THE HOD HYPOTHESIS AND A SUPERCOMPACT CARDINAL

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Abstract. In this paper, we prove that: if $\kappa$ is supercompact and the HOD Hypothesis holds, then there is a proper class of regular cardinals in $V_\kappa$ which are measurable in HOD. From [11], Woodin also proved this result. As a corollary, we prove Woodin’s Local Universality Theorem. This work shows that under the assumption of the HOD Hypothesis and supercompact cardinals, large cardinals in $V$ are reflected to be large cardinals in HOD in a local way, and reveals the huge difference between HOD-supercompact cardinals and supercompact cardinals under the HOD Hypothesis.

1. Introduction

The HOD Hypothesis is an important hypothesis about HOD proposed by W.Hugh Woodin in [10], which says that there is a proper class of regular cardinals that are not $\omega$-strongly measurable in HOD (see Definition 3.3 and Definition 3.4). In [5], Woodin uses the term “The HOD Conjecture” to denote the same statement as the HOD Hypothesis. For this paper, our main references are [5] and [11]; all basic facts about the HOD Hypothesis we know are in [5] and [10]. Our notations are standard, see [3] and [4].

Examining under which hypothesis HOD and $V$ are close to each other and how HOD and $V$ can be pushed apart via forcing is a very interesting area of research. From [1], via forcing, behaviors of large cardinals from $V$ can become disordered in HOD. A natural question is whether the HOD Hypothesis has some effect on the behavior of large cardinals from $V$ in HOD. We want to know whether under the HOD Hypothesis, behaviors of large cardinals from $V$ become more regular in HOD; Especially, whether and how, under the HOD Hypothesis, large cardinals in $V$ can be transferred into HOD. In this paper, we answer this question for one supercompact cardinal and prove the following main result: if $\kappa$ is supercompact and the HOD Hypothesis holds, then there is a proper class of regular cardinals below $\kappa$ which are measurable in HOD. From [11], Woodin also proved this theorem. Woodin proved the Global Universality Theorem in [5, Theorem 201] and announced his Local Universality Theorem in [11]. As a corollary of the above main result, we have Woodin’s Local Universality Theorem (see Corollary 4.5).

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This paper is organized in the following way: In Section 2, we discuss the three main motivations for the HOD Hypothesis; In Section 3, we give a systematical and self-contained introduction to the HOD Hypothesis and its basic facts which would be used in later passages; In Section 4, we prove our main result Theorem 4.3; In Section 5, we conclude with some natural and interesting questions.

2. The Motivation of the HOD Hypothesis

The inner model program for one supercompact cardinal, the limits of the large cardinal hierarchy and the HOD Dichotomy Theorem are the three main motivations for the HOD Hypothesis.

(1) Inner model theory has a long and complex history, starting with Jensen’s work on L from the 1960’s. There is a large variety of inner models (by ‘inner models’ we mean transitive models of ZFC containing all the ordinals), and one natural classification criterion for them is their structural simplicity and their invariance with respect to extensions of the universe via forcing. In one extreme we have L. L has a well–understood fine structure and all models of set theory with the same ordinals have exactly the same version of L. It follows that we can decide most natural questions in mathematics by working in L (more accurately put, by working in the theory ZFC + V = L). In the other extreme, we have the universe, V, which in a typical theory T of the form ZFC + large cardinals is quite underdetermined. HOD is also, to a large extent, such an underdetermined inner model. Given the above, it would seem that L would be a natural choice for our universe. L has a serious drawback, though, which is that it can contain only very weak large cardinals.

The main goal of inner model theory is to build, under suitable assumptions¹ inner models containing suitable large cardinals but with as many of the nice structural properties of L as possible (in particular, it would be desirable to be able to run a ‘fine–structural’ analysis of these models). Also, these inner models would be typically supposed to be as small as possible (in the sense of containing, besides all ordinals, just the bare minimum amount of information that would enable them to accommodate the large cardinal hypothesis at hand). Inner models of large cardinals in this sense are always so–called extender models, i.e., models constructed in the same way as L but incorporating in the construction certain (carefully chosen) approximations to the relevant elementary embeddings that we would like the final model to capture. The strongest large cardinal hypothesis within reach of the present inner model theory is some accumulation of Woodin cardinals. This is much stronger than, say, the existence of a measurable, but far weaker than, for example, the existence of a supercompact cardinal².

A surprising fact due to Woodin is that if the inner model program can be extended to prove that if there is a supercompact cardinal then there is a

¹There will be for example forcing extensions satisfying this same theory T but disagreeing with the ground model about the truth value of, for example, the Continuum Hypothesis.
²For example, but not only, under the assumption that the relevant large cardinal axiom holds in V.
³Supercompact cardinals figure prominently in many consistency proofs in higher set theory; famously, in the consistency proof of the maximal forcing axiom Martin’s Maximum due to Foreman–Magidor–Shelah in 1984, for example.
so-called L–like weak extender model for a supercompact cardinal, then that
L–like model accommodates all large cardinal axioms that have ever been con-
sidered[4] and is close to V in a certain well–defined sense. If that construction
of an L–like model is definable (so the model is contained in HOD), then HOD
must necessarily be close to V in the relevant sense, and in particular the HOD
Hypothesis must be true. Therefore, if the inner model program can be ex-
tended to the level of one supercompact cardinal, then the HOD Hypothesis
must be true (if there is a supercompact cardinal). This motivates the HOD
Hypothesis, in the sense that the HOD Hypothesis is a good test question for
the success of the inner model program for one supercompact cardinal and, by
the comments above, for the happy conclusion of the inner model program.

(2) If the HOD Hypothesis is provable, then one can in a natural hierarchy of
large cardinal axioms give a threshold for inconsistency, against just ZF as
background theory, which closely parallels Kunen’s inconsistency in the ZFC
context.

**Theorem 2.1.** (Woodin, [5]) (ZF) Assume “ZF + there is a supercompact
cardinal” implies the HOD Hypothesis. Suppose δ is an extendible cardinal and
λ > δ. Then there is no non-trivial elementary embedding j : Vλ+2 → Vλ+2.

It is a matter of fact that most large cardinal hypotheses can be naturally
stated in terms of the existence of elementary embeddings of the form j : (V, ∈)
→ (M, ∈) different from the identity, where M is some transitive model.
The closer the structure M is to V, the stronger is the large cardinal situation
posed. Usually, the relevant large cardinal is the critical point of j (i.e., the
least ordinal κ such that κ < j(κ)). The above has been traditionally a general
template for generating large cardinal axioms and explains, in many cases,
why most large cardinal axioms considered to date tend to build a linearly
ordered hierarchy with respect to consistency strength[5]. For example, κ is a
supercompact cardinal if and only if for every ordinal λ it holds that κ is the
critical point of some elementary embedding j : (V, ∈) → (M, ∈), where M is a
transitive class closed under λ–sequences (i.e., for every sequence (ai : i < λ),
if each ai is in M, then (ai : i < λ) ∈ M). A natural upper limit for large
cardinal axioms given by the above template is therefore the situation where M
is actually all of V; in other words, the statement that there is an elementary
embedding j : (V, ∈) → (V, ∈) which is not the identity[6]. The existence of
such an elementary embedding was proposed by W. Reinhardt in his doctoral

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[4]Woodin uses the term Ultimate–L to refer to the hypothetical inner model that includes
supercompact cardinals and therefore all large cardinals. This Ultimate–L would be robust enough
with respect to forcing that one would be able to answer essentially all natural questions by
working in V = Ultimate–L. This would make a very strong case for adopting the axiom V =
Ultimate–L. The construction of this Ultimate–L, if possible, would be a natural culmination of
the inner model program the way it is currently understood.

