A NOTE ON FIRST-ORDER ARITHMETIC

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ABSTRACT. This paper describes a system $S'$ obtained by modifying first-order arithmetic $S$ to 'parameterise' the individual variables so that under any interpretation of $S'$, the individual variables range over all and only the individuals assigned to the numerals under this interpretation. Since $S'$ contains Peano arithmetic and is recursively axiomatised we can modify Gödel's technique to define a Gödel sentence for $S'$, say $(\forall x)R(x)$. $S'$ may be shown to be inconsistent since $(\forall x)R(x)$ must be an $S'$ theorem. Since the syntax of $S'$ and $S$ are identical however the inconsistency of $S$ itself is implied by this result.

1. Introduction

This paper describes a system $S'$ obtained by modifying first-order arithmetic to 'parameterise' the individual variables so that under any interpretation of $S'$, the individual variables range over all and only the individuals assigned to the numerals under this interpretation. Since $S'$ contains Peano arithmetic and is recursively axiomatised we can modify Gödel's technique to define a Gödel sentence for $S'$, say $(\forall x)R(x)$. $S'$ may be shown to be inconsistent since $(\forall x)R(x)$ must be an $S'$ theorem. Since the syntax of $S'$ and $S$ are identical however the inconsistency of $S$ itself is implied by this result.

Unless otherwise indicated, when the 'completeness' of a first-order theory is mentioned herein it is the semantic completeness of the theory that is referred to. This point, together with some comments on the notation / symbolism of this paper, are briefly discussed in the following subsection, which most readers should nevertheless be able to skip without loss.

1.1. A digression on notation and terminology. To avoid confusion between use of a formula (under some interpretation) and mention of the formula (viewed as an uninterpreted sequence of signs) I primarily rely on the intelligence of the reader though I sometimes use quotation symbols. While use of corners for quasi-quotation would improve precision in some contexts I generally refrain from this so that confusion will not arise with use of corners below for the 'encoding' of some $S$ or $S'$ expression within $S$ or $S'$; that is, if $B_n(x_{i_n})$ is associated with the number $n$, then "$B_n(x_{i_n})$" is the numeral $\overline{n}$. For informal metalinguistic reasoning I generally use '∨', '∧', '⟨', '⟩', '←', '→', '(Ey)', '(x)', $\{x|F(x)\}$ for (respectively): disjunction, negation, conjunction, the (material or formal) conditional, the (material or formal) biconditional, existential and universal quantification, and 'the class of objects $x$ such that $F(x)$ holds'; in discussing the arithmetisation of the syntax of $S$ or $S'$ I sometimes however follow [2] and use the symbolism of the object language to facilitate comparison with material drawn from [2]. The following section describes

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the system of interest $S'$. In the interest of brevity the following definitions are used:

**Semantic completeness:** A formal theory $T$ is *semantically complete* if every well-formed formula of $T$ that is logically valid is a $T$-theorem (formally provable within $T$ using $T$’s axioms and rules of inference).

**Propositional completeness:** A formal theory $T$ exhibits *propositional completeness* if, for the interpretation $\mathfrak{M}$ of interest, every (well-formed) sentential formula $B$ of $T$ is either: (1) a $T$-theorem or formally refutable within $T$ or (2) there exists a second sentential formula $C$ of $T$ such that both the following hold: (i) $C$ is either a $T$-theorem or formally refutable within $T$ and (ii) $C$ is assigned the same ‘meaning’ as $B$ under $\mathfrak{M}$.

**Syntactic completeness:** A formal theory $T$ is *syntactically complete* if every (well-formed) sentential formula $B$ of $T$ is either a $T$-theorem or formally refutable within $T$.

While the notion of propositional completeness is not used in this paper, I note that it is useful in considering the significance of Gödel’s incompleteness results; without this notion, it is difficult to state succinctly that the results raise the question of whether the syntactical incompleteness of the systems implies propositional incompleteness. The following section describes the system of interest $S'$.

2. The system $S'$

For brevity I assume familiarity with Mendelson’s first order number theory $S$ [2]; my description of $S'$ will accordingly be somewhat terse (to improve readability I use ‘0’ instead of the official $S$ numeral for zero ‘$a_1$’). As should be clear from the definition below, the notion of a ‘first-order language’ used throughout is strictly speaking a notion of a first-order language in an extended sense - in this extended sense, a first-order language may contain, an addition to the individual constants, ‘parameterised variables’ rather than the variables that are primitive symbols of first-order languages in the restricted sense ([2], §2.2). Since parameterised variables are neither constants, nor predicate nor function symbols, their addition to the language of $S$ does not imply the need for a revision of the general notion of an interpretation of a first-order language ([2], §2.2). It does however require a modification in the definition of the function $s^*$ used to specify whether a well formed formula $B$ is ‘satisfied’ at a particular sequence of objects in $D$. ($s^*$ assigns an individual $d$ in the domain of an interpretation $M$ to each term of a first order language $L$.)

