Flat Connections for Characters in Irrational Conformal Field Theory

M. B. Halpern* and N. Sochen*

Department of Physics†
University of California at Berkeley
and
Theoretical Physics group
Lawrence Berkeley Laboratory
Berkeley, CA 94720, U.S.A.

Abstract
Following the paradigm on the sphere, we begin the study of irrational conformal field theory (ICFT) on the torus. In particular, we find that the affine-Virasoro characters of ICFT satisfy heat-like differential equations with flat connections. As a first example, we solve the system for the general $g/h$ coset construction, obtaining an integral representation for the general coset characters. In a second application, we solve for the high-level characters of the general ICFT on simple $g$, noting a simplification for the subspace of theories which possess a non-trivial symmetry group. Finally, we give a geometric formulation of the system in which the flat connections are generalized Laplacians on the centrally-extended loop group.

* e-mail: mbhalpern@lbl.gov, halpern@physics.berkeley.edu
* e-mail: sochen@asterix.lbl.gov
† This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-90-21139.
1. Introduction

Affine Lie algebra, or current algebra on $S^1$, was discovered independently in mathematics [1] and physics [2]. It is now understood that affine Lie algebra underlies both rational conformal field theory (RCFT) and irrational conformal field theory (ICFT), which includes RCFT as a small subspace,

$$\text{ICFT} \supset\supset \text{RCFT}. \quad (1.1)$$

At present, the only known path into the space of ICFT’s is the general affine-Virasoro construction whose conformal stress tensors have the form [3,4],

$$T(L) = L^{ab*} J_a J_b^* \quad (1.2)$$

where $J_a$, $a = 1 \ldots \dim g$ are the currents of affine $g$ and $L^{ab}$ is a solution of the Virasoro master equation [3,4]. The space of solutions of the master equation, called affine-Virasoro space, includes the affine-Sugawara constructions [2,5,6], the coset constructions [2,5,7] and a vast number of unitary constructions with irrational central charge [8]. See Reference [9] for a brief survey of affine-Virasoro space and its partial classification by graph theory.

One of the most prominent features of affine-Virasoro space is K-conjugation covariance [2,5,7,8], which says that affine-Virasoro constructions come in commuting K-conjugate pairs, $T = T(L)$ and $\tilde{T} = T(\tilde{L})$, which sum to the affine-Sugawara construction $T_g$ on $g$,

$$\tilde{T} + T = T_g, \quad \tilde{c} + c = c_g. \quad (1.3)$$

Thus, each K-conjugate pair of ICFT’s naturally forms a biconformal field theory [10,11], complete with biprimary and bisecondary fields, whose biconformal correlators must be factorized to obtain the conformal correlators of the individual ICFT’s.

Recently, dynamical equations for the biconformal correlators, the affine-Virasoro Ward identities [11,12], have been obtained for ICFT on the sphere. These may be expressed as generalized Knizhnik-Zamolodchikov equations [13],

$$\partial_i A(\bar{z}, z) = A(\bar{z}, z) \overline{W}_i(\bar{z}, z), \quad \partial_i A(\tilde{z}, \bar{z}) = A(\tilde{z}, \bar{z}) W_i(\tilde{z}, z) \quad (1.4a)$$
$$A_g(\bar{z}) = A(\bar{z}, z) \quad (1.4b)$$

for the biconformal correlators $A$. The affine-Virasoro connections $\overline{W}, W$ are flat connections and the affine-Sugawara correlators $A_g$ are obtained from the biconformal correlators.
when $z = \bar{z}$. These equations have been solved for the correlators of the general $g/h$ coset construction \cite{1,2,3}, providing a first-principle derivation of the coset blocks first proposed by Douglas \cite{4}. Moreover, the equations have been solved for the general ICFT at high level \cite{12,13} on simple $g$, although the resulting high-level conformal correlators have not yet been analyzed at the level of conformal blocks. See Ref. \cite{15} for a review of the affine-Virasoro Ward identities, including the general solution \cite{12} which exhibits braiding for all ICFT.

In this paper, we begin the study of ICFT on the torus, following the paradigm on the sphere. In particular, we study the affine-Virasoro characters (the bicharacters),

$$\chi(T, \bar{\tau}, \tau, g) = Tr_T \left( \tilde{L}(0) - \bar{c}/24 q L(0) - c/24 g \right)$$

(1.5)

where $\tilde{L}(0)$ and $L(0)$ are the zero modes of a K-conjugate pair of stress tensors on affine $g$. Here, the trace is over the affine irrep $V_T$, the source $g \in G$ is an element of the Lie group, and, as on the sphere, the affine-Sugawara (or affine) characters are obtained from the bicharacters at $\bar{\tau} = \tau$,

$$\chi_g(T, \tau, \tau, g) = \chi(T, \tau, \tau, g) = Tr_T \left( q L(0) - c/24 g \right)$$

(1.6)

Given (1.6) as a boundary condition, we find that the bicharacters are the unique solutions of heat-like differential equations with flat connections $\tilde{D}$ and $D$,

$$\partial_{\bar{\tau}} \chi(T, \bar{\tau}, \tau, g) = \tilde{D}(\bar{\tau}, \tau, g) \chi(T, \bar{\tau}, \tau, g)$$

(1.7a)

$$\partial_{\tau} \chi(T, \bar{\tau}, \tau, g) = D(\bar{\tau}, \tau, g) \chi(T, \bar{\tau}, \tau, g)$$

(1.7b)

whose existence and properties are the central subject of this paper. These systems include and generalize the heat equations for the affine-Sugawara characters obtained by Bernard \cite{16} and by Eguchi and Ooguri \cite{17}.

Following the development on the sphere, we solve the system first for the simple case of $h$ and the $g/h$ coset constructions, obtaining a new integral representation for the general coset characters.

In a second application, we solve for the high-level form of the general bicharacters on simple $g$, noting a simplification on the subspace of $H$-invariant CFT’s \cite{18},

$$\text{ICFT} \supset H\text{-invariant CFT’s} \supset \text{Lie} h\text{-invariant CFT’s} \supset \text{RCFT}$$

(1.8)
The set of $H$-invariant CFT’s is the subset of all ICFT’s which possess a residual global symmetry group $H$, which may be a finite group or a Lie group. Those theories with a Lie invariance are called the Lie $h$-invariant CFT’s [18], which include the affine-Sugawara constructions and the coset constructions as a small subspace.

The simplification occurs when the source is chosen in the symmetry group of the theory, and, as seen in the hierarchy [18], this simplification occurs for the coset constructions as well. Using intuition gained from the cosets, we factorize the high-level bicharacters of the Lie $h$-invariant CFT’s to obtain a set of high-level candidate characters for this class of ICFT’s. The set of candidate characters correctly includes the high-level form of the coset characters and should be further analyzed with respect to modular covariance in the general case.

Finally we give a geometric formulation of the system on an affine source, where the flat connections are generalized Laplacians on the centrally-extended loop group. These Laplacians involve new first-order differential representations of affine Lie algebra.

### 2. The General Affine-Virasoro Construction

In this Section, we review some aspects of ICFT which will be relevant in the development below.

The general affine-Virasoro construction begins with the currents $J_a$ of untwisted affine Lie $g$ [1,2],

\[
[J_a(m), J_b(n)] = i f_{ab}^c J_c(m + n) + m G_{ab} \delta_{m+n,0} \tag{2.1a}
\]

\[
a, b = 1 \ldots \dim g, \quad m, n \in \mathbb{Z} \tag{2.1b}
\]

where $f_{ab}^c$ and $G_{ab}$ are respectively the structure constants and general Killing metric of $g = \oplus_I g_I$. To obtain invariant levels $x_I = 2k_I/\psi_I^2$ of $g_I$ with dual Coxeter numbers $\tilde{h}_I = Q_I/\psi_I^2$, take

\[
G_{ab} = \oplus_I k_I \eta_{Iab} \quad , \quad f_{ac}^d f_{bd}^c = - \oplus_I Q_I \eta_{Iab} \tag{2.2}
\]

where $\eta_{Iab}$ and $\psi_I$ are respectively the Killing metric and highest root of $g_I$.

The stress tensors of the general affine-Virasoro construction are elements of the enveloping algebra of the affine algebra [3,4],

\[
T(L) = L_{a}^{b*} J_a J_b^* = \sum_{m \in \mathbb{Z}} L(m) z^{-m-2} \tag{2.3}
\]
where $L^{ab} = L^{ba}$ is called the inverse inertia tensor in analogy with the spinning top. In order that the modes $L(m)$ generate the Virasoro algebra,

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12} m(m^2 - 1) \delta_{m+n,0}$$  \hspace{1cm} (2.4)

the inverse inertia tensor must satisfy the Virasoro master equation [3,4],

$$L^{ab} = 2 L^{ac} G_{cd} L^{db} - L^{cd} L^{ef} f^{a}_{ce} f^{b}_{df} - L^{cd} f^{a}_{ce} f^{(a}_{df} L^{b)ce}$$  \hspace{1cm} (2.5a)

$$c(L) = 2G_{ab} L^{ab}$$  \hspace{1cm} (2.5b)

where $c(L)$ is the central charge of the CFT. The master equation has been identified [19] as an Einstein-like system on the group manifold with central charge $c = \text{dim } g - 4R$, where $R$ is the Einstein curvature scalar.

A. Affine-Sugawara constructions [2,4]. The affine-Sugawara construction $L_g$ is

$$L^{ab}_g = \sum_k \frac{n_k^{ab}}{2k + Q_k} , \hspace{1cm} c_g = \sum_k \frac{x_k \text{dim } g_k}{x_k + h_k}$$  \hspace{1cm} (2.6)

for arbitrary levels of affine $g$, and similarly for $L_h$ when $h \subset g$.

B. K-conjugation covariance [2,5,7]. When $L$ is a solution of the master equation on $g$, then so is the K-conjugate partner of $L$, called $\tilde{L}$,

$$\tilde{L}^{ab} = L^{ab}_g - L^{ab} , \hspace{1cm} c(\tilde{L}) = c_g - c(L)$$  \hspace{1cm} (2.7)

while the corresponding stress tensors $T \equiv T(L)$ and $\tilde{T} \equiv T(\tilde{L})$ form a commuting pair of Virasoro operators which sum to the affine-Sugawara construction,

$$\tilde{T} + T = T_g , \hspace{1cm} \tilde{c} + c = c_g , \hspace{1cm} T(z)\tilde{T}(w) = \text{regular}$$  \hspace{1cm} (2.8)

As the simplest examples of K-conjugation, the $g/h$ coset constructions $\tilde{T} = T_{g/h} = T_g - T_h$ are the K-conjugate partners of $T_h$ on $g$.

More generally, each breakup $T_g = \tilde{T} + T$ naturally defines a biconformal field theory [10,11], with two commuting Virasoro operators. The breakup also suggests that the affine-Sugawara construction is a tensor product CFT, formed by tensoring the conformal theories of $\tilde{T}$ and $T$. In practice, we face the inverse problem, namely to factorize [10,11,12,13] the affine-Sugawara blocks into the conformal blocks of $\tilde{T}$ and $T$. 

4
C. \(T, J\) commutator. The commutator of the stress tensor with the currents is \(3\),

\[
[L(m), J_a(n)] = -nM(L)_a^b J_b(m + n) + N(L)_a^{bc}(J_b J_c^*)_{m+n}
\]

and similarly for the K-conjugate theory with the substitution \(L(m) \rightarrow \tilde{L}(m), L^{ab} \rightarrow \tilde{L}^{ab}\).

D. High-level solutions. At high-level on simple \(g\), the high-level smooth solutions of the master equation have the form \(20, 21\),

\[
\tilde{L}^{ab} = \frac{1}{2k} \eta^{ac} \tilde{P}_c^b + O(k^{-2}) \quad , \quad L^{ab} = \frac{1}{2k} \eta^{ac} P_c^b + O(k^{-2})
\]

\[
\tilde{c} = \text{rank} \tilde{P} + O(k^{-1}) \quad , \quad c = \text{rank} P + O(k^{-1}) \quad , \quad c_g = \text{dim} g + O(k^{-1})
\]

where \(\tilde{P}\) and \(P\) are the high-level projectors of the \(\tilde{L}\) and \(L\) theories respectively,

\[
\tilde{P}^2 = \tilde{P} \quad , \quad P^2 = P \quad , \quad \tilde{P} + P = 1 \quad , \quad \tilde{P} P = P \tilde{P} = 0
\]

In the partial classification of ICFT by graph theory \(22, 29\), the projectors are the adjacency matrices of the graphs, each of which labels a level family of ICFT’s.

E. Symmetry groups in ICFT \(18\). The generic ICFT on \(g\) has no residual global symmetry \(22\), but it is useful to distinguish the subspace of \(H\)-invariant CFT’s, which are all ICFT’s with a residual global symmetry \(H\),

\[
\text{ICFT} \supset H\text{-invariant CFT’s} \supset \text{Lie} h\text{-invariant CFT’s} \supset \text{RCFT}
\]

The \(H\)-invariant CFT’s on \(g\) satisfy

\[
\tilde{L} = w \tilde{L} w^{-1} \quad , \quad L = w L w^{-1} \quad , \quad \forall w \in H \subset \text{Aut}G
\]

where \(G\) is the Lie group whose algebra is \(g\). If \(H\) is also a Lie group, with Lie algebra \(h \subset g\), we have the proper subspace of Lie \(h\)-invariant CFT’s, which satisfy

\[
[T^{\text{adj}}_A, \tilde{L}] = [T^{\text{adj}}_A, L] = 0 \quad , \quad A = 1, \ldots, \text{dim} h
\]

where \(T^{\text{adj}}_a\), \(a = 1, \ldots, \text{dim} g\) is the adjoint representation of \(g\). As seen in the hierarchy \((2.12)\), the \(h\)-invariant CFT’s include \(h\) and the \(g/h\) coset constructions as a small subspace.
3. The Affine-Virasoro Characters

For each K-conjugate pair \( \tilde{T}, T \) of affine-Virasoro constructions on compact \( g \), the affine-Virasoro (or biconformal) characters are defined as,

\[
\chi(T, \tilde{\tau}, \tau, h) = Tr_T \left( \tilde{q} \tilde{L}^{(0)} - \tilde{c}/24 q L^{(0)} - c/24 h \right)
\]  

(3.1)

where \( q = e^{2\pi i \tau} (\tilde{q} = e^{2\pi i \tilde{\tau}}) \) with \( \text{Im} \tau > 0 \) (\( \text{Im} \tilde{\tau} > 0 \)), and

\[
\tilde{L}(0) = \tilde{L}^{ab}(J_a(0)J_b(0) + 2 \sum_{n>0} J_a(-n)J_b(n))
\]  

(3.2a)

\[
L(0) = L^{ab}(J_a(0)J_b(0) + 2 \sum_{n>0} J_a(-n)J_b(n))
\]  

(3.2b)

are the zero modes of \( \tilde{T} \) and \( T \). For flexibility below, we specify the source \( h \) in (3.1) to be an element of the compact Lie group \( H \subset G \), which may be parametrized, for example, as

\[
h = e^{i\beta A(x)J_A(0)} , \quad A = 1, \ldots, \dim h
\]  

(3.3)

where \( x^i, i = 1, \ldots, \dim h \) are coordinates on the \( H \) manifold. As special cases, we may then choose, if desired, the standard sources on \( G \) or Cartan \( G \) employed in Refs. [16] and [17] respectively.

