GENERAL HIGHER-ORDER MAJORIZATION-MINIMIZATION
ALGORITHMS FOR (NON)CONVEX OPTIMIZATION

ION NECOARA∗ AND DANIELA LUỘ†

Abstract. Majorization-minimization algorithms consist of successively minimizing a sequence of upper bounds of the objective function so that along the iterations the objective function decreases. Such a simple principle allows to solve a large class of optimization problems, even nonconvex and nonsmooth. We propose a general higher-order majorization-minimization algorithmic framework for minimizing an objective function that admits an approximation (surrogate) such that the corresponding error function has a higher-order Lipschitz continuous derivative. We present convergence guarantees for our new method for general optimization problems with (non)convex and/or (non)smooth objective function. For convex (possibly nonsmooth) problems we provide global sublinear convergence rates, while for problems with uniformly convex objective function we obtain locally faster superlinear convergence rates. We also prove global stationary point guarantees for general nonconvex (possibly nonsmooth) problems and under Kurdyka-Łojasiewicz property of the objective function we derive local convergence rates ranging from sublinear to superlinear for our majorization-minimization algorithm. Moreover, for unconstrained nonconvex problems we derive convergence rates in terms of first- and second-order optimality conditions.

Key words. (Non)convex optimization, majorization-minimization, higher-order methods, convergence rates. LATEX

AMS subject classifications. 90C25, 90C06, 65K05.

1. Introduction. The principle of successively minimizing upper bounds of the objective function is often called majorization-minimization [15, 16, 25]. Most techniques, e.g., gradient descent, utilize convex quadratic majorizers based on first-order oracles in order to guarantee that the majorizer is easy to minimize. Despite the empirical success of first-order majorization-minimization algorithms to solve difficult optimization problems, the convergence speed of such methods is known to slow down close to saddle points or in ill-conditioned landscapes [18, 23]. Higher-order methods are known to be less affected by these problems [2, 5, 6, 19]. In this work, we focus our attention on higher-order majorization-minimization methods to find (local) minima of (potentially nonsmooth and nonconvex) objective functions. At each iteration, these algorithms construct and optimize a local (Taylor) model of the objective using higher-order derivatives with an additional step length penalty term that depends on how well the model approximates the real objective.

Contributions. This paper provides an algorithmic framework based on the notion of higher-order upper bound approximations of the (non)convex and/or (non)smooth objective function, leading to a general higher-order majorization-minimization algorithm, which we call GHOM. Then, we present convergence guarantees for the GHOM algorithm for general optimization problems when the upper bounds approximate the objective function up to an error that is $p \geq 1$ times differentiable and has a Lipschitz continuous $p$ derivative; we call such upper bounds higher-order surrogate functions. More precisely, on general (possibly nonsmooth) convex problems our general higher-order majorization-minimization (GHOM) algorithm achieves global sublinear convergence rate for the function values. When we apply our method to

∗Automatic Control and Systems Engineering Department, University Politehnica Bucharest, 060042 Bucharest, Romania; Gheorghe Mihoc-Caius Iacob Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, 050711 Bucharest, Romania. ion.necoara@upb.ro.
†Automatic Control and Systems Engineering Department, University Politehnica Bucharest, Romania. daniela.lupu@upb.ro.
optimization problems with uniformly convex objective function, we obtain faster local superlinear convergence rates in multiple criteria: function values, distance of the iterates to the optimal point and in the minimal norms of subgradients. Then, on general (possibly nonsmooth) nonconvex problems we prove for GHOM global asymptotic stationary point guarantees and convergence rates in terms of first-order optimality conditions. We also characterize the convergence rate of GHOM algorithm locally in terms of function values under the Kurdyka-Lojasiewicz (KL) property of the nonconvex objective function. Our result show that the convergence behavior of GHOM ranges from sublinear to superlinear depending on the parameter of the underlying KL geometry. Moreover, on smooth unconstrained nonconvex problems we derive convergence rates in terms of first and second-order optimality conditions. In Table 1 we summarize the main convergence results of this paper.

Besides providing a unifying framework for the design and analysis of higher-order majorization-minimization methods, in special cases, where complexity bounds are known for some particular higher-order tensor algorithms, our convergence results recover the existing bounds. More precisely, our convergence results recover for \( p = 1 \) the convergence bounds for the (proximal) gradient \([1,16,25]\) and Gauss-Newton \([9,24]\) type algorithms from the literature. For \( p > 1 \) and convex composite objective functions (see Example 2.5), we recover the global convergence results from \([20]\) and the local convergence rates from \([10]\). For \( p > 1 \) and nonconvex unconstrained optimization problems (see Example 2.3), we recover the convergence results from \([2,5]\) and for problems with simple constraints we obtain similar rates as in \([6]\). However, for other examples (such as, Examples 2.2, 2.6, in the convex case; Examples 2.2, 2.5, 2.6 and the second part of 2.7, in the nonconvex case) our convergence results seem to be new. In fact our unifying algorithmic framework is inspired in part by the recent work on higher-order Taylor-based methods for convex optimization \([19,20]\), but it yields a more general update rule and is also appropriate for nonconvex optimization.
Note that there is a major difference between the Taylor expansion and the model approximation based on a general majorization-minimization framework. Taylor expansion is unique. Conversely, majorization-minimization approach may admit many upper bound models for a given objective function and every model leads to a different optimization method. Another major difference between our approach and the existing works such as [1,2,5,6,9,19,20] is that we assume Lipschitz continuity of the $p$-derivative of the error function, while the other papers assume directly the Lipschitz continuity of the $p$-derivative of the objective function. Hence, our convergence proofs are very different from the existing works.

1.1. Related work. Higher-order methods are popular due to their performance in dealing with ill conditioning and fast rates of convergence [2,5–7,12,19–21]. For example, in [19] the following unconstrained convex problem was considered:

$$ f^* = \min_{x \in \mathbb{R}^n} f(x), $$

where $f$ is convex, $p$ times continuously differentiable and with the $p$ derivative Lipschitz continuous of constant $L^f_p$ (see Section 1.2 for a precise definition). Then, Nesterov proposed in [19] the following higher-order Taylor-based iterative method for finding an optimal solution of the convex problem (1.1):

$$ x_{k+1} = \arg\min_{y \in \mathbb{R}^n} g(y; x_k) \left( := \sum_{i=0}^{p} \frac{1}{i!} \nabla^i f(x_k)[y-x_k]^i + \frac{M_p}{(p+1)!} \|y-x_k\|^{p+1} \right), $$

where $M_p \geq L^f_p$. Hence, at each iteration one needs to construct and optimize a local Taylor model of the objective with an additional regularization term that depends on how well the model approximates the real objective. This is a natural extension of the cubic regularized Newton’s method extensively analyzed e.g., in [7,21,28]. Under the above settings [19] proves the following convergence rate in function values for (1.2):

$$ f(x_k) - f^* \leq O \left( \frac{1}{k^p} \right) \quad \forall k \geq 1. $$

Extensions of this method to composite convex problems and to objective functions with Holder continuous higher-order derivatives have been given in [20] and [12], respectively, inexact variants were analyzed e.g., in [22] and local superlinear convergence results were given recently in [10].

Further, for the unconstrained nonconvex case [2,5] provides convergence rates for a similar algorithm. The basic assumptions are that $f$ is $p$ times continuously differentiable having the $p$ derivative smooth with constant $L^f_p$ and bounded below. For example, [5] proposes an adaptive regularization algorithm (called AR$p$), which requires building a higher-order model based on an appropriate regularization of the $p$ Taylor approximation $g(y; x_k)$. Basically, at each iteration the AR$p$ algorithm approximately computes a (local) minimum of the model $g(y; x_k)$ that must satisfy certain second-order optimality conditions:

$$ x_{k+1} \approx \arg\min_{y \in \mathbb{R}^n} g(y; x_k). $$

For the AR$p$ algorithm [5] proves the best known convergence rate for this class of problems ($f$ nonconvex with $p$ derivative smooth) in terms of the first- and second-order model conditions:
order optimality conditions:

\[
\min_{i=1:k} \max \left( -\lambda_{\min}^{\frac{p+1}{p}}(\nabla^2 f(x_i)), \|\nabla f(x_i)\|_{\ast}^{\frac{p+1}{p}} \right) \leq O \left( \max \left( \left( \frac{1}{k} \right)^{\frac{p}{p+1}}, \left( \frac{1}{k} \right)^{\frac{p}{p+1}+1} \right) \right).
\]

Extension of these results to smooth optimization problems with simple constraints were given recently in [6]. Furthermore, several studies demonstrate that special geometric properties of the objective function, such as gradient dominance condition or Kurdyka-Lojasiewicz (KL) property [3], can enable faster convergence of the ARp algorithm (e.g., when \( p = 2 \), see for example [21, 28]. Notably, all these approaches require optimizing a \((p+1)\)-th order polynomial, which is known to be a difficult problem. Recently, [19, 20] introduced an implementable method for convex functions (see also Lemma 1.5 below). In particular, for \( p = 2 \) and \( p = 3 \) there are efficient methods from optimization for minimizing the corresponding Taylor-based upper approximation (1.2), see e.g., [7, 8, 13, 19, 20].

**Majorization-minimization algorithms** approximate at each iteration the objective function by a majorizing function that is easy to minimize [4, 15]. Most techniques, e.g., gradient descent, utilize convex quadratic majorizers in order to guarantee that the model at each iteration is easy to minimize. The framework of first-order majorization-minimization methods, i.e. methods that are using only gradient information to build the upper model, has been analyzed widely in the literature of the recent two decades, see e.g., [4, 14–16, 25]. However, to the best of our knowledge there are no results on the convergence behavior of general higher-order majorization-minimization algorithms, i.e. methods that are using higher-order derivatives to build the upper model. In this paper we provide a general framework for the design of higher-order majorization-minimization algorithms and derive a complete convergence analysis for them (global and local convergence rates) covering a large class of optimization problems, that is convex or nonconvex, smooth or nonsmooth.

**Content.** The paper is organized as follows: Section 1.2 presents notation and preliminaries; in Section 2 we define our higher-order majorization-minimization framework and the corresponding algorithm; in Section 3 we derive global and local convergence results for our scheme in the convex settings, while the convergence analysis for non-convex problems is given in Section 4.

1.2. Notations and preliminaries. We denote a finite-dimensional real vector space with \( E \) and by \( E^* \) its dual space composed by linear functions on \( E \). Using a self-adjoint positive-definite operator \( D : E \to E^* \) (notation \( D = D^* > 0 \)), we can endow these spaces with *conjugate Euclidean norms*, see also [19]:

\[
\|x\| = \langle Dx, x \rangle^{1/2}, \quad x \in E, \quad \|h\|_* = \langle h, D^{-1}h \rangle^{1/2}, \quad h \in E^*.
\]

Let \( H \) be a \( p \) multilinear form on \( E \). The value of \( H \) in \( h_1, \ldots, h_p \in E \) is denoted \( H[h_1, \ldots, h_p] \). The abbreviation \( H[h]_p \) is used when \( h_1 = \cdots = h_p = h \) for some \( h \in E \). The norm of \( H \) is defined in the standard way:

\[
\|H\| := \max_{\|h_1\| = \cdots = \|h_p\| = 1} |H[h_1, \ldots, h_p]|.
\]

If the form \( H \) is symmetric, it is known that the maximum in the above definition can be achieved when all the vectors are the same:

\[
\|H\| = \max_{\|h\| = 1} |H[h]_p|.
\]
We denote the Taylor approximation of $\psi$ around $x$ of order $p$ by:

$$\psi^{(p)}(y; x) = \psi(x) + \sum_{i=1}^{p} \frac{1}{i!} \nabla^i \psi(x)(y - x)^i \quad \forall y \in \mathbb{E}.$$  

It is known that if (1.3) holds, then the residual between function value and its Taylor approximation can be bounded [19]:

$$|\psi(y) - \psi^{(p)}(y; x)| \leq \frac{L_p}{(p+1)!} \|y - x\|^{p+1} \quad \forall x, y \in \mathbb{E}.$$  

If $p \geq 2$, we also have the following inequalities valid for all $x, y \in \mathbb{E}$:

$$\|\nabla \psi(y) - \nabla \psi(y; x)\| \leq \frac{L_p}{p!} \|y - x\|^p,$$

$$\|\nabla^2 \psi(y) - \nabla^2 \psi(y; x)\| \leq \frac{L_p}{(p-1)!} \|y - x\|^{p-1}.$$  

Next, we provide several examples of functions that have known Lipschitz continuous $p$ derivatives (also called $p$ smooth functions), see Appendix for proofs.

**Example 1.2.** For the power of Euclidean norm $\psi_{p+1}(x) = \|x - x_0\|^{p+1}$, with $p \geq 1$, the Lipschitz continuous condition (1.3) holds with $L_p = (p+1)!$.

**Example 1.3.** For given $a_i \in \mathbb{E}$, $1 \leq i \leq m$, consider the log-sum-exp function:

$$\psi(x) = \log \left( \sum_{i=1}^{m} e^{a_i x} \right), \quad x \in \mathbb{R}^n.$$  

Then, for the Euclidean norm $\|x\| = \langle Dx, x \rangle^{1/2}$ for $x \in \mathbb{R}$ and $D := \sum_{i=1}^{m} a_i a_i^*$ (assuming $D > 0$, otherwise we can reduce dimensionality of the problem), the Lipschitz continuous condition (1.3) holds with $L_p^D = 1$ for $p = 1$, $L_p^D = 2$ for $p = 2$ and $L_p^D = 3$ for $p = 3$. Note that for $m = 2$ and $a_1 = 0$, we recover the general expression of logistic regression function, which is a loss function widely used in machine learning [16].

