The Geodesic Diameter of Polygonal Domains

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\textbf{Abstract}

This paper studies the geodesic diameter of polygonal domains having $h$ holes and $n$ corners. For simple polygons (i.e., $h = 0$), it is known that the geodesic diameter is determined by a pair of corners of a given polygon and can be computed in linear time. For general polygonal domains with $h \geq 1$, however, no algorithm for computing the geodesic diameter was known prior to this paper. In this paper, we present first algorithms that compute the geodesic diameter of a given polygonal domain in worst-case time $O(n^{7.73})$ or $O(n^7 \log n + h))$. The algorithms are based on our new geometric observations, part of which states as follows: the geodesic diameter of a polygonal domain can be determined by two points in its interior, and in that case there are at least five shortest paths between the two points.

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1 Introduction

A polygonal domain \( P \) with \( h \) holes and \( n \) corners \( V \) is a connected and closed subset of \( \mathbb{R}^2 \) of genus \( h \) whose boundary \( \partial P \) consists of \( h + 1 \) simple closed polygonal chains of \( n \) total line segments. The holes and the outer boundary of \( P \) are regarded as obstacles so that any feasible path in \( P \) is not allowed to cross the boundary \( \partial P \). The geodesic distance \( d(p, q) \) between any two points \( p, q \) in a polygonal domain \( P \) is defined as the (Euclidean) length of a shortest obstacle-avoiding path between \( p \) and \( q \).

In this paper, we address the geodesic diameter problem in polygonal domains. The geodesic diameter \( \text{diam}(P) \) of a polygonal domain \( P \) is defined as \( \text{diam}(P) := \max_{s,t \in P} d(s,t) \). A pair \((s,t)\) of points in \( P \) that realizes the geodesic diameter \( \text{diam}(P) \) is called a diametral pair. The geodesic diameter problem is to find the value of \( \text{diam}(P) \) and a diametral pair.

For simple polygons (i.e., \( h = 0 \)), the geodesic diameter has been extensively studied and fully understood. Chazelle [6] provided the first \( O(n^2) \)-time algorithm computing the geodesic diameter of a simple polygon, and Suri [18] presented an \( O(n \log n) \)-time algorithm that solves the all-geodesic-farthest neighbors problem, computing the farthest neighbor of every corner and thus finding the geodesic diameter. At last, Hershberger and Suri [11] showed that the diameter can be computed in linear time using their fast matrix search technique.

On the other hand, to the best of our knowledge, no algorithm for computing \( \text{diam}(P) \) has yet been discovered when \( P \) is a polygonal domain having one or more holes \((h \geq 1)\). Mitchell [14] has posed an open problem asking an algorithm for computing the geodesic diameter \( \text{diam}(P) \). However, even for the corner-to-corner diameter \( \max_{u,v \in V} d(u,v) \), only known is a brute-force algorithm that takes \( O(n^2 \log n) \) time, checking all the geodesic distances between every pair of corners.\(^1\)

This fairly wide gap between simple polygons and polygonal domains is seemingly due to the uniqueness of the shortest path between any two points; it is well known that there is a unique shortest path between any two points in a simple polygon [9]. Using this uniqueness, one can show that the diameter is indeed realized by a pair of corners in \( P \); that is, \( \text{diam}(P) = \max_{u,v \in V} d(u,v) \) if \( h = 0 \) [11,18]. For general polygonal domains with \( h \geq 1 \), however, this is not the case. In this paper, we exhibit several examples where the diameter is realized by non-corner points on \( \partial P \) or even by interior points of \( P \). (See Figure 1 and Appendix A.) This observation also shows an immediate difficulty in devising any exhaustive algorithm since the search space like \( \partial P \) or the whole domain \( P \) is not discrete.

The status of the geodesic center problem is also similar. The geodesic center is defined to be a point in \( P \) that minimizes the maximum geodesic distance from it to any other point of \( P \). Asano and Toussaint [3] introduced the first \( O(n^4 \log n) \)-time algorithm for computing the geodesic center of a simple polygon, and Pollack, Sharir and Rote [17] improved it to \( O(n \log n) \) time. As with the diameter problem, there is no known algorithm for general polygonal domains. Note that computing the geodesic center involves computing the geodesic diameter because the geodesic center may be determined by the midpoint of a shortest path defining the geodesic diameter. See O’Rourke and Suri [16] and Mitchell [14] for more references on the geodesic diameter/center problem in simple polygons and polygonal domains.

Since the geodesic diameter/center of a simple polygon is determined by its corners, one can exploit the geodesic farthest-site Voronoi diagram of the corners \( V \) to compute the diameter/center, which can be built in \( O(n \log n) \) time [2]. Recently, Bae and Chwa [4] presented an \( O(nk \log^3(n + k)) \)-time algorithm for computing the geodesic farthest-site Voronoi diagram of \( k \) sites in a general polygonal domain. This can be used to compute the geodesic diameter \( \max_{p,q \in S} d(p,q) \) of a finite set \( S \) of points in \( P \), but cannot be exploited for computing \( \text{diam}(P) \) without any characterization of the geodesic diameter of polygonal domains with \( h \geq 1 \). Moreover, when \( S = V \), this approach is no better than the brute-force \( O(n^2 \log n) \)-time algorithm for computing the corner-to-corner diameter \( \max_{u,v \in V} d(u,v) \).

In this paper, we present the first algorithms that compute the geodesic diameter of a given polygonal domain in \( O(n^{7.73}) \) or \( O(n^7 (\log n + h)) \) time in the worst case. We also show that for small constant \( h \)

\(^1\)Personal communication with Mitchell.
the time bound has been improved to $O$. Schevon \cite{15} proved that if the geodesic diameter on a convex domains, $\left\| \cdot \right\|$ for fixed neighborhoods, and the boundary $\partial A$ of a set $A$; unless stated otherwise, all of them are supposed to be derived with respect to the standard topology on $\mathbb{R}^d$ with the Euclidean norm $\| \cdot \|$ for fixed $d \geq 1$. We denote the straight line segment joining two points $a, b$ by $\overline{ab}$.

We are given as input a polygonal domain $\mathcal{P}$ with $h$ holes and $n$ corners. More precisely, $\mathcal{P}$ consists of an outer simple polygon in the plane $\mathbb{R}^2$ and a set of $h$ ($\geq 0$) disjoint simple polygons inside the outer polygon. As a subset of $\mathbb{R}^2$, $\mathcal{P}$ is the region contained in its outer polygon excluding the interior of the holes; thus $\mathcal{P}$ is a bounded, closed subset of $\mathbb{R}^2$. The boundary $\partial \mathcal{P}$ of $\mathcal{P}$ is regarded as a series of obstacles so that any feasible path inside $\mathcal{P}$ is not allowed to cross $\partial \mathcal{P}$. Note that some portion or the whole of a feasible path may go along the boundary $\partial \mathcal{P}$. The length of a path is the sum of the Euclidean lengths of its segments. It is well known from earlier work that there always exists a shortest (feasible) path between any two points $p, q \in \mathcal{P}$ \cite{13}. The geodesic distance, denoted by $d(p, q)$, is then defined to be the length of a shortest path between $p \in \mathcal{P}$ and $q \in \mathcal{P}$.

**Shortest path map.** Let $V$ be the set of all corners of $\mathcal{P}$ and $\pi(s, t)$ be a shortest path between $s \in \mathcal{P}$ and $t \in \mathcal{P}$. Then, it is represented as a sequence $\pi(s, t) = (s, v_1, \ldots, v_k, t)$ for some $v_1, \ldots, v_k \in V$; that is, a polygonal chain through a sequence of corners \cite{13}. Note that possibly we may have $k = 0$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Three polygonal domains where the geodesic diameter is determined by a pair $(s^*, t^*)$ of non-corner points; Gray-shaded regions depict the interior of the holes and dark gray segments depict the boundary $\partial \mathcal{P}$. Recall that $\mathcal{P}$, as a set, contains its boundary $\partial \mathcal{P}$. (a) Both $s^*$ and $t^*$ lie on $\partial \mathcal{P}$. There are three shortest paths between $s^*$ and $t^*$. In this polygonal domain, there are two (symmetric) diametral pairs. (b) $s^* \in \partial \mathcal{P} \setminus V$ and $t^* \in \text{int} \mathcal{P}$. Three triangular holes are placed in a symmetric way. There are four shortest paths between $s^*$ and $t^*$. (c) Both $s^*$ and $t^*$ lie in the interior $\text{int} \mathcal{P}$. Here, the five holes are packed like jigsaw puzzle pieces, forming narrow corridors (dark gray paths) and two empty, regular triangles. Observe that $d(u_1, v_1) = d(u_1, v_2) = d(u_2, v_2) = d(u_2, v_3) = d(u_3, v_3) = d(v_3, v_1)$. $s^*$ and $t^*$ lie at the centers of the triangles formed by the $u_i$ and the $v_i$, respectively. There are six shortest paths between $s^*$ and $t^*$. More details on this example can be found in Appendix A.2.}
\end{figure}

2 Preliminaries

Throughout the paper, we frequently use several topological concepts such as open and closed subsets, neighborhoods, and the boundary $\partial A$ and the interior $\text{int} A$ of a set $A$; unless stated otherwise, all of them are supposed to be derived with respect to the standard topology on $\mathbb{R}^d$ with the Euclidean norm $\| \cdot \|$ for fixed $d \geq 1$. We denote the straight line segment joining two points $a, b$ by $\overline{ab}$.
In this section, we give some analysis on local maxima of the lower envelope of convex functions, which provides a key observation for our further discussions on the geodesic diameter and diametral pairs.

