The Definitions of Special Geometry

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ABSTRACT

The scalars in vector multiplets of N=2 supersymmetric theories in 4 dimensions exhibit ‘special Kähler geometry’, related to duality symmetries, due to their coupling to the vectors. In the literature there is some confusion on the definition of special geometry, which we want to clear up, for rigid as well as for local N=2.

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1 Introduction

In four dimensions, the duality transformations are transformations between the field strengths of spin-1 fields. The kinetic terms of the vectors (we only treat abelian vectors) are generically

\[ L_1 = \frac{1}{4} (\text{Im} N_{\Lambda \Sigma}) F_{\mu \nu}^\Lambda F^{\mu \nu \Sigma} - \frac{i}{8} (\text{Re} N_{\Lambda \Sigma}) \epsilon_{\mu \nu \rho \sigma} F_{\mu \nu}^\Lambda F_{\rho \sigma}^{\Sigma}, \]

where \( \Lambda, \Sigma = 1, ..., m \), and \( N_{\Lambda \Sigma} \) depend on the scalars. The transformations of interest preserve the set of field equations and Bianchi identities and form a group \( Sp(2m, \mathbb{R}) \), defined by matrices

\[ S = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad \text{where} \quad S^T \Omega S = \Omega \quad \text{and} \quad \Omega = \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix}. \]

Under the action of this group the coupling constant matrix \( N \) should also transform:

\[ \tilde{N} = (C + DN)(A + BN)^{-1}, \]

which demands a relation between the dualities and the scalar sector.

A relation between the scalars and the vectors arises naturally in \( N = 2 \) supersymmetry vector multiplets (or larger \( N \) extensions). The geometry defined by the couplings of the complex scalars in the supergravity version has been given the name 'special Kähler geometry'. The similar geometry in rigid supersymmetry has been called 'rigid special Kähler'.

The above paragraph determines the concept of a special Kähler manifold. Various authors have proposed a definition which does not refer explicitly to supersymmetry. However, these definitions are not completely equivalent, as we will explain. We will give new definitions, and have proven their equivalence. Proofs and examples are worked out in detail in 

2 Rigid special geometry

The vector multiplets of $N = 2$ supersymmetry are constrained chiral superfields. The construction of an action starts by introducing a holomorphic function $F(X)$ of the lowest components of the multiplets, $X^A$, where $A = 1, ..., n$ labels the different multiplets. The kinetic terms of the scalars then define a Kähler manifold with Kähler potential

$$K(z, \bar{z}) = i \left( X^A \frac{\partial}{\partial X^A} \bar{F}(\bar{X}) - \bar{X}^A \frac{\partial}{\partial \bar{X}^A} F(X) \right).$$

The $X^A$ are functions of $z^\alpha$ for $\alpha = 1, ..., n$, which are arbitrary coordinates of the scalar manifolds. The straightforward choice $z^\alpha = X^A$ is called 'special coordinates'. The coupling of the $n$ vectors with the scalars is described as in Eq. 1 by the tensor

$$N_{AB} = \frac{\partial}{\partial X^A} \frac{\partial}{\partial X^B} \bar{F}.$$  

Note that the positivity condition for the Kähler metric is the same as the requirement of negative definiteness of $\text{Im} \, N_{AB}$, which guarantees the correct sign for the kinetic energies of the vectors (this condition is preserved by symplectic transformations Eq. 3).

One can check that Eq. 5 leads to the transformation Eq. 3 for $N$ if we combine $X^A$ and $F_A \equiv \partial F/\partial X^A$ in a symplectic vector. Remark that then also the Kähler potential Eq. 4 gets the symplectic form

$$K = i \langle V, \bar{V} \rangle \quad \text{where} \quad V = \begin{pmatrix} X^A \\ F_A \end{pmatrix} \quad \text{and} \quad \langle V, W \rangle \equiv V^T \Omega W.$$ 

The symplectic transformations do not always lead to the same action. However, the scalar manifold remains always the same (the Kähler potential is invariant).

This leads to our first definition of a rigid special Kähler manifold. It is a Kähler manifold with on every chart holomorphic functions $X^A(z)$ and a holomorphic function $F(X)$ such that the Kähler potential can be written as in Eq. 4. Where charts $i$ and $j$ overlap, there should be transition functions:

$$\left( \begin{array}{c} X \\ \partial F \end{array} \right)_{(i)} = e^{ic_{ij}} M_{ij} \left( \begin{array}{c} X \\ \partial F \end{array} \right)_{(j)} + b_{ij},$$

with $c_{ij} \in \mathbb{R}$; $M_{ij} \in Sp(2n, \mathbb{R})$ and $b_{ij} \in \mathbb{Q}^{2n}$. These overlap functions should satisfy the cocycle condition on overlaps of 3 charts.

