CHARACTERISTIC CLASSES IN SYMPLECTIC TOPOLOGY

(Preliminary Version)

OCTOBER 10, 1994
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with an appendix of Ludmil Katzarkov
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0.1. From the cohomological point of view the symplectomorphism group \( \text{Sympl}(M) \) of a symplectic manifold is “tamer” than the diffeomorphism group. The existence of invariant polynomials in the Lie algebra \( \text{sympl}(M) \), the symplectic Chern-Weil theory, and the existence of Chern-Simons-type secondary classes are first manifestations of this principles. On a deeper level live characteristic classes of symplectic actions in periodic cohomology and symplectic Hodge decompositions.

The present paper is called to introduce theories and constructions listed above and to suggest numerous concrete applications. These includes: nonvanishing results for cohomology of symplectomorphism groups (as a topological space, as a topological group and as a discrete group), symplectic rigidity of Chern classes, lower bounds for volumes of Lagrangian isotopies, the subject started by Givental, Kleiner and Oh, new characters for Torelli group and generalizations for automorphism groups of one-relator groups, arithmetic properties of special values of Witten zeta-function and solution of a conjecture of Brylinski. The Appendix, written by L. Katzarkov, deals with fixed point sets of finite group actions in moduli spaces.

0.2. Here is a more detailed description of the paper. Chapter 1 starts with the definition of the set \( p_k \) of invariant polynomials in \( \text{sympl}(M) \). Using these, we define special cohomology classes in \( H^{2k-1}_{\text{top}}(\text{Sympl}(M)) \), and prove nonvanishing results if \( M \) admits a certain symmetry. In particular, we prove

**Theorem (1.3)** If a compact simply-connected symplectic manifold \( M \) admits an effective symplectic action of a compact nonabelian simple group \( G \), then \( H^3_{\text{top}}(\text{Sympl}(M), \mathbb{R}) \neq 0 \).

**Theorem (1.4)** The embedding \( \text{PSU}(n + 1) \to \text{Sympl}(%CP^n) \) is “totally nonhomologous to zero”. In particular, \( \sum_{i=0}^{\infty} b_i(\text{Sympl}(%CP^n)) \geq 2^n \).

We then describe a “symplectic Chern-Weil Theory” and prove as an application

**Theorem (1.5)** Symplectic rigidity of Chern Classes. Let \( E_i \to M_i \) be Hermitian vector bundles, \( i = 1, 2 \). Let \( PE_i \) be the projectivization of \( E_i \).

Let \( f : PE_1 \to PE_2 \) be a fiber-like symplecticomorphism, covering a map \( \varphi : M_1 \to M_2 \). Then \( \varphi_*(c_k(E_2)) = c_k(E_1) \) for \( k \geq 2 \).
In Chapter 2 we discuss the cohomology of $\text{Sympl}(M)$ made discrete. Our main tools are the general theory of regulators, described in [Re1], and extended Bloch-Beilinson regulator for Frechet-Lie groups constructed in [Re3]. The construction of [Re3] is deeply related to Karoubi’s MK-Theory [Ka1], [Ka2], [Sou]. One of the most interesting results is the following

**Theorem 2.2** There exists a regulator

$$\pi_3(B\text{Sympl}^\delta(\mathbb{C}P^2))^+ \to \mathbb{R}/\mathbb{Z}$$

which makes the diagram

$$\begin{array}{ccc}
\pi_3(B\text{PSU}^\delta(3))^+ & \xrightarrow{\text{stabilization}} & K_3(\mathbb{C}) \\
\downarrow & & \| \\
\pi_3(B\text{Sympl}^\delta(\mathbb{C}P^2)) & \to & \mathbb{R}/\mathbb{Z}
\end{array}$$

commutative. Here $B$ is the Beilinson-Karoubi regulator $K_{2n-1}(\mathbb{C}) \to \mathbb{C}/\mathbb{Z}$ ([Ka1]).

In Chapter 3 we define a secondary characteristic class, $\rho : H_{2k-1}(L, \mathbb{Z}) \to \mathbb{R}/\mathbb{Z}$ for any Lagrangian submanifold of a symplectic manifold $M$ with integral symplectic periods. The class $\rho$ shows useful properties (isoperimetric estimates, rationality, rigidity etc.), which allow us to derive a following application.

**Theorem (3.5)** Let $X \subset \mathbb{P}^n$ be a smooth projective variety defined over $\mathbb{R}$, $\dim X = 2k - 1$ and let $M = X(\mathbb{C})$ and $L = X(\mathbb{R})$. Suppose the homomorphism $H_{2k-1}(L) \to H_{2k-1}(\mathbb{R}\mathbb{P}^n) = \mathbb{Z}_2(k \geq 2)$ is nontrivial. Then for any metric on $M$ and any Lagrangian homotopy $L_t$ of $L$ in $M$, $\text{Vol}(L_t)$ stays bound away from zero by a constant $\gamma(M)$.

In Chapter 4 we use these ideas to define a character of the symplectomorphism group $\text{Sympl}_h(M)$, acting trivially in homology:

$$\chi : \text{Sympl}_h(M) \to \text{Hom}(H_{\text{odd}}(M, \mathbb{Z}), \mathbb{R}/\mathbb{Z}),$$

which has some similarity with the famous Futaki invariant [Fu]. Taking $M$ to be the modular space of stable vector bundles over a Riemannian surface, we get a character for the Torelli group $M_g \supset I_g \to (\mathbb{R}/\mathbb{Z})^{2g}$. 

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It would be extremely interesting to understand the relation of our character to the Johnson homomorphism [J1] and the Birman-Craggs homomorphism [J1],[BC].

The most interesting feature of our character is that it generalizes to any one-relator group $\Gamma = F_2\langle r \rangle$, where $r \in F_2'$, and for the generalized Torelli group $I(\Gamma) = \text{Ker}(\text{Out}(\Gamma) \to \text{Aut}H_1(\Gamma))$ one gets a character

$$I(\Gamma) \to (\mathbb{R}/\mathbb{Z})^{2g}.$$ 

In Chapter 5 we describe the multiplicative transfer of Eves and derive the following result:

**Theorem (5.3, special case).** Suppose $\mathbb{Z}_p$ acts symplectically on a compact symplectic manifold $M$ with integer periods. If $\dim \text{Fix}(\mathbb{Z}_p) < \frac{\dim M}{p}$, then $\text{Vol } M$ is divisible by $p$.

In Chapter 6 we apply this result to the representation variety $\mathcal{M}_g = \text{Hom}(\pi_1(C_g), G)/G$, where $C_g$ is a Riemann surface of genus $g$, and $G$ is a simple compact Lie group. If $\mathbb{Z}_p$ acts non-freely in $C_g$, then the induced action of $\mathbb{Z}_p$ in $\mathcal{M}_g$ yields the condition of above theorem. This is proved in Appendix 7, written by L. Katzarkov. The volume $\text{Vol } \mathcal{M}_g$ has been computed by E. Witten [W]:

$$\text{Vol } M = \zeta^G_W(2g - 2),$$

where $\zeta^G_W(s) = \sum_{\alpha \in G} \frac{1}{(\dim \alpha)^{s}}$.

So we have:

**Theorem (6.1) (Von Staudt theorem for Witten zeta-function).** For any $p|m$ such that $p(p - 1) \leq m$ the number

$$W_G(2m) = \frac{(3m)!}{(2m)!} \zeta^G_W(2m)$$

is divisible by $p$.

In Chapter 7 we introduce other characteristic classes of symplectic actions $G \to \text{Sympl}(M)$, this time lying in $HC_{\text{per}}^c(\Omega(M))$. This space has been extensively studied by Bryllinski [Br] who found the dimension filtration $F_0 \subset F_1 \subset \cdots \subset F_\infty ( = HC_{\text{even}}^c(\text{resp } HC_{\text{odd}}^c))$, such that $F_{k+1}/F_k = H^{2k+i}(M, \mathbb{R})$, where $i = 0, 1$ respectively.
Our main contribution is a new “symplectic Hodge decomposition”

\[ H^{\text{even}} = \oplus V_i \]
\[ HC^{\text{odd}} = \oplus W_i \]

by eigenspaces of the “weight operator” \( T \) (see 7.3). In Kähler case there is a canonical \( sl(2, \mathbb{R}) \) action in \( HC^{\text{even}} = \oplus H^{2k}(M, \mathbb{R}) \) and \( HC^{\text{odd}} = \oplus H^{2k+1}(M, \mathbb{R}) \). The dimension operator \( S \) corresponds to \( \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \) whereas our operator \( T \) corresponds to \( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \). The two operators \( S, T \) generate all the \( sl(2, \mathbb{R}) \) - action.

In general symplectic case the dimension operator fails to exist, and what is left from it is the dimension filtration. The relation of dimensional filtration to the weight operator \( T \) is a deep and attractive matter. We study at depth the case of four-dimensional \( M \), and find a negative solution to the Brylinski conjecture [Br]:

Conjecture (J.- L. Brylinski). Any cohomology class in \( H^*(M, \mathbb{R}) \) can be represented by a “harmonic” form \( \omega \in \Omega^k(M) \) with \( d\omega = \delta \omega = 0 \).

Here \( \delta \) is the operator of Koszul-Brylinski.

There are four appendices to the paper. Appendix \( \aleph \) deals with transfer for non-free actions of finite groups. We outline an approach using the classical Dold-Thom Theorem \( H_i(X) = \pi_i(SX) \) and show that the multiplicative transfer fails to exist.

Appendix \( \wp \) describes the analogue of the formalism of Chapter 6 in the setting of noncommutative Poisson manifolds (algebras \( A \) over a field \( k \) with an element \( \mu \in HH^2(A) \) satisfying \([\mu, \mu] = 0\), where \([ \cdot, \cdot ] : HH^i \otimes HH^j \to HH^{i+j-1} \) is the Gerstenhaber bracket).

In appendix \( \wp \) we study geometric structures on representation varieties.

\[ V_{\Gamma}^G = \text{Hom}(\Gamma, G) \text{ and } X_{\Gamma}^G = V_{\Gamma}^G / G, \]

where \( \Gamma \) is a f.g. group and \( G \) is a Lie group. We define and study a map

\[ FD : H_*(\Gamma, \mathbb{R}) \otimes I_G(\mathfrak{g}) \to \Omega^*_{cl}(X_{\Gamma}^G) \]

and secondary characteristic maps

\[ \tilde{K}_i^{\text{alg}}(\Gamma) \to H^{i-1-2s}(X_{\Gamma}^G, \mathbb{R}/\mathbb{Z}), \ s \geq 0 \]
\[ \tilde{K}_i^{\text{alg}}(\Gamma) \to \text{Hom}(\text{all}(i-1) - \text{currents}(X_{\Gamma}^G), \mathbb{R}/\mathbb{Z}). \]
We outline a program of excitingly promising applications to 3-manifold invariants.

Appendix 7, written by Ludmil Katzarkov, identifies the fixed point set of a finite group action in the moduli space with the moduli space of parabolic vector bundles.

0.3 Acknowledgements. It is my pleasure to thank all the people, with whom I discussed various aspects of symplectic theory, related to this paper. The material of Chapter 1 has been directly affected by discussions with and the papers by H. Hofer [Ho] and further discussions with Th. Delzant, A. Reyman, J-P. Sikorav, C. Kassel. The relation of the regulator maps in Chapter 2 to the $MK$-theory has become clear after discussions with M. Karoubi. The various aspects of multiplicative transfer were discussed with B. Kahn and D. Blank. M. S. Narasimhan kindly introduced me to the Witten’s paper [W] and W. Nahm explained it to me, as well as his brilliant constructions in $TQFT$. Finally, S. Seshadri and L. Katzarkov are responsible for lemma 6.4 and Appendix 7. I also wish to thank various institutions where these discussions became possible: MPI, Bonn, Université Louis-Posteur, Strasbourg, Université Paris-VII, ICTP, Trieste and Université Paul Sabatier, Toulouse. Special thanks are due to P. Deligne for the illuminating correspondence concerning the secondary classes.
List of open Problems

1. Let $M$ be a compact simply-connected symplectic manifold. Is it true, that $H^3_{\text{top}}(\text{Sympl}(M), \mathbb{R}) \neq 0$? (yes, if $M$ admits a symplectic action of a compact non-abelian Lie group, see 1.3).

2. Does there exist a class in $H^{2i-1}(\text{Sympl}^\delta(\mathbb{C}P^N), \mathbb{R}/\mathbb{Z})$, $N \gg i$, which restricts to the Chern-Simons class under the inclusion $PSU(N+1) \subset \text{Sympl}(\mathbb{C}P^N)$? (see 2.2).

3. Let $M_g$ be the mapping class group. Is there a regulator $v : K_3(M_g) \to \mathbb{R}/\mathbb{Z}$ in spirit of chapter 2?

4. Let $r \in [F_g, F_g]$ be a balanced word. Compute the volume of the moduli space $\mathcal{M} = \text{Hom}(\Gamma, G)/G$, where $\Gamma = F_{2g}/\{r\}$ and $G$ a compact simple nonabelian Lie group. Can one derive a Von Staudt Theorems from this computation?

5. What is the connection of the homomorphism $I_g : (\mathbb{R}/\mathbb{Z})^{2g}$ defined in the section 4.4 to the Johnson’s homomorphism?

6. Let $F$ be a totally real number field. Does there exists a Witten-Dedekind zeta-function $\zeta_F^W(s)$? Is there a Von Staudt Theorem for special values of $\zeta_F^W(s)$?

5. Does the Brylinski conjecture hold for dimensions less than half dimension of the symplectic manifold? (See 7.4.2. for details).

