Constructing Quantum Mechanics from a Clifford substructure of the relativistic point particle

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Abstract

We show that the quantized free relativistic point particle can be understood as a string in a Clifford space which generates the space-time coordinates through its inner product. The generating algebra is preserved by a unitary symmetry which becomes the symmetry of the quantum states. We start by resolving the space-time canonical variables of the point particle into inner products of Weyl spinors with components in a Clifford algebra. Next, we show that a system of $N$ particles has a $U(N)$ symmetry that mixes the Clifford coordinates and momenta belonging to different particles. The inner products of these variables are assembled into Hermitian matrices $X$ and $P$ which are employed in defining a general unitarily invariant dynamical system. When $X$ and $P$ commute, this system can be gauged back into the original system of independent particles. When they do not commute, the system becomes irreducible and infinite and generates a space-time canonical system formally identical to Matrix Mechanics. The continuum limit is identified as a particular parametrization of a relativistic string in Clifford space.

1 Introduction

There are two reasons why Clifford algebras are interesting for a deeper understanding of the relationship between quantum mechanics and space-time structure. The first one is that in even dimensions they come with a built-in unitary symmetry in their generating algebra which can serve as a basis for the unitary symmetry of quantum states. This is the subject matter of this paper. The second reason, as argued in [1], is that quantum mechanics can be understood as a local theory on such a non-commutative space. This resolves the apparent paradox that quantum mechanics, though formulated as a local theory, shows non-local behavior. We shall briefly elaborate on this second point.

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Unlike other statistical theories, quantum mechanics employs probabilities that are not primary quantities, but are expressed in terms of underlying linear complex amplitudes. When applied to experiments like the single particle double slit experiment, this leads to interference terms in the probabilities which signal an apparent non-local behavior. This does not show that quantum mechanics per se is non-local, only that it is non-local with respect to space-time (or any equivalent commutative space). If space-time was generated by an underlying space with the structure group $SL(2,C)$, it is not difficult to imagine that the double homomorphism $SL(2,C) \Rightarrow SO(1,3)$ would permit a local interpretation of the interference terms. For mathematical reasons such a space would necessarily have to be non-commutative and therefore Bell’s theorem [2] would not apply. In [1] (see also [3,4,5]), we studied a model of this kind. The space-time coordinates $x^\mu$ of the relativistic point particle were resolved into spinors with components in a Clifford algebra according to

$$\sigma^{\mu\nu} x^\mu = c^A \cdot c^B$$

where $\sigma$ are the Pauli matrices, $c^A$ transforms like a two-component Weyl spinor and $\cdot$ is the inner product of the Clifford algebra. The easiest way to understand how this can affect locality is to consider the transition amplitudes in terms of paths in Clifford space. In [1] we showed that, in a suitable parametrization, the simplest possible classical trajectories for the relativistic point particle are

$$c^A = a^A \tau \quad x^\mu = b^\mu \tau^2 \quad \tau \in \mathbb{R}$$

The trajectories in Clifford space cover the trajectories in space-time twice with $c(\tau)$ and $c(-\tau)$ corresponding to the same space-time point $x(\tau^2)$. In the quantum regime, however, paths are not restricted in this way and can contain points corresponding to different positions in space at the same proper time $\tau^2$. When in the double slit experiment a particle travels from one point to another through two slits, there are only two alternative sets of paths in space-time but four alternative sets of paths in Clifford space. The transition amplitude in Clifford space is the sum of four parts and is identical to the space-time transition probability. The two interference terms in the transition probability are simply the amplitudes for the particle to travel along a path in Clifford space which passes through both slits at opposite values of $\tau$. According to this view, the resolution of the locality problem follows from the topology of the Lorentz group and is implemented by a non-commutative representation.

The Clifford model differs in one respect from conventional physics, in that the world line of a particle cannot be extended into both the infinite past and the infinite future as measured in proper time. In space-time, the endpoint of the trajectory would appear as a singular point, but in Clifford space it merely represents a ‘turning point’ where the space-time trajectory is being reproduced for the second time.
2 Mathematical preliminaries

It is well known that a null vector can be resolved into a product of two Weyl spinors

\[ x^{A\dot{B}} = c^A \cdot c^{\ast \dot{B}} \quad x^{\mu} x_{\mu} = 0 \]  

(2)

where \( x^{A\dot{B}} \) and \( x^{\mu} \) are related through the equivalence between real four-vectors and second-rank hermitian spinors

\[ V^\mu = \frac{1}{2} \sigma^\mu_{\dot{A}\dot{B}} V^{A\dot{B}} \quad V^{A\dot{B}} = \sigma^A_{\mu} V^\mu \]  

(3)

and \( \sigma_\mu \) are the four hermitian Pauli matrices. To resolve non-null vectors, we need something like

\[ x^{A\dot{B}} = c^A \cdot c^{\ast \dot{B}} \]  

(4)

where \( \cdot \) is a product which belongs to some non-commutative algebra. This problem can be compared to the somewhat similar problem of resolving the Lorentz metric \( \eta_{\mu\nu} \) into vectors. The well known solution is \( \eta_{\mu\nu} = \frac{1}{2} \{ \gamma_\mu, \gamma_\nu \} \) where the Dirac matrices \( \gamma_\mu \) generate the Clifford algebra \( Cl(1,3,\mathbb{R}) \). The components of any real symmetric \( 4 \times 4 \) matrix of signature \( (1,3) \) can therefore be expressed as the inner products (anti-commutators) of vectors (real linear combinations of \( \gamma \) matrices) belonging to \( Cl(1,3,\mathbb{R}) \). Real Clifford algebras are associated with real quadratic forms, but there is no similar connection between hermitian sesquilinear forms and complex Clifford algebras \( Cl(\mathbb{C}) \) [6]. Instead we must use even-dimensional real Clifford algebras written in complex form. Consider a future directed time-like vector \( x^\mu \). A unitary transformation followed by a non-uniform scaling can reduce \( x^{A\dot{B}} \) to a diagonal matrix with ones in the diagonal and can be effected by a suitable linear transformation of \( c^A \) so that (4) becomes

\[ c_i \cdot c^*_j = \delta_{ij} \]  

(5)

This can be compared to the algebra of creation and annihilation operators for two fermions

\[ \{ a_i, a_j^\dagger \} = \delta_{ij} \cdot 1 \quad \{ a_i, a_j \} = 0 \quad i, j = 1,2 \]  

