An intrinsic characterization of $p$-symmetric Heegaard splittings

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Abstract

We show that every $p$-fold strictly-cyclic branched covering of a $b$-bridge link in $S^3$ admits a $p$-symmetric Heegaard splitting of genus $g = (b - 1)(p - 1)$. This gives a complete converse to a result of Birman and Hilden, and gives an intrinsic characterization of $p$-symmetric Heegaard splittings as $p$-fold strictly-cyclic branched coverings of links.

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1 Introduction

The concept of $p$-symmetric Heegaard splittings has been introduced by Birman and Hilden (see [3]) in an extrinsic way, depending on a particular embedding of the handlebodies of the splitting in the ambient space $E^3$. The definition of such particular splittings was motivated by the aim to prove that every closed, orientable 3-manifold of Heegaard genus $g \leq 2$ is a 2-fold covering of $S^3$ branched over a link of bridge number $g + 1$ and that, conversely, the 2-fold covering of $S^3$ branched over a link of bridge number $b \leq 3$ is a closed, orientable 3-manifold of Heegaard genus $b - 1$ (compare also [2]).

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A genus \( g \) Heegaard splitting \( M = Y_g \cup \phi Y'_g \) is called \( p \)-symmetric, with \( p > 1 \), if there exist a disjoint embedding of \( Y_g \) and \( Y'_g \) into \( E^3 \) such that \( Y'_g = \tau(Y_g) \), for a translation \( \tau \) of \( E^3 \), and an orientation-preserving homeomorphism \( \rho : E^3 \to E^3 \) of period \( p \), such that \( \rho(Y_g) = Y_g \) and, if \( \mathcal{G} \) denotes the cyclic group of order \( p \) generated by \( \rho \) and \( \Phi : \partial Y'_g \to \partial Y_g \) is the orientation-preserving homeomorphism \( \Phi = \tau^{-1}|_{\partial Y_g} \phi \), the following conditions are fulfilled:

1. \( Y_g / \mathcal{G} \) is homeomorphic to a 3-ball;
2. \( \text{Fix}(\rho^h|_{Y_g}) = \text{Fix}(\rho^h|_{Y'_g}) \) for each \( 1 \leq h \leq p - 1 \);
3. \( \text{Fix}(\rho|_{Y_g}) / \mathcal{G} \) is an unknotted set of arcs in the ball \( Y_g / \mathcal{G} \);
4. there exists an integer \( p_0 \) such that \( \Phi \rho|_{\partial Y_g} \Phi^{-1} = (\rho|_{\partial Y_g})^{p_0} \).

Remark 1 By the positive solution of the Smith Conjecture \[4\] it is easy to see that necessarily \( p_0 \equiv \pm 1 \mod p \).

The map \( \rho' = \tau \rho \tau^{-1} \) is obviously an orientation-preserving homeomorphism of period \( p \) of \( E^3 \) with the same properties as \( \rho \), with respect to \( Y'_g \), and the relation \( \Phi \rho|_{\partial Y_g} \Phi^{-1} = (\rho'|_{\partial Y'_g})^{p_0} \) easily holds.

The \( p \)-symmetric Heegaard genus \( g_p(M) \) of a 3-manifold \( M \) is the smallest integer \( g \) such that \( M \) admits a \( p \)-symmetric Heegaard splitting of genus \( g \).

The following results have been established in \[2\]:

1. Every closed, orientable 3-manifold of \( p \)-symmetric Heegaard genus \( g \) admits a representation as a \( p \)-fold cyclic covering of \( S^3 \), branched over a link which admits a \( b \)-bridge presentation, where \( g = (b - 1)(p - 1) \).

2. The \( p \)-fold cyclic covering of \( S^3 \) branched over a knot of braid number \( b \) is a closed, orientable 3-manifold \( M \) which admits a \( p \)-symmetric Heegaard splitting of genus \( g = (b - 1)(p - 1) \).

Note that statement 2 is not a complete converse of 1, since it only concerns knots and, moreover, \( b \) denotes the braid number, which is greater than or equal to (often greater than) the bridge number. In this paper we...
fill this gap, giving a complete converse to statement 1. Since the coverings
involved in 1 are strictly-cyclic (see next section for details on strictly-cyclic
branched coverings of links), our statement will concern this kind of cov-
erings. More precisely, we shall prove in Theorem 2 that a p-fold strictly-
cyclic covering of $S^3$, branched over a link of bridge number $b$, is a closed,
orientable 3-manifold $M$ which admits a $p$-symmetric Heegaard splitting of
genus $g = (b - 1)(p - 1)$, and therefore has $p$-symmetric Heegaard genus $g_p(M) \leq (b - 1)(p - 1)$. This result gives an intrinsic interpretation of $p$-
symmetric Heegaard splittings as $p$-fold strictly-cyclic branched coverings of
links.

