Abstract

We examine effective field theories (EFTs) with a continuum sector in the presence of gravity. We first explain, via arguments based on central charge and species scale, that an EFT with a free continuum cannot consistently couple to standard (i.e. 4D Einstein) gravity. It follows that EFTs with a free, or nearly-free, continuum must either have a finite number of degrees of freedom or nonstandard gravity. We demonstrate the latter through holographically-defined continuum models, focusing on a class of 5D dilaton-graviton systems giving rise to a gapped continuum (i.e. the linear dilaton background). In the simplest version of the model we find an $R^{-2}$ deviation from the Newtonian potential. At finite temperature, we find an energy density with $a^{-5}$ scaling law (i.e. $w = \frac{2}{3}$) in the brane Friedmann equation, induced by the horizon in the bulk. We also present a slightly more evolved model for which these exotic deviations transition into those from pure AdS. Brane cosmology in dilaton-gravity backgrounds could be explored along these lines.
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1 Introduction

Among the multitude of effective field theories (EFT) extending the Standard Model (SM) of particles physics, models involving a continuum sector stand out as an intriguing possibility. Of course, any weakly coupled Poincaré-invariant quantum field theory (QFT) features a continuum in its spectral distributions. But beyond this standard case, a nearly-free continuum can also emerge in theories with some nontrivial underlying dynamics. Such a continuum can for example appear from gauge sectors with large number of degrees of freedom, or from EFTs whose background features a brane (i.e. domain wall or defect) living in a higher dimensional spacetime. Such nontrivial continuum sectors are the subject of this note.

From a more phenomenological viewpoint we may also write, from a bottom-up approach, a continuum model with arbitrary spectral functions describing the phenomena that are potentially observables in a given set of experiments, as allowed by the rules of the EFT paradigm. Phenomenologically, the underlying dynamics of the continuum may or may not matter, depending on the situation. In particular, in certain cases, it may be sufficient to use an effective model in which the continuum has properties analogous to an ordinary free field. This is called a generalized free field [1]. This approach applies for example to processes observable at colliders, such as “SM → continuum → SM” and “SM → continuum” for which only the two-point function of the continuum is needed. Regarding the latter class of processes, we emphasize that even though a continuum does not have well defined asymptotic states, such processes make sense as inclusive ones, for which no measurement of the continuum final state is required.

The EFT of a free continuum works fine for scattering processes observable at a collider. But, are there other physical observables for which the description of a continuum as a generalized free field does not apply? The answer is positive: whenever interactions with gravity are considered, the underlying dynamics of the continuum does matter. Clarifying the interplay of continuum models with gravity is the first aim of this note. This investigation will then naturally lead us to explore aspects of gravity in holographic models of continuum, which is the second aim of this note.

Our analysis is structured as follows. In Sec. 2 we lay out the formalism, and introduce the notion of generalized free field. In Sec. 3 we review the arguments (both old and new) that prevent a generalized free field to consistently couple to standard gravity. As an interesting aside we give an argument for the species scale that is valid for conformal field theories (CFTs) with any central charge and coupling. We then discuss the classes of models that give rise to a gravity-compatible continuum. It turns out that, apart from the conformal case, continuum models are best studied holographically, via 5-dimensional braneworld models. In Sec. 4.3 we lay out the basic holographic framework, review necessary QFT aspects and show how to compute the deviations from the standard Friedmann equation and Newtonian potential. In Sec. 5 we solve two versions of a specific dilaton-gravity background (both analytically and numerically) that features a gapped continuum and investigate gravity aspects. Section 6 contains a summary. Finally App. A contains further discussions on the transition between discretum and continuum, and App. C in-
cludes a piecewise solving of the asymptotically-AdS linear dilaton background.

**Previous literature:** The perspective of continuum models as a phenomenological possibility was first highlighted in Refs. [2, 3]. Among the subsequent references we mention only a few, as an introduction into the literature [4–8]. Aspects of cosmology with a conformal sector have been investigated in, e.g. Refs. [9–13], and in Refs. [14–16] in case of large $N$ weakly coupled CFT. A continuum as a mediator in the dark sector has been investigated in Refs. [17, 18]. Finally a proposal of “continuum dark matter” was recently made in Refs. [19, 20], that needs to be put in perspective with the arguments and results of the present note.

## 2 Continuum models

In this section we discuss some basic aspects of continuum models and introduce the notion of free continuum limit.

### 2.1 Continuum EFT

We consider an EFT described by the following four-dimensional Lagrangian,

$$
\mathcal{L} = \mathcal{L}_{\text{particles}}[\varphi] + \mathcal{L}_{\text{continuum}}[\Phi] + b\tilde{O}[\varphi]O[\Phi].
$$

This Lagrangian contains in general irrelevant operators $^3$. The fundamental fields $\varphi, \Phi$ can in principle have any spin. $O$ and $\tilde{O}$ are in general composite operators made of the corresponding fundamental fields. For simplicity those are assumed to be scalars.

The $\Phi$ sector is assumed to feature a nontrivial continuum — in a sense defined further below and in Sec. 2.2. In the $\varphi$ sector, the spectral functions are assumed to describe stable, or narrow, particles as occurs in weakly coupled QFT. The two sectors interact which each other only via the $\tilde{O}[\varphi]O[\Phi]$ operator. From the viewpoint of an observer able to probe the particle sector $\mathcal{L}_{\text{particle}}[\varphi]$, the continuum sector is probed by the $\varphi$ fields through the $\tilde{O}O$ operator. Thus in the correlators of the $\varphi$ fields, the continuum sector manifests itself via subdiagrams made out of the correlators of $O[\Phi]$, i.e. $\langle O(x_1)O(x_2)\rangle$, $\langle O(x_1)O(x_2)O(x_3)\rangle$, ... $^4$. Our interest precisely lies in these correlators of the continuum sector.

For any two-point (2pt) correlator one can always introduce a spectral representation of the form $^5$

$$
\langle O(x)O(0) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \int_C ds \frac{i \rho(s)}{p^2 - s + i\epsilon}
$$

where $\rho(s)$ is the spectral distribution and the contour $C$ encloses non-analyticities of the correlator in momentum space. In our conventions the momentum is timelike for $p^2 > 0$. The non-analyticities can either be poles or branch cuts along $\mathbb{R}_+$. On the domain

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$^3$The $\tilde{O}O$ operator is often taken as an irrelevant operator i.e. $\text{dim}[b] < 0$. A detailed parametrization is unnecessary for our purposes.

$^4$Time-ordering is left implicit, we use the usual shortcut notation $\langle O(x)O(0) \rangle = \langle \Omega|T\{O(x)O(0)\}|\Omega \rangle$.

$^5$This follows from Cauchy’s integral formula $f(a) = \frac{1}{2\pi i} \oint dz \frac{f(z)}{z-a}$. 
corresponding to a branch cut the spectral density \( \rho(s) \) is a smooth function. In this most generic case we refer to the \( \Phi \) sector as a \textit{continuum}. As a more particular case, it may be that the support of \( \rho(s) \) be a discrete set of points. Similarly it is also possible that the function be made of a series of narrow resonances such that the branch cut can be approximated by a set of points. In such cases the spectral distribution describes a countable set of standard 4D particles, and we refer to the \( \Phi \) sector more specifically as a \textit{discretum}.

2.1.1 Interactions

It is useful to classify the interactions encoded in the continuum sector.

i) There are fundamental interactions between the \( \Phi \) fields, encoded inside the Lagrangian \( L_{\text{continuum}} \). We denote collectively these interactions by the coupling \( g \). These fundamental interactions may be either weak or strong.

ii) The continuum interacts with the particle sector via the \( \tilde{O}[\varphi]O[\Phi] \) operator. This implies that local operators of the form

\[
L_{\text{continuum}} \supset g_{O,n} (O[\Phi](x))^n
\]

are generically present in the continuum sector. Analogous ones with an arbitrary number of derivatives also exist. All these operators are in general present due to the quantum dynamics in the \( \varphi \) sector. In general, even if these operators are set to zero at a given scale, they are generated at a different scale due to renormalization group (RG) running. We refer collectively to these local operators as \( O^n \) and the corresponding coupling as \( g_{O} \).

2.1.2 Realizations

In principle \textit{any} interacting QFT can realize the setup of Eq. (2.1). For example, for a weakly coupled interacting QFT the continuous part of the spectral distribution \( \rho \) encodes a multiparticle continuum, and possibly a resonance. However, our interest lies in theories that can give rise to a \textit{free continuum} when some parametric limit is taken in the model (see Sec. 2.2). A non-trivial dynamics is needed for such a limit to occur. It is realized in at least the two following classes of theories.

i) \textit{Gauge theories with a large number of colors} \( N \). For simplicity we assume that the \( \Phi \) fields are in the adjoint representation, so that the standard large \( N \) scaling applies [21]. The theory may, in principle, have either weak or strong t’Hooft coupling \( \lambda \equiv g^2N \). As a particular case, the gauge theory may be at a conformal fixed point in which case it is a CFT. For \( \lambda \ll 1 \) this occurs at a Banks-Zaks fixed point if the theory has number of flavors within the conformal window. At \( \lambda \ll 1 \) stringy effects are expected to emerge for large \( N \) (see \textit{e.g.} Refs. [22–28]) while at \( \lambda \gg 1 \) the stringy effects are expected to decouple [29, 30].
ii) Holographic theories. These arise from EFTs living in a 5D background (with arbitrary metric) featuring a flat 3-brane. In such a setup an effective Lagrangian of the form of Eq. (2.1) appears from the viewpoint of an observer placed on the brane. The $\varphi$ field is identified as a brane-localized mode with standard 4D spectral distribution, which mixes with a continuum controlled by the 5D dynamics (see Sec. 5 for more details). In such models there are both bulk and brane-localized local interactions, that we denote by $g_{\text{bulk}}$ and $g_{\text{brane}}$. We also refer to this setup as a “braneworld” in the context of cosmology.

2.2 The free continuum (GFT) limit

We are interested in taking a parametric limit for which a free continuum arises in the general Lagrangian of Eq. (2.1). Our notion of free continuum is equivalent to the one described by a generalized free theory (GFT), hence we are using either naming depending on context.

GFTs have been studied in the context of QFT and CFT (see e.g. Refs. [1, 31, 32]). In a GFT the connected part of the correlators of $\mathcal{O}$ vanishes. As a result the odd correlators are zero while the even correlators are given by the disconnected contributions which are just a product of 2pt correlators. For example for the 4pt correlator we have

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \langle \mathcal{O}(x_1)\mathcal{O}(x_2) \rangle \langle \mathcal{O}(x_3)\mathcal{O}(x_4) \rangle + \text{permutations}. \quad (2.4)$$

We define the free continuum (i.e. GFT) limit as the limit for which the fundamental interactions of the continuum sector vanish,

$$\mathcal{L}_{\text{continuum}} \bigg|_{g \to 0} \to \mathcal{L}_{\text{GFT}} \quad (2.5)$$

while the spectral density does not become discrete (i.e. remains supported on $\mathbb{R}$ and not only on a discrete set of points when $g \to 0$). This definition of the free continuum limit automatically excludes the trivial case of an interacting QFT with finite degrees of freedom, since in that case for $g \to 0$ the multiparticle continuum vanishes and the spectral density of $\mathcal{O}[\Phi]$ becomes discrete. Thus some nontrivial dynamics in $\mathcal{L}_{\text{continuum}}$ is required for a free continuum to emerge at $g \to 0$.

Our definition of GFT allows for the existence of the local interactions $\mathcal{O}^n$. Thus in our definition the GFT correlators can have $O(g_\mathcal{O})$ contributions. This is however a minor point in the rest of our analysis, as we will obtain the same conclusions as if $g_\mathcal{O} = 0$.

