Gluon polarization tensor in a thermo-magnetic medium

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We compute the gluon polarization tensor in a thermo-magnetic environment in the weak and strong magnetic field cases, both at zero and at high temperature. The magnetic field effects are introduced using Schwinger’s proper time method. Thermal effects are computed in the Hard Thermal Loop approximation. We find that for the zero temperature case and a non-vanishing quark mass, the coefficients of the tensor structures describing the polarization tensor develop an imaginary part corresponding to the threshold for quark-antiquark pair production. These coefficients are infrared finite and simplify considerably when the quark mass vanishes. In the high temperature case, the quark mass can be safely ignored. Nevertheless, we explicitly maintain this mass finite to separate the unrenormalized vacuum and matter contributions. We discuss how knowledge of this coefficients is useful in particular to study the thermo-magnetic evolution of the strong coupling in the four regimes hereby treated.

PACS numbers: 12.38.Bx, 11.10.Wx,25.75.-q, 13.40.-f
Keywords: QCD, Gluon Polarization Tensor, Magnetic Fields, Hard Thermal Loops.

I. INTRODUCTION

The properties of strongly interacting matter immersed in a magnetized medium have been the subject of intense research over the last years. The motivation for this activity stems from several fronts: On the one hand, lattice QCD (LQCD) $^1$ has shown that for temperatures above the chiral restoration pseudo-critical temperature, the quark-antiquark condensate decreases and that this temperature itself also decreases, both as functions of the field intensity. This result, dubbed inverse magnetic catalysis (IMC), has sparked a large number of explanations $^2-12$. On the other hand, it has been argued that intense magnetic fields can be produced in peripheral heavy-ion collisions. Possible signatures of the presence of such fields in the interaction region can be the chiral magnetic effect $^13$ or the enhanced production of prompt photons $^14,15$. Moreover, magnetic fields can have an impact on the properties of compact astrophysical objects, such as neutron stars $^{16}$.

The dispersive properties for gluons propagating in a magnetized medium are encoded in the gluon polarization tensor. This tensor has been computed and studied in detail for the case of intense magnetic fields where the lowest Landau Level (LLL) approximation can be used $^{19,20}$. However, several physical situations may involve magnetic fields whose intensities are smaller compared to other energy scales, such as the temperature. For this reason it is desirable to study the properties of the gluon polarization tensor in several regimes such as the weak and strong field cases, both at zero and finite temperature.

In this work we undertake such task. We make a thorough study of the gluon polarization tensor in the presence of either a weak or a strong constant magnetic field, at zero and finite temperature. This paper is organized as follows. In Sec. III we compute the gluon polarization tensor in the presence of a uniform and constant magnetic field, both at zero and high temperature, writing in each case the most general tensor structure and discussing its properties. In Sec. III we summarize and discuss our results. We reserve for the appendices the calculation details for each of the regimes where the gluon polarization tensor is computed.

II. THERMO-MAGNETIC GLUON POLARIZATION TENSOR

We proceed to compute the gluon polarization tensor at one-loop order in the presence of a magnetic field, both in vacuum and in a thermal bath. In order to have a more complete picture of its thermo-magnetic behavior, we perform the computation in four different regimes implementing a hierarchy among the energy (squared) scales relevant for the problem, namely, the gluon’s momentum squared $q^2$, the temperature squared, $T^2$ and the strength of the magnetic field $|eB|$. The hierarchy considered is the weak field limit $|eB| < |q^2|$ and the strong field limit $|eB| > |q^2|$, both at zero and at high temperature. In general, the one-loop contribution to the
The gluon polarization tensor, depicted in Fig. 1, is given by

\[ \Pi_{(ab)}^{\mu\nu} = -\frac{\alpha_s}{2\pi} \int \frac{d^4k}{(2\pi)^4} \text{Tr} \{ i g t_\mu \gamma^\nu i S_F(k) i g t_\alpha \gamma^\mu i S_F(k - q) \}, \]

where \( g \) is the coupling constant, \( S_F(k) \) is the quark propagator and \( t_a, b \) are the generators of the color group.

Since the quark anti-quark pair in the loop interact with the magnetic field, the quark propagator is modified from its vacuum expression and is written as

\[ S_F(x, x') = \Phi(x, x') \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-x')} S(k), \]

where \( \Phi(x, x') \) is called the Schwinger phase factor. The latter accounts for the loss of Lorentz invariance in the presence of the magnetic field. For the present calculation, the phase factor can be gauged away \[11, 21\] and we need just work with the translationally invariant part of the fermion propagator. The latter is given by

\[ iS(k) = \int_0^\infty \frac{ds}{\cos(qfBs)} e^{is(k^2 - k_\perp^2 \cos(qfBs) - m_f^2 - \frac{q^2}{k^2} \kappa_\parallel)} \times \left\{ \cos(qfBs) + \gamma_1 \gamma_2 \sin(qfBs) \right\}(m_f + \kappa_\parallel) \]

where \( q_f \) and \( m_f \) are the value of the quark’s electric charge and mass, respectively. Hereafter, we use the following notation for the parallel and perpendicular (with respect to the magnetic field) pieces of the scalar product of two four-vectors \( a^\mu \) and \( b^\mu \)

\[ (a \cdot b)_\parallel = a_0 b_0 - a_3 b_3, \]
\[ (a \cdot b)_\perp = a_1 b_1 + a_2 b_2, \]

such that

\[ a \cdot b = (a \cdot b)_\parallel - (a \cdot b)_\perp. \]

Gauge invariance requires that the gluon polarization tensor be transverse. However, the breaking of Lorentz symmetry makes this tensor to split into three transverse structures, such that the gluon polarization tensor (omitting the diagonal color factor \( \delta_{ab} \)) can be written as \[22\] (see also Ref. \[23\])

\[ \Pi^{\mu\nu} = P^{\parallel} \Pi_{\parallel}^{\mu\nu} + P^{\perp} \Pi_{\perp}^{\mu\nu} + P^{0} \Pi_{0}^{\mu\nu}, \]

where

\[ \Pi_{\parallel}^{\mu\nu} = g_{\parallel}^{\mu\nu} - \frac{q^\mu q^\nu}{q_\parallel^2}, \]
\[ \Pi_{\perp}^{\mu\nu} = g_{\perp}^{\mu\nu} - \frac{q^\mu q^\nu}{q_\perp^2}, \]
\[ \Pi_{0}^{\mu\nu} = g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2} - (\Pi_{\parallel}^{\mu\nu} + \Pi_{\perp}^{\mu\nu}). \]

Notice that the three tensor structures in Eq. (7) are orthogonal to each other, hence, their coefficients in Eq. (6) can be expressed as

\[ P^{\parallel} = \Pi^{\parallel}_{\mu\nu}, \]
\[ P^{\perp} = \Pi^{\perp}_{\mu\nu}, \]
\[ P^{0} = \Pi^{0}_{\mu\nu}. \]

We now proceed to compute each of the coefficients in Eq. (8) in the weak and strong field limits.