The shortest path map SPM(s) for a fixed \( s \in \mathcal{P} \) is a decomposition of \( \mathcal{P} \) into cells such that every point in a common cell can be reached from \( s \) by shortest paths of the same combinatorial structure. Each cell \( \sigma_s(v) \) of SPM(s) is associated with a corner \( v \in V \) or \( s \) itself, which is the last corner of \( \pi(s, t) \) for any \( t \) in the cell \( \sigma_s(v) \). In particular, the cell \( \sigma_s(s) \) is the set of points \( t \) such that \( \pi(s, t) \) passes through no corner in \( V \) and thus \( d(s, t) = \|s - t\| \). Each edge of SPM(s) is an arc on the boundary of two incident cells \( \sigma_s(v_1) \) and \( \sigma_s(v_2) \) and thus determined by two corners \( v_1, v_2 \in V \cup \{s\} \). Similarly, each vertex of SPM(s) is determined by at least three corners \( v_1, v_2, v_3 \in V \cup \{s\} \). Note that for fixed \( s \in \mathcal{P} \) a point that locally maximizes \( d_s(t) := d(s, t) \) lies at either (1) a vertex of SPM(s), (2) an intersection between the boundary \( \partial \mathcal{P} \) and an edge of SPM(s), or (3) a corner in \( V \).

The shortest path map SPM(s) has \( O(n) \) complexity can be computed in \( O(n \log n) \) time using \( O(n \log n) \) working space \(^{[12]}\). For more details on shortest path maps, see \(^{[12]}-[14]\).

### Path-length function

If \( \pi(s, t) \neq \overline{st} \), then there are two corners \( u, v \in V \) such that \( \pi(s, t) \) is formed as the union of a shortest path from \( u \) to \( v \) and two segments \( su \) and \( tv \). Note that \( u \) and \( v \) are not necessarily distinct. In order to realize such a path, we assert that \( s \) is visible from \( u \) and \( t \) is visible from \( v \); thus, \( s \in \text{VP}(u) \) and \( t \in \text{VP}(v) \), where \( \text{VP}(p) \) for any \( p \in \mathcal{P} \) is defined to be the set of all points \( q \in \mathcal{P} \) such that \( pq \subset \mathcal{P} \). The set \( \text{VP}(p) \) is also called the visibility profile of \( p \) in \( \mathcal{P} \) \(^{[7]}\).

We now define the path-length function \( \text{len}_{u,v} : \text{VP}(u) \times \text{VP}(v) \to \mathbb{R} \) for any fixed pair of corners \( u, v \in V \) to be

\[
\text{len}_{u,v}(s, t) := \|s - u\| + d(u, v) + \|v - t\|.
\]

Then, \( \text{len}_{u,v}(s, t) \) represents the length of the path from \( s \) to \( t \) that has the fixed combinatorial structure, entering \( u \) from \( s \) and exiting \( v \) to \( t \). Also, unless \( d(s, t) = \|s - t\| \) (equivalently, \( s \in \text{VP}(t) \)), the geodesic distance \( d(s, t) \) can be expressed as the pointwise minimum of some path-length functions:

\[
d(s, t) = \min_{u \in \text{VP}(s), \ v \in \text{VP}(t)} \text{len}_{u,v}(s, t).
\]

Consequently, we have two possibilities for a diametral pair \( (s^*, t^*) \); either we have \( d(s^*, t^*) = \|s^* - t^*\| \) or the pair \( (s^*, t^*) \) is a local maximum of the lower envelope of several path-length functions.

### 3 Local Maxima on the Lower Envelope of Convex Functions

In this section, we give some analysis on local maxima of the lower envelope of convex functions, which provides a key observation for our further discussions on the geodesic diameter and diametral pairs.

We start with a basic observation on the intersection of hemispheres on a unit sphere in the \( d \)-dimensional space \( \mathbb{R}^d \). For any fixed positive integer \( d \), let \( S^{d-1} := \{x \in \mathbb{R}^d \mid \|x\| = 1\} \) be the unit sphere in \( \mathbb{R}^d \) centered at the origin. A closed (or open) hemisphere on \( S^{d-1} \) is defined to be the intersection of \( S^{d-1} \) and a closed (open, respectively) half-space of \( \mathbb{R}^d \) bounded by a hyperplane that contains the origin. We call a \( k \)-dimensional affine subspace of \( \mathbb{R}^d \) a \( k \)-flat. Note that a hyperplane in \( \mathbb{R}^d \) is a \( (d - 1) \)-flat and a line in \( \mathbb{R}^d \) is a 1-flat. Also, the intersection of \( S^{d-1} \) and a \( k \)-flat through the origin in \( \mathbb{R}^d \) is called a great \( (k - 1) \)-sphere on \( S^{d-1} \). Note that a great 1-sphere is called a great circle and a great 0-sphere consists of two antipodal points. Then, we observe the following.

**Lemma 1** For any two positive integers \( d \) and \( m \leq d \), a set of any \( m \) closed hemispheres on \( S^{d-1} \) has a nonempty common intersection. Moreover, if the intersection has an empty interior relative to \( S^{d-1} \), then it includes a great \( (d - m) \)-sphere on \( S^{d-1} \).

**Proof.** Proof can be found in Appendix B.

Using Lemma 1 we prove the following theorem, which is the goal of this section.
Theorem 1 Let $F$ be a finite family of real-valued convex functions defined on an open and convex subset $C \subseteq \mathbb{R}^d$ and $g(x) := \min_{f \in F} f(x)$ be their pointwise minimum. Suppose that $g$ attains a local maximum at $x^* \in C$ and there are exactly $m$ functions $f_1, \ldots, f_m \in F$ such that $m \leq d$ and $f_i(x^*) = g(x^*)$ for all $i = 1, \ldots, m$. If none of the $f_i$ attains a local minimum at $x^*$, then there exists a $(d + 1 - m)$-flat $\varphi \subset \mathbb{R}^d$ through $x^*$ such that $g$ is constant on $\varphi \cap U$ for some neighborhood $U \subset \mathbb{R}^d$ of $x^*$ with $U \subset C$.

Proof. Let $x^* \in C$ and $m$ be as in the statement. For each $i$, consider the sublevel set $L_i := \{x \in C \mid f_i(x) \leq f_i(x^*)\}$. Since each $f_i$ is convex and $x^*$ does not minimize $f_i$, the set $L_i$ is convex and $x^*$ lies on the boundary $\partial L_i$ of $L_i$. Therefore, there exists a supporting hyperplane $h_i$ to $L_i$ at $x^*$. Denote by $h_i^\ominus$ the closed half-space that is bounded by $h_i$ and does not contain $L_i$. Note that $f_i(x^*) \leq f_i(x)$ for any $x \in h_i^\ominus \cap C$ and $f_i(x^*) < f_i(x)$ for any $x \in (h_i^\oplus \setminus h_i) \cap C$. Let $H_i := \{x - x^* \mid x \in h_i^\ominus, \|x - x^*\| = 1\}$ be a closed hemisphere on the unit sphere $S^{d-1}$ centered at the origin.

Since $g(x^*) = f_i(x^*)$ for any $i \in \{1, \ldots, m\}$ and $x^*$ is a local maximum of $g$, the intersection $\bigcap H_i$ has an empty interior relative to $S^{d-1}$; otherwise, there exists $y \in S^{d-1}$ such that $f_i(x^* + \lambda y) > f_i(x^*)$ for any $i \in \{1, \ldots, m\}$ and any $\lambda > 0$ with $x^* + \lambda y \in C$. Hence, by Lemma 1, $\bigcap H_i$ has a nonempty intersection including a great $(d - m)$-sphere $G$ on $S^{d-1}$. Let $\varphi$ be the corresponding $(d - m + 1)$-flat in $\mathbb{R}^d$ through $x^*$ defined as $\varphi := \{x^* + \lambda y \in \mathbb{R}^d \mid y \in G \text{ and } \lambda \in \mathbb{R}\}$. Consider the restriction $f_i|_{\varphi \cap C}$ of $f_i$ on $\varphi \cap C$. Since $f_i$ is convex and $\varphi$ is an affine subspace (thus convex), $f_i|_{\varphi \cap C}$ is also convex and their pointwise minimum $g|_{\varphi \cap C}$ attains a local maximum at $x^*$. Furthermore, each $f_i|_{\varphi \cap C}$ attains a local minimum at $x^*$; since $\varphi \subseteq h_i^\ominus$, we have $f_i(x^*) \leq f(x)$ for any point $x \in \varphi \cap C$. Hence, $g|_{\varphi \cap C}$ also attains a local minimum at $x^*$ since $g(x^*) = f_i(x^*)$ for any $i \in \{1, \ldots, m\}$. Consequently, $g$ is locally constant at $x^*$ on $\varphi$; more precisely, there is a sufficiently small neighborhood $U \subset \mathbb{R}^d$ of $x^*$ with $U \subset C$ such that $g$ is constant on $U \cap \varphi$, completing the proof. \]

Remark that the theorem should have its own interest and find an application in problems of maximizing the pointwise minimum of several convex functions.

4 Properties of Geodesic-Maximal Pairs

We call a pair $(s^*, t^*) \in \mathcal{P} \times \mathcal{P}$ maximal if $(s^*, t^*)$ is a local maximum of the geodesic distance function $d$. That is, $(s^*, t^*)$ is maximal if and only if there are two neighborhoods $U_s, U_t \subset \mathbb{R}^2$ of $s^*$ and of $t^*$, respectively, such that for any $s \in U_s \cap \mathcal{P}$ and any $t \in U_t \cap \mathcal{P}$ we have $d(s^*, t^*) \geq d(s, t)$. For any pair $(s, t)$, let $\Pi(s, t) = \{\pi_1, \ldots, \pi_m\}$ be the set of all distinct shortest paths from $s$ to $t$, where $m$ denotes the number of shortest paths. Let $u_i$ and $v_i$ be the first and the last corners in $V$ along $\pi_i$ from $s$ to $t$, and let $V_s := \{u_1, \ldots, u_m\}$ and $V_t := \{v_1, \ldots, v_m\}$.

Let $E$ be the set of all sides of $\mathcal{P}$ without their endpoints and $B$ be their union. Note that $B = \partial \mathcal{P} \setminus V$, the boundary of $\mathcal{P}$ except the corners $V$. The goal of this section is to prove the following theorem, which is the main geometric result of this paper.

Theorem 2 Suppose that $(s^*, t^*)$ is a maximal pair in $\mathcal{P}$ and $\Pi(s^*, t^*)$, $V_s^*$, and $V_t^*$ be defined as above. Then, we have the following implications.

