VACUUM STATIC SPACES WITH POSITIVE ISOTROPIC CURVATURE

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ABSTRACT. In this paper, we study vacuum static spaces with positive isotropic curvature. We prove that if \((M^n, g, f)\), \(n \geq 4\), is a compact vacuum static space with positive isotropic curvature, then up to finite cover, \(M\) is isometric to a sphere \(S^n\) or the product of a circle \(S^1\) with an \((n-1)\)-dimensional sphere \(S^{n-1}\).

1. Introduction

Let \((M^n, g)\) be an \(n\)-dimensional Riemannian manifold with \(n \geq 4\). The Riemannian metric \(g = \langle \ , \ \rangle\) can be extended either to a complex bilinear form \(\langle \ , \ \rangle\) or a Hermitian inner product \(\langle \ , \ \rangle\) on each complexified tangent space \(T_pM \otimes \mathbb{C}\) for \(p \in M\). A complex 2-plane \(\sigma \subset T_pM \otimes \mathbb{C}\) is totally isotropic if \((Z, Z) = 0\) for any \(Z \in \sigma\). For any 2-plane \(\sigma \subset T_pM \otimes \mathbb{C}\), we can define the complex sectional curvature of \(\sigma\) with respect to \(\langle \ , \ \rangle\) by

\[
K_\mathbb{C}(\sigma) = \langle \mathcal{R}(Z \wedge W), Z \wedge W \rangle,
\]

where \(\mathcal{R} : \Lambda^2 T_pM \to \Lambda^2 T_pM\) is the curvature operator and \(\{Z, W\}\) is a unitary basis for \(\sigma\) with respect to \(\langle \ , \ \rangle\).

A Riemannian \(n\)-manifold \((M^n, g)\) is said to have positive isotropic curvature (PIC in short) if the complex sectional curvature on isotropic planes is positive, that is, for any totally isotropic 2-plane \(\sigma \subset T_pM \otimes \mathbb{C}\),

\[
K_\mathbb{C}(\sigma) > 0.
\]

In view of (1.1), the condition (1.2) can be formulated in purely real terms: For orthonormal vectors \(\{e_1, e_2, e_3, e_4\}\) in \(T_pM\) with \(\sigma = \text{span}\{e_1 + ie_2, e_3 + ie_4\}\), letting

\[
Z = e_1 + ie_2, \quad W = e_3 + ie_4,
\]

we have

\[
2K_\mathbb{C}(\sigma) = \langle \mathcal{R}(Z \wedge W), Z \wedge W \rangle = R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234}.
\]

Hence \((M^n, g)\) has PIC if and only if for any four orthonormal frame \(\{e_1, e_2, e_3, e_4\}\),

\[
R_{1313} + R_{1414} + R_{2323} + R_{2424} > 2R_{1234}.
\]

If \((M, g)\) has positive curvature operator, then it has PIC [24]. Thus, a standard sphere \((S^n, g_0)\) has PIC. Also, if the sectional curvature of \((M, g)\) is pointwise strictly quarter-pinched, then \((M, g)\) has PIC [24]. It is well-known that the product metric on \(S^{n-1} \times S^1\) has also PIC and the connected sum of manifolds with PIC admits a PIC metric [25]. For existence of Riemannian metrics with PIC on compact manifolds which fiber over the circle, see [20]. On the other hand, PIC implies that \(g\) has positive scalar curvature [25]. Recently, there are many known results on the structure of Riemannian manifolds with positive isotropic curvature [4, 6, 7, 12, 13, 28, 29].

In this paper, we consider vacuum static spaces having positive isotropic curvature. An \(n\)-dimensional complete Riemannian manifold \((M, g)\) is said to be a static space with a perfect fluid if there exists a smooth non-trivial function \(f\) on \(M\) satisfying

\[
D_g df - \left( r_g - \frac{s_g}{n-1} \right) f = \frac{1}{n} \left( \frac{s_g}{n-1} f + \Delta_g f \right) g.
\]

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where $D_g df$ is the Hessian of $f$, $r_g$ is the Ricci tensor of $g$ with its scalar curvature $s_g$, and $\Delta_g f$ is the (negative) Laplacian of $f$. In particular, if
\begin{equation}
\Delta_g f = -\frac{s_g}{n-1} f,
\end{equation}
$(M, g)$ is said to be a vacuum static space. In this case, the equation \(1.4\) reduces to
\begin{equation}
D_g df - \left( r_g - \frac{s_g}{n-1} g \right) f = 0.
\end{equation}
Using the traceless Ricci tensor $z_g = r_g - \frac{4}{n-1} g$, \(1.6\) can be written as
\begin{equation}
f z_g = D_g df + \frac{s_g f}{n(n-1)} g.
\end{equation}
It is easy to see that if a smooth function $f$ on $M$ is a solution of the vacuum static equation \(1.4\), then $f$ is an element of the kernel space, $\ker s_g^{\ast\ast}$, of the operator $s_g^{\ast\ast}$, where $s_g^{\ast\ast}$ is the $L^2$-adjoint operator of the linearized scalar curvature $s_g$ with respect to the metric $g$ which is given by
\begin{equation}
s_g^{\ast\ast}(\phi) = D_g df - (\Delta_g \phi) g - \phi r_g.
\end{equation}
for any smooth function $\phi$ on $M$ (cf. \[2\]). By taking the divergence of \(1.5\), we have $\frac{1}{2} f ds_g = 0$, which implies that $s_g$ is constant on $M$ since there are no critical points of $f$ in $f^{-1}(0)$ (see, for example, \[3\]). When $M$ is compact, it is known \[3\] that a compact vacuum static space is either isometric to a Ricci-flat manifold with $\ker s_g^{\ast\ast} = \mathbb{R} \cdot 1$ and $s_g = 0$, or the scalar curvature $s_g$ is a strictly positive constant and $\frac{s_g}{n-1}$ is an eigenvalue of the Laplacian. Also, it turns out \[8\] that the warped product manifold $(M \times_f \mathbb{R}, g \pm f^2 dt^2)$ is Einstein when $f \in \ker s_g^{\ast\ast}$.

When the static space $(M, g)$ is locally conformally flat, Kobayashi and Obata \[19\] proved that, around the hypersurface $f^{-1}(c)$ for a regular value $c$, the metric $g$ is isometric to a warped product metric of constant curvature. For vacuum static spaces, Fischer and Marsden \[10\] conjectured that only compact vacuum static space is a standard sphere, but it is known \[15\] that flat torus and round spheres, product $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ and warped product $\mathbb{S}^1 \times \Sigma^{n-1}$ are also compact vacuum static spaces (for references, see \[9\], \[17\], \[21\], \[20\]) for locally conformally flat case. Furthermore, it is known \[27\] that $\mathbb{S}^1(\sqrt{\frac{2}{n-2}}) \times \Sigma^{n-1}$ for Einstein manifold $\Sigma$ with scalar curvature $(n-1)(n-2)$ is a compact vacuum static space, which may not be locally conformally flat.

Some rigidity results related to vacuum static spaces have been found. For example, Qing and Yuan showed \[27\] that, if $(M, g, f)$ is a Bach-flat vacuum static space with compact level sets of $f$, then it is either Ricci-flat, isometric to $\mathbb{S}^n$, $\mathbb{H}^n$, or a warped product space. Recall that the Bach tensor (see Section 2 for definition) discussed first by Bach in \[1\] is deeply related to general relativity and conformal geometry (cf. \[22\]), and in dimension $n = 4$, it is well known \[2\] that the Bach tensor is conformally invariant and arises as a gradient of the total Weyl curvature functional, which is given by the integral of the square norm of the Weyl tensor. On the other hand, Kim and Shin \[18\] proved a local classification of four-dimensional vacuum static spaces with harmonic curvature. We say that a Riemannian manifold $(M, g)$ has harmonic curvature if $\text{div } R = 0$, or equivalently, that the Ricci tensor $r_g$ is a Codazzi tensor.

In this paper, we prove the following result.

