JACOBIAN PAIRS

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Abstract. We study meromorphic jacobian pairs, i.e., pairs of polynomials in one variable, with coefficients meromorphic series in a second variable, whose jacobian relative to the two variables depends only on the second variable. We pose two meromorphic jacobian conjectures about such pairs one of which is in terms of an invariant of the pair which we call the beta invariant. These conjectures are shown to imply the bivariate algebraic jacobian conjecture which predicts that two bivariate polynomials generate the polynomial ring if their jacobian is a nonzero constant. Actually we define the beta invariant for any two meromorphic curves, i.e., polynomials in one variable with coefficients which are meromorphic series in another variable. One of our basic techniques is the beta-jacobian identity which relates the beta invariant to the jacobian of the meromorphic curves. When the pair of meromorphic curves consists of a curve and its derivative, the beta invariant is reduced to the beta-bar invariant of the curve, and the beta-jacobian identity is reduced to the betabar-derivative identity. When the curve is analytic, i.e., given by a bivariate power series, the betabar-derivative identity gives rise to the conductor-derivative identity which is related to Dedekind’s formula expressing the derivative ideal as a product of the conductor and the different. In turn, when the curve is algebraic, i.e., when the power series is a bivariate polynomial, the betabar-derivative identity gives rise to the betabar-genus identity which connects the betabar invariant of the curve to its genus. In the complex case the betabar invariant is related to the rank of the first homology group and the numbers of Milnor and Tjurina. As another technique for studying the jacobian conjectures we revisit the Newton polygon.

Section 1: Introduction

Let $F = F(X, Y)$ and $G = G(X, Y)$ be meromorphic curves, i.e., let $F$ and $G$ be members of $R = k((X))[Y]$ = the polynomial ring in $Y$ over the meromorphic series field $k((X))$ where $k$ is an algebraically closed field of characteristic zero. We want to study the jacobian $J(F, G) = F_X G_Y - F_Y G_X$ of $F$ and $G$ relative to $X$ and $Y$ where subscripts indicate partial derivatives. To do this we shall consider factors of $F$ in $R^2 = \text{the set of all irreducible monic polynomials (of positive degree) in } Y$ over $k((X))$. In general we shall use the terminology of \cite{AA1} and \cite{AA2}. 

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Let

\[ N = \deg Y \]

and note that: if \( F \neq 0 \) then \( N \) is a nonnegative integer, and if \( F = 0 \) then \( N = -\infty \). Let \( F_0 = F_0(X) \in k((X)) \) be defined by saying that if \( F = 0 \) then \( F_0 = 0 \), and if \( F \neq 0 \) then \( F_0 = \) the coefficient of \( Y^N \) in \( F \). We say that \( F \) is \( k \)-monic to mean that \( 0 \neq F_0 \in k \). We can write

\[ F = F_0 \prod_{1 \leq j \leq \chi(F)} F_j \quad \text{where} \quad F_j = F_j(X,Y) \in R^2 \text{ with } \deg Y F_j = N_j \]

for \( 1 \leq j \leq \chi(F) \), and \( \chi(F) \) is a nonnegative integer with: \( \chi(F) > 0 \iff N > 0 \). We call \( \chi(F) \) the \textbf{branch number} of \( F \). For any integer \( \nu > 0 \) which is divisible by \( N_1, \ldots, N_{\chi(F)} \), by Newton’s Theorem we can write

\[ F(X^\nu, Y) = F_0(X^\nu) \prod_{1 \leq i \leq N} (Y - z_i(X)) \quad \text{with} \quad z_i(X) \in k((X)) \]

for \( 1 \leq i \leq N \), and upon letting

\[ M = \deg Y G \]

we define the \textbf{intersection multiplicity} of \( F \) with \( G \) by putting

\[
\int(F,G) = \begin{cases} 
\frac{M}{\nu} \ord_X F_0(X^\nu) + (1/\nu) \sum_{i=1}^N \ord_X G(X^\nu, z_i(X)) & \text{if } F \neq 0 \neq G \\
0 & \text{if } F = 0 \neq G \in k((X)) \text{ or } G = 0 \neq F \in k((X)) \\
\infty & \text{if } F = 0 \neq G \not\in k((X)) \text{ or } G = 0 \neq F \not\in k((X)) \\
\infty & \text{if } F = 0 = G 
\end{cases}
\]

and by noting that this is independent of the choice of \( \nu \). We follow the convention that a sum of a finite number of quantities is \( \infty \) if each of them is either a real number or \( \infty \) and at least one of them is \( \infty \). Moreover the product of \( \infty \) and a positive real number is \( \infty \). Also the sum of an empty family is zero, and the product of an empty family is one. Note that if \( FG \neq 0 \) then

\[ \int(F,G) = \ord_X (\text{Res}_Y(F,G)) \]

where \( \text{Res}_Y(F,G) \) is the \( Y \)-resultant of \( F \) and \( G \), and hence always

\[ \int(F,G) = \int(G,F) = \text{an integer or } \infty. \]

We define \( \gcd(F,G) = 1 \) to mean that \( \int(F,G) \neq \infty \) and we note that then

\[ \gcd(F,G) = 1 \iff \begin{cases} 
eq \text{either } F \neq 0 \neq G \text{ and } G \not\subseteq F_j R \text{ for } 1 \leq j \leq \chi(F), \\
or F = 0 \neq G \in k((X)), \\
or G = 0 \neq F \in k((X)). 
\end{cases} \]

For a moment suppose that \( F \in R^2 \); then for \( 1 \leq i \leq N \) we have \( \int(F,G) = \ord_X G(X^\nu, z_i(X)) \) and: \( \int(F,G) \geq 0 \iff \ord_X G(X^\nu, z_i(X)) - \lambda \geq 0 \) for some \( \lambda \in k \); note that if this \( \lambda \) exists then it is unique and is independent of \( i \) as well as \( \nu \), and we call it the \textbf{residue} of \( G \) at \( F \) and denote it by \( \text{res}(F,G) \); also note that the said \( \lambda \), if it exists, can also be characterized as the unique element of \( k \) such that \( \int(F,G - \lambda) > 0 \); if \( \int(F,G) < 0 \) then we put \( \text{res}(F,G) = \infty \). Thus

\[
\begin{cases} 
eq \text{if } F \in R^2 \text{ then } \lambda = \text{res}(F,G) \in k \cup \{ \infty \} \text{ is such that:} \\
\lambda = \infty \iff \int(F,G) < 0, \text{ and} \\
\lambda \in k \iff \int(F,G) \geq 0 \iff \int(F,G - \lambda) > 0.
\end{cases}
\]
In the general case of $F \in R$, for any $\lambda \in k$ we define the $\lambda$-alpha invariant of $G$ relative to $F$ and the $\lambda$-beta invariant of $G$ relative to $F$ by putting
\[
\alpha_{\lambda}(F, G) = \{ j : 1 \leq j \leq \chi(F) \text{ with } \text{res}(F_j, G) = \lambda \}
\]
and
\[
\beta_{\lambda}(F, G) = \sum_{j \in \alpha_{\lambda}(F, G)} \text{int}(F_j, G - \lambda)
\]
respectively, and we define the alpha invariant of $G$ relative to $F$ and the beta invariant of $G$ relative to $F$ by putting
\[
\alpha(F, G) = \{ \lambda \in k : \alpha_{\lambda}(F, G) \neq \emptyset \}
\]
and
\[
\beta(F, G) = \sum_{\lambda \neq \alpha(F, G)} \beta_{\lambda}(F, G)
\]
respectively.

Note that although the invariant $\alpha_{\lambda}(F, G)$ depends on the way the factors $F_j$ of $F$ are labelled, the other three invariants do not. We could have avoided this dependence by defining $\alpha_{\lambda}(F, G)$ to be the obvious effective divisor of $R$, where by an effective divisor of $R$ we mean a nonnegative-integer-valued map of $R^\times$ with finite support. We would have then started by associating to $F$ the effective divisor $F_D$ which sends any $\Phi \in R^\times$ to the number of $j$ with $1 \leq j \leq \chi(F)$ for which $F_j = \Phi$, and so on.

Note that for any $\lambda \in k$ we have
\[
|\alpha_{\lambda}| \leq \chi(F) < \infty
\]
where $| |$ denote size (= cardinality) and
\[
\left\{ \begin{array}{l}
\beta_{\lambda}(F, G) \text{ is a nonnegative integer or } \infty \text{ such that: }\\
\beta_{\lambda}(F, G) = 0 \iff \alpha_{\lambda}(F, G) = \emptyset, \text{ and }\\
\beta_{\lambda}(F, G) = \infty \iff G - \lambda \in F_j R \text{ for some } j \in \alpha_{\lambda}(F, G).
\end{array} \right.
\]
Moreover
\[
\alpha(F, G) = \{ \lambda \in k : \text{res}(F_j, G) = \lambda \text{ for some } j \}
\]
and hence
\[
|\alpha(F, G)| \leq \chi(F) - |\{ j : 1 \leq j \leq \chi(F) \text{ with } \text{int}(F_j, G) < 0 \}| < \infty
\]
and
\[
\left\{ \begin{array}{l}
\beta(F, G) \text{ is a nonnegative integer or } \infty \text{ such that: }\\
\beta(F, G) = 0 \iff \alpha(F, G) \subset \{ 0 \}, \text{ and }\\
\beta(F, G) = \infty \iff G - \lambda \in F_j R \text{ for some } 0 \neq \lambda \in \alpha(F, G) \text{ and } j \in \alpha_{\lambda}(F, G).
\end{array} \right.
\]
Defining the minimal-intersection of $F$ with $G$ by putting
\[
\text{minint}(F, G) = \min_{\mu \in k^\times} \text{int}(F, G - \mu)
\]
(where min stands for glb = greatest lower bound) we see that
\[
\left\{ \begin{array}{l}
\text{minint}(F, G) \text{ is an integer or } \infty, \\
\text{and: } \text{minint}(F, G) = \infty \iff F = 0 \neq G \notin k((X)).
\end{array} \right.
\]
and
\[
\alpha(F, G) = \{ \lambda \in k : \text{int}(F, G - \lambda) > \minint(F, G) \} \\
= \{ \lambda \in k : \text{int}(F, G - \lambda) \neq \minint(F, G) \}
\]
and
\[
\beta(F, G) = \sum_{0 \neq \lambda \in \alpha(F, G)} \left[ \text{int}(F, G - \lambda) - \minint(F, G) \right] \\
= \text{a nonnegative integer or } \infty.
\]

By a relative irregular value of \( G \) (on \( F \)) we mean an element of \( \alpha(F, G) \). We call \( \alpha(F, G) \) the relative irregular value set of \( G \) (on \( F \)), and we call \( \alpha_\lambda(F, G) \) the relative irregular \( \lambda \)-label set of \( G \) (on \( F \)). We call \( \beta(F, G) \) the relative excess intersection of \( G \) (on \( F \)), and we call \( \beta_\lambda(F, G) \) the relative excess intersection of \( G \) (on \( F \)) at \( \lambda \). [We may think of the \( \alpha_\lambda \)'s, with \( \lambda \neq 0 \), as branches of \( F \) close to \( G \), each of them giving a residue which when subtracted from \( G \) gives a higher intersection multiplicity, and the \( \beta \)'s are the sums of these higher intersection multiplicities.]

For any \( \lambda \in k \) we put
\[
\overline{\alpha}_\lambda(F) = \alpha_\lambda(F_Y, F)
\]
and
\[
\overline{\beta}_\lambda(F) = \beta_\lambda(F_Y, F)
\]
and call these the \( \lambda \)-alphabet invariant of \( F \) and the \( \lambda \)-betabar invariant of \( F \) respectively. Also we put
\[
\overline{\alpha}(F) = \alpha(F_Y, F)
\]
and
\[
\overline{\beta}(F) = \beta(F_Y, F)
\]
and call these the alphabet invariant of \( F \) and the betabar invariant of \( F \) respectively. By an irregular value of \( F \) we mean an element of \( \overline{\alpha}(F) \). We call \( \overline{\alpha}(F) \) the irregular value set of \( F \), and we call \( \overline{\alpha}_\lambda(F) \) the irregular \( \lambda \)-label set of \( F \). We call \( \overline{\beta}(F) \) the excess intersection of \( F \), and we call \( \overline{\beta}_\lambda(F) \) the excess intersection of \( F \) at \( \lambda \). The objects \( \overline{\alpha}(F) \) and \( \overline{\beta}(F) \) were considered in \[Ass\] where they were denoted by \( I(F) \) and \( A_F \) respectively.

In Section 2 we shall prove an identity for \( \overline{\beta}(F, G) \) involving \( \text{int}(F, J(F, G)) \) which we call the beta-jacobian identity, and from this we shall deduce an identity for \( \overline{\beta}(F, G) \) involving \( \text{int}(F, F_Y) \) which we call the beta-derivative identity, and in turn from this we shall deduce an identity for \( \overline{\beta}(F) \) involving \( \text{int}(F, F_Y) \) which we call the betabar-derivative identity and which was directly proved in \[Ass\].

The beta-jacobian identity gives a geometric characterization of the beta invariant without digging into the branch structure. The beta-derivative identity says that if \( G \) has a special relationship with \( F \) then the beta-invariant has a simpler expression than the one given by the beta-jacobian identity.

In Sections 3 and 4 we shall deduce further corollaries of the beta-jacobian identity for analytic curves (given by power series in \( X, Y \)) and algebraic curves (given by polynomials in \( X, Y \)) respectively. In Sections 5 and 6 we shall consider curves with no irregular value and one irregular value respectively. In Section 7 we shall make two meromorphic jacobian conjectures and indicate how they imply the usual bivariate algebraic jacobian conjecture, which predicts that if \( F \) and \( G \) are algebraic curves, i.e., if they are members of the bivariate polynomial ring \( k[X, Y] \),
then: $0 \neq J(F, G) \in k \iff k[X, Y] = k[F, G]$; note that the implication $\Leftarrow$ is obvious since it follows by the chain rule for jacobians. In Section 8 we shall settle some cases of the first meromorphic jacobian conjecture. In Sections 9 and 10, as a tool for studying the second meromorphic jacobian conjecture, we shall revisit the Newton Polygon. In particular we shall prove a parallelness property for the Newton Polygons of two meromorphic curves whose jacobian depends only on one of the variables. In Section 11 we shall relate some of the invariants studied in Sections 1 to 6 with homology rank and the numbers of Milnor and Tjurina.

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Section 2: The beta-jacobian identity

Let $E$ be the square free part of $F \in R$, i.e., in the notation of Section 1, $E = F_0 \prod_{1 \leq i \leq \beta(F)} F_{a(i)}$ where $a(1), \ldots , a(b(F))$ are distinct integers amongst $1, \ldots , \chi(F)$ such that $F_{a(1)}, \ldots , F_{a(b(F))}$ are all the distinct elements amongst $F_1, \ldots , F_{\chi(F)}$; let us call $E$ the radical of $F$ and denote it by rad$(F)$. Recall that for $G \in R$ we have: GCD$(F, G) = 1 \iff$ int$(F, G) \neq \infty$. Now if $F \neq 0$ then clearly: GCD$(F, F_Y) = 1 \iff$ rad$(F) = F$. Using these concepts, we shall now prove:

The beta-jacobian identity (2.1). Let $F \in R$ be k-monic of Y-degree N, and let $G \in R$ be such that GCD$(F, G - c) = 1$ for all $c \in k$. Then for $E = \text{rad}(F)$ we have:

$$\text{int}(F, J(E, G)) = \text{int}(F, G) + \text{int}(F, E_Y) - N + \beta(F, G)$$

where each term is an integer.

PROOF. For $N = 0$ this is obvious because each term is reduced to 0. For a moment suppose $N = 1$. Then $E(X, Y) = F(X, Y) = F_0(Y - z(X))$ with $0 \neq F_0 \in k$ and $z(X) \in k((X))$. Clearly $J(E, G) = -F_0 z_X (G_Y (X, Y) - F_0 G_X (X, Y)$ and by substituting $z(X)$ for $Y$ in the Right Hand Side of this equation it becomes equal to $-F_0 H_X (X)$ where $H(X) = G(X, z(X))$. Therefore $\text{int}(F, J(E, G)) = \text{ord}_X H_X (X)$. Also obviously $\text{int}(F, G) = \text{ord}_X H(X)$ and $\text{int}(F, E_Y) = 0$. If $\text{ord}_X H(X) \neq 0$ then $\beta(F, G) = 0 = \text{ord}_X H_X (X) - \text{ord}_X H(X) + 1$, and if $\text{ord}_X H(X) = 0$ then $H(X) = \lambda + \overline{F}(X)$ where $0 \neq \lambda \in k$ and $0 \neq \overline{F}(X) \in k[[X]]$ with $\overline{F}(0) = 0$ and hence again $\beta(F, G) = \text{ord}_X \overline{F}(X) = \text{ord}_X H_X (X) - \text{ord}_X H(X) + 1$. Thus always $\beta(F, G) = \text{ord}_X H_X (X) - \text{ord}_X H(X) + 1$ and hence $\text{int}(F, J(E, G)) = \text{int}(F, G) + \text{int}(F, E_Y) - N + \beta(F, G)$ where each term is an integer.