In (3.1), the trace is over the integrable affine irrep \( V_T \) whose affine primary states \( |R_T\rangle \) correspond to matrix irrep \( T \) of \( g \). In an \( L \)-basis of \( T \) [10][11], these are called the \( L^{ab} \)-broken affine primary states, which satisfy,

\[
J_a(m)|R_T\rangle = \delta_{m,0}\lambda_{m,0}|R_T\rangle , \quad m \geq 0
\]  

(3.4a)

\[
\tilde{L}^{ab}(T_aT_b)_{\alpha}^\beta = \tilde{\Delta}_\alpha(T)\delta_{\alpha}^\beta , \quad L^{ab}(T_aT_b)_{\alpha}^\beta = \Delta_\alpha(T)\delta_{\alpha}^\beta
\]  

(3.4b)

\[
\tilde{L}(0)|R_T\rangle = \tilde{\Delta}_\alpha(T)|R_T\rangle , \quad L(0)|R_T\rangle = \Delta_\alpha(T)|R_T\rangle
\]  

(3.4c)

\[
\tilde{\Delta}_\alpha(T) + \Delta_\alpha(T) = \Delta_g(T)
\]  

(3.4d)

where \( \tilde{\Delta}_\alpha(T), \Delta_\alpha(T) \) and \( \Delta_g(T) \) are the conformal weights of the broken affine primaries under \( \tilde{T}, T \) and \( T_g \) respectively. More generally, \( L \)-bases are the eigenbases of the conformal weight matrices, such as (3.4b), which occur at each level of the irrep. In what follows, we often refer to the affine-Virasoro characters as the bicharacters.

In this Section, we confine our remarks to some simple properties of the bicharacters.
1. N-point correlators. The bicharacters (3.1) are only the simplest (zero point) correlators on the torus. Although we will not pursue this here, N-point correlators on the torus can be defined, as on the sphere, by insertion of biconformal fields [10,11] in the trace.

2. K-conjugation invariance. It is clear on inspection that the bicharacters satisfy the K-conjugation invariance,

$$\chi(T, \tilde{\tau}, \tau, h)|_{\tau \leftrightarrow \tilde{\tau}} = \chi(T, \tilde{\tau}, \tau, h)$$

under exchange of the K-conjugate CFT's.

3. Affine-Sugawara boundary condition. Since $\tilde{T} + T = T_g$ and $\tilde{c} + c = c_g$, the affine-Virasoro characters reduce to the affine-Sugawara (or affine) characters,

$$\chi_g(T, \tau, h) = \chi(T, \tau, \tau, h) = Tr_T \left( q^{L_g(0)-c_g/24} h \right)$$

on the affine-Sugawara line $\tilde{\tau} = \tau$.

4. Small $\tilde{q}$ and $q$. In order to obtain the leading terms of the bicharacters when $\tilde{q}$ and $q$ are small, we need the identities,

$$h|R_T^\alpha = |R_T^\beta h(T)^\beta^\alpha$$

$$hJ_a(n)h^{-1} = \Omega(h)_a^b J_b(n) \quad , \quad \Omega(h) = h(T^{adj})^{-1}$$

where $h(T)$ is the corresponding element of $H \subset G$ in matrix irrep $T$ of $g$. Then, with (3.4) and the affine algebra (2.1), we may easily compute the contributions of the lowest states,

$$\chi(T, \tilde{\tau}, \tau, h) = \sum_{\alpha=1}^{\dim T} q^{\tilde{\Delta}_\alpha(T)-\tilde{c}/24} q^{\Delta_\alpha(T)-c/24} h(T)^\alpha + \cdots$$

$$\chi(T = 0, \tilde{\tau}, \tau, h) = 1 + \sum_{A=1}^{\dim g} q^{\tilde{\Delta}_A-\tilde{c}/24} q^{\Delta_A-c/24} h(T^{adj})^A + \cdots$$

For the vacuum bicharacter in (3.8b), the computation of the non-leading terms was performed in the L-basis $J_A(-1)|0\rangle$ of the one-current states, so that $\tilde{\Delta}_A$ and $\Delta_A$ (with $\tilde{\Delta}_A + \Delta_A = \Delta_g = 1$) are the conformal weights of these states under $\tilde{T}$ and $T$. 

7
4. The Affine-Virasoro Ward Identities

4.1. Statement and Strategy

In this Section, we establish and study the affine-Virasoro Ward identities for the bicharacters, which have the form,

\[
\tilde{\partial}^q \partial^p \chi(T, \tilde{\tau}, \tau, h) \mid_{\tilde{\tau}=\tau} = D_{qp}(\tau, h) \chi_g(T, \tau, h)
\]

(4.1a)

\[
\partial \equiv \partial_\tau = 2\pi iq\partial_q, \quad \tilde{\partial} \equiv \partial_{\tilde{\tau}} = 2\pi i\tilde{q}\partial_{\tilde{q}}
\]

(4.1b)

where \(\chi_g\) is the affine-Sugawara character given in eq. (3.6).

The \(h\)-differential operators \(D_{qp}(\tau, h)\), called the affine-Virasoro connection moments, are defined by the formula,

\[
D_{qp}(\tau, h) \chi_g(T, \tau, h) = (2\pi i)^{q+p} Tr_T \left( q^{L_g(0)} - c_g/24 (\tilde{L}(0) - \tilde{c}/24)^q (L(0) - c/24)^p h \right)
\]

(4.2)

where the zero modes \(L(0), \tilde{L}(0)\) of the K-conjugate stress tensors are given in eq. (3.2). Note that the quantities on the right side of (4.2) are averages in the affine-Sugawara theory, so the connection moments may be computed in principle by the methods of Refs. [23,16].

Our strategy to obtain these results is as follows: According to the definition (3.1), the left side of (4.1) is equal to the right side of (4.2), so the Ward identities (4.1) follow if the right side of (4.2) is proportional to \(\chi_g\). This will emerge in the following method for the computation of the connection moments.

4.2. Proof by Computational Scheme

The following inductive algebraic scheme for the computation of affine-Sugawara averages is equivalent to the methods of Refs. [23,16].

We organize the problem in terms of the basic quantity,

\[
Tr_T \left( q^{L_g(0)} J_a(-n) O h \right), \quad n \in \mathbb{Z}
\]

(4.3)

where \(O\) is any vector in the enveloping algebra, and, for simplicity, we restrict the source \(h\) to those subgroups for which \(G/H\) is a reductive coset space. In this case, we may choose a basis \(a = 1, \ldots, \dim g = (A, I), \) in which

\[
f_{AB}^C = G_{AI} = 0
\]

(4.4a)

\[
A = 1, \ldots, \dim h, \quad I = 1, \ldots, \dim g/h
\]

(4.4b)
Then we have the relations,

\[ q^{L_{\Phi}(0)} J_a(-n) = q^n J_a(-n) q^{L_{\Phi}(0)} \]  

(4.5a)

\[ \Omega(h)_a^b = \left( \begin{array}{cc} \rho(h)B & 0 \\ 0 & \sigma(h)I \end{array} \right) \]  

(4.5b)

\[ hJ_A(-n) = \rho(h)B J_B(-n)h \quad , \quad hJ_I(-n) = \sigma(h)I^I J_J(-n)h \]  

(4.5c)

from (3.7) and (2.1). With these relations and cyclicity of the trace, the current in (4.3) can be moved first to the left, then to the right of the trace and finally to the left of the source. Rewriting \( OJ_a \) as the original product \( J_a O \) plus the commutator, we may solve the relation for the original quantities (4.3) to obtain the basic identities,

\[ Tr_T \left( q^{L_{\Phi}(0)} J_A(-n) O h \right) = \left( \frac{q^n \rho(h)}{1 - q^n \rho(h)} \right) A B Tr_T \left( q^{L_{\Phi}(0)} [O, J_B(-n)] h \right) , \quad n \neq 0 \]  

(4.6a)

\[ Tr_T \left( q^{L_{\Phi}(0)} J_I(-n) O h \right) = \left( \frac{q^n \sigma(h)}{1 - q^n \sigma(h)} \right) I J Tr_T \left( q^{L_{\Phi}(0)} [O, J_J(-n)] h \right) , \quad n \in \mathbb{Z} . \]  

(4.6b)

The identity (4.6a) does not hold for \( n = 0 \) since \( 1 - \rho(h) \) is not invertible. With these relations and the affine algebra we may iteratively reduce the number of currents in the trace by one, except for the zero modes \( J_A(0) \) of the \( h \)-currents.

To include the zero modes of the \( h \) current, we introduce \( h \)-vielbeins \( \bar{e}_i^A \) and \( e_i^A \),

\[ e_i(h) = -ih \partial_i h^{-1} = \bar{e}_i^A(h) J_A(0) \quad , \quad e_i(h) = -ih^{-1} \partial_i h = e_i^A(h) J_A(0) \]  

(4.7)

where \( i, A = 1, \ldots, \text{dim} \ h \), and Lie derivatives on \( h \),

\[ \bar{E}_A = -i \bar{e}_A^i \partial_i \quad , \quad E_A = -i e_A^i \partial_i \]  

(4.8a)

\[ [\bar{E}_A, \bar{E}_B] = i f_{AB}^C \bar{E}_C \quad , \quad [E_A, E_B] = i f_{AB}^C E_C \quad , \quad [\bar{E}_A, E_B] = 0 \]  

(4.8b)

where \( \bar{e}_A^i \) and \( e_A^i \) are the inverse \( h \)-vielbeins. From the definitions in (4.8a) we find that

\[ \bar{E}_A(h) h = -J_A(0) h \quad , \quad E_A(h) h = h J_A(0) \]  

(4.9a)

\[ \bar{E}_A(h) h(T) = -T_A h(T) \quad , \quad E_A(h) h(T) = h(T) T_A \]  

(4.9b)

and, using (4.3), we obtain the basic identity for the zero modes of the \( h \)-currents,

\[ Tr_T \left( q^{L_{\Phi}(0)} J_A(0) O h \right) = E_A(h) Tr_T \left( q^{L_{\Phi}(0)} O h \right) . \]  

(4.10)
Taken together, the relations (4.6) and (4.10) allow a reduction by one in the number of currents in any affine-Sugawara average.

Iterating this step, the averages on the right side of (4.2) may be reduced to differential operators on the one-current averages,

\[ \text{Tr}_T \left( q^{Lg(0)-c_g/24} J_A(0) h \right) = E_A(h) \chi_g(T, \tau, h) \]

and

\[ \text{Tr}_T \left( q^{Lg(0)-c_g/24} J_I(0) h \right) = 0 \]

which are proportional to the affine-Sugawara characters. This completes the proof of the affine-Virasoro Ward identities (4.1).

As an example, we have computed the first moment \( D_{01} \) of the \( L \) theory, using

\[ D_{01}(\tau, h) \chi_g(T, \tau, h) = 2\pi i \text{Tr}_T \left( q^{Lg(0)-c_g/24} (L(0) - c/24) h \right) \]

\[ L(0) = L^{ab} J_a(0) J_b(0) + 2L^{ab} \sum_{n>0} J_a(-n) J_b(n) \]

\[ = L^{AB} J_A(0) J_B(0) + L^{AI} (J_A(0) J_I(0) + J_I(0) J_A(0)) + L^{IJ} J_I(0) J_J(0) \]

\[ + 2 \sum_{n>0} (L^{AB} J_A(-n) J_B(n) + L^{AI} (J_A(-n) J_I(n) + J_I(-n) J_A(n)) + L^{IJ} J_I(-n) J_J(n)) \]

and the identities (4.6–11). The result is

\[ D_{01}(L, \tau, h) = 2\pi i \left( -c/24 + L^{AB} E_A(h) E_B(h) + L^{IJ} \left( \frac{\sigma(h)}{1 - \sigma(h)} \right)^{IK} (i f_{JK}^A E_A(h)) \right. \]

\[ + 2L^{AB} \sum_{n>0} \left( \frac{q^n \rho(h)}{1 - q^n \rho(h)} \right)_A^C (i f_{BC}^D E_D(h) + nG_{BC}) \]

\[ + 2L^{IJ} \sum_{n>0} \left( \frac{q^n \sigma(h)}{1 - q^n \sigma(h)} \right)_I^K (i f_{JK}^A E_A(h) + nG_{JK}) \left( i f_{JK}^A E_A(h) + nG_{JK} \right) \] .

Similarly, the result for \( D_{10} \) is obtained from (4.13) by the substitution \( L \rightarrow \tilde{L} \) and \( c \rightarrow \tilde{c} \).

4.3. The Affine-Sugawara Characters

Adding the (1,0) and (0,1) Ward identities, we find the heat equation for the affine-Sugawara characters,
\begin{equation}
\partial \chi_g(T, \tau, h) = D_g(\tau, h)\chi_g(T, \tau, h) \tag{4.14a}
\end{equation}

\begin{align*}
D_g(\tau, h) &= D_{01} + D_{10} \\
&= 2\pi i \left\{ -c_g/24 + L_g^{AB} E_A(h)E_B(h) + L_g^{IJ} \left( \frac{\sigma(h)}{1 - \sigma(h)} \right)_I^K (if_{JK}^A E_A(h)) \\
&\quad + 2L_g^{AB} \sum_{n>0} \left( \frac{q^n\rho(h)}{1 - q^n\rho(h)} \right) A^C E_D(h) + nG_{BC} \\
&\quad + 2L_g^{IJ} \sum_{n>0} \left( \frac{q^n\sigma(h)}{1 - q^n\sigma(h)} \right) I^K (if_{JK}^A E_A(h) + nG_{JK}) \right\} \tag{4.14b}.
\end{align*}

The affine-Sugawara characters can also be understood as the simplest examples of affine-Virasoro characters, obtained by choosing the simplest K-conjugate pair $\tilde{L} = 0$ and $L = L_g$. The heat-like equations in this case,

\begin{align*}
\partial \chi(T, \tilde{\tau}, \tau, h) &= D_g(\tau, h)\chi(T, \tilde{\tau}, \tau, h) \\
\tilde{\partial} \chi(T, \tilde{\tau}, \tau, h) &= 0 \tag{4.15a} \\
\chi(T, \tilde{\tau}, \tau, h) &= \chi(T, \tau, \tau, h) = \chi_g(T, \tau, h) \tag{4.15b}.
\end{align*}

are equivalent to (4.14).

