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Arithmetical proofs of strong normalization results for the symmetric $\lambda\mu$-calculus

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Abstract. The symmetric $\lambda\mu$-calculus is the $\lambda\mu$-calculus introduced by Parigot in which the reduction rule $\mu'$, which is the symmetric of $\mu$, is added. We give arithmetical proofs of some strong normalization results for this calculus. We show (this is a new result) that the $\mu\mu'$-reduction is strongly normalizing for the un-typed calculus. We also show the strong normalization of the $\beta\mu\mu'$-reduction for the typed calculus: this was already known but the previous proofs use candidates of reducibility where the interpretation of a type was defined as the fix point of some increasing operator and thus, were highly non arithmetical.

1 Introduction

Since it has been understood that the Curry-Howard isomorphism relating proofs and programs can be extended to classical logic, various systems have been introduced: the $\lambda_c$-calculus (Krivine [12]), the $\lambda\text{exn}$-calculus (de Groote [6]), the $\lambda\mu$-calculus (Parigot [18]), the $\lambda\text{Sym}$-calculus (Barbanera & Berardi [1]), the $\lambda\Delta$-calculus (Rehof & Sorensen [24]), the $\lambda\mu\tilde{\mu}$-calculus (Curien & Herbelin [3]), ...

The first calculus which respects the intrinsic symmetry of classical logic is $\lambda\text{Sym}$. It is somehow different from the previous calculi since the main connector is not the arrow as usual but the connectors or and and. The symmetry of the calculus comes from the de Morgan laws.

The second calculus respecting this symmetry has been $\lambda\mu\tilde{\mu}$. The logical part is the (classical) sequent calculus instead of natural deduction.

Natural deduction is not, intrinsically, symmetric but Parigot has introduced the so called Free deduction [17] which is completely symmetric. The $\lambda\mu$-calculus comes from there. To get a confluent calculus he had, in his terminology, to fix the inputs on the left. To keep the symmetry, it is enough to keep the same terms and to add a new reduction rule (called the $\mu'$-reduction) which is the symmetric rule of the $\mu$-reduction and also corresponds to the elimination of a cut. We get then a symmetric calculus that is called the symmetric $\lambda\mu$-calculus.

The $\mu'$-reduction has been considered by Parigot for the following reasons. The $\lambda\mu$-calculus (with the $\beta$-reduction and the $\mu$-reduction) has good properties: confluence in the un-typed version, subject reduction and strong normalization in the typed calculus. But this system has, from a computer science point of view, a drawback: the unicity of the representation of data is lost. It is known that, in the $\lambda$-calculus, any term of type $N$ (the usual type for the integers) is $\beta$-equivalent to a Church integer. This no more true in the $\lambda\mu$-calculus and we can find normal terms of type $N$ that are not Church integers. Parigot has remarked that by adding the $\mu'$-reduction and some simplification rules the unicity of the representation of data is recovered and subject reduction is preserved, at least for the simply typed system, even though the confluence is lost.
Barbanera & Berardi proved the strong normalization of the $\lambda^{sym.}$-calculus by using candidates of reducibility but, unlike the usual construction (for example for Girard’s system $F$), the definition of the interpretation of a type needs a rather complex fix-point operation. Yamagata [25] has used the same technic to prove the strong normalization of the symmetric $\lambda\mu$-calculus where the types are those of system $F$ and Parigot, again using the same ideas, has extended Barbanera & Berardi’s result to a logic with second order quantification. These proofs are thus highly non arithmetical.

We consider here the $\lambda\mu$-calculus with the rules $\beta$, $\mu$ and $\mu'$. It was known that, for the un-typed calculus, the $\mu$-reduction is strongly normalizing (see [23]) but the strong normalization of the $\mu\mu'$-reduction for the un-typed calculus was an open problem raised long ago by Parigot. We give here a proof of this result. Studying this reduction by itself is interesting since a $\mu$ (or $\mu'$)-reduction can be seen as a way “to put the arguments of the $\mu$ where they are used” and it is useful to know that this is terminating. We also give an arithmetical proof of the strong normalization of the $\beta\mu\mu'$-reduction for the simply typed calculus. We finally show (this is also a new result) that, in the un-typed calculus, if $M_1, ..., M_n$ are strongly normalizing for the $\beta\mu\mu'$-reduction, then so is $(x \ M_1 \ ... \ M_n)$.

The proofs of strong normalization that are given here are extensions of the ones given by the first author for the simply typed $\lambda$-calculus. This proof can be found either in [7] (where it appears among many other things) or as a simple unpublished note on the web page of the first author (www.lama.univ-savoie.fr/~david).

The same proofs can be done for the $\lambda\mu\tilde{\mu}$-calculus and these proofs are, in fact, much simpler for this calculus since some difficult problems that appear in the $\lambda\mu$-calculus do not appear in the $\lambda\mu\tilde{\mu}$-calculus: this is mainly due to the fact that, in the latter, there is a right-hand side and a left-hand side (the terms and the environments) whereas, in the $\lambda\mu$-calculus, this distinction is impossible since a term on the right of an application can go on the left of an application after some reductions. The proof of the strong normalization of the $\mu\tilde{\mu}$-reduction can be found in [22]. The proof is done (by using candidates of reducibility and a fix point operator) for a typed calculus but, in fact, since the type system is such that every term is typable, the result is valid for every term. A proof of the strong normalization of the $\lambda\mu\tilde{\mu}$-typed calculus (again using candidates of reducibility and a fix point operator) can also be found there. Due to the lack of space, we do not give our proofs of these results here but they will appear in [11].

The paper is organized as follows. In section 2 we give the syntax of the terms and the reduction rules. An arithmetical proof of strong normalization is given in section 3 for the $\mu\mu'$-reduction of the un-typed calculus and, in section 4, for the $\beta\mu\mu'$-reduction of the simply typed calculus. In section 5, we give an example showing that the proofs of strong normalization using candidates of reducibility must somehow be different from the usual ones and we show that, in the un-typed calculus, if $M_1, ..., M_n$ are strongly normalizing for the $\beta\mu\mu'$-reduction, then so is $(x \ M_1 \ ... \ M_n)$. We conclude with some future work.

2 The symmetric $\lambda\mu$-calculus

2.1 The un-typed calculus

The set (denoted as $T$) of $\lambda\mu$-terms or simply terms is defined by the following grammar where $x, y, ...$ are $\lambda$-variables and $\alpha, \beta, ...$ are $\mu$-variables:

$$T ::= x \mid \lambda x T \mid (T \ T) \mid \mu \alpha T \mid (\alpha \ T)$$
Note that we adopt here a more liberal syntax (also called de Groote’s calculus) than in the original calculus since we do not ask that a $\mu\alpha$ is immediately followed by a $(\beta M)$ (denoted $[\beta]M$ in Parigot’s notation).

**Definition 1.** Let $M$ be a term.

1. $\text{ctxy}(M)$ is the number of symbols occurring in $M$.
2. We denote by $N \leq M$ (resp. $N < M$) the fact that $N$ is a sub-term (resp. a strict sub-term) of $M$.
3. If $\overline{P}$ is a sequence $P_1, ..., P_n$ of terms, $(M \overline{P})$ will denote $(M P_1 ... P_n)$.

### 2.2 The typed calculus

The types are those of the simply typed $\lambda\mu$-calculus i.e. are built from atomic formulas and the constant symbol $\perp$ with the connector $\to$. As usual $\neg A$ is an abbreviation for $A \to \perp$.

