Modelling the effect of wind farming on mesoscale flow - Part 1: Flow model

A J Brand
ECN Wind Energy, P.O. Box 1, NL 1755 RG Petten, Netherlands
brand@ecn.nl

Abstract. This paper presents a new method for determining the interaction between a wind farm and the prevailing wind for wind energy siting studies. It is shown that neutral planetary boundary layer flow with wind farming essentially is steady and two-dimensional; and that the convective forces, the Coriolis forces and the vertical and spanwise gradients of the turbulent momentum fluxes all have the same order of magnitude. In addition it is shown that a numerical representation in the form of backward differences allows for an implicit solution of the two horizontal velocity components in vertical direction, iterating on the turbulent viscosity, in combination with a marching solution in the horizontal directions.

1. Introduction

Offshore wind farms tend to be placed closer together over the years, as already illustrated by OWEZ and Q7-WF (separated 15 km) in the Netherlands or Horns Rev I and II (separated 23 km) in Denmark. Since these separation distances are between 5 and 10 times the wind farm's horizontal scale, the velocity deficit due to an upstream wind farm may be considerable [1]. If so, energy production loss and mechanical load increase are expected to be significant. For this reason wind farm wake studies have gained attention recently.

In this paper we present a new method for determining the interaction between a wind farm and the prevailing wind. First, section 2 gives a brief description of prior work on modelling wind farm wakes. Next, section 3 presents the new flow model, and finally, in section 4 we summarize the new model and introduce the future developments.

2. Prior work

A wind farm wake study requires simulation of mesoscale atmospheric flow together with energy extraction/redistribution due to wind turbines. The studies that have been published so far can be subdivided into two categories: self-similar approaches and mesoscale approaches. In a self-similar approach [2][3] the convective force and the spanwise turbulent flux gradients are assumed to dominate the flow, allowing for standard wake-like solutions. In a mesoscale approach, on the other
hand, the flow is assumed to be dominated by the Coriolis force and the vertical turbulent flux gradients, opening the door to either extra surface drag approaches [4] or more generic mesoscale approaches [5][6][7]. As will be shown in section 3.2.1 of this paper, neither the self-similar wake approach nor the extra surface drag approach is valid because over the separation distance between wind farms the convective and the Coriolis forces are of equal order of magnitude so that neither can be neglected. Although this was already implicitly recognized in the more generic approaches, these studies lack realistic formulations for the turbulence and the wind turbines.

3. Flow model

3.1. Overview

In section 3.2 first we derive the governing equations from the planetary boundary layer equations and the continuity equation by introducing length and velocity scales, modelling Reynolds stresses with turbulent viscosity and wind turbines with body forces. Since these equations are essentially steady and two-dimensional (the vertical wind speed scale is several orders of magnitude smaller than a horizontal wind speed scale), the pressure gradients can be treated as geostrophic wind speed components. We subsequently cast the governing equations in non-dimensional form, employing linear transformations in the horizontal directions and an exponential transformation in the vertical direction. Next, in section 3.3 we derive a numerical representation of the non-dimensional governing equations by using finite differences. The resulting scheme is implicit in vertical direction, iterating between the two horizontal velocity components and the turbulent viscosity, and allows for a marching solution in the horizontal directions. We then address the turbulence parameterization (Baldwin-Lomax) in section 3.4, and the wind turbine parameterization (via the rotor thrust) in section 3.5. Finally, we present the boundary conditions (no-slip at the bottom and geostrophic at the top of the numerical domain; vanishing but non-zero turbulent viscosity both at the bottom and the top) and initial conditions (logarithmic x-wise and linear y-wise velocity profiles; corresponding turbulent viscosity profile) in section 3.6.

3.2. Governing equations

3.2.1. Dimensional form

The mean (in the sense of Reynolds averaged) flow in the neutral planetary boundary layer is described by the momentum equations [8, section 5.2.1]:

\[
\frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u} u}{\partial x} + \frac{\partial \bar{u} v}{\partial y} + \frac{\partial \bar{u} w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{\partial \bar{u}' u'}{\partial x} - \frac{\partial \bar{u}' v'}{\partial y} - \frac{\partial \bar{u}' w'}{\partial z} + f_x, \tag{1}
\]

\[
\frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{u} u}{\partial x} + \frac{\partial \bar{v} v}{\partial y} + \frac{\partial \bar{v} w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{\partial \bar{v}' u'}{\partial x} - \frac{\partial \bar{v}' v'}{\partial y} - \frac{\partial \bar{v}' w'}{\partial z} + f_y, \tag{2}
\]

\[
\frac{\partial \bar{w}}{\partial t} + \frac{\partial \bar{u} u}{\partial x} + \frac{\partial \bar{v} v}{\partial y} + \frac{\partial \bar{w} w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} - \frac{\partial \bar{w}' u'}{\partial x} - \frac{\partial \bar{w}' v'}{\partial y} - \frac{\partial \bar{w}' w'}{\partial z} + f_z \tag{3}
\]

in combination with the continuity equation:

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial z} = 0 ; \tag{4}
\]
where

- \( \overline{u}, \overline{v}, \overline{w} \) are the components of the mean velocity;
- \( u', v', w' \) are the velocity fluctuations;
- \( \rho \) is the air density;
- \( \overline{p} \) is the mean pressure;
- \( f_s \) is the Coriolis parameter;
- \( \overline{f}_x, \overline{f}_y, \overline{f}_z \) are the components of a mean body force representing wind turbines; and
- the covariances represent the turbulent momentum fluxes.

