The Zero Dynamics of Nonlinear Stochastic Systems: Stabilisation and Output Tracking in the Ideal Case

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Abstract: This paper introduces the notion of zero dynamics and presents results of local stabilisation and output tracking for single-input single-output nonlinear stochastic systems described by stochastic differential equations. For this class of systems we define the zero dynamics when the stochastic relative degree is strictly smaller than the order of the system. We show that, under suitable conditions on the zero dynamics, the equilibrium at the origin can be stabilised via a coordinate change and a nonlinear state feedback. In an analogous way, we show that it is possible to achieve local asymptotic output tracking of a reference signal. We validate the theory through a numerical example.

1 INTRODUCTION

Nonlinear feedback control of dynamical systems is a vast research field and major results in the past decades have contributed to its advance. In this context, particular relevance is given to systems that can be transformed into normal forms. The normal form of a nonlinear control system consists in a representation of the dynamics in a coordinate frame in which the differential equations are simpler to handle for analysis and control purposes. This facilitates the design of nonlinear control laws for stabilisation, tracking and estimation, see e.g. Isidori (1995). In Isidori et al. (1981) the normal form for deterministic systems was firstly introduced as a way to address the problem of static state-feedback non-interacting control, whereas further results can be found, e.g., in Zeit (1983), Bestle and Zeit (1983) and Krener (1987). The problem of linearisation via feedback was proposed and solved for single-input single-output systems in Brockett (1978) and for multi-input multi-output systems in Jakubczyk and Respondek (1980). The concept of zero dynamics was introduced in Byrnes and Isidori (1984), and it was used to address the problem of stabilisation in Byrnes and Isidori (1988).

This paper is intended to extend the analysis of the zero dynamics to a broad class of nonlinear stochastic systems, specifically systems described by stochastic differential equations. The use of stochastic differential equations to model uncertain systems allows embedding model uncertainties in the dynamics in the form of noisy coefficients, see Øksendal (2003). Applications of stochastic systems theory are broad and vary from optimal stopping to production planning, finance, technology diffusion and research funding Øksendal (2003), Yong and Zhou (1999). Although some notions of normal form for stochastic systems are present in the literature, they were not introduced for the purpose of controlling the systems. For example, a normal form was proposed in Arnold and Imkeller (1998) and, subsequently, in Arnold (2003). Therein a theory of normal forms is introduced by employing Stratonovich calculus on systems described by purely diffusive terms. The normal form is obtained via a coordinate transformation which requires anticipating the noise over a short time scale. Coordinate transformations for stochastic systems were also dealt with in, e.g., Gaeta and Rodriguez Quintero (1999), where the authors introduce symmetries for stochastic differential equations, and Roberts (2008), where fast and slow dynamics can be separated using a normal form.

In this paper we employ a notion of normal form with the purpose of controlling a general class of nonlinear stochastic single-input single-output systems. The notion of normal form that we adopt in this work was firstly introduced in Mellone and Scarciotti (2019a). Therein the authors first define the concept of relative degree for stochastic systems, which parallels the deterministic version presented, e.g., in Isidori (1995), and then use this concept to derive a coordinate transformation that brings the system to a normal form which is particularly meaningful for control design. Under the assumption of full relative degree the authors also propose a linearising state feedback in the unrealistic hypothesis that the noise affecting the system is measurable.

Extending the results of Mellone and Scarciotti (2019a), in this paper we investigate the case of systems with relative degree smaller than their order. In particular, we define the concept of zero dynamics for nonlinear stochastic systems. Moreover, we present results concerning the local stabilisation and output tracking of nonlinear systems which extensively rely on the concept of the zero dynamics. We hereby point out that the control laws that we introduce depend explicitly on the white noise process affecting the system. Clearly, this hypothesis is not practically reasonable as white noise cannot be physically measured. However, this treatise is intended as a fundamental preliminary step for the solution of a practically implementable version of these results. In fact, following the ideas presented, e.g.,
in Mellone and Scarciotti (2019b) to solve practically the problem of output regulation for linear stochastic systems. We propose in the accompanying paper Mellone and Scarciotti (2020) an approximate, yet practical, control architecture which does not depend on measurements of the noise. The rest of the paper is organised as follows. In Section 2 we recall some preliminary notions related to stochastic systems. In Section 3 we introduce the concept of zero dynamics and present the main results of the section, namely the local stabilisation of an equilibrium point and the local asymptotic tracking of reference inputs. In Section 4 we show a numerical example that illustrates the theoretical results of Section 3. Finally, Section 5 contains some concluding remarks.

