Propositional Knowledge Representation in Restricted Boltzmann Machines

Son N. Tran
The Australian E-Health Research Center, CSIRO
son.tran@csiro.au

Abstract

Representing symbolic knowledge into a connectionist network is the key element for the integration of scalable learning and sound reasoning. Most of the previous studies focus on discriminative neural networks which unnecessarily require a separation of input/output variables. Recent development of generative neural networks such as restricted Boltzmann machines (RBMs) has shown a capability of learning semantic abstractions directly from data, posing a promise for general symbolic learning and reasoning. Previous work on Penalty logic show a link between propositional logic and symmetric connectionist networks, however it is not applicable to RBMs. This paper proposes a novel method to represent propositional formulas into RBMs/stack of RBMs where Gibbs sampling can be seen as maximising satisfiability. It also shows a promising use of RBMs to learn symbolic knowledge through maximum likelihood estimation.

1 Introduction

In AI research, there have been much debate over symbolism and connectionism as they are two key opposed paradigms for information processing [Smolensky, 1987; Minsky, 1991]. The former has been known as the foundation language of AI which captures higher level of intelligence with explainable and reasoning capability. The latter is getting more attention due to its indisputable advantages in scalable learning and dealing with noisy data. Despite their difference, there is a strong argument that combination of the two should offer joint benefits [Smolensky, 1995; Valiant, 2006; Garcez et al., 2008]. The last two decades have witnessed consistent efforts in developing neural-symbolic integration systems [Towell and Shavlik, 1994; d’Avila Garcez and Zaverucha, 1999; Penning et al., 2011; França et al., 2014; Tran and Garcez, 2016]. Such systems are well known not only for better reasoning but also for more efficient learning. The key success here is lying on a mechanism to represent symbolic knowledge in a connectionist network. This is also useful for knowledge extraction [Towell and Shavlik, 1993; d’Avila Garcez et al., 2001], i.e. to seek for symbolic representation of the networks.

Centred in the earliest neural-symbolic systems is the artificial neural networks which can be seen as a black box of input-output mapping function. However, such systems are limited in knowledge representation and reasoning due to their discriminative architecture. Different from discriminative neural networks which separate the variables in a domain into input and target variables, the generative counterparts treat all variables in a domain equally, hence are more useful for symbolic reasoning. Also, recent emergence of representation learning shows that unsupervised connectionist networks with latent variables such as restricted Boltzmann machines (RBMs) can learn semantic patterns from large amount of data efficiently [Smolensky, 1986; Hinton, 2002]. Notably, by stacking those generative networks one on top of the others, i.e. to construct a deep networks, we can not only extract different level of abstractions from domain’s data but also achieve better performance [Hinton et al., 2006; Lee et al., 2009]. This poses a desire for a study of symbolic knowledge representation in RBMs.

The most related literature to this work is Penalty logic [Pinkas, 1995]. Penalty logic is an extension of propositional logic which is equivalent to symmetric connectionist networks (SCNs), including Hoffeld networks and Boltzmann machines. However, we claim that it is difficult to apply Penalty logic to restricted Boltzmann machines, a simplified version of Boltzmann machines where there is no connections between units in the same visible/hidden layers. Indeed, in order to convert the energy function from high-order to quadratic form more hidden variables may need to be created, this results in adding more connections within the hidden layer. Moreover, the equivalence between a Penalty logic and SCNs is asymmetric, i.e the formulas represent a SCN are in different form from the formulas represented by it. Also, an unanswered question is how Penalty logic can benefit from stochastic learning in SCNs. Several attempts have been made recently to integrate symbolic representation and RBMs [Penning et al., 2011; Tran and Garcez, 2016]. Despite achieving good results they are still heuristic and lack a supporting theory.

In this paper we answer two questions: “How to represent propositional knowledge in RBMs?”; and “Is it possible to learn propositional knowledge from RBMs?”. First, we
show that any propositional formula can be represented in an RBM where symbolic reasoning is equivalent to minimising the energy function. The idea is to convert a formula into disjunctive normal form (DNF) with only one conjunctive clause holds given a preferred model. We then extend this to show how to represent a propositional knowledge program (a set of weighted propositional formulas) in an RBM and a stack of RBMs, as known as deep belief networks (DBNs). Here, inference with Gibbs sampling can be seen as maximising satisfaction. Finally, we show that it is possible to learn an RBM to approximate an unknown formula by applying maximum likelihood estimation over a set of preferred models (training samples). Although this is not always guaranteed it reveals a promising use of RBMs to approximate a symbolic program from training data.

The most exciting perspective of this work may be that by creating a bridge between the grand old propositional logic and the emerging representation learning networks, RBMs, it offers a vital basis for further exploration on unsupervised neural-symbolic integration and reasoning, and perhaps also on explanation of the effectiveness of deep architectures.

2 Background

2.1 Restricted Boltzmann Machines

Connectionist systems normally refer to a set of models made by interconnected networks of computational neurons (also called ‘units’) [Smolensky, 1987; Hinton, 1989]. An symmetric connectionist network is a neural network with bidirectional connections which is characterised by a quadratic function called energy. The weights of the connections are stored in a symmetric matrix. A SCN behaves as a memory where information are stored in lower energy states. Inferences in such system may be viewed as searching for the minimum energy, which is either deterministic as in Hofffield networks (HN) [Hopfield, 1982], or stochastic as in Boltzmann machines (BM) [Hinton, 1989].

