Indefinite boundary value problems on graphs *

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Abstract

We consider the spectral structure of indefinite second order boundary-value problems on graphs. A variational formulation for such boundary-value problems on graphs is given and we obtain both full and half-range completeness results. This leads to a max-min principle and as a consequence we can formulate an analogue of Dirichlet-Neumann bracketing and this in turn gives rise to asymptotic approximations for the eigenvalues.

1 Introduction

Let G be an oriented graph with finitely many edges, say K, each of unit length, having the path-length metric. Suppose that n of the edges have positive weight, 1, and K − n of

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the edges have negative weight, $-1$. We consider the second-order differential equation

$$ly := -\frac{d^2y}{dx^2} + q(x)y = \lambda By,$$  (1.1)

on $G$, where $q$ is real valued and essentially bounded on $G$ and $By(x) = b(x)y(x)$ with

$$b(x) := \begin{cases} 
1, & \text{for } x \text{ on edges with positive weight.} \\
-1, & \text{for } x \text{ on edges with negative weight.}
\end{cases}$$

At the vertices or nodes of $G$ we impose formally self-adjoint boundary conditions, see [6] for more details regarding the self-adjointness of boundary conditions.

A variational formulation for a class of indefinite self-adjoint boundary-value problems on graphs is given, see [4] and [9] for background on Sturm-Liouville problems with indefinite weight, and [5] concerning variational principles in Krein spaces. We then study the nature of the spectrum of this variational problem and obtain both full and half-range completeness results. A max-min principle for indefinite Sturm-Liouville boundary-value problems on directed graphs is then proved which enables us to develop an analogue of Dirichlet-Neumann bracketing for the eigenvalues of the boundary-value problem and consequently to obtain eigenvalue asymptotics.

In parallel to the variational aspects of boundary-value problems on graphs studied here and on trees in [21], the work of Pokornyi and Pryadiev, and Pokornyi, Pryadiev and Al-Obeid, in [17] and [18], should be noted for the extension of Sturmian oscillation theory to second order operators on graphs. The idea of approximating the behaviour of eigenfunctions and eigenvalues for a boundary-value problem on a graph by the behaviour of associated problems on the individual edges, used here, was studied in the definite case in [2], [11] and [22].

An extensive survey of the physical systems giving rise to boundary-value problems on graphs can be found in [15] and the bibliography thereof. Second order boundary-value problems on finite graphs arise naturally in quantum mechanics and circuit theory, [3] [12]. Multi-point boundary-value problems and periodic boundary-value problems can be considered as particular cases of boundary-value problems on graphs, [7].

In Section 2 the boundary-value problem, which forms the topic of this paper, is stated and allowable boundary conditions discussed. An operator formulation is given along with definitions of the various function spaces used. A variational reformulation of the boundary-value problem together with the definition of co-normal (elliptic) boundary conditions is given in Section 3. Here we also show that a function is a variational eigenfunction if and only if it is a classical eigenfunction. In Section 4 we study the spectrum of the variational problem. The main result of this section is that an eigenfunction is in the positive cone, with respect to the $B$ (indefinite inner product), if and only if the corresponding eigenvalue is positive and similarly for the negative cone. Following the approach used by Beals in [4] we prove both full and half-range completeness in Section 5, see Theorem 5.3 and Theorem 5.5. In Section 6 a max-min characterization of the
eigenvalues of the boundary value problem is given which is then used in Section 7 to obtain a variant of Dirichlet-Neumann bracketing of the eigenvalues. Hence eigenvalue asymptotics are found. Dirichlet-Neumann bracketing for elliptic partial differential equations can be found in [8].

2 Preliminaries

Denote the edges of the graph $G$ by $e_i$ for $i = 1, \ldots, K$. As $e_i$ has length 1, $e_i$ can be considered as the interval $[0, 1]$, where 0 is identified with the initial point of $e_i$ and 1 with the terminal point.

We recall, from [11], the following classes of function spaces:

\[
L^2(G) := \bigoplus_{i=1}^{K} L^2(0, 1),
\]

\[
H^m(G) := \bigoplus_{i=1}^{K} H^m(0, 1), \quad m = 0, 1, 2, \ldots,
\]

\[
H^m_0(G) := \bigoplus_{i=1}^{K} H^m_0(0, 1), \quad m = 0, 1, 2, \ldots,
\]

\[
C^\omega(G) := \bigoplus_{i=1}^{K} C^\omega(0, 1), \quad \omega = \infty, 0, 1, 2, \ldots,
\]

\[
C^\omega_0(G) := \bigoplus_{i=1}^{K} C^\omega_0(0, 1), \quad \omega = \infty, 0, 1, 2, \ldots.
\]

The inner product on $H^m(G)$ and $H^m_0(G)$, denoted $(\cdot, \cdot)_m$, is defined by

\[
(f, g)_m := \sum_{i=1}^{K} \sum_{j=0}^{m} \int_0^1 f|^{(j)}_{e_i} \overline{g|^{(j)}_{e_i}} \, dt =: \sum_{j=0}^{m} \int_G f|^{(j)} \overline{g|^{(j)}} \, dt.
\] (2.2)

Note that $L^2(G) = H^0(G) = H^0_0(G)$. For brevity we will write $(\cdot, \cdot) = (\cdot, \cdot)_0$, $\|f\|_m = (f, f)_m$ and $\|f\| = \|f\|_0$.

The differential equation [11] on the graph $G$ can be considered as the system of equations

\[
- \frac{d^2 y_i}{dx^2} + q_i(x)y_i = \lambda b_i(x)y_i, \quad x \in [0, 1], \quad i = 1, \ldots, K,
\] (2.3)

where $q_i$, $b_i$ and $y_i$ denote $q|_{e_i}$, $b|_{e_i}$ and $y|_{e_i}$.

As in [11], the boundary conditions at the node $\nu$ are specified in terms of the values of $y$ and $y'$ at $\nu$ on each of the incident edges. In particular, if the edges which start at
\[ \sum_{j \in \Lambda_s(\nu)} [\alpha_{ij}y_j + \beta_{ij}y_j'](0) + \sum_{j \in \Lambda_e(\nu)} [\gamma_{ij}y_j + \delta_{ij}y_j'](1) = 0, \quad i = 1, \ldots, N(\nu), \quad (2.4) \]

where \( N(\nu) \) is the number of linearly independent boundary conditions at node \( \nu \). For formally self-adjoint boundary conditions \( N(\nu) = \#(\Lambda_s(\nu)) + \#(\Lambda_e(\nu)) \) and \( \sum_{\nu} N(\nu) = 2K \), see [6, 16] for more details.

