Dynamic risk measures with fluctuation of market volatility under Bochne-Lebesgue space

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Abstract

Starting from the global financial crisis to the more recent disruptions brought about by geopolitical tensions and public health crises, the volatility of risk in financial markets has increased significantly. This underscores the necessity for comprehensive risk measures capable of capturing the complexity and heightened fluctuations in market volatility. This need is crucial not only for new financial assets but also for the traditional financial market in the face of a rapidly changing financial environment and global landscape. In this paper, we consider the risk measures on a special space $L^{p(\cdot)}$, where the variable exponent $p(\cdot)$ is no longer a given real number as in the conventional risk measure space $L^p$, but rather a random variable reflecting potential fluctuations in volatility within financial markets. Through further development of axioms related to this class of risk measures, we also establish dual representations for them.

Keywords: dynamic risk measure; market volatility; Bochne-Lebesgue space; time consistency

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1 Introduction

Risk measures were initially applied in insurance for pricing insurance contracts, then have evolved to become a foundational element for decision-making and management across various domains within operations research, management science, and mathematical finance. In the realm of operations research, these measures provide valuable insights into optimizing decision strategies under uncertainty. Within management science, risk measures are indispensable tools for evaluating the potential impact of various choices on overall organizational objectives.

As economic landscapes evolve, traditional risk metrics may prove insufficient in capturing the complexities and nuances of contemporary risks. In particular, from the global financial crisis of 2008 to the more recent disruptions caused by geopolitical tensions and public health crises, which have underscored the need for robust tools and methodologies to assess and mitigate risks effectively. In addition, the financial markets continue to evolve, incorporating new technologies, new instruments, and regulatory frameworks, the demand for advanced risk measurement techniques has become even more pronounced. The traditional approaches which are foundational but are being complemented and, in some cases, supplanted by sophisticated models that account for non-linearities, tail risks, and the dynamic nature of volatility. Besides, it becomes increasingly evident that risk measures play a pivotal role not only in risk management but also in shaping investment strategies, influencing regulatory frameworks, and guiding decision-making across a spectrum of financial activities.

The seminal work of Artzner et al. (1999) has bestowed upon the field of risk assessment a set of four pivotal axioms that stand as the cornerstones of coherence for any reputable risk measure. Building upon this foundational framework, Föllmer and Schied (2002), in tandem with the pioneering efforts of Frittelli and Rosazza-Gianin (2002) expanded the purview of risk measures. They introduced the broader class which are convex risk measures by dropping one of the coherency axioms. Song and Yan (2009) gave an overview of representation theorems for various static risk measures. Shushi and Yao (2020) proposed two multivariate risk measures based on conditional expectation and derived explicit formulae for exponential dispersion models. Zuo and Yin (2022) considered the multivariate tail covariance for generalized skew-elliptical distributions. Cai et al. (2022) defined a new multivariate conditional Value-at-Risk risk measure based on the minimization of the expectation of a multivariate loss function. While these advancements have introduced sophisticated risk measures, it’s important to highlight that their theoretical foundation frequently exists within a static framework. The conven-
tional depiction of these theories operates within a fixed temporal frame, offering a foundational understanding of risk.

Over the past two decades, not only has the study of static risk measures flourished, but also dynamic theories of risk measurement have developed into a thriving and mathematically refined area of research. Dynamic risk measures represent a sophisticated and evolving field within risk management, extending the analysis beyond static frameworks to account for temporal changes in risk. Unlike traditional static risk measures that provide a snapshot assessment, dynamic risk measures recognize the fluid nature of financial markets and aim to capture how risk evolves over time. Introduced by Riedel (2004), dynamic coherent risk measures offer a framework that allows for a more nuanced understanding of risk dynamics. This advancement enables a comprehensive assessment of risk in the context of changing market conditions and evolving investment portfolios. Additionally, the introduction of dynamic convex risk measures by Detlefsen and Scandolo (2005) further enriched the field, providing insights into the time consistency properties of risk measures over different time horizons. Cheridito et al. (2006) considered dynamic coherent, convex monetary and monetary risk measures for discrete-time processes modelling the evolution of financial values. Acciaio et al. (2012) extended dynamic convex risk measures in Cheridito et al. to take the timing of cash flow into consideration. Sun and Hu (2018) introduced a new class of set-valued risk measures which satisfies cash sub-additivity and investigated dynamic set-valued cash sub-additive risk measures. Wang et al. (2021) constructed some general continuous-time equilibrium dynamic risk measures through using a adapted solution to a backward stochastic Volterra integral equation. Chen et al. (2018) and Sun et al. (2018) extended convex risk measures to loss-based cases. More recent research on dynamic risk measures reference in Chen et al. (2021), Chen and Feinstein (2022), Mastrogiacomo and Rosazza (2022), Yoshioka and Yoshioka (2024)

Nowadays, as the digital economy and cryptocurrencies develop rapidly, they have a great impact on the financial market. The volatility of cryptocurrencies is a distinctive characteristic defined by rapid and substantial price fluctuations within relatively short periods. Compared to traditional financial assets, cryptocurrencies such as Bitcoin and Ethereum are considered as asset which can be used for speculative purpose, hence it can lead to extreme volatility and bubbles (see Fry and Cheah, 2016). Factors contributing to this volatility include market sentiment, regulatory developments, technological advancements, and macroeconomic conditions. Apart from extreme volatility of cryptocurrencies, different orders of risk data are mixed in a short period of time can occur when multiple levels or types of risk factors are simultaneously influencing the financial market. For ex-
ample, major economic events, such as financial crises, geopolitical tensions, or central bank policy announcements, can trigger rapid and varied responses across different asset classes. This simultaneous impact can result in mixed risk signals. In addition, sudden and unexpected shocks to the market, whether related to economic indicators, corporate news, or global events, can lead to a convergence of various risk factors that creating a mixed picture of risk data where Yang et al. (2023) found that an increase in economic policy uncertainty in the China and US exacerbates fluctuation in the global oil price, particularly during times of crisis. Besides, high correlations between different asset classes or markets can lead to a synchronization of risk data. For example, during periods of heightened risk aversion, equities, currencies, and commodities may all exhibit increased volatility simultaneously. Therefore, the need for comprehensive risk measures that can capture the complexity and increasing fluctuation of market volatility is significant not only for new financial assets but also for traditional financial market with rapidly changing financial environment and global landscape.

Although there are many different studies of risk measure, a common aspect of these studies is the space of financial positions was considered as the space or subspace of $L^p$. It is crucial to note that our focus diverges from this conventional space. As financial markets grow in complexity, traditional position spaces may prove inadequate in capturing market volatility. This has raised awareness of the urgent need for more appropriate risk measures under a financial systems with much greater volatility. It is necessary to explore a new position space that can depict the mixed occurrence of different orders of risk data. From our point of view, to better capture the increasing fluctuation of market volatility should base on risk measures with reasonable position spaces and these spaces should allow for higher volatility in the risk positions within them. A comprehensive study on risk measures in a novel space could address the issue of increasing fluctuation of market volatility and also it can provide some theoretic foundation for major importance to maintain stability of the financial systems. This paper aims to provide axiomatic theories of risk measures on a special space which reflects complexities of market volatility in the contemporary economic environment.

By contrast with some classical literature on convex risk measures (such as Frittelli and Rosazza-Gianin (2002), Detlefsen and Scandolo (2005)), we aim to extend risk measures to a new space: the variable exponent Bochner–Lebesgue space, which is denoted by $L^{p(\cdot)}$. Under this space, the order $p(\cdot)$ is no longer a fixed positive number like $L^p$, but a measurable function. Variable exponent Lebesgue spaces were first described by Orlicz(1931). More recent studies on variable exponent Lebesgue spaces include those of Almeida et al. (2008), Diening et al. (2009), Harjulehto et al. (2010), Hästö (2009), Wei and Xu (2019), Kaltenbach
and Ruzicka (2021), Benavides and Japón (2021) and reference therein. To the best of our knowledge, this is the first time that variable exponent Lebesgue space has been applied to risk measures to capture increasing fluctuation of market volatility.

