A Graph Theory Approach for Regional Controllability of Boolean Cellular Automata.

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**ABSTRACT**

Controllability is one of the central concepts of modern control theory that allows a good understanding of a system’s behaviour. It consists in constraining a system to reach the desired state from an initial state within a given time interval. When the desired objective affects only a sub-region of the domain, the control is said to be regional. The purpose of this paper is to study a particular case of regional control using cellular automata models since they are spatially extended systems where spatial properties can be easily defined thanks to their intrinsic locality. We investigate the case of boundary controls on the target region using an original approach based on graph theory. Necessary and sufficient conditions are given based on the Hamiltonian Circuit and strongly connected component. The controls are obtained using a preimage approach.

**KEYWORDS**

Regional Controllability; Deterministic Cellular Automata; Graph Theory; Hamiltonian Circuit; Strongly connected component.

1. Introduction

Control theory is a branch of mathematics that deals with the behaviour of dynamical systems studied in terms of inputs and outputs. With the recent developments in computing, communications, and sensing technologies, the scope of control theory is rapidly evolving to encompass the increasing complexity of real-life phenomena. Controllability and observability are two major concepts of control theory that have been extensively developed during the last two centuries. The concept of controllability refers to the ability of designing control inputs so as to steer the state of the system to desired values within an interval time $[0, T]$ while the observability describes whether the internal state variables of the system can be externally measured. These concepts are being increasingly useful in a wide range of applications such as: biology, biochemistry, biomedical engineering, ecology, economics etc. \cite{12}. Controllable and observable systems have been characterized so far using the Kalman condition in the linear case. The aim of this paper is to find a general way to give a necessary and
sufficient condition for controllability of complex systems via cellular automata models. We concentrate in this work on regional controllability via boundary actions on the target region $\omega$ that consists in achieving an objective only in a subdomain of the lattice when some specific actions are exerted on the target region boundaries.

The concept of controllability has been widely studied for both finite and infinite dimensional systems. As in many practical problems one is interested in achieving some objectives only on a restricted given sub-region, the notion of controllability has been extended to the so-called regional controllability concept that has been introduced by El Jai and Zerrik and well studied in several works.

For distributed parameter systems, the term regional has been used to refer to control problems in which the desired state is only defined and may be reachable on a portion of the domain $\Omega$.

As for controllability issue, one can consider controls applied on the boundary of the domain or, in the case of regional controllability, on the boundaries of the considered subregion. The controls will steer the system from an initial state to a desired target on a subregion $\omega$ during a fixed time $T$.

Boundary regional controllability problems for distributed parameter systems have been mainly described so far, by partial differential equations and considered for linear or nonlinear, continuous or discrete systems. In this paper, we propose to investigate these problems by using cellular automata as they have been often considered as a good alternative to partial differential equations.

Cellular automata (CA for short) are discrete dynamical systems considered as the simplest models of spatially extended systems. They are widely used for studying the mathematical properties of discrete systems and for modelling physical systems. However, control of systems described by CA remains very difficult. The techniques for controlling discrete systems are quite different from those used in continuous ones, since discrete systems are in general strongly nonlinear and the usual linear approximation cannot be directly applied. We restrict our study to the case of some CA rules.

A Boolean cellular automaton is a collection of cells that can be in one of two states, on and off, 1 or 0, $S = \{0, 1\}$. The states of each cell varies in time depending on the states of their neighbourhood and a local transition function that defines the connections between the cells. This function can be deterministic or probabilistic, synchronous or asynchronous, linear or nonlinear.

We focus in this study, on a particular case of deterministic and synchronous transition rule that calculates the output of each cell, at each time step as a function of the current state of the cell and the states of its two immediate neighbours. A CA configuration or global state defines the image representing the states at time $t$, of the whole lattice cells. The CA evolution is described by the succession of configurations at different time steps.

This evolution in the case of Boolean deterministic CA can be represented by an oriented graph where the vertices corresponds to the configurations obtained from the binary representation converted to decimal. There is an arc between two vertices $v_1$ and $v_2$ if the configuration corresponding to $v_2$ can be obtained from the configuration obtained from $v_1$ where the local transition function is applied.

Some regional control problems has been studied using CA, see for instance. In [16], the control of 1D and 2D additive CA has been studied by exploring a numerical approach based on genetic algorithms. The regional control problem has been studied on deterministic cellular automata in [13] and on probabilistic CA in [16]. In [18] an approach based on Markov Chain has been used to prove the regional controllability of CA.
controllability of linear 1D and 2D cellular automata instead of using the well known Kalman criterion.