To prove the general case, let the notation be as in Section 1. Let $\hat{E} = \hat{E}(X, Y) = E(X^\nu)$ and $\hat{F} = \hat{F}(X, Y) = F(X^\nu)$ and $\hat{G} = \hat{G}(X, Y) = G(X^\nu)$. Let $\hat{F}_0 = F_0$ and $\hat{N}_0 = 0$, and for $1 \leq i \leq N$ let $\hat{F}_i = F_i(X, Y) = Y - z_i(X)$ and $\hat{N}_i = 1$. Let \(\{u(1, 1), \ldots , u(1, v(1))\}, \ldots , \{u(w, 1), \ldots , u(w, v(w))\}\) be a partition of \(\{0, \ldots , N\}\) into disjoint nonempty subsets such that $\hat{F}_{u(1, 1)}, \ldots , \hat{F}_{u(w, 1)}$ are exactly all the distinct elements amongst $\hat{F}_0, \ldots , \hat{F}_N$, and for all $i, j'$ we have $\hat{F}_{u(i, j)} = \hat{F}_{u(i, j')}$. Then

$$\hat{F} = \prod_{1 \leq i \leq w} \hat{F}_{u(i, j)} \quad \text{and} \quad \hat{E} = \prod_{1 \leq i \leq w} \hat{F}_{u(i, 1)}.$$
By the above two cases we see that for \(1 \leq i \leq w\) and \(1 \leq j \leq v(i)\) we have
\[
\text{int}(\hat{F}_{u(i,j)}, J(\hat{F}_{u(i,1)}, \hat{G})) = \text{int}(\hat{F}_{u(i,j)}, \hat{G}) + \text{int}(\hat{F}_{u(i,j)}, \hat{F}_{u(i,1)}Y) - \hat{N}_{u(i,j)} + \beta(\hat{F}_{u(i,j)}, \hat{G})
\]
where each term is an integer. Adding \(\text{int}(\hat{F}_{u(i,j)}, \hat{E}/\hat{F}_{u(i,1)})\) to both sides of the above equation, and noting that by various product rules and such we have
\[
\text{int}(\hat{F}_{u(i,j)}, \hat{E}/\hat{F}_{u(i,1)}) + \text{int}(\hat{F}_{u(i,j)}, J(\hat{F}_{u(i,1)}, \hat{G})) = \text{int}(\hat{F}_{u(i,j)}, (\hat{E}/\hat{F}_{u(i,1)})J(\hat{F}_{u(i,1)}, \hat{G}))
\]
and
\[
\text{int}(\hat{F}_{u(i,j)}, \hat{E}/\hat{F}_{u(i,1)}) + \text{int}(\hat{F}_{u(i,j)}, \hat{F}_{u(i,1)}Y) = \text{int}(\hat{F}_{u(i,j)}, (\hat{E}/\hat{F}_{u(i,1)})\hat{F}_{u(i,1)}Y)
\]
we get
\[
\text{int}(\hat{F}_{u(i,j)}, J(\hat{E}, \hat{G})) = \text{int}(\hat{F}_{u(i,j)}, \hat{G}) + \text{int}(\hat{F}_{u(i,j)}, \hat{E}Y) - \hat{N}_{u(i,j)} + \beta(\hat{F}_{u(i,j)}, \hat{G})
\]
where each term is an integer. Summing the above equation over \(1 \leq i \leq w\) and \(1 \leq j \leq v(i)\), and noting that
\[
\sum_{1 \leq i \leq w} \sum_{1 \leq j \leq v(i)} \text{int}(\hat{F}_{u(i,j)}, J(\hat{E}, \hat{G})) = \text{int}(\hat{F}, J(\hat{E}, \hat{G}))
\]
and
\[
\sum_{1 \leq i \leq w} \sum_{1 \leq j \leq v(i)} \text{int}(\hat{F}_{u(i,j)}, \hat{G}) = \text{int}(\hat{F}, \hat{G})
\]
and
\[
\sum_{1 \leq i \leq w} \sum_{1 \leq j \leq v(i)} \text{int}(\hat{F}_{u(i,j)}, \hat{E}Y) = \text{int}(\hat{F}, \hat{E}Y)
\]
and
\[
\sum_{1 \leq i \leq w} \sum_{1 \leq j \leq v(i)} \hat{N}_{u(i,j)} = N
\]
and
\[
\sum_{1 \leq i \leq w} \sum_{1 \leq j \leq v(i)} \beta(\hat{F}_{u(i,j)}, \hat{G}) = \beta(\hat{F}, \hat{G}),
\]
we get
\[
\text{int}(\hat{F}, J(\hat{E}, \hat{G})) = \text{int}(\hat{F}, \hat{G}) + \text{int}(\hat{F}, \hat{E}Y) - N + \beta(\hat{F}, \hat{G})
\]
where each term is an integer. Now the desired identity follows from the above equation by noting that clearly
\[\text{int}(\hat{F}, \hat{G}) = \text{int}(F, G)\nu\quad\text{and}\quad\text{int}(\hat{F}, \hat{E}_Y) = \text{int}(F, E_Y)\nu\]
and
\[\beta(\hat{F}, \hat{G}) = \beta(F, G)\nu\]
and, upon letting \(\hat{J}(X, Y)\) be obtained by substituting \(X^\nu\) for \(X\) in the expression of \(J(E, G)\) as a member of \(k((X))[Y]\), by the chain rule for jacobians we have
\[\hat{J}(E, G) = \hat{J}(X, Y) J(X^\nu, Y) = \hat{J}(X, Y)\nu X^{\nu-1}\]
and hence
\[\text{int}(\hat{F}, J(\hat{E}, \hat{G})) = \text{int}(F, J(E, G))\nu + N(\nu - 1).\]

Now we shall deduce:

**The beta-derivative identity (2.2).** Let \(F \in R\) be \(k\)-monic of \(Y\)-degree \(N\), and let \(G \in R\) be such that \(\text{GCD}(F, G - c) = 1\) for all \(c \in k\) and, in the notation of Section 1, \(G_Y \in F_j R\) for \(1 \leq j \leq \chi(F)\). Then we have
\[\text{int}(F, G_X) = \text{int}(F, G) - N + \beta(F, G)\]
where each term is an integer.

**PROOF.** Let \(E = \text{rad}(F)\) and recall that \(J(E, G) = E_X G_Y - E_Y G_X\); since by assumption \(G_Y \in F_j R\) for \(1 \leq j \leq \chi(F)\), we get \(\text{int}(F, J(E, G)) = \text{int}(F, E_Y) + \text{int}(F, G_X)\); therefore the beta-derivative identity follows by subtracting \(\text{int}(F, E_Y)\) from both sides of the beta-jacobian identity.

Next we shall deduce:

**The betabar-derivative identity (2.3).** Let \(F \in R\) be \(k\)-monic of \(Y\)-degree \(N > 0\) with \(\text{GCD}(F_Y, F - c) = 1\) for all \(c \in k\). Then we have
\[\text{int}(F_X, F_Y) = \text{int}(F, F_Y) - N + 1 + \overline{\beta}(F)\]
where each term is an integer.

**PROOF.** This follows by taking \((F_Y, F)\) for \((F, G)\) in the beta-derivative identity.

**Remark (2.4).** Let the notation be as in Section 1, with \(F\) and \(G\) in \(R\). Note that for generic \(\lambda \in k\), i.e., for all except a finite number of \(\lambda \in k\) we have
\[\text{int}(F_j, G - \lambda) \leq 0\] for all \(j\).
This motivates calling \(G\) **generic** (relative to \(F\)) to mean that
\[\text{int}(F_j, G) \leq 0\] for all \(j\).
It is clear that
\[(2.4.1)\quad G - \lambda \text{ is generic for almost all } \lambda \in k\]
(where almost all means for all except a finite number of) and
\[(2.4.2)\quad G \text{ is generic } \iff \text{int}(F, G) = \text{minint}(F, G)\]
(2.4.3) $G$ generic $\iff \int(F,G) = \minint(F,G) \iff 0 \not\in \alpha(F,G)$

and

(2.4.4) $G$ generic $\Rightarrow \beta(F,G) = \sum_{\lambda \in \kappa} [\int(F,G - \lambda) - \int(F,G)]$

which motivates calling $\beta(F,G)$ the relative excess intersection of $G$.

**Section 3: The conductor-derivative formula**

Let $R_0 = k[[X,Y]]$. We say that $F = F(X,Y) \in R$ is $k$-distinguished if it is $k$-monic of $Y$-degree $N$ and, for $0 \leq i < N$, the coefficient of $Y^i$ in it has positive $X$-order; note that then $0 \neq F \in R_0$ and: $N > 0 \iff F(0,0) = 0$. For any $0 \neq F = F(X,Y) \in R_0$ with $F(0,0) = 0$, we let $B(F) = R_0/(FR_0)$ and call it the local ring of $F$; clearly $B(F)$ is a one-dimensional local ring. Recall that the conductor $C$ of a ring $B$ is the largest ideal in $B$ which remains an ideal in the integral closure $B^*$ of $B$ in its total quotient ring; the length of $C$ is the maximum length $n$ of strictly increasing chains of ideals $C = C_0 < C_1 < \cdots < C_n < B$ in $B$; if there is no maximum we take the length to be $\infty$; otherwise it is a nonnegative integer; note that $n = 0 \iff C = B \iff B = B^*$. For $0 \neq F = F(X,Y) \in R_0$ with $F(0,0) = 0$ we let $\delta(F)$ denote the length of the conductor of $B(F)$ and call it the conductor-length of $F$; note that if $F$ is $k$-distinguished with $\rad(F) = F$ then $\delta(F)$ is a nonnegative integer.

We shall now prove two lemmas and then we shall prove the conductor-derivative formula.

**Lemma (3.1).** Let $F = F(X,Y) \in R_0$ be $k$-distinguished of $Y$-degree $N \geq 0$ and let $G = G(X,Y) \in R_0$ be such that $G(0,0) = 0$. Then $\alpha(F,G) \subset \{0\}$ and $\beta(F,G) = 0$.

**PROOF.** In the notation of Section 1, $F$ distinguished $\Rightarrow z_i(X) \in k[[X]]$ with $z_i(0) = 0$ for $1 \leq i \leq N$, and hence $\alpha(F,G) \subset \{0\}$ and therefore $\beta(F,G) = 0$.

**Lemma (3.2).** Let $F = F(X,Y) \in R_0$ be $k$-distinguished of $Y$-degree $N > 0$. Then $\overline{\alpha}(F) \subset \{0\}$ and $\overline{\beta}(F) = 0$.

**PROOF.** Follows from (3.1).

**The conductor-derivative formula (3.3).** Let $F \in R_0$ be $k$-distinguished of $Y$-degree $N > 0$ with $\rad(F) = F$. Then

(3.3.1) $\int(F_X,F_Y) = \int(F,F_Y) - N + 1$

and

(3.3.2) $\int(F,F_Y) - N + 1 = 2\delta(F) - \chi(F) + 1$

and

(3.3.3) $\int(F_X,F_Y) = 2\delta(F) - \chi(F) + 1$

where all the terms in the above three equations are integers.
Section 4: The affine beta-jacobian identity

Let $R_2$ be the affine coordinate ring of the affine plane over $k$, i.e., let $R_2$ be the polynomial ring $k[X,Y]$, and note that then

$$R_2 = k[X,Y] \subset K[[X]][Y] = R_0 \cap R \subset K((X))(Y).$$

We are particularly interested in the case when $F(X,Y) = f(X^{-1}, Y)$ with $f(X,Y)$ in $R_2$; when this is so we shall say that $F$ is the \textbf{meromorphic associate} of $f$ or $f$ is the \textbf{polynomial associate} of $F$, and we indicate it by writing $F \sim_m f$ or $f \sim_p F$. Likewise when $F(X,Y) = f(X^{-1}, Y)$ and $G(X,Y) = g(X^{-1}, Y)$ we shall indicate it by writing $(F,G) \sim_m (f,g)$ or $(f,g) \sim_p (F,G)$.

Referring for details to Appendix I of Abhyankar’s Montreal Lecture Notes [Ab1] and Lectures 5 to 19 of Abhyankar’s Engineering Book [Ab4], let us review some definitions and facts about the \textbf{intersection multiplicity} $\int(f,g; A)$ of two plane curves $f$ and $g$ in the \textbf{affine plane} $\mathcal{A} = k^2$ over $k$, i.e., of two members $f = f(X,Y)$ and $g = g(X,Y)$ of $R_2$. Let

$$N = \deg_{(X,Y)} f \quad \text{and} \quad M = \deg_{(X,Y)} g$$

where $\deg_{(X,Y)} f$ denotes the $(X,Y)$-degree of $f$, i.e., the total degree of $f$; note that $N$ is a nonnegative integer or $-\infty$ according as $f \neq 0$ or $f = 0$. Let us write $\gcd(f,g) = 1$ or $\gcd(f,g) \neq 1$ according as the curves $f$ and $g$ do not or do have a \textbf{common component}, i.e., according as the polynomials $f$ and $g$ do not or do have a \textbf{nonconstant} (= not belonging to $k$) common factor in $R_2$. If $\gcd(f,g) \neq 1$ then we put $\int(f,g; \mathcal{A}) = \infty$. If $\gcd(f,g) = 1$ then the curves $f$ and $g$ have a finite number of common points $Q_i = (u_i, v_i)$ for $1 \leq i \leq n$ in $\mathcal{A}$, i.e., $f(u_i, v_i) = 0 = g(u_i, v_i)$ for $1 \leq i \leq n$, and we put

$$\int(f,g; \mathcal{A}) = \sum_{1 \leq i \leq n} \int(f,g; Q_i)$$

where for any $Q = (u, v) \in \mathcal{A}$ we define the \textbf{intersection multiplicity} $\int(f,g; Q)$ of $f$ and $g$ at $Q$ thus. Given any $Q = (u, v) \in \mathcal{A}$, if $f \neq 0$ then by the Weierstrass Preparation Theorem we can uniquely write

$$f(X + u, Y + v) = \tilde{f}_Q(X,Y)f_Q(X,Y)$$

where

$$\tilde{f}_Q(X,Y) \in k[[X,Y]] \text{ with } \tilde{f}_Q(0,0) \neq 0$$

and

$$f_Q(X,Y) = X^aY^b + c_1(X)Y^{b-1} + \cdots + c_b(X)$$

with nonnegative integers $a, b$ and elements $c_1(X), \ldots, c_b(X)$ in $k[[X]]$ for which $c_1(0) = \cdots = c_b(0) = 0$, and we call $f_Q = f_Q(X,Y) \in k[[X,Y]]$ the \textbf{incarnation} of $f$ at $Q$; if $f = 0$ then we put $f_Q = f_Q(X,Y) = 0$. Now for any $Q \in \mathcal{A}$ we define

$$\int(f,g; Q) = \int(f_Q,g_Q).$$
It can be shown that in terms of \( k \)-vector space dimension we always have
\[
\text{int}(f, g; \mathcal{A}) = [R_2/(f, g)R_2 : k].
\]
(4.1)

To extend the above discussion to the **intersection multiplicity** \( \text{int}(f, g; \mathcal{P}) \) of \( f \) and \( g \) in the **projective plane** \( \mathcal{P} \) over \( k \), recall that \( \mathcal{P} \) is the disjoint union of \( \mathcal{A} \) with the line at infinity \( \mathcal{L}_\infty \) given by
\[
\mathcal{L}_\infty = \{ (\infty, v) : v \in k \cup \{ \infty \} \}.
\]
If \( f \neq 0 \) then we define its **homogenization** \( f_h = f_h(X, Y, Z) \in k[X, Y, Z] \) by putting
\[
f_h(X, Y, Z) = Z^N f(X/Z, Y/Z)
\]
and if \( f = 0 \) then we put \( f_h = f_h(X, Y, Z) = 0 \). For any \( Q = (\infty, b) \in \mathcal{L}_\infty \) we define the **incarnation** \( f_Q = f_Q(X, Y) \in k[[X, Y]] \) of \( f \) at \( Q \) by putting
\[
f_Q(X, Y) = \begin{cases} (f_h(1, Y, X))(0, b) & \text{if } b \in k \\ (f_h(X, 1, Y))(0, b) & \text{if } b = \infty \end{cases}
\]
and we define \( \text{int}(f, g; Q) \) by putting
\[
\text{int}(f, g; Q) = \text{int}(f_Q, g_Q).
\]

Now \( Q = (u, v) \in \mathcal{A} \) is a common point of \( f \) and \( g \), i.e., \( f(u, v) = g(u, v) = 0 \), iff \( = (\text{if and only if}) f_Q(0, 0) = 0 = g_Q(0, 0) \); by analogy we call \( Q \in \mathcal{L}_\infty \) a **common point** of \( f \) and \( g \) iff \( f_Q(0, 0) = g_Q(0, 0) \). If \( \gcd(f, g) \neq 1 \) then we put \( \text{int}(f, g; \mathcal{P}) = \infty \).

If \( \gcd(f, g) = 1 \) then the curves \( f \) and \( g \) have a finite number of common points \( Q_i = (u_i, v_i) \) for \( 1 \leq i \leq m \) in \( \mathcal{P} \), i.e., \( f_Q_i(0, 0) = g_Q_i(0, 0) \) for \( 1 \leq i \leq m \), and we put
\[
\text{int}(f, g; \mathcal{P}) = \sum_{1 \leq i \leq m} \text{int}(f, g; Q_i).
\]

Note that by **Bezout’s Theorem**
\[
\text{int}(f, g; \mathcal{P}) = \begin{cases} NM & \text{if } \gcd(f, g) = 1 \text{ and } f \neq 0 = g \\ 0 & \text{if } \gcd(f, g) = 1 \text{ and } f = 0 \neq g \text{ or } f \neq 0 = g \\ \infty & \text{if } \gcd(f, g) \neq 1. \end{cases}
\]
(4.2)

For simplicity of notation let us put
\[
(\infty) = (\infty, 0) \in \mathcal{L}_\infty.
\]

We say that \( f \) is **\( Y \)-monic** to mean that
\[
f \neq 0 \text{ and } \deg_{(X,Y)}[f(X,Y) - f_0 Y^N] < N \text{ for some } 0 \neq f_0 \in k
\]
and we note that
\[
f \text{ is } Y \text{-monic } \Rightarrow \text{ the } (X, Y)\text{-degree of } f \text{ coincides with its } Y\text{-degree}
\]
and
\[
f \text{ is } Y \text{-monic } \Leftrightarrow \{ Q \in \mathcal{L}_\infty : f_Q(0, 0) = 0 \} \subset \{ (\infty) \}.
\]

It can be shown that
\[
\begin{cases} \text{if } (F, G) \sim_m (f, g) \text{ where } f \text{ is } Y \text{-monic and } \gcd(f, g) = 1, \\
\text{then } \text{int}(F, G) = -\text{int}(f, g; \mathcal{A}). \end{cases}
\]
(4.3)

If \( f \neq 0 \) and \( \deg_{(X,Y)}[f(X,Y) - f_0 Y^N] < N \) for some \( 0 \neq f_0 \in k \), then by the **top coefficient** of \( f \) we mean the nonzero element of \( k \) which is the
coefficient of the highest power of \(X\) in \(f_0\); if \(f = 0\) then we take 0 to be the
top coefficient of \(f\). If \(f \in k\) then we define the radical of \(f\) in \(A\) by putting
\(\text{rad}(f; A) = f\), and if \(f \not\in k\) then we factor \(f\) as a product of positive powers of
mutually nonassociate irreducible members of \(R_2\) by writing \(f = f_1^{e_1} \cdots f_r^{e_r}\) and
we put \(\text{rad}(f; A) = cf_1 \cdots f_r\) where \(0 \neq c \in k\) is so chosen that \(f\) and \(\text{rad}(f; A)\)
have the same top coefficient.

In analogy with the alpha and beta invariants, for any \(f, g\) in \(R_2\) we define the
affine alpha invariant \(\alpha(f, g; A)\) and the affine beta invariant \(\beta(f, g; A)\) of \(f\)
relative to \(g\), and in analogy with the alphabar and beta-bar invariants, for any
\(f \in R_2\) we define the affine alphabar invariant \(\overline{\alpha}(f; A)\) and the affine betabar
invariant \(\overline{\beta}(f; A)\) of \(f\) thus.

First, for any \(f, g\) in \(R_2\), we define the maximal-intersection of \(f\) with \(g\) by
putting
\[
\text{maxint}(f, g; A) = \max_{\mu \in k} \text{int}(f, g - \mu; A)
\]
(where max stands for lub = least upper bound) and we note that by Bezout
\[
\begin{align*}
\text{maxint}(f, g; A) &= \text{a nonnegative integer or } \infty, \\
\text{and: maxint}(f, g; A) &= \infty \iff \gcd(f, g - c; A) \neq 1 \text{ for some } c \in k
\end{align*}
\]
and we put
\[
\alpha(f, g; A) = \{\lambda \in k : \text{int}(f, g - \lambda; A) < \text{maxint}(f, g; A)\}
\]
and
\[
\beta(f, g; A) = \sum_{0 \neq \lambda \in \alpha(f, g; A)} \left[\text{maxint}(f, g; A) - \text{int}(f, g; A)\right]
\]
and we note that then
\[
\beta(f, g; A) = \text{a nonnegative integer or } \infty.
\]

By a relative affine irregular value of \(g\) (on \(f\)) we mean an element of \(\alpha(f, g; A)\).
We call \(\alpha(f, g; A)\) the relative affine irregular value set of \(g\) (on \(f\)), and we call
\(\beta(f, g; A)\) the relative deficit intersection of \(g\) (on \(f\)).

Next, for any \(f\) in \(R_2\), we put
\[
\overline{\alpha}(f; A) = \alpha(f_Y, f; A)
\]
and
\[
\overline{\beta}(f; A) = \beta(f_Y, f; A).
\]

By an affine irregular value of \(f\) we mean an element of \(\overline{\alpha}(f; A)\). We call \(\overline{\alpha}(f; A)\)
the affine irregular value set of \(f\), and we call \(\overline{\beta}(f; A)\) the deficit intersection of \(f\).

As a consequence of (4.3) we see that:

\[
\begin{align*}
\text{if } (F, G) &\sim_m (f, g) \text{ where } f \text{ is } Y\text{-monic} \\
\text{and } \gcd(f, g - c) &= 1 \text{ for all } c \in k, \\
\text{then } \alpha(f, g; A) &= \alpha(F, G) = \text{a finite set} \\
\text{and } \beta(f, g; A) &= \beta(F, G) = \text{a nonnegative integer}.
\end{align*}
\]
and

\[
\begin{cases}
\text{if } F \sim_m f \text{ where } f \text{ is } Y\text{-monic of } Y\text{-degree } N > 0 \\
\quad \text{and } \gcd(f, f - c) = 1 \text{ for all } c \in k, \\
\quad \text{then } \overline{\pi}(f; A) = \overline{\pi}(F) = \text{a finite set} \\
\quad \text{and } \overline{\beta}(f; A) = \overline{\beta}(F) = \text{a nonnegative integer}.
\end{cases}
\] (4.5)

Now as consequences of (2.1), (2.2), (2.3) we shall deduce (4.6), (4.7), (4.8) respectively.

