Stabilisation and state trajectory tracking problem for nonlinear control systems in the presence of disturbances

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ABSTRACT
In this paper, we consider the problem of stabilisation and tracking of desired state trajectory for a wide range of nonlinear control problems with disturbances. We present the sufficient conditions for the existence of $C^k$ state-feedback controllers and the process of their mathematical designing is described.

1. Introduction and problem formulation
We will consider the stabilisation and state trajectory tracking problem for the nonlinear control systems in the general form:

$$\dot{x} = f(x, u, w(t)), \quad t \geq 0,$$

where $f : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \rightarrow \mathbb{R}^n$, $x = (x_1, \ldots, x_n)^T$ is the state vector, $\dot{x}$ is the time derivative of $x$, $u = (u_1, \ldots, u_m)^T$ is the control input variable manipulated by the controller and $w(t)$ represents the total disturbance (unmodelled system dynamics, deterministic parameter uncertainty, overall external disturbances that affect the system, etc.).

We will assume that $f$ is a $C^k$ function ($k \geq 2$) in the variables $(x, u) \in \mathbb{R}^n \times \mathbb{R}^m$, continuous on $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ and that for every initial state $x(0)$ and input $u$ there exists a unique solution defined on $[0, \infty)$.

Notations and assumptions: In this paper, the following notations will be used:

- $|\cdot|$ – the Euclidean norm of a vector;
- $||\cdot||_F$ – the Frobenius matrix norm (or another norm with the properties of submultiplicativity and compatibility with a vector norm);
- $(\cdot)^T$ – a vector transpose;
- $\exp(\cdot)$ – the exponential function with base $e$ (the Euler’s number);
- $(x_d, u_d)$ – a desired state and input trajectory of Equation (1);
- $(e, v)$ – a tracking error, the difference between desired and actual trajectory, $e = x - x_d$ and $v = u - u_d$;
- $F(e, v, w(t)) = f(e + x_d, v + u_d, w(t)) - f(x_d, u_d, w(t))$ – the vector field of error dynamics;
- $A(t) = JF|_{(0,0)}$ – the Jacobian matrix of a vector-valued function $F$ with respect to the variable $e$ and evaluated at $(e, v) = (0, 0)$ [clearly, $J_F|_{(e,v)=(0,0)} = J_{f|_{(e,v)=(0,0)}}$];
- $(\cdot, t)$ – the Frobenius matrix norm (or another norm with the properties of submultiplicativity and compatibility with a vector norm);
- $(\cdot)^T$ – a vector transpose;
- $\exp(\cdot)$ – the exponential function with base $e$ (the Euler’s number);
- $(x_d, u_d)$ – a desired state and input trajectory of Equation (1);

Some of these notations are used also in the ‘tilde’ version in the Section 3 with an analogous meaning as above.

- $E(t)$ – a secondary error function, $E(t) = E(t) - e(t), t \geq 0$.

A large class of control problems consists of stabilisation and tracking control for state trajectory in the presence of the disturbances. If we are given a desired state
trajectory \( x = x_d(t), \ t \geq 0 \), satisfying Equation (1) for an input \( u = u_d(t) \), the goal is to construct a feedback compensator which locally asymptotically stabilises the system to this trajectory. Using the transformation \( e = x - x_d \) and \( v = u - u_d \), the problem of tracking the desired state trajectory may be reformulated in terms of tracking error dynamics which can be written, in general, as a time-varying control system:

\[
\dot{e} = \dot{x} - \dot{x}_d = f(x, u, w(t)) - f(x_d, u_d, w(t)) = f(e + x_d, v + u_d, w(t)) - f(x_d, u_d, w(t)) := F(e, v, w(t)).
\]

The main goal of this paper is to design a state-feedback control law \( u^* = u_d + v^*(x - x_d, w(t), t) \) [or in terms of tracking error, \( v^* = v^*(e, w(t), t) \)], such that the solution \( x(t) \) of Equation (1) \( |e(t) = F(e, v, w(t))| \) asymptotically tracks the desired state trajectory \( x_d(t) \) [\( e = 0 \)], in the presence of initial state error \( x(0) \neq x_d(0) \) [\( e(0) \neq 0 \)] and the disturbances \( w(t) \), that is,

\[
x(t) \to x_d(t) \ [e(t) \to 0] \ \text{as} \ t \to \infty.
\]

In some cases, for the general control systems (1) and the general time-varying disturbance inputs, it might not be feasible to achieve asymptotic disturbance rejection but only a disturbance attenuation, formulated as a requirement to achieve ultimate boundedness of the tracking error with a prescribed tolerance:

\[
|x(t) - x_d(t)| = |e(t)| \leq \varepsilon, \ \text{for all} \ t \geq T,
\]

where \( \varepsilon \) is a specified (small) positive constant.

