TORIC VARIETIES WITH HUGE GROTHENDIECK GROUP

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ABSTRACT. In each dimension $n \geq 3$ there are many projective simplicial toric varieties whose Grothendieck groups of vector bundles are at least as big as the ground field. In particular, the conjecture that the Grothendieck groups of locally trivial sheaves and coherent sheaves on such varieties are rationally isomorphic fails badly.

1. Introduction

The question whether the natural homomorphism $K_0(X) \to G_0(X)$ between the Grothendieck groups of locally trivial sheaves and coherent sheaves on a complete simplicial toric variety $X$ is a rational isomorphism has attracted an attention of researchers for some time (see [BV]). More generally, it has been conjectured that such a map is an isomorphism after tensoring with $\mathbb{Q}$ for arbitrary quasi-projective orbifold (see, for instance, [C, Section 7.12]). The main result of this paper says that this is far from being the case even for projective simplicial toric varieties in dimensions 3 and higher.

More precisely, it was shown in [BV] that $K_0(X)_{\mathbb{Q}} \to G_0(X)_{\mathbb{Q}}$ is a surjection for a simplicial toric variety $X$. If $X$ is quasiprojective then by Riemann-Roch for singular varieties [BFM] we have an isomorphism $G_0(X)_{\mathbb{Q}} \to A_*(X)_{\mathbb{Q}}$. On the other hand the Chow groups of $X$ are finitely generated by [FMSS]. Therefore, $G_0(X)$ has a finite rank. Fix arbitrary field $k$, not necessarily algebraically closed and satisfying the conditions $\text{char } k = 0$ and $\dim \mathbb{Q} k = \infty$. Theorem 5 below gives many examples of projective simplicial toric $k$-varieties $X$ such that $\text{rank } K_0(X) \geq \dim \mathbb{Q} k$.

A word on notation and terminology. Throughout the paper $k$ is assumed to be a field of the mentioned type. An affine monoid means a finitely generated submonoid of a free abelian group. It is called positive if there are no non-trivial units, and simplicial if the cone, spanned by the monoid in the ambient Euclidean space, is simplicial (i.e. spanned by linearly independent vectors). An affine monoid $M$ is normal if the implication $(c \in \mathbb{N}, x \in \text{gp}(M), cx \in M) \Rightarrow (x \in M)$ holds, $\text{gp}(M)$ being the group of differences of $M$. (Here we use additive notation.) $\mathbb{Z}_+$ refers to the additive monoid of non-negative integral numbers, $\mathbb{Z}_-, \mathbb{R}_+$ and $\mathbb{R}_-$ are defined similarly and $\mathbb{N} = \{1, 2, \ldots \}$. A free monoid is the one isomorphic to $\mathbb{Z}_+^n$ for some $n \in \mathbb{N}$. A cone $C \subseteq \mathbb{R}^n$ is called unimodular if the submonoid $C \cap \mathbb{Z}^n \subset \mathbb{Z}^n$ is generated by a part of a basis of $\mathbb{Z}^n$. As usual, the rank of an abelian group $H$ means $\dim \mathbb{Q}(\mathbb{Q} \otimes H)$. Finally, we put $r = n - 1$.

2000 Mathematics Subject Classification. 14M25, 14C35, 19A99.

The work was done during the author’s visit to Université Louis Pasteur, Strasbourg (Autumn 2001). He was also supported by INTAS grant 99-00817 and TMR grant ERB FMRX CT-97-0107.
Acknowledgement. This paper was made possible by the author’s visit to Université Louis Pasteur (Strasbourg) and the many discussions with Abdallah Al Amrani there. Our initial attempt was to construct such examples among weighted projective spaces – a question which is motivated by the previous works [Al1, Al2] and which remains unanswered. Eventually, I have not been able to convince Abdallah Al Amrani to coauthor the present paper.

Also, I am grateful to the referee for figuring out a somewhat inaccurate use of the Witt(k)-action on nil-K-theory in the first version of the paper.

