More two-distance counterexamples to Borsuk’s conjecture from strongly regular graphs

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1 Abstract

In [1] (2013) and finally [2] Andriy V. Bondarenko showed how to construct a two-distance counterexample to Borsuk’s conjecture from any strongly regular graph whose vertex set is not the union of at most \(f+1\) cliques (sets of pairwise adjacent vertices) where \(f\) is the multiplicity of the second-largest eigenvalue of its adjacency matrix.

He applied that construction to those two graphs that he had been able to prove to fulfill the condition: From the \(G_2(4)\) graph (on 416 vertices) he got a 65-dimensional two-distance counterexample. From the \(F_{i23}\) graph (on 31671 vertices) he got a 782-dimensional one and, by considering certain induced subgraphs, counterexamples in dimensions 781, 780 and 779.

This article presents two other strongly regular graphs fulfilling the condition, on 28431 and on 2401, resp., vertices. It gives dedicated counterexamples in dimensions from 781 down to 764 derived from the bigger graph (that turned out to be an induced subgraph of the \(F_{i23}\) graph) and a 240-dimensional counterexample derived from the smaller graph.

Several contained propositions rely on the results of (often extensive) computations, mainly within the computer algebra system GAP. The source package contains (almost) all used source files.

2 Introduction

2.1 Borsuk’s conjecture

In [3] (1933) Karol Borsuk asked whether each bounded set in the \(n\)-dimensional Euclidean space can be divided into \(n+1\) parts of smaller diameter. The diameter of a set is defined as the supremum (least upper bound) of the distances of contained points. Implicitly, the whole set is assumed to contain at least two points.

The hypothesis that the answer to that question is positive became famous under the name Borsuk’s conjecture.

Beginning with Jeff Kahn and Gil Kalai in 1993, several authors have proved that in certain (almost all) high dimensions such a division is not generally possible.

2.2 Preliminaries

We consider simple loopless finite undirected graphs.

For any graph \(\Gamma = (V, E)\), any \(a \in V\) and \(W \subseteq V\):

- \(N(\Gamma, a, W) = \{b \in W : (a, b) \in E\}\),
- \(n(\Gamma, a, W) = |N(\Gamma, a, W)|\) and
- \(\Gamma[W]\) is the subgraph of \(\Gamma\) induced by \(W\), i.e., the graph whose vertex set is \(W\) and whose edge set is \(E \cap (W \times W)\).
Saying that a vertex $j$ is a non-neighbour of a vertex $i$ means that $i$ and $j$ are neither neighbours nor identical.

For a positive integer $d$ and a point set (vector set) $X$, the propositions “$X$ is in dimension $d$” and “$X$ is d-dimensional” both mean that the exact affine dimension of $X$ is at most $d$.

2.3 Strongly regular graphs

A graph $\Gamma = (V, E)$ is called strongly regular with parameter set $(v, k, \lambda, \mu)$, or shortly a srg$(v, k, \lambda, \mu)$, iff $|V| = v$ and for all $i, j \in V$

$$|\{h \in V : (h, i) \in E \wedge (h, j) \in E\}| = \begin{cases} k & \text{if } i = j \\ \lambda & \text{if } (i, j) \in E \\ \mu & \text{otherwise} \end{cases}$$

2.4 Euclidean representations of strongly regular graphs

This construction follows [1].

Let $\Gamma = (V, E)$ a srg$(v, k, \lambda, \mu)$ and $A$ its $(0,1)$-adjacency matrix. Then $A$ has exactly 3 different eigenvalues: $k$ of multiplicity 1, the second-largest eigenvalue of multiplicity $f = \frac{1}{2} \left( v - 1 - \frac{2k + (v - 1)(\lambda - \mu)}{\sqrt{(\lambda - \mu)^2 + 4(k - \mu)}} \right)$, and the smallest eigenvalue

$$s = \frac{1}{2} \left( \lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)} \right)$$

In the remaining part of this article we use these notations: $I$ is the identity matrix of size $v$, $y$ is $A - sI$, $y_i$, where $i \in V$, are the columns of $y$, and $y_{i,j}$, where $i, j \in V$, are the entries of $y$, and for any $W \subseteq V$, the corresponding point set $P(W)$ is $\{y_i : i \in W\}$.

Remark: Bondarenko used these $y_i$ as intermediate values to derive sets of $z_i$ and finally $x_i$ by synchronous scaling and moving in order to get all points onto the unit sphere around the origin but this is not necessary (here) and could lead to non-integer coordinate values.

The given properties of the eigenvalues imply $\dim P(V) = f$.

For $i, j \in V$

$$y_{i,j} = \begin{cases} -s & \text{if } i = j \\ 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

For $i \in V$: $y_i$ consists of one -s (at position $i$), $k$ 1’s, and $v - k - 1$ 0’s; its norm is the square root of $s^2 + k$.