In this paper, we first introduce convex risk measures on \( L^p(\cdot) \). With the help of these convex risk measures on \( L^p(\cdot) \), we consider dynamic and cash sub-additive risk measures on \( L^p(\cdot) \). By further developing the axioms related to these classes of risk measures, we are able to derive their dual representations. Moreover, the optimized certainty equivalent on variable exponent Bochner–Lebesgue spaces is investigated as an example. The main contribution of this paper are risk measures on the variable exponent Bochner–Lebesgue space \( L^p(\cdot) \), particularly dynamic risk measures on the variable exponent Bochner–Lebesgue space \( L^p(\cdot) \). The remainder of this paper is organized as follows. In Sect. 2, we briefly review the definition and main properties of variable exponent Bochner–Lebesgue spaces. In Sect. 3, we consider the dual representation of convex risk measures on variable exponent Bochner–Lebesgue spaces. Sect. 4 discusses a class of specific risk measures known as optimized certainty equivalents. Finally, in Sect. 5, we study dual representation of dynamic risk measures on the variable exponent Bochner–Lebesgue spaces. The related time consistency is also studied.

\section{Preliminaries}

In this section, we briefly introduce the definition and main properties of variable exponent Bochner–Lebesgue spaces and some preliminaries that are used throughout this paper.

Let \( (\Omega, \mathcal{F}, \mu) \) be a \( \sigma \)-finite complete measurable space, and \( E \) be a given non-reflexive Banach space with zero element \( \theta \) and its dual space \( E^* \) has Radon-Nikodym property. In the rest of this article we assume that \( E^* \) is partially ordered by a given cone \( K_0 \) and \( E \) is partially ordered by the dual wedge \( K \) of \( K_0 \), where \( K := \{ f \in E | \langle X, f \rangle \geq 0 \text{ for any } X \in K_0 \} \) is the positive dual cone of \( K_0 \). We say that \( K \) is a wedge if it satisfies \( K + K \subseteq K \) and \( \lambda K \subseteq K \) for any \( \lambda \geq 0 \).

In fact, the \( K \subseteq E \) to be the wedge of the financial positions which implies safety conditions for a set of investors. The wedge \( K_0 \subseteq E^* \) is actually the set of ‘admissible’ prices with respect to this wedge \( K \) and being the set of spot-price functionals which assigns a positive price for any such position \( X \in K \).

Throughout this paper, we suppose that the numeraire asset \( z \) of \( E \) is some interior point in \( K \). The asset \( z \) is actually either a ‘reference cash stream’, see Stoica (2006) or a ‘relatively secure cash stream’ as in Jaschke and Küchler (2001).
Remark 2.1. In this paper, the partial order relation $\geq_K$ is defined as follows. For any $X, Y \in E$,
\[ X \geq_K Y \iff X - Y \in K. \]

As pointed out by Konstantinides and Kountzakis (2011), the wedge under which $E$ is partially ordered is consisted by those positions whose ownership indicates safety for the previously mentioned set of investors. In fact, $K$ can also be viewed as the solvency set of financial positions, which denotes the way that a set of investors jointly interpret the common notion of the cost of financial positions.

Example 2.1. If Banach space $E = L^\infty(\Omega, \mathcal{F}, \mu)$, the financial explanation of the cone $K$ becomes easier. In this case $K$ is defined by $K \subseteq \mathbb{R}$ and $\mathbb{R}^+ \subseteq K$. In fact, the frictions between the markets are modeled by the solvency cone $K$. Typically, The cone $K$ includes all positions of reference instruments which can be exchanged, by paying transaction costs, into reference positions with nonnegative entries only. If $K = \mathbb{R}^+$, then there isn’t any possible exchange.

The Banach-space-valued Bochner–Lebesgue spaces with variable exponents were first introduced by Cheng and Xu (2013). Now recall the definition and related properties of this special space. We denote the set of all $\mathcal{F}$-measurable functions $p(\cdot) : \Omega \to [1, \infty]$ by $\mathcal{S}(\Omega, \mu)$. These functions are said to be variable exponent on $\Omega$. For a function $p(\cdot) \in \mathcal{S}(\Omega, \mu)$, we define $p'(\cdot) \in \mathcal{S}(\Omega, \mu)$ by $1/p(y) + 1/p'(y) = 1$. The following definitions and properties come from Cheng and Xu (2013).

Definition 2.1. A function $f : \Omega \to E$ is strongly $\mathcal{F}$-measurable if there exists a sequence $\{f_n\}_{n \geq 1}$ converging to $f$ $\mu$-almost everywhere.

Definition 2.2. The Bochner–Lebesgue space with variable exponent, which is denoted by $L^{p(\cdot)}(\cdot)(\Omega, E)$, is the collection of all strongly $\mathcal{F}$-measurable functions $f : \Omega \to E$ endowed with the norm
\[ \|f\|_{L^{p(\cdot)}(\Omega, E)} := \inf \{\lambda > 0, \rho_{p(\cdot)}(f/\lambda) \leq 1\} \]
where
\[ \rho_{p(\cdot)}(f) := \int_\Omega \|f(y)\|^{p(y)}d\mu(y) \] and $p(\cdot) \in \mathcal{S}(\Omega, \mu)$.

Remark 2.2. If $E$ is a reflexive Banach space, then the dual of $L^{p(\cdot)}(\Omega, E)$ is characterized by the mapping $g \mapsto V_g$ with $g \in L^{p'(\cdot)}(\Omega, E^*)$ and $V_g \in (L^{p(\cdot)}(\Omega, E))^*$ as follows:
\[ \langle V_g, f \rangle = \int_\Omega \langle g, f \rangle d\mu \] for any $f \in L^{p(\cdot)}(\Omega, E)$.

See Cheng and Xu (2013).
3 Convex risk measures on $L^{p(\cdot)}$

Recently, many external factors including changes in international situations, increase of war risk and significant environmental changes, all cause the financial market becomes much more volatile than before, and the financial market also has different orders of risk data mixed over a short period of time. Therefore, it is valuable to study the risk measures on variable exponent Bochner–Lebesgue spaces. Under this position space, the order risk position is no longer a fixed positive number, but a measurable function. The characteristics of this space are able to characterise risk positions in the above volatile financial market.

The main purpose of this paper is to study the dynamic risk measures on variable exponent Bochner–Lebesgue spaces. This section firstly considers convex risk measures on $L^{p(\cdot)}$, which will be used later for the dynamic risk measures. In the absence of ambiguity, we denote the variable exponent Bochner–Lebesgue space by $L^{p(\cdot)} := L^{p(\cdot)}(\Omega, E)$. Let $T$ be a discrete time horizon which can reach infinity and consider a filtered probability space $(\Omega, F, (F_t)_{t=0}^T, \mathbb{P})$ with $\{\emptyset, \Omega\} = F_0 \subset F_1 \subset \ldots \subset F_T = F$. Let $L^{p(\cdot)}(F_t)$ be the space of all strongly $F_t$-measurable functions $f_t$ which satisfy Definition 2.2. Note that $L^{p(\cdot)} = L^{p(\cdot)}(F_T)$. We denote

$$L^{p(\cdot)}(K) := \{f \in L^{p(\cdot)}| f : \Omega \to K\},$$
and

$$L^{p(\cdot)}(K_0) := \{g \in L^{p(\cdot)}| g : \Omega \to K_0\}.$$ We then denote the space of all essentially bounded $F_t$-measurable random variables by $L^{\infty}_t := L^{\infty}(\Omega, F_t, \mu)$. The definition of a convex risk measure on $L^{p(\cdot)}$ is introduced by using an axiomatic approach.

