ON SOME METRIC TOPOLOGIES
ON PRIVÁLOV SPACES ON THE UNIT DISK

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Summary. Let $N^p (1 < p < \infty)$ be the Privalov class $N^p$ of holomorphic functions on the open unit disk $\mathbb{D}$ in the complex plane. In 1977 M. Stoll proved that the class $N^p$ equipped with the topology given by the metric $\lambda_p$ defined by

$$\lambda_p(f, g) = \left( \int_0^{2\pi} \left( \log(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|) \right)^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad f, g \in N^p,$$

becomes an $F$-algebra. In the recent overview paper by Meštrović and Pavičević (2017) a survey of some known results on the topological structures of the Privalov spaces $N^p (1 < p < \infty)$ and their Fréchet envelopes $F^p$ are presented.

In this article we continue a survey of results concerning the topological structures of the spaces $N^p (1(p < \infty)$. In particular, for each $p > 1$, we consider the class $N^p$ as the space $M^p$ equipped with the topology induced by the metric $\rho_p$ defined as

$$\rho_p(f, g) = \left( \int_0^{2\pi} \log^p(1 + M(f - g)(\theta)) \frac{d\theta}{2\pi} \right)^{1/p}, \quad f, g \in M^p,$$

where $Mf(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|$. On the other hand, we consider the class $N^p$ with the metric topology introduced by Meštrović, Pavičević and Labudović (1999) which generalizes the Gamelin-Lumer’s metric which is generally defined on a measure space $(\Omega, \Sigma, \mu)$ with a positive finite measure $\mu$. The space $N^p$ with the associated modular in the sense of Musielak and Orlicz becomes the Hardy-Orlicz class. It is noticed that the all considered metrics induce the same topology on the space $N^p$.

1 INTRODUCTION AND PRELIMINARY RESULTS

Let $\mathbb{D}$ denote the open unit disk in the complex plane and let $\mathbb{T}$ denote the boundary of $\mathbb{D}$. Let $L^q(\mathbb{T}) (0 < q \leq \infty)$ be the familiar Lebesgue spaces on the unit circle $\mathbb{T}$. The Privalov class $N^p (1 < p < \infty)$ consists of all holomorphic functions $f$ on $\mathbb{D}$ for which

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \left( \log^+ \frac{|f(re^{i\theta})|}{|z|} \right)^p \frac{d\theta}{2\pi} < +\infty,$$

where for $z \in \mathbb{C}$, $\log^+ |z| = \max(\log |z|, 0)$ if $z \neq 0$ and $\log^+ 0 = 0$. These classes were firstly considered by I.I. Privalov in [40, p. 93], where $N^p$ is denoted as $A_q$.

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Notice that for \( p = 1 \) the condition (1) defines the Nevanlinna class \( N \) of holomorphic functions on \( D \). Recall that the Smirnov class \( N^+ \) is the set of all functions \( f \in N \) such that
\[
\lim_{r \to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f^*(e^{i\theta})| \frac{d\theta}{2\pi} < +\infty,
\]
where \( f^* \) is the boundary function of \( f \) on \( T \), i.e.,
\[
f^*(e^{i\theta}) = \lim_{r \to 1^-} f(re^{i\theta})
\]
is the radial limit of \( f \) which exists for almost every \( e^{i\theta} \in T \). Recall that the classical Hardy space \( H^q \) \((0 < q \leq \infty)\) consists of all functions \( f \) holomorphic on \( D \) such that
\[
\sup_{0 \leq r < 1} \int_0^{2\pi} |f(re^{i\theta})|^q \frac{d\theta}{2\pi} < \infty
\]
if \( 0 < q < \infty \), and which are bounded when \( q = \infty \):
\[
\sup_{z \in D} |f(z)| < \infty.
\]
It is known that (see \([36] \) and \([25] \))
\[
N^r \subset N^p \quad (r > p), \quad \bigcup_{q > 0} H^q \subset \bigcap_{p > 1} N^p, \quad \text{and} \quad \bigcup_{p > 1} N^p \subset M \subset N^+ \subset N,
\]
where the above containment relations are proper.

