The diameter vulnerability of two-dimensional optimal circulant networks

E A Monakhova and O G Monakhov
Institute of Computational Mathematics and Mathematical Geophysics SB RAS, Ac. Lavrentiev ave, 6, Novosibirsk, 630090, Russia
E-mail: monakhov@rav.sscc.ru

Abstract. This paper studies the effect of changing the diameter of the network in circulant networks of dimension two with unreliable elements (nodes, links). The well-known \((\Delta, D, D', s)\)-problem is to find \((\Delta, D)\)-graphs with maximum degree \(\Delta\) and diameter \(D\) such that the subgraphs obtained from the original graph by deleting any set of up to \(s\) vertices (edges) have diameter at most \(D'\). For a family of optimal circulants of degree four we found the ranges of the orders of the graphs that preserve the diameter of the graph for one (two) vertex or edge failures. It is proved that in the investigated circulant networks in case of failure of one or two edges (vertices), the diameter can increase by no more than one, and in case of failure of three elements by no more than two (edge failures) or three (vertex failures). It is shown that two-dimensional optimal circulants, in comparison with two-dimensional tori, have a better diameter in case of element failures.

1. Introduction
In conditions of impact on the communication network of destructive factors, overloads and deadlocks of different types, the use of reserve shortest paths providing the ability to bypass blocked sections of the network, and in this regard, the study of properties of networks to save the length of the maximum path (diameter) in the graph becomes important. This means that the subgraph obtained from the original graph after vertex or edges failures, still has a small diameter and good routings. The review [1] discusses reliability and fault-tolerant properties of networks, in particular those concerning diameter vulnerability in case of node or link failures. Diameter vulnerability issues were investigated for various classes of networks in many works, see [1]. When communicating in networks it is required that element failures not to significantly impact to performance of the entire system. For example, to realize such requirement as the existence of a path between two nodes preserved after network failures with a length not exceeding some fixed value. This problem is known as the \((\Delta, D, D', s)\)-problem – search for large graphs of maximum \(\Delta\) and minimum \(\delta\) of degrees and diameter \(D\) such that the subgraphs remaining after deleting any sets of vertices (edges) up to \(s\) vertices (edges), \(1 \leq s \leq \delta - 1\), have a diameter at most \(D'\) [1]. It is proved that the problem of determining the maximum possible diameter for the graph obtained from any connected graph \(G\) by deleting \(s\) edges, is NP-complete [2]. For general graphs of diameter \(D\) for \(s = 1\) \(D' \leq 2D\), the boundary is reached in odd cycles. In [2] it is proved that the deleting \(s > 1\) edges in connected graphs of diameter \(D \geq 2\) gives \(D' \leq 3D - 1\) for \(s = 2\), and \(D' \leq 4D - 2\) for \(s = 3\), and these boundaries are reached. In [3] it is shown that \(D' \leq D + 1\) for \(s = 1\) for \(n\)-dimensional hypercubes. On the investigation
of estimates of diameter vulnerability when deleting edges in other families of graphs, see for example [2, 4, 5]. It should also be noted that the author of [6] proposed a general approach to the classification of networks from the point of view of their robustness in the event of element failures, including the effect of failures on the metric characteristics of the network, and in [7] an example of another possible approach to calculating the reliability of a network with unreliable elements and a limitation on the diameter of the network is given. For results including the probabilities of network element failures see also recent survey [8].

The \((\Delta, D, D', s)\)-problem for circulant graphs has not yet been fully investigated. In this paper, we give an estimate for the diameter vulnerability in the case of one vertex or edge failure in undirected circulant graphs of any dimension, and investigate a solution to the \((\Delta, D, D', s)\)-problem with \(s \geq 1\) for circulant graphs of degree four. Undirected circulant graph \(C(N; s_1, \ldots, s_k)\) has the set of vertices \(V = \mathbb{Z}_N = \{0, 1, \ldots, N-1\}\) and the set of edges \(E = \{(v, v \pm s_i (\text{mod} \ N)) | v \in V, i = 1, k\}\). Here \(S = \{s_1, s_2, \ldots, s_k\}, 1 \leq s_1 < \ldots < s_k < N\), is the set of generators, \(N\) is the order of \(C, k\) is its dimension. Note that \(\Delta = \delta\) for the class of circulants. The diameter of \(C\) is \(D = \max_{u,v \in V} l(u, v)\), where \(l(u, v)\) is the length of the shortest path between vertices \(u\) and \(v\). The average distance of \(C(N; S)\) is \(\bar{D} = (1/N(N - 1)) \sum_{i,j} l(i, j)\).

