Complexity of virtual 3-manifolds

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Abstract. Virtual 3-manifolds were introduced by Matveev in 2009 as natural generalizations of classical 3-manifolds. In this paper, we introduce a notion of complexity for a virtual 3-manifold. We investigate the values of the complexity for virtual 3-manifolds presented by special polyhedra with one or two 2-components. On the basis of these results, we establish the exact values of the complexity for a wide class of hyperbolic 3-manifolds with totally geodesic boundary.

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§1. Introduction

Matveev [1] introduced the concept of a virtual 3-manifold, which generalized the classical concept of a 3-manifold. A virtual manifold is an equivalence class of so-called special polyhedra. Each virtual manifold determines a 3-manifold with nonempty boundary and, possibly, $RP^2$-singularities. Many properties and invariants of 3-manifolds, such as Turaev-Viro invariants [2], extend to virtual manifolds.

The complexity of a manifold is an important invariant in the theory of 3-manifolds, see [3]. The problem of calculating the complexity is very difficult. To this day, exact values of the complexity are known only for a finite number of tabulated manifolds [4], [5] and for certain infinite families of manifolds with boundary [6]–[9], closed manifolds [10], [11] and manifolds with cusps [12]–[14]. Estimates of the complexity were obtained for some infinite families of manifolds in [15]–[20].

In this paper we introduce the complexity of a virtual 3-manifold. We investigate the values of the complexity for virtual manifolds presented by special polyhedra with one or two 2-components. On the basis of these results we establish the exact values of the complexity for a wide class of hyperbolic 3-manifolds with totally geodesic boundary.

The paper is organized as follows. In §2 we present some basic facts from the theory of simple and special polyhedra, define a virtual 3-manifold and discuss

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invariants of virtual manifolds. In §3 we introduce the notion of the complexity of a virtual manifold, investigate some of its properties and establish two-sided estimates for the complexity. We also state Theorem 6 and establish Lemma 4, which are devoted to computing the complexity of virtual manifolds presented by special polyhedra with two 2-components. Section 4 is completely devoted to the proof of Theorem 6. For the reader’s convenience we include the definitions of the $\varepsilon$-invariant and Turaev-Viro invariants, which play a key role in the proof. In §5 we discuss the relationship between our results about the complexity of virtual manifolds and the complexity of classical 3-manifolds. We also prove Theorem 9, giving exact values of the complexity for hyperbolic 3-manifolds with totally geodesic boundary that have special spines with two 2-components. In §6 we give examples of two infinite families of 3-manifolds satisfying the conditions of Theorem 9.

§2. Virtual 3-manifolds

First, we recall some basic facts from the theory of simple and special polyhedra developed by Matveev (see [3], Ch. 1). A compact two-dimensional polyhedron $P$ is said to be simple if the link of each point $x \in P$ is homeomorphic either to a circle (such a point $x$ is called nonsingular), to a graph consisting of two vertices and three edges joining them (such a point $x$ is called a triple point), or to the complete graph $K_4$ with four vertices (such a point $x$ is called a true vertex). Connected components of the set of all nonsingular points are called 2-components of $P$, while connected components of the set of all triple points are called triple lines of $P$. The set of singular points of $P$ (that is, the union of triple lines and true vertices) is called the singular graph of $P$. Every simple polyhedron is naturally stratified. In this stratification each 2-dimensional stratum (a 2-component) is a connected component of the set of nonsingular points, strata of dimension 1 are (open or closed) triple lines, and strata of dimension 0 are true vertices.

A simple polyhedron is special if each of its 1-dimensional strata is an open 1-cell, and each of its 2-components is an open 2-cell. A singular graph of a special polyhedron has at least one true vertex and is a 4-regular graph. Therefore, it is natural to call triple lines of a special polyhedron edges.

For a special polyhedron $P$ with at least two true vertices, a $T$-move on $P$ removes a proper subpolyhedron $E_T \subset P$, shown on the left of Figure 1, and glues on a proper subpolyhedron $E'_T \subset P$ instead, as shown on the right of Figure 1. Note that $T$ increases by 1 both the number of true vertices and the number of 2-components of $P$, while the inverse move $T^{-1}$ decreases both these numbers by 1. The moves $T$ and $T^{-1}$ do not change the Euler characteristic of the special polyhedron.

For each 2-component $\xi$ of a special polyhedron $P$ there is a characteristic map, $f: D^2 \rightarrow P$, which carries the interior of the disc $D^2$ onto $\xi$ homeomorphically and whose restriction to $S^1 = \partial D^2$ is a local embedding. We will call the curve $f|_{\partial D^2}: \partial D^2 \rightarrow P$ (and its image $f|_{\partial D^2}(\partial D^2)$) the boundary curve $\partial \xi$ of $\xi$. Let $A^2 \cup D^2$ denote the annulus $S^1 \times I$ with a disc $D^2$ attached along its middle circle, and denote by $M^2 \cup D^2$ a Möbius band with a disc $D^2$ attached along its middle circle. We say that the boundary curve $\partial \xi$ has a trivial (nontrivial)
normal bundle if the characteristic map $f : D^2 \to P$ of $\xi$ extends to a local embedding $f(A^2 \cup D^2) : A^2 \cup D^2 \to P$ (to a local embedding $f(M^2 \cup D^2) : M^2 \cup D^2 \to P$, respectively).

We say that the boundary curve $\partial \xi$ is short if it traverses precisely three true vertices of $P$ and passes through each of them only once.

We now state a condition on a special polyhedron which allows us to apply the move $T^{-1}$ to it.

Remark 1. A special polyhedron allows a move $T^{-1}$ if and only if at least one of its boundary curves is short and has a trivial normal bundle.

The concept of a virtual manifold was introduced in [1].

Definition 1. We say that two special polyhedra are equivalent if one of them can be transformed into the other one by a finite sequence of moves $T^{\pm 1}$. A virtual 3-manifold is the equivalence class $[P]$ of a special polyhedron $P$.

