OPTIMAL DESIGN PROBLEMS GOVERNED BY THE NONLOCAL \( p \)-LAPLACIAN EQUATION

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Abstract. In the present work, a nonlocal optimal design model has been considered as an approximation of the corresponding classical or local optimal design problem. The new model is driven by the nonlocal \( p \)-Laplacian equation, the design is the diffusion coefficient and the cost functional belongs to a broad class of nonlocal functional integrals. The purpose of this paper is to prove the existence of an optimal design for the new model. This work is complemented by showing that the limit of the nonlocal \( p \)-Laplacian state equation converges towards the corresponding local problem. Also, as in the paper by F. Andrés and J. Muñoz [J. Math. Anal. Appl. 429:288–310], the convergence of the nonlocal optimal design problem toward the local version is studied. This task is successfully performed in two different cases: when the cost to minimize is the compliance functional, and when an additional nonlocal constraint on the design is assumed.

1. Introduction. Formulation of the problem. The nonlocal formulation is one of the main alternatives to reformulate problems of Applied Mathematics [15, 46]. The particular context of diffusion modelling, including anomalous diffusion and turbulence, is a field where many efforts have been done to develop rigorous procedures aiming at explaining phenomena related to Thermodynamics and Kinetic Theory of Molecular Motion [43, 35, 16, 36, 27, 8, 38]. In connection with this task it is widely accepted the interest generated by the nonlocal \( p \)-Laplacian operator. This type of equation has been employed as an accurate tool to model different sorts of diffusion. See for instance [44, 45, 28].

The nonlocal setting has attracted the attention of a large number of researchers. To a great extent, this is caused by the capability of the model in order to reproduce...
phenomena in many of the branches of science. Another motivation to apply non-local models is because of the smoothness requirement that solutions of the state equation must satisfy. One of the main reasons to use the nonlocal formulation is to overcome the difficulty this strong condition entails. So, instead of considering partial differential equations to carry out the modelization, a certain type of integral equation is used. Besides, the functional to be minimized in optimal control, the cost, is often replaced by its corresponding nonlocal form. The integral differential equation, even the cost, are characterized by the presence of a kernel that has basically the role of a filter for the states. This kernel depends on a parameter $\delta$, called horizon, that plays the role of internal length scale for the interaction at two different points of the set where we are solving the equation. Thus, the above parameter serves to measure the degree of nonlocality and this is usually done with the peculiarity that the final constitutive model allows some type of discontinuous states. For consistency, it is to be expected that the nonlocal formulation coincides, somehow, with the classical formulation when $\delta$ tends to zero. Unfortunately, there is no clear way to proceed in order to prove that the limit of a sequence of states (and controls) converge to a limit obeying a concrete local state equation. A crucial point in this process relies on the fact that the analyzed sequences are minimizing ones. The consequences of such a choice of sequences and the particular format of the cost functional are, in some cases, crucial for the $G$-convergence study of the state equation. Another relevant aspect is the regularity expected for the controls when we have in mind a bound on the size of the mesh of our numerical approximation. If under certain circumstances it is assumed a uniform bound for the oscillations of the optimal sequences, then the $G$-convergence for the state equation is a goal that could likely be achieved.

The present article refers only to the stationary diffusion case. It is devoted to the analysis of optimal design problems whose state equation is the nonlocal $p$-Laplacian. The diffusion coefficient plays the role of the control or design, and the boundary condition is the corresponding nonlocal boundary Dirichlet constraint. The purpose of the paper is twofold: the goal of the first part is the existence of a nonlocal optimal design. This objective is achieved for a fairly general class of cost functionals. The aim of the second part is the convergence of the nonlocal problem toward the local optimal design one. In a first stage we prove $G$-convergence of the state equation to the classical $p$-Laplacian. Then, we face the study of convergence for the optimal design problem. There are two cases where the convergence is accomplished: the first occurs when the cost we want to minimize is the compliance functional. This case gives rise to an important optimal problem in Diffusion or in Conductivity Theory where, for some purposes, the compliance functional is considered as the energy dissipated in the domain (see [18, 1, 20]). The second instance for which the convergence is proved is for a type of problem whose formulation includes some additional nonlocal constraint for the controls (see (46) below). As we shall see, the additional ingredient is to assume a uniform bound on the controls in order to prevent abrupt oscillations and to achieve some sort of convergence that could facilitate the pass to the limit.

To summarize, and as a corollary of the above comments, we could underline the importance of the contributions of this research in the following points:
1. The obtainment of a general existence result of nonlocal optimal designs provides the opportunity to ascertain how accurate is the model if we want to reproduce a specific physical model of diffusion.
2. The existence result we have derived establishes several frameworks where we can perform different numerical schemes to approximate optimal designs.
3. The proposed nonlocal setting serves to tackle the difficulties of the $G$-convergence process, and therefore, it facilitates the recovering of the corresponding local problem when the horizon tends to zero.
4. The pass to the limit to recover the local setting provides meaningful information about the qualitative behavior of the sequences of minimizers.
5. This is especially significant when we deal with the compliance functional. In that case, the convergence of the nonlocal optimal design problem allows for proving an existence result in the local setting (Corollary 1). This is the extension to the case $p > 1$ of the theorem proved in the celebrated article [17] for $p = 2$.

Even though some papers analyzing this topic have already appeared within the specialized literature, there is not much about the case in which the state equation is the nonlocal $p$-Laplacian, with $p \neq 2$. Concerning such a nonlinear nonlocal equation, we recall essential advances [6, 7, 13, 32]. This type of formulation has served to model interesting physical situations in which the fundamental issues of existence and uniqueness of solution, were previously guaranteed. See [2, 30, 25, 26, 10] for a framework quite similar to the one described here. In relation to the existence of optimal controls (or designs), some articles have dealt with this aspect with a certain generality [21, 4, 22, 23, 12]. About the analysis of $G$-convergence or $\Gamma$-convergence the reader can consult [40, 47, 21, 34, 33, 5]. Even though there are not so many results about explicit computations of the limit, we can find some theoretical advances. In this sense we must underline [34, 11, 9].

Before proceeding we will make a few remarks concerning the concrete problems to be solved and the assumed hypotheses.

1.1. Notation and hypotheses. In the following, we make precise some notation and the spaces where we shall work.

1. The domain $\Omega$ is a bounded Lipschitz domain in $\mathbb{R}^N$ and

$$\Omega_\delta = \Omega \cup \{ \cup_{p \in \partial \Omega} B(p, \delta) \},$$

where $B(x, r)$ is the notation for an open ball centered at $x \in \mathbb{R}^N$ and radius $r > 0$, and $\delta \leq \delta_0$ where $\delta_0 > 0$ is a given small number.

2. About the source term of the elliptic equation that we are going to solve: we assume

$$f \in L_0^{p'}(\Omega_\delta) = \left\{ g : \Omega_\delta \to \mathbb{R} \mid g \in L^{p'}(\Omega), \ g = 0 \text{ in } \Omega_\delta - \Omega \right\},$$

where $p' = \frac{p}{p-1}$ and $p > 1$.

3. The set of designs participating in the minimization principle is

$$\mathcal{H} = \{ h : \Omega_\delta \to \mathbb{R} \mid h(x) \in [h_{\text{min}}, h_{\text{max}}] \text{ a.e. } x \in \Omega, \ h = 0 \text{ in } \Omega_\delta - \Omega \},$$

where $h_{\text{min}}$ and $h_{\text{max}}$ are positive constants such that $0 < h_{\text{min}} < h_{\text{max}}$.

