A New Fuzzy $H_\infty$ Filter Design for Nonlinear Time-Delay Systems with Mismatched Premise Membership Functions

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Abstract: This paper is concerned with the fuzzy $H_\infty$ filter design issue for nonlinear systems with time-varying delay. To overcome the shortcomings of the conventional methods with matched preconditions, the fuzzy $H_\infty$ filter to be designed and the T-S fuzzy model are assumed to have different premise membership functions and number of rules, thus, greater design flexibility and robustness to uncertainty can be achieved. However, such design will also make the derived results conservative, to relax the result, a novel integral inequality which is tighter than the traditional inequalities derived from the Leibniz-Newton formula is applied, besides, a fuzzy Lyapunov function and the information of the membership functions are also introduced. All the design methods are presented in LMI-based conditions. Finally, two numerical examples are given to prove the effectiveness and superiority of the proposed approach.

Keywords: Fuzzy $H_\infty$ filter, Mismatched conditions, New integral inequality, Time Delay, T-S fuzzy model

1. INTRODUCTION

It is well known that time-delay can cause instability and deteriorate the performance of the systems, also it is inevitable in practical control systems. So the study of control systems with time-delay is critical, and researchers have devoted great efforts to investigate them (Cao and Frank [2000], Jeung et al. [1998], Wu et al. [2004], Wang et al. [2020]). And in this paper, we will mainly investigate nonlinear systems with time-varying delay.

Moreover, filtering technology is playing a crucial role in signal processing, and over the development of several decades, fruitful research results have been obtained (Kong and Dai [2016], Bernstein and Haddad [1989], Gao and Wang [2003], Kong and Dai [2016], Fridman and Shaked [2001], Ma et al. [2017]), among them, $H_\infty$ filtering (Fridman and Shaked [2001], Lin et al. [2008], Ma [2022], Qin et al. [2021]) has attracted widespread attention, as $H_\infty$ filter has no particular requirement for external noise signal and it is not sensitive to uncertainty. In the past few years, researchers has proposed various techniques to improve the performance of the $H_\infty$ filter, to mention a few, in (Lin et al. [2008]), a delay-dependent fuzzy $H_\infty$ filter design method was proposed for T-S fuzzy-model-based system with time-varying delay. However, in this paper, the Lyapunov-Krasovskii function candidate was chosen as a single Lyapunov function, to obtain more relaxed results, the literature (Zhang et al. [2009]) adopted a fuzzy Lyapunov function to analyze the stability condition. In (Su et al. [2009]), the fuzzy $H_\infty$ filter design approach was improved through estimating the upper bound of the derivative of Lyapunov function without ignoring any useful terms. On the basis of (Su et al. [2009]), literature (Huang et al. [2011]) proposed a technique to obtain more accurate upper bound of the derivative of Laypunov function.

However, when designing $H_\infty$ filter for nonlinear systems with time-delay, all the proposed methods require matching conditions, i.e, the fuzzy filter and the fuzzy model are assumed to have same membership functions, such assumption can facilitate the filter design, but on the other hand, it will also limit the design flexibility and make the designed filter lose robustness to the system with uncertain membership functions. To address this problem, we will apply the imperfect premise matching method (Lam and Narimani [2009], Kong and Dai [2018]) to design fuzzy $H_\infty$ filter, which means the fuzzy filter to be designed and the fuzzy model will be allowed to have different premise membership functions and number of fuzzy rules, thus successfully address the issue of traditional $H_\infty$ filter design methods. Nevertheless, such design will also make the designed filter tend to be more conservative. So in this paper, we will adopt a fuzzy Lyapunov function to derive the fuzzy $H_\infty$ filter, which can increase the number
of free-weighting matrices, and relax the conservatism of the result.

Besides, as $H_\infty$ filter has to ensure the filtering error system is asymptotically stable, the stability analysis is necessary, and in this process, the inequalities derived from the Leibniz-Newton formula is often adopted, like in literature (Huang et al. [2011], Lin et al. [2008], He et al. [2007], Qu et al. [2009], Lin et al. [2007], Ma et al. [2017]). Though the $H_\infty$ filter design issue can be solved, the derived results are conservative, and there is little room left to improve. So in this paper, we aim to use a new integral inequality to substitute the conventional inequalities derived from the Leibniz-Newton formula, and to further relax the designed fuzzy $H_\infty$ filter.

What’s more, usually the existing fuzzy $H_\infty$ filter design methods are membership functions independent (Kong and Dai [2017, 2016], Huang et al. [2011], Lin et al. [2008], He et al. [2007]), to the best of our knowledge, the membership functions dependent $H_\infty$ filter design approach is yet to be thoroughly investigated, so in order to further reduce the conservativeness, we will consider the information of the membership functions in the criterion.

