Spin 1 particle in the magnetic monopole potential, nonrelativistic approximation. Minkowski and Lobachevski spaces

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Abstract

Spin 1 particle is treated in presence of the Dirac magnetic monopole. Separation of the variables is done in the matrix 10-component Duffin–Kemmer–Petiau wave equation. In the radial equations, the nonrelativistic approximation has been performed, there arise system of three second order interrelated differential equations, associated with spin 1 particle in the Pauli description. These three equations can be disconnected with the use of special linear transformation making the mixing matrix diagonal. As result, there arise three separated differential equations which contain roots $A_k$ of cubic algebraic equation as parameters. The algorithm permits extension to the presence of additional external spherically symmetrical fields, the Coulomb and oscillator potential are considered in detail. For both cases, there are found three series of energy spectra of the spin 1 particle, $\epsilon = \epsilon (A_k, j, n)$. Special attention is given to states with minimal value of total conserved moment $j$.

This approach is extended to the case of Lobachevsky geometry background. Three corresponding of second order differential interrelated equations, which cannot be disconnected in presence of the monopole. Progress is possible only in presence of pure Coulomb and pure oscillator fields, both problems are treatable in the frames of the general Heun equation. With the use of special requirement, reasonable from physical standpoint energy spectra have been obtained.

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1 Introduction

Spin significantly influences on behavior of quantum-mechanical particles in the field of Dirac monopole – see [1, 2]. Till now, mainly the particles of spin 0 and 1/2 were investigated. The case of

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Dirac particle exhibits existence of very peculiar solutions which can be associated with bound states of the particle in the external magnetic monopole field, existence of similar states for spin 1 particle was demonstrated in [8]). However, other solutions for spin 1 case were not constructed.

The progress in the study of spin 1 theory in presence of the Dirac monopole can be reached when using the non-relativistic approximation. Besides, it is possible to take into account additionally Coulomb and oscillator potentials.

2 Separation of the variables in DKP-equation

Spin 1 particle in monopole potential will be treated on the base of the matrix approach in the frames of the tetrad formalism (see notation in [7]); the DKP equation reads (we use the spherical tetrad [8])

\[
\left[i \beta^0 \partial_t + i \left( \beta^3 \partial_r + \frac{1}{r} (\beta^1 j^{31} + \beta^2 j^{32}) \right) + \frac{1}{r} \Sigma_{\theta,\phi}^k - M \right] \Phi(x) = 0 \ ,
\]

where

\[
\Sigma_{\theta,\phi}^k = i \beta^1 \partial_\theta + \beta^2 \frac{i \partial_\phi + (i j^{12} - k) \cos \theta}{\sin \theta} .
\]

Parameter \( k = eg/\hbar c \) is quantized according to the Dirac rule [1]: \(|k| = 1/2, 1, 3/2, 2, ... \)

It is convenient to use so-called cyclic representation for DKP-matrices [8, 9]; correspondingly, the third projection of spin \( iJ^{12} \) has a diagonal structure

\[
iJ^{12} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & t_3 & 0 & 0 \\
0 & 0 & t_3 & 0 \\
0 & 0 & 0 & t_3
\end{pmatrix} , \quad t_3 = \begin{pmatrix}
+1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{pmatrix} .
\]

The components of the total conserved moment [1] in spherical tetrad are determined by the formulas [8]:

\[
J^k_1 = l_1 + \frac{\cos \phi}{\sin \theta} (iJ^{12} - \kappa) , \quad J^k_2 = l_2 + \frac{\sin \phi}{\sin \theta} (iJ^{12} - \kappa) , \quad J^k_3 = l_3 .
\]

In accordance with the general method [8], wave functions of the spin 1 particle with quantum numbers \((\epsilon, j, m)\) are constructed within the substitution

\[
\Phi_{\epsilon jm}(x) = e^{-i\epsilon t} \left[ f_1(r) D_k, f_2(r) D_{k-1}, f_3(r) D_{k}, f_4(r) D_{k+1}, f_5(r) D_{k-1}, f_6(r) D_k, f_7(r) D_{k+1}, f_8(r) D_{k-1}, f_9(r) D_k, f_{10}(r) D_{k+1} \right] ,
\]

where symbol \( D_{\sigma} \) stands for the Wigner [9] functions \( D^j_{m,\sigma}(\phi, \theta, 0) \). Below, when producing equations for ten radial functions \( f_1, \ldots, f_{10} \), we will need recurrent relations [9]:

\[
\partial_\theta D_{k-1} = a D_{k-2} - c D_k , \quad \frac{m - (k - 1) \cos \theta}{\sin \theta} D_{k-1} = -a D_{k-2} - c D_k ,
\]

\[
\partial_\theta D_k = (c D_{k-1} - d D_{k+1}) , \quad \frac{m - k \cos \theta}{\sin \theta} D_k = -c D_{k-1} - d D_{k+1} ,
\]

\[
\partial_r D_k = \left( \frac{m + k \cos \theta}{\sin \theta} D_k - r \right) , \quad \frac{m + (k - 1) \cos \theta}{\sin \theta} D_k = \left( r \right) .
\]
\[ \partial_\theta D_{k+1} = (d D_k - b D_{k+2}) , \quad -m - (k + 1) \cos \theta \sin \theta D_{k+1} = -d D_k - b D_{k+2} , \]

where
\[ a = \frac{1}{2} \sqrt{(j + k - 1)(j + k + 2)} , \quad b = \frac{1}{2} \sqrt{(j - k - 1)(j + k + 2)} , \quad c = \frac{1}{2} \sqrt{(j + k)(j - k + 1)} , \quad d = \frac{1}{2} \sqrt{(j - k)(j + k + 1)} . \]

From (2.1), it follows the radial system
\[ -i (\frac{d}{dr} + \frac{1}{r}) f_2 - i \frac{\sqrt{2}c}{r} f_3 - M f_8 = 0 , \]
\[ i (\frac{d}{dr} + \frac{1}{r}) f_4 + i \frac{\sqrt{2}d}{r} f_3 - M f_{10} = 0 , \]
\[ i c f_5 + i (\frac{d}{dr} + \frac{1}{r}) f_8 + i \frac{\sqrt{2}c}{r} f_9 - M f_2 = 0 , \]
\[ i c f_7 - i (\frac{d}{dr} + \frac{1}{r}) f_{10} - i \frac{\sqrt{2}d}{r} f_9 - M f_4 = 0 , \]
\[ -i e f_2 + \frac{\sqrt{2}c}{r} f_1 - M f_5 = 0 , \]
\[ -i e f_4 + \frac{\sqrt{2}d}{r} f_1 - M f_7 = 0 , \]
\[ -(\frac{d}{dr} + \frac{2}{r}) f_6 - \frac{\sqrt{2}}{r} (c f_5 + d f_7) - M f_1 = 0 , \]
\[ i e f_7 + \frac{\sqrt{2}c}{r} (c f_5 + d f_7) - M f_3 = 0 , \]
\[ i \frac{\sqrt{2}}{r} (c f_2 - d f_4) - M f_9 = 0 , \quad -i e f_3 - \frac{d}{dr} f_1 - M f_6 = 0 . \] (2.5)

Pauli criterium [8] allows for \( j \) to take values (we are to examine separately the value \( k = \pm 1/2 \) and all remaining ones \( k \):
\[ k = \pm 1/2 , \quad j = \mid k \mid , \mid k \mid + 1, ... \]
\[ k = \pm 1, \pm 3/2, ... \quad j = \mid k \mid - 1, \mid k \mid , \mid k \mid + 1, ... \] (2.6)

In both cases, states with \( j = \mid k \mid - 1 \) should be treated with special caution, because system (1.4) is meaningless in these cases.

Let us turn to states with \( j = \mid k \mid - 1 \). For states with \( j = 0 \) (it is possible when \( k = \pm 1 \)), the corresponding wave function does not depend on the angular variables \( \theta, \phi \). Let it be \( k = +1 \) and \( j = 0 \), then the initial substitution is
\[ \Phi^{(0)}(t, r) = e^{-i \epsilon t} ( 0, f_2, 0, 0, f_5, 0, 0; f_8, 0, 0 ) . \] (2.7)

It is readily checked that the angular operator \( \Sigma_{\theta, \phi} \) acts on \( \Phi^{(0)} \) as zero operator: \( \Sigma_{\theta, \phi} \Phi^{(0)} = 0 \). This results in only three nontrivial radial equations
\[ i \epsilon f_5 + i (\frac{d}{dr} + \frac{1}{r}) f_8 - M f_2 = 0 , \]
\[ -i \epsilon f_2 - M f_5 = 0 , \quad -i (\frac{d}{dr} + \frac{1}{r}) f_2 - M f_8 = 0 . \] (2.8)
From whence it follows

\[ f_5 = -i \frac{\epsilon}{M} f_2, \quad f_8 = -i \frac{\epsilon}{M} \left( \frac{d}{dr} + \frac{1}{r} \right) f_2; \]

and for \( F_2(r) \), determined by substitution \( f_2(r) = r^{-1} F_2(r) \), we get the following equation

\[ \left( \frac{d^2}{dr^2} + \epsilon^2 - M^2 \right) F_2 = 0. \tag{2.9} \]

It is exactly the same equation which arises in the electro case in similar situation \[8\].

The case of \( j = 0, k = -1 \) can be treated in similar manner. The initial substitution is

\[ \Phi^{(0)}(t, r) = e^{-i \epsilon t} ( 0, 0, f_4, 0, 0, 0, 0, f_7, 0, 0, f_{10} ); \tag{2.10} \]

radial equations are

\[ i \epsilon f_7 - i \left( \frac{d}{dr} + \frac{1}{r} \right) f_{10} - M f_4 = 0, \]
\[ -i f_4 - M f_7 = 0, \quad i \left( \frac{d}{dr} + \frac{1}{r} \right) f_4 - M f_{10} = 0. \tag{2.11} \]

As result, we derive

\[ f_7 = -i \frac{\epsilon}{M} f_4, \quad f_{10} = i \frac{\epsilon}{M} \left( \frac{d}{dr} + \frac{1}{r} \right) f_2, \]
\[ \left( \frac{d^2}{dr^2} + \epsilon^2 - M^2 \right) F_4 = 0, \quad f_4 = \frac{1}{r} F_4. \tag{2.12} \]

Now, let us consider the case of minimal values \( j = |k| - 1 \) with \( k = \pm 3/2, \pm 2, \ldots \). Firstly, assume \( k \) be positive – in this case we should start with the substitution

\[ k \geq 3/2, \quad \Phi^{(0)} = e^{-i \epsilon t} ( 0, f_2 D_{k-1}, 0, 0; f_5 D_{k-1}, 0, 0; f_8 D_{k-1}, 0, 0 ). \tag{2.13} \]

With the use of recurrent relations \[9\]

\[ \partial_\theta D_{k-1} = \sqrt{\frac{k-1}{2}} D_{k-2}, \quad \frac{-m - (k - 1) \cos \theta}{\sin \theta} D_k = -\sqrt{\frac{k-1}{2}} D_{k-2}, \]

we derive

\[ \Sigma_{\theta, \phi} \Phi^{(0)} = 0. \]

Therefore, radial system for \( f_2, f_5, f_8 \) must coincide with \eqref{eq:2.8}. The case \( j = |k| - 1 \) with negative \( k \) looks similarly:

\[ k \leq -3/2, \quad \Phi^{(0)} = e^{-i \epsilon t} ( 0, 0, 0, f_4 D_{k+1}, 0, 0, f_7 D_{k+1}, 0, 0, f_{10} D_{k+1}); \tag{2.14} \]

the identity holds \( \Sigma_{\theta, \phi} \Phi^{(0)} = 0 \) again, and the radial system coincides with \eqref{eq:2.11}.

Thus, all states with minimal values \( j, j = |k| - 1 \), lead to the simple solutions of exponential form. Till now, they are only states which are treatable and solvable analytically.

To proceed further in the next section, we will perform nonrelativistic approximation in the radial system, after that the problem arising can be solved exactly.
3 Pauli approximation

Let us employ the method of producing non-relativistic equations previously used in [8]. First, in (1.4) let us separate four equations which permit to exclude non-dynamical variables

\[-(\frac{d}{dr} + \frac{1}{r}) f_6 - \frac{\sqrt{2}}{r} (c f_5 + d f_7) = M f_1 , \quad -(\frac{d}{dr} + \frac{1}{r}) f_2 - i\frac{\sqrt{2}c}{r} f_3 = M f_8 ,
\]
\[i\frac{\sqrt{2}}{r} (c f_2 - d f_4) = M f_9 , \quad i(\frac{d}{dr} + \frac{1}{r}) f_4 + i\frac{\sqrt{2}d}{r} f_3 = M f_10 . \quad (3.1)\]

From remaining six equations, multiplied by M:

\[i\epsilon M f_5 + i(\frac{d}{dr} + \frac{1}{r}) M f_8 + i\frac{\sqrt{2}c}{r} M f_9 - M^2 f_2 = 0 ,
\]
\[i\epsilon M f_6 + \frac{\sqrt{2}i}{r} (c M f_8 + d M f_10) - M^2 f_3 = 0 ,
\]
\[i\epsilon M f_7 - i(\frac{d}{dr} + \frac{1}{r}) M f_10 - i\frac{\sqrt{2}d}{r} M f_9 - M^2 f_4 = 0 ,
\]
\[-i\epsilon M f_2 + \frac{\sqrt{2}c}{r} M f_1 - M^2 f_5 = 0 ,
\]
\[-i\epsilon M f_3 - \frac{d}{dr} M f_1 - M^2 f_6 = 0 ,
\]
\[-i\epsilon M f_4 + \frac{\sqrt{2}d}{r} M f_1 - M^2 f_7 = 0 , \quad (3.2)\]

one exclude the variables [3.1]:

\[i\epsilon M f_5 + i(\frac{d}{dr} + \frac{1}{r}) \left[-i(\frac{d}{dr} + \frac{1}{r}) f_2 - i\frac{\sqrt{2}c}{r} f_3\right] + i\frac{\sqrt{2}c}{r} \left[i\frac{\sqrt{2}}{r} (c f_2 - d f_4)\right] - M^2 f_2 = 0 ,
\]
\[i\epsilon M f_6 + \frac{\sqrt{2}i}{r} \left[-c \left(-i(\frac{d}{dr} + \frac{1}{r}) f_2 - i\frac{\sqrt{2}c}{r} f_3\right) + d \left(i(\frac{d}{dr} + \frac{1}{r}) f_4 + i\frac{\sqrt{2}d}{r} f_3\right)\right] - M^2 f_3 = 0 ,
\]
\[i\epsilon M f_7 - i(\frac{d}{dr} + \frac{1}{r}) \left[i(\frac{d}{dr} + \frac{1}{r}) f_4 + i\frac{\sqrt{2}d}{r} f_3\right] - i\frac{\sqrt{2}d}{r} \left[i\frac{\sqrt{2}}{r} (c f_2 - d f_4)\right] - M^2 f_4 = 0 ,
\]
\[-i\epsilon M f_2 + \frac{\sqrt{2}c}{r} \left[-(\frac{d}{dr} + \frac{2}{r}) f_6 - \frac{\sqrt{2}}{r} (c f_5 + d f_7)\right] - M^2 f_5 = 0 ,
\]
\[-i\epsilon M f_3 - \frac{d}{dr} \left[-(\frac{d}{dr} + \frac{2}{r}) f_6 - \frac{\sqrt{2}}{r} (c f_5 + d f_7)\right] - M^2 f_6 = 0 ,
\]
\[-i\epsilon M f_4 + \frac{\sqrt{2}d}{r} \left[-(\frac{d}{dr} + \frac{2}{r}) f_6 - \frac{\sqrt{2}}{r} (c f_5 + d f_7)\right] - M^2 f_7 = 0 . \quad (3.3)\]