Notation. The symbol $N$ denotes the set of integer numbers, while $R$ and $C$ denote the fields of real and complex numbers, respectively; by adding the subscript "\(<0\)" (\(\geq0\), "\(=0\)"") to any symbol indicating a set of numbers, we denote that subset of numbers with negative (non-negative, zero) real part. The symbol $\partial_k^n$ is used as a shorthand for the operator $\partial^n/\partial x^n$, while $\alpha^{(n)}$ indicates the $n$-th time derivative of $\alpha$. $(\nabla, F, P)$ is a probability space given by the set $\nabla$, the $\sigma$-algebra $F$ defined on $\nabla$ and the probability measure $P$ on the measurable space $(\nabla, F)$. A stochastic process with state space $\mathbb{R}^n$ is a family $\{x_t, t \in \mathbb{R}\}$ of $\mathbb{R}^n$-valued random variables, i.e. for every fixed $t \in \mathbb{R}$, $x_t(\cdot)$ is an $\mathbb{R}^n$-valued random variable and, for every fixed $w \in \nabla$, $(w(x))$ is an $\mathbb{R}^n$-valued function of time (Arnold, 1974, Section 1.8). For ease of notation, we often indicate a stochastic process $\{x_t, t \in \mathbb{R}\}$ simply with $x_t$ (this is common in the literature, see e.g. Arnold (1974)). With a slight abuse of notation, any subscript different from the symbol "\(r\)" corresponds the coordinate component of the vector $x_t$, e.g. $x_t$ is the $i$-th component of the vector $x_t$. Let $C_0^\infty(\mathbb{R})$ denote the space of all infinitely differentiable functions on $\mathbb{R}$ with compact support. A generalised stochastic process is a random generalised function in the sense that a random variable $\psi(\varphi)$ is assigned to every $\varphi \in C_0^\infty$, where $\psi$ is, with probability 1, a generalised function (Arnold, 1974, Section 3.2). The symbol $W_t$ indicates a standard Wiener process, also referred to as Brownian motion, whereas $\xi_t = W_t$ indicates the generalised white noise obtained by differentiating $W_t$. $W_t$ and $\xi_t$ are defined on the probability space $(\nabla, F, P)$.

2 PRELIMINARIES

In this section we recall some preliminary notions related to stochastic differential equations and the concept of stochastic relative degree. Consider the nonlinear single-input, single-output stochastic system expressed in the shorthand integral notation

$$
\begin{align*}
 dx_t &= f(x_t) + g(x_t)u dt + (l(x_t) + m(x_t)u)dW_t, \\
 y_t &= h(x_t),
\end{align*}
$$

(1)

with $x_t \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y_t \in \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^n$, $l : \mathbb{R}^n \to \mathbb{R}^m$, $m : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}$ smooth functions, i.e. they admit continuous partial derivatives of any order. We assume that, for a fixed initial condition $x_{t_0}$, the solution of (1) is unique. For the reasons reported in (Arnold, 1974, Section 10.3), we can rewrite equation (1) in the differential notation

$$
\begin{align*}
\dot{x}_t &= f(x_t) + g(x_t)u + (l(x_t) + m(x_t)u)\xi_t, \\
\dot{y}_t &= h(x_t),
\end{align*}
$$

