Original Paper

$G$-monopole invariants on some connected sums of 4-manifolds

Chanyoung Sung

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Abstract On a smooth closed oriented 4-manifold $M$ with a smooth action of a finite group $G$ on a Spin$^c$ structure, $G$-monopole invariant is defined by “counting” $G$-invariant solutions of Seiberg–Witten equations for any $G$-invariant Riemannian metric on $M$. We compute $G$-monopole invariants on some $G$-manifolds. For example, the connected sum of $k$ copies of a 4-manifold with nontrivial mod 2 Seiberg–Witten invariant has nonzero $\mathbb{Z}_k$-monopole invariant mod 2, where the $\mathbb{Z}_k$-action is given by cyclic permutations of $k$ summands.

Keywords Seiberg–Witten equations · $G$-monopole invariant · Group action

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1 Introduction

Let $M$ be a smooth closed oriented manifold of dimension 4. A second cohomology class of $M$ is called a monopole class if it arises as the first Chern class of a Spin$^c$ structure for which the Seiberg–Witten equations

$$\begin{aligned}
D_A \Phi &= 0 \\
F_A^+ &= \Phi \otimes \Phi^* - \frac{|\Phi|^2}{2} \text{Id},
\end{aligned}$$

admit a solution for every choice of a Riemannian metric. Clearly a basic class, i.e. the first Chern class of a Spin$^c$ structure with a nonzero Seiberg–Witten invariant is a monopole class. However, ordinary Seiberg–Witten invariants which are gotten by the intersection theory on the moduli space of solutions $(A, \Phi)$ of the above equations is trivial in many important cases, for example connected sums of 4-manifolds with $b_2^+ > 0$. 
Bauer and Furuta [4,5] made a breakthrough in detecting a monopole class on certain connected sums of 4-manifolds. Their new brilliant idea is to generalize the Pontryagin-Thom construction to a proper map between infinite-dimensional spaces, which is the sum of a linear Fredholm map and a compact map, and take some sort of a stably-framed bordism class of the Seiberg–Witten moduli space as an invariant. However its applications are still limited in that this new invariant which is expressed as a stable cohomotopy class is difficult to compute, and we are seeking after further refined invariants of the Seiberg–Witten moduli space.

In the meantime, sometimes we need a solution of the Seiberg–Witten equations for a specific metric rather than any Riemannian metric. The case we have in mind is the one when a manifold $M$ and its Spin$^c$ structure $s$ admit a smooth orientation-preserving action by a compact Lie group $G$ and we are concerned with finding a solution of the Seiberg–Witten equations for any $G$-invariant metric.

Thus for a $G$-invariant metric on $M$ and a $G$-invariant perturbation of the Seiberg–Witten equations, we consider the $G$-monopole moduli space $X$ consisting of their $G$-invariant solutions modulo gauge equivalence. One can easily see that the ordinary moduli space $\mathfrak{M}$ is acted on by $G$, and $X$ turns out to be a subset of its $G$-fixed point set. The intersection theory on $X$ will give the $G$-monopole invariant $SW^G_{M,s}$ defined first by Ruan [20], which encodes the information of the given $G$-action along with $M$, and may be sometimes sharper than the ordinary Seiberg–Witten invariant $SW_{M,s}$. To be precise, we need the dimension $b^+_2(M)^G$ of the maximal dimension of subspaces of $G$-invariant 2nd cohomology classes of $M$, where the intersection form is positive-definite to be bigger than 1. In view of this, the following definition is relevant:

**Definition 1** Suppose that $M$ admits a smooth action by a compact Lie group $G$ preserving the orientation of $M$.

A second cohomology class of $M$ is called a $G$-monopole class if it arises as the first Chern class of a $G$-equivariant Spin$^c$ structure for which the Seiberg–Witten equations admit a $G$-invariant solution for every $G$-invariant Riemannian metric of $M$.

When the $G$-monopole invariant is nonzero, its first Chern class has to be a $G$-monopole class. As explain in [25], the cases we are aiming at are those for finite $G$. If a compact connected Lie group $G$ has positive dimension and is not a torus $T^k$, then $G$ contains a Lie subgroup isomorphic to $S^3$ or $S^3/\mathbb{Z}_2$, and hence $M$ admitting an effective action of such $G$ must have a $G$-invariant metric of positive scalar curvature by the well-known Lawson-Yau theorem [13]. Therefore when $b^+_2(M)^G > 1$, $M$ has no $G$-monopole class for such $G$. On the other hand, the Seiberg–Witten invariants of a 4-manifold with an effective $S^1$ action were extensively studied by Baldridge [1–3].

Using $G$-monopole invariants, we find $G$-monopole classes in some connected sums which have vanishing Seiberg–Witten invariants:

**Theorem 1.1** Let $M$ and $N$ be smooth closed oriented connected 4-manifolds satisfying $b^+_2(M) > 1$ and $b^+_2(N) = 0$, and $\bar{M}_k$ for any $k \geq 2$ be the connected sum $M \# \cdots \# M \# N$ where there are $k$ summands of $M$.

Suppose that a finite group $G$ with $|G| = k$ acts effectively on $N$ in a smooth orientation-preserving way such that it is free or has at least one fixed point, and that $N$ admits a Riemannian metric of positive scalar curvature invariant under the $G$-action and a $G$-equivariant Spin$^c$ structure $\tilde{s}_N$ with $c_1^2(\tilde{s}_N) = -b_2(N)$.

Define a $G$-action on $\bar{M}_k$ induced from that of $N$ permuting $k$ summands of $M$ glued along a free orbit in $N$, and let $\bar{s}$ be the Spin$^c$ structure on $\bar{M}_k$ obtained by gluing $\tilde{s}_N$ and a Spin$^c$ structure $\tilde{s}$ of $M$. 

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Then for any $G$-action on $\tilde{s}$ covering the above $G$-action on $\tilde{M}_k$, $SW_{M, \tilde{s}}^G \mod 2$ is non-trivial if $SW_{M, s} \mod 2$ is nontrivial.

Note that if a smooth closed manifold $X$ has a smooth effective action of a compact Lie group $G$, then the fixed-point set $X^g$ under $g \in G$ is, if nonempty, an embedded submanifold each component of which has positive codimension. Thus $N$ in the above theorem always has a free orbit under $G$. The precise computation of $SW_{M, \tilde{s}}^G \mod 2$ will be given in Sect. 3, and we will also give some examples of such $N$ in the last section. The condition on $N$ may be generalized a bit more.

This article is a refined publish version of original results announced in the archive [26]. In a subsequent paper [27], we will use $G$-monopole invariants to detect smooth exotic actions of finite groups on 4-manifolds. The existence of a $G$-monopole class also has applications to Riemannian geometry such as $G$-invariant Einstein metrics and $G$-Yamabe invariant, which are dealt with in [25]. For a relation between $G$-monopole invariants and ordinary Seiberg-Witten invariants, the readers are referred to [18].

2 $G$-monopole invariant

Let $M$ be a smooth closed oriented 4-manifold. Suppose that a compact Lie group $G$ acts on $M$ smoothly preserving the orientation, and this action lifts to an action on a Spin$^c$ structure $s$ of $M$. Once there is a lifting, any other lifting differs from it by an element of $\text{Map}(G \times M, S^1)$. We fix a lifting and put a $G$-invariant Riemannian metric $g$ on $M$. Then the associated spinor bundles $W_\pm$ are also $G$-equivariant, and we let $\Gamma(W_\pm)^G$ be the set of its $G$-invariant sections. When we put $G$ as a superscript on a set, we always mean the subset consisting of its $G$-invariant elements. Thus $\mathcal{A}(W_+)^G$ is the space of $G$-invariant connections on $\det(W_+)$, which is identified as the space of $G$-invariant purely-imaginary valued 1-forms $\Gamma(\Lambda^1(M; i\mathbb{R}))^G$, and $G^G = \text{Map}(M, S^1)^G$ is the group of $G$-invariant gauge transformations.

The perturbed Seiberg–Witten equations give a smooth map

$$H : \mathcal{A}(W_+)^G \times \Gamma(W_+)^G \times \Gamma(\Lambda^2_+(M; i\mathbb{R}))^G \rightarrow \Gamma(W_-)^G \times \Gamma(\Lambda^2_+(M; i\mathbb{R}))^G$$

defined by

$$H(A, \Phi, \varepsilon) = \left( D_A\Phi, F_A^+ + |\Phi|^2 + \frac{1}{2} |\Phi|^2 \text{Id} + \varepsilon \right),$$

where the domain and the range are endowed with $L^2_{l+1}$ and $L^2_l$ Sobolev norms for a positive integer $l$ respectively, and $D_A$ is a Spin$^c$ Dirac operator. The $G$-monopole moduli space $\mathcal{X}_\varepsilon$ for a perturbation $\varepsilon$ is then defined as

$$\mathcal{X}_\varepsilon := H_{\varepsilon}^{-1}(0)/\mathcal{G}^G,$$

where $H_{\varepsilon}$ denotes $H$ restricted to $\mathcal{A}(W_+)^G \times \Gamma(W_+)^G \times \{ \varepsilon \}$.