The equations (1), (2), (3) and (4) constitute a system of 4 equations with 4 unknowns, which can be solved once boundary conditions are set. We come back to the boundary conditions in section 3.3.

In order to estimate the magnitude of the individual terms in the equations (1), (2), (3) and (4) we introduce length and velocity scales [8, section 2.4] that correspond to wind farming in the planetary boundary layer (table 1). A length scale a distance related to motions or objects in the planetary boundary layer and a velocity scale is the velocity variation over a given length scale.

**Table 1. Scales in the planetary boundary layer with wind farming**

| Scale | Value | Magnitude |
|-------|-------|-----------|
| \( z_0 \) Surface roughness length | 1 mm - 1 cm |
| \( D \) Rotor diameter | 100 m |
| \( S_t \) Turbine separation | 10 D |
| \( S_f \) Wind farm size | 10 km |
| \( L_x \) Wind farm separation | 100 km |
| \( L_y \) Wind farm wake width | 100 km |
| \( L_z \) Planetary boundary layer height | 1 km |
| \( U_x \) x-Velocity variation | 10 m/s |
| \( U_y \) y-Velocity variation | 10 m/s |
| \( U_z \) Vertical velocity variation | 0.1 \( U_x \) \( L_z / L_x \) |
| \( \Delta p \) Pressure variation | 1 hPa |
| \( \Delta T \) Temporal variation | 1 day |
| \( u \) Velocity variation | 1 m/s |
| \( \rho \) Air density | 1 kg/m³ |
| \( f_s \) Coriolis parameter | \( 10^{-4} \) 1/s |

First we address the length scales (figure 1). The vertical length scales include those of the surface layer (proportional to the surface roughness length), the turbine layer (proportional to the turbine hub height and therefore to the rotor diameter) and the top layer (proportional to the planetary boundary layer height). Since we consider wind farming in the planetary boundary layer, the vertical length scale \( L_z \) of our flow problem is proportional to the height of the planetary boundary layer. The horizontal length scales include the length of the turbine near wake (proportional to the rotor diameter), the turbine far wake and the related turbine separation (up to 10D), the horizontal scale of a wind farm (typically consisting of 10 rows/columns), and the wind farm wake (up to 10 horizontal wind farm scales). Since we consider motions in the planetary boundary layer due to wind farm wakes, our flow problem has two horizontal length scales: the x-wise length scale \( L_x \) which we define to be proportional to the streamwise separation between wind farms, and the y-wise length scale \( L_y \) proportional to the width of the wake of a wind farm. Now the vertical length scale \( L_z \) is smaller than...
the horizontal length scales $L_x$ and $L_y$, and the y-wise length scale $L_y$ is of the same order of magnitude as (but smaller than) the x-wise length scale $L_x$. Table 1 gives the typical values.

![Diagram of length scales](image)

**Figure 1.** Vertical (left) and horizontal (right) length scales

As to the velocity scales (figure 2), in our flow problem the y-wise velocity scale $U_y$ is of the same order of (but smaller than) the x-wise velocity scale $U_x$. Since flow on planetary boundary layer scale approximately is horizontal ($U_z$ is smaller than $U_x$ and $U_y$), inflow in x-direction is mainly balanced by outflow in y-direction. In other words: the horizontal velocity scales $U_x$ and $U_y$ are of opposite sign. A scale analysis of the continuity equation (table 2) now gives an upper bound for $U_z$:

$$U_z \ll \frac{L_z}{L_x} U_x.$$ 

In the following $U_z$ is an order of magnitude smaller than this upper bound. Yet another velocity scale is the turbulence velocity scale $u$, which is of the order of 10% of a horizontal velocity scale. The typical values of the velocity scales are presented in table 1.

