Determinantal Point Processes
and Fermionic Fock Space

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In this note, we construct a canonical embedding of the space $L^2$ over a determinantal point process to the fermionic Fock space. Equivalently, we show that a determinantal process is the spectral measure for some commutative group of Gaussian operators in the fermionic Fock space.

1. Determinantal processes

1.1. Determinantal processes. Let $K(x, y)$ be a function of two real variables. Consider the integral operator

$$K f(x) = \int_{\mathbb{R}} K(x, y) f(y) \, dy$$

in $L^2(\mathbb{R})$.

We assume that $K(x, y)$ satisfies the following conditions

1. $K(x, y) = K(y, x)$, i.e. $K^* = K$.
2. $0 \leq K \leq 1$, i.e.,

$$\langle K f, f \rangle_{L^2} \geq 0, \quad \langle (1 - K) f, f \rangle_{L^2} \geq 0$$

for all $f \in L^2(\mathbb{R})$.
3. the function $K(x, y)$ is $C^\infty$-smooth.

Denote by $\Omega$ space of all countable (or finite) subsets in $\mathbb{R}$. We call these sets by configurations.

We define a probability measure on $\Omega$ by the following rule.

Let $x_1, \ldots, x_n$ be points of $\mathbb{R}$. Consider infinitesimal small intervals $[x_j, x_j + dx_j]$ near these points. Denote by $\Xi = \Xi(x_j, dx_j)$ the following event (set): each interval $[x_j, x_j + dx_j]$ contains a point of $\omega \in \Omega$. We require that the probability of the event $\Xi = \Xi(x_j, dx_j)$ is

$$\left\{ \det_{1 \leq i \leq n, 1 \leq j \leq n} K(x_i, x_j) \right\} dx_1 \ldots dx_n.$$ 

The self-consistence of this definition is not self-obvious, however it is self-consistent, see the comprehensive Soshnikov’s survey [10].

We denote by $\mu(\omega) = \mu_K(\omega)$ the measure obtained in this way.

1.2. Examples. References. The first process of this kind was discovered in the famous work of Dyson [5] in 1962, in his case

$$K(x, y) = \frac{\sin(x - y)}{\pi(x - y)}$$

The corresponding process is named the sine-process (this process is one of possible limit distributions of eigenvalues of unitary $N \times N$ unitary matrices as $N \to \infty$).
Many other processes of this kind were discovered later, see a collection of processes having natural origins in [10].

Recently, A. Borodin and G. Olshanski in their works on infinite-dimensional harmonic analysis discovered a collection of new processes of this kind, see [2]-[4] (but they do not satisfy the conditions that are necessary for construction of this paper).

Lytvynov [7] earlier realized some determinantal processes as spectral measures for quasi-free states for canonical commutation relations (he considered convolution type kernels $K(x-y)$).

In many interesting cases $\mathcal{K}$ is an orthogonal projector in $L^2$. Obviously, the Dyson sine-kernel satisfies this condition.

1.3. Multiplicative functionals. Let $a(x)$ be a $C^\infty$-smooth function with a compact support on $\mathbb{R}$. Denote by $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ the operator

$$Af(x) = (1 + a(x))f(x). \quad (1.1)$$

We also consider the functional $\Psi[a](\omega)$ on $\Omega$ defined by

$$\Psi_a(\omega) = \prod_{x_j \in \omega} (1 + a(x_j)).$$

Proposition.

$$\int_{\Omega} \Psi_a(\omega) \, d\mu(\omega) = \det(1 + \mathcal{K}(A - 1)). \quad (1.2)$$

Lemma. The operator $\mathcal{K}(A - 1)$ is contained in the trace class.

Proof. Consider the operator

$$(\mathcal{K}(A - 1))^* (\mathcal{K}(A - 1)) = (A^* - 1)\mathcal{K}^2(A - 1).$$

We have

$$(A^* - 1)\mathcal{K}^2(A - 1) + (A^* - 1)(\mathcal{K} - \mathcal{K}^2)(A - 1) = (A^* - 1)\mathcal{K}(A - 1). \quad (1.3)$$

The kernel of the operator in the right hand side is

$$S(x, y) = \overline{a(x)} K(x, y) a(y) \quad (1.4)$$

By our last assumption, the integral is a smooth function. The kernel $S(x, y)$ is a compactly supported smooth function and hence it is a kernel of an operator with rapidly decreasing singular values.

