MODULI SPACES OF MEROMORPHIC CONNECTIONS, QUIVER VARIETIES, AND INTEGRABLE DEFORMATIONS

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Abstract. This is a note in which we first review symmetries of moduli spaces of stable meromorphic connections on trivial vector bundles over the Riemann sphere, and next discuss symmetries of their integrable deformations as an application. In the study of the symmetries, a realization of the moduli spaces as quiver varieties is given and plays an essential role.

Introduction

This note is designed to review symmetries of moduli spaces of meromorphic connections on trivial vector bundles over \( \mathbb{P}^1 \) and give an application of these symmetries for their isomonodromic deformations. In the series of works by Okamoto [30], it was clarified that Painlevé equations have affine Weyl group symmetries. After these pioneering works, many studies about symmetries of Painlevé type equations are successfully developed in connection with the algebraic geometry, representation theory of affine Lie algebras and so on (see Noumi and Yamada [29], Sakai [32], Sasano [34] and their references for instance). On the other hand, the recent work of Kawakami, Nakamura and Sakai [21] suggests that many known Painlevé type equations are uniformly obtained from isomonodromic deformations of linear ordinary differential equations. In this note, inspired by their work, we shall introduce a study of symmetries of isomonodromic deformations from those of moduli spaces of meromorphic connections.

In the first section we shall explain some relationship among moduli spaces \( M(B) \) (see \( \S1.4 \) for the definition) of stable meromorphic connections on trivial bundles with at most unramified irregular singularities over \( \mathbb{P}^1 \), quiver varieties \( M^{\text{reg}}_\lambda(Q, \alpha) \) (see \( \S1.3 \)) and integrable deformations. One of the most remarkable facts which will be introduced there is the following.

**Theorem 0.1.** For any moduli space \( M(B) \), there exists a quiver \( Q \), a dimension vector \( \alpha \) and a complex parameter \( \lambda \) such that we have an open embedding

\[
M(B) \hookrightarrow M^{\text{reg}}_\lambda(Q, \alpha).
\]

Here the embedding is an isomorphism if the number of irregular singular points is less than or equal to 1. This embedding is found by Crawley-Boevey [6] when all singular points are regular singular, by Boalch [3] and Yamakawa with the author [15] when only one singular point is irregular singular and the others are regular singular, and by the author [13] for general cases.
From the geometry of quiver varieties, the above embedding leads us to a natural proof of the fact:

**Theorem 0.2.** The moduli space $\mathcal{M}(\mathcal{B})$ has a structure as a connected complex symplectic manifold.

Compare with the results by Boalch [2] and Inaba-Saito [16], etc. This fact should be essential for the theory of isomonodromic deformations because it is believed that isomonodromic deformations should have descriptions as Hamiltonian systems over the moduli spaces.

Also we can determine the condition under which the moduli spaces are nonempty.

**Theorem 0.3.** The explicit necessary and sufficient condition for $\mathcal{M}(\mathcal{B}) \neq \emptyset$ is determined by means of the root system of the quiver $Q$.

The above fact is also called a *additive Deligne-Simpson problem*, see Kostov [25], Crawley-Boevey [6], Boalch [3], Yamakawa with the author [15] and the author [13], etc.

Further, there is another advantage of this realization of moduli spaces as quiver varieties, which shall be explained in the second section. Namely, quiver varieties naturally have Weyl group actions generated by the reflection functors. Thus we can translate these Weyl group symmetries of quiver varieties to those of the moduli spaces. On the other hand, the moduli spaces themselves have symmetries generated by middle convolutions. Thus we shall compare these symmetries of middle convolutions and of quivers. Afterward, the orbits under these Weyl groups are in our interest. It will be shown that a kind of a finiteness of the fundamental domains under these actions:

**Theorem 0.4.** If we fix the dimension of moduli spaces, then there exist only finite numbers of fundamental spectral types (see Definition 1.30) of moduli spaces.

This was proved by Oshima [31] when all singularities are regular singular and by Oshima and the author [14] for general cases. We also give the classifications of fundamental spectral types of dimension 2 and 4 for example.

Finally we shall translate the symmetries of moduli spaces into those of isomonodromic deformations. Namely, Haraoka and Filipuk [10] showed that deformation equations of isomonodromic deformations of linear Fuchsian differential equations are invariant under middle convolutions. This theory is generalized for isomonodromic deformations of irregular singular differential equations by Boalch [3] and Yamakawa [11]. As a corollary of these facts, it will be shown that there exists fundamental domains of isomonodromic deformations under the action of middle convolutions. Moreover we shall see the finiteness of spectral types of fundamental isomonodromic deformations of a fixed dimension and the classifications of them for lower dimensional cases. As an application of these, it will be shown that the spectral types appeared in the paper of Kawakami-Nakamura-Sakai [21] in which they considered confluences of 4-dimensional isomonodromic deformations is the complete list of fundamental spectral types of dimensional 4, and moreover we classify their Weyl group symmetries explicitly.
We should mention related previous works. In [39], Yamakawa gave another realization of moduli spaces as generalized quiver varieties associated with non-symmetric root systems and clarified the symmetries of moduli spaces from his quivers. In [4], Boalch gave a formulation of isomonodromic deformations as Hamiltonian systems on quiver varieties and studied the symmetries of them though he only considered meromorphic connections with only one irregular singular point and an arbitrary number of regular singular points. In this note we do not deal with Hamiltonian equations of isomonodromic deformations directly. However our study of the symmetries of isomonodromic deformations of meromorphic connections with general unramified irregular singularities should induce the symmetries of their Hamiltonian equations. Also our study after these works will shed a light over the importance of quiver varieties for the study of isomonodromic deformations.

1. Moduli spaces of meromorphic connections, quiver varieties and integrable deformations

For a commutative ring $R$, $M(n, R)$ denotes the set of $n \times n$ matrices with coefficients in $R$ and $\text{GL}(n, R) \subset M(n, R)$ consists of invertible elements. The sheaves of holomorphic functions and meromorphic functions on a complex manifold $X$ are written by $\mathcal{O}_X$ and $\mathcal{M}_X$ respectively. In particular when $X = \mathbb{P}^1$, we write $\mathcal{O} = \mathcal{O}_{\mathbb{P}^1}$ and $\mathcal{M} = \mathcal{M}_{\mathbb{P}^1}$ for short. Let us denote the ring of convergent (resp. formal) power series of $z$ by $\mathbb{C}\{z\}$ (resp. $\mathbb{C}\{\!\{z\}\!}$). Their total quotient fields are written by $\mathbb{C}\{\!\{z\}\!\}$ and $\mathbb{C}\{\!\{z\}\!\}$ respectively.

In this section we first give a quick review of meromorphic connections on vector bundles over $\mathbb{P}^1$ and their relation with systems of first order linear ordinary differential equations defined on $\mathbb{P}^1$. Next we introduce moduli spaces of stable meromorphic connections on trivial vector bundles over $\mathbb{P}^1$ and give their realizations as quiver varieties. Finally we consider integrable deformations of connections in these moduli spaces.

1.1. Gauge equivalences of differential equations. We recall gauge transformations of systems of first order linear ordinary differential equations defined locally on $\mathbb{P}^1$ and moreover recall Hukuhara-Turrittin-Levelt normal forms of local differential equations under formal gauge transformations.

Let $U$ be an open subset of $\mathbb{P}^1$ and $z$ a local coordinate on $U$.

**Definition 1.1** (gauge transformation). For a linear differential equation

$$\frac{d}{dz} Y = AY$$

with $A \in M(n, \mathcal{M}(U))$ and $X \in \text{GL}(n, \mathcal{M}(U))$, we define a new differential equation $\frac{d}{dz} \tilde{Y} = B\tilde{Y}$ by

$$B := XAX^{-1} + \left(\frac{d}{dz} X\right) X^{-1}.$$  

We call $B$ the **meromorphic gauge transformation** of $A$ by $X$ and write $B =: X[A]$. In particular if $X \in \text{GL}(n, \mathcal{O}(U))$, we say the **holomorphic gauge transformation**.
Here we note that if a vector $Y$ is a solution of $\frac{d}{dz}Y = AY$ then $\tilde{Y} = XY$ is a solution of $\frac{d}{dz}\tilde{Y} = BY$ for $B = X[A]$.

Let us take $a \in U$ and choose a local coordinate $z$ which is zero at $a$. Then the stalks $\mathcal{O}_a$ and $\mathcal{M}_a$ at $a$ can be identified with $\mathbb{C}\{z\}$ and $\mathbb{C}\{z\}$}. We can similarly define holomorphic and meromorphic gauge transformations of a local differential equation $\frac{d}{dz}Y = AY$ with $A \in M(n, \mathcal{M}_a)$. In this case we can moreover define formal gauge transformations, namely we say $X[A]$ is the **formal holomorphic gauge transformation of $A$ by $X$** if $X \in \text{GL}(n, \mathbb{C}(z))$ and **formal meromorphic gauge transformation if $X \in \text{GL}(n, \mathbb{C}(z))$**.

For a local differential equation $\frac{d}{dz}Y = AY$ with $A \in M(n, \mathbb{C}(z))$, it is known that there exists a normal form under the formal meromorphic gauge transformations as follows.

**Definition 1.2** (Hukuhara-Turrittin-Levelt normal form). By *Hukuhara-Turrittin-Levelt normal form or HTL normal form* for short, we mean an element in $M(n, \mathbb{C}(z))$ of the form

$$\text{diag} \left( q_1(z^{-1})I_{n_1} + R_1z^{-1}, \ldots, q_m(z^{-1})I_{n_m} + R_mz^{-1} \right)$$

where $q_i(s) \in \mathbb{C}s\{s\}$ satisfying $q_i \neq q_j$ if $i \neq j$, and $R_i \in M(n_i, \mathbb{C})$.

The following is a fundamental fact of the local formal theory of differential equations with irregular singularities.

**Theorem 1.3** (Hukuhara-Turrittin-Levelt, see [36] for instance). For any $A \in M(n, \mathbb{C}(z))$, there exists a field extension $\mathbb{C}(t) \supset \mathbb{C}(z)$ with $t' = z$, $r \in \mathbb{Z}_{>0}$ and $X \in \text{GL}(n, \mathbb{C}(t))$ such that $X[A]$ is an HTL normal form in $M(n, \mathbb{C}(t))$. We call this $X[A]$ the normal form of $A$.

In the above theorem, we may assume the field extension is minimal, i.e.,

$$r = \min \left\{ s \mid \text{normal form } X[A] \in M(n, \mathbb{C}(z^s)) \right\}.$$

Then the normal form

$$\text{diag} \left( q_1(t^{-1})I_{n_1} + R_1t^{-1}, \ldots, q_m(t^{-1})I_{n_m} + R_m t^{-1} \right)$$

of $A$ is unique up to permutations of $\{q_i(t)I_{n_i} + R_i t^{-1}\}_{i=1,\ldots,m}$ and replacing $R_i$ for $R_i + \alpha I_{n_i}$ with $\alpha \in \frac{1}{r} \mathbb{Z}$.

### 1.2. Meromorphic connections

Let us recall the notion of meromorphic connections and see their relationship with differential equations. For $f = \sum_{i>\infty}^\infty a_i z^i \in \mathbb{C}(z)$, the order is

$$\text{ord}(f) := \min \{ i \mid a_i \neq 0 \}.$$ 

If $f = 0$, we formally put $\text{ord}(f) = \infty$. For a meromorphic function $f$ locally defined near $a \in \mathbb{P}^1$, we denote the germ of $f$ at $a$ by $f_a$. We may see $f_a \in \mathbb{C}\{z_a\} \subset \mathbb{C}(z_a)$ by setting $z_a = z - a$ if $a \in \mathbb{C}$ and $z_a = 1/z$ if $a = \infty$ where we take $z$ as the standard coordinate of $\mathbb{C}$. Then define

$$\text{ord}_a(f) := \text{ord}(f_a).$$

For a meromorphic 1-form $\omega$ defined on $\mathbb{P}^1$, the order $\text{ord}_a(\omega)$ can be defined as follows. Set $U_1 = \mathbb{P}^1 \setminus \{\infty\}$ and $U_2 = \mathbb{P}^1 \setminus \{0\}$. Let $z_i$ be coordinates of $U_i$. 

\(i = 1, 2\), such that \(z_1(0) = z_2(\infty) = 0\) and \(z_2 = 1/z_1\) in \(U_1 \cap U_2\). Then there exist meromorphic functions \(f_i\) on \(U_i\) such that
\[
\omega = f_i \, dz_i
\]
on \(U_i\) for \(i = 1, 2\). Here we note that
\[
z_1^2 f_1(z_1) = -f_2(1/z_1)
\]for \(z_1 \in U_1 \cap U_2 = \mathbb{C}\setminus\{0\}\). Then define
\[
\text{ord}_a(\omega) := \text{ord}_a(f_i)
\]for \(a \in U_i, i = 1, 2\).

Let us fix a collection of points \(a_0, \ldots, a_p \in \mathbb{P}^1\) and set \(S := k_0 a_0 + \cdots + k_p a_p\) as an effective divisor with \(k_0, \ldots, k_p > 0\). For \(a \in \mathbb{P}^1\) let \(S(a)\) be the coefficient of \(a\) in \(S\), i.e.,
\[
S(a) := \begin{cases} k_i & \text{if } a = a_i \text{ for } i = 0, \ldots, p, \\ 0 & \text{otherwise}. \end{cases}
\]

For an open set \(U \subset \mathbb{P}^1\) we define \(\Omega_S(U)\) to be the set of all meromorphic 1-forms \(\omega\) on \(U\) satisfying \(\text{ord}_a(\omega) \geq -S(a)\) for any \(a \in U\). This correspondence defines the sheaf \(\Omega_S\) by the natural restriction mappings.

Let \(\mathcal{E}\) be a locally free sheaf of rank \(n\) on \(\mathbb{P}^1\), namely a sheaf of \(\mathcal{O}\)-modules satisfying that for any \(a \in \mathbb{P}^1\) there exists an open neighbourhood \(V \subset \mathbb{P}^1\) such that \(\mathcal{E}|_V \cong \mathcal{O}^n|_V\). We may sometimes regard \(\mathcal{E}\) as a holomorphic vector bundle over \(\mathbb{P}^1\).

**Definition 1.4** (meromorphic connection). A **meromorphic connection** is a pair \((\mathcal{E}, \nabla)\) of a locally free sheaf \(\mathcal{E}\) and a morphism \(\nabla: \mathcal{E} \to \mathcal{E} \otimes \Omega_S\) of sheaves of \(\mathbb{C}\)-vector spaces satisfying
\[
\nabla(fs) = df \otimes s + f \otimes \nabla(s)
\]
for all \(f \in \mathcal{O}(U), s \in \mathcal{E}(U)\) and open subsets \(U \subset \mathbb{P}^1\).

Let \(U \subset \mathbb{P}^1\) be an open subset which gives a local trivialization of \(\mathcal{E}\) and \(z\) a local coordinate of \(U\). Then if we fix an identification \(\mathcal{E}|_U \cong \mathcal{O}^n|_U\), we can write \(\nabla = d - A' dz\) by \(A \in M(n, \mathcal{M}(U))\) on \(U\). Note that if we write \(\nabla = d - A' dz\) by another identification \(\mathcal{E}|_U \cong \mathcal{O}^n|_U\), then \(A'\) can be obtained by a holomorphic gauge transformation of \(A\), namely there exists \(X \in \text{GL}(n, \mathcal{O}(U))\) such that
\[
A' = X[A].
\]
Thus we may say that \((\mathcal{E}, \nabla)\) defines a holomorphic gauge equivalent class of a local differential equation
\[
\frac{d}{dz} Y = AY
\]
on \(U \subset \mathbb{P}^1\).