The typing rules are given by figure 1 below where $\Gamma$ is a context, i.e. a set of declarations of the form $x : A$ and $\alpha : \neg A$ where $x$ is a $\lambda$ (or intuitionistic) variable, $\alpha$ is a $\mu$ (or classical) variable and $A$ is a formula.

```
$$
\begin{array}{c}
\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x M : A \to B} \quad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash (MN) : B} \\
\frac{\Gamma, \alpha : \neg A \vdash M : \perp}{\Gamma \vdash \mu\alpha M : A} \\
\end{array}
$$
```

Figure 1.

Note that, here, we also have changed Parigot’s notation but these typing rules are those of his classical natural deduction. Instead of writing $M : (A_1^{x_1}, ..., A_n^{x_n}, C_1^{\alpha_1}, ..., C_m^{\alpha_m})$ we have written $x_1 : A_1, ..., x_n : A_n, \alpha_1 : \neg C_1, ..., \alpha_m : \neg C_m \vdash M : B$.

**Definition 2.** Let $A$ be a type. We denote by $lg(A)$ the number of arrows in $A$.

### 2.3 The reduction rules

The cut-elimination procedure (on the logical side) corresponds to the reduction rules (on the terms) given below. There are three kinds of cuts.

- A **logical cut** occurs when the introduction of the connective $\to$ is immediately followed by its elimination. The corresponding reduction rule (denoted by $\beta$) is:

  $$(\lambda x M \ N) \triangleright M[x := N]$$

- A **classical cut** occurs when $\perp_e$ appears as the left premiss of a $\rightarrow_e$. The corresponding reduction rule (denoted by $\mu$) is:

  $$(\mu\alpha M \ N) \triangleright \mu\alpha M[\alpha =_r N]$$

where $M[\alpha =_r N]$ is obtained by replacing each sub-term of $M$ of the form $(\alpha U)$ by $(\alpha (U N))$. This substitution is called a $\mu$-substitution.
A symmetric classical cut occurs when \( \perp \) appears as the right premiss of a \( \rightarrow \).

The corresponding reduction rule (denoted by \( \mu' \)) is:

\[
(M \mu \alpha N) \vdash \mu \alpha N[\alpha =_l M]
\]

where \( N[\alpha =_l M] \) is obtained by replacing each sub-term of \( N \) of the form \( (\alpha U) \) by \( (\alpha (M U)) \). This substitution is called a \( \mu' \)-substitution.

**Remarks**

1. It is shown in [18] that the \( \beta\mu \)-reduction is confluent but neither \( \mu\mu' \) nor \( \beta\mu' \) is. For example \((\mu ax \mu by)\) reduces both to \(\mu ax\) and to \(\mu by\). Similarly \((\lambda xz \mu by)\) reduces both to \(x\) and to \(\mu by\).

2. The reductions on terms correspond to the elimination of cuts on the proofs.

   - The \( \beta \)-reduction is the usual one.
   - The \( \mu \)-reduction is as follows. If \( M \) corresponds to a proof of \( \perp \) assuming \( \alpha : \neg(A \rightarrow B) \) and \( N \) corresponds to a proof of \( A \), then \( M[\alpha =_r N] \) corresponds to the proof \( M \perp \) assuming \( \alpha : \neg B \) but where, each time we used the hypothesis \( \alpha : \neg(A \rightarrow B) \) with a proof \( U \) of \( A \rightarrow B \) to get \( \perp \), we replace this by the following proof of \( \perp \). Use \( U \) and \( N \) to get a proof of \( B \) and then \( \alpha : \neg B \) to get a proof of \( \perp \).

   - Similarly, the \( \mu' \)-reduction is as follows. If \( N \) corresponds to a proof of \( \perp \) assuming \( \alpha : \neg A \) and \( M \) corresponds to a proof of \( A \rightarrow B \), then \( N[\alpha =_l M] \) corresponds to the proof \( N \perp \) assuming \( \alpha : \neg B \) but where, each time we used the hypothesis \( \alpha : \neg A \) with a proof \( U \) of \( A \) to get \( \perp \), we replace this by the following proof of \( \perp \). Use \( U \) and \( M \) to get a proof of \( B \) and then \( \alpha : \neg B \) to get a proof of \( \perp \).

3. Unlike for a \( \beta \)-substitution where, in \( M[x := N] \), the variable \( x \) has disappeared it is important to note that, in a \( \mu \) or \( \mu' \)-substitution, the variable \( \alpha \) has not disappeared. Moreover its type has changed. If the type of \( N \) is \( A \) and, in \( M \), the type of \( \alpha \) is \( \neg(A \rightarrow B) \) it becomes \( \neg B \) in \( M[\alpha =_r N] \). If the type of \( M \) is \( A \rightarrow B \) and, in \( N \), the type of \( \alpha \) is \( \neg A \) it becomes \( \neg B \) in \( N[\alpha =_l M] \).

In the next sections we will study various reductions: the \( \mu\mu' \)-reduction in section 3 and the \( \beta\mu' \)-reduction in sections 4, 5. The following notions will correspond to these reductions.

**Definition 3.** Let \( \triangleright \) be a notion of reduction and \( M \) be a term.

1. The transitive (resp. reflexive and transitive) closure of \( \triangleright \) is denoted by \( \triangleright^+ \) (resp. \( \triangleright^\ast \)).
2. If \( M \) is in \( SN \) i.e. \( M \) has no infinite reduction, \( \eta(M) \) will denote the length of the longest reduction starting from \( M \) and \( \eta_c(M) \) will denote \((\eta(M), cxtg(M))\).
3. We denote by \( N \prec M \) the fact that \( N \leq M' \) for some \( M' \) such that \( M \triangleright^+ M' \) and either \( M \triangleright^+ M' \) or \( N < M' \). We denote by \( \preceq \) the reflexive closure of \( \prec \).

**Remarks**

- It is easy to check that the relation \( \preceq \) is transitive and that \( N \preceq M \) iff \( N \leq M' \) for some \( M' \) such that \( M \triangleright^+ M' \).
- If \( M \in SN \) and \( N \prec M \), then \( N \in SN \) and \( \eta_c(N) < \eta_c(M) \). It follows that the relation \( \preceq \) is an order on the set \( SN \).
- Many proofs will be done by induction on some \( k \)-uplet of integers. In this case the order we consider is the lexicographic order.
3 The $\mu\mu'$-reduction is strongly normalizing

In this section we consider the $\mu\mu'$-reduction, i.e. $M \triangleright M'$ means $M'$ is obtained from $M$ by one step of the $\mu\mu'$-reduction. The main points of the proof of the strongly normalization of $\mu\mu'$ are the following.

- We first show (cf. lemma 6) that a $\mu$ or $\mu'$-substitution cannot create a $\mu$.
- It is easy to show (see lemma 8) that if $M \in SN$ but $M[\sigma] \not\in SN$ where $\sigma$ is a $\mu$ or $\mu'$-substitution, there are an $\alpha$ in the domain of $\sigma$ and some $M' \prec M$ such that $M'[\sigma] \in SN$ and (say $\sigma$ is a $\mu$-substitution) $(M'[\sigma] \sigma(\alpha)) \not\in SN$. This is sufficient to give a simple proof of the strongly normalization of the $\mu$-reduction. But this is not enough to do a proof of the strongly normalization of the $\mu\mu'$-reduction. We need a stronger (and more difficult) version of this: lemma 9 ensure that, if $M, N \in SN$ then the real cause of non $SN$ is, in some sense, $[\alpha = r, P]$. Having these results, we show, essentially by induction on $\etac(M) + \etac(N)$, that if $M, N \in SN$ then $(M N) \not\in SN$. The point is that there is, in fact, no deep interactions between $M$ and $N$ i.e. in a reduct of $(M N)$ we always know what is coming from $M$ and what is coming from $N$.