For different $i, j \in V$

$$\|y_i - y_j\|^2 = \begin{cases} 2 \times (k - \lambda - 1 + (-s - 1)^2) = 2 \times (k - \lambda + s^2 + 2s) & \text{if } (i, j) \in E \\ 2 \times (k - \mu + s^2) & \text{otherwise} \end{cases}$$

The distance square for the non-adjacent case exceeds the distance square for the adjacent case by $2 \times (\lambda - \mu - 2s) = 2 \times \sqrt{(\lambda - \mu)^2 + 4(k - \mu)}$

If the graph is not complete, this excess is positive and we can conclude:

For any two different $i, j \in V$, the distance of $y_i$ and $y_j$ is smaller than the diameter of the complete vector set if and only if $i$ and $j$ are neighbours. Thus, for each $W \subseteq V$ the diameter of the corresponding point set $P(W)$ is smaller than that of $P(V)$ if and only if $W$ is a clique. And $P(V)$ can be divided into $x$ parts of smaller diameter if and only if $V$ can be divided into $x$ cliques.
2.4.1 Euclidean representations for vertex subsets

We will often consider \( P(W) \) for \( W \subset V \). After recognising that \( P(V) \) is a counterexample to Borsuk’s conjecture, we will try to find ones in a smaller dimensions. For that purpose, we will several times prove for subsets \( W_2 \subset W_1 \) of \( V \), that \( \dim P(W_2) \leq \dim P(W_1) - 1 \). This inequality is fulfilled iff there are \( x \in \mathbb{R}^v \) and \( c \in \mathbb{R} \) such that

\[
\forall i \in W_2 : \langle x, y_i \rangle = c \quad \text{and} \quad \exists i \in W_1 : \langle x, y_i \rangle \neq c.
\]

As it turned out to be useful in several cases, for any \( i \in V \) its non-neighbourhood \( \{ j \in V : i \neq j \land (i, j) \notin E \} \) induces a lower-dimensional vector set because for \( i, j \in V \)

\[
\langle y_i, y_j \rangle = \begin{cases} 
  s^2 + k & \text{if } i = j \\
  \lambda - 2s & \text{if } (i, j) \in E \\
  \mu & \text{otherwise}
\end{cases}
\]

and we know from above that \( \lambda - 2s - \mu = \sqrt{(\lambda - \mu)^2 + 4(k - \mu)} \) equals 0 only if the graph is complete (and thus any non-neighbourhood empty).

2.5 Bondarenko’s concrete results

In [1] and [2], A. Bondarenko considered the \( G_2(4) \) graph, a srg(416, 100, 36, 20), and the \( F_{123} \) graph, a srg(31671, 3510, 693, 351), whose corresponding point sets are of dimensions 65 and 782, resp. He proved that the sizes of contained cliques cannot exceed 5 and 23, resp. Because \( 416/5 > 83 \) and \( 31671/23 = 1377 \), the corresponding point sets cannot be divided into less than 84 and 1377, resp., parts of smaller diameter.

For each of those two graphs, he also gave a construction of a family of point sets which are dedicated counterexamples for all dimensions that are larger than that of the respective initial one.

In addition, he considered the vertices of the \( F_{123} \) graph that are non-neighbours of 1, 2 and 3, resp., pairwise adjacent fixed vertices. The corresponding point sets are in dimensions 781, 780 and 779, resp., contain (at least) 28160, 25344 and 23040, resp., elements, and thus cannot be divided into less than 1225, 1102 and 1002, resp., parts of smaller diameter.

Remark: As the reduction of the decrement of the number of remaining vertices indicates, the sizes of the common non-neighbourhoods of 4 and 5, resp., pairwise adjacent vertices are also large enough to provide counterexamples to Borsuk’s conjecture in dimensions 778 and 777, resp., but for the concrete numbers (of points and parts) one would have to do actual countings on the base of actually chosen vertices.

3 Regular partitions and induced subgraphs

In the case of the \( G_2(4) \) graph, the non-neighbourhood of a fixed vertex contains exactly 315 vertices. The corresponding point set is in dimension 64 and cannot be divided into less than 63 parts, but this does not prove the point set to be a counterexample.

But, as shown in [9] and [10], one can construct a 64-dimensional counterexample from a certain regular partition of the vertex set of the \( G_2(4) \) graph.

As explained in the following, the \( F_{123} \) graph does have the properties needed for an analogous construction, finally allowing to derive more than a dozen different counterexamples in dimensions below of 782.

