PROBLEMS 85 AND 87 OF BIRKHOF F’S \textit{LATTICE THEORY}

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Abstract. We solve problems 85 and 87 from Birkhoff’s book \textit{Lattice Theory} \cite{1}.

1. Introduction

A partially ordered set (or \textit{poset} for short) is a set \( X \) with a binary relation \( \leq \) that is reflexive, transitive, and anti-symmetric (i.e., \( x, y \in X \) with \( x \leq y \) and \( y \leq x \) implies \( x = y \)). Often, a poset is denoted by \( (X, \leq) \). A subset \( D \subseteq X \) is called a \textit{down-set} if it is “closed under going down”, that is \( d \in D, x \in X, x \leq d \) jointly imply \( x \in D \). A special case of a down-set is the set \( \downarrow_P x = \{ y \in X : y \leq x \} \) for \( x \in X \). (Sometimes we just write \( \downarrow x \) if the poset \( P \) is clear from the context.) Down-sets of this form are called \textit{principal}. If \( S \subseteq X \) we say \( S \) has a \textit{smallest element} \( s_0 \in S \) if \( s_0 \leq s \) for all \( s \in S \). Note that anti-symmetry of \( \leq \) implies that a smallest element is unique (if it exists at all!). Similarly, we define a largest element. Moreover, we set \( S^u = \{ x \in X : x \geq s \text{ for all } s \in S \} \) to be the \textit{set of upper bounds} of \( S \). The set of lower bounds \( S^\ell \) is defined analogously.

We say that a subset \( S \subseteq X \) of a poset \( (X, \leq) \) has an \textit{infimum} or \textit{largest lower bound} if

\begin{enumerate}
  \item \( S^\ell \neq \emptyset \), and
  \item \( S^\ell \) has a largest element.
\end{enumerate}

Again, an infimum (if it exists) is unique by anti-symmetry of the ordering relation, and it is denoted by \text{inf}(S) or \( \bigwedge_X S \). The dual notion (everything taken “upside down” in the poset) is called \textit{supremum} and is denoted by \text{sup}(S) or \( \bigvee_X S \). The infimum of the empty set is defined to be the largest element of \( X \) if it has one, and the supremum is the smallest element of \( X \).

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A poset \((X, \leq)\) in which infima and suprema exist for all \(S \subseteq X\) is called a complete lattice. A lattice has suprema and infima for finite non-empty subsets. If \((X, \leq)\) is a poset and \(x, y \in X\) we use the following notation
\[
x \lor y := \bigvee_{x} \{x, y\},
\]
and \(x \land y\) is defined analogously. To emphasize the binary operations \(\lor, \land\), a lattice \((L, \leq)\) is sometimes written as \((L, \lor, \land)\). A lattice is distributive if for all \(x, y, z \in L\) we have
\[
x \land (y \lor z) = (x \land y) \land (x \land z).
\]

**Definition 1.1.** Given a poset \((X, \leq)\), the interval topology \(\tau_i(X)\) is given by the subbase
\[
S = \{X \setminus (\downarrow x) : x \in X\} \cup \{X \setminus (\uparrow x) : x \in X\}.
\]

Finally we give the notion of breadth of a lattice.

**Definition 1.2.** Let \(n \in \mathbb{N}\) be a positive integer. For a complete lattice \((L, \leq)\) we say that it has breadth \(\leq n\) if for any finite set \(F\) there is \(A \subseteq F\) with \(|A| \leq n\) such that \(\inf(A) = \inf(F)\). We say \(L\) has finite breadth if there is a positive integer \(n \in \mathbb{N}\) such that \(L\) has breadth \(\leq n\). Otherwise we say that \(L\) has infinite breadth.

\section{Problem 85}

Here is the statement of this problem:

Is every complete morphism (i.e., for arbitrary joins and meets) of complete lattices continuous with respect to star-convergence? in the interval topology?

For the notion of star-convergence, we have to introduce some further notions. We start with the answer to the second part of the question, which is about the interval topology.