In particular, suppose that \(\mathcal{E}\) is trivial, i.e., \(\mathcal{E} \cong \mathcal{O}^n\) and set \(U_1 = \mathbb{P}^1 \setminus \{\infty\}\) and \(U_2 = \mathbb{P}^1 \setminus \{0\}\) as before. Then if we fix a trivialization \(\mathcal{E} \cong \mathcal{O}^n\), we have \(\nabla = d - A(z_1) dz_1\) on \(U_1\) with \(A(z_1) = (\alpha_{i,j}(z_1))_{i,j=1,\ldots,n} \in M(n, \mathbb{C}(z))\)
satisfying $ \text{ord}_a(\alpha_{i,j}) \geq -S(a)$ for all $a \in U_1$. Similarly on $U_2$ we have
$\nabla = d - B(z_2)d_{z_2}$. Since $\mathcal{E}$ is trivial,
$$A(z_1)d_{z_1} = B(z_2)d_{z_2} \text{ on } U_1 \cap U_2.$$
Namely,
$$B(z_2) = -\frac{A(1/z_2)}{z_2^2}.$$ 
This is nothing but the coordinate exchange $t = \frac{1}{z}$ for a differential equation
$$\frac{d}{dz} Y = A(z)Y \mapsto -t \frac{d}{dt} Y = A(1/t)Y.$$ 
Thus a meromorphic connection $(\mathcal{E}, \nabla)$ with a trivial bundle $\mathcal{E}$ on $\mathbb{P}^1$corresponds to a meromorphic differential equation $d/dz Y = AY$ with $A = (\alpha_{i,j})_{i,j=1,...,n} \in M(n, \mathbb{C}(z))$ satisfying $\text{ord}_a(\alpha_{i,j}) \geq -S(a)$ for all $a \in \mathbb{P}^1$, and vice versa. This correspondence is unique up to the choice of $\mathcal{E} \cong \mathcal{O}^n$, i.e., $\text{GL}(n, \mathbb{C})$-action.

1.3. Quiver varieties. In this subsection we shall introduce quiver varieties which are defined by symplectic reductions of representation spaces of quivers.

1.3.1. Complex symplectic manifold and symplectic reduction. Before seeing the definition of quiver varieties, let us recall complex symplectic manifolds.

**Definition 1.5** (complex symplectic manifold). Let $M$ be an even dimensional complex manifold and $\omega$ a closed nondegenerate holomorphic 2-form on $M$. Then $\omega$ is called a symplectic form on $M$ and the pair $(M, \omega)$ is called a complex symplectic manifold.

**Example 1.6** (cotangent bundle of $\mathbb{C}^n$). Let us see that the holomorphic cotangent bundle $T^*\mathbb{C}^n$ of $\mathbb{C}^n$ is a complex symplectic manifold. Since $T^*\mathbb{C}^n \cong \mathbb{C}^n \times (\mathbb{C}^n)^*$, the coordinate system $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$ and the dual coordinate $(\xi_1, \ldots, \xi_n)$ of $(\mathbb{C}^n)^*$ defines a coordinate system of $T^*\mathbb{C}^n$. Then the 2-form of $T^*\mathbb{C}^n$ defined by
$$\omega := \sum_{i=1}^{n} dz_i \wedge d\xi_i$$
is closed since $\omega = -d\alpha$,
$$\alpha := \sum_{i=1}^{n} \xi_i dz_i.$$ 
This $\omega$ is called the canonical symplectic form of $T^*\mathbb{C}^n$.

Let us define a moment map on $(M, \omega)$. Suppose that $M$ has an action of a complex Lie group $G$. Let $\mathfrak{g}$ be the Lie algebra of $G$ and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. Let us denote the exponential map by
$$\exp: \mathfrak{g} \rightarrow G.$$ 
Then any $\xi \in \mathfrak{g}$ defines the holomorphic vector field $\xi_M \in TM$ by the action of the one parameter subgroup
$$m \mapsto \exp(-t\xi) \cdot m \quad m \in M, \quad t \in \mathbb{R}.$$ 
Then a moment map is defined as follows.
Definition 1.7 (moment map). Let \((M, \omega)\) be a complex symplectic manifold with a \(G\)-action as above. Then a \(G\)-equivariant map
\[
\mu: M \rightarrow g^*
\]
is called moment map if it satisfies
\[
d(\mu, \xi)(v) = \omega(v, \xi_M)
\]
for all \(v \in TM\). Here \(\langle \ , \ \rangle: g \times g^* \rightarrow \mathbb{C}\) is the canonical pairing.

If we assume that the action of \(G\) is free and proper, namely stabilizers in \(G\) of \(m\) are trivial for all \(m \in M\) and the action map
\[
G \times M \rightarrow M \times M
\]
\[(g, m) \mapsto (g \cdot m, m)\]
is a proper map. In this case it is known that the homogeneous space \(M/G\) becomes a manifold. Let us take \(\xi \in (g^*)^G\), \(G\)-invariant under the coadjoint action, and consider \(\mu^{-1}(\xi)\). Then \(\mu^{-1}(\xi)/G\) becomes a complex manifold and moreover has a symplectic form \(\omega\) defined as follows. For \(p \in \mu^{-1}(\xi)/G\) and \(u, v \in T_p(\mu^{-1}(\xi)/G)\), take \(p \in \pi^{-1}(p)\), the inverse image of the projection \(\pi: \mu^{-1}(\xi) \rightarrow \mu^{-1}(\xi)/G\), and \(u, v \in T_p\mu^{-1}(\xi)\) so that \((\pi_\ast)_p(u) = u\) and \((\pi_\ast)_p(v) = v\). Here \(\pi_\ast: T\mu^{-1}(\xi) \rightarrow T(\mu^{-1}(\xi)/G)\) is the differentiation of \(\pi\). Then we define \(\omega_p(u, v) := \omega_p(u, v)\). It can be shown that \(\omega\) is well-defined because \(\mu\) is a moment map.

Definition 1.8 (symplectic reduction, Marsden-Weinstein reduction). The symplectic manifold \((\mu^{-1}(\xi)/G, \omega)\) is called a symplectic reduction or Marsden-Weinstein reduction of \((M, \omega)\) under the action of \(G\).

Sometimes we drop the assumption that the action of \(G\) is free and proper, and call \(\mu^{-1}(\xi)/G\) a symplectic reduction too though it may have singularities.

1.3.2. Representations of quiver and quiver variety. Now let us recall representations of quivers

Definition 1.9 (quiver). A quiver \(Q = (Q_0, Q_1, s, t)\) is the quadruple consisting of \(Q_0\), the set of vertices, and \(Q_1\), the set of arrows connecting vertices in \(Q_0\), and two maps \(s, t: Q_1 \rightarrow Q_0\), which associate to each arrow \(\rho \in Q_1\) its source \(s(\rho) \in Q_0\) and its target \(t(\rho) \in Q_0\) respectively.

Definition 1.10 (representation of quiver). Let \(Q\) be a finite quiver, i.e., \(Q_0\) and \(Q_1\) are finite sets. A representation \(M\) of \(Q\) is defined by the following data:

1. To each vertex \(a\) in \(Q_0\), a finite dimensional \(\mathbb{C}\)-vector space \(M_a\) is attached.
2. To each arrow \(\rho: a \rightarrow b\) in \(Q_1\), a \(\mathbb{C}\)-linear map \(x_\rho: M_a \rightarrow M_b\) is attached.

We denote the representation by \(M = (M_a, x_\rho)_{a \in Q_0, \rho \in Q_1}\). The collection of integers defined by \(\text{dim} M = (\text{dim}_\mathbb{C} M_a)_{a \in Q_0}\) is called the dimension vector of \(M\).
For a fixed vector $\alpha \in (\mathbb{Z}_{\geq 0})^{Q_0}$, the representation space is

$$\text{Rep}(Q, V, \alpha) = \bigoplus_{\rho \in Q_1} \text{Hom}_\mathbb{C}(V_{s(\rho)}, V_{t(\rho)}),$$

where $V = (V_a)_{a \in Q_0}$ is a collection of finite dimensional $\mathbb{C}$-vector spaces with $\dim_\mathbb{C} V_a = \alpha_a$. If $V_a = \mathbb{C}^{\alpha_a}$ for all $a \in Q_0$, we simply write

$$\text{Rep}(Q, \alpha) = \bigoplus_{\rho \in Q_1} \text{Hom}_\mathbb{C}(\mathbb{C}^{\alpha_0(\rho)}, \mathbb{C}^{\alpha_{t(\rho)}}).$$

To each $(x_\rho)_{\rho \in Q_1} \in \text{Rep}(Q, V, \alpha)$, the representation $(V_a, x_\rho)_{a \in Q_0, \rho \in Q_1}$ associates. Thus we may identify $(x_\rho)_{\rho \in Q_1}$ with $(V_a, x_\rho)_{a \in Q_0, \rho \in Q_1}$.

The representation space $\text{Rep}(Q, V, \alpha)$ has an action of $\prod_{a \in Q_0} \text{GL}(V_a)$. For $(x_\rho)_{\rho \in Q_1} \in \text{Rep}(Q, V, \alpha)$ and $g = (g_a) \in \prod_{a \in Q_0} \text{GL}(V_a)$, then $g \cdot (x_\rho)_{\rho \in Q_1} \in \text{Rep}(Q, V, \alpha)$ consists of $x'_\rho = g_{t(\rho)} x_\rho g_{s(\rho)}^{-1} \in \text{Hom}_\mathbb{C}(V_{s(\rho)}, V_{t(\rho)}),$.

Let $M = (M_a, x_\rho^M)_{a \in Q_0, \rho \in Q_1}$ and $N = (N_a, x_\rho^N)_{a \in Q_0, \rho \in Q_1}$ be representations of a quiver $Q$. Then $N$ is called the subrepresentation of $M$ if we have the following:

1. There exists a direct sum decomposition $M_a = N_a \oplus N'_a$ for each $a \in Q_0$.
2. For each $\rho : a \to b \in Q_1$, the equality $x_\rho^M |_{N_a} = x_\rho^N$ holds.

In this case we denote $N \subset M$. Moreover if

3. for each $\rho : a \to b \in Q_1$, we have $x_\rho^M |_{N_a} \subset N'_b$,

then we say $M$ has a direct sum decomposition $M = N \oplus N'$ where $N' = (N'_a, x_\rho^M |_{N'_a})_{a \in Q_0, \rho \in Q_1}$. The representation $M$ is said to be irreducible if $M$ has no subrepresentations other than $M$ and $\{0\}$. Here $\{0\}$ is the representation of $Q$ which consists of zero vector spaces and zero linear maps. On the other hand if any direct sum decomposition $M = N \oplus N'$ satisfies either $N = \{0\}$ or $N' = \{0\}$, then $M$ is said to be indecomposable.

Let us recall the double of a quiver $Q$.

**Definition 1.11** (double quiver). Let $Q = (Q_0, Q_1)$ be a finite quiver. Then the double quiver $\overline{Q}$ of $Q$ is the quiver obtained by adjoining the reverse arrow $\rho^* : b \to a$ to each arrow $\rho : a \to b$. Namely $\overline{Q} := (\overline{Q}_0 := Q_0, \overline{Q}_1 := Q_1 \cup Q^*_1)$ where $Q^*_1 := \{\rho^* : t(\rho) \to s(\rho) \mid \rho \in Q_1\}$. Let us note that for each $\rho \in Q_1$ we can identify

$$\text{Hom}_\mathbb{C}(\mathbb{C}^{\alpha_0(\rho)}, \mathbb{C}^{\alpha_{t(\rho)}})^* \cong \text{Hom}_\mathbb{C}(\mathbb{C}^{\alpha_0(\rho^*)}, \mathbb{C}^{\alpha_{t(\rho^*)}})$$

by the trace pairing. Thus the representation space $\text{Rep}(\overline{Q}, \alpha)$ can be identified with the cotangent bundle

$$T^* \text{Rep}(Q, \alpha) \cong \text{Rep}(\overline{Q}, \alpha).$$

In this case the canonical symplectic form is given by

$$\omega(x, y) = \sum_{\rho \in Q_1} (\text{tr}(x_\rho y_{\rho^*}) - \text{tr}(x_{\rho^*} y_\rho)).$$
Thus we can see $\text{Rep}(\mathbb{Q}, \alpha)$ as a complex symplectic manifold with the action of

$$G := \prod_{a \in \mathbb{Q}_0} \text{GL}(\alpha_a, \mathbb{C}).$$

Then the following map is a moment map:

$$\mu_\alpha : \text{Rep}(\mathbb{Q}, \alpha) \to \prod_{a \in \mathbb{Q}_0} M(\alpha_a, \mathbb{C})$$

whose images $(\mu_\alpha(x)_a)_{a \in \mathbb{Q}_0}$ are given by

$$\mu_\alpha(x)_a = \sum_{t(\rho) = a} x_\rho x_{\rho'} - \sum_{s(\rho) = a} x_{\rho'} x_\rho.$$

Now we are ready to define quiver varieties.

**Definition 1.12 (quiver variety).** Let us take a collection of complex numbers $\lambda = (\lambda_a) \in \mathbb{C}^{\mathbb{Q}_0}$. Then a quiver variety is the symplectic reduction

$$\mathfrak{M}_\lambda(Q, \alpha) := \mu^{-1}(\lambda)/G.$$

Note that we can regard $\mathfrak{M}_\lambda(Q, \alpha)$ as the affine quotient $\text{Specm} \mathbb{C}[\mu^{-1}(\lambda)]^G$ by the theory of Kempf-Ness [22] and Kirwan [24]. Here $\mathbb{C}[\mu^{-1}(\lambda)]$ is the coordinate ring of $\mu^{-1}(\lambda)$.

Since this variety might have singularities, we moreover consider the regular part of this variety defined as follows.

**Definition 1.13.** We say that $x \in \text{Rep}(\mathbb{Q}, \alpha)$ is **stable** if

1. the orbit $G \cdot x$ is closed,
2. stabilizer of $x$ in $G/\mathbb{C}^\times$ is finite.

Here we note that the diagonal subgroup $\mathbb{C}^\times \subset G$ acts trivially on $\text{Rep}(\mathbb{Q}, \alpha)$.

It is known that the stability of $x$ assures that the morphism

$$\sigma_x : G \to \text{Rep}(\mathbb{Q}, \alpha)$$

$$g \mapsto g \cdot x$$

is proper.

In our case moreover the stability can be rephrased by the irreducibility of representations.

**Theorem 1.14 (King [23]).** $x \in \text{Rep}(\mathbb{Q}, \alpha)$ is stable if and only if $x$ is an irreducible representation.

Thus let us consider the (possibly empty) subspace

$$\mu^{-1}(\lambda)_{\text{irr}} := \{ x \in \mu^{-1}(\lambda) \mid x \text{ is irreducible} \}.$$

Then the action of $G/\mathbb{C}^\times$ on this space is proper and moreover free (see King [23]). Thus the homogeneous space

$$\mathfrak{M}_{\lambda}^{\text{reg}}(Q, \alpha) := \mu^{-1}(\lambda)_{\text{irr}}/G$$

can be seen as a complex manifold with the symplectic structure, i.e., a complex symplectic manifold. We call this manifold a quiver variety too.
Remark 1.15. The above quiver varieties are special ones of Nakajima quiver varieties which enjoy rich geometric properties and applications for representation theory and theoretical physics and so on (see [27] for instance).

