The Branch Set of a Quasiregular Mapping

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Abstract

We discuss the issue of branching in quasiregular mapping, and in particular the relation between branching and the problem of finding geometric parametrizations for topological manifolds. Other recent progress and open problems of a more function theoretic nature are also presented.

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1. Branched coverings

A continuous mapping \( f : X \to Y \) between topological spaces is said to be a branched covering if \( f \) is an open mapping and if for each \( y \in Y \) the preimage \( f^{-1}(y) \) is a discrete subset of \( X \). The branch set \( B_f \) of \( f \) is the closed set of points in \( X \) where \( f \) does not define a local homeomorphism.

Nonconstant holomorphic functions between connected Riemann surfaces are examples of branched coverings. From the Weierstrassian (power series) point of view this property of holomorphic functions is almost immediate. It is a deeper fact, due to Riemann, that the same conclusion can be drawn from the mere definition of complex differentiability, or, equivalently, from the Cauchy-Riemann equations. Most of this article discusses the repercussions of this fact.

2. Quasiregular mappings

In a 1966 paper [27], Reshetnyak penned a definition for mappings of bounded distortion or, as they are more commonly called today, quasiregular mappings.
These are nonconstant mappings \( f : \Omega \rightarrow \mathbb{R}^n \) in the Sobolev space \( W^{1,n}_{loc}(\Omega; \mathbb{R}^n) \), where \( \Omega \subset \mathbb{R}^n \) is a domain and \( n \geq 2 \), satisfying the following requirement: there exists a constant \( K \geq 1 \) such that
\[
|f'(x)|^n \leq K J_f(x)
\] (2.1)
for almost every \( x \in \Omega \), where \( |f'(x)| \) denotes the operator norm of the (formal) differential matrix \( f'(x) \) with \( J_f(x) = \det f'(x) \) its Jacobian determinant. One also speaks about \( K \)-quasiregular mappings if the constant in (2.1) is to be emphasized.\(^1\)

Requirement (2.1) had been used as the analytic definition for quasiconformal mappings since the 1930s, with varying degrees of smoothness conditions on \( f \). Quasiconformal mappings are by definition quasiregular homeomorphisms, and Reshetnyak was the first to ask what information inequality (2.1) harbours per se. In a series of papers in 1966–69, Reshetnyak laid the analytic foundations for the theory of quasiregular mappings. The single deepest fact he discovered was that quasiregular mappings are branched coverings (as defined above). It is instructive to outline the main steps in the proof for this remarkable assertion, which akin to Riemann’s result exerts significant topological information from purely analytic data. For the details, see, e.g., [28], [29], [18].

To wit, let \( f : \Omega \rightarrow \mathbb{R}^n \) be \( K \)-quasiregular. Fix \( y \in \mathbb{R}^n \) and consider the preimage \( Z = f^{-1}(y) \). One first shows that the function \( u(x) = \log |f(x) - y| \) solves a quasilinear elliptic partial differential equation
\[
-\text{div}A(x, \nabla u(x)) = 0, \quad A(x, \xi) \cdot \xi \simeq |\xi|^n,
\] (2.2)
in the open set \( \Omega \setminus Z \) in the weak (distributional) sense. In general, \( A \) in (2.2) depends on \( f \), but its ellipticity only on \( K \) and \( n \). For holomorphic functions, i.e., for \( n = 2 \) and \( K = 1 \), equation (2.2) reduces to the Laplace equation \(-\text{div}\nabla u = 0\).

Now \( u(x) \) tends to \(-\infty\) continuously as \( x \) tends to \( Z \). Reshetnyak develops sufficient nonlinear potential theory to conclude that such polar sets, associated with equation (2.2), have Hausdorff dimension zero. It follows that \( Z \) is totally disconnected, i.e., the mapping \( f \) is light. This is the purely analytic part of the proof. The next step is to show that nonconstant quasiregular mappings are sense-preserving. This part of the proof mixes analysis and topology. What remains is a purely topological fact that sense-preserving and light mappings between connected oriented manifolds are branched coverings.

Initially, Reshetnyak’s theorem served as the basis for a higher dimensional function theory. In the 1980’s, it was discovered by researchers in nonlinear elasticity. In the following, we shall discuss more recent, different types of applications.