For the sake of obtaining satisfying fuzzy $H_\infty$ filter, we are going to employ the mismatching method to design fuzzy filter, and to relax the results, a novel integral inequality which is tighter than conventional inequalities will be applied, besides the information of the membership functions and a fuzzy Lyapunov function technique will be introduced into the criteria as well. And the rest of this paper will be organized as follows: Section 2 systematically describes the problem we are going to solve and gives a relevant lemma about the new integral inequality. In Section 3, the detailed $H_\infty$ design methods are presented. And Section 4 uses three simulation examples to illustrate the effectiveness of the designed methods. Finally, some conclusions are given in Section 5.

2. PRELIMINARIES

Consider a nonlinear system involving time-varying delay, which is described by the following p-rule T-S fuzzy model:

**Plant Rule i**: IF $\psi_i(t)$ is $\mathcal{M}_1^i$ and $\psi_2(t)$ is $\mathcal{M}_2^i$ and ... and $\psi_m(t)$ is $\mathcal{M}_m^i$, THEN

$$\begin{align*}
\dot{x}(t) &= A_i x(t) + A_{ir} x(t - \tau(t)) + B_i w(t), \\
y(t) &= C_i x(t) + C_{ir} x(t - \tau(t)) + D_i w(t), \\
z(t) &= E_i x(t) + E_{ir} x(t - \tau(t)), \\
x(t) &= \chi(t), \quad \forall t \in [-\tau_0, 0],
\end{align*}$$

where $i = 1, 2, \ldots, p$, $\alpha_1, \alpha_2, \ldots, \alpha_m$ is the premise variable. $\mathcal{M}_i^a$ is the fuzzy term of rule i which corresponds to the function $\psi_i$, $m$ is a positive integer. And $x(t)$ in $\mathbb{R}^n$ is the system state, $z(t)$ in $\mathbb{R}^p$ is the unmatched noise signal which is assumed to be arbitrary and satisfy $w(t) \in [0, \infty]$, $A_i, A_{ir}, B_i, C_i, C_{ir}, D_i, E_i, E_{ir}$ are given system matrices. Time delay $\tau(t)$ is a continuously differentiable function, satisfying the conditions followed:

$$0 \leq \tau(t) < h, \quad \dot{\tau}(t) \leq \rho.$$  (2)

By fuzzy blending, the system dynamics can be presented as

$$\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{p} \phi_i(t) [A_i x(t) + A_{ir} x(t - \tau(t)) + B_i w(t)], \\
y(t) &= \sum_{i=1}^{p} \phi_i(t) [C_i x(t) + C_{ir} x(t - \tau(t)) + D_i w(t)], \\
z(t) &= \sum_{i=1}^{p} \phi_i(t) [E_i x(t) + E_{ir} x(t - \tau(t))].
\end{align*}$$

where

$$\phi_i(t) = \begin{cases} 
1, & \phi_i(t) \geq 0, \\
0, & \text{otherwise}
\end{cases}$$

and $\phi_i(t)$ is the normalized membership function; $\mu_{\mathcal{M}_i^a}(\psi_i(x(t)))$ is the grade of membership function which corresponds to the fuzzy term $\mathcal{M}_i^a$.

In order to eliminate the drawbacks of the conventional PDC methodology, in this paper, the fuzzy $H_\infty$ filter to be designed is allowed to have different premise membership functions and number of rules from the fuzzy model. Hence, we assume that the number of the fuzzy filter to be designed is $c$, and it can be described as:

**Filter Rule j**: IF $g_1(t)$ is $\mathcal{N}_1^j$ and $g_2(t)$ is $\mathcal{N}_2^j$ and ... and $g_c(t)$ is $\mathcal{N}_c^j$, THEN

$$\begin{align*}
\dot{x}_j(t) &= A_{fj} x_f(t) + B_{fj} y(t), \\
z_j(t) &= C_{fj} x_f(t),
\end{align*}$$

where $x_f(t) \in \mathbb{R}^n$ and $z_f(t) \in \mathbb{R}^q$ are the state and output of the fuzzy $H_\infty$ filter respectively. And $A_{fj}, B_{fj}, C_{fj}$ are the filter matrices of that will be designed.

Similarly, through fuzzy blending, the fuzzy $H_\infty$ filter to be designed can be presented as

$$\begin{align*}
\dot{x}_f(t) &= \sum_{j=1}^{c} n_j(g(t))(A_{fj} x_f(t) + B_{fj} y(t)), \\
z_f(t) &= \sum_{j=1}^{c} n_j(g(t))C_{fj} x_f(t),
\end{align*}$$

where

$$g(t) = (g_1(t), \ldots, g_c(t)), \quad \sum_{j=1}^{c} n_j(g(t)) = 1, \quad n_j(g(t)) \geq 0, \quad n_j(g(t)) = \begin{cases} 
1, & \mu_{\mathcal{N}_j^{a}}(g_j(x_f(t))) \\
0, & \text{otherwise}
\end{cases}$$

for all $j$, $n_j(g(t))$ is the normalized membership function. $\mu_{\mathcal{N}_j^{a}}(g_j(x_f(t))) (\beta = 1, 2, \ldots, 6)$ is the grade of membership functions which corresponds to the fuzzy term $\mathcal{N}_j^a$.