Definition 4. - The set of simultaneous substitutions of the form $[\alpha_1 = s_1 P_1 \ldots, \alpha_n = s_n P_n]$ where $s_i \in \{l, r\}$ will be denoted by $\Sigma$.

- For $s \in \{l, r\}$, the set of simultaneous substitutions of the form $[\alpha_1 = s_1 P_1 \ldots, \alpha_n = s_n P_n]$, we denote by $dom(\sigma)$ (resp. $im(\sigma)$) the set $\{\alpha_1, \ldots, \alpha_n\}$ (resp. $\{P_1, \ldots, P_n\}$).

- Let $\sigma \in \Sigma$. We say that $\sigma \in SN$ iff for every $N \in im(\sigma)$, $N \in SN$.

Lemma 5. If $(M N) \triangleright^* \mu \alpha P$, then either $M \triangleright^* \mu \alpha M_1$ and $M_1[\alpha = r] N \triangleright^* P$ or $N \triangleright^* \mu \alpha N_1$ and $N_1[\alpha = l] M \triangleright^* P$.

Proof By induction on the length of the reduction $(M N) \triangleright^* \mu \alpha P$.

Lemma 6. Let $M$ be a term and $\sigma \in \Sigma$. If $M[\sigma] \triangleright^* \mu \alpha P$, then $M \triangleright^* \mu \alpha Q$ for some $Q$ such that $\etac[\sigma] \triangleright^* P$.

Proof By induction on $M$. $M$ cannot be of the form $(\beta M')$ or $\lambda x M'$. If $M$ begins with a $\mu$, the result is trivial. Otherwise $M = (M_1 M_2)$ and, by lemma 2, either $M_1[\sigma] \triangleright^* \mu \alpha R$ and $R[\alpha = \gamma M_2[\sigma]] \triangleright^* P$ or $M_2[\sigma] \triangleright^* \mu \alpha R$ and $R[\alpha = l M_1[\sigma]] \triangleright^* P$. Look at the first case (the other one is similar). By the induction hypothesis $M_1[\sigma] \triangleright^* Q$ for some $Q$ such that $Q[\etac[\sigma] \triangleright^* R$ and thus $M \triangleright^* \mu \alpha Q[\alpha = r M_2]$. Since $Q[\alpha = r M_2[\sigma]] = Q[\etac[\sigma] \triangleright^* R[\alpha = r M_2[\sigma]] \triangleright^* P$ we are done.

Lemma 7. Assume $M, N \in SN$ and $(M N) \not\in SN$. Then either $M \triangleright^* \mu \alpha M_1$ and $M_1[\alpha = r] N \not\in SN$ or $N \triangleright^* \mu \beta N_1$ and $N_1[\beta = l M] \not\in SN$.

Proof By induction on $\etac(M) + \etac(N)$. Since $(M N) \not\in SN$, $(M N) \triangleright P$ for some $P$ such that $P \not\in SN$. If $P = (M' N)$ where $M \triangleright M'$ we conclude by the induction hypothesis since $\etac(M') + \etac(N) < \etac(M) + \etac(N)$. If $P = (M N')$ where $N \triangleright N'$ the proof is similar. If $M = \mu \alpha M_1$ and $P = \mu \alpha M_1[\alpha = r, N]$ or $N = \mu \beta N_1$ and $P = \mu \beta N_1[\beta = l M]$ the result is trivial.

Lemma 8. Let $M$ be term in $SN$ and $\sigma \in \Sigma$ be in $SN$. Assume $M[\sigma] \not\in SN$. Then, for some $(\alpha P) \preceq M$, $P[\sigma] \in SN$ and, if $s = l$ (resp. $s = r$), $(\sigma(\alpha) P[\sigma]) \not\in SN$ (resp. $(P[\sigma] \sigma(\alpha)) \not\in SN$).

Proof We only prove the case $s = l$ (the other one is similar). Let $M_1 \preceq M$ such that $M_1[\sigma] \not\in SN$ and $\etac(M_1)$ is minimal. By the minimality, $M_1$ cannot be $\lambda x M_2$ or $\mu \alpha M_2$. It cannot be either $(N_1 N_2)$ because otherwise, by the minimality, the $N_1[\sigma]$ would be in $SN$ and thus, by lemma 5 and lemmas 9, we would have, for example,
\( N_1 \vdash^+ \mu \alpha N'_1 \) and \( N'_1[\sigma][\alpha \Rightarrow N_2[\sigma]] = N'_1[\alpha \Rightarrow N_2][\sigma] \not\in SN \) but this contradicts the minimality of \( M_1 \) since \( \eta(N'_1[\alpha \Rightarrow N_2]) < \eta(M_1) \). Then \( M_1 = (\alpha P) \) and the the minimality of \( M_1 \) implies that \( P[\sigma] \in SN \).

\[ \square \]

**Remark**

From these results it is easy to prove, by induction on the term, the strong normalization of the \( \mu \)-reduction. It is enough to show that, if \( M, N \in SN \), then \((M \ N) \in SN \). Otherwise, we construct below a sequence \((M_n)\) of terms and a sequence \((\sigma_n)\) of substitutions such that, for every \( i \), \( \sigma_i \) has the form \( [\alpha_1 \Rightarrow N_1, ..., \alpha_n \Rightarrow N_n] \). \( M_i[\sigma_i] \not\in SN \) and \( M_{i+1} \not\Rightarrow M_i \). The sequence \((M_i)\) contradicts the fact that \( M \in SN \). Since \((M \ N) \not\in SN \), by lemma 8, \( M \vdash^+ \mu \alpha M_1 \) and \( M_1[\alpha \Rightarrow N] \not\in SN \). Assume we have constructed \( M_i \) and \( \sigma_i \). Since \( M_i[\sigma_i] \not\in SN \), by lemma 8, there is \( M'_i \not\Rightarrow M_i \) such that \( M'_i[\sigma_i] \in SN \) and \((M'_i[\sigma] N) \not\in SN \). By lemmas 4 and 5, \( M'_i \vdash^+ \mu \alpha M_{i+1} \) and \( M_{i+1}[\sigma_i + \alpha \Rightarrow N] \not\in SN \).

In the remark above, the fact that \((M \ N) \not\in SN \) gives an infinite \( \mu \)-reduction in \( M \). This not the same for the the \( \mu \mu \)-reduction and, if we try to do the same, the substitutions we get are more complicated. In particular, it is not clear that we get an infinite sequence either of the form \( \ldots \prec M_2 \prec M_1 \prec M \) or of the form \( \ldots \prec N_2 \prec N_1 \prec N \). Lemma 9 below will give the answer since it will ensure that, at each step, we may assume that the cause of non \( SN \) is the last substitution.

**Lemma 9.** Let \( M \) be a term and \( \sigma \in \Sigma_s \). Assume \( \delta \) is free in \( M \) but not free in \( Im(\sigma) \). If \( M[\sigma] \in SN \) but \( M[\sigma][\delta =_r P] \not\in SN \), there is \( M' \prec M \) and \( \sigma' \) such that \( M'[\sigma'] \in SN \) and, if \( s = r \), \( (M'[\sigma'] P) \not\in SN \) and, if \( s = i \), \( (P M'[\sigma']) \not\in SN \).

**Proof**

Assume \( s = r \) (the other case is similar). Let \( Im(\sigma) = \{N_1, ..., N_k\} \).

Assume \( M, \delta, \sigma, P \) satisfy the hypothesis. Let \( U = \{U / U \not\in M\} \) and \( V = \{V / V \not\in N_i \text{ for some } i\} \).

