Quantum Group Symmetries in Conformal Field Theory

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Abstract Quantum groups play the role of hidden symmetries of some two-dimensional field theories. We discuss how they appear in this role in the Wess-Zumino-Witten model of conformal field theory.

1. WZW model

Wess-Zumino-Witten (WZW) model [1] occupies an important place in the conformal field theory: it is a model of two-dimensional quantum massless fields exactly soluble for energy spectrum and correlation functions; it is also a generating theory for a rich family of other conformal field theories which may be obtained from the WZW one by various versions of the so-called coset construction [2][3]. It is a 1 + 1-dimensional analogue of 0 + 1-dimensional particle on the group \( G \) (for simplicity, we shall assume \( G \) to be a compact matrix group). The motion of the particle on \( G \) is described by the classical equation

\[
\partial_t (g \partial_t g^{-1}) = 0. \tag{1}
\]

On the quantum level, the system may be solved by harmonic analysis on \( G \). The quantum space of states is \( L^2(G) \) and carries the representation of \( G \times G \).

\[
L^2(G) = \bigoplus \lambda V_\lambda \otimes V_\lambda \tag{2}
\]

where \( V_\lambda \) denotes the irreducible highest weight (HW) \( \lambda \) representation of \( G \). The Hamiltonian is proportional to the quadratic Casimir operator. The multiplication by matrix elements of \( g \in G \) may be expressed in the realization (2) of \( L^2(G) \) as a bilinear combination of Clebsch-Gordan coefficients for the tensor product with the fundamental representation, i.e. as a bilinear expression in intertwiners of representations of \( G \).

The WZW model may be viewed as describing the particle on the loop group \( LG \). The classical equation of motion is

\[
\partial_- (g \partial_+ g^{-1}) = 0 \tag{3}
\]

where \( \partial_\pm \equiv \partial/\partial(x^1 \pm x^0) \) and we shall consider the cylindrical geometry with \( x^1 \) taken mod \( 2\pi \). The quantum theory is solved by harmonic analysis

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on the Kac-Moody group $\hat{LG}$, the central extension of the loop group. The space of states is
\[ \bigoplus_{\lambda \text{ int.}} V_{k,\lambda} \otimes V_{k,\lambda} \]  
(4)
where the level $k$, a fixed positive integer, is the value of the central charge of $\hat{LG}$ and $\lambda$ runs through the finite ($k$-dependent) set of the so called integrable highest weights. The symmetry of the model is $\hat{LG} \times \hat{LG}$ where the factors act on, respectively, left- and right-moving degrees of freedom. The spaces $V_{k,\lambda}$ and $\bar{V}_{k,\lambda}$ combine diagonally in eq. (4) due to the coupling between the left- and right-movers which share a finite number of degrees of freedom. The field operators of the WZW theory are bilinear combinations of intertwiners of representations of $\hat{LG}$.

There are several ways to see manifestations of the hidden quantum group symmetry in the WZW model.\footnote{See also [4]-[6] for the related work concerning the minimal or Liouville theories and [7]-[9] for an attempt at a more general, less model dependent approach relating the quantum-group-like symmetries to the theory of superselection sectors and field statistics in the spirit of [10][11]}

The labeling of the representations $V_{k,\lambda}$ is similar to that of the “good” representations $[12]$ of the Drinfeld-Jimbo quantum group $[13][14] U_q(\mathcal{G})$ for $q = e^{\pi i/(k+\hbar)}$ ($\hbar$ is the dual Coxeter number of $\mathcal{G}$, the Lie algebra of $G$). The exchange algebra of the intertwining operators is given by the $6j$-symbols of $U_q(\mathcal{G})$ $[15]-[17]$. In the free field description of the WZW theory $[18][19]$, the quantum group describes the homology of the contours of screening charge integrals $[20]-[23]$. Let $j(x)$ be the current giving the infinitesimal version of a representation $\pi$ of $\hat{LG}$ by

