Generalized Paley graphs equienergetic with their complements

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ABSTRACT
We consider generalized Paley graphs $\Gamma_1^k(q)$, generalized Paley sum graphs $\Gamma_1^+(k,q)$, and their corresponding complements $\overline{\Gamma_1}^k(q)$ and $\overline{\Gamma_1}^+(k,q)$, for $k = 3, 4$. Denote by $\Gamma_1^k$ either $\Gamma_1^k(q)$ or $\Gamma_1^+(k,q)$. We compute the spectra of $\Gamma_1^3(q)$ and $\Gamma_1^4(q)$ and from them we obtain the spectra of $\Gamma_1^+(3,q)$ and $\Gamma_1^+(4,q)$ also. Then we show that, in the non-semiprimitive case, the spectrum of $\Gamma_1^k(p^t)$ and $\Gamma_1^+(p^t)$ with $p$ prime can be recursively obtained, under certain arithmetic conditions, from the spectrum of the graphs $\Gamma_1^3$ and $\Gamma_1^4$ for any $t \in \mathbb{N}$, respectively. Using the spectra of these graphs we give necessary and sufficient conditions on the spectrum of $\Gamma_1^k$ such that $\Gamma_1^k$ and $\overline{\Gamma_1}^k$ are equienergetic for $k = 3, 4$. In a previous work we have classified all bipartite regular graphs $\Gamma_{\text{bip}}$ and all strongly regular graphs $\Gamma_{\text{srg}}$ which are complementary equienergetic, i.e. $\{\Gamma_{\text{bip}}, \overline{\Gamma_{\text{bip}}})$ and $\{\Gamma_{\text{srg}}, \overline{\Gamma_{\text{srg}}})$ are equienergetic pairs of graphs. Here we construct infinite pairs of equienergetic non-isospectral regular graphs $\{\Gamma, \Gamma\}$ which are neither bipartite nor strongly regular.

1. Introduction

Recently, in [1], we studied regular graphs equienergetic with their complements and we characterized all bipartite graphs and all strongly regular graphs which are equienergetic with their complements. This includes the case of semiprimitive generalized Paley graphs $\Gamma(k,q)$, which are known to be strongly regular. We recall that $\Gamma(k,q)$ is semiprimitive if $-1$ is a power of $p$ modulo $k$. Thus, if $\Gamma(k,q)$ is semiprimitive, it is either a classical Paley graph $\Gamma(2,q)$ with $q \equiv 1 \pmod {4}$ or else we have $k > 2, q = p^m$ with $m$ even and $k \mid p^t + 1$ for some $t \mid \frac{m}{2}$. Here, by considering generalized Paley graphs of the form $\Gamma(3,q)$ and $\Gamma(4,q)$ which are not semiprimitive, we will produce infinite pairs of complementary equienergetic regular graphs which are neither bipartite nor strongly regular.
**Spectrum and energy**

Let $\Gamma$ be a graph of $n$ vertices. The eigenvalues of $\Gamma$ are the eigenvalues $\{\lambda_i\}_{i=1}^n$ of its adjacency matrix. The *spectrum* of $\Gamma$, denoted $\text{Spec}(\Gamma) = \{[\lambda_{i1}]^{e_1}, \ldots, [\lambda_{is}]^{e_s}\}$, is the set of all the different eigenvalues $\{\lambda_{ij}\}$ of $\Gamma$ counted with their multiplicities $\{e_{ij}\}$.

The *energy* of $\Gamma$ is defined by

$$E(\Gamma) = \sum_{i=1}^n |\lambda_i| = \sum_{j=1}^s e_{ij} |\lambda_{ij}|.$$  \hspace{1cm} (1)

We refer to the books [2] or [3] for a complete viewpoint of spectral theory of graphs and to [4] for a survey on energy of graphs (see also the book [5] which contains many open problems related to energy of graphs).

Let $\Gamma_1$ and $\Gamma_2$ be two graphs with the same number of vertices. The graphs are said *isospectral* if $\text{Spec}(\Gamma_1) = \text{Spec}(\Gamma_2)$ and *equienergetic* if

$$E(\Gamma_1) = E(\Gamma_2).$$

It is clear by the definitions that isospectrality implies equienergeticity, but the converse does not hold in general. There are many papers on these problems (see for instance [1, 6–11] and the references therein). If a graph $\Gamma$ and its complement $\bar{\Gamma}$ are equienergetic we will say, as in [12], that they are *complementary equienergetic* graphs. Self-complementary graphs are trivially complementary equienergetic, so the interest is put on non self-complementary graphs.

**Generalized Paley (sum) graphs**

Let $G$ be a finite abelian group and $S$ a subset of $G$ with $0 \notin S$. The *Cayley graph* $X(G, S)$ is the directed graph whose vertex set is $G$ and $v, w \in G$ form a directed edge of $\Gamma$ from $v$ to $w$ if $w - v \in S$. Since $0 \notin S$ then $\Gamma$ has no loops. Analogously, the *Cayley sum graph* $X^+(G, S)$ has the same vertex set $G$ but now $v, w \in G$ are connected in $\Gamma$ by an arrow from $v$ to $w$ if $v + w \in S$. We will use the notation $X^*(G, S)$ when we want to consider both $X(G, S)$ and $X^+(G, S)$. Notice that if $S$ is symmetric, that is $-S = S$, then $X^*(G, S)$ is an $|S|$-regular simple (undirected without multiple edges) graph. However, the graph $X^+(G, S)$ may contain loops. In this case, there is a loop on vertex $x$ provided that $x + x \in S$. For an excellent survey of spectral properties of general Cayley graphs we refer the reader to [13]. In [10], Theorem 2.7, we showed that if $G$ is abelian and $S$ is a symmetric subset of $G$ not containing zero, then $X(G, S)$ and $X^+(G, S)$ are equienergetic graphs. Also, in Proposition 2.10 in [10] we give sufficient conditions for $X(G, S)$ and $X^+(G, S)$ to be non-isospectral.
We are interested in generalized Paley graphs

\[ \Gamma(k, q) = X(F_q, R_k) \quad \text{with} \quad R_k = \{x^k : x \in F_q^*\}, \]

that is when \( G \) is the finite field \( F_q \) with \( q \) elements and \( S \) is the set of non-zero \( k \)th powers of \( F_q \), and the generalized Paley sum graph

\[ \Gamma^+(k, q) = X^+(F_q, R_k). \]

We will refer to them simply as \( GP \)-graphs and \( GP^+ \)-graphs, respectively (or \( GP^* \)-graphs for both indistinctly). When \( k = 1 \), the graph \( \Gamma(1, q) \) is just the complete graph in \( q \)-vertices \( K_q \) and when \( k = 2 \) the graphs \( \Gamma(2, q) \) correspond to the classical Paley graphs \( P(q) \) if \( q = 1 \mod 4 \). The next \( GP \)-graphs to consider, aside from general families such as the semiprimitive ones or the Hamming \( GP \)-graphs, are those with \( k = 3, 4 \), that is \( \Gamma(3, q) \) and \( \Gamma(4, q) \). Notice that for \( q \) even, we have that \( \Gamma^+(k, q) = \Gamma(k, q) \). On the other hand, when \( q \) is odd, it can be seen that \( \Gamma^+(k, q) \) always has loops, since in this case we can find exactly \( |R_k| \) elements \( x \in F_q \) such that \( x + x \in R_k \) (multiplication by 2 is a bijection in \( F_q \) for \( q \) odd). In particular, \( \Gamma^+(k, q) \) has loops for \( k = 1, 2, 3, 4 \) and \( q \) odd.

Generalized Paley graphs have been extensively studied in the past few years. Lim and Praeger studied their automorphism groups and characterized all \( GP \)-graphs which are Hamming graphs [14]. Also, Pearce and Praeger characterized those \( GP \)-graphs which are Cartesian decomposable [15]. Recently, Chi Hoi Yip has studied their clique number ([16, 17], see also [18]). Both classic Paley graphs and generalized Paley graphs have been used to find linear codes with good decoding properties [19–21]. They can also be seen as particular regular maps in Riemann surfaces [22, 23]. The number of walks in \( GP \)-graphs are related with the number of solutions of diagonal equations over finite fields [24]. Moreover, the diameter of \( \Gamma(k, q) \), if it exists, coincides with the Waring number \( g(k, q) \) (see [25, 26]). Under some mild restrictions, the spectrum of \( GP \)-graphs determines the weight distribution of their associated irreducible codes [27, 28].

**Cameron’s hierarchy**

There is a hierarchy of regularity conditions on graphs due to Cameron [29]. For a non-negative integer \( t \) and sets \( S_1 \) and \( S_2 \) of at most \( t \) vertices, let \( C(t) \) be this graph property: if the induced subgraphs on \( S_1 \) and \( S_2 \) are isomorphic, then the number of vertices joined to every vertex in \( S_1 \) is equal to the number of vertices joined to every vertex in \( S_2 \). A graph satisfying property \( C(t) \) is called \( t \)-tuple regular. Conditions \( C(t) \) are stronger as \( t \) increases. A graph satisfies \( C(1) \) if and only if it is regular and it satisfies \( C(2) \) if and only if it is strongly regular. If a graph satisfies \( C(3) \) then it is the pentagon \( C_5 \) or it has the parameters of a pseudo Latin square, a negative Latin square or a Smith type graph (see [30]). Up to complements, there are only two known examples of graphs satisfying \( C(4) \) but not \( C(5) \), the Schlafli graph and the McLaughlin graph. Finally, the hierarchy is finite; if a graph satisfies \( C(5) \) then it satisfies \( C(t) \) for any \( t \). The only such graphs are \( aK_m \) and its complement for \( a, m \geq 1 \), the pentagon \( C_5 \) and the \( 3 \times 3 \) square lattice \( L(K_{3,3}) \) (see [31]).

**Outline and results**

We now summarize the main results in the paper. In Section 2, we give the spectrum of \( \Gamma^+(3, q) \) and \( \Gamma^+(4, q) \) from the spectral relationship between irreducible cyclic codes and
GP-graphs (see [27, 28]), and the spectral relationship between Cayley graphs and Cayley sum graphs (see [10]). Namely, in Theorems 2.2 and 2.4 we compute the spectrum of $\Gamma(3, q)$ and $\Gamma(4, q)$ using results in [27] while in Theorems 2.8 and 2.9 we obtain, from the spectrum of $\Gamma(3, q)$ and $\Gamma(4, q)$, the spectrum of the sum graphs $\Gamma^+(3, q)$ and $\Gamma^+(4, q)$, using results in [10]. In the non-semiprimitive case, that is when $p \equiv 1 \pmod{k}$, the spectra of $\Gamma^*(3, q)$ and $\Gamma^*(4, q)$ are given in terms of certain integer solutions of quadratic diophantine equations. More precisely, solutions of the equations

$$p^t = X^2 + 27Y^2 \quad \text{for} \quad k = 3 \quad \text{and} \quad p^{2t} = X^2 + 4Y^2 \quad \text{for} \quad k = 4,$$

with $t \in \mathbb{N}$.

In Section 3 we show that, under certain conditions, the spectrum of $\Gamma(3, q)$ and $\Gamma(4, q)$ in the non-semiprimitive case can be obtained recursively from the spectrum of the graphs $\Gamma(3, q')$ and $\Gamma(4, q')$ with $q' < q$. More precisely, in Theorem 3.1 we proved that if $p$ is a prime with $p \equiv 1 \pmod{3}$ then the spectra of $\Gamma(3, p^{3\ell})$ and $\Gamma(3, p^{3s})$ determine the spectrum of $\Gamma(3, p^{3t})$ for every $\ell \geq 1$ and $0 \leq s < t$, where $t$ is the minimal integer such that the first equation in (3) has integer solutions $(x, y)$ with $(x, p) = 1$. In Theorem 3.4 we showed that if $2$ is a cubic residue modulo a prime $p \equiv 1 \pmod{3}$ then the spectrum of $\Gamma(3, p)$ determines the spectrum of $\Gamma(3, p^{3\ell})$ for every $\ell \in \mathbb{N}$. Also, in Theorem 3.6 we prove that the spectrum of $\Gamma(4, p)$ always determines the spectrum of $\Gamma(4, p^{4\ell})$ for every $\ell \in \mathbb{N}$. In all these 3 theorems we give the corresponding spectrum explicitly in terms of the integer solution of the corresponding base equation (for instance $p = X^2 + 27Y^2$ and $p^2 = X^2 + 4Y^2$ in Theorems 3.4 and 3.6, respectively).

In Section 4 we focus on the energy of the graphs $\Gamma^*(3, q)$ and $\Gamma^*(4, q)$. In Proposition 4.1 we study their energies, computing them explicitly in the semiprimitive case and giving lower and upper bounds in the non-semiprimitive case. Then, in Theorem 4.2 we give conditions on $\Gamma^*(k, q)$ to be equienergetic to its complement $\overline{\Gamma}^*(k, q)$. In fact, we show that the graphs $\Gamma(k, q)$ and $\overline{\Gamma}(k, q)$ are equienergetic if and only if among the non-principal eigenvalues of $\Gamma(k, q)$ exactly one of them is positive.

Finally, in Theorems 5.1 and 5.3, by using the results in Sections 3 and 4, we construct infinite pairs of complementary equienergetic graphs of the form $\{\Gamma^*(k, q), \overline{\Gamma}^*(k, q)\}$ for $k = 3$ and $4$ and $q = p^m$ which are neither bipartite nor strongly regular. This is relevant since we have previously obtained complementary equienergetic pairs of regular graphs which are either bipartite or strongly regular (see Sections 4–7 in [1]). In fact, we have characterized all such complementary equienergetic pairs of regular graphs. That is to say that, in terms of Cameron’s hierarchy, we understand complementary equienergetic graphs in the families $C(2), C(3)$ and $C(5)$; and for bipartite graphs in $C(1)$. For general (i.e. non-bipartite) graphs in $C(1)$ the problem seems to be out of scope. The pairs obtained in this work are precisely non-bipartite regular graphs which are not strongly regular, that is non-bipartite graphs in $C(1) \setminus C(2)$.

