Unitarity and Complete Reducibility of Certain Modules over Quantized Affine Lie Algebras

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Abstract:

Let $U_q(\hat{G})$ denote the quantized affine Lie algebra and $U_q(G^{(1)})$ the quantized nontwisted affine Lie algebra. Let $\mathcal{O}_{\text{fin}}$ be the category defined in section 3. We show that when the deformation parameter $q$ is not a root of unit all integrable representations of $U_q(\hat{G})$ in the category $\mathcal{O}_{\text{fin}}$ are completely reducible and that every integrable irreducible highest weight module over $U_q(G^{(1)})$ corresponding to $q > 0$ is equivalent to a unitary module.
1 Introduction

Quantum (super)groups, or more precisely quantum universal enveloping (super)algebras or for short quantized (super)algebras, are defined as q deformations of classical universal enveloping algebras of finite-dimensional simple Lie (super)algebras \([1][2][3]\). The definition for the quantized finite-dimensional simple Lie algebras can be extended to infinite-dimensional affine Lie algebras, or even to arbitrary Kac-Moody algebras, with symmetrizable, generalized Cartan matrices in the sense of Kac\([4]\). In this paper we shall be concerned with the case of quantized affine Lie algebras.

Quantized affine Lie algebras and their representations are important in, among others, the so-called Yang-Baxterization method for obtaining spectral parameter dependent solutions to the quantum Yang-Baxter equation \([5][6]\) and the q-deformed WZNW CFT’s\([7][8]\). It is known\([9][10]\) that most algebras and representations of interest in physics and mathematics have corresponding q deformations. In particular, for quantized simple Lie algebras, all finite-dimensional representations are known to be completely reducible\([3]\). However, for quantized affine Lie algebras, only some isolated results are available. In particular, the complete reducibility and unitarity of representations have not been clarified. In fact the latter remains unproved even for the quantized finite-dimensional simple Lie algebra case.

In this paper we will address the problems of complete reducibility and unitarity of certain representations for quantized affine Lie algebras. Our main results are the proofs of complete reducibility and unitarity of some important modules over the quantized affine Lie algebras.

The paper is set up in the following way. After recalling, in section 2, some basic facts on quantized affine Lie algebras, in section 3 we investigate representations of the quantized affine Lie algebras. The main result of this section is theorem 3.1 which says that all integrable representations of the quantized affine Lie algebras with weight spectrum bounded from above are completely reducible. In section 4 we prove the unitarity of every integrable irreducible highest weight module over quantized nontwisted affine Lie algebras; our main results are stated in theorem 4.2, 4.4 and corollary 4.3. We conclude, in section 5, with a brief discussion of our main results.

2 Preliminaries

We start with the definition of the quantum affine Lie algebra \(U_q(\hat{G})\). Let \(A^0 = (a_{ij})_{1 \leq i,j \leq r}\) be a symmetrizable Cartan matrix. Let \(G\) stand for the finite-dimensional simple Lie algebra with symmetric Cartan matrix \(A^0_{\text{sym}} = (a_{ij}^{\text{sym}}) = (\alpha_i, \alpha_j),\ i,j = 1,2,...,r\), where \(r\) is the rank of \(G\). Let \(A = (a_{ij})_{0 \leq i,j \leq r}\) be a symmetrizable, generalized Cartan matrix in the sense of Kac. Let \(\hat{G}\) denote the affine Lie algebra associated with the corresponding symmetric Cartan matrix \(A_{\text{sym}} = (a_{ij}^{\text{sym}}) = (\alpha_i, \alpha_j),\ i,j = 0,1,...,r\). The quantum algebra \(U_q(\hat{G})\) is defined to be a Hopf
algebra with generators: \( \{ e_i, f_i, q^{h_i} \ (i = 0, 1, ..., r), \ q^d \} \) and relations,

\[
q^h.q^{h'} = q^{h+h'} \quad (h, h' = h_i \ (i = 0, 1, ..., r), \ d)
\]

\[
q^h e_i q^{-h} = q^{(h,\alpha_i)} e_i, \quad q^h f_i q^{-h} = q^{-(h,\alpha_i)} f_i
\]

\[
\{ e_i, f_j \} = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}
\]

\[
1 - a_{ij} \sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = 0 \quad (i \neq j)
\]

\[
1 - a_{ij} \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0 \quad (i \neq j)
\]

where

\[
e_i^{(k)} = e_i^{k} \frac{1}{[k]q}, \quad f_i^{(k)} = f_i^{k} \frac{1}{[k]q}, \quad [k] = \frac{q^k - q^{-k}}{q - q^{-1}}
\]

The algebra \( U_q(\hat{G}) \) is a Hopf algebra with coproduct, counit and antipode similar to the case of \( U_q(G) \):

\[
\Delta(q^h) = q^h \otimes q^h, \quad h = h_i, \ d
\]

\[
\Delta(e_i) = q^{-h_i/2} \otimes e_i + e_i \otimes q^{h_i/2}
\]

\[
\Delta(f_i) = q^{-h_i/2} \otimes f_i + f_i \otimes q^{h_i/2}
\]

\[
S(a) = -q^{h_p}aq^{-h_p}, \quad a = e_i, f_i, h_i, d
\]

where \( \rho \) is the half-sum of the positive roots. We have omitted the formula for counit since we do not need them.

For quasitriangular Hopf algebras, there exists a distinguished element

\[
u = \sum_i S(b_i) a_i
\]

where \( a_i \) and \( b_i \) are coordinates of the universal \( R \)-matrix \( R = \sum_i a_i \otimes b_i \). One can show that \( u \) has inverse

\[
u^{-1} = \sum_i S^{-2}(b_i) a_i
\]

and satisfies

\[
S^2(a) = uau^{-1}, \quad \forall a \in U_q(\hat{G})
\]

\[
\Delta(u) = (u \otimes u)(R^T R)^{-1}
\]

where \( R^T = T(R) \), \( T \) is the twist map: \( T(a \otimes b) = b \otimes a \), \( \forall a \in U_q(\hat{G}) \).

**Proposition 2.1**: \( \Omega = u q^{2h_p} \) belongs to the center of \( U_q(\hat{G}) \), i.e. it is a Casimir operator.

We may equivalently work with the coproduct \( \bar{\Delta} \) and antipode \( \bar{S} \) defined by

\[
\bar{\Delta}(q^h) = q^h \otimes q^h, \quad h = h_i, \ d
\]
\[
\Delta(e_i) = q^{h_i/2} \otimes e_i + e_i \otimes q^{-h_i/2} \\
\Delta(f_i) = q^{h_i/2} \otimes f_i + f_i \otimes q^{-h_i/2} \\
S(a) = -q^{-h \rho} a q^{h \rho}, \quad a = e_i, f_i, h_i, d
\] (7)

Corresponding to the coproduct and antipode (7) we have another form of the \( R \)-matrix, denoted as \( \bar{R} \). If we write \( \bar{R} = \sum_i \bar{a}_i \otimes \bar{b}_i \), then we have

\[
\bar{u} = \sum_i \bar{S}(\bar{b}_i) \bar{a}_i, \quad \bar{u}^{-1} = \sum_i \bar{S}^{-2}(\bar{b}_i) \bar{a}_i
\] (8)

which satisfy

\[
\bar{S}^2(a) = \bar{u} a \bar{u}^{-1}, \quad \forall a \in U_q(\hat{G}) \\
\bar{\Delta}(\bar{u}) = (\bar{u} \otimes \bar{u}) (\bar{R}^T \bar{R})^{-1}
\] (9)

**Proposition 2.2:** \( \bar{\Omega} = \bar{u}^{-1} q^{-2h \rho} \) is the Casimir operator of \( U_q(\hat{G}) \) with coproduct and antipode given by (7).

