Attractive Bose-Einstein Condensates in three dimensions under rotation: Revisiting the problem of stability of the ground state in harmonic traps

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Abstract

We study harmonically trapped ultracold Bose gases with attractive interparticle interactions under external rotation in three spatial dimensions and determine the critical value of the attraction strength where the gas collapses as a function of the rotation frequency. To this end we examine the stationary state in the corotating frame with a many-body approach as well as within the Gross-Pitaevskii theory of systems in traps with different anisotropies. In contrast to recently reported results [N. A. Jamaludin, N. G. Parker, and A. M. Martin, Phys. Rev. A 77, 051603(R) (2008)], we find that the collapse is not postponed in the presence of rotation. Unlike repulsive gases, the properties of the attractive system remain practically unchanged under rotation in isotropic and slightly anisotropic traps.

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Bose-Einstein condensates (BECs) have occupied a central role in the study of atomic and quantum physics since their first experimental realization. In particular, attractive condensates [1], i.e., systems whose bosons attract each other, are distinguished due to their peculiar features. Namely, in a trapped three-dimensional attractive gas whose number of particles or the strength of the interparticle interaction exceeds a threshold value, the kinetic energy cannot balance the (negative) interaction energy and so the gas implodes and collapses [2, 3]. However, in the range below the critical interaction strength (or critical particle number), there can exist metastable states, i.e., states that will survive the collapse for some finite time. The collapse of attractive BECs and situations where it can be hindered have been the subject of much interest already two decades ago, see, for instance, Refs. [4, 5]. It is furthermore known that the attractive gas, once prepared in a vortex configuration, will be more stable against collapse [3, 5, 6]. More recently, fragmented metastable excited states [7] and ground states with definite nonzero angular momentum [8] have also been found to postpone the collapse.

On the other hand, it is known that rotating (stirring) condensates is a way of imprinting angular momentum in a gas and nucleating vortices [9, 10]. In repulsive gases rotating with a frequency smaller than the trapping frequency there can exist configurations where the system is well described by a stationary state with some finite nonzero vorticity. Vortices [11], vortex lattices [9] and highly correlated – fractional quantum Hall – states [12], as well as giant-vortices [13] have all been experimentally observed in repulsive gases. In sharp contrast, the behavior of the attractive system under rotation is quite different [14–16]. The question of how rotation would affect the stability and collapse of the attractive condensate in harmonic traps has recently been addressed [17]. In Ref. [17] it has been found at the Gross-Pitaevskii (GP) mean-field (MF) level that the attractive gas can be stabilized against collapse for rotation frequencies smaller than the trap frequency. These findings have motivated us to attack the same problem at the many-body (MB) level. We show herein that rotating an attractive condensate, confined by a harmonic isotropic or slightly anisotropic trap, with a frequency below the trap frequency, does not have an impact on the stability as well as on the angular momentum of the ground state. We then analyze the problem on the GP (MF) level and find as well that no stabilization of the ground state occurs. We stress at this point that it has been previously shown that there is no stabilization of the attractive gas in an isotropic anharmonic trap with a slight anharmonicity for rotation frequencies.
The system. We consider an attractive BEC of $N$ atoms of mass $m$, confined by a generally anisotropic trapping potential

$$V(r) = \frac{1}{2} m \omega^2 \left[ (1 - \varepsilon)x^2 + (1 + \varepsilon)y^2 + \zeta^2 z^2 \right] = V_0(x, y, z) - \varepsilon V_a(x, y), \quad (1)$$

where $\omega$, $\varepsilon$ and $\zeta$ are real nonnegative parameters that determine the frequencies of the trap and its deformation, namely $\omega_x = \omega \sqrt{1 - \varepsilon}$, $\omega_y = \omega \sqrt{1 + \varepsilon}$ and $\omega_z = \omega \zeta$, $V_0 = \frac{m \omega^2}{2}(x^2 + y^2 + \zeta^2 z^2)$ is the axially symmetric part of the potential and $V_a = \frac{m \omega^2}{2}(x^2 - y^2)$ the ‘rotating’ anisotropy. Since we are interested in the rotating problem we will work in the corotating frame of reference, where the MB Hamiltonian takes on the time-independent appearance:

$$\hat{H} = \sum_i^{N} \left[ -\frac{\hbar^2}{2m} \nabla^2(r_i) + V(r_i) - \Omega \hat{L}_z(r_i) \right] + \lambda_0 \sum_{i<j}^{N} \delta(r_i - r_j), \quad (2)$$

where $\Omega$ is the frequency of the rotation around the z-axis, $\hat{L}_z$ the z-projection of the angular momentum operator and $\lambda_0$ measures the interaction strength and takes on negative values for attraction. We set hereafter $\hbar = m = \omega = 1$ so as to work in dimensionless units.

