Universal Aspects of QCD-like Theories

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Abstract

In these lectures I review some basic examples of how the concepts of universality and scaling can be used to study aspects of the chiral and the deconfinement transition, if not in QCD directly but in QCD-like theories. As an example for flavor dynamics I discuss a quark-hadron model to describe the phase diagram of two-color QCD with the functional renormalization group. Universal aspects of deconfinement are illustrated mainly in the \(2+1\) dimensional SU(\(N\)) gauge theories with second order transition where many exact results from spin models can be exploited.

Keywords: strongly interacting matter, two-color QCD, functional RG, universality, scaling, duality, deconfinement

1. Introduction

Strongly interacting matter fuels the stars and makes up almost the entire mass of the luminous universe. The underlying theory of quarks and gluons, Quantum Chromodynamics (QCD), completely specifies the interactions. However, these are so complex and nonlinear that they have yet to be fully understood. Indeed, it is these strong interactions which under normal conditions confine quarks and gluons into the interior of hadrons. Understanding the generation of their masses, the confinement of quarks and gluons, the different phases of QCD at extreme temperatures or densities and the transitions between them are some of the great challenges in physics. At temperatures in familiar units close to \(2 \times 10^{12} \text{ K} \sim 170 \text{ MeV}\), for example, confined hadronic matter undergoes a transition into an unconfined state which is a nearly perfect fluid, the quark-gluon plasma. Degenerate neutron matter presumably exists in neutron stars at similarly extreme pressures of above \(1.6 \times 10^{33} \text{ Pa} \sim 10 \text{ MeV/fm}^3\). Compared to the conditions around us, both are of course beyond any imagination. The conditions in the finite temperature deconfinement transition are almost 10 orders of magnitude hotter than the surface of the Earth and still more than 5 orders of magnitude hotter than the center of the Sun. They bring us back in the evolution of the early universe within 10 \(\mu\)s after the Big Bang, and they are being recreated in the heavy-ion collision experiments at RHIC and LHC [1]. The pressure inside neutron stars for comparison corresponds to the weight of about 100 solar masses pressing on one square meter. Such conditions at comparatively moderate temperatures require heavy-ion collisions at much lower beam energies, corresponding to center-of-mass energies per colliding nucleon pair in the GeV range instead of TeV at the LHC. This energy range was pioneered at the AGS and it will be the goal of the HADES and CBM experiments at FAIR where penetrating probes such as lepton pairs will be used to study the ultradense conditions in the early stages of the collisions, in particular [2, 3].

Nevertheless, the same characteristic features are being discussed in the QCD phase diagram as those, e.g., in the phase diagram of water under more normal conditions of pressure and temperature. Conventionally one plots the QCD phase diagram in the plane of temperature over baryon chemical potential. An unconventionally conventional pressure versus temperature phase diagram for strongly interacting matter is shown in the...
sketch of Fig. 1. A number of color superconducting and other ordered condensed matter phases are expected to exist in the high pressure region perhaps comparable to the sixteen or so known crystalline phases of water. There is the liquid-gas transition to nuclear matter with a critical endpoint at a temperature of around 15 MeV, and just as in the phase diagram of water there may be a second critical endpoint at a higher pressure where the conjectured first-order line for chiral symmetry restoration ends. But also the possibility of an at least approximate triple point in the QCD phase diagram has been discussed [4], which is a point of three-phase coexistence as in water at the low pressure end of the freezing temperature around 610 Pa.

In order to understand these main characteristic features in the phase diagram of strongly interacting matter, it has proven to be very useful to deform QCD by not only varying the individual quarks' masses but also the numbers of their different flavors and colors. Such deformations are particularly useful when they lead to second-order phase transitions. Then we can apply the powerful concepts of universality, scaling and finite-size scaling which provide by now standard tools that are straightforward applications of the renormalization group from statistical physics. The classic example is to consider chiral symmetry restoration with only two flavors of (nearly) massless quarks. In the chiral limit, the low temperature phase is characterized by chiral symmetry breaking with the quark condensate \langle \bar{q}q \rangle_T as the order parameter analogous to the spontaneous magnetization in a ferromagnet. Just as the magnetization, the condensate melts with temperature due to thermal fluctuations. For low temperatures this is very well described by chiral effective field theory [5] based on a non-linear realization of chiral symmetry in terms of the (would-be) Goldstone bosons which is determined entirely by the geometry of the coset \( G/H \) or vacuum manifold of the symmetry breaking \( G \rightarrow H \). With massless quarks at two-loop, for example,

\[
\langle \bar{q}q \rangle_T = \langle \bar{q}q \rangle_T(0) \left[ 1 - \frac{T^2}{8f^2} - \frac{1}{6} \left( \frac{T^2}{f^2} \right)^2 + O(T^6) \right].
\]  

As temperature increases, however, such asymptotic expansions necessarily break down eventually. In contrast, the restoration of chiral symmetry at the analogue of the Curie temperature in the ferromagnet is very well described by a linear sigma model. The global chiral symmetry is \( G = SU(2)_R \times SU(2)_L \), corresponding to independent flavor rotations of the right and left-handed components of the two massless quarks. This symmetry is locally the same as \( SO(4) \), the group of rotations in 4 Euclidean dimensions. It is broken spontaneously in the ordered phase down to the diagonal, vector-like isospin symmetry \( H = SU(2)_V \simeq SO(3) \). Therefore, scaling and universality predict that the singularities of the thermodynamic observables at this critical point, the endpoint of the first-order line in the plane of external field and temperature, are all described by the critical exponents in the class of the O(4) Heisenberg ferromagnet in three dimensions [7]. In particular, with the analogue of the external magnetic field being the explicitly symmetry breaking quark mass \( m_q \), and \( t = T/T_c - 1 \) the reduced temperature, this implies that

\[
\langle \bar{q}q \rangle_T \sim (-t)^\beta, \quad \text{for } t \rightarrow 0^- \text{ at } m_q = 0, \quad \text{and}
\]

\[
\langle \bar{q}q \rangle_T \sim m_q^{1/\delta}, \quad \text{for } m_q \rightarrow 0, \quad \text{where}
\]

\[
\beta = 0.38(1), \quad \delta = 4.82(1).
\]

Scaling and hyperscaling relations which hold below the upper critical dimension \( d = 4 \) then determine all other critical exponents, which is referred to as two-exponent scaling. More generally, the behavior of the thermodynamical observables in the vicinity of the critical point is governed by universal scaling functions. For example, the magnetic equation of state here is of the form

\[
\langle \bar{q}q \rangle_T \sim m_q^{1/\delta} E(t m_q^{1/\delta}),
\]

with \( E(0) = 1 \) and \( E(y) \sim (-y)^\gamma \) for \( y \rightarrow -\infty \) [8].

The study of this O(4) universality and scaling in the quark-meson model with the functional renormalization group (FRG) by now also has a long history [9, 10]. There is recent evidence, however, that the true scaling window might actually be very small, and that it might not include the region of physical pion masses in which only an apparent scaling may have been observed [11]. Finite volume effects were also investigated in the two-
flavor quark-meson model and found to be under very good control with finite-size scaling [11, 12].

When a third, strange quark is included, the situation changes depending on its mass. With three massless or nearly massless quarks, the transition must be of first order because of the axial $U_A(1)$ anomaly [13]. Near its physical value, there is nowadays good evidence from lattice simulations at imaginary chemical potential, with nearly massless quarks, the transition must be of first order. With three massless or flavor quark-meson model and found to be under very good control with finite-size scaling [11, 12].

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conveniently accommodate SU(3) as a subgroup. All its representations are real, so there is no fermion-sign problem even for a single flavor and lattice simulations at finite baryon density are possible [19]. Moreover, if the $G_2$ gauge symmetry can be broken down to SU(3) via the Higgs mechanism in a controlled way, one can study how the sign problem gradually reemerges.

We should eventually be able to completely understand the phase diagrams of QCD’s closest relatives without the fermion-sign problem, and this should be a worthwhile exercise also for QCD itself. QCD-like theories can be used to study universal aspects but also as benchmarks for functional methods which work for QCD as well but which rely on truncations or model assumptions. While two-color QCD with its bosonic baryons has been studied extensively in the past, as will be reviewed in the next section, studies of the phase diagram of $G_2$ gauge theory are still in their infancy. They will be an important next step in understanding a sign-problem-free variant of QCD with fermionic baryons.

The other special role that is being played by the center of the gauge group is via the global center symmetry and its spontaneous breaking in the deconfinement transition of the pure gauge theory. Again, if we change the number of colors or the dimension of space from three to two, this transition becomes second order and we can gain a very precise understanding of it from universality, scaling and finite-size scaling. This will be discussed in Section 3.

2. Quark-meson-diquark model of two-color QCD

Quantum Chromodynamics with two colors (QC$_2$D) has been well studied for many years within chiral effective field theory and random matrix theory [20–28], in lattice simulations [29–36], and the Nambu–Jona-Lasinio model [37–46]. In this section we focus on the (Polyakov-)quark-meson-diquark model for studying the phase diagram of QC$_2$D with the functional renormalization group [47], including fluctuations due to collective baryonic excitations.

The most important differences between two and three colors both follow from the property of the SU(2) gauge group of QC$_2$D that its representations are either real or pseudo-real. This leads to an antunitary symmetry in the Dirac operator [21]. As one consequence, the fermion determinant remains real for non-vanishing baryon or quark chemical potential, $\mu \neq 0$, as it does for adjoint quarks in any-color QCD, or in the $G_2$ gauge theory with fundamental fermions. Thus, at least for an even number of degenerate fundamental quark flavors in QC$_2$D there is no fermion-sign problem and the phase diagram is amenable to Monte-Carlo simulations.

Another consequence of the reality or pseudo-reality of the quarks’ color representation is a Pauli-Gürsey symmetry which allows to combine quarks and charge-conjugated antiquarks into enlarged flavor multiplets. As a result, for vanishing chemical potential and quark mass, $\mu = m_q = 0$, the usual SU($N_f$) × SU($N_f$) × U(1)$_B$ chiral and baryon number symmetries are replaced by an extended SU(2$N_f$) flavor symmetry. A similarly extended flavor symmetry is also known from (even numbers of) fermions in 2 + 1 dimensions with U(1) or SU($N$) gauge fields with $N \geq 3$. There it is due to the fact that there is no physical helicity and that the 4-dimensional spinor representation of Dirac fermions is reducible in the 3-dimensional space-time. The resulting extended U(2$N_f$) flavor symmetry is relevant for the expected semimetal-insulator transition in QED$_3$ with not too many flavors, estimated for $N_f \lesssim 4$, or for the description of the electronic excitations around the Dirac points of graphene at half-filling which are also described by $N_f = 2$ such 4-spinors of 3-dimensional Dirac fermions in a single layer or $N_f = 4$ in a double layer, for example [48]. With (pseudo-)real gauge fields such as SU(2) or $G_2$ in 3 dimensions one obtains an even larger extended flavor symmetry [24].

As usual, the extended flavor symmetry of our 4-dimensional QCD-like theories is (spontaneously) broken by a (dynamical) Dirac mass. The breaking patterns are somewhat different from the usual, however. In the pseudo-real case of the fundamental two-color quarks the (non-anomalous) extended SU(2$N_f$) gets broken down to the (2$N_f + 1$)$N_f$ dimensional compact symplectic group Sp($N_f$). In the real cases of adjoint SU($N$) or fundamental $G_2$ quarks, for example, the corresponding breaking would be SU(2$N_f$) → Spin(2$N_f$), the double cover of the proper rotation group SO(2$N_f$).

For $N_f = 2$ the extended SU(4) flavor symmetry group and its Sp(2) subgroup combining the isospin and baryon number symmetries of QC$_2$D are locally isomorphic to the rotation groups SO(6) and SO(5), respectively. The coset is given by $S^3$, the unit sphere in six dimensions, and a spontaneously generated Dirac mass will lead to five instead of the usual three Goldstone bosons in this case, the three pions plus a scalar diquark-antidiquark pair.

Moreover, for $N_f = 2$ these color-singlet scalar diquarks play a dual role, as would-be-Goldstone bosons and bosonic baryons at the same time. While this thus represents the perhaps most important difference as compared to the real world, it also makes it much easier to investigate the effects of baryonic degrees of free-
dom on the phase diagram in functional approaches. In that sense the quark-meson-diquark (QMD) model can be considered as a first step towards their inclusion in a ‘quark-meson-baryon’ model for real QCD with $N_c = 3$.

For the same reason the QMD model of QC$_2$D provides a relativistic analogue of the BEC-BCS crossover in ultracold fermionic quantum-gases, which has also been described successfully with functional renormalization group methods [49–51]. In contrast to non-relativistic models of the BEC-BCS crossover, an interesting additional constraint thereby arises from the Silver Blaze property [52]: When a relativistic chemical potential $\mu$ is coupled to degrees of freedom with a mass gap $\Delta$, as temperature approaches zero, the partition function and hence thermodynamic observables must actually become independent of the chemical potential as long as $\mu < \Delta$. In general, it is not guaranteed that this constraint is automatically satisfied in non-perturbative approaches such as functional renormalization group studies which rely on truncations, but it is yet another example of valuable extra information to devise intelligent truncation schemes [47].

One main virtue of the QMD model, however, is to explicitly demonstrate the impact of baryonic degrees of freedom on the phase diagram by comparing it with the corresponding purely mesonic model, as representative of typical three-color QCD model calculations. For this comparison it is more appropriate to think of the vacuum diquark mass as the baryon mass $m_B$ rather than the pion mass $m_\pi$. In QC$_2$D with its extended flavor symmetry they are the same, but the essential aspect of this assignment is that a continuous phase transition at zero temperature occurs at a critical quark chemical potential $\mu_c = m_B/N_c$. Except for the scale separation between $m_\pi/2$ and $m_B/3$ in the real world, this transition is then to be compared to the liquid-gas transition of nuclear matter in QCD with three colors which is of first order, involves the binding energy, and thus occurs somewhat below $\mu = m_B/N_c$.

As temperature increases the liquid gas transition ends, turning into a crossover with continuously varying but nevertheless probably still relatively abruptly increasing baryon density along some narrow region. This rapid increase is generally expected to lead to the strong chemical-potential dependence of the chemical freeze-out line observed in heavy ion collisions at center-of-mass energies below about 10 GeV per nucleon pair, the baryonic freeze-out [53, 54]. One might conclude that the phase transition line for diquark condensation, where a rapidly increasing baryon density develops, would be the origin of a corresponding baryonic freeze-out line in two-color QCD, with $N_c = N_f = 2$ arguably not necessarily further from reality than the large $N_c$ limits. As in the latter, one might then even identify a two-color version of quarkyonic matter [4, 17, 35, 42].

Finally, it is worth noting that the model, the functional renormalization group equations and the techniques to solve them have a broad scope of applications beyond two-color QCD. One example is QCD with two light flavors at finite isospin chemical potential, which has been studied with the NJL model in mean-field plus random phase approximation (RPA) [55, 56]. There is a precise equivalence between the corresponding quark-meson model with isospin chemical potential and the quark-meson-diquark model of two-color QCD discussed here. Besides changing $N_c$ this merely involves reducing the number of would-be-Goldstone bosons from five to three again, retaining only one of the three degenerate pions for the neutral one, and reinterpretting the diquark-antidiquark pair of QC$_2$D as the charged pions of QCD with isospin chemical potential [57]. Similar models are also studied in the context of color superconductivity [58–60]. The capacity to numerically solve functional renormalization group equations on higher dimensional grids in field space is generally useful for competing symmetries, as in a quark-meson model study of the axial anomaly with scale dependent ‘t Hooft couplings, for example [61].

### 2.1. Extended flavor symmetry and model Lagrangian

As all half-odd integer representations of SU(2), its fundamental representation is pseudo-real because it is isomorphic to its complex conjugate representation with the isometry given explicitly by $S = i\sigma_2$, $S^2 = -1$. Therefore, charge conjugation of the gauge fields in QC$_2$D can be undone by the constant SU(2) gauge transformation $S = i\sigma_2$. With $T^a = \sigma^a/2$ for the fundamental color generators in SU(2), one has

$$T^a T^a = T^{a*} = -ST^a S^{-1}. \tag{4}$$

We will reserve $\sigma_1$, $(\tau_i)$ for the Pauli matrices in spinor (flavor) space from now on. Together with the charge conjugation matrix $C$ in spinor space, likewise with $C^2 = -1$, and complex conjugation denoted by $K$ one then defines an antiunitary symmetry $T = SCK$ with $T^2 = +1$ as time-reversal invariance in quantum mechanics which leaves the Dirac operator invariant.

For comparison, the irreducible representations of the proper rotations, as the adjoint representations of all SU($N$), are examples of real representations with hermitian generators that satisfy

$$T^a T^a = T^{a*} = -T^a. \tag{5}$$
More generally this corresponds to an isometry $S$ for complex conjugation as in (4) but now with $S^2 = 1$. For all real color representations, as those of the adjoint groups $SU(N)/Z_N$ or the exceptional Lie groups mentioned in the introduction such as $G_2$, one thus has $T^2 = -1$, correspondingly. The Dirac operator then has an antiunitary symplectic symmetry which results in a two-fold degeneracy of its eigenstates and a positive fermion determinant even for a single flavor. This leads to the classification of the Dirac operator by the Dyson index $\beta$ of random matrix theory [20, 21], with $\beta = 1$ for fermions in pseudo-real color representations corresponding to the Gaussian orthogonal ensemble, $\beta = 4$ in real color representations corresponding to the Gaussian symplectic, and $\beta = 2$ corresponding to the Gaussian unitary ensemble otherwise. The Dirac operator in the fundamental color representation of $QC_2D$ (in the $N$-ality $N/2$ representations of the $SU(N)'s$ with even $N$) falls in the first class ($\beta = 1$). Those of $G_2$, $F_4$ and $E_8$ (and the $N$-ality zero representations of $SU(N)$) fall in the second ($\beta = 4$).

Following [21], let’s start from the standard kinetic part of the Euclidean $QC_2D$ Lagrangian, in the chiral basis,

$$L_{\text{kin}} = \bar{\psi}D\psi\Psi = \bar{\psi}^L_i\sigma_\mu D_\mu \psi^L_i - \bar{\psi}^R_i\sigma_\mu D_\mu \psi^R_i,$$  

with hermitian $\gamma$-matrices, $\sigma_\mu = (-i, \vec{\sigma})$, and $\psi_{R/L}$ as independent Grassmann variables with $\psi^*_\mu \equiv \psi^T_{R/L}$. The covariant derivative is $D_\mu = \partial_\mu + iA_\mu$, and the coupling is absorbed in the gauge fields $A_\mu = A_\mu^a T^a$. The two terms in (6) get interchanged under the antiunitary symmetry $T$. One can apply a corresponding transformation to only one of the two terms to change its sign, however. Using $(-i\sigma_2)$ for the chiral $R$-component of the charge rotation matrix $C$ in the second term, say, by changing variables to $\psi_R = -i\sigma_2\bar{\psi}^L_i\Psi$ and $\bar{\psi}_R = -i\sigma_2\bar{\psi}_L$, one can therefore re-express

$$L_{\text{kin}} = \bar{\psi}^T_i\sigma^\mu D_\mu \Psi$$  

in terms of $2N_f$ 4-dimensional spinors $\Psi = (\psi_L^\dagger, \bar{\psi}_R^\dagger)^T$ and $\bar{\Psi}^T = (\bar{\psi}_L^\dagger, \psi_R^\dagger)$. Because $L_{\text{kin}}$ is now block diagonal, the $SU(2N_f)$ symmetry in the space combining flavor and transformed chiral components is manifest in this form. With the same transformation of variables

the quarks’ Dirac-mass term becomes

$$m\bar{\psi}\psi = \frac{m}{2}(\bar{\Psi}^T i\sigma_2 S \Sigma_0 \Psi - \Psi^T i\sigma_2 S \Sigma_0 \Psi^*),$$  

where the symplectic matrix

$$\Sigma_0 = \begin{pmatrix} 0 & I_{N_f} \\ -I_{N_f} & 0 \end{pmatrix}$$  

acts in the $2N_f$-dimensional extended flavor space and transforms as $\Sigma_0 \rightarrow U^T \Sigma_0 U$, because it is antisymmetric according to the $N_f(2N_f - 1)$-dimensional antisymmetric rank 2 tensor representation of SU($2N_f$). The invariance group of $\Sigma_0$ as bilinear form on complex $2N_f$-vectors is the compact symplectic group Sp($N_f$) which is the intersection of the unitary $U(2N_f)$ and the symplectic $Sp(2N_f, \mathbb{C})$, therefore sometimes also referred to as USp($2N_f$). An explicit (dynamical) Dirac mass thus explicitly (spontaneously) breaks the original SU($2N_f$) down to Sp($N_f$).

For real color representations ($\beta = 4$), going through the same steps with replacing $S \rightarrow I$ [21], the mass term is a symmetric color singlet and the corresponding flavor matrix is therefore symmetric, likewise. It then belongs to the $N_f(2N_f + 1)$-dimensional symmetric rank 2 tensor representation of SU($2N_f$), and the invariance group, generated by the $N_f(2N_f - 1)$ antisymmetric hermitian $2N_f \times 2N_f$ matrices, is SO($2N_f$) or its double cover Sp($2N_f$) for fermionic states.