6. Are there compact symplectic four-manifolds $M$, for which the odd spectrum of $M$ (see 7.4.2) is different from $\{\pm 1\}$?

7. Does there exist a homology three-sphere $M$, for which $FD([M]) \in \Omega^3_{c\ell}(V^{SU(n)}_{\pi_1(M)})$ is nonzero? (see Appendix 2)

8. Let $M$ be a Seifert homology sphere. Prove that $FD([M]) = 0$.

9. Let $M$ be as in problem 8. Compute $P(M) \in \otimes \mathbb{C}/\mathbb{Z}$. Is it true that $P(M) \in \otimes \mathbb{Q}/\mathbb{Z}$?
Invariant Polynomials and Cohomology of the
Symplectomorphism Group, Symplectic Chern-Weil Theory
and Characteristic Classes of Symplectic Fibrations

In this chapter we will show that the symplectomorphism group $\text{Sympl}(M)$ of a symplectic manifold $M$ behaves “cohomologically” very much like a simple compact Lie group, the fundamental reason being the existence of a sequence of invariant polynomials $p_k$ in the Lie algebra $\text{sympl}(M)$ of Hamiltonian vector fields. We will construct cohomology classes both in $H^*_{\text{top}}(\text{Sympl}(M))$ (which we call Cartan classes) and in $H^*_{\text{sympl}}(\text{BSympl}(M))$ (which we call symplectic Chern classes) and prove nonvanishing results in case $M$ has a certain degree of symmetry. The last result is the symplectic rigidity of usual Chern classes under fiberlike symplectomorphisms, the relation somewhat similar to defining relation in the Atiah-Adams $J$-groups in topological $K$-theory.

Recall that some class in $H^1_{\text{top}}(\text{Sympl}(M))$ has been constructed by Eugen Calabi and Alan Weinstein [We], see also [Gi 2]. As the reader will see, in some sense our symplectic Chern classes are higher dimensional analogues of the Calabi-Weinstein class (or invariant)

If $M$ is acted upon symplecticaly by a Lie group $G$, then the inverse image of our classes in $H^*_{\text{top}}(\text{Sympl}(M))$ gives characteristic classes of this symplectic action lying in $H^*_{\text{top}}(G)$. On the other hand, in chapter 7 we will construct also some characteristic classes in $H^*(M, \mathbb{R})$, using the canonical complex of Brylinski-Kassel.

1.1 Throughout this chapter the underlying symplectic manifold $M$ is assumed to be compact and simply-connected. Any symplectic vector field $X$ has a Hamiltonian $f_X$ normalized by the condition $\int_M f_X = 0$. In other words, the canonical extension of Lie algebras

$$0 \rightarrow \mathbb{R} \rightarrow C^\infty(M) \rightarrow \text{sympl}(M) \rightarrow 0$$

splits.

**Definition:** (1.1). The Cartan-Killing bilinear form $(X,Y)$ on $\text{sympl}(M)$ is defined by

$$(X,Y) = \int_M f_X \cdot f_Y$$
Lemma 1.1. The Cartan-Killing form is invariant, that is, for any $X \in \text{sympl}(M)$, $[X, \cdot]$ is a skew-adjoint operator.

Proof: The flow of $X$ preserves the Liouville measure, hence the $L^2$-product in $C^\infty(M)$, so $f \mapsto Xf$ is skew-adjoint. By (1.1) we can identify $\text{sympl}(M)$ with functions with integral zero, equivariantly with respect to the $\text{Sympl}(M)$ - action, hence the result.

1.2. Cartan Form: Recall that whenever a Lie algebra $\mathfrak{g}$ over a field $k$ is given an invariant symmetric bilinear form $(X,Y)$, the formula $\alpha(X,Y,Z) = ([X,Y],Z)$ defines an invariant 3-form. For $k = \mathbb{R}, \mathbb{C}$ and $\mathfrak{g} = \text{Lie}(G)$, a Frechet-Lie group, the biinvariant 3-form on $\mathfrak{g}$, corresponding to $\alpha$, is closed. For $\mathfrak{g}$ compact finite-dimension semisimple, the class of this form gives a generator of $H^3_{\text{top}}(\mathfrak{g}, \mathbb{R})$.

Definition (1.2): The invariant 3-form on $\text{Sympl}(M)$, corresponding to $\alpha$, will be called the Cartan form, and its class is $H^3(\text{Sympl}(M))$ will be called the Cartan class.

1.3 Higher Forms: We define the $n$-th elementary polynomial $p_k$ on $\text{sympl}(M)$ by

$$p_k(X) = \int_M f_X^k.$$ 

The argument of 1.1 shows that $p_k$ is an invariant polynomial on $\text{sympl}(M)$. For a Frechet-Lie group $G$, let $\theta \in \Omega^1(G, \mathfrak{g})$ be the Mauer-Cartan form. We define $\mu_k \in \Omega^{2k}(\text{Sympl}(M))$ by $\mu_k = p_k(\theta, [\theta, \theta], \cdots [\theta, \theta])$. The computation of [ChS] shows that $\mu_k$ is biinvariant, hence closed.

Definition (1.3): The class $\lambda_k = [\mu_k] \in H^k_{\text{top}}(\text{Sympl}(M), \mathbb{R})$ will be called higher Cartan cohomology classes of $\text{Sympl}(M)$. Here $H^\ast_{\text{top}}(\text{Sympl}(M))$ stands for the cohomology of $\text{Sympl}(M)$ as a topological space.

Suppose now we have an effective symplectic action of a compact semisimple group $G$ on $M$, that is, a Lie groups homomorphism $\pi : G \to \text{Sympl}(M)$. The composition of $p_k$ with $\pi_* : \mathfrak{g} \to \text{Sympl}(M)$ gives an invariant polynomial in $\mathfrak{g}$, positive, if $k$ is even. In particular, $p_2 \circ \pi_*$ is an invariant quadratic form, which should be proportional to the Cartan-Killing form of $\mathfrak{g}$. We deduce the following theorem.

Theorem (1.3). For an effective symplectic action $\pi$ of a compact semisimple Lie group $G$
on $M$ the inverse image $\pi^*(\lambda_2) \in H^3(G, \mathbb{R})$ is nonzero. In particular, $H^3_{\text{top}}(\text{Sympl}(M), \mathbb{R}) \neq 0$.

1.4: For an arbitrary action, one does not know whether the polynomials $p_k \circ \pi_*$ are algebraically independent. However, for a homogeneous action one often can compute these polynomials explicitly. Let, for instance, $M = \mathbb{C}P^n$ with canonical Kähler symplectic structure and let $G = SU(n+1)$ with a standard action on $M$. The Hamiltonian of a vector field $X$, corresponding to $A \in \mathfrak{su}(n+1)$ is a Hermitian quadratic form $\alpha \mapsto (AZ, Z)$ where $Z \in S^{2n+1}$ represents a point in $\mathbb{C}P^n$. Therefore by Fubini

$$p_k \circ \pi_*(A) = \text{const} \cdot \int_{S^{2n+1}} (AZ, Z)^k dz.$$  

We claim that $p_k \circ \pi_*$ generate invariant polynomial ring of $\mathfrak{su}(n+1)$. Indeed, start with the identity $\int_0^\infty e^{-ar^2}rdr = \frac{1}{2a}$. Differentiating by a $n$ times we get $\int_0^\infty e^{-ar^2} \cdot r^{2r+1} = \text{const} \cdot a^{-n-1}$. Now, let $B$ be a positive Hermitian operator in $\mathbb{C}^{n+1}$. Integrating in polar coordinates, we get

$$\int_{\mathbb{C}^{n+1}} e^{-(Bz,z)} = \int_{S^{2n+1}} dv \int_0^\infty e^{-r^2(Bv,v)}r^{2n+1} drdv = \text{const} \cdot \int_{S^{2n+1}} (Bv, v)^{-n-1}.$$  

On the other hand, the first integral is $\text{const} \cdot (\det B)^{-1}$, so

$$\int_{S^{2n+1}} (Bv, v)^{-n-1} = \text{const}(\det B)^{-1}$$

Take $t > 0$ big enough and replace $B$ by $B + t \cdot E$ to get

$$\int_{S^{2n+1}} t^{-n-1}(1 + t^{-1}(Bv, v))^{-n-1} = \text{const} \cdot (\det^{-1}(B + tE))$$

or

$$t^{-n-1} \sum_{k=0}^{\infty} \int_{S^{2n+1}} \binom{-n-1}{k} t^{-k}(Bv, v)^k = \text{const} \cdot t^{-n}\det^{-1}(E + 1/tB).$$

Hence the knowledge of $\pi_* \circ p_k(B)$ determines the values of all elementary symmetric polynomials in eigenvalues of $B$. The result then follows from the Stone-Weierstrass theorem and the fact that all $p_k \circ \pi_*$ are homogeneous polynomials.

In view of 1.3 this results in the following theorem.
**Theorem (1.4).** The embedding $PSU(n+1) \to Sympl(\mathbb{C}P^n)$ is “totally nonhomologous to zero”, that is induces an injective map in real homology. In particular, $\sum_{i=0}^{\infty} b_i(Sympl(\mathbb{C}P^n)) \geq 2^n$.

It is very likely that the same is true for any simple Lie group $G$ and any coadjoint orbit $M$ of $G$.

1.5: Let $\pi : Q \to F \to M$ be a smooth fibration with fiber-like symplectic structure $\omega_x$, $x \in M$. We assume it to be symplectically locally trivial, that is, we assume local diffeomorphisms $\pi^{-1}(U) \cong U \times Q$, such that for any $x \in U$, the push forward symplectic form $\omega = \varphi_* \omega_x$ on $Q$ does not depend on $x$. One can say that $\pi$ is a fibration associated to a principal fibration $P$ with a structure group $Sympl(Q, \omega)$. Our goal is to construct Chern classes $c_i \in H^{2i}(M, \mathbb{R})$.

We first define a connection in $P$ in a usual way as a $sympl(Q)$ valued 1-form with usual properties. A connection exists in any principal bundle with a Frechet-Lie structure group $G$ over a finitely-dimensional manifold, because the Atiyah regulator $A \in H^1(\Omega^1 \otimes adP)$ is zero, since $\Omega^1 \otimes adP$ is a fine sheaf. Now, the invariant polynomials $p_k$, introduced in 1.3 give rise in a usual way to Chern classes $(\ast)$, independent on the choice of connection. In particular, this implies immediately the following result.

**Theorem (1.5). Symplectic rigidity of Chern Classes.** Let $E_i \to M_i$ be Hermitian vector bundles, $i = 1, 2$. Let $PE_i$ be the projectivization of $E_i$.

Let $f : PE_1 \to PE_2$ be a fiber-like symplectictomorphism, covering a map $\varphi : M_1 \to M_2$. Then $\varphi_*(c_k(E_2)) = c_k(E_1)$ for $k \geq 2$.

**Remark:** Since the projectivization of a line bundle is always trivial, this is obviously false for the first Chern class.

**Proof:** The construction above gives Chern classes in $BSympl(Q)$ for any symplectic $Q$. For $PE_i$ we have inclusion of structure groups $PSU_n \to BSympl(\mathbb{C}P^{n-1})$, and the argument of 1.3. shows that the inverse map in cohomology is surjective. The statement...
of the theorem follows from the diagram

\[
\begin{array}{ccc}
M_1 & \rightarrow & M_2 \\
\searrow & & \swarrow \\
B PSU_n & \rightarrow & B Sympl(\mathbb{C}P^{n-1})
\end{array}
\]

2. Cohomology of the Symplectomorphisms Group Made Discrete:

Relations to the Regulators in Algebraic \(K\)-Theory

2.1: We need to discuss briefly the formalism of [Re3]. Let \(G\) be a Frechet-Lie group over \(\mathbb{R}\) or \(\mathbb{C}\) with the Lie algebra \(g\). Assume \(g/\mathfrak{g} \cdot \mathfrak{g} = 0\). Consider the standard complex

\[
k \rightarrow g^* \rightarrow \wedge^2 g^* \rightarrow \cdots
\]

which is a DGA. These are associated homotopy groups \(\pi_i(g)\) [Re 3] with standard structures (Whitehead bracket, Quillen spectral sequence etc) and a natural graded Lie algebra homomorphism

\[
(2.1) \quad \pi_i(G^{\text{top}}) \rightarrow \pi_i(g)
\]

([Re 3], section 1).

In case \(G = SL(A)\), \(A\) is a commutative Frechet Algebra, \(\pi_i(G^{\text{top}}) = K_{i-1}(A), \pi_i(g) = HC_{i-1}(A)\) and (2.1) becomes the Karoubi-Connes Chern character. In particular if \(M\) is a compact manifold and \(A = C^\infty(M)\), then \(HC_{i-1}(C^\infty(M)) = \frac{\Omega^{i-1}(M)}{d\Omega^{i-2}(M)} \oplus H^{i-3}(M, k) \oplus H^{i-5}(M, k) \oplus \cdots\) and (2.1) becomes a usual Chern character.

Now, one defines an algebraic \(K\)-theory of \(G\) by \(K^\text{alg}_i(G) = \pi_i((BC^\delta)^+).\) The argumented algebraic \(K\)-theory is defined as a kernel of the natural map \(K^\text{alg}_i \rightarrow K^\text{top}_i:\n\]

\[
0 \rightarrow K^\text{alg}_i(G) \rightarrow K^\text{alg}_i(G) \rightarrow \pi_{i-1}(G^{\text{top}}).
\]

The regulator map is a homomorphism

\[
r : K^\text{alg}_i(G) \rightarrow \text{coker}(\pi_i(G^{\text{top}}) \rightarrow \pi_i(g)).
\]

In case \(G = SL(A)\) the right side is \(HC_{i-1}(A)/JmK^{\text{top}}(A),\) and if \(A = C^\infty(M)\) and \(i \rightarrow \infty\) then the right side becomes essentially \(H^{\text{even}}(M, \mathbb{R})/H^{\text{even}}(M, \mathbb{Z})\) or \(H^{\text{odd}}(M, \mathbb{R})/H^{\text{odd}}(M, \mathbb{Z}).\)
This agrees with Bloch-Beilinson regulator [Bl] [Be] if M is smooth complex projective variety.

