EXTENDING VECTOR BUNDLES ON CURVES

SIDDHARTH MATHUR

Abstract. Given a curve in a (smooth) projective variety $C \subset X$, we show that a vector bundle $V$ on $C$ can be extended to a $(\mu$-stable) vector bundle on $X$ if $\text{rank}(V) \geq \dim(X)$ and $\text{det}(V)$ extends to $X$.

1. Introduction

Understanding which vector bundles on a subvariety extend to an ambient variety is a well studied problem in algebraic geometry. A famous example is the Grothendieck-Lefschetz theorems which considers the case of a complete intersection in a projective variety. A particularly striking consequence of this work is that complete intersections $X \subset P^n$ with dimension $\dim(X) \geq 3$ always have a Picard group which is freely generated by $\mathcal{O}_{P^n}(1)|_X$. However, there are many counterexamples to this statement when the hypothesis on the dimension is dropped: indeed, all elliptic curves can be realized as ample divisors in $P^2$ but they have a non-finitely generated Picard group. The purpose of this paper is to show that this is the only obstruction when the subvariety is a curve and the rank of the vector bundle is sufficiently large.

**Theorem 1.** Let $(X, \mathcal{O}_X(1))$ be a projective variety over an algebraically closed field $k$, $C \subset X$ a 1-dimensional closed subvariety and $V$ a vector bundle on $C$ with $\text{rank}(V) \geq \dim(X)$. Then $V$ extends to $X$ if and only if $\text{det}(V)$ extends to $X$. If $X$ is assumed to be smooth and $\text{det}(V)$ extends to $X$, then $V$ may be extended to a $\mu$-stable vector bundle.

The proof involves Bertini-type arguments and the theory of elementary transformations due to Maruyama (see [3] for a gentle introduction to the technique).

**Acknowledgments** I would like to thank Jack Hall, Andrew Kresch and David Stapl eton for helpful comments. This research was conducted in the framework of the research training group GRK 2240: Algebro-geometric Methods in Algebra, Arithmetic and Topology, which is funded by the Deutsche Forschungsgemeinschaft.

2. Some Lemmas

**Lemma 2.** Let $Y$ be a 1-dimensional scheme which is finite-type and separated over $k$ and suppose $Z \subset Y$ is a finite set containing the associated points of $Y$. Moreover, let $V$ denote a nontrivial rank $r$ vector bundle on $Y$ which is globally generated, then there exists a Cartier divisor $i : D \to Y$ which doesn’t meet $Z$ as well as an exact sequence

$$0 \to V^* \to \mathcal{O}_Y^\oplus r \to i_* \mathcal{O}_D \to 0$$

**Proof:** By [5, Lemma 27] there is a Cartier divisor $i : D \subset Y$ not meeting $Z$, a line bundle $L$ on $D$ and an exact sequence

$$0 \to \mathcal{O}_Y^\oplus r \to V \to i_* L \to 0$$
Since $D$ is a finite scheme, $L$ is trivial and so we have

$$0 \to \mathcal{O}_Y^\oplus \to V \to i_*\mathcal{O}_D \to 0$$

Now we apply the functor $\mathcal{H}om(-, \mathcal{O}_Y)$ to this sequence to obtain:

$$0 \to V^\vee \to \mathcal{O}_Y^\oplus \to \mathcal{E}xt^1(i_*\mathcal{O}_D, \mathcal{O}_Y) \to 0$$

Indeed, the next term $\mathcal{E}xt^1(V, \mathcal{O}_Y)$ in the long exact sequence vanishes since $V$ is locally free. To determine $\mathcal{E}xt^1(i_*\mathcal{O}_D, \mathcal{O}_Y)$ one can apply the functor $\mathcal{H}om(-, \mathcal{O}_Y)$ to the ideal exact sequence

$$0 \to \mathcal{O}_Y(-D) \to \mathcal{O}_Y \to i_*\mathcal{O}_D \to 0$$

to see that

$$\mathcal{E}xt^1(i_*\mathcal{O}_D, \mathcal{O}_Y) \simeq \text{coker}(s_D : \mathcal{O}_Y \to \mathcal{O}_Y(D)) \simeq i_*\mathcal{O}_D \otimes \mathcal{O}_Y(D)$$

where $s_D$ is the section corresponding to $D$. Since $D$ is a finite scheme this is just isomorphic to $i_*\mathcal{O}_D$.