1.3.3. Some geometry of quiver varieties. As we noted before, the complex symplectic manifold \( M_{\lambda}^{\text{reg}}(Q, \alpha) \) is possibly empty. Thus next we see a necessary and sufficient condition for the non-emptiness of \( M_{\lambda}^{\text{reg}}(Q, \alpha) \) obtained by Crawley-Boevey in [5].

In order to explain the condition, recall the root system of a quiver \( Q \) (cf. [17]). Let \( Q \) be a finite quiver. From the Euler form \( \langle \alpha, \beta \rangle := \sum_{a \in Q_0} \alpha_a \beta_a - \sum_{\rho \in Q_1} \alpha_{s(\rho)} \beta_{t(\rho)} \), a symmetric bilinear form and quadratic form are defined by \( (\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle \), \( q(\alpha) := \frac{1}{2}(\alpha, \alpha) \) and set \( p(\alpha) := 1 - q(\alpha) \). Here \( \alpha, \beta \in \mathbb{Z}^Q_0 \).

For each vertex \( a \in Q_0 \), define \( \epsilon_a \in \mathbb{Z}^Q_0 \) \((a \in Q_0)\) so that \( (\epsilon_a)_a = 1 \), \( (\epsilon_a)_b = 0 \), \((b \in Q_0 \setminus \{a\})\). We call \( \epsilon_a \) a fundamental root if the vertex \( a \) has no edge-loop, i.e., there is no arrow \( \rho \) such that \( s(\rho) = t(\rho) = a \). Denote by \( \Pi \) the set of fundamental roots. For a fundamental root \( \epsilon_a \), define the fundamental reflection \( s_a \) by \( s_a(\alpha) := \alpha - (\alpha, \epsilon_a) \epsilon_a \) for \( \alpha \in \mathbb{Z}^Q_0 \).

The group \( W \subset \text{Aut} \mathbb{Z}^Q_0 \) generated by all fundamental reflections is called the Weyl group of the quiver \( Q \). Note that the bilinear form \((\ , \)\) is \( W \)-invariant. Similarly we can define the reflection \( r_a : \mathbb{C}^{Q_0} \to \mathbb{C}^{Q_0} \) by \( r_a(\lambda)_b := \lambda_b - (\epsilon_a, \epsilon_b) \lambda_a \) for \( \lambda \in \mathbb{C}^{Q_0} \) and \( a, b \in Q_0 \). Define the set of real roots by \( \Delta^{\text{re}} := \bigcup_{w \in W} w(\Pi) \).

For an element \( \alpha = (\alpha_a)_{a \in Q_0} \in \mathbb{Z}^Q_0 \) the support of \( \alpha \) is the set of \( \epsilon_a \) such that \( \alpha_a \neq 0 \), and denoted by \( \text{supp} \left( \alpha \right) \). We say the support of \( \alpha \) is connected if the subquiver consisting of the set of vertices \( a \) satisfying \( \epsilon_a \in \text{supp} \left( \alpha \right) \) and all arrows joining these vertices, is connected. Define the fundamental set \( F \subset \mathbb{Z}^Q_0 \) by \( F := \left\{ \alpha \in (\mathbb{Z}_{\geq 0})^{Q_0} \setminus \{0\} \mid (\alpha, \epsilon) \leq 0 \text{ for all } \epsilon \in \Pi, \text{ support of } \alpha \text{ is connected} \right\} \).

Then define the set of imaginary roots by \( \Delta^{\text{im}} := \bigcup_{w \in W} w(F \cup -F) \).

Then the root system is \( \Delta := \Delta^{\text{re}} \cup \Delta^{\text{im}} \).
An element $\Delta^+ := \alpha \in \Delta \cap (\mathbb{Z}_{\geq 0})^Q$ is called a positive root.

Now we are ready to see Crawley-Boevey’s theorem. For a fixed $\lambda = (\lambda_a) \in \mathbb{C}^Q_0$, the set $\Sigma_\lambda$ consists of the positive roots satisfying

1. $\lambda \cdot \alpha := \sum_{a \in Q_0} \lambda_a \alpha_a = 0$,

2. if there exists a decomposition $\alpha = \beta_1 + \beta_2 + \cdots$, with $\beta_i \in \Delta^+$ and $\lambda \cdot \beta_i = 0$, then $p(\alpha) > p(\beta_1) + p(\beta_2) + \cdots$.

**Theorem 1.16** (Crawley-Boevey. Theorem 1.2 in [5]). Let $Q$ be a finite quiver and $\overline{Q}$ the double of $Q$. Let us fix a dimension vector $\alpha \in (\mathbb{Z}_{\geq 0})^Q_0$ and $\lambda \in \mathbb{C}^Q_0$. Then $\mu^{-1}(\lambda)^{irr} \subset \text{Rep}(\overline{Q}, \alpha)$ is nonempty if and only if $\alpha \in \Sigma_\lambda$. Furthermore, in this case $\mu^{-1}(\lambda)$ is an irreducible algebraic variety and $\mu^{-1}(\lambda)^{irr}$ is dense in $\mu^{-1}(\lambda)$.

Moreover Crawley-Boevey showed the following geometric properties of quiver varieties.

**Theorem 1.17** (Crawley-Boevey Corollary 1.4 in [5]). If $\alpha \in \Sigma_\lambda$ then the quiver variety $\mathfrak{M}_\lambda(Q, \alpha)$ is a reduced and irreducible variety of dimension $2p(\alpha)$.

Combining these results, we have the following non-emptiness condition of regular parts of quiver varieties.

**Corollary 1.18** (Crawley-Boevey [5]). The quiver variety $\mathfrak{M}_\lambda^{reg}(Q, \alpha)$ is non-empty if and only if $\alpha \in \Sigma_\lambda$. Furthermore in this case, it is a connected complex symplectic manifold of dimension $2p(\alpha)$.

### 1.4. Moduli spaces of stable meromorphic connections on trivial bundles

Let us define moduli spaces of meromorphic connections on trivial bundles following Boalch’s paper [2] (see also [15]).

Let

$$B = \text{diag}(q_1(z^{-1})I_{n_1} + R_1 z^{-1}, \cdots q_m(z^{-1})I_{n_m} + R_m z^{-1}) \in \text{GL}(n, \mathbb{C}(z))$$

be an HTL normal form. The equivalent class of $B$ under formal holomorphic gauge transformations is

$$\mathcal{O}_B := \{ X[B] \mid X \in \text{GL}(n, \mathbb{C}[z]) \}.$$

Let us consider another equivalent class of $B$ called the truncated orbit of $B$. We write $\tilde{B}$ for the image of $B$ by the projection

$$\iota: M(n, \mathbb{C}(z)) \longrightarrow M(n, \mathbb{C}(z))/\mathbb{C}[z]).$$

And define the truncated orbit of $B$ by the adjoint action of $\text{GL}(n, \mathbb{C}[z])$ on $M(n, \mathbb{C}(z))/\mathbb{C}[z])$:

$$\mathcal{O}_B^{ru} := \text{Ad}(\text{GL}(n, \mathbb{C}[z]))(\tilde{B}).$$

Let us note that

$$\iota(X[B]) = \iota(XBX^{-1}) + \iota\left( \frac{d}{dz} X \cdot X^{-1} \right)$$

$$= \iota(XBX^{-1})$$

$$= X\tilde{B}X^{-1}.$$
Thus $\iota$ gives the well-defined map

$$\iota : O_B \longrightarrow O_B^{\text{tr}}.$$  

Under a good situation, we can moreover show the following.

**Proposition 1.19.** If eigenvalues of $R_i$ never differ by any integer for each $i = 1, \ldots, m$, then

$$\iota^{-1}(O_B^{\text{tr}}) = O_B$$

**Proof.** The Frobenious method (see [36] for instance) shows that if for $A \in M(n, \mathbb{C}[[z]])$ there exists $X \in \text{GL}(n, \mathbb{C}[z])$ such that $\iota(XAX^{-1}) = \tilde{B}$, then there exists $X' \in \text{GL}(n, \mathbb{C}[z])$ such that $X'[A] = B$. □

For a meromorphic connection $(\mathcal{E}, \nabla)$ on $\mathbb{P}^1$, we write $\nabla_a \in O_B$ (resp. $O_B^{\text{tr}}$) for $a \in \mathbb{P}^1$ if a local trivialization $\nabla = d - A \, dz$ near $a$ satisfies $A \in O_B$ (resp. $O_B^{\text{tr}}$). It is independent of choices of trivializations. Let $S = k_0 a_0 + \ldots + k_p a_p$ be an effective divisor on $\mathbb{P}^1$ as before. Define a set of meromorphic connections on $\mathbb{P}^1$

$$\text{Triv}^{(n)}_S := \left\{ (\mathcal{E}, \nabla) \bigg| \begin{array}{l} \mathcal{E} : \text{trivial of rank } n, \\ \nabla : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_S \end{array} \right\}.$$  

We say $(\mathcal{E}, \nabla) \in \text{Triv}^{(n)}_S$ is stable if there exists no nontrivial proper subspace $\mathcal{W} \subset \mathbb{C}^n$ such that the subbundle $\mathcal{W} := W \otimes O \subset \mathbb{C}^n \otimes O = \mathcal{E}$ is closed under $\nabla$, i.e.,

$$\nabla(\mathcal{W}) \subset \mathcal{W} \otimes \Omega_S.$$

Let $B = (B_0, \ldots, B_p) \in M(n, \mathbb{C}[[z]])^{p+1}$ be a collection of HTL normal forms satisfying $\text{ord}(B_i) = k_i$ for all $i = 0, \ldots, p$. Then the moduli space of stable meromorphic connections on trivial bundles is

$$\mathcal{M}(B) := \left\{ (\mathcal{E}, \nabla) \in \text{Triv}^{(n)}_S \bigg| \begin{array}{l} (\mathcal{E}, \nabla) : \text{stable,} \\ \nabla_a \in O_B^{\text{tr}} \text{ for all } i = 0, \ldots, p \end{array} \right\} / \text{GL}(n, \mathbb{C}).$$

Here $\text{GL}(n, \mathbb{C}) = \text{GL}(n, O(\mathbb{P}^1))$ acts on $\text{Triv}^{(n)}_S$ as holomorphic gauge transformations.

A Möbius transformation allows us to set $a_0 = \infty \in \mathbb{P}^1$. Then by a trivialization $\mathcal{E} \cong O^n$ we can identify $(\mathcal{E}, \nabla) \in \text{Triv}^{(n)}_S$ with a meromorphic differential equation defined on $\mathbb{P}^1$,

$$\frac{d}{dz} Y = \left( \sum_{i=1}^p \sum_{\nu=1}^{k_i} A^{(i)}_{\nu} \frac{1}{(z-a_i)^\nu} + \sum_{2 \leq \nu \leq k_0} A^{(0)}_{\nu} z^{\nu-2} \right) Y$$

up to $\text{GL}(n, \mathbb{C})$-action, i.e.,

$$\frac{d}{dz} Y = A(z) Y \mapsto \frac{d}{dz} Y' = gA(z)g^{-1}Y' \quad (g \in \text{GL}(n, \mathbb{C})).$$

The stability of $(\mathcal{E}, \nabla)$ corresponds to the irreducibility of the differential equation, namely, we say the above differential equation is irreducible if there is no proper subspace of $\mathbb{C}^n$ other than $\{0\}$ which is invariant under all $A^{(i)}_{\nu}$, $i = 0, \ldots, p$, $\nu = 1, \ldots, k_i$. Here we set

$$A^{(0)}_{1} := - \sum_{i=1}^p A^{(i)}_{1}.$$
Thus we can regard $\mathfrak{M}(B)$ as a moduli space of meromorphic differential equations on $\mathbb{P}^1$,

$$\mathfrak{M}(B) = \left\{ \frac{d}{dz} Y = \left( \sum_{i=1}^{m} \sum_{\nu=1}^{k_i} \frac{A^{(i)}_{\nu}}{(z-a_i)^{\nu}} + \sum_{2 \leq \nu \leq k_0} A^{(0)}_{\nu} \right) Y : \text{irreducible} \right\} / \text{GL}(n, \mathbb{C}).$$

If we take another effective divisor $S' = k_0 b_0 + \cdots + k_p b_p$ with the same $k_i$ as $S$, then we can identify $\text{Triv}^{(n)}_S$ and $\text{Triv}^{(n)}_{S'}$ as follows. We may assume $b_0 = \infty$ by applying a Möbius transformation if necessary. Then replacing $a_i$ with $b_i$ in

$$d - \left( \sum_{i=1}^{p} \sum_{\nu=1}^{k_i} \frac{A^{(i)}_{\nu}}{(z-a_i)^{\nu}} + \sum_{2 \leq \nu \leq k_0} A^{(0)}_{\nu} \right) dz \in \text{Triv}^{(n)}_S$$

for $i = 1, \ldots, p$, we obtain an element in $\text{Triv}^{(n)}_{S'}$.

Thus we may regard

$$\mathfrak{M}(B) =$$

$$\left\{ A = (A^{(i)}(z))_{0 \leq i \leq p} \in \prod_{i=0}^{p} \mathcal{O}^{\text{tr}u}_{B^{(i)}} \left| \begin{array}{c} A \text{ is irreducible}, \\ \sum_{i=0}^{p} \text{pr}_{\text{res}}(A^{(i)}(z)) = 0 \end{array} \right\} / \text{GL}(n, \mathbb{C})$$

which is free from locations of $a_i$ in $\mathbb{P}^1$. Here

$$\text{pr}_{\text{res}} \left( \sum_{j=1}^{k} A_j z^{-j} \right) := A_1.$$

For $(\mathcal{E}, \nabla) \in \text{Triv}^{(n)}_S$ and a resulting element $A = (\sum_{j=1}^{k_i} A_j^{(i)} z^{-j})_{0 \leq i \leq p} \in \prod_{i=0}^{p} \mathcal{O}^{\text{tr}u}_{B^{(i)}}$ through a trivialization $\mathcal{E} \cong \mathcal{O}^n$, we write

$$(\mathcal{E}, \nabla) \sim A.$$

1.5. **Moduli spaces of connections and quiver varieties.** We shall give a realization of the moduli space $\mathfrak{M}(B)$ as a quiver variety. Let us suppose that $B^{(0)}, \ldots, B^{(p)}$ are written by

$$B^{(i)} = \text{diag} \left( q^{(i)}_1 (z^{-1}) I_{q^{(i)}_1}, R^{(i)}_1 z^{-1}, \ldots, q^{(i)}_{m^{(i)}} (z^{-1}) I_{q^{(i)}_{m^{(i)}}}, R^{(i)}_{m^{(i)}} z^{-1} \right)$$

and choose complex numbers $\xi^{[i,j]}_1, \ldots, \xi^{[i,j]}_{l^{[i,j]}}$ so that

$$\prod_{k=1}^{l^{[i,j]}} (R^{(i)}_j - \xi^{[i,j]}_k) = 0$$

for $i = 0, \ldots, p$ and $j = 1, \ldots, m^{(i)}$. Set

$$k_i := -\max_{j=1, \ldots, m^{(i)}} \{ \text{ord}(q^{(i)}_j (z^{-1})) \}.$$
for each $i = 0, \ldots, p$. Set

$$I_{\text{irr}} := \{i \in \{0, \ldots, p\} \mid m^{(i)} > 1\} \cup \{0\}$$

and

$$I_{\text{reg}} := \{0, \ldots, p\}\setminus I_{\text{irr}}.$$  

If $m^{(i)} = 1$, then all coefficients of $z^{-j}$ for $j > 1$ in $B^{(i)}$ are scalar matrices.  
Thus there exists a gauge transformation $X \in \text{GL}(n, \mathbb{C}(x))$ such that 
$X[B^{(i)}] = \text{pr}_{\text{res}}(B^{(i)})z^{-1}$ and $X[B^{(j)}] = B^{(j)}$ for $j \neq i$.  
Therefore we may assume $k_i = 1$ if $m^{(i)} = 1$.  
Under these gauge transformations, we may say that $I_{\text{irr}}$ consists of $a_i$ with $k_i > 1$, so called irregular singular points and $\infty$, and $I_{\text{reg}}$ consists of $a_i$ with $k_i = 1$, so called regular singular points other than $\infty$.  