2. Basic configuration

Assume $M$ is a positive normal affine monoid, rank $M = r$. (We put rank $M = \text{rank} \text{gp}(M)$.) By fixing an isomorphism in $\text{gp}(M) \approx \mathbb{Z}^r$ we can make the identification $\mathbb{Z} \oplus M = \mathbb{Z}^n$. There is a free basis $\{x_1, \ldots, x_r\}$ of $\text{gp}(M)$ such that $\mathbb{R}_+ M \subset \mathbb{R}_+ x_1 + \cdots + \mathbb{R}_+ x_r$, both cones being considered in $\mathbb{R}^r$. In fact, passing to the dual cone $(\mathbb{R}_+ M)^{\text{op}} \subset (\mathbb{R}^r)^{\text{op}}$ we only need the existence of a basis of the dual group $(\mathbb{Z}^r)^{\text{op}}$ inside this dual cone and this is easily seen (see, for instance, [G3 Lemma 2.3(d)]).

Put $e = (1,0) \in \mathbb{Z} \oplus M$ and consider the sequence of submonoids $M = M_0, M_1, M_2, \ldots \subset \mathbb{Z}^n$ defined by $M_i = (\mathbb{R} \oplus \mathbb{R}_+ M) \cap (\mathbb{Z}(x_1 - ie) + \cdots + \mathbb{Z}(x_r - ie))$, $i \in \mathbb{Z}_+$.

Lemma 1. (a) $\mathbb{Z} \oplus M = \mathbb{Z} e + M_0 = \mathbb{Z} e + M_1 = \mathbb{Z} e + M_2 = \cdots$, all being direct sums,

(b) $\mathbb{Z}_+ e + M_0 \subset \mathbb{Z}_+ e + M_1 \subset \cdots$ and $\bar{M} := \bigcup_{i=0}^{\infty} (\mathbb{Z}_+ e + M_i) = (\mathbb{Z} \oplus M) \setminus \{-ie\}_{i \in \mathbb{N}}$,

(c) there are isomorphisms $\alpha_i : M_i \rightarrow M_{i+1}$, $i \in \mathbb{Z}_+$, making the diagrams

$$
\begin{array}{ccc}
\mathbb{Z}_+ e + M_i & \xrightarrow{\subset} & \mathbb{Z}_+ e + M_{i+1} \\
\downarrow^{1+\alpha_i} & & \downarrow^{1+\alpha_{i+1}} \\
\mathbb{Z}_+ e + M_{i+1} & \xrightarrow{\subset} & \mathbb{Z}_+ e + M_{i+2}
\end{array}
$$

commutative.

This is easily proved. The isomorphisms $\alpha_i$, for instance, are the corresponding restrictions of the automorphism $\mathbb{Z}^n \rightarrow \mathbb{Z}^n$, $e \mapsto e$, $x_j \mapsto x_j - e$ (equivalently, $x_j - ie \mapsto x_j - (i+1)e$), $j \in [1,r]$.

Fix arbitrarily two affine normal submonoids $N_+ \subset \mathbb{Z}_+ e + M$ and $N_- \subset \mathbb{Z}_- e + M_1$ so that the following conditions hold: (i) $e \subset N_+$, $-e \subset N_-$, (ii) $N_- \cap M_1 = \{0\}$, and (iii) $\mathbb{Z} e + N_+ = \mathbb{Z} e + N_- = \mathbb{Z} e + M$. This is possible: fix two rational cones $C_+ \subset \mathbb{R}_+ e + \mathbb{R}_+ M$ and $C_- \subset \mathbb{R}_- e + \mathbb{R}_+ M_1$ so that $C_+$ is bounded by the facets of $\mathbb{R}_+ e + \mathbb{R}_+ M$, containing $e$, and one more hyperplane through the origin, and similarly for the cone $C_-$ with respect to $\mathbb{R}_- e + \mathbb{R}_+ M_1$ under the following additional requirement: the facet of $C_-$ that does not contain $-e$ intersects $\mathbb{R}_+ M_1$ only at the origin. Then $N_+ = C_+ \cap \mathbb{Z}^n$ and $N_- = C_- \cap \mathbb{Z}^n$.

A triple of the type $\mathcal{C} = (M, N_+, N_-)$ will be called a basic configuration.
3. Noncomplete case

Let $\mathcal{C}$ be a basic configuration. By $X(\mathcal{C})$ we denote the scheme obtained by gluing $\text{Spec}(k[N_+])$ and $\text{Spec}(k[N_-])$ along their common open subscheme $\text{Spec}(k[\mathbb{Z} \oplus M])$. This is a toric variety whose fan has two maximal $n$-dimensional cones – the dual cones $(\mathbb{R}_+ N_+)^{op}, (\mathbb{R}_+ N_-)^{op} \subset (\mathbb{R}^n)^{op}$. The quasi-projectivity of $X(\mathcal{C})$ is discussed in Section 4 below.