Let \( \alpha_{ij} = 0 = \beta_{ij} \) for \( i = 1, \ldots, N(\nu) \) and \( j \not\in \Lambda_s(\nu) \) and similarly let \( \gamma_{ij} = 0 = \delta_{ij} \) for \( i = 1, \ldots, N(\nu) \) and \( j \not\in \Lambda_e(\nu) \). The boundary conditions (2.4) considered over all nodes \( \nu \), after possible relabelling, may thus be written as

\[ \sum_{j=1}^{K} [\alpha_{ij}y_j(0) + \gamma_{ij}y_j(1)] = 0, \quad i = 1, \ldots, J, \quad (2.5) \]

\[ \sum_{j=1}^{K} [\alpha_{ij}y_j(0) + \beta_{ij}y_j'(0) + \gamma_{ij}y_j(1) + \delta_{ij}y_j'(1)] = 0, \quad i = J + 1, \ldots, 2K, \quad (2.6) \]

where all possible Dirichlet-like terms are in (2.5), i.e. if (2.6) is written in matrix form then Gauss-Jordan reduction will not allow any pure Dirichlet conditions linearly independent of (2.5) to be extracted.

The boundary-value problem (2.3)-(2.4) on \( G \) can be formulated as an operator eigenvalue problem in \( L^2(G) \), [1, 6, 20], for the closed densely defined operator \( BL \), where

\[ Lf := -f'' + qf \quad (2.7) \]

with domain

\[ \mathcal{D}(L) = \{ f \mid f, f' \in AC, Lf \in L^2(G), \ f \text{ obeying (2.4)} \}. \quad (2.8) \]

The formal self-adjointness of (2.4) relative to \( L \) ensures that \( L \) is a closed densely defined self-adjoint operator in \( L^2(G) \), see [13, 16, 23], and that \( BL \) is self-adjoint in \( H_K \) where \( H_K \) is \( L^2(G) \) with indefinite inner product \[ [f, g] = (Bf, g) \].

From [11] we have that the operator \( L \) is lower semibounded in \( L^2(G) \).

### 3 Variational Formulation

In this section we give a, variational formulation for the boundary-value problem (2.3)-(2.4) or equivalently for the eigenvalue problem associated with the operator \( BL \).
Definition 3.1 (a) Let \( \mathcal{D}(F) = \{ y \in \mathcal{H}^1(G) \mid y \text{ obeys (2.5)} \} \), where

\[
\int_{\partial G} y \, d\sigma := \sum_{i=1}^{K} [y_i(1) - y_i(0)] = \int_{G} y' \, dt.
\]

(b) We say that the boundary conditions on a graph are co-normal or elliptic with respect to \( l \) if there exists \( f \) defined on \( \partial G \), such that \( x \in \mathcal{D}(F) \) has

\[
\int_{\partial G} (fx + x') \eta \, d\sigma = 0, \quad \text{for all} \quad y \in \mathcal{D}(F)
\]

if and only if \( x \) obeys (2.6).

(c) If the boundary conditions are co-normal and \( f \) is as in (b) and \( \mathcal{D}(F) \) is as in (a), then we define the sesquilinear form \( F(x, y) \) for \( x, y \in \mathcal{D}(F) \) by

\[
F(x, y) := \int_{\partial G} f x \eta \, d\sigma + \int_{G} (x' \eta' + xq \eta) \, dt. \quad (3.9)
\]

We note that ‘Kirchhoff’, Dirichlet, Neumann and periodic boundary conditions are all co-normal, but this class does not include all self-adjoint boundary-value problems on graphs.

The following lemma shows that a function is a variational eigenfunction if and only if it is a classical eigenfunction.

Lemma 3.2 Suppose that (2.5)-(2.6) are co-normal boundary conditions with respect to \( l \) of (1.1). Then \( u \in \mathcal{D}(F) \) satisfies \( F(u, v) = \lambda(Bu, v) \) for all \( v \in \mathcal{D}(F) \) if and only if \( u \in \mathcal{H}^2(G) \) and \( u \) obeys (1.1), (2.5)-(2.6).

Proof: Assume that \( u \in \mathcal{H}^2(G) \) and \( u \) obeys (1.1), (2.5)-(2.6). Then for each \( v \in \mathcal{D}(F) \)

\[
F(u, v) = \int_{\partial G} f u \bar{v} \, d\sigma + \int_{G} (u' \bar{v'} + qu \bar{v}) \, dt
\]

\[
= \int_{\partial G} f u \bar{v} \, d\sigma + \int_{G} ((u')' - u'' \bar{v} + qu \bar{v}) \, dt
\]

\[
= \int_{\partial G} f u \bar{v} \, d\sigma + \int_{G} (u' \eta') \, dt + \lambda(Bu, v)
\]

\[
= \int_{\partial G} (f u + u') \eta \, d\sigma + \lambda(Bu, v).
\]

The assumption that (2.5)-(2.6) are co-normal boundary conditions with respect to \( l \) gives that \( u \in \mathcal{D}(F) \) and

\[
\int_{\partial G} (f u + u') \eta \, d\sigma = 0, \quad \text{for all} \quad v \in \mathcal{D}(F),
\]

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completing the proof this in case.

Now assume \( u \in \mathcal{D}(F) \) satisfies \( F(u, v) = \lambda(Bu, v) \) for all \( v \in \mathcal{D}(F) \). As \( C_0^\infty(G) \subset \mathcal{D}(F) \), it follows that
\[
F(u, v) = \lambda(Bu, v), \quad \text{for all } v \in C_0^\infty(G).
\]
Hence \( F(u, \cdot) \) can be extended to a continuous linear functional on \( L^2(G) \). In particular, since \( q \in L^\infty(G) \), this gives that
\[
\partial u' \in L^2(G) \subset L^1_{loc}(G)
\]
where \( \partial \) denotes the distributional derivative. Then, by [20, Theorem 1.6, page 44], \( u' \in AC \) and \( u'' \in L^1_{loc}(G) \) allowing integration by parts. Thus
\[
l u = -u'' + qu \in L^1_{loc}(G)
\]
and consequently \( l u = \lambda Bu \in L^2(G) \). Now \( q \in L^\infty(G) \) and \( \mathcal{D}(F) \subset L^2(G) \), giving \( u, u'' \in L^2(G) \) and hence \( u \in \mathcal{H}^2(G) \).

The definition of \( \mathcal{D}(F) \) ensures that (2.5) holds. Integration by parts gives
\[
\int_{\partial G} (fu + u') \overline{y} \, d\sigma = 0, \quad \text{for all } y \in \mathcal{D}(F),
\]
which, from the definition of \( f \) and the constraints on the class of boundary conditions, is equivalent to \( u \) obeying (2.6).