**Example 1.4.** For a $p$ differentiable function $\psi$ its $p$ Taylor approximation $T_p^{\psi}(\cdot; x)$ at any point $x$ has the $p$ derivative Lipschitz with constant $L_p^{T_p^{\psi}} = 0$. Moreover, its $p$ Taylor approximation has also the $p - 1$ derivative Lipschitz with $L_{p-1}^{T_p^{\psi}} = \|\nabla^p \psi(x)\|$. Finally, in the convex case Nesterov proved in [19] a remarkable result which states that an appropriately regularized Taylor approximation of a convex function is a convex multivariate polynomial.

**Lemma 1.5.** [19] Assume $\psi$ convex and $p > 2$ differentiable function having the $p$ derivative Lipschitz with constant $L_p^\psi$. Then, the regularized Taylor approximation:

$$g(y, x) = T_p^{\psi}(y; x) + \frac{M_p}{(p+1)!} \|y - x\|^{p+1}$$

is also a convex function in $y$ provided that $M_p \geq pL_p^\psi$.  

5
As discussed in the introduction section, in higher-order tensor methods one usually needs to minimize at each iteration a regularized higher-order Taylor approximation. Thus, for the convex case, we dispose of a large number of powerful methods [7,8,13,20] for finding the solution of the corresponding subproblem from each iteration. Further, let us introduce the class of uniformly convex functions [20,21] which will play a key role in the local convergence analysis of our algorithm in the convex settings. We denote \( \mathbb{R} = \mathbb{R} \cup \{ \infty \} \) and for a given proper function \( \psi : \mathbb{E} \rightarrow \mathbb{R} \) its domain is \( \text{dom} \psi = \{ x \in \mathbb{E} : \psi(x) < \infty \} \).

**Definition 1.6.** A function \( \psi : \mathbb{E} \rightarrow \mathbb{R} \) is uniformly convex of degree \( q \geq 2 \) if there exists a positive constant \( \sigma_q > 0 \) such that:

\[
(1.7) \quad \psi(y) \geq \psi(x) + \langle \psi^x, y - x \rangle + \frac{\sigma_q}{q} \| x - y \|^q \quad \forall x, y \in \text{dom} \psi,
\]

where \( \psi^x \) is an arbitrary vector from the subdifferential \( \partial \psi(x) \) at \( x \).

Minimizing both sides of (1.7), we also get [20]:

\[
(1.8) \quad \psi^* = \min_{y \in \mathbb{E}} \psi(y) \geq \psi(x) - \frac{1}{q} \left( \frac{1}{\sigma_q} \right)^{\frac{1}{q}} \| \psi^x \|_q \| x \|^q \quad \forall x \in \text{dom} \psi.
\]

Note that for \( q = 2 \) in (1.7) we recover the usual definition of a strongly convex function. Moreover, (1.8) for \( q = 2 \) is the main property used when analyzing the convergence behavior of first-order methods [17]. One important class of uniformly convex functions is given next (see e.g., [20]).

**Example 1.7.** For \( q \geq 2 \) let us consider the convex function \( \psi(x) = \frac{1}{q} \| x - \bar{x} \|^q \), where \( \bar{x} \) is given. Then, \( \psi \) is uniformly convex of degree \( q \) with \( \sigma_q = 2^{2-q} \).

For nonconvex functions we have a more general notion than uniform convexity, called the Kurdyka-Lojasiewicz (KL) property, which captures a broad spectrum of the local geometries that a nonconvex function can have [3].

**Definition 1.8.** A proper and lower semicontinuous function \( \psi : \mathbb{E} \rightarrow \overline{\mathbb{R}} \) satisfies Kurdyka-Lojasiewicz (KL) property if for every compact set \( \Omega \subseteq \text{dom} \psi \) on which \( \psi \) takes a constant value \( \psi^* \) there exist \( \delta, \epsilon > 0 \) such that one has:

\[
\kappa(\psi(x) - \psi^*) \cdot \text{dist}(0, \partial \psi(x)) \geq 1 \quad \forall x : \text{dist}(x, \Omega) \leq \delta, \psi^* < \psi(x) < \psi^* + \epsilon,
\]

where \( \partial \psi(x) \) is the (limiting) subdifferential of \( \psi \) at \( x \) and \( \kappa : [0, \epsilon] \rightarrow \mathbb{R} \) is a concave differentiable function satisfying \( \kappa(0) = 0 \) and \( \kappa' > 0 \).

Denote for any \( x \in \text{dom} \psi \):

\[
S(x) = \text{dist}(0, \partial \psi(x)) \left( := \inf_{\psi^* \in \partial \psi(x)} \| \psi^x \|_\ast \right).
\]

If \( \partial \psi(x) = \emptyset \), we set \( S(x) = \infty \). Note that \( \partial \psi(x) \) is a closed set for \( \psi \) convex function. For nonconvex function \( \psi \), \( \partial \psi(x) \) denotes the limiting subdifferential of \( \psi \) at \( x \), see e.g., [27] for the definition. When \( \kappa \) takes the form \( \kappa(t) = \sigma_q^{\frac{1}{q}} \frac{q}{q-1} t^{\frac{q}{q-1}} \), with \( q > 1 \) and \( \sigma_q > 0 \) (which is our interest here), the KL property establishes the following local geometry of the nonconvex function \( \psi \) around a compact set \( \Omega \):

\[
(1.9) \quad \psi(x) - \psi^* \leq \sigma_q S(x)^q \quad \forall x : \text{dist}(x, \Omega) \leq \delta, \psi^* < \psi(x) < \psi^* + \epsilon.
\]
Note that the relevant aspect of the KL property is when \( \Omega \) is a subset of critical points for \( \psi \), i.e. \( \Omega \subseteq \{ x : 0 \in \partial \psi(x) \} \), since it is easy to establish the KL property when \( \Omega \) is not related to critical points.

**Example 1.9.** The KL property holds for a large class of functions including semi-algebraic functions (e.g., real polynomial functions), vector or matrix (semi)norms (e.g., \( \| \cdot \|_p \), with \( p \geq 0 \) rational number), logarithm functions, exponential functions and uniformly convex functions, see [3] for a comprehensive list.

Finally, for a function \( \psi \), we denote its sublevel set at a given \( x_0 \) by:

\[
L_{\psi}(x_0) = \{ x \in E : \psi(x) \leq \psi(x_0) \}.
\]

**2. General higher-order majorization-minimization algorithm.** In what follows, we study the following general optimization problem:

\[
(2.1) \quad \min_{x \in \text{dom} f} f(x),
\]

where \( f : E \to \bar{\mathbb{R}} \) is a proper lower semicontinuous (non)convex function and \( \text{dom} f \) is a nonempty closed convex set. We assume that a solution \( x^* \) exists for problem (2.1), hence the optimal value is finite and \( f \) is bounded from below by some \( f^* > -\infty \).

In the convex case we consider \( f^* = f(x^*) \) (the optimal value). Since \( f \) is extended valued, it allows the inclusion of constraints. Note that even in this general context the Fermat’s rule remains unchanged, that is \( x^* \in \text{dom} f \) is a (local) minimum of \( f \) at \( x^* \) [27]. Moreover, in our convergence analysis below we assume that the sublevel set \( L_f(x_0) \) is bounded. Then, there exists \( R > 0 \) such that:

\[
\| x - x^* \| \leq R \quad \forall x \in L_f(x_0).
\]

These basic assumptions are standard in the literature, see e.g., [2, 6, 7, 19, 21]. The main approach we adopt in solving the optimization problem (2.1) is to use a class of functions that approximates well the objective function \( f \) but are easier to minimize. We call this class higher-order surrogate functions. The main properties of our surrogate function are defined next:

**Definition 2.1.** Given \( f : E \to \bar{\mathbb{R}} \) and \( p \geq 1 \), we call an extended valued function \( g(\cdot; x) : E \to \bar{\mathbb{R}} \) a \( p \) higher-order surrogate of \( f \) at \( x \in \text{dom} f \) if \( \text{dom} g(\cdot; x) = \text{dom} f \) and it has the properties:

(i) the surrogate function is bounded from below by the original function:

\[
g(y; x) \geq f(y) \quad \forall y \in \text{dom} f.
\]

(ii) the error function \( h(y; x) = g(y; x) - f(y) \), with \( \text{dom} f \subseteq \text{int} (\text{dom} h) \), is \( p \) differentiable and has \( p \) derivative smooth with Lipschitz constant \( L_h^p \) on \( \text{dom} f \).

(iii) the derivatives of the error function \( h \) satisfy:

\[
\nabla^i h(x; x) = 0 \quad \forall i = 0 : p,
\]

where \( i = 0 \) means that \( h(x; x) = 0 \), or equivalently \( g(x; x) = f(x) \).

Note that [16] provided a similar definition, but only for a first-order surrogate function and used in the context of stochastic optimization. Next, we give several non-trivial examples of higher-order surrogate functions (for details see Appendix).
Example 2.2. (proximal functions). For a general (possibly nonsmooth and non-convex) function \( f : \mathbb{E} \to \mathbb{R} \), one can consider for any \( M_p > 0 \) the following \( p \geq 1 \) higher-order surrogate function:

\[
g(y; x) = f(y) + \frac{M_p}{(p+1)!} \| y - x \|^{p+1} \quad \forall x, y \in \text{dom} \, f.
\]

In this case, the error function \( h(y; x) = g(y; x) - f(y) \) has the Lipschitz constant \( L^h_p = M_p \) on \( \mathbb{E} \). Indeed, the first property of the surrogate is immediate. Next we need to prove that the error function has \( p \) derivative Lipschitz, i.e., \( h(y, x) = g(y, x) - f(y) = \frac{M_p}{(p+1)!} \| y - x \|^{p+1} \), which according to Example 1.2 has the \( p \) derivative Lipschitz with constant \( L^h_p = M_p \). For the last property of a surrogate, we notice that \( \nabla^i(\| y - x \|^{p+1})_{y=x} = 0 \) for all \( i = 0 : p \). Thus, \( \nabla^i h(x; y) = 0 \quad \forall i = 0 : p \).

Example 2.3. (smooth derivative functions). For a function \( f : \mathbb{E} \to \mathbb{R} \) that is \( p \geq 1 \) times differentiable and with the \( p \) derivative Lipschitz of constant \( L^f_p \) on \( \mathbb{E} \), one can consider the following \( p \) higher-order surrogate function:

\[
g(y; x) = T^f_p(y; x) + \frac{M_p}{(p+1)!} \| y - x \|^{p+1} \quad \forall x, y \in \mathbb{E},
\]

where \( M_p \geq L^f_p \). In this case, the error function \( h \) has \( L^h_p = M_p + L^f_p \).

Remark 2.4. If the function \( f \) additionally satisfies \( f(y) \geq T^f_p(y; x) \) for all \( y \), then we can improve the Lipschitz constant for the error function \( h \) from Example 2.3 to \( L^h_p = M_p \). Indeed, using the third property of a \( p \) higher-order surrogate function, we have \( T^f_p(y; x) = 0 \) for all \( y \) and thus:

\[
|h(y; x) - T^h_p(y; x)| = |g(y; x) - f(y)| = \left| T^f_p(y; x) + \frac{M_p}{(p+1)!} \| y - x \|^{p+1} - f(y) \right|
\]

\[
= \left| f(y) - T^f_p(y; x) - \frac{M_p}{(p+1)!} \| y - x \|^{p+1} \right|.
\]

We observe that the last term in the equation above is positive and due to the additional condition \( f(y) \geq T^f_p(y; x) \), we also have that \( f(y) - T^f_p(y; x) \geq 0 \). Hence, using \( |a - b| \leq \max\{a, b\} \) for any two positive scalars \( a \) and \( b \), from (2.2) we further get:

\[
|h(y; x) - T^h_p(y; x)| \leq \max \left( f(y) - T^f_p(y; x), \frac{M_p}{(p+1)!} \| y - x \|^{p+1} \right)
\]

\[
\leq \max \left( \frac{L^f_p}{(p+1)!} \| y - x \|^{p+1}, \frac{M_p}{(p+1)!} \| y - x \|^{p+1} \right) = \frac{M_p}{(p+1)!} \| y - x \|^{p+1}.
\]

Therefore, in this case \( L^h_p = M_p \). Functions that satisfy the additional condition from Remark 2.4, are e.g., the convex functions for \( p = 1 \) (since in this case from convexity of \( f \) we automatically have \( f(y) \geq T^f_1(y; x) \)) or the quadratic functions for \( p = 2 \) (since in this case \( f(y) = T^f_2(y; x) \)). Recall that if \( f \) is convex function and \( M_p \) is chosen conveniently, then the higher-order surrogate function \( g \) from Example 2.3 is also convex (see Lemma 1.5).

Example 2.5. (composite functions). Let \( f_1 : \mathbb{E} \to \bar{\mathbb{R}} \) and \( f_2 : \mathbb{E} \to \bar{\mathbb{R}} \) has the \( p \geq 1 \) derivative smooth with constant
$L^F_1$ on $\text{dom} f_1 \subset \text{int}(\text{dom} f_2)$. Then, for the composite function $f = f_1 + f_2$ one can take the following $p$ higher-order surrogate:

$$g(y; x) = f_1(y) + T^f_{p^2}(y; x) + \frac{M_p}{(p + 1)!} \| y - x \|^{p + 1} \quad \forall x, y \in \text{dom} f \ (:= \text{dom} f_1),$$

where $M_p \geq L^F_1$. Moreover, error function $h$ has Lipschitz constant $L^h = M_p + L^F_1$. If additionally $f_2$ satisfies $f_2(y) \geq T^f_{L^f_1}(y; x)$ for all $y$, then the Lipschitz constant for the error function $h$ can be improved to $L^h = M_p$ (see Remark 2.4).