A simplified version of stochastic SCN, i.e. Boltzmann machines, is called restricted Boltzmann machine (RBM) where units in the same layers are disconnected [Smolensky, 1986; Hinton, 2002]. For the convenience of presentation the symmetric weight matrix is reduced to the weights between visible and hidden units, denoted as $W$. The energy function of RBMs is:

$$E_{RBM}(x, h) = - \sum_{i,j} w_{ij} x_i h_j - \sum_i a_i x_i - \sum_j b_j h_j$$ (1)

Compare to BMs, learning in RBMs is easier due to the efficiency mechanism. More importantly, it has shown that by stacking several RBMs, one on top of another we can not only extract different level of abstractions from domain’s data but also achieve better performance [Hinton et al., 2006; Lee et al., 2009; Mohamed et al., 2012]. Recently, several attempts have been made to extract and encode symbolic knowledge into RBMs [Penning et al., 2011; Tran and Garcez, 2016]. However, it is not theoretically clear how such knowledge, as known as Confidence rules, represent the RBMs formally.

2.2 Penalty Logic

Penalty logic is among the earliest works to study symbolic representation of neural networks [Pinkas, 1991a; Pinkas, 1991b; Pinkas, 1995]. Different from the others which focus on feedforward neural networks, Penalty logic explains the relation between propositional formulas and SCNs.

The Penalty logic formulas, as known as penalty logic well-formed formulas (PLOFF), is defined as a finite set of pairs $(\rho, \varphi)$, in each propositional well-formed formula (WFF) $\varphi$ is associated with a real value $\rho$ called penalty. PLOFF define a violation rank function $V_{rank}$ which is the sum of the penalties of violated formulas. The preferred model is a truth assignment $x$ that has minimum total penalty. Applied to classification, for example, to decide the truth-value of a target proposition $y$ given an assignment $x$ of the other propositions, one will choose the value of $y$ that minimises $V_{rank}(x, y)$.

Reasoning with Penalty logic is shown to be equivalent to minimising energy function in SCNs. This is the key foundation to form the link between propositional knowledge and connectionist networks. The equivalence is defined as:

$$V_{rank}(x) = E_{rank}(x) + \text{constant}$$ (2)

where $E_{rank}(x) = \min_h E(x, h)$ is the energy function minimised over all hidden variables.

Let us use BMs as an example. In [Pinkas, 1995] Gadi demonstrates the capability of Penalty logic to represent propositional knowledge in a BMs. An energy function is constructed by converting every WFF into a conjunction of triple form. More hidden variables may be added to convert the energy function to a quadratic form and also to guarantee the equivalence. Knowledge extraction, i.e representing a BM in PLOFF, can be done by eliminating all hidden variables to turn the energy function back to high-order form and translating every product term into a conjunction. For learning propositional formulas, a BM is constructed incrementally from truth assignments which is equivalent to a k-CNF formula.

However, applying Penalty logic to RBMs is difficult even though, as a restricted version of BMs, RBMs are also SCNs. First, in order to represent propositional formulas to RBMs extra hidden variables are needed. This removes the connections within visible layer but, as a price, would create connections within hidden layer. Second, knowledge extraction with Penalty Logic is computationally expensive. Even eliminating one single hidden unit in an energy function is already exponential over the number of visible units connecting to it. This will make the extraction of knowledge from a complex domains intractable. Third, because there does not always exist an RBM to represent a propositional knowledge the learning method in [Pinkas, 1995] does not always hold.

3 Propositional Calculus and RBMs

Since symmetric connectionist networks is the generalisation of RBMs we can apply Penalty logic to represent propositional knowledge in that restricted variant. However, it is unnecessarily complicated where, according to the proposed algorithm in [Pinkas, 1995], it needs to construct a higher-order BM and then transform its energy function to quadratic
form by adding more hidden variables. The latter step must be repeated until an energy term has at most one visible variable. This paper introduces a much simpler method to represent a well-formed formula in an RBM. We show that this could be accomplished by converting WFFs into disjunctive normal form, as detailed below, instead of conjunctions of sub-formulas as in Penalty Logic [Pinkas, 1999b; Pinkas, 1995].

In propositional logic, any WFF $\varphi$ can be represented in disjunctive normal form (DNF) [Russell and Norvig, 2003]:

$$\varphi = \bigvee_j (\bigwedge_{t \in T_j} x_t \land \bigwedge_{k \in K_j} \neg x_k)$$

where each $(\bigwedge_{t \in T_j} x_t \land \bigwedge_{k \in K_j} \neg x_k)$ is called a “conjunctive clause”. Here we denote the literals as $x_t$, $x_k$ and $T_j$ and $K_j$ are the set of positive literals and the set of negative literals respectively.

**Definition 1.**

- A “strict DNF” (SDNF) is a DNF where at most one conjunctive clause is true.
- A “full DNF” is a DNF where each variable must appear at least once in every conjunctive clause.