The moves $T$ and $T^{-1}$ do not change the Euler characteristic $\chi(P)$ of a special polyhedron $P$ nor the number of its 2-components whose boundary curves have nontrivial normal bundles. This implies the following lemma.

Lemma 1. The Euler characteristic $\chi(P)$ of a special polyhedron $P$ and the number of 2-components of $P$ whose boundary curves have nontrivial normal bundles are invariants of the virtual 3-manifold $[P]$.

Every special polyhedron $P$ determines a 3-manifold $W(P)$ with a nonempty boundary and $RP^2$-singularities (see [1] and [3], §1.1.5).

Definition 2. A compact three-dimensional polyhedron $W$ is called a 3-manifold with $RP^2$-singularities if the link of any point of $W$ is either a 2-sphere, a 2-disc, or $RP^2$. The set of points of $W$ whose links are 2-discs form the boundary $\partial W$ of $W$.

We shall now describe the construction of the manifold $W(P)$ following [1]. Replacing each true vertex of $P$ by a 3-ball and each triple line of $P$ by a handle of index 1, we obtain a (possibly, non-orientable) handlebody $H$ such that each 2-component $\xi$ of $P$ meets $\partial H$ along a single circle $c = \xi \cap \partial H$. The rest of the construction involves two cases, depending on the type of the normal bundle of
the boundary curve \( \partial \xi \). If the normal bundle of \( \partial \xi \) is trivial, then a regular neighbourhood \( N(c) \) of \( c \) in \( \partial H \) is an annulus. In this case we thicken the disc \( \xi \setminus (\xi \cap H) \) to an index 2 handle. If the normal bundle of \( \partial \xi \) is nontrivial, then a regular neighbourhood \( N(c) \) of \( c \) in \( \partial H \) is a Möbius strip. In this case we attach to the boundary \( \partial H \) of \( H \) a disc \( D^2 \) along the circle \( \partial N(c) \) and replace \( \xi \setminus (\xi \cap H) \) with a cone over the projective space \( RP^2 = N(c) \cup D^2 \). Repeating this process for all 2-components of \( P \) we obtain a 3-manifold \( W(P) \) with nonempty boundary and \( RP^2 \)-singularities.

The following theorem shows that each virtual manifold determines a 3-manifold with \( RP^2 \)-singularities and nonempty boundary.

**Theorem 1** (see [1], Theorem 3). The correspondence \( P \to W(P) \) induces a well defined surjection \( \varphi : \mathcal{V} \to \mathcal{W} \) from the set \( \mathcal{V} \) of all virtual 3-manifolds onto the set \( \mathcal{W} \) of all 3-manifolds with \( RP^2 \)-singularities and nonempty boundary.

If the boundary curves of all 2-components of a special polyhedron \( P \) have trivial normal bundles, then by construction \( W(P) \) is a genuine 3-manifold without singularities. Moreover, \( P \) is embedded into \( W(P) \) so that \( W(P) \setminus P \) is homeomorphic to \( \partial W(P) \times (0,1) \). In this case \( P \) is a spine of \( W(P) \) in the following sense.

**Definition 3.** Let \( M \) be a connected compact 3-manifold. A compact two-dimensional polyhedron \( P \subset M \) is called a spine of \( M \) if either \( \partial M \neq \emptyset \) and \( M \setminus P \) is homeomorphic to \( \partial M \times (0,1) \) or \( \partial M = \emptyset \) and \( M \setminus P \) is homeomorphic to an open ball.

A spine of a disconnected 3-manifold is the union of spines of its connected components.

A spine of a 3-manifold is said to be special if it is a special polyhedron.

**Theorem 2** (see [21]). Any compact 3-manifold has a special spine.

The importance of the role the moves \( T \) and \( T^{-1} \) play is clear from the following theorem.

**Theorem 3** (see [3], Theorem 1.2.5). Let \( P_1 \) and \( P_2 \) be special polyhedra having at least two true vertices each. Then the following hold.

1) If \( P_1 \) and \( P_2 \) are special spines of the same 3-manifold, then \( P_1 \) can be transformed into \( P_2 \) by a finite sequence of moves \( T^{\pm 1} \).

2) If \( P_1 \) can be transformed into \( P_2 \) by a finite sequence of moves \( T^{\pm 1} \) and if one of \( P_1 \) or \( P_2 \) is a special spine of a 3-manifold, then so is the other.

**Remark 2.** Let \( P \) be a special spine of a 3-manifold \( M \). Suppose that \( P \) has at least two true vertices. Theorem 3 implies that the equivalence class \([P]\) of \( P \) consists precisely of the special spines of \( M \) with at least two true vertices.

**Remark 3.** Consider the restriction of the map \( \varphi : \mathcal{V} \to \mathcal{W} \) in Theorem 1 to the set of classes of special spines with at least two true vertices. Theorems 2 and 3 imply that this restriction is a bijection on the set of compact 3-manifolds with nonempty boundary.

Turaev-Viro invariants [2] play an important role in the theory of 3-manifolds. The following theorem justifies their use in the study of virtual manifolds.

**Theorem 4** (see [1], Theorem 4). Turaev-Viro invariants of 3-manifolds extend to the class of virtual 3-manifolds.
§ 3. Complexity of virtual 3-manifolds

Definition 4. The complexity $\text{cv}[P]$ of a virtual 3-manifold $[P]$ is equal to $k$ if the equivalence class $[P]$ contains a special polyhedron with $k$ true vertices and contains no special polyhedra with a smaller number of true vertices.

We establish some properties of the complexity $\text{cv}[P]$.

Lemma 2. Let $P$ be a special polyhedron. Then the following statements hold.

1) $\text{cv}[P] \geq 1$.

2) The following conditions are equivalent:
   (a) $\text{cv}[P] = 1$;
   (b) $P$ has exactly one true vertex;
   (c) $[P] = \{P\}$.

3) If $P$ has exactly two true vertices then $\text{cv}[P] = 2$.