\textsuperscript{a}We always suppose here and below that the metric is positive definite.
The previous definition makes use of the prepotential function $F(X)$, which is not a symplectic invariant object. In the second definition, inspired by Strominger’s definition for local special geometry, we only use objects which have a meaning in the symplectic bundle.

**Definition 2:** A Kähler manifold $\mathcal{M}$ such that there exists a complex $U(1)$-line bundle $\mathcal{L}$ with constant transition functions, and an $ISp(2n, \mathbb{R})$-vector bundle (in the sense of Eq. 3, thus with complex inhomogeneous part) $\mathcal{H}$ over $\mathcal{M}$. It should have a holomorphic section $V$ of $\mathcal{L} \otimes \mathcal{H}$, such that the Kähler form is given by Eq. 6, and

$$\langle \partial_\alpha V, \partial_\beta V \rangle = 0 \quad (8)$$

The last equation has no equivalent in $\mathcal{H}$. To argue that it is appropriate, define

$$U_\alpha = \partial_\alpha V = \partial_\alpha \left( \frac{X^A}{F_A} \right) = \left( e^A_\alpha \right). \quad (9)$$

Then $g_{\alpha \bar{\beta}} = i \langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle$. The positivity requirement for this metric implies invertibility of $e^A_\alpha$. Then the coupling constant matrix $\mathcal{N}$, Eq. 3, can be written as $\mathcal{N}_{AB} \equiv (e^A_\alpha)^{-1} h_{\alpha B}$. The condition Eq. 3 is the requirement for the matrix $\mathcal{N}$ to be symmetric, and is therefore necessary. In turn it implies the integrability $F_A = \partial_A F(X)$, and thus the existence of the function $F$, which is essential in the first definition.

The **third definition** is convenient for defining special geometry in the moduli space of Riemann surfaces. It starts from a manifold with in each chart $2n$ closed 1-forms $U_\alpha$:

$$\partial_\alpha U_\beta = 0 \quad \partial_{[\alpha} U_{\beta]} = 0 \quad (10)$$

such that

$$\langle U_\alpha, U_\beta \rangle = 0 \quad \langle U_\alpha, \bar{U}_{\bar{\beta}} \rangle = -ig_{\alpha \bar{\beta}} \quad (11)$$

The first of equations 11 replaces Eq. 3, while the second one defines the metric in this approach. In the transition functions, the inhomogeneous term (see Eq. 3) is no longer present. It reappears in the integration from $U_\alpha$ to $V$ by Eq. 11.

Rigid special geometries can be constructed by considering Riemann surfaces parametrised by $n$ complex moduli. The $U_\alpha$ are then identified with integrals of $n$ holomorphic 1-forms over a $2n$-dimensional canonical homology basis of 1-cycles. Most of the equations of the third definition are then automatically satisfied. The non-trivial one is the second one of Eq. 11.
3 (Local) Special geometry

The name special geometry has been reserved for the manifold determined by the scalars of vector multiplets in $N=2$ supergravity. The first construction of these actions made use of superconformal tensor calculus. This setup, which involves a local dilatation and an extra $U(1)$ gauge symmetry, gave insight in the structure of these manifolds. E.g. the (auxiliary) gauge field of the $U(1)$ turns into the connection of the Kähler transformations $K(z,\bar{z}) \rightarrow K(z,\bar{z}) + f(z) + \bar{f}(\bar{z})$. Also, this construction involves an extra vectormultiplet (labelled with 0), whose scalar fields $X^0$ can be gauge fixed by this dilatation and extra $U(1)$, while its vector becomes the physical graviphoton. This structure with $(n+1)$ vectormultiplets with scaling invariance is at the basis of the formulations below.

The first construction started (as in rigid susy) from chiral multiplets, and thus leads in the same way to a holomorphic prepotential $F(X_I)$, where ($I=0,1,...,n$). But because of the scaling symmetry, there is now an extra requirement: $F(X)$ should be homogeneous of weight 2, where $X$ has weight 1. The $X^I$ thus span an $n+1$ dimensional complex projective space. There are only $n$ complex physical scalars, an overall factor is not relevant. One can choose coordinates $z^\alpha$, with $\alpha=1,...,n$, such that the $X^I$ are proportional to $n+1$ holomorphic functions of these: $X^I \propto Z^I(z^\alpha)$. One can then choose a gauge condition for the dilatations. To decouple the kinetic terms of the scalars and the graviton one chooses:

$$i(\bar{X}^IF_I - \bar{F}_IX^I) = i\langle \bar{V}, V \rangle = 1 \quad \text{with} \quad V \equiv \begin{pmatrix} X^I \\ F_I \end{pmatrix}, \quad (12)$$