**Proposition 2.3:** The Casimir operators \( \Omega \) and \( \bar{\Omega} \) have the properties: \( \Omega = S(\Omega), \quad \bar{\Omega} = \bar{S}(\bar{\Omega}) \).

We define a conjugate operation \( \dagger \) and an anti-involution \( \theta \) on \( U_q(\hat{G}) \) by

\[
d^\dagger = d, \quad h_i^\dagger = h_i, \quad e_i^\dagger = f_i, \quad f_i^\dagger = e_i, \quad i = 0, 1, \ldots r \]
\[
\theta(q^h) = q^{-h}, \quad \theta(e_i) = f_i, \quad \theta(f_i) = e_i, \quad \theta(q) = q^{-1}
\] (10)

which extend uniquely to an algebra anti-automorphism and anti-involution on all of \( U_q(\hat{G}) \), respectively, so that \( (ab)^\dagger = b^\dagger a^\dagger \), \( \theta(ab) = \theta(b) \theta(a) \), \( \forall a, b \in U_q(\hat{G}) \).

Throughout the paper, we assume that \( q \) is not a root of unit and use the notations:

\[
(n)_q = \frac{1 - q^n}{1 - q}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \quad q_\alpha = q^{(\alpha, \alpha)}
\]
\[
\exp_q(x) = \sum_{n \geq 0} \frac{x^n}{(n)_q!}, \quad (n)_q! = (n)_q(n-1)_q \ldots (1)_q
\]
\[
(ad_q x_\alpha) x_\beta = [x_\alpha, x_\beta]_q = x_\alpha x_\beta - q^{(\alpha, \beta)} x_\beta x_\alpha
\] (11)

### 3 Complete Reducibility

Representations of \( U_q(\hat{G}) \) are known to be isomorphic (as a linear space) to corresponding reps of \( U(\hat{G}) \) \([10]\). Let

\[
U_q(\hat{G}) = U_q(\mathcal{N}_-) \otimes U_q(\mathcal{H}) \otimes U_q(\mathcal{N}_+)
\] (12)

be the triangular decomposition of \( U_q(\hat{G}) \), generated by the \( f_i \)'s, \( q^h \) \( (h \in \mathcal{H}) \) and \( e_i \)'s, respectively. Let \( V \) denote an \( U_q(\hat{G}) \)-module. We say that \( V \) is \( \mathcal{H} \)-diagonalizable if

\[
V = \bigoplus_{\lambda \in \mathcal{H}^*} V_\lambda
\] (13)
where, $V_{\lambda} = \{ v \in V | q^h v = q^{\lambda(h)} v, h \in H \}$ be the weight spaces corresponding to the weight $\lambda$. Following Kac\cite{kac}, we introduce the notations

$$\Pi(V) = \{ \lambda \in H^* | V_{\lambda} \neq 0 \}, \quad D(\lambda) = \{ \mu \in H^* | \mu \leq \lambda, \lambda \in H^* \}$$

(14) and consider the category $\mathcal{O}_{\text{fin}}$ defined by

**Definition 3.1:** $\mathcal{O}_{\text{fin}}$ is the category of $U_q(\hat{G})$ modules $V$ which are $H$-diagonalizable with finite dimensional weight spaces and in which there exist a finite number of elements $\lambda_1, \ldots, \lambda_s \in H^*$ such that

$$\Pi(V) \subset \bigcup_{i=1}^s D(\lambda_i)$$

(15) implies that weights in $\mathcal{O}_{\text{fin}}$ are bounded from above.

**Example:** Highest weight modules.

We say that an $U_q(\hat{G})$-module $V$ is a highest weight module with the highest weight $\Lambda$ if there exists a non-zero vector $v \in V$ such that

$$e_i(v) = 0 \quad (i = 0, 1, \ldots, r), \quad q^h v = q^{\Lambda(h)} v \quad (h \in H)$$

$$V = U_q(\hat{G})v$$

(16) The vector $v$ is called as highest weight vector. We deduce from (12) and (16),

$$V = \bigoplus_{\lambda \leq \Lambda} V_{\lambda}, \quad V_{\Lambda} = Cv, \quad \dim V_{\lambda} < \infty$$

(17) which implies that a highest weight module lies in $\mathcal{O}_{\text{fin}}$.

**Proposition 3.1 (Lusztig \cite{lusztig}:** For any $\Lambda \in H^*$, there exists a unique, up to isomorphism, irreducible highest weight $U_q(\hat{G})$-module $L(\Lambda)$ with highest weight $\Lambda$.

Following Kac\cite{kac}, we have the following

**Definition 3.2:** Let $V$ be an $U_q(\hat{G})$-module. A vector $v \in V_{\lambda}$ is called primitive of weight $\lambda$ if there exists a submodule $U$ in $V$ such that

$$v \notin U; \quad e_i v \in U$$

(18) The weight $\lambda$ is called a primitive weight.

**Example:** If $V$ is an irreducible $U_q(\hat{G})$-module, so that the only proper submodule is $U = \{0\}$, then a weight vector is primitive implies that $v \neq 0$ and $e_i v = 0$.

We now state two propositions (3.2, 3.3, below) and one lemma (3.1, below) analogous to Kac’s classical results (compare Kac\cite{kac}, proposition 9.3, lemma 9.5 and proposition 9.9, respectively).

**Proposition 3.2:** Let $V$ be a non-zero $U_q(\hat{G})$-module from $\mathcal{O}_{\text{fin}}$.

a) $V$ contains a non-zero weight vector $v$ such that $e_i(v) = 0$; in particular $V$ contains a primitive
Proof:

One can show that the universal $\Omega$ in proposition 2.1 takes the form

\[ \Omega = q^{-\sum_{i=1}^{r} H_i^i} \]

\[ \times \sum_{t} S(b'_t) q^{-\sum_{i=1}^{r} H_i^i} \cdot a'_t \cdot q^{2h_\psi} \]

The condition (iii) means that the $L(\Lambda)$ exhaust all irreducible modules from $O_{\text{fin}}$.

**Lemma 3.1:** Let $V$ be an $U_q(\hat{G})$-module from $O_{\text{fin}}$. If for any two primitive weights $\lambda$ and $\mu$ of $V$, the inequality $\lambda \geq \mu$ implies $\lambda = \mu$, then the module $V$ is completely reducible.

**Proof:** The proposition (3.2) and lemma (3.1) are proved exactly as in Kac ([K], proposition 9.3 and lemma 9.5). \( \square \)

Let $Q = \sum_{i=0}^{r} Z\alpha_i$ denote the root lattice and set $Q_+ = \sum_{i=0}^{r} Z_+\alpha_i$.