2.2: In [Re3] we applied this formalism to construct Chern-Simons classes for Diffδ(S^2) and Diffδ(S^1). The construction goes as follows: one starts with a cohomology class μ ∈ H^i(g) and defines the group of periods A_μ by A_μ = μ(Jmπ_i(G^{top}). Then μ ◦ r defies a homomorphism from K^alg_i(G) to k/A_μ. Sometimes it is possible to know directly that A_μ ⊆ Z or A_μ ⊆ Q. In particular, let G = Sympl(M) where we keep the restrictions of Chapter 1 on M and let q be a polynomial in pk, defined in 1.3. Then q(θ, [θ, θ] · · · [θ, θ]) defines first cohomology class of g = sympl(M), second, a cohomology class in H^*_top(Sympl(M), R) and third, a regulator 

q ◦ r : K^alg(Sympl(M)) → R/A_q,

where A_q is a group of values of the just defined in H^*_top(Sympl(M), R) class on the Hurewitz image of π*(Sympl(M)) in H_*(Sympl(M), Z). In particular, let M = (CP^2, can). Then one knows that that Sympl(M) has PSU(3) as a homotopical retract. Choose q in such a way, that it restricts on the second Chern polynomial of SU(3) under the pull-back map coming from the inclusion PSU(3) → Sympl(M). Then we get A_q = Z, so the following result is true.

**Theorem 2.2.** There exists a regulator

π_3(BSymplδ(CP^2))^+ → R/Z

which makes the diagram

\[
\begin{array}{ccc}
π_3(BPSU^δ(3))^+ & \xrightarrow{\text{stabilization}} & K_3(\mathbb{C}) \xrightarrow{Re(B)} R/\mathbb{Z} \\
\downarrow & & \| \\
π_3(BSymplδ(CP^2))^+ & → & R/\mathbb{Z}
\end{array}
\]

commutative. Here B is the Beilinson - Karoubi regulator K_{2n-1}(\mathbb{C}) → \mathbb{C}/\mathbb{Z}. [Ka1].

Following the framework of [CheS, Re1] one would expect that in fact one has a class in H^3(Symplδ(CP^2), R/Z) which extends the Chern-Simons class in H^3(PSU(3), R/Z), but the author cannot prove that at the moment. Due to the lack of knowledge about the topology of Sympl(CP^n) for n > 2 we cannot say whether the period group of the higher analog of q lies in Z or Q, (which would give an extension to H^{2i-1}(Symplδ(CP^n)) of the Chern-Simons class in H^{2i-1}(PSU(n), R/Z), comp. problem 2 in the problem list)
2.3: There is another interesting class in $H^2(B\text{Sympl}^\delta(M))$, whose definition has been sketched in [Re1], as follows. Consider a flat fibration $M \to \mathcal{F} \to B\text{Sympl}^\delta(M)$. Any element in $H_2(B\text{Sympl}^\delta(M),\mathbb{Z})$ is given by a map of a surface $\Sigma^g$ to $M$. The pullback of $\mathcal{F}$ to $\Sigma$ is a flat (hence smooth) fibration with a parallel two-form $\omega$, coming from the symplectic structure of $M$. Since $\pi_1(M) = 0$, it has a smooth section $s$. The pullback $s^*\omega$ is a two-form on $\Sigma$ and one has a number $(s^*\omega, [\Sigma])$. One checks, following the general theory of regulators in [Re1], that this define a cohomology class in $H^2(B\text{Sympl}^\delta(M),\mathbb{R}/\mathbb{A})$, where $\mathbb{A}$ is the group of periods of $\omega$, that is, a $(\omega, H_2(M,\mathbb{Z})) \subset \mathbb{R}$. In particular, if $\omega$ is an integer form, we come to a class in $H^2(B\text{Sympl}^\delta(M),\mathbb{R}/\mathbb{Z})$. The Bockstein image of this class in $H^3_{\text{tors}}(B\text{Sympl}^\delta(M),\mathbb{Z})$ is just a transgression of $[\omega]$ in $H^2(M,\mathbb{Z})$ in the flat fibration written above.

If one has a symplectic action of a Lie group $G$ on $M$, say $\pi : G \to \text{Sympl}(M)$ one immediately arrives to the secondary characteristic class in $H^2(G^\delta,\mathbb{R}/\mathbb{A})$. In notations of [Re1], this is $\text{Bor}(\pi,\omega).

Example (2.3): Let $G = SL(2,\mathbb{R})$ acting on $\mathcal{H}^2$. Then the class in $H^2(SL^\delta(2,\mathbb{R}),\mathbb{R})$ is just the Euler class.

A more interesting example is given by a compact group $G$ acting symplectically on a coadjoint orbit $P \subset \mathfrak{g}$ with the canonical (Kirillov) symplectic form normalized such that $A = \mathbb{Z}$. One gets a class in $H^2(G^\delta,\mathbb{Z})$. For instance, let $G = SU(2)$ or $SO(3)$ and $P = CP^1 \approx S^2$. Since the Euler class in $H^3(SO^\delta(3),\mathbb{R}^3)$ is nonzero, by the remark above we know at least that our class is also nonzero. On the other hand, it is rigid in the sense of [Che-S]. So by [Re1], and because $K_2(\mathbb{F})$ is torsion for any number field $\mathbb{F}$, our class is “locally torsion”, that is, its value on any element of $H_2(BSU^\delta,\mathbb{Z})$ lies in $\mathbb{Q}/\mathbb{Z}$.

2.4: The most interesting situation is that of the mapping class group $M_g$ acting symplectically in the moduli space $\mathcal{M}_g$ (see 4.4). One immediately gets a class in $H^2(M_g,\mathbb{R}/\mathbb{Z})$. If we knew the topology of $\text{Sympl}(\mathcal{M}_g)$, we could try to define a class in $\text{Hom}(K_3(M_g),\mathbb{R}/\mathbb{Z})$ as in 2.2. This would be useful for three-manifolds invariants (cf. problem 3).
3. Secondary Invariants of Lagrangian Submanifolds

3.1: Throughout this chapter, \( M \) stands for a symplectic manifold of dimension \( 2n \). We will work with dimension \( k \leq n \), satisfying the following conditions:

\[
H_{2k-1}(M, \mathbb{Z}) = 0
\]

We also assume that the class of the symplectic form \( \omega \) is integer, that is, lies in the image of \( H^2(M, \mathbb{Z}) \) in \( H^2(M, \mathbb{R}) \).

Let \( L \) be an immersed compact Lagrangian submanifold of \( M \). Let \( z \in H_{2k-1}(L, \mathbb{Z}) \) be represented by a singular chain \( c \). Since \( H_{2k-1}(M, \mathbb{Z}) = 0 \), one finds a chain, \( b \), in \( C_{2k}(M) \), such that \( \partial b = c \). Put \( \rho(z) = \int b \omega^k \mod \mathbb{Z} \). If \( b' \) is another chain with \( \partial b' = c \), we get \( \partial(b - b') = 0 \), so \( \int b \omega^k - \int b' \omega^k \in \mathbb{Z} \), hence \( \rho(z) \) does not depend on the choice of \( b \).

On the other hand, if \( c' \) is another chain, representing \( z \), then choose a chain \( a \in C_{2k}(L) \) with \( \partial a = c - c' \). A union \( b \cup a \) will span \( c' \), and \( \int b \omega^k = \int b \omega^k + \int a \omega^k = \int b \omega^k \), since the latter term in zero. Hence \( \rho : H_{2k-1}(L, \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z} \) is a well-defined homomorphism.

3.2: Here is another description of the invariant \( \rho \). Let \((\ell, \theta)\) be a unitary line bundle with a connection \( \theta \), such that the curvature of \( \theta \) equals \( \omega \). It is well-known that such bundles exist. For \( L \) Lagrangian, the restriction \( \ell|_L \) is flat. According to the general theory of characteristic classes [Che-S], there is a secondary class in \( H^{2k-1}(L, \mathbb{R}/\mathbb{Z}) \), corresponding to \( c^k_1 \). In particular, there is a homomorphism \( H_{2k-1}(L, \mathbb{Z}) \rightarrow \mathbb{R}/\mathbb{Z} \), which coincides with \( \rho \). To prove the last statement, let us find a map \( M \xrightarrow{\varphi} \mathbb{C}P^N, N \gg 1 \), such that the pullback of the hyperplane line bundle \( \mathcal{O}(1) \) with canonical connection will be \((\ell, \theta)\). This is possible by Narasimhan-Ramanan. For \( z \) and \( c \) as before, the value of the secondary class on \( z \) is given as follows [Ch-S]: one span \( c \) in \( \mathbb{C}P^N \) and integrate the \( k \)-th power of the Fubini-Study symplectic form across the spanning bubble. Since there exists a spanning bubble already in \( M \), the two definitions coincide.

3.3: Here are the basic properties of the invariant \( \rho \).

3.3.1.: If \( \omega \) is exact and \( k \geq 2 \), then \( \rho = 0 \). Indeed, let \( \omega = d\alpha \), then for \( z, c \) and \( k \) as above we have \( \int_b \omega^k = \int_b d(\alpha \cdot \omega^{k-1}) = \int_c \alpha \cdot \omega^{k-1} = 0 \)
3.3.2.: (Rigidity). Let \( \varphi : L \times I \to M \) be a smooth family of Lagrangian immersion of a manifold \( L \) to \( M \). Put \( L_t = \varphi(0, t) \). If \( k \geq 2 \), then \( \rho(L_t) \) is constant in \( \text{Hom}(H_{2k-1}(L, \mathbb{Z}), \mathbb{R}/\mathbb{Z}) \).

**Proof:** Let \( z, c, b \) be as above with respect to \( L_0 \). The chain \( \varphi(c \times I) \cup b \) spans the image of \( c \) under \( \varphi(\cdot, 1) \). So \( \rho_1(z) = \int b \omega_k + \int_{(c \times I)} (\varphi^* \omega)^k \). If \( k \geq 2 \), then the latter integrand is pointwise zero, which proves the statement.

3.3.3.: (Isoperimetric estimate.) Let \( M \) be compact. Fix a Riemannian metric \( a_M \).

There exists a constant \( \gamma(M) \), such that

\[ \rho(z) \leq \gamma \cdot \|z\| \]

Here \( \|z\| \) is a volume (mass) norm in induced metric an \( L \).

**Proof:** This is an immediate corollary of Gromov-Eliashberg isoperimetric film theorem \([Gr-El]\).

3.3.4.: (Rationality, comp. \([Re1]\), \([Re2]\)). For any \( z \in H_{2k-1}(L, \mathbb{Z}), \rho(z) \in \mathbb{Q} \).

**Proof:** Fix \((\ell, \theta)\) as in 5.2. Let \( U \) be the unit circle subbundle of \( \ell \). Consider the diagram

\[
\begin{align*}
H^2(L, \mathbb{Z}) &\to H^2(L, \mathbb{R}) \\
\uparrow &\quad \uparrow \\
H^2(M, \mathbb{Z}) &\to H^2(M, \mathbb{R})
\end{align*}
\]

The image of \( c_1(\ell) \) in \( H^2(M, \mathbb{R}) \) coincides with the class of \( \omega \). This implies that \( c_1(\ell|L) \) is torsion. Hence for \( N \) big enough, \( c_1(\ell^N|L) = 0 \). Relabel \( \ell^N \) by \( \ell \) and \( N \omega \) by \( \omega \). We seek to prove that (in new notation) \( \rho = 0 \). But now there exists a smooth section \( S \) of \( U|L \).

Since the pull-back of \( \omega \) in \( U \) is exact, \( \rho = 0 \) by 3.3.1.

Observe that we prove in fact that \( N \rho(z) \in \mathbb{Z} \) where \( N \) is the order of \( c_1(\ell) \) in \( H^2(L, \mathbb{Z})_{\text{tors}} \).

3.4: In this section we will furnish a computation of the invariant \( \rho \) for the standard Lagrangian embedding \( \mathbb{R}P^n \to \mathbb{C}P^n \). Let \( \ell \) be the hyperplane bundle over \( \mathbb{C}P^n \).

**Lemma 3.4:** \( w_2(\ell|\mathbb{R}P^n) \neq 0 \).

**Proof:** The restriction \( \ell|_{\mathbb{R}P^n} \) is isomorphic to \( \tau \otimes \mathbb{C} \), where \( \tau \) is the tautological real line bundle over \( \mathbb{R}P^n \). So \( w_2(\ell) = w_1^2(\tau) \neq 0 \).