2. Spectrum

Here we will compute the spectrum of the graphs $\Gamma(k, q)$ and $\Gamma^+(k, q)$ for $k = 3, 4$ using previous results obtained by the authors.

We begin with the spectrum of the GP-graphs $\Gamma(3, q)$ and $\Gamma(4, q)$, which will follow from a spectral correspondence with the weight distribution of certain cyclic codes $C(3, q)$
and $C(4, q)$ obtained in [27], that we now recall. Let $p$ be a prime and $q = p^m$ for some $m$. For $k \mid q - 1$ consider the $p$-ary irreducible cyclic codes

$$C(k, q) = \left\{ c_\gamma = \left( \text{Tr}_{q/p}(\gamma^k) \right)^{n-1} : \gamma \in \mathbb{F}_q \right\}$$

(4)

where $\omega$ is a primitive element of $\mathbb{F}_q$ over $\mathbb{F}_p$. These are the codes with zero $\omega^{-k}$ and length

$$n = \frac{q - 1}{N} \quad \text{with} \quad N = \gcd \left( \frac{q - 1}{p - 1}, k \right).$$

(5)

Then, we have the following result.

**Theorem 2.1 ([27]):** Let $q = p^m$ with $p$ prime and $k \in \mathbb{N}$ such that $k \mid \frac{q - 1}{p - 1}$. Put $n = \frac{q - 1}{k}$. Let $\Gamma(k, q)$ and $C(k, q)$ be as in (2) and (4). Thus, we have:

(a) The eigenvalue $\lambda_\gamma$ of $\Gamma(k, q)$ and the weight of $c_\gamma \in C(k, q)$ are related by the expression

$$\lambda_\gamma = n - \frac{p}{p - 1} w(c_\gamma).$$

(6)

(b) If $\Gamma(k, q)$ is connected the multiplicity of $\lambda_\gamma$ is the frequency $A_{w(c_\gamma)}$ for all $\gamma \in \mathbb{F}_q$. In particular, the multiplicity of $\lambda_0 = n$ is $A_0 = 1$.

It is known that some structural properties of graphs can be read from the spectrum. For instance, it is a classic result that a regular graph $\Gamma$ is connected if and only if its principal eigenvalue has multiplicity one and that it is bipartite if and only if the spectrum is symmetric. Also, $\Gamma$ is strongly regular if and only if it is connected with three different eigenvalues (disregarding multiplicities).

In the next two theorems we give the spectrum of the GP-graphs $\Gamma(3, q)$ and $\Gamma(4, q)$.

**Theorem 2.2:** Let $q = p^m \geq 5$ with $p$ prime such that $3 \mid \frac{q - 1}{p - 1}$ and put $n = \frac{q - 1}{3}$. Thus, the graph $\Gamma(3, q)$ is connected with integral spectrum given as follows:

(a) If $p \equiv 1 \pmod{3}$ then $m = 3t$ for some $t \in \mathbb{N}$ and

$$\text{Spec}(\Gamma(3, q)) = \left\{ [n], \left[ \frac{a \sqrt{q} - 1}{3} \right]^n, \left[ -\frac{1}{2} (a + 9b) \sqrt{q} - 1 \right]^n, \right.$$

$$\left. \left[ -\frac{1}{2} (a - 9b) \sqrt{q} - 1 \right]^n \right\}$$

where $a, b$ are integers uniquely determined by

$$4 \sqrt{q} = a^2 + 27b^2, \quad a \equiv 1 \pmod{3} \quad \text{and} \quad (a, p) = 1.$$
(b) If \( p \equiv 2 \pmod{3} \) then \( m = 2t \) for some \( t \in \mathbb{N} \) and

\[
\text{Spec}(\Gamma(3,q)) = \begin{cases} 
\{ [n]^1, \left[ \frac{\sqrt{q-1}}{3} \right]^{2n}, \left[ -\frac{2}{3} \sqrt{q-1} \right]^n \} & \text{for } m \equiv 0 \pmod{4}, \\
\{ [n]^1, \left[ \frac{2}{3} \sqrt{q-1} \right]^n, \left[ -\sqrt{q-1} \right]^{2n} \} & \text{for } m \equiv 2 \pmod{4}.
\end{cases}
\]

In particular, \( \Gamma(3,q) \) is a strongly regular graph in this case.

**Proof:** Let \( q = p^m \). First note that condition 3 | \( \frac{q-1}{p-1} = p^{m-1} + \cdots + p + 1 \) implies that \( m = 3t \) if \( p \equiv 1 \pmod{3} \) and \( m = 2t \) if \( p \equiv 2 \pmod{3} \).

We will apply Theorem 2.1 to the code \( C(3,q) \). The spectrum of \( C(3,q) \) is given in Theorems 19 and 20 in [32], with different notations (\( r \) for our \( q \), \( N \) for our \( k \), etc.). By (a) in Theorem 2.1, the eigenvalues of \( \Gamma(3,q) \) are given by

\[
\lambda_i = \frac{q-1}{3} - \frac{p}{p-1} w_i
\]

where \( w_i \) are the weights of \( C(3,q) \).

If \( p \equiv 1 \pmod{3} \), by Theorem 19 in [32], the four weights of \( C(3,q) \) are \( w_0 = 0 \),

\[
w_1 = \frac{(p-1)(q - a \sqrt{q})}{3p}, \\
w_2 = \frac{(p-1)(q + 1/2(a + 9b) \sqrt{q})}{3p}, \\
w_3 = \frac{(p-1)(q + 1/2(a - 9b) \sqrt{q})}{3p}
\]

with frequencies \( A_0 = 1 \) and \( A_1 = A_2 = A_3 = \frac{q-1}{3} \); where \( a \) and \( b \) are the only integers satisfying \( 4 \sqrt{q} = a^3 + 27b^3 \), \( a \equiv 1 \pmod{3} \) and \( (a,p) = 1 \).

On the other hand, if \( p \equiv 2 \pmod{3} \), by Theorem 20 in [32], the three weights of \( C(3,q) \) are

\[
w_0 = 0, \quad w_1 = \frac{(p-1)(q - \sqrt{q})}{3p}, \quad w_2 = \frac{(p-1)(q + 2 \sqrt{q})}{3p},
\]

with frequencies \( A_0 = 1, A_1 = \frac{2(q-1)}{3} \) and \( A_2 = \frac{q-1}{3} \) if \( m \equiv 0 \pmod{4} \) while

\[
w_0 = 0, \quad w_1 = \frac{(p-1)(q - 2 \sqrt{q})}{3p}, \quad w_2 = \frac{(p-1)(q + \sqrt{q})}{3p},
\]

with frequencies \( A_0 = 1, A_1 = \frac{q-1}{3} \) and \( A_2 = \frac{2(q-1)}{3} \) if \( m \equiv 2 \pmod{4} \). By introducing (9), (10) and (11) in (8), we get the eigenvalues in (a) and (b) of the statement.

To compute the multiplicities, we use (b) of Theorem 2.1. The hypothesis that \( \Gamma(3,q) \) is connected is equivalent to the fact that \( n = \frac{q-1}{3} \) is a primitive divisor of \( q-1 \) (see the Introduction, after (2)). We now show that this is always the case for \( q \geq 5 \).

Suppose that \( p \geq 5 \). Then, we have that \( p^{m-1} - 1 \leq \frac{q-1}{3} \), since this inequality is equivalent to \( 3p^{m-1} - 3 \leq p^m - 1 \) which holds for \( p \geq 5 \). This implies that \( n \) is greater that
$p^a - 1$ for all $a < m$ and, hence, $n$ is a primitive divisor of $q - 1$. Now, if $p = 2$ we only have to check that $n$ does not divide $2^{m-1} - 1$, since $n > 2^{m-2} - 1$. Notice that $n \mid 2^{m-1} - 1$ if and only if $2^m - 1 \mid 3(2^m - 1)$, which can only happen if $m = 2$. If $m > 2$ we have $3(2^m - 1) \equiv 2^{m-1} - 2 \equiv 0 \pmod{2^m - 1}$ as we wanted. The prime $p = 3$ is excluded by hypothesis.

Thus, by part (b) of Theorem 2.1, the multiplicities of the eigenvalues of $\Gamma (3, q)$ are the frequencies of the weights of $C(3, q)$, and we are done. 

Note that in case (a) of the previous theorem, the non-principal eigenvalues are of the form

$$\lambda = \frac{\alpha \sqrt{q} - 1}{3} \quad \text{with} \quad \alpha = a, \frac{1}{2}(a - 9b), \frac{1}{2}(a + 9b)$$

where $(a, b)$ are the solutions of $4\sqrt{q} = X^2 + 27Y^2$ with $a \equiv 1 \pmod{3}$ and $(a, p) = 1$. Furthermore, since $\sqrt{p^{4t+2}} = p\sqrt{p^4}$, for $p = 2$ we have a relation between the eigenvalues of $\Gamma (3, 2^t)$, $\Gamma (3, 2^{t+2})$ and $\Gamma (3, 2^{t+4})$. Namely,

$$\lambda_3(\Gamma (3, 2^t)) = \lambda_3(\Gamma (3, 3^{2t+2})) \quad \text{and} \quad \lambda_2(\Gamma (3, 3^{2t+2})) = \lambda_2(\Gamma (3, 3^{2t+4})),$$

where $\lambda_1 > \lambda_2 > \lambda_3$.

Example 2.3: (i) Let $p = 7$ and $m = 3$, hence $q = 7^3 = 343$. Since $p \equiv 1 \pmod{3}$, we must find $a, b \in \mathbb{Z}$ such that $28 = a^2 + 27b^2$, $a \equiv 1 \pmod{3}$ and $(a, 7) = 1$. Clearly $a = b = 1$ satisfy these conditions. By Theorem 2.2 (i) we have $\text{Spec}(\Gamma (3, 7^3)) = \{[114]^1, [9]^{114}, [2]^{114}, [-12]^{114}\}$.

(ii) Let $p = 2$. By Theorem 2.2 (ii) we have for instance $\text{Spec}(\Gamma (3, 2^4)) = \{[5]^1, [1]^10, [-2]^5\}$, $\text{Spec}(\Gamma (3, 2^6)) = \{[21]^1, [5]^21, [-3]^42\}$ and $\text{Spec}(\Gamma (3, 2^8)) = \{[85]^1, [5]^170, [-3]^85\}$. For $p = 5$ we have $\text{Spec}(\Gamma (3, 5^2)) = \{[8]^1, [3]^8, [-2]^16\}$ and $\text{Spec}(\Gamma (3, 5^4)) = \{[208]^1, [24]^416, [-17]^208\}$.

\[\text{Theorem 2.4:}\] Let $q = p^m$ with $p$ prime such that $4 \mid \frac{q - 1}{p - 1}$ and $q \geq 5$ with $q \neq 9$. Let $n = \frac{q - 1}{4}$. Thus, the graph $\Gamma (4, q)$ is connected with integral spectrum given as follows:

(a) If $p \equiv 1 \pmod{4}$ then $m = 4t$ for some $t \in \mathbb{N}$ and

$\text{Spec}(\Gamma (4, q)) = \{[n]^1, \left[\frac{\sqrt{q} + 4d \sqrt{q} - 1}{4}\right]^n, \left[\frac{\sqrt{q} - 4d \sqrt{q} - 1}{4}\right]^n, \left[\frac{-\sqrt{q} + 2c \sqrt{q} - 1}{4}\right]^n, \left[\frac{-\sqrt{q} - 2c \sqrt{q} - 1}{4}\right]^n\}$

where $c, d$ are integers uniquely determined by

$$\sqrt{q} = c^2 + 4d^2, \quad c \equiv 1 \pmod{4} \quad \text{and} \quad (c, p) = 1. \quad (12)$$
(b) If \( p \equiv 3 \pmod{4} \) then \( m = 2t \) for some \( t \in \mathbb{N} \) and

\[
\text{Spec}(\Gamma(4, q)) = \begin{cases} 
[n]^1, \left[\frac{\sqrt{q-1}}{4}\right]^{3n}, \left[\frac{-3\sqrt{q-1}}{4}\right]^n & \text{for } m \equiv 0 \pmod{4}, \\
n^1, \left[\frac{3\sqrt{q-1}}{4}\right]^n, \left[\frac{-\sqrt{q-1}}{4}\right]^{3n} & \text{for } m \equiv 2 \pmod{4}.
\end{cases}
\]

In particular, \( \Gamma(4, q) \) is a strongly regular graph in this case.

**Proof:** The proof is similar to the one of Theorem 2.2. We apply Theorem 2.1 to the code \( C(4, q) \) since the spectrum of this code is given in Theorem 21 in [32]. Thus, we skip the details and only show that if \( q \geq 5 \) with \( q \neq 9 \), then \( \frac{q-1}{4} \) is a primitive divisor of \( q-1 \) and, hence, \( \Gamma(4, q) \) is connected.