(2)

as long as $\xi_t$ is understood as a generalised white noise, (Arnold, 1974, Section 10.3). Given the equivalence of the two representations in the framework of generalised stochastic processes, in the remainder of the paper equations (1) and (2) are used interchangeably, as convenient, to refer to the same underlying nonlinear stochastic system. Recall that the derivative of $h$ along the vector field $f$, which is called Lie derivative and is indicated with the symbol $L_f h$, is defined as

$$
L_f h(x) = \partial_x [h] f(x) = \sum^n_{i=1} \frac{\partial h}{\partial x_i} f_i(x).
$$

We indicate the derivative of $h$ first along the vector field $f$ and then along the vector field $g$ as $L_g L_f h(x) = \partial_x [L_f h] g(x)$. We use the recursive relation $L_f^k h(x) = \partial_x [L_f^{k-1} h] f(x)$, with $L_f^0 h(x) = h(x)$, to indicate the $k$-th differentiation of $h$ along $f$. We now recall three operators, firstly introduced in (Mellone and Scarciotti, 2019a, Section III). The first one, which indicates the second derivative of $h$ along the vector fields $f$ and $g$, is defined as

$$
\nabla^2_g h(x) = (g(x))^T \partial^2_x h f(x) = \sum^n_{j=1} g_j(x) \sum^n_{i=1} \frac{\partial^2 h}{\partial x_i \partial x_j} f_i(x).
$$

Similarly to the Lie derivative, we use the notation $\nabla^2_g \nabla^2_f h(x) = (m(x))^T \partial^2_x \nabla^2_g h l(x)$, and $\nabla^2_g \nabla^2_f h(x) = (g(x))^T \partial^2_x \nabla^2_g \nabla^2_f h f(x)$, to indicate the reiterated operators. The second operator is $S_f h$, which is employed to define the stochastic Lie derivative of $h$ along the drift vector field $f$ and the diffusion vector field $l$, namely

$$
S_f h(\xi_t, x) = L_f h(x) + L_l h(x)\xi_t + \frac{1}{2} \nabla^2_f h(x).
$$

If $S_f h(\xi_t, x) = \nabla^2_f h(x)$ is a deterministic expression, i.e. the white noise does not appear explicitly or equivalently $L_l h \equiv 0$, then, similarly to the deterministic Lie derivative, we use the notation $L_f \nabla_f^2 h(\xi_t, x) = S_f S_f^2 h(\xi_t, x)$. Iteratively, if $S_f^{k-1} h(\xi_t, x) = S_f S_f^{k-1} h(x)$ is deterministic, $S_f^k h(\xi_t, x) = S_f S_f^{k-1} h(x)$, with $S_f h(x) = h(x)$ by definition. Finally, we define a third stochastic differential operator

$$
\nabla^m h(\xi_t, x) = L_g h(x) + L_m h(x)\xi_t + \nabla^m g h(x).
$$

By using Itô’s formula, it is easy to see that the first derivative of the output of system (2) is given by

$$
y_t^{(1)} = S_f h(\xi_t, x_t) + \nabla^m A h(\xi_t, x_t) u_t + \frac{1}{2} \nabla^m g h(x_t) u_t^2.
$$

Using the previous definitions, we recall the concept of stochastic relative degree.

Definition 1. (Stochastic Relative Degree. Mellone and Scarciotti (2019a)) Assume that there exists $\bar{r}$ such that

$$
L_f^1 S_f^k h(\xi_t, x) = 0,
$$

(3)

for all $k \in \{0, 1, ..., \bar{r} - 2\}$ and for all $x$ in a neighbourhood of $x$. System (2) is said to have stochastic relative degree $\bar{r}$ at a point $\bar{x}$ if $\bar{r} = \bar{r}$ and (a) all the following conditions are satisfied

$$
0 = L_g S_f^k h(\xi_t, x),
$$

$$
0 = L_m S_f^k h(\xi_t, x),
$$

$$
0 = \nabla^m g S_f^k h(\xi_t, x),
$$

where $\nabla^m$ indicates the $m$-th derivative of a function with respect to its arguments. We denote $r^m$ as the stochastic relative degree of system (2) with respect to the variable $x$.
for all $x$ in a neighborhood of $\bar{x}$ and all $k \in \{0, 1, ..., r-2\}$.