After that, let us use translate equations [3.3] to the more symmetrical notation

\[(f_2, f_3, f_4) \rightarrow (\Phi_1, \Phi_2, \Phi_3) , \quad (f_5, f_6, f_7) \rightarrow (E_1, E_2, E_3) ; \quad (3.4)\]
thus we derive

\[ i\epsilon ME_1 + i(\frac{d}{dr} + \frac{1}{r}) \left[-i(\frac{d}{dr} + \frac{1}{r})\Phi_1 - i\frac{\sqrt{2}c}{r}\Phi_2 \right] + i\frac{\sqrt{2}c}{r} \left[i\frac{\sqrt{2}}{r}(c\Phi_1 - d\Phi_3)\right] - M^2\Phi_1 = 0 , \]

\[ i\epsilon ME_2 + \frac{\sqrt{2}i}{r} \left[-c \left(-i(\frac{d}{dr} + \frac{1}{r})\Phi_1 - i\frac{\sqrt{2}c}{r}\Phi_2 \right) + d \left(i(\frac{d}{dr} + \frac{1}{r})\Phi_3 + i\frac{\sqrt{2}d}{r}\Phi_2 \right)\right] - M^2\Phi_2 = 0 , \]

\[ i\epsilon ME_3 - i(\frac{d}{dr} + \frac{1}{r}) \left[i(\frac{d}{dr} + \frac{1}{r})\Phi_3 + i\frac{\sqrt{2}d}{r}\Phi_2 \right] - i\frac{\sqrt{2}d}{r} \left[i\frac{\sqrt{2}}{r}(c\Phi_1 - d\Phi_3)\right] - M^2\Phi_3 = 0 , \]

\[ -i\epsilon M\Phi_1 + \frac{\sqrt{2}c}{r} \left[-(\frac{d}{dr} + \frac{2}{r})E_2 - \frac{\sqrt{2}}{r}(cE_1 + dE_3)\right] - M^2E_1 = 0 , \]

\[ -i\epsilon M\Phi_2 - \frac{d}{dr} \left[-(\frac{d}{dr} + \frac{2}{r})E_2 - \frac{\sqrt{2}}{r}(cE_1 + dE_3)\right] - M^2E_2 = 0 , \]

\[ -i\epsilon M\Phi_3 + \frac{\sqrt{2}d}{r} \left[-(\frac{d}{dr} + \frac{2}{r})E_2 - \frac{\sqrt{2}}{r}(cE_1 + dE_3)\right] - M^2E_3 = 0 . \]  

(3.5)

Big \( \Psi_j \) and small \( \psi_j \) constituents are introduced by the linear relations [8]:

\[ \Psi_j = \Phi_j + iE_j , \quad \psi_j = \Phi_j - iE_j . \]  

(3.6)

After regrouping the above equations (3.5) in pairs, and in the same time separating the rest energy with the help of formal change \( \epsilon = M + E \), we get

\[ i(M + E)ME_1 + i(\frac{d}{dr} + \frac{1}{r}) \left[-i(\frac{d}{dr} + \frac{1}{r})\Phi_1 - i\frac{\sqrt{2}c}{r}\Phi_2 \right] \]

\[ + i\frac{\sqrt{2}c}{r} \left[i\frac{\sqrt{2}}{r}(c\Phi_1 - d\Phi_3)\right] - M^2\Phi_1 = 0 , \]

\[ -i(M + E)\Phi_1 + \frac{\sqrt{2}c}{r} \left[-(\frac{d}{dr} + \frac{2}{r})E_2 - \frac{\sqrt{2}}{r}(cE_1 + dE_3)\right] - M^2E_1 = 0 , \]  

(3.7)

\[ i(M + E)ME_2 + \frac{\sqrt{2}i}{r} \left[-c \left(-i(\frac{d}{dr} + \frac{1}{r})\Phi_1 - i\frac{\sqrt{2}c}{r}\Phi_2 \right) \right. \]

\[ + d \left(i(\frac{d}{dr} + \frac{1}{r})\Phi_3 + i\frac{\sqrt{2}d}{r}\Phi_2 \right)\right] - M^2\Phi_2 = 0 , \]

\[ -i(M + E)\Phi_2 - \frac{d}{dr} \left[-(\frac{d}{dr} + \frac{2}{r})E_2 - \frac{\sqrt{2}}{r}(cE_1 + dE_3)\right] - M^2E_2 = 0 , \]  

(3.8)
\[ i(M + E)ME_3 - i\left(\frac{d}{dr} + \frac{1}{r}\right) \left[ i\left(\frac{d}{dr} + \frac{1}{r}\right)\Phi_3 + \frac{i\sqrt{2d}}{r}\Phi_2 \right] \]
\[ - \frac{i\sqrt{2d}}{r} \left[ \frac{i\sqrt{2}}{r}(c\Phi_1 - d\Phi_3) \right] - M^2\Phi_3 = 0 , \]
\[ -i(M + E)M\Phi_3 + \frac{\sqrt{2d}}{r} \left[ -\left(\frac{d}{dr} + \frac{2}{r}\right)E_2 - \frac{\sqrt{2}}{r}(cE_1 + dE_3) \right] - M^2E_3 = 0 . \tag{3.9} \]

Let us consider the pair \textit{3.7}:
\[ iM^2E_1 + iEME_1 + \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\Phi_1 + \frac{\sqrt{2c}}{r}\frac{d}{dr}\Phi_2 - \frac{2c}{r^2}(c\Phi_1 - d\Phi_3) - M^2\Phi_1 = 0, \]
\[ -iM^2\Phi_1 - iEM\Phi_1 - \frac{\sqrt{2c}}{r}\left(\frac{d}{dr} + \frac{2}{r}\right)E_2 - \frac{2c}{r^2}(cE_1 + dE_3) - M^2E_1 = 0 ; \]
translating it to the big and small components
\[ M^2(\Psi_1 - \psi_1) + EM(\Psi_1 - \psi_1) + \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)(\Psi_1 + \psi_1) + \frac{\sqrt{2c}}{r}\frac{d}{dr}(\Psi_2 + \psi_2) - \]
\[ - \frac{2c}{r^2}[c(\Psi_1 + \psi_1) - d(\Psi_3 + \psi_3)] - M^2(\Psi_1 + \psi_1) = 0 , \]
\[ -iM^2(\Psi_1 + \psi_1) - iEM(\Psi_1 + \psi_1) + \frac{i\sqrt{2c}}{r}\left(\frac{d}{dr} + \frac{2}{r}\right)(\Psi_2 - \psi_2) + \]
\[ + \frac{2c}{r^2}[c(\Psi_1 - \psi_1) + d(\Psi_3 - \psi_3)] + iM^2(\Psi_1 - \psi_1) = 0 , \]
and regrouping the terms we arrive at
\[ - M^2\psi_1 + EM(\Psi_1 - \psi_1) + \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)(\Psi_1 + \psi_1) + \frac{\sqrt{2c}}{r}\frac{d}{dr}(\Psi_2 + \psi_2) - \]
\[ - \frac{2c}{r^2}[c(\Psi_1 + \psi_1) - d(\Psi_3 + \psi_3)] = 0 , \]
\[ -M^2\psi_1 - EM(\Psi_1 + \psi_1) + \frac{\sqrt{2c}}{r}\left(\frac{d}{dr} + \frac{2}{r}\right)(\Psi_2 - \psi_2) + \]
\[ + \frac{2c}{r^2}[c(\Psi_1 - \psi_1) + d(\Psi_3 - \psi_3)] = 0 . \tag{3.10} \]
Subtracting the second equation from the first, we obtain
\[ \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} + 2EM \right)\Psi_1 - \frac{2\sqrt{2c}}{r^2}\Psi_2 - \frac{4c^2}{r^2}\Psi_1 = 0 ; \tag{3.11} \]
this equation contains only big components and represent one of equations in the Pauli approximation.

Let us consider eqs. \textit{3.9}:
\[ iM^2E_3 + iEME_3 + \left(\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr}\right)\Phi_3 + \frac{\sqrt{2d}}{r}\frac{d}{dr}\Phi_2 + \frac{2d}{r^2}(c\Phi_1 - d\Phi_3) - M^2\Phi_3 = 0 , \]
\[ -iM^2\Phi_3 - iEM\Phi_3 - \frac{\sqrt{2d}}{r}\left(\frac{d}{dr} + \frac{2}{r}\right)E_2 - \frac{2d}{r^2}(cE_1 + dE_3) - M^2E_3 = 0 .\]
Translating them to the big and small components (and separating the rest energy)

\[
M^2(\Psi_3 - \psi_3) + EM(\Psi_3 - \psi_3) + \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right)(\Psi_3 + \psi_3) + \frac{\sqrt{2d}}{r} \frac{d}{dr}(\Psi_2 + \psi_2) + \\
\frac{2d}{r^2} [c(\Psi_1 + \psi_1) - d(\Psi_3 + \psi_3)] - M^2(\Psi_3 + \psi_3) = 0 ,
\]

\[
-iM^2(\Psi_3 + \psi_3) - iEM(\Psi_3 + \psi_3) + i\frac{\sqrt{2d}}{r} \left( \frac{d}{dr} + \frac{2}{r} \right)(\Psi_2 - \psi_2) - \\
\frac{2d}{r^2} [c(\Psi_1 - \psi_1) + d(\Psi_3 - \psi_3)] + iM^2(\Psi_3 - \psi_3) = 0 .
\]

after some regrouping the terms we have

\[
-2M^2\psi_3 + EM(\Psi_3 - \psi_3) + \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right)(\Psi_3 + \psi_3) + \frac{\sqrt{2d}}{r} \frac{d}{dr}(\Psi_2 + \psi_2) \\
\frac{2d}{r^2} [c(\Psi_1 + \psi_1) - d(\Psi_3 + \psi_3)] = 0 ,
\]

\[
-2M^2\psi_3 - EM(\Psi_3 + \psi_3) + \frac{\sqrt{2d}}{r} \left( \frac{d}{dr} + \frac{2}{r} \right)(\Psi_2 - \psi_2) \\
\frac{2d}{r^2} [c(\Psi_1 - \psi_1) + d(\Psi_3 - \psi_3)] = 0 .
\]

Subtracting the second equation from the first, we get

\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2EM \right) \Psi_3 - \frac{2\sqrt{2d}}{r^2} \Psi_2 - \frac{4d^2}{r^2} \Psi_3 = 0 ;
\]

(3.13)

it is the second differential equation in the Pauli approximation.

Finally, let us consider the pair (3.8):

\[
iM^2E_2 + iEME_2 - \frac{\sqrt{2d}}{r} \left( \frac{d}{dr} + \frac{1}{r} \right)\Phi_1 - \frac{2c^2}{r^2} \Phi_2 - \\
- \frac{\sqrt{2d}}{r} \left( \frac{d}{dr} + \frac{1}{r} \right)\Phi_3 - \frac{2d^2}{r^2} \Phi_2 - M^2\Phi_2 = 0 ,
\]

\[
-iM^2\Phi_2 - iEM\Phi_2 + \frac{d}{dr} \left( \frac{d}{dr} + \frac{2}{r} \right) E_2 + \frac{d}{dr} \left( \frac{d}{dr} + \frac{2}{r} \right)(cE_1 + dE_3) - M^2E_2 = 0 .
\]

They give

\[
M^2(\Psi_2 - \psi_2) + EM(\Psi_2 - \psi_2) - \frac{\sqrt{2d}}{r} \left( \frac{d}{dr} + \frac{1}{r} \right)(\Psi_1 + \psi_1) - \frac{2c^2}{r^2}(\Psi_2 + \psi_2) - \\
- \frac{\sqrt{2d}}{r} \left( \frac{d}{dr} + \frac{1}{r} \right)(\Psi_3 - \psi_3) - \frac{2d^2}{r^2}(\Psi_2 + \psi_2) - M^2(\Psi_2 + \psi_2) = 0 ,
\]

\[
-iM^2(\Psi_2 + \psi_2) - iEM(\Psi_2 + \psi_2) - i\frac{d}{dr} \left( \frac{d}{dr} + \frac{2}{r} \right)(\Psi_2 - \psi_2) - \\
- i\frac{d}{dr} \left[ c(\Psi_1 - \psi_1) + d(\Psi_3 - \psi_3) \right] + iM^2(\Psi_2 - \psi_2) = 0 ,
\]

8
or differently

\[-2M^2 \psi_2 + EM (\Psi_2 - \psi_2) - \frac{\sqrt{2}c}{r} \left( \frac{d}{dr} + \frac{1}{r} \right) (\Psi_1 + \psi_1) - \frac{2c^2}{r^2} (\Psi_2 + \psi_2)\]

\[-\frac{\sqrt{2}d}{r} \left( \frac{d}{dr} + \frac{1}{r} \right) (\Psi_3 - \psi_3) - \frac{2d^2}{r^2} (\Psi_2 + \psi_2) = 0 , \]

\[-2M^2 \psi_2 - EM (\Psi_2 + \psi_2) - \frac{d}{dr} \left( \frac{d}{dr} + \frac{2}{r} \right) (\Psi_2 - \psi_2)\]

\[-\frac{d}{dr} \left( c(\Psi_1 - \psi_1) + d(\Psi_3 - \psi_3) \right) = 0 . \quad (3.14)\]

Subtracting the second equation from the first, we produce

\[\left( 2EM - \frac{2c^2}{r^2} - \frac{2d^2}{r^2} + \frac{2d}{r} \frac{d}{dr} - \frac{2}{r^2} \right) \Psi_2 + \frac{\sqrt{2}c}{r} \left( - \frac{d}{dr} - \frac{1}{r} + \frac{d}{dr} - \frac{1}{r} \right) \Psi_1 + \frac{\sqrt{2}d}{r} \left( - \frac{d}{dr} - \frac{1}{r} + \frac{d}{dr} - \frac{1}{r} \right) \Psi_3 = 0 ;\]

from this we arrive at

\[\left( \frac{d^2}{dr^2} + \frac{2d}{r} + 2EM \right) \Psi_2 - \frac{2(c^2 + d^2 + 1)}{r^2} \Psi_2 - \frac{2 \sqrt{2}c}{r^2} \Psi_1 - \frac{2 \sqrt{2}d}{r^2} \Psi_3 = 0 . \quad (3.15)\]

It is the third equation of the Pauli system.

4 Solutions of the radial Pauli equations

No we are to examine radial system in the Pauli approximation

\[\left( \frac{d^2}{dr^2} + \frac{2d}{r} + 2EM \right) \Psi_1 - \frac{2 \sqrt{2}c}{r^2} \Psi_2 - \frac{4c^2}{r^2} \Psi_1 = 0 , \]

\[\left( \frac{d^2}{dr^2} + \frac{2d}{r} + 2EM \right) \Psi_2 - \frac{2(c^2 + d^2 + 1)}{r^2} \Psi_2 - \frac{2 \sqrt{2}c}{r^2} \Psi_1 - \frac{2 \sqrt{2}d}{r^2} \Psi_3 = 0 , \]

\[\left( \frac{d^2}{dr^2} + \frac{2d}{r} + 2EM \right) \Psi_3 - \frac{2 \sqrt{2}d}{r^2} \Psi_2 - \frac{4d^2}{r^2} \Psi_3 = 0 . \quad (4.1)\]

It is convenient to introduce the shortening notation

\[\frac{1}{2} r^2 \left( \frac{d^2}{dr^2} + \frac{2d}{r} + 2EM \right) = \Delta \]

then system (4.1) is written as

\[\Delta \Psi = A \Psi , \quad A = \begin{vmatrix} 2c^2 & \sqrt{2}c & 0 \\ \sqrt{2}c & (c^2 + d^2 + 1) & \sqrt{2}d \\ 0 & \sqrt{2}d & 2d^2 \end{vmatrix} , \quad \Psi = \begin{vmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{vmatrix} ; \quad (4.2)\]
the nonrelativistic wave function is determined by the relations

$$\Phi_{cjm}(x) = e^{-i t} \begin{pmatrix} \Psi_1(r) D_{k-1} \\ \Psi_2(r) D_k \\ \Psi_3(r) D_{k+1} \end{pmatrix} = e^{-i t} \begin{pmatrix} (\Phi_1 + i E_1) D_{k-1} \\ (\Phi_2 + i E_2) D_k \\ (\Phi_3 + i E_3) D_{k+1} \end{pmatrix}.$$  

To solve the system (4.2), we are to diagonalize the matrix $A$. Equations for determining corresponding transformation $S$ and diagonal elements $A_1, A_2, A_3$ have the form

$$\begin{vmatrix} 2c^2 & \sqrt{2c} & 0 \\ \sqrt{2c} & (c^2 + d^2 + 1) & \sqrt{2d} \\ 0 & 2d^2 & s_{11} \ s_{12} \ s_{13} \end{vmatrix} = \begin{vmatrix} A_1 \ 0 \ 0 \end{vmatrix}.$$  

(4.3)

Arising equations can be divided into three groups:

$$2c^2 s_{11} + \sqrt{2c} s_{21} = s_{11} A_1 ,$$

$$\sqrt{2c} s_{11} + (c^2 + d^2 + 1) s_{21} + \sqrt{2d} s_{31} = s_{21} A_1 ;$$

$$\sqrt{2c} s_{12} + \sqrt{2c} s_{22} = s_{12} A_2 ,$$

$$\sqrt{2c} s_{12} + (c^2 + d^2 + 1) s_{22} + \sqrt{2d} s_{32} = s_{22} A_2 ;$$

$$\sqrt{2c} s_{13} + \sqrt{2c} s_{23} = s_{13} A_3 ,$$

$$\sqrt{2c} s_{13} + (c^2 + d^2 + 1) s_{23} + \sqrt{2d} s_{33} = s_{23} A_3 ;$$

$$\sqrt{2c} s_{13} + \sqrt{2c} s_{23} + \sqrt{2d} s_{33} = s_{33} A_3 .$$  

(4.4)