3.1 A two-step construction of subsets

Let \( \Gamma = (V, E) \) a strongly regular graph, \( A \) its adjacency matrix, \( s \) the smallest eigenvalue of \( A \), \( y = A - sI \), \( W \subseteq V \), \( \{ B_1, B_2, B_3, C \} \) a partition of \( V \), \( (B_1 \cup B_2) \cap W \neq \emptyset \), \( B_3 \cap W \neq \emptyset \), such that
We have $\forall g, h \in \{1, 2, 3\} : g \neq h \rightarrow \forall i \in B_g : n(\Gamma, i, B_h) = 0$

(2) $\forall \Gamma_i \in C : n(\Gamma, i, B_1) = n(\Gamma, i, B_2) = n(\Gamma, i, B_3)$

Recall that for $i, j \in V$

$$y_{i,j} = \begin{cases} -s & \text{if } i = j \\ 1 & \text{if } (i, j) \in E \\ 0 & \text{otherwise} \end{cases}$$

For $h \in \{1, 2, 3\}$ let $x_h$ the vector with index set $V$ whose entries $x_{h,j}$, $j \in V$, are 1 if $j \in B_h$ and 0 otherwise. Consequently, for any $h \in \{1, 2, 3\}$ and $i \in V$

$$\langle x_h, y_i \rangle = \sum_{j \in B_h} y_{i,j} = \begin{cases} n(\Gamma, i, B_h) - s > 0 & \text{if } j \in B_h \\ n(\Gamma, i, B_h) & \text{if } j \in C \\ n(\Gamma, i, B_h) = 0 & \text{otherwise} \end{cases}$$

Let $p = x_1 - x_2$. Combined with (1) and (2) this implies $\forall i \in V$

$$\langle p, y_i \rangle = \langle x_1, y_i \rangle - \langle x_2, y_i \rangle = \begin{cases} (n(\Gamma, i, B_1) - s) - 0 = n(\Gamma, i, B_2) - s > 0 & \text{if } i \in B_1 \\ 0 - (n(\Gamma, i, B_2) - s) = s = n(\Gamma, i, B_2) < 0 & \text{if } i \in B_2 \\ 0 - 0 = 0 & \text{if } i \in B_3 \\ n(\Gamma, i, B_1) - n(\Gamma, i, B_2) = 0 & \text{if } i \in C \end{cases}$$

We have $\{i \in W : \langle p, y_i \rangle = 0\} = (B_3 \cup C) \cap W = W \setminus (B_1 \cup B_2) \subset W$.

Thus, $\dim P((B_3 \cup C) \cap W) \leq \dim P(W) - 1$.

Let $q = x_1 + x_2 - 2x_3$. Combined with (1) and (2) this implies $\forall i \in V$

$$\langle q, y_i \rangle = \langle x_1, y_i \rangle + \langle x_2, y_i \rangle - 2\langle x_3, y_i \rangle = \begin{cases} (n(\Gamma, i, B_1) - s) + 0 - 2 \times 0 = n(\Gamma, i, B_1) - s & \text{if } i \in B_1 \\ 0 + (n(\Gamma, i, B_2) - s) - 2 \times 0 = n(\Gamma, i, B_2) - s & \text{if } i \in B_2 \\ 0 + 0 - 2 \times (n(\Gamma, i, B_1) - s) = 2 \times (s - n(\Gamma, i, B_1)) & \text{if } i \in B_3 \\ n(\Gamma, i, B_1) + n(\Gamma, i, B_2) - 2 \times n(\Gamma, i, B_3) = 0 & \text{if } i \in C \end{cases}$$

We have $\{i \in (B_3 \cup C) \cap W : \langle q, y_i \rangle = 0\} = C \cap W \subset (B_3 \cup C) \cap W$.

Thus, $\dim P(C \cap W) \leq \dim P((B_3 \cup C) \cap W) - 1$.

### 3.2 Application to the $G_2(4)$ graph

As discussed in [9] and [10], one can find (many but equivalent) partitions $\{B_1, B_2, B_3, C\}$ of the vertex set $V$ with $|B_1| = |B_2| = |B_3| = 32$ such that the conditions for the construction are fulfilled in the case $W = V \wedge b = 20 \wedge c = 8$. The first reduction step results in a 64-dimensional point set of size 352 that cannot be divided into less than 71 parts of smaller diameter. It was the first known 64-dimensional counterexample to Borsuk’s conjecture. The second reduction step (considered in [9]) results in a 63-dimensional point set of size 320 that can be divided into 64 parts of smaller diameter and therefore is no counterexample.

### 3.3 Application to the $F_{23}$ graph

One can find a partition $\{B_1, B_2, B_3, C\}$ of the vertex set $V$ with $|B_1| = |B_2| = |B_3| = 1080$ such that the conditions for the construction are fulfilled in the case $W = V \wedge b = 351 \wedge c = 120$.

This proof (of this fact) is just an adaption of the first part of the section 4 of [10].