**Proposition 2.** Let $\mathcal{C} = (M, N_+, N_-)$ be a basic configuration in which the monoid $M$ is simplicial and non-free. Then $\text{rank } K_0(X(\mathcal{C})) \geq \dim_k k$.

We will use the fact that the $K$-groups of [TT] agree with those of Quillen for quasi-projective schemes over an affine scheme.

**Proof.** We will use multiplicative notation for the monoid operation with $X = e$. By [TT] Theorem 8.1 we have the exact sequence $K_1(k[N_+]) \oplus K_1(k[N_-]) \to K_1(k[M][X, X^{-1}]) \to K_0(X(\mathcal{C}))$ and, hence, it is enough to show that $\text{rank } \text{Coker } (K_1(k[N_+]) \oplus K_1(k[N_-]) \to K_1(k[M][X, X^{-1}])) \geq \dim_k k$.

On the other hand, $k[M][X, X^{-1}] = k[M_1][X, X^{-1}]$ by Lemma III(a) and

$$K_1(k[M_1][X, X^{-1}]) = K_1(k[M_1]) \oplus NK_1^+(k[M_1]) \oplus NK_1^-(k[M_1]) \oplus K_0(k[M_1])$$

by the Fundamental Theorem [Ba, Ch.7,§7]. Here $K_1(k[M_1]) \oplus NK_1^+(k[M_1]) = K_1(k[M_1][X])$ and $K_1(k[M_1]) \oplus NK_1^-(k[M_1]) = K_1(k[M_1][X^{-1}])$. Because of the inclusions $k[N_+] \subset k[M][X]$ and $k[N_-] \subset k + X^{-1}k[M][X^{-1}]$ the group

$$\text{Coker } (K_1(k[N_+]) \oplus K_1(k[N_-]) \to K_1(k[M][Z]))$$

surjects onto

$$\text{Coker } (K_1(k[M][X]) \oplus NK_1^-(k[M_1]) \to K_1(k[M_1][X, X^{-1}])).$$

But the latter group contains $\text{Coker } (K_1(k[M][X]) \to K_1(k[M_1][X]))$ as a direct summand because, firstly, the homomorphism $k[M][X] \to k[M_1][X, X^{-1}]$ factors through $k[M_1][X] \to k[M_1][X, X^{-1}]$ and, secondly, the groups $K_1(k[M_1][X])$ and $NK_1^-(k[M_1])$ inside $K_1(k[M_1][X, X^{-1}])$ are complementary direct summand. It is therefore enough to show the inequality

$$\text{rank } \text{Coker } (K_1(k[M][X]) \to K_1(k[M_1][X])) = \text{rank } \text{Coker } (K_1(k[M][X])/k^* \to K_1(k[M_1][X])/k^*) \geq \dim_k k.$$

(We view the multiplicative group $k^*$ as a natural direct summand of the $K_1$-groups. We can fix a grading $k[M_1][X] = k \oplus A_1 \oplus A_2 \oplus \cdots$ so that all elements of $M_1$ as well as the variable $X$ are homogeneous. This grading restricts to a grading on $k[M][X]$.)

By Weibel [W], generalizing the Bloch-Stienstra operations on nil-$K$-theory [EI S] to the relative situation in arbitrary graded rings, the both groups $K_1(k[M][X])/k^*$ and $K_1(k[M_1][X])/k^*$ are modules over the ring of big Witt vectors Witt($k$). The *ghost map* establishes a ring isomorphism Witt($k$) $\cong \Pi_{1}^{\infty}k$. Thus there is a copy of $k$ inside Witt($k$). In particular, due to functoriality of the Witt($k$)-action, the homomorphism

$$K_1(k[M][X])/k^* \to K_1(k[M_1][X])/k^*$$
is a homomorphism of $k$-vector spaces. Thus everything boils down to non-surjectivity of this homomorphism or, equivalently, to non-surjectivity of

\[ K_1(k[M][X]) \to K_1(k[M_1][X]). \]