Suppose that \( p \geq 5 \). Then, we have that \( p^{m-1} - 1 \leq \frac{q-1}{4} \) since this inequality is equivalent to \( 4p^{m-1} - 4 \leq p^m - 1 \), which is true because \( p \geq 5 \). This implies that \( n \) is greater that \( p^a - 1 \) for all \( a < m \) and hence \( n \) is a primitive divisor of \( q-1 \). Now, if \( p = 3 \) we only have to check that \( n \) does not divide \( 3^{m-1} - 1 \), since \( n > 3^{m-2} - 1 \). Notice that \( n \mid 3^{m-1} - 1 \) if and only if \( 3^m - 1 \mid 4(3^{m-1} - 1) \), which can only happen if \( m = 2 \). If \( m > 2 \) in this case \( 4(3^{m-1} - 1) \equiv 3^{m-1} - 3 \not\equiv 0 \pmod{3^{m-1}} \) as we wanted. The prime \( p = 2 \) is excluded by hypothesis. \( \blacksquare \)

In case (a) of the previous theorem, the non-principal eigenvalues are of the form

\[
\lambda = \frac{\alpha \sqrt{q} - 1}{4} \quad \text{with} \quad \alpha = \sqrt{q} \pm 4d, -\sqrt{q} \pm 2c
\]

where \((c, d)\) are integer solutions of \( 4\sqrt{q} = X^2 + 27Y^2 \) with \( a \equiv 1 \pmod{3} \) and \((a, p) = 1\).

Also, since \( \sqrt{p^{2t+2}} = p\sqrt{p^t} \), for \( p = 3 \) we have the relations

\[
\lambda_3(\Gamma(3, 3^{2t})) = \lambda_3(\Gamma(3, 3^{2t+2})) \quad \text{and} \quad \lambda_2(\Gamma(3, 3^{2t+2})) = \lambda_2(\Gamma(3, 3^{2t+4}))
\]

between the eigenvalues of \( \Gamma(3, 3^{2t}) \), \( \Gamma(3, 3^{2t+2}) \) and \( \Gamma(3, 3^{2t+4}) \), where \( \lambda_1 > \lambda_2 > \lambda_3 \).

**Remark 2.5:** We now make two observations relative to Theorems 2.2 and 2.4.

(i) In parts (b) of these theorems the graphs \( \Gamma(3, q) \) and \( \Gamma(4, q) \) are semiprimitive (see the Introduction). The spectrum of semiprimitive generalized Paley graphs \( \Gamma(k, q) \) was studied and computed in [27]. Thus, parts (b) in these theorems can be obtained as particular cases of Theorem 3.3 in [27] with \( k = 3, 4 \).

(ii) By the theorems, it is implicit that the equations \( 4p^t = X^2 + 27Y^2 \) for \( p \equiv 1 \pmod{3} \) and \( p^{2t} = X^2 + 4Y^2 \) for \( p \equiv 1 \pmod{4} \) always have integer solutions \((x, y)\) with \((x, p) = 1\), where \( x \equiv 1 \pmod{3} \) for the first equation and \( x \equiv 1 \pmod{4} \) for the second one. This is known from number theory results.

**Example 2.6:** (i) Let \( q = 5^4 = 625 \), that is \( p = 5 \) and \( m = 4 \). Since \( p \equiv 1 \pmod{4} \), we have to find integers \( c, d \) such that \( 25 = c^2 + 4d^2 \), \( c \equiv 1 \pmod{4} \) and \( (c, 5) = \)
1. One can check that \((c, d) = (-3, 2)\) satisfy these conditions and hence by \((i)\) in Theorem 2.4, the spectrum of \(\Gamma(4, 625)\) is given by

\[
\text{Spec}(\Gamma(4, 625)) = \left\{ [156]^1, [16]^1, [1]^1, [-4]^1, [-14]^1 \right\}.
\]

(ii) Let \(p = 3\). By \((b)\) in Theorem 2.4 we have \(\text{Spec}(\Gamma(4, 3^4)) = \{ [20]^1, [2]^60, [-7]^20 \}\),
\(\text{Spec}(\Gamma(4, 3^6)) = \{ [182]^1, [20]^182, [-7]^546 \}\) and \(\text{Spec}(\Gamma(4, 3^8)) = \{ [1640]^1, [20]^1920, [-61]^1640 \}\).

\[\diamondsuit\]

**Remark 2.7:** Notice that from items \((a)\) of the previous two theorems, if we denote by \(\lambda_1, \ldots, \lambda_k\) the non-principal eigenvalues of \(\Gamma(k, q)\), with \(k = 3\) or \(4\), in the order listed in the theorems, we then have that \(a = \frac{3\lambda_1 A + 1}{p^2}, b = \frac{\lambda_3 - \lambda_2}{3p^2}, c = \frac{\lambda_3 - \lambda_4}{3p^2}\), and \(d = \frac{\lambda_1 - \lambda_2}{2p^2}\), respectively.

We now give the spectrum of the sum GP-graphs \(\Gamma^+(3, q)\) and \(\Gamma^+(4, q)\) for \(q\) odd, since for \(q\) even \(\Gamma^+(3, q) = \Gamma(3, q)\) and \(\Gamma^+(4, q) = \Gamma(4, q)\). The spectra of \(\Gamma^+(3, q)\) and \(\Gamma^+(4, q)\) will be obtained from the corresponding ones of \(\Gamma(3, q)\) and \(\Gamma(4, q)\) using the same techniques as in [10].

**Theorem 2.8:** Let \(q = p^m\) with \(p\) an odd prime and \(m \in \mathbb{N}\) such that \(3 \mid \frac{q-1}{p^2}\) and \(q \geq 5\) and put \(n = \frac{q-1}{3}\). Thus, the graph \(\Gamma^+(3, q)\) is connected with integral spectrum given by:

(a) If \(p \equiv 1 \pmod{3}\) then \(m = 3t\) for some \(t \in \mathbb{N}\) and

\[
\text{Spec}(\Gamma^+(3, q)) = \left\{ [n]^1, \left[ \pm \frac{a\sqrt{q} - 1}{3} \right]^\frac{n}{2}, \left[ \pm \frac{-\frac{1}{2}(a + 9b)\sqrt{q} - 1}{3} \right]^\frac{n}{2}, \left[ \pm \frac{-\frac{1}{2}(a - 9b)\sqrt{q} - 1}{3} \right]^\frac{n}{2} \right\}
\]

where \(a, b\) are integers uniquely determined by conditions \((7)\).

(b) If \(p \equiv 2 \pmod{3}\) then \(m = 2t\) for some \(t \in \mathbb{N}\) and

\[
\text{Spec}(\Gamma^+(3, q)) = \begin{cases} 
[n]^1, \left[ \pm \frac{\sqrt{q} - 1}{3} \right]^\frac{n}{2}, & \text{for } m \equiv 0 \pmod{4}, \\
[n]^1, \left[ \pm \frac{\frac{1}{2}\sqrt{q} - 1}{3} \right]^\frac{n}{2}, & \text{for } m \equiv 2 \pmod{4}.
\end{cases}
\]

**Proof:** It is a direct consequence of Proposition 2.10 in [10] and Theorem 2.2, since \(\mathbb{F}_q\) has no trivial real characters for \(q\) odd.

In the same way, we have the following result.

**Theorem 2.9:** Let \(q = p^m\) with \(p\) an odd prime and \(m \in \mathbb{N}\) such that \(4 \mid \frac{q-1}{p-1}\) and \(q \geq 5\) with \(q \neq 9\) and put \(n = \frac{q-1}{4}\). Thus, the graph \(\Gamma^+(4, q)\) is connected with integral spectrum given by:
(a) If \( p \equiv 1 \pmod{4} \) then \( m = 4t \) for some \( t \in \mathbb{N} \) and

\[
\text{Spec}(\Gamma^+(4, q)) = \left\{[n]^1, \left[\frac{\pm \sqrt{q} + 4d \sqrt{q} - 1}{4}\right]^2, \left[\frac{\pm \sqrt{q} - 4d \sqrt{q} - 1}{4}\right]^2, \left[\frac{- \sqrt{q} + 2c \sqrt{q} - 1}{4}\right]^2, \left[\frac{- \sqrt{q} - 2c \sqrt{q} - 1}{4}\right]^2\right\}
\]

where \( c, d \) are integers uniquely determined by conditions (12).

(b) If \( p \equiv 3 \pmod{4} \) then \( m = 2t \) for some \( t \in \mathbb{N} \) and

\[
\text{Spec}(\Gamma^+(4, q)) = \left\{[n]^1, \left[\frac{\pm \sqrt{q} + 1}{4}\right]^2, \left[\pm \frac{3\sqrt{q} + 1}{4}\right]^2\right\} \text{ for } m \equiv 0 \pmod{4},
\]

\[
\left\{[n]^1, \left[\frac{-3\sqrt{q} - 1}{4}\right]^2, \left[\pm \frac{3\sqrt{q} - 1}{4}\right]^2\right\} \text{ for } m \equiv 2 \pmod{4}.
\]

**Proof:** Again we use Proposition 2.10 in [10] and Theorem 2.2.

As in Remark 2.7, we can put the integers \( a, b, c, d \) appearing in Theorems 2.8 and 2.9 in term of the eigenvalues.

Of course, from the spectrum of \( \Gamma = \Gamma(k, q) \) for \( k = 3, 4 \) one can obtain the spectrum of the complementary graph \( \overline{\Gamma} \) and the spectra of the associated line graphs \( L(\Gamma), L(\overline{\Gamma}) \) and of their complements \( \overline{L(\Gamma)} \) and \( \overline{L(\overline{\Gamma})} \).

We finish the section with some comments on Theorems 2.2, 2.4, 2.8 and 2.9 which will be referred to simply as ‘the theorems’.

**Remark 2.10:**

(i) The integers \( b \) and \( d \) in ‘the theorems’ are determined up to sign. However, by symmetry, the eigenvalues in these theorems are not affected by these choices of sign.

(ii) From ‘the theorems’ we know that all the graphs \( \Gamma^*(3, q), \Gamma^*(4, q) \) are connected and non-bipartite. The semiprimitive graphs \( \Gamma(k, q) \) with \( k = 3, 4 \), i.e. those in items (b) of Theorems 2.2 and 2.4, are strongly regular while the non-semiprimitive graphs (all other graphs in these four theorems) are non strongly regular.

(iii) By ‘the theorems’, \( \{\Gamma(3, q), \Gamma^+(3, q)\} \) and \( \{\Gamma(4, q), \Gamma^+(4, q)\} \) are pairs of equienergetic non-isospectral graphs which are non-bipartite. In the non-semiprimitive case \( q \equiv 1 \pmod{k} \) with \( k = 3, 4 \) they are also non strongly regular graphs.

### 3. Derived spectrum on field extensions

In this section we show that, under certain mild hypothesis, we can give the spectrum of \( \Gamma(k, p^{k\ell}) \) for \( k = 3, 4 \), with \( p \) any prime number congruent to 1 mod \( k \) and \( \ell > 1 \), in the non-semiprimitive case (i.e. for \( p \equiv 1 \pmod{k} \)) from the spectra of \( \Gamma(k, p^{k\ell}) \). More precisely, by (a) in Theorem 2.2, if \( q = p^{3\ell} \) and \( p \equiv 1 \pmod{3} \) then the eigenvalues of \( \Gamma(3, q) \) and \( \Gamma(3, q^n) \) are respectively given in terms of certain integer solutions \( (a, b) \) and
(a_n, b_n) of the equations
\[ 4q = X^2 + 27Y^2 \quad \text{and} \quad 4q'' = X^2 + 27Y^2. \]

We will show that we can recursively obtain integer solutions \((a_n, b_n)\) of \(4q'' = X^2 + 27Y^2\) of the required form in infinitely many field extensions \(\mathbb{F}_q^\ell\) of \(\mathbb{F}_q\) from an initial integer solution \((a, b)\) of \(q = x^2 + 27y^2\) of the required form. Similarly for \(\Gamma (4, q)\) and \(\Gamma (4, q'')\) by using Theorem 2.4, although this case is much easier.

For \(k = 3\), we can give a more general result. Equations of the form \(4p^{3t} = X^2 + 27Y^2\) always have integers solutions (see Remark 2.5), but nothing can be said about the solutions of the equation \(p^{3t} = X^2 + 27Y^2\). However, assuming that there is a minimal \(t\) such that \(p^{3t} = X^2 + 27Y^2\) has an integer solution we can provide the spectrum of \(\Gamma (3, p^{3(t\ell+s)})\) for any \(\ell \in \mathbb{N}\) and \(0 \leq s < t\).

**Theorem 3.1:** Let \(p\) be a prime with \(p \equiv 1 \pmod{3}\). If there is a minimal \(t \in \mathbb{N}\) such that
\[ p^t = X^2 + 27Y^2 \tag{13} \]
has integral solutions \(x, y \in \mathbb{Z}\) with \((x, p) = 1\), then the spectra of \(\Gamma (3, p^{3(t\ell+s)})\), with \(\ell \geq 1\) and \(0 \leq s < t\), is determined by the spectra of the GP-graphs \(\Gamma (3, p^{3\ell})\) and \(\Gamma (3, p^{3s})\).

**Proof:** Let \(t\) be minimal in \(\mathbb{N}\) such that (13) has an integral solution \(x, y\) with \((x, p) = 1\). Notice that if \((x, y)\) is a solution of (13) then \((x, -y)\) and \((-x, \pm y)\) are also solutions. Also, from (13) we have that \(x^2 \equiv 1 \pmod{3}\) since \(p \equiv 1 \pmod{3}\) and hence \(x \equiv \pm 1 \pmod{3}\). Thus, we will choose one solution \((x_0, y_0)\), with \(x_0 \in \{\pm x\}\) and \(y_0 \in \{\pm y\}\), such that \(x_0 \equiv 1 \pmod{3}\).

