Coalgebraic Trace Semantics for Büchi and Parity Automata

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\section*{Abstract}
Despite its success in producing numerous general results on state-based dynamics, the theory of coalgebra has struggled to accommodate the Büchi acceptance condition—a basic notion in the theory of automata for infinite words or trees. In this paper we present a clean answer to the question that builds on the “maximality” characterization of infinite traces (by Jacobs and Cîrstea): the accepted language of a Büchi automaton is characterized by two commuting diagrams, one for a least homomorphism and the other for a greatest, much like in a system of (least and greatest) fixed-point equations. This characterization works uniformly for the nondeterministic branching and the probabilistic one; and for words and trees alike. We present our results in terms of the parity acceptance condition that generalizes Büchi’s.

\section*{1 Introduction}

\textbf{Büchi Automata} Automata are central to theoretical computer science. Besides their significance in formal language theory and as models of computation, many formal verification techniques rely on them, exploiting their balance between expressivity and tractable complexity of operations on them. See e.g. \cite{34,14}. Many current problems in verification are about nonterminating systems (like servers); for their analyses, naturally, automata that classify infinite objects—such as infinite words and infinite trees—are employed.

The Büchi acceptance condition is the simplest nontrivial acceptance condition for automata for infinite objects. Instead of requiring finally reaching an accepting state—which makes little sense for infinite words/trees—it requires accepting states visited infinitely often. This simple condition, too, has proved both expressive and computationally tractable: for the word case the Büchi condition can express any \(\omega\)-regular properties; and the emptiness problem for Büchi automata can be solved efficiently by searching for a lasso computation.

\textbf{Coalgebras} Studies of automata and state-based transition systems in general have been shed a fresh categorical light in 1990’s, by the theory of coalgebra. Its simple modeling of state-based dynamics—as a coalgebra, i.e. an arrow \(c: X \to FX\) in a category \(C\)—has produced numerous results that capture mathematical essences and provide general techniques. Among its basic results are: behavior-preserving maps as homomorphisms; a final coalgebra as a fully abstract domain of behaviors; coinduction (by finality) as definition and proof principles; a general span-based definition of bisimulation; etc. See e.g. \cite{18,27}. More advanced results are on: coalgebraic modal logic (see e.g. \cite{9}); process algebras and congruence formats (see e.g. \cite{20}); generalization of Kleene’s theorem (see e.g. \cite{27}); etc.
Büchi Automata, Coalgebraically  In the coalgebra community, however, two important phenomena in automata and/or concurrency have been known to be hard to model—many previous attempts have seen only limited success. One is internal ($\tau$-)transitions and weak (bi)similarity; see e.g. recent [13]. The other one is the Büchi acceptance condition.

Here is a (sketchy) explanation why these two phenomena should be hard to model coalgebraically. The theory of coalgebra is centered around homomorphisms as behavior-preserving maps; see the diagram on the right. Deep rooted in it is the idea of local matching between one-step transitions in $c$ and those in $d$. This is what fails in the two phenomena: in weak bisimilarity a one-step transition in $c$ is matched by a possibly multi-step transition in $d$; and the Büchi acceptance condition—stipulating that accepting states are visited infinitely often, in the long run—is utterly nonlocal.

There have been some works that study Büchi acceptance conditions (or more general parity or Muller conditions) in coalgebraic settings. One is [7], where they rely on the lasso characterization of nonemptiness and use Sets$^2$ as a base category. Another line is on coalgebra automata (see e.g. [35]), where however Büchi/parity/Muller acceptance conditions reside outside the realm of coalgebras inspired by these works, and also by our work in [10] on alternating fixed points and coalgebraic model checking, the current paper introduces a coalgebraic modeling of Büchi and parity automata based on systems of fixed-point equations.

Contributions We present a clean answer to the question of “Büchi automata, coalgebraically,” relying on the previous work on coalgebraic infinitary trace semantics [17] [8] and fixed-point equations [16]. Our modeling, hinted in [1], features: 1) accepting states as a partition of a state space; and 2) explicit use of $\mu$ and $\nu$—for least/greatest fixed points—in diagrams. We state our results for the parity condition (that generalizes the Büchi one).

Our framework is generic: its leading examples are nondeterministic and (generative) probabilistic tree automata, with the Büchi/parity acceptance condition.

Our contributions are: 1) coalgebraic modeling of automata with the Büchi/parity conditions; 2) characterizing their accepted languages by diagrams with $\mu$'s and $\nu$'s (tr$^p$ in [1]); and 3) proving that the characterization indeed captures the conventional definitions. The last “sanity-check” proves to be intricate in the probabilistic case, and our proof—relying on previous [8] [24]—identifies the role of final sequences [30] in probabilistic processes.

With explicit $\mu$'s and $\nu$'s—that specify in which homomorphism, among many that exist, we are interested—we depart from the powerful reasoning principle of finality (existence of a unique homomorphism). We believe this is a necessary step forward, for the theory of coalgebra to take up long-standing challenges like the Büchi condition and weak bisimilarity. Our characterization [1]—although it is not so simple as the uniqueness argument by finality—seems useful, too: we have obtained some results on fair simulation notions between Büchi automata [31], following the current work.  

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1 More precisely: a coalgebra automaton is an automaton (with Büchi/parity/Muller acceptance conditions) that classifies words and trees (as generalization of coalgebras). A coalgebra automaton itself is not described as a coalgebra; nor is its acceptance condition.
Organization of the Paper. In §2 we provide backgrounds on: the coalgebraic theory of trace in a Kleisli category [17, 8] (where we explain the diagram on the left in 1); and systems of fixed-point equations. In §3 we present a coalgebraic modeling of Büchi/parity automata and their languages. Coincidence with the conventional definitions is shown in §4 for the nondeterministic setting, and in §5 for the probabilistic one.

Most proofs are deferred to the appendix.

Future Work. Here we are based on the coalgebraic theory of trace and simulation [24, 17, 15, 29]; it has been developed under the trivial acceptance condition (any run that does not diverge, i.e. that does not come to a deadend, is accepted). The current paper is about accommodating the Büchi/parity conditions in the trace part of the theory; for the simulation part we also have exploited the current results to obtain sound fair simulation notions for nondeterministic Büchi tree automata and probabilistic Büchi word automata [31].

On the practical side our future work mainly consists of proof methods for trace/language inclusion, a problem omnipresent in formal verification. Simulations—as one-step, local witnesses for trace inclusion—have been often used as a sound (but not necessarily complete) proof method that is computationally more tractable; with the observations in [31] we are naturally interested in them. Possible directions are: synthesis of simulation matrices between finite systems by linear programming, like in [30]; synthesis of simulations by other optimization techniques for program verification (where problem instances are infinite due to the integer type); and simulations as a proof method in interactive theorem proving.

2 Preliminaries

2.1 Coalgebras in a Kleisli Category

We assume some basic category theory, most of which is covered in [18].

The conventional coalgebraic modeling of systems—as a function \(X \to FX\)—is known to capture branching-time semantics (such as bisimilarity) [18, 25]. In contrast accepted languages of Büchi automata (with nondeterministic or probabilistic branching) constitute linear-time semantics; see [32] for the so-called linear time-branching time spectrum.

For the coalgebraic modeling of such linear-time semantics we follow the “Kleisli modeling” tradition [24, 17, 15]. Here a system is parametrized by a monad \(T\) and an endofunctor \(F\) on Sets: the former represents the branching type while the latter represents the (linear-time) transition type; and a system is modeled as a function of the type \(X \to TFX\).

A function \(X \to TFX\) is nothing but an \(\bar{F}\)-coalgebra \(X \to pY\) in the Kleisli category \(K(\mathcal{T})\)—where \(\bar{F}\) is a suitable lifting of \(F\). This means we can apply the standard coalgebraic machinery to linear-time behaviors, by changing the base category from Sets to \(K(\mathcal{T})\).