The affine beta-jacobian identity (4.6). Let \( f \in R_2 \) be \( Y\)-monic of \( Y\)-degree \( N \), and let \( g \in R_2 \) be such that \( \gcd(f, g - c) = 1 \) for all \( c \in k \). Then for \( e = \rad(f; A) \) we have

\[\int(f, g; A) + \int(f, eY; A) = \int(f, J(e, g); A) + \beta(f, g; A) + N\]

where each term is a nonnegative integer.

PROOF. Let \((F, G) \sim_m (f, g)\); then \( F \in R \) is \( k\)-monic of \( Y\)-degree \( N \), and \( G \in R \) is such that \( \gcd(F, G - c) = 1 \) for all \( c \in k \). Let \( E \sim_m e \); then \( E = \rad(F) \) and \( EY \sim_m eY \). Let \( J = J(E, G); \) also let \( J' = J(e, g) \) and \( J' \sim_m j' \); then \( J' \) is the jacobian of \( E \) and \( G \) relative to \( X^{-1} \) and \( Y \) and hence by the chain rule for jacobians we see that \( J \) equals \( J' \) times the jacobian of \( X^{-1} \) and \( Y \) relative to \( X \) and \( Y \) which is clearly equal to \( -X^{-2} \). Also obviously \( \int(F, -X^{-2}) = -2N \) and \( \int(F, J) = \int(F, -X^{-2}J') = \int(F, -X^{-2}) + \int(F, J') \). Therefore \( \int(F, J) = -2N + \int(F, J') \), and substituting this in (2.1) we get

\[-2N + \int(F, J') = \int(F, G) + \int(F, EY) + N + \beta(F, G)\]

and by sending some terms from one side to the other we obtain

\[-\int(F, G) - \int(F, EY) = -\int(F, J') + \beta(F, G) + N\]

and in view of (4.3) and (4.4) this yields the desired result.

The affine beta-derivative identity (4.7) Let \( f \in R_2 \) be \( k\)-monic of \( Y\)-degree \( N \), and let \( g \in R_2 \) be such that \( \gcd(f, g - c) = 1 \) for all \( c \in k \) and, in the notation of Section 1, for \((F, G) \sim_m (f, g)\) we have \( G_Y \in F_jR \) for \( 1 \leq j \leq \chi(F) \). Then

\[\int(f, g; A) = \int(f, gX; A) + \beta(f, g; A) + N\]

where each term is a nonnegative integer.

PROOF. Now \( F \in R \) is \( k\)-monic of \( Y\)-degree \( N \), and \( G \in R \) is such that \( \gcd(F, G - c) = 1 \) for all \( c \in k \) and, in the notation of Section 1, \( G_Y \in F_jR \) for \( 1 \leq j \leq \chi(F) \). Let \( g' = gX \) and \( G' \sim_m g' \); then \( G' \) is the derivative of \( G \) with respect to \( X^{-1} \) and hence by the chain rule for derivatives we see that \( G_X \) equals \( G' \) times the derivative of \( X^{-1} \) with respect to \( X \) which is clearly equal to \( -X^{-2} \). Also obviously \( \int(F, -X^{-2}) = -2N \) and \( \int(F, G_X) = \int(F, -X^{-2}G') = \int(F, -X^{-2}) + \int(F, G') \). Therefore \( \int(F, G_X) = -2N + \int(F, G') \), and substituting this in (2.2) we get

\[-2N + \int(F, G') = \int(F, G) - N + \beta(F, G)\]
and by sending some terms from one side to the other we obtain
\[-\text{int}(F, G) = -\text{int}(F, G') + \beta(F, G) + N\]
and in view of (4.3) and (4.4) this yields the desired result.

**The affine betabar-derivative identity (4.8).** Let \( f \in R_2 \) be \( Y \)-monic of \( Y \)-degree \( N > 0 \) with \( \gcd(f_Y, f - c) = 1 \) for all \( c \in k \). Then we have
\[\text{int}(f, f_Y; A) = \text{int}(f_X, f_Y; A) + \beta(f; A) + (N - 1)\]
where each term is a nonnegative integer.

**PROOF.** Let \( F \sim_m f \); then \( F \in R \) is \( k \)-monic of \( Y \)-degree \( N > 0 \) with \( \gcd(F, G - c) = 1 \) for all \( c \in k \); also clearly \( F_Y \sim_m f_Y \). Let \( f' = f_X \) and \( F' \sim_m f' \); then \( F' \) is the derivative of \( F \) with respect to \( X^{-1} \) and hence by the chain rule for derivatives we see that \( F_Y \) equals \( F' \) times the derivative of \( X^{-1} \) with respect to \( X \) which is clearly equal to \( -X^{-2} \). Also obviously \( \text{int}(-X^{-2}, F_Y) = -2(N - 1) \) and \( \text{int}(F_X, F_Y) = \text{int}(-X^{-2}F', F_Y) = \text{int}(-X^{-2}, F_Y) + \text{int}(F', F_Y) \). Therefore \( \text{int}(F_X, F_Y) = -2(N - 1) + \text{int}(F', F_Y) \), and substituting this in (2.3) we get
\[-2(N - 1) + \text{int}(F', F_Y) = \text{int}(F, F_Y) - N + 1 + \beta(F)\]
and by sending some terms from one side to the other we obtain
\[-\text{int}(F, F_Y) = -\text{int}(F', F_Y) + \beta(F) + (N - 1)\]
and in view of (4.3) and (4.5) this yields the desired result.

As a consequence of (4.8) we shall now prove:

**The projective betabar-derivative identity (4.9).** Let \( f \in R_2 \) be \( Y \)-monic of \( Y \)-degree \( N > 0 \) with \( \gcd(f_Y, f - c) = 1 \) for all \( c \in k \). Then we have
\[\text{int}(f, f_Y; P) = \text{int}(f_X, f_Y; P) + \beta(f; A) + 2(N - 1)\]
where each term is a nonnegative integer.

**PROOF.** By (3.2) we have \( \beta(f_{(\infty)}) = 0 \) and hence by the betabar-derivative identity (2.3) we get
\[\text{int}(f_{(\infty)}, f_{(\infty)Y}) = \text{int}(f_{(\infty)X}, f_{(\infty)Y}) + (N - 1)\]
and adding this to (4.8) we obtain the desired result by noting that obviously \( f_Y \) is \( Y \)-monic and hence
\[(4.9.1) \quad \text{int}(f, f_Y; A) + \text{int}(f_{(\infty)}, f_{(\infty)Y}) = \text{int}(f, f_Y; P)\]
and
\[(4.9.2) \quad \text{int}(f_X, f_Y; A) + \text{int}(f_{(\infty)X}, f_{(\infty)Y}) = \text{int}(f_X, f_Y; P).\]

Finally as a consequence of (4.9) we shall prove:

**The betabar-conductor identity (4.10).** Let \( f \in R_2 \) be \( Y \)-monic of \( Y \)-degree \( N > 0 \) with \( \gcd(f_Y, f - c) = 1 \) for all \( c \in k \). Then we have
\[(N - 1)(N - 2) + [\chi(f_{(\infty)}) - 1] = \text{int}(f_X, f_Y; A) + 2\beta(f_{(\infty)}) + \beta(f; A)\]
where each term is a nonnegative integer.
NOTE. We regard \((N - 1)(N - 2)\) as a term. Also we regard a square bracketed expression as a term. For instance \([\chi(f_1) - 1]\) is a term.

PROOF. By Bezout we have \(\text{int}(f, f_Y; P) = N(N - 1)\), and hence by (4.9) and (4.9.2) we get

\[(N - 1)(N - 2) = \text{int}(f_X, f_Y; A) + \text{int}(f_\infty X, f_\infty Y) + \mathfrak{m}(f; A)\]

and from this our assertion follows by using (3.3.3) applied to \(F = f_\infty\).

Section 5: No Irregular Value

For nonconstant \(f \in R_2\) we let \(B(f; A) = R_2/(fR_2)\) and call it the affine coordinate ring of \(f\). Clearly \((fR_2) \cap k = \{0\}\) and hence we may identify \(k\) with a subfield of \(B(f; A)\). We define \(\delta(f; A)\) and \(\delta(f; P)\) by putting

\[\delta(f; A) = \sum_{Q \in A} \delta(f_Q)\]

and

\[\delta(f; P) = \sum_{Q \in P} \delta(f_Q)\]

and we note that if \(\text{rad}(f; A) = f\) then these are nonnegative integers; otherwise they are \(\infty\). In case \(f\) is irreducible (in \(R_2\)), the genus of the quotient field of \(B(f; A)\) over \(k\) is a nonnegative integer which we denote by \(\gamma(f)\). Again if \(f\) is irreducible then by the number of places of \(f\) at infinity, denoted by \(\chi_\infty(f)\), we mean the number of DVRs (= discrete valuations rings = one dimensional regular local rings) which do not contain \(B(f; A)\) but whose quotient field coincides with the quotient field of \(B(f; A)\). In the general case, by factoring \(f\) as a product of irreducible factors, i.e., writing \(f = f_1 \cdots f_r\) where \(f_1, \ldots, f_r\) are irreducible members of \(R_2\), we call \(\chi_\infty(f_1) + \cdots + \chi_\infty(f_r)\) the number of places of \(f\) at infinity, and denote it by \(\chi_\infty(f)\); note that this is always a positive integer; also note that

\[
\begin{align*}
\text{if } f & \text{ is Y-monic then } \chi_\infty(f) = \chi(f_\infty) \text{ and} \\
\text{ upon letting } F \sim_m f & \text{ we have } \chi(F) = \chi_\infty(f).
\end{align*}
\]

Let us call \(f\) a uniline if \(f\) is irreducible with \(\gamma(f) = 0 = \delta(f; A)\) and \(\chi_\infty(f) = 1\), and let us call \(f\) a unihyperbola if \(f\) is irreducible with \(\gamma(f) = 0 = \delta(f; A)\) and \(\chi_\infty(f) = 2\). Also let us call \(f\) a multihyperbola if in the above notation we have \(\text{int}(f_i, f_j; A) = 0\) for \(1 \leq i < j \leq r\) and \(f_i\) is a unihyperbola for \(1 \leq i \leq r\), and let us call \(f\) a multihyperbolic line if in the above notation we have \(\text{int}(f_i, f_j; A) = 0\) for \(1 \leq i < j \leq r\), \(f_j\) is a uniline for some \(j \in \{1, \ldots, r\}\), and \(f_i\) is a unihyperbola for all \(i \in \{1, \ldots, j - 1, j + 1, \ldots, r\}\). Clearly

\[
\begin{align*}
f & \text{ is a uniline } \Leftrightarrow B(f; A) \approx k[X] \\
& \text{ } \Leftrightarrow f \text{ is an irreducible multihyperbolic line}
\end{align*}
\]

and

\[
\begin{align*}
f & \text{ is a unihyperbola } \Leftrightarrow B(f; A) \approx k[X, X^{-1}]
\end{align*}
\]
where ≈ denotes \( k \)-isomorphism of rings. By the \textbf{epimorphism theorem} (see \textit{Ab2}) we know that

\[
\begin{aligned}
(5.1) & \quad \begin{cases}
 f \text{ is a uniline} \\
 \iff 
 \text{for some } k \text{-automorphism } \sigma \text{ of } R_2 \text{ we have } \sigma(f) = X
\end{cases}
\end{aligned}
\]

and

\[
\begin{aligned}
(5.2) & \quad \begin{cases}
 f \text{ is a multihyperbolic line which is not a uniline} \\
 \iff 
 \text{for some } k \text{-automorphism } \sigma \text{ of } R_2 \\
 \quad \quad \text{we have } \sigma(f) = X(1 + X h(X, Y)) \\
 \quad \quad \text{where } 0 \neq h(X, Y) \in R_2 \text{ is such that} \\
 \quad \quad \quad \quad 1 + X h(X, Y) \text{ is a multihyperbola.}
\end{cases}
\end{aligned}
\]

The well-known \textbf{genus formula} tells us that

\[
(5.3) \quad \begin{cases}
 \text{if } f \in R_2 \text{ is irreducible of } (X, Y)\text{-degree } N > 0 \text{ then} \\
 2\gamma(f) + 2\delta(f; \mathcal{P}) = (N - 1)(N - 2).
\end{cases}
\]

Likewise, continuing the assumption of nonconstant \( f \in R_2 \), a well-known consequence of \textbf{Bertini’s Theorem} tells us that

\[
(5.4) \quad \begin{cases}
 \text{rad}(f - c; \mathcal{A}) = f - c \text{ for all } c \in k \\
 \implies f - \lambda \text{ is irreducible for almost all } \lambda \in k
\end{cases}
\]

and as supplements to this we note the easy to prove facts which say that

\[
(5.5) \quad \begin{cases}
 \text{rad}(f - c; \mathcal{A}) = f - c \text{ for all } c \in k \\
 \iff \text{int}(f_X, f_Y; \mathcal{A}) < \infty
\end{cases}
\]

and

\[
(5.6) \quad \begin{cases}
 \text{int}(f_X, f_Y; \mathcal{A}) = 0 \implies \text{int}(f_X, f_Y; \mathcal{A}) < \infty, \\
 \quad \text{and} \\
 \text{int}(f_X, f_Y; \mathcal{A}) = 0 \iff \delta(f - c; \mathcal{A}) = 0 \text{ for all } c \in k
\end{cases}
\]

and

\[
(5.7) \quad \begin{cases}
 \text{if } f \text{ is } Y\text{-monic then:} \\
 \text{rad}(f - c; \mathcal{A}) = f - c \text{ for all } c \in k \\
 \iff \gcd(f_Y, f - c; \mathcal{A}) = 1 \text{ for all } c \in k.
\end{cases}
\]

We shall now deduce some consequences of (5.1) to (5.7).

\textbf{The betabar-genus identity (5.8).} \textit{Let } \( f \in R_2 \text{ be irreducible } Y\text{-monic of } Y\text{-degree } N > 0 \text{ with } \gcd(f_Y, f - c) = 1 \text{ for all } c \in k. \text{ Then}

\[
(5.8.1) \quad 2\gamma(f) + 2\delta(f; \mathcal{A}) + [\chi(f(\infty)) - 1] = \text{int}(f_X, f_Y; \mathcal{A}) + \beta(f; \mathcal{A})
\]

\textit{where each term is a nonnegative integer. Moreover,

\[
(5.8.2) \quad \begin{cases}
 \text{if int}(f_X, f_Y; \mathcal{A}) = 0 \\
 \text{then } 2\gamma(f) + [\chi(f(\infty)) - 1] = \beta(f; \mathcal{A})
\end{cases}
\]
where each term is a nonnegative integer. Finally,

\[(5.8.3) \begin{cases} 
\text{if } \int(f_X, f_Y; A) = 0 = \overline{\beta}(f; A) \\
\text{then } f \text{ is a uniline.}
\end{cases}\]

**PROOF.** \((5.8.1)\) follows from \((4.10)\) and \((5.3)\) by noting that \(\delta(f; P) = \delta(f; A) + \delta(f;_{\infty})\). Moreover, \((5.8.2)\) follows from \((5.8.1)\) by noting that \(\int(f_X, f_Y; A) = 0 \Rightarrow \delta(f; A) = 0\). Since \(\gamma(f)\) and \(\chi(f;_{\infty}) - 1\) are nonnegative integers, it follows that if \(2\gamma(f) + [\chi(f;_{\infty}) - 1] = 0\) then we must have \(\gamma(f) = 0 = [\chi(f;_{\infty}) - 1]\); therefore, since \(\int(f_X, f_Y; A) = 0 \Rightarrow \delta(f; A) = 0\), \((5.8.3)\) follows from \((5.8.2)\).

**No Irregular Value Theorem (5.9).** Let \(f \in R_2\) be \(Y\)-monic of \(Y\)-degree \(N > 0\). Then

\[\int(f_X, f_Y; A) = 0 = |\overline{\alpha}(f; A)| \iff f \text{ is a uniline.}\]

\[\iff f - c \text{ is a uniline for all } c \in k.\]

**PROOF.** We shall give a circular proof by showing that LHS \(\Rightarrow\) MHS \(\Rightarrow\) RHS \(\Rightarrow\) LHS where MHS = Middle Hand Side = the condition “\(f\) is a uniline.” Assuming \(\int(f_X, f_Y; A) = 0\), by \((5.4)\) to \((5.7)\) there exists \(\lambda \in k\) such that for \(f' = f - \lambda\) we have that \(f' \in R_2\) is irreducible \(Y\)-monic of \(Y\)-degree \(N > 0\) such that \(\int(f'_X, f'_Y) = 0\) and \(\gcd(f'_X, f'_Y) = 1\) for all \(c \in k\); also assuming \(\overline{\alpha}(f; A) = 0\), we see that \(\overline{\beta}(f'; A) = 0\), and hence \(f'\) is a uniline by \((5.8.3)\), and therefore \(f\) is a uniline by \((5.1)\). By \((5.1)\) we know that if \(f\) is a uniline then \(f - c\) is a uniline for all \(c \in k\). Finally, upon letting \(f_c = f - c\), assume that \(f_c\) is a uniline for all \(c \in k\). Then, for any \(c \in k\), clearly \(f_c \in R_2\) is irreducible \(Y\)-monic of \(Y\)-degree \(N > 0\) with \(\gamma(f_c) = 0 = [\chi(f_c;_{\infty}) - 1]\), and by \((5.5)\) to \((5.7)\) we see that \(\int(f_X, f_Y) = 0\) and \(\gcd(f_X, f - c') = 1\) for all \(c' \in k\), and hence by \((5.8.2)\) we get \(\overline{\beta}(f_c; A) = 0\). Therefore \(\int(f_X, f_Y; A) = 0 = |\overline{\alpha}(f; A)|\).

**Remark (5.10).** As hinted in the above proof, from the definitions of the affine invariants \(\overline{\alpha}, \overline{\beta}, \alpha, \beta\) it immediately follows that for any \(f\) in \(R_2\) we have

\[\overline{\alpha}(f; A) \subset \{0\} \iff \overline{\beta}(f; A) = 0\]

and

\[\overline{\alpha}(f; A) = \emptyset \iff \overline{\beta}(f - c; A) = 0\]

for all \(c \in k\)

and for any \(f, g\) in \(R_2\) we have

\[\alpha(f, g; A) \subset \{0\} \iff \beta(f, g; A) = 0\]

and

\[\alpha(f, g; A) = \emptyset \iff \beta(f, g - c; A) = 0\]

for all \(c \in k\).

**Section 6: One Irregular Value**

Here are some consequences of \((5.1)\) to \((5.3)\) and \((5.5)\) to \((5.7)\) which do not use \((5.4)\).
**Product Identity (6.1).** Let $f \in R_2$ be $Y$-monic of $Y$-degree $N > 0$ with $\gcd(f_Y, f) = 1$. Let $f = \prod_{1 \leq i \leq r} f_i$ be a factorization of $f$ where $f_i \in R_2$ is $Y$-monic of $Y$-degree $N_i > 0$ for $1 \leq i \leq r$. Then

$$\int(f, f_Y; A) - N = 2 \sum_{1 \leq i < j \leq r} \int(f_i, f_j; A)$$

$$+ \sum_{1 \leq i \leq r} [(N_i - 1)(N_i - 2) - 2\delta(f_i(\infty)) - \chi(f_i(\infty)) - 2]$$

(6.1.1)

where each term is an integer. Moreover,

$$\begin{cases}
\text{if } \int(f_X, f_Y; A) = 0 \text{ and } f_i \text{ is irreducible for } 1 \leq i \leq r \\
\text{then } \int(f, f_Y; A) - N = \sum_{1 \leq i \leq r} [2\gamma(f_i) + \chi(f_i(\infty)) - 2] \\
\text{and } \delta(f_i; A) = 0 \text{ for } 1 \leq i \leq r
\end{cases}$$

(6.1.2)

where each term is an integer.