**Example 2.6. (bounded derivative functions).** We consider a function $f : \mathbb{E} \rightarrow \mathbb{R}$ that is $p + 1$ times continuously differentiable, where $p \geq 1$. Assume that the $p + 1$ derivative of $f$ is upper bounded by a $p + 1$ constant symmetric multilinear form $H$, i.e., $(\nabla^{p+1} f(x)|h|^p, h) \leq \langle H|h|^p, h \rangle \forall h, x \in \mathbb{E}$ (notation $\nabla^{p+1} f(x) \ll H$). In this case one can consider the following $p$ higher-order surrogate function:

$$g(y; x) = T^f_{p^2}(y; x) + \frac{1}{(p + 1)!} \langle H|y - x|^p, y - x \rangle \quad \forall x, y \in \mathbb{E}.$$

Moreover, the error function $h$ has the Lipschitz constant $L^h = \| H \| + L^F_1$. Note that the class of functions considered in this example is different from that from Example 2.3 as one can find functions with the $p$ derivative smooth but the $p + 1$ derivative may not exist everywhere.

**Example 2.7. (composition of functions).** Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a differentiable function having Lipschitz continuous Jacobian matrix with constant $L^F_1$, $\phi : \mathbb{R}^m \rightarrow \mathbb{R}$ be a differentiable function having the first derivative smooth with constant $L^\phi_1$ and $f_1 : \mathbb{R}^n \rightarrow \mathbb{R}$ be a proper closed convex function with $\text{dom} f_1$ compact convex set. Then, for the function $f(x) = f_1(x) + \phi(F(x))$ one can take the first-order surrogate:

$$g(y; x) = f_1(y) + \phi(F(x) + \nabla F(x)(y - x)) + \frac{M}{2} \| y - x \|^2 \quad \forall x, y \in \text{dom} f_1,$$

where $M \geq L^\phi_1 L^F_1$ and for the error function the Lipschitz constant is

$$L^h_1 = M + L^F_1 \max_{y \in \text{dom} f_1} \| \nabla \phi(F(y)) \| + 2L^\phi_1 \left( \max_{y \in \text{dom} f_1} \| \nabla F(y) \| \right)^2.$$ 

If $F$ is simple, e.g., separable of the form $F(x) = [F_1(x_1) \cdots F_n(x_n)]$, then one can take the following first-order surrogate:

$$g(y; x) = f_1(y) + \phi(F(x)) + \langle \nabla \phi(F(x)), F(y) - F(x) \rangle + \frac{M}{2} \| F(y) - F(x) \|^2,$$

where $M \geq L^\phi_1$ and $L^h_1 = 2L^F_1 \max_{y \in \text{dom} f_1} \| \nabla \phi(F(y)) \| + 2ML^F_1 \max_{y \in \text{dom} f_1} \| F(y) \| + (M + L^\phi_1) \left( \max_{y \in \text{dom} f_1} \| \nabla F(y) \| \right)^2$. Note that in both cases minimizing $g$ is easy since either $g$ is convex or $F$ has a simple structure (e.g., separable). Moreover, the second surrogate function (2.4) resembles the augmented Lagrangian method of multipliers [23, 27].

The reader can find other examples of higher-order surrogate functions depending on the structure of the objective function $f$ in (2.1) and we believe that this paper opens a window of opportunity for higher-order algorithmic research. In the following we define our General Higher-Order Majorization-Minimization (GHOM) algorithm:
Algorithm 2.1 Algorithm GHOM

Given $x_0 \in \text{dom } f$ and $p \geq 1$, for $k \geq 0$ do:

1. Compute the $p$ surrogate function $g(y; x_k)$ of $f$ near $x_k$

2. Compute a stationary point $x_{k+1}$ of the subproblem:

\begin{equation}
(2.5) \quad \min_{y \in \text{dom } g} g(y; x_k),
\end{equation}

satisfying the following descent property

\begin{equation}
(2.6) \quad g(x_{k+1}; x_k) \leq g(x_k; x_k) \quad \forall k \geq 0.
\end{equation}

Note that in the convex case we can use very efficient methods from convex optimization to find the global solution $x_{k+1}$ of the subproblem at each iteration, see e.g., [7,8,13,20] and also Lemma 1.5. In the nonconvex case, our convergence analysis below requires only the computation of a stationary point for the subproblem (2.5) satisfying the descent (2.6). Note that almost all nonconvex optimization algorithms are able to identify stationary points of nonconvex problems. Moreover, in our convergence analysis below we can relax the stationary point condition, that is we can require $x_{k+1}$ to satisfy $\|g(x_{k+1})\| \leq \theta \|x_{k+1} - x_k\|^p$ for some $\theta > 0$, where $g(x_{k+1}) \in \partial g(x_{k+1}; x_k)$. For simplicity of the exposition however, we assume below that $x_{k+1}$ is a stationary point of the subproblem (2.5).

3. Convergence analysis of GHOM for convex optimization. In this section we analyze the global and local convergence of algorithm GHOM under various assumptions on the convexity of the objective function $f$. Note that when the function $f$ is convex we assume that $x_{k+1}$ is a global minimum of the $p$ higher-order surrogate $g$ at $x_k$. For the particular surrogate given in Example 2.5 similar convergence analysis has been given in [20], but [20] requires $M_p \geq pL_p^2$, while in our analysis it is sufficient to have $M_p \geq L_p^2$. Moreover, from our knowledge the other examples of surrogate functions (Examples 2.2, 2.6, 2.7) are not investigated in the literature.

3.1. Global sublinear convergence of GHOM. In this section we derive global rate of convergence of order $O(1/k^p)$ for GHOM in terms of function values when (2.1) is a general convex optimization problem.

**Theorem 3.1.** Consider the optimization problem (2.1). Suppose the objective function $f$ is proper, lower semicontinuous, convex and admitting at each point $x \in \text{dom } f$ a $p \geq 1$ higher-order surrogate function $g(\cdot; x)$ as given in Definition 2.1. Then, the sequence $(x_k)_{k \geq 0}$ generated by algorithm GHOM has the following global sublinear convergence rate:

\begin{equation}
(3.1) \quad f(x_k) - f(x^*) \leq \frac{L_p R^{p+1}}{p! \left(1 + \frac{k}{p+1}\right)^p}.
\end{equation}

**Proof.** Since $h(y, x)$, defined as the error between the $p$ higher-order surrogate $g(y; x)$ and the function $f$, has the $p$ derivative smooth with Lipschitz constant $L_p$, then from (1.4) we get:

\[ h(y; x) \leq T_p^h(y; x) + \frac{L_p}{(p+1)!} \|y - x\|^{p+1} \quad \forall x, y \in \text{dom } f. \]
However, based on the condition (iii) from Definition 2.1 the Taylor approximation of \( h \) of order \( p \) at \( x \) satisfies:

\[
T^h_p(y; x) = h(x; x) + \sum_{i=1}^{p} \frac{1}{i!} \nabla^i h(x; x)[y - x]^i = 0.
\]

Therefore, we obtain:

\[
h(y; x) = g(y; x) - f(y) \leq T^h_p(y; x) + \frac{L^h_p}{(p + 1)!} \| y - x \|^{p+1},
\]

which implies that

\[
g(y; x) \leq f(y) + \frac{L^h_p}{(p + 1)!} \| y - x \|^{p+1} \quad \forall x, y \in \text{dom } f.
\]

Since the surrogate function \( g \) at \( x_k \) satisfies \( \text{dom } g(\cdot; x_k) = \text{dom } f \), it is bounded from below by \( f \) and \( x_{k+1} \) is a global minimum of \( g \), we further get:

\[
f(x_{k+1}) \leq g(x_{k+1}; x_k) = \min_{y \in \text{dom } f} g(y; x_k) \leq \min_{y \in \text{dom } f} \left( f(y) + \frac{L^h_p}{(p + 1)!} \| y - x_k \|^{p+1} \right).
\]

Since we assume \( f \) to be convex, we can choose \( y = x_k + \alpha (x^* - x_k) \), with \( \alpha \in [0, 1] \), and obtain further:

\[
f(x_{k+1}) \leq \min_{\alpha \in [0, 1]} f(x_k + \alpha (x^* - x_k)) + \frac{L^h_p}{(p + 1)!} \| x_k + \alpha (x^* - x_k) - x_k \|^{p+1}
\]

\[
\leq \min_{\alpha \in [0, 1]} f(x_k) - \alpha (f(x_k) - f(x^*)) + \frac{L^h_p}{(p + 1)!} \alpha^{p+1} \| x_k - x^* \|^{p+1}.
\]

Let us show that GHOM is a decent algorithm. Indeed, from (2.6) we have:

\[
f(x_k) = g(x_k; x_k) \geq g(x_{k+1}; x_k) \geq f(x_{k+1}) \quad \forall k \geq 0.
\]

Hence, all the iterates \( x_k \) are in the level set \( \mathcal{L}_f(x_0) \) and thus satisfy \( \| x_k - x^* \| \leq R \) for all \( k \geq 0 \). Subtracting the optimal value on both side of (3.4) and recalling the fact that the sublevel set of \( f \) at \( x_0 \) is assumed bounded, we obtain:

\[
f(x_{k+1}) - f(x^*) \leq \min_{\alpha \in [0, 1]} (1 - \alpha)(f(x_k) - f(x^*)) + \frac{L^h_p R^{p+1}}{(p + 1)!} \alpha^{p+1}.
\]

For simplicity, we denote \( \Delta_k = f(x_k) - f(x^*) \). We consider two cases in (3.5) (see also [19, 20]):

First case: if \( \Delta_k > \frac{L^h_p R^{p+1}}{p!} \), then optimal point is \( \alpha^* = 1 \) and \( \Delta_{k+1} \leq \frac{L^h_p R^{p+1}}{p!} \).

Second case: if \( \Delta_k \leq \frac{L^h_p R^{p+1}}{p!} \), then optimal point is \( \alpha^* = \sqrt{\frac{\Delta_k p!}{R^{p+1} L^h_p}} \) and obtain:

\[
\Delta_{k+1} \leq \Delta_k \left( 1 - c \Delta_k^{\frac{1}{p}} \right), \quad \Delta_k^{\frac{1}{p}} \leq \Delta_k^{\frac{1}{p}} \left( 1 - c \Delta_k^{\frac{1}{p}} \right)^{-\frac{1}{p}},
\]

\[
\Delta_k^{\frac{1}{p}} \geq \Delta_k^{\frac{1}{p}} \left( 1 + \frac{c}{p} \Delta_k^{\frac{1}{p}} \right) = \Delta_k^{\frac{1}{p}} + \frac{c}{p},
\]

\[11\]
where \( c = \frac{1}{p+1} \sqrt{\frac{p!}{R_p + 1}} \) and the last inequality is given by \((1 - x)^{-p} \geq 1 + px\) for \( x \in [0, 1] \), see e.g., [23]. We now apply recursively the previous inequalities, starting with \( k = 1 \). If \( \Delta_0 \geq \frac{L_p^h R_p + 1}{p+1} \), we are in the first case and then \( \Delta_1 \leq \frac{L_p^h R_p + 1}{(p+1)!} \). Then, we will subsequently be in the second case for all \( k \geq 2 \) and

\[
\Delta_k \leq \Delta_1 \left( 1 + \frac{(k-1)c}{p} \Delta_1^\frac{1}{k} \right)^{-p} \leq \frac{L_p^h R_p + 1}{(p+1)!} \left( 1 + \frac{(k-1) \Delta_1^\frac{1}{k}}{p+1} \right)^{-p}.
\]

Otherwise, if \( \Delta_0 \leq \frac{L_p^h R_p + 1}{p} \), then we are in the second case and obtain:

\[
\Delta_k \leq \Delta_0 \left( 1 + \frac{k c}{p} \Delta_0^\frac{1}{k} \right)^{-p} \leq \frac{L_p^h R_p + 1}{p!} \left( 1 + \frac{k}{p+1} \right)^{-p}.
\]

These prove the statement of the theorem. \( \square \)

Note that the convergence results from [19, 20] assume Lipschitz continuity of the \( p \) derivative of the objective function \( f \), while Theorem 3.1 assumes Lipschitz continuity of the \( p \) derivative of the error function \( h = g - f \). Hence, our proof is different from [19, 20]. Moreover, our convergence rate (3.1) recovers the usual convergence rates \( O(1/k^p) \) of higher-order Taylor-based methods in the unconstrained convex case [19] (Example 2.3) and composite convex case [20] (Example 2.5), respectively. Therefore, Theorem 3.1 provides a unified convergence analysis for higher-order majorization-minimization algorithms, that covers in particular (composite) convex problems, under possibly more general assumptions than in [19, 20]. In fact there is a major difference between the Taylor expansion approach from [19, 20] and the model approximation based on our general majorization-minimization approach. Taylor expansion yields unique approximation model around a given point while in the majorization-minimization approach one may consider many upper bound models and every model leads to a different optimization method.

### 3.2. Local superlinear convergence of GHOM

Next, by assuming uniform convexity on \( f \), we prove that GHOM can achieve faster rates locally. More precisely we get local superlinear convergence rates for GHOM in several optimality criteria: function values, distance of the iterates to the optimal point and in the norm of minimal subgradients. For this we first need some auxiliary results.