(6)

Defining \( e_i = i(a_i + a_i^\dagger) \), \( e_{2+i} = a_i - a_i^\dagger \), \( i = 1,2 \), the commutation relations (6) become

\[ \{ e_i, e_j \} = -2 \delta_{ij} \quad i, j = 1,\ldots,4 \]  

(7)

which generate the Clifford algebra \( Cl(0,4,\mathbb{R}) \). This suggests that a solution to (4) would be to use spinors with values in the split Clifford algebra \( Cl(4,4,\mathbb{R}) \) and to let \( \cdot \) be the inner product (anti-commutator) of this algebra. Since any null vector can be created by letting \( c \) contain only a single generator (giving (2)), we do not need to consider degenerate algebras. This expectation is borne out by the following proposition.
Let $V_C$ be a $2n$-dimensional complex linear space with complex conjugation $^*$ and $H$ an $n \times n$ Hermitian matrix of arbitrary signature. Then the components of $H$ can be expressed as

$$H_{ij} = c_i \cdot c_j^* \quad c_i \cdot c_j = 0 \quad i, j = 1, \ldots, n$$

where $c_i$ belong to $V_C$ and $\cdot$ is the inner product

$$a \cdot b \equiv \frac{1}{2}\{a, b\}$$

of the Clifford algebra $Cl(2n, 2n, \mathbb{R})$ on the $4n$ dimensional real linear space $V_R$ which corresponds to $V_C$.

Proof. Let $e_i$, $f_i$, $i = 1, \ldots, n$ be a basis for $V_C$ and $g_i = i(e_i + e_i^*)$, $g_{n+i} = e_i - e_i^*$, $h_i = i(f_i + f_i^*)$, $h_{n+i} = f_i - f_i^*$, $i = 1, \ldots, n$ a basis for $V_R$. Let $g_i$ and $h_i$ generate the Clifford algebra $Cl(2n, 2n, \mathbb{R})$ on $V_R$ through

$$g_i \cdot g_j = 2\delta_{ij} \quad h_i \cdot h_j = -2\delta_{ij} \quad g_i \cdot h_j = 0 \quad i, j = 1, \ldots, 2n \quad (8)$$

Then the basis $e_i$, $f_i$ for $V_C$ satisfies

$$e_i \cdot e_i^* = -\delta_{ij} \quad f_i \cdot f_j^* = \delta_{ij} \quad e_i \cdot e_j = f_i \cdot f_j = 0 \quad i, j = 1, \ldots, n \quad (9)$$

We can create any $n \times n$ diagonal matrix of plus or minus ones by setting $c_i$ equal to either $f_i$ or $e_i$. A zero in the $k$-th entry of the diagonal can be created by $c_k = e_k + f_k$. A non-uniform scaling followed by a unitary transformation can transform this diagonal matrix into any desired $n \times n$ hermitian matrix with the same signature and can be effected by a suitable complex linear transformation of the $c_i$'s.

We shall resolve both the coordinates and momenta of the point particle into Clifford spinors

$$x^{A\dot{B}} = c^A \cdot c^{\ast \dot{B}} \quad p_{A\dot{B}} = d^*_A \cdot d_{\dot{B}} \quad (10)$$

but we also need the Clifford algebra to be large enough that the inner products $c^A \cdot d^*_B$ are algebraically independent of $x$ and $p$. This can for example be accomplished by enlarging $Cl(4, 4, \mathbb{R})$ to $Cl(8, 8, \mathbb{R})$ and then generating $x$ and $p$ by each their own $Cl(4, 4, \mathbb{R})$ subalgebra. This makes $c \cdot d$ vanish. The second step is to choose two Clifford elements $h_i$ whose inner products with both $c$ and $d$ vanish, and to make the substitution

$$c^A \rightarrow c^A + A^i_{\dot{i}} h_i \quad d^*_A \rightarrow d^*_A + B_{i\dot{A}} h_i^* \quad (11)$$

This will only change $x$ and $p$ by additive matrices that will not constrain them, and the two matrices $A$ and $B$ can be adjusted to produce any desired value of $c \cdot d$. Apart from this requirement, the dimension of the single-particle Clifford algebra is not of any importance in this paper.
Note that $c^A$ and $d_A^*$ have the same commutation properties but transform differently under $\text{SL}(2, \mathbb{C})$. The complex conjugation symbol $^*$ can therefore not be omitted, as it often is, because it specifies the commutation properties of the element in question. It is tacitly assumed that the inner product of elements of the same kind vanishes, and this will not be written out explicitly.

The variation of a real function $f$ which depends on $c^A$ only through an inner product can be expressed on the form

$$\delta f = \frac{\partial f}{\partial c^A} \bullet \delta c^A + \frac{\partial f}{\partial c^* B} \bullet \delta c^* B \quad (12)$$

which defines the ‘derivative’ of $f$ with respect to $c$. This will serve as a convenient notation.

3 Clifford substructure of the relativistic point particle

Let the space-time coordinates and momenta of the relativistic point particle be resolved into Clifford spinors according to (10). The equations of motion are obtained from the condition that the reparametrization invariant action

$$I = 4\sqrt{m} \int \sqrt{\frac{1}{2} \frac{dc^A}{d\tau} \cdot \frac{dc^* B}{d\tau} + \frac{dc_A^*}{d\tau} \cdot \frac{dc_B^*}{d\tau}} \ d\tau \ d\tau \quad (13)$$

is stationary under arbitrary variations of $c(\tau)$. The momenta conjugate to $c$ are

$$d_A^* \equiv \frac{\partial L}{\partial \dot{c}^A} = \sqrt{m} \left( \frac{1}{2} \frac{dc^E}{d\tau} \cdot \frac{dc^* F}{d\tau} - \frac{dc^* E}{d\tau} \cdot \frac{dc^F}{d\tau} \right) + \frac{1}{2} \left( \frac{dc_A^*}{d\tau} \cdot \frac{dc_B^*}{d\tau} \right) \times \left( \frac{dc^* A}{d\tau} \cdot \frac{dc^* B}{d\tau} \right) \quad (14)$$

and as expected, the Hamiltonian vanishes. A straightforward calculation using the four-vector rule

$$V_{AE} V^{B\dot{E}} = \frac{1}{2} \delta_A^B V_{\dot{F}E} V^{\dot{F}\dot{E}} \quad (15)$$

shows that the conjugate momenta $d_A^*$ satisfy the constraint

$$p^\mu p_\mu - m^2 = 0 \quad (16)$$

where $p_\mu$ are the space-time momenta defined in (10). This happens to be the same constraint as would have been obtained from the space-time action $\int \sqrt{x^2} \, d\tau$. According to constrained dynamics, the Hamiltonian is proportional to the constraint

$$H(p, e(\tau)) = e(\tau)(p^\mu p_\mu - m^2) \quad (17)$$
where \( e(\tau) \) is an einbein. This Hamiltonian can also be obtained from the Polyakov action

\[
I = \int 3e(\tau)^{-\frac{1}{3}} \sqrt{\frac{1}{2} \frac{dc^A}{d\tau} \cdot \frac{dc^{*B}}{d\tau} \cdot \frac{dc_A}{d\tau} \cdot \frac{dc^{*}_B}{d\tau} + m^2 e(\tau)} \ d\tau
\]  

