A note on the carré du champ on the Poisson space

Ronan Herry

July 27, 2020

Abstract

The goal of this short note is to establish, in complete generality, the representation for the carré du champ operator associated with the Ornstein-Uhlenbeck semi-group on the Poisson space in terms of the add-one and drop-one operators (see Proposition 2 below).

Keywords: Carré du champ; Dirichlet forms; Poisson point process.

MSC Classification: 60G55; 60H07; 60J46.

Poisson setting

We fix \((Z, \mathcal{Z})\) a measurable space equipped with a \(\sigma\)-finite measure \(\nu\). In particular, we do not make any topological assumptions on \((Z, \mathcal{Z})\). We consider \(\mathcal{M}\) be the space of all countable sums of \(\mathbb{N}\)-valued measures on \((Z, \mathcal{Z})\). The space \(\mathcal{M}\) is endowed with the \(\sigma\)-algebra \(\mathcal{M}\), generated by the cylindrical mappings \(\xi \in \mathcal{M} \mapsto \xi(B) \in \mathbb{N} \cup \{\infty\}, \ B \in \mathcal{Z}\).

The Poisson point process with intensity \(\nu\) is the only probability \(\Pi\) on \(\mathcal{M}\) such that the Mecke equation holds:

\[
\hat{u}(\eta, z) d\Pi(\eta) = \hat{u}(\eta + \delta_z, z) d\Pi(\eta) \nu(dz),
\]

for all measurable \(u : \mathcal{M} \times \mathcal{Z} \to [0, \infty]\). Poisson processes with \(\sigma\)-finite intensity exist [6, Theorem 3.6]. Note that, if, in the previous equation, \(f\) is replaced by a measurable function with values in \(\mathbb{R}\), the previous formula still holds provided both sides of the identity are finite when we replace \(f\) by \(|f|\). Integration with respect to \(\Pi\) will also be denoted by the probabilistic notation \(E_\Pi\).

The add and drop operators

Given \(z \in \mathcal{Z}\) and \(F : \mathcal{M} \to \mathbb{R}\) measurable, we let

\[
D^+_z F(\eta) = F(\eta + \delta_z) - F(\eta);
\]

\[
D^-_z F(\eta) = (F(\eta) - F(\eta - \delta_z)) 1_{z \in \eta}.
\]

The operator \(D^+\) (resp. \(D^-\)) is called the add operator (resp. drop operator). Due to the Mecke formula (1), these operations do not depend on the choice of the representative of \(f\) \(\Pi\)-almost surely.

Lemma 1. Let \(F \in \mathcal{L}^\infty(\Pi),\) then \(D^+ F \in \mathcal{L}^\infty(\Pi \otimes \nu)\).

Proof. First of all, \(\delta : Z \ni z \mapsto \delta_z \in \mathcal{M}\) is measurable (if \(A\) is of the form \(\{\eta(B) = k\}\) for some \(B \in \mathcal{Z}\), then the pre-image by \(\delta\) of \(A\) is \(B\), if \(k = 1\); and the pre-image is empty, if \(k > 1\)). Hence, \(D^+ F\) is bi-measurable. Now let

\[
U = \{t \in \mathbb{R}, \text{ such that } \Pi(F \geq t) = 0\};
\]

\[
V = \{t \in \mathbb{R}, \text{ such that } (\Pi \otimes \nu)(F + D^+_z F \geq t) = 0\}.
\]
By assumption $U \neq \emptyset$, and we want to show that $V \neq \emptyset$. Take $t \in U$, by the Mecke formula (1), we have that
\[
\int \int 1_{\{F + D_z^+ F \geq t\}} \nu(dz) \Pi(d\eta) = \int \int 1_{\{F \geq t\}} \eta(dz) \Pi(d\eta) = 0.
\]
Hence $t \in V$, this concludes the proof. \hfill \square