[5]Typically, if j : (V, ∈) → (M, ∈) is an elementary embedding with critical point κ and M is
“sufficiently close to V”, then M thinks that κ is the critical point of an elementary embedding
i : (M, ∈) → (N, ∈) in which the target model N is in principle “less close to M” than M was to
V. By elementarily of j and since κ < j(κ) it follows that, in V, there is a λ < κ which is the
critical point of an elementary embedding i : (V, ∈) → (N, ∈), where N has the second (weaker)
degree of closure relative to V.

[6]The way it is expressed here, this is a second order statement, but there are various ways to
make sense of this in a first order way.
dissertation from 1967. A few years later (in 1971), Kenneth Kunen proved in a landmark result that such elementary embeddings cannot possibly exist. One hypothesis used crucially in Kunen’s proof—and in all other alternative proofs found afterwards—is that $V$ satisfies the Axiom of Choice. In fact, after more than 40 years it is not yet known whether the nonexistence of a nontrivial elementary embedding $j : (V, ∈) → (V, ∈)$ can be proved just assuming $V \models \text{ZF}$. Theorem 2 suggests that proving the HOD Hypothesis would have a huge foundational significance, in that it would provide a route to showing that there are no nontrivial elementary embeddings from $V$ to $V$ even if AC fails.

(3) The HOD Dichotomy says that either HOD is close to $V$ or else HOD is far from $V$.

**Theorem 2.2.** *(Woodin, HOD Dichotomy Theorem, Theorem 2 in [9]) Assume that $δ$ is an extendible cardinal. Then exactly one of the following holds.
(a) For every singular cardinal $γ > δ$, $γ$ is singular in HOD and $γ^+ = (γ^+)_{\text{HOD}}$.
(b) Every regular cardinal greater than $δ$ is measurable in HOD.

Note that the two possible scenarios, (a) and (b), given by the HOD Dichotomy Theorem look indeed very different (i.e., (b) looks like a quite small subset of the logical negation of (a)). In fact, (a) says that HOD is close to $V$ in the way that $L$ is close to $V$ when $0^\#$ does not exist, and (b) says that HOD is small compared to $V$ also in very much the same way that $L$ is small compared to $V$ when $0^\#$ exists. The HOD Dichotomy Theorem 2.2 motivates the HOD Hypothesis: The HOD Hypothesis rules out possibility (b) and therefore says that only (a) can be the case and therefore HOD is always close to $V$.

3. The HOD Hypothesis

In this section, we give a self-contained exposition of Woodin’s results about the HOD Hypothesis. Intuitively, the HOD Hypothesis just says that HOD is close to $V$ in a certain sense.

The following Theorem 3.2 is very important and we use it several times in this paper. Firstly, we list some important facts about forcing with respect to HOD which are used to prove Theorem 3.2.

**Proposition 3.1.** *(1) *(Forklore, [9]) If $P$ is a weakly homogeneous and ordinal definable poset in $V$ and $G$ is a $V$-generic filter on $P$, then $\text{HOD}_{V[G]} ⊆ \text{HOD}_V$.
(2) *(9 Lemma 4, Theorem 5) If $κ > ω$ is an regular cardinal, $P$ is a poset with $|P| < κ, G$ is a $P$-generic over $V$, then in $V[G], V$ is $Σ_2$ definable from $V \cap P(κ)$.

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7 As a matter of fact, no other inconsistency in the realm of large cardinal hypotheses has been discovered. It could well be that the existence of a non-trivial elementary embedding $j : (V, ∈) → (V, ∈)$ in the absence of choice is consistent. There could even be a rich hierarchy of consistent large cardinal hypotheses extending the hypothesis that there is such an elementary embedding, and therefore incompatible with choice; this would indicate that the Axiom of Choice eventually fails as we climb up the large cardinal hierarchy in very much the same way that $V = L$ fails as we climb up the large cardinal hierarchy (specifically, when we pass the $0^\#$ barrier).

8 More precisely, Jensen’s Dichotomy Theorem for $L$ says that exactly one of the following holds: (1) $L$ is correct about singular cardinals and computes their successors correctly or (2) Every uncountable cardinal is inaccessible in $L$. Theorem 2.2 can therefore be seen as a generalization for HOD of Jensen’s theorem for $L$.
For every ordinal \( \kappa \), there exists \( B \in \text{HOD} \) such that \( \text{HOD} \models B \) is a complete Boolean algebra, and for any \( E \subseteq \kappa \), there exists a HOD-generic filter \( G \) on \( B \) such that \( \text{HOD}[E] \subseteq \text{HOD}_{\{E\}} = \text{HOD}[G] \).

(4) (Vopěnka, [9, Theorem 15.43]) Let \( G \) be generic on \( B \). If \( M \) is a model of \( \text{ZFC} \) such that \( V \subseteq M \subseteq V[G] \), then there exists a complete subalgebra \( D \subseteq B \) such that \( M = V[D \cap G] \).

**Theorem 3.2.** (Woodin, [10], Corollary 7) Let \( \mathbb{P} \in \text{OD} \) be a weakly homogeneous poset. Suppose \( G \) is a \( V \)-generic filter on \( \mathbb{P} \). Then \( \text{HOD}^V \) is a generic extension of \( \text{HOD}^{V[G]} \).

**Proof.** Let \( \kappa \) be an uncountable regular cardinal such that \( |\mathbb{P}| < \kappa \). By Proposition 3.1.2, \( V \) is definable in \( V[G] \) from \( S \) where \( S = \mathcal{P}(\kappa) \cap V \). In \( V \), let \( \delta = |S| \) and \( E \) be a binary relation on \( \delta \) such that the Mostowski collapse of \( (\delta, E) \) is \( (\text{trcl}(\{S\}), \in) \). Then \( \text{HOD}^V \subseteq \text{HOD}_{\{E\}}^{V[G]} \). By Proposition 3.1.3, there is a \( \text{HOD}^{V[G]} \)-generic filter \( H \) on a Vopěnka algebra such that \( \text{HOD}_{\{E\}}^{V[G]} = \text{HOD}_{\{E\}}^{V[G][H]} \). By Proposition 3.1.1, \( \text{HOD}^{V[G]} \subseteq \text{HOD}^V \). Since \( \text{HOD}^{V[G]} \subseteq \text{HOD}^V \subseteq \text{HOD}_{\{E\}}^{V[G]} = \text{HOD}^{V[G][H]} \), by Proposition 3.1.4, \( \text{HOD}^V \) is a generic extension of \( \text{HOD}^{V[G]} \). \( \square \)

**Definition 3.3.** (Woodin, [10], Definition 189) Let \( \lambda \) be an uncountable regular cardinal. Then \( \lambda \) is \( \omega \)-strongly measurable in \( \text{HOD} \) iff there is \( \kappa < \lambda \) such that \( (2^\kappa)^\text{HOD} < \lambda \) and there is no partition \( \langle S_\alpha \mid \alpha < \kappa \rangle \) of \( \text{cof}(\omega) \cap \lambda \) into stationary sets such that \( \langle S_\alpha \mid \alpha < \kappa \rangle \in \text{HOD} \).

**Definition 3.4.** (Woodin, [10], Definition 3.42) The HOD Hypothesis denotes the following statement: there is a proper class of regular cardinals that are not \( \omega \)-strongly measurable in \( \text{HOD} \).

In Woodin’s recent paper [10], the HOD Conjecture denotes the following statement as in Definition 3.5. In Woodin’s old paper such as [9, 5] and [10], the HOD Conjecture denotes the same statement as the HOD Hypothesis.

**Definition 3.5.** (Woodin, [10], Definition 3.48) The HOD Conjecture denotes the following statement: the theory \( \text{ZFC} \) + “there exists a supercompact cardinal” proves the HOD Hypothesis.

**Theorem 3.6.** (Woodin, [9], Lemma 10) Suppose \( \kappa \) is \( \omega \)-strongly measurable in \( \text{HOD} \). Then \( \kappa \) is measurable in \( \text{HOD} \).