Diameter and average distance estimate the maximum and average structural delays in the network, as well as connectivity and survivability of a network \([9, 10, 11]\). Research has shown that the best structures of multiprocessor systems according to various criteria functioning with an equal number of nodes and communication lines are structures with minimum \(D\) and \(\bar{D}\), the so-called optimal ones \([9, 12]\). This work is devoted to the study of the \((\Delta, D, D', s)\)-problem with \(s \geq 1\) for the family of two-dimensional optimal circulants \([9, 13]\). Should note that the questions of refusal of vertices in graphs of the family studied in \([14]\). But unlike \([14]\), where the authors used fixed shortest paths between the vertices of the graph and determined the diameter of the so-called surviving route graph on the example of one subset of the family, we use the whole set of shortest paths in graphs and present a complete solution of \((\Delta, D, D', s)\)-problem for the studied family in case of vertices and edges failures, defined up to the orders of graphs of the family.

2. Diameter estimation in \(k\)-dimensional circulants in the presence of one failure

For a vertex \(v, 0 < v < N\), of a circulant graph \(C(N; s_1, s_2, \ldots, s_k)\) of dimension \(k > 1\) we define from the zero vertex vector of shortest paths \(P(0, v) = (p_1(v), p_2(v), \ldots, p_k(v))\) as a vector having \(k\) coordinates, where \(|p_i(v)|, i = 1, k\), specifies the number of steps along the edges corresponding to the generator \(s_i\) \((-s_i)\) in the shortest path from 0 to \(v\), the + \((-)\) sign sets the movement along \(s_i\) \((-s_i)\), moreover, \(l(0, v) = \sum_{i=1}^{k} |p_i(v)|\). Since the circulant is a vertex-transitive graph, then \(l(0, v) = l(0, N - v)\), \(P(0, v) = -P(0, N - v)\).

As noted, in case of node or link failures in a network, it is important that there are redundant shortest paths in the network that provide the ability to bypass a failed element without increasing the path length. For some pairs of vertices \((u, v)\) in the circulant graph there may exist several alternative shortest path vectors, each of which defines its own subset of reserved shortest paths differing in different order passing the generators included in the shortest path from \(u\) to \(v\). Since circulant graphs are Cayley graphs of Abelian groups, then the number of reserve shortest paths from 0 to vertex \(v\) for each separate alternative path to \(v\) defines

**Lemma 1** Let \(C(N; s_1, \ldots, s_k)\), where \(k \geq 2\), be a circulant graph and let \(v, 0 < v < N\), be a vertex of \(C\) having a vector of shortest paths \(P(0, v) = (p_1(v), p_2(v), \ldots, p_k(v))\). Then the number of reserve shortest paths from 0 to \(v\) is equal to

\[
\frac{(|\sum_{i=1}^{k} |p_i(v)||)!}{\prod_{i=1}^{k} |p_i(v)||!}.
\]
Consider the operation of a communication network modeled by a circulant graph, under the presence of a single failure of a node (vertex) or communication line (edge). Further, everywhere under the deletion of the edge corresponding to generator $\pm s_i$, $i = 1, \ldots, k$, it is understood that there is no edge in the graph, corresponding to the transition from $v$ to $v \pm s_i (\text{mod} \ N)$. Let us show that in a circulant of any dimension $k \geq 2$ in the presence of one node (link) failure length of the shortest path between two nodes of the network can increase by a maximum of two, and this in turn determines maximum increase in the diameter of the circulant graph by one.