4 Nature of the spectrum

The operator \( L \) is self-adjoint in \( L^2(G) \) with spectrum consisting of pure point spectrum and accumulating only at \(+\infty\). In addition, we assume that \( L \) is positive definite, thus the spectrum of \( L \) may be denoted \( 0 < \rho_1 \leq \rho_2 \leq \ldots \) where \( \lim_{n \to \infty} \rho_n = \infty \). Since \( L \) is positive definite and the spectrum consists only of point spectrum, \( L^{-1} \) exists and is a compact operator see, [10, p.24], moreover
\[
L^{-1}y(t) = \int_G g(t, \tau)y(\tau) \, d\tau, \quad (4.10)
\]
where \( g(t, \tau) \) is the Green’s function of \( L \). Thus \( L^{-1}B \) is a compact operator. Consider the eigenvalue problem
\[
\mu y = L^{-1}By, \quad y \in L^2(G),
\]
where \( \mu = \frac{1}{\lambda} \). Since \( L^{-1}B \) is compact it has only discrete spectrum except possibly at \( \mu = 0 \) and the only possible accumulation point is \( \mu = 0 \). In addition, \( \mu = 0 \) is not an eigenvalue of \( L^{-1}B \) since \( 0 \) is not an eigenvalue of \( L^{-1} \). Thus \( L^{-1}B \) has countably
ininitely many eigenvalues, all non-zero, but accumulating at 0. From \((4.10)\) it follows that
\[
L^{-1}B y(t) = \int_G g(t, \tau) B y(\tau) d\tau = \int_G \tilde{g}(t, \tau) y(\tau) d\tau,
\]
where \(\tilde{g}(t, \tau) = g(t, \tau)b(\tau)\). Hence \(BL\) has discrete spectrum only, with possible accumulation point at \(\infty\) in the complex plane. The spectrum is also countably infinite and, as 0 is not an eigenvalue of \(L\), 0 is also not an eigenvalue of \(BL\).

**Lemma 4.1** The space \(\mathcal{D}(F)\) is a Hilbert space with inner product \(F\). The norm generated by \(F\) on \(\mathcal{D}(F)\) is equivalent to the \(H^1(G)\) norm, making \(\mathcal{D}(F)\) a closed subspace of \(H^1(G)\).

**Proof:** By \((3.9)\), \([1, \text{ Preliminaries}]\) and the trace theorem, see \([1, \text{ p. 38}]\) we have that there exist constants \(K, c > 1\) such that
\[
\frac{1}{c} ||x||^2_{H^1(G)} \leq F(x, x) + K ||x||^2 \leq c ||x||^2_{H^1(G)}.
\]

Thus the sesquilinear form \(F(x, y) + K(x, y)\) is an inner product on \(\mathcal{D}(F)\). From \((4.11)\) we get directly that
\[
\frac{1}{c} F(x, x) + K ||x||^2 \leq ||x||^2_{H^1(G)} = c (F(x, x) + K ||x||^2),
\]
making \(F(x, y) + K(x, y)\) and \((x, y)_{H^1(G)}\) equivalent inner products on \(\mathcal{D}(F)\).

We now show that \(F(x, y)\) is an inner product on \(\mathcal{D}(F)\) and is equivalent to the inner product \(F(x, y) + K(x, y)\) on \(\mathcal{D}(F)\). As \(\rho_1\) is the least eigenvalue of \(L\) on \(L^2(G)\),
\[
(Ly, y) = \rho_1(y, y) = \rho_1 ||y||^2,
\]
for all \(y \in \mathcal{D}(L) \subset \mathcal{D}(F)\). Since \(F(y, y) = (Ly, y)\), for all \(y \in \mathcal{D}(L)\), we get
\[
F(y, y) \geq \rho_1 ||y||^2,
\]
for \(y \in \mathcal{D}(L)\).

Now, \(\mathcal{D}(L)\) is dense in \(\mathcal{D}(F)\) for \(\mathcal{D}(F)\) with norm \(||x||^2 := F(x, x) + K(x, x)\). Thus, by continuity,
\[
||y||^2_F := F(y, y) \geq \rho_1 ||y||^2,
\]
for all \(y \in \mathcal{D}(F)\), showing that \(|| \cdot ||_F\) is a norm on \(\mathcal{D}(F)\) and that \(F(x, y)\) is an inner product on \(\mathcal{D}(F)\). In addition
\[
\left(1 + \frac{K}{\rho_1}\right) ||y||^2_F = F(y, y) + KF(y, y) \geq F(y, y) + K(y, y) \geq F(y, y) = ||y||^2_F,
\]
where \(K\) is as given above. Thus \(F(x, y) + K(x, y)\) and \(F(x, y)\) are equivalent inner products on \(\mathcal{D}(F)\) and since \(F(x, y) + K(x, y)\) and \((x, y)_{H^1(G)}\) are equivalent inner products on \(\mathcal{D}(F)\) we have that \(F(x, y)\) and \((x, y)_{H^1(G)}\) are equivalent inner products on \(\mathcal{D}(F)\).
We now show that, with the $F$ inner product, $\mathcal{D}(F)$ is a Hilbert space. For this, we need only show that $\mathcal{D}(F)$ is closed in $H^1(G)$. The map $\hat{T} : H^1(G) \to \mathbb{C}^J$ given by

$$
\hat{T} : y \to \left( \sum_{j=1}^{K} [\alpha_{ij}y_j(0) + \gamma_{ij}y_j(1)] \right)_{i=1,...,J},
$$

is continuous by the trace theorem, see [1], and thus the kernel of $\hat{T}$, $\text{Ker}(\hat{T}) = \mathcal{D}(F)$ is closed.

\textbf{Theorem 4.2} The spectrum of (1.1), (2.5)-(2.6) is real and all eigenvalues are semi-simple.

\textbf{Proof:} As $\mathcal{D}(L)$ is dense in $\mathcal{D}(F)$, $L$ is a densely defined operator in $\mathcal{D}(F)$. Now $F(x,y) := (Lx,y)$ for all $x \in \mathcal{D}(L)$ and $y \in \mathcal{D}(F)$.

Let $\tilde{L} := L^{-1}B$, then $\tilde{L} : L^2(G) \to \mathcal{D}(L)$ and is thus a map from $\mathcal{D}(F)$ to $\mathcal{D}(L)$.

Since $B$ and $L$ are self adjoint in $L^2(G)$ we get

$$
F(\tilde{L}x,y) = F(L^{-1}Bx,y) = (Bx,y) = (x,By) = (By,x) = F(\tilde{L}y,x) = F(x,\tilde{L}y).
$$

for $x,y \in \mathcal{D}(F)$.

So $\tilde{L}$ is self adjoint in $\mathcal{D}(F)$ (with respect to $F$). Thus, in $\mathcal{D}(F)$, $\tilde{L}$ has only real spectrum and all eigenvalues are semi-simple. Therefore, by Lemma 3.2 the pencil $Lx = \lambda Bx$ has only real spectrum and all eigenvalues are semi-simple. ■

Let

$$
[f,g] := \sum_{i=1}^{n} \int_{0}^{1} f|_{e_i} \bar{g}|_{e_i} dt - \sum_{i=n+1}^{K} \int_{0}^{1} f|_{e_i} \bar{g}|_{e_i} dt = (Bf,g), \quad (4.12)
$$

then $L^2(G)$, with the indefinite inner product given by (4.12), is a Krein space which we denote by $H_K$.

We now define the positive, $C^+$, and negative, $C^-$, cones of $H_K$ by

$$
C^+ := \{ y \in H_K | [y,y] > 0 \}, \quad C^- := \{ y \in H_K | [y,y] < 0 \}.
$$
Theorem 4.3  For $L$ positive definite in $L^2(G)$ and $y$ an eigenfunction of (1.1), (2.5)-(2.6) corresponding to the eigenvalue $\lambda$ we have $y \in C^+$ if and only if $\lambda > 0$, and $y \in C^-$ if and only if $\lambda < 0$.