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Using 3.2, we may view $\rho$ as an element of $H^{2k-1}(\mathbb{RP}^n, \mathbb{R}/\mathbb{Z})$. Moreover, by 3.3.4 where we can take $N = 2$, $\rho$ lives in $H^{2k-1}(\mathbb{RP}^n, \mathbb{Z} \cdot \frac{1}{2}/\mathbb{Z})$. Consider the diagram

$$
\begin{array}{c}
H^{2k-1}(\mathbb{RP}^n, \mathbb{Z} \cdot \frac{1}{2}/\mathbb{Z}) \\ \downarrow \quad \beta \quad \downarrow \\
H^{2k-1}(\mathbb{RP}^n, \mathbb{R}/\mathbb{Z}) & \cong & H^{2k}(\mathbb{RP}^n, \mathbb{Z}) & \cong & H^{2k}(\mathbb{RP}^n, \mathbb{Z}_2) \\
\end{array}
$$

where $\beta$ means Bockstein. By [Che-S], the image of $\rho$ as an element of $H^{2k-1}(\mathbb{RP}^n, \mathbb{R}/\mathbb{Z})$ is $c_1^k(b)$, whose reduction mod 2 is $\omega_2^k(\ell) \neq 0$ by the previous lemma. So as an element of $H^{2k-1}(\mathbb{RP}^n, \mathbb{Z} \cdot \frac{1}{2}/\mathbb{Z}) = H^{2k-1}(\mathbb{R}P^k, \mathbb{Z})$, $\rho \neq 0$. We get the following statement.

**Proposition (3.4).** The value of $\rho$ on the generator of $H_{2k-1}(\mathbb{RP}^n, \mathbb{Z})$ is $\frac{1}{2}$ (mod $\mathbb{Z}$).

Now, let $M$ be a symplectic submanifold of $\mathbb{C}P^n$ and suppose some $\mathbb{RP}^n \subset \mathbb{C}P^n$ intersects $M$ transversally by a Lagrangian submanifold $L$. This is always the case if $M$ is a smooth projective variety over $\mathbb{R}$. Then we get immediately that if the map $H_{2k-1}(L, \mathbb{Z}) \rightarrow H_{2k-1}(\mathbb{RP}^n, \mathbb{Z})$ is nontrivial, then $\rho(L) \neq 0$.

3.5. Piecing together 3.3.3, 3.3.3, and 3.4 we come to the following theorem.

**Theorem (3.5).** Let $X \subset \mathbb{P}^n$ be a smooth projective variety defined over $\mathbb{R}$, $\dim X = 2k - 1$ and let $M = X(\mathbb{C})$ and $L = X(\mathbb{R})$. Suppose the homomorphism $H_{2k-1}(L) \rightarrow H_{2k-1}(\mathbb{RP}^n) = \mathbb{Z}_2(k \geq 2)$ is nontrivial. Then for any metric on $M$ and any Lagrangian homotopy $L_t$ of $L$ in $M$, Vol $(L_t)$ stays bound away from zero by a constant $\gamma(M)$.

For $X = \mathbb{P}^n$ this (with a sharp $\gamma(M)$) is a theorem of Kleiner-Oh based on the fixed points theorem of Givental [Gi1].

4. Futaki-type characters of symplectomorphisms groups with application to the structure of the Torelli group and Automorphisms Group of One-relator Groups

4.1.: We will apply ideas of the previous chapter to define a character of $\text{Sympl}_h(M)/\text{Sympl}_0(M)$ for a symplectic manifold $M$ with integer symplectic class. Here $\text{Sympl}_h(M)$ is the kernel
of the natural homomorphism $\operatorname{Sympl}(M) \to \operatorname{Aut}H_\ast(M, \mathbb{Z})$, and $\operatorname{Sympl}_0(M)$ is the connected through Hamiltonian isotopy component of identity on $\operatorname{Sympl}(M)$. The character we define values in $\operatorname{Hom}(H_{\text{odd}}(M, \mathbb{Z}), \mathbb{R}/\mathbb{Z})$. Let $f \in \operatorname{Sympl}_h(M)$ and $z \in H_{2k-1}(M, \mathbb{Z})$. Represent $z$ by a chain $c$ and consider a chain $f(c) - c$. Since $[f(c)] = [c]$, there exists a chain $b$ such that $\partial b = f(c) - c$. Put

\begin{equation}
\chi(f, z) = \int_b \omega^k (\text{mod} \mathbb{Z})
\end{equation}

As usual, this does not depend on the choice of $b$. If $c'$ is another chain with $[c'] = z$ find $a$ such that $\partial a = c' - c$. Then we have $\partial (fa - a + b) = c' - c$,

$$
\int_{fa-a+b} \omega^k = \int_b \omega^k - \int_a \omega^k + \int_a (f^*\omega)^k = \int_b \omega^k.
$$

We see that $\chi(f, z)$ is well-defined. Now, for $f, g \in \operatorname{Sympl}_h(M)$ let

\begin{align*}
\partial b &= gc - c \quad \text{and} \quad \partial b = fc - c, \quad \text{then} \quad \partial (fb) &= \\
&= fgc - c, \quad \text{so} \quad \partial (fb \cup b') = fgc - c \quad \text{and} \\
\chi(fg, z) &= \int_{fb \cup b'} \omega^k = \int_b \omega^k + \int_{b'} (f^*\omega)^k = \int_b \omega^k + \int_{b'} \omega^k = \\
&= \chi(f, z) + \chi(g, z).
\end{align*}

We resume this calculation in the following theorem

**Theorem 4.1.** There exists a group homomorphism

$$
\chi : \operatorname{Sympl}_h(M)/\operatorname{Sympl}_0(M) \to \operatorname{Hom}(H_{\text{odd}}(M, \mathbb{Z}), \mathbb{R}/\mathbb{Z})
$$

defined by (4.1).

**Proof:** Recall that, $\operatorname{Sympl}_0(M) \subset [\operatorname{Sympl}_h(M), \operatorname{Sympl}_h(M)]$. Since the right hand side group is abelian, the homomorphism of $\operatorname{Sympl}_h(M)$ defined above factors through the quotient group.

**4.2. Examples:** Let $M$ be a torus $(S^1 \times S^1, \text{can})$. We get a character

$$
\operatorname{Sympl}_h(M)/\operatorname{Sympl}_0(M) \to \mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z}
$$
which is in fact an isomorphism. Moreover, $\text{Sympl}_h(M)$ splits as a semidirect product $\text{Sympl}_0(M) \rtimes (\mathbb{R}/\mathbb{Z} \oplus \mathbb{R}/\mathbb{Z})$. In a similar fashion, let $M$ be a surface of genus $g$ with an area form with integer integral. Then we have a character

$$\text{Sympl}_h(M)/\text{Sympl}_0(M) \to J^1(M) = \frac{H^1(M, \mathbb{R})}{H^1(M, \mathbb{Z})}.$$  

4.3. Remark: Let $M$ be any compact manifold with an integer volume form $\nu$. Let $\text{Diff}^\nu(M)$ be a group of volume-preserving diffeomorphism, and $\text{Diff}^\nu_h(M)$ be the kernel of the natural map $\text{Diff}^\nu(M) \to \text{Aut}(H_1(M), \mathbb{Z})$. Then one has a character

$$\text{Diff}^\nu_h(M) \to J^1(M),$$

defined exactly as above. For ergodic diffeomorphisms this catches invariants under conjugacy in $\text{Diff}(M)$, see [Re3].

4.4.: In this section we apply the previously developed technique to the study of the mapping class group $M_g$. Recall that this is the group of outer automorphisms of the surface group $\Pi_g = \pi_1(C_g)$ (fundamental group of a closed surface of genus $g$). Another view at $M_g$ is as a quotient group of the diffeomorphisms of $C_g$, by the subgroup of diffeomorphisms isotopic to identity.

Fix a Riemann surface structure on $C_g$ and consider the moduli $\mathcal{M}_g$ space of rank two stable holomorphic vector bundles of degree one over $C_g$. By the theorem of Narasimhan-Seshadri, this space is canonically diffeomorphic to the representation variety $\text{Hom}(\Pi_g, PSU(2))/PSU(2)$ of representations with nontrivial second Stiefel-Whitney class. The latter space has a canonical Goldman’s symplectic structure [Gold], and the natural action of $M_g$ is symplectic.

The Torelli group $I_g$ is the kernel of the natural surjective homomorphism $M_g \to Sp(2g, \mathbb{Z}) \subset \text{Aut}H_1(C_g, \mathbb{Z})$. It was studied in [J1], [J2] [J3], and the most fundamental discovery made there was the existence of the homomorphism $I_g \to \mathbb{Z}^{(2g)}$ [J1], now known as Johnson’s homomorphism.

To see the relation to our formalism, we recall a theorem of Newstead which says that the action of $I_g$ is homology of $\mathcal{M}_g$ is trivial [N]. Therefore we have a homomorphism $I_g \to \text{Sympl}_h(\mathcal{M}_g)$.
By the calculation of the previous section, we obtain a map

\[ I_g \rightarrow \text{Hom}(H_{\text{odd}}(\mathcal{M}_g), \mathbb{R}/\mathbb{Z}). \]

Now, the third homology \( H_3(\mathcal{M}) \) is canonically isomorphic to \( H_1(C_g) \), c.f. [A-B], [H-N]. So we arrive to a character

\[ I_g \rightarrow \text{odd} \wedge H^1(C_g) \otimes \mathbb{R}/\mathbb{Z} = (\mathbb{R}/\mathbb{Z})^{2g}. \]

For the present the author does not have enough tools to compute the image of this character and to understand the connection with the Johnson’s homomorphism.

4.5: We wish to extend the characteristic homomorphism \( I_g \rightarrow (\mathbb{R}/\mathbb{Z})^{2g} \) of the previous section to the situation of the automorphism groups of one-regulator groups. Let \( F_{2g} \) be a free group in \( 2g \) generator and let \( r \subset F_{2g} \) be a balanced word that is, \( r \) lies in the commutator \([F_{2g}, F_{2g}]\). Consider a one-reator group \( \Gamma = F_{2g}/\{r\} \). We denote \( M(\Gamma) = \text{Aut}(\Gamma) \) and \( I(\Gamma) \) a kernel of the homomorphism \( M(\Gamma) \rightarrow \text{Aut}(H_1(\Gamma, \mathbb{Z})) \). Our goal is to construct abelian quotients of \( I(\Gamma) \).

Fix a compact semisimple non-abelian Lie group \( G \) and consider the character variety \( \mathcal{M}_G(\Gamma) = \text{Hom}(\Gamma, G)/G \). The Fourier-Donaldson transform of appendix \( \mathcal{J} \) gives a closed two-form \( \omega = FD(z) \in \Omega^2_{\text{cl}}(\mathcal{M}_{\text{reg}}(\Gamma)) \), where \( z \) is the generator of \( H_2(\Gamma, \mathbb{Z}) = \mathbb{Z} \). This form is invariant under the natural \( M(\Gamma) \)-action and nondegenerate.

We put for simplicity \( G = PSU(2) \) and consider the component of \( \mathcal{M}_G(\Gamma) \) with the Steifel-Whitney number one, which we relabel \( \mathcal{M}(\Gamma) \). We make the following regularity assumption:

**Regularity condition** (4.5) A one-reator group \( \Gamma = F_{2g}/\{r\} \) is called regular, if the character variety \( \mathcal{M}(\Gamma) \) is smooth.

The surface group \( r = [x_1, x_2] \cdots [x_{2g-1}, x_{2g}] \) is regular, but the author does not know a general criterion to distinct regular groups. Up to the end of this chapter, we consider only regular one-reator groups.

4.6.: We wish to define a map \( H_1(\Gamma, \mathbb{Z}) \rightarrow H^3(\mathcal{M}(\Gamma), \mathbb{Z}) \). Fix an element \( r \in \Gamma \) and consider the evaluation map (on \( \gamma \)):

\[ \text{Hom}(\Gamma, G) \rightarrow G. \]
The pull-back of the generator of $H^3(G, \mathbb{Z}) = \mathbb{Z}$ gives an element in $H^3(\text{Hom}(\Gamma, G), \mathbb{Z})$. It is immediate to check that the restriction of this element on orbits of the adjoint $G$-action in $\text{Hom}(\Gamma, G)$ vanishes. So the Leray-Serre spectral sequence of the fibration

$$G \rightarrow \text{Hom}(\Gamma, g) \rightarrow \mathcal{M}(\Gamma)$$

predicts that this element comes from the unique element in $H^3(\mathcal{M}(\Gamma), \mathbb{Z})$. One checks at once that first, one gets a group homomorphism $\Gamma \rightarrow H^3(\mathcal{M}, \mathbb{Z})$ and second, that $\mu(\lambda \gamma \lambda^{-1}) = \mu(\gamma)$ for every $\gamma, \lambda$. So the homomorphism $\tilde{b}$ factors through $H_1(\Gamma)$.

The homomorphism $b$ is in fact injective. To see that, fix a system of generators $(x_1, \cdots, x_{2g})$ of $F_{2g}$ and consider a map $G \rightarrow \text{Hom}(\Gamma, G) \rightarrow \mathcal{M}(\Gamma)$ defined by $g \mapsto (\downarrow^{x_1}, \cdots, \downarrow^{x_i}, \cdots, \downarrow^{x_{2g}})$. Since $r$ is a balanced word, this is well-defined. This gives an element in $H_3(\mathcal{M}(\Gamma), \mathbb{Z})$, say $\varphi_i$ and one gets $b(\bar{x}_i)(z_j) = \delta_{ij}$.

One obviously gets an extended homomorphism

$$\wedge^{\text{odd}} H_1(\Gamma) \rightarrow H^{\text{odd}}(\mathcal{M}(\Gamma), \mathbb{Z}).$$

Now, the action of $I(\Gamma)$ on the image of this homomorphism is trivial, and, applying Theorem 4.1, we arrive to a homomorphism

$$I(\Gamma) \rightarrow \text{Hom}(\wedge^{\text{odd}} H_1(\Gamma), \mathbb{R}/\mathbb{Z}) \approx (\mathbb{R}/\mathbb{Z})^{2g}.$$ 

If $\Gamma$ is the surface group, we recover the characteristic homomorphism of $4.4$. 