**PROOF.** By the product rule for derivatives we get

$$\int(f, f_Y; A) = 2 \sum_{1 \leq i < j \leq r} \int(f_i, f_j; A) + \sum_{1 \leq i \leq r} \int(f_i, f_Y; A)$$

and for $1 \leq i \leq r$ we have

$$\int(f_i, f_Y; A) = N_i(N_i - 1) - \int(f_i(\infty), f_i(\infty)Y) \quad \text{by Bezout}$$

$$= N_i(N_i - 1) - 2\delta(f_i(\infty)) + \chi(f_i(\infty)) - N_i \quad \text{by (3.3.2)}$$

$$= (N_i - 1)(N_i - 2) - 2\delta(f_i(\infty)) + \chi(f_i(\infty)) + (N_i - 2) \quad \text{by simplifying}$$

and by substituting this value of $\int(f_i, f_Y; A)$ in the RHS of the above equation we get (6.1.1) by noting that $N = \sum_{1 \leq i \leq r} N_i$. Moreover, if $\int(f_X, f_Y) = 0$ then for $1 \leq i < j \leq r$ we have $\int(f_i, f_j; A) = 0$, and for $1 \leq i \leq r$ we have $\delta(f_i; A) = 0$ and $\delta(f_i(\infty)) = \delta(f_i; P)$, and hence (6.1.2) follows from (5.3) and (6.1.1).

**Product Lemma (6.2).** Let $f \in R_2$ be $Y$-monic of $Y$-degree $N > 0$ with $\int(f, f_Y; A) = N - 1$ and $\int(f_X, f_Y; A) = 0$. Then $f$ is a multihyperbolic line.

**PROOF.** By (6.1.2) we have $-1 = \sum_{1 \leq i \leq r} [2\gamma(f_i) + \chi(f_i(\infty)) - 2]$ and $\delta(f_i; A) = 0$ for $1 \leq i \leq r$. Therefore, since $\gamma(f_i)$ and $\chi(f_i(\infty)) - 1$ are nonnegative integers for $1 \leq i \leq r$, just by numerical considerations, we conclude that there is a unique $j \in \{1, \ldots, r\}$ such that $\gamma(f_j) = 0 = \chi(f_j(\infty)) - 1$ and $\gamma(f_j) = 0 = \chi(f_j(\infty)) - 2$ for all $i \in \{1, \ldots, j - 1, j + 1, \ldots, r\}$. Thus $f$ is a multihyperbolic line.

**Product Theorem (6.3).** Let $f \in R_2$ be $Y$-monic of $Y$-degree $N > 0$ such that $\int(f_X, f_Y; A) = 0 = \beta(f; A)$ and $\gcd(f_Y, f - c; A) = 1$ for all $c \in k$. Then $\int(f, f_Y; A) = N - 1$ and $f$ is a multihyperbolic line.

**PROOF.** Follows from (4.8) and (6.2).
One Irregular Value Theorem (6.4). Let $f \in \mathbb{R}^2$ be $Y$-monic of $Y$-degree $N > 0$ with $\text{int}(f_X, f_Y; \mathcal{A}) = 0 = |\overline{\alpha}(f; \mathcal{A})| - 1$, and let $f' = f - \lambda$ with $\{\lambda\} = \overline{\alpha}(f; \mathcal{A})$. Then $f'$ is a multihyperbolic line which is not a uniline.

PROOF. Since $\text{int}(f_X, f_Y; \mathcal{A}) = 0$, by (5.5) to (5.7) we see that $f' \in \mathbb{R}^2$ is $Y$-monic of $Y$-degree $N > 0$ such that $\text{int}(f'_X, f'_Y; \mathcal{A}) = 0$ and $\gcd(f'_Y, f' - c; \mathcal{A}) = 1$ for all $c \in k$. Since $|\overline{\alpha}(f; \mathcal{A})| - 1 = 0$ and $f' = f - \lambda$ with $\{\lambda\} = \overline{\alpha}(f; \mathcal{A})$, we also see that $\overline{\beta}(f'; \mathcal{A}) = 0$. Therefore by (6.3) we conclude that $f'$ is a multihyperbolic line, and by (5.9) we see that it is not a uniline.

Remark (6.5). In the proof of (6.1) we could have tried to use (4.8) by assuming the condition $\gcd(f_Y, f - c) = 1$ for all $c \in k$. tried to use (4.8) in its proof. This did not work because, as shown by the following example, this condition is not inherited by the factors of $f$. Namely, by taking $f = f_1f_2$ with $f_1 = X^2Y + 1$ and $f_2 = Y$, we see that $\gcd(f_Y, f - c) = 1$ for all $c \in k$ does not imply $\gcd(f_1Y, f_1 - c) = 1$ for all $c \in k$.

Section 7: Two Conjectures

Let
\[ k^\times = k \setminus \{0\} \]
and, as in Abhyankar’s previous lectures, let $\oplus$ denote an unspecified element of $k^\times$. Now here are two meromorphic jacobian conjectures for $F, G$ in $\mathbb{R}$ of $Y$-degrees $N$ and $M$ respectively.

CONJECTURE I: $J(F, G) = \oplus X^{-2} \Rightarrow \beta(F, G) = 0 \notin \overline{\alpha}(F)$.

CONJECTURE II: $J(F, G) = \oplus X^{-2} \Rightarrow$ either $M|N$ or $N|M$.

Remark (7.1). Note that Conjecture II is not true if we allow the coefficients of $F$ and $G$ to be fractional meromorphic series. For example, if $F = Y^3 + (3/2)X^{-1/2}Y$ and $G = Y^2 + X^{-1/2}$ then $J(F, G) = (-3/4)X^{-2}$ but $M = 2$ and $N = 3$.

Remark (7.2). In Remark (8.9) of the next section we shall show that both these meromorphic conjectures imply the algebraic jacobian conjecture which predicts that if $f, g$ in $\mathbb{R}_2$ are such that $J(f, g) = \oplus$ then $k[f, g] = \mathbb{R}_2$.

Section 8: Thoughts on Conjecture I

Before turning to Conjecture I, let us observe that for any $f, g$ in $\mathbb{R}_2$ we have

\begin{align*}
\text{(8.1)} \quad J(f, g) = \oplus \text{ and } f \text{ is a uniline } & \Rightarrow k[f, g] = \mathbb{R}_2 \\
\text{(8.2)} \quad J(f, g) = \oplus \text{ and } f \text{ is a multihyperbolic line } & \Rightarrow f \text{ is a uniline}
\end{align*}
and

\[(8.3)\quad k[f, g] = R_2 \Rightarrow |\alpha(f, g; A)| = 0.\]

Out of this (8.1) and (8.2) follow from the parallelness of the Newton polygons of \(f\) and \(g\) proved in \[Ab3, Ab5\]. The third assertion (8.3) follows by noting that if \(k[f, g] = R_2\) then for every \(c \in k\) we clearly have \(k[f, g - c] = R_2\) and hence \(\text{int}(f, g - c; A) = 1\), and therefore \(|\alpha(f, g; A)| = 0\).

Now let us prove the following two polynomial analogue of (5.8.3).

**No Deficit Intersection Theorem (8.4).** Let \(f \in R_2\) be \(Y\)-monic of \(Y\)-degree \(N > 0\), and let \(g \in R_2\) be such that \(J(f, g) = \Phi\) and \(\beta(f, g; A) = 0 \notin \overline{\pi}(f; A)\). Then \(f\) is a uniline, and hence in particular \(k[f, g; A] = 1\); therefore, \(|\alpha(f, g; A)| = 0\).

**PROOF.** Since \(J(f, g) = \Phi\), we get \(\text{rad}(f; A) = f\) and \(\text{int}(f, J(f, g); A) = 0\) with \(\gcd(f, g - c; A) = 1\) for all \(c \in k\). Therefore, since \(\beta(f, g; A) = 0\), by (4.6) we obtain

\[\text{int}(f, g; A) + \text{int}(f, f_Y; A) = N.\]

Since \(J(f, g) = \Phi\), we also get \(\text{int}(f_X, f_Y; A) = 0\) with \(\gcd(f_Y, f - c; A) = 1\) for all \(c \in k\). Therefore by (4.8) we obtain

\[\text{int}(f, f_Y; A) = \overline{\beta}(f; A) + (N - 1).\]

The above two displays tell us that

\[\text{int}(f, g; A) + \overline{\beta}(f; A) = 1.\]

Since \(\text{int}(f, g; A)\) and \(\overline{\beta}(f; A)\) are nonnegative integers, we must have \(\overline{\beta}(f; A) = 0\) or \(1\). For a moment suppose that \(\overline{\beta}(f; A) = 1\); then, since \(0 \notin \overline{\pi}(f; A)\), we must have \(|\overline{\pi}(f; A)| = 1\); therefore, upon letting \(f' = f - \lambda\) with \(\{\lambda\} = \overline{\pi}(f; A)\), by (6.4) we see that \(f'\) is a multihyperbolic line which is not a uniline; clearly \(J(f, g) = \Phi \Rightarrow J(f', g) = \Phi\) which is a contradiction by (8.2). Thus we must have \(\overline{\beta}(f; A) = 0\). Therefore \(f\) is a uniline by (5.8.3). Hence \(k[f, g] = R_2\) by (8.1), and \(|\alpha(f, g; A)| = 0\) by (8.3).

Here is a two polynomial analogue of (5.9).

**No Affine Irregular Value Theorem (8.5).** Let \(f \in R_2\) be \(Y\)-monic of \(Y\)-degree \(N > 0\), and let \(g \in R_2\) be such that \(J(f, g) = \Phi\) and \(|\alpha(f, g; A)| = 0 \notin \overline{\pi}(f; A)\). Then \(\beta(f, g; A) = 0\) and \(k[f, g] = R_2\).

**PROOF.** By (5.10) we see that \(|\alpha(f, g; A)| = 0 \notin \overline{\pi}(f; A) \Rightarrow \beta(f, g; A) = 0 \notin \overline{\alpha}(f; A)\) and by (8.4) we see that \(\beta(f, g; A) = 0 \notin \overline{\alpha}(f; A) \Rightarrow f\) is a uniline.

Finally here is a two polynomial analogue of (6.4).

**One Affine Irregular Value Theorem (8.6).** Let \(f \in R_2\) be \(Y\)-monic of \(Y\)-degree \(N > 0\), and let \(g \in R_2\) be such that \(|\alpha(f, g; A)| - 1 = 0 \notin \overline{\pi}(f; A)\). Then \(J(f, g) \notin k^\times\).
PROOF. Let \( g' = g - \lambda \) with \( \{\lambda\} = \alpha(f, g; A) \). Then \( g' \in R_2 \) with \( \beta(f, g'; A) = 0 \not\in \overline{\tau}(f; A) \) and hence by (8.4) we get \( J(f, g') \not\in k^\times \). Therefore \( J(f, g) \not\in k^\times \).

**Corollary (8.7).** Let \( f \in R_2 \) be \( Y \)-monic of \( Y \)-degree \( N > 0 \), and let \( g \in R_2 \). Then we have the following.

(8.7.1) If \( g \not\in k \) and \( \gcd(g, f - \mu) = 1 \) for all \( \mu \in k \) then \( |\alpha(f, g; A)| < \chi_\infty(f) \).

(8.7.2) If \( 0 \not\in \overline{\tau}(f; A) \) and \( \chi(F) = 2 \) then \( J(f, g) \not\in k^\times \).

**PROOF.** In view of (8.5) and (8.6), (8.7.2) follows from (8.7.1) by noting that if \( g \in k \) then \( J(f, g) = 0 \not\in k^\times \), and if \( f \) and \( g - \mu \) for some \( \mu \in k \) have a nonconstant common factor \( \theta \) in \( R_2 \) then \( J(f, g - \mu) = 0 \) is divisible by \( \theta \) and hence \( J(f, g) \not\in k^\times \). To prove (8.7.1), assume that \( g \not\in k \) and \( \gcd(f, g - \mu) = 1 \) for all \( \mu \in k \). Then by replacing \( g \) by \( g - c \) for some \( c \in k \) we may suppose that \( \int(f, g; A) = \max \int(f, g; A) \). Then \( \int(f, g; A) \) is a positive integer and, upon letting \( (F, G) \sim_m (f, g) \), for all \( \lambda \in k \) we have \( \int(F, G - \lambda) = -\int(f, g - \lambda; A) \). Therefore by the inequality for \( |\alpha(F, G)| \) given in Section 1 we conclude that \( |\alpha(f, g; A)| < \chi_\infty(f) \).

**Remark (8.8).** Let us note that by a well-known construction, given any finite number of nonzero elements \( f_1, \ldots, f_r \) in \( R_2 \), we can find a \( k \)-automorphism \( \sigma \) of \( R_2 \) such that \( \sigma(f_1), \ldots, \sigma(f_r) \) are \( Y \)-monic. Clearly it suffices to prove this for their product \( f = f_1 \ldots f_r \). If \( f \in k[Y] \) then we can take \( \sigma \) to be identity. So assume \( f \not\in k[Y] \), let \( p \geq 0 \) be the \((X, Y)\)-degree of \( f \), let \( m > 0 \) be the \( X \)-degree of \( f \), let \( a(Y) \) with \( 0 \neq a(Y) \in k[Y] \) be the coefficient of \( X^m \) in \( f \), and let \( n \geq 0 \) be the \( Y \)-degree of \( a(Y) \). Take integer \( q > p \) and let \( \sigma \) be the \( k \)-automorphism of \( R_2 \) given by \( \sigma(X) = X + Y^q \) and \( \sigma(Y) = Y \). Then \( \sigma(f) \) is \( Y \)-monic of \( Y \)-degree \( t = n + mq \). Note that if \( r = 2 \) with \( f_i \not\in k[Y] \) for \( 1 \leq i \leq 2 \) and for their corresponding degrees \( p_i, m_i, n_i, t_i \) we have \( m_1/m_2 = n_1/n_2 \) then we have \( t_1/t_2 = n_1/n_2 \).

**Remark (8.9).** The AJC = the Algebraic Jacobian Conjecture predicts that for any \( f, g \in R_2 \) with \( J(f, g) = \oplus \) we have \( k[f, g] = R_2 \). To relate this to the two meromorphic jacobian conjectures, given any \( f, g \in R_2 \), upon taking \( F, G \in R \) with \( (F, G) \sim_m (f, g) \), and upon letting \( J = J(F, G) \) and \( J' = J(f, g) \), as in the proof of (4.6), by the chain rule for jacobians we get \( J = -X^{-2}J' \). To show that Conjecture I implies AJC, in view of the first sentence of the above Remark (8.8), we may assume that \( f \in R_2 \) is \( Y \)-monic of \( Y \)-degree \( N > 0 \), and \( g \in R_2 \) such that \( J(f, g) = \oplus \), and we want to show that \( k[f, g] = R_2 \); now clearly \( J = \oplus X^{-2} \); by (4.4) and (4.5) we also have \( \beta(f, g; A) = \beta(F, G) \) and \( \overline{\tau}(f; A) = \overline{\tau}(F) \) and hence by Conjecture I we get \( \beta(f, g; A) = 0 \not\in \overline{\tau}(f; A) \); therefore by (8.4) we conclude that \( k[f, g] = R_2 \). Before dealing with Conjecture II, let us note that from what is shown in [Ab3, Ab5] it follows that AJC is equivalent to a certain variation AJC* of it. To state this, for \( 1 \leq i \leq 2 \) let \( f_i = f_i(X, Y) \in R_2 \setminus k[Y] \), let \( p_i \) and \( m_i \) be the \((X, Y)\)-degree and \( X \)-degrees of \( f_i \), let \( a_i \) be \( a_i(Y) \) be the coefficient of \( X^{m_i} \) in \( f_i \), let \( n_i \) be the \( Y \)-degree of \( a_i(Y) \), and assume that \( p_i = n_i + m_i > m_i \) and \( \deg_{(X, Y)}[f_i(X, Y) - a_i(Y)X^{m_i}] < p_i \). Now AJC* says that, for any such pair \( f_1, f_2 \), if \( J(f_1, f_2) = \oplus \) and \( m_1/m_2 = n_1/n_2 \) then either \( p_1/p_2 \) or \( p_2/p_1 \) (This is a very iffy proposition because, as is shown in [Ab3, Ab5], if AJC were true then such a pair \( f_1, f_2 \) cannot exist). In view of the last sentence of the
above Remark (8.8), to prove AJC* it suffices to prove AJC** which says that if $f, g$ in $R_2$ are $Y$-monic of $Y$-degrees $N > 0$ and $M > 0$ such that $J(f, g) = 4\oplus$ then either $M|N$ or $N|M$. Clearly Conjecture II implies AJC**.

Section 9: Usual Newton Polygon

As a tool for dealing with Conjecture II, we shall now revisit the Usual Newton Polygon as developed in [Ab3].

Recall that $R = k((X))[[Y]]$ where $k$ is an algebraically closed field of characteristic zero, and $R^\sharp$ is the set of all irreducible monic polynomials of positive degree in $Y$ over $k((X))$. As in Section 1, for any $F = F(X, Y) \in R$ of $Y$-degree $N$ and branch number $\chi(F)$ we have

$$F = F_0 \prod_{1 \leq j \leq \chi(F)} F_j$$

where

$$F_0 = F_0(X) \in k((X)) \quad \text{and} \quad F_j = F_j(X, Y) \in R^\sharp \text{ with } \deg_Y F_j = N_j$$

and for any integer $\nu > 0$ which is divisible by $N_1, \ldots, N_{\chi(F)}$ as Newton factorization of $F$ we have

$$F(X^\nu, Y) = F_0(X^\nu) \prod_{1 \leq i \leq N} (Y - z_i(X)) \quad \text{with} \quad z_i(X) \in k((X)).$$

Note that

$$\text{int}(F, Y) = \ord_X F(X, 0)$$

and let us define the final root order of $F$ by putting

$$\hat{O}(F) = \max\{\ord_X z_i(X) : 1 \leq i \leq N\}$$

with the understanding that if $N \leq 0$ then $O(F) = -\infty$. For any $c$ which is a rational number or $\infty$ we define the vertical label of $F$ at $c$ and the starred vertical label of $F$ at $c$ to be the nonnegative integers $L(F, c)$ and $L^*(F, c)$ obtained by putting

$$L(F, c) = |\{i \in \{1, \ldots, N\} : \ord_X z(i) \geq c\nu\}|$$

and

$$L^*(F, c) = \begin{cases} |\{i \in \{1, \ldots, N\} : \ord_X z(i) > c\nu\}| & \text{if } c \neq \infty \\ |\{i \in \{1, \ldots, N\} : \ord_X z(i) \geq c\nu\}| & \text{if } c = \infty \end{cases}$$

with the understanding that if $N \leq 0$ then $L(F, c) = 0 = L^*(F, c)$. We define the final vertical label and the postfinal vertical label of $F$ by putting

$$\hat{L}(F) = L(F, \hat{O}(F)) \quad \text{and} \quad \hat{L}(F) = L^*(F, \hat{O}(F)).$$

For any integer $a$ we denote the coefficient of $X^a$ in $F$ by $\text{coef}_X(F, a)$, i.e., taking summation over all integers $a$ we have

$$F(X, Y) = \sum \text{coef}_X(F, a)X^a \quad \text{with} \quad \text{coef}_X(F, a) \in k[Y]$$

and we extend this by putting

$$\text{coef}_X(F, a) = 0 \text{ if } a \text{ is either } \infty \text{ or a rational number which is not an integer}.$$ 

Note that the $X$-initial coefficient and the $X$-initial form of $F$ are given by

$$\text{inco}_X F = \text{coef}_X(F, \ord_X F)$$
where we call $L$ and $22$ BY SHREERAM S. ABHYANKAR AND ABDALLAH ASSI constitute the UNP($F$) of nonnegative integers the exception that $0$ is the size of the above set, and $0$ and we call $O$ and we note that this is a rational number or $\infty$ we have

$$\text{coef}_{X} (y, a) \in k.$$  

For any $G = G(X, Y) \in R$ of $Y$-degree $M$ and branch number $\chi(G)$ we have

$$G = G_{0} \prod_{1 \leq l \leq \chi(G)} G_{l}$$

where

$$G_{0} = G_{0}(X) \in k((X)) \quad \text{and} \quad G_{l} = G_{l}(X, Y) \in R^{3} \text{ with } \deg_{Y} G_{l} = M_{l}$$

and assuming the integer $\nu > 0$ to be divisible by $N_{1}, \ldots, N_{\chi(F)}, M_{1}, \ldots, M_{\chi(G)}$, in addition to the Newton factorization of $F$, as Newton Factorization of $G$ we have

$$G(X^{\nu}, Y) = G_{0}(X^{\nu}) \prod_{1 \leq e \leq M} (Y - w_{e}(X)) \quad \text{with} \quad w_{e}(X) \in k((X))$$

We define the normalized contact of $F$ and $G$ by putting

$$\text{noc}(F, G) = (1/\nu) \max \{\text{ord}_{X} (z_{i}(X) - w_{e}(X)) : 1 \leq i \leq N \text{ and } 1 \leq e \leq M\}$$

and we note that this is a rational number or $\pm \infty$ and: it is $-\infty \iff N \leq 0$ or $M \leq 0$, and: it is $\infty \iff F_{j} = G_{l}$ for some $j$ and $l$ with $1 \leq j \leq \chi(F)$ and $1 \leq l \leq \chi(G)$. We also define the restricted normalized contact of $F$ and $G$ by putting

$$\text{rnoc}(F, G) = (1/\nu) \max \{\text{ord}_{X} (z_{i}(X) - w_{e}(X)) : 1 \leq i \leq N \text{ and } 1 \leq e \leq M \text{ with } z_{i}(X) \neq w_{e}(X)\}$$

and we note that this is a rational number or $-\infty$ and: it is $-\infty \iff F_{1} = \cdots = F_{\chi(F)} = G_{1} = \cdots = G_{\chi(G)}$ and $N_{1} = \cdots = N_{\chi(F)} = 1 = M_{1} = \cdots = M_{\chi(G)}$.