**Proposition 3.3:** a) Let $V(\Lambda)$ be an $U_q(\hat{G})$-module with highest weight $\Lambda$. If $2(\Lambda+\rho, \beta) \neq (\beta, \beta)$ for every $\beta \in Q_+, \beta \neq 0$, then $V(\Lambda)$ is irreducible.

b) Let $V$ be an $U_q(\hat{G})$-module from $O_{\text{fin}}$. If for any two primitive weights $\lambda$ and $\mu$ of $V$, such that $\lambda - \mu = \beta > 0$, one has $2(\lambda + \rho, \beta) \neq (\beta, \beta)$, then $V$ is completely reducible.

**Proof:** We mimic Kac’s proof in the classical case. To this end, we first prove the following

**Lemma 3.2:** Let $V$ be an $U_q(\hat{G})$-module.

a) If there exists $v \in V$ such that $e_i v = 0$ for all $i = 0, 1, ..., r$ and $q^h v = q^{\Lambda(h)} v$ for some $\Lambda \in H^*$ and all $h \in H$, then

\[ \Omega v = q^{-(\Lambda, \Lambda+2\rho)} v \]  

(19)

b) If, furthermore, $V = U_q(\hat{G})v$, then

\[ \Omega |_V = q^{-(\Lambda, \Lambda+2\rho)} I_V \]  

(20)

**Proof:** One can show that the universal $R$-matrix $R$ of $U_q(\hat{G})$ can be written in the form

\[ R = \left( I \otimes I + \sum_{t} a'_t \otimes b'_t \right) \cdot q^{\sum_{i=1}^{r} H_i^i \otimes H^i + c \otimes d + d \otimes c} \]  

(21)

where $\{a'_t\}$ and $\{b'_t\}$ are the basis of the subalgebras of $U_q(\hat{G})$ generated by $\{e_i q^{-h_i/2}\}$ and $\{q^{h_i/2} f_i\}$, $i = 0, 1, ..., r$, respectively; $c = h_0 + h_\psi$, $\psi$ is the highest root of $\hat{G}$; $\{H_i\}$ and $\{H^i\}$ ($i = 1, 2, ..., r$) satisfy

\[ \sum_{i=1}^{r} \Lambda(H_i)\Lambda'(H^i) = (\Lambda_0, \Lambda'_0), \quad \forall \Lambda = (\Lambda_0, \kappa, \sigma), \Lambda' = (\Lambda'_0, \kappa', \sigma') \in H^* \]  

(22)

So the Casimir $\Omega$ in proposition 2.1 takes the form

\[ \Omega = q^{-\sum_{i=1}^{r} H_i^i H_i - dc - cd - 2h_\rho} + \sum_{t} S(b'_t) q^{-\sum_{i=1}^{r} H_i^i H_i - dc - cd} \cdot a'_t \cdot q^{2h_\psi} \]  

(23)
Acting on \( v \), only the first term survives, 
\[
\Omega v = q^{-\sum_{i=1}^{r} H_i \cdot \rho - 2h_p} v = q^{-(\Lambda, \Lambda + 2\rho)} v
\] 
where use has been made of (22) and \((\Lambda, \Lambda') = (\Lambda_0, \Lambda'_0) + \kappa\sigma' + \sigma\kappa'\). This proves a). Part b) follows from (24) and proposition 2.1. \( \square \)

**Corollary 3.1:**
a) If \( V \) is a highest weight \( U_q(\hat{G}) \) module with highest weight \( \Lambda \), then
\[
\Omega = q^{-|\Lambda + \rho|^2 - |\rho|^2} I_V
\]
b) If \( V \) is an \( U_q(\hat{G}) \)-module from \( O_{\text{fin}} \) and \( v \) is a primitive vector with weight \( \lambda \), then there exists a submodule \( U \subset V \) such that \( v \not\in U \) and
\[
\Omega v = q^{-|\Lambda + \rho|^2 - |\rho|^2} v \pmod{U}
\]

Now we are in the position to prove proposition 3.3. Assume that \( V(\Lambda) \) is reducible. Then proposition 3.2b) implies that there exists a primitive weight \( \lambda = \Lambda - \beta \), where \( \beta > 0 \) and thus from corollary 3.1a) we have
\[
q^{- (\Lambda, \Lambda + 2\rho)} = q^{- (\Lambda - \beta, \Lambda - \beta + 2\rho)}
\] 
which gives \( 2(\Lambda + \rho, \beta) = (\beta, \beta) \) since \( q \) is not a root of unity. This leads to a contradiction and thus we prove a).

We now prove b). We may assume that the \( U_q(\hat{G}) \)-module is indecomposable. Then, locally, the Casimir operator \( \Omega \) has the same spectrum on \( V \). We thus obtain from corollary 3.1b)
\[
q^{-|\lambda + \rho|^2 - |\rho|^2} = q^{-|\mu + \rho|^2 - |\rho|^2}
\]
for any two primitive weights \( \lambda \) and \( \mu \). Since \( q \) is not a root of unity, the above equation gives \( |\lambda + \rho|^2 = |\mu + \rho|^2 \) for any two primitive weights \( \lambda \) and \( \mu \). Therefore, we must have \( \lambda = \mu \). Indeed, if this is not the case, then we deduce \( 2(\lambda + \rho, \beta) = (\beta, \beta) \), which contradicts the condition of the proposition. Now point b) follows from lemma 3.1. \( \square \)

**Definition 3.3:** An \( U_q(\hat{G}) \)-module \( V \) is called integrable if \( V \) is \( \mathcal{H} \)-diagonalizable and if \( e_i \) and \( f_i \) \( (i = 0, 1, ..., r) \) are locally nilpotent endomorphisms of \( V \).

Let \( \Pi(\Lambda) \) denote the set of weights of the \( U_q(\hat{G}) \)-module \( L(\Lambda) \) and \( D_+ = \{ \lambda \in \mathcal{H}^* | (\lambda, \alpha_i) \geq 0, \ 0 \leq i \leq r \} \) the set of dominant integral weights.

**Proposition 3.5:**
a) Let \( V \) be an \( U_q(\hat{G}) \)-module from \( O_{\text{fin}} \) and \( \lambda \) be a primitive weight. If \( V \) is integrable, then \( \lambda \in D_+ \).
b) The highest weight \( U_q(\hat{G}) \)-module \( L(\Lambda) \) with highest weight \( \Lambda \) is integrable iff \( \Lambda \in D_+ \).

**Proof:** Part a) is proved following the same arguments as in Lusztig ([10], proposition 3.2) and part b) follows from ([8] [11]). \( \square \)
We now state our main result (complete reducibility theorem) in this section.

**Theorem 3.1:** Every integrable $U_q(\hat{G})$-module $V$ from $O_{\text{fin}}$ is completely reducible, that is, is isomorphic to a direct sum of modules $L(\Lambda), \Lambda \in D_+$.