In the following, we give a detailed proof that $\mathcal{X}_\varepsilon$ for generic $\varepsilon$ and finite $G$ is a smooth compact manifold, because some statements in [6,20] are incorrect or without proofs.

**Lemma 2.1** The quotient map

$$p : (\mathcal{A}(W_+)^G \times (\Gamma(W_+)^G - \{0\})/\mathcal{G}^G \rightarrow (\mathcal{A}(W_+)^G \times (\Gamma(W_+)^G - \{0\})/\mathcal{G}$$

is bijective, and hence $\mathcal{X}_\varepsilon$ is a subset of the ordinary Seiberg–Witten moduli space $\mathcal{M}_\varepsilon$. 

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Proof Obviously $p$ is surjective, and to show that $p$ is injective, suppose that $(A_1, \Phi_1)$ and $(A_2, \Phi_2)$ in $\mathcal{A}(W_+)^G \times (\Gamma(W_+)^G - \{0\})$ are equivalent under $\gamma \in G$. Then

$$A_1 = A_2 - 2d \ln \gamma, \quad \text{and} \quad \Phi_1 = \gamma \Phi_2.$$  

By the first equality, $d \ln \gamma$ is $G$-invariant.

Let $S$ be the subset of $M$ where $\gamma$ is $G$-invariant. By the continuity of $\gamma$, $S$ must be a closed subset. Since $S$ contains a nonempty subset $$\{ x \in M | \Phi_1(x) \neq 0 \},$$ $S$ is nonempty. It suffices to show that $S$ is open. Let $x_0 \in S$. Then we have that for any $g \in G$,

$$g^* \ln \gamma(x_0) = \ln \gamma(x_0), \quad \text{and} \quad g^* d \ln \gamma = d \ln \gamma,$$

which implies that $g^* \ln \gamma = \ln \gamma$ on an open neighborhood of $x_0$ on which $g^* \ln \gamma$ and $\ln \gamma$ are well-defined. By the compactness of $G$, there exists an open neighborhood of $x_0$ on which $g^* \ln \gamma$ is well-defined for all $g \in G$, and $\ln \gamma$ is $G$-invariant. This proves the openness of $S$.

\hfill \Box

As in the ordinary Seiberg–Witten moduli space, the transversality is obtained by a generic perturbation $\varepsilon$:

**Lemma 2.2** $H$ is a submersion at each $(A, \Phi, \varepsilon) \in H^{-1}(0)$ for nonzero $\Phi$.

**Proof** Obviously $dH$ restricted to the last factor of the domain is onto the last factor of the range. Using the surjectivity in the ordinary setting, for any element $\psi \in \Gamma(W_-)^G$, there exists an element $(a, \varphi) \in \mathcal{A}(W_+) \times \Gamma(W_+)$ such that $dH(a, \varphi, 0) = \psi$. The average

$$(\bar{a}, \bar{\varphi}) := \int_G h^*(a, \varphi) \, d\mu(h) := \left( \int_G h^* a \, d\mu(h), \int_G h^* \varphi \, d\mu(h) \right)$$

using a unit-volume $G$-invariant metric on $G$ is an element of $\mathcal{A}(W_+)^G \times \Gamma(W_+)^G$. It follows from the smoothness of the $G$-action that every $h^*(a, \varphi)$ and hence $(\bar{a}, \bar{\varphi})$ belong to the same Sobolev space as $(a, \varphi)$. Moreover it still satisfies

$$dH(\bar{a}, \bar{\varphi}, 0) = \int_G dH(h^*(a, \varphi, 0)) \, d\mu(h) = \int_G h^* dH((a, \varphi, 0)) \, d\mu(h) = \int_G h^* \psi \, d\mu(h) = \psi,$$

where we used the fact that $dH$ is a $G$-equivariant differential operator. This completes the proof. \hfill \Box

Assuming that $b_2^+(M)^G$ is nonzero, $\mathcal{X}_\varepsilon$ consists of irreducible solutions. By the above lemma, $\bigcup \mathcal{X}_\varepsilon$ is a smooth submanifold, and applying Smale-Sard theorem to the projection map onto $\Gamma(\Lambda^2_+ (M; i\mathbb{R}))^G$, $\mathcal{X}_\varepsilon$ for generic $\varepsilon$ is also smooth. (Nevertheless $\mathcal{X}_\varepsilon$ for that $\varepsilon$ may not be smooth in general. Its obstruction is explained in [6].) From now on, we will always...
assume that a generic $\varepsilon$ is chosen so that $\mathcal{X}_\varepsilon$ is smooth, and often omit the notation of $\varepsilon$, if no confusion arises.

Its dimension and orientability can be obtained in the same way as the ordinary Seiberg–Witten moduli space. The linearization $dH$ is deformed by a homotopy to

$$d^+ + 2d^* : \Gamma(A^1)^G \rightarrow \Gamma(A^0 \oplus \Lambda_+^2)^G$$

and

$$D_A : \Gamma(W_+)^G \rightarrow \Gamma(W_-)^G$$

so that the virtual dimension of $\mathcal{X}$ is equal to

$$\dim H_1(M; \mathbb{R})^G - b_2^+(M)^G - 1 + 2(\dim C(\ker D_A)^G - \dim C(\text{coker} D_A)^G),$$

and its orientation can be assigned by fixing the homology orientation of $H_1(M; \mathbb{R})^G$ and $H^2_+(M; \mathbb{R})^G$. When $G$ is finite, one can use Lefschetz-type formula to explicitly compute the last term $\text{ind}^G D_A$ in the above formula. For more details, one may consult [6].

**Theorem 2.3** When $G$ is finite, $\mathcal{X}_\varepsilon$ for any $\varepsilon$ is compact.

**Proof** Following the proof for the ordinary Seiberg–Witten moduli space, we need the $G$-equivariant version of the gauge fixing lemma.

**Lemma 2.4** Let $\mathcal{L}$ be a $G$-equivariant complex line bundle over $M$ with a hermitian metric, and $A_0$ be a fixed $G$-invariant smooth unitary connection on it.

Then for any $l \geq 0$ there are constants $K, C > 0$ depending on $A_0$ and $l$ such that for any $G$-invariant $L^2_1$ unitary connection $A$ on $\mathcal{L}$ there is a $G$-equivariant $L^2_{l+1}$ change of gauge $\sigma$ so that $\sigma^*(A) = A_0 + \alpha$ where $\alpha \in L^2_1(T^*M \otimes i\mathbb{R})^G$ satisfies

$$d^*\alpha = 0, \quad ||\alpha||^2_{L^2_1} \leq C||F^+_A||^2_{L^2_{l+1}} + K.$$

**Proof** We know that a gauge-fixing $\sigma$ with the above estimate always exists, but we need to prove the existence of $G$-invariant $\sigma$. Write $A$ as $A_0 + \alpha$ where $\alpha \in L^2_1(T^*M \otimes i\mathbb{R})^G$. Let $a = a^{\text{harm}} + df + d^*\beta$ be the Hodge decomposition. By the $G$-invariance of $a$, so are $a^{\text{harm}}, df,$ and $d^*\beta$. Applying the ordinary gauge fixing lemma to $A_0 + d^*\beta$, we have

$$||d^*\beta||^2_{L^2_1} \leq C'|||F^+_A||^2_{L^2_1} + K' = C'|||F^+_A||^2_{L^2_{l+1}} + K'$$

for some constants $C', K' > 0$. Defining a $G$-invariant $i\mathbb{R}$-valued function $f_{av} = \frac{1}{|G|} \sum_{g \in G} g^*f$, we have

$$df = \frac{1}{|G|} \sum_{g \in G} g^*df = d(f_{av}) = -d \ln \exp(-f_{av}),$$

and hence $df$ can be gauged away by a $G$-invariant gauge transformation $\exp(-f_{av})$. Write $a^{\text{harm}}$ as $(n|G| + m)a^h$ for $m \in [0, |G|)$ and an integer $n \geq 0$, where $a^h \in H^1(M; \mathbb{Z})^G$ is not a positive multiple of another element of $H^1(M; \mathbb{Z})^G$. There exists $g \in \mathcal{G}$ such that $a^h = -d \ln g$. In general $g$ is not $G$-invariant, but

$$|G|a^h = \sum_{g \in G} g^*a^h = -d \ln \prod_{g \in G} g^*g.$$
and hence $n |G| a^h$ can be gauged away by a $G$-invariant gauge transformation $\prod_{g \in G} g^* g^n$. In summary, $A_0 + a$ is equivalent to $A_0 + ma^h + d^a \beta$ after a $G$-invariant gauge transformation, and

$$||ma^h + d^a \beta||_{L^2}^2 \leq (||ma^h||_{L^2} + ||d^a \beta||_{L^2})^2$$

$$\leq |G|^2 ||a^h||_{L^2}^2 + 2 |G|| ||a^h||_{L^2}||d^a \beta||_{L^2}^2 + ||d^a \beta||_{L^2}^2$$

$$\leq 3 |G|^2 ||a^h||_{L^2}^2 + 3 ||d^a \beta||_{L^2}^2$$

$$= K'' + 3 C' ||F^+_A||_{L^2_{-1}}^2 + 3 K'$$

for a constant $K'' > 0$. This completes the proof. \hfill \square

Now the rest of the compactness proof proceeds in the same way as the ordinary case using the Weitzenböck formula and standard elliptic and Sobolev estimates. For details the readers are referred to [14]. \hfill \square