**Lemma 3.2.** Let the assumptions of Theorem 3.1 hold. Then, there exist subgradients \( f^{x_{k+1}} \in \partial f(x_{k+1}) \), where the sequence \((x_k)_{k \geq 0}\) is generated by the algorithm GHOM, such that the following relation holds:

\[
\|f^{x_{k+1}}\|_* \leq \frac{L_p^h}{p!} \|x_{k+1} - x_k\|^p \quad \forall k \geq 0.
\]

**Proof.** From the definition of the \( p \) higher-order surrogate function, we know that the error function \( h \) has the \( p \) derivative Lipschitz with constant \( L_p^h \). Thus, we have the inequality (1.5) for \( y = x_{k+1} \), that is:

\[
\|\nabla h(x_{k+1}; x_k) - \nabla T^h_p(x_{k+1}; x_k)\|_* \leq \frac{L_p^h}{p!} \|x_{k+1} - x_k\|^p.
\]

Since the Taylor approximation of \( h \) at \( x_k \) of order \( p \), \( T^h_p(y; x_k) \), is zero, we get:

\[
\|\nabla h(x_{k+1}; x_k)\|_* \leq \frac{L_p^h}{p!} \|x_{k+1} - x_k\|^p.
\]
Further, from the optimality of the point $x_{k+1}$ we have that $0 \in \partial g(x_{k+1}; x_k)$. Thus, since the error function $h(y; x_k) = g(y; x_k) - f(y)$ is differentiable, we obtain from calculus rules [27] that:

$$-\nabla h(x_{k+1}; x_k) \in \partial f(x_{k+1}).$$

Returning with this relation in the previous inequality, we obtain (3.6) by simply defining $f^{k+1} = -\nabla h(x_{k+1}; x_k)$.

**Lemma 3.3.** Let $f$ be proper lower semicontinuous convex function and admitting a $p \geq 1$ higher-order surrogate function $g(y; x)$ at each point $x \in \text{dom} f$ as given in Definition 2.1 such that the error function $h = g - f$ is convex. Then, there exist subgradients $f^{k+1} \in \partial f(x_{k+1})$, where the sequence $(x_k)_{k \geq 0}$ is generated by the algorithm GHOM, such that the following relation holds:

$$\langle f^{x_{k+1}}, x_k - x_{k+1} \rangle \geq 0 \quad \forall k \geq 0.$$ (3.7)

**Proof.** Since the error function $h$ is $p \geq 2$ differentiable satisfying $h(y; x_k) \geq 0$ and $h(x_k; x_k) = 0$, then using further the convexity of $h(\cdot; x_k)$, we have:

$$0 = h(x_k; x_k) \geq h(x_k+1; x_k) + \langle \nabla h(x_k+1; x_k), x_k - x_{k+1} \rangle$$

for $h \geq 0$. From Lemma 3.2 we have $-\nabla h(x_k+1; x_k) \in \partial f(x_{k+1})$. We get our statement by simply defining $f^{x_{k+1}} = -\nabla h(x_{k+1}; x_k)$. □

Now we are ready to prove the superlinear convergence of the GHOM algorithm in function values for general uniformly convex objective functions.

**Theorem 3.4.** Let $f$ be uniformly convex function of degree $q \geq 2$ with constant $\sigma_q$ and admitting a $p \geq 1$ higher-order surrogate function $g(\cdot; x)$ at each point $x \in \text{dom} f$ as given in Definition 2.1 such that the error function $h = g - f$ is convex. Then, the sequence $(x_k)_{k \geq 0}$ generated by algorithm GHOM has the following convergence rate in function values:

$$f(x_{k+1}) - f(x^*) \leq (q-1)q^{-\frac{p+1}{q-p+1}} \left( \frac{1}{\sigma_q} \right)^{\frac{q}{2}} \left( \frac{L^h_p}{p!} \right)^{\frac{q}{q-p+1}} (f(x_k) - f(x^*))^{\frac{q}{q-p+1}}.$$ (3.8)

**Proof.** For any $k \geq 0$ and $f^{x_{k+1}} \in \partial f(x_{k+1})$, we have:

$$f(x_k) - f(x^*) \geq f(x_k) - f(x_{k+1})$$

(1.7)

$$\geq \langle f^{x_{k+1}}, x_k - x_{k+1} \rangle + \frac{\sigma_q}{q} \|x_k - x_{k+1}\|^q$$

(3.7)

$$\geq \frac{\sigma_q}{q} \|x_k - x_{k+1}\|^q \geq \frac{\sigma_q}{q} \left( \frac{p!}{L^p_h} \|f^{x_{k+1}}\|_p \right)^{\frac{q}{p}}$$

(3.6)

$$\geq \frac{\sigma_q}{q} \left( \frac{p!}{L^p_h} \right)^{\frac{q}{p}} \left( \frac{q\sigma_q^{\frac{q}{q-p+1}}}{q-1} (f(x_{k+1}) - f(x^*)) \right)^{\frac{q}{q-p+1}}.$$ (3.8)

which proves the statement of the theorem. □
Note that if \( p > q - 1 \), then GHOM has local superlinear convergence rate since \( \frac{p}{q-1} > 1 \) and from Theorem 3.1 we have \( f(x_k) - f(x^*) \to 0 \). E.g., if \( q = 2 \) (strongly convex function) and \( p = 2 \), then the local rate of convergence is quadratic. If \( q = 2 \) and \( p = 3 \), then the local rate of convergence is cubic. If \( q = 2 \) and \( p = 1 \) we recover the usual linear convergence rate, etc. Note that by choosing appropriately \( M_p \) and \( \mathcal{H} \) in Examples 2.2, 2.3, 2.5 and 2.6, we indeed obtain error functions \( h \) that are convex. However, if we remove the convexity assumption on \( h \) we can still prove local superlinear convergence for GHOM in function values, but the rate is slightly worse. This result is stated next.

**Theorem 3.5.** Let \( f \) be uniformly convex function of degree \( q \in [2, p + 1] \) with constant \( \sigma_q \) and admitting a \( p \) surrogate function \( g(\cdot; x) \) at each point \( x \in \text{dom} f \) as given in Definition 2.1. Then, the sequence \( (x_k)_{k \geq 0} \) generated by algorithm GHOM has the following local superlinear convergence rate:

\[
(3.9) \quad f(x_{k+1}) - f(x^*) \leq \frac{L_p^h}{(p+1)!} \left( \frac{q}{\sigma_q} \right)^{\frac{p+1}{q}} (f(x_k) - f(x^*))^{\frac{p+1}{q}}.
\]

**Proof.** If \( f \) is uniformly convex, then it has a unique optimal point \( x^* \). Moreover, from (3.3), we have:

\[
(3.10) \quad f(x_{k+1}) \leq \min_{y \in \text{dom} f} (f(y)) + \frac{L_p^h}{(p+1)!} \|y - x_k\|^{p+1}
\]

On the other hand, since \( f \) is uniformly convex of degree \( q \), then using (1.7) for \( f \) and the fact that \( 0 \in \partial f(x^*) \), we get:

\[
(3.11) \quad f(x_k) - f(x^*) \geq \frac{\sigma_q}{q} \|x_k - x^*\|^q.
\]

Combining the inequalities (3.10) and (3.11), we further obtain:

\[
f(x_{k+1}) - f(x^*) \leq \frac{L_p^h}{(p+1)!} \|x_k - x^*\|^{p+1} \leq \frac{L_p^h}{(p+1)!} \left( \frac{q}{\sigma_q} (f(x_k) - f(x^*)) \right)^{\frac{p+1}{q}}
\]

\[
= \left( \frac{L_p^h}{(p+1)!} \left( \frac{q}{\sigma_q} \right)^{\frac{p+1}{q}} (f(x_k) - f(x^*))^{\frac{p+1}{q}} - 1 \right) (f(x_k) - f(x^*)).
\]

If \( q < p + 1 \), we have that \( \beta_k = \frac{L_p^h}{(p+1)!} \left( \frac{q}{\sigma_q} \right)^{\frac{p+1}{q}} (f(x_k) - f(x^*))^{\frac{p+1}{q}} - 1 \) converges to zero since from Theorem 3.1 we have \( f(x_k) - f(x^*) \to 0 \), thus proving the local superlinear convergence in function values of the sequence \( (x_k)_{k \geq 0} \) generated by GHOM. \( \square \)

**Remark 3.6.** Note that using the inequalities (3.10) and (3.11) in the convergence rates (3.8) and (3.9), respectively, we immediately obtain local superlinear convergence also for \( \|x_k - x^*\| \). Since the derivations are straightforward, we omit them.

Finally, we show local superlinear convergence for the sequence of minimal norms of subgradients of \( f \), which we recall that we denoted by:

\[
S(x_k) := \text{dist}(0, \partial f(x_k)) = \inf_{f^x \in \partial f(x_k)} \|f^x\|_*
\]
Theorem 3.7. Under the assumptions of Theorem 3.5 the sequence \((x_k)_{k \geq 0}\) generated by GHOM has the following convergence rate:

\[
S(x_{k+1}) \leq \frac{L_p^h}{p!} \left( \frac{q}{\sigma_q} \right)^{\frac{q}{q-1}} S(x_k)^{\frac{q}{q-1}}.
\]

Proof. Since GHOM is a descent method and \(f\) is uniformly convex, we have:

\[
f(x_k) \geq f(x_{k+1}) \geq f(x_k) + \langle f^{x_k}, x_{k+1} - x_k \rangle + \frac{\sigma_q}{q} \|x_{k+1} - x_k\|^q,
\]

where \(f^{x_k} \in \partial f(x_k)\). Using the Cauchy-Schwarz inequality we further get:

\[
0 \geq \langle f^{x_k}, x_{k+1} - x_k \rangle + \frac{\sigma_q}{q} \|x_{k+1} - x_k\|^q \geq -\|f^{x_k}\|_* \|x_{k+1} - x_k\| + \frac{\sigma_q}{q} \|x_{k+1} - x_k\|^q,
\]

or equivalently

\[
\|f^{x_k}\|_* \geq \frac{\sigma_q}{q} \|x_{k+1} - x_k\|^{q-1} \quad \forall k \geq 0.
\]

Now, since \(\partial f(x_k)\) is compact set, taking \(f^{x_k}\) such that \(\|f^{x_k}\|_* = S(x_k)\) and then using Lemma 3.2, we get:

\[
S(x_{k+1}) \leq \|f^{x_{k+1}}\| \leq \frac{L_p^h}{p!} \|x_{k+1} - x_k\|^p \leq \frac{L_p^h}{p!} \left( \frac{q}{\sigma_q} \|f^{x_k}\|_* \right)^{\frac{q}{q-1}} \leq \frac{L_p^h}{p!} \left( \frac{q}{\sigma_q} \right)^{\frac{q}{q-1}} S(x_k)^{\frac{q}{q-1}},
\]

which proves the statement of the theorem.

In the next section (see Theorem 4.2) we prove that the sequence of (sub)gradients generated by the GHOM algorithm converges to zero globally for convex problems, i.e. \(S(x_k) \to 0\) as \(k \to \infty\). Hence, since for \(q \in [2, p+1]\) we have \(\frac{q}{q-1} > 1\), we conclude from Theorem 3.7 that the sequence of minimal norms of subgradients generated by GHOM has also local superlinear convergence rate in the convex case.

4. Convergence analysis of GHOM for nonconvex optimization. In this section we analyze the convergence behavior of algorithm GHOM for general or structured nonconvex optimization. Note that in the nonconvex case we only assume that \(x_{k+1}\) is a stationary point of the subproblem \((2.5)\) satisfying the descent \((2.6)\).

4.1. Global convergence of GHOM for general nonconvex optimization. We assume a general (possibly nonconvex) proper lower semicontinuous objective function \(f : E \to \mathbb{R}\). Recall that even in this general setting a necessary first-order optimality condition for a (local) optimum \(x^*\) of \(f\) is to have \(0 \in \partial f(x^*)\). Now we are ready to analyze the convergence behavior of GHOM under these general settings. We first derive some auxiliary result:

Lemma 4.1. Let \(\tilde{h}\) be a function \(p \geq 2\) differentiable and with the \(p\) derivative smooth with constant \(L_p^h\). For any \(x, y \in \text{dom} f\) and scalar \(M_p^h \geq L_p^h\) let us define:

\[
H = \nabla^2 \tilde{h}(x) + \sum_{i=3}^{p} \frac{1}{(i-1)!} \nabla^i \tilde{h}(x)(y-x)^{i-2} + \frac{M_p^h}{p!} \|y-x\|^{p-1} D.
\]
Then, we have the following bounds on the matrix $H$:

$$
\int_0^1 \nabla^2 \tilde{h}(x + \tau (y - x))d\tau \leq H
$$

$$
\leq \int_0^1 \left( \nabla^2 \tilde{h}(x + \tau (y - x)) + \frac{M^\tilde{h}_p + L^\tilde{h}_p}{(p - 1)!} \tau^{p-1} \|y - x\|^{p-1}D \right) d\tau.
$$

**Proof.** We note that:

$$
H = \int_0^1 \left( \nabla^2 \tilde{\tilde{h}}(x) + \sum_{i=3}^{p} \frac{\tau^{i-2}}{(i-2)!} \nabla^i \tilde{h}(x) \|y - x\|^{i-2} + \frac{M^\tilde{h}_p}{(p - 1)!} \|y - x\|^{p-1}D \right) d\tau
$$

$$
= \int_0^1 \left( \nabla^2 \tilde{\tilde{h}}(x) + \frac{M^\tilde{h}_p}{(p - 1)!} \|y - x\|^{p-1}D \right) d\tau
$$

$$
\leq \int_0^1 \left( \nabla^2 \tilde{\tilde{h}}(x) + \frac{M^\tilde{h}_p + L^\tilde{h}_p}{(p - 1)!} \|y - x\|^{p-1}D \right) d\tau.
$$

Similarly, we have:

$$
H \geq \int_0^1 \left( \nabla^2 \tilde{\tilde{h}}(x + \tau (y - x)) + \frac{M^\tilde{h}_p - L^\tilde{h}_p}{(p - 1)!} \|y - x\|^{p-1}D \right) d\tau
$$

$$
\geq \int_0^1 \nabla^2 \tilde{\tilde{h}}(x + \tau (y - x))d\tau,
$$

since $M^\tilde{h}_p \geq L^\tilde{h}_p$ and $D \succeq 0$. These conclude our statement. \qed

**Theorem 4.2.** Let us assume that the objective function $f : \mathbb{E} \rightarrow \mathbb{R}$ is proper, lower semicontinuous, (possibly nonconvex) and admits $p \geq 1$ higher-order surrogate function $g(\cdot; x)$ at any $x \in \text{dom} f$ as given in Definition 2.1. Then, $(f(x_k))_{k \geq 0}$ monotonically decreases and the sequence $(x_k)_{k \geq 0}$ is bounded and satisfies the asymptotic stationary point condition $S(x_k) \rightarrow 0$ as $k \rightarrow \infty$.