Thus, we get three homogeneous linear systems of one the same structure:

$$\begin{vmatrix} (2c^2 - A_1) & \sqrt{2c} & 0 \\ \sqrt{2c} & (c^2 + d^2 + 1 - A_1) & \sqrt{2d} \\ 0 & \sqrt{2d} & (2d^2 - A_1) \end{vmatrix} = \begin{vmatrix} s_{11} \\ s_{21} \\ s_{31} \end{vmatrix} = 0 ,$$

$$\begin{vmatrix} (2c^2 - A_2) & \sqrt{2c} & 0 \\ \sqrt{2c} & (c^2 + d^2 + 1 - A_2) & \sqrt{2d} \\ 0 & \sqrt{2d} & (2d^2 - A_2) \end{vmatrix} = \begin{vmatrix} s_{12} \\ s_{22} \\ s_{32} \end{vmatrix} = 0 ,$$

$$\begin{vmatrix} (2c^2 - A_3) & \sqrt{2c} & 0 \\ \sqrt{2c} & (c^2 + d^2 + 1 - A_3) & \sqrt{2d} \\ 0 & \sqrt{2d} & (2d^2 - A_3) \end{vmatrix} = \begin{vmatrix} s_{13} \\ s_{23} \\ s_{33} \end{vmatrix} = 0 .$$  

(4.5)

Diagonal elements coincide with the roots of the cubic algebraic equation

$$\begin{vmatrix} (2c^2 - A) & \sqrt{2c} & 0 \\ \sqrt{2c} & (c^2 + d^2 + 1 - A) & \sqrt{2d} \\ 0 & \sqrt{2d} & (2d^2 - A) \end{vmatrix} = 0 ,$$

it reduces to

$$(c^2 + d^2 - 1)4c^2 d^2 + A [-4c^2 d^2 - 2(c^2 + d^2)^2] + A^2 [(c^2 + d^2 + 1) + 2(c^2 + d^2)] - A^3 = 0 .$$  

(4.6)
With shortening notation
\[ c^2 + d^2 = \frac{j(j+1) - k^2}{2} = M > 0 , \quad 4c^2d^2 = \frac{j^2 - k^2}{2} \left( \frac{(j+1)^2 - k^2}{2} \right) = N \geq 0 \]
(note that \( M^2 - N = (c^2 - d^2)^2 > 0 \), eq. (4.6) reads
\[
A^3 + rA^2 + sA + t = 0 , \quad r = -(3M + 1) , \quad s = (N + 2M^2) , \quad t = -(M - 1)N . \tag{4.7}
\]
The roots obey the algebraic identity
\[ \lambda^3 + r\lambda^2 + s\lambda + t = (\lambda - A_1)(\lambda - A_2)(\lambda - A_3) = 0 , \]
from whence it follows
\[
r = -(A_1 + A_2 + A_3) \implies 3M + 1 = A_1 + A_2 + A_3 > 0 , \quad s = A_1A_2 + A_1A_3 + A_2A_3 \implies N + 2M^2 = A_1A_2 + A_1A_3 + A_2A_3 > 0 , \quad t = -A_1A_2A_3 \implies (M - 1)N = A_1A_2A_3 \geq 0 . \tag{4.8}
\]
Making the change of the variable
\[ B = A + \frac{r}{3} , \quad A = B + (M + \frac{1}{3}) , \]
we obtain the cubic equation in a reduced form
\[ B^3 + pB + q = 0 , \quad p = \frac{3s - r^2}{3} , \quad q = \frac{2r^3}{27} - \frac{rs}{3} + t , \tag{4.9}
\]
where
\[
p = -\left( (c^2 - d^2)^2 + 2(c^2 + d^2) + \frac{1}{3} \right) < 0 , \quad q = -\left( \frac{4}{3}(c^2 - d^2)^2 + \frac{2}{3}(c^2 + d^2) + \frac{2}{27} \right) < 0 . \tag{4.10}
\]
Note that
\[ c^2 - d^2 = \frac{k}{2} , \quad c^2 + d^2 = \frac{j(j+1) - k^2}{2} , \]
and expressions for \( p \) and \( q \) can be presented simpler:
\[
p = -\left( j(j+1) - \frac{3}{4}k^2 + \frac{1}{3} \right) < 0 , \quad q = -\left( \frac{1}{3}j(j+1) + \frac{2}{27} \right) < 0 . \tag{4.11}
\]
Let us write down some results of numerical calculation for the roots \( A_1, A_2, A_3 \) at different \( j \):
\[
\begin{array}{cccccccc}
k = \pm 1/2 & j = 3/2 & j = 5/2 & j = 7/2 & j = 9/2 & j = 11/2 \\
0.31 & 1.79 & 4.28 & 7.77 & 12.27 \\
1.73 & 4.24 & 7.75 & 12.25 & 17.75 \\
4.21 & 7.72 & 12.23 & 17.73 & 24.24
\end{array}
\]
When all these values are real and positive. In general, properties of the roots are determined by the sign of the discriminant $D$:

$$D = \frac{p}{3}^3 + \left(\frac{q}{2}\right)^2. \quad (3.8a)$$

When $D < 0$, all the roots must be real-valued. Discriminant can be presented as

$$D = -\left(\frac{j(j+1)}{3} - \frac{k^2}{12} + \frac{1}{9}\right)^3 + \left(\frac{j(j+1)}{6} + \frac{1}{27}\right)^2. \quad (4.12)$$

The sign of this parameter $D$ can be verified, with the use of simple substitutions:

\[
\begin{align*}
\text{when } k > 0 & \quad n = 1, 2, 3, \ldots \\
D &= -\frac{1}{72} k^5 n - \frac{17}{144} k^4 n^2 - \frac{17}{144} k^4 n - \frac{11}{27} k^3 n^3 - \frac{11}{18} k^3 n^2 - \frac{17}{36} k^2 n^4 - \\
&\quad -\frac{17}{18} k^2 n^3 - \frac{2}{9} k n^5 - \frac{5}{9} k n^4 - \frac{1}{54} k n - \frac{1}{108} n^2 - \frac{11}{54} k n - \frac{7}{12} k^2 n^2 - \\
&\quad -\frac{1}{9} k^2 n - \frac{13}{27} k n^3 - \frac{1}{6} k n^2 - \frac{1}{432} k^4 - \frac{13}{108} n^4 - \frac{1}{18} n^3 - \frac{1}{1728} k^6 - \frac{1}{144} k^5 - \frac{1}{27} n^6 - \frac{1}{9} n^5 < 0; \\
\text{when } k < 0 & \quad n = 1, 2, 3, \ldots \\
D &= -\frac{1}{72} k^5 n - \frac{17}{144} k^4 n^2 - \frac{17}{144} k^4 n + \frac{11}{27} k^3 n^3 + \frac{11}{18} k^3 n^2 - \frac{17}{36} k^2 n^4 - \\
&\quad -\frac{17}{18} k^2 n^3 + \frac{2}{9} k n^5 + \frac{5}{9} k n^4 + \frac{1}{54} k n - \frac{1}{108} n^2 + \frac{11}{54} k n - \frac{7}{12} k^2 n^2 - \\
&\quad -\frac{1}{9} k^2 n + \frac{13}{27} k n^3 + \frac{1}{6} k n^2 - \frac{1}{432} k^4 - \frac{13}{108} n^4 - \frac{1}{18} n^3 - \frac{1}{1728} k^6 + \frac{1}{144} k^5 - \frac{1}{27} n^6 - \frac{1}{9} n^5 < 0.
\end{align*}
\]
Due to inequality $D < 0$, the reduced cubic equation has only real roots. Their analytical form can be found in a standard way. Indeed, let introduce two auxiliary quantities

$$
\rho = \sqrt{-\frac{p^3}{27}}, \quad \cos \phi = -\frac{q}{2\rho};
$$

then the roots of reduced equation are given by

$$
\begin{align*}
B_1 &= 2\rho^{1/3} \cos \frac{\phi}{3} = 2\sqrt{-\frac{p}{3}} \cos \frac{\phi}{3}, \\
B_2 &= 2\rho^{1/3} \cos \left(\frac{\phi}{3} + \frac{2\pi}{3}\right) = 2\sqrt{-\frac{p}{3}} \cos \left(\frac{\phi}{3} + \frac{2\pi}{3}\right), \\
B_3 &= 2\rho^{1/3} \cos \left(\frac{\phi}{3} + \frac{4\pi}{3}\right) = 2\sqrt{-\frac{p}{3}} \cos \left(\frac{\phi}{3} - \frac{2\pi}{3}\right),
\end{align*}
$$

(4.13)

where

$$
\begin{align*}
\sqrt{-\frac{p}{3}} &= \sqrt{j(j+1) - k^2} + \frac{k^2}{12} + \frac{1}{9}, \\
\rho &= \sqrt{-\frac{p^3}{27}} = \left(\frac{j(j+1) - k^2}{3} + \frac{k^2}{12} + \frac{1}{9}\right)^{3/2}, \\
\cos \phi &= -\frac{q/2}{\rho} = \frac{\frac{1}{2} j(j+1) + \frac{1}{9}}{\left(\frac{j(j+1) - k^2}{3} + \frac{k^2}{12} + \frac{1}{9}\right)^{3/2}}.
\end{align*}
$$

(4.14)

In turn, the roots $A_j$ are determined by the formulas

$$
\begin{align*}
A_1 &= \left(\frac{j(j+1) - k^2}{2} + \frac{1}{3}\right) + 2\sqrt{\frac{j(j+1) - k^2}{3} + \frac{k^2}{12} + \frac{1}{9}} \cos \frac{\phi}{3}, \\
A_2 &= +\left(\frac{j(j+1) - k^2}{2} + \frac{1}{3}\right) + 2\sqrt{\frac{j(j+1) - k^2}{3} + \frac{k^2}{12} + \frac{1}{9}} \cos \left(\frac{\phi}{3} + \frac{2\pi}{3}\right), \\
A_3 &= +\left(\frac{j(j+1) - k^2}{2} + \frac{1}{3}\right) + 2\sqrt{\frac{j(j+1) - k^2}{3} + \frac{k^2}{12} + \frac{1}{9}} \cos \left(\frac{\phi}{3} - \frac{2\pi}{3}\right).
\end{align*}
$$

(4.15)
5 Explicit form of three types of solutions

Let us turn back to (4.4). The matrix $S_{ij}$ is determined by 9 equations (in fact, each sub-system contains only two independents equations):

\begin{align*}
2c^2s_{11} + \sqrt{2}cs_{21} &= s_{11}A_1, \\
\sqrt{2}cs_{11} + (c^2 + d^2 + 1)s_{21} + \sqrt{2}ds_{31} &= s_{21}A_1, \\
\sqrt{2}ds_{21} + 2d^2s_{31} &= s_{31}A_1; \\
2c^2s_{12} + \sqrt{2}cs_{22} &= s_{12}A_2, \\
\sqrt{2}cs_{12} + (c^2 + d^2 + 1)s_{22} + \sqrt{2}ds_{32} &= s_{22}A_2, \\
\sqrt{2}ds_{22} + 2d^2s_{32} &= s_{32}A_2; \\
2c^2s_{13} + \sqrt{2}cs_{23} &= s_{13}A_3, \\
\sqrt{2}cs_{13} + (c^2 + d^2 + 1)s_{23} + \sqrt{2}ds_{33} &= s_{23}A_3, \\
\sqrt{2}ds_{23} + 2d^2s_{33} &= s_{33}A_3. \tag{5.1}
\end{align*}

Let us omit the third equation in each sub-system and set

$s_{11} = 1, \quad s_{22} = 1, \quad s_{33} = 1;$

this results in

\begin{align*}
s_{21} &= -\frac{(2c^2 - A_1)}{\sqrt{2}c}, \quad s_{31} = -\frac{d}{(2d^2 - A_1)} \frac{(2c^2 - A_1)}{c}, \\
s_{12} &= -\frac{\sqrt{2}c}{(2c^2 - A_2)}, \quad s_{32} = -\frac{\sqrt{2}d}{(2d^2 - A_2)}, \\
s_{13} &= -\frac{c}{(2c^2 - A_3)} \frac{(2d^2 - A_3)}{d}, \quad s_{23} = -\frac{(2d^2 - A_3)}{\sqrt{2}d}. \tag{5.2}
\end{align*}

Defined by (5.2) transformation $S_{ij}$

\[
S\Psi' = \begin{vmatrix} 1 & s_{12} & s_{13} \\ s_{21} & 1 & s_{23} \\ s_{31} & s_{32} & 1 \end{vmatrix} \begin{vmatrix} \Psi'_1 \\ \Psi'_2 \\ \Psi'_3 \end{vmatrix} = \begin{vmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{vmatrix} \tag{5.3}
\]

will reduce the radial system to the diagonal form

\[
\Delta = \begin{vmatrix} \Psi'_1 \\ \Psi'_2 \\ \Psi'_3 \end{vmatrix} = \begin{vmatrix} A_1 & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & A_3 \end{vmatrix} \begin{vmatrix} \Psi'_1 \\ \Psi'_2 \\ \Psi'_3 \end{vmatrix}. \tag{5.4}
\]

Because equations in (5.4) are not connected with each other, one can study three linearly independent solutions as

\[
\begin{vmatrix} \Psi'_1 \\ 0 \\ 0 \end{vmatrix}, \quad \begin{vmatrix} 0 \\ \Psi'_2 \\ 0 \end{vmatrix}, \quad \begin{vmatrix} 0 \\ 0 \\ \Psi'_3 \end{vmatrix}. \tag{5.5}
\]
One can turn back to the initial basis, then constructed solutions will look as follows

\[
S \left| \begin{array}{c} 
\Psi_1' \\ 0 \\
0 \\ 0 \\
\end{array} \right|, \quad S \left| \begin{array}{c} 
0 \\ \Psi_2' \\
0 \\ \Psi_3' \\
\end{array} \right|, \quad S \left| \begin{array}{c} 
0 \\ 0 \\
\end{array} \right|.
\]

(5.6)

Thus, the problem has been reduced to the second order differential equation of the general form

\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2EM - \frac{L(L+1)}{r^2} \right) f(r) = 0,
\]

(5.7)

where

\[
L(L + 1) = 2A = \{2A_1, 2A_2, 2A_3\}, \quad L = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2A}.
\]

(5.8)

Note that positive roots \(A = A_1, A_2, A_3\) ensure existence of positive values for parameter \(L\) (upper sign in (5.8)).

