THE BROWN-PETERSON SPECTRUM IS NOT $E_{2(p^2+2)}$ AT ODD PRIMES

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ABSTRACT. Recently, Lawson has shown that the 2-primary Brown-Peterson spectrum does not admit the structure of an $E_{12}$ ring spectrum, thus answering a question of May in the negative. We extend Lawson’s result to odd primes by proving that the $p$-primary Brown-Peterson spectrum does not admit the structure of an $E_{2(p^2+2)}$ ring spectrum. We also show that there can be no map $MU \to BP$ of $E_{2p+3}$ ring spectra at any prime.

1. Introduction

Two of the most influential themes in modern homotopy theory are the study of structured ring spectra, in particular $E_\infty$ ring spectra, and chromatic homotopy theory, which had its genesis in computations with the Adams-Novikov spectral sequence based on the $p$-primary Brown-Peterson spectrum $BP$ [MRW77]. In [May75], May asked a fundamental question about the interaction between these two programs.

Question 1.1. Does the Brown-Peterson spectrum admit a model as an $E_\infty$ ring spectrum?

This question has been seminal in the development of the theory of structured ring spectra. In an unpublished preprint [Kri95], Kriz developed the theory of topological André-Quillen cohomology in an attempt to prove that $BP$ does admit the structure of an $E_\infty$ ring spectrum. While his attempt to apply his theory to $BP$ suffered from an error, the careful study of what exactly went wrong became the seed of a new attempt by Lawson to answer May’s question in the negative; recently, this project reached maturity in Lawson’s proof [Law17] that $BP$ does not admit an $E_\infty$ multiplication at the prime $p = 2$.

In this paper, we prove in Theorem 1.2 that $BP$ does not admit an $E_\infty$ multiplication at odd primes. Our technique is akin to Lawson’s and relies on the computation of a certain secondary power operation in the dual Steenrod algebra. The fundamental input to this computation is the calculation of a certain tom Dieck-Quillen power operation in the coefficient ring of the complex cobordism spectrum $MU$. To make this calculation, we generalized the method of the appendix of [Law17] to odd primes.

For further motivation and background, we refer the reader to the introduction of [Law17].
1.1. Statement of the Results. We prove two main results: one limiting the coherence of multiplicative structures on the Brown-Peterson spectrum and related spectra at odd primes, and another giving a stronger limitation on the coherence of complex orientations of such spectra.

Since the first theorem reduces to Theorem 1.1.2 of [Law17] at the prime \( p = 2 \), we are able to state it for all primes.

**Theorem 1.2.** Neither the Brown-Peterson spectrum \( BP \), nor the truncated Brown-Peterson spectra \( BP\langle n \rangle \) for \( n \geq 4 \), nor any of their \( p \)-adic completions admit the structure of an \( E_{2(p^2+2)} \) ring spectrum.

We will prove Theorem 1.2 at the end of Section 3.

The \( p = 2 \) case of the second theorem is not proven in [Law17], though a sketch of an argument is given in Remark 4.4.4. Making use of results of [Law17] at the prime \( p = 2 \), we prove it for all primes.

**Theorem 1.3.** Neither the Brown-Peterson spectrum \( BP \), nor the truncated Brown-Peterson spectra \( BP\langle n \rangle \) for \( n \geq 3 \), nor any of their \( p \)-adic completions admit an \( E_{2p+3} \)-map from the complex cobordism spectrum \( MU \).

We will prove Theorem 1.3 at the end of Section 2.

1.2. Outline of the Paper. In Section 2.1 we carry out the computations of \( MU \) power operations that we will need; the main result is Theorem 2.1. In Section 2.2 we generalize results of [Law17] to convert the \( MU \) power operations of Theorem 2.1 into Dyer-Lashof operations in \( \pi_\ast(\mathcal{H}_p \wedge_{MU} \mathcal{H}_p) \), thus obtaining Theorem 2.9.

At the end of this section, we apply these results to obtain Theorem 1.3.

In Section 3.1 we state some relations satisfied by the action of the Dyer-Lashof operations on \( H_\ast(\mathcal{H}_p; \mathbb{F}_p) \) and \( H_\ast(\mathcal{H}_p; \mathbb{F}_p) \). In Section 3.2 we write down the relation defining the secondary operation of interest and show that it is defined on \( -\xi_1 \in H_\ast H \). Finally, in Section 3.3 we compute this secondary operation on \( -\xi_1 \) to be a nonzero multiple of \( \tau_4 \) modulo the \( \xi_i \) by applying juggling formulae and a Peterson-Stein relation to reduce to Theorem 2.9. We then deduce Theorem 1.2.

1.3. Questions. Our work raises several interesting questions. While Theorems 1.2 and 1.3 provide upper bounds on the coherence of multiplicative structures on \( BP \) that are functions of \( p \), the best known lower bounds [BM13] and [CM15], which state that \( BP \) is an \( E_2 \)-algebra and admits an \( E_2 \) orientation \( MU \to BP \), do not depend on the prime \( p \). So one is led to ask whether these coherence bounds are independent of \( p \).

**Question 1.4.** Let \( \text{coh}_{BP}(p) \) denote the largest integer \( n \) such that the \( p \)-primary \( BP \) admits the structure of an \( E_n \) ring spectrum. Is \( \text{coh}_{BP}(p) \) constant in \( p \)? If not, how does it vary with \( p \)?

In another direction, we may ask about \( E_\infty \) structures on the truncated Brown-Peterson spectra \( BP\langle n \rangle \). While Theorem 1.2 rules out the possibility of such structures for \( n \geq 4 \), the only known positive results state that \( BP\langle 1 \rangle \) always admits
an $E_\infty$ structure (since it is the Adams summand) and that $BP(2)$ admits an $E_\infty$ structure at the primes 2 and 3 \cite{HL10 \cite{LN12}. What about the remaining cases?

**Question 1.5.** At which of the primes $p \geq 5$ does the height 2 truncated Brown-Peterson spectrum $BP(2)$ admit an $E_\infty$ multiplication?

**Question 1.6.** At which primes does the height 3 truncated Brown-Peterson spectrum $BP(3)$ admit an $E_\infty$ multiplication?

The author expects the answer to Question 1.5 to be that $BP(2)$ does admit an $E_\infty$ multiplication at all primes, and expects the answer to Question 1.6 to be that at all primes $BP(3)$ does not admit an $E_\infty$ multiplication.

**Remark 1.7.** The above questions are not quite well-defined: there are many generalized truncated Brown-Peterson spectra $BP\langle n \rangle$ which are no a priori equivalent. However, Angeltveit and Lind \cite{AL17} have shown that all choices of $BP\langle n \rangle$ are equivalent after $p$-completion, so that Question 1.5 and Question 1.6 are well-defined after $p$-completion.

1.4. **Conventions.** We work throughout at a fixed odd prime $p$. We will let $H$ denote the mod $p$ Eilenberg-MacLane spectrum $HF_p$ and let $H_* (X)$ denote mod $p$ homology.

We work with EKMM spectra \cite{EKMM97}, the linear isometries $E_\infty$-operad and the little $n$-cubes $E_n$-operads. However, to prove Proposition 2.14, we will pass to underlying $\infty$-categories \cite{Lur09} and work with $E_n$-ring spectra in the sense of \cite{Lur}. The comparison between these two perspectives is justified by Theorem 7.10 of \cite{PS14}.

1.5. **Generators of the Homology and Homotopy of MU.** For the convenience of the reader, we review the relations between various sets of elements of $\pi_* (MU)$, $H_* (MU; \mathbb{Z})$ and $\pi_* (MU) \otimes \mathbb{Q}$ that we will need to make use of and compare.