Assuming $N > 0$, to enlarge the pair $\tilde{O}(F), L(F)$ into the Usual Newton Polygon of $F$, we arrange the set $\{(1/\nu)\text{ord}_{X} z_{i}(X) : 1 \leq i \leq N\}$ as an increasing sequence

$$O_{1}(F) < O_{2}(F) < \ldots < O_{\nu(F)}(F)$$

with preaugmentation $O_{0}(F) = \text{ord}_{X} F_{0}(X)$

and we call $O_{i}(F)$ the $i$-th root order of $F$ and $\nu(F)$ the index of $F$; note that $\nu(F)$ is the size of the above set, and $O_{0}(F), O_{1}(F), O_{2}(F), \ldots, O_{\nu(F)}(F)$ are integers with the exception that $O_{\nu(F)}(F)$ may be $\infty$. Next we introduce the decreasing sequence of nonnegative integers

$$L_{1}(F) > L_{2}(F) > \cdots > L_{\nu(F)}(F) \geq L_{\nu(F)+1}(F)$$

with $L_{i}(F) = L(F, O_{i}(F))$ for $1 \leq i \leq \nu(F)$

and $L_{\nu(F)+1}(F) = \tilde{L}(F)$

where we call $L_{i}(F)$ the $i$-th vertical label of $F$. The above two sequences together constitute the UNP($F$) = the Usual Newton Polygon of $F$. 
To relate the UNP with the customary picture in the real plane, continuing with the assumption of $N > 0$, for any $c$ which is a rational number or $\infty$, upon letting

\begin{align*}
(\prec c) &= \{1 \leq i \leq N : \ord_X z_i(X) < c\}, \\
(= c) &= \{1 \leq i \leq N : \ord_X z_i(X) = c\}, \\
(\succ c) &= \{1 \leq i \leq N : \ord_X z_i(X) > c\},
\end{align*}

we define the horizontal level $\Lambda(F,c)$ of $F$ at $c$ and the starred horizontal level $\Lambda^*(F,c)$ of $F$ at $c$ by putting

\begin{align*}
\Lambda(F,c) &= O_0(F) + \left[ \sum_{i \in (=c)} \left(1/\nu\right) \ord_X z_i(X) \right] + \lfloor(= c)\rfloor \\
\Lambda^*(F,c) &= O_0(F) + \left[ \sum_{i \in (=c)} \left(1/\nu\right) \ord_X z_i(X) \right] + \lfloor(= c)\rfloor + \lfloor(> c)\rfloor
\end{align*}

with the understanding that $0$ times $\infty$ is $0$, and we define the polynomial $0 \neq P^{(F,c)} = P^{(F,c)}(Y) \in k[Y]$ of $F$ at $c$ by putting

$$
P^{(F,c)}(Y) = \inco F_0(X) \left[ \prod_{i \in (=c)} \inco z_i(X) \right] \left[ \prod_{i \in (=c)} (Y - \inco z_i(X)) \right] Y^{(> c)}.
$$

We define the final horizontal level and the postfinal horizontal level of $F$ by putting

$$
\overset{\sim}{\Lambda}(F) = \begin{cases} 
\Lambda(F,O_{\iota(F)-1}(F)) & \text{if } \iota(F) \neq 1 \\
O_0(F) & \text{if } \iota(F) = 1
\end{cases}
$$

and

$$
\widetilde{\Lambda}(F) = \Lambda(F,\overline{O}(F))
$$

and we define final polynomial of $F$ by putting

$$
\widetilde{P}^{(F)} = \widetilde{P}^{(F)}(Y) = P^{(F,\overline{O}(F))}(Y).
$$

We introduce the sequence

$$
\Lambda_{i}(F) = O_0(F) \quad \text{and} \quad \Lambda_{i}(F) = \Lambda(F,O_{i-1}(F)) \quad \text{for } 2 \leq i \leq \iota(F) + 1
$$

where we call $\Lambda_{i}(F)$ the $i$-th horizontal label of $F$, and we introduce the sequence

$$
P_{i}^{(F)} = P^{(F,O_{i}(F))} \quad \text{for } 1 \leq i \leq \iota(F)
$$

where we call $0 \neq P_{i}^{(F)} = P_{i}^{(F)}(Y) \in k[Y]$ the $i$-th polynomial of $F$.

Now the CNP($F$) = the Customary Newton Polygon of $F$ consists of the $\iota(F)$ segments in the real plane, where, for $1 \leq i \leq \iota(F)$, the $i$-th segment or side of CNP($F$) joins the point $(\Lambda_i(F), L_i(F))$ to the point $(\Lambda_{i+1}(F), L_{i+1}(F))$, with the understanding that if $\overline{O}(F) = \infty$ then the last segment or side is the half-infinite horizontal line emanating from the point $(\overset{\sim}{\Lambda}(F), L(F))$ and going to infinity on the right. For $1 \leq i \leq \iota(F)$ we embellish the $i$-th side of CNP($F$) by placing the $i$-th polynomial $0 \neq P_{i}^{(F)} = P_{i}^{(F)}(Y) \in k[Y]$ of $F$ on it. Alternatively, CNP($F$) may be constructed thus. Its first vertex is $(\Lambda_1(F), L_1(F)) = (O_0(F), N)$. The first side is the line of slope $O_1(F)$ starting at the first vertex and ending at height $L_2(F)$ giving us the second vertex. Inductively, the $i$-the side is defined to be the
line of slope $O_i(F)$ starting at the $i$-th vertex $(\Lambda_i(F), L_i(F))$ and ending at height $L_{i+1}(F)$ giving us the $(i+1)$-th vertex whose horizontal coordinate is defined to be $\Lambda_{i+1}(F)$. Letting this side continue to height zero, the horizontal coordinate of the point so obtained is $\Lambda^*(F, O_i(F))$. Note that we are interpreting the slope of a side to be the tangent of the angle it makes with the $Y$-axis. This heuristic-geometric paragraph is not a logical part of the paper.

Continuing with the assumption of $N > 0$, clearly we have

\begin{equation}
\begin{aligned}
\hat{O}(F) &= O_i(F) \quad \text{and} \quad \hat{P}^{(F)} = P_i(F) F^{(F, \hat{O}(F))} \\
\text{and deg}_Y \hat{P}^{(F)} &= \hat{L}(F) = L_i(F) \quad \text{with} \quad \Lambda_1(F) = N
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\bar{A}(F) &= \Lambda_i(F)+1 \\
\text{ord}_Y \bar{A}^{(F)} &= \bar{L}(F) = L_i(F) + 1
\end{aligned}
\end{equation}

and for $1 \leq i \leq \iota(F)$ we have

\begin{equation}
\begin{aligned}
\Lambda_{i+1}(F) &= \Lambda_i(F) + (L_i(F) - L_{i+1}(F))O_i(F) \\
&= O_0(F) + \sum_{1 \leq j \leq i} (L_j(F) - L_{j+1}(F))O_i(F)
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\text{deg}_Y P_i^{(F)} &= L_i(F) \quad \text{and} \quad \text{ord}_Y P_i^{(F)} = L_{i+1}(F)
\end{aligned}
\end{equation}

and for any $c$ which is a rational number or $\infty$ we have

\begin{equation}
\begin{aligned}
\Lambda(F, c) = \begin{cases}
\Lambda_1(F) & \text{if } c < O_1(F) \\
\Lambda_{i+1}(F) & \text{if } O_i(F) \leq c < O_{i+1}(F) \text{ with } 1 \leq i < \iota(F) \\
\Lambda_{i+1}(F) & \text{if } O_i(F) (\iota(F)) \leq c
\end{cases}
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\Lambda^*(F, c) &= \Lambda(F, c) + c(> c) \quad \text{with } (> c) \text{ as above}
\end{aligned}
\end{equation}

and

\begin{equation}
\begin{aligned}
\text{deg}_Y P^{(F, c)} &= L(F, c) \quad \text{and} \quad \text{ord}_Y P^{(F, c)} = L^*(F, c).
\end{aligned}
\end{equation}

Moreover, since ord is additive and inco is multiplicative, for any rational number $c$ for which $c\nu$ is an integer, we have

\begin{equation}
\begin{aligned}
\text{ord}_X F(X^{c\nu}, Y^{X^{c\nu}}) &= \Lambda^*(F, c) \nu \quad \text{with} \quad \text{inco}_X F(X^{c\nu}, Y^{X^{c\nu}}) = P^{(F, c)}(Y).
\end{aligned}
\end{equation}

Assuming $N > 0$ and $M > 0$, for $0 \leq j \leq \min(\iota(F), \iota(G))$ we say that UNP($F$) and UNP($G$) are $j$-step parallel, in symbols we write UNP($F$)$||_j$UNP($G$), if

\begin{equation}
\begin{aligned}
MO_0(F) &= NO_0(G), \quad \text{and for } 1 \leq i \leq j \text{ we have} \\
O_i(F) &= O_i(G) \quad \text{and} \quad ML_i(F) = NL_i(G).
\end{aligned}
\end{equation}

Moreover, we say that UNP($F$) and UNP($G$) are parallel, in symbols we write UNP($F$)$||$UNP($G$), if

\begin{equation}
\begin{aligned}
\iota(F) &= \iota(G) \quad \text{and} \quad \text{UNP}(|F)$$||_\iota(F)$UNP($G$).
Likewise, we say that \( \text{UNP}(F) \) is **smaller** than \( \text{UNP}(G) \), in symbols we write \( \text{UNP}(F) < \text{UNP}(G) \), if
\[
\begin{cases}
\hat{O}(F) < \hat{O}(G) \text{ with } \hat{L}(G) = 1, \\
\text{either } i(F) = i(G) \text{ with } \text{UNP}(F)||_{i(F)-1}\text{UNP}(G) \text{ and } M\hat{L}(F) = N\hat{L}(G), \\
\text{or } i(F) = i(G) - 1 \text{ with } \text{UNP}(F)||_{i(F)}\text{UNP}(G)
\end{cases}
\]
and we note that
\[
(*) \quad \text{UNP}(F) < \text{UNP}(G) \Rightarrow L^*(G, \hat{O}(F)) = 1.
\]
Finally, we say that \( \text{UNP}(F) \) and \( \text{UNP}(G) \) are **pseudoparallel**, in symbols we write \( \text{UNP}(F)||\text{UNP}(G) \), if either \( \text{UNP}(F)||\text{UNP}(G) \) or \( \text{UNP}(F) < \text{UNP}(G) \) or \( \text{UNP}(G) < \text{UNP}(F) \); note that these three conditions are mutually exclusive. In view of (9.1), (9.2) and (9.4), by the definition of parallelness we see that
\[
(9.10) \quad \begin{cases}
\text{if } \text{UNP}(F)||\text{UNP}(G) \text{ then } \\
(M)\text{int}(F, Y) = (N)\text{int}(G, Y) \text{ and } M\hat{L}(F) = N\hat{L}(G).
\end{cases}
\]
**Calculation.** Continuing with the assumption that \( N > 0 \) and \( M > 0 \), let \( c \) be a rational number such that \( cv \) is an integer. Let
\[
(9.11) \quad \tilde{F} = \tilde{F}(X, Y) = F(X^\nu, YX^{cv}) \quad \text{and} \quad \tilde{G} = \tilde{G}(X, Y) = G(X^\nu, YX^{cv})
\]
and similarly let
\[
(9.12) \quad \begin{cases}
\tilde{J} = \tilde{J}(X, Y) \text{ be obtained by substituting } \\
(X^\nu, YX^{cv}) \text{ for } (X, Y) \text{ in } J(F, G).
\end{cases}
\]
Clearly \( J(X^\nu, YX^{cv}) = \nu X^{cv+\nu-1} \) and hence by the chain rule for jacobians we get
\[
(9.13) \quad J(\tilde{F}, \tilde{G}) = \nu X^{cv+\nu-1}\tilde{J}(X, Y).
\]
Now
\[
(9.14) \quad \tilde{F}(X, Y) = X^aP(Y) + \text{ (terms of } X\text{-degree } > a)\]
where
\[
(9.15) \quad a = \text{ord}_XF(X^\nu, YX^{cv}) \text{ with } 0 \neq P = P(Y) = \text{inco}_XF(X^\nu, YX^{cv}) \in k[Y]
\]
and
\[
(9.16) \quad \tilde{G}(X, Y) = X^bQ(Y) + \text{ (terms of } X\text{-degree } > b)\]
where
\[
(9.17) \quad b = \text{ord}_XG(X^\nu, YX^{cv}) \text{ with } 0 \neq Q = Q(Y) = \text{inco}_XG(X^\nu, YX^{cv}) \in k[Y].
\]
Letting ’**denote** Y-derivative’ we have
\[
(9.18) \quad J(\tilde{F}, \tilde{G}) = X^{a+b-1}(aPQ' - bQP') + \text{ (terms of } X\text{-degree } > a + b - 1)\]
and hence
\[
(9.19) \quad \begin{cases}
\text{ord}_XJ(\tilde{F}, \tilde{G}) \geq a + b - 1, \\
\text{with: } \text{ord}_XJ(\tilde{F}, \tilde{G}) > a + b - 1 \iff aPQ' - bQP' = 0
\end{cases}
\]
and by (9.13) we see that
\[
(9.20) \quad \text{if } J(F, G) \in k((X)) \text{ then } aPQ' - bQP' \in k.
\]
Related Polynomials. To analyze (9.19) and (19.20), let us recall the concept of related polynomials developed in [Ab3]. Given any 

\[ 0 \neq P = P(Y) \in k[Y] \quad \text{and} \quad 0 \neq Q = Q(Y) \in k[Y] \]

with \( \deg_Y P = n \) and \( \deg_Y Q = m \), we say that \( P \) and \( Q \) are related to mean that \( P^m = Q^n \). Recall that upon letting

\[ P = P(Y) = P_0 \prod_{1 \leq i \leq p} (Y - u_i)^{r_i} \quad \text{and} \quad Q = Q(Y) = Q_0 \prod_{1 \leq j \leq q} (Y - v_j)^{s_j} \]

with \( P_0 \neq 0 \neq Q_0 \) in \( k \), pairwise distinct elements \( u_1, \ldots, u_p \) in \( k \), pairwise distinct elements \( v_1, \ldots, v_q \) in \( k \), nonnegative integers \( p, q \), and positive integers \( r_1, \ldots, r_p, s_1, \ldots, s_q \), we have

\[ \text{rad}(P) = P_0 \prod_{1 \leq i \leq p} (Y - u_i) \quad \text{and} \quad \text{rad}(Q) = Q_0 \prod_{1 \leq j \leq q} (Y - v_j) \]

and note that

\[ PQ \] has a multiple root

\[ \iff \text{either } r_i \geq 2 \text{ for some } i \text{ or } s_j \geq 2 \text{ for some } j \text{ or } u_i = v_j \text{ for some } i, j. \]

Clearly

(9.21) if \( m + n \neq 0 \) and \( P \) and \( Q \) are related then \( n \neq 0 \neq m \)

and by a standard argument we see that

\[
\begin{cases}
\text{if } m + n \neq 0 \text{ then: } P \text{ and } Q \text{ are related } \iff \\
\text{ rad}(P) = \oplus \text{ rad}(Q) \text{ and upon relabelling } v_1, \ldots, v_q \\
\text{ so that } u_i = v_i \text{ for } 1 \leq i \leq p = q \\
\text{ we have } mu_i = nv_i \text{ for } 1 \leq i \leq p 
\end{cases}
\]

(9.22)

and

\[
\begin{cases}
\text{if } m + n \neq 0 \text{ and } a, b \text{ are integers} \\
\text{ such that } aPQ' - bQP' = 0 \text{ with either } a < 0 \text{ or } b < 0 \\
\text{ then } P^{-b} = \oplus Q^{-a} \text{ with } a < 0 \text{ and } b < 0 \\
\text{ and } P \text{ and } Q \text{ are related with } ma = nb.
\end{cases}
\]

(9.23)

Clearly

\[
\begin{cases}
\text{if } a, b \text{ are integers} \\
\text{ such that } aPQ' - bQP' \in k^\times \text{ with either } a < 0 \text{ or } b < 0 \\
\text{ then } PQ \text{ has no multiple root}
\end{cases}
\]

(9.24)

and hence by (9.23) we see that

\[
\begin{cases}
\text{if } PQ \text{ has a multiple root and } a, b \text{ are integers} \\
\text{ such that } aPQ' - bQP' \in k \text{ with either } a < 0 \text{ or } b < 0 \\
\text{ then } aPQ' - bQP' = 0 \\
\text{ and } P^{-b} = \oplus Q^{-a} \text{ with } a < 0 \text{ and } b < 0 \\
\text{ and } P \text{ and } Q \text{ are related with } ma = nb.
\end{cases}
\]

(9.25)
Calculation Continued. Reverting to the definition of $a, b, P, Q$ given in (9.15) and (9.17), we continue to let \( \deg_Y P = n \) and \( \deg_Y G = m \). By (9.2), (9.4), (9.6), (9.7) and (9.9) we see that

\[
(9.26) \quad a \leq \text{int}(F, Y)\nu \quad \text{and} \quad b \leq \text{int}(G, Y)\nu
\]

and hence

\[
(9.27) \quad \begin{cases} 
\text{if either } \text{int}(F, Y) < 0 \text{ or } \text{int}(G, Y) < 0 \\
\text{then either } a < 0 \text{ or } b < 0.
\end{cases}
\]