**A. Weak field approximation**

In the weak field limit, \( |q_f B| \ll q_f \), the quark propagator can be written as \[21\]

\[ iS(k) = \int_0^\infty \frac{ds}{\cos(qfBs)}\frac{\cos(qfBs) + \gamma_1 \gamma_2 \sin(qfBs)}{(m_f + \kappa_\parallel)} (m_f + \kappa_\parallel) \]

When using this propagator, the first nontrivial magnetic contribution to the gluon polarization tensor turns out to be of order \( O(|q_f B|^2) \). The relevant diagrams are depicted in Fig. 2. The photon lines represent the coupling of the magnetic field to the quark-antiquark pair in the loop. There are two kinds of contributions to order \( O(|q_f B|^2) \): The first one comes from the product of the linear terms in Eq. (9), and the second one results from the product of the vacuum and the quadratic terms. We call the first term \( \Pi^{(11)}_{\nu\mu} \) and the second term \( \Pi^{(20)}_{\nu\mu} \). Their explicit expressions are
\[ i \Pi^{\mu \nu}_{(11)} = -\frac{4g^2}{2} \sum_f q_f^2 B^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu (k-q)^\nu + k^\nu (k-q)^\mu + [g^{\mu \nu} - g_{\perp}^{\mu \nu}] (m_f^2 - (k-q)_\parallel \cdot k_\parallel)}{[(k-q)^2 - m_f^2] [k^2 - m_f^2]^2} \]
\[ i \Pi^{\mu \nu}_{(20)} = -\frac{8g^2}{2} \sum_f q_f^2 B^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{[k^2 - m_f^4][(k-p)^2 - m_f^2]} \times \left[ k^\mu (k-q)_\parallel + k^\nu (k-q)_\parallel \right] k_\perp^2 - (k_\perp^2 - m_f^2) [k_\perp^2 (k_\perp^2 - q^2) + k_\perp^2 (k_\perp^2 - q^2) - g^{\mu \nu} k_\perp \cdot (k-q)_\perp] . \tag{10} \]

The gluon polarization tensor in this limit is given by \( i \Pi^{\mu \nu} = i \Pi^{\mu \nu}_{(11)} + 2i \Pi^{\mu \nu}_{(20)} \). It is easy to show that the required contractions with the polarization tensor are given by

\[ \Pi^{\mu \nu}_{(11)} = \frac{4g^2}{2} \sum_f q_f^2 B^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu (k\cdot q) - 2(k-q)_\parallel}{[(k-q)^2 - m_f^2] [k^2 - m_f^2]^2} . \]
\[ \Pi^{\mu \nu}_{(11)} = \frac{4g^2}{2} \sum_f q_f^2 B^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu - (k\cdot q) - m_f^2}{[(k-q)^2 - m_f^2] [k^2 - m_f^2]^2} . \]
\[ \Pi^{\mu \nu}_{(20)} = \frac{8g^2}{2} \sum_f q_f^2 B^2 \int \frac{d^4k}{(2\pi)^4} \frac{k^\mu - m_f^2}{[(k-q)^2 - m_f^2] [k^2 - m_f^2]^2} \times \left( k_\perp^2 - m_f^2 \right) \frac{2(k\cdot q) (k\cdot q)_\perp}{q_\perp^2} - 3(k\cdot q)_\perp + k_\perp^2 \left( 2k_\perp^2 + 3(k\cdot q)_\parallel \right) - \frac{k_\perp^2}{q_\perp^2} (2k\cdot q) (k\cdot q)_\parallel . \tag{11} \]

Introducing Feynman’s parametrization and after integrating over the four-momentum \( k \) (see Appendix A), the contractions with \( \Pi^{\mu \nu}_{(11)} \) and \( \Pi^{\mu \nu}_{(20)} \) result in the following expressions

\[ \Pi^{\mu \nu}_{(11)} = \frac{g^2}{2} \sum_f q_f^2 B^2 \int_0^1 dx \frac{x(1-x)}{[m_f^2 - x(1-x)q^2]^2} \left( q_\parallel^2 x(x - 1) - m_f^2 \right) , \]
\[ \Pi^{\mu \nu}_{(11)} = \frac{g^2}{2} \sum_f q_f^2 B^2 \int_0^1 dx \frac{x(1-x)}{[m_f^2 - x(1-x)q^2]^2} \left( 2m_f^2 - q_\perp^2 x(x - 1) \right) , \]
\[ \Pi^{\mu \nu}_{(11)} = \frac{g^2}{2} \sum_f q_f^2 B^2 \int_0^1 dx \frac{x(1-x)}{[m_f^2 - x(1-x)q^2]^2} \left( m_f^2 \frac{2q_\parallel^2 + q_\perp^2}{q_\perp^2 - q_\perp^2} \right) \]
\[ \Pi^{\mu \nu}_{(20)} = \frac{g^2}{2} \sum_f q_f^2 B^2 \int_0^1 dx \frac{(1-x)^3}{[m_f^2 - x(1-x)q^2]^3} \left( 3m_f^4 + m_f^2 x(q_\parallel^2(2x - 3) + q_\perp^2(x + 1)) - q_\perp^2(x - 1)x^3(q_\parallel^2 + 3q_\perp^2) \right) , \]
\[ \Pi^{\mu \nu}_{(20)} = \frac{g^2}{2} \sum_f q_f^2 B^2 \int_0^1 dx \frac{(1-x)^3}{[m_f^2 - x(1-x)q^2]^3} \left( 2m_f^4 + m_f^2 x(2q_\parallel^2 x(x - 1) + 3q_\perp^2 x) - q_\perp^2 x^3(x - 1)(3q_\parallel^2 + 3q_\perp^2) \right) , \]
\[ \Pi^{\mu \nu}_{(20)} = \frac{g^2}{2} \sum_f q_f^2 B^2 \int_0^1 dx \frac{(1-x)^3}{[m_f^2 - x(1-x)q^2]^3} \times \left\{ \frac{1}{q_\parallel^2 - q_\perp^2} [m_f^4(2q_\parallel^2 - 3q_\perp^2) + m_f^2 x(2q_\parallel^2 x(x - 1) + q_\parallel^2 q_\perp^2(5x + 3) - q_\perp^2(x + 1)) + 2q_\parallel^2 q_\perp^2(x - 1)x^3(q_\parallel^2 - q_\perp^2)] \right\} . \tag{12} \]
Figure 3. Coefficients $P^\parallel (a, b)$, $P^\perp (c, d)$ and $P^0 (e, f)$ in the weak field limit at $T = 0$ as functions of $y = q^2/m_f^2$ for two fixed values of $y_{\parallel} = q_{\parallel}^2/m_f^2$ and $y_{\perp} = q_{\perp}^2/m_f^2$ for $m_f \neq 0$. The coefficient are normalized to the quantity $A = (q^2/2) \sum_f (q_f^2 B^2)/m_f^2).