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5. Digression: Evens multiplicative transfer and the fixed point theory

5.1: Let $X \xrightarrow{\pi} Y$ be a finite degree covering of CW-complexes. Fix a field $\mathbb{F}$. A (classical) transfer map is a homomorphism $t : H^*(X, \mathbb{F}) \to H^*(Y, \mathbb{F})$ with the following properties:

i) $t \circ \pi^*$ is a multiplication by $d$

ii) if $\pi$ is a regular covering with a Galois group $\Gamma$, then $\pi^* \circ t(z) = \sum_{\gamma \in \Gamma} \gamma z$.

In Appendix $\aleph$, we give a construction of transfer, which works for non-free actions of $\Gamma$ in $X$, using Thom-Dold theorem, in spirit of recent work of Karoubi [Ka3], [Ka4].

5.2: We will describe here the “multiplicative” transfer of Evens [Ev]. This construction proved very useful in group cohomology (see [Ben] and references therein), Galois cohomology [Kahn] and algebraic geometry [F-M]. We confine ourselves to regular coverings only. In notations of 8.1., the multiplicative transfer, also denoted by $t$, is map $t : H^*(X, \mathbb{F}) \to H^*(Y, \mathbb{F})$ such that

\[(\ast) \quad \pi^* \circ t(z) = \prod_{\gamma \in \Gamma} (1 + \gamma z)\]

Developing the product in RHS of (\ast) we get all “homogeneous transfers” $t_i : H^i(X, \mathbb{F}) \to H^i(Y, \mathbb{F})$ such that $t_1$ is the classical “additive” transfer and $t_d$ satisfies

\[(5.2) \quad \pi^* \circ t_d(z) = \prod_{\gamma \in \Gamma} \gamma z\]

We sketch very briefly how can one define the map $t$. Consider the spectral sequence of the covering $X \to Y$, with the $E^2$ term $E_2^{i,j} = H^i(\Gamma, H^j(X))$. So the first column is $E_2^{0,j} = H_{inv}^j(X)$. Now, the claim actually is that “constructible” elements in $H_{inv}^j(X)$ survive in $E_\infty$. By constructible elements we mean elementary symmetric polynomials in $\gamma z, \gamma \in \Gamma$, where $z \in H^*(X)$ is a fixed homogeneous element. To prove that, we notice first that such classes is functorial by a $\Gamma$-space $X$, that is, if $X \xrightarrow{\lambda} \bar{X}$ is a map of $\Gamma$-spaces and $\bar{\varphi} \in H_{inv}(\bar{X})$ survives in $E_\infty$, then $\lambda^*\bar{\varphi}$ survives as well. Now, consider the embedding $X \xrightarrow{\lambda} X \times X \times \ldots \times X$ by $x \mapsto (\gamma \mapsto \gamma x)$. This is $\Gamma$-equivariant, if we consider the
permutation action of $\Gamma$ in the latter space. Denote the latter space by $W$ and consider the spectral sequence for $S_d$-equivariant cohomology with $E^2_{i,j} = H^i(S_d, H^j(W)) \Rightarrow H^{i+j}_{S_d}(W)$. Suppose we know that this spectral sequence degenerates, that is, $H^i_{inv}(W)$ survive in $E_\infty$. Then for any $\varphi \in H^i_{inv}(W)$, $\lambda^* \bar{z}$ survives in the spectral sequence of the covering $X \to Y$, since the equivariant cohomology of $X$ is just $H^*(Y)$, because the action of $\Gamma$ in $X$ is free. Now, take $\bar{z}$ to be the average (under the action of $S_d$) of $z \times z \times \ldots \times z \times 1 \times \ldots \times 1$. Then $\lambda^* \bar{z}$ is just $i$-th symmetric polynomial in $\gamma z$, and we are done. So what is left is to prove that the spectral sequence above degenerates. For a group $\Gamma$ and a complex $C$ with $\Gamma$-action one defines the equivariant cohomology of $C$ in a usual way, as $H^*_\Gamma(C) = H^*(C \otimes P)_{inv}$ where $P$ is the free resolution of $\mathbb{F}$ as a trivial $\Gamma$-module. Now, $C^{sing}(X \times \ldots \times X) \otimes P$ is homotopy equivalent $C^{sing}(X) \otimes \ldots \otimes C^{sing}(X) \otimes P$ as $\mathbb{F}[S_d]$- complexes, by an equivariant version of Eilenberg - Silber theorem, and the latter complex is homotopically equivalent to $H^*(X) \otimes \ldots \otimes H^*(X) \otimes P$. So what we need to prove that for any complex of vector spaces over $\mathbb{F}$, say $V$, the spectral sequence for the $S_d$ equivariant cohomology of $\underbrace{V \otimes \ldots \otimes V}_d$ degenerates. This is easy and left to the reader.

5.3: There is a nice application of the multiplicative transfer of the fixed point theory, developed for $\mathbb{Z}_2$ - action by Bredon, see [Bred]. Start with a free action of $\mathbb{Z}_p$ in a compact orientable manifold $X^n$. If $p = 2$ we assume that the action is orientation-preserving. We assume $p|n$.

**Lemma (5.3).** For any $z \in H^{n/p}(X, \mathbb{Z})$ one has

$$p|\left( \prod_{\gamma \in \mathbb{Z}_p} \gamma z, [X] \right)$$

**Proof:** Reducing mod $p$, we reduce the statement to $\prod_{\gamma \in \mathbb{Z}_p} \gamma z = 0$ in $H^n(X, \mathbb{F}_p)$. Let $Y = X/\mathbb{Z}_p$, then by (5.2), $\prod_{\gamma \in \mathbb{Z}_p} \gamma z = \pi^*(t z)$. But $\pi_* = 0$ on $H^n(Y, \mathbb{F}_p)$ and we are done.

Now we will show that the statement of the lemma is still true if the fixed point set of $X$ is “small”.

**Theorem (5.3).** Let $X^n$ be a compact oriented manifold with $\mathbb{Z}_p$-action, preserving ori-
entation. Assume \( \dim(\text{Fix}(X)) < \frac{n}{p} \). Then for any \( z \in H^{n/p}(X, \mathbb{Z}) \) one has

\[
p|\left( \prod_{\gamma \in \mathbb{Z}_p} \gamma z, [X] \right)
\]

PROOF: Let \( W^n \) be a result of the following surgery: remove a tubular neighbourhood of \( \text{Fix}(X) \), say \( N \), and glue two copies of \( X \setminus N \) along \( \partial N \). Then \( W^n \) carries a free \( \mathbb{Z}_p \) -action. Now, let \( Z \) be a homology class in \( H_{n-n/p}(X, \mathbb{Z}) \), Poincaré -dual to \( z \). We can always represent \( Z \) by a chain, whose support is disjoint from \( \text{Fix}(X) \), by transversality. So \( Z \) survives under surgery and defines a homology class in \( W \). The intersection number \( \bigcap_{\gamma \in \Gamma} \gamma Z \) stays the same and is divisible by \( p \) by the previous lemma, Q.E.D.

5.4.: An immediate application is the reverse statement: if for some \( z \in H^{n/p}(X) \), the number \( \left( \prod_{\gamma \in \mathbb{Z}_p} \gamma z, [X] \right) \) is not divisible by \( p \), then \( \dim(\text{Fix}(X)) \geq n/p \). This applies for spaces like Grassmanians and their products, because the action of \( \mathbb{Z}_p \) in cohomology is often shown to be trivial for any action, and intersection numbers are computable (see [Re5] for other applications).

6. Volumes of moduli spaces and Van Staudt theorem for Witten’s zeta-function

6.1: We now return our attention to the remarkable symplectic manifold \( \mathcal{M} \), the representation variety of the surface group, which was in use in Chapter 4, and apply the ideas of the previous chapter to derive interesting number-theoretic statements concerning the special values of the Witten’s zeta function.

\[
\zeta^W_G(s) = \sum_{\alpha} \frac{1}{(\dim \alpha)^s}
\]

Here, one fixes a compact simple non-abelian Lie group \( G \), and the summation in (9.1) goes over all irreducible characters of \( G \). In case \( G = SU(2) \), \( \zeta^W_G(s) \) is just the Riemann zeta-function. Witten proved in [W], that the normalized values of \( \zeta^W_G \) at positive even integers, namely \( W^G_G(2m) = \frac{(2\pi)^2(3m)!}{2} \zeta^W_G(2m) \) are integers. Our main result deals with the divisibility of these integers by prime divisors of \( m \), as follows.
Theorem (6.1). For any $p|m$ such that $p(p-1) \leq m$ the number
\[ W_G(2m) = \frac{(3m)!}{(2m)!} \zeta^W(2m) \]
is divisible by $p$.

In case $G = SU(2)$ one has $W(2m) = \frac{(3m)!}{(2m)!} B_{2m}$ and the statement of the Theorem 6.1. is a consequence of the classical Von Staudt theorem on divisibility of Bernoulli numbers. So one views our result, the Theorem 6.1, as a natural extension of the Von Staudt theorem for Witten zeta-function.

6.2: Before going into a proof, we describe its main idea. Consider a closed surface $C$ of a genus $g = m + 1$. For any $p|m$, “sufficiently small” there exists an action of $\mathbb{Z}_p$ on $C$ which is not free. Fix an invariant conformal structure on $C$. Then we have an induced action of $\mathbb{Z}_p$ in $M$, the representation variety $M = \text{Hom}(\pi_1(C), G)/G$. The fixed point set $\text{Fix}(M)$ is essentially the moduli space of parabolic vector bundles, and the dimension calculation shows that $\dim(\text{Fix}(M))$ is strictly less than $\frac{1}{p} \dim M$ (here one uses that the action of $\mathbb{Z}_p$ on $C$ is not free). So by the Theorem 5.3, for any $\alpha \in H^{\dim M/p}(M, \mathbb{Z})$, the number $(\prod_{\gamma \in \mathbb{Z}_p} \gamma z, [M])$ is divisible by $p$. Now one looks at the symplectic class $[\omega]$ of $M$ and takes $z$ to be $[\omega]^{\dim M/2p}$. Since the action of the mapping class group in $M$ is symplectic, the product $\prod_{\gamma \in \mathbb{Z}_p} \gamma z$ is actually $z^p = [\omega]^{\dim M/2}$, and the number $(\prod_{\gamma \in \mathbb{Z}_p} \gamma z, [M])$ is just $( [\omega]^{\dim M/2}, [M])$, the volume of the moduli space. The value of this number is now well known due to Witten’s computation [W].

6.3. Lemma. Let $C$ be a surface of genus $g$. For any prime $p|(g-1)s.t.p(p-1) < 2(g-1)$ there exists a non-free action of $\mathbb{Z}_p$ on $C$.

Proof: Take $\bar{k}$ such that $0 \leq |\chi_0| = \frac{|\chi|}{p} - (p-1) \bar{k} \leq p - 1$, where $\chi = 2 - 2g$. Take a Riemann surface $\Sigma$ of Euler characteristic $\chi_0$ which supports a meromorphic function $f$ with exactly $\frac{1}{2} \bar{k} \cdot p$ simple zeros and poles. Now let $C \subset \Sigma \times \mathbb{P}^1$ be defined by the equation $y^p = f(x)$. It is a smooth surface with a holomorphic map $C \to \Sigma$, having precisely $\bar{k} \cdot p$ ramification points of ramification index $(p-1)$. So by the Hurwitz formula, $\chi(C) = \chi$. The group $\mathbb{Z}_p$ acts on $C$ by the formula $(x, y) \mapsto (x, e^{\frac{2\pi i}{p}} y)$. 

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Now consider the induced action of $\mathbb{Z}_p$ on the character variety $\mathcal{M} = \text{Hom}(\pi_1(C_g), G)/G$, where $G$ is a simple compact group. The crucial fact we need is contained in the following lemma.

**Lemma (6.4).** Suppose $\mathbb{Z}_p$ acts non-freely on $C_g$. Then

$$\dim(\text{Fix}(\mathcal{M})) < \frac{\dim \mathcal{M}}{p}$$

**Proof:** The ideology behind the proof is that $\text{Fix}(\mathcal{M})$ is the moduli space of parabolic vector bundles over the manifold $C_g/\mathbb{Z}_p$. The details are given in Appendix 7.

### 6.5

Now we consider the symplectic class $[\omega] \in H^2(\mathcal{M})$ which is a first Chern class of the “Theta-bundle” ([BNR]) and therefore an integer class. Since the action of the mapping class group as $\mathcal{M}$ is symplectic, the class $[\omega]$ is invariant under the $\mathbb{Z}_p$-action. Put $z = [\omega]^{\frac{\dim \mathcal{M}'}{2p}}$ and apply Theorem 8.3.

We get

$$p|([\omega]^{\frac{\dim \mathcal{M}'}{2p}}, [\mathcal{M}] = W_G(2g - 2),$$

which completes the proof of Theorem 6.1.

### 6.6

We will discuss here some extensions of the Theorem 6.1. For $\ell \geq 2$ and $r$ such that $p|r$ we can consider the nonfree action of $\mathbb{Z}_p$ on an orbifold of genus $g$ with $r$ ramification points of index $\ell$. The representation variety of such an orbifold is given by equation

$$\begin{cases}
x_1^\ell = 1 \\
\vdots \\
x_r^\ell = 1 \\
x_1 \cdots x_r = [y_1, y_2] \cdots [y_{2g-1}, y_{2g}]$

and has a dimension $2r + (2g - 2)\dim G$. Again, this is a compact Kähler manifold with an integer symplectic class. Its volume was computed by Witten [W]. The same argument as above gives $p|([\omega]^{\frac{\dim \mathcal{M}'}{2p}}, [\mathcal{M}]$ in this case. In case $G = SU(2)$ this gives a version of Von Staudt Theorem for Hurwitz zeta-function.