The integral homology $H_* (MU; \mathbb{Z})$ is generated by elements $b_i$ which are the images of the duals of $c_1^i$ under $H_* (CP^\infty; \mathbb{Z}) \to H_* (BU; \mathbb{Z}) \cong H_* (MU; \mathbb{Z})$. If we define the Newton polynomials in $b_i$ inductively by $N_1 (b) = b_1$ and

$$N_n (b) = b_1 N_{n-1} (b) - t_2 N_{n-2} (b) + \cdots + (-1)^{n-2} b_{n-1} N_1 (b) + (-1)^{n-1} n b_n,$$

then $N_n (b)$ generates the group of primitive elements in $H_* (MU; \mathbb{Z})$. Furthermore, $N_n (b) \equiv (-1)^{n-1} n b_n$ modulo decomposables. As we will see in Section 3.1, there are convenient formulae for the action of the Dyer-Lashof operations on $N_n (b)$.

The homotopy $\pi_* (MU)$ of $MU$ is generated by elements $x_i$ whose images under the Hurewicz map are $h(x_i) \equiv q^i b_i$ modulo decomposables when $i = q^n - 1$ for some prime $q$ and $h(x_i) \equiv b_i$ modulo decomposables otherwise.

We may view the class of $CP^n$ as an element of $\pi_{2n} (MU)$: then the $CP^n$ do not generate $\pi_* (MU)$, though they are generators of $\pi_* (MU) \otimes \mathbb{Q}$. Under the isomorphism $\pi_* (MU) \otimes \mathbb{Q} \cong H_* (MU; \mathbb{Q})$ induced by the Hurewicz map, $CP^n \equiv -(n + 1) b_n$ modulo decomposables.
The logarithm of of the formal group $F$ on $\pi_*(MU)$ may be expressed in terms of the $\mathbb{CP}^n$:
$$\ell_F(x) = \sum \frac{\mathbb{CP}^{n-1} x^n}{n}.$$ 

1.6. When are the Dyer-Lashof Operations Defined? To obtain the precise bounds on $E_n$ structures of Theorem 1.2 and Theorem 1.3 we need to know when a Dyer-Lashof operation $Q^k$ is defined on an element $x \in \pi_n R$ for $R$ an $E_n$-$H$-algebra.

**Theorem 1.8** (BMMS86, Theorems 3.1 and 3.3). Let $R$ be an $E_n$-$H$-algebra. Then the operation $Q^s$ is defined on an element $x \in \pi_n R$ when $2s - \deg(x) \leq n - 1$; however, these operations only satisfy the expected properties (e.g. linearity, Cartan formula) when $2s - \deg(x) \leq n - 2$.

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2. Power Operations

2.1. A Power Operation in the Homotopy of $MU$. Recall that the $H_\infty$-structure on $MU$ equips the even $MU$-cohomology of a space $X$ with a power operation
$$P_p : MU^{2s}(X) \to MU^{2p^s}(X \times B\Sigma_p).$$
The inclusion $C_p \hookrightarrow \Sigma_p$ induces an injection
$$MU^*(B\Sigma_p) \hookrightarrow MU^*(BC_p),$$
and there is a canonical isomorphism
$$MU^*(BC_p) \cong MU^*[\alpha]/[p]F(\alpha).$$
We may therefore view this power operation applied to $X$ a point as a map
$$P : MU^{2s} \to MU^{2p^s}[\alpha]/[p]F(\alpha).$$
Our goal in this section will be to make the following computation of the composition of $P$ and the map $r_* : MU^*[\alpha]/[p]F(\alpha) \to BP^*[\alpha]/[p]F(\alpha)$ induced by the standard complex orientation of $BP$.

**Theorem 2.1.** Let
$$\chi = \prod_{i=1}^{p-1} [i]_F(\alpha) \in MU^*(BC_p) \cong MU^*[\alpha]/[p]F(\alpha)$$
denote the $MU$-Euler class of the real reduced regular representation of $C_p$.

Then, modulo decomposables in $BP^*$, the following equalities hold:
$$r_* \left( \chi^{2(p-1)} P(\mathbb{CP}^{2(p-1)}) \right) \equiv v_3 \alpha^{p^3-1-2(p-1)} + O(\alpha^{p^3})$$
and
\[ r_\ast \left( \chi^{p(p-1)} P(\mathbb{C}P^{p(p-1)}) \right) \equiv -v_3\alpha^{p^2-1-p(p-1)} + O(\alpha^p) \]

**Remark 2.2.** As we will see in the proof of Proposition 2.4, \( P(\mathbb{C}P^n) \) may be computed purely algebraically in terms of the formal group law of \( MU \).

**Remark 2.3.** It follows immediately from Theorem 2.1 that the standard complex orientation of \( BP \) is not \( H_\infty \).

To do this, we first make a reduction. Since we are working modulo decomposables and \( v_1 \) and \( v_2 \) cannot appear for degree reasons, the above may be checked after applying the map \( q : BP^* \to \mathbb{Z}_p[v_3]/(v_3^3) \) that sends \( v_3 \) to \( v_3 \) and \( v_i \) to 0 for \( i \neq 3 \). So to prove Theorem 2.1 it suffices to prove the following proposition.

**Proposition 2.4.** There are equalities
\[ q \circ r_\ast \left( \chi^{2(p-1)} P(\mathbb{C}P^{2(p-1)}) \right) = v_3\alpha^{p^2-1-2(p-1)} \]
and
\[ q \circ r_\ast \left( \chi^{p(p-1)} P(\mathbb{C}P^{p(p-1)}) \right) = -v_3\alpha^{p^2-1-p(p-1)}. \]

In the appendix of [Law17], Lawson shows how this computation may be made internally to \( \mathbb{Z}_p[v_3]/(v_3^3) \) and the induced formal group law. Since this formal group law is much simpler than the formal group law of \( BP \), the computation that we need to make simplifies dramatically and so becomes tractable.

**Proof of Proposition 2.4.** Let us first review the method of the appendix of [Law17], revising it along the way to make it apply to odd primes. Set
\[ \langle p \rangle_F(x) = \frac{[p]_F(x)}{x}. \]
The power operation \( P : MU^{2*}(X) \to MU^{2p*}(X)[[\alpha]]/[p]_F(\alpha) \) satisfies the following properties:

1. \( P(uv) = P(u)P(v) \)
2. \( P(u) = u^p \) modulo \( \alpha \)
3. \( P(u + v) = P(u) + P(v) \) modulo \( \langle p \rangle_F(\alpha) \)
4. On the orientation class \( x \in \widetilde{MU}^2(\mathbb{C}P^\infty), P(x) = g(x, \alpha), \) the Euler class of the external tensor product of the canonical representation of \( S^1 \) with the regular representation of \( C_p \).

More explicitly, there is a formula
\[ g(x, \alpha) = x \prod_{i=1}^{p-1} (x + [i]_F(\alpha)). \]
The above properties along with naturality imply that the composite
\[ \Psi : MU^* \to MU^*[\alpha]/[p]_F(\alpha) \]
of \( P \) with the quotient map \( MU^*[[\alpha]]/[p]F(\alpha) \to MU^*[[\alpha]]/(p)F(\alpha) \) is a ring homomorphism and that the power series \( g(x, \alpha) \) defines an isogeny \( F \to \Psi^*F \). Since \( MU^* \) and \( MU^*[[\alpha]]/(p)F(\alpha) \) are torsion-free, \( F \) and \( \Psi^*F \) admit logarithms

\[
\ell_F(x) = \sum \frac{\mathbb{C} \mathbb{P}^{n-1} x^n}{n}
\]
and

\[
\ell_{\Psi^*F}(x) = \sum \frac{\Psi((\mathbb{C} \mathbb{P}^{n-1}) x^n)}{n}.
\]

By lifting the \( \Psi(\mathbb{C} \mathbb{P}^n) \) to \( MU^*[[\alpha]] \), we may view these as power series in \( MU^*[[x]] \) and \( MU^*[[x, \alpha]] \).