2 Main results

Let $\beta = \{(p_k(t), t) \mid 1 \leq k \leq 2n, t \in [0, 1]\} \subset E^2 \times [0, 1]$ be a geometric
2n-string braid of $E^3$ [1], where $p_1, \ldots, p_{2n} : [0, 1] \to E^2$ are continuous
maps such that $p_k(t) \neq p_{k'}(t)$, for every $k \neq k'$ and $t \in [0, 1]$, and such that
\{p_1(0), \ldots, p_{2n}(0)\} = \{p_1(1), \ldots, p_{2n}(1)\}. We set $P_k = p_k(0)$, for each $k = 1, \ldots, 2n$, and $A_i = (P_{2i-1}, 0), B_i = (P_{2i}, 0), A_i' = (P_{2i-1}, 1), B_i' = (P_{2i}, 1)$,
for each $i = 1, \ldots, n$ (see Figure 1). Moreover, we set $F = \{P_1, \ldots, P_{2n}\}$,
$F_1 = \{P_1, P_3, \ldots, P_{2n-1}\}$ and $F_2 = \{P_2, P_4, \ldots, P_{2n}\}$.

The braid $\beta$ is realized through an ambient isotopy $\hat{\beta} : E^2 \times [0, 1] \to
E^2 \times [0, 1], \hat{\beta}(x, t) = (\beta_t(x), t)$, where $\beta_t$ is an homeomorphism of $E^2$
such that $\beta_0 = \text{Id}_{E^2}$ and $\beta_t(P_i) = p_i(t)$, for every $t \in [0, 1]$. Therefore, the braid $\beta$
naturally defines an orientation-preserving homeomorphism $\hat{\beta} = \beta_1 : E^2 \to
E^2$, which fixes the set $F$. Note that $\beta$ uniquely defines $\beta$, up to isotopy of
$E^2$ mod $F$.

Connecting the point $A_i$ with $B_i$ by a circular arc $\alpha_i$ (called top arc)
and the point $A_i'$ with $B_i'$ by a circular arc $\alpha_i'$ (called bottom arc), as in
Figure 1, for each $i = 1, \ldots, n$, we obtain a $2n$-plat presentation of a link
$L$ in $E^3$, or equivalently in $S^3$. As is well known, every link admits plat
presentations and, moreover, a $2n$-plat presentation corresponds to an $n$-
bridge presentation of the link. So, the bridge number $b(L)$ of a link $L$ is the
smallest positive integer $n$ such that $L$ admits a representation by a $2n$-plat.
For further details on braid, plat and bridge presentations of links we refer to [1].

**Remark 2** A $2n$-plat presentation of a link $L \subset E^3 \subset S^3 = E^3 \cup \{\infty\}$
shuffles a $(0, n)$-decomposition $F$ $(S^3, L) = (D, A_n) \cup_{\phi'} (D', A_n')$ of the
link, where $D$ and $D'$ are the 3-balls $D = (E^2 \times \mathbb{R} - \infty, 0) \cup \{\infty\}$ and $D' =
(E^2 \times [1, +\infty) \cup \{\infty\}, A_n = \alpha_1 \cup \cdots \cup \alpha_n, A_n' = \alpha_1' \cup \cdots \cup \alpha_n'$ and $\phi' : \partial D \to \partial D'$.
Figure 1: A $2n$-plat presentation of a link.

is defined by $\phi'(\infty) = \infty$ and $\phi'(x,0) = (\tilde{\beta}(x),1)$, for each $x \in \mathbb{R}^2$.

If a $2n$-plat presentation of a $\mu$-component link $L = \bigcup_{j=1}^{\mu} L_j$ is given, each component $L_j$ of $L$ contains $n_j$ top arcs and $n_j$ bottom arcs. Obviously, $\sum_{j=1}^{\mu} n_j = n$. A $2n$-plat presentation of a link $L$ will be called special if:

(1) the top arcs and the bottom arcs belonging to $L_1$ are $\alpha_1, \ldots, \alpha_{n_1}$ and $\alpha'_1, \ldots, \alpha'_{n_1}$ respectively, the top arcs and the bottom arcs belonging to $L_2$ are $\alpha_{n_1+1}, \ldots, \alpha_{n_1+n_2}$ and $\alpha'_{n_1+1}, \ldots, \alpha'_{n_1+n_2}$ respectively, ..., the top arcs and the bottom arcs belonging on $L_\mu$ are $\alpha_{n_1+\ldots+n_{\mu-1}+1}, \ldots, \alpha_{n_1+\ldots+n_\mu} = \alpha_n$ and $\alpha'_{n_1+\ldots+n_{\mu-1}+1}, \ldots, \alpha'_{n_1+\ldots+n_\mu} = \alpha'_n$ respectively;

(2) $p_{2i-1}(1) \in \mathcal{F}_1$ and $p_{2i}(1) \in \mathcal{F}_2$, for each $i = 1, \ldots, n$.