When the subgroup $H \subset G$ is taken to be $G$ itself, we recover Bernard’s result [16] on a $G$-source,

\begin{align*}
\partial \chi_g(T, \tau, g) &= D_g(\tau, g)\chi_g(T, \tau, g) \tag{4.16a} \\
D_g(\tau, g) &= 2\pi i \left\{ -c_g/24 + L_g^{ab} E_a(g)E_b(g) \\
&\quad + 2L_g^{ab} \sum_{n>0} \left( \frac{q^n\Omega(g)}{1 - q^n\Omega(g)} \right) e^c (if_{bc}^d E_d(g) + nG_{bc}) \right\} \tag{4.16b}
\end{align*}

where $L_g^{ab}$ is given in eq. (2.6). For simple $g$, Bernard also gives the alternate form of the $g$-connection,

\begin{equation}
D_g(\tau, g) = \frac{2\pi i}{2k + Q_g} \left\{ -(2k + Q_g)c_g/24 + \eta^{ab} E_a(g)E_b(g) \\
&\quad + 2\eta^{ab} (E_a(g) \log \Pi(\tau, \Omega(g))) E_b(g) - 2kq\partial_q \log \Pi(\tau, \Omega(g)) \right\} \tag{4.17a}
\end{equation}

\begin{equation}
\Pi(\tau, M) = \prod_{n=1}^{\infty} \det(1 - q^nM) \tag{4.17b}
\end{equation}

which follows from (4.9b). The $\Pi$-function in (4.17b), which satisfies $\Pi(\tau, A\oplus B) = \Pi(\tau, A)\Pi(\tau, B)$, was first studied by Fegan [24].
For use below, we list a number of simple properties of the affine-Sugawara characters.

A. Evolution operator of \( g \). It is convenient to define the (invertible) evolution operator of \( g \),

\[
\Omega_g(\tau, \tau_0, h) = T e^{\int_{\tau_0}^{\tau} d\tau' D_g(\tau', h)}
\]

(4.18)

where \( T \) denotes \( \tau \)-ordered product. This operator is the unique solution of the heat equation and boundary condition,

\[
\partial_\tau \Omega_g(\tau, \tau_0, h) = D_g(\tau, h) \Omega_g(\tau, \tau_0, h) \quad \Omega_g(\tau, \tau_0, h) = 1 \quad \Omega_g(\tau, \tau_0, h) = \Omega_g^{-1}(\tau_0, \tau, h)
\]

(4.19a) (4.19b)

and hence \( \Omega_g \) determines the evolution of the affine-Sugawara characters,

\[
\chi_g(T, \tau, h) = \Omega_g(\tau, \tau_0, h) \chi_g(T, \tau_0, h).
\]

(4.20)

B. Conserved quantities. In addition to the heat equation (4.14), the affine-Sugawara characters also satisfy a number of \( h \)-differential equations, whose generic form is

\[
C^g(T, \tau, h) \chi_g(T, \tau, h) = 0
\]

(4.21a)

\[
C^g(T, \tau', h) = \Omega_g(\tau', \tau, h) C^g(T, \tau, h) \Omega_g(\tau, \tau', h)
\]

(4.21b)

The \( h \)-differential operators \( C^g \) are the conserved quantities of the affine-Sugawara characters. The simplest example of such relations is the \( T \)- and \( \tau \)-independent \( h \)-global Ward identity

\[
Q_A^g(h) \chi_g(T, \tau, h) = 0 \quad Q_A^g(h) = \bar{E}_A(h) + E_A(h) \quad A = 1, \ldots, \dim h
\]

(4.22)

but there are other “spatial” equations [17] of the form (4.21) which follow from the existence of singular vectors in the affine Verma module \( V_T \). The global Ward identity (4.22) tells us that, if desired, we may replace \( E_A \to -\bar{E}_A \) in \( D_g \) and more generally on the right side of any term in \( D_{qp} \).

C. Explicit form. The explicit form of the affine-Sugawara characters for integrable representation \( T \) of simple \( g \) is [10],

\[
\chi_g(T, \tau, h) = \frac{1}{\prod(\tau, \rho(h))} \sum_{T'} N^g_{\tau T'} Tr(h(T')) q^{\Delta(T') - \frac{c_g}{2}}
\]

(4.23)
where the $\Pi$-function is defined in eq. (4.17), and the sum is over the set of all unitary irreps $T$ of $g$. The coefficients in the sum satisfy

$$\mathcal{N}_{T'}^T = \begin{cases} \det \omega & \text{if } \lambda(T') = \omega(\lambda(T) + \rho) - \rho + (x + \tilde{h}_g) \sigma \\ 0 & \text{otherwise} \end{cases}$$

(4.24)

where $\lambda(T)$ is the highest weight of irrep $T$, $\omega$ is some element in the Weyl group of $g$, $\sigma$ is some element of the coroot lattice, $\rho$ is the Weyl vector, $x$ is the invariant level and $\tilde{h}_g$ is the dual Coxeter number. For $g = \oplus_I g_I$ and $T = \oplus_I T_I$, the affine-Sugawara characters are $\chi_g(T) = \prod_I \chi_{g_I}(T_I)$.

4.4. General Properties of the Connection Moments

The following properties of the connection moments $D_{qp}$ are easily established from their definition in (4.2).

A. Representation independence. Since the computational scheme in subsection 4.2 is independent of irrep $T$ of $g$, the connection moments $D_{qp}(\tilde{\tau}, \tau, h)$ are independent of irrep $T$. This means that the representation dependence of the bicharacters is determined entirely by their affine-Sugawara boundary condition $\chi(T, \tau, \tau, h) = \chi_g(T, \tau, h)$.

B. $\tilde{L}$ and $L$ dependence. The one-sided connection moments,

$$D_{q0}(\tilde{L}) \quad D_{0p}(L)$$

(4.25)

are functions only of $\tilde{L}$ and $L$ as shown, while the mixed moments $D_{qp}(\tilde{L}, L)$ with $q, p \geq 1$ are functions of both $\tilde{L}$ and $L$.

C. K-conjugation covariance. Under exchange of the K-conjugate ICFT’s, the connection moments exhibit the K-conjugation covariance,

$$D_{qp}(\tilde{L}, L) = D_{pq}(L, \tilde{L})$$

(4.26a)

$$D_{q0}(\tilde{L}) = D_{0q}(L)|_{L \to \tilde{L}}$$

(4.26b)

D. Consistency Relations. Define the $g$-covariant derivatives,

$$d_g f \equiv \partial f + f D_g(\tau) \quad \bar{d}_g f \equiv \bar{\partial} f + f D_g(\bar{\tau})$$

(4.27)

on any $f(\tau, \tau, h)$. Then the connection moments satisfy the consistency relations,

$$d_g D_{qp} = D_{q+1,p} + D_{q,p+1} \quad D_{00} = 1$$

(4.28)
in analogy to the consistency relations on the sphere. When \( q = p = 0 \), these relations reproduce the identity \( D_g = D_{10} + D_{01} \) in eq. (4.14). Following the development on the sphere, the consistency relations can be solved at each fixed value of \( q + p \) to express all \( D_{qp} \) in terms of the canonical sets \( \{ D_g, D_{0p} \} \) or \( \{ D_g, D_{q0} \} \).

E. Other relations. It appears that all the relations known for the connection moments on the sphere \([11,12]\) have their close counterparts on the torus. Among these, we list only the translation sum rule,

\[
\sum_{r,s=0}^{\infty} \frac{(\tau - \tau_0)^{r+s}}{r!s!} D_{r+q,s+p}(\tau_0) = D_{qp}(\tau) \Omega_g(\tau, \tau_0, h)
\] (4.29)

where \( \Omega_g \) is the evolution operator of \( g \), and the partially-factorized form of the bicharacters,

\[
\chi(T, \tilde{\tau}, \tau, h) = \sum_{q,p=0}^{\infty} \frac{(\tilde{\tau} - \tau_0)^q}{q!} C_{qp}(T, \tau_0, h) \frac{(\tau - \tau_0)^p}{p!} \quad (4.30a)
\]

\[
C_{qp}(T, \tau_0, h) \equiv D_{qp}(\tau_0, h) \chi_g(T, \tau_0, h)
\] (4.30b)

which follows from the Ward identities using (4.29). The right side of (4.30a) is independent of the regular reference point \( \tau_0 \).

Following the development on the sphere \([11]\), the eigenvectors of the matrix \( C_{qp}(\tau_0) \) give a factorization of the bicharacters,

\[
\chi(T, \tilde{\tau}, \tau, h) = \sum_{\nu} \chi_{\tilde{L}}^{\nu}(T, \tilde{\tau}, \tau_0, h) \chi_{L}^{\nu}(T, \tau, \tau_0, h)
\] (4.31)

into candidate conformal characters \( \chi_{\tilde{L}} \) and \( \chi_{L} \) of the \( \tilde{L} \) and the \( L \) theory respectively. On the sphere, the corresponding conformal correlators of \( \tilde{L} \) and \( L \) are covariant under the braid group, so the analogous form (4.31) should be studied for modular covariance on the torus. We will return to the subject of factorization in Sections 7 and 9.

5. Flat Connections on the Torus

Following Ref. [13], we may reexpress the Ward identities (4.1) as heat-like differential equations with flat connections.
Using Taylor series in $\tilde{\tau}$ or $\tau$ and the Ward identities, we first write the bicharacters in the two equivalent forms,

$$\chi(T, \tilde{\tau}, \tau, h) = \tilde{B}(\tilde{\tau}, \tau, h)\chi_g(T, \tau, h) = B(\tilde{\tau}, \tau, h)\chi_g(T, \tilde{\tau}, h) \quad (5.1a)$$

$$\tilde{B}(\tilde{\tau}, \tau, h) = \sum_{q=0}^{\infty} \frac{(\tilde{\tau} - \tau)^q}{q!} D_{q0}(\tau, h), \quad B(\tilde{\tau}, \tau, h) = \sum_{p=0}^{\infty} \frac{(\tau - \tilde{\tau})^p}{p!} D_{0p}(\tilde{\tau}, h). \quad (5.1b)$$

These forms show explicitly that the affine-Virasoro characters $\chi$ are completely determined given the affine-Sugawara characters $\chi_g$ and the connection moments, which appear in the (invertible) $h$-differential operators $\tilde{B}$ and $B$.

Then, by differentiation of $\chi(T, \tilde{\tau}, \tau, h)$, we obtain the heat-like differential equations for the bicharacters,

$$\tilde{D} = \partial \tilde{B} \tilde{B}^{-1}, \quad D = \partial B B^{-1} \quad (5.2c)$$

where the $h$-differential operators $\tilde{D}$ and $D$ are the affine-Virasoro connections of the heat-like system. Eq. (5.2a) defines the connections as non-linear functionals of the connection moments, and the connections may also be evaluated in principle from the formulae,

$$\tilde{D}(\tilde{\tau}, \tau, h)\chi(T, \tilde{\tau}, \tau, h) = 2\pi i \text{Tr}_T \left( \tilde{q}^{\tilde{L}(0)-c/24} q^{L(0)-c/24} (\tilde{L}(0) - c/24) h \right) \quad (5.3a)$$

$$D(\tilde{\tau}, \tau, h)\chi(T, \tilde{\tau}, \tau, h) = 2\pi i \text{Tr}_T \left( \tilde{q}^{\tilde{L}(0)-c/24} q^{L(0)-c/24} (L(0) - c/24) h \right) \quad (5.3b)$$

which follow from (3.1) and the heat-like system (5.2).

We note that, like the connection moments, the operators $\tilde{B}$, $B$ and the connections $\tilde{D}$, $D$ are independent of the representation $T$.

Following the development of Ref. [13], we find close analogues of the general relations known for the connections on the sphere.

A. Flat connections. To see that the connections $\tilde{D}, D$ are flat, define covariant derivatives,

$${\tilde d} f \equiv \partial f + f \tilde{D}, \quad {d} f \equiv \partial f + f D, \quad \forall f . \quad (5.4)$$

Then the flatness condition,

$${d}\tilde{D} = \tilde{d}D \quad (5.5)$$
is obtained by differentiation of the heat-like equations. The same condition is also obtained by differentiation of (5.3), using the heat-like equations.

B. Inversion formula. Note that the covariant derivatives commute

\[ [d, \tilde{d}] = 0 \]  

because the connections are flat, and that the inversion formula for the connection moments,

\[ D_{qp}(\tau, h) = \tilde{d}^q d^p 1|_{\tilde{\tau} = \tau} \]  

follows by multiple differentiation of the heat-like system. The formula (5.7), which is the inverse of (5.2c), expresses the connection moments in terms of the flat connections. As examples, we use the formula to list the first few moments,

\[ D_{00}(\tau) = 1 \]  
\[ D_{10}(\tau) = \tilde{D}(\tau, \tau) \quad , \quad D_{01}(\tau) = D(\tau, \tau) \]  
\[ D_{20}(\tau) = (\partial \tilde{D} + D^2)|_{\tilde{\tau} = \tau} \quad , \quad D_{02}(\tau) = (\partial D + D^2)|_{\tilde{\tau} = \tau} \]  
\[ D_{11}(\tau) = (\partial D + D\tilde{D})|_{\tilde{\tau} = \tau} = (\partial \tilde{D} + \tilde{D} D)|_{\tilde{\tau} = \tau}. \]  

As on the sphere, we note that the pinched connections (at $\tilde{\tau} = \tau$) are always equal to the first connection moments.