**Remark** If $G$ is not finite, $\mathcal{X}_s$ may not be compact.

For example, consider $M = S^1 \times Y$ with the trivial Spin$^c$ structure and its obvious $S^1$ action, where $Y$ is a closed oriented 3-manifold. For any $n \in \mathbb{Z}$, $nd\theta$ where $\theta$ is the coordinate on $S^1$ is an $S^1$-invariant reducible solution. Although $nd\theta$ is gauge equivalent to 0, but never via an $S^1$-invariant gauge transformation which is an element of the pull-back of $C^\infty(Y, S^1)$. Therefore as $n \to \infty$, $nd\theta$ diverges modulo $G_{S^1}$, which proves that $\mathcal{X}_0$ is non-compact.

In the rest of this paper, we assume that $G$ is finite. Then note that $G$ induces smooth actions on

$$\mathcal{C} := \mathcal{A}(W_+) \times \Gamma(W_+),$$

$$\mathcal{B}^s = (\mathcal{A}(W_+) \times (\Gamma(W_+) - \{0\}))/G,$$

and also the Seiberg–Witten moduli space $\mathcal{M}$ whenever it is smooth.

Since $\mathcal{X}_s$ is a subset of $\mathcal{M}_s$, (actually a subset of the fixed locus $\mathcal{M}_s^G$ of a $G$-space $\mathcal{M}_s$), we can define the $G$-monopole invariant $SW_{M,s}^G$ by integrating the same universal cohomology classes as in the ordinary Seiberg–Witten invariant $SW_{M,s}$. Thus using the $\mathbb{Z}$-algebra isomorphism

$$\mu_{M,s} : \mathbb{Z}[H_0(M; \mathbb{Z})] \otimes \wedge^* H_1(M; \mathbb{Z})/\text{torsion} \to H^s(\mathcal{B}^s; \mathbb{Z}),$$

we define it as a function

$$SW_{M,s}^G : \mathbb{Z}[H_0(M; \mathbb{Z})] \otimes \wedge^* H_1(M; \mathbb{Z})/\text{torsion} \to \mathbb{Z}$$

$$\alpha \mapsto \langle [\mathcal{X}], \mu_{M,s}(\alpha) \rangle,$$

which is set to be 0 when the degree of $\mu_{M,s}(\alpha)$ does not match $\dim \mathcal{X}$. To be specific, for $[c] \in H_1(M, \mathbb{Z}),$

$$\mu_{M,s}([c]) := \text{Hol}^c_{\mathbb{R}}([d\theta])$$

where $[d\theta] \equiv 1 \in H^1(S^1, \mathbb{Z})$ and $\text{Hol}_c : \mathcal{B}^s \to S^1$ is given by the holonomy of each connection around $c$, and $\mu_{M,s}(U)$ for $U \equiv 1 \in H_0(M, \mathbb{Z})$ is given by the first Chern class of the $S^1$-bundle

$$\mathcal{B}^s = (\mathcal{A}(W_+) \times (\Gamma(W_+) - \{0\}))/G_o$$
over $B^*$ where $G_o = \{ g \in G | g(o) = 1 \}$ is the based gauge group for a fixed base point $o \in M$. (The $S^1$-bundles obtained by choosing a different base point are all isomorphic by the connectedness of $M$.)

As in the ordinary case, a different choice of a $G$-invariant metric and a $G$-invariant perturbation $\varepsilon$ gives a cobordant $X$ so that $SW^G_{M,s}$ is independent of such choices, if $b_2^+ (M)^G > 1$. When $b_2^+ (M)^G = 1$, one should get an appropriate wall-crossing formula.

When $M^c$ happens to be smooth for a $G$-invariant perturbation, the induced $G$-action on it is a smooth action, and hence $M^G$ is a smooth submanifold. Moreover if the finite group action is free, then $\pi : M \to M/G$ is a covering, and $s$ is the pull-back of a Spin$^c$ structure on $M/G$, which is determined up to the kernel of $\pi^* : H^2(M/G, \mathbb{Z}) \to H^2(M, \mathbb{Z})$, and all the irreducible solutions of the upstairs is precisely the pull-back of the corresponding irreducible solutions of the downstairs:

**Theorem 2.5** [17, 21] Let $M$, $s$, and $G$ be as above. Under the assumption that $G$ is finite and the action is free, for a $G$-invariant generic perturbation

$$X_{M,s} = M_{M/G,s'}$$

and

$$M^G_{M,s} \simeq \coprod_{c \in \ker \pi^*} M_{M/G,s'+c},$$

where the second one is a homeomorphism in general, and $s'$ is the Spin$^c$ structure on $M/G$ induced from $s$ and its $G$-action.

Finally we remark that the $G$-monopole invariant may change when a homotopically different lift of the $G$-action to the Spin$^c$ structure is chosen.

### 3 Connected sums and $G$-monopole invariant

For $(\tilde{M}_k, \tilde{s})$ described in Theorem 1.1, there is at least one $G$-action lifted to $\tilde{s}$ coming from the given $G$-action on $(N, s_N)$ and the $G$-equivariant gluing of $k$-copies of $(M, s)$. In general, there may be homotopically inequivalent liftings of the $G$-action on $\tilde{M}_k$ to $\tilde{s}$.

Take a $G$-invariant metric of positive scalar curvature on $N$. In order to do the connected sum with $k$ copies of $M$, we perform a Gromov–Lawson type surgery [10, 23] around each point of a free orbit of $G$ keeping the positivity of scalar curvature to get a Riemannian manifold $\tilde{N}$ with cylindrical ends with each end isometric to a Riemannian product of a round $S^3$ and $\mathbb{R}$. We suppose that this is done in a symmetric way so that the $G$-action on $\tilde{N}$ is isometric.

On $M$ part, we put any metric and perform a Gromov–Lawson surgery with the same cylindrical end as above. Let’s denote this by $\tilde{M}$. Now chop the cylinders at sufficiently large length and then glue $\tilde{N}$ and $k$-copies of $\tilde{M}$ along the boundary to get a desired $G$-invariant metric $g_k$ on $\tilde{M}_k$. Sometimes we mean $(\tilde{M}_k, g_k)$ by $\tilde{M}_k$.

**Theorem 3.1** Let $(\tilde{M}_k, \tilde{s})$ in Theorem 1.1 be endowed with $g_k$ as above. Then for any sufficiently large cylindrical length and some generic perturbation, $X_{\tilde{M}_k, \tilde{s}}$ is diffeomorphic to $M_{\tilde{M}, \tilde{s}} \times T^1$, where $v = \dim H_1 (N; \mathbb{R})^G$.

**Proof** First we consider the case when the $G$-action on $N$ has a fixed point.

Let’s first figure out the ordinary moduli space $M_{\tilde{M}_k}$ of $(\tilde{M}_k, \tilde{s})$. Let $M_{\tilde{N}}$ and $M_{N}$ be the moduli spaces of finite-energy solutions of Seiberg–Witten equations on $(\tilde{M}, s)$ and $(\tilde{N}, s_N)$ respectively. From now on, $\{ \cdot \}$ of a configuration $\cdot$ denotes its gauge equivalence class.
By the gluing theory\footnote{For more details, one may consult [12,19,22,24,28].} of Seiberg–Witten moduli space, which is now a standard method in gauge theory, $\mathcal{M}_M$ is diffeomorphic to $\hat{\mathcal{M}}_M$. In $\hat{\mathcal{M}}_M$, we use a compact-supported self-dual 2-form for a generic perturbation.

Since $\hat{N}$ has a metric of positive scalar curvature and the property that $b_2^+(\hat{N}) = 0$ and $c_1^2(s_{\hat{N}}) = -b_2(\hat{N})$, $\hat{N}$ also has no gluing obstruction even without perturbation so that $\mathcal{M}_N$ is diffeomorphic to $\mathcal{M}_{\hat{N}} = \mathcal{M}^{red}_N$, which can be identified with the space of $L^2$-harmonic 1-forms on $\hat{N}$ modulo gauge, i.e.

$$H^1_{cpt}(\hat{N}, \mathbb{R})/H^1_{cpt}(\hat{N}, \mathbb{Z}) \simeq T^{b_1(N)}.$$  

(Here by $T^0$ we mean a point, and $\mathcal{M}^{red} \subset \mathcal{M}$ denotes the moduli space of reducible solutions.)