**Proof.** Using the properties of the surrogate (see Definition 2.1) and the descent property (2.6), we obtain:

$$
f(x_k) = g(x_k; x_k) \geq g(x_{k+1}; x_k) \geq f(x_{k+1}) \quad \forall k \geq 0.
$$

This relation guarantees that $(f(x_k))_{k \geq 0}$ is nonincreasing sequence and thus convergent, since $f$ is assumed to be bounded from below by $f^*$. Moreover, since we also assume the level set $L_f(x_0)$ bounded (see Section 2), then it follows that the sequence generated by GHOM $(x_k)_{k \geq 0} \subset L_f(x_0)$ is also bounded. Using further the definition of the error function $h$:

$$
(4.1) \quad 0 \leq h(x_{k+1}; x_k) = g(x_{k+1}; x_k) - f(x_{k+1}) \leq f(x_k) - f(x_{k+1}).
$$

Telescoping the previous relation for $k = 0 : \infty$, we get:

$$
0 \leq \sum_{k=0}^{\infty} h(x_{k+1}; x_k) \leq f(x_0) - f^* < \infty.
$$
Therefore, the positive term of the series, \((h(x_{k+1}; x_k))_{k\geq 0}\), necessarily converges to 0. Since \(g = h + f\) is also proper and lower semicontinuous, then from \(x_{k+1}\) being a stationary point w.r.t. \(g\) we have \(0 \in \partial g(x_{k+1}; x_k)\) the (limiting) subdifferential of \(g(\cdot; x_k)\) at \(x_{k+1}\). Since \(h\) is differentiable, from calculus rules we also have that:

\[
(4.2) \quad -\nabla h(x_{k+1}; x_k) \in \partial f(x_{k+1}).
\]

For proving the asymptotic stationary point condition we consider two cases: \(p \geq 2\) and \(p = 1\). For \(p \geq 2\) let us define the function \(\tilde{h}(y) = h(y; x_k)\), which, according to the Definition 2.1, has the \(p\) derivative smooth. Further, let us choose the point:

\[
y_{k+1} = \arg \min_{y \in \mathbb{R}} T^h_p(y; x_{k+1}) + \frac{M^h_p}{(p+1)!} \|y - x_{k+1}\|^{p+1},
\]

where \(M^h_p > L^h_p\). It is important to note that in practice we do not need to compute \(y_{k+1}\). Using the (global) optimality of \(y^{k+1}\), we have:

\[
T^h_p(y_{k+1}; x_{k+1}) + \frac{M^h_p}{(p+1)!} \|y_{k+1} - x_{k+1}\|^{p+1}
\leq T^h_p(x_{k+1}; x_{k+1}) + \frac{M^h_p}{(p+1)!} \|x_{k+1} - x_{k+1}\|^{p+1}
\]

\[
= h(x_{k+1}; x_k) + \sum_{i=1}^{p} \frac{1}{i!} \nabla^i h(x_{k+1}; x_k) [x_{k+1} - x_k]^i = h(x_{k+1}; x_k).
\]

Moreover, writing explicitly the left term of the previous inequality, we get:

\[
h(x_{k+1}; x_k) + \sum_{i=1}^{p} \frac{1}{i!} \nabla^i h(x_{k+1}; x_k) [y_{k+1} - x_{k+1}]^i + \frac{M^h_p}{(p+1)!} \|y_{k+1} - x_{k+1}\|^{p+1}
\]

\[
\leq h(x_{k+1}; x_k).
\]

Thus, we have:

\[
(4.3) \quad \sum_{i=1}^{p} \frac{1}{i!} \nabla^i h(x_{k+1}; x_k) [y_{k+1} - x_{k+1}]^i + \frac{M^h_p}{(p+1)!} \|y_{k+1} - x_{k+1}\|^{p+1} \leq 0.
\]

From relation (1.4) we also obtain:

\[
\tilde{h}(y) = h(y; x_k) \leq T^h_p(y; x_{k+1}) + \frac{L^h_p}{(p+1)!} \|y - x_{k+1}\|^{p+1} \quad \forall y.
\]

We rewrite this relation for our chosen point \(y_{k+1}\):

\[
h(y_{k+1}; x_k) \leq T^h_p(y_{k+1}; x_{k+1}) + \frac{L^h_p}{(p+1)!} \|y_{k+1} - x_{k+1}\|^{p+1}
\]

\[
= h(x_{k+1}; x_k) + \sum_{i=1}^{p} \frac{1}{i!} \nabla^i h(x_{k+1}; x_k) [y_{k+1} - x_{k+1}]^i + \frac{L^h_p}{(p+1)!} \|y_{k+1} - x_{k+1}\|^{p+1}
\]

\[
(4.3) \quad \leq h(x_{k+1}; x_k) - \frac{M^h_p - L^h_p}{(p+1)!} \|y_{k+1} - x_{k+1}\|^{p+1}.
\]
Recalling that $h$ is nonnegative, then it follows that:

\[(4.4)\quad 0 \leq h(x_{k+1}; x_k) - \frac{M_p^h - L_p^h}{(p+1)!} \|y_{k+1} - x_{k+1}\|^{p+1}.
\]

This leads to:

\[h(x_{k+1}; x_k) \geq \frac{M_p^h - L_p^h}{(p+1)!} \|y_{k+1} - x_{k+1}\|^{p+1} \geq 0.
\]

Since $(h(x_{k+1}; x_k))_{k \geq 0}$ converges to 0, then necessarily $(y_{k+1} - x_{k+1})_{k \geq 0}$ converges to 0 and is bounded, since $h$ is continuous and $(x_k)_{k \geq 0}$ is bounded. Consequently, the sequence $(y_k)_{k \geq 0}$ is also bounded. Moreover, from the optimality conditions for $y_{k+1}$, we have:

\[(4.5)\quad \nabla h(x_{k+1}; x_k) + H_{k+1}[y_{k+1} - x_{k+1}] = 0,
\]

where we denote the matrix

\[(4.6)\quad H_{k+1} = \nabla^2 h(x_{k+1}; x_k) + \sum_{i=3}^{p} \frac{1}{(i-1)!} \nabla^i h(x_{k+1}; x_k)[y_{k+1} - x_{k+1}]^{i-2}
\]

\[+ \frac{M_p^h}{p!} \|y_{k+1} - x_{k+1}\|^{p-1} D.
\]

From Lemma 4.1 we have that:

\[(4.7)\quad \int_0^1 \nabla^2 h(x_{k+1} + \tau(y_{k+1} - x_{k+1}); x_k) d\tau \leq H_{k+1}
\]

\[\leq \int_0^1 \left( \nabla^2 h(x_{k+1} + \tau(y_{k+1} - x_{k+1}); x_k) + \frac{M_p^h + L_p^h}{(p+1)!} \tau^{p-1} \|y_{k+1} - x_{k+1}\|^{p-1} D \right) d\tau.
\]

Since the sequences $(x_k)_{k > 0}$ and $(y_k)_{k > 0}$ are bounded and $h$ is $p \geq 2$ times continuously differentiable, then $\nabla^2 h(x_{k+1} + \tau(y_{k+1} - x_{k+1}); x_k)$ is bounded for $\tau \in [0, 1]$. Moreover, $y_{k+1} - x_{k+1} \to 0$ as $k \to \infty$ and it is bounded. Therefore, $H_{k+1}$ is bounded and consequently from (4.5) it follows that

\[(4.8)\quad \nabla h(x_{k+1}; x_k) \to 0 \text{ as } k \to \infty.
\]

For the case $p = 1$ we can just take $y_{k+1} = x_{k+1} - 1/L_1^h \nabla h(x_{k+1}; x_k)$. Then, using that $h(\cdot; x_k)$ has gradient Lipschitz with constant $L_1^h$ we obtain [18]:

\[0 \leq h(y_{k+1}; x_k) \leq h(x_{k+1}; x_k) - \frac{1}{2L_1^h} \|\nabla h(x_{k+1}; x_k)\|^2,
\]

which further yields

\[\frac{1}{2L_1^h} \|\nabla h(x_{k+1}; x_k)\|^2 \leq h(x_{k+1}; x_k) \to 0 \text{ as } k \to \infty,
\]

since we have already proved that the sequence $h(x_{k+1}; x_k)$ converges to zero. Therefore, also in the case $p = 1$ we have (4.8) valid. Finally, using (4.8) in (4.2), it follows that:

\[0 \leq S(x_{k+1}) = \inf_{f^{x_{k+1}} \in \partial f(x_{k+1})} \|f^{x_{k+1}}\|_* \leq \|\nabla h(x_{k+1}; x_k)\|_* \to 0 \text{ as } k \to \infty,
\]

i.e. the sequence $(x_k)_{k \geq 0}$ satisfies the asymptotic stationary point condition for the general nonconvex problem (2.1).
Note that the main difficulty in the previous proof is to handle \( h \) having \( p \geq 2 \) derivative smooth. We overcome this difficulty by introducing a new sequence \((y_k)_{k \geq 0}\) and proving that it has similar properties as the sequence \((x_k)_{k \geq 0}\) generated by GHOM. The previous result proves only asymptotic convergence for \( S(x_k) \). Therefore, in the nonconvex settings the requirements from Definition 2.1 on the surrogate function \( g \) do not seem to be reach enough to enable convergence rates. However, it is well-known that for unconstrained smooth problems the surrogate from Example 2.3 allows to derive convergence rates \( O\left(k^{-\frac{p}{p+1}}\right) \) for \( \|\nabla f(x_k)\| \), see e.g., [5]. At a closer look one can notice that the surrogate \( g \) of Example 2.3 induces the following inequality on the error function \( h \):

\[
h(y; x) = T_p^f(y; x) + \frac{M_p}{(p+1)!} \|y - x\|^{p+1} - f(y) \geq \frac{M_p - L_p}{(p+1)!} \|y - x\|^{p+1} \quad \forall x, y \in E,
\]

where \( M_p > L_p \). In fact, using the same reasoning as before, it is easy to see that such a relation holds for all the surrogate functions from Examples 2.2, 2.3, 2.5 and 2.6. Hence, if we additionally assume that for some \( \Delta > 0 \) our surrogate function satisfies the following inequality in terms of the error function:

\[
(4.9) \quad \Delta \|y - x\|^{p+1} \leq h(y; x) \quad \forall x, y \in \text{dom } f,
\]

then we can strengthen the results from Theorem 4.2, i.e. we get convergence rates.

**Theorem 4.3.** Let the assumptions from Theorem 4.2 hold. Additionally, assume that our surrogate function satisfies along the iterations of GHOM the relation (4.9). Then, the sequence \((x_k)_{k \geq 0}\) generated by GHOM satisfies the following convergence rate in terms of first-order optimality conditions:

\[
\min_{i=1,k} S(x_i) \leq \frac{L_p^h}{p!} \left( \frac{(f(x_0) - f^*)}{k \cdot \Delta} \right)^{\frac{p}{p+1}}.
\]

**Proof.** Since the error function \( h \) has the \( p \) derivative Lipschitz, we have:

\[
\|\nabla h(x_{k+1}; x_k) - \nabla T_p^h(x_{k+1}; x_k)\|_* \leq \frac{L_p^h}{p!} \|x_{k+1} - x_k\|^p.
\]

Using that \( \nabla T_p^h(x_{k+1}; x_k) = 0 \) (according to Definition 2.1 (iii)), we get:

\[
\|\nabla h(x_{k+1}; x_k)\| \leq \frac{L_p^h}{p!} \|x_{k+1} - x_k\|^p.
\]

Combining this relation with our assumptions, we further obtain:

\[
f(x_k) - f(x_{k+1}) \overset{(4.1)}{\geq} h(x_{k+1}; x_k) \overset{(4.9)}{\geq} \Delta \|x_{k+1} - x_k\|^{p+1}
\]

\[
= \Delta \left( \|x_{k+1} - x_k\|^p \right)^{\frac{p+1}{p}} \geq \Delta \left( \frac{p!}{L_p^h} \|\nabla h(x_{k+1}; x_k)\|_* \right)^{\frac{p+1}{p}}.
\]

Telescoping from \( i = 0 : k - 1 \) the previous inequality and using that \( -\nabla h(x_{k+1}; x_k) \in \)
\[ \partial f(x_{k+1}) \text{ (see (4.2))}, \] we further get:
\[
f(x_0) - f^* \geq \Delta \sum_{i=0}^{k-1} \left( \frac{p^i L_p^i}{\lambda_p^i} \| \nabla h(x_{i+1}; x_i) \| \right)^{\frac{p+1}{p}}
\]
\[
\geq \Delta \left( \frac{p^i L_p^i}{\lambda_p^i} \right)^{\frac{p+1}{p}} \sum_{i=0}^{k-1} S(x_{i+1})^{\frac{p+1}{p}}
\]
\geq k \cdot \Delta \left( \frac{p^i L_p^i}{\lambda_p^i} \right)^{\frac{p+1}{p}} \min_{i=0:k-1} S(x_{i+1})^{\frac{p+1}{p}}.
\]

After rearranging the terms, we obtain our statement.