In this paper, we pursue our investigation on regional control problems of deterministic CA by using a graph theory approach. The evolution of a controlled CA can be represented by an oriented graph where the vertices represent the possible configurations in the controlled region \( \omega \) which are related to each other by arcs. A couple of vertices \((v_1, v_2)\) (CA configurations restricted to \( \omega \)) are related by arc if it exists a boundary control \((\ell, r)\) such that \(v_2\) is reachable starting from \(v_1\). In this paper we focus on the problem of regional controllability by applying the controls on the boundaries of the region \( \omega \). We prove the regional controllability by checking the existence of Hamiltonian Circuit which allows us to give a sufficient condition and necessary condition. We address also the problem of decidability of regional controllability by looking for a strongly connected component in the graph related to the controlled CA. A necessary and sufficient condition for regional controllability is then obtained. Finally, the controls required on the boundaries of \( \omega \) that ensure the regional controllability are obtained using a method for generating the preimages.

The paper is organized as follows. Section 2 provides necessary definitions and section 3 presents the problem of regional controllability. Section 4 is devoted to the formulation of the problem using transition graphs and section 5 gives necessary and sufficient conditions for regional controllability. It first deals with the existence of a Hamiltonian Circuit in the graph representing the Boolean CA global evolution and then the decidability criterion of regional controllability by establishing a relation with strongly connected component is given. According to this criterion, we give a classification of selected rules in the one-dimensional CA case. In section 6, we introduce a method to trace the configurations where a regional control is possible using a method based on preimages. Finally, a conclusion will be given in section 7.

2. Basic definitions

**Definition 2.1.** [14] A cellular automaton (CA for short) is defined by a tuple \( A = (\mathcal{L}, S, \mathcal{N}, f) \) where:

1. \( \mathcal{L} \) is a cellular space which consists in a regular paving of domain \( \Omega \) of \( \mathbb{R}^d \), \( d = 1, 2 \).
2. \( S \) is a finite set of possible states.
3. \( \mathcal{N} \) is a function that defines the neighborhood of a cell \( c \). We denote:

\[
\mathcal{N} : \mathcal{L} \rightarrow \mathcal{L}^r \\
c \rightarrow \mathcal{N}(c) = (c_{i_1}, c_{i_2}, \ldots, c_{i_r})
\]

where \( c_{i_j} \) is a cell for \( j = 1, \ldots, r \) and \( r \) is the size of the neighborhood \( \mathcal{N}(c) \) of the cell \( c \).

4. \( f \) is the transition function that allows to compute the state of a cell at time \( t + 1 \) according to the state of its neighborhood at time \( t \). It is defined as follow:

\[
f : S^r \rightarrow S \\
s_t(\mathcal{N}(c)) \rightarrow f(s_t(\mathcal{N}(c))) = s_{t+1}(c)
\]
where \( s_t(c) \) is the state of a cell \( c \) at time \( t \) and \( s_t(N(c)) = \{ s_t(c'), c' \in N(c) \} \) is the state of the neighborhood of \( c \).

**Definition 2.2.**
- The configuration of a CA at time \( t \) corresponds to the set \( \{ s_t(c), c \in \mathcal{L} \} \).
- The global dynamics of a CA is given by the function:
  \[
  F: S^L \rightarrow S^L
  \]
  \[
  \{ s_t(c), c \in \mathcal{L} \} \rightarrow \{ s_{t+1}(c), c \in \mathcal{L} \}
  \]
  \( F \) maps a configuration of CA at time \( t \) a new configuration at time \( t + 1 \).
- We denote by \( \omega \) the region we want to control. We have: \( \omega = \{ c_1, \ldots, c_n \} \).

**Definition 2.3.** An elementary CA (ECA) is a one-dimensional cellular automaton constituted by an array of cells that take their states in \( \{0, 1\} \) and change it depending only on the states of their neighbours. One can say that the neighborhood of a cell is given by the cell itself and its right and left nearest neighbours.

Since there are \( 2^3 = 8 \) possible binary states for a cell and its two immediate neighbours, there are a total of \( 2^8 = 256 \) ECA, each of which can be indexed with an 8-bit binary number [19,20].

### 3. Problem Statement

Let us consider:
- a 1D-cellular domain \( \mathcal{L} \) of \( N \) cells,
- a discrete time horizon \( I = \{0, 1, \ldots, T\} \),
- a sub-domain \( \omega \) that defines the controlled region where we want to drive the CA towards a given configuration.
  It will contain \( n \) cells denoted by \( c_i, i = 1, 2, \ldots, n, n < N \).
- \( \overline{\omega} = \{ c_0, c_1, \ldots, c_n, c_{n+1} \} = \omega \cup \{ c_0, c_{n+1} \} \) where \( \{ c_0, c_{n+1} \} \) are the boundary cells of \( \omega \) where we apply control.