1. The sequence of kernels $(k_\delta)_{\delta \leq \delta_0}$ is a family of radial functions such that for every $\delta \leq \delta_0$ the following properties hold:
We also need to specify the functional space $X_0$ we shall deal with. For $p > 1$ we define the space $X = X(\delta)$ as

$$X = \{ u \in L_p^0(\Omega_\delta) : B(u, u) < \infty \},$$

where $B$ is the operator defined in $X \times X$ by means of the formula

$$B(u, v) = \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x' - x|) \frac{|u(x') - u(x)|^p}{|x' - x|^p} \times (u(x') - u(x)) (v(x') - v(x)) \, dx' \, dx.$$
for any admissible pair \((h,u)\) (the set of admissibility will be defined later), where the integrand is a function \(F : \Omega_\delta \times \Omega_\delta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) verifying the following conditions:

1. \(F (\cdot,\cdot, u_1, u_2)\) is measurable for any \((u_1, u_2) \in \mathbb{R} \times \mathbb{R}\)
2. \(F (x_1, x_2, \cdot, \cdot)\) is a lower semicontinuous function for any \((x_1, x_2) \in \Omega_\delta \times \Omega_\delta\).

Next, we describe the problems, both local and nonlocal:

1.2. The local design problem. One of our goals is to solve or approximate the minimization problem \((\mathcal{P}_{loc})\) described as follows:

\[
\inf_{(h,u) \in \mathcal{A}_{loc}} J_{loc} (h,u) \tag{7}
\]

where, the cost is the functional

\[
J_{loc} (h,u) = \int_{\Omega} G (x, h, u, \nabla u) \, dx \tag{8}
\]

and \(G\) is certain function that would be appropriately defined. The admissibility set is

\[
\mathcal{A}_{loc} = \{(h,u) \in \mathcal{H} \times W^{1,p}_0 (\Omega) : (h,u) \text{ solves (10)}\} \tag{9}
\]

and the local state equation is

\[
- \text{div} \left( h |\nabla u|^{p-2} \nabla u \right) = f, \text{ in } W^{1,p}_0 (\Omega). \tag{10}
\]

It is said that \((h,u)\) solves (10) if

\[
\int_{\Omega} h (x) |\nabla u|^{p-2} \nabla u (x) \nabla w (x) \, dx = \int_{\Omega} f (x) w (x) \, dx \tag{11}
\]

for any \(w \in W^{1,p}_0 (\Omega)\).

In connection with this equation we shall consider the operator \(b_h (\cdot,\cdot)\) defined in \(W^{1,p}_0 (\Omega) \times W^{1,p}_0 (\Omega)\) through the formula

\[
b_h (u,v) = \int_{\Omega} h (x) |\nabla u|^{p-2} \nabla u (x) \nabla v (x) \, dx. \tag{12}
\]

1.3. The nonlocal design problem. The beginning of our investigation is the analysis of the following minimization problem \((\mathcal{P}_\delta)\):

\[
\inf_{(h,u) \in \mathcal{A}_\delta} J_\delta (h,u) \tag{13}
\]

where the cost functional is defined as

\[
J_\delta (h,u) = \int_{\Omega_\delta} \int_{\Omega_\delta} F (x', x, u', u) \, dx' \, dx \tag{14}
\]

with \(F\) fulfilling the above properties.

The admissibility set is

\[
\mathcal{A}_\delta = \{(h,u) \in \mathcal{H} \times X_0 : (h,u) \text{ solves (15)}\} \tag{14}
\]

and the nonlocal state equation is

\[
B_h (u,w) = (f,w)_{X_\delta^0 \times X_0} \tag{15}
\]
for any \( w \in X_0 \), that is
\[
\int_{\Omega_h} \int_{\Omega_h} H(x', x) k_3(|x' - x|) \\
\times \frac{|u(x') - u(x)|^{p-2}(u(x') - u(x))(w(x') - w(x))}{|x' - x|^p} \ dx'dx = \int_{\Omega_h} f \ dx dx
\] (16)

for any \( w \in X_0 \), where \( H(x', x) = \frac{h(x') + h(x)}{2} \).

For every \( u \in X_0 \) (with \( h \) and \( \delta \) fixed), we define the operator
\[
\mathcal{L}_h(u)(x) = -2 \int_{\Omega_h} H(x', x) k_3(|x' - x|) \frac{|u(x') - u(x)|^{p-2}(u(x') - u(x))}{|x' - x|^p} \ dx',
\]
whose action on any element of \( X_0 \) is defined by the formula
\[
\langle \mathcal{L}_h(u), w \rangle_{X'_h \times X_h} = \int_{\Omega} \mathcal{L}_h(u)(x) \ w(x) \ dx.
\]

Then \( \mathcal{L}_h(u) \in X'_0 \) and \( \langle \mathcal{L}_h(u), w \rangle_{X'_h \times X_h} = B_h(u, w) \).

**Remark 1.** Both in the local and the nonlocal problem, Section 6 addresses a case in which \( H \) includes an additional nonlocal constraint.

1.4. **Results and organization of the paper.**

1. First of all, in Section 2 we shall recall some essential tools.
2. The first result derived from this research is the existence of solutions for the nonlocal optimal design when the compliance is the cost functional. See Theorem 3.1 in Section 3.
3. The aim of Section 4 is to generalize the above existence result for a wide class of cost functionals (see Theorem 4.2). This entails a \( G \)-convergence result for the state equation when the horizon is fixed (see Theorem 4.1).
4. A previous result about the \( G \)-convergence of the nonlocal \( p \)-Laplacian to the local version is proved in Section 5 (Theorem 5.1). Then, a convergence result of the nonlocal design problem (\( P_3 \)) towards the local design problem (\( P_{loc} \)) is proved when the compliance functional is assumed to be the cost functional. See Theorem 5.2. As a by-product, we prove an existence result for the local problem (Corollary 1).
5. Section 6 contains the analysis of the convergence of general nonlocal design problems under the additional assumption of a nonlocal bound on the controls. See Theorems 6.1 and 6.2.

2. **Preliminaries.** Here we review some technical tools we are going to use.

1. The key points in our research are the existence and uniqueness of solution to the local state equation, and the corresponding minimization energy functional. A remarkable fact that will be employed later is the characterization of (10) by means of the Dirichlet principle:

**Theorem 2.1.** Assume \( h \in \mathcal{H} \) and \( f \in L^p' (\Omega) \) are fixed. Then there exists a unique solution \( u \in W^{1,p}_0(\Omega) \) to the problem (10) and it is characterized as the solution of the minimization principle
\[
\min_{w \in W^{1,p}_0(\Omega)} E_{\text{loc}}(w),
\]
where
\[ E_{\text{loc}}(w) \doteq \frac{1}{p} b_h(w, w) - \int_{\Omega} f(x) w(x) \, dx. \]
Moreover,
\[ E_{\text{loc}}(u) = \left( \frac{1}{p} - 1 \right) b_h(u, u). \]

For the proof, we just need to adapt (because we have to include \( h \)) and follow the detailed account given in [19, Theorems 17.1 and 17.5]. Notice that one could assume only \( f \in W^{-1,p'}(\Omega) \) and get the same existence result.

2. Compactness embedding: the embedding
\[ X_0 \subset L^p_0(\Omega_\delta) \]
is compact. To clarify this statement we assume there is a positive constant \( C \) such that,
\[ B_{h_j}(u_j, u_j) \leq C \text{ for every } j \tag{17} \]
for a sequence of controls \( (h_j)_j \) and the corresponding sequence of states \( (u_j)_j \).

If we take into account the nonlocal Poincaré inequality
\[ C \|u_j\|_{L^p} \leq B_{h_j}(u_j, u_j) \tag{18} \]
(see [31, Theorem 1, p. 798]), and we pay attention to the hypotheses on the kernel, then we are allowed to write
\[ \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{|u_j(x') - u_j(x)|^p}{|x' - x|^{N+sp}} \, dx' \, dx \leq C. \tag{19} \]
From (17) and (18) we infer the existence of \( u \in L^p(\Omega_\delta) \) such that, for a subsequence of \( (u_j)_j \), still denoted by \( u_j, u_j \rightharpoonup u \) weakly in \( L^p(\Omega_\delta) \).

From (19) it is well-known that we can extract a subsequence of \( (u_j)_j \), denoted also by \( (u_j)_j \), strongly convergent to \( u \) in \( L^p(\Omega_\delta) \) (see [24, Section 7] for the details).

