Integrability of Exceptional Hydrodynamic Type Systems

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Abstract
In this paper we consider non-diagonalisable hydrodynamic type systems integrable by the Extended Hodograph Method. We restrict our consideration to non-diagonalisable hydrodynamic reductions of the Mikhalëv equation. We show that families of these hydrodynamic type systems are reducible to the Heat hierarchy. Then we construct new particular explicit solutions for the Mikhalëv equation.

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1 Introduction

The theory of two-dimensional integrable quasilinear systems of first order by the Tsarev Generalised Hodograph Method was established in [12]. Such systems we call semi-Hamiltonian hydrodynamic type systems. They have all pairwise distinct roots of characteristic polynomials of velocity matrices\(^1\) and they are diagonalisable.

In this paper we deal with integrable hydrodynamic type systems, which have just one root. These systems are non-diagonalisable. However, they have infinitely many hydrodynamic conservation laws, commuting flows and particular solutions. Instead of Generalised Hodograph Method we apply the Extended Hodograph Method presented in [11].

In a general case even diagonalisable hydrodynamic type systems with pairwise distinct roots are not semi-Hamiltonian, i.e. they are non-integrable. Classification and integration of hydrodynamic type systems with double roots, triple roots, etc. become much more complicated from combinatorial point of view. For instance, in the two-component case we have two different sub-cases: two distinct roots and a one common root; in the three-component case we have three different sub-cases: three distinct roots, a one double root and a one triple root; in the four-component case we have five different sub-cases: four distinct roots, a one double root, two double roots, a one triple root and a one quadruple root, etc. This is one of many reasons: why integrable hydrodynamic type systems with multiple roots were not investigated earlier.

Another important reason was absence of any interesting examples known in the theory of integrable systems. Just very recently (see [5]), such a family of hydrodynamic type systems was extracted from integrable hydrodynamic chain (see [8])

\[
c_{k,t} = c_{k+1,x} + c_1 c_{k,x}, \quad k = 1, 2, \ldots
\]  

(1)

It was shown in [8] that this chain possesses infinitely many integrable diagonalisable hydrodynamic reductions with pairwise distinct roots. The authors of [5] proved that this chain also possesses infinitely many integrable hydrodynamic reductions with multiple roots. Then the case with a single root was deeply investigated in [6], where the authors constructed a link to the Heat hierarchy. In this paper we continue to consider hydrodynamic reductions with a single root. We apply the Extended Hodograph Method for linearisation of these hydrodynamic type systems to the Heat hierarchy, i.e.

\[
 u_t = u_{xx}, \quad u_y = u_{xxx}, \quad u_z = u_{xxxx}, \ldots,
\]

and we derive corresponding solutions for the remarkable Mikhal'ev equation (see [7]).

The paper is organised as follows: in Section 2 we discuss integrable hydrodynamic chains (2), which possess infinitely many non-diagonalisable hydrodynamic reductions with multiple roots. In Section 3 we apply the Extended Hodograph Method for linearisation of multi-component hydrodynamic reductions with a single multiple root to the

\(^1\)Everywhere below we shall write just “root” for simplicity.
remarkable Heat hierarchy. In Subsection 3.1 we consider the simplest two-component case integrable by the classical hodograph transformation. In Subsection 3.2 we reduce two commuting three-component hydrodynamic type systems with a single triple root to the Heat equation together with its first higher commuting flow. In Subsection 3.3 we reduce an arbitrary number of commuting multi-component hydrodynamic type systems with a single quadruple root to the Heat hierarchy. In Section 4 we show that the integrable three-dimensional linearly degenerate Mikhalëv system simultaneously possesses multi-component non-diagonalisable hydrodynamic reductions with a single multiple root. In the three-component case we construct corresponding particular solutions presented in explicit forms. Finally in Conclusion 5 we discuss existence of such non-diagonalisable hydrodynamic reductions for other integrable hydrodynamic chains.