We are interested in the problem of regional controllability via actions exerted on the boundary of the target region \( \omega \) which is a part of the cellular automaton space as illustrated in Figure 1 for \( n = 3 \). Our aim is to determine the values at the boundary cells in order to obtain a specific behaviour on \( \omega \).

**Definition 3.1.** [16] Let \( s_d \in S^\omega \) (\( s_d : \omega \rightarrow S \)) be a desired profile to be reached on \( \omega \subset \mathcal{L} \). The CA is said to be regionally controllable for \( \omega \) at time \( T \) if there exists a control sequence \( u = (u_0, \ldots, u_{T-1}) \) where \( u_i = (u_i(c_0), u_i(c_{n+1})) \), \( i = 0, \ldots, T - 1 \) such as:

\[
\begin{align*}
  s_T &= s_d \quad \text{on} \quad \omega
\end{align*}
\]

where \( s_T \) is the final configuration at time \( T \) and \( s_d \) is the desired configuration.

**Notation 1.** Let us introduce the following notations:
Figure 1. Regional control of one-dimensional CA.

\[
\ell^t = u_t(c_0) \\
r^t = u_t(c_{n+1}) \\
s^t_i = s_t(c_i)
\]

\(\forall i, 1 \leq i \leq n.\)

\((l \cdot x \cdot r)\) is the concatenation operation describing the CA state on \(\omega\) where \(x = s(c_1), \ldots, s(c_n)\), \(\ell = u(c_0)\) and \(r = u(c_{n+1})\).

**Problem 1.** Starting from an initial condition and for a given desired configuration \(s_d\), the considered problem of regional controllability consists in finding the control required on the boundaries \(\{c_0, c_{n+1}\}\), in order to get at time \(T\), the configuration \(s_d^n\) in the controlled region \(\{c_1, \ldots, c_n\}\), such that \(s_d(c_i) = s_T(c_i)\ \forall i = 1, \ldots, n\), for a given time horizon \(T\).

**Example 3.2.** Consider the Wolfram’s rule 90 for which the evolution can completely be described by a table mapping the next state from all possible combinations of three inputs \((s_{-1}, s_0, s_{+1})\) according to the sum modulo 2 of the state values of the cells to its left and to its right \(s_{-1} \oplus s_{+1}\):

\[
f_{90}: \\
111 \mapsto 0 \\
110 \mapsto 1 \\
101 \mapsto 0 \\
100 \mapsto 1 \\
011 \mapsto 1 \\
010 \mapsto 0 \\
001 \mapsto 1 \\
000 \mapsto 0
\]

For instance, with \(n = 6\) (cf. Figure 2), if we assume starting at time 0 with an initial configuration \(\{s_1^0, s_2^0, s_3^0, s_4^0, s_5^0, s_6^0\}\) = \(\{011001\}\) on \(\omega = \{c_1, \ldots, c_6\}\) and given a desired null state on \(\omega\), there exists a control \(u = (u_0, u_1, u_2)\) where \(u_0 = (\ell^0, r^0) = \ldots\)
Figure 2. The evolution of CA Wolfram rule 90 on the region $\omega = \{c_1, \cdots, c_6\}$ starting with the same initial configuration; on the left without control and on the right with control.

$(0,1), u_1 = (l^1, r^1) = (1, 0), u_2 = (l^2, r^2) = (1, 0)$ that are applied on cells $c_0, c_7$, such that the final CA configuration on $\omega$ obtained at time $T = 3$ from the evolution of rule 90, is $\{s_3^1, s_2^3, s_4^3, s_3^3, s_0^0\} = \{000000\}$.

Remark 1. The same problem can be defined on two-dimensional CA. For example, we can apply the control on one side of the boundary or on the whole boundary cells of the controlled region $\omega$ in order to get the desired state inside $\omega$ (see Figure 3).

Example 3.3. Consider the following local evolution rule of a two dimensional CA given by the function:

$$s^{t+1}(c_{i,j}) = s^t(c_{i-1,j}) \oplus s^t(c_{i+1,j}) \oplus s^t(c_{i,j-1}) \oplus s^t(c_{i,j+1})$$

We consider a controlled region given by the square $\omega = \{c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}\}$.