It is well known (see, e.g., \([4] \) p. 26)) that a function \( f \in N^+ \) has a unique factorization of the form
\[
f(z) = B(z)S_\mu(z)F(z), \quad z \in D,
\]
where \( B \) is the Blaschke product with respect to zeros \( \{z_k\} \subset D \) of \( f \), \( S_\mu \) is a singular inner function and \( F \) is an outer function in \( N^+ \), i.e.,
\[
B(z) = z^m \prod_{k=1}^{\infty} \frac{|z_k|}{z_k} \cdot \frac{z_k - z}{1 - \overline{z_k}z}, \quad z \in D,
\]
with \( \sum_{k=1}^{\infty} (1 - |z_k|) < \infty \), \( m \) a nonnegative integer,
\[
S_\mu(z) = \exp \left( - \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} d\mu(t) \right)
\]
with positive singular measure \( d\mu \) and
\[
F(z) = \lambda \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log |F^*(e^{it})| \, dt \right), \tag{2}
\]
where \( \lambda \in \mathbb{C} \) with \( |\lambda| = 1 \) and \( \log |F^*| \in L^1(T) \).

Recall that a function \( I \) of the form
\[
I(z) = B(z)S_\mu(z), \quad z \in D,
\]
is called an inner function. Furthermore, it is well known that \( |I^*(e^{it})| = 1 \) for almost every \( e^{it} \in T \) and hence, \( |f^*(e^{it})| = |F^*(e^{it})| \) for almost every \( e^{it} \in T \).

I.I. Privalov \([40] \) p. 98 (also see \([25] \) Theorem 5.3)) proved that a function \( f \) holomorphic on \( D \) belongs to the class \( N^p \) if and only if \( f = IF \), where \( I \) is an inner function on \( D \) and \( F \) is an outer function given by (2) such that \( \log^+ |f^*| \in L^p(T) \) (or equivalently, \( \log^+ |F^*| \in L^p(T) \)).
M. Stoll [44, Theorem 4.2] showed that the space $N^p$ (with the notation $(\log^+ H)^{\alpha}$ in [44]) equipped with the topology given by the metric $\lambda_p$ defined by
\[
\lambda_p(f, g) = \left( \int_0^{2\pi} \left( \log(1 + |f^*(e^{i\theta}) - g^*(e^{i\theta})|) \right)^p \frac{d\theta}{2\pi} \right)^{1/p}, \quad f, g \in N^p,
\]
becomes an $F$-algebra. Recall that the function $\lambda_1 = \lambda$ defined on the Smirnov class $N^+$ by (3) with $p = 1$ induces the metric topology on $N^+$. N. Yanagihara [45] proved that under this topology, $N^+$ is an $F$-space.

Furthermore, in connection with the spaces $N^p$ ($1 < p < \infty$), Stoll [44] (also see [5] and [29, Section 3]) also studied the spaces $F^q$ ($0 < q < \infty$) (with the notation $F_1/q$ in [44]), consisting of those functions $f$ holomorphic on $D$ for which
\[
\lim_{r \to 1} (1 - r)^{1/q} \log M\infty(r, f) = 0,
\]
where
\[
M\infty(r, f) = \max_{|z| \leq r} |f(z)|.
\]
Stoll [44, Theorem 3.2] also proved that the space $F^q$ with the topology given by the family of seminorms $\{||| \cdot |||_{q,c} \}_{c > 0}$ defined for $f \in F^q$ as
\[
|||f|||_{q,c} = \sum_{n=0}^{\infty} |\hat{f}(n)| e^{-cn^{1/(q+1)}} < \infty
\]
for each $c > 0$, where $\hat{f}(n)$ is the $n$-th Taylor coefficient of $f$, is a countably normed Fréchet algebra. By a result of C.M. Eoff [5, Theorem 4.2], $F^p$ is the Fréchet envelope of $N^p$ and hence, $F^p$ and $N^p$ have the same topological duals.

Following H.O. Kim ([13] and [14]), the class $M$ consists of all holomorphic functions $f$ on $D$ for which
\[
\int_0^{2\pi} \log^+ M f(\theta) \frac{d\theta}{2\pi} < \infty,
\]
where
\[
M f(\theta) = \sup_{0 \leq r < 1} |f(re^{i\theta})|
\]
is the maximal radial function of $f$.