**Malliavin derivative**

For a random variable $F$, we write $F \in \text{Dom } D$ whenever: $F \in \mathcal{L}^2(\Pi)$ and
\[
\int_{\mathcal{Z}} \int (D_z^+ F(\eta))^2 \Pi(d\eta) \nu(dz) < \infty.
\]
Given $F \in \text{Dom } D$, we write $DF$ to denote the random mapping $DF: \mathcal{Z} \ni z \mapsto D_z^+ F$. We regard $D$ as an unbounded operator $\mathcal{L}^2(\Pi) \rightarrow \mathcal{L}^2(\Pi \otimes \nu)$ with domain $\text{Dom } D$. The operator $D$ is closed [5, Lemma 3] and thus $\text{Dom } D$ is Hilbert when equipped with the scalar product
\[
(F, G) \rightarrow \Pi(FG) + (\Pi \otimes \nu)(DFDG).
\]

**The divergence operator**

We consider the divergence operator $\delta = D^*: \mathcal{L}^2(\Pi \otimes \nu) \rightarrow \mathcal{L}^2(\nu)$, that is the unbounded adjoint of $D$. Its domain $\text{Dom } \delta$ is composed of random functions $u \in \mathcal{L}^2(\Pi \otimes \nu)$ such that there exists a constant $c > 0$ such that
\[
\left| \int \int D_z^+ F(\eta) u(\eta, z) \nu(dz) \Pi(d\eta) \right| \leq c \sqrt{\Pi(F^2)}, \quad \forall F \in \text{Dom } D.
\]
For $u \in \text{Dom } \delta$, the quantity $\delta u \in \mathcal{L}^2(\Pi)$ is completely characterised by the duality relation
\[
\mathbb{E}_\Pi G \delta u = \int \int u(\eta, z) D_z F(\eta) \Pi(d\eta) \nu(dz), \quad \forall F \in \text{Dom } D. \tag{2}
\]
From [5, Theorem 5], we have the following Skorokhod isometry. For $u \in \mathcal{L}^2(\Pi \otimes \nu)$, $u \in \text{Dom } \delta$ if and only if $\int (D_z^+ u(\eta, z'))^2 \Pi(d\eta) \nu(dz) \nu(dz') < \infty$ and, in that case:
\[
\mathbb{E}_\Pi (\delta u)^2 = \int \int u(\eta, z)^2 \nu(dz) \Pi(d\eta) + \int \int D_z^+ u(\eta, z') D_z^+ u(\eta, z) \Pi(d\eta) \nu(dz) \nu(dz'). \tag{3}
\]
The Skorokhod isometry implies the following Heisenberg commutation relation. For all $u \in \text{Dom } \delta$, and all $z \in Z$ such that $z' \mapsto D_z^+ u(z') \in \text{Dom } \delta$:
\[
D_z \delta u(\eta) = u(\eta, z) + \delta D_z^+ u(\eta, \cdot).
\]
From [5, Theorem 6], we have the following pathwise representation of the divergence: if $u \in \text{Dom } \delta \cap \mathcal{L}^1(\Pi \otimes \nu)$, then
\[
\delta u(\eta) = \int u(\eta, z) \eta(dz) - \int u(\eta, z) \nu(dz). \tag{4}
\]
Note that $\text{Dom } \delta \cap \mathcal{L}^1(\Pi \otimes \nu)$ is dense in $\text{Dom } \delta$. 

2
The Ornstein-Uhlenbeck generator

The Ornstein-Uhlenbeck generator $L$ is the unbounded self-adjoint operator on $\mathcal{L}^2(\Pi)$ verifying

$$\text{Dom } L = \{ F \in \text{Dom } D, \text{ such that } DF \in \text{Dom } \delta \} \text{ and } L = -\delta D.$$  