Any propositional well-formed formula can be presented in a SDNF. Indeed, suppose that $\varphi$ is a WFF in disjunctive normal form. If $\varphi$ is not SDNF then there exist some groups of conjunctive clauses which are true given a preferred assignment. We can always convert this group of conjunctive clauses to a full DNF which is also a SDNF.

**Definition 2.** A WFF $\varphi$ is said to equivalent to a SCN if and only if for any model $x$, $s_\varphi(x) = -AE_{\text{rank}}(x) + B$, where $s_\varphi(x)$ is the truth value of $\varphi$ given $x$; $A > 0$ and $B$ are fixed real numbers; $E_{\text{rank}}(x) = \min_k E(x, h)$ is the energy ranking function of $N$ minimised over all hidden units.

This definition of equivalence is similar to $\square$ in Penalty Logic. Here the equivalence guarantees that all preferred models of a WFF would also minimise the energy of the network. In addition, by construction, non-preferred models of the formula for false would result in maximum energy of the network.

**Lemma 1.** Any SDNF $\varphi$ can be mapped onto a SCN with energy function $E = -\sum_j \prod_{t \in T_j} x_t \prod_{k \in K_j} (1 - x_k)$ where $T_j$, $K_j$ are respectively the sets of positive and negative propositions of each conjunctive clause $j$ in the SDNF.

**Proof.** By definition, $\varphi = \bigvee_j (\bigwedge_{t \in T_j} x_t \land \bigwedge_{k \in K_j} \neg x_k)$. Each conjunctive clause $\bigwedge_{t \in T_j} x_t \land \bigwedge_{k \in K_j} \neg x_k$ corresponds to $\prod_{t \in T_j} x_t \prod_{k \in K_j} (1 - x_k)$ which maps to 1 if and only if $x_t = 1$ (true) and $x_k = 0$ (false) for all $t \in T_j$ and $k \in K_j$. Since $\varphi$ is a SDNF, that is true if and only if one conjunctive clause is true, then the sum $\sum_j \prod_{t \in T_j} x_t \prod_{k \in K_j} (1 - x_k) = 1$ if and only if the assignment of truth-values for $x_t$, $x_k$ is a preferred model of $\varphi$. Hence, there exists a SCN where energy function $E = -\sum_j \prod_{t \in T_j} x_t \prod_{k \in K_j} (1 - x_k)$ such that $s_{\varphi}(x) = -E_{\text{rank}}(x)$.

**Example 1.** The XOR formula $(x \oplus y) \leftrightarrow z$ can be converted into a SDNF as:

$$\varphi = (\neg x \land \neg y \land \neg z) \lor (\neg x \land y \land z) \lor (x \land \neg y \land z) \lor (x \land y \land \neg z)$$

For each conjunctive clause, for example $x \land y \land \neg z$ we create a term $xy(1 - z)$ and add it to the energy function. After all terms are added, we have the energy function for $N$:

$$E(x, y, z) = -(1 - x)(1 - y)z - xy(1 - z) - z(x - 1)y - (1 - x)yz$$

The correspondence between $\varphi$ and $N$ exists because $s_\varphi(x, y, z) = -E_N(x, y, z)$. If we expand the energy function above we can see it is equivalent to the energy function used by Penalty Logic minus one.

**Theorem 1.** Any SDNF $\varphi$ can be mapped onto an equivalent RBM with energy function $E = -\sum_j h_j (\sum_t x_t - \sum_k x_k - |T_j| + \epsilon)$ where $T_j$, $K_j$ are respectively the sets of positive and negative propositions of each conjunctive clause $j$ in the SDNF.

**Proof.** We have seen in Lemma 1 that any SDNF $\varphi$ can be mapped onto energy function $E = -\sum_j \prod_{t \in T_j} x_t \prod_{k \in K_j} (1 - x_k)$. Let us denote $|T_j|$ as the number of positive propositions in a conjunctive clause $j$. For each term $\tilde{e}_j(x) = -\prod_{t \in T_j} x_t \prod_{k \in K_j} (1 - x_k)$ we can construct an energy term with an hidden variable $h_j$ as:

$$\tilde{e}_j(x, h_j) = h_j (|T_j| - \sum_{t \in T_j} x_t + \sum_{k \in K_j} x_k - \epsilon)$$

with $0 < \epsilon < 1$ such that $\tilde{e}_j(x) = \frac{s_{\varphi}(x) - E_{\text{rank}}(x)}{\epsilon}$. This equation holds because $|T_j| - \sum_{t \in T_j} x_t + \sum_{k \in K_j} x_k - \epsilon = -\epsilon$ if and only if $x_t = 1$ and $x_k = 0$ for all $t \in T_j$ and $k \in K_j$, which makes $\min_{h_j} \tilde{e}_j(x, h_j) = -\epsilon$ with $h_j = 1$. Otherwise $|T_j| - \sum_{t \in T_j} x_t + \sum_{k \in K_j} x_k - \epsilon > 0$ and then $\min_{h_j} \tilde{e}_j(x, h_j) = 0$ with $h_j = 0$. By repeating the process on every term $\tilde{e}(x)$ we can conclude that any SDNF $\varphi$ is equivalent with an RBM with the energy function:

$$E = -\sum_j h_j (\sum_t x_t - \sum_k x_k - |T_j| + \epsilon)$$

where: $s_{\varphi}(x) = -\frac{1}{\epsilon}E_{\text{rank}}(x)$

In what follows we show how to construct an RBM from a formula.

