Model Categories for Orthogonal Calculus

David Barnes

Joint work with Peter Oman
The input

We want to study functors from vector spaces (with inner products) to spaces.

**examples**

| $V \mapsto S^V$ | $V \mapsto E(V)$ for $E$ a spectrum |
|-----------------|--------------------------------------|
| $V \mapsto BO(V)$ | $V \mapsto B\text{Top}(V)$ |
| $V \mapsto B\text{Diff}^b(M \times V)$ | $V \mapsto \text{Emb}(M, V)$ |

We want to make sure that the functors we study take into account the topology on the space of linear isometries from $V$ to $W$. 
Continuous functors

Definition

Let $\mathcal{E}_0$ be the category with

- objects the continuous functors from the category of inner product spaces and isometries to the category of based spaces
- morphisms the natural transformations

If $E \in \mathcal{E}_0$ then for each inner product space $V$ there is a topological space $E(V)$. For each pair $V, W$ there is a map

$$\sigma_E(V, W) : \mathcal{I}(V, W) \vee E(V) \to E(W)$$

The maps $\sigma_E(V, W)$ should be associative and unital.
The filtration

For $E \in \mathcal{E}_0$, the orthogonal calculus creates a tower of fibrations in $\mathcal{E}_0$.

For each $V$, $D_nE(V) \to T_nE(V) \to T_{n-1}E(V)$ is a homotopy fibre sequence.
For any $E$ there is a canonical map,

$$\rho_E(V) : E(V) \rightarrow (\tau_n E)(V) := \text{holim}_{0 \neq U \subseteq \mathbb{R}^{n+1}} E(V \oplus U)$$

**Definition**

A functor $E$ is **$n$-polynomial** if the above map is a weak homotopy equivalence for all $V$.

The homotopy fibre of $\rho_E$ is the $(n+1)^{st}$-derivative of $E$.

**Definition**

$$T_n E = \text{hocolim}(E \rightarrow \tau_n E \rightarrow \tau_n^2 E \rightarrow \tau_n^3 E \rightarrow \ldots)$$
Examples

A functor $E$ is 0-polynomial if and only if $E(0) \simeq E(V)$ for all $V$. For $F \in \mathcal{E}_0$, $T_0 F$ is the space $\hocolim_k F(\mathbb{R}^k)$.

The first derivative of $BO(V)$ is the homotopy fibre of

$$BO(V) \to BO(V \oplus \mathbb{R})$$

This is $S^V$, so the first derivative is the sphere spectrum. The second derivative is the desuspension of the sphere spectrum.

The first derivative of $B \text{Top}(V)$ is Waldhausen’s $A$-theory of a point.
If \( f : E \to F \) is a map in \( \mathcal{E}_0 \) and \( F \) is \( n \)-polynomial then (up to homotopy) this map factors through as \( E \to T_n E \). Any \((n - 1)\)-polynomial functor is \( n \)-polynomial, so there is a map \( T_n E \to T_{n-1} E \).

**Definition**

An \textbf{\( n \)-homogeneous} functor is an \( n \)-polynomial functor \( F \) such that \( T_{n-1} F \simeq \ast \). \( D_n E \) is the homotopy fibre of \( T_n E \to T_{n-1} E \).

**Theorem (Weiss)**

The category of \( n \)-homogeneous functors (up to homotopy) is equivalent to the category of spectra with an \( O(n) \)-action, (up to homotopy).
The orthogonal tower

For \( F \in \mathcal{E}_0 \) let \( \Psi^n_F \) be the \( O(n) \)-spectrum corresponding to \( D_nF \).

\[
\begin{align*}
T_3 F(V) & \leftarrow \Omega^\infty [EO(3) + \wedge_{O(3)} (\Psi^3_F \wedge S^3 V)] \\
T_2 F(V) & \leftarrow \Omega^\infty [EO(2) + \wedge_{O(2)} (\Psi^2_F \wedge S^2 V)] \\
T_1 F(V) & \leftarrow \Omega^\infty [EO(1) + \wedge_{O(1)} (\Psi^1_F \wedge S^1 V)] \\
F(V) & \longrightarrow T_0 F(V)
\end{align*}
\]

Where \( nV = \mathbb{R}^n \otimes V \).
The category $\mathcal{E}_0$ has a **projective model structure** where fibrations and weak equivalences are defined objectwise.

There is an *$n$-polynomial model structure* on $\mathcal{E}_0$ where the weak equivalences are the $T_n$-equivalences and the cofibrations are the same as for the projective model structure.