\begin{lemma}
Let $C \subset Y$ be the inclusion of a proper 1-dimensional closed subscheme in a quasiprojective scheme $Y$ over $k$. Suppose that $E$ is a rank $r$ vector bundle on $C$ whose determinant extends to $Y$, then there is an ample line bundle $L$ on $Y$ with the following properties

1. $E \otimes L|_C$ is globally generated,
2. det$(E \otimes L|_C)$ is ample, and
3. if det$(E \otimes L|_C) = \mathcal{O}_C(D)$ for some effective Cartier divisor $D \subset C$ not containing any associated point of $Y$, then there is an ample Cartier divisor $H \subset Y$ with $H \cap C = D$ scheme-theoretically. If $Y'$ is smooth and projective we may also take $H$ to be smooth away from $D$.
\end{lemma}

\begin{proof}
The proof is the same as [5, Lemma 28] except we need to show that if $Y$ is smooth and projective, then we may take $H$ to be smooth away from $D$. Follow the argument in [5] verbatim except we choose an ample line bundle $L$ on $Y$ satisfying an additional property: the global sections of $L' \otimes L^\oplus \otimes I_D$ should separate tangent vectors on $Y \setminus D$ (here $I_D$ denotes the ideal sheaf of $D$ in $Y$). Then, to conclude, we may choose a section of

$$s_{H'} + H^0(Y, L' \otimes L^\oplus \otimes I_D) \subset H^0(Y, L' \otimes L^\oplus)$$

whose vanishing is smooth away from $D$ by [4, Proposition 4.5.12] (applied to $Y \setminus D$).
\end{proof}

\begin{lemma}
Let $(X, \mathcal{O}_X(1))$ denote a smooth projective variety over $k$ and let $H$ be an ample Cartier divisor which is smooth away from a finite subscheme $D \subset H$. Fix integers $r$ and $\rho$, then there exists an $N_0$ such that for all $N \geq N_0$ and any $\psi \in \text{Hom}(\mathcal{O}_X^\oplus, \mathcal{O}_H(N))$ there is a nonempty open

$$U_N \subset \mathbb{A}(\psi + \text{Hom}(\mathcal{O}_X^\oplus, \mathcal{O}_H(N)) \otimes I_D) \subset \mathbb{A}(\text{Hom}(\mathcal{O}_X^\oplus, \mathcal{O}_H(N)))$$

so all $\phi \in U_N$ do not factor through a torsion free quotient $F$ of $\mathcal{O}_X^\oplus$ with $\mu(F) \leq \rho$ and $\text{rk}(F) < r$.
\end{lemma}

\begin{proof}
Consider the family $\mathcal{F}$ of all torsion free quotients of $\mathcal{O}_X^\oplus$ with $\mu(F) \leq \rho$. By [2, Lemma 1.7.9], there is a finite type Quot-scheme $Q$ of $\mathcal{O}_X^\oplus$ so that each $F \in \mathcal{F}$ appears as a fiber of the universal quotient on $Q$:

$$\mathcal{O}_X^{\oplus Q} \to F_{\text{univ}}$$

Let $p_2 : Q \times X \to X$ denote the projection and apply the functor $(p_1)_*(\text{Hom}(\cdot, p_2^*\mathcal{O}_H(N)))$ for sufficiently large $N > 0$ to obtain an inclusion of vector bundles over $Q$

$$g : \mathbb{A}((p_1)_*\text{Hom}(F_{\text{univ}}, p_2^*\mathcal{O}_H(N))) \hookrightarrow \mathbb{A}(\text{Hom}(\mathcal{O}_X^{\oplus Q}, \mathcal{O}_H(N))) \times_k Q$$
which at a point \( q \in Q \) is the inclusion of homomorphisms \( \psi : \mathcal{O}_X^{\oplus r} \to \mathcal{O}_H(N) \) which factor through \((F_{\text{univ}})_q\). Thus, to prove the lemma it suffices to show there is a \( N_0 \) so that for every \( N \geq N_0 \)

\[
\dim \mathbb{A}((p_1), \text{Hom}(F_{\text{univ}}, \mathcal{O}_H(N))) < \dim (\mathbb{A}(\text{Hom}(\mathcal{O}_X^{\oplus r}, \mathcal{O}_H(N)) \otimes I_D))
\]