**Construction 1.** An RBM can be constructed from a WFF by:

- Convert a WFF into SDNF.
- For all conjunctive clause $j$, $\bigwedge_{t \in T_j} x_t \land \bigwedge_{k \in K_j} \neg x_k$.
  - Create a hidden unit $h_j$.
  - Create a connection between all visible unit $i$ ($t \in T_j$) and the hidden unit $j$ with a weight $w_{ij} = 1$.
  - Create a connection between all visible unit $k$ ($k \in K_j$) and the hidden unit $j$ with a weight $w_{kj} = -1$.
  - Set the bias $b_j = -|T_j| + \epsilon$ with $0 < \epsilon < 1$ for the hidden unit $j$.

**Example 2.** Applying Theorem 1 we can construct an RBM for a XOR function in Example 1 as in Figure 1a. In this example we choose $\epsilon = 0.5$. The energy function of this RBM is:

$$E = xh_1 + yh_1 + z(h_1 - 0.5h_2 - xh_2 - yh_2 + zh_2 + 1.5h_4)$$

$$- xh_3 + yh_3 - zh_3 + 1.5h_3 + xh_4 - yh_4 - zh_4 + 1.5h_4$$
For comparison, we also construct an RBM for XOR using Penalty Logic. In this case, it is possible because we can represent the formula into a conjunction of triple form without adding more hidden variables. First, we compute a higher-order energy function:

\[ E^{p} = 4xyz - 2xy - 2xz - 2yz + x + y + z \]

then we transform it to a quadratic form by adding a hidden variable:

\[ E^{p} = -8xh_1 - 8y_h_1 + 12h_1 - 4xh_2 + 4y_h_2 + 2h_2 - 4y_h_3 - 4xh_3 + 6h_3 - 4xh_4 - 4xh_2 + 6h_4 + 3x + y + z \]

This is still not an energy function of an RBM, so we keep adding hidden variables until the energy function becomes:

\[ E^{p} = -8xh_1 - 8y_h_1 + 12h_1 - 4xh_2 + 4y_h_2 + 2h_2 - 4y_h_3 - 4xh_3 + 6h_3 - 4xh_4 - 4xh_2 + 6h_4 + 3x + y + z \]

which is the RBMs in Figure 1b.

4 Representing Propositional Knowledge in RBMs

The previous section explains how to map a propositional formula onto an unsupervised form of statistical calculus, the RBMs. We now generalise the symbolic presentation from conjunctive clauses to confidence rules [Penning et al., 2011; Tran and Garcez, 2012; Tran and d’Avila Garcez, 2013; Tran and Garcez, 2016] to show the equivalence between RBMs and a propositional knowledge, i.e. a set of weighted propositional formulas. We then show that by stacking the rules in layer-wise fashion we can representing propositional knowledge in DBNs.

4.1 Confidence Rules

We can present a set of formulas \( \Phi = \{\gamma_1, ..., \gamma_N\} \) in RBMs by applying Theorem 1 to the formula \( \varphi = \gamma_1 \land ... \land \gamma_N \). For a set of weighted formulas \( \Phi = \{\alpha_1 : \gamma_1, ..., \alpha_N : \gamma_N\} \) that cannot be straightforwardly extended, we need to construct a conjunctive clause to a Confidence rule, as defined in [Tran and Garcez, 2016]. Confidence rule is a if-and-only-if clause associated with a positive real values called “confidence value” as:

\[ c_j : h_j \leftrightarrow \bigwedge_{t \in T_j} x_t \land \bigwedge_{k \in K_j} \neg x_k \]

Confidence rules have been used as an intermediate language for symbolic knowledge extraction and encoding knowledge bases into RBMs/DBNs [Penning et al., 2011; Tran and Garcez, 2016]. However, there is no formal study to show the relations between Confidence rules and RBMs as the empirical results in previous work suggest. Based on the Theorem 1 we now can claim that one can represent a weighted knowledge base onto an RBM by converting it into a set of Confidence rules.

**Proposition 1.** A set of Confidence rules can be represented in an RBM.

**Proof.** Similar to the proof of Theorem 1 we can show that a rule \( h_j \leftrightarrow \bigwedge_{t \in T_j} x_t \land \bigwedge_{k \in K_j} \neg x_k \) is equivalent to the energy function \( c_j(x, h_j) = h_j(T_j) = \sum_{t \in T_j} x_t + \sum_{k \in K_j} x_k - \epsilon \).

The confidence rule \( c_j : h_j \leftrightarrow \bigwedge_{t \in T_j} x_t \land \bigwedge_{k \in K_j} \neg x_k \) therefore is equivalent to the energy \( c_j(x, h_j). \) As the result, each confidence rule we can add a hidden unit to an RBM and assign the weights \( w_{tj} = c_j \) and \( w_{kj} = -c_j \) with all \( t \in T_j \) and \( k \in K_j \). The bias for this hidden unit is \( c_j(T_j) - \epsilon \). The set of Confidence rules and the RBM is equivalent such that \( s_{\Phi} = -\frac{1}{\epsilon} E_{rank} \).