The Hamiltonian of Eq. $(2)$ admits exact solutions in the absence of interaction, i.e., when $\lambda_0 = 0$. In the case of isotropic system ($\varepsilon = 0$, $\zeta = 1$) and in the limit of fast rotation ($\Omega \to \omega$) the energy levels are organized into what is known as Landau Levels. The same holds true for weak interparticle interactions [18–20]. Thus, in the fast rotation and weak interaction limit the Lowest Landau Level (LLL) is particularly designated for the description of the ground state of the system. The orbitals that comprise the (scaled) LLL have the form $\psi_k^{LLL}(r) = N_k r^k e^{-r^2/2\sigma^2} Y_{mk}(\theta, \phi)$, $k = 0, 1, 2, \ldots$, where $Y_{mk}^k$ is the spherical harmonic with $l = m_l = k$ and $N_k$ is the normalization constant. The scaling parameter $\sigma$ defines the width of the Gaussian part and will be treated variationally, i.e., so as to minimize the total energy. Of course, if $\lambda_0 = 0$ then $\sigma = 1$. At the resonance, $\Omega_r = \omega$, all (infinitely many) orbitals of this set become degenerate in energy. The above orbitals can also be expressed in Cartesian coordinates as appropriate linear combinations of the solutions $\phi_i(x, y, z) = \varphi_{n_x}(\omega_x, x) \varphi_{n_y}(\omega_y, y) \varphi_{n_z}(\omega_z, z)$ of the three-dimensional harmonic oscillator, i.e., the scaled Hermite-Gauss functions, $\varphi_{n_x}(\omega_x, x) = \frac{(\omega_x/\sigma \omega_{n_x})^{1/4}}{\sqrt{2^{n_x} n_x!}} H_{n_x} \left( \frac{\sqrt{\omega_{n_x}} x}{\sigma} \right) e^{-\omega_x x^2/2\sigma^2}$, where $H_n(\ldots)$ denotes the Hermite polynomial of degree $n$. Namely, for the isotropic case $\omega_x = \omega_y = \omega_\varepsilon = \omega_z = \omega_\zeta =$
\( \omega_y = \omega_z \), we rewrite the orbitals as

\[
\psi_k(r) = \sum_{n_x+n_y=k} c_i \phi_i, \quad (3)
\]

with \( c_i = \langle \psi_k^{LLL} | \phi_i \rangle \), \( n_x + n_y = k \), \( k = 0, 1, 2, \ldots \), and \( i = i(n_x, n_y) = \frac{1}{2} \left[ n_x + 3n_y + (n_x + n_y)^2 \right] \) is a function that we employ to uniquely map the pair \( \{ n_x, n_y \} \) to the single parameter \( i \). Once we depart from the isotropy of the trap the infinite degeneracy, now at \( \Omega_r = \omega_x < \omega_y \), is not lifted [21] and the above LLL states are not solutions of the anisotropic system. Since the radial symmetry of the trap is broken the orbitals do not possess exact angular symmetries and one cannot express the solutions in terms of pure spherical harmonics anymore. Instead, one should resort to the orbitals \( \psi_k \) expressed as a mixture of functions \( \phi_i \). The same transformation coefficients \( c_i \), that are defined above for the isotropic case, can also be used for generic \( \omega_x \neq \omega_y \neq \omega_z \). This transformation maps the functions from the Hermite-Gauss representation to that with nonzero (expectation value of) orbital angular momentum. Of course, for \( \varepsilon = 0 \) and \( \zeta = 1 \) (i.e., for isotropic traps) the mapped orbitals give back the spherical harmonics. The expectation values of \( \hat{L}_z \) for the orbitals \( \psi_k \), for \( \varepsilon \) small enough, are \( \langle \psi_k | \hat{L}_z | \psi_k \rangle \equiv l_k = \left[ 1 + \frac{\varepsilon^2}{8} + O(\varepsilon^4) \right] k \), with \( k = 0, 1, 2, \ldots \). Note that, when \( \varepsilon \neq 0 \), the orbital set \( \{ \psi_k \} \) is also not an exact solution of the noninteracting anisotropic Hamiltonian, since the linear combination \( \psi_k = \sum c_i \phi_i \) mixes nondegenerate states. However, in the limit of small \( \varepsilon \), this choice is justified on account of working with single-particle states \( \psi_k \) that have nonzero (expectation value of) angular momentum \( l_k \) and thus allows for a possible coupling to the rotation.