When lifting \( g(x, \alpha) \) and \( \chi \) to the power series ring, we will find it convenient to replace the integers \( i = 1, \ldots, p - 1 \) in their formulae with the \((p - 1)\)st roots of unity. We therefore work in the tensor product

\[
MU^*_p = MU^* \otimes_{\mathbb{Z}} \mathbb{Z}_p
\]
from this point on. Let \( \omega \in \mathbb{Z}_p \) denote a primitive \((p - 1)\)st root of unity. The formal group \( F \) admits the structure of a \( \mathbb{Z}_p \)-module over \( MU^*_p \), so that there are endomorphisms \([\omega^i]F(x) \) of \( F \). Since \( \omega, \ldots, \omega^{p-1} \) form a set of representatives for \( 1, \ldots, p - 1 \) modulo \( p \), we find that

\[
(1) \quad \chi \equiv \prod_{i=1}^{p-1} [\omega^i]_F(\alpha)
\]
and that

\[
(2) \quad g(x, \alpha) \equiv x \prod_{i=1}^{p-1} (x + F [\omega^i]_F(\alpha)).
\]

We lift \( g(x, \alpha) \) and \( \chi \) to \( MU^*_p[[x, \alpha]] \) and \( MU^*_p[[\alpha]] \) using Equation (1) and Equation (2).

Now, by taking the derivative of the equation \( g(x, \alpha) + \Psi^*F g(y, \alpha) \equiv g(x + F y, \alpha) \) in \( MU^*_p[[x, \alpha]]/(p)F(\alpha) \) with respect to \( y \) and evaluating at \( y = 0 \), we obtain the equation

\[
\frac{g'(0, \alpha)}{(\ell_{\Psi^*F})'(g(x, \alpha))} \equiv \frac{g'(x, \alpha)}{(\ell_F)'(x)}
\]
in \( MU^*_p[[x, \alpha]]/(p)F(\alpha) \). The implies the existence of an equality

\[
g'(x, \alpha) \cdot (\ell_{\Psi^*F})'(g(x, \alpha)) = \chi \cdot (\ell_F)'(x) + h(x, \alpha) \cdot (p)F(\alpha)
\]
in \( MU^*_p[[x, \alpha]] \) for some \( h(x, \alpha) \in MU^*_p[[x, \alpha]] \).

Next we make a substitution \( x = \chi y \) and write \( g(\chi y, \alpha) = \chi^2 k(y, \alpha) \) for some \( k(y, \alpha) \in MU^*_p[[y, \alpha]] \) with linear term \( y \). This implies that a composition inverse \( k^{-1}(y, \alpha) \) of \( k(y, \alpha) \) exists. We therefore obtain an equation

\[
\chi \cdot k'(y, \alpha) \cdot (\ell_{\Psi^*F})'(\chi^2 k(y, \alpha)) = \chi \cdot (\ell_F)'(\chi y) + h(\chi y, \alpha) \cdot (p)F(\alpha),
\]
which implies that the equation
\[(\ell_{\Psi F})'(\chi^2 y) = (\ell_F)'(\chi k^{-1}(y, \alpha)) \cdot (\kappa^{-1})'(y, \alpha) + \tilde{h}(y, \alpha) \cdot \langle p \rangle F(\alpha)\]
also holds for some \(\tilde{h}(y, \alpha)\). Letting \(f_n(\alpha)\) equal the coefficient of \(y^n\) in
\[(\ell_F)'(\chi k^{-1}(y, \alpha)) \cdot (\kappa^{-1})'(y, \alpha),\]
this implies that
\[\Psi(\mathbb{C}P^n)\chi^{2n} = f_n(\alpha) + \tilde{h}_n(\alpha) \cdot \langle p \rangle F(\alpha)\]
for some \(\tilde{h}_n(\alpha) \in MU_p^{*}[\alpha]\).

Now suppose that we are given a map \(MU_p^{*} \to R^{*}\) with \(R^{*}\) torsion-free. Then we may compute the image of \(f_n(\alpha)\) in \(R^{*}\) by applying the process above to \(R^{*}\) and its induced formal group law. Furthermore, there exists a polynomial \(h_n(\alpha)\) in \(\alpha\) so that
\[f_n(\alpha) - h_n(\alpha) \cdot \langle p \rangle F(\alpha) \equiv (\mathbb{C}P^n)^p\]
in \(R^{*}[\alpha]/(\chi^{2n}\alpha)\) by property (2) of the power operation \(P\) listed above. Since \(R^{*}\) is torsion-free, then \(\langle p \rangle F(\alpha)\) is not a zero divisor in \(R^{*}[\alpha]\), so that \(h_n(\alpha)\) is uniquely determined in \(R^{*}\) and therefore may be computed there.

In conclusion, we obtain that
\[f(\chi^n P(\mathbb{C}P^n)) \equiv \chi^{-n}(f_n(\alpha) - h_n(\alpha) \cdot \langle p \rangle F(\alpha))\]
in \(R^{*}[\alpha]/\langle p \rangle F(\alpha)\).

We begin by noticing that, since the formal group law \(F\) of \(\mathbb{Z}_p[v_3]/(v_3^2)\) is \(p\)-typical, \([\omega^i]F(x) = \omega^i x\). Therefore
\[\chi = \prod_{i=1}^{p-1} \omega^i \alpha = -\alpha^{p-1}\]
and
\[g(x, \alpha) = x \prod_{i=1}^{p-1} (x + F (\omega^i \alpha)).\]

Our first order of business is to compute \(g(x, \alpha)\). To do this, we begin by noting that the logarithm is
\[\ell_F(x) = x + \frac{v_3}{p} x^{p^3},\]
which implies that
\[x + F y = x + y + \frac{v_3}{p} (x^{p^3} + y^{p^3} - (x + y)^{p^3}).\]
We also note that
\[\langle p \rangle F(\alpha) = p\alpha - (p^{p^3-1} - 1)v_3 \alpha^{p^3},\]
so that
\[\langle p \rangle F(\alpha) = p - (p^{p^3-1} - 1)v_3 \alpha^{p^3-1}.\]
We therefore compute
\[ g(x, \alpha) = x \prod_{i=1}^{p-1} (x + F(\omega^i \alpha)) \]
\[ = x \prod_{i=1}^{p-1} (x + \omega^i \alpha) \left[ 1 + \frac{v_3}{p} \sum_{j=1}^{p-1} \frac{(x + \omega^j \alpha)^{p^3} - x^{p^3} - (\omega^j \alpha)^{p^3}}{x + \omega^j \alpha} \right] \]
\[ \equiv x \prod_{i=1}^{p-1} (x + \omega^i \alpha) + O(x^{p^3}) \mod [p]_F(x) \]
\[ = x(x^{p-1} - \alpha^{p-1}) + O(x^{p^3}) \]
\[ = \chi x + x^p + O(x^{p^2}) \]
where we have used the fact that \( pv_3 \alpha = 0 \) modulo \([p]_F(\alpha)\). Now we need to compute the coefficients \( f_n(\alpha) \) of \((\ell_F)'(\chi k^{-1}(y, \alpha)) \cdot (k^{-1})'(y, \alpha)\).

We first obtain \( k(y, \alpha) \) by change variables in \( g(x, \alpha) \)
\[ k(y, \alpha) = y - \chi^{p-2} y^p + O(y^{p^2}) \]
Applying Lagrange inversion, we find that
\[ k^{-1}(y, \alpha) = y + \sum_{n=1}^{p} \frac{\binom{np}{n}}{n(p-1)} x^n y^{(p-1)n+1} + O(y^{p^2}) \]
and therefore that
\[ (k^{-1})'(y, \alpha) = 1 + \sum_{n=1}^{p} \binom{np}{n} x^n y^{(p-1)n} + O(y^{p-1}) \]
Next, we note that the \((\ell_F)'(\chi k^{-1}(y, \alpha))\) term does not contribute because \( (\ell_j)'(x) = 1 + O(x^{p^3-1}) \)
and therefore
\[ (\ell_F)'(\chi k^{-1}(y, \alpha)) = 1 + O(y^{p^3-1}). \]
We conclude that
\[ f_{i(p-1)}(\alpha) = \binom{ip}{i} \chi^i(\alpha) \]
for \( i = 2, \ldots, p \).