Define inductively the sets \( \Sigma_m \) and \( \Sigma_n \) of substitutions by the following rules:

- \( \rho \in \Sigma_m \) if \( \rho = \emptyset \) or \( \rho = \rho' + [\delta =, V[\tau]] \) for some \( V \in V \), \( \tau \in \Sigma_n \) and \( \rho' \in \Sigma_m \)
- \( \tau \in \Sigma_n \) if \( \tau = \emptyset \) or \( \tau = \tau' + [\alpha =, U[\rho]] \) for some \( U \in U \), \( \rho \in \Sigma_m \) and \( \tau' \in \Sigma_n \)

Denote by \( C \) the conclusion of the lemma, i.e. there is \( M' \prec M \) and \( \sigma' \) such that \( M'[\sigma'] \in SN \), and \((M'[\sigma'] P) \not\in SN \).

We prove something more general.

(1) Let \( U \in U \) and \( \rho \in \Sigma_m \). Assume \( U[\rho] \in SN \) and \( U[\rho][\delta =_r P] \not\in SN \). Then, \( C \) holds.

(2) Let \( V \in V \) and \( \tau \in \Sigma_n \). Assume \( V[\tau] \in SN \) and \( V[\tau][\delta =_r P] \not\in SN \). Then, \( C \) holds.

The conclusion \( C \) follows from (1) with \( M \) and \( \sigma \). The properties (1) and (2) are proved by a simultaneous induction on \( \eta, U[\rho] \) (for the first case) and \( \eta, V[\tau] \) (for the second case).

Look first at (1).

- if \( U = \lambda x U' \) or \( U = \mu \alpha U' \): the result follows from the induction hypothesis with \( U' \) and \( \rho \).
- if \( U = (U_1 U_2) \): if \( U_1[\rho][\delta =_r P] \not\in SN \) for \( i = 1 \) or \( i = 2 \), the result follows from the induction hypothesis with \( U_i \) and \( \rho \). Otherwise, by lemma 4 and 5, say \( U_1 \vdash^+ \mu \alpha U'_1 \) and, letting \( U' = U_1[\alpha =_r U_2] \), \( U'[\rho][\delta =_r P] \not\in SN \) and the result follows from the induction hypothesis with \( U' \) and \( \rho \).
- if \( U = (\delta U_1) \): if \( U_1[\rho][\delta =_r P] \in SN \), then \( M' = U_1 \) and \( \sigma' = \rho[\delta =_r P] \) satisfy the desired conclusion. Otherwise, the result follows from the induction hypothesis with \( U_1 \) and \( \rho \).
- if \( U = (\alpha U_1) \): if \( \alpha \not\in dom(\rho) \) or \( U_1[\rho][\delta =_r P] \not\in SN \), the result follows from the induction hypothesis with \( U_1 \) and \( \rho \). Otherwise, let \( \rho(\alpha) = V[\tau] \). If \( V[\tau][\delta =_r P] \not\in SN \)
Proof (1) is trivial. (2) is as in Lemma 5.

Lemma 11. When we will use Lemma 18, the additional fact that we already know that, if \( M \) is small, then \( M \), \( N \) the additional hypothesis will be available.

\[ \text{induction on the type, when we will use lemma 18, the addition hypothesis will be lost. Let e.g.} \]

\[ M \vdash \lambda \delta N [\delta = \tau] \notin S N \text{ where } \tau' = \tau + [\beta =_I U_1[\rho]]. \]

The result follows from the induction hypothesis with \( U_2 \) and \( \rho' \).

\[ V \vdash \mu \beta V_1 \text{ and } V_1[\tau'][\delta = \tau] \notin S N \text{ where } \tau' = \tau + [\beta =_I U_1[\rho]]. \]

The result follows from the induction hypothesis with \( V_1 \) and \( \tau' \) (with (2)).

The case (2) is proved in the same way. Note that, since \( \delta \) is not free in the \( N_i \), the case \( b = (\delta V_1) \) does not appear.

Theorem 10. Every term is in \( S N \).

Proof By induction on the term. It is enough to show that, if \( M, N \in S N \), then \( (M \cdot N) \in S N \). We prove something more general: let \( \sigma \) (resp. \( \tau \)) be in \( \Sigma_r \) (resp. \( \Sigma_l \)) and assume \( M[\sigma], N[\tau] \in S N \). Then \( (M[\sigma] \cdot N[\tau]) \in S N \). Assume it is not the case and choose some elements such that \( M[\sigma], N[\tau] \in S N \), \( (M[\sigma] \cdot N[\tau]) \notin S N \) and \( (\eta(M) + \eta(N), cxtg(M) + cxtg(N)) \) is minimal. By Lemma 3 either \( M[\sigma] \vdash \mu \delta M_1 \) and \( M_1[\delta =_r N[\tau]] \notin S N \) or \( N[\tau] \vdash \mu \beta N_1 \) and \( N_1[\beta =_I M[\sigma]] \notin S N \). Look at the first case (the other one is similar). By Lemma 3, \( M \vdash \mu \delta M_2 \) for some \( M_2 \) such that \( M_2[\sigma] \vdash \mu \delta M_1 \). Thus, \( M_2[\sigma][\delta =_r N[\tau]] \notin S N \). By Lemma 4 with \( M_2[\sigma] \) and \( N[\tau] \), let \( M' \prec M_2 \) and \( \sigma' \) be such that \( M'[\sigma'] \in S N \), \( (M'[\sigma'] \cdot N[\tau]) \notin S N \). This contradicts the minimality of the chosen elements since \( \eta c(M') < \eta c(M) \).

4 The simply typed symmetric \( \lambda \mu \)-calculus is strongly normalizing

In this section, we consider the simply typed calculus with the \( \beta \mu \mu' \)-reduction i.e. \( M \vdash M' \) means \( M' \) is obtained from \( M \) by one step of the \( \beta \mu \mu' \)-reduction. To prove the strong normalization of the \( \beta \mu \mu' \)-reduction, it is enough to show that, if \( M, N \in S N \), then \( M[x := N] \) also is in \( S N \). This is done by induction on the type of \( N \). The proof very much looks like the one for the \( \mu \)-reduction and the induction on the type is used for the cases coming from a \( \beta \)-reduction. The two new difficulties are the following.

- A \( \beta \)-substitution may create a \( \mu \), i.e. the fact that \( M[x := N] \vdash \mu \alpha Q \) does not imply that \( M \vdash \mu \alpha Q \). Moreover the \( \mu \) may come from a complicated interaction between \( M \) and \( N \) and, in particular, the alternation between \( M \) and \( N \) can be lost. Let e.g. \( M = (M_1 (x (\lambda y \lambda y' \lambda \alpha M_4) M_2 M_3)) \) and \( N = \lambda x (z N_1) \). Then \( M[x := N] \vdash \mu \alpha M_4 \cdot M_3) \vdash \mu \alpha M_3[\alpha =_I M_3][\alpha =_I M_3] \). To deal with this case, we need to consider some new kind of \( \mu \mu' \)-substitutions (see definition 13). Lemma 13 gives the different ways in which a \( \mu \) may appear. The difficult case in the proof (when a \( \mu \) is created and the control between \( M \) and \( N \) is lost) will be solved by using a typing argument.

- The crucial lemma (Lemma 9) is essentially the same as the one (Lemma 8) for the \( \mu \mu' \)-reduction but, in its proof, some cases cannot be proved “by themselves” and we need an argument using the types. For this reason, its proof is done using the additional fact that we already know that, if \( M, N \in S N \) and the type of \( N \) is small, then \( M[x := N] \) also is in \( S N \). Since the proof of Lemma 9 is done by induction on the type, when we will use Lemma 8, the additional hypothesis will be available.

Lemma 11. 1. If \( (M \cdot N) \vdash \lambda \gamma M_1 \) and \( M_1[y := N] \vdash \lambda \gamma P \).