**Many-body approach.** We study our system at the MB level, i.e., beyond a MF description. To this end we follow the *Configuration Interaction* (CI) expansion, a general variational MB method that allows the system to fragment and takes into consideration fluctuations of the states. For details on this method and the construction of the configuration space the reader is referred to the literature, e.g., [8, 22–24]. The MB wave function \( |\Psi\rangle \) of the system is expanded over a set of functions \( |\Phi_i\rangle \) (permanents),

\[
|\Psi\rangle = \sum C_i |\Phi_i\rangle, \quad (4)
\]

each describing a MF state of a condensed or fragmented Bose gas of \( N \) atoms. The permanents are built over a certain set of \( M \) single-particle functions (orbitals). In this work
$M = 4$ and the set of orbitals comprise the LLL and its anisotropic extension, as described above. The permanents can be written in an occupation-number-representation as $|\Phi_i\rangle = |\vec{n}\rangle = |n_0, n_1, \ldots, n_{M-1}\rangle$, where it is meant that $n_i$ bosons occupy the $\phi_i$ orbital, satisfying $\sum_i n_i = N$. The Hamiltonian of the problem is then represented as a matrix $\mathcal{H}$ over the permanents $|\Phi_i\rangle$ and diagonalized. The eigenvalues $E_i$ of $\mathcal{H}$ are the energies of the states. The eigenvectors $\{C_i\}$ of $\mathcal{H}$ provide us with the wave functions with which one can compute various quantities like the natural occupation numbers $\rho_i$ of the ground and excited states, with $\sum_i \rho_i = N$. Note that the natural orbitals and the orbitals described and used in the expansion above coincide. This holds for the isotropic case (due to the symmetry of the problem) and has been found (numerically) to be well satisfied for the slightly anisotropic case discussed below. From the natural occupations we can calculate the total angular momentum of the ground state as $L = \sum_{l=0}^{3} l \rho_l$. By varying the parameter $\sigma$ (i.e., the Gaussian width of the orbitals), we minimize the energies per particle $\epsilon = E/N$ as a function of the rotation frequency $\Omega$, for some fixed value of $\lambda = |\lambda_0|(N - 1)$ and determine the optimal value $\sigma_0$. The analysis of the system that follows is always done for optimal states, i.e., at $\sigma_0$. The number of particles is hereafter set to $N = 12$.

We denote with $\lambda_c$ the critical value of the parameter $\lambda$ where the ground state of the condensate ceases to exist. This is calculated as the largest value of $\lambda$ where there is a (local) minimum in the energy $E$ as a function of $\sigma$. The absence of such a minimum denotes a collapsed state (see also [7, 8, 25, 26]). We are interested in the dependence of $\lambda_c$ on the rotation frequency $\Omega$. In Fig. 1 we plot the critical value $\lambda_c$ against $\Omega$ for the isotropic $\varepsilon = 0$, $\zeta = 1$ and the slightly anisotropic case $\varepsilon = 0.1$, $\zeta = 1$.\footnote{Note that even smaller trap anisotropies are sufficient to nucleate vortices in experimental setups, like $\varepsilon = 0.025$ for instance, in the rotating repulsive gas of Ref. [10].} The values of $\Omega$ range from 0 to $\Omega_r = \omega_x = \sqrt{1 - \varepsilon}$. At exactly the resonance frequency $\Omega_r$, the energy diverges and the gas becomes mechanically unstable. We notice no change in the stability of the ground state of the isotropic system as the rotation frequency $\Omega$ increases from 0 up to $\Omega_r$, and only a negligible increase in $\lambda_c(\Omega)$ of less than 0.1% for $\varepsilon = 0.1$. The value of the critical parameter $\lambda_c = 8.425(9)$ remains unchanged when $\varepsilon = 0$ for the whole allowed region of $\Omega$, and marginally increases from $\lambda_c(0) = 8.436(2)$ to $\lambda_c(\Omega_r) = 8.440(7)$ for $\varepsilon = 0.1$.\footnote{Here and hereafter, when we write $\lambda_c(\Omega_r)$ it is meant, mathematically, $\lambda_c(\Omega)$ in the limit of the resonance frequency $\Omega \rightarrow \Omega_r$. The same is meant for other system’s properties at the resonance frequency.}
These results, obtained at the MB level, obviously contradict the GP results of Ref. [17] (see analysis and discussion below).