3. Compact operator: let \( (h_\delta, u_\delta)_\delta \) be a sequence of admissible pairs for the nonlocal problem such that the uniform estimation
\[ B_{h_\delta}(u_\delta, u_\delta) \leq C, \]
is fulfilled (where \( C \) is a positive constant). Then, from \( (u_\delta)_\delta \) we can extract a subsequence, labelled by \( u_\delta \), such that \( u_\delta \rightharpoonup u \) strongly in \( L^p_0(\Omega) \) and \( u \in W^{1,p}_0(\Omega) \). See [14, Th. 4] and [39, Th. 1.2].

4. One of the main foundations is the existence and uniqueness of solution to the state equation. As in the local case, the solution coincides with the minimizer of the corresponding functional energy.

**Theorem 2.2.** Assume that \( h \) and \( \delta \) are fixed. If \( f \in L^p(\Omega) \) then there exists a unique solution \( u_\delta \in X_0 \) to the problem (15) and it is characterized as the solution of the minimization principle
\[ \min_{w \in X_0} E_\delta(w) \]
where
\[ E_\delta(w) \doteq \frac{1}{p} B_h(w, w) - \int_{\Omega_\delta} f(x) w(x) \, dx. \]
Moreover,

\[ E_\delta (u_\delta) = \left( \frac{1}{p} - 1 \right) B_h (u_\delta, u_\delta). \]

The proof of this result is standard and can be consulted in [11, Corollary 2.4] or [30, Theorems 3.9 and 3.12]. Again, as in the local case, \( f \) can be assumed to be in \( X_0' \) to obtain the same result of existence.

3. Existence of nonlocal optimal designs for the compliance case. The first existence result treats a particular cost, the compliance (see [17] and [4]). This functional is defined as

\[ J_\delta (h, u) = \int_\Omega f (x) u (x) \, dx \]

where \((h, u)\) is any pair solution to the problem (15). It is clear that

\[ J_\delta (h, u) = \int_{\Omega_\delta} \int_{\Omega_\delta} F (x', x, u', u) \, dx' \, dx \]

if we consider the nonlocal integrand

\[ F (x', x, u', u) = \frac{f (x) u (x) + f (x') u (x')}{{\Vert h \Vert}^2}. \]

**Theorem 3.1.** If \( F \) is defined as in (21) and \( f \) is assumed to be in \( L^{p'} (\Omega) \), then there exists an optimal design to the problem (13).

**Proof.** Let \((h_j, u_j)\) be a minimizing sequence of pairs for the problem (13), that is

\[ \lim_{j \to \infty} J_\delta (h_j, u_j) = \inf_{(h, u) \in A_\delta} J_\delta (h, u). \]

Let \( h \) be the weak-\* limit in \( L^\infty (\Omega) \) of \( h_j \) and let \( u \) be the corresponding nonlocal state.

Since each pair \((h_j, u_j)\) is the solution in \( \mathcal{H} \times X_0 \) of the variational problem (15), then

\[ u_j = \arg \min_{X_0} \left\{ \frac{1}{p} B_h (w, w) - \int_{\Omega_\delta} f w \, dx \right\} \]

Thanks to Theorem 2.2 we have

\[
\left( \frac{1}{p} - 1 \right) \int_{\Omega_\delta} \int_{\Omega_\delta} H_j \frac{k_\delta (|x' - x|)}{|x' - x|^p} |u_j (x') - u_j (x)|^p \, dx' \, dx \\
= \frac{1}{p} \int_{\Omega_\delta} \int_{\Omega_\delta} H_j \frac{k_\delta (|x' - x|)}{|x' - x|^p} |u_j (x') - u_j (x)|^p \, dx' \, dx - \int_{\Omega_\delta} f u_j \, dx \\
\leq \frac{1}{p} \int_{\Omega_\delta} \int_{\Omega_\delta} H_j \frac{k_\delta (|x' - x|)}{|x' - x|^p} |u (x') - u (x)|^p \, dx' \, dx - \int_{\Omega_\delta} f u \, dx
\]

If we pass to the limit we get

\[
\lim_{j \to \infty} \left( \frac{1}{p} - 1 \right) \int_{\Omega_\delta} \int_{\Omega_\delta} H_j \frac{k_\delta (|x' - x|)}{|x' - x|^p} |u_j (x') - u_j (x)|^p \, dx' \, dx \\
\leq \frac{1}{p} \lim_{j \to \infty} \int_{\Omega_\delta} \int_{\Omega_\delta} H_j \frac{k_\delta (|x' - x|)}{|x' - x|^p} |u (x') - u (x)|^p \, dx' \, dx - \int_{\Omega_\delta} f u \, dx \\
= \frac{1}{p} \int_{\Omega_\delta} \int_{\Omega_\delta} H \frac{k_\delta (|x' - x|)}{|x' - x|^p} |u (x') - u (x)|^p \, dx' \, dx - \int_{\Omega_\delta} f u \, dx
\]
which is true because
\[
\int_{\Omega_h} \frac{k_{h} (|x' - x|)}{|x' - x|^p} |u (x') - u (x)|^p \, dx' \in L^1 (\Omega)
\]
and \( h_j \rightharpoonup h \) weak-\* limit in \( L^\infty (\Omega) \). Notice the above limit exists since the sequence of solutions of the state equation, \((u_j)\), is uniformly bounded in \( L^p \) and therefore
\[
\lim_{j \to \infty} \int_{\Omega_h} \int_{\Omega_h} H_j \frac{k_{h} (|x' - x|)}{|x' - x|^p} |u_j (x') - u_j (x)|^p \, dx' \, dx = \lim_{j \to \infty} \int_{\Omega_h} f u_j \, dx.
\]
In either case, we could take \( \text{lim inf} \) instead and proceed in the same way.

If we take into account that the pair \((h, u)\) minimizes the functional
\[
\frac{1}{p} \int_{\Omega_h} \int_{\Omega_h} H \frac{k_{h} (|x' - x|)}{|x' - x|^p} |w (x') - w (x)|^p \, dx' \, dx - \int_{\Omega_h} f w \, dx, \quad w \in X_0,
\]
then it is clear that the above estimate yields the following inequality:
\[
\left( \frac{1}{p} - 1 \right) \lim_{j \to \infty} \int_{\Omega_h} \int_{\Omega_h} H_j \frac{k_{h} (|x' - x|)}{|x' - x|^p} |u_j (x') - u_j (x)|^p \, dx' \, dx
\leq \left( \frac{1}{p} - 1 \right) \int_{\Omega_h} \int_{\Omega_h} H \frac{k_{h} (|x' - x|)}{|x' - x|^p} |u (x') - u (x)|^p \, dx' \, dx
\]
which is equivalent to
\[
\lim_{j \to \infty} \int_{\Omega_h} \int_{\Omega_h} H_j \frac{k_{h} (|x' - x|)}{|x' - x|^p} |u_j (x') - u_j (x)|^p \, dx' \, dx
\geq \int_{\Omega_h} \int_{\Omega_h} H \frac{k_{h} (|x' - x|)}{|x' - x|^p} |u (x') - u (x)|^p \, dx' \, dx,
\]
or in other words,
\[
\lim_{j \to \infty} J_\delta (h_j, u_j) \geq J_\delta (h, u).
\]
By using the fact that \((h_j, u_j)\) is a minimizing sequence, we obtain the equality
\[
\lim_{j \to \infty} J_\delta (h_j, u_j) = J_\delta (h, u)
\]
and thereby \((h, u)\) is an optimal solution of the problem. \( \square \)

4. **Existence of nonlocal optimal designs: A general case.** The cost functional we optimize is
\[
J_\delta (h, u) = \int_{\Omega_h} \int_{\Omega_h} F (x', x, u', u) \, dx' \, dx
\]
where \((h, u)\) is any admissible pair and \( F \) is assumed to verify the properties given at Subsection 1.1.