**Proof:** We check that if $\lambda$ and $\mu$ are primitive weights such that $\lambda - \mu = \beta$, where $\beta \in Q_+/\{0\}$, then

$$2(\lambda + \rho, \beta) \neq (\beta, \beta)$$

(29)

This can easily be done as follows. By means of proposition 3.5a) and the fact that $(\rho, \beta) > 0$ for all $\beta \in Q_+/\{0\}$, we have

$$2(\lambda + \rho, \beta) - (\beta, \beta) = (\lambda + (\lambda - \mu) + 2\rho, \beta) = (\lambda + \mu + 2\rho, \beta) > 0$$

(30)

The theorem then follows from proposition 3.3b). $\square$

### 4 Unitarity

In this section we will focus our attention on quantized *nontwisted* affine Lie algebras $U_q(\mathcal{G}(1))$. Analogous conclusions are true for the *twisted* case. We first introduce the following

**Definition 4.0:** An $U_q(\mathcal{G}(1))$-module $V$ is called *unitary* if $V$ can be equipped with an inner product $\langle | \rangle$ such that, for all $a \in U_q(\mathcal{G}(1))$

$$\langle a^\dagger v | w \rangle = \langle v | aw \rangle, \ \forall v, w \in V$$

(31)

Equivalently, if $\pi$ is the representation of $U_q(\mathcal{G}(1))$ afforded by $V$, then $V$ is called unitary provided

$$\pi(a^\dagger) = \pi(a)\dagger, \ \forall a \in U_q(\mathcal{G}(1))$$

(32)

where $\dagger$ on the r.h.s. denotes Hermitian conjugate.

**Lemma 4.0:** Every integrable highest weight $U_q(\mathcal{G}(1))$-module $L(\Lambda)$ carries a unique, up to a constant factor, and well-defined nondegenerate inner product $\langle | \rangle$. With respect to this inner product, $L(\Lambda)$ decomposes into an orthogonal direct sum of weight spaces.

**Proof:** This can be easily proved following the similar arguments as in Kac ([4], proposition 9.4). $\square$

**Proposition 4.0 (Kac):** Let $\Lambda \in D_+$ and $\lambda \in \Pi(\Lambda)$. Then $|\Lambda + \rho|^2 - |\lambda + \rho|^2 \geq 0$ and equality holds iff $\lambda = \Lambda$.

4.1. Let $\hat{\mathcal{G}} = sl(2)^{(1)}$. Fix a normal ordering in the positive root system $\Delta_+$ of $sl(2)^{(1)}$:

$$\alpha, \alpha + \delta, ..., \alpha + n\delta, ..., \delta, 2\delta, ..., m\delta, ..., ..., \beta + l\delta, ..., \beta$$

(33)
where $\alpha$ and $\beta$ are simple roots and $l, m, n \geq 0$; $\delta = \alpha + \beta$ is the minimal positive imaginary root. Let us introduce standard generators

$$E_\alpha = e^\alpha q^{-h_\alpha/2}, \quad E_\beta = e^\beta q^{-h_\beta/2}$$

$$F_\alpha = q^{h_\alpha/2} f_\alpha, \quad F_\beta = q^{h_\beta/2} f_\beta$$

(34) then,

$$S(E_\alpha^\dagger) = -F_\alpha, \quad S(E_\beta^\dagger) = -F_\beta$$
$$S(F_\alpha^\dagger) = -E_\alpha, \quad S(F_\beta^\dagger) = -E_\beta$$

(35)

Construct Cartan-Weyl generators $E_\gamma, F_\gamma = \theta(E_\gamma), \gamma \in \Delta_+$ of $U_q(sl(2)^{(1)})$ as follows[12]: We define

$$\tilde{E}_\delta = [(\alpha, \alpha)]_q^{-1} [E_\alpha, E_\beta]_q$$
$$E_{\alpha+n\delta} = (-1)^n (\text{ad}\tilde{E}_\delta)^n E_\alpha$$
$$E_{\beta+n\delta} = (\text{ad}\tilde{E}_\delta)^n E_\beta,$$

$$\tilde{E}_{n\delta} = [(\alpha, \alpha)]_q^{-1} [E_{\alpha+(n-1)\delta}, E_\beta]_q$$

(36)

where $[\tilde{E}_{n\delta}, \tilde{E}_{m\delta}] = 0$ for any $n, m > 0$. For any $n > 0$ there exists a unique element $E_{n\delta}$ which satisfies $[E_{n\delta}, E_{m\delta}] = 0$ for any $n, m > 0$ and the relation

$$\tilde{E}_{n\delta} = \sum_{k_1 p_1 + \ldots + k_m p_m = n} \frac{(q^{(\alpha,\alpha)} - q^{-(\alpha,\alpha)}) \sum_{\pi_1} \ldots \sum_{\pi_m} (E_{k_1 \delta})^{p_1} \ldots (E_{k_m \delta})^{p_m}}{p_1! \ldots p_m!}$$

(37)

Then the vectors $E_\gamma$ and $F_\gamma = \theta(E_\gamma), \gamma \in \Delta_+$ defined above are the Cartan-Weyl generators for $U_q(sl(2)^{(1)})$. Moreover,

**Theorem 4.1** (Khoroshkin-Tolstoy[12]): The universal $R$-matrix for $U_q(sl(2)^{(1)})$ may be written as

$$R = \left( \Pi_{n \geq 0} \exp_{qa}((q - q^{-1})(E_{\alpha+n\delta} \otimes F_{\alpha+n\delta})) \right) \cdot \exp \left( \sum_{n \geq 0} \frac{n[n]}{q_a^{-1}} (q_a - q_a^{-1}) (E_{n\delta} \otimes F_{n\delta}) \right) \cdot \left( \Pi_{n \geq 0} \exp_{qa}((q - q^{-1})(E_{\beta+n\delta} \otimes F_{\beta+n\delta})) \right) \cdot q^{2 h_\alpha \otimes h_\alpha + c \otimes d + d \otimes c}$$

(38)

where $c = h_\alpha + h_\beta$. The order in the product (38) coincides with the chosen normal order (33).

We have

**Lemma 4.1:**

$$S(E_{\alpha+n\delta}^\dagger) = -q^n(\alpha,\beta) F_{\alpha+n\delta}, \quad S(F_{\beta+n\delta}^\dagger) = -q^n(\alpha,\beta) F_{\beta+n\delta}$$
we use the Casimir \( \Omega = u q^{2} \). We have, from the R-matrix \( \mathbb{R} \),

\[
S(F_{\alpha+n\delta}^\dagger) = -q^{-n(\alpha,\beta)} E_{\alpha+n\delta} \quad S(F_{\beta+n\delta}^\dagger) = -q^{-n(\alpha,\beta)} E_{\beta+n\delta}
\]

\[
S(\tilde{E}_{\alpha+n\delta}^\dagger) = -q^{-n(\alpha,\beta)} \tilde{F}_{n\delta} \quad S(\tilde{E}_{\beta+n\delta}^\dagger) = -q^{-n(\alpha,\beta)} F_{n\delta}
\]

\[
S(\tilde{F}_{n\delta}^\dagger) = -q^{-n(\alpha,\beta)} \tilde{E}_{n\delta} \quad S(F_{n\delta}^\dagger) = -q^{-n(\alpha,\beta)} E_{n\delta}
\]

**Proof:** The proof follows, from \( \mathbb{R} \), \( \mathbb{I} \) and \( \mathbb{J} \) and similar relations for \( F_{\gamma} = \theta(E_{\gamma}) \), by induction in \( n \). \( \square \)