For a given initial configuration given by $\{s^0_{1,1}, s^0_{1,2}, s^0_{2,1}, s^0_{2,2}\} = \{1, 0, 0, 0\}$ on $\omega$, we first let the system evolve without applying controls and get the final configuration $\{s^1_{1,1}, s^1_{1,2}, s^1_{2,1}, s^1_{2,2}\} = \{0, 1, 1, 0\}$ at time $T = 1$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Regional Control of two-dimensional CA}
\end{figure}
\[
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

T=0

\[
\begin{bmatrix}
- & - & - & - \\
- & 0 & 1 & - \\
- & 1 & 0 & - \\
- & - & - & - \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
- & - & - & - \\
- & 1 & 1 & - \\
- & 1 & 1 & - \\
- & - & - & - \\
\end{bmatrix}
\]

T=1

Figure 4. Evolution of the CA rule example 3.3 on \(\omega\) in the autonomous and controlled cases on the left and right matrices respectively.

We look for controls applied on the boundary cells of \(\omega\) in order to obtain a desired configuration consisting of all 1s on \(\omega\), this can be obtained by controls illustrated in red in Figure 4.

Remark 2. We can define asymmetric controls i.e, we keep a part of the boundaries fixed (for instance at 0) and act on a subset of the boundary of the controlled region (red cells) in order to get the desired state (see Figure 5).

\[
\]

Figure 5. Regional control of two dimensional CA with asymmetric controls

The above examples of CA in one and two dimensional cases show that it is possible to steer a system from an initial state to a desired target on a subregion of the domain by acting on its boundary. The aim of this paper is to generalize this results and find necessary and sufficient conditions for the regional controllability of some Boolean CA rules. The proposed method in the following will be based on transition graphs.
4. Transition graph approach and regional controllability problem

In this section, we describe the main tool on which this paper is based: the transition graph $\Upsilon$. It was inspired by the one introduced in ref [18] where the authors have built a transformation matrix based on all possible state combinations of the CA to show the transition steps. Let us describe here the construction of $\Upsilon$ and the transformation matrix $C$ that is the associate adjacency matrix of $\Upsilon$.

Recall that the evolution of controlled CA for one step can be represented by a directed graph where the vertices represent the configurations and the arcs represent the transition from a configuration to another one in one step i.e. by applying the global transition function $F$. Consider an Elementary CA where the controlled region $\omega$ is of size $|\omega|$ and controls are applied on its two boundary cells $\{c_0, c_{n+1}\}$. When considering the restriction of $F$ on $S^{[\omega]}$, there exists a bijection between $S^{[\omega]}$ and the set of integers $[0 : 2^{|\omega|} - 1]$ that represents CA configurations on $\omega$ as $|\omega|$-bit binary numbers. Let $\lambda$ be a vertex labelling such that for every vertex $v$, $\lambda(v)$ is the Boolean conversion of vertex $v$.

We define the transition graph $\Upsilon = (V, A)$ as follow where the vertices $V$ corresponds to each possible configuration of the region $\omega$ and $A$ is the set of arcs. Let $v_1$ and $v_2$ be two vertices in $V$, there is an arc from the vertex $v_1$ to the vertex $v_2$ if there exists a control $u = (\ell, r) \in \{(0, 0); (0, 1); (1, 0); (1, 1)\}$ such that $\lambda(v_2)$ is equal to $F_{[\omega]}(\ell \cdot \lambda(v_1) \cdot r)$, where the $\lambda(v_1)$ denotes the configuration in the controlled region at time $t$ and $\lambda(v_2)$ denotes the configuration in the controlled region at time $t + 1$.

We denote by $C$ the transition matrix which is the associate adjacency matrix of the graph $\Upsilon$. The transition matrix is built as a Boolean matrix of size $2^{|\omega|} \times 2^{|\omega|}$. There is a 1 at position $(i,j)$, the $i$th row and $j$th column, if there is an arc between vertices $i$ and $j$ for all $i,j$ in $[0 : 2^{|\omega|} - 1]$. Otherwise it stays at 0.

We proceed as follow to construct the transition graph $\Upsilon$ (the algorithm is given in the appendix). For each vertex $v$, we compute the four configurations (represented by $u_1, u_2, u_3, u_4$) obtained by the application of the global transition function $F_{[\omega]}$ to the four possible configurations obtained by the concatenation of the controls $((0, 0); (0, 1); (1, 0); (1, 1))$ on the extremities of $v$. Then we add an arc from $v$ to each of the four $u_i$. In total, the time complexity to build $\Upsilon$ is $O(|V|)$ i.e. $O(2^{|\omega|})$ where $|\omega|$ is the size of the controlled region $\omega$ in the CA. The space complexity is the size of the Boolean matrix $C$: $O(|V| \times |V|) = O(2^{|\omega|} \times 2^{|\omega|})$. Note that the number of arcs is at most $4 \times |V|$.