(18)

which recovers (13) when the equations of motion for \( e(\tau) \) are substituted back into the action. The momenta conjugate to \( c \), are

\[
d^*_A = e(\tau)^{-\frac{1}{3}} (\frac{1}{2} \frac{dE}{d\tau} \cdot \frac{dc^{*F}}{d\tau} \cdot \frac{dc_F}{d\tau} - \frac{2}{3} (\frac{dc_A}{d\tau} \cdot \frac{dc^{*}_B}{d\tau}) \frac{dc^{*B}}{d\tau})
\]  

(19)

which can be used to determine the Hamiltonian density

\[
H(c, d) = d^*_A \cdot \frac{dc^A}{d\tau} + c.c. - L
\]  

(20)

where \( c.c. \) denotes the complex conjugate of the previous term and \( L \) is the Lagrangian in (18). A straightforward calculation gives

\[
p^\mu p_\mu = \frac{1}{2} d^*_A \cdot d^{*B} d_A \cdot d^*_B = e(\tau)^{-\frac{1}{3}} \sqrt{\frac{1}{2} \frac{dc^A}{d\tau} \cdot \frac{dc^{*B}}{d\tau} \cdot \frac{dc_A}{d\tau} \cdot \frac{dc^{*}_B}{d\tau}}
\]  

(21)

which, when applied to (20), gives the Hamiltonian (17) of constrained dynamics. Hence the first order (Hamiltonian) form of the action (13) is

\[
I = \int d^*_A \cdot \frac{dc^A}{d\tau} + c.c. - e(\tau)(p^\mu p_\mu - m^2) \ d\tau
\]  

(23)

This action has a global \( SL(2,\mathbb{C}) \) and \( U(1) \) gauge symmetry with the conserved Noether charges

\[
J_{AB} \equiv d^*_A \cdot c_B + d^*_B \cdot c_A \quad j \equiv i(d^*_A \cdot c^A - c.c.)
\]  

(24)

To obtain the correct space-time equations of motion, it is necessary to assume (as an initial value condition) that they vanish

\[
d^*_A \cdot c_B + d^*_B \cdot c_A = 0
\]  

(25)

\[
d^*_A \cdot c^A - c.c. = 0
\]  

(26)

Since all skew symmetric second rank tensors are proportional to \( \epsilon_{AB} \), (25) gives

\[
d^*_A \cdot c_B = \mu(\tau) \epsilon_{AB} \quad \mu(\tau) \equiv \frac{1}{2} d^*_E \cdot c^E
\]  

(27)
with \(^{(26)}\) saying that \(\mu(\tau)\) is real. We shall refer to this condition as the ‘Noether condition’. The canonical equations of motion are obtained by independent variation of \(c\) and \(d\)

\[
\frac{d c_A}{d\tau} = \frac{\partial H}{\partial d_A^*} = \frac{\partial H}{\partial p_{AE}} d_E^* \quad \frac{d d_A}{d\tau} = -\frac{\partial H}{\partial x^{AE}} c_A^p ( = 0 ) \quad (28)
\]

Taking the inner product of these equations with \(c_A^* \dot{B}\) and \(d_B \dot{\mu}\) gives

\[
\frac{dx^{A\dot{B}}}{d\tau} = 2 \frac{\partial H}{\partial p_{AB}} c^{A\dot{B}} \cdot d_E \quad \frac{dp_{A\dot{B}}}{d\tau} = -2 \frac{\partial H}{\partial x^{AB}} c^{A\dot{B}} \cdot d_B ( = 0 ) \quad (29)
\]

which by use of the Noether condition \(^{(27)}\) become

\[
\frac{dx^{A\dot{B}}}{d\tau} = 2 \frac{\partial H}{\partial p_{AB}} \mu(\tau) \quad \frac{dp_{A\dot{B}}}{d\tau} = -2 \frac{\partial H}{\partial x^{AB}} \mu(\tau) ( = 0 ) \quad (30)
\]

In the parametrization

\[
\tau(\tau) = \frac{1}{2m\mu(\tau)} \quad (31)
\]

these equations reduce to the canonical equations of motion

\[
\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial p^\mu} \quad \frac{dp_\mu}{d\tau} = -\frac{\partial H}{\partial x^\mu} ( = 0 ) \quad \mathcal{H}(x, p) = \frac{1}{2m}(\mu^\mu p_\mu - m^2) \quad (32)
\]

for a relativistic point particle with proper time \(\tau\). This proper time is not defined at points where \(\mu\) vanishes. There will be just one such point and it represents a ‘turning point’ where the space-time trajectory has an endpoint and the underlying trajectory in Clifford space starts to reproduce it for the second time. From \(^{(28)}\) and the Hamiltonian constraint \(^{(16)}\), we obtain an explicit expression for \(\mu(\tau)\)

\[
\frac{d}{d\tau} \mu(\tau) = \frac{d}{d\tau}(\frac{1}{2} d_E^* \cdot c_E) = e(\tau)m^2 \quad \mu(\tau) = \int_{\tau_0}^\tau m^2 e(t) \, dt \quad (33)
\]

Hence \(\mu(\tau)\) is determined by the mass of the particle and the ‘turning point’ \(\tau_0\) of its motion.