As is well-known, approximate solutions on $\bar{\hat{M}}_k$ are obtained by chopping-off solutions on each $\hat{M}$ and $\hat{N}$ at a sufficiently large cylindrical length and then grafting them to $\bar{\hat{M}}_k$ via a sufficiently slowly-varying partition of unity in a $G$-invariant way. More precise prescription of grafting is as follows. First, let’s name $k$ parts of $\bar{\hat{M}}_k$. Choose one of $k$ parts and we call it the 1st part. To assign other $k-1$ parts, let’s denote $G$ by $\{\sigma_1, \sigma_2, \ldots, \sigma_k = e\}$ where $e$ is the identity element. Since each $M$ part of $\bar{\hat{M}}_k$ is the image of the 1st $M$ part under exactly one of $\sigma_i \in G$, lets call it the $i$-th $M$ part. Now choosing an identification of the Spin$^c$ structure on the 1st $M$ part with that of $\hat{M}$, and the identifications of Spin$^c$ structures on other $M$ parts with that of $\hat{M}$ can be done using the $G$-action on $\hat{s}$. Once there is such identification, we can graft a cut-off solution on $\hat{M}$ to each $M$ part of $\bar{\hat{M}}_k$.

In taking cut-offs of solutions on $\hat{N}$, we use a special gauge-fixing condition. Fix a $G$-invariant connection $\eta_0$ such that $[\eta_0] \in \mathcal{M}_{\hat{N}}$, which exists by taking the $G$-average of any reducible solution, and take compact-supported closed 1-forms $\beta_i$ which generate $H^1_{cpt}(\hat{N}; \mathbb{Z})$ and vanish on the cylindrical gluing regions. Any element $[\eta] \in \mathcal{M}_{\hat{N}}$ can be expressed as

$$\eta = \eta_0 + \sum_i c_i \beta_i$$

for $c_i \in \mathbb{R}/\mathbb{Z}$, and the gauge equivalence class of its cut-off

$$\tilde{\eta} := \rho \eta = \rho \eta_0 + \sum_i c_i \beta_i$$

using a $G$-invariant cut-off function $\rho$ which is equal to 1 on the support of every $\beta_i$ is well-defined independently of the mod $\mathbb{Z}$ ambiguity of each $c_i$.

Similarly, for the cut-off procedure to be well-defined independently of the choice of a gauge representative on $\mathcal{M}_M$, one needs to take a gauge-fixing so that homotopy classes of gauge transformations on $\hat{M}$ are parametrized by $H^1_{cpt}(\hat{M}; \mathbb{Z})$, whose elements are gauge transformations constant on gluing regions. Thus the gluing produces a smooth map from

$$\left( \prod_{i=1}^k \mathcal{M}_M \right) \times \mathcal{M}_{\hat{N}} := (\mathcal{M}_M \times \cdots \times \mathcal{M}_M) \times \mathcal{M}_{\hat{N}}$$
to a so-called approximate moduli space $\tilde{\mathcal{M}}_{M_k}$ in $\mathfrak{M}^{\ast}_{M_k}$. This gluing map is one to one, because of the unique continuation principle [12] of Seiberg–Witten equations. From the unobstructedness of gluing, $\tilde{\mathcal{M}}_{M_k} \subset \mathfrak{M}^{\ast}_{M_k}$ is a smoothly embedded submanifold diffeomorphic to

$$\left(\prod_{i=1}^{k} \mathcal{M}_i^0 / S^1\right) \times \tilde{\mathcal{M}}_{\tilde{N}} = \left(\prod_{i=1}^{k} \mathcal{M}_i^\ast\right) \tilde{\times} T^{k-1} \times T^{b_1(N)},$$

where $\mathcal{M}_i^0$ is the based moduli space fibering over $\mathcal{M}_i$ with fiber $\mathcal{G}_o / \mathcal{G} = S^1$, and $\tilde{\times}$ means a $T^{k-1}$-bundle over $\prod_{i=1}^{k} \mathcal{M}_i^\ast$.

As the length of the cylinders in $\tilde{M}_k$ increases, approximate solutions get close to genuine solutions exponentially. Once we choose smoothly-varying normal subspaces to tangent spaces of $\mathcal{M}_M \subset \mathfrak{M}_M^\ast$, the Newton method [7] gives a diffeomorphism

$$\Upsilon : \tilde{\mathcal{M}}_{\tilde{M}_k} \to \mathcal{M}_{M_k}$$

given by a very small isotopy along the normal directions. A bit more explanation will be given in Lemma 3.3.

An important fact for us is that the same $k$ copies of a compactly supported self-dual 2-form can be used for the perturbation on $M$ parts, while no perturbation is put on the $N$ part. Along with the $G$-invariance of the Riemannian metric $g_k$, the perturbed Seiberg–Witten equations for $(\tilde{M}_k, g_k)$ are $G$-equivariant so that the induced smooth $G$-action on $B^\ast_{M_k}$ maps $\mathcal{M}_{M_k}$ to itself.

Let’s describe elements of $\tilde{\mathcal{M}}_{\tilde{M}_k}$ for $(\tilde{M}_k, g_k)$ more explicitly. For $[\xi] \in \mathcal{M}_{M_k}$, let $\tilde{\xi}$ be an approximate solution for $\xi$ cut-off at a large cylindrical length, and $\tilde{\xi}(\theta)$ be its gauge-transform under the gauge transformation by $e^{i\theta} \in C^\infty(\tilde{M}, S^1)$. (From now on, the tilde $\tilde{\cdot}$ of a solution will mean its cut-off.) Any element in $\tilde{\mathcal{M}}_{\tilde{M}_k}$ can be written as an ordered $(k+1)$-tuple

$$[\xi(\theta_1), \ldots, \xi_{k-1}(\theta_{k-1}), \xi_k(0), \eta]$$

for each $[\xi_i] \in \mathcal{M}_i$ and constants $\theta_i$’s, where the $i$-th term for $i = 1, \ldots, k$ represents the approximate solution grafted on the $i$-th $M$ part, and the last term is a cut-off of $\eta \in \mathcal{M}_{\tilde{N}}^{\text{red}}$.

In fact, there is a bijective correspondence

$$\tilde{\mathcal{M}}_{\tilde{M}_k}$$

\[ \leftrightarrow \]

$$\{(\xi(\theta_1), \ldots, \xi_{k-1}(\theta_{k-1}), \xi_k(0), \eta) \mid [\eta] \in \mathcal{M}_N, [\xi_i] \in \mathcal{M}_i, \theta_i \in [0, 2\pi) \forall i\}. \quad (3.2)$$

**Lemma 3.2** The $G$-action on $B^\ast_{M_k}$ maps $\tilde{\mathcal{M}}_{\tilde{M}_k}$ to itself.

**Proof** The $G$-action on $(\tilde{M}_k, \tilde{s})$ can be obviously extended to an action on the Spin$^c$ structure of $\tilde{N} \cup \bigsqcup_{i=1}^{k} \tilde{M}$ and also its moduli space of finite-energy monopoles. Let $\sigma \in G$. By the $G$-invariance of $\rho$,

$$\sigma^* \eta = \sigma^* (\rho \eta) = \rho \sigma^* \eta = \tilde{\sigma^* \eta}.$$

Since $\sigma^* \beta_i$ also gives an element of $H^1_{cpt}(\tilde{N}; \mathbb{Z})$, let’s set $\sigma^* \beta_i$ be cohomologous to $\sum_j d_{ij} \beta_j$ for each $i$. Thus
\[ \tilde{\sigma}^* \eta = \rho \sigma^* \left( \eta_0 + \sum_i c_i \beta_i \right) = \rho \eta_0 + \sum_i c_i \sigma^* \beta_i \]

is gauge-equivalent to

\[ \rho \eta_0 + \sum_{i,j} c_i d_{ij} \beta_j \]

which is the cut-off of \( \eta_0 + \sum_{i,j} c_i d_{ij} \beta_j \).

The \( G \)-action on the first \( k \) components of (3.1) just permutes them. Thus a constant gauge transform of \( \sigma^* (\tilde{\xi}_1(\theta_1), \ldots, \tilde{\xi}_{k-1}(\theta_{k-1}), \tilde{\xi}_k(0), \tilde{\eta}) \) is also of the type (3.1).

Moreover we may assume that the map \( \Upsilon \) is \( G \)-equivariant by the following lemma.

**Lemma 3.3** \( \Upsilon \) can be made \( G \)-equivariant, and the smooth submanifold \( M^G \hat{M}_k \) pointwisely fixed under the action is isotopic to \( \hat{M}^G \hat{M}_k \), the fixed point set in \( \hat{M}_k \).

**Proof** To get a \( G \)-equivariant \( \Upsilon \), we need to choose a smooth normal bundle of \( \hat{M} \hat{M}_k \subset B^*_M \) in a \( G \)-equivariant way. This can be achieved by taking the \( G \)-average of any smooth Riemannian metric defined in a small neighborhood of \( \hat{M} \hat{M}_k \).