Next, to analyze the MB results, we chose $\lambda = 3$ as a representative value of the interaction parameter of an isotropic system ($\varepsilon = 0$) with noncollapsed ground state and calculated the energy per particle $\epsilon$, the angular momentum per particle $L/N$ and the natural occupations $\rho_i$, $i = 0, \ldots, 3$ for the ground state. We found that the above quantities remain constant for any $\Omega \in [0, \Omega_r)$. The state remains condensed ($\rho_0 = N$), carries no angular momentum ($L/N = 0$) and has energy $\epsilon = 1.396(8)$. For the anisotropic case of $\varepsilon = 0.1$ we also found that the above quantities practically do not change. Namely, $\rho_0$ marginally decreases from $\rho_0(0) \simeq 12$ to $\rho_0(\Omega_r) = 11.998(9)$, and the rest of the natural occupations change from $\rho_1(0) \simeq 10^{-8}, \rho_2(0) \simeq 10^{-5}, \rho_3(0) \simeq 10^{-13}$ to $\rho_1(\Omega_r) \simeq 10^{-7}, \rho_2(\Omega_r) = 10^{-3}, \rho_3(\Omega_r) \simeq 10^{-12}$. There is an insignificant decrease in the energy [from $\epsilon(0) = 1.395(8)$ to $\epsilon(\Omega_r) = 1.395(4)$] and a corresponding increase in the angular momentum [from $L(0)/N \simeq 0$ to $L(\Omega_r)/N = 2 \cdot 10^{-4}$].

The fact that the ground state of the isotropic system is found to be fully (i.e., 100%) condensed deserves some discussion. The total absence of depletion and fluctuations in this case is explained if one considers the MB orbital set used: since each orbital $\psi_{LLL}^k$ has different angular symmetry a coupling between the different modes is forbidden due to the symmetry of the problem. Any nonzero occupation of the $i = 1, 2, 3$ orbitals would result in the change of the total angular momentum of the system. Naturally, such a coupling is induced in the system when the anisotropy $\varepsilon$ is turned on and hence the occupations $\rho_i, i = 1, 2, 3$ can be nonzero. Nonetheless, as we have found above, for attractive systems in three-dimensional isotropic and slightly anisotropic traps, coupling of the ground zero-angular-momentum state to excited-states with nonvanishing angular momentum essentially does not occur, even for rotation frequencies as high as the resonance frequency $\Omega_r$. In other words, for rotating attractive BECs none of the $\psi_{LLL}^{k>0}$ (or, for slightly anisotropic traps, $\psi_{k>0}$) states becomes the state lowest-in-energy, even for rotation frequencies as high as the resonance frequency $\Omega_r$.

Do the above findings change for a MB basis set that does allow for ground-state depletion? The answer is negative. Having used, in place of the LLL, the set consisting of the $s, p_+, p_0$ and $p_-$ orbitals (see in this respect Ref. [8]), we found the ground state of the isotropic system slightly depleted (i.e., about 98% condensed for $\lambda \simeq \lambda_c$), but its angular momentum zero for all rotation frequencies up to the resonance frequency $\Omega_r$. Side by side,
the depletion and fluctuations of the ground state do not depend on the rotation frequency. Importantly, the critical value $\lambda_c$ for the collapse does not depend on the rotation frequency as well. The same conclusion holds for slightly anisotropic traps ($\varepsilon = 0.1$). In summary, we have shown by a MB approach that the critical value of the interaction for collapse, $\lambda_c$, of rotating three-dimensional attractive BECs does not depend on the frequency of rotation.

The fact that the ground states of both the isotropic ($\varepsilon = 0$) and the slightly anisotropic system ($\varepsilon = 0.1$) were found at the MB level to be essentially fully condensed for any rotation frequency $\Omega$ smaller than the resonance frequency $\Omega_r$, means that the GP theory should be valid here and reproduce the MB conclusions.