How is the free continuum/GFT limit realized in the classes of models listed in Sec. 2.1.2?

i) A GFT emerges from a gauge theory by taking the limit of infinite number of colors $N \to \infty$ at constant ‘t Hooft coupling — notice that $g \to 0$ in this limit. Indeed, by normalizing the 2pt function coefficient such that it does not scale with $N$, standard large $N$ scaling arguments imply that the connected correlators scale as powers of $1/N$. In the $N \to \infty$ limit the odd correlators are $O(1/N)$ and the even correlators are given by the free disconnected result plus $O(1/N)$ terms. This matches the properties
of a GFT, hence
\[ \mathcal{L}_\text{gauge} \bigg|_{N \to \infty, \lambda \text{ fixed}} \to \mathcal{L}_\text{GFT} . \] (2.6)

We could similarly write this limit for the full Lagrangian including the \( \mathcal{O}^n \) interactions.

ii) A GFT emerges from a holographic setup by sending the bulk couplings to zero. Indeed in this limit the bulk propagators are free, hence the higher point correlators factorize into 2pt propagators. The holographic theory inherits this property, therefore the holographic theory is a GFT. The brane couplings contribute solely to the \( \mathcal{O}^n \) interactions — which are allowed in our definition of GFT. In summary for the full holographic Lagrangian we schematically have
\[ \mathcal{L}_\text{hol} \bigg|_{g_{\text{bulk}} \to 0} \to \mathcal{L}_\text{particles}[\varphi] + \mathcal{L}_\text{GFT}[\Phi] + c\hat{\mathcal{O}}[\varphi]\mathcal{O}[\Phi] . \] (2.7)

### 2.2.1 Continuous mass representation

In the GFT the correlators of \( \mathcal{O} \) can be described with a diagrammatic expansion using perturbation theory in the \( g_\mathcal{O} \) couplings. The resulting diagrams are built from the \( \mathcal{O}^n \) vertices, connected by lines which are the propagators of \( \mathcal{O} \), i.e. the 2pt free correlator
This is just the usual picture of Feynman diagrams, with GFT propagators instead of ordinary propagators.

We can thus view the continuum sector as a set of fields \( \Phi \equiv \{ \varphi_s \} \) whose only interactions are those encoded in the \( \mathcal{O}^n \) operators. The domain for the \( s \) label is determined below. These \( \varphi_s \) fields must reproduce the propagator of \( \mathcal{O} \). Using the spectral representation introduced in Eq. (2.2), this is possible if the \( \mathcal{O}[\varphi_s] \) operator is

\[
\mathcal{O}[\varphi_s] = \int_0^\infty ds \sqrt{\rho(s)} \varphi_s
\]

with

\[
\mathcal{L}[\varphi_s]_{\text{continuum}} \supset \int_0^\infty ds \mathcal{L}[\varphi_s(x)]_{\text{free}}, \quad \mathcal{L}[\varphi_s(x)]_{\text{free}} = \frac{1}{2} (\partial \mu \varphi_s)^2 - \frac{s}{2} (\varphi_s)^2.
\]

The \( \varphi_s(x) \) are ordinary free fields with squared mass \( s \) and propagator\(^6\)

\[
\langle \varphi_s(x) \varphi_s'(0) \rangle = \delta(s - s') \int \frac{d^4 p}{(2\pi)^4} \frac{ie^{-ipx}}{p^2 - s + i\epsilon}.
\]

Eq. (2.10) together with the definition (2.8) reproduces the spectral representation Eq. (2.2) of the \( \langle \mathcal{O}(x) \mathcal{O}(0) \rangle \) correlator. Similar developments can be found in Refs. \([33, 34]\).

The higher point correlators of \( \mathcal{O} \) follow trivially since they inherit the properties of the free fields \( \varphi_s(x) \). Namely, the odd correlators of \( \mathcal{O} \) vanish, up to \( O(g_\mathcal{O}) \), and the even correlators tend to the free disconnected result, up to \( O(g_\mathcal{O}) \), as required for a GFT. For example in the 4pt case, one obtains Eq. (2.4).

3 Consistency with gravity

In the previous section we have introduced the notion of free continuum \( i.e. \) of GFT. Here we expand the explanation of why a GFT is not compatible with gravity. In this section gravity means 4D Einstein gravity. Some of the arguments already existed previously, and are hereby reviewed, while others are new to the best of our knowledge. We then discuss gravity-compatible realizations and sketch some basic cosmological consequences.

3.1 Arguments from OPE

In this section we provide arguments based on the operator product expansion (OPE).

3.1.1 From CFT (review)

We start with a gauge theory with arbitrary \( \text{t} \)Hooft coupling, focusing on the conformal case. In conformal theories, there is a rigorous claim that the simultaneous existence of a generalized free field and the stress-energy tensor are incompatible, unless the generalized free field is an ordinary free field (see \( e.g. \) Ref. \([31, 32]\)).

\(^6\)In terms of quantization rules, one introduces creation and annihilation operators of fields \( \varphi_s \) such that \[ [a_{p,s}, a^\dagger_{p',s'}] = (2\pi)^3 2p_0 \theta(p_0) \delta^{(3)}(p - p') \delta(s - s'). \]
A version of the proof of this well-known result goes as follows. Let us assume that a conformal theory contains a generalized free field $\mathcal{O}$ and a stress tensor $T_{\mu\nu}$. The stress tensor has dimension $d$ and spin 2. The 4pt function of $\mathcal{O}$, given in Eq. (2.4), contains information about the spectrum and OPE coefficients of $\mathcal{O}$. It can be shown [32] that Eq. (2.4) implies that the OPE of $\mathcal{O}(x)\mathcal{O}(0)$ only contains bilinear operators built from derivatives of $\mathcal{O}$, e.g., $\mathcal{O} \Box^n \mathcal{O}$. Such operators have dimension $2\Delta + n$. These facts put together imply that there can be a stress tensor in the OPE of $\mathcal{O}(x)\mathcal{O}(0)$ only if $2\Delta + 2 = d$, hence requiring $\Delta = (d - 2)/2$, which corresponds to the ordinary free field (in which case one has $\Phi(0)\Phi(x) \supset 1/2 \eta^{\mu\nu} T_{\mu\nu}$). Otherwise, i.e., if $\Delta > 1$, there cannot be $T_{\mu\nu}$ in the OPE of $\mathcal{O}(x)\mathcal{O}(0)$. The latter feature implies, by symmetry of the OPE coefficients, that $\mathcal{O}$ is absent of the $T_{\mu\nu}(x)\mathcal{O}(0)$ OPE. This is inconsistent with translation invariance, which requires that $\mathcal{O}$ must appear in this OPE with a nonzero coefficient. We thus reach a contradiction. The contradiction is solved if either the generalized free field $\mathcal{O}$ or $T_{\mu\nu}$ are absent from the conformal theory.

3.1.2 From continuous mass representation

The fact that a GFT with stress tensor is inconsistent can also be directly seen from the continuum mass representation defined in Sec. 2.2.1. The $\rho(s)$ distribution is in general supported over $[0, \infty)$, but the argument also applies if the distribution is truncated to an interval such as $[0, \Lambda^2)$, as may occur in an EFT. In the presence of the free continuum described by the set of free fields $\varphi_s$, we can formally derive a stress tensor from the Lagrangian Eq. (2.9), which gives $T_{\mu\nu} = \int ds T_{\mu\nu}[\varphi_s]$. We can then compute the correlator of this generalized free stress tensor with itself and focus on the traceless part. The result is proportional to $\int ds \delta(0) x^2 d$. Since $\int ds \delta(0) = \infty$, the central charge is infinite. The infinite central charge effectively sends to zero the coefficient involving $T_{\mu\nu}$ in the OPE of $\mathcal{O}(x)\mathcal{O}(0)$. This leads to a contradiction with translation invariance, as in the CFT proof above. The argument given here extends beyond the CFT case, and holds whether or not there are $\mathcal{O}^n$ local operators.

3.2 From species scale

Let us consider the GFT in the presence of dynamical gravity via the action $S = S_{\text{grav}} + \int d^4x \sqrt{g} L_{\text{continuum}}$. This is in general a low-energy EFT describing subPlanckian gravity interacting with matter, together with classical black holes. What is the UV cutoff scale of this EFT?

Even though the strength of gravity is set by the reduced Planck mass $M_{\text{Pl}}$, the actual validity scale of the EFT may be lower. Using an argument based on classical black hole lifetime, Ref. [35] established the bound $\Lambda \sim M_{\text{Pl}}/\sqrt{N_{\text{sp}}}$ where $N_{\text{sp}}$ is the number of species of matter in the theory. This argument relies on Hawking radiation and thus assumes that the species are stable or narrow particles.

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7 The same feature is true in an ordinary free field theory, and thus the same conclusion can be obtained using the continuum mass representation.

8 This is consistent with the viewpoint of the GFT as a CFT with $N \to \infty$, which also gives infinite central charge.
Here we will verify that the species bound can be extended, beyond weak coupling, to a CFT with arbitrary central charge $c$ and arbitrary 't Hooft coupling $\lambda$. This is an aside result that we use it to strengthen our analysis and which is also interesting in itself.

### 3.2.1 Species scale for CFT with arbitrary central charge

Let us consider $S = S_{\text{grav}} + \int d^4x \sqrt{g} L_{\text{CFT}}$. We want to determine the UV cutoff scale $\Lambda$ of the theory. Let us assume there is no cosmological constant and let us put the CFT at finite temperature $T$. The energy density is given by $\rho_{\text{CFT}} = \pi^2 \zeta c T^4$, with $\zeta = 2$ and $\zeta = \frac{3}{2}$ at weak and strong coupling, respectively. For simplicity we drop the $\pi^2 \zeta$ factor in the following. As a result of this energy density, spacetime expands with a Hubble rate

$$H \sim \sqrt{\frac{c}{M_{\text{Pl}}}} T^2. \quad (3.1)$$

The associated volume for a Hubble patch is $1/H^3$. But this volume is bounded from below by the cutoff of the theory, as it cannot be smaller than the volume $(\Delta x)^3 = 1/\Lambda^3$, which amounts to a Hubble rate $H = \Lambda$. The corresponding momentum scale is of order $\Lambda$ and, since temperature is proportional to the average momentum scale, we can say that this Hubble rate is attained for $T \sim \Lambda$. Therefore the UV cutoff is determined by the condition $H|_{T=\Lambda} \sim \Lambda$, which gives

$$\Lambda \sim \frac{M_{\text{Pl}}}{\sqrt{c}}. \quad (3.2)$$

In the case of weakly coupled stable species we have $c \rightarrow N_{\text{sp}}$ which recovers the usual formula from Ref. [35].

### 3.2.2 Application to GFT

Having ensured that the species scale applies to any CFT, we turn to the GFT. Viewing the GFT as the limit of a CFT with $c \sim N^2 \rightarrow \infty$, we can see that the number of species in the GFT goes to infinity. Therefore $\Lambda \rightarrow 0$, and so there is no energy regime where gravity is weakly coupled! This means that a GFT coupled to gravity simply does not exist.

The same conclusion is obtained when considering the continuous mass representation. For any $\Delta > 1$ there is an infinite number of degrees of freedom $N_{\text{sp}} = \infty$, hence $c = \infty$, which implies $\Lambda \rightarrow 0$.

From all of the above arguments we conclude that gravity cannot couple to the GFT because the latter has infinitely too many degrees of freedom. Notice that, in contrast, a CFT has a finite number of species $c \sim N^2$, and thus in that case the UV cutoff $\Lambda$ is nonzero. Notice also that all the arguments would be avoided in the case of an ordinary free field ($\Delta = \frac{d-2}{2}$, $\rho(s) \propto \delta(s - m^2)$); however this is excluded in our definition of free continuum (see Sec. 2.2). Thus, in other words, the coupling to gravity would enforce the generalized free field to be an ordinary free field.