Then let us define a quiver $Q$ as follows. Set

$$Q_0^{\text{irr}} := \left\{[i, j] \mid i \in I_{\text{irr}}, j = 1, \ldots, m^{(i)} \right\},$$

$$Q_0^{\text{leg}} := \left\{[i, j, k] \mid i = 0, \ldots, p, j = 1, \ldots, m^{(i)}, k = 1, \ldots, e_{[i,j]} - 1 \right\}.$$  

Then the set of vertices of $Q$ is the disjoint union

$$Q_0 := Q_0^{\text{irr}} \sqcup Q_0^{\text{leg}}.$$  

Also set

$$Q_1^{0 \rightarrow I_{\text{irr}}} := \left\{\rho_{[0, j]}^{[i, j]} : [0, j] \rightarrow [i, j'] \mid j = 1, \ldots, m^{(0)}, i \in I_{\text{irr}}\setminus\{0\}, j' = 1, \ldots, m^{(i)} \right\},$$

$$Q_1^{B^{(i)}} := \left\{\rho^{[k]}_{[i,j],[i,j']} : [i, j] \rightarrow [i, j'] \mid 1 \leq j < j' \leq m^{(i)}, 1 \leq k \leq d_i(j, j') \right\},$$

$$Q_1^{\text{leg}^{(i)}} := \left\{\rho_{[i,j,k]}^{[i,j,k]} : [i, j, k] \rightarrow [i, j, k - 1] \mid j = 1, \ldots, m^{(i)}, k = 2, \ldots, e_{[i,j]} - 1 \right\},$$

$$Q_1^{\text{leg}^{(i)} \rightarrow B^{(i)}} := \left\{\rho_{[i,j,1]}^{[i,j,1]} : [i, j, 1] \rightarrow [i, j] \mid j = 1, \ldots, m^{(i)} \right\},$$

$$Q_1^{\text{leg}^{(i)} \rightarrow 0} := \left\{\rho_{[0, j]}^{[1, 1]} : [0, j] \rightarrow [0, j] \mid i \in I_{\text{reg}}, j = 1, \ldots, m^{(0)} \right\}.$$  

Here $d_i(j, j') := \deg c_{[i]}(q_i^{(j)}(z) - q_i^{(j)}(z)) - 2$.  
Then the set of arrows of $Q$ is the disjoint union

$$Q_1 := Q_1^{0 \rightarrow I_{\text{irr}}} \sqcup \bigsqcup_{i \in I_{\text{irr}}} \left(Q_1^{B^{(i)}} \sqcup Q_1^{\text{leg}^{(i)} \rightarrow B^{(i)}} \sqcup Q_1^{\text{leg}^{(i)}}\right) \sqcup \bigsqcup_{i \in I_{\text{reg}}} \left(Q_1^{\text{leg}^{(i)} \rightarrow 0} \sqcup Q_1^{\text{leg}^{(i)}}\right).$$
Example 1.20. Let us consider the following $B = (B^{(0)}, B^{(1)}, B^{(2)})$.

\[
B^{(0)} = \begin{pmatrix}
    a_4^{(0)} & a_4^{(0)} \\
    a_4^{(0)} & b_4^{(0)}
\end{pmatrix} z^{-4} + \begin{pmatrix}
    a_3^{(0)} & a_3^{(0)} \\
    b_3^{(0)} & c_3^{(0)}
\end{pmatrix} z^{-3} \\
+ \begin{pmatrix}
    a_2^{(0)} & b_2^{(0)} \\
    c_2^{(0)} & d_2^{(0)}
\end{pmatrix} z^{-2} + \begin{pmatrix}
    \xi_1^{[0,1]} & \xi_1^{[0,2]} \\
    \xi_1^{[0,3]} & \xi_1^{[0,4]}
\end{pmatrix} z^{-1},
\]

\[
B^{(1)} = \begin{pmatrix}
    a_2^{(1)} & a_2^{(1)} \\
    a_2^{(1)} & b_2^{(1)}
\end{pmatrix} z^{-2} + \begin{pmatrix}
    \xi_1^{[1,1]} & \xi_1^{[1,1]} \\
    \xi_1^{[1,1]} & \xi_1^{[1,1]}
\end{pmatrix} z^{-1},
\]

\[
B^{(2)} = \begin{pmatrix}
    \xi_1^{[2,1]} & \xi_1^{[2,1]} \\
    \xi_1^{[2,1]} & \xi_1^{[2,1]}
\end{pmatrix} z^{-1}.
\]

Here any distinct two of $\{a_j^{(i)}, b_j^{(i)}, c_j^{(i)}, d_j^{(i)}\}$ stand for distinct complex numbers and $\xi_k^{[i,j]} \neq \xi_{k'}^{[i,j]}$ if $k \neq k'$.

Then we can associate the following quiver to this $B$.

Let $\alpha = (\alpha_a)_{a \in Q_0} \in \mathbb{Z}^{Q_0}$ be the vector,

\[
\alpha_{[i,j]} := n_j^{(i)} \quad \text{and} \quad \alpha_{[i,j,k]} := \text{rank } \prod_{l=1}^k (R_j^{(i)} - \xi^{[i,j]})_k.
\]
Also define \( \lambda = (\lambda_a)_{a \in Q_0} \in \mathbb{C}^{Q_0} \) by

\[
\begin{align*}
\lambda_{[i,j]} &: = -\xi_{[i,j]} \\
\lambda_{[0,j]} &: = -\xi_{[0,j]} - \sum_{i \in \text{reg}} \xi_{[i,1]} \\
\lambda_{[i,j,k]} &: = \xi_{[i,j]} - \xi_{[k+1]}
\end{align*}
\]

for \( i \in I_{\text{irr}} \setminus \{0\}, j = 1, \ldots, m^{(i)} \),

\[
\lambda_{[0,j]} = -\xi_{[0,j]} - \sum_{i \in \text{reg}} \xi_{[i,1]}
\]

for \( j = 1, \ldots, m^{(0)} \),

\[
\lambda_{[i,j,k]} = \xi_{[i,j]} - \xi_{[k+1]}
\]

for \( i = 0, \ldots, p, j = 1, \ldots, m^{(i)} \),

\[
\lambda_{[i,j,k]} = \xi_{[i,j]} - \xi_{[k+1]}
\]

for \( k = 1, \ldots, e_{[i,j]} - 1 \).

Also define a sublattice of \( \mathbb{Z}^{Q_0} \),

\[
\mathcal{L} = \left\{ \beta \in \mathbb{Z}^{Q_0} \middle| \sum_{j=1}^{m^{(0)}} \beta_{[0,j]} = \sum_{j=1}^{m^{(i)}} \beta_{[i,j]} \text{ for all } i \in I_{\text{irr}} \setminus \{0\} \right\}.
\]

Set \( \mathcal{L}^+ = \mathcal{L} \cap (\mathbb{Z}_{\geq 0})^{Q_0} \).

1.5.1. \( \mathcal{M}(B) \) and a quiver variety. Now we shall give an identification of \( \mathcal{M}(B) \) with a subspace of the quiver variety \( \mathcal{M}_\lambda(Q, \alpha) \). Before seeing this, we introduce \( \mathcal{L} \)-irreducible representations in \( \mu^{-1}(\lambda) \) which are defined by a weaker condition than the irreducibility.

**Definition 1.21** (\( \mathcal{L} \)-irreducible). If \( x \in \mu^{-1}(\lambda) \) has no nontrivial proper subrepresentation \( \{0\} \neq y \subset x \) in \( \mu^{-1}(\lambda) \) with \( \text{dim} y \in \mathcal{L} \), then \( x \) is said to be \( \mathcal{L} \)-irreducible.

Then we have the following bijection from \( \mathcal{M}(B) \) onto a subset of the quiver variety \( \mathcal{M}_\lambda(Q, \alpha) \).

**Theorem 1.22.** There exists a bijection

\[
\Phi_B : \mathcal{M}(B) \longrightarrow \mathcal{M}_\lambda(Q, \alpha)^{\text{dif}}
\]

where

\[
\mathcal{M}_\lambda(Q, \alpha)^{\text{dif}} := \left\{ x \in \mu^{-1}(\lambda) \middle| \det \begin{pmatrix} x_{[i,j]} \end{pmatrix}_{1 \leq j \leq m^{(i)} \atop 1 \leq i \leq m^{(0)}} \neq 0, i \in I_{\text{irr}} \setminus \{0\} \right\} / G.
\]

For a meromorphic connection \((\mathcal{E}, \nabla) \in \mathcal{M}(B)\) we sometimes write

\[
(\mathcal{E}, \nabla) \in \mathcal{M}_\lambda(Q, \alpha)^{\text{dif}}
\]

under the above identification.

Let us recall the construction of the above \( \Phi_B \) which is first obtained by Crawley-Boevey in \([9]\) when \( k_0 = \cdots = k_p = 1 \), by Boalch in \([3]\) when \( k_0 = 3 \) and \( k_1 = \cdots = k_p = 1 \), by Yamakawa and the author in \([15]\) when \( k_0 \geq 1 \) and \( k_1 = \cdots = k_p = 1 \) and finally by the author in \([13]\) for general \( k_0, \ldots, k_p \in (\mathbb{Z}_{\geq 0})^{p+1} \).
First we decompose truncated orbits $O_{B_1^{(i)}}^{\text{tr}}$ as follows. Set
\[
G_{k_i}^\circ := \left\{ I_n + \sum_{i=1}^{k_i-1} g_i z^i \in \text{GL}(n, \mathbb{C}[z]/z^{k_i}\mathbb{C}[z]) \right\},
\]
\[
(O_{B_1^{(i)}}^{\text{tr}})^a := \text{Ad}(G_{k_i}^\circ)(B_1^{(i)}) \subset M(n, \mathbb{C}[[z]])/\mathbb{C}[z],
\]
\[
H_i := \left\{ \text{diag}(h_1, \ldots, h_{m(i)}) \big| h_j \in \text{GL}(n_j^{(i)}), j = 1, \ldots, m(i) \right\}.
\]
for $i = 0, \ldots, p$. Then we can decompose $O_{B_1^{(i)}}^{\text{tr}}$ as follows.

**Proposition 1.23** (see Lemma 2.4 in [2] and Proposition 3.1 in [12]). For each $i = 0, \ldots, p$, we have the bijection
\[
\text{GL}(n, \mathbb{C}) \times H_1 \overset{\text{Ad}_{H_1}}{\longrightarrow} (O_{B_1^{(i)}}^{\text{tr}})^a \overset{\cong}{\longrightarrow} gAg^{-1}.
\]
Here $\text{Ad}_{H_1}(O_{B_1^{(i)}}^{\text{tr}})^a := \left\{ h(O_{B_1^{(i)}}^{\text{tr}})^ah^{-1} \big| h \in H_1 \right\}$ and $\text{GL}(n, \mathbb{C}) \times H_1(O_{B_1^{(i)}}^{\text{tr}})^a := \left( \text{GL}(n, \mathbb{C}) \times (O_{B_1^{(i)}}^{\text{tr}})^a \right) / \sim$ by the identification $(g, A) \sim (gh^{-1}, hAh^{-1})$ for $h \in H_1$.

Thus it suffices to investigate the structure of $(O_{B_1^{(i)}}^{\text{tr}})^a$.

Fix $i \in \text{Irr}$ and write $B_1^{(i)} = B_1^{(i)} + \cdots + B_{k_i}^{(i)} z^{-k_i}$. Let $\bigoplus_{i=1}^{m(i)} V_{(s,t)}^{(i)}$ be the decomposition of $\mathbb{C}^n$ as simultaneous eigen-spaces of $(B_{s+1}^{(i)}, B_{s+2}^{(i)}, \ldots, B_{k_i}^{(i)})$ for $s = 1, \ldots, k_i - 1$.

Define surjections $\pi: J_s^{(i)} := \{1, \ldots, m_s^{(i)}\} \rightarrow J_{s+1}^{(i)} := \{1, \ldots, m_{s+1}^{(i)}\}$ so that $V_{(s,t)}^{(i)} \subset V_{(s+1, \pi_s(t))}$. Fix a total ordering $\prec$ of $J_1 = \{1 < 2 < \cdots < m^{(i)}\}$. Then inductively define total orderings on $J_s$, $s = 2, \ldots, k_i - 1$ so that
\[
\text{if } t_1 \prec t_2, \text{ then } \pi_s(t_1) \prec \pi_s(t_2), \quad t_1, t_2 \in J_s.
\]

According to the ordering on each $J_s^{(i)}$, $s = 1, \ldots, k_i - 1$, let us define parabolic subalgebras of $M(n, \mathbb{C})$ as below,
\[
(p_s^{(i)})^+ := \bigoplus_{t_1, t_2 \in J_s^{(i)}, t_1 \geq t_2} \text{Hom}_{\mathbb{C}}(V_{(s,t_1)}^{(i)}, V_{(s,t_2)}^{(i)}),
\]
\[
(p_s^{(i)})^- := \bigoplus_{t_1, t_2 \in J_s^{(i)}, t_1 \leq t_2} \text{Hom}_{\mathbb{C}}(V_{(s,t_1)}^{(i)}, V_{(s,t_2)}^{(i)}),
\]
and similarly nilpotent subalgebras
\[
(u_s^{(i)})^+ := \bigoplus_{t_1, t_2 \in J_s^{(i)}, t_1 \geq t_2} \text{Hom}_{\mathbb{C}}(V_{(s,t_1)}^{(i)}, V_{(s,t_2)}^{(i)}),
\]
\[
(u_s^{(i)})^- := \bigoplus_{t_1, t_2 \in J_s^{(i)}, t_1 \leq t_2} \text{Hom}_{\mathbb{C}}(V_{(s,t_1)}^{(i)}, V_{(s,t_2)}^{(i)}),
\]
for $s = 1, \ldots, k_i - 1$. Then let us define subsets of $G_{k_i}^0$,

$$\mathcal{P}_{k_i}^\pm := \left\{ \sum_{s=0}^{k_i-1} P_s z^s \in G_{k_i}^0 \mid P_s \in (p_{s+1}^{(i)})^\pm, s = 0, \ldots, k_i - 1 \right\},$$

$$\mathcal{U}_{k_i}^\pm := \left\{ \sum_{s=0}^{k_i-1} U_s z^s \in G_{k_i}^0 \mid U_s \in (u_{s+1}^{(i)})^\pm, s = 0, \ldots, k_i - 1 \right\}.$$

Also define

$$\Sigma_{k_i}^\pm := \left\{ \sum_{s=1}^{k_i-1} U_s z^s \mid U_s \in (u_{s+1}^{(i)})^\pm, s = 0, \ldots, k_i - 1 \right\},$$

$$(\Sigma_{k_i}^\pm)^* := \left\{ \sum_{s=1}^{k_i-1} U_s z^{-s-1} \mid U_s \in (u_{s+1}^{(i)})^\pm, s = 0, \ldots, k_i - 1 \right\}.$$

Here we put $(p_{k_i}^{(i)})^\pm := M(n, \mathbb{C})$ and $(u_{k_i}^{(i)})^\pm := \{0\}$.

Then we have the following decomposition of $G_{k_i}^0$.