Under the assumption that tracking error remains small, we can linearise this system around its equilibrium state \( (e, v) = (0, 0) \),

\[
\frac{de}{dt} \approx A(t)e + B(t)v, \quad A(t) = JF|_{(0, 0)} , \quad B(t) = J_dF|_{(0, 0)}.
\]

If the matrices \( A(t), B(t) \) are constant ones and in the absence of time-varying disturbances, for the controllable linearised error system (2) the classical linear control design techniques provide linear feedback control laws \( v^* = -Ke \) which asymptotically stabilise \( e = 0 \) for the closed-loop system. The problem reduces to calculating a suitable gain matrix \( K \) such that \( A - BK \) is Hurwitz stable. Moreover, any of these feedbacks also locally asymptotically stabilise \( e = 0 \) for the original nonlinear system as follows from the Grobman–Hartman theorem about the local behaviour of dynamical systems in the neighbourhood of a hyperbolic equilibrium point (Pergo, 2001, p. 127). The location of the closed-loop eigenvalues in the open left half-plane may be chosen according to the general principle of obtaining fast convergence to zero of the tracking error with a reasonable control effort. On the other hand, if system (2) has unstable uncontrollable eigenvalues, then smooth stabilisability is not possible, not even locally. As usual, the critical case is encountered when the linearisation has uncontrollable eigenvalues with zero real part.

### 2. Analysis of the case \( m = n \)

The analysis of the problem we divide into two cases: \( m = n \) and \( m < n \). We will not deal with the case \( n < m \), which means that there are more independent actuators than state vector components, that is, the control problem is redundant. The more challenging topics in this area are the design of local stabilising control laws for the nonlinear systems with more degrees of freedom than control inputs. We will deal with this issue in Section 3.

The result regarding the case \( m = n \) is formulated in the following theorem.

**Theorem 2.1 (The case \( m = n \)):** Let us consider the nonlinear control system (1) and the pair \((x_d(t), u_d(t))\). Let for all \( t \geq 0 \)

\[
(A1) \ |x_d(t)| \leq y_{x_d}, \ |u_d(t)| \leq y_{u_d} \ \text{and} \ |w(t)| \leq y_w \ \text{for some nonnegative constants} \ y_{x_d}, y_{u_d} \ \text{and} \ y_w ;
\]

\[
(A2) \ \text{rank of the matrix} \ B(t) \ \text{is equal to} \ n ;
\]

\[
(A3) \ |B^{-1}(t)| \leq \beta_1 \exp[\lambda_+ t] \ \text{for some constants} \ \beta_1 > 0 \ \text{and} \ \lambda_+ \geq 0 , \ \text{where}
\]

\[
\begin{align*}
|B^{-1}(t)|_p & \leq \beta_1 \exp[\lambda_+ t] \iff |B(t)|_p \geq \frac{\sqrt{n}}{\beta_1} \exp[-\lambda_+ t] ;
\end{align*}
\]

\[
(A4) \ |B^{-1}(t)A(t)|_p \leq \beta_2 \exp[\lambda_+ t], \ \text{for some constant} \ \beta_2 \ \geq 0 ;
\]

\[
(A5) \ \text{there exist the constants} \ \alpha > 0, \ \beta_3 \geq 0 \ \text{and} \ \kappa > 0 \ \text{such that} \ |r(e, 0, w(t))| \leq \beta_3 |e|^\alpha \ \text{for} \ |e| \leq \kappa \ \text{(defining} \ \alpha = 1 \ \text{if} \ \beta_3 = 0) , \ \text{where}
\]

\[
r(e, 0, w(t)) = f(x + x_d, u_d, w(t)) - f(x_d, u_d, w(t)) - A(t)e .
\]

Then there exists a \( C^\infty \) in the variable \( x - x_d \) control law \( u^* = u_d(t) + v^*(x - x_d, w(t)) \), which makes the desired trajectory \( x_d(t) \) of system (1) locally asymptotically (exponentially) stable, that is, \( e(t) = x(t) - x_d(t) \to 0 \ \text{for} \ t \to \infty \) and for \( x(0) \) satisfying \( |x(0) - x_d(0)| < \delta \) for some \( \delta > 0 \).
Specifically, for an arbitrary semi-simple Hurwitz matrix $\Delta$ in the Jordan canonical form with the eigenvalues $\lambda_i$, $i = 1, \ldots, n$ satisfying

$$2 \max \left\{ \frac{\lambda_1 - \alpha}{\alpha}, \frac{\lambda_n - \alpha}{\alpha} \right\} + \max_{i=1,\ldots,n} |\text{Re} \lambda_i| < 0$$

we have

$$x(t) - x_d(t) = \exp[\Delta t](x(0) - x_d(0))$$

for all $t \geq 0$.