2 The Integrable Hydrodynamic Chain

Integrable hydrodynamic chain (1) possesses infinitely many higher commuting flows (see [8])

\[ c_{k,t_{n+1}} = \sum_{m=0}^{n} a_m c_{k+n-m,x}, \quad k = 1, 2, ..., \quad n = 0, 1, 2, ..., \]  

(2)

where all polynomial functions \( a_k(c) \) can be found from the expansion \((\lambda \to 0)\)

\[ 1 + \sum_{m=1}^{\infty} a_m \lambda^m = \exp \left( \sum_{m=1}^{\infty} c_m \lambda^m \right). \]  

(3)

This means that polynomial functions \( a_k(c) \) are nothing but the well-known Bell polynomials (up to appropriate factorial multipliers).

For instance,

\[ a_0 = 1, \quad a_1 = c_1, \quad a_2 = c_2 + \frac{1}{2} c_1^2, \quad a_3 = c_3 + c_1 c_2 + \frac{1}{6} c_1^3. \]  

(4)

Corresponding commuting hydrodynamic chains are

\[ c_{k,t} = c_{k+1,x} + c_1 c_{k,x}, \quad c_{k,y} = c_{k+2,x} + c_1 c_{k+1,x} + \left( c_2 + \frac{1}{2} c_1^2 \right) c_{k,x}, \]  

(5)

\[ c_{k,z} = c_{k+3,x} + c_1 c_{k+2,x} + \left( c_2 + \frac{1}{2} c_1^2 \right) c_{k+1,x} + \left( c_3 + c_1 c_2 + \frac{1}{6} c_1^3 \right) c_{k,x}, \]  

(6)

where we denoted \( x = t_1, \quad t = t_2, \quad y = t_3, \quad z = t_4. \)

The differential of (3) leads to recursive consequences \((n = 1, 2, ...)\)

\[ da_{n+1} = \sum_{m=0}^{n} a_m dc_{n+1-m}. \]

Thus (see (2)),

\[ c_{1,t_{n+1}} = \sum_{m=0}^{n} a_m c_{n+1-m,x} = a_{n+1,x}. \]
For instance,
\[ c_{1,x} = a_{1,x}, \quad c_{1,t} = a_{2,x}, \quad c_{1,y} = a_{3,x}, \quad c_{1,z} = a_{4,x}. \] (7)

Thus, (2) can be written in the form
\[ c_{k,t_{n+1}} = c_{k+n,x} + \sum_{m=1}^{n} \Phi_m c_{k+n-m,x}, \] (8)

where \( \Phi \) is a potential function for conservation laws (7), i.e. \( a_k = \Phi_k \). Here \( \Phi_m \equiv \partial \Phi / \partial t_m \).

**Remark:** Infinitely many conservation law densities \( \sigma_k(c) \) can be found from the expansion \( (\lambda \to 0) \)
\[ 1 + \sum_{m=1}^{\infty} \sigma_m \lambda^m = \exp \left( -\sum_{m=1}^{\infty} c_m \lambda^m \right). \] (9)

This means, that (cf. (3))
\[ a_1 + \sigma_1 = 0, \quad a_2 + a_1 \sigma_1 + \sigma_2 = 0, \quad a_3 + a_2 \sigma_1 + a_1 \sigma_2 + \sigma_3 = 0, \ldots \]

Taking into account (4), one can obtain again Bell polynomials (with another choice of multipliers)
\[ \sigma_1 = -c_1, \quad \sigma_2 = -c_2 + \frac{1}{2} c_1^2, \quad \sigma_3 = -c_3 + c_1 c_2 - \frac{1}{6} c_1^3, \ldots \]

Corresponding conservation laws (see [8]) are
\[ \sigma_{k,t_{n+1}} = \left( \sum_{s=0}^{n} a_s \sigma_{k+n-s} \right)_x, \quad k = 1, 2, \ldots, n = 0, 1, 2, \ldots \]

The crucial observation made in [5] is that the reduction \( c_{N+1} = 0 \) reduces hydrodynamic chain (1) to \( N \) component non-diagonalisable hydrodynamic type systems, which have infinitely many conservation laws, commuting flows and infinitely many particular solutions.