Remark 3. Note that we have taken the binary representation for the controlled region in a reverse order (the least significant bit is the first one). For instance, for a controlled region of size 3, we note: $\lambda(100) = 1, \lambda(110) = 3$ and $\lambda(001) = 4$

Example 4.1. For instance, consider the rule 30 where the controlled region is of size $|\omega| = 2$ for more simplicity. The corresponding graph is represented in Figure 6.

The corresponding table for rule 30 is:
Consider vertex 2 whose binary conversion is 01. We have that

\[ F|_\omega(0010) = F|_\omega(0011) = 11 \] and

\[ F|_\omega(1010) = F|_\omega(1011) = 01. \]

Therefore there are two arcs from 2: (2, 2) and (2, 3) as the binary conversion of 3 is 11.

**Figure 6.** Transition graph \( \Upsilon \) for the CA rule 30 where the region to be controlled is of size 2 and \( \lambda(0) = 00, \lambda(1) = 10, \lambda(2) = 01, \lambda(3) = 11. \)

**Example 4.2. Rule 90 when the controlled region is of size \(|\omega| = 3.\)**

Another example is given in Figure 7. We have considered the rule 90 where the controlled region is of size 3.

**Figure 7.** Transition graph and its adjacency matrix for the CA rule 90 where the controlled region is of size 3.

\[ C_{90} = \begin{pmatrix}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
2 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
3 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
4 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
5 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
6 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\
7 & 1 & 1 & 0 & 0 & 1 & 1 & 0
\end{pmatrix}. \]
5. Characterizing regional controllability for Boolean deterministic CA

5.1. Necessary and sufficient Condition: Hamiltonian Circuit

In this section we prove the regional controllability for one-dimensional and two-dimensional CA using a method based on the existence of a Hamiltonian circuit. The CA is regionally controllable if all the states are reachable in the target region (starting from each vertex we can achieve another vertex in finite number of steps). The existence of a Hamiltonian circuit ensures that all vertices (configurations) are visited once and ensures that it exists a time $T$ such as all the configurations are reachable.

**Definition 5.1.** A Hamiltonian circuit of a graph $G = (V,A)$ is a simple directed path of $G$ that includes every vertex exactly once.

**Notation 2.** We introduce the notation $a \xrightarrow{} b$. This means that $b$ is reachable from $a$ i.e. there is a directed path starting from $a$ to $b$. In other words, there exist vertices $v_1, v_2, \ldots, v_i$ such that $(a, v_1), (v_1, v_2), \ldots, (v_{i-1}, v_i), (v_i, b)$ are arcs in $A$.

**Theorem 5.2.** A Cellular Automaton is regionally controllable iff there exists a $t$ such that the graph associated to the transformation matrix $C^t$ contains a Hamiltonian circuit.

**Proof.** Let us start with the first implication. Let $\Upsilon = (V,A)$ be the transition graph built in Section 4 for a CA with a controlled region of size $|\omega|$ and $V = \{v_1, \ldots, v_{2^{|\omega|}}\}$. The graph $\Upsilon$ will be represented by an adjacency matrix $C$. Let $G_1$ be the transition graph associated to the matrix $C^t$. The proof is based on the following property in graph theory: the $(i,j)$th entry of the matrix $C^t$ corresponds to the number of paths of length $t$ from vertex $i$ to $j$.

Assume that the CA is regionally controllable at time $T \geq t$. Then some configurations can be reached in less than $T$ steps from any other one. That means that each pair of vertices are linked by a directed path of length at most equal to $T$. Therefore, $C^T$ will be strictly positive as reported in the theorem in [18] which states that the CA is regionally controllable if there exists a power $T$ such that $C^T > 0$. The associated graph $G_T$ to the matrix $C^t$ is therefore a complete graph and it is trivial to find a Hamiltonian circuit in a complete graph which implies that there exists $t \leq T$ such that the graph related to the matrix $C^t$ contains a Hamiltonian Circuit and the direct implication holds.