**Definition 1.** $S'$ is the system that results from Mendelson’s $S$ when the definition of the function $s^*$ is modified to parameterise the individual variables (‘$x_1$’, ..., ‘$x_n$’, ...) so that under any interpretation these ranges over all and only the individuals assigned to the numerals:

1. The symbols, formation and inference rules, and logical and proper axioms of $S'$ are the same as those for $S$;
2. (Definition of $s'^*$) For any interpretation $M$ of $S'$ the function $s'^*$ is equal to the corresponding function $s^*$ defined for $S$ at $M$ except that where ‘$x_i$’
is any variable the following rule is adopted (c.f. \[2\]: §2.2):

\[
(s^* (x_i)) = \begin{cases} 
  s_i & \text{if for some numeral } \overline{n}: s_i = s^*(\overline{n}) \\
  s^*(0) & \text{otherwise}
\end{cases}
\]

For brevity, I will speak of any interpretation of the language of S as being identical with the corresponding interpretation of the language of $S'$ (see Proposition \[2\]). For the proof that $S'$ is inconsistent four metatheorems concerning the system are used, the first of which is the following:

**Proposition 1.** The formation and inference rules of $S'$ are (primitive) recursive in Gödel’s sense (\[1\]: 610); that is, once ‘we replace the primitive signs in some way by natural numbers’.

Proof Sketch. For a full proof of Proposition \[1\] an explicit arithmetisation of the syntax of $S'$ must be exhibited. Since the syntax of $S'$ and S are identical however, Mendelson’s existing arithmetisation of the syntax of S (\[2\]: §3.4) demonstrates that $S'$ is primitive recursive.

I turn now to the second metatheoretical proposition regarding $S'$ required for the proof that $S'$ is inconsistent which is this:

**Proposition 2.** The metatheory of an arbitrary first order theory $K$ presented in \[2\]: Chapter Two can be shown, by the arguments presented therein, to hold for $S'$ as well; in particular, the demonstrations imply that:

1. The notions of ‘truth’, ‘falsity’, and ‘satisfaction’ of an $S'$ formula under an interpretation yield the expected properties when applied to $S'$ provided they hold for an arbitrary first order theory $K$;
2. $S'$ is semantically complete (\[2\] Corollary 2.18) and a wff $B$ ‘is true in every denumerable model’ of $S'$ if and only if $\vdash_{S'} B$ (\[2\] Corollary 2.20a).

Proof. As the syntax of S and $S'$ are identical it is sufficient to establish that the key semantic metatheorems (Propositions I-XI \[2\]: §2.2) - concerning the properties of the notions ‘truth’, ‘falsity’ and ‘satisfaction’ of an $S'$ formula(s) under an interpretation - hold for $S'$ if they hold for an arbitrary first order theory $K$. To establish this proposition some notation is useful. Let $B$ be any $S'$ formula (hence also an S formula); let $\mathfrak{F}(i, B, M)$ be the metatheoretical proposition that Proposition i holds for $B$ under the interpretation M. For example, if M is the standard interpretation then $\mathfrak{F}(1a, (0 = 0), M)$ is the claim that $(0 = 0)$ is false for this interpretation if and only if $\neg (0 = 0)$ is true for this interpretation. Similarly, let $\mathfrak{S}(i)$ be the metatheoretical proposition that Proposition $i$ holds for every S formula under every interpretation and $\mathfrak{F}(i, B, M)$ be the metatheoretical proposition that Proposition $i$ holds for the S formula $B$ under an S interpretation M. Proposition $\[2\]$ holds since:

**Lemma 1.** For each of the indexed metatheoretical propositions $i$ and every interpretation $M$, for any $S'$ formula $B$ there exists an $S$ interpretation $M'$ such that:

\[
\mathfrak{S}(i) \rightarrow \mathfrak{F}(i, B, M') \rightarrow \mathfrak{F}(i, B, M)
\]

The proof by contradiction of Lemma \[1\] is straightforward. Suppose that the $i$th semantic metatheme fails for an $S'$ formula for the interpretation M. Let the S
interpretation $M'$ be chosen as follows. The domain $D'$ of $M'$ shall be the smallest subset of $D$ that includes $(0)^M$ that is closed with respect to the relation $(f_1^1)^m$; let $(f_1^1)^{m'}$, $(f_2^1)^{m'}$ and $(f_2^2)^{m'}$ be $(f_1^1)^m$ restricted to $D'$, $(f_2^1)^m$ restricted to $D'$ and $(f_2^2)^m$ restricted to $D'$ respectively; finally, let $(A_1^2)^{m'}$ be $(A_1^2)^m$ restricted to $D'$. Then the hypothesis $F(i, B, M)$ implies $\overline{F}(i, B, M')$ as required. □

The third metatheoretical proposition regarding $S'$ required for the proof that $S'$ is inconsistent is this:

**Corollary 1.** $S'$ contains Peano arithmetic in the sense that every recursive function (or relation) is representable (expressible) in $S'$.