Since \( (\mathbb{C}[v_3])^p = 0 \) in \( \mathbb{Z}_{(p)[v_3]}/(v_3^2) \) for all \( n > 0 \), we find that
\[ h_{i(p-1)}(\alpha) = \binom{ip}{p} \chi^i(\alpha). \]
We therefore compute
\[
q \circ r_* \left( \chi^{i(p-1)} P(\mathbb{C}P^{i(p-1)}) \right) = \chi^{-i(p-1)} \cdot (f_2(p-1)(\alpha) - h_2(p-1)(\alpha) \cdot (p) F(\alpha) \\
= \chi^{-i(p-1)} \cdot (-h_i(p-1)(\alpha)) \cdot (- (p^{i-1} - 1)v_3 \alpha^{p^3-1}) \\
= -h_i(p-1)(\alpha)v_3 \alpha^{i(p-1)-1}(p-1) \\
= -\left(\frac{ip}{p}\right)v_3 \alpha^{p^3-1-i(p-1)}
\]
where we have used the fact that $pv_3\alpha \equiv 0$ modulo $[p] E(\alpha)$.

Finally, we apply the congruences $\frac{(2p)}{p} \equiv -1$ and $\frac{(p^3)}{p} \equiv 1$ mod $p$ to deduce that
\[
q \circ r_* \left( \chi^{2(p-1)} P(\mathbb{C}P^{2(p-1)}) \right) \equiv v_3 \alpha^{p^3-1-2(p-1)}
\]
and
\[
q \circ r_* \left( \chi^{p(p-1)} P(\mathbb{C}P^{p(p-1)}) \right) \equiv -v_3 \alpha^{p^3-1-p(p-1)}
\]

\[\square\]

**Remark 2.5.** We understand that Zeshen Gu has independently been working on computations of the above type.

**Remark 2.6.** In future work we will come back to these methods and use them to completely determine the action of the Dyer-Lashof action on $\pi_*(H \wedge_{MU} H)$ and consequently obtain $MU$-Nishida relations for the $MU$-homology $H^*_{MU}(R) = \pi_*(R \wedge_{MU} H)$ of $MU$-$E_n$-algebras $R$, thus addressing Problems 1.3.2 and 1.3.3 of [Law17].

### 2.2. A Dyer-Lashof Operation in the $MU$-Dual Steenrod Algebra

In this section, we apply Theorem 2.1 to compute certain Dyer-Lashof operations in $\pi_*(H \wedge_{MU} H)$, which we call the $MU$-dual Steenrod algebra. We begin by determining the structure of $\pi_*(H \wedge_{MU} H)$ as an algebra.

**Proposition 2.7.** The algebra $\pi_*(H \wedge_{MU} H)$ is isomorphic to an exterior algebra $\Lambda_{F_p}(\tau_i) \otimes \Lambda_{F_p}(\sigma m_i \mid i \neq p^k - 1)$, and the map $H \wedge H \to H \wedge_{MU} H$, upon taking homotopy, induces the map

$$
\Lambda_{F_p}(\tau_i) \otimes F_p[\xi_i] \to \Lambda_{F_p}(\tau_i) \otimes \Lambda_{F_p}(\sigma m_i \mid i \neq p^k - 1)
$$

sending $\tau_i$ to $\tau_i$ and $\xi_i$ to zero.

**Proof.** By comparison of the K"unneth spectral sequence

$$
\text{Tor}^H_{*,*}(H_*, H_* H) \Rightarrow \pi_*(H \wedge_{MU} H)
$$

with the other K"unneth spectral sequence

$$
\text{Tor}^H_{*,*}(H_*, H_* H) \Rightarrow \pi_*(H \wedge_{MU} H),
$$

we find that the first K"unneth spectral sequence collapses at the $E^2$-page. Since $\text{Tor}^H_{*,*}(H_*, H_* H)$ is isomorphic to $\Lambda_{F_p}(\tau_i) \otimes \Lambda_{F_p}(\sigma m_i \mid i \neq p^k - 1)$, the description
of $\pi_*(H \wedge_{MU} H)$ follows. The assertion about the map $H \wedge H \rightarrow H \wedge_{MU} H$ follows from the naturality of the Künneth spectral sequence.

**Remark 2.8.** Note that the second Künneth spectral sequence above gives an alternative description of $\pi_*(H \wedge_{MU} H)$ as $\Lambda_{F_p}(\sigma x_i)$. Furthermore, Lawson [Law17] shows that for $x \in \pi_n R$ for $n \geq 1$, there is a distinguished choice of $\sigma x \in \pi_*(H \wedge R H)$: he shows that there is a map $H_*(SL_1(R)) \rightarrow \pi_{*+1}(H \wedge R H)$ which sends the Hurewicz image of $x \in \pi_n R \cong \pi_n SL_1(R)$ to a distinguished choice of $\sigma x$.

Furthermore, this map $\sigma: \pi_n R \rightarrow \pi_{*+1}(H \wedge R H)$ annihilates decomposables. Whenever we write $\sigma x$ for $x \in \pi_n R$, we will be referring to this distinguished choice of $\sigma x$.

**Theorem 2.9.** In $\pi_*(H \wedge_{MU} H)$, we have

$$Q^{p^2+p-1}(\sigma CP^{(p-1)}) = \sigma x_{p^3-1},$$

and

$$Q^{p^2+1}(\sigma CP^{(p-1)}) = -\sigma x_{p^3-1}.$$ 

This follows immediately from Theorem 2.1 and the following theorem.

**Theorem 2.10.** Let $y \in \pi_{2n} MU$ and suppose that

$$\chi^n P(y) = \sum_{i=0}^{\infty} c_i \alpha^i$$

for some elements $c_i \in \pi_{2(n+i)} MU$. Then the action of the Dyer-Lashof operations on $\pi_* H \wedge_{MU} H$ are determined by the equation

$$Q^k(\sigma y) = \sigma c_{k(p-1)}.$$ 

This theorem follows from the analysis of Section 4 of [Law17] once Lemma 4.2.5 and Proposition 4.4.1 are supplemented with the following odd-primary analogues.

**Lemma 2.11.** For a space $X$ with $p$th extended power $D_p(X)$, the composite diagonal map

$$H_*(X) \otimes H_*(B \Sigma_p) \rightarrow H_*(X \times B \Sigma_p) \rightarrow H_*(D_p(X))$$

on mod-$p$ homology is given by

$$x \otimes \beta_n \mapsto \sum_{j \geq 0} Q^{j+n}(P_j x)$$

and

$$x \otimes \gamma_n \mapsto \sum_{j \geq 0} \beta Q^{j+n}(P_j x) - \sum_{j \geq 0} Q^{j+n}(P_j \beta x)$$

where $\beta_n$ is dual to $u^n$ in $H^*(B \Sigma_p) \cong \mathbb{F}_p[u] \otimes \Lambda_{\mathbb{F}_p}[v]$, $\gamma_n$ is dual to $u^{n-1} v$, $P_j$ is the homology operation dual to $P^j$, and $P_j \beta$ is the homology operation dual to $\beta P^j$.

**Proof.** This follows from the definition of the Dyer-Lashof operations and Proposition 9.1 of [May70].
Proposition 2.12. Let $p$ be an odd prime. Then the multiplicative Dyer-Lashof operations in the Hopf ring of an $E_\infty$-ring satisfy the following identity whenever $y$ is in the homology of the path component of zero:

$$\hat{Q}^s([1] \# y) \equiv [1] \# \hat{Q}^s(y) \pmod{\# \text{ and } \circ \text{ decomposables}}.$$  

We first prove a lemma.