**Table 2.** Scale analysis of the continuity equation

| Continuity | $\frac{\partial \bar{u}}{\partial x}$ | $\frac{\partial \bar{v}}{\partial y}$ | $\frac{\partial \bar{w}}{\partial z}$ |
|------------|-------------------------------------|-------------------------------------|-------------------------------------|
| Scale      | $U_x$ | $U_y$ | $U_z$ |
| Scale      | $L_x$ | $L_y$ | $L_z$ |
| Magnitude [1/s] | $10^{-4}$ | $10^{-4}$ | $\ll 10^{-4}$ |
Figure 2. Length and velocity scales in the planetary boundary layer with wind farming

Table 3. Scale analysis of the momentum equation

| x-Momentum | \(\frac{\partial u}{\partial t}\) | u \(\frac{\partial u}{\partial x}\) | v \(\frac{\partial u}{\partial y}\) | w \(\frac{\partial u}{\partial z}\) | \(\frac{1}{\rho} \frac{\partial p}{\partial x}\) | f_y u | \(\frac{\partial u'u'}{\partial x}\) | \(\frac{\partial u'v'}{\partial y}\) | \(\frac{\partial u'w'}{\partial z}\) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-------|-----------------|-----------------|-----------------|
| Scale       | \(\frac{U_x}{\Delta T}\) | \(\frac{U_x}{L_x}\) | \(\frac{U_x}{L_y}\) | \(\frac{U_x}{L_z}\) | \(\frac{\Delta p}{\rho L_x}\) | f_y U_x | \(\frac{u^2}{L_x}\) | \(\frac{u^2}{L_y}\) | \(\frac{u^2}{L_z}\) |
| Magnitude [m/s^2] | 10^{-4} | 10^{-3} | 10^{-3} | 10^{-4} | 10^{-3} | 10^{-5} | 10^{-5} | 10^{-3} |

| y-Momentum | \(\frac{\partial v}{\partial t}\) | u \(\frac{\partial v}{\partial x}\) | v \(\frac{\partial v}{\partial y}\) | w \(\frac{\partial v}{\partial z}\) | \(\frac{1}{\rho} \frac{\partial p}{\partial y}\) | f_y u | \(\frac{\partial v'u'}{\partial x}\) | \(\frac{\partial v'v'}{\partial y}\) | \(\frac{\partial v'w'}{\partial z}\) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-------|-----------------|-----------------|-----------------|
| Scale       | \(\frac{U_y}{\Delta T}\) | \(\frac{U_y}{L_x}\) | \(\frac{U_y}{L_y}\) | \(\frac{U_y}{L_z}\) | \(\frac{\Delta p}{\rho L_y}\) | f_y U_y | \(\frac{u^2}{L_x}\) | \(\frac{u^2}{L_y}\) | \(\frac{u^2}{L_z}\) |
| Magnitude [m/s^2] | 10^{-4} | 10^{-3} | 10^{-3} | 10^{-4} | 10^{-3} | 10^{-5} | 10^{-5} | 10^{-3} |

| z-Momentum | \(\frac{\partial w}{\partial t}\) | u \(\frac{\partial w}{\partial x}\) | v \(\frac{\partial w}{\partial y}\) | w \(\frac{\partial w}{\partial z}\) | \(\frac{1}{\rho} \frac{\partial p}{\partial z}\) | \(\frac{\partial w'u'}{\partial x}\) | \(\frac{\partial w'v'}{\partial y}\) | \(\frac{\partial w'w'}{\partial z}\) |
|-------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| Scale       | \(\frac{U_z}{\Delta T}\) | \(\frac{U_z}{L_x}\) | \(\frac{U_z}{L_y}\) | \(\frac{U_z}{L_z}\) | \(\frac{g}{L_x}\) | \(\frac{u^2}{L_y}\) | \(\frac{u^2}{L_z}\) |
| Magnitude [m/s^2] | 10^{-7} | 10^{-6} | 10^{-6} | 10^{-7} | 10 | 10^{-5} | 10^{-5} | 10^{-3} |
The subsequent scale analysis of the momentum equations (1), (2) and (3) is presented in table 3. By neglecting the small terms, it follows that also in the case of wind farming the neutral planetary boundary layer equations are essentially steady and two-dimensional:

\[
\begin{align*}
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= + f_y (\tilde{v} - v) - \frac{\partial \tilde{u} \tilde{v}}{\partial y} - \frac{\partial \tilde{u} \tilde{w}}{\partial z} + \tilde{f}_x \\
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= - f_y (\tilde{u} - u) - \frac{\partial \tilde{v} \tilde{v}}{\partial y} - \frac{\partial \tilde{v} \tilde{w}}{\partial z} + \tilde{f}_y \\
\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} &= 0 ,
\end{align*}
\]

where \( u_g \) and \( v_g \) by definition are the components of the geostrophic velocity reached far from the surface. But in contrast to standard geostrophic flow the convective force and the \( y \)-wise turbulent momentum flux gradients are maintained, which rules out extra surface drag approaches. Also note that in contrast to general wake flow the Coriolis force and the vertical turbulent momentum flux gradients are maintained, ruling out self-similar wake approaches.