**Proposition 2.** For every weighted knowledge base \( \Phi = \{\alpha_1 : \gamma_1, ..., \alpha_N : \gamma_N\} \) there exists an equivalent RBM such that \( s_{\Phi} = -\frac{1}{\epsilon} E_{rank} \).

**Proof.** Every weighted formula \( \alpha_n : \gamma_n \) can be converted into a set of of Confidence rules and a disjunctive clause of hidden literals as:

\[ \gamma_n = (\bigvee_j h_j) \bigwedge_{j}(h_j \leftrightarrow \bigwedge_{t \in T_j} x_t \land \bigwedge_{k \in K_j} \neg x_k) \]

We combine the rules that have the same sets \( T_j \) and \( K_j \) to a single one and its confidence value is the sum of the rules’ confidence values. The final set of Confidence rules are used to construct an RBM as shown in Proposition 1.

**Example 3.** Nixon diamond.

1000 : n → r Nixon is a republican.
1000 : n → q Nixon is also a quaker.
10 : r → ¬p republicans tend not to be pacifist.
10 : q → p quakers tend to be pacifist.

The energy function of the RBM is:

\[ E = 1000h_1n + 1000h_1r - 500h_4 - 1000h_2r + 500h_2 + 1000h_3n + 1000h_3q - 500h_3 - 1000h_4 + 500h_4 - 10h_{sp} + 10h_{sn} + 5h_5 + 10h_{ap} - 5h_4 + 10h_{aq} + 10h_{tp} - 5h_7 - 10h_{ap} + 5h_8 \]

We now show a theoretical idea of using RBMs to support satisfiability inference.

**Proposition 3.** Given an RBM which is constructed from a weighted knowledge base, inference with Gibbs sampling is equivalent to minimising the total weighted satisfiability.

\( s_{\Phi} \) in this case is the sum of confidence values from satisfied rules.
Proof. The proof is straightforward, since the energy function of the RBM and the satisfiability of the knowledge base is inversely proportional then every step of Gibbs sampling to reduce the energy function will also increase the total satisfiability.

Here, we can take the advantage of RBMs in that inference is efficient. One can see this satisfiability solver as a reconstruction from impaired data, where instead of searching the hypothesis space which is NP-complete one can approximate the solutions with Gibbs sampling. This can be done by clamping the variables which have been assigned with truth values then iteratively inferring the hidden variables and the unassigned ones until the RBM reach stationary state.

4.2 Stacking of Confidence rules

In many cases some conjunctive clauses of a WFF share common literals which can be grouped and replaced by a hidden literal. For example, let us consider the WFF \( \varphi = \chi_1 \land \chi_2 \land (\chi_3 \oplus \chi_4) \) which can be converted to be:

\[
\varphi = (\chi_1 \land \chi_2 \land \neg \chi_3 \land \chi_4) \lor (\chi_1 \land \chi_2 \land \chi_3 \land \neg \chi_4)
\]

As being shown in Proposition 1 we can represent three Confidence rules \((h_1 \leftrightarrow (\chi_1 \land \chi_2)), (h_2 \leftrightarrow \neg (\chi_1 \land \chi_4))\) and \((h_3 \leftrightarrow (\chi_3 \land \neg \chi_4))\) in an RBM. The DNF of hidden literals can be seen a higher level formula and can be represented in another RBM on top of the previous one. Alternatively, we can repeat the process on the DNF to construct a multiple layer of RBMs, one on top of another. Now we extend this to show that it is possible to represent a knowledge base in a stack of RBMs (also known as a DBN).

Theorem 2. Any weighted knowledge base can be approximated represented in a stack of RBMs.

Proof. (Sketch) Convert all WFFs from the knowledge base into SDNFs. Define a set of sub-clauses of conjunctions with at most K literals such that every conjunctive clause in the SDNFs can be the conjunction of at least one sub-clause. Convert every SDNF into a set of Confidence rules and a higher level DNF of hidden literals as shown above. Assign each Confidence rules with small confidence values \(c_{+0}\) from which we create an RBM as shown in Proposition 1.

Repeat this process on the DNFs until the next higher level DNFs of hidden literals is a disjunctive clause. At this point, we can construct the top RBM as shown in Proposition 2.

The energy ranking function of the DBN will be a sum of the energy ranking functions from all RBMs. Since the lower RBMs have small weights (due to the small confidence values of the lower Confidence rules) and the top RBM is equivalent to the knowledge base then the total energy ranking function of the DBN is to the satisfiability of the knowledge base. In particular \(E_{\phi} = -\frac{1}{2}E_{\text{rank}} + MC_{+0} \approx -\frac{1}{2}E_{\text{rank}}\) where \(M\) is the number of satisfied Confidence rules at lower levels.