### 3.3 Holographic theory vs GFT

We have shown that a GFT (as defined in Sec. 2.2) is obtained from a holographic setup by setting all the bulk interactions to zero. This definition implies that 5D gravity is
removed when taking the GFT limit, $M_{\text{Pl},5} \to \infty$. In such a limit we have a gravity-less bulk which can be trivially integrated out. Conversely, a holographic setup with gravity provides automatically a continuum compatible with gravity. However the price to pay is that gravity in the holographic theory is intrinsically 5D, implying that the graviton itself has a continuum component, such that gravity deviates from 4D Einstein gravity.

Let us comment about the case of AdS background (e.g. the RS2 setup [37]). In this case the AdS/CFT correspondence applies. By the correspondence the $g_{\text{bulk}}$ coupling goes as some power of $1/N$, and hence the GFT limit is consistent from either the AdS or the CFT viewpoints since $(g_{\text{bulk}} \to 0) \Leftrightarrow (N \to \infty)$. We can also note that, even without gravity, the AdS theory always has a 5D stress tensor. What really changes when taking $g_{\text{bulk}} \to 0$ is that the graviton field is removed. The CFT operator dual to this bulk field is the stress tensor, which is thus removed upon taking $g_{\text{bulk}} \to 0$. This is in agreement with the argument of Sec. 3.1.1.

3.4 Gravity-compatible continuum models

Along with the arguments of Secs. 3.1 and 3.2 we have established that a free continuum i.e. a GFT is incompatible with 4D Einstein gravity. We now consider theories lying in the neighborhood of this limiting case in theory space (see Fig. 1). Such neighboring theories feature some notion of free or nearly-free continuum, and some ingredient making the continuum EFT compatible with gravity — associated to loopholes in the no-go arguments of Secs. 3.1 and 3.2.

By examining the latter arguments, we can identify the following logical possibilities for EFTs neighbors to the excluded case of GFT+4D Einstein gravity: a) The EFT has large but finite number of degrees of freedom, and b) Gravity differs from 4D Einstein gravity. Following these lines we then identify the following three (possibly overlapping) classes of theories giving rise to free or nearly-free continuum models consistent with gravity.

i) The continuum is really a discretum.

It is possible that the free continuum be an approximation of a free discretum. Indeed both are indistinguishable to a finite precision experiment unable to resolve the discretum spacing. In this case the underlying degrees of freedom are countable and their number is finite since they are bounded by a gravity-induced UV cutoff. Thus the central charge is finite and inconsistencies with gravity are avoided. In the bottom-up EFT of a free continuum this can be simply obtained by making the spectral distribution discrete in the continuous mass representation of Sec. 2.2.1.

ii) The continuum is a large-$N$ gauge correlator.

For finite number of colors $N$ the central charge is finite, thus inconsistencies with gravity are avoided. In that case the continuum is nearly-free since it has nontrivial connected correlators which are $1/N$-suppressed but nonzero. At strong coupling a discretum may arise at low energy if the theory enters a confining phase, hence providing a realization of i).

\footnote{A situation reproduced by Little String Theories [36].}
iii) The continuum is holographic.

In this case the underlying dynamics is intrinsically 5D even though it is seen from a brane viewpoint. The matter continuum arising in the 4D holographic theory automatically couples consistently to gravity. The counterpart is that gravity itself has a continuum component. Thus gravity in the holographic theory is not 4D Einstein gravity. The holographic framework can also realize the above ones in specific cases, as for certain backgrounds a discrete KK spectrum arises, hence realizing i), and for pure AdS background ii) is realized via the AdS/CFT correspondence.

For convenience we refer to the continuum from both i) and ii) as a nearly-free continuum. For i), “nearly” applies to “continuum”, which really is a discretum, while for ii), “nearly” applies to “free”, since the continuum has small but nonzero nontrivial correlators.

We can now observe that for any kind of EFT with a free or nearly-free continuum consistently coupled to gravity, substantial deviations must appear in the gravity sector. These are the deviations that would blow up and make the theory inconsistent when taking the limit of a free continuum coupled to 4D Einstein gravity.

This fact is evident for holographic models, class iii), in which gravity automatically deviates from 4D Einstein gravity. But this fact also occurs in the classes of models i) and ii) because, in any event, the graviton propagator is dressed by insertions of \( \langle TT \rangle \) correlators which are proportional to the central charge. This is a physical QFT correction to the Newton law of gravity. In the limit of large central charge the correction to the graviton propagator blows up, inducing thus large effects on the gravity sector.

In summary we can state that consistent models of a free or nearly-free continuum must feature deviations in the gravity sector as a general feature. This is pictured in Fig. 1. We investigate such effects in a concrete framework in the upcoming sections.

3.5 Cosmological implications

In this section we qualitatively discuss the expected cosmological effects in the classes of gravity-compatible continuum models listed in Sec. 3.4. Along the same lines as the observations made there, such models must have significant impact on standard cosmology since they feature either a large number of degrees of freedom or deviations from gravity that blow up when approaching the forbidden limit of GFT+4D Einstein gravity (see Fig. 1). Here we thus discuss basic cosmological aspects of the classes of models i) ii) and iii), i.e. discretum, large \( N \) gauge theories and holographic theories.

A cosmological discretum is a fairly intuitive possibility. In that case the continuum is really made out of a set of 4D particles with standard properties and thus their contributions to the Friedmann equation are clear. For example, at temperature lower than the mass gap \( \sigma \), the tower of particles is nonrelativistic and can be a candidate for dark matter. Such a scenario has been studied at length, see e.g. Refs. [38, 39].

The cosmological implications of a hidden large \( N \) gauge theory are trickier, because in general we do not know the equation of state \( p = w \rho \), except in the following particular cases. First, the gauge theory may transition to a confined phase at low temperature, in which case the confined case is described by a discretum EFT already discussed above.
Second, the gauge theory may be at a conformal fixed point, in which case it is a CFT whose properties are very constrained by symmetries. Let us review this well-known particular case. The hot CFT behaves as dark radiation because scale invariance implies $T_{\text{CFT},\mu} = 0$ which in turn implies $p = \rho/3$, i.e. $w = 1/3$. Since the hidden CFT has many ($\sim N^2$) degrees of freedom, the temperature $T_h$ must be much lower than the one of the visible sector, otherwise the CFT energy density $\rho_h = \zeta \pi^2 N^2 T_h^4$ overwhelms the visible one, which amounts to a large amount of dark radiation, excluded by observations. Hence one requires $\rho_h \lesssim \rho$. Since $N \gg 1$, such a requirement on $\rho_h$ implies that the temperature of the hidden CFT should be much lower than the visible one, $T_h/T_{\text{vis}} \sim g_*/g_{\text{vis}}^{1/4} N^{-1/2} \ll 1$. For more general gauge theories a similar reasoning applies at a more qualitative level, yielding the generic prediction that a large $N$ hidden sector must be ultracold in order to not spoil cosmology. However, we cannot say more because we do not know the equation of state for such an energy density. A cosmological continuum model, apart from the CFT case, is thus best studied via holography.

We now turn to holographic continuum models. The cosmology of some of these models has been well studied. The simplest, and best studied, cosmological scenario is the one for which the bulk is exactly AdS$_5$ everywhere, which furthermore exactly mirrors the scenario of hot CFT reviewed above (see e.g. Refs. [40–45]). The key point is that at finite temperature a horizon develops in the bulk, with AdS-Schwarzschild metric (AdS-S). The presence of the horizon crucially modifies the effective Friedmann equation projected on the brane with a term which, from the standpoint of the brane observer, behaves as dark radiation. This effective radiation term arising from the bulk geometry matches the CFT result $\rho_h$ for strong 't Hooft coupling. We summarize such a remarkable feature as

$$\text{AdS-Schwarzschild horizon} \Leftrightarrow w = \frac{1}{3} \text{ (dark radiation).} \quad (3.3)$$

Departing from the pure AdS case there are plenty of possible background geometries, in particular “soft-wall” backgrounds appearing from the 5D gravity-dilaton system, see e.g. Refs. [46–56]. Some of these backgrounds give rise to a continuum in the 4D holographic theory. Continuum models from the gravity-dilaton framework will be the focus of the rest of the paper.

4 The holographic continuum: gravity and Friedmann equation

Our focus here is on holographically defined continuum models. Such models are particularly attractive as everything is calculable since the 5D QFT is weakly coupled. In this section we lay out the overall framework for holographic models of continuum. The setup is reminiscent of braneworld models (see e.g. Ref. [57]). Namely, we consider a five-dimensional spacetime with a flat 3-brane (i.e. domain wall or defect) living on it, and evaluate the effective theory for an observer living in the brane worldvolume (see Fig. 2). In such a setup the 5D excitations are integrated out and form a continuum from the standpoint of the brane observer.
Since the overarching theme of the paper is the consistency of continuum EFT with gravity, we will be especially interested in the gravity side of the holographic continuum models. Thus two concrete objects of study stand out.

- **The gravitational potential**
  At any scale for which a continuum is present in the holographic EFT, something nontrivial has to occur in the gravity sector to ensure consistency with gravity. Thus some deviation from Newtonian gravity can be expected at such scales. This can also be qualitatively understood in terms of the existence of a stress tensor in the continuum sector. Such a stress tensor, whose existence is ensured in the holographic setup, dresses the 4D graviton, yielding a modification of the Newtonian potential.

- **The Friedmann equation**
  The equation of state in the continuum sector is in general nontrivial and unknown (see also the discussion in Sec. 3.5). However in holographic models this equation of state is encoded into the geometry of the 5D background. This appears at the level of the effective Friedmann equation seen by a brane observer, which contains nontrivial information about the bulk geometry — and thus about the equation of state.

In summary we expect deviations to both the Newtonian potential and Friedmann equation.

### 4.1 The five-dimensional background

We consider a five-dimensional spacetime with a flat 3-brane (i.e. domain wall). The 5D coordinates are denoted by $M, N, \ldots$ indices, while the 4D coordinates on the 3-brane $M$ are denoted by $\mu, \nu \ldots$ indices.

We consider the action of the graviton-dilaton system

$$S = \int d^5x \sqrt{g} \left( -\frac{M_5^3}{2} \mathcal{R} + \frac{1}{2} (\partial_M \phi)^2 - V(\phi) + \Lambda_5 \right) + M_5^3 \int_{\text{brane}} d^4x \sqrt{\bar{g}} \Lambda_b + S_{\text{matter}} + \ldots,$$

with $\phi$ the dilaton field, $M_5$ the fundamental 5D Planck scale, $\Lambda_5$ the 5D cosmological constant, $\Lambda_b$ the brane tension, and $\bar{g}_{\mu\nu}$ the induced metric on the brane. $S_{\text{matter}}$ encodes the quantum fields living on this background. The ellipses encode a term giving rise to a vacuum expectation value (VEV) for $\phi$ that does not need to be explicitly specified here. For concreteness we can assume it is fixed on the boundary of the 5D spacetime. \(^{10}\)

The 5D metric is given the form

$$ds^2 = g_{MN} x^M x^N = \omega^2(z) \left( -f(z)dt^2 + dx^2 + \frac{1}{f(z)} dz^2 \right)$$

$$= -n^2(r)dt^2 + \frac{r^2}{\ell^2} dx^2 + b^2(r) dr^2.$$

The coordinate frame in the first line shows that the metric is conformally related to the flat space Schwarzschild metric. The functions $\omega(z)$ and $f(z)$ are referred to, respectively, \(^{10}\) Alternatively a potential giving rise to the dilaton VEV may be localized on the brane of Eq. (4.1). Such a possibility is inequivalent to the model considered here and will be studied in a future work.
as the warp and blackening factors. The coordinate frame in the second line are convenient for brane cosmology. Along any constant slice of \( r \), the \( r = \ell \omega(z) \) coordinate acts like a scale factor.