Classically, $\text{Dom } L$ is endowed with the Hilbert norm $E_\Pi \left( F^2 + (LF)^2 \right)$. The eigenvalues of $L$ are the non-positive integers and for $q \in \mathbb{N}$ the eigenvectors associated to $-q$ are the so-called iterated Poisson stochastic integrals of order $q$ (see [5] for details). The kernel of $L$ coincides with the set of constants and the pseudo-inverse of $L$ is defined on the quotient $\mathcal{L}^2(\Pi) \setminus \ker L$, that is the space of centered square integrable random variables. For $F \in \mathcal{L}^2(\Pi)$ with $\Pi(F) = 0$, we have $LL^{-1}F = F$. Moreover, if $F \in \text{Dom } L$, we have $L^{-1}LF = F$. As a consequence of (3), $\text{Dom } D^2 = \text{Dom } L$.

The Dirichlet form

We refer to [3, Chapter 1] for more details about the formalism of Dirichlet forms. The introduction of [1] also provides an overview of the subject. For every $F, G \in \text{Dom } D$, we let $\mathcal{E}(F, G) = (\Pi \otimes \nu)(DFDG)$. Since by [5, Lemma 3], the operator $D$ is closed, $\mathcal{E}$ is a Dirichlet form with domain $\text{Dom } \mathcal{E} = \text{Dom } D$. Moreover, in view of the integration by parts (2), the generator of $\mathcal{E}$ is given by $L$. By [3, Chapter I Section 3], $\mathcal{A} := \text{Dom } D \cap \mathcal{L}^\infty(\Pi)$ is an algebra with respect to the pointwise multiplication; $\text{Dom } D$ and $\mathcal{A}$ are stable by composition with Lipschitz functions; $\mathcal{A}$ is stable by composition with $C^k(\mathbb{R}^d)$ functions ($k \in \mathbb{N}$).

The carré du champ operator

For every $F \in \mathcal{A}$, we define the functional carré du champ of $F$ as the linear form $\Gamma(F)$ on $\mathcal{A}$, defined by

$$\Gamma(F)[\Phi] = \mathcal{E}(F, F\Phi) - \frac{1}{2}\mathcal{E}(F^2, \Phi), \quad \text{for all } \Phi \in \mathcal{A}.$$  

From [3, Proposition I.4.1.1],

$$0 \leq \Gamma(F)[\Phi] \leq \|\Phi\|_{\mathcal{L}^\infty(\Pi)} \mathcal{E}(F), \quad \text{for all } F, \Phi \in \mathcal{A}.$$  

This allows us to extend the definition of the linear form $\Gamma(F)$ to all $F \in \text{Dom } \mathcal{E}$. For $F \in \text{Dom } \mathcal{E}$, we write that $F \in \text{Dom } \Gamma$ if the linear form $\Gamma(F)$ can be represented by a measure absolutely continuous with respect to $\Pi$; in that case we denote its density by $\Gamma(F)$. In other words, $F \in \text{Dom } \Gamma$ if and only if there exists a non-negative $\Gamma(F) \in \mathcal{L}^1(\Pi)$ such that

$$\Gamma(F)[\Phi] = E_\Pi \Gamma(F)\Phi, \quad \text{for all } \Phi \in \mathcal{A}.$$  

From the general theory, we know that $\text{Dom } \Gamma$ is a closed sub-linear space of $\text{Dom } \mathcal{E}$. In the Poisson case, we prove the following representation of the carré du champ that is a consequence of Lemma 4.

**Proposition 2.** We have that $\text{Dom } \Gamma = \text{Dom } D$ and, for all, $F \in \text{Dom } D$:

$$\Gamma(F) = \frac{1}{2} \int (D_z^+ F)^2 \nu(dz) + \frac{1}{2} \int (D_z^- F)^2 \eta(dz).$$