\subsection{Interval topology.}

**Proposition 2.1.** A complete homomorphism between complete lattices is continuous in the interval topology.

**Proof.** Let \(L\) and \(M\) be complete lattices and let \(f : L \to M\) be a complete lattice homomorphism. Then \(f\) is order-preserving. Let \(x, y \in M\) be such that \(x \leq y\). Either \(f^{-1}([x, y])\) is empty or else take \(a = \inf(f^{-1}([x, y]))\) and \(b = \sup(f^{-1}([x, y]))\). Then \(a \leq b\). Then
\[
f(a) = f(\inf(f^{-1}([x, y]))) = \inf(f(f^{-1}([x, y]))) \geq x,
\]
and similarly 
\[ f(b) = f(\sup(f^{-1}([x, y]))) = \sup(f(f^{-1}([x, y]))) \leq y, \]
and \( f(a) \leq f(b) \) so \( x \leq f(a) \leq f(b) \leq y \) and hence \( f(a), f(b) \) are in \([x, y]\).

Let \( c \in [a, b] \). Then \( x \leq f(a) \leq f(c) \leq f(b) \leq y \), so \([a, b]\) is a subset of \( f^{-1}([x, y])\). But \( f^{-1}([x, y]) \) is a subset of \([a, b]\). Thus \( f^{-1}([x, y]) = [a, b] \).

Hence \( f^{-1} \) takes subbasic closed sets to subbasic closed sets or the empty set. Therefore \( f \) is continuous when \( L \) and \( M \) have the interval topology. \( \square \)

2.2. Star-convergence. For this part of the question we need the notion of order convergence expressed with filters (Birkhoff uses nets, and filters offer an equivalent, but more concise approach to convergence [3]).

Let \((P, \leq)\) be a poset. By a set filter \( F \) on \( P \) we mean a collection of subsets of \( P \) such that:
- \( \emptyset \notin F; \)
- \( A, B \in F \) implies \( A \cap B \in F; \)
- \( U \in F, U' \subseteq P \) and \( U' \supseteq U \) implies \( U' \in F. \)

If \( F \) is a set filter, then we set \( F^u = \bigcup \{ F^u : F \in F \} \) and define \( F^\ell \) similarly.

For \( x \in P \) and \( F \) a set filter on \( P \) we write
\[ F \to x \text{ iff } \bigwedge F^u = x = \bigvee F^\ell \]
and say \( F \) order-converges to \( x \).

If \( B \) is a collection of subsets of \( P \) such that
- \( \emptyset \notin B, \)
- for \( A, B \in B \) there is \( C \in B \) with \( C \subseteq A \cap B, \)
then we call \( B \) a filter base. The filter generated by \( B \) is the collection of sets that contain some member of \( B \).

If \( F \subseteq G \) are filters on \( P \) we say that \( G \) is a super-filter of \( F \).

Finally, we say that a filter \( F \) star-converges to \( x \in P \) if for every super-filter \( F' \) of \( F \) there is a super-filter \( G \) of \( F' \) such that \( F \to x \).

If \( X, Y \) are sets and \( F \) is a filter on \( X \) then it is easy to verify that \( B_f := \{ f(F) : F \in F \} \) is a filter base in \( Y \). We define \( f(F) \) to be the filter generated by \( B_f \).

The positive answer to the star-convergence part of question 85 follows from the following two lemmas:

**Lemma 2.2.** Let \( L, M \) be complete lattices and let \( f : L \to M \) be a complete lattice homomorphism. Suppose that \( F \) is a filter on \( L \) and \( x \in L \) such that \( F \to x \). Then \( f(F) \to f(x) \).
Proof. We prove that $\bigwedge_M f(F)^u = f(x)$.

The tool we use is Fact 1.1(1) from [2], which states that

$$x \in F^u \iff \downarrow x \in F.$$ 

So assuming $F \to x$ in the lattice $L$, we get $\downarrow x \in F$. Therefore $f(\downarrow x) \in B_f$.