\[ \pi(X) = \int_0^{2\pi} \text{tr}(j(s)X(s))\,ds \]  
(5)
for $X \in \mathcal{LG}$. In representation of central charge $k$, current $j$ satisfies the commutation relations

\[ [j(x)_1, j(y)_2] = (j(x)_1C - Cj(y)_2) \delta(x-y) - (ik/2\pi)C \delta'(x-y) \]  
(6)
where $j_1 \equiv j \otimes \text{Id}$, $j_2 \equiv \text{Id} \otimes j$ and $C = \sum a \otimes t^a \in \mathcal{G} \otimes \mathcal{G}$, $\text{tr} t^a t^b = \delta^{ab}$. Eq. (6) may be viewed as the defining relation of the Kac-Moody algebra $\hat{L}\mathcal{G}$. Various aspects of the $U_q(\mathcal{G})$ symmetry in the WZW model may be formally explained by postulating that the monodromy operators

\[ M(x) = Pe^{\int_x^{x+2\pi} j(s)\,ds} \]  
(7)
satisfy the $\mathcal{U}_q(\mathcal{G})$ relations
\begin{equation}
M_1 R^+ M_2 (R^+)^{-1} = R^- M_2 (R^+)^{-1} M_1
\end{equation}
where $R^\pm$ is a pair of solutions of the quantum Yang-Baxter equation
\begin{equation}
R^\pm_{12} R^\pm_{13} R^\pm_{23} = R^\pm_{23} R^\pm_{13} R^\pm_{12}.
\end{equation}
Operators (7) appeared (implicitly) in [24] and were explicitly discussed as
generators of $\mathcal{U}_q(\mathcal{G})$ in [25]. The problem is that, as written above, they are very singular objects due to the short distance singularity of currents $j(x)$.

2. Lattice Kac-Moody algebra

A possible way out from the above difficulty has been proposed in a series of papers [26]-[29] which introduced the lattice version of the Kac-Moody algebra. This allows to build the regularized version of the WZW model preserving essentially all the symmetries of the continuum WZW theory and at the same time making explicit the quantum group symmetry hidden in the continuum version of the theory.

View $\mathbf{Z}_N$ as an $N$-point lattice in $S^1$. The lattice Kac-Moody algebra $\mathcal{K}_N$ proposed in [26],[27] is given by matrix generators $J(n), n \in \mathbf{Z}_N$, satisfying the quadratic relations
\begin{align*}
J(n)_1 J(n)_2 &= R^+ J(n)_2 J(n)_1 R^-, \\
J(n)_1 R^- J(n-1)_2 &= J(n-1)_2 J(n)_1, \\
J(n)_1 J(m)_2 &= J(m)_2 J(n)_1 \quad \text{for } |n-m| > 1.
\end{align*}
The other relations are the deformations of the $\text{det} = 1$ conditions for $\text{SL}(n)$ etc. For $\text{SL}(2)$ they read $J_{11}(n)J_{22}(n) - q^{-1}J_{21}(n)J_{12}(n) = q^{1/2}$ (here the subscripts "ij" refer to the matrix element of $J(n)$). The $\text{SL}(2)$ $R$-matrices are
\begin{align*}
R^+ &= q^{1/2} \begin{pmatrix}
q^{-1} & 0 & 0 & 0 \\
0 & 1 & q^{-1} - q & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & q^{-1}
\end{pmatrix}, \\
R^- &= q^{-1/2} \begin{pmatrix}
q & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & q^{-1} & 1 & 0 \\
0 & 0 & 0 & q
\end{pmatrix}.
\end{align*}
Operators $J(n)$ should be thought of as regularized versions of parallel transport operators $\text{P} \exp[(k + \hbar)^{-1} \int_{2\pi n/N}^{2\pi(n+1)/N} j(s) \, ds]$ [26].