**Proposition 1.24** (Lemma 3.5 in [15]). Take $i \in I_{irr}$. For any $g \in G_{k_i}^0$, there uniquely exist $u_- \in \mathcal{U}_{k_i}^-$ and $p_+ \in \mathcal{P}_{k_i}^+$ such that $g = u_- p_+$.

For $A \in (\mathcal{O}_{B(i)}^{\irr})^0$, take $g \in G_{k_i}^0$ so that $g^{-1} A g = B^{(i)}$ and decompose $g = u_- p_+$ as above. Define

$$Q := u_- - I_n, \quad P := u_-^{-1} A|_{U_{k_i}^+}.$$

Notice that we can show that these $P$ and $Q$ are independent of the choice of $g$. Conversely Theorem 3.6 in [15] tells us that $A - \text{pr}_{\text{res}}(A)$ is uniquely determined by these $P$ and $Q$.

Now we are ready to define the map $\Phi_B$. Take $A = (A^{(i)}(z))_{0 \leq j \leq p} \in \mathfrak{M}(B)$ and define $x = \Phi_B(A)$ as follows. As we saw in Proposition 1.23 choose $g_i \in \text{GL}(n, \mathbb{C})$ and $A^{(i)}(z) \in (\mathcal{O}_{B(i)}^{\irr})^0$ so that $g_i A^{(i)}(z) g_i^{-1} = A^{(i)}(z)$ for $i \in I_{irr}$. Then define

for $\rho^{[0,j]}_{[i,j]} \in Q_1^{0 \rightarrow I_{irr}}$,

$$X_{\rho^{[0,j]}_{[i,j]}} := (g_i^{-1})_{[i,j], [0,j]},$$

$$X_{(\rho^{[0,j]}_{[i,j]})^*} := (g_i \tilde{A}^{(i)})_{[0,j], [i,j]}.$$

for $\rho^{[k]}_{[i,j],[i',j']} \in Q_1^{I_{irr}}$, $i \in I_{irr},$

$$X_{\rho^{[k]}_{[i,j],[i',j']}} := (P^{(i)})_{[i,j],[i',j]},$$

$$X_{(\rho^{[k]}_{[i,j],[i',j']})^*} := (Q^{(i)})_{[i',j],[i,j]}.$$

Here $\tilde{A}^{(i)} := \text{pr}_{\text{res}}(\tilde{A}^{(i)}(z))$ and $X_{[i,j],[i',j']}$ denotes $\text{Hom}_{\mathbb{C}}(V_{[i,j]}, V_{[i',j']})$ component of $X \in M(n, \mathbb{C})$. Furthermore, $P^{(i)} = \sum_{k=1}^{k_i-1} (P^{(i)})_{[k]} z^{k-1}$ and $Q^{(i)} = \sum_{k=1}^{k_i-1} (Q^{(i)})_{[k]} z^k$ are defined from $\tilde{A}^{(i)}(z)$ by the above equations (1).
Also define
\[
x_{\rho[i,j,k]} : \text{Im} \prod_{i=1}^{k} (\tilde{A}(i)_{j,j} - \xi_i^{[i,j]} I_{n_{l(i)}}) \mapsto \text{Im} \prod_{i=1}^{k-1} (\tilde{A}(i)_{j,j} - \xi_i^{[i,j]} I_{n_{l(i)}}),
\]
\[
x_{(\rho[i,j,k])^*} := (\tilde{A}(i)_{j,j} - \xi_i^{[i,j]} I_{n_{l(i)}}) \bigg|_{\text{Im} \prod_{i=1}^{k-1} (\tilde{A}(i)_{j,j} - \xi_i^{[i,j]} I_{n_{l(i)}})},
\]
for \( i = 0, \ldots, p, j = 1, \ldots, m^{(i)} \) and \( k = 2, \ldots, e_{[i,j]} - 1 \). Here \( X_{j,j} \) denote \( \text{End}_C(V^{(i)}_{(j,j)}) \)-components of \( X \) for \( j = 1, \ldots, m^{(i)} \). For \( i \in I_{\text{irr}} \) and \( j = 1, \ldots, m^{(i)} \),
\[
x_{\rho[i,j,1]} : \text{Im} (\tilde{A}_{1} - \xi_1^{[i,1]} I_{n_{1}^{(i)}}) \mapsto V^{(i)}_{(1,j)}.
\]
\[
x_{(\rho[i,j,1])^*} := (\tilde{A}_{1} - \xi_1^{[i,1]} I_{n_{1}^{(i)}}) \bigg|_{V^{(i)}_{(1,j)}}.
\]

1.5.2. Open embedding of \( \mathfrak{M}(B) \) into a quiver variety. Now we notice that
\[
\mathfrak{M}_\lambda(Q, \alpha)^{\text{dif}} \not\subset \mathfrak{M}_\lambda(Q, \alpha)^{\text{reg}}
\]
in general, since the \( \mathcal{L} \)-irreducibility is weaker than the ordinary irreducibility. Thus it seems to be possible that \( \mathfrak{M}(B) \cong \mathfrak{M}_\lambda(Q, \alpha)^{\text{dif}} \) has singularities.

To consider this problem, we introduce an operation on \( \mathfrak{M}(B) \) called addition.

**Definition 1.25** (addition). For a collection of complex numbers
\[
v = (v_1, \ldots, v_p) \in \mathbb{C}^p,
\]
the addition translates a differential equation
\[
\frac{d}{dz} Y = \left( \sum_{i=1}^{p} \sum_{\nu=1}^{k_i} \frac{A^{(i)}_{\nu}}{(z - a_i)^{\nu}} + \sum_{2 \leq \nu \leq k_0} A^{(0)}_{\nu} z^{\nu-2} \right) Y \in \mathfrak{M}(B)
\]
to
\[
\frac{d}{dz} Y = \left( \sum_{i=1}^{p} \sum_{\nu=1}^{k_i} \frac{A^{(i)}_{\nu}}{(z - a_i)^{\nu}} + \frac{v_i I_n}{z - a_i} + \sum_{2 \leq \nu \leq k_0} A^{(0)}_{\nu} z^{\nu-2} \right) Y \in \mathfrak{M}(B + v).
\]
Here \( B + v := (B_i + v_i I_n z^{-1})_{i=0, \ldots, p} \) with \( v_0 := -\sum_{i=1}^{p} v_i \).

Thus for \( v \in \mathbb{C}^p \) the addition defines the bijection
\[
\text{Add}_v : \mathfrak{M}_\lambda(Q, \alpha)^{\text{dif}} \longrightarrow \mathfrak{M}_{\lambda + \psi}(Q, \alpha)^{\text{dif}}
\]
where $\bar{v} = (\bar{v}_a)_{a \in Q_0}$ is defined as follows,

$$
\bar{v}_{[i,j]} := v_i \quad \text{for all } [i, j] \in Q_0^{irr} \text{ and } i \in I_{irr} \setminus \{0\},
$$

$$
\bar{v}_{[0,j]} := v_0 + \sum_{k \in I_{reg}} v_k \quad \text{for all } j = 1, \ldots, m^{(0)},
$$

$$
\bar{v}_{[i,j,k]} := 0 \quad \text{for all } [i, j, k] \in Q_0^{reg}.
$$

Then we can find a nice $v$ such that $\text{Add}_e$ sends $\mathcal{M}_\lambda(Q, \alpha)^{\text{diff}}$ into the quiver variety $\mathcal{M}_{\lambda + \bar{v}}^{\text{reg}}(Q, \alpha)$.

**Theorem 1.26.** Let $\mathcal{M}_\lambda(Q, \alpha)^{\text{diff}}$ be as above. Then there exists $v \in \mathbb{C}^p$ such that

$$
\mathcal{M}_{\lambda + \bar{v}}^{\text{reg}}(Q, \alpha)^{\text{diff}} =
\left\{ x \in \mu^{-1}(\lambda + \bar{v}) \mid \begin{array}{c}
\text{x is irreducible,} \\
\det \left( x_{\rho^{[0,j]}_{[i,j']}} \right)_{1 \leq j \leq m^{(0)}} \neq 0, i \in I_{irr} \setminus \{0\}
\end{array} \right\} / G
\subset \mathcal{M}_{\lambda + \bar{v}}^{\text{reg}}(Q, \alpha)
$$

**Proof.** Lemma 5.9 in [13] shows that there exists $v \in \mathbb{C}^p$ such that $\lambda' = \lambda + \bar{v}$ satisfies the following. If $\beta \in (\mathbb{Z}_{\geq 0})^Q_0$ satisfies that $\beta \leq \alpha$, i.e., $\beta_a \leq \alpha_a$ for all $a \in Q_0$, and $\lambda' \cdot \beta = 0$, then $\beta \in L$. Thus any subrepresentation $y$ of $x \in \mathcal{M}_{\lambda}(Q, \alpha)$ satisfies that $\sqrt{y} \in L$, that is, the $L$- irreducibility implies the irreducibility. □

Thus we have an open embedding of $\mathcal{M}(B)$ into the regular part of a quiver variety.

**Theorem 1.27.** Let us take $B = (B^{(i)})_{0 \leq i \leq p}$, a collection of HTL normal forms, the quiver $Q$, $\alpha \in (\mathbb{Z}_{\geq 0})^Q_0$ and $\lambda \in \mathbb{C}^Q_0$ as above. Then there exists $v \in \mathbb{C}^p$ and an injection

$$
\Phi : \mathcal{M}(B) \hookrightarrow \mathcal{M}_{\lambda + \bar{v}}^{\text{reg}}(Q, \alpha)
$$

such that

$$
\Phi(\mathcal{M}(B)) =
\left\{ x \in \mathcal{M}_{\lambda + \bar{v}}^{\text{reg}}(Q, \alpha) \mid \begin{array}{c}
\det \left( x_{\rho^{[0,j]}_{[i,j']}} \right)_{1 \leq j \leq m^{(0)}} \neq 0, i \in I_{irr} \setminus \{0\}
\end{array} \right\}.
$$

In particular if $I_{irr} = \{0\}$, then $v = 0$ and $\Phi$ is bijective.

**Corollary 1.28.** If $\mathcal{M}(B) \neq \emptyset$, then it can be seen as a connected symplectic complex manifold of dimension $2p(\alpha)$.

**Proof.** If $\mathcal{M}(B) \neq \emptyset$, then $\mathcal{M}_{\lambda + \bar{v}}^{\text{reg}}(Q, \alpha) \neq \emptyset$ and thus $\mathcal{M}_{\lambda + \bar{v}}^{\text{reg}}(Q, \alpha) \neq \emptyset$ with $v \in \mathbb{C}^p$ chosen as in the previous theorem. Thus by Theorems 1.16 and 1.17 it suffices to check the connectedness.

Theorem 1.17 says that $\mu^{-1}(\lambda + \bar{v})$ is an irreducible variety. Let us recall that

$$
\mu^{-1}(\lambda + \bar{v})^{\text{sta}} := \{ x \in \mu^{-1}(\lambda + \bar{v}) \mid x \text{ is stable under } G \}.
$$
is an open subset of $\mu^{-1}(\lambda + \bar{v})$ (see Proposition 5.15 in [26] for instance). Since $\mu^{-1}(\lambda + \bar{v})^{\text{sta}} = \mu^{-1}(\lambda + \bar{v})^{\text{irr}}$,

\[
\mu^{-1}(\lambda + \bar{v})^{\text{det}} := \left\{ x \in \mu^{-1}(\lambda + \bar{v})^{\text{irr}} \left| \text{det} \left( x_{[p,j]} \right)_{1 \leq j \leq m(0)} \neq 0, i \in I_{\text{irr}} \setminus \{0\} \right. \right\}
\]

is also an open subset of $\mu^{-1}(\lambda + \bar{v})$. Since open subsets of an irreducible topological space are connected, $\mu^{-1}(\lambda + \bar{v})^{\text{det}}$ is connected. Moreover $\mathcal{M}_{\lambda + \bar{v}}(Q, \alpha)^{\text{dif}}$ is the image of the continuous projection from $\mu^{-1}(\lambda + \bar{v})^{\text{det}}$, thus it is connected not only as an algebraic variety but also as an analytic space by GAGA. □

1.5.3. Non-emptiness of $\mathcal{M}(\mathcal{B})$. We close this subsection by giving a necessary and sufficient condition for $\mathcal{M}(\mathcal{B}) \neq \emptyset$. Define a set $\Sigma^{\text{dif}}_{\lambda}$ consists of $\beta \in \mathcal{L}^{+}$ satisfying

1. $\beta$ is a positive root of $Q$ and $\beta \cdot \lambda = 0$,
2. for any decomposition $\beta = \beta_{1} + \cdots + \beta_{r}$ where $\beta_{i} \in \mathcal{L}^{+}$ are positive roots of $Q$ satisfying $\beta_{i} \cdot \lambda = 0$, we have

\[
p(\beta) > p(\beta_{1}) + \cdots + p(\beta_{r}).
\]

**Theorem 1.29** (Non-emptiness of moduli spaces. Theorem 0.9 in [13]). The moduli space $\mathcal{M}(\mathcal{B}) \neq \emptyset$ if and only if $\alpha \in \Sigma^{\text{dif}}_{\lambda}$.

Let us recall the spectral type of $\mathcal{B}$ defined by the above $\alpha$. Consider the inductive limit

\[
\mathbb{Z}^{\infty} := \lim_{\longrightarrow} \mathbb{Z}^{n}
\]

defined by inclusions $\phi_{i,i+1}: \mathbb{Z}^{i} \ni (a_{1}, \ldots, a_{i}) \mapsto (a_{1}, \ldots, a_{i}, 0) \in \mathbb{Z}^{i+1}$ for $i = 1, 2, \ldots$.

**Definition 1.30** (spectral type and index of rigidity). The spectral type of $\mathcal{B}$ is the pair

\[
\left( m_{\alpha}, (d_{i}(j, j'))_{i=0, \ldots, p \atop 1 \leq j < j' \leq m^{(i)}(0)} \right)
\]

where

\[
m_{\alpha} = \left( m_{[i,j,k,l]}, \ldots, m_{[i,j,e_{[i,j]]]} \right)_{0 \leq i \leq p \atop 1 \leq j \leq m^{(i)}} \in \bigoplus_{i=0}^{p} \bigoplus_{j=1}^{m^{(i)}} \mathbb{Z}^{\infty}
\]

is defined by

\[
m_{[i,j,k,l]} = \cdots = \sum_{j=1}^{m^{(p)}(0)} \sum_{k=1}^{m^{(i)}} m_{[i,j,k,l]} = \alpha_{[i,j,k-1]} - \alpha_{[i,j,k]}
\]

where

\[
\alpha_{[i,j,k]} = \begin{cases} \alpha_{[i,j]} & \text{if } i \in I_{\text{irr}}, \\ \sum_{k=1}^{m^{(i)}(0)} \alpha_{[0,k]} & \text{if } i \in I_{\text{reg}} \end{cases}
\]

and $\alpha_{[i,j,e_{[i,j]}]} = 0$. Sometimes we write $m_{\alpha} = (m_{\alpha}, d_{i}(j, j'))$ for short.

The index of rigidity of $m_{\alpha}$ is defined by

\[
\text{id}_m := 2q(\alpha).
\]
Here we note that we do not distinguish $m_t$ and
\[
\left( (m_{[\sigma(i), s(j), t(1)]}, \cdots, m_{[\sigma(i), s(j), t(e_{[\sigma(i), s(j)]})]}) \right)_{0 \leq i \leq p, \ 1 \leq j \leq m_t} = \\
(d_{[\sigma(i)]}(s(j), s(j')))_{i=0, \ldots, p, \ 1 \leq j < j' \leq m_t}
\]
for any permutations $\sigma \in \mathfrak{S}_{p+1}$, $s \in \mathfrak{S}_{m_t}$ and $t \in \mathfrak{S}_{e_{[i,j]}}$ for $i = 0, \ldots, p$, $j = 1, \ldots, m_t$.