In these cases, (3) and (9) reduce to the form, respectively
\[ 1 + \sum_{m=1}^{\infty} a_m \lambda^m = \exp \left( \sum_{m=1}^{N} c_m \lambda^m \right), \quad 1 + \sum_{m=1}^{\infty} \sigma_m \lambda^m = \exp \left( -\sum_{m=1}^{N} c_m \lambda^m \right). \] (10)

The potential function \( \Phi_{(N)} \) can be found in quadratures
\[ d\Phi_{(N)} = \sum_{m=1}^{N} a_m dt_m. \] (11)
3 The Extended Hodograph Method

Integration of $N$ component hydrodynamic type systems by the Extended Hodograph Method is based on the preliminary computation of auxiliary $N-2$ commuting flows. In a general case, this is very complicated task even for semi-Hamiltonian hydrodynamic type systems, because usually commuting hydrodynamic flows can be found solving some linear system of equations with variable coefficients in partial derivatives (for further details, see [12]). However, in our case they already are extracted from higher commuting flows (2) by the same reduction $c_{N+1} = 0$.

Examples:

I. If $N = 2$, then $c_3 = 0$. Then we have just the two-component hydrodynamic type system (see (5))

\[
\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_t = \begin{pmatrix} c_1 & 1 \\ 0 & c_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}_x.
\]

II. If $N = 3$, then $c_4 = 0$. Then we have two three-component commuting hydrodynamic type systems (see (5) again)

\[
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_t = \begin{pmatrix} c_1 & 1 & 0 \\ 0 & c_1 & 1 \\ 0 & 0 & c_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_x,
\]

\[
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_y = \begin{pmatrix} \frac{1}{2}c_1^2 + c_2 & c_1 & 1 \\ 0 & \frac{1}{2}c_1^2 + c_2 & c_1 \\ 0 & 0 & \frac{1}{2}c_1^2 + c_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_x.
\]

III. If $N = 4$, then $c_5 = 0$. Then we have three four-component commuting hydrodynamic type systems (see (5), (6))

\[
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}_t = \begin{pmatrix} a_1 & 1 & 0 & 0 \\ 0 & a_1 & 1 & 0 \\ 0 & 0 & a_1 & 1 \\ 0 & 0 & 0 & a_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}_x,
\]

\[
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}_y = \begin{pmatrix} a_2 & a_1 & 1 & 0 \\ 0 & a_2 & a_1 & 1 \\ 0 & 0 & a_2 & a_1 \\ 0 & 0 & 0 & a_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}_x,
\]

\[
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}_z = \begin{pmatrix} a_3 & a_2 & a_1 & 1 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix}_x,
\]

where (we remind, see (4)) $a_1 = c_1$, $a_2 = c_2 + \frac{1}{2}c_1^2$ and $a_3 = c_3 + c_1c_2 + \frac{1}{6}c_1^3$.

So, now the construction of first $N-1$ commuting hydrodynamic type systems becomes obvious. Due to a triangular form of all above square matrices, each of them possesses just a single root (i.e. the $N$th matrix has a single root $a_N$).
3.1 The Hodograph Method

Here we apply a classical hodograph method for integration of two-component hydrodynamic type system (12). As usual such a system can be written in the potential form

\[ d\xi_1 = c_1 dx + \left(c_2 + \frac{1}{2} c_1^2\right) dt, \quad d\xi_2 = \left(c_2 - \frac{1}{2} c_1^2\right) dx - \frac{1}{3} c_1^3 dt. \]

Then one can introduce the auxiliary potentials

\[ d\tilde{\xi}_1 = xdc_1 + td \left(c_2 + \frac{1}{2} c_1^2\right), \quad d\tilde{\xi}_2 = x \left(c_2 - \frac{1}{2} c_1^2\right) - \frac{1}{3} td(c_1^3). \]

Now field variables \( c_1 \) and \( c_2 \) became new independent variables, while independent variables \( x \) and \( t \) became new field variables. Existence of the potential functions \( \tilde{\xi}_1 \) and \( \tilde{\xi}_2 \) means that their second order derivatives must coincide, i.e.

\[ \frac{\partial x}{\partial c_2} = \frac{\partial t}{\partial c_1} - c_1 \frac{\partial t}{\partial c_2}, \quad \frac{\partial x}{\partial c_1} = -c_1 \frac{\partial t}{\partial c_1}. \tag{14} \]

The compatibility condition \( \partial(\partial x/\partial c_2)/\partial c_1 = \partial(\partial x/\partial c_1)/\partial c_2 \) yields the Heat equation

\[ \frac{\partial t}{\partial c_2} = \frac{\partial^2 t}{\partial c_1 \partial c_1}. \tag{15} \]

Once solution of this equation is found, another function \( x(c_1, c_2) \) can be found in quadratures (see (14)), i.e.

\[ d(x + c_1 t) = tdc_1 + \frac{\partial t}{\partial c_1} dc_2. \]

Thus, here we show that two-component hydrodynamic type system (12) is connected with the Heat equation (15) by the hodograph transformation.