Considering the complex number
\[ z_{x,y} := x + 3\sqrt{3}iy, \tag{14} \]
we have that \(\|z_{x,y}\|^2 = x^2 + 27y^2 = p^t\) and hence
\[ p^{t\ell} = \|z_{x,y}\|^{2\ell} = \|z_{x,y}^\ell\|^2 \tag{15} \]
for any \(\ell \in \mathbb{N}\). Now, we will express \(z_{x,y}^\ell\) in the form given in (14). For any \(\ell \in \mathbb{N}\) put
\[ z_{x,y}^\ell := z_{x_{\ell-1},y_{\ell-1}} = x_{\ell-1} + 3\sqrt{3}iy_{\ell-1} \]
where \(z_{x,y}^1 = z_{x,y}\) and \(x_0 = x, y_0 = y\). For instance, \(x_1 + 3\sqrt{3}iy_1 = z_{x,y}^2 = (x^2 - 27y^2) + 3\sqrt{3}i(2xy)\) so \(x_1 = x^2 - 27y^2\) and \(y_1 = 2xy\). By the relation \(z_{x,y}^{\ell+1} = z_{x,y}z_{x,y}^\ell\), one sees that the sequence \(\{(x_\ell, y_\ell)\}_{\ell \in \mathbb{N}_0}\) is thus recursively defined as follows: let \(x_0 = x, y_0 = y\) and for
any \( \ell > 0 \) take

\[
x_\ell = x_0x_{\ell-1} - 27y_0y_{\ell-1} \quad \text{and} \quad y_\ell = x_0y_{\ell-1} + x_{\ell-1}y_0.
\]  

(16)

Note that if \( \ell \geq 1 \), starting from \( y_\ell = x_0y_{\ell-1} + x_{\ell-1}y_0 \) and changing the \( y_j \)'s recursively we obtain the following equality

\[
y_\ell = y_0 \left( x_0^\ell + \sum_{i=0}^{\ell-1} x_ix_0^{\ell-1-i} \right).
\]  

(17)

So, from the second equation in (16) and (17) we get

\[
x_{\ell-1} = \frac{1}{y_0} (y_\ell - x_0y_{\ell-1}) = \sum_{i=0}^{\ell-1} x_ix_0^{\ell-1-i} - x_0 \sum_{i=0}^{\ell-2} x_ix_0^{\ell-2-i}.
\]  

(18)

From the first equation in (16), by using (17) and (18) and the fact that \( p^t = x^2 + 27y^2 \), for all \( \ell > 1 \) we obtain that

\[
x_\ell = x_0 \sum_{i=0}^{\ell-1} x_ix_0^{\ell-1-i} - p^t \sum_{i=0}^{\ell-2} x_ix_0^{\ell-2-i} - 27y_0^2x_0^{\ell-1}.
\]  

(19)

Claim 1: \( x_\ell \equiv 1 \pmod{3} \) and \( (x_\ell, p) = 1 \) for all \( \ell \in \mathbb{N}_0. \)

Proof of the claim: Clearly, \( x_0 \equiv 1 \pmod{3} \) since we choose \( x \) satisfying this property. From the first equation in (16) we have that \( x_\ell \equiv x_{\ell-1} \pmod{3} \) and hence the first statement follows by induction.

On the other hand, we have that \( (x_0, p) = 1 \) by hypothesis. Notice that \( x_1 = x^2 - 27y^2 \) and \( y_1 = 2xy. \) By taking into account that \( p^t = x^2 + 27y^2, \) we obtain that \( x_1 = 2x^2 - p^t \equiv 2x^2 \pmod{p}. \) Since \( p > 3 \) is prime and \( (x, p) = 1, \) we obtain that \( x_1 \not\equiv 0 \pmod{p} \) and thus \( (x_1, p) = 1. \) We now prove that \( (x_\ell, p) = 1 \) for any \( \ell \geq 2 \) by contradiction.

Suppose that the second statement of the claim is false, so there exists a minimum \( L > 1 \) such that \( p \mid x_L, \) that is \( x_L \equiv 0 \pmod{p}. \) By (19), we obtain that

\[
x_0 \sum_{i=0}^{L-1} x_ix_0^{L-1-i} - 27y_0^2x_0^{L-1} \equiv x_L \equiv 0 \pmod{p},
\]

and using that \( 27y_0^2 = p^t - x_0^2 \) we get

\[
x_0 \sum_{i=0}^{L-1} x_ix_0^{L-1-i} + x_0^{L+1} = x_0 \left( x_0^L + \sum_{i=0}^{L-1} x_ix_0^{L-1-i} \right) \equiv 0 \pmod{p}.
\]

(20)
Notice that
\[ \sum_{i=0}^{L-1} x_i x_0^{L-1-i} = x_{L-1} + \sum_{i=0}^{L-2} x_i x_0^{L-1-i}. \]

By applying (19) with \( \ell = L - 1 \) we arrive at
\[ x_{L-1} \equiv x_0 \sum_{i=0}^{L-2} x_i x_0^{L-2-i} - 27y_0^2 x_0^{L-2} \equiv x_0 x_0^{L-1} + x_0^2 \quad (\text{mod } p) \]
where we again used that \( 27y_0^2 = p^t - x_0^2 \). Thus, we have that
\[ 2x_{L-1} \equiv x_{L-1} + \sum_{i=0}^{L-2} x_i x_0^{L-1-i} + x_0^t \equiv x_0^t + \sum_{i=0}^{L-1} x_i x_0^{L-1-i} \equiv 0 \quad (\text{mod } p), \]
by (20). Hence \( x_{L-1} \equiv 0 \) (mod \( p \)) since \( (2, p) = 1 \), which contradicts the minimality of \( L \).
Therefore \( (x_0, p) = 1 \) for all \( \ell \in \mathbb{N}_0 \). This proves the claim. \( \diamond \)

Notice that by (15) and Claim 1, we have obtained a double sequence of integers \( \{(x_\ell, y_\ell)\}_{\ell \in \mathbb{N}_0} \) such that
\[ p^{4(\ell+1)} = x_\ell^2 + 27y_\ell^2 \quad \text{with} \quad x_\ell \equiv 1 \quad (\text{mod } 3) \quad \text{and} \quad (x_\ell, p) = 1. \]

We now seek for solutions of the equation \( 4p^{t\ell} = X^2 + 27Y^2 \). Since \( p > 3 \), by defining
\[ a_{\ell,0} = -2x_{\ell-1} \quad \text{and} \quad b_{\ell,0} = -2y_{\ell-1} \]
for \( \ell > 0 \) we get a sequence of integers \( \{(a_{\ell,0}, b_{\ell,0})\}_{\ell \in \mathbb{N}} \) satisfying
\[ 4p^{t\ell} = a_{\ell,0}^2 + 27b_{\ell,0}^2 \quad \text{with} \quad (a_{\ell,0}, p) = 1 \quad \text{and} \quad a_{\ell,0} \equiv 1 \quad (\text{mod } 3), \]
which are just the conditions (7) in Theorem 2.2. Therefore, if we put \( n_{\ell,0} = \frac{p^{3\ell} - 1}{3} \), then the spectrum of \( \Gamma(3, p^{3\ell}) \) is given by
\[ \text{Spec } \Gamma(3, p^{3\ell}) = \left\{ \left[ n_{\ell,0} \right], \left[ \frac{a_{\ell,0}p^{t\ell} - 1}{3} \right]^{n_{\ell,0}}, \left[ \frac{-1}{2}(a_{\ell,0} + 9b_{\ell,0})p^{t\ell} - 1 \right]^{n_{\ell,0}} \right\} \quad (21) \]

Moreover, the sequence \( \{(a_{\ell,0}, b_{\ell,0})\}_{\ell \in \mathbb{N}} \) satisfies the recursions
\[ a_{\ell+1,0} = x_0 a_{\ell,0} - 27y_0 b_{\ell,0} \quad \text{and} \quad b_{\ell+1,0} = x_0 b_{\ell,0} + y_0 a_{\ell,0}. \quad (22) \]

This implies that the spectrum of \( \Gamma(3, p^{3(\ell+1)}) \) can be determined by the spectrum of \( \Gamma(3, p^{3\ell}) \), recursively. Thus, the spectrum of \( \Gamma(3, p^{3\ell}) \) is determined by the spectrum of \( \Gamma(3, p^3) \) by induction, as desired.
Now assume that \( s \in \{1, \ldots, t-1\} \) (the case \( s = 0 \) was treated before), and let \( a_{0,s}, b_{0,s} \in \mathbb{Z} \) with \( a_{0,s} \equiv 1 \pmod{3} \) and \( (a_{0,s}, p) = 1 \) such that

\[
4p^s = a_{0,s}^2 + 27b_{0,s}^2 = \|z_{a_{0,s},b_{0,s}}\|^2
\]

with \( z_{a_{0,s},b_{0,s}} = a_{0,s} + 3\sqrt{3}ib_{0,s} \). Hence, we have that

\[
4p^\ell+s = \|z_{a_{0,s},b_{0,s}}\|^2\|z_{x_{\ell-1},y_{\ell-1}}\|^2 = \|z_{a_{0,s},b_{0,s}}z_{x_{\ell-1},y_{\ell-1}}\|^2 = \|z_{a_{\ell,s},b_{\ell,s}}\|^2
\]

where \( \{(a_{\ell,s}, b_{\ell,s})\}_{\ell \in \mathbb{N}_0} \) also satisfies the recursions

\[
a_{\ell,s} = a_{0,s}x_{\ell-1} - 27b_{0,s}y_{\ell-1} \quad \text{and} \quad b_{\ell,s} = a_{0,s}y_{\ell-1} + b_{0,s}x_{\ell-1},
\]

with \( x_{\ell}, y_{\ell} \) recursively defined as in (16).

**Claim 2:** \( a_{\ell,s} \equiv 1 \pmod{3} \) and \( (a_{\ell,s}, p) = 1 \) for all \( \ell \in \mathbb{N}_0 \).

**Proof of the claim:** For simplicity, here we put \( a_{\ell} \) and \( b_{\ell} \) instead of \( a_{\ell,s} \) and \( b_{\ell,s} \), respectively. Clearly \( a_{\ell} \equiv 1 \pmod{3} \). On the other hand, if \( (a_{\ell}, p) > 1 \) then we have that \( p \mid b_{\ell} \), so there are two cases: \( v_p(a_{\ell}) = v_p(b_{\ell}) \) or \( v_p(a_{\ell}) \neq v_p(b_{\ell}) \). Suppose first that \( v_p(a_{\ell}) \neq v_p(b_{\ell}) \) for some \( \ell \geq 1 \). Thus, since the numbers \( a_0, b_0, x_{\ell-1} \) are mutually coprime with \( p \) and \( p > 3 \), we have that \( v_p(a_{\ell}) = v_p(a_0a_{\ell}) \neq v_p(27b_0b_{\ell}) = v_p(b_{\ell}) \) and similarly \( v_p(a_{\ell}) = v_p(x_{\ell-1}a_{\ell}) \neq v_p(27y_{\ell-1}b_{\ell}) = v_p(b_{\ell}) \). In this way we get

\[
\min\{v_p(a_{\ell}), v_p(b_{\ell})\} = v_p(a_0a_{\ell} + 27b_0b_{\ell}) = v_p(4x_{\ell-1}p^s) = s,
\]

\[
\min\{v_p(a_{\ell}), v_p(b_{\ell})\} = v_p(x_{\ell-1}a_{\ell} + 27y_{\ell-1}b_{\ell}) = v_p(a_0p^{2\ell}) = \ell t.
\]

Thus \( s = \ell t \), which is absurd since \( s < t \). Hence, we must have that \( v_p(a_{\ell}) = v_p(b_{\ell}) \) for all \( \ell \geq 1 \).

On the other hand, from the recursions (16) and (23) we have that

\[
a_{\ell} = x_0a_{\ell-1} - 27y_0b_{\ell-1} \quad \text{and} \quad b_{\ell} = x_0b_{\ell-1} + y_0a_{\ell-1}.
\]

Combining both recursions we get

\[
a_{\ell+1} = x_0a_{\ell} - 27y_0(x_0b_{\ell-1} + y_0a_{\ell-1}) = x_0a_{\ell} - 27y_0^2a_{\ell-1} - 27x_0y_0b_{\ell-1},
\]

and, by using that \( p^t = x_0^2 + 27y_0^2 \), we arrive at

\[
a_{\ell+1} = x_0a_{\ell} - p^ta_{\ell-1} + x_0^2a_{\ell-1} - 27x_0y_0b_{\ell-1} = 2x_0a_{\ell} - p^ta_{\ell-1}.
\]

Now, recall that \( v_p(a_{\ell}) \leq s < t \) for all \( \ell \geq 0 \) and so \( v_p(2x_0a_{\ell}) \neq v_p(p^ta_{\ell-1}) \). Thus, we have that

\[
v_p(a_{\ell+1}) = \min\{v_p(a_{\ell}), t + v_p(a_{\ell-1})\} = v_p(a_{\ell})
\]

for all \( \ell \geq 1 \). Hence, it is enough to see that \( v_p(a_1) = 0 \). Let \( M = v_p(a_1) = v_p(b_1) \). First notice that \( M < s \). Indeed, if \( a_1 = p^s\hat{x} \) and \( b_1 = p^s\hat{y} \), thus

\[
p^{2s}(\hat{x}^2 + 27\hat{y}^2) = 4p^s
\]

which cannot happen since \( s > 1 \). On the other hand, by (24) we have that

\[
a_1^2 = x_0^2a_0^2 - 54x_0a_0y_0b_0 + 27^2y_0^2b_0^2 = (p^t - 27y_0^2)(4p^s - 27b_0^2) - 54x_0a_0y_0b_0 + 27^2y_0^2b_0^2
\]


and so we obtain

\[ a_1^2 = 4p^{t+s} - 4 \cdot 27p^s y_0^2 - 27p^t b_0^2 + 27^2 y_0^2 b_0^2 - 54x_0 a_0 y_0 b_0 + 27^2 y_0^2 b_0^2. \]

Thus, using that \( a_1 = a_0 x_0 - 27 b_0 y_0 \), we get

\[ a_1^2 = 4p^{t+s} - 4 \cdot 27p^s y_0^2 - 27p^t b_0^2 - 54y_0 b_0 a_1. \]

Since \( t + s, t, s, M \) are all different, in fact \( M < s < t < t + s \), and \( y_0, b_0 \) are both coprime with \( p \), we obtain that

\[ 2M = v_p(a_1^2) = \min\{t + s, t, s, M\} = M. \]

Hence \( M = 0 \), as desired. So, we obtain that \( (a_\ell, p) = 1 \) for all \( \ell \geq 1 \). This proves the claim.