**Analysis within the Gross-Pitaevskii approach.** We now want to turn from the MB to the GP (MF) description and address the same question, namely how the stability of the attractive gas is affected as the system is rotated externally. The GP theory assumes that all particles reside in the same single-particle state and hence the wave function for the state of the whole system is given by a single permanent $\Psi_{GP} = \prod_i^N \psi_{GP}(r_i)$. The GP orbital $\psi_{GP}$ for the ground state of the rotating gas should be represented with an ansatz that takes into consideration orbitals with nonzero angular momentum, as done in the MB treatment. To this end we expand $\psi_{GP}$ as a linear combination

$$\psi_{GP}(r, \sigma) = \sum_k b_k \psi_k(r, \sigma),$$

(5)

where the basis $\psi_k$ is the same as the one used in the MB computations reported above. The coefficients $b_k$ and parameter $\sigma$ (Gaussian width of the orbitals) are determined variationally with the normalization constraint $\sum_k |b_k|^2 = 1$ and the summation running over from $k = 0$ to $k = 3$. We calculate the expectation value $E = \langle \Psi_{GP} | \hat{H} | \Psi_{GP} \rangle$ with the above GP ansatz and minimize it with respect to the parameters $b_i, i = 1, 2, 3$ and $\sigma$, for different values of the interaction parameter $\lambda = |\lambda_0|(N - 1)$ and for given values of $\Omega \in [0, \sqrt{1 - \varepsilon})$ and the (small) trap anisotropy $\varepsilon$. The expectation value of the angular momentum operator for $\psi_{GP}$ is $l = \sum_{i,j} \mathcal{L}_{ij} b_i^* b_j$, $i, j = 1, \ldots, 4$, where the matrix elements $\mathcal{L}_{ij} = \langle \psi_i | \hat{L}_z | \psi_j \rangle$ are given, in second order approximation, as $\mathcal{L}_{ii} = l_{i-1} = \left(1 + \frac{\varepsilon^2}{8}\right) \cdot (i - 1)$, $\mathcal{L}_{13} = \mathcal{L}_{31} = \frac{\varepsilon}{2\sqrt{2}}$, $\mathcal{L}_{24} = \mathcal{L}_{42} = \frac{\varepsilon}{2\sqrt{2}}$, and the rest of the elements are zero. The total angular momentum is $L = NL$.

We calculate the critical value of the interaction $\lambda_c$ as a function of the rotation frequency
Ω, for the cases of \( \varepsilon = 0, \ \zeta = 1 \) and \( \varepsilon = 0.1, \ \zeta = 1 \). As anticipated from the MB analysis, we again found no essential change in the stability of the gas, as \( \Omega \) varies from 0 to \( \Omega_r = \sqrt{1 - \varepsilon} \). Namely, in the isotropic case, the GP ansatz of Eq. (5) yields the value \( \lambda_c = 8.425(9) \) which coincides with that obtained from the MB analysis, and remains fixed for any \( \Omega \in [0, \Omega_r) \).

In the case of anisotropic trap (\( \varepsilon = 0.1 \)) we found \( \lambda_c(0) = 8.436(3) \) and a negligible increase as \( \Omega \) increases, i.e., \( \lambda_c(\Omega_r) \simeq 1.0005 \cdot \lambda_c(0) \). We then fix the interaction parameter to \( \lambda = 3 \) as before. In the isotropic case, \( \varepsilon = 0 \), the energy \( \epsilon = 1.396(8) \) and angular momentum \( L/N = 0 \) remain constant for all \( \Omega \in [0, \Omega_r) \) and, as above, the values coincide with those of the MB ansatz. For \( \varepsilon = 0.1 \), we found \( \epsilon(0) = 1.395(8), \ \epsilon(\Omega_r) = 1.391(8), \ L(0)/N \simeq 0 \) and \( L(\Omega_r)/N = 0.045(2) \). Namely, the energies found in the MB and GP approaches are almost identical while the angular momentum computed within the GP theory at the resonance frequency \( \Omega_r \) is somewhat above the value that the MB theory gives. Nonetheless, both values of angular momentum can be considered practically zero. In conclusion, the rotation does not increase the stability of the ground state described by the GP ansatz of Eq. (5).