To prove the converse one, let us assume that $G_1$ contains a Hamiltonian circuit. This means that there is a directed path that goes through all the vertices once. Therefore there exists an order $i_1, i_2, \ldots, i_{2^{|\omega|}}$ such that: $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_{2^{|\omega|}-1}}, v_{i_{2^{|\omega|}}}), (v_{i_{2^{|\omega|}}}, v_{i_1})$ are arcs in $A$. And then we have:

$$v_i \xrightarrow{} v_j \forall \ i, j \in \{1, \ldots, 2^{|\omega|}\} \ and \ i \neq j$$

Thus, $\exists T \geq t$ such that all the vertices (configurations) are reachable. And the theorem holds.

As the problem of proving the existence of a Hamiltonian circuit in a graph is NP-complete, the time complexity can be exponential in the number of vertices of the transformation graph. We improve this criterion in the next section with a solution in polynomial time that gives a necessary and sufficient condition.
5.2. Necessary and Sufficient Condition: Strongly connected component

**Definition 5.3.** [21] A strongly connected component (SCC for short) of a directed graph $G$ is a maximal set of vertices $C \subset V$ such that for every pair of vertices $v_1$ and $v_2$ in $C$, there is a directed path from $v_1$ to $v_2$ and a directed path from $v_2$ to $v_1$.

**Theorem 5.4.** A CA is regionally controllable for a given rule iff the transition graph $\Upsilon$ associated to the rule has only one SCC.

**Proof.** Let $\Upsilon = (V,A)$ be the transition graph built in Section 4 from a controlled region of size $|\omega|$.

Assume that the graph contains only one SCC. There exists a directed path which relates each pair of vertices of the graph. Hence there is a sequence of controls that permits to go from every configuration to any other one. The CA is then controllable on $\omega$ and the direct implication holds.

Assume now that the graph $\Upsilon$ contains more than one SCC, let say it contains two. Then, the set of vertices can be divided in two sets related to each SCC such as:

$$V_1 = \{v_1, \ldots, v_k\} \quad V_2 = \{v_{k+1}, \ldots, v_{2|\omega|}\}$$

and there is no arc between $V_1$ and $V_2$. Therefore, there is no control that allows to obtain a configuration represented in $V_1$ from a configuration in $V_2$ according to the construction of $\Upsilon$. It is impossible since the CA is regionaly controllable and so the converse implication holds.

**Time complexity** To find the SCCs, we have used Tarjan’s algorithm [22] which has a linear time complexity: $O(|V| + |A|)$ on the graph $\Upsilon = (V,A)$. If we consider a controlled region $\omega$ of size $|\omega|$ and since in that case $|A| \leq 4|V|$, then the time complexity is $O(|V|) = O(2^{|\omega|})$.

**Remark 4.** The regional controllability depends on the rule and the size of the controlled region. The size of the controlled region for the same rule has an impact on the number of SCCs. According to Theorem 5.4 by changing the size of CA, a rule can be sometimes regionally controllable and sometimes not.

In Table 1 the results of our simulations are highlighted.

| Rules | Decidability Criterion | number of SCC |
|-------|------------------------|---------------|
| 0,255 | not controllable        | 64 for $|\omega| = 6$ and 16 for $|\omega| = 4$ |
| 1     | not controllable        | 8 for $|\omega| = 4$ and 1 for $|\omega| = 2$ |
| 60,90,102,150,170 | controllable | 1 |
| 204   | not controllable        | $> 1$ |
| 2,4,8,16 | not controllable | $> 1$ |
| 105,195,165,153,85 | controllable for all the sizes of CA | 1 |
| 22    | controllable for $|\omega| = 2$, not controllable $|\omega| = 5$ | 1 for $|\omega| = 2$ and $> 1$ for $|\omega| = 5$ |
| 26    | controllable $2 \leq |\omega| \leq 3$, not controllable $|\omega| = 4$ | 1 for $2 \leq |\omega| \leq 3$ , 2 for $|\omega| = 4$ |
| 233   | not controllable        | $> 1$ |
| 3     | controllable $|\omega| = 2$, otherwise not controllable | 1 for $|\omega| = 2$ , otherwise $> 1$ |

To illustrate the obtained results, we shall give in the following section, some examples in both one and two dimensional cellular automata.
5.3. Examples

Example 5.5. Let us consider the two linear Wolfram rules 150 and 90 \[23\]. For a controlled region of \(|\omega| = 4\), the graph of the matrix \(C\) associated to these two rules is illustrated in the figure Fig. [5.5] It contains one strongly connected component which means that there exists a time \(T\) where each configuration is reachable.

![One SCC associated to the graph of rule 150.](image1)

![One SCC associated to the graph of rule 90.](image2)

**Figure 8.** Graphs of the matrices \(C_{150}\) and \(C_{90}\) respectively.

Example 5.6. Wolfram Rule 0 is not controllable neither its Boolean complement rule 255 as they converge to a fixed point. The graph of their matrix \(C\) contains more than one strongly connected component and the previous theorem states that these rules are not regionally controllable for every region \(\omega\).