On the other hand, if $\mathbb{Z}_p$ acts in a free group $F_{2g}$ and fixes a balanced word $r \in [F_{2g}, F_{2g}]$ which we arrive to an action of $\mathbb{Z}_p$ is the character variety $\mathcal{M} = \text{Hom}(F_{2g}, \langle \{r\}, G \rangle)/G$. It seems appealing to compute the volume of $\mathcal{M}$ and to apply the technique above (cf. problem 4).
7. Canonical complex I: A Symplectic Hodge Theory and Brylinski’s Conjecture.

7.1 In this chapter we will review some of the constructions of chapter 1 from the different standpoint of canonical complex of J.-L. Brylinski. We will also construct some interesting characteristic classes of a symplectic action $G \to Sympl(M)$ lying in $H^*(M, \mathbb{R})$. Recall that in Chapter 1, we constructed for such an action the classes $\pi^*\lambda_k \in H^{2k-1}_{top}(G, \mathbb{R})$.

7.2 Let $(M^{2m}, \omega)$ be a symplectic manifold. The de Rham complex $\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \cdots$ may be given another differential $\delta : \Omega^{k+1} \to \Omega^k$, satisfying $d\delta + \delta d = 0$. To define it, we first introduce the symplectic Hodge $\ast : \Omega^k \to \Omega^{m-k}$ in a fashion, close to the Riemannian case, namely, for a symplectic space $V^{2m}$ we have a canonical bilinear form on $\wedge^k V$, symmetric if $k$ is even and skew-symmetric if $k$ is odd. So one has an $\wedge^k V \approx (\wedge^k V)^\ast$. On the other hand, there exists a coupling $\wedge^k V \otimes \wedge^{2m-k} V \to \wedge^{2m} V \approx \mathbb{R}$, thus an isomorphism $\wedge^{2m-k} V \approx (\wedge^k V)^\ast$. Putting this together, we arrive to an isomorphism $\ast : \wedge^k V \to \wedge^{2m-k} V$. Now, one defines $\delta$ by $\delta = \ast d\ast$.

The two differentials $d$ and $\delta$ make $\Omega(M)$ into a “complex mixe” in terminology of C. Kassel [Kas]). Following Brylinski and Kassel, one defines the canonical cohomology as $H^*(\Omega, \delta)$. It is easy to show an isomorphism $H^*(\Omega, \delta) = H^{2n-*}(\Omega, d) = H^{2n-*}_{dR}(M, \mathbb{R})$. Next, one defined periodic cyclic homology $HC_{per}(\Omega)$ as a homology of periodic Tsygan-Loday-Quillen bicomplex, that is, the homology of a periodic complex

$$\bigoplus_{k=0}^{\infty} \Omega^{2k} \xrightarrow{d+\delta} \bigoplus_{k=0}^{\infty} \Omega^{2k+1} \xrightarrow{d+\delta} \bigoplus_{k=0}^{\infty} \Omega^{2k} \to \cdots$$

Brylinski shows [Br], [Br-Ge] that the spectral sequence of this bicomplex degenerates, and therefore there exists a filtration, which we will call the dimension filtration

$$0 = F_0 \subset F_1 \subset \cdots \subset F_{\infty} = HC_{even},$$
$$0 = F_0 \subset \cdots \subset F_{\infty} = HC_{odd}$$

with $F_{k+1}/F_k \approx H^{2k+i}(M, \mathbb{R}), i = 0, 1$ respectively.
The fundamental goal of this chapter is to introduce a new canonical structure, namely a decomposition

\[ HC^{\text{even}} = \oplus V_i, \]

and

\[ HC^{\text{odd}} = \oplus W_i \]

which we call symplectic Hodge decomposition, SH-decomposition for short. It owes its existence to first-order symplectic identities, parallel to Kähler identities described below in 10.3.1.

The most essential value of this new decomposition is that it catches partially the features of the Hodge Theory in the Kähler case. A natural question arises; what is the relative between the dimension filtration and the SH-decomposition? An answer is, roughly, that these structures fit nicely exactly when one has a full-strength Hodge Theory for \( H^*(M) \). A very typical example of a relation between the dimension filtration and the SH-decomposition may be translated to the following conjecture of J. L. Brylinski ([Br]):

**Brylinski Conjecture** Assume \( M \) is compact. The any cohomology class in \( H^*(M, \mathbb{R}) \) may be represented by a “harmonic” form, i.e. a form \( \omega \in \Omega^k(M) \) such that \( d\omega = \delta\omega = 0 \).

Brylinski proved this conjecture in two cases:

(i) \( k = 1 \)

(ii) \( M \) is Kähler.

We will show this conjecture is in general wrong, and this is exactly the responsibility of a failure for a nice connection between the dimension filtration and the SH-decomposition. One starts to think that the SH-decomposition is the most one could expect to exist for a general symplectic manifold.

### 7.3.1

Here we will introduce the first-order symplectic identities. Define \( L \)-operator \( L : \Omega^k \to \Omega^{k+2} \) in a usual way \( L(\mu) = \mu \wedge \omega \). Define a \( \Lambda \)-operator \( \Lambda : \Omega^k \to \Omega^{k-2} \) by \( \Lambda = *L* \), equivalently, \( \Lambda(\mu) = Z|\mu \), where \( Z \) is a bivector field, dual to \( \omega \). Now, we claim

**First order symplectic identities (10.3).**

\[
[L, \delta] = d,
\]

\[
[\Lambda, d] = \delta.
\]
Proof: A conceptual proof will be given in appendix \( \mathbb{II} \) within the framework of noncommutative calculus of Getzler-Daletski-Tsygan. An impatient reader may prefer to make a direct calculation in local canonical coordinates.

Warning: If \( M \) is Kähler, then our operators \( \delta \) and \( \Lambda \) differs from those in classical Hodge theory (whereas \( L, d \) are the same).

7.3.2 We define a weight operator \( \tilde{T} : \Omega^\text{even} \to \Omega^\text{even} \), resp. \( \tilde{T} : \Omega^\text{odd} \to \Omega^\text{odd} \) by \( \tilde{T} = L + \Lambda \).

Lemma 10.3.2. \( [\tilde{T}, d + \delta] = d + \delta \)

Proof: \( [\tilde{T}, d + \delta] = [L + \Lambda, d + \delta] = [L, d] + [L, \delta] + [\Lambda, d] + [\Lambda, \delta] = d + \delta \).

7.3.3 Corollary. The operator \( \tilde{T} \) descends to an operator

\[
T : HC^\text{even} \to HC^\text{even}, \text{resp.} T : HC^\text{odd} \to HC^\text{odd}
\]

Proof: Let \( \varphi \) be a cycle, i.e. \( (d + \delta)z = 0 \). Then \( (d + \delta)\tilde{T}z = \tilde{T}(d + \delta)z - (d + \delta)z = 0 \), so \( \tilde{T}z \) is a cycle. Similarly, if \( z = (d + \delta)y \), then \( \tilde{T}z = \tilde{T}(d + \delta)y = (d + \delta)\tilde{T}y + (d + \delta)y = (d + \delta)(\tilde{T}y + y) \), a coboundary. Q.E.D.

7.3.4. Proposition. The operator \( T \) is semisimple with integer eigenvalues.

Proof: Brylinski proves in [Br] an identity, which in our language reads \( \exp(2\pi i \tilde{T}) = id \). Therefore \( \exp(2\pi iT) = id \), so \( T \) is a generator of an \( SO(2) \)-action, hence a result.

7.3.5 Definition: The eigenspace decomposition of the operator \( T : HC^\text{even} = \oplus V_i \), resp. \( HC^\text{odd} = \oplus W_i \) will be called the symplectic Hodge decomposition, SH-decomposition for short.

7.4. The goal of this section is to explore examples of the relation between the dimension filtration and the SH-decomposition. We start with the Kähler case and then proceed to general four-dimensional symplectic manifolds. We will see how the Brylinski conjecture will enter naturally in the picture.

7.4.1.: In this subjection we assume that the symplectic manifold \( M \) under study is compact and Kähler. In this case the Hodge Theory immediately gives a canonical decomposition. \( HC^\text{even} = \bigoplus_{k=0}^{n} H_{dR}^{2k}(M, \mathbb{R}) \), \( HC^\text{odd} = \bigoplus H_{dR}^{2k+1}(M, \mathbb{R}) \). Moreover, the operators \( L \)
and Λ are both well-defined on $HC^{\text{even}}$ and $HC^{\text{odd}}$ and with the dimension operator $S$ form the basis of an $S1(2, \mathbb{R})$-action. The three operators $S, L, \Lambda$ correspond respectively to elements $egin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $egin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $egin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of the Lie algebra $sl(2, \mathbb{R})$. The operator $L + \Lambda$ corresponds to $egin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, which is conjugate to $egin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. So its eigenvalues range in $-n, \cdots, n$. So in Kähler case there are two gradings given by eigenspaces of the dimension operator $S$ and the weight operator $T$. The two operators $S$ and $T$ generate a $sl(2, \mathbb{R})$-action in $HC^\text{per}$.

7.4.2.: In this subsection we assume $\dim M = 4$ and consider the first nontrivial case of dimension filtration and the weight decomposition, namely, in $HC^{\text{odd}}(\Omega)$. Everywhere below lowcase Greek letters stand for one-forms and uppercase letters stand for 3-forms. A cycle in $\Omega^{\text{odd}}$ is a pair $(\rho, P)$ such that $\delta \rho = 0, dP = 0, d\rho + \delta P = 0$. A map $[(\rho, p)] \to [P]$ is a well-defined homomorphism $HC^{\text{odd}} \to H^3(M, \mathbb{R})$. Indeed, a coboundary in $\Omega^{\text{odd}}$ is given by $(d\lambda_0 + \delta \lambda_2, d\lambda_2 + \delta \lambda_4)$, where $\lambda_i \in \Omega^i$. But $\delta L_4 = [\Lambda, d]L_4 = -d\Lambda \lambda_4$ is exact, as well as $d\lambda_2$, so $[P]$ is well-defined.

The map $HC^{\text{odd}} \to H^3(M, \mathbb{R})$ is on. Indeed, let $P$ be closed and consider $(P, P)$. We have $dP = 0, \delta (P) = \ast dP = 0$, and $(d \ast P + \delta P) = (d \ast P) + d \ast P = 0$. The latter identity needs explanation. Since $dP = 0$, we have $\delta \ast P = 0$, or $0 = [\Lambda, d](\ast P) = \Lambda d(\ast P)$, the second term being zero, since $\ast P$ is one-dimensional. Now, in $\Omega^2$ the operator $\ast$ acts as identity on $\mathbb{R} \cdot \omega$ and as $-id$ on the kernel of $\Lambda$, so indeed $(d \ast P) + d \ast P = 0$. That means $(\ast P, P)$ is a cycle in $\Omega^{\text{odd}}$.

The kernel of the surjective map $HC^{\text{odd}} \to H^3(M, \mathbb{R}) \to 0$ should be isomorphic to $H^1(M, \mathbb{R})$. In fact, the embedding $0 \to H^1(M, \mathbb{R}) \to HC^{\text{odd}}$ is given by a formula $[\rho] \to [(\rho, 0)]$. Indeed, let $d\rho = 0$, then $\delta \rho = [\Lambda, d] \rho = 0$ (this has been noticed by Brylinski, in connection to his conjecture, comp. and [ ], p. ), so $(\rho, 0)$ is a cycle. If $\rho = df$, then $(\rho, 0) = (d + \delta)(f, 0)$, so the map $H^1(M, \mathbb{R}) \to HC^{\text{odd}}$ is well-defined, and immediately seen to be injective. By dimension considerations, the sequence

$$0 \to H^1(M, \mathbb{R}) \to HC^{\text{odd}} \to H^3(M, \mathbb{R}) \to 0$$

should be exact.
Now comes the first surprise: there is no natural splitting of this exact sequence in general. Let us see why the correspondence \( P \to [(\ast, P, P)] \) considered above does not define a map \( H^3(M, \mathbb{R}) \to HC^{\text{odd}} \). What we need in fact to have such a map is that the image of a coboundary, say \( d\lambda_2 \), would be a coboundary. Now, the image of \( d\lambda_2 \) is \((\delta \ast \lambda_2, d\lambda_2) = (d + \delta)(\ast \lambda_2, 0) + (0, d(\lambda_2 - \ast \lambda_2))\), and there is no reason why the second term should be \((d + \delta)\)-coboundary.

We will see now that the existence of the splitting would follow from the hard Lefschetz, that is, that \( L : H^1(M, \mathbb{R}) \to H^3(M, \mathbb{R}) \) is an isomorphism. The hard Lefshetz theorem is valid for Kähler manifolds, but not for general compact symplectic manifolds.

Consider the composition \( 0 \to H^1(M, \mathbb{R}) \xrightarrow{i} HC^{\text{odd}} \xrightarrow{\ast} HC^{\text{odd}} \), defined on the level of forms by \([\rho] \to [(0, \ast \rho)]\). Observe that \( \ast \) is well-defined in \( HC^{\text{per}} \), since if commutes with \( d + \delta \) on the level of forms. Let \( \psi : H^1(M, \mathbb{R}) \to HC^{\text{odd}} \) is the composed map. We have a diagram

\[
\begin{array}{ccc}
H^1(M, \mathbb{R}) & \xrightarrow{\psi} & HC^{\text{odd}} \\
\downarrow L & & \downarrow \\
H^3(M, \mathbb{R}) & \xrightarrow{\ast} & 0
\end{array}
\]

since \( \ast \rho = L\rho \) on 1-forms. So if \( L \) would be an isomorphism, then \( \psi \circ L^{-1} : H^3(M, \mathbb{R}) \to HC^{\text{odd}} \) would be a desired splitting.