(VV) $s^* \in V$, $t^* \in V$ implies $|\Pi(s^*, t^*)| \geq 1, |V_s^*| \geq 1, |V_t^*| \geq 1$;

(VB) $s^* \in V$, $t^* \in B$ implies $|\Pi(s^*, t^*)| \geq 2, |V_s^*| \geq 1, |V_t^*| \geq 2$;

(VI) $s^* \in V$, $t^* \in \text{int} \mathcal{P}$ implies $|\Pi(s^*, t^*)| \geq 3, |V_s^*| \geq 1, |V_t^*| \geq 3$;

(BB) $s^* \in B$, $t^* \in B$ implies $|\Pi(s^*, t^*)| \geq 3, |V_s^*| \geq 2, |V_t^*| \geq 2$;

(BI) $s^* \in B$, $t^* \in \text{int} \mathcal{P}$ implies $|\Pi(s^*, t^*)| \geq 4, |V_s^*| \geq 2, |V_t^*| \geq 3$;

(II) $s^* \in \text{int} \mathcal{P}$, $t^* \in \text{int} \mathcal{P}$ implies $|\Pi(s^*, t^*)| \geq 5, |V_s^*| \geq 3, |V_t^*| \geq 3$.

Moreover, each of the above bounds is best possible by examples.
In the following, we thus define the sequence collinear. In this degenerate case, the path-length functions under the degenerate case. If \((u, v)\) minimum at \(C\) first three corners are collinear (b) For points in a small disk \(B\) centered at \(s^*\) with \(B \subset \text{VP}(u_i^*) \cup \text{VP}(u_i)\), the function \(h_i^*\) measures the length of the shortest path from \(u_i^*\) to each.

To see the tightness of the bounds, we present examples with remarks in Figure 1 and Appendix A. In particular, one can easily see the tightness of the bounds on any two corners \(u, v \in V\). This assumption does not affect Theorem 2 since multiple shortest paths between corners in \(V\) can only increase \(|\Pi(s^*, t^*)|\). Note that this assumption implies that the pairs \((u_i, v_i)\) are distinct, while the \(u_i\) (also the \(v_i\)) are not necessarily distinct. We thus have \(|V_{s^*}| \leq m\), \(|V_{t^*}| \leq m\), and \(|\{(u_i, v_i) \mid 1 \leq i \leq m\}| = m\), where \(m = |\Pi(s^*, t^*)|\).

The following lemma proves the bounds on \(|V_{s^*}|\) and \(|V_{t^*}|\) of Theorem 2. Proofs of the lemmas presented in this section can be found in Appendix B.

**Lemma 2** Let \((s^*, t^*)\) be a maximal pair. Then, \(|V_{s^*}| \geq 2\) if \(t^* \in B\); \(|V_{t^*}| \geq 3\) if \(t^* \in \text{int}\(\mathcal{P}\)\). Moreover, if \(t^* \in e \in E\), then there exists \(v \in V_{s^*}\) such that \(v\) is off the line supporting \(e\); if \(t^* \in \text{int}\(\mathcal{P}\)\), then \(t^*\) lies in the interior of the convex hull of \(V_{s^*}\).

**Lemma 2** immediately implies the lower bound on \(|\Pi(s^*, t^*)|\) when \(s^* \in V\) or \(t^* \in V\) since \(|\Pi(s^*, t^*)| \geq \max\{|V_{s^*}|, |V_{t^*}|\}\). This finishes the proof for Cases (V–). Note that Case (VV) is trivial.

From now on, we assume that both \(s^*\) and \(t^*\) are not corners in \(V\). This assumption, together with Lemma 2 implies multiple shortest paths between \(s^*\) and \(t^*\), and thus \(d(s^*, t^*) > |s^* - t^*|\). Hence, as discussed in Section 2, any maximal pair falling into one of Cases (BB), (BI), and (II) appears as a local maximum of the lower envelope of some path-length functions.

**Case (II): When both \(s^*\) and \(t^*\) lie in \(\text{int}\(\mathcal{P}\)\).** We will apply Theorem 1 to prove Theorem 2 for Case (II). For the purpose, we find \(m = |\Pi(s^*, t^*)|\) convex functions \(f_i\) defined on a convex neighborhood \(C\) of \((s^*, t^*)\) such that the following requirements are satisfied: (i) the pointwise minimum \(g\) of the \(f_i\) coincides with the geodesic distance \(d\) on \(C\), (ii) \(f_i(s^*, t^*) = g(s^*, t^*) = d(s^*, t^*)\) for any \(i \in \{1, \ldots, m\}\), (iii) \(g\) attains a local maximum at \((s^*, t^*) \in C\), and (iv) none of the \(f_i\) attains a local minimum at \((s^*, t^*)\).

If there are exactly \(m\) pairs \((u, v)\) of corners such that \(\text{len}_{u,v}(s^*, t^*) = d(s^*, t^*)\), then we can apply Theorem 1 simply with the \(m\) path-length functions \(\text{len}_{u_i,v_i}\). Unfortunately, this is not always the case; a single shortest path \(\pi_i \in \Pi(s^*, t^*)\) may result in several pairs \((u, v)\) of corners with \(u, v \in \pi_i\) such that \((u, v) \neq (u_i, v_i)\) and \(\text{len}_{u,v}(s^*, t^*) = d(s^*, t^*)\). This happens only when either \(u, u_i, s^*\) or \(v, v_i, t^*\) are collinear. In this degenerate case, the path-length functions \(\text{len}_{u_i,v_i}\) violate the first requirement above. In the following, we thus define the merged path-length functions that satisfy all the requirements even under the degenerate case.

Recall that the combinatorial structure of each shortest path \(\pi_i \in \Pi(s^*, t^*)\) can be represented by a sequence \((u_i = u_i, 1, \ldots, u_i, k = v_i)\) of corners in \(V\). We define \(u_i^j\) to be one of the \(u_{i,j}\) as follows: If \(s^*\) does not lie on the line \(\ell \subset \mathbb{R}^2\) through \(u_i\) and \(u_{i,2}\), then \(u_i^j := u_i\); otherwise, if \(s^* \in \ell\), then \(u_i^j := u_{i,j}\), where \(j\) is the largest index such that for any open neighborhood \(U \subset \mathbb{R}^2\) of \(s^*\) there exists a point \(s \in (U \cap \text{VP}(u_{i,j})) \setminus \ell\). Note that such \(u_i^j\) always exists, and if no three of \(V\) are collinear, then
we always have either \( u'_i = u_i \) or \( u'_i = u_{i,2} \); Figure 2(a) illustrates how to determine \( u'_i \). Also, we define \( v'_i \) in an analogous way. Let \( h^s_i \) and \( h^t_i \) be two functions defined as

\[
\begin{align*}
    h^s_i(s) & := \begin{cases} 
    \|s - u'_i\| & \text{if } s \in \text{VP}(u'_i), \\
    \|s - u_i\| + \|u_i - u'_i\| & \text{if } s \in \text{VP}(u_i) \setminus \text{VP}(u'_i); 
    \end{cases} \\
    h^t_i(t) & := \begin{cases} 
    \|t - v'_i\| & \text{if } t \in \text{VP}(v'_i), \\
    \|t - v_i\| + \|v_i - v'_i\| & \text{if } t \in \text{VP}(v_i) \setminus \text{VP}(v'_i). 
    \end{cases}
\end{align*}
\]

Then, the merged path-length function \( f_i : D_i \to \mathbb{R} \) is defined as

\[
f_i(s, t) := h^s_i(s) + \text{d}(u'_i, v'_i) + h^t_i(t),
\]

where \( D_i := (\text{VP}(u'_i) \cup \text{VP}(u_i)) \times (\text{VP}(v'_i) \cup \text{VP}(v_i)) \subseteq \mathcal{P} \times \mathcal{P} \). We consider \( \mathcal{P} \times \mathcal{P} \) as a subset of \( \mathbb{R}^4 \) and each pair \((s, t) \in \mathcal{P} \times \mathcal{P}\) as a point in \( \mathbb{R}^4 \). Also, we denote by \((s_x, s_y)\) the coordinates of a point \( s \in \mathcal{P} \) and we write \( s = (s_x, s_y) \) or \( (s, t) = (s_x, s_y, t_x, t_y) \) by an abuse of notation. Observe that \( f_i(s, t) = \min\{\text{len}_{u_i, v_i}(s, t), \text{len}_{u'_i, v'_i}(s, t)\} \) for any \((s, t) \in D_i\) if we define \( \text{len}_{u_i, v_i}(s, t) = \infty \) when \( s \not\in \text{VP}(u) \) or \( v \not\in \text{VP}(v) \); see Figure 2(b).

We first show the convexity of the functions \( f_i \).

**Lemma 3** For any \( i \in \{1, \ldots, m\} \) and any convex subset \( C \subset D_i \), \( f_i \) is convex on \( C \).

Observe that each of the \( f_i \) is indeed not strictly convex. Figure 3 illustrates one such line in \( \mathbb{R}^4 \) for a fixed \((s, t) \in D_i\) that \( f_i \) stays constant when \((s, t)\) moves locally along the line. We show that such a line in \( \mathbb{R}^4 \) is unique for any fixed \((s, t) \in D_i\).

**Lemma 4** For any \( i \in \{1, \ldots, m\} \) and any \((s, t) \in \text{int}D_i\), there exists a unique line \( \ell_i \subset \mathbb{R}^4 \) through \((s, t)\) such that \( f_i \) is constant on \( \ell_i \cap U \) for some neighborhood \( U \) of \((s, t)\) with \( U \subset D_i \). Moreover, \( f_i \) is constant on \( \ell_i \cap C \) for any convex neighborhood \( C \) of \((s, t)\) with \( C \subset D_i \).

Now, we let \( g : \cap D_i \to \mathbb{R} \) be the pointwise minimum of the \( f_i \) defined as \( g(s, t) = \min_i f_i(s, t) \) for any \((s, t) \in \cap D_i\). Note that the intersection \( \cap D_i \) contains a nonempty interior and \((s^*, t^*) \in \text{int} \cap D_i \) by our construction. We show that the \( f_i \) satisfy the aforementioned requirements to apply Theorem 1.