**Remark 1** By the vertex transitivity of circulants, if a vertex (edge) fails, we consider, without loss of generality, the vertex 0 as the root vertex and find the distances from 0 to all the remaining vertices of the graph. In view of the commutativity of circulants when deleting a vertex (edge) of the graph the distance from 0 to $v$ can increase only if between 0 and $v$ there is a single shortest path and the deleted vertex (edge) is included in this path. Lemma 1 implies that this path must contain generators of only one form $s_i$ (or $-s_i$) for any $i \in \{1, k\}$.

**Lemma 2** Let $k \geq 2$ and $C(N; s_1, \ldots, s_k)$ be a circulant graph. If an edge (vertex) is deleted in $C$, then the length of the shortest path between two vertices can increase by a maximum of two.

**Proof.** Consider a circulant graph $C(N; s_1, s_2, \ldots, s_k)$, where $k \geq 2$. Without loss of generality, let it be necessary to get from vertex 0 to vertex $v$ and, for definiteness, the shortest path from 0 to $v$ contains a fragment with two generators of the form $s_1$: $0 \rightarrow s_1 \rightarrow 2s_1 \rightarrow \ldots \rightarrow v$. The length of this fragment is two in the absence of failures. We consider two cases, depending on the fault type.

**Case 1.** Let the edge between 0 and $w = s_1$ be deleted. Commutability property of circulants allows us to construct $k - 1$ different paths of length four $0 \rightarrow s_1 \rightarrow s_1 + s_1 \rightarrow s_1 \rightarrow 2s_1$. Here, the generators $s_1$, $s_1$, $-s_1$, $s_1$ are used sequentially and $i \in \{2, \ldots, k\}$.

**Case 2.** Let the vertex $w = s_1$ be deleted. Similarly, we obtain $k - 1$ different paths of length four $0 \rightarrow s_1 \rightarrow s_1 + s_1 \rightarrow s_1 + 2s_1 \rightarrow 2s_1$. Here the generators $s_1$, $s_1$, $s_1$, $-s_i$ are used and $i \in \{2, \ldots, k\}$.

In both cases, the use of generators $-s_i$ gives other $k - 1$ different bypass paths. Thus, with a single failure of an edge (vertex) the total path length from 0 to $v$ can increase by a maximum of two. Q.E.D.

**Theorem 1** Let $C(N; s_1, \ldots, s_k)$, where $k \geq 2$, be a circulant graph of diameter $D > 1$, and let $D'$ be the diameter of the graph $C'$ obtained from $C$ by deleting a vertex (edge). Then $D \leq D' \leq D + 1$.

**Proof.** Let there be a circulant graph $C(N; s_1, \ldots, s_k)$, $k \geq 2$, of diameter $D > 1$ and a vertex (edge) $r$ is deleted from it. According to remark 1, we take vertex 0 as a reference point, $r \neq 0$, and determine the distances to those vertices $v \neq r$ of the graph, which have a single shortest path from 0. Lemma 1 implies that this path contains generators of only one type $s_i$ (or $-s_i$) for any $i \in \{1, k\}$ and, by Lemma 2, has length $D - 1$ or $D$. Let this path include the vertex (edge) $r$. In the case $v = (D - 1)s_i$, the path from 0 to $v$ bypassing $r$ by Lemma 2 gives a possible increase in the diameter of $C'$ up to $D + 1$.

Consider now the vertex with the number $w = Ds_i (\text{mod} \ N)$ (the case $w = -Ds_i (\text{mod} \ N)$ is considered similarly). Because we consider any circulant graph $C$, then two options are possible.

**Case 1.** The path from 0 to $w$ of length $D$, consisting of $D$ edges $s_i$, is not the shortest path in $C$. In this case, there is a shortest path from 0 to $w$ of length less than $D$, not passing through $r$. Then the diameter of $C'$ does not increase in comparison with $C$.