**Definition 3.1.** A continuous function $\varrho^K : L^{p(\cdot)} \to \mathbb{R}$ is said to be a $p(\cdot)$-convex risk measure (with partial order $\leq_K$) if it satisfies the following conditions.

**A1** Monotonicity: for any $f_1, f_2 \in L^{p(\cdot)}$, $f_1 \leq_K f_2$ a.s. implies $\varrho^K(f_1) \geq \varrho^K(f_2)$;

**A2** Translation invariance: for any $m \in \mathbb{R}$ and $f \in L^{p(\cdot)}$, $\varrho^K(f+ mz) = \varrho^K(f) − m$;

**A3** Convexity: for any $f_1, f_2 \in L^{p(\cdot)}$ and $\lambda \in (0, 1)$, $\varrho^K(\lambda f_1 + (1 − \lambda)f_2) \leq \lambda \varrho^K(f_1) + (1 − \lambda)\varrho^K(f_2)$.

**Remark 3.1.** In **A1**, $f_1 \leq_K f_2$ means $f_1(\omega) \leq_K f_2(\omega)$ for any $\omega \in \Omega$ where the partial order $\leq_K$ is defined in Remark 2.1. The interior point $z$ of $K$ in **A2** is considered to be the numeraire asset, which means that $mz \in E$ for any $m \in \mathbb{R}$. Before the dual representation of the $p(\cdot)$-convex risk measures is being studied, the acceptance sets should be defined.
Definition 3.2. The acceptance set of the $p(\cdot)$-risk measure (with partial order $\leq_K$) $\varrho^K$ is defined as

$$A_{\varrho^K} := \{ f \in L^p(\cdot) | \varrho^K(f) \leq 0 \}$$

and we denote $A^0_{\varrho^K}$ by

$$A^0_{\varrho^K} := \{ g \in (L^p(\cdot))^* | \langle g, f \rangle \geq 0 \text{ for any } f \in A_{\varrho^K} \}.$$ 

Remark 3.2. It is relatively easy to check that $A_{\varrho^K}$ is a convex set if $\varrho^K$ satisfies the convexity property. $A^0_{\varrho^K}$ can be considered as the positive polar cone of $A_{\varrho^K}$.

Now, the dual representation of $p(\cdot)$-convex risk measures is provided and which will be used in the proof of $p(\cdot)$-dynamic risk measures in Sect. 5.

Theorem 3.1. If $\varrho^K : L^p(\cdot) \rightarrow \mathbb{R}$ is a $p(\cdot)$-convex risk measure (with partial order $\leq_K$), then for any $f \in L^p(\cdot)$,

$$\varrho^K(f) = \sup_{g \in Q_p(\cdot)} \{ \langle g, -f \rangle - \alpha(g) \}$$

where

$$Q_p(\cdot) := \{ g \in (L^p(\cdot))^* | \left( \frac{dg}{d\mu} \right) z = 1, \frac{dg}{d\mu} \in L^{p'(\cdot)}(K_0) \},$$

and $\alpha : Q_p(\cdot) \rightarrow \mathbb{R}$ is the penalty function while the minimal penalty function $\alpha_{\min}$ is denoted by

$$\alpha_{\min}(g) := \sup_{f \in L^p(\cdot)} \{ \langle g, -f \rangle - \varrho^K(f) \} = \sup_{f \in A_{\varrho^K}} \{ \langle g, -f \rangle \}.$$ 

Proof. For any $g \in Q_p(\cdot)$, denoting

$$\alpha(g) = \sup_{f \in L^p(\cdot)} \{ \langle g, -f \rangle - \varrho^K(f) \}$$

and

$$\alpha_{\min}(g) = \sup_{f \in A_{\varrho^K}} \{ \langle g, -f \rangle \}.$$ 

We now show that $\alpha(g) = \alpha_{\min}(g)$ for any $g \in Q_p(\cdot)$. For any $f \in A_{\varrho^K}, \langle g, -f \rangle - \varrho^K(f) \geq \langle g, -f \rangle$. Hence,

$$\sup_{f \in L^p(\cdot)} \{ \langle g, -f \rangle - \varrho^K(f) \} \geq \sup_{f \in A_{\varrho^K}} \{ \langle g, -f \rangle - \varrho^K(f) \} \geq \sup_{f \in A_{\varrho^K}} \{ \langle g, -f \rangle \}.$$
That is $\alpha(g) \geq \alpha_{\min}(g)$. Then for any $f \in L^{p(\cdot)}$, consider $f_1 = f + g^K(f)z \in A_{g^K}$. Thus,

$$\alpha_{\min}(g) \geq \langle g, -f_1 \rangle = \langle g, -f \rangle - g^K(f)\langle g, z \rangle = \langle g, -f \rangle - g^K(f)\int_{\Omega} \frac{dg}{d\mu} z d\mu = \langle g, -f \rangle - g^K(f).$$

That is $\alpha(g) \leq \alpha_{\min}(g)$. As $\alpha(g) \geq \alpha_{\min}(g)$ and $\alpha(g) \leq \alpha_{\min}(g)$, hence we have $\alpha(g) = \alpha_{\min}(g)$, and it is easy to check that

$$g^K(f) \geq \sup_{g \in Q_{p(\cdot)}} \{\langle g, -f \rangle - \alpha(g)\}.$$

Next, we show that the above inequality only holds in the case of equality. Suppose there is some $f_0 \in L^{p(\cdot)}$ such that

$$g^K(f_0) > \sup_{g \in Q_{p(\cdot)}} \{\langle g, -f_0 \rangle - \alpha(g)\}.$$

Hence, there exists some $m \in \mathbb{R}$ such that

$$g^K(f_0 + mz) = g^K(f_0) - m > 0,$$

which means that $f_0 + mz \notin A_{g^K}$. As $\{f_0 + mz\}$ is a singleton set, it is also a convex set. Meanwhile, $A_{g^K}$ is also a closed convex set because $g^K$ is a $p(\cdot)$-convex risk measure. Then, by the Strong Separation Theorem for convex sets, there exists some $\pi \in (L^{p(\cdot)})^*$ such that

$$\langle \pi, f_0 + mz \rangle > \sup_{f \in A_{g^K}} \langle \pi, f \rangle. \quad (3.1)$$

By Remark 2.2, $\langle \pi, f \rangle = \int_{\Omega} \langle h, f \rangle d\mu$, where $h \in L^{p(\cdot)}(\Omega, E^*)$. It is easy to check that $\langle \pi, f \rangle = \int_{\Omega} \langle h, f \rangle d\mu \leq 0$ for any $f \in L^{p(\cdot)}(K)$. Then, we have that $-h \in L^{p(\cdot)}(K_0)$. For any $-\pi \in Q_{p(\cdot)}$, $\langle h, z \rangle = -1$. Thus, by (3.1), it gives

$$\langle \pi, f_0 + mz \rangle > \sup_{f \in A_{g^K}} \langle \pi, f \rangle \Rightarrow \int_{\Omega} \langle h, f_0 + mz \rangle d\mu > \sup_{f \in A_{g^K}} \int_{\Omega} \langle h, f \rangle d\mu \Rightarrow \int_{\Omega} \langle (h, f_0) - m \rangle d\mu > \sup_{f \in A_{g^K}} \int_{\Omega} \langle h, f \rangle d\mu.$$
⇒ \int_{\Omega} \langle h, f_0 \rangle d\mu - m > \sup_{f \in \mathcal{A}_{\kappa}} \int_{\Omega} \langle h, f \rangle d\mu

⇒ \langle \pi, f_0 \rangle - \sup_{f \in \mathcal{A}_{\kappa}} \langle \pi, f \rangle > m

⇒ \langle \pi, f_0 \rangle - \alpha(\pi) > m.