3.2 The Three Component Case

Here we apply the Extended Hodograph Method for integration of two commuting three-component hydrodynamic type systems (13). These systems also can be written in the potential form

\[ d\xi_1 = c_1 dx + \left(c_2 + \frac{1}{2} c_1^2\right) dt + \left(c_3 + c_1 c_2 + \frac{1}{6} c_1^3\right) dy, \]

\[ d\xi_2 = \left(c_2 - \frac{1}{2} c_1^2\right) dx + \left(c_3 - \frac{1}{3} c_1^3\right) dt + \left(\frac{1}{2} c_2^2 - \frac{1}{2} c_1^2 c_2 - \frac{1}{8} c_1^4\right) dy, \]

\[ d\xi_3 = \left(c_3 - c_1 c_2 + \frac{1}{6} c_1^3\right) dx + \left(-\frac{1}{2} c_2^2 - \frac{1}{2} c_2 c_1^2 + \frac{1}{8} c_1^4\right) dt + \left(-c_1 c_2^2 + \frac{1}{20} c_1^5\right) dy. \]

Then one can introduce the auxiliary potentials

\[ d\tilde{\xi}_1 = \left(x + tc_1 + yc_2 + \frac{1}{2} yc_1^2\right) dc_1 + (t + yc_1) dc_2 + y dc_3, \]
Taking into account identities

So, in the component case, we have three independent variables, while independent variables \(x, t\) and \(y\) became new field variables. Existence of the potential functions \(\xi_1, \xi_2\) and \(\xi_3\) means that their second order derivatives must coincide, i.e.

\[
\frac{\partial x}{\partial c_1} = \left(\frac{1}{2} c_1^2 - c_2\right) \frac{\partial y}{\partial c_1}, \quad \frac{\partial x}{\partial c_2} = -c_1 \frac{\partial y}{\partial c_1} + \left(\frac{1}{2} c_2^2 - c_2\right) \frac{\partial y}{\partial c_2}, \quad \frac{\partial t}{\partial c_1} = \frac{\partial y}{\partial c_1} - c_1 \frac{\partial y}{\partial c_2}, \quad \frac{\partial t}{\partial c_2} = \frac{\partial y}{\partial c_2} - c_1 \frac{\partial y}{\partial c_3}.
\]

The compatibility conditions \(\partial(\partial x/\partial c_k)/\partial c_m = \partial(\partial x/\partial c_m)/\partial c_k, \partial(\partial t/\partial c_k)/\partial c_m = \partial(\partial t/\partial c_m)/\partial c_k\) yield the system

\[
\frac{\partial y}{\partial c_2} = \frac{\partial^2 y}{\partial c_1 \partial c_1}, \quad \frac{\partial y}{\partial c_3} = \frac{\partial^2 y}{\partial c_1 \partial c_2}, \quad \frac{\partial^2 y}{\partial c_1 \partial c_3} = \frac{\partial^2 y}{\partial c_2 \partial c_2}.
\]

This means that the function \(y(c_1, c_2, c_3)\) satisfies simultaneously the Heat equation and its first higher commuting flow

\[
\frac{\partial y}{\partial c_2} = \frac{\partial^2 y}{\partial c_1 \partial c_1}, \quad \frac{\partial y}{\partial c_3} = \frac{\partial^2 y}{\partial c_1 \partial c_2}, \quad (16)
\]

and the functions \(t(c_1, c_2, c_3)\) and \(x(c_1, c_2, c_3)\) can be found in quadratures

\[
d(t + yc_1) = \frac{\partial y}{\partial c_2} dc_3 + \frac{\partial y}{\partial c_1} dc_2 + ydc_1,
\]

\[
d \left[ x + tc_1 + y \left( c_2 + \frac{1}{2} c_1^2 \right) \right] = \frac{\partial y}{\partial c_1} dc_3 + ydc_2 + (t + yc_1) dc_1.
\]

Thus, here we show that two three-component hydrodynamic type systems (13) are connected with the Heat equation and its first commuting flow (16) by the extended hodograph transformation \(c_k(x, t, y) \rightarrow x(c_1, c_2, c_3), t(c_1, c_2, c_3), y(c_1, c_2, c_3)\).