Lemma 2.13. In the situation of Proposition 2.12, for any $x$ there exist elements $z_i$ for $0 < i < |x|$ such that the additive Dyer-Lashof operations satisfy

$$Q^s(x) = Q^s[1] \circ x + \sum Q^s_i[1] \circ z_i.$$  

Therefore $Q^s(x)$ is $\circ$-decomposable for any $x$ and any $s > 0$.

Proof. This follows from the formula

$$Q^s[1] \circ x = \sum Q^{s+i}([1] \circ P_i x)$$

of II.1.6 of [CLM76] by inducting on the degree of $x$. □

Proof of Proposition 2.12. We apply the mixed Cartan formula, which states that

$$\hat{Q}^s(x \# y) = \sum_{s_0 + \cdots + s_p = s} \sum \hat{Q}^s_0(x_0 \otimes y_0) \# \cdots \# \hat{Q}^s_p(x_p \otimes y_p)$$

where

$$\Delta_{p+1}(x \otimes y) = \sum (x_0 \otimes y_0) \otimes \cdots \otimes (x_p \otimes y_p)$$

and where

$$\hat{Q}^s_p(x \otimes y) = \hat{Q}^s(\epsilon(x) y),$$

$$\hat{Q}^s_0(x \otimes y) = \hat{Q}^s(x \epsilon(y)),$$

and for $0 < i < p$ we put $m_i = \frac{1}{p} \binom{p}{i}$ so that

$$\hat{Q}^s_i(x \otimes y) = [m_i] \circ \left( \sum Q^j(x_1 \circ \cdots \circ x_i \circ y_1 \circ \cdots \circ y_{p-i}) \right).$$

where $\Delta_i = \sum x_1 \otimes \cdots \otimes x_i$ and $\Delta_{p-i} = \sum y_1 \otimes \cdots \otimes y_{p-i}$.

Applying this to the case that $x = [1]$ and $y$ is in the homology of the path component of zero, we first note that this is $\#$-decomposable and hence zero unless all of but one of the terms lies in degree 0, i.e. unless all of the $y_i = [0]$ and $s_i = 0$ for all but one $i$.

Using Lemma 2.13, we further deduce that all of the terms with $s_i \neq 0$ for some $0 < i < p$ are zero. Finally, we note that $\hat{Q}^s_p([1] \otimes y) = \hat{Q}^s([1]) = 0$ for $s > 0$, so that in fact the only term left is

$$\hat{Q}^s_0([1] \otimes y) \# \hat{Q}^s_0([1] \otimes [0]) \# \cdots \# \hat{Q}^s_0([1] \otimes [0]) = \hat{Q}^s y \# [1].$$

1Here we follow [Law17] in our notation for the two products in a Hopf ring: $\#$ is the additive product and $\circ$ the multiplicative.
All that remains is to show that the multiplicity of this term is one, i.e. that
\[(1 \otimes y) \otimes (1 \otimes [0]) \otimes \cdots \otimes (1 \otimes [0])\]
appears with coefficient one in \(\Delta_{p+1}(1 \otimes y)\).
That this term appears with coefficient \(p + 1 \equiv 1\) in \(\Delta_{p+1}(1 \otimes x)\) follows from the fact that \(\Delta_{p+1}(1) = [1] \otimes \cdots \otimes [1]\) and that \(x \otimes [0] \otimes \cdots \otimes [0]\) appears in \(\Delta_{p+1}(x)\) with coefficient one.

Our next goal is to deduce Theorem 1.3 from Theorem 2.9 by noting that the Dyer-Lashof operations exhibited therein are incompatible with the existence of a highly structured map \(H \wedge MU \rightarrow H \wedge BP\). We begin by showing that a highly structured map \(MU \rightarrow BP\) would induce a (slightly less) highly structured map \(H \wedge MU \rightarrow H \wedge BP\).

**Proposition 2.14.** Let \(R\) be an \(E_\infty\)-ring and let \(A \rightarrow B\) denote a map of \(E_n\)-rings augmented over \(R\). Then there exists a natural map \(R \wedge A \rightarrow R \wedge B\) of \(E_n-1(R \wedge R)\)-algebras.

**Proof.** Let \(C\) denote the \(\infty\)-category \(\text{Alg}^{E_n-1}_{R}\) of \(E_n-1-R\)-algebras, equipped with the symmetric monoidal structure induced by that of \(\text{Mod}_R\). Then the bar construction defines a functor \(\text{Bar} : \text{Alg}(\mathcal{C})_R \rightarrow \mathcal{C}\) by Example 5.2.2.3 of [Lur]. By Theorem 5.1.2.2 of [Lur], \(\text{Alg}(\mathcal{C})\) is equivalent to \(\text{Alg}^{E_n}_{R}\), so that \(\text{Bar}\) defines a functor from augmented \(E_n\)-rings to \(E_n-1-R\)-algebras.

Since the forgetful functor \(\mathcal{C} \rightarrow \text{Mod}_R\) preserves sifted colimits by Proposition 3.2.3.1 of [Lur], \(\text{Bar}\) is computed in \(R\)-modules and so \(\text{Bar}(\cdot) \cong R \wedge -\) as functors into \(R\)-modules.

This implies the existence of a natural map \(R \wedge_A R \rightarrow R \wedge_B R\) of \(E_n-1(R \wedge R)\)-modules. Applying the functor \(-\wedge (R \wedge R)\) yields the desired map \(R \wedge A \rightarrow R \wedge B\) of \(E_n-1(R \wedge R)\)-algebras. \(\square\)

We are now ready to prove Theorem 1.3. In this proof, we allow \(p\) to be 2: in this case, Theorem 2.9 may be replaced by Corollary 4.4.3 of [Law17]. At \(p = 2\), Lawson indicated in Remark 4.4.4 of [Law17] that the following argument should work in the case of \(BP\).

**Proof of Theorem 1.3.** For the sake of simplicity of notation, we prove Theorem 1.3 for \(BP\). The proof for \(BP\langle n\rangle\) with \(n \geq 3\) is analogous. Taking the \(p\)-completion changes nothing because we are only using the mod \(p\) homology. If the K"unneth spectral sequences
\[
\text{Tor}^{H_{BP}}_{*,*}(H_*H) \Rightarrow \pi_*(H \wedge BP H)
\]
and
\[
\text{Tor}^{\pi_*}_{*,*}(H_*H) \Rightarrow \pi_*(H \wedge BP H)
\]
collapse at the \(E^2\)-term. So there are isomorphisms \(\pi_*(H \wedge BP H) \cong \Lambda_{FP}(\tau_i)\) and \(\pi_*(H \wedge BP H) \cong \Lambda_{FP}(\sigma v_i)\).

Suppose that there were a map of \(E_{2p+3}\)-rings \(MU \rightarrow BP\). By the naturality of Postnikov towers of \(E_{2p+3}\)-rings, this is a map of \(E_{2p+3}\)-algebras augmented over \(H\).
Then Proposition 2.14 implies that this induces a map $H \wedge_{MU} H \to H \wedge_{BP} H$ of $E_{2p+2}(H \wedge H)$-algebras. Forgetting the action of the left $H$, we obtain a map of $E_{2p+2}$-$H$-algebras.

We claim that the induced map $\Lambda_{p}(\sigma x_{k}) \cong H \wedge_{MU} H \to H \wedge_{BP} H \cong \Lambda_{p}(\sigma v_{k})$ sends $\sigma x_{p-1}$ to a nonzero multiple of $\sigma v_{k}$. Assuming this, we obtain a contradiction with the operation $Q_{p}^{x_{p-1}} \sigma x_{p} = C_{2} \sigma x_{p-1}$ of Theorem 2.9 because $\sigma x_{p}$ goes to zero in $\Lambda_{p}(\sigma v_{k})$ for degree reasons. This operation is preserved by maps of $E_{2p+2}$-$H$-algebras by Theorem 1.8.