By (9.8) we see that

\[
(9.28) \quad \begin{cases} 
(i) \ \deg_Y P > 0 \iff \deg_Y P \geq \hat{L}(F) \iff c \leq \hat{O}(F), \\
(ii) \ \ord_Y P > 0 \iff \ord_Y P \geq \hat{L}(F) \iff c < \hat{O}(F), \\
(iii) \ \deg_Y P > \ord_Y P \iff c \in \{O_1(F), \ldots, O_{i(F)}(F)\}, \\
(iv) \ \deg_Y Q > 0 \iff \
\ord_Y Q \geq \hat{L}(G) \iff c \leq \hat{O}(G), \\
(vi) \ \deg_Y Q > \ord_Y Q \iff c \in \{O_1(G), \ldots, O_{i(G)}(G)\}, \\
(vii) \ c < O_{i(F)}-1 \text{ with } \iota(F) \geq 2 \Rightarrow \ord_Y P \geq 2, \\
(viii) \ c < O_{i(G)}-1 \text{ with } \iota(G) \geq 2 \Rightarrow \ord_Y Q \geq 2,
\end{cases}
\]

and hence

\[
(9.29) \quad \begin{cases} 
\text{if either } (i) \ c < \min(\hat{O}(F), \hat{O}(G)), \\
\text{or (ii) } c < \hat{O}(G) \text{ with } \hat{L}(G) \geq 2, \\
\text{or (iii) } c < O_{i(G)}-1(\hat{G}) \text{ with } \iota(G) \geq 2, \\
\text{or (iv) } c < \hat{O}(F) \text{ with } \hat{L}(F) \geq 2, \\
\text{or (v) } c < O_{i(F)}-1(\hat{F}) \text{ with } \iota(F) \geq 2, \\
\text{then } \ord_Y PQ \geq 2,
\end{cases}
\]

and

\[
(9.30) \quad \begin{cases} 
\text{if either } c \in \{O_1(F), \ldots, O_{i(F)}(F)\} \setminus \{O_1(G), \ldots, O_{i(G)}(G)\} \\
\text{or } c \in \{O_1(G), \ldots, O_{i(G)}(G)\} \setminus \{O_1(F), \ldots, O_{i(F)}(F)\} \\
\text{then } P \text{ and } Q \text{ are not related.}
\end{cases}
\]

Main Lemma (9.31). Let \( F \) and \( G \) in \( R \) be of \( Y \)-degrees \( N > 0 \) and \( M > 0 \) respectively, and assume that \( J(F, G) \in \mathfrak{k}(X) \) and \( \text{UNP}(F)||\text{UNP}(G) \). Also assume that either \( \text{int}(F, Y) < 0 \) or \( \text{int}(G, Y) < 0 \). Then \( \text{UNP}(F)||\text{UNP}(G) \), and for \( 1 \leq i < \min(\iota(F), \iota(G)) \) the polynomials \( P_i(F) \) and \( P_i(G) \) are related. Moreover, if \( \hat{O}(F) = \hat{O}(G) \) then \( \text{UNP}(F)||\text{UNP}(G) \) and hence in particular \( M \text{int}(F, Y) = (N)\text{int}(G, Y) \) and \( M\hat{L}(F) = N\hat{L}(G) \).

PROOF. By induction on \( j \) we shall show that, given any integer \( j \) with \( 0 \leq j \leq \min(\iota(F), \iota(G)) \), we have (1.1) to (6.1) stated below. By taking \( j = \min(\iota(F), \iota(G)) \) this will establish the lemma.

(1.1) If \( j < \min(\iota(F), \iota(G)) \) then we have: \( \text{UNP}(F)||\text{UNP}(G) \), and the polynomials \( P_i(F) \) and \( P_i(G) \) are related for \( 1 \leq i \leq j \).
(2.) If \( \hat{O}(F) = \hat{O}(G) \) and \( j = \min(\iota(F), \iota(G)) \) then we have: \( j = \iota(F) = \iota(G) \), UNP(\( F \)|\( j \))UNP(\( G \)), the polynomials \( P_i^{(F)} \) and \( P_i^{(G)} \) are related for \( 1 \leq i < j \), and \( (M) \text{int}(F, Y) = (N) \text{int}(G, Y) \).

(3.) If \( \hat{O}(F) < \hat{O}(G) \) and \( j = \min(\iota(F), \iota(G)) = \iota(G) \) then we have: \( j = \iota(F) \), UNP(\( F \)|\( j \))UNP(\( G \)), the polynomials \( P_i^{(F)} \) and \( P_i^{(G)} \) are related for \( 1 \leq i < j \), \( M \bar{L}(F) = N \bar{L}(G) \), and \( \bar{L}(G) = 1 \).

(4.) If \( \hat{O}(F) < \hat{O}(G) \) and \( j = \min(\iota(F), \iota(G)) = \iota(F) \) then we have: \( j = \iota(F) \neq \iota(G) \), UNP(\( F \)|\( j \))UNP(\( G \)), the polynomials \( P_i^{(F)} \) and \( P_i^{(G)} \) are related for \( 1 \leq i < j \), \( M \bar{L}(F) = N \bar{L}(G) \), and \( \bar{L}(F) = 1 \).

(5.) If \( \hat{O}(F) < \hat{O}(F) \) and \( j = \min(\iota(F), \iota(G)) = \iota(F) \) then we have: \( j = \iota(F) = \iota(G) = \iota(G) \), UNP(\( F \)|\( j \))UNP(\( G \)), the polynomials \( P_i^{(F)} \) and \( P_i^{(G)} \) are related for \( 1 \leq i < j \), \( M \bar{L}(F) = N \bar{L}(G) \), and \( \bar{L}(F) = 1 \).

(6.) If \( \hat{O}(G) < \hat{O}(F) \) and \( j = \min(\iota(F), \iota(G)) = \iota(G) \) then we have: \( j = \iota(G) = \iota(F) - 1 \), UNP(\( F \)|\( j \))UNP(\( G \)), the polynomials \( P_i^{(F)} \) and \( P_i^{(G)} \) are related for \( 1 \leq i < j \), \( M \bar{L}(F) = N \bar{L}(G) \), and \( \bar{L}(F) = 1 \).

By hypothesis this holds for \( j = 0 \). So let \( j > 0 \) and assume for \( j - 1 \). Note that now (2) to (6) are vacuous, and so in proving (1) to (6) we shall only use (1) and that we shall do without mentioning it explicitly; we shall also tacitly use the fact that \( M \bar{L}(F) = N \bar{L}(G) \) which in case of \( j > 1 \) follows from (9.5) and the relatedness of \( P_{j-1}^{(F)} \) and \( P_{j-1}^{(G)} \), and is obvious in case of \( j = 1 \) because \( L_1(F) = N \) and \( L_1(G) = M \). Since either \( \text{int}(F, Y) < \text{int}(G, Y) < 0 \), upon letting \( c = \min(O_j(F), O_j(G)) \) we see that \( c \) is a rational number such that \( cq \) is an integer. So we may use the above Calculation, and then by (9.20) we see that \( \alpha P Q - \beta Q P \in k \) and by (9.27) we see that \( a < 0 \) and \( b < 0 \). If \( j < \min(\iota(F), \iota(G)) \) then by (9.29) we see that \( P Q \) has a multiple root and therefore by (9.25) and (9.30) we see that \( P = P_j^{(F)} \) and \( Q = P_j^{(G)} \) are related with \( O_j(F) = O_j(G) \); this proves (1.). If \( \hat{O}(F) = \hat{O}(G) \) and \( j = \min(\iota(F), \iota(G)) \) then obviously \( j = \iota(F) = \iota(G) \) with \( O_j(F) = \hat{O}(F) = \hat{O}(G) = O_j(G) = \hat{O}(G) \) and hence by (9.10) we see that \( (M) \text{int}(F, Y) = (N) \text{int}(G, Y) \); this proves (2.). If \( \hat{O}(F) < \hat{O}(G) \) and \( j = \min(\iota(F), \iota(G)) = \iota(G) \) then by (9.30) we see that \( P \) and \( Q \) are not related and hence by (9.25) and (9.29) we get \( j = \iota(F) \) and \( \bar{L}(G) = 1 \); this proves (3.). If \( \hat{O}(F) < \hat{O}(G) \) and \( j = \min(\iota(F), \iota(G)) \neq \iota(G) \) then \( j = \iota(F) < \iota(G) \) and by (9.28) we see that \( P \) and \( Q \) are not related and hence by (9.25) and (9.29) we get \( j = \iota(F) = \iota(G) - 1 \) with \( O_j(F) = O_j(G) \) and \( \bar{L}(G) = 1 \); this proves (4.). Interchanging \( F \) and \( G \) in the proof of (3.) and (4.) we get (5.) and (6.).

**Definition (9.32).** Using (9.31), in (9.33) to (9.38) we shall show that, under certain condition, \( J(F, G) \in k((X)) \) implies that most branches of \( F \) and \( G \) can be partitioned into **packets** \( (F_1, \ldots, F_r, G_1, \ldots, G_s) \) whose members are **pseudocog-nates** of each other, i.e., their roots coincide up to the last characteristic term. To define these concepts more precisely, let us review some relevant terms.

For any \( f \in k^5 \) of \( Y \)-degree \( n \), the **newtonian sequence of characteristic exponents** of \( f \) relative to \( n \), denoted by

\[
m(f) = m_i(f)_{0 \leq i \leq h(m(f)) + 1}
\]
is defined on pages page 3-4 of [AA2], where the GCD-sequence
\[ d(m(f)) = d_i(m(f))_{0 \leq i \leq h(d(m(f)))+2} \]
of \( m(f) \) is also defined. For simplicity of notation we put
\[ h(f) = h(m(f)) = h(m(d(f))) \quad \text{and} \quad d(f) = d(m(f)) \]
and
\[ \tilde{d}(f) = d_{h(f)}(f) \quad \text{and} \quad \tilde{m}(f) = \begin{cases} m_{h(f)}(f) & \text{if } h(f) \neq 0 \\ -\infty & \text{if } h(f) = 0 \end{cases} \]
Note that then
\[ d_{h(f)+1}(f) = 1 \quad \text{and} \quad m_{h(f)+1}(f) = \infty \]
and
\[ d_0(f) = 0 \quad \text{and} \quad m_0(f) = n = d_1(f) \]
and
\[ h(f) = 0 \iff f(X,Y) = Y \iff \tilde{d}(f) = 0 \iff \tilde{m}(f) = -\infty. \]
Also note that:
(9.32.0) \( \text{noc}(f, f') = \tilde{m}(f)/n. \)
For any \( c \) which is a rational number or \( \infty \) we define the \( c \)-position of \( f \) to be the
unique nonnegative integer \( p(f, c) \leq h(f) \) such that \( m_i(f)/n < c \) for \( 1 \leq i \leq p(f, c), \)
and \( c \leq m_j(f)/n \) for \( p(f, c) < j \leq h(f) \). We also define the quantities
\[ \tilde{d}(f, c) = d_{p(f,c)+1}(f) \quad \text{and} \quad \tilde{m}(f, c) = m_{p(f,c)+1}(f) \]
and
\[ t(f, c) = \begin{cases} \text{the minimal monic polynomial of} \\ \sum_{i \leq cn} \text{coef}_X(\eta(X), i)X^i \text{ over } k((X^n)) \end{cases} \]
were \( \eta(X) \) is a root of \( f(X^n, Y) \) in \( k((X)) \)
where we call \( t(f, c) = t(f, c)(X,Y) \in R^2 \) the \( c \)-normalized truncation of \( f \).
Note that then
\[ \tilde{d}(f, c) = n/\deg_Y t(f, c) \]
and
\[ \begin{cases} \text{if either } h(f) \neq 0 \text{ with } m_{h(f)}(f)/n < c \text{ or } h(f) = 0, \\ \text{then } p(f, c) = h(f) \text{ with } \tilde{m}(f, c) = \infty \text{ and } \tilde{d}(f, c) = 1 \end{cases} \]
and
\[ \begin{cases} \text{if } h(f) \neq 0 \text{ with } m_{h(f)}(f)/n = c, \\ \text{then } p(f, c) = h(f) - 1 \text{ with } \tilde{m}(f, c) = \tilde{m}(f) \text{ and } \tilde{d}(f, c) = \tilde{d}(f). \end{cases} \]

Let \( f' \in R^2 \) be of \( Y \)-degree \( n' \). We say that \( f \) is a cognate of \( f' \) if \( \text{noc}(f, f') = \tilde{m}(f)/n = \tilde{m}(f')/n' \). We say that \( f \) is an overcognate of \( f' \) if \( h(f) \neq 0 \neq h(f') \)
with \( \text{noc}(f, f') > \max(\tilde{m}(f)/n, \tilde{m}(f')/n') \). We say that \( f \) is a subcognate of \( f' \) if \( \tilde{m}(f)/n < \text{noc}(f, f') = \tilde{m}(f')/n' \). Note that:
(9.32.1) if \( f \) is a cognate (resp: overcognate) of \( f' \) then \( f' \) is a cognate (resp: overcognate) of \( f \);
(9.32.2) if \( f \) is a cognate of \( f' \) then \( h(f) \neq 0 \neq h(f') \);
(9.32.3) if \( h(f) = 0 = h(f') \text{ then } f \) is an overcognate of \( f' \);
(9.32.4) if \( f \) is an overcognate of \( f' \) then \( \tilde{m}(f)/n = \tilde{m}(f')/n' \).
(9.32.5) if \( f \) is a cognate or overcognate of \( f' \) then \( n = n' \) and \( h(f) = h(f')\) with \( m(f) = m(f') \) and \( d(f) = d(f') \) and \( d'(f) = d'(f') \), and for any rational number \( c \) we have \( p(f, c) = p(f', c) \) with \( \hat{m}(f, c) = \hat{m}(f', c) \) and \( \hat{d}(f, c) = \hat{d}(f', c) \);

(9.32.6) if \( h(f) + 1 = h(f') \) with \( \text{noc}(f, f') = \hat{m}(f')/n' \) then \( f \) is a subcognate of \( f' \); and

(9.32.7) if \( f \) is a subcognate of \( f' \) then \( h(f) + 1 = h(f') \) and \( m_i(f)/n = m_i(f')/n' \) for \( 1 \leq i \leq h(f) \) with \( \text{cof}_X(\eta(X), \hat{m}(f')n/n') = 0 \) where \( \eta(X) \) is a root of \( f(X^n, Y) \) in \( K((X)) \).

Also note that if \( h(f') \neq 0 \) and \( f \) is the \( \hat{d}(f') \)-th approximate root of \( f' \) in the sense of [Ab2] then \( f \) is a subcognate of \( f' \); consequently we may think of a subcognate of \( f' \) as a \textbf{last pseudoapproximate root} of \( f' \). Moreover, if either \( f \) is a cognate of \( f' \), or \( f \) is an overcognate of \( f' \), or \( f \) is a subcognate of \( f' \), \( f' \) is a subcognate of \( f \), then we may think of \( f \) and \( f' \) as being \textbf{pseudocognates} of each other.

By an \textbf{equilateral sequence} in \( R^3 \) we mean a sequence \( (f_i)_{1 \leq i \leq r} \) of members of \( R^3 \), with integer \( r > 1 \), for which there exists a (necessarily unique) rational number \( c \) such that for all \( i \neq j \) in \( \{1, \ldots, r\} \) we have \( \text{noc}(f_i, f_j) = c \) and for all \( i \in \{1, \ldots, r\} \) we have \( \eta_i(f_i, f_i) \leq c \); we call \( c \) the \textbf{diameter} of the sequence. By a \textbf{cognate sequence} in \( R^3 \) we mean an equilateral sequence \( (f_i)_{1 \leq i \leq r} \) in \( R^3 \) such that for all \( i \neq j \) in \( \{1, \ldots, r\} \) we have that \( f_i \) is a cognate of \( f_j \). By an \textbf{overcognate sequence} in \( R^3 \) we mean an equilateral sequence \( (f_i)_{1 \leq i \leq r} \) in \( R^3 \) such that for all \( i \neq j \) in \( \{1, \ldots, r\} \) we have that \( f_i \) is an overcognate of \( f_j \). By a \textbf{subcognate sequence} in \( R^3 \) we mean an equilateral sequence \( (f_i)_{1 \leq i \leq r} \) in \( R^3 \) for which there exists a unique \( i' \) in \( \{1, \ldots, r\} \) such that for all \( j \neq i' \) in \( \{1, \ldots, r\} \) we have that \( f_i \) is a subcognate of \( f_j \) and for all \( i \neq j \) in \( \{1, \ldots, r\} \setminus \{i'\} \) we have that \( f_i \) is a cognate of \( f_j \); we call \( f_{i'} \) the \textbf{special branch} of the sequence. By an \textbf{equicognate sequence} in \( R^3 \) we mean an \textbf{equilateral sequence} in \( R^3 \) which is either a cognate sequence in \( R^3 \) or an overcognate sequence in \( R^3 \) or a subcognate sequence in \( R^3 \).

Note that for an equicognate sequence \( (f_i)_{1 \leq i \leq r} \) in \( R^3 \) with diameter \( c \) we have that:

(9.32.8) if the sequence is cognate then for \( 1 \leq i \leq r \) we have
\[
\hat{d}(f_i, c) = \hat{d}(f_i) = \hat{d}(f_1) \quad \text{and} \quad d_1(f_i) = d_1(f_1);
\]

(9.32.9) if the sequence is overcognate then for \( 1 \leq i \leq r \) we have
\[
\hat{d}(f_i, c) = 1 \quad \text{and} \quad d_1(f_i) = d_1(f_1);
\]

(9.32.10) and if the sequence is subcognate and we have labelled the branches so that the special branch is \( f_r \) then for \( 1 \leq i < r \) we have

\[
\begin{align*}
1 = \hat{d}(f_r, c) & \leq \hat{d}(f_i, c) = \hat{d}(f_i) = \hat{d}(f_1) \\
\text{and} \quad d_1(f_r) = d_1(f_i)/\hat{d}(f_i) = d_1(f_1)/\hat{d}(f_1).
\end{align*}
\]

Let \( F \) and \( G \) in \( R \) be \( Y \)-monic of \( Y \)-degrees \( N > 0 \) and \( M > 0 \) respectively. By a \textbf{bisquence} of \( (F, G) \) we mean a pair of families \( (F_j)_{j \in J}, (G_i)_{i \in J'} \) where \( J \) and \( J' \) are nonempty subsets of \( \{1, \ldots, \chi(F)\} \) and \( \{1, \ldots, \chi(G)\} \) respectively. This induces the sequence \( (f_i)_{1 \leq i \leq r} \) in \( R^3 \) where \( r = |J| + |J'| \) and, upon letting \( j_1 < \cdots < j_{|J|} \) and \( l_1 < \cdots < l_{|J'|} \) be the increasing labellings of \( J \) and \( J' \) respectively, we have \( f_i = F_j \), for \( 1 \leq i \leq |J| \) and \( f_{j_i+l_i} = G_i \), for \( 1 \leq i \leq |J'| \). The bisquence is
said to be **equilateral**, ..., **equicognate** if the induced sequence is respectively **equilateral**, ..., **equicognate**. By the **diameter** of an equilateral bisequence we mean the diameter of the induced sequence. By the **special** branch of a subcognate bisequence we mean the special branch of the induced sequence.

An equilateral bisequence \((F_j)_{j \in J}, (G_l)_{l \in J'}\) of \((F, G)\), whose diameter is \(c\), is said to be **saturated** if:

- (9.32.11) there is no \(i \in \{1, \ldots, \chi(F)\} \setminus J\) such that either for some \(j \in J\) we have \(\text{nocc}(F_i, F_j) \geq c\) or for some \(l \in J'\) we have \(\text{nocc}(F_i, G_l) \geq c\);
- (9.32.12) there is no \(i'\in \{1, \ldots, \chi(G)\} \setminus J'\) such that either for some \(j \in J\) we have \(\text{nocc}(G_{i'}, F_j) \geq c\) or for some \(l \in J'\) we have \(\text{nocc}(G_{i'}, G_l) \geq c\);
- (9.32.13) and for all \(j \in J\) and \(l \in J'\) we have \(\text{nocc}(F_j, G_l) = c\) and \(\text{nocc}(F_j, G_l) = c\).