In contrast to the original system the equations (5), (6) and (7) constitute an overdetermined system: 3 equations with 2 unknowns. Although one might be tempted to discard the continuity equation, we retain it because doing so we can derive an explicit solution of the momentum equations in a given point. Since evidently the velocity must obey conservation of mass, the solution to the momentum equations must be corrected in such a way that the velocity satisfies continuity while remaining close to that solution. This is achieved by using the Lagrange multiplier method [9, section 7.7], which essentially provides the third unknown.

Another scale analysis, the Rossby-number similarity approach [10, section 3.2.1], shows that close to the surface the \( x \)-wise velocity component of geostrophic flow is much larger than the \( y \)-wise velocity component. At first approximation the near-surface wind is therefore in the \( x \)-direction. For this reason in the following the \( x \)-direction is referred to as the streamwise direction, and the \( y \)-direction as the spanwise direction. Note this outcome collaborates the definition of the length scales \( L_x \) and \( L_y \) above.

In order to close the momentum equations (5) and (6) we represent the turbulent momentum fluxes by a mean turbulent viscosity \( k_m \) in combination with gradients of the mean velocity components:

\[
\begin{align*}
\tilde{u}'v' &= -k_m \left( \frac{\partial \tilde{u}}{\partial y} + \frac{\partial \tilde{v}}{\partial x} \right) , \quad \tilde{u}'w' \approx -k_m \frac{\partial \tilde{u}}{\partial z} , \quad \tilde{v}'v' \approx -2k_m \frac{\partial \tilde{v}}{\partial y} \quad \text{and} \quad \tilde{v}'w' \approx -k_m \frac{\partial \tilde{v}}{\partial z} .
\end{align*}
\]

The approximations here originate from a scale analysis of the turbulent fluxes (table 4) employing the length and velocity scales introduced above (table 1). Note the equations are not closed completely because the mean turbulent viscosity remains; its parameterization is addressed in section 3.4.

The body forces in the momentum equations (5) and (6) require another form of closure because these forces, which represent wind turbines, ultimately depend on the horizontal velocity. We treat this in section 3.5.
Table 4. Scale analysis of the turbulent fluxes

| Turbulent fluxes | $\bar{u}'\bar{v}'$ | $\bar{u}'\bar{w}'$ | $\bar{v}'\bar{v}'$ | $\bar{v}'\bar{w}'$ | $\bar{w}'\bar{w}'$ |
|------------------|------------------|------------------|------------------|------------------|------------------|
| $\frac{\partial \bar{u}}{\partial y}$ | $\frac{\partial \bar{v}}{\partial x}$ | $\frac{\partial \bar{u}}{\partial z}$ | $\frac{\partial \bar{w}}{\partial x}$ | $\frac{\partial \bar{v}}{\partial y}$ | $\frac{\partial \bar{w}}{\partial z}$ |
| $\bar{U}_x$ | $\bar{U}_y$ | $\bar{U}_z$ | $\bar{U}_x$ | $\bar{U}_y$ | $\bar{U}_z$ |
| Scale | $L_x$ | $L_y$ | $L_z$ | $L_x$ | $L_y$ |
| Magnitude [1/s] | $10^{-4}$ | $10^{-4}$ | $10^{-2}$ | $10^{-7}$ | $10^{-4}$ | $10^{-2}$ | $10^{-7}$ |

By inserting the expressions (8) for the turbulent momentum fluxes into the momentum equations (5) and (6), and by applying the retained continuity equation (7), we obtain:

\[
-u \frac{\partial \bar{v}}{\partial y} + v \frac{\partial \bar{u}}{\partial y} = f_y (\bar{v} - \bar{v}_y) + \frac{\partial \bar{k}_m}{\partial y} \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) + k_m \left( \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{v}}{\partial y \partial x} + \frac{\partial^2 \bar{u}}{\partial z^2} \right) + \frac{\partial \bar{k}_m}{\partial z} \frac{\partial \bar{u}}{\partial z} + \bar{f}_x, \tag{9}
\]

\[
- u \frac{\partial \bar{v}}{\partial x} - v \frac{\partial \bar{u}}{\partial x} = f_x (\bar{u} - \bar{u}_x) + 2 \frac{\partial \bar{k}_m}{\partial y} \frac{\partial \bar{u}}{\partial y} + k_m \left( 2 \frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 \bar{v}}{\partial z^2} \right) + \frac{\partial \bar{k}_m}{\partial z} \frac{\partial \bar{v}}{\partial z} + \bar{f}_y, \tag{10}
\]

and

\[
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0. \tag{11}
\]

The momentum equations (9) and (10), together with the continuity equation (11) as the constraint, are the governing equations for neutral planetary boundary layer flow with wind farming.