2. If \( (M \cdot N) \vdash \mu \alpha P \), then either \( (M \vdash \lambda \gamma M_1 \) and \( M_1[y := N] \vdash \lambda \gamma P \) or \( (M \vdash \mu \alpha M_1 \) and \( M_1[\alpha =_I M] \vdash \mu \alpha P \) or \( (N \vdash \mu \alpha N_1 \) and \( N_1[\alpha =_I M] \vdash \mu \alpha P \).

Proof (1) is trivial. (2) is as in Lemma 8.
Lemma 12. Let $M \in SN$ and $\sigma = [x_1 := N_1, \ldots, x_k := N_k]$. Assume $M[\sigma] \triangleright^* \lambda y P$. Then, either $M \triangleright^* \lambda y P_1$ and $P_1[\sigma] \triangleright^* P$ or $M \triangleright^* (x_1 \overline{Q})$ and $(N_i \overline{Q[\sigma]}) \triangleright^* \lambda y P$.

Proof. By induction on $\eta(M)$. The only non immediate case is $M = \eta(SN)$. By lemma [3], there is a term $R_1$ such that $R[\sigma] \triangleright^* \lambda z R_1$ and $R_1[z := S[\sigma]] \triangleright^* \lambda y P$. By the induction hypothesis (since $\eta(R) < \eta(M)$), we have two cases to consider.

1. $R \triangleright^* \lambda z R_2$ and $R_2[\sigma] \triangleright^* R_1$, then $R_2[z := S[\sigma]] \triangleright^* \lambda y P$. By the induction hypothesis (since $\eta(R_2[z := S]) < \eta(M)$),

   - either $R_2[z := S] \triangleright^* \lambda y P_1$ and $P_1[\sigma] \triangleright^* P$; but then $M \triangleright^* \lambda y P_1$ and we are done.
   - or $R_2[z := S] \triangleright^* (x_i \overline{Q})$ and $(N_i \overline{Q[\sigma]}) \triangleright^* \lambda y P$, then $M \triangleright^* (x_i \overline{Q})$ and again we are done.

2. $R \triangleright^* (x_i \overline{Q})$ and $(N_i \overline{Q[\sigma]}) \triangleright^* \lambda z R_1$. Then $M \triangleright^* (x_i \overline{Q} S)$ and the result is trivial. $\square$

Definition 13. An address is a finite list of symbols in $\{l, r\}$. The empty list is denoted by $[]$ and, if $a$ is an address and $s \in \{l, r\}$, $[s :: a]$ denotes the list obtained by putting $s$ at the beginning of $a$.

- Let $a$ be an address and $M$ be a term. The sub-term of $M$ at the address $a$ (denoted as $M_a$) is defined recursively as follows: if $M = (P Q)$ and $a = [r :: b]$ (resp. $a = [l :: b]$) then $M_a = Q_b$ (resp. $P_b$) and undefined otherwise.

- Let $M$ be a term and $a$ be an address such that $M_a$ is defined. Then $M(a = N)$ is the term $M$ where the sub-term $M_a$ has been replaced by $N$.

- Let $M, N$ be some terms and $a$ be an address such that $M_a$ is defined. Then $N[\alpha = a M]$ is the term $N$ in which each sub-term of the form $(\alpha U)$ is replaced by $(\alpha M(a = U))$.

Remarks and examples

- Let $N = \lambda x (\alpha (\lambda y (x \mu \beta (\alpha y))))$, $M = (M_1 (M_2 M_3))$ and $a = [r :: l]$. Then $N[\alpha = a M] = \lambda x (\alpha (M_1 (\lambda y (\mu \beta (\alpha (\lambda y M_3))))))$.

- Let $M = (P ((R (x T)) Q))$ and $a = [r :: l :: r :: l]$. Then $N[\alpha = a M] = N[\alpha = T][\alpha = R][\alpha = Q][\alpha = P]$.

- Note that the sub-terms of a term having an address in the sense given above are those for which the path to the root consists only on applications (taking either the left or right son).

- Note that $[\alpha = a M]$ is not the same as $[\alpha = l M]$ but $[\alpha = l M]$ is the same as $[\alpha = r (M N)]$ where $N$ does not matter. More generally, the term $N[\alpha = a M]$ does not depend of $M_a$.

- Note that $M(a = N)$ can be written as $M'[x_a := N]$ where $M'$ is the term $M$ in which $M_a$ has been replaced by the fresh variable $x_a$ and thus (this will be used in the proof of lemma [3]) if $M_a$ is a variable $x, (\alpha U)[\alpha = a M] = (\alpha M_1[y := U[\alpha = M]])$ where $M_1$ is the term $M$ in which the particular occurrence of $x$ at the address $a$ has been replaced by the fresh name $y$ and the other occurrences of $x$ remain unchanged.

Lemma 14. Assume $M, N \in SN$ and $(M N) \not\in SN$. Then, either $(M \triangleright^* \lambda y P$ and $P[y := N] \not\in SN)$ or $(M \triangleright^* \mu \alpha P$ and $P[\alpha = r N] \not\in SN)$ or $(N \triangleright^* \mu \alpha P$ and $P[\alpha = l M] \not\in SN)$.

Proof. By induction on $\eta(M) + \eta(N)$. $\square$

In the rest of this section, we consider the typed calculus. To simplify the notations, we do not write explicitly the type information but, when needed, we denote by $\text{type}(M)$ the type of the term $M$.

Lemma 15. If $\Gamma \vdash M : A$ and $M \triangleright^* N$ then $\Gamma \vdash N : A$.

Proof. Straight forward.
Lemma 16. Let $n$ be an integer, $M \in SN$, $\sigma = [x_1 := N_1, ..., x_k := N_k]$ where $\lg(type(N_i)) = n$ for each $i$. Assume $M[\sigma] \triangleright^* \mu\alpha P$. Then,

1. either $M \triangleright^* \mu\alpha P_1$ and $P_1[\sigma] \triangleright^* P$
2. or $M \triangleright^* Q$ and, for some $i$, $N_i \triangleright^* \mu\alpha N'_i$ and $N'_i[\alpha = a Q[\sigma]] \triangleright^* P$ for some address $a$ in $Q$ such that $Q_a = x_i$.
3. or $M \triangleright^* Q, Q_a[\sigma] \triangleright^* \mu\alpha N'$ and $N'[\alpha = a Q[\sigma]] \triangleright^* P$ for some address $a$ in $Q$ such that $\lg(type(Q_a)) < n$.

Proof. By induction on $\eta \gamma(M)$. The only non immediate case is $M = (R S)$. Since $M[\sigma] \triangleright^* \mu\alpha P$, the application $(R[\sigma] \ S[\sigma])$ must be reduced. Thus there are three cases to consider.

- It is reduced by a $\mu\eta$-reduction, i.e. there is a term $S_1$ such that $S[\sigma] \triangleright^* \mu\alpha S_1$ and $S_1[\alpha = l R[\sigma]] \triangleright^* P$. By the induction hypothesis:
  - either $S \triangleright^* \mu\alpha Q$ and $Q[\sigma] \triangleright^* S_1$, then $M \triangleright^* \mu\alpha Q[\alpha = l R]$ and $Q[\alpha = l R][\sigma] \triangleright^* P$.
  - or $S \triangleright^* Q$ and, for some $i$, $N_i \triangleright^* \mu\alpha N'_i$, $Q_a = x_i$ for some address $a$ in $Q$ and $N'_i[\alpha = a Q[\sigma]] \triangleright^* S_1$. Then $M \triangleright^* (R Q) = Q'$ and letting $b = [r :: a]$ we have $N'_i[\alpha = a Q'[\sigma]] \triangleright^* P$.
  - or $S \triangleright^* Q, Q_a[\sigma] \triangleright^* \mu\alpha N'$ for some address $a$ in $Q$ such that $\lg(type(Q_a)) < n$ and $N'[\alpha = a Q[\sigma]] \triangleright^* S_1$. Then $M \triangleright^* (R Q) = Q'$ and letting $b = [r :: a]$ we have $N'[\alpha = a Q'[\sigma]] \triangleright^* P$ and $\lg(type(Q'_a)) < n$.