Note that if \( V = \text{HOD} \), then no cardinals can be \( \omega \)-strongly measurable in \( \text{HOD} \). So \( \kappa \) is measurable in \( \text{HOD} \) does not imply \( \kappa \) is \( \omega \)-strongly measurable in \( \text{HOD} \).

**Definition 3.7.** (Woodin, [5], Definition 132) Suppose \( N \) is a proper class inner model of \( V \) and \( \delta \) is a supercompact cardinal. Then \( \delta \) is \( N \)-supercompact if for all \( \lambda > \delta \), there exists an elementary embedding \( j : V \to M \) such that \( \text{crit}(j) = \delta, j(\delta) > \lambda, M^{\lambda_\delta} \subseteq M \) and \( j(N \cap V_\delta) \cap V_\lambda = N \cap V_\lambda \).

**Theorem 3.8.** (Woodin, [5], Lemma 188) Suppose that \( \delta \) is an extendible cardinal. Then \( \delta \) is \( \text{HOD} \)-supercompact.

Note that if \( \delta \) is extendible, then \( \delta \) is a limit of \( \text{HOD} \)-supercompact cardinals.

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9The notion of \( N \)-supercompactness is a generalization of supercompactness. \( \kappa \) is supercompact does not imply that \( \kappa \) is \( \text{HOD} \)-supercompact.
Definition 3.9. (Woodin, [9, Definition 15]) Suppose $N$ is a transitive class, $\text{Ord} \subseteq N$ and $N \models \text{ZFC}$. $N$ is a weak extender model for $\delta$ supercompact if for every $\gamma > \delta$ there exists a normal fine $\delta$-complete measure $U$ on $P_\delta(\gamma)$ such that $N \cap P_\delta(\gamma) \in U$ and $U \cap N \subseteq N$.

The notion of “weak extender model for $\delta$ supercompact” is very important in the study of Inner Model Theory for one supercompact cardinal. Woodin speculates that the extension to the level of one supercompact cardinal should yield as a theorem that if $\delta$ is supercompact then there exists $N \subseteq \text{HOD}$ such that $N$ is a weak extender model for $\delta$ supercompact (c.f. [8]).

Theorem 3.10. (Woodin, [5, Theorem 138]) Suppose $N$ is a weak extender model for $\delta$ supercompact and $\gamma > \delta$ is a singular cardinal. Then $\gamma$ is singular in $N$ and $\gamma^+ = (\gamma^+)^N$.

Theorem 3.11. (Magidor, [4, Theorem 22.10]) $\delta$ is supercompact if and only if for every $\kappa > \delta$, there exist $\alpha < \delta$ and an elementary embedding $j : V_{\alpha+1} \to V_\kappa$ with critical point $\delta$ such that $j(\delta) = \delta$.

The following theorem is a generalization of Magidor’s characterization of supercompactness and an alternative formulation of “weak extender model for $\delta$ supercompact” in terms of suitable elementary embeddings.

Theorem 3.12. (Woodin, [9, Theorem 21]) Let $N$ be a proper class inner model of $\text{ZFC}$. Then the following are equivalent:

1. $N$ is a weak extender model for $\delta$ supercompact.
2. For every $\kappa > \delta$, there exist $\alpha < \delta$ and an elementary embedding $j : V_{\alpha+1} \to V_\kappa$ such that:
   (a) $\text{crit}(j) = \delta$ and $j(\delta) = \delta$;
   (b) $j \upharpoonright (N \cap V_\alpha) \in N$ and $j(N \cap V_\alpha) = N \cap V_\kappa$.

As a corollary of Theorem 3.12, we have the following universality theorem for weak extender model for $\delta$ supercompact.

Theorem 3.13. (Woodin, [5, Theorem 144]) Suppose $N$ is a weak extender model for $\delta$ supercompact and $\gamma > \delta$ is a cardinal of $N$. If $j : (H_{\gamma^+})^N \to (H_{j(\gamma^+)})^N$ is an elementary embedding with $\text{crit}(j) \geq \delta$. Then $j \in N$.

Theorem 3.14. (Woodin, [5, Theorem 193]) Suppose the HOD Hypothesis holds and $\delta$ is HOD-supercompact. Then HOD is a weak extender model for $\delta$ supercompact.

From [5, Lemma 136], if $N$ is a weak extender model for $\delta$ supercompact, then $\delta$ is $N$-supercompact. So if the HOD Hypothesis holds, then “HOD is a weak extender model for $\delta$ supercompact” is equivalent to $\delta$ is HOD-supercompact.

From Theorem 3.12, Theorem 3.13 and Theorem 3.14, if the HOD Hypothesis holds and $\kappa$ is extendible (or $\kappa$ is HOD-supercompact) in $V$, then $\kappa$ is supercompact in HOD.

The following remarkable Universality Theorem follows from Theorem 3.14 and Theorem 3.13.

\footnote{Theorem 3.12 is a reformulation of [9, Theorem 21] in terms of Magidor’s characterization of supercompactness.}

\footnote{Theorem 3.14 is a reformulation of [5, Theorem 193].}
Theorem 3.15. (Woodin, Global Universality Theorem, ([3] Theorem 201)) Suppose the HOD Hypothesis holds and $\delta$ is HOD-supercompact. If $j: \text{HOD} \cap V_{\lambda+1} \rightarrow M \subseteq \text{HOD} \cap V_{j(\gamma)+1}$ is an elementary embedding with $\text{crit}(j) \geq \delta$. Then $j \in \text{HOD}$.

As a corollary of Theorem 3.15 if the HOD Hypothesis holds and $\delta$ is HOD-supercompact, then there is no non-trivial elementary embedding $j: \text{HOD} \rightarrow \text{HOD}$ such that $\delta \leq \text{crit}(j)$.

The following definition of super-HOD cardinal is isolated from the proof of Theorem 3.15 in [3] Theorem 193. From Theorem 3.15, Definition 3.16 provides a different and equivalent definition of HOD-supercompact cardinal.

Definition 3.16. Define that $\kappa$ is a super-HOD cardinal if for any $\lambda > \kappa$ and any $a \in V_\lambda$, there exist $j: V_{\lambda_0+\omega} \rightarrow V_{\lambda+\omega}, a_0 \in V_{\lambda_0}$ and $\kappa_0 < \lambda_0 < \kappa$ such that $\text{crit}(j) = \kappa_0, j(\kappa_0) = \kappa, j(a_0) = a$ and $j(\text{HOD} \cap V_{\lambda_0}) = \text{HOD} \cap V_\lambda$.

Lemma 3.17. Suppose $\lambda > \kappa$ are uncountable regular cardinals, $|V_\lambda| = \lambda$ and $\text{HOD} \cap V_\lambda = (\text{HOD})^{V_\lambda}$. Then the following two statements are equivalent:

1. There exists an elementary embedding $j: V \rightarrow M$ such that $\text{crit}(j) = \kappa, j(\kappa) > \lambda, M^{V_\lambda} \subseteq M$ and $j(\text{HOD} \cap V_\lambda) \cap V_\lambda = \text{HOD} \cap V_\lambda$.

2. There exists a normal fine $\kappa$-complete ultrafilter $U$ on $P_\kappa(V_\lambda)$ such that $Z \in U$ where $Z = \{X < V_\lambda: \text{the transitive collapse of } X \text{ is } V_\theta \text{ for some } \theta \text{ such that } \text{HOD} \cap V_\theta = (\text{HOD})^{V_\theta}\}$.