Therefore, if we put \( n_{\ell,s} = \frac{p^{3(t+s)} - 1}{3} \) then, by Theorem 2.2, the spectrum of \( \Gamma(3, p^{3(t+s)}) \) is given by

\[
\text{Spec } \Gamma(3, p^{3(t+s)}) = \left\{ [n] \mid \left[ \frac{ap^{t+s} - 1}{3} \right]^n, \left[ \frac{-\frac{1}{2} (a + 9b) p^{t+s} - 1}{3} \right]^n, \left[ \frac{-\frac{1}{2} (a - 9b) p^{t+s} - 1}{3} \right]^n \right\}
\]

where we write \( n = n_{\ell,s}, a = a_{\ell,s} \) and \( b = b_{\ell,s} \) for simplicity.

In order to prove that the spectrum of \( \Gamma(3, p^{3(t+s)}) \) is determined by the spectra of \( \Gamma(3, p^{3\ell}) \) and \( \Gamma(3, p^{3t}) \), it is enough to put \( a_{\ell,s} \) and \( b_{\ell,s} \) in terms of \( a_{0,s}, b_{0,s} \) and \( x_0, y_0 \). Notice that the sequence of \( b_{\ell,s} \)'s satisfy the same recurrence as in (25), i.e.

\[ b_{\ell+1,s} = 2x_0 b_{\ell,s} - p^t b_{\ell-1,s}. \]

By solving this two terms linear recurrence and recalling that \( a_{1,s} = a_{0,s} x_0 - 27 b_{0,s} y_0 \) and \( b_{1,s} = a_{0,s} y_0 + x_0 b_{0,s} \), we obtain that

\[
a_{\ell,s} = \frac{1}{2} (a_{0,s} + 3\sqrt{3}b_{0,s}i) (x_0 + 3\sqrt{3}y_0i) \ell + \frac{1}{2} (a_{0,s} - 3\sqrt{3}b_{0,s}i) (x_0 - 3\sqrt{3}y_0i) \ell,
\]

\[
b_{\ell,s} = \frac{1}{2} (b_{0,s} - \frac{\sqrt{3}}{9} a_{0,s}i) (x_0 + 3\sqrt{3}y_0i) \ell + \frac{1}{2} (b_{0,s} + \frac{\sqrt{3}}{9} a_{0,s}i) (x_0 - 3\sqrt{3}y_0i) \ell.
\]

In this way, for every \( \ell \in \mathbb{N} \), \( a_{\ell,s} \) and \( b_{\ell,s} \) can be put in terms of \( a_{0,s}, b_{0,s} \) and \( x_0, y_0 \) only, as we wanted to show. \( \blacksquare \)

**Remark 3.2:** As in the case \( s > 0 \), from (22) for \( s = 0 \) we have that both \( \{a_{\ell,0}\} \) and \( \{b_{\ell,0}\} \) satisfy the recursion

\[ c_{\ell+2,0} = 2x_0 c_{\ell+1,0} - p^t c_{\ell,0} \]

for any \( \ell > 0 \). Finally, since \( a_{2,0} = a_{1,0} x_0 - 27 b_{1,0} y_0 \) and \( b_{2,0} = a_{1,0} y_0 + x_0 b_{1,0} \), we obtain that (27) also holds for \( s = 0 \) in the following way

\[
a_{0,0} = \frac{1}{2} (a_{1,0} + 3\sqrt{3}b_{1,0}i) (x_0 + 3\sqrt{3}y_0i)^{\ell-1} + \frac{1}{2} (a_{1,0} - 3\sqrt{3}b_{1,0}i) (x_0 - 3\sqrt{3}y_0i)^{\ell-1},
\]
\[
\begin{align*}
b_{\ell,0} &= \frac{1}{2}(b_{1,0} - \sqrt{3}a_{1,0}i)(x_0 + 3\sqrt{3}y_0)i^{\ell-1} + \frac{1}{2}(b_{1,0} + \sqrt{3}a_{1,0}i)(x_0 - 3\sqrt{3}y_0)i^{\ell-1}.
\end{align*}
\]
Moreover, by taking into account that \(a_{1,0} = -2x_0 \) and \(b_{1,0} = -2y_0 \) we obtain that
\[
\begin{align*}
a_{\ell,0} &= -(x_0 + 3\sqrt{3}y_0)i^{\ell} - (x_0 - 3\sqrt{3}y_0)i^{\ell}, \\
b_{\ell,0} &= -(y_0 - \frac{\sqrt{3}}{9}x_0)i(x_0 + 3\sqrt{3}y_0)i^{\ell-1} - \left(y_0 + \frac{\sqrt{3}}{9}x_0i\right)(x_0 - 3\sqrt{3}y_0)i^{\ell-1},
\end{align*}
\]
where \(x_0 \) and \(y_0 \) are the solutions of \(p^t = X^2 + 27Y^2 \) with \((x_0, p) = 1 \) and \(x_0 \equiv 1 \mod 3 \).

Notice that (27) can be written in the notation of (14) as
\[
\begin{align*}
a_{\ell,s} &= \frac{1}{2}(z_{a,b}z_{x,y}^s + \bar{z}_{a,b}\bar{z}_{x,y}^s) = Re(z_{a,b}z_{x,y}^s), \\
b_{\ell,s} &= \frac{1}{2}(z_{b,\bar{a}}\bar{z}_{x,y}^s + \bar{z}_{b,\bar{a}}\bar{z}_{x,y}^s) = Re(z_{b,\bar{a}}\bar{z}_{x,y}^s),
\end{align*}
\]
where \((a, b, x, y) = (a_{0,s}, b_{0,s}, x_0, y_0) \) and \(\bar{a} = \frac{1}{27}a \). Similarly, expressions (28) can be written as the real part of a complex number.

Theorem 2.2 provides the spectrum of \(\Gamma(3, q) \) explicitly, with \(q = p^3r \) and \(p \) a prime of the form \(3h + 1 \), in terms of an integer solution \((a, b) \) of the equation \(4q = X^2 + 27Y^2 \) with \(a \equiv 1 \mod 3 \) and \((a, p) = 1 \). If one wants the spectrum of \(\Gamma(3, q^\ell) \), one needs to obtain an integer solution \((a_t, b_t) \) of the equation \(4q^t = X^2 + 27Y^2 \) with \(a_t \equiv 1 \mod 3 \) and \((a_t, p) = 1 \). This may be tedious and operationally costly. However, Theorem 3.1 ensures that we can obtain all the spectra of the GP-graphs \(\Gamma(3, q^\ell) \) from a unique base solution of the first equation \(q^t = X^2 + 27Y^2 \). Moreover, the spectrum of \(\Gamma(3, p^t) \) is given explicitly in the previous proof by expressions (26) and (27).

**Example 3.3:** Let \(p = 7 \). We look for the minimal \(t \) such that \(7^t = X^2 + 27Y^2 \) has an integer solution \((x, y) \) with \(x \equiv 1 \mod 3 \) and \((x, 7) = 1 \). In this case \(t = 3 \) with solution \((x_0, y_0) = (10, 3) \), since \(7^3 = 10^2 + 27 \cdot 3^2 \) (for \(t = 1 \) there are no integer solutions and for \(t = 2 \) we have the trivial solution \((7, 0) \) but it is not of the required form). Consider \(s = 1 \). By (26), the spectrum of \(\Gamma(3, 7^{3(\ell+3)}) \) for any \(\ell \in \mathbb{N} \) is given by
\[
\text{Spec } \Gamma(3, 7^{3(\ell+3)}) = \left\{ [n_\ell]^{\text{Spec } \Gamma(3, 7^{3(\ell+3)})} \right\},
\]
where \(n_\ell = \frac{7^{\ell+3}-1}{3} \) and the numbers \(a_\ell \) and \(b_\ell \) are, by (27), as follows
\[
\begin{align*}
a_\ell &= \frac{1}{2}(1 - 3\sqrt{3}i)(10 - 9\sqrt{3}i)i\ell + \frac{1}{2}(1 + 3\sqrt{3}i)(10 + 9\sqrt{3}i)\ell, \\
b_\ell &= \frac{1}{2}(1 + \frac{\sqrt{3}i}{9})(10 - 9\sqrt{3}i)\ell + \frac{1}{2}(1 - \frac{\sqrt{3}i}{9})(10 + 9\sqrt{3}i)\ell,
\end{align*}
\]
where \(a_0 = 1 \) and \(b_0 = 1 \) since \(4 \cdot 7 = 1^2 + 27 \cdot 1^2 \).

In Table 1 we give the spectrum of \(\Gamma(3, 7^{3(\ell+3)}) \) for the first five values of \(\ell \). For simplicity we only list the non-principal eigenvalues \(\{\lambda_1, \lambda_2, \lambda_3\} \) without the multiplicities and separately the principal eigenvalues.
The principal eigenvalues are given by \( n_0 = 114, \ n_1 = 4613762400, \ \ n_2 = 186181954694428002, \ \ n_3 = 751311343023075269595416 \) and \( n_4 = 303181226709953713606735006629714. \) Note that we do not have to solve \( 4 \cdot 7^{3\ell+1} = X^2 + 27Y^2 \) for each value of \( \ell \geq 1. \)

Recall that an integer \( a \) is a cubic residue modulo a prime \( p \) if \( a \equiv x^3 \pmod{p} \) for some integer \( x. \) By Euler’s criterion, \( a \) is a cubic residue mod \( p, \) with \( (a, p) = 1, \) if and only if

\[
a^{p-1} \equiv 1 \pmod{p}
\]

where \( d = (3, p - 1). \) We have the following direct consequence of Theorem 3.1.

**Theorem 3.4:** Let \( p \) be a prime with \( p \equiv 1 \pmod{3}. \) If \( 2 \) is a cubic residue modulo \( p, \) then the spectrum of \( \Gamma(3, p^3) \) determines the spectrum of \( \Gamma(3, p^{3\ell}) \) for every \( \ell \in \mathbb{N}. \) In this case, the spectrum of \( \Gamma(3, p^{3\ell}) \) is given by

\[
\text{Spec} \ \Gamma(3, p^{3\ell}) = \left\{ [n_\ell]^1, \left\lfloor \frac{a_\ell p^\ell - 1}{3} \right\rfloor, \left\lfloor \frac{-\frac{1}{2}(a_\ell + 9b_\ell)p^\ell - 1}{3} \right\rfloor, \left\lfloor \frac{-\frac{1}{2}(a_\ell - 9b_\ell)p^\ell - 1}{3} \right\rfloor \right\}.
\]

with \( n_\ell = \frac{p^{3\ell} - 1}{3} \) and where \( a_\ell, b_\ell \) are the numbers \( a_{\ell,0}, b_{\ell,0} \) given in (28)

\[
a_{\ell,0} = -(x_0 + 3\sqrt{3}y_0i)^\ell - (x_0 - 3\sqrt{3}y_0i)^\ell,
\]

\[
b_{\ell,0} = -(y_0 - \sqrt{3}y_0i)(x_0 + 3\sqrt{3}y_0i)^{\ell-1} - (y_0 + \sqrt{3}y_0i)(x_0 - 3\sqrt{3}y_0i)^{\ell-1},
\]

where \( x_0 \) and \( y_0 \) are the solutions of \( p = X^2 + 27Y^2 \) with \( (x_0, p) = 1 \) and \( x_0 \equiv 1 \pmod{3}. \)

**Proof:** A classic result in number theory, conjectured by Euler and first proved by Gauss using cubic reciprocity, asserts that (see for instance [33])

\[
p = x^2 + 27y^2 \quad \text{for some } x, y \in \mathbb{Z} \quad \iff \quad \begin{cases} p \equiv 1 \pmod{3} \quad \text{and}, \\ 2 \text{ is a cubic residue modulo } p. \end{cases}
\]

By hypothesis we have that \( p \equiv 1 \pmod{3} \) and \( 2 \) is a cubic residue modulo \( p, \) so there exist \( x, y \in \mathbb{Z} \) such that \( p = x^2 + 27y^2. \) Moreover, since either \( x \) or \( -x \) is congruent to \( 1 \pmod{p}, \) we choose the solution \((z, y), \) where \( z \in \{\pm 1\} \) with \( z \equiv 1 \pmod{3}. \) Thus, the assertion follows directly from Theorem 3.1 with \( t = 1 \) and \( s = 0. \)
Theorem 3.6: If $p$ is a prime with $p \equiv 1 \pmod{4}$, then the spectrum of $\Gamma(4, p^{4\ell})$ is determined by the spectrum of $\Gamma(4, p^{3\ell})$ for every $\ell \in \mathbb{N}$. Moreover, the spectrum of $\Gamma(4, p^{4\ell})$ is given by

$$
\text{Spec}(\Gamma(4, p^{4\ell})) = \left\{ [n\ell]^1, \left[ \frac{p^{2\ell} + 4d_\ell p^\ell - 1}{4} \right]^{n\ell}, \left[ \frac{p^{2\ell} - 4d_\ell p^\ell - 1}{4} \right]^{n\ell} \right\},
$$

where $d_\ell$ is the number of distinct $d$-tuples that sum to $\ell$. Note that by (30), if $p \equiv 1 \pmod{3}$ as in the theorem, 2 is a cubic residue modulo $p$ if and only if $2^{p-1}/9 \equiv 1 \pmod{p}$. That is, if $p = 3k + 1$ then $2^k \equiv 1 \pmod{3k + 1}$. Thus, the first primes $p$ of the form $3k + 1$ for which 2 is a cubic residue are 31 and 43 since $2^{10} \equiv 1 \pmod{31}$ and $2^{14} \equiv 1 \pmod{43}$. In fact, $4^3 \equiv 1 \pmod{31}$ and $20^3 \equiv 1 \pmod{43}$.

