \theta = 0 \). Thereby we may conclude that the free energy \( F(\mu) \equiv -T \ln Z \) does not depend on \( \phi \) and differs from \( F(0) \) simply by ‘renormalization’ \( \eta(0) \rightarrow \eta(\mu) \). Moreover, the independence \( F(\mu) \) of \( \phi \) signals about an implicit charge symmetry \( (\Omega' \leftrightarrow \Omega^* \leftrightarrow \theta \leftrightarrow -\theta) \) in the model which is formally broken in \( \Phi \). This, in particular, leads to \( \langle \Omega |\text{Im} \Phi | \rangle \sim \frac{\partial F(\mu)}{\partial \phi} = 0 \).

Therefore, one may write for the free energy \( F(\mu) \simeq -VT \max \{ \Phi(0, r_0), \Phi(0, 0) \} \), where the saddle point \( r_0 \) is defined by \( \frac{\delta \Phi(0, r_0)}{\delta r_0} = 0 \). Failing to solve such an equation precisely, we studied it in the areas \( r << 1 \) and \( 1 - r << 1 \). We find that the border line between ‘ordered’ \( (r_0 = \langle |\Omega| \rangle \neq 0) \) and ‘disordered’ \( (\langle |\Omega| \rangle = 0) \) phases can be roughly presented \(^3\) as \( \frac{1}{4} \gamma + \eta \simeq 1 \).

\[ \langle \bar{\psi} \psi \rangle = \langle \bar{\psi}(+) \psi(+) \rangle + \langle \bar{\psi}(-) \psi(-) \rangle = \frac{T}{V} \frac{\partial \ln Z}{\partial m} \frac{1}{2} \left[ T \left( 2e^{-\frac{m}{T}} - 1 - e^{-\frac{m}{T}} \right) - \frac{\partial g(\Omega)}{\partial \Omega} \cdot \text{Re}(\Omega) \right]. \]

### 3. Conclusions

This paper considers finite temperature QCD on an anisotropic lattice \( \langle \xi >> 1 \rangle \) in the approximation \( SU(N) \simeq Z(N) \).

We argue that at least in above approximation the imaginary part of the action plays a marginal role and the free energy at finite density differs from the one at \( \mu = 0 \) mainly by ‘renormalization’ of the real part of the action: \( \eta(0) \rightarrow \eta(\mu) \).

The chiral condensate in this model does not turn to zero at any \( \frac{\mu}{T} \neq 0 \) which indicates chiral symmetry breakdown. \( \langle \bar{\psi} \psi \rangle \) strongly depends on \( \langle |\Omega| \rangle \) for \( \frac{|\mu| - m}{T} \gg 1 \), so the gluon environment plays an essential role in this region.

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