3. The branch set

Branched coverings between surfaces behave locally like analytic functions according to a classical theorem of Stoilow. By a theorem of Chernavskii, for every
\(^{1}\)The definition readily extends for mappings between connected oriented Riemannian \( n \)-manifolds.
$n \geq 2$, the branch set of a discrete and open mapping between $n$-manifolds has topological dimension at most $n - 2$. For branched coverings between 3-manifolds, the branch set is either empty or has topological dimension 1 [24], but in dimensions $n \geq 5$ there are branched coverings between $n$-manifolds with branch set of dimension $n - 4$, cf. Section 7.2.

The branch set of a quasiregular mapping is a somewhat enigmatic object in dimensions $n \geq 3$. It can be very complicated, containing for example many wild Cantor sets of classical geometric topology [14], [15]. There is currently no theory available that would explain or describe the geometry of allowable branch sets, cf. Problems 2 and 4 in Section 7.

In the next three sections, we shall discuss the problem of finding bi-Lipschitz parametrizations for metric spaces. It will become clear only later how this problem is related to the branch set.

4. Bi-Lipschitz parametrization of spaces

A homeomorphism $f : X \to Y$ between metric spaces is bi-Lipschitz if there exists a constant $L \geq 1$ such that

$$L^{-1}d_X(a, b) \leq d_Y(f(a), f(b)) \leq Ld_X(a, b)$$

for each pair of points $a, b \in X$. It appears to be a difficult problem to decide when a given a metric space $X$ can be covered by open sets each of which is bi-Lipschitz homeomorphic to an open set in $\mathbb{R}^n$, $n \geq 2$. If this is the case, let us say, for brevity and with a slight abuse of language, that $X$ is locally bi-Lipschitz equivalent to $\mathbb{R}^n$.

Now a separable metrizable space is a Lipschitz manifold (in the sense of charts) if and only if it admits a metric, compatible with the given topology, that makes the space locally bi-Lipschitz equivalent to $\mathbb{R}^n$ [22]. The problem here is different from characterizing Lipschitz manifolds among topological spaces, for the metric is given first, cf. [8], [39], [40], [41].

To get a grasp of the difficulty of the problem, consider the following example: There exist finite 5-dimensional polyhedra that are homeomorphic to the standard 5-sphere $S^5$, but not locally bi-Lipschitz equivalent to $\mathbb{R}^5$. This observation of Siebenmann and Sullivan [38] is based on a deep result of Edwards [9], which asserts that the double suspension $\Sigma^2 H^3$ of a 3-dimensional homology sphere $H^3$, with nontrivial fundamental group, is homeomorphic to the standard sphere $S^5$. (See also [6].) One can think of $X = \Sigma^2 H^3$ as a join $X = S^1 \ast H^3$, and it is easy to check that the complement of the suspension circle $S^1$ in $X$ is not simply connected. Consequently, every homeomorphism $f : X \to S^5$ must transfer $S^1$ to a closed curve $\Gamma = f(S^1)$ whose complement in $S^5$ is not simply connected. A general position argument and Fubini’s theorem imply that, in this case, the Hausdorff dimension of $\Gamma$ must be at least 3. Hence $f$ cannot be Lipschitz. In fact, $f$ cannot be H"{o}lder continuous with any exponent greater than $1/3$. It is not known what other obstructions there are for a homeomorphism $X \to S^5$, cf. [16, Questions 12–14].

See [33] and [37] for surveys on parametrization and related topics.

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2See [23] for a recent survey on dimension theory and branched coverings.
5. Necessary conditions

What are the obvious necessary conditions that a given metric space $X$ must satisfy, if it were to be locally bi-Lipschitz equivalent to $\mathbb{R}^n$, $n \geq 2$? Clearly, $X$ must be an $n$-manifold. Next, bi-Lipschitz mappings preserve Hausdorff measure in a quantitative manner, so in particular $X$ must be $n$-rectifiable in the sense of geometric measure theory; moreover, locally the Hausdorff $n$-measure should assign to each ball of radius $r > 0$ in $X$ a mass comparable to $r^n$. Let us say that $X$ is metrically $n$-dimensional if it satisfies these geometric measure theoretic requirements.