Example 3.5: Let $p = 31$. We know that 2 is a cubic residue modulo 31 and in this case we have $31 = 2^2 + 27 \cdot 1^2$. We take the solutions $x_0 = -2$ and $y_0 = 1$ of $31 = X^2 + 37Y^2$. By Theorem 3.4 and (28) in Remark 3.2 for every $\ell > 0$ we have that the spectrum of $\Gamma(4, 31^3)$ determines the spectrum of $\Gamma(4, 31^{3\ell})$ for every $\ell$ and, from (31), it is given by

$$
\text{Spec} \Gamma(3, 31^{3\ell}) = \left\{ [n\ell]^1, \left[ \frac{a_{31\ell} - 1}{3} \right]^{n\ell}, \left[ -\frac{1}{2} (a_{31\ell} + 9b_{31\ell}) 31^{3\ell} - 1 \right]^{n\ell} \right\},
$$

with $n\ell = \frac{31^{3\ell} - 1}{3}$ and where $a_{\ell}, b_{\ell}$ are the numbers $a_{\ell,0}, b_{\ell,0}$ given in (28), that is

$$
a_{\ell} = -(-2 + 3\sqrt{3}i)^\ell - (-2 - 3\sqrt{3}i)^\ell,
$$

$$
b_{\ell} = - \left( 1 + \frac{2\sqrt{3}}{9}i \right) (-2 + 3\sqrt{3}i)^{\ell-1} - \left( 1 - \frac{2\sqrt{3}}{9}i \right) (-2 - 3\sqrt{3}i)^{\ell-1}.
$$

In Table 2 we give the spectrum of $\Gamma(3, 31^\ell)$ for the first five values of $\ell$ (we follow the same notation as in Example 3.3) where the principal eigenvalues are $n_0 = 114$, $n_1 = 4613762400$, $n_2 = 186181954694428002$, $n_3 = 7513113430422032756295954416$ and $n_4 = 30318122670995371360673506629714$.

The following result for GP-graphs $\Gamma(4, q)$ is in the same vein of Theorem 3.1 for GP-graphs $\Gamma(3, q)$. However, these results are somehow different, since in this case we do not need the assumption of an initial solution of an equation (as in Theorem 3.1), although it covers different cases.

Table 2. First values of $a_{\ell}$ and $b_{\ell}$ for $\Gamma(3, 31^{3\ell})$.

| $\ell$ | $a_{\ell}$ | $b_{\ell}$ | $q$ |
|--------|------------|------------|-----|
| 1      | 4          | -2         | 31  |
| 2      | 46         | 8          | 31^2 |
| 3      | -308       | 30         | 31^3 |
| 4      | -194       | -368       | 31^4 |
| 5      | 10324      | 542        | 31^5 |

Note that by (30), if $p \equiv 1 \pmod{3}$ as in the theorem, 2 is a cubic residue modulo $p$ if and only if $2^{p-1}/9 \equiv 1 \pmod{p}$. That is, if $p = 3k + 1$ then $2^k \equiv 1 \pmod{3k + 1}$. Thus, the first primes $p$ of the form $3k + 1$ for which 2 is a cubic residue are 31 and 43 since $2^{10} \equiv 1 \pmod{31}$ and $2^{14} \equiv 1 \pmod{43}$. In fact, $4^3 \equiv 1 \pmod{31}$ and $20^3 \equiv 1 \pmod{43}$. Moreover, the spectrum of $\Gamma(4, p^{3\ell})$ is given by

$$
\text{Spec}(\Gamma(4, p^{3\ell})) = \left\{ [n\ell]^1, \left[ \frac{p^{2\ell} + 4d_\ell p^\ell - 1}{4} \right]^{n\ell}, \left[ \frac{p^{2\ell} - 4d_\ell p^\ell - 1}{4} \right]^{n\ell} \right\},
$$

where $d_\ell$ is the number of distinct $d$-tuples that sum to $\ell$. Note that by (30), if $p \equiv 1 \pmod{3}$ as in the theorem, 2 is a cubic residue modulo $p$ if and only if $2^{p-1}/9 \equiv 1 \pmod{p}$. That is, if $p = 3k + 1$ then $2^k \equiv 1 \pmod{3k + 1}$. Thus, the first primes $p$ of the form $3k + 1$ for which 2 is a cubic residue are 31 and 43 since $2^{10} \equiv 1 \pmod{31}$ and $2^{14} \equiv 1 \pmod{43}$. In fact, $4^3 \equiv 1 \pmod{31}$ and $20^3 \equiv 1 \pmod{43}$.
It is well known that the equation
\[ p^2 = X^2 + 4Y^2 \]
with \( p \equiv 1 \) (mod 4) always has a solution \((x, y)\) satisfying \((x, p) = 1\) (see Remark 2.5). Let \( c_1, d_1 \) be the solution of the above equation with \( c_1 \equiv 1 \) (mod 4) and \((c_1, p) = 1\). Notice that if we take \( z_{x,y} = x + 2iy \), then \( p^2 = \|z_{c_1,d_1}\|^2 \), so we have that
\[ p^{2\ell} = \|z_{c_1,d_1}\|^2. \]
As in the proof of Theorem 3.1, we can put \( z_{c_1,d_1}^\ell =: z_{c_\ell,d_\ell} \), where \( c_\ell, d_\ell \) are defined recursively as follows
\[ c_{\ell+1} = c_1 c_\ell - 4d_1 d_\ell \quad \text{and} \quad d_{\ell+1} = c_1 d_\ell + d_1 c_\ell. \]
Both sequences \( \{c_\ell\}_{\ell \in \mathbb{N}_0} \) and \( \{d_\ell\}_{\ell \in \mathbb{N}_0} \) also satisfy the recursion
\[ r_{\ell+1} = 2c_1 r_\ell - p^2 r_{\ell-1}. \]
Proceeding similarly as in the proof of Theorem 3.1, we can show that
\[ d_{\ell+1} = d_1 \left( \sum_{i=1}^{\ell} c_1 c_1^{\ell-i} + c_1^\ell \right) \quad \text{and} \]
\[ c_\ell = \frac{1}{d_1} (d_{\ell+1} - c_1 d_\ell) = \sum_{i=1}^{\ell} c_1 c_1^{\ell-i} - c_1 \sum_{i=1}^{\ell-1} c_1 c_1^{\ell-1-i} \]
so we have that
\[ c_{\ell+1} = c_1 \sum_{i=1}^{\ell} c_1 c_1^{\ell-i} - p^2 \sum_{i=1}^{\ell-1} c_1 c_1^{\ell-1-i} - 4d_1^2 c_1 c_1^{\ell-1}. \]
As in proof of Theorem 3.1, we can show that \((c_\ell,p) = 1\) and \( c_\ell \equiv 1 \) (mod 4). Hence, by Theorem 2.4 we have that the spectrum of \( \Gamma(4,p^{4\ell}) \) is given as in the statement.

In order to prove that the spectrum of \( \Gamma(4,p^{4\ell}) \) is determined by the spectrum of \( \Gamma(4,p^4) \), it is enough to put every \( c_\ell \) and \( d_\ell \) in terms of \( c_1 \) and \( d_1 \) only. By solving the linear recurrence (35) and by recalling that \( c_2 = c_1^2 - 4d_1^2 \) and \( d_2 = 2c_1d_1 \), we obtain that \( c_\ell \) and \( d_\ell \) are as given in (33). Therefore, the spectrum of \( \Gamma(4,p^{4\ell}) \) is determined by the spectrum of \( \Gamma(4,p^4) \), as desired. ■
Example 3.7: Let \( p = 5 \). Since \( 5^2 = 3^2 + 4 \cdot 2^2 \), we take \( c_1 = -3 \) and \( d_1 = 2 \). The spectrum of \( \Gamma(4, 5^4\ell) \) is given for any \( \ell \in \mathbb{N} \) by

\[
\text{Spec}(\Gamma(4, 5^4\ell)) = \left\{ \left[ \frac{52\ell + 44d_15\ell - 1}{4} \right]^{n\ell}, \left[ \frac{52\ell - 4d_1p\ell - 1}{4} \right]^{n\ell}, \left[ -\frac{52\ell + 2c_15\ell - 1}{4} \right]^{n\ell}, \left[ -\frac{52\ell - 2c_15\ell - 1}{4} \right]^{n\ell}\right\}
\]

with \( n\ell = \frac{5\ell - 1}{4} \) and where, by (33),

\[
c_\ell = \frac{1}{2}(-3 + 4\ell) + \frac{1}{2}(-3 - 4\ell) \quad \text{and} \quad d_\ell = -\frac{1}{4}(-3 + 4\ell) + \frac{1}{4}(-3 - 4\ell).
\]

In Table 3 we give the spectrum of \( \Gamma(4, 5^4\ell) \) for the first five values of \( \ell \) (we follow the same notation as in Example 3.3)

| \( \ell \) | \( c_\ell \) | \( d_\ell \) | Non-principal eigenvalues of \( \Gamma(4, 5^4\ell) \) |
|---|---|---|---|
| 1 | -3 | 2 | \{16, -4, -14, 1\} |
| 2 | -7 | -12 | \{-144, 456, -244, -69\} |
| 3 | 117 | 22 | \{6656, 1156, 3406, -11219\} |
| 4 | -527 | 168 | \{202656, -7344, -262344, 67031\} |
| 5 | 237 | -1558 | \{-2427344, 7310156, -2071094, -2811719\} |

where \( n_1 = 156, n_2 = 97656, n_3 = 61035156, n_4 = 38146972656 \) and \( n_5 = 23841857910156 \) are the principal eigenvalues.

4. Energy

In this section we first study the energy of the graphs \( \Gamma = \Gamma^*(k, q) \) for \( k = 3, 4 \) where \( q = p^r \) with \( p \) prime. Then, we give conditions on \( \Gamma \) ensuring that \( \Gamma \) and \( \tilde{\Gamma} \) are complementary equienergetic (i.e. they have the same energy), without computing the energies of \( \Gamma \) and \( \tilde{\Gamma} \). It turns out that we will only need to know the sign of the eigenvalues of \( \Gamma \).

We begin by studying the energies of \( \Gamma^*(3, q) \) and \( \Gamma^*(4, q) \). We will give the exact values in the semiprimitive case \( p \equiv -1 \mod k \) with \( k = 3, 4 \). However, in the non-semiprimitive case \( p \equiv 1 \mod k \) we can only give bounds.

Proposition 4.1: For the energies of \( \Gamma^*(3, q) \) and \( \Gamma^*(4, q) \) we have the following:

(a) If \( p \equiv 1 \mod k \) with \( k = 3 \) or \( k = 4 \) then

\[
n\left(1 + \frac{1}{3}|2a\sqrt[3]{q} + 1|\right) \leq E(\Gamma^*(3, q)) \leq n\left(1 + \frac{2}{3}(|a|\sqrt[3]{q} + 1) + 3|b|\right),
\]

\[
n(\sqrt[4]{q} + 1) \leq E(\Gamma^*(4, q)) \leq n(\sqrt[4]{q} + 1 + (|c| + 2|d|)\sqrt[4]{q}).
\]

(b) If \( p \equiv -1 \mod k \) then we have:
(a) (i) If \( p \equiv 2 \pmod{3} \) then

\[
E(\Gamma^*(3, q)) = \begin{cases} 
2n\sqrt{q} + 1 & \text{if } m \equiv 0 \pmod{4}, \\
3 & \text{if } m \equiv 2 \pmod{4}.
\end{cases}
\]

(b) (ii) If \( p \equiv 3 \pmod{4} \) then

\[
E(\Gamma^*(4, q)) = \begin{cases} 
3n\sqrt{q} + 1 & \text{if } m \equiv 0 \pmod{4}, \\
2 & \text{if } m \equiv 2 \pmod{4}.
\end{cases}
\]

**Proof:** It is clear from (iii) in Remark 2.10 that the graphs \( \Gamma(k, q) \) and \( \Gamma^+(k, q) \) are equienergetic for \( k = 3, 4 \). So, it is enough to compute the energy for the graphs \( \Gamma(3, q) \) and \( \Gamma(4, q) \). The result follows by the definition of energy in (1) and by applying Theorems 2.2 and 2.4. The equalities in parts (i) and (ii) in (b) of the statement are straightforward consequences of part (b) of the aforementioned theorems. The bounds in part (a) of the statement are deduced from part (a) of these theorems, by applying the triangle inequality of real numbers. \( \blacksquare \)

### 4.1. Equienergy

We now study when the graphs \( \Gamma(3, q) \) and \( \Gamma(4, q) \) are equienergetic to their corresponding complements \( \tilde{\Gamma}(3, q) \) and \( \tilde{\Gamma}(4, q) \). In the case that \( \Gamma(k, q) \) is semiprimitive with \( k = 3, 4 \), the answer is already known. In fact, we have that \( q = p^2 \) with \( p \) prime satisfying \( p \equiv -1 \pmod{k} \) and, from Proposition 6.6 in [10], we obtain that \( \{\Gamma(3, p^2), \tilde{\Gamma}(3, p^2)\} \) and \( \{\Gamma(4, p^2), \tilde{\Gamma}(4, p^2)\} \) are pairs of equienergetic non-isospectral complementary graphs if and only if \( t \) is odd. That is,

\[
\{\Gamma(3, p^{4s+2}), \tilde{\Gamma}(3, p^{4s+2})\} \quad \text{and} \quad \{\Gamma(4, p^{4s+2}), \tilde{\Gamma}(4, p^{4s+2})\}
\]

are pairs of equienergetic non-isospectral complementary graphs for any \( s \in \mathbb{N} \).