Proof. 1) This follows from the definition, since each special polyhedron has at least one true vertex.

2) The claim is obvious since each one-vertex special polyhedron is the only element in its equivalence class.

3) If $P$ has exactly two true vertices, then $\text{cv}[P] \leq 2$. The second part of the lemma implies that $\text{cv}[P] \neq 1$. This completes the proof.

Note the following relationship between the number of 2-components of a special polyhedron and the number of boundary components of the corresponding manifold with $RP^2$-singularities.

Lemma 3. Let $P$ be a special polyhedron with $d$ 2-components and let $b$ denote the number of boundary components of the 3-manifold $W(P)$ with $RP^2$-singularities. Then $d \geq b$.

By the above construction, the 3-manifold $W(P)$ with $RP^2$-singularities has a nonempty boundary, that is, $b \geq 1$. The proof of Lemma 3 is similar to that of Lemma 2.1 in [6].

Using Lemma 3 we establish two-sided estimates for the complexity.

Theorem 5. Let $P$ be a special polyhedron with $n \geq 2$ true vertices and with $d$ 2-components. Let $b$ denote the number of boundary components of the 3-manifold $W(P)$ with $RP^2$-singularities. Then

$$n - (d - b) \leq \text{cv}[P] \leq n.$$ 

Proof. The definition of complexity implies the right-hand inequality $\text{cv}[P] \leq n$. We will prove the left-hand inequality. Since the special graph of any special polyhedron is 4-regular, $\chi(P) = n - 2n + d = d - n$. We choose a special polyhedron in the class $[P]$ with $\text{cv}[P]$ true vertices and denote it by $P'$. Let $m$ be the number of 2-components of $P'$. Then $\chi(P') = m - \text{cv}[P]$. The equality $\chi(P') = \chi(P)$ implies that $\text{cv}[P] = n - d + m$. Since the map $\varphi: \nu \rightarrow \nu'$ considered above is well defined, the manifolds $W(P')$ and $W(P)$ are homeomorphic. Applying Lemma 3 to the polyhedron $P'$ we get $m \geq b$. Therefore, $\text{cv}[P] \geq n - d + b$, which completes the proof of the theorem.
Corollary 1. If, under the assumptions of Theorem 5, \( d = b \), then \( cv[P] = n \). In particular, if the special polyhedron \( P \) has exactly one 2-component, then \( cv[P] = n \).

The following theorem allows us to calculate the complexity of virtual manifolds presented by special polyhedra with two 2-components. Recall that the boundary curve of a 2-component is said to be short if it passes through three true vertices and passes through each of them exactly once.

**Theorem 6.** Let \( P \) be a special polyhedron with \( n \geq 2 \) true vertices and with two 2-components whose boundary curves are not short. Then \( cv[P] = n \).

The proof of Theorem 6 occupies §4 below.

**Lemma 4.** Let \( P \) be a special polyhedron with \( n \geq 2 \) true vertices and with two 2-components. Suppose that the boundary curve of at least one of the 2-components is short. Then:

1) If the short boundary curve has a trivial normal bundle, then \( cv[P] = n - 1 \).

2) If both boundary curves have nontrivial normal bundles, then \( cv[P] = n \).

**Proof.**
1) By Remark 1 we can apply the move \( T^{-1} \) to \( P \). This gives a special polyhedron \( P' \) equivalent to \( P \). Therefore, \( cv[P] = cv[P'] \). By construction, the polyhedron \( P' \) has \( n - 1 \) true vertices and one 2-component. By Corollary 1 we have \( cv[P'] = n - 1 \).

2) Since \( P \) has two 2-components, Lemma 3 implies that the number of boundary components of the manifold \( W(P) \) is either 1 or 2. Then \( n - 1 \leq cv[P] \leq n \) by Theorem 5. We will now show that \( cv[P] \neq n - 1 \).

Suppose, on the contrary, that \( cv[P] = n - 1 \), that is, \( P \) is equivalent to a special polyhedron \( P' \) with \( n - 1 \) true vertices. Since all polyhedra in the same equivalence class have the same Euler characteristic, \( \chi(P') = \chi(P) = 2 - n \). Therefore, the polyhedron \( P' \) has exactly one 2-component.

If both boundary curves of \( P \) have nontrivial normal bundles, then Lemma 1 implies that \( P' \) has two 2-components whose boundary curves have nontrivial normal bundles. This contradicts the fact that \( P' \) has only one 2-component. This completes the proof.

**Remark 4.** Suppose that \( P \) is a special polyhedron with \( n \geq 2 \) true vertices and with two 2-components. Using simple combinatorial arguments it is easy to show that the boundary curves of the 2-components of \( P \) cannot both be short.

§4. The proof of Theorem 6

First note that if \( n = 2 \) then \( cv[P] = 2 \) by part 3) of Lemma 2. Therefore, we will assume that \( n \geq 3 \).

An argument analogous to that in the proof of Lemma 4, part 2) proves the inequalities \( n - 1 \leq cv[P] \leq n \). It remains to show that \( cv[P] \neq n - 1 \).

Suppose, on the contrary, that \( cv[P] = n - 1 \), that is, in the class \([P]\) there is a special polyhedron \( P' \) with \( n - 1 \) true vertices and hence with one 2-component.

A simple subpolyhedron contained in a special polyhedron is proper if it is distinct from the empty set and from the whole polyhedron. The following lemma describes all proper simple subpolyhedra of \( P \).
Lemma 5. The polyhedron $P$ has exactly one proper simple subpolyhedron.