(b) one of the following conditions is satisfied

\[ 0 \neq L_x^i S_f^{-1} h(\xi_t, \bar{x}) + m G_t^i S_f^{-1} h(\xi_t, \bar{x}), \]
\[ 0 \neq L_x^i m S_f^{-1} h(\xi_t, \bar{x}), \]
\[ 0 \neq m G_m^i S_f^{-1} h(\xi_t, \bar{x}). \]

**Assumption 1.** Let $r$ be the stochastic relative degree of system (2) at $\bar{x}$. Assume that the vector rows $\partial_i [h(\bar{x})], \partial_i [S_f h(\bar{x})], \ldots, \partial_i [S_f^{-1} h(\bar{x})]$ are linearly independent.

Suppose that the relative degree of system (2) at $\bar{x}$ is $r \leq n$. Since Assumption 1 and the standing assumption (3) hold \(^1\), then there exist functions $\phi_i(x)$, $i = r+1, ..., n$ such that $\Phi_t$ given by

\[ \Phi_t(x) = [h(x)^i S_f h(x) \ldots (S_f^{-1} h(x) \phi_{r+1}(x) \ldots \phi_n(x))^r]^T, \]

is a local diffeomorphism in a neighborhood of $\bar{x}$ such that the dynamics of system (2), written in the new coordinates $\bar{z}_i = \Phi(x_t)$, is expressed by (Mellone and Scarciotti (2019a))

\[ \hat{\bar{z}}_i = \bar{z}_{i+1}, \]
\[ \hat{\bar{z}}_r = c(\bar{z}_i, \bar{z}_t) + b(\bar{z}_i, \bar{z}_t) u + a(\bar{z}_i) u^2, \]
\[ \hat{\bar{z}}_j = p_j(\bar{z}_i, \bar{z}_t) + q_j(\bar{z}_i, \bar{z}_t) u + s_j(\bar{z}_i) u^2, \]
\[ y_t = \bar{z}_1, \]

with $c(\bar{z}_i, \bar{z}_t) = \xi_i(z_i, z_t) = \xi_j(z_i, z_t) = \xi_j(z_i, z_t)$, $a(\bar{z}_i) = \xi_i(z_i, z_t)$, $b(\bar{z}_i, \bar{z}_t) = \xi_i(z_i, z_t)$, $p_j(\bar{z}_i, \bar{z}_t) = \xi_j(z_i, z_t)$, $q_j(\bar{z}_i, \bar{z}_t) = \xi_j(z_i, z_t)$, $s_j(\bar{z}_i) = \xi_j(z_i, z_t)$.

In particular, observe that the dependence of the coefficients $c$, $b$, $p_j$ and $q_j$ on the white noise $\xi_t$ is affine, i.e. they can be decomposed as

\[ c(\bar{z}_i, \bar{z}_t) = c_i(\bar{z}_i) + c_i(\bar{z}_t), \]
\[ b(\bar{z}_i, \bar{z}_t) = b_i(\bar{z}_i) + b_i(\bar{z}_t), \]
\[ p_j(\bar{z}_i, \bar{z}_t) = p_{j,0}(\bar{z}_i) + p_{j,1}(\bar{z}_i), \]
\[ q_j(\bar{z}_i, \bar{z}_t) = q_{j,0}(\bar{z}_i) + q_{j,1}(\bar{z}_i), \]

where all the quantities are uniquely defined as a consequence of the previous observation. For compactness, we define $p = [p_{j,0} \ldots p_{n,0}]^T$, $q = [q_{j,0} \ldots q_{n,0}]^T$ and $s = [s_{n+1} \ldots s_n]^T$. When designing the control input of system (4) we need to distinguish between two cases.

**A)** $a(\bar{z}_i) \equiv 0$ in a neighborhood $U_1$ of $\bar{x}$. Then, by definition of relative degree, there exists a neighborhood $U_2$ of $\bar{x}$ such that $b(\bar{z}_i, \bar{z}_t) \neq 0$ in $\Phi(U_2)$. Let $U = U_1 \cap U_2$.