By replacing $-\pi$ by $g_0$, we have

$$\langle g_0, -f_0 \rangle - \alpha(g_0) > m.$$ 

This is a contradiction, because in this case

$$m > \sup_{g \in \mathcal{Q}_{p(\cdot)}} \{ \langle g, -f_0 \rangle - \alpha(g) \} \geq \langle g_0, -f_0 \rangle - \alpha(g_0) > m.$$ 

The contradiction arises from the assumption that some $f_0 \in L^{p(\cdot)}$ exists such that

$$g^K(f_0) > \sup_{g \in \mathcal{Q}_{p(\cdot)}} \{ \langle g, -f_0 \rangle - \alpha(g) \}.$$ 

Hence, we have

$$g^K(f) = \sup_{g \in \mathcal{Q}_{p(\cdot)}} \{ \langle g, -f \rangle - \alpha(g) \}.$$ 

For the opposite direction, it is relatively simple to check that $g^K$ satisfies the properties of a $p(\cdot)$-convex risk measure. This completes the proof of Theorem 3.1.

**Remark 3.3.** The $p(\cdot)$-convex risk measures are extensions of convex risk measures in Föllmer and Schied (2002). In fact, if the position space $L^{p(\cdot)}$ degenerates into the ordinary $L^p$ space, then the dual representation in Theorem 3.1 also becomes the ordinary dual representation of convex risk measures.

An example of the degenerated dual representation of $p(\cdot)$-convex risk measure in Theorem 3.1 on $L^1$ is given in below. However, before deriving the degenerated dual representation of $p(\cdot)$-convex risk measure on $L^1$, a cone $C_0$ in $L^\infty$ needs to be found so that the numeraire asset $e$ is an interior point of $C$, where $C$ is the positive dual cone of $C_0$. In fact, this cone $C$ can be seen as the degeneration of $K$ for $L^{p(\cdot)}$. According to Proposition 2.4 and what is mentioned in Jameson (1970), we have that $e$ is an interior point of $C$ by considering the cone $C_0 = \{ h \in L^\infty(\Omega, \mathcal{F}, \mu) | \int_{\Omega} h d\mu \geq \frac{1}{2} \| h \|_{\infty} \}$. 

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Example 3.1. Let $L^p(\cdot)$ degenerates into $L^1(\Omega, \mathcal{F}, \mu)$, $z$ degenerates into $e$ and $L^\infty(\Omega, \mathcal{F}, \mu)$ is partially ordered by the cone $C_0 = \{ h \in L^\infty(\Omega, \mathcal{F}, \mu) \mid \int_{\Omega} h \, d\mu \geq \frac{1}{2} \|h\|_\infty \}$, then $e \in \text{int}(C)$ and we may suppose that $L^\infty(\Omega, \mathcal{F}, \mu)$ is partially ordered by the wedge $C$. Then by Feinstein and Rudloff (2013) and Feinstein and Rudloff (2015), we have the risk measures $\rho^K_{\text{deg}} : L^1(\Omega, \mathcal{F}, \mu) \to \mathbb{R}$ represented in the way that Theorem 3.1 indicates,

$$\rho^K_{\text{deg}}(X) = \sup_{\mathcal{Y} \in \mathcal{H}} \{ \mathcal{Y}(-X) - \beta(\mathcal{Y}) \},$$

where

$$\mathcal{H} = \{ \mathcal{Y} \in C_0 \mid \mathcal{Y}(e) = 1 \}.$$

A special example of $p(\cdot)$-convex risk measures which is so called OCE, is discussed in the next section. Finally, in Sect. 5, the $p(\cdot)$-convex risk measures are used to study the dual representation of the $p(\cdot)$-dynamic risk measures.

4 Optimized Certainty Equivalent on $L^p(\cdot)$

In this section, a special class of $p(\cdot)$-convex risk measures that is the Optimized Certainty Equivalent (OCE) is studied and it will be used as an example of dynamic risk measures in Sect. 5. The OCE was first introduced by Ben-Tal and Teboulle (1986) and later developed by the same researchers Ben-Tal and Teboulle (2007). In this section, the OCE on variable exponent Bochner–Lebesgue spaces $L^p(\cdot)$ is defined. Further, we establish its main properties and show that how it can be used to generate $p(\cdot)$-convex risk measures. Note that the OCE can be used as an application of convex risk measures. Other applications of risk measures such as the application in insurance, see Wang and Peng (2017).

Definition 4.1. Let $u : E \to [-\infty, +\infty]$ be a closed, concave, and non-decreasing (partially ordered by $K$) function. Suppose that $u(\theta) = 0$, where $\theta$ is the zero element of $E$. We denote the set of such $u$ by $U$.

Remark 4.1. For any $u \in U$ and $f \in L^p(\cdot)$, we denote by $u(f) : \Omega \to \mathbb{R}$ and $\mathbb{E}u(f)$ the expectation of $u(f)$ with respect to a probability measure $\mu$.

Definition 4.2. For any $u \in U$ and $f \in L^p(\cdot)$, the OCE of some uncertain outcome $f$ is defined by the map $S_u : L^p(\cdot) \to \mathbb{R},$

$$S_u(f) = \sup_{\eta \in \mathbb{R}} \{ \eta + \mathbb{E}u(f - \eta z) \}$$

where the domain of $S_u$ is defined as $\text{dom}S_u = \{ f \in L^p(\cdot) \mid S_u(f) > -\infty \} \neq \emptyset$ and $S_u$ is finite on $\text{dom}S_u$. 

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Theorem 4.1. For any $u \in U$, the following properties hold for $S_u$:

(a) For any $f \in L^p(\cdot)$ and $m \in \mathbb{R}$, $S_u(f + m z) = S_u(f) + m$;

(b) For any $f_1, f_2 \in L^p(\cdot)$, $f_1 \leq_K f_2$ a.s. implies that $S_u(f_1) \leq S_u(f_2)$;

(c) For any $f_1, f_2 \in L^p(\cdot)$ and $\lambda \in (0, 1)$, $S_u(\lambda f_1 + (1 - \lambda) f_2) \geq \lambda S_u(f_1) + (1 - \lambda) S_u(f_2)$.

Proof.

(a) For any $f \in L^p(\cdot)$, $m \in \mathbb{R}$,
\[
S_u(f + m z) = \sup_{\eta \in \mathbb{R}} \{\eta + \mathbb{E} u(f + m z - \eta z)\} \\
= m + \sup_{\eta \in \mathbb{R}} \{\eta - m + \mathbb{E} u(f - (\eta - m) z)\} \\
= m + S_u(f).
\]

(b) For any $f_1, f_2 \in L^p(\cdot)$ with $f_1 \leq_K f_2$, we have $f_1 - \eta z \leq_K f_2 - \eta z$. As $u$ is non-decreasing, we have
\[
S_u(f_1) = \sup_{\eta \in \mathbb{R}} \{\eta + \mathbb{E} u(f_1 - \eta z)\} \leq \sup_{\eta \in \mathbb{R}} \{\eta + \mathbb{E} u(f_2 - \eta z)\} = S_u(f_2).
\]

(c) For any $f_1, f_2 \in L^p(\cdot)$ and $\lambda \in (0, 1)$,
\[
S_u(\lambda f_1 + (1 - \lambda) f_2) = \sup_{\eta \in \mathbb{R}} \{\eta + \mathbb{E} u(\lambda f_1 + (1 - \lambda) f_2 - \eta z)\}.
\]