Using Eqs. (12) into Eq. (3), one can compute the coefficients of the transverse structures in Eq. (6). For the general case when $m_f \neq 0$, these coefficients are given by large expressions. Here we restrict ourselves to provide examples of their behavior as functions of $q^2 = q_{\parallel}^2 - q_{\perp}^2$. Since the magnetic field breaks Lorentz invariance, Fig. (3) shows these coefficients as functions of $y = q^2/m_f^2$ for fixed values of either $y_{\parallel} = q_{\parallel}^2/m_f^2$ or $y_{\perp} = q_{\perp}^2/m_f^2$. The procedure to explicitly find these coefficients is described in Appendix A. For the case $m_f = 0$, the coefficients become considerably simpler and are given by
Following expression is obtained by adding these two contributions, resulting in the strong field limit, namely
\[
\Pi_{\mu\nu}(m_f - \vec{k})\Pi_{\mu\nu}(m_f - (\vec{k} - \vec{q})_||) O^\pm = m_j^2 Tr[\gamma^\nu (\vec{O}^\pm) \gamma^\mu (\vec{O}^\pm)] + Tr[\gamma^\nu \vec{k} \Pi_{\mu\nu} (\vec{O}^\pm) + Tr[\gamma^\nu \vec{k} \Pi_{\mu\nu} (\vec{O}^\pm)]
\]
Substituting Eqs. (15) and (15) into Eq. (17), we obtain

\[
i\Pi_{\mu\nu} + i\Pi^\mu_{\nu} = 2g^2 \sum_f \int \frac{d^2 k_\perp}{(2\pi)^2} e^{-\frac{q^2}{2\gamma_f m_l^2}} e^{-\frac{(q_\perp - k_\perp)^2}{2\gamma_f m_l^2}}
\]
\[
\times \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{1}{|k_\parallel - m_j^2|} \frac{|q_\parallel - k_\parallel|^2}{|q_\parallel - k_\parallel|^2}
\]
\[
\times \{Tr[\gamma^\nu (m_f - \vec{k}) \Pi_{\mu\nu} (m_f - (\vec{k} - \vec{q})_||) O^\pm] + Tr[\gamma^\nu (m_f - \vec{k}) \Pi_{\mu\nu} (m_f - (\vec{k} - \vec{q})_||) O^-\}
\]
\[
+ Tr[\gamma^\nu (m_f - \vec{k}) \Pi_{\mu\nu} (m_f - (\vec{k} - \vec{q})_||) O^+\}
\]

After integrating over the transverse components of the four-momentum, the expression for the polarization tensor becomes

\[iS_{\text{LLL}}(k) = 2e^{-\frac{\gamma_1^2}{\gamma_2^2} k_\parallel + m_f^2} O^\pm, \quad \text{(14)}\]

where

\[O^\pm = \frac{1}{2} \{1 \pm i\gamma_1 \gamma_2 \text{sign}(q_f B)\}. \quad \text{(15)}\]

Figure 4 shows the diagrams contributing to the calculation. These represent one and the other possible electric charge flow direction within the loop, which, in the presence of the magnetic field and in the LLL, have both to be accounted for. Using Eq. (14), into the Eq. (11), the explicit expression for diagram (a) in Fig. 4 is given by

\[i\Pi_{\mu\nu} + i\Pi^\mu_{\nu} = 2g^2 \sum_f \int \frac{d^2 k_\perp}{(2\pi)^2} e^{-\frac{q^2}{2\gamma_f m_l^2}} e^{-\frac{(q_\perp - k_\perp)^2}{2\gamma_f m_l^2}} \]
\[
\times \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{1}{|k_\parallel - m_j^2|} \frac{|q_\parallel - k_\parallel|^2}{|q_\parallel - k_\parallel|^2}
\]
\[
\times \{Tr[\gamma^\nu (m_f - \vec{k}) \Pi_{\mu\nu} (m_f - (\vec{k} - \vec{q})_||) O^\pm] + Tr[\gamma^\nu (m_f - \vec{k}) \Pi_{\mu\nu} (m_f - (\vec{k} - \vec{q})_||) O^-\}
\]
\[
+ Tr[\gamma^\nu (m_f - \vec{k}) \Pi_{\mu\nu} (m_f - (\vec{k} - \vec{q})_||) O^+\}
\]
\[
\text{(16)}\]