**Case 2.** The path from 0 to $w$ of length $D$, consisting of $D$ edges $s_i$, is the shortest path in $C$ and, by Lemma 2, unique. In this case, $w$ is adjacent to $2k - 1$ vertices located at a distance of $D$ from 0, for which the shortest path from 0 of length $D$ does not pass through $r$. Thus, an another path from 0 to $w$ has length at most $D + 1$ and, accordingly, the diameter of $C'$ can increase at most per unit. Q.E.D.
3. Family of two-dimensional optimal circulants

Two-dimensional circulant graphs (networks) are intensively studied in connection with various practical applications in graph theory, cryptography, as communication models of complex systems. Circulant graphs of dimension two are known as topologies of various multiprocessor systems, including promising networks-on-chip (NoCs) topologies [13, 15]. For the first time the author proved in [16] that for any number of vertices $N$ two-dimensional circulant graphs of the form (1) simultaneously have minima diameter and average distance, which is realized in NoCs based on analytically calculated the shortest paths to this family and its application. In [13], an efficient routing algorithm for the family of (2) was realized in NoCs based on analytically calculated the shortest paths vectors.

The value of $D^{*} = \lfloor (-1 + \sqrt{2N - 1})/2 \rfloor$ is found in [16, 17, 18], the value of $D^{*} = D^{*}(1 - (2(D^{*})^2 - 2)/3(N - 1))$ is defined in [17]. The works [9, 13] contain a complete list of references to this family and its application. In [13], an efficient routing algorithm for the family of circulants of the form (1) is realized in NoCs based on analytically calculated the shortest paths vectors.

Measures of resistance (vulnerability) of a graph $G$ to failures studied in literature, are also vertex-connectivity $\delta$ – the minimum cardinality of a set of vertices whose deletion from $G$ results in a disconnected graph, and edge-connectivity $\lambda$ – the minimum cardinality of a set of edges whose deletion from $G$ results in a disconnected graph [18, 19]. In [18] it is proved that circulant of the form (1), where $N > 6$, is optimal also with respect to vertex and edge connections, that is, $\delta = \lambda = 2k = 4$. Thus, the number $2k - 1 = 3$ corresponds to the maximum possible number of deleted edges (vertices), when the graph remains connected. This article solves the problem of diameter vulnerability in graphs of the family with any number of vertices after deleting $s = 1/2k - 1$ elements. Also, based on the analysis of the structure of family graphs, the ranges of variation of their orders are defined for which the upper and lower bounds for the change of diameter vulnerability in graphs of the family with any number of vertices after deleting $s = (N - 1)/2$ elements.

Further, for convenience, we will use the description (2) found in [16] of graphs of the family (1) with generators represented as functions of diameter $D > 1$ of the graph:

$$C(N; d, d + 1), \text{ where } d \text{ is the nearest integer to } (-1 + \sqrt{2N - 1})/2. \quad (1)$$

The value of $D^{*} = \lfloor (-1 + \sqrt{2N - 1})/2 \rfloor$ is found in [16, 17, 18], the value of $D^{*} = D^{*}(1 - (2(D^{*})^2 - 2)/3(N - 1))$ is defined in [17]. The works [9, 13] contain a complete list of references to this family and its application. In [13], an efficient routing algorithm for the family of circulants of the form (1) is realized in NoCs based on analytically calculated the shortest paths vectors.

Theorem 2 [16] For any integer $N \geq 5$, the optimal two-dimensional circulant of order $N$ is

$$C(N; d, d + 1), \text{ where } d \text{ is the nearest integer to } (-1 + \sqrt{2N - 1})/2. \quad (1)$$

The value of $D^{*} = \lfloor (-1 + \sqrt{2N - 1})/2 \rfloor$ is found in [16, 17, 18], the value of $D^{*} = D^{*}(1 - (2(D^{*})^2 - 2)/3(N - 1))$ is defined in [17]. The works [9, 13] contain a complete list of references to this family and its application. In [13], an efficient routing algorithm for the family of circulants of the form (1) is realized in NoCs based on analytically calculated the shortest paths vectors.