Note that similar convergence rates as in Theorem 4.3 have been obtained in [5] for unconstrained problems and recently for problems with simple constraints [6] using the surrogate of Example 2.3. The result of Theorem 4.3 is more general as it covers more complicated objective functions (e.g., general composite models) and many other types of surrogate functions (see Examples 2.2, 2.3, 2.5 and 2.6). Further, for nonconvex unconstrained problems we can further derive convergence rates for the sequence \((x_k)_{k \geq 0}\) generated by GHOM in terms of first- and second-order optimality criteria. We use the notation \(a^+ = \max(a, 0)\).

**Theorem 4.4.** Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a \( p > 1 \) times differentiable function with smooth \( p \) derivative having Lipschitz constant \( L_p^f \). In this case, according to Example 2.3, we can consider the surrogate function:
\[ g(y; x) = T_p^f (y; x) + \frac{M_p}{(p+1)!} \| y - x \|^{p+1}, \]

with \( M_p > L_p^f \). Additionally assume that \( x_{k+1} \) is a local minimum of subproblem (2.5). Then, the sequence \((x_k)_{k \geq 0}\) generated by GHOM satisfies the following convergence rate in terms of first and second-order optimality conditions:

\[
\min_{i=1:k} \max \left( \left( -\lambda_{\min}^{-1} \left( \nabla^2 f(x_i) \right) \right)^{\frac{p+1}{p}} \lambda_{\max}^{\frac{p+1}{p}} (D) \left( \frac{M_p + L_p^f}{(p+1)!} \| \nabla f(x_i) \| \right)^{\frac{p+1}{p}} \right)
\]
\[
\leq \frac{p(p+1)}{(p-1)!} \frac{(M_p + L_p^f)^{\frac{p+1}{p+1}}}{M_p - L_p^f} \cdot \lambda_{\max}^{\frac{p+1}{p+1}} (D) \cdot \frac{k}{k} \cdot (f(x_0) - f^*).
\]

**Proof.** From the descent property (2.6), we have: \( g(x_k; x_k) \geq g(x_{k+1}; x_k) \). Further, taking into account the property (iii) of the surrogate function, we get:
\[
f(x_k) \geq f(x_k) + \sum_{i=1}^{p} \frac{1}{i!} \nabla^i f(x_k) |x_{k+1} - x_k|^i + \frac{M}{(p+1)!} ||x_{k+1} - x_k||^{p+1}.
\]

If we define \( r_k = ||x_{k+1} - x_k|| \), then we further get:
\[
- \frac{M}{(p+1)!} r_{k+1}^{p+1} \geq \sum_{i=1}^{p} \frac{1}{i!} \nabla^i f(x_k) |x_{k+1} - x_k|^i.
\]
\[\]
In view of relation (1.4), we have:
\[ f(x_{k+1}) = T_p^f(x_{k+1}; x_k) + \frac{L_p f}{(p+1)!} r_{k+1}^{p+1} \]
\[ \leq f(x_k) + \sum_{i=1}^{p+1} \nabla_i f(x_k)[x_{k+1} - x_k] + \frac{L_p f}{(p+1)!} r_{k+1}^{p+1} \]
\[ \leq f(x_k) - \frac{M_p - L_p f}{(p+1)!} r_{k+1}^{p+1}. \]

Hence, we obtain the following descent relation:
\[ f(x_k) - f(x_{k+1}) \geq \frac{M_p - L_p f}{(p+1)!} r_{k+1}^{p+1}. \]

Further, from the optimal conditions for \( x_{k+1} \) we obtain:
\[ \nabla g(x_{k+1}; x_k) = \nabla T_p^f(x_{k+1}; x_k) + \frac{M_p}{p!} ||x_{k+1} - x_k||^{p-1} D(x_{k+1} - x_k) = 0. \]

Using inequality (1.5) for \( f \), we get:
\[ ||\nabla f(x_{k+1}) - \nabla T_p^f(x_{k+1}; x_k)||_* \leq \frac{L_p}{p!} ||x_{k+1} - x_k||^p. \]

This yields the following relation:
\[ ||\nabla f(x_{k+1})||_* \leq ||\nabla T_p^f(x_{k+1}, x_k)||_* + \frac{L_p f}{p!} r_{k+1}^p \]
\[ \leq || - \frac{M_p}{p!} r_{k+1}^{p-1} D(x_{k+1} - x_k)||_* + \frac{L_p f}{p!} r_{k+1}^p \leq \frac{M_p + L_p f}{p!} r_{k+1}^p \]

or equivalently
\[ \left( \frac{p!}{M_p + L_p f} \right)^{p+1} \frac{r_{k+1}^{p+1}}{||\nabla f(x_{k+1})||_*^p} \leq r_{k+1}^{p+1}. \]

Moreover, since we assume that \( x_{k+1} \) is a local minimum of \( g(\cdot; x_k) \), we have:
\[ \nabla^2 g(x_{k+1}; x_k) \succ 0. \]

Computing explicitly the above expression, we obtain:
\[ \nabla^2 T_p^f(x_{k+1}; x_k) + \frac{M_p (p-1)}{p!} r_{k+1}^{p-3} D(x_{k+1} - x_k)(x_{k+1} - x_k)^T D + \frac{M_p}{p!} r_{k+1}^{p-1} D \succ 0. \]

However, from (1.6) we have:
\[ \nabla^2 f(x_{k+1}) + \frac{L_p f}{(p-1)!} r_{k+1}^{p-1} D \succ \nabla^2 T_p^f(x_{k+1}; x_k). \]

Returning with the above relation in (4.14) we get:
\[ \nabla^2 f(x_{k+1}) + \frac{M_p + pL_p f}{p!} r_{k+1}^{p-1} D + \frac{M_p (p-1)}{p!} r_{k+1}^{p-3} D(x_{k+1} - x_k)(x_{k+1} - x_k)^T D \succ 0. \]
Rearranging the terms, we obtain:

\[-\nabla^2 f(x_{k+1}) \leq \frac{M_p + pL_p f(x_{k+1})^p}{p!} r_{k+1}^{p-1} D + \frac{M_p(p-1)}{p!} r_{k+1}^{p-3} D(x_{k+1} - x_k)(x_{k+1} - x_k)^T D \]

\[\leq \frac{M_p + pL_p f(x_{k+1})^p}{p!} r_{k+1}^{p-1} D + \frac{M_p(p-1)}{p!} r_{k+1}^{p-3} 2^2 r_{k+1}^{p-1} D \leq \frac{M_p + L_p f^p}{(p-1)!} r_{k+1}^{p-1} D.\]

Tacking the maximum eigenvalue, we obtain:

\[\lambda_{\text{max}}(-\nabla^2 f(x_{k+1})) \leq \frac{M_p + L_p f^p}{(p-1)!} r_{k+1}^{p-1} \lambda_{\text{max}}(D) \]

or 

\[-\lambda_{\text{min}}(\nabla^2 f(x_{k+1})) \leq \frac{M_p + L_p f^p}{(p-1)!} r_{k+1}^{p-1} \lambda_{\text{max}}(D).\]

Finally, using the notation \(a^+ = \max(a, 0)\) and the fact that if \(a \leq b\) for some \(b \geq 0\), then also \(a^+ \leq b\), the above inequality yields:

\[(4.15) \quad \left(\frac{(p-1)!}{(M_p + L_p f^p)\lambda_{\text{max}}(D)}\right)^{\frac{p+1}{p+2}} (-\lambda_{\text{min}}(\nabla^2 f(x_{k+1})))^{\frac{p+1}{p+2}} \leq r_{k+1}^{p+1}.\]

By combining (4.13) and (4.15) we get the following compact form:

\[\zeta_{k+1} := \max \left( \left(\frac{(p-1)!}{(M_p + L_p f^p)\lambda_{\text{max}}(D)}\right)^{\frac{p+1}{p+2}} (-\lambda_{\text{min}}(\nabla^2 f(x_{k+1})))^{\frac{p+1}{p+2}}, \right.\]

\[\left. \left(\frac{p!}{M_p + L_p f^p}\right)^{\frac{p+1}{p}} \|\nabla f(x_{k+1})\|_r^{\frac{p+1}{p}} \right) \leq r_{k+1}^{p+1}.\]

Telescoping (4.11), we get:

\[f(x_0) - f^* \geq \frac{M_p - L_p f^p}{(p+1)!} \sum_{i=0}^{k-1} r_{i+1}^{p+1} \geq \frac{M_p - L_p f^p}{(p+1)!} \sum_{i=0}^{k-1} \zeta_{i+1} \geq \frac{k(M_p - L_p f^p)}{(p+1)!} \min_{i=1:k} \zeta_i.\]

Rearranging the terms we get the statement of the theorem.

Note that [2, 6, 7] obtains similar convergence results for their Taylor-based algorithm as in Theorem 4.4. However, the update \(x_{k+1}\) in these papers must usually satisfy some second-order optimality conditions. E.g., in [7] the authors require:

\[\max \{0, -\lambda_{\text{min}}(\nabla^2 g(x_{k+1}; x_k)) \} \leq \theta \|x_{k+1} - x_k\|^{p-1} \quad \text{for some} \ \theta > 0 \quad \text{and} \quad \|\nabla g(x_{k+1}; x_k)\| \leq \theta \|x_{k+1} - x_k\|^p.\]

In the previous theorem we do not require such optimality conditions on \(x_{k+1}\) and thus our proof is different from [2, 6, 7].

### 4.2. Local convergence of GHOM under the KL property.

The KL property was first analyzed in details in [3] and then widely applied to analyze the convergence behavior of various first-order [1, 3] and second-order [11, 28] algorithms for nonconvex optimization. However, to the best of our knowledge there are no studies analyzing the convergence rate of higher-order majorization-minimization algorithms under the KL property. The main difficulty comes from the fact that \(f\) satisfies the KL property while the error function \(h\) is assumed smooth and it is hard to establish connections between them. In the next theorem we connect the geometric property of nonconvex function \(f\) with the smooth property of \(h\) and establish local convergence of GHOM in the full parameter regime of the KL property.

Let us denote the set of limit points of the sequence \((x_k)_{k \geq 0}\) generated by algorithm GHOM with \(\Omega(x_0)\).
Hence, all the conditions of the KL property from Definition 1.8 are satisfied and we taking value $x$ decreases and converges to $f$. On the other hand let 

\[(4.16)\]

Assume also that either $f$ is continuous or $x_{k+1}$ is a local minimum of subproblem (2.5). Then, $\Omega(x_0)$ is compact set and $f$ is constant on $\Omega(x_0)$.

Proof. From Theorem 4.2 we have that the sequence $(x_k)_{k \geq 0}$ is bounded, hence the set of limit points $\Omega(x_0)$ is also bounded. Closedness of $\Omega(x_0)$ also follows by observing that $\Omega(x_0)$ can be viewed as an intersection of closed sets, i.e. $\Omega(x_0) = \cap_{j \geq 0} \overline{\Omega(x_j)}$. Hence, $\Omega(x_0)$ is a compact set and $\text{dist}(x_k, \Omega(x_0)) \to 0$ as $k \to \infty$. Further, let us also show that $f(\Omega(x_0))$ is constant. From Theorem 4.2 we have that $(f(x_k))_{k \geq 0}$ is monotonically decreases and since $f$ is assumed bounded from below by $f^* > -\infty$, it converges, let us say to $f_* > -\infty$, i.e. $f(x_k) \to f_*$ as $k \to \infty$. On the other hand let $x_*$ be a limit point of the sequence $(x_k)_{k \geq 0}$. This means that there is a subsequence $(x_{kj})_{j \geq 0}$ such that $x_{kj} \to x_*$ as $j \to \infty$. If $f$ is continuous, then $\lim_{j \to \infty} f(x_{kj}) = f(x_*)$ and $f(x_*) = \lim_{j \to \infty} f(x_k) = f_*$. If $x_{k+1}$ is a local minimum of (2.5), then from the lower semicontinuity of $f$ we always have:

\[
\liminf_{j \to \infty} f(x_{kj}) \geq f(x_*).
\]

Since we assume $x_{kj}$ to be a local minimum of $g(\cdot;x_{kj-1})$, there exists $\delta_j > 0$ such that $g(x_{kj};x_{kj-1}) \leq g(y;x_{kj-1})$ for all $\|y - x_{kj}\| \leq \delta_j$. As $x_{kj} \to x_*$, then there exists $j_0$ such that for all $j \geq j_0$ we have $\|x_* - x_{kj}\| \leq \delta_j$ and consequently $g(x_{kj};x_{kj-1}) \leq g(x_*;x_{kj-1})$. Since $g = h + f$, we get $h(x_{kj};x_{kj-1}) + f(x_{kj}) \leq h(x_*;x_{kj-1}) + f(x_*)$ for all $j \geq j_0$. Using that $h(\cdot;x_{kj-1})$ is continuous and taking lim sup, we get:

\[
\limsup_{j \to \infty} f(x_{kj}) \leq f(x_*).
\]

Hence, we get $\lim_{j \to \infty} f(x_{kj}) = f(x_*)$ and $f(x_*) = \lim_{j \to \infty} f(x_{kj}) = f_*$. In conclusion, we have $f(\Omega(x_0)) = f_*$. $\square$

From previous lemma we note that all the conditions of the KL property from Definition 1.8 are satisfied.