The study on the class $M$ on the disk $D$ has been extensively investigated by H.O. Kim in [13] and [14], V.I. Gavrilov and V.S. Zaharyan [9] and M. Nawrocky [39]. Kim [14, Theorems 3.1 and 6.1] showed that the space $M$ with the topology given by the metric $\rho$ defined by
\[
\rho(f, g) = \int_0^{2\pi} \log(1 + M(f - g)(\theta)) \frac{d\theta}{2\pi}, \quad f, g \in M,
\]
becomes an $F$-algebra. Furthermore, Kim [14, Theorems 5.2 and 5.3] gave an incomplete characterization of multipliers of $M$ into $H^\infty$. Consequently, the topological dual of $M$ is not exactly determined in [14], but as an application, it was proved in [14, Theorem 5.4] (also cf. [39, Corollary 4]) that $M$ is not locally convex space. Furthermore, the space $M$ is not locally bounded ([14, Theorem 5.5] and [39, Corollary 5]).

Nevertheless, that as noticed above, the class $M$ is essentially smaller than the class $N^+$, M. Nawrocky [39] showed that the class $M$ and the Smirnov class $N^+$ have the
same corresponding locally convex structure which was already established by N. Yanagihara for the Smirnov class in \[45\] and \[46\]. More precisely, it was proved in \[39\] Theorems 1 that the Fréchet envelope of the class \(M\) can be identified with the space \(F^+\) of holomorphic functions on the open unit disk \(\mathbb{D}\) such that

\[
\|f\|_c := \sum_{n=0}^{\infty} |\hat{f}(n)| e^{-cn\sqrt{\pi}} < \infty
\]

for each \(c > 0\), where \(\hat{f}(n)\) is the \(n\)-th Taylor coefficient of \(f\). Notice that \(F^+\) coincides with the space \(F^1\) defined above. It was shown in \[46\] (also see \[45\]) that \(F^+\) is actually the containing Fréchet space for \(N^+\) (also see \[43\]). Moreover, Nawrocky \[39, Theorem 1\] characterized the set of all continuous linear functionals on \(M\) which by a result of Yanagihara \[45\] coincides with those on the Smirnov class \(N^+\).

Motivated by the mentioned investigations of the classes \(M\) and \(N^+\), and the fact that the classes \(N^p\) \((1 < p < \infty)\) are generalizations of the Smirnov class \(N^+\), in \[20, Chapter 6\] and \[22\] the first author of this paper investigated the classes \(M^p\) \((1 < p < \infty)\) as generalizations of the class \(M\). Accordingly, the class \(M^p\) \((1 < p < \infty)\) consists of all holomorphic functions \(f\) on \(\mathbb{D}\) for which

\[
\int_0^{2\pi} \left(\log^+ Mf(\theta)\right)^p \frac{d\theta}{2\pi} < \infty.
\]

Obviously,

\[
\bigcup_{p>1} M^p \subset M.
\]

By analogy with the topology defined on the space \(M\) \((\[13\] and \[14\])}, the space \(M^p\) can be equipped with the topology induced by the metric \(\rho_p\) defined as

\[
\rho_p(f, g) = \left( \int_0^{2\pi} \log^p \left(1 + M(f - g)(\theta)\right) \frac{d\theta}{2\pi} \right)^{1/p},
\]

with \(f, g \in M^p\).

After Privalov, the study of the spaces \(N^p\) \((1 < p < \infty)\) was continued in 1977 by M. Stoll \[44\] (with the notation \((\log^+ H)^\alpha\) instead of \(N^p\) in \[44\]). Further, the linear topological and functional properties of these spaces were extensively investigated by C.M. Eoff in \[5\] and \[6\], N. Mochizuki \[36\], Y. Iida and N. Mochizuki \[12\], Y. Matsugu \[17\], J.S. Choa \[2\], J.S. Choa and H.O. Kim \[3\], A.K. Sharma and S.-I. Ueki \[42\] and in works \[19\]-\[35\] of authors of this paper; typically, the notation varied and Privalov was mentioned in \[17\], \[21\]-\[24\], \[29\]-\[32\], \[34\], \[33\] and \[42\]. In particular, it was proved in \[21, Corollary\] that \(N^p\) is not locally convex space and in \[30, Theorem 1.1\] that \(N^p\) is not locally bounded space. We refer the recent monograph \[10, Chapters 2, 3 and 9\] by V.I. Gavrilov, A.V. Subbotin and D.A. Efimov for a good reference on the spaces \(N^p\) \((1 < p < \infty)\).