C. Evolution operators. It follows from (5.1) and (5.2c) that the operators $\tilde{B}, B$ in (5.1) are the (invertible) evolution operators of the flat connections,

\[ \tilde{B}(\tilde{\tau}, \tau, h) = \tilde{T} e^{\int_{\tilde{\tau}}^{\tau} d\tilde{\tau}' D(\tilde{\tau}', \tau, h)} \]  
\[ B(\tilde{\tau}, \tau, h) = T e^{\int_{\tilde{\tau}}^{\tau} d\tau' D(\tilde{\tau}, \tau', h)} \]  
\[ \partial \tilde{B} = \tilde{D} \tilde{B} \quad , \quad \partial B = DB \]  
\[ \tilde{B}(\tau, \tau, h) = B(\tilde{\tau}, \tilde{\tau}, h) = 1. \]  

where $\tilde{T}$ and $T$ are $\tilde{\tau}$ and $\tau$ ordering respectively. Moreover, the two forms of the bicharacter in (5.1a) show that the evolution operators of the flat connections are related by the evolution operator of $g$,

\[ \tilde{B}(\tilde{\tau}, \tau, h) = B(\tilde{\tau}, \tau, h) \Omega_g(\tilde{\tau}, \tau, h) \quad , \quad B(\tilde{\tau}, \tau, h) = \tilde{B}(\tilde{\tau}, \tau, h) \Omega_g(\tau, \tilde{\tau}, h) \]  

16
and hence the evolution operator of $g$ is composed of the evolution operators of the flat connections,

$$\Omega_g(\tilde{\tau}, \tau, h) = B^{-1}(\tilde{\tau}, \tau, h) \tilde{B}(\tilde{\tau}, \tau, h) \quad (5.11)$$

The identities (5.10) also imply the differential relations

$$(\tilde{d}_g - \tilde{D}) B = (d_g - D) \tilde{B} = 0 \quad (5.12)$$

which supplement the differential relations in (5.9c).

D. $\tilde{L}$ and $L$ dependence. According to (5.11) and (4.26), the evolution operators $\tilde{B}(\tilde{L})$ and $B(L)$ and the connections $\tilde{D}(\tilde{L})$ and $D(L)$ are functions of only $\tilde{L}$ and $L$ as shown.

E. K-conjugation covariance. The evolution operators and connections satisfy the K-conjugation covariance,

$$B(L, \tilde{\tau}, \tau, h)|_{\tilde{L} \to \tilde{L}} = \tilde{B}(\tilde{L}, \tilde{\tau}, \tau, h) \quad , \quad D(L, \tilde{\tau}, \tau, h)|_{\tilde{L} \to \tilde{L}} = \tilde{D}(\tilde{L}, \tilde{\tau}, \tau, h) \quad . (5.13)$$

F. Non-local conserved quantities [13]. Following subsection 4.3 and the development on the sphere, we find a non-local conserved quantity $C(T, \tilde{\tau}, \tau, h)$,

$$C(T, \tilde{\tau}, \tau, h) \chi(T, \tilde{\tau}, \tau, h) = 0 \quad (5.14a)$$

$$C = BC_g(\tilde{\tau}) B^{-1} = \tilde{B} C_g(\tau) \tilde{B}^{-1} \quad (5.14b)$$

for each of the conserved quantities $C_g(T, \tau, h)$ of the affine-Sugawara character on an $h$ source. These non-local conserved quantities are the lift of the conserved quantities $C_g$ into the space of ICFT’s.

Because they are related by a similarity transformation, the algebra of the non-local quantities is the same as the algebra of the $C_g$’s. For example, we have the non-local conserved generators $Q_A(\tilde{\tau}, \tau, h)$ of $h \subset g$,

$$Q_A = BQ_A^g B^{-1} = \tilde{B} Q_A^g \tilde{B}^{-1} \quad (5.15a)$$

$$Q_A \chi = 0 \quad , \quad [Q_A, Q_B] = i f_{ABC} Q_C \quad (5.15b)$$

where $Q_A^g = \tilde{E}_A + E_A$, $A = 1, \ldots, \text{dim } h$ are the global generators of $h \subset g$. As on the sphere, the non-local generators of $h \subset g$ degenerate to the global generators of $h \subset g$ when $h$ is an ordinary Lie symmetry of the K-conjugate pair. We will check this explicitly for $h$ and the $g/h$ coset constructions in the following section, but it is also true for all the Lie $h$-invariant CFT’s [18].

We also find non-local conserved quantities $C$ associated to the $C_g$’s of the null states of the affine modules [17]. These conserved quantities provide further differential relations on the bicharacters, beyond the heat equations, but we will not need their explicit form here. Instead, we encode their information in the bicharacters by the choice of the correct affine-Sugawara characters $\chi_g$ in the affine-Sugawara boundary condition.
6. Coset Constructions

6.1. Choosing the Source

In this Section, we study the bicharacters of $h$ and the $g/h$ coset constructions \cite{2,5,7},

$$\tilde{L} = L_{g/h} \quad , \quad L = L_h$$

(6.1)

with $G/H$ a reductive coset space. In this case, we are able to solve the system exactly by choosing the subgroup $H$ of the source $h$ to be the same subgroup involved in the $G/H$ coset. Then, the bicharacter and its heat-like equations have the form,

$$\chi(T, \tilde{\tau}, \tau, h) = Tr_T \left( q^{L_{g/h}(0) - c_{g/h}/24} q^{L_h(0) - c_h/24} \right)$$

(6.2a)

$$\tilde{\partial} \chi = \tilde{D}(L_{g/h}) \chi \quad , \quad \partial \chi = D(L_h) \chi$$

(6.2b)

where $\tilde{D}(L_{g/h})$ and $D(L_h)$ are the connections of $g/h$ and $h$ respectively. Evaluation of the bicharacters of $h$ and $g/h$ on sources larger than $H$ is more complicated, as discussed below and in the following Section.

6.2. The Subgroup Connection

We evaluate $D(L_h)$ from the general form \cite{3b}, which now reads,

$$D(L_h, \tau, h) \chi(T, \tilde{\tau}, \tau, h) = 2\pi i Tr_T \left( q^{L_{g/h}(0) - c_{g/h}/24} q^{L_h(0) - c_h/24} (L_h(0) - c_h/24) \right)$$

(6.3a)

$$L_h(0) = L_h^{AB} (J_A(0)J_B(0) + 2 \sum_{n>0} J_A(-n)J_B(n)) \quad , \quad L_{g/h}(0) = L_g(0) - L_h(0)$$

(6.3b)

where $A, B = 1, \ldots, \dim h$ and $L_g(0)$ is given in eq. (2.6).

In this case, we are able to follow the strategy of subsection 4.2 exactly. We need the relations,

$$q^{L_h(0)} J_A(-n) = q^n J_A(-n) q^{L_h(0)}$$

(6.4a)

$$\tilde{q}^{L_{g/h}(0)} J_A(-n) = J_A(-n) \tilde{q}^{L_{g/h}(0)}$$

(6.4b)

$$hJ_A(-n) = \rho(h) A^B J_B(-n) h$$

(6.4c)

$$E_A h = h J_A(0)$$

(6.4d)
where (6.4a, b) records that the $h$-currents $J_A$ have conformal weights 1 and 0 respectively under $T_h$ and $T_{g/h}$. Then, we obtain the formulae,

$$\text{Tr}_T \left( \tilde{q}^{L_{g/h}(0)} q^{L_h(0)} J_A(-n) \mathcal{O} h \right) = \left( \frac{q^n \rho(h)}{1 - q^n \rho(h)} \right)^B_A \text{Tr}_T \left( \tilde{q}^{L_{g/h}(0)} q^{L_h(0)} [\mathcal{O}, J_B(-n)] h \right), \quad n \neq 0 \quad (6.5a)$$

$$\text{Tr}_T \left( \tilde{q}^{L_{g/h}(0)} q^{L_h(0)} J_A(0) \mathcal{O} h \right) = E_A \text{Tr}_T \left( \tilde{q}^{L_{g/h}(0)} q^{L_h(0)} \mathcal{O} h \right) \quad (6.5b)$$

for any operator $\mathcal{O}$. Note that, had we chosen a source in $G$ or a non-reductive coset space, eq. (6.4c) would have contained extra terms with coset currents $J_I$, so that averages of the $h$-currents no longer satisfy closed equations. We will return to the subject of sources larger than $H$ in Section 9.

Using (6.3) and (6.5), we obtain the exact $h$-connection,

$$D(L_h, \tau, h) = 2\pi i \left\{ -c_h/24 + L_h^{AB} E_A(h) E_B(h) + 2L_h^{AB} \sum_{n>0} \left( \frac{q^n \rho(h)}{1 - q^n \rho(h)} \right)^C_A \left( i \hat{f}_{BC} D_E(h) + nG_{BC} \right) \right\}. \quad (6.6)$$

The $h$-connection is not a function of $\tilde{\tau}$, so that, according to eq. (5.8d), the $h$-connection is equal to the first connection moment of the $h$-theory,

$$D_h(\tau, h) \equiv D(L_h, \tau, h) = D_{01}(L_h, \tau, h) \quad . \quad (6.7)$$

This relation is easily checked by comparison with the general first moment result (4.13) in this case.

We also observe the embedding relation between the connections of $g$ and $h$,

$$D_h(L_h, \tau, h) = D_g(L_g, \tau, g)|_{L_g \to L_h} \quad (6.8)$$

which follows on comparison of eqs. (4.16d) and (6.6). Following Ref. [13], such embedding relations can be used to compute the connections of all the affine-Sugawara nests on $g \supset h_1 \ldots \supset h_n$, but we will limit our work here to the simplest case of the coset constructions.
6.3. The Coset Connection

Having determined the $h$-connection $D(L_h) = D_h$ by direct computation, we may obtain the coset connection $\tilde{D}(L_{g/h})$ by solving the flatness condition (5.5), which now reads

$$\partial \tilde{D}(L_{g/h}) = [D_h, \tilde{D}(L_{g/h})]$$

(6.9)

because $\tilde{D}D_h = 0$. The flatness condition (6.9) determines the $\tau$ dependence of the coset connection as

$$\tilde{D}(L_{g/h}, \tilde{\tau}, \tau, h) = \Omega_h(\tau, \tilde{\tau}, h)\tilde{D}(L_{g/h}, \tilde{\tau}, \tilde{\tau}, h)\Omega_h^{-1}(\tau, \tilde{\tau}, h)$$

(6.10)

where $\Omega_h$ is the (invertible) evolution operator of $h$, which satisfies,

$$\Omega_h(\tau, \tilde{\tau}, h) = T e^{\int_\tau^{\tilde{\tau}} d\tau' D_h(\tau', h)}$$

(6.11a)

$$\partial \Omega_h(\tau, \tilde{\tau}, h) = D_h(\tau, h)\Omega_h(\tau, \tilde{\tau}, h)$$

$$\tilde{\partial} \Omega_h(\tau, \tilde{\tau}, h) = -\Omega_h(\tau, \tilde{\tau}, h)D_h(\tilde{\tau}, h)$$

(6.11b)

$$\Omega_h(\tilde{\tau}, \tilde{\tau}, h) = 1$$

$$\Omega_h^{-1}(\tau, \tilde{\tau}, h) = \Omega_h(\tilde{\tau}, \tau, h)$$

(6.11c)

in analogy to the evolution operator of $g$. The quantity $\tilde{D}(L_{g/h}, \tilde{\tau}, \tau, h)$ in (6.10) is the pinched coset connection, whose form is known from eqs. (5.8b) and (5.3b),

$$\tilde{D}(L_{g/h}, \tilde{\tau}, \tau, h) = D_{10}(L_{g/h}, \tilde{\tau}, h) = D_g(\tilde{\tau}, h) - D_h(\tilde{\tau}, h) \equiv D_{g/h}(\tilde{\tau}, h)$$

(6.12)

Combining eqs. (6.10) and (6.12), we obtain the coset connection,

$$\tilde{D}(L_{g/h}, \tilde{\tau}, \tau, h) = \Omega_h(\tau, \tilde{\tau}, h)D_{g/h}(\tilde{\tau}, h)\Omega_h^{-1}(\tau, \tilde{\tau}, h)$$

(6.13)

which, as on the sphere [11–13], is an $h$-dressing of the first coset connection moment.

6.4. Connection Moments of $h$ and $g/h$

Having obtained the $h$ and $g/h$ connections $D(L_h) = D_h$ and $D(L_{g/h})$, we may compute the connection moments $D_{qp}$ of $h$ and $g/h$ from the inversion formula (5.7),

$$D_{qp} = \tilde{d}^q d^p 1|_{\tilde{\tau}=\tau}$$

(6.14)

In this case, the inversion formula simplifies to the factorized form,

$$D_{qp}(\tau) = D_{0p}^h(\tau)D_{q0}^{g/h}(\tau)$$

$$D_{0p}^h(\tau) \equiv d^p 1|_{\tilde{\tau}=\tau}$$

$$D_{q0}^{g/h}(\tau) \equiv \tilde{d}^q 1|_{\tilde{\tau}=\tau}$$

(6.15a)
because $\partial D_h = 0$. The result (6.13) is analogous to the factorization of the connection moments of $h$ and $g/h$ on the sphere [1][2].

Computation of the moments from the factorized form (6.13) is particularly simple, and we list the examples,

$$D^h_{01} = D_h, \quad D^{g/h}_{10} = D_{g/h}$$  (6.16a)
$$D^h_{02} = \partial D_h + D^2_h, \quad D_{11} = D_h D_{g/h}$$  (6.16b)
$$D^{g/h}_{20} = \partial D_{g/h} + D^2_{g/h} + [D_{g/h}, D_h]$$  (6.16c)

through order $q + p \leq 2$.

6.5. The Bicharacters of $h$ and $g/h$

Given the flat connections $D(L_h)$ and $D(L_{g/h})$, the bicharacters $\chi(\tilde{\tau}, \tau)$ are the unique solution to the heat-like system (5.2b) with the affine-Sugawara boundary condition $\chi(\tau, \tau) = \chi_g(\tau)$.

To find this solution quickly, use $D = D_h$ and eq. (5.10) to obtain the evolution operators of the flat connections,

$$B(\tilde{\tau}, \tau, h) = \Omega_h(\tau, \tilde{\tau}, h), \quad \tilde{B}(\tilde{\tau}, \tau, h) = \Omega_h(\tau, \tilde{\tau}, h)\Omega_g(\tilde{\tau}, \tau, h)$$  (6.17)

Then the bicharacters of $h$ and $g/h$,

$$\chi(T, \tilde{\tau}, \tau, h) = \Omega_h(\tau, \tilde{\tau}, h)\chi_g(T, \tilde{\tau}, h)$$  (6.18)

follow immediately as a special case of the general result (5.1d).