3.2.2. Non-dimensional form

Next we introduce the horizontal length scale $D$ (proportional to the wind turbine rotor diameter), the vertical length scale $z_0$ (proportional to the surface roughness length), and the velocity scale $G$ (proportional to the geostrophic wind velocity), and define:

- the components $U$ and $V$ of the non-dimensional mean velocity:
  $\bar{u} = G \bar{U}$ and $\bar{v} = G \bar{V}$ with $G^2 = u_g^2 + v_g^2$;
- the non-dimensional coordinates $X$, $Y$ and $S$:
  $x = D X$, $y = D Y$ and $z = z_0 \exp(S)$,
  anticipating on a rectangular horizontal grid and a logarithmic vertical grid;
- the non-dimensional mean turbulent viscosity $K_m$:
  $\bar{k}_m = D G K_m$;
- the components $F_x$ and $F_y$ of the non-dimensional mean body force:
  $\bar{F}_x = \frac{G^2}{D} F_x$ and $\bar{F}_y = \frac{G^2}{D} F_y$. 

The Science of Making Torque from Wind IOP Publishing
Journal of Physics: Conference Series 75 (2007) 012043 doi:10.1088/1742-6596/75/1/012043

7
By inserting these definitions into the equations (9), (10) and (11) we obtain the non-dimensional form of the governing equations:

\[
- U \frac{\partial V}{\partial Y} + \left( V - \frac{\partial K_n}{\partial Y} \right) \frac{\partial U}{\partial Y} = \int_\alpha (V - V_s) + \frac{\partial K_n}{\partial Y} \frac{\partial V}{\partial Y} + K_n \frac{\partial^2 U}{\partial Y^2} + \frac{\partial^2 V}{\partial Y^2} + D_n \frac{\partial^2 U}{\partial S^2} + D_n \left( -K_n + \frac{\partial K_n}{\partial S} \right) \frac{\partial U}{\partial S} + F,
\]

(12)

\[
U \frac{\partial V}{\partial X} - \frac{\partial U}{\partial X} = -\int_\alpha (U - U_s) + 2 \frac{\partial K_n}{\partial Y} \frac{\partial U}{\partial Y} + 2K_n \frac{\partial^2 U}{\partial Y^2} + D_n \frac{\partial^2 V}{\partial S^2} + D_n \left( -K_n + \frac{\partial K_n}{\partial S} \right) \frac{\partial V}{\partial S} + F,
\]

(13)

where

\[
\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = 0
\]

(14)

and

\[
f = \frac{f D}{G} \quad \text{and} \quad D = \frac{D \exp(-S)}{z_0}
\]

are non-dimensional parameters.

3.3. Numerical representation

Anticipating on an explicit solution in a given point we discretize the equations (12) and (13) in the grid point \([i,j,k]\) with backward differences employing grid sizes \(\Delta X, \Delta Y, \text{ and } \Delta S\). Doing so, the non-linear terms cancel:

\[
- U \frac{\partial V}{\partial Y} + \left( V - \frac{\partial K_n}{\partial Y} \right) \frac{\partial U}{\partial Y} \approx U[i,j,k] \frac{V[i,j-1,k]}{\Delta Y} - V[i,j,k] \frac{U[i,j,k]}{\Delta Y} - \frac{\partial K_n}{\partial Y} \frac{U[i,j-1,k]}{\Delta Y} \cdot \frac{1}{\Delta Y} \left( V[i,j-1,k] - \frac{\partial K_n}{\partial Y} \right) U[i,j,k] - \frac{U[i,j-1,k]}{\Delta Y} \frac{V[i,j,k]}{\Delta Y} + \frac{\partial K_n}{\partial Y} \frac{U[i,j-1,k]}{\Delta Y}
\]

and

\[
U \frac{\partial V}{\partial X} - \frac{\partial U}{\partial X} \approx -\frac{V[i-1,j,k]}{\Delta X} U[i,j,k] + \frac{U[i-1,j,k]}{\Delta X} V[i,j,k],
\]

where the approximations originate from neglecting the truncation errors. Omitting the identifier \([i,j]\) for ease of notation, the resulting difference equations have the following form:
\[
A_i[k] U[k] + B_k[k] V[k] = C_i[k] U[k-1] + D_k[k] U[k-2] + E_i[k]
\]

\[
A_i[k] U[k] + B_k[k] V[k] = C_i[k] V[k-1] + D_k[k] V[k-2] + E_i[k];
\]

which is equivalent to:

\[
\begin{pmatrix}
D_k[k] & 0 \\
0 & D_k[k]
\end{pmatrix}
\begin{pmatrix}
U[k-2] \\
V[k-2]
\end{pmatrix}
+ \begin{pmatrix}
C_i[k] & 0 \\
0 & C_i[k]
\end{pmatrix}
\begin{pmatrix}
U[k-1] \\
V[k-1]
\end{pmatrix}
+ \begin{pmatrix}
-A_i[k] & -B_k[k] \\
-A_i[k] & -B_k[k]
\end{pmatrix}
\begin{pmatrix}
U[k] \\
V[k]
\end{pmatrix}
= \begin{pmatrix}
-E_i[k] \\
-E_i[k]
\end{pmatrix}.
\]  

(15)

The system (15) is the already announced explicit solution of the momentum equation in a given point. The coefficients A, B, C, D and E "only" depend on:

- the parameters of the numerical domain,
- the turbulent viscosity in the point \([i,j,k]\) and its spanwise and vertical gradients,
- the components of the body force, and
- the velocity components in the backward points \([i,j-1,k], [i-1,j,k], [i-1,j-1,k], [i,j-2,k]\).