To be the system stabilisable by feedback control laws, the assumption $m = n$ may be justified in some cases or significantly simplifies and accelerates the calculations:

1. Let us consider a driftless $C^1$ nonholonomic control system

   $$\dot{x} = \sum_{i=1}^{m} g_i(x)u_i$$

   and a constant desired state trajectory $x_d(t) = x_d \in \mathbb{R}^n$ for $u_d(t) = 0$. For tracking error dynamics we obtain the system of differential equations:

   $$\dot{e} = \sum_{i=1}^{m} g_i(e + x_d)v_i = F(e, v).$$

   (3)

There exists a stabilising linear control law $v^* = -Ke$ for $\dot{e} = F(e, v)$ provided the unstable eigenvalues of the linearised system are controllable and there exists no stabilising control law if the linearised system has an unstable eigenvalue which is uncontrollable. The Brockett's topological result adapted to error system (3) states that a necessary condition for the existence of a $C^1$ feedback control $v = v^*(e)$ that makes $\dot{e} = 0$ locally asymptotically stable is that the image of $F(e, v) = \sum_{i=1}^{m} g_i(e + x_d)v_i$ contains an open neighborhood of $e = 0$. If the vectors $g_i(e + x_d)$ are linearly independent at $e = 0$, then $m = n$ is a necessary and sufficient condition for $C^1$ stabilisability of error system at $e = 0$ (Brockett, 1983, p. 187).

For the system (3) we have $A = (0)$ and if the vectors $g_i(e + x_d)$ are linearly independent at $e = 0$, then the assumption $m = n$ implies controllability of a linearisation and the error system (3) is locally asymptotically stabilisable with a linear feedback control $v^* = -Ke$. Note that the Brockett’s theorem does not apply to time-varying feedback laws of the form $v = v(e, t)$. More on the topic of stabilisation and trajectory tracking of the nonholonomic systems can be found in Ge, Zhuping, and Lee (2003), Tian and Li (2002), Shi, Yu, and Khoo (2016) or in the book Jarzebowska (2012).

2. For the linear time-invariant (LTI) error systems $\dot{e} = Ae + Bv$ with invertible matrix $B$ we obtain the simple formula for a calculation of state-feedback gain matrix $K$, namely

$$A - BK = \Delta \Rightarrow K = B^{-1}(A - \Delta),$$

where $\Delta$ is preassigned Hurwitz matrix, globally asymptotically stabilising error dynamics at $e = 0$. The equality $K = B^{-1}(A - \Delta)$ is a LTI version of Theorem 2.1.

In the last decades the stabilisation and tracking problems have been intensively studied. Both problems – stabilisation of the nonlinear control systems and state trajectory tracking represent the important class of problems in engineering practice.

For example, in Pan and Wang (2015) a flatness-based robust active disturbance rejection control technique scheme with tracking differentiator is proposed for the problem of stabilisation and tracking control of the $\chi - Z$ inverted pendulum, which is widely used in laboratories to implement and validate new ideas emerging in the control engineering.

The paper (Chipofya, Lee, & Chong, 2015) presents a solution to stabilisation and trajectory tracking of a quadrotor system using a model predictive controller designed using a special type of orthonormal functions. A time-varying adaptive controller at the torque level is designed in the paper (Wang & Miao, 2015) to simultaneously solve the stabilisation and the tracking problem of unicycle mobile robots with unknown dynamic parameters.

Recently in Xiong, Lai, and Wu (in press), by employing the technique based on energy attenuation, the authors present a stable control strategy for the planar active-passive-active-active manipulator to move the end-effector to a given target position. The manipulator is modelled as second-order nonholonomic planar under-actuated control system.

In all of these papers, and many others that have appeared in the literature, the specific problems are investigated and the corresponding techniques are developed. In contrast, in this paper we attempted to state the results as generally as possible. Therefore, one can expect that stronger results can be obtained for some special forms of Equation (1) using other methods. One such case is analysed at the end of this paper in Remark 3.2.

To the best of our knowledge, there has been very limited research, if any, regarding the general approach to trajectory tracking problem for nonlinear systems with
(time-varying) disturbances, and thus, this topic does not seem to have been well studied until now.

Now we proceed to the proof of Theorem 2.1.

2.1 Proof of Theorem 2.1

Let $\Delta$ be as yet an arbitrary Hurwitz matrix in the Jordan canonical form and without loss of generality we will assume that for every eigenvalue $\lambda_i$, $i = 1, \ldots, n$ its algebraic multiplicity is equal to the geometric multiplicity, that is, $\Delta$ is a semi-simple matrix. Although $m = n$ we will keep in the proof the original notation, $n$ and $m$.