Indeed, in that case the morphism

\[
\pi_1 \circ g : \mathbb{A}((p_1), \text{Hom}(F_{\text{univ}}, \mathcal{O}_H(N))) \to \mathbb{A}(\text{Hom}(\mathcal{O}_X^{\oplus r}, \mathcal{O}_H(N)))
\]

cannot have an image containing a coset of \( \mathbb{A}(\text{Hom}(\mathcal{O}_X^{\oplus r}, \mathcal{O}_H(N)) \otimes I_D) \).

For all large \( N \):

\[
\dim \mathbb{A}((p_1), \text{Hom}(F_{\text{univ}}, \mathcal{O}_H(N))) \leq \dim(Q) + \max_{q \in Q} \{ \chi((F_{\text{univ}})_q^\vee(N)|_H) \}
\]

and for every \( q \in Q \)

\[
\chi(\mathcal{O}_X^{\oplus r}(N)|_H) = \chi((F_{\text{univ}})_q(N)|_H) + p_q(N)
\]

for some nonconstant polynomial \( p_q(t) \) since \((F_{\text{univ}})_q(N)\) is a quotient with rank \( < r \). Moreover, since \( D \) is a finite subscheme, there is a fixed constant \( d > 0 \) with

\[
\chi(\mathcal{O}_X^{\oplus r}(N)|_H \otimes I_D) + d = \chi(\mathcal{O}_X^{\oplus r}(N)|_H)
\]

for all large \( N \). Thus, since \( \{ p_q(t) \}_{q \in Q} \) is a finite set of polynomials there is a \( N_0 > 0 \) so that

\[
\dim(Q) + \max_{q \in Q} \{ \dim(\chi((F_{\text{univ}})_q(N)|_H)) \} < \dim (\text{Hom}(\mathcal{O}_X^{\oplus r}, \mathcal{O}_H(N)) \otimes I_D)
\]

for all \( N \geq N_0 \), as desired. \( \square \)

3. Proof of the main theorem

**Proof of Theorem** Let \( \mathcal{O}_X(1) \) denote an ample line bundle on \( X \). By replacing \( V \) with \( V(m) = V \otimes \mathcal{O}_X(1)^{\otimes m}|_C \) for \( m \gg 0 \) we may suppose that the conclusion of Lemma 3 holds. Let \( L \) denote a fixed extension of \( \det(V) \). In particular, \( V \) is globally generated. Thus, by Lemma 3 there exists a Cartier divisor \( i : D \to C \) missing the associated points of \( X \) which may lie on \( C \) and an exact sequence

\[
0 \to V^\vee \to \mathcal{O}_C^{\oplus r} \to i_* \mathcal{O}_D \to 0
\]

By adjunction, the surjection on the right, call it \( \phi_D \), is determined by the induced map of sheaves on \( D \):

\[
\phi_D : \mathcal{O}_C^{\oplus r}|_D = \mathcal{O}_D^{\oplus r} \to \mathcal{O}_D
\]

To prove the theorem it suffices to show \( V^\vee \) extends to \( X \). Observe that \( \det(\mathcal{O}_D) = \mathcal{O}_C(D) = L|_C \) and that \( D \subset C \) is a Cartier divisor missing the associated points of \( X \). Thus, we may apply the full conclusion of Lemma 3. In particular, there is an effective ample Cartier divisor \( H \subset X \) with \( H \cap C = D \) scheme-theoretically. If \( X \) is smooth, we may take \( H \) to be smooth away from \( D \).