Proof. It suffices to check that $Z \in U$ iff $j(\text{HOD} \cap V_\lambda) \cap V_\lambda = \text{HOD} \cap V_\lambda$. Note that $Z \in U$ iff $\{j(x): x \in V_\lambda\} \in j(Z)$. Note that $j(Z) = \{X < V_j(\lambda): \text{the transitive collapse of } X \text{ is } V_\theta \text{ for some } \theta \text{ such that } j(\text{HOD} \cap V_\lambda) \cap V_\theta = (\text{HOD})^{V_\theta}\}$. Since the transitive collapse of $\{x: x \in V_\lambda\}$ is $V_\lambda$ and $\text{HOD} \cap V_\lambda = (\text{HOD})^{V_\lambda}$, we have $\{j(x): x \in V_\lambda\} \in j(Z)$ iff $j(\text{HOD} \cap V_\lambda) \cap V_\lambda = (\text{HOD})^{V_\lambda} = \text{HOD} \cap V_\lambda$.

Theorem 3.18. The following three statements are equivalent:

1. $\kappa$ is HOD supercompact.

2. $\kappa$ is a super-HOD cardinal.

3. For any $\lambda > \kappa$, there exists a normal fine $\kappa$-complete ultrafilter $U$ on $P_\kappa(V_\lambda)$ such that $Z \in U$ where $Z = \{X < V_\lambda: \text{the transitive collapse of } X \text{ is } V_\theta \text{ for some } \theta \text{ such that } \text{HOD} \cap V_\theta = (\text{HOD})^{V_\theta}\}$.

Proof. (1) $\Rightarrow$ (2): From [5] Lemma 133, if $\delta$ is HOD-supercompact, then $\delta$ is super-HOD cardinal.

(2) $\Rightarrow$ (3): Suppose $\kappa$ is super-HOD cardinal, $\lambda > \kappa$, and $V_\lambda$ is a $\Sigma_2$ elementary substructure of $V$ such that $|V_\kappa| = \kappa$ and $\text{HOD} \cap V_\kappa = (\text{HOD})^{V_\kappa}$. We show that there exists a normal fine $\kappa$-complete ultrafilter $U$ on $P_\kappa(V_\lambda)$ such that $Z \in U$ where $Z = \{X < V_\lambda: \text{the transitive collapse of } X \text{ is } V_\theta \text{ for some } \theta \text{ such that } \text{HOD} \cap V_\theta = (\text{HOD})^{V_\theta}\}$.

Since $\kappa$ is super-HOD, there exists $\pi < \lambda < \kappa$ and an elementary embedding $\pi: V_{\lambda+\omega} \rightarrow V_{\lambda+\omega}$ such that $\text{crit}(\pi) = \pi, \pi(\lambda) = \lambda$, and $\pi(\text{HOD} \cap V_\lambda) = \text{HOD} \cap V_\lambda$. Let $U$ be the $\pi$-complete normal fine ultrafilter on $P_\pi(V_\lambda)$ given by $\pi$. Thus $U \in V_{\lambda+\omega}$. Then $\pi(U)$ is a $\kappa$-complete normal fine ultrafilter on $P_\kappa(V_\lambda)$.

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The isolation of this statement as the equivalence of HOD-supercompact cardinal is due to Woodin from [11].
It suffices to show that $Z \in \pi(U)$. Let $\pi(Z) = Z$ and $\sigma_\pi = \{\pi(a) : a \in V_\lambda\}$. Since $\text{HOD} \cap V_\lambda = (\text{HOD})^{V_\lambda}$ and $\pi(\text{HOD} \cap V_\lambda) = \text{HOD} \cap V_\lambda$, we have $\text{HOD} \cap V_\lambda = (\text{HOD})^{V_\lambda}$. Thus $\sigma_\pi \in Z$. Note that $\pi(Z) = Z$ and $\sigma_\pi = \{\pi(a) : a \in V_\lambda\}$. Since $HOD \cap V_\lambda = (HOD)^{V_\lambda}$ and $\pi(HOD \cap V_\lambda) = HOD \cap V_\lambda$, we have $\pi(HOD \cap V_\lambda) = (HOD)^{V_\lambda}$. Thus $\sigma_\pi \in Z$. Note that $Z \in U$ if $\sigma_\pi \in \pi(Z)$. So $\pi(Z) \in Z$ and hence $Z \in \pi(U)$.

(3) $\Rightarrow$ (1): Follows from Lemma 3.17 since we can only consider $\lambda > \kappa$ such that $|V_\lambda| = \lambda$ and $HOD \cap V_\lambda = (HOD)^{V_\lambda}$. □

**Definition 3.19.** For regular cardinals $\delta < \kappa$, we say $(\delta, \kappa)$ is a HOD-partition pair if there exists a partition $\langle S_\alpha | \alpha < \delta \rangle \in HOD$ of $\{\alpha < \kappa | \text{cf}(\alpha) = \omega\}$ into pairwise disjoint stationary sets.

If $V = HOD$, then for any regular cardinals $\delta < \kappa$, $(\delta, \kappa)$ is a HOD-partition pair. Note that the HOD Hypothesis implies that for any $\delta$ there is regular cardinal $\kappa > \delta$ such that $(\delta, \kappa)$ is a HOD-partition pair.

**Theorem 3.20.** (Woodin, [5, Theorem 195]) If $\delta$ is HOD-supercompact and for any $\gamma > \delta$ there is regular cardinal $\lambda > \gamma$ such that $(\gamma, \lambda)$ is a HOD-partition pair. Then HOD is a weak extender model for $\delta$ supercompact.

The following Theorem 3.21 and Theorem 3.22 are a reformulation and summary of Woodin’s results in [5] (eg. [5, Theorem 212], [9, Theorem 19], etc.).

**Theorem 3.21.** (Woodin, [5]) Suppose $\delta$ is HOD-supercompact. Then the following are equivalent:

1. The HOD Hypothesis.
2. HOD is a weak extender model for $\delta$ supercompact.
3. There exists a weak extender model $N$ for $\delta$ supercompact such that $N \subseteq HOD$.
4. Every singular cardinal $\gamma > \delta$ is singular in HOD and $\gamma^+ = (\gamma^+)^{\text{HOD}}$.
5. There is a proper class of regular cardinals that are not $\omega$-strongly measurable in HOD.
6. For any $\gamma > \delta$ there is regular cardinal $\lambda > \gamma$ such that $(\gamma, \lambda)$ is a HOD-partition pair.

Proof. By Theorem 3.14, (1) $\Rightarrow$ (2). By Theorem 3.10, (2) $\Rightarrow$ (4). By Theorem 3.6, (5) $\Rightarrow$ (1). By Theorem 3.20, (6) $\Rightarrow$ (2). It is a theorem in ZFC that (2) $\Rightarrow$ (3), (1) $\Rightarrow$ (6), (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5). By Theorem 3.12 and Theorem 3.11, $\delta$ is supercompact in HOD if and only if HOD is a weak extender model for $\delta$ supercompact.

(4) $\Rightarrow$ (1): if (4) holds, then $\{\gamma^+: \gamma > \delta \text{ is a singular cardinal}\}$ is a proper class of regular cardinals which are not $\omega$-strongly measurable in HOD. By the similar argument, we have (4) $\Rightarrow$ (5).

Finally, it suffices to show that (3) $\Rightarrow$ (1). By Theorem 3.10, (3) $\Rightarrow$ (4). Since (4) $\Rightarrow$ (1), we have (3) $\Rightarrow$ (1). □

**Theorem 3.22.** (Woodin, [5]) Suppose $\delta$ is extendible. Then the following are equivalent:

1. The HOD Hypothesis.
2. There exists a regular cardinal $\kappa \geq \delta$ such that $\kappa$ is not measurable in HOD.
3. There exists a regular cardinal $\kappa \geq \delta$ such that $(\delta, \kappa)$ is a HOD-partition pair.
4. For any cardinal $\kappa$, if $\kappa$ is HOD-supercompact, then HOD is a weak extender model for $\kappa$-supercompact.