Proof: Let $y$ be an eigenfunction corresponding to $\lambda$. Using the fact that any element, $y$, of $H_K$ may be written in the form $y = \{f, g\}$ or $y = f \oplus g$, where $f = (y|e_1, \ldots, y|e_n)$ has $n$ components and $g = (y|e_{n+1}, \ldots, y|e_K)$ has $K - n$ components, we get that

$C^+ = \{\{f, g\} | ||f||^2_{L^2(G^+)} > ||g||^2_{L^2(G^-)}\}$,

and

$C^- = \{\{f, g\} | ||f||^2_{L^2(G^+)} < ||g||^2_{L^2(G^-)}\}$.

Here $G^+$ denotes the subgraph of $G$ where the weights are positive and $G^-$ denotes the subgraph of $G$ where the weights are negative.

Since $L > 0$ and $y = \{f, g\}$,

$0 < (Ly, y) = (\lambda By, y) = \lambda[y, y] = \lambda(||f||^2_{L^2(G^+)} - ||g||^2_{L^2(G^-)})$.

Hence, $y \in C^+$ if and only if $\lambda > 0$, and $y \in C^-$ if and only if $\lambda < 0$.  \[\blacksquare\]

5 Full and half-range completeness

In this section we prove both half and full range completeness of the eigenfunctions of (1.1), (2.5)- (2.6). In the case presented here the proof is simpler than that of Beals [4], but it is assumed that the problem is left definite, i.e. $L$ is a positive operator.

Recall that, by Lemma 4.1, $D(F)$ is a Hilbert space. Define

$\tilde{F}[u](v) := F(u, v)$

then $\tilde{F} : D(F) \rightarrow D(F)'$, where $D(F)'$ is the conjugate dual of $D(F)$, i.e. the space of continuous conjugate-linear maps from $D(F)$ to $\mathbb{C}$.

Lemma 5.1  $\tilde{F}$ is an isomorphism from $D(F)$ to $D(F)'$.

Proof: If $F(u_1, v) = F(u_2, v)$, for all $v \in D(F)$, then $u_1 = u_2$ since $F$ is an inner product on $D(F)$, see Lemma 4.1. Thus $\tilde{F}$ is one to one.

Now, for $\hat{v} \in D(F)'$ we have that $\hat{v}(x) = F(v, x)$ for some $v \in D(F)$ by the Theorem of Riesz, [19]. So $\hat{v}(x) = \tilde{F}[v](x)$ giving that $\tilde{F}[\hat{v}] = \hat{v}$. Hence $\tilde{F}$ is onto.
Also $\tilde{F}$ and $\tilde{F}^{-1}$ are everywhere defined maps on a Hilbert space and are thus continuous as a consequence of the principle of uniform boundedness (Banach Steinhaus theorem), [19].

So $\tilde{F}$ is an isomorphism from $\mathcal{D}(F)$ to $\mathcal{D}(F)'$.

Define $T[u](v) := (Bu, v)$ for $u, v \in \mathcal{D}(F)$. Then $T : \mathcal{D}(F) \rightarrow \mathcal{D}(F)'$ is compact since $\mathcal{D}(F)$ is compactly embedded in $L^2(G)$ and $Bu \in L^2(G)$ with the mapping $Bu \mapsto (Bu, \cdot)$ from $L^2(G)$ to $L^2(G)'$ continuous. Thus $S := \tilde{F}^{-1}T$ is a compact map with $S : \mathcal{D}(F) \rightarrow \mathcal{D}(F)$.

**Lemma 5.2** The compact operator $S$ on $\mathcal{D}(F)$ is self-adjoint with respect to the inner product $F$.

**Proof:** For $u, v \in \mathcal{D}(F)$

$$F(Su, v) = \tilde{F}[Su](v) = T[u](v) = (Bu, v) = (u, Bv).$$

Similarly

$$F(Sv, u) = \tilde{F}[Sv](u) = (Su, v) = (u, Su).$$

As $S$ is a compact self-adjoint operator on $\mathcal{D}(F)$ and as $0$ is not an eigenvalue of $S$, the eigenfunctions, $(u_n)$, of $S$, with eigenvalues $(\lambda_n^{-1})$, can be chosen so that $(u_n)$ is an orthonormal basis for $\mathcal{D}(F)$.

**Note:** The equation $Su_n = \lambda_n^{-1}u_n$ is equivalent to the equation $Lu_n = \lambda_n Bu_n$, in the sense that if

$$\lambda_n Su_n = u_n,$$

then, by the definition of $S$,

$$\lambda_n(\tilde{F}^{-1}T)u_n = u_n.$$

Applying $\tilde{F}$ to the above gives

$$\lambda_n Tu_n = \tilde{F}u_n.$$

Thus

$$\lambda_n T[v](u_n) = \tilde{F}[v](u_n),$$

for all $v \in \mathcal{D}(F)$. From the definition of $T$, this gives

$$\lambda_n (Bu, u_n) = \tilde{F}[v](u_n).$$

Hence

$$\lambda_n (Bu, u_n) = F(v, u_n)$$

for all $v \in \mathcal{D}(F)$. Using Lemma 3.2 we we obtain that

$$\lambda_n (Bu, u_n) = (v, Lu_n).$$
Therefore
\[(v, \lambda_n Bu_n - Lu_n) = 0,\]
for all \(v \in \mathcal{D}(F)\), and by the density of \(\mathcal{D}(F)\) in \(L^2(G)\), this yields
\[Lu_n = \lambda_n Bu_n.\]
It is easy to show that if \(Lu_n = \lambda_n Bu_n\), then \(Su_n = \lambda_n^{-1}u_n\).

In summary, we have the following theorem:

**Theorem 5.3 (Full range completeness)** The eigenfunctions \((y_n)\) of (1.1), (2.5)-(2.6) form a Riesz basis for \(L^2(G)\) and can be chosen to form an orthonormal basis for \(\mathcal{D}(F)\) (with respect to the \(F\) inner product). In addition \((y_n)\) is orthogonal with respect to \([\cdot, \cdot]\).

*Proof:* Since \(S\) is a compact self-adjoint operator on the Hilbert space \(\mathcal{D}(F)\), the eigenvectors can be chosen to form an orthonormal basis in \(\mathcal{D}(F)\). As shown in the note above the variational eigenfunctions coincide with those of \(L^{-1}B\) (with eigenvalues mapped by \(\lambda \mapsto \frac{1}{\lambda}\) and where 0 is not in the point spectrum). Thus the eigenfunctions of \(L^{-1}B\) can be chosen to form an orthonormal basis for \(\mathcal{D}(F)\) and as \(\mathcal{D}(F)\) is dense in \(L^2(G)\) they form a Riesz basis for \(L^2(G)\).