![rule 0, |\(\omega| = 6.](image3)

![rule 255, |\(\omega| = 4.](image4)

**Figure 9.** Graphs related to the matrices \(C_0\) and \(C_{255}\) respectively.

Example 5.7. Let us consider now a two-dimensional cellular automaton. Its local
evolution is given by the transition function:

\[ s_{t+1}(c_{i,j}) = s_t(c_{i-1,j}) \oplus s_t(c_{i+1,j}) \oplus s_t(c_{i,j-1}) \oplus s_t(c_{i,j+1}) \]

that is also denoted by rule 170 using Wolfram’s formalism. We impose asymmetric controls by setting all cells on the boundaries to 0 except for the two red colored cells illustrated in Fig. 5.

The obtained graph of the matrix \( C_{170} \) for \( |\omega| = 2 \times 2 \), \( \omega = \{c_{1,1}, c_{1,2}, c_{2,1}, c_{2,2}\} \), contains one strongly connected component and so the CA is regionally controllable.

**Example 5.8.** Finally, an example with rule 1 is given to show that the decidability criterion for regional controllability may change for the same rule, according to the size of \( \omega \). With a region of size \( |\omega| = 6 \), the graph of the matrix \( C_1 \) contains more than one SCC while for \( |\omega| = 2 \) it contains only one strongly connected component. Consequently, the CA is not regionally controllable in the first case and regionally controllable in the second one.

6. Pre-images of a regional controlled area

Let \( \{s_1^i, s_2^i, \ldots, s_n^i\} \) be the configuration at time \( i \) of the region to be controlled. The idea is to find a boundary control given as a sequence \( (\ell^0, r^0), (\ell^1, r^1), \ldots, (\ell^{T-1}, r^{T-1}) \) so as to obtain a desired configuration \( \{s_1^T, s_2^T, \ldots, s_n^T\} \) at time \( T \) from an initial one \( \{s_1^0, s_2^0, \ldots, s_n^0\} \).

Let us define in what follows some needed notions and present the data structure required to solve the problem.
6.1. Distance function

We define the distance function

\[ \Delta_i : \text{vertex} \mapsto \text{list of vertices} \]

that associates to each vertex \( v \in \Upsilon \), the list of vertices from which \( v \) can be reached within a path of length exactly \( i \).

Where \( \Upsilon \) is the transition graph introduced in Section 4.

Therefore \( \Delta_i(v) \) gives all initial configurations from which the desired configuration \( v \) can be reached in exactly \( i \) steps with the application of control.

Let \( T \) be the time at which we want to reach a desired state. Representing \( \Delta \) as a map, we shall consecutively construct the functions \( \Delta_i \) by searching the predecessors from \( \Delta_{i-1} \). We start by \( \Delta_1 \) and go until \( \Delta_T \). The algorithm is given in the appendix.

If \( \delta \) is the maximum number of predecessors of a vertex, the time complexity is

\[ O(T \times \delta \times |V|) = O(T \times \delta \times 2^{|\omega|}). \]

6.2. Path controllability

We address two problems in this section.

**Problem 2.** Find one state configuration that can be driven to a desired state configuration \( b_f \) in \( k \) steps and the relative control sequence.

To solve this problem, we construct the distance function \( \Delta_i \) for \( 1 \leq i \leq k \). Then we consider the ancestors \( b_i \) at distance \( k \) of \( b_f \) (stored in \( \Delta_k(b_f) \)). If there is no ancestor, that means that it is not possible to reach this state in \( k \) steps. Otherwise, we can find the path of length \( k \) with end extremity \( b_f \). To do so, we pick one predecessor of \( b_f \), say \( b_{k-1} \), that can be reached in \( k - 1 \) steps, \( i.e. \) one among the vertices in the list \( \Delta_{k-1}(b_f) \). Then we search the first one among the predecessors of \( b_{k-1} \), say \( b_{k-2} \), that can be reached in \( k - 2 \) steps, \( i.e. \) one among the vertices in the list \( \Delta_{k-2}(b_f) \) and so forth until we find \( b_1 \). We obtain the path of configurations \( b_1, \ldots, b_{k-1}, b_f \). It just remains to find the appropriate control by applying the rules to each configuration extended with the boundaries \((0,0), (0,1), (1,0), (1,1)\).
In total, once the distance function is built, it takes $O(k)$ time.