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13Theorem 212 in [5] assumes that $\delta$ is extendible. In fact it suffices to assume that $\delta$ is HOD-supercompact.
(5) There exists a regular cardinal $\kappa \geq \delta$ such that $\kappa$ is not $\omega$-strongly measurable in HOD.

Proof. By Theorem 3.14 and Theorem 8.20 $(1) \Rightarrow (2)$. It is a theorem in ZFC that $(1) \Rightarrow (3)$.

By Theorem 8.28 and Theorem 8.21 $(4) \Rightarrow (1)$. $(2) \Rightarrow (1)$: Let $I$ be the set of regular cardinals $\gamma$ such that there exists $\eta > \gamma$ such that $V_\eta \models$ ZFC and $V_\eta \models \gamma$ is not $\omega$-strongly measurable in HOD. Note that if $\gamma \in I$, then $\gamma$ is not $\omega$-strongly measurable in HOD. Since $\kappa$ is not measurable in HOD, $\kappa$ is not $\omega$-strongly measurable in HOD in $V_\eta$ for sufficiently large $\eta$ and hence $\kappa \in I$. Let $\eta$ be the witness of $\kappa \in I$. Since $\delta$ is extendible, for any $\alpha$, there exists $j : V_{\eta+1} \rightarrow V_{j(\eta)+1}$ such that $\text{crit}(j) = \delta$ and $j(\delta) > \alpha$. Then $j(\eta)$ witnesses that $j(\kappa) \in I$ and $j(\kappa) > \alpha$.

$(3) \Rightarrow (4)$: Suppose there exists a regular cardinal $\kappa > \delta$ such that $(\delta, \kappa)$ is a HOD-partition pair and $\kappa$ is HOD-supercompact. Let $\theta > \kappa$ be large enough such that $\text{HOD} \cap 2^\kappa = \text{HOD}^{V_\delta} \cap 2^\kappa$. Let $j : V_{\theta+1} \rightarrow V_{j(\theta)+1}$ be an elementary embedding such that $\text{crit}(j) = \delta$ and $j(\delta) > \kappa$. Let $\varphi(\delta)$ denote the statement: for any regular $\lambda < \delta$ there exists regular $\gamma > \lambda$ such that $(\lambda, \gamma)$ is a HOD-partition pair. Note that $V \models \varphi(\delta)$. Since $\text{HOD} \cap 2^\kappa = \text{HOD}^{V_\delta} \cap 2^\kappa$, $V_\delta \models \varphi(\delta)$. By elementarity of $j$, since $\text{HOD}^{V_\delta} \subseteq \text{HOD}$, for any $\lambda < j(\delta)$ there exists $\gamma > \lambda$ such that $(\lambda, \gamma)$ is a HOD-partition pair. Since $j$ can be chosen with $j(\delta)$ arbitrarily large, it follows that for any $\lambda$ there exists $\gamma > \lambda$ such that $(\lambda, \gamma)$ is a HOD-partition pair. By Theorem 8.20 HOD is a weak extender model for $\kappa$-supercompact. 

In the following, we discuss some basic facts about forcing with respect to the HOD Hypothesis. It is not hard to force statements listed in Theorem 3.21 and Theorem 3.22. Suppose $\kappa$ is a supercompact cardinal and $\phi$ is any statement listed in Theorem 3.21 and Theorem 3.22. Then one can force that $V \neq \text{HOD}$, $\kappa$ is supercompact and $\phi$ holds as follows: First force to make $\kappa$ indestructible with the appropriate preparatory forcing, then force $V = \text{HOD}$ and finally add a Cohen real; In the final model, $V \neq \text{HOD}$, $\kappa$ is supercompact and $\phi$ holds.

It is not hard to force the HOD Hypothesis since $V = \text{HOD}$ implies the HOD Hypothesis. It is a folklore that relative to ZFC, we can force $V = \text{HOD}$ by a proper class forcing notion. For nearly any known large cardinal notion $\phi$, relative to $\text{ZFC} + \phi^+$ we can force that $V = \text{HOD}$ and $\phi$ holds. There is a simple way to force that $V \neq \text{HOD}$ and the HOD Hypothesis holds: First force $V = \text{HOD}$ and then add a Cohen real.

Lemma 3.23. Suppose $\delta$ is HOD-supercompact, $P \in V_\delta$ and $G$ is $P$-generic over $V$. If $P$ is weakly homogeneous and ordinal definable, then $V[G] \models \text{The HOD Hypothesis}$ if and only if $V[G] \models \text{The HOD Hypothesis}$.

Proof. From Theorem 8.14 the HOD Hypothesis is equivalent to the statement: every singular cardinal $\gamma > \delta$ is singular in HOD and $\gamma^+ = (\gamma^+)^{\text{HOD}}$. Suppose $V \models \text{The HOD Hypothesis}$. We show that $V[G] \models \text{The HOD Hypothesis}$. Suppose in $V[G], \gamma > \delta$ is singular. Then in $V, \gamma > \delta$ is singular. Since $V \models \text{The HOD Hypothesis}$, $\gamma$ is singular in $\text{HOD}^V$. By Proposition 3.2, $\text{HOD}^V$ is a $\delta$-c.c. generic extension of $\text{HOD}^{V[G]}$. Then $\gamma$ is singular in $\text{HOD}^{V[G]}$. Note that $(\gamma^+)^{V[G]} = (\gamma^+)^{\text{HOD}} = (\gamma^+)^{\text{HOD}^{V[G]}}$. So in $V[G]$, if $\gamma > \delta$ is singular, then $\gamma$ is singular in HOD and $\gamma^+ = (\gamma^+)^{\text{HOD}}$. By a similar argument, we can show that if $V[G] \models \text{The HOD Hypothesis}$, then $V \models \text{The HOD Hypothesis}$. 

$\square$
Proposition 3.24. ([9] Corollary 8) Suppose \( \delta \) is HOD-supercompact, \( G \) is \( \mathbb{P} \)-generic over \( V \) and \( \mathbb{P} \in V_\delta \). Then \( V \models \) The HOD Hypothesis if and only if \( V[G] \models \) The HOD Hypothesis.

Proof. Take \( \kappa < \delta \) be an inaccessible cardinal such that \( \mathbb{P} \in V_\kappa \). Let \( I \) be a \( V[G] \)-generic filter on \( Coll(\omega, \kappa) \) and \( J \) be a \( V \)-generic filter on \( Coll(\omega, \kappa) \) such that \( V[G][I] = V[J] \). Since \( Coll(\omega, \kappa) \) is ordinal definable and weakly homogeneous, by Lemma 3.23, the HOD Hypothesis is absolute between \( V[G] \) and \( V[G][I] \), as well as between \( V \) and \( V[J] \). So the HOD Hypothesis is absolute between \( V \) and \( V[G] \). \( \square \)

Corollary 3.25. \( \square \)

1. Suppose \( \delta \) is HOD-supercompact. Then \( V \models \) The HOD Hypothesis iff for any partial order \( \mathbb{P} \in V_\delta, V^\mathbb{P} \models \) The HOD Hypothesis.

2. If there exists a proper class of HOD-supercompact cardinals, then \( V \models \) The HOD Hypothesis if for any partial order \( \mathbb{P} \in V_\delta, V^\mathbb{P} \models \) The HOD Hypothesis.

Definition 3.26. The Strong HOD Hypothesis denotes the statement: there is a proper class of regular cardinals which are not measurable in HOD.

By Theorem 3.6, the Strong HOD Hypothesis implies the HOD Hypothesis. By Theorem 3.21, if there exists an HOD-supercompact cardinal, then the HOD Hypothesis is equivalent to the Strong HOD Hypothesis. From Corollary 3.25, if \( \delta \) is HOD-supercompact, then \( V \models \) The Strong HOD Hypothesis if for any partial order \( \mathbb{P} \in V_\delta, V^\mathbb{P} \models \) The Strong HOD Hypothesis. The difficulty in forcing the failure of the Strong HOD Hypothesis comes from the difficulty in making the successors of singular cardinals measurable in HOD.