A smooth Riemannian metric on a Hilbert manifold is a smoothly varying bounded positive-definite symmetric bilinear forms on its tangent spaces. In order to have a well-defined exponential map as a diffeomorphism on a neighborhood of the origin, we want the metric to be “strong” in the sense that the metric on each tangent space induces the same topology as the original Hilbert space topology. (For a proof, see [11]).

Since \( \hat{M} \hat{M}_k \) is compact, we use a partition of unity on it to glue together obvious Hilbert space metrics in local charts, thereby constructing a smooth Riemannian metric in a neighborhood of \( \hat{M} \hat{M}_k \) in a Hilbert manifold \( B^*_M \). Taking its average under the \( G \)-action, we get a desired Riemannian metric, which is easily checked to be strong.

Taking the orthogonal complement to the tangent bundle of \( \hat{M} \hat{M}_k \) under the above-obtained metric, we get its normal bundle which is trivial by being infinite-dimensional. In the same way as the finite dimensional case, the inverse function theorem implies that a small neighborhood of the zero section in the normal bundle is mapped diffeomorphically into \( B^*_M \) by the exponential map. Thus we can view a small neighborhood of \( \hat{M} \hat{M}_k \) as \( \hat{M} \hat{M}_k \times \mathbb{H} \) where \( \mathbb{H} \) is the Hilbert space isomorphic to the orthogonal complement of the tangent space of \( \hat{M} \hat{M}_k \) at any point.

Applying the Newton method, \( \Upsilon \) is pointwisely a vertical translation along \( \mathbb{H} \) direction. Now the assertion follows from the \( G \)-invariance of the normal directions.

As a preparation for finding \( G \)-fixed points of \( \hat{M} \hat{M}_k \),

**Lemma 3.4** \( \hat{M}^G \hat{N} \) is diffeomorphic to \( T^\nu \), the space of \( G \)-invariant \( L^2 \)-harmonic 1-forms on \( \hat{N} \) modulo \( \mathbb{Z} \).

**Proof** Let \( [\eta] \in \hat{M}^G \hat{N} \), i.e. \([\sigma^* \eta] = [\eta]\) for any \( \sigma \in G \). Then

\[ \tilde{\eta} := \frac{1}{k} \sum_{\sigma \in G} (\sigma)^* \eta \]

satisfies that \( \sigma^* \tilde{\eta} = \tilde{\eta} \) for any \( \sigma \in G \), and \( \tilde{\eta} \) is cohomologous to \( \eta \) so that \([\tilde{\eta}] = [\eta]\).
When \( v \neq 0 \), let \( b_1, \ldots, b_{b_1(N)} \in H_1(N; \mathbb{Z}) \) be a basis of \( H_1(N; \mathbb{R}) \) such that \( b_1, \ldots, b_v \in H_1(N; \mathbb{Z})^G \), where we used that
\[
\text{rank}(H_1(N; \mathbb{Z})^G) = \dim H_1(N; \mathbb{R})^G,
\]
simply because \( G \) also acts on \( H_1(N; \mathbb{Z}) \). Also let \( b_1^*, \ldots, b_{b_1(N)}^* \in H^1_{\text{cpt}}(\hat{N}; \mathbb{R}) \) be the corresponding dual cohomology classes under the isomorphism
\[
H^1_{\text{cpt}}(\hat{N}; \mathbb{R}) \simeq H_1(N; \mathbb{R})^*.
\]
Since \( b_i^*(b_j) = \delta_{ij} \) for all \( i, j = 1, \ldots, b_1(N) \), a simple linear algebra shows that \( b_1^*, \ldots, b_v^* \) are not only in \( H^1_{\text{cpt}}(\hat{N}; \mathbb{Z})^G \), but also form a basis of \( H^1_{\text{cpt}}(\hat{N}; \mathbb{R})^G \). Therefore \( \mathcal{M}_N^G \) is a \( v \)-dimensional torus spanned by \( b_1^*, \ldots, b_v^* \). When \( v = 0 \), \( \mathcal{M}_N^G \) is a point. \( \square \)

As an easy case,

**Lemma 3.5** If \( G = \mathbb{Z}_k \), then \( \mathcal{X}_{\hat{M}_k} \) is diffeomorphic to \( k \) copies of \( \mathcal{M}_M \times T^v \), where \( T^0 \) means a point.

**Proof** Let \( \sigma \) be a generator of \( \mathbb{Z}_k \), and take the numbering of elements of \( G = \{ \sigma_1, \ldots, \sigma_k = e \} \) such that \( \sigma_i = \sigma^i \).

Thus the condition for a fixed point is that
\[
(\tilde{\xi}_k(0), \tilde{\xi}_1(\theta_1), \ldots, \tilde{\xi}_{k-1}(\theta_{k-1}), \sigma^*\eta) \equiv (\tilde{\xi}_1(\theta_1), \ldots, \tilde{\xi}_{k-1}(\theta_{k-1}), \tilde{\xi}_k(0), \tilde{\eta})
\]
modulo gauge transformations. By (3.2) this implies
\[
[\tilde{\xi}_1] = [\tilde{\xi}_2] = \cdots = [\tilde{\xi}_k] \in \mathcal{M}_M^1, \quad \text{and} \quad [\sigma^*\eta] = [\eta] \in \mathcal{M}_N^1.
\]
and
\[
0 \equiv \theta_1 + \theta, \quad \theta_1 \equiv \theta_2 + \theta, \ldots, \theta_{k-1} \equiv 0 + \theta \mod 2\pi
\]
for some constant \( \theta \in [0, 2\pi) \). Summing up the above \( k \) equations gives
\[
0 \equiv k\theta \mod 2\pi,
\]
and hence
\[
\theta = 0, \frac{2\pi}{k}, \ldots, \frac{2(k-1)\pi}{k},
\]
which lead to the corresponding \( k \) solutions
\[
[(\tilde{\xi}((k-1)\theta), \tilde{\xi}((k-2)\theta), \ldots, \tilde{\xi}(\theta), \tilde{\xi}(0), \tilde{\eta})].
\]
where we let \( \xi_i = \xi \) for all \( i \) and \( [\eta] \in \mathcal{M}_N^1 \). Therefore \( \mathcal{X}_{\hat{M}_k} \) is diffeomorphic to \( k \) copies of \( \mathcal{M}_M \times \mathcal{M}_N^1 \simeq \mathcal{M}_M \times T^v \). \( \square \)

**Lemma 3.6** If \( G = \mathbb{Z}_k \), then \( \mathcal{X}_{\hat{M}_k} \) is diffeomorphic to \( \mathcal{M}_M \times T^v \).

**Proof** For \( \xi \in \mathcal{M}_M, \eta \in \mathcal{X}_N \), and \( \theta \) as above, let
\[
\tilde{\xi}_a = (\tilde{\xi}((k-1)\theta), \tilde{\xi}((k-2)\theta), \ldots, \tilde{\xi}(\theta), \tilde{\xi}(0), \tilde{\eta}),
\]
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and denote \( T([\tilde{\Xi}_\theta]) \) by \([\Xi_\theta]\). From the above lemma, we have that
\[
\sigma^* \tilde{\Xi}_\theta = e^{i\theta} \cdot \tilde{\Xi}_\theta,
\]
where \( \sigma \) is a generator of \( \mathbb{Z}_k \), and \( \cdot \) denotes the gauge action.

We will show that \( k-1 \) copies of \( \tilde{\mathcal{M}}_M \times T^\nu \) corresponding to nonzero \( \theta \) do not belong to \( \tilde{\mathcal{X}}_{\tilde{M}_k} \). Let \( \theta = \frac{2\pi}{k}, \ldots, \frac{2(k-1)\pi}{k} \). By the \( \mathbb{Z}_k \)-equivariance of \( T \), \([\sigma^* \Xi_\theta] = \sigma^*[\Xi_\theta] \), and so write
\[
\sigma^* \Xi_\theta = e^{i\theta} \cdot \Xi_\theta \quad \text{for} \quad e^{i\theta} \in \text{Map}(\tilde{M}_k, S^1).
\]
By taking the cylindrical length sufficiently large, \( e^{i\theta} \) can be made arbitrarily close to the constant \( e^{i\theta} \) in a Sobolev norm and hence \( \mathcal{C}^0 \)-norm too by the Sobolev embedding theorem. (The Sobolev embedding constant does not change, if the cylindrical length gets large, because the local geometries remain unchanged.)

Assume to the contrary that \( \sigma^*(g \cdot \Xi_\theta) = g \cdot \Xi_\theta \) for some \( g \in G \). Then combined with that
\[
\sigma^*(g \cdot \Xi_\theta) = \sigma^*(g) \cdot \sigma^*(\Xi_\theta) = (\sigma^*(g)) \cdot (e^{i\theta} \cdot \Xi_\theta) = (\sigma^*(g))e^{i\theta} \cdot \Xi_\theta,
\]
it follows that
\[
g \cdot \Xi_\theta = (\sigma^*(g))e^{i\theta} \cdot \Xi_\theta,
\]
which implies that
\[
\sigma^*(g) = ge^{-i\theta}, \tag{3.4}
\]
where we used the continuity of \( g \) and the fact that the spinor part of \( \alpha \) is not identically zero on an open subset by the unique continuation property.