For convenience we introduce the following notation for $m$. The each number $d_t(i,j') + 1$ is expressed by the number of parentheses ( ) between the sequences $m_{[i,j,1]}$, $m_{[i,j,2]}$, $\cdots$ and $m_{[i,j',1]}$, $m_{[i,j',2]}$, $\cdots$. For instance, if
\[
m_{i, j} = \cdots m_{[i,j,1]} m_{[i,j,2]} \cdots m_{[i,j,k,i,j]},
\]
then we can see the double parenthesis $((\text{between } m_{[i,j,1]} \cdots) \text{, and } m_{[i,j',1]} \cdots)$.

This means $d_t(i, j') = 1$. Let us see an example. Consider the case where $p = 1$, $(m(0), m(1)) = (2, 3)$, $(c_{[0,1]}, c_{[0,2]}, c_{[1,1]}, c_{[1,2]}, c_{[2,1]}) = (1, 2, 1, 1, 2)$ and $(d_0(1,2), d_1(1,2), d_2(2,3), d_1(1,3)) = (0, 0, 1, 1)$.

Then $m = ((m_{[i,j,1]} \cdots, m_{[i,j,k,i,j]})_{0 \leq i \leq p}$ is written by
\[
(m_{[0,1,1]}m_{[0,2,1]}m_{[0,2,2]}), \quad (m_{[1,1,1]}m_{[1,2,1]}m_{[1,3,1]}m_{[1,3,2]}).
\]

1.6. Integrable deformation. Let us introduce integrable admissible deformations of connections following Boalch [2] and Yamakawa [41].

Let $T$ be a contractible complex manifold and $a_i : T \to \mathbb{P}^1 \times T$, $i = 0, \ldots, p$, holomorphic sections of the fiber bundle $\pi : \mathbb{P}^1 \times T \to T$. Moreover assume that
\[
a_i(t) \neq a_j(t) \text{ if } i \neq j
\]
in each fiber $\mathbb{P}^1_t := \mathbb{P}^1 \times \{t\}$. Let us consider a moduli space $\mathfrak{M}(B)$ and the corresponding quiver variety $\mathfrak{M}_\lambda(Q, \alpha)^{\text{dif}}$. We further impose the non-resonance condition,
\[
\lambda_{[i,j,k]} \notin \mathbb{Z} \setminus \{0\} \text{ for all } [i, j, k] \in \mathbb{Q}_0^{\text{leg}}.
\]

Definition 1.31 (admissible deformation). Let $V$ be an $n$-dimensional $\mathbb{C}$-vector space and set $E = \mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{C}} V$. Fix a point $t_0 \in T$ and $(E_t, \nabla) \in \mathfrak{M}(B) \neq \emptyset$. Then a family $\left( (E_t = \mathcal{O}_{\mathbb{P}^1} \otimes_{\mathbb{C}} V, \nabla_t) \right)_{t \in T}$ is called an admissible deformation of $(E, \nabla)$ if

1. $(E_{t_0}, \nabla_{t_0}) = (E, \nabla)$,
2. $(E_t, \nabla_t) \in \mathfrak{M}_\lambda(Q, \alpha)^{\text{dif}}$ for all $t \in T$,
3. the mapping $\nabla \ni t \mapsto (E_t, \nabla_t) \in \mathfrak{M}_\lambda(Q, \alpha)^{\text{dif}}$ is analytic.

In this case we call $(\nabla_t)_{t \in T}$ the admissible deformation with the spectral data $(Q, \alpha, \lambda)$.

The dimension of the admissible deformation is $\dim(\mathfrak{M}_\lambda(Q, \alpha)) = 2p(\alpha)$.

Definition 1.32 (integrable deformation). Let $((E_t, \nabla_t))_{t \in T}$ be an admissible deformation of $(E, \nabla) \in \mathfrak{M}(B)$. If there exists a flat meromorphic connection $\nabla$ on $\mathcal{O}_{\mathbb{P}^1 \times \Delta} \otimes_{\mathbb{C}} V$ with poles on $\bigcup_{i=0}^p a_i(\Delta)$ such that $\nabla|_{\mathbb{P}^1_t} = \nabla_t$, then $(E_t, \nabla_t)$ is integrable.
then we call the family \( (\mathcal{E}_t, \nabla_t)_{t \in \mathbb{T}} \) an \textit{integrable admissible deformation} of \( (\mathcal{E}, \nabla) \). In this case such \( \nabla \) is called a \textit{flat extension} of \( (\nabla_t)_{t \in \mathbb{T}} \).

2. **Middle convolutions, Weyl groups and integrable deformations**

In the previous section, we saw that moduli spaces of stable meromorphic connections are realized as quiver varieties on which integrable admissible deformations of connections are defined. As it is known, quiver varieties have Weyl group symmetries generated by reflection functors (see \cite{7} and \cite{28}). Similarly on the moduli space side, we also have the symmetries generated by middle convolutions. In this section, we see the relationship between middle convolutions and Weyl group of quivers and give a classification of their symmetries in certain lower dimensional cases. And we see the symmetries of integrable deformations as an application.

2.1. **A review of middle convolutions.** Let us give a review of middle convolutions on differential equations with irregular singular points. The middle convolution is originally defined by Katz in \cite{19} and reformulated as an operation on Fuchsian systems by Dettweiler-Reiter \cite{9}, see also \cite{8} and Völklein’s paper \cite{37}. There are several studies to generalize the middle convolution to non-Fuchsian differential equations, see \cite{1, 20, 35, 40} for example. Among them we shall give a review of middle convolutions following \cite{40}.

Let us take \( (\mathcal{E}, \nabla) \in \mathfrak{M}(\mathbf{B}) \). Choose a trivialization \( \mathcal{E} \cong \mathcal{O}^n \) and write

\[
\nabla = d - \left( \sum_{i=1}^{p} \sum_{\nu=1}^{k_i} \frac{A_{\nu}^{(i)}}{(z-a_{i})^\nu} + \sum_{2 \leq \nu \leq \kappa_0} A_{\nu}^{(0)} z^{\nu-2} \right) dz.
\]

Set

\[
\mathbf{A} = \left( \sum_{j=1}^{k_i} A^{(i)}_j z^{-j} \right)_{0 \leq i \leq p} \in \prod_{i=0}^{p} \mathcal{O}_{B(i)}
\]

where \( A^{(i)}_1 := - \sum_{i=1}^{p} A^{(i)}_1 \). Then we may regard

\[
\mathfrak{M}(\mathbf{B}) = \left\{ \mathbf{A} = \left( \sum_{j=1}^{k_i} A^{(i)}_j z^{-j} \right)_{0 \leq i \leq p} \in \prod_{i=0}^{p} \mathcal{O}_{B(i)} \mid \mathbf{A} \text{ is irreducible}, \sum_{i=0}^{p} A^{(i)}_1 = 0 \right\} / \text{GL}(n, \mathbb{C}).
\]

Let us construct a 5-tuple \( (V, W, T, Q, P) \) consisting of \( \mathbb{C} \)-vector spaces \( V, W \) and \( T \in \text{End}_\mathbb{C}(W) \), \( Q \in \text{Hom}_\mathbb{C}(W, V) \), \( P \in \text{Hom}_\mathbb{C}(V, W) \). Set \( V = \mathbb{C}^n \) and \( \tilde{W}_i = V^{\oplus k_i} \) for \( i = 0, \ldots, p \). Then define

\[
\tilde{Q}_i := (A^{(i)}_{k_i}, A^{(i)}_{k_i-1}, \ldots, A^{(i)}_1) \in \text{Hom}_\mathbb{C}(\tilde{W}_i, V),
\]

\[
\tilde{P}_i := \begin{pmatrix}
0 & \ldots & 0 \\
\vdots & \ddots & \ldots \\
0 & 0 & \text{Id}_V
\end{pmatrix} \in \text{Hom}_\mathbb{C}(V, \tilde{W}_i), \quad \hat{N}_i := \begin{pmatrix}
0 & \text{Id}_V & 0 \\
\vdots & \ddots & \ldots \\
0 & 0 & \text{Id}_V
\end{pmatrix} \in \text{End}_\mathbb{C}(\tilde{W}_i).
\]
Setting $\widehat{W} := \bigoplus_{i=0}^{p} \widehat{W}_i$, $\widehat{T} := (\widehat{N}_i)_{0 \leq i \leq p} \in \bigoplus_{i=0}^{p} \text{End}_C(\widehat{W}_i) \subset \text{End}_C(\widehat{W})$, $\widehat{Q} := (\widehat{Q}_i)_{0 \leq i \leq p} \in \bigoplus_{i=0}^{p} \text{Hom}_C(\widehat{W}_i, V) = \text{Hom}_C(W, V)$ and $\widehat{P} := (\widehat{P}_i)_{0 \leq i \leq p} \in \bigoplus_{i=0}^{p} \text{Hom}_C(V, \widehat{W}_i) = \text{Hom}_C(V, \widehat{W})$, we have a 5-tuple $(V, \widehat{W}, \widehat{T}, \widehat{Q}, \widehat{P})$. Further setting

$$
\widehat{A}_i := \begin{pmatrix}
A^{(i)}_{k_i} & A^{(i)}_{k_{i-1}} & \cdots & A^{(i)}_1 \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & \ddots & \ddots \\
\end{pmatrix} \in \text{End}_C(\widehat{W}_i),
$$

we define $W_i := \widehat{W}_i/\text{Ker}\widehat{A}_i$ and $W := \bigoplus_{i=0}^{p} W_i$. Then $T, Q, P$ are the maps induced from $\widehat{T}, \widehat{Q}, \widehat{P}$ respectively.

**Definition 2.1.** The 5-tuple $(V, W, T, Q, P)$ given above is called the canonical datum for $A \in \prod_{i=0}^{p} O_{B^{(i)}}$.

Fix $t \in \{0, \ldots, p\}$, take a polynomial $q_t(z^{-1}) = \sum_{j=1}^{k_t} q^{(t)}_j z^{-j} \in z^{-1} C[z^{-1}]$ and define an operation, called addition which already appeared in some special cases before. For an element $A = (A_i(z^{-1}))_{0 \leq i \leq p} \in \prod_{i=0}^{p} O_{B^{(i)}}$, we define $\text{Add}_{q_t(z-1)}^{(t)}(A) := (A'_i(z^{-1}))_{0 \leq i \leq p}$ by

$$
A'_i(z^{-1}) := \begin{cases} 
A_i(z^{-1}) & \text{if } i \neq t, \\
A_t(z^{-1}) - q_t(z^{-1}) & \text{if } i = t.
\end{cases}
$$

Then $\text{Add}_{q_t(z-1)}^{(t)}(A) \in \prod_{i=0}^{p} O_{(B'_t)^{(i)}}$ where

$$
(B'_t)^{(i)} := \begin{cases} 
B^{(i)} & \text{if } i \neq t, \\
B^{(i)} - q_t(z^{-1}) & \text{if } i = t.
\end{cases}
$$

Set

$$
\mathcal{J}_i := \{[i, j] \mid j = 1, \ldots, m^{(i)}\} \quad \text{for } i = 0, \ldots, p
$$

and

$$
\mathcal{J} := \prod_{i=0}^{p} \mathcal{J}_i.
$$

Then let us define

$$
\text{Add}_i := \prod_{i=0}^{p} \text{Add}_{q^{(i)}_t(z^{-1}) + \xi_i^{[i,j]} z^{-1}},
$$

for $i = ([i, j])_{0 \leq i \leq p} \in \mathcal{J}$.

Suppose that we can choose $i \in \mathcal{J}$ so that $\xi_i := \sum_{j=0}^{p} \xi^{[i,j]} \neq 0$. Let $(V, W, T, Q, P)$ be the canonical datum of $\text{Add}_i(A)$. Following Example 3 in [30], we construct a new 5-tuple $(V', W, T, Q', P')$ as follows. Note that $QP = -\xi_i \text{Id}_V$. Thus $Q$ and $P$ are surjective and injective respectively. Let us set $V' = \text{Coker} P$ and $Q': W \to V'$, the natural projection. Then we have the split exact sequence

$$
0 \to V \xrightarrow{P} W \xrightarrow{Q'} V' \to 0.
$$
Note that \((-\xi_1^{-1}Q)P = \text{Id}_V\). Let \(P' : V' \to W\) be the injection such that \(Q'(\xi_1^{-1}P') = \text{Id}_{W'}\). Then we have a 5-tuple \((V', W, T, Q', P')\).

Next we set \(Q'_i\) (resp. \(P'_i\)) to be the \(\text{Hom}_{C}(W_i, V)\) (resp. \(\text{Hom}_{C}(V, W_i)\)) component of \(Q'\) (resp. \(P'\)). Also set \(N_i\) to be the \(\text{End}_{C}(W_i)\)-component of \(T\). Define

\[
(A')_j^{(i)} := Q'_i N_j^{i-1} P'_i
\]

and \(A' = (A'_i(z^{-1}))_{0 \leq i \leq p}\) where \(A'_i(z^{-1}) = \sum_{j=1}^{k_i} (A')_j^{(i)} z^{-j}\). We note that

\[
\sum_{i=0}^{p} (A')_1^{(i)} = Q'P' = \xi \text{Id}_{V'}.
\]

Finally let us set

\[
A'' := \text{Add}_{-1}^{-1} \circ \text{Add}_{2, z^{-1}}(A').
\]

Then \(A'' = (A''_i(z^{-1}))_{0 \leq i \leq p}\) satisfies that \(\sum_{i=0}^{p} \text{pr}_{\text{res}} A''_i(z^{-1}) = 0\). Let us denote \(A''\) by \(mc_1(A)\) and call the operator \(mc_1\) the middle convolution at \(i\).

Let us recall basic properties of middle convolutions.

**Proposition 2.2 (see Yamakawa [40]).** Let us take \(A = (A_i(x^{-1}))_{0 \leq i \leq p} \in \prod_{i=0}^{p} \mathcal{O}_{B(i)}\) satisfying \(\sum_{i=0}^{p} \text{pr}_{\text{res}} A_i(z^{-1}) = 0\). Suppose we can choose \(i \in J\) so that \(\xi_1 \neq 0\).

1. If \(A\) is irreducible, then \(mc_1(A)\) is irreducible.
2. If \(A\) is irreducible,

\[
mc_1 \circ mc_1(A) \sim A,
\]

i.e., there exists \(g \in \text{GL}(n, \mathbb{C})\) such that \(mc_1 \circ mc_1(A) = gA g^{-1} := (gA_i(z^{-1})g^{-1})_{0 \leq i \leq p}\).