### 4 System of N particles with a U(N) symmetry

Assuming that the Clifford algebra for the point particle is \(Cl(2n, 2n, \mathbb{R})\), we can accommodate \(N\) particles in \(Cl(2nN, 2nN, \mathbb{R})\) in such a way that all inner products between Clifford coordinates and momenta belonging to different particles vanish. The generating algebra

\[
e_i^p \cdot e_j^q = \delta_{ij}\delta_{pq} \text{sign}(p) \quad e_i^p \cdot e_j^q = 0 \quad i, j = 1, \ldots, N \quad p, q = 1, \ldots, 2n \quad (34)
\]
where $\text{sign}(p)$ denotes the sign of $e^p \cdot e^{*p}$, is preserved by the $U(N)$ unitary transformation

$$e^p_i \rightarrow U_{ih} e^p_h \quad U_{ih} U_{jh}^* = \delta_{ij}$$

(35)

If we assemble the canonical variables $c^A_i$ and $d^{*A}_i$, $i = 1 \ldots, N$ of the $N$ particles into the ket- and bra-vectors $\rangle C^A$ and $\langle D^A$ respectively, then the corresponding space-time coordinates and momenta are elements of the $N \times N$ diagonal matrices

$$X^{AB} = \rangle C^A \bullet \langle D^B \quad P_{AB} = \rangle \dot{C}^A \bullet \langle \dot{D}^A$$

(36)

which trivially satisfy the commutation relations

$$[X^\mu, X^\nu] = [P_\mu, P_\nu] = [X^\mu, P_\nu] = 0$$

(37)

The equations of motion for this dynamical system can be derived from the sum

$$I = \int \frac{d}{d\tau} (\rangle C^A \bullet \langle D_A + \text{c.c.} - H) \, d\tau \quad H \equiv e(\tau)(P^\mu P_\mu - m^2 \cdot 1)$$

(38)

of the single-particle actions (23). The Noether condition (27) becomes

$$\rangle C^A \bullet \langle D_B = \mu(\tau) \delta^A_B \cdot 1$$

(39)

We observe that (38) and (39) are preserved by the global $U(N)$ transformations

$$\rangle C^A \rightarrow U \rangle C^A \quad \langle D_A \rightarrow \langle U \dot{D}^A$$

(40)

which produce the similarity transformations

$$X^\mu \rightarrow U X^\mu U^\dagger \quad P_\mu \rightarrow U P_\mu U^\dagger$$

(41)

of the Hermitian matrices $X^\mu$ and $P_\mu$. Such transformations create off-diagonal entries in $X$ and $P$ which correspond to artificial couplings between Clifford coordinates and momenta belonging to different particles.

The motion of a classical point particle can be described by a set of integral curves in the phase space $(x^\mu, p_\mu)$. From the foregoing it follows that these integral curves consist of eigenvalues of the Hermitian matrices $X^\mu$ and $P_\mu$ which are the dynamical variables of a unitarily invariant system.

5 Matrix Mechanics

We have seen that $N$ independent particles in Clifford space leads to a unitarily invariant dynamical system. The reverse problem is to determine under which conditions such a type of system can be gauged back into a set of independent particles. To address this problem, we must define a unitarily invariant system.
which relaxes one or more assumptions associated with independence. A natural starting point is to define an action principle for the whole system which is equivalent to the combined action principles for the particles it contains. To this end, we observe that to make \( N \) single-particle actions stationary is equivalent to making all time-independent linear combinations of them stationary. This corresponds to the action

\[
I = \int \sum_{i=1}^{N} \phi_i L_i \, d\tau \quad L_i = d^*_i A \cdot \frac{d\xi_i^A}{d\tau} + c.c. - H(p_i, e(\tau)) \tag{42}
\]

where the coefficients \( \phi_i \) are arbitrary real constants, and \( L_i \) are the single-particle Lagrangians. In section 6, the \( \phi \)'s will be given a geometrical interpretation as a dilaton field. When \( \Phi \) denotes the \( N \times N \) diagonal matrix with \( \phi_i \) along its diagonal, the action (42) can be written as

\[
I = \int Tr \left( \Phi \left( \frac{dC^A}{d\tau} \cdot \frac{D_A}{D_A} + h.c. - H \right) \right) d\tau \quad H \equiv e(\tau)(P^\mu P_\mu - m^2 \cdot 1) \tag{43}
\]

where \( P_\mu \) is diagonal. This action is preserved by the unitary transformation (40) with \( \Phi \) transforming according to

\[
\Phi \rightarrow U \Phi U^\dagger \quad \Phi^\dagger = \Phi \tag{44}
\]

The diagonal matrices \( \Phi \) and \( P_\mu \) trivially satisfy the unitarily invariant conditions

\[
\frac{d\Phi}{d\tau} = 0 \tag{45}
\]

\[
[\Phi, P_\mu] = 0 \tag{46}
\]

\[
[P_\mu, P_\nu] = 0 \tag{47}
\]

Conversely, these conditions ensure that the action (43) can be gauged back into (42). The conserved \( SL(2, \mathbb{C}) \) and \( U(N) \) Noether charges corresponding to the action (43), are

\[
J_{AB} = Tr \left( \Phi \left( C^A \cdot D_B + C_B \cdot D^A \right) \right) \quad j = i(\Phi \left( C^A \cdot D^A - h.c. \right) \tag{48}
\]

Requiring that they vanish for all values of \( \Phi \), gives (39). The equations of motion are obtained by requiring the action (43) to be stationary for all \( \Phi \) which satisfy (45). By independent variation of \( C \) and \( D \), we obtain

\[
\frac{d}{d\tau} C^A = \frac{\partial H}{\partial P_A^E} D^E \quad \frac{d}{d\tau} D_A = - C^E \frac{\partial H}{\partial X^A_E} = 0 \tag{49}
\]
Taking the inner product on both sides of these equations with $C^B$ and $\tilde{D}_B$ and applying (39) and the reparametrization (31), we obtain

$$ \frac{dX^\mu}{d\tau} = \frac{\partial H}{\partial P_\mu} \quad \frac{dP_\mu}{d\tau} = - \frac{\partial H}{\partial X^\mu} \quad (= 0) \quad H \equiv \frac{1}{2m}(P^\mu P_\mu - m^2 \cdot 1) \quad (50) $$