According to (3) and (6), and define the augmented state vector as $\zeta(t) = [x^2(t), x_f(t)]^T$ and $e(t) = z(t) - z_f(t)$, we can obtain the $H_\infty$ filtering system as follows:
Lemma 1 (Hong et al. [2015])

deduction of the main results.
What's more, the following lemma is useful for the later

\[
H_R Fuzzy
\]

where

\[
Ω = \Delta
\]

(6) satisfying the following two conditions:

\begin{align*}
\Xi_1(t) & = \frac{1}{h} \begin{bmatrix}
-\Omega(t) & O(t) & 0 & 0 \\
* & -\Omega(t) & 0 & 0 \\
* & * & 0 & 0 \\
* & * & * & 0
\end{bmatrix} \\
\Xi_2(t) & = \frac{3}{h} \begin{bmatrix}
-\Omega(t) & -O(t) & 20(t) & 0 \\
* & -\Omega(t) & 20(t) & 0 \\
* & * & -40(t) & 0 \\
* & * & * & 0
\end{bmatrix} \\
\Xi_3(t) & = \frac{5}{h} \begin{bmatrix}
-\Omega(t) & O(t) & 60(t) & -60(t) \\
* & -\Omega(t) & -60(t) & 60(t) \\
* & * & -36O(t) & 360(t) \\
* & * & * & 0
\end{bmatrix}
\end{align*}

\[\Theta_3(t) = \begin{bmatrix}
\theta_1 & MA_\epsilon(t) & 0 & 0 & MB(t) \\
* & -N(t) & 0 & 0 & 0 \\
* & * & 0 & 0 & 0 \\
* & * & * & -\gamma^2 & 0
\end{bmatrix},\]

\[\theta_1 = \text{Sym}[MA_\epsilon(t)] + N(t),\]

\[\Gamma_1(t) = [\hat{A}(t) \hat{A}(t) 0 0 \hat{B}(t)]\]

\[\Gamma_2(t) = [\hat{E}(t) \hat{E}_\tau(t) 0 0 0] .\]

As fuzzy $H_\infty$ filter has to guarantee the asymptotical

stability of the filtering system, we will first analyze the

stability of the system (8).

Lemma 2 For system (8), the constants $h, \rho$ and $\gamma > 0$ are

prescribed, it will be asymptotically stable with $w(t) \equiv 0,$

and satisfy the $H_\infty$ performance condition (10), if there

exist matrices $M \in \mathbb{R}^{2n \times 2n} > 0, N(t) \in \mathbb{R}^{2n \times 2n} > 0,$

$O(t) \in \mathbb{R}^{2n \times 2n} > 0,$ such that the following inequality

is feasible.

\[
\Xi_1(t) = \frac{1}{h} \begin{bmatrix}
-\Omega(t) & O(t) & 0 & 0 \\
* & -\Omega(t) & 0 & 0 \\
* & * & 0 & 0 \\
* & * & * & 0
\end{bmatrix},
\]

\[\Xi_2(t) = \frac{3}{h} \begin{bmatrix}
-\Omega(t) & -O(t) & 20(t) & 0 \\
* & -\Omega(t) & 20(t) & 0 \\
* & * & -40(t) & 0 \\
* & * & * & 0
\end{bmatrix} ,
\]

\[\Xi_3(t) = \frac{5}{h} \begin{bmatrix}
-\Omega(t) & O(t) & 60(t) & -60(t) \\
* & -\Omega(t) & -60(t) & 60(t) \\
* & * & -36O(t) & 360(t) \\
* & * & * & 0
\end{bmatrix} ,
\]

\[\theta_1 = \text{Sym}[MA_\epsilon(t)] + N(t),\]

\[\Gamma_1(t) = [\hat{A}(t) \hat{A}(t) 0 0 \hat{B}(t)]\]

\[\Gamma_2(t) = [\hat{E}(t) \hat{E}_\tau(t) 0 0 0] .\]

proof To relax the result, the Lyapunov-Krasovskii func-
tional candidate can be defined as fuzzy Lyapunov function:

\[
V(t) = \varepsilon^T(t)M\varepsilon(t) + \int_{t-\tau(t)}^{t} \varepsilon^T(s)N(s)\varepsilon(s)ds \\
+ \int_{-h}^{0} \int_{t+s}^{t} \varepsilon^T(s)O(s)\varepsilon(s)dsd\theta,
\]

where

\[M > 0, \quad N(t) = \sum_{i=1}^{p} \phi(\psi(t))N_i, \quad O(t) = \sum_{i=1}^{p} \phi(\psi(t))O_i.\]