The 3-brane is a hypersurface located at \( r = r_0 \). With the above coordinates the induced metric \( \bar{g}_{\mu \nu} \) reads

\[
\bar{d}s^2 = \bar{g}_{\mu \nu} dx^\mu dx^\nu = -dt^2 + \frac{r_0^2}{\ell^2} dx^2 ,
\]

where we have introduced the brane cosmic time \( dt = n(r_0) d\tau \). According to this metric, if the brane moves along \( r \) in the extra dimension, i.e. if \( r_0 = r_0(t) \), the observer perceives expansion of the 4D universe.

The 5D equations of motion for metric factors and the dilaton field are given in conformal coordinates by \[58\]

\[
\frac{f''(z)}{f'(z)} + 3 \frac{\omega'(z)}{\omega(z)} = 0 , \tag{4.5}
\]

\[
\omega'(z) \omega(z) - 2 \left( \frac{\omega'(z)}{\omega(z)} \right)^2 + \bar{\phi}'(z)^2 = 0 , \tag{4.6}
\]

\[
\frac{\omega'(z)}{\omega(z)} \left( \frac{f'(z)}{\omega(z)} + \frac{1}{4} f(z) \right) - \frac{1}{6M_5^2} \frac{\omega(z)^2}{f(z)} V(\bar{\phi}) - \frac{1}{4} \bar{\phi}'(z)^2 = 0 , \tag{4.7}
\]

\[
\bar{\phi}''(z) + \left( 3 \frac{\omega'(z)}{\omega(z)} + \frac{f'(z)}{f(z)} \right) \bar{\phi}'(z) + \frac{1}{3M_5^2} \frac{\omega(z)^2}{f(z)} \frac{\partial V}{\partial \bar{\phi}} = 0 , \tag{4.8}
\]

where for convenience we have defined the dimensionless scalar field \( \bar{\phi} \equiv \phi / (\sqrt{3}M_5^{3/2}) \). The general solutions contain five integration constants. However it turns out that one of the equations, e.g. Eq. (4.7), acts as an algebraic constraint on the integration constants hence there are only four independent constants. Some of the integration constants have no physical meaning and can be fixed without loss of generality i.e. amount to “gauge redundancies”, while others have physical meaning. A detailed discussion is provided in App. B.

4.2 The effective Friedmann equation

The effective Einstein equation seen by an observer standing on the brane is computed from the 5D Einstein equation, projected on the 3-brane via the Gauss equation together with the Israel junction condition, which relates the extrinsic curvature to the brane-localized stress tensor. The outcome takes the form \[41\]

\[
R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R = \Lambda_4 g_{\mu \nu} + \frac{1}{M_{Pl}^2} \pi_{\mu \nu} + \frac{1}{M_5^6} \pi_{\mu \nu} - \dot{C}_{\mu \nu} . \tag{4.9}
\]

Here \( R_{\mu \nu} \) is the Ricci tensor projected on the brane. \( T_{\mu \nu} \) is the stress energy tensor on the brane. The effective 4D cosmological constant \( \Lambda_4 \) contains contributions from both the 5D cosmological constant and from the brane tension, which are tuned so that \( \Lambda_4 = 0 \). The \( \pi_{\mu \nu} \) tensor is a quadratic combination of brane-localized stress tensors, thus it comes with
and a $M_5^{-6}$ factor. Finally, $\hat{C}_{\mu\nu}$ is a term built from the projection of the 5D Weyl tensor — i.e. the traceless part of the Riemann tensor — on the brane.

The $\pi_{\mu\nu}$ tensor is a term induced by the extrinsic curvature terms in the Gauss equation and contains quadratic combinations of the stress tensor $T^2$. We can see that, at energy densities $\rho \lesssim M_5^3/M_{Pl}^2$, the effect of $\pi_{\mu\nu}$ is negligible with respect to the standard $M_{Pl}^{-2}T_{\mu\nu}$ term of Einstein equation.

We plug the metric of Eq. (4.3) into Eq. (4.9), for a brane at $r = r_0$. Focusing on $\rho \ll M_5^6/M_{Pl}^2$, tuning to zero the 4D cosmological constant, and focussing on the $(0,0)$ component of Eq. (4.9) we obtain the effective Friedmann equation on the brane

$$3M_{Pl}^2 \left( \frac{\dot{r}_0}{r_0} \right)^2 = \rho - M_{Pl}^2 \mathcal{C}(r_0) + O \left( \frac{\rho^2 M_{Pl}^2}{M_5^6} \right),$$

(4.10)

where $\rho$ in this equation is the 4D energy density localized at the brane. Here $\dot{r}_0 \equiv \frac{dr_0}{dt}$ where $t$ is the cosmic time for the brane observer. The $\mathcal{C}(r)$ term depends on the $C_{5050}$ component of the Weyl tensor,

$$\mathcal{C}(r_0) = \frac{1}{n^2(r_0)b^2(r_0)} C_{5050}(r_0)$$

$$= \frac{1}{2b(r_0)^2} \left[ \frac{n''(r_0)}{n(r_0)} - \left( \frac{b'(r_0)}{b(r_0)} \right) \left( \frac{n'(r_0)}{n(r_0)} - \frac{1}{r_0} \right) \right].$$

(4.11)

The second line is the result obtained with the metric of Eq. (4.3).

From Eq. (4.10) we can see that all the effects of the 5D geometry at energies below $M_5^6/M_{Pl}^2$ are encapsulated into the $\mathcal{C}$ term. We thus have a geometric effect which translates from the brane viewpoint as an effective energy term, that we refer to as the Weyl energy. The scaling of $\mathcal{C}(r_0)$ in $r_0$ determines the equation of state of the Weyl energy. The Weyl tensor measures deviation from conformality and it vanishes if $f \to 0$ identically in Eq. (4.2).

If $f$ vanishes at a given point $f(z_h) = 0$, the hypersurface $z = z_h$ is a horizon whose temperature and entropy are given by $T_h = |f'(z_h)|/(4\pi)$ and $S = V_3 \omega^3(z_h)/(4G_5)$, respectively, where $V_3$ is the volume in the 3D space and $G_5$ is the Newton constant in 5D. The presence of the Weyl energy in the brane Friedmann equation is thus associated with the temperature of the horizon in the bulk. In Sec. 5 we will compute the horizon temperature for completeness and to compare with the literature. Regarding the entropy, its general formula in the brane cosmology coordinates for any model is $S = V_3 \left( \frac{2\pi}{T} \right)^3/(4G_5)$.

4.3 QFT overview

In this section we consider quantum fields living over the 5D background encoded in the term $S_{\text{matter}}$ in Eq. (4.1). We review some essential properties of bulk QFT as seen from a brane, that are needed to establish the general picture of a holographic continuum model.

Our focus is on the fields living in the $r \leq r_0$, i.e. $z \geq z_0$, region of the bulk. We assume that the fields have Neumann boundary conditions (BC) on the brane, i.e. the fields are

$$T_h = \frac{1}{4\pi} \sqrt{\frac{dx^2(\tau)}{dr} \frac{dx^2(\tau)}{dr}} r = r_h$$

where $\chi(r) \equiv 1/b(r)$.
allowed to fluctuate on the brane. The fields are described by a Lagrangian in the 5D bulk, but additionally there can always be operators localized on the brane. In fact those are always generated by loop effects (see Ref. [59] for explicit results). Thus following the EFT paradigm such operators should be included in the brane Lagrangian in a first place.

Let us now consider a generic bulk field $\Phi$ with value $\Phi_0 = \Phi|_{M}$ on the brane. The field propagates in the bulk, but the brane-localized operators would influence its propagation. In fact, on general grounds, a brane-to-brane propagator takes the form $G = [G_0^{-1} + B]^{-1}$, where $G_0 \equiv G|_{B=0}$ and $B$ is the bilinear insertion induced by the brane-localized operators [59] and dressing $G_0$. In momentum space, both $B$ and possibly $G_0^{-1}$ contain an analytic piece $\propto p^2$, which amounts to having an isolated 4D free mode in the spectrum.

The wavefunction of this mode is typically localized near the brane. Singling out this 4D localized mode, the propagator can be written as

$$\langle \Phi_0(x')\Phi_0(0) \rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ipz} G_p, \quad G_p = \frac{iZ_0}{p^2 - m_0^2 + b_0\Pi(p^2)}, \quad (4.12)$$

where $Z_0$ is a wavefunction renormalization effect, and the $\Pi(p^2)$ term is non-analytical. For sufficiently smooth background, as the one we will consider here, $\Pi(p)$ has a branch cut along some region of the $p^2 > 0$ axis, i.e. it is a continuum. This term encodes the contributions of all the rest of the bulk modes to the brane-to-brane propagator. We thus have split the denominator into a 4D free piece and a continuum piece.

We can see that the structure of Eq. (4.12) amounts to the one of a 4D free propagator dressed by insertions due to mixing with a continuum (see Fig. 2). This is the same structure as the $\langle \varphi(x)\varphi(0) \rangle$ propagator of the continuum EFT in Eq. (2.1) dressed by $\langle O(\varphi) \rangle$ insertions, upon identifying $O[\varphi] \propto \varphi$ and $\langle O(p)O(-p) \rangle \propto \Pi(p)$. 13

This shows explicitly that the holographic setup leads to a continuum model of the kind described by the generic continuum EFT given in Eq. (2.1). The crucial gain with respect to the generic continuum Lagrangian is that, here, the setup dictates exactly how the law of gravity is modified. Before focusing on gravity we discuss qualitatively some other QFT aspects which are useful for the overall understanding of the model.

### 4.3.1 Spectrum and continuum final state

The continuum piece $\Pi(p^2)$ may, or may not, be supported at the pole location given by $p^2 - m_0^2 + b_0\Pi(p^2) \equiv 0$. In analogy with familiar weakly coupled QFT we can distinguish

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12 A field with Dirichlet boundary condition would contribute to the brane correlators only via internal lines. This is not the focus of the present study.

13 The notion of mixing can be understood more explicitly as follows. In the set of all degrees of freedom of $\Phi$, we can single out those which do not fluctuate on the brane, i.e. have Dirichlet BC. Writing $\Phi = \Phi_0 K_\varphi(z) + \int f^A\Phi_D^A f^A(z)$, with $K_\varphi$ the amputated brane-to-bulk propagator and $f^A_D$ the continuous basis of Dirichlet modes, the set $(K_\varphi, f^A_D)$ forms a complete basis which is orthogonal — in the sense that the quadratic action is diagonal in $(\Phi_0, \Phi_D^A)$ [59]. In this basis $\Phi_0$ has a nontrivial propagator, Eq. (4.12), i.e. a nontrivial spectral distribution. However one could instead, as introduced in Ref. [60], trade the $K_\varphi(z)$ component for $K_{\text{perm}}(z)$, in which case the associated degree of freedom $\Phi_0 \equiv \varphi$ simply is a 4D free field. In that case the propagator of $\varphi$ is trivial, but in counterpart the $(\varphi, \Phi_D)$ basis is not orthogonal (see [60]), and therefore there is a mixing between $\varphi$ and $\Phi_D$. The form of Eq. (4.12) is understood as a manifestation of this mixing.