**Theorem 1.1.** Let $(M^n, g, f)$, $n \geq 4$, be a compact vacuum static space with positive isotropic curvature. Then, up to finite cover, $M$ is isometric to a sphere $\mathbb{S}^n$ or a product of a circle with an $(n-1)$-dimensional sphere $\mathbb{S}^{n-1}$.

From the PIC condition, we can show that the Ricci tensor $\text{Ric}_g$ is radially flat, that is, $\text{Ric}_g(\nabla f, X) = 0$ for any vector field $X$ orthogonal to $\nabla f$. This property plays an important role in our proof of main result as it does in the critical point case \(1.5\). We say that a compact Riemannian manifold $(M^n, g)$ satisfies the critical point equation if
\begin{equation}
(1 + f)z_g = D_g df + \frac{s_g f}{n(n-1)} g,
\end{equation}
which looks very like the vacuum static equation \(1.7\). Even though these two equations look very similar, there are also some different properties between them. One thing is that on the vacuum static space, intuitively the level set $f^{-1}(0)$ is important and it is known that this level set is totally geodesic. Whereas in the case of critical point equation, the level set $f^{-1}(-1)$ has some geometric structures (cf. \[15\]).
Using the fact that the Ricci tensor is radially flat, we can show that the metric can be rewritten as a warped product form, and then prove that \((M,g)\) must be Bach-flat, \(B = 0\). So, applying one of main result in \([27]\), we conclude that, up to finite cover, either \(M\) is isometric to a standard sphere \(S^n\), or a warped product \(S^1 \times \xi S^{n-1}\) of a circle with an \((n-1)\)-dimensional sphere \(S^{n-1}\). We can show that the minimum set of the potential function \(f\) is either a point or a hypersurface. Thus the former corresponds to the sphere, and the latter case does to the warped product. Finally, in case of the warped product \(S^1 \times \xi S^{n-1}\), by using a maximum principle, we show the warping function \(\xi \equiv 1\), which implies it is in fact product manifold \(S^1 \times S^{n-1}\). We would like mention that, in the critical point equation case, the nontrivial potential function \(f\) satisfying (1.8) attains only one maximum point and only one minimum point, and so it has only two critical points. From this, we can apply a result in \([30]\) directly and conclude our result. But in vacuum static spaces, it happens that the minimum set of the potential function \(f\) satisfying (1.8) could be a hypersurface. So, we cannot apply a result in \([30]\) directly, and when the minimum set is a hypersurface in vacuum static spaces, we prove a rigidity result through a little more technical argument.

**Notations:** Hereafter, for convenience and simplicity, we denote curvatures \(\text{Ric}_g, z_g, s_g\), and the Hessian and Laplacian of \(f, D_g df, \Delta_g\) by \(r, z, s\), and \(D f, \Delta\), respectively, if there is no ambiguity.

### 2. Preliminaries

In this section, we introduce a structural 3-tensor \(T\) which plays an important role in the proof of our main theorem, and give some relations of it to the Bach tensor and the Cotton tensor. We here present the Bach tensor and the Cotton tensor for self-containedness.

Let \((M^n, g)\) be a Riemannian manifold of dimension \(n\) with the Levi-Civita connection \(D\). First of all, we start with a differential operator acting on the space of symmetric 2-tensors. Let \(b\) be a symmetric 2-tensors on \(M\). The differential \(D^D b\) is defined by

\[
D^D b(X, Y, Z) = D_X b(Y, Z) - D_Y b(X, Z)
\]

for any vectors \(X, Y\) and \(Z\). The **Cotton tensor** \(C \in \Gamma(M^2 \otimes T^* M)\) is defined by

\[
C = D^D \left( r - \frac{s}{2(n-1)} g \right) = D^D r - \frac{1}{2(n-1)} ds \wedge g.
\]

It is well-known that, for \(n = 3\), \(C = 0\) if and only if \((M^3, g)\) is locally conformally flat. Moreover, for \(n \geq 4\), the vanishing of Weyl tensor \(\mathcal{W}\) implies the vanishing of the Cotton tensor \(C\), while \(C = 0\) corresponds to the Weyl tensor being harmonic, i.e., \(\delta \mathcal{W} = 0\) because of the following identity \([2]\): for a Riemannian \(n\)-manifold \((M^n, g)\), the Weyl tensor \(\mathcal{W}\) satisfies the following

\[
\delta \mathcal{W} = -\frac{n-3}{n-2} D^D \left( r - \frac{s}{2(n-1)} g \right) = -\frac{n-3}{n-2} C
\]

under the following identification

\[
\Gamma(T^* M \otimes \Lambda^2 M) \cong \Gamma(\Lambda^2 M \otimes T^* M).
\]

The **Bach tensor** \(B\) on a Riemannian manifold \((M^n, g)\), \(n \geq 4\), is defined by

\[
B = \frac{1}{n-3} \delta D \delta \mathcal{W} + \frac{1}{n-2} \hat{\mathcal{W}} r.
\]

When the scalar curvature \(s\) is constant, we have \(\mathcal{W} r = \hat{\mathcal{W}} z\). For a symmetric 2-tensor \(b\) on a Riemannian manifold \((M^n, g)\), we define \(\mathcal{W} b\) by

\[
\mathcal{W} b(X, Y) = \sum_{i=1}^n b(\mathcal{W}(X, E_i) Y, E_i) = \sum_{i, j=1}^n \mathcal{W}(X, E_i, Y, E_j) b(E_i, E_j)
\]

for any local frame \(\{E_i\}\). From (2.2), we have

\[
C = -\frac{n-2}{n-3} \delta \mathcal{W} \quad \text{and} \quad \delta C = -\frac{n-2}{n-3} \delta D \delta \mathcal{W}.
\]

Thus, one can see that the Bach tensor satisfies

\[
B = \frac{1}{n-2} (-\delta C + \hat{\mathcal{W}} z).
\]
Furthermore, the following property holds in general (cf. [5]) for \( n \geq 4 \): for any vector field \( X \),
\[
(n - 2)\delta B(X) = -\frac{n - 4}{n - 2}(i_X C, z).
\]

The complete divergence of the Bach tensor has the following form [14]:
\[
\delta \delta B = -\frac{n - 4}{(n - 2)^2}\left(\frac{1}{2}|C|^2 - (\delta C, z)\right).
\]

Now let \((M, g, f)\) be a vacuum static space satisfying (1.6) or equivalently (1.7). We define a 3-tensor \( T \) on \((M, g, f)\) by
\[
T = \frac{1}{n - 2} df \wedge z + \frac{1}{(n - 1)(n - 2)} i_{\nabla f} z \wedge g.
\]

Here \( i_{\nabla f} \) denotes the usual interior product to the first factor defined by \( i_{\nabla f} z(X) = z(\nabla f, X) \) for any vector \( X \). As the Cotton tensor \( C \), the cyclic summation of \( T_{ijk} \) is vanishing. The tensor \( T \) looks very similar as in [5] (cf. [19]), where the authors used this in classifying complete Bach flat gradient shrinking Ricci solitons. In [15] (cf. [14], [33]), we derived some useful relations of the Bach tensor \( B \) and the Cotton tensor \( C \) to the tensor \( T \) and its divergence \( \delta T \). In view of (1.7) and (1.8), one can see that these relations are still valid in vacuum static spaces if we just replace the function \( 1 + f \) in CPE case by \( f \). Here, we just enumerate some identities about these tensors which are needed and refer [15] for proofs.

The following identities hold for vacuum static spaces [14], [33]:

(i)
\[
(n - 2)fB = -i_{\nabla f} C + \frac{n - 3}{n - 2} \tilde{C} + (n - 1)\delta T,
\]
where \( \tilde{C} \) is a 2-tensor defined as
\[
\tilde{C}(X, Y) = C(Y, \nabla f, X)
\]
for any vectors \( X, Y \).