Finally, if \((y_n)\) is an orthonormal basis of \(\mathcal{D}(F)\) of eigenfunctions then
\[\delta_{n,m} = F(y_n, y_m) = \lambda_n(By_n, y_m) = \lambda_n[y_n, y_m].\]
Hence \((y_n)\) is orthogonal with respect to \([\cdot, \cdot]\). \(\blacksquare\)

Let \(P_{\pm}\) be the positive and negative spectral projections of \(S\). Note that \(\text{Ker}(S) = \{0\}\). The projections, \(P_{\pm}\), are then defined by the property
\[P_{\pm}u_n = \begin{cases} u_n, & \pm \lambda_n > 0 \\ 0, & \pm \lambda_n < 0 \end{cases},\]

hence
\[|S| = S(P_+ - P_-) = (P_+ - P_-)S.\]

On \(\mathcal{D}(F)\) we introduce the inner product \((u, v)_S = F(|S|u, v)\) with related norm \(|u|_S = (u, u)_S^{1/2}\).

We must now show that this norm is equivalent to the \(L^2(G)\) norm, \(|u| = (u, u)^{1/2}\).

The operator \(B\) is a self-adjoint operator in \(L^2(G)\) and \(B\) has spectral projections \(Q_{\pm}\), where
\[Q_{\pm}u(x) = \begin{cases} u(x), & b(x) = \pm 1 \\ 0, & b(x) = \mp 1 \end{cases}.\]
Thus $|B| = I = B(Q_+ + Q_-) = (Q_+ + Q_-)B$ is just the identity map, and $|T|$ is the map from $\mathcal{D}(F)$ to $\mathcal{D}(F)'$ induced by $|B|$, i.e. $|T|[u](v) = (u, v)$. But $T[u](v) := (Bu, v)$ for all $u, v \in \mathcal{D}(F)$, and thus can be extended to $u, v \in L^2(G)$, i.e.

$$T : L^2(G) \to L^2(G)' \hookrightarrow \mathcal{D}(F)'.$$

In this sense $TQ_\pm : L^2(G) \to \mathcal{D}(F)'$ is compact.

Also $T(Q_+ + Q_-)[u](v) = (B(Q_+ + Q_-)u, v) = (u, v) = |T|[u](v)$ for all $u, v \in L^2(G)$ and thus for $u, v \in \mathcal{D}(F)$. We now observe that $Q_\pm T : \mathcal{D}(F) \to \mathcal{D}(F)'$, using the extension of $T$ to $L^2(G)$, is well defined as $Q_\pm T[u](v) = T[u](Q_\pm v) = (Bu, Q_\pm v) = (Q_\pm Bu, v) = (BQ_\pm u, v)$ making $TQ_\pm = Q_\pm 'T$. Hence

$$|T| = T(Q_+ - Q_-) = (Q_+ ' - Q_- 'T).$$

**Theorem 5.4** The norms $|| \cdot ||_S$ and $|| \cdot ||$ are equivalent on $\mathcal{D}(F)$.

*Proof:* Considered as an operator in the subspace $P_+(\mathcal{D}(F))$, $S$ is a positive operator. Let $y \in \mathcal{D}(L)$. Since $L$ is a positive operator and $\mathcal{D}(F)$ is compactly embedded in $L^2(G)$ we have that there exists some constant $C > 0$ such that

$$\langle Ly, y \rangle = F(y, y) \geq C(y, y), \quad (5.13)$$

for all $y \in \mathcal{D}(L)$. Also

$$||Q_\pm y||^2 \leq ||y||^2. \quad (5.14)$$

Combining (5.13) and (5.14) we obtain that

$$C||Q_\pm y||^2 \leq C(y, y) \leq (Ly, y), \quad (5.15)$$

for $y \in \mathcal{D}(L)$. Let $(y_n)$ be an orthonormal basis of eigenfunctions of $S$ in $\mathcal{D}(F)$ where $y_n$ has eigenvalue $\lambda_n$ with $0 < \lambda_1 < \lambda_2 < \ldots$ and $0 > \lambda_{-1} > \lambda_{-2} > \ldots$. Now

$$P_+(\mathcal{D}(F)) = \langle y_1, y_2, \ldots \rangle,$$

and $Ly_n = \lambda_n B y_n$ for all $n = 1, 2, \ldots$.

Let $y \in P_+(\mathcal{D}(L))$ then $y = \sum_{n=1}^{\infty} \alpha_n y_n$ where $\alpha_n \in \mathbb{C}, n \in \mathbb{N}$. From (5.15) we have that

$$||Q_\pm y||^2 \leq \frac{1}{C}(Ly, y).$$

Using the orthogonality of $(y_n)$ we get

$$\frac{1}{C}(Ly, y) = \sum_{n=1}^{\infty} |\alpha_n|^2 \frac{\lambda_n}{C}(By_n, y_n),$$

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thus
\[ ||Q_+ y||^2 \leq \frac{\lambda_1}{C} \sum_{n=1}^{\infty} |\alpha_n|^2 (By_n, y_n). \]

But
\[ \sum_{n=1}^{\infty} |\alpha_n|^2 (By_n, y_n) = (By, y), \]

hence
\[ ||Q_+ y||^2 \leq \frac{\lambda_1}{C} (By, y) \]
\[ = \frac{\lambda_1}{C} T[y](y) \]
\[ = \frac{\lambda_1}{C} \tilde{F}[S y](y) \]
\[ = \frac{\lambda_1}{C} F(S y, y) \]
\[ = \frac{\lambda_1}{C} F(|S| y, y). \]

So
\[ ||Q_+ y||^2 \leq \frac{\lambda_1}{C} ||y||_S^2 \]
and setting \( \sqrt{\frac{\lambda_1}{C}} := k > 0 \) gives
\[ ||Q_+ y|| \leq k ||y||_S. \] (5.16)

Similarly
\[ ||Q_- y||^2 \leq \frac{\lambda_1}{C} ||y||_S^2 \]
i.e.
\[ ||Q_- y|| \leq k ||y||_S. \] (5.17)

Since \( \mathcal{D}(L) \) is dense in \( \mathcal{D}(F) \), (5.16) and (5.17) hold on all \( P_+(\mathcal{D}(F)) \), so as \( ||y||^2 = ||Q_+ y||^2 + ||Q_- y||^2 \) we have \( ||y|| \leq \sqrt{2k} ||y||_S \) for all \( y \in P_+(\mathcal{D}(F)) \).

Working on \( P_-(\mathcal{D}(F)) \) yields a similar estimate but with \( \lambda_1 \) replaced by \(-\lambda_1\). Thus there exists a constant \( C_1 > 0 \) so that for all \( y \in \mathcal{D}(F) \),
\[ ||y|| \leq C_1 ||y||_S. \] (5.18)

To obtain (5.19), the reverse of (5.18), we observe that
\[ ||y||_S = F(|S| y, y) = F((SP_+ - SP_-) y, y). \]
But \(SP_\pm = P_\pm S\) so

\[
||y||^2_S = F(Sy, P_+ y - P_- y) = \tilde{F}[Sy](P_+ y - P_- y) = T[y](P_+ y - P_- y) = |T|[Q_+ y - Q_- y](P_+ y - P_- y).
\]

Using Hölder’s inequality we obtain that

\[
|T|[Q_+ y - Q_- y](P_+ y - P_- y) \leq ||y||||Q_+ y - Q_- y||||P_+ y - P_- y||.
\]

Thus

\[
||y||^2_S \leq ||Q_+ y - Q_- y||||P_+ y - P_- y|| = ||y||||P_+ y - P_- y||.
\]

By (5.18)

\[
||y||^2_S \leq C_1||y||||P_+ y - P_- y||s.
\]

Now

\[
||P_+ y - P_- y||s = F(|S|(P_+ - P_-)y, (P_+ - P_-)y) = F(Sy, (P_+ - P_-)y) = F((P_+ - P_-)Sy, y) = F(|S|y, y),
\]

giving

\[
||y||^2_S \leq C_1||y||||y||s,
\]

therefore

\[
||y||_S \leq C_1||y||. \tag{5.19}
\]

Combining (5.18) and (5.19) gives

\[
\frac{1}{C_1}||y||_S \leq ||y|| \leq C_1||y||_S
\]

and thus the two norms are equivalent in \(D(F)\).