It is not difficult to find examples of metrically $n$-dimensional manifolds that are not locally bi-Lipschitz equivalent to $\mathbb{R}^n$. The measure theory allows for cusps and folds that are not tolerated by bi-Lipschitz parametrizations. Further geometric constraints are necessary; but, unlike in the case of the measure theoretic conditions, it is not obvious what these constraints should be. A convenient choice is that of local linear contractibility: locally each metric ball in $X$ can be contracted to a point inside a ball with the same center but radius multiplied by a fixed factor.\footnote{See [36] for analytic implications of this condition.}

Still, a metrically $n$-dimensional and locally linearly contractible metric $n$-manifold need not be locally bi-Lipschitz equivalent to $\mathbb{R}^n$. The double suspension of a homology 3-sphere with nontrivial fundamental group as described in the previous section serves as a counterexample. In 1996, Semmes [34], [35] exhibited examples to the same effect in all dimensions $n \geq 3$, and recently Laakso [21] crushed the last hope that the above conditions might characterize at least 2-dimensional metric manifolds that are locally bi-Lipschitz equivalent to $\mathbb{R}^2$. However, unlike the examples of Edwards and Semmes, Laakso’s metric space cannot be embedded bi-Lipschitzly in any finite dimensional Euclidean space. Thus the following problem remains open:

\textbf{Problem 1} Let $X$ be a topological surface inside some $\mathbb{R}^N$ with the inherited metric. Assume that $X$ is metrically 2-dimensional and locally linearly contractible. Is $X$ then locally by-Lipschitz equivalent to $\mathbb{R}^2$?

In conclusion, perhaps excepting the dimension $n = 2$, more necessary conditions are needed in order to characterize the spaces that are locally bi-Lipschitz equivalent to $\mathbb{R}^n$.\footnote{There are interesting and nontrivial sufficient conditions known, but these are far from being necessary [42], [43], [2], [3], [5].} The idea to use Reshetnyak’s theorem in this connection originates in two papers by Sullivan [40], [41], and is later developed in [17]. Recall that in this theorem topological conclusions are drawn from purely analytic data. Now imagine that such data would make sense in a space that is not a priori Euclidean. Then, if one could obtain a branched covering mapping into $\mathbb{R}^n$, manifold points would appear, at least outside the branch set. We discuss the possibility to develop this idea in the next section.

6. Cartan-Whitney presentations
Let $X$ be a metrically $n$-dimensional, linearly locally contractible $n$-manifold that is also a metric subspace of some $\mathbb{R}^N$. Suppose that there exists a bi-Lipschitz homeomorphism $f : X \to f(X) \subset \mathbb{R}^n$. Then $f$ pulls back to $X$ the standard coframe of $\mathbb{R}^n$, providing almost everywhere defined (essentially) bounded differential $1$-forms $\rho_i = f^*dx_i$, $i = 1, \ldots, n$. To be more precise here, by Kirzbraun’s theorem, $f$ can be extended to a Lipschitz mapping $\overline{f} : \mathbb{R}^N \to \mathbb{R}^n$, and the $1$-forms

$$\rho_i = \overline{f}^*dx_i = d\overline{f}_i, \quad i = 1, \ldots, n \quad (6.1)$$

are well defined in $\mathbb{R}^N$ as flat $1$-forms of Whitney. Flat forms are forms with $L^\infty$-coefficients such that the distributional exterior differential of the form also has $L^\infty$-coefficients. The forms in (6.1) are closed, because the fundamental relation $d\overline{f}^* = f^*d$ holds true for Lipschitz maps.

According to a theorem of Whitney [45, Chapter IX], flat forms $(\rho_i)$ have a well defined trace on $X$, and on the measurable tangent bundle of $X$, essentially because of the rectifiability.\footnote{There is a technical point about orientation which we ignore here [17, 3.26].} Because $f = \overline{f}|X$ has a Lipschitz inverse, there exists a constant $c > 0$ such that

$$*(\rho_1 \wedge \cdots \wedge \rho_n) \geq c > 0 \quad (6.2)$$

almost everywhere on $X$, where the Hodge star operator $*$ is determined by the chosen orientation on $X$.

Condition (6.2) was turned into a definition in [17]. We say that $X$ admits local Cartan-Whitney presentations if for each point $p \in X$ one can find an $n$-tuple of flat $1$-forms $\rho = (\rho_1, \ldots, \rho_n)$ defined in an $\mathbb{R}^N$-neighborhood of $p$ such that condition (6.2) is satisfied on $X$ near the point $p$.

**Theorem 1** [17] Let $X \subset \mathbb{R}^N$ be a metrically $n$-dimensional, linearly locally contractible $n$-manifold admitting local Cartan-Whitney presentations. Then $X$ is locally bi-Lipschitz equivalent to $\mathbb{R}^n$ outside a closed set of measure zero and of topological dimension at most $n - 2$.