A monad \(\mathcal{T} = (T, \eta, \mu)\) on a category \(\mathcal{C}\) induces the Kleisli category \(K(\mathcal{T})\). The objects of \(K(\mathcal{T})\) are the same as \(\mathcal{C}\)'s; and for each pair \(X, Y\) of objects, the homset \(K(\mathcal{T})(X, Y)\) is given by \(\mathcal{C}(X, TY)\). An arrow \(f \in K(\mathcal{T})(X, Y)\)—that is \(X \to TY\) in \(\mathcal{C}\)—is called a Kleisli arrow and is denoted by \(f : X \to Y\) for distinction. Given two successive Kleisli arrows \(f : X \to Y\) and \(g : Y \to Z\), their Kleisli composition is given by \(\mu_Z \circ Tg \circ f : X \to Z\) (where

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2 Another eminent approach to coalgebraic linear-time semantics is the Eilenberg-Moore one (see e.g. [19, 11]): notably in the latter a system is expressed as \(X \to FTX\). The Eilenberg-Moore approach can be seen as a categorical generalization of determinization or the powerset construction. It is however not clear how determinization serves our current goal (namely a coalgebraic modeling of the Büchi/parity acceptance conditions).
The technical requirement of being standard Borel—meaning that it arises from a polynomial functor on \( \text{Sets} \)—is restricted to be polynomial; this essentially means that we are dealing with systems that generate trees over some ranked alphabet (with additional \( T \)-branching).

**Definition 2.3** (Tree\( _\Sigma \)). An (infinite) \( \Sigma \)-tree, as in the standard definition, is a possibly infinite tree whose nodes are labeled with the ranked alphabet \( \Sigma \) and whose branching degrees are consistent with the arity of labels. The set of \( \Sigma \)-trees is denoted by Tree\( _\Sigma \).

**Lemma 2.4.** Let \( \Sigma \) be a ranked alphabet, and \( F_\Sigma = \prod_{\sigma \in \Sigma} \langle \_ \rangle^{\sigma |} \) be the corresponding polynomial functor on \( \text{Sets} \). The set Tree\( _\Sigma \) of (infinite) \( \Sigma \)-trees carries a final \( F_\Sigma \)-coalgebra.

The same holds in \( \text{Meas} \), for countable \( \Sigma \) and the corresponding polynomial functor \( F_\Sigma \).

We collect some standard notions and notations for such trees in Appendix A.

It is known [13, 24] that for \((C, T) \in \{(\text{Sets}, \mathcal{P}), (\text{Meas}, \mathcal{G})\}\) and polynomial \( F \) on \( C \), there is a canonical distributive law [22] \( \lambda: FT \Rightarrow TF \), making \( TF \) a natural transformation compatible with \( T \)'s monad structure. Such \( \lambda \) induces a functor \( F: \mathcal{K}(T) \to \mathcal{K}(T) \) that makes the diagram (2) commute.

Using this lifting \( \mathcal{T} \) of a polynomial functor \( F \) on \( C \) to \( \mathcal{K}(T) \), an arrow \( c: X \to TFX \) in \( C \)—that is how we model an automaton—can be regarded as an \( \mathcal{T} \)-coalgebra \( c: X \to FX \) in \( \mathcal{K}(T) \).

Then the dynamics of \( A \)—ignoring its initial and accepting states—is modeled as an \( \mathcal{T} \)-coalgebra \( c: X \to FX \) in \( \mathcal{K}(T) \) where: \( F = \langle a, b \rangle \times \langle \_ \rangle \), \( X = \{x_1, x_2\} \) and \( c: X \to FX \) is the function \( c(x_1) = c(x_2) = \{(a, x_1), (b, x_2)\} \). The information on initial and accepting states is redeemed later in §3.1.
Finite trace

Finite trace semantics (3) in Table 1: typically the results apply to generalizing the theory in [21]. Here we review the former; it underpins our developments in (5) later; this is deemed as a summary of existing results on coalgebraic trace semantics.

Example 2.5. Let \( M \) be the Markov chain on the right. The dynamics of \( M \) is modeled as an \( \mathcal{F} \)-coalgebra \( c: X \to \mathcal{F}X \) in \( \mathcal{K}(\mathcal{G}) \) where: \( F = \{a, b\} \times \{x_1, x_2\} \) with the discrete measurable structure, and \( c: X \to \mathcal{G}FX \) is the (measurable) function defined by \( c(x) \{\{a, x_1\}\} = c(x) \{\{b, x_2\}\} = 1/2 \), and \( c(x) \{\{d, x\}'\} = 0 \) for the other \( \{d, x\}' \in \{a, b\} \times X \).

Later we will equip Markov chains with accepting states and obtain (generative) probabilistic Büchi automata. Their probabilistic accepted languages will be our subject of study.

Remark 2.6. Due to the use of the sub-Giry monad is that, in \( f: X \to Y \) in \( \mathcal{K}(\mathcal{G}) \), the probability \( f(x)(Y) \) can be smaller than 1. The missing \( 1 - f(x)(Y) \) is understood as that for divergence. In the nondeterministic case \( f: X \to Y \) in \( \mathcal{K}(\mathcal{P}) \) diverges at \( x \) if \( f(x) = \emptyset \).

This is in contrast with a system coming to halt generating a 0-ary symbol (such as \( \checkmark \) in [5] later); this is deemed as successful termination.

2.2 Coalgebraic Theory of Trace

The above “Kleisli” coalgebraic modeling has produced some general results on: linear-time process semantics (called trace semantics); and simulations as witnesses of trace inclusion, generalizing the theory in [21]. Here we review the former; it underpins our developments later. A rough summary is in Table 1: typically the results apply to \( T \in \{\mathcal{P}, \mathcal{D}, \mathcal{G}\} \)—where \( \mathcal{D} \) is the subdistribution monad on \( \text{Sets} \), a discrete variant of \( \mathcal{G} \)—and polynomial \( F \). In what follows we present these previous results in precise terms, sometimes strengthening the assumptions for the sake of presentation. The current paper’s goal is to incorporate the Büchi acceptance condition in the right column of Table 1.

Firstly, finite trace semantics—linear-time behaviors that eventually terminate, such as the accepted languages of finite words for NFAs—is captured by finality in \( \mathcal{K}(T) \).

Theorem 2.7 ([15]). Let \( T \in \{\mathcal{P}, \mathcal{D}\} \) and \( F \) be a polynomial functor on \( \text{Sets} \). An initial \( F \)-algebra \( A \to A \) in \( \text{Sets} \) yields a final \( \mathcal{F} \)-coalgebra in \( \mathcal{K}(T) \), as in (3) in Table 1. ▲

The carrier \( A \) of an initial \( F \)-algebra in \( \text{Sets} \) is given by finite words/trees (over the alphabet that corresponds to \( F \)). The significance of Thm. 2.7 is that: for many examples, the unique homomorphism \( \text{tr}(c) \) induced by finality [3] captures the finite trace semantics of the system \( c \). Here the word “finite” means that we collect only behaviors that eventually terminate.

What if we are also interested in nonterminating behaviors, like the infinite word \( b^\omega = \ldots bbb \) accepted by the automaton in Example 2.5? There is a categorical characterization of such infinitary trace semantics too, although proper finality is now lost.

Theorem 2.8 ([17, 8, 29]). Let \( (C, T) \in \{(\text{Sets}, \mathcal{P}), (\text{Meas}, \mathcal{G})\} \) and \( F \) be a polynomial functor on \( C \). A final \( F \)-coalgebra \( \zeta: Z \to Z \) in \( C \) gives rise to a weakly final \( \mathcal{F} \)-coalgebra in \( \mathcal{K}(T) \), as in (4) in Table 1. Moreover, the coalgebra \( \mathcal{J}_\zeta \) additionally admits the greatest homomorphism \( \text{tr}^\infty(c) \) with respect to the pointwise order \( \subseteq \) in the homsets of \( \mathcal{K}(T) \) (given by inclusion for \( T = \mathcal{P} \), and by pointwise \( \leq \) on subprobability measures for \( T = \mathcal{G} \)). That is: for each homomorphism \( f \) from \( c \) to \( \mathcal{J}_\zeta \) we have \( f \subseteq \text{tr}^\infty(c) \). ▲
In many examples the greatest homomorphism \( \text{tr}^\infty(c) \) captures the infinitary trace semantics of the system \( c \). (Here by infinitary we mean both finite and infinite behaviors.) For example, for the system \( \mathcal{G} \) where \( \nu \) denotes successful termination, its finite trace semantics is \( \{\varepsilon, a, aa, \ldots\} \) whereas its infinitary trace semantics is \( \{\varepsilon, a, aa, \ldots\} \cup \{a^\omega\} \). The latter is captured by the diagram \( \mathcal{I} \), with \( T = \mathcal{P} \) and \( F = \{\nu\} + \{a\} \times \{\_\} \).