To prove the claim, we use the fact that $\text{Tor}^{H,\Lambda}_{*,*}(H, H, H)$ is concentrated in homological degree zero and is therefore just $H \otimes_{H, BP} H$. The induced map

$$H \otimes_{H, MU} H \to H \otimes_{H, BP} H$$

is automatically surjective; therefore the induced map of Künneth spectral sequences is surjective on the $E^{2}$ and therefore on the $E^{\infty}$ term because it collapses at the $E^{2}$-term. We conclude that the map on indecomposables is surjective, which is equivalent to the claim.

$$\square$$

3. A Secondary Power Operation in the Dual Steenrod Algebra

In this section, we define and compute a secondary power operation in the dual Steenrod algebra and then show that Theorem 1.2 follows from this computation. We make free use of the formalism of Toda brackets in categories enriched over pointed topological spaces developed in Section 2 of [Law17], including the juggling, additivity and Peterson-Stein formulae of Propositions 2.3.5 and 2.4.3.

Notation 3.1. Given a set $S$ of formal variables with gradings, we let $\mathbb{P}_{H}^{H}(S)$ denote the free $E_{\infty}$-$H$-algebra on the wedge of spheres $\bigvee_{x \in S} S^{[x]}$ and let $x \in \pi_{[x]}(\mathbb{P}_{H}^{H}(S))$ denote the homotopy element corresponding to the fundamental class $t_{[x]} \in \pi_{[x]}(S^{[x]})$.

Let $x$ be a formal variable with degree $2(p-1)$ and let $\mathbb{F}_{H}^{2(p^{2}+2)}(x)$ denote the free $E_{2(p^{2}+2)}$-$H$-algebra on $x$. Then we will let $\mathcal{D}$ denote the category $\text{Alg}_{H}^{E_{2(p^{2}+2)}}/\mathbb{E}_{H}^{2(p^{2}+2)}(x)$ of $E_{2(p^{2}+2)}$-$H$-algebras under $\mathbb{P}_{H}^{2(p^{2}+2)}(x)$. This is a topological category, so the category $\mathcal{C} = \mathcal{D}_{\pm}$ of possibly pointed or augmented objects ([Law17], Definition 2.2.2) in this category is enriched over pointed topological spaces.

Whenever we take brackets in the below, it will be in the category $\mathcal{C}$. Given a set of graded elements $S$, we always view $\mathbb{P}_{H}^{2(p^{2}+2)}(x, S)$ as an element of $\mathcal{C}$ via the augmentation $\mathbb{P}_{H}^{2(p^{2}+2)}(x, S) \to \mathbb{P}_{H}^{2(p^{2}+2)}(x)$ sending $x$ to $x$ and all of the elements of $S$ to 0.

Notation 3.2. In the following, we will make our computations in the exterior quotient $\Lambda_{p}(\tau_{0}, \tau_{1}, \ldots)$ of the dual Steenrod algebra $H_{*}H$; we call this quotient $E_{*}$. 

THE BROWN-PETERSON SPECTRUM IS NOT $E_{2(p^{2}+2)}$ AT ODD PRIMES 13
3.1. **Dyer-Lashof operations in** $H_*(MU)$ **and** $H_*(H)$. We will need to be able to compute Dyer-Lashof operations in $H_*(MU)$ and $H_*(H)$. We will find the description of this action in terms of Newton polynomials convenient for our purposes, so we review how this works. Our choice to describe the action in this way was heavily influenced by Section 5 of [Bak15].

We define the mod $p$ Newton polynomials $N_n(t) = N_n(t_1, \ldots, t_n) \in \mathbb{F}_p[t_1, \ldots, t_n]$ by setting $N_1(t) = t_1$ and inductively letting

$$N_n(t) = t_1N_{n-1}(t) - t_2N_{n-2}(t) + \cdots + (-1)^{n-2}t_{n-1}N_1(t) + (-1)^{n-1}nt_n.$$  

Then the following useful relation holds:

$$N_{pn}(t) = (N_n(t))^p \mod p.$$  

We let $N_n(b) \in H_*MU$ be defined by setting $t_n = b_n$, and let $N_n(\xi) \in H_*MU$ be defined by setting $t_{p^k-1} = \xi_k$ and the other $t_n$ to zero. Writing out the recurrence for $N_{p^k-1}(\xi)$ shows that $N_{p^k-1}(\xi) = -\overline{\xi}_k$ where $x \mapsto \overline{x}$ is the conjugation in the Hopf algebra $H_*H$.

Kochman [Koc73] showed that the action of the Dyer-Lashof operations on $N_n(b)$ is described by the formula:

$$Q^rN_n(b) = (-1)^{r+n}{r-1 \choose n-1}N_{n+r(p-1)}(b).$$  

Since the orientation $MU \to H$ maps $b_{p^k-1} \to \xi_k$ and the other $b_n$ to zero, it maps $N_n(b)$ to $N_n(\xi)$ and so we also have:

$$Q^rN_n(\xi) = (-1)^{r+n}{r-1 \choose n-1}N_{n+r(p-1)}(\xi).$$  

Using $N_{p^k-1}(\xi) = -\overline{\xi}_k$, we get:

$$Q^r\overline{\xi}_k = (-1)^{r+1}{r-1 \choose p^k-2}N_{p^k-1+r(p-1)}(\xi).$$  

Using the above formulae, we may deduce the following two propositions by direct calculation.

**Proposition 3.3.** In the dual Steenrod algebra $H_*H$, the following identities hold:

- $Q^p\overline{\xi}_1 = (\overline{\xi}_1^{-1})^pQ^p\overline{\xi}_1$
- $Q^p\overline{\xi}_1 = 0$ for $i = 1, \ldots, p - 2$
- $Q^{p^2+p-1}\overline{\xi}_1 = -(Q^p(\overline{\xi}_1))^p$
- $Q^{p-1+p}(\overline{\xi}_1^{-1}) = -(\overline{\xi}_1)^p$
- $Q^{p^2+p}Q^i\overline{\xi}_1 = 0$ for $i = 1, \ldots, p - 1$
- $Q^{2p}\overline{\xi}_1 = -\overline{\xi}_1(Q^p(\overline{\xi}_1))$
Proposition 3.4. The following identities hold in $H_*(MU)$:

\[
Q^{p^2}N_{p-1}(b) = \frac{1}{2}Q^{p^2-1}N_{2(p-1)}(b)
\]

\[
Q^{p^2+i}N_{p-1}(b) = 0 \text{ for } i = 1, \ldots, p - 2
\]

\[
Q^{p^2+p-1}N_{p-1}(b) = -(Q^p(N_{p-1}(b)))^p
\]

\[
Q^{p^2-p+1}(N_{p-1}(b)^p-1) = -(N_{p-1}(b))^{(p-2)p}(N_{2(p-1)}(b))^p
\]

\[
Q^{p^2+pi}Q^pN_{p-1}(b) = 0 \text{ for } i = 1, \ldots, p - 1
\]

\[
Q^{2p}N_{p-1}(b) = -\frac{1}{2}Q^{2p-1}(N_{2(p-1)}(b))
\]

3.2. A Relation Among Power Operations. We will define the secondary operation of interest to us in terms of the following relation between primary power operations.