**Corollary 4.1:**

\[
S(F_{\alpha+n\delta}^\dagger) = -q^{-n(\alpha,\beta)} S^2(\tilde{E}_{\alpha+n\delta}^\dagger) = -q^{-n(\alpha,\beta)-(\alpha+n\delta,2\rho)} E_{\alpha+n\delta}
\]

\[
S(F_{\beta+n\delta}^\dagger) = -q^{-n(\alpha,\beta)} S^2(\tilde{E}_{\beta+n\delta}^\dagger) = -q^{-n(\alpha,\beta)-(\beta+n\delta,2\rho)} E_{\beta+n\delta}
\]

\[
S(F_{n\delta}^\dagger) = -q^{-n(\alpha,\beta)} S^2(\tilde{E}_{n\delta}^\dagger) = -q^{-n(\alpha,\beta)-(n\delta,2\rho)} E_{n\delta}
\]

**Theorem 4.2:** Every integrable highest weight module \( L(\Lambda) \) over \( U_q(sl(2)) \) corresponding to \( q > 0 \) is equivalent to a unitary module.

**Proof:** In the limit \( q \rightarrow 1 \), the \( U_q(sl(2)) \)-module \( L(\Lambda) \) reduces to the corresponding module of \( U(sl(2)) \) \( \mathbb{R} \) and thus is equivalent to a unitary module according to Kac \( \mathbb{I} \). We now show that for \( 0 < q < 1 \) and \( q > 1 \) the module \( L(\Lambda) \) is equivalent to a unitary module. By lemma 4.0, one only need to show that if \( < \mathbf{v} \mathbf{v} > 0 \) is a highest weight vector, then the restriction of \( < \mathbf{v} \mathbf{v} > \) to \( L(\Lambda) \) is positive definite for each weight \( \lambda \) in \( L(\Lambda) \). We prove this by induction on \( \text{ht}(\Lambda - \lambda) \) (the height of \( (\Lambda - \lambda) \)). Let \( \lambda \in \Pi(\Lambda)/\{\Lambda\} \).

(i). For \( 0 < q < 1 \):

we use the Casimir \( \Omega = u q^{-2h_c} \). We have, from the R-matrix \( \mathbb{R} \),

\[
u = \sum_{\{l,n,k\}} A_{l,n,k}(q) S(F_{\beta}^{k_0} ... S(F_{\beta+M\delta}^{k_M} ... S(F_{L\delta}^{n_L} ... S(F_{\delta}^{n_1})
\]

\[
\ldots S(F_{\alpha+N\delta})^{n_N} ... S(F_{\alpha}^{l_{\alpha}} q^{-\frac{1}{2}} h_o h_o - cd - dc(E_{\alpha})^{l_\alpha} ... (E_{\alpha+N\delta})^{n_N} ... 
\]

\[
(E_{\delta})^{n_1} ... (E_{L\delta})^{n_L} ... (E_{\beta+M\delta})^{k_M} ... (E_{\beta})^{k_0}
\]

where \( \{l\} = \{l_0, l_1, \ldots, l_N, \ldots\} \), \( \{n\} = \{n_1, n_2, \ldots, n_L, \ldots\} \), \( \{k\} = \{k_0, k_1, \ldots, k_M, \ldots\} \); the constants \( A_{l,n,k}(q) \) are given by

\[
A_{l,n,k} = \frac{(q - q^{-1})^{l_0 + l_1 + \ldots + l_{N} + \ldots} (q - q^{-1})^{k_0 + k_1 + \ldots + k_M + \ldots}}{(l_0)_{q_o} ^1 \ldots (l_N)_{q_o} ^1 \ldots (k_0)_{q_o} ^1 \ldots (k_M)_{q_o} ^1 ^1 \ldots n_1 \ldots n_L \ldots}
\]

\[
= \frac{(q - q^{-1})^{l_0 + l_1 + \ldots + l_{N} + \ldots} (q - q^{-1})^{k_0 + k_1 + \ldots + k_M + \ldots}}{(l_0)_{q_o} ^1 \ldots (l_N)_{q_o} ^1 \ldots (k_0)_{q_o} ^1 \ldots (k_M)_{q_o} ^1 ^1 \ldots n_1 \ldots n_L \ldots}
\]

and satisfy

\[
(-1)^{l_0 + l_1 + \ldots + l_{N} + \ldots} (-1)^{k_0 + k_1 + \ldots + k_M + \ldots} A_{l,n,k}(q) > 0 \quad \text{for} \quad 0 < q < 1
\]
Then the corollary 4.1 implies that

\[ \Omega = q^{-\frac{1}{2}h_\alpha h_\alpha - cd - dc - 2h_\rho} + \sum'_{\{l,n,k\}} \hat{A}_{l,n,k}(q) (E^\dagger_\beta)^{k_0} \cdots (E^\dagger_{\beta + M\delta})^{k_M} \cdots \]

\[ \cdot (E^\dagger_{L\delta})^{nL} \cdots (E^\dagger_{\lambda^1 n})^{n_1} \cdots (E^\dagger_{\alpha + N\delta})^{n_{\alpha N}} \cdots (E^\dagger_{\alpha})^{l_0} q^{-\frac{1}{2}h_\alpha h_\alpha - cd - dc} (E_\alpha)^{l_0} \cdots (E_{\alpha + N\delta})^{l_N} \]

\[ \cdot \cdots (E^\delta)_{n_1} \cdots (E_{L\delta})^{n_L} \cdots (E_{\beta + M\delta})^{k_M} \cdots (E_\beta)^{k_0} q^{-2h_\rho} \]

where \( \hat{A}_{l,n,k}(q) = (-1)^{l_0 + \cdots + l_N} \cdots (-1)^{n_1 + \cdots + n_L} \cdots (-1)^{k_0 + \cdots + k_M} \cdots A_{l,n,k}(q) \) and so by \([43]\)

\[ \hat{A}_{l,n,k}(q) > 0 \quad \text{for } 0 < q < 1 \] (45)

and \( \sum' \) denotes the sum over all \( \{l,n,k\} \neq \{0,0,0\} \).

Computing \( <v|\Omega|v> \) in two different ways, we obtain

\[ (q^{-|\lambda + \rho|^2 - |\rho|^2} - q^{-|\lambda + \rho|^2 - |\rho|^2}) <v|v> = \sum'_{\{l,n,k\}} \hat{A}_{l,n,k}(q) q^{-2(\lambda,\rho)} \cdot (E_\alpha)^{l_0} \cdots (E_{\alpha + N\delta})^{l_N} \]

\[ \cdot (E^\delta)_{n_1} \cdots (E_{L\delta})^{n_L} \cdots (E_{\beta + M\delta})^{k_M} \cdots (E_\beta)^{k_0} q^{-2h_\rho} <E_\alpha)^{l_0} \cdots (E_{\alpha + N\delta})^{l_N} \cdots \]

By the inductive assumption the r.h.s. of \([44]\) is non-negative thanks to eq.\([43]\). Using proposition 4.0 we deduce that \( <v|v> \geq 0 \) for \( 0 < q < 1 \). Since \( <v|v> \) is non-degenerate on \( L(\Lambda)_\lambda \) we conclude that for \( 0 < q < 1 \) it is positive definite on \( L(\Lambda) \).