2.3 Equational Systems for Alternating Fixed Points

Nested, alternating greatest and least fixed points—as in a \( \mu \)-calculus formula \( \nu u_2.\mu u_1. (p \land u_2) \lor \Box u_1 \)—are omnipresent in specification and verification. For their relevance to the Büchi/parity acceptance condition one can recall the well-known translation of LTL formulas to Büchi automata and vice versa (see e.g. [34]). To express such fixed points we follow [10, 3] and use equational systems—we prefer them to the textual \( \mu \)-calculus-like presentations.

▶ Definition 2.9 (equational system). Let \( L_1, \ldots, L_n \) be posets. An equational system \( E \) over \( L_1, \ldots, L_n \) is an expression

\[
u_1 =_n f_1(u_1, \ldots, u_n), \ldots, \nu_n =_n f_n(u_1, \ldots, u_n)
\]

where: \( u_1, \ldots, u_n \) are variables, \( \eta_1, \ldots, \eta_n \in \{\mu, \nu\} \), and \( f_i : L_1 \times \cdots \times L_n \to L_i \) is a monotone function. A variable \( u_i \) is a \( \mu \)-variable if \( \eta_i = \mu \); it is a \( \nu \)-variable if \( \eta_i = \nu \).

The solution of the equational system \( E \) is defined as follows, under the assumption that \( L_i \)'s have enough supremums and infimums. It proceeds as: 1) we solve the first equation to obtain an interim solution \( u_1 = l(1)_1(u_2, \ldots, u_n) \); 2) it is used in the second equation to eliminate \( u_1 \) and yield a new equation \( u_2 =_n f_2(u_2, \ldots, u_n) \); 3) solving it again gives an interim solution \( u_2 = l(2)_2(u_3, \ldots, u_n) \); 4) continuing this way from left to right eventually eliminates all variables and leads to a closed solution \( u_n = l(n)_n \in L_n \); and 5) by propagating these closed solutions back from right to left, we obtain closed solutions for all of \( u_1, \ldots, u_n \).

A precise definition is found in Appendix B.

It is important that the order of equations matters: for \( (u =_\mu v, v =_\nu u) \) the solution is \( u = v = \top \) while for \( (v =_\nu u, u =_\nu v) \) the solution is \( u = v = \bot \).

Whether a solution is well-defined depends on how “complete” the posets \( L_1, \ldots, L_n \) are. It suffices if they are complete lattices, in which case every monotone function \( L_i \to L_i \) has greatest/least fixed points (the Knaster-Tarski theorem). This is used in the nondeterministic setting: note that \( \mathcal{P} \mathcal{Y} \), hence the homset \( \mathcal{K}(\mathcal{P})(X, Y) \), are complete lattices.

▶ Lemma 2.10. The system \( E \) has a solution if each \( L_i \) is a complete lattice. ◀

This does not work in the probabilistic case, since the homsets \( \mathcal{K}(\mathcal{G})(X, Y) = \text{Meas}(X, \mathcal{G}Y) \) with the pointwise order—on which we consider equational systems—are not complete lattices. For example \( \mathcal{G}Y \) lacks the greatest element in general; even if \( Y = 1 \) (when \( \mathcal{G}1 \cong [0, 1] \)), the homset \( \mathcal{K}(\mathcal{G})(X, 1) \) can fail to be a complete lattice. See Example B.2. Our strategy is: 1) to apply the following Kleene-like result to the homset \( \mathcal{K}(\mathcal{G})(X, 1) \); and 2) to “extend” fixed points in \( \mathcal{K}(\mathcal{G})(X, 1) \) along a final F-sequence. See §5.1 later.

▶ Lemma 2.11. The equational system \( E \) has a solution if: each \( L_i \) is both a pointed \( \omega \)-cpo and a pointed \( \omega^{\text{op}} \)-cpo; and each \( f_i \) is both \( \omega \)-continuous and \( \omega^{\text{op}} \)-continuous. ◀

In Appendix B we have additional lemmas on “homomorphisms” of equational systems and preservation of solutions. They play important roles in the proofs of the later results.
3 Coalgebraic Modeling of Büchi/Parity Automata and Its Trace Semantics

Here we present our modeling of Büchi/parity automata. We shall do so axiomatically with parameters \( C, T \) and \( F \)—much like in \( \S 2.1 \). Our examples cover: both nondeterministic and probabilistic branching; and automata for trees (hence words as a special case).

\begin{itemize}
  \item Assumptions 3.1. In what follows a monad \( T \) and an endofunctor \( F \), both on \( C \), satisfy:
    \begin{itemize}
      \item The base category \( C \) has a final object \( 1 \) and finite coproducts.
      \item The functor \( F \) has a final coalgebra \( \zeta : Z \to FZ \) in \( C \).
      \item There is a distributive law \( \lambda : FT \Rightarrow TF \) (22), hence \( F : C \to C \) is lifted to \( \overline{F} : K\ell(T) \to K\ell(T) \). See (2).
      \item For each \( X, Y \in K\ell(T) \), the homset \( K\ell(T)(X, Y) \) carries an order \( \subseteq_{X,Y} \) (or simply \( \subseteq \)).
      \item Kleisli composition \( \circ \) and cotupling \( \underline{\rightarrow} \) are monotone with respect to the order \( \subseteq \).
      \item The latter gives rise to an order isomorphism \( K\ell(T)(X_1 + X_2, Y) \cong K\ell(T)(X_1, Y) \times K\ell(T)(X_2, Y) \), where \( + \) is inherited along a left adjoint \( J : C \to K\ell(T) \).
      \item \( F : K\ell(T) \to K\ell(T) \) is locally monotone: for \( f, g \in K\ell(T)(X, Y) \), \( f \subseteq g \) implies \( Ff \subseteq Fg \).
    \end{itemize}
  \end{itemize}

\begin{itemize}
  \item Example 3.2. The category \( \text{Sets} \), the powerset monad \( \mathcal{P} \) (Def. 2.1) and a polynomial functor \( F \) on \( \text{Sets} \) (Def. 2.2) satisfy Assm. 3.1. Here for \( X, Y \in K\ell(\mathcal{P}) \), an order \( \subseteq_{X,Y} \) is defined by: \( f \subseteq g \) if \( f(x) \subseteq g(x) \) for each \( x \in X \).
  \item Example 3.3. The category \( \text{Meas} \), the sub-Giry monad \( \mathcal{G} \) (Def. 2.1) and a polynomial functor \( F \) on \( \text{Meas} \) (Def. 2.2) satisfy Assm. 3.1. For \( X, Y \in K\ell(\mathcal{G}) \), a natural order \( \subseteq_{(X,\mathcal{G}_X),(Y,\mathcal{G}_Y)} \) is defined by: \( f \subseteq g \) if \( f(x)(A) \leq g(x)(A) \) (in \( [0,1] \)) for each \( x \in X \) and \( A \in \mathcal{G}_Y \).
\end{itemize}

3.1 Coalgebraic Modeling of Büchi/Parity Automata

The Büchi and parity acceptance conditions have been big challenges to the coalgebra community, because of their nonlocal and asymptotic nature (see §1). One possible modeling is to take the distinction between \( \bigcirc \) vs. \( \otimes \) or different priorities in the parity case—as state labels. This is much like in the established coalgebraic modeling of deterministic automata as \( 2 \times (\otimes) \)-coalgebras (see e.g. 29, 18). Here the set \( 2 \) tells if a state is accepting or not.