(ii) For any vector field \( X \),
\[
\delta \delta T(X) = \frac{1}{n - 2} f(i_X C, z) + (i_X T, z).
\]

(iii)
\[
fC = \tilde{i}_{\nabla f} W - (n - 1)T
\]
and
\[
\delta(\tilde{i}_{\nabla f} W) = -\frac{n - 3}{n - 2} \tilde{C} + f\tilde{W} z.
\]

Here \( \tilde{i}_X \) is the interior product to the last factor defined by \( \tilde{i}_X W(Y, Z, U) = W(Y, Z, U, X) \) for any vectors \( Y, Z, \) and \( U \).

**Lemma 2.1.** (14, 33) Let \((g, f)\) be a non-trivial solution of the vacuum static equation, Then
\[
|T|^2 = \frac{2}{(n - 2)^2} |\nabla f|^2 \left( |z|^2 - \frac{n}{n - 1} |i_N z|^2 \right).
\]

The following result can be obtained from a result of Qing and Yuan in [27].

**Theorem 2.2.** Let \((M^n, g, f)\) be a compact vacuum static space with positive scalar curvature. If \( T = 0 \), then up to a finite quotient and appropriate scaling, \( M \) is either isometric to a sphere \( S^n \), or a warped product, \( S^1 \times \Sigma^{n-1} \), \( g = dt^2 + g_{\Sigma} \), of a circle and a compact Einstein manifold \( \Sigma \) with positive constant Ricci curvature.

**Proof.** It is proved in [14] that a vacuum static space \((M^n, g, f)\) with \( T = 0 \) is Bach-flat. The conclusion follows from a result due to Qing and Yuan [27]. \( \square \)
3. Vacuum Static Spaces with PIC

Throughout this section we assume that \((M, g, f)\) is a vacuum static space with positive isotropic curvature. In this section, we will prove that \(z(\nabla f, X) = 0\) for any vector field \(X\) orthogonal to \(\nabla f\), which plays an important role in showing our main result. Considering \(i_{\nabla f}z\) as a 1-form, we define a 2-form \(\omega\) by

\[
\omega := df \wedge i_{\nabla f}z.
\]

We prove that \(\omega = 0\) (Theorem 3.6) when \((g, f)\) is a non-trivial solution of the vacuum static equation and \((M, g)\) has positive isotropic curvature. However, since a proof for this property is very similar as in the critical point equation \([15]\), we just give outline briefly without details as mentioned in Section 2 above. Replacing the function \(1 + f\) in CPE case by the function \(f\), one can prove almost all properties which are discussed here.

**Lemma 3.1.** We have

\[
(3.1) \quad \omega = (n - 1)i_{\nabla f}T = -f i_{\nabla f}C.
\]

**Lemma 3.2.** Let \(\{E_1, E_2, \ldots, E_n\}\) be a local frame with \(E_1 = N = \frac{\nabla f}{|\nabla f|}\). Then

\[
\omega = 0 \quad \text{if and only if} \quad i_{\nabla f}C(N, E_j) = 0 \quad (j \geq 2).
\]

**Lemma 3.3.** As a 2-form, we have the following

\[
i_{\nabla f}C = div_{\nabla f}z.
\]

**Lemma 3.4.** \(\omega\) is a closed 2-form, i.e., \(d\omega = 0\).

Now, let \(\Omega = \{p \in M \mid \omega_p \neq 0 \text{ on } T_pM\}\). Then \(\Omega\) is an open subset of \(M\). Adapting an argument in \([34]\), we can show the following similarly as in \([15]\).

**Lemma 3.5.** Suppose that \(\omega_p \neq 0\) at \(p \in M\). Then

\[
(3.2) \quad |D\omega|^2(p) \geq |\delta \omega|^2(p).
\]

Now, we are ready to prove the following main result in this section.

**Theorem 3.6.** Let \((g, f)\) be a nontrivial solution of the vacuum static equation on a compact manifold \(M\). If \((M, g)\) has positive isotropic curvature, then the 2-form \(\omega = df \wedge i_{\nabla f}z\) is vanishing.

**Proof.** It suffices to prove that \(\Omega = \emptyset\), where \(\Omega = \{p \in M \mid \omega_p \neq 0 \text{ on } T_pM\}\). Suppose, on the contrary, that \(\Omega \neq \emptyset\). For \(p \in \Omega\), let \(\Omega_0\) be a connected component of \(\Omega\) containing \(p\). Note that \(\Delta \omega = -d\delta \omega\) by Lemma 3.4.

It follows from the Bochner-Weitzenböck formula for 2-forms (cf. \([23], [31]\)) that

\[
\frac{1}{2} \Delta |\omega|^2 = \langle \Delta \omega, \omega \rangle + |D\omega|^2 + \langle E(\omega), \omega \rangle,
\]

where \(E(\omega)\) is a (local) 2-form containing isotropic curvature terms as its coefficients. In particular, if \((M, g)\) has positive isotropic curvature, by Proposition 2.3 in \([34]\) (cf. \([25], [28]\)) we have

\[
(3.4) \quad \langle E(\omega), \omega \rangle > 0.
\]

Therefore, integrating (3.3) over \(\Omega_0\), we obtain

\[
\frac{1}{2} \int_{\Omega_0} \Delta |\omega|^2 = \int_{\Omega_0} \langle \Delta \omega, \omega \rangle + |D\omega|^2 + \int_{\Omega_0} \langle E(\omega), \omega \rangle.
\]

Since \(\omega = 0\) on the boundary \(\partial \Omega_0\) and \(\omega\) is a closed form by Lemma 3.4 we have

\[
0 = \int_{\Omega_0} |D\omega|^2 - |\delta \omega|^2 + \langle E(\omega), \omega \rangle.
\]

However, by Lemma 3.5 and the inequality (3.4), the above equation is impossible if \(\omega\) is nontrivial. Hence, we may conclude that \(\omega = 0\), or \(\Omega_0 = \emptyset\). \(\square\)
4. Vacuum Static Spaces with $\omega = 0$

Let $(g, f)$ be a non-trivial solution of the vacuum static equation satisfying $\omega = df \wedge i_{\nabla f} z = 0$. In this section, we investigate some geometric properties on both tensors $z, T$ and the function $f$. First, since $\omega = 0$, it is easy to see that
\begin{equation}
(4.1) \quad z(\nabla f, X) = 0
\end{equation}
for any vector $X$ orthogonal to $\nabla f$ by plugging $(\nabla f, X)$ into $\omega$. Thus, we may write
\[ i_{\nabla f} z = \alpha df, \quad \text{where } \alpha = z(N, N), \quad N = \frac{\nabla f}{|\nabla f|} \]
as a 1-form. Note that the function $\alpha$ is well-defined only on the set $M \setminus \text{Crit}(f)$, where $\text{Crit}(f)$ is the set of all critical points of $f$. However since $|\alpha| \leq |z|$, $\alpha$ can be extended to a $C^0$ function on the whole $M$. See Lemma 3.1. For (4.3), we note from the vacuum static equation that
\[ \delta(i_{\nabla f} z) = -f|z|^2. \]

So, we can obtain (4.3) by taking the divergence of (4.2).

For a nontrivial solution $(g, f)$ of the vacuum static equation, assuming $\omega = df \wedge i_{\nabla f} z = 0$, we can show that both the function $\alpha = z(N, N)$ and $|z|^2$ are constant on each level set $f^{-1}(c)$, and there are no critical points of $f$ except maximum and minimum points of $f$ in $M$. To show fiberwise constancy of the functions $\alpha$ and $|z|^2$, we first compute the divergence of the tensor $T$.

Lemma 4.2. (34) Let $(g, f)$ be a non-trivial solution of the vacuum static equation (1.0). Then
\begin{equation}
(4.4) \quad (n - 1)(n - 2)\delta T = \frac{n - 2}{n - 1} s f z - (n - 2) D_{\nabla f} z + \bar{C} + n f z \circ z - f|z|^2 g.
\end{equation}

Here $\circ z$ is defined by
\[ z \circ z(X, Y) = \sum_{i=1}^{n} z(X, E_i)z(Y, E_i) \]
for a local frame $\{E_i\}_{i=1}^{n}$ of $M$.