3. Let us define elements in \(M((n'_j)^{(i)}, \mathbb{C})\) by

\[
(R')^{(i)}_j := \begin{cases} R^{(i)}_j + (d_i(j, j_i) + 2) \xi_1 I_{n_j}^{(i)} & \text{if } i \neq 0, \\ R^{(0)}_j + d_0(j, j_0) \xi_1 I_{n_j}^{(0)} & \text{if } i = 0 \end{cases}
\]

for all \(i \in \{0, \ldots, p\}\) and \(j \in \{1, \ldots, m^{(i)}\} \setminus \{j_i\}\). Here we put \((n'_j)^{(i)} := n_j^{(i)}\). Further define \((R')^{(i)}_{j_i} \in M((n'_j)^{(i)}, \mathbb{C})\) for \(i = 1, \ldots, p\) so that equations

\[
\text{rank} ((R')^{(i)}_{j_i} - \xi_1^{[i,j_i]} \prod_{k=1}^{l} ((R')^{(i)}_{j_k} - \xi_1^{[k,j_k]}) = \text{rank} \prod_{k=1}^{l} (R^{(i)}_{j_k} - \xi_1^{[k,j_k]})
\]

hold for all \(l = 2, \ldots, e_{i,j_i}\). Similarly define \((R')^{(0)}_{j_0} \in M((n'_j)^{(0)}, \mathbb{C})\) so that equations

\[
\text{rank} ((R')^{(0)}_{j_0} - \xi_1^{[0,j_0]} + 2 \xi_1 \prod_{k=1}^{l} ((R')^{(0)}_{j_k} - \xi_1^{[k,j_k]} + \xi_1) = \text{rank} \prod_{k=1}^{l} (R^{(0)}_{j_k} - \xi_1^{[k,j_k]})
\]

hold for all \(l = 2, \ldots, e_{[0,j_0]}\). Here we put

\[
(n')^{(i)}_{j_i} := n_j^{(i)} + \text{dim}_{\mathbb{C}}W - 2n.
\]
Finally define
\[(B')^{(i)} := \text{diag}\left(q^{(i)}_1 (x^{-1}) I_{(n')_1}, \ldots, q^{(i)}_{m(i)} (x^{-1}) I_{(n')_{m(i)}}, (R^{(i)}_m)^{-1} \right)\]
for \(i = 0, \ldots, p\). Then \(mc_1(A) \in \bigoplus_{i=0}^p C_{(B')^{(i)}}\).

**Remark 2.3.** Let us note that the description of \(A'' = mc_1(A)\) depends on the choice of the coordinate systems of \(W_1\) and \(V'\) in the canonical data. Thus \(mc_1\) defines the following well-defined bijection
\[mc_1: \mathcal{M}(B) \rightarrow \mathcal{M}(B')\]
where \(B' = ((B')^{(i)})_{0 \leq i \leq p}\).

### 2.2. Middle convolutions on representations of a quiver

For a vertex with no edge-loop in \(Q_0\), it is known that there exists a bijection
\[s_a: \mathcal{M}_\lambda(Q, \alpha) \rightarrow \mathcal{M}_{r_a(\lambda)}(Q, s_a(\alpha))\]
if \(\lambda \neq 0\), so-called reflection functor see [7] and [28]. In this section, we shall define an analogy of the reflection functors for the subspace \(\mathcal{M}_\lambda(Q, \alpha)^{\text{dif}} \subset \mathcal{M}_\lambda(Q, \alpha)\) by using middle convolutions. We notice that a reflection functor does not necessarily preserve the subset \(\mathcal{M}_\lambda(Q, \alpha)^{\text{dif}}\), namely it may happen that
\[s_a \left( \mathcal{M}_\lambda(Q, \alpha)^{\text{dif}} \right) \not\subset \mathcal{M}_{r_a(\lambda)}(Q, s_a(\alpha))^{\text{dif}}\]
for some \(a \in Q_0\). However as we saw in Remark 2.3 a middle convolution \(mc_1\) can be seen as a transformation of moduli spaces \(\mathcal{M}(B) \cong \mathcal{M}_\lambda(Q, \alpha)^{\text{dif}}\). Thus it can be seen as a transformation of quiver varieties \(\mathcal{M}_\lambda(Q, \alpha)^{\text{dif}}\) as below.

For \(i = ([i, j])_{0 \leq i \leq p} \in \mathcal{J}\), let us define \(\epsilon_i \in Z_0\) by
\[(\epsilon_i)_a := \begin{cases} 1 & \text{if } a = [i, j], \ i \in I_{\text{irr}}, \\ 0 & \text{otherwise.} \end{cases}\]
We note that \(\epsilon_i\) for \(i \in \mathcal{J}\) are positive real roots of \(Q\). Let us define
\[s_i(\beta) := \beta - (\beta, \epsilon_i)\epsilon_i\]
for \(i \in \mathcal{J}\) and \(\beta \in Z_0\). Also define \(r_i(\mu)\) for \(\mu \in C_0\) by
\[r_i(\mu)|_{[i, j]} := \begin{cases} \mu_{[i, j]} & \text{if } [i, j] \neq [0, 0], \\ \mu_{[0, 0]} - 2\mu_i & \text{if } [i, j] = [0, 0], \end{cases}\]
\[r_i(\mu)|_{[i, j, k]} := \begin{cases} \mu_{[i, j, k]} & \text{if } [i, j, k] \neq [i, j, 1], \\ \mu_{[i, j, 1]} + \mu_i & \text{if } [i, j, k] = [i, j, 1]. \end{cases}\]

Then Proposition 2.22 tells us the following.

**Theorem 2.4.** Let us consider \(\mathcal{M}(B) \neq \emptyset\) and the corresponding quiver variety \(\mathcal{M}_\lambda(Q, \alpha)^{\text{dif}}\) under the bijection in Theorem 1.22. Suppose that we can take \(i = ([i, j]) \in \mathcal{J}\) so that \(\lambda_i := \sum_{j \in I_{\text{irr}}} \lambda_{[i, j]} = -\xi_i \neq 0\). Then there exists a bijection
\[s_1: \mathcal{M}_\lambda(Q, \alpha)^{\text{dif}} \rightarrow \mathcal{M}_{r_i(\lambda)}(Q, s_1(\alpha))^{\text{dif}}\]
Remark 2.5. For each $[i, j, k] \in Q_0^{\text{lg}}$, the ordinary reflection functor of quiver varieties gives a bijection

$$s_{i,j,k} : \mathcal{M}_\lambda(Q, \alpha)^{\text{dif}} \rightarrow \mathcal{M}_{\tau_{i,j,k}(\lambda)}(Q, s_{i,j,k}(\alpha))^{\text{dif}}$$

if $\lambda_{[i,j,k]} \neq 0$.

Let us define an analogue of fundamental set of the root lattice $\mathbb{Z}Q_0$,

$$\tilde{F} := \left\{ \beta \in \mathcal{L}^+ \setminus \{0\} \mid (\beta, \epsilon_a) \leq 0 \text{ for all } a \in J \cup Q_0^{\text{lg}} \text{ support of } \beta \text{ is connected} \right\}$$

called $\mathcal{L}$-fundamental set. Then we can see that $\tilde{F}$ can be seen as a fundamental domain under the action of the group

$$W^{\text{mc}} := \left\langle s_1, s_{[i,j,k]} \mid i \in J, [i, j, k] \in Q_0^{\text{lg}} \right\rangle.$$  

**Theorem 2.6.** For $\mathcal{M}_\lambda(Q, \alpha)^{\text{dif}} \neq \emptyset$, there exists $w \in W^{\text{mc}}$ such that

$$w \left( \mathcal{M}_\lambda(Q, \alpha)^{\text{dif}} \right) = \mathcal{M}_{\lambda'}(Q, \alpha')^{\text{dif}}$$

with

$$\left\{ \begin{array}{ll}
\alpha' \in \tilde{F} & \text{if } q(\alpha) \leq 0, \\
\alpha' = \epsilon_i & \text{for some } i \in J \text{ otherwise.}
\end{array} \right.$$

**Proof.** See Lemma 7.2, Theorem 7.9 and Theorem 7.10 in [13]. □

We introduce a condition for $\lambda$ which will be used in the latter section.

**Definition 2.7.** For $\mathcal{M}_\lambda(Q, \alpha)^{\text{dif}} \neq \emptyset$, we say $\lambda$ is fractional if

$$\lambda'_i := \sum_{i \in I_{\text{irr}}} \lambda'_{[i,j,i]} \notin \mathbb{Z}$$

for all $i \in J$ and $\lambda' \in \{ \epsilon_{r(\lambda)} \mid r \in \left\langle r_{[i,j,k]} \mid [i, j, k] \in Q_0^{\text{lg}} \right\rangle \}$.

Moreover if there exists a sequence $a_1, a_2, \ldots, a_t \in J \cup Q_0^{\text{lg}}$ such that

$$r_{a_k} \circ r_{a_{k-1}} \circ \cdots \circ r_{a_1}(\lambda)$$

are fractional for all $k = 1, \ldots, t$ and $w = s_{a_t} s_{a_{t-1}} \cdots s_{a_1}$ where $w \in W^{\text{mc}}$ is chosen as in Theorem 2.6 then we say that $\lambda$ has a fractional reduction.

**Remark 2.8.** Let $B = (B^{(i)})_{0 \leq i \leq p}$ be a collection of HTL normal forms

$$B^{(i)} = \text{diag} \left( q_1^{(i)} (z^{-1}) I_{n_1^{(i)}} + R_1^{(i)} z^{-1}, \ldots, q_m^{(i)} (z^{-1}) I_{n_m^{(i)}} + R_m^{(i)} z^{-1} \right)$$

such that $\mathcal{M}(B) \neq \emptyset$. Then $\lambda$ of $\mathcal{M}(B) \cong \mathcal{M}_\lambda(Q, \alpha)^{\text{dif}}$ is fractional if and only if $\sum_{i=0}^p \xi_i \notin \mathbb{Z}$ where $\xi_i$ is an arbitrary eigenvalue of $pr_{\text{res}}(B^{(i)})$ for each $i = 0, \ldots, p$. 

---

**Proof.** We retain the notation in Proposition 2.2 and put $B' = ((B')^{(i)})_{0 \leq i \leq p}$.
2.3. **The lattice \( \mathcal{L} \) as a Kac-Moody root lattice.** As we saw in Theorem 1.29 if \( \mathcal{M}(\mathcal{B}) \cong \mathcal{M}(\mathcal{Q}(\alpha)) \), then \( \alpha \) must be in \( \mathcal{L} \cap \Delta \) where \( \Delta \) is the set of roots in \( \mathbb{Z}^Q \). This inclines us to see \( \mathcal{L} \cap \Delta \) as an analogy of the set of roots of the lattice \( \mathcal{L} \) which may not be a true Kac-Moody root lattice.

**Definition 2.9** (symmetric Kac-Moody root lattice). We call a \( \mathbb{Z} \)-lattice \( \mathcal{L} := \bigoplus_{i \in I} \mathbb{Z} \alpha_i \) with a finite index set \( I \) a symmetric Kac-Moody root lattice, when \( \mathcal{L} \) has the following bilinear form

\[
(\alpha_i, \alpha_i) = 2 \quad (i \in I), \\
(\alpha_i, \alpha_j) = (\alpha_j, \alpha_i) \in \mathbb{Z}_{\leq 0} \quad (i, j \in I, i \neq j).
\]

For each \( \alpha_i, \ (i \in I) \) which is called a simple root, we can define the simple reflection by

\[
s_{\alpha_i}(\beta) := \beta - (\beta, \alpha_i)\alpha_i
\]

for \( \beta \in \mathcal{L} \). The Weyl group \( \mathcal{W} \in \text{Aut}_\mathbb{Z}(\mathcal{L}) \) is the group generated by all simple reflections \( s_{\alpha_i}, i \in I \).

We can attach \( \mathcal{L} \) to a diagram, called the Dynkin diagram, regarding simple roots as vertices and connecting \( \alpha_i, \alpha_j \) by \( (\alpha_i, \alpha_j) \) edges if \( i \neq j \).

Notions of real roots, fundamental set and imaginary roots and so on are also defined in the same way as we saw in §1.3.3. Then for our quiver \( \mathcal{Q} \), the \( \mathbb{Z}^Q_0 \) is a symmetric Kac-Moody root lattice.

It can be checked that \( \mathcal{L} \) is generated by \( \{ \epsilon_a | a \in \mathcal{J} \cup \mathcal{Q}^\text{leg}_0 \} \) over \( \mathbb{Z} \) and \( \mathcal{W}^\text{mc} = \langle s_a | a \in \mathcal{J} \cup \mathcal{Q}^\text{leg}_0 \rangle \). This may lead us to believe that \( \mathcal{L} \) can be seen as a root lattice with the set of simple roots \( \{ \epsilon_a | a \in \mathcal{J} \cup \mathcal{Q}^\text{leg}_0 \} \) and the Weyl group \( \mathcal{W}^\text{mc} \). However elements in \( \{ \epsilon_a | a \in \mathcal{J} \cup \mathcal{Q}^\text{leg}_0 \} \) are not independent over \( \mathbb{Z} \) in general. Thus we shall introduce a new lattice \( \mathcal{\hat{L}} \) of which \( \mathcal{L} \) can be seen as a quotient. Let us note that

\[
(\epsilon_i, \epsilon_j) = 2 - \sum_{0 \leq p \leq k, j_i, j'_i} (d_i(j_i, j'_i) + 2),
\]

\[
(\epsilon_i, \epsilon_{[i,j,k]}) = \begin{cases} -1 & \text{if } j = j_i \text{ and } k = 1, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
(\epsilon_{[i,j,k]}, \epsilon_{[i',j',k']}) = \begin{cases} 2 & \text{if } [i, j, k] = [i', j', k'], \\ -1 & \text{if } (i, j) = (i', j') \text{ and } |k - k'| = 1, \\ 0 & \text{otherwise} \end{cases}
\]

for \( i, i' \in \mathcal{J} \) and \([i, j, k], [i', j', k'] \in \mathcal{Q}^\text{leg}_0\). Thus we consider a new lattice \( \mathcal{\hat{L}} \) generated by the set of indeterminate

\[
\mathcal{C} = \{ c_a | a \in \mathcal{J} \cup \mathcal{Q}^\text{leg}_0 \},
\]

and define a symmetric bilinear form \((\ , \ )\) on \( \mathcal{\hat{L}} \) in accordance with equations 2, 3 and 4. Then \( \mathcal{\hat{L}} \) becomes a symmetric Kac-Moody root lattice and we have a projection

\[
\Xi: \mathcal{\hat{L}} \rightarrow \mathcal{L}
\]
where for $\gamma = \sum_{c \in C} \gamma_c c \in \hat{L}$, the image $\Xi(\gamma) = (\beta_a)_{a \in Q_0}$ is given by
\[
\beta_{[i,j]} = \sum_{\{i = (i,j) \in J \mid j_i = j\}} \gamma_c, \\
\beta_{[i,j,k]} = \gamma_{c_{(i,j,k)}}.
\]

**Proposition 2.10** (Theorem 3.6 in [12]). We have the following.

1. The map $\Xi$ is an isometry, that is, $(\gamma, \gamma') = (\Xi(\gamma), \Xi(\gamma'))$ for any $\gamma, \gamma' \in \hat{L}$.

2. The map $\Xi$ is injective if and only if
\[
\#\{i \in \{0, \ldots, p\} \mid m(i) > 1, i = 0, \ldots, p\} \leq 1.
\]

3. The map $\Xi$ is $W^{mc}$-equivariant, that is, for $\gamma \in \hat{L}$ and $a \in J \cup Q_{0}^{\text{leg}}$, we have
\[
\Xi(s_a(\gamma)) = s_a(\Xi(\gamma)).
\]

This proposition tells us that $\hat{L}$ is a “lift” of $L$ to a Kac-Moody root lattice with the Weyl group $W^{mc}$.

The kernel of $\Xi$ is a big space in general. Thus if we consider the inverse image of an element $\beta \in \hat{L}$, it is convenient to restrict $\Xi$ to some smaller space as follows. Fix $\beta \in \hat{L}$ and set $J_\beta := \{(i,j) \in J \mid \beta_{[i,j]} \neq 0 \text{ for all } i \in I_{\text{irr}}\}$ and $(Q_{0}^{\text{leg}})_\beta := Q_{0}^{\text{leg}} \cap \text{supp}(\beta)$. Then define
\[
(J \cup Q_{0}^{\text{leg}})_\beta := J_\beta \cup (Q_{0}^{\text{leg}})_\beta
\]
and a sublattice and subgroup
\[
\hat{L}_\beta := \sum_{\{a \in (J \cup Q_{0}^{\text{leg}})_\beta\}} \mathbb{Z}c_a, \\
W_{\beta}^{mc} := \{s_a \mid a \in (J \cup Q_{0}^{\text{leg}})_\beta\}.
\]

Denote the set of all positive elements in $\hat{L}_\beta$ by $\hat{L}_\beta^+$. We write the restriction of $\Xi$ on $\hat{L}_\beta$ by $\Xi_\beta$.