A key to our current modeling, however, is that accepting states should rather be specified by a partition \( X = X_1 + X_2 \) of a state space, with \( X_1 = \{ \bigcirc \} \)-s and \( X_2 = \{ \otimes \} \)-s. This idea smoothly generalizes to parity conditions, too, by \( X_i = \{ \text{states of priority } i \} \). Equipping such partitions to coalgebras (with explicit initial states, as in §2.3) leads to the following.

Henceforth we state results for the parity condition, with Büchi being a special case.

\begin{itemize}
  \item Definition 3.4 (parity \((T,F)\)-system). A parity \((T,F)\)-system is given by a triple \( \mathcal{X} = ( (X_1,\ldots,X_n), c : X \Rightarrow F\mathcal{X}, s : 1 \rightarrow X ) \) where \( n \) is a positive integer, and:
    \begin{itemize}
      \item \( (X_1,\ldots,X_n) \) is an \( n \)-tuple of objects in \( C \) for states (with their priorities), and we define \( X = X_1 + \cdots + X_n \) (a coproduct in \( C \));
      \item \( c : X \Rightarrow F\mathcal{X} \) is an arrow in \( K\ell(T) \) for dynamics; and
      \item \( s : 1 \rightarrow X \) is an arrow in \( K\ell(T) \) for initial states.
    \end{itemize}
  \end{itemize}

For each \( i \in [1,n] \) we define \( c_i : X_i \Rightarrow F\mathcal{X} \) to be the restriction \( c \circ \kappa_i : X_i \rightarrow F\mathcal{X} \) along the coprojection \( \kappa_i : X_i \inin X \), in case the maximum priority is \( n = 2 \), a parity \((T,F)\)-system is referred to as a Büchi \((T,F)\)-system.
3.2 Coalgebraic Trace Semantics under the Parity Acceptance Condition

On top of the modeling in Def. 3.4 we characterize accepted languages—henceforth referred to as trace semantics—of parity \((T, F)\)-systems. We use systems of fixed-point equations; this naturally extends the previous characterization of infinitary traces (i.e. under the trivial acceptance conditions) by maximality (Thm. 2.8; see also (1)).

Definition 3.5 (trace semantics of parity \((T, F)\)-systems). Let \(X = ((X_1, \ldots, X_n), c, s)\) be a parity \((T, F)\)-system. It induces the following equational system \(E_X\), where \(\zeta : Z \rhd FZ\) is a final coalgebra in \(C\) (see Asm. 3.1). The variable \(u_i\) ranges over the poset \(\mathcal{K}(T)(X_i, Z)\).

\[
E_X := \begin{align*}
  u_1 & = \mu \ (J\zeta)^{-1} \circ F[u_1, \ldots, u_n] \circ c_1 \in \mathcal{K}(T)(X_1, Z) \\
  u_2 & = \nu \ (J\zeta)^{-1} \circ F[u_1, \ldots, u_n] \circ c_2 \in \mathcal{K}(T)(X_2, Z) \\
  \vdots \\
  u_n & = \eta_n \ (J\zeta)^{-1} \circ F[u_1, \ldots, u_n] \circ c_n \in \mathcal{K}(T)(X_n, Z)
\end{align*}
\]

Here \(\eta_i = \mu\) if \(i\) is odd and \(\eta_i = \nu\) if \(i\) is even. The functions in the equations are seen to be monotone, thanks to the monotonicity assumptions on cotupling, \(F\) and \(\circ\) (Asm. 3.1).

We say that \((T, F)\) constitutes a parity trace situation, if \(E_X\) has a solution for any parity \((T, F)\)-system \(X\), denoted by \(\text{tr}_p(X) : X_1 \rightarrow Z, \ldots, \text{tr}_p(X) : X_n \rightarrow Z\). The composite

\[
\text{tr}_p(X) := \left( \begin{array}{c}
  \frac{1 \dashv}{\rightarrow} X = X_1 + X_2 + \cdots + X_n \\
  \frac{\text{tr}_p(X),\text{tr}_p(X),\ldots,\text{tr}_p(X)}{\rightarrow} \frac{1}{\rightarrow} Z
\end{array} \right)
\]

is called the trace semantics of the parity \((T, F)\)-system \(X\).

If \(X\) is a Büchi \((T, F)\)-system, the equational system \(E_X\)—with their solutions \(\text{tr}_p(X)\) and \(\text{tr}_n(X)\) in place—can be expressed as the following diagrams (with explicit \(\mu\) and \(\nu\)). See (1).

\[
\begin{array}{cccc}
  FX & = & FZ \\
  \smash{\xrightarrow{\text{tr}_p(c_1),\text{tr}_p(c_2)}} & \smash{\equiv[J\zeta]_c} & \smash{\xrightarrow{\text{tr}_n(c_1),\text{tr}_n(c_2)}} & \smash{\equiv[J\zeta]_c} \\
  X_1 & \equiv[J\zeta]_c & \smash{\xrightarrow{\text{tr}_p(c_1)}} & Z \\
  \smash{\xrightarrow{\text{tr}_n(c_1),\text{tr}_n(c_2)}} & \smash{\equiv[J\zeta]_c} & \smash{\xrightarrow{\text{tr}_n(c_2)}} & Z
\end{array}
\]

4 Coincidence with the Conventional Definition: Nondeterministic

The rest of the paper is devoted to showing that our coalgebraic characterization (Def. 3.5) indeed captures the conventional definition of accepted languages. In this section we study the nondeterministic case; we let \(C = \text{Sets}, T = \mathcal{P}\), and \(F\) be a polynomial functor.

We first have to check that Def. 3.5 makes sense. Existence of enough fixed points is obvious because \(\mathcal{K}(\mathcal{P})(X_i, Z)\) is a complete lattice (Lem. 2.10). See also Example 3.2.

Theorem 4.1. \(T = \mathcal{P}\) and a polynomial \(F\) constitute a parity trace situation (Def. 3.5).

Here is the conventional definition of automata [14].

Definition 4.2 (NPTA). A nondeterministic parity tree automaton (NPTA) is a quadruple

\[
\mathcal{X} = ((X_1, \ldots, X_n), \Sigma, \delta : X \rightarrow \mathcal{P}(\biguplus_{s \in \Sigma} X^{\lceil i \rceil}), s \in \mathcal{P}X),
\]

where \(X = X_1 + \cdots + X_n\), each \(X_i\) is the set of states with the priority \(i\), \(\Sigma\) is a ranked alphabet (with the arity map \(\lceil : \Sigma \rightarrow \mathbb{N}\)), \(\delta\) is a transition function and \(s\) is the set of initial states.
The accepted language of an NPTA $\mathcal{X}$ is conventionally defined in the following way. Here we are sketchy due to the lack of space; precise definitions are in Appendix A.

A (possibly infinite) $(\Sigma \times X)$-labeled tree $\rho$ is a run of an NPTA $\mathcal{X} = (\vec{X}, \Sigma, \delta, \iota)$ if: for each node with a label $(\sigma, x)$, it has $|\sigma|$ children and we have $(\sigma, (x_1, \ldots, x_{|\sigma|})) \in \delta(x)$ where $x_1, \ldots, x_{|\sigma|}$ are the $X$-labels of its children. For a pedagogical reason we do not require the root $X$-label to be an initial state. A run $\rho$ of an NPTA $\mathcal{X}$ is accepting if any infinite branch $\pi$ of the tree $\rho$ satisfies the parity acceptance condition (i.e. $\max\{i \mid \pi \text{ visits states in } X_i \text{ infinitely often}\}$ is even). The sets of runs and accepting runs of $\mathcal{X}$ are denoted by $\text{Run}_{\mathcal{X}}$ and $\text{AccRun}_{\mathcal{X}}$, respectively.