The tracking error dynamics $\dot{e} = F(e, v, w(t))$ can be expressed using the Taylor series expansion of $F$ in the form

$$
\dot{e} = [A(t)e + B(t)v + r(e, v, w(t))] - \Delta e + \Delta e,
$$

with an invertible for all $t \geq 0$ matrix $B(t)$. Now we show that for every $t_0 \geq 0$ fixed, there exist the open neighbourhoods $\Omega_e(t_0)$ of $0 \in \mathbb{R}^n$, $\Omega_v(t_0)$ of $0 \in \mathbb{R}^m$, and the $C^k$ function $v^*(e, w(t_0))$ mapping the set $\Omega_e(t_0)$ to $\Omega_v(t_0)$ such that

$$
A(t_0)e + B(t_0)v^*(e, w(t_0)) + r(e, v^*(e, w(t_0)), w(t_0)) - \Delta e = 0
$$

for all $e \in \Omega_e(t_0)$. Equating the term inside the square brackets in Equation (4) to zero, we have

$$
v = -B^{-1}(t_0) [A(t_0)e + r(e, v, w(t_0))] - \Delta e =: K_e(v).
$$

Fix any $e$ sufficiently near $e = 0$. Then, $K_e(v)$ is a function of $v$ only and we may apply the following contraction mapping lemma.

**Lemma 2.1** (Compare with Hartman, 2002, p. 404): Let $B_a = \{z \in \mathbb{R}^q : |z| < a\}$ denotes the open ball of radius $a$ centred on the origin in $\mathbb{R}^q$. If the function $g : B_a \rightarrow \mathbb{R}^q$ obeys

1) there is constant $\Gamma < 1$ such that $|g(z_1) - g(z_2)| < \Gamma |z_1 - z_2|$ for all $z_1, z_2 \in B_a$;

2) $|g(0)| < (1 - \Gamma)a$,

then the equation $z = g(z)$ has exactly one solution $z^*$, and $z^* \in B_a$.

Now we check that the assumptions of Lemma 2.1 are satisfied. First observe that

$$
r(e, v, w(t_0)) = F(e, v, w(t_0)) - A(t_0)e - B(t_0)v
$$

and so, because $A(t_0), B(t_0)$ are the linear transformations, $r \in C^k$ and also

$$
J_br|_{(0, 0)} = 0
$$

due to the fact that the remainder $r$ contains only higher-order terms of $e$ and $v$. Let $\eta > 0$ is a constant. By continuity, we may choose $a(t_0) > 0$ sufficiently small that

$$
||J_br|_{(e, v)}||_F \leq \frac{\exp[-\lambda_{2}t_0]}{\beta_1 + \eta}
$$

and consequently

$$
||B^{-1}(t_0)J_br|_{(e, v)}||_F \leq \frac{1}{\beta_1}||J_br|_{(e, v)}||_F
$$

$$
\leq \frac{\beta_1}{\beta_1 + \eta} =: \Gamma < 1
$$

and

$$
|K_{e=0}(v_1) - K_{e=0}(v_2)| \leq ||B^{-1}(t_0)J_br|_{(e, v)}||_F|v_1 - v_2|
$$

$$
\leq \Gamma|v_1 - v_2|
$$

whenever $|e|, |v|, |v_1|, |v_2|$ are all smaller than $a(t_0)$. It is important to note that $\Gamma$ is a constant independent of $t$. Also observe that $K_{e=0}(0) = 0$ and so we can choose $a'(t_0) \in (0, a(t_0))$, so that

$$
|K_{e=0}(0)| < (1 - \Gamma)a(t_0)
$$

whenever $|e| < a'(t_0)$.

We conclude from contraction mapping lemma that, assuming $B(t_0)$ is invertible, there exist $a(t_0), a'(t_0) > 0$ such that, for each $e$ obeying $|e| < a'(t_0)$ the system of equations

$$
A(t_0)e + B(t_0)v + r(e, v, w(t_0)) - \Delta e = 0
$$

has exactly one solution, $v^*(e, w(t_0))$, satisfying

$$
|v^*(e, w(t_0))| < a(t_0),
$$

$B_{a'(t_0)} \subset \Omega_v(t_0)$ and $B_{a(t_0)} \subset \Omega_v(t_0)$. Because $F \in C^k$ in the variables $(e, v)$ also the function $v^*(e, w(t_0))$ is $C^k$ in the variable $e$. Moreover, $v^*$ can be estimated in the following way:

$$
|v^*(e, w(t_0))| = |K_{e=0}(v^*(e, w(t_0))) - K_{e=0}(0)|
$$

$$
\leq \text{Lip}(K_{e=0})|v^*(e, w(t_0))| + |K_{e=0}(0)|
$$

$$
\leq \Gamma|v^*(e, w(t_0))| + |-B^{-1}(t_0)[A(t_0)e + r(e, 0, w(t_0)) - \Delta e]|$$
In other words, the fulfilment of this condition will ensure that the decay of \(e(t)\) and \(v^*(e(t), w(t))\) for \(t \to \infty\) with \(e(t) = \exp[\Delta t]e(0)\) is at least of the order \(O(\exp[-\lambda_s t])\) or faster.

The equivalence in the Assumption (A3) tells us that \(||B(t)||_f\) is on \([0, \infty)\) bounded below by some exponential function of the form \(c_1 \exp[-c_2 t]\) with the constants \(c_1 > 0\) and \(c_2 \geq 0\) and follows from the inequality:

\[
\sqrt{n} = ||B(t)B^{-1}(t)||_f \leq ||B(t)||_f ||B^{-1}(t)||_f.
\]

Theorem 2.1 is proved.