For $N_f = 2$ flavors the enlarged flavor symmetry group is SU($4$), not U($4$) because of the axial anomaly, it replaces the usual chiral and baryon number symmetries SU($2)_L \times SU(2)_R \times U(1)_B$. Just as this extended flavor SU($4$) shares its 15 dimensional Lie algebra with the group of rotations in 6 dimensions, SO($6$), its Sp($2$) subgroup leaving the $\beta = 1$ Dirac-mass term invariant has the 10 dimensional Lie algebra of SO($5$) (in fact they are both the universal covers of the respective rotation groups). So the symmetry breaking patterns by Dirac mass terms are locally the same as SO($6$) $\rightarrow$ SO($5$) and SO($6$) $\rightarrow$ SO($4$) for $\beta = 1$ and $\beta = 4$, respectively.

On the other hand, at finite chemical potential $\mu$, but still with massless quarks, $m = 0$, the SU($2N_f$) symmetry is broken explicitly by [21]

$$\mu\bar{\psi}\gamma_0\psi = \mu\bar{\Psi}B_0\Psi,$$

with $B_0 = \begin{pmatrix} I_{N_f} & 0 \\ 0 & -I_{N_f} \end{pmatrix}$, down to SU($N_f)_L \times SU(N_f)_R \times U(1)_B$. For $N_f = 2$, in terms of the rotation groups, this symmetry breaking pattern is locally the same as SO($6$) $\rightarrow$ SO($4$) $\times$ SO($2$).

When both $\mu$ and $m$ are non-zero, the unbroken flavor symmetry is of course given by the common subgroup
SU(2)F × U(1)B of the two limiting cases: (i) \( \mu \to 0 \) at finite \( m \) with either Sp(2) (\( \beta = 1 \)) or Spin(4) (\( \beta = 4 \)) symmetry, and (ii) \( m \to 0 \) at finite \( \mu \) with SU(Nf)F × SU(Nf)B × U(1)B in either case, as discussed above.

Whether the resulting SU(2)F × U(1)B is actually closer to the combined isospin plus baryon-number symmetries or the standard chiral symmetry, naturally depends on the relative sizes of the quarks’ Dirac mass \( m \) and their chemical potential \( \mu \). More precisely, for quark chemical potential \( \mu < m_\pi/2 \), the chiral symmetry breaking pattern essentially remains the \( \mu = 0 \) one and the vacuum alignment is said to be \( \langle \bar{q}q \rangle \)-like. There is only an explicit breaking of the combined isospin/baryon-number symmetry proportional to \( \mu \): according to Sp(2) → SU(2)F × U(1)B which is the same as SO(5) → SO(3) × SO(2) for the bosonic states of QC2D with its pseudo-real fundamental quarks (\( \beta = 1 \)), or according to Spin(4) = SU(2) × SU(2) → SU(2)F × U(1)B for G2 with real fundamental quarks or other theories with quarks in real color representations (\( \beta = 4 \)). If the chemical potential is small, the respective enlarged isospin/baryon-number symmetries are approximately realized. The vacuum is unchanged and the degeneracies in the spectrum simply split up by quark number proportional to \( \mu \). For example, among the would-be-Goldstone bosons the masses \( m_{\Lambda^c} \) of diquark and antidiquark in QC2D split from those of the pion as \( m_{\Lambda^c} = m_\pi \pm 2\mu \).

When \( m_{\Lambda^c} = 0 \), i.e., for \( \mu \geq m_\pi/2 \), a diquark condensate develops at zero temperature and the vacuum alignment starts rotating from being \( \langle \bar{q}q \rangle \)-like to becoming more and more \( \langle \bar{q}q \rangle \)-like as \( \mu \) is further increased. The chiral condensate then rapidly decreases and chiral symmetry gets restored to the approximate SU(2)L × SU(2)R \( \cong \) SO(4). In this phase, the baryon number U(1)B is spontaneously broken and the remaining isospin SU(2)F changes from the approximately realized enlarged isospin/baryon-number Sp(2) symmetry to becoming an approximate standard chiral SU(2)L × SU(2)R symmetry.\(^2\) Both are only approximate symmetries, the first is weakly broken by the small \( \mu \) (but there is a large dynamical quark mass), while the second is weakly broken by the small current quark mass \( m \) (at large \( \mu \)). They change in a crossover. In fact, the dynamical contribution to the quark mass changes in this crossover from being predominantly the original Dirac mass, with a condensate of tightly bound light diquarks, to becoming a dynamical Majorana mass as in BCS theory. This vacuum realignment in the diquark-condensation phase with superfluidity of the bosonic baryons in QC2D is the analogue of the BEC-BCS crossover in ultracold fermionic quantum gases.

At zero temperature, for \( 2\mu < m_\pi \), below the onset of the condensation of diquarks as bosonic baryons, the baryon density remains zero and the thermodynamic observables must be independent of \( \mu \). Because this is far from obvious to verify explicitly in actual calculations, it has been named the Silver Blaze Problem [52]. In order to be able to excite any states at zero temperature, and with a gap in the spectrum, the relativistic chemical potential needs to be increased beyond the mass gap in the correlations to which it couples. Here, with a continuous zero-temperature quantum phase transition at \( \mu_B \equiv 2\mu = m_\pi \) this gap is simply given by the lightest baryon mass in vacuum which because of the extended flavor symmetry in QC2D coincides with the pion mass, \( m_B = m_\pi \). This latter property is of course special to \( N_f = 2 \) or other theories with quarks in pseudo-real (without fermionic baryons) or real color representations (with fermionic baryons, but where the lightest baryon with mass \( m_B = m_\pi \) is still bosonic). The Silver Blaze property must hold as it does here, however, also when there are no bosonic baryons as in QCD up to a quark chemical potential of the order of \( m_B/N_f \) (reduced by \( 1/N_f \) of the binding energy per nucleon when the transition is of first order).

At finite temperature, a qualitative picture emerges for the phase diagram of QC2D as sketched in Fig. 3. The solid line in the \( T = 0 \) plane represents the continuous zero-temperature transition with diquark conden-

\(^2\)In our example with \( N_f = 2 \) quarks in a real color representation it might seem that there is no change in the approximate symmetry, it is Spin(4) = SU(2) × SU(2) for small chemical potential below the crossover and SU(2)L × SU(2)R above. They may look the same but they are two different subgroups of the original SU(4), the first one contains U(1)B as a subgroup and the second one does not.
sation which is of mean-field type. Because the quark mass \( m_q \) scales quadratically with the pion mass, it will occur along a parabola \( m_q \propto \mu_B^2 \). The thick dashed lines represent the corresponding second-order transitions at finite temperature in fixed \( m_q \) planes of the \( O(2) \) universality. The thick line along the temperature axis is the magnetic first-order transition in the \( \mu_B = 0 \) plane which might end in a multicritical point. When viewed in the \( \mu_B = 0 \) plane, this is the critical endpoint in the \( O(6) \) universality class for the chiral phase transition in QC2D with its extended SU(4) flavor symmetry. In the \( m_q = 0 \) plane, the vacuum alignment will always be \( \langle qq \rangle \)-like, for no-matter-how-small \( \mu > 0 \). Therefore, in this plane one only has the second-order O(2) line which, if it ends in the same point, would make it multicritical.

The construction of the QMD model for QC2D starts from the flavor structure of the standard chiral condensate and the quark mass term which is of the form \( \bar{\Psi}^T \Sigma \Psi \). It therefore transforms under the full flavor SU(4) according to the six-dimensional antisymmetric representation in the decomposition \( 4 \oplus 6 \oplus 10 \).

The other components belonging to the same multiplet are obtained from transformations

\[
\Psi \rightarrow U \Psi, \quad U = \exp(i \pi^a X^a) \in \text{SU}(4)/\text{Sp}(2). \quad (11)
\]

Then, \( \Psi^T \Sigma \Psi \rightarrow \Psi^T \Sigma \Psi \), where, from Cartan’s immersion theorem, the whole coset \( \text{SU}(4)/\text{Sp}(2) \cong S^5 \) is obtained in this way via \( \Sigma \equiv U^T \Sigma_0 U \). The coset elements \( \Sigma \) are in turn parameterized by six-dimensional unit vectors \( \vec{n} \) as \( \Sigma = \vec{n} \vec{\Sigma} \), with \( \Sigma_i = \Sigma_j = 2 \delta_{ij} \) and \( \Sigma = (\Sigma_0, \Sigma_3 X^3) \) such that \( X^a, a = 1 \ldots 5 \), form a basis for the coset generators [25]. Thus, one verifies explicitly that the vector \( \Psi^T \Sigma \Psi \) transforms as a (complex) six-dimensional vector under SO(6).

The QMD model for QC2D is therefore defined by coupling the real SO(6) vector of quark bilinears \((\Psi^T \Sigma \Psi + \text{h.c.})\) to a vector of mesonic fields \( \vec{\phi} = (\sigma, \vec{\pi}, \text{Re} \Delta, \text{Im} \Delta^T) \) formed by the scalar \( \sigma \) meson, the pseudoscalar pions \( \vec{\pi} \) and the scalar diquark-antidiquark pair \( \Delta \). This yields the Lagrangian,

\[
\mathcal{L}_{\text{QMD}} = \phi \left( \partial \phi + g \sigma + i \tau_2 \sigma \right) - \frac{1}{2} \left( \phi^T i \tau_2 S \Sigma \Psi - \Psi^T i \tau_2 S \Sigma \Psi \phi \right) + \frac{1}{2} \left( \phi \Delta \phi \right) + V(\vec{\phi}), \quad (12)
\]

where \( V(\vec{\phi}) \) is the meson and diquark potential. A non-vanishing chemical potential not only couples to the quarks but also to the bosonic diquarks. Rewriting Eq. (12) in terms of the original variables, one obtains

\[
\mathcal{L}_{\text{QMD}} = \phi \left( \partial \phi + g \sigma + i \tau_2 \sigma \right) - \frac{1}{2} \left( \phi^T i \tau_2 S \Sigma \Psi - \Psi^T i \tau_2 S \Sigma \Psi \phi \right) + \frac{1}{2} \left( \phi \Delta \phi \right) + V(\vec{\phi}), \quad (12)
\]

with \( C = \gamma^0 \gamma^0 \) and a flavor- and color-blind Yukawa coupling \( g \). With

\[
V(\vec{\phi}) = \frac{1}{4} (\vec{\phi}^2 - v^2)^2 - c \sigma, \quad (14)
\]

one obtains the corresponding O(6) linear sigma model; and in the limit \( \lambda \rightarrow \infty \), the bosonic part of \( \mathcal{L}_{\text{QMD}} \) is then equivalent to the leading-order \( \chi PT \) Lagrangian of Refs. [21] with the identifications \( v = f_s = 2F \) and \( c = f_s m_q^2 = 2F m_q^2 \). The coefficient of the leading term in \( \mu \) of the \( \chi PT \) Lagrangian, which is \( \mu^2 \text{tr}(\Sigma^T \Sigma) \) with \( B = U^T B U \), was fixed from gauging the flavor SU(4) in [20]. Here it simply follows from \(-2 \mu^2 |\Delta|^2 \) as part of the kinetic term of the complex scalar diquark field \( \Delta \) with chemical potential \( \mu_B = 2 \mu \). This implies in particular, that the meson/diquark potential \( V(\vec{\phi}) \) itself, up to the explicit breaking by \(- c \sigma \), which in principle needs to be only SO(4) x SO(2) invariant at finite \( \mu \), must remain SO(6) invariant, however, at this leading order, \( O(\mu^2) \), and therefore at \( O(\mu^2) \) in the fields, likewise. We can thus only have an SO(6) invariant mass term in \( V(\vec{\phi}) \).

Gauge field dynamics and confinement effects can be modeled also in QC2D by including a constant Polyakov-loop variable as a background field as in the NJL model [42], and analogous to what is commonly done in the so-called Polyakov-loop-extended quark-meson models of three-color QCD [62-64]. To this end one introduces a constant temporal background gauge field \( A_\mu = A_\mu(t) \), which is furthermore assumed to be in the Cartan subalgebra as in the Polyakov gauge, i.e., for the color SU(2) of QC2D simply given by \( A_0 = T^3 a_0 \). This leads to the Polyakov loop variable

\[
\Phi = \frac{1}{2} \text{Tr} e^{i \beta a_0} = \cos(\beta a_0), \quad (15)
\]

to model a thermal expectation value of the color-traced Polyakov loop at an inverse temperature \( \beta = 1/T \), as an order parameter for the deconfinement transition at vanishing chemical potential. With the covariant derivative \( D_\mu = \partial_\mu - i \phi \partial_\mu A_\mu \), replacing the ordinary one in the fermionic part of the QMD model in Eq. (13), this leads to a contribution of the form \(- i \phi \gamma^\mu T^2 a_0 \phi \) for
a Polyakov-loop-extended quark-meson-diquark model (PQMD) Lagrangian.

It is often convenient to rewrite the fermionic part of the Lagrangian in yet another form, \( \mathcal{L}_F = \bar{\Psi} \gamma^R \mathcal{L}_{\text{QMD}} \Psi \), where \( \psi_i (\psi_i) \) explicitly denote the red (green) color components of \( \psi \) and \( \psi^C \equiv C\bar{\psi}^T \) per flavor, and

\[
S^{-1} = \left( \begin{array}{cc}
\delta - \gamma^0 (a_0 + p) + g_i (\sigma_i \gamma^5 \bar{\tau}^\mu) \\
-\gamma^5 \Delta \\
\delta - \gamma^0 (a_0 - p) + g_i (\sigma_i \gamma^5 \tau^\mu)
\end{array} \right).
\]

The PQMD model Lagrangian then becomes

\[
\mathcal{L}_{\text{PQMD}} = \bar{\Psi} \gamma^R \mathcal{L}_{\text{QMD}} + \mathcal{U}_{\text{pol}}(\Phi) + \frac{1}{2} (\partial_\mu \phi^2)^2 + \frac{1}{2} (\bar{\tau}^\mu \pi^\mu)^2 + \mathcal{V}(\phi),
\]

where \( \mathcal{U}_{\text{pol}}(\Phi) \) is the Polyakov-loop potential [42] which is commonly fitted to lattice results, but which can also be computed with functional methods [65, 66]. In contrast to the three-color case the Polyakov-loop potential is a function of one single real variable \( \Phi \) here, even in the presence of a diquark condensate.

2.2. Mesonic and baryonic fluctuations with the functional renormalization group

The phases of \( \text{QC}_2 \text{D} \) and the competing dynamics of the fluctuations of the order parameters due to collective mesonic and baryonic excitations are most conveniently described within the framework of the functional renormalization group (FRG) [67–73]. The central object in the Wilsonian RG flow equation as pioneered by Wetterich [74], is the scale \( k \) dependent effective average action \( \Gamma_k[\Phi] \), where \( \Phi \) generically represents the set of all local fields in the theory. This scale-dependent effective average action interpolates between the microscopic classical action at some ultraviolet (UV) cutoff scale \( k = \Lambda \), at which fluctuations of essentially all momentum modes are suppressed, and the effective action of the full quantum theory in the infrared (IR), for \( k \to 0 \), which then includes all quantum and thermal fluctuations. The scale-dependence is described by the Wetterich flow equation,

\[
\partial_t \Gamma_k \equiv \kappa \partial_t \Gamma_k[\Phi] = \frac{1}{2} \text{Tr} \left[ \partial_t R_k (\Gamma_k^{(2)} + R_k)^{-1} \right],
\]

which involves a momentum- and scale-dependent regulator \( R_k \), whose precise form is not fixed but leaves a considerable flexibility. The role of the regulator \( R_k \) is to suppress the fluctuations of modes with momenta below the renormalization scale \( k \), and the flow equation is UV as well as IR finite. \( \Gamma_k^{(2)}[\Phi] \) are the second functional derivatives of the effective average action with respect to all the fields at scale \( k \). The functional trace represents a one-loop integration typically evaluated in momentum space and includes the sum over all fields and their internal and space-time indices as well, with standard modifications for fermionic fields. It contains the full field and \( k \)-dependent propagators of the regulated theory with cutoff \( R_k \), the inverse of \( \Gamma_k^{(2)}[\Phi] + R_k \). In order to solve the flow equation an initial microscopic action \( S = \Gamma_{k=\Lambda} \) at some UV scale \( \Lambda \) has to be specified. Truncating the effective action to a specific form, the functional equation can be converted into a closed set of (integro-)differential equations, but will in general also introduce some regulator dependence in the flow. The choice of an optimized regulator minimizes this regulator dependence for physical observables. As bosonic (fermionic) regulators \( R_{k, \text{b}} (R_{k, \text{ff}}) \) we choose

\[
R_{k, \text{b}} (\bar{p}) = (k^2 - \bar{p}^2) \theta(k^2 - \bar{p}^2),
\]

\[
R_{k, \text{ff}} (\bar{p}) = -i \bar{p} \cdot \vec{\gamma} \sqrt{\frac{k^2}{\bar{p}^2} - 1} \theta(k^2 - \bar{p}^2),
\]

which are three-momentum analogues of the optimized Litim regulators [75]. With this choice the three-momentum integration becomes trivial and the remaining Matsubara sums can be evaluated analytically. Furthermore, this choice leaves the semilocal U(1)-symmetry of the Lagrangian unaffected, analogous to [50], where the chemical potential acts like the zero-component of an Abelian gauge field.

At leading-order in a derivative expansion, all wave-function renormalization factors are neglected and only the scale-dependent effective potential \( U_k \) is taken into account. The ansatz for the effective average action then simply reads \( \Gamma_k = \int d^4x \mathcal{L}_{\text{PQMD}}|_{\bar{p} \to \bar{p}_k} \). This means that in \( \mathcal{L}_{\text{PQMD}} \) from Eq. (17) the meson/diquark potential \( V(\phi) \) of the O(6) linear sigma model from Eq. (14) is replaced by \( U_k \). The explicit symmetry breaking term \(-c r \sigma \) does not affect the flow and is thus not part of \( U_k \) but added after the RG evolution to the full effective potential again. At \( \mu = 0 \), the scale-dependent \( U_k \) then only depends on the modulus of \( \phi = (\sigma, \bar{r}, \text{Re} \Delta, \text{Im} \Delta)^T \). At non-vanishing chemical potential, however, we only have SO(4) \( \times \) SO(2) symmetry and must therefore allow it to depend on two invariants, i.e., \( U_k \equiv U_k(\rho^2, d^2) \) where \( \rho^2 = \sigma^2 + \bar{r}^2 \), and \( d^2 = |\Delta|^2 \) as before. For \( \mu \to 0 \) we recover the full extended SO(6) invariance, of
course, so that \( U_k \) then depends only on the combination \( \phi^2 = \rho^2 + d^2 \) again.