Last, we re-examine the attractive rotating gas using a different GP ansatz that has been previously used in the literature, namely the ansatz of Ref. [17] (see also references therein). The authors of Ref. [17] considered a GP ansatz for the ground state of the system, which they expressed – depending on the geometry of the confining potential – either as a Gaussian-sech single-particle wave function:

\[
\phi(r) = \left[ N(2l_x l_y l_z \pi)^{-1} \right]^{1/2} e^{-x^2/2l_x^2} e^{-y^2/2l_y^2} sech \left( \frac{z}{l_z} \right) e^{i\alpha xy} \tag{6}
\]

or as a Gaussian:

\[
\phi(r) = \left[ N(l_x l_y l_z)^{-1} \pi^{-3/2} \right]^{1/2} e^{-x^2/2l_x^2} e^{-y^2/2l_y^2} e^{-z^2/2l_z^2} e^{i\alpha xy}. \tag{7}
\]

The parameters \( l_x, l_y, l_z \) and \( \alpha \) were to be determined variationally. The phase \( \alpha xy \) put in Eqs. (6)-(7) is referred to as the ‘quadrupolar flow’ term; a nonzero value of \( \alpha \) increases the energy of the isotropic system in this state.

In Ref. [17] it is found that, using the ansatz of Eq. (6) or (7), the stability of the gas is significantly increased with increasing frequency \( \Omega \). However, an algebraic error in the above work is responsible for this (erroneous) behavior of the energy per particle \( \epsilon \) as a function of the rotation frequency \( \Omega \). Already in Eq. (4) of Ref. [17] there is a sign error; redoing carefully the calculations we convinced ourselves that in the true expression the sign
in the last term of the integrand is a plus instead of a minus. This sign error gives rise to an extra (negative) term in the GP energy functional which overestimates the dependence of the energy on $\Omega$ and artificially reduces the energy of the system (see the Appendix for more details).

In fact, also with the ansätze of Eqs. (6)-(7) the external rotation does not practically affect the stability of the condensate, in the sense that the critical value of the interaction parameter $\lambda_c$ does not essentially change with $\Omega$. We have verified this by calculating the energies and critical parameters $\lambda_c$ by varying all three parameters $l_x, l_y, l_z$ of Eqs. (6)-(7) for $\zeta = 0, 1, 5$ and $\varepsilon = 0, 0.1$, as it is originally done in Ref. [17] (the parameter $\alpha$ is expressed as a function of $l_x$ and $l_y$ and absorbed into the GP energy functional as in [17]). The Gaussian-sech ansatz of Eq. (6) is used in the case $\zeta = 0$, while the Gaussian ansatz of Eq. (7) is used when $\zeta = 1$ and 5. The critical $\lambda_c$ of the radially symmetric systems (i.e., $\varepsilon = 0$, $\zeta = 0, 1, 5$) remains fixed, while in the slightly anisotropic systems (i.e., $\varepsilon = 0.1$, $\zeta = 0, 1, 5$) $\lambda_c$ does not increase more than $0.2\%$ as $\Omega$ increases from 0 to $\Omega_r$. The computed values of $\lambda_c$ for different values of $\zeta$ and $\varepsilon$ are presented in Table I. We then fix $\lambda = 3$ and calculate the energies and angular momenta of the ground state. For $\varepsilon = 0$, irrespective of the choice of $\zeta$, the energy has been found to be independent of the rotation frequency $\Omega$. For instance, we found $\epsilon(\varepsilon = 0, \zeta = 1) = 1.396(8)$. For $\varepsilon = 0.1$ the energy $\epsilon(\Omega_r)$ was found to decrease by $1.5\%$ for $\zeta = 0$, by $0.32\%$ for $\zeta = 1$, and by $0.04\%$ for $\zeta = 5$ with respect to the corresponding energy $\epsilon(0)$ of the nonrotating system. Last, the expectation values of the angular momentum of the two ansätze are (almost) exactly zero for $\varepsilon = 0$ ($\varepsilon = 0.1$), regardless of the value of $\zeta$.

Finally, we point out that, for the cases examined, the optimal value of parameter $\alpha$ is practically zero. Note that the minimization of the GP energy functional with the ansatz of Eq. (5) yields a distribution for the coefficients $b_i$ ($b_0 \simeq 1$ and $b_i \simeq 0$, $i = 1, 2, 3$) which essentially includes only the first of the LLL Gaussian-shaped orbital. I.e., the ansätze of Eqs. (7) and (5) essentially coincide with the respective orbital of isotropic systems.