The key feature of the problem is in the following result of convergence:

**Theorem 4.1.** Let \( f \) be in \( L^p (\Omega) \). From any sequence of admissible pairs \((h_j, u_j) \in A_\delta\) we can extract a subsequence, still denoted by \((h_j, u_j)\), and a pair \((h, u) \in A_\delta\) such that \( h_j \rightharpoonup h \) weak-\* in \( L^\infty (\Omega) \) and \( u_j \rightharpoonup u \) strongly in \( L^p (\Omega) \).
The theorem establishes that \((u_j)\), the sequence of solutions of the state equation whose control is \(h_j\), weakly (eventually, strongly thanks to the compactness of the operators \(B_{h_j}\)) converges in \(L^p\) to \(u\), the solution to the state equation whose associated control is \(h\), the weak-\(^\ast\) limit in \(L^\infty\) of \(h_j\). In other words,

\[
\lim_{j \to \infty} \min_{X_0} \left\{ \frac{1}{p} B_{h_j} (w, w) - \int_{\Omega_\delta} fwdx \right\} = \min_{w \in X_0} \left\{ \frac{1}{p} B_h (w, w) - \int_{\Omega_\delta} fwdx \right\} \tag{23}
\]

\[
= \frac{1}{p} B_h (u, u) - \int_{\Omega_\delta} fudx. \tag{24}
\]

**Proof.** Let \((h_j, u_j)\) be any admissible sequence of pairs. Then

\[u_j = \arg \min_{X_0} \left\{ \frac{1}{p} B_{h_j} (w, w) - \int_{\Omega_\delta} fwdx \right\}.
\]

Since there is a subsequence such that \(h_j \rightharpoonup h\) weak-\(^\ast\) in \(L^\infty (\Omega)\), then we can consider the corresponding state \(u\). Thus \((h, u) \in A\) and therefore

\[u = \arg \min_{X_0} \left\{ \frac{1}{p} B_h (w, w) - \int_{\Omega_\delta} fwdx \right\}.
\]

As in the previous section, we realize that the less or equal part of (23) is proved:

\[
\lim_{j \to \infty} \min_{X_0} \left\{ \frac{1}{p} B_{h_j} (w, w) - \int_{\Omega_\delta} fwdx \right\} \leq \min_{X_0} \left\{ \frac{1}{p} B_h (w, w) - \int_{\Omega_\delta} fwdx \right\}
\]

\[
= \frac{1}{p} \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{H_{k_\delta} (|x' - x|)}{|x' - x|^p} |u(x') - u(x)|^p dx'dx - \int_{\Omega_\delta} fudx
\]

\[
= \min_{X_0} \left\{ \frac{1}{p} B_h (w, w) - \int_{\Omega_\delta} fwdx \right\}.
\]

To check the reverse inequality we take into account the following facts:

1. Since \(u_j\) is a uniformly bounded sequence in \(L^p (\Omega)\) and there is a constant \(C > 0\) such that for every \(j\)

\[B_{h_j} (u_j, u_j) \leq C,
\]

then we can find a subsequence of \(u_j\) (which is still denoted by \(u_j\)) strongly convergent in \(L^p\) to a function \(u^* \in L^p_0 (\Omega_\delta)\).

2. Define now the measures \(\mu_j\) and \(\mu\), both on \(\Omega_\delta \times \Omega_\delta\), by means of the formulae

\[\mu_j (A) = \int A H_j dx'dx
\]

and

\[\mu (A) = \int A H dx'dx,
\]

where \(A\) is any measurable set included in \(\Omega_\delta \times \Omega_\delta\). Then it is immediate to be convinced of this setwise convergence:

\[\lim_{j \to \infty} \mu_j (A) = \mu (A).
\]
3. Since, additionally, we have the a.e. convergence
\[ \lim_{j \to \infty} \frac{k_\delta (|x'| - |x|)}{|x'| - |x|^p} |u_j (x') - u_j (x)|^p = \frac{k_\delta (|x'| - |x|)}{|x'| - |x|^p} |u^* (x') - u^* (x)|^p, \] (25)
we are allowed to use the Generalized Fatou’s Lemma (see [42, 29]) and thus, to write
\[
\liminf_{j \to \infty} \int_{\Omega} \int_{\Omega} H \frac{k_\delta (|x'| - |x|)}{|x'| - |x|^p} |u_j (x') - u_j (x)|^p d\nu' d\nu
\geq \int_{\Omega} \int_{\Omega} H \frac{k_\delta (|x'| - |x|)}{|x'| - |x|^p} |u^* (x') - u^* (x)|^p d\nu' d\nu.
\]

4. Consequently, we easily derive the reverse inequality:
\[
\liminf_{j \to \infty} \left(\frac{1}{p} B_{h_j} (u_j, u_j) - \int_{\Omega} f u_j d\nu\right)
\geq \frac{1}{p} \int_{\Omega} \int_{\Omega} H \frac{k_\delta (|x'| - |x|)}{|x'| - |x|^p} |u^* (x') - u^* (x)|^p d\nu' d\nu - \int_{\Omega} f u^* d\nu
\geq \frac{1}{p} \int_{\Omega} \int_{\Omega} H \frac{k_\delta (|x'| - |x|)}{|x'| - |x|^p} |u (x') - u (x)|^p d\nu' d\nu - \int_{\Omega} f u d\nu
= \min_{\mathcal{X}_0} \left\{ \frac{1}{p} B_h (w, w) - \int_{\Omega} f w d\nu \right\}.
\]

By gathering the above estimates we conclude that \( u = u^* \) and the \( G \)-convergence result (23)-(24) holds.

**Remark 2.** In particular we have proved the limit
\[
\lim_{j \to \infty} B_{h_j} (u_j, u_j) = B_h (u, u),
\]
or equivalently
\[
\lim_{j \to \infty} \int_{\Omega} \int_{\Omega} H \frac{k_\delta (|x'| - |x|)}{|x'| - |x|^p} |u_j (x') - u_j (x)|^p d\nu' d\nu
= \int_{\Omega} \int_{\Omega} H \frac{k_\delta (|x'| - |x|)}{|x'| - |x|^p} |u (x') - u (x)|^p d\nu' d\nu.
\]

**Remark 3.** If we consider \( f \in X_0' \), then, in an abstract setting, we could reformulate Theorem 4.1 in the following way: let \( A_j : X_0 \to X_0' \) be the sequence of operators defined by means of \( w \to A_j (w) \) where
\[
(A_j (w), v) = B_{h_j} (w, v), \quad v \in X_0.
\]
If for each \( f \in X_0' \) the expression \( A_j^{-1} f = u_j \) means that \( A_j (u_j) = f \), then the existence and uniqueness of solution to this latter equation is ensured by Theorem 2.2. This solution can be characterized as the solution of the minimization principle defined in the left part of (23).

It is said that the sequence of operators \( (A_j)_j \) \( G \)–converges if \( j \to +\infty \), to the operator \( A_0 \), which is defined via the formula
\[
(A_0 (w), v) = B_h (w, v), \quad v \in X_0,
\]
if for any \( f \in X_0' \) the following convergence holds
\[
A_j^{-1} f \rightharpoonup A_0^{-1} f \text{ weakly in } X_0, \quad (26)
\]
where \( u = A_0^{-1} f \) denotes the only solution in \( X_0 \) of \( A_0 (u) = f \) (the minimizer of the right part of (23)).

In the present research, Theorem 4.1 just states \( A_j^{-1} f \rightharpoonup A_0^{-1} f \) weakly in \( L^p \). A more accurate analysis could provide the proof of the weak convergence in \( X_0 \) of (26). As far as the authors know, this result has only been proved for certain kind of kernels (see [11, 9]). See also [4, Th. 6] for the case \( p = 2 \).

Now, we are ready to state and prove our main result:

**Theorem 4.2.** For any given \( \delta \), there exists a solution \((h_\delta, u_\delta) \in A_\delta\) of the problem (13).