Assume the latter map is surjective. We have a filtered union representation $k[\bar{M}] = \bigcup_j k[M_j][X]$, $\bar{M}$ as in Lemma II(b). In particular, $K_1(k[\bar{M}]) = \lim_i (K_1(k[M][X]) \to \cdots \to K_1(k[M_i][X]) \to \cdots)$. By Lemma II(c) the mapping $K_1(k[M_i][X]) \to K_1(k[M_{i+1}][X])$ for every index $i$ is the same, up to an isomorphic transformation, as the initial homomorphism $K_1(k[M][X]) \to K_1(k[M_1][X])$. Therefore, by our surjectivity assumption all these mappings are surjective. In particular, the limit map $K_1(k[M][X]) \to K_1(k[M])$ is also surjective. But it is injective as well because so is the composite map $K_1(k[M][X]) \to K_1(k[M]) \to K_1(k[M][X, X^{-1}])$. The diagram

\[
\begin{array}{ccc}
  k[\bar{M}] & \xrightarrow{\subset} & k[M][X, X^{-1}] \\
  \downarrow & & \downarrow \\
  k[X] & \xrightarrow{\subset} & k[X, X^{-1}]
\end{array}
\]

is a Cartesian square by Lemma II(b). Here the vertical maps are defined by $m \mapsto 0 \in k$ for all non-trivial elements $m \in M$. Using the equality $K_1(k[M][X]) = K_1(k[\bar{M}])$ the associated Milnor Mayer-Vietoris sequence reads as

\[ K_1(k[M][X]) \to K_1(k[X]) \oplus K_1(k[M][X, X^{-1}]) \to K_1(k[X, X^{-1}]) \]

(Notice, just surjectivity of $K_1(k[M][X]) \to K_1(k[M])$ is enough for this sequence.) Therefore, the Fundamental Theorem implies $N K_1^{-}(k[M]) = 0$, i. e. $K_1(k[M]) = K_1(k[M][X^{-1}])$. Since $k[M]$ is graded ring whose 0th component is $k$ the Swan-Weibel homotopy trick (see below) implies $K_1(k) = K_1(k[M])$, i. e. $SK_1(k[M]) = 0$ – a contradiction by [G2].

The mentioned homotopy trick says that for a graded ring $\Lambda = \Lambda_0 \oplus \Lambda_1 \oplus \cdots$ and a functor $F$ from rings to abelian groups the following implication holds $F(\Lambda) = F(\Lambda[Z]) \Rightarrow F(\Lambda_0) = F(\Lambda)$, $Z$ being an indeterminate (see [An]).

**Corollary 3.** Let $\mathcal{Z}(\mathcal{F})$ be a quasiprojective $n$-dimensional toric variety, defined by a fan $\mathcal{F}$ in the dual space $(\mathbb{R}^n)^{\text{op}}$, and $\mathcal{C} = (M, N_+, N_-)$ be a basic configuration in which the monoid $M$ is simplicial and nonfree. Assume the dual cones $(\mathbb{R}_+ N_+)^{\text{op}}, (\mathbb{R}_+ N_-)^{\text{op}} \subset (\mathbb{R}^n)^{\text{op}}$ are among the maximal cones of $\mathcal{F}$ and all other maximal cones of $\mathcal{F}$ are unimodular. Then rank $K_0(\mathcal{Z}) \geq \dim_k k$.

**Proof.** We have the open cover $\mathcal{Z} = X(\mathcal{C}) \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_s$, where the $\mathcal{A}_j$ are the smooth affine toric varieties (i. e. of type $\mathbb{A}_a^b \times \mathbb{T}_b^a$, $a + b = n$) that correspond to the mentioned unimodular maximal cones. By [TT] Theorem 8.1 for each $j \in [1, s]$ we get the exact sequence

\[ K_0(X(\mathcal{C}) \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_j) \to K_0(X(\mathcal{C}) \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_{j-1}) \oplus K_0(\mathcal{A}_j) \to K_0(U_j), \]
where $U_j = (X(\mathcal{C}) \cup A_1 \cup \cdots \cup A_{j-1}) \cap A_j$. Since the $U_j$ are open subschemes of $A_j$ and $K_0(A_j) = \mathbb{Z}$ we have $K_0(U_j) = \mathbb{Z}$, $j \in [1, s]$. Therefore, the induction from $j = 1$ to $j = s$ shows that $\text{rank} K_0(\mathcal{Z}) \geq \dim_{\mathbb{Q}} k$. (Here one needs $\dim_{\mathbb{Q}} k = \infty$.) \hfill \Box