It is clear that, because of (2), the homeomorphism $\tilde{\beta}$, associated to a $2n$-string braid $\beta$ defining a special plat presentation, keeps fixed both the sets $\mathcal{F}_1$ and $\mathcal{F}_2$. Although a special plat presentation of a link is a very particular case, we shall prove that every link admits such kind of presentation.

**Proposition 1** Every link $L$ admits a special $2n$-plat presentation, for each $n \geq b(L)$.

**Proof.** Let $L$ be presented by a $2n$-plat. We show that this presentation is equivalent to a special one, by using a finite sequence of moves on the plat presentation which changes neither the link type nor the number of plats. The moves are of the four types $I$, $I'$, $II$ and $II'$ depicted in Figure 2. First
of all, it is straightforward that condition (1) can be satisfied by applying a suitable sequence of moves of type $I$ and $I'$. Furthermore, condition (2) is equivalent to the following: $(2')$ there exists an orientation of $L$ such that, for each $i = 1, \ldots, n$, the top arc $\alpha_i$ is oriented from $A_i$ to $B_i$ and the bottom arc $\alpha'_i$ is oriented from $B'_i$ to $A'_i$. Therefore, choose any orientation on $L$ and apply moves of type $II$ (resp. moves of type $II'$) to the top arcs (resp. bottom arcs) which are oriented from $B_i$ to $A_i$ (resp. from $A'_i$ to $B'_i$).}

A $p$-fold branched cyclic covering of an oriented $\mu$-component link $L = \bigcup_{j=1}^{\mu} L_j \subset S^3$ is completely determined (up to equivalence) by assigning to each component $L_j$ an integer $c_j \in \mathbb{Z}_p - \{0\}$, such that the set $\{c_1, \ldots, c_\mu\}$ generates the group $\mathbb{Z}_p$. The monodromy associated to the covering sends each meridian of $L_j$, coherently oriented with the chosen orientations of $L$ and $S^3$, to the permutation $(1 2 \cdots p)^{c_j} \in \Sigma_p$. Multiplying each $c_j$ by the same invertible element of $\mathbb{Z}_p$, we obtain an equivalent covering.
Following [3] we shall call a branched cyclic covering:

a) strictly-cyclic if $c_{j'} = c_{j''}$, for every $j', j'' \in \{1, \ldots, \mu\}$,

b) almost-strictly-cyclic if $c_{j'} = \pm c_{j''}$, for every $j', j'' \in \{1, \ldots, \mu\}$,

c) meridian-cyclic if $\gcd(b, c_j) = 1$, for every $j \in \{1, \ldots, \mu\}$,

d) singly-cyclic if $\gcd(b, c_j) = 1$, for some $j \in \{1, \ldots, \mu\}$,

e) monodromy-cyclic if it is cyclic.

The following implications are straightforward:

\[ a) \Rightarrow b) \Rightarrow c) \Rightarrow d) \Rightarrow e) \]

Moreover, the five definitions are equivalent when $L$ is a knot. Similar definitions and properties also hold for a $p$-fold cyclic covering of a 3-ball, branched over a set of properly embedded (oriented) arcs.

It is easy to see that, by a suitable reorientation of the link, an almost-strictly-cyclic covering becomes a strictly-cyclic one. As a consequence, it follows from Remark 1 that every branched cyclic covering of a link arising from a $p$-symmetric Heegaard splitting – according to Birman-Hilden construction – is strictly-cyclic.

Now we show that, conversely, every $p$-fold branched strictly-cyclic covering of a link admits a $p$-symmetric Heegaard splitting.

**Theorem 2** A $p$-fold strictly-cyclic covering of $S^3$ branched over a link $L$ of bridge number $b$ is a closed, orientable 3-manifold $M$ which admits a $p$-symmetric Heegaard splitting of genus $g = (b-1)(p-1)$. So the $p$-symmetric Heegaard genus of $M$ is

\[ g_p(M) \leq (b-1)(p-1). \]

**Proof.** Let $L$ be presented by a special $2b$-plat arising from a braid $\beta$, and let $(S^3, L) = (D, A_b) \cup (D', A'_b)$ be the $(0, b)$-decomposition described in Remark 2. Now, all arguments of the proofs of Theorem 3 of [2] entirely apply and the condition of Lemma 4 of [2] is satisfied, since the homeomorphism $\tilde{\beta}$ associated to $\beta$ fixes both the sets $F_1$ and $F_2$. □

As a consequence of Theorem 2 and Birman-Hilden results, there is a natural one-to-one correspondence between $p$-symmetric Heegaard splittings and $p$-fold strictly-cyclic branched coverings of links.
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