Choose a fixed point \( p \in \tilde{M}_k \) under the \( \mathbb{Z}_k \)-action. Then evaluating (3.4) at the point \( p \) gives
\[
g(p) = g(p)e^{-i\theta(p)},
\]
which is close to \( g(p)e^{-i\theta} \), thereby yielding a desired contradiction.

It remains to show that \( \tilde{\mathcal{M}}_M \times T^\nu \) corresponding to \( \theta = 0 \) belongs to \( \mathcal{A}(W_+^G) \times (\Gamma(\mathcal{W}_+^G - \{0\})) / G^G \). Let \( \Xi_0 = \tilde{\Xi}_0 + (a, \varphi) \) where \( a \in \Gamma(\Lambda^1(\tilde{M}_k; i\mathbb{R})) \) satisfies the Lorentz gauge condition \( d^*a = 0 \). Since
\[
\sigma^* \Xi_0 = \tilde{\Xi}_0 + (\sigma^*a, \sigma^*\varphi)
\]
belongs to the same gauge equivalence class as \( \Xi_0 \), and
\[
d^*(\sigma^*a) = \sigma^*(d^*a) = 0
\]
using the isometric action of \( G \), we have that \( \sigma^*a \equiv a \) modulo \( H^1(\tilde{M}_k; \mathbb{Z}) = \mathbb{Z}^{b_1(\tilde{M}_k)} \).

Applying the obvious identity \( (\sigma^*)^k = \text{Id} \), it follows that \( \sigma^*a = a \). This implies that \( \sigma^* \Xi_0 \) is a constant gauge transform \( e^c \cdot \Xi_0 \) of \( \Xi_0 \). If \( e^c \neq 1 \), it leads to a contradiction by the same method as above using the existence of a fixed point. Therefore \( \sigma^* \Xi_0 = \Xi_0 \) as desired, and we conclude that \( \tilde{\mathcal{X}}_{\tilde{M}_k} \) is equal to \( \tilde{\mathcal{M}}_M \times T^\nu \). 

\footnote{This and the next two paragraphs are the only three places where we use the condition that the action on \( N \) has a fixed point, which was assumed in the beginning of the proof of current theorem.}
Now we will consider the case of any finite group $G$. We will show that $\mathfrak{X}_{\tilde{M}_k}$ is diffeomorphic to $\Upsilon(S)$, where

$$S := \{(\tilde{\xi}(0), \ldots, \tilde{\xi}(0), \eta) | [\eta] \in \mathcal{M}_N^G, [\xi] \in \mathcal{M}_M, \theta_i \in [0, 2\pi) \forall i\}.$$ 

Since $(\tilde{\xi}(0), \ldots, \tilde{\xi}(0), \tilde{\eta}(0))$ is $G$-invariant, $\Upsilon(\{(\tilde{\xi}(0), \ldots, \tilde{\xi}(0), \tilde{\xi}(0), \tilde{\eta})\})$ is also represented by a $G$-invariant element by the same method as the above paragraph using the existence of a fixed point. Hence $\Upsilon(S) \subset \mathfrak{X}_{\tilde{M}_k}$.

To show the reverse inclusion, first note that any element of $\mathfrak{X}_{\tilde{M}_k}$ can be written as $\Upsilon(\{(\tilde{\xi}(\theta_1), \ldots, \tilde{\xi}(\theta_{k-1}), \tilde{\xi}(0), \tilde{\eta})\})$ for $[\xi] \in \mathcal{M}_M$. We only need to show all $\theta_i$ are zero, and $[\eta] \in \mathcal{M}_N^G$. For $\sigma_i \in G$, let $(\sigma_i)$ be the cyclic subgroup generated by $\sigma_i$, and $\mathfrak{X}_{\tilde{M}_k, (\sigma_i)}$ be the $\langle \sigma_i \rangle$-monopole moduli space. Since $\mathfrak{X}_{\tilde{M}_k}$ is a subset of $\mathfrak{X}_{\tilde{M}_k, (\sigma_i)} \subset \mathcal{M}_M$, we can use the above lemma to deduce that $\theta_i = 0$, and $[\eta] \in \mathcal{M}_N^{(\sigma_i)}$. Since $i$ is arbitrary, we get a desired conclusion.

Finally let’s prove the theorem when the action on $N$ is free. In this case, directly from Theorem 2.5 and the gluing theory, we have diffeomorphisms

$$\mathfrak{X}_{\tilde{M}_k, \tilde{s}} = \mathcal{M}_{M \# N/G, \tilde{s} \# \tilde{s}_N^G}^N \simeq \mathcal{M}_{M, \tilde{s}}^N \times \mathcal{M}_{M, \tilde{s}}^{G, \tilde{s}_N^G} \simeq \mathcal{M}_{M, \tilde{s}} \times T^v,$$

where $\tilde{s}_N^G$ is the Spin$^c$ structure on $N/G$ induced from $s_N$ and its $G$ action induced from that of $\tilde{s}$. This completes all the proof. $\square$

Now we come to the main theorem which implies Theorem 1.1.

**Theorem 3.7** Let $(\tilde{M}_k, \tilde{s})$ be as in Theorem 1.1 and $d \geq 0$ be an integer. If $v := \text{dim } H_1(N; \mathbb{R})^G = 0$, then for $A = 1$ or $a_1 \wedge \cdots \wedge a_j$

$$SW^G_{\tilde{M}_k, \tilde{s}}(U^d A) \equiv SW_{M, \tilde{s}}(U^d A) \mod 2,$$

where $U$ denotes the positive generator of the zeroth homology of $\tilde{M}_k$ or $M$, and each $a_i \in H_1(M; \mathbb{Z})/\text{torsion}$ also denotes any of $k$ corresponding elements in $H_1(M_k; \mathbb{Z})$ by abuse of notation.

If $v \neq 0$, then

$$SW^G_{\tilde{M}_k, \tilde{s}}(U^d A \wedge b_1 \wedge \cdots \wedge b_v) \equiv SW_{M, \tilde{s}}(U^d A) \mod 2,$$

where $A$ is as above, and $b_1, \ldots, b_v \in H_1(N; \mathbb{Z})$ is a basis of $H_1(N; \mathbb{R})^G$.

**Proof** As before, let’s first consider the case when the action has a fixed point. We continue to use the same notation and context as the previous theorem. $\square$

**Lemma 3.8** The $\mu$ cocycles on $\mathcal{M}_M \times T^v$ and $\mathfrak{X}_{\tilde{M}_k}$ coincide, i.e.

$$\mu_M(a_i) = \mu_{\tilde{M}_k}(a_i), \quad \mu_N(b_i) = \mu_{\tilde{M}_k}(b_i), \quad \mu_M(U) = \mu_{\tilde{M}_k}(U)$$

where the equality means the identification under the above diffeomorphism.

**Proof** The first equality comes from that the holonomy maps $\text{Hol}_{a_i}$ defined on $\mathcal{M}_M$ and $\tilde{M}_{M_k}$ are just the same, when the representative of $a_i$ is chosen away from the gluing regions. Using the isotopy between $\mathcal{M}_M^G$ and $\tilde{M}_{M_k}^G$, the induced maps $\text{Hol}_{a_i}^*$ from $H^1(S^1; \mathbb{Z})$ to $H^1(\mathcal{M}_M; \mathbb{Z})$ and $H^1(\tilde{M}_{M_k}; \mathbb{Z})$ are the same so that

$\square$
\[ \mu_M(a_i) = H \Omega^{n_i}([d\theta]) = \mu_{\tilde{M}_k}(a_i) \]

for each \(i\). Likewise for the second equality.

For the third equality, note that the \(S^1\)-fibrations on \(\mathcal{M}_{\tilde{M}_k} \times T^v\) and \(\mathcal{M}_{M_k}^G\) induced by the \(G/G_o\) action are isomorphic in an obvious way, where the \(T^v\) part is fixed under the \(G/G_o\) action. Since the identity between \(\mathcal{M}_{\tilde{M}_k}\) and \(\mathcal{M}_{M_k}\) can be extended to the \(S^1\)-fibrations induced by the \(G/G_o\) action, those \(S^1\)-fibrations are isomorphic. In the same way using gluing theory, there are isomorphisms of \(S^1\)-fibrations on \(\mathcal{M}_M\), its approximate moduli space \(\mathcal{M}_M\), and \(\mathcal{M}_{\tilde{M}_k}\). Therefore we have an isomorphism between those \(S^1\)-fibrations on \(\mathcal{M}_M \times T^v\) and \(\mathcal{X}_{\tilde{M}_k}\).