**Proof.** Let \((h_j, u_j)\) be a minimizing sequence. We can extract a subsequence from \( h_j \) that converges weak-* in \( L^\infty \) to \( h \). Besides, there is a subsequence of \( u_j \) strongly convergent in \( L^p \) to a function \( u \). Moreover, by Theorem 4.1 we know the pair \((h, u)\) is admissible. By applying Fatou's Lemma we derive the inequality
\[
\lim_{j \to \infty} \int_{\Omega} \int_{\Omega} F(x', x, u', u_j) \, dx' \, dx \geq \int_{\Omega} \int_{\Omega} F(x', x, u', u) \, dx' \, dx,
\]
which amounts to say the pair \((h, u)\) is a solution of the design problem (13). \( \square \)

5. G-Convergence: Approximation to the local compliance case.

5.1. **G-convergence for the state equation.** A previous key issue is the convergence of the nonlocal state equation to the local one. As we shall check, the proof reveals and contains the main features for proofs of the remaining results of the paper.

In the present context \( G \)-convergence essentially reads as in the previous section. The sequence of operators is given through the parameter \( \delta \):

\[
(A_\delta (w), v) = B_h (w, v), \quad v \in X_0.
\]

The same comments from the above section apply in this setting. However, there is a difference in how the limit problem is defined: now, the design \( h \) is fixed, the index is the horizon \( \delta \), and the limit operator is \( A_0 \), where \((A_0 (w), v) = b_h (w, v), \quad v \in W_0^{1, p} (\Omega)\). Under these circumstances, the sequence \((A_\delta)\) \( G \)-converges to \( A_0 \) if for any \( f \in X'_0 \) the weak convergence \( A_\delta^{-1} f \rightharpoonup A_0^{-1} f \) holds weakly in \( X_0 \) if \( \delta \to 0 \). Next theorem, as in the previous section, establishes this convergence only in \( L^p \).

**Theorem 5.1.** Assume \( h \) is a fixed control, \( f \in L^p (\Omega) \) and \((u_\delta)_\delta \) is the sequence of solution of (15). Then there exists a subsequence of \((u_\delta)_\delta \), which will be not relabeled, and a function \( u \in W_0^{1, p} (\Omega) \) such that \( u_\delta \to u \) strongly in \( L^p (\Omega) \) if \( \delta \to 0 \), and \((h, u)\) solves (10). Or equivalently
\[
\lim_{\delta \to 0} \min_{w \in X_0} \left\{ \frac{1}{p} B_h (w, w) - \int_{\Omega} f(x) w(x) \, dx \right\} = \min_{v \in W_0^{1, p} (\Omega)} \left\{ \frac{1}{p} b_h (v, v) - \int_{\Omega} f(x) v(x) \, dx \right\}.
\] (27)

**Proof.** Let \( m_\delta \) and \( m \) be the minima from the right and left parts of (27), respectively.
1. We notice

\[ m = \frac{1}{p} \int_{\Omega} h(x) |\nabla u(x)|^p \, dx - \int_{\Omega} f(x) \, u(x) \, dx. \]

Also we know

\[ m_\delta = \frac{1}{p} \int_{\Omega_\delta} \int_{\Omega_\delta} H(x', x) k_\delta(|x' - x|) \left( \frac{|u_\delta(x') - u_\delta(x)|}{|x' - x|^p} \right) \, dx' \, dx \]

\[ - \int_{\Omega_\delta} f(x) \, u_\delta(x) \, dx. \]

Since the \( u_\delta \) are the solutions, \( (u_\delta)_\delta \) is bounded in \( L^p \) and \( \lim_{\delta \to 0} m_\delta \) exists (at least for a subsequence). Besides, it is clear that

\[ \lim_{\delta \to 0} m_\delta \leq \lim_{\delta \to 0} \frac{1}{p} \int_{\Omega_\delta} \left( \int_{\Omega_\delta} k_\delta(|x' - x|) \frac{|u(x') - u(x)|^p}{|x' - x|^p} \, dx' \right) h(x) \, dx \]

\[ \leq \int_{\Omega} h(x) |\nabla u(x)|^p \, dx. \quad (28) \]

(see \cite{14, Corollary 1}). If we assume \( u \) is smooth and we previously fix a small radius \( r > 0 \), then there is a positive constant \( C \) such that the following inequalities hold for any \( x \in \Omega \):

\[ \int_{\Omega \setminus \Omega_\delta} k_\delta(|x' - x|) \frac{|u(x') - u(x)|^p}{|x' - x|^p} \, dx' \]

\[ \leq \int_{\Omega \setminus B(x, r)} k_\delta(|x' - x|) \frac{|u(x') - u(x)|^p}{|x' - x|^p} \, dx' \]

\[ \leq C \int_{\mathbb{R}^n \setminus B(0, r)} k_\delta(|z|) \, dz \quad (29) \]

By performing the decomposition

\[ \int_{\Omega_\delta \times \Omega_\delta} \, dx' \, dx = \int_{\Omega \times \Omega} \, dx' \, dx + 2 \int_{\Omega \times \Omega_\delta} \, dx' \, dx + \int_{\Omega_\delta \times \Omega \setminus \Omega_\delta} \, dx' \, dx \]

and using (28)-(29), and hypothesis (2), next identity follows:

\[ \lim_{\delta \to 0} \int_{\Omega_\delta} \left( \int_{\Omega_\delta} k_\delta(|x' - x|) \frac{|u(x') - u(x)|^p}{|x' - x|^p} \, dx' \right) h(x) \, dx \]

\[ = \int_{\Omega} h(x) |\nabla u(x)|^p \, dx. \quad (30) \]

If we proceed by a density argument in \( W_0^{1,p}(\Omega) \) and we take into account the estimation

\[ \lim_{\delta \to 0} \int_{\Omega_\delta} \left( \int_{\Omega_\delta} k_\delta(|x' - x|) \frac{|U(x') - U(x)|^p}{|x' - x|^p} \, dx' \right) h(x) \, dx \]

\[ \leq C \|\nabla U\|_{L^p(\Omega)}^p \quad (31) \]
for every $u \in W_0^{1,p}(\Omega)$ (see [14, Theorem 1]), then the same assertion, the one given in (30), remains true for any $u \in W_0^{1,p}(\Omega)$. Indeed: for the proof we take $(U_j)_j \subset C_0^\infty(\Omega)$, a sequence strongly convergent to $U$ in $W_0^{1,p}(\Omega)$ and we perform the decomposition

\[ I = \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x' - x|) \frac{|U(x') - U(x)|^p}{|x' - x|^p} h(x) \, dx' \, dx = I_1 + I_2 \quad (32) \]

where

\[ I_1 = \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x' - x|) \left( \frac{|U(x') - U(x)|^p}{|x' - x|^p} - \frac{|U_j(x') - U_j(x)|^p}{|x' - x|^p} \right) \, dx' \, dx \]

and

\[ I_2 = \int_{\Omega_\delta} \int_{\Omega_\delta} k_\delta(|x' - x|) \frac{|U_j(x') - U_j(x)|^p}{|x' - x|^p} \, dx' \, dx. \]

We shall use the elementary inequality

\[ \int |a|^p - |b|^p \, d\mu(x', x) \leq 2pR^{p-1} \left( \int |a - b|^p \, d\mu(x', x) \right)^{1/p} \quad (33) \]

where

\[ R = \max \left\{ \int |a|^p \, d\mu, \int |b|^p \, d\mu \right\} \]

with

\[ a = \frac{(U(x') - U(x))}{|x' - x|}, \quad b = \frac{(U_j(x') - U_j(x))}{|x' - x|} \]

and $d\mu(x', x) = k_\delta(|x' - x|) h(x) \, dx' \, dx$. We observe the inequality (31) and the strong convergence of the sequence $(U_j)_j$ ensure the constant $R$ is finite. Then, by using (33), (30) and the strong convergence, it is straightforward to deduce

\[ \lim_{j \to \infty} \lim_{\delta \to 0} |I_1| \leq 2p \lim_{j \to \infty} \lim_{\delta \to 0} \left( \int_{\Omega_\delta} \int_{\Omega_\delta} \frac{|U(x') - U(x)|}{|x' - x|} - \frac{|U_j(x') - U_j(x)|}{|x' - x|} \right)^p \times k_\delta(|x' - x|) h(x) \, dx' \, dx \]^1/p \quad (34) \]

\[ \leq 2p \lim_{j \to \infty} C \| \nabla U - \nabla U_j \|_{L^p(\Omega)} = 0 \]