It is necessary to mention that in the Lyapunov function,

the matrix $M$ is presented as constant matrix, but $N(t)$

and $O(t)$ are expressed as matrix function, this is because if

matrix $M$ is denoted as matrix function, it will be difficult to
determine the upper bound of $||\phi(x(t))||.$
Differentiating (13) along the trajectories of system (8) yields:

\[ \dot{V}(t) = 2\varepsilon^T(t)\dot{\varepsilon}(t) - (1 - \dot{\tau}(t))\varepsilon^T(t - \tau(t))N(t)\varepsilon(t - \tau(t)) + h\dot{\varepsilon}(t)^T\dot{O}(t)\dot{\varepsilon}(t) - \int_{t-h}^{t} \dot{\varepsilon}(s)^T\dot{O}(t)\dot{\varepsilon}(s)ds. \]  

(15)

Applying Lemma 1, we can derive

\[ -\int_{t-h}^{t} \dot{\varepsilon}(s)^T(O(t)\dot{\varepsilon}(s)ds \leq -\int_{t-\tau(t)}^{t} \dot{\varepsilon}(s)^T(O(t)\dot{\varepsilon}(s)ds \leq \xi^T(t) \left[ \tau(t)F_1O(t)^{-1}F_1^T + \frac{\tau(t)}{3}F_2O(t)^{-1}F_2^T + \frac{\tau(t)}{5}F_3O(t)^{-1}F_3^T + \text{Sym}\{F_1\Pi_1 + F_2\Pi_2 + F_3\Pi_3\} \right] \xi(t) \]

\[ < \xi^T(t) \left[ hF_1O(t)^{-1}F_1^T + \frac{h}{3}F_2O(t)^{-1}F_2^T + \frac{h}{5}F_3O(t)^{-1}F_3^T + \text{Sym}\{F_1\Pi_1 + F_2\Pi_2 + F_3\Pi_3\} \right] \xi(t) \]

\[ = \xi^T(t)(\Theta_1(t) + \Theta_2)\xi(t), \]

where

\[ \xi(t) = [\varepsilon^T(t) \varepsilon^T(t - \tau(t))] \frac{1}{\tau(t)} \int_{t-\tau(t)}^{t} \dot{\varepsilon}(s)ds \theta_2 w(t)^T, \]

\[ \theta_2 = \frac{2}{\tau(t)} \int_{t-\tau(t)}^{t} \int_{t-\tau(t)}^{s} \dot{\varepsilon}(u)du w(t)^T, \]

\[ \Theta_1(t) = hF_1O(t)^{-1}F_1^T + \frac{h}{3}F_2O(t)^{-1}F_2^T + \frac{h}{5}F_3O(t)^{-1}F_3^T + \text{Sym}\{F_1\Pi_1 + F_2\Pi_2 + F_3\Pi_3\} \]

\[ \Theta_2 = \text{Sym}\{F_1\Pi_1 + F_2\Pi_2 + F_3\Pi_3\} \]

\[ e_i = [0_{2n \times (i-1)2n} I_{2n} 0_{2n \times (5-i)2n}^T], \quad i = 1, 2, 3, 4, 5, \]

\[ \Pi_1 = e_1 - e_2, \quad \Pi_2 = e_1 + e_2 - 2e_3, \]

\[ \Pi_3 = e_1 - e_2 - 6e_3 + 6e_4. \]

Based on inequality (2) and equation (8), we can derive

\[ \dot{V}(t) < 2\varepsilon^T(t)M[A(t)e(t) + A_\tau(t)e(t - \tau(t)) + \bar{B}(t)w(t)] - (1 - \rho)\varepsilon^T(t - \tau(t))N(t)\varepsilon(t - \tau(t)) + h\dot{\varepsilon}(t)^T\dot{O}(t)\dot{\varepsilon}(t) + \xi^T(t)(\Theta_1(t) + \Theta_2)\xi(t), \]

and through a straightforward computation we can obtain:

\[ \dot{V}(t) + e^T(t)e(t) - \gamma^2(w^T(t)w(t)) < \xi^T(t)(\Theta_1(t) + \Theta_2) + \Theta_3(t) + h\Gamma_1^T(\Omega_1(t) + \Gamma_2^T\Gamma_2(t))\xi(t), \]

(16)

where \( \Theta_3(t), \Gamma_1(t), \Gamma_2(t) \) are defined in (12).

From the inequality (17), it can be inferred that if

\[ \Theta_1(t) + \Theta_2 + \Theta_3(t) + h\Gamma_1^T(\Omega_1(t) + \Gamma_1^T\Gamma_1(t)) + \Gamma_2^T\Gamma_2(t) < 0, \]

(18)

the following inequality holds

\[ \dot{V}(t) + e^T(t)e(t) - \gamma^2(w^T(t)w(t)) < 0. \]

(19)

Further, we can derive

\[ \int_0^L ([|c(t)|]^2 - \gamma^2|w(t)|^2)dt + V(t)|_{t=L} - V(t)|_{t=0} \leq 0, \]

for \( V(t)|_{t=0} = 0, \) and \( V(t)|_{t=L} \geq 0. \) The inequality (20) can be easily converted to inequality (10), which implies the \( H_\infty \) performance requirement is satisfied.
λ(t) will be true.

invertible via invoking small perturbation if it is necessary.