The idea will be to extend the elementary transformation of \( \mathcal{O}_C^{\oplus r} \) on \( C \) along \( D \) to an elementary transformation of \( \mathcal{O}_X^{\oplus r} \) on \( X \) along \( H \). To make this precise, first fix an isomorphism \( g_1 : \mathcal{O}_X(1)|_D \to \mathcal{O}_D \) and note that this induces isomorphisms \( g_N : \mathcal{O}_X(N)|_D \cong \mathcal{O}_D \) for every \( N > 0 \). Our goal is to find a surjective morphism \( \psi : \mathcal{O}_H^{\oplus r} \to \mathcal{O}_X(N)|_H \) for some \( N > 0 \) so that the following diagram commutes:
Once we have found such a \( \psi \), compose it with the natural adjunction morphism to obtain \( \psi': \mathcal{O}_X \to \mathcal{O}_H \to \mathcal{O}_X|_H \). Now consider the associated elementary transformation on \( X \):

\[
0 \to W \to \mathcal{O}_X \to \mathcal{O}_X|_H \to 0
\]

and observe that because \( H \) is a Cartier divisor, \( W \) must be locally free. Upon restriction to \( C \) the isomorphism \( g_N: \mathcal{O}_X|_D \to \mathcal{O}_D \) induces a morphism of short exact sequences:

\[
\begin{array}{ccc}
0 & \longrightarrow & W|_C \\
\downarrow \cong & & \downarrow \text{id} \\
0 & \longrightarrow & V^\vee
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{O}_X|_C \\
\downarrow \cong & & \downarrow \text{id} \\
0 & \longrightarrow & \mathcal{O}_D
\end{array}
\]

thereby producing an isomorphism \( W|_C \cong V^\vee \), as desired. Note that the isomorphisms \( g_N: \mathcal{O}_X|_D \cong \mathcal{O}_D \) we have fixed determine isomorphisms

\[
\text{Hom}(\mathcal{O}_H, \mathcal{O}_X|_H) \otimes \mathcal{O}_D \cong \text{Hom}(\mathcal{O}_H, \mathcal{O}_D)
\]

As such, the rest of the proof is devoted to finding a \( \psi \) which restricts to \( \phi_D \) via an isomorphism \( g_N \).

Next, take \( N \) to be large enough so that the short exact sequence on \( H \)

\[
0 \to \text{Hom}(\mathcal{O}_H, \mathcal{O}_X|_H) \otimes I_D \to \text{Hom}(\mathcal{O}_H, \mathcal{O}_X|_H) \to \text{Hom}(\mathcal{O}_H, \mathcal{O}_D) \to 0
\]

remains exact after taking global sections so that we may lift \( \phi_D \in \text{H}^0(H, \text{Hom}(\mathcal{O}_H, \mathcal{O}_D)) \) to a section \( \psi_D \in \text{H}^0(H, \text{Hom}(\mathcal{O}_H, \mathcal{O}_X|_H)) \). The issue is that \( \psi_D: \mathcal{O}_H \to \mathcal{O}_X|_H \) may not be surjective away from \( D \). We will rectify this by adding a factor from \( \text{H}^0(H, \text{Hom}(\mathcal{O}_H, \mathcal{O}_X|_H) \otimes I_D) \) which doesn’t change the behavior of \( \psi_D \) along \( D \).

After perhaps increasing \( N \), fix a basis

\[
\psi_1, ..., \psi_n \in \text{H}^0(H, \text{Hom}(\mathcal{O}_H, \mathcal{O}_X|_H) \otimes I_D)
\]

so that at any point \( p \in H \setminus D \), there is a collection of \( r \) sections among the \( \psi_1, ..., \psi_n \) which form a basis for the vector space \( \text{H}^0(H, \text{Hom}(\mathcal{O}_H, \mathcal{O}_X|_H)) \otimes k(p) \). Viewing the sections \( \psi_D, \psi_1, ..., \psi_n \) in \( \text{H}^0(H, \text{Hom}(\mathcal{O}_H, \mathcal{O}_X|_H)) \) we set \( A^n_k = \text{Spec} \ k[x_1, ..., x_n] \) and consider the universal section

\[
\psi_{\text{univ}} = \psi_D + \Sigma_{i=1}^n \lambda_i \psi_i
\]

of \( \psi_D + \text{H}^0(H, \mathcal{O}_X|_H) \otimes I_D \) pulled back to \( A^n_k \times H \). Thus, by construction, the universal section restricts to the section \( \psi_\alpha = \psi_D + \Sigma_{i=1}^n \lambda_i \psi_i \) over \( \alpha = (a_1, ..., a_n) \in A^n_k(k) \).