The function $\text{rt} : \text{Run}_{\mathcal{X}} \to X$ is defined to return the root $X$-label of a run. For each $X' \subseteq X$, we define $\text{Run}_{\mathcal{X}, X'}$ by $\{ \rho \in \text{Run}_{\mathcal{X}} \mid \text{rt}(\rho) \in X'\}$; the set $\text{AccRun}_{\mathcal{X}, X'}$ is similar. The map $\text{DelSt} : \text{Run}_{\mathcal{X}} \to \text{Tree}_{\Sigma}$ takes a run, removes all $X$-labels and returns a $\Sigma$-tree.

**Definition 4.3** (Lang($\mathcal{X}$) for NPTAs). Let $\mathcal{X}$ be an NPTA. Its accepted language $\text{Lang}(\mathcal{X})$ is defined by $\text{DelSt}(\text{AccRun}_{\mathcal{X}, s})$.

The following coincidence result for the nondeterministic setting is fairly straightforward. A key is the fact that accepting runs are characterized—among all possible runs—using an equational system that is parallel to the one in Def. 3.5.

**Lemma 4.4.** Let $\mathcal{X} = (\vec{X}, \Sigma, \delta, \iota)$ be an NPTA, and $l_1^{\text{sol}}, \ldots, l_n^{\text{sol}}$ be the solution of the following equational system, whose variables $u_1, \ldots, u_n$ range over $\mathcal{P}(\text{Run}_{\mathcal{X}})$:

\begin{equation}
\begin{aligned}
u_1 &= q_1 \triangleright_{\text{X}} (u_1 \cup \cdots \cup u_n) \cap \text{Run}_{\mathcal{X}, X_1}, & \ldots & \quad u_n = q_n \triangleright_{\text{X}} (u_1 \cup \cdots \cup u_n) \cap \text{Run}_{\mathcal{X}, X_n}
\end{aligned}
\end{equation}

Here, $\triangleright_{\text{X}} : \mathcal{P}(\text{Run}_{\mathcal{X}}) \to \mathcal{P}(\text{Run}_{\mathcal{X}})$ is given by $\triangleright_{\text{X}} R := \{((\sigma, x), (r_1, \ldots, r_{|\sigma|})) \in \text{Run}_{\mathcal{X}} \mid \sigma \in \Sigma, x \in X, r_i \in R\}$ (see the figure above); $X = X_1 + \cdots + X_n$; and $\eta_i$ is $\mu$ (for odd $i$) or $\nu$ (for even $i$). Then the $i$-th solution $l_i^{\text{sol}}$ coincides with $\text{AccRun}_{\mathcal{X}, X_i}$.

We shall translate the above result to the characterization of accepted trees (Lem. 4.5). In its proof (that is deferred to the appendix) Lem. B.3 on homomorphisms of equational systems—plays an important role.

**Lemma 4.5.** Let $\mathcal{X} = (\vec{X}, \Sigma, \delta, \iota)$ be an NPTA, and let $l_1^{\text{sol}}, \ldots, l_n^{\text{sol}}$ be the solution of the following equational system, where $u'_i$ ranges over the complete lattice $(\mathcal{P}(\text{Tree}_{\Sigma}))^{X_i}$:

\begin{equation}
\begin{aligned}
u_1 &= q_1 \triangleright_{\text{X}} (u'_1 \uplus \cdots \uplus u'_n) \mid X_1, & \ldots & \quad u'_n = q_n \triangleright_{\text{X}} (u'_1 \uplus \cdots \uplus u'_n) \mid X_n
\end{aligned}
\end{equation}

Here $\eta_i$ is $\mu$ (for odd $i$) or $\nu$ (for even $i$); $(\_ \mid X_i) : (\mathcal{P}(\text{Tree}_{\Sigma}))^X \to (\mathcal{P}(\text{Tree}_{\Sigma}))^{X_i}$ denotes domain restriction; and the function $\triangleright_{\text{X}} : (\mathcal{P}(\text{Tree}_{\Sigma}))^X \to (\mathcal{P}(\text{Tree}_{\Sigma}))^{X_i}$ is given by

$(\triangleright_{\text{X}} \triangleright_{\text{X}})_{X_i}(x) := \{((\sigma, (\tau_1, \ldots, \tau_{|\sigma|})), (\sigma, (x_1, \ldots, x_{|\sigma|})) \in \delta(x), \tau_i \in T(x_i)\}.$

Then we have a coincidence $l_i^{\text{sol}} = \text{DelSt'}(\text{AccRun}_{\mathcal{X}, X_i})$, where the function $\text{DelSt'} : \mathcal{P}(\text{Run}_{\mathcal{X}}) \to (\mathcal{P}(\text{Tree}_{\Sigma}))^X$ is given by $\text{DelSt'}(\rho) := \text{DelSt}(\{\rho \in \text{R} \mid \text{rt}(\rho) = x\})$. Recall that $\text{rt}$ returns a run’s root $X$-label.

**Theorem 4.6** (coincidence, in the nondeterministic setting). Let $\mathcal{X} = ((X_1, \ldots, X_n), \Sigma, \delta, \iota)$ be an NPTA, and $F_{\Sigma} = \prod_{\sigma \in \Sigma}(\_ \mid \sigma)^{\text{sol}}$ be the polynomial functor on Sets that corresponds to $\Sigma$. Then $\mathcal{X}$ is identified with a parity $(\mathcal{P}, F_{\Sigma})$-system; moreover $\text{Lang}(\mathcal{X})$ (in the conventional sense of Def. 3.3) coincides with the coaugebraic trace semantics $\text{tr}^\sigma(\mathcal{X})$ (Def. 3.5). Note here that $\text{Tree}_{\Sigma}$ carries a final $F_{\Sigma}$-coalgebra (Lem. 2.4).
5 Coincidence with the Conventional Definition: Probabilistic

In the probabilistic setting the coincidence result is much more intricate. Even the well-definition of parity trace semantics (Def. 3.5) is nontrivial: the posets $\mathcal{K}(\mathcal{G})(X, 1)$ of our interest are not complete lattices, and they even lack the greatest element $\top$. Therefore neither of Lem. 2.10 or 2.11 ensures a solution of $E_\mathcal{G}$ in Def. 3.5. As we hinted in §2.3 our strategy is: 1) to apply the Lem. 2.11 to the homset $\mathcal{K}(\mathcal{G})(X, 1)$; and 2) to “extend” fixed points in $\mathcal{K}(\mathcal{G})(X, 1)$ along a final $F$-sequence. Implicit in the proof details below, in fact, is a correspondence between: abstract categorical arguments along a final sequence; and concrete operational intuitions on probabilistic parity automata.

In this section we let $C = \text{Meas}$, $T = \mathcal{G}$ (Def. 2.1), and $F$ be a polynomial functor.

Remark 5.1. The class of probabilistic systems of our interest are generative (as opposed to reactive) ones. Their difference is eminent in the types of transition functions:

- $X \longrightarrow \mathcal{G}(A \times X)$ (word) $X \longrightarrow \mathcal{G}(\prod_{\sigma \in \Sigma} X^{[\sigma]})$ (tree) for generative;
- $X \longrightarrow (\mathcal{G}X)^A$ (word) $X \longrightarrow \prod_{\sigma \in \Sigma} \mathcal{G}(X^{[\sigma]})$ (tree) for reactive.

A generative system (probabilistically) chooses which character to generate; while a reactive one receives a character from the environment. Reactive variants of probabilistic tree automata have been studied e.g. in [6], following earlier works like [21] on reactive probabilistic word automata. Further discussion is in Appendix C.5.

5.1 Trace Semantics of Parity $(\mathcal{G}, F)$-Systems is Well-Defined

In the following key lemma—that is inspired by the observations in [8, 26, 29]—a typical usage is for $X_A = X_1 + \cdots + X_i$ and $X_B = X_{i+1} + \cdots + X_n$.

Lemma 5.2. Let $\mathcal{X} = ((X_1, \ldots, X_n), s, c)$ be a parity $(\mathcal{G}, F)$-system, and suppose that we are given a partition $X = X_A + X_B$ of $X := X_1 + \cdots + X_n$.