From (32), we can obtain:

\[ \tilde{\Theta}_3(t) = \begin{bmatrix} \tilde{\Theta}_3(t) & 0 & 0 \\ 0 & \tilde{\Theta}_3(t) & 0 \\ 0 & 0 & \tilde{\Theta}_3(t) \end{bmatrix}, \]

and in this case, a feasible \( H_\infty \) filter can be presented as

\[ A_f(t) = \tilde{M}_{22}^{-1} \mathcal{A}(t), \quad B_f(t) = \tilde{M}_{22}^{-1} \mathcal{B}(t), \quad C_f(t) = \mathcal{E}(t). \]

(26)

Theorem 1 For system (8), the constants \( h, \rho, \nu \) and \( \gamma > 0 \) are prescribed, it will be asymptotically stable with \( w(t) \equiv 0 \), and satisfy the \( H_\infty \) performance condition (10), if there exist matrices

\[ \tilde{M} = \begin{bmatrix} M_{11} & \tilde{M}_{22} \\ \ast & \ast \end{bmatrix} > 0, \]

\( \tilde{N}_i \in \mathbb{R}^{2n \times 2n} > 0, \tilde{O}_i \in \mathbb{R}^{2n \times 2n} > 0 \), such that the following LMIs are feasible.

\[ \hat{\Theta}_{ij} = \begin{bmatrix} \hat{\Theta}_{ij} & \sqrt{\tilde{N}_i^T} \tilde{f}_{ij} & \tilde{N}_{ij}^T \\ \hat{\Theta}_{ij} & \ast & \ast \\ \tilde{N}_{ij} & 0 & -1 \end{bmatrix} < 0 \]

(36)

where

\[ \hat{\Theta}_{ij} = (\tilde{\Xi}_{1i} + \tilde{\Xi}_{2i} + \tilde{\Xi}_{3i}) + \hat{\Theta}_{3ij}, \]

\[ \tilde{\Xi}_{1i} = \frac{1}{h} \begin{bmatrix} -\tilde{O}_i & 0 & 0 & 0 \\ -\tilde{O}_i & 0 & 0 & 0 \\ 0 & \ast & \ast & 0 \\ 0 & \ast & \ast & 0 \end{bmatrix}, \]

\[ \tilde{\Xi}_{2i} = \frac{3}{h} \begin{bmatrix} -\tilde{O}_i & 0 & 0 & 0 \\ -\tilde{O}_i & 0 & 0 & 0 \\ 0 & \ast & \ast & 0 \\ 0 & \ast & \ast & 0 \end{bmatrix}, \]

\[ \tilde{\Xi}_{3i} = \frac{5}{h} \begin{bmatrix} -\tilde{O}_i & 0 & 0 & 0 \\ -\tilde{O}_i & 0 & 0 & 0 \\ 0 & \ast & \ast & 0 \\ 0 & \ast & \ast & 0 \end{bmatrix}, \]

Besides, as \( \tilde{M}_{22} = M_{12} M_{22}^{-1} M_{12}^T \), through an equivalent transformation \( M_{12}^{-T} M_{22}^T \), we can obtain an admissible fuzzy \( H_\infty \) realization as:

\[ A_f(t) = M_{12}^{-1} M_{12}^{-T} M_{22}^T, \quad B_f(t) = M_{12}^{-1} \mathcal{B}(t), \quad C_f(t) = \mathcal{E}(t) M_{12}^{-T} M_{22}^T. \]

(33)

Thus, we complete the proof of Lemma 3.

Lemma 3 provides a feasible \( H_\infty \) filter for system (3), however, it is necessary to point out that the equation (26) cannot be directly applied to fuzzy filter design. To address this problem, we will transfer the conditions in Lemma 3 into a finite set of LMIs.
\[ \lambda_{2ij} = \begin{bmatrix} M_{11}A_{ri} + \mathcal{B}_{ri}C_{ri} + 0 \\ M_{22}A_{ri} + \mathcal{B}_{ri}C_{ri} + 0 \end{bmatrix}, \]
\[ \lambda_{3ij} = \begin{bmatrix} M_{11}B_{ri} + \mathcal{B}_{ri}D_{ri} \\ M_{22}B_{ri} + \mathcal{B}_{ri}D_{ri} \end{bmatrix}, \]
\[ i = 1, 2, \ldots, p, j = 1, 2, \ldots, c. \]

And in this case, the parameters of the fuzzy filter can be presented as:
\[ A_{fj}' = M_{22}^{-1}A_{fj}, \quad B_{fj}' = M_{22}^{-1}B_{fj}, \quad C_{fj}' = C_{fj}. \]  
(37)

**Proof** Since
\[ \sum_{i=1}^{p} \phi_i(\psi(t)) = \sum_{i=1}^{p} \sum_{j=1}^{c} \phi_i(\psi(t))n_j(g(t)) = 1, \]
and in terms of (9) and (14), we can derive
\[ \breve{\Omega}(t) = \sum_{i=1}^{p} \sum_{j=1}^{c} \phi_i(\psi(t))n_j(g(t)) \breve{\Omega}_{ij}, \]  
(38)

where \( \breve{\Omega}_{ij} \) is defined in (36).