The contribution from diagram (b) in Fig. 4 is obtained by replacing $O^+ \rightarrow O^-$. The polarization tensor is obtained by adding these two contributions, resulting in the following expression

\[i\Pi_{\mu\nu} + i\Pi^\mu_{\nu} = 2g^2 \sum_f \int \frac{d^2 k_\perp}{(2\pi)^2} e^{-\frac{q^2}{2\gamma_f m_l^2}} e^{-\frac{(q_\perp - k_\perp)^2}{2\gamma_f m_l^2}} \]
\[
\times \int \frac{d^2 k_\parallel}{(2\pi)^2} \frac{1}{|k_\parallel - m_j^2|} \frac{|q_\parallel - k_\parallel|^2}{|q_\parallel - k_\parallel|^2}
\]
\[
\times \{Tr[\gamma^\nu (m_f - \vec{k}) \Pi_{\mu\nu} (m_f - (\vec{k} - \vec{q})_||) O^\pm] + Tr[\gamma^\nu (m_f - \vec{k}) \Pi_{\mu\nu} (m_f - (\vec{k} - \vec{q})_||) O^-\]
\[
+ Tr[\gamma^\nu (m_f - \vec{k}) \Pi_{\mu\nu} (m_f - (\vec{k} - \vec{q})_||) O^+\}],
\]
\[
\text{(17)}\]
\[ i \Pi^\mu_\alpha + i \Pi^\mu_\beta = g^2 \sum_f \left( \frac{|q_f B|}{2\pi} e^{\frac{-q^2_f}{\pi\sigma_f m_f^2}} \right) \int_0^1 dx \int \frac{d^d l_\parallel}{(2\pi)^d} \frac{2l_\parallel^\mu - g_\parallel^\mu \Gamma(1 - d/2) - 2x(1 - x)q_f^\alpha q_f^\nu + g_\parallel^\mu (m_f^2 + x(1 - x)q_f^2)}{[l_\parallel^2 - \Delta]^2}, \]  

where in the integrand we have already discarded linear terms in \( l_\parallel \) which give a vanishing contribution. In order to compute the momentum integrals, we recall the following well known relations

\[
\int \frac{d^d l_\parallel}{(2\pi)^d} \frac{1}{[l_\parallel^2 - \Delta]^n} = \begin{cases} \frac{(-1)^n}{(4\pi)^{d/2}} \Gamma(n - d/2)(1/\Delta)^{n-d/2} & \text{for } n \neq 2d/2, \\ \frac{(-1)^{n-1} d \Gamma(n - d/2 - 1)}{(4\pi)^{d/2}} (1/\Delta)^{n-d/2-1} & \text{for } n = 2d/2. \end{cases}
\]

Using these into Eq. (22), we get

\[
i \Pi^\mu_\alpha + i \Pi^\mu_\beta = g^2 \sum_f \left( \frac{|q_f B|}{2\pi} e^{\frac{-q^2_f}{\pi\sigma_f m_f^2}} \right) \int_0^1 dx \int \frac{d^d l_\parallel}{(2\pi)^d} \frac{1}{[l_\parallel^2 - \Delta]^n} \left\{ g_\parallel^\mu \Gamma(1 - d/2)(1/\Delta)^{1-d/2} - g_\parallel^\mu \Gamma(1 - d/2)(1/\Delta)^{1-d/2} + 2x(1 - x)q_f^\alpha q_f^\nu \Gamma(2 - d/2)(1/\Delta)^{2-d/2} - g_\parallel^\mu (m_f^2 + x(1 - x)q_f^2) \Gamma(2 - d/2)(1/\Delta)^{2-d/2} \right\}. \tag{24}
\]

Equation (24) is apparently divergent when taking the limit \( d \to 2 \). To show that this is not the case, we first combine the terms proportional to the tensor structure \( g_\parallel \), namely, the first, second and fourth terms in Eq. (24), to obtain

\[
g_\parallel^\mu \left[ \Gamma(1 - d/2)(1/\Delta)^{1-d/2} - \frac{d}{2} \Gamma(1 - d/2)(1/\Delta)^{1-d/2} - (m_f^2 + x(1 - x)q_f^2) \Gamma(2 - d/2)(1/\Delta)^{2-d/2} \right] \\
= g_\parallel^\mu \left[ (1 - d/2) \Gamma(1 - d/2) \Delta - (m_f^2 + x(1 - x)q_f^2) \Gamma(2 - d/2) \right](1/\Delta)^{2-d/2} \\
= g_\parallel^\mu \left[ (m_f^2 - x(1 - x)q_f^2) - (m_f^2 + x(1 - x)q_f^2) \right] \Gamma(2 - d/2)(1/\Delta)^{2-d/2} \\
= -2q_f^\alpha q_f^\nu \Gamma(2 - d/2)(1/\Delta)^{2-d/2}. \tag{25}
\]

Substituting Eq. (25) into Eq. (24), we get

\[
i \Pi^\mu_\alpha + i \Pi^\mu_\beta = g^2 \sum_f \left( \frac{|q_f B|}{2\pi} e^{\frac{-q^2_f}{\pi\sigma_f m_f^2}} \right) \int_0^1 dx \int \frac{d^d l_\parallel}{(2\pi)^d} \frac{1}{[l_\parallel^2 - \Delta]^n} \left\{ g_\parallel^\mu \left[ -m_f^2 + x(1 - x)q_f^2 + m_f^2 + x(1 - x)q_f^2 \right] - 2x(1 - x)q_f^\alpha q_f^\nu \right\}(1/\Delta)^{2-d/2} \\
= g^2 \sum_f \left( \frac{|q_f B|}{2\pi} e^{\frac{-q^2_f}{\pi\sigma_f m_f^2}} \right) \int_0^1 dx \left(1 - x\right)(1/\Delta)^{2-d/2}. \tag{26}
\]

Notice that Eq. (26) is now free of divergences when taking the limit \( d \to 2 \). We thus obtain

\[
i \Pi^\mu_\alpha + i \Pi^\mu_\beta = ig^2 \sum_f \left( \frac{|q_f B|}{4\pi^2} e^{\frac{-q^2_f}{\pi\sigma_f m_f^2}} \right) \frac{q_f^\alpha q_f^\nu}{q_f^2} \int_0^1 dx \left(1 - x\right)(1/\Delta). \tag{27}
\]

from where it is seen that the emerging tensor structure is equal to \( \Pi^\mu_\parallel \). Therefore, in the strong field limit, the
coefficients $P^\perp$ and $P^\parallel$ are equal to zero.

The integral over the $x$ variable can be expressed in terms of the function

$$I(y_\parallel) = y_\parallel \int_0^1 dx \frac{x(1-x)}{1-x(1-x)y_\parallel},$$

where $y_\parallel \equiv q_\parallel^2/m_f^2$. For the case $m_f = 0$, $I = -1$ and the gluon polarization tensor is given by

$$i\Pi^{\mu\nu} \bigg|_{m_f=0} = (i\Pi^{\mu\nu}_{a} + i\Pi^{\mu\nu}_{b}) \bigg|_{m_f=0} = -ig^2 \left\{ g^{\mu\nu} - \frac{q^{\mu}_0 q^{\nu}_0}{q^2_\parallel} \right\} \times \sum f \left( \frac{|q_f B|}{4\pi^2} \right) e^{-\frac{q^2_\parallel}{2m_f^2}}.$$  (28)

This result coincides (albeit for the case of the photon polarization tensor) with the one found in Ref. [19]. Figure 5 shows the behavior of the function $I(y_\parallel)$ for the general case when $m_f^2 \neq 0$. Notice that this function develops an imaginary part for $y_\parallel \geq 4$, corresponding to the threshold for quark-antiquark production.