Working out the second functional derivatives of the effective action, one obtains in momentum space with constant fields, and coordinates such that \( \sigma = \rho, \vec{z} = 0 \), \( \text{Re} \Delta = d \), \( \text{Im} \Delta = 0 \), the inverse bosonic propagator \( \Gamma^{(2)}_{k,B} + R_{k,B} \), with

\[
\Gamma^{(2)}_{k,B} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

where short-hand index notations for the derivatives of the potential with respect to the fields are defined as

\[
U_{k,d} \equiv \frac{\partial U_k}{\partial d^2}, \quad U_{k,\rho} \equiv \frac{\partial U_k}{\partial \rho^2}, \quad \text{and} \quad U_{k,\rho d} \equiv \frac{\partial^2 U_k}{\partial \rho^2 \partial d^2}, \quad \text{etc.}
\]

The inverse fermion-propagator can essentially be read off from Eq. (16). With the above replacements for constant fields it is diagonal in flavor and given by, \( \Gamma^{(2)}_{k,F} + R_{k,F} \), with

\[
\Gamma^{(2)}_{k,F} = \begin{pmatrix}
\langle p^2 + 2U_{k,\rho} \rangle & 0 & 0 & 0 & 0 \\
0 & \langle p^2 + 2U_{k,\rho} \rangle & 0 & 0 & 0 \\
0 & 0 & \langle p^2 + 2U_{k,\rho} \rangle & 0 & 0 \\
0 & 0 & 0 & \langle p^2 + 2U_{k,\rho} \rangle & 0 \\
0 & 0 & 0 & 0 & \langle p^2 + 2U_{k,\rho} \rangle
\end{pmatrix},
\]

for each flavor. With the regulators from Eqs. (19) the \( k \)-dependent propagators are obtained upon inverting these expressions for \( \Gamma^{(2)}_{k,B/F} + R_{k,B/F} \). When these propagators are inserted into the Wetterich equation, Eq. (18), the integration over the spatial momentum components in the explicit loop, with the three-dimensional optimized regulators in Eqs. (19), can be performed rather straightforwardly. For example, for the bosonic contribution to the flow, with bosonic Matsubara frequencies \( p_0 = \omega_n = 2\pi T n \) for periodic boundary conditions in the zero-component \( p_0 \) of the loop momentum at finite temperature we have in the subspace of the three pions,

\[
\left( \Omega^{(2)}_{k,B}(p) + R_{k,B}(p) \right)_{ij} = \left( \omega_n^2 + \max(p_i^2, k_i^2) + 2U_{k,\rho} \right) \delta_{ij}
\]

with \( i, j = 1, \ldots, 3 \) for the upper-left \( 3 \times 3 \) block of \( \Gamma^{(2)}_{k,B} \) in Eq. (20). In the subspace of the \( \sigma \) meson and the diquark-antidiquark pair, which mix when the diquark condensate \( d = |\Delta| \) is non-zero, one analogously obtains a \( 3 \times 3 \) matrix corresponding to the lower-right block in Eq. (20) with the same replacements, \( p_0 \to \omega_n \) and \( \vec{p} \to \max(p^2, k^2) \). If we call the inverse of this \( 3 \times 3 \) submatrix

\[
A(\omega_n, \max(p^2, k^2)) = \left( \Gamma^{(2)}_{k,B/F} + R_{k,B/F} \right)^{-1},
\]

the total bosonic contribution to the flow equation for the effective potential \( U_k \) (up to a 4-dimensional spacetime-volume factor \( V \) the same as that for the effective action in this truncation), is then given by the following Matsubara sum

\[
\partial_\tau U_{k,B} = \frac{1}{2 V} \text{Tr} \left( \partial_\tau R_{k,B} \left( \Gamma^{(2)}_{k,B/F} + R_{k,B/F} \right)^{-1} \right)
\]

\[
= \frac{1}{2 T} \sum_{n=-\infty}^{\infty} p^3 d p \int_0^\infty k^2 \left( \omega_n^2 + k^2 + 2U_{k,\rho} \right)
\]

\[
+ \text{tr} A(\omega_n, k^2)
\]

\[
= \frac{k^5 T}{6 \pi^2} \sum_{n=-\infty}^{\infty} \left( \frac{3}{\omega_n^2 + k^2 + 2U_{k,\rho}} + \text{tr} A(\omega_n, k^2) \right).
\]

The trace of \( A \) is the ratio of the sum of three \( 2 \times 2 \) determinants of submatrices of \( \left( \Gamma^{(2)}_{k,B/F} + R_{k,B/F} \right)_{\sigma,\Delta} \) over the full determinant of the \( 3 \times 3 \) block \( \left( \Gamma^{(2)}_{k,B/F} + R_{k,B/F} \right)_{\sigma,\Delta} \) corresponding to the sigma and diquark-antidiquark directions in field space in Eq. (20). Therefore, the numerator of \( \text{tr} A \) is a quadratic polynomial in \( \omega_n^2 \) and the denominator a cubic one,

\[
\text{tr} A(\omega_n, k^2) = \frac{3(\omega_n^2)^2 + \alpha_1 \omega_n^2 + \omega_0}{(\omega_n^2)^3 + \beta_2 \omega_n^2 + \beta_1 \omega_n + \beta_0}
\]

The coefficients of these polynomials are probably not
very illuminating. They are worked out to be [47],
\[ \alpha_0 = 3k^4 \]
where the fermionic Matsubara frequencies \( \omega_n \) are used for antiperiodic boundary conditions. The square of the denominator is the square of the denominator and the derivatives of the potential. This mixing of the renormalization scale, the chemical potential, the fields, and the derivatives of the potential. The mixing of the sigma with the diquark sector in the diquark condensation phase, where \( \sigma = |\Delta| \neq 0 \) and baryon number is no-longer conserved, is what makes the equations somewhat more complicated than usual. But the structure behind these lengthy expressions is actually not as bad as it might be at first appear. With a partial-fraction decomposition of \( \text{tr} A \), for example, the Matsubara sum can still be computed analytically from the residue theorem in the standard way [76, 77]. With the roots of the polynomial in the denominator denoted as \( \omega_n^2 = -e_i^2, \ i = 1, ..., 3 \), this yields for the bosonic flow
\[ \partial_t U_{t,\beta} = \frac{k^5}{12\pi^2} \left\{ \frac{3}{E_k^2} \coth\left(\frac{E_k^2}{2T}\right) \right\} + \sum_{i=1}^3 \frac{e_i^2}{(\omega_i^2 - e_i^2)(\omega_i^2 - e_i^2)} \frac{1}{e_i} \coth\left(\frac{e_i}{2T}\right), \]
where \( E_k^2 = \sqrt{q^2 + 2m^2} \).

The fermionic contribution to the flow can be worked out analogously. The derivative of the regulator \( R_{t,\beta} \) in Eqs. (19) cuts off the spatial loop-momentum integration at \( p^2 = k^2 \), and for \( p^2 < k^2 \) the fermionic two-point function \( \Gamma_{t,\beta}^{(2)} + R_{t,\beta} \) is obtained from Eq. (21) upon replacing \( -i \vec{p} \cdot \gamma_k \) with \( -i \vec{p} \cdot \gamma_k / |\vec{p}| \) therein. Inverting the resulting fermion matrix, this then yields for the trace over Dirac (and flavor) indices for \( \vec{p}^2 \leq k^2 \),
\[ \text{tr} \left( -i \vec{p} \cdot \gamma_k \right) \left( \Gamma_{t,\beta}^{(2)} + R_{t,\beta} \right)^{-1} = \]
\[ 16k^2 \left( (p_0 + a_0) - \mu^2 + k^2 + g^2(d^2 + \mu^2) \right)^2 \approx \left( (p_0 + a_0)^2 + E_k^2 \right)^2 \left( (p_0 + a_0)^2 + E_k^2 \right)^2 \]
One factor of \( N_f = 2 \) hereby arises from the sum of the two flavors. The square of the denominator is the determinant of the fermion matrix \( \Gamma_{t,\beta}^{(2)} + R_{t,\beta} \) per flavor, whose eigenvalues are given by the eight combinations of different signs of \( \pm i(p_0 + a_0) \pm E_k^2 \), with
\[ E_k^2 = \sqrt{q^2 + (\epsilon_k + \mu)^2}, \quad \text{and} \quad \epsilon_k = \sqrt{k^2 + g^2\rho^2}. \]
For \( \vec{p}^2 > k^2 \) one would simply have to replace \( k^2 \) by \( \vec{p}^2 \) in these expressions again, as one would for \( \Gamma_{t,\beta}^{(2)} \) without the regulator, e.g., for a mean-field analysis of the model, see Ref. [47]. These momenta are not needed for the optimized fermionic flow either, however, which then readily follows to be given by,
\[ \partial_t U_{t,\beta} = -\frac{1}{V} \text{tr} \left( \partial_t \left( \Gamma_{t,\beta}^{(2)} + R_{t,\beta} \right)^{-1} \right) \]
\[ = -\frac{8k^5 T}{3\pi^2} \sum_{n=-\infty}^{\infty} \frac{\text{coth}\left(\frac{\mu}{2\pi}\right)}{E_k^2} \left( 1 + \frac{\mu}{\sqrt{k^2 + g^2\rho^2}} \right) \left( 1 - 2N_\Phi(E_k^2, T, \Phi) \right), \]
where the fermionic Matsubara frequencies \( p_0 = \nu_n = (2n+1)\pi T \) are used for antiperiodic boundary conditions in imaginary time. When their sum is evaluated, one finally obtains,
\[ \partial_t U_{t,\beta} = -\frac{k^5}{3\pi^2} \sum_{n=-\infty}^{\infty} \frac{2}{E_k^2} \left( 1 + \frac{\mu}{\sqrt{k^2 + g^2\rho^2}} \right) \left( 1 - 2N_\Phi(E_k^2, T, \Phi) \right). \]
Here, \( N_\Phi(E/T, \Phi) \) are Polyakov-loop enhanced quark occupation numbers for QC\(_2\)D,
\[ N_\Phi(E/T, \Phi) = \frac{1 + \Phi e^{E/T}}{1 + 2\Phi e^{E/T} + e^{2E/T}} \]
which reduce to the Fermi-Dirac and Bose-Einstein distributions for \( \Phi = 1 \) and \( \Phi = -1 \), respectively. It is also common to use [42],
\[ \phi_\Phi(E/T) \equiv 1 - 2N_\Phi(E/T, \Phi) = \frac{\sinh(E/T)}{\Phi + \cosh(E/T)}, \]
which satisfies
\[ \phi_\Phi(x) = \begin{cases} \tanh(x/2) & x = 1 - 2\rho_f(x), \quad \Phi = 1 \\ \tanh(x) & x = 1 - 2\rho_f(2x), \quad \Phi = 0 \\ \coth(x/2) & x = 1 + 2\rho_f(x), \quad \Phi = -1 \end{cases} \]
where \( \rho_f(x) = \frac{1}{e^x + 1} \), and \( \rho_b(x) = \frac{1}{e^x - 1} \).

The SU(2) center sector with \( \Phi = -1 \) corresponds to using periodic quarks in QC\textsubscript{2D} and leads to a bosonic excitation spectrum. It is explicit in Eq. (30) that nothing changes, if one changes to periodic quarks and changes the background according to such a center flip \( \Phi \rightarrow -\Phi \) at the same time. The first amounts to replacing \( v_n \rightarrow \omega_n \) and the second to \( a_0 \rightarrow a_0 + \pi T \) which together leave the fermionic flow invariant.\(^3\) This is a manifestation of the so-called Roberge-Weiss symmetry [78]. For \( \Phi = 0 \), in a center-symmetric background with \( a_0 = \pi T/2 \mod \pi \), the thermal excitation energies are twice the quark energies which is occasionally interpreted as modeling confinement.

The full and final flow equation for the effective potential in the PQMD model for QC\textsubscript{2D} [47] in the leading order derivative expansion is given by the sum of the bosonic and the fermionic flows in Eqs. (27) and (31),

\[
\partial_t U_k = \partial_t U_{k,B} + \partial_t U_{k,F} \, .
\]

A significant complexity in this flow equation for the effective potential \( U(\rho, d) \) is introduced by the presence of two fields \( \rho \) and \( d \) corresponding to the standard chiral quark condensate and the diquark condensate, respectively, which have a mutual influence on one another with physically important implications.

### 2.3. SO(6) symmetric flow

In the normal hadronic phase without diquark condensation, we may set \( d = |\Delta| = 0 \) to obtain an explicitly SO(6) symmetric flow for \( U_k(\Phi) \). If we set \( U_{k,\Phi} = U_{k,\Phi} = U_{k,\Phi} \), Eq. (35) then reduces to a more familiar looking form,

\[
\partial_t U_k = \frac{k^5}{12\pi^2} \left\{ \frac{3}{E_k^2} \coth \left( \frac{E_k^2}{2T} \right) + \frac{1}{E_k^2} \coth \left( \frac{E_k^2 - 2\mu}{2T} \right) + \frac{1}{E_k^2} \coth \left( \frac{E_k^2 + 2\mu}{2T} \right) \right. \\
\left. - \frac{16}{\epsilon_k} \left( 1 - N_q(\epsilon_k - \mu; T, \Phi) - N_d(\epsilon_k + \mu; T, \Phi) \right) \right\} ,
\]

with equal single-particle energies for mesons and diquarks,

\[
E_k^2 = E_k^2 = \sqrt{k^2 + 2U_{k,\Phi}} ,
\]

while for the sigma meson one has

\[
E_k^2 = \sqrt{k^2 + 2U_{k,\Phi} + 4\varphi^2U_{k,\Phi}} ,
\]

and for the quarks \( \epsilon_k = \sqrt{k^2 + \varphi^2\varphi^2} \) as above. Except for the change in the number of active degrees of freedom contributing to this flow, and the isospin-like chemical potential coupling to one pseudo-Goldstone boson pair, the SO(6) symmetric flow equation here is entirely analogous to the one of the PQM model for QCD with three colors, see e.g., [64, 79, 80]. In the PQM model for QCD with isospin chemical potential, however, one must allow for pion condensation and then arrives at a flow equation for two competing fields [57] analogous to Eq. (35), with Eqs. (27) and (31).

A further simplification occurs for \( \mu = 0 \) and in a trivial gauge-field background, \( a_0 = 0 \) corresponding to \( \Phi = 1 \),

\[
\partial_t U_k = \frac{k^5}{12\pi^2} \left\{ \frac{5}{E_k^2} \coth \left( \frac{E_k^2}{2T} \right) + \frac{1}{E_k^2} \coth \left( \frac{E_k^2}{2T} \right) \right. \\
\left. - \frac{16}{\epsilon_k} \tanh \left( \frac{\epsilon_k}{2T} \right) \right\} .
\]

Because \( \mu = 0 \) the diquarks are now fully degenerate with the pions which leaves us with the \( N_c = 2 \) analogue of the familiar three-color QM model flow equation [81, 82] except that there are now five pseudo-Goldstone bosons instead of the usual three pions.

As in the studies of O(4) universality and scaling in the three-color QM model [9–11, 83, 84], one can use this flow equation to analogously check the symmetry

---

\(^3\)In Eq. (30) we can actually continuously change \( v_n \rightarrow v_n + c \) and \( a_0 \rightarrow a_0 - c \) without effect. It might appear that we can thus change the quarks’ boundary conditions by a general U(1) phase as we could in QED with unbroken displacement symmetry. This is not the case, however, because \( v_n \rightarrow v_n + c \) would rotate the boundary conditions of the red quarks in the same way as those of the green antiquarks, i.e., those of the red ones in the opposite direction of those of the green ones. Thus this U(1) rotates about the \( T_2 \)-direction of SU(2) in color and there is no contradiction.
The corresponding temperature dependent screening masses as extracted from the resulting effective potential also look qualitatively exactly the same as in the QM model and are shown in Fig. 5 for the same set of zero-temperature pion masses as in Fig. 4.

If we dismiss for the moment the effects of a finite density of the bosonic baryons of QC$_2$D, and the corresponding fluctuations due to the collective baryonic excitations at finite density, we may go ahead and solve the one-dimensional SO(6) symmetric flow equation in Eq. (36) at finite quark chemical potential $\mu$ and finite $T$. The resulting quark condensate is shown in the three-dimensional plot in Fig. 6. The corresponding phase diagram is shown in Fig. 7.

The most important aspect of these results is that they are again qualitatively exactly as in the quark-meson model for QCD. There is a low-temperature 1$^{st}$-order transition line with an endpoint at around $\mu \approx 2.5 m_\sigma$, and a chiral crossover which is here indicated by the dashed lines marking the half-value of the vacuum condensate. As also known from studies of the quark-meson model [82], the inclusion of fluctuations in the order parameter to account for collective excitations with the FRG leads to the capacity to describe a low temperature phase transition to bound quark matter. As shown in Figs. 6 and 7, chiral symmetry is not fully restored but only jumps at the transition and gradually decreases further into the quark-matter phase. This is conceptually different from the restoration of chiral sym-
symmetry at the first-order line in mean-field studies which would lead at the quark level to the analogue of Lee-Wick matter as in the chiral Walecka models [87], but which is a mean-field artifact.

Another nice feature of the SO(6) symmetric FRG result at finite $\mu$ is that the zero-temperature quark condensate remains precisely constant until $\mu$ reaches the critical value for the transition to bound quark matter as seen in Fig. 6. There is no Silver-Blaze problem in the SO(6) symmetric flow from Eq. (36). The vacuum does not change at all until the quantum phase transition to quark matter is reached.

These results are all nice and well to illustrate general features of quark-meson models, but they fail of course to describe an essential part of the dynamics which are the effects of finite baryon density and collective baryonic excitations. In QC$_2$D this means to correctly describe the diquark condensation phase with superfluidity of the bosonic baryons and the BEC-BCS crossover in the cold ‘quarkionic’ two-color quantum gas. The advantage here as compared to the three-color case is that we can understand these effects quite well and study in detail what we need to do to include them. Moreover, it is not only much more straightforward to include the baryonic degrees of freedom in form of the diquarks of QC$_2$D than the true fermionic baryons in the real world with three colors, but it can also be considered a valuable warm-up exercise for a successful quark-diquark description of baryonic degrees of freedom in QCD.

The emphasis in ‘quarkionic’ is on replacing the fermions in the ultracold fermionic gases by the colored quarks which are unphysical. If it is on baryonic excitations as the physical ones in confined quark matter, quarkyonic may be also appropriate.

Figure 6: The chiral condensate from the SO(6) symmetric RG flow over quark chemical potential and temperature. The $\mu = 0$ plane corresponds to the data shown in Fig. 4 for the $m_\pi = 138$ MeV pion mass.

2.4. Diquark condensation

At zero temperature, the vacuum and the mass spectrum must remain independent of the (relativistic) chemical potential $\mu$ until the quantum phase transition is reached. To demonstrate that this phenomenological feature holds in the functional integral representation of the grand potential has been called the Silver Blaze problem after the story by Arthur Conan Doyle in which the dog did not bark when the horse named Silver Blaze was abducted [52]. This “curious incident of the dog in the night-time” doing nothing led Sherlock Holmes to solve the case. The solution to the Silver Blaze problem for QCD with isospin chemical potential [52] equally holds for QC$_2$D with baryon chemical potential. However, the relevant quantum phase transition in QC$_2$D is not the QM model transition to quark matter, but that of diquark condensation at $\mu = m_\pi/2$ with the spontaneous breaking of baryon number and baryon superfluidity.

Diquark condensation at $\mu = m_\pi/2$ thus occurs way before any quark matter or chiral transition at around $\mu \approx 2.5m_\pi$ as seen in the essentially purely mesonic model above. This is well known from chiral effective field theory [21], lattice simulations [31] and mean-field studies of the NJL [40] or the PNJL model [42] as mentioned at the beginning of this section. In order to describe it within the FRG one needs to solve the full flow equation (35) with including the competing fluctuations in the two directions of field space corresponding to the chiral and diquark condensates, respectively, both at the same time. This was done numerically on a two-dimensional grid in field space in Ref. [47].

Of course, it can also be described qualitatively already at mean-field level in the QMD model [47].

Figure 7: QMD model phase diagram without diquark condensation and baryonic fluctuations. Shown are results from [47] for the SO(6) symmetric flow (RG) and a typical mean-field diagram including vacuum contributions (MF).
resulting meson and diquark mass spectrum is shown in Fig. 8. Just as the dichotomy between chiral symmetry breaking and the Goldstone theorem is manifest within the mean-field/RPA framework or in the rainbow/ladder truncation to Dyson-Schwinger equations, the Silver Blaze problem is absent here as well. The meson masses at zero temperature remain constant at finite \( \mu \) below the transition. It can be shown analytically that the diquark and antidiquark masses are given exactly by \( m_\Delta = m_\pi \pm 2 \mu \) there, and the independence of the fermion determinant on \( \mu \) for \( \mu < m_\pi/2 \) follows.

It is important to stress that at the mean-field level, this exact feature of the theory holds only for the pole masses in the meson and diquark propagators as obtained from the random-phase approximation. This is true in the (P)QMD model as it is in the (P)NJL model. The differences between the two are not significant and only quantitative in nature in this approximation. The corresponding pole masses are defined as the zeroes of the bosonic two-point function,

\[
\det \Gamma_B^{(2)}(p) = 0, \quad \text{for} \quad -p^2 = m_k^2, \quad k = 1, \ldots, 6. \tag{42}
\]

Defining

\[
\Gamma_B^{(2)}(p) = \Gamma_B^{(2)}(p) + \Pi_B(p), \tag{43}
\]

where \( \Gamma_B^{(2)} \) and \( \Pi_B \) are the tree-level and vacuum-polarization contributions, respectively, and with the bosonic potential \( V(\phi) \) from the linear-sigma model, Eq. (14), in the normal phase without diquark condensation the pole-mass conditions (42) are then simply given by the solutions of the following equations for \( \omega \),

\[
m_\pi^2 = \omega^2 - m^2 + 3 \lambda \sigma^2 + \Pi_\pi(\omega, T),
\]

\[
m_\sigma^2 = \omega^2 - m^2 + \lambda \sigma^2 + \Pi_\sigma(\omega, T),
\]

\[
m_\Delta^2 = (\omega \pm 2 \mu)^2 - m^2 + \lambda \sigma^2 + \Pi_\Delta(\omega, T). \tag{44}
\]

Up to the chemical potential entries in \( \Gamma_B^{(2)}(\mu) \) the bosonic two-point function is diagonal in the normal phase with \( d = 0 \), and we have introduced \( \Pi_\pi(\omega, T) \) for the six diagonal entries of \( \Pi_B(p) \) with \( p = (-i \omega, 0) \) at finite \( T \).