Besides, thanks to (30) and to the strong convergence of $(U_j)_j$ again, then it is easy to check that

\[ \lim_{j \to \infty} \int_{\Omega} h(x) |\nabla U_j(x)|^p \, dx = \int_{\Omega} h(x) |\nabla U(x)|^p \, dx \quad (35) \]

From (32)-(35) we deduce what we desired to prove,

\[ \lim_{\delta \to 0} I_1 = \int_{\Omega} h(x) |\nabla U(x)|^p \, dx \text{ for any } U \in W_0^{1,p}(\Omega). \]
Consequently

\[
\lim_{\delta \to 0} m_\delta \leq \min_{v \in W^{1,p}_0(\Omega)} \left\{ \frac{1}{p} b_h(v, v) - \int_\Omega f(x) v(x) \, dx \right\}.
\]

2. The use of a couple of specific results will lead us to the reverse inequality. The first one is the Ponce inequality, a result derived in [39, pag. 12] and extended to measurable sets in [5, Appendix A], [37, Th. 1]. It can be read as follows: from \((u_\delta)_\delta\) we can extract a subsequence, still denoted by \((u_\delta)_\delta\), and we can find \(u^* \in W^{1,p}(\Omega)\) such that, \(u_\delta \to u^*\) strongly in \(L^p(\Omega)\) and let \(G\) be any measurable set in \(\Omega\), then

\[
\lim_{\delta \to 0} \int_G \int \kappa_\delta(|x' - x|) \frac{|u_\delta(x') - u_\delta(x)|}{|x' - x|^p} \, dx' \, dx \geq \int_G |\nabla u^*(x)|^p \, dx.
\]

If at this point we assume that \(h\) is a simple function, \(h(x) = \sum_{i=1}^m h_i 1_{B_i}(x)\), where \(\{B_i\}\) is a finite covering of disjoint measurable sets of \(\Omega\) and \((h_i)_i\) is a set of numbers such that \(h_{\text{min}} \leq h \leq h_{\text{max}}\), then it can be easily checked that

\[
\lim_{\delta \to 0} \int_{\Omega_i} \int_{\Omega_i} h(x', x) \kappa_\delta(|x' - x|) \frac{|u_\delta(x') - u_\delta(x)|}{|x' - x|^p} \, dx' \, dx \
\geq \sum_{i=1}^m h_i \int_{B_i} |\nabla u^*(x)|^p \, dx = \int_{\Omega} h(x) |\nabla u^*(x)|^p \, dx.
\]

Since any measurable function \(h\) can be approximated by \(s_n\), an increasing sequence of simple functions, then

\[
\lim_{\delta \to 0} \int_{\Omega_i} \int_{\Omega_i} h(x', x) \kappa_\delta(|x' - x|) \frac{|u_\delta(x') - u_\delta(x)|}{|x' - x|^p} \, dx' \, dx \
\geq \lim_{\delta \to 0} \int_{\Omega_i} s_n(x) \int_{\Omega_i} \kappa_\delta(|x' - x|) \frac{|u_\delta(x') - u_\delta(x)|}{|x' - x|^p} \, dx' \, dx \
\geq \int_{\Omega} s_n(x) |\nabla u^*(x)|^p \, dx.
\]

It remains to apply the Monotone Convergence Theorem to state

\[
\lim_{\delta \to 0} \int_{\Omega} \int_{\Omega} h(x', x) \kappa_\delta(|x' - x|) \frac{|u_\delta(x') - u_\delta(x)|}{|x' - x|^p} \, dx' \, dx \
\geq \int_{\Omega} h(x) |\nabla u^*(x)|^p \, dx.
\]

(36)

The second specific result directly deals with the compactness stated at point 3 in Section 2: indeed, by using the Nonlocal Poincaré inequality we easily check \(B_h(u_\delta, u_\delta) \leq C\) for any \(\delta\). This implies the existence of a subsequence of \((u_\delta)_\delta\) strongly convergent in \(L^p\) to some function \(u \in W^{1,p}_0(\Omega)\). Now, the proof of the reverse inequality is straightforward because thanks to the above convergence and (36) we get

\[
\lim_{\delta \to 0} m_\delta \geq \frac{1}{p} b_h(u, u) - \int_\Omega f(x) u(x) \, dx = m,
\]

which implies \(u = u^*\) and, whereby, the proof of the theorem finishes.
5.2. Convergence of the nonlocal optimal design problem. We analyze the particular situation of the compliance functional, namely the optimal design problem whose cost functional is

\[ F(x', x, u', u) = \frac{f(x, u(x)) + f(x', u(x'))}{2|\Omega_\delta|} \]

For each \( \delta \), Theorem 4.2 ensures the existence of a pair \((h_\delta, u_\delta) \in A_\delta\), solution to the problem (13). Under these circumstances we know \( h_\delta \rightharpoonup h \) weak-\(*\) in \( L^\infty(\Omega) \).

Let \( u \in W^{1,p}_0(\Omega) \) be the corresponding local state of \( h \), which means

\[- \text{div} (h |\nabla u|^{p-2} \nabla u) = f, \quad \text{in} \quad W^{1,p}_0(\Omega)\]

Then

\[ J_{loc}(h, u) = \int_{\Omega} f(x) u(x) \, dx = \int_{\Omega} h(x)|\nabla u(x)|^p \, dx, \quad (37) \]

if \((h, u) \in A_{loc}\), and

\[ J_\delta(h_\delta, u_\delta) = \int_{\Omega} f(x) u_\delta(x) \, dx \]

\[ = \int_{\Omega_\delta} \int_{\Omega_\delta} H_\delta \frac{k_\delta(|x'| - |x|)}{|x'-x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx \]

if \((h_\delta, u_\delta) \in A_\delta\).

**Theorem 5.2.** Let \((h_\delta, u_\delta)\) be the sequence of solutions of the optimal design problem

\[ \min_{(h, u) \in A_\delta} \int_{\Omega} f(x) u(x) \, dx. \]

Then there is a pair \((h, u) \in H \times W^{1,p}_0(\Omega)\) and a subsequence of indices \( \delta \) such that:

1. \( h_\delta \rightharpoonup h \) weak-\(*\) in \( L^\infty(\Omega) \) if \( \delta \to 0 \) and there is \( u \in W^{1,p}_0(\Omega) \) such that \( (h, u) \in A_{loc} \)
2. \( \lim_{\delta \to 0} J_\delta(h_\delta, u_\delta) = J_{loc}(h, u) \)
3. The equality

\[ \lim \min_{\delta \to 0} \left\{ \frac{1}{p} B_h \delta(w, w) - \int_{\Omega_\delta} f(x) w(x) \, dx \right\} \]

\[ = \min_{v \in W^{1,p}_0(\Omega)} \left\{ \frac{1}{p} b_h(v, v) - \int_{\Omega} f(x) v(x) \, dx \right\}, \quad (39) \]

holds and \((h, u)\) is the solution of the classical problem of optimal design

\[ \min_{(h, u) \in A_{loc}} \int_{\Omega} f(x) u(x) \, dx. \]