Let $\Gamma = (V, E)$ the $F_{23}$ graph and $\Sigma$ the $F_{24}$ graph, a srg$(306936,31671,3510,3240)$. It is well-known (cf. [2]) that $\Gamma$ occurs as the local graph of $\Sigma$. Let $x_0$ and $x_1$ two nonadjacent vertices of $\Sigma$. We can identify the set of the 31671 neighbours of $x_0$ with $V$; consequently, the common neighbours of $x_0$ and $x_1$ form a 3240-subset, say $B$ of $V$. 

This proof (of this fact) is just an adaption of the first part of the section 4 of [10].
The graph $\Sigma$ has a triple cover $3 \cdot \Sigma$ on 920808 vertices. It is distance-transitive with intersection array
\{31671, 28160, 2160, 1; 1, 1080, 28160, 31671\}. We see that $B$ is the disjoint union of three mutually nonadjacent subsets of size 1080. We call them $B_1$, $B_2$ and $B_3$. Let $C = V \setminus B$. According to [11], $3 \cdot \Sigma$ is tight, and (see in particular Figure A.4 in [11]) the partition $\{B_1, B_2, B_3, C\}$ possesses the required properties.

3.3.1 (Repeated) application of the two-step subset construction

As in the just given proof, we take the $Fi_{23}$ graph $\Sigma$ as the initial object, choose a vertex $x_0$ of $\Sigma$, get the $Fi_{23}$ graph $\Gamma$ as the subgraph induced by the neighbourhood of $x_0$. We start with $\Gamma$ and its vertex set $V$ and perform one or more two-step subset constructions, which we will call rounds and number from 1 onwards.

In round $k$, we choose a vertex $x_k$ of $\Sigma$ that is not adjacent to $x_0$ and different from $x_0, \ldots, x_{k-1}$. As seen above, one can get the partition $\{B_1, B_2, B_3, C\}$ of $V$ where $C$ is the non-neighbourhood of $x_k$ in $V$ and $B_1, B_2$ and $B_3$ are the components of the neighbourhood of $x_k$ in $V$. We use this partition to derive from a subset $Z_{2k-2}$ of $V$ the subsets $Z_{2k-1} = Z_{2k-2} \cap (B_3 \cup C)$ and $Z_{2k} = Z_{2k-2} \cap C$.

Let $Z_0 = V$.

Because the numbering of the three components of the neighbourhood of $x_k$ in $V$ is not automatically determined, one has to add the information which of the components should be denoted as $B_3$ in order to define $Z_{2k-1}$ completely.

Notice that $\dim P(Z_0) = \dim P(V) = 782$ and

$$Z_{2k} \neq Z_{2k-1} \neq Z_{2k-2} \implies \dim P(Z_{2k}) < \dim P(Z_{2k-1}) < \dim P(Z_{2k-2}).$$

Clearly, the dimension decrements are at least 1.

4 The computations

4.1 Prehistory

In subsection 3.3 of [6] and of [7] the authors in particular derived from the group $O(7, 3)$, also known as $O_7(3)$, a $\text{sr}(28431, 3150, 621, 315)$, denoted $\Gamma^6_6$, and a $\text{sr}(28431, 2880, 324, 288)$, denoted $\Gamma^3_6$, which seemed to be the first known SRGs with the respective parameter sets.

The appendix of [7] contains a script for the GAP computer algebra system, helping the readers to get hands on both graphs. In contrast to the body of [7], the appendix is accessible without restriction. The versions 1 to 4 of this article described a (mainly computational) partial exploration of $\Gamma^6_6$ and derived counterexamples to Borsuk’s conjecture in dimensions down to 774 from it.

In the survey preprint [6], the authors wrote in particular (without a proof or a dedicated reference) that $\Gamma^6_6$ is an induced subgraph of the $Fi_{23}$ graph, more precisely the subgraph of the $Fi_{24}$ graph induced by the intersection of the neighbourhood of a random vertex and the non-neighbourhood of a random non-neighbour of that vertex. In terms from the previous section this means that $\Gamma^6_6$ is (isomorphic to) $\Gamma^6_6[V_2]$, independent of the choice of $x_0$ and $x_1$ (implying that counterexamples to Borsuk’s conjecture derived from $\Gamma^6_6$ are also counterexamples derived from the $Fi_{23}$ graph).

The computations considered herein include a successful check of this assertion.

4.2 Computation outline and results

Preliminary note: It is common to speak of the $Fi_{23}$ graph and the $Fi_{24}$ graph, thereby abstracting from the concrete (labellings of the) vertex sets and considering isomorphic graphs as equal, even
identical. In order to express that we consider individual graphs and not isomorphism classes of graphs, we will speak of \( F_{i23} \) graphs and \( F_{i24} \) graphs in the remainder of this section.