As with chiral symmetry breaking and the occurrence of massless pions in the chiral limit, one then readily verifies that the gap equation for the diquark condensate along the bifurcation line \( \mu_c(T) \), which defines the onset of U(1)\(_b\) breaking and diquark condensation in the \((T, \mu)\)-plane, leads to the condition,

\[
\left. \left( -m^2 + \lambda \sigma^2 - 4 \mu^2 + \Pi_\Delta(0, T) \right) \right|_{\mu = \mu_c(T)} = 0, \tag{45}
\]

because the corresponding derivative of the effective potential is equal to \( \Pi_\Delta(0, T) = \Pi_\Delta(0, T) \). This implies that a solution with diquark-mass zero exists in Eqs. (44) along the same line \( \mu_c(T) \) in the \((T, \mu)\)-plane.\(^5\)

One can further verify analytically that \( \Pi_\Delta(\omega, T) = \Pi_\pi(\omega, T) \) for \( \mu = 0 \) as it must from SO(5) symmetry, and, more significantly, that at zero temperature,

\[
\Pi_\Delta(\omega, 0) = \Pi_\pi(\omega \pm 2 \mu, 0). \tag{46}
\]

Therefore, the \( T = 0 \) solutions to Eqs. (44) obey

\[
m_\Delta = m_\pi \pm 2 \mu. \tag{47}
\]

Together with the \( \mu \)-independence of the pion and sigma masses, this guarantees that there is no Silver Blaze problem at mean-field/RPA level as seen in Fig. 8, and that the zero-temperature quantum phase transition occurs at \( \mu = \mu_c(0) = m_\pi/2 \).

In contrast, in quark-meson models one usually assigns the physical masses of the mesons to the so-called screening masses to fix the model parameters. These are obtained as the eigenvalues of the Hessian of the effective potential in the leading-order derivative expansion. They agree with the eigenvalues of

\[
\Gamma_B^{(2)}(0) = \Gamma_B^{(2)}(0) + \Pi_B(0), \tag{48}
\]

and with the correct analyticity properties [76, 77] in the continuation from discrete Matsubara to continuous real frequencies this equivalence holds at all temperatures.

\(^5\)This Goldstone mode persists everywhere into the diquark condensation phase but the simple pole-mass conditions (44) are no-longer valid in this form there.
However, it is straightforward to verify [47] that the screening masses have an enormous Silver Blaze problem. Their behavior with chemical potential is utterly unphysical. That of the sigma meson does not remain independent of \( \mu \) at \( T = 0 \) in the normal phase as it should. The diquark and antidiquark screening masses remain degenerate throughout the normal phase at all \( T \), and they both vanish at \( \mu_\text{c}(T) \). However, \( \mu_\text{c}(T) \) is not half the screening mass of the pion but half its pole mass. One would have thought that the difference is small, as we only have to extrapolate from \( p^2 = 0 \) to \( p^2 = -m_\pi^2 \) in the momentum dependence of the two-point function to get from one to the other, but this is not the case. In the QMD model of QC\( ^2 \)D, where the quantum phase transition at \( \mu_\text{c} \) is the BCS limit at large \( \mu \), one can verify explicitly that one of the three modes in the diquark/sigma sector remains exactly massless in the superfluid phase, also at finite temperature. This is of course the Goldstone boson corresponding to the spontaneously broken U(1)\(_b\) baryon number. Another one becomes degenerate with the pions for large values of the chemical potential, eventually, reflecting the restoration of chiral symmetry. They combine into an SO(4) multiplet as the chiral condensate vanishes for large \( \mu \).

Fig. 8 also shows the corresponding mass formulas from the leading order chiral Lagrangian and the linear sigma model along with results of an RPA mass calculation where the mixing terms in the sigma/diquark sector were neglected. In those results all off-diagonal terms in \( \Gamma_{\mu \lambda}^{(2)} \) other than the chemical potential entries in \( \Gamma_{\mu \mu}^{(2)} \) were set to zero by hand to demonstrate that it is these terms which lead to an avoided crossing in the sigma/diquark channel. In particular, without this mixing, the masses show more clearly that it really is the original sigma which becomes degenerate with the pion at high density as required by chiral symmetry. The transition marks the BEC-BCS crossover in which the massive diquark mode changes from its original would-be-Goldstone nature (shifted by \( 2\mu \)) to the heavy Higgs mode in the BCS limit at large \( \mu \).

From the discussion above, it is clear that it is best to fix the pion mass to the zero temperature quantum phase transition in the full FRG calculation of the effective potential from Eqs. (35), (27) and (31). A corresponding pole mass can be calculated from the flow pion pole-mass is equal to \( m_\pi = 2\mu \) above the onset of diquark condensation at \( 2\mu = m_\pi \). Moreover, one verifies explicitly that one of the three modes in the diquark/sigma sector remains exactly massless in the superfluid phase, also at finite temperature. This is of course the Goldstone boson corresponding to the spontaneously broken U(1)\(_b\) baryon number. Another one becomes degenerate with the pions for large values of the chemical potential, eventually, reflecting the restoration of chiral symmetry. They combine into an SO(4) multiplet as the chiral condensate vanishes for large \( \mu \).

Figure 9: Zero temperature condensates from full flow compared to mean-field and \( \chi^\text{PT} \) results, and lattice data (from [31]).

Figure 10: The chiral and diquark condensates from the full 2d RG flow over quark chemical potential and temperature. The \( \mu = 0 \) plane corresponds to the data shown in Fig. 8 for the \( m_\pi = 138 \text{ MeV} \) pion mass, the 2d grid solution with explicit diquark fluctuations agrees with that of the SO(6) symmetric flow as shown in Fig. 6 there.

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of the bosonic propagator but it is not needed once we know they must agree. The resulting zero temperature chiral and diquark condensates are shown in Fig. 9. The linear sigma model expressions for the $T = 0$ condensates $\sigma = \langle \bar{q}q \rangle$ and $\Delta = \langle q\bar{q} \rangle$ are \[ \frac{\sigma}{\sigma_0} = \begin{cases} 1 & \text{for } \mu < \mu_c \\ \frac{1}{x} & \text{for } \mu > \mu_c \end{cases} \]
\[ \frac{|\Delta|}{\sigma_0} = \begin{cases} 0 & \text{for } \mu < \mu_c \\ \sqrt{1 - \frac{1}{y^2} + \frac{2(\Delta/\sigma_0)^2}{y^2}} & \text{for } \mu > \mu_c \end{cases} \] (49)
where $x = 2\mu/m_{\pi}$ and $y = m_{\sigma}/m_{\pi}$. The only difference between these and the leading order $\chi$PT result \[21\], \[ i.e. \] in the non-linear sigma model, is the $y$-dependent term in the diquark condensate which reduces to the $\chi$PT formula for $y \to \infty$. In this limit the vacuum realignment from $\langle \bar{q}q \rangle$-like to $\langle q\bar{q} \rangle$-like is described by a simple rotation with constant $\langle \bar{q}q \rangle^2 + \langle q\bar{q} \rangle^2$.

2.5. Phase diagram of the QMD model for QC2D

The solution to the full flow Eq. (35), with Eqs. (27) and (31) on a two-dimensional grid in field space can be extended into the entire $(T,\mu)$-plane to demonstrate the effect of baryon density on the chiral condensate in QC2D \[47\]. For vanishing gauge-field background, $\Phi = 1$, the result is summarized in Fig. 10. This figure, from the full two-dimensional flow for an effective potential with the reduced SO(4)$\times$SO(2) symmetry, should be compared to Fig. 6 from the flow for the SO(6) symmetric effective potential, to appreciate the influence of finite baryon density.

For small chemical potential in the normal phase the results agree very well. Considering that the full solution does include independent mesonic and baryonic fluctuations also in this regime, albeit the latter around $\Delta = 0$, this is perhaps not quite so obvious as one might think at first. It confirms, however, that a purely mesonic model can produce reliable results so long as the baryon density remains zero or sufficiently small, \[ i.e. \] the part of the phase diagram relevant for the mesonic freeze-out. Those results are unambiguously determined by the chiral symmetry breaking pattern, here in the O(6) universality class. Allowing additional interactions with lower symmetry has no effect on the flow in this regime.

Once the quark-chemical potential approaches half the baryon mass, corresponding to $m_B/N_c$, however, the rapidly increasing baryon density equally rapidly suppresses the chiral condensate. With the proper inclusion of the collective baryonic excitations, there is no trace left of the chiral first-order transition and the critical endpoint of the purely mesonic model. At least at zero temperature, the baryon density is an order parameter for $N_f = N_c = 2$ as well, and it rapidly increases at the finite temperature transition line separating the normal from the superfluid phase which should give rise to the two-color analogue of the baryonic freeze-out.

The corresponding phase diagrams from the two RG solutions with and without finite baryon density are compared in Fig. 11. The dashed lines again denote the half-value of the chiral condensate in the vacuum as an indication of the chiral crossover line.

The onset of diquark condensation and superfluidity of these bosonic baryons at low temperatures also marks the line at which the residual SU(2)$_V$ flavor symmetry starts changing in nature from the approxi-
mating $\text{Sp}(2)$ symmetry in the normal phase to becoming the approximate $\text{SU}(2)_L \times \text{SU}(2)_R$ quasi-restored chiral symmetry. Because they are both explicitly broken and only approximate symmetries, this vacuum realignment is a crossover. The quark mass with large chiral condensate in the normal phase starts out as the predominantly spontaneously generated Dirac mass, and the bosonic baryons undergo Bose-Einstein condensation in form of a dilute gas of strongly bound diquarks at the onset of diquark superfluidity. As their density increases, the underlying quark mass rotates into a spontaneous Majorana mass leading to a BCS-like pairing. This is the relativistic analogue in two-color QCD of the BEC-BCS crossover in ultracold fermionic quantum gases. It is indicated in Fig. 12 by additional dashed lines in the superfluid phase where the quarks’ Dirac-mass equals their chemical potential. For a more comprehensive discussion within the NJL model, see Ref. [46].

In Figure 12 the phase diagram from the full RG solution is compared to a QMD model mean-field result from Ref. [47]. As temperature increases, the line of the diquark-condensation phase transition, which should be in the O(2) universality class, in the FRG solution with fluctuations differs more and more from the mean-field result. The first-order transition line at high temperatures in the mean-field approximation is washed out by the fluctuations, and the associated tricritical point which was as also predicted from next-to-leading order $\chi PT$ [23] turns out to be a mean-field artifact.

To summarize, the particular advantages of using two instead of the usual three colors here were twofold:

– to prepare for the inclusion of diquark correlations and baryonic degrees of freedom in a covariant quark-diquark description, e.g., by a corresponding quark-meson-baryon model for real QCD as a next step;

– to be able to test non-perturbative functional methods and models against exact results and lattice simulations.

The main results from the tests discussed in this section include: the verification of O(6) scaling at $\mu = 0$; the demonstration of the relevance of pole masses to correctly describe the zero temperature quantum phase transition of two-color QCD in the FRG framework, and the failure of the usual screening masses to be capable of that; and finally but most importantly, the non-existence of a chiral first-order transition and critical endpoint at finite baryon density.

3. Universal aspects of deconfinement

The phases of gauge theories can be classified according to the behavior of the flux of a static fundamental charge. In a Coulomb phase, the electric flux spreads out and extends to infinity because of Gauss’ law. The Coulomb cloud of the charge is isotropic and of long-range nature because the photon is massless. With a mass gap, there are no such long-range Coulomb fields. Then the distinction between Higgs and confinement phases depends on whether charge is conserved or not. If it is, there is a kind of Gauss law and the flux of the test charge must go somewhere. Because of the mass gap, it will get squeezed into a string to minimize the cost of free energy. In a Higgs phase, on the other hand, charge is not conserved, there is no Gauss law, the condensate screens the charge and its flux lines peter out. This rough classification is sketched in Fig. 13. In the language of local quantum field theory, in terms of local field systems to measure the colored quantum numbers of the elementary degrees of freedom but not necessarily with physical asymptotic states associated to these fields (which would be called field-particle duality), this is described by the Kugo-Ojima criterion. This criterion specifies the same two conditions for confinement in the local field theory language: (i) there must be a mass gap to avoid the analogue of the Coulomb cloud in QED, but (ii) all global gauge charges must be conserved to have a Gauss law and to avoid the Higgs mechanism.

When the charges of global gauge invariance are unbroken one derives more generally that every gauge-invariant localized state is a singlet under these unbroken global gauge charges. Thus, without (electric) Higgs mechanism, QED and QCD have in common that any localized physical state must be chargeless/colorless. The extension to all physical states is possible only with a mass gap, however. Without that, as in QED, non-local charged states which are gauge-invariant can arise as limits of local ones which are not. The Hilbert space decomposes into so-called superselection sectors of physical states with different charges. A sector of (total) charge one differs from the neutral sector by the presence of one charged particle, and the
energy difference between the ground states of both sectors defines its mass. With a mass gap in QCD, on the other hand, color-electric charge superselection sectors cannot arise: every gauge-invariant state can be approximated by gauge-invariant localized ones (which are colorless). One concludes that every gauge-invariant state must also be a color singlet.

To prepare a charged sector, one starts in a finite spatial volume with suitable boundary conditions. The free energy difference between the charged and the neutral sector then approaches a constant finite value, zero, or it tends to infinity in the infinite volume limit, in a Coulomb, a Higgs, or a confined phase, respectively.

This is all well established and tested in pure gauge theories with infinitely heavy test charges but without dynamical (color-)charged particles. Beyond that, in particular at finite temperature it is not so clear, however. A mass gap can change gradually so the high temperature deconfined and the low temperature confined phases can be analytically connected as in full QCD with a crossover. But even at zero temperature where there should be a clear distinction between differently charged superselection sectors, there is not. The methods to fix the total color charge which will be described below only work for the pure SU(N) gauge theory without dynamical quarks, at least when QCD is studied as an isolated theory. An alternative where these methods could be applied might be to go beyond that and, in particular, include the quarks’ fractional electric charges as in the Standard Model. This possibility will be discussed briefly in Subsection 3.5.

In QCD alone, a non-zero total fundamental color charge cannot be screened completely without Higgs mechanism when dynamical quarks are present either, but with string breaking it should become a boundary effect and the energy difference to the neutral sector should reduce to the corresponding string-breaking scale in the thermodynamic limit. Moreover, one might expect that the Kugo-Ojima criterion should predict symmetry breaking and thus a phase boundary between confined and Higgs phases, for example, but there are counter examples on the lattice for that as well [88].

In pure gauge theories where the methods to fix differently charged sectors work well, we often also have a duality between magnetic and electric sectors. For example, antiperiodic (spatial) boundary conditions to mimic the presence of a mirror charge in the neighboring volume along the direction of the flux. This is a bit more complicated than the simple anti/C-periodic boundary conditions in the Abelian case but it has the analogous interpretation in terms of mirror charges. It is reviewed in Subsec. 3.2. Moreover, combinations of C-periodic and twisted boundary conditions can be used to calculate the mass of the ’t Hooft-Polyakov monopole in SU(N) with adjoint Higgs at least for even N [91].

3.1. Center vortices and spin interfaces

The finite temperature deconfinement transition in SU(N) gauge theories in \( d + 1 \) dimensions is very well understood in terms of the spontaneous breakdown of their global \( Z_N \) center symmetry [92]. This symmetry is faithfully represented by the fundamental Polyakov loops \( P(\vec{z}) \) which live on the \( d \) spatial dimensions and describe static fundamental charges. Under the global \( Z_N \) center symmetry they transform like spins \( s_i \) in a \( d \) dimensional \( q \)-state Potts model with \( q = N \) and Hamiltonian [93],

\[
\mathcal{H} = - J \sum_{\langle i,j \rangle} \delta_{s_i,s_j} - H \sum_i \delta_{s_i,0} , \quad s_i = 0, 1, \ldots , q - 1 ,
\]

with nearest neighbor coupling \( J \). A non-zero external field \( H \), inversely related to the quark mass \( m_q \), may be included to mimic the leading effect of heavy dynamical quarks. When \( 1/m_q = 0 \), the Polyakov loop develops a non-zero expectation value only in the deconfined, \( Z_N \)-broken phase, while the expectation value of \( P(\vec{z}) \) vanishes in the disordered, confined phase much like the spontaneous magnetization in the spin model.

This is well described in terms of spacelike center vortices which play the role of spin interfaces. They separate regions where the Polyakov loop differs by
a phase $\varepsilon \in \mathbb{Z}_N$, and their proliferation disorders the Polyakov loop and leads to confinement. The suppression of spatial center vortices at high temperatures coincides with the ordering of the Polyakov loop, and their free energy offers an elegant order parameter for the transition [94].

In order to be able to apply the powerful tools of universality and scaling near a critical point, here we are particularly interested in cases where this transition is of 2nd order. For pure QCD with $N = 3$ colors in $3 + 1$ dimensions this is of course not the case. The transition is first order, but only just. If we either reduce the number of colors to $N = 2$ or the dimensions to $2 + 1$, or both, we obtain Yang-Mills theories with a second order deconfinement transition within the universality class of $q$-state Potts models with $q = 2$ (Ising) in $d = 3$ dimensions or with $q = 2$ and $3$ in $d = 2$ [95]. The $q = 4$ Potts model in 2 dimensions is interesting because it is known [96] to have a 2nd order transition and to fall precisely on the separatrix $q_c(d)$ with respect to the order of the transition in the $(q,d)$-plane as shown in Fig. 14, where only 1st order transitions occur for $q > q_c(d)$, i.e., $q_c(2) = 4$. The corresponding SU(4) gauge theory in 2+1 dimensions has been studied for example in [97–99]. The conclusion in [98] was that the transition is weakly 1st order, unlike the $q = 4$ Potts case. It nevertheless seems that there is a rather wide range of intermediate length scales where at least approximate Potts scaling can be observed [97, 100]. One might then like to understand, for example, why among the wider class of $\mathbb{Z}_q$-symmetric Ashkin-Teller models with continuously varying critical exponents, it is the standard $q = 4$ Potts scaling that is relevant here, and whether this can be derived from an effective Polyakov-loop model analogous to the known cases with $N = 2$ and 3, along the lines of Refs. [101–103].

One general property of the 2$d$ Potts models is that they are selfdual for all $q$. We will see below, how this selfduality is reflected in a duality between the spacelike center vortices and the confining electric fluxes of the gauge theory.

In the pure SU($N$) gauge theories ‘t Hooft’s twisted boundary conditions fix the total $\mathbb{Z}_N$-valued center flux, i.e., the total number modulo $N$ of center vortices in the various planes of the Euclidean spacetime box [90]. At finite temperature $T$, the free energy differences of these twisted ensembles and the periodic one define the corresponding center-vortex free energies. If they tend to zero in the thermodynamic limit, then these vortices condense which leads to an area law for those Wilson loops that feel their disordering phases in the corresponding plane of the twist. In a $1/T \times L^d$ box, we have to distinguish between temporal and magnetic twist. The latter is defined in purely spatial planes and corresponds to the $\mathbb{Z}_N$-valued magnetic flux $\vec{n}$ of static center monopoles, with the direction of $\vec{n}$ perpendicular to the spatial plane of the twist. Their free energy tends to zero with $L \rightarrow \infty$ as $\exp(-\sigma_s(T)L^2)$ at all $T$ [104], corresponding to the area law for spatial Wilson loops with spatial string tension $\sigma_s(T)$.

Relevant for the deconfinement transition are only the temporal twists. These are characterized by $d$-dimensional vectors $\vec{k}$ of integers mod $N$, i.e., with components $k_i \in \mathbb{Z}_N$ representing the twist in the temporal plane of orientation $(0,i)$, with total center flux $\exp(2\pi i k_i/N)$ through that plane. See Fig. 15 for an illustration of such a vortex ensemble in SU(2). We denote the partition functions of the various ensembles with temporally twisted boundary conditions by $Z(k)$. The corresponding center-vortex free energies per $T$ are then given by

$$F_k(\vec{k}) \equiv -\ln R_k(\vec{k}) \equiv -\ln (Z_k(\vec{k})/Z_k(0)),$$

where $Z_k(0)$ stands for the periodic ensemble.

Intuitively, center vortices can lower their free energy by spreading out. As temperature is increased, however, the temporal ones can no-longer spread arbitrarily but get squeezed more and more until the phase transition is reached above which they are completely suppressed. In the vicinity of a second order deconfinement transition these vortex free energies show the universal behavior of interfaces in the respective $d$-dimensional Potts model. Interfaces in the spin models are typically introduced as frustrations along which the coupling of adjacent spins favors cyclically shifted spin states rather than parallel ones for the usual ferromagnetic couplings $J > 0$. They form $d - 1$ dimensional surfaces dual
for all cyclic boundary conditions and are conveniently studied by introducing analogous cyclically shifted boundary conditions,

\[ s_{x+c_1} = s_x + c_1 \mod q \],

with \( c_i = 0, 1, \ldots, q - 1 \). (52)

Here, the interface free energies per temperature,

\[ F_I(\vec{c}) = -\ln R_q(\vec{c}) = -\ln (Z_q(\vec{c})/Z_q(0)) \],

are obtained from ratios \( R_q(\vec{c}) \) of Potts model partition functions \( Z_q(\vec{c}) \) with cyclically shifted boundary conditions labeled by \( \vec{c} \) over the periodic one, \( Z_q(0) \). Interfaces are suppressed in the low temperature ordered phase, and these ratios \( R_q(\vec{c}) \) tend to zero in the thermodynamic limit. Interfaces in the spin model below \( T_c \) thus behave as the temporal center vortices in the ordered, deconfined phase of SU(\( N \)) above \( T_c \). Complementary to that, in the disordered(confined) phase above(below) \( T_c \), it is the interface(vortex) free energies that tend to zero such that the ratios \( R_q, R_k \) approach 1 for all boundary conditions. Only at the critical temperature \( T = T_c \) do these ratios converge to non-trivial and universal values.