6 The case of minimal \(j\)

Let us turn to the system (2.8)

\[
i \varepsilon E_1 + i \left( \frac{d}{dr} + \frac{1}{r} \right) H_1 - M \Phi_1 = 0,
\]

\[
- i \varepsilon M \Phi_1 - M E_1 = 0 , \quad - i \left( \frac{d}{dr} + \frac{1}{r} \right) \Phi_1 - M H_1 = 0.
\]

(6.1)

To make non-relativistic approximation, one excludes nondynamical component \(H_1\):

\[
i \varepsilon ME_1 + \left( \frac{d}{dr} + \frac{1}{r} \right) \left( \frac{d}{dr} + \frac{1}{r} \right) \Phi_1 - M^2 \Phi_1 = 0 ,
\]

\[
- i \varepsilon M \Phi_1 - M^2 E_1 = 0.
\]

(6.2)

Now, introduce big and small components, we obtain

\[
(M + E)M(\Psi_1 - \psi_1) + \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right)(\Psi_1 + \psi_1) - M^2(\Psi_1 + \psi_1) = 0 ,
\]

\[
-(M + E)M(\Psi_1 + \psi_1) + M^2(\Psi_1 - \psi_1) = 0
\]

or differently

\[
-2M^2\psi_1 + EM(\Psi_1 - \psi_1) + \left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} \right)(\Psi_1 + \psi_1) = 0 ,
\]

\[
-2M^2\psi_1 - ME(\Psi_1 + \psi_1) = 0.
\]

Subtracting the second equation from the first

\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2EM \right) \Psi_1 = 0 ;
\]
we get the needed nonrelativistic equation

\[
\Psi_1 = \frac{1}{r} f_1(r), \quad \left( \frac{d^2}{dr^2} + 2EM \right) f_1 = 0, \quad f_1 = e^{\pm \sqrt{-2ME}r} .
\] (6.3)

Special solution in the relativistic case which could be associated with a bound state at \(0 < \epsilon < M\), now in the nonrelativistic approximation takes the form (remember that \(\epsilon = M + E\)) associated with the bound state as well

\[
\Psi_1 = e^{-\sqrt{-2EM}r} .
\]

7 **Vector particle in presence of the Coulomb field**

The above formalism permits allowing for the Coulomb potential – it is reached by one formal change in the differential equation (5.7) as shown below

\[
\left( \frac{d^2}{dr^2} + 2 \frac{d}{r} \frac{d}{dr} + 2M(E + \frac{\alpha}{r}) - \frac{L(L + 1)}{r^2} \right) f(r) = 0 ,
\] (7.1)

where

\[
L(L + 1) = 2A = \{2A_1, 2A_2, 2A_3\} , \quad L = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2A} .
\]

Near the point \(r = 0\), possible solutions behave in accordance with the relationships

\[
\left( \frac{d^2}{dr^2} + 2 \frac{d}{r} \frac{d}{dr} - \frac{L(L + 1)}{r^2} \right) f = 0 , \quad f \sim r^a ,
\]

\[
a(a - 1) + 2a - L(L + 1) = 0 \quad \Longrightarrow \quad a = L, -L - 1 .
\]

To describe bound states, from two possibilities \(f \sim r^L\) and \(f \sim r^{-L-1}\), one must use the first

\[
r \to 0 , \quad f \sim r^L , \quad L = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2A} .
\]

At infinity, solutions associated with bound states behave according to

\[
\left( \frac{d^2}{dr^2} + 2 \frac{d}{r} \frac{d}{dr} + 2ME \right) f = 0 , \quad f \sim e^{br} , \quad b^2 + 2ME = 0 , \quad b = -\sqrt{-2ME} .
\]

Now, let us construct exact solutions of the equation (7.1) in all domain of the variable \(r\). It is convenient to translate the task to the new variable \(x = \sqrt{-2ME} r\), then it reads

\[
\left( \frac{d^2}{dx^2} + 2 \frac{d}{x} \frac{d}{dx} - \frac{L(L + 1)}{x^2} - 1 - \frac{\alpha \sqrt{-2ME}}{Ex} \right) f = 0 .
\] (7.2)

Let us use the shortening notation:

\[
- \frac{\alpha \sqrt{-2ME}}{E} = B > 0 , \quad \left( \frac{d^2}{dx^2} + 2 \frac{d}{x} \frac{d}{dx} - 1 + \frac{B}{x} - \frac{L(L + 1)}{x^2} \right) f = 0 .
\] (7.3)
With the use of substitution $f = x^L e^{-x} F(x)$, we arrive at the equation

$$x F'' + (2L - 2x + 2) F' + (-2L - 2 + B) F = 0.$$  

In the variable $z = 2x$, it reads

$$z \frac{d^2 F}{dz^2} + (2L + 2 - z) \frac{dF}{dz} + \left(\frac{B}{2} - L - 1\right) F = 0,$$

which can be identified with confluent hypergeometric equation $z F'' + (c - z) F' - \alpha F = 0$. Using the polynomial condition, $\alpha = -n$, $n = 0, 1, 2, \ldots$, we get the quantization rule for energy levels

$$-\frac{B}{2} + L + 1 = -n \quad \implies \quad \frac{B}{2} = n + L + 1.$$

Taking into account expression for $B$, we get

$$-\frac{\alpha \sqrt{-2M E}}{2E} = n + L + 1 \quad \implies \quad E = -\frac{1}{2} \frac{\alpha^2 M}{(n + L + 1)^2}.$$  

(7.4)

There exist three series of energy levels in accordance with existence three possible values for $L$:

$$L_i = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2A_i}, \quad E_i = -\frac{1}{2} \frac{\alpha^2 M}{(n + L_i + 1)^2}.$$  

(7.5)

The case of minimal value of $j$ – see (6.3), can be treated in the same manner:

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2EM + \frac{\alpha}{r}\right) \Psi_1 = 0;$$  

(7.6)

because this equation follows from (7.1) at the formal change $L \implies L = 0$, we obtain solution and energy spectrum straightforwardly:

$$z = 2\sqrt{-2ME} r, \quad f = e^{-z/2} F(z),$$

$$z \frac{d^2 F}{dz^2} + (2 - z) \frac{dF}{dz} + \left(\frac{B}{2} - 1\right) F = 0, \quad E = -\frac{1}{2} \frac{\alpha^2 M}{(n + 1)^2}.$$  

(7.7)

8 Particle in presence of the oscillator potential

Let us take into account additionally the presence of the oscillator potential – then we have the differential equation

$$\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} + 2M(E - \frac{k r^2}{2}) - \frac{L(L + 1)}{r^2}\right) f = 0.$$  

(8.1)

In the new variable $x = \sqrt{Mr^2}$, it reads

$$\left(\frac{d^2}{dx^2} + \frac{3}{2x} \frac{d}{dx} - \frac{1}{4} + \frac{E \sqrt{M}}{2 \sqrt{r} x} - \frac{L(L + 1)}{4x^2}\right) f(x) = 0,$$  

(8.2)
where
\[ L(L + 1) = 2A = \{2A_1, 2A_2, 2A_3\}, \quad L = -\frac{1}{2} + \sqrt{\frac{1}{4} + 2A} > 0. \]

Using the substitution \( f(x) = x^a e^{-bx} F(x) \), from eq. (8.2) we get
\[
x \frac{d^2 F}{dx^2} + \frac{1}{2} (4a + 3 - 4bx) \frac{dF}{dx} + \\
+ \left( \frac{a(4a + 2) - L(L + 1)}{4x} - \frac{3}{2} b - 2ab + \frac{E \sqrt{M}}{2 \sqrt{k}} + (b^2 - \frac{1}{4})x \right) F = 0.
\]

At \( b = +1/2, a = +L/2 \), this equation becomes simpler
\[
x \frac{d^2 F}{dx^2} + (L + 3/2 - x) \frac{dF}{dx} - \left( \frac{3}{4} + \frac{L}{2} - \frac{E \sqrt{M}}{2 \sqrt{k}} \right) F = 0,
\]
it is confluent hypergeometric equation for \( F(A, C; x) \) with parameters
\[
A = \frac{1}{2} \left( \frac{3}{2} + L - E \sqrt{\frac{M}{k}} \right), \quad C = L + \frac{3}{2}.
\]

Quantization condition \( A = -n \) leads to the energy spectrum
\[
E_i = \frac{1}{2} \sqrt{\frac{k}{M}} (\frac{3}{2} + L_i + 2n), \quad L_i = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 2A_i}.
\]  (8.3)

For the minimal value of \( j \) we have the following equation
\[
\left( \frac{d^2}{dr^2} + \frac{2}{r} \frac{dr}{dx} + 2EM - \frac{k_j^2}{2} \right) \Psi_1 = 0;
\]  (8.4)

which is solved as follows:
\[
x = \sqrt{Mr}^2, \quad f(x) = e^{-x/2} F(x), \quad \frac{d^2 F}{dx^2} + (3/2 - x) \frac{dF}{dx} - \left( \frac{3}{4} - \frac{E \sqrt{M}}{2 \sqrt{k}} \right) F = 0, \quad E = \frac{1}{2} \sqrt{\frac{k}{M}} (\frac{3}{2} + 2n). \]  (8.5)

9 Spin 1 particle in the Lobachevsky space

Now, let us extended the above approach to the case of Lobachevsky geometry background. In spherical coordinates of the Lobachevsky space
\[
dS^2 = c^2 dt^2 - dr^2 - sh^2 r \left( d\theta^2 + sh^2 \theta d\phi^2 \right),
\]  (9.1)
in the tetrad chosen as shown
\[
\epsilon^a_{(0)} = (1, 0, 0, 0), \quad \epsilon^a_{(3)} = (0, 1, 0, 0),
\]
DKP equation takes the form (see the notation in [7, 8])
\[
\left[ i \beta^0 \partial_t + i \left( \beta^3 \partial_r + \frac{1}{\tanh r} (\beta^1 j^{31} + \beta^2 j^{32}) \right) + \frac{1}{\sin \theta} \Sigma_{\theta,\phi}^k - M \right] \Phi(x) = 0 ,
\] (9.2)
where the angular operator \( \Sigma_{\theta,\phi}^k \) is defined as
\[
\Sigma_{\theta,\phi}^k = i \beta^1 \partial_\theta + \beta^2 \frac{i}{\sin \theta} (\beta^1 j^{12} - k) \cos \theta .
\]
\( \beta^a \) stand for \((10 \times 10)\)-matrices of DKP; \( j^{ab} = \beta^a \beta^b - \beta^b \beta^a \); parameter \( k = eg/hc \) is the quantized according to the Dirac rule [1]: \( |k| = 1/2, 1, 3/2, 2, ... \); \( e \) and \( g \) represent electric and magnetic charges respectively.

It is convenient to rewrite eq. (9.2) shorter
\[
\hat{W}_0(x) \Phi^0(x) = 0 .
\] (9.3)
Analysis has showed that treatment of the task in Lobachevsky space (even in the nonrelativistic approximation) turned out to be very difficult. However, complete analytical study can be successfully performed if one makes one formal change in the equation (9.3): it is enough to take slightly different equation
\[
\left[ i \beta^0 \partial_t + i \left( \beta^3 \partial_r + \frac{1}{\tanh r} (\beta^1 j^{31} + \beta^2 j^{32}) \right) + \frac{1}{\sin \theta} \Sigma_{\theta,\phi}^k - M \right] \Phi(x) = 0 .
\] (9.4)
Transition from (9.2) to (9.4) can be described through inserting in eq. (9.3) an additional interaction term
\[
\left[ \hat{W}_0(x) + \Delta(r) \right] \Phi(x) = 0 , \quad \hat{W}_0 \rightarrow \hat{W}(x) , \quad \Phi_0(x) \rightarrow \Phi(x) ,
\] (9.5)
where the term is taken as follows
\[
\Delta(r) = i \frac{1 - \cosh r}{\sinh r} (\beta^1 j^{31} + \beta^2 j^{32}) .
\] (9.6)
It is helpful to get the form of this term in the Cartesian tetrad gauge. In solving this task we will need Cartesian basis of the DKP-matrices [7, 8]
\[
\beta^i = \begin{vmatrix}
0 & 0 & e_j & 0 \\
0 & 0 & 0 & \tau_j \\
\tilde{e}_j & 0 & 0 & 0 \\
0 & -\tau_j & 0 & 0
\end{vmatrix} , \quad e_1 = (i, 0, 0) , \quad e_2 = (0, i, 0) , \quad e_3 = (0, 0, i) ,
\]
\[
\tau_1 = \begin{vmatrix}
0 & 0 & 0 \\
0 & 0 & -i \\
0 & i & 0
\end{vmatrix} , \quad \tau_2 = \begin{vmatrix}
0 & 0 & i \\
0 & 0 & 0 \\
-i & 0 & 0
\end{vmatrix} , \quad \tau_3 = \begin{vmatrix}
0 & -i & 0 \\
i & 0 & 0 \\
0 & 0 & 0
\end{vmatrix} .
\]
For generators $J^{ab}$, we have expressions

$$J^{31} = \beta^3 \beta^1 - \beta^1 \beta^3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & -i\tau_2 & 0 & 0 \\ 0 & 0 & -i\tau_2 & 0 \\ 0 & 0 & 0 & -i\tau_2 \end{vmatrix}, \quad J^{32} = \beta^3 \beta^1 - \beta^1 \beta^3 = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 0 & i\tau_1 & 0 & 0 \\ 0 & 0 & i\tau_1 & 0 \\ 0 & 0 & 0 & i\tau_1 \end{vmatrix};$$

then calculate the term

$$(\beta^1 J^{31} + \beta^2 J^{32}) = \begin{vmatrix} 0 & 0 & 2e_3 & 0 \\ 0 & 0 & 0 & \tau_3 \\ 0 & 0 & 0 & 0 \\ 0 & -\tau_3 & 0 & 0 \end{vmatrix}.$$ 

Thus, the term (9.6) can be presented as

$$\Delta(r) = i \frac{1 - \cosh r}{\sinh r} \begin{vmatrix} 0 & 0 & 2e_3 & 0 \\ 0 & 0 & 0 & \tau_3 \\ 0 & 0 & 0 & 0 \\ 0 & -\tau_3 & 0 & 0 \end{vmatrix}; \quad (9.7)$$

in turn, this relation can be rewritten as

$$\Delta(r) = \frac{1 - \cosh r}{\sinh r} (\beta^1 S^2 - \beta^2 S^1), \quad (9.8)$$

where

$$iJ^{23} = S^1, \quad iJ^{31} = S^2, \quad iJ^{12} = S^3.$$ 

Now, we should take into account that wave functions in spherical and Cartesian tetrads are referred to each other with the help of the following orthogonal transformation [7, 8]

$$\Phi_{\text{spher}} = S\Phi_{\text{Cart}}, \quad W_0(x) + \Delta(r)\Phi_{\text{spher}}(x) = 0,$$

$$S^{-1}\left[W_0(x) + \Delta(r)\right]S\Phi_{\text{Cart}}(x) = 0,$$

$$W_0^{\text{Cart}}(x) + \Delta^{\text{Cart}}(x)\Phi_{\text{Cart}}(x) = 0, \quad \Delta^{\text{Cart}}(x) = S^{-1}\Delta(r)O, \quad (9.9)$$

where the matrix $S$ is determined by the formulas $S:

$$S = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & U & 0 & 0 \\ 0 & 0 & U & 0 \\ 0 & 0 & 0 & U \end{vmatrix}, \quad O = \begin{vmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi & -\sin \theta \\ \cos \phi & \sin \phi & 0 \\ \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \phi \end{vmatrix}.$$ 

Let us calculate $\Delta^{\text{Cart}}(x)$

$$\Delta^{\text{Cart}}(x) = i \frac{1 - \cosh r}{\sinh r} \begin{vmatrix} 0 & 0 & 2e_3 O & 0 \\ 0 & 0 & 0 & O^{-1}\tau_3 O \end{vmatrix}, \quad (9.10)$$
Allowing for identities
\[
2e_3O = 2i(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = 2i \vec{n},
\]
we derive
\[
O^{-1}\tau_3 O = \begin{vmatrix}
0 & -\cos \theta & \sin \theta \sin \phi \\
\cos \theta & 0 & -\sin \theta \cos \phi \\
-\sin \theta \sin \phi & \sin \theta \cos \phi & 0
\end{vmatrix} = i(n_1 \tau_1 + n_2 \tau_2 + n_3 \tau_3),
\]
we derive
\[
\Delta^{\text{Cart}}(x) = \frac{\cosh r - 1}{\sinh r} \begin{vmatrix}
0 & 0 & 2\vec{n} & 0 \\
0 & 0 & 0 & \vec{n}\vec{\tau} \\
0 & -\vec{n}\vec{\tau} & 0 & 0
\end{vmatrix}.
\]

It is readily verified that this expression for \(\Delta^{\text{Cart}}(x)\) can be presented in a more symmetrical way as follows
\[
\Delta^{\text{Cart}}(x) = \frac{1 - \cosh r}{\sinh r} \left[ \vec{n} \cdot (\vec{\beta} \times \vec{S}) \right].
\]

This term can be interpreted as a special and additional interaction of spin 1 particle with the background geometry of the Lobachevsky space.