Let us recall that in our recent overview paper \[52\] it was given a survey of some known results on different topologies on the Privalov classes \(N^p\) \((1 < p < \infty)\) and their Fréchet envelopes \(F^p\) \((1 < p < \infty)\) on the open unit disk. Here we give a survey on related extended results involving some other metrics and the induced topologies on the classes \(N^p\).

The remainder of this overview paper is organized in three sections. For any fixed \(p > 1\), in Section 2 we present some results concerning the topological and functional
structures on the classes $M^p$ ($1 < p < \infty$). Section 3 is devoted to the consideration of the Privalov class $N^p$ as a closed subspace of some Orlicz space. In this setting $N^p$ with the associated modular in the sense of Musielak and Orlicz becomes the Hardy-Orlicz class whose topology coincides with both metric topologies $\lambda_p$ and $\rho_p$. Concluding remarks are presented in the last section.

2 THE $\rho_p$-METRIC TOPOLOGY ON PRIVALOV SPACE $N^p$

Here we focus our attention to certain results from [20, Chapter 6] and [22] concerning the classes $M^p$ ($1 < p < +\infty$). In [22] it is proved the following basic result.

Theorem 1 ([22, Theorem 2]). The function $\rho_p$ defined on $M^p$ as

$$
\rho_p(f, g) = \left( \int_0^{2\pi} \log^p (1 + M(f - g)(\theta)) \frac{d\theta}{2\pi} \right)^{1/p}, \quad f, g \in M^p,
$$

(5)

is a translation invariant metric on $M^p$. Further, the space $M^p$ is a complete metric space with respect to the metric $\rho_p$.

Remark 1. Notice that the expression (5) with $p = 1$ defines the metric $\rho_1 = \rho$ on the class $M$ (given by (4)) introduced by H.O. Kim in [13] and [14]. As noticed above, it was proved in [14] that the metric $\rho$ induces the topology on $M$ under which $M$ is also an $F$-algebra.

Moreover, the following two statements are also proved in [20, Chapter 6].

Theorem 2 ([22, Theorem 11]). $M^p = N^p$ for each $p > 1$; that is, the spaces $M^p$ and $N^p$ coincide.

Theorem 3 ([22, Theorem 15]). $M^p$ with the topology given by the metric $\rho_p$ defined by (5) becomes an $F$-space.

Using Theorem 3 and the open mapping theorem (see, e.g., [41, Corollary 2.12 (b)]), the following result was also proved in [22].

Theorem 4 ([22, Theorem 16]). For each $p > 1$ the classes $M^p$ and $N^p$ coincide, and the metric spaces $(M^p, \rho_p)$ and $(N^p, \lambda_p)$ have the same topological structure, where the metrics $\rho_p$ and $\lambda_p$ are given on $M^p$ and $N^p$ by (5) and (3), respectively.

As an immediate consequence of Theorem 4 and [22, Lemma 8], we obtain the following assertion.

Proposition 1. The convergence with respect to the metric $\rho_p$ given by (5) on the space $M^p$ is stronger than the metric of uniform convergence on compact subsets of the disk $\mathbb{D}$.

Remark 2. For an outer function $h$ let $H^2(|h|^2)$ denote the closure of the (analytic) polynomials in the space $L^2(|h|^2 d\theta)$. By using the famous Beurling’s theorem for the Hardy space $H^2$ ([11], also see [11, Ch. 7, p. 99]), it was proved in [6] (also see [27, Section 1]) that the class $N^p$ can be represented as a union of certain weighted Hardy classes. Using this representation, the following two topologies are defined on $N^p$ in [27]: the usual locally convex inductive limit topology, which we shall call the Helson topology and denote by $\mathcal{H}_p$, in which a neighborhood base for 0 is given by those
balanced convex sets whose intersection with each \( H^2(|h|^2) \) is a neighborhood of zero in \( \mathcal{B}^2(|h|^2) \), and a not locally convex topology, denoted by \( I_p \), in which a neighborhood base for zero is given by all sets whose intersection with each space \( H^2(|h|^2) \) is a neighborhood of zero. It was proved in [27, Theorem E] (cf. [20, Chapter 3]) that the topology \( \mathcal{H}_p \) coincides with the metric topology induced on \( N^p \) by the Stoll’s metric topology \( \lambda_p \) given by (3). Moreover, it was proved in [6] that the topology \( I_p \) coincides with the metric topology \( \lambda_p \) and by Theorem 4, \( I_p \) also coincides with the metric topology \( \rho_p \), which are not locally convex. Hence, \( I_p \) is strictly stronger than \( \mathcal{H}_p \). The analogous results for the space \( N^\pm \) are proved by J.E. McCarthy in [18].