**C. Thermo-magnetic polarization tensor in the HTL and LLL approximations**

We now proceed to calculate the polarization tensor in the strong field limit within the Hard Thermal Loop approximation. At finite temperature, the third of Eqs. (27) splits into two structures, due to the presence of the vector $u^\mu$, which defines the medium’s reference frame. These are the (three dimensionally) transverse $\Pi^{\mu\nu}_{T}$ and longitudinal $\Pi^{\mu\nu}_{L}$ structures. Therefore, the gluon polarization tensor can be written as [22]

$$\Pi^{\mu\nu} = P^{T}\Pi^{\mu\nu}_{T} + P^{L}\Pi^{\mu\nu}_{L} + P^{\parallel}\Pi^{\mu\nu}_{\parallel} + P^{\perp}\Pi^{\mu\nu}_{\perp},$$  (30)

with

$$\Pi^{\mu\nu}_{T} = -g^{\mu\nu} + \frac{q^{\mu}_0 q^{\nu}_0}{q^2_\parallel} (q^{\mu}_0 u^{\nu} + u^{\mu} q^{\nu}_0) - \frac{1}{q^2_\parallel} (q^{\mu}_0 q^{\nu}_0 + q^2 u^{\mu} u^{\nu}),$$

$$\Pi^{\mu\nu}_{L} = -g^{\mu\nu} + \frac{q^{\mu}_0 q^{\nu}_0}{q^2_\parallel} (q^{\mu}_0 u^{\nu} + u^{\mu} q^{\nu}_0) + \frac{1}{q^2_\parallel} (q^{\mu}_0 q^{\nu}_0 + q^2 u^{\mu} u^{\nu})$$

$$\Pi^{\mu\nu}_{\parallel} = g^{\parallel\mu\nu} - \frac{q^{\parallel\mu}_0 q^{\parallel\nu}_0}{q^2_\parallel},$$

$$\Pi^{\mu\nu}_{\perp} = g^{\perp\mu\nu} - \frac{q^{\perp\mu}_0 q^{\perp\nu}_0}{q^2_\parallel},$$  (31)

where, in the medium’s reference frame, $u^\mu = (1, 0, 0, 0)$. Since the tensor structures in Eq. (31) are orthogonal to each other, the coefficients in Eq. (30) can be simply expressed as

$$P^{T} = \Pi^{T}_{\mu\nu} \Pi^{\mu\nu}_{T},$$

$$P^{L} = \Pi^{L}_{\mu\nu} \Pi^{\mu\nu}_{L},$$

$$P^{\parallel} = \Pi^{\parallel}_{\mu\nu} \Pi^{\mu\nu}_{\parallel},$$

$$P^{\perp} = \Pi^{\perp}_{\mu\nu} \Pi^{\mu\nu}_{\perp}.$$  (32)

In order to compute the coefficients in Eq. (32), we follow a procedure similar to the one that lead to Eq. (20). This implies using that, in the strong field limit, transverse and parallel structures factorize, and also that temperature effects are obtained, in the Matsubara formalism, from the time-like component of the integration four-vector. Therefore all temperature effects are comprised to the parallel pieces of the integrals. Explicitly,

$$P^{T} = g^2 \sum f \left( \frac{|q_f B|}{2\pi^2} \right) e^{-\frac{q^2_\parallel}{2m_f^2}}$$

$$\times \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{|k^2_\perp - m_f^2||(k - q)^2 - m_f^2|}$$

$$\times \left\{ -4k_0 (k \cdot q)_\parallel + 2q_\parallel k_0 + 2q^2_\parallel q_0 - 2m_f^2 q_0 \right\} \frac{q^2_0}{q^2},$$  (33)

$$P^{L} = g^2 \sum f \left( \frac{|q_f B|}{2\pi^2} \right) e^{-\frac{q^2_\parallel}{2m_f^2}}$$

$$\times \int \frac{d^2 k_\perp}{(2\pi)^2} \frac{1}{|k^2_\perp - m_f^2||(k - q)^2 - m_f^2|}$$

$$\times \left\{ -4k_0 (k \cdot q)_\parallel + 2q_\parallel k_0 + 2q^2_\parallel q_0 - 2m_f^2 q_0 \right\} \frac{q^2_0}{q^2}$$

$$+ \left[ 2(k \cdot q)^2_\perp - q^2_\parallel (k \cdot q)_\parallel - q^2_\parallel m_f^2 + q^4_\parallel m_f^2 \right] \frac{q^2_0}{q^2} q^2,$$  (34)
\[ P^\parallel = g^2 \sum_f \frac{(\pi |q f B|)}{2\pi^2} e^{-\frac{q^2}{2\pi^2\mu^2}} \times \int \frac{d^2 k}{(2\pi)^2} \frac{k^2}{|k^2 - m_f^2|} \left[ \frac{2(k \cdot q)^2}{q^2} + (k \cdot q) + m_f^2 \right], \quad (35) \]

Within the Matsubara formalism, we transform the integrals to Euclidean space by means of a Wick rotation, namely

\[ \int \frac{d^4 k}{(2\pi)^4} f(k) \to i T \int_{-\infty}^{\infty} \frac{d^3 k}{(2\pi)^3} f(i\omega_n, \vec{k}), \quad (37) \]

where the integral over the time-like component of the fermion momentum has been discretized and we introduced the fermion Matsubara frequencies \( \omega_n = (2n + 1)\pi T, k_0 = i\omega_n \) and \( q_4 = iq_0 = \omega \). In order to get all the coefficients, we need to compute the sum over Matsubara frequencies and the integral over \( k_3 \). The procedure is explicitly shown in Appendix [B]