**B)** There exists a neighborhood $U$ of $\bar{x}$ such that $a(\bar{z}_i) \neq 0$ in $\Phi(U)$.

Note that, by the definition of relative degree, no other cases are possible. If a system is such that case **A** is satisfied, then the control input does not appear quadratically in the $r$-th derivative of the output for $x_t$ in $U$, whereas if **B** is satisfied, then the square of the input does appear in the $r$-th derivative of the output. The distinction between these two cases will be used in the remainder to define two separate control strategies, depending on which case is satisfied for the system under consideration.

### 3 THE ZERO DYNAMICS OF A STOCHASTIC SYSTEM

Suppose that the stochastic relative degree $r$ of system (2) is strictly less than $n$ at $\bar{x}$. For compactness, set $\xi_t = [x_1 \ldots x_r]^T$ and $\eta_t = [x_{r+1} \ldots x_n]^T$. From the theory of normal forms of deterministic systems, recall that necessarily $\xi_t = 0$ at $\bar{x}$, whereas it is straightforward to observe that the value of $\eta_t$ at $\bar{x}$ can be arbitrarily chosen for stochastic systems as well. Therefore let $\Phi(x_t)$ be such that $\eta_t = 0$ as well at $\bar{x}$, which makes zero an equilibrium of system (4). We hereby extend the definition of zero dynamics to nonlinear stochastic systems, i.e. the internal dynamics of the system when the input and the initial conditions constrain the output to be identically zero.

**Definition 2.** (Zero Dynamics) The stochastic differential equation

\[ \eta_t = \dot{c}(\xi_t, 0, \eta_t) + \dot{q}(\xi_t, 0, \eta_t) \eta u_{z,t} + s(0, \eta_t) \eta^2, \]

with $\eta_0$ in a neighbourhood of zero and $u_{z,t}$, if it exists, given by either (A)

\[ u_{z,t} = -c(\xi_t, 0, \eta_t), \]

or (B)

\[ u_{z,t} = -b(\xi_t, 0, \eta_t) \frac{\eta}{2a(0, \eta_t)}, \]

as long as $b(\xi_t, 0, \eta_t)^2 - 4a(0, \eta_t)c(\xi_t, 0, \eta_t)$ $\geq 0$, is called the zero dynamics of system (2).

**Remark 1.** By selecting $u$ as in Definition 2, we are enlarging the class of systems (2) to allow control laws which depend explicitly on $\xi_t$. In general, this yields a closed-loop system that is not equivalent to (1). To avoid this, we hereby define a control law to be admissible if it preserves the equivalence of the closed-loop systems (1) and (2), i.e. the dependence of the dynamics in (2) is affine in $\xi_t$. To address the most general case, we refrain from making additional assumptions which would be sufficient, although restrictive, to guarantee the admissibility of $u$ and we leave this condition to be checked a posteriori. See Section 4.1 for an example.

**Remark 2.** Since $u_{z,t}$ is uniquely dependent on the state $\eta_t$ and the noise $\xi_t$, the zero dynamics is autonomous and can be denoted by

\[ \dot{\eta}_t = \dot{p}(\xi_t, 0, \eta_t) = p(\xi_t, 0, \eta_t) + q(\xi_t, 0, \eta_t) u_{z,t} + s(0, \eta_t) u^2_{z,t}, \]

**Remark 3.** It might be possible to choose $\phi_{r+1}, \ldots, \phi_n$ such that $q = 0$ and $s = 0$ in a neighbourhood of zero, which reduces the zero dynamics to

\[ \dot{\eta}_t = \dot{p}(\xi_t, 0, \eta_t) = p(\xi_t, 0, \eta_t). \]

While for nonlinear deterministic systems it is always possible to find functions $\phi_j$ such that the dynamics of $\eta_t$ does not explicitly depend on the control input (Isidori, 1995, Proposition 4.1.3), for stochastic systems this property is currently under investigation.