Lemma 4.3. (34) Suppose that $\omega = 0$. Then the functions $\alpha$ and $|z|^2$ are both constant on each level set $f^{-1}(c)$.

Lemma 4.4. (32) $\alpha$ is nonpositive on $M$.

Lemma 4.5. Let $(g, f)$ be a non-trivial solution of the vacuum static equation on an $n$-dimensional compact manifold $M$ with $\omega = 0$. Then there are no critical points of $f$ except at its maximum and minimum points of $f$.

Proof. Note that $|\nabla f|^2$ is constant on each level sets of $f$. From the Bochner-Weitzenböck formula (cf. 23, 34) together with (1.7), we have
\begin{equation}
(4.5) \quad \frac{1}{2} \Delta |\nabla f|^2 - \frac{1}{2} \frac{\nabla f(|\nabla f|^2)}{f} = |Ddf|^2.
\end{equation}

By the maximum principle, the function $|\nabla f|^2$ cannot have its local maximum in $M_0 := \{x \in M \mid f(x) < 0\}$ nor in $M^0 := \{x \in M \mid f(x) > 0\}$. In other words, $|\nabla f|^2$ may attain its local maximum only on the set $f^{-1}(0)$.

Let $p \in M$ be a critical point of $f$ with $f(p) = c$ other than minimum or maximum points of $f$ so that $\nabla f(p) = 0$. Then the function $|\nabla f|^2$ must have a local maximum between $f^{-1}(\min)$ and $f^{-1}(c)$, or between $f^{-1}(c)$ and $f^{-1}(\max)$. This contradicts the argument mentioned above. \qed
The following result shows that the structure of minimum set or maximum set of the potential function is either a single point or a totally geodesic stable minimal hypersurface. In [15], we show that, in case of CPE with positive isotropic curvature, both minimum and maximum sets are single points.

**Theorem 4.6.** Let \((g, f)\) be a non-trivial solution of the vacuum static equation on an \(n\)-dimensional compact manifold \(M\) with \(\omega = 0\). Let \(\min_M f = a\). Then either \(f^{-1}(a)\) contains only a single point, or \(f^{-1}(a)\) is a totally geodesic stable minimal hypersurface of \(M\). The same property also holds for \(f^{-1}(b)\) with \(\max_M f = b\). Further, if \(f^{-1}(a)\) is a single point, then so dose \(f^{-1}(b)\) and vice versa. In case that \(f^{-1}(a)\) is a single point, every level set \(f^{-1}(t)\) except \(a\) and \(b\) is a hypersurface and is homotopically a sphere \(\mathbb{S}^{n-1}\).

**Proof.** Let \(\min_M f = a\). We may assume \(a < 0\) from the maximum principle together with (1.5). By Lemma [4.5] and Morse theory, the inverse set \(f^{-1}(a)\) is a single point, every level set \(f^{-1}(t)\) except \(a\) and \(b\) is a hypersurface of \(M\) (\(a\) is a regular value of \(f\)).

Suppose that \(f^{-1}(a)\) is a hypersurface. Then for a sufficiently small \(\epsilon > 0\), \(f^{-1}(a + \epsilon)\) has two connected components, say \(\Sigma^+\), \(\Sigma^-\). Note that \(\Sigma^+ = \Sigma^- = f^{-1}(a)\).

Let \(\nu\) be a unit normal vector field on \(\Sigma := f^{-1}(a)\). Then \(\nu\) can be extended smoothly to a vector field \(\Xi\) defined on a tubular neighborhood of \(f^{-1}(a)\) such that \(\Xi|f^{-1}(a) = \nu\) and \(\Xi|\Sigma^+ = N = \frac{\nabla f}{|\nabla f|}\). \(\Xi|\Sigma^- = -N = -\frac{\nabla f}{|\nabla f|}\).

Note that
\[
\lim_{\epsilon \to 0^+} N = \lim_{\epsilon \to 0^-} (-N) = \nu.
\]

Next, on the hypersurface \(f^{-1}(a + \epsilon)\) near \(f^{-1}(a)\), the Laplacian of \(f\) is given by
\[
\Delta f = \Delta' f + Ddf(N, N) + m(N, \nabla f) = Ddf(N, N) + m(N, \nabla f),
\]
where \(\Delta'\) and \(m\) denote the Laplacian and mean curvature of \(f^{-1}(a + \epsilon)\), respectively. In particular, on the minimum set \(f^{-1}(a)\), we have
\[
\Delta f = Ddf(\nu, \nu).
\]

Let \(p \in f^{-1}(a)\) be any point. Choosing an orthonormal basis \(\{e_1 = \nu(p), e_2, \ldots, e_n\}\) on \(T_p M\), it follows from (4.6) together with \(a = \min_M f\) that
\[
f z_p(e_i, e_i) - \frac{sf}{n(n-1)} = Ddf_p(e_i, e_i) = 0
\]
for each \(i = 2, \ldots, n\).

Now we claim that the minimum set \(\Sigma = f^{-1}(a)\) is totally geodesic, and in particular the mean curvature is vanishing, \(m = 0\). In fact, fix \(i\) \((i = 2, 3, \ldots, n)\), say \(i = 2\), and let \(\gamma : [0, t] \to M\) be a unit speed geodesic such that \(\gamma(0) = p \in \Sigma\) and \(\gamma'(0) = e_2 \in T_p \Sigma\). Then we have
\[
D_{\gamma'} N = \langle \gamma', N \rangle N \left( \frac{1}{|\nabla f|} \right) \nabla f + \frac{1}{|\nabla f|} \left[ \frac{f z(\gamma', \cdot)}{n(n-1)} - \frac{sf}{n(n-1)} \gamma' \right].
\]

Note that
\[
N \left( \frac{1}{|\nabla f|} \right) = -\frac{1}{|\nabla f|^2} \left( f \alpha - \frac{sf}{n(n-1)} \right).
\]

Let
\[
\gamma' = \langle \gamma', N \rangle N + \gamma''^T,
\]
where \(\gamma''(t)^T\) is the tangential component of \(\gamma'(t)\) to \(f^{-1}(\gamma(t))\), and substituting these into (4.8), we obtain
\[
|\nabla f|D_{\gamma'} N = f z(\gamma''^T, \cdot) - \frac{sf}{n(n-1)} \gamma''^T.
\]

Taking the covariant derivative in the direction \(N\), we have
\[
Ddf(N, N)D_{\gamma'} N + |\nabla f|D_N D_{\gamma'} N = |\nabla f|z(\gamma''^T, \cdot) + f D_N[z(\gamma''^T, \cdot)] - \frac{s}{n(n-1)} |\nabla f| \gamma''^T - \frac{sf}{n(n-1)} D_N \gamma''^T.
\]

Letting \(t \to 0^+\), we obtain
\[
-\frac{sa}{n-1} D_{e_2} \nu = a D_{\nu}[z(\gamma''^T, \cdot)] - \frac{sa}{n(n-1)} D_{\nu} \gamma''^T \bigg|_p.
\]
because the covariant derivative depends only on the point $p$ and initial vector $e_2$. Now since $z(N,X) = 0$ for $X \perp N$, we may assume that $\{e_i\}_{i=2}^n$ diagonalizes $z$ at the point $p$ so that

$$z(D\nu\gamma'[t],e_2) = 0.$$  \hfill (4.10)