**Proof.** We split the proof in three parts:

1. The first part of the theorem is immediate.
2. We firstly prove

\[ \lim_{\delta \to 0} J_\delta(h_\delta, u_\delta) \geq J_{loc}(h, u). \]
If we use Theorems 2.1 and 2.2, and the arguments employed in part 1 of Theorem 5.1, then we easily derive the following chain of inequalities:

\[
\lim_{\delta \to 0} \left( \frac{1}{p} - 1 \right) \int_{\Omega_\delta} \int_{\Omega_\delta} H_\delta \frac{k_\delta (|x' - x|)}{|x' - x|^p} |u_\delta (x') - u_\delta (x)|^p \, dx' \, dx
\]

\[
= \lim_{\delta \to 0} \min_{w \in X_0} \left\{ \frac{1}{p} B_{k_\delta} (w, w) - \int_{\Omega_\delta} f \, dx \right\}
\]

\[
= \lim_{\delta \to 0} \left( \frac{1}{p} B_{k_\delta} (u_\delta, u_\delta) - \int_{\Omega_\delta} f u_\delta (dx) \right)
\]

\[
\leq \lim_{\delta \to 0} \left( \frac{1}{p} B_{k_\delta} (u, u) - \int_{\Omega_\delta} f u (dx) \right)
\]

\[
= \frac{1}{p} \int_{\Omega} h (x) |\nabla u (x)|^p \, dx - \int_{\Omega_\delta} f u (dx)
\]

\[
= \min_{w \in W^{1,p}_0 (\Omega)} \left\{ \frac{1}{p} \int_{\Omega} h (x) |\nabla w (x)|^p \, dx - \int_{\Omega_\delta} f w (dx) \right\}
\]

\[
= \left( \frac{1}{p} - 1 \right) \int_{\Omega} h (x) |\nabla u (x)|^p \, dx.
\]

Notice, here we have just used the convergence

\[
\lim_{\delta \to 0} B_{k_\delta} (u, u) = \int_{\Omega} h (x) |\nabla u (x)|^p \, dx.
\] (40)

This statement is easy to derive if we make use of Corollary 1 from [14, Pag. 3], which ensures

\[
\int_{\Omega} k_\delta \frac{|u (x') - u (x)|^p \, dx' \, dx}{|x' - x|^p} \to |\nabla u (x)|^p \text{ in } L^1 (\Omega). \tag{41}
\]

Indeed, the same is true if we put \( \Omega_\delta \) instead of \( \Omega \): we simply realize that

\[
B_{k_\delta} (u, u) = \int_{\Omega} h_\delta (x) \left( \int_{\Omega_\delta} k_\delta \frac{|u_\delta (x') - u_\delta (x)|^p \, dx'}{|x' - x|^p} \right) \, dx
\]

and

\[
\int_{\Omega_\delta} k_\delta \frac{|u_\delta (x') - u_\delta (x)|^p \, dx'}{|x' - x|^p} \, dx'
\]

\[
= \int_{\Omega} k_\delta \frac{|u_\delta (x') - u_\delta (x)|^p \, dx'}{|x' - x|^p} \, dx' + r (\delta)
\] (42)

with \( 0 \leq r (\delta) \leq C \int_{\mathbb{R}^N \setminus B (0, r)} k_\delta (|z|) \, dz \) (see 29). Since \( r (\delta) \downarrow 0 \), \( h_\delta \to h \) weak-* in \( L^\infty \) and (41), then (40) is true.

We have proved

\[
\lim_{\delta \to 0} \int_{\Omega_\delta} \int_{\Omega_\delta} H_\delta \frac{k_\delta (|x' - x|)}{|x' - x|^p} |u_\delta (x') - u_\delta (x)|^p \, dx' \, dx
\]

\[
\geq \int_{\Omega} h (x) |\nabla u (x)|^p \, dx
\]
and consequently
\[ \lim_{\delta \to 0} J_\delta (h_\delta, u_\delta) = \lim_{\delta \to 0} \int_{\Omega_\delta} \int_{\Omega_\delta} H \frac{k_\delta (|x' - x|)}{|x' - x|^p} |u_\delta (x') - u_\delta (x)|^p dx'dx \]
\[ \geq \int_{\Omega} h (x) |\nabla u (x)|^p dx = J_{loc} (h, u) \]

Now, we take into account that \((h_\delta, u_\delta)\) is a minimizer for the nonlocal optimal design problem, so that
\[ \lim_{\delta \to 0} J_\delta (h_\delta, u_\delta) \leq \lim_{\delta \to 0} \int_{\Omega_\delta} \int_{\Omega_\delta} H \frac{k_\delta (|x' - x|)}{|x' - x|^p} |u_\delta^* (x') - u_\delta^* (x)|^p dx'dx \]
\[ = \int_{\Omega} h (x) |\nabla u^* (x)|^p = J_{loc} (h, u^*) = J_{loc} (h, u) \]

where \(u_\delta^* \in X_0\) is the underlying state of the control \(h\) for the nonlocal problem. Since \(u_\delta^* \to u^*\) strongly in \(L^p (\Omega)\) and \(u^*\) must be the state associated with \(h\), then \(u^* = u\) and therefore (thanks to Theorem 2.2)
\[ \lim_{\delta \to 0} J_\delta (h_\delta, u_\delta) = J_{loc} (h, u) \]

Then, we have proved \(\lim_{\delta \to 0} J_\delta (h_\delta, u_\delta) = J_{loc} (h, u)\).

3. The above limit automatically implies that (39) holds and \((h, u) \in \mathcal{A}_{loc}\). To verify that \((h, u)\) is a minimizer of \(J_{loc}\) we take any pair \((\tilde{h}, \tilde{u}) \in \mathcal{A}_{loc}\), and we consider the only underlying nonlocal state \(\bar{u}_\delta\) that corresponds to \(\tilde{h}\). Then
\[ J_{loc} (h, u) = \lim_{\delta \to 0} J_\delta (h_\delta, u_\delta) \geq \lim_{\delta \to 0} J_\delta (h_\delta, u_\delta) = J_{loc} (h, u) \]

where the first equality is due to Theorem 5.1 and the inequality is because \((h_\delta, u_\delta)\) minimizes \(J_\delta\).

If we focus our attention on the local problem, then, from the previous result, we can extract this remarkable result on the existence of optimal designs:

**Corollary 1.** There exists a solution to the problem
\[ \min_{(h, u) \in \mathcal{A}_{loc}} \int_{\Omega} f (x) u (x) dx. \]

6. **Convergence under an additional assumption on the controls.** We just extend the results obtained in [5] to the case of the \(p\)-Laplacian.
We explore problem \((\mathcal{P}_\delta)\) given by
\[ \inf_{(h, u) \in \mathcal{A}_\delta} J_\delta (h, u) \] where
\[ J_\delta (h, u) = \int_{\Omega} G (x, u (x), h (x)) dx \]
\[ G : \Omega \times \mathbb{R} \times [h_{\min}, h_{\max}] \rightarrow \mathbb{R} \text{ is a uniformly Lipschitz continuous function with respect to } u \text{ and } h. \text{ The set of admissibility is} \]
\[ \mathcal{A}_\delta = \{(h, u) \in H_x(r_0) \times X_0 : (h, u) \text{ solves (15)}\} \]  
where
\[ H_x(r_0) = \{h \in H : B_{r_0}(h, h) < \varepsilon\}, \]
\[ B_{r_0} \text{ is the nonlocal operator defined as} \]
\[ B_{r_0}(h, h) = \int_{\Omega_{r_0}} \int_{\Omega_{r_0}} \frac{k_{r_0}(|x'| - x|)}{|x'| - x|} |h(x') - h(x)|^p dx' dx \]
and, \( \varepsilon, r_0 \) and \( q > 1 \) are given positive constants.