From a certain \( F_{i24} \) graph \( \Sigma \), we get a \( F_{i23} \) graph \( \Gamma \) as the subgraph induced by the neighbourhood of vertex 1. We choose eight different non-neighbours \( e_1, \ldots, e_8 \) of vertex 1 and build three lists from these external (with respect to \( \Gamma \)) vertices of \( \Sigma \), namely \( L_1 = (e_1, e_2, e_3, e_6, e_7) \), \( L_2 = (e_1, e_2, e_3, e_4, e_5, e_6, e_7) \) and \( L_3 = (e_1, e_2, e_3, e_4, e_5, e_6, e_8) \).

For each of these three lists, we do this:

We take the contained vertices of \( \Sigma \) in the given order and perform the described two-step subset construction rounds. In each performed round \( k \), we number the components of the neighbourhood of the used external vertex in a way such that \( |Z_{2k-1}| \) is as large as possible, i.e.,

\[
|Z_{2k-2} \cap (B_3 \cup C)| \geq |Z_{2k-2} \cap (B_1 \cup C)| \text{ and } |Z_{2k-2} \cap (B_3 \cup C)| \geq |Z_{2k-2} \cap (B_2 \cup C)|.
\]

We check that \( |Z_{2k-1}| < |Z_{2k}| < |Z_{2k-2}| \). This allows us to conclude that dim \( P(Z_i) \leq 782 - i \) for all constructed sets \( Z_i \).

Because the first three external vertices in those three lists are the same, not just the initial \( Z_0 = V \) is independent of the used list but also the subsets \( Z_1, \ldots, Z_6 \). Their respective computed sizes are 29511, 28431, 26487, 25515, 23571, 22599.

The sizes of the then following subsets \( Z_7 \) and so on are:

- For \( L_1 \): 21111, 20367, 19119, 18405.
- For \( L_2 \): 20979, 20169, 18549, 17739, 16611, 16047, 15075, 14553.
- For \( L_3 \): 20979, 20169, 18549, 17739, 16611, 16047, 15075, 14589.

As mentioned above, Bondarenko proved in particular that the clique number (largest clique size) of the \( F_{i23} \) graph is 23. The results of in some cases rather time-consuming computations imply in particular: The clique numbers of the subgraphs induced by \( Z_2 \) and \( Z_6 \) are 21 and 19, resp., and for \( L_2 \) and \( L_3 \), the clique number of the subgraph induced by \( Z_{10} \) is 18.

These results also establish upper bounds for the clique sizes of the subgraphs induced by the other constructed subsets.

If \( W \) is one of the constructed subsets of \( V \), \( s \) (an upper bound of) the clique number of \( \Gamma[W] \) and an integer \( p < |W|/s \), then \( \Gamma[W] \) cannot be divided into \( p \) (or less) cliques.

The following table presents actual values for each of the constructed subsets of \( V \). In each row, the summarizing value in the last cell is the maximum of the values of \( p \) given in preceding cells in the same row and the values in the last column in following rows.

| \( Z \) | Dim | \( L_1 \) | \( L_2 \) | \( L_3 \) | > |
|---|---|---|---|---|---|
| 1 | 781 | \( e_1 \) | 29511/23 > 1283 | \( e_1 \) | 29511/23 > 1283 |
| 2 | 780 | e2 | 28431/21 > 1353 | e2 | 28431/21 > 1353 |
| 3 | 779 | e3 | 26487/21 > 1261 | e3 | 26487/21 > 1261 |
| 4 | 778 | e4 | 25515/21 > 1214 | e4 | 25515/21 > 1214 |
| 5 | 777 | e5 | 23571/21 > 1122 | e5 | 23571/21 > 1122 |
| 6 | 776 | e6 | 22599/19 > 1189 | e6 | 22599/19 > 1189 |
| 7 | 775 | e7 | 21111/19 > 1111 | e7 | 21079/19 > 1104 |
| 8 | 774 | e8 | 20367/19 > 1071 | e8 | 20169/19 > 1061 |
| 9 | 773 | e9 | 19119/19 > 1006 | e9 | 18549/19 > 976 |
| 10 | 772 | e10 | 18405/19 > 968 | e10 | 17739/18 > 985 |
| 11 | 771 | e11 | 16611/18 > 922 | e11 | 16611/18 > 922 |
| 12 | 770 | e12 | 16047/18 > 891 | e12 | 16047/18 > 891 |
| 13 | 769 | e13 | 15087/18 > 838 | e13 | 15075/18 > 837 |
| 14 | 768 | e14 | 14553/18 > 808 | e14 | 14589/18 > 810 |
The essence in a sentence:
The Euclidean representations of the subgraphs of \( \Gamma \) induced by the constructed subsets of \( V \) establish point sets in dimensions
\[ 781, 780, 779, 778, 777, 776, 775, 774, 773, 772, 771, 770, 769 \text{ and } 768, \text{ resp.,} \]
that cannot be divided into
\[ 1353, 1353, 1261, 1214, 1189, 1111, 1006, 985, 922, 891, 838 \text{ and } 810, \text{ resp.,} \]
parts of smaller diameter.