**Problem 3.** Find all the needed controls and the intermediate states of the controlled region required to obtain a desired state $b_f$ in exactly $k$ steps (i.e.) from a state configuration $b_1$.

For this problem, instead of checking if the list of ancestors is not empty (and taking one among the vertices), we need to check if among the ancestors there is the initial configuration $b_1$.

Therefore, the time complexity is $O(k \times \delta)$.

7. Conclusion

We have studied in this paper the problem of regional controllability of Boolean cellular automata focusing on actions performed on the boundary of a target region. We established some necessary and sufficient conditions using graph theory tools. We showed that the existence of a Hamiltonian circuit or a strongly connected component guarantees the regional controllability. To obtain the control that allows the system to reach the desired state during a given time horizon and starting from a given initial condition, an efficient algorithm for generating preimages was used. Several examples of Elementary cellular automata has been considered. The obtained control is not unique at this stage and the problem of optimality will be addressed afterward. A first problem of regional controllability in minimal time is currently under study.

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We present in the appendix, the algorithms to construct the data structures used in the paper.

Construction of the transition graph $\Upsilon$.

**Algorithm transGraph($d_\omega$, $F$)**

| Line | Description |
|------|-------------|
| 1. | $d \leftarrow d_\omega$ |
| 2. | $d_\Upsilon \leftarrow 2^d$ |
| 3. | $\Upsilon \leftarrow [0]_{|d_\Upsilon| \times |d_\Upsilon|}$ (zero matrix of size $d_\Upsilon$) |
| 4. | $\Upsilon \leftarrow [0]_{|d_\Upsilon| \times |d_\Upsilon|}$ (for every configuration $i$) |
| 5. | $\forall 0 \leq i < d_\Upsilon$ (for every configuration $i$) |
| 6. | $\lambda(v_1) \leftarrow F(0 \cdot \hat{i}_d \cdot 0)$ |
| 7. | $\lambda(v_2) \leftarrow F(0 \cdot \hat{i}_d \cdot 1)$ |
| 8. | $\lambda(v_3) \leftarrow F(1 \cdot \hat{i}_d \cdot 0)$ (add the boundary controls and apply the rule) |
| 9. | $\lambda(v_4) \leftarrow F(1 \cdot \hat{i}_d \cdot 1)$ |
| 10. | $\forall 0 \leq j < d_\Upsilon$ (for every configuration $j$) |
| 11. | if $\lambda(v_1), \lambda(v_2), \lambda(v_3)$ or $\lambda(v_4)$ equal $\hat{j}_d$ (add an edge if there is a boundary control) |
| 12. | $\Upsilon(i, j) \leftarrow 1$ for $i$ that leads to $j$ |
| 13. | return $\Upsilon$ |

Construction of the distance function.

**Algorithm distanceFunction($G_\Upsilon$, $k$, $v$)**

| Line | Description |
|------|-------------|
| 1. | Let $d_\Upsilon$ be the number of vertices of $G_\Upsilon$ |
| 2. | Let $\Delta$ be an empty matrix of size $d_\Upsilon \times k$ |
| 3. | for each vertex $v$ in $G_\Upsilon$ |
| 4. | $\Delta(v, 1) \leftarrow$ predecessors($G_\Upsilon$, $v$) |
| 5. | for $1 < \ell \leq k$ |
| 6. | for each vertex $v$ in $G_\Upsilon$ |
| 7. | for each vertex $u$ in $\Delta(v, \ell - 1)$ |
| 8. | add($\Delta(v, \ell)$, predecessors($G_\Upsilon$, $u$)) |
| 9. | return $\Delta$ |

Finding the control in $k$ steps to reach the desired state.

**Algorithm pathControllability($G_\Upsilon$, $k$, $v_{init}$, $v_{desired}$)**

| Line | Description |
|------|-------------|
| 1. | $p \leftarrow [v_{desired}]$ |
| 2. | $\Delta \leftarrow$ distanceFunction($G_\Upsilon$, $k$, $v_{desired}$) |
| 3. | if $v_{init} \notin \Delta(v, k)$ |
| 4. | return $p$ |
| 5. | $\text{pred} \leftarrow v_{desired}$ |
| 6. | for $i$ from $k$ to $1$ |
| 7. | $\text{predList} \leftarrow$ predecessors($G_\Upsilon$, $\text{pred}$) |
| 8. | $\text{pred} \leftarrow \text{find}(x \in \text{predList}, v_{init} \in \Delta(x, i))$ |
| 9. | add($\text{pred}$, $p$) |
| 10. | return $p$ |