We define a function $\Gamma : \mathcal{K}(\mathcal{G})(X, Z) \to \mathcal{K}(\mathcal{G})(X, 1)$ by $\Gamma(g) = \int_Z \circ g$, where $!: Z \to 1$ is the unique function of the type. Its variants $\Gamma_A : \mathcal{K}(\mathcal{G})(X_A, Z) \to \mathcal{K}(\mathcal{G})(X_A, 1)$ and $\Gamma_B : \mathcal{K}(\mathcal{G})(X_B, Z) \to \mathcal{K}(\mathcal{G})(X_B, 1)$ are defined similarly.

For arbitrary $g_B : X_B \to Z$, we define $\mathfrak{S}_g$ and $\mathfrak{S}_g^a$ as the following sets of “fixed points”:

$$\mathfrak{S}_g := \left\{ g_A : X_A \to Z \bigg| \begin{array}{l} F X \xrightarrow{[g_A, g_B]} F Z \\ e_{\mathcal{A}} \xrightarrow{h_A} Z \end{array} = \begin{array}{l} F 1 \\ e_{\mathcal{A}} \xrightarrow{h_A} 1 \end{array} \right\}$$

$$\mathfrak{S}_g^a := \left\{ h_A : X_A \to 1 \bigg| \begin{array}{l} F X \xrightarrow{[h_A, \Gamma(g_B)]} F 1 \\ e_{\mathcal{A}} \xrightarrow{h_A} 1 \end{array} = \begin{array}{l} F 1 \\ e_{\mathcal{A}} \xrightarrow{h_A} 1 \end{array} \right\}$$

(11)

Then $\Gamma_A$ restricts to a function $\mathfrak{S}_g \to \mathfrak{S}_g$. Moreover, the restriction is an order isomorphism, with its inverse denoted by $\Delta^g : \mathfrak{S}_g \cong \mathfrak{S}_g^a$. △
In the proof of the last lemma (deferred to the appendix), the inverse $\Delta^{\eta_n}$ is defined by “extending” $h_A : X_A \to 1$ to $X_A \to Z$, along the final $F$-sequence $1 \leftarrow F1 \leftarrow \cdots$ (more precisely: the image of the sequence under the Kleisli inclusion $J : \text{Meas} \to \mathcal{K}(\mathcal{G})$). We are ready to prove existence of $E_X$’s solution (Def. 3.5).

Lemma 5.3. Assume the same setting as in Lem. 5.2. We define $\Phi_X : \mathcal{K}(\mathcal{G})(X, Z) \to \mathcal{K}(\mathcal{G})(X, Z)$ and $\Psi_X : \mathcal{K}(\mathcal{G})(X, 1) \to \mathcal{K}(\mathcal{G})(X, 1)$, respectively, by

$$
\Phi_X(g) := J\zeta^{-1} \circ Fg \circ c \quad \text{and} \quad \Psi_X(h) := Jl \circ Fh \circ c ;
$$

these are like the diagrams in (11), except that the latter are parametrized by $X_A, X_B, g_B$.

Now consider the following equational systems, where: $\eta_i = \mu$ if $i$ is odd and $\eta_i = \nu$ if $i$ is even; $u_i$ ranges over $\mathcal{K}(\mathcal{G})(X_i, Z)$; and $u'_i$ ranges over $\mathcal{K}(\mathcal{G})(X_i, 1)$.

$$
E = \begin{bmatrix}
    u_1 =_{\eta_1} \Phi_X([u_1, \ldots, u_n]) \circ \kappa_1 \\
    \vdots \\
    u_n =_{\eta_n} \Phi_X([u_1, \ldots, u_n]) \circ \kappa_1
\end{bmatrix} \quad \quad E' = \begin{bmatrix}
    u'_1 =_{\eta_1} \Psi_X([u'_1, \ldots, u'_n]) \circ \kappa_1 \\
    \vdots \\
    u'_n =_{\eta_n} \Psi_X([u'_1, \ldots, u'_n]) \circ \kappa_1
\end{bmatrix} \quad \quad (12)
$$

We claim that the equational systems have solutions $(l_1^{\text{sol}}, \ldots, l_n^{\text{sol}})$ and $(l_1^{\text{sol}}, \ldots, l_n^{\text{sol}})$; and moreover, we have $\Gamma(\text{tr}(X)) = \Gamma([l_1^{\text{sol}}, \ldots, l_n^{\text{sol}}]) = [l_1^{\text{sol}}, \ldots, l_n^{\text{sol}}]$.

Theorem 5.4. $T = \mathcal{G}$ and a polynomial $F$ constitute a parity trace situation (Def. 3.5).

Remark 5.5. The process-theoretic interpretation of the isomorphism $\mathfrak{S}^{\eta_n} \cong \mathfrak{G}^{\eta_n}$ is interesting. Let us set $X_A = X$ and $X_B = \emptyset$ for simplicity. The greatest element on the left is the infinitary trace semantics (i.e. accepted languages under the trivial acceptance condition), as in Thm. 2.3 (cf. Table 1). The corresponding greatest element on the right—a function $h_A : X_A \to \mathcal{G}1 \cong [0, 1]$, assigns to each state $x \in X$ the probability with which a run from $x$ does not diverge (recall from Rem. 2.6 that the sub-Giry monad $\mathcal{G}$ allows divergence probabilities). The accepted language under the parity condition is in general an element of $\mathfrak{S}^{\eta_n}$ that is neither greatest nor least; the corresponding element in $\mathfrak{G}^{\eta_n}$ assigns to each state the probability with which it generates a accepting run (over any $\Sigma$-tree).

5.2 Probabilistic Parity Tree Automata and Its Languages

Definition 5.6 (PPTA). A (generative) probabilistic parity tree automaton (PPTA) is

$$
\mathcal{X} = ((X_1, \ldots, X_n), \Sigma, \delta : X \to \mathcal{G}([\bigcup_{s \in \Sigma} X^{[s]}]), s \in \mathcal{G}X) ,
$$

where $X = X_1 + \cdots + X_n$, each $X_i$ is a countable set and $\Sigma$ is a countable ranked alphabet. The subdistribution $s$ over $X$ is for the choice of initial states.

In Def. 5.6 the size restrictions on $X$ and $\Sigma$ are not essential: restricting to discrete $\sigma$-algebras, however, makes the following arguments much simpler.

We shall concretely define accepted languages of PPTAs, continuing $\mathfrak{G}$ and deferring precise definitions to Appendix A. This is mostly standard; a reactive variant is found in [3].

Definition 5.7 (Tree$\Sigma$ and Run$\mathcal{X}$). Let $\Sigma$ be a ranked alphabet; Tree$\Sigma$ is the set of $\Sigma$-trees. A finite $(\Sigma \cup \{\ast\})$-labeled tree $\lambda$, with its branching degrees compatible with the label arities, is called a partial $\Sigma$-tree. Here the new symbol $\ast$ (“continuation”) is deemed to be $0$-ary. The cylinder set associated to $\lambda$, denoted by $\text{Cyl}_\Sigma(\lambda)$, is the set of (non-partial) $\Sigma$-trees that have $\lambda$ as their prefix (in the sense that a subtree is replaced by $\ast$). The (smallest) $\sigma$-algebra on Tree$\Sigma$ generated by the family $\{\text{Cyl}_\Sigma(\lambda) \mid \lambda$ is a partial $\Sigma$-tree$\}$ will be denoted by $\mathfrak{S}_\Sigma$. 

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A run of a PPTA $X$ with state space $X$ is a (possibly infinite) $(\Sigma \times X)$-labeled tree whose branching degrees are compatible with the arities of $\Sigma$-labels. $\text{Run}_X$ denotes the set of runs. The measurable structure $\mathfrak{F}_X$ on $\text{Run}_X$ is defined analogously to $\mathfrak{F}_\Sigma$: a partial run $\xi$ of $X$ is a suitable $(\Sigma \cup \{\ast\}) \times X$-labeled tree; it generates a cylinder set $\text{Cyl}_X(\xi) \subseteq \text{Run}_X$; and these cylinder sets generate the $\sigma$-algebra $\mathfrak{F}_X$. Finally, the set $\text{AccRun}_X$ of accepting runs consists of all those runs all branches of which satisfy the (usual) parity acceptance condition (namely: $\max\{i \mid \pi \text{ visits states in } X_i \text{ infinitely often}\}$ is even).