3.3 The Multi Component Case

So, in the \(N\) component case, we have \(N\) time variables \(t_m\) and \(N\) field variables \(c_k\). Taking into account identities

\[
\sum_{m=1}^{N} \frac{\partial t_k}{\partial c_m} \frac{\partial c_m}{\partial t_n} = \delta^n_k, \quad \sum_{m=1}^{N} \frac{\partial t_m}{\partial c_n} \frac{\partial c_k}{\partial t_m} = \delta^c_k,
\]

one can linearise \(N\) component hydrodynamic type systems (see (2) with the reduction \(c_{N+1} = 0\)), i.e.
\[
\frac{\partial t_k}{\partial c_{N-n}} = \sum_{m=k}^{N} \sigma_{m-k} \frac{\partial t_m}{\partial c_{m-n}}, \quad n = 0, \ldots, k-1; \quad \frac{\partial t_k}{\partial c_{N-n}} = \sum_{m=n+1}^{N} \sigma_{m-k} \frac{\partial t_m}{\partial c_{m-n}}, \quad n = k, \ldots, N-1,
\]

where \(k = 1, 2, \ldots, N-1\).

If \(k = N-1\), the compatibility conditions for the sub-system

\[
\frac{\partial t_{N-1}}{\partial c_n} = \frac{\partial t_N}{\partial c_{n-1}} - c_1 \frac{\partial t_N}{\partial c_n}, \quad n = 2, \ldots, N; \quad \frac{\partial t_{N-1}}{\partial c_1} = -c_1 \frac{\partial t_N}{\partial c_1}
\]

lead to \(N-1\) commuting flows of the Heat hierarchy

\[
\frac{\partial t_N}{\partial c_2} = \frac{\partial^2 t_N}{\partial c_1^2}, \quad \frac{\partial t_N}{\partial c_3} = \frac{\partial^3 t_N}{\partial c_1^3}, \ldots, \quad \frac{\partial t_N}{\partial c_N} = \frac{\partial^N t_N}{\partial c_1^N}. \quad (17)
\]

Once some particular common solution of these equations is found, all other time variables can be found in quadratures, i.e.

\[
dt_{N-n} + \sum_{m=1}^{n} a_m \partial t_{N-n+m} = \sum_{m=n+1}^{N} \frac{\partial t_N}{\partial c_{m-n}} dc_m, \quad n = 1, \ldots, N-1.
\]

Moreover, the dependencies \(t_k(c)\) can be expressed explicitly. Indeed,

\[
t_k = \sum_{m=0}^{N-k} \sigma_m \frac{\partial U_N}{\partial c_{k+m}}, \quad k = 1, \ldots, N, \quad (18)
\]

where the function \(U_N\) satisfies the reduced Heat hierarchy (17)

\[
\frac{\partial U_N}{\partial c_2} = \frac{\partial^2 U_N}{\partial c_1^2}, \quad \frac{\partial U_N}{\partial c_3} = \frac{\partial^3 U_N}{\partial c_1^3}, \ldots, \quad \frac{\partial U_N}{\partial c_N} = \frac{\partial^N U_N}{\partial c_1^N}.
\]

Thus, the potential function \(\Phi_{(N)}\) (see (11)) also can be found explicitly, i.e.

\[
\Phi_{(N)} = -\sum_{m=1}^{N} \sigma_m \frac{\partial U_N}{\partial c_m} - U_N. \quad (19)
\]

4 The Three Dimensional Mikhalëv System

In previous papers [8] and [10] diagonalisable hydrodynamic reductions and dispersive reductions were investigated for the Mikhalëv system (see [7])

\[
a_{1,t} = a_{2,x}, \quad a_{1}a_{2,x} + a_{1,y} = a_{2}a_{1,x} + a_{2,t}, \quad (20)
\]

whose Lax pair is

\[
p_t = [(\lambda + a_1)p]_x, \quad p_y = [(\lambda^2 + a_1\lambda + a_2)p]_x. \quad (21)
\]
The substitution (see (9), $\lambda \to \infty$)

$$p = \exp \left( -\sum_{m=1}^{\infty} c_m \lambda^{-m} \right)$$

into (21) yields two first commuting hydrodynamic chains (5).