**Lemma 5** The functions \( f_i \) and their pointwise minimum \( g \) satisfy the following conditions.

(i) There exists a convex neighborhood \( C \subset \mathbb{R}^4 \) of \((s^*, t^*)\) with \( C \subseteq \cap D_i \) such that \( \text{d}(s, t) = g(s, t) \) for any \((s, t) \in C\).

(ii) \( f_i(s^*, t^*) = g(s^*, t^*) = \text{d}(s^*, t^*) \) for any \( i \in \{1, \ldots, m\} \).

(iii) \( g \) attains a local maximum at \((s^*, t^*)\).

(iv) None of the \( f_i \) attains a local minimum at \((s^*, t^*)\).

Now, we take a convex neighborhood \( C \subset \mathbb{R}^4 \) of \((s^*, t^*)\) that is as described in Lemma 5. We restrict \( f_1, \ldots, f_m \) and \( g \) on \( C \). Then, each \( f_i \) is convex by Lemma 3 and, by Lemma 5, the \( m \) functions \( f_i \) and their pointwise minimum \( g \) satisfy the conditions of Theorem 1 with open convex domain \( C \subset \mathbb{R}^4 \).

Suppose that \( m < 5 \). Then, Theorem 1 implies that there exists at least one line \( \ell \subset \mathbb{R}^4 \) through \((s^*, t^*)\) such that \( g \) is constant on \( \ell \cap C \). On the other hand, Lemma 4 implies that there is a unique line
Lemma 6 There are at most two indices \( i \in \{1, \ldots, m\} \) such that \( \ell = \ell_i \).

The last task is to check two possibilities; one or two of the \( f_i \) are constant on \( \ell \cap C \). Also, recall that \( m \geq 3 \) by Lemma 2. Without loss of generality, we first assume that \( \ell = \ell_1 \neq \ell_i \) for any \( i \geq 2 \). Along \( \ell_1 \), each \( f_i \) with \( i \geq 2 \) is not constant but convex. Since \( f_1(s, t) = g(s, t) = \min_{i} f_i(s, t) \) for any \((s, t) \in \ell \cap C\), by Lemma 4, \( f_i \) with \( i \geq 2 \) must strictly increase from \((s^*, t^*)\) in both directions along \( \ell \). Thus, for any \((s, t) \in \ell \cap C\) with \((s, t) \neq (s^*, t^*)\), we have a strict inequality \( g(s, t) = f_1(s, t) < f_i(s, t) \). Then, by Lemmas 3 and 4 at any such \((s, t) \in \ell \cap C\) there is a direction in which \( f_1 \) strictly increases: more precisely, for any arbitrarily small neighborhood \( U \subset C \) of \((s^*, t^*)\), \( g(s, t) = f_1(s, t) < f_i(s, t) \) for \((s, t) \in \partial U \cap \ell \) and thus there exist a sufficiently small neighborhood \( U' \subset C \) of \((s, t) \) and \((s', t') \in U' \) such that \( f_1(s, t) < f_1(s', t') < f_i(s', t') \) for any \( i \geq 2 \), which implies that \( g(s^*, t^*) = g(s, t) < g(s', t') \), a contradiction to that \( g \) attains a local maximum at \((s^*, t^*)\).

Thus, two of the \( f_i \) must be constant on \( \ell \cap C \). We assume that \( \ell = \ell_1 = \ell_2 \neq \ell_i \) for \( i \geq 3 \). In this case, for any \((s, t) \in \ell \cap C\) with \((s, t) \neq (s^*, t^*)\), we have a strict inequality \( g(s, t) = f_1(s, t) = f_2(s, t) < f_i(s, t) \). Then \( f_i \) has a direction from \((s, t) \) in which both of \( f_1 \) and \( f_2 \) strictly increase by Lemmas 3 and 4. Since \( g(s, t) = f_1(s, t) = f_2(s, t) < f_i(s, t) \) for any \( i \geq 3 \), we get a contradiction analogously to the above.

Hence, we achieve a bound \( m = |\Pi(s^*, t^*)| \geq 5 \), as claimed in Case (II) of Theorem 2.

Case (BB): When both \( s^* \) and \( t^* \) lie on \( B \). In this case, we assume that \( s^* \in e_s \in E \) and \( t^* \in e_t \in E \). Let \( p \) be an endpoint of \( e_s \) and \( l_p \) be the length of \( e_s \). We denote by \( s(\zeta) \) the unique point on \( e_s \) such that \( |s(\zeta) - p| = \zeta \) for any \( 0 < \zeta < l_s \). Here, we consider \( s: (0, l_s) \to e_s \) as a bijective map between a real open interval \((0, l_s) \subset \mathbb{R} \) and a segment \( e_s \subset \mathbb{R}^2 \) except its endpoints. Analogously, we also define \( t(\zeta) \). Let \( \zeta^* \) and \( \zeta^*_t \) be real numbers such that \( s^* = s(\zeta^*) \) and \( t^* = t(\zeta_t^*) \).

The outline of proof is analogous to the above discussion for Case (II). We redefine \( f_i: D_i \to \mathbb{R} \) as
\[
f_i(s(\zeta), \zeta) := h_i^s(s(\zeta)) + d(u'_i, v'_i) + h_i^t(t(\zeta)),
\]
where \( D_i := s^{-1}((VP(u'_i) \cup VP(u_i)) \cap e_s) \times t^{-1}((VP(v'_i) \cup VP(v_i)) \cap e_t) \). We consider \( D_i \) as a subset of \( \mathbb{R}^2 \) and each pair \((s(\zeta), \zeta) \in D_i \) as a point in \( \mathbb{R}^2 \). Also, let \( g(s(\zeta), \zeta) := \min_i f_i(s(\zeta), \zeta) \) for any \((s(\zeta), \zeta) \in \ell \cap D_i \).

The convexity of \( f_i \) on any convex subset of \( D_i \) is deduced from Lemma 9. Analogously to Lemma 5, one can show that (i) there exists a convex neighborhood \( C \subset \mathbb{R}^2 \) of \((\zeta^*_s, \zeta^*_t)\) with \( C \subset \ell \cap D_i \) such that \( g(s(\zeta), \zeta) = \min_i f_i(s(\zeta), \zeta) = d(s(\zeta), t(\zeta)) \) for any \((s(\zeta), \zeta) \in C \), (ii) \( f_i(\zeta^*_s, \zeta_t^*) = g(\zeta^*_s, \zeta^*_t) = d(s(\zeta^*_s), t(\zeta^*_t)) \) for any \( i \in \{1, \ldots, m\} \), (iii) \( g \) attains a local maximum at \((\zeta^*_s, \zeta^*_t)\). Also, observe that (iv) none of the \( f_i \) attains a local minimum at \((\zeta^*_s, \zeta^*_t)\) if \( m < 3 \). Assume that some \( f_i \) attains a local minimum at \((\zeta^*_s, \zeta^*_t)\) and \( m < 3 \). This happens only when \( s^* = s(\zeta^*) \) is the perpendicular foot of \( u_i \) on \( e_s \) and \( t^* = t(\zeta^*_t) \) is the perpendicular foot of \( v_i \) on \( e_t \). In this case, there always exists a direction along \( e_s \) such that if we move \( s^* \) in the direction, then \( h_i^s \) strictly increases for every \( 1 \leq i \leq m < 3 \), which contradicts the assumption that \( s^*, t^* \) is maximal.

In addition, we observe the following.

Lemma 7 If there exists a line \( \ell \subset \mathbb{R}^2 \) such that \( f_i \) is constant on \( \ell \cap C \), then \( u_i \) lies on the line supporting \( e_s \) and \( v_i \) lies on the line supporting \( e_t \).

We assume that \( m < 3 \), and restrict the functions \( f_1 \) and \( g \) on \( C \subset \mathbb{R}^2 \). Then, Theorem 1 implies that there exists a line \( \ell \subset \mathbb{R}^2 \) through \((\zeta^*_s, \zeta^*_t)\) such that \( g \) is constant on \( \ell \cap C \). Moreover, Lemma 4 implies that at least one of the \( f_i \) is constant on \( \ell \cap C \). Assume that only \( f_1 \) is constant on \( \ell \cap C \). Then, by the same argument as above for Case (II), for any point \((s(\zeta), \zeta) \in \ell \cap C \) with \((s(\zeta), \zeta) \neq (\zeta^*_s, \zeta^*_t)\), we have a strict inequality \( g(s(\zeta), \zeta) = f_1(s(\zeta), \zeta) < f_i(s(\zeta), \zeta) \) for \( i \geq 2 \), leading to a contradiction: for any arbitrarily
small neighborhood \( U \subset C \) of \((\zeta_s^*, \zeta_t^*)\), \(g(\zeta_s, \zeta_t) = f_1(\zeta_s, \zeta_t) < f_1(\zeta_s^*, \zeta_t^*)\) for \((\zeta_s, \zeta_t) \in \partial U \cap \ell\) and thus there exist a sufficiently small neighborhood \( U' \subset C \) of \((\zeta_s, \zeta_t)\) and \((\zeta_s^*, \zeta_t^*) \in U'\) such that \(f_1(\zeta_s, \zeta_t) < f_1(\zeta_s^*, \zeta_t^*) < f_1(\zeta_s^*, \zeta_t^*)\), which implies that \(d(s^*, t^*) = g(\zeta_s^*, \zeta_t^*) = g(\zeta_s, \zeta_t) < g(\zeta_s^*, \zeta_t^*)\).

Hence, both \(f_1\) and \(f_2\) are constant on \(\ell \cap C\). Then, by Lemma 4, \(u_1, u_2,\) and \(s^*\) are collinear and \(v_1, v_2,\) and \(t^*\) are collinear. Since \(m < 3\), this situation violates the second part of Lemma 2. Thus, we get a contradiction again, concluding that \(m = |\Pi(s^*, t^*)| \geq 3\) for Case (BB) when both \(s^*\) and \(t^*\) lie on \(\mathcal{B}\).