**Summary and Outlook.** We have studied the stability under rotation of attractive ultracold Bose gases, confined by an isotropic as well as a slightly anisotropic harmonic trap. The problem has been mapped to and calculated in the corotating frame. Both many-body and Gross-Pitaevskii approaches revealed that the rotation does not affect the stability of
the gas against the collapse. Namely, the maximum value of the interaction strength $\lambda_c$ where the attractive gas collapses remains essentially unchanged as the rotation frequency $\Omega$ varies within the extreme values 0 and $\Omega_r = \omega_x = \sqrt{1 - \varepsilon}$, where $\omega_x$ is the frequency of the trap in the direction of the weakest confinement.

We have found here on both the MB and the GP (MF) levels that the ground state of the rotating attractive system carries zero (or almost zero in the anisotropic case) angular momentum for the whole range of the allowed values of $\Omega$. Obviously, no vortex states are created. In the MB treatment, this means that no transition between LLL states of different angular symmetries has been found for the rotating attractive system. In the GP (MF) analysis this means that no symmetry broken states were found to be energetically favorable as the rotation frequency $\Omega$ increases from 0 to $\Omega_r$. In both MB and GP approaches, the energy of the ground state remains practically unchanged and the attractive gas is condensed in the nodeless s-orbital as the rotation frequency increases. Hence, the GP description agrees well with the MB computation. These results conflict the findings of Ref. [17].

We revisited then the problem using the ansatz that incorporates a ‘quadrupolar flow’ term, used in Ref. [17]. The energy and stability of the system were again found not to be affected by the rotation of the trap. The resolution of this discrepancy lies in a sign error in the expectation value of the Hamiltonian of Ref. [17], which leads to a qualitatively different behavior of the properties of the system as a function of the rotation frequency $\Omega$. Our results are in agreement with findings in the literature for isotropic harmonic (see [16]) and isotropic anharmonic traps with slight anharmonicity [14, 15].

We should, finally, stress that the present variational approach to the stationary ground state is not an extensive study of the rotating attractive gas. Even though we can rule out the stability-enhancement of the stationary ground state and the vortex nucleation in the attractive gas rotating with a frequency $\Omega$ smaller than the resonance frequency $\Omega_r$, there is more physics beyond that. For instance, the stability of low-lying excited states with nonzero total angular momentum is expected to depend on $\Omega$. Indeed, we have some numerical indication for such a dependence in a MB treatment of the problem involving ground states of $L > 0$.  

\[3\]
ground-state symmetry will not change as $\Omega$ increases. On the other hand, a ground state initially with nonzero $L$ will be affected by the rotation. The critical parameter $\lambda_c$ in that case could increase as a function of $\Omega$, before the latter reaches the extreme value $\Omega_r$, and this will further stabilize the rotating $L \neq 0$ state against collapse. It is still to be investigated whether and for which parameters’ values crossings of energy levels and symmetry changing of the (noncollapsed) rotating ground-state might occur. Last, based on the found absence of symmetry breaking of the ground state in the examined region $\Omega < \Omega_r$ and the divergence of the energy and angular momentum for $\Omega \geq \Omega_r$, we may speculate that, in the rotating attractive gas a vortex ground state— if at all can exist— may only appear as a giant-vortex (i.e., a single vortex at the center of the trap whose radius and vorticity are increasing functions of time) and this only for rotation with frequency $\Omega \geq \Omega_r$. A time-dependent many-body treatment, for instance using the multiconfigurational time-dependent Hartree for bosons (MCTDHB) method [27] that has described successfully many-body dynamics of attractive BECs [28], should shed light on this interesting problem and uncover the response mechanisms of attractive gases to rotations.

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Appendix: The Gross-Pitaevskii energy functional with the ‘Quadrupolar Flow’ Ansatz

We re-derive and discuss the expression for the energy functional of Ref. [17]. The GP energy functional in the co-rotating frame reads:

$$E = \int \left[ \frac{\hbar^2}{2m} |\nabla \phi|^2 + V(\mathbf{r})|\phi|^2 + \frac{\lambda_0}{2} |\phi|^4 + i\hbar \Omega (\phi^* \frac{\partial \phi}{\partial y} + \phi \frac{\partial \phi^*}{\partial x}) \right] d\mathbf{r},$$