4. Projective case

Given a basic configuration $\mathcal{C}$ in which the monoid $M$ is simplicial. Then $X(\mathcal{C})$ can be embedded into a projective simplicial toric variety as an equivariant open subscheme. A quick way to do so is given by the following construction. Consider the intersection $\Delta(\mathcal{C}) = (e + \mathbb{R}_+ N_-) \cap \mathbb{R}_+ N_+$. This is a rational $n$-simplex having a pair of corners, spanning respectively the cones $\mathbb{R}_+ N_+$ and $e + \mathbb{R}_+ N_-$. Let $X(\mathcal{C})$ be the following projective toric variety. We have the $(n + 1)$-dimensional cone $C(\Delta(\mathcal{C})) = \bigcup_{x \in \Delta(\mathcal{C})} \mathbb{R}_+(x, 1) \subset \mathbb{R}^{n+1}$ and the associated monoid ring $k[C(\Delta(\mathcal{C}))] \cap \mathbb{Z}^{n+1}$, which carries the graded structure $k[C(\Delta(\mathcal{C}))] \cap \mathbb{Z}^{n+1} = k \oplus C_1 \oplus C_2 \oplus \cdots$ given by the last coordinate in the exponent vectors of monomials. Then $X(\mathcal{C}) = \text{Proj} \left( k[C(\Delta(\mathcal{C}))] \cap \mathbb{Z}^{n+1} \right)$. Its standard affine charts are $\text{Spec}(k[C_v \cap \mathbb{Z}^n])$, $v \in \text{vert}(\Delta(\mathcal{C}))$, where the $C_v \subset \mathbb{R}^n$ are the corner cones spanned by $\Delta(\mathcal{C})$ at its vertices $v$ and then shifted by $-v$. The maximal cones in the fan of $X(\mathcal{C})$ are just the dual cones $(-v + C_v)^{\text{op}} \subset (\mathbb{R}^n)^{\text{op}}$, $v \in \text{vert}(\Delta(\mathcal{C}))$ [F. Section 1.5]. This fan contains $(\mathbb{R}_+ N_+)^{\text{op}}$ and $(\mathbb{R}_+ N_-)^{\text{op}}$ as adjacent maximal cones whose common facet is exactly $(\mathbb{R}_+ + \mathbb{R}_+ M)^{\text{op}}$.

The difficulty in applying Corollary 3 to $X(\mathcal{C})$ is that the cones $-v + C_v$, $v \in \text{vert}(\Delta(\mathcal{C}))$, different from $\mathbb{R}_+ N_+$ and $\mathbb{R}_+ N_-$, are in general not unimodular. We overcome this difficulty by resolving the corresponding toric singularities without affecting $\text{Spec}(k[N_+])$ and $\text{Spec}(k[N_-])$. As we will see, sometimes this is possible.

Call a basic configuration $\mathcal{C} = (M, N_+, N_-)$ admissible if $M$ is simplicial and all facets of $(\mathbb{R}_+ N_+)^{\text{op}}$ and $(\mathbb{R}_+ N_-)^{\text{op}}$, except maybe their common facet, are unimodular.

Lemma 4. Assume $\mathcal{C}$ is an admissible basic configuration. Then there exists an equivariant open embedding of $X(\mathcal{C})$ into a projective simplicial toric variety $\mathcal{Z}$ whose affine charts that do not come from $X(\mathcal{C})$ are all smooth.

Proof. We apply the standard equivariant resolution to $X(\mathcal{C})$ [F. Section 2.6] except we do not touch the cones $(\mathbb{R}_+ N_+)^{\text{op}}, (\mathbb{R}_+ N_-)^{\text{op}}$. This is possible because the admissibility condition on $\mathcal{C}$ guarantees that the cone-subdividing elements $z \in (\mathbb{Z}^n)^{\text{op}}$ we produce in the resolution process do not belong to the boundary $\partial((\mathbb{R}_+ N_+)^{\text{op}} \cup (\mathbb{R}_+ N_-)^{\text{op}})$, that is $z \in (\mathbb{R}^n)^{\text{op}} \setminus ((\mathbb{R}_+ N_+)^{\text{op}} \cup (\mathbb{R}_+ N_-)^{\text{op}})$. The projectivity condition is not lost in this process because we essentially have barycentric (rather, stellar) subdivisions — they are projective — and a composition of projective subdivisions is again projective [KKMS Ch.3, §1]. \hfill \Box

Now we are ready to prove

Theorem 5. For each $n \geq 3$ there are projective simplicial toric varieties $\mathcal{Z}$ for which $\text{rank} K_0(\mathcal{Z}) \geq \dim_{\mathbb{Q}} k$.