To prove Theorem 1 fix a point $p \in X$, and let $\rho = (\rho_1, \ldots, \rho_n)$ be a Cartan-Whitney presentation near $p$. The requirement that $\rho$ be flat together with inequality (6.2) can be seen as a quasiregularity condition for forms.\footnote{In fact, (6.2) resembles a stronger, Lipschitz version of (2.1) studied in [26], [40], [15].} We define a mapping

$$f(x) = \int_{[p,x]} \rho \quad (6.3)$$

for $x$ sufficiently near $p$, where $[p,x]$ is the line segment in $\mathbb{R}^N$ from $p$ to $x$, and claim that Reshetnyak’s program can be run under the stipulated conditions on $X$. In particular, we show that for a sufficiently small neighborhood $U$ of $p$ in $X$, the map $f : U \to \mathbb{R}^n$ given in (6.3) is a branched covering which is locally bi-Lipschitz outside its branch set $B_f$, which furthermore is of measure zero and of topological dimension at most $n - 2$. It is important to note that $\rho$ is not assumed to be closed, so that $df \neq 0$ in general.
In executing Reshetnyak’s proof, we use recent advances of differential analysis on nonsmooth spaces [13], [20], [36], as well as the theory developed simultaneously in [15]. Incidentally, we avoid the use of the Harnack inequality for solutions, and therefore a deeper use of equation (2.2); this small improvement to Reshetnyak’s argument was found earlier in a different context in [12].

Theorem 1 provides bi-Lipschitz coordinates for $X$ only on a dense open set. In general, one cannot have more than that. The double suspension of a homology 3-sphere, as discussed in Section 4, can be mapped to the standard 5-sphere by a finite-to-one, piecewise linear sense-preserving map. By pulling back the standard coframe by such map, we obtain a global Cartan-Whitney presentation on a space that is not locally bi-Lipschitz equivalent to $\mathbb{R}^5$. Similar examples in dimension $n = 3$ were constructed in [14], [15], by using Semmes’s spaces [34], [35]. On the other hand, we have the following result:

**Theorem 2** Let $X \subset \mathbb{R}^N$ be a metrically 2-dimensional, linearly locally contractible 2-manifold admitting local Cartan-Whitney presentations. Then $X$ is locally bi-Lipschitz equivalent to $\mathbb{R}^2$.

Theorem 2 is an observation of M. Bonk and myself. We use Theorem 1 together with the observation that, in dimension $n = 2$, the branch set consists of isolated points, which can be resolved. The resolution follows from the measurable Riemann mapping theorem together with the recent work by Bonk and Kleiner [4]. While Theorem 2 presents a characterization of surfaces in Euclidean space that admit local bi-Lipschitz coordinates, we do not know whether the stipulation about the existence of local Cartan-Whitney presentations is really necessary (compare Problem 1 and the discussion preceding it).

For dimensions $n \geq 3$, it would be interesting to know when there is no branching in the map (6.3). In [17], we ask if this be the case when the flat forms $(\rho_i)$ of the Cartan-Whitney presentation belong to a Sobolev space $H^{1,2}_{loc}$ on $X$. The relevant example here is the map $(r, \theta, z) \mapsto (r, 2\theta, z)$, in the cylindrical coordinates of $\mathbb{R}^n$, which pulls back the standard coframe to a frame that lies in the Sobolev space $H^{2,\epsilon}_{loc}$ for each $\epsilon > 0$. Indeed, it was shown in [11] that in $\mathbb{R}^n$ every (Cartan-Whitney) pullback frame in $H^{1,2}_{loc}$ must come from a locally injective mapping.

7. Other recent progress and open problems

In his 1978 ICM address, Väisälä [44] asked whether the branch set of a $C^1$-smooth quasiregular mapping is empty if $n \geq 3$. It was known that $C^{n/(n-2)}$-smooth quasiregular mappings have no branching when $n \geq 3$. The proof in [Ri, p. 12] of this fact uses quasiregularity in a rather minimal way. In this light, the following recent result may appear surprising:

**Theorem 3** [1] For every $\epsilon > 0$ there exists a degree two $C^{3-\epsilon}$-smooth quasiregular mapping $f : S^3 \to S^3$ with branch set homeomorphic to $S^1$.

We are also able to improve the previous results as follows:
Theorem 4 [1] Given \( n \geq 3 \) and \( K \geq 1 \), there exist \( \epsilon = \epsilon(n, K) > 0 \) and \( \epsilon' = \epsilon'(n, K) > 0 \) such that the branch set of every \( K \)-quasiregular mapping in a domain in \( \mathbb{R}^n \) has Hausdorff dimension at most \( n - \epsilon \), and that every \( C^{m/(n-2) - \epsilon'} \)-smooth \( K \)-quasiregular mapping in a domain in \( \mathbb{R}^n \) is a local homeomorphism.