Algebra $\mathcal{K}_N$ contains quantum group $\mathcal{U}_q(\mathcal{G})$ as the monodromy which is now, unlike in eq. (7), defined by a regular expression:
\begin{equation}
M(n) = J(n + N - 1)J(n + N - 2) \cdots J(n).
\end{equation}
$M(n)$ satisfy relations (8). For the $SL(2)$ case, we may write

$$M = q^{3/2} \begin{pmatrix} q^{-S^3} & (q^{-1} - q)S^+ \\ 0 & q^{-1} \end{pmatrix} \begin{pmatrix} q^{S^3} & 0 \\ (q - q^{-1})S^- & q^{-S^3} \end{pmatrix}$$  \hspace{1cm} (11)$$

where

$$[S^3, S^\pm] = \pm S^\pm,$$

$$[S^+, S^-] = (q^{2S^3} - q^{-2S^3})/(q - q^{-1}).$$

Algebras $\mathcal{K}_N$ may be also viewed as interpolating between quantum group $\mathcal{U}_q(\mathcal{G})$ and the enveloping algebra $\mathcal{U}(\widehat{\mathcal{L}} \mathcal{G})$ when $N$ goes from 1 to $\infty$.

3. Conformal invariance on lattice

The algebraic manifestation of the conformal invariance underlying the continuum Kac-Moody algebras is the action of the orientation-preserving diffeomorphisms of the circle $D \in \text{Diff}_+ S^1$ on the currents

$$j(x) \mapsto \frac{dD(x)}{dx} j(D(x))$$

which induces automorphisms of $\widehat{\mathcal{L}} \mathcal{G}$. The latter are unitarily implementable in the HW representations:

$$\frac{dD(x)}{dx} j(D(x)) = U_D j(x) U_D^{-1}.$$  \hspace{1cm} (13)$$

$D \mapsto U_D$ is a projective representation of $\text{Diff}_+ S^1$ giving, on the infinitesimal level, a representation of the Virasoro algebra.

This structure essentially descends to the lattice. Let

$$D : \mathbb{Z} \to \mathbb{Z}$$

be an increasing map such that $D(n + N') = D(x) + N$ $(N' \leq N)$. Thus $D$ describes blocking of intervals of lattice $\mathbb{Z}_N$ into those of $\mathbb{Z}_{N'}$. $D$ induces a “block spin” homomorphism of the lattice Kac-Moody algebras $\mathcal{D} : \mathcal{K}_{N'} \to \mathcal{K}_N$

$$\mathcal{D}(J(n)) = J(D(n + 1) - 1) \cdots J(D(n) + 1)J(D(n)).$$  \hspace{1cm} (12)$$

On the lattice, the conformal transformations are represented by a (local) renormalization group!

If we relabel algebras $\mathcal{K}_N$ by arbitrary triangulations $T$ of $S^1$ (i.e. splittings of $S^1$ into intervals $l_n$) renaming the generators as $J(l_n)$ then the block spin homomorphisms give rise to homomorphisms $\iota_{T':T} : \mathcal{K}_T \to \mathcal{K}_{T'}$
for $T'$ finer than $T$. One may define the continuum limit algebra $\mathcal{K}_\infty$ as the inductive limit of algebras $\mathcal{K}_T$. Any $D \in \text{Diff}_+ S^1$ defines an isomorphism $D : \mathcal{K}_T \to \mathcal{K}_{D(T)}$ which descends to $\mathcal{K}_\infty$. As a result, $\text{Diff}_+ S^1$ acts by automorphisms on $\mathcal{K}_\infty$ showing that the block spin homomorphisms indeed encode the conformal invariance.

In continuum, the generators of the Virasoro algebra implementing conformal invariance in the HW representations of the Kac-Moody algebra are given in terms of current $j(x)$ by the Sugawara construction. On the lattice, the block spin homomorphisms $D$ may be also implemented in the class of representations that we shall study below. If $D$ is a rigid rotation of $\mathbb{Z}_N$ then we may expect that the implementing maps are expressible in terms of generators of $\mathcal{K}_N$. Such lattice Sugawara expressions for the (Minkowski time) transfer matrix are not known yet except for the $U(1)$ case, see below.

4. Free field representations of the Kac-Moody algebras

In continuum, there are various ways to approach the construction of the HW representations $V_{k,\lambda}$ of the Kac-Moody algebras.