We are ready for the evaluation of the Seiberg–Witten invariant on \(\mathcal{X}_{\tilde{M}_k}\). Suppose \(v \neq 0\). Let \(l_1, \ldots, l_{b_1(N)}\) be loops representing homology classes \(b_1, \ldots, b_{b_1(N)}\) respectively. Then \(b_i^v\) introduced in Lemma 3.4 restricts to a nonzero element of \(H^1(l_j; \mathbb{Z})\) iff \(i = j\). Moreover \(b_i^v\) is a generator of \(H^1(l_j; \mathbb{Z})\), and hence \(\{\mu(b_1), \ldots, \mu(b_v)\}\) is a standard generator of the 1st cohomology of \(T^v \simeq \mathbb{R}(b_1^v, \ldots, b_v^v)/\mathbb{Z}(b_1^v, \ldots, b_v^v)\). Combining the fact that \(\mu(b_1) \wedge \cdots \wedge \mu(b_v)\) is a generator of \(H^v(T^v; \mathbb{Z})\) with the above identification of \(\mu\)-cocycles, we can conclude that

\[ SW_{\tilde{M}_k, \bar{s}}(U^d A \wedge b_1 \wedge \cdots \wedge b_v) \equiv SW_{M_k, \bar{s}}(U^d A) \mod 2 \]

for \(A = 1\) or \(a_1 \wedge \cdots \wedge a_j\). The case of \(v = 0\) is just a special case.

When the action is free, the theorem is obvious from the identification \(\mathcal{X}_{\tilde{M}_k, \bar{s}} = \mathcal{M}_M \#_{\mathcal{G}} \mathcal{M}_N\).

Remark If the diffeomorphism between \(\mathcal{X}_{\tilde{M}_k}\) and \(\mathcal{M}_M \times T^v\) is orientation-preserving, then \(G\)-monopole invariants and Seiberg–Witten invariants are exactly the same. We conjecture that the diffeomorphism between \(\mathcal{X}_{\tilde{M}_k}\) and \(\mathcal{M}_M \times T^v\) is orientation-preserving, when the homology orientations are appropriately chosen.

One may try to prove \(\mathcal{X}_{\tilde{M}_k} \simeq \mathcal{M}_M \times T^v\) by gluing \(G\)-monopole moduli spaces directly. But the above method of proof by gluing ordinary moduli spaces also shows that for \(G = \mathbb{Z}_k\), \(\mathcal{M}_{M_k}^{\mathbb{Z}_k}\) is diffeomorphic to \(k\) copies of \(\mathcal{M}_M \times T^v\). Lemma 3.8 is also true for any other component of \(\mathcal{M}_M^{\mathbb{Z}_k}\).

### 4 Examples of \((N, s_N)\) of Theorem 1.1

In this section, \(G, H\) and \(K\) denote compact Lie groups. Let’s recall some elementary facts on equivariant principal bundles.

**Definition 2** A principal \(G\) bundle \(\pi : P \to M\) is said to be \(K\)-equivariant if \(K\) acts left on both \(P\) and \(M\) in such a way that

1. \(\pi\) is \(K\)-equivariant:
   \[ \pi(k \cdot p) = k \cdot \pi(p) \]
   for all \(k \in K\) and \(p \in P\),
2. the left action of \(K\) commutes with the right action of \(G\):
   \[ k \cdot (p \cdot g) = (k \cdot p) \cdot g \]
   for all \(k \in K\), \(p \in P\), and \(g \in G\).

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If $H$ is a normal subgroup of $G$, then one can define a principal $G/H$ bundle $P/H$ by taking the fiberwise quotient of $P$ by $H$. Moreover if $P$ is $K$-equivariant under a left $K$ action, then there exists the induced $K$ action on $P/H$ so that $P/H$ is $K$-equivariant.

**Lemma 4.1** Let $P$ and $\tilde{P}$ be a principal $G$ and $\tilde{G}$ bundle respectively over a smooth manifold $M$ such that $\tilde{P}$ double-covers $P$ fiberwise. For a normal subgroup $H$ containing $\mathbb{Z}_2$ in both $\tilde{G}$ and $S^1$ where the quotient of $\tilde{G}$ by that $\mathbb{Z}_2$ gives $G$, let

$$\tilde{P} \otimes_H S^1 := (\tilde{P} \times_M (M \times S^1))/H$$

be the quotient of the fiber product of $\tilde{P}$ and the trivial $S^1$ bundle $M \times S^1$ by $H$, where the right $H$ action is given by

$$(p, (x, e^{i\vartheta})) \cdot h = (p \cdot h, (x, e^{i\vartheta}h^{-1})).$$

Suppose that $M$ and $P$ admit a smooth $S^1$ action such that $P$ is $S^1$-equivariant. Then a principal $\tilde{G} \otimes_H S^1$ bundle $\tilde{P} \otimes_H S^1$ is also $S^1$-equivariant by lifting the action on $P$. In particular, any smooth $S^1$-action on a smooth spin manifold lifts to its trivial Spin$^c$ bundle so that the Spin$^c$ structure is $S^1$-equivariant.

**Proof** Any left $S^1$ action on $P$ can be lifted to $\tilde{P}$ uniquely at least locally commuting with the right $G$ action. If the monodromy is trivial for any orbit, then the $S^1$ action can be globally well-defined on $\tilde{P}$, and hence on $\tilde{P} \otimes_H S^1$, where the $S^1$ action on the latter $S^1$ fiber can be any left action, e.g. the trivial action, commuting with the right $S^1$ action.

If the monodromy is not trivial, it has to be $\mathbb{Z}_2$ for any orbit, because the orbit space is connected. In that case, we need the trivial $S^1$ bundle $M \times S^1$ with an “ill-defined” $S^1$ action with monodromy $\mathbb{Z}_2$ defined as follows.

First consider the double covering map from $M \times S^1$ to itself defined by $(x, z) \mapsto (x, z^2)$. Equip the downstairs $M \times S^1$ with the left $S^1$ action which acts on the base as given and on the fiber $S^1$ by the multiplication as complex numbers. Then this downstairs action can be locally lifted to the upstairs commuting with the right $S^1$ action. Most importantly, it has $\mathbb{Z}_2$ monodromy as desired. Explicitly, $e^{i\vartheta}$ for $\vartheta \in [0, 2\pi)$ acts on the fiber $S^1$ by the multiplication of $e^{i\frac{\vartheta}{2}}$. Combining this with the local action on $\tilde{P}$, we get a well-defined $S^1$ action on $\tilde{P} \otimes_H S^1$, because two $\mathbb{Z}_2$ monodromies are cancelled each other.

Once the $S^1$ action on $\tilde{P} \otimes_H S^1$ is globally well-defined, it commutes with the right $\tilde{G} \otimes_H S^1$ action, because the local $S^1$ action on $\tilde{P} \times S^1$ commuted with the right $\tilde{G} \times S^1$ action.

If $S^1$ acts on a smooth manifold, the orthonormal frame bundle is always $S^1$-equivariant under the action. Then by the above result any $S^1$ action on a smooth spin manifold lifts to the trivial Spin$^c$ bundle which is (spin bundle) $\otimes_{\mathbb{Z}_2} S^1$.

**Lemma 4.2** Let $P$ be a flat principal $G$ bundle over a smooth manifold $M$ with a smooth $S^1$ action. Suppose that the action can be lifted to the universal cover $\tilde{M}$ of $M$. Then it can be also lifted to $P$ so that $P$ is $S^1$-equivariant.

**Proof** For the covering map $\pi : \tilde{M} \to M$, the pull-back bundle $\pi^*P$ is the trivial bundle $\tilde{M} \times G$. By letting $S^1$ act on the fiber $G$ trivially, $\pi^*P$ can be made $S^1$-equivariant. For the deck transformation group $\pi_1(M)$, $P$ is determined by an element of $\text{Hom}(\pi_1(M), G)$. Any deck transformation acts on each fiber $G$ as the left multiplication of a constant in $G$ so that it commutes with not only the right $G$ action but also the left $S^1$ action which is trivial on the fiber $G$. Therefore the $S^1$ action on $\pi^*P$ projects down to an $S^1$ action on $P$. To see whether this $S^1$ action commutes with the right $G$ action, it is enough to check for the local $S^1$ action, which can be seen upstairs on $\pi^*P$.  

\(\square\)
Lemma 4.3  On a smooth closed oriented 4-manifold $N$ with $b_2^+(N) = 0$, any Spin$^c$ structure $s$ satisfies
\[ c_1^2(s) \leq -b_2(N), \]
and the choice of a Spin$^c$ structure $s_N$ satisfying $c_1^2(s_N) = -b_2(N)$ is always possible.