Proof. As we mentioned above, the Turaev-Viro invariants extend to virtual manifolds. To prove the lemma, we use one of the resulting invariants, known as the $\varepsilon$-invariant, which is the homologically trivial part of the order 5 Turaev-Viro invariant. We give the definition of the $\varepsilon$-invariant following [3]. Consider a special polyhedron $R$ and let $\mathcal{F}(R)$ be the set of all simple subpolyhedra of $R$ including $R$ and the empty set. Set $\varepsilon = (1 + \sqrt{5})/2$, a solution of the equation $\varepsilon^2 = \varepsilon + 1$. With each $Q \in \mathcal{F}(R)$ we associate its $\varepsilon$-weight by the formula

$$w_\varepsilon(Q) = (-1)^{V(Q)}\varepsilon^{\chi(Q) - V(Q)},$$

where $V(Q)$ is the number of true vertices of the simple polyhedron $Q$ and $\chi(Q)$ is its Euler characteristic. Set

$$t(R) = \sum_{Q \in \mathcal{F}(R)} w_\varepsilon(Q).$$

As was shown in [3], the number $t(R)$ is invariant under the moves $T^{\pm 1}$. We call $t(R)$ the $\varepsilon$-invariant of the virtual manifold $[R]$ and denote it by $t[R]$.

We return to the proof of the lemma and use the $\varepsilon$-invariant to show that $P$ has at least one proper simple subpolyhedron. If this is not the case, then

$$t(P) = (-1)^n \varepsilon^{2-2n} + 1 \quad \text{and} \quad t(P') = (-1)^{n-1} \varepsilon^{3-2n} + 1.$$ 

Since $\varepsilon = (1 + \sqrt{5})/2$, the values $t(P)$ and $t(P')$ are distinct. This contradicts the fact that $P' \in [P]$. Thus, $P$ has at least one proper simple subpolyhedron.

We will show that this subpolyhedron is unique. Since any simple subpolyhedron is compact, if it contains a point of a 2-component, then it contains the whole of it. Thus, to describe a simple subpolyhedron of $P$ it is enough to indicate which 2-components of $P$ it includes (its triple points and true vertices will then be determined uniquely). Since $P$ only has two 2-components, it has at most two proper simple subpolyhedra, and each of these contains exactly one 2-component. Since $P$ is connected, it has an edge traversed twice by the boundary curve of one of the 2-components and traversed once by the boundary curve of the other 2-component. The latter 2-component cannot be contained in a proper simple subpolyhedron of $P$. This completes the proof.

Let $Q$ denote the proper simple subpolyhedron of $P$ whose existence was proved in Lemma 5. Let $V(Q)$ be the number of true vertices of $Q$.

Lemma 6. The number of true vertices of $Q$ satisfies $V(Q) = n - 3$.

Proof. Since $Q$ is a proper simple subpolyhedron of $P$, we have $V(Q) < n$. Clearly,

$$t[P] = t(P) = (-1)^n \varepsilon^{2-2n} + (-1)^{V(Q)}\varepsilon^{\chi(Q) - V(Q)} + 1.$$

As in the proof of Lemma 5, the calculation of the same $\varepsilon$-invariant $t[P]$ on the polyhedron $P'$ gives $t[P'] = t(P') = (-1)^{n-1} \varepsilon^{3-2n} + 1$. It is easy to see that $t(P) = t(P')$ if and only if

$$(-1)^{V(Q)}\varepsilon^{\chi(Q) - V(Q)} = (-1)^{n-1}[\varepsilon^{3-2n} + \varepsilon^{2-2n}].$$

This equality holds if and only if the following conditions hold:
(i) $V(Q)$ and $n - 1$ have the same parity;
(ii) the inequality $\varepsilon \chi(Q) - V(Q) = \varepsilon^3 - 2n + \varepsilon^2 - 2n$ holds.
Since $\varepsilon^3 - 2n + \varepsilon^2 - 2n = \varepsilon^2 - 2n(\varepsilon + 1) = \varepsilon^2 - 2n\varepsilon^2 = \varepsilon^4 - 2n$, condition (ii) is equivalent to
\[\chi(Q) + 2n = 4 + V(Q).\] (4.1)

Now we show that $V(Q) \neq n - 1$. Denote the 2-components of $P$ by $\alpha$ and $\beta$, assuming for definiteness that $\alpha$ is contained in $Q$, and $\beta$ is not. Suppose, on the contrary, that $V(Q) = n - 1$. Then the boundary curve $\partial \beta$ passes through only one true vertex of $P$ and hence through one edge of $P$ (see Figure 2). Hence $Q = P \setminus \beta$. Therefore, $\chi(Q) = \chi(P) - 1 = 1 - n$. Obviously, equality (4.1) does not hold for these values of $V(Q)$ and $\chi(Q)$. This contradiction implies that $V(Q) \neq n - 1$.

![Figure 2. The boundary curve $\partial \beta$ passes through one true vertex and one edge of the special polyhedron.](image)

Recall that the special polyhedron $P$ has a natural cell decomposition: a 2-cell is a connected component of the set of nonsingular points, a 1-cell is an edge, and a 0-cell is a true vertex. The cell decomposition of $P$ induces a cell decomposition of its simple subpolyhedron $Q$. Denote the number of 0-cells of $Q$ by $k_0$ and the number of 1-cells of $Q$ by $k_1$. Hence $k_0 \leq n$ and $k_1 \leq 2k_0$. Since $Q$ has exactly one 2-cell (that is the 2-component $\alpha$), we have $\chi(Q) = k_0 - k_1 + 1 \geq 1 - k_0 \geq 1 - n$. Hence, using (4.1) we obtain $V(Q) \geq n - 3$. But $V(Q)$ has the same parity as $n - 1$ and $V(Q) \neq n - 1$, and so $V(Q) = n - 3$. The proof is complete.

Recall that by the choice of notation in the proof of Lemma 6, the 2-component $\beta$ of the polyhedron $P$ is not contained in the simple subpolyhedron $Q$.

**Lemma 7.** The boundary curve $\partial \beta$ of the 2-component $\beta$ of the polyhedron $P$ has the following properties:
(a) $\partial \beta$ passes along each edge of $P$ at most once;
(b) $\partial \beta$ passes through three true vertices of $P$ and goes through each of them at most twice.