**Case (BI): When \(s^* \in B\) and \(t^* \in \text{int} \mathcal{P}\).** We assume that \(s^* \in e_s \in E\) and \(t^* \in \text{int} \mathcal{P}\). Define \(s(\zeta_s)\) as done in Case (BB) with \(s(\zeta_s^*) = s^*.\) We redefine the function \(f_i: D_i \to \mathbb{R}\) to be

\[
f_i(\zeta_s, t_x, t_y) := h_i(s(\zeta_s)) + d(u_i', v_i') + h_i(t_x, t_y),
\]

where \(D_i := s^{-1}((\text{VP}(u_i') \cup \text{VP}(v_i')) \cap e_s) \times (\text{VP}(v_i') \cup \text{VP}(v_i))\) is a subset of \(\mathbb{R}^3\). Let \(g(\zeta_s, t_x, t_y) := \min_i f_i(\zeta_s, t_x, t_y)\) for any \((\zeta_s, t_x, t_y)\) in \(\bigcap D_i\).

Analogously to Lemmas 3 and 5, each \(f_i\) is convex on any convex subset of \(D_i\) and there exists a convex neighborhood \(C \subset \mathbb{R}^3\) of \((\zeta_s^*, t_x^*, t_y^*)\) with \(C \subset \bigcap D_i\) such that the four requirements are satisfied. Suppose that \(m = |\Pi(s^*, t^*)| < 4.\) Then, Theorem 2 implies that there exists a line \(\ell \subset \mathbb{R}^3\) through \((\zeta_s^*, t_x^*, t_y^*) \in \mathbb{R}^3\) such that \(g\) is constant on \(\ell \cap C\), thus at least one of the \(f_i\) is constant on \(\ell \cap C\) by Lemma 4.

If only one of the \(f_i\) is constant on \(\ell \cap C,\) then we have a contradiction as done in Cases (II) and (BB). Thus, assume that \(f_1\) and \(f_2\) are constant on \(\ell \cap C\). By Lemmas 4 and 7, \(v_1, v_2,\) and \(t^*\) should be collinear. Further, by the second part of Lemma 2, \(t^*\) must lie in the interior of the convex hull of \(V_{t^*}.\) In order to have an interior point of the convex hull on the segment between \(v_1\) and \(v_2,\) we need at least two more points. Nonetheless, we have \(|V_{t^*}| \leq |\Pi(s^*, t^*)| < 4,\) a contradiction. Thus, we have \(m = |\Pi(s^*, t^*)| \geq 4\) for Case (BI), as claimed.

Finally, we complete a proof of Theorem 2. The claimed bounds on \(|V_{s^*}|\) and \(|V_{t^*}|\) are shown by Lemma 2 and the bounds on \(|\Pi(s^*, t^*)|\) are shown case by case as above.  

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5 Computing the Geodesic Diameter

Since a diametral pair is in fact maximal, it falls into one of the cases shown in Theorem 2. In order to find a diametral pair we examine all possible scenarios accordingly. 

**Cases (V-),** where at least one point is a corner in \(V,\) can be handled in \(O(n^2 \log n)\) time by computing \(\text{SPM}(v)\) for every \(v \in V\) and traversing it to find the farthest point from \(v,\) as discussed in Section 2. We thus focus on Cases (BB), (BI), and (II), where a diametral pair consists of two non-corner points.

From the computational point of view, the most difficult case corresponds to Case (II) of Theorem 2 in particular, \(|\Pi(s^*, t^*)| = 5\) in which 10 corners of \(V\) are involved, resulting in \(|V_{t^*}| = |\Pi(s^*, t^*)| = 5\) (see Appendix A.3). Note that we do not need to take a special care for the case of \(|\Pi(s^*, t^*)| > 5.\) By Theorem 2 and its proof, it is guaranteed that there are five distinct pairs \((u_1, v_1), \ldots, (u_5, v_5)\) of corners in \(V\) such that \(\text{len}_{u_i, v_i}(s^*, t^*) \neq d(s^*, t^*)\) for any \(i \in \{1, \ldots, 5\}\) and the system of equations \(\text{len}_{u_1, v_1}(s, t) = \cdots = \text{len}_{u_5, v_5}(s, t)\) indeed determines a 0-dimensional zero set, corresponding to a constant number of candidate pairs in \(\text{int} \mathcal{P} \times \text{int} \mathcal{P}.\) Moreover, each path-length function \(\text{len}_{u_i, v_i}\) is an algebraic function of degree at most 4. Thus, given five distinct pairs \((u_i, v_i)\) of corners, we can compute all candidate pairs \((s, t)\) in \(O(1)\) time by solving the system. Then, for each candidate pair we compute the geodesic distance between the pair to check its validity. Since the geodesic distance between any two points \(s, t \in \mathcal{P}\) can be computed in \(O(n \log n)\) time [12], we obtain a brute-force \(O(n^{11} \log n)\)-time algorithm, checking \(O(n^{10})\) candidate pairs obtained from all possible combinations of 10 corners in \(V.\)
As a different approach, one can exploit the SPM-equivalence decomposition of $\mathcal{P}$, which subdivides $\mathcal{P}$ into regions such that the shortest path map of any two points in a common region are “topologically equivalent” [7]. It is not difficult to see that if $(s, t)$ is a pair of points that equalizes any five path-length functions, then both $s$ and $t$ appear as vertices of the decomposition. However, the currently best upper bound on the complexity of the SPM-equivalence decomposition is $O(n^{10})$ [7], and thus this approach hardly leads to a remarkable improvement.

Instead, we do the following for Case (II) with $|V_{s^*}| = 5$. We choose any five corners $u_1, \ldots, u_5 \in V$ (as a candidate for the set $V_{s^*}$) and overlay their shortest path maps $\text{SPM}(u_i)$. Since each $\text{SPM}(u_i)$ has $O(n)$ complexity, the overlay consists of $O(n^2)$ cells. Then, any cell of the overlay is the intersection of five cells associated with $v_1, \ldots, v_5 \in V$ in $\text{SPM}(u_1), \ldots, \text{SPM}(u_5)$, respectively. Choosing a cell of the overlay, we get five (possibly, not distinct) $v_1, \ldots, v_5$ and thus a constant number of candidate pairs by solving the system $\text{len}_{u_1,v_1}(s, t) = \cdots = \text{len}_{u_5,v_5}(s, t)$. We iterate this process for all possible tuples of five corners $u_1, \ldots, u_5$, obtaining a total of $O(n^7)$ candidate pairs in $O(n^7 \log n)$ time. Note that the other subcases with $|V_{s^*}| \leq 4$ can be handled similarly, resulting in $O(n^6)$ candidate pairs.

In order to test the validity of each candidate pair $(s, t)$, we check the geodesic distance $d(s, t)$ using a two-point query structure of Chiang and Mitchell [7]: for a fixed parameter $0 \leq \delta \leq 1$ and any fixed $\epsilon > 0$, we can construct, in $O(n^{5+10\delta+\epsilon})$ time, a data structure that supports $O(n^{1-\delta} \log n)$-time two-point shortest path queries. Then, the total running time is $O(n^7 \log n) + O(n^{5+10\delta+\epsilon}) + O(n^7) \times O(n^{1-\delta} \log n)$. We set $\delta = \frac{1}{11}$ to optimize the running time to $O(n^7 + \frac{n^7}{2})$.

Also, we can use an alternative two-point query data structure whose performance is sensitive to the number $h$ of holes [7]: after $O(n^5)$ preprocessing time using $O(n^5)$ storage, two-point queries can be answered in $O(\log n + h)$ time. Using this alternative structure, the total running time of our algorithm becomes $O(n^7 (\log n + h))$. Note that this method outperforms the previous one when $h = O(n^\frac{1}{11})$.

The other cases can be handled analogously with strictly better time bound. For Case (BI), we handle only the case of $|\Pi(s^*, t^*)| = 4$ with $|V_{s^*}| = 3$ or 4. For the subcase with $|V_{t^*}| = 4$, we choose any four corners from $V$ as $v_1, \ldots, v_4$ as a candidate for $V_{t^*}$ and overlay their shortest path maps $\text{SPM}(v_i)$. The overlay, together with $V$, decomposes $\partial \mathcal{P}$ into $O(n)$ intervals. Then, each such interval determines $u_1, \ldots, u_4$ as above, and the side $e_s \in E$ on which $s^*$ should lie. Now, we have a system of four equations on four variables: three from the corresponding path-length functions $\text{len}_{u_i,v_i}$ which should be equalized at $(s^*, t^*)$ and the fourth from the supporting line of $e_s$. Solving the system, we get a constant number of candidate maximal pairs, again by Theorem 2 and its proof. In total, we obtain $O(n^5)$ candidate pairs. The other subcase with $|V_{t^*}| = 3$ can be handled similarly, resulting in $O(n^4)$ candidate pairs. As above, we can exploit two different structures for two-point queries. Consequently, we can handle Case (BI) in $O(n^{5+\frac{n^7}{2}+\epsilon})$ or $O(n^5 (\log n + h))$ time.

In Case (BB) when $s^*, t^* \in \mathcal{B}$, we handle the case of $|\Pi(s^*, t^*)| = 3$ with $|V_{s^*}| = 2$ or 3. For the subcase with $|V_{s^*}| = 3$, we choose three corners as a candidate of $V_{s^*}$ and take the overlay of their shortest path maps $\text{SPM}(u_i)$. It decomposes $\partial \mathcal{P}$ into $O(n)$ intervals. Then, each such interval determines three corners $v_1, v_2, v_3$ forming $V_t$ and a side $e_t \in E$ on which $t^*$ should lie. Note that we have only three equations so far; two from the three path-length functions and the third from the line supporting to $e_t$. Since $s^*$ also should lie on a side $e_s \in E$ with $e_s \neq e_t$, we need to fix such a side $e_s$ that $\bigcap_{1 \leq i \leq 3} \text{VP}(u_i)$ intersects $e_s$. In the worst case, the number of such sides $e_s$ is $O(n)$. Thus, we have $O(n^5)$ candidate pairs for Case (BB); again, the other subcase with $|V_{s^*}| = 2$ contributes to a smaller number $O(n^4)$ of candidate pairs. Testing each candidate pair can be performed as above, resulting in $O(n^{5+\frac{n^7}{2}+\epsilon})$ or $O(n^5 (\log n + h))$ total running time.