As a result, if \( \breve{\Omega}_{ij} < 0 \) holds, \( \breve{\Omega}(t) < 0 \) can be derived. And according to Lemma 3, we know that \( \breve{\Omega}(t) < 0 \) means the filtering system \( S \) is asymptotically stable and satisfy the \( H_\infty \) performance condition. Hence, the proof of Theorem 1 is accomplished.

**Theorem 2** For system (8), the constants \( h, \rho, \upsilon \) and \( \gamma > 0 \) are prescribed, it will be asymptotically stable with \( w(t) \equiv 0 \), and satisfies the \( H_\infty \) performance condition (10), if there exist matrices:
\[ \bar{\Omega} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix} > 0, \]  
(40)

\[ \bar{\breve{\Omega}}_i \in \mathbb{R}^{2n \times 2n} > 0, \quad \bar{\breve{\Omega}}_j \in \mathbb{R}^{2n \times 2n} > 0, \quad M_{ij} \in \mathbb{R}^{(10n+2) \times (10n+2)}, \quad M_{ij} \in \mathbb{R}^{(10n+2) \times (10n+2)}, \]  
such that the following LMIs are feasible:
\[ \bar{\breve{\Omega}}_i - M_{ij} + Q_{ij} + \sum_{r=1}^{p} \sum_{s=1}^{c} \bar{\breve{d}}_{ij} M_{rs} - \sum_{a=1}^{p} \sum_{b=1}^{c} \bar{d}_{ab} M_{ab} < 0, \]
\[ i = 1, 2, \ldots, p, j = 1, 2, \ldots, c, \]  
(41)

where \( \breve{\Omega}_{ij} \) is defined in (36). And in this case, the parameters of the fuzzy filter can also be presented as (37).

**Proof** In this part, for the convenience of notations, we denote
\[ \phi_i(\psi(t)) = \phi_i, \quad n_j(g(t)) = n_j, \quad \phi_i(\psi(t))n_j(g(t)) = d_{ij}, \]  
(42)

and assume \( \bar{d}_{ij} \) and \( d_{ij} \) are the lower bound and upper bound of \( d_{ij} \).

From Lemma 2 and Lemma 3, we have
\[ \hat{V}(t) + \varepsilon^2 \hat{w}^T(t)w(t) < \xi^T(t) \bar{\breve{\Omega}}(t) \xi(t), \]  
(43)
so we can derive
\[ \xi^T(t) \bar{\breve{\Omega}}(t) \xi(t) = \sum_{i=1}^{p} \sum_{j=1}^{c} d_{ij} \xi^T(t) \Omega_{ij} \xi(t) \]
\[ \leq \sum_{i=1}^{p} \sum_{j=1}^{c} d_{ij} \xi^T(t) \Omega_{ij} \xi(t) + \sum_{i=1}^{p} \sum_{j=1}^{c} (\bar{d}_{ij} - d_{ij}) \xi^T(t) M_{ij} \xi(t) \]
\[ + \sum_{i=1}^{p} \sum_{j=1}^{c} (d_{ij} - d_{ij}) \xi^T(t) Q_{ij} \xi(t) \]
\[ = \sum_{i=1}^{p} \sum_{j=1}^{c} d_{ij} \xi^T(t) (\Omega_{ij} - M_{ij} + Q_{ij}) \xi(t) \]
\[ + \sum_{i=1}^{p} \sum_{j=1}^{c} \bar{d}_{ij} \xi^T(t) M_{ij} \xi(t) - \sum_{i=1}^{p} \sum_{j=1}^{c} d_{ij} \xi^T(t) Q_{ij} \xi(t) \]
\[ = \sum_{i=1}^{p} \sum_{j=1}^{c} d_{ij} \xi^T(t) (\Omega_{ij} - M_{ij} + Q_{ij}) + \sum_{i=1}^{p} \sum_{j=1}^{c} \bar{d}_{ij} \xi^T(t) Q_{ij} \xi(t) \]
\[ - \sum_{a=1}^{p} \sum_{b=1}^{c} \bar{d}_{ab} Q_{ab} \xi^T(t). \]  
(44)

Consequently, if LMIs (41) holds, we can get \( \breve{\Omega}(t) < 0 \), which means both the \( H_\infty \) performance condition (10) and the asymptotically stable requirement can be satisfied. Thus, the proof of Theorem 2 is finished.