We extend $\Gamma$ to a bilinear map

$$\Gamma(F, G) = \frac{1}{2} \int D_z^+ F D_z^+ G \nu(dz) + \frac{1}{2} \int D_z^- F D_z^- G \eta(dz), \quad \forall F, G \in \text{Dom } D.$$
Remark 1. This representation of $\Gamma$ using the add-one and drop-one operators is, at the formal level, well-known in the literature: it appears without a proof in the seminal paper [2, p. 191]. One of the main assumptions of [2] is the existence of an algebra of functions contained in $\text{Dom} \mathcal{L}$, the so called standard algebra. In the case of a Poisson point process, it is not clear what to choose for the standard algebra (note that $\mathcal{L} = \text{Dom} \mathcal{E} \cap L^\infty(\Pi)$ is not included in $\text{Dom} \mathcal{L}$). [4] derives the formula without relying on the notion of standard algebra. However, since [4] follows the strategy of [2], [4] has to assume a restrictive assumption on $F$: $F \in \text{Dom} \mathcal{L}$ and $F^2 \in \text{Dom} \mathcal{L}$. In particular, the authors of [4] did not obtain that $\text{Dom} \Gamma = \text{Dom} \mathcal{E}$. This is why, following [3], we use the formalism of Dirichlet forms to compute the carré du champ and obtain a representation for the carré du champ under minimal assumptions.

The energy bracket

Given two elements $u \in \mathcal{L}^2(\nu \otimes \Pi)$ and $v \in \mathcal{L}^2(\nu \otimes \Pi)$, we define the energy bracket of $u$ and $v$: it is the function

$$ [u, v]_\Gamma(\eta) = \frac{1}{2} \int u(\eta, z)v(\eta, z)\nu(dz) + \frac{1}{2} \int u(\eta - \delta_z, z)v(\eta - \delta_z, z)\eta(dz). $$

The energy bracket can be compared with the two related objects:

$$ [u, v]_+(\eta) = \int u(\eta, z)v(\eta, z)\nu(dz); $$

$$ [u, v]_-(\eta) = \int u(\eta - \delta_z, z)v(\eta - \delta_z, z)\eta(dz). $$

Note that $[u, v]_+$ is simply the scalar product of $u$ and $v$ in $\mathcal{L}^2(\nu)$. By the Cauchy-Schwarz inequality $[u, v]_+ \in \mathcal{L}^1(\Pi)$, and by the Mecke formula:

$$ E_{\Pi}[u, v]_\Gamma = E_{\Pi}[u, v]_+ = E_{\Pi}[u, v]_-.$$

If $F$ and $G \in \text{Dom} \mathcal{D}$, by Proposition 2, we have that

$$ \Gamma(F, G) = |DF, DG|_\Gamma. $$

The fact that the carré du champ is not $\nu(DFDG)$ is characteristic of non-local Dirichlet forms.

A formula for the divergence

Since the operator $\mathcal{D}$ is not a derivation, [7, Proposition 1.3.3] (obtained in the setting of Malliavin calculus for Gaussian processes) does not hold. We however have the following Poisson counterpart.

Lemma 3. Let $F \in \text{Dom} \mathcal{D}$ and $u \in \text{Dom} \mathcal{\delta}$ such that $Fu \in \text{Dom} \mathcal{\delta}$. Then,

$$ \mathcal{\delta}(Fu) = F\mathcal{\delta}u - |DF, u|_-.$$

Proof. Let $G \in \mathcal{A} = \text{Dom} \mathcal{D} \cap \mathcal{L}^\infty(\Pi)$, and assume moreover that $u \in \mathcal{L}^1(\Pi \otimes \nu)$. By integration by parts and the Mecke formula, we find that

$$ E_{\Pi}G\delta(Fu) = \int F(\eta)u(\eta, z)\mathcal{D}_zG(\eta)\nu(dz)\Pi(d\eta) $$

$$ = \int G(\eta)\left(F(\eta - \delta_z)u(\eta - \delta_z, \eta)\Pi(d\eta) - \int G(\eta)F(\eta)u(\eta, z)\nu(dz)\Pi(d\eta) - \int F(\eta)u(\eta, z)\nu(dz)\Pi(d\eta) \right). $$