- It is reduced by a $\mu\eta$-reduction. This case is similar to the previous one.

- It is reduced by a $\beta$-reduction, i.e. there is a term $U$ such that $R[\sigma] \triangleright^* \lambda y U$ and $U[y := S[\sigma]] \triangleright^* P$. By lemma 13 there are two cases to consider.
  - either $R \triangleright^* \lambda y R_1$ and $R_1[\sigma][y := S[\sigma]] = R_1[y := S[\sigma]] \triangleright^* P$. The result follows from the induction hypothesis since $\eta(R_1[y := S]) < \eta(M)$.
  - or $R \triangleright^* (x_i R'_i)$. Then $Q = (x_i R'_i S)$ and $a = []$ satisfy the desired conclusion since then $\lg(type(M)) < n$. $$

Definition 17. Let $A$ be a type. We denote by $\Sigma_A$ the set of substitutions of the form $[\alpha_1 =_{a_1}, M_1, ..., \alpha_n =_{a_n}, M_n]$ where the type of the $\alpha_i$ is $\neg A$.

Remark
Since in such substitutions the type of the variables changes, when we consider the term $N[\alpha]$, where $\sigma \in \Sigma_A$, we mean that the type of the $\alpha_i$ is $A$ in $\sigma$ i.e. before the substitution. Also note that considering $N[\alpha = a M]$ implies that the type of $M_a$ is $A$.

Lemma 18. Let $n$ be an integer and $A$ be a type such that $\lg(A) = n$. Let $N, P$ be terms and $\tau \in \Sigma_A$. Assume that,

- for every $M, N \in SN$ such that $\lg(type(N)) < n$, $M[x := N] \in SN$.
- $N[\tau] \in SN$ but $N[\tau][\delta = a P] \notin SN$.
- $\delta$ is free and has type $\neg A$ in $N$ but $\delta$ is not free in $Im(\tau)$.

Then, there is $N' \prec N$ and $\tau' \in \Sigma_A$ such that $N'[\tau'] \in SN$ and $P[a = N'[\tau']] \notin SN$.

Proof. Essentially as in lemma 3. Denote by (H) the first assumption i.e. for every $M, N \in SN$ such that $\lg(type(N)) < n$, $M[x := N] \in SN$.

Let $\tau = [\alpha_1 =_{a_1}, M_1, ..., \alpha_n =_{a_n}, M_n]$, $U = \{U / U \in N\}$ and $V = \{V / V \in U \}$ for some $i$. Define inductively the sets $\Sigma_m$ and $\Sigma_n$ of substitutions by the following rules:

- $\rho \in \Sigma_n$ if $\rho = \emptyset$ or $\rho = \rho' + [\alpha = a V[\sigma]]$ for some $V \in V$, $\sigma \in \Sigma_m$, $\rho' \in \Sigma_n$ and $\alpha$ has type $\neg A$. 


\[\sigma \in \Sigma_n \text{ iff } \sigma = \emptyset \text{ or } \sigma = \sigma' + \{x := U[\rho]\} \text{ for some } U \in \mathcal{U}, \rho \in \Sigma_n, \sigma' \in \Sigma_m \text{ and } x \text{ has type } A.\]

Denote by C the conclusion of the lemma. We prove something more general.

(1) Let \(U \in \mathcal{U}\) and \(\rho \in \Sigma_n\). Assume \(U[\rho] \in SN\) and \(U[\rho][\delta \mapsto a] P \notin SN\). Then, C holds.

(2) Let \(V \in \mathcal{V}\) and \(\sigma \in \Sigma_m\). Assume \(V[\sigma] \in SN\) and \(V[\sigma][\delta \mapsto a] P \notin SN\). Then, C holds.

The conclusion C follows from (1) with \(N\) and \(\tau\). The properties (1) and (2) are proved by a simultaneous induction on \(\eta c(U[\rho])\) (for the first case) and \(\eta c(V[\tau])\) (for the second case).

The proof is as in lemma 3. The new case to consider is, for \(V[\sigma]\), when \(V = (V_1, V_2)\) and \(V_1[\sigma][\delta \mapsto a] P \in SN\).

- Assume first the interaction between \(V_1\) and \(V_2\) is a \(\beta\)-reduction. If \(V_1 \triangleright^* \lambda x V'_1\), the result follows from the induction hypothesis with \(V'_1[x := V_2[\sigma]]\). Otherwise, by lemma 12, \(V_1 \triangleright^* (x[\overline{W}]\). Let \(\sigma(x) = U[\rho]\). Then \((U[\rho][\overline{W}[\sigma]]) \triangleright^* \lambda y Q\) and \(Q[y := V_2[\sigma]]\). But, since the type of \(x\) is \(A\), the type of \(y\) is less than \(A\) and since \(Q[\delta \mapsto a] P\) and \(V_2[\sigma][\delta \mapsto \alpha\sigma]\) are in \(SN\) this contradicts (H).

- Assume next the interaction between \(V_1\) and \(V_2\) is a \(\mu\) or \(\mu'\)-reduction. We consider only the case \(\mu\) (the other one is similar). If \(V_1 \triangleright^* \mu a V'_1\), the result follows from the induction hypothesis with \(V'_1[\alpha := V_2]\). Otherwise, by lemma 14, there are two cases to consider.

- \(V_1 \triangleright^* Q, Q_c = x\) for some address \(c\) in \(Q\) and \(x \in dom(\sigma)\), \(\sigma(x) = U[\rho]\), \(U \triangleright^* \mu a U\) and \(U[\rho][\alpha := \lambda Q]\). Let \(\delta = (\alpha, V_2[\sigma][\delta \mapsto a]\). \(P \notin SN\). Let \(V' = (Q_1, V_2)\) and \(b = l :: c\). The result follows then from the induction hypothesis with \(U_1[\rho]\) where \(\rho' = \rho + [\alpha := b V'[\sigma]]\).

- \(V_1 \triangleright^* Q, Q_c = x\) for some address \(c\) in \(Q\) such that \(lg(type(Q_c)) < n\), \(R[\alpha := c Q[\sigma][\delta \mapsto a] P][\alpha := r, V_2[\sigma][\delta \mapsto a] P]\notin SN\). Let \(V' = (Q_1, V_2)\) where \(Q'\) is the same as \(Q\) but \(Q_c\) has been replaced by a fresh variable \(y\) and \(b = l :: c\). Then \(R[\alpha := c V'[\sigma][\delta \mapsto a] P]\notin SN\). Let \(R'\) be such that \(R' \prec R, R'[\alpha := b V'[\sigma][\delta \mapsto a] P]\notin SN\) and \(\eta c(R')\) is minimal. It is easy to check that \(R' = (\alpha R'), R'[\alpha := c V'[\sigma][\delta \mapsto a] P]\notin SN\) where \(\sigma' = \sigma + y := R'[\alpha := c V'[\sigma]]\). If \(V'[\sigma][\delta \mapsto a] P\notin SN\), we get the result by the induction hypothesis since \(\eta c(V'[\sigma]) < \eta c(V[\sigma])\). Otherwise this contradicts the assumption (H) since \(V'[\sigma][\delta \mapsto a] P\), \(R'[\alpha := c V'[\sigma][\delta \mapsto a] P]\notin SN\), \(V'[\sigma][\delta \mapsto a] P[y := R'[\alpha := c V'[\sigma]]\notin SN\) and the type of \(y\) is less than \(n\).