4.3 Computation environment and preconditions

All computations described herein have been done on a system with Intel Pentium(R) Dual-Core E5500 at 2.80 GHz and 4 GB RAM, running (Linux distribution) Lubuntu 20.04 (64 bit)), and with the computer algebra system GAP ([15], version 4.10.2) and the GAP packages GRAPE ([17], version 4.8.5) and AtlasRep ([18], version 2.1.0) installed.

Certain older versions of AtlasRep may work as well, but version 1.5.1-2 (installed with GAP 4.10.2) turned out to be too old. For some tasks, in particular some used in the computation considered here, GRAPE employs the program Dreadnaut (from the popular graph theoretic software nauty (by Brendan McKay and Adolfo Piperno, [16]). The installation of GRAPE includes the installation of (an older version of) Dreadnaut. For version 4.8.5 of GRAPE, in contrast to older versions, using files instead of (memory based) strings is default in the transmission of graph data to Dreadnaut, in order to avoid insufficient memory, in particular in the case of large graphs as treated here. Originally, GRAPE 4.8.5 refuses to run on GAP versions before 4.11. If you (as me) have GAP version 4.10.2 installed and can’t or don’t want to upgrade to a newer version, you can remove that obstacle by modifying the file PackageInfo.g coming with GRAPE 4.8.5: In the assignment to the record Dependencies, change the string value assigned to the field GAP from “\( > = 4.11 \)” to “\( > = 4.10.2 \)”.

As recorded in [13], L. H. Soicher, the author of GRAPE, replied to my respective question, that he did not see an argument against this work-around.

4.4 Input files

Nine (plain ASCII text) files are expected to be in the working directory before and during the computations. While GA36.g is from the mentioned appendix of [7], the others are included in the source package of this article.

Script files for the GAP system:

\texttt{GA36OUT.g, Fi23OUT.g, Fi24BOR.g, GA36.g, Fi23.g, Fi24.g and WRIGRA.g.}

(The latter four files are to be called from within one of the first three files).

Command files for Dreadnaut:

\texttt{Fi24L, Fi23.DRE and Fi24LR1_GA36.DRE}

Times given in the remainder of this section are CPU core running times, measured on the computer system described above.

4.5 Task 1: GAP processes \texttt{Fi24BOR.g}

The first action is a call of \texttt{Fi24.g}, in order to construct a \( F_{i24} \) graph \( \Phi_{24} \) from the finite group \( F_{i24}' \) and assign it to the variable \texttt{Graf}.

The (labels of) 8 certain different non-neighbours (in the outline denoted \( e_1, \ldots, e_8 \)) of vertex 1 of \( \Phi_{24} \) are assigned to the variable \texttt{Ext}.

Three lists of indices of those external (with respect to the neighbourhood of vertex 1) vertices are stored in the variable \texttt{elis}, reflecting the definition of the lists \( L_1, L_2 \) and \( L_3 \) in the outline.
For each of those three lists, the function `check_subset_counts` is called, to construct the partitions of the neighbourhood of vertex 1 induced by the external vertices, construct the vertex subsets (in the outline denoted $Z_1$ and so on), calculate their cardinalities and print the results (onto the screen, at least in the default case). (21 seconds)

Then an offer to start writing graph data into files and checking clique size bounds appears. The decision is left to the user because the execution could take long and needs a considerable amount of memory space (mostly for GAP itself, much less for the called program Dreadnaut). (2:06 hours and up to 1700 MB main memory space)

There are seven stages (passes). With the exception of the first one, they are ordered by increasing system demands (in order to get as much as possible work done when the system becomes overstressed).

4.5.1 Stage 1
Builds the subgraph $G$ of $\Phi_{24}$ induced by the vertex set consisting of the neighbourhood of vertex 1 and the eight external vertices and assigns it to the variable `Graf`. (44 minutes)
That graph contains the complete information that is relevant for the following stages. Its subgraph induced by the vertices from 1 to 31671 is a $F_{23}$ graph. The external vertices $e_1, \ldots, e_8$ are mapped to the other vertices (from 31672 to 31679), keeping their order.
Tests have shown that this action considerably reduces the memory space demands and the running time of the whole task.