Example 4. Let us consider the following knowledge base:

\[
5 : x_1 \land x_2 \land (\chi_3 \oplus \chi_4), 10 : x_1 \land x_2 \land x_3 \land ((\neg x_2 \lor x_5) \rightarrow x_4)
\]

We convert them into the SDNFs with hidden units:

\[
5 : (h_1 \land h_2) \lor (h_1 \land h_3) \land (h_2 \leftrightarrow \neg \chi_3 \land \chi_4) \land (h_3 \leftrightarrow x_2 \land \neg \chi_4)
\]

\[
10 : (h_1 \land h_4) \lor (h_2 \land h_3) \land (h_1 \leftrightarrow x_1 \land \chi_2) \land (h_3 \leftrightarrow x_2 \land \neg \chi_4) \land (h_4 \leftrightarrow x_3 \land \neg \chi_5) \land (h_5 \leftrightarrow x_1 \land \neg \chi_5)
\]

from which first level of Confidence rules with small confidence values can be constructed as:

\[
c_{-0} : h_1^{(+)} \leftrightarrow (x_1 \land x_2), c_{-0} : h_2^{(+)} \leftrightarrow (\neg x_3 \land \chi_4) \land c_{-0} : h_3^{(+)} \leftrightarrow (x_3 \land \neg x_4)
\]

\[
and two DNFs 5 : (h_1 \land h_2) \lor (h_1 \land h_3), 10 : (h_1 \land h_4) \lor (h_3 \land h_5) \land (h_2 \leftrightarrow \neg h_1\land h_4) \land (h_1 \leftrightarrow h_2\lor h_3) \land (h_2 \leftrightarrow h_1\land h_4) \land (h_1 \leftrightarrow h_2) \land (h_3 \leftrightarrow h_2) \land (h_1 \leftrightarrow h_2) \land (h_3 \leftrightarrow h_2)
\]

This hierarchical rules can be represented in a DBNs by constructing RBMs for every level, one after another.

5 Knowledge Approximation

We have shown how to represent propositional knowledge in RBMs/stack of RBMs where finding the maximum satisfiability is equivalent to minimising the energy functions. For completeness, we are now in a position to discuss the capability of using RBM to obtain knowledge from data. As being shown in [Pinkas, 1995], one can use a symbolic learning rules to create a BM from truth assignments incrementally. We can apply the similar idea to construct an RBMs from the truth assignments. However, more interest is on how to obtain symbolic knowledge using the learning capability of connectionist networks rather than using symbolic learning rules to construct it. Learning in a connectionist network is seen as an approximation of parameters over a set of preferred models \(D = \{x^{(n)}|n = 1, \ldots, N\}\) from an unknown formula \(\varphi^*\). To start, let us consider the case where the set is complete, i.e. it consists of all preferred models. We will show that learning an RBM to represent the SDNF of \(\varphi^*\) is possible in this case even though it is not guaranteed.

Proposition 4. Learning an RBM to represent a formula from its complete set of preferred model is possible but not guaranteed.

Proof. The gradient of the negative log-likelihood of the RBM is:

\[
\frac{\partial \ell}{\partial \theta} = E_h[\frac{\partial E(x, h)}{\partial \theta}] - E_h[\frac{\partial E(x, h)}{\partial \theta}] |_{h, x}
\]

(5)

This function is not convex, i.e. there does not exist an apparent global minimum, therefore we are not always guaranteed to learn an RBM to represent the formula if it exists. At a local minimum such gradient is close to 0 for all parameters, which means:

\[
\frac{\partial \ell}{\partial w_{ij}} = -\frac{1}{N} \sum_{x \in D} x_i p(h_j|x) + \sum_{x} x_i p(h_j|x) p(x) \approx 0
\]

(6)
A solution for this is:

\[ p(h_j | x)p(x) \approx \begin{cases} \\
\frac{p(h_j | x)}{p(x)} & \text{if } x \in D \\
0 & \text{otherwise} \end{cases} \quad (7) \]

The solution can be achieved by either having \( p(h_j | x) \approx 0 \) or \( p(x) \approx 0 \) for all \( x \notin D \) and \( p(x) \approx \frac{1}{N} \) for \( x \in D \).

Since \( p(x) = \frac{1}{N} \sum_h \exp(-E(x,h)) \) then for a training sample (preferred model) \( x \) we have:

\[ \sum_{x \in D} \sum_h \exp(-E(x', h)) \approx N \sum_h \exp(-E(x, h)) \quad (8) \]

This means the solution can be achieved if \( \sum_h \exp(-E(x, h)) \) is equally large for all \( x \in D \), and much smaller otherwise. We can further factorize this sum to be \( \sum_h \exp(-E(x, h)) \propto \prod_j (1 + \exp(\sum_i w_{ij}x_i + b_j)) \).