In two dimensions these universal numbers,

\[ 0 < R_{q,m,n}^{(m,n)} < 1 \], \( R_{q,te}^{(m,n)} = R_q(\vec{c}) \big|_{T_c} \),

for all cyclic boundary conditions \( \vec{c} = (m,n) \) and all Potts models with 2\( m \)\( n \) order transitions, i.e., for \( q = 2, 3 \) and 4, follow from the exact expressions of the corresponding partition functions at criticality obtained in Ref. [105]. In terms of Jacobi elliptic theta functions,

\[ \theta_3(z) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 z}, \]

(55)

and with the definitions (for square lattices),

\[ T_q(m,n) = \frac{\sqrt{3} q}{N} \theta_3(x_q m^2) \theta_3(x_q n^2) \]

where

\[ x_q = 1 - \arccos(\sqrt{q}/2) = \left\{ \frac{\sqrt{3}}{4}, \frac{\sqrt{5}}{6}, 1 \right\} \]

for \( q = 2, 3, 4 \), and \( N = \Gamma[1/4]^2/(2\pi^{3/2}) \),

the partition functions of the \( q \)-state Potts models with different cyclic boundary conditions at criticality can be expressed [105], for the Ising model with \( q = 2 \) as,

\[ Z_{2,c}^{(0)} = 4T_3(4,4) - T_3(1,1) \],

(57)

\[ Z_{2,c}^{(1)} = Z_{2,c}^{(0)} = Z_{2,c}^{(0)} + 2T_3(2,1) - 4T_3(2,2) \],

\[ Z_{2,c}^{(1)} = Z_{2,c}^{(0)} - 2Z_{2,c}^{(0)} \],

for the \( q = 3 \) state Potts model as,

\[ Z_{3,c}^{(0)} = 6T_3(6,6) - 3T_3(3,3) + 2T_3(2,2) - T_3(1,1) \],

(58)

\[ Z_{3,c}^{(1)} = Z_{3,c}^{(0)} = Z_{3,c}^{(0)} + 3T_3(3,1) - 6T_3(6,2) \],

\[ Z_{3,c}^{(1)} = Z_{3,c}^{(0)} = Z_{3,c}^{(0)} + Z_{3,c}^{(0)} \]

\[ Z_{3,c}^{(1)} = Z_{3,c}^{(0)}/2 - Z_{3,c}^{(0)} \],

and the distinct ones for \( q = 4 \) as

\[ Z_{4,c}^{(0)} = 6T_4(2,2) - T_4(1,1) \],

(59)

\[ Z_{4,c}^{(1)} = 2T_4(2,2) - T_4(1,1) + 4T_4(4,1) - 4T_4(4,2) \],

\[ Z_{4,c}^{(1)} = 4T_4(4,2) + T_4(1,1) - 4T_4(2,1) \],

\[ Z_{4,c}^{(2)} = -2T_4(2,2) - T_4(1,1) + 4T_4(2,1) \],

\[ Z_{4,c}^{(2)} = Z_{4,c}^{(2)} - 4T_4(4,1) + 4T_4(4,2) \],

\[ Z_{4,c}^{(2,2)} = 6T_4(2,2) + 3T_4(1,1) - 8T_4(2,1) \].

Generally, the partition functions obey \( Z_{q,c}^{(m,n)} = Z_{q,c}^{(m,n)} \) and \( Z_{q,c}^{(q-n,m)} = Z_{q,c}^{(n,m)} \). The ratios \( R_{q,c}^{(m,n)} = Z_{q,c}^{(m,n)}/Z_{q,c}^{(0,0)} \) readily follow.

These universal ratios can be used to determine the critical couplings \( \beta_c \) of the transition in the 2 + 1 dimensional gauge theories with high precision [106], by requiring that \( \beta = \beta_c \) at \( R_c = R_{q,c} \) with \( q = N \) and the corresponding boundary conditions. For SU(2) in 2 + 1 dimensions, for example, where,

\[ R_{2,c}^{(1,0)} = \frac{1}{23/4 + 1} \], \( R_{2,c}^{(1,1)} = \frac{3/4}{23/4 + 1} \),

(60)

from the 2d square Ising model, this was used in [106] to determine the critical couplings of the deconfinement transition for lattices with up to \( N_t = 9 \) links in the Euclidean time direction with a precision typically two orders of magnitude better than previous literature values where they were available [99, 107, 108]. In particular, the high precision allows to reliably determine the subleading \( 1/N_t \)\( \rightarrow \)corrections to the linearly increasing behavior of \( \beta_c \) with \( N_t \) near the continuum limit in 2 + 1
dimensions,
\[ \beta_c(N_g) = 1.5028(21) N_i + 0.705(21) - 0.718(49) \frac{1}{N_t} + \cdots. \tag{61} \]

The slope of the leading term determines the critical temperature \( T_c \) in units of the dimensionful continuum coupling \( g_3^2 \), of the \((2+1)d\) theory, yielding \( T_c/g_3^2 = 0.3757(4) \). By standard arguments, from this result one can also read off the temperature dependence of the coupling at a given \( N_i \) [106],
\[ \beta(t) = \beta_c + 4 N_i \frac{T_c}{g_3^2} t - 0.270(2) \frac{g_3^2}{N_i} \frac{t}{T_c} \left( 1 + t \right) + O(1/N_i^2), \tag{62} \]
\[ t = T/T_c - 1. \]

For each fixed \( N_i \), we thus have precise control of the temperature \( T = 1/(a N_i) \), where \( a \) is the lattice spacing, by varying the lattice coupling \( \beta \). The physical length of the spatial volume then follows from
\[ L = a N_x = \frac{N_x}{N_i T_c} = \frac{N_x}{N_i T_c} \frac{1}{1 + t(\beta)}. \tag{63} \]

3.2. Twisted boundary conditions, electric and magnetic center flux

As mentioned above, the different choices of twisted b.c.'s in pure SU(\( N \)) gauge theories at finite temperature are labeled by two \( Z_N \)-valued vectors, \( \vec{m} \) and \( \vec{k} \), and fix the total numbers of vortices mod \( N \) through the orthogonal planes of the Euclidean \( 1/T \times L^d \) spacetime box as sketched in Fig. 16. There are \( d(d-1)/2 \) purely spatial planes and the \( d(d-1)/2 \)-dimensional vector \( \vec{m} \) denotes the total conserved, \( Z_N \)-valued and gauge-invariant magnetic flux through those, as generated by a static center monopole. The ones through the \( d \) temporal planes labeled by the \( d \)-dimensional vector \( \vec{k} \) are the universal partners of the Potts interfaces.

In order to understand the inequivalent choices for imposing boundary conditions on the gauge fields \( A \), which are blind to the center \( Z_N \) of \( SU(N) \), one first chooses \( A(x) \) to be periodic with the lengths \( L_\mu \) of the system in each direction \( \mu \) up to gauge transformation \( \Omega_p(x) \in SU(N) \), which is physically equivalent to periodic boundary conditions,
\[ A_\nu(x+L_\mu \hat{\mu} - i \partial_\nu) \equiv A_\nu^\Omega(x) \equiv \Omega_p(x) (A_\nu(x) - i \partial_\nu) \Omega_p^\dagger(x). \tag{64} \]

In the \( 1/T \times L^d \) box at finite temperature, we use
\[ L_0 = \beta = 1/T, \quad \text{and} \quad L_i = L, \quad i = 1, \ldots, d. \tag{65} \]

Figure 16: \( Z_2 \)-vortex, e.g., as in nematic liquid crystals or pure SU(2) gauge theories (left). Center vortices in the \( d+1 \) dimensional gauge theory lower their free energy by spreading out: magnetic ones through the \( d(d-1)/2 \) spatial \( \vec{m} \)-planes spread at all \( T \), while those through the \( d \) temporal \( \vec{k} \)-planes get squeezed with temperature (right).

Then, compatibility of two successive translations in a \((\mu, \nu)\)-plane requires that (no summation of indices)
\[ \Omega_\mu(x + L_\mu \hat{\mu} \vec{n}_\nu) \Omega_\nu(x) = Z_{\mu \nu} \Omega_\nu(x + L_\mu \hat{\mu} \vec{n}_\nu) \Omega_\mu(x) \tag{66} \]
with \( Z_{\mu \nu} = e^{2\pi i n_{\mu \nu}/N} \), and \( n_{\mu \nu} = -n_{\nu \mu} \in Z_N \).

The total number modulo \( N \) of center vortices in a \((\mu, \nu)\)-plane is specified in each sector by the corresponding component of the twist tensor \( n_{\mu \nu} \). In \( d = 3 \) spatial dimensions magnetic center flux \( \vec{m} \) through the box is given by \( n_{ij} = \epsilon_{ijm} m_j \), in \( d = 2 \) by \( n_{ij} = \epsilon_{ijm} m_j \). The time components \( n_{ij} \equiv k_i \) define the temporal twist \( \vec{k} \in Z_N^d \).

With these inequivalent choices of boundary conditions, the finite-volume theory in \( 3+1 \) dimensions decomposes into sectors of fractional Chern-Simons number \( \langle \nu + \vec{k} \cdot \vec{m} / N \rangle \) [109] with states labelled by \( (\vec{k}, \vec{m}, \nu) \), where the integer \( \nu \) is the usual instanton winding number. These sectors are connected by homotopically non-trivial gauge transformations \( \Omega(\vec{k}, \nu) \).

\[ \Omega(\vec{k}', \nu') | (\vec{k}, \vec{m}, \nu) \rangle = e^{i \vec{k} \cdot \vec{m}} e^{i \vec{k} \cdot \vec{m}}. \tag{67} \]

Note that such a gauge transformation must be multivalued in \( SU(N) \) and hence singular to change the twist. A Fourier transform of the twist sectors \( Z_\nu(\vec{k}, \vec{m}, \nu) \), which generalizes that for \( \theta \)-vacua as Bloch waves from \( \nu \)-vacua in two ways, by replacing \( \nu \rightarrow \nu + \vec{k} \cdot \vec{m} / N \) for fractional winding numbers, and with an additional \( Z_N^d \)-Fourier transform w.r.t. the temporal twist \( \vec{k} \), yields,
\[ Z_\nu(\vec{e}, \vec{m}, \nu \theta) = \frac{1}{N^3} \sum_{\vec{k}, \nu} e^{i \vec{k} \cdot \vec{m}} Z_\nu(\vec{k}, \vec{m}, \nu) \tag{68} \]

Up to a geometric phase,
\[ \omega(\vec{k}, \nu) = 2\pi \vec{e} \cdot \vec{k} / N + \theta(\nu + \vec{k} \cdot \vec{m} / N) \tag{69} \]
the states in the new sectors are then invariant under the non-trivial $\Omega[k, \nu]$ also,

$$\Omega[k, \nu](\vec{e}, \vec{m}, \theta) = \exp(-i\omega(\vec{k}, \nu))|\vec{e}, \vec{m}, \theta\rangle. \quad (70)$$

Their partition functions $Z_\nu$ are classified, in addition to their magnetic flux $\vec{m}$ and vacuum angle $\theta$, by their $Z_N$-valued gauge-invariant electric flux in the $\vec{e}$-direction [90, 110]. Here, we do not consider finite $\theta$ and omit the argument $\theta$ in the partition functions henceforth. Magnetic flux is irrelevant for the deconfinement transition also, because for all combinations of magnetic flux $\vec{m}$ with either electric flux $\vec{e}$ or temporal twist $\vec{k}$ [104].

$$Z_\nu(\vec{e}, \vec{m}) \xrightarrow{L \to \infty} Z_\nu(\vec{0}), \quad \text{and} \quad Z_\nu(\vec{k}, \vec{m}) \xrightarrow{L \to \infty} Z_\nu(\vec{k}, \vec{0}). \quad (71)$$

There is no analogue of magnetic center flux in the 3d spin systems, and center monopoles always condense because they are screened by the spatial string tension which is non-zero on either side of the deconfinement transition, likewise. Different combinations of temporal $\vec{k}$ and spatial $\vec{m}$-twists, with different fractional topological charge, can however be used to measure the topological susceptibility without cooling [111].

The essential structure here is, however, that the temporally twisted vortex ensembles $Z_\nu(\vec{k})$ in the $d+1$ dimensional pure SU($N$) gauge theory are related to those with fixed units of electric flux $Z_\nu(\vec{0})$ (and a perhaps more intuitive physical interpretation) by a $d$-dimensional discrete $Z_N$-Fourier transform [90],

$$Z_\nu(\vec{e}) = \frac{1}{N^d} \sum_{\vec{k} \in \mathbb{Z}^d/N} e^{2\pi i \vec{e} \cdot \vec{k}/N} Z_\nu(\vec{k}), \quad (72)$$

no matter what the magnetic flux is, which is why the argument $\vec{m}$ is dropped here and in the following also.

The role of the electric flux ensembles is best understood in terms of the translationally invariant flux between a fundamental color charge at some point $\vec{x}$ in the finite volume and its mirror charge at $\vec{x} + dL$ in a neighboring volume in the direction of the flux $\vec{e}$ [112],

$$\frac{Z_\nu(\vec{e})}{Z_\nu(\vec{0})} = \frac{1}{N} \left| \text{tr} \left( P(\vec{x}) P(\vec{x} + dL) \right) \right|_{\text{no-flux}}, \quad (73)$$

where the $P(\vec{x})$’s are untraced fundamental Polyakov loops, including any potentially non-trivial transition function $\Omega(\vec{x})$ in the time direction. With time-ordering from right to left,

$$P(\vec{x}) = \Omega(0, \vec{x}) T \left( \exp \left[ i \int_0^t A_0(t, \vec{x}) \, dt \right] \right). \quad (74)$$

The gauge field at $t = \beta$ differs from that at $t = 0$ by the gauge transformation $\Omega(0, \vec{x})$ and we must undo this to make $\text{tr} \left( P(\vec{x}) \right)$ gauge invariant also under non-periodic gauge transformations $g(x)$ which change the transition functions as

$$\Omega'(x) = g(x + L \beta \vec{m}) \Omega(0, x) \cdot g'(x). \quad (75)$$

With this, the Polyakov line in Eq. (74) transforms as

$$P'(\vec{x}) = g(0, \vec{x}) P(0, \vec{x}) g^{-1}(0, \vec{x}), \quad (76)$$

and its trace is invariant under such gauge transformations even when $g(\beta, \vec{x}) \neq g(0, \vec{x})$.

In order to derive Eq. (73) we first decompose the total electric flux vector $\vec{e} \equiv \vec{e}_i \vec{e}_i$, with $e_i = 1, \ldots, N = 1$, and perform successive translations by $L$ in the direction of the individual unit vectors $\vec{e}_i$. For each step we use the boundary conditions (64) for the gauge fields $A_\mu$, and the cocycle condition (66) for the transition functions $\Omega_{\mu}$, to show that in a fixed twist sector with boundary conditions $\vec{k}$, and $Z_\nu = e^{2\pi i \vec{e} \cdot \vec{k}/N}$,

$$P(\vec{x} + L \vec{e}_i) = \Omega(0, \vec{x} + L \vec{e}_i) \Omega(\vec{x}) \cdot P(\vec{x}) \Omega^{-1}(0, \vec{x}) = Z_{\vec{e}_i} \Omega(0, \vec{x}) \Omega(0, \vec{x}) \Omega^{-1}(0, \vec{x}), \quad (77)$$

where we have used $P(\vec{x}) = T \left( \exp \left[ \int_0^\beta A_0(t, \vec{x}) \, dt \right] \right)$ to denote the normal Polyakov line as suitable for periodic boundary conditions. If we repeat this step for an arbitrary translation by $L \vec{e} \in L \mathbb{Z}^d$, we pick up a product of center elements for the twists along the way,

$$P(\vec{x} + L \vec{e}) = e^{2\pi i \vec{e} \cdot \vec{k}/N} \Omega(0, \vec{x}) \Omega(0, \vec{x}) \Omega^{-1}(0, \vec{x}), \quad (78)$$

and $\Omega(0, \vec{x})$ stands for the product of spatial transition functions when going from $\vec{x}$ to $\vec{x} + L \vec{e}$. Changing the order of two spatial steps in different directions, $\vec{e}_i \leftrightarrow \vec{e}_j$, in the path from $\vec{x}$ to $\vec{x} + L \vec{e}$, multiplies $\Omega$ by the corresponding spatial twist $Z_{\vec{e}_j}$. Because $Z_{\vec{e}_i} Z_{\vec{e}_j} = 1$, their order does therefore not matter in Eq. (78). We have,

$$P(\vec{x}) \Omega^{-1}(0, \vec{x}) = e^{2\pi i \vec{e} \cdot \vec{k}/N} \mathbb{1}, \quad (79)$$

in an ensemble with fixed temporal twist $\vec{k}$. When all boundary conditions are spatially periodic, $\Omega(0, \vec{x}) = 1$.

We can then insert the unity in Eq. (79) into the partition functions with fixed boundary conditions and sum over all temporally twisted sectors to establish Eq. (73) from Eq. (72). The subscript ‘no-flux’ in Eq. (73) indicates that the expectation value is meant to be taken in the enlarged ensemble $Z_\nu(0) = \sum_\vec{k} Z(\vec{k})/N^d$ corresponding to this sum. If there is no magnetic flux, $\vec{m} = 0$,
we can always apply time-independent (non-periodic) gauge transformations with
\[ g(\vec{x} + L\vec{e}_i) = g(\vec{x}) \Omega^i(0, \vec{x}) \Rightarrow \Omega^i(0, \vec{x}) = 1 \, . \] (80)

We can thus assume without loss that \( \Omega(0, \vec{x}) = 1 \), as in Eq. (73) which then remains manifestly invariant under spatially periodic gauge transformations in this form. When \( \vec{m} \neq 0 \) we can still use Eq. (73) for a single direction, but for diagonal electric fluxes in a plane with magnetic twist we must include the combination of spatial transition functions \( \Omega(0, \vec{x}) \) in the expectation value on the right as in Eq. (79) which is then, however, invariant under non-periodic gauge transformations as well.\(^6\)

Eq. (73) emphasizes the physical interpretation of the electric flux sectors, as providing the necessary mirror charges for the sectors of SU\((N)\) with non-vanishing total color charge, dual to those with fixed boundary conditions. Unlike the standard product tr\((P(\vec{x})P^i(\vec{x} + \vec{R}_i))\) of static charges at distance \( R \) in a periodic ensemble, which is not gauge invariant, the electric fluxes (73) determine the gauge-invariant color-singlet free energy or potential between static charges at distance \( L[\vec{e}] \). They contain no ultraviolet divergent perimeter terms and no short-distance Coulomb contributions either.

3.3. Electric fluxes and selfduality in 2 + 1 dimensions

Independent of universality and scaling, exact maps between the spin systems and their dual theories, in terms of disorder variables on the dual lattice, are provided by Kramers-Wannier duality [113]. In 2 dimensions, the dual of a \( Z_N \) spin model is a \( Z_N \) spin model again. In particular, the 2\(d\) \(q\)-state Potts models (50) are selfdual for all \( q \). In 3 dimensions, the spin models are dual to \( Z_N \) gauge theories. The best known example is the 3\(d\) Ising model whose dual partner is the \( Z_2 \) gauge theory, and this system is relevant for the dual universal behavior of center vortex and electric flux ensembles in the 3 + 1 dimensional SU(2) gauge theory [112, 114]. In 4 dimensions, the dual of a \( Z_N \) gauge model is a \( Z_N \) gauge model again [113].

One particular aspect of the SU\((N)\) gauge theories in 2 + 1 dimensions with \( N = 2, 3 \) and 4, that has been overlooked until recently, follows from the selfduality of the corresponding 2\(d\) \(q\)-state Potts models and universality. The center vortex and electric flux ensembles of the 2 + 1 dimensional gauge theories are mirror images of one another within the universal scaling window around criticality [100, 115, 116]. This is obvious from the fact that, as we will see, the exact duality transformation of the 2\(d\) \(q\)-state Potts models in a finite volume is precisely of the structure of ’t Hooft’s relation between the temporal center vortex and electric flux ensembles in Eq. (72) discussed above.