Over the complement \( U = H \setminus D \subset H \), consider the closed locus of non-surjective maps

\[
Z = \{ (\alpha, u) \mid \psi_\alpha \otimes k(u) \text{ is not surjective} \} \subset A^n_k \times U
\]
For any $u \in U$ the fiber $Z_u$ has codimension $r$ since $\psi_1, \ldots, \psi_n$ generate $\text{Hom}(\mathcal{O}_H^{\oplus r}, \mathcal{O}_X(N)|_H)$ at all $u \in U$. Indeed, there is a surjective linear map

$$\pi : A^n_k(u) \to \text{Hom}_{k(u)}(k(u)^r, k(u)) = \text{Hom}(\mathcal{O}_H^{\oplus r}, \mathcal{O}_X(N)|_H) \otimes k(u)$$

and only the zero map doesn’t have full rank. Therefore the fiber $\text{ker}(\pi) = Z_u$ has dimension $n - r$ so the dimension of $Z$ (and the closure of its image $\overline{p_1(Z)}$ in $A^n_k$) is at most

$$n - r + \dim(H) < n$$

because $r = \text{rank}(V) \geq \dim(X) > \dim(H)$.

Thus, there is a point $\underline{c} = (c_1, \ldots, c_n) \in A^n_k(k)$ avoiding $\overline{p_1(Z)}$, and we claim that the corresponding section of $\text{Hom}(\mathcal{O}_H^{\oplus r}, \mathcal{O}_X(N)|_H)$ works as desired. Indeed, a point avoiding $\overline{p_1(Z)}$ corresponds to a section

$$\psi_{\underline{c}} = \psi_D + \sum_{i=1}^nc_i\psi_i \in H^0(H, \text{Hom}(\mathcal{O}_H^{\oplus r}, \mathcal{O}_X(N)|_H))$$

which is a surjective linear map for every $u \in U$ (since $\underline{c}$ is not in $\overline{Z}$). Moreover, on $D$, we have $\psi_{\underline{c}}|_D = \psi_D|_D = \phi_D$. Also, $\phi_D$ is surjective so Nakayama’s lemma implies $\psi_{\underline{c}}$ is surjective over all of $H$. Thus, the kernel of $\psi'_{\underline{c}} : \mathcal{O}_X^{\oplus r} \to \mathcal{O}_H^{\oplus r} \to \mathcal{O}_H(N)$ is a vector bundle $\tilde{W}$ on $X$ extending $V^\vee$.

If $X$ is smooth, Lemma [4] says we may enlarge $N$ further so that there is a $\underline{c} \in A^n_k \setminus \overline{p_1(Z)}$ so that the corresponding surjection $\psi_{\underline{c}} : \mathcal{O}_X^{\oplus r} \to \mathcal{O}_H(N)$ has the additional property that it doesn’t factor through a torsion free $F$ with $\text{rk}(F) < r$ and $\mu(F) \leq \frac{r}{r-1}\deg(\mathcal{O}(H))$. That the resulting kernel is $\mu$-stable follows from the argument in [2, Proposition 5.2.5].

References

[1] Ofer Gabber, Qing Liu, Dino Lorenzini, et al. Hypersurfaces in projective schemes and a moving lemma. *Duke Mathematical Journal*, 164(7):1187–1270, 2015.

[2] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge University Press, 2010.

[3] Masaki Maruyama. Elementary transformations in the theory of algebraic vector bundles. In *Algebraic geometry*, pages 241–266. Springer, 1982.

[4] Siddharth Mathur. *Some Theorems on the Resolution Property and the Brauer map*. PhD thesis, University of Washington, 2018.

[5] Siddharth Mathur. Experiments on the Brauer map in High Codimension. *arXiv e-prints*, page arXiv:2002.12205, February 2020.

[6] The Stacks Project Authors. *stacks project*. [http://stacks.math.columbia.edu](http://stacks.math.columbia.edu), 2017.

Mathematisches Institut, Heinrich-Heine-Universität, 40204 Düsseldorf, Germany. URL: [https://sites.google.com/view/sidmathur/home](https://sites.google.com/view/sidmathur/home)