We are ready to disprove the Brylinski conjecture (see 17.2) in general.

**Proposition (7.4.2.).** The Brylinski conjecture fails for compact 4-dimensional symplectic manifolds.

**Proof:** Suppose any class in \( H^3(M, \mathbb{R}) \) is given by a form \( P \) such that \( dP = \delta P = 0 \). Put \( \ast P = \rho \) and notice that \( d\rho = \ast \delta P = 0 \). On the other hand, \( \ast = L \) on \( \Omega^1 \), so \([P] = L[\rho]\), which means that hard Lefshetz holds for \( H^1(M) \), which is not generally the case, see Gompf [Gom]. In fact, the same argument shows that the conjecture fails for symplectic manifold \( M^{2m}, m \geq 2 \) an all dimensions larger than \( m \). It remains open for dimensions less then \( m \).

We return to our analysis of \( HC^{\text{odd}}(\Omega) \) of four-dimensional \( M \). Our next goal is to understand the action of the operator \( T \). For elements of the type \([\ast P, P]\), \( dP = 0 \) we
compute $T[(\ast P, P)] = (\ast P, P)$. On the other hand, for elements of the type $[-\rho, \ast \rho], d\rho = 0$, i.e. those which lie in $(\psi - i)(H^1(M, \mathbb{R}))$, we compute $T[(\rho, \ast \rho)] = -(\rho, \ast \rho)$. If $L : H^1(M, \mathbb{R}) \to H^3(M, \mathbb{R})$ is an isomorphism, then $\psi - i : H^1(M, \mathbb{R}) \to HC_{\text{odd}}$ is injective, since the composition $H^1(M, \mathbb{R}) \xrightarrow{\psi - i} HC_{\text{odd}} \to H^3(M, \mathbb{R})$ is just $L$. Since moreover the different eigenspaces are disjoint, we conclude by dimension consideration that $HC_{\text{odd}} = \{T = id\} \oplus \{T = -id\}$. For a general compact symplectic manifold, $M$ let us call an even (resp. odd) spectrum of $M$ the set of eigenvalues of $T$ in $HC_{\text{even}}$ (resp. $HC_{\text{odd}}$). What we have in fact proved is that the odd spectrum of $M^4$ is $\{\pm 1\}$ if the hard Lefshetz holds for $H^1(M, \mathbb{R})$.

7.5. In this section we will use the canonical complex to define characteristic classes of symplectic group actions. Let $G$ be a Lie group, acting symplectically on a simply-connected symplectic manifold $M$. We assume that obstruction class in $H^2(g, \mathbb{R})$ vanishes, (for example, $G$ is semisimple), so the action is Hamiltonian and the moment map $M \xrightarrow{\mu} g^*$ is defined. If the image of $\mu$ had been consisted of just one orbit $P$ we would have had a characteristic map in cohomology $H^*(P) \to H^*(M)$. Yet in general this is not the case, one still has a characteristic map defined below.

Consider the homomorphism of Frechet algebras $C^\infty(g^*) \xrightarrow{\alpha} C^\infty(M)$. Notice that $\alpha$ is a Poisson map, that is, a homomorphism of Poisson algebras. So, according to Appendix 2, we have a map of mixed complexes $(\text{HH}(C^\infty(g^*)), d, \delta) \to (\text{HH}(C^\infty(M)), d, \delta)$. Observe that $\text{HH}_{\ast}(C^\infty(N)) = \Omega^\ast(N)$ for any smooth manifold $N$. The map above induces a homomorphism in cyclic homology of the mixed complexes:

$$HC(\Omega(g^*), d, \delta) \xrightarrow{\sim} HC(\Omega(M), d, \delta).$$

The right hand side group has been computed in [B-G]: $HC_i(\Omega(M), d, \delta) = \oplus H^{i-2k}(M, \mathbb{R})$. The left hand side group have been computed for semisimple groups by Feigin-Tsygan [F-T], see also Kassel [Kas]. For any canonical generator $t_\ell$ of $HC(\Omega(g^*), d, \delta)$, given in [F-T], one gets a characteristic class

$$\chi(t_\ell) \in \oplus H^{\ell-2k}(M, \mathbb{R})$$
7.6 Let us see what the construction above gives for $G = SU(2)$ (or $G = SO(3)$). Identify $\mathfrak{g}$ with $\mathbb{R}^3$ with a usual Lie bracket. Consider an element $\omega_{00} + \omega_{11}$ of the cyclic bicomplex

\[
\begin{array}{c}
\Omega^2 & \leftarrow & \delta & \Omega^3 \\
\uparrow & & & \uparrow d \\
\Omega^1 & \leftarrow & \delta & \Omega^2 \rightarrow \Omega^3 \\
d & \uparrow & & \\
\Omega^0 & \leftarrow & \delta & \Omega^1 \leftarrow \Omega^2 \\
\end{array}
\]

where $\omega_{00} = x^2 + y^2 + z^2$ and $\omega_{11} = xdydz - ydx dz + zdx d z$. One checks immediately using the explicit formulas for $\delta$ in [Br], that $\omega_{00} + \omega_{11}$ is a cycle. So for any symplectic $SU(2)$ - manifold $M$ one gets a characteristic class

$$
\chi(\omega_{00} + \omega_{11}) \in \bigoplus H^{2k}(M, \mathbb{R}).
$$

This class is nontrivial already for a coadjoint orbit $S^2 \subset \mathbb{R}^3$.

**Appendix \$\$: Thom-Dold theorem and transfer for nonfree -actions.

Our purpose here is to suggest a generalization of the classical transfer operation, as follows.

**Theorem.** Let $\Gamma$ be a finite group, acting (maybe nonfree) as a CW-complex $X$. Let $Y = X/\Gamma$ and let $p : X \to Y$ be a natural map. There exists a homomorphism

$$
t : H_*(Y, \mathbb{Z}) \to H_*(X, \mathbb{Z})
$$

such that

(i) $p_* \circ t = |\Gamma|.id$

(ii) $t \circ p_*(z) = \sum_{\gamma \in \Gamma} \gamma_* z$

**Proof:** The idea of the proof is to deduce the construction from the classical Dold-Thom Theorem [D-T]. This theorem states that $H_*(X, \mathbb{Z}) = \pi_*(SX)$ (an infinite symmetric power of $X$). We refer to Karoubi [Ka4] for the detailed discussion. Now, the map $A \to p^{-1}(A)$, where $A$ is a finite subset of $Y$, extends to a well-defined map $SY \to SX$,
and the corresponding homomorphism $\pi_*(SY) \to \pi_*(SX)$ is $t$. It is immediate to check the properties (i) and (ii) of the Theorem.

What makes this generalization of transfer really interesting is in the fact that the multiplicative transfer, described in Chap. 5, has no chance to exist for the maps $X \to X/\Gamma = Y$, if the action of $\Gamma$ in $X$ is not free. Indeed, consider a Galois conjugation $\tau$ in $\mathbb{C}P^2$. The quotient $\mathbb{C}P^2/\tau$ is known to be $S^4$ ([G-M]). If there would be a multiplicative transfer $H^2(\mathbb{C}P^2, \mathbb{F}_2) \to H^4(S^4, \mathbb{F}_2)$ such that $p^* \circ t = z \cdot \tau z$, then we would get $z^2 = z \cdot \tau z = p^*(tz) = 0$, since $\deg p : \mathbb{C}P^2 \to S^4$ is 2.
Appendix 2: Canonical Complex, II: Non-commutative Poisson manifolds

1. Basic notations Through this appendix we fix a ground field $k$ of characteristic zero. All algebras, homomorphisms etc. will be defined over $k$.

A noncommutative manifold $X$ is just a $k$-algebra $A$. Manifolds form a category, opposite to the category of $k$-algebras (arrows reversed). The de Rham complex $\Omega^*(X)$ is defined by $\Omega^i(X) = HH_i(A)$ with the Connes-Tsygan differential $d : \Omega^i(X) \rightarrow \Omega^{i+1}(X)$. This gives a functor manifolds $\mapsto$ complexes over $k$. The de Rham cohomology $H^*_{dR}(X)$ is defined as the cohomology of the de Rham complex. Again, one has a functor manifolds $\mapsto$ graded spaces over $k$.

Recall the standard definitions of the Hochschild homology and cohomology:

$HH_i(A) = \text{Tor}_i A \times A^0(A,A)$, $HH^i(A) = \text{Ext}_A^i(A,A)$. Intuitively, an element in $HH_i(A)$ is a differential form on $X$, whereas an element of $HH^i(A)$ is a polyvector field on $X$. One has the following formal calculus on $X$:

**Substitution** There is a homomorphism $HH^i(A) \otimes HH_j(A) \rightarrow HH_{j-i}(A)$. If $z \in HH^i(A)$ and $\omega \in HH_j(A) = \Omega^j(X)$ we denote this operation $i_z \omega$.

**Multiplication of polyvectors** The graded space $\oplus HH^i(A)$ is given a structure of graded commutative algebra over $k$. The map $z \mapsto i_z$ is an algebra homomorphism $\oplus HH^i(A) \rightarrow \text{End}_k(\oplus \Omega^j(X))$.

**Gerstenhaber Lie bracket** The shifted graded space $\oplus_i HH^{i+1}(A)$ is given a structure of a graded Lie algebra over $k$. Moreover, the multiplication and the Lie bracket fit into a structure of a graded Poisson algebra.

**Lie derivative** Define $\mathcal{L}_z \omega = di_z \omega + i_z d\omega$. The map $z \mapsto \mathcal{L}_z$ is a Lie algebra homomorphism $\oplus_i HH^{i+1}(A) \rightarrow \text{End}_k(\oplus \Omega^j(X))$. We refer to Gerstenhaber [Ger], Getzler [Get] and Daletsky and Tsygan [D-T] for the detailed description.

2. Poisson structure Recall a deformational description of the second Hochschild cohomology $HH^2(A)$. Suppose one has a deformation $A \otimes A \rightarrow A[[\varepsilon]]$ of an original product in $A$, which extends to an associative product in $A[[\varepsilon]]$. Then the term containing $\varepsilon$ is a
Hochschild 2-cocycle on $A$, say $\tilde{\mu} : A \otimes A \rightarrow A$. Now, informally speaking, the deformation of the product induces the deformation of the differential in the complex, computing Hochschild homology:

$$A \xleftarrow{\beta} A \otimes A \xleftarrow{\beta} A \otimes A \otimes A \xleftarrow{\cdots}$$

say $b_\varepsilon$, with the property $b_\varepsilon^2 = 0$. Put $L\tilde{\mu} = \frac{d}{d\varepsilon}(b_\varepsilon)$, then $L\mu b = -bL\mu$, so $L\mu$ descend to the Hochschild homology. This is, of course, just the Lie derivative of section 1.

**Definition:** A noncommutative Poisson manifold $(X, \{ \} )$ is a $k$-algebra $A$ with an element $\mu \in HH^2(A)$ satisfying the integrability condition $[\mu, \mu] = 0$.

**Theorem.** The de Rham complex $\Omega(X)$ of a Poisson manifold $X$ is a mixed complex with respect to the Connes - Tsygan differential $d$ and the Lie derivative $L\mu$.

**Proof:** It is an immediate corollary of the property $[d, L\mu] = 0$ (the graded bracket!), which follows from $d^2 = 0$ and the definition of $L\mu$, and the fact that $\mu \mapsto L\mu$ is a graded Lie algebra homomorphism.

In case $X$ is an “honest” Poisson manifold (so $A = C^\infty(X)$) and $\mu$ is given by the Poisson bracket, the operator $L\mu : \Omega^{i+1}(X) \rightarrow \Omega^i(X)$ is just the operation $\delta$ of the Chapter 7. Moreover, the operation $i_\mu$ is just $\Lambda$ in this case. The formula $i_\mu d + di_\mu = L\mu$ becomes $[\Lambda, d] = \delta$.

Following the definitions of the Chapter 7, one defines the canonical homology and periodic cyclic homology of the double complex $\Omega^{i-j}(X)$, called $HC(\Omega(X), d, L\mu)$, or $HC(\Omega(X))$ for short.

**3. Lie algebra actions and characteristic classes** Fix a Lie algebra of over $k$. A Hamiltonian action of $g$ on $X$ is a homomorphism of algebras

$$k[g^*] \xrightarrow{\pi} A$$

such that $\pi_*(\{ \}) = \mu \circ \pi$, as elements of $HH^2(k[g^*], A)$. Here $\{ \}$ is the usual Poisson bracket on $k[g^*]$.

Consider an induced map $\Omega^i(g^*) \rightarrow \Omega^i(X)$. One checks immediately that it is a map of mixed complexes. So one gets a characteristic map

$$HC(\Omega(g^*)) \xrightarrow{\chi} HC(\Omega(X)),$$
a noncommutative version of the map defined in Chapter 7. If \( g \) is a finite-dimensional semisimple Lie algebra, then the canonical generators in \( HC(\Omega(g^*)) \) give characteristic classes in \( HC(\Omega(X)) \).
Appendix I: Fourier-Donaldson transform in group cohomology
and geometric structures of representation varieties
with application to three-manifolds

1. Let $\Gamma$ be a finitely-generated group and let $G$ be a real Lie group. The representation variety $V^R_\Gamma$ is just $\text{Hom}(\Gamma, R)$. If $G$ is algebraic, then $V^R_\Gamma$ is a (real) algebraic variety, so we get a functor

$$\text{f.g. groups } \times \text{ algebraic groups } \rightarrow \text{ algebraic varieties}$$

Observe that $V^G_\Gamma$ is compact if $G$ is. Our first goal is to define a homomorphism

$$(1) \quad FD : \mathcal{I}_G(g) \otimes H_*(\Gamma, \mathbb{R}) \xrightarrow{FD} \Omega^*_{cl}((V^G_\Gamma)_{reg}),$$

which is a natural transformation of functors. Here $\mathcal{I}_G(g)$ stands for the algebra of invariant polynomials on $g$, and $\Omega^*_{cl}$ stands for closed forms on nonsingular part of $V^G_\Gamma$.