1. In the algebraic approach [34] one starts from the concept of Verma modules of $\hat{L}\mathcal{G}$ and analyzes their reducibility studying the structure of singular vectors with the Kac-Kazhdan determinant formula, constructing the BGG resolution etc.

2. In the geometric Borel-Weil type approach [35] one constructs $V_{k,\lambda}$ as the space of holomorphic sections of a line bundle over $LG/T$ ($T$ is the Cartan subgroup of $G$).

3. Finally, more recently, another algebraic approach to the HW representations of $\hat{L}\mathcal{G}$ has been obtained [18][19][36][37] by representing current $j(x)$ by free fields and constructing $V_{k,\lambda}$ as a cohomology of a complex of Fock space (Wakimoto) modules of the Kac-Moody algebra.

From those three approaches at least the last one carries over to the lattice so let us sketch some of its essential points. We shall stick to the $SL(2)$ case. The free fields one uses to represent $j(x)$ are chiral scalar field $\phi(x)$ satisfying

$$[\phi(x), \phi(y)] = \frac{\pi i}{2(k+2)} \text{sgn}(x-y)$$

or the corresponding $u(1)$ current $\partial \phi$ ($\phi(x+2\pi) = \phi(x) + \pi r/(k+2)$ where $r$ is the momentum) and the $\beta \gamma$ system $\beta(x), \gamma(x)$ (periodic in $x$) s.t.

$$[\beta(x), \gamma(y)] = \frac{2\pi i}{k+2} \delta(x-y)$$

Its generators are $P \exp\left[ \int (k+h)^{-1} j(s) ds \right]$ which are singular in the standard continuum Kac-Moody algebra.
(all other commutators vanish). These fields act in the standard Fock spaces \( \mathcal{F}_r \) (labeled by the eigenvalues of \( r \)).

\[
j = \frac{k+2}{2\pi} \left( i\partial \phi^+ : \beta \gamma : \quad \partial \beta - 2i\beta \partial \phi^- : \beta^2 \gamma : \quad -i\partial \phi^- : \beta \gamma : \right)
\]  

(15)
gives the \( \widetilde{sl}(2) \) currents and turns the Fock spaces into the (Wakimoto) \( \widetilde{sl}(2) \)-modules. For \( r = 1, \ldots, k+1 \), using the (regularized) powers of the screening charge integral

\[
Q(x) = \int_x^{x+2\pi} \gamma(s) : e^{2i\phi(s)} : ds,
\]  

(16)
one obtains the following (x-independent) complex of Fock space modules (\( p \equiv k + 2 \))

\[
\cdots \rightarrow \mathcal{F}_{r+2p} \rightarrow \mathcal{F}_r \rightarrow \mathcal{F}_{r-p} \rightarrow \mathcal{F}_{r-2p} \rightarrow \cdots
\]  

(17)
whose cohomology is concentrated at \( \mathcal{F}_r \) and provides the HW representation of \( \widetilde{sl}(2) \) of level \( k \) and spin \( \frac{1}{2}(r-1) \) [19]. The intertwining chiral fields are

\[
h = \left( \begin{array}{c}
\beta : e^{-i\phi} : Q \\
\beta : e^{-i\phi} : e^{-i\phi}:
\end{array} \right).
\]  

(18)
Each row of the matrix defines an operator between the complexes (17) which projected onto their cohomology gives the component with magnetic number \( \pm \frac{1}{2} \) of the spin \( \frac{1}{2} \) primary field of the WZW model [38]. The real fields of the WZW theory are bilinear combinations of fields \( h \) and their right-moving partners.

5. Lattice free fields

One may build the representations of the lattice Kac-Moody algebra \( \mathcal{K}_N \) using (deformed) free fields. Below, \( q^{1/2} = e^{\pi i/(2(k+2))} \) so that \( q \) is a primitive \( 2p \)-th root of unity. We shall need lattice versions of the \( u(1) \) current and of the \( \beta\gamma \) system.