**Proof.** By Proposition 3.24 and Corollary 3.25 respectively ([2]: §3.3): every recursive function (relation) is representable (respectively, expressible) in $S$. Since the syntax of $S$ and $S'$ is identical $S$, every $S$ theorem is an $S'$ theorem. Thus $S'$ also contains Peano arithmetic in this sense. □

The final metatheorem required to establish the inconsistency of $S'$ is that the system contains a Gödel sentence, say $(\forall x)R$. The existence of such a sentence follows from the fact that $S'$ is recursively axiomatised and (essentially) contains Peano arithmetic ([1] Theorem VI). The following section provides a summary of one method of defining this formula. The reader familiar with any standard presentation of Gödel’s first incompleteness theorem may skim or skip this section.

### 3. A Gödel sentence for $S'$

In defining a Gödel sentence for $S'$ I follow Gödel’s original approach, rather than using the diagonalisation Lemma ([2] §3.5), to facilitate comparison with the former source. In this section, all the the symbols for (arithmetised) metamathematical functions / relations (‘Gd’ etc) should, unless otherwise stated, have subscripts indicating that it is the function / relation for $S'$ that is used or mentioned (e.g. ‘Pf$_S'$’)

**Lemma 2.** If $S'$ is consistent then there exists an $S'$ formula $R$, with the free variable ‘$x_1$’, such that:

1. $\not\vdash_{S'} (\forall x_1)R$
2. For any $S'$ numeral $\pi$: $\vdash_{S'} R(\pi)$.

**Proof.** Adapting [1], let $Q$ be the relation of numbers such that (where Neg, Num, Pf, and Sub are the primitive recursive number theoretic functions / relations defined at [2] §3.4):

(3.1) $Q(x, y) \leftrightarrow Pf\{x, \text{Sub}[y, \text{Num}(y), 21]\}$

As the relation $Q$ may be shown to be (primitive) recursive, there exists (by Corollary 1 and 2 Corollary 3.25) an $S'$ formula $Q$ (associated via the arithmetisation function $g$ with a number $q$) with the free variables ‘$x_1$', ‘$x_2$'; such that, for all 2-tuples of natural numbers $(x, y)$:

(3.2) $Pf\{x, \text{Sub}[y, \text{Num}(y), 21]\} \rightarrow \vdash_{S'} Q(\pi, \gamma)$
(3.3) $Pf\{x, \text{Sub}[y, \text{Num}(y), 21]\} \rightarrow \vdash_{S'} \neg Q(\pi, \gamma)$
Let $p$ be the number associated via the arithmetisation function $g$ with the $S'$ formula $(\forall x_1)Q$. Put

\[ r = \text{Sub}[q, \text{Num}(p), 29] \]

(3.4) Hence: $g[(\forall x_1)R] = \text{Sub}[p, \text{Num}(p), 29]$

(3.5)

Using the definitions provided, Lemma 2 is equivalent to the proposition that, if $S'$ is consistent then:

(3.6) $(x)(Eu)\text{Pf}\{u, \text{Sub}[r, \text{Num}(x), 21]\}$

(3.7) $(Eu)\text{Pf}\{u, \text{Sub}[p, \text{Num}(p), 29]\}$

This follows from the proof of [1] Theorem VI:

(1) If $S'$ is consistent, then $(Eu)\text{Pf}\{u, \text{Sub}[p, \text{Num}(p), 29]\}$. Otherwise for some natural number $x$, $\text{Pf}\{x, \text{Sub}[p, \text{Num}(p), 29]\}$ yields:

(a) From 3.3 $\vdash_{S'} \neg Q(x, p)$ hence (in view of 3.4): $\vdash_{S'} \neg R(x)$

(b) From 3.5 $\vdash_{S'} (\forall x_1)R$ hence $\vdash_{S'} R(x)$

(2) If $S'$ is consistent, then $(x)(Eu)\text{Pf}\{u, \text{Sub}[r, \text{Num}(x), 21]\}$ since:

(a) From the generalisation of 3.2 and the definition of $r$:

$$ (x)(Eu)\text{Pf}\{x, \text{Sub}[p, \text{Num}(p), 29]\} \rightarrow (Eu)\text{Pf}\{u, \text{Sub}[r, \text{Num}(x), 21]\} \]$$

(3.8)

(b) 3.8 and an instance of the first-order theorem $( (x)[F(x) \rightarrow G(x)] \rightarrow [(x)F(x) \rightarrow (x)G(x)] )$ yield (via MP):

$$ (x)\text{Pf}\{x, \text{Sub}[p, \text{Num}(p), 29]\} \rightarrow (x)(Eu)\text{Pf}\{u, \text{Sub}[r, \text{Num}(x), 21]\} \]$$

(3.9)

(c) 3.9 thus follows from 3.8 and 3.9 via MP.

\[ \square \]

If $(\forall x)R$ is a Gödel sentence for $S'$ then it can also be shown that, if $S'$ is $\omega$-consistent then $\not\vdash_{S'} (\exists x)\neg R$ (2 Proposition 3.37); this proposition is not required for what follows however. The following section shows that the above implies that $S'$ is inconsistent and moreover that the standard interpretation is not a model of $S$.

### 4. A PROOF THAT S IS NOT CONSISTENT

As a preliminary step to the main result of this paper I firstly show that:

**Proposition 3.** $S'$ is inconsistent

**Proof.** (1) By the completeness of $S'$ (2 Corollary 2.20a), there must be a (non-standard) denumerable models of $S'$, say $M$, such that $\models_M \neg[(\forall x)R]$. For otherwise, $(\forall x)R$ would be true in every denumerable model and hence by the completeness of $S'$ we would have $\vdash_{S'} (\forall x)R$, contradicting the hypothesis that $S'$ is consistent (Lemma 2).

(2) Let $M$ then be a denumerable model of $S'$ such that $\models_M \neg[(\forall x)R]$ holds; it may be seen as follows that this implies the contradiction that $\models_M (\forall x)R$ also holds:
(a) By the proof of Lemma 2, the consistency of $S'$ implies that, for every numeral $\overline{n}$ we have $\vdash_{S'} R(\overline{n})$ hence $\models_M R(\overline{n})$. But since the range of $x$ is restricted to the objects in the domain that are named by the numerals, this implies that $\models_M (\forall x) R$ holds.

(3) To avoid the contradiction that $\neg \models (\forall x) R$ and $(\forall x) R$ are both true for some model $M$ we must thus conclude that $(\forall x) R$ holds for all denumerable models of $S'$; hence by the completeness of $S'$ it is an $S'$ theorem and thus $S'$ is inconsistent.

$\square$

Since the syntax of $S$ and $S'$ are identical, we have as a consequence of Proposition 3:

Corollary 2. $S$ is not consistent

I turn now to a brief discussion of the significance of these results.

5. Conclusions

For brevity I will assume, as appears to be true, that the above proof only exploits features of $S$ that are common to any orthodox, formalist account of first-order number theory. If one assumes that the metatheory of such systems is free from paradox then the result is genuinely puzzling; from this perspective, it appears obvious that all of the logical and proper axioms of $S$ are true under the standard interpretation and the inference rules preserve truth under an interpretation. The conclusion provides strong support for the claim that the metatheory is subject to paradox and that formalism thus fails to provide an adequate account of classical mathematics.

The idea that the metatheory is subject to paradox might be paraphrased as follows: when the informal metatheory of formalist first-order arithmetic is made precise a genuine contradiction arises. This does not imply that a certain contradiction is true; on the contrary, if the above proof is examined, it is clear that the law of contradiction is relied on in the informal metatheory at certain points. It would be confused or dishonest to suggest that the law of contradiction can be denied and the above proof simultaneously affirmed. The implication is rather that the devices that formalists rely on for the avoidance of paradox are not adequate to the task. On this view, the intuitions that underly the formalist theory of arithmetic are false in the sense that: given any statement of this view, some of the sentences involved are ’meaningless’ / do not express a proposition.

References

[1] Kurt Gödel, ’Über formal unentscheidbare Sätze der Principia mathematica und verwandter Systeme I’, Monatshefte für Mathematik und Physik 38, 173-198. English translation by J. van Heijenoort as ’On formally undecidable propositions of Principia Mathematica and related systems I’, 1931, From Frege to Gödel: A Source Book in Mathematical Logic (Jean van Heijenoort, ed.), Harvard University Press, Cambridge, Mass., 1967, pp. 596–616.

[2] Elliott Mendelson, Introduction to Mathematical Logic, Fifth ed., Chapman & Hall, London, 2010.