Proof  If $b_2(N) = 0$, it is obvious. The case of $b_2(N) > 0$ can be seen as follows. Using Donaldson’s theorem [8,9], we diagonalize the intersection form $Q_N$ on $H^2(N; \mathbb{Z})/\text{torsion}$ over $\mathbb{Z}$ with a basis \{\(\alpha_1, \ldots, \alpha_{b_2(N)}\)\} satisfying $Q_N(\alpha_i, \alpha_i) = -1$ for all $i$. Then for any Spin$^c$ structure $s$, the rational part of $c_1(s)$ should be of the form
\[ \sum_{i=1}^{b_2(N)} a_i \alpha_i \]
where each $a_i \equiv 1 \mod 2$, because
\[ Q_N(c_1(s), \alpha) \equiv Q_N(\alpha, \alpha) \mod 2 \]
for any $\alpha \in H^2(N; \mathbb{Z})$. Consequently $|a_i| \geq 1$ for all $i$ which means
\[ c_1^2(s) = \sum_{i=1}^{b_2(N)} -a_i^2 \leq -b_2(N), \]
and we can get a Spin$^c$ structure $s_N$ with
\[ c_1(s_N) \equiv \sum_i \alpha_i \mod \text{torsion} \]
by tensoring any $s$ with a line bundle $L$ satisfying
\[ 2c_1(L) + c_1(s) \equiv \sum_i \alpha_i \mod \text{torsion}, \]
completing the proof. \(\square\)

Theorem 4.4  Let $X$ be one of
\[ S^4, \quad \mathbb{CP}_2, \quad S^4 \times (L_1 \# \cdots \# L_n), \quad \text{and} \quad \widetilde{S^1 \times L} \]
where each $L_i$ and $L$ are quotients of $S^3$ by free actions of finite groups, and $\widetilde{S^1 \times L}$ is the manifold obtained from the surgery on $S^1 \times L$ along an $S^1 \times \{\text{pt}\}$.

Then for any integer $l \geq 0$ and any smooth closed oriented 4-manifold $Z$ with $b_2^+(Z) = 0$ admitting a metric of positive scalar curvature,
\[ X \# klZ \]
satisfies the properties of $N$ with $G = \mathbb{Z}_k$ in Theorem 1.1, where the Spin$^c$ structure of $X\#klZ$ is given by gluing any Spin$^c$ structure $s_X$ on $X$ and any Spin$^c$ structure $s_Z$ on $Z$ satisfying $c_1^2(s_X) = -b_2(X)$ and $c_1^2(s_Z) = -b_2(Z)$ respectively.

Proof  First, we will define $\mathbb{Z}_k$ actions preserving a metric of positive scalar curvature. In fact, our actions on $X$ will be induced from such $S^1$ actions.

For $X = S^4$, one can take a $\mathbb{Z}_k$-action coming from a nontrivial action of $S^1 \subset SO(5)$ preserving a round metric. In this case, one can choose a free action or an action with fixed points also.
If $X = \mathbb{C}P_2$, then one can use the following action:

$$j \cdot [z_0, z_1, z_2] = \left[z_0, e^{\frac{2jm_1}{k}}z_1, e^{\frac{2jm_2}{k}}z_2\right]$$

(4.1)

for $j \in \mathbb{Z}_k$, where integers $m_1$ and $m_2$ satisfy that $\gcd(m_1, k)$ and $\gcd(m_2, k)$ are relatively prime. It preserves the Fubini-Study metric and has at least 3 fixed points $\{1, 0, 0\}, \{0, 1, 0\}, \{0, 0, 1\}$.

Before considering the next example, recall that every finite group acting freely on $S^3$ is in fact conjugate to a subgroup of $SO(4)$, and hence its quotient 3-manifold admits a metric of constant positive curvature. This follows from the well-known result of G. Perelman. (See [15, 16].)

In $S^1 \times (L_1# \cdots #L_n)$, the action is defined as a rotation along the $S^1$-factor, which is obviously free and preserves a product metric. By endowing $L_1# \cdots #L_n$ with a metric of positive scalar curvature via the Gromov–Lawson surgery [10], $S^1 \times (L_1# \cdots #L_n)$ has a desired metric.

Finally the above-mentioned $S^1$ action on $S^1 \times L$ can be naturally extended to $\widetilde{S^1 \times L}$, and moreover the Gromov–Lawson surgery [10] on $S^1 \times \{pt\}$ produces an $S^1$-invariant metric of positive scalar curvature. Its fixed point set is $\{0\} \times S^2$ in the attached $D^2 \times S^2$.

Now $X#kIZ$ has an obvious $\mathbb{Z}_k$-action induced from that of $X$ and a $\mathbb{Z}_k$-invariant metric which has positive scalar curvature again by the Gromov–Lawson surgery.

It remains to prove that the above $\mathbb{Z}_k$-action on $X#kIZ$ can be lifted to the Spin$^c$ structure obtained by gluing the above $s_X$ and $s_Z$. For this, we will only prove that any such $s_X$ is $\mathbb{Z}_k$-equivariant. Then one can glue $k$ copies of $IZ$ in an obvious $\mathbb{Z}_k$-equivariant way. Recalling that the $\mathbb{Z}_k$ action on $X$ actually comes from an $S^1$ action, we will actually show the $S^1$-equivariance of $s_X$ on $X$.

On $S^4$, the unique Spin$^c$ structure is trivial. Any smooth $S^1$ action on $S^4$ which is spin can be lifted its trivial Spin$^c$ structure by Lemma 4.1.

Any smooth $S^1$ action on $\mathbb{C}P_2$ is uniquely lifted to its orthonormal frame bundle $F$, and any Spin$^c$ structure on $\mathbb{C}P_2$ satisfying $c_1^2 = -1$ is the double cover $P_1$ and $P_2$ of $F \oplus P$ and $F \oplus P^*$ respectively, where $P$ is the principal $S^1$ bundle over $\mathbb{C}P_2$ with $c_1(P) = [H]$ and $P^*$ is its dual. Note that there is a base-preserving diffeomorphism between $P$ and $P^*$ whose total space is $S^5$. Obviously the action (4.1) is extended to $S^5 \subset \mathbb{C}^3$ commuting with the principal $S^1$ action of the Hopf fibration. By Lemma 4.1 the $S^1$-action can be lifted to $P_i \otimes s_i$ in an $S^1$-equivariant way, which is isomorphic to $P_i$ for $i = 1, 2$.

In case of $S^1 \times (L_1# \cdots #L_n)$, any Spin$^c$ structure is the pull-back from $L_1# \cdots #L_n$, and satisfies $c_1^2 = 0 = -b_2$. Because the tangent bundle is trivial, a free $S^1$-action is obviously defined on its trivial spin bundle. Then the action can be obviously extended to any Spin$^c$ structure, because it is pulled-back from $L_1# \cdots #L_n$.  

\textbf{Lemma 4.5} $S^1 \times L$ is a rational homology 4-sphere, and

$$H^2(S^1 \times L; \mathbb{Z}) = H_1(L; \mathbb{Z}).$$

Its universal cover is $(|\pi_1(L)| - 1)S^2 \times S^2$ where $0(S^2 \times S^2)$ means $S^4$.

\textbf{Proof} Since the Euler characteristic is easily computed to be 2 from the surgery description, and $b_1(S^1 \times L) = b_1(L) = 0$, it follows that $S^1 \times L$ is a rational homology 4-sphere.
By the universal coefficient theorem,
\[
H^2(\hat{S}^1 \times L; \mathbb{Z}) = \text{Hom}(H_2(\hat{S}^1 \times L; \mathbb{Z}), \mathbb{Z}) \oplus \text{Ext}(H_1(\hat{S}^1 \times L; \mathbb{Z}), \mathbb{Z})
\]
\[
= H_1(\hat{S}^1 \times L; \mathbb{Z})
\]
\[
= H_1(L; \mathbb{Z}).
\]

The universal cover is equal to the manifold obtained from \(S^1 \times S^3\) by performing surgery along \(S^1 \times \{ |\pi_1(L)\text{ points in } S^3| \}\), and hence it must be \(|\pi_1(L)| - 1\) \(S^2 \times S^2\). \(\square\)

By the above lemma, there are \(|H_1(L; \mathbb{Z})|\) \(\text{Spin}^c\) structures on \(\hat{S}^1 \times L\), all of which are torsion to satisfy \(c_1^2 = 0 = -b_2(\hat{S}^1 \times L)\). Since any \(S^1\) bundle on \(\hat{S}^1 \times L\) is flat, and the \(S^1\)-action on \(\hat{S}^1 \times L\) can be obviously lifted to its universal cover, Lemma 4.2 says that any \(S^1\) bundle is \(S^1\)-equivariant under the \(S^1\) action.

By the construction, \(\hat{S}^1 \times L\) is spin, and hence the trivial \(\text{Spin}^c\) bundle is \(S^1\)-equivariant by Lemma 4.1. Any other \(\text{Spin}^c\) structure is given by the tensor product over \(S^1\) of the trivial \(\text{Spin}^c\) bundle and an \(S^1\) bundle, both of which are \(S^1\)-equivariant bundles. Therefore any \(\text{Spin}^c\) bundle of \(\hat{S}^1 \times L\) is \(S^1\)-equivariant.

This completes all the proof. \(\square\)

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