(ii) For \( q > 1 \):

In this case we work with the coproduct and antipode \([7]\). Let us introduce the standard generators

\[ \tilde{E}_\alpha = e_\alpha q^{h_\alpha / 2} \quad \tilde{E}_{\beta} = e_\beta q^{h_\beta / 2} \]

\[ \tilde{F}_\alpha = q^{-h_\alpha / 2} f_\alpha \quad \tilde{F}_{\beta} = q^{-h_\beta / 2} f_\beta \] (47)

then

\[ \tilde{S}(\tilde{E}^\dagger_\alpha) = -\tilde{F}_\alpha \quad \tilde{S}(\tilde{E}^\dagger_{\beta}) = -\tilde{F}_{\beta} \]

\[ \tilde{S}(\tilde{F}^\dagger_\alpha) = -\tilde{E}_\alpha \quad \tilde{S}(\tilde{F}^\dagger_{\beta}) = -\tilde{E}_{\beta} \] (48)

and we have

**Lemma 4.2:**

\[ \tilde{S}(\tilde{F}_\alpha + n \delta) = -q^{n(\alpha,\beta) + (\alpha + n \delta, 2 \rho)} \tilde{E}_{\alpha + n \delta} \]

\[ \tilde{S}(\tilde{F}_{\beta} + n \delta) = -q^{n(\alpha,\beta) + (\beta + n \delta, 2 \rho)} \tilde{E}_{\beta + n \delta} \]

\[ \tilde{S}(\tilde{F}_{\delta}) = -q^{n(\alpha,\beta) + (n \delta, 2 \rho)} \tilde{E}_{\delta} \] (49)

**Proof:** Similar to the proof of lemma 4.1. □
Now we define inductively,
\[
\tilde{E}_\delta = [(\alpha, \alpha)]_q^{-1} [E_\alpha, \tilde{E}_\beta]_{q^{-1}}
\]
\[
\tilde{E}_{\alpha+n\delta} = (-1)^n \left( \text{ad} \tilde{E}_\delta \right)^n \tilde{E}_\alpha
\]
\[
\tilde{E}_{\beta+n\delta} = \left( \text{ad} \tilde{E}_\delta \right)^n \tilde{E}_\beta, ...
\]
\[
\tilde{E}_{n\delta} = [(\alpha, \alpha)]_q^{-1} [E_{\alpha+(n-1)\delta}, \tilde{E}_\beta]_{q^{-1}}
\]
\[
\tilde{E}_{n\delta} = \sum_{k_1 p_1 + ... + k_m p_m = n} \frac{\left( q^{-\langle\alpha, \alpha\rangle} - q^{\langle\alpha, \alpha\rangle} \right) \sum_{i=1}^{n-1} (\tilde{E}_{k_1\delta})^{p_1} ... (\tilde{E}_{k_m\delta})^{p_m}}{p_1! ... p_m!} \tag{50}
\]
and similarly for \(\tilde{F}_\gamma = \theta(\tilde{E}_\gamma)\). Then we immediately obtain, from the \(R\)-matrix \(\tilde{R}\), our matrix \((\tilde{R}^T)^{-1}\):
\[
(\tilde{R}^T)^{-1} = \sum_{\{1, n, k\}} A_{1, n, k}(q^{-1}) (\tilde{E}_\alpha)^{l_0} ... (\tilde{E}_{\alpha+N\delta})^{l_N} ... (\tilde{E}_\delta)^{n_1} ... (\tilde{E}_{L\delta})^{n_L} ...
\]
\[
\ldots (\tilde{E}_{\beta+M\delta})^{k_M} ... (\tilde{E}_\beta)^{k_0} \otimes (\tilde{F}_\alpha)^{l_0} ... (\tilde{F}_{\alpha+N\delta})^{l_N} ... (\tilde{F}_\delta)^{n_1} ... (\tilde{F}_{L\delta})^{n_L} ...
\]
\[
\ldots \ldots (\tilde{F}_{\beta+M\delta})^{k_M} ... (\tilde{F}_\beta)^{k_0} \cdot q^{-\frac{1}{2} h_a \otimes h_a - c \otimes d - d \otimes c} \tag{51}
\]
where the constants \(A_{1, n, k}(q^{-1})\) satisfy
\[
(-1)^{l_0 + ... + l_N + ... (-1)^{n_1 + ... + n_L + ... (-1)^{k_0 + ... + k_M + ... A_{1, n, k}(q^{-1}) > 0, \quad \text{for } q > 1} \tag{52}
\]
We deduce, from \(\tilde{R}\),
\[
\tilde{R} = (I \otimes \tilde{S}) \tilde{R}^{-1} = \sum_{\{1, n, k\}} A_{1, n, k}(q^{-1}) (\tilde{F}_\alpha)^{l_0} ... (\tilde{F}_{\alpha+N\delta})^{l_N} ... (\tilde{F}_\delta)^{n_1} ... (\tilde{F}_{L\delta})^{n_L} ...
\]
\[
\ldots (\tilde{F}_{\beta+M\delta})^{k_M} ... (\tilde{F}_\beta)^{k_0} \otimes q^{\frac{1}{2} h_a \otimes h_a + c \otimes d + d \otimes c} \cdot \tilde{S}(\tilde{E}_\beta)^{k_0} ... \tilde{S}(\tilde{F}_{\beta+M\delta})^{k_M} ...
\]
\[
\ldots \ldots \tilde{S}(\tilde{E}_{L\delta})^{n_L} ... \tilde{S}(\tilde{E}_\delta)^{n_1} ... \tilde{S}(\tilde{E}_{\alpha+N\delta})^{l_N} ... \tilde{S}(\tilde{E}_\alpha)^{l_0} \tag{53}
\]
Thus we obtain the following Casimir operator
\[
\tilde{\Omega} = \tilde{S}(\tilde{\Omega}) = \tilde{S}(\tilde{u}^{-1} q^{-2 h_\rho}) = \tilde{q}^{2 h_\rho} \tilde{S}(\tilde{u}^{-1})
\]
\[
= \tilde{q}^{2 h_\rho} \sum_{\{1, n, k\}} A_{1, n, k}(q^{-1}) \tilde{S}(\tilde{F}_\beta)^{k_0} ... \tilde{S}(\tilde{F}_{\beta+M\delta})^{k_M} ...
\]
\[
\cdot \tilde{S}(\tilde{E}_\alpha)^{l_0} ... \tilde{S}(\tilde{E}_{\alpha+N\delta})^{l_N} ... \tilde{S}(\tilde{F}_\delta)^{n_1} ... \tilde{S}(\tilde{F}_{L\delta})^{n_L} ...
\]
\[
\cdot \tilde{S}(\tilde{E}_{\beta+M\delta})^{k_M} ... \tilde{S}(\tilde{F}_\beta)^{k_0} = \tilde{q}^{\frac{1}{2} h_a \otimes h_a + c \otimes d + d \otimes c}
\]
\[
\cdot (\tilde{E}_\alpha)^{l_0} ... (\tilde{E}_{\alpha+N\delta})^{l_N} ... (\tilde{E}_\delta)^{n_1} ... (\tilde{E}_{L\delta})^{n_L} ...
\]
\[
\ldots (\tilde{E}_{\beta+M\delta})^{k_M} ... (\tilde{E}_\beta)^{k_0} \tag{54}
\]
which, using the lemma 4.2, takes the form
\[
\tilde{\Omega} = \tilde{q}^{\frac{1}{2} h_a \otimes h_a + c \otimes d + d \otimes c + 2 h_\rho} + \sum_{\{1, n, k\}} A_{1, n, k}(q^{-1}) (\tilde{E}_\beta)^{k_0} ... (\tilde{E}_{\beta+M\delta})^{k_M} ...
\]
\[
\cdot (\tilde{E}_{L\delta})^{n_L} ... (\tilde{E}_\delta)^{n_1} ... (\tilde{E}_{\alpha+N\delta})^{l_N} ... (\tilde{E}_\alpha)^{l_0} \cdot \tilde{q}^{\frac{1}{2} h_a \otimes h_a + c \otimes d + d \otimes c} (\tilde{E}_\alpha)^{l_0} ... (\tilde{E}_{\alpha+N\delta})^{l_N}
\]
\[
\ldots (\tilde{E}_{\beta+M\delta})^{k_M} ... (\tilde{E}_\beta)^{k_0} \tag{55}
\]
where \( \hat{A}_{1,n,k}(q^{-1}) = (-1)^{\ell_0 + \ldots + \ell_N + \ldots (-1)^{n_1 + \ldots + n_L + \ldots (-1)^{k_0 + \ldots k_M + \ldots} A_{1,n,k}(q^{-1}) \) and so by (52)