We take $\eta = \lambda \eta_1 + (1 - \lambda) \eta_2$. Then,
\[
S_u(\lambda f_1 + (1 - \lambda) f_2) \\
= \sup_{\eta_1, \eta_2 \in \mathbb{R}} \{\lambda \eta_1 + (1 - \lambda) \eta_2 + \mathbb{E} u(\lambda (f_1 - \eta_1 z) + (1 - \lambda) (f_2 - \eta_2 z))\} \\
\geq \sup_{\eta_1, \eta_2 \in \mathbb{R}} \{\lambda \eta_1 + (1 - \lambda) \eta_2 + \lambda \mathbb{E} u(f_1 - \eta_1 z) + (1 - \lambda) \mathbb{E} u(f_2 - \eta_2 z)\} \\
= \sup_{\eta_1, \eta_2 \in \mathbb{R}} \{\lambda (\eta_1 + \mathbb{E} u(f_1 - \eta_1 z)) + (1 - \lambda) (\eta_2 + \mathbb{E} u(f_2 - \eta_2 z))\} \\
= \lambda S_u(f_1) + (1 - \lambda) S_u(f_2).
\]

This completes the proof of Theorem 4.1.
Theorem 4.2. The function \( g^K \), defined as \( g^K(f) := -S_u(f) \) for any \( f \in L^p(\cdot) \), is a \( p(\cdot) \)-convex risk measure.

Proof: The proof of Theorem 4.2 is straightforward from Theorem 4.1.

Proposition 4.1. For any \( u \in U, \alpha \in \mathbb{R}^+, \) and \( f \in L^p(\cdot) \), the OCE \( S_u(f) \) is sub-homogeneous, i.e.

(a) \( S_u(\alpha f) \leq \alpha S_u(f) \), \quad \forall \alpha > 1; \\
(b) \( S_u(\alpha f) \geq \alpha S_u(f) \), \quad \forall 0 \leq \alpha \leq 1.

Proof. Denote \( S(\alpha) := \frac{1}{\alpha}S_u(\alpha f) \), then,

\[
S(\alpha) = \frac{1}{\alpha}S_u(\alpha f) = \sup_{\eta \in \mathbb{R}} \left\{ \eta + \frac{1}{\alpha}u(\alpha(f - \eta z)) \right\}.
\]  

(4.1)

Next, we show that \( S(\alpha) \) is non-increasing in \( \alpha > 0 \) for any \( f \in L^p(\cdot) \). For \( \alpha_2 \geq \alpha_1 \geq 0 \), we have

\[
\frac{u(\alpha_2 t) - u(\alpha_1 t)}{\alpha_2 - \alpha_1} \leq \frac{u(\alpha_1 t) - u(\theta)}{\alpha_1 - 0}, \quad \text{for any } t \in E
\]

by the concavity of \( u \). As \( u(\theta) = 0 \), we have

\[
\frac{1}{\alpha_2}u(\alpha_2 t) \leq \frac{1}{\alpha_1}u(\alpha_1 t).
\]

Then, from (4.1), we have \( S(\alpha_1) \geq S(\alpha_2) \), which clearly implies (a) and (b).

Proposition 4.2. (Second-order stochastic dominance) We denote \( C_u(f) \) by \( uC_u(f) := E u(f) \) for any \( u \in U \) and \( f \in L^p(\cdot) \). We also assume that the supremum in the definition of \( S_u \) is attained. Then, for any \( f_1, f_2 \in L^p(\cdot) \),

\[
S_u(f_1) \geq S_u(f_2) \quad \text{if and only if} \quad C_u(f_1) \geq C_u(f_2).
\]

Proof. At first, we show the “if” part. If \( C_u(f_1) \geq C_u(f_2) \), we have \( Eu(f_1) \geq Eu(f_2) \) by the fact that \( u \) is non-decreasing. From the definition of \( S_u \), it follows that \( S_u(f_1) \geq S_u(f_2) \). Then we show the “only if” part. Let \( \ell_{f_1}, \ell_{f_1} \) be the points where the suprema of \( S_u(f_1) \) and \( S_u(f_2) \) are attained, respectively. Then, for any \( u \in U \),

\[
S_u(f_1) = \ell_{f_1} + Eu(f_1 - \ell_{f_1} z) \geq \ell_{f_2} + Eu(f_2 - \ell_{f_2} z) \geq \ell_{f_1} + Eu(f_2 - \ell_{f_1} z),
\]

where the first inequality comes from \( S_u(f_1) \geq S_u(f_2) \). Therefore, for any \( u \in U \),

\[
Eu(f_1 - \ell_{f_1} z) \geq Eu(f_2 - \ell_{f_1} z), \quad \text{which implies } Eu(f_1) \geq Eu(f_2).
\]

Then, \( C_u(f_1) \geq C_u(f_2) \).
5 Dynamic risk measures on $L^p(\cdot)$

In this section, we extend the definition of $p(\cdot)$-convex risk measures in Sect. 3 to a dynamic setting. Similar to the axiomatic approach in Detlefsen and Scandolo (2005), we first define the conditional $p(\cdot)$-convex risk measures.

**Definition 5.1.** A map $\varrho^K_t : L^p(\cdot) \to L^\infty_t$ is called a conditional $p(\cdot)$-convex risk measure (with partial order $\leq^K_t$) if it satisfies the following properties for all $f, f_1, f_2 \in L^p(\cdot)$:

i. Monotonicity: $f_1 \leq^K_t f_2$ a.s. implies $\varrho^K_t(f_1) \geq \varrho^K_t(f_2)$;

ii. Conditional cash invariance: for any $m_t \in L^\infty_t$, $\varrho^K_t(f + m_t) = \varrho^K_t(f) - m_t$;

iii. Conditional convexity: for any $\lambda \in L^\infty_t$ with $\lambda \in [0, 1]$, $\varrho^K_t(\lambda f_1 + (1 - \lambda) f_2) \leq \lambda \varrho^K_t(f_1) + (1 - \lambda) \varrho^K_t(f_2)$;

iv. Normalization: $\varrho^K_t(\theta) = 0$, $\varrho^K_t(f) < \infty$.

**Remark 5.1.** Note that any element in $L^\infty_t := L^\infty(\Omega, \mathcal{F}_t, \mu)$ is a random variable, where $\mathcal{F}_t$ is a sub-$\sigma$-algebra of $\mathcal{F}$. As stated by Detlefsen and Scandolo [?], if the additional information is described by a sub-$\sigma$-algebra $\mathcal{F}_t$ of the total information $\mathcal{F}_T$, then a conditional risk measure is a map assigning an $\mathcal{F}_t$-measurable random variable $\varrho^K_t(f)$, representing the conditional riskiness of $f$, to every $\mathcal{F}_T$-measurable function $f$, representing a final payoff.

The acceptance set of a conditional $p(\cdot)$-convex risk measure $\varrho^K_t$ is defined as

$$\mathcal{A}_t := \{ f \in L^p(\cdot) | \varrho^K_t(f) \leq 0 \} \text{ for any } 0 \leq t \leq T. \quad (5.1)$$

The corresponding stepped acceptance set is defined as

$$\mathcal{A}_{t,t+s} := \{ f \in L^p(\cdot)(\mathcal{F}_{t+s}) | \varrho^K_t(f) \leq 0 \} \text{ for any } 0 \leq t < t + s \leq T. \quad (5.2)$$

**Proposition 5.1.** The acceptance set $\mathcal{A}_t$ of a conditional $p(\cdot)$-convex risk measure $\varrho^K_t$ has the following properties:

1. Conditional convexity: for any $f_1, f_2 \in \mathcal{A}_t$, and an $\mathcal{F}_t$-measurable function $\alpha$ with $0 \leq \alpha \leq 1$, we have $\alpha f_1 + (1 - \alpha) f_2 \in \mathcal{A}_t$;

2. Solidity: for any $f_1 \in \mathcal{A}_t$ with $f_1 \leq^K_t f_2$ a.s. implies $f_2 \in \mathcal{A}_t$;

3. Normalization: $\theta \in \mathcal{A}_t$. 

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Proof. It is easy to check properties 1–3 by using Definition 5.1.