Measures of resistance (vulnerability) of a graph $G$ to failures studied in literature, are also vertex-connectivity $\delta$ – the minimum cardinality of a set of vertices whose deletion from $G$ results in a disconnected graph, and edge-connectivity $\lambda$ – the minimum cardinality of a set of edges whose deletion from $G$ results in a disconnected graph [18, 19]. In [18] it is proved that circulant of the form (1), where $N > 6$, is optimal also with respect to vertex and edge connections, that is, $\delta = \lambda = 2k = 4$. Thus, the number $2k - 1 = 3$ corresponds to the maximum possible number of deleted edges (vertices), when the graph remains connected. This article solves the problem of diameter vulnerability in graphs of the family with any number of vertices after deleting $s = 1/2k - 1$ elements. Also, based on the analysis of the structure of family graphs, the ranges of variation of their orders are defined for which the upper and lower bounds for the change of diameter vulnerability in graphs take place in case of failures. For a better understanding of the structure of graphs of the family, in [13], one can find their geometric model, which defines the dense packing (tessellation) of all graphs of the family on the plane $\mathbb{Z}^2$.

Further, for convenience, we will use the description (2) found in [16] of graphs of the family (1) with generators represented as functions of diameter $D > 1$ of the graph:

$$C(N; s_1, s_2) = \begin{cases} C(N; D - 1, D) & \text{if } N_{D-1} < N \leq 2D^2, \\ C(N; D, D + 1) & \text{if } 2D^2 < N \leq N_D. \end{cases} \quad (2)$$

Here $N_D = 2D^2 + 2D + 1$. For example, below there are all graphs of the family (2) with $D = 5$:

$$C(N; s_1, s_2) = \begin{cases} C(N; 4, 5) & \text{for } 42 \leq N \leq 50, \\ C(N; 5, 6) & \text{for } 51 \leq N \leq 61. \end{cases}$$

4. Solving $(\Delta, D, D', 1)$-problem

The diameter vulnerability of graphs of the family (2) was studied in [17] with the following output:

Theorem 3 ([17], theorem 6) The diameter of the graph obtained by deleting one vertex or one edge in $C(N; D^{*}, D^{*} + 1)$ is at most $D^{*} + 1$.

Let us extend the result of Theorem 3. To do this, we find the ranges of orders of graphs of the family (2) for which the diameter of the graph is conserved or increases by one if one vertex or edge fails. The following remark is general when considering one vertex (edge) failure in graphs of the family (2).
Remark 2 In case of a vertex (edge) failure in graphs of the family (2) the distance from 0 to \( v \), according to remark 1, can increase if the shortest path from 0 to \( v \) contains generators of only one from four possible forms \( \pm s_i, \ i = 1, 2 \), and by Lemma 2 has length \( D - 1 \) or \( D \).

Therefore, to determine the diameter of the graph obtained by deleting a vertex \( r \) (edge \( e \)), it is necessary and sufficient to define the lengths of the shortest paths from 0 to \( v = (D - 1)s_i \) and \( w = Ds_i, \ i = 1, 2 \). Due to the symmetry of circulants, it is not required to consider the cases of deleting edges \( e = -s_i \) or vertices \( N - r \).

4.1. Deleting one vertex
Consider the case of deletion of any vertex in a circulant of the form (2).

Theorem 4 Let \( C \) be a circulant graph of the form (2) of diameter \( D > 2 \), and let \( r \) be a vertex of \( C \). Then the maximum diameter of the graph \( C' \) obtained from \( C \) by deleting \( r \) is

\[
D'_1 = \begin{cases} D & \text{if } N_{D-1} < N \leq 2D^2, \\ D + 1 & \text{if } 2D^2 < N \leq N_D. \end{cases}
\]

Proof. According to remark 2, it suffices to define lengths of the shortest paths from 0 to vertices \( v = (D - 1)s_i \) and \( w = Ds_i \) that do not pass through vertex \( r = ms_i, \ i = 1, 2 \).

Case 1. Let \( N_{D-1} < N \leq 2D^2 \), then \( s_1 = D - 1, s_2 = D \).

We consider the cases of \( i = 1 \) and \( i = 2 \) separately.