5.1. \( u(1) \) CURRENT

The lattice \( u(1) \) algebra \( \mathcal{U}_N \) is generated by invertible elements \( \Theta(n) \), \( n \in \mathbb{Z}_N \), with the relations

\[
\Theta(n) \Theta(n+1) = q^{1/2} \Theta(n+1) \Theta(n)
\]  

(19)
(all other generators commute). Think about $\Theta(n)$ as of a regularized version of $\exp\left[i\frac{2\pi(n+1)}{N}\partial\phi(s)ds\right]$. $U_N$ becomes a $*$-algebra if we put $\Theta(n)^* = \Theta(n)^{-1}$.

Consider, for $N$ odd, the representations of $U_N$ acting in the space spanned by orthonormal vectors $|\alpha\rangle$ where $\alpha \equiv (\alpha_0, \alpha_2, \ldots, \alpha_{N-1})$, $\alpha_{2n} \in \mathbb{Z}_{4p}$, $\sum \alpha_{2n} = 0$:

$$\Theta(2n) |\alpha\rangle = z_{2n} q^{\alpha_{2n}/2} |\alpha\rangle \text{ for } 2n < N-1,$$

$$\Theta(2n+1) |\alpha\rangle = z_{2n+1}[(\alpha_0, \ldots, \alpha_{2n} + 1, \alpha_{2n+2} - 1, \ldots, \alpha_{N-1})],$$

$$\Theta(N-1) |\alpha\rangle = z_{N-1} q^{N-2+(N+1)/4} [\alpha_0 - 1, \alpha_2, \ldots, \alpha_{N-3}, \alpha_{N-1} + 1]$$

where $|z_n| = 1$.

**Proposition.** [39] The above formulae give irreducible $*$-representations of $U_N$. They are equivalent iff they correspond to the same eigenvalues of the central elements $\Theta(n)^{4p} = z(n)^{4p}$ and $\Pi \equiv q^{-1/2}\Theta(N-1)\cdots\Theta(1)\Theta(0) = \prod z_n$. Every irreducible $*$-representation of $U_N$ is equivalent to one of the above.

Below, we shall study uniquely representations with $\Theta(n)^{4p} = 1$ (we could have included this condition into the defining relations of $U_N$). In those representations, the rigid rotation of $\mathbb{Z}_N$ by two units may be implemented by elements $U \in U_N$:

$$U \Theta(n+1) = \Theta(n-1) U$$

where

$$U = (4p)^{-N/2} \sum_{\alpha_n \in \mathbb{Z}_{4p}} q^{\frac{1}{2} \alpha_0 \alpha_{N-1}} \Theta(N-1)^{\alpha_{N-1}} \cdots \Theta(1)^{\alpha_1} \Theta(0)^{\alpha_0}. \quad (20)$$

Notice that, except for the boundary term, $U$ is a product of local expressions. Eq. (20) gives the lattice version of the Sugawara construction of the WZW Hamiltonian for the abelian group.

The case of $N$ even is similar.

### 5.2. $\beta\gamma$ System

Consider algebra $\mathcal{B}$ with generators $B$, $\Gamma$ and relation

$$q B \Gamma - q^{-1} \Gamma B = q - q^{-1}. \quad (21)$$

$B^p$ and $\Gamma^p$ generate the center of $\mathcal{B}$ and we have different classes of representations depending on the eigenvalues of those elements.
1. If in an irreducible representation $\Gamma^p \neq 0$ then $\Gamma$ is invertible and we may introduce
$$D = \Gamma^{-1} - B$$ satisfying
$$\Gamma D = q^2 D \Gamma.$$ If $D = 0$ we get the 1-dimensional representation of $B$. If $D \neq 0$, the representations of $B$ are periodic: we may find a basis $|s\rangle$, $s \in \mathbb{Z}_p$, s.t.
$$\Gamma |s\rangle = \zeta_1 |s + 1\rangle, \quad D |s\rangle = \zeta_2 q^{-2s} |s\rangle.$$ The periodic representations are characterized by the eigenvalues of $\Gamma^p = \zeta_1^p$ and of $B^p = \zeta_1^{-p} + \zeta_2^p$.