Two important remarks are due to be made. First, we will present several approaches to define the homomorphism (1). One approach is an extension of Atiyah-Bott-Goldman reduction procedure [AB], [Gol]. The other one is closer in spirit to the philosophy of this paper and is based on Dennis trace map in algebraic $K$-theory. Second, one replace the representation variety $V^G_\Gamma$ by the character variety $X^G_\Gamma = V^G_\Gamma/G$, with certain caution. We refer to [J-M] for rather a delicate yoga how to deal with $X^G_\Gamma$.

We will go on and define a secondary homomorphism

$$\tilde{K}^{\text{alg}}_i(\Gamma) \rightarrow H^{i-1-2s}(V^G_\Gamma, \mathbb{R}/\mathbb{Z}),$$

$s \geq 1$, and its modification for $i = 0$. Here $K^{\text{alg}}_i(\Gamma) = \pi_i((BG)^+)$ and $\tilde{K}^{\text{alg}}_i(\Gamma) = \ker FD : K^{\text{alg}}_i(\Gamma) \rightarrow K^{\text{top}}_i(V^G_\Gamma)$ provided $\Gamma$ is “good” (say perfect). One may take $K^{\text{alg}}_i(\Gamma)$ to be the homology bordism group of homomorphisms $\pi_1(\Sigma^i) \rightarrow \Gamma$, where $\Sigma^i$ is a homology $i$-sphere, see [Ka], [Vo], [Hau]. We will sketch very briefly applications to three-manifold invariants, which are sort of higher Casson invariants on one hand and higher Chern-Simons invariants on the other.
2. We start with a definition of the homomorphism (1). Let \( f \in I_G(\mathfrak{g}) \), let \( \alpha \in H_i(\Gamma, \mathbb{R}) \), let \( \pi \in V^G_\Gamma \) and let \( Y_1, \ldots, Y_i \in T_{\text{zar}}(V^G_1) \). One identifies \( T_{\text{zar}}(V^G_1) \) with \( Z^1(\Gamma, \mathfrak{g}) \) where \( \Gamma \), acts in \( \mathfrak{g} \) as \( \text{Ad} \circ \pi \). Then we put

\[
FD(f, z)(Y_1, \ldots, Y_i) = ([f(Y_1, \ldots, Y_i)], z).
\]

Here \( f(Y_1, \ldots, Y_i) \in Z^i(\Gamma, \mathbb{R}) \), and \([f(Y_1, \ldots, Y_i)]\) is its cohomology class.

Since the actual tangent space to \( \pi \in (V^G_\Gamma)_{\text{reg}} \) is a subspace of the Zariski tangent space, we get a well-defined \( i \)-form on \((V^G_\Gamma)_{\text{reg}}\). To prove that this form is closed, we apply a trick of Atiyah-Bott-Goldman, namely, we choose a manifold \( M \) with \( \pi_1(M) = \Gamma \), and consider a space of all connections in a trivial \( G \)-bundle \( P \) over \( M \). This is an affine space, say \( \mathcal{A} \), and a tangent space to a point of \( \mu \in \mathcal{A} \) is just \( \Omega^1_{\text{inv}}(P, \mathfrak{g}) \) with the usual invariance properties and vanishing condition. Then the composition

\[
\Omega^1(P, \mathfrak{g}) \wedge \cdots \wedge \Omega^1(P, \mathfrak{g}) \rightarrow \Omega^i(P, \otimes^i \mathfrak{g}) \overset{f}{\rightarrow} \Omega^i(P, \mathbb{R})
\]

gives a constant \( \Omega^i(P, \mathbb{R}) \) - valued \( i \)-form on \( \mathcal{A} \).

Now, we restrict this form on the subvariety \( \mathcal{A}_f \) flat of flat connections, satisfying \( d\mu + \frac{1}{2}[\mu, \mu] = 0 \). An immediate check shows:

(i) The image of any polyvector in LHS of (2), tangent to the subvariety of flat bundles, is a closed \( G \)-invariant form, vanishing an vertical vectors, that is, coming from a closed form in \( \Omega^i_{\text{celf}}(M, \mathbb{R}) \).

(ii) The composition

\[
\Omega^1(P, \mathfrak{g}) \wedge \cdots \wedge \Omega^1(P, \mathfrak{g}) \rightarrow \Omega^i_{\text{celf}}(M, \mathbb{R}) \rightarrow H^i(M, \mathbb{R}) \overset{\cap \mathbb{R}}{\rightarrow} \mathbb{R}
\]

is a \( i \)-form on \( \mathcal{A}_{\text{flat}} \) which is invariant under the gauge group \( \mathcal{G} = \text{Map}(M, G) \) and vanishes a vectors, tangent to orbits.

Therefore one gets a reduced closed form on \( \mathcal{A}_{\text{flat}}/\mathcal{G} = \text{Hom}(\Gamma, G)/G \). Its pull-back to \( \text{Hom}(\Gamma, G) = V^G_\Gamma \) obviously coincides with \( FD(f, z) \), which is therefore closed.

3. Example If \( G \) is semisimple, \( f \) is the Cartan-Killing quadratic form and \( \Gamma \) is a surface group, let \( z \) be a generator \( H_2(\Gamma, \mathbb{Z}) = \mathbb{Z} \), we obtain the (Goldman’s) 2-form in the representation variety \( V^G_1 \). It is known to become nondegenerate after descend to the character variety \( X^G_1 \). The mapping class group obviously acts symplectically in \( X^G_1 \).
(ii) for $G$ and $f$ as above let $\Gamma$ be one-relator group $F_{2g}/\{r\}$, where $\{r\} \in \{F_{2g}, F_g\}$ is a balanced word. Then $H_2(\Gamma, \mathbb{Z}) = \mathbb{Z} [\gamma]$, and so we get a closed 2-form in $V^G_\Gamma$, which descends to $X^G_\Gamma$. The outer automorphism group $\text{Out}(\Gamma)$ acts on $X^G_\Gamma$, preserving this form.

(iii) for $G = SU(2)$ and $f(A) = Tr A^3$ let $\Gamma$ be a fundamental group of a three-manifold $M$. Take $z$ to be the fundamental class of $M$, then $FD(f, z)$ is a closed three-form an $V^G_\Gamma$, which descends to $X^G_\Gamma$.

(iv) Let $M$ be a closed Riemannian four-manifold. Let $P$ is a principal $G$-bundle over $M$. Consider a moduli space of $M$ of self-dual connections in $P$. We obviously have $\mathcal{M} \supseteq \mathcal{A}_{\text{flat}}/G = X^G_\Gamma$. There is a $H^2(M, \mathbb{R})$-valued closed two-form in $\mathcal{M}$, introduced by Donaldson [D]. This form is defined exactly as in section 2 with slight modifications made. Its restriction to $X^G_\Gamma$ essentially coincide with our form.

4. Hochschild homology description In this section $G \subset GL(n, k)$ is a linear algebraic group over $k = \mathbb{R}, \mathbb{C}$, and $f$ comes from standard invariant polynomials (defining the Chern classes). We wish to give an alternative description of the FD-map $H_*(\Gamma, k) \rightarrow \Omega^*_{\text{reg}}(V^G_\Gamma)$. We may assume that $G$ is the group $SL_n$. Observe that $V^G_{GL_n}$ is an affine algebraic variety and there exists an evaluation homomorphism

$$\Gamma \rightarrow GL_n(k[V^G_{GL_n}]).$$

So we get an induced map

$$H_*(\Gamma) \rightarrow H_*(GL_n(k[V^G_{GL_n}])).$$

Now, for any $k$-algebra there is a Dennis trace map $H_*(GL_n(A)) \overset{D}{\rightarrow} HH_*(A)$, so we get a composition map

$$H_*(\Gamma) \rightarrow HH_*(k[V^G_{GL_n}]) \rightarrow HH_*(k[V^G_{GL_n}]_{\text{reg}}).$$

But for any smooth affine variety $V$, $HH_*(k[V]) = \Omega^*(V)$, so we come to a map

$$H_*(\Gamma) \rightarrow \Omega^*((V^G_{GL_n})_{\text{reg}}).$$
Now, the image of $H_*(GL_n(A))$ in $HH_*(A)$ lies in the subgroup of cycles with respect to the Connes - Tsygan differential. This follows from the description of the Hochschild homology of a group algebra, see [F - T]. Since under the map $HH_*(k[V]) \approx \Omega^*(V)$ the Connes - Tsygan differential becomes de Rham differential, we get that the image of $H_*(\Gamma)$ lies in the space of closed forms.

5. **Secondary homomorphism** We assume now that $\Gamma$ is perfect that is, $H_1(\Gamma, \mathbb{Z}) = 0$. Consider a map

$$K^\text{alg}_i(\Gamma) = \pi_i((B\Gamma)^+) \to K^\text{alg}(V_{GL_n}^\Gamma)$$

defined through the group homomorphism $\Gamma \to GL_n(k[V_{GL_n}^\Gamma])$. There exists a natural map $K^\text{alg}_i(V_{GL_n}^\Gamma) \to K^\text{top}_i(V_{GL_n}^\Gamma)$, which makes the diagram

$$
\begin{array}{ccc}
K^\text{alg}_i(V_{GL_n}^\Gamma) & \xrightarrow{FD} & \Omega^i_{\text{cl}}((V_{GL_n}^\Gamma)_{\text{reg}}) \\
\ downarrow_{\lambda} & & \downarrow \\
K^\text{top}_i(V_{GL_n}^\Gamma) & \xrightarrow{(ch)_i} & H^i((V_{GL_n}^\Gamma)_{\text{reg}})
\end{array}
$$

commutative. Let $\tilde{K}^\text{alg}_i(\Gamma)$ be the kernel of the composite map $K^\text{alg}_i(\Gamma) \to K^\text{top}_i(V_{GL_n}^\Gamma)$. Then one gets homomorphism

$$\tilde{K}^\text{alg}_i(\Gamma) \to H^{i-1-2s}(V_{GL_n}^\Gamma, k/\mathbb{Z})$$

for $s \geq 1$, and

$$\tilde{K}^\text{alg}_i(\Gamma) \to \text{Hom}\left(\frac{\text{all } (i-1)-\text{currents}}{\text{exact } (i-1)-\text{currents}}, k/\mathbb{Z}\right)$$

for $s = 0$.

This follows immediately from the extension of Bloch-Beilinson regulator maps, as suggested in [Re3]. Alternatively, one may use Karoubi’s MK - theory [Ka1], [Ka2], [Sou].

6. **Three-manifolds invariants.** Let $M^3$ be a homology sphere, and let $\Gamma = \pi_1(M^3)$. Take $G = SU(n)$ and consider the closed three-form $FD([M])$ on $V_{GL_n}^\Gamma$. Assume $3|\dim V_{GL_n}^\Gamma$ and the codim $(V_{GL_n}^\Gamma)_{\text{sing}} \geq 2$. Define

$$\rho(M) = \int_{V_{GL_n}^\Gamma} (FD([M]))^{1/3 \dim V_{GL_n}^\Gamma}.$$

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This is a natural “higher” Casson invariant for $M$. More generally, we can consider a $K^\text{top}(V^G_\Gamma)$-valued invariant $\lambda([M])$ where $[M]$ is considered as an element in the homology bordism group of $(B\Gamma)^+$, which maps to $K_3^\text{alg}(\Gamma)$.

Now, if $\lambda([M])$ vanishes, we get secondary invariants in $H^1(V^G_\Gamma, \mathbb{C}/\mathbb{Z})$ and $\text{Hom}(\frac{\text{two-currents}}{\text{exact two-currents}}, \mathbb{C}/\mathbb{Z})$. Moreover, if $FD([M]) \in \Omega^3_{\text{ct}}(V^G_\Gamma)$ vanishes, the latter secondary invariant lies actually in $H^2(V^G_\Gamma, \mathbb{C}/\mathbb{Z})$. Call it $BB([M])$. Assume $\dim V^G_\Gamma$ is even and define

$$P(M) = ((BB([M]))^{1/2\dim V^G_\Gamma}, [V^G_\Gamma]) \in \bigotimes_{\mathbb{Z}}^{1/2\dim V^G_\Gamma} \mathbb{C}/\mathbb{Z}.$$

A very good example is given by a Seifert homology spheres and $G = SU(2)$. The representation variety $V^G_\Gamma$ consists of smooth rational projective varieties over $\mathbb{C}$ which have, due to the solution of the Fintushel - Stern conjecture, only ever-dimensional homology $[\text{Ba-Ok}] [\text{Ki-Kl}]$. There is a good reason to believe that $FD([M]) = 0$ and $P(M)$ is a well-defined invariant. Observe that $P(M)$ is a “higher” Chern-Simons invariant. Starting from the rationality theorem for the Chern - Simons invariant $[\text{Re1}] [\text{Re2}]$ one may ask if $P(M)$ is valued in $\bigotimes_{\mathbb{Q}}\mathbb{Z}$.

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