\[
\hat{A}_{1,n,k}(q^{-1}) > 0 \quad \text{for} \quad q > 1
\] (56)

Computing \(<v|\Omega|v>\) in two different ways as above, we obtain

\[
\left(q^{\lambda+\rho^2-|\rho|^2} - q^{\lambda+\rho^2-|\rho|^2}\right) <v|v> = \sum_{\{\ell,n,k\}} \hat{A}_{1,n,k}(q^{-1}) q^{2(\lambda,\rho)} <(\hat{E}_\alpha)^{\ell_0} \ldots (\hat{E}_{\alpha+N\delta})^{\ell_N} \ldots \cdot (\hat{E}_\delta)^{n_1} \ldots (\hat{E}_{\delta+M\delta})^{k_M} \ldots (\hat{E}_\beta)^{k_0} v q^{\frac{1}{2}h_d h_a + cd + dc} \\
\cdot (\hat{E}_\alpha)^{l_0} \ldots (\hat{E}_{\alpha+N\delta})^{l_N} \ldots (\hat{E}_\delta)^{n_L} \ldots (\hat{E}_{\delta+M\delta})^{k_M} \ldots (\hat{E}_\beta)^{k_0} v >
\] (57)

By the inductive hypothesis we have that the r.h.s. of (57) is non-negative for \( q > 1 \) thanks to the formula (57). Therefore, we deduce from proposition 4.0 that \(<v|v>\geq 0\) for \( q > 1 \). Then the non-degeneracy of \(<|>\) on \( L(\Lambda) \) implies that \(<v|v> > 0\) for \( q > 1 \). \( \square \)

4.2. General case: \( \hat{G} = G^{(1)} \). Fix some order in the positive root system \( \Delta_+ \) of \( G^{(1)} \), which satisfies an additional condition,

\[
\alpha + n\delta \leq k\delta \leq (\delta - \beta) + l\delta
\] (58)

where \( \alpha, \beta \in \Delta^0, \quad \Delta^0_+ \) is the positive root system of \( G \); \( k, l, n \geq 0 \) and \( \delta \) is the minimal positive imaginary root.

Let us as before introduce standard generators,

\[
E_i = e_i q^{-h_i/2}, \quad F_i = q^{h_i/2} f_i, \quad i = 0, 1, \ldots, r
\] (59)

then we have

\[
S(E_i^\dagger) = -F_i, \quad S(F_i^\dagger) = -E_i
\] (60)

Cartan-Weyl generators \( E_\gamma \) and \( F_\gamma = \theta(E_\gamma) \), \( \gamma \in \Delta_+ \) may be constructed inductively as follows (12). We start from the simple roots. If \( \gamma = \alpha + \beta, \quad \alpha < \gamma < \beta, \) is a root and there are no other positive roots \( \alpha' \) and \( \beta' \) between \( \alpha \) and \( \beta \) such that \( \gamma = \alpha' + \beta' \), then we set

\[
E_\gamma = [E_\alpha, E_\beta]_q = E_\alpha E_\beta - q^{(\alpha,\beta)} E_\beta E_\alpha
\] (61)

When we get the root \( \delta \), we use the following formula for roots \( \gamma + n\delta \) and roots \( (\delta - \gamma) + n\delta, \) for \( \gamma \in \Delta^0_+ \),

\[
\hat{E}_\delta^{(i)} = [(\alpha_i, \alpha_i)]_q^{-1} [E_{\alpha_i}, E_{\delta - \alpha_i}]_q, \\
E_{\alpha_i + n\delta} = (-1)^n \left( \text{ad} \hat{E}_\delta^{(i)} \right)^n E_{\alpha_i}, \\
E_{\delta - \alpha_i + n\delta} = \left( \text{ad} \hat{E}_\delta^{(i)} \right)^n E_{\delta - \alpha_i}, \ldots, \\
\hat{E}_{\delta}^{(n)} = [(\alpha_i, \alpha_i)]_q^{-1} [E_{\alpha_i + (n-1)\delta}, E_{\delta - \alpha_i}]_q
\] (62)
Then we repeat the above inductive procedure to obtain other real root vectors $E_{\gamma+n\delta}, E_{\delta-\gamma+n\delta}$, $\gamma \in \Delta_+^0$. Finally, the imaginary root vectors $E_{n\delta}^{(i)}$ are defined through $\tilde{E}_{n\delta}^{(i)}$ by the relation (37) with $\alpha$ there changing to $\alpha_i$. Then, the above operators $E_{n\delta}^{(i)}, F_{n\delta}^{(i)} = \theta(E_{n\delta}^{(i)})$ ($i = 1, 2, ..., r$), $E_\gamma$, $F_\gamma = \theta(E_\gamma)$ are the Cartan-Weyl generators of $U_q(G^{(1)})$. Moreover