We proceed as in the previous section: if \( (h_\delta, u_\delta) \delta \) is the sequence of minimizer of the problem \( (P_\delta) \) given in (43), then there is a subsequence of \( (u_\delta) \delta \) and a function \( u^* \in W^{1,p}_{0} (\Omega) \) such that \( u_\delta \rightarrow u^* \) strongly in \( L^p(\Omega) \). Also, since \( (h_\delta) \delta \subset H_x(\delta_0) \) the term \( B_{r_0}(h_\delta, h_\delta) \) is uniformly bounded and hence, there exists \( h \in L^p \) and a subsequence of \( (h_\delta) \delta \), such that \( (h_\delta) \delta \rightarrow h \text{ strongly in } L^p(\Omega) \) (see point 2 at Section 2). Clearly \( h \in H_x(r_0) \) but we do not know whether \( (h, u^*) \) solves (10). In such a case \( (h, u^*) \) would belong to the space
\[ \mathcal{A}_{loc} = \{(g, w) \in H_x(r_0) \times W^{1,p}_{0}(\Omega) : (g, w) \text{ solves (10)}\}. \]
and the pair would be a solution of the optimal design problem \( (P_{loc}) \) defined as
\[ \inf_{(g, w) \in \mathcal{A}_{loc}} J_{loc}(h, u). \]

More precisely, we have the following theorem:

**Theorem 6.1.** If \( (h_\delta, u_\delta) \) is a sequence of solutions for the problem (43) then there exists \( (h, u) \in \mathcal{A}_{loc} \), and a subsequence of \( \delta' \)'s such that

1. \( h_\delta \rightarrow h \text{, a.e. in } \Omega \) and \( u_\delta \rightarrow u \text{ strongly in } L^p(\Omega) \), and
\[ \lim_{\delta \rightarrow 0} J_{\delta}(h_\delta, u_\delta) = J_{loc}(h, u) \]
2. The pair \( (h, u) \) is a solution of the local design problem (48).

To prove this theorem we need convergence for the state equation:

**Theorem 6.2.** Assume that \( (h_\delta) \delta \subset H_x(r_0) \) converges to \( h \in H_x(r_0) \) a.e. \( x \in \Omega \). If \( (u_\delta) \delta \) is the sequence of states that corresponds to \( (h_\delta) \delta \), then there is a subsequence of \( (u_\delta) \delta \) (which is going to be denoted \( (u_\delta) \delta \) too) and \( u^* \in W^{1,p}_{0}(\Omega) \) such that \( u_\delta \rightarrow u^* \text{ strongly in } L^p(\Omega) \). Moreover, if \( u \) is the underlying state of \( h \), then
\[ \lim_{\delta \rightarrow 0} \min_{w \in X_0} \left\{ \frac{1}{p} \int_{\Omega} \int_{\Omega_{\delta}} H_\delta(x', x) k_\delta(|x' - x|) \frac{|w(x') - w(x)|^p}{|x' - x|^p} dx' dx - \int_{\Omega_{\delta}} f(x) w(x) dx \right\} \]
\[ = \min_{v \in W^{1,p}_{0}(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} h(x) |\nabla v(x)|^p dx - \int_{\Omega} f(x) v(x) dx \right\} \]
and $u = u^*$, which means

$$
\min_{v \in W^{1,p}_0(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} h(x) |\nabla v(x)|^p \, dx - \int_{\Omega} f(x) v(x) \, dx \right\} = \frac{1}{p} \int_{\Omega} h(x) |\nabla u^*(x)|^p \, dx - \int_{\Omega} f(x) u^*(x) \, dx.
$$

\textbf{Proof.} Since $(u_\delta)_{\delta}$ is the sequence of minimizers of $\frac{1}{p} B_{h_{\delta}} (\cdot, \cdot) - (f, \cdot)_{L^2 \times L^2}$ then $u_\delta$, at least for a subsequence, strongly converges to some $u^* \in W^{1,p}_0(\Omega)$. If we examine the functional

$$
B_{h_{\delta}} (u_\delta, u_\delta) = \int_{\Omega \delta} \int_{\Omega \delta} H_{\delta} (x', x) k_{\delta} (|x' - x|) \frac{|u_\delta (x') - u_\delta (x)|^p}{|x' - x|^p} \, dx' \, dx
$$

and we define the sequence of measures

$$
\mu_{\delta} (A) = \int_A \int_{B(x, \delta)} k_{\delta} (|x' - x|) \frac{|u_\delta (x') - u_\delta (x)|^p}{|x' - x|^p} \, dx' \, dx,
$$

then it is clear that

$$
\liminf_{\delta \to 0} \mu_{\delta} (A) \geq \liminf_{\delta \to 0} \int_A \int_{B(x, \delta)} k_{\delta} (|x' - x|) \frac{|u_\delta (x') - u_\delta (x)|^p}{|x' - x|^p} \, dx' \, dx.
$$

Besides, by using Ponce inequality for measurable sets (see Theorem 1 in ([37])) we have

$$
\lim_{\delta \to 0} \mu_{\delta} (A) \geq \int_A |\nabla u^*(x)|^p \, dx.
$$

Consequently, we have proved $\lim_{\delta \to 0} \mu_{\delta} (A) \geq \mu_0 (A)$, where $\mu_0$ is the measure defined by means of $\mu_0 (A) = \int_A |\nabla u^*(x)|^p \, dx$. Since $h_{\delta}$ converges pointwise to some $h \in H^1_c (\mathbb{R}^n)$, we are in position to apply the Generalized Fatou’s (see [29, Th. 2.2]) Lemma and obtain

$$
\liminf_{\delta \to 0} B_{h_{\delta}} (u_\delta, u_\delta) = \liminf_{\delta \to 0} \int_{\Omega \delta} h_{\delta} (x) \, d\mu_{\delta} (x)
\geq \int_{\Omega} h(x) \, d\mu_0 (x)
= \int_{\Omega} h(x) |\nabla u^*(x)|^p \, dx.
$$

The above inequality leads us to

$$
\lim_{\delta \to 0} \min_{w \in X_0} \left\{ \frac{1}{p} B_{h_{\delta}} (w, w) - \int_{\Omega \delta} f(x) w(x) \, dx \right\}
\geq \frac{1}{p} \int_{\Omega} h(x) |\nabla u^*(x)|^p \, dx - \int_{\Omega} f(x) u^*(x) \, dx
\geq \min_{v \in W^{1,p}_0(\Omega)} \left\{ \frac{1}{p} \int_{\Omega} h(x) |\nabla v(x)|^p \, dx - \int_{\Omega} f(x) v(x) \, dx \right\}.
$$
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To verify the reverse inequality we come back to the proof of inequality (42):

\[
\lim_{\delta \to 0} \min_{w \in X_0} \left\{ \frac{1}{p} B_{\delta} (w, w) - \int_{\Omega_\delta} fwdx \right\} \\
\leq \lim_{\delta \to 0} \left( \frac{1}{p} B_{\delta} (u, u) - \int_{\Omega_\delta} fwdx \right) \\
= \frac{1}{p} \int_{\Omega} h(x) |\nabla u(x)|^p \, dx - \int_{\Omega} fwdx \\
= \min_{w \in W^{1,p}_0 (\Omega)} \left\{ \frac{1}{p} \int_{\Omega} h(x) |\nabla w(x)|^p \, dx - \int_{\Omega} fwdx \right\},
\]

which finishes the proof of the theorem. \qed

Proof of Theorem 6.1. It remains to check two points:

1. \( \lim_{\delta \to 0} J_\delta (h_\delta, u_\delta) = J_{\text{loc}} (h, u) \) and,
2. \( (h, u) \) is a solution to \( \inf_{(g, w) \in A_{\text{loc}}} J_{\text{loc}} (h, u) \).

1 is trivial because of the continuity of the integrand \( G \). Concerning part 2 we notice that for any control \( \tilde{h} \)

\[
\lim_{\delta \to 0} J_\delta (h_\delta, u_\delta) \leq \lim_{\delta \to 0} J_\delta \left( \tilde{h}, \tilde{u}_\delta \right)
\]

where \( \tilde{u}_\delta \) is the state associated with \( \tilde{h} \). Since \( \tilde{u}_\delta \) strongly converges to \( \tilde{u} \) in \( L^p \) and
the local state of \( \tilde{h} \) is \( \tilde{u} \)

\[
\lim_{\delta \to 0} J_\delta \left( \tilde{h}, \tilde{u}_\delta \right) = J_{\text{loc}} \left( \tilde{h}, \tilde{u} \right).
\]

We have arrived at

\[
J_{\text{loc}} (h, u) \leq J_{\text{loc}} \left( \tilde{h}, \tilde{u} \right)
\]

for any pair \( (\tilde{h}, \tilde{u}) \in A_{\text{loc}}. \) \qed

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