#### 3.1 Local asymptotic stabilisation

In this section we show that under the assumption that the zero dynamics is almost surely asymptotically stable
it is possible to stabilise the equilibrium at the origin via nonlinear feedback. Consider the transformed system (4). Let 
\[ v(\zeta_i) = -(d_0z_1 + d_2z_2 + \ldots + d_{r-1}z_r) \]
and the feedback control law be given by \( u_t = u_t^{stab} \), where \( u_t^{stab} \) is either (A) or (B)

\[ u_t^{stab} = -\frac{c(\zeta_i, \zeta, \eta) - v(\zeta)}{b(\zeta_i, \zeta, \eta)}, \]

or (B)

\[ u_t^{stab} = (2a(\zeta_i, \eta_i)^{-1} \left[ b(\zeta_i, \zeta, \eta) \right] \right), \]

where \( d_i, \) with \( i = 0, \ldots, r - 1, \) are, if possible, such that \( b^2 - 4a(\epsilon - v) \geq 0. \)

Define the tracking error \( e_t := y_t - y(t) \). Then it is easy to see that, under the control \( u_t^{track} \), the tracking error has dynamics \( e_t^{(r)} + d_1e_t^{(r-1)} + \ldots + d_re_t^{(1)} + d_0e_t \), which can be forced exponentially to zero by suitably selecting the coefficients \( d_i \). We are also interested in analysing the boundedness of the states \( z_i \) and of the internal variable \( \eta_t \), under the control \( u_t^{track} \), when the reference output and its first \( r - 1 \) time derivatives are bounded. Define \( \zeta(t) = [y_t(t) \ldots y_t^{(r-1)}(t)]^T \) and \( \theta_t = [e_t \ldots e_t^{(r-1)}]^T \). Then the following result holds.

**Theorem 2.** Consider system (2) and let Assumptions 1 and 2 hold. Suppose \( y_R(t), y_R^{(1)}(t), \ldots, y_R^{(r-1)}(t) \) are bounded. Let \( \eta_{R,i} \) be the solution of

\[ \dot{\eta}_{R,i} = \tilde{p}(\zeta_i, \zeta, \eta_i), \]

and \( \dot{\eta}_{R,0} = 0 \) and let \( \tilde{p}_R \) be Lipschitz continuous. Moreover, assume that there exists a strict Lyapunov function \( V(x,t) \) for (7) such that \( \dot{V}(x,t) \) and \( \frac{\partial V}{\partial x}(x,t) \) are bounded for all \( x \in U \) and \( t \geq 0 \). Suppose that the roots of the polynomial \( \Lambda(s) \) in (6) have negative real part. Then for sufficiently small \( \epsilon_R > 0 \), if

\[ |z_i(t) - y_R^{(i)}(t)| < \epsilon_R, \quad 1 \leq i \leq r, \quad \|\eta_t - \eta_{R,i}\| < \epsilon_R, \]

then for all \( \epsilon > 0 \) there exists \( \delta > 0 \) such that

\[ |z_i(t) - y_R^{(i)}(t)| < \delta \rightarrow |z_i(t) - y_R^{(i)}(t)| < \epsilon, \]

\[ 1 \leq i \leq r, \quad \text{for all } t \geq \delta \geq 0, \]

\[ \|\eta_t - \eta_{R,i}\| < \delta \rightarrow \|\eta_t - \eta_{R,i}\| < \epsilon, \quad \text{for all } t \geq \delta \geq 0, \]

almost surely, i.e., the response \( z_i \) and \( \eta_t \), \( t \geq \delta \geq 0, \) of system (2) under the control law \( u_t^{track} \) is bounded almost surely.

The previous theorem solves the local asymptotic output tracking problem, i.e., the output \( y_t = z_1 \) asymptotically converges to \( y_R \) while the state \( z_i \) remains bounded almost surely.