4. The HOD Hypothesis and a supercompact cardinal

In [1], very large cardinals such as supercompact cardinals in \( V \) are forced not to exhibit their large cardinal properties in HOD: they can be very small (not even weakly compact) in HOD. A reasonable natural question would be how far this can be taken, that is whether there exists a supercompact cardinal in \( V \) which is not only not even weakly compact in HOD but also has no other cardinals in HOD which exhibit large cardinal behavior. In the following, we prove in Theorem 4.3 that under the HOD Hypothesis the answer is no: if \( \kappa \) is supercompact and the HOD Hypothesis holds, then there is a proper class of regular cardinals below \( \kappa \), which are measurable in HOD.

The main idea of Theorem 4.3 is as follows. Suppose \( \kappa \) is supercompact and the HOD Hypothesis holds. Take \( \alpha < \kappa \) and \( \lambda > \kappa \) such that \( \lambda \) is a limit of regular cardinals which are not \( \omega \)-strongly measurable in HOD and \( HOD \cap V_\lambda = HOD^{V_\lambda} \).

To find a measurable cardinal between \( \alpha \) and \( \kappa \) in HOD, we need find elementary embeddings \( \pi_1 : V_{\lambda+1} \rightarrow V_{\lambda+1} \) and \( \pi_2 : V_{\lambda+1} \rightarrow V_{\lambda+1} \) such that \( \text{crit}(\pi_1) = \kappa_1, \alpha < \kappa_1 < \kappa \) and \( \pi_3(HOD \cap V_{\lambda_1}) \subseteq HOD \) where \( \pi_3 = \pi_2 \circ \pi_1 \), which we will get in Theorem 4.4. We want to show that \( \kappa_1 \) is measurable in HOD. To do this, it suffices to show that any \( \gamma < \lambda_1, \pi_3 \models (HOD \cap V_\gamma) \models \text{HOD}. \) Suppose \( \pi_3(\gamma) = \gamma \).

Take \( \delta > |V_{\gamma+\omega+1}| \) such that \( \delta < \lambda \) and \( \delta \) is not \( \omega \)-strongly measurable in HOD. By the HOD Hypothesis, there exists \( (S_\alpha \mid \alpha < |V_{\gamma+\omega}|) \in HOD \) which is a partition of \( S_\delta \) into stationary sets in \( \delta \). Then applying Lemma 4.2 to \( \pi_3 \) and \( (S_\alpha \mid \alpha < |V_{\gamma+\omega}|) \), we have \( \tau_{\theta_0} = \tau_3 \upharpoonright |V_{\gamma+\omega}|. \) From \( (S_\alpha \mid \alpha < |V_{\gamma+\omega}|) \in HOD \), we can show that

\footnote{This corollary strengthens theorem 214 in [5].}
\( \pi_3 \upharpoonright |V_{\pi_{\omega}}| \in \text{HOD}. \) Finally, from \( \pi_3(\text{HOD} \cap V_{\lambda}) \subseteq \text{HOD} \) and \( \pi_3 \upharpoonright |V_{\pi_{\omega}}| \in \text{HOD} \), by a standard argument, we can show that \( \pi_3 \upharpoonright (\text{HOD} \cap V_{\pi}) \in \text{HOD} \).

The following Theorem 4.1 gives a new formulation of supercompactness which is important in the proof of our main result Theorem 4.3. Compared to Magidor’s characterization of supercompactness in Theorem 4.11, the new component of this formulation is the coherence condition for \( \pi_1 \) in Theorem 4.1(3). From [11], Woodin also proved Theorem 4.1.

The idea behind Theorem 4.1 is as follows. Let \( j_0 : V \rightarrow M_0 \) be the witness embedding for \( \kappa \)-supercompactness such that \( M_0 \) is closed under \( V_{\omega+1} \)-sequences. Then \( j_0(j_0) : M_0 \rightarrow M_1. \) Take \( j = j_0(j_0) \circ j_0. \) Then \( j : V \rightarrow M_1. \) Then we can get the coherence of the intermediate embeddings in \( V \) via showing in \( M_1 \) the coherence of the intermediate embeddings \( \pi_1 = j_0 \upharpoonright V_{\lambda_1} \) and \( \pi_2 = j_0(\pi_1) = j_0(j_0) \upharpoonright j_0(V_{\lambda_1}). \)

**Theorem 4.1.** \( \kappa \) is supercompact if and only if for all \( \lambda > \kappa, \) any \( \alpha < \kappa \) and for all \( N \subseteq V_{\kappa}, \) there exist \( \kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < \kappa, \) and elementary embeddings \( \pi_1 : V_{\lambda_1+1} \rightarrow V_{\lambda_2+1} \) and \( \pi_2 : V_{\lambda_2+1} \rightarrow V_{\lambda_1+1} \) such that

1. \( \alpha < \kappa_1, \text{crit}(\pi_1) = \kappa_1 \) and \( \text{crit}(\pi_2) = \kappa_2; \)
2. \( \pi_2(\kappa_2) = \kappa \) and \( \pi_1(\kappa_1) = \kappa_2; \) and
3. \( \pi_1(N \cap V_{\lambda_1}) = N \cap V_{\lambda_2} \) and \( \pi_2(N \cap V_{\lambda_2}) = N \cap V_{\lambda_1}. \)

**Proof.** Fix \( \lambda > \kappa, \alpha < \kappa \) and \( N \subseteq V_{\kappa}. \) Take \( j_0 : V \rightarrow M_0 \) such that \( \text{crit}(j_0) = \kappa \) and \( M_0 \) is closed under \( V_{\omega+1} \)-sequences. Then \( j_0(j_0) : M_0 \rightarrow M_1 \) and \( M_1 \) is closed under \( j_0(V_{\omega+1}) \)-sequences in \( M_0. \) Let \( j = j_0(j_0) \circ j_0. \) Then \( j : V \rightarrow M_1. \) It suffices to show in \( M_1 \) that there exist \( \kappa_1 < \lambda_1 < \kappa_2 < \lambda_2 < j(\kappa), \) \( \pi_1 : V_{\lambda_1+1} \rightarrow V_{\lambda_2+1} \) and \( \pi_2 : V_{\lambda_2+1} \rightarrow V_{j(\lambda)+1} \) such that

1. \( j(\alpha) = \alpha < \kappa_1, \text{crit}(\pi_1) = \kappa_1 \) and \( \text{crit}(\pi_2) = \kappa_2; \)
2. \( \pi_2(\kappa_2) = j(\kappa) \) and \( \pi_1(\kappa_1) = \kappa_2; \) and
3. \( \pi_1(j(N) \cap V_{\lambda_1}) = j(N) \cap V_{\lambda_2} \) and \( \pi_2(j(N) \cap V_{\lambda_2}) = j(N) \cap V_{j(\lambda)}. \)

Let \( \kappa_1 = \kappa, \lambda_1 = \lambda, \kappa_2 = j_0(\kappa), \lambda_2 = j_0(\lambda), \) \( \pi_1 = j_0 \upharpoonright V_{\lambda_1+1} \) and \( \pi_2 = j_0(\pi_1) = j_0(j_0) \upharpoonright j_0(V_{\lambda_1+1}). \) Then \( \pi_1 : V_{\lambda_1+1} \rightarrow V_{j_0(\lambda)+1} \) and \( \pi_2 : V_{j_0(\lambda)+1} \rightarrow V_{j(\lambda)+1}. \) Since \( M_1 \) is closed under \( V_{\omega+1} \)-sequences in \( V, \pi_1 \in M_1. \) Since \( M_1 \) is closed under \( j_0(V_{\omega+1}) \)-sequences in \( M_0, \pi_2 \in M_1. \) It is easy to check that (1) and (2) hold. We only check (3) as follows. Since \( N \subseteq V_{\kappa} \) and \( \text{crit}(j_0) = \kappa, j_0(N) = N. \) Note that \( \pi_1(j(N) \cap V_{\lambda_1}) = j_0(j_0(j_0(N))) \cap V_{j_0(\lambda)} = j_0(j_0(N)) \cap V_{j_0(\lambda)} = j_0(N) \cap V_{j_0(\lambda)}. \) By the similar argument, we have \( \pi_2(j(N) \cap V_{j_0(\lambda)}) = j(N) \cap V_{j(\lambda)}. \)

The following lemma is isolated from Woodin’s proof of Theorem 3.14 in [3, Theorem 193]. Since Lemma 4.2 will be used in the proof of Theorem 4.3, we prove it with details here. The technique in the proof of Lemma 4.2 also appears in Woodin’s lemma in [2, Theorem 11].