Remark 2.1:

1. The Assumptions (A1)–(A5) of Theorem 2.1 guarantee that the error dynamics \(\tilde{e}(t)\) can be designed in the form \(\exp[\Delta t]e(0)\) with the Hurwitz matrices \(\Delta\) satisfying the inequality (8). Conversely, if the Assumptions (A1)–(A5) (or some of these assumptions) are not fulfilled, the decay of the error \(e(t)\) to zero for \(t \to \infty\) can be slower than \(\exp[\Delta t]e(0)\) for an arbitrary Hurwitz matrix \(\Delta\).

2. Theorem 2.1 provides the sufficient conditions for the local asymptotic (exponential) stabilisability of the desired state trajectory and thus the region of attraction may be larger than the area we obtain from the inequality (5).

3. Analysis of the case \(m < n\)

Analysis of the case of underactuated control system (1) with \(m < n\) is much more complicated than the previous one where \(m = n\). The major problem to be overcome is a non-invertibility of the matrix \(B(t)\). We will proceed as follows: let us consider the augmented problem to the original problem (1), namely

\[
\begin{align*}
\dot{x} &= f(x, \tilde{u}, w(t)) + l_{m+1}(t)\tilde{u}_{m+1} + \cdots + l_n(t)\tilde{u}_n \\
&=: \tilde{f}(x, \tilde{u}, w(t)),
\end{align*}
\]

where \(l_i(t), i = m + 1, \ldots, n, \geq 0\) are the column vectors specified below and \(\tilde{u}_{m+1}, \ldots, \tilde{u}_n\) are the auxiliary control variables. Now, we define the pair \((\tilde{x}_d, \tilde{u}_d)\) of the augmented system (9) associated with \((x_d, u_d)\) of the original system as

\[
(\tilde{x}_d, \tilde{u}_d) = (x_d, (u_d^T, 0, \ldots, 0)^T),
\]

and the error dynamics

\[
\begin{align*}
\dot{\tilde{e}} &= \tilde{f}(\tilde{e} + \tilde{x}_d, \tilde{u} + \tilde{u}_d, w(t)) - \tilde{f}(\tilde{x}_d, \tilde{u}_d, w(t)) \\
&= f(\tilde{e} + x_d, \tilde{u}_1, u_d, |_{i=1, \ldots, m}, w(t)) \\
&+ \sum_{i=m+1}^{n} l_i(t)\tilde{u}_i - f(x_d, u_d, w(t)) \\
&=: \tilde{F}(\tilde{e}, \tilde{u}, w(t)).
\end{align*}
\]
with an initial state \( \tilde{e}(0) = e(0) \). The vectors \( l_i(t), i = m + 1, \ldots, n \) are selected so that

\[
\tilde{B}(t) = \left( B(t) : l_{m+1}(t) : \ldots : l_n(t) \right)
\]

is an invertible matrix for all \( t \geq 0 \). On the basis of Theorem 2.1 there exists a feedback control law \( \tilde{v}^*(\tilde{e}, w(t)) \) which locally asymptotically stabilises the error dynamics of the augmented system at \((0, 0)\) associated with the pair \((\tilde{x}_d, \tilde{u}_d)\). Now, we can define the feedback control law \( v^* = (v^*_1, \ldots, v^*_m)^T \) for an error dynamics of the original system as

\[
v^*_j(e, w(t), t) = \tilde{v}^*_j(e, w(t)), \quad j = 1, \ldots, m \tag{10}
\]

where the terms \( \tilde{v}^*_j(\tilde{e}, w(t)), j = m + 1, \ldots, n, \) eventually occurring in the argument list of the functions on the right side of the equality (10) are replaced by

\[
\tilde{v}^*_j(t) = \tilde{v}^*_j \left( \exp [\tilde{\Delta}t] e(0), w(t) \right),
\]

where the eigenvalues \( \tilde{\lambda}_i, i = 1, \ldots, n \) of the semi-simple matrix \( \tilde{\Delta} \) in the Jordan canonical form satisfy

\[
2 \max \left\{ \frac{\tilde{\lambda}_i}{\alpha}, \tilde{\lambda}_i \right\} + \max_{i=1,\ldots,n} \{ \text{Re} \tilde{\lambda}_i \} < 0. \tag{11}
\]

Let us define for all \( t \geq 0 \) the secondary error function \( E(t) \) as \( \tilde{e}(t) - e(t) \) and calculate \( \dot{E} \). We obtain

\[
\dot{E} = \dot{\tilde{e}} - \dot{e} = \tilde{F}(\tilde{e}, \tilde{v}^*(\tilde{e}), w(t)) - F(e, \tilde{v}^*(e, t), w(t)) = \tilde{F}(\tilde{e}, \tilde{v}^*(\tilde{e}), w(t)) - F(\tilde{e} - E, \tilde{v}^*(\tilde{e} - E, t), w(t)) =: H(E, \tilde{e}, w(t), t),
\]

that is, the dynamics of secondary error function \( E \) is governed by the initial state problem:

\[
\dot{E} = H(E, \tilde{e}, w(t), t), \quad E(0) = 0, \quad \text{where} \quad \tilde{e}(t) = \exp[\tilde{\Delta}t] e(0). \tag{12}
\]