Mikhail’ev system (20) also can be obtained from (5). Indeed, one can take first two equations from the first hydrodynamic chain

$$c_{1,t} = c_{2,x} + c_1 c_{1,x}, \quad c_{2,t} = c_{3,x} + c_1 c_{2,x},$$

and just one equation from the second hydrodynamic chain

$$c_{1,y} = c_{3,x} + c_1 c_{2,x} + \left( c_2 + \frac{1}{2} c_1^2 \right) c_{1,x}.$$  

Eliminating the field variable $c_3$, one can obtain the two-component three-dimensional system

$$c_{1,t} = c_{2,x} + c_1 c_{1,x}, \quad c_{2,t} = c_{1,y} - \left( c_2 + \frac{1}{2} c_1^2 \right) c_{1,x},$$

which is equivalent to Mikhail’ev system (20), where (see (4)) $a_1 = c_1, a_2 = c_2 + c_1^2/2$.

Taking into account (see (8)) $a_1 = \Phi_x, a_2 = \Phi_t$, Mikhail’ev system (20) reduces to the Mikhail’ev equation

$$\Phi_x \Phi_{xt} + \Phi_{xy} = \Phi_t \Phi_{xx} + \Phi_{tt}. \quad (22)$$

In a general case, computation of particular solutions based on the method of two-dimensional diagonalisable reductions is a very complicated task, because some intermediate calculations contain integrations of linear systems with variable coefficients in partial derivatives (for further details, see [2]). However, in the case of non-diagonalisable hydrodynamic reductions considered in the previous Section, corresponding solutions of Mikhail’ev equation (22) can be written in implicit form.

Without loss of generality and for simplicity here we consider the three-component case (13) only. For better presentation below we denote $c_1 = X, c_2 = T$ and $c_3 = Y$. Now we utilise formulas (18) and (19).

**Lemma:** The particular solution of three-dimensional Mikhail’ev equation (22) selected by non-diagonalisable hydrodynamic reduction (13) can be presented in the implicit form

$$\Phi(x, t, y) = \left( Y - XT + \frac{1}{6} X^3 \right) \frac{\partial U}{\partial Y} + \left( T - \frac{1}{2} X^2 \right) \frac{\partial U}{\partial T} + X \frac{\partial U}{\partial X} - U,$$

where the function $U(X, T, Y)$ is a solution of the pair of commuting Heat equations

$$\frac{\partial U}{\partial T} = \frac{\partial^2 U}{\partial X^2}, \quad \frac{\partial U}{\partial Y} = \frac{\partial^3 U}{\partial X^3}, \quad (23)$$

and the functions $X(x, t, y), T(x, t, y), Y(x, t, y)$ can be found from

$$y = \frac{\partial U}{\partial Y}, \quad t = \frac{\partial U}{\partial T} - X \frac{\partial U}{\partial Y}, \quad x = \frac{\partial U}{\partial X} - X \frac{\partial U}{\partial T} + \left( \frac{1}{2} X^2 - T \right) \frac{\partial U}{\partial Y}.$$
**Example I:** If \((\kappa)\) is an arbitrary constant
\[
U = \exp(\kappa X + \kappa^2 T + \kappa^3 Y),
\]
them Mikhalëv equation (22) has a particular solution
\[
\Phi(x, t, y) = \frac{y}{\kappa^3} \ln \frac{y}{\kappa^3} - \frac{11}{6} \frac{y}{\kappa^2} + \frac{1}{\kappa} \left(x - \frac{t^2}{2y}\right) + \frac{t^3}{3y^2} - \frac{x}{y},
\]
where
\[
X = \kappa^{-1} - \frac{t}{y}, \quad T = \frac{1}{2} \kappa^{-2} - \frac{x}{y} + \frac{t^2}{2y^2}, \quad Y = \kappa^{-3} \left(\ln \frac{y}{\kappa^3} - \frac{3}{2}\right) + \kappa^{-2} \frac{t}{y} + \kappa^{-1} \left(x - \frac{t^2}{2y^2}\right)
\]
and \(U = y\kappa^{-3}\).