An equilateral bisequence \((F_j)_{j \in J}, (G_l)_{l \in J'}\) of \((F, G)\), whose diameter is \(c\), is said to be **balanced** if it is saturated and:

- (9.32.14) for all \(j \in J\) and \(l \in J'\) we have \(\text{int}(F_j, G_l) < 0\) and \(\text{int}(F_i, G_l) < 0\) with
  \[
  \frac{\text{int}(F_j, G_l)}{\text{int}(F_i, G_l)} = \frac{NM_i}{MN_j};
  \]
  (9.32.15) and we have
  \[
  \frac{\sum_{j \in J} \hat{d}(F_j, c)}{\sum_{l \in J'} \hat{d}(G_l, c)} = \frac{N}{M};
  \]

An equilateral bisequence \((F_j)_{j \in J}, (G_l)_{l \in J'}\) of \((F, G)\) is said to be **well-balanced** if it is balanced and for it:

- (9.32.16) the degrees satisfy the equation
  \[
  \sum_{j \in J} N_j \sum_{l \in J'} M_l = \frac{N}{M};
  \]
- (9.32.17) the intersection multiplicities satisfy the equation
  \[
  \sum_{j \in J} \text{int}(F_j, G_l) = \sum_{l \in J'} \text{int}(F_i, G_l);
  \]
- (9.32.18) there exist unique negative rational numbers \(N'\) and \(M'\) with \(MN' = NM\) such that for all \(j \in J\) and \(l \in J'\) we have \(N_j = N' \text{int}(F_j, G)\) and \(M_l = M' \text{int}(F_i, G_l)\); and
- (9.32.19) upon letting \(E = \max((N_j)_{j \in J}, (M_l)_{l \in J'})\) and \(D = \min(N_j)_{j \in J}\) and \(D' = \min(M_l)_{l \in J'}\), we have that:
  - (i) if \(D \neq E\) then there is a unique \(s \in J\) such that \(N_s|E\) with \(N_s < E = N_j = M_l\) for all \(j \in J\setminus\{s\}\) and \(l \in J'\), and
  - (ii) if \(D' \neq E\) then there is a unique \(s' \in J'\) such that \(M_{s'}|E\) with \(M_{s'} < E = N_j = M_l\) for all \(j \in J\) and \(l \in J'\setminus\{s'\}\).

Finally, by a **packet** of \((F, G)\) we mean a balanced equicognate bisequence \((F_j)_{j \in J}, (G_l)_{l \in J'}\) of \((F, G)\). Note that then a packet has properties (9.32.11) to (9.32.19), and its induced sequence has properties (9.32.8) to (9.32.10). In view of the last 3 references, (i) or (ii) occur exactly when the packet is subcognate with \(E \neq 1\), and then \(F_i\) or \(G_{s'}\) is the special branch. In all other cases, i.e., if the packet is subcognate with \(E = 1\), or the packet is cognate, or the packet is overcognate, then for all \(j \in J\) and \(l \in J'\) we have \(N_j = E = G_l\).
**First Corollary (9.33).** Let $F$ and $G$ in $R$ be $Y$-monic of $Y$-degrees $N > 0$ and $M > 0$ respectively, and assume that $J(F,G) \in k((X))$. Let $y(X) \in k((X))$ be such that $\text{ord}_XF(X^\nu,y(X)) < 0$. Let

$$
\begin{aligned}
c &= \frac{1}{\nu} \max_{1 \leq i \leq N} \text{ord}_X (y(X) - z_i(X)), \\
I &= \{ i : 1 \leq i \leq N \text{ with } \text{ord}_X (y(X) - z_i(X)) = cv \}, \\
J &= \{ j : 1 \leq j \leq \chi(F) \text{ with } F_j(X^\nu, z_i(X)) = 0 \text{ for some } i \in I \},
\end{aligned}
$$

and

$$
\begin{aligned}
c' &= \frac{1}{\nu} \max_{1 \leq e \leq M} \text{ord}_X (y(X) - w_e(X)), \\
I' &= \{ e : 1 \leq e \leq M \text{ with } \text{ord}_X (y(X) - w_e(X)) = c'v \}, \\
J' &= \{ l : 1 \leq l \leq \chi(G) \text{ with } G_l(X^\nu, w_e(X)) = 0 \text{ for some } e \in I' \}.
\end{aligned}
$$

Then $c$ is a rational number such that $cv$ is an integer, and we have the following.

(9.33.1) If $c < c'$ then $|I(G)| = 1$.

(9.33.2) If $c = c'$ then $\text{ord}_X G(X^\nu, y(X)) < 0$ and

$$
\frac{\text{ord}_X (F(X^\nu, y(X)))}{\text{ord}_X G(X^\nu, y(X))} = \frac{N}{M} = \frac{\sum_{j \in J} \hat{d}(F_j, c)}{\sum_{l \in J'} \hat{d}(G_l, c)}.
$$

(9.33.3) If $c = c'$, and there exists $e \in I'$ such that for all $i \in I$ we have $\text{coef}_X(z_i(X), cv) \neq \text{coef}_X(w_e(X), cv)$, then

$$
\begin{aligned}
&\text{for all } i \in I \text{ and } e \in I' \text{ we have } \text{coef}_X(z_i(X), cv) \neq \text{coef}_X(w_e(X), cv), \\
&\text{for all } i \neq i' \in I \text{ we have } \text{coef}_X(z_i(X), cv) \neq \text{coef}_X(z_{i'}(X), cv), \text{ and} \\
&\text{for all } e \neq e' \in I' \text{ we have } \text{coef}_X(w_e(X), cv) \neq \text{coef}_X(w_{e'}(X), cv).
\end{aligned}
$$

PROOF. Let $\tau = cv$ and $\tau' = c'v$. Since $\text{ord}_XF(X^\nu, y(X)) < 0$, it follows that $c$ is a rational number such that $\tau$ is an integer. Let

$$
\begin{aligned}
\bar{X} &= X^\nu \quad \text{and} \quad \bar{Y} = Y + y(X)
\end{aligned}
$$

and

$$
\begin{aligned}
\overline{F} &= \overline{F}(X,Y) = F(\bar{X}, \bar{Y}) \quad \text{and} \quad \overline{G} = \overline{G}(X,Y) = G(\bar{X}, \bar{Y}).
\end{aligned}
$$

By the chain rule for jacobians we get $J(\overline{F}, \overline{G}) \in k((X))$. Clearly $\text{int}(\overline{F}, \overline{Y}) = \text{ord}_X \overline{F}(X^\nu, y(X))$ and hence $\text{int}(\overline{F}, \overline{Y}) < 0$. Moreover $\hat{O}(\overline{F}) = \tau$ and $\hat{O}(\overline{G}) = \tau'$. Now by taking $\overline{F}, \overline{G}$ for $F, G$ in (9.31) we see that if $\hat{O}(\overline{F}) < \hat{O}(\overline{G})$ then $L^*(\overline{G}, \hat{O}(\overline{F})) = 1$. From this it follows that $\text{ord}_X (y(X) - w_e(X)) > \tau$ for at most one $e$ in $\{1, \ldots, M\}$, which proves (9.33.1).

Henceforth assume that $c = c'$. Again by taking $\overline{F}, \overline{G}$ for $F, G$ in (9.31) we see that $\text{UNP}(\overline{F})||\text{UNP}(\overline{G})$ and $\text{int}(\overline{G}, Y) < 0$ with

$$
\frac{\text{int}(\overline{F}, Y)}{\text{int}(\overline{G}, Y)} = \frac{N}{M} = \frac{\hat{L}(\overline{F})}{\hat{L}(\overline{G})}.
$$

Clearly $\text{int}(\overline{G}, Y) = \text{ord}_X G(X^\nu, y(X))$ and hence we get the first equality of (9.33.2). Also clearly

$$
\begin{aligned}
\hat{L}(\overline{F}) = |I| = \sum_{j \in J} \hat{d}(F_j, c) \quad \text{and} \quad \hat{L}(\overline{G}) = |I'| = \sum_{l \in J'} \hat{d}(G_l, c).
\end{aligned}
$$
and hence get the second equality of (9.33.2). Taking $\overline{F}, \overline{G}, \overline{r}, 1$ for $F, G, c, \nu$ in the above Calculation we have $\overline{F} = \overline{F}(X, Y) = \overline{F}(X, YX^\nu)$ and $\overline{G} = \overline{G}(X, Y) = \overline{G}(X, YX^\nu)$. Clearly the members of the sets

$$\{\text{coef}_X(z_i(X), \overline{r}) - \text{coef}_X(y(X), \overline{r}) : i \in I\}$$

and

$$\{\text{coef}_X(w_e(X), \overline{r}) - \text{coef}_X(y(X), \overline{r}) : e \in I'\}$$

are precisely the roots of $P(Y)$ and $Q(Y)$. Now assume that there exists $\epsilon \in I'$ such that for all $i \in I$ we have $\text{coef}_X(z_i(X), cv) \neq \text{coef}_X(w_e(X), cv)$. Then $\text{coef}_X(w_e(X), \overline{r}) - \text{coef}_X(y(X), \overline{r})$ belongs to the second set but not to the first set, and hence $P$ and $Q$ are not related. Therefore (9.33.3) follows from (9.25) and (9.27).

**Second Corollary (9.34).** Let $F$ and $G$ in $R$ be $Y$-monic of $Y$-degrees $N > 0$ and $M > 0$ respectively, and assume that $J(F, G) \in k((X))$. For an integer $v$ with $1 \leq v \leq \chi(G)$ let $c = \text{noc}(F, G_v)$ and assume that $\text{int}(F, G_v) < 0$. Then $c$ is a rational number such that $cv$ is an integer, and upon letting

$$J = \{j : 1 \leq j \leq \chi(F) \text{ with } \text{noc}(F_j, G_v) \geq c\}$$

and

$$J' = \{j : 1 \leq j \leq \chi(G) \text{ with } \text{noc}(G_{j1}, G_v) \geq c\}$$

we have that $(F_j)_{j \in J}, (G_j)_{j \in J'}$ is a balanced equilateral bisquence of $(F, G)$ with diameter $c$. In particular $J \neq \emptyset$ and $v \in J'$.

**PROOF.** Since $\text{int}(F, G_v) < 0$, we must have $c \neq \infty$. Therefore $c$ is a rational number and upon letting $\overline{r}$ we see that $\overline{r}$ is an integer. Clearly $J \neq \emptyset$ and $v \in J'$. Also clearly, for every $j \in J$ we have $\text{noc}(F_j, G_v) = c$. We can take $\epsilon$ with $1 \leq \epsilon \leq M$ such that $G_v(X^\nu, w_\epsilon(X)) = 0$. Now by taking $w_\epsilon(X)$ for $y(X)$ in (9.33.1) we see that $\text{rnoc}(G_v, G_v) \leq c$ and for all $l \neq v$ in $J'$ we have $\text{noc}(G_i, G_v) = c$. We can take $\lambda \in k$ such that for $1 \leq i \leq N$ and $1 \leq e \leq M$ we have

$$\text{coef}_X(z_i(X), \overline{r}) \neq \lambda \neq \text{coef}_X(w_e(X), \overline{r}).$$

Henceforth let $y(X) \in k((X))$ be defined by putting

$$y(X) = \lambda X^\nu + \sum_{a \neq \overline{r}} \text{coef}_X(w_\epsilon(X), a)X^a.$$

Since $\text{noc}(F, G_v) = c$, we get

$$\text{ord}_X F(X^\nu, y(X)) = (\nu/M_v)\text{int}(F, G_v).$$

Therefore $\text{ord}_X F(X^\nu, y(X))) < 0$. It follows that we are in the situation of (9.33) with $c = c'$. Consequently by (9.33.2) we see that $\text{ord}_X G(X^\nu, y(X)) < 0$ and

$$(1) \frac{\text{ord}_X F(X^\nu, y(X))}{\text{ord}_X G(X^\nu, y(X))} = \frac{N}{M} = \frac{\sum_{j \in J} \hat{d}(F_j, c)}{\sum_{i \in J'} \hat{d}(G_i, c)}.$$

Clearly $\epsilon \in I'$ and for all $i \in I$ we have $\text{coef}_X(z_i(X), cv) \neq \text{coef}_X(w_\epsilon(X), cv)$. Therefore by (9.33.3) we see that $(F_j)_{j \in J}, (G_j)_{j \in J'}$ is a saturated equilateral bisquence of $(F, G)$ with diameter $c$. In particular, for all $i \in I$ and $e \in I'$ we have

$$y(X) = \lambda X^\nu + \sum_{a \neq \overline{r}} \text{coef}_X(z_i(X), a)X^a.$$
and

\[ y(X) = \lambda X^r + \sum_{a \neq r} \text{coef}_X(w_a(X), a)X^a. \]

Likewise, for all \( j \in J \) and \( l \in J' \) we have \( \text{noc}(F,G_l) = c \) and \( \text{noc}(F_j,G) = c \), and hence

\[
\begin{align*}
\text{ord}_X F(X^\nu, y(X)) &= (\nu/M_l) \text{int}(F,G_l) \\
\text{and} \\
\text{ord}_X G(X^\nu, y(X)) &= (\nu/N_j) \text{int}(F,G). 
\end{align*}
\]

By (1) and (2) we get (9.32.14) and (9.33.15), and hence our bisequence is balanced.

**Equicognate Lemma (9.35).** Let \( F \) and \( G \) in \( R \) be \( Y \)-monic of \( Y \)-degrees \( N > 0 \) and \( M > 0 \) respectively. Then any balanced equicogonate bisequence of \((F,G)\) is well-balanced.

**PROOF.** Let \((F_j)_{j \in J}, (G_l)_{l \in J'}\) be a balanced equicognate bisequence of \((F,G)\) with diameter \( c \). By (9.32.8) to (9.32.10) there exists a unique positive integer \( B \) such that for all \( j \in J \) and \( l \in J' \) we have \( N_j = B\hat{d}(F_j,c) \) and \( M_l = B\hat{d}(G_l,c) \). Hence by (9.32.15) we get

\[
\frac{\sum_{i \in J'} N_j}{\sum_{i \in J'} M_l} = \frac{N}{M}.
\]

By (9.32.14) there exist unique negative rational numbers \( N' \) and \( M' \) with \( MN' = N'M \) such that for all \( j \in J \) and \( l \in J' \) we have \( N_j = N' \text{int}(F_j,G) \) and \( M_l = M' \text{int}(F,G_l) \). Namely, to obtain the existence of \( M' \), fixing some \( j \in J \) and letting \( M' = \frac{M(N_j)}{\text{int}(F,G_j)} \), by (9.32.14) we see that for all \( l \in J' \) we have \( \frac{M_l}{\text{int}(F,G_j)} = M' \). By symmetry we get the existence of \( N' \). The uniqueness follows in a similar manner. This proves (9.32.18). From this and the above display we see that

\[
\sum_{j \in J} \text{int}(F_j,G) = \sum_{l \in J'} \text{int}(F,G_l).
\]

In view of (9.32.8) to (9.32.10), upon letting \( E = \max((N_j)_{j \in J}, (M_l)_{l \in J'}) \) and \( D = \min(N_j)_{j \in J} \) and \( D' = \min(M_l)_{l \in J'} \), we have that:

(i) if \( D \neq E \) then there is a unique \( s \in J \) such that \( N_s < E = N_j = M_l \) for all \( j \in J \setminus \{s\} \) and \( l \in J' \), and

(ii) if \( D' \neq E \) then there is a unique \( s' \in J' \) such that \( M_{s'} < E = N_j = M_l \) for all \( j \in J \) and \( l \in J' \setminus \{s'\} \).

Therefore the bisequence is well-balanced.

**Equilateral Lemma (9.36).** Every equilateral sequence in \( R^2 \) is equicognate.

**PROOF.** Let \((f_i)_{1 \leq i \leq r}\) be an equilateral sequence in \( R^2 \) of diameter \( c \), and let \( n_i \) be the \( Y \)-degree of \( f_i \). By (9.32.0) we see that \( \hat{m}(f_i)/n_i \leq c \) for \( 1 \leq i \leq r \). If \( \hat{m}(f_i)/n_i = c \) for \( 1 \leq i \leq r \) then clearly the sequence is cognate. Assuming the contrary, after suitable relabelling we may henceforth suppose that \( \hat{m}(f_1)/n_1 < c \).

If \( \hat{m}(f_i)/n_i < c \) for \( 2 \leq i \leq r \) then clearly the sequence is overcognate, and if \( \hat{m}(f_i)/n_i = c \) for \( 2 \leq i \leq r \) then clearly the sequence is subcognate. Assuming the contrary, we must have \( r \geq 3 \) and after suitable relabelling we may henceforth also suppose that \( \hat{m}(f_2)/n_2 < c = \hat{m}(f_3)/n_3 \). Now both \( f_1 \) and \( f_2 \) are subconjugates of
f_3$, and hence by applying the coefficient equation of (9.32.7) to the pairs $(f_1, f_3)$ and $(f_2, f_3)$ we get $\text{noc}(f_1, f_3) > \epsilon$. This contradicts the fact that $\epsilon$ is the diameter of our equilateral sequence. Thus our sequence is equicoginate.

**First Packet Lemma (9.37).** Let $F$ and $G$ in $R$ be $Y$-monic of $Y$-degrees $N > 0$ and $M > 0$ respectively, and assume that $J(F, G) \in k((X))$. Then we have the following.

(9.37.1) If $l^*$ is an integer with $1 \leq l^* \leq \chi(G)$ and $\text{int}(F, G_{l^*}) < 0$, then upon letting

$$
\begin{align*}
J &= \{ j : 1 \leq j \leq \chi(F) \text{ with } \text{noc}(F, G_{l^*}) \geq c \}, \\
J' &= \{ j : 1 \leq j \leq \chi(G) \text{ with } \text{noc}(G_{l^*}) \geq c \},
\end{align*}
$$

we have that $c$ is a rational number such that $cv$ is an integer and $(F_j)_{j \in J}, (G_{l^*})_{l \in J'}$ is a packet of $(F, G)$ with diameter $c$, and hence in particular $J \neq \emptyset$ and $l^* \in J'$.

(9.37.2) If $j^*$ is an integer with $1 \leq j^* \leq \chi(F)$ and $\text{int}(F_{j^*}, G) < 0$, then upon letting

$$
\begin{align*}
J &= \{ j : 1 \leq j \leq \chi(F) \text{ with } \text{noc}(F_{j^*}, G) \geq c \}, \\
J' &= \{ j : 1 \leq j \leq \chi(G) \text{ with } \text{noc}(F_{j^*}, G) \geq c \},
\end{align*}
$$

we have that $c$ is a rational number such that $cv$ is an integer and $(F_{j^*})_{j \in J}, (G_{l^*})_{l \in J'}$ is a packet of $(F, G)$ with diameter $c$, and hence in particular $j^* \in J$ and $J' \neq \emptyset$.

**PROOF.** (9.37.1) follows from (9.34) to (9.36). By symmetry, (9.37.2) follows from (9.37.1).

**Second Packet Lemma (9.38).** Let $F$ and $G$ in $R$ be $Y$-monic of $Y$-degrees $N > 0$ and $M > 0$ respectively, and assume that $J(F, G) \in k((X))$. Let $J = \{ j : 1 \leq j \leq \chi(F) \text{ with } \text{int}(F_j, G) < 0 \}$ and $J' = \{ j : 1 \leq j \leq \chi(G) \text{ with } \text{int}(F, G_j) < 0 \}$. Then we have the following.

(9.38.1) There exists a nonnegative integer $r$ together with disjoint partitions \( J = \bigsqcup_{1 \leq i \leq r} J(i) \) and \( J = \bigsqcup_{1 \leq i \leq r} J(i) \) of $J$ and $J'$ into pairwise disjoint nonempty subsets such that $(F_j)_{j \in J(i)}, (G_{l^*})_{l \in J(i)}$ is a packet of $(F, G)$ for $1 \leq i \leq r$. Moreover these partitions are unique up to order.