Furthermore, we have that

\[
H \in C^k \quad \text{in the variables} \quad E \quad \text{and} \quad \tilde{e}, \quad \text{and}
\]

\[
H(0, \tilde{e}, t) = \sum_{i=m+1}^n l_i(t) \tilde{v}^*_i(\tilde{e}(t), w(t)).
\]

Summarising the above considerations, we have just proved the following theorem.

**Theorem 3.1 (The case \( m < n \))**: Let us consider the nonlinear control system (1) and the pair \((x_d(t), u_d(t))\). Let for all \( t \geq 0 \)

\[
\begin{aligned}
\dot{x}_1 &= x_1 w(t) + x_2 + u_1, \\
x_d(t) &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
\dot{u}_d(t) &= \begin{pmatrix} 0 \end{pmatrix},
\end{aligned}
\]

and its error dynamics along \((x_d, u_d)\)

\[
\begin{aligned}
\dot{e}_1 &= e_1 w(t) + e_2 + v_1, \\
\dot{e}_2 &= e_2 + v_1.
\end{aligned}
\]

Example 3.1: As an illustrative example to demonstrate the applicability of the proposed approach, let us consider the control system

\[
\dot{x}_1 = x_1 w(t) + x_2 + u_1, \\
x_d(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \\
\dot{u}_d(t) = \begin{pmatrix} 0 \end{pmatrix},
\]

and its error dynamics along \((x_d, u_d)\)

\[
\begin{aligned}
\dot{e}_1 &= e_1 w(t) + e_2 + v_1, \\
\dot{e}_2 &= e_2 + v_1.
\end{aligned}
\]
Augmenting the original system vector field with the vector $l_{2(t)}\tilde{\mathbf{u}}_2 = (1, 0)^T \tilde{\mathbf{u}}_2$ we obtain
\[
\begin{align*}
\dot{\tilde{e}}_1 &= \tilde{e}_1 w(t) + \tilde{e}_2 + \tilde{v}_1 + \tilde{v}_2 \\
\dot{\tilde{e}}_2 &= \tilde{e}_2 + \tilde{v}_1
\end{align*}
\]
or
\[
\begin{align*}
\begin{pmatrix} \dot{\tilde{e}}_1 \\ \dot{\tilde{e}}_2 \end{pmatrix} &= \begin{pmatrix} w(t) & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{v}_1 \\ \tilde{v}_2 \end{pmatrix} \\
&\quad - \begin{pmatrix} \tilde{\lambda}_1 & 0 \\ 0 & \tilde{\lambda}_2 \end{pmatrix} \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix} + \begin{pmatrix} \tilde{\lambda}_1 & 0 \\ 0 & \tilde{\lambda}_2 \end{pmatrix} \begin{pmatrix} \tilde{e}_1 \\ \tilde{e}_2 \end{pmatrix}.
\end{align*}
\]

On the basis of what was said in the introductory part of this article, without loss of generality we consider here a diagonal matrix $\tilde{\Delta}$ with the provisionally arbitrary, negative real eigenvalues $\tilde{\lambda}_i$, $i = 1, 2$ as follows from the inequality (11) for $\bar{\lambda}_n = 0$. The expression in the square brackets is equal to zero for
\[
\tilde{u}^*(\tilde{e}, w(t)) := \begin{pmatrix} -\tilde{e}_2 + \tilde{\lambda}_2 \tilde{e}_2 \\ -\tilde{e}_1 w(t) + \tilde{\lambda}_1 \tilde{e}_1 - \tilde{\lambda}_2 \tilde{e}_2 \end{pmatrix}.
\]

Hence, by the equality (10), the state-feedback for the original error system is $\tilde{u}_1^* (\tilde{e}, w) = -\tilde{e}_2 + \tilde{\lambda}_2 \tilde{e}_2$ and we have the following error dynamics of original and augmented system:
\[
\begin{align*}
\tilde{\dot{e}} &= \begin{pmatrix} \tilde{\lambda}_1 & \tilde{\lambda}_2 \\ \tilde{\lambda}_2 & \tilde{\lambda}_2 \end{pmatrix} \tilde{e}, \\
\dot{\tilde{e}} &= \begin{pmatrix} \tilde{\lambda}_1 w(t) + \tilde{\lambda}_2 \tilde{e}_2 \\ \tilde{\lambda}_2 \tilde{e}_2 \end{pmatrix}.
\end{align*}
\]

