Newton polygons and curve gonialities

Filip Cools
joint work with Wouter Castryck

K.U. Leuven

Tropical Geometry workshop (Edinburgh)
April 5, 2012
Contents

1 Introduction

2 An upper bound for the gonality

3 Proving sharpness: a graph-theoretic attack
Introduction

- \( f \in \mathbb{C}[x^{\pm 1}, y^{\pm 1}] \): irreducible Laurent polynomial
- \( \Delta(f) \): its Newton polygon
  - i.e. if
    \[
    f = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij} x^i y^j,
    \]
    then
    \[
    \Delta(f) = \text{Conv}\{(i,j) \in \mathbb{Z}^2 \mid c_{ij} \neq 0\} \subset \mathbb{R}^2
    \]
- \( C(f) \): curve in \( \mathbb{T}_\mathbb{C}^2 = (\mathbb{C} \setminus \{0\})^2 \) defined by \( f \)

Theorem

(Baker, 1893) The (geometric) genus of \( C(f) \) is bounded by the number of \( \mathbb{Z}^2 \)-points in the interior of \( \Delta(f) \).

(Khovanskii, 1977) Generically, this bound is attained.
Examples

\[ f = y^2 - x^3 - Ax - B \text{ with } B \neq 0 \]

\[ \Delta(f) \]
\[ \#(\Delta \cap \mathbb{Z}^2) = 1 \]
the genus of \( C(f) \) is equal to one
iff \( 4A^3 + 27B^2 \neq 0 \)

\[ f = y^2 - h(x) \text{ with } \deg h = 2g + 1 \text{ and } h(0) \neq 0 \]

\[ \Delta(f) \]
\[ \#(\Delta \cap \mathbb{Z}^2) = g \]
the genus of \( C(f) \) is equal to \( g \) iff
\( h(x) \) has no multiple roots
Central question of this talk

Question

Does there exist a similar combinatorial interpretation for the gonality?

- gonality = minimal degree of a non-constant rational map to $\mathbb{P}^1_C$
- hyperelliptic = gonality 2 (by definition)
Central question of this talk

- **A lattice polygon** is the convex hull in $\mathbb{R}^2$ of a finite number of $\mathbb{Z}^2$-points (also called lattice points).

- **The genus** of a two-dimensional lattice polygon $\Delta$ is the (geometric) genus of the curve defined by a generic Laurent polynomial $f$ with $\Delta(f) = \Delta$.

  Notation: $g(\Delta)$. By the foregoing: $g(\Delta) = \#(\Delta^\circ \cap \mathbb{Z}^2)$.

- **The gonality** of a two-dimensional lattice polygon $\Delta$ is the gonality of the curve defined by a generic Laurent polynomial $f$ with $\Delta(f) = \Delta$.

  Notation: $\gamma(\Delta)$. Well-defined by a semi-continuity argument.

Question (reformulated)

*Does there exist a purely combinatorial interpretation for $\gamma(\Delta)$?*
1 Introduction
2 An upper bound for the gonality
3 Proving sharpness: a graph-theoretic attack
Some terminology and easy facts

- A $\mathbb{Z}$-affine transformation is a map

$$\varphi : \mathbb{R}^2 \to \mathbb{R}^2 : (x, y) \mapsto (x, y)A + b$$

with $A \in \text{GL}_2(\mathbb{Z})$ and $b \in \mathbb{Z}^2$.

- Two lattice polygons $\Delta$ and $\Delta'$ are equivalent if there is a $\mathbb{Z}$-affine transformation $\varphi$ such that $\varphi(\Delta) = \Delta'$. (Notation: $\Delta \equiv \Delta'$)

- A $\mathbb{Z}$-affine transformation $\varphi$ acts on $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ as

$$f = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij}(x, y)^{(i,j)} \quad \mapsto \quad \varphi(f) = \sum_{(i,j) \in \mathbb{Z}^2} c_{ij}(x, y)^{(i,j)}.\quad \Delta(\varphi(f)) = \varphi(\Delta(f)) \text{ and } C(f) \cong C(\varphi(f)) \quad \text{ICMS Edinburgh} \quad 8 / 21$$
The lattice width as an upper bound

- The **lattice width** of a non-empty lattice polygon $\Delta$ is the minimal $d$ for which there is a $\mathbb{Z}$-affine transformation $\varphi$ such that

  $$
  \varphi(\Delta) \subset \{(x, y) \in \mathbb{R}^2 | 0 \leq y \leq d\}.
  $$

- Notation: $lw(\Delta)$.

- Convention: $lw(\emptyset) = -1$.

- **Easy fact:** $\gamma(\Delta) \leq lw(\Delta)$.
  - Let $f$ be a generic Laurent polynomial with $\Delta(f) = \Delta$.
  - Let $\varphi$ be a $\mathbb{Z}$-affine transformation realizing $lw(\Delta)$.
  - $C(f) \cong C(\varphi(f))$, so it suffices to deal with $C(\varphi(f))$.
  - Then $C(\varphi(f)) \rightarrow \mathbb{A}^1_{\mathbb{C}} \subset \mathbb{P}^1_{\mathbb{C}}$ : $(x, y) \mapsto x$ is of degree at most $d$. 

Sharp?

- Counterexample 1

- $\gamma(\Delta) = d - 1$ (Namba, 1979: gonality of smooth plane curves)

- $lw(\Delta) = d$, since every edge contains $d + 1$ lattice points
Sharpe?