Definition 5.2. A sequence \((g^K_t)_{t=0}^T\) is called a dynamic \(p(\cdot)\)-convex risk measure if each \(g^K_t\) is a conditional \(p(\cdot)\)-convex risk measure for any \(0 \leq t \leq T\).

We now study the dual representation of a conditional \(p(\cdot)\)-convex risk measure. At first, the notion of the \(\mathcal{F}_t\)-conditional inner product related to \(L^p(\cdot)\) should be defined.

Definition 5.3. For any \(f \in L^p(\cdot)\) and \(g \in (L^p(\cdot))^*\), we define the \(\mathcal{F}_t\)-conditional inner product \(\langle g, -f \rangle_t\) by
\[
\int_{\Omega} \langle g, -f \rangle_t dP = \langle g, -f \rangle_t,
\]
where \(P \in \{P \in (\Omega, \mathcal{F}_t)|P\text{ is absolutely continuous w.r.t. } \mu \text{ and } P = \mu \text{ on } \mathcal{F}_t\}\). We also define the minimal penalty function \(\alpha_{t}^{\min}\) as
\[
\alpha_{t}^{\min}(g) := \text{ess sup}_{f \in \mathcal{A}_t} \langle g, -f \rangle_t.
\]

Lemma 5.1. For any \(g \in Q_{p(\cdot)}\) and \(0 \leq t \leq T\),
\[
\int_{\Omega} \alpha_{t}^{\min}(g) dP = \sup_{f \in \mathcal{A}_t} \langle g, -f \rangle_t.
\]

Proof. We first show that there exists a sequence \((f_n)_{n \in \mathbb{N}}\) in \(\mathcal{A}_t\) such that
\[
\text{ess sup}_{f \in \mathcal{A}_t} \langle g, -f \rangle_t = \lim_{n \to \infty} \langle g, -f_n \rangle_t.
\]
Indeed, for any \(f_1, f_2 \in \mathcal{A}_t\), we define \(\widehat{f} := f_1 I_B + f_2 I_{B^c}\) where \(B := \{\langle g, -f_1 \rangle_t \geq \langle g, -f_2 \rangle_t\}\). By property 1 of Proposition 5.1, we know that \(\widehat{f} \in \mathcal{A}_t\). Hence, by the definition of \(\widehat{f}\),
\[
\langle g, -\widehat{f} \rangle_t = \max\{\langle g, -f_1 \rangle_t, \langle g, -f_2 \rangle_t\}.
\]
Thus, (5.6) holds. We now have
\[
\int_{\Omega} \alpha_{t}^{\min}(g) dP = \int_{\Omega} \text{ess sup}_{f \in \mathcal{A}_t} \langle g, -f \rangle_t dP.
\]
\[
\int_{\Omega} \lim_{n \to \infty} \langle g, -f_n \rangle_t dP = \lim_{n \to \infty} \int_{\Omega} \langle g, -f_n \rangle_t dP = \lim_{n \to \infty} \langle g, -f_n \rangle_t \leq \sup_{f \in A_t} \langle g, -f \rangle_t.
\]

The converse inequality can be checked easily.

The following theorem gives the dual representation of conditional \( p(\cdot) \)-convex risk measures.

**Theorem 5.1.** Suppose \( g^K_t \) is a conditional \( p(\cdot) \)-convex risk measure (with partial order \( \leq_K \)). Then, the following statements are equivalent.

1. \( g^K_t \) has the robust representation
   \[
   g^K_t(f) = \mathrm{ess} \sup_{g \in Q_{p(\cdot)}} \{ \langle g, -f \rangle_t - \alpha(t)(g) \} \quad \text{for any } f \in L^{p(\cdot)},
   \]
   where
   \[
   Q_{p(\cdot)} := \left\{ g \in \left( L^{p(\cdot)} \right)^* : \left( \frac{dg}{d\mu}, z \right) = 1, \frac{dg}{d\mu} \in L^{p(\cdot)}(K_0) \right\},
   \]
   and \( \alpha(t) \) is the penalty function from \( Q_{p(\cdot)} \) to the set of \( \mathcal{F}_t \)-measurable random variables such that \( \mathrm{ess} \sup_{g \in Q_{p(\cdot)}} \{-\alpha(t)(g)\} = 0 \);

2. \( g^K_t \) has a robust representation in terms of the minimal function, i.e.
   \[
   g^K_t(f) = \mathrm{ess} \sup_{g \in Q_{p(\cdot)}} \{ \langle g, -f \rangle_t - \alpha^\text{min}_t(g) \} \quad \text{for any } f \in L^{p(\cdot)};
   \]

3. \( g^K_t \) is continuous from above under \( K \), i.e.
   \[
   f_n \searrow f \Rightarrow g^K_t(f_n) \nearrow g^K_t(f).
   \]

**Proof.** (2) \( \Rightarrow \) (1) is obvious. We first prove (1) \( \Rightarrow \) (3). Using Lemma 5 of Cheng and Xu (2013), suppose that \( f_n \searrow f \). Then, by the monotonicity of \( g^K_t \), we have \( g^K_t(f_n) \nearrow g^K_t(f) \).

Next, we show (3) \( \Rightarrow \) (2). The inequality
\[
g^K_t(f) \geq \mathrm{ess} \sup_{g \in Q_{p(\cdot)}} \{ \langle g, -f \rangle_t - \alpha^\text{min}_t(g) \}
\]
is a direct consequence of the definition of $\alpha_{t}^{\text{min}}$. Now, we need only show the inverse inequality. To this end, we define a map $\tilde{\rho}^{K} : L^{p}(\cdot) \to \mathbb{R}$ as $\tilde{\rho}^{K}(f) = \int_{\Omega} g^{K}_{t}(f) dP$. It is easy to check that $\tilde{\rho}^{K}$ is a $p(\cdot)$-convex risk measure as defined in Sect. 3 which is continuous from above. Hence, by Theorem 3.1, we know that $\tilde{\rho}^{K}$ has the dual representation

$$
\tilde{\rho}^{K}(f) = \sup_{g \in Q_{p(\cdot)}} \{ \langle g, -f \rangle - \alpha(g) \}, \quad f \in L^{p}(\cdot),
$$

where the minimum penalty function $\alpha_{\text{min}}$ is given by $\alpha_{\text{min}}(g) := \sup_{f \in A_{t}^{\tilde{\rho}^{K}}} \{ \langle g, -f \rangle \}$.