The second assertion in Theorem 4 follows from the first, by way of Sard-type techniques. The first assertion was known earlier in a local form where \( \epsilon > 0 \) was dependent on the local degree [31]. Our improvement uses [31] together with the work [30] by Rickman and Srebro.

The methods in [1] fall short in showing the sharpness of Theorem 4 in dimensions \( n \geq 4 \) in two technical aspects. First, we would need to construct a quasiconformal homeomorphism of \( \mathbb{R}^n \) to itself that is uniformly expanding on a codimension two affine subspace; moreover, such a map needs to be smooth outside this subspace. In \( \mathbb{R}^3 \), it is easier to construct a mapping with expanding behavior on a line; moreover, every quasiconformal homeomorphism in dimension three can be smoothened (with bounds) outside a given closed set [19].

We finish with some open problems related to branching and quasiregular mappings. The problems are neither new nor due to the author.

Problem 2 What are the possible values for the topological dimension of the branch set of a quasiregular mapping?

By suspending a covering map \( H^3 \to S^3 \), where \( H^3 \) is as in Section 4, and using Edwards's theorem, one finds that there exists a branched covering \( S^5 \to S^5 \) that branches exactly on \( S^1 \subset S^5 \). It is not known whether there exists a quasiregular mapping \( S^5 \to S^5 \) with similar branch set. If no such map existed, we would have an interesting implication to a seemingly unrelated parametrization problem; it would follow that no double suspension of a homology 3-sphere with nontrivial fundamental group admits a quasisymmetric homeomorphism onto the standard 5-sphere, cf. [38], [16, Question 12].

By work of Bonk and Kleiner [4], the bi-Lipschitz parametrization problem in dimension \( n = 2 \) is equivalent to an analytic problem of characterizing, up to a bounded factor, the Jacobian determinants of quasiconformal mappings in \( \mathbb{R}^2 \). An affirmative answer to Problem 1 in Section 5 would give an affirmative answer to the following problem.

Problem 3 (Compare [16, Question 2]) Is every \( A_1 \)-weight in \( \mathbb{R}^2 \) locally comparable to the Jacobian determinant of a quasiconformal mapping?

An \( A_1 \)-weight is a nonnegative locally integrable function whose mean-value over each ball is comparable to its essential infimum over the ball. See [7], [32], [3], [16] for further discussion of this and related problems.

Problem 4 [16, Question 28] Is there a branched covering \( f : S^n \to S^n \), for some \( n \geq 3 \), such that for every pair of homeomorphisms \( \phi, \psi : S^n \to S^n \), the mapping \( \phi \circ f \circ \psi \) fails to be quasiregular?
Branched coverings constructed by using the double suspension are obvious candidates for such mappings. In [15, 9.1], we give an example of a branched covering \( f : S^3 \to S^3 \) such that for every homeomorphism \( \psi : S^3 \to S^3 \), \( f \circ \psi \) fails to be quasiregular. The example is based on a geometric decomposition space arising from Bing’s double [34].

We close this article by commenting on the lack of direct proofs for some fundamental properties of quasiregular mappings related to branching. For example, it is known that for each \( n \geq 3 \) there exists \( K(n) > 1 \) such that every \( K(n) \)-quasiregular mapping is a local homeomorphism [25], [28, p. 232]. All known proofs for this fact are indirect, exploiting the Liouville theorem, and in particular there is no numerical estimate for \( K(n) \). It has been conjectured that the winding mapping \((r, \theta, z) \mapsto (r, 2\theta, z)\) is the extremal here (cf. Section 6). Thus, if one uses the inner dilatation \( K_I(f) \) of a quasiregular mapping, then conjecturally \( K_I(f) < 2 \) implies that \( B_f = \emptyset \) for a quasiregular mapping \( f \) in \( \mathbb{R}^n \) for \( n \geq 3 \) [29, p. 76].

Ostensibly different, but obviously a related issue, arises in search of Bloch’s constant for quasiregular mappings. Namely, by exploiting normal families, Eremenko [10] recently proved that for given \( n \geq 3 \) and \( K \geq 1 \), there exists \( b_0 = b_0(n, K) > 0 \) such that every \( K \)-quasiregular mapping \( f : \mathbb{R}^n \to S^n \) has an inverse branch in some ball in \( S^n \) of radius \( b_0 \). No numerical estimate for \( b_0 \) is known. More generally, despite the deep results on value distribution of quasiregular mappings, uncovered by Rickman over the past quarter century, the affect of branching on value distribution is unknown, cf. [29, p. 96].

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