Case 1.1. Deleting vertex \( r = ms_1, \ m = 1, D - 2 \).

There are paths from 0 to \( v = (D - 1)s_1 \) and to \( w = Ds_1 \) of length at most \( D \), not passing through \( r \), namely:

\[
0 \to s_2 \to 2s_2 \to ... \to (D - 1)s_2 \to (D - 1)s_2 - s_1 = v,
\]

\[
0 \to s_2 \to 2s_2 \to ... \to (D - 1)s_2 = w.
\]

Thus, \( D'_1 = D \). The same paths can be written in an abbreviated form:

\[
0 + (D - 1)s_2 - s_1 = v,
\]

\[
0 + (D - 1)s_2 = w.
\]

We will use such a representation of the paths below, with one of the possible paths to a vertex up to the order of passage of different generators.

Case 1.2. Deleting vertex \( r = ms_2, \ m = 1, D - 2 \).

In this case, we need to consider both vertices \( v = (D - 1)s_2 \) and \( w = Ds_2 \). There is a path from 0 to \( v \) of length \( D \) that does not go through \( r \):

\[
0 + Ds_1 = v.
\]

Let us show that there is a path from 0 to \( w \) that does not pass through \( r \), of length not more than \( D \).

We set \( N = N_{D-1} + i \), where \( i = \overline{1,D - 1} \), for \( N_{D-1} < N < N_{D-1} + D \). Then the required path is:

\[
N - (D - 1 - i)s_1 - is_2 = w.
\]

We set \( N = 2D^2 - D + i \), where \( i = \overline{1,D} \), for \( N_{D-1} + D \leq N \leq 2D^2 \). Then the required path is:

\[
N - (D - i)s_1 - is_2 = w.
\]

Thus, \( D'_1 = D \).

Case 2. Let \( 2D^2 < N \leq 2D^2 + D + 1 \), then \( s_1 = D, s_2 = D + 1 \).
It is enough to consider the deletion of vertices $r = ms_1$, $m = \overline{1, D-2}$. Set $N = 2D^2 + i$, where $i = \overline{1, D+1}$. All shortest paths from 0 to vertex $v = (D-1)s_1$ have length $D + 1$:

$$N - (D + 1 - i)s_1 - is_2 = v,$$

$$0 + s_2 + (D - 1)s_1 - s_2 = v.$$

By Theorem 1, all other vertices of the graph also have paths from 0 that do not pass through $r$ and have length at most $D + 1$. Thus, $D_1' = D + 1$.

**Case 3.** Let $2D^2 + D < N \leq N_D$, then $s_1 = D$, $s_2 = D + 1$.

To prove the theorem, it suffices to consider the deleting vertices $r = ms_1$, $m = \overline{1, D-2}$. Set $N = N_D - i$, where $i = \overline{0, D}$. The length of the shortest paths from 0 to $w = Ds_1$ that do not pass through $r$ is $D + 1$:

$$N - is_1 - (D + 1 - i)s_2 = w,$$

$$0 + Ds_2 - s_1 = w.$$

By Theorem 1, all other vertices of the graph also have paths from 0 that do not pass through $r$ and have length at most $D + 1$. Thus, $D_1' = D + 1$. Q.E.D.

For two-dimensional optimal graphs of diameter two, values of $D_1'$ are obtained by an exhaustive enumeration of possible failures of one vertex.

**4.2. Deleting one edge**

We consider the deletion of two types of edges, corresponding to generators $s_1$ and $s_2$.

**Lemma 3** Let $C$ be a circulant graph of the form (2) of diameter $D > 1$, and let $e$ be an edge of $C$, corresponding to generator $s_1$. Then the maximum diameter of the graph $C'$ obtained from $C$ by deleting $e$ is

$$D_1' = \begin{cases} D & \text{if } N_{D-1} < N \leq 2D^2, \\ D + 1 & \text{if } 2D^2 < N \leq N_D. \end{cases}$$

**Proof.** In accordance with remark 2, it suffices to consider the vertices $v = (D-1)s_1$ and $w = Ds_1$. The proof of the lemma corresponds to the proof given in Sections 1.1, Sections 2, and 3 of Theorem 4, with the change of vertices $r$ to the edge $e$ connecting the vertices $r_m$ and $r_{m+1} = r_m + s_1$, where $r_0 = 0$, $m = \overline{0, D-2}$. Q.E.D.