\[\square\]

**Lemma 19.** If \(M, N \in SN\), then \(M[x := N] \notin SN\).

**Proof.** We prove something a bit more general: let \(A\) be a type, \(M, N_1, ..., N_k\) be terms and \(\tau_1, ..., \tau_n\) be substitutions in \(\Sigma_A\). Assume that, for each \(i\), \(N_i\) has type \(A\) and \(N_i[\tau_i] \in SN\). Then \(M[x_1 := N_1[\tau_1]], ..., x_k := N_k[\tau_k] \in SN\). This is proved by induction on \((lg(A), \eta(M), cxy(M), \Sigma \eta(N_i), \Sigma cxy(N_i))\) where, in \(\Sigma \eta(N_i)\) and \(\Sigma cxy(N_i)\), we count each occurrence of the substituted variable. For example if \(k = 1\) and \(x_1\) has \(n\) occurrences, \(\Sigma \eta(N_i) = n \cdot \eta(N_i)\).

If \(M\) is \(\lambda M_1\) or \(\mu M_1\) or \(\mu a M_1\) or a variable, the result is trivial. Assume then that \(M = (M_1 M_2)\). Let \(\sigma = \{x_i := N_i[\tau_i], ..., x_k := N_k[\tau_k]\}\). By the induction hypothesis, \(M_1[\sigma], M_2[\sigma] \in SN\). By lemma 14, there are 3 cases to consider.

- \(M_1[\sigma] \triangleright^* \lambda y P\) and \(P[y := M_2[\sigma]] \notin SN\). By lemma 12 there are two cases to consider.

  - \(M_1 \triangleright^* \lambda Q\) and \(Q[\sigma] \triangleright^* P\). Then \(Q[y := M_2][\sigma] = Q[\sigma][y := M_2[\sigma]] \triangleright^* P[y := M_2][\sigma]\) and, since \(\eta(Q[y := M_2]) < \eta(M)\), this contradicts the induction hypothesis.

...
• $M_1 \vdash x_i \overline{Q}$ and $(N_i \overline{Q[\sigma]}) \vdash \lambda y P$. Then, since the type of $N_i$ is $A$, $lg(type(y)) < lg(A)$. But $P, M_2[\sigma] \in SN$ and $P[y := M_2[\sigma]] \not\in SN$. This contradicts the induction hypothesis.

\[ M_1[\sigma] \vdash \mu \alpha P \text{ and } P[\alpha =_r M_2[\sigma]] \not\in SN. \] By lemma [12], there are three cases to consider.

• $M_1 \vdash Q \mu \alpha Q$ and $Q[\sigma] \vdash P$. Then, $Q[\alpha =_r M_2[\sigma]] = Q[\alpha =_r M_2[\sigma]] \vdash P[\alpha =_r M_2[\sigma]]$ and, since $\eta(Q[\alpha =_r M_2[\sigma]]) < \eta(M)$, this contradicts the induction hypothesis.

• $M_1 \vdash Q, N_1[\tau_1] \vdash \mu \alpha L'$ and $Q_\alpha = x_i$ for some address $a$ in $Q$ such that $L'[\alpha =_a Q[\sigma]] \vdash P$ and thus $L'[\alpha =_b M'[\sigma]] \not\in SN$ where $b = (l : a)$ and $M' = (Q M_2)$. By lemma [13], there is an $L_1 < L$ and $\tau'$ such that $L_1[\tau'] \in SN$ and $M'[\sigma][b = L_1[\tau']] \not\in SN$. Let $M''$ be $M'$ where the variable $x_i$ at the address $b$ has been replaced by the fresh variable $y$ and let $\sigma_1 = \sigma + y := L_1[\tau']$. Then $M''[\sigma_1] = M'[\sigma][b = L_1[\tau']] \not\in SN$.

If $M_1 \vdash Q$ we get a contradiction from the induction hypothesis since $\eta(M'') < \eta(M)$. Otherwise, $M''$ is the same as $M$ up to the change of name of a variable and $\sigma_1$ differs from $\sigma$ only at the address $b$. At this address, $x_i$ was substituted in $\sigma$ by $N_1[\tau_1]$ and in $\sigma_1$ by $L_1[\tau']$ but $\eta(L_1[\tau']) < \eta(N_1)$ and thus we get a contradiction from the induction hypothesis.

• $M_1 \vdash Q, Q_\alpha[\sigma] \vdash \mu \alpha L$ for some address $a$ in $Q$ such that $lg(type(Q_\alpha)) < lg(A)$ and $L[\alpha =_a Q[\sigma]] \vdash P$. Then, $L[\alpha =_b M'[\sigma]] \not\in SN$ where $b = (l : a)$ and $M' = (Q M_2)$. By lemma [13], there is an $L'$ and $\tau'$ such that $L'[\tau'] \in SN$ and $M'[\sigma][b = L'[\tau']] \not\in SN$. Let $M'''$ be $M''$ where the variable $x_i$ at the address $b$ has been replaced by the fresh variable $y$. Then $M'''[\sigma][y := L'[\tau']] = M'[\sigma][b = L'[\tau']] \not\in SN$.

But $\eta(M''') < \eta(M)$ and $\text{ctxy}(M''') < \text{ctxy}(M)$ since, because of its type, $Q_\alpha$ cannot be a variable and thus, by the induction hypothesis, $M''''[\sigma] \in SN$. Since $M'''[\sigma][y := L'[\tau']] \not\in SN$ and $lg(type(L')) < lg(A)$, this contradicts the induction hypothesis.

$M_2[\sigma] \vdash \mu \alpha P$ and $P[\alpha =_l M_1[\sigma]] \not\in SN$. This case is similar to the previous one. \hfill \square

\textbf{Theorem 20.} Every typed term is in $SN$.

\textbf{Proof} \quad By induction on the term. It is enough to show that if $M, N \in SN$, then $(M N) \in SN$. Since $(M N) = (x y)[x := M][y := N]$ where $x, y$ are fresh variables, the result follows by applying theorem [19] twice and the induction hypothesis. \hfill \square

5 \quad Why the usual candidates do not work?

In [21], the proof of the strong normalization of the $\lambda\mu$-calculus is done by using the usual (i.e. defined without a fix-point operation) candidates of reducibility. This proof could be easily extended to the symmetric $\lambda\mu$-calculus if we knew the following properties for the un-typed calculus:

1. If $N$ and $(M[x := N] P)$ are in $SN$, then so is $(\lambda x M N P)$.
2. If $N$ and $(M[\alpha =_r N] P)$ are in $SN$, then so is $(\mu\alpha M N P)$.
3. If $P$ are in $SN$, then so is $(x P)$. 


These properties are easy to show for the $\beta\mu$-reduction but they were not known for the $\beta\mu'$-reduction.

The properties (1) and (2) are false. Here is a counter-example. Let $M_0 = \lambda x(x P \ \|) \ \|$ and $M_1 = \lambda x(x P \ \|)$ where $\| = \lambda x\lambda yy$, $\| = \lambda x\lambda x\lambda x$ and $P = \lambda x\lambda y\lambda z \ (y \ (z \ \|) \ (z \ \|) \ \|) \ \| \ \| \lambda A \ A \ A \ \|$. Let $M = \lambda f(f \ (x \ M_1) \ (x \ M_0))$, $M' = \lambda f(f (\beta x(x(M_1))) (\beta x(x(M_0))))$ and $N = (\alpha \ \lambda x(\alpha z))$. Then,

1. $M[x := \mu_\alpha N] \notin SN$ but $(\lambda xM \ \mu_\alpha N) \notin SN$.
2. $M'[\beta =_r \mu_\alpha N] \notin SN$ but $(\mu_\beta M' \ \mu_\alpha N) \notin SN$.