3 THE SPACE \( N^p \) AS THE HARDY-ORLICZ CLASS

In this section we give a short survey about Privalov classes \( N^p \) \((1 < p < +\infty)\) as the Hardy-Orlicz classes. Related results are mainly obtained in [33]. Let \((\Omega, \Sigma, \mu)\) be a measure space, i.e., \( \Omega \) is a nonempty set, \( \Sigma \) is a \( \sigma \)-algebra of subsets of \( \Omega \) and \( \mu \) is a nonnegative finite complete measure not vanishing identically. Denote by \( L^p(\mu) = L^p(\Omega, \Sigma, \mu) \) \((0 < p \leq \infty)\) the familiar Lebesgue spaces on \( \Omega \). For each real number \( p > 0 \) \([33] \) it was considered the class \( L^+_p(\mu) = \lambda^+_p \) of all (equivalence classes of) \( \Sigma \)-measurable complex-valued functions \( f \) defined on \( \Omega \) such that the function \( \log^+ |f| \) belongs to the space \( L^p \), i.e.,

\[
\int_\Omega (\log^+ |f(x)|)^p \, d\mu < +\infty,
\]

where \( \log^+ |a| = \max(\log |a|, 0) \). Clearly, \( L^+_q \subset L^+_p \) for \( q > p \) and \( \bigcup_{p>0} L^p \subset \bigcap_{p>0} L^+_p \) \([33] \) Section 2]. For each \( p > 0 \) the space \( L^+_p \) is an algebra with respect to the pointwise addition and multiplication. For each \( p > 0 \) we define the metric \( d_p \) on \( L^+_p \) by

\[
d_p(f, g) = \inf_{t>0} \left[ t + \mu(\{x \in \Omega : |f(x) - g(x)| \geq t\}) \right]
+ \int_\Omega \left( (\log^+ |f(x)|)^p - (\log^+ |g(x)|)^p \right) \, d\mu. \tag{6}
\]

Recall that the space \( L^+_1 \) was introduced by T. Gamelin and G. Lumer in [8] p. 122] (also see [7, p. 122], where \( L^+_1 \) is denoted as \( L(\mu) \)). Note that the metric \( d_p \) given by (6) with \( p = 1 \) coincides with the Gamelin-Lumer’s metric \( d \) defined on \( L^+_1 \). It was proved in [8, Theorem 1.3, p. 122] (also see [7, Theorem 2.3, p. 122]) that the space \( L^+_1 \) with the topology given by the metric \( d_1 \) becomes a topological algebra. The following result is a generalization of the corresponding result for the case \( p = 1 \) given in [8, p. 122] (also see [7, p. 122]).

Theorem 5 ([33, Theorem 2.1]). \textit{The space } \( L^+_p \) \textit{with the metric } \( d_p \) \textit{given by (6) is a topological algebra, i.e., a topological vector space with a complete metric in which multiplication is continuous.}

By the inequality

\[
(\log(1 + |z|))^p \leq 2^{\max(p-1,0)} \left( (\log 2)^p + (\log^+ |z|)^p \right), \quad z \in \mathbb{C},
\]
it follows that a function $f$ belongs to the space $L^+_p$ if and only if
\[
\|f\|_p := \left( \int_\Omega \left( \log(1 + |f(x)|) \right)^p d\mu \right)^{1/p} < \infty. \tag{7}
\]

Furthermore [33, Section 2], the function $\sigma_p$ defined as
\[
\sigma_p(f, g) = (\|f - g\|_p)^{\min(1, p)}, \quad f, g \in L^+_p, 0 < p \leq 1, \tag{8}
\]
is a translation invariant metric on $L^+_p$ for all $p > 0$. Notice that in the case of Privalov space $N^p$ ($1 < p < \infty$), the metric $\sigma_p$ given by (8) coincides with Stoll’s metric $\lambda_p$ defined by (3).