(9.38.2) If $\text{int}(F, G) \neq 0$ and for $1 \leq j \leq \chi(F)$ and $1 \leq l \leq \chi(G)$ we have $\text{int}(F_j, G) \leq 0$ and $\text{int}(F, G_l) \leq 0$, then $J \neq \emptyset \neq J'$ and in the notation of (9.38.1) we have $r > 0$.

**PROOF.** (9.38.1) follows from (9.37). (9.38.2) is obvious.

**Remark (9.39).** Let $F$ and $G$ in $R$ be $Y$-monic of $Y$-degrees $N > 0$ and $M > 0$ respectively. To elucidate the hypothesis of (9.38.2), and in analogy with the notion of mimint introduced in Section 1, we define the **strict minimal intersection** of $F$ and $G$ by putting

$$
\text{sminint}(F, G) = \min_{u, v \in k^2} \text{int}(F - u, G - v).
$$

Now if $\text{int}(F, G) = \text{sminint}(F, G) \neq 0$ then clearly the hypothesis of (9.38.2) is satisfied, i.e., $\text{int}(F, G) \neq 0$ and for $1 \leq j \leq \chi(F)$ and $1 \leq l \leq \chi(G)$ we have
int\((F_1, G_1) \leq 0\) and int\((F, G) \leq 0\), Continuing the discussion of Remark (2.4), let us say that the pair \((F, G)\) is \textbf{generic} to mean that \(\text{int}(F, G) = \text{sminint}(F, G)\).

By taking indeterminates \(U, V\) over \(R\), we can consider intersection multiplicities in \(k^*(X)[Y]\) where \(k^*\) is an algebraic closure of \(k(U, V)\). Then, assuming \(\text{GCD}(F, G) = 1\), we get

\[
\text{Res}_Y(F - U, G - V) = \Theta(U, V) X^i + \text{terms of } X\text{-degree} > i
\]

where

\[
i = \text{int}(F - U, G - V) \quad \text{and} \quad 0 \neq \Theta(U, V) \in k[U, V].
\]

It follows that

\[
\text{int}(F - U, G - V) = \text{sminint}(F, G)
\]

and hence for any \((u, v) \in k^2\) we have:

\[
(F - u, G - v) \text{ is generic} \iff \Theta(u, v) \neq 0.
\]

It can be shown that if \((F, G) \sim_m (f, g)\) with \(f, g\) in \(k[X, Y]\) then the field degree \([k(X, Y); k(f, g)]\) equals the intersection multiplicity \(\text{int}(F - U, G - V)\). Moreover, if \(f, g\) is an automorphic pair, i.e., if \(k[f, g] = k[X, Y]\), then by [Ab2] we see that \(F, G\) are irreducible over \(k((X))\) and their roots coincide up to the last characteristic term, i.e., they are pseudocognates of each other in the sense of (9.32). This motivates (9.38) where we showed that, under certain condition, \(J(F, G) \in k((X))\) implies that most branches of \(F\) and \(G\) can be partitioned into packets \((F_1, \ldots, F_r, G_1, \ldots, G_s)\) whose members are pseudocognates of each other. Thus (9.38) may be viewed as a small contribution to the jacobian problem.

**Section 10: Enhanced Newton Polygon**

To simplify the statement of Main Lemma (9.31), in the notation of Section 9, assuming \(N > 0\), we let the ENP\((F) = \text{the Enhanced Newton Polygon of } F\) to consist of the two sequences

\[
(O_i(F))_{1 \leq 0 \leq \iota(F)} \quad \text{and} \quad (P_i(F))_{1 \leq i \leq \iota(F)}.
\]

By (9.5) it follows that

\[
\text{ENP}(F) \text{ determines UNP}(F).
\]

Assuming \(N > 0\) and \(M > 0\), for \(0 \leq j \leq \min(\iota(F), \iota(G))\) we say that ENP\((F)\) and ENP\((G)\) are \textbf{j-step parallel}, in symbols we write ENP\((F)||_j\text{ENP}(G)\), if

\[
\begin{cases}
M_{O_0}(F) = M_{O_0}(G), \\
O_i(F) = O_i(G) \text{ for } 1 \leq i \leq j, \text{ and} \\
P_i(F) \text{ and } P_i(G) \text{ are related for } 1 \leq i < j.
\end{cases}
\]

Moreover, we say that ENP\((F)\) and ENP\((G)\) are \textbf{parallel}, in symbols we write ENP\((F)||\text{ENP}(G)\), if

\[
\iota(F) = \iota(G) \text{ and } \text{ENP}(F)||_{\iota(F)}\text{ENP}(G).
\]
Likewise, we say that $\text{ENP}(F)$ is smaller than $\text{ENP}(G)$, in symbols we write $\text{ENP}(F) < \text{ENP}(G)$, if

$$
\begin{cases}
\hat{\mathcal{O}}(F) < \hat{\mathcal{O}}(G) \text{ with } \deg_y \hat{\mathcal{P}}^{(G)} = 1, \text{ and } \\
either \iota(F) = \iota(G) \text{ with } \text{ENP}(F)||_{\iota(F)-1}\text{ENP}(G) \\
\text{and } M\deg_y \hat{\mathcal{P}}^{(F)} = N\deg_y \hat{\mathcal{P}}^{(G)}, \\
or \iota(F) = \iota(G) - 1 \text{ with } \text{ENP}(F)||_{\iota(F)}\text{ENP}(G).
\end{cases}
$$

Finally, we say that $\text{ENP}(F)$ and $\text{ENP}(G)$ are pseudoparallel, in symbols we write $\text{ENP}(F)||\text{ENP}(G)$, if either $\text{ENP}(F)||\text{ENP}(G)$ or $\text{ENP}(F) < \text{ENP}(G)$ or $\text{ENP}(G) < \text{ENP}(F)$. By (9.5) it follows that

$$
\text{ENP}(F)||\text{ENP}(G) \iff \text{UNP}(F)||\text{UNP}(G) \text{ and } P_{i}^{(F)} \text{ and } P_{i}^{(G)} \text{ are related for } 1 \leq i < j
$$

and

$$
\text{ENP}(F) < \text{ENP}(G) \iff \text{UNP}(F) < \text{UNP}(G) \text{ and } P_{i}^{(F)} \text{ and } P_{i}^{(G)} \text{ are related for } 1 \leq i < \iota(G) - 1.
$$

Consequently, (9.31) may be restated in the following equivalent form:

**Main Proposition (10.5).** Let $F$ and $G$ in $R$ be of $Y$-degrees $N > 0$ and $M > 0$ respectively, and assume that $J(F,G) \in k((X))$ and $M\text{O}_0(F) = N\text{O}_0(G)$. Also assume that either $\text{int}(F,Y) < 0$ or $\text{int}(G,Y) < 0$. Then $\text{ENP}(F)||\text{ENP}(G)$. Moreover, if $\hat{\mathcal{O}}(F) = \hat{\mathcal{O}}(G)$ then $\text{ENP}(F)||\text{ENP}(G)$ and hence in particular $(M)\text{int}(F,Y) = (N)\text{int}(G,Y)$ and $M\hat{\mathcal{L}}(F) = N\hat{\mathcal{L}}(G)$.

**Remark (10.6).** Let us define the average or postfinal root order of $F$ by putting

$$
\hat{\mathcal{O}}(F) = (1/N)\text{ord}_Y F(X,0).
$$

Now the equation $(M)\text{int}(F,Y) = (N)\text{int}(G,Y)$ in (9.10), (9.31) and (10.5) may be restated as saying $\hat{\mathcal{O}}(F) = \hat{\mathcal{O}}(G)$. We may postaugument the $O$-sequence by declaring that $O_{\iota(F)-1}(F) = \hat{\mathcal{O}}(F)$ and noting that then: $\text{ENP}(F)||\text{ENP}(G) \Rightarrow \text{UNP}(F)||\text{UNP}(G) \Rightarrow \iota(F) = \iota(G)$ and $O_{j}(F) = O_{j}(G)$ for $0 \leq j \leq \iota(F) + 1$. Finally, we may close-up $\text{CNP}(F)$ by its $(\iota(F) + 1)$-th line whose slope is $\hat{\mathcal{O}}(F)$ and which joins the point $(\hat{\Lambda}_1(F), L_1(F))$ to the point point $(\hat{\Lambda}(F), \hat{L}(F))$, with the understanding that if $\hat{\mathcal{O}}(F) = \infty$ then this is the half-infinite horizontal line emanating from the point $(\Lambda_1(F), L_1(F))$ and going to infinity on the right. This $(\iota(F) + 1)$-th line may be called the hypotenuse of $\text{CNP}(F)$. 
Remark (10.7). The whole game of the Newton Polygon may be redone by starting with the polynomial $P^{(F,c)} = P^{(F,c)}(Y) = \text{inco}_X F(X^\nu, YX^\nu) \in k[Y]$ and noting that $O_i(F) < \cdots < O_\tau(F)$ are exactly those rational numbers $c$ for which this is a true polynomial, i.e., has at least two terms. Now put $P_i^{(F)} = P^{(F,O_i)}$ with $L_i(F) = \deg_Y P_i^{(F)}$ and note that $L_{i+1}(F) = \text{ord}_Y P_i^{(F)}$. Special adjustments have to be made if $F(X,0) = 0$.

Section 11: Concordance with Homology Rank and the Numbers of Milnor and Tjurina

As we said in Section 3, the proof of Formula (3.3.2) follows from Dedekind's Theorem which says that (the ideal generated by) the derivative equals the conductor times the different; see page 65 of [Ab4] where it is paraphrased in the geometric theorem which says that (the ideal generated by) the derivative equals the conductor of the singular locus. Identity (4.8) may be thought of as a modified version of this, and may be codified in the algebraic aphorism: the affine derived size equals the modified affine conductor size plus the modified affine different size plus the degree minus one. Thus for any $f \in R_2$ we call $\text{int}(f,f_Y;\mathcal{A})$ and $\text{int}(f_X,f_Y;\mathcal{A})$ the affine derived size of $f$ and the modified affine conductor size of $f$ respectively; in the algebraic aphorism we are calling $\overline{\beta}(f;\mathcal{A})$ the modified affine different size of $f$. As abbreviations we put

$$\epsilon(f;\mathcal{A}) = \text{int}(f,f_Y;\mathcal{A}) \quad \text{and} \quad \mu(f;\mathcal{A}) = \text{int}(f_X,f_Y;\mathcal{A})$$

where these are nonnegative integers or infinity. Now (4.8) says that if $f \in R_2$ is $Y$-monic of $Y$-degree $N > 0$ with $\gcd(f_Y,f-c;\mathcal{A}) = 1$ for all $c \in k$ then

$$(11.1) \quad \epsilon(f;\mathcal{A}) = \mu(f;\mathcal{A}) + \overline{\beta}(f;\mathcal{A}) + (N - 1)$$

where all the terms are nonnegative integers. By analogy, for any $F \in R$ we put

$$\epsilon(F) = \text{int}(F,F_Y) \quad \text{and} \quad \mu(F) = \text{int}(F_X,F_Y)$$

where these are integers or infinity and we call them the derived size of $F$ and the modified conductor size of $F$ respectively. For any $f \in R_2$ we put

$$\mu_0(f;\mathcal{A}) = \sum_{\{Q \in \mathcal{A} : f_Q(0,0) = 0\}} \text{int}(f_X,f_Y;Q)$$

and

$$\overline{\mu}(f;\mathcal{A}) = \sum_{\{Q \in \mathcal{A} : f_Q(0,0) \neq 0\}} \text{int}(f_X,f_Y;Q)$$

and call these the restricted conductor size of $f$ and the corestricted conductor size of $f$ and we note that then

$$\mu(f;\mathcal{A}) = \mu_0(f;\mathcal{A}) + \overline{\mu}(f;\mathcal{A}) = \sum_{\lambda \in k} \mu_0(f - \lambda;\mathcal{A})$$

where all these quantities are nonnegative integers or infinity. For any $f \in R_2$ we put

$$\rho(f) = \overline{\mu}(f;\mathcal{A}) + \overline{\beta}(f;\mathcal{A})$$

and we call $\rho(f)$ the rank of $f$ and note that it is a nonnegative integer or infinity. Now a paraphrase of (11.1) says that if $f$ is $Y$-monic of $Y$-degree $N > 0$ with
gcd($f_Y, f - c; \mathcal{A}$) = 1 for all $c \in k$ then
\begin{equation}
(11.2)
\epsilon(f; \mathcal{A}) = \mu_0(f) + \rho(f) + (N - 1)
\end{equation}
where all the terms are nonnegative integers. For any $F \in R$ we put
\[\chi(F) = \chi(F) - 1\]
and call this the **decreased branch number** of $F$, and we note that it is an integer $\geq -1$. For any $f \in R_2$ we put
\[\chi(f; \mathcal{A}) = \sum_{\{Q \in \mathcal{A} : f_Q(0,0) = 0\}} \chi(f_Q)\]
and
\[\chi(f; \mathcal{P}) = \sum_{\{Q \in \mathcal{P} : f_Q(0,0) = 0\}} \chi(f_Q)\]
and call these the **decreased affine branch number** of $f$ and the **decreased projective branch number** of $f$ respectively, and note that they are nonnegative integers or infinity. If $f \in R_2$ is $Y$-monic of $Y$-degree $N > 0$ with $\gcd(f_Y, f - c; \mathcal{A}) = 1$ for all $c \in k$ then by (11.6) we see that
\begin{equation}
(11.3)
\mu_0(f; \mathcal{A}) + \chi(f; \mathcal{A}) = 2\delta(f; \mathcal{A})
\end{equation}
where all the terms are nonnegative integers, and hence by (4.10) we get
\begin{equation}
(11.4)
(N - 1)(N - 2) + \chi(f; \mathcal{P}) = 2\delta(f; \mathcal{P}) + \rho(f)
\end{equation}
where all the terms are nonnegative integers. In view of the genus formula (5.3), by (11.4) we see that if $f \in R_2$ is irreducible $Y$-monic of $Y$-degree $N > 0$ with $\gcd(f_Y, f - c; \mathcal{A}) = 1$ for all $c \in k$ then
\begin{equation}
(11.5)
\rho(f) = 2\gamma(f) + \chi(f; \mathcal{P})
\end{equation}
and therefore in this situation our rank $\rho(f)$ coincides with Abhyankar-Sathaye’s rank $\tau(f)$ introduced in their paper [ASa]. With the assumptions as in (11.5), as was pointed out in [ASa], in the complex case, $\rho(f)$ coincides with the **rank of the first homology group** of $f$, i.e., of the point-set $\{(u, v) \in \mathbb{C}^2 : f(u, v) = 0\}$.

Formula (3.3.3) can be paraphrased by saying that if $F \in R_0 = k[[X, Y]]$ is $k$-distinguished of $Y$-degree $N > 0$ with $\rad(F) = F$ then
\begin{equation}
(11.6)
\mu(F) + \chi(F) = 2\delta(F)
\end{equation}
where all the terms are nonnegative integers. In the complex case, a topological proof of (11.6) was given by Milnor [Mil], and $\mu(F)$ is sometimes called the **Milnor number** of $F$.

Continuing with $F \in R_0$ which is $k$-distinguished of $Y$-degree $N > 0$ with $\rad(F) = F$, and recalling that $B(F) = R_0/(FR_0)$, we define the nonnegative integer $\tau(F)$ by putting
\[\tau(F) = \text{the length of the ideal in } B(F) \text{ generated by the images of } F_X \text{ and } F_Y\]
and we call this the **torsion size** of $F$. It is easily seen that $\mu(F)$ is the length of the ideal in $R_0$ generated by $F_X$ and $F_Y$, and hence $\mu(F) \geq \tau(F)$. In [Zar] it is shown that if $F$ is irreducible in $R_0$ then $\tau(F)$ is the length of the torsion submodule of the module of differentials of $B(F)$. In that paper, Zariski gives an interesting characterization of those irreducible $F$ for which $\tau(F) = 2\delta(F)$. In the complex case, $\tau(F)$ is sometimes called the **Tjurina number** of $F$. 

\[\text{gcd} (f_Y, f - c; \mathcal{A}) = 1 \text{ for all } c \in k \text{ then}\]
\[\epsilon (f; \mathcal{A}) = \mu_0 (f) + \rho (f) + (N - 1)\]
Now let us prove the:

\[(11.7)\]

\[
F, G \text{ in } R_0 \text{ where } F \text{ is } k\text{-distinguished}
\]

\[
\Rightarrow \text{int}(F, G) = [R_0/(F, G)R_0 : k] = \text{int}(G, F).
\]

Now let us prove the:

**Supplemented conductor-derivative formula (11.8).** Let \( F \in R_0 \) be \( k\)-distinguished of \( Y\)-degree \( N > 0 \) with \( \text{rad}(F) = F \). Let \( H = H(X, Y) \in R_0 \) be such that \( H = UF \) where \( U = U(X, Y) \in R_0 \) with \( U(0,0) \neq 0 \). Let \( V = V(X, Y) \in R_0 \) and \( W = W(X, Y) \in R_0 \) with \( V(0,0) \neq 0 \neq W(0,0) \) be such that \( VH_Y \) is \( k\)-distinguished of \( Y\)-degree \( N - 1 \), \( WH_X = 0 \) if \( H_X = 0 \), and if \( H_X \neq 0 \) then \( WH_X = X^a[Y^b + c_1(X)Y^{k-1} + \cdots + c_b(X)] \) with nonnegative integers \( a, b \) and elements \( c_1(X), \ldots, c_b(X) \) in \( k[[X]] \) for which \( c_1(0) = \cdots = c_b(0) = 0 \). (\( V \) and \( W \) exists by the Weierstrass Preparation Theorem). Then

\[
\text{int}(H_X, VH_Y) = \text{int}(F_X, F_Y) = \text{int}(F, F_Y) - N + 1
\]

(11.8.1)

and

\[
\text{int}(H, VH_Y) - N + 1 = \text{int}(F, F_Y) - N + 1 = 2\delta(F) - \chi(F) + 1
\]

and

\[
\text{int}(WH_X, VH_Y) = \text{int}(F_X, F_Y) = 2\delta(F) - \chi(F) + 1
\]

(11.8.3)

where all the terms in the above three items are integers.

**PROOF.** By taking \((VH_Y, H)\) for \((F, G)\) in (2.2) we see that

\[
\text{int}(VH_Y, H_X) = \text{int}(H_Y, H) - N + 1 + \beta(VH_Y, H)
\]

where each term is an integer. By (3.1) we have \( \beta(VH_Y, H) = 0 \), and clearly \( \text{int}(VH_Y, H_X) = \text{int}(H_X, VH_Y) \) and \( \text{int}(VH_Y, H) = \text{int}(H, VH_Y) \), and hence by the above display we get

\[
\text{int}(H_X, VH_Y) = \text{int}(H, VH_Y) - N + 1
\]

(1)

where each term is an integer. By taking \( U = 1 \) in (1) we get the equation

\[
\text{int}(F_X, F_Y) = \text{int}(F, F_Y) - N + 1
\]

(2)

where each term is an integer. In view of (11.7), by the derivative formula \( H_Y = U_Y F + UF_Y \) we see that

\[
\text{int}(H, VH_Y) = \text{int}(F, F_Y).
\]

(3)

In view of (11.7), by (1), (2), and (3) we get the equations

\[
\text{int}(WH_X, VH_Y) = \text{int}(H_X, VH_Y) = \text{int}(F_X, F_Y)
\]

(4)

and the equations (11.8.1). By Dedekind’s Theorem (see pages 65 and 150 of [Ab4]) we have

\[
\text{int}(F, F_Y) - N + 1 = 2\delta(F) - \chi(F) + 1.
\]

By (3) and (5) we get (11.8.2). By (2), (4), and (5) we get (11.8.3).
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