The following result gives a condition for \( \Gamma(k, q) \) such that \( \Gamma(k, q) \) and \( \tilde{\Gamma}(k, q) \) are equienergetic graphs in the non-semiprimitive case. We recall that the principal eigenvalue of a regular graph is its degree of regularity.

**Theorem 4.2:** Let \( q = p^m \) with \( p \) prime, \( m \in \mathbb{N} \) and \( k = 3, 4 \) such that \( k \mid \frac{q-1}{p-1} \) and \( (k, q) \) is not a semiprimitive pair. Then, the graphs \( \Gamma(k, q) \) and \( \tilde{\Gamma}(k, q) \) are equienergetic if and only if among the non-principal eigenvalues of \( \Gamma(k, q) \) exactly one is positive. In this case, the graphs \( \{\Gamma(k, q), \Gamma^+(k, q), \tilde{\Gamma}(k, q)\} \) are mutually equienergetic and non-isospectral.

**Proof:** We begin by showing that \( \Gamma(k, q) \) and \( \tilde{\Gamma}(k, q) \) are equienergetic graphs if and only if \( \Gamma(k, q) \) has only one positive non-principal eigenvalue for \( k = 3, 4 \).
Let us first consider $k = 3$. Since $(3, q)$ is not semiprimitive, then $p \equiv 1 \pmod{3}$ and $m = 3t$ for some $t \in \mathbb{N}$. Moreover, by $(a)$ in Theorem 2.2, the spectrum of $\Gamma(3, q)$ is

$$\text{Spec } \Gamma(3, q) = \left\{ \left[ n \right]^{1}, \left[ \frac{a\sqrt{q} - 1}{3} \right]^{n}, \left[ \frac{-\frac{1}{2}(a + 9b)\sqrt{q} - 1}{3} \right]^{n}, \left[ \frac{-\frac{1}{2}(a - 9b)\sqrt{q} - 1}{3} \right]^{n} \right\}$$

where $n = \frac{q - 1}{3}$ and the integers $a, b$ satisfy $4\sqrt{q} = a^2 + 27b^2$ with $a \equiv 1 \pmod{3}$ and $(a, p) = 1$. Hence, if we denote by $A_3 = \{ \lambda \in \text{Spec}(\Gamma) : \lambda \neq n \}$ the set of non-principal eigenvalues of $\Gamma(3, q)$, then its energy can be written as

$$E(\Gamma(3, q)) = n \left( 1 + \sum_{\lambda \in A_3} |\lambda| \right). \quad (37)$$

Now, recall that the eigenvalues of $\bar{\Gamma}(3, q)$ are $q - 1 - n = (k - 1)n = 2n$ and $-1 - \lambda$ for $\lambda$ a non-principal eigenvalue of $\Gamma(3, q)$. Hence, we have

$$E(\bar{\Gamma}(3, q)) = 2n + n \sum_{\lambda \in A_3} |\lambda + 1|.$$

Since every element in $A_3$ is a non-zero integer we have

$$|\lambda + 1| = |\lambda| + \text{sign}(\lambda)$$

for all $\lambda \in A_3$, where $\text{sign}(t)$ denotes the sign of an integer $t$ (i.e. $\text{sign}(t) = 1$ if $t > 0$ and $\text{sign}(t) = -1$ if $t < 0$). Thus, the energy of $\bar{\Gamma}(3, q)$ takes the following form

$$E(\bar{\Gamma}(3, q)) = n \left( 2 + \sum_{\lambda \in A_3} |\lambda| + \sum_{\lambda \in A_3} \text{sign}(\lambda) \right). \quad (38)$$

Therefore, by (37) and (38), we have that $E(\Gamma(3, q)) = E(\bar{\Gamma}(3, q))$ if and only if

$$\sum_{\lambda \in A_3} \text{sign}(\lambda) = -1.$$

This can only happen if $\Gamma(3, q)$ has only one non-principal positive eigenvalue and the other two non-principal eigenvalues are negative. In fact, $\# A_3 = 3$ since $b$ in (7) cannot be 0 (if $b = 0$ in (7), then $(a, p) > 1$, which is absurd).

The case $k = 4$ is similar. If we denote by $A_4$ the set of non-principal eigenvalues of $\Gamma(4, q)$, proceeding as before (we omit the details) we get that $E(\Gamma(4, q)) = E(\bar{\Gamma}(4, q))$ if and only if

$$\sum_{\lambda \in A_4} \text{sign}(\lambda) = -2.$$

This can only happen if among the non-principal eigenvalues there is exactly one which is positive. This is because $\# A_4 = 4$ since $d \neq 0$ in (12).

Now, assume we are in the case that the graphs $\Gamma(k, q)$ and $\bar{\Gamma}(k, q)$ are equienergetic. They are also non-isospectral since they have different degrees of regularity. Notice that
Corollary 4.3: Let \( q = p^m \) with \( p \) prime, \( m \in \mathbb{N} \), and let \( k = 3, 4 \) such that \( k \mid \frac{q-1}{p-1} \) and \( (k, q) \) is not a semiprimitive pair. Suppose that \( a, b \) and \( c, d \) are pairs of integers satisfying conditions (7) and (12), respectively.

(i) If \( a > 9|b| \) or else if \( a < 0 \) and \( -a < 9|b| \) then \( \{\Gamma(3, q), \Gamma^+(3, q), \Gamma(3, q)\} \) are equienergetic and non-isospectral.

(ii) If \( 2|c| < \sqrt[3]{q} < 4|d| \) then the graphs \( \{\Gamma(4, q), \Gamma^+(4, q), \Gamma(4, q)\} \) are equienergetic and non-isospectral. In particular, \( \{\Gamma(4, q), \Gamma^+(4, q), \Gamma(4, q)\} \) are equienergetic and non-isospectral when \( |c| < \frac{2\sqrt[3]{|d|}}{3} \).

Proof: By Theorem 4.2, it is enough to see that among the non-principal eigenvalues of \( \Gamma(k, q), k = 3, 4 \), exactly one is positive and the other ones are all negative.

(i) By Theorem 2.2, the spectra of \( \Gamma(3, q) \) is given by

\[
\text{Spec}(\Gamma(3, q)) = \left\{ [\mathbb{N}]^1, \left[ \frac{a \sqrt[3]{q} - 1}{3} \right]^n, \left[ -\frac{1}{2} (a + 9b) \sqrt[3]{q} - 1 \right]^n, \left[ -\frac{1}{2} (a - 9b) \sqrt[3]{q} - 1 \right]^n \right\}
\]

with \( a, b \) satisfying (7). Now, if \( a > 9|b| > 0 \) then \( a - 9b > 0 \) and \( a + 9b > 0 \), this implies that

\[
a \sqrt[3]{q} - 1 > 0, \quad \frac{-\frac{1}{2} (a + 9b) \sqrt[3]{q} - 1}{3} < 0, \quad \text{and} \quad \frac{-\frac{1}{2} (a - 9b) \sqrt[3]{q} - 1}{3} < 0.
\]

On the other hand, if \( a < 0 \) and \( -a < 9|b| \) we have that either

\[
a \sqrt[3]{q} - 1 < 0, \quad \frac{-\frac{1}{2} (a + 9b) \sqrt[3]{q} - 1}{3} < 0, \quad \text{and} \quad \frac{-\frac{1}{2} (a - 9b) \sqrt[3]{q} - 1}{3} > 0
\]

for \( b > 0 \) or else

\[
a \sqrt[3]{q} - 1 < 0, \quad \frac{-\frac{1}{2} (a + 9b) \sqrt[3]{q} - 1}{3} > 0, \quad \text{and} \quad \frac{-\frac{1}{2} (a - 9b) \sqrt[3]{q} - 1}{3} < 0
\]

for \( b < 0 \). Hence, the conditions given by (i) assure that exactly one of the non-principal eigenvalues is positive.

(ii) By Theorem 2.4, the spectra of \( \Gamma(4, q) \) is given by

\[
\text{Spec}(\Gamma(4, q)) = \left\{ [\mathbb{N}]^1, \left[ \sqrt[4]{q} + \frac{4d \sqrt[4]{q} - 1}{4} \right]^n, \left[ \sqrt[4]{q} - \frac{4d \sqrt[4]{q} - 1}{4} \right]^n, \right\}
\]

\[
+ \left\{ \left[ -\frac{\sqrt[4]{q} + 2c \sqrt[4]{q} - 1}{4} \right]^n, \left[ -\frac{\sqrt[4]{q} - 2c \sqrt[4]{q} - 1}{4} \right]^n \right\}
\]

with \( c, d \) satisfying conditions (12). In a similar way as in (i), one can check that the condition \( 2|c| < \sqrt[3]{q} < 4|d| \) assures that one of the eigenvalues of \( \Gamma(4, q) \) is positive and the other ones are negative. Therefore, we have that \( \{\Gamma(4, q), \Gamma^+(4, q), \Gamma(4, q)\} \)
are equienergetic and non-isospectral. For the last sentence, notice that the conditions $2|c| < \sqrt{q}$ and $\sqrt{q} < 4|d|$ are equivalent to $3c^2 < 4d^2$ and $c^2 < 12d^2$, respectively, since $\sqrt{q} = c^2 + 4d^2$. Thus, if $|c| < \frac{\sqrt{2}}{\sqrt{3}}|d|$ then $2|c| < \sqrt{q}$ and $\sqrt{q} < 4|d|$. Therefore, we obtain that \{\(\bar{\Gamma}(4, q), \Gamma^+(4, q), \Gamma(4, q)\)\} are equienergetic and non-isospectral in this case, as asserted.

We now illustrate the previous proposition and corollary.

**Example 4.4:** Here we give complementary equienergetic pairs of graphs \(\Gamma(k, q)\) with \(k = 3, 4\).

(i) Let \(p = 7\) and \(m = 6\). Hence, \(q = 7^6 = 117,649, p \equiv 1 \pmod{3}\) and the pair \((3, 7^6)\) is not semiprimitive. The integers \(a = 13\) and \(b = 1\) satisfy (7) in Theorem 2.2, since \(4\sqrt{q} = 13^2 + 27 \cdot 1^2 = 196\). Thus, the integers \(a, b\) satisfy the hypothesis of Corollary 4.3 and hence we known that \(\bar{\Gamma}(3, 7^6)\) and \(\bar{\Gamma}(3, 7^6)\) are equienergetic and non-isospectral, without the need to compute the spectrum. The spectrum of \(\Gamma(3, 7^6)\) is given by

\[
\text{Spec}(\Gamma(3, 7^6)) = \{[n]^1, [212]^n, [-180]^n, [-33]^n\}
\]

where \(n = \frac{q-1}{3} = 39216\). Therefore, we see that there is only one positive non-principal eigenvalue and Theorem 4.2 also shows that \(\Gamma(3, 7^6)\) and \(\bar{\Gamma}(3, 7^6)\) are equienergetic and non-isospectral.

(ii) Let \(p = 5, m = 8\) and \(q = 5^8 = 390625\). Thus \(p \equiv 1 \pmod{4}\) and \((4, 5^8)\) is not a semiprimitive pair of integers. The integers \(c = -7\) and \(d = 12\) satisfy (12) in Theorem 2.4, since \(625 = \sqrt{q} = 7^2 + 4 \cdot 12^2\), and hence we have

\[
\text{Spec}(\Gamma(4, 5^8)) = \{[n]^1, [456]^n, [-69]^n, [-144]^n, [-244]^n\}
\]

where \(n = \frac{q-1}{4} = 97656\). Therefore, the spectra of \(\Gamma(3, 7^6)\) has only one positive non-principal eigenvalue and by Theorem 4.2 we have that \(\Gamma(4, 5^8)\) and \(\bar{\Gamma}(4, 5^8)\) are equienergetic and non-isospectral. This also follows more easily using Corollary 4.3 directly, since \(c\) and \(d\) satisfy \(|c| < 2\sqrt{\frac{27}{3}}|d|\).

\(\checkmark\)

**Example 4.5:** Now, we give a graph \(\Gamma(3, q)\) which it not complementary equienergetic. Let \(p = 7\) and \(m = 3\), hence \(q = 7^3 = 343\). Since \(p \equiv 1 \pmod{3}\), we have to find integers \(a, b\) such that \(28 = 4\sqrt{q} = a^2 + 27b^2\), \(a \equiv 1 \pmod{3}\) and \((a, 7) = 1\). Clearly \(a = b = 1\) satisfy these conditions. By (a) in Theorem 2.2 we have \(\text{Spec}(\Gamma(3, 7^3)) = \{[114]^1, [9]^{114}, [2]^{114}, [-12]^{114}\}\) and thus we get \(\text{Spec}(\bar{\Gamma}(3, 7^3)) = \{[2 \cdot 114]^1, [-10]^{114}, [-3]^{114}, [11]^{114}\}\). Hence,

\[
E(\Gamma(3, 7^3)) = 114 (1 + 9 + 2 + 12) \neq 114 (2 + 10 + 3 + 11) = E(\bar{\Gamma}(3, 7^3)).
\]

Therefore, \(E(\Gamma(3, 7^3)) \neq E(\bar{\Gamma}(3, 7^3))\). Notice that the graph \(\Gamma(3, 7^3)\) has more that one positive non-prinicipal eigenvalue. Also, the conditions for \(a\) and \(b\) in Corollary 4.3 do not hold.