Equations (43)-(47) describe a general class of unitarily invariant dynamical systems which includes, but is not limited to, systems of independent particles. Systems of independent particles are obtained by adding the commutation relations

$$ [X^\mu, X^\nu] = [X^\mu, P_\nu] = 0 \quad (51) $$

allowing all off-diagonal entries of $X$ and $P$, that is all couplings between different particles, to be gauged away in the same unitary frame. (51) can be generalized by observing that from the equations of motion (50) and the commutativity (47) of the space-time momenta, it follows that the skew symmetric tensor

$$ J^{\mu\nu} \equiv X^\mu P^\nu - X^\nu P^\mu \quad (52) $$

is a constant of motion. Constraints on $J^{\mu\nu}$ are therefore compatible with the equations of motion. A natural choice is to let $J^{\mu\nu}$ be Hermitian and satisfy the $SO(1,3)$ Lie algebra

$$ [J^{\mu\nu}, J^{\rho\sigma}] = ik(\eta^{\nu\rho} J^{\mu\sigma} - \eta^{\mu\rho} J^{\nu\sigma} - \eta^{\nu\sigma} J^{\mu\rho} + \eta^{\mu\sigma} J^{\nu\rho}) \quad (53) $$

which is accomplished by the commutation relations

$$ [X^\mu, X^\nu] = 0 \quad (54) $$

$$ [X^\mu, P_\nu] = ik\delta^\mu_\nu \cdot 1 \quad (55) $$

For $k \neq 0$, the couplings between different particles can no longer be gauged away and we obtain an infinite and irreducible system of coupled tracks. It should be noted that according to (55), $X$ diverges when $P$ approaches diagonality. This does not affect the action (43), it being independent of $X$. The commutation relations (55) allow the derivatives of $\mathcal{H}$ to be written as commutators, turning (50) into

$$ \frac{dX^\mu}{d\tau} = \frac{i}{k} [\mathcal{H}, X^\mu] \quad \frac{dP_\mu}{d\tau} = \frac{i}{k} [\mathcal{H}, P_\mu] \quad (= 0) \quad (56) $$

These equations together with the commutation relations (47), (51) and (55) are formally identical to Matrix Mechanics in the Heisenberg picture.

Matrix Mechanics can conveniently be expressed in a 'picture'-independent form by means of an auxiliary gauge connection which turns the global unitary symmetry into a local one and which couples only to the (vanishing) $U(N)$ Noether charge. This gauge connection transforms according to
Γ \rightarrow UΓU^\dagger - i\frac{dU}{d\tau}U^\dagger \quad \bar{\Gamma}(\bar{\tau}) = \Gamma(\tau)\frac{d\tau}{d\bar{\tau}} \quad (57)
and defines the gauge covariant derivatives
\nabla_\tau \hat{V} = \left( \frac{d}{d\tau} - i\Gamma(\tau)\right) \hat{V} \quad \nabla_{\bar{\tau}} \hat{V} = \left( \frac{d}{d\bar{\tau}} - i\bar{\Gamma}(\bar{\tau})\right) \hat{V} \quad (58)
which turn (56) into
\nabla_{\bar{\tau}} X^\mu = i\int H, X^\mu \quad \nabla_{\bar{\tau}} P^\mu = i\int [H, P^\mu] \quad (= 0) \quad (59)\n
The Heisenberg picture corresponds to the gauge \( \Gamma = 0 \). In the gauge \( \Gamma(\tau) = -\frac{1}{k}H \), the commutators on the left and right hand sides of (55) cancel out and \( X \) and \( P \) become stationary. This gauge therefore corresponds to the Schrödinger picture.

Note that in the classical system \( k = 0 \), both the number of tracks and the initial values of the canonical variables are arbitrary and have to be put in by hand, whereas in the non-classical system the canonical commutation relations (55) determine both the number of tracks (as being infinite) and automatically provide an infinite range of (stationary) eigenvalues.

6 The state vector

The dynamical system constructed in the foregoing generates a set of tracks which can be used to describe a physical point particle. Let us first consider the classical system \( k = 0 \). In the gauge where \( X \) is diagonal, all tracks are decoupled from each other and we expect that when, for example, the space-time position \( x \) of the particle is being measured at some time \( \tau \), a good measurement will return a value \( x_i(\tau) \) belonging to one of these tracks. Expressed in a gauge invariant manner, this is equivalent to saying that it will return an eigenvalue of \( X(\tau) \). The result of a measurement can be represented as a gauge invariant expectation value \( E \) in terms of a state vector \( |s> \)

\[ E(C^A) \equiv <s| C^A \quad E(X^{AB}) \equiv E(C^A) \bullet E(C^B) = <s|X^{AB}|s> \quad (60) \]

\( \hat{C} (\tau) \) can be expanded in terms of the Clifford coordinates \( c_i(\tau) \) which generate the eigenvalues of \( X \)

\[ C^A (\tau) = \sum_i x_r(\tau))c_r^A(\tau) \quad c_r^A(\tau) \bullet c_s^B(\tau) = \delta_{rs}x_s^{AB}(\tau) \quad (61) \]

where \( |x_i(\tau)> \) denotes the eigenvectors of \( X^\mu(\tau) \) with eigenvalues \( x_i^\mu(\tau) \). For short, we shall also refer to \( c_i(\tau) \) as eigenvalues. It follows that, if the expectation value \( E(\hat{C}) \) is going to return the correct value \( c_i \) of a measurement,
the state vector $|s\rangle$ must be set equal to the eigenvector $|x_i\rangle$. Conversely, if the expectation value coincides with an eigenvalue of $X$, we would expect a good measurement to return this value. To serve the purpose of predicting the outcome of future measurements, the state vector must be subject to a time evolution. In classical dynamics it is natural to assume that after a measurement has been performed, the expectation value must stay on the track corresponding to this measurement. According to the foregoing, the classical system can be gauged into a set of decoupled tracks with $X(\tau)$ being diagonal and $\Gamma = 0$. In this gauge the eigenvectors $|x_i\rangle$ can be chosen to be constants of motion and the state vector must therefore also be a constant of motion. This leads to the gauge invariant time evolution

$$\nabla_\tau |s\rangle \equiv \left(\frac{d}{d\tau} - i\Gamma\right)|s\rangle = 0 \quad (62)$$

For the classical system, these measurement principles merely represent a different way of formulating the traditional initial value problem. They do, however, also apply to the non-classical system, irrespective of the fact that the way they were derived is no longer valid. In the non-classical system where $X$ and $P$ do not commute, the assumption that measurements must be expressed through a state vector, imposes restrictions on which type of measurements that can be performed. The time evolution (62) also holds true, as follows from the fact that the state vector is known to be a constant of motion in the Heisenberg picture $\Gamma = 0$. To help appreciate the difference between the classical and the non-classical systems, we expand the expectation value $E(C)$ in terms of the eigenvalues $c_i$

$$E(C^A(\tau)) \equiv <s| C^A(\tau) = <s| x_i(\tau) > c_i(\tau) \quad (63)$$

In the classical system, in the gauge $\Gamma = 0$, both $<s|$ and $|x_i>$ are constants of motion and hence the expectation value $E(C(\tau))$ is equal to one of the eigenvalues $c_i(\tau)$. The outcome of a measurement is therefore predictable. This is not surprising since it was used to derive the time evolution of the state vector. In the non-classical system, in the gauge $\Gamma = 0$, the state vector is also stationary, but the eigenvectors $|x_i>$ undergo a unitary time evolution. After a measurement has been performed, the expectation value therefore drifts into a complex linear combination of different eigenvalues $c_i(\tau)$. Accordingly, the outcome of a measurement is no longer predictable, but instead occurs with statistical frequencies given by the Born rule.