After performing the sum and integral, removing the infinity in the \( \overline{\text{MS}} \) scheme, introducing the ultraviolet renormalization scale \( \mu \) and coming back to Minkowski space, we obtain

\[ P^T = \frac{g^2}{4\pi^2} \sum_f |q f B| e^{-\frac{q^2}{2\pi^2\mu^2}} \left[ \frac{q^2}{q^2} \right] \times \left[ \ln \left( \frac{2\pi \mu^2}{m_f^2} \right) + \ln \left( \frac{m_f^2}{\pi^2 T^2} \right) \right], \quad (38) \]

\[ P^L = -\frac{g^2}{4\pi^2} \sum_f |q f B| e^{-\frac{q^2}{2\pi^2\mu^2}} \left[ \frac{q_0^2(q_3^2 + q_4^2)}{q_0^2 q_4^2} + 2q_4^2 \right] \times \left[ \ln \left( \frac{2\pi \mu^2}{m_f^2} \right) + \ln \left( \frac{m_f^2}{\pi^2 T^2} \right) \right], \quad (39) \]

\[ P^\parallel = \frac{g^2}{4\pi^2} \sum_f |q f B| e^{-\frac{q^2}{2\pi^2\mu^2}} \left[ \frac{q_3^2 + q_4^2}{q_3^2} \right] \times \left[ \ln \left( \frac{2\pi \mu^2}{m_f^2} \right) + \ln \left( \frac{m_f^2}{\pi^2 T^2} \right) \right], \quad (40) \]

where we have written the vacuum and matter contributions separate. This explicit separation becomes important when discussing the renormalization group evolution properties of these structures. For the present purposes, we note that we could have combined the logarithmic terms in Eqs. (38)–(40) to get rid of the fermion mass dependence, finding a finite result in the massless limit, as corresponds to a calculation in the spirit of the HTL approximation.

III. SUMMARY AND DISCUSSION

In this work we have computed the gluon polarization tensor in a thermo-magnetic medium. The computation has been performed including the magnetic field effects by means of Schwinger’s proper time method. We have studied the weak and strong field limits, both at zero and high temperature. The latter case has been implemented within the HTL approximation.

For the \( T = 0 \) case, we have computed the coefficients of the tensor structures describing the polarization tensor including the case where the quark mass is not vanishing. This leads to complicated expression that nevertheless can be handled both analytically and numerically. We have shown that in this case, the coefficients of the tensor structures develop an imaginary part corresponding to the threshold for quark-antiquark pair production, namely for a gluon momentum squared such that \( q^2 \gtrsim 4m_f^2 \). On the contrary, when \( m_f = 0 \), these coefficients are real and their expressions simplify considerably.

For large \( T \), we have implemented the calculation in the LLL approximation. We have explicitly separated the vacuum and matter contribution and carried out the renormalization program in the \( \overline{\text{MS}} \) scheme. This is particularly useful since for a renormalization group analysis of the gluon polarization tensor, suited to extract the behavior of the strong coupling in a thermo-magnetic environment, it is necessary to make use of the unrenormalized matter contribution. Also, the HTL analysis in the LLL approximation renders itself to study both cases, where either the magnetic field or the temperature, are the largest of the energy scales. This is possible since in this approximation, temperature and magnetic field effects factorize, due to the dimensional reduction in the LLL.

The results of this work can be used to study the thermo-magnetic evolution of the strong coupling in the four regimes considered, namely the zero and high temperature for the weak and strong field cases. This is work that will soon be reported elsewhere.

ACKNOWLEDGEMENTS

This work was supported by Consejo Nacional de Ciencia y Tecnología grant number 256494, by the National Research Foundation (South Africa), by Fondecyt (Chile) grant numbers 1170107, 1150471, 11508427, Conicyt/PIA/Basal (Chile) grant number FB0821. R. Z.
would like to thank support from CONICYT FONDECYT Iniciación under grant number 11160234. D. M. acknowledges support from a PAPIIT-DGAPA-UNAM fellowship.

Appendix A: Integrals for the polarization tensor in the weak field limit

To find the expressions for the integrals in Eqs. (11) after performing the shift over the integration variable $k \to i$, we observe that all of these can be expressed from an integral whose general form is

$$I_{mn}^{pj}(q) = \int_0^1 dx (1 - x)^p x^j J_{mn}(x; q), \quad \text{(A1)}$$

with

$$J_{mn}(x; q) = i (-1)^m + r \int \frac{d^2 l_\parallel d^2 l_\perp}{(2\pi)^2} \frac{l_\parallel^2 m l_\perp^2}{l_\parallel^2 + l_\perp^2 + \Delta^2}, \quad \text{(A2)}$$

where, in order to compute the integrals over the parallel and transverse components of the loop momentum, a Wick rotation has been performed. Also, $\Delta = m^2_j - (1 - x) q^2$ and $l_\parallel, l_\perp = k_\parallel, k_\perp - x q_\parallel, q_\perp$. Notice that the integrals over the magnitude of the vectors $l_\perp$ and $l_\parallel$ in Eq. (A2) can be explicitly computed in terms of the expression for Euler’s beta function $E$

$$B(x, y) = 2 \int_0^\infty dt t^{2x-1} (1 + t^2)^{-x-y}, \quad \text{(A3)}$$

from where we obtain

$$J_{mn}(x; q) = \frac{i (-1)^m + r}{16\pi^2} B(m + 1, r - m - n - 2) \times B(n + 1, r - n - 1) \frac{1}{\Delta^{r-m-2}}. \quad \text{(A4)}$$

The integral over $x$ in Eq. (A1) can be performed from integrals whose general expression is

$$K_{mj}^{pj} = \int_0^1 dx \frac{(1 - x)^p x^j}{\Delta^h}, \quad \text{(A5)}$$

where for our case $p = 1, 3, j = 0, 1, 2, 3, h = 2, 3$ and $a = \pm 1$, and where, after extracting the corresponding power of $m^2_j$, we have introduced the dimensionless quantity $\Delta = 1 - x(1 - x)$, with $y = q^2/m^2_j$.