**Remark 3**

It can be seen that Theorem 2 includes the information of the membership functions while Theorem 1 does not. As a result, Theorem 2 is less conservative than Theorem 2. Whereas, on the other hand, Theorem 2 also includes complex matrices \( M_{ij}, Q_{ij}, \) \((i = 1, \ldots, p, j = 1, \ldots, c)\), which means it will be more difficult to implement in engineering applications. So both Theorem 1 and Theorem 2 have their own significance in practice.

### 4. SIMULATION

In this section, two simulation examples will be provided to demonstrate the effectiveness and superiority of the designed criteria.

#### 4.1 Example 1

Consider a time-delay system in (3) with
\[ A_1 = \begin{bmatrix} -2.1 & 0.1 \\ 1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1.1 & 0 \\ -0.2 & -1.1 \end{bmatrix}, \]
\[ A_{r1} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix}, \quad A_{r2} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix}, \]
\[ B_1 = \begin{bmatrix} 1 \\ -0.2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}, \]
\[ C_1 = [1 \ 0], \quad C_2 = [0.5 \ -0.6], \]
\[ C_{r1} = [-0.8 \ 0.6], \quad C_{r2} = [-0.2 \ 1], \]
\[ D_1 = 0.3, \quad D_2 = -0.6, \]
\[ E_1 = [1 \ -0.5], \quad E_2 = [-0.2 \ 0.3], \]
\[ E_{r1} = [0.1 \ 0], \quad E_{r2} = [0 \ 0.2]. \]
and the membership functions of the fuzzy model and the fuzzy filter are chosen as
\[
\phi_1(\psi(t)) = 1 - \frac{0.5}{1 + e^{-3\tau}}, \\
\phi_2(\psi(t)) = 1 - \phi_1(\psi(t)), \\
n_1(g(t)) = 0.7 - \frac{0.5}{1 + e^{4\tau}}, \\
n_2(g(t)) = 1 - n_1(g(t)).
\]
Assume \((\rho, \upsilon, h) = (0.2, 1, 0.5)\), using the LMIs (41) presented in Theorem 2, the minimum attenuation level \(\gamma = 0.18\) can be acquired. Theorem 2 will demonstrate the validity of the proposed method, we will respectively apply Theorem 2 in this paper and the methods presented in Huang et al. [2011], Lin et al. [2008], Su et al. [2009], Zhang et al. [2009], Zhou and He [2015] to discuss the results. The corresponding results are listed in Table 1-3.}

### Table 2. The minimum attenuation level \(\gamma\) for \(\nu = 5\)

| method          | \(h = 0.5\) | \(h = 0.6\) | \(h = 0.8\) | \(h = 1\) |
|-----------------|-------------|-------------|-------------|-------------|
| Lin et al. [2008]| 0.34        | 0.34        | 0.35        | 0.37        |
| Su et al. [2009]| 0.24        | 0.24        | 0.25        | 0.26        |
| Zhang et al. [2009]| 0.24        | 0.24        | 0.25        | 0.26        |
| Huang et al. [2011]| 0.24        | 0.24        | 0.25        | 0.26        |
| Zhou and He [2015]| 0.23        | 0.24        | 0.24        | 0.25        |
| Th. 2            | 0.16        | 0.17        | 0.17        | 0.17        |

### Table 3. The minimum attenuation level \(\gamma\) for \(\nu = 20\)

| method          | \(h = 0.5\) | \(h = 0.6\) | \(h = 0.8\) | \(h = 1\) |
|-----------------|-------------|-------------|-------------|-------------|
| Lin et al. [2008]| 0.37        | 0.45        | 1.01        | --          |
| Su et al. [2009]| 0.26        | 0.32        | 0.70        | --          |
| Zhang et al. [2009]| 0.26        | 0.28        | 0.44        | --          |
| Huang et al. [2011]| 0.25        | 0.26        | 0.35        | 0.45        |
| Zhou and He [2015]| 0.23        | 0.24        | 0.25        | 0.25        |
| Th. 2            | 0.17        | 0.17        | 0.17        | 0.17        |

**Remark 4** The less conservative results can be obtained with the approach proposed in this paper mainly because of three reasons. First, the new integral inequality (11) is employed to derive stability condition, which is tighter than those derived from the Leibniz-Newton formula. Besides, our criterion is membership functions dependent while the others are membership functions independent. Also, we adopt fuzzy Lyapunov function, we can increase the number of free-weighting matrices, and further relax the result.