Using that $F(\eta - \delta_z)u(\eta - \delta_z, \eta) = F(\eta)u(\eta - \delta_z, \eta) - \mathcal{D}_zFu(\eta - \delta_z, z)$, we conclude by (4) that

$$ E_{\Pi}G\delta(Fu) = E_{\Pi}GF\delta u - E_{\Pi}G|DF, u|_\eta. $$

We conclude by density. \qed
Algebraic relations for the add and drop operators

Some immediate algebra yields:

\[ D^*_z F^2 = 2FD^*_z F + (D^*_z F)^2; \]  
(5)

\[ D^-_z F^2 = 2FD^-_z F - (D^-_z F)^2. \]  
(6)

An integrated chain rule for the energy

Recall that we write \( \mathcal{A} \) for the algebra \( \mathbb{Dom} \mathcal{E} \cap L^\infty(\Pi) \). We now remark that even if \( D \) is not a derivation, the Dirichlet energy \( \mathcal{E} \) acts as a derivation.

Lemma 4. Let \( F \) and \( G \in \mathcal{A} \), and \( u \in L^2(\Pi \otimes \nu) \). Then,

\[ \mathcal{E}\Pi [D(FG), u|_\Gamma] = \mathcal{E}\Pi F[DG, u|_\Gamma] + \mathcal{E}\Pi G[DF, u|_\Gamma]. \]

In particular, with \( H \in \mathbb{Dom} D \):

\[ \mathcal{E}(FG, H) = \mathcal{E}\Pi F[DG, DH|_\Gamma] + \mathcal{E}\Pi G[DF, DH|_\Gamma]. \]

This establishes Proposition 2.

Remark 2. The formula Lemma 4 for \( \mathcal{E} \) cannot be iterated. In particular, consistently with the fact that \( L \) is not a diffusion, Lemma 4 does not imply \( \mathcal{E}(\phi(F), G) = \mathcal{E}\Pi \phi'(F)[DF, DG|_\Gamma] \).

Proof. Since \( F \in L^\infty(\Pi) \), by Lemma 1, we have that \( DF \in L^\infty(\Pi \otimes \nu) \); and by assumption, \( DF \in L^2(\Pi \otimes \nu) \). A similar result holds for \( G \), and we find that \( DFDG \) is square integrable.

By the Mecke formula, and (5) and (6), we can write:

\[ \mathcal{E}\Pi [D(FG), u|_\Gamma] = \mathcal{E}\Pi F[DG, u|_\Gamma] + \mathcal{E}\Pi G[DF, u|_\Gamma] + \frac{1}{2} \mathcal{E}\Pi \int D^*_z F \otimes DG \otimes u(z) \nu(dz) \]

\[ - \frac{1}{2} \mathcal{E}\Pi \int (1 - D^-_z)F(1 - D^-_z)G(1 - D^-_z)u(z) \eta(dz). \]

By the Mecke formula, the two terms on the two last lines cancel out. This proves the first part of the claim. To establish Proposition 2, we simply write, for \( F \) and \( \Phi \in \mathcal{A} \):

\[ \mathcal{E}(F, F\Phi) - \frac{1}{2} \mathcal{E}(F^2, \Phi) = \mathcal{E}\Pi F[DF, D\Phi|_\Gamma] + \mathcal{E}\Pi \Phi[DF, DF|_\Gamma] - \mathcal{E}\Pi F[DF, D\Phi|_\Gamma]. \]

This shows that \( \mathbb{Dom} \Gamma \supset \mathcal{A} \) and that

\[ \Gamma(F)[\Phi] = \mathcal{E}\Pi [DF, DF|_\Gamma] \Phi. \]

We extend this expression to \( \mathbb{Dom} \mathcal{E} = \mathbb{Dom} D \). This concludes the proof.

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R.H. is a member of Institut für Angewandte Mathematik, Bonn Universität
Endenicher Allee 60, D-53115, Bonn, Germany.
ronan.herry@live.fr
ORCid: 0000-0001-6313-1372