The duality transformation and the selfduality of the \(q\)-state Potts models in 2 dimensions has long been known for infinite systems. A particularly simple proof based on the random bond-cluster representation was given in [93]. Far less is known in a finite volume with translationally invariant boundary conditions, however, where ensembles with different boundary conditions in general mix under duality transformations. One case where this is known is the 3\(d\) Ising/\(Z_2\)-gauge theory system [117–119]. For the Potts model, using the random cluster methods developed in [105], we were able to obtain the following exact duality transformations for all \( q \) on a finite 2\(d\) torus as discrete 2\(d\) Fourier transforms over all ensembles \( Z_q^{(m,n)} \) with cyclically shifted boundary conditions \( \vec{c} = (m,n) \) [115],

\[ Z_q^{(m,n)}(\vec{K}) = \left( e^{\vec{K} - 1} \right)^{N_{\text{sites}}} - 1 \sum_{m,n} e^{2\pi i (m+n)/q} Z^{(m,n)}(\vec{K}) , \]

\[ m,n,r,s = 0,1,\ldots,q-1 , \] (81)

where \( N_{\text{sites}} \) is the total number of sites of the 2\(d\) lattice (i.e., \( N_{\text{sites}} = N^2 \) on an \( N \times N \) square lattice), \( K = J/T \) is the coupling per temperature, and \( \vec{K} \) its dual obtained from

\[ (e^{\vec{K}} - 1)(e^{\vec{K}} - 1) = q , \] (82)

as usual, with temperature mirrored around criticality at

\[ K = \vec{K} = K_c = \ln(1 + \sqrt{q}) . \] (83)

For \( q = 2 \), and with \( K \rightarrow K/2 \), conventionally, this duality relation reduces to an analogous known result for the Ising model [120]. With \( q = 3 \) it agrees with a result obtained for the planar or vector Potts model [121] which is equivalent to the standard one in that case.

Before we turn to the sketch of the proof, note that the structure of the finite-volume duality transformation in the Potts models (81) is precisely the same as that in the relation between the temporal center vortex and electric flux ensembles in Eq. (72). The temporal center vortices are the universal partners of the interfaces in the spin model, \( c.f. \), Fig. 15, and the electric fluxes are their duals, with \( d = 2 \) again universally related to ensembles with interfaces, but at the dual temperature, swapped
around criticality. In $d = 3$ spatial dimensions the pattern is the same [114], the center vortex free energies in SU(2) share their universal behavior with those of the interfaces in the 3d Ising model, and electric fluxes correspond to ensembles of the dual 3d $Z_2$ gauge theory with anti-periodic boundary conditions. But they are not exact mirror images of one another in $d = 3$.

The general formula (81), for all $q$-state Potts models on a 2d torus, was first given to my knowledge in [115]. I have not published its proof nor have I seen it in the literature before. So I will sketch it here, in the rest of this subsection which will not be needed later on and which the reader may thus choose to skip ad libitum.

The proof uses mostly standard techniques [93, 122], and it is based on the setup and results of Ref. [105]. We start from the Hamiltonian (50) without external field ($H = 0$) but with cyclic interfaces such that

$$-\beta H^{(m,n)} = K \sum_{ij} \delta(s_{ij}, G_{ij}^{(m,n)}) ,$$

where the notation is such that we have introduced matrices $G_{ij}^{(m,n)}$ on the links connecting nearest neighbors $(i, j)$, and $\delta(s, s') = \delta_{s,s'} = 1$ when $s = s'$ and zero otherwise. For the cyclic interfaces we introduce straight lines (seams) $L_x$ and $L_y$ in the x and y directions on the dual lattice and use

$$G_{ij}^{(m,n)} = \begin{cases} S^m, & (i, j) \in L_x^\ast, \\ S^n, & (i, j) \in L_y^\ast, \\ \mathbb{I}, & \text{otherwise} , \end{cases}$$

(85)

where $S$ is a $q \times q$ matrix for an elementary cyclic shift,

$$S = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} ,$$

(86)

such that along the x-direction the spins get shifted $m$ times at the links dual to (i.e., crossing) $L_x$ and $n$ times in the y-direction at those dual to $L_y$. The cyclically shifted boundary conditions implemented in this way are special cases of the general permutations of spin states across the boundary considered in Ref. [105]. We will use the random-cluster representation of the Potts models on a finite torus from there. With

$$v = e^K - 1$$

(87)

the random-cluster representation of the partition function with $(m, n)$-boundary conditions then follows from

$$Z^{(m,n)}_q = \sum_{[\bar{\sigma}]} \exp \left\{ K \sum_{\langle ij \rangle} \delta(s_{ij}, G_{ij}^{(m,n)}) \right\} = \sum_{\text{graphs}} v^{N_0} q^{N_c} .$$

This is the standard random-cluster representation of the infinite Potts models [93, 122, 123]. It is valid also on the 2d torus in this form as long as we only consider the cyclic shifts of the spin states across the seams in Eq. (85) as will be shown below. The product in the second line is multiplied out and the spin sum represented as a sum over all graphs where a link with aligned spins, $\delta(s_{ij}, G_{ij}^{(m,n)}) = 1$, is said to be occupied while those with $\delta(s_{ij}, G_{ij}^{(m,n)}) = 0$ are empty. The possible configurations are then identified with graphs consisting of clusters connected by occupied links, whereby each disconnected site counts as its own cluster. So all sites are part of a cluster. Each occupied link (or bond) in a graph contributes a factor $v$, and their total number in a given graph is called $N_0$. Each of the $N_c$ clusters in a graph can be flipped independently as a whole and thus gives a combinatoric weight $q$ to the particular graph in the partition sum.

On the 2d torus one needs to classify the clusters by the possible winding numbers $(\omega_1, \omega_2)$ of the non-selfintersecting closed loops around the torus that one can draw on them:

- If one cluster extends across both boundaries in such a way that independent $(0, 1)$ and $(1, 0)$ loops are possible, the graph is said to have a torus cluster (TC) of which it can have at most one.

- A cylinder cluster (CC) extends across the boundaries in such a way that only non-trivial loops with winding numbers $(\omega_1, \omega_2)$ are possible, where $\omega_1$ and $\omega_2$ are coprime. We call the number of cylinder clusters in a graph $N_{\text{CC}}$, and they must all have the same pair of winding numbers.

- And finally, island-type clusters are the ones on which only $(0, 0)$ loops are possible. A graph with only island-type clusters is called a torus lake (TL) because the empty links form stacks which extend across the boundaries in both directions such that independent $(0, 1)$ and $(1, 0)$ loops are possible on the dual lattice, perpendicular to these stacks.

Generally, the empty links on the original lattice $L$ are dual to links which can be connected to form a cluster on the dual lattice $L^\ast$. Therefore, a lake on the original lattice is a cluster on the dual lattice [105].

With this classification, and more general seams corresponding to arbitrary and independent permutations
The first sum is over all graphs with cylinder clusters, the second over all graphs with one torus cluster and the third over all graphs with only island-type clusters in a torus lake. See Ref. [105] for more details.

In the second line (90) we have used that \( G_x = S^m \) and \( G_y = S^n \) here, and that the weight of an \((\omega_1, \omega_2)\) cylinder cluster from [105] then reduces to

\[
C(G_x, G_y; \omega_1, \omega_2) = \text{tr}(G_x^{\omega_1} G_y^{\omega_2}) = \text{tr}(S^{m\omega_1} S^{n\omega_2}),
\]

(91)

whereby, with the Kronecker delta modulo \( q \),

\[
\text{tr}(S^{m\omega_1} S^{n\omega_2}) = q \delta_q(m \omega_1 + n \omega_2)
\]

(92)

\[
= \left\{ \begin{array}{ll} q, & m \omega_1 + n \omega_2 \equiv 0 \mod q, \\ 0, & \text{otherwise}. \end{array} \right.
\]

Moreover, unlike the more general case with arbitrary permutations \( G_x, G_y \) in [105], for the cyclic ones in Eq. (85) considered here, a torus cluster is only possible when \( m = n = 0 \), i.e., with periodic boundary conditions. But then it can have \( q \) different states just as any other island-type cluster, therefore

\[
T(m, n) \equiv T(S^m, S^n) = \left\{ \begin{array}{ll} q, & m = n = 0, \\ 0, & \text{otherwise}. \end{array} \right.
\]

(93)

This means that when we restrict to cyclically shifted boundary conditions, all clusters that are possible for such given \((m, n)\) boundary conditions have the same weight, \( v^m q^N \), and this establishes that the infinite volume form of the cluster representation in Eq. (88) remains formally valid. One only needs to figure out the allowed cylinder and torus cluster and torus lake graphs as in Eq. (90) which is the starting point for the duality transformation in the finite volume.

The first task in obtaining the dual cluster representation is to re-express the number of occupied links \( N_b \) and the number of clusters \( N_c \) in Eq. (90) in terms of the corresponding numbers on the dual lattice \( L^* \).

To achieve this, recall that for an arbitrary 2d lattice, the Euler formula relates the number of vertices \( V \), edges \( E \), faces \( F \), and open ends \( N_{\text{open}} \) to the genus \( g \) of the surface on which it is drawn via the Euler characteristic,

\[
\chi = V - E + F = 2 - 2g - N_{\text{open}}.
\]

(94)

The Euler characteristic of the torus with \( g = 1 \) and without open ends, \( N_{\text{open}} = 0 \), is \( \chi = 0 \). Moreover, \( V = N_s \) is the number of sites of \( L \), and \( E \) is the total number of links, that is the sum of occupied ones \( N_b \) and empty ones which is identical to the number \( N_b^* \) of occupied ones on the dual lattice \( L^* \). Thus \( E = N_b + N_b^* \). Finally, the number of faces is the number of sites on \( L^* \), i.e., \( F = N^*_f \). Therefore

\[
N_b = N_s + N^*_f - N_b^*.
\]

(95)

It also follows from the Euler relation on the torus with \( \chi = 0 \) that the total number of sites \( N_s \) of the lattice \( L \) is given by the number of clusters \( N_c \) in a graph (recall that every site is part of a cluster) plus the number of occupied links \( N_b \) minus the number of independent non-homotopic (and non-selfintersecting) loops \( N_l \) on the clusters of the graph. Therefore, the number of clusters \( N_c \) in a graph is expressed as

\[
N_c = N_s - N_b + N_l.
\]

(96)

All TC graphs, with one torus cluster on the original lattice, are graphs with one torus lake on the dual lattice and vice versa, i.e., \( \text{(TC)} = \{L^*\} \) and \( \text{(TL)} = \{C^*\} \).

On a TL graph, where all clusters are islands whose coastlines have \((0, 0)\) winding numbers, one can walk around every pond which is a lake with \((0, 0)\) coastline inside a cluster. Therefore \( N_l = N_{\text{ponds}} \). On a TC graph,
one can walk around all ponds individually, but walking around all of them is homotopic to a point. In addition there are the \((0, 1)\) and \((1, 0)\) loops, thus \(N_L = N_{\text{ponds}} + 1\). Together with the fact that the islands of a TL graph are the ponds of its dual graph \(T_L^*\) and vice versa, this implies for

\[
\{T_L\} = \{T_L^*\} : \quad N_L = N_{\text{ponds}} = N_c^* - 1 , \quad (97)
\]

\[
\{T_C\} = \{T_C^*\} : \quad N_L = N_{\text{ponds}} + 1 = N_c^* + 1 . \quad (98)
\]

The ponds become islands on the dual graph. So \(N_{\text{ponds}}\) is equal to the number of dual clusters \(N_c^*\) on TL\(^*\) graphs, while \(N_c^*\) is the number of islands (and thus \(N_{\text{ponds}}\)) plus one on TL\(^*\) graphs.

If we insert the identities (97) together with the Euler relation (95) into Eq. (96), this leads to

\[
\{T_L\} = \{T_L^*\} : \quad N_L = -N_c^* + N_c^* + 1 - 1 , \quad (99)
\]

\[
\{T_C\} = \{T_C^*\} : \quad N_L = -N_c^* + N_c^* + 1 . \quad (100)
\]

CC graphs are selfdual. They contain an equal number of cylinder clusters and cylinder lakes, and all of them have the same winding numbers \((\omega_1, \omega_2)\). The dual graph is the one with clusters and lakes swapped, and it thus has the same number of cylinder clusters, \(N_{CC} = N_{CC}^*\), with the same set of winding numbers. On a CC graph one can walk non-homotopically around every pond and along the \((\omega_1, \omega_2)\) loop of every cylinder cluster, therefore,

\[
\{CC\} = \{CC^*\} : \quad N_L = N_{\text{ponds}} + N_{CC} = N_{\text{islands}}^* + N_{CC}^* = N_c^* ,
\]

\[
N_c = -N_c^* + N_c^* + N_c^* . \quad (101)
\]

Now we use the relations in Eqs. (95), (98), (99), and (100), to re-express \(N_L\) and \(N_c\) in Eq. (90) in terms of the corresponding numbers \(N_c^*\) on the dual lattice \(L^*\). Labeling the graphs in the sums by the respective dual cluster configurations, this yields for the partition function with \((m, n)\) boundary conditions,

\[
Z_q^{(m,n)} = v^{N_c^* + N_c^*} q^{-N_c^*} \left\{ \sum_{(CC)} \left( \frac{q}{v} \right)^{N_c^*} q^{N_c^*} \delta_q(m \omega_1 + n \omega_2) + \delta_{m,0} \delta_{n,0} \sum_{(T_L)} \left( \frac{q}{v} \right)^{N_c^*} q^{N_c^* + 1} + \sum_{(T_C)} \left( \frac{q}{v} \right)^{N_c^*} q^{N_c^* - 1} \right\} ,
\]

and its discrete Fourier transform over \(m, n = 0, \ldots, q - 1\) becomes,

\[
\sum_{m, n} e^{2 \pi i (m \omega_1 + n \omega_2) / q} Z_q^{(m,n)} = v^{N_c^* + N_c^*} q^{-N_c^*} \left\{ \sum_{m, n} e^{2 \pi i (m \omega_1 + n \omega_2) / q} \sum_{(CC)} \left( \frac{q}{v} \right)^{N_c^*} q^{N_c^* - 1} \delta_q(m \omega_1 + n \omega_2) \right. \]

\[
+ \left. \sum_{(T_L)} \left( \frac{q}{v} \right)^{N_c^*} q^{N_c^*} + q^2 \delta_{m,0} \delta_{n,0} \sum_{(T_C)} \left( \frac{q}{v} \right)^{N_c^*} q^{N_c^* - 2} \right\} . \quad (102)
\]

We have factorized another \(q\) in Eq. (102) to get the right power \(N_c^*\) in the middle term for which the sums over \(m\) and \(n\) collapse to the single \(m = n = 0\) term. The additional factor \(q^2\) in front of the last term in Eq. (102) arises because the TL graphs are independent of the boundary conditions \(m\) and \(n\) and therefore their sums yield \(q \delta_{m,0}\) and \(q \delta_{n,0}\), respectively. In the first term, the graph sum extends over all kinds of graphs with cylinder clusters for all pairs of coprime windings \(\omega_1\) and \(\omega_2\), and the periodic Kronecker delta,

\[
\delta_q(m \omega_1 + n \omega_2) = \frac{1}{q} \sum_{t=0}^{q-1} e^{-2 \pi i (m \omega_1 + n \omega_2) t / q} \quad (103)
\]

selects the right ones for each pair of boundary conditions \((m, n)\). The duality transformation is thus completed upon realizing that for any pair of coprime \((\omega_1, \omega_2)\), we can write,

\[
\sum_{m, n} e^{2 \pi i (m \omega_1 + n \omega_2) / q} e^{-2 \pi i (m \omega_1 + n \omega_2) t / q} = \sum_{t=0}^{q-1} q \delta_q((\omega_1 t - r) \omega_1 t - s) = q^2 \delta_q((\omega_2 t - r) = q^2 \delta_q((\omega_2 t - \omega_1 s) . \quad (104)
\]

The last step might not be immediately obvious. It is verified by showing the following:\(^7\)

\(^7\)I thank Nils Stridthoff for verifying that these two conditions are indeed satisfied.
(a) There is at most one \( t \) in \([0, \ldots, q−1]\) which solves the two conditions \( \omega_1 t = r \mod q \) and \( \omega_2 t = s \mod q \) at the same time. If it does, then the condition \( \omega_2 t − \omega_1 s = 0 \mod q \) is also satisfied.

(b) If there is no solution \( t \) to the first two conditions, then the new condition \( \omega_2 r − \omega_1 s = 0 \mod q \) does not have one either.

Therefore, we have shown that the terms in brackets on the right in Eq. (102) agree with the random cluster representation (90) of a Potts model partition function \( Z_q^{(\omega_1, \omega_2)} \) with boundary conditions \((−s, r)\), and with the replacement \( ν \rightarrow q/ν \) which is equivalent to replacing the coupling per temperature \( K \) by its dual \( \tilde{K} \) according to their duality relation in Eq. (82). Analogously rewriting the prefactor on the right in Eq. (102) with Eq. (82),

\[
n^{(\omega_1, \omega_2)} = (e^K − 1)^{N_s} = \frac{q^{N_s}}{(e^K − 1)^{N_s}}, \tag{105}
\]

and rearranging Eq. (102) to solve for the terms in brackets on the right, equalling \( Z_q^{(\omega_1, \omega_2)}(\tilde{K}) \), then finally yields the selfduality relation (81) of the 2d q-state Potts models (50) for all \( q \). The result here is actually a bit more general than that given in (81) where we have used \( N_{\text{sites}} = N_s = N_q \) for a 2d square lattice. More generally, the duality relation valid for any lattice \( L \) on a 2d torus without open ends, and symmetric in \( L \) and \( L^* \), here follows from Eq. (102) as

\[
(e^K − 1)^{N_s} Z_q^{(\omega_1, \omega_2)}(\tilde{K}) = (e^K − 1)^{N_s} \frac{1}{q} \sum_{m,n} e^{2\pi i (m+sn)/q} Z_q^{(m,n)}(K). \tag{106}
\]

At criticality, with \( K = \tilde{K} \), the prefactors cancel and one verifies that the values of the partition functions for the various cyclic boundary conditions in Eqs. (57), (58), and (59) reproduce themselves under the discrete Fourier transform (106) for \( q = 2, 3 \) and \( 4 \) as they must. The finite-volume duality relation is valid, however, for all \( q \), including \( q > 4 \) with first order transition.

In the 2+1 dimensional gauge theories, temperature is the same on both sides of the \( Z_q \)-Fourier transform in Eq. (72). As a consequence of the selfduality of the corresponding spin models, however, the free energies of spatial center vortices and those of the confining electric fluxes are mirror images of one another within the universal scaling window around the second order phase transition.

### 3.4. Universality and finite-size scaling

The ratios of partition functions with \((\tilde{k}, \tilde{m})\)-twisted and periodic boundary conditions in the SU(\(N\)) gauge theories are obtained from multiplying coclosed stacks \( \Omega'(\mu, \nu) \) of plaquettes \( U_\Omega \) by the center elements such that one plaquette in every plane of orientation \((\mu, \nu)\) is multiplied by the \( Z_{\nu} = e^{2\pi i n_{\nu}/N} \) corresponding to the twist tensor \( n_{\nu} \) (c.f., Eqs. (66)),

\[
\frac{Z_k(\tilde{k}, \tilde{m})}{Z_k(0, 0)} = \int \prod dU \exp \left[ -\beta \sum_{\nu} \frac{1}{N} \text{Re}(Z_U \text{tr} U_{\Omega}) \right] \frac{1}{\int \prod dU \exp \left[ -\beta \sum_{\nu} \frac{1}{N} \text{Re}(\text{tr} U_{\Omega}) \right]},
\]

\[
Z_\Omega = \left\{ \begin{array}{ll}
Z_{\nu}, & \square \in \Omega'(\mu, \nu), \\
1, & \text{otherwise}.
\end{array} \right. \tag{107}
\]

These coclosed stacks of plaquettes \( \Omega'(\mu, \nu) \) are dual to non-homotopic closed surfaces or lines on the dual lattices in \( 3+1 \) or \( 2+1 \) dimensions, respectively.

If the total number of plaquettes in all stacks \( \Omega'(\mu, \nu) \) is \( N_{\Omega} \), the ratio of partition functions in (107) can be converted into a product of \( N_{\Omega} \) ratios, each of which represents the expectation value of the \( n^\nu \)-twisted plaquette \( Z_{\nu}^{(0)} U_{\Omega}^{(0)} \) in an ensemble \( Z_{\nu}^{(n-1)} \) with \( n = 2 \) already twisted ones, where \( n \) runs lexicographically through the plaquettes of one \( \Omega'(\mu, \nu) \) stack after another, to implement twisted b.c.’s in all \((\mu, \nu)\)-planes according to the non-zero components of the twist tensor \( n_{\nu} \).

\[
\frac{Z_k(\tilde{k}, \tilde{m})}{Z_k(0, 0)} = \prod_{n=1}^{N_{\Omega}} \frac{Z_k^{(0)}}{Z_k^{(n-1)}} = \prod_{n=1}^{N_{\Omega}} \left\{ e^{\xi_n \text{Re}(1-Z_{\nu}^{(0)} U_{\Omega}^{(0)})} \right\},
\]

\[
Z_k^{(0)} \equiv Z_k(0, 0) \quad Z_k^{(N_{\Omega})} \equiv Z_k(\tilde{k}, \tilde{m}). \tag{108}
\]

In this way, one ratio of partition functions requires \( N_{\Omega} \) independent Monte-Carlo simulations [112, 124].

For the temporal \( \tilde{k}\)-twists one introduces coclosed stacks \( \Omega'(\mu, \nu) \) aligned with the time direction between one pair of adjacent time-slices, where they are dual to spatial lines in \( 2+1 \) and surfaces in \( 3+1 \) dimensions, just as the respective spin interfaces in 2 and 3 dimensions as illustrated in Fig. 15 above.