Now, suppose we have an RBM \((W^*, b^*)\) to represent the unknown formula. Here we consider the case that the RBM is equivalent to a set of Confidence rules with equally large confidence-values \( c_{+\infty} \). This RBM would allow only one hidden unit to be activated for a preferred model, while deactivates all hidden units for a non-preferred model. Similar to Theorem and Proposition, we can show that a hidden neuron is activated with an input message \( c_{+\infty} \epsilon \gg 0 \); and it is de-activated with an input message at most as \( c_{+\infty} (\epsilon - 1) \ll 0 \). Therefore we can choose \( c_{+\infty} \) large enough to meet the solution because:

\[ \prod_j (1 + \exp(\sum_i w_{ij}x_i + b_j)) \approx \begin{cases} \\
\exp(c_{+\infty}) & \text{if } x \in D \\
1 & \text{otherwise} \end{cases} \quad (9) \]

It should be noted that, this is one of many possible local minima so we are not guaranteed to learn the rules. Therefore, in order to achieve this we may need to add more constraints which favour our desired minimum.

**Example 5. Learning XOR:** We train an RBM on XOR truth table. In order to increase a chance to converge the wanted local optimum we only use four hidden units, as the same number of conjunctive clauses from DNF of XOR. We omit the visible biases as they are not needed. Interestingly, by using Contrastive Divergence [Hinton, 2002] we can learn RBMs that represent exactly our theory. One is shown below:

\[
\begin{aligned}
W &= \begin{bmatrix} -7.9283 & 5.6875 & -6.7200 & 6.2166 \\
-6.8593 & 6.0078 & 6.2068 & -6.7347 \\
-6.9774 & -6.5855 & 6.0395 & 6.3059 \\
\end{bmatrix} \\
b &= [5.7909, -6.3370, -6.3476, 6.2276]^T
\end{aligned}
\]

The weight matrix and biases above resemble four Confidence rules that represent XOR. It can be seen that in each column vector of the weight matrix the strength of the weights are similar and can be approximated as a Confidence value \( c_j \) which is their average. If we run the training for long time, the Confidence values are increasing as the optimised function is approaching a local/global minimum, see Figure 2.

This is what we expect when we mentioned about large \( c_{+\infty} \).

By extending Proposition, we can also show that learning RBMs or DBNs can be seen as an approximation of a set of weighted formulas \( \Phi^* \). Intuitively, one can see the learning of RBMs as lowering the energy of the training sample while raising the energy of the others. Under a particular condition, this will converge to a set of Confidence rules that assign minimum energy to the preferred models (training samples) and much higher energy to the others. We exemplify this on a car evaluation task after pruning the small weights we obtain a mixing of sound and unsound rules. However, interestingly we observe that the sound rules tend to have higher confidence values and some of them look similar as the results of Valliant’s elimination [Valiant, 1984]. For example: \( h_1 \leftrightarrow \text{low\_safety} \land \text{car\_is\_unacceptable}, h_2 \leftrightarrow 2\text{\_seats} \land \text{car\_is\_unacceptable} \) are sound and only contain 2 variables from total 6 variables of the domain.

### 6 Discussion

This work addresses two main problems in unifying symbolic logic and neural networks when restricted Boltzmann machines is employed. First, we show how to represent a propositional logic program in an unsupervised energy-based connectionist networks. Second, we show the possibility of using statistical learning in RBMs to approximate propositional knowledge. The results from this work can set up a theoretical foundation for further exploration of unsupervised neural-symbolic integration, reasoning and knowledge learning and extraction.

A promising application of this work can be neural-symbolic integration and reasoning where background knowledge is encoded into RBMs/DBNs for better learning and more effective reasoning. Here, we may take the advantage of RBMs where inference is efficient. One exciting future work we are interested to investigate is to integrate first-order logic into RBMs/DBNs.

In practice, learning the exact rules would need to constrain a weight to be discrete, i.e \( w_{ij} \in \{-c_j, 0, c_j\} \) and a bias must be \( c_j (\|T_j\| - \epsilon) \). Even though learning discrete neural network can be done efficiently with Expectation Back-propagation [Soudry et al., 2014], the difficulty here is the stochastic sampling since the negative log-likelihood is generally intractable. This may be solved by carefully herd- the learning process using deterministic method [Welling, 2009]. Alternative approach for knowledge approximation is to extract rules from trained RBMs. For examples, in [Penning et al., 2011], the extraction is done by deterministic sampling from each activated hidden unit. In [Tran and Garcez, 2016], the rules are extracted to minimise the L2-norm of their difference with the network’s weights. These methods

https://archive.ics.uci.edu/ml/datasets/Car+Evaluation
are heuristic, i.e. do not guarantee the minimum energies for the preferred models, although we can still extract meaningful but incomplete knowledge. We hope that, based on the result of this work further study can find better extraction algorithms.

References

[Avila Garcez and Zaverucha, 1999] Artur S. Avila Garcez and Gerson Zaverucha. The connectionist inductive learning and logic programming system. Applied Intelligence, 11(1):5977, July 1999.

d’Avila Garcez et al., 2001] A.S. d’Avila Garcez, K. Broda, and D.M. Gabbay. Symbolic knowledge extraction from trained neural networks: A sound approach. Artificial Intelligence, 125:155–207, 2001.

[Fransa et al., 2014] Manoel V. M. Fransa, Gerson Zaverucha, and Artur S. d’AvilaGarcez. Fast relational learning using bottom clause propositionalization with artificial neural networks. Machine Learning, 94(1):81–104, 2014.