Recall that two metrics (or norms) defined on the same space will be called equivalent if they induce the same topology on this space.

**Theorem 6** ([33, Theorem 2.3]). The metric $d_p$ given by (6) defines the topology for $L^+_p$ which is equivalent to the topology defined by the metric $\sigma_p$ given by (8).

**Remark 4.** It was pointed out in [33, Remark, Section 2] that using the same argument applied in the proof of Theorem 2.3 of [33], it is easy to show that the metrics $\sigma_p$ and $d_p$ are equivalent with the metric $\delta_p$ given on $L^+_p$ by
\[
\delta_p(f, g) = \inf_{t > 0} \left[ t + \mu \left( \{x \in \Omega : |f(x) - g(x)| \geq t \} \right) \right]
+ \left( \int_\Omega \left| \log^+ |f(x)| - \log^+ |g(x)| \right|^p d\mu \right)^{1/\max(p,1)},
\quad f, g \in L^+_p. \tag{9}
\]

**Remark 5.** In [45, Remark 5, p. 460] M. Hasumi pointed out that the Yanagihara’s metric $\lambda = \lambda_1$ on the Smirnov class (given by (3) with $p = 1$) defines the topology on the space $L^+_1 = L(\mu)$ which is equivalent to the metric topology $d_1 = d$ (given by (6) with $p = 1$).

As a consequence of Theorems 5 and 6, it can be obtained the following result.

**Theorem 7** ([33, Corollary 2.4]). For each $p > 0$ the space $L^+_p$ with the topology given by the metric $\sigma_p$ is an $F$-algebra, i.e., a topological algebra with a complete translation invariant metric $\sigma_p$.

**Remark 6.** In view of Theorem 7, note that $L^+_p$ may be considered as the generalized Orlicz space $L^w_p$ with the constant function $w(t) \equiv 1$ on $[0, 2\pi)$ defined in [33, Section 6].

The real-valued function $\psi : [0, \infty) \mapsto [0, \infty)$ defined as $\psi(t) = (\log(1 + t))^p$, is continuous and nondecreasing in $[0, \infty)$, equals zero only at 0, and hence it is a $\varphi$-function (see, e.g., [37, p. 4, Examples 1.9]). Moreover, $\psi$ is a log-convex function since it can be represented in the form $\psi(x) = \Psi(\log x)$ for $x > 0$, where $\Psi(u) := \max(u^p, 0)$ $(u \in [0, \infty))$ is a convex function on the whole real axis, satisfying the condition $\lim_{u \to +\infty} \frac{\Psi(u)}{u} = +\infty$. Notice that convex $\varphi$-functions are a particular case of log-convex $\varphi$-functions.
Further, observe that [33 Section 4] the space \( L_p^+ (dt/(2\pi)) = L_p^+ (p > 0) \), consisting of all complex-valued functions \( f \), defined and measurable on \([0, 2\pi)\), for which

\[
\|f\|_p := \left( \int_0^{2\pi} \left( \log(1 + |f(t)|) \right)^p \frac{dt}{2\pi} \right)^{1/p} < +\infty
\]

(10)
is the Orlicz class (see [37 p. 5]; cf. [33 Section 4]), whose generalization was given in [33 Section 4]. It follows by the dominated convergence theorem that the class \( L_p^+ \) coincides with the associated Orlicz space (see [37 Definition 1.4, p. 2]) consisting of those functions \( f \in L_p^+ \) such that

\[
\int_0^{2\pi} \left( \log(1 + |f(t)|) \right)^p \frac{dt}{2\pi} \to 0 \quad \text{as} \quad c \to 0 + .
\]

Since \( \sigma_p(f, g) = \left( \|f - g\|_p \right)^{\min(p, 1)} \) is an invariant metric on \( L_p^+ \), the function \( \| \cdot \|_p \) given by (10) is a modular in the sense of Definition 1.1 in [37 p. 1], where \( \sigma_p \) is the metric defined by (8). For any function \( f \in L_p^+ \), by the monotone convergence theorem, it follows that \( \lim_{c \to 0} \|cf\| = 0 \) and thus \( (L_p^+, \sigma_p) \) is a modular space in the sense of Definition 1.4 in [37 p. 2]. In other words, the function \( \| \cdot \|_p \) is an F-norm. It is known (see [37 Theorem 1.5, p. 2 and Theorem 7.7, p. 35]) that the functional \( | \cdot |_p \) defined as