2. The case $\Gamma^p = 0$ but $B^p \neq 0$ may be treated similarly.

3. Finally, if $\Gamma^p = 0$ and $B^p = 0$, we either have a trivial representation or a HW representation in $p$-dimensional space spanned by states $|s\rangle$, $s = 0, 1, \ldots, p - 1$, with the action
$$\Gamma |s\rangle = \begin{cases} |s + 1\rangle & \text{if } s < p - 1, \\ 0 & \text{if } s = p - 1, \end{cases}$$
$$B |s\rangle = \begin{cases} (1 - q^{-2s}) |s - 1\rangle & \text{if } s > 0, \\ 0 & \text{if } s = 0. \end{cases}$$

The lattice $\beta\gamma$ system is obtained by taking a copy of algebra $B$ for each lattice site:
$$B_N = \bigotimes_{n \in \mathbb{Z}_N} B.$$ We shall denote its generators as $B(n)$ and $\Gamma(n)$. $B_N$ may be represented in the space spanned by vectors $\langle s \rangle$, $B(n)$, $\Gamma(n)$ acting on the $n$-th component of $\underline{s} \equiv (s_0, s_1, \ldots, s_{N-1})$. Below, we shall only consider the HW representations of $B_N$ so that we could include $B(n)^p = \Gamma(n)^p = 0$ into the defining relations of $B_N$. The periodic representations might be also of interest but we shall not study them here.

In the HW representation, the rigid rotation of $\mathbb{Z}_N$ by one unit is implemented by element $U \in B_N$,
$$U = p^{-N} \sum_{\alpha_n, \beta_n \in \mathbb{Z}_p} \prod_{n=0}^{N-1} (1 - q^{-2\beta_n})^{-1} (1 - q^{-2(\beta_n - 1)})^{-1} \cdots (1 - q^{-2})^{-1} \cdot \Gamma(N - 1)^{\beta_0} (1 - \Gamma(N - 1)B(N - 1))^{\alpha_{N-1}} B(N - 1)^{\beta_{N-1}} \cdots \cdots \Gamma(1)^{\beta_2} (1 - \Gamma(1)B(1))^{\alpha_1} B(1)^{\beta_1} \Gamma(0)^{\beta_1} (1 - \Gamma(0)B(0))^{\alpha_0} B(0)^{\beta_0},$$

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5 I owe this observation to L. Faddeev and A. Volkov
6 This possibility was pointed to me by R. Kashaev
such that
\[ U \mathcal{B}(n+1) = \mathcal{B}(n)U, \]
\[ U \Gamma(n+1) = \Gamma(n)U. \]

5.3. Lattice Wakimoto representation

It will be convenient to put together the lattice \( u(1) \) fields and the \( \beta\gamma \) system in a somewhat twisted way defining their action on vectors \( |\alpha, s\rangle \equiv |\alpha\rangle \otimes |s\rangle \) by
\[ \Theta(n) |\alpha, s\rangle = q^{s_{n+1} - s_n} (\Theta(n) |\alpha\rangle) \otimes |s\rangle, \]
\[ \mathcal{B}(n) |\alpha, s\rangle = |\alpha\rangle \otimes (\mathcal{B}(n) |s\rangle), \]
\[ \Gamma(n) |\alpha, s\rangle = |\alpha\rangle \otimes (\Gamma(n) |s\rangle). \]

Taking different representations of \( \mathcal{U}_N \) and the HW representation of \( \mathcal{B}_N \), we obtain this way irreducible representations (labelled by the eigenvalue \( z \) of \( \Pi \)) of algebra \( \mathcal{U}_N \otimes \mathcal{B}_N \), a twisted tensor product of \( \mathcal{U}_N \) and \( \mathcal{B}_N \). We shall denote by \( \mathcal{H}_z \) the corresponding representation space.