For Case (BB), however, one can exploit a two-point query structure only for boundary points on $\partial \mathcal{P}$. The two-point query structure by Bae and Okamoto [5] indeed builds an explicit representa-
tion of the graph of the lower envelope of the path-length functions \( \text{len}_{u,v} \) restricted on \( \partial P \times \partial \bar{P} \) in \( O(n^5 \log n \log^3 n) \) time. Since \( |\Pi(s^*, t^*)| \geq 3 \) in Case (BB), such a pair appears as a vertex on the lower envelope. Hence, we are done by traversing all the vertices of the lower envelope.

The following table summarizes the discussion so far.

| Case     | Independent of \( h \) | Dependent on \( h \) |
|----------|------------------------|----------------------|
| (VV), (VB), (VI) | \( O(n^2 \log n) \) | \( O(n^2 \log n) \) |
| (BB)     | \( O(n^5 \log n \log^3 n) \) | \( O(n^5 \log n + h) \) |
| (BI)     | \( O(n^{7+\frac{7}{11}\epsilon}) \) | \( O(n^8 \log n + h) \) |
| (II)     | \( O(n^{7+\frac{7}{11}\epsilon}) \) | \( O(n^8 \log n + h) \) |

As Case (II) being a bottleneck, we conclude the following.

**Theorem 3** Given a polygonal domain having \( n \) corners and \( h \) holes, the geodesic diameter and a diametral pair can be computed in \( O(n^{7+\frac{7}{11}\epsilon}) \) or \( O(n^7 \log n + h) \) time in the worst case, where \( \epsilon \) is any fixed positive number.

We can avoid some difficult cases when \( h \) is a small constant based on a simple observation: If there are two distinct shortest paths between \( s \) and \( t \) in \( P \), then we know that there is at least one hole in the region closed by the two paths. In general, if \( k \) shortest paths exist between any two points of \( P \), we can conclude \( h \geq k - 1 \). By contraposition, if \( h < k - 1 \), then there cannot exist two points that have \( k \) or more distinct shortest paths between them.

**Theorem 4** Given a polygonal domain having \( n \) corners and \( h \) holes, the geodesic diameter and a diametral pair can be computed in the following worst-case time bound, depending on \( h \).

- \( O(n) \) time, if \( h = 0 \) (by Hershberger and Suri [11]),
- \( O(n^2 \log n) \) time, if \( h = 1 \),
- \( O(n^5 \log n) \) time, if \( h = 2 \) or 3,
- \( O(n^7 \log n + h) \) time, if \( 4 \leq h = O(n^{\frac{5}{11}}) \),
- \( O(n^{7+\frac{7}{11}\epsilon}) \) time, otherwise.

### 6 Concluding Remarks

We have presented first algorithms that compute the geodesic diameter of a given polygonal domain. They are based on our new geometric observations on local maxima of the geodesic distance function, which show tight lower bounds on the number of shortest paths between a maximal pair. It is worth noting that with analysis in Section 5 the number of geodesic-maximal pairs is shown to be at most \( O(n^7) \). On the other hand, one can easily construct a simple polygon in which the number of maximal pairs is \( \Omega(n^2) \). An interesting question would be what the maximum possible number of maximal pairs in a polygonal domain is.

Though in this paper we have focused on exact geodesic diameters only, an efficient algorithm for finding an approximate geodesic diameter would be also interesting. Notice that any point \( s \in P \) and its farthest point \( t \in P \) yield a 2-approximate diameter; that is, \( \text{diam}(P) \leq 2 \max_{t \in P} \text{d}(s, t) \) for any \( s \in P \).

Also, based on a standard technique using a rectangular grid with a specified parameter \( 0 < \epsilon < 1 \), one can easily obtain a \( (1 + \epsilon) \)-approximate diameter in \( O((\frac{n}{\epsilon^2} + \frac{n^2}{\epsilon}) \log n) \) time. \(^5\) Breaking the quadratic bound in \( n \) for the \( (1 + \epsilon) \)-approximate diameter seems a challenge at this stage. Thus, we pose the following problem: For any or some \( 0 < \epsilon < 1 \), is there any algorithm that finds a \( (1 + \epsilon) \)-approximate diametral pair in \( O(n^{2-\delta} \cdot \text{poly}(1/\epsilon)) \) time for some positive \( \delta > 0 \)?

\(^4\)More precisely, in \( O(n^4 \lambda_m(n) \log n) \) time, where \( \lambda_m(n) \) stands for the maximum length of a Davenport-Schinzel sequence of order \( m \) on \( n \) symbols.

\(^5\)We thank Hee-Kap Ahn for giving us the idea of this approximation algorithm.
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APPENDIX

A More Examples and Remarks

In this section, we show more constructions of polygonal domains and their diametral pairs with interesting remarks. In the figures, we keep the following rules: the boundary $\partial P$ is depicted by dark gray segments and the interior of holes by light gray region. A diametral pair is given as $(s^*, t^*)$ and shortest paths between $s^*$ and $t^*$ are described as black dashed polygonal chains.

A.1 Examples where at least one point of a diametral pair lies on $\partial P$

![Figure 4: (a–c) Polygonal domains whose geodesic diameter is determined by a corner $s^*$ and (d–g) variations of the construction (c). (a) When both $s^*$ and $t^*$ are corners; (b) When $t^*$ is a point on $\partial P$; (c) When $t^* \in \text{int} P$. This polygonal domain consists of two holes, forming a narrow corridor and three shortest paths between $s^*$ and $t^*$. Here, we have $d(s^*, v_1) = d(s^*, v_2) = d(s^*, v_3)$ and $t^*$ is indeed the vertex of $\text{SPM}(s^*)$ defined by $v_1, v_2, v_3$; (d) Variation of (c) with all convex holes; (e) Three shortest paths are not enough to determine a boundary-interior diametral pair; (f) If we add one more hole, then the diameter is determined by $s^* \in B$ and $t^* \in \text{int} P$ with four shortest paths; (g) A polygonal domain made by attaching two copies of (e) and modifying it to have $d(u_1, v_1) = d(u_2, v_2) = d(u_3, v_3)$. Observe that, in this polygonal domain, the diameter is determined by two boundary points with three shortest paths.](image)

Note that, as expected, every example in Figure 4 satisfies Theorem 2. An interesting construction is Figure 4(g), where neither of the two centers of $\triangle u_1u_2u_3$ and of $\triangle v_1v_2v_3$ appears in any diametral pair. Also note that Figure 4(d) consists of convex holes only. We think that any complicated construction can be “convexified” in a similar fashion. This would suggest that computing the diameter in polygonal domains with convex holes only might be as difficult as the general case.
A.2 A proof for Figure 1(c): Case (II) with 6 shortest paths

Claim 1 In the polygonal domain described in Figure 1(c), \((s^*, t^*)\) is the unique diametral pair.

Proof of Claim. Recall that by construction of the problem instance, the triangles \(\triangle u_1u_2u_3\) and \(\triangle v_1v_2v_3\) are regular and \(d(u_1, v_1) = d(u_2, v_2) = d(u_3, v_3) = d(u_3, v_1) = L\), for some arbitrarily large value \(L > 0\). Also, \(s^*\) and \(t^*\) are the centers of \(\triangle u_1u_2u_3\) and \(\triangle v_1v_2v_3\), respectively.

We assume that both triangles \(\triangle u_1u_2u_3\) and \(\triangle v_1v_2v_3\) are inscribed in a unit circle. Then, we have \(d(s^*, t^*) = 2 + L\). For any point \(s\) on any shortest path between \(u_i\) and \(v_j\), it is easy to see that \(d(s, t) \leq \sqrt{3} + L < d(s^*, t^*)\) for every point \(t \in P\). Thus, any point on those paths cannot contribute to the diameter.

1. First, observe that \(\max_{t \in \triangle v_1v_2v_3} d(s^*, t) = \max_{s \in \triangle u_1u_2u_3} d(s, t^*) = d(s^*, t^*)\).

2. For any \(s \in \triangle u_1u_2u_3\), its farthest point \(t \in \triangle v_1v_2v_3\) is on the angle bisector of some \(v_i\). Consider any \(s \in \triangle u_1u_2u_3\). Without loss of generality we assume that \(\|s - u_1\| \leq \min_i\{\|s - u_i\|\}\). Then, both the shortest paths to \(v_1\) and to \(v_2\) from \(s\) pass through \(u_1\). We thus have \(d(s, v_1) = d(s, v_2)\) by construction and its farthest point \(t \in \triangle v_1v_2v_3\) must be in the angle bisector of \(v_3\). By symmetry, the same property holds when the closest corner from \(s^*\) is either \(u_2\) or \(u_3\).

Conversely, for any \(t\), its farthest point \(s \in \triangle u_1u_2u_3\) must be on a bisector of some \(u_i\). In any diametral pair \((s, t)\), we have that \(t\) is the farthest point of \(s\) (and vice versa), so both must be on one of the angle bisectors.

3. If \((s, t)\) is a diametral pair, then \(s \in \overline{u_is^*}\) and \(t \in \overline{v_jt^*}\), for some \(i\) and \(j\). Suppose that \(s\) lies on the bisector of \(u_1\) but not in between \(u_1\) and \(s^*\). We then have \(\|s - u_2\| = \|s - u_3\| < \|s - u_1\|\) and \(d(s, v_1) = d(s, v_2) = d(s, v_3) = \|s - u_2\| + L\) by construction. This implies that \(t^*\) is the farthest point of such \(s\). Since \(\|s - u_2\| < 1\) and thus \(d(s, t^*) < 2 + L\), \((s, t^*)\) is not a diametral pair.

4. Now, pick any point \(s \in \overline{u_is^*}\) with \(s \neq s^*\). Suppose that \(t \in \triangle v_1v_2v_3\) is the farthest point from \(s\). We know that \(t \in \overline{v_3t^*}\) by above discussions. In this case, we have four shortest paths between \(s\) and \(t\) through \((u_1, v_1), (u_1, v_2), (u_2, v_3), \) and \((u_3, v_3)\); the other two are strictly longer unless \(s = s^*\). Thus, by Theorem 2 such \(s \in \overline{u_is^*}\) with \(s \neq s^*\) and its farthest point \(t\) cannot form a maximal pair. By symmetry, the other cases where \(s \in \overline{u_is^*}\) can be handled.