4.5.2 Stage 2
Writes the data of the subgraph of $G$ induced by the first 31671 vertices (isomorphic to the local graph of $\Phi_{24}$) into the file `F124L.DRE`, using the input format of Dreadnaut. (31671 vertices, 301134276 bytes)

4.5.3 Stage 3
Writes the data of the subgraph of $G$ induced by those of the first 31671 vertices that are non-neighbours of the first external vertex into the file `F124LR1.DRE`, using the input format of Dreadnaut. (28431 vertices, 240295550 bytes)

4.5.4 Stage 4
Writes the data of the subgraph of $G$ induced by those of the first 31671 vertices that are common non-neighbours of the first 5 external vertices into the file `F124LR5.A`, using the ASCII version of the DIMACS graph data file format. (17739 vertices, 221837476 bytes)

4.5.5 Stage 5
Writes the data of the subgraph of $G$ induced by those of the first 31671 vertices that are common non-neighbours of the first 3 external vertices into the file `F124LR3.A`, using the ASCII version of the DIMACS graph data file format. (22599 vertices, 368011696 bytes)

4.5.6 Stage 6
Checks that the subgraph of $G$ induced by those of the first 31671 vertices that are common non-neighbours of the first 5 external vertices does not have a clique of size 19.
4.5.7 Stage 7

Checks that the subgraph of $G$ induced by those of the first 31671 vertices that are common non-neighbours of the first 3 external vertices does not have a clique of size 20.

4.6 Task 2: GAP processes GA36OUT.g

Constructs $\Gamma_3^{(3)}$ from the finite group $O_8^+(3)$, calculates and shows the index number of its local graph (at vertex 1) and writes its data into the file GA36.DRE, using the input format of Dreadnaut. (239663676 bytes, 2:25 minutes)

4.7 Task 3: GAP processes Fi23OUT.g

Constructs a $F_{i23}$ graph from the finite group $F_{i23}$ and writes its data into the file $Fi23.DRE$, using the input format of Dreadnaut. (300751232 bytes, 3:47 minutes, about 1 GB main memory space)

4.8 Task 4: Dreadnaut processes Fi24L

Reads the data of two graphs from the files $Fi24L.DRE$ and $Fi23.DRE$, resp., and checks that those two graphs are isomorphic, by canonical labelling of both graphs and checking the results, assigned to the variables $h$ and $h'$, for identity. (Duration of reading in and labelling: 96.59 seconds and 58.25 seconds, resp.)

4.9 Task 5: Dreadnaut processes Fi24LR1_GA36.DRE

Reads the data of two graphs from the files $Fi24LR1.DRE$ and $GA36.DRE$, resp., and checks that those two graphs are isomorphic, by canonical labelling of both graphs and checking the results, assigned to the variables $h$ and $h'$, for identity. (Duration of reading in and labelling: 829.77 seconds seconds and 725.48 seconds, resp., much longer than for the analogous operations in task 4 although the graphs are smaller here)

4.10 Tasks 7 and 8 (optional): Independent calculation of clique numbers

Apply a clique number computing program that accepts input files in the ASCII version of the DIMACS graph data format to one or both of $Fi24LR3.A$ and $Fi24LR5.A$ generated in stages 4 and 5, resp., of task 1. This way, the clique numbers could be found (or verified) independent of GAP but probably at the cost of (much) more computer runtime if that program does not make use of symmetries as much as GAP does. In the tests, the popular Cliquer software (by Sampo Niskanen and Patric C. Östergård) has been used.

5 A counterexample from a srg(2401,240,59,20)

The chapter Individual graph descriptions of [4] contains a section on rank 3 graphs on 2401 vertices. One of those graphs is a srg(2401, 240, 59, 20). These parameters imply that the multiplicity $f$ of the second-largest eigenvalue is 240. Thus, the Euclidean representation is 240-dimensional. According to the information given in that subsection, the largest clique size of that graph is 9. Because $9 \times 266 < 2401$, that graph can not be divided into less than 267 cliques, and its Euclidean representation can not be divided into less than 267 parts of smaller diameter.

Information on a construction of that graph is given in preceding chapters of [4]. The vertices are the $7^4$ vectors of $GF(7)^4$, and two vertices are adjacent iff their difference is in a certain set of 240
vectors from a projective (40, 4, 12, 5) set of points in the projective space PG(3,7). An explanation of this type of correspondences can be found in the popular early survey paper [5].

5.1 Derived counterexamples in lower dimensions?

The subgraph induced by the non-neighbours of a random vertex contains exactly $2401 - 1 - 240 = 2160$ vertices. The dimension of its Euclidean representation is $240 - 1 = 239$. It would establish a counterexample to Borsuk’s conjecture if a partition into 240 cliques would be impossible. Because $2160 = 240 \times 9$, one can not prove this by a single division.