Using the heat equations (4.14) and (6.11) for $\Omega_h$ and $\chi_g$, it is easy to check directly that the bicharacters (6.18) solve the heat-like system (5.2b).

We find,

$$\partial \chi = D(L_h)\chi$$  (6.19a)
$$\tilde{\partial} \chi = \Omega_h(\tau, \tilde{\tau}, h)D_{g/h}(\tilde{\tau}, h)\chi_g(T, \tilde{\tau}, h)$$  (6.19b)
$$= \Omega_h(\tau, \tilde{\tau}, h)D_{g/h}(\tilde{\tau}, h)\Omega_h^{-1}(\tau, \tilde{\tau}, h)\chi(T, \tilde{\tau}, \tau, h)$$  (6.19c)
$$= D(L_{g/h})\chi$$  (6.19d)

where the flat connections $D_h = D(L_h)$ and $\tilde{D}(L_{g/h})$ are given in eqs. (6.6) and (6.13).
6.6. Non-local Conserved Quantities

Using the $h$-transformation properties of the adjoint representation,

$$E_A \rho(h)_B^C = \rho(h)_B^D (T_A^\text{adj})_D^C, \quad E_A \rho(h)_B^C = -(T_A^\text{adj})_B^D \rho(h)_D^C$$  \hfill (6.20)

we verify the $h$-invariance of the $h$-evolution operator and the connections of $h$ and $g/h$,

$$[Q_A^g(h), D_h(\tau, h)] = [Q_A^g(h), \Omega_h(\tau, \tilde{\tau})] = [Q_A^g(h), \tilde{D}(L_{g/h}, \tilde{\tau}, \tau)] = 0$$ \hfill (6.21)

where $Q_A^g(h) = \hat{E}_A(h) + E_A(h)$ are the global generators of $h \subset g$. Then, the non-local conserved generators (5.15) of $h$ and $g/h$,

$$Q_A(\tilde{\tau}, \tau, h) \chi(T, \tilde{\tau}, \tau, h) = 0, \quad A = 1, \ldots, \dim h$$ \hfill (6.22a)

$$Q_A(\tilde{\tau}, \tau, h) = \Omega_h(\tau, \tilde{\tau}, h) Q_A^g(h) \Omega_h^{-1}(\tau, \tilde{\tau}, h) = Q_A^g(h)$$ \hfill (6.22b)

are equal to the global generators of $h \subset g$. If we choose the source in $G$, we still find $Q_A = Q_A^g$ because $\hat{B}$ and $B$ are $h$-invariant. On the other hand, the extra conserved coset generators $Q_I = B Q_I^g B^{-1}$, $I = 1, \ldots, \dim g/h$ remain non-local on the $G$ source, in parallel with results on the sphere $\mathbb{R}$.

7. Integral Representation for Coset Characters

To further analyze the bicharacters of $h$ and $g/h$, we introduce the $h$-characters for integrable representation $T^h$ on an $h$ source,

$$\chi_h(T^h, \tau, h) = Tr_{\tau} \left( q^{L_h(0) - c_h/24} h \right)$$ \hfill (7.1a)

$$= \frac{1}{\Pi(\tau, \rho(h))} \sum_{T^{\tau h}} N_{T^{\tau h}}^h Tr(h(T^{\tau h})) q^{\Delta_h(T^{\tau h}) - c_h/24}$$ \hfill (7.1b)

where $h$ is a simple subalgebra of $g$, the sum is over all the unitary irreps of $h$ and $N_{T^{\tau h}}^h$ is the $h$-analogue of $N_{T^{\tau h}}^T$ in (4.24). The connection between the characters of $T$ (irrep of $g$) and $T^h$ (irrep of $h$) is

$$Tr(h(T)) = \sum_{T^h} m(T, T^h) Tr(h(T^h))$$ \hfill (7.2)

where $m(T, T^h)$ is the multiplicity of irrep $T^h$ in irrep $T$.  \hfill 22
The $\hat{h}$-characters satisfy the heat and evolution equations,

$$\partial \chi_h(T^h, \tau, h) = D_h(\tau, h) \chi_h(T^h, \tau, h)$$  \hspace{1cm} (7.3a)$$
$$\chi_h(T^h, \tau', h) = \Omega_h(\tau', \tau, h) \chi_h(T^h, \tau, h)$$  \hspace{1cm} (7.3b)$$

where $D_h$ is the same $h$-connection which controls the $\tau$ dependence of the bicharacter. It is therefore reasonable to assume that the bicharacter lives in the space spanned by the $\hat{h}$-characters,

$$\chi(T, \tilde{\tau}, \tau, h) = \sum_{T_h} '\chi_{g/h}(T, T^h, \tilde{\tau}) \chi_h(T^h, \tau, h)$$  \hspace{1cm} (7.4)$$

which solves the bicharacter equation $\partial \chi = D_h \chi$ so long as the coset characters $\chi_{g/h}$ are independent of the source. In (7.4), the primed sum is over the integrable representations $T^h$ of $h$ at the induced level of the subalgebra.

At $\tau = \tilde{\tau}$, the factorized form (7.4) implies the known factorization of the affine-Sugawara characters $[25, 26, 27]$,

$$\chi_g(T, \tilde{\tau}, h) = \sum_{T_h} '\chi_{g/h}(T, T^h, \tilde{\tau}) \chi_h(T^h, \tilde{\tau}, h)$$  \hspace{1cm} (7.5)$$

and, conversely, using (7.3) in the bicharacter solution (6.18), we recover (7.4) in the steps,

$$\chi(T, \tilde{\tau}, \tau, h) = \Omega_h(\tau, \tilde{\tau}, h) \sum_{T_h} '\chi_{g/h}(T, T^h, \tilde{\tau}) \chi_h(T^h, \tilde{\tau}, h)$$  \hspace{1cm} (7.6a)$$
$$\quad = \sum_{T_h} '\chi_{g/h}(T, T^h, \tilde{\tau}) \Omega_h(\tau, \tilde{\tau}, h) \chi_h(T^h, \tilde{\tau}, h)$$  \hspace{1cm} (7.6b)$$
$$\quad = \sum_{T_h} '\chi_{g/h}(T, T^h, \tilde{\tau}) \chi_h(T^h, \tau, h)$$  \hspace{1cm} (7.6c)$$

To obtain (7.6b), we used the fact that the coset characters are independent of the source.

To check that the factorized form (7.4) also satisfies the $\bar{D}$ equation, follow the steps,

$$\Omega_h(\tilde{\tau}, \tau, h)(\bar{\partial} - \bar{D}(L_{g/h}, \tilde{\tau}, \tau, h)) \chi(T, \tilde{\tau}, \tau, h)$$  \hspace{1cm} (7.7a)$$
$$\quad = \sum_{T_h} ' (\bar{\partial} \chi_{g/h}(T, T^h, \tilde{\tau}) - \chi_{g/h}(T, T^h, \tilde{\tau}) D_{g/h}(\tilde{\tau}, h)) \chi_h(T^h, \tilde{\tau}, h)$$  \hspace{1cm} (7.7b)$$
$$\quad = (\bar{\partial} - D_g(T, \tilde{\tau}, h)) \chi_g(T, \tilde{\tau}, h) = 0$$  \hspace{1cm} (7.7c)
where we used the form \( (6.13) \) of the coset connection \( \tilde{D}(L_{g/h}) \) and the heat equations on \( g \) and \( h \). The equation \( \tilde{\partial}\chi = \tilde{D}\chi \) is then satisfied because \( \Omega_h \) is invertible.

We may now obtain linear differential equations for the coset characters as follows. From the definition (4.24) of the coefficients \( N_{g/h}^{\tau h} \), it follows that \( 28 \),

\[
N_{g/h}^{\tau h} N_{g/h}^{\tau'h} = \delta(T^h, T'^h)|N_{g/h}^{\tau h}|
\]

where \( \delta \) is Kronecker delta, and we know that

\[
\int dh \; Tr(h^* (T'^h)) Tr(h(T^h)) = \delta(T'^h, T^h)
\]

(7.9)

where \( dh \) is Haar measure on \( h \). Using (7.8) and (7.9), we verify the orthonormality relation for \( h \)-characters,

\[
\int dh \chi_h^\dagger(T'^h, \tau, h) \chi_h(T^h, \tau, h) = \delta(T'^h, T^h)
\]

(7.10a)

\[
\chi_h^\dagger(T^h, \tau, h) \equiv \frac{\Pi(\tau, \rho(h))}{f(T^h, q)} \sum_{T'^h} N_{T'^h}^T Tr(h^* (T'^h)) q^{\Delta_h(T'^h)} + c_h/24
\]

(7.10b)

\[
f(T^h, \tau) \equiv \sum_{T'^h} |N_{T'^h}^T| q^{2\Delta_h(T'^h)}
\]

(7.10c)

With \( h \to g \), \( \rho(h) \to \Omega(g) \) and \( T'^h \to T \), these relations apply as well for the \( \tilde{g} \)-character \( \chi_g(T, \tau, g) \) on a \( G \) source.

Then, integrating eq. (7.10b) with \( \int dh \chi_h^\dagger(T'^h, \tilde{\tau}, h) \), we obtain the coset equations,

\[
\tilde{\partial}\chi_{g/h}(T, T'^h, \tilde{\tau}) = \sum_{T'^h} 'w[L_{g/h}, T^h, T'^h, \tilde{\tau}] \chi_{g/h}(T, T'^h, \tilde{\tau})
\]

(7.11a)

\[
w[L_{g/h}, T^h, T'^h, \tilde{\tau}] = \int dh \chi_h^\dagger(T^h, \tilde{\tau}, h) D_{g/h}(\tilde{\tau}, h) \chi_h(T'^h, \tilde{\tau}, h)
\]

(7.11b)

where \( D_{g/h} \), given in eq. (6.12), is the first connection moment of the coset construction. These equations are the analogue of the coset equations in the block basis on the sphere \( [11] \), and we note that, as on the sphere, the \( c \)-function coset coefficients \( w[L_{g/h}] \) in (7.11b) are an \( h \)-dressing of the first coset connection moment.

The correct solutions of the coset equations (which respect the affine cutoff of \( \tilde{g} \) and \( \hat{h} \)) are most easily obtained by the same projection on eq. (7.11). We obtain an integral representation for the general \( g/h \) coset character,

\[
\chi_{g/h}(T, T'^h, \tilde{\tau}) = \int dh \chi_h^\dagger(T^h, \tilde{\tau}, h) \chi_g(T, \tilde{\tau}, h)
\]

(7.12a)

\[
= \frac{\tilde{q}^{-\gamma_{g/h}}}{f(T'^h, \tilde{\tau})} \sum_{T', T'^h} N_{T'}^T N_{T'^h}^T \left( \int dh \frac{Tr(h^* (T'^h)) Tr(h(T'))}{\Pi(\tilde{\tau}, \sigma(h))} \right) \tilde{q}^{\Delta_{g}(T') + \Delta_h(T'^h)}
\]

(7.12b)
The general result (7.12a), which we have been unable to find in the literature, holds for semi-simple \( g \) and simple \( h \). In form, this result is the analogue of the formula \( C_{g/h} = \mathcal{F}_g \mathcal{F}_h^{-1} \) for the coset blocks on the sphere \([14,11,12]\). The special case in (7.12b) is the explicit form of (7.12a) for simple \( g \).

8. High-level Affine-Virasoro Characters

8.1. High-level Systematics

In this Section, we consider the high-level affine-Virasoro characters for the broad class of ICFT’s which are high-level smooth on simple \( g \). In this case, the high-level forms of the inverse inertia tensors are \([20,21]\),

\[
\tilde{L}^{ab} = \frac{\tilde{P}^{ab}}{2k} + O(k^{-2}) \quad , \quad L^{ab} = \frac{P^{ab}}{2k} + O(k^{-2}) \quad (8.1a)
\]

\[
\tilde{c} = \text{rank} \tilde{P} + O(k^{-1}) \quad , \quad c = \text{rank} P + O(k^{-1}) \quad (8.1b)
\]

where \( \tilde{P} \) and \( P \) are the high-level projectors of the \( \tilde{L} \) and the \( L \) theory respectively.

According to \([4.2], (8.1) \) and the affine algebra \([2.1]\), we see that each new factor \( \tilde{L} \) or \( L = O(k^{-1}) \) in \( D_{qp} \) comes with two more currents and hence with the possibility of one more current contraction, proportional to the central term \( G_{ab} = k\eta_{ab} \). It follows that all \( D_{qp} \) begin at the same order of \( k^{-1} \). Since \( D_{00} = 1 \), we conclude that \( D_{qp}, \tilde{D} \) and \( D \) are power series in \( k^{-1} \) with leading terms,

\[
\{ D_{qp}, \tilde{D}, D \} = O(k^0) \quad (8.2)
\]

which come entirely from current contractions.

8.2. High-level Flat Connections

More explicitly, we will evaluate the high-level form of the flat connection \( \tilde{D} \), using the high-level form of (5.3),

\[
\tilde{D}(\tilde{\tau}, \tau, g)\chi(T, \tilde{\tau}, \tau, g) = \frac{2\pi i Tr_T}{k} \left( \eta^{L(0)} - \text{rank} \tilde{P} / 24 \eta^{L(0)} - \text{rank} P / 24 \left( \frac{\tilde{P}^{ab}}{k} \sum_{n > 0} J_a(-n) J_b(n) - \frac{\text{rank} \tilde{P}}{24} g \right) \right) \quad (8.3)
\]
for general source \( g \in G \). Here, we have already used the high-level form of \( \tilde{L}^{ab} \) and we have neglected the zero-mode contribution of the currents, which is automatically higher order.