The following result is much like [3, Lem. 36] and hardly novel.

**Lemma 5.8.** The set $\text{AccRun}_X$ of accepting runs is an $\mathfrak{F}_X$-measurable subset of $\text{Run}_X$. ▶

In the following NoDiv$^X(x)$ is the probability with which an execution from $x$ does not diverge: since we use the sub-Giry monad (Def. 5.6), a PPTA can exhibit divergence.

**Definition 5.9 ($\mu^\text{Run}_X$ over $\text{Run}_X$).** Let $X = ((X_1, \ldots, X_n), \Sigma, \delta, s)$ be a PPTA.

Firstly, for each $k \in \mathbb{N}$, let $\text{NoDiv}^X_{X,k} : X \to [0, 1]$ ("no divergence in $k$ steps") be defined inductively by: $\text{NoDiv}^X_{X,0}(x) := 1$ and

$$
\text{NoDiv}^X_{X,k+1}(x) := \sum_{(\xi, (x_1, \ldots, x_{|\xi|})) \in \prod_{i \in [1,|\xi|]} X_i^{|\xi|}} \delta(x)(\xi, (x_1, \ldots, x_{|\xi|})) \cdot \prod_{i \in [1,|\xi|]} \text{NoDiv}^X_{X,k}(x_i).
$$

We define $\text{NoDiv}^X(x) := \bigwedge_{k \in \mathbb{N}} \text{NoDiv}^X_{X,k}(x)$.

Secondly we define a subprobability measure $\mu^\text{Run}_X$ over $\text{Run}_X$. It is given by

$$
\mu^\text{Run}_X(\text{Cyl}_X(\xi)) := s(\text{rt}(\xi)) \cdot P_X(\xi) \quad \text{for each partial run } \xi, \text{ where } P_X(\xi) \text{ is given by }$

$$
P_X(\xi) :=
\begin{cases}
\text{NoDiv}_X(x) & \text{if } \xi = ((s, x)); \\
\delta(x)(\sigma, (\text{rt}(\xi_1), \ldots, \text{rt}(\xi_{|\xi|}))) \cdot \prod_{i \in [1,|\xi|]} P_X(\xi_i) & \text{if } \xi = ((\sigma, x), (\xi_1, \ldots, \xi_{|\xi|})).
\end{cases}
$$

The above extends to a measure thanks to Carathéodory’s theorem. See Lem. [3, 7].

Thirdly we introduce a measure $\mu^\text{Tree}_X$ over $\Sigma^\text{Tree}$ ("which trees are generated by what probabilities"). It is a push-forward measure of $\mu^\text{Run}_X$ along $\text{DelSt} : \text{Run}_X \to \Sigma^\text{Tree}$:

$$
\mu^\text{Tree}_X(\Sigma^\text{Tree}(\lambda)) := \mu^\text{Run}_X(\text{DelSt}^{-1}(\Sigma^\text{Tree}(\lambda)) \cap \text{AccRun}_X) \quad \text{for each partial } \Sigma\text{-tree } \lambda.
$$

Since $X$ is countable DelSt is easily seen to be measurable.

Finally, the accepted language $\text{Lang}(X) \subseteq \mathcal{G}(\Sigma^\text{Tree})$ of $X$ is defined by $\mu^\text{Tree}_X$ in the above.

### 5.3 Coincidence between Conventional and Coalgebraic Languages

**Lemma 5.10.** Let $X = ((X_1, \ldots, X_n), \Sigma, \delta, s)$ be a PPTA with $X = \bigsqcup_i X_i$, and $\Psi^X : [0, 1]^X \to [0, 1]^X$, $\Psi^X(p)(x) := \sum_{(\sigma, x_1, \ldots, x_{|\sigma|}) \in \prod_{i \in [1,|\sigma|]} X_i^{|\sigma|}} \delta(x)(\sigma, (x_1, \ldots, x_{|\sigma|})) \cdot \prod_{i \in [1,|\sigma|]} p(x_i)$.

Let us define $\mu^\text{Tree}_X(x) := \mu^\text{Tree}_X(\xi(x))$ where $\xi(x)$ is the PPTA obtained from $X$ by changing its initial distribution $s$ into the Dirac distribution $\delta_x$; $\mu^\text{Run}_X$ is similar. We define $\text{AccProb}_X : X \to [0, 1]$—it assigns to each state the probability of generating an accepting run—by $\text{AccProb}_X(x) := \mu^\text{Run}_X(\text{AccRun}_X)$.

Consider the following equational system, where $u'_i$ ranges over $K(I(\mathcal{G}))(X_i, 1)$, and $(\_ \mid X_i)$ denotes domain restriction.

$$
u_1 =_n \Psi^X([u'_1, \ldots, u'_n]) \mid X_1, \ldots, u_n =_n \Psi^X([u'_1, \ldots, u'_n]) \mid X_n
$$

We claim: 1) the system has a solution $u_1^{\text{sol}}, \ldots, u_n^{\text{sol}}$; and 2) $[u_1^{\text{sol}}, \ldots, u_n^{\text{sol}}] = \text{AccProb}_X$. ▶
Its proof (in the appendix) relies on Lem. B.4 on homomorphisms of equational systems.

Theorem 5.11 (coincidence, in the probabilistic setting). Let $\mathcal{X} = ((X_1, \ldots, X_n), \Sigma, \delta, s)$ be a PPTA, and $X = X_1 + \cdots + X_n$, and $F_{\Sigma}$ be the polynomial functor on $\text{Meas}$ that corresponds to $\Sigma$. Then $\mathcal{X}$ is identified with a parity $(G, F_{\Sigma})$-system; moreover its coalgebraic trace semantics $\text{tr}^p(\mathcal{X})$ (Def. 3.5) coincides with the (probabilistic) language $\text{Lang}(\mathcal{X})$ concretely defined in Def. 5.9. Precisely: $\text{tr}^p(\mathcal{X})(\bullet)(U) = \text{Lang}(\mathcal{X})(U)$ for any measurable subset $U$ of $\text{Tree}_{\Sigma}$, where $\bullet$ is the unique element of 1 in $\text{tr}^p(\mathcal{X}): 1 \to G(\text{Tree}_{\Sigma})$.

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A Tree, Run, and Accepting Run

Here are some supplementary definitions on (conventional notions) of nondeterministic/probabilistic tree automata. See first §4 and §5.2.

Remark A.1. We let \(\mathbb{N}^*\) and \(\mathbb{N}^\omega\) denote the sets of finite and infinite sequences over natural numbers, respectively. We let \(\mathbb{N}^\infty := \mathbb{N}^* \cup \mathbb{N}^\omega\). Concatenation of finite/infinite sequences, and/or characters are denoted simply by juxtaposition. Given an infinite sequence \(\pi = \pi_1\pi_2\ldots \in \mathbb{N}^\omega\) (here \(\pi_i \in \mathbb{N}\)), its prefix \(\pi_1\ldots\pi_n\) is denoted by \(\pi_{\leq n}\).

The following formalization of trees and related notions is standard, with its variations used e.g. in [6]. A sequence \(w \in \mathbb{N}^*\) is understood as a position in a tree.

Definition A.2 (\(\Sigma\)-tree). Let \(\Sigma\) be a ranked alphabet, with each element \(\sigma \in \Sigma\) coming with its arity \(|\sigma| \in \mathbb{N}\). A \(\Sigma\)-tree \(\tau\) is given by a nonempty subset \(\text{Dom}(\tau) \subseteq \mathbb{N}^*\) (called the domain of \(\tau\)) and a labeling function \(\tau : \text{Dom}(\tau) \to \Sigma\) that are subject to the following conditions.