The lattice version of the Wakimoto realization (15) of the Kac-Moody currents by free fields is given by the formulae
\[ J_{11}(n) = \Theta(n) + q^{-1/2} \Theta(n)^{-1} \mathcal{B}(n+1) \Gamma(n), \]
\[ J_{12}(n) = -\Theta(n) \mathcal{B}(n) + q^{-1/2} \Theta(n)^{-1} \mathcal{B}(n+1) (1 - \Gamma(n) \mathcal{B}(n)), \] (22)
\[ J_{21}(n) = q^{1/2} \Theta(n)^{-1} \Gamma(n), \]
\[ J_{22}(n) = q^{1/2} \Theta(n)^{-1} (1 - \Gamma(n) \mathcal{B}(n)) \]
which define a homomorphism from the lattice Kac-Moody algebra \( \mathcal{K}_N \) to \( \mathcal{U}_N \otimes \mathcal{B}_N \) and turn each representation space \( \mathcal{H}_z \) of the latter into a (Wakimoto) \( \mathcal{K}_N \)-module. These modules are irreducible if \( z^{2p} \neq 1 \). For \( z = q^r \), \( r = -p + 1, \ldots, p - 1, p \), their reducibility may be studied by adapting to the lattice the cohomological constructions of [19].

6. Bernard-Felder cohomology

We shall have to adjoin to the \( u(1) \) algebra \( \mathcal{U}_N \) the zero mode \( \Psi(0) \) s.t.
\[ \Theta(0) \Psi(0) = q^{-1/2} \Psi(0) \Theta(0), \]
\[ \Theta(N-1) \Psi(0) = q^{-1/2} \Psi(0) \Theta(N-1) \]
and all other commutators are trivial. \( \Psi(0) \), which divides the eigenvalue of \( \Pi \) by \( q \), may be implemented in the sum of representation spaces of \( \mathcal{U}_N \) with \( \Pi = q^r \). We shall also let it act in \( \oplus \mathcal{H}_{q^r} \) by
\[ \Psi(0) |\alpha, s\rangle = q^{sr} (\Psi(0) |\alpha\rangle) \otimes |s\rangle. \]
More symmetrically, we may construct the lattice $u(1)$ vertex operator ($\sim e^{i\phi(2\pi n/N)}$)

$$\Psi(n) = \Theta(n-1)\ldots\Theta(1)\Theta(0)\Psi(0) \quad \text{for } n > 0,$$
$$\Psi(n) = \Theta(n)^{-1}\ldots\Theta(-2)^{-1}\Theta(-1)^{-1}\Psi(0) \quad \text{for } n < 0.$$ \quad (23)

The screening charge integral is now defined by

$$Q(n) = \Pi^{-1} \sum_{m=n}^{n+N-1} \Gamma(m)\Psi(m)^2,$$ \quad (24)

compare eq. (16). It is related to the lower left matrix element of the monodromy matrix (10) in the Wakimoto realization (22):

$$Q(n) = q M_{21}(n) \Psi(n)^2$$

and thus (see eq. (11)) to the $U_q(sl(2))$ lowering operator. The relation of the screening charge integrals to quantum groups has been observed in [20] and was developed in [21]-[23] into a theory of topological realizations of quantum groups.

$Q(n)$ acts as an operator from $\mathcal{H}_z$ to $\mathcal{H}_{z'}$, $z' = q^{-2}z$, in a nilpotent way: $Q(n)^p = 0$. Besides, for $r = 0, 1, \ldots, p$, powers of $Q(n)$ define complexes of $K_N$-modules

$$0 \rightarrow \mathcal{H}_{q^{-r}} \xrightarrow{Q^{p-r}} \mathcal{H}_{q^r} \xrightarrow{Q_r} \mathcal{H}_{q^{-r}} \rightarrow 0,$$

$$0 \rightarrow \mathcal{H}_{q^r} \xrightarrow{Q} \mathcal{H}_{q^{-r}} \xrightarrow{Q^{p-r}} \mathcal{H}_{q^r} \rightarrow 0.$$ 