**Lemma 4.2.** Suppose \( \kappa \) is an uncountable regular cardinal, \( \gamma < \kappa, \) and \( \langle S_\alpha : \alpha < \gamma \rangle \) is a partition of \( \text{cof}(\omega) \cap \kappa \) into stationary sets. Let \( j \) be an elementary embedding with critical point \( \delta \) such that \( j(\delta) < \gamma. \) Let \( j(\tau, \pi) = (\gamma, \kappa, \pi) \) and \( j(S_\alpha : \alpha < \gamma) = (S_\alpha : \alpha < \gamma). \) For \( \eta < \pi \) such that \( \text{cof}(\eta) > \omega, \) let \( \sigma_\eta = \{ \alpha < \pi : \text{cof}(\eta) > \omega \}. \) Let \( \tau_\eta = \{ \tau_\eta : \eta < \kappa, \text{cof}(\eta) > \omega \}. \) Let \( \eta_0 = \sup\{ j(\varepsilon) : \varepsilon < \pi \}. \) Then \( \tau_\eta_0 = \{ j(\alpha) : \alpha < \pi_\eta \} \) (i.e. for \( \theta < \gamma, \theta \in \text{ran}(j) \) if and only if \( S_\theta \cap \eta_0 \) is stationary in \( \eta_0 \)).
Proof. Note that \((S_\alpha : \alpha < \gamma)\) is a partition of \(cof(\omega) \cap \pi\) into stationary sets. For \(\eta < \kappa, cof(\eta) > \omega, \tau_\eta = \{\alpha < \gamma : S_\alpha \cap \eta\) is stationary in \(\eta\}\). Note that \(\eta_0 < \kappa\) and \(cof(\eta_0) = \pi > \omega\).

It is easy to check that for any club \(C \subseteq \eta_0\) there exists a club \(D \subseteq \pi\) such that \(\{j(\varepsilon) : \varepsilon \in D, cof(\varepsilon) = \omega\} \subseteq \{\varepsilon \in C : cof(\varepsilon) = \omega\}\).

We first show that \(\tau_{\eta_0} \subseteq \{j(\alpha) : \alpha < \gamma\}\). Suppose \(\beta \in \tau_{\eta_0}\). Then \(S_\beta \cap \eta_0\) is stationary in \(\eta_0\). Let \(C\) be any club in \(\eta_0\). Then there exists \(\varepsilon \in C \cap S_\beta\). Let \(\pi\) be the preimage of \(\varepsilon\) under \(j\). Note that \(\bigcup_{\alpha < \gamma} S_\alpha = \pi \cap cof(\omega)\). So there exists \(\alpha < \gamma\) such that \(\pi \subseteq S_\alpha\). Then \(\varepsilon \in S_{j(\alpha)}\). Since \(\varepsilon \in S_{j(\alpha)} \cap S_\beta, \beta = j(\alpha)\).

Next we show that \(\{j(\alpha) : \alpha < \gamma\} \subseteq \tau_{\eta_0}\). Suppose there exists \(\pi < \gamma\) such that \(j(\pi) = \alpha\) but \(\alpha \notin \tau_{\eta_0}\). Then there exists club \(C_\alpha \subseteq \eta_0\) such that \(C_\alpha \cap S_\alpha = \emptyset\). Then there exists a club \(D_\alpha \subseteq \pi\) such that \(\{j(\varepsilon) : \varepsilon \in D_\alpha, cof(\varepsilon) = \omega\} \subseteq \{\varepsilon \in C_\alpha : cof(\varepsilon) = \omega\}\). Then there exists \(\zeta \in D_\alpha \cap S_\pi\) such that \(j(\zeta) \in C_\alpha\). So \(j(\zeta) \in C_\alpha \cap S_\alpha\) which leads to a contradiction. \(\square\)

Now we prove the main result Theorem 4.3. The idea of Theorem 4.3 comes from Theorem 3.1 and proof of Theorem 3.14 in theorem [5, Theorem 193]. We first prove this main result and then give a summary of the proof in the end.

**Theorem 4.3.** Suppose \(\kappa\) is supercompact and the HOD Hypothesis holds. Then for each \(\alpha < \kappa\), there exists \(\gamma\) such that \(\alpha < \gamma < \kappa\) and \(\gamma\) is measurable in \(\text{HOD}\).

Proof. Fix \(\alpha < \kappa\). Take \(\lambda > \kappa\) such that \(|\mathcal{V}_\lambda| = \lambda\), \(\lambda\) is a limit of regular cardinals which are not \(\omega\)-strongly measurable in \(\text{HOD}\) and \(\text{HOD} \cap \mathcal{V}_\lambda = \text{HOD}^{\mathcal{V}_\lambda}\). Let \(N = \text{HOD} \cap \mathcal{V}_\kappa\). By Theorem 3.1 there exist \(\kappa_1 < \lambda_1 < \kappa_2 < \kappa_3 < \kappa\), and elementary embeddings \(\pi_1 : \mathcal{V}_{\lambda_1+1} \to \mathcal{V}_{\lambda_2+1}\) and \(\pi_2 : \mathcal{V}_{\lambda_2+1} \to \mathcal{V}_{\lambda_3+1}\) such that \(crit(\pi_1) = \kappa_1, \alpha < \kappa_1, \pi_1(\text{HOD} \cap \mathcal{V}_{\lambda_1}) = \text{HOD} \cap \mathcal{V}_{\lambda_2}\) and \(\pi_2(\text{HOD} \cap \mathcal{V}_{\lambda_2}) = \text{HOD} \cap \mathcal{V}_{\lambda_3} \cap \mathcal{V}_\kappa = \text{HOD} \cap \mathcal{V}_{\kappa'}.\) Let \(\pi_3 = \pi_2 \circ \pi_1\). We want to show that \(\kappa_1\) is measurable in \(\text{HOD}\). Since \(\text{crit}(\pi_3) = \kappa_1\), it suffices to show that for any \(\gamma < \lambda_1, \pi_3 \models (\text{HOD} \cap V_\omega) \in \text{HOD}\).