**Assertion:** $\nu(z(\gamma'(t),\gamma'(t)))|_{t=0} = 0.$

Define $\varphi(t) := f \circ \gamma(t)$. Note that $\varphi'(0) = 0$ and also $\varphi''(0) = Ddf(e_2,e_2) = 0$ by (4.7). Since $\Sigma$ is the minimum set of $f$, $\varphi'(t)$ is nondecreasing when $\varphi(t)$ is sufficiently close to $a = \min f$ and so $\varphi''(t) \geq 0$ for sufficiently small $t > 0$. So

$$\varphi''(t) = Ddf(\gamma'(t),\gamma'(t)) = \varphi(t)z(\gamma'(t),\gamma'(t)) - \frac{s}{n(n-1)}\varphi(t) \geq 0$$  \hfill (4.11)

for sufficiently small $t > 0$. However, by (4.7), we have $z(\gamma'(t),\gamma'(t)) > 0$ for sufficiently small $t > 0$. So,

$$0 < z(\gamma'(t),\gamma'(t)) \leq \frac{s}{n(n-1)}$$

for sufficiently small $t$. Defining $\xi(t) = z(\gamma'(t),\gamma'(t)) - \frac{s}{n(n-1)}$, we have $\xi(t) \leq 0$ and $\xi(0) = 0$ by (4.7). Thus,

$$\xi'(0) = \left. \frac{d}{dt} \right|_{t=0} z(\gamma'(t),\gamma'(t)) \leq 0.$$

Now considering a smooth extension of $\gamma$ to the interval $(-\epsilon,0]$, we can see that $\frac{d}{dt}|_{t=0} z(\gamma'(t),\gamma'(t))$ cannot be negative since $\Sigma$ is the minimum set of $f$. In other words, we must have

$$\left. \frac{d}{dt} \right|_{t=0} z(\gamma'(t),\gamma'(t)) = 0.$$

\[\square\]

Let $\{N,E_2,\cdots,E_n\}$ be a local frame around $p$ such that $E_i(p) = e_i$ for $2 \leq i$ and $E_2 = \frac{\gamma''}{|\gamma''|}$. Then, for $i \geq 2$,

$$z(E_i,\cdot) = \sum_{j=2}^n z(E_i,E_j)E_j$$

and

$$D_N[z(\gamma',\cdot)] = D_N\left[ z(\gamma'^\top,E_j)E_j \right] = N \left( z(\gamma'^\top,E_j) \right) E_j + z(\gamma'^\top,E_j)D_N E_j.$$

So,

$$(D_N[z(\gamma',\cdot)],E_2) = N \left( z(\gamma'^\top,E_2) + z(\gamma'^\top,E_j)D_N E_2 \right).$$

Note that $z(\gamma',\gamma') = |\gamma''|^2 z(\gamma'^\top,E_2) + (\gamma',N)^2 \alpha$ with $\alpha = z(N,N)$. Since $|\gamma''|$ attains its maximum at $p$ and $(\gamma',N)^2$ attains its minimum at $p$, we have

$$\nu[z(\gamma',\gamma')] = |\gamma''|^2 \nu[z(\gamma'^\top,E_2)] + (\gamma',N)^2 \nu(\alpha)(p) = \nu[z(\gamma'^\top,E_2)]\bigg|_p,$$

which shows that

$$\nu[z(\gamma'^\top,E_2)]\bigg|_p = 0.$$  \hfill (4.12)

by **Assertion.** Letting $t \to 0$ and applying (4.12), we obtain

$$(D_{\nu}[z(\gamma'^\top,\cdot)],E_2)\big|_p = \nu[z(\gamma'^\top,E_2)]\big|_{t=0} + z(e_2,e_2)(D_{\nu}E_2,E_2)|_p = 0.$$  \hfill (4.13)

Thus, by (4.9) and (4.10) again, we have

$$-\frac{sa}{n(n-1)}(D_{e_i}\nu,e_i) = a(D_{\nu}[z(\gamma'^\top,\cdot)],E_2)|_p - \frac{sa}{n(n-1)}(D_{\nu}E_2,E_2)|_p = 0.$$

Since $i = 2$ is, in fact, arbitrary, we have

$$(D_{e_i}\nu,e_i) = 0$$

for any $i \geq 2$. Hence, the square norm of the second fundamental form $A$ is given by

$$|A|^2(p) = \sum_{i=2}^n |(D_{e_i}e_i)|^2 = \sum_{i=2}^n (D_{e_i}\nu,e_i)^2 = 0,$$
which shows $f^{-1}(a)$ is totally geodesic.

Finally, from (4.6) together with the vacuum static equation (1.4), we have

$$-rac{s f}{n - 1} = D df(\nu, \nu) = f z(\nu, \nu) - \frac{s f}{n(n - 1)}.$$  

That is,

$$z_p(\nu, \nu) = -\frac{s f}{n f} = -\frac{s}{n}$$

on the set $\Sigma = f^{-1}(a)$. Thus,

$$r(\nu, \nu) = \text{Ric}(\nu, \nu) = 0$$

and obviously the stability operator for hypersurfaces with vanishing second fundamental form becomes

$$\int_{\Sigma} |\nabla \varphi|^2 \geq 0$$

for any function $\varphi$ on $\Sigma$. Consequently, $\Sigma = f^{-1}(a)$ is a totally geodesic stable minimal hypersurface of $M$. The very similar argument shows that the same property holds for $f^{-1}(b)$ with $\max_M f = b$. Final arguments follow from Theorem 5.3.

**Theorem 4.7.** Let $(g, f)$ be a non-trivial solution of the vacuum static equation on an $n$-dimensional compact manifold $M$ with positive isotropic curvature, and let $\min_M f = a$. Suppose that $f^{-1}(a)$ is not a single point. Then $f^{-1}(a)$ is homeomorphic to $S^{n-1}$ up to finite cover.

**Proof.** Let $\Sigma = f^{-1}(a)$. It follows from Lemma 4.4 and Theorem 4.6 that $\Sigma$ is a connected, totally geodesic stable minimal hypersurface of $M$. Thus $\Sigma$ also satisfies vacuum static equation and has positive isotropic curvature with respect to the induced metric for $n \geq 5$. In the proof of Theorem 4.6, we see that $z \neq 0$ on the set $\Sigma$ since $z(\nu, \nu) = -\frac{s}{2}$. Since $|z|$ is constant on $\Sigma$, we have $|z| > 0$ on $\Sigma$.

Let $\eta$ is a harmonic 1-form on $\Sigma$ with respect to the induced metric. Let $\xi := \frac{1}{2} f \nabla f + \frac{s}{2n(n-1)} f^2$. Then

$$d\varphi = f i\nabla f z = 0$$

on $\Sigma$, $\xi$ is a vanishing 2-form on $\Sigma$, i.e., $\xi = 0$. Nonetheless, since

$$\Delta \varphi = -\delta d\varphi = z(\nabla f, \nabla f) - f^2 i\nabla f z$$

on the set $\Sigma$, we have

$$\begin{align*}
0 &= \delta \xi = (\Delta \varphi) \eta - (\delta \xi) d\varphi - [\nabla \varphi, \eta]^b \\
&= -f^2 |z|^2 \eta,
\end{align*}$$

which implies that $\eta = 0$ on $\Sigma$. In other words, $\Sigma$ has no harmonic 1-forms with respect to the induced metric. Since $\Sigma$ has positive isotropic curvature with respect to the induced metric, $\Sigma$ turns out to be homeomorphic to $S^{n-1}$ up to finite cover [24].

In fact, we see that the maximal set $\Sigma' = f^{-1}(b)$ with $b = \max_M f$ is also a totally geodesic stable minimal hypersurface in $M$ (see Theorem 4.6 and Theorem 5.3 below). In particular, it follows from Lemma 4.5 that every level hypersurface of $f$ is homotopic to each other. So, $M$ can be considered as a fibration $M \to S^1$ with fiber $\Sigma = f^{-1}(a)$. Here $S^1$ is two copies of the closed interval $[a, b]$ and $M$ is obtained from two copies of $[a, b] \times \Sigma$ by gluing two copies of $\{a\} \times \Sigma$ and two copies of $\{b\} \times \Sigma'$. From this fibration, we have the following exact sequence of fundamental groups

$$1 \to \pi_1(M) \to \pi_1(\Sigma) \to 1$$

and so $\pi_1(M) = \mathbb{Z} \oplus \pi_1(\Sigma)$. It is well-known that [12] that for any compact Riemannian manifold of dimension $n \geq 5$ with PIC, the fundamental group does not contain a subgroup isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$, which means $\pi_1(\Sigma)$ must be finite. When $n = 4$, Chen-Tang-Zhu [24] gave a complete classification on compact 4-manifolds with PIC. In our case that $f^{-1}(a)$ is hypersurface, since $M$ satisfies the vacuum static equation, $(\mathbb{R} \times S^3)/\Gamma$ is the only one possible choice, where $\Gamma$ is a cocompact fixed point free discrete isometric subgroup of the standard $\mathbb{R} \times S^3$. This complete the proof.
In vacuum static spaces, it is well-known that the zero set of the potential function $f$ is a totally geodesic hypersurface. If $f \geq 0$, it follows from $\Delta f = -\frac{f}{n-1}$ that $f$ should be constant. So, we may assume that $\min_M f < 0$.