---
two cases. If the pole lies in a region where $\Pi(p^2)$ is zero, the 4D mode described by the propagator Eq. (4.12) is stable. It thus contributes as a Dirac delta function to the spectral distribution and is identified as a particle in the Hilbert space of the 4D theory. In contrast if the pole lies in a region where $\Pi(p^2)$ is nonzero, the 4D mode acquires a width given by $m_0\Gamma = \text{Im} \Pi(m_0^2)$ and thus amounts to a resonance as first noted in Ref. [61]. This striking feature means that the 4D mode has a nonzero probability to convert into the continuum.

We notice here a key difference between continuum and discretum. If $\Pi$ was a discretum, e.g. $\Pi(p^2) \sim \sum_i a_i \delta(p^2 - m_i^2)$, the isolated 4D mode would remain exactly stable. Such a propagator would simply describe a mixing between the 4D mode and the discretum. This, in a sense, is because a free particle cannot just convert into another one with different mass. In contrast, the mass of the continuum is a continuous variable, thus it can be arbitrarily close to $m_0$. As a result there is a well-defined probability for the 4D mode to convert into the continuum. 14

The spectral function contains the necessary information to describe a continuum final state. In practice, in a given diagram one can simply take a unitarity cut on the generic brane-to-brane propagator Eq. (4.12). In particular, in the case of a stable particle the result takes the form

$$\text{Disc}[G_{p}] = Z_0 \left( 2\pi \delta(p^2 - m_0^2) - i \frac{b_0 \text{Disc}[\Pi(p^2)]}{|p^2 - m_0^2 + b_0 \Pi(p^2)|^2} \right), \quad (4.13)$$

where Disc computes the discontinuity across $p^2 > 0$, as defined in Sec. 4.4. In Eq. (4.13), Disc[$G_{p}$] is real and Disc[$\Pi(p^2)$] is imaginary. We can see from Eq. (4.13) that the final state can either be the stable 4D mode, or transition via a 4D propagator into the continuum. In the notation of the generic Lagrangian Eq. (2.1), this amounts to a “$\varphi^* \rightarrow$ continuum” process, see Fig. 2.

4.3.2 Finite temperature

The sector of brane-localized 4D modes can form a thermal bath. In such a case we can simply say that there is finite temperature on the brane. The conversion processes highlighted in the above section appear in the collision term of the Boltzmann equation of the 4D modes. They describe a sustained flux of radiation into the continuum of bulk modes, dumping energy into the bulk. In a sense these processes are responsible for “heating up” the bulk since, when falling deep enough in the bulk, they create a horizon which, in Eq. (4.2), is encoded in the blackening factor $f(z)$ (see e.g. Ref. [43]). Such processes, and the overall coupled dynamics, have been studied in a number of references, at various degrees of refinement, using both the 5D and dual 4D viewpoints, see e.g. Refs. [12, 43–45, 64, 65]. Similar calculations could similarly be done in the linear dilaton background.

14At a deeper level, a continuum does not have the properties required to build the familiar asymptotic multiparticle states of flat space, and may thus obey other rules. In the AdS case, for example, the continuum amounts to the normalizable bulk modes of AdS, that we know are perfectly stable (see e.g. Ref. [62]). Diagrams with AdS modes, such as $1 \rightarrow 2$, for example, only induce a mixing of the bulk modes, and thus amount in familiar terms to a radiation process rather than a decay process that would remove the initial mode from the spectrum (see e.g. [63]).
Figure 2. Overview of the holographic continuum framework. The brane and the horizon are respectively at $r = r_0$ and $r = r_h$. The blackening factor profile is pictured by the dotted line. The brane effective action features isolated 4D modes, identified in the analytical part of the self-energy. These are represented by the simple lines on the brane. The non-analytical part of the self-energy corresponds to the bulk modes, here represented by the double lines. The mixing between the modes is represented by the blue vertices. In particular the 4D graviton mode is dressed by a continuum component, which implies a deviation from the Newtonian potential. The production of bulk modes from the brane feeds the horizon in the bulk, which in turn induces a Weyl energy term in the brane Friedmann equation.

although this is not the focus of the present work. Here we take the horizon coordinate $r_h$ as a free parameter, constant in time. This assumption is compatible with the typical cosmological history, for which the above mentioned processes are efficient at very high energy but then quickly lose efficiency when the temperature drops, resulting in a constant $r_h$.

4.4 Gravitational potential

The graviton propagator should, following the above discussions, describe a massless 4D mode with bilinear mixing to a continuum. We denote the general propagator as

$$\langle h_0^{\alpha\beta}\left(p\right)h_0^{\rho\sigma}\left(-p\right)\rangle = G_p^2 \theta^{\alpha\beta\rho\sigma}. \quad (4.14)$$

where the superindex 2 in the propagator refers to the spin of the graviton. The polarization structure $\theta^{\alpha\beta\rho\sigma}$ is given below. What is the gravitational potential resulting from the propagator, Eq. (4.14)? To obtain it we write the spectral representation of the propagator
as [66],
\[ \langle h_0^\alpha \beta (p) h_0^{\rho \sigma} (-p) \rangle = \frac{1}{2 \pi i} \int_0^\infty ds \text{Disc}_s \left[ \frac{G^2_s}{s - p^2 - i \epsilon} \right] \theta^{\alpha \beta \rho \sigma} , \] (4.15)
where \( \text{Disc}_s [g(s)] \) is the discontinuity of \( g(s) \) across the branch cut along the real line, \( s \in \mathbb{R}^+ \):
\[ \text{Disc}_s [g(s)] = \lim_{\epsilon \to 0} (g(s + i \epsilon) - g(s - i \epsilon)) , \quad \epsilon > 0 . \] (4.16)
In this representation the tensor structures are those of the standard Fierz-Pauli propagators [67],
\[ \theta^{\alpha \beta \rho \sigma} = 0 = \frac{1}{2} \left( \eta^{\alpha \rho} \eta^{\beta \sigma} + \eta^{\alpha \sigma} \eta^{\beta \rho} - \eta^{\alpha \beta} \eta^{\rho \sigma} \right) , \] (4.17)
\[ \theta^{\alpha \beta \rho \sigma} > 0 = \frac{1}{2} \left( P^{\alpha \rho} P^{\beta \sigma} + P^{\alpha \sigma} P^{\beta \rho} \right) - \frac{1}{3} P^{\alpha \beta} P^{\rho \sigma} , \] (4.18)
with \( P^{\alpha \beta} = \eta^{\alpha \beta} - \frac{p^{\alpha} p^\beta}{s} \).

The potential can be directly obtained using the spectral representation Eq. (4.15) (see e.g. [68] and also [18]). One picks point sources at rest such that\( T_{1,2}^\alpha \delta_0^\beta \). Performing the \( d^3q \) integral yields a general representation of the long-range potential as
\[ V_N(R) = -\frac{m_1 m_2}{\pi M_3^3} \int_0^\infty ds \text{Disc}_s \left[ G^{\sqrt{s}} \right] \frac{e^{-\sqrt{s}R}}{R} \theta^{0000} , \] (4.19)
where \( \theta^{0000} = \frac{1}{2} \), \( \theta^{0000} > 0 = \frac{2}{3} \).

If \( G^2_p = \frac{1}{p^2 + i \epsilon} \) we have \( \text{Disc}_s \left[ \frac{1}{s + i \epsilon} \right] = 2 \pi \delta(s) \), which reproduces the standard Newtonian potential. The continuum term will induce a deviation to this potential. In the following we compute explicitly the continuum-induced deviation in specific 5D backgrounds.

5 The holographic gapped continuum

We focus on a specific version of the dilaton-gravity setup called the linear dilaton (LD). This background has the fascinating property that it naturally realizes the notion of a gapped continuum that was proposed phenomenologically in Ref. [2]. We assume the presence of a thermal bath on the brane, inducing a horizon in the bulk via QFT processes as described in Sec. 4.3 and Fig. 2.

5.1 AdS-Schwarzschild (review)

The well-known case of pure AdS background is recovered in the case where \( V(\phi) = 6 M_3^3 / \ell^2 \) with \( M_3^3 = M_{Pl}^3 / \ell \), and the dilaton has no VEV, i.e. \( \phi(z) = \text{cte} \). For \( f \neq 1 \) the background is AdS-Schwarzschild, i.e. hot AdS. In the cosmological context this amounts to the RS2 model [37] at finite temperature. In that case one has, in conformal coordinates
\[ f_{\text{AdS}}(z) = 1 - \frac{z^4}{z_h^4} , \quad \omega_{\text{AdS}}(z) = \frac{\ell}{z} , \] (5.1)
for any value of $z$ and, in brane cosmology coordinates

$$n_{\text{AdSS}}(r) = \frac{r}{\ell} \left(1 - \frac{r_h}{r}\right)^{1/2}, \quad b_{\text{AdSS}}(r) = \frac{\ell}{r} \left(1 - \frac{r_h}{r}\right)^{-1/2},$$  \hspace{1cm} (5.2)

where we have used the relation $\frac{\ell}{z} = \frac{r}{\ell}$. Finally, the temperature of the black hole is

$$T_{\text{AdSS}} = \frac{r_h}{\pi \ell^2}.$$  \hspace{1cm} (5.3)

### 5.1.1 Deviation from the Friedmann equation

We find

$$C_{\text{AdSS}}(r_0) = \frac{3 r_0^4}{\ell^2 r_0^3}.$$  \hspace{1cm} (5.4)

This indicates that the Weyl energy behaves as 4D radiation — in accordance with the discussion in Sec. 3.5. Notice that the Weyl energy is regular at the Schwarzschild horizon.

### 5.1.2 Deviation from the Newtonian potential

In AdS the reduced brane-to-brane graviton propagator takes the form (see e.g. Ref. [69])

$$G_{\text{AdSS},p}^2 = \frac{i}{p^4} + i \left(2\gamma - 1 + 2 \log \left(\sqrt{-p^2\ell/2}\right)\right) \frac{\ell^2}{4} + O(p^2\ell^2),$$  \hspace{1cm} (5.5)

where we are using that $|p|\ell \ll 1$. The discontinuity is found to be

$$\text{Disc}_s[G_{\text{AdSS},\sqrt{s}}^2] = 2\pi \delta(s) + \frac{\pi \ell^2}{2}.$$  \hspace{1cm} (5.6)

After substituting in Eq. (4.19) we obtain the gravitational potential

$$V_N(R) = -\frac{m_1 m_2}{M_{\text{Pl}}^2 R} \left(1 + \frac{2}{3} \frac{\ell^2}{R^2} + O \left(\frac{\ell^4}{R^4}\right)\right).$$  \hspace{1cm} (5.7)

The $\ell^2/R^3$ deviation is the manifestation of the continuum $\Pi(p)$ which mixes with the 4D graviton. This is the well known behavior found in [37], with the exact coefficient obtained in [68].

### 5.2 Linear dilaton

The linear dilaton model (LD) is defined by the potential [70]

$$V(\phi) = \frac{9 M_5^3}{2\ell^2} e^{2\phi},$$  \hspace{1cm} (5.8)

where $M_5^3 = 3\eta M_{\text{Pl}}^2$. Here $\eta$ is a scale characterizing a mass gap (up to some $O(1)$ multiplicative factor) for the fields living over the LD background.