In other words, the powers of $Q(n)$ above (independent, in fact, of $n$) commute with the action of $K_N$ as given by eqs. (22). One may show that the above complexes are exact in the middle. We conjecture that the remaining cohomology

$$\mathcal{H}_{q^r} \supset \ker Q^r \cong \mathcal{H}_{q^r}' \cong \mathcal{H}_{q^{-r}}/\im Q^r \cong \mathcal{H}_{q^{-r}}'$$ \quad (25)

and

$$\mathcal{H}_{q^{-r}} \supset \ker Q^{p-r} \cong \mathcal{H}_{q^{-r}}'' \cong \mathcal{H}_{q^r}/\im Q^{p-r} \cong \mathcal{H}_{q^r}''$$ \quad (26)

gives irreducible representations of $K_N$. This is true, for example, for $N = 1$ when $K_N$ reduces to $U_q(sl(2))$. 

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7. WZW theory on lattice

The chiral intertwining operators on the lattice are
\[ h(n) = \begin{pmatrix}
(\Pi - \Pi^{-1}) \Psi(n) + B(n) \Psi(n)^{-1} Q(n) & B(n) \Psi(n)^{-1} Q(n) \\
\Psi(n)^{-1} Q(n) & \Psi(n)^{-1} Q(n)
\end{pmatrix}. \tag{27}\]

In comparison with the expression (18), notice additional term in the upper left corner of (27) killed by Wick ordering renormalization in the continuum. Intertwiners (27) map space \( \oplus \mathcal{H}_q \) into itself and descend to cohomology (25), (26).

The above constructions have been adapted to the left-moving sector of the WZW theory. In the ones for the right-moving sector (distinguished below by the bar), we should replace \( q \) by \( \bar{q} = q^{-1} \). The space of states of the lattice WZW theory may be taken as
\[ \mathcal{H}' = \bigoplus_{r=1}^{p-1} \mathcal{H}_q^r \otimes \bar{\mathcal{H}}_q^r \]
or as \( \mathcal{H}'' \) using the doubly-primed spaces. The real fields of the lattice WZW model are then
\[ g(n, m) = h(n) \bar{h}(m)^{-1} \tag{28} \]
acting in spaces \( \mathcal{H}' \) or \( \mathcal{H}'' \), \( g(n, m) = g(n + N, m + M) \).

It should be stressed that the resulting theory lives on Minkowski lattice (\( n, m \) in eq. (28) are integer valued light-cone variables). The model has essentially all the symmetries of the continuum WZW theory (in deformed form), including the conformal covariance. Its relation to quantum groups is explicit. Let us conclude by listing some open problems.

1. As mentioned above, the lattice counterpart of the Sugawara construction of energy-momentum is not known apart from the abelian case.

2. The continuum limit is not easy to understand even for the \( u(1) \) case. One should expect it to take place in a weak form, for vacuum expectation values or traces of products of operators. Studying the first ones would require a choice of vacuum on the lattice (among many states invariant under lattice translations). The traces (e.g. the characters of the lattice Kac-Moody algebra) seem more accessible.

3. In the above constructions we did not study the unitarity properties of representations of \( K_N \) or of the intertwining fields. One knows that in the continuum this is difficult within the free field approach, even when the cohomology of the Fock space modules gives unitary HW representations of the Kac-Moody algebra. It remains to be seen if there exists a (deformed?) version of unitarity properties on the lattice.
4. There is a close relation between the two-dimensional (chiral) WZW model and the three dimensional Chern-Simons theory \cite{40}. For example, the space of the fixed-time Chern-Simons states on the disc is the basic representation of the Kac-Moody algebra \cite{41}. The Chern-Simons theory seems to possess lattice versions given by constructions \cite{42}\cite{43} of 3-manifold invariants from quantum groups. The lattice WZW model described above should be related to the latter. It seems that the clue to understanding this relation should be the fusion of lattice Kac-Moody algebras generalizing the coproduct for quantum groups and allowing to glue spaces of Chern-Simons states for more complicated topologies.

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