Suppose \(\pi_3(\gamma) = \gamma\). Take \(\delta > |\mathcal{V}_{\gamma+\omega+1}|\) such that \(\delta < \lambda\) and \(\delta\) is not \(\omega\)-strongly measurable in \(\text{HOD}\). Let \(S^\delta_\alpha = \{\alpha < \delta : cf(\alpha) = \omega\}\). Since \(\delta\) is not \(\omega\)-strongly measurable in \(\text{HOD}\), there exists \(\langle S_\alpha : \alpha < |\mathcal{V}_{\gamma+\omega}| \rangle \in \text{HOD}\) which is a partition of \(S^\delta_\alpha\) into stationary sets in \(\delta\). Let \(a = \langle \gamma, \delta, \langle S_\alpha : \alpha < |\mathcal{V}_{\gamma+\omega}| \rangle \rangle\). Note that \(a \in \mathcal{V}_{\lambda} \cap \text{HOD}\). Since \(\text{HOD} \cap \mathcal{V}_\lambda = \text{HOD}^{\mathcal{V}_\lambda}\), \(a\) is definable in \(\mathcal{V}_\lambda\). Let \(\pi_3(\delta) = \delta\) and \(\pi_3(\langle S_\alpha \mid \alpha < |\mathcal{V}_{\gamma+\omega}| \rangle) = \langle S_\alpha \mid \alpha < |\mathcal{V}_{\gamma+\omega}| \rangle\). Then \(\langle S_\alpha \mid \alpha < |\mathcal{V}_{\gamma+\omega}| \rangle \in \text{HOD}^{\mathcal{V}_\lambda}\) is a partition of \(S^\delta_\alpha\) into stationary sets in \(\delta\). For \(\eta < \delta\) such that \(cf(\eta) > \omega\), let \(\sigma_\eta = \{\alpha < |\mathcal{V}_{\gamma+\omega}| : S_\alpha \cap \eta\) is stationary in \(\eta\}\). Note that \(\langle \sigma_\eta : \eta < \delta, cf(\eta) > \omega\rangle \in \text{HOD}^{\mathcal{V}_\lambda}\). Let \(\langle \eta, \eta < \delta, cf(\eta) > \omega\rangle = \pi_3(\langle \sigma_\eta : \eta < \delta, cf(\eta) > \omega\rangle)\). For each \(\eta < \delta\) such that \(cf(\eta) > \omega\), \(\tau_\eta = \{\alpha < |\mathcal{V}_{\gamma+\omega}| : S_\alpha \cap \eta\) is stationary in \(\eta\}\). Then \(\langle \eta, \eta < \delta, cf(\eta) > \omega\rangle \in \text{HOD}^{\mathcal{V}_\lambda} = \text{HOD} \cap \mathcal{V}_\lambda\). Let \(\eta_0 = \sup(\pi_3(\xi) : \xi < \delta)\). By Lemma 3.2 we have \(\tau_{\eta_0} = \pi_3(x) \models (\text{HOD} \cap \mathcal{V}_\omega) \in \text{HOD}\).

To show that \(\pi_3 \models (\text{HOD} \cap \mathcal{V}_\omega) \in \text{HOD}\), it suffices to show that \(\pi_3(x) \models (\text{HOD} \cap \mathcal{V}_\omega) \in \text{HOD}\). Take \(j \models \text{HOD} \cap \mathcal{V}_\lambda\) such that \(j : \theta \to \text{HOD} \cap \mathcal{V}_\lambda\) is a surjection for some \(\theta < |\mathcal{V}_{\gamma+\omega}|\). For \(x \in \text{HOD} \cap \mathcal{V}_\lambda, \pi_3(x) = \pi_3(j(\xi)) = \pi_3(j(\pi_3(\xi)))\) for some \(\xi \in \theta\). Since \(\pi_3(\xi) \in \text{HOD}\), we have \(\pi_3 \models |\mathcal{V}_{\gamma+\omega}| \in \text{HOD}\). Since \(\pi_3 \models (\text{HOD} \cap \mathcal{V}_\omega) \in \text{HOD}\), we have \(\pi_3 \models (\text{HOD} \cap \mathcal{V}_\omega) \in \text{HOD}\). \(\square\)
There are four key points in the proof of Theorem 4.3: (1) from Theorem 4.1 we get an embedding $\pi_3 : V_{\lambda+1} \to V_{\lambda+1}$ such that $\text{crit}(\pi_3) = \kappa_1$ and $\pi_3(\text{HOD}\cap V_{\lambda_1}) \subseteq \text{HOD}$; (2) from the HOD Hypothesis, we can get a partition $\langle S_\alpha \mid \alpha < \gamma_{\omega+1}\rangle$ in HOD of $\delta$ into stationary subsets; (3) from Lemma 4.2, $\tau_{\gamma_0} = \{ \pi_3(x) : x \in \gamma_{\omega+1} \}$, and from $\langle S_\alpha \mid \alpha < \gamma_{\omega+1}\rangle \in \text{HOD}$, we can show that $\pi_3 \upharpoonright |V_{\gamma+\omega}| \in \text{HOD}$; (4) from $\pi_3(\text{HOD} \cap V_{\lambda_1}) \subseteq \text{HOD}$ and $\pi_3 \upharpoonright |V_{\gamma+\omega}| \in \text{HOD}$, by a standard argument, we can show that $\pi_3 \upharpoonright (\text{HOD} \cap V_\gamma) \in \text{HOD}$.

**Corollary 4.4.** Suppose $\kappa$ is supercompact and the HOD Hypothesis holds. Then $V_\kappa \models$ there is a proper class of regular cardinals which are measurable in HOD.

The following Local University Theorem follows from proof of Theorem 4.3 and is a reformulation of Woodin’s original version announced in [11].

**Corollary 4.5.** (Woodin, Local Universality Theorem) Suppose $\kappa$ is supercompact and the HOD Hypothesis holds. Then for each $\alpha < \kappa$, there exists an elementary embedding $j : V_{\lambda+1} \to V_{j(\lambda)+1}$ such that

1. $\text{crit}(j) = \pi$, $\alpha < \pi < \lambda < \kappa$ and $j(\lambda) < \kappa$;
2. $j \upharpoonright (\text{HOD} \cap V_\lambda) \in \text{HOD}$ and
3. $j(\text{HOD} \cap V_\lambda) = \text{HOD} \cap V_{j(\lambda)}$.

From [11], Woodin essentially proved that if $\delta$ is $N$-supercompact and $N$ is a weak extender model for $\delta$-supercompact, then any measurable cardinal $\kappa \geq \delta$ is measurable in $N$. As a corollary, if $\delta$ is HOD-supercompact and the HOD Hypothesis holds, then any measurable cardinal $\kappa \geq \delta$ is measurable in HOD. Comparing this result with Theorem 4.3 and Global Universality Theorem with Local Universality Theorem, we can see the huge difference between HOD-supercompact cardinals and supercompact cardinals under the HOD Hypothesis even if HOD-supercompact cardinals and supercompact cardinals seem to be close in the large cardinal hierarchy: under the assumption of the HOD Hypothesis and HOD-supercompact cardinals, large cardinals in $V$ are reflected to be large cardinals in HOD in a global way; however, under the assumption of the HOD Hypothesis and supercompact cardinals, large cardinals in $V$ are reflected to be large cardinals in HOD in a local way.

### 5. Questions

Theorems 3.21 and Theorem 3.22 have established the equivalence of the HOD Hypothesis under the assumption of HOD-supercompact cardinals and extendible cardinals. A natural question is whether we can establish the equivalence of the HOD Hypothesis only assuming supercompact cardinals. Especially, if $\kappa$ is supercompact, whether the HOD Hypothesis is the equivalent to the statement: for each $\alpha < \kappa$, there exists $\gamma$ such that $\alpha < \gamma < \kappa$ and $\gamma$ is measurable in HOD? It seems for me the HOD Hypothesis expresses the global property of large cardinals in HOD. In this paper, we prove the forward direction. I conjecture that the backward direction does not hold. The difficulty in proving this conjecture comes from the difficulty in forcing the failure of the HOD Hypothesis. Under the assumption of only supercompactness, as far as we know, we do not know any equivalence of the HOD Hypothesis. Woodin conjectured in [11] that if $\delta$ is supercompact then

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$^{16}$I would like to thank the referee for pointing out this question to me.
the HOD Hypothesis is equivalent to the existence of a weak extender model $N$ for $\delta$ supercompact such that $N \subseteq \text{HOD}$, which as far as we know is an open problem.

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