**Remark:** While the Extended Hodograph Method admits linearisation of sets of \(N-1\) commuting hydrodynamic flows, the Tsarev Generalised Hodograph Method allows to extract infinitely many particular solutions from higher commuting dispersive chains (8). For instance, non-diagonalisable commuting hydrodynamic type systems (13) have infinitely many hydrodynamic conservation laws
\[
\left(\frac{\partial F}{\partial c_2}\right)_t = \left(\frac{c_1 \partial F}{\partial c_2} + \frac{\partial F}{\partial c_1}\right)_x, \quad \left(\frac{\partial F}{\partial c_2}\right)_y = \left[\left(c_2 + \frac{1}{2} c_1^2\right) \frac{\partial F}{\partial c_2} + c_1 \frac{\partial F}{\partial c_1} - F\right]_x,
\]
where the function \(F(c_1, c_2, c_3)\) satisfies (cf. (16))
\[
\frac{\partial F}{\partial c_2} = - \frac{\partial^2 F}{\partial c_1 c_1}, \quad \frac{\partial F}{\partial c_3} = \frac{\partial^3 F}{\partial c_1 \partial c_1 \partial c_1},
\]
and infinitely many higher commuting flows
\[
\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} U & \frac{\partial U}{\partial c_1} & \frac{\partial U}{\partial c_3} \\ 0 & U & \frac{\partial U}{\partial c_3} \\ 0 & 0 & U \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix},
\]
where the function \(U(X, T, Y)\) satisfies (23).

**Statement:** The Heat hierarchy
\[
\frac{\partial U}{\partial c_2} = \frac{\partial^2 U}{\partial c_1 \partial c_1}, \quad \frac{\partial U}{\partial c_3} = \frac{\partial^3 U}{\partial c_1 \partial c_1 \partial c_1}, \quad \frac{\partial U}{\partial c_4} = \frac{\partial^4 U}{\partial c_1 \partial c_1 \partial c_1 \partial c_1}, \ldots
\]
has infinitely many polynomial solutions \(U_k(c)\) which are Bell polynomials, determined by (see (3) and (9))
\[
1 + \sum_{m=1}^{\infty} U_m \lambda^m = \exp \left(\sum_{m=1}^{\infty} c_m \lambda^m\right).
\]

**Example II:** A general solution of Mikhalëv equation (22) is determined by (see (11), (18), (19))
\[
\Phi = c_1 \frac{\partial U}{\partial c_1} + \left(c_2 - \frac{1}{2} c_1^2\right) \frac{\partial U}{\partial c_2} + \left(c_3 - c_1 c_2 + \frac{1}{6} c_1^3\right) \frac{\partial U}{\partial c_3} - U,
\]
where (we remind that the function $U(X, T, Y)$ satisfies (23))

$$y = \frac{\partial U}{\partial c_3}, \quad t = \frac{\partial U}{\partial c_2} - c_1 \frac{\partial U}{\partial c_3}, \quad x = \frac{\partial U}{\partial c_1} - c_1 \frac{\partial U}{\partial c_2} + \left(-c_2 + \frac{1}{2} c_1^2\right) \frac{\partial U}{\partial c_3}.$$ 

So, infinitely many polynomial solutions $\tilde{U}_k(c)$ can be extracted from the expansion (cf. (10))

$$1 + \sum_{m=1}^{\infty} \tilde{U}_m \lambda^m = \exp \left( \sum_{m=1}^{3} c_m \lambda^m \right).$$

For instance (here we again use the notation $c_1 = X, c_2 = T, c_3 = Y$),

$$\tilde{U}_1 = X, \quad \tilde{U}_2 = T + \frac{1}{2} X^2, \quad \tilde{U}_3 = Y + XT + \frac{1}{6} X^3,$$

$$\tilde{U}_4 = XY + \frac{1}{2} T^2 + \frac{1}{2} X^2 T + \frac{1}{24} X^4, \quad \tilde{U}_5 = TY + \frac{1}{2} X^2 Y + \frac{1}{2} XT^2 + \frac{1}{2} X^3 T + \frac{1}{120} X^5.$$