The both summands in the left-hand side of (1.3) are positive operators. Hence the eigenvalues of the first summands are rapidly decrease. Hence the Fredholm determinant in (1.2) is well defined. Obviously, our conditions for $\mathcal{K}$ and $a(x)$ are surplus (for instance, $a(x)$ can be a piece-wise smooth function with jumps), but some restrictions in this place are necessary.

Lemma. Linear combinations of functions $\Psi_a$ are dense in $L^2(\Omega)$.
Proof. Let $X_1, \ldots, X_l$ be disjoint segments on $\mathbb{R}$. Denote by $\mathcal{E}(X_1; \alpha_1| \ldots | X_l, \alpha_l) \subset \Omega$ the following event: $X_j$ contains $\alpha_j$ points of $\omega \in \Omega$. Denote by $\chi_{\alpha_1, \ldots, \alpha_l}(\omega)$ the indicator of this set, i.e.,

$$
\chi_{\alpha_1, \ldots, \alpha_l}(\omega) = \begin{cases}
1, & \text{if } \omega \in \mathcal{E}(X_1; \alpha_1| \ldots | X_l, \alpha_l) \\
0, & \text{otherwise}
\end{cases}
$$

Fix complex $r_j$. Let

$$
a(x) = a[r_1, \ldots, r_l](x) = \begin{cases}
r_j - 1, & x \in X_j \\
0, & x \notin \bigcap X_j
\end{cases}
$$

Then

$$
\Psi_a[r_1, \ldots, r_l](\omega) = \sum_{\alpha_1, \ldots, \alpha_l} r_1^{\alpha_1} \ldots r_l^{\alpha_l} \chi_{\alpha_1, \ldots, \alpha_l}(\omega)
$$

Evidently, function $\chi_{\alpha_1, \ldots, \alpha_l}(\omega)$ can be obtained as limits of linear combinations of $\Psi_a[r_1, \ldots, r_l](\omega)$ (for instance, by differentiation in parameters $r_j$).

1.4. Sketch of proof of (1.2). Let us explain how to prove (1.2). First, assume that we have not $\mathbb{R}$ but a finite set $R = \{1, 2, \ldots, n\}$. Let $K = K(i, j)$ be a symmetric function on $R \times R$ (i.e., $K$ is a $n \times n$-matrix).

Denote by $\Omega$ the space of all the finite subsets in $R$. Let $I \subset R$. Let $\Xi(I) \subset \Omega$ be set (‘event’) of all $J \supset I$. We assume that probability $p(\Xi(I))$ is

$$
p(I) = \sum_{J \supset I} (-1)^{|J \setminus I|} \det_{l, m \in I} K(l, m)
$$

where $|J \setminus I|$ denotes the number of elements in $J \setminus I$.

If $K \geq 0$, $1 - K \geq 0$; this defines a probability measure (otherwise, probabilities can be negative).

Now let $a = a(j)$ be a function on $R$. Define a functional on $\Omega$ by

$$
\Psi_a(I) = \prod_{j \in I} (1 + a(j))
$$

Find the mean of this function over $\Omega$,

$$
\mathbb{M} \Psi_a = \sum_{I \subset R} \left\{ \prod_{j \in I} (1 + a(j)) \sum_{J \supset I} (-1)^{|J \setminus I|} \det_{l, m \in J} K(l, m) \right\} = \\
= \sum_{J \supset I} \left\{ \det_{l, m \in J} K(l, m) \cdot \sum_{I \subset J} (-1)^{|J \setminus I|} \prod_{j \in I} (1 + a(j)) \right\} = \\
= \sum_{I \subset R} \left\{ \det_{l, m \in I} K(l, m) \cdot \prod_{j \in I} a(j) \right\} = \det_{p, q \in R} (1 + K(p, q)a(q))
$$
The continuous case is the same, we only must consider Fredholm determinants and follow convergence.