Proof. In view of Corollary 3 and Lemma 4 we only need to show the existence of admissible configurations $\mathcal{C} = (M, N_+, N_-)$ in which $M$ is not free.
First we observe that there are simplicial rational \( n \)-cones whose all facets except one are unimodular and the distinguished facet is not unimodular (see Example 6 below). Fix such a cone \( C \) in the dual space \((\mathbb{R}^n)\text{op}\) (with respect to the dual lattice \((\mathbb{Z}^n)\text{op}\)). Let \( F \subset C \) be the nonunimodular facet. The dual cone \( C\text{op} \) is a rational simplicial cone whose one edge, say \( l \), corresponds to \( F \) (under the duality). Put \( N_+ = C\text{op} \cap \mathbb{Z}^n \) and denote by \( e \) the generator of \( \mathbb{Z}^n \cap l \approx \mathbb{Z}_+ \). By [G1, Theorem 1.8] \( \mathbb{Z}_+ e + N_+ = \mathbb{Z} e + M' \) for some rank \( r \) simplicial normal monoid \( M' \). We have \( (\mathbb{R} e + \mathbb{R}_+ M')\text{op} = F \). As in Lemma 4 we can find a free basis \( \{x_1, \ldots, x_r\} \) of \( \text{gp}(M') \) such that \( \mathbb{R}_+ M' \subset \mathbb{R}_+ x_1 + \cdots + \mathbb{R}_+ x_r \). If a natural number \( c \) is big enough then the monoid \( M = \mathbb{Z}^n \cap (\mathbb{R}_+(x_1 - ce) + \cdots + \mathbb{R}_+(x_r - ce)) \) satisfies the conditions \( \mathbb{R}_+ N_+ \subset \mathbb{R}_+ e + \mathbb{R}_+ M \) (equivalently, \( N_+ \subset \mathbb{Z}_+ e + M \)) and \( \mathbb{R}_+ N_+ \cap \mathbb{R}_+ M = 0 \). We fix such \( c \) and define the monoid \( M_1 \) to be \( \mathbb{Z}^n \cap (\mathbb{R}_+(x_1 - (c+1)e) + \cdots + \mathbb{R}_+(x_r - (c+1)e)) \). To construct \( N_- \) we make the identification of the monoids \( \mathbb{Z}_+ e + M \) and \( \mathbb{Z}_+ e + M_1 \) along the isomorphism induced by \( e \mapsto -e, x_j - ce \mapsto x_j - (c+1)e \ (j \in [1, r]) \). Then \( N_- \) is by definition the corresponding copy of \( N_+ \). It follows from the construction that \((M, N_+, N_-)\) is an admissible basic configuration. (Say, the non-freeness of \( M \) follows from the non-unimodularity of \( F \), i.e. non-smoothness of \( \text{Spec}(k[\mathbb{Z} e + M]) \)).

\[ \square \]

**Example 6.** For each natural number \( r \geq 2 \) consider the simplicial normal nonfree monoid \( M_r = \mathbb{Z}_+(1, \ldots, 1) + \sum_{j=1}^r \mathbb{Z}_+(r e_j) \subset \mathbb{Z}_+^r \), where \( e_j \) is the \( j \)th standard basic vector in \( \mathbb{R}^r \) (\( j \in [1, r] \)). The facets of \( \mathbb{R}_+^n \setminus \mathbb{R}_+ M_r \) are unimodular with respect to the sublattice \( \text{gp}(M_r) \subset \mathbb{Z}^r \). Therefore, all facets of the cone \( \mathbb{R}_+(\mathbb{Z}_+ \oplus M_r) = \mathbb{R}_+^n \), except exactly one, are unimodular with respect to the lattice \( \text{gp}(\mathbb{Z}_+ \oplus M_r) = \mathbb{Z} \oplus \text{gp}(M_r) \subset \mathbb{R}^n \).

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