Together this implies that if the turbulent viscosity is computed separately (see section 3.4), the body force is known (see section 3.5) and the indicated backward velocities are available, the system (15) allows for an implicit solution of U and V in a vertical at \([i,j]\) once appropriate boundary conditions are set.

At the bottom of the numerical domain \((k = 1: z = z_0)\) the boundary condition is no slip, so that, on omitting the identifier \([i,j]\) for ease of notation, we get:

\[
\begin{pmatrix}
C_i[3] & 0 \\
0 & C_i[3]
\end{pmatrix}
\begin{pmatrix}
U[2] \\
V[2]
\end{pmatrix}
+ \begin{pmatrix}
-A_i[3] & -B_i[3] \\
-A_i[3] & -B_i[3]
\end{pmatrix}
\begin{pmatrix}
U[3] \\
V[3]
\end{pmatrix}
= \begin{pmatrix}
-E_i[3] \\
-E_i[3]
\end{pmatrix}.
\]

At the top \((k = k_{max}: z = h_{geo})\) that is the height where the velocity first reaches the geostrophic value) the velocity is geostrophic, so that:

\[
\begin{pmatrix}
D_k[k_{max}] & 0 \\
0 & D_k[k_{max}]
\end{pmatrix}
\begin{pmatrix}
U[k_{max}-2] \\
V[k_{max}-2]
\end{pmatrix}
+ \begin{pmatrix}
C_i[k_{max}] & 0 \\
0 & C_i[k_{max}]
\end{pmatrix}
\begin{pmatrix}
U[k_{max}-1] \\
V[k_{max}-1]
\end{pmatrix}
= \begin{pmatrix}
-E_i[k_{max}] \\
-E_i[k_{max}]
\end{pmatrix} + \begin{pmatrix}
A_i[k_{max}] & B_i[k_{max}]
\end{pmatrix}
\begin{pmatrix}
U_g \\
V_g
\end{pmatrix}
\]

where \(U_g = u_g / G\) and \(V_g = v_g / G\) are the non-dimensional components of the geostrophic velocity.

By employing for ease of notation the matrix-vector equivalent of the system (15):

\[
D[k] U[k-2] + C[k] U[k-1] + A B[k] U[k] = E[k],
\]

the momentum system in the vertical at \([i,j]\) is:
\[ \mathbf{M} \mathbf{u} = \zeta ; \]  

with the square \(2(k_{\text{max}}-2) \times 2(k_{\text{max}}-2)\) matrix

\[
\mathbf{M} = \begin{pmatrix}
C[3] & AB[3] & 0 & 0 & \ldots & 0 & 0 & 0 \\
D[4] & C[4] & AB[4] & 0 & \ldots & 0 & 0 & 0 \\
0 & D[5] & C[5] & AB[5] & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & D[k_{\text{max}}-1] & C[k_{\text{max}}-1] & AB[k_{\text{max}}-1] \\
0 & 0 & 0 & 0 & \ldots & 0 & D[k_{\text{max}}] & C[k_{\text{max}}] \\
\end{pmatrix}
\]

and the vectors

\[
\mathbf{u} = \begin{pmatrix}
U[2] \\
U[3] \\
U[4] \\
\vdots \\
U[k_{\text{max}}-2] \\
U[k_{\text{max}}-1]
\end{pmatrix}
\quad \text{and} \quad
\zeta = \begin{pmatrix}
E[3] \\
E[4] \\
E[5] \\
\vdots \\
E[k_{\text{max}}-1] \\
E[k_{\text{max}}] - AB[k_{\text{max}}] \mathbf{U}
\end{pmatrix}.
\]

Since the velocity depends on the turbulent viscosity, the solution procedure iterates between solving the matrix-vector system (16) and computing the turbulent viscosity profile. The procedure starts in the vertical at \([i+1,j+2]\) so that initial velocity conditions are needed at the inlet plane \(i = 1\) and the two planes at \(j = 1\) and \(j = 2\), and initial turbulent viscosity conditions also at \([i,j]\). Once a vertical profile of \(U\) and \(V\) is computed in \([i,j]\), the procedure proceeds with the profiles in \([i,j+1]\), \([i,j+1]\) etc. until \(j_{\text{max}}\) is reached. Next follow the planes at \(i+1\), \(i+2\) etc up to \(i_{\text{max}}\). The solution therefore marches in the two horizontal directions.