Once the reduced temperature \( t(\beta) \) is determined in terms of the lattice coupling \( \beta \), as from Eq. (62) for \( 2+1 \) dimensional SU(2), and with the spatial lattice size \( L \) from Eq. (63), a finite-size-scaling analysis can be performed. In the vicinity of the \( 2^d \) order phase transition generalized couplings such as the vortex-ensemble ratios \( R_k(\tilde{k}) \), for sufficiently large \( L \), only depend on the ratio of \( L \) and the large correlation length \( \xi \) which in the infinite volume diverges as

\[
\xi = f^q |t|^{-v} + \cdots, \quad t \to 0^+. \tag{109}
\]

with the correlation-length critical exponent \( v \), where for the \( 2d \) Potts models \( \nu = 4/3 \) for \( q = 1 \) (percolation), \( \nu = 1 \) for \( q = 2 \) (Ising model), \( \nu = 5/6 \) for \( q = 3 \)
slope \( \sigma \) calculate the deviations of lattice size is needed for any desired accuracy, one can

\[
F_I(N, K) = -\ln R_{q=2}^{(1,0)}(N, K)
\]

\[-2\ln(1 + \sqrt{2}) x = \sigma_I N \]

Figure 17: The universal scaling function \( f_I(x) \) calculated for a (1,0) interface corresponding to one antiperiodic direction from the ratio exact finite volume partition functions \( Z_{q=2}^{(1,0)}/Z_{q=2}^{(0,0)} \) of the \( N \times N \) square lattice Ising model in Ref. [125].

and \( \nu = 2/3 \) for \( q = 4 \) [122]. Here we are particularly interested in the ratios \( R_q^{(m,n)} \) which, up to finite size corrections \( \sim N^{-\omega} \) on an \( N \times N \) square lattice, depend on \( x = N^{1/\nu} t \) in a universal way. The corresponding universal scaling functions \( f_I(x) \) for the interface free energies per temperature, suppressing the indices \( (m,n) \) for the boundary conditions, are given by

\[
F_I(N, K) = f_I(N^{1/\nu} t) + c_I N^{-\omega} + \cdots, \quad \text{with} \quad f_I(0) = -\ln R_{\nu}, \quad (110)
\]

\[
f_I(x) = \sigma_0 (-x)^\mu + \cdots, \quad x \to -\infty,
\]

where \( K = K_c/(t+1), \quad K_c = \ln(1 + \sqrt{2}), \quad \text{and} \quad R_{\nu} \)

stands for the universal ratio \( R_q^{(m,n)} \) with the particular combination of boundary conditions that is being used in the \( q = 2, 3 \) and \( 4 \) state Potts models as per Eqs. (57), (58) or (59).

For the 2d square Ising model, with \( \nu = 1 \), the finite volume partition functions \( Z_{q=2}^{(m,n)}(N, K) \) have been obtained exactly for all combinations of periodic and anti-periodic boundary conditions in Ref. [125]. We can use these results for the ratios \( R_q^{(1,0)} \) and \( R_q^{(0,1)} \) to obtain the corresponding universal scaling functions \( f_I(x) \) from Eq. (110). In Fig. 17, we plot \( F_I(N, K) \) for the (1,0) interface, for example, over \( x = Nt \) for \( N = 100 \) and \( N = 1000 \). We have tested that the difference between \( F_I(N, K) \) and \( f_I(x) \) vanishes as \( N^{-\omega} \) with \( \omega = 2 \) [106]. The corresponding small residual finite-volume effects are observed, e.g., in the deviation of the \( N = 100 \) result around \( x = -5 \) that for \( N = 1000 \). To check which lattice size is needed for any desired accuracy, one can calculate the deviations of \( F_I(N, K) \) from the asymptotic slope \( \sigma_0 \) for \( x \to -\infty \) which is also known analytically.

Generally, the interface tensions (per temperature) in the \( d \) dimensional spin models are defined in the ordered phase as

\[
\sigma_I = \lim_{N \to \infty} N^{-(d-1)} F_I(N, K), \quad K > K_c. \quad (111)
\]

Comparison of this definition with Eqs. (110) thus entails that, if \( \sigma_I \) is finite in this limit, the interface tensions near criticality must behave as

\[
\sigma_I = \sigma_0 (t-\mu) + \cdots, \quad \text{with} \quad \mu = (d-1)\nu, \quad t \to 0^+. \quad (112)
\]

The exponent of the interface tension \( \mu \) is tied to the correlation length exponent \( \nu \) by one of the so-called hyperscaling relations. In particular, for the 2d square Ising model with one antiperiodic direction, with \( \mu = \nu = 1 \) and thus \( \sigma_I = \sigma_0 t + \cdots \), the asymptotic slope \( \sigma_0 \) follows from Onsager’s famous result [126], valid for all \( 0 \leq T \leq T_c \) in the ordered phase with \( N \to \infty \), for the tension of a straight (1,0) interface in the thermodynamic limit,

\[
\sigma_I^{(1,0)} = 2K + \ln \tanh K. \quad (113)
\]

When the spin-coupling per temperature \( K = J/T \) is expanded about criticality at \( K_c = \ln(1 + \sqrt{2})/2 \) this leads to,

\[
\sigma_0^{(1,0)} = 2 \ln(1 + \sqrt{2}), \quad \text{and this agrees with the asymptotic slope of} \quad f_I(x) \quad \text{for} \quad x \to -\infty \quad \text{in Fig. 17 as obtained from the exact finite volume partition functions} \quad Z_{q=2}^{(1,0)} \quad \text{and} \quad Z_{q=2}^{(0,0)} \quad \text{of Ref. [125]}. \]

Many alternative and simplified derivations of Onsager’s result (113) were given since then and can be found in text books [122, 127]. A particularly interesting one for us is given in Baxter’s book [122], as it provides the tension of a diagonal (1,1) interface. In our notations his result can be written as

\[
\sigma_0^{(1,1)} = 2 \ln \sinh(2K). \quad (115)
\]

As the free energies so far, these are all interface tensions per temperature. The proper free energy of an interface is given by \(-T \ln R_q = T \sigma_I N\). The corresponding proper interface tensions,

\[
\Sigma(\theta) \equiv \theta \sigma_I, \quad \theta = T/J = K^{-1}, \quad (116)
\]

per coupling \( J \) over temperature, for \( \theta = 0 \) to \( \theta_c = K_c^{-1} \), following from Eqs. (113) and (115) are compared in Fig. 18. At \( T = 0 \), every antiferromagnetic bond-coupling costs \( 2J \) in energy. The diagonal or zigzag

\[\text{---We use the typical Ising model conventions for} \ J \ \text{such that} \ K \to K/2 \ \text{here as compared to the general Potts model formulae above.} \]
interface is twice as long as the straight one. It needs twice as many antiferromagnetic couplings, hence, in units of $J$, $\Sigma(0) = 2$ for the straight and 4 for the diagonal interface. In the isotropic limit, when the correlation length is much larger than the lattice spacing, on the other hand, the underlying lattice does not matter anymore and the diagonal interface through a large finite volume is only a factor of $\sqrt{2}$ longer than the straight one, i.e., from Eq. (115) around $K = K_c$, \[
\sigma_{0}^{(1,1)} = 2 \sqrt{2} \ln(1 + \sqrt{2}) = \sqrt{2} \sigma_{0}^{(1,0)}. \tag{117}
\]
The same square-root ratios near $T_c$ are observed for the 3d Ising interfaces [128], \[
\sigma_{0}^{(1,0)} : \sigma_{0}^{(1,1)} : \sigma_{0}^{(1,1,1)} \sim 1 : \sqrt{2} : \sqrt{3}. \tag{118}
\]
The spin-interface tension $\sigma_I$ corresponds to the dual string tension $\sigma$ for spatial center vortices in $2 + 1$ and spatial 't Hooft loops in $3 + 1$ dimensions for which a dual area law holds in the high-temperature $Z_N$-broken phase. The same square root ratios (118) are observed for the dual string tension in SU(2) [112, 114]. From the duality Eq. (72), on can show that they then must also hold for the electric fluxes in the confined $Z_N$-disordered phase below $T_c$, and they are interpreted as the smoking gun of string formation: The free energy of orthogonal fluxes is minimized when their length is, i.e., when diagonal strings through the volume form.

At least for the $2 + 1$ dimensional SU(2) gauge theory string formation just below the deconfinement transition temperature is thus proven from universality and the observed square-root ratios of the exact Ising model interface tensions in the isotropic limit.

Unlike the spin models, however, whose interface tensions at $T = 0$ follow ratios $1 : 2 : 3$ as for isotropic flux, the electric fluxes in the SU($N$) gauge theories are expected to also show the square-root ratios for string formation at zero temperature. The numerical evidence from early Monte-Carlo results on the lattice is affirmative that this is indeed the case [129].

The $(2+1)d$ SU(2) vortex-ensemble ratios $R_{\xi}(k)$ were calculated on $N_t \times N_s^2$ lattices for $N_t = 2$ to 10, each with spatial sizes up to $N_s = 96$ and in a suitable window of lattice couplings $\beta_{\xi}$ around $\beta_c$ to test the finite-size scaling (FSS) in Refs. [106, 115, 116]. For each fixed $N_t$ one generally observes very good scaling of the available data for all $N_s$ when plotted over the FSS variable $x = \frac{L}{L_{\xi}}$. With $\nu = 1$, the physical spatial length $L$ from Eq. (63) in units of $T_c^{-1}$, and the reduced temperature $t(\beta)$ from the lattice coupling $\beta$ via Eq. (62), we define, \[
x \equiv LT_c t = \frac{N_t}{N_t + 1} \frac{t(\beta)}{1 + t(\beta)} \tag{119}
\]
For the resulting vortex free energies $F_{\xi}$, for $k = (1,0)$ and $(1,1)$, we then obtain accurate one-parameter fits via \[
F_{\xi}(x) = f_{J}(\lambda x), \tag{120}
\]
to the exact universal scaling functions $f_{J}(x)$ for $(1,0)$ and $(1,1)$ b.c.'s computed from the results of Ref. [125] as in Fig. 17. This determines the single non-universal parameter $\lambda \equiv \lambda(N_t)$ which is dimensionless and relates the SU(2) FSS variable to the Ising one, \[
\lambda_{\text{Ising}} = -\lambda(N_t) x_{\text{SU(2)}} \tag{121}
\]
whereby the minus sign reflects the interchange of high and low temperature phases between the two.

How hard it is to precisely determine the dual string tension $\sigma$ directly from the data for the vortex free energy is shown in Fig. 19. We need to get to asymptotically large $x$, c.f., Eqs. (110), but that requires larger and
larger $L = aN_t$ in order to stay within the universal scaling window for sufficiently small $t$. Note that $x = 4$ here roughly amounts to $x \approx -6$ in Fig. 17 where $N_t = 100$ is not large enough to suppress finite-size corrections even in the Ising model. The computational costs in the $d + 1$ dimensional gauge theory roughly increase as $N_t^{d+2(d-1)}$, so it becomes rather expensive to feed the FSS corrections to the dual string tension by brute force, and this was observed even more so in $d = 3$ [112].

Here, our one-parameter fits to the vortex free energies as per $F_0(x) = f_0(-\lambda x)$ to the exact universal scaling functions help tremendously because we can obtain accurate data at comparatively low computational costs for these fits from small $x$ values, i.e., small volumes. Once $\lambda$ is fixed, however, we can compute the dual string tension and, as we will see, also the string tension for electric flux from these.

This is because with the hyperscaling relations $\mu = (d - 1)\nu$, the product of interface tension and correlation length, $\sigma_t \xi^{d-1}$ is independent of $t$ near criticality. In fact, this product is another example of a universal amplitude ratio [8]. To be precise, if we use the so-called exponential correlation length in the disordered phase of the spin model above $T_c$, $\xi_{\text{gap}}$, then the universal constant is given by

$$\sigma_t(\xi_{\text{gap}})^{d-1} = R^*_t \xi_{\text{gap}}^{\nu(d-1)} \Rightarrow \xi_{\text{gap}} = \left(\frac{\sigma_i}{R^*_t}\right)^{1/(d-1)}$$

(122)

where $R^*_t = \left\{\begin{array}{ll} 1, & \text{for } q = 2, 3, 4, \text{ in } d = 2, \\ 0.40(1), & \text{for } q = 2, \text{ in } d = 3. \end{array}\right.$

The $d = 2$ results are again exact [93, 122, 127], while the $3d$ Ising ratio is determined numerically, e.g., see the review in Ref. [8].

The exponential correlation length corresponding to $\xi_{\text{gap}}$ on the gauge theory side is that of the Polyakov loops in the confined phase below $T_c$, which is given by the string tension $\sigma$ and temperature as

$$\frac{1}{\xi} = \frac{\sigma}{T}.$$  (123)

Therefore, we have a universal relation between the string tension per $T$ below $T_c$, playing the role of $\xi_{\text{gap}}^{-1}$ in Eq. (122), and the dual string tension $\tilde{\sigma}$ above $T_c$, as the interface tension $\sigma_i$ in Eq. (122),

$$\tilde{\sigma} = R^*_r (\sigma/T)^{d-1}, \text{ around criticality, or}$$

$$\sigma = \rho T^{1/(d-1)}, \text{ for } T < T_c, \text{ and}$$

$$\tilde{\sigma} = R^*_r (\rho T_c)^{1/(d-1)} T^{1/(d-1)}.$$  (124)

where we have introduced an unknown non-universal constant $\rho$ which drops out from the universal ratio, and we have replaced one factor $T$ by $T_c$ in the string tension in second line as we work at leading order in the reduced temperature $t$ here. Instead of determining the constant $\rho$ from the vortex free energies in the high temperature phase at asymptotically large $x = (T_c L)^{1/t}$, here we can use the exact universal scaling function $f_0(x)$ from the Ising model: We know the asymptotic slope of $f_0(x)$ which is $2 \ln(1 + \sqrt2)$, therefore, from

$$F_0(x) = f_0(-\lambda x) \Rightarrow \rho = 2\lambda \ln(1 + \sqrt2).$$  (125)

follows from the hard hexagon model [122], and for $q = 4$ probably from the Baxter-Wu (triangular three-spin) model [122, 130], but I am unsure about the status of a mathematical proof of the latter.
Once we extract $\lambda$ from our one-parameter fits at small $x$ where we have very accurate data, this then equally accurately determines the string tension and its dual around $T_c$. Moreover, we can determine this single non-universal parameter $\lambda$ for $N_t = 4, 5 \ldots 10$ which allows a polynomial fit,

$$\lambda(N_t) = \lambda_0 + b/N_t + c/N_t^2,$$

with the extrapolated result $\lambda_0 = 1.354(25)$ [115]. This then determines the leading behavior of the continuum string tension and its dual around the phase transition,

$$\sigma = \lambda_0 T^2 \ln(1 + \sqrt{2}) |t| + \cdots, \quad t \to 0^-, \quad \text{and}$$

$$\tilde{\sigma} = \lambda_0 T^2 \ln(1 + \sqrt{2}) t + \cdots, \quad t \to 0^+.$$

Once we have fixed the single non-universal parameter $\lambda$, we can plot the calculated center vortex free energies $F_k$ or the corresponding ratios $R_k(k)$ in comparison with $\exp(-f_j(-\lambda x))$ as in Fig. 20 for $k = (1, 0)$. The data for all different spatial lattice sizes collapse beautifully onto the universal curve, the finite-size scaling description works rather well. Together with the corresponding $\kappa = (1, 1)$ result for two orthogonal center vortices we can perform the $2d$ $Z_2$-Fourier transform (72) to obtain the ratios of electric flux $\tilde{e}$ over the no-flux ensemble in Eq. (73),

$$R_e(\tilde{e}) \equiv \frac{Z_e(\tilde{e})}{Z_e(0)} = \frac{1}{N} \left\langle \text{tr} \left( P(\tilde{e}) P(\tilde{e} + \tilde{\epsilon} L) \right) \right\rangle_{\text{no flux}}.$$

These are equally well described by $\exp(-f_j(\lambda x))$ without refitting the non-universal parameter $\lambda$, they are mirror images of the $R_k(k)$ in a surprisingly large scaling window around criticality as seen in the right panel of Eq. (73), where we swapped $x \to -x$ for the electric fluxes to demonstrate that. This is the manifestation of the selfduality of the corresponding $2d$ spin model.

Moreover, if we rescale the FSS variables by the parameters $\lambda(N_t)$ determined from the one-parameter fits for the individual $N_t = 4, \ldots 10$ data sets, the scaling of the data over all spatial lattice sizes $N_t$ can be extended to include different $N_t$ lattices. With this renormalization, one thus essentially obtains good continuum results already with the coarsest $N_t = 4$ lattice here, as seen in the zoom-in plot around criticality in Fig. 21. The rescaled $x$-range there, roughly corresponds to $x = [-1.33, 1.33]$ in Fig. 20, and the dashed line is obtained from the universal scaling function as $\exp[-f_j(x)]$ and thus now independent of $N_t$. The slight deviations of the vortex free energy from this curve near $x = 1$ seen here, are predominantly due to the leading finite-size corrections $\sim L^{-n}$, analogous to what is being observed in the ordered phase of the Ising model in Fig. 17 at negative $x$.

The selfduality of the underlying Potts model is also observed in the $2d$ $SU(3)$ gauge theory [100], as shown in Fig. 22. The dashed line in this figure marks the universal ratio $R_{3,c}^{(1,0)}$ from Eqs. (58). The two independent ones that one can form on a symmetric lattice for $q = 3$ evaluate to

$$R_{3,c}^{(1,0)} = R_{3,c}^{(2,0)} = 0.30499982 \ldots, \quad \text{and}$$

$$R_{3,c}^{(1,1)} = R_{3,c}^{(2,2)} = R_{3,c}^{(1,2)} = 0.19500018 \ldots.$$

Even though the data in Fig. 22 was obtained on a quite small $N_t = 2$ and $N_c = 24$ lattice, the intersection point of the vortex and electric flux ensembles $R_0$ and $R_e$ is almost right on the universal line. To exploit selfduality, where possible, turns out to be the fastest converging method to determine critical couplings by far.
There is a long history of methods to extract critical couplings or temperatures from simulations in finite volumes, going back to using pairwise intersections of Binder cumulants on successively larger lattices [131]. Hasenbusch later demonstrated that the ratios of partition functions with different boundary conditions could be used in the same way to obtain a much more rapid convergence with very good estimates already from rather small lattices [132]. At criticality, these ratios tend to the universal values 0 from rather small lattices [132]. At criticality, these ratios tend to the universal values 0 < \( R_c \) < 1 in the thermodynamic limit. In [106] it was therefore shown how to obtain critical couplings for gauge theories from intersecting the ratios \( R_k \) of finite volume partition functions with these universal fixed points, once their values are known. For (2+1)d SU(2) this led to an even much faster convergence than their pairwise intersections. At the time we thought this is the best method, but now look at the comparison of the results from [106] with the intersection points from selfduality in Fig. 23, again for SU(2) in 2 + 1 dimensions as a benchmark. Already with \( N_t = 16 \), and without any extrapolation, the result from selfduality is within the errors of the extrapolated best infinite volume result, \( \beta_c = 6.53661(13) \) from Ref. [106], as indicated by the narrow grey band in Fig. 23.

The reason for this impressive result is that the leading finite-size corrections do not change the position of the intersection point in \( \beta \). They only move it upwards along a straight vertical line with increasing \( L \) in plots such as the one of Fig. 22.

To see this explicitly, first use a finite-size scaling ansatz for the vortex ensemble ratios \( R_k \) as functions of the lattice coupling \( \beta \) around criticality of the form

\[
R_k(\beta) = R_c + b (\beta - \beta_c) N_s^{\alpha / \nu} + c N_s^{\alpha / \nu} + \cdots ,
\]

Defining pseudocritical couplings \( \beta_c(N_s, N_t) \) in a finite volume by requiring \( R_k(\beta) = R_c \), for the method of [106] leads to,

\[
\beta_c(N_s, N_t) = \beta_c(N_t) - (c/|b|) N_s^{-\alpha / \nu} + \cdots .
\]

These extrapolate to \( \beta_c(N_t) \) from large spatial lattice sizes \( N_s \) at fixed numbers of time slices \( N_t \). As a byproduct this method gives numerical estimates of the correction to scaling exponent \( \alpha \).

With selfduality, however, one must then have \( R_k(\beta) = R_k(\beta) \) for like values of electric flux \( \mathcal{F} \) and and temporal twist \( \tilde{k} \) at \( \beta = \beta_c \). In fact, it is straightforward to verify that then,

\[
R_k(\beta) = R_c - b (\beta - \beta_c) N_s^{\alpha / \nu} + c N_s^{\alpha / \nu} + \cdots ,
\]

with the same coefficients \( b \) and \( c \) as in (130). Therefore, the leading finite-size corrections to \( \beta_c \) when defined by \( R_c = R_k \) cancel. At criticality,

\[
R_c(\beta_c) = R_c(\beta_c) = R_c + c N_s^{\alpha / \nu} + \cdots ,
\]

so the leading corrections only move the intersection point of \( R_k(\beta) \) and \( R_c(\beta) \) vertically without shifting the so defined critical coupling.