By Lemma 5.1, we have

$$
\int_{\Omega} \alpha_{t}^{\text{min}}(g) dP = \sup_{f \in A_{t}} \langle g, -f \rangle
$$

for any $g \in Q_{p(\cdot)}$. As $\tilde{\rho}^{K}(f) \leq 0$ for all $f \in A_{t}$,

$$
\int_{\Omega} \alpha_{t}^{\text{min}}(g) dP = \sup_{f \in A_{t}} \langle g, -f \rangle \leq \alpha(g)
$$

for any $g \in Q_{p(\cdot)}$. Thus, we have

$$
\int_{\Omega} g_{t}^{K}(f) dP = \tilde{\rho}^{K}(f)
$$

$$
= \sup_{g \in Q_{p(\cdot)}} \{ \langle g, -f \rangle - \alpha(g) \}
$$

$$
\leq \sup_{g \in Q_{p(\cdot)}} \left\{ \int_{\Omega} \langle g, -f \rangle dP - \int_{\Omega} \alpha_{t}^{\text{min}}(g) dP \right\}
$$

$$
= \sup_{g \in Q_{p(\cdot)}} \left\{ \int_{\Omega} \langle (g, -f)_{t} - \alpha_{t}^{\text{min}}(g) \rangle dP \right\}
$$

$$
\leq \int_{\Omega} \text{ess sup}_{g \in Q_{p(\cdot)}} \{ \langle g, -f \rangle_{t} - \alpha_{t}^{\text{min}}(g) \} dP.
$$

Thus, (5.8) holds.

**Remark 5.2.** The conditional $p(\cdot)$-convex risk measures (with partial order $\leq_{K}$) are extensions of conditional convex risk measures introduced in Detlefsen and Scandolo (2005). In fact, if the position space $L^{p(\cdot)}$ degenerates into the ordinary $L^{p}$ space, then the dual representation in Theorem 5.1 will also degenerates into the dual representation of conditional convex risk measures in Detlefsen and Scandolo (2005).
Now, with the definition and dual representation, we consider the time consistency of dynamic $p(\cdot)$-convex risk measures.

**Definition 5.4.** A dynamic $p(\cdot)$-convex risk measure $(\varrho^K_t)_{t=0}^T$ is said to be time consistent if, for all $f_1, f_2 \in L^p(\cdot)$ and $0 \leq t < t + s \leq T$,

$$\varrho^K_{t+s}(f_1) \leq \varrho^K_{t+s}(f_2) \Rightarrow \varrho^K_t(f_1) \leq \varrho^K_t(f_2). \quad (5.10)$$

**Remark 5.3.** Time consistency means that if two payoffs will have the same riskiness tomorrow in every state of nature, then the same conclusion should be drawn today.

**Theorem 5.2.** Let $(\varrho^K_t)_{t=0}^T$ be a dynamic $p(\cdot)$-convex risk measure (with partial order $\leq_K$) such that each $\varrho^K_t$ is continuous from above. Then, the following conditions are equivalent for any $0 \leq t < t + s \leq T$:

1). $(\varrho^K_t)_{t=0}^T$ is time consistent;

2). $A_t = A_{t,t+s} + A_{t+s};$

3). $\varrho^K_t(-\varrho^K_{t+s}(f)z) = \varrho^K_t(f)$ for any $f \in L^p(\cdot)$.

**Proof.** We first show the equivalence between 1) and 3). Suppose that 3) holds and $\varrho^K_{t+s}(f_1) \leq \varrho^K_{t+s}(f_2)$ for any $f_1, f_2 \in L^p(\cdot)$. Then, by the monotonicity of $\varrho^K_t$,

$$\varrho^K_t(f_1) = \varrho^K_t(-\varrho^K_{t+s}(f_1)z) \leq \varrho^K_t(-\varrho^K_{t+s}(f_1)z) = \varrho^K_t(f_2).$$

Next, suppose that $(\varrho^K_t)_{t=0}^T$ is time consistent, and set $f_2 := -\varrho^K_{t+s}(f_1)z$ and then by Definition 5.1, we get that $-\varrho^K_{t+s}(f_1)z = -\varrho^K_{t+s}(f_2)z$ for any $f_1 \in L^p(\cdot)$. Thus,

$$\varrho^K_t(f_1) = \varrho^K_t(f_2) = \varrho^K_t(-\varrho^K_{t+s}(f_1)z).$$

We now show the equivalence between 2) and 3). To this end, suppose that 3) holds and let $f_1 \in A_{t,t+s}$, $f_2 \in A_{t+s}$. Then, setting $f := f_1 + f_2$, we have

$$\varrho^K_{t+s}(f) = \varrho^K_{t+s}(f_1 + f_2) = \varrho^K_{t+s}(f_2) - \frac{f_1}{z} \leq -\frac{f_1}{z}.$$

Thus, by the monotonicity of $\varrho^K_t$, we know that

$$\varrho^K_t(f) = \varrho^K_t(-\varrho^K_{t+s}(f)z) \leq \varrho^K_t(f_1) \leq 0,$$

which implies

$$A_t \supseteq A_{t,t+s} + A_{t+s}.$$
For the inverse relation, let \( f \in \mathcal{A}_t \) and define \( f_2 := f + \varphi^{K}_{t+s}(f)z \), \( f_1 := f - f_2 = -\varphi^{K}_{t+s}(f)z \). Then, by the conditional cash invariance of \( \varphi^{K}_t \), it is easy to check that \( f_1 \in \mathcal{A}_{t,t+s}, f_2 \in \mathcal{A}_{t+s}, \) which implies

\[
\mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s}.
\]

Let us now suppose that 2) holds and \( f \in \mathcal{A}_t \). It is easy to check that \( f + \varphi^{K}_{t+s}(f)z \in \mathcal{A}_{t+s} \). Then, with \( \mathcal{A}_t \subseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \), we have \( -\varphi^{K}_{t+s}(f)z \in \mathcal{A}_{t,t+s} \). Hence, we know that \( \varphi^{K}_t(-\varphi^{K}_{t+s}(f)z) \leq 0 \), which implies

\[
\varphi^{K}_t(-\varphi^{K}_{t+s}(f)z) \leq \varphi^K_t(f).
\]

Now, we need only show the inverse inequality. Indeed, for any \( f \in L^p(\cdot) \) such that \( -\varphi^{K}_{t+s}(f)z \in \mathcal{A}_{t,t+s} \), we have \( \varphi^{K}_t(-\varphi^{K}_{t+s}(f)z) \leq 0 \). It is easy to check that \( f + \varphi^{K}_{t+s}(f)z \in \mathcal{A}_{t+s} \). Thus, by \( \mathcal{A}_t \supseteq \mathcal{A}_{t,t+s} + \mathcal{A}_{t+s} \), we have \( f \in \mathcal{A}_t \), which implies

\[
\varphi^{K}_t(-\varphi^{K}_{t+s}(f)z) \geq \varphi^K_t(f).
\]

The case for the recursive property strongly relies on the validity of conditional cash invariance for \( \varphi^{K}_t \), and hence on the interpretation as conditional capital requirements. In fact, if \( \varphi^{K}_{t+s}(f) \) is the conditional capital requirement that has to be set aside at date \( t + s \) in view of the final payoff \( f \), then the risky position is equivalently described, at date \( t \), by the payoff \( \varphi^{K}_t(-\varphi^{K}_{t+s}(f)z) \) occurring in \( t + s \).

We end this section with a special example of conditional \( p(\cdot) \)-convex risk measures (with partial order \( \leq_K \)).

**Example 5.1.** (Conditional OCE) Let \( u : E \to \mathbb{R} \) be a closed, concave, and non-decreasing (partially ordered by \( K \)) function and suppose that \( u(\theta) = 0 \), where \( \theta \) is the zero element of \( E \). Then, for any \( f \in L^p(\cdot) \), the conditional OCE of some uncertain outcome \( f \) is defined by the map \( S_u : L^p(\cdot) \to L^\infty_t \):

\[
S_u(f) = \text{ess sup}_{\eta \in L^\infty_t} \left\{ \eta + \mathbb{E}[u(f - \eta z)] | \mathcal{F}_t \right\}.
\]

Thus, by Definition 5.1, it is easy to check that the function \( \varphi^K_t \) defined as \( \varphi^K_t(f) := -S_u(f) \) for any \( f \in L^p(\cdot) \) is a conditional \( p(\cdot) \)-convex risk measure.
6 Conclusion

In this paper, we introduced risk measures on a unique variable exponent Bochner–Lebesgue space denoted as $L^{p(\cdot)}$ and then explored dynamic and cash sub-additive risk measures in this space. Through further refinement of the axioms associated with these risk measures, we derived their dual representations. Additionally, we delved into the investigation of the optimized certainty equivalent on variable exponent Bochner–Lebesgue spaces as an illustrative example. The study of this paper gives a new set of risk measures to capture the fluctuation of volatility of financial markets and further investigation on how these risk measures can be applied to financial markets can be carried on.