Theorem 4.6. Let the objective function $f$ be proper and lower semicontinuous (possibly nonconvex), satisfy the KL property (1.9) for some $q > 1$ and admit $p \geq 1$ higher-order surrogate function $g(\cdot;x)$ at any $x \in \text{dom} f$ as given in Definition 2.1. Assume also that either $f$ is continuous or $x_{k+1}$ is a local minimum of subproblem (2.5). Then, the sequence $(x_k)_{k \geq 0}$ generated by GHOM satisfies:

1. If $q > p + 1$, then $f(x_k)$ converges locally to $f_*$ at a superlinear rate.
2. If $q = p + 1$, then $f(x_k)$ converges locally to $f_*$ at a linear rate.
3. If $q < p + 1$, then $f(x_k)$ converges locally to $f_*$ at a sublinear rate.

Proof. Note that the set of limit points $\Omega(x_0)$ of the sequence $(x_k)_{k \geq 0}$ generated by GHOM is compact, $\text{dist}(x_k, \Omega(x_0)) \to 0$ as $k \to \infty$ and $f(\Omega(x_0))$ is constant taking value $f_*$ (see Lemma 4.5). Moreover, we have that $(f(x_k))_{k \geq 0}$ monotonically decreases and converges to $f_*$. Then, for any $\delta, \epsilon > 0$ there exists $k_0$ such that:

\[
x_k \in \{x : \text{dist}(x, \Omega(x_0)) \leq \delta, f_* < f(x) < f_* + \epsilon\} \quad k \geq k_0.
\]

Hence, all the conditions of the KL property from Definition 1.8 are satisfied and we can exploit the KL inequality (1.9). Combining (4.1) with (4.4) we get:

\[
(4.16) \quad \frac{M_p - L_p^h}{(p + 1)!} \|y_{k+1} - x_{k+1}\|^{p+1} \leq f(x_k) - f(x_{k+1}) \quad \forall k \geq 0,
\]

23
for any fixed $M_h^h > L_p^h$. From $x_{k+1}$ being a stationary point of subproblem (2.5), we have $0 \in \partial g(x_{k+1}; x_k)$ (Fermat’s rule [27]) and from $h = g - f$ we get $-\nabla h(x_{k+1}; x_k) \in \partial f(x_{k+1})$. Then, using (4.5) and the definition of $H_{k+1}$ from (4.6), we obtain:

\[
S(x_{k+1}) = \text{dist}(0, \partial f(x_{k+1})) \leq \|\nabla h(x_{k+1}; x_k)\|_*, = \|H_{k+1}(y_{k+1} - x_{k+1})\|_*
\]

(4.17)

where $c = \max_{k \geq 0} \|H_{k+1}\| < \infty$ (see (4.7)). From KL property (1.9), we further get:

\[
f(x_{k+1}) - f_* \leq \sigma_q S(x_{k+1})^q \leq \sigma_q c^q \|y_{k+1} - x_{k+1}\|^q
\]

(4.16)

\[
\leq \sigma_q c^q \left( \frac{(q+1)!}{M_p^h - L_p^h} \right)^\frac{q}{p+1} (f(x_k) - f(x_{k+1})) \frac{1}{p+1} \forall k \geq k_0.
\]

(4.17)

Let us denote $\Delta_k = f(x_k) - f_*$ and $\bar{C} = \sigma_q c^q \left( \frac{(q+1)!}{M_p^h - L_p^h} \right)^\frac{q}{p+1}$. Then, we obtain:

\[
\Delta_{k+1} \leq C(\Delta_k - \Delta_{k+1}) \frac{1}{p+1} \forall k \geq k_0.
\]

(4.18)

If we denote $\Delta_k = C^\frac{p+1}{q+1} \tilde{\Delta}_k$, we further get the recurrence:

\[
\tilde{\Delta}_{k+1}^\frac{p+1}{q+1} \leq \tilde{\Delta}_k - \tilde{\Delta}_{k+1} \forall k \geq k_0.
\]

(4.19)

We distinguish the following cases:

Case (i): $q > p + 1$. Then, from (4.19) we have:

\[
\tilde{\Delta}_{k+1} \leq \frac{1}{1 + \tilde{\Delta}_k^\frac{p+1}{q+1}} \tilde{\Delta}_k,
\]

and since $f(x_k) \to f_*$ it follows that $\tilde{\Delta}_{k+1} \to 0$ and hence $\Delta_{k+1}^\frac{p+1}{q+1} \to \infty$ as $\frac{p+1}{q+1} < 0$. Therefore, in this case $f(x_k)$ locally converges to $f_*$ at a superlinear rate.

Case (ii): $q = p + 1$. Then, from (4.18) we have:

\[
(1 + C)\Delta_{k+1} \leq C\Delta_k,
\]

hence in this case $f(x_k)$ locally converges to $f_*$ at a linear rate.

Case (iii): $q < p + 1$. Then, from (4.19) we have:

\[
\Delta_{k+1}^\frac{p+1}{q+1} \leq \Delta_k - \Delta_{k+1}, \text{ with } \frac{p+1}{q} > 0,
\]

and from Lemma 11 in [20] we have for some constant $\alpha > 0$:

\[
\tilde{\Delta}_k \leq \frac{\tilde{\Delta}_k}{(1 + \alpha (k - k_0))^{\frac{p+1}{q+1}}} \forall k \geq k_0.
\]

Therefore, in this case $f(x_k)$ locally converges to $f_*$ at a sublinear rate. □
5. Conclusions. This paper has explored the convergence behaviour of higher-order majorization-minimization algorithms for minimizing (non)convex functions that admit a surrogate model such that the corresponding error function has a $p \geq 1$ higher-order Lipschitz continuous derivative. Under these settings we derived global convergence results for our algorithm in terms of function values or first-order optimality conditions. Faster local rates of convergence were established under uniform convexity or KL property of the objective function. Moreover, for unconstrained nonconvex problems we derived convergence rates in terms of first- and second-order optimality conditions. As a future direction, it is interesting to extend these results to the stochastic settings, e.g., minimizing finite sum objective functions using a stochastic variant of algorithm GHOM.

6. Appendix. In this appendix, for completeness and to facilitate a better understanding of our theory, we provide the proofs of some known results.

6.1. Smoothness verification. Proof of Example 1.2: This result was given in [26]. For consistency we also provide a proof. Note that we have the following polynomial description of the $q$ derivative of $f$:

\begin{equation}
\nabla^q f_{p+1}(x)[h]^q = \|x - x_0\|^{p+1-q}g_{q,p+1}(\tau_h(x)),
\end{equation}

where $h \in \mathbb{E}$ is an arbitrary unit vector and

$$
\tau_h(x) := \begin{cases} 
\langle D(x-x_0)h, \frac{(x-x_0)}{\|x-x_0\|} \rangle, & \text{if } x \neq x_0 \\
0, & \text{if } x = x_0.
\end{cases}
$$

The polynomial $g_{q,p+1}$ is a combination of the previous polynomial $g_{q-1,p+1}$ and its derivative $g'_{q-1,p+1}$:

$$
g_{q,p+1}(\tau) := (1 - \tau^2) g_{q-1,p+1}(\tau) + (p - q + 2)\tau g_{q-1,p+1}(\tau) \quad \forall q \geq 1.
$$

When $q = 0$, $g_{q,p+1}(\tau)$ is set to 1. For (1.3) to hold it is sufficient to show that:

$$
|\nabla^{p+1} f_{p+1}(x)[h]^{p+1}| \leq (p + 1)! \quad \forall x, h \in \mathbb{E}.
$$

Considering (6.1), we have:

$$
\nabla^{p+1} f_{p+1}(x)[h]^{p+1} = g_{p+1,p+1}(\tau_h(x)).
$$

From Cauchy-Schwartz inequality we obtain that $|\tau_h(x)| \leq 1$ and therefore:

$$
|\nabla^{p+1} f_{p+1}(x)[h]^{p+1}| = |g_{p+1,p+1}(\tau_h(x))| \leq \max_{\tau \in [-1,1]} |g_{p+1,p+1}(\tau)|.
$$

However, by induction we can easily prove that (see also Proposition 4.5 in [26]):

$$
\max_{[-1,1]} |g_{p+1,p+1}| = \prod_{i=0}^{p} (p + 1 - i) = (p + 1)!,
$$

which concludes the statement of Example 1.2.
Proof of Example 1.3: This results has been proved in [22]. Let us denote for simplicity $\kappa(x) = \sum_{i=1}^{m} e^{(a_{i}, x)}$. Then, for all $x \in \mathbb{E}$ and $h \in \mathbb{E}$, we have:

$$\langle \nabla f(x), h \rangle = \frac{1}{\kappa(x)} \sum_{i=1}^{m} e^{(a_{i}, x)} (a_{i}, h),$$

$$\langle \nabla^2 f(x)h, h \rangle = \frac{1}{\kappa(x)} \sum_{i=1}^{m} e^{(a_{i}, x)} (\langle a_i, h \rangle - \langle \nabla f(x), h \rangle)^2 \leq \sum_{i=1}^{m} (a_i, h)^2 = \|h\|^2.$$

Taking maximum over $\|h\| = 1$ in the previous expression we get that $\|\nabla^2 f(x)\| \leq 1$, hence $L_f^1 = 1$. Similarly, for $p = 2$ we have:

$$\nabla^3 f(x)[h]^3 = \frac{1}{\kappa(x)} \sum_{i=1}^{m} e^{(a_{i}, x)} (\langle a_i, h \rangle - \langle \nabla f(x), h \rangle)^3 \leq \langle \nabla^2 f(x), h \rangle \max_{1 \leq i,j \leq m} \langle a_i - a_j, h \rangle \leq 2\|h\|^3.$$

Taking again maximum over $\|h\| = 1$ in the previous expression we obtain that $\|\nabla^3 f(x)\| \leq 2$, hence $L_f^2 = 2$. Finally, for $p = 3$ we have:

$$\nabla^4 f(x)[h]^4 = \frac{1}{\kappa(x)} \sum_{i=1}^{m} e^{(a_{i}, x)} (\langle a_i, h \rangle - \langle \nabla f(x), h \rangle)^4 - 3 \langle \nabla^2 f(x), h \rangle^2 \leq \nabla^3 f(x)[h]^3 \max_{1 \leq i,j \leq m} \langle a_i - a_j, h \rangle \leq 4\|h\|^4.$$

Proceeding as before, i.e. taking maximum over $\|h\| = 1$ in the previous expression, we get that $\|\nabla^4 f(x)\| \leq 4$, hence $L_f^3 = 4$. These prove the statements of Example 1.3.

Proof of Example 1.4: Since $\nabla^p T_f^p(y; x) = \nabla^p f(x)$ (i.e. the $p$ derivative is constant for all $y$), we have:

$$\|\nabla^p T_f^p(y; x) - \nabla^p T_f^p(z; x)\| = \|\nabla^p f(x)\| = 0 \leq L_f^p \|y - z\| \quad \forall y, z.$$

for any $L_f^p \geq 0$. Moreover, the $p$ Taylor approximation of $f$ has also the $p - 1$ derivative Lipschitz with constant $L_f^{p-1} = \|\nabla^{p-1} f(x)\|$. These prove the statements of Example 1.4.

Proof of Lemma 1.5: This key result has been proved in [19]. Note that for any $p \geq 2$ we have:

$$\nabla^2 \left( \frac{1}{p} \|x\|^p \right) = (p - 2)\|x\|^{p-4} D x x^* D + \|x\|^{p-2} D \geq \|x\|^{p-2} D. \tag{6.2}$$

Fixing an arbitrary $x$ and $y$ from $\text{dom } f$, we have for any direction $d \in \mathbb{E}$ the following:

$$\langle (\nabla^2 f(y) - \nabla^2 T_f^p(y; x))d, d \rangle \leq \|\nabla^2 f(y) - \nabla^2 T_f^p(y; x)\| \|d\|^2 \leq \frac{L_f^p}{(p-1)!} \|y - x\|^{p-1} \|d\|^2. \tag{1.6}$$

This implies that:

$$\nabla^2 f(y) - \nabla^2 T_f^p(y; x) \preceq \frac{L_f^p}{(p-1)!} \|y - x\|^{p-1} D. \tag{6.3}$$
Further, from the convexity of \( f \) and \( M_p \geq pL_p^f \), we get:
\[
0 \leq \nabla^2 f(y) \overset{(6.3)}{=} \nabla^2 T_p^f(y; x) + \frac{L_p}{(p-1)!} y - x \parallel y - x \parallel^{p-1} D
\]
\[
\overset{(6.2)}{=} \nabla^2 T_p^f(y; x) + \frac{pL_p}{(p+1)!} \nabla^2 (\parallel y - x \parallel^{p+1})
\]
\[
\leq \nabla^2 T_p^f(y; x) + \frac{M_p}{(p+1)!} \nabla^2 (\parallel y - x \parallel^{p+1}) = \nabla^2 g(y; x).
\]

Thus, \( g(y; x) \) is convex in \( y \). This proves the statement of Lemma 1.5.