From the possible values of $p$ and $j$, we notice that the integrands in Eq. (A5) contain, at most, a polynomial of degree 3. Therefore, we can write these integrals in terms of integrals of the form

$$K_{mj}^{mh} = \int_0^1 dx \frac{x^m}{\Delta^h}, \quad \text{(A6)}$$

where $m = 0, \ldots, 7$. When $m \neq 2n - 1$, the general expression for this integral is given in terms of a recurrence relation as (see Eq. 160.28 in Ref. [24])

$$K_{mj}^{mh} = \int_0^1 dx \frac{x^m}{\Delta^h} = -\frac{x^{m-1}}{(2h - m - 1)y \Delta^{h-1}} \bigg|_0^1 - \frac{(m - 1)}{(2h - m - 1)y \Delta^{h-2}} \int_0^1 dx \frac{x^{m-2}}{\Delta^h} + \frac{(h - m)}{(2h - m - 1)} \int_0^1 dx \frac{x^{m-1}}{\Delta^h} \quad \text{(A7)}$$

For the case when $m = 2n - 1$, we have (see Eq. 160.29 in Ref. [24])

$$K_{mj}^{mh} = \int_0^1 dx \frac{x^{2n-1}}{\Delta^h} = \frac{1}{y} \int_0^1 dx \frac{x^{2n-3}}{\Delta^{h-1}} - \frac{1}{y} \int_0^1 dx \frac{x^{2n-3}}{\Delta^h} + \int_0^1 dx \frac{x^{2n-2}}{\Delta^h}. \quad \text{(A8)}$$

In this manner the solution is in turn given in terms of integrals of the type

$$K_{mj}^h = \int_0^1 dx \frac{1}{\Delta^h}, \quad \text{(A9)}$$

whose explicit expression is given itself in terms of a recurrence relation by (see Eq. 160.09 of Ref. [24])

$$K_{mj}^h = \int_0^1 dx \frac{1}{\Delta^h} = \frac{y(2x - 1)}{(h - 1)(4y - y^2)\Delta^{h-1}} \bigg|_0^1 + \frac{(2h - 3)2y}{(h - 1)(4y - y^2)} \int_0^1 dx \frac{1}{\Delta^{h-1}}. \quad \text{(A10)}$$

Therefore, the solution is reduced to finding the expression for the integral

$$K_0 = \int_0^1 dx \frac{1}{\Delta}. \quad \text{(A11)}$$

The behavior of this last integral (see Eq. 160.01 of Ref. [24]) depends on whether $y$ is negative or positive. In the former case there are no divergences in the integration interval and the explicit expression for Eq. (A11) is

$$K_{0, y < 0} = \frac{4}{y\sqrt{4y^2 - 1}} \arctan\left(\frac{1}{\sqrt{4y^2 - 1}}\right). \quad \text{(A12)}$$

In the latter, the value of the integral requires knowledge of the location of the zeros of the denominator, given by

$$x_{1, 2} = \frac{1}{2} \left(1 \pm \sqrt{1 - 4y^2}\right). \quad \text{(A13)}$$
For \( y > 0 \) the solution is divided in three cases

\[
K_{0, y > 0}^{y > 4} = \frac{2}{y \sqrt{1 - 4y^{-1}}} \ln \left[ \frac{1}{2} \left( 1 - y \sqrt{1 - 4y^{-1}} \right) - 1 \right],
\]
\[
K_{0, y > 0}^{y < 4} = \frac{4}{y \sqrt{4y^{-1} - 1}} \arctan \left( \frac{1}{\sqrt{4y^{-1} - 1}} \right),
\]
\[
K_0^{y = 4} = -1.
\]

Figure 5 shows the behavior of the real (solid black line) and imaginary (dash red line) parts of \( K_0 \), Eq. (A11), as a function of \( y \). This function presents a discontinuity at \( y = 4 \) that corresponds to the threshold for quark-antiquark production.

Figure 6. Behavior of \( K_0 \) as a function of \( y \). Shown are the real (solid black line) and imaginary (red dashed line) parts. This function presents a discontinuity at \( y = 4 \) that corresponds to the threshold for quark-antiquark production.

Appendix B: Integrals for the polarization tensor in the LLL and HTL approximations

Here we show the explicit steps that lead to Eqs. (38)–(40). First, we introduce temperature effects into Eqs. (33)–(35) using the Matsubara formalism of thermal field theory, obtaining

\[
P^L = g^2 T \sum_{n=-\infty}^{\infty} \sum_{f} \left( \frac{\pi|q_f B|}{2\pi^2} \right) e^{-\frac{q_f^2}{2m_f^2}} \int \frac{dk_3}{(2\pi)} \times \left\{ \frac{1}{|\tilde{\omega}_n^2 + k_3^2 + m_f^2|(|\tilde{\omega}_n - \omega)^2 + (k_3 - q_3)^2 + m_f^2|} \right\}
\]
\[
\times \left\{ \left( 4\tilde{\omega}_n(-\tilde{\omega}_n\omega - k_3q_3) + 2\tilde{\omega}_n(\omega_n^2 + q_3^2) + 2(\tilde{\omega}_n\omega^2 + k_3^2 + 2\tilde{\omega}_n k_3 q_3) \right) \frac{\omega^2}{q^2} \right\}
\]
\[
\times \left\{ \left( \tilde{\omega}_n^2 + 2\tilde{\omega}_n\omega - k_3^2 + \tilde{\omega}_n\omega + k_3 q_3 - m_f^2 q_3^2 \right) \right\},
\]
\[
(B2)
\]
\[
P^\parallel = g^2 T \sum_{n=-\infty}^{\infty} \sum_{f} \left( \frac{\pi|q_f B|}{2\pi^2} \right) e^{-\frac{q_f^2}{2m_f^2}} \int \frac{dk_3}{(2\pi)} \times \left\{ \frac{1}{|\tilde{\omega}_n^2 + k_3^2 + m_f^2|(|\tilde{\omega}_n - \omega)^2 + (k_3 - q_3)^2 + m_f^2|} \right\}
\]
\[
\times \left\{ \left( \tilde{\omega}_n^2 + \tilde{\omega}_n\omega + k_3 q_3 \right) \frac{2}{q^2} \left( \tilde{\omega}_n k_3 q_3 - k_3^2 \right) \right\}
\]
\[
\times \left\{ \left( \tilde{\omega}_n^2 + 2\tilde{\omega}_n\omega - k_3^2 - \tilde{\omega}_n\omega - k_3 q_3 \right) \right\},
\]
\[
(B3)
\]