## 10 Separation of the variables in presence of magnetic monopole

In spherical coordinates and tetrad of the Lobachevsky space let us consider the modified (see previous Section) DKP equation
\[
\left[ i\beta^0 \partial_t + i \left( \beta^3 \partial_r + \frac{1}{\sinh r} (\beta^1 j^{31} + \beta^2 j^{32}) \right) + \frac{1}{\sinh r} \Sigma^k_\theta,\phi - M \right] \Phi = 0.
\]

It is convenient to use so-called cyclic representation for DKP-matrices [9, 10], where the third projection of the spin, \(iJ^{12}\), has a diagonal structure
\[
iJ^{12} = \begin{vmatrix}
0 & 0 & 0 & 0 \\
0 & t_3 & 0 & 0 \\
0 & 0 & t_3 & 0 \\
0 & 0 & 0 & t_3
\end{vmatrix}, \quad t_3 = \begin{vmatrix}
+1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -1
\end{vmatrix}.
\]

The components of the total conserved moment in the basis (10.1) are determined by the formulas [3]:
\[
J^k_1 = l_1 + \cos \phi \sin \theta (iJ^{12} - \kappa), \quad J^k_2 = l_2 + \sin \phi \sin \theta (iJ^{12} - \kappa), \quad J^k_3 = l_3.
\]

In accordance with the general method [3], wave functions of the particle with quantum numbers \((\epsilon, j, m)\) are constructed within the substitution
\[
\Phi_{\epsilon jm}(x) = e^{-i\epsilon t} \left[ f_1(r) D_k, f_2(r) D_{k-1}, f_3(r) D_k, f_4(r) D_{k+1}, f_5(r) D_{k-1}, f_6(r) D_k, f_7(r) D_{k+1}, f_8(r) D_{k-1}, f_9(r) D_k, f_{10}(r) D_{k+1} \right],
\]

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where symbol $D_{\sigma}$ stands for Wigner $[9]$ functions $D^{j_m,\sigma}_{\phi,\theta,0}$. Below, when producing equations for radial functions $f_1, \ldots, f_{10}$, we will need recurrent relations $[9]$:  

\[
\begin{align*}
\partial_\theta D_{k-1} &= a D_{k-2} - c D_k, & \quad -m - (k - 1) \cos \theta & \sin \theta \quad \frac{D_{k-1}}{D_k} = -a D_{k-2} - c D_k, \\
\partial_\theta D_k &= (c D_{k-1} - d D_{k+1}), & \quad -m - k \cos \theta & \sin \theta \quad \frac{D_k}{D_{k-1}} = -c D_{k-1} - d D_{k+1}, \\
\partial_\theta D_{k+1} &= (d D_k - b D_{k+2}), & \quad -m - (k + 1) \cos \theta & \sin \theta \quad \frac{D_{k+1}}{D_k} = -d D_k - b D_{k+2},
\end{align*}
\]

where

\[
\begin{align*}
a &= \frac{1}{2} \sqrt{(j + k - 1)(j - k + 2)}, & \quad b &= \frac{1}{2} \sqrt{(j - k - 1)(j + k + 2)}, \\
c &= \frac{1}{2} \sqrt{(j + k)(j - k + 1)}, & \quad d &= \frac{1}{2} \sqrt{(j - k)(j + k + 1)}.
\end{align*}
\]

From \((10.1)\), it follows the radial equations

\[
\begin{align*}
-(\frac{d}{dr} + \frac{2}{\sin r}) f_6 - \frac{\sqrt{2}}{\sin r} (c f_5 + d f_7) - M f_1 &= 0, \\
i \epsilon f_5 + i(\frac{d}{dr} + \frac{1}{\sin r}) f_8 + i \frac{\sqrt{2}}{\sin r} f_9 - M f_2 &= 0, \\
i \epsilon f_6 + \frac{\sqrt{2} i}{\sin r} (-c f_8 + d f_{10}) - M f_3 &= 0, \\
i \epsilon f_7 - i(\frac{d}{dr} + \frac{1}{\sin r}) f_{10} - i \frac{\sqrt{2} d}{\sin r} f_9 - M f_4 &= 0, \\
-i \epsilon f_2 + \frac{\sqrt{2}}{\sin r} f_1 - M f_5 &= 0, \\
-i \epsilon f_3 - \frac{d}{dr} f_1 - M f_6 &= 0, \\
-i \epsilon f_4 + \frac{\sqrt{2} d}{\sin r} f_1 - M f_7 &= 0, \\
-i(\frac{d}{dr} + \frac{1}{\sin r}) f_2 - i \frac{\sqrt{2}}{\sin r} f_3 - M f_8 &= 0, \\
i \frac{\sqrt{2}}{\sin r} (c f_2 - d f_4) - M f_9 &= 0, \\
i(\frac{d}{dr} + \frac{1}{\sin r}) f_4 + i \frac{\sqrt{2} d}{\sin r} f_3 - M f_{10} &= 0.
\end{align*}
\]

The quantum number $j$ takes the values

\[k = \pm 1/2, \quad j = |k|, |k| + 1, \ldots; \]
\[k = \pm 1, \pm 3/2, \ldots, \quad j = |k|, |k|, |k| + 1, \ldots\]

The cases of the type $j = |k| - 1$ are special. For instance, let $k = +1$ and $j = 0$, then the substitution should be

\[\Phi^{(0)}(t, r) = e^{-i \epsilon t} (0, f_2, 0, 0, f_5, 0, 0; f_8, 0, 0);\]
and further we derive three radial equations

\[ f_5 = -i \frac{\epsilon}{M} f_2 , \quad f_8 = -i \frac{1}{M} \left( \frac{d}{dr} + \frac{1}{\text{sh} \, r} \right) f_2 , \]

\[ \left( \frac{d^2}{dr^2} + \frac{2}{\text{sh} \, r} \frac{d}{dr} + \frac{1 - \text{ch} \, r}{\text{sh}^2 r} + \epsilon^2 - M^2 \right) f_2 = 0 . \]  \hspace{1cm} (10.6)

By special substitution, we simplify the problem:

\[ f_2(r) = \frac{1 + \text{ch} \, r}{2 \, \text{sh} \, r} F_2(r) , \quad \left( \frac{d^2}{dr^2} + \epsilon^2 - M^2 \right) F_2 = 0 , \]  \hspace{1cm} (10.7)

which coincides with equation arising in the flat space [3]. The case \( j = 0 \), \( k = -1 \) looks much the same.

11 Nonrelativistic approximation in radial equations

Let us employ the method of producing non-relativistic equations previously used in [3, 10]. First, in (1.4) let us separate four equations which permit to exclude non-dynamical variables

\[ -\left( \frac{d}{dr} + \frac{2}{\text{sh} \, r} \right) f_6 - \frac{\sqrt{2}}{\text{sh} \, r} (c f_5 + d f_7) = M f_1 , \]

\[ -i \left( \frac{d}{dr} + \frac{1}{\text{sh} \, r} \right) f_2 - i \frac{\sqrt{2} c}{\text{sh} \, r} f_3 = M f_8 , \quad i \frac{\sqrt{2}}{\text{sh} \, r} (c f_2 - d f_4) = M f_9 , \]

\[ i \left( \frac{d}{dr} + \frac{1}{\text{sh} \, r} \right) f_4 + i \frac{\sqrt{2} d}{\text{sh} \, r} f_3 = M f_{10} . \]  \hspace{1cm} (11.1)
Excluding these component $f_1, f_8, f_9, f_{10}$ from remaining six equations, we get

\[
\begin{align*}
&i\epsilon Mf_5 + i\left(\frac{d}{dr} + \frac{1}{\text{sh} \, r}\right) \left[-i\left(\frac{d}{dr} + \frac{1}{\text{sh} \, r}\right)f_2 - i\sqrt{2}c \frac{\text{sh} \, r}{\text{sh} \, r}f_3\right] \\
&\quad + i\sqrt{2}c \frac{\text{sh} \, r}{\text{sh} \, r} \left[-i\left(\frac{d}{dr} + \frac{1}{\text{sh} \, r}\right)f_2 - i\sqrt{2}c \frac{\text{sh} \, r}{\text{sh} \, r}f_3\right] - M^2 f_2 = 0, \\
&\quad + i\sqrt{2}c \frac{\text{sh} \, r}{\text{sh} \, r} \left[-c \left[-i\left(\frac{d}{dr} + \frac{1}{\text{sh} \, r}\right)f_2 - i\sqrt{2}c \frac{\text{sh} \, r}{\text{sh} \, r}f_3\right] + d \left[i\left(\frac{d}{dr} + \frac{1}{\text{sh} \, r}\right)f_4 + i\sqrt{2}d \frac{\text{sh} \, r}{\text{sh} \, r}f_3\right]\right] - M^2 f_3 = 0, \\
&\quad + d \left[i\left(\frac{d}{dr} + \frac{1}{\text{sh} \, r}\right)f_4 + i\sqrt{2}d \frac{\text{sh} \, r}{\text{sh} \, r}f_3\right] - M^2 f_4 = 0, \\
&-i\epsilon Mf_6 + \sqrt{2}c \frac{\text{sh} \, r}{\text{sh} \, r} \left[-\left(\frac{d}{dr} + \frac{2}{\text{sh} \, r}\right)f_6 - \sqrt{2}c \frac{\text{sh} \, r}{\text{sh} \, r}(cf_5 + df_7)\right] M^2 f_5 = 0, \\
&-i\epsilon Mf_6 - \frac{d}{dr} \left[-\left(\frac{d}{dr} + \frac{2}{\text{sh} \, r}\right)f_6 - \sqrt{2}c \frac{\text{sh} \, r}{\text{sh} \, r}(cf_5 + df_7)\right] - M^2 f_6 = 0, \\
&\quad -i\epsilon Mf_7 + \sqrt{2}d \frac{\text{sh} \, r}{\text{sh} \, r} \left[-\left(\frac{d}{dr} + \frac{2}{\text{sh} \, r}\right)f_6 - \sqrt{2}c \frac{\text{sh} \, r}{\text{sh} \, r}(cf_5 + df_7)\right] - M^2 f_7 = 0.
\end{align*}
\]

Let us translate these equations to more symmetrical designations

\[
(f_2, f_3, f_4) - > (\Phi_1, \Phi_2, \Phi_3), \quad (f_5, f_6, f_7) - > (E_1, E_2, E_3); \quad (11.2)
\]
this results in

\[ i\epsilon ME_1 + i\left(\frac{d}{dr} + \frac{1}{\text{sh } r}\right) \left[ -i\left(\frac{d}{dr} + \frac{1}{\text{sh } r}\right)\Phi_1 - i\sqrt{2}c \text{ sh } r\Phi_2 \right] + \frac{\sqrt{2}c}{\text{sh } r} \left[ \frac{\sqrt{2}r}{\text{sh } r}(c\Phi_1 - d\Phi_3) \right] - M^2\Phi_1 = 0 , \]

\[ i\epsilon ME_2 + \frac{\sqrt{2}i}{\text{sh } r} \left[ -c \left( -i\left(\frac{d}{dr} + \frac{1}{\text{sh } r}\right)\Phi_1 - i\sqrt{2}c \Phi_2 \right) \right] + d \left( i\left(\frac{d}{dr} + \frac{1}{\text{sh } r}\right)\Phi_3 + i\sqrt{2}d \frac{\text{sh } r}{\Phi_2} \right) - M^2\Phi_2 = 0 , \]

\[ i\epsilon ME_3 - i\left(\frac{d}{dr} + \frac{1}{\text{sh } r}\right) \left[ i\left(\frac{d}{dr} + \frac{1}{\text{sh } r}\right)\Phi_3 + i\sqrt{2}d \frac{\text{sh } r}{\Phi_2} \right] - \frac{\sqrt{2}d}{\text{sh } r} \left[ \frac{\sqrt{2}r}{\text{sh } r}(c\Phi_1 - d\Phi_3) \right] - M^2\Phi_3 = 0 , \]

\[ -i\epsilon\Phi_1 + \sqrt{2} \frac{c}{\text{sh } r} \left[ -\frac{1}{\text{sh } r} \left(\frac{d}{dr} + \frac{2}{\text{sh } r}\right)E_2 - \frac{\sqrt{2}}{\text{sh } r}(cE_1 + dE_3) \right] - M^2E_1 = 0 , \]

\[ -i\epsilon\Phi_2 - \frac{d}{dr} \left[ -\frac{1}{\text{sh } r} \left(\frac{d}{dr} + \frac{2}{\text{sh } r}\right)E_2 - \frac{\sqrt{2}}{\text{sh } r}(cE_1 + dE_3) \right] - M^2E_2 = 0 , \]

\[ -i\epsilon\Phi_3 + \sqrt{2} \frac{d}{\text{sh } r} \left[ -\frac{1}{\text{sh } r} \left(\frac{d}{dr} + \frac{2}{\text{sh } r}\right)E_2 - \frac{\sqrt{2}}{\text{sh } r}(cE_1 + dE_3) \right] - M^2E_3 = 0 . \] (11.3)

Big \( \Psi_k \) and small \( \psi_k \) components are introduced by the linear combinations \( \Phi_k = \frac{\Psi_k + \psi_k}{2}, \quad iE_k = \frac{\Psi_k - \psi_k}{2} . \) (11.4)

After regrouping the above equations (11.3) in pairs, and in the same time separating the rest energy with the help of the formal change \( \epsilon = (M + E) \), we get

\[ i(M + E)ME_1 + i\left(\frac{d}{dr} + \frac{1}{\text{sh } r}\right) \left[ -i\left(\frac{d}{dr} + \frac{1}{\text{sh } r}\right)\Phi_1 - i\sqrt{2}c \text{ sh } r\Phi_2 \right] + \frac{\sqrt{2}c}{\text{sh } r} \left[ \frac{\sqrt{2}r}{\text{sh } r}(c\Phi_1 - d\Phi_3) \right] - M^2\Phi_1 = 0 , \]

\[ -i(M + E)\Phi_1 + \sqrt{2} \frac{c}{\text{sh } r} \left[ -\frac{1}{\text{sh } r} \left(\frac{d}{dr} + \frac{2}{\text{sh } r}\right)E_2 - \frac{\sqrt{2}}{\text{sh } r}(cE_1 + dE_3) \right] - M^2E_1 = 0 , \] (11.5)
$$i(M + E)M E_2 + \frac{\sqrt{2}i}{\text{sh } r} \left[-i \left(-i \frac{d}{dr} + \frac{1}{\text{sh } r}\right) \Phi_1 - i \frac{\sqrt{2}c}{\text{sh } r} \Phi_2\right]$$

$$+ d \left(i \frac{d}{dr} + \frac{1}{\text{sh } r}\right) \Phi_3 + i \frac{\sqrt{2}d}{\text{sh } r} \Phi_2 \right) - M^2 \Phi_2 = 0 ,$$

$$-i(M + E)M \Phi_2 - \frac{d}{dr} \left[-(\frac{d}{dr} + \frac{2}{\text{sh } r}) E_2 - \frac{\sqrt{2}}{\text{sh } r}(cE_1 + dE_3)\right] - M^2 E_2 = 0 , \quad (11.6)$$

$$i(M + E)M E_3 - i(\frac{d}{dr} + \frac{1}{\text{sh } r}) \left[i(\frac{d}{dr} + \frac{1}{\text{sh } r}) \Phi_3 + i \frac{\sqrt{2}d}{\text{sh } r} \Phi_2 \right]$$

$$-i \frac{\sqrt{2}d}{\text{sh } r} \left[ i \frac{\sqrt{2}}{\text{sh } r} (c\Phi_1 - d \Phi_3) \right] - M^2 \Phi_3 = 0 ,$$

$$-i(M + E)M \Phi_3 + \frac{\sqrt{2}d}{\text{sh } r} \left[-(\frac{d}{dr} + \frac{2}{\text{sh } r}) E_2 - \frac{\sqrt{2}}{\text{sh } r}(cE_1 + dE_3)\right] - M^2 E_3 = 0 . \quad (11.7)$$

Let us consider eqs. (11.5):

$$iM^2 E_1 + iEM E_1 + \left(\frac{d^2}{dr^2} + \frac{2}{\text{sh } r} \frac{d}{dr} + \frac{1 - \text{ch } r}{\text{sh}^2 r}\right) \Phi_1 +$$

$$+ \frac{\sqrt{2}c}{\text{sh } r} \left(\frac{d}{dr} + \frac{1 - \text{ch } r}{\text{sh } r}\right) \Phi_2 - \frac{2c}{\text{sh}^2 r}(c\Phi_1 - d \Phi_3) - M^2 \Phi_1 = 0 ,$$

$$-iM^2 \Phi_1 - iEM \Phi_1 - \frac{\sqrt{2}c}{\text{sh } r} \left(\frac{d}{dr} + \frac{2}{\text{sh } r}\right) E_2 - \frac{2c}{\text{sh}^2 r}(cE_1 + dE_3) - M^2 E_1 = 0 .$$

After translating them to the big and small components, we obtain

$$M^2(\Psi_1 - \psi_1) + EM(\Psi_1 - \psi_1) + \left(\frac{d^2}{dr^2} + \frac{2}{\text{sh } r} \frac{d}{dr} + \frac{1 - \text{ch } r}{\text{sh}^2 r}\right) (\Psi_1 + \psi_1) +$$

$$+ \frac{\sqrt{2}c}{\text{sh } r} \left(\frac{d}{dr} + \frac{1 - \text{ch } r}{\text{sh } r}\right) (\Psi_2 + \psi_2) -$$

$$- \frac{2c}{\text{sh}^2 r} \left[c(\Psi_1 + \psi_1) - d (\Psi_3 + \psi_3)\right] - M^2 (\Psi_1 + \psi_1) = 0 ,$$

$$-iM^2(\Psi_1 + \psi_1) - iEM(\Psi_1 + \psi_1) + i \frac{\sqrt{2}c}{\text{sh } r} \left(\frac{d}{dr} + \frac{2}{\text{sh } r}\right)(\Psi_2 - \psi_2) +$$

$$+ i \frac{2c}{\text{sh}^2 r} \left[c(\Psi_1 - \psi_1) + d (\Psi_3 - \psi_3)\right] + iM^2(\Psi_1 - \psi_1) = 0 .$$
Regrouping the terms we arrive at

\[-2M^2\psi_1 + EM(\Psi_1 - \psi_1) + \left( \frac{d^2}{dr^2} + \frac{2}{sh\:r} \frac{d}{dr} + \frac{1 - ch\:r}{sh^2\:r} \right)(\Psi_1 + \psi_1) \]

\[+ \sqrt{2c} \left( \frac{d}{dr} + \frac{1 - ch\:r}{sh\:r} \right)(\Psi_2 + \psi_2) \]

\[- \frac{2c}{sh^2\:r} [c(\Psi_1 + \psi_1) - d(\Psi_3 + \psi_3)] = 0 , \]

\[-2M^2\psi_1 - EM(\Psi_1 + \psi_1) + \sqrt{2c} \left( \frac{d}{dr} + \frac{2}{sh\:r} \right)(\Psi_2 - \psi_2) \]

\[+ \frac{2c}{sh^2\:r} [c(\Psi_1 - \psi_1) + d(\Psi_3 - \psi_3)] = 0 . \tag{11.8} \]

Subtracting the second equation from the first, we obtain

\[\left( \frac{d^2}{dr^2} + \frac{2}{sh\:r} \frac{d}{dr} + \frac{1 - ch\:r}{sh^2\:r} + 2EM \right)\Psi_1 - \sqrt{2c} \frac{1 + ch\:r}{sh^2\:r} \Psi_2 - \frac{4c^2}{sh^2\:r} \Psi_1 = 0 ; \tag{11.9} \]

this equation contains only the big components and represents one of equations in the Pauli approximation.