**Proof.** We will show that the simple subpolyhedron $Q$ contains all the true vertices of $P$ (they are not necessarily true vertices in $Q$) and all its edges. That is,
in the above notation, we have $k_0 = n$ and $k_1 = 2n$. Indeed, if $k_0 < n$ then $k_1 < 2n - 3$. Hence $\chi(Q) = k_0 - k_1 + 1 > k_0 - 2n + 4$. Since $V(Q) = n - 3$, by (4.1) we have $\chi(Q) = 1 - n$. Therefore, $k_0 < n - 3$, which contradicts the condition $k_0 \geq V(Q) = n - 3$. Thus $k_0 = n$. Then the equalities $k_0 - k_1 + 1 = 1 - n$ and $k_0 = n$ imply that $k_1 = 2n$.

Since the simple subpolyhedron $Q$ contains all the edges of $P$, the curve $\partial \beta$ passes along each edge of $P$ at most once, and therefore passes through each true vertex of $P$ at most twice. Since $V(Q) = n - 3$, the curve $\partial \beta$ passes through exactly three true vertices of $P$. The proof is complete.

We apply a move $T$ to the polyhedron $P'$ in an arbitrary way. Denote the resulting special polyhedron by $P''$. The polyhedron $P''$ has $n$ true vertices and two 2-components and belongs to the equivalence class $[P]$.

Lemma 8. The virtual 3-manifolds $[P]$ and $[P'']$ have distinct Turaev-Viro invariants.

Proof. We will show that the order 7 Turaev-Viro invariants of these virtual manifolds differ. We recall the construction of Turaev-Viro invariants [2] using the exposition in [3] and calculate the value of the order 7 invariant for $[P]$ and $[P'']$.

Let $R$ be a special polyhedron, $V(R)$ its set of vertices, and $U(R)$ its set of 2-components. A colouring of $R$ in the colour palette $C = \{0, 1, \ldots, r - 2\}$, where $r \geq 3$ is an integer, is a map $\zeta : U(R) \rightarrow C$. An unordered triple $i, j, k$ of colours from the palette $C$ is admissible if

- $i + j \geq k$, $j + k \geq i$, $k + i \geq j$;
- $i + j + k$ is even;
- $i + j + k \leq 2r - 4$.

A colouring $\zeta$ of a special polyhedron $R$ is admissible if the colours of any three 2-components adjacent to the same edge form an admissible triple. The set of all admissible colourings of $R$ in the palette $C$ will be denoted by $\text{Col}(R)$.

We will describe all admissible colourings of the special polyhedra $P$ and $P''$ when $r = 7$. It was established in Lemma 7 that the boundary curve $\partial \beta$ passes through exactly three true vertices of $P$. Let $m$ denote the number of those vertices traversed by the curve $\partial \beta$ exactly twice. By the assumptions of Theorem 6 the boundary curves of the 2-components of $P$ are not short, and therefore the curve $\partial \beta$ traverses at least one true vertex of $P$ twice. Thus $1 \leq m \leq 3$. As in the proof of Lemma 6, we let $\alpha$ denote the second 2-component of $P$. We divide all the edges of $P$ into two types. An edge is of type I, if only the curve $\partial \alpha$ passes along it. An edge is of type II if the curve $\partial \alpha$ passes along it twice and the curve $\partial \beta$ passes along it once.

Recall that the special polyhedron $P''$ is obtained by applying the move $T$ to the special polyhedron $P'$. Therefore, the polyhedron $P''$, like $P$, has two 2-components, and the boundary curve of one of its 2-components is short. Denote this 2-component of $P''$ by $\delta$ and the second 2-component of $P''$ by $\gamma$. As with $P$, we can divide all the edges of $P''$ into two types. An edge is of type I, if only the curve $\partial \gamma$ passes along this edge, and of type II if the curve $\partial \gamma$ passes along it twice and the curve $\partial \delta$ passes along it once.

Let $(k, \ell)$ denote a colouring of $P$ such that the 2-component $\alpha$ is painted the colour $k$ and the 2-component $\beta$ is painted the colour $\ell$. Analogously, let $(k, \ell)$
denote a colouring of $P''$ such that the 2-component $\gamma$ is painted the colour $k$ and the 2-component $\delta$ is painted the colour $\ell$. A colouring $(k, \ell)$ is called monochrome if $k = \ell$. Consider the palette $\mathcal{C} = \{0, 1, 2, 3, 4, 5\}$ for $r = 7$. From the definition of an admissible colouring it follows immediately that each of the polyhedra $P$ and $P''$ has four admissible colourings. More precisely,

$$\text{Col}(P) = \text{Col}(P'') = \{(0, 0), (2, 2), (2, 0), (2, 4)\}.$$ 

We recall the relevant notation from the theory of quantum invariants. Let $q$ be a $2r$th root of unity such that $q^2$ is a primitive root of unity of degree $r$. For a nonnegative integer $n$ set

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{and} \quad [n]! = [n][n-1] \ldots [2][1].$$

In particular, $[1]! = [1] = 1$. By definition, $[0]! = 1$.

To each colour $i \in \mathcal{C}$ we assign a weight

$$w_i = (-1)^i [i + 1].$$

In particular,

$$w_0 = 1, \quad w_2 = [3] \quad \text{and} \quad w_4 = [5].$$

(4.2)

For a nonnegative integer $m$ we write $\hat{m} = m/2$. For an admissible triple $i, j, k$ put

$$\Delta(i, j, k) = \left(\frac{[i + j - k][j + k - i][k + i - j]}{[i + j + k + 1]}\right)^{1/2}.$$ 

In particular,

$$\Delta(0, 0, 0) = 1, \quad \Delta(2, 2, 2) = \frac{1}{([4]!)^{1/2}},$$

$$\Delta(2, 2, 0) = \frac{1}{[3]^{1/2}} \quad \text{and} \quad \Delta(2, 2, 4) = \frac{[2]!}{([5]!)^{1/2}}.$$ 