Let \(H_S\) be the completion of \(D(F)\) with respect to \(||\cdot||_S\).

**Theorem 5.5 (Half-range completeness)** For \(Q_+\) and \(Q_-\) as previously defined \(\{Q_+ y_n, \lambda_n > 0\}\) is a Riesz basis \(L^2(G^+)\) and \(\{Q_- y_n, \lambda_n < 0\}\) is a Riesz basis \(L^2(G^-)\).

**Proof:** To prove the half-range completeness we show that \(\{y_n, \lambda_n > 0\}\) and \(\{y_n, \lambda_n < 0\}\) are Riesz bases for \(Q_+ P_+(H_S)\) and \(Q_- P_-(H_S)\) respectively via showing that \(V := Q_+ P_+ + Q_- P_-\) is an isomorphism from \(H_S\) to \(L^2(G)\), see [4].

Let \(u, v \in D(F)\), then

\[
(Q_\pm u, P_\pm v)_S = (Q_\pm u, P_\pm v) \tag{5.20}
\]
and

\[(Q_{\pm} u, P_{\mp} v)_S = -(Q_{\pm} u, P_{\mp} v).\]  

(5.21)

To see this, as \(S\) is self-adjoint with respect to \(F\) so is \(|S|\), we have, for example,

\[(Q_+ u, P_- v)_S = F(|S|Q_+ u, P_- v) = F(Q_+ u, |S|P_- v) = F(Q_+ u, S(P_+ - P_-)P_- v) = F(SQ_+ u, -P_- v) = -F(SQ_+ u, P_- v) = -(Q_+ u, P_- v),\]

because \(F(SQ_+ u, P_- v) = (BQ_+ u, P_- v)\) and \(Q_+ u(x) = 0\) when \(b(x) = -1\) and \(Q_+ u(x) = u(x)\) when \(b(x) = 1\).

Now, as \(P_{\pm}\) are self-adjoint with respect to \([\cdot, \cdot]\),

\[
||u||^2_S = F(|S|u, u) \\
= F((P_+ - P_-)Su, u) \\
= F(Su, (P_+ - P_-)u) \\
= (Bu, (P_+ - P_-)u) \\
= ((Q_+ - Q_-)u, (P_+ - P_-)u) \\
= (Q_+ u, P_+ u) + (Q_- u, P_- u) - (Q_+ u, P_- u) - (Q_- u, P_+ u).
\]

For \(u \in \mathcal{D}(F)\),

\[
||Vu||^2 = (Q_+ P_+ u, Q_+ P_+ u) + (Q_- P_- u, Q_- P_- u) + (Q_- P_- u, Q_+ P_+ u) + (Q_+ P_+ u, Q_- P_- u) \\
= (Q_+ P_+ u, Q_+ P_+ u) + (Q_- P_- u, Q_- P_- u) \\
= (Q_+ (I - P_-)u, (I - Q_-)P_+ u) + (Q_- (I - P_+)u, (I - Q_+)P_- u) \\
= (Q_+ u, P_+ u) - (Q_- P_- u, P_+ u) + (Q_- u, P_- u) - (Q_+ P_+ u, P_- u) \\
= ||u||^2_S + (Q_+ u, P_- u) + (Q_- u, P_+ u) - (Q_+ P_- u, P_+ u) - (Q_- P_+ u, P_- u).
\]

Setting \(W := Q_+ P_+ + Q_- P_-\), since \(Q_+ - Q_- = B\) and \(P_{\pm}\) are self-adjoint and orthogonal with respect to \([\cdot, \cdot]\), we obtain

\[
||Vu||^2 \quad = ||u||^2_S + (Q_- P_- u, Q_- P_- u) + (Q_+ P_+ u, Q_+ P_+ u) \\
= ||u||^2_S + ||Wu||^2.
\]

As \(||\cdot||\) and \(||\cdot||_S\) are equivalent norms on \(\mathcal{D}(F)\), the above equality holds for \(u \in H_S\) and shows that the bounded operator \(V\) has closed range and kernel \((0)\).

Equations (5.20) and (5.21) show that, as mappings from \(H_S\) to \(L^2(G)\), \(V\) and \(W\) have adjoints \(V^* = P_+ Q_+ + P_- Q_-\) and \(W^* = -P_+ Q_- - P_- Q_+\). But \(V^*\) and \(W^*\) obey, by the same reasoning as above,

\[
||V^* u||^2_S = ||W^* u||^2_S + ||u||^2. \quad (5.22)
\]
Thus $V^*$ is one to one and therefore $V$ is an isomorphism. Hence we have proved the theorem. □

6 Max-Min Property

In this section we give a maximum-minimum characterization for the eigenvalues of indefinite boundary-value problems on graphs. We refer the reader to [8, page 406] and [24] where analogous results for partial differential operators were considered.

In the following theorem $\{v_1, \ldots, v_n\}^\perp$ will denote the orthogonal complement with respect to $[\cdot, \cdot] = (B\cdot, \cdot)$ of $\{v_1, \ldots, v_n\}$. In addition, as is customary, it will be assumed that the eigenvalues, $\lambda_n > 0$, $n \in \mathbb{N}$, of (1.1), (2.5)-(2.6), are listed in increasing order and repeated according to multiplicity, and that the eigenfunctions, $y_n$, are chosen so as to form a complete orthonormal family in $L^2(G) \cap C^+$. More precisely, as in Theorem 5.3, $(y_n, n \in \mathbb{N}) \setminus \{0\}$ can be chosen so as to form an orthonormal basis for $D(F)$ and thus for $L^2(G)$ with respect to $B$. In particular, $(y_n)_{n \in \mathbb{N}}$ is then an orthonormal basis for $L^2(G) \cap C^+$ with respect to $B$ (i.e. $[\cdot, \cdot]$). The case of $L^2(G) \cap C^-$ is similar, so for the remainder of the paper we will restrict ourselves to $L^2(G) \cap C^+$.