To proceed, we need the high-level form of the \( T, J \) commutator in (2.9),

\[
[L(0), J_a(-n)] = n(P J(-n))_a, \quad [\tilde{L}(0), J_a(-n)] = n(\tilde{P} J(-n))_a
\]

where \( (P J)_a \equiv P^b_a J_b \) and \( P^b_a \equiv \eta_{ac} P^{cb} \). This gives the high-level analogue of eq. (4.5a),

\[
\tilde{q} L(0) q L(0) J_a(-n) = kn \left\{ \left( \frac{\tilde{q}^n P + q^n P}{1 - (\tilde{q}^n P + q^n P)\Omega(g)} \right)_a \right\} \chi(T, \tilde{\tau}, \tau, g)
\]

and then we may follow the usual procedure to express the \( J_a J_b \) trace in terms of the commutator of the two currents. Keeping only the contraction term in the commutator, we obtain the relation,

\[
\text{Tr}_r \left( (\tilde{q} L(0) - c/24 q L(0)) J_a(-n) J_b(n) g \right) = kn \left\{ \left( \frac{\tilde{q}^n P + q^n P}{1 - (\tilde{q}^n P + q^n P)\Omega(g)} \right)_a \right\} \chi(T, \tilde{\tau}, \tau, g)
\]

where \( \Omega(g) = g^{-1}(T^{adj}) \) is the adjoint representation of \( G \).

Using (8.3) and (8.4), we read off the leading terms of the flat connections,

\[
\tilde{D}(\tilde{L}, \tilde{\tau}, \tau, g) = 2\pi i \left( \sum_{n > 0} n \text{Tr} \left( \frac{X_n}{1 - X_n} \tilde{P} \right) - \frac{\text{rank} \tilde{P}}{24} \right) + O(k^{-1})
\]

\[
D(L, \tilde{\tau}, \tau, g) = 2\pi i \left( \sum_{n > 0} n \text{Tr} \left( \frac{X_n}{1 - X_n} P \right) - \frac{\text{rank} P}{24} \right) + O(k^{-1})
\]

\[
X_n(\tilde{\tau}, \tau, g) \equiv (\tilde{q}^n P + q^n P)\Omega(g)
\]

where the result for \( D \) follows from that for \( \tilde{D} \) and the K-conjugation covariance of the connections in (5.13). We note in particular that the leading terms (8.7) in the flat connections are functions, so that their differential structure begins at \( O(k^{-1}) \).

It is instructive to check that these connections are flat, that is \( d\tilde{D} = dD \). Using the high-level forms in (8.7), we know that

\[
[D, \tilde{D}] = 0
\]

because the high-level connections are functions. Then, we need only check that the high-level connections are abelian flat,

\[
\partial \tilde{D} = \tilde{\partial} D
\]
This property, and hence the flatness of the high-level connections, follows from the identities,
\[ q \partial_q Tr \left\{ \left( \frac{X_n}{1 - X_n} \right) \tilde{P} \right\} = \tilde{q} \partial_{\tilde{q}} Tr \left\{ \left( \frac{X_n}{1 - X_n} \right) P \right\} = n(\tilde{q}q)^n Tr \left\{ \tilde{P} \Omega \frac{1}{1 - X_n} P \Omega \frac{1}{1 - X_n} \right\} \]

(8.10)

which are easily checked by differentiation. We remark that the high-level flat connections of ICFT on the sphere are also abelian-flat [13].

8.3. High-level Bicharacters

Having determined the high-level flat connections $\tilde{D}$ and $D$, we may integrate eq. (5.3) to obtain the high-level evolution operators $\tilde{B}$ and $B$ of the flat connections,
\[ \tilde{B}(\tilde{\tau}, \tau, g) = \left( \frac{q}{\tilde{q}} \right)^{\frac{\text{rank}P}{24}} \prod_{n=1}^{\text{rank}P} e^{2\pi i n \int_{\tau}^{\tilde{\tau}} d\tilde{\tau}' Tr \left\{ \left( \frac{X_n(\tilde{\tau}', \tau, g)}{1 - X_n(\tilde{\tau}', \tau, g)} \right) \tilde{P} \right\}} + O(k^{-1}) \]  

(8.11a)

\[ B(\tilde{\tau}, \tau, g) = \left( \frac{\tilde{q}}{q} \right)^{\frac{\text{rank}P}{24}} \prod_{n=1}^{\text{rank}P} e^{2\pi i n \int_{\tau}^{\tilde{\tau}} d\tau' Tr \left\{ \left( \frac{X_n(\tilde{\tau}', \tau, g)}{1 - X_n(\tilde{\tau}', \tau, g)} \right) P \right\}} + O(k^{-1}) \]  

(8.11b)

\[ X_n(\tilde{\tau}, \tau, g) = \left( e^{2\pi i n \tilde{\tau}} \tilde{P} + e^{2\pi i n \tau} P \right) \Omega(g) . \]  

(8.11c)

Finally, we may substitute the results (8.11) into eq. (5.1a) to obtain the high-level forms of the low-spin affine-Virasoro characters,
\[ \chi(T, \tilde{\tau}, \tau, g) = \left( \frac{q}{\tilde{q}} \right)^{\frac{\text{rank}P}{24}} \prod_{n=1}^{\text{rank}P} e^{2\pi i n \int_{\tau}^{\tilde{\tau}} d\tilde{\tau}' Tr \left\{ \left( \frac{X_n(\tilde{\tau}', \tau, g)}{1 - X_n(\tilde{\tau}', \tau, g)} \right) \tilde{P} \right\}} \frac{Tr(g(T))}{\Pi(\tau, \Omega(g))} \]  

(8.12a)

\[ = \left( \frac{\tilde{q}}{q} \right)^{\frac{\text{rank}P}{24}} \prod_{n=1}^{\text{rank}P} e^{2\pi i n \int_{\tau}^{\tilde{\tau}} d\tau' Tr \left\{ \left( \frac{X_n(\tilde{\tau}', \tau, g)}{1 - X_n(\tilde{\tau}', \tau, g)} \right) P \right\}} \frac{Tr(g(T))}{\Pi(\tilde{\tau}, \Omega(g))} \]  

(8.12b)

where low spin means that the invariant Casimir of irrep $T$ is $O(k^0)$ (and so the conformal weights of irrep $T$ are $O(k^{-1})$). To obtain this result, we also used the high-level form of the low-spin affine-Sugawara characters,
\[ \chi_g(T, \tau, g) = q^{-\frac{\text{dim}g}{24}} \frac{Tr(g(T))}{\Pi(\tau, \Omega(g))} \]  

(8.13)

which follows from their explicit form in (4.23.) The results (8.12) and (8.13) are the leading terms of the high-level asymptotic expansion of these quantities.

As a check on the high-level bicharacters (8.12), we note the simple intuitive result at unit source,
\[ \chi(T, \tilde{\tau}, \tau, g = 1) = \frac{\text{dim}T}{\eta(\tilde{\tau})^{\text{rank}P} \eta(\tau)^{\text{rank}P}} \]  

(8.14)

where $\eta$ is the Dedekind $\eta$-function.
9. When the Source is the Symmetry Group

9.1. Source Dependence of $h$ and $g/h$

We return to the case of $h$ and the $g/h$ coset constructions, now on a general source $g \in G$, whose high-level connections and bicharacters are included in the results above. The answers for $h$ and $g/h$ can be obtained for any of the results of Section 8 by choosing,

$$\tilde{P} = P_{g/h} = 1 - P_h, \quad P = P_h$$

(9.1)

where $P_h$ is the projector onto $h \subset g$.

Comparing (8.7) and (6.6), we see in particular that the $h$-connection $D(L_h, \tilde{\tau}, \tau, g)$ on a $G$ source is a function of $\tilde{\tau}$ and $\tau$, while the $h$-connection $D(L_h, \tau, h)$ on an $H$ source is a function only of $\tau$. Correspondingly, all the results for $h$ and $g/h$ are more complicated for the $G$ source, and, in particular the factorization (7.4) is obscured on the general source.

This is an interesting complication for $h$ and $g/h$, which should be studied in the future. In the present paper, we limit ourselves to understanding that the simplification on an $H$ source is due to the $h$-symmetry of the K-conjugate pair $h$ and $g/h$.

When the source is restricted to $h \in H \subset G$, we may use the $h$-invariance of the projectors,

$$[\Omega(h), P_{g/h}] = [\Omega(h), P_h] = 0$$

(9.2)

to simplify the high-level connections of $h$ and $g/h$. Then the connections (8.7) reduce to the forms,

$$\tilde{D}(L_{g/h}, \tilde{\tau}, h) = 2\pi i \left( \sum_{n>0} n Tr \left( \frac{q^n \Omega(h)}{1 - q^n \Omega(h)} P_{g/h} \right) - \frac{\text{rank} P_{g/h}}{24} \right) + O(k^{-1})$$

(9.3a)

$$D(L_h, \tau, h) = 2\pi i \left( \sum_{n>0} n Tr \left( \frac{q^n \Omega(h)}{1 - q^n \Omega(h)} P_h \right) - \frac{\text{rank} P_h}{24} \right) + O(k^{-1})$$

(9.3b)

which are functions only of $\tilde{\tau}$ and $\tau$ respectively. When $H$ is further restricted so that $G/H$ is a reductive coset space, it is easy to check that the results (9.3) agree with the high-level form of the exact results in eqs. (6.6) and (6.13).

We are now prepared to exploit this simplification in a much larger class of ICFT’s.
9.2. The H-invariant CFT’s

As reviewed in Section 2, the K-conjugate pairs h and g/h are only the simplest examples of the much larger class of ICFT’s known as the H-invariant CFT’s [18].

\[
\text{ICFT} \supset \supset \text{H-invariant CFT’s} \supset \supset \text{Lie} \text{-invariant CFT’s} \supset \supset \text{RCFT} . \quad (9.4)
\]

The space of H-invariant CFT’s is the set of all ICFT’s with a residual global symmetry group H, and the H-symmetry, which is the symmetry group of ˜L and L, may be a finite group or a Lie group.

The simplification seen for h and g/h in subsection 9.1 extends to all the H-invariant CFT’s. The basic point is that the bicharacters of any K-conjugate pair of H-invariant CFT’s are H-invariant when the source h is chosen in \( H \subset G \),

\[
\chi(T, \tilde{\tau}, \tau, h_0hh_0^{-1}) = \chi(T, \tilde{\tau}, \tau, h) \quad \forall h_0 \in H \subset G 
\]

while a larger source breaks the H-symmetry. The relation (9.5) follows from (3.1) and (2.13).

At high-level on simple g, we can study this simplification in further detail. According to eqs. (2.10) and (2.13), the high-level form of the H-invariance of the K-conjugate pair reads,

\[
[\Omega(h), \tilde{P}] = [\Omega(h), P] = 0 \quad \forall h \in H \subset G \quad . \quad (9.6)
\]

Then, choosing the source h in the symmetry group H of the pair, we may use (9.6) in (8.7) to obtain the flat connections of all the H-invariant CFT’s,

\[
\tilde{D}(\tilde{L}, \tilde{\tau}, h) = 2\pi i \left( \sum_{n>0} nTr \left( \frac{\tilde{q}^n\Omega(h)}{1-\tilde{q}^n\Omega(h)} \tilde{P} \right) - \frac{\text{rank}\tilde{P}}{24} \right) + O(k^{-1}) \quad (9.7a)
\]

\[
D(L, \tau, h) = 2\pi i \left( \sum_{n>0} nTr \left( \frac{q^n\Omega(h)}{1-q^n\Omega(h)} P \right) - \frac{\text{rank}P}{24} \right) + O(k^{-1}) \quad . \quad (9.7b)
\]

Note that, on the H-source, these connections are functions only of \( \tilde{\tau} \) and \( \tau \) respectively.

The connections (9.7) can be further simplified by introducing the generalized Π-function,

\[
\Pi(M, \tau, \Omega(h)) \equiv \prod_{n=1}^{\infty} e \left( \text{Tr} \left( M \log(1-q^n\Omega(h)) \right) \right) \quad (9.8a)
\]

\[
\Pi(M, \tau, \Omega(h))\Pi(N, \tau, \Omega(h)) = \Pi(M+N, \tau, \Omega(h)) \quad , \quad \Pi(1, \tau, \Omega(h)) \equiv \Pi(\tau, \Omega(h)) \quad (9.8b)
\]

\[
\Pi(\tilde{P} \text{ or } P, \tau, \Omega(h_0hh_0^{-1})) = \Pi(\tilde{P} \text{ or } P, \tau, \Omega(h)) \quad (9.8c)
\]
where \( h_0 \in H \) and \( \Pi(\tau, \Omega(h)) \) is the simple \( \Pi \)-function in (4.17f). Then, the connections can be written as

\[
\tilde{D}(\tilde{L}, \tilde{\tau}, h) = -(2\pi i \frac{\text{rank} \tilde{P}}{24} + \tilde{\partial} \log \Pi(\tilde{P}, \tilde{\tau}, \Omega(h))) + O(k^{-1}) \quad (9.9a)
\]

\[
D(L, \tau, h) = -(2\pi i \frac{\text{rank} P}{24} + \partial \log \Pi(P, \tau, \Omega(h))) + O(k^{-1}) \quad . \quad (9.9b)
\]

Using this form, it is easy to obtain the evolution operators of the flat connections,

\[
\tilde{B}(\tilde{\tau}, \tau, h) = \left(\frac{\tilde{q}}{q}\right)^{\text{rank} \tilde{P}} \frac{\Pi(\tilde{P}, \tau, \Omega(h))}{\Pi(\tilde{P}, \tilde{\tau}, \Omega(h))} + O(k^{-1}) \quad (9.10a)
\]

\[
B(\tilde{\tau}, \tau, h) = \left(\frac{q}{\tilde{q}}\right)^{\text{rank} P} \frac{\Pi(P, \tilde{\tau}, \Omega(h))}{\Pi(P, \tau, \Omega(h))} + O(k^{-1}) \quad (9.10b)
\]

by integrating eq. (5.9).

Finally, we obtain the high-level, low-spin bicharacters of the \( H \)-invariant CFT’s,

\[
\chi(T, \tilde{\tau}, \tau, h) = \frac{1}{k} \left(\frac{\tilde{q}}{q}\right)^{\text{rank} \tilde{P}} \frac{Tr(h(T))}{\Pi(\tilde{P}, \tilde{\tau}, \Omega(h))} \frac{1}{\frac{\text{rank} P}{24} \Pi(P, \tau, \Omega(h))} \quad (9.11)
\]

from (5.1), (8.13) and (9.10), using (9.8d) in the form

\[
\Pi(\tau, \Omega(h)) = \Pi(\tilde{P}, \tau, \Omega(h)) \Pi(P, \tau, \Omega(h)) \quad . \quad (9.12)
\]

With eq. (9.8d), we explicitly verify the \( h \)-invariance (9.5) of the bicharacters in (9.11).