Consider a regular neighbourhood $N(v)$ of a true vertex $v$ of a special polyhedron $R$. The intersection of $N(v)$ with the union of 2-components of $R$ consists of six discs which, as in [3], are called the wings of $N(v)$. Two wings are opposite if their closures in $N(v)$ meet only in $v$. It is clear that the wings of $N(v)$ are divided into three pairs of opposite wings. Note that any colouring of $R$ induces a colouring of the wings of $N(v)$. If the colouring of $R$ is admissible and the three pairs of opposite wings are painted the colours $i$ and $l$, $j$ and $m$, $k$ and $n$ (see Figure 3), then with the vertex $v$ we associate a quantum $6j$-symbol

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_v \in \mathcal{C}.$$ 

The $6j$-symbol is defined by the formula

$$\begin{vmatrix} i & j & k \\ l & m & n \end{vmatrix}_v = \sum_{z} (-1)^{z}[z + 1]! A_{z, (i \; j \; k)} B_{z, (i \; j \; k)} C_{z, (i \; j \; k)}.$$
where

$$A(i, j, k, l, m, n) = (\sqrt{-1})^{(i+j+k+l+m+n)} \Delta(i, j, k) \Delta(i, m, n) \Delta(j, l, n) \Delta(k, l, m),$$

$$B(z, i, j, k, l, m, n) = [z - \widehat{i} - \widehat{j} - \widehat{k}]! [z - \widehat{i} - \widehat{m} - \widehat{n}]! [z - \widehat{j} - \widehat{l} - \widehat{n}]! [z - \widehat{k} - \widehat{l} - \widehat{m}]!,$$

$$C(z, i, j, k, l, m, n) = [\widehat{i} + \widehat{j} + \widehat{l} + \widehat{m} - z]! [\widehat{i} + \widehat{k} + \widehat{l} + \widehat{n} - z]! [\widehat{j} + \widehat{k} + \widehat{m} + \widehat{n} - z]!$$

and the sum is taken over all the integers $z$ such that $a \leq z \leq b$, where

$$a = \max\{\widehat{i} + \widehat{j} + \widehat{k}, \widehat{i} + \widehat{m} + \widehat{n}, \widehat{j} + \widehat{l} + \widehat{n}, \widehat{k} + \widehat{l} + \widehat{m}\},$$

$$b = \min\{\widehat{i} + \widehat{j} + \widehat{l} + \widehat{m}, \widehat{i} + \widehat{k} + \widehat{l} + \widehat{n}, \widehat{j} + \widehat{k} + \widehat{m} + \widehat{n}\}.$$
\[ A \left( \begin{array}{ccc} 2 & 2 & 4 \\ 2 & 2 & 4 \end{array} \right) = \frac{([2]!)^4}{([5]!)^2}, \quad B \left( \begin{array}{ccc} 2 & 2 & 4 \\ 2 & 2 & 4 \end{array} \right) = 1, \quad C \left( \begin{array}{ccc} 2 & 2 & 4 \\ 2 & 2 & 4 \end{array} \right) = 1. \]

Thus,
\[ \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} = 1, \quad \begin{array}{ccc} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array} = 1, \quad \begin{array}{ccc} 2 & 2 & 2 \\ 2 & 2 & 2 \end{array} = 1, \quad \begin{array}{ccc} 2 & 2 & 4 \\ 2 & 2 & 4 \end{array} = 1. \quad (4.3) \]

In general, the weight of an admissible colouring \( \zeta \in \text{Col}(R) \) is defined by the rule
\[ w(\zeta, R) = \prod_{v \in V(R)} \left[ \begin{array}{ccc} i & j & k \\ l & m & n \end{array} \right] \prod_{v \in U(R)} w_{\zeta(u)}. \]

The weight of the special polyhedron \( R \) is defined as the sum of the weights of all its admissible colourings:
\[ w(R) = \sum_{\zeta \in \text{Col}(R)} w(\zeta, R). \]

By Theorem 4 the weight \( w(R) \) of \( R \) is the order \( r \) Turaev-Viro invariant of the virtual manifold \( [R] \).

Now we look at our polyhedra \( P \) and \( P'' \) again. By definition, the weight of \( P \) is the sum of the weights of all admissible colourings:
\[ w(P) = w((0, 0), P) + w((2, 0), P) + w((2, 2), P) + w((2, 4), P), \]

and similarly,
\[ w(P'') = w((0, 0), P'') + w((2, 0), P'') + w((2, 2), P'') + w((2, 4), P''). \]

Since the special polyhedra \( P \) and \( P'' \) have the same number of true vertices, the weights of their monochrome colourings obviously coincide:
\[ w((0, 0), P) = w((0, 0), P'') \quad \text{and} \quad w((2, 2), P) = w((2, 2), P''). \]

Next we compare the weights of the colouring \((2, 0)\). Recall that \( m \), where \( 1 \leq m \leq 3 \), is the number of true vertices of \( P \) traversed by the curve \( \partial \beta \) exactly twice. Taking (4.2) into account we obtain
\[ w((2, 0), P) = \left[ \begin{array}{ccc} 2 & 2 & 2 \\ 2 & 2 & 2 \end{array} \right]^{3-m} \left[ \begin{array}{ccc} 2 & 2 & 0 \\ 2 & 2 & 0 \end{array} \right]^m \left[ \begin{array}{ccc} 2 & 2 & 2 \\ 2 & 2 & 2 \end{array} \right]. \]

It follows from (4.3) that
\[ w((2, 0), P) = w((2, 0), P''). \]
Now we compare the weights of the colouring $(2, 4)$:

$$w((2, 4), P) = \left| \begin{array}{ccc} 2 & 2 & 2 \\ 2 & 2 & 4 \\ 2 & 2 & 4 \end{array} \right| \left| \begin{array}{ccc} n-3 & m & 2 \\ 2 & 2 & 4 \end{array} \right| \left| \begin{array}{ccc} 3-m & 3 \end{array} \right|,$$

$$w((2, 4), P'') = \left| \begin{array}{ccc} 2 & 2 & 2 \\ 2 & 2 & 4 \\ 2 & 2 & 4 \end{array} \right| \left| \begin{array}{ccc} n-3 & 2 & 3 \end{array} \right| \left| \begin{array}{ccc} 2 & 2 & 4 \end{array} \right| \left| \begin{array}{ccc} 3 \end{array} \right| \left| \begin{array}{ccc} 3 \end{array} \right|.$$

Using formulae (4.3) and the condition $m > 1$ we obtain

$$w((2, 4), P) \neq w((2, 4), P'')$$

and hence

$$w(P) \neq w(P'').$$

Thus, the virtual manifolds $[P]$ and $[P'']$ differ by the order 7 Turaev-Viro invariant. This completes the proof.