**Lemma 4.8.** Let $(g, f)$ be a non-trivial solution of the vacuum static equation on an $n$-dimensional compact manifold $M$. Then any connected component of $f^{-1}(0)$ is a totally geodesic hypersurface of $M$.

**Lemma 4.9.** Suppose that $\omega = 0$. Then

1. for vectors $X, Y$ orthogonal to $\nabla f$,
   \[i_{\nabla f}T(X, Y) = \frac{|\nabla f|^2}{n-2} (z + \frac{\alpha}{n-1} g)(X, Y)\]

2. $i_{\nabla f}T(\nabla f, X) = i_{\nabla f}T(X, \nabla f) = 0$ for any vector $X$.

**Proof.** If $\omega = 0$, then $i_{\nabla f}z = 0$ and so
   \[T = \frac{1}{n-2} df \wedge \left(z + \frac{\alpha}{n-1} g\right).\]

For the curvature tensor $R$ with $N = \nabla f / |\nabla f|$, $R_N$ is defined as follows
   \[R_N(X, Y) = R(X, N, Y, N)\]
for any vector fields $X$ and $Y$. For the Weyl curvature tensor $W$, $W_N$ is similarly defined. It follows from (4.10) that
   \[|\nabla f|^2 W_N = f i_{\nabla f}C + (n-1) i_{\nabla f}T.\]

**Lemma 4.10.** Let $(g, f)$ be a non-trivial solution of the vacuum static equation with $\omega = 0$. Then
   \[\frac{s}{n(n-1)} g = R_N + \frac{f}{|\nabla f|^2} i_{\nabla f}C + \left(\frac{s}{n(n-1)} - \alpha\right) \frac{df}{|df|} \otimes \frac{df}{|df|}\]

**Proof.** Let
   \[\Phi := \frac{s}{n(n-1)} g - z - \frac{f}{|\nabla f|^2} i_{\nabla f}C.\]
For vector fields $X, Y$ with $X \perp \nabla f$ and $Y \perp \nabla f$, from the curvature decomposition
   \[R = \frac{s}{2n(n-1)} g \otimes g + \frac{1}{n-2} z \otimes g + W\]
we can obtain
   \[R_N(X, Y) = \frac{s}{n(n-1)} g(X, Y) + \frac{1}{n-2} z(X, Y) + \frac{\alpha}{n-2} g(X, Y) + W_N(X, Y).\]
Since, by Lemma 4.9 together with (4.14)
   \[W_N(X, Y) = -\frac{f}{|\nabla f|^2} i_{\nabla f}C(X, Y) - \frac{n-1}{n-2} z(X, Y) - \frac{\alpha}{n-2} g(X, Y),\]
we have
   \[R_N(X, Y) = \Phi(X, Y).\]
Now, let $X$ and $Y$ are arbitrary tangent vector fields. Then $X$ and $Y$ can be decomposed into
   \[X = X_1 + \langle X, N \rangle N, \quad Y = Y_1 + \langle Y, N \rangle N\]
with $\langle X_1, N \rangle = 0 = \langle Y_1, N \rangle$. Thus,
   \[R_N(X, Y) = R_N(X_1, Y_1) = \Phi(X_1, Y_1) = \Phi(X, Y) - \langle X, N \rangle \langle Y, N \rangle \Phi(N, N)\]
   \[= \Phi(X, Y) + \left(\alpha - \frac{s}{n(n-1)}\right) \frac{df}{|df|} \otimes \frac{df}{|df|}(X, Y).\]
\[\square\]
5. Vacuum Static Spaces with Positive Isotropic Curvature

In this section, we prove that if \( (g, f) \) is a non-trivial solution of the vacuum static equation with \( \omega = df \wedge \iota_{\nabla f} z = 0 \), then \( (M, g) \) is isometric to a standard sphere. However, by Theorem 1.2 of [32] it suffices to prove that \( (M, g) \) has harmonic curvature. To do this, we introduce a warped product metric involving \( \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} \) as a fiber metric on each level set \( f^{-1}(c) \). By investigating properties on this warped product metric, and deducing its relations to the given solution metric \( g \), we prove the vanishing of the tensor \( T \).

Consider a warped product metric \( \bar{g} \) on \( M \) by

\[
\bar{g} = \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} + |\nabla f|^2 g_\Sigma,
\]

where \( g_\Sigma \) is the restriction of \( g \) to a level set \( \Sigma = f^{-1}(c) \) for a regular value \( c \) of \( f \). Note that, from Theorem 4.6, the metric \( \bar{g} \) is smooth on \( M \) except only \( f^{-1}(a) \cup f^{-1}(b) \).

The following lemma shows that \( \nabla f \) is a conformal Killing vector field with respect to the metric \( \bar{g} \).

**Lemma 5.1.** Let \( (g, f) \) be a non-trivial solution of the vacuum static equation on an \( n \)-dimensional compact manifold \( M \) with \( \omega = 0 \). Then

\[
\frac{1}{2} L_{\nabla f} \bar{g} = N(|\nabla f|) \bar{g} = \frac{1}{n} (\bar{\Delta} f) \bar{g}.
\]

Here \( L \) denotes the Lie derivative.

**Proof.** Note that, by (1.17) we have

\[
\frac{1}{2} L_{\nabla f} g = D_g df = f z - \frac{s f}{n(n-1)} g.
\]

Moreover, the following identity holds:

\[
N(|\nabla f|) = f \alpha - \frac{s f}{n(n-1)}.
\]

From these properties together with the definition of Lie derivative, we obtain

\[
\frac{1}{2} L_{\nabla f} (df \otimes df)(X, Y) = D(df(X, \nabla f)) df(Y) + df(X) D(df(Y, \nabla f))
\]

\[
= 2 \left( f \alpha - \frac{s f}{n(n-1)} \right) df \otimes df(X, Y).
\]

Therefore,

\[
\frac{1}{2} L_{\nabla f} \left( \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} \right) = N(|\nabla f|) \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|}.
\]

Since

\[
\frac{1}{2} L_{\nabla f}(|\nabla f|^2 g_\Sigma) = \frac{1}{2} \nabla f(|\nabla f|^2) g_\Sigma = D(df(\nabla f, \nabla f)) g_\Sigma = N(|\nabla f|)|\nabla f|^2 g_\Sigma,
\]

we can conclude that

\[
\frac{1}{2} L_{\nabla f} \bar{g} = \bar{D} df = N(|\nabla f|) \bar{g}.
\]

In particular, we have \( \bar{\Delta} f = n N(|\nabla f|) \).

**Corollary 5.2.** Let \( (g, f) \) be a non-trivial solution of the vacuum static equation on an \( n \)-dimensional compact manifold \( M \) with \( \omega = 0 \). Then level hypersurfaces given by \( f \) are homothetic to each other.

**Proof.** By Lemma 5.1 we can choose a local coordinate system \( (u^1) \) in a neighborhood of any hypersurface \( \Sigma := f^{-1}(c) \), \( a < c < b \) such that

\[
\bar{g} = (du^1)^2 + |\nabla f|^2 \eta_{ij}(u^2, \ldots, u^n) du^i \otimes du^j,
\]

where \( du^1 = \frac{df}{|\nabla f|} \) and the functions \( \eta_{ij} \) depend only on \( u^2, \ldots, u^n \) (cf. [10] or [30]). Comparing this to (5.1), we have

\[
g_\Sigma = \eta_{ij}(u^2, \ldots, u^n) du^i \otimes du^j.
\]

These show that level hypersurfaces are homothetic to each other and to \( \Sigma \) with \( \eta_{ij}(u^2, \ldots, u^n) du^i \otimes du^j \) as metric form.