### 3.4. Turbulence parameterisation

The mean turbulent viscosity \(k_{u}\) in the momentum equations (9) and (10) is parameterized by using the algebraic Baldwin-Lomax model [11, section 3.4.2]:

\[
k_{u} = \min[k_{u}, k_{u}, \bar{\omega}],
\]

where \(k_{u}\) and \(k_{u}\) are the inner and outer layer turbulent viscosity, respectively.

The inner layer turbulent viscosity depends on the mixing length \(\lambda_{\text{mix}}\) and the mean vorticity \(\bar{\omega}\):

\[
k_{u} = \lambda_{\text{mix}}^{2} \bar{\omega} \quad \text{with} \quad \lambda_{\text{mix}} = \kappa \left[ 1 - \exp \left( \frac{z}{A_{\omega}} \right) \right] \quad \text{and} \quad \bar{\omega} \approx \left( \frac{\partial \bar{u}}{\partial y} - \frac{\partial \bar{v}}{\partial x} \right)^{2} + \left( \frac{\partial \bar{u}}{\partial z} \right)^{2} + \left( \frac{\partial \bar{v}}{\partial z} \right)^{2},
\]
where

\[ A_v = \frac{v}{u^*} \]

is the Van Driest coefficient rescaled to the surface roughness length \( z_0 \); \( v \) is the kinematic viscosity and \( u^* \) is the friction velocity. Note the model uses the Van Driest function in order to limit the growth of the mixing length.

The outer layer turbulent viscosity \( \overline{k_{m,v}} \) depends on the wake turbulent viscosity \( \overline{k_{n,v}} \) and employs the Klebanoff function so that the turbulent viscosity decreases with distance from the surface in the outer layer:

\[
\overline{k_{m,v}} = \alpha C_c \left[ 1 + C_{k1} \left( \frac{C_{k1}}{h_{\text{mix}}} \right)^3 \right] \overline{k_{n,v}}.
\]

The wake turbulent viscosity is the smaller of two viscosities that are defined on basis of a turbulent mixing velocity \( \nu_{\text{mix}} \), the distance \( h_{\text{mix}} \) above the surface where the turbulent mixing velocity is largest, and the velocity \( U_{\text{mix}} \) at \( h_{\text{mix}} \):

\[
\overline{k_{n,v}} = \min \left[ \overline{k_{m,v1}}, \overline{k_{m,v2}} \right] \quad \text{with} \quad \overline{k_{m,v1}} = \nu_{\text{mix}} h_{\text{mix},\text{max}} \quad \text{and} \quad \overline{k_{m,v2}} = C_w h_{\text{mix}} \frac{U_{\text{mix}}}{\nu_{\text{mix,\text{max}}}},
\]

where

\[
\nu_{\text{mix,\text{max}}} = \frac{1}{\kappa} \max \left[ \nu_{\text{mix}} \right] \quad \text{with} \quad \nu_{\text{mix}} = \frac{\lambda_{\text{mix}} \omega}{\kappa} \quad \text{and} \quad U_{\text{mix}} = \overline{u} \left( h_{\text{mix}} \right) + \overline{v} \left( h_{\text{mix}} \right).
\]

Note that although the resulting turbulent viscosity is zero at the surface it has a small but non-zero value at the bottom of the numerical domain (\( k = 1: z = z_0 \)). Also at the top of this domain, where the velocity first reaches the geostrophic value, turbulent viscosity is small but non-zero.

**Table 5. Coefficients in the Baldwin-Lomax model**

| \( \kappa \) | \( \alpha \) | \( C_c \) | \( C_{k1} \) | \( C_{k2} \) | \( C_w \) | \( A_v^* \) |
|---|---|---|---|---|---|---|
| 0.4 | 0.0168 | 1.6 | 0.3 | 5.5 | 1 | 26 |

For the moment the Baldwin-Lomax algebraic parameterization is preferred over one or two-equation parameterizations (like the \( \kappa \epsilon \) or the \( \k_\omega \) model) because, apart from being simple to implement, it is a turbulent boundary layer parameterization containing the elementary physics (mean vorticity as the primary source of turbulent viscosity so that it varies with vertical as well as streamwise and spanwise velocity gradients).
3.5. Wind turbine parameterization

The body force in the momentum equation is the force exerted on the flow by wind turbines, which apart from the sign is equal to the thrust on the rotors. The rotor thrust essentially depends on the velocity induced by the rotor of a wind turbine, which in turn is related to the power production and thus to the horizontal velocity at hub height. Evidently the body force is only present in a grid point where a wind turbine rotor is located or the grid points where a rotor is interpolated; it is zero in all other grid points.