6.2. Surrogate properties verification. Verification of Example 2.2: The first property of the surrogate function is straightforward. Next we need to verify that the error function has derivative \( p \) Lipschitz:
\[
h(y, x) = g(y, x) - f(y) = \frac{M_p}{(p+1)!} \parallel y - x \parallel^{p+1},
\]
which according to Example 1.2 has the \( p \) derivative Lipschitz with constant \( L_h^f = M_p \).

For the last property of a surrogate function, we notice that \( \nabla^i (\parallel y - x \parallel^{p+1}) \rvert_{y=x} = 0 \) for all \( i = 0 : p \). Thus, we have that \( \nabla^i h(x; x) = 0 \ \forall i = 0 : p \), i.e. condition (iii) from Definition 2.1.

Verification of Example 2.3: In order to prove condition (i) from Definition 2.1 we use (1.4), i.e.:
\[
f(y) \leq T_p^f(y; x) + \frac{L_p}{(p+1)!} \parallel y - x \parallel^{p+1}
\]
\[
\leq T_p^f(y; x) + \frac{M_p}{(p+1)!} \parallel y - x \parallel^{p+1} = g(y; x),
\]
where the last inequality holds since \( M_p \geq L_p^f \). The second property of a \( p \) higher-order surrogate function requires that the error function \( h(y; x) = g(y; x) - f(y) \) has the \( p \) derivative smooth, where:
\[
h(y; x) = T_p^f(y; x) + \frac{M_p}{(p+1)!} \parallel y - x \parallel^{p+1} - f(y).
\]

We observe that the first term, \( T_p^f \), has the \( p \) derivative smooth with Lipschitz constant 0 due to Example 1.4. The \( p \) derivative smoothness of the second term is covered by Example 1.2 and has the Lipschitz constant equal to \( M_p \). The last term, \( f \), has \( p \) derivative Lipschitz with constant \( L_p^f \) according to our assumption. Since \( h \) is a sum of these three functions, then it is also \( p \) derivative smooth with Lipschitz constant \( L_h^f = M_p + L_p^f \). Hence, condition (ii) from Definition 2.1 also holds. For the last property of a surrogate function, we notice that \( \nabla^i T_p^f(y; x) \rvert_{y=x} = \nabla^i f(x) \) and \( \nabla^i (\parallel y - x \parallel^{p+1}) \rvert_{y=x} = 0 \) for all \( i = 0 : p \). Thus, we have that \( \nabla^i h(x; x) = 0 \ \forall i = 0 : p \), i.e. condition (iii) from Definition 2.1.

Verification of Example 2.5: From relation (1.4) and \( M_p \geq L_p^f \), we have:
\[
(6.4) \quad f_2(y) \leq T_p^{f_2}(y; x) + \frac{L_p^{f_2}}{(p+1)!} \parallel y - x \parallel^{p+1} \leq T_p^{f_2}(y; x) + \frac{M_p}{(p+1)!} \parallel y - x \parallel^{p+1}.
\]
Adding \( f_1(y) \) on both sides of (6.4) we obtain:

\[
    f(y) = f_1(y) + f_2(y) \leq f_1(y) + T_p^{f_2}(y; x) + \frac{M_p}{(p+1)!} \| y - x \|^{p+1} = g(y; x),
\]

which leads to condition (i) from Definition 2.1. Further, for proving the second property of a surrogate function we write explicitly the expression of \( h \):

\[
    h(y; x) = f_1(y) + T_p^{f_2}(y; x) + \frac{M_p}{(p+1)!} \| y - x \|^{p+1} - f_1(y) - f_2(y) \\
    = T_p^{f_2}(y; x) + \frac{M_p}{(p+1)!} \| y - x \|^{p+1} - f_2(y).
\]

We observe that the first term, \( T_p^{f_2} \), has the \( p \) derivative smooth with Lipschitz constant 0 according to Example 1.4. The \( p \) derivative smoothness of the second term is covered by Example 1.2 and has the Lipschitz constant equal to \( M_p \). The last term, \( f_2 \), has the \( p \) derivative Lipschitz with constant \( L_p^{f_2} \) according to our assumption. Since \( h \) is a sum of these three functions, then it has also the \( p \) derivative smooth with the Lipschitz constant \( L_p^h = M_p + L_p^{f_2} \). Hence, property (ii) from Definition 2.1 also holds.

For the last property of a surrogate function, we write explicitly the expression of \( \nabla^i h(x; x) \) for \( i = 0 : p \):

\[
    \nabla^i h(x; x) = \nabla^i T_p^{f_2}(y; x)|_{y=x} + \frac{M_p}{(p+1)!} \nabla^i (\| y - x \|^{p+1})|_{y=x} - \nabla^i f_2(x) \\
    = \nabla^i T_p^{f_2}(y; x)|_{y=x} - \nabla^i f_2(x) = 0.
\]

The last equality is given by the fact that \( \nabla^i T_p^{f_2}(y; x)|_{y=x} = \nabla^i f_2(x) \) for all \( i = 0 : p \). This proves that the property (iii) from Definition 2.1 also holds.

**Verification of Example 2.6:** From Taylor’s theorem we have that there \( \exists t \in [0, 1] \) such that:

\[
    f(y) = T_p^f(x; t(y-x)) (\nabla^{p+1} f(x + t(y-x))(y-x)^{p}, y-x),
\]

and using that \( \nabla^{p+1} f(x) \preceq H \) we obtain:

\[
    f(y) \leq T_p^f(x; t) + \frac{1}{(p+1)!} (H[y-x]^p, y-x) = g(y; x).
\]

Thus, the first property of a higher-order surrogate function holds. Further, we write explicitly the error function \( h \) as:

\[
    h(y; x) = T_p^f(y; x) + \frac{1}{(p+1)!} (H[y-x]^p, y-x) - f(y).
\]

We observe that the first term, \( T_p^f \), is \( p \) derivative smooth with Lipschitz constant 0 due to Example 1.4. The \( p \) derivative smoothness of the second term is covered by Example 1.2 and has the Lipschitz constant equal to \( ||H|| \). The last term, \( f \), has the \( p \) derivative Lipschitz with constant \( L_p^f \) according to our assumption (boundedness of the \( p+1 \) derivative of \( f \) implies Lipschitz continuity of the \( p \) derivative of \( f \)). Since \( h \) is a sum of these three functions, then \( h \) has also the \( p \) derivative smooth with
Lipschitz constant \( L_0^h = \|H\| + L_f^x \). For the last property of a surrogate function, we write the expression of \( \nabla^i h(x; x) \) explicitly:

\[
\nabla^i h(x; x) = \nabla^i T_p^f(y; x)_{|y=x} + \frac{1}{(p+1)!} \nabla^i ((H[y-x]^p, y-x))_{|y=x} - \nabla^i f(x)
\]

\[
= \nabla^i T_p^f(y; x)_{|y=x} - \nabla^i f(x) = 0 \quad \forall i = 0 : p,
\]

since \( \nabla^i T_p^f(y; x)_{|y=x} = \nabla^i f(x) \) and \( \nabla^i ((H[y-x]^p, y-x))_{|y=x} = 0 \) for all \( i = 0 : p \).

Thus, the last property of a higher-order surrogate function (i.e. condition (iii) from Definition 2.1) also holds.

**Verification of Example 2.7:** Since \( \phi \) is convex over \( \mathbb{R}^m \) it follows that it is Lipschitzian relative to any compact set from \( \mathbb{R}^m \) (see e.g., Theorem 10.4 in [27]). Then, there exists \( 0 < L_0^\phi < \infty \) such that:

\[
\phi(F(y)) - \phi(F(x) + \nabla F(x)(y-x)) \leq L_0^\phi \|F(y) - F(x) + \nabla F(x)(y-x)\|
\]

\[
\leq L_0^\phi L_1^F \|y-x\|^2 \quad \forall x, y \in \text{dom} f_1,
\]

where the second inequality follows from \( \nabla F \) being Lipschitz. This proves that the surrogate function (2.3) satisfies \( g(y; x) \geq f(x) \). The Lipschitz property of the Jacobian matrix \( \nabla F \) implies immediately that the second surrogate (2.4) also majorizes \( f \). The other properties follow from the simple observations:

\[
\nabla h(y; x) = \nabla F(x)\nabla\phi(F(x) + \nabla F(x)(y-x)) + M(y-x) - \nabla F(y)\nabla\phi(F(y)),
\]

\[
\nabla h(y; x) = \nabla F(y)\nabla\phi(F(x)) + M\nabla F(y)(F(y) - F(x)) - \nabla F(y)\nabla\phi(F(y)),
\]

respectively.

**Acknowledgments.** The research leading to these results has received funding from the NO Grants 2014–2021, under project ELO-Hyp, contract nr. 24/2020.

**REFERENCES**

[1] H. Attouch and J. Bolte, *On the convergence of the proximal algorithm for nonsmooth functions involving analytic features*, Mathematical Programming, 116(1-2): 5–16, 2009.

[2] E.G. Birgin, J.L. Gardenghi, J.M. Martínez, S.A. Santos and P.L. Toint, *Worst-case evaluation complexity for unconstrained nonlinear optimization using high-order regularized models*, Mathematical Programming, 163(1): 359–368, 2017.

[3] J. Bolte, A. Daniilidis and A. Lewis, *The Lojasiewicz inequality for nonsmooth subanalytic functions with applications to subgradient dynamical systems*, SIAM Journal on Optimization, 17: 1205–1223, 2007.

[4] J. Bolte and E. Pauwels, *Majorization-minimization procedures and convergence of SQP methods for semi-algebraic and tame programs*, Mathematics of Operations Research, 41: 442–465, 2016.

[5] C. Cartis, N.M. Gould and P.L. Toint, *A concise second-order complexity analysis for unconstrained optimization using high-order regularized models*, Optimization Methods and Software, 35: 243–256, 2020.

[6] C. Cartis, N.M. Gould, and P L. Toint, *Sharp worst-case evaluation complexity bounds for arbitrary-order nonconvex optimization with inexpensive constraints*, SIAM Journal on Optimization, 30: 513–541, 2020.

[7] C. Cartis, N.M. Gould and P.L. Toint, *Adaptive cubic regularisation methods for unconstrained optimization: Parts I & II*, Mathematical Programming, 127(2): 245-295, 2011.

[8] Y. Carmon and J.C. Duchi, *Gradient descent finds the cubic-regularized nonconvex Newton step*, SIAM Journal on Optimization, 29(3): 2146–2178, 2019.

[9] D. Drusvyatskiy and C. Paquette, *Efficiency of minimizing compositions of convex functions and smooth maps*, Mathematical Programming, 178 (1-2): 503–558, 2019.
[10] N. Doikov and Yu. Nesterov, Local convergence of tensor methods, Mathematical Programming, doi: 10.1007/s10107-020-01606-x, 2021.
[11] P. Frankel, G. Garrigos and J. Peypouquet, Splitting methods with variable metric for Kurdyka–Lojasiewicz functions and general convergence rates, Journal of Optimization Theory and Applications, 165(3): 874–900, 2015.
[12] G.N. Grapiglia and Yu. Nesterov, Tensor methods for minimizing functions with Hölder continuous higher-order derivatives, SIAM Journal Optimization: 30(4), 2750–2779, 2020.
[13] G.N. Grapiglia and Yu. Nesterov, On inexact solution of auxiliary problems in tensor methods for convex optimization, Optimization Methods and Software: 36(1), 145–170, 2021.
[14] D.R. Hunter and K. Lange, A tutorial on MM algorithms, The American Statistician, 58: 30–37, 2004.
[15] K. Lange, D.R. Hunter and I. Yang, Optimization transfer using surrogate objective functions, Journal of Computational and Graphical Statistics, 9: 1–20, 2000.
[16] J. Mairal, Incremental majorization-minimization optimization with application to large-scale machine learning, SIAM Journal on Optimization, 25(2): 829–855, 2015.
[17] I. Necoara, Yu. Nesterov and F. Glineur, Linear convergence of first-order methods for non-strongly convex optimization, Mathematical Programming, 175: 69–107, 2019.
[18] Y. Nesterov, Introductory lectures on convex optimization: A basic course, Springer, 2004.
[19] Y. Nesterov, Implementable tensor methods in unconstrained convex optimization, Mathematical Programming, doi.org/10.1007/s10107-019-01449-1, 2019.
[20] Y. Nesterov, Inexact basic tensor methods for some classes of convex optimization problems, Optimization Methods and Software, to appear, 2021.
[21] Yu. Nesterov and B.T. Polyak, Cubic regularization of Newton method and its global performance, Mathematical Programming, 108(1): 177–205, 2006.
[22] Y. Nesterov and N. Doikov, Inexact tensor methods with dynamic accuracies, International Conference on Machine Learning, 2020.
[23] B.T. Polyak, Introduction to optimization, Optimization Software, 1987.
[24] E. Pauwels, The value function approach to convergence analysis in composite optimization, Operations Research Letters, 44: 790–795, 2016.
[25] M. Razaviyayn, M. Hong and Z.-Q. Luo, A unified convergence analysis of block successive minimization methods for nonsmooth optimization, SIAM Journal on Optimization, 23(2): 1126–1153, 2013.
[26] A. Rodomanov and Y. Nesterov, Smoothness parameter of power of Euclidean norm, Journal of Optimization Theory and Applications, 185(2): 303–326, 2020.
[27] R.T. Rockafellar and R.J.-B. Wets, Variational Analysis, Springer, 2004.
[28] Y. Zhou, Z. Wang and Y. Liang, Convergence of cubic regularization for nonconvex optimization under KL property, Neural Information Processing Systems Conference, 2018.