Hence, \((s^*, t^*)\) is a unique diametral pair and the geodesic diameter is \(2 + L\). 

\[\square\]
A.3 Diametral pair of Case (II) with exactly 5 shortest paths

Here, we present a polygonal domain in which the diameter is determined by two interior points and exactly five shortest paths between them. This proves the tightness of Case (II) in Theorem 2.

![Diagram showing a polygonal domain with five shortest paths](image)

Figure 6: A schematic diagram of a polygonal domain in which $|V_s| = |V_t| = 3$ and $\Pi(s^*, t^*) = 5$.

Figure 6 shows a schematic description of a polygonal domain $\mathcal{P}$. We assume that only the position of the vertices $u_i$ and the $v_i$ are geometrically precise. Thus, we construct the problem instance such that we have $u_1 = u_2 = u_3$, $v_1 = v_5$, and $v_3 = v_4$, and the convex hulls of the $u_i$ and of the $v_i$ form isosceles triangles $\Delta_u$ and $\Delta_v$. Each of $\Delta_u$ and $\Delta_v$ is inscribed in a unit circle centered at $c_u$ and $c_v$. Moreover, the bases of both triangles are horizontal and the angles opposite to the bases are $18^\circ$ and $112^\circ$, respectively. Note that the side lengths of the triangles $\Delta_u$ and $\Delta_v$ are as follows: $\|u_1 - u_4\| = 1.97537\cdots$ and $\|u_4 - u_5\| = 0.61803\cdots$; $\|v_2 - v_1\| = 1.11833\cdots$ and $\|v_1 - v_3\| = 1.85436\cdots$.

In this configuration, we set the constants as follows: letting $L := d(u_1, v_1) = d(u_3, v_3)$ be some sufficiently large number, we set $d(u_2, v_2) = L + 0.5$ and $d(u_4, v_4) = d(u_5, v_3) = L + 0.2$. Note that this configuration can be realized with four obstacles in a similar way as Figure 1c.

Since we have fixed all necessary parameters, we have a fully explicit description of the $\text{len}_{u_i, v_j}$. Due to the difficulty of finding an exact analytical solution, we used numerical methods to solve the system of equations $\text{len}_{u_1, v_1}(s, t) = \cdots = \text{len}_{u_5, v_5}(s, t)$. We have found that there is a unique solution $(s^*, t^*)$ such that $s^* \in \Delta_u$ and $t^* \in \Delta_v$; we obtained $s^* = c_u + (0, -0.102795\cdots)$, $t^* = c_v + (0, 0.555361\cdots)$ and $d(s^*, t^*) = 2.047433734\cdots + L$. (See Figure 6)

We first checked that $(s^*, t^*)$ is indeed a maximal pair based on the following lemma, which can be shown using elementary linear algebra together with the convexity of the path-length functions.

**Lemma 8** Suppose that $(s, t)$ is a solution to the system $\text{len}_{u_1, v_1}(s, t) = \cdots = \text{len}_{u_5, v_5}(s, t)$. If any four of the five gradients $\nabla \text{len}_{u_i, v_i}$ at $(s, t)$ are linearly independent (as vectors in a 4-dimensional space) and one of them is represented as a linear combination of the other four with all “negative” coefficients, then $(s, t)$ is a local maximum of the pointwise minimum of the five functions $\text{len}_{u_i, v_i}$.

Next, to see that $(s^*, t^*)$ is a diametral pair, we have run our algorithm for each of Cases (BB), (BI), and (II); as a result, there are 44 candidate pairs, including $(s^*, t^*)$, falling into those cases among which at most 11 are maximal and only $(s^*, t^*)$ is diametral. Note that the pair $(s^*, t^*)$ is the only candidate pair of Case (II). Also, observe that any point on the shortest path between $u_i$ and $v_i$ cannot belong to a diametral pair. This implies that none of the $u_i$ and the $v_i$ belongs to a diametral pair and thus that there is no diametral pair in Cases (V–). In addition, we also sampled about 350,000 points uniformly from each of $\Delta_u$ and $\Delta_v$, and evaluated the geodesic distances of the 350,000 pairs.

Note that one can modify the construction to have $|V_s| = |V_t| = |\Pi(s^*, t^*)| = 5$. For the purpose, we can split $u_1$, $u_2$, $u_3$ into three close corners (analogously for corners, $v_1$, $v_5$ and $v_3, v_4$). The splitting process should preserve the differences between the distances $d(u_i, v_i)$ for all $i = 1, \ldots, 5$. We also have tested such an example in the same way as above and concluded that a solution equalizing the five path-length functions is indeed a diametral pair.
B Proofs Omitted from the Body

Proof of Lemma 1 We only give a proof for the second statement, which implies the first. The case of $d = 1$ is trivial, so we assume $d > 1$. Let $H_1, \ldots, H_m$ be any $m$ closed hemispheres on $S^{d-1}$, and $h_i$ be the hyperplane through the origin in $\mathbb{R}^d$ such that $H_i$ lies in a closed half-space supported by $h_i$. In this proof, we denote by $\widetilde{H}_i$ the open hemisphere, defined to be $\widetilde{H}_i = H_i \setminus h_i$. Also, let $H_j := \bigcap_{1 \leq i \leq j} H_i$ and $\overline{H}_j := \bigcap_{1 \leq i \leq j} \overline{H}_i$.

Suppose that $H_m = \emptyset$. Let $k$ be the smallest integer such that $\overline{H}_k = \emptyset$. By definition, $k \geq 2$ and $\overline{H}_{k-1} \neq \emptyset$. Since the intersection of any $k - 1$ non-parallel hyperplanes of $\mathbb{R}^d$ includes a $(d - k + 1)$-flat and each $h_i$ contains the origin, $\bigcap_{1 \leq i \leq k-1} h_i$ includes a $(d - k + 1)$-flat through the origin and thus $\overline{H}_{k-1}$ includes a great $(d - k)$-sphere $G$ on $S^{d-1}$. Since $x \in G$ implies $-x \in G$ for any $x \in S^{d-1}$, we must have $G \subseteq h_k$, in order to have an empty intersection $\overline{H}_k$. This implies that $\bigcap_{1 \leq i \leq k} h_i$ also includes a $(d - k + 1)$-flat through the origin, and further that $\bigcap_{1 \leq i \leq m} h_i$ includes a $(d - m + 1)$-flat through the origin. We hence conclude that $H_m = \bigcap_{1 \leq i \leq m} H_i$ includes a great $(d - m)$-sphere on $S^{d-1}$. □

Proof of Lemma 2 Since $(s^*, t^*)$ is a maximal pair, $d_{s^*}(t) := d(s^*, t)$ is maximized at $t^*$ over a small neighborhood $U \subset \mathcal{P}$ of $t^*$. Thus, as discussed in Section 2 if $t^* \notin \mathcal{P}$, then $t^*$ must either be a vertex of $\text{SPM}(s^*)$ or an intersection point between an edge of $\text{SPM}(s^*)$ and $\partial \mathcal{P}$. If $t^* \in \text{int} \mathcal{P}$, then $t^*$ should fall into the former case and we have at least three corners $v_1, v_2, v_3 \in V$ determining a vertex of $\text{SPM}(s^*)$. If $t^* \in B$, then $t^*$ may fall into either case. Even in the latter case, $t^*$ must lie on $B$ and we have at least two corners $v_1, v_2 \in V$ determining an edge of $\text{SPM}(s^*)$.

The second part of the lemma can be shown as follows. If $t^* \in \text{int} \mathcal{P}$ but lies out of the interior of the convex hull of $V_{t^*}$, then we can find another point $t \in U$ for any neighborhood $U \subset \text{int} \mathcal{P}$ of $t^*$ such that $||t - v_i|| > ||t^* - v_i||$ for every $v_i \in V_{t^*}$, implying that $d(s^*, t) > d(s^*, t^*)$. If $t^* \in E$ but every $v_i \in V_{t^*}$ lies on the supporting line $\ell$ of $e$, then when we move $t^*$ in a perpendicular direction to $\ell$, we obtain a strictly larger distance as above. (Remark that a similar argument can be found also in [13] Lemma 2.2.) □

Proof of Lemma 3 Since the sum of convex functions is a convex function, it suffices to show that $h_i^s$ and $h_i^t$ are convex. More precisely, for any $(s_1, t_1), (s_2, t_2) \in C$ and $0 \leq \lambda \leq 1$, we have

$$f_i(\lambda(s_1, t_1) + (1 - \lambda)(s_2, t_2)) = h_i^s(\lambda s_1 + (1 - \lambda)s_2) + d(u_i', v_i')) + h_i^t(\lambda t_1 + (1 - \lambda)t_2) \leq \lambda h_i^s(s_1) + (1 - \lambda)h_i^s(s_2) + d(u_i', v_i') + \lambda h_i^t(t_1) + (1 - \lambda)h_i^t(t_2) = \lambda f_i(s_1, t_1) + (1 - \lambda)f_i(s_2, t_2)$$

if $h_i^s$ and $h_i^t$ are convex.

We thus show the convexity of $h_i^s$ on any convex subset $C_s \subset \text{VP}(u_i') \cup \text{VP}(u_i)$. The convexity of $h_i^t$ can be shown in the same way. There are two cases: $u_i' = u_i$ or $u_i' \neq u_i$. For the former case, $h_i^s$ is convex on $C_s$ since it measures the Euclidean distance between $u_i$ and a given point in $C_s$. Letting $\ell_0$ be the line through $u_i, u_i'$, and also $s^*$, for the latter case, $C_s$ may be partitioned by $\ell_0$ into two regions $A_1$ and $A_2$, where $A_1 = C_s \cap \text{VP}(u_i')$ and $A_2 = C_s \setminus A_1$. Note that $h_i^s$ is convex on $A_1$ and on $A_2$. Thus, we are done by checking every point on $\ell_0 \cap C_s$.