(A.1)

where $\lambda_0$ measures the strength of the interaction, $m$ is the mass of the particle and $\Omega$ is the frequency of the rotation around the z-axis. In Ref. [17] there is an algebraic error in the above expression (Eq. (4) of Ref. [17]). There, the sign of the last term of the integrand
is a minus instead of a plus. This sign error remains further in the calculations of Ref. [17] and is seen in Eq. (8) and (10) therein. Indeed, in the last term of Eq. (10) of Ref. [17] the ‘-2’ term has to be omitted and so the corrected expression would read:

$$
\epsilon_G = \frac{1}{4} \left[ \frac{1}{\gamma_x^2} + \frac{1}{\gamma_y^2} + \frac{1}{\gamma_z^2} + (1 - \varepsilon)\gamma_x^2 + (1 + \varepsilon)\gamma_y^2 + \varepsilon^2\gamma_z^2 \right] - \frac{k}{\sqrt{2\pi}\gamma_x\gamma_y\gamma_z} - \frac{\Omega^2}{4} \left( \gamma_x^2 - \gamma_y^2 \right)^2
$$

(A.2)

(for the $\alpha > 0$ branch), where $\gamma_{x,y} = l_{x,y}/\sqrt{\hbar/m\omega}$ and $k = |\lambda_0|N/4\pi$. The same correction is required for Eq. (8) of Ref. [17] as well. The presence of this extra term gives rise to an artificial dependence of the critical value of $\lambda_c$ on the rotation frequency $\Omega$, qualitatively different from the correct one. Indeed, a first order expansion of the (correct) energy, Eq. (A.2), around $\gamma_x = \gamma_y$, i.e., for small deformations, will result in an expression of the energy that does not depend on the frequency $\Omega$. According to this, for zero or small ellipticity $\varepsilon$ of the trapping potential, the resulting shape of the orbital $\phi$ is symmetric around the z-axis, i.e., $\gamma_x = \gamma_y$, and the energy of the system, as well as the critical interaction strength, practically do not depend on the frequency $\Omega$. On the other hand, the (incorrect) energy $\epsilon_G$ as it is calculated in Ref. [17] strongly depends on $\Omega$. Furthermore, it can be easily seen that the ‘quadrupolar flow’ ansatz of either Eq. (6) or (7) gives, for small $\varepsilon$, an expectation value of almost zero angular momentum $\langle \hat{L}_z \rangle = \frac{1}{2} \left( \frac{l_x^2 - l_y^2}{l_x^2 + l_y^2} \right) mN \varepsilon \rightarrow 0 = 0$, and hence cannot describe any state with nonzero angular momentum that can in principal increase the stability of the system. For zero or small $\varepsilon$ the energy and the critical parameter $\lambda_c$ cannot change as a function of $\Omega$ and this reflects the cylindrical symmetry of the ansätze used, since $l_x \simeq l_y$ if $\varepsilon \simeq 0$. 
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FIG. 1: (Color online). Many-body calculations for the critical parameter $\lambda_c$ as a function of the frequency of the external rotation $\Omega$, for the cases of isotropic [$\varepsilon = 0$, $\zeta = 1$; lower (red) line] and slightly anisotropic [$\varepsilon = 0.1$, $\zeta = 1$; upper (blue) line] confining traps. The critical interaction $\lambda_c$ remains practically unaffected (note the scale!) for the whole region of $\Omega \in [0, \Omega_r = \sqrt{1 - \varepsilon})$. The number of particles is $N = 12$. See text for more details. All quantities are dimensionless.

| $\lambda_c$ | $\zeta=0$ | $\zeta=1$ | $\zeta=5$ |
|------------|----------|----------|----------|
| $\varepsilon=0$ | 9.547(7) | 8.425(9) | 5.522(8) |
| $\varepsilon=0.1$ | 9.554(9) | 8.429(0) | 5.523(2) |

TABLE I: Critical parameter $\lambda_c$ for different values of the anisotropy $\varepsilon$ and the z-deformation $\zeta$ of the trapping potential calculated in the GP theory, using the ‘quadrupolar flow’ ansätze, see Eqs. (6)-(7) and text below it. The parameter $\lambda_c$ in the radially symmetric ($\varepsilon = 0$) cases does not depend on the frequency $\Omega$ of the rotation, while the change in $\lambda_c$ as $\Omega$ varies from 0 (for which $\lambda_c$ values are collected in the table) to $\Omega_r = \sqrt{1 - \varepsilon}$ is negligible (less than 0.2%) when the trap is slightly anisotropic ($\varepsilon = 0.1$). All quantities are dimensionless.