Let us consider eqs. (11.7)

\[iM^2E_3 + iEME_3 + \left( \frac{d^2}{dr^2} + \frac{2}{sh\:r} \frac{d}{dr} + \frac{1 - ch\:r}{sh^2\:r} \right)\Phi_3 + \]

\[+ \frac{\sqrt{2d}}{r} \left( \frac{d}{dr} + \frac{1 - ch\:r}{sh\:r} \right)\Phi_2 + \frac{2d}{sh^2\:r}(c\Phi_1 - d\Phi_3) - M^2\Phi_3 = 0 , \]

\[-iM^2\Phi_3 - iEM\Phi_3 - \sqrt{2d} \frac{d}{dr} + \frac{2}{sh\:r}E_2 - \]

\[- \frac{2d}{sh^2\:r}(cE_1 + dE_3) - M^2E_3 = 0 . \]

Translating them to the big and small components (and separating the rest energy), we get

\[M^2(\Psi_3 - \psi_3) + EM(\Psi_3 - \psi_3) + \left( \frac{d^2}{dr^2} + \frac{2}{sh\:r} \frac{d}{dr} + \frac{1 - ch\:r}{sh^2\:r} \right)(\Psi_3 + \psi_3) + \]

\[+ \frac{\sqrt{2d}}{r} \left( \frac{d}{dr} + \frac{1 - ch\:r}{sh\:r} \right)(\Psi_2 + \psi_2) + \]

\[+ \frac{2d}{sh^2\:r}(c(\Psi_1 + \psi_1) - d(\Psi_3 + \psi_3)) - M^2(\Psi_3 + \psi_3) = 0 , \]

\[-iM^2(\Psi_3 + \psi_3) - iEM(\Psi_3 + \psi_3) + i\sqrt{2d} \frac{d}{dr} + \frac{2}{sh\:r}(\Psi_2 - \psi_2) + \]

\[+ i\frac{2d}{sh^2\:r}(c(\Psi_1 - \psi_1) + d(\Psi_3 - \psi_3)) + iM^2(\Psi_3 - \psi_3) = 0 ; \]
whence after regrouping the terms we get

\[-2M^2\psi_3 + EM(\Psi_3 - \psi_3) + \left( \frac{d^2}{dr^2} + \frac{2}{sh^2 r} \right) \left( \Psi_3 + \psi_3 \right) - \frac{d}{dr} \left( \frac{2}{sh^2 r} \right) \left( \Psi_2 + \psi_2 \right) - \frac{1}{sh^2 r} \left( \Psi_3 + \psi_3 \right) \]

\[-2M^2\psi_3 - EM(\Psi_3 + \psi_3) + \left( \frac{d^2}{dr^2} + \frac{2}{sh^2 r} \right) \left( \Psi_2 - \psi_2 \right) - \frac{1}{sh^2 r} \left( \Psi_3 + \psi_3 \right) = 0 \; ,
\]

Subtract the second equation from the first:

\[-2M^2\psi_3 + EM(\Psi_3 - \psi_3) + \left( \frac{d^2}{dr^2} + \frac{2}{sh^2 r} \right) \left( \Psi_3 + \psi_3 \right) - \frac{d}{dr} \left( \frac{2}{sh^2 r} \right) \left( \Psi_2 + \psi_2 \right) - \frac{1}{sh^2 r} \left( \Psi_3 + \psi_3 \right) = 0 \; ;
\]

it is the second differential equation in the Pauli approximation.

Finally, let us consider eqs. (11.6)

\[iM^2E_2 + iEME_2 - \frac{\sqrt{2}c}{sh^2 r} \left( \frac{d}{dr} \right) \Phi_1 - \frac{2c^2}{sh^2 r} \Phi_2 = 0 \; ,
\]

\[-iM^2\Phi_2 - iEM\Phi_2 + \left( \frac{d^2}{dr^2} + \frac{2}{sh^2 r} \right) \left( \Phi_2 + \psi_2 \right) = 0 \; ,
\]

They give

\[M^2(\Psi_2 - \psi_2) + EM(\Psi_2 - \psi_2) - \frac{\sqrt{2}c}{sh^2 r} \left( \frac{d}{dr} \right) (\Psi_1 + \psi_1) - \frac{2c^2}{sh^2 r} (\Psi_2 + \psi_2) - \frac{1}{sh^2 r} \left( \Psi_3 + \psi_3 \right) = 0 \; ,
\]

\[-iM^2(\Psi_2 + \psi_2) - iEM(\Psi_2 + \psi_2) - \frac{2d^2}{sh^2 r} \left( \Psi_2 + \psi_2 \right) - M^2(\Psi_2 + \psi_2) = 0 \; ,
\]

\[-iM^2(\Psi_2 + \psi_2) - iEM(\Psi_2 + \psi_2) - \frac{2d^2}{sh^2 r} \left( \Psi_2 + \psi_2 \right) = 0 \; ,
\]

differently they look

\[-2M^2\psi_2 + EM(\Psi_2 - \psi_2) - \frac{\sqrt{2}c}{sh^2 r} \left( \frac{d}{dr} \right) (\Psi_1 + \psi_1) - \frac{2c^2}{sh^2 r} (\Psi_2 + \psi_2) - \frac{1}{sh^2 r} \left( \Psi_3 + \psi_3 \right) = 0 \; ;
\]

\[28\]
\[
- \sqrt{2} d \left( \frac{d}{dr} + \frac{1}{\text{sh}r} \right) (\Psi_3 + \psi_3) - \frac{2d^2}{\text{sh}^2r} (\Psi_2 + \psi_2) = 0 ,
\]
\[
-2M^2 \psi_2 - EM (\Psi_2 + \psi_2) - \left( \frac{d^2}{dr^2} + \frac{2}{\text{sh}r} \frac{d}{dr} - \frac{2 \text{ch}r}{\text{sh}^2r} \right) (\Psi_2 - \psi_2) -
\]
\[
- \frac{\sqrt{2}}{\text{sh}r} \frac{d}{dr} \left( \text{ch}r \text{sh}^2r \right) [c(\Psi_1 - \psi_1) + d(\Psi_3 - \psi_3)] = 0 .
\]

Subtracting the second equation from the first, we derive
\[
\left( \frac{d^2}{dr^2} + \frac{2}{\text{sh}r} \frac{d}{dr} + \frac{1 - \text{ch}r}{\text{sh}^2r} + 2EM \right) \Psi_2 - \frac{\sqrt{2}c(1 + \text{ch}r)}{\text{sh}^2r} \Psi_1 = 0 .
\]

It is the third equation of the Pauli system.

Thus, the complete system of nonrelativistic equations is
\[
\left( \frac{d^2}{dr^2} + \frac{2}{\text{sh}r} \frac{d}{dr} + \frac{1 - \text{ch}r}{\text{sh}^2r} + 2EM \right) \Psi_1 - \sqrt{2}c \frac{1 + \text{ch}r}{\text{sh}^2r} \Psi_2 - \frac{4c^2}{\text{sh}^2r} \Psi_1 = 0 ,
\]
\[
\left( \frac{d^2}{dr^2} + \frac{2}{\text{sh}r} \frac{d}{dr} + \frac{1 - \text{ch}r}{\text{sh}^2r} + 2EM \right) \Psi_3 - \sqrt{2}d \frac{1 + \text{ch}r}{\text{sh}^2r} \Psi_2 - \frac{4d^2}{\text{sh}^2r} \Psi_3 = 0 ,
\]
\[
\left( \frac{d^2}{dr^2} + \frac{2}{\text{sh}r} \frac{d}{dr} + \frac{1 - \text{ch}r}{\text{sh}^2r} + 2EM \right) \Psi_2 - \frac{2(c^2 + d^2)}{\text{sh}^2r} \Psi_2 - \frac{\sqrt{2}c(1 + \text{ch}r)}{\text{sh}^2r} \Psi_1 - \frac{\sqrt{2}d(1 + \text{ch}r)}{\text{sh}^2r} \Psi_3 = 0 .
\]

With the help of identity
\[
\left( \frac{d^2}{dr^2} + \frac{2}{\text{sh}r} \frac{d}{dr} + \frac{1 - \text{ch}r}{\text{sh}^2r} + 2EM \right) \frac{\text{ch}r + 1}{\text{sh}r} F = \frac{\text{ch}r + 1}{\text{sh}r} \left( \frac{d^2}{dr^2} + 2ME \right) F ,
\]

one can translate eqs. (11.13) to a simpler form:
\[
\left( \frac{d^2}{dr^2} + 2EM - \frac{4c^2}{\text{sh}^2r} \right) F_1 = \frac{\text{ch}r + 1}{\text{sh}r} \left( \sqrt{2}c F_2 \right) ,
\]
\[
\left( \frac{d^2}{dr^2} + 2EM - \frac{2(c^2 + d^2)}{\text{sh}^2r} \right) F_2 = \frac{\text{ch}r + 1}{\text{sh}r} \left( \sqrt{2}c F_1 + F_2 + \sqrt{2}d F_3 \right) ,
\]
\[
\left( \frac{d^2}{dr^2} + 2EM - \frac{4d^2}{\text{sh}^2r} \right) F_3 = \frac{\text{ch}r + 1}{\text{sh}r} \left( \sqrt{2}d F_2 \right) .
\]

Unfortunately, this system turns to be very difficult for solving, the method used in the flat space [3] cannot be applied here. However, we can solve exactly the case of minimal value of \( j = |k| - 1 \) in presence of additional Coulomb or oscillator potentials.
12 Minimal value of \( j \), Coulomb and oscillator potentials

The case of minimal value \( j \) (in presence of magnetic charge) can be solved analytically. Let us take into account the Coulomb potential \([11]\) – then the main equation looks

\[
\left( \frac{d^2}{dr^2} + \left( \epsilon + \frac{\alpha}{\ell} r \right)^2 - M^2 \right) F_2 = 0 .
\]  

(12.1)

In the variable \( x = 1 - e^{-2r} \), with substitution \( F_1 = x^a (1 - x)^b f(x) \), at positive

\[
a = \frac{1 + \sqrt{1 - 4\alpha^2}}{2}, \quad b = \frac{1}{2} \sqrt{-(\epsilon + \alpha)^2 + M^2},
\]

we get the hypergeometric equation

\[
x (1 - x) f'' + [2a - (2a + 2b + 1)x] f' - \left[ (a + b)^2 + \left( \frac{\epsilon}{2} - \frac{\alpha}{2} \right)^2 - \frac{M^2}{4} \right] f = 0
\]

Quantization rule \( \alpha = -n \) provides us with the energy spectrum

\[
\epsilon = \frac{M}{\sqrt{1 + \alpha^2 / \nu^2}} \sqrt{1 - \frac{\alpha^2 + \nu^2}{M^2}}, \quad \nu = n + \frac{1}{2} \sqrt{1 - 4\alpha^2}.
\]  

(12.2)

In usual units it reads (\( R \) stands for a curvature radius of the Lobachevsky space)

\[
E = \frac{mc^2}{\sqrt{1 + \alpha^2 / \nu^2}} \sqrt{1 - \frac{\hbar^2}{m^2 c^2 R^2 (\alpha^2 + \nu^2)}},
\]  

(12.3)

Solutions constructed are good only when the restriction below holds

\[
\nu^2 < \frac{m^2 c^2 R^2}{\hbar^2} - \alpha^2;
\]  

(12.4)

which means that the number of bound states is finite.

Now, for states with minimal \( j \), let us take into account the oscillator potential \([11]\)

\[
\left( \frac{d^2}{dr^2} + 2M (E - K \frac{\ell^2}{2} r) \right) F_2 = 0 .
\]  

(12.5)

Solutions are constructed in hypergeometric functions

\[
E = N \sqrt{\frac{K}{M} + \left( \frac{1}{2M} \right)^2 - \frac{1}{2M} (N^2 + \frac{1}{4})} .
\]  

(12.6)

To have solutions tending to zero at infinity, one must impose restriction in the form of inequality

\[
2n + \frac{3}{2} < \sqrt{1 + 4KM} .
\]

In usual units, the energy spectrum reads

\[
\epsilon = \hbar \left( N \sqrt{\frac{k}{m} + \frac{h^2}{4m^2 R^4} - \frac{\hbar^2}{2m R^2} (N^2 + \frac{1}{4})} \right), \quad N = 2n + \frac{3}{2} ;
\]  

(12.7)

the above imposed inequality looks as follows

\[
2n + \frac{3}{2} < \frac{1}{2} \sqrt{1 + \frac{4km}{\hbar^2} R^4} ;
\]  

(12.8)

the number of discrete energy levels is finite and governed by the curvature radius.
13 Spin 1 particle in absence of the monopole background

In absence of the monopole, identity $d = c = \frac{1}{2} \sqrt{j(j+1)}$ holds, and the task becomes simpler

\[
\left( \frac{d^2}{dr^2} + 2EM \right) F_1 - \sqrt{2}c \frac{1 + ch \, r}{sh^2 r} F_2 - \frac{4c^2}{sh^2 r} F_1 = 0 , \\
\left( \frac{d^2}{dr^2} + 2EM \right) F_3 - \sqrt{2}c \frac{1 + ch \, r}{sh^2 r} F_2 - \frac{4c^2}{sh^2 r} F_3 = 0 ; \\
\left( \frac{d^2}{dr^2} + 2EM \right) F_2 - \frac{4c^2}{sh^2 r} F_2 - \frac{(1 + ch \, r)}{sh^2 r} F_2 - \frac{\sqrt{2}c(1 + ch \, r)}{sh^2 r} F_1 - \frac{\sqrt{2}c(1 + ch \, r)}{sh^2 r} F_3 = 0 .
\] (13.1)

In this case, one can diagonalize additionally the space reflection operator

\[
P = (-1)^{j+1}, \quad F_2 = 0, \quad F_3 = -F_1 ; \quad P = (-1)^j, \quad F_3 = +F_1.
\] (13.2)

Correspondingly, the system is divided into two ones (consisting of one and two equations):

\[
\left( \frac{d^2}{dr^2} + 2EM - \frac{4c^2}{sh^2 r} \right) F_1 = 0 ,
\] (13.3)

and

\[
\tilde{\Delta} \begin{vmatrix} F_1 \\ F_2 \end{vmatrix} = \begin{vmatrix} 0 & \nu \\ 2\nu & 1 \end{vmatrix} \begin{vmatrix} F_1 \\ F_2 \end{vmatrix}, \quad \nu = c\sqrt{2} , \quad (13.4)
\]

where

\[
\tilde{\Delta} = \frac{sh^2 r}{1 + ch \, r} \left( \frac{d^2}{dr^2} - \frac{j(j+1)}{sh^2 r} + 2EM \right) ,
\]

In the case of sub-system of two equations, let us diagonalize the mixing matrix

\[
F = SF', \quad \tilde{\Delta} F' = S^{-1} A SF', \quad S = \begin{vmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{vmatrix}, \quad \tilde{\Delta} F' = \begin{vmatrix} j + 1 & 0 \\ 0 & -j \end{vmatrix} F' ;
\] (13.5)

so we have two separated equations

\[
\left( \frac{d^2}{dr^2} + 2EM - \frac{j(j+1)}{sh^2 r} - \frac{1 + ch \, r}{sh^2 r} (j+1) \right) F'_1 = 0 , \\
\left( \frac{d^2}{dr^2} + 2EM - \frac{j(j+1)}{sh^2 r} + \frac{1 + ch \, r}{sh^2 r} j \right) F'_2 = 0 .
\] (13.6)

The above three equations are of one the same type; in the variable $y = (ch \, r + 1)/2$ they read

\[
\left( y(y-1) \frac{d^2}{dy^2} + (y - \frac{1}{2}) \frac{d}{dy} + 2ME - \frac{j(j+1)}{4y(y-1)} + \frac{\mu}{2(y-1)} \right) F = 0 ,
\] (13.7)
where $\mu = 0, -j - 1, +j$. Their solutions are constructed in hypergeometric functions:

\[
F = y^a (1 - y)^b F(\alpha, \beta, \gamma; y) , \quad \gamma = 2a + 1, \\
\alpha = a + b + i\sqrt{2ME} , \quad \beta = a + b - i\sqrt{2ME} ,
\]

where

\[
a = \frac{j + 1}{2}, \quad a' = -\frac{j}{2} ; \\
\mu = 0 , \quad b = \frac{j + 1}{2}, \quad b' = -\frac{j}{2} ; \\
\mu = -j - 1 , \quad b = \frac{j + 2}{2}, \quad b' = -\frac{j + 1}{2} ; \\
\mu = +j , \quad b = \frac{j}{2}, \quad b' = -\frac{j + 1}{2} .
\]

To the value $y = 1$ corresponds the point $r = 0$; so to get finite at $r = 0$ solutions, we must take $b > 0$. Asymptotic behavior at infinity is given by

\[
f(y) = \frac{\Gamma(\gamma)\Gamma(\beta - \alpha)}{\Gamma(\gamma - \alpha)\Gamma(\beta)} (-e^r)^{-i\sqrt{2EM}} + \frac{\Gamma(\gamma)\Gamma(\alpha - \beta)}{\Gamma(\gamma - \beta)\Gamma(\alpha)} (-e^r)^{+i\sqrt{2EM}} .
\]

Therefore, we have constructed real standing wave solutions, regular at the $r = 0$.