**Remark 4.** The white noise \( \xi \) explicitly appears in the control laws \( u_t^{stab} \) and \( u_t^{track} \) implying that in general its knowledge is needed in order to compute the control input. This is impossible in practice. However, the results presented in this paper lay the theoretical grounds to approach the stabilisation and tracking problem in a practically viable way. Specifically, in Melone and Scarciotti (2020) we show how to obtain a causal estimation of the increments of the Brownian motion and, subsequently, how to use them to approximate the locally stabilising control law \( u_t^{stab} \). The practical counterpart of the control law \( u_t^{track} \) will be the topic of forthcoming papers.

## 4 Example

Consider the nonlinear stochastic system

\[ \dot{x}_1 = \begin{bmatrix} 2x_2(1 + x_1) + 2x_3 + x_1s_2 - 2x_1s_2 \frac{x_1}{c_2} \ 
-2x_2 \ 
2x_3 + x_1s_2 - 2x_1s_2 \frac{x_1}{c_2} \ 
\left[ c_3 \right] + \frac{x_1}{c_2} \ 
\end{bmatrix} u + \left[ \xi \right] \]
with the output $y_t = x_1 + s_2 - x_3$, where $s_i = \sin(x_i)$ and $c_i = \cos(x_i)$. We study the system in a neighbourhood $U = (-\pi/2, \pi/2)^3$ of zero. We look for a change of coordinates $\Phi$ to bring the system into the normal form. By setting $z_1 = y_t$ simple computations yield $z_2 = \dot{z}_1 = -s_2 - 2x_3$. As neither the control input nor the white noise appear in the expression of $z_2$, the standing assumption (3) is satisfied and the relative degree of the system is larger than 1. Therefore, we compute the second derivative of the output, which results in

$$
\dot{z}_2 = 2s_2 - 4x_3 - 2x_1s_2 + \frac{6x_1^2s_2}{c_2} + 4x_1\xi_t - 2e^{x_3}u - 2x_3^2\xi_t
$$

$$
= \tilde{c}(\xi_t, x_t) + \tilde{b}(\xi_t, x_t)u,
$$

with $\tilde{c}(\xi_t, x_t) = c(\xi_t, \Phi(x_t))$ and $\tilde{b}(\xi_t, x_t) = b(\xi_t, \Phi(x_t))$. As the control input appears on the right-hand side and $\tilde{b}$ is non-zero at $\bar{x} = 0$, the relative degree of the system at the origin is 2. To complete the definition of the change of coordinates $\Phi$, we set $z_3 = x_1 - x_3$, which in turn yields

$$
z_3 = s_2 - 2x_3 + \frac{2x_1^2s_2}{c_2} + 2x_1\xi_t - \tilde{p}(\xi_t, x_t) = p(\xi_t, \Phi(x_t)).
$$

(9)

We now want to study the stability of the zero dynamics. Define $\zeta_t = [z_1 \ z_2]^\top$ and $\eta = z_3$. We set $\zeta_t = 0$ and, using the definition of $z_3$, we obtain the system of equations

$$
0 = x_1 + s_2 - x_3, \quad 0 = -s_2 - 2x_3, \quad z_3 = x_1 - x_3,
$$

which yields

$$
x_1 = \frac{3}{2} z_3, \quad s_2 = -z_3, \quad x_3 = \frac{1}{2} z_3.
$$

Substituting these values in (9), we get

$$
\dot{\eta}_t = p(\xi_t, 0, \eta) = -2\eta_t + \frac{9\eta^3}{2(\eta^2 - 1)} + 3\eta_t \xi_t.
$$

The linear approximation of this system is

$$
\dot{\eta}_t = A_\eta \eta_t + F_\eta \eta_t \xi_t = -2\eta_t + 3\eta_t \xi_t,
$$

which is asymptotically stable almost surely if $2A_\eta - F_\eta^2 < 0$, see, e.g., Gard (1988). Since this condition is verified, we conclude that the zero dynamics is locally asymptotically stable almost surely.