This comes from the fact that $(M_0 \ M_0)$ and $(M_1 \ M_1)$ are in $SN$ but $(M_1 \ M_0)$ and $(M_0 \ M_1)$ are not in $SN$. More details can be found in [11].

The third property is true and its proof is essentially the same as the one of the strong normalization of $\mu\mu'$. This comes from the fact that, since $(x \ M_1 ... M_n)$ never reduces to a $\lambda$, there is no "dangerous" $\beta$-reduction. In particular, the $\beta$-reductions we have to consider in the proofs of the crucial lemmas, are uniquely those that appear in the reductions $M \leq M'$. We give this proof below.

**Lemma 21.** The term $(x \ M_1 ... M_n)$ never reduces to a term of the form $\lambda y M$.

**Proof** By induction on $n$. Use lemma [1].

**Definition 22.** Let $M_1, ..., M_n$ be terms and $1 \leq i \leq n$. Then, the term $M$ in which every sub-term of the form $(\alpha U)$ is replaced by $(\alpha \ (x \ M_1 ... M_{i-1} \ U \ M_{i+1} ... M_n))$ will be denoted by $M[\alpha =_i (M_1 ... M_n)]$.

We will denote by $\Sigma_k$ the set of simultaneous substitutions of the form $[\alpha_1 =_{i_1} (M_1^1 \ ... \ M_n^1), \ldots, \alpha_k =_{i_k} (M_1^k \ ... \ M_n^k)]$.

**Remark** These substitutions are special cases of the one defined in section [1] (see definition [4]). For example $M[\alpha =_2 (M_1 \ M_2 \ M_3)] = M[\alpha =_1 (x \ M_1)] [\alpha =_r \ M_3] = M[\alpha =_a (x \ M_1 \ M_2 \ M_3)]$ where $a = [\| \vdash \|]$.

**Lemma 23.** Assume $(x \ M_1 ... M_n) \beta^* \mu_\alpha M$. Then, there is an $i$ such that $M_i \beta^* \mu_\alpha P$ and $P[\alpha =_i (M_1 ... M_n)] \beta^* M$.

**Proof** By induction on $n$.

- $n = 1$. By lemma [1], $M_1 \beta^* \mu_\alpha P$ and $P[\alpha =_1 x] = P[\alpha =_1 (M_1)] \beta^* M$.
- $n \geq 2$. Assume $(x \ M_1 ... M_{n-1} \ M_n) \beta^* \mu_\alpha M$. By lemmas [1] and [2] (either $(x \ M_1 ... M_{n-1} \ M_n) \beta^* \mu_\alpha N$ and $N[\alpha =_r \ M_n] \beta^* M$. By the induction hypothesis, there is an $i$ such that $M_i \beta^* \mu_\alpha P$ and $P[\alpha =_i (M_1 ... M_{n-1} \ M_n)] \beta^* N[\alpha =_r \ M_n] \beta^* M$. Or $M_i \beta^* \mu_\alpha N$ and $N[\alpha =_i (x \ M_1 ... M_{n-1})] \beta^* M$. Then $N[\alpha =_i (x \ M_1 ... M_{n-1})] = N[\alpha =_n (M_1 ... M_{n-1} \ M_n)] \beta^* M$.)

**Lemma 24.** Assume $M_1, ..., M_n \in SN$ and $(x \ M_1 ... M_n) \notin SN$. Then, there is an $i$ such that $M_i \beta^* \mu_\alpha U$ and $U[\alpha =_i (M_1 ... M_n)] \notin SN$.

**Proof** By induction on $n$.

- $i = 1$. By lemma [1], $M_1 \beta^* \mu_\alpha U$ and $U[\alpha =_i (x \ M_1 ... M_{k-1})] \notin SN$. Then, $i = k$ satisfies the desired property since $U[\alpha =_k (M_1 ... M_n)] = U[\alpha =_i (x \ M_1 ... M_{k-1})] [\alpha =_r M_k ... M_{k+1}] [\alpha =_r M_n]$.
- $(x \ M_1 ... M_{k-1}) \beta^* \mu_\alpha P$ and $P[\alpha =_r M_k] \notin SN$. By lemma [2], let $i \leq k - 1$ be such that $(x \ M_1 ... M_{k-1}) \beta^* \mu_\alpha U$ and $U[\alpha =_i (M_1 ... M_{k-1})] \beta^* P$. Then $U[\alpha =_i (M_1 ... M_{k-1})] [\alpha =_r M_k] [\alpha =_r M_{k+1}] [\alpha =_r M_n]$ reduces to $P[\alpha =_r M_k] [\alpha =_r M_{k+1}] [\alpha =_r M_n]$. \qed
Lemma 25. Let $M$ be a term and $\sigma \in \Sigma_x$. If $M[\sigma] \vdash^* \mu \alpha P$ (resp. $M[\sigma] \vdash^* \lambda x P$), then $M \vdash^* \mu Q$ (resp. $M \vdash^* \lambda x Q$) for some $Q$ such that $Q[\sigma] \vdash^* P$.

Proof As in lemma 23. □

Lemma 26. Let $M$ be a term and $\sigma \in \Sigma_x$. Assume $\delta$ is free in $M$ but not free in $\text{Im}(\sigma)$. If $M[\sigma] \in SN$ but $M[\sigma][\delta = (P_1 \ldots P_n)] \not\in SN$, there is $M' \prec M$ and $\sigma'$ such that $M'[\sigma'] \in SN$ and $(x P_1 \ldots P_{i-1} M'[\sigma'] P_{i+1} \ldots P_n) \not\in SN$.

Proof As in lemma 23. □

Theorem 27. Assume $M_1, \ldots, M_n$ are in $SN$. Then $(x \ M_1 \ldots M_n) \in SN$.

Proof We prove a more general result: Let $M_1, \ldots, M_n$ be terms and $\sigma_1, \ldots, \sigma_n$ be in $\Sigma_x$. If $M_1[\sigma_1], \ldots, M_n[\sigma_n] \in SN$, then $(x M_1[\sigma_1] \ldots M_n[\sigma_n]) \in SN$. The proof is done exactly as in theorem 14 using lemmas 24, 25 and 26. □

6 Future work

- Parigot has introduced other simplification rules in the $\lambda\mu$-calculus. They are as follows: $(\alpha \ M) \rightarrow_\rho M[\beta := \alpha]$ and, if $\alpha$ is not free in $M$, $\mu(\alpha \ M) \rightarrow_\theta M$. It would be interesting to extend our proofs to these reductions. The rule $\theta$ causes no problem since it is strongly normalizing and it is easy to see that this rule can be postponed (i.e. if $M \rightarrow^* \mu \beta \ M_1$ then $M \rightarrow^* \mu \beta \ M_2 \rightarrow^* M_1$ for some $M_2$). However it is not the same for the rule $\rho$ which cannot be postponed. Moreover a basic property (if $M[\alpha = N] \vdash^* \mu \beta P$, then $M \vdash^* \mu \beta Q$ for some $Q$ such that $Q[\alpha = N] \vdash^* P$) used in the proofs is no more true if the $\rho$-rule is used. It seems that, in this case, the $\mu$ can only come either from $M$ or from $N$ i.e. without deep interaction between $M$ and $N$ and thus that our proofs can be extended to this case but, due to the lack of time, we have not been able to check the details.

- We believe that our technique, will allow to give explicit bounds for the length of the reductions of a typed term. This is a goal we will try to manage.

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