2.3.1. Finiteness of spectral types. As we saw in Theorem 2.6, quiver varieties $\mathfrak{M}_\Lambda(Q, \alpha)^{\text{diff}}$ with $\alpha \in \hat{F}$ are fundamental elements under the action of $W^{mc}$. We shall see that a kind of finiteness of the set $\hat{F}$. First let us introduce the shape of $\beta \in \hat{L}$.

**Definition 2.11** (shape). Fix a Kac-Moody root lattice $L = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ and $\alpha = \sum_{i \in I} m_i \alpha_i \in L$. For the Dynkin diagram of the support of $\alpha$, we attach each coefficient $m_i$ of $\alpha$ to the vertex corresponding to $\alpha_i$, then we obtain the diagram with the coefficients, which we call the shape of $\alpha$.

For example, if $\alpha = m_1 \alpha_{i_1} + m_2 \alpha_{i_2} + m_3 \alpha_{i_3} \in L$ with the diagram of the support $\alpha_{i_1} \alpha_{i_2} \alpha_{i_3}$, the diagram with coefficients is $m_1 \alpha_{i_1} m_2 \alpha_{i_2} m_3 \alpha_{i_3}$.

By using this we define shapes of elements in $\hat{L}$ as follows.

**Definition 2.12.** For $\beta \in \hat{L}$, the shape of $\beta$ is the set of shapes of elements in $\Xi_\beta^{-1}(\beta) \subset \hat{L}_\beta$. 
We say that \( m \) is effective if there exists \( B \in \Ht \) such that \( \mathcal{M}(B) \cong \mathcal{M}(Q, \alpha)^{\text{dif}} \neq \emptyset \) and \( m = m_\alpha \). A spectral type \( m = m_\alpha \) is said to be basic if \( \alpha \in \hat{F} \). Also we say that \( m \) is reduced if \( \alpha \) is reduced. We say that \( m \) is fundamental if \( m \) is effective, basic and reduced. By the shape of \( m \), we mean the shape of \( \alpha \).

Then we can show the following finiteness of basic spectral types.

**Theorem 2.16** (Theorem 8 in [14]). Let us fix an integer \( q \in 2\mathbb{Z}_{\leq 0} \). Then there exist only finite number of fundamental spectral types \( m \) satisfying \( \text{id}_m = q \).

Let us see the cases \( q = 0 \) and \( -2 \) for example. The first case is \( q = 0 \).
Theorem 2.17 (Theorem 9 in [14]). Shapes of fundamental spectral types \( m \) satisfying \( \text{id}(m) = 0 \) are one of the following.

\[
\begin{array}{cccccc}
 & & & & & \\
1 & 2 & 3 & 2 & 1 & \\
& & & & & \\
1 & 2 & 3 & 4 & 2 & 1 \\
& & & & & 1
\end{array}
\]

We simply write sets \( \{ x_a \mid a \in \mathbb{Z} \} \) and \( \{ x \} \) by \( x_a (a \in \mathbb{Z}) \) and \( x \), respectively. For the first 4 star shaped graphs, corresponding spectral types are given in Remark 2.18. In the above list of shapes, we omit the spectral types (is isomorphic to one of the Weyl groups of the following types, \( \alpha \).

If \( \mathcal{M}_\lambda(Q, \alpha)^{\text{diff}} \neq \emptyset \) and \( \alpha \in \tilde{F} \) with \( q(\alpha) = 0 \), then by the above list of shapes of \( \alpha \), we can check that \( \alpha \) is invariant under \( W_\alpha^{\text{mc}} \), i.e., \( w(\alpha) = \alpha \) for any \( w \in W_\alpha^{\text{mc}} \). Then

\[
s_a : \mathcal{M}_\lambda(Q, \alpha)^{\text{diff}} \to \mathcal{M}_{r_a(\lambda)}(Q, \alpha)^{\text{diff}}
\]

for each \( a \in (\mathcal{J} \cup \mathcal{Q}^{\text{leg}})_{\alpha} \) defines a \( W_\alpha^{\text{mc}} \)-action on the parameter space

\[
\sum_{a \in (\mathcal{J} \cup \mathcal{Q}^{\text{leg}})_{\alpha}} \mathbb{C}c_a \to \sum_{a \in (\mathcal{J} \cup \mathcal{Q}^{\text{leg}})_{\alpha}} \mathbb{C}c_a
\]

see also Proposition 3.7 in [12]. Here if \( \lambda_\alpha = 0 \), i.e., \( s_a \) on \( \mathcal{M}_\lambda(Q, \alpha)^{\text{diff}} \) is not well-defined, we formally set \( s_a = \text{id} \) and \( r_a = \text{id} \). By the above theorem, \( W_\alpha^{\text{mc}} \) is isomorphic to one of the Weyl groups of the following types,

\[
E_8^{(1)}, E_7^{(1)}, E_6^{(1)}, D_4^{(1)}, A_3^{(1)}, A_2^{(1)}, A_1^{(1)}, A_1^{(1)} \times A_1^{(1)}.
\]

Remark 2.18. In the above list of shapes, we omit the spectral types for star-shaped diagrams. For these cases spectral types are obtained as follows. Consider a shape

\[
\begin{array}{cccc}
n_0 & n_1 & n_2 & \cdots \\
n_{i,1} & n_{i,0} & n_{i,2} & \cdots \\
n_{2,1} & n_{2,0} & n_{2,2} & \cdots \\
& n_{p,0} & n_{p,2} & \cdots \\
& & & \cdots \\
& & & \cdots 
\end{array}
\]

and put \( m_{(i,1)} := n_0 - n_{i,1} \),

\[
m_{(i,j)} := n_{i,j} - n_{i,j+1}, m_{(i,0)} := \sum_{k \leq p} n_{k,1} - n_0 \quad \text{and} \quad m_{(0)} := \sum_{i=0}^p n_{i,1} -
\]
Then the shape corresponds to the following 5 types.

\[ m_{(0,1)}m_{(0,2)} \ldots, m_{(1,1)}m_{(1,2)} \ldots, \ldots, m_{(p,1)}m_{(p,2)} \ldots, \]
\[ m_{(0)}n_0, (m_{(0)}m_{(0,3)} \ldots) \ldots (m_{(p,2)}m_{(p,3)} \ldots), \]
\[ m_{(1,0)}m_{(1,1)} \ldots, (m_{(0,2)}m_{(0,3)} \ldots) \ldots (m_{(i-1,2)} \ldots)(m_{(i+1,2)} \ldots), \]
\[ ((m_{(i,1)}m_{(i,2)} \ldots))(m_{(0,2)}m_{(0,3)} \ldots) \ldots (m_{(i-2,2)} \ldots)(m_{(i+2,2)} \ldots), \]
\[ (n_0)((m_{(0,2)}m_{(0,3)} \ldots) \ldots (m_{(p,2)}m_{(p,3)} \ldots)). \]

Next let us see the case \( q = -2 \).

**Theorem 2.19** (Theorem 10 in [14]). *Shapes of fundamental spectral types \( m \) satisfying \( \text{idx} m = -2 \) are one of the following.*

\[
\begin{align*}
(1) & \quad 1 - a \\
W^{\text{inv}} & = \emptyset
\end{align*}
\]

\[
\begin{align*}
(1) & \quad 1 - a \\
2 - a & \quad (a \in \mathbb{Z})
\end{align*}
\]

\[
\begin{align*}
(1)(1), (1)(1) & \quad (a \in \mathbb{Z})
\end{align*}
\]

\[
\begin{align*}
(1)(11), (1)(11) & \quad (1)(1), (2)(1)
\end{align*}
\]

\[
\begin{align*}
2 - a & \quad 2 - a \\
W^{\text{inv}} & = A_3
\end{align*}
\]

\[
\begin{align*}
2 - a & \quad 2 - a \\
W^{\text{inv}} & = A_1 \times A_1 \times A_1
\end{align*}
\]

\[
\begin{align*}
(2)(2), (1)(111) & \quad (a \in \mathbb{Z})
\end{align*}
\]

\[
\begin{align*}
(2)(2), (1)(111) & \quad (2)(2), (1)(11)
\end{align*}
\]
Here we simply denote the sets \( \{ x_a \mid a \in \mathbb{Z} \} \) and \( \{ y \} \) by \( x_a \) (\( a \in \mathbb{Z} \)) and \( y \), respectively. For the spectral types of the star shaped graphs, see Remark
Here for fundamental spectral types $m = m_\alpha$ 

$$W^{\text{inv}} := \{ s_\alpha | \ s_\alpha (\alpha) = \alpha, \ a \in (\mathcal{F} \cup \mathcal{Q}_0^{(\emptyset)})_\beta \} \subset W^{\text{mc}}_\alpha.$$ 

Plain circles in the Dynkin diagrams correspond to simple roots $c_a$ such that $s_\alpha (\beta) = \beta$ and dotted circles correspond to $s_\alpha (\beta) \neq \beta$.

As well as the case $q = 0$, in the case of $q = -2$, $\mathcal{M}_{\lambda}(Q, \alpha)^{\text{dif}} \neq \emptyset$ with $\alpha \in \mathbb{F}$ such that $q (\alpha) = -2$ has a $W^{\text{inv}}$-action on the parameter space.

$$r_a : \sum_{a \in (\mathcal{J} \cup \mathcal{Q}_0^{(\emptyset)})_\alpha} \mathbb{C} c_a \rightarrow \sum_{a \in (\mathcal{J} \cup \mathcal{Q}_0^{(\emptyset)})_\lambda} \mathbb{C} c_a$$

2.4. Integrable deformations and middle convolutions. We shall discuss symmetries of integrable deformations coming from symmetries of quiver varieties $\mathcal{M}_\lambda(Q, \alpha)^{\text{dif}} \simeq \mathcal{M}(B)$. When $Q$ is a simply-laced quiver and $I_{\text{irr}} = \{0\}$, Boalch gave a formulation of integrable deformations as Hamiltonian systems over quiver varieties and discuss their symmetries in [2]. The following results might be seen as a generalization of his work.

Theorem 2.20 (Yamakawa. Corollary 3.17 in [41], cf. Haroaka-Filipuk [10] and Boalch [4]). Let $(\nabla_t)_{t \in \mathbb{T}}$ be an admissible integrable deformation with the spectral data $(Q, \lambda, \alpha)$ where $\alpha \in \Sigma_\lambda^{\text{dif}}$. Let us take $A(t) \in \prod_{i=0}^p O^{\text{tr}}_{B(i)}$ such that $A(t) \sim \nabla_t$ for $t \in \mathbb{T}$. Suppose that $\lambda$ is fractional. Then for each $i \in \mathcal{I}$, there exists an admissible integrable deformation $(\nabla^i_t)_{t \in \mathbb{T}}$ with the spectral data $(Q, r_i(\lambda), s_i(\alpha))$ such that $\nabla^i_t \sim mc_t(A(t))$ for all $t \in \mathbb{T}$.

Proof. See Corollary 3.17 and (ii) in Remark 3.18 in [41].

Remark 2.21. In the above theorem, we only discussed middle convolutions. Similarly relations between Fourier-Laplace transformations and integrable deformations are also important and found in [4, 41, 38] and [39].

Definition 2.22. We say that an admissible integral deformation $(\nabla_t)_{t \in \mathbb{T}}$ with $(Q, \lambda, \alpha)$ is fundamental when $\alpha \in \mathbb{F} \cap \Sigma_\lambda^{\text{dif}}$ and $\alpha$ is reduced.

Then Theorem 2.20 and Theorem 2.20 show the following.

Theorem 2.23. Let $(\nabla_t)_{t \in \mathbb{T}}$ be an admissible integrable deformation with the spectral data $(Q, \lambda, \alpha)$ where $\alpha \in \Sigma_\lambda^{\text{dif}}$ and $q(\alpha) \leq 0$. Suppose that $\lambda$ is fractional and moreover has a fractional reduction. Then $(\nabla_t)_{t \in \mathbb{T}}$ can be reduced to a fundamental admissible integral deformation by a finite iteration of middle convolutions and additions.

For an admissible integrable deformation $(\nabla_t)_{t \in \mathbb{T}}$ with a spectral data $(Q, \lambda, \alpha)$, we call $m_{\alpha}$ the spectral type of $(\nabla_t)_{t \in \mathbb{T}}$ and also $\lambda$ the spectral parameter.

Theorem 2.24. Let us fix an integer $d \in 2\mathbb{Z}_{>0}$ There exists only finite spectral types of fundamental admissible integrable deformations of dimension $d$.
Proof. This directly follows from Theorems 2.16 and 2.20.

We have the classification of spectral types of admissible deformations of dimension $d = 2$ and 4.

**Theorem 2.25.** Spectral types of basic admissible integrable deformations of dimension $d = 2, 4$ are listed in Theorem 2.17 (resp. Theorem 2.19) for $d = 2$ (resp. $d = 4$). Moreover generic spectral parameters have $W^{mc}$-actions (resp. $W_{inv}$-action) for $d = 2$ (resp. $d = 4$). Here we say that a spectral parameter $\lambda \in \mathbb{C}^{Q_0}$ is generic when for any sequence $a_1, \ldots, a_l \in J \cup Q_0^{leg}$, $\lambda' = r_{a_1} \circ \cdots \circ r_{a_2} \circ r_{a_1}(\lambda)$ satisfies non-resonance condition.

In Theorems 2.17 and 2.19, we saw the cases where several different spectral types correspond to a same shape. For example,

\[
\begin{align*}
&\begin{array}{c}
\circ \\
\circ \\
\circ \\
\end{array} \\
&((1)(1))(1)(1)(1)) \\
&(1)(1), 11, 11, 11 \\
&(1)(1)(1), 21, 21
\end{align*}
\]

Then we identify two spectral types of dimension 2 or 4 if they have the same shape. This identification might be justified by Boalch’s simply-laced isomonodromy theory [4]. By Theorems 2.17 and 2.19, we can check that the identification happens only when the corresponding quiver $Q$ in the spectral data $(Q, \lambda, \alpha)$ is simply-laced and $I_{irr} = \{0\}$. Thus integrable deformations with spectral types of the same shape are isomorphic by Theorem 1.1 in [4] in the sense of the paper [4].

In [21], Kawakami, Nakamura, and Sakai considered isomonodromic deformations of linear differential equations which obtained by the confluent process from Fuchsian differential equations with 4 accessory parameters classified by Oshima in [31]. And they gave explicit Hamiltonian equations of the isomonodromic deformations after Sakai’s computation in the Fuchsian cases (see [33]). Then under the above identification of spectral types, Theorem 2.19 shows that the list spectral types appeared in their paper [21] is the complete list of fundamental spectral types of dimension 4.

**Theorem 2.26.** Under the above identification of spectral types, if we exclude the spectral types corresponding to differential equations which have only 3 regular singular points and no other singularities, then the list of spectral types appeared in Section 1.3 of [21] is complete list of spectral types of integrable deformations of dimension 4.

Proof. Compare the list of spectral types in [21] and the shapes listed in Theorem 2.19. □

Moreover Theorem 2.25 assures that integrable deformations considered in [21] have $W^{inv}$-symmetries listed in Theorem 2.19.

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