14 Spin 1 particle in the Coulomb attractive force

In presence of the external Coulomb field, equations (13.3) and (13.6) take the more complicated form:

\[
\left( \frac{d^2}{dr^2} + 2M(E + \frac{\alpha}{\tanh r}) - \frac{j(j + 1)}{\text{sh}^2 r} \right) F_1 = 0 ; 
\]

\[
\left( \frac{d^2}{dr^2} + 2M(E + \frac{\alpha}{\tanh r}) - \frac{j(j + 1)}{\text{sh}^2 r} - \frac{1 + \text{ch} r}{\text{sh}^2 r} (j + 1) \right) F'_1 = 0 , \\
\left( \frac{d^2}{dr^2} + 2M(E + \frac{\alpha}{\tanh r}) - \frac{j(j + 1)}{\text{sh}^2 r} + \frac{1 + \text{ch} r}{\text{sh}^2 r} j \right) F'_2 = 0 .
\]

The first equation (14.1) turns to be much more simple than two others in (14.2); it is solved in hypergeometric functions and gives the energy levels (in usual units)

\[
\epsilon = -mc^2 \frac{\alpha^2}{2(j + 1 + n)^2} - \frac{\hbar^2}{mR^2} \frac{(j + 1 + n)^2}{2} .
\]

Two others can be reduced to the Heun function. Applying only the first of the two conditions of having polynomial solutions (so we do not construct polynomials) we have arrived at yet two series of energy levels:

\[
E = -\frac{Ma^2}{2(j + 3/2 + n/2)^2} - \frac{(j + 3/2 + n/2)^2}{2M} ,
\]
\[
E = -\frac{M\alpha^2}{2(j + 1/2 + n/2)^2} - \frac{(j + 1/2 + n/2)^2}{2M}.
\] (14.5)

In eq. (14.1), let us introduce a new variable
\[
x = 1 - e^{-2r}, \quad \frac{d}{dr} = 2(1 - x) \frac{d}{dx}, \quad \frac{d^2}{dr^2} = 4(1 - x)^2 \frac{d^2}{dx^2} - 4(1 - x) \frac{d}{dx},
\]

\[
\operatorname{sh} r = \frac{x}{2\sqrt{1 - x}}, \quad \operatorname{ch} r = \frac{2 - x}{2\sqrt{1 - x}}, \quad \tanh r = \frac{x}{2 - x},
\]

then it read
\[
\left[(1 - x)^2 \frac{d^2}{dx^2} - (1 - x) \frac{d}{dx} + \frac{M}{2} (E + \alpha \frac{2 - x}{x}) - j(j + 1) \frac{1 - x}{x^2}\right]F_1 = 0.
\] (14.6)

With the substitution \( F_1 = x^a(1 - x)^b f(x) \), we get
\[
x(1 - x)f'' + [2a - (2a + 2b + 1)x] f' + \left[\frac{a(a - 1) - j(j + 1)}{x} + (b^2 + \frac{M}{2} (E + \alpha)) \frac{1}{1 - x} - a^2 - 2ab - b^2 - M \frac{2}{2} (E - \alpha)\right] f = 0.
\]

Requiring
\[
a(a - 1) - j(j + 1) = 0 \quad \implies \quad a = j + 1, -j,
\]
\[
b^2 + M \frac{2}{2} (E + \alpha) = 0 \quad \implies \quad b = \pm \sqrt{-M \frac{2}{2} (E + \alpha)},
\] (14.7)

we obtain a simpler equation
\[
x(1 - x)f'' + [2a - (2a + 2b + 1)x] f' + \left[-a^2 - 2ab - b^2 - M \frac{2}{2} (E - \alpha)\right] f = 0.
\]

which is of hypergeometric type
\[
x(1 - x) \frac{d^2}{dx^2} f + \left[\gamma + (\alpha + \beta + 1)x\right] \frac{df}{dx} - \alpha \beta; F = 0, \quad \alpha + \beta = 2a + 2b,
\]
\[
\alpha \beta = (a + b)^2 + M \frac{2}{2} (E - \alpha).
\]

To describe bound states, we take
\[
a = j + 1, \quad b = \sqrt{-M \frac{2}{2} (E + \alpha)}, \quad \alpha = a + b - \sqrt{-M \frac{2}{2} (E - \alpha)}, \quad \beta = a + b + \sqrt{-M \frac{2}{2} (E - \alpha)}.
\] (14.8)
Polynomial condition \( \alpha = -n \) gives

\[
j + 1 + \sqrt{-\frac{M}{2} (E + \alpha)} - \sqrt{-\frac{M}{2} (E - \alpha)} = -n ;
\]

whence it follows

\[
N \equiv j + 1 + n = \sqrt{-\frac{M}{2} (E - \alpha)} - \sqrt{-\frac{M}{2} (E + \alpha)} .
\]

and further we derive the formula for energy levels

\[
E = -\frac{M \alpha^2}{2N^2} - \frac{N^2}{2M} , \quad N = j + 1 + n .
\] (14.9)

In usual units this formula read

\[
\epsilon = -mc^2 \frac{\alpha^2}{2N^2} - \frac{\hbar^2}{mR^2} \frac{N^2}{2} , \quad (\alpha = \frac{e^2}{\hbar c} = \frac{1}{137}) .
\] (14.10)

Let us give some details of deriving the spectra (14.4) and (14.5). In case (14.2), one can use the variable \( z = \text{th}(r/2) \), eqs. will read

\[
\begin{align*}
&\left[ \frac{d^2}{dz^2} - \frac{2z}{1-z^2} \frac{d}{dz} + 8M(E + \alpha) \frac{1+2j}{2z} \right] F_1' = 0 , \quad (14.11) \\
&\left[ \frac{d^2}{dz^2} - \frac{2z}{1-z^2} \frac{d}{dz} + 8M(E + \alpha) \frac{1+2j}{2z} \right] F_2' = 0 . \quad (14.12)
\end{align*}
\]

Singular points are \( z = 0, \infty, \pm 1 \); two of them are physical \( r = 0 \) \((z = 0)\), \( r = \infty \) \((z = +1)\).

In eq. (14.11), let us use a simplifying substitution

\[
F_1' (z) = \frac{f_1'(z)}{\sqrt{z^2 - 1}} , \quad f_1' = z^A (1-z)^B (-1-z)^C f_1,
\]

then

\[
\begin{align*}
\frac{d^2 f_1}{dz^2} + & \left[ \frac{2A}{z} - \frac{2B}{1-z} - \frac{2C}{-1-z} \right] \frac{df_1}{dz} + \left[ \frac{A(A-1) - (j+1)(j+2)}{z^2} \right. \\
+ & \frac{(2B-1)^2 + 8M(E+\alpha)}{4(z-1)^2} + \frac{(2C-1)^2 + 8M(E-\alpha)}{4(z+1)^2} + \left. \frac{-2A(B-C) + 4M \alpha}{z} \right] f_1 = 0 .
\end{align*}
\]

At \( A, B, C \) taken according to

\[
A (A-1) - (j+1)(j+2) = 0 \quad \Longrightarrow \quad A = -(j+1), j+2, \\
(2B-1)^2 + 8M(E+\alpha) = 0 \quad \Longrightarrow \quad B = \frac{1}{2} \pm \sqrt{-2M(E+\alpha)} , \\
(2C-1)^2 + 8M(E-\alpha) = 0 \quad \Longrightarrow \quad C = \frac{1}{2} \pm \sqrt{-2M(E-\alpha)} ,
\]

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the last equation becomes simpler

$$\frac{d^2 f_1}{dz^2} + \left( \frac{2A}{z} + \frac{2B}{z-1} + \frac{2C}{z+1} \right) \frac{df_1}{dz} + \left[ \frac{-2A(B-C)+4M\alpha}{z} - \frac{4B(2A+C)+3+4j-8M(E+\alpha)}{4(1-z)} \right. \left. + \frac{-4C(2A+B)-3-4j+8M(E-\alpha)}{4(z+1)} \right] f_1 = 0; \quad (14.13)$$

it can be recognized as the general Heun equation for

$$\frac{d^2 H}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z+1} \right) \frac{dH}{dz} + \frac{\alpha \beta z - q}{z(z-1)(z-a)} H = 0, \quad (\gamma + \delta + \epsilon = \alpha + \beta + 1)$$

or differently

$$\frac{d^2 H}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z-1} + \frac{\epsilon}{z+1} \right) \frac{dH}{dz} + \left( \frac{-q}{\alpha} \frac{1}{z} + \frac{q - \alpha \beta}{(a-1)} \frac{1}{z-1} + \frac{\alpha \beta a - q}{a(a-1)} \frac{1}{z-a} \right) H = 0$$

with parameters

$$a = -1, \quad q = 4M\alpha - 2A(B - C),$$

$$\gamma = 2A, \quad \delta = 2B, \quad \epsilon = 2C,$$

$$\alpha = -\frac{1}{2} + A + B + C + \frac{1}{2} \sqrt{(2A-1)^2 + (2B-1)^2 + (2C-1)^2 + 8(2ME-j-1)},$$

$$\beta = -\frac{1}{2} + A + B + C - \frac{1}{2} \sqrt{(2A-1)^2 + (2B-1)^2 + (2C-1)^2 + 8(2ME-j-1)}, \quad (14.14)$$

Note identity

$$\frac{1}{2} \sqrt{(2A-1)^2 + (2B-1)^2 + (2C-1)^2 + 8(2ME-j-1)} = j + \frac{1}{2};$$

correspondingly, parameters $\alpha, \beta$ read

$$\alpha = -j - 1 + (A + B + C), \quad \beta = j + (A + B + C).$$

We will use quantization condition in the form (at this we do not arrive at polynomials) $\beta = -n.$

It turn out that appropriate is the following choice for $A, B, C:

$$A = j + 2, \quad B = \frac{1}{2} + \sqrt{-2M(E+\alpha)}, \quad C = \frac{1}{2} - \sqrt{-2M(E-\alpha)}.$$

$$A + B + C = j + 3 - \left( \sqrt{-2ME-2M\alpha} + \sqrt{-2ME+2M\alpha} \right);$$

then parameters $\alpha, \beta$ are given by

$$\alpha = 2 - \left( \sqrt{-2ME-2M\alpha} + \sqrt{-2ME+2M\alpha} \right),$$

$$\beta = 2j + 3 - \left( \sqrt{-2ME-2M\alpha} + \sqrt{-2ME+2M\alpha} \right) = -n. \quad (14.15)$$
The quantization rule takes the form

$$-\sqrt{-2ME - 2M\alpha} + \sqrt{-2ME + 2M\alpha} = 2j + 3 + n.$$  

from whence we get the formula for energy levels

$$E = -\frac{M\alpha^2}{2(j + 3/2 + n/2)^2} - \frac{(j + 3/2 + n/2)^2}{2M}.$$  

(14.16)

In similar manner, eq. (14.12) provides us with the energy levels

$$E = -\frac{M\alpha^2}{2(j + 1/2 + n/2)^2} - \frac{(j + 1/2 + n/2)^2}{2M}.$$  

(14.17)

Thus we have found three series of energy levels: (14.3), (14.16), (14.17). The presence of \(n'\) and \(n'/2\) is due to the use of different variables in solving respective differential equations, \(z = \text{th} \frac{\tau}{2}\) and \(x = 1 - e^{-2r}\) related by quadratical relations:

$$x = \frac{2\text{th} r}{1 + \text{th} r}, \quad \text{th} r = \frac{2z}{1 + z^2}, \quad x = \frac{4z(1 + z^2)}{(1 + z^2)^2 + 4z^2} = \frac{4(z + z^{-1})}{4 + (z + z^{-1})}.$$  

15 Spin 1 particle in the oscillator field

In the presence of the oscillator potential, radial equations take the form

$$P = (-1)^{j+1},$$

$$\left(\frac{d^2}{dr^2} + 2M(E - \frac{K \text{th}^2 r}{2}) - \frac{j(j + 1)}{\text{sh}^2 r}\right) F_1 = 0,$$  

(15.1)

and

$$P = (-1)^j,$$

$$\left(\frac{d^2}{dr^2} + 2M(E - \frac{K \text{th}^2 r}{2}) - \frac{j(j + 1)}{\text{sh}^2 r} - \frac{1 + \text{ch} \frac{r}{2}}{\text{sh}^2 r}(j + 1)\right) F_1' = 0,$$

$$\left(\frac{d^2}{dr^2} + 2M(E - \frac{K \text{th}^2 r}{2}) - \frac{j(j + 1)}{\text{sh}^2 r} + \frac{1 + \text{ch} \frac{r}{2}}{\text{sh}^2 r}(j + 1)\right) F_2' = 0.$$  

(15.2)

Let us translate them to a new variable

$$x = \text{ch} r, \quad \frac{d}{dr} = \sqrt{x^2 - 1} \frac{d}{dx}, \quad \frac{d^2}{dr^2} = (x^2 - 1) \frac{d^2}{dx^2} + x \frac{d}{dx},$$

which results

$$P = (-1)^{j+1},$$

$$\left((x^2 - 1) \frac{d^2}{dx^2} + x \frac{d}{dx} + 2M(E - \frac{K x^2}{2} - \frac{1}{x^2}) - \frac{j(j + 1)}{x^2 - 1}\right) F_1 = 0;$$  

(15.3)
\[ P = (-1)^j, \]
\[
\left( (x^2 - 1) \frac{d^2}{dx^2} + x \frac{d}{dx} + 2M(E - \frac{K x^2 - 1}{2 x^2}) - \frac{j(j + 1)}{x^2 - 1} - \frac{1 + x}{x^2 - 1} (j + 1) \right) F_1' = 0, \\
\left( (x^2 - 1) \frac{d^2}{dx^2} + x \frac{d}{dx} + 2M(E - \frac{K x^2 - 1}{2 x^2}) - \frac{j(j + 1)}{x^2 - 1} + \frac{1 + x}{x^2 - 1} j \right) F_2' = 0. \quad (15.4)
\]

For equation (15.3) one can use the variable \( x^2 = y \):

\[
\frac{d}{dx} = 2\sqrt{y} \frac{d}{dy}, \quad \frac{d^2}{dx^2} = 4y \frac{d^2}{dy^2} + 2 \frac{d}{dy};
\]

then it looks

\[
\left( 4y(y - 1) \frac{d^2}{dy^2} + (4y - 2) \frac{d}{dy} + 2ME - MK \frac{y - 1}{y} - \frac{j(j + 1)}{y - 1} \right) F_1 = 0. \quad (15.5)
\]