The classical and non-classical systems are related through Ehrenfest’s theorem. The time evolution of the expectation value $E(C)$ is

$$\frac{d}{d\tau} E(C^A) = \langle \nabla_\tau <s| C^A + <s| \nabla_\tau C^A \rangle = <s| \frac{i}{2\hbar}[H, X^A] D^E \quad (64)$$

Taking the inner product of this equation with $E(C^B)$ and using (63) and (61),
we obtain after a reparametrization the time evolution of the expectation value of the space-time coordinates

$$\frac{d}{d\tau} E(X^\mu) = \langle s | \frac{i}{\hbar} [\mathcal{H}, X^\mu] | s \rangle$$  \hspace{1cm} (65)$$

This is Ehrenfest’s theorem which could also have been obtained directly from (59) by use of (60) and (62).

In the non-relativistic limit, the proper time \(\tau\) is equal to the expectation value of \(X^0\) which represents the ‘physical’ time \(t \equiv \langle s | X^0 | s \rangle\)

$$\frac{dt}{d\tau} = \langle s | \nabla_\tau X^0 | s \rangle = \frac{1}{m} \langle s | P^0 | s \rangle \approx 1$$  \hspace{1cm} (66)$$

where we have used the time evolution of the state vector and the equations of motion for \(X^0\). Restricting the equations of motion (56) to \(\mu = 1, 2, 3\), the Hamiltonian \(\mathcal{H}\) effectively reduces to the non-relativistic Hamiltonian

$$\tilde{H} = \frac{1}{2m} (P_x^2 + P_y^2 + P_z^2)$$  \hspace{1cm} (67)$$

Taken together with the corresponding commutation relations, this system is identical to that of non-relativistic Matrix Mechanics. The Schrödinger picture corresponds to the non-relativistic gauge condition \(\mathcal{T}(t) = -\frac{1}{\tau} \tilde{H}\) which turns the time evolution (62) of the state vector into the matrix form of the Schrödinger equation.

### 7 The relativistic string in Clifford space

To obtain the continuum limit of the dynamical system constructed in section 4, we start by mapping the generators \(e^p_i\) of the generating algebra (34) for \(N = \infty\) into the Clifford elements

$$f^p(\sigma) = \sum_{i=1}^{\infty} g_i(\sigma) e^p_i$$  \hspace{1cm} (68)$$

where \(g_i(\sigma)\) are complex functions of a real parameter \(\sigma\). These functions are chosen so that \(f\) satisfies

$$f^p(\sigma) \bullet f^{*q}(\sigma') = \delta_n(\sigma - \sigma') \text{sign}(p) \delta^{pq} \quad f(\sigma) \bullet f(\sigma') = 0$$  \hspace{1cm} (69)$$

where \(\delta_n(\sigma), n = 1, \ldots\) is a sequence of positive even functions which converges to the Dirac delta function \(\delta(\sigma)\) for \(n \to \infty\). \(f^p(\sigma)\) can be regarded as a ket-vector \(\tilde{f}^p\) with a continuous index \(\sigma\), and correspondingly \(\delta_n(\sigma - \sigma')\) can be regarded as a real symmetric matrix \(\delta_n\) with continuous indices \(\sigma\) and \(\sigma'\). In this notation, the algebra (69) is preserved by the pseudo-unitary transformation

$$\tilde{f}^p \rightarrow U \tilde{f}^p$$  \hspace{1cm} (70)$$
which preserve the metric $\delta_n$

$$U \delta_n U^\dagger = \delta_n$$ (71)

In the continuum limit $n \to \infty$ these pseudo-unitary transformations become unitary transformations. The Clifford coordinates $c(\tau, \sigma)$ are defined as integral transforms of $f$

$$c^A(\tau, \sigma) \equiv \int a_p^A(\tau, \sigma, \sigma') f^p(\sigma') d\sigma'$$ (72)

and represent a string in Clifford space. The dynamics of this string will be derived from an action principle which is preserved by arbitrary reparametrizations $(\tau, \sigma) \to (\tau', \sigma')$.

It is well known that for a string which resides in space-time, the Lorentz metric $\eta_{\mu\nu}$ induces a metric on the worldsheet through the tangent derivatives $\partial_{\alpha} x^\mu$. For a string which resides in Clifford space, we use the complex vectors

$$V_\alpha^\mu \equiv \sigma_{AB}^\mu c^A \cdot \partial_{\alpha} c^B$$ (73)

which have the real part $\partial_{\alpha} x^\mu$. These vectors induce the hermitian tensor

$$g_{\alpha\beta} = V_\alpha^\mu V_\beta^\nu \eta_{\mu\nu} \quad g_\alpha^\beta = g_{\beta\alpha}$$ (74)

on the Clifford worldsheet, which can be decomposed into a real symmetric tensor $h_{\alpha\beta}$ and a real scalar $\phi$

$$g_{\alpha\beta} = h_{\alpha\beta} + i\phi \sqrt{h} \epsilon_{\alpha\beta}, \quad h_{\alpha\beta} \equiv g_{(\alpha\beta)}, \quad \phi \equiv -\frac{1}{2} i h^{-\frac{1}{2}} \epsilon^{\alpha\beta} g_{\alpha\beta}, \quad h \equiv |\det(h_{\alpha\beta})|$$ (75)

The reparametrization invariant string generalization of the Polyakov point particle action (18), is

$$I = \int \sqrt{W} \left[ (W^\mu W_\mu) - m^2 \right] d\phi \sqrt{h} d\tau d\sigma$$ (76)