We have proved that the equivalent polyhedra $P$ and $P''$, and hence, $P$ and $P'$, have distinct Turaev-Viro invariants. This contradiction shows that the original assumption that there exists a special polyhedron $P' \in [P]$ with $n-1$ true vertices is false. This completes the proof of Theorem 6.

§ 5. Complexity of manifolds

In this section we discuss the relationship between the results above on the complexity of virtual 3-manifolds with the complexity theory of classical 3-manifolds.

A compact polyhedron $P$ is said to be almost simple if the link of each of its points can be embedded into the complete graph $K_4$ with four vertices. The points whose links are homeomorphic to $K_4$ are the true vertices of $P$. A spine of a 3-manifold is said to be almost simple if it is an almost simple polyhedron.

**Definition 5.** The complexity $c(M)$ of a compact 3-manifold $M$ is the non-negative integer $n$ such that $M$ has an almost simple spine with $n$ true vertices and has no almost simple spines with fewer true vertices.

The following theorem shows that the complexity of a 3-manifold can be bounded above by the complexity of the corresponding virtual manifold.

**Theorem 7.** Let $P$ be a special spine of a 3-manifold $M$. Then $c(M) \leq cv[P]$.

**Proof.** By Definition 5, the complexity of $M$ does not exceed the number of true vertices of any almost simple spine of $M$. Consequently, $c(M)$ does not exceed the number of true vertices of any special spine of $M$ as all special polyhedra are almost simple. Therefore, $c(M) \leq cv[P]$.

**Remark 5.** The inequality in Theorem 7 is strict for some compact 3-manifolds. For example, if $P$ is a special spine of a manifold $M$ of complexity 0, then $c(M) < cv[P]$ because by Lemma 2 we have $cv[P] \geq 1$. The closed manifolds of complexity 0 are $S^3$, $\mathbb{RP}^3$, and the lens space $L_{3, 1}$. The orientable irreducible and boundary irreducible 3-manifolds with nonempty boundary and complexity 0 are described in [22].
We describe a class of 3-manifolds for which the inequality in Theorem 7 becomes equality.

**Theorem 8.** Let $P$ be a special spine of an irreducible boundary irreducible 3-manifold $M$ such that $c(M) \neq 0,1$ and all proper annuli in $M$ are inessential. Then $c(M) = cv[P]$.

**Proof.** Let $\hat{P}$ be a minimal almost simple spine of $M$, that is, an almost simple spine with $c(M)$ true vertices. Since $M$ is irreducible and boundary irreducible and since all proper annuli in $M$ are inessential, Theorem 2.2.4 in [3] implies that there is a special spine $P_1$ of $M$ having the same number of true vertices as $\hat{P}$, namely $c(M)$. The special spines $P_1$ and $P$ have at least two true vertices each, because $c(M) > 2$. Therefore, by Remark 2 we have $[P_1] = [P]$ and $cv[P] = cv[P_1] = c(M)$.

It is known that all hyperbolic 3-manifolds are irreducible, have irreducible boundary and contain no essential annuli. Therefore, Theorem 8 implies the following corollary.

**Corollary 2.** Let $P$ be a special spine of a hyperbolic 3-manifold $M$ with totally geodesic boundary such that $c(M) \neq 0,1$. Then $c(M) = cv[P]$.

The next result follows from Theorem 6 and Corollary 2.

**Theorem 9.** Let $P$ be a special spine of a hyperbolic 3-manifold $M$ with totally geodesic boundary such that $P$ has $n > 2$ true vertices and two 2-components. If the boundary curves of both 2-components of $P$ are not short, then $c(M) = n$.

The complexity of manifolds which have special spines with $n > 2$ true vertices and only one 2-component was calculated in [6] in terms of ideal triangulations. We reformulate this result in terms of spines.

**Theorem 10** (see [6], Theorem 1.2). Let $P$ be a special spine of a 3-manifold $M$ such that $P$ has $n > 2$ true vertices and one 2-component. Then $c(M) = n$.

§6. Two examples

We give two examples of infinite series of 3-manifolds and their special spines satisfying the assumptions of Theorem 9. The complexity of these manifolds was previously computed by the authors in [7] and [9].

6.1. Paoluzzi-Zimmermann manifolds. We describe a two-parameter family of 3-manifolds $M_{n,k}$ with nonempty boundary, constructed by Paoluzzi and Zimmermann [23]. For an integer $n \geq 4$ consider an $n$-gonal bipyramid $B_n$ (see Figure 4), which is the union of pyramids $NV_0V_1\ldots V_{n-1}$ and $SV_0V_1\ldots V_{n-1}$, which meet each other only along the common $n$-gonal base $V_0V_1\ldots V_{n-1}$. Let $k$ be an integer such that $0 \leq k < n$ and $\gcd(n, 2-k) = 1$. For each $i = 0,\ldots, n-1$ consider a transformation $y_i$ which identifies the face $V_iV_{i+1}N$ with the face $SV_{i+k}V_{i+k+1}$ (indices are taken mod $n$ and the vertices are glued to each other in the order in which they are written).
The identifications \( \{y_0, y_1, \ldots, y_{n-1}\} \) define equivalence relations on the sets of faces, edges, and vertices of the bipyramid. It is easy to see that the faces are partitioned into pairs of equivalent faces, the edges constitute one equivalence class and so do the vertices (this is guaranteed by the above conditions on \( k \)). Denote the associated quotient space by \( M_{n,k}^* \). It is an orientable pseudomanifold with one singular point since \( \chi(M_{n,k}^*) = 1 - 1 + n - 1 = n - 1 \neq 0 \). Cutting off an open conical neighbourhood of the singular point of \( M_{n,k}^* \) we obtain a compact manifold \( M_{n,k} \) with one boundary component.