**Theorem 6.1** Suppose $(L\varphi, \varphi) > 0$ for all $\varphi \in D(L) \setminus \{0\}$, and for $v_j \in L^2(G) \cap C^+$, $j = 1, 2, \ldots$, let

$$d_{n+1}(v_1, \ldots, v_n) = \inf \left\{ \frac{F(\varphi, \varphi)}{(B\varphi, \varphi)} \bigg| \varphi \in \{v_1, \ldots, v_n\}^\perp \cap D(F) \setminus \{0\}, (B\varphi, \varphi) > 0 \right\}. \tag{6.23}$$

Then

$$\lambda_{n+1} = \sup \{d_{n+1}(v_1, \ldots, v_n) \mid v_1, \ldots, v_n \in L^2(G) \cap C^+\}, \tag{6.24}$$

for $n = 0, 1, \ldots$, and this maximum-minimum is attained if and only if $\varphi = y_{n+1}$ and $\varphi = y_j$, $i = 1, \ldots, n$, where $y_j$ is an eigenfunction of $L$ with eigenvalue $\lambda_j$, and $(y_j)$ is a $B$-orthogonal family.

**Proof:** Let $v_1, \ldots, v_n \in L^2(G) \cap C^+$. As span$\{y_1, \ldots, y_{n+1}\}$ is $n + 1$ dimensional and span$\{v_1, \ldots, v_n\}$ is at most $n$ dimensional there exists $\varphi$ in span$\{y_1, \ldots, y_{n+1}\} \setminus \{0\}$ having

$$(B\varphi, v_i) = 0, \quad \text{for all} \quad i = 1, \ldots, n.$$

In particular, this ensures that $\varphi \in D(F)$ as each $y_i$ is in $D(F)$.

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Denote \( \varphi = \sum_{k=1}^{n+1} c_k y_k \), then

\[
F(\varphi, \varphi) = \sum_{i,k=1}^{n+1} c_i \bar{c}_k F(y_i, y_k) = \sum_{i=1}^{n+1} |c_i|^2 F(y_i, y_i)
\]

\[
= \sum_{i=1}^{n+1} |c_i|^2 (Ly_i, y_i)
\]

\[
= \sum_{i=1}^{n+1} |c_i|^2 (\lambda_i B y_i, y_i)
\]

\[
= \sum_{i=1}^{n+1} |c_i|^2 \lambda_i (B y_i, y_i)
\]

\[
\leq \lambda_{n+1} \sum_{i=1}^{n+1} |c_i|^2 (B y_i, y_i)
\]

\[
= \lambda_{n+1} (B \varphi, \varphi),
\]

thus showing that

\[
d_{n+1}(v_1, \ldots, v_n) \leq \lambda_{n+1} \quad \text{for all } v_1, \ldots, v_n \in L^2(G) \cap C^+.
\]

Hence

\[
\sup \{d_{n+1}(v_1, \ldots, v_n) \mid v_1, \ldots, v_n \in L^2(G) \cap C^+ \} \leq \lambda_{n+1}.
\]

Now suppose \( \lambda_{n+1} > d_{n+1}(y_1, \ldots, y_n) \). Then there exists \( u \in \mathcal{D}(F) \setminus \{0\} \), \( u \in \{y_1, \ldots, y_n\}^\perp \), such that \( B(u, u) = 1 \) and

\[
F(u, u) < d_{n+1}(y_1, \ldots, y_n) + \frac{1}{2} (\lambda_{n+1} - d_{n+1}(y_1, \ldots, y_n)). \quad (6.25)
\]

By Theorem [5.3] we can write \( u = \sum_{j \notin \{0, \ldots, n\}} \alpha_j y_j \). Therefore

\[
F(u, u) = \sum_{i, j \notin \{0, \ldots, n\}} \alpha_i \bar{\alpha}_j F(y_i, y_j)
\]

\[
= \sum_{i \notin \{0, \ldots, n\}} |\alpha_i|^2 F(y_i, y_i)
\]

\[
= \sum_{i \notin \{0, \ldots, n\}} |\alpha_i|^2 (Ly_i, y_i)
\]

\[
= \sum_{i \notin \{0, \ldots, n\}} |\alpha_i|^2 (\lambda_i B y_i, y_i).
\]
Now as \( \lambda_i(By_i, y_i) = F(y_i, y_i) > 0 \) for all \( i \), we have

\[
F(u, u) = \sum_{i > n} |\alpha_i|^2 \lambda_i(By_i, y_i) + \sum_{i \leq -1} |\alpha_i|^2 \lambda_i(By_i, y_i)
\]

\[
\geq \sum_{i > n} |\alpha_i|^2 \lambda_i(By_i, y_i)
\]

\[
\geq \lambda_{n+1} \sum_{i > n} |\alpha_i|^2 (By_i, y_i)
\]

\[
= \lambda_{n+1} \left( B \sum_{i > n} \alpha_i y_i, \sum_{j > n} \alpha_j y_j \right)
\]

\[
= \lambda_{n+1} (BP_u, P_u).
\]

Combining the above with \( (6.25) \) and noting that \( (Bu, u) = 1 \), gives

\[
\lambda_{n+1} - \frac{1}{2} (\lambda_{n+1} - d_{n+1}(y_1, \ldots, y_n)) > \lambda_{n+1} (BP_u, P_u).
\]

Thus

\[
(Bu, u) - \frac{\lambda_{n+1} - d_{n+1}(y_1, \ldots, y_n)}{2\lambda_{n+1}} = 1 - \frac{\lambda_{n+1} - d_{n+1}(y_1, \ldots, y_n)}{2\lambda_{n+1}} > (BP_u, P_u).
\]

Using the self-adjointness of the projections \( P_\pm \) with respect to \([\cdot, \cdot]\) now gives

\[
(BP_- u, P_- u) > \frac{\lambda_{n+1} - d_{n+1}(y_1, \ldots, y_n)}{2\lambda_{n+1}} > 0.
\]

But \( P_- u \in C^- \), so we have a contradiction and therefore \( \lambda_{n+1} \leq d_{n+1}(y_1, \ldots, y_n) \).

We have shown that \( \lambda_{n+1} = d_{n+1}(y_1, \ldots, y_n) \), \( (6.24) \) holds and \( d_{n+1} \) attains its supremum for \((y_1, \ldots, y_n)\). Also a direct computation gives \( F(y_{n+1}, y_{n+1}) = \lambda_{n+1} (By_{n+1}, y_{n+1}) \).

It remains to be shown that if \( u \in \mathcal{D}(F) \) is such that the maximum is attained for \( u, v_1, \ldots, v_n \) then \( u \) is an eigenfunction with eigenvalue \( \lambda = d_{n+1}(v_1, \ldots, v_n) \).

Let \( u \in \mathcal{D}(F) \) with \( (Bu, u) = 1 \) and

\[
J(\varphi, \epsilon) = \frac{F(u + \epsilon \varphi, u + \epsilon \varphi)}{(Bu + \epsilon \varphi, u + \epsilon \varphi)} \text{ for all } \varphi \in \mathcal{D}(F), \epsilon \in \mathbb{R}, |\epsilon| \text{ small}.
\]

Differentiation with respect to \( \epsilon \) of \( J(\varphi, \epsilon) \) gives

\[
0 = \frac{\partial}{\partial \epsilon} J(\varphi, \epsilon)|_{\epsilon=0} = 2\Re[F(\varphi, u) - d_{n+1}(v_1, \ldots, v_n)(B \varphi, u)],
\]

for all \( \varphi \in \mathcal{D}(F) \) and \( (Bu, u) = 1 \). Since everything in the above expression is real we obtain that

\[
F(\varphi, u) = d_{n+1}(v_1, \ldots, v_n)(B \varphi, u), \quad (6.26)
\]
for all $\varphi \in \mathcal{D}(F)$ and $(Bu, u) = 1$.