Since $f$ is order-preserving, we get $\downarrow M f(x) \supseteq f(\downarrow x)$, which implies $\downarrow M f(x) \in f(F)$ because $B_f$ is a filter base for $f(F)$.

Using the other direction of the equivalence stated above, we get $\bigwedge (f(F))^u = f(x)$.

Lemma 2.3. If $G \supseteq F$ are filters on a set $X$ and $f : X \to Y$ is any map, then $f(G) \supseteq f(F)$.

3. Problem 87

Here is the statement of this problem:

Can a lattice of infinite breadth be a Hausdorff lattice in its interval topology?

We will show that $2^\omega$ is such an example. (We order $2 = \{0, 1\}$ by $0 < 1$ and set $2^\omega$ to be the set of all functions $f : \omega \to 2$, ordered pointwise.)

First, we look at the interval topology of $2^\omega$.

Lemma 3.1. Let $(P_k)_{k \in K}$ be a family of posets. The interval topology $\tau_i = \tau_i(\prod_{k \in K} P_k)$ on $P = \prod_{k \in K} P_k$ equals the product topology $\tau_p$ of the topological spaces $(P_k, \tau_i(P_k))$.

Proof. Take a subbasic element of $U \in \tau_i$ and show that it is a member of $\tau_p$. Without loss of generality we let $U = P \setminus (\uparrow (x_k)_{k \in K})$ where $x_k \in P_k$. Note that $\uparrow (x_k)_{k \in K}$ is a product of closed sets in the spaces $(P_k, \tau_i(P_k))$, therefore it is closed in the product topology, so $U \in \tau_p$.

Conversely, for some $j \in K$ let $U = \pi_j^{-1}(U_j)$ be subbasic in $\tau_p$ where $\pi_j : P \to P_j$ is the projection map and $U_j = P_j \setminus \uparrow x^*$ for some $x^* \in P_j$. Then

$$U = \bigcup \{P \setminus (\uparrow (z_k)_{k \in K}) : (z_k)_{k \in K} \in P \text{ and } z_j = x^*\}. $$

So $U \in \tau_i$. □

Corollary 3.2. The interval topology on $2^\omega$ is Hausdorff.
Proof. The lemma shows that the interval topology is just the product topology of the (discrete) Hausdorff topology on \( 2 = \{0, 1\} \), and the product topology of Hausdorff spaces is always Hausdorff.

\[ \square \]

**Lemma 3.3.** The complete lattice \( 2^\omega \) has infinite breadth.

**Proof.** For \( m \in \omega \) we let \( e_m : \omega \to 2 = \{0, 1\} \) be the function where \( e_m(m) = 0 \) and \( e_m(k) = 1 \) for \( m \neq k \).

In order to show that for any positive \( n \in \mathbb{N} \) the complete lattice \( 2^\omega \) does not have breadth \( \leq n \), we consider the finite set

\[ F = \{e_0, \ldots, e_n\}. \]

So \( \inf(F) \in 2^\omega \) is the function \( r : \omega \to 2 \) such that \( r(k) = 0 \) for \( k \leq n \) and \( r(k) = 1 \) otherwise.

Note that \( F \) has \( n+1 \) elements, and that for no subset of \( \mathcal{A} \subseteq F \) with \( \mathcal{A} \neq F \) do we have \( \inf(\mathcal{A}) = \inf(F) \).  

\[ \square \]

So corollary 3.2 and lemma 3.3 answer question 87.

### References

[1] G. Birkhoff, *Lattice Theory*, third edition, p. 253.

[2] D. van der Zypen, *Order convergence and compactness*, Cah. Topol. Geom. Diff. Cat. (2004), 45(4), 297-300.

[3] [https://en.wikipedia.org/wiki/Net_(mathematics)#Relation_to_filters](https://en.wikipedia.org/wiki/Net_(mathematics)#Relation_to_filters)

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