Proposition 3.5. Let $R$ be an $E_{2(p^2+2)}$-$H$-algebra and $x \in \pi_{2(p-1)}(R)$. Define classes $a_i$, $i = 0, \ldots, p - 1$; $b, c_i, i = 1, \ldots, p$ in $\pi_*(R)$ by the following formulae:

\[
a_0 = Q^{p^2}x - (x^{p-1})^pQ^p x
\]

\[
a_i = Q^{p^2+i}x \text{ for } i = 1, \ldots, p - 2
\]

\[
a_{p-1} = Q^{p^2+p-1}x + (Q^p x)^p
\]

\[
b = Q^{p^2-p+1}(x^{p-1}) + x^{p^2}
\]

\[
c_i = Q^{p^2+pi}Q^p x \text{ for } i = 1, \ldots, p - 1
\]

\[
c_p = Q^{2p}x + (Q^p x)x^p
\]

Then the following identity holds:

\[
0 = Q^{p^3+p}a_0 + \sum_{i=1}^{p-2}(-1)^iQ^{p^3+p-i}a_i + Q^{p^3+1}a_{p-1} + \\
Q^pQ^{p^2}bQ^p x + \sum_{i=1}^{p-1}(Q^{p^2-p-i+1}(x^{p-1}))^p c_i + (x^{p-1})^{p^2}Q^{2p^2-p}c_p
\]

Proof. This is defined for $E_{2(p^2+2)}$-$H$-algebras by Theorem [LS] because the operation which takes the greatest $n$ to be defined on $E_n$-$H$-algebras is the $Q^{p^3+p}$ in $Q^{p^3+p}a_0$. Since $|a_0| = 2(p-1)(p^2+1)$, we conclude that this is defined and satisfies the usual properties whenever

\[
n \geq 2(p^3 + p) - 2(p-1)(p^2 + 1) + 2 = 2(p^2 + 2).
\]
The desired identity reduces to the following identities, which may be deduced
from the Adem relations, the instability relations, and the Cartan formula:

\[
\begin{align*}
Q^{p^3+p}Q^{p^2}x &= \sum_{i=1}^{p-1}(-1)^{i+1}Q^{p^3+p-i}Q^{p^2+i}x \\
Q^{p^3+1}(Q^p)x^p &= 0 \\
Q^{p^3+p}(x^{p-1})Q^{p}x &= \sum_{i=0}^{p}(Q^{p^2-p-i+1}(x^{p-1}))p^i Q^px \\
Q^{2p^2}Q^{p}x &= Q^{2p^2-p}Q^{2p}x \\
Q^{2p^2-p}(x^{p}Q^{p}x) &= x^{p^2}Q^{p^2}Q^{p}x.
\end{align*}
\]

\[\square\]

Let the symbols \(a_i, i = 0, \ldots, p - 1; b; c_j, j = 1, \ldots, p\) have the gradings of the
the elements in Proposition 3.5, and let \(d\) have the grading of the relation there
described. Then the relation above determines maps

\[
Q : \mathbb{P}^{2(p^2+2)}_H(x, a_0, \ldots, a_{p-1}, b, c_0, \ldots, c_{p-1}) \to \mathbb{P}^{2(p^2+2)}_H(x)
\]

and

\[
R : \mathbb{P}^{2(p^2+2)}_H(x, d) \to \mathbb{P}^{2(p^2+2)}_H(x, a_0, \ldots, a_{p-1}, b, c_0, \ldots, c_{p-1})
\]
such that the composition \(Q \circ R\) is nullhomotopic.

**Proposition 3.6.** The bracket \(\langle \overline{\xi}_1, Q, R \rangle\) is defined in \(H \ast H\) and has zero indeter-
minacy in the quotient \(E_\ast = \Lambda_{F_p}(\tau_0, \tau_1, \ldots)\) of \(H \ast H\).

**Proof.** To show that the bracket is defined, we need to show that \(Q(\overline{\xi}_1) = 0\). This
is equivalent to Proposition 3.5

The indeterminacy comes from degree \(2p^3 + 2p^2 + 2p + 1\) homotopy operations
applied to \(\overline{\xi}_1\) and from the image of the suspended operation \(\sigma R\). All homotopy
operations are generated by multiplication, addition, the operations \(Q^n\) and \(\beta Q^n\),
and the Browder bracket. Since \(H\) is \(E_\infty\), the Browder bracket always vanishes.
The rest of these operations preserve the subalgebra of \(H \ast H\) generated by the \(\overline{\xi}_i\)
and therefore the first sort of indeterminacy is trivial in \(E_\ast\).

Up to indecomposables, \(\sigma R\) is equal to \(Q^{p^3+p}\sigma a_0 + \sum_{i=1}^{p-2}(-1)^i Q^{p^3+p-i} \sigma a_i + Q^{p^3+\sigma a_{p-1}},\) where the \(\sigma a_i\) are variables in degree one higher than \(a_i\). So \(|\sigma a_i| = (p^2 + i + 1)(p - 1) + 1, i = 0, \ldots, p - 1\). Since \(E_\ast\) is decomposable in these degrees, we
conclude as above that the second sort of indeterminacy must be decomposable in
\(E_\ast\). Since there are no nonzero decomposables in \(E_\ast\) in degree \(2p^4 - 1\), we conclude
that the indeterminacy must actually be trivial in \(E_\ast\).

\[\square\]
3.3. Computation of the Secondary Operation. To compute this operation, we will first juggle it into a functional operation for the map $H \wedge MU \to H \wedge H$. To this end, we define maps:

$$\mu : \mathbb{P}^{2(p^2+2)}_H(x, a_0, \ldots, a_{p-1}, b, c_0, \ldots, c_{p-1}) \to \mathbb{P}^{2(p^2+2)}_H(x, y_{2(p-1)})$$

$$\nu : \mathbb{P}^{2(p^2+2)}_H(x, d) \to \mathbb{P}^{2(p^2+2)}_H(x, y_{2(p-1)})$$

$$\alpha : \mathbb{P}^{2(p^2+2)}_H(x, d) \to \mathbb{P}^{2(p^2+2)}_H(x, w_1, \ldots, w_{p-2}, c_1, \ldots, c_{p-1}, z_{p^2(p-1)}, z_{(2p+1)(p-1)})$$

$$\beta : \mathbb{P}^{2(p^2+2)}_H(x, w_1, \ldots, w_{p-2}, c_1, \ldots, c_{p-1}, z_{p^2(p-1)}, z_{(2p+1)(p-1)}) \to \mathbb{P}^{2(p^2+2)}_H(x, y_{2(p-1)})$$

with the $y_i$ and the $z_i$ in grading $2i$ and the $w_i$ in grading $2(p-1)(p^2+i+2)$, by:

$$\mu(a_0) = Q^{p^2-1}y_{2(p-1)} - (x^{p-1})^pQ^p x$$

$$\mu(a_i) = 0 \text{ for } i \neq 0$$

$$\mu(b) = -2x^{(p-2)}p y_{2(p-1)} + x^p$$

$$\mu(c_i) = 0 \text{ for } i \neq p$$

$$\mu(c_p) = -Q^{2p-1}y_{2(p-1)} + (Q^p x) x^p$$

$$\nu(d) = -Q^p z_{p^3-1}$$

$$\alpha(d) = \sum_{i=1}^{p-2} \sigma_i Q^{p^3+p-(i+1)}w_i - z_{p^2(p-1)}^{p^3} Q^{p^2} Q^{p^2} x$$

$$- \sum_{i=1}^{p-1} (Q^{p^2-p-i+1}(x^{p-1}))^p c_i - (x^{p-1})^{p^2} Q^{2p^2-p} z_{(2p+1)(p-1)}$$

$$\beta(w_i) = Q^{p^2+i} y_{2(p-1)}$$

$$\beta(c_i) = Q^{p^2+i} Q^p x$$

$$\beta(z_{p^2(p-1)}) = \frac{1}{2} x^{p(p-2)} y_{2(p-1)}^{p^2} Q^{p^2-p+1} x^{p-1}$$

$$\beta(z_{(2p+1)(p-1)}) = Q^{2p-1} y_{2(p-1)} + Q^{2p} x$$

Here we choose the $\sigma_i$ such that

$$Q^{p^3+p} Q^{p^2-1} y_{2(p-1)} = \sum_{i=1}^{p-2} \sigma_i Q^{p^3+p-(i+1)} Q^{p^2+i} y_{2(p-1)} - Q^{p^3} Q^{p^2+p-1} y_{2(p-1)}.$$
where \( f \) is the map defined by sending \( x \) to \( -N_{p-1}(b) \) and \( y_{2(p-1)} \) to \(-\frac{N_{2(p-1)}(b)}{2}\).