\[
|f|_p = \inf \left\{ \varepsilon > 0 : \int_0^{2\pi} \left( \log \left( 1 + \frac{|f(t)|}{\varepsilon} \right) \right)^p \frac{dt}{2\pi} \leq \varepsilon \right\}, \quad f \in L_p^+ , \quad (11)
\]
is a complete F-norm on \( L_p^+ \). Furthermore (see [16 p. 54]), if we denote by \( L_p^0 \) the class of all functions \( f \) such that \( \alpha f \in L_p^+ \) for every \( \alpha > 0 \), then \( L_p^0 \) is the closure of the space of all continuous functions on \([0, 2\pi)\) in the space \( (L_p^+, | \cdot |_p) \).

We also give the following two results.

**Theorem 8** ([33 Theorem 4.1]). The F-norms \( \| \cdot \|_p \) and \( | \cdot |_p \) (given by (10) and (11), respectively), induce the same topology on the space \( L_p^+ \). In other words, the norm and modular convergences are equivalent.

**Proposition 2** ([33 Corollary 4.2]). There does not exist a nontrivial continuous linear functional on the space \( (L_p^+, \| \cdot \|_p) \).

Note that [33 Section 5] the algebra \( N^p \) may be considered as the Hardy-Orlicz space with the Orlicz function \( \psi : [0, \infty) \to [0, \infty) \) defined as \( \psi(t) = ( \log(1 + t) )^p \). These spaces were firstly studied in 1971 by R. Leśniewicz [15]. For more information on the Hardy-Orlicz spaces, see [37 Ch. IV, Sec. 20]. Identifying a function \( f \in N \) with its boundary function \( f^* \), by [16 3.4, p. 57], the space \( N^p \) is identical with the closure of the space of all functions holomorphic on the open unit disk \( \mathbb{D} \) and continuous on \( \mathbb{D} : |z| \leq 1 \) in the space \( (L_p^+(dt/2\pi) \cap N, | \cdot |_p) \), where \( dt/2\pi \) is the usual normalized Lebesgue measure on the unit circle \( \mathbb{T} \). Using this fact, Theorem 4 and Theorem 6, the main surveyed results of this paper can be summarized as follows.

**Theorem 9** ([33 Theorem 1]). For each \( p > 1 \) Privalov class is the Hardy-Orlicz space with the Orlicz function \( \psi(t) = ( \log(1 + t) )^p (t \in [0, 2\pi)) \). Moreover, the metrics \( \lambda_p, \rho_p, \)
The metrics $d_p$, $\delta_p$, and the functional $| \cdot |_p$ (defined respectively by (3), (5), (6), (9) and (11)) induce the same topology on $N^p$ under which $N^p$ becomes an $F$-algebra.

4 CONCLUSION

This paper continues an overview of topologies on the Privalov spaces $N^p (1 < p < +\infty)$ induced by different metrics. Notice that the class $N^p$ equipped with the topology given by the metric $\lambda_p$ introduced by M. Stoll becomes an $F$-algebra. The same statement is also true for the class $M^p$ with respect to the $\rho_p$-metric topology. These facts are used in [20] to prove that for each $p > 1$ the classes $M^p$ and $N^p$ coincide and the metric spaces $(M^p, \rho_p)$ and $(N^p, \lambda_p)$ have the same topological structure. In Section 3 we give a short survey about Privalov spaces $N^p (1 < p < +\infty)$ whose topology is induced by the generalized Gamelin-Lumer’s metric $d_p$ defined on the space $L^\beta_p (dt/(2\pi))$. Notice that the space $L^\beta_p (dt/(2\pi))$ coincides with the Orlicz class associated to the log-convex $\varphi$-function $\psi(t) = (\log(1 + t))^p (t \in [0, +\infty))$. Accordingly, it follows that for each $p > 1$ Privalov space is the Hardy-Orlicz space with the Orlicz function $\psi(t) = (\log(1 + t))^p (t \in [0, 2\pi))$. Moreover, the metrics $\lambda_p$, $\rho_p$ and $d_p$ induce the same topology on $N^p$ under which $N^p$ becomes an $F$-algebra. We believe that presented results would be useful for future research on related topics, as well as for some applications in Functional and Complex Analysis.

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