**Theorem 4.3** (Khoroshkin-Tolstoy[12]): The universal $R$-matrix $U_q(G^{(1)})$ may be written in the following form,

$$R = \left( \prod_{\gamma \in \Delta^e_+, \gamma < \delta} \exp_{q,\gamma} \left( \frac{q - q^{-1}}{C_{\gamma}(q)} E_\gamma \otimes F_\gamma \right) \right) \cdot \exp \left( \sum_{n > 0} \sum_{i,j=1}^{r} C_n^{ij}(q)(q - q^{-1})(E_{n\delta}^{(i)} \otimes F_{n\delta}^{(j)}) \right) \cdot \left( \prod_{\gamma \in \Delta^e_+, \gamma > \delta} \exp_{q,\gamma} \left( \frac{q - q^{-1}}{C_{\gamma}(q)} E_\gamma \otimes F_\gamma \right) \right) \cdot q^{\sum_{i,j=1}^{r} (a_{\gamma/m})^{ij} h_i \otimes h_j + c \odot d \odot c} \quad (63)$$

where $c = h_0 + h_q$, $\psi$ is the highest root of $G$: $(C_{ij}(q)) = (C_{ji}(q))$, $i, j = 1, 2, ..., r$, is the inverse of the matrix $(B_{ij}^{(n)}(q))$, $i, j = 1, 2, ..., r$ with

$$B_{ij}^{(n)}(q) = (-1)^{n(1-\delta_{ij})} q_{ij}^{-n} \frac{q_{ij}^{n} - q_{ji}^{-n}}{q - q^{-1}}, \quad q_{ij} = q^{(\alpha_i, \alpha_j)}, \quad q_i \equiv q^{\alpha_i} \quad (64)$$

and $C_\gamma(q)$ is a normalizing constant defined by

$$[E_\gamma, F_\gamma] = \frac{C_\gamma(q)}{q - q^{-1}} \left( q^{h_\gamma} - q^{-h_\gamma} \right), \quad \gamma \in \Delta^e_+ \quad (65)$$

The order in the product of the $R$-matrix coincides with the chosen normal ordering [58] in $\Delta_+$. We now state an important

**Remark:** $C_\gamma(q)$ have the following general property

$$C_\gamma(q) = C_\gamma(q^{-1}) > 0 \quad \text{for} \quad q > 0, \quad q \neq 1 \quad (66)$$

as shown in our previous paper[13].

We have the following

**Lemma 4.3:** For any $\alpha \in \Delta_+^0$,

$$S(E_\alpha^+) = -q^{(\alpha, \alpha - 2\rho)/2} F_\alpha, \quad S(F_\alpha^+) = -q^{-(\alpha, \alpha - 2\rho)/2} E_\alpha \quad (67)$$

**Proof:** We prove them by induction. The results obviously are valid for $\alpha = \alpha_i$, $i = 1, 2, ..., r$, a simple root since we have $(\alpha_i, \alpha_i - 2\rho) = 0$. Now we show that the results are also true for $E_{\alpha + \beta} = [E_\alpha, E_\beta]_q$ and $F_{\alpha + \beta} = [F_\beta, F_\alpha]_q^{-1}$. We have

$$S(E_{\alpha + \beta}^+) = S(F_\alpha^+) S(E_\beta^+) - q^{(\alpha, \beta)} S(E_\beta^+) S(E_\alpha^+) \quad (68)$$
which by the inductive assumption gives

$$S(E_{\alpha+\beta}^\dagger) = q^{(a,\alpha-2p)/(\alpha,\beta-2p)/2} \left( F_\alpha F_\beta - q^{(a,\beta)} F_\beta F_\alpha \right)$$

$$= q^{(a,\beta)+(\alpha,\alpha-2p)/(\alpha,\beta-2p)/2} \left( F_\beta F_\alpha - q^{-(a,\beta)} F_\alpha F_\beta \right)$$

$$= -q^{(a+\beta,\alpha+\beta-2p)/2} F_{\alpha+\beta}$$

Similarly, we have

$$S(F_{\alpha+\beta}^\dagger) = -q^{-(a+\beta,\alpha+\beta-2p)/2} F_{\alpha+\beta}$$

This completes our proof. 

**Corollary 4.2:** For any $\alpha \in \Delta_+^0$,

$$S(E_{\delta-\alpha}^\dagger) = -q^{(\delta-\alpha,\delta-\alpha-2p)/2} F_{\delta-\alpha}$$,

$$S(F_{\delta-\alpha}^\dagger) = -q^{-(\delta-\alpha,\delta-\alpha-2p)/2} F_{\delta-\alpha}$$

**Proof:** The results are true for $\alpha = \psi$, the highest root. Then the results follow, from lemma 4.3 and $E_{\delta-\alpha} = [E_\beta, E_{\delta-(\alpha+\beta)}]_q$ and $F_{\delta-\alpha} = [F_{\delta-(\alpha+\beta)}, F_\beta]_q^{-1}$, by induction as above. 

**Lemma 4.4:** For any $\alpha \in \Delta_+^0$,

$$S(E_{\alpha+n\delta}^\dagger) = -q^{(a,\alpha-2p)/(\alpha,\delta-\alpha-2p)/2} E_{\alpha+n\delta}$$

$$S(F_{\alpha+n\delta}^\dagger) = -q^{-(a,\alpha-2p)/(\alpha,\delta-\alpha-2p)/2} E_{\alpha+n\delta}$$

$$S(E_{\delta-\alpha+n\delta}^\dagger) = -q^{(\delta-\alpha,\delta-\alpha-2p)/(\alpha,\delta-\alpha-2p)/2} E_{\delta-\alpha+n\delta}$$

$$S(F_{\delta-\alpha+n\delta}^\dagger) = -q^{-(\delta-\alpha,\delta-\alpha-2p)/(\alpha,\delta-\alpha-2p)/2} E_{\delta-\alpha+n\delta}$$

$$S(E_{n\delta}^{(i)}^\dagger) = -q^{n(\delta,\rho)} E_{n\delta}$$,

$$S(F_{n\delta}^{(i)}^\dagger) = -q^{n(\delta,\rho)} E_{n\delta}$$

**Proof:** The proof follows from (17), (22) and lemma 4.3 and corollary 4.2, by induction in $n$. 

Our main result is:

**Theorem 4.4:** Every integrable highest weight module $L(\Lambda)$ over $U_q(\mathfrak{g}^{(1)})$ corresponding to $q > 0$ is equivalent to a unitary module.

**Proof:** The proof is similar to the one of theorem 4.2 for $U_q(sl(2)^{(1)})$ case thanks to lemma 4.3, corollary 4.2, lemma 4.4 and remark (36). 

**Corollary 4.3:** Every integrable highest weight module $L(\Lambda)$ over $U_q(\mathfrak{g})$ corresponding to $q > 0$ is equivalent to a unitary module.

## 5 Concluding Remarks

To summarize, in this paper we have investigated the complete reducibility and unitarity of certain modules over the quantized affine Lie algebras. In our proofs the Casimir operator (thus
the universal $R$-matrix) plays a key role and our approach is actually a modest imitation of Kac’s one\cite{4} and the one in \cite{14} used in proving the similar results for the classical affine Lie algebras and finite-dimensional simple Lie superalgebras, respectively.

The complete reducibility theorem 3.1 implies that the tensor product $L(\Lambda) \otimes L(\Lambda')$ of the integrable irreducible highest weight modules $L(\Lambda)$ and $L(\Lambda')$ is completely reducible and the irreducible components are integrable highest weight representations. This makes possible the computation of link polynomials \cite{15} \cite{16} associated with the quantized affine Lie algebras. The point is that the universal $R$-matrix for the quantized affine Lie algebras can be shown to satisfy the conjugation rule, $R^\dagger = R^T$; therefore, braid generators are diagonalizable\cite{14} on $L(\Lambda) \otimes L(\Lambda')$, regardless of multiplicity. Our results also allow us to construct generalized Gelfand invariants \cite{17} of the quantized affine Lie algebras. All details will be reported in a separate publication.

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