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Competing interests

The authors declare no conflict of interest.

References

[1] Acciaio B, Föllmer H, Penner I. Risk assessment for uncertain cash flows: Model ambiguity, discounting ambiguity, and the role of bubbles[J]. Finance and Stochastics. 2012, 16: 669-709.

[2] Almeida, A, Hasanov, J, Samko, S. Maximal and potential operators in variable exponent Morrey spaces[J]. Georgian Mathematical Journal. 2008, 15, 195-208

[3] Artzner P, Delbaen F, Eber JM, Heath D. Coherent measures of risk[J]. Mathematical Finance. 1999, 9(3): 203-228.
[4] Benavides TD, Japón MA. Fixed point properties and reflexivity in variable Lebesgue spaces[J]. Journal of Functional Analysis. 2021, 15; 280(6):108896.

[5] Ben-Tal A, Teboulle M. Expected utility, penalty functions, and duality in stochastic nonlinear programming[J]. Management Science. 1986, 32(11):1445-66.

[6] Ben-Tal A, Teboulle M. An old-new concept of convex risk measures: The optimized certainty equivalent[J]. Mathematical Finance. 2007, 17(3):449-76.

[7] Cai J, Jia H, Mao T. A multivariate CVaR risk measure from the perspective of portfolio risk management[J]. Scandinavian Actuarial Journal. 2022, 2022(3): 189-215.

[8] Chen Y H, Sun F, Hu, Y J. Coherent and convex loss-based risk measures for portfolio vectors[J]. Positivity. 2018, 22(1), 399-414.

[9] Chen Y, Feinstein Z. Set-valued dynamic risk measures for processes and for vectors[J]. Finance and Stochastics. 2022, 26(3):505-33.

[10] Chen Z, Lin Y, Xiao Z, Zhang G. Laws of Large Numbers for Dynamic Coherent Risk Measures[J]. Journal of Mathematical Finance. 2021, 12(1):301-23.

[11] Cheng C, Xu J. Geometric properties of Banach space valued Bochner-Lebesgue spaces with variable exponent[J]. Journal of Mathematical Inequalities. 2013, 7(3), 461-475

[12] Cheridito P, Delbaen F, Kupper M. Dynamic Monetary Risk Measures for Bounded Discrete-Time Processes[J], Electronic Journal of Probability. 2006, 11(3), 57-106.

[13] Detlefsen K, Scandolo G. Conditional and dynamic convex risk measures[J]. Finance and Stochastics. 2005, 9: 539-561.

[14] Diening L, Hästö P, Roudenko S. Function spaces of variable smoothness and integrability[J]. Journal of Functional Analysis. 2009, 15; 256(6):1731-68.

[15] Feinstein Z, Rudloff B. Time consistency of dynamic risk measures in markets with transaction costs[J]. Quantitative Finance. 2013, 13(9):1473-89.

[16] Feinstein Z, Rudloff B. Multi-portfolio time consistency for set-valued convex and coherent risk measures[J]. Finance and Stochastics. 2015, 19(1):67-107.
[17] Föllmer H, Schied A. Convex measures of risk and trading constraints[J]. Finance and Stochastics. 2002, 6: 429-447.

[18] Frittelli M, Gianin E R. Putting order in risk measures[J]. Journal of Banking & Finance. 2002, 26(7): 1473-1486.

[19] Fry J, Cheah ET. Negative bubbles and shocks in cryptocurrency markets[J]. International Review of Financial Analysis. 2016,1;47:343-52.

[20] Harjulehto P, Hästö P, Lê Út V Nuortio, M. Overview of differential equations with non-standard growth[J]. Nonlinear Analysis: Theory, Methods & Applications. 2010,15;72(12):4551-74.

[21] Hästö PA. Local-to-global results in variable exponent spaces[J]. Mathematical Research Letters. 2009,16(2):263-78.

[22] Jameson G. Ordered Linear Spaces, Lecture Notes in Math. 1970, vol. 141, Springer-Verlag.

[23] Jaschke S, Küchler U. Coherent risk measures and good-deal bounds[J]. Finance and Stochastics. 2001, 5:181-200.

[24] Konstantinides D G., Kountzakis C E. Risk measures in ordered normed linear spaces with non-empty cone-interior[J]. Insurance: Mathematics and Economics. 2011, 48(1):111-22.

[25] Kaltenbach A, Ruzicka M. Variable exponent Bochner–Lebesgue spaces with symmetric gradient structure[J]. Journal of Mathematical Analysis and Applications. 2021, 503(2): 125355.

[26] Mastrogiacomo E, Rosazza Gianin E. Dynamic capital allocation rules via BSDEs: an axiomatic approach[J]. Annals of Operations Research. 2022, 28:1-24.

[27] Orlicz W. Über konjugierte Exponentenfolgen[J]. Studia Mathematica. 1931;3(1):200-11.

[28] Riedel F. Dynamic coherent risk measures[J]. Stochastic Processes and Their Applications. 2004, 112(2): 185-200.

[29] Shushi T, Yao J. Multivariate risk measures based on conditional expectation and systemic risk for exponential dispersion models[J]. Insurance: Mathematics and Economics. 2020, 93: 178-186.
[30] Song Y, Yan J. An overview of representation theorems for static risk measures[J]. Science in China Series A: Mathematics. 2009, 52(7): 1412-1422.

[31] Sun F, Chen Y H, Hu Y J. Set-valued loss-based risk measures[J]. Positivity. 2018, 22(3), 859-871.

[32] Sun F, Hu Y. Set-valued cash sub-additive risk measures[J]. Probability in the Engineering and Informational Sciences. 2019, 33(2):241-57.

[33] Stoica G. Relevant coherent measures of risk[J]. Journal of Mathematical Economics. 42, 794-806, 2006.

[34] Wang H, Sun J, Yong J. Recursive utility processes, dynamic risk measures and quadratic backward stochastic Volterra integral equations[J]. Applied Mathematics & Optimization. 2021, 84(1):145-90.

[35] Wang W Y, Peng X C. Reinsurer’s optimal reinsurance strategy with upper and lower premium constraints under distortion risk measures[J]. Journal of Computational and Applied Mathematics. 2017, 315, 142-160.

[36] Wei H, Xu J. Simultaneous Approximations in Banach Space-Valued Bochner–Lebesgue Spaces with Variable Exponent[J]. Numerical Functional Analysis and Optimization. 2019, 40(1): 19-33.

[37] Yang T, Zhou F, Du M, Du Q, Zhou S. Fluctuation in the global oil market, stock market volatility, and economic policy uncertainty: a study of the US and China[J]. The Quarterly Review of Economics and Finance. 2023, 87: 377-87.

[38] Yoshioka H, Yoshioka Y. Assessing fluctuations of long-memory environmental variables based on the robustified dynamic Orlicz risk[J]. Chaos, Solitons & Fractals. 2024, 80:114336.

[39] Zuo B, Yin C. Multivariate tail covariance risk measure for generalized skew-elliptical distributions[J]. Journal of Computational and Applied Mathematics. 2022, 410: 114210.