**Proof.** The proof of the identity \( \mu R = Q + \beta \alpha \) follows directly from the relations

\[
Q^{p^3 + p} Q^{p^2 - 1} y_{2(p-1)} = \sum_{i=1}^{p-2} \sigma_i Q^{p^3 + p - (i+1)} Q^{p^2 + i} y_{2(p-1)} - Q^{p^3} Q^{p^2 + p - 1} y_{2(p-1)}
\]

and

\[
Q^{p^3 + p} ((x^{p-1})^p Q^p x) = \sum_{i=0}^{p} (Q^{p^2 - p - i + 1} (x^{p-1}))^p Q^{p^2 + pi} Q^p x.
\]

The right triangle of the diagram commutes because \( \bar{\xi}_1 = -N_{p-1}(\xi) \) and hence \( p(-N_{p-1}(b)) = \bar{\xi}_1 \). The left square commutes by Proposition 3.4.

\( \square \)

**Proposition 3.8.** There is an equality \( \langle \bar{\xi}_1, Q, R \rangle \equiv -Q^p(\langle p, f, \nu \rangle) \) in \( E_\ast \).

**Proof.** Exactly as in [Law17], the juggling relations for brackets imply the following sequence of identities because each term is defined

\[
\langle \bar{\xi}_1, Q, R \rangle = \langle p N_{p-1}(b), Q, R \rangle \subset \langle p, N_{p-1}(b) Q, R \rangle = \langle p, f \mu, R \rangle \supset \langle p, f, \mu R \rangle = \langle p, f, Q \nu + \beta \alpha \rangle \subset \langle p, f, Q \nu \rangle + \langle p, f, \beta \alpha \rangle \supset \langle p, f, Q \nu \rangle + \langle p, f, \beta \alpha \rangle.
\]

To show that we have equality up to decomposables in \( E_\ast \), it suffices to show that the indeterminacy of “local maxima” \( \langle p, N_{p-1}(b) Q, R \rangle \) and \( \langle p, f, Q \nu \rangle + \langle p, f, \beta \alpha \rangle \) are decomposable in \( E_\ast \). The total indeterminacy of these two brackets is made up of elements of three kinds. The first are in the image of \( H_\ast MU \rightarrow H_\ast H \), which maps to zero in \( E_\ast \). The second are in the image of \( \sigma R \), which we already dealt with in the proof of Proposition 3.6. Finally, there are elements in the images of \( \sigma(Q \nu) \) and \( \sigma(\beta \alpha) \). These are either decomposable or multiples of Dyer-Lashof operations

\[
\begin{align*}
\mathbb{P}_H^{2(p^2+2)}(x, a_0, \ldots, a_{p-1}, b, c_0, \ldots, c_{p-1}) \quad \xrightarrow{Q} & \quad \mathbb{P}_H^{2(p^2+2)}(x) \\
& \downarrow \mu \\
\mathbb{P}_H^{2(p^2+2)}(x, y_{2(p-1)}) \quad \xrightarrow{f} & \quad H \land MU \\
& \downarrow \phi \\
& \quad \xrightarrow{p} H \land H
\end{align*}
\]
applied to a class in degree $2(p - 1) + 1$; there are no nonzero indecomposables in $E_*$ in this degree.

Finally, we note that $\alpha$ applied to any set of classes in $H_*H$ is decomposable in $E_*$ because $E_*$ has no nonzero indecomposables in the degrees of $w_i, i = 1, \ldots, p - 1$. Therefore the second term is zero modulo decomposables.

Since there are no nonzero decomposables in degree $2p^4 - 1$ of $E_*$, we conclude that this holds on the nose in $E_*$. \qed

Finally, we note that $\alpha$ applied to any set of classes in $H_*H$ is decomposable in $E_*$ because $E_*$ has no nonzero indecomposables in the degrees of $w_i, i = 1, \ldots, p - 1$.

Therefore the second term is zero modulo decomposables. Since there are no nonzero decomposables in degree $2p^4 - 1$ of $E_*$, we conclude that this holds on the nose in $E_*$. \qed

Finally, we compute the bracket $\langle p, f, Q \rangle$ by means of Theorem 2.9.

**Proposition 3.9.** There is an equality $\langle p, f, Q \rangle \equiv C\tau_3$ in $E_*$ for some nonzero $C \in \mathbb{F}_p$.

**Proof.** By noting that each pair of maps in the diagram

$$
P_H^{2(p^2+2)}(x, y_{p^2-1}) \xrightarrow{\overline{Q}} P_H^{2(p^2+2)}(x, y_{2(p-1)}) \xrightarrow{f} H \wedge MU \xrightarrow{p} H \wedge H \xrightarrow{i} H \wedge MU$$

compose to a nullhomotopic map in $C$, we find that we are allowed to apply the Peterson-Stein formula to obtain the equality

$$i\langle p, f, Q \rangle = -(i, p, f)\overline{Q}.$$

By Proposition 2.6.5 of [Law17], $\sigma(-\frac{N_2(p-1)(b)}{2}) \in \langle i, p, f \rangle$. Since

$$-\frac{N_2(p-1)(b)}{2} \equiv b_{2(p-1)} \equiv -\frac{CP^{2(p-1)}}{2p-1} \equiv CP^{2(p-1)}$$

modulo decomposables, where we view $CP^n$ as an element of homology via the Hurewicz map, we have $\sigma(-\frac{N_2(p-1)(b)}{2}) = \sigma CP^{2(p-1)}$. By Theorem 2.9 $\overline{Q}$ applied to this is $-Q^{p^2+p-1}\sigma CP^{2(p-1)} = -\sigma\tau_3$. Since $i$ is an isomorphism modulo decomposables in this degree, we conclude that $C\tau_3 \equiv \langle p, f, Q \rangle$ modulo decomposables for some nonzero $C \in \mathbb{F}_p$, as desired.

As before, we upgrade this from a result modulo decomposables in $E_*$ to a precise result in $E_*$ by noting that there are no nonzero decomposables in degree $2p^3 - 1$ of $E_*$.

\qed

**Corollary 3.10.** There exists a nonzero $C \in \mathbb{F}_p$ and an equality $\langle \xi_1, Q, R \rangle \equiv C\tau_4$ in $E_*$.

**Proof.** Combine Propositions 3.8 and 3.9. \qed

Since maps of $E_{2(p^2+2)}$-ring spectra must preserve secondary power operations by Proposition 2.1.10 of [Law17], we immediately obtain the following corollary.

**Corollary 3.11.** Let $R$ be an $E_{2(p^2+2)}$-ring spectrum and let $R \rightarrow H$ be a map of $E_{2(p^2+2)}$-ring spectra. Then if the induced map on homology $H_*R \rightarrow H_*H$ is injective in degrees less than or equal to $(2p^2 + 1)(p - 1)$ and contains $\xi_1$ in its image, then $\tau_4$ must also be in the image of the composite $H_*R \rightarrow H_*H \rightarrow E_*$.

We conclude by deducing Theorem 1.2 from Corollary 3.11.
Proof of Theorem 1.2. Assume that $BP$ were an $E_{2(p^2+2)}$-ring spectrum. Since the Postnikov tower of an $E_n$-ring spectrum naturally lifts to a tower of $E_n$-ring spectra, there is a map of $E_{2(p^2+2)}$-ring spectra

$$BP \to \tau_{\leq 0}BP \cong H\mathbb{Z}(p) \to H$$

which induces the inclusion

$$\mathbb{F}_p[\xi_1, \xi_2, \ldots] \hookrightarrow \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \ldots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \ldots]$$

upon taking homology. In particular, the map is injective and contains $\xi_1$ in its image. However, $\tau_4$ cannot be in the image of $H_*BP \to H_*H \to E_*$ because this composite is zero.

The case of $BP(n)$ for $n \geq 4$ is analogous, using the fact that

$$H_*(BP(n)) \cong \Lambda_{\mathbb{F}_p}(\tau_{n+1}, \tau_{n+2}, \ldots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \ldots].$$

Finally, taking $p$-completions makes no difference because we are only working with mod $p$ homology in the first place. □

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