The results (9.10) and (9.11) can also be verified directly from eqs. (8.11) and (8.12).

Similarly, for the special case of \( h \) and \( g/h \) with \( G/H \) a reductive coset space, we may use the identities,

\[
\Pi(P_{g/h}, \tilde{\tau}, \Omega(h)) = \Pi(\tilde{\tau}, \sigma(h)) \quad , \quad \Pi(P_h, \tau, \Omega(h)) = \Pi(\tau, \rho(h)) \quad (9.13)
\]

to check (9.10) and (9.11) against the high-level forms of the exact results in (6.17) and (6.18).

9.3. Candidate Characters for the Lie \( h \)-invariant CFT’s

To obtain the characters of the individual ICFT’s, it is necessary to factorize the biconformal characters,

\[
\chi(T, \tilde{\tau}, \tau, h) = \sum_{\nu} \chi_L^\nu(T, \tilde{\tau}, h) \chi_L^\nu(T, \tau, h) \quad (9.14)
\]
into the conformal characters $\chi_{\tilde{L}}$ and $\chi_L$ of the $\tilde{L}$ and the $L$ theory respectively. As on the sphere $[12]$, there are many factorizations, or bases, of the form (9.14), but we are interested only in those factorizations for which the conformal characters exhibit modular covariance. See Ref. [12] for an analogous factorization of the bicorrelators of ICFT on the sphere, in which the conformal correlators of $\tilde{L}$ and $L$ are covariant under the braid group.

Here, we make a modest beginning in this direction, obtaining high-level candidate characters for the Lie $h$-invariant CFT’s $[18]$, which form the subset of all $H$-invariant CFT’s with $H$ a Lie group. This class of ICFT includes $h$ and the $g/h$ coset constructions as a small subspace. For all these theories, We know from (2.14) and (9.3) that,

$$[T_{\text{adj}}^A, \tilde{L}] = [T_{\text{adj}}^A, L] = 0 \quad A = 1, \ldots, \dim h$$

(9.15a)

$$(\bar{E}_A + E_A)\chi(T, \tilde{\tau}, \tau, h) = 0$$

(9.15b)

and we may hope to follow the intuition gained from $h$ and the $g/h$ coset constructions.

More precisely, we restrict ourselves to the Lie $h$-invariant CFT’s with simple $h \subset g$. Then we know $[18]$ that one of the theories, say $\tilde{L}$, has a local Lie $h$-invariance (like $L_{g/h}$)

$$[J_A(m), \tilde{L}(n)] = 0 \quad m, n \in \mathbb{Z}$$

(9.16)

while its K-conjugate partner (like $L_h$) carries only the corresponding global invariance,

$$[J_A(0), L(n)] = 0 \quad m \in \mathbb{Z}$$

(9.17)

In this case, as for $h$ and $g/h$, we may adopt as a working hypothesis that all the source dependence of the bicharacters is associated to the global theory.

At high-level on simple $g$, the low-spin bicharacters of the Lie $h$-invariant CFT’s are given by the result (9.11) with

$$[T_{\text{adj}}^A, \tilde{P}] = [T_{\text{adj}}^A, P] = 0$$

(9.18)

so each factor in the high-level bicharacters of the Lie $h$-invariant CFT’s on the $H$ source are explicitly $h$-invariant. Then we may $h$-character expand the local theory $\tilde{L}$ in (9.11) to obtain the factorized bicharacters,

$$\chi(T, \tilde{\tau}, \tau, h) = \sum_{k, T^h} \chi_{\tilde{L}}(T, T^h, \tilde{\tau})\chi_L(T, T^h, \tau, h)$$

(9.19)
where the sum is over all unitary irreps $T^h$ of $h$, and
\[
\chi_L(T, T^h, \tilde{\tau}) = \frac{1}{k} \int dh \frac{Tr(h^*(T^h))Tr(h(T))}{q^{\frac{\text{rank} P}{24}} \Pi(\tilde{P}, \tilde{\tau}, \Omega(h))}
\]
\[
\chi_L(T^h, \tau, h) = \frac{Tr(h(T^h))}{q^{\frac{\text{rank} P}{24}} \Pi(P, \tau, \Omega(h))}
\]
are the high-level candidate characters for the Lie $h$-invariant CFT's.

As a check on the candidate characters (9.20), we reconsider the simple case of $h$ and the $g/h$ coset constructions, with $G/H$ a reductive coset space. In this case, the candidate characters reduce to the high-level characters of $h$ and $g/h$,
\[
\chi_{L_{g/h}}(T, T^h, \tilde{\tau}) = \frac{1}{k} \int dh \frac{Tr(h^*(T^h))Tr(h(T))}{q^{\frac{\text{dim}(g/h)}{24}} \Pi(\tilde{\tau}, \sigma(h))}
\]
\[
\chi_{L_h}(T^h, \tau, h) = \frac{Tr(h(T^h))}{q^{\frac{\text{dim} h}{24}} \Pi(\tau, \rho(h))}
\]
which agree with the high-level forms of the exact results in (7.12) and (7.13).

The next step is to test the candidate characters for modular covariance, or to further decompose the candidates until modular covariance is obtained. This investigation is beyond the scope of the present paper, but we may set the stage with some simple remarks.

We know that the modular transformation $\tau \rightarrow -\frac{1}{\tau}$ mixes low spin with all spin, and we have checked in examples that, at high level, this transformation is dominated by high spin (of order the level for $SU(2)$). Thus, high-spin candidate characters are also needed to study modular covariance at high level. Because all representation dependence of the bicharacters comes from the affine-Sugawara characters $\chi_g(T)$, such high-spin candidate characters for the Lie-$h$ invariant ICFT's can be obtained as above, from the high-level form of the high-spin affine-Sugawara characters.

Although this program is technically involved, it is expected that chiral modular covariant characters and non-chiral modular invariants exist in ICFT, just as braid-covariant correlators have been found on the sphere [12]. This expectation has further support in the case of the high-level smooth ICFT's studied here, because diffeomorphism-invariant world-sheet actions [21,29,30] are known for the generic theory of this type.
10. A Geometric Formulation

The characters studied in the sections above were defined with a conventional Lie source, but we wish to point out the geometric form that our problem takes on an affine source \( \hat{\gamma} \), in the centrally-extended loop group \( \hat{LG} \) of affine \( g \).

We write the affine source as
\[
\hat{\gamma}(x,y) = e^{iy\hat{k}} \hat{g}(x)
\]
(10.1)
where \( y \) and \( x^{\alpha\mu} \), \( \alpha = 1, \ldots, \text{dim} \ g, \mu \in \mathbb{Z} \) are the coordinates on the loop group manifold and \( \hat{k} \) is the level operator, or central element. The \( y \)-independent factor \( \hat{g} \) can be chosen in many bases such as,
\[
\hat{g}(x) = \exp(i \sum_{am} \beta_{am}(x) J_a(m))
\]
(10.2)
or normal-ordered forms such as the Borel decompositions in Refs. [31] and [32]. In practice, one may wish to choose a basis of \( \hat{g} \) which simplifies the Laplacians in the formulation below.

Define the affine-Virasoro characters on the affine source as
\[
\chi(T, \tilde{\tau}, \tau, \hat{\gamma}) = Tr_T \left( \hat{q}^{L(0)} - \frac{c}{24} q^{L(0)} - \frac{1}{24} \hat{\gamma} \right) .
\]
(10.3)
Then, following the development in the earlier sections, we introduce left and right invariant vielbeins, inverse vielbeins and affine Lie derivatives on the loop group as follows,
\[
e_\Lambda = -i \hat{\gamma}^{-1} \partial_\Lambda \hat{\gamma} = e^L_\Lambda J_L , \quad \mathcal{E}_L = -ie^L_\Lambda \partial_\Lambda , \quad \mathcal{E}_L \hat{\gamma} = \hat{\gamma} J_L
\]
(10.4a)
\[
\bar{e}_\Lambda = -i \hat{\gamma} \partial_\Lambda \hat{\gamma}^{-1} = \bar{e}^L_\Lambda J_L , \quad \bar{E}_L = -i \bar{e}^L_\Lambda \partial_\Lambda , \quad \bar{E}_L \hat{\gamma} = -J_L \hat{\gamma}
\]
(10.4b)
\[
J_L = (J_a(m), \hat{k}) , \quad \mathcal{E}_L = (\mathcal{E}_a(m), \mathcal{E}_y) , \quad \bar{E}_L = (\bar{E}_a(m), \bar{E}_y) \quad \Lambda = (\alpha \mu, y) , \quad L = (am, y)
\]
(10.4c)
\[
\text{where } [J_L, J_M] = if_{LM}^N J_N \text{ is the affine algebra and the vielbeins } e^L_\Lambda , \bar{e}^L_\Lambda \text{ and inverse vielbeins } e^L_\Lambda , \bar{e}^L_\Lambda \text{ are independent of the operators } J_L \text{. The affine Lie derivatives } \mathcal{E}_L \text{ and } \bar{E}_L \text{ in (10.4) satisfy two commuting affine algebras with central elements } \mathcal{E}_y \text{ and } \bar{E}_y \text{ respectively.}
With these tools, it is straightforward to see that the bicharacters (10.3) satisfy the heat-like equations,

\[ \tilde{\partial} \chi = \tilde{D} \chi , \quad \partial \chi = D \chi \] (10.5a)

\[ \tilde{D}(\hat{\gamma}) = -2\pi i \tilde{\Delta}(\hat{\gamma}) = 2\pi i \tilde{L}^{ab}(\mathcal{E}_a(0)\mathcal{E}_b(0) + 2 \sum_{m>0} \mathcal{E}_a(-m)\mathcal{E}_b(m)) \] (10.5b)

\[ D(\hat{\gamma}) = -2\pi i \Delta(\hat{\gamma}) = 2\pi i \mathcal{L}^{ab}(\mathcal{E}_a(0)\mathcal{E}_b(0) + 2 \sum_{m>0} \mathcal{E}_a(-m)\mathcal{E}_b(m)) \] (10.5c)

and the usual affine-Sugawara boundary condition,

\[ \chi(T, \tau, \hat{\tau}, \hat{\gamma}) = \chi(T, \tau, \hat{\gamma}) = Tr_T (q^{L_g(0)} - c_g/24\hat{\gamma}) \] (10.6a)

\[ \partial \chi_g = D_g \chi , \quad D_g = \tilde{D} + D \] (10.6b)

\[ D_g(\hat{\gamma}) = -2\pi i \Delta_g(\hat{\gamma}) = 2\pi i \mathcal{L}_g(\mathcal{E}_a(0)\mathcal{E}_b(0) + 2 \sum_{m>0} \mathcal{E}_a(-m)\mathcal{E}_b(m)) \] (10.6c)

where \( \chi_g(T, \tau, \hat{\gamma}) \) are the affine-Sugawara characters on the affine source. The bicharacters and the affine-Sugawara characters also satisfy an analogous heat-like system with \( \mathcal{E}_a(m) \to \tilde{\mathcal{E}}_a(m) \).

The objects \( \tilde{\Delta}, \Delta \) and \( \Delta_g \) are three mutually-commuting Laplacians on the centrally-extended loop group. It is easy to verify that the affine-Virasoro connections \( \tilde{D}, D \) are flat, \( \partial \tilde{D} + D\tilde{D} = \partial \tilde{D} + \tilde{D}D \), and moreover,

\[ \partial \tilde{D} = \partial \tilde{D} = 0 , \quad [\tilde{D}, D] = 0 \] (10.7)

so the connections are also abelian flat.

In further detail, we find from (10.1) that

\[ e_{\alpha\mu} = -i\hat{g}^{-1}\partial_{\alpha\mu}\hat{g} = e_{\alpha\mu}^{am} J_a(m) + e_{\alpha\mu}^{y} \hat{k} \] (10.8a)

\[ e_{y}^{L} = \delta_{y}^{L} , \quad e_{\alpha\mu}^{L} \text{ is independent of } y \] (10.8b)

\[ e_{y}^{\Lambda} = \delta_{y}^{\Lambda} , \quad e_{am}^{\Lambda} \text{ is independent of } y \] (10.8c)

\[ e_{am}^{\alpha\mu} e_{\alpha\mu}^{bn} = \delta_{am}^{bn} , \quad e_{am}^{\alpha\mu} e_{\beta\nu}^{bn} = \delta_{am}^{\beta\nu} , \quad e_{am}^{y} = -e_{am}^{\alpha\mu} e_{\alpha\mu}^{y} \] (10.8d)

and similarly for the \( \bar{e} \)'s. This gives the explicit forms for the left and right invariant Lie derivatives,

\[ \mathcal{E}_y = -i\partial_y , \quad \mathcal{E}_a(m) = -i(e_{am}^{\alpha\mu} \partial_{\alpha\mu} + e_{am}^{y} \partial_y) \] (10.9a)

\[ \bar{\mathcal{E}}_y = i\partial_y , \quad \bar{\mathcal{E}}_a(m) = -i(\bar{e}_{am}^{\alpha\mu} \partial_{\alpha\mu} + \bar{e}_{am}^{y} \partial_y) \] (10.9b)
where the sign difference of $E_y$ and $\tilde{E}_y$ comes from $\tilde{e}_y^A = -\delta_y^A$.