The geometrical interpretation of this action is obtained from the equations of motion for the metric $h_{\alpha\beta}$

$$(W^\mu W_\mu)^{\frac{1}{2}} W_{AB} \partial_{(\alpha} c^A \cdot \partial_{\beta)} c^B - \frac{1}{2} (3(W^\mu W_\mu)^{\frac{1}{2}} - m^2) h_{\alpha\beta} = 0$$ (77)

Contracting this equation with $h^{\alpha\beta}$ gives $(W^\mu W_\mu)^{\frac{1}{2}} = m^2$ and thereby

$$I = 2m^2 \int \phi \sqrt{h} d\tau d\sigma$$ (78)

which, apart from a dilaton field $\phi$, is proportional to the area of the worldsheet.
To write the action (76) in an explicit covariant first order form, we use Dedonder-Weyl covariant canonical variables [7]. The multi-momenta conjugate to \( c \) are
\[
d^\alpha_A = \frac{\partial L}{\partial (\partial_\alpha c^A)} = (W^\nu W_\nu)^{-\frac{3}{2}} W_\mu \sigma^\mu_{AB} h^{\alpha\beta} \partial_\beta c^B \phi \sqrt{h}
\] (79)
where \( L \) denotes the Lagrangian density in (76). With a redefinition \( d \rightarrow \phi \sqrt{h} d \) of the momenta, this leads to the expressions
\[
\frac{1}{2} h_{\alpha\beta} d^{*\alpha}_A \cdot d^{*\beta}_B = \sqrt{W^\mu W_\mu}
\] (80)
from which we obtain the Dedonder-Weyl covariant Hamiltonian density
\[
H \equiv \phi \sqrt{h} d^{*\alpha}_A \cdot \partial_\alpha c^A + c.c. - L = (p^\mu p_\mu + m^2) \phi \sqrt{h}
\] (82)
and hence the first order form
\[
I = \int \left( d^{*\alpha}_A \cdot \partial_\alpha c^A + c.c. - (p^\mu p_\mu + m^2) \right) \phi \sqrt{h} \, d\tau d\sigma
\] (84)
of the Polyakov action (76). By independent variation of \( d \) and \( c \), we obtain the equations of motion
\[
\partial_\alpha c^A = h_{\alpha\beta} p^{AE} d^\beta_E
\] (85)
\[
\partial_\alpha (\phi \sqrt{h} d^{*\alpha}_A) = 0
\] (86)
To turn the dilaton into a dynamical field, we add to the Lagrangian the Weyl invariant term \( \kappa \sqrt{h} h^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi \). The equations of motion for \( h_{\alpha\beta} \) then become
\[
\frac{\partial \mathcal{L}}{\partial h_{\alpha\beta}} = -2\phi \sqrt{h} p^\mu \frac{1}{2} \sigma^A_B d^{*\alpha}_A \cdot d^{*\beta}_B - \kappa \sqrt{h} h^{\alpha\gamma} h^{\beta\delta} \partial_\gamma \phi \partial_\delta \phi + \frac{1}{2} \mathcal{L} h^{\alpha\beta} = 0
\] (87)
\[
\mathcal{L} = \left( d^{*\alpha}_A \cdot \partial_\alpha c^A + c.c. - (p^\mu p_\mu + m^2) \right) \phi \sqrt{h} + \kappa \sqrt{h} h^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi
\] (88)
Upon inserting the expression (85) for \( \partial c \) into (88), the trace of (87) gives the mass shell equation
\[
p^\mu p_\mu - m^2 = 0
\] (89)
Note that the kinetic term for $\phi$ does not contribute to the trace because it is Weyl invariant. (87) determines the metric algebraically. The metric could be turned into a dynamical field by adding the term $\phi \sqrt{h} h^{\alpha\beta} R_{\alpha\beta}$ to the Lagrangian in accordance with the Jackiw-Teitelboim 2d model of gravity [8, 9]. However, in the second order formulation it would break Weyl invariance and violate the mass shell equation (89), and in the first order formulation, where the affinity is varied independently of the metric, $\phi$ would become a constant [10]. Varying the action with respect to $\phi$ gives the equations of motion

$$2\kappa \frac{1}{\sqrt{h}} \partial_\alpha (\sqrt{h} h^{\alpha\beta} \partial_\beta \phi) = d_A^{*\alpha} \cdot \partial_\alpha c^A + c.c. - (p^\mu p_\mu + m^2)$$

(90)

When (85) and (89) are applied to the right hand side of (90), it becomes $m^2$.

The equations of motion (86) determine the multi-momenta only up to an arbitrary worldsheet scalar $\chi_A$

$$\phi \sqrt{h} d_A^{*\alpha} \to \phi \sqrt{h} d_A^{*\alpha} + \epsilon^{\alpha\beta} \partial_\beta \chi_A$$

(91)

The class of solutions which correspond to the discrete model in section 4 is obtained by choosing $\chi_A$ so that $d_A^{*\beta}$ satisfies the reparametrization invariant condition

$$\mu^\alpha \epsilon_{\alpha\beta} d_A^{*\beta} = 0$$

(92)

$$\mu^\alpha \equiv \frac{1}{4} (d_A^{*\alpha} \cdot c^A + c.c.)$$

(93)

This condition will be imposed as a constraint on the action principle by adding the terms $\lambda^A \cdot \mu^\alpha \epsilon_{\alpha\beta} d_A^{*\beta} + c.c.$ to the Lagrangian, where $\lambda^A$ is a Lagrange multiplier. The equations of motion (88) and (89) then become

$$\partial_\alpha (\phi \sqrt{h} d_A^{*\alpha}) = \frac{-1}{2} d_A^{*\beta} (\lambda^E \cdot \epsilon_{\beta\gamma} d_E^{*\gamma})$$

(94)

$$\partial_\alpha (\phi \sqrt{h} d_A^{*\alpha} c^A) = \frac{-1}{2} d_A^{*\beta} (\lambda^E \cdot \epsilon_{\beta\gamma} d_E^{*\gamma})$$

(95)

In the discrete system, the vanishing of the Noether charges was imposed as an initial value condition. Correspondingly, the conserved $SL(2, \mathbb{C})$ and $U(1)$ Noether currents

$$J_{AB}^{\alpha} \equiv \phi \sqrt{h} (d_A^{*\alpha} \cdot c_B + d_B^{*\alpha} \cdot c_A) \quad J^\alpha \equiv i \phi \sqrt{h} (d_A^{*\alpha} \cdot c^A - c.c.)$$