We notice that since in the HTL approximation terms proportional to \( \tilde{\omega}_n \) and to \( k_3 \) in the numerators do not contribute, the calculation of Eqs. (13)–(15) involves only two kinds of sums over the Matsubara frequencies. These are explicitly given by [27]

\[
T \sum_{n=-\infty}^{\infty} \frac{1}{|\tilde{\omega}_n - \omega)^2 + (k_3 - q_3)^2 + m_f^2|}
\]
\[
= \frac{1}{2E_2} (-1 + 2\tilde{f}(E_2)),
\]
\[
(B4)
\]

and

\[
T \sum_{n=-\infty}^{\infty} \frac{1}{|\tilde{\omega}_n^2 + k_3^2 + m_f^2|(|\tilde{\omega}_n - \omega)^2 + (k_3 - q_3)^2 + m_f^2|}
\]
\[
= \sum_{s_1,s_2=\pm 1} \frac{-s_1 s_2}{4E_1 E_2} \left( \frac{1 - \tilde{f}(s_1 E_1) - \tilde{f}(s_2 E_2)}{i\omega - s_1 E_1 - s_2 E_2} \right),
\]
\[
(B5)
\]

with \( E_1^2 = k_3^2 + m_f^2 \), \( E_2^2 = (k_3 - q_3)^2 + m_f^2 \) and \( \tilde{f}(x) = 1/(e^{x/T} + 1) \). Using Eqs. (B4) and (B5) and continuing within the HTL approximation, we notice that the external momentum \( q \) is a soft energy scale compared with the temperature, and therefore we can neglect \( q^2 \) and \( k \cdot q \) in each of the integrand’s numerators. We can therefore compute all the sums appearing in Eq. (B1)–Eq. (B5).
and we obtain
\[ P^T = g^2 \sum_f \left( \frac{|q_f B|}{4\pi^2} \right) e^{-\frac{q_f^2}{2\pi^2 m_f}} \int dk_3 \left( \frac{1}{E_1} - 2\tilde{f}(E_1) \right) \] (B6)
\[ P^L = g^2 \sum_f \left( \frac{|q_f B|}{4\pi^2} \right) e^{-\frac{q_f^2}{2\pi^2 m_f}} \times \left\{ \int dk_3 \left( -\frac{1}{E_1} + 2\tilde{f}(E_1) \right) \left( \frac{\omega^2 + q_f^2}{q_f^2 q_f^2} \right) \right\} \] (B7)
\[ P^\parallel = g^2 \sum_f \left( \frac{|q_f B|}{4\pi^2} \right) e^{-\frac{q_f^2}{2\pi^2 m_f}} \times \left\{ \int dk_3 \left( -\frac{1}{2E_1} + \tilde{f}(E_1) \right) \left( \frac{2\omega^2 - q_f^2}{q_f^2} \right) \right\} + \int dk_3 \left( \frac{1}{2E_1} - \tilde{f}(E_1) \right) \left( \frac{-\omega^2 + q_f^2}{q_f^2} \right) \right\} \] (B8)

We notice that we have two contributions: The first one is the vacuum contribution and the second one is the matter contribution. The vacuum contribution is divergent. Using dimensional regularization, this contribution has the following expression
\[ \int \frac{dk_3}{E_1} = \int \frac{1}{(k_3^2 + m_f^2)^{1/2}} \left( \ln \left( \frac{2\pi \mu^2}{m_f^2} \right) + 2\gamma_E + \frac{1}{\epsilon} \right) \] (B9)

with \( \mu \) the ultraviolet renormalization scale. When removing the infinity in the \( \overline{\text{MS}} \) scheme, we get
\[ \int \frac{dk_3}{E_1} = \int \frac{1}{(k_3^2 + m_f^2)^{1/2}} \left( \ln \left( \frac{2\pi \mu^2}{m_f^2} \right) + 2\gamma_E \right) \] (B10)

We now turn to compute the matter contribution, which is divergence-free. For this we have
\[ \int \frac{dk_3 f(E_1)}{E_1} = \int \frac{1}{(k_3^2 + m_f^2)^{1/2}} e^{\frac{1}{\sqrt{(k_3^2 + m_f^2)^2/T + 1}}} \] (B11)

Using the general expression from Ref. 28
\[ f_n(\tilde{y}) = \frac{1}{\Gamma(n)} \int_0^{\infty} dx x^{-1} \frac{1}{\sqrt{x^2 + \tilde{y}^2} e^{\sqrt{x^2 + \tilde{y}^2} + 1}} \] (B12)

we identify Eq. (B11) as corresponding to the case with \( n = 1 \) and \( \tilde{y} = m_f/T \). In the limit where \( \tilde{y} \) is small, Eq. (B12) becomes
\[ f_1(m_f/T) = -\frac{1}{2} \ln \left( \frac{m_f}{\pi T} \right) - \frac{1}{2} \gamma_E + \ldots \] (B13)

therefore we obtain
\[ \int \frac{dk_3 f(E_1)}{E_1} = -\left( \ln \left( \frac{m_f}{\pi T} \right) + \gamma_E \right) \] (B14)

Using the above expression, we get
\[ P^T = \frac{g^2}{4\pi^2} \sum_f |q_f B| e^{-\frac{q_f^2}{2\pi^2 m_f}} \left[ \frac{q_f^2}{q_f^2} \right] \] (B15)
\[ \times \left[ \ln \left( \frac{2\pi \mu^2}{m_f^2} \right) + \ln \left( \frac{m_f^2}{\pi^2 T^2} \right) \right] \]
\[ P^L = -\frac{g^2}{4\pi^2} \sum_f |q_f B| e^{-\frac{q_f^2}{2\pi^2 m_f}} \left[ \frac{\omega^2(q_f^2 + q_f^2)}{q_f^2 q_f^2} \right] \] (B16)
\[ \times \left[ \ln \left( \frac{2\pi \mu^2}{m_f^2} \right) + \ln \left( \frac{m_f^2}{\pi^2 T^2} \right) \right] \]
\[ P^\parallel = \frac{g^2}{4\pi^2} \sum_f |q_f B| e^{-\frac{q_f^2}{2\pi^2 m_f}} \left[ \frac{q_f^2 - \omega^2}{q_f^2} \right] \] (B17)
\[ \times \left[ \ln \left( \frac{2\pi \mu^2}{m_f^2} \right) + \ln \left( \frac{m_f^2}{\pi^2 T^2} \right) \right] \]

finally to obtain Eqs. (33)–(40) we perform an analytical continuation back to Minkowski space by means of the transformation \( i\omega \to k_0 \).

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