This model has a solution at zero temperature which is given in conformal coordinates by

$$\omega_{\text{LD}}(z) = e^{-\eta(z-\ell)}, \quad \bar{\phi}_{\text{LD}}(z) = \eta(z - \ell) + \log(\eta\ell),$$  \hspace{1cm} (5.9)
with \( f_{LD}(z) = 1 \). The solution at finite temperature is given by the same expressions of Eq. (5.9), with the blackening factor

\[
f_{LD}(z) = 1 - e^{3\eta(z-z_h)} .
\] (5.10)

In the brane cosmology coordinates, the black hole solution can be written as

\[
n_{LD}(r) = \frac{r}{\ell} \left( 1 - \frac{r_h^3}{r^3} \right)^{1/2} , \quad b_{LD}(r) = \frac{1}{\eta \ell} \left( 1 - \frac{r_h^3}{r^3} \right)^{-1/2} , \quad \bar{\phi}_{LD}(r) = - \log \left( \frac{r}{\eta \ell^2} \right) .
\] (5.11)

The black hole temperature in the LD background is

\[
T_{LD} = \frac{3\eta}{4\pi} .
\] (5.13)

These results are consistent with the borderline solution between confining and non-confining geometries reported in [71].

5.2.1 Deviation from the Friedmann equation

We find

\[
\mathcal{C}_{LD}(r_0) = - \frac{9}{4} \eta^2 \ell^2 \frac{r_h^3}{r_0^5} ,
\] (5.14)

which is a non-standard Weyl energy. In terms of the scale factor for the brane observer \( a \propto r \), the Weyl energy scales as \( a^{-5} \) in the effective Friedmann equation. In terms of the parameter of the equation of state \( p = w \rho \), the Weyl energy has \( w = \frac{2}{3} \), which should be interpreted as a sort of exotic radiation.

5.2.2 Deviation from the Newtonian potential

In the LD background the reduced brane-to-brane graviton propagator is [70]

\[
G_{LD,p}^2 = - \frac{1}{2\sigma^2} \frac{i}{\sqrt{1 - \frac{p^2}{\sigma^2}} - 1} ,
\] (5.15)

where \( \sigma = 3\eta/2 \). This expression has both a pole at \( p^2 = 0 \) and a branch cut along \( p^2 \geq \sigma^2 \). The denominator can also be put in the form \( p^2 - 2\sigma^2 \left[ \sqrt{1 - \frac{p^2}{\sigma^2}} - 1 + \frac{p^2}{2\sigma^2} \right] \) which reproduces the form shown in Eq. (4.12). The first term is the 4D pole with \( m_0 = 0 \). The second term corresponds to the pure continuum part which is non-analytical above \( p^2 \geq \sigma^2 \) and \( O(p^4) \) near \( p \sim 0 \). We obtain the discontinuity

\[
\text{Disc}_s[G_{LD,p}^2] = 2\pi \delta(s) + \frac{\sqrt{\frac{\sigma^2}{s}} - 1}{s} \theta(s \geq \sigma^2) .
\] (5.16)

As expected the graviton spectral distribution features a massless pole and a gapped continuum.
Substituting into Eq. (4.19) we obtain the gravitational potential
\[ V_N(R) = -\frac{m_1 m_2}{M_{Pl}^2 R} \left(1 + \Delta(R)\right), \] (5.17)
with
\[ \Delta(R) = \frac{2}{3\pi} \int_{\sigma^2}^{\infty} ds \sqrt{\frac{s}{\pi}} - \frac{1}{s} e^{-\sqrt{s}R} \approx \begin{cases} \frac{4}{3\pi \sigma R} & \text{if } R \ll \frac{1}{\sigma} \\ O(e^{-\sigma R}) & \text{if } R \gg \frac{1}{\sigma} \end{cases}. \] (5.18)
We see that the deviation from the Newtonian potential appears essentially below the distance scale \(1/\sigma\) corresponding to the inverse mass gap. The deviation to the potential goes as \(\propto 1/R^2\), unlike the AdS case, where it goes as \(1/R^3\).

### 5.3 Linear dilaton with AdS asymptotics (LDA)

We consider a modification of the LD background by assuming an AdS asymptotic behavior in the UV. The model is defined by the superpotential \[ W(\bar{\phi}) = \frac{6 M_5^3}{\ell} \left(1 + e^{\bar{\phi}}\right), \] (5.19)
which leads to the following scalar potential
\[ V(\bar{\phi}) = \frac{1}{6 M_5^3} \left(-\frac{1}{4} \left(\frac{\partial W}{\partial \bar{\phi}}\right)^2 + W(\bar{\phi})^2\right) = \frac{6 M_5^3}{\ell^2} \left(1 + 2 e^{\bar{\phi}} + \frac{3}{4} e^{2\bar{\phi}}\right). \] (5.20)
The metric we are considering is, using proper coordinates,
\[ ds^2 = e^{-2A(y)} \left(-h(y) d\tau^2 + d\mathbf{x}^2\right) + \frac{dy^2}{h(y)}, \] (5.21)
\[ = -n(r)^2 d\tau^2 + \frac{r^2}{\ell^2} d\mathbf{x}^2 + b(r)^2 dr^2. \] (5.22)
The solution of the background equation of motion is
\[ A_{LDA}(y) = \frac{y}{\ell} \log \left(1 - \frac{y}{y_s}\right), \quad \bar{\phi}_{LDA}(y) = -\log \left(\frac{y_s - y}{\ell}\right), \quad h_{LDA}(y) = 1 - \int_{-\infty}^{y_s} dy e^{A(y)} \int_{-\infty}^{y} dy e^{4A(y)}, \] (5.23)
where \(y_s\) is the location of a naked singularity, which would correspond to \(z_s \to \infty\) in conformal coordinates.

In the brane cosmology coordinates the solution is given by \(15\)
\[ n_{LDA}(r) = \frac{r}{\ell} \sqrt{h_{LDA}(y(r))}, \quad b_{LDA}(r) = \frac{1}{\sqrt{h_{LDA}(y(r))}} \frac{\ell}{r} \frac{W(r/\eta^2)}{1 + W(r/\eta^2)}, \] (5.25)
\[ \bar{\phi}_{LDA}(r) = -\log \mathcal{W} \left(\frac{r}{\eta^2}\right), \]
\[ \frac{r}{\ell} = \left(1 - \frac{y}{y_s}\right) e^{-y/\ell} \quad \text{or equivalently} \quad y = y_s - \ell \cdot \mathcal{W} \left(\frac{r}{\eta^2}\right). \] (5.24)
where \(\mathcal{W}(z)\) is the Lambert function.

---
\(15\) The relation between the proper “\(y\)” and the brane cosmology “\(r\)” coordinates in the LDA model of Sec. 5.3 is
with
\[ \eta = \frac{1}{y_\text{s}} e^{-y_\text{s}/\ell}, \]  
(5.26)

and \( W(z) \) is the principal branch of the Lambert function. As in the LD model of Sec. 5.2, in the LDA model the graviton spectrum has a mass gap \( \sigma = 3\eta/2 \).

5.3.1 Deviation from the Friedmann equation

The Weyl contribution to the Friedmann equation is then computed by using Eq. (4.11). Explicit analytical results for the temperature and Weyl energy in the LDA model are provided in App. B, cf. Eq. (B.16). Using the piecewise approximation of the metric given in App. C we can compute a more transparent analytical approximation to the Weyl tensor. Details are given in App. C. We find that the Weyl energy behaves as

\[
C_{\text{LDA}}(r_0) \approx \begin{cases} 
-9\eta^2 \ell^2 r_0^2 / r_0^3 & \text{if } r_0 \ll \eta \ell^2 \\
-3\eta r_0^3 / r_0^3 & \text{if } r_0 \gg \eta \ell^2
\end{cases}.
\]  
(5.27)

The behavior of \( C(r_0) \) computed exactly with the LDA model of Eq. (5.19) is displayed in Fig. 3. It matches the analytical behavior obtained in (5.27) and illustrates the transition at \( r_0 \approx \eta \ell^2 \) between the two regimes, AdS-like for \( r_0 > \eta \ell^2 \), and LD-like for \( r_0 < \eta \ell^2 \).

5.3.2 Deviation from the Newtonian potential

The equations of motion on the LDA background do not have exact solutions. However an approximation is easily obtained by considering two regimes. As can be seen in App. C, the metric is approximately AdS for \( z \ll 1/\sigma \) and LD for \( z \gg 1/\sigma \). On the other hand, at the
Figure 4. Results within the LDA model of Sec. 5.3. We display the discontinuity of the graviton propagator (left panel), and the deviation from the Newtonian potential (right panel). The analytical result displayed in the right panel corresponds to the correction term inside the bracket in Eq. (5.29). In this figure we have considered the piecewise approximation of the LDA model given in App. C.

level of propagation we know that AdS propagators, expressed in \((p, z)\) space with given spacelike momentum \(q\) (with \(q^2 = -p^2\)), are exponentially suppressed beyond \(z \sim 1/q\). That is, the propagator only knows about the \(z \lesssim 1/\sigma\) region of the bulk. This fact implies that if \(\sqrt{s} \gg \sigma\) the spectral function should not know about the LD part of the background, and thus be approximately AdS. On the other hand, for \(\sqrt{s} \ll \sigma\) the propagator should know about the LD background. But since the LD background induces a mass gap at \(\sigma\), the dominance of the LD background implies that the continuum vanishes. This is consistent with the spectral function obtained in our approximation, in which the continuum part starts at \(\sqrt{s} = \sigma\).

In summary we can approximate the discontinuity of the graviton propagator as

\[
\text{Disc}_\sqrt{s}^2[G^2_{\text{LDA, }\sqrt{s}}] \approx 2\pi\delta(s) + \frac{\pi l^2}{2} \theta\left(s \geq \sigma^2\right) .
\]  

(5.28)

The Newtonian potential is easily computed by plugging Eq. (5.28) into Eq. (4.19), giving

\[
V_N(R) \approx -\frac{m_1 m_2}{M_{\text{Pl}}^2} R \left(1 + \frac{2l^2}{3R^2} e^{-\sigma R} (1 + \sigma R)\right) .
\]  

(5.29)

We can see that for \(R \ll 1/\sigma\) the expression reduces to the AdS one, Eq. (5.7). On the other hand for \(R > 1/\sigma\) the potential is exponentially suppressed — as a consequence of the mass gap induced by the LD background. We also evaluate numerically in Fig. 4 the results of \(\text{Disc}_s^2[G^2_{\text{LDA, }\sqrt{s}}]\) and \(V_N(R)\) by considering the piecewise approximation of the metric in App. C. Nontrivial oscillations occur near the threshold that cannot be captured analytically. Despite this detail the numerical evaluation of the potential accurately reproduces the analytical behavior.
5.4 Discussion

We have found that the deviations from the Newtonian potential and Friedmann equation appearing in the LD background completely differ from those occurring in the AdS background.

The deviation from the Newtonian potential induced in the LD background goes as $1/(\sigma R^2)$ and is gapped at $R \sim 1/\sigma$. In contrast, the deviation from gravity in the AdS background goes as $\ell^2/R^3$ and is ungapped. We can use the landscape of Yukawa-like fifth force searches to bound the deviation. We find that the relevant bound is the one from micron-scale fifth force experiments [72]. The order of magnitude bound is

$$\frac{1}{\sigma} \lesssim 60 \mu m$$

or $\sigma \gtrsim 2$ meV.

The Weyl energy term induced by the bulk horizon in the effective Friedmann equation on the brane has the equation of state parameter $w = \frac{2}{3}$. This exotic energy term is determined by a combination of parameters of the model. However we can simply write the Friedmann equation with

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 \left[ \Omega_{\text{rad}} \left(\frac{a_0}{a}\right)^4 + \Omega_{\text{mat}} \left(\frac{a_0}{a}\right)^3 + \Omega_{\text{Weyl}} \left(\frac{a_0}{a}\right)^5 \right]$$

where $H_0$, $a_0$ and $\Omega_i$ are the Hubble constant, scale factor and fractions of energy at present times, respectively, and then bound $\Omega_{\text{Weyl}}$. Requiring that the exotic term be negligible at the BBN time or earlier, at which the matter term is also negligible, we obtain

$$\frac{\Omega_{\text{Weyl}}}{\Omega_{\text{rad}}} \ll 10^{-9}.$$  \hspace{1cm} (5.31)

For the LDA background the deviations feature transitions between asymptotically LD and AdS regimes. For the deviation from the Newtonian potential, the AdS regime emerges in the UV i.e. for small $R$, while the LD regime shows up in the IR i.e. for large $R$. In the Friedmann equation, the Weyl energy behaves as in LD for high temperature while it behaves as in AdS for low temperature.