4.1 Local asymptotic stabilisation

We first show an example of stabilisation of the equilibrium of the system at the origin. We set $v(\zeta_t) = -6z_1 - z_2$ and we apply the control law $u_t^{stab}$ (case (A)) with $\tilde{c}(\xi_t, \zeta_t, \eta) = \tilde{c}(\xi_t, \Phi^{-1}(z_1))$ and $b(\xi_t, \zeta_t, \eta) = \tilde{b}(\xi_t, \Phi^{-1}(z_1))$. This control law is admissible because in (8) the expression $(g + m(\zeta))u_t^{stab} = [\frac{1}{2} \ 0 \ 0 \ \frac{1}{2}] g(\xi_t, x_t) - \tilde{v}(x_t)$, with $\tilde{v}(x_t) = v(\Phi(x_t))$, is affine in $\zeta_t$ (and so it can be rewritten in the form (1)). Moreover, the control law brings the system to the form (5), where the matrix $A$ has eigenvalues $\{-2, -3\}$. The assumptions of Theorem 1 are satisfied and therefore the closed-loop system has an almost surely asymptotically stable equilibrium at the origin.

Figure 1 shows the time histories of the state of the system for the case of local asymptotic stabilisation. In particular, Figure 1(a) shows the time history of the state in the coordinates $x_t$ and Figure 1(b) shows the time history of the state in the coordinates $z_t$. While all the three components of $x_t$ display a noisy behaviour, the change of coordinates $\Phi$ decouples the noise from the first two components of the vector $z_t$ and projects this onto the third component $z_3$. This is in line with the standing assumption (3). Moreover, Figure 1(b) confirms that the state $\zeta_t = [z_1 \ z_2]^\top$ has linear, deterministic and asymptotically stable dynamics, whereas $\eta_t = z_3$ has nonlinear, stochastic yet asymptotically stable dynamics.

4.2 Local asymptotic output tracking

We now consider an example of local asymptotic output tracking. We assume that the reference signal $y_R(t) = 0.01\sin(2t)$ has to be tracked by the output $y_t$. In this case we set $v(\zeta_t, y_R(t)) = -0.04\sin(2t) - 6(z_1 - 0.01\sin(2t)) - 5(z_2 - 0.02\cos(2t))$ and we apply the control law $u_t^{track}$ (case (A)) with $c(\xi_t, \zeta_t, \eta) = \tilde{c}(\xi_t, \Phi^{-1}(z_1))$ and $b(\xi_t, \zeta_t, \eta) = \tilde{b}(\xi_t, \Phi^{-1}(z_1))$. This control law is admissible and asymptotically stabilises the tracking error $e_t$.

Figure 2 shows the time histories of the state of the system for the case of local asymptotic output tracking. In particular, Figure 2(a) shows the time history of the state in the coordinates $x_t$ and Figure 2(b) shows the time history of the state in the coordinates $z_t$. Similarly to the case of stabilisation, the isomorphism $\Phi$ and the control $u_t^{track}$ project the noise on the third component $z_3$ of the state in the coordinates $z_t$, while $z_1$ and $z_2$ display a linear deterministic behaviour. Moreover, the control is such that the output $y_t = z_1$ converges asymptotically to the reference $y_R$ while the internal variable $z_3$ remains bounded almost surely.
Fig. 2. Local asymptotic output tracking of system (8).

5 CONCLUSIONS

In this paper we have introduced the notion of zero dynamics of nonlinear stochastic systems described by a general class of stochastic differential equations and we have employed it to address the problems of local stabilisation and asymptotic output tracking. We have showed that, under the assumption of an almost surely asymptotically stable zero dynamics, a change of coordinates and a nonlinear state feedback solve the problems. Finally, we have demonstrated the validity of the theory with a numerical example.

In this work we have stressed that the nonlinear state feedback yielding stabilisation and tracking require the knowledge of the white noise affecting the system. Although this is impossible in practical circumstances, the treatise in this paper is a fundamental preliminary step towards a practical solution of the problems. This is the subject of the accompanying paper Mellone and Scarciotti (2020).

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