**Theorem 5.3.** Let \( (g, f) \) be a non-trivial solution of the vacuum static equation on an \( n \)-dimensional compact manifold \( M \) with positive isotropic curvature, Then \( (M, g) \) is Bach-flat.
Proof. Since $\omega = 0$ by Theorem 3.6, it follows from Lemma 4.10 that

$$s \frac{1}{n(n-1)} g_{ij} = R_{ij} + \left( z - \alpha \frac{df}{|df|} \otimes \frac{df}{|df|} \right) + \frac{f}{|\nabla f|^2} |\nabla f|^2 C$$

for each level hypersurface $\Sigma$, and

$$g = \frac{df}{|df|} \otimes \frac{df}{|df|} + \frac{n(n-1)}{s} \left[ R_{ij} + \left( z - \alpha \frac{df}{|df|} \otimes \frac{df}{|df|} \right) + \frac{f}{|\nabla f|^2} |\nabla f|^2 C \right].$$

Therefore, by Corollary 5.2, the metric $g$ can also be written as

$$g = \frac{df}{|df|} \otimes \frac{df}{|df|} + \xi(f)^2 g_0,$$

where $g_0 = g_{ij}$ is the induced metric on $f^{-1}(0)$ (a connected component, totally geodesic, topologically $S^{n-1}$ up to finite cover).

Since

$$\frac{1}{2} \mathcal{L}_{\nabla f} \left( \frac{df}{|df|} \otimes \frac{df}{|df|} \right) = N(|\nabla f|) \frac{df}{|df|} \otimes \frac{df}{|df|}$$

and

$$\frac{1}{2} \mathcal{L}_{\nabla f} (\xi^2 g_0) = \xi(\nabla f, \nabla \xi) g_0 = \xi(|\nabla f|^2) \frac{df}{|df|} \otimes \frac{df}{|df|}$$

we have

$$\frac{1}{2} \mathcal{L}_{\nabla f} g = N(|\nabla f|) \frac{df}{|df|} \otimes \frac{df}{|df|} + \xi(|\nabla f|^2) \frac{df}{|df|} \otimes \frac{df}{|df|}.$$

On the other hand, we have

$$\frac{1}{2} \mathcal{L}_{\nabla f} g = Df = f z - \frac{s f}{n(n-1)} g$$

$$= N(|\nabla f|) \frac{df}{|df|} \otimes \frac{df}{|df|} + f z - f \alpha \frac{df}{|df|} \otimes \frac{df}{|df|} - \frac{s f}{n(n-1)} \xi^2 g_0.$$

Comparing this to [3.3], we obtain

$$\left( \xi(|\nabla f|^2) \frac{df}{|df|} + \frac{s f}{n(n-1)} \xi^2 \right) g_0 = f \left( z - \alpha \frac{df}{|df|} \otimes \frac{df}{|df|} \right)$$

as $(n-1) \times (n-1)$ matrices.

Now, let $\{E_1, E_2, \cdots, E_n\}$ be a local frame with $E_1 = N$. Then, we have

$$\xi(|\nabla f|^2) \frac{df}{|df|} = f z(E_i, E_i) - \frac{s f}{n(n-1)} \xi^2$$

for each $2 \leq j \leq n$. Summing up these, we obtain

$$(n-1) \xi(|\nabla f|^2) \frac{df}{|df|} = -f \alpha - \frac{s f}{n} \xi^2.$$ 

Substituting this into [3.3], we get

$$- \frac{\alpha}{n-1} g_0 = z - \frac{\alpha}{n-1} \frac{df}{|df|} \otimes \frac{df}{|df|}$$

as $(n-1) \times (n-1)$ matrices. This implies that, on each level hypersurface $f^{-1}(c)$, we have

$$z(E_i, E_j) = - \frac{\alpha}{n-1}$$

for $2 \leq j \leq n$. Hence,

$$|z|^2 = \alpha^2 + \frac{\alpha^2}{n-1} = \frac{n \alpha^2}{n-1} = \frac{n}{n-1} |\nabla f|^2,$$

since $z(N, E_i) = 0$ for $i \geq 2$. As a result, it follows from Lemma 5.3 that $T = 0$. So, $(M, g)$ is a Bach-flat vacuum static space. \hfill $\square$

Theorem 5.4. Let $(g, f)$ be a non-trivial solution of the vacuum static equation on an $n$-dimensional compact manifold $M$ with positive isotropic curvature, and let $\min_M f = a$. Suppose $f^{-1}(a)$ is a single point. Then $M$ is, up to finite cover, isometric to $S^n$. 
Theorem 5.5. Let \((g, f)\) be a non-trivial solution of the vacuum static equation on an \(n\)-dimensional compact manifold \(M\) with positive isotropic curvature, then up to finite cover, either \(M\) is isometric to \(S^n\) or \(S^1 \times S^{n-1}\).

**Proof.** By Theorem 5.3, \((M, g)\) is a Bach-flat vacuum static space. By applying a result due to Qing and Yuan [27] together with the fact that the scalar curvature is positive, \((M, g)\), up to finite cover and scaling, is isometric \(S^n\), or to the warped product \(S^1 \times S^{n-1}\), where \(S^{n-1}\) is an \((n-1)\)-dimensional Einstein manifold with positive scalar curvature.

(i) \(f^{-1}(a)\) is a single point.

In this case, the maximal level set \(f^{-1}(b)\) is also a single point, and so \((M, g)\) is isometric to \(S^n\) up to a finite cover. In fact, applying a result in [30] together with Lemma 5.1 we can see that \(M\) is conformal to an \(n\)-dimensional spherical space.

(ii) Either \(f^{-1}(a)\) is a hypersurface and \(f^{-1}(b)\) is discrete, or \(f^{-1}(a)\) is discrete and \(f^{-1}(b)\) is a hypersurface.

We claim that this case does not happen. Suppose that \(f^{-1}(a)\) is a hypersurface and \(f^{-1}(b)\) is discrete. Since \(f\) has critical points only on the set \(f^{-1}(a) \cup f^{-1}(b)\), \(f^{-1}(b)\) must be two points and \(M\) is homotopically a sphere \(S^n\). The proof shown as above, we can see that \(M\), up to a finite cover, is isometric to a sphere and so \(\alpha = 0\) on the whole \(M\). However, since \(\alpha = \frac{s}{n}\) on the set \(f^{-1}(a)\), we have a contradiction.

(iii) Both \(f^{-1}(a)\) and \(f^{-1}(b)\) are hypersurfaces.

In this case, we have that \(\alpha = \frac{s}{n}\) on \(f^{-1}(a)\) and \(f^{-1}(b)\). However, it follows from (5.8) together with \(\nabla f = 0\) on \(f^{-1}(a)\) and \(f^{-1}(b)\) that

\[ \alpha = -\frac{s}{n}\xi^2 \]

which implies \(\xi = 1\) on \(f^{-1}(a)\) and \(f^{-1}(b)\).

In the next, we will show that \(\xi = 1\) on \(M\) which complete the proof.

In the rest of this section, we will prove that \(\xi \equiv 1\) which implies that \(g\) is a product metric. For a nontrivial solution \(f\) of the vacuum static equation \(\nabla f = a\), recall that

\[ \min_{x \in M} f(x) = a, \quad \max_{x \in M} f(x) = b \]

and \(\xi\) is a function satisfying (5.5). Note that in the proof of Theorem 4.6 we have

\[ \alpha = -\frac{s}{n} \]

on the set \(f^{-1}(a)\) and \(f^{-1}(b)\) when they are hypersurfaces. First we have the following.