Finally, let us briefly comment further about the behaviour of the Weyl energy in the LDA model. In terms of cosmological evolution, there are two cases corresponding to whether the scale factor today $r_{0,\text{now}}$ is smaller or larger than $\sigma\ell^2$. If $r_{0,\text{now}} < \sigma\ell^2$, the Weyl energy has $r_0^{-5}$ scaling, i.e. $w = \frac{2}{3}$ until the present days. In contrast if $r_{0,\text{now}} > \sigma\ell^2$ there is a transition at some point in the history of the Universe, when $r_{0,\text{transition}} \sim \sigma\ell^2$. Before this time, the Weyl energy behaves with $r_0^{-5}$ scaling but after this time it has $r_0^{-4}$ scaling. In other words, the Weyl energy turns into a radiation term at late times. The bound at BBN times from Eq. (5.31) applies if $r_{0,\text{BBN}} < \sigma\ell^2$. Otherwise, the Weyl energy turns into dark radiation before BBN happens, and standard BBN bounds on 4D dark radiation apply, see e.g. Refs. [43, 45].

These cosmological braneworld scenarios would deserve further investigation.
6 Summary

Here we summarize the logical steps and results of our study. Our interest lies in theories giving rise to a free continuum in some parametric limit. A theory featuring a free continuum is referred to as a GFT. In our definition of GFT we allow for local interactions of the continuum, as this has no impact on the results. A free continuum sector emerges in the limit of theories with nontrivial dynamics, such as the $N \to \infty$ limit of gauge theories or the $g_{\text{bulk}} \to 0$ limit of braneworld models. Additionally, a GFT may be seen as an approximation of a discretum. Standard Poincaré-invariant (i.e. no brane) weakly coupled QFTs do not give rise to a continuum in the free limit, thus these are excluded from our study.

There is a priori no obvious principle to prevent us from writing an EFT featuring a free continuum. However we argue that such an EFT is incompatible with standard gravity. One line of argument is to show that the continuum sector has no stress tensor, or that the central charge is infinite. An axiomatic version of this fact is known for CFT and reviewed here. Using the continuous mass representation we obtain a similar conclusion for any non-conformal free continuum. Another line of reasoning relies on the species scale of gravity. The species scale is usually given for stable particles. Here, as a side result, we present a finite-temperature-based argument that generalizes the species scale in terms of the central charge of any CFT. Using the species scale we argue that the free continuum sector amounts to an infinite number of species, and thus that the cutoff of the EFT is zero.

These arguments imply that a free continuum in the presence of standard gravity cannot exist. We then consider the neighborhood of this point in theory space, that evade the no-go arguments either because the number of degrees of freedom is finite or gravity is nonstandard. This is the case of the classes of theories already listed above: a discretum, gauge theories with finite $N$, holographic theories. We point out that a common feature of all these models is that they must feature significant deviations in the gravity sector — these are the effects blowing up when approaching the GFT+4D Einstein gravity point.

We focus on holographic theories giving rise to a continuum. We consider a class of 5D gravity-dilaton models giving rise to a gapped holographic continuum. We lay out — together with a review of QFT aspects needed for an overall understanding of the holographic model — the necessary formalism to compute the effective Friedmann equation and the Newtonian potential. When brane-localized fields are at finite temperature, a horizon forms in the bulk. We solve the pure linear dilaton background (LD) at finite temperature analytically and the asymptotically LD at finite temperature using both analytical approximations and exact numerical solving.

In the pure linear dilaton background, we find that the Newtonian potential features a $\sim 1/(\sigma R^2)$ deviation and has a mass gap at $R \sim 1/\sigma$. This is in sharp contrast with the deviation in the AdS background. We find that the Friedmann equation features an anomalous “Weyl” energy with $a^{-5}$ scaling. This is summarized as

\[
\text{Linear dilaton horizon } \iff w = \frac{2}{3} \text{ (exotic energy)}
\]
in terms of the equation of state parameter. This is, again, in contrast with the AdS case for which the Weyl energy scales as dark radiation \( w = \frac{1}{3} \). We also study a somewhat more evolved linear dilaton background with AdS asymptotics near the boundary. The Newtonian potential is found to be essentially like the AdS one, but with a gap at \( R = 1/\sigma \) like in the LD case. The Weyl energy features a transition from the LD regime \( (w = \frac{2}{3}) \) at high temperature to the AdS regime \( (w = \frac{1}{3}) \) at low temperature.

A general lesson from our study of the holographic models is that the cosmology of continuum models is highly nontrivial. This, in a sense, is because it necessarily involves the underlying dynamics giving rise to the continuum. Here we have studied a particular case of the dilaton-gravity background. A host of solutions remains to be explored. A more detailed study of the cosmological history of these braneworld models certainly deserves further study. These exciting directions are left for future work.

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A On the transition between discretum and continuum

In this appendix we expand on the possibility of a discretum EFT, studying its validity range using general arguments.

For concreteness we assume that the discretum arises as the low-energy limit of a confining large \( N \) Yang-Mills theory with large ‘t Hooft coupling \( \lambda \gg 1 \). We choose the spectral distribution of the free propagator to be

\[
\rho_d^{\text{free}}(s) = \sum_{n=0}^{\infty} \rho_n \delta(p^2 - s_n),
\]

(A.1)
where the $s_n$ are intervals with some spacing set by some typical scale $\xi^2$. The sum starts at $n_0$, with $s_{n_0} = \sigma^2$. We refer to $\xi$ as the mode spacing and $\sigma$ as the mass gap of the spectrum. The gap $\sigma$ is either 0 or $\geq \xi$. In the following we assume $\sigma \gg \xi$. The conclusions are trivially extended to the cases $\sigma = 0, \sigma = O(\xi)$ by substituting $\sigma$ by $\xi$ in the arguments.

The free propagator takes the form

$$\langle O(p)O(-p)\rangle_{\text{free}} = i \sum_{n_0}^{\infty} \frac{\rho_n}{p^2 - s_n}. \quad (A.2)$$

It encodes a series of free 4D particles. Similarly to Sec. 2.2.1 the model is equivalently written with a set of canonically normalized 4D fields $\{\phi_n\}$ with $\phi_{s_n} \equiv \phi_n$ and the operator $O = \sum_{n=n_0}^{\infty} \sqrt{\rho_n} \phi_n$.

A similar picture is also obtained from holographic models with a discrete spectrum [73]. In the context of phenomenological continuum models, some aspects of the discretum EFT were discussed in Ref. [4].

Assuming that the discretum arises from confinement of gauge theory, the $\phi_n$ fields can be understood as glueball fields. The couplings among the $\phi_n$ fields are then controlled by powers of $1/N$ [21]. The modes encoded into the full 2pt function are thus narrow — in accordance with our definition of discretum. The discretum EFT has a validity cutoff scale $\Lambda$, above which the gauge theory description takes over, and above which $O(x)O(0)$ is a continuum. This means that in the spectral distribution of the 2pt function there should be a transition, between the discrete and the continuous regime, as a function of the mass variable $s$. What can we learn from general considerations about the transition scale $\Lambda$?

We can reason in terms of degrees of freedom. On very general grounds, the number of degrees of freedom should decrease when the RG flow goes toward the IR. The UV theory (i.e. the deconfined gauge theory) has $\sim N^2$ degrees of freedom. Hence the low-energy effective theory can have at most $\sim N^2$ degrees of freedom. Hence the heaviest field of the EFT has at most a squared mass of $s_{n_0 + N^2}$. Moreover, since the $\phi_n$ fields are by assumption regularly spaced, the cutoff has to be of order of the heaviest field of the discretum EFT in order to truncate the heavier ones. We conclude that the transition scale is constrained to be

$$\Lambda \lesssim \sqrt{s_{n_0 + N^2}}. \quad (A.3)$$

We can also reason in terms of interactions. Using large-$N$ counting rules for glueballs, the 3pt interaction of the $\phi_n$ fields has $1/N$ strength and we can then evaluate the width of an individual field $\phi_n$. A very rough estimate is $\Gamma_n \sim \sqrt{s_n(n-n_0)/N^2}$, where $n-n_0$ counts the lighter states available for decay. Therefore the $\phi_n$ fields become broad (i.e. $\Gamma_n \sim \sqrt{s_n}$) at $n \sim n_0 + N^2$, which signals a breakdown of the EFT. We conclude that the cutoff cannot be higher than $\sqrt{s_{n_0 + N^2}}$. This estimate matches the one from the number of degrees of freedom.

A more refined estimate can also be obtained using input from holographic models, and in particular, and for simplicity, using a pure AdS two brane model (see Ref. [73]). In that case the spacing is $s_n = n^2\xi^2$, we have $\sigma = 0$, and we know that the width estimate is rather $\Gamma_n \sim \sqrt{s_n}/N^2$ because the selection rules set by the residual symmetries constrain
the decay channels. We also know that the transition scale is reached when the modes tend to overlap with each other, in which case not only the diagonal width $\Gamma_n$, but the full self-energy matrix that mixes all the $\varphi_n$ would become relevant [73, 74]. At that scale the $\varphi_n$’s merge, giving rise to a continuum. The estimate of the transition scale in this case is $\tilde{\Lambda} \sim \sqrt{sN^2} = N^2\xi$. This matches the previous one when using $\sigma = 0$.

**B Solutions of the equations of motion**

We present in this appendix the most general solutions of Eqs. (4.5)-(4.8), both in conformal coordinates and in brane cosmology coordinates. The integration constants are discussed.

**B.1 AdS-Schwarzschild**

In the conformal frame the solution of the equations of motion is given by

$$f_{\text{AdSS}}(z) = C_A^2 \left( 1 - \frac{(z - z_0)^4}{(z_h - z_0)^4} \right), \quad \omega_{\text{AdSS}}(z) = C_A \frac{\ell}{z - z_0}, \quad \bar{\phi}_{\text{AdSS}}(z) = C_\phi,$$

where $C_A$, $C_\phi$, $z_0$, and the horizon position $z_h$ are integration constants. In the brane cosmology frame one finds

$$n_{\text{AdSS}}(r) = C_A \frac{r}{\ell} \sqrt{1 - \frac{r_h^4}{r^4}}, \quad b_{\text{AdSS}}(r) = \frac{1}{\ell} \frac{1}{\sqrt{1 - \frac{r_h^4}{r^4}}}, \quad \bar{\phi}_{\text{AdSS}}(r) = C_\phi,$$

and the relation between both coordinates is given by $\omega_{\text{AdSS}}(z) = \frac{z}{r}$. We can already notice that $z_0$ is an irrelevant shift symmetry in the coordinate $z$ which does not have counterpart in the brane cosmology frame. We can thus set $z_0 = 0$ without loss of generality. The $C_\phi$ constant is also physically irrelevant because $V(\Phi) = \text{cte.}$ The temperature and Weyl energy turn out to be

$$T_h = C_A \frac{r_h}{\pi \ell^2}, \quad C_{\text{AdSS}}(r_0) = -\frac{3}{\ell^2} \frac{r_h^4}{r_0^4}.$$

The $C_A$ constant affects the horizon temperature but not the Weyl energy. It can be eliminated by a constant rescaling of the coordinates, so that we can set $C_A = 1$, which gives the usual AdS-Schwarzschild metric of section 5.1. The Weyl energy depends on $r_h$, the horizon position. This is the only physically meaningful integration constant.