The following lemma is proved similarly.

**Lemma 4** Let $C$ be a circulant graph of the form (2) of diameter $D > 1$, and let $e$ be an edge of $C$ corresponding to generator $s_2$. Then the maximum diameter of the graph $C'$ obtained from $C$ by deleting $e$ is

$$D_1' = \begin{cases} D & \text{if } N_{D-1} < N < 2D^2 + D, \\ D + 1 & \text{if } 2D^2 + D \leq N \leq N_D. \end{cases}$$

As you can see, the length of the ranges of the graphs orders, for which the diameter of the graph does not change when an edge fails, depends on type of deleted edges ($s_1$ or $s_2$). If we take the minimum value of order of the graph for which the failure of an edge does not increase the diameter, then according to Lemmas 3 and 4 and Theorem 1, we have:

**Theorem 5** Let $C$ be a circulant graph of the form (2) of diameter $D > 1$, and let $e$ be any edge of $C$. Then the maximum diameter of the graph $C'$ obtained from $C$ by deleting $e$ is

$$D_1' = \begin{cases} D & \text{if } N_{D-1} < N \leq 2D^2, \\ D + 1 & \text{if } 2D^2 < N \leq N_D. \end{cases}$$
5. Solution of \((\Delta, D, D', s)\)-problem for \(s = 2\) and \(s = 3\)

When investigating the diameter vulnerability of graphs of the family (2) in case of failures of two or three vertices (edges), the following results were obtained.

5.1. Deleting two elements

Consider the diameter vulnerability in circulant graphs of the family (2) when two vertices (two edges) are deleted.

**Theorem 6** Let \(C\) be a circulant graph of the form (2) of diameter \(D > 4\) and \(N \neq 42\). Then the maximum diameter of the graph \(C'\) obtained from \(C\) by deleting two vertices is

\[
D'_2 = \begin{cases} 
D & \text{if } N = N_{D-1} + 1, \\
D+1 & \text{if } N_{D-1} + 1 < N \leq N_D.
\end{cases}
\]

**Theorem 7** Let \(C\) be a circulant graph of the form (2) of diameter \(D > 1\). Then the maximum diameter of the graph \(C'\) obtained from \(C\) by deleting two edges is

\[
D'_2 = \begin{cases} 
D & \text{if } N = N_{D-1} + 1, \\
D+1 & \text{if } N_{D-1} + 1 < N \leq N_D.
\end{cases}
\]

We present schemes of proofs of Theorems 6 and 7 due to limitations on the size of the article. Taking into account Remark 2 and the results of Theorems 4 and 5, we have identified configurations of two failed elements that cause maximum damage to the network diameter. For such configurations, it is shown that in the range of variation of the number of vertices of graphs of the family (2) from \(N_{D-1} + 1\) to \(N_D\), there is a single value of \(N = N_{D-1} + 1\), at which the network diameter is preserved for any configurations of two failures. For all other values of \(N\) from the given range, there is at least one vertex in every graph of the family (2) for which the shortest distance to vertex 0 for the considered failures is \(D + 1\).

5.2. Deleting three elements

Consider the diameter vulnerability in circulant graphs of the family (2) after deleting three edges (three vertices). The following property holds.