and $F_I \equiv \partial F(X)/\partial X^I$. One obtains an action where the scalars describe a Kähler manifold and $V = e^{K/2} v(z)$, where $v$ is holomorphic (its upper components are $Z^I(z)$, see above, and the lower are $\bar{F}_I(Z(z))$) and $K$ is the Kähler potential

$$K(z,\bar{z}) = -\log \left[ i\bar{Z}^I \frac{\partial}{\partial Z^I} F(Z) - iZ^I \frac{\partial}{\partial Z^I} \bar{F}(\bar{Z}) \right] = -\log \left[ i\langle \bar{v}(\bar{z}), v(z) \rangle \right]. \quad (13)$$

The first definition of a special manifold then starts from an $n$-dimensional Hodge-Kähler manifold with on any chart complex projective coordinates $Z^I(z)$, and a holomorphic function $F(Z^I)$ homogeneous of second degree, such that the Kähler potential is given by Eq. (13). On overlaps of charts $i$ and $j$, one should have

$$\left( \begin{array}{c} Z \\ \partial F \end{array} \right)_{(i)} = e^{f_{ij}(z)} M_{ij} \left( \begin{array}{c} Z \\ \partial F \end{array} \right)_{(j)}, \quad (14)$$
with $f_{ij}$ holomorphic and $M_{ij} \in Sp(2n + 2, \mathbb{R})$. The overlap functions satisfy the cocycle condition on overlaps of 3 charts.

Note that local special geometry involves local holomorphic transition functions. This is related to the presence of the gauge field of the local $U(1)$ in the superconformal approach.

The **second definition** is, as in the rigid case, an intrinsic symplectic formulation. It starts from an $n$-dimensional Hodge-Kähler manifold $\mathcal{M}$ with $\mathcal{L}$ a complex line bundle whose first Chern class equals the Kähler form $\mathcal{K}$. There is a $Sp(2(n + 1), \mathbb{R})$ vector bundle $\mathcal{H}$ over $\mathcal{M}$, and a holomorphic section $v(z)$ of $\mathcal{L} \otimes \mathcal{H}$, such that the Kähler potential is given by the last expression in Eq. (13), and such that

$$\langle v, \partial_\alpha v \rangle = 0 ; \quad \langle \partial_\alpha v, \partial_\beta v \rangle = 0 .$$ (15)

This definition is a rewriting of Strominger’s definition, where, however, Eq. (15) was missing. This condition is necessary to have a symmetric matrix $N_{IJ}$ (see \(\text{[6]}\) for the definition of $N$ in this context). We have constructed counterexamples \(\text{[10]}\) to show the necessity of both conditions in general. Actually, for $n > 1$ the first one is a consequence of the other constraints. For $n = 1$ the second condition is empty.

In \(\text{[6]}\) it was shown that the prepotential $F(X)$ may not exist\(\text{[10]}\). Therefore one might wonder about the equivalence of the two definitions. It has now been proven that in all such cases there exists a symplectic transformation to a form where $F$ exists\(\text{[14]}\). Although this does not invalidate the necessity of the formulation without a prepotential for physical purposes (e.g. to have actions invariant under larger parts of the isometry group and gaugings thereof), it implies that to describe the scalar manifolds, one can choose in each chart such a symplectic basis. This is the essential step in proving the equivalence of the two definitions.

There is a third definition (inspired by \(\text{[5, 6, 7]}\)), which is analogous to the third definition in the rigid case. It can be formulated using the $2(n + 1) \times 2(n + 1)$ matrix built from the symplectic vectors $V, \bar{V}, U_\alpha, \bar{U}_{\bar{\alpha}}$ (the latter two now being defined by covariant derivatives on the first two involving also the $U(1)$ connection). This formulation is then appropriate for expressing the moduli space of Calabi-Yau surfaces as a special manifold. This matrix is then identified with the integrals of $(3, 0), (2, 1), (1, 2)$ and $(0, 3)$ forms over 3-cycles.

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\(\text{[6]}\) This case includes some physically important theories.
4 Remark and conclusions

Special geometry is sometimes defined by giving the curvature formulas. Whereas these equations are always valid, we have not investigated (and know of no proof elsewhere) in how far they constitute a sufficient condition.

We have given several equivalent definitions of special geometry. It is clear that the symplectic transformations, inherited from the dualities on vectors, are crucial for the scalar manifolds. We have seen that some equations are missing in nearly all earlier proposals. The missing equations are mostly related to the requirement of symmetry \( N_{IJ} \), which is a necessary requirement for the discussion of symplectic transformations.

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