**Lemma 5.6.** If the function \(\alpha + \frac{s}{n}\) has a minimum on \(M \setminus \{f^{-1}(a) \cup f^{-1}(b)\}\), then it attains its minimum on the set \(f^{-1}(0)\).

**Proof.** Recall that \(\alpha + \frac{s}{n}\) is not constant on the set \(f^{-1}(a) \cup f^{-1}(b)\). Assume that \(\alpha + \frac{s}{n}\) is not constant (If \(\alpha + \frac{s}{n}\) is constant, then it is trivial since \(\alpha + \frac{s}{n} = 0\) on \(M\)). Suppose that \(\alpha + \frac{s}{n}\) attains its minimum on the set \(f^{-1}(c)\)

with \(a < c < b\). Then

\[ \alpha + \frac{s}{n} < 0, \quad N(\alpha) = 0 \quad \text{and} \quad NN(\alpha) \geq 0. \]

From \(f|z|^2 = \nabla f(\alpha) - \frac{s f(\alpha)}{n-1}\) or \(\frac{n}{n-1} f \alpha^2 = \nabla f(\alpha) - \frac{s f(\alpha)}{n-1}\) (recall that \(T = 0\) and \(|z|^2 = \frac{n}{n-1} \alpha^2\)), we have, on the set \(f^{-1}(c)\),

\[ \frac{nc}{n-1} \alpha^2 + \frac{s c \alpha}{n-1} = 0, \quad \text{i.e.,} \quad \frac{nc}{n-1} \alpha (\alpha + \frac{s}{n}) = 0, \]

which implies that \(c = 0\).

**Lemma 5.7.** Let \((g, f)\) be a non-trivial solution of the vacuum static equation on an \(n\)-dimensional compact manifold \(M\) with \(\omega = 0\). Then

\[ \alpha + \frac{s}{n} \geq 0 \]

on \(M\).
Lemma 5.8 (cf. [2]). Let \((M^n, g)\) be a Riemannian manifold with constant scalar curvature \(s_g\). Then
\[
\delta d^D z = D^* D z + \frac{n}{n-2} z \circ z + \frac{s}{n-4} z - \frac{1}{n-2} |z|^2 g.
\]

Lemma 5.9. Let \((g, f)\) be a non-trivial solution of the vacuum static equation on an \(n\)-dimensional compact manifold \(M\) with \(\omega = 0\). Suppose that \(f^{-1}(a)\) is a hypersurface. If \(\alpha + \frac{s}{n} \geq 0\) on \(M\), then
\[
\alpha + \frac{s}{n} = 0
\]
on the whole \(M\).

Proof. We have \(T = 0, C = 0\) and so \(\delta W z = 0\). So, it follows from Lemma 5.8 that
\[
-(D^* D z, z) = \frac{s}{n-1} |z|^2 + \frac{n}{n-2} \langle z \circ z, z \rangle.
\]

Thus
\[
\frac{1}{2} \Delta |z|^2 = -(D^* D z, z) + |D z|^2 = \frac{s}{n-1} |z|^2 + \frac{n}{n-2} \langle z \circ z, z \rangle + |D z|^2.
\]

Since \(T = 0\), we have
\[
z(N, N) = \alpha, \quad z(E_i, E_i) = - \frac{\alpha}{n-1}
\]
and so
\[
|z|^2 = \frac{n}{n-1} \alpha^2, \quad \langle z \circ z, z \rangle = \alpha^3 - \frac{1}{(n-1)^2} \alpha^3.
\]

Substituting these into (5.10), we obtain
\[
\frac{1}{2} \Delta |z|^2 = \left( \frac{n}{n-1} \right) \alpha^2 \left( \frac{s}{n} + \alpha \right) + |D z|^2.
\]

Since \(|z|\) is a subharmonic function, it is constant and so is \(\alpha\) because \(|z|^2 = \frac{n}{n-1} \alpha^2\). Since \(\alpha = -\frac{s}{n}\) on \(f^{-1}(a)\), we have
\[
\alpha = -\frac{s}{n}
\]
on the whole \(M\).

Remark 5.10. We would like to mention that under the hypotheses of Lemma 5.8, \(\alpha\) cannot be zero identically on \(M\).

Lemma 5.11. Let \((g, f)\) be a non-trivial solution of the vacuum static equation on an \(n\)-dimensional compact manifold \(M\) with \(\omega = 0\). If both \(f^{-1}(a)\) and \(f^{-1}(b)\) are hypersurfaces, then
\[
\xi \equiv 1
\]
on the whole \(M\).
Proof. First, we claim that $\xi = 1$ on the set $f^{-1}(0)$. By Lemma 5.11 and Lemma 5.9, we have
\[
\text{Ric}_M(N, N) = \alpha + \frac{s}{n} = 0 \quad \text{on } M.
\]
Since $f^{-1}(0)$ is totally geodesic, we have
\[
s_0 = s - 2\text{Ric}_M(N, N) = s,
\]
where $s_0$ is the scalar curvature of $f^{-1}(0)$ with respect to the induced metric. Next, it follows from (5.10) that
\[
s = s_0 - 2(n - 1)\frac{\xi''}{\xi} - (n - 1)(n - 2)\frac{\xi''^2}{\xi^2},
\]
where $\xi'$ and $\xi''$ denote the derivatives of $\xi$ with respect to $f$. Thus, we obtain
\[
2(n - 1)\frac{\xi''}{\xi} - (n - 1)(n - 2)\frac{\xi''^2}{\xi^2} = 0
\]
on the set $f^{-1}(0)$. Now from (5.10), it is easy to see that
\[
\text{Ric}_M(N, N) = -(n - 1)\frac{\xi''}{\xi}
\]
on the set $f^{-1}(0)$. Consequently, we have
\[
\xi'' = 0, \quad \xi' = 0
\]
on the set $f^{-1}(0)$. From (5.8), we have
\[
\alpha + \frac{s}{n}\xi' = -(n - 1)|\nabla f|^2\xi \cdot \xi'
\]
on the set $M \setminus f^{-1}(0)$. By letting $x \to p \in f^{-1}(0)$ and applying L'Hospital rule, we obtain
\[
\alpha + \frac{s}{n}\xi' = -(n - 1)|\nabla f|^2\xi'' = 0.
\]
So, on the set $f^{-1}(0)$, we have
\[
\alpha + \frac{s}{n}\xi' = 0 = \alpha + \frac{s}{n}
\]
which implies that $\xi = 1$ on the set $f^{-1}(0)$.

Now we claim that $\xi = 1$ on the whole $M$. Since $\alpha = -\frac{s}{n}$, from (5.5), we have
\[
(n - 1)\xi|\nabla f|^2\xi' = \frac{sf}{n}(1 - \xi^2).
\]
Taking the derivative with respect to $f$, we have
\[
F(\xi'', \xi') = \frac{s}{n}(1 - \xi^2),
\]
where
\[
F(\xi'', \xi') = (n - 1)|\nabla f|^2\xi'' + (n - 1)|\nabla f|^2(\xi')^2 + (n - 1)(|\nabla f|^2)\xi'\xi'' + \frac{2sf}{n}\xi'.
\]
From (5.13) again, we have
\[
f F(\xi'', \xi') = \frac{sf}{n}(1 - \xi^2) = (n - 1)|\nabla f|^2\xi',
\]
i.e.,
\[
F(\xi'', \xi') - (n - 1)\frac{|\nabla f|^2\xi'}{f} = 0.
\]
Applying the maximum principle to $\xi$ in (5.14) on the set $a < f < b$, we obtain
\[
\sup_{a < f < 0} \xi = \sup_{\{f = a\} \cup \{f = b\}} \xi = 1
\]
and
\[
\inf_{a < f < 0} \xi = \inf_{\{f = a\} \cup \{f = b\}} \xi = 1.
\]
Hence $\xi = 1$ on the set $a < f < b$. The same argument shows that $\xi = 1$ on the set $0 < f < b$. Consequently $\xi = 1$ on $M$. 

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16 SEUNGSU HWANG AND GABJIN YUN*