- Counterexample 2

- $\gamma(\Delta) \leq 3$ (by Brill-Noether Theorem, curves of genus 4 are at most 3-gonal)

- $lw(\Delta) = 4$, because the interior polygon contains an interior $\mathbb{Z}^2$-point itself
The interior polygon

Let \( \Delta \) be a two-dimensional lattice polygon. The convex hull of the interior lattice points is called the **interior polygon** of \( \Delta \).

**Notation:** \( \Delta^{(1)} \)

**Theorem (–, Lubbes & Schicho)**

\[
\text{lw}(\Delta^{(1)}) = \text{lw}(\Delta) - 2, \text{ unless } \Delta \equiv \text{Conv}\{(0,0), (d,0), (0,d)\} \text{ for } d \geq 2, \\
\text{in which case } \text{lw}(\Delta) = d \text{ and } \text{lw}(\Delta^{(1)}) = d - 3.
\]

Thus in fact \( \gamma(\Delta) \leq \text{lw}(\Delta^{(1)}) + 2 \). This rules out Counterexample 1 as an exceptional case. **Counterexample 2** is more fundamental.

**Algorithm for computing \( \text{lw}(\Delta) \).**

**Conjecture**

\[
\gamma(\Delta) = \text{lw}(\Delta^{(1)}) + 2, \text{ unless } \Delta \equiv \text{Conv}\{(2,0), (0,2), (-2,-2)\}, \text{ in which case } \gamma(\Delta) = 3.
\]
1 Introduction
2 An upper bound for the gonality
3 Proving sharpness: a graph-theoretic attack
The metric graph $\Gamma(h)$

- Let $\Delta \subset \mathbb{R}^2$ be a lattice polygon.
- Let $\Delta_1, \ldots, \Delta_r \subset \Delta$ be a regular subdivision.
- Let $h : \Delta \to \mathbb{R}$ be an upper-convex piece-wise linear function such that its restrictions to $\Delta_1, \ldots, \Delta_r$ are linear. Assume that $h(\Delta \cap \mathbb{Z}^2) \subset \mathbb{Z}$.

Definition metric graph $\Gamma(h)$:
- vertices $v_1, \ldots, v_r$
- number of edges between $v_i$ and $v_j$ is the integral length of $\Delta_i \cap \Delta_j$
- length of an edge between $v_i$ and $v_j$ is the greatest common divisor of the $2 \times 2$-minors of

$$
\begin{pmatrix}
  a_{i1} & a_{i2} & 1 \\
  a_{j1} & a_{j2} & 1 
\end{pmatrix},
$$

where $(a_{k1}, a_{k2}, 1)$ is a primitive normal vector to the graph of $h|_{\Delta_k}$.
Note that the edge \((v_1, v_2)\) has length equal to 2 since the corresponding \(2 \times 3\)-matrix is
\[
\begin{pmatrix}
1 & -1 & 1 \\
-1 & -1 & 1
\end{pmatrix}.
\]
**Lower bound for $\gamma(\Delta)$**

- Given a metric graph $\Gamma$, denote by $\gamma(\Gamma)$ the gonality of $\Gamma$, i.e.
  \[ \gamma(\Gamma) = \min \{ d \mid \exists D \in \text{Div}_d(\Gamma) : r_{BN}(D) \geq 1 \}. \]

**Theorem**

*If $h : \Delta \to \mathbb{R}$ gives rise to a regular subdivision (as above), then*

\[ \gamma(\Gamma(h)) \leq \gamma(\Delta). \]

**Idea of proof:**

- Let $\text{Tor}(\tilde{\Delta})$ be the toric threefold corresponding to $h$
- Consider the toric degeneration of $\text{Tor}(\Delta)$ to $\bigcup_{i=1}^{r} \text{Tor}(\Delta_i)$
- View $C(f)$ as a generic hyperplane section of the toric surface $\text{Tor}(\Delta)$ and let it degenerate
- Use Baker’s Specialization Lemma (might need to blow-up some boundary $\mathbb{T}^1$’s at the bottom of $\tilde{\Delta}$!)
We expect that it is always possible to obtain equality:

**Conjecture**

There always exists a height function $h : \Delta \rightarrow \mathbb{R}$ such that

$$\gamma(\Gamma(h)) = \gamma(\Delta).$$

**Example:** our Counterexample 2.

\[\gamma(\Delta) = \gamma(\Gamma(h)) = 3.\]
A proof of the following combinatorial statement would solve it all:

**Conjecture**

There always exists a height function $h : \Delta \to \mathbb{R}$ such that

$$\gamma(\Gamma(h)) = \text{lw}(\Delta^{(1)}) + 2,$$

except if $\Delta \equiv \text{Conv}\{(2, 0), (0, 2), (-2, -2)\}$.

Example of a lattice polygon $\Delta$ for which we can prove the above conjecture:
A specific guess for the height function $h$: the “union skin” subdivision of $\Delta$

Example: $\gamma(\Delta) = \gamma(\Gamma(h)) = \text{lw}(\Delta) = 8$
Question: does the metric graph $\Gamma(h)$ (corresponding to the union skin subdivision of $\Delta$) have the same Clifford index and dimension as the generic curve $C(f)$?

Example: $\Delta = \text{Conv}\{(3, 0), (0, 3), (-3, -3)\}$

- $C(f)$ has a $g_9^3$ since it is the intersection of two cubics in $\mathbb{P}^3$. The Clifford index is $9 - 2.3 = 3 < 6 - 2.1 = 4$ and the Clifford dimension is 3.
- $\Gamma(h)$ also has a $g_9^3$. 
Thanks for listening!