#### 4.2 Example 2

Consider a system in the form of (3) with
\[
A_1 = \begin{bmatrix} -2.1 & 0.1 \\ 1 & -2 \end{bmatrix}, A_2 = \begin{bmatrix} -1.9 & 0 \\ -0.2 & -1.1 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
A_{r1} = \begin{bmatrix} -1.1 & 0.1 \\ -0.8 & -0.9 \end{bmatrix}, A_{r2} = \begin{bmatrix} -0.9 & 0 \\ -1.1 & -1.2 \end{bmatrix}, \\
A_{r3} = \begin{bmatrix} 0.5 & 0 \\ 1 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} 1 \\ -0.2 \end{bmatrix}, B_2 = \begin{bmatrix} 0.3 \\ 0.1 \end{bmatrix}, B_3 = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}, \\
C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0.5 & 0.6 \end{bmatrix}, C_3 = \begin{bmatrix} 0.5 & -0.6 \end{bmatrix}, \\
C_{r1} = \begin{bmatrix} 0.8 & 0.6 \end{bmatrix}, C_{r2} = \begin{bmatrix} -0.2 & 1 \end{bmatrix}, C_{r3} = \begin{bmatrix} 0 & 0.5 \end{bmatrix}, \\
D_1 = 0.3, D_2 = -0.6, D_3 = 0.5, \\
E_1 = \begin{bmatrix} 1 & -0.5 \end{bmatrix}, E_2 = \begin{bmatrix} -0.2 & 0.3 \end{bmatrix}, E_3 = \begin{bmatrix} 0.6 & 1 \end{bmatrix}, \\
E_{r1} = \begin{bmatrix} 0.1 & 0 \end{bmatrix}, E_{r2} = \begin{bmatrix} 0.2 & 0 \end{bmatrix}, E_{r3} = \begin{bmatrix} 0.2 & 0.1 \end{bmatrix},
\]
and the membership functions are defined as
\[
\phi_1(\psi(t)) = 1 - \frac{0.6}{1 + e^{-3\tau}}, \\
\phi_2(\psi(t)) = 1 + e^{-3\tau}, \\
\phi_3(\psi(t)) = 1 - \phi_1(\psi(t)) - \phi_2(\psi(t)), \\
n_1(g(t)) = 0.7 - \frac{0.5}{1 + e^{4\tau}}, \\
n_2(g(t)) = 1 - n_1(g(t)).
\]
Then Theorem 1 and Theorem 2 will be adopted to compute the minimum attenuation level $\gamma$, the corresponding results are listed in Table 4-5.

Table 4. The minimum attenuation level $\gamma$ for $h = 0.5$

| method | $\nu = 0.7$ | $\nu = 1$ | $\nu = 2$ | $\nu = 5$ | $\nu = 10$ | $\nu = 20$ |
|--------|-------------|------------|------------|------------|------------|------------|
| Th. 1  | 0.40        | 0.27       | 0.24       | 0.23       | 0.22       | 0.24       |
| Th. 2  | 0.20        | 0.16       | 0.13       | 0.11       | 0.09       | 0.10       |

Table 5. The minimum attenuation level $\gamma$ for $h = 0.8$

| method | $\nu = 0.7$ | $\nu = 1$ | $\nu = 2$ | $\nu = 5$ | $\nu = 10$ | $\nu = 20$ |
|--------|-------------|------------|------------|------------|------------|------------|
| Th. 1  | 4.53        | 0.55       | 0.26       | 0.24       | 0.24       | 0.25       |
| Th. 2  | 0.64        | 0.42       | 0.15       | 0.12       | 0.11       | 0.11       |

From Table 4-5, we can clearly see that Theorem 2 can produce much smaller minimum attenuation level $\gamma$ than Theorem 1, which means Theorem 2 is more relaxed than Theorem 1. And this is because Theorem 2 takes the information of the membership functions into consideration while Theorem 1 does not. Besides, in the simulation process, Theorem 2 spent much longer time to compute the minimum attenuation level $\gamma$ than Theorem 1, the reason for this phenomenon is that LMIs (41) in Theorem 1 is simpler than the LMIs (41) in Theorem 2, which also implies Theorem 1 will be more realizable than Theorem 2 in practice.

Remark 5 It is essential to point out that in Example 2, the fuzzy model has 3 fuzzy rules while the fuzzy filter have 2 fuzzy rules, and the membership functions of the fuzzy model and the fuzzy filter are totally different. Such design means the number of fuzzy rules and the membership functions of the fuzzy filter can be freely chosen, thus, we can lower the implementation cost via choosing simpler membership functions, and successfully deal with the situation with uncertainty through avoiding unknown membership functions.

5. CONCLUSIONS

In this paper, the fuzzy $H_\infty$ filter design problem has been investigated for nonlinear time-delay systems. The T-S fuzzy model has been used to describe the dynamics of the system, and two LMI-based criteria have been derived. Unlike conventional fuzzy $H_\infty$ filter, the designed fuzzy filter has been allowed to freely choose the premise membership functions and the number of rules, ergo, robustness to uncertainty and lower implementation cost can be realized. Besides, to reduce the conservatism of the derived results, a novel integral inequality which is tighter than other existing ones has been introduced, and a fuzzy Lyapunov function and the information of the membership functions have been taken into account, too. Finally, three examples have demonstrated the validity of the designed fuzzy $H_\infty$ filter.

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