The results for SU(3) from the \( R_c = R_k \) intersection points are compared to the extrapolated pseudocritical couplings from intersecting \( R_k(\beta) \) with its universal critical value \( R_c \) via Eq. (131) in Tab. 1.

In 2+1 dimensions the critical couplings grow linearly with \( N_t \) to leading order at large \( N_s \), c.f. Eq. (61), and the critical temperature in units of the dimensionful continuum coupling \( g_3 \) is given by the slope,

\[
\beta_c(N_t)/(2N_t) = (T_c / g_3^2) N_t + \cdots .
\]

From the values for \( \beta_c \) with \( N_t = 4, 6 \) and 8 in Tab. 1, one obtains for SU(3) [100],

\[
\frac{T_c}{g_3^2} = 0.5475(3)
\]

corresponding to \( T_c / \sqrt{\sigma} = 0.9938(9) \) with a zero temperature string tension \( \sqrt{\sigma}/g_3^2 = 0.5509(4) \) from a weighted average of the four values in [133]. This is consistent with \( T_c / \sqrt{\sigma} = 0.9994(40) \) from [99].

Moreover, because the spatial center vortex free energies \( F_k \) for sufficiently large \( L \) depend only on \( L^{1/\nu} \), and \( t \propto (\beta - \beta_c) \) at leading order in \( N_t \), when expanding

\[
F_k(\beta) = -\ln R_c + d(N_t)(\beta - \beta_c) + \cdots ,
\]
| \( N_t \) | \( \beta_\nu(R_\nu = R_t) \) | Lit. | \( \beta_\nu(R_\nu = R_t) \) | \( \beta_\nu(R_\nu = R_t) \) |
|---|---|---|---|---|
| 2 | 8.15309(11) | 8.1489(31) | 8.15309(11) | 14.7262(9) |
| 4 | 14.7262(9) | 14.7194(45) | 14.717(17) | - |
| 6 | 21.357(25) | - | 21.34(4) | - |
| 8 | 27.84(12) | - | - | - |

Table 1: SU(3) critical couplings from selfduality (weighted means), and intersection with the universal value (extrapolated) from [100], previous literature values from [99], [108].

one expects the slope at \( \beta_c \) to behave as

\[
d(N_t) \sim N_t^{1/y_c}.
\]

(137)

Our current best estimates from fitting these slopes are: \( \nu = 0.99(3) \) for SU(2) where one expects \( \nu = 1 \) for the 2d Ising model, \( \nu = 0.818(24) \) for SU(3) where \( \nu = 5/6 \approx 0.833 \) for the 2d 3-state Potts model, and \( \nu = 0.673(9) \) for SU(4) as compared to \( \nu = 2/3 \) for the \( q = 4 \) Potts model. An update on our results including more on SU(4) will be published elsewhere.

For SU(4), the \( Z_4 \) center symmetry alone does not uniquely specify the effective spin model to describe the dynamics of Polyakov loops. SU(4) is a rank-three group and has three fundamental representations, 4, \( \bar{4} \) and 6. So even the simplest effective Polyakov-loop model will consist of two distinct real terms, with nearest neighbor couplings between loops in 4/\( \bar{4} \) representations and between loops in the 6 representation, \textit{c.f.} Ref. [134]. Depending on the relative weight between the two, the corresponding spin model could be any of the \( Z_4 \)-symmetric Ashkin-Teller models with three energy levels per link and continuously varying critical exponents between the \( q = 4 \) Potts model class with \( \nu = 2/3 \) and \( \beta = 1/12 \), and that of the planar or vector Potts model, or simply the clock model, which corresponds to two non-interacting Ising models in this case with \( \nu = 1 \) and \( \beta = 1/8 \). In between these limits the model corresponds to two interacting Ising models with spins \( s_i \) and \( \sigma_i \) and it thus has two order parameters, the usual magnetic ones \( \langle s_i \rangle \) or \( \langle \sigma_i \rangle \), which are the same in the isotropic spin model with critical exponent \( \beta_m = (2 - y)/(24 - 16y) \), and an electric one \( \langle s_i \sigma_i \rangle \) with exponent \( \beta_e = (12 - 8y)^{-1} \), with \( y = 0 \) for the standard Potts model \( (\beta_e = \beta_m) \), and \( y = 1 \) for the non-interacting case \( (\beta_e = 2\beta_m) \) of the clock model [122].

The present conclusion from the studies in Refs. [97–99] is that the deconfinement transition in \( (2+1d) \) SU(4) gauge theory is weakly 1\textsuperscript{st} order. We do observe, however, at least approximately at the length scales corresponding to our spatial lattice volumes, a universal scaling which seems closest to the standard \( q = 4 \) Potts case [100]. The critical couplings from the universal amplitude ratios in Eqs. (59) are fully consistent with those determined independent of Potts scaling and universality, \textit{e.g.}, for \( N_t = 4 \) we find \( \beta_c = 26.294(2) \) from Eqs. (59) as compared to \( \beta_c = 26.283(9) \) independent of that. The correlation length exponent is also consistent with \( 2/3 \) and the data scales nicely for all volumes up to \( N_t = 80 \). This maybe still too small to see the 1\textsuperscript{st} order nature, and the conclusions may well depend on \( N_t \) as well. But even if the transition is weakly 1\textsuperscript{st} order in the infinite volume \( (N_t \to \infty) \) and continuum limits \( (N_t \to \infty) \), it seems legitimate to ask, why it is the standard Potts model universality that is observed before those limits are reached, and why not any of the other Ashkin-Teller models, and can we derive the correct effective Polyakov-loop model to demonstrate that?

3.5. Fractional electric charge and quark confinement

Center symmetry is explicitly broken and the finite temperature transition of QCD becomes a smooth crossover when dynamical quarks in the fundamental representation are included. This crossover at zero chemical potential is very well studied on the lattice [135, 136]. In presence of fields such as the quarks in QCD which faithfully represent the center of the gauge group, there are no twisted boundary conditions on the torus and therefore no center vortex ensembles to study. The detailed understanding of the deconfinement phase transition in the pure gauge theory might appear to be a rather academic exercise then.

This neglects the electric charge of quarks, however, which is commonly expected to only require small perturbative corrections. On the other hand, the inclusion of the quarks’ fractional electric charge is well known to lead to a global \( Z_4 \) symmetry of the full fermion and Higgs sector of the Standard Model. This \( Z_4 \) symmetry combines the centers of the color and electroweak gauge groups, see also [137], and [138] for a review. Since quarks carry fractional electric charges \( Q = \frac{1}{3}e \) or \(-\frac{1}{3}e\), their color and electromagnetic phases in the pair of combined transformations,

\[
(e^{i2\pi/3}, e^{i2\pi Q/e}), (e^{-i2\pi/3}, e^{-i2\pi Q/e}) \in SU(3)_c \times U(1)_{em},
\]

cancel precisely. They act trivially on all other particles in the Standard Model, which are blind to the center of SU(3) and carry integer electric charge. Electric charge \( Q \) is related to hypercharge \( Y \) and the third component of weak isospin \( t_3 \) by \( Q/e = t_3 + Y/2 \), with \( e^{i2\pi y} \equiv -1 \in SU(2) \) and \( Y \) quantized in units of 1/3. The symmetry
is therefore generated by

\[(e^{2\pi i/3}, -1, e^{i\pi}) \in SU(3) \times SU(2) \times U(1)_Y, \quad (138)\]

which gives six elements including the identity.

A global symmetry brings with it the possibility of a phase transition characterized by spontaneous symmetry breaking. In this case, one expects the transition to be driven by topological defects (vortices) that carry both color and electromagnetic flux, that is, center vortices with an additional electromagnetic Dirac string [139, 140]. Could the physical realization of this symmetry have non-trivial implications for confinement and the phase structure of QCD as part of the Standard Model then? We have started to address this question recently in a toy model with half-integer electrically charged quarks in a two-color QCD world with electromagnetism [141, 142].

The starting point is QC_2D plus electromagnetism with 2 flavors of Wilson fermions in 3 + 1 dimensions. By including ‘up’ and ‘down’ quarks with fractional charges \(x \in \mathbb{Z}/2\) relative to the \(U(1)_{em}\) gauge action, we obtain a model with a global \(Z_2\) symmetry. The lattice action is

\[
S = -\sum_{\mathbf{x}} \left( \frac{\beta_{\text{col}}}{2} \text{tr} U_{\mathbf{x}} + \beta_{\text{em}} \cos \theta_{\mathbf{x}} \right) + S_{f,W}, \quad (139)
\]

where \(S_{f,W}\) is the usual Wilson fermion action with the distinction that parallel transporters for quarks are products of SU(2) color matrices and \(U(1)_{em}\) phases,

\[
U_{\mu}(x)e^{i\theta_{\mu}(x)/2}, \quad U_{\mu}(x) \in SU(2), \quad \theta_{\mu}(x) \in (-2\pi, 2\pi). \quad (140)
\]

The SU(2) plaquettes \(U_{\mathbf{x}}\) and \(U(1)_{em}\) plaquette angles \(\theta_{\mathbf{x}}\) are formed from \(U_{\mu}\) and \(\theta_{\mu}\) in the usual way. In this model, ‘fractional charge’ means that the parallel transporters for quarks contain half the \(U(1)_{em}\) angle relative to the \(\theta_{\mu}\)'s that appear in the plaquette angle \(\theta_{\mathbf{x}}\). That is, an \(e^{i\theta_{\mu}/2} = -1\) electromagnetic link for quarks appears as an \(e^{i\theta_{\mathbf{x}}} = +1\) link in the \(U(1)_{em}\) gauge action. An important point is that the ‘volume’ of the compact \(U(1)_{em}\) is determined by the quarks, which carry the smallest quantum of electric charge. The range of \(\theta_{\mu}(x)\) is chosen such that we integrate over all possible electromagnetic transporters for the quarks, amounting to a double counting in the \(U(1)_{em}\) gauge action and for all integer charged particles. This is consistent with the premise that our compact \(U(1)_{em}\) is the result of symmetry breaking in a SU(3) \(\rightarrow SU(2) \times U(1)_{em}\) unified theory, e.g., see Ref. [143].

It is clear from Eq. (140) that a color center element \(-1 \in SU(2)\) combined with an electromagnetic phase \(\theta_{\mathbf{x}}/2\) \(\in U(1)_{em}\) would be a superconductor. The results in [141, 142] indicate that the usual ordering of color links by the dynamical quarks is negated by the inclusion of fractional electric charge. This is understood by analogy with the spin systems. In QCD alone, terms in a loop expansion of the fermion determinant that wind around the temporal direction favor the center sector in which the traced Polyakov loop \(P_{\text{col}} = 1\). They break center symmetry and lead to an ordering in much the same way as an external magnetic field \(H\) does for the Potts model spins in Eq. (50). The inclusion of fractional electric charge in our model bestows quark loops with an additional \(U(1)_{em}\) phase which may undo this effect. Consider, for example, the hopping expansion of the Wilson fermion action for \(N_f = 4\) time slices to leading order in the hopping parameter, \(O(\kappa^4)\),

\[
S_{\text{eff}} = -16\kappa^4 \left( \sum_{\mathbf{x}} \cos \frac{\theta_{\mathbf{x}}}{2} \cdot \text{tr} U_{\mathbf{x}} \right) + 8 \sum_{\mathbf{x}} \cos \left( P_{\text{col}}(\mathbf{x}) \right) \Re \text{Re} P_{\text{col}}(\mathbf{x}) + \ldots . \quad (141)
\]

If the \(U(1)_{em}\) Polyakov loop angle for quarks \(P_{\text{col}}(\mathbf{x}) = \sum_{t=0}^{N_t-1} \theta_{t}(t, \mathbf{x})/2\) is disordered, then \(P_{\text{col}} = 1\) sector is no longer favored and the SU(2) center symmetry is dynamically restored. The effect is analogous to placing a Potts model in a fluctuating magnetic field, or to the Peccei-Quinn mechanism [144], in which the coupling
of the CP violating term in QCD to an axion field allows for the dynamical restoration of CP symmetry.

Since the parallel transporters for quarks possess a $Z_2$ degree of freedom that the U(1)$_{em}$ gauge action is blind to, this is possible even in the Coulomb phase for integer electric charges. Indeed, the SU(2) Polyakov loop in the toy model with half-integer charged dynamical Wilson quarks is indistinguishable from the quenched result at values of the hopping parameter, or the quark mass, that would otherwise cause a significant amount of ordering in standard 2-color QCD. As shown in Figure 24, this is equally true when the U(1)$_{em}$ links are totally disordered in the confined phase $\beta_{em} \lesssim 1.01$, and in the Coulomb phase at $\beta_{em} \gtrsim 1.01$ of the U(1)$_{em}$. Since the U(1)$_{em}$ gauge action is blind to the $Z_2 \subset$ U(1)$_{em}$ disorder as seen by the quarks, i.e., $e^{\beta_{em} f^2} = \pm 1$, it is unable to remove it.

The quenched result in Fig. 24 reflects the second order nature of the phase transition of pure SU(2) in the universality class of the 3d Ising/$Z_2$ gauge system with typical finite-size corrections. It clearly turns into a crossover in full QC$_2$D at $\kappa = 0.15$ as in the Ising model with a symmetry-breaking external field. When the half-integer electromagnetic charges are included, the SU(2) Polyakov loop falls straight back onto the quenched curve indicating that there might be true disorder in the vacuum when QCD is embedded into the Standard Model.

This is not the full story, however. A comparison of the $\pi$ and $\rho$ meson masses in the toy model and in QC$_2$D with equal hopping parameter $\kappa$ and SU(2) gauge coupling $\beta_{col}$ as in [145] reveals that the mass scales have changed dramatically in the toy model [142]. One plausible interpretation is that there is strong $Z_2$ disorder even for $\beta_{em} \rightarrow \infty$ which adds to the strength of the confining potential. This effect might be unphysical. One needs to go back and verify scaling of the masses (or correlation lengths) in physical units as the continuum limit is approached which might not even exist without unification in the toy model.

Meanwhile, we have explored the phase diagram at the leading order $O(\kappa^2)$ in the hopping expansion for $N_f = 4$, with the effective action in Eq. (141), which allows us to cheaply check our intuition about the resulting couplings of SU(2) and U(1)$_{em}$ links, without having to worry about the chiral limit for $\kappa$. We thus take the leading-order hopping expansion at face value, as a model beyond its range of applicability as an approximation to the full toy model. This effective model then shares its qualitative features with a fractionally charged fundamental Higgs model [146].

Fig. 25 shows the results for the SU(2) Polyakov loop in the effective SU(2) × U(1) model with the leading order interactions in Eq. (141), where the U(1)$_{em}$ phases $e^{\beta_{em} \theta/2}$ are restricted to ±1, which amounts to $\beta_{col} \rightarrow \infty$ in the U(1)$_{em}$ gauge action. As $\kappa$ increases, the disorder-order deconfinement transition of the SU(2) Polyakov loop moves to smaller values of $\beta_{col}$ and sharpens dramatically. Note that in the combined limit $\kappa$, $\beta_{em} \rightarrow \infty$, the plaquette-plaquette coupling in Eq. (141) forces the SU(2) plaquettes to take the values ±1. For very large $\kappa$ the transition line should therefore terminate with the first order bulk transition of the 4d $Z_2$ gauge theory at $\beta_{col} \sim 0.44$ [147]. As the SU(2) plaquette is driven to unity for large $\beta_{col}$, the U(1)$_{em}$ plaquettes corresponding to quark loops, $\cos(\theta_{eff}/2)$, receive an effective coupling that suppresses $Z_2$ disorder. The relevant U(1)$_{em}$ Polyakov loop $\cos F_{\theta/2}$ remains disordered at small values of $\kappa$, but transitions to unity for large $\beta_{col}$ and $\kappa$ (not shown). Due to the $Z_2$ disorder transition, the model might have three distinct phases, the confined phase for small $\kappa$ and small $\beta_{col}$, the deconfined one for small $\kappa$ but $\beta_{col}<\beta_{col}$ and a Higgs phase in the corner where both couplings $\kappa$ and $\beta_{col}$ are large. We currently investigate the same effect of $Z_2$ disorder from half-integer electric charge in the phase diagram of fundamental SU(2) Higgs models [146].

Given the fractional electric charge of quarks and the existence of a global symmetry that relates the centers of the color and electroweak gauge groups, it may be misleading to study non-perturbative phenomena such as confinement and perhaps also dynamical mass generation in QCD alone. In the toy two-color model, the coupling of Wilson quarks with half-integer electric charge relative to a compact U(1)$_{em}$ has a dramatic effect on the color sector. This is not due to electromagnetic fluctuations which should have small effects deep in the Coulomb phase for large $\beta_{em}$, but it can be due
to a frozenler $Z_2$ disorder that the low energy theory inherits from the grand unified theory. Such frozen disorder could have an effect akin to a random external field in Potts models [148] which are known to have phase transitions where they would not otherwise have them, with constant external fields.

Here the SU(2) Polyakov loop shows an order-disorder transition that is consistent with the spontaneous center symmetry breaking transition of the pure gauge theory at values of the hopping parameter where center symmetry is clearly explicitly broken in standard two-color QCD. This happens despite the fact that static quarks are now described by the combined SU(2) and half-integer U(1)$_{\text{em}}$ loops which continue to show a crossover behavior. The scales of both transitions are still widely separated and this problem needs to be solved in such models.

Combined twisted boundary conditions are nevertheless possible with more than one gauge group and the corresponding center vortex ensembles might become relevant for a deconfinement phase transition in the standard model if its hidden center-like symmetry is relevant to the real world. Whether this is the case or not seems hard to decide without grand unification. Meanwhile one might want to find out whether SU(3) × $Z_3$ models of the kind of our two-color toy model in the $\beta_{\text{em}} \to \infty$ limit are at all suited as effective low energy theories for hadron physics. If we did have absolute confinement in the real world, separated by a true phase transition from the quark-gluon plasma, this would obviously make a conceptually and phenomenologically important difference as compared to our present understanding that quarks are only exponentially suppressed in a hadronic phase that is analytically connected to the high-temperature plasma.

4. Concluding remarks

Many interesting facets of strongly interacting matter can be studied in QCD-like theories. Changing the number of quark flavors, their masses or the number of colors often leads to simplifications or idealizations which allow to understand qualitative features and mechanisms from such powerful concepts as universality, scaling and finite-size scaling. They allow to relate phenomena from diverse areas of physics by often surprising analogies. The liquid-gas transition of nuclear matter, Heisenberg ferromagnets for chiral symmetry breaking, ultracold fermionic quantum gases with BEC-BCS crossover for two-color QCD at finite baryon density, the very well studied spin models and their dualities for the deconfinement transition and the electric fluxes in the pure gauge theories are some of the famous examples touched upon in these lecture notes.

The description of critical phenomena in statistical physics and quantum field theory is today synonymous with the renormalization group. Valuable introductory accounts of various aspects of the renormalization group have been given at this 49th Schladming Winter School. I have presented some selected examples of applications of basic renormalization group concepts towards a better understanding of chiral dynamics at finite density, quark confinement, the different phases of strongly interacting matter at extremes of temperature or density, and the nature of the transitions between them.

Arguably, two of the most promising non-perturbative tools to study the QCD phase diagram at the moment are the functional renormalization group and lattice simulations, especially in combination. These two complement each other very well. Non-perturbative functional methods in general rely on additional assumptions. These can be guided by intuition, derive from fundamental requirements such as causality, locality or gauge invariance, or they can be backed up by results from other approaches as external input. This is what lattice simulations can provide where they are possible without fermion sign problem. Therefore, it will be extremely valuable to investigate and ultimately completely understand the phase diagrams of QCD-like theories such as two-color QCD or the $G_2$ gauge theory which don’t have this problem.

A lot of work in this direction has been done in two-color QCD as reviewed in Section 2. Here we are naturally interested in the functional renormalization group treatment which we have seen to be well capable of describing the effects of finite baryon density and the competing dynamics of collective mesonic and baryonic fluctuations. The bosonic nature of the baryons in two-color QCD with a BEC-BCS crossover provides interesting analogies, but I have argued that the impact of the finite baryon density on the chiral transition reveals a genuine effect. There are reasons to believe that there might not be enough chiral symmetry breaking left for a chiral phase transition and a critical endpoint at finite baryon density in the QCD phase diagram either. Suitable theories with fermionic baryons but without fermion sign problem seem to be the logical next step to study at finite density. The $G_2$ gauge theory is such a theory and the studies of its phase diagram have only just begun. The challenge there will be to disentangle finite density effects due to bosonic from those due to the fermionic baryons, an ideal task for the next combined effort with the functional renormalization group and lattice Monte-Carlo simulations.
The confinement problem is obviously a hard one. But a better understanding on a fundamental level even in the simple 2 + 1 dimensional SU(N) gauge theories from exact results seems worthwhile to work our way up from there. I found some of these results such as the selfduality reflected in the center vortex and electric flux ensembles quite enlightening and perhaps also unexpected. The models may not be realistic but the results are quite powerful, and who knows, perhaps we will eventually find that it wasn’t wasted but that combined vortices and disorder in the vacuum may be relevant when QCD is embedded in the Standard Model.

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