Similarly, by using (a) in Theorem 2.4 one can get that \(E(\Gamma(4, 5^4)) \neq E(\bar{\Gamma}(4, 5^4))\). \(\checkmark\)
5. Infinite pairs of complementary equienergetic graphs

In this final section we give infinite families of complementary equienergetic non-isospectral graphs \( \{ \Gamma^+(k, q), \Gamma^-(k, q) \} \) for \( k = 3, 4 \), where \( q \) is a power of a prime \( p \), for infinite different prime numbers \( p \). Since we will take \((k, q)\) a non-semiprimitive pair of integers, the involved graphs will be all neither bipartite nor strongly regular. This complements the results obtained in [1] where we characterized all bipartite graphs and all strongly regular graphs which are complementary equienergetic.

We begin with GP-graphs of the form \( \Gamma(3, q) \).

**Theorem 5.1:** Let \( p \) be a prime with \( p \equiv 1 \pmod{3} \). If there is some \( t \in \mathbb{N} \) such that \( p^t = x^2 + 27y^2 \) for some \( x, y \in \mathbb{Z} \) with \((x, p) = 1\), then there are infinitely many \( \ell \in \mathbb{N} \), such that \( \{ \Gamma(3, p^{3(\ell+1)}), \bar{\Gamma}(3, p^{3(\ell+1)}) \} \) are equienergetic and non-isospectral for any \( s \in \{0, 1, \ldots, t-1\} \).

**Proof:** Let \( x, y, t \) be integers satisfying \( p^t = x^2 + 27y^2 \) with \( t > 0 \) and \((x, p) = 1\). We split the proof in two cases, \( s = 0 \) and \( s > 0 \).

(a) First, assume that \( s = 0 \). Since \( p > 3 \) is prime, we have that necessarily \( x \neq 0 \pmod{3} \), so we can choose \( x \) such that \( x \equiv 1 \pmod{3} \).

If we put \( z_{x,y} = x + 3\sqrt{3}iy \), then \( p^t = \|z_{x,y}\|^2 \). From the proof of Theorem 3.1, the spectra of \( \Gamma(3, p^{3\ell}) \) is given by

\[
\text{Spec } \Gamma(3, p^{3\ell}) = \left\{ [n_\ell]^{1/3}, \left[ \frac{a_\ell p^{\ell} - 1}{3} \right]^{n_\ell}, \left[ \frac{-\frac{1}{2}(a_\ell + 9b_\ell)p^{\ell} - 1}{3} \right]^{n_\ell} \right\}
\]

with \( n_\ell = \frac{p^{3\ell} - 1}{3} \) and \( a_\ell = -2x_\ell - 1 \) and \( b_\ell = -2y_\ell - 1 \), where \( x_\ell \) and \( y_\ell \) are recursively defined by

\[
x_\ell = x_0x_{\ell-1} - 27y_0y_{\ell-1} \text{ and } y_\ell = x_0y_{\ell-1} + y_0x_{\ell-1} \quad (39)
\]

for \( \ell > 0 \), where \( x_0 = x, y_0 = y \). By Corollary 4.3, the GP-graph \( \Gamma(3, p^{3\ell}) \) is equienergetic with its complement, in the following cases:

(i) if \( a_\ell > 0 \) and \( a_\ell > 9|b_\ell| \) or (ii) if \( a_\ell < 0 \) and \( -a_\ell < 9|b_\ell| \). (40)

If \( a_\ell, b_\ell \) are both negative numbers, then item (ii) written in terms of \( x_\ell, y_\ell \) is equivalent to \( x_{\ell-1} < 9y_{\ell-1} \) with \( x_{\ell-1}, y_{\ell-1} > 0 \). By taking into account that \( x_{\ell-1} = \text{Re}(z_{x_{\ell-1}, y_{\ell-1}}) \) and \( y_{\ell-1} = \frac{\sqrt{3}}{9} \text{Im}(z_{x_{\ell-1}, y_{\ell-1}}) \) and the fact that \( z_{x_{\ell-1}, y_{\ell-1}} = z_{x,y}^\ell \) if we take \( z := z_{x,y} \) then

\[
x_{\ell-1} = \text{Re}(z^\ell) \text{ and } y_{\ell-1} = \frac{\sqrt{3}}{9} \text{Im}(z^\ell).
\]

So, item (ii), in these terms, implies that

\[
\frac{\pi}{6} = \arctan \left( \frac{\sqrt{3}}{3} \right) < \text{Arg}(z^\ell) < \frac{\pi}{2} \Rightarrow E(\Gamma(3, p^{3\ell})) = E(\bar{\Gamma}(3, p^{3\ell})). \quad (41)
\]

Notice that \((y_\ell, p) = 1\) for all \( \ell \in \mathbb{N}_0 \), since \( p \mid x_\ell \) otherwise. So, we have that \( \text{Im} z^\ell \neq 0 \) for all \( \ell \in \mathbb{N}_0 \). This implies that \( \text{Arg} z \notin \mathbb{Q}\pi \), so we obtain that the classes of \( \text{Arg} z^\ell \) are dense in
\( \mathbb{R} / 2\pi \mathbb{Z} \) when \( \ell \) runs over \( \mathbb{N}_0 \). Therefore, there are infinite values \( \ell \in \mathbb{N}_0 \) satisfying (41), so there are infinite values \( \ell \in \mathbb{N}_0 \) such that \( \Gamma(3, p^{3\ell}) \) is equienergetic with its complement, as asserted.

(b) Now assume that \( s \in \{1, \ldots, t - 1\} \), and let \( a_{0,s}, b_{0,s} \in \mathbb{Z} \) with \( a_{0,s} \equiv 1 \pmod{3} \) and \( (a_{0,s}, p) = 1 \) such that

\[
4p^s = a_{0,s}^2 + 27b_{0,s}^2 = ||z_{a_{0,s}, b_{0,s}}||^2.
\]

As in the proof of Theorem 3.1, if \( z_{a_{\ell,s}, b_{\ell,s}} = a_{\ell,s} + 3\sqrt{3}b_{\ell,s}i \) where \( a_{\ell,s} \) and \( b_{\ell,s} \) are defined recursively by

\[
a_{\ell,s} = a_{0,s}x_{\ell-1} - 27b_{0,s}y_{\ell-1} \quad \text{and} \quad b_{\ell,s} = a_{0,s}y_{\ell-1} + b_{0,s}x_{\ell-1},
\]

then

\[
4p^{s+1} = a_{\ell,s}^2 + 27b_{\ell,s}^2 = ||z_{a_{\ell,s}, b_{\ell,s}}||^2
\]

with \( a_{\ell,s} \equiv 1 \pmod{3} \) and \( (a_{\ell,s}, p) = 1 \), thus \( a_{\ell,s} \), \( b_{\ell,s} \) give the spectra of \( \Gamma(3, p^{3(s+1)}) \). Moreover, we have that

\[
z_{a_{\ell,s}, b_{\ell,s}} = z_{a_{0,s}, b_{0,s}}z_{x_0, y_0}^\ell.
\]

As before, by Corollary 4.3 the GP-graph \( \Gamma(3, p^{3(s+1)}) \) is equienergetic with its complement in the two cases in (40), with \( a_\ell = a_{\ell,s} \) and \( b_\ell = b_{\ell,s} \). In this case, when \( a_{\ell,s}, b_{\ell,s} \) are both positive integers, the item (i) ensures that

\[
0 < \text{Arg}(z_{a_{\ell,s}, b_{\ell,s}}) < \arctan \left( \frac{\sqrt{3}}{3} \right) = \frac{\pi}{6} \quad \Rightarrow \quad E(\Gamma(3, p^{3(s+1)})) = E(\Gamma(3, p^{3(s+1)})).
\]

(42)

Since \( \text{Arg}(z_{x_0, y_0}^\ell) \) is dense in \( \mathbb{R} / 2\pi \mathbb{Z} \) when \( \ell \) runs over \( \mathbb{N} \) and \( z_{a_{\ell,s}, b_{\ell,s}} = z_{a_{0,s}, b_{0,s}}z_{x_0, y_0}^\ell \), hence we obtain that \( \text{Arg}(z_{a_{\ell,s}, b_{\ell,s}}) \) is dense in \( \mathbb{R} / 2\pi \mathbb{Z} \) when \( \ell \) runs over \( \mathbb{N} \). Therefore, there are infinite values \( \ell \in \mathbb{N} \) satisfying (42), so there are infinite values \( \ell \in \mathbb{N} \) such that \( \Gamma(3, p^{3(s+1)}) \) is equienergetic with its complement, as desired.

As a direct consequence of the previous theorem we obtain the following result.

**Corollary 5.2:** Let \( p \) be a prime with \( p \equiv 1 \pmod{3} \) such that 2 is a cubic residue modulo \( p \). Then, there are infinite \( \ell \in \mathbb{N} \), such that \( \{\Gamma(3, p^{3\ell}), \Gamma(3, p^{3\ell})\} \) are equienergetic.

**Proof:** Take \( t = 1 \). If 2 is a cubic residue modulo \( p \) then \( p = X^2 + 27Y^2 \) has integer solutions \( x, y \) with \( (x, p) = 1 \), by (32). Now, taking \( s = 0 \) in the previous theorem we get the desired result.

We now give the analogous of Theorem 5.1 for GP-graphs of the form \( \Gamma(4, q) \).

**Theorem 5.3:** Let \( p \) be a prime with \( p \equiv 1 \pmod{4} \). Then, there exist infinitely many \( \ell \in \mathbb{N} \) such that \( \{\Gamma(4, p^{4\ell}), \Gamma(4, p^{4\ell})\} \) are equienergetic.

**Proof:** If we put \( z_{x,y} = x + 2yi \), then \( p^2 = ||z_{c_1, d_1}||^2 \), where \( c_1 \) and \( d_1 \) are integral solutions of \( p^2 = X^2 + 4Y^2 \) with \( c_1 \equiv 1 \pmod{4} \) and \( (c_1, p) = 1 \). From the proof of Theorem 3.6, the spectra of \( \Gamma(3, p^{4\ell}) \) is given by

\[
\text{Spec}(\Gamma(4, p^{4\ell})) = \left\{ [n_{\ell}]^{1}, \left[ \frac{p^{2\ell} + 4d_1p^{\ell} - 1}{4} \right]^{n_{\ell}}, \left[ \frac{p^{2\ell} - 4d_1p^{\ell} - 1}{4} \right]^{n_{\ell}} \right\},
\]
\[
\begin{align*}
+ \left[ -p^{2\ell} + 2c_\ell p^\ell - 1 \right]^{n_\ell} & \quad , \\
- p^{2\ell} - 2c_\ell p^\ell - 1 \right]^{n_\ell} & \quad ,
\end{align*}
\]
where \( n_\ell = \frac{p^{4\ell} - 1}{4} \) and \( c_\ell, d_\ell \) are recursively defined as follows
\[
c_{\ell+1} = c_1 c_\ell - 4d_1 d_\ell \quad \text{and} \quad d_{\ell+1} = c_1 d_\ell + d_1 c_\ell.
\]
By Corollary 4.3, the graph \( \Gamma(4, p^{4\ell}) \) is equienergetic with its complement if it satisfies \(|c_\ell| < \frac{2\sqrt{3}}{3} |d_\ell| \). If \( c_\ell \) and \( d_\ell \) are both positive integers, using that \( z_{c_\ell, d_\ell} = z_{c_1, d_1}^{\ell} \), we obtain that
\[
\frac{\pi}{3} = \arctan(\sqrt{3}) < \text{Arg}(z_{c_1, d_1}^{\ell}) < \frac{\pi}{2} \quad \Rightarrow \quad E(\Gamma(4, p^{4\ell})) = E(\bar{\Gamma}(4, p^{4\ell})).
\] (43)

The same argument as in the case \( s = 0 \) in Theorem 5.1 ensures that the classes of \( \text{Arg}(z_{c_1, d_1}^{\ell}) \) are dense in \( \mathbb{R}/2\pi \mathbb{Z} \) when \( \ell \) runs over \( \mathbb{N} \). Therefore, there are infinite values \( \ell \in \mathbb{N} \) satisfying (43), so there are infinite values \( \ell \in \mathbb{N} \) such that \( \Gamma(4, p^{4\ell}) \) is equienergetic with its complement, as we wanted to see.

As a final remark, we want to stress that in [1] we have classified all bipartite regular graphs \( \Gamma_{\text{bip}} \) and all strongly regular graphs \( \Gamma_{\text{srg}} \) which are complementary equienergetic, i.e. \( \{\Gamma_{\text{bip}}, \bar{\Gamma}_{\text{bip}}\} \) and \( \{\Gamma_{\text{srg}}, \bar{\Gamma}_{\text{srg}}\} \) are equienergetic pairs of non-isospectral graphs. Here, using non-semiprimitive GP-graphs of the form \( \Gamma(3, q) \) and \( \Gamma(4, q) \), Theorems 5.1 and 5.3 ensure that there are infinitely many pairs of equienergetic non-isospectral regular graphs \( \{\Gamma, \bar{\Gamma}\} \) which are neither bipartite nor strongly regular.

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