We are primarily interested in the reduced bicharacters $\chi(\hat{g})$, which satisfy

$$\chi(T, \tilde{\tau}, \tau, \hat{g}) = e^{i\hat{g}k} \chi(T, \tilde{\tau}, \tau, \hat{g}) \quad (10.10a)$$

$$\chi(T, \tilde{\tau}, \tau, \hat{g}) = T_{\tau} \ell q^{L(0)-c/24} q^{L(0)-c/24} \hat{g} \quad (10.10b)$$

where $\hat{k}$ is replaced by the level $k$ in the reduced quantities. It follows from (10.3) and (10.10a) that the reduced bicharacters satisfy the heat-like system,

$$\partial\chi(\hat{g}) = \partial_{\chi}(\hat{g}) = D(\hat{g})\chi(\hat{g}) \quad (10.11a)$$

$$\tilde{\partial}\chi(\hat{g}) = \tilde{D}(\hat{g})\chi(\hat{g}) \quad (10.11b)$$

$$\tilde{D}(\hat{g}) = -2\pi i \tilde{\Delta}(\hat{g}) = 2\pi i \tilde{L}^{c} (E_a(0)E_b(0) + 2 \sum_{m>0} E_a(-m)E_b(m)) \quad (10.11c)$$

$$D(\hat{g}) = -2\pi i \Delta(\hat{g}) = 2\pi i L^{c} (E_a(0)E_b(0) + 2 \sum_{m>0} E_a(-m)E_b(m))$$

where the reduced affine Lie derivatives $E_a(m)$ and $\tilde{E}_a(m)$ are

$$E_a(m) = -ie^{\alpha\mu}_{am}(\partial_{\alpha\mu} + i\epsilon_{\alpha\mu}) \quad (10.12a)$$

$$\tilde{E}_a(m) = -i\tilde{e}^{\alpha\mu}_{am}(\partial_{\alpha\mu} + i\tilde{\epsilon}_{\alpha\mu}) \quad (10.12b)$$

These differential operators satisfy

$$E_a(m)\hat{g} = \hat{g}J_a(m) \quad \tilde{E}_a(m)\hat{g} = -J_a(m)\hat{g} \quad (10.13a)$$

$$[E_a(m), E_b(n)] = if^c_{ab} E_c(m+n) + mk\eta_{ab}\delta_{m+n,0} \quad (10.13b)$$

$$[\tilde{E}_a(m), \tilde{E}_b(n)] = if^c_{ab} \tilde{E}_c(m+n) - mk\eta_{ab}\delta_{m+n,0} \quad (10.13c)$$

$$[E_a(m), \tilde{E}_b(n)] = 0 \quad (10.13d)$$

and we remark that the the left and right invariant operators satisfy the affine algebra of $g$ at level $k$ and $-k$ respectively. It follows that $Q_a(m) \equiv E_a(m) + \tilde{E}_a(m)$ satisfies the loop algebra of $g$.

We emphasize that the result (10.13a) is a class of new representations of the affine algebra, one for each basis choice of $\hat{g}$. An example of this system, for a particular basis of affine $su(2)$, has been studied in Ref. [33]. As an example on all affine $g$, one may choose the basis

$$\hat{g}(x) = \exp \left( i x^{\alpha\mu} e^{\alpha\mu}_{am}(0) J_a(m) \right) \quad (10.14)$$
Then we obtain the explicit forms of the left invariant quantities,

\[ e_{\alpha \mu}^a(x) = M(x)_{am} e_{bn}^{\alpha \mu}(0) \quad , \quad e_{\alpha \mu}^y(x) = M(x)_{am}^y \]  

(10.15a)

\[ e_{\alpha \mu}^y(x) = e_{\alpha \mu}^{am}(0) M^{-1}(x)_{am}^y \]  

(10.15b)

\[ M(x) = \frac{\log \hat{g}(\hat{T}^{adj}, x)}{\hat{g}(\hat{T}^{adj}, x) - 1} , \quad \hat{g}(\hat{T}^{adj}, x) = \exp \left( ix^{\alpha \mu} e_{\alpha \mu}^{am}(0) \hat{T}^{adj}_{am} \right) \]  

(10.15c)

\[ (\hat{T}^{adj}_{am})_{bn}^{cr} = \delta_{m+n,r} (T^{adj}_a)_{bc}^r , \quad (\hat{T}^{adj}_{am})_{bn}^y = n \eta_{ab} \delta_{m+n,0} \]  

(10.15d)

and similarly for the right invariant quantities, where the non-zero elements of the affine adjoint matrix \((\hat{T}^{adj}_L)_M^N = -if_{LM}^N\) are given in (10.15d).

The reduced system (10.11), taken with the usual affine-Sugawara boundary condition,

\[ \chi_g(T, \tau, \hat{g}) = \chi(T, \tau, \hat{g}) , \quad \partial \chi_g(\hat{g}) = D_g(\hat{g}) \chi_g , \quad D_g(\hat{g}) = \hat{D}(\hat{g}) + D(\hat{g}) = -2\pi i \Delta g(\hat{g}) \]  

(10.16)

can be further analyzed using the machinery developed in the earlier sections. In particular, the reduced objects \(\tilde{\Delta}(\hat{g}), \Delta(\hat{g})\) and \(\Delta_g(\hat{g})\), which represent \(-\hat{L}(0), -L(0)\) and \(L_g(0)\) respectively, are three mutually commuting generalized Laplacians on the centrally-extended loop group, and the reduced connections \(\hat{D}, D\) are flat and abelian flat. Moreover, the reduced bicharacters \(\chi(\hat{g})\) are uniquely determined from eqs. (5.1) and (5.9),

\[ \chi(T, \tilde{\tau}, \tau, \hat{g}) = e^{-2\pi i (\tilde{\tau} - \tau) \tilde{\Delta}(\hat{g})} \chi_g(T, \tau, \hat{g}) = e^{-2\pi i (\tau - \tilde{\tau}) \Delta(\hat{g})} \chi_g(T, \tilde{\tau}, \hat{g}) \]  

(10.17)

in terms of the reduced affine-Sugawara characters (10.16), whose explicit form we will not obtain in this paper.

The form of the solution (10.17) is seen more clearly by introducing the simultaneous eigenvectors \(\psi_n(T, \hat{g})\) of the three Laplacians,

\[ -\tilde{\Delta} \psi_n = \tilde{\lambda}_n \psi_n , \quad -\Delta \psi_n = \lambda_n \psi_n , \quad -\Delta_g \psi_n = \lambda^g_n \psi_n \]  

(10.18a)

\[ \tilde{\lambda}_n + \lambda_n = \lambda^g_n \]  

(10.18b)

in the Hilbert space of affine irrep \(T\), where the eigenvalues \(\tilde{\lambda}_n(T), \lambda_n(T)\) and \(\lambda^g_n(T)\) are the conformal weights of the states \(\psi_n(T, \hat{g})\) under the stress tensors \(\hat{T}, T\) and \(T_g\). The basis (10.18) is therefore the simultaneous \(L\)-basis (see Section 3) for all levels of the affine irrep \(T\). For unitary theories, \(\hat{L}(0), L(0)\) and \(L_g(0)\) are hermitean in an inner product with a non-negative norm, so, in the representation above, there is an induced inner
product $\langle A|B \rangle$ in which the Laplacians are hermitean and the eigenvectors are orthonormal $\langle \psi_m(T)|\psi_n(T) \rangle = \delta_{m,n}$.

Using these eigenvectors, we find the unique solution of the heat-like system,

$$\chi(T, \tilde{\tau}, \tau, \tilde{g}) = \sum_n \left( \frac{\tilde{q}}{q_0} \right)^{\lambda_n(T)} \psi_n(T) (\langle \psi_n(T)|\chi_g(T, \tau_0) \rangle) \quad \text{(10.19a)}$$

$$\chi_g(T, \tau, \tilde{g}) = \chi(T, \tau, \tau, \tilde{g}) = \sum_n \left( \frac{q}{q_0} \right)^{\lambda_n(T)} \psi_n(T) (\langle \psi_n(T)|\chi_g(T, \tau_0) \rangle) \quad \text{(10.19b)}$$

where $q_0 = \exp(2\pi i \tau_0)$ is a regular reference point.

The solution (10.19a) for the bicharacters follows from eq. (10.17), using the expansion (10.19b) of the affine-Sugawara characters, and it is easy to check that the full solution (10.19a, b) solves the heat-like equations (10.11a) and the affine-Sugawara boundary condition (10.16). Moreover, the bicharacters and the affine-Sugawara characters are independent of the reference point, as they should be. For example, one finds

$$\partial_{\tau_0} \chi = \sum_n \left( \frac{\tilde{q}}{q_0} \right)^{\lambda_n} \psi_n (\langle \psi_n| \left( -2\pi i \lambda_n^g + D_g(T, \tau_0, \hat{g}) \right) \chi_g(T, \tau_0, \hat{g}) \rangle) = 0 \quad \text{(10.20)}$$

where hermiticity of $D_g$ is used in the last step.

The bicharacters of the earlier sections can be obtained from these bicharacters by restricting the affine source to a Lie source. The advantage of the geometric formulation is that we now have the flat connections in closed form, which may be useful in the investigation of global properties such as factorization and modular covariance.

11. Conclusions

Irrational conformal field theory (ICFT) includes rational conformal field theory as a small subspace. So far, the only known path into the space of ICFT's is the general affine-Virasoro construction [3,4] on the currents of affine Lie $g$. Recently, dynamical equations for the correlators of ICFT have been obtained on the sphere [11,12], where they are understood as generalized Knizhnik-Zamolodchikov equations with flat connections [13].

In this paper we have begun the study of ICFT on the torus, following the paradigm on the sphere. In particular, we have defined the affine-Virasoro characters, or bicharacters, which involve the two commuting Virasoro operators of any affine-Virasoro construction, and we have shown that the bicharacters satisfy heat-like equations with flat connections.
As a first example of the formulation, we have solved the system completely for the simple case of \( h \) and the \( g/h \) coset constructions, obtaining a new integral representation for the general coset characters.

In a second application, we have solved for the high-level bicharacters of the general ICFT on simple \( g \), and proposed a set of high-level candidate characters for the Lie \( h \)-invariant CFTs \( \mathcal{I}_R \), which is the set of all ICFTs with a Lie Symmetry.

A next step is to analyze the high-level candidate characters with respect to the modular group. For this investigation, one should begin with some of the many explicit examples of Lie-\( h \) invariant CFT’s, beyond the coset constructions. We did not attempt to factorize the general theory on a source larger than its symmetry group, which is an important problem because the generic ICFT has no residual symmetry. As a first step in this direction, one should study the factorization of \( h \) and \( g/h \) when the source is larger than \( H \).

Finally, we noted a geometric formulation of the system on an affine source, where the flat connections are generalized Laplacians on the centrally-extended loop group. These Laplacians involve new first-order differential representations of affine Lie algebra.

**Acknowledgments**

We thank I. Bars, D. Bernard, E. Kiritsis, W. Taylor and S. Yankielowicz for helpful remarks, and we are indebted to H. Ooguri for a discussion which stimulated this investigation.

This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY-90-21139.
References

[1] V.G. Kač, Anal. App. 1 (1967) 328;  
R.V. Moody, Bull. Am. Math. Soc. 73 (1967) 217.
[2] K. Bardakçi and M.B. Halpern, Phys. Rev. D3 (1971) 2493.
[3] M.B. Halpern and E. Kiritsis, Mod. Phys. Lett. A4 (1989) 1373;  
Erratum ibid. A4 (1989) 1797.
[4] A. Yu Morozov, A.M. Perelomov, A.A. Rosly, M.A. Shifman and A.V. Turbiner, Int.  
J. Mod. Phys. A5 (1990) 803.
[5] M.B. Halpern, Phys. Rev. D4 (1971) 2398.
[6] E. Witten, Commun. Math. Phys. 92 (1984) 455;  
V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. B247 (1984) 83;  
G. Segal, unpublished.
[7] P. Goddard, A. Kent and D. Olive, Phys. Lett. B152 (1985) 88.
[8] M.B. Halpern, E.B. Kiritsis, N.A. Obers, M. Porrati and J.P. Yamron, Int. J. Mod.  
Phys. A5 (1990) 2275.
[9] M.B. Halpern, “Recent Developments in the Virasoro Master Equation”, in the  
proceedings of the Stony Brook conference, Strings and Symmetries 1991, World Scientific,  
1992.
[10] M.B Halpern, Ann. of Phys. 194 (1989) 247.
[11] M.B Halpern and N.A. Obers, Int. J. Mod. Phys. A9 (1994) 265.
[12] M.B. Halpern and N.A. Obers, Int. J. Mod. Phys. A9 (1994) 419.
[13] M.B. Halpern and N.A. Obers, Flat Connections and Non-Local Conserved Quantities  
in Irrational Conformal Field Theory Berkeley/Ecole Polytechnique preprint UCB-  
PTH-93/33, CPTH-A277.1293, hep-th/9312050.
[14] M.R. Douglas, “G/H Conformal Field Theory”, Caltech preprint, CALT-68-1453,  
1987, unpublished.
[15] M.B. Halpern, “Recent Progress in Irrational Conformal Field Theory”, Berkeley  
preprint, UCB-PTH-93/25, hep-th/9309087, 1993. To appear in the proceedings of  
the Berkeley conference, Strings 1993.
[16] D. Bernard, Nucl. Phys. B303 (1988) 77.
[17] T. Eguchi and H. Ooguri, Nucl. Phys. B313 (1989) 492.
[18] M.B. Halpern, E. Kiritsis and N.A. Obers, The Lie h-Invariant Conformal Field The-  
ories and the Lie h-invariant Graphs, in Infinite Analysis, Part A, Advanced Series in  
Mathematical Physics 16, Proceedings of the RIMS Project, World Scientific, 1992.
[19] M.B. Halpern and J.P. Yamron, Nucl. Phys. B332 (1990) 411.
[20] M.B. Halpern and N.A. Obers, Nucl. Phys. B345 (1990) 607.
[21] M.B. Halpern and J.P. Yamron, Nucl. Phys. B351 (1991) 333.
[22] M.B. Halpern and N.A. Obers, Commun. Math. Phys. 138 (1991) 63.
[23] T. Eguchi and H. Ooguri, Nucl. Phys. B282 (1987) 308.
[24] H. Fegan, Trans. Am. Math. Soc. 246 (1978) 339; J. Diff. Geom. 13 (1978) 589.
[25] V.G. Kač and M. Wakimoto, Adv. Math. 70 (1988) 156.
[26] P. Goddard, A. Kent and D. Olive, Commun. Math. Phys. 103 (1986) 105.
[27] D. Gepner and Z. Qiu, Nucl. Phys. B285 [FS19] (1987) 423.
[28] K. Gawędzki and A. Kupiainen, Nucl. Phys. B320 (1989) 625.
[29] A.A Tseytlin, “On a ‘Universal’ Class of WZW-Type conformal Models”, CERN-TH.7068/93, hep-th/9311062, 1993.
[30] J. de Boer, K. Clubok and M.B. Halpern, Linearized Form of the Generic Affine-Virasoro Action, UCB-PTH-93/34, LBL-34938, ITP-SB-93-88, hep-th/9312094.
[31] D. Bernard and G. Felder, Commun. Math. Phys. 127 (1990) 145.
[32] W. Taylor, Berkeley PhD Thesis Coadjoint Orbits and Conformal Field Theory, 1993, UCB-PTH-93/26, LBL-34507, hep-th/9310040.
[33] V. Aldaya and J. Navarro-Salas, Commun. Math. Phys. 113 (1987) 375.