(96)

are assumed to vanish on some space-like curve on the worldsheet. In a parametrization where $\mu^2 = 0$, it follows from the constraint (92) that $d^2 = 0$ and therefore also $J^2 = J^2 = 0$. In such a parametrization, the Noether charges $J^1$ and $J^1$ become constants of motion and must vanish everywhere. Hence the Noether currents $J^\alpha$ and $J^\alpha$ vanish everywhere, leading to the Noether condition

$$d_A^{*\alpha} \cdot c^B = \mu^\alpha \delta^B_A$$

(97)
where $\mu^\alpha$ is the real vector (93). Taking the inner product of both sides of (94) with $c^* B$ and contracting with $\mu^\alpha$, the $\lambda$-terms vanish and it reduces to

$$\mu^\alpha \partial_\alpha x^\mu = 2 (h_{\alpha \beta} \mu^\alpha \mu^\beta) p^\mu$$

(98)

Consider the vector

$$v^\gamma \equiv (h_{\alpha \beta} \mu^\alpha \mu^\beta)^{-1} \mu^\gamma$$

(99)

In a domain where $\mu^\alpha$ is regular, $v^2$ and thereby $\mu^2$ can be made to vanish through a reparametrization. Any subsequent reparametrization of the form $\tau \to \tau(\tau, \sigma), \sigma \to \tau(\sigma)$ preserves $v^2 = 0$ and can, since the weight of $v$ is different from 1, be used to make $v^1 = 2m$. In this parametrization, the equations of motion (98) become

$$\frac{\partial x^\mu}{\partial \tau} = \frac{1}{m} p^\mu$$

(100)

and (105) reduces to

$$\frac{\partial}{\partial \tau} (\phi \sqrt{h} d^\alpha_A) = 0$$

(101)

From (101) and the mass shell equation (89), it follows that $\phi^2 h_{11}^{-1}$ and consequently $\sqrt{h_{11} d^\alpha_A}$ and $p^\mu$ are constants of motion. Accordingly, the constraint (92) leads to the reparametrization invariant equations of motion for $x^\mu$ and $p_\mu$

$$\nu^\alpha \partial_\alpha x^\mu = 2 p^\mu \quad \nu^\alpha \partial_\alpha p_\mu = 0$$

(102)

We are now in a position to compare the discrete system based on the action (42) with the continuous system based on the action (84) subject to the constraint (92). The equations of motion and constraints corresponding to the action (42), are

$$\frac{dc^A}{d\tau} = e(\tau) p^{AE} d_i E \quad \frac{dd_i A}{d\tau} = 0 \quad p_i^{AB} \equiv d^*_i A \bullet d^*_i B$$

(103)

$$p^\mu _\mu - m^2 = 0 \quad d^*_i A \bullet e_i^B = \mu(\tau) \delta^B_A$$

(104)

with no summation over $i$. For the continuous system, we shall use the parametrization $v^2 = \partial_\sigma v^1 = 0$ which allows for an arbitrary time-reparametrization $\tau \to \tau'(\tau)$. We resolve the metric into zweibeins $h_{\alpha \beta} = \eta^{ab} e^\alpha_a e^\beta_b$ and choose local zweibein frames in which $e^2_i = 0$. Defining $\tilde{d}_E \equiv e^1_i d^1_i E$ which is a scalar under time-reparametrization, the continuous system can be written as

$$\frac{\partial c^A}{\partial \tau} = e_1 p^A E \tilde{d}_E \quad \frac{\partial \tilde{d}_A}{\partial \tau} = 0 \quad p^{AB} \equiv \tilde{d}^A \bullet \tilde{d}^B$$

(105)

$$p^\mu _\mu - m^2 = 0 \quad \tilde{d}^A \bullet e_i^B = e_1^1 \mu^1 \delta^B_A$$

(106)

From the parametrization condition $\partial_\sigma v^1 = 0$ it follows that $h_{11} \mu^1 = (e_1^1)^2 \mu^1$ is a function of $\tau$ only, so that the two systems lead to the same equations of motion for $x^\mu$ and $p_\mu$, in accordance with our previous finding.
8 Clifford strings and the ontology of quantum mechanics

Interpreting quantum mechanics [11] is made more difficult by the mathematical leap between the space-time description of classical and of quantum objects. In the Clifford space description, both the classical and the quantized point particle are understood as strings and obey the same three measurement principles:

(i) the result of a measurement is an eigenvalue of the observable
(ii) the results of measurements performed at the same parameter-time can be expressed as expectation values corresponding to a single state vector
(iii) the state vector is gauge covariantly constant in time (Schrödinger equation)

These principles are derived from the classical string but also apply to the quantum string where they have the well-known ‘unexpected’ consequences. The classical string is reducible and can (in a suitable gauge) be described as a set of tracks in Clifford phase space, the points of which generate eigenvalues of both $X$ and $P$. These tracks are independent of each other in the sense that they are integral curves corresponding to the same equations of motion. The role of the state vector is hereby reduced to the trivial one of selecting an integral curve. This makes it possible to describe not only the outcome, but also the object of the measurement as a point particle. The general case of the irreducible Clifford string however, shows, that such a picture is misleading. The object of a measurement is a Clifford string, not a point particle. The point particle is invoked only to describe the outcome of a measurement.

9 Conclusion

We have shown that the quantized free relativistic point particle can be understood as a particular parametrization of a relativistic string in Clifford space. In obtaining this result, we considered only the dynamical degrees of freedom corresponding to the constraint (92).

There are good reasons to believe that a four-dimensional Minkowski space does not suffice to accommodate the particle physics of the Standard Model. The Clifford model discussed in the foregoing is limited to a four-dimensional Minkowski space because it is based on complex Weyl spinors. Since Weyl spinors are an integral part of the model, it is difficult to see how the dimension of space-time can be increased without replacing the complex numbers with a higher dimensional Algebra. The complex numbers correspond to the Clifford algebra $Cl(0, 1, \mathbb{R})$. Increasing the dimension, we find $Cl(0, 2, \mathbb{R})$ which corresponds to the quaternions and $Cl(0, 3, \mathbb{R})$ which can be deformed into the octonions. For algebraic reasons [12, 13, 14], such spinors would be expected to generate a six-dimensional and a ten-dimensional Minkowski space respectively.
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