There is an *$n$-homogeneous model structure* on $\mathcal{E}_0$ where the weak equivalences are the $D_n$-equivalences and the fibrations are the same as for the *$n$-polynomial model structure*. 

\[
\begin{array}{c}
n\text{-homog--}\mathcal{E}_0 \\
1 \downarrow \quad \uparrow 1 \quad 1 \\
n\text{-poly--}\mathcal{E}_0 \quad \xleftarrow{1} \quad \mathcal{E}_0 \\
1 \downarrow \quad \uparrow 1 \\
(n-1)\text{-poly--}\mathcal{E}_0
\end{array}
\]
The category $\mathcal{E}_0$ has a **projective model structure** where fibrations and weak equivalences are defined objectwise.

There is an **$n$-polynomial model structure** on $\mathcal{E}_0$ where the weak equivalences are the $T_n$-equivalences and the cofibrations are the same as for the projective model structure.

There is an **$n$-homogeneous model structure** on $\mathcal{E}_0$ where the weak equivalences are the $D_n$-equivalences and the fibrations are the same as for the $n$-polynomial model structure.
The intermediate category $O(n)\mathcal{E}_n$

**Definition**

- The objects are the $O(n)$-equivariant objects of $\mathcal{E}_0$ which are equipped with associative and unital $O(n)$-equivariant maps $S^n W \wedge X(V) \to X(V \oplus W)$.
- The morphisms the natural transformations that are compatible with the structure maps.

If $Y$ is an orthogonal spectrum with an $O(n)$-action then we can define an object of $O(n)\mathcal{E}_n$ via $(\alpha_n Y)(V) = Y(nV)$. This gives a right adjoint $\alpha_n : O(n)\mathcal{I}S \to O(n)\mathcal{E}_n$.

The forgetful functor followed by taking orbits gives a functor $\text{Res}_0^n / O(n) : O(n)\mathcal{E}_n \to \mathcal{E}_0$. This functor has a right adjoint.
Differentiation as a right adjoint

Define $nS \in \mathcal{E}_0$ to be the functor $V \mapsto S^{nV}$. This is a commutative ring object in $\mathcal{E}_0$. An object of $O(n)\mathcal{E}_n$ is, in particular, an $nS$-module.

An object $E \in \mathcal{E}_0$ gives an object of $O(n)\mathcal{E}_n$ via $\text{Nat}_{\mathcal{E}_0}(nS, E)$. This functor is right adjoint to $\text{Res} / O(n)$.

**Lemma**

The $O(n)$-space $\text{Nat}_{\mathcal{E}_0}(nS, E)(V)$ is weakly equivalent to the homotopy fibre of

$$\rho_E(V) : E(V) \to (\tau_nE)(V) := \text{holim}_{0 \neq U \subseteq \mathbb{R}^{n+1}} E(V \oplus U)$$
The \( n \)-stable model structure

We need a model structure on \( O(n) \mathcal{E}_n \). The most important step is to decide on the weak equivalences.

**Definition**

For \( k \in \mathbb{Z} \) define \( n\pi_k(X) \) to be the colimit of

\[
[S^k, X(0)] \rightarrow [S^{k+n}, S^n \wedge X(0)] \rightarrow [S^{k+n}, X(\mathbb{R})] \rightarrow \ldots
\]

There is a model structure on \( O(n) \mathcal{E}_n \) where the weak equivalences are those maps \( f \) such that \( n\pi_k(f) \) is an isomorphism for all \( k \in \mathbb{Z} \).

This model category is stable, proper and cofibrantly generated.
There are Quillen equivalences

\[ n\text{-homog} \mathcal{E}_0 \overset{\text{Res/}\mathcal{O}(n)}{\leftrightarrow} \mathcal{O}(n)\mathcal{E}_n \overset{\beta_n}{\leftrightarrow} \mathcal{O}(n)\mathcal{I}\mathcal{S} \]

\[ \overset{\text{Nat}(nS,\_)}{\leftrightarrow} \overset{\alpha_n}{\leftrightarrow} \]

Hence the homotopy theory of homogeneous functors is entirely captured by spectra with an \( \mathcal{O}(n) \)-action.

For \( F \in \mathcal{E}_0 \), the \( \mathcal{O}(n) \)-spectrum object \( \Psi^n_F \) of the orthogonal tower is \( \text{L} \beta_n \text{R} \text{Nat}(nS, F) \)
Advantages of the model structures

We now have a better description of the functors and categories we are using. Rather than using the category of spectra with $O(n)$-action we can just use $O(n)\mathcal{E}_n$.

Applications

- Stable orthogonal calculus, studies functors from inner product spaces to spectra.
- Can do orthogonal calculus for any cellular, proper topological model category.
- Rational orthogonal calculus, using the rational model structure on topological spaces. Homogeneous functors are then classified in terms of modules over $H^*(BSO(n))[C2]$.
- Same pattern to Goodwillie calculus which has an intermediate category analogous to $O(n)\mathcal{E}_n$. 