1.5. Coherent states. The formula (1.2) implies the following identity

\[ \langle \Psi_1, \Psi_2 \rangle_\mathcal{H}^2 = \det(1 + \mathcal{K}[AA^* - 1]) \]  

(1.5)

2. Fermionic Fock space

2.1. The space of semiinfinite forms. Let \( H = V \oplus W \) be a Hilbert space. Denote by \( \Pi \) the projector to the subspace \( W \). Let \( e_0, e_{-1}, e_{-2}, \ldots \) be an orthonormal basis in \( V \), let \( e_1, e_2, \ldots \) be an orthonormal basis in \( V \).

A good basic monomial is a product having the form

\[ e_{k_1} \wedge e_{k_2} \wedge \ldots, \quad k_1 < k_2 < \ldots \]

such that \( k_j = j \) starting some place.

Example. The vectors \( e_1 \wedge e_2 \wedge \ldots \) and \( e_{-6} \wedge e_2 \wedge e_3 \ldots \) are good monomials, and \( e_0 \wedge e_1 \wedge e_2 \ldots \) is a bad monomial.

We denote by \( \Lambda(H; \Pi) \) the Hilbert space, whose orthonormal basis consists of good monomials (see [1], [6]).

Remark. The space \( \Lambda \) is the so-called space of seminfinite forms. It is a subspace in the fermionic Fock space. To obtain the whole Fock space in the usual sense, we must allow \( k_j = j + \alpha \) starting some place (\( \alpha \) is a constant depending on a monomial), see [8], IV.1.

2.2. Group of symmetries of \( \Lambda \). Denote by \( G = G(H; \Pi) \) the group of block matrices

\[ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : V \oplus W \rightarrow V \oplus W \]  

(2.1)

satisfying the conditions

1*. \( g \) is a bounded operator
2*. \( b, c \) are Hilbert-Schmidt operators
3*. \( d - 1 \) is a trace class operator.

This group acts in the space \( \Lambda(H; \Pi) \) by the usual change of variables (\( e_j \mapsto ge_j \)). We denote these operators by \( \lambda(g) \). Obviously,

\[ \lambda(g_1)\lambda(g_2) = \lambda(g_1g_2) \]  

(2.2)

If \( g \) is unitary operator, then \( \lambda(g) \) also is unitary. Otherwise, \( \lambda(g) \) can be unbounded, nevertheless all the operators \( \lambda(g) \) have a common invariant domain of definiteness, see [8], IV.1.

Remark. The condition 3* can be omitted, but we must require the Fredholm index of the operator \( d - 1 \) to be zero. Under this condition the operators \( \lambda(g) \) can be defined, but their definition is not obvious (since matrix elements of \( \lambda(g) \) in this case are divergent series). Also the identity (2.2) breaks down, and we obtain a projective (not a linear representation), see [8].
2.3. Coherent states. Denote by $\Upsilon$ the vacuum vector

$$\Upsilon = e_1 \wedge e_2 \wedge e_3 \cdots \in \bigwedge$$

Write the matrix $g = \{g_{kl}\} \in G(H; \Pi)$ as a usual infinite matrix in the basis $e_j$.

Simultaneously, we preserve the block notation

Consider the vector

$$\lambda(g) \Upsilon = \bigwedge_{j=1}^{\infty} \left( \sum g_{jk} e_k \right)$$

Obviously (see, for instance, [9]),

$$\langle \lambda(g) \Upsilon, \lambda(\tilde{g}) \Upsilon \rangle_\Lambda = \det(e^c \tilde{c}^* + d^d \tilde{d}^*)$$

(2.3)

Denote by $\Pi : H \rightarrow H$ the orthogonal projector to the subspace $W$. Then the last formula can be written in the form

$$\langle \lambda(g) \Upsilon, \lambda(\tilde{g}) \Upsilon \rangle_\Lambda = \det(1 + \Pi(\tilde{g} \tilde{g}^* - 1))$$

(2.4)

3. Correspondence. I.

First, let the operator $\mathcal{K} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ be a projector, $\mathcal{K}^2 = \mathcal{K}$.

3.1. Multiplications by functions. Let $H = L^2(\mathbb{R})$. Let $\Pi = \mathcal{K}$, then $W$ is the image of $\mathcal{K}$, $V$ is the kernel of $\mathcal{K}$.