Now $F(u, u) > 0$ therefore $d_{n+1}(v_1, \ldots, v_n)(Bu, u) > 0$ which, since $(Bu, u) = 1$, gives $d_{n+1}(v_1, \ldots, v_n) > 0$. From (6.26), for $\varphi \in C_0^\infty(G)$, we get that

$$(L\varphi, u) - d_{n+1}(v_1, \ldots, v_n)(B\varphi, u) = 0,$$

giving

$$(\varphi, (l - d_{n+1}(v_1, \ldots, v_n)B)u) = 0.$$ 

Hence, by the proof of Lemma 3.2, $u \in H^2(G) \cap \mathcal{D}(F)$ and obeys (1.1) and (2.5). We must still show that $u$ obeys the boundary condition (2.6).

From the proof of Lemma 3.2 we see that, for $\varphi \in \mathcal{D}(F)$,

$$F(u, \varphi) = \int_{\partial G} (fu + u')\bar{\varphi}d\sigma + d_{n+1}(v_1, \ldots, v_n)(Bu, \varphi).$$

This together with (6.26) gives that

$$0 = \int_{\partial G} (fu + u')\bar{\varphi}d\sigma$$

for all $\varphi \in \mathcal{D}(F)$.

As, (6.27) holds for all $\varphi \in \mathcal{D}(F)$, $u$ obeys (2.6), giving that $u$ is an eigenfunction of (1.1), (2.5)-(2.6) with eigenvalue $\lambda = d_{n+1}(v_1, \ldots, v_n)$. ■

7 Eigenvalue Bracketing and Asymptotics

If the boundary conditions (2.5)-(2.6) are replaced by the Dirichlet condition $y = 0$ at each node of $G$, i.e.

$$y_i(1) = 0 \quad \text{and} \quad y_i(0) = 0, \quad i = 1, \ldots, K,$$

(7.1)

then the graph $G$ becomes disconnected with each edge $e_i$ becoming a component subgraph, $G_i$, with Dirichlet boundary conditions at its two nodes (ends). The boundary value problem on each sub-graph $G_i$ is equivalent to a Sturm-Liouville boundary value problem on $[0, 1]$ with Dirichlet boundary conditions. Depending on whether the edge has positive or negative weight the resulting boundary value problem is

$$-y_i'' + q_iy_i = \mu y_i, \quad i = 1, \ldots, n,$$

(7.2)

or

$$-y_i'' + q_iy_i = -\mu y_i, \quad i = n + 1, \ldots, K,$$

(7.3)
with boundary conditions (7.1).

Let \( \lambda_1^D \leq \lambda_2^D \leq \ldots \) be the eigenvalues (repeated according to multiplicity) of the system (7.1) with (7.2) and (7.3) for which the eigenvectors are in \( L^2(G) \cap C^+ \). Let \( \Lambda_1^D < \Lambda_2^D < \ldots \) be the eigenvalues of the system (7.1) with (7.2) and (7.3) not repeated by multiplicity. Denote by \( \nu_j^D \) the dimension of the maximal positive (with respect to \([\cdot, \cdot]\)) subspace of the eigenspace \( E_j^D \) to \( \Lambda_j^D \).

Observe that if \( \mu \) is an eigenvalue of the system (7.1) with (7.2) and (7.3), with multiplicity \( \nu \) and eigenspace \( E \), then there are precisely \( \nu \) indices \( i_1, \ldots, i_\nu \) such that

\[
- y''_i + q_i y_i = \mu y_i, \quad (7.4)
\]

with boundary conditions (7.1). In particular, if

\[
Y_j^i := \begin{cases} 0, & j \neq i, \\ y_i, & j = i, \end{cases}
\]

where \( j \in \{1, \ldots, K\} \), then \( Y_1^i, \ldots, Y_K^i \) are eigenfunctions to (7.1) with (7.2) and (7.3) and form a basis for \( E \), which is orthogonal with respect to both \((\cdot, \cdot)\) and \([\cdot, \cdot]\). Hence, by \cite{[14]} Corollary 10.1.4, the maximal \( B \)-positive subspace of \( E \) has dimension

\[
\nu^+ = \# (\{i_1, \ldots, i_\nu\} \cap \{1, \ldots, n\}).
\]

I.e. \( \nu^+ \) is the multiplicity of \( \mu \) as an eigenvalue of (7.1) with (7.2).

Hence \( \lambda_j^D \) is the \( j \)th eigenvalue of (7.1) with (7.2), i.e. of (1.1) with (7.1) considered only on \( G^+ \).

Similarly if we consider the equation (2.3) with the non-Dirichlet conditions

\[
y'_i(1) = f(1)y_i(1) \quad \text{and} \quad y'_i(0) = f(0)y_i(0), \quad i = 1, \ldots, K, \quad (7.5)
\]

where \( f \) is given in (3.9), then, as in the Dirichlet case, above, \( G \) decomposes into a union of disconnected graphs \( G_1, \ldots, G_K \). Again, depending on whether the edge has positive or negative weight, we have the equation

\[
- y''_i + q_i y_i = \mu y_i, \quad i = 1, \ldots, n, \quad (7.6)
\]

or

\[
- y''_i + q_i y_i = -\mu y_i, \quad i = n + 1, \ldots, K, \quad (7.7)
\]

with boundary conditions (7.5). Let \( \lambda_1^N \leq \lambda_2^N \leq \ldots \) be the eigenvalues (repeated according to multiplicity) of the system (7.5) with (7.6) and (7.7) for which the eigenvectors are in \( L^2(G) \cap C^+ \). By the same reasoning as above, \( \lambda_j^N \) is the \( j \)th eigenvalue of (7.5) with (7.6), i.e. of (1.1) with (7.5) considered only on \( G^+ \).
Thus, from Theorem 6.1 and [11] we have that, in $\mathcal{L}^2(G) \cap C^+$, the eigenvalues of (2.3), (2.5)-(2.6) are ordered by

$$\lambda_n^N \leq \lambda_n \leq \lambda_n^D, \quad n = 1, 2, \ldots.$$  \hfill (7.8)

The asymptotics for $\lambda_n^N$ and $\lambda_n^D$ are well known, in particular, using the results in [11] for (1.1) on $G^+$, with (7.1) and (7.5) we obtain the following theorem:

**Theorem 7.1** Let $G$ be a compact graph with finitely many nodes. If the boundary value problem (2.3), (2.5)-(2.6) has co-normal (elliptic) boundary conditions, then the eigenvalues in $\mathcal{L}^2(G) \cap C^+$ obey the asymptotic development

$$\sqrt{\lambda_n} = \frac{n\pi}{\text{length}(G^+)} + O(1), \quad \text{as } n \to \infty.$$

By formally replacing $\lambda$ by $-\lambda$ in (1.1) a similar result is obtained for $\mathcal{L}^2(G) \cap C^-$.

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