Now we construct a special spine \( P_{n,k} \) of \( M_{n,k} \). Cut \( B_n \) into \( n \) tetrahedra \( T_i = NSV_iV_{i+1}, \) where \( i = 0, 1, \ldots, n - 1 \). For each \( T_i \) consider the union \( R_i \) of the links of all four vertices of \( T_i \) in the first barycentric subdivision. The pseudomanifold \( M_{n,k}^* \) can be obtained by gluing the tetrahedra \( T_0, \ldots, T_{n-1} \) via the identifications \( \{y_0, y_1, \ldots, y_{n-1}\} \). This gluing determines a pseudotriangulation \( \mathcal{T} \) of \( M_{n,k}^* \) and induces a gluing of the corresponding polyhedra \( R_i, i = 0, \ldots, n - 1, \) together. This gluing yields a special spine \( P_{n,k} = \bigcup_i R_i \) of \( M_{n,k} \). Since each \( R_i \) is homeomorphic to a cone over \( K_4 \), the spine \( P_{n,k} \) has exactly \( n \) true vertices.

**Corollary 3** (see [7]). For every integer \( n \geq 4 \) we have \( c(M_{n,k}) = n \).

**Proof.** By the construction of the special spine \( P_{n,k} \), its 2-components are in a one-to-one correspondence with the edges of the pseudotriangulation \( \mathcal{T} \). Since \( \mathcal{T} \) has two edges, \( P_{n,k} \) has two 2-components. In addition, the boundary curves of both these 2-components are not short because each of them traverses all \( n \geq 4 \) true vertices of the spine.

Paoluzzi and Zimmermann [23] proved that the manifolds \( M_{n,k} \) are hyperbolic and have totally geodesic boundary. Thus the manifolds \( M_{n,k} \) and their special spines \( P_{n,k} \) satisfy the assumptions of Theorem 9. This implies the corollary.

### 6.2. Manifolds from [9]

We describe a family of 3-manifolds \( N_n \) constructed in [9] and having nonempty boundary. Let \( s \) be a nonnegative integer and let \( n = 5 + 4s \). We construct a plane 4-regular graph \( G_n \) with decorated vertices and edges as follows. The graph \( G_n \) has \( n \) vertices, two loops and \( n - 1 \) double
edges. At each vertex of the graph, over- and underpasses are specified (just as for a crossing in a knot diagram), and each edge is assigned an element of the cyclic group $\mathbb{Z}_3 = \{0, 1, 2\}$. The decorated graph $G_5$ is shown in Figure 5.

![Figure 5. The decorated graph $G_5$.](image)

The graph $G_5$ has a block structure: it is composed of three subgraphs $A$, $C$ and $E$ shown in Figure 6. We express this fact as $G_5 = A \cdot C \cdot E$. Each of the graphs $A$ and $E$ has one 4-valent vertex and one 2-valent vertex. The graph $C$ has one 4-valent vertex and two 2-valent vertices. The decorations of the vertices and edges of the graphs $A$, $C$ and $E$ are induced by the decoration of the graph $G_5$.

![Figure 6. Subgraphs $A$, $C$ and $E$ of $G_5$.](image)

Next, we define graphs $B$ and $D$ as shown in Figure 7. The graphs $B$ and $D$ have the same combinatorial structure as the graph $C$ and the same decorations of the vertices; however, they differ in the decorations of their edges.

![Figure 7. Graphs $B$ and $D$.](image)

Let $G_n$ be the decorated graph composed successively of the subgraph $A$, $s$ copies of the subgraph $B$, the subgraph $C$, $s$ copies of the subgraph $D$ and the subgraph $E$. In other words, $G_n = A \cdot B^s \cdot C \cdot D^s \cdot E$. The graph $G_9$ is shown in Figure 8.

![Figure 8. Graph $G_9$.](image)

Note that the graphs $G_n$ belong to the class of o-graphs defined in [24]. That paper presents an algorithm which constructs a special polyhedron from an arbitrary o-graph. According to the algorithm, to obtain the polyhedron determined by $G_n$ we have to replace the subgraphs $A$, $B$, $C$, $D$ and $E$ by the similarly named blocks shown in Figures 9 and 10. As a result of this gluing of blocks, we obtain a special polyhedron. It is proved in [24] that this polyhedron is a special spine of a compact orientable 3-manifold with nonempty boundary. Let $N_n$ and $P_n$ be...
a manifold and its special spine, respectively, constructed from the o-graph $G_n$ via this algorithm from [24].

**Corollary 4** (see [9]). *For each $n = 5 + 4s$, where $s$ is a nonnegative integer, the complexity of the manifold $N_n$ is equal to $n$."

**Proof.** According to the algorithm above, the number of true vertices of the spine $P_n$ is equal to the number of vertices of the o-graph $G_n$, that is, to $n$. The curves shown in the pictures of the blocks (see Figures 9 and 10) are joined into closed curves. It can easily be verified that for any $n$ the number of closed curves is equal to two. Since these closed curves correspond to the boundary curves of the 2-components of the spine $P_n$, we find that $P_n$ has two 2-components. It follows from the construction of the spine that the boundary curves of both 2-components of $P_n$ are not short. Moreover, it was proved in [9] that the manifolds $N_n$ are hyperbolic with totally geodesic boundary. Therefore, the manifolds $N_n$ and their special spines $P_n$ satisfy the assumptions of Theorem 9. This proves the corollary.

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