Let $a(x)$ be a smooth function on $\mathbb{R}$ with a compact support (as above). Let $A : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ be the same operator

$$Af(x) = (1 + a(x))f(x)$$

as above.

**Proposition.** $A \in G(L^2; \mathcal{K})$.

**Proof.** It is sufficient to show that $\mathcal{K}(A - 1)$, $(A - 1)\mathcal{K}$ are trace class operators. This was shown in 1.3. \qed

**Remark.** Let Diff be the group of compactly supported diffeomorphisms of $\mathbb{R}$. The operators

$$T(q)f(x) = f(q(x))q'(x), \quad q \in \text{Diff}$$

also are contained in $G(L^2; \mathcal{K})$. Hence the group Diff also acts in $\bigwedge(L^2, \mathcal{K})$.

3.2. Coherent states. Denote

$$\varphi_a = \lambda(A) \Upsilon \in \bigwedge(L^2(\mathbb{R}), \mathcal{K})$$

By (2.4), we have

$$\langle \varphi_a, \varphi_{\tilde{a}} \rangle_{\bigwedge(L^2, \mathcal{K})} = \det(1 + \mathcal{K}(A \tilde{A}^* - 1))$$

(3.1)
We observe that the inner products (1.5) and (3.1) coincide. The system $\Psi_a$ spans $L^2(\Omega)$ and hence we obtain a canonical isometric embedding

$$U : L^2(\Omega) \to \bigwedge (L^2(\mathbb{R}; \mathcal{X})), \quad U\Psi_a = \psi_a$$

Apparently, this operator is one-to-one correspondence, but I do not know a proof.

4. Correspondence. II.

Now $K(x, y)$ is an arbitrary kernel satisfying the conditions given in 1.1.

4.1. Multiplications by functions. Consider the space $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$.

Consider the projector $\mathcal{L}$ in this space given by

$$\mathcal{L} = \begin{pmatrix} 1 - \mathcal{K} & \sqrt{\mathcal{K} - \mathcal{K}^2} \\ \sqrt{\mathcal{K} - \mathcal{K}^2} & \mathcal{K} \end{pmatrix} : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$$

Let a function $a(x)$ and the operator $A : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the same as above. Denote by $g = g_a$ the operator

$$g = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : L^2(\mathbb{R}) \oplus L^2(\mathbb{R}) \to L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$$

Proposition. $g \in G(L^2 \oplus L^2, \mathcal{L})$.

Proof. It is sufficient to show that the operator $\mathcal{L}(g - 1)$ is a trace class operator. For this purpose, we evaluate

$$\mathcal{L}(g - 1)^* \mathcal{L}(g - 1) = (g^* - 1)\mathcal{L}(g - 1) = \begin{pmatrix} 0 & 0 \\ 0 & (A^* - 1)\mathcal{K}(A - 1) \end{pmatrix}$$

Again, we obtain an operator with the kernel (1.4). \hfill \Box

4.2. Coherent states. Denote

$$\varphi_a = \wedge (g_a) \mathbf{y}$$

By (2.3),

$$\langle \varphi_a, \varphi_a \rangle_{\wedge (L^2 \oplus L^2, \mathcal{L})} = \det(1 + \mathcal{L}(g_A g_{A^*} - 1)) = \det \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 - \mathcal{K} & \sqrt{\mathcal{K} - \mathcal{K}^2} \\ \sqrt{\mathcal{K} - \mathcal{K}^2} & \mathcal{K} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A A^* - 1 \end{pmatrix} \right\} = \det \{ 1 + \mathcal{K}(A A^* - 1) \}$$

Finally, we have

$$\langle \varphi_a, \varphi_a \rangle_{\wedge (L^2 \oplus L^2, \mathcal{L})} = \det \{ 1 + \mathcal{K}(A A^* - 1) \}$$

(4.1)

Again, the inner products (1.5 and (4.1) coincide. Thus we obtain a canonical isometric embedding

$$U : L^2(\Omega) \to \bigwedge (L^2(\mathbb{R}) \oplus L^2(\mathbb{R}), \mathcal{L}), \quad U\Psi_a = \psi_a$$

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