Corresponding solutions of Mikhalëv equation (22) become ($k = 4, 5, ...$)

$$\Phi(k) = X \tilde{U}_{k-1} + \left(T - \frac{1}{2} X^2\right) \tilde{U}_{k-2} + \left(Y - XT + \frac{1}{6} X^3\right) \tilde{U}_{k-3} - \tilde{U}_k,$$

where

$$x = \tilde{U}_{k-1} - X \tilde{U}_{k-2} + \left(-T + \frac{1}{2} X^2\right) \tilde{U}_{k-3}, \quad t = \tilde{U}_{k-2} - X \tilde{U}_{k-3}, \quad y = \tilde{U}_{k-3}.$$

In the simplest nontrivial case $\tilde{U}_4$, we have three commuting hydrodynamic type system (cf. (13))

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_t = \begin{pmatrix} \tilde{U}_1 & 1 & 0 \\ 0 & \tilde{U}_1 & 1 \\ 0 & 0 & \tilde{U}_1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_x,$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_y = \begin{pmatrix} \tilde{U}_2 & \tilde{U}_1 & 1 \\ 0 & \tilde{U}_2 & \tilde{U}_1 \\ 0 & 0 & \tilde{U}_2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_x,$$

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_z = \begin{pmatrix} \tilde{U}_3 & \tilde{U}_2 & \tilde{U}_1 \\ 0 & \tilde{U}_3 & \tilde{U}_2 \\ 0 & 0 & \tilde{U}_3 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}_x.$$

According to [4], [12] one can obtain the system\(^2\)

$$\tilde{U}_3 = x + t \tilde{U}_1 + y \tilde{U}_2, \quad \tilde{U}_2 = t + y \tilde{U}_1, \quad \tilde{U}_1 = y.$$

\(^2\)The Tsarev Generalised Hodograph Method reformulated by Yu. Kodama and J. Gibbons for a three-component case means: $x \delta_k^t + t \delta_k^u + y V_k^v = W_k^v$, where $U_k^u$ and $V_k^v$ are velocity matrices of two commuting hydrodynamic type systems, while $W_k^v$ is a velocity matrix of any other commuting higher commuting flow. $\delta_k^t$ is the Kronecker delta.
Then (see (11), (18), (19))

\[ d\Phi = \tilde{U}_1 dx + \tilde{U}_2 dt + \tilde{U}_3 dy \]

can be integrated explicitly. A corresponding solution of Mikhalëv equation (22) is

\[ \Phi = xy + \frac{1}{2} t^2 + ty^2 + \frac{1}{4} y^4. \]

The next case \( \tilde{U}_5 \) leads to the non-polynomial solution of Mikhalëv equation (22)

\[ \Phi = yt + \frac{1}{5} X^5, \]

where (here \( Y = t + \frac{1}{3} X^3, T = y - \frac{1}{2} X^2 \))

\[ X^4 = 4x + 2y^2. \]

So, the existence of non-diagonalisable hydrodynamic reductions for integrable three-dimensional quasilinear systems of first order is also a new interesting phenomenon. Thus, examination of multi-dimensional quasilinear systems of first order should be continued by this reason in further publications.

5 Conclusion

Strictly hyperbolic hydrodynamic type systems have pairwise distinct roots. In this paper we considered the opposite situation: hydrodynamic type systems with multiple roots. The most degenerate case, just a one multiple root. This means that such hydrodynamic type systems no longer are hyperbolic. They are parabolic. The transformation to the Heat equation hierarchy is a natural illustration of this parabolic phenomenon.

An effective mechanism of integration of hydrodynamic type systems with pairwise distinct roots based on existence of special coordinate system, i.e. the Riemann invariants. In the case considered in this paper the coordinate system \( c_k \) is a most convenient from computational point of view. These coordinates have a natural interpretation, in these coordinates the Heat hierarchy is written as an infinite set of commuting linear equations in partial derivatives with constant coefficients.

A straightforward computation shows that, for instance, the Benney hydrodynamic chain does not possess such non-diagonalisable hydrodynamic reductions (hydrodynamic type systems with multiple roots). So, we come to the observation: just linearly degenerate hydrodynamic chains (they have at least one \( N \) component linearly degenerate hydrodynamic reduction for each \( N \), for further details see [1], [3], [9]) possess hydrodynamic reductions with multiple roots. Inspection of other linearly degenerate hydrodynamic chains should be made in a separate paper.

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