[Garcez et al., 2008] Artur S. d’Avila Garcez, Lus C. Lamb, and Dov M. Gabbay. Neural-Symbolic Cognitive Reasoning. Springer Publishing Company, Incorporated, 2008.

[Hinton et al., 2006] Geoffrey E. Hinton, Simon Osindero, and Yee-Whye Teh. A fast learning algorithm for deep belief nets. Neural Comput., 18(7):15271554, July 2006.

[Hinton, 1989] Geoffrey E. Hinton. Connectionist learning procedures. Artificial Intelligence, 40(1-3):185–234, September 1989.

[Hinton, 2002] Geoffrey E. Hinton. Training products of experts by minimizing contrastive divergence. Neural Comput., 14(8):1771–1800, August 2002.

[Hopfield, 1982] J. J. Hopfield. Neural networks and physical systems with emergent collective computational abilities. Proceedings of the National Academy of Sciences of the United States of America, 79(8):2554–2558, 1982.

[Lee et al., 2009] Honglak Lee, Peter T. Pham, Yan Largman, and Andrew Y. Ng. Unsupervised feature learning for audio classification using convolutional deep belief networks. In NIPS, pages 1096–1104, 2009.

[Minsky, 1991] Marvin Minsky. Logical versus analogical or symbolic versus connectionist or neat versus scruffy. AI Magazine, 12(2):34–51, 1991.

[Mohamed et al., 2012] Abdel-rahman Mohamed, George Dahl, and Geoffrey Hinton. Acoustic modeling using deep belief networks. IEEE Transactions on Audio, Speech & Language Processing, 20(1):14–22, 2012.

[Penning et al., 2011] Leo de Penning, Artur S. d’Avila Garcez, Lus C. Lamb, and John-Jules Ch Meyer. A neural-symbolic cognitive agent for online learning and reasoning. In IJCAI, pages 1653–1658, 2011.

[Pinkas, 1991a] Gadi Pinkas. Propositional non-monotonic reasoning and inconsistency in symmetric neural networks. In Proceedings of the 12th International Joint Conference on Artificial Intelligence - Volume 1, IJCAI’91, pages 525–530, San Francisco, CA, USA, 1991. Morgan Kaufmann Publishers Inc.

[Pinkas, 1991b] Gadi Pinkas. Symmetric neural networks and propositional logic satisfiability. Neural Comput., 3(2):282–291, June 1991.

[Pinkas, 1995] Gadi Pinkas. Reasoning, nonmonotonict and learning in connectionist networks that capture propositional knowledge. Artificial Intelligence, 77(2):203–247, September 1995.

[Russell and Norvig, 2003] Stuart Russell and Peter Norvig. Knowledge, reasoning, and planning. In Artificial Intelligence: A Modern Approach. Pearson Education, 2003.

[Smolensky, 1986] Paul Smolensky. Information processing in dynamical systems: Foundations of harmony theory. In Rumelhart, D. E. and McClelland, J. L., editors, Parallel Distributed Processing: Volume 1: Foundations, pages 194–281. MIT Press, Cambridge, 1986.

[Smolensky, 1987] P. Smolensky. Connectionist ai, symbolic ai, and the brain. Artificial Intelligence Review, 1(2):95–109, 1987.

[Smolensky, 1995] Paul Smolensky. Constituent structure and explanation in an integrated connectionist/symbolic cognitive architecture. In C. Mcdonald, editor, Connectionism: Debates on Psychological Explanation, pages 221–290. Blackwell, Cambridge, 1995.

[Son Tran and Garcez, 2012] Son Tran and Artur Garcez. ICML logic extraction from deep belief networks. In ICML 2012 Representation Learning Workshop, Edinburgh, July 2012.

[Soudry et al., 2014] Daniel Soudry, Itay Hubara, and Ron Meir. Expectation backpropagation: Parameter-free training of multilayer neural networks with continuous or discrete weights. In Advances in Neural Information Processing Systems 27, pages 963–971. Curran Associates, Inc., 2014.

[Towell and Shavlik, 1993] Geoffrey G. Towell and Jude W. Shavlik. The extraction of refined rules from knowledge-based neural networks. In Machine Learning, page 71101, 1993.

[Towell and Shavlik, 1994] Geoffrey G. Towell and Jude W. Shavlik. Knowledge-based artificial neural networks. Artificial Intelligence, 70(1-2):119–165, 1994.

[Tran and d’Avila Garcez, 2013] Son N. Tran and Artur d’Avila Garcez. Knowledge extraction from deep belief networks for images. In IJCAI-2013 Workshop on Neural-Symbolic Learning and Reasoning, 2013.

[Tran and Garcez, 2016] Son Tran and Artur Garcez. Deep logic networks: Inserting and extracting knowledge from deep belief networks. IEEE Transactions on Neural Networks and Learning Systems, PP(99):1–13, 2016.

[Valiant, 1984] L. G. Valiant. A theory of the learnable. Commun. ACM, 27(11):1134–1142, November 1984.

[Valiant, 2006] Leslie G. Valiant. Knowledge infusion. In Proceedings, The Twenty-First National Conference on
[Welling, 2009] Max Welling. Herding dynamical weights to learn. In ICML, page 141, 2009.