The body force due to a wind turbine is determined as follows. First, the vertical profile of the velocity components $U$ and $V$ is computed with the body force set to zero, yielding the "undisturbed" horizontal velocity $U_{hub}$ at hub height and the corresponding direction $\varphi_{hub}$:

$$U_{hub} = \vec{u}(h_{hub}) + \vec{v}(h_{hub}) \quad \text{and} \quad \varphi_{hub} = \arctan\left(\frac{u(h_{hub})}{v(h_{hub})}\right).$$

Then, by using the power curve $P(U)$ of the wind turbine, via the implicit relation between the power coefficient and the induction factor, the induction factor $a$ is computed:

$$a \left(1-a\right) = \frac{2 \rho}{\pi} \frac{P(U_{hub})}{U_{hub}^2 D^2}.$$

The induction factor subsequently gives the streamwise and the spanwise component of the non-dimensional body force:

$$f_x = f \cos(\varphi_{hub}) \quad \text{and} \quad f_y = f \sin(\varphi_{hub}) \quad \text{with} \quad f = \frac{T}{\rho G^2 D^2} = \frac{\pi}{8} \left(\frac{U_{hub}}{G}\right)^2 C_T = \frac{\pi}{2} \left(\frac{U_{hub}}{G}\right)^2 a(1-a),$$

which components are interpolated between the relevant grid points. Note that alternatively the thrust coefficient can be computed directly if the thrust curve $C_T(U)$ of the wind turbine is available.

3.6. Initial and boundary conditions

3.6.1. Boundary conditions Mean velocity boundary conditions have already been introduced in section 3.3.1: zero at the bottom of the numerical domain (corresponding to $z_0$) and geostrophic at its top (corresponding to $h_{geo}$). The boundary conditions for the mean turbulent viscosity are included in its parameterization (see section 3.4): vanishing but non-zero at both the bottom and the top of the numerical domain.

3.6.2. Initial conditions Velocity initial conditions are needed in the inlet plane $i = 1$ and in the planes $j = 1$ and $j = 2$. These are inspired by the Rossby-number similar planetary boundary layer velocity profiles [10, section 3.2.1], and comprise a logarithmic profile for the non-dimensional streamwise velocity

$$U(z) = \left(1 - \frac{\ln(z/h_{geo})}{\ln(z_i/h_{geo})}\right) \frac{v_z}{G}$$

(17)
and a linear profile for the non-dimensional spanwise velocity:

$$V(z) = \frac{z - z_s}{h_{geo} - z_s} G$$

where $z$ is the distance from the surface.

The initial profile for the mean turbulent viscosity, also needed in the vertical at $[i, j]$, is obtained by applying the Baldwin-Lomax model to the profiles (17) and (18).

4. Summary and future

It has been shown that neutral planetary boundary layer flow with wind farming essentially is steady and two-dimensional; and that the convective forces, the Coriolis forces and the vertical and spanwise gradients of the turbulent momentum fluxes all have the same order of magnitude. In addition it has been shown that a numerical representation in the form of backward differences allows for an implicit solution of the two horizontal velocity components in vertical direction, iterating on the turbulent viscosity, and a marching solution in the horizontal directions.

Subsequently the solution procedure will be further developed. Insights that will be derived from the method will include the modification of the wind profile due to a hypothetical upstream wind farm, particularly the impact of separation distance from and layout (spacing, hub height) of the wind farm.

Acknowledgements

This work was performed in the framework of the Dutch Ministry of Economic Affairs BSIK programme We@Sea, project "Windenergiecentrale Noordzee - Parkinteractie" (We@Sea/BSIK 2005/002).

References

[1] Christiansen M B and Hasager C B 2005 Wake studies around a large offshore wind farm using satellite and airborne SAR In: 31st Int. Symp on Remote Sensing of Environment, St Petersburg, Russian Federation
[2] Frandsen S et al. 2004 The necessary distance between large wind farms offshore - Study Risø National Laboratory, Report Risø-R-1518(EN)
[3] Hegberg T 2004 Turbine interaction in large offshore wind farms - Atmospheric boundary layer above a wind farm Report ECN-C--04-033
[4] Hegberg T 2002 The effect of large wind farms on the atmospheric boundary layer In: Proc. Global Wind Power Conference 2002, Paris, France
[5] Liu M-K et al. 1983 Mathematical model for the analysis of wind-turbine wakes J. Energy, Vol. 7, No. 1, pp. 73-78
[6] Baidya Roy S et al. 2004 Can large wind farms affect local meteorology? J. Geoph. Research, Vol. 109, D19101
[7] Rooijmans P 2004 Impact of a large-scale offshore wind farm on meteorology - Numerical simulations with a mesoscale circulation model Universiteit Utrecht, Masters thesis
[8] Holton J R 1992 An Introduction to dynamic meteorology (3rd ed.) Academic Press
[9] Ferziger J H and Peric M 1997 Computational methods for fluid dynamics (2nd ed.) Springer
[10] Garratt J R 1994 The atmospheric boundary layer Cambridge University Press
[11] Willeox D C 1998 Turbulence modelling for CFD (2nd ed.) DCW Industries Inc.