**Theorem 8** Let \(C\) be a circulant graph of the form (2) of diameter \(D > 2\). Then the maximum diameter of the graph \(C'\) obtained from \(C\) by deleting three edges is

\[
D'_3 = \begin{cases} 
D + 1 & \text{if } N_{D-1} < N \leq 2D^2, \\
D + 2 & \text{if } 2D^2 < N \leq N_D.
\end{cases}
\]

**Theorem 9** Let \(C\) be a circulant graph of the form (2) of diameter \(D > 4\). Then the maximum diameter of the graph \(C'\) obtained from \(C\) by deleting three vertices is

\[
D'_3 = \begin{cases} 
D + 2 & \text{if } N_{D-1} < N \leq 2D^2 + 1, \\
D + 3 & \text{if } 2D^2 + 1 < N \leq N_D.
\end{cases}
\]

The proofs of Theorems 8 and 9 are based on the results of Theorems 6 and 7. We present schemes of proofs of these theorems. For Theorem 8 we have identified some types of configurations of three failed edges that cause maximum increase of the network diameter. For each type of configurations, it is shown that in the range of variation of the number of vertices of graphs of the family (2) from \(N_{D-1} + 1\) to \(2D^2\), the network diameter is increased by one, and for other values of \(N\) from the given range the network diameter is increased by two. The proof of Theorem 9 is similar to the proof of Theorem 8, but in this case we have identified some types of configurations of three failed vertices that cause maximum increase of the network diameter.
Note that for all graphs of the family (2) of small diameters from 2 to 5, which are not reflected in Theorems 6-9, we found the exact values of $D'_2$ and $D'_3$ using a complete enumeration of the set of possible failures of two (three) elements. The worst graphs in terms of diameter stability among them turned out to be graphs with orders $11 \leq N \leq 13$ of diameter two, which gave an increment of the diameter in the case of failures of three edges equal to $2D$.

6. Comparison of the diameter vulnerability of circulants and tori

Two-dimensional torus (2D-torus) is a popular topology used for multiprocessor systems and NoCs [20], and also, like circulants, refer to Cayley graphs of Abelian groups. Optimal 2D circulants and 2D tori were compared for the diameter and average distance (see, for example, [21]) and the advantage of circulants was shown. In this paper, an additional comparison is made of 2D tori and graphs of the family of optimal circulants in terms of diameter vulnerability in case of vertex and edge failures. Some of the results obtained are shown in figure 1 and figure 2. Estimates of diameter vulnerability for two-dimensional tori were obtained by us using modeling in the Wolfram Mathematica 10. The given plots reflect the results of comparing two-dimensional circulants and tori by diameters of the graphs obtained after deleting $s = 1$ and $s = 3$ vertices in the initial graphs. In figure 1 and figure 2 designations $1fn$ and $3fn$ mean the number of fault nodes. Graphs with orders $N \leq 400$ were compared. For tori and circulants, the values of $N = m^2$ are used, since tori exist only for values of $N = m \times n$, in contrast to circulants, which exist for any $N$. The comparison results show that even in case of refusals $s$, $1 \leq s \leq 3$, vertices (edges) in the considered classes of graphs, circulants still have a smaller diameter than tori with the same number of vertices and degrees. Thus, the reliability and performance of circulants during transmission data at failures (deadlocks, overloads) are higher than those of tori.

![Figure 1. Diameter vulnerability of tori and circulants for $s = 1$](image1.jpg)

![Figure 2. Diameter vulnerability of tori and circulants for $s = 3$](image2.jpg)

7. Conclusion

The estimates of the diameter vulnerability in case of node and link failures were obtained for the family of optimal (in terms of diameter and average distance) two-dimensional circulant networks of any order. It is proved that in the investigated circulant networks for failure of one or two edges (vertices), the diameter can increase no more, than one $1$, and in case of failure of three elements – no more than two (edge failures) or three (vertex failures). Based on the analysis of the structure of graphs of the considered family, the ranges of their orders were found

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1 except some graphs of diameter three with two vertex failures
at which the maxima and minima of the found estimates are reached. Comparison of optimal
two-dimensional circulants with 2D tori showed the best values of the diameter of circulant
networks in case of node and communication link failures and, thus, their higher suitability for
modeling reliable communication networks. Further research is related to the study of diameter
vulnerability of circulants of large dimensions \(k > 2\) by the example of extremal families of
undirected circulant networks known in the literature.

**Acknowledgments**
The work has been carried out under the budget project of ICMMG SB RAS N 0251-2021-0005.

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