With substitution \( F_1(y) = y^a(1 - y)^b f(y) \), we get

\[
y(1 - y)f'' + \left[ 2a + \frac{1}{2} - (2a + 2b + 1)y \right] f' - \left( \frac{EM}{2} - \frac{KM}{4} + a^2 + b^2 + 2ab \right) f
\]
\[
- \frac{1}{y} \left( -a^2 + \frac{a}{2} + \frac{KM}{4} \right) + \frac{1}{1 - y} \left( b^2 - \frac{b}{2} - \frac{j(j + 1)}{4} \right) = 0.
\]

Let it be

\[
a = \frac{1}{4} \pm \frac{1}{4} \sqrt{1 + 4KM}, \quad b = -\frac{j}{2}, \frac{1 + j}{2}; \quad (15.6)
\]

then we obtain a more simple equation

\[
y(1 - y)f'' + \left[ 2a + \frac{1}{2} - (a + b + 1)y \right] f' - \left( \frac{2EM - KM}{4} + (a + b)^2 \right) f = 0
\]

of the hypergeometric type

\[
x(1 - x) \frac{d^2}{dx^2} f + \left[ \gamma + (\alpha + \beta + 1)x \right] \frac{df}{dx} - \alpha\beta F = 0, \quad \gamma = 2a + 1,
\]
\[
\alpha = a + b - \sqrt{-\frac{2EM}{4} + \frac{KM}{4}}, \quad \beta = a + b + \sqrt{-\frac{2EM}{4} + \frac{KM}{4}},
\]
\[
F_1(y) = y^a(1 - y)^b F(\alpha, \beta, \gamma, y) = (\text{ch } r)^{2a}(-\text{sh } r)^{2b} F(\alpha, \beta, \gamma, \text{ch}^2 r). \quad (15.7)
\]

To construct solutions associated with bound states, we should take

\[
2b = (1 + j) > 0, \quad 2a = \frac{1 - \sqrt{1 + 4KM}}{2} < 0; \quad (15.8)
\]
at this the above parameters read

\[ \alpha = 1 - \frac{\sqrt{1 + 4KM}}{2} + \frac{j + 1}{2} - \frac{1}{2}\sqrt{-2EM + KM} , \]
\[ \beta = 1 - \frac{\sqrt{1 + 4KM}}{2} + \frac{j + 1}{2} + \frac{1}{2}\sqrt{-2EM + KM} . \]  

(15.9)

Polynomial condition has the form \( \alpha = -n \), which leads to

\[ \frac{1 - \sqrt{1 + 4KM}}{2} + 1 + j - \sqrt{-2EM + KM} = -2n , \]

or (let \( N = 2n + j + \frac{3}{2} \))

\[ N - \frac{\sqrt{1 + 4KM}}{2} = \sqrt{-2EM + KM} ; \]  

(15.10)

from whence it follows

\[ E = N\sqrt{\frac{K}{M}} + (\frac{1}{2M})^2 - \frac{1}{2M}(N^2 + \frac{1}{4}) . \]  

(15.11)

It should be emphasized yet another property of the constructed solutions. Indeed, allowing for the structure of the complete solution

\[ f(y) = y^a (1 - y)^b F(\alpha, \beta, \gamma, y) \]

it is readily seen an inequality which ensures vanishing these solution at infinity

\[ a + b + n < 0 \implies \frac{1 - \sqrt{1 + 4KM}}{4} + \frac{j + 1}{2} + n < 0 ; \]  

(15.12)

this gives the upper boundary for possible quantum number of bound states

\[ 2n + j + \frac{3}{2} < \frac{\sqrt{1 + 4KM}}{2} . \]  

(15.13)

In order to translate these relations to usual units, it ia enough to take into account that

\[ M = \frac{mcR}{\hbar} , \quad E = \frac{\epsilon R}{\hbar} , \quad K = \frac{kR^3}{\hbar} ; \]

and further we obtain

\[ \frac{\epsilon}{\hbar} = N\sqrt{\frac{kR^3}{\hbar} \frac{\hbar}{mcR} + \left(\frac{\hbar}{2mcR}\right)^2} - \frac{\hbar}{2mcR}(N^2 + \frac{1}{4}) , \]

that is

\[ \epsilon = \hbar \left( N \sqrt{\frac{k}{m} + \frac{\hbar^2}{4m^2R^4} - \frac{\hbar}{2mR^2}(N^2 + \frac{1}{4})} \right) , \quad N = 2n + j + \frac{3}{2} . \]  

(15.14)
In the limit of flat space ($R \to \infty$), this formula leads to the known result
\[ \epsilon = \hbar \sqrt{\frac{k}{m}} N . \]

In usual units, restriction on the quantum numbers (15.13) read
\[ 2n + j + \frac{3}{2} < \frac{1}{2} \sqrt{1 + \frac{4km}{\hbar^2} R^4} . \] (15.15)

Now, let us turn to the first equation in (15.4). Let us start with the substitution

\[ F'_{1} = x^{A}(1 - x)^{B}(-1 - x)^{C} f_{1} , \] (15.16)
then we get
\[
\begin{align*}
\frac{d^2 f_{1}}{dx^2} + \left( \frac{2A}{x} + \frac{4B + 1}{2(x - 1)} + \frac{4C + 1}{2(x + 1)} \right) \frac{df_{1}}{dx} + \left[ \frac{A(A - 1) - MK}{x^2} + \frac{2B(2B - 1) - (j + 1)(j + 2)}{4(x - 1)^2} + \frac{2C(2C - 1) - j(j + 1)}{4(x + 1)^2} \right. \\
- \frac{2A(B - C)}{x} + \frac{(2A + C)(4B + 1) + B + 4ME + (j + 1)^2}{4(x - 1)} \\
- \left. \frac{(2A + B)(4C + 1) + C + 4ME + (j + 1)^2}{4(x + 1)} \right] f_{1} = 0 .
\end{align*}
\]

At $A, B, C$ chosen as
\[
A(A - 1) - MK = 0 \quad \Rightarrow \quad A = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 + 4MK} ,
\]
\[
2B(2B - 1) - (j + 1)(j + 2) = 0 \quad \Rightarrow \quad B = \frac{1}{2} - \frac{j}{2} , \quad 1 + \frac{j}{2} ,
\]
\[
2C(2C - 1) - j(j + 1) = 0 \quad \Rightarrow \quad C = \frac{j}{2} , \quad 1 + \frac{j}{2} ,
\] (15.17)
the above equation becomes simpler
\[
\begin{align*}
\frac{d^2 f_{1}}{dx^2} + \left( \frac{2A}{x} + \frac{4B + 1}{2(x - 1)} + \frac{4C + 1}{2(x + 1)} \right) \frac{df_{1}}{dx} + \left[ \frac{2A(C - B)}{x} + \frac{(2A + C)(4B + 1) + B + 4ME + (j + 1)^2}{4(x - 1)} \\
- \frac{(2A + B)(4C + 1) + C + 4ME + (j + 1)^2}{4(x + 1)} \right] f_{1} = 0 ,
\end{align*}
\] (15.18)
which coincides with the general Heun equation
\[
\frac{d^2 H}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z - 1} + \frac{\epsilon}{z + 1} \right) \frac{dH}{dz} + \frac{\alpha \beta z - q}{z(z - 1)(z - a)} H = 0 , \quad (\gamma + \delta + \epsilon = \alpha + \beta + 1) \]
or differently
\[
\frac{d^2 H}{dz^2} + \left( \frac{\gamma}{z} + \frac{\delta}{z - 1} + \frac{\epsilon}{z + 1} \right) \frac{dH}{dz} + \]
with parameters
\[ a = -1, \quad q = -2A(B - C), \]
\[ \gamma = 2A, \quad \delta = 2B + \frac{1}{2}, \quad \epsilon = 2C + \frac{1}{2}. \]
\[
\alpha = A + B + C + \frac{1}{2} \sqrt{4A(A - 1) + 2B(2B - 1) + 2C(2C - 1) - 2(j + 1)^2 - 8ME},
\]
\[
\beta = A + B + C - \frac{1}{2} \sqrt{4A(A - 1) + 2B(2B - 1) + 2C(2C - 1) - 2(j + 1)^2 - 8ME}. \quad (15.19)
\]
Allowing for the identity
\[
\frac{1}{2} \sqrt{4A(A - 1) + 2B(2B - 1) + 2C(2C - 1) - 2(j + 1)^2 - 8ME} = \sqrt{-M(2E - K)},
\]
for parameters \( \alpha, \beta \) we obtain expressions
\[
\alpha = A + B + C + \sqrt{-M(2E - K)}, \quad \beta = A + B + C - \sqrt{-M(2E - K)}. \quad (15.20)
\]
As a formal quantization condition, we will use one of the known polynomial relations (because of the use only one of two those, we do not construct actually polynomials): \( \beta = -n \). At
\[
A = \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4MK}, \quad B = 1 + \frac{j}{2}, \quad C = \frac{1}{2} + \frac{j}{2}, \quad (15.21)
\]
the above condition \( \beta = -n \) takes the form
\[
j + 2 - \frac{1}{2} \sqrt{1 + 4MK} - \sqrt{-M(2E - K)} = -n.
\]
From whence we arrive at the energy spectrum (let \( N = 2 + j + n \))
\[
E = N \sqrt{\frac{K}{M} + \left( \frac{1}{2M} \right)^2 - \frac{1}{2M} (N^2 + \frac{1}{4})}. \quad (15.22)
\]
Now, let us consider the second equation in (15.1):
\[
\left( x^2 - 1 \right) \frac{d^2}{dx^2} + x \frac{d}{dx} + 2M \left( E - \frac{K x^2 - 1}{2 x} + \frac{j(j + 1)}{x^2 - 1} + \frac{1 + x}{x^2 - 1} \right) f'_2 = 0.
\]
With the substitution
\[
F'_2 = x^A(1 - x)^B(-1 - x)^C f_2, \quad (2.7.13a)
\]
we get
\[
\frac{d^2 f_2}{dx^2} + \left( \frac{2A}{x} + \frac{4B + 1}{2(x - 1)} + \frac{4C + 1}{2(x + 1)} \right) \frac{df_2}{dx} + \left[ \frac{A(A - 1) - MK}{x^2} + \frac{2B(2B - 1) - j(j - 1)}{4(x - 1)^2} + \frac{2C(2C - 1) - j(j + 1)}{4(x + 1)^2} \right] f_2 = 0.
\]
\[-\frac{2A(B - C)}{x} + \frac{(2A + C)(4B + 1) + B + 4ME + j^2}{4(x - 1)} - \frac{(2A + B)(4C + 1) + C + 4ME + j^2}{4(x + 1)}\] \[f_2 = 0.\]

At \(A, B, C\) taken as

\[
A(A - 1) - MK = 0 \quad \implies \quad A = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4MK},
\]

\[
2B(2B - 1) - j(j - 1) = 0 \quad \implies \quad B = \frac{1}{2} - \frac{j}{2}, \quad j.
\]

\[
2C(2C - 1) - j(j + 1) = 0 \quad \implies \quad C = -\frac{j}{2}, \quad \frac{1}{2} + \frac{j}{2},
\]

the above equation will simplify

\[
\frac{d^2 f_2}{dx^2} + \left(\frac{2A}{x} + \frac{4B + 1}{2(x - 1)} + \frac{4C + 1}{2(x + 1)}\right) \frac{df_2}{dx} + \left[\frac{-2A(B - C)}{x} + \frac{(2A + C)(4B + 1) + B + 4ME + j^2}{4(x - 1)} - \frac{(2A + B)(4C + 1) + C + 4ME + j^2}{4(x + 1)}\right] f_2 = 0
\]

and it coincides with the general Heun equation

\[
\frac{d^2 H}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z - 1} + \frac{\epsilon}{z + 1}\right) \frac{dH}{dz} + \frac{\alpha \beta z - q}{z(z - 1)(z - a)} H = 0, \quad (\gamma + \delta + \epsilon = \alpha + \beta + 1)
\]

or differently

\[
\frac{d^2 H}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z - 1} + \frac{\epsilon}{z + 1}\right) \frac{dH}{dz} + \left(\frac{-q}{a} \frac{1}{z} + \frac{q - \alpha \beta}{a - 1} \frac{1}{z - 1} + \frac{\alpha \beta a - q}{a(a - 1)} \frac{1}{z - a}\right) H = 0,
\]

with parameters

\[
a = -1, \quad q = 2A(C - B),
\]

\[
\gamma = 2A, \quad \delta = 2B + \frac{1}{2}, \quad \epsilon = 2C + \frac{1}{2},
\]

\[
\alpha = A + B + C + \frac{1}{2} \sqrt{4A(A - 1) + 2B(2B - 1) + 2C(2C - 1) - 2j^2 - 8ME},
\]

\[
\beta = A + B + C - \frac{1}{2} \sqrt{4A(A - 1) + 2B(2B - 1) + 2C(2C - 1) - 2j^2 - 8ME}.
\]

Allowing for the identity

\[
\frac{1}{2} \sqrt{4A(A - 1) + 2B(2B - 1) + 2C(2C - 1) - 2j^2 - 8ME} = \sqrt{-M(2E - K)},
\]

for parameters \(\alpha, \beta\) we obtain expressions

\[
\alpha = A + B + C + \sqrt{-M(2E - K)}, \quad \beta = A + B + C - \sqrt{-M(2E - K)}.
\]
We will use the above rule to produce energy spectrum, \( \beta = -n \). At

\[
A = \frac{1}{2} - \frac{1}{2} \sqrt{1 + 4MK}, \quad B = \frac{j}{2}, \quad C = \frac{1}{2} + \frac{j}{2},
\]

it gives

\[
j + 1 - \frac{1}{2} \sqrt{1 + 4MK} - \sqrt{-M(2E - K)} = -n.
\]

from whence it follows the formulas for energies levels (again let it be \( N = 1 + j + n \))

\[
E = N \sqrt{\frac{K}{M}} + \left( \frac{1}{2M} \right)^2 - \frac{1}{2M} \left( N^2 + \frac{1}{4} \right).
\]

Thus, collecting results together, we write down three series for energy values:

\[
\begin{align*}
\epsilon &= \hbar \left( N \sqrt{\frac{k}{m} + \frac{h^2}{4m^2 R^4}} - \frac{h}{2mR^2} \left( N^2 + \frac{1}{4} \right) \right), \quad N = 2n + j + \frac{3}{2}, \\
\epsilon &= \hbar \left( N \sqrt{\frac{k}{m} + \frac{h^2}{4m^2 R^4}} - \frac{h}{2mR^2} \left( N^2 + \frac{1}{4} \right) \right), \quad N = 2n' + j + 2, \\
\epsilon &= \hbar \left( N \sqrt{\frac{k}{m} + \frac{h^2}{4m^2 R^4}} - \frac{h}{2mR^2} \left( N^2 + \frac{1}{4} \right) \right), \quad N = 2n'' + j + 1.
\end{align*}
\]

16 Conclusion

The quantum-mechanical particle with spin 1 is explored in the field of magnetic charge. In the relativistic Duffin–Kemmer–Petiau we have separated the variables using the technique of D-Wigner functions, three quantum numbers \( \epsilon, j, m \): energy, the square and the third projection of the generalized total angular momentum, arise. System of 10 radial equations is complex and not amenable to analytical solution. Transition to the nonrelativistic Pauli approximation is performed – the problem is reduced to a system of three differential equations of the second order for three radial functions. The equations in the system can be separated and the problem is reduced to the study of the three ordinary differential equations of the same structure, each of these contains as a parameter a root \( A_k, k = 1, 2, 3 \) of cubic equation arising in solving the problem of bringing mixing matrix in the system of equations to the diagonal form. The analysis is generalized to presence of spherically symmetric external fields, the presence the external Coulomb field, and the external oscillator potential, are studied in detail. For each case, there are found three series of energy spectra of spin 1 particles \( \epsilon = \epsilon (A_k, j, n) \). This consideration is extended to the presence of external spherically symmetrical fields, in particular, Coulomb and oscillator ones. There are found energy spectrum and exact solutions in terms of hypergeometric functions.

In the same manner, spin 1 particle is treated in presence of Lobachevsky geometry background in nonrelativistic approximation. After separation of the variables the problem is reduced to the system of second order differential interrelated equations, which cannot be disconnected in presence of the monopole. However, in absence of the monopole field, equations have been solved exactly, for instance, in presence of Coulomb and oscillator potentials. Energy spectra have been found and solutions constructed in terms of hypergeometric and Heun functions.
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