**B.2 Linear dilaton**

In the LD model, the solution of the equations of motion in conformal frame writes

$$f_{\text{LD}}(z) = C_A^2 \left( 1 - e^{3C_S(z - z_0)} \right), \quad \omega_{\text{LD}}(z) = e^{-C_S(z - z_0)}, \quad \bar{\phi}_{\text{LD}}(z) = C_S(z - z_0) + \log(C_A C_S \ell),$$

where $C_A$, $C_S$, $z_0$, and $z_h$ are integration constants. As in the AdSS case, there is an irrelevant shift symmetry in $z$, so that we can set $z_0 = 0$ without loss of generality. The $C_S$ constant has physical meaning. It is identified with the $\eta$ scale (see main text) which
is proportional to the mass gap in the spectrum (see also e.g. [70, 75, 76] for discussions).

In the brane cosmology frame one finds

\[ n_{\text{LD}}(r) = C_A \frac{r}{\ell} \sqrt{1 - \frac{r_h^3}{r^3}}, \quad b_{\text{LD}}(r) = C_B \frac{1}{\sqrt{1 - \frac{r_h^3}{r^3}}}, \quad \bar{\phi}_{\text{LD}}(r) = -\log \left( C_B \frac{r}{\ell} \right). \] (B.5)

In these coordinates the solution involves three integration constants: \( C_A, C_B \) and \( r_h \). The relation between both coordinates is

\[ e^{-C_S(z - z_0)} = \frac{r}{\ell}, \quad C_B = \frac{1}{C_A C_S \ell}. \] (B.6)

The \( C_A \) constant affects the horizon temperature but not the Weyl energy. Unlike the AdSS case, \( C(r_0) \) depends on both the horizon position and another integration constant, \( C_B \). This constant corresponds to the boundary value of \( b(r) \), i.e. \( C_B = \lim_{r \to \infty} b(r) \). Moreover, \( C_B \) contributes to the scalar VEV via \( -\log C_B \). In the present work we have taken the hypothesis that the scalar VEV is set by a mechanism independent of the brane at \( r = r_0 \), e.g. at the boundary of spacetime. As a result \( C_B \) is independent of \( r_0 \). It follows that the Weyl energy scales as \( \frac{1}{r_0^5} \). Since \( C_B \) is inversely proportional to \( C_A \), then \( C_A \) is also independent of \( r_0 \). In particular, \( C_A = 1 \) leads to the zero temperature solution near the boundary, i.e. far from the black hole horizon. In summary, we can set

\[ C_A = 1, \quad C_S = \eta, \quad C_B = \frac{1}{C_A C_S \ell} = \frac{1}{\eta \ell}, \] (B.7)

which leads to the solution of section 5.2.

### B.3 Linear dilaton with AdS asymptotics

In the LDA model, the solution in proper coordinates “\( y \)” is

\[ A_{\text{LDA}}(y) = C_a \frac{y - y_0}{\ell} - \log \left( 1 - \frac{y - y_0}{y_h} \right), \] (B.8)

\[ h_{\text{LDA}}(y) = \frac{1}{C_a^2} \left( 1 - \frac{\int_{-\infty}^{y_0} dy e^{A_{\text{LDA}}(y)}}{\int_{-\infty}^{y_h} dy e^{A_{\text{LDA}}(y)}} \right), \] (B.9)

\[ \bar{\phi}_{\text{LDA}}(y) = -\log \left( \frac{y_h - (y - y_0)}{\ell} \right) - \log C_a, \] (B.10)

with \( C_a, y_0, y_s \) and \( y_h \) as integration constants, and their relation with the scale \( \eta \) is

\[ \eta = \frac{1}{y_s} e^{-C_a y_s/\ell}. \] (B.11)

In the brane cosmology frame we find

\[ n_{\text{LDA}}(r) = C_A \frac{r}{\ell} \sqrt{h_{\text{LDA}}(y(r))}, \] (B.12)

\[ b_{\text{LDA}}(r) = \frac{1}{C_a \sqrt{h_{\text{LDA}}(y(r))}} \frac{1}{r} \left( \frac{\ell}{1 + W(C_B \frac{r}{\ell})} \right), \] (B.13)

\[ \bar{\phi}_{\text{LDA}}(r) = -\log W \left( C_B \frac{r}{\ell} \right). \] (B.14)
The relation between both frames is
\[
\frac{r}{\ell} = e^{-C_a(y-y_0)/\ell} \left( 1 - \frac{y-y_0}{y_s} \right).
\]  
(B.15)

The integration constants in the brane cosmology coordinates are \(C_A, C_a, C_B\) and \(r_h \equiv r(y_h)\), and their relation with \(\eta\) is \(C_B = C_a/(\eta \ell)\). Finally, the temperature and Weyl energy turn out to be
\[
T_h = \frac{C_A}{C_a^2 4\pi \chi_h^3 T_h}, \quad C_{\text{LDA}}(r_0) = -\frac{1}{C_a^4 4 r_0^4} \frac{1}{I_h} \left( 1 + \frac{1}{W(\chi_0)} \right),
\]  
(B.16)

where
\[
I_h \equiv \int_{\chi_h}^{\infty} \frac{dx}{x^5} \frac{W(x)}{1+W(x)} = \frac{1}{3\chi_h^4} \left[ W(\chi_h) - 2W(\chi_h)^2 + 8W(\chi_h)^3 + 32\chi_h^4 \text{Ei}(-4W(\chi_h)) \right],
\]  
(B.17)

and
\[
\chi_0 \equiv C_a \frac{r_0}{\eta \ell^2}, \quad \chi_h \equiv C_a \frac{r_h}{\eta \ell^2},
\]  
(B.18)

while \(\text{Ei}(z)\) is the exponential integral function. Using \(W(\chi_h) \overset{\chi_h \ll 1}{\sim} \chi_0\) and \(W(\chi_0) \overset{\chi_0 \gg 1}{\sim} \log \chi_0\), one finds the asymptotic behaviors of the Weyl energy \(C_{\text{LDA}}(r_0) \overset{r_0 \ll \eta \ell^2}{\sim} 1/r_0^5\) and \(C_{\text{LDA}}(r_0) \overset{r_0 \gg \eta \ell^2}{\sim} 1/r_0^4\).

Regarding the fixing of the integration constants, we can set \(y_0 = 0\) as \(y_0\) is an irrelevant shift symmetry in the \(y\) coordinate. We will fix the other integration constants as
\[
C_A = 1, \quad C_a = 1, \quad C_B = \frac{C_a}{\eta \ell} = \frac{1}{\eta \ell},
\]  
(B.19)

which allows to connect with the zero temperature solution near the boundary. This leads to the solution of section 5.3.

### C Linear dilaton background with AdS asymptotics: approximate analytical solution

Here we present analytical results obtained within an approximate version of the LDA of Sec. 5.3. We consider the following realization of the model \cite{77}
\[
V(\tilde{\phi}) = \frac{6M_3^3}{\ell^2} \theta(\tilde{v}_1 - \tilde{\phi}) + \frac{9M_3^3}{2\ell^2} e^{2(\tilde{\phi} - \tilde{v}_0)} \theta(\tilde{\phi} - \tilde{v}_1),
\]  
(C.1)

where \(\theta(x)\) is the step function, while \(\tilde{v}_0\) is the value of the dimensionless scalar field \(\tilde{\phi}\) at the UV brane and \(\tilde{v}_1\) is a value of the field which determines the transition region from AdS to LD.\(^{16}\)

\(^{16}\)We can understand the value of \(\tilde{v}_1\) as coming from a hypothetical IR brane located at \(z = z_1\) with a potential fixing the value of the field \(\tilde{\phi}\) at \(\tilde{v}_1\).
The solution at zero temperature of this model can be written in conformal coordinates

\[
\omega_{\text{LDA}}(z) = \frac{\ell}{z} \theta(z_1 - z) + \frac{\ell}{z_1} e^{-\eta(z - z_1)} \theta(z - z_1), \quad \text{(C.2)}
\]

\[
\phi_{\text{LDA}}(z) = \tilde{\nu}_0 \theta(z_1 - z) + \left[\tilde{\nu}_0 + \eta(z - z_1)\right] \theta(z - z_1), \quad \text{(C.3)}
\]

with \( f_{\text{LDA}}(z) = 1 \). In this model \( z_1 = 1/\eta \). The solution at finite temperature is given by the same expressions of Eqs. (C.2)-(C.3), with the blackening factor

\[
f_{\text{LDA}}(z) = \begin{cases} 
\left(1 - \frac{r^4}{r_c^4}\right) \theta(z_1 - z) + \left(1 - \frac{r^4}{r_1^4}\right) \theta(r - r_1) & z_h \leq z_1, \\
\left(1 - \frac{r^4}{r_c^4}\right) \theta(z_1 - z) + \left[1 - e^{3\eta(z - z_h)}\right] \theta(z - z_1) & z_1 < z_h,
\end{cases} \quad \text{(C.4)}
\]

where \( z_c \equiv e^{\frac{3}{4}\eta(z_h - z_1)} z_1 \). This solution is valid for \( 0 \leq z \leq z_h \). In the brane cosmology coordinates, the black hole solution writes

\[
n_{\text{LDA}}(r) = \frac{r}{\ell} f_{\text{LDA}}^{1/2}(r), \quad \text{(C.5)}
\]

\[
b_{\text{LDA}}(r) = \frac{1}{\eta \ell} \theta(r_1 - r) + \frac{r}{\tilde{\nu}_0} \theta(r - r_1) f_{\text{LDA}}^{-1/2}(r), \quad \text{(C.6)}
\]

\[
\phi_{\text{LDA}}(r) = \left[\tilde{\nu}_0 - \log \left(\frac{r}{\eta \ell^2}\right)\right] \theta(r_1 - r) + \tilde{\nu}_0 \theta(r - r_1), \quad \text{(C.7)}
\]

where

\[
f_{\text{LDA}}(r) = \begin{cases} 
\left(1 - \frac{r^4}{r_c^4}\right) \theta(r_1 - r) + \left(1 - \frac{r^4}{r_1^4}\right) \theta(r - r_1) & r_h < r_1, \\
\left(1 - \frac{r^4}{r_1^4}\right) \theta(r - r_1) & r_1 \leq r_h,
\end{cases} \quad \text{(C.8)}
\]

with \( r_c \equiv (r_1 r_h^3)^{1/4} \) and \( r_1 = \eta \ell^2 \). The solution in these coordinates is valid for \( r_h \leq r \). The temperature and entropy of the black hole in the LDA are

\[
T_{\text{LDA}} = \frac{3\eta}{4\pi} \theta(r_1 - r_h) + \frac{r_h}{\pi \ell^2} \theta(r_h - r_1), \quad S_{\text{LDA}} = \frac{V_3}{4G_5} \left(\frac{r_h}{r}\right)^3, \quad \text{(C.9)}
\]

respectively. The graviton spectrum has a mass gap \( \sigma = 3\eta/2 \). Finally, we find

\[
C_{\text{LDA}}(r_0) = \begin{cases} 
-\frac{9}{4} \eta^2 \ell^2 \frac{r_0^4}{r_0^4} \theta(r_1 - r_0) - 3\eta \frac{r_0^4}{r_0^4} \theta(r_0 - r_1) & r_h < r_1, \\
-\frac{3}{r_0^4} \frac{r_0^4}{r_h^4} & r_1 \leq r_h,
\end{cases} \quad \text{(C.10)}
\]

valid for \( r_h \leq r_0 \).

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