Thus, for the secondary error function $E(t)$, $E(t) = \tilde{e}(t) - e(t)$, we get the initial state problem:
\[
\begin{align*}
\dot{E} &= \begin{pmatrix} \tilde{\lambda}_1 \tilde{e}_1 - e_1 w(t) - \tilde{\lambda}_2 \tilde{e}_2 \\ \tilde{\lambda}_2 (\tilde{e}_2 - e_2) \end{pmatrix} \\
&= \begin{pmatrix} \tilde{\lambda}_1 \tilde{e}_1 - (\tilde{e}_1 - E_1) w(t) - \tilde{\lambda}_2 (\tilde{e}_2 - E_2) \\ \tilde{\lambda}_2 E_2 \end{pmatrix}, \\
E(0) &= 0,
\end{align*}
\]

where $\tilde{e}_i = \exp[\tilde{\lambda}_i t] e_i(0)$, $i = 1, 2$. The values of $\tilde{\lambda}_i$ may be modified in subsequent analysis of the dynamics for $E$. Because $E_2(t) \equiv 0$, we will focus on the differential equation for the first component of secondary error function $E$,
\[
\begin{align*}
\dot{E}_1 - w(t)E_1 &= h_1(t), \\
h_1(t) &= : \tilde{\lambda}_1 \tilde{e}_1 - w(t) \tilde{e}_1 \\
&\quad - \tilde{\lambda}_2 \tilde{e}_2 \to 0 \text{ for } t \to \infty.
\end{align*}
\]

and its solution for an initial state $E_1(0) = 0$,
\[
E_1(t) = \exp \left[ \int_0^t w(s)ds \right] \int_0^t h_1(\tau) \times \exp \left[ - \int_0^\tau w(s)ds \right] d\tau.
\]

Now let $w(t)$ and $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ are such that
\[
\exp \left[ - \int_0^t w(s)ds \right] \to \infty \text{ and } \frac{h_1(t)}{w(t)} \to 0 \text{ for } t \to \infty.
\]

Then, using the L’Hospital’s rule to evaluate asymptotics of $E_1(t)$, we get
\[
\lim_{t \to \infty} E_1(t) = \lim_{t \to \infty} \frac{\int_0^t h_1(\tau) \exp \left[ - \int_0^\tau w(s)ds \right] d\tau}{\exp \left[ - \int_0^t w(s)ds \right]} = - \lim_{t \to \infty} \frac{h_1(t)}{w(t)} = 0
\]
for all $e(0)$. Thus, on the basis of Theorem 3.1, part (ii), the desired trajectory $x_d(t) = 0$ is locally asymptotically stable solution (even globally, $\delta_0 = \infty$) of the original system with $u = u_1^*(x_1, x_2) = -x_2 + \tilde{\lambda}_2 x_2$.

It is worth noting that, for $w(t) \equiv 0$, we obtain from the equality (13) that $\lim_{t \to \infty} E_1(t) = \int_0^\infty h_1(\tau)d\tau$ and the value of this integral can be made arbitrarily small by appropriate choice of $\tilde{\lambda}_1$, $\tilde{\lambda}_2$ and $\delta (\tilde{\lambda}_1, \tilde{\lambda}_2)$, which implies only the local stability (in the sense of Lyapunov) of $x_d$, and this fact is in accordance with the Brockett’s necessary condition for the existence of a $C^1$ closed-loop control law of the form $u'(x)$ that locally asymptotically stabilises the nonlinear control system to an equilibrium point.

Remark 3.1: This example illustrates the main benefit and power of Theorem 3.1 which lies in the possible reduction of the order of the analysed state tracking problem from the order $n$ up to $n - m$, not only for linear systems as we will see in the following remark. It can be expected that treatment with this reduced-order system will be substantially easier than with the original problem.

Remark 3.2: As already mentioned above, Theorems 2.1 and 3.1 capture a broad class of nonlinear control problems, therefore, it can be expected that the stronger results can be obtained for some special classes of Equation (1). For example, let us consider two-input chained system $\dot{x} = (u_1, u_2, x_2 u_1)^T$. The chained systems, a special class of the nonholonomic systems, are often at the centre of
interest due to their applicability to modelling the feedback techniques for car-like robotic systems, see, e.g. De Luca, Oriolo, and Samson (1998).

By employing the method developed in this paper, for \( l_3 = (0, 0, 1)^T \) we obtain the state-feedback

\[
\tilde{u}^*(\tilde{e}) = \begin{pmatrix}
\tilde{\lambda}_1 \tilde{e}_1 \\
\tilde{\lambda}_2 \tilde{e}_2 \\
\tilde{\lambda}_3 \tilde{e}_3
\end{pmatrix} - \begin{pmatrix}
\tilde{\lambda}_1 \tilde{e}_1 \\
\tilde{\lambda}_2 \tilde{e}_2 \\
\tilde{\lambda}_3 \tilde{e}_3
\end{pmatrix} = \begin{pmatrix}
\tilde{\lambda}_1 E_1 \\
\tilde{\lambda}_2 E_2 \\
\tilde{\lambda}_3 E_3
\end{pmatrix},
\]