Pick any $s \in \ell_0 \cap C_s$ and any line $\ell \subset \mathbb{R}^2$ through $s$. Let $\theta$ be the angle between $\ell_0$ and $\ell$. If we restrict the domain of $h_i^s$ on $\ell \cap C_s$, then one can check with basic calculus that both the derivatives of $||s - u_i|| + ||u_i - u_i'||$ and of $||s - u_i'||$ are equal to $c \cos \theta$ at $s$ for some constant $c$. Hence, $h_i^s$ is smooth and convex along $\ell$. Since we have taken any line $\ell$ through any point on $\ell_0 \cap C_s$, this suffices to prove the convexity of $h_i^s$ on $C_s$. □
Proof of Lemma 4. There are four cases according to the position of \( s \) and \( t \): either \( s \in \text{VP}(u'_i) \) or \( s \in \text{VP}(u'_i) \setminus \text{VP}(u'_j) \); either \( t \in \text{VP}(u'_j) \) or \( t \in \text{VP}(v'_i) \setminus \text{VP}(v'_j) \). We give a proof only for the case of \( s \in \text{VP}(u'_i) \) \( \setminus \text{VP}(u'_j) \) and \( t \in \text{VP}(v'_i) \setminus \text{VP}(v'_j) \) but the proofs for the other cases are almost identical.

Recall that \( s \neq u_i \) and \( t \neq v_i \). Any ray (or half-line) \( \gamma \subset \mathbb{R}^4 \) with endpoint \( (s, t) \) can be determined by three parameters \( (\theta_s, \theta_t, \lambda) \) with \( 0 \leq \theta_s, \theta_t \leq \pi \) and \( \lambda \geq 0 \) as follows: Let \( \gamma_s \) and \( \gamma_t \) be the projections of \( \gamma \) onto the \( (s_x, s_y) \)-plane and the \( (t_x, t_y) \)-plane, respectively. Note that \( \gamma_s \) is a ray in the \( (s_x, s_y) \)-plane with endpoint \( s \) and \( \gamma_t \) is a ray in the \( (t_x, t_y) \)-plane with endpoint \( t \). Let \( \theta_s \) be the smaller angle at \( s \) made by \( \gamma_s \) and another ray starting from \( s \) in direction away from \( u_i \). Define \( \theta_t \) analogously. Then, the derivative of \( f_i \) at \( (s, t) \) along \( \gamma \) is represented as \( c(\cos \theta_s + \lambda \cos \theta_t) \) for some constants \( \lambda \geq 0 \) and \( c > 0 \). Also, the second derivative of \( f_i \) at \( (s, t) \) along \( \gamma \) is derived as \( c \left( \frac{\sin^2 \theta_s}{\|s-u_i\|} + \lambda \frac{\sin^2 \theta_t}{\|t-v_i\|} \right) \).

Suppose that \( f_i \) is constant along \( \gamma \) locally around \( (s, t) \). Then, its first and second derivatives should be zero in a small neighborhood \( U \subset \mathbb{R}^4 \) of \( (s, t) \) with \( U \subset D_i \). First, we observe that \( \lambda \) should be positive; if \( \lambda = 0 \), then \( t \) is fixed while \( s \) moves along \( \gamma_s \) so that \( f_i \) does not stay constant. Since every term of the second derivative is nonnegative and \( \lambda > 0 \), we get \( (\theta_s, \theta_t) = (\pi, \pi) \) or \( (\pi, 0), (0, \pi), (0, 1), (\pi, 0) \). We hence obtain only two solutions \( (\theta_s, \theta_t, \lambda) = (0, \pi, 1) \) or \( (\pi, 0, 1) \). The two rays \( \gamma \) corresponding to these two solutions form a unique line \( \ell_i \subset \mathbb{R}^4 \) through \( (s, t) \) such that \( f_i \) is constant on \( \ell_i \cap U \).

Now, we pick any \( (s', t') \in \ell_i \cap \text{int}D_i \) such that both \( (s, t) \) and \( (s', t') \) lie on a common connected component of \( \ell_i \cap \text{int}D_i \). Then, we observe that \( f_i(s', t') = f_i(s, t) \). Since any convex neighborhood \( C \) of \( (s, t) \) with \( C \subset D_i \) intersects \( \ell_i \) in a single connected component, for any \( (s', t') \in \ell_i \cap C \), it holds that \( f_i(s', t') = f_i(s, t) \), proving the second part of the lemma.

Remark that \( \ell_i \cap \text{int}D_i \) may consist of more than one connected components; a typical situation where \( \ell_i \cap \text{int}D_i \) is disconnected is when \( \ell_i \) passes through \( (u_i, v_i) \) or \( (s_i, t_i) \) on \( \partial D_i \) for some \( s'' \in \text{VP}(u'_j) \cup \text{VP}(u'_j) \) and some \( t'' \in \text{VP}(v'_j) \cup \text{VP}(v'_j) \). See Figure 3 for more intuitive and geometric description on \( \ell_i \).

Proof of Lemma 5. (i) In this proof, we define \( \text{len}_{u,v}(s, t) = \infty \) if \( s \notin \text{VP}(u) \) or \( t \notin \text{VP}(v) \). By definition, there exists a small neighborhood \( U_i \subset D_i \) of \( (s^*, t^*) \) such that \( f_i(s, t) = \min \{ \text{len}_{u_i,v_i}(s, t), \text{len}_{u_i,v'_i}(s, t), \text{len}_{u'_i,v_i}(s, t), \text{len}_{u'_i,v'_i}(s, t) \} = \min_{u,v \in \Pi_i} \text{len}_{u,v}(s, t) \) for all \( (s, t) \in U_i \). We claim that there exists a convex neighborhood \( C \subset \bigcap_i U_i \) such that for any \( (s, t) \in C \)

\[
\text{d}(s, t) = \min_{1 \leq i \leq m} f_i(s, t) = g(s, t).
\]

To prove our claim, assume to the contrary that for every convex neighborhood \( C \subset \mathbb{R}^4 \) of \( (s^*, t^*) \) in \( \mathbb{R}^4 \) there exist a pair \( (u, v) \) of corners and \( (s, t) \in C \) such that \( \text{d}(s, t) = \text{len}_{u,v}(s, t) < \min_i f_i(s, t) \). Note that none of the shortest paths \( \pi_i \in \Pi(s^*, t^*) \) between \( s^* \) and \( t^* \) passes through both of \( u \) and \( v \) since otherwise \( \text{len}_{u,v}(s, t) = \min_{i} f_i(s, t) \).

Consider a sequence \( C_1, C_2, \ldots \) of neighborhoods of \( (s^*, t^*) \) in \( \mathbb{R}^4 \) that converges to the singleton \( \{(s^*, t^*)\} \). Since there are only \( n^2 \) pairs of corners, there exist a fixed pair \( (u_0, v_0) \) of corners and a subsequence \( C_{k_1}, C_{k_2}, \ldots \) converging to the singleton \( \{(s^*, t^*)\} \) such that none of the \( \pi_i \) passes through both of \( u_0, v_0 \) and for any integer \( j > 0 \) there is a point \( (s_j, t_j) \in C_{k_j} \) with

\[
\text{d}(s_j, t_j) = \text{len}_{u_0,v_0}(s_j, t_j) < \min_{1 \leq i \leq m} f_i(s_j, t_j).
\]

Since \( \lim_{j \to \infty} (s_j, t_j) = (s^*, t^*) \), it holds that \( \lim_{j \to \infty} \text{d}(s_j, t_j) = \lim_{j \to \infty} \min_i f_i(s_j, t_j) = \text{d}(s^*, t^*) \). By the sandwich theorem, we thus have

\[
\lim_{j \to \infty} \text{len}_{u_0,v_0}(s_j, t_j) = \text{len}_{u_0,v_0}(s^*, t^*) = \text{d}(s^*, t^*).
\]

This implies the existence of the \((m+1)\)-st shortest path between \( s^* \) and \( t^* \) since none of the \( \pi_i \in \Pi(s^*, t^*) \) contains both \( u_0 \) and \( v_0 \), a contradiction.
(ii) The claim follows from the fact that $f_i(s^*, t^*) = \text{len}_{u_i,v_i}(s^*, t^*)$.

(iii) From (i), $g(s, t) = d(s, t)$ for all $(s, t) \in C$. Since $d$ attains a local maximum at $(s^*, t^*)$, so does $g$.

(iv) Consider any pair $(s, t)$ such that $s \in s^* u_i$ and $t \in t^* v_i$ but $s \neq s^*$ and $t \neq t^*$. Then, $f_i(s, t) = \text{len}_{u_i,v_i}(s, t) < f_i(s^*, t^*)$. Hence, there is no neighborhood $U \subset \mathbb{R}^4$ of $(s^*, t^*)$ such that $f_i(s^*, t^*) \leq f_i(s, t)$ for any $(s, t) \in U$, implying that $(s^*, t^*)$ is not a local minimum of any $f_i$. 

**Proof of Lemma 6** Let $\ell_i \subset \mathbb{R}^4$ be the unique line through $(s^*, t^*)$ such that $f_i$ is constant on $\ell_i \cap C$. The uniqueness of $\ell_i$ is proven by Lemma 4.

Observe from the proof of Lemma 4 that the projection of $\ell_i$ onto the $(s_x, s_y)$-plane is the line through $s^*$ and $u_i$. Also, the projection of $\ell_i$ onto the $(t_x, t_y)$-plane is the line through $t^*$ and $v_i$. Hence, $\ell_i = \ell_j$ implies that $u_i, u_j, s^*$ are collinear and $v_i, v_j, t^*$ are collinear. First, since the pairs $(u_i, v_i)$ are all distinct, we have $u_i \neq u_j$ or $v_i \neq v_j$. If $u_i = u_j$ and $v_i \neq v_j$, one can easily check that $\ell_i \neq \ell_j$. We thus have $u_i \neq u_j$ and $v_i \neq v_j$. Moreover, $s^*$ must lie in between $u_i$ and $u_j$ and $t^*$ must lie in between $v_i$ and $v_j$ by definition; if $u_j$ lies in between $u_i$ and $s^*$, then the first corner of $\pi_i$ from $s^*$ becomes $u_j$ since the three are collinear. Therefore, for each $i \in \{1, \ldots, m\}$, there is at most one index $j \in \{1, \ldots, m\}$ such that $i \neq j$ and $\ell_i = \ell_j$, completing the proof.

**Proof of Lemma 7** The lemma immediately follows from Lemma 4 and its proof.