5.2 Computations

The source package of this article contains the Pascal source file SRG2401E.PAS, written on the base of the construction of the graph considered here as given in [4]. The projective point set has been pre-calculated and is given as constant p3to7. The variable A to store the adjacency relation is initialized during runtime.

For the tests, the Free Pascal compiler has been used, started by entering the command line

```
fpc -Mtp SRG2401E.PAS
```

The vast majority of the running time has been used by an (unsuccessful) search for a 10-clique and by a check that the graph is indeed strongly regular with the given parameter set.

The respective durations were about 17 and 51 seconds on the computer system described above, using compiler version 3.0.4 for x86_64. They were about 47 and 170 seconds on a 1 GHz PIII running MS Windows 98SE, using compiler version 2.4.4 for i386.

The search for a 10-clique would take much longer if not the symmetry of the graph caused by its construction using a difference set would allow to consider just the cliques containing vertex 1.

The strong regularity does not have to be checked computationally because it can be easily concluded from the construction and the fact that the instruction `check_intersections(12,5)` has been passed. But I do not know of a mathematical proof of the fact that that graph does not have a 10-clique.

References

[1] Andriy V. Bondarenko, On Borsuk’s conjecture for two-distance sets, arXiv:math.MG/1305.2584v2.

[2] Andriy Bondarenko, On Borsuk’s Conjecture for Two-Distance Sets, Discrete Comput. Geom., Springer 2014, DOI 10.1007/s00454-014-9579-4

[3] Karol Borsuk, Drei Sätze über die $n$-dimensionale euklidische Sphäre, Fund. Math., 20 (1933), 177-190.

[4] Andries E. Brouwer, Hendrik Van Maldeghem, Strongly regular graphs, a preprint downloaded 2021-06-17 from https://homepages.cwi.nl/~aeb/math/srg/ru3/srgw.pdf, listed as fragments of a text on strongly regular graphs in section 2021 of https://www.win.tue.nl/~aeb/preprints.html

[5] R. Calderbank, W. M. Kantor, The geometry of two weight codes, Bull. London Math. Soc. 18 (1986), 97-122.

[6] Dean Crnković, Sanja Rukavina, Andrea Švob, On some distance-regular graphs with many vertices, arXiv:math.CO/1809.10197v2.
[7] Dean Crnković, Sanja Rukavina, Andrea Švob, *On some distance-regular graphs with many vertices*, J. Algebr. Comb., Vol. 51, Iss. 4, June 2020, 641-652, DOI 10.1007/s10801-019-00888-5

[8] Xavier L. Hubaut, *Strongly regular graphs*, Discrete Math., Vol. 13 (1975), 357-381.

[9] Thomas Jenrich, *A 64-dimensional two-distance counterexample to Borsuk’s conjecture*, arXiv:math.MG/1308.0206v6

[10] Thomas Jenrich, Andries E. Brouwer, *A 64-dimensional counterexample to Borsuk’s conjecture*, Electronic Journal of Combinatorics, 21(4):Paper 4.29, 3, 2014

[11] Aleksandar Jurišić, Jack Koolen, Paul Terwilliger, *Tight Distance-Regular Graphs*, J. Algebr. Comb., Vol. 12 (2000), 163-197

[12] Dmitrii V. Pasechnik, *Geometric Characterization of the Sporadic Groups Fi22, Fi23, and Fi24*, J. of Comb. Theory, Series A, Vol. 68 (1994), 100-114

[13] Robert A. Wilson, *The finite simple groups*, Springer, 2009

[14] Issue #28 on GAP package GRAPE: Why does grape version 4.8.5 demand GAP version 4.11? [https://github.com/gap-packages/grape/issues/28](https://github.com/gap-packages/grape/issues/28)

[15] The GAP Group, *GAP (Groups, Algorithms, Programming)*, a System for Computational Discrete Algebra, Version 4.10.2 of 19-Jun-2019, [http://www.gap-system.org](http://www.gap-system.org).

[16] Brendan McKay, Adolfo Piperno, *nauty and Traces*, [http://pallini.di.uniroma1.it/index.html](http://pallini.di.uniroma1.it/index.html), [http://cs.anu.edu.au/~bdm/nauty/](http://cs.anu.edu.au/~bdm/nauty/).

[17] Leonard H. Soicher, *GRAPE (GRaph Algorithms using PErmutation groups)*, a GAP package, Version 4.8.5 (Released 26/03/2021), [http://www.gap-system.org/Packages/grape.html](http://www.gap-system.org/Packages/grape.html).

[18] Robert A. Wilson, Richard A. Parker, Simon Nickerson, John N. Bray, Thomas Breuer, *AtlasRep*, a GAP Interface to the Atlas of Group Representations, Version 2.1.0 (Released 10/05/2019), [https://www.gap-system.org/Packages/atlasrep.html](https://www.gap-system.org/Packages/atlasrep.html).

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