locally asymptotically stabilising augmented system with the arbitrary real numbers \( \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 < 0 \) (\( \tilde{\lambda}_s = 0 \)). The corresponding secondary error dynamics is

\[
\dot{\tilde{E}} = \begin{pmatrix}
\tilde{\lambda}_1 \tilde{e}_1 \\
\tilde{\lambda}_2 \tilde{e}_2 \\
\tilde{\lambda}_3 \tilde{e}_3
\end{pmatrix} - \begin{pmatrix}
\tilde{\lambda}_1 \tilde{e}_1 \\
\tilde{\lambda}_2 \tilde{e}_2 \\
\tilde{\lambda}_3 \tilde{e}_3
\end{pmatrix} = \begin{pmatrix}
\tilde{\lambda}_1 E_1 \\
\tilde{\lambda}_2 E_2 \\
\tilde{\lambda}_3 E_3 - \tilde{\lambda}_1 (\tilde{e}_1 - E_1) (\tilde{e}_2 - E_2)
\end{pmatrix}.
\]

The initial state \( E(0) = 0 \) implies that \( E_1 = E_2 \equiv 0 \) and

\[
\dot{E}_3 = \tilde{\lambda}_3 \tilde{e}_3 - \tilde{\lambda}_1 \tilde{e}_1 \tilde{e}_2, \quad \tilde{e}_i(t) = \exp[\tilde{\lambda}_i t] e_i(0), \quad i = 1, 2, 3.
\]

Integrating this between 0 and \( t \) we have

\[
E_3(t) = \int_0^t \left( \tilde{\lambda}_3 \tilde{e}_3(0) \exp[\tilde{\lambda}_3 \tau] - \tilde{\lambda}_1 e_1(0) e_2(0) \exp[(\tilde{\lambda}_1 + \tilde{\lambda}_2) \tau] \right) d\tau
\]

\[
\approx \frac{\tilde{\lambda}_1 e_1(0) e_2(0)}{\tilde{\lambda}_1 + \tilde{\lambda}_2} - e_3(0).
\]

As follows from Theorem 3.1, part (i), the feedback control law \( u^*(x) = (\tilde{\lambda}_1 x_1, \tilde{\lambda}_2 x_2)^T \) locally stabilises (in the sense of Lyapunov) the origin for the closed-loop system. We can verify it directly from the explicit solution:

\[
x_1(t) = x_1(0) \exp[\tilde{\lambda}_1 t] \\
x_2(t) = x_2(0) \exp[\tilde{\lambda}_2 t] \\
x_3(t) = x_3(0) - \frac{\tilde{\lambda}_1 x_1(0) x_2(0)}{\tilde{\lambda}_1 + \tilde{\lambda}_2} \left( 1 - \exp[(\tilde{\lambda}_1 + \tilde{\lambda}_2) t] \right),
\]

and so the result is in accordance with Theorem 3.1, part (i).

On the other side, the Brockett’s necessary condition fails to be satisfied for our system as no point of the form \((0, 0, v), v \neq 0\) is in the image of \( f \), and therefore, the system under consideration cannot be asymptotically stabilised to the origin by using the control laws of the form \( u = u^*(x) \). As is presented in the papers (Murray, Walsh, & Sastry, 1992) and generally for the chained systems in Teel, Murray, and Walsh (1992), the local asymptotic stabilisation to the origin is achieved by the time-varying control law with sinusoids:

\[
u_1(x, t) = -x_1 + x_3 \sin t, \quad u_2(x, t) = -x_2 - x_3^2 \cos t.
\]

4. Conclusions

The problem of interest is that of regulating the tracking error \( e(t) \) around zero by attenuating and possibly rejecting the disturbances. The general framework of control design strategy has been proposed to stabilise (asymptotically or in the sense of Lyapunov) nonlinear control system around the desired state trajectory. We have shown that for \( m = n \) the control system with the disturbances can be stabilised at the desired state trajectory by applying the \( C^k \) state-feedback control law of the form \( u = u_d(t) + u^*(x - x_d, w(t)) \) that the desired state trajectory \( x_d(t) \) is a locally asymptotically stable solution of the system \( \dot{x} = f(x, u^*, w(t)) \).

A weaker result was obtained for the underactuated systems \((m < n)\), when the local stabilisability of control system (asymptotic or in the sense of Lyapunov) in the neighbourhood of desired state trajectory depends on the properties of secondary error system (12).

Further efforts could be directed toward weakening the conditions imposed on \( \|B^{-1}(t)\|_e \) and \( \|B^{-1}(t)A(t)\|_e \) by using more general error dynamics system \( \dot{e} = \Delta(e) \), \( \Delta(0) = 0 \) instead of the linear system \( \dot{e} = \Delta e \) with the limiting rate of convergence to \( e = 0 \), which may be insufficient for some nonlinear control systems.

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