1. \(\text{Dom}(\tau)\) is prefix-closed: for any \(w \in \mathbb{N}^*\) and \(i \in \mathbb{N}\), \(w_i \in \text{Dom}(\tau)\) implies \(w \in \text{Dom}(\tau)\). See Fig. 1.
2. \(\text{Dom}(\tau)\) is lower-closed: for any \(w \in \mathbb{N}^*\) and \(i,j \in \mathbb{N}\), \(w_j \in \text{Dom}(\tau)\) and \(i \leq j\) imply \(w_i \in \text{Dom}(\tau)\). See Fig. 1.
3. The branching degrees are consistent with the label arities: for any \(w \in \text{Dom}(\tau)\), let \(\sigma = \tau(w)\). Then \(w_0,w_1,\ldots,w(|\sigma| - 1)\) belong to \(\text{Dom}(\tau)\), and \(w_i \notin \text{Dom}(\tau)\) for any \(i\) such that \(|\sigma| \leq i\). See Fig. 2.

The set of all \(\Sigma\)-trees shall be denoted by \(\text{Tree}_\Sigma\).

The following definitions are almost standard in the tree-automata literature, too. A notable difference here, that is for a pedagogical reason, is that the root of a run is not required to be a initial state. That is also natural in our current coalgebraic study; in the coalgebraic contexts initial states are usually unspecified.

Definition A.3 (run). A run \(\rho\) of an NPTA (Def. 4.2) \(\mathcal{A} := ((X_1,\ldots,X_n),\Sigma,\delta,s)\) is a (possibly infinite) tree whose nodes are \((\Sigma \times X)\)-labeled—here \(X = X_1 + \cdots + X_n\)—subject to the following conditions.

---

3 We shall use the same notation \(\tau\) for a tree itself and its labeling function. Confusion is unlikely.
1. (Tree) The nonempty subset \( \text{Dom}(\rho) \subseteq \mathbb{N}^* \) that is subject to the same conditions (of being prefix-closed and lower-closed) as for \( \Sigma \)-trees (Def. A.2).

2. (Branching degree) The labeling function \( \rho: \text{Dom}(\rho) \rightarrow \Sigma \times X \) is such that, if \( \rho(w) = (\sigma, x) \), then \( w \) has precisely \( |\sigma| \) successors \( w0, w1, \ldots, w(|\sigma| - 1) \in \text{Dom}(\rho) \).

3. (Transition) Successors are reachable by a transition, in the sense that \( \sigma_x \) successors \( \delta(xw, w) \in \text{DelSt}(\text{Dom}(\rho)) \), where \( \rho(w) \) is labeled with \( (\sigma_w, x_w) \), and \( \rho(w) \) is labeled with \( (\sigma_{w_i}, x_{w_i}) \) for any \( 0 \leq i < |\sigma| \).

The set of all runs of the NPTA \( \mathcal{X} \) is denoted by \( \text{Run}_{\mathcal{X}} \).

A run \( \rho \) of a PPTA (Def. 5.6) is defined similarly, though it is required to satisfy only Cond. 1–2 in the above. This relaxed condition is natural—for impossible transitions we simply assign the probability 0. The set of all runs of a PPTA \( \mathcal{X} \) is also denoted by \( \text{Run}_{\mathcal{X}} \).

The map that takes a run \( \rho \in \text{Run}_{\mathcal{X}} \), removes its \( \Sigma \)-labels (i.e. applies the first projection to each label), and returns the resulting \( \Sigma \)-labeled tree (that is easily seen to be a \( \Sigma \)-tree, Def. A.2), is denoted by \( \text{DelSt}: \text{Run}_{\mathcal{X}} \rightarrow \text{Tree}_{\Sigma} \). We say that a run \( \rho \) is over the \( \Sigma \)-tree \( \text{DelSt}(\rho) \).

A branch of a tree is a maximal path from its root \( \varepsilon \).

**Definition A.4** (branch). Let \( \tau \) be a \( \Sigma \)-tree. An (infinitary) branch of \( \tau \) is either:

- an infinite sequence \( \pi = \pi_1\pi_2\ldots \in \mathbb{N}^\omega \) (where \( \pi_i \in \mathbb{N} \)) such that any finite prefix \( \pi_{\leq n} = \pi_1\ldots\pi_n \) of it belongs to \( \text{Dom}(\tau) \); or
- a finite sequence \( \pi = \pi_1\ldots\pi_n \in \mathbb{N}^* \) (where \( \pi_i \in \mathbb{N} \)) that belongs to \( \text{Dom}(\tau) \) and such that \( \pi_0 \not\in \text{Dom}(\tau) \) (meaning that \( \pi \) is a leaf of \( \tau \), and that \( \tau(\pi) \) is a 0-ary symbol).

The set of all branches of a \( \Sigma \)-tree \( \tau \) is denoted by \( \text{Branch}(\tau) \).

**Definition A.5** (accepting run). A run \( \rho \) of an NPTA (or a PPTA) \( \mathcal{X} = ((X_1, \ldots, X_n), \Sigma, \delta, s) \) is said to be accepting if any branch \( \pi \in \text{Branch}(\rho) \) of \( \mathcal{X} \) satisfies either of the following conditions:

- the branch \( \pi \) is an infinite sequence \( \pi = \pi_1\pi_2\ldots \in \mathbb{N}^\omega \), and the \( X \)-labels \( x_{\pi_1}, x_{\pi_1\pi_2}, \ldots \) along the branch satisfies the parity acceptance condition, that is, \( \max\{i \in [1, n] \mid x_{\pi_1\ldots\pi_i} \in X_i \text{ for infinitely many } k \in \omega \} \text{ is even} \); or
- the branch \( \pi \) is a finite sequence \( \pi = \pi_1\ldots\pi_m \in \mathbb{N}^* \).

The set of all accepting runs over \( \mathcal{X} \) is denoted by \( \text{AccRun}_{\mathcal{X}} \).

**Definition A.6** (partial \( \Sigma \)-tree, partial run). A partial \( \Sigma \)-tree \( \lambda \) is a finite prefix tree of a \( \Sigma \)-tree \( \tau \) that is proper, in the sense that if a node \( w \) of \( \tau \) is in \( \lambda \) then all the siblings of the node \( w \) are also in \( \lambda \). Its branching degrees are compatible of arities of the \( \Sigma \)-labels, and its leaves are labeled by an additional symbol \( * \) ("continuation") or a 0-ary symbol \( \sigma \).

Precisely: a partial \( \Sigma \)-tree \( \lambda \) is given by a subset \( \text{Dom}(\lambda) \subseteq \mathbb{N}^* \) together with a labeling function \( \lambda: \text{Dom}(\lambda) \rightarrow (\Sigma \cup \{\star\}) \), such that:

1. \( \text{Dom}(\lambda) \) is a nonempty and finite subset of \( \mathbb{N}^* \), that is prefix-closed and lower-closed, in the sense of Def. A.2.

2. (Properness) Let \( w \in \text{Dom}(\lambda) \). The labeling function \( \lambda \) satisfies:
   - if \( \lambda(w) = \sigma \), then \( w0, w1, \ldots, w(|\sigma| - 1) \in \text{Dom}(\lambda) \) and \( wi \not\in \text{Dom}(\lambda) \) for any \( i \geq |\sigma| \) (like in Def. A.2); and
   - if \( \lambda(w) = \star \), then \( wi \not\in \text{Dom}(\lambda) \) for any \( i \in \mathbb{N} \).

Similarly, a partial run \( \xi \) of an NPTA \( \mathcal{X} = ((X_1, \ldots, X_n), \Sigma, \delta, s) \) is a finite tree subject to the following.

1. Its domain \( \text{Dom}(\xi) \) is a nonempty, finite, prefix-closed and lower-closed subset of \( \mathbb{N}^* \).
2. (Properness) A labeling function \( \xi : \text{Dom}(\xi) \to (\Sigma \cup \{\ast\}) \times X \) such that, for each \( w \in \text{Dom}(\xi) \):
- if \( \xi(w) = (\sigma,x) \), then \( w0,w1,\ldots,w(|\sigma|-1) \in \text{Dom}(\xi) \) and \( wi \notin \text{Dom}(\xi) \) for any \( i \geq |\sigma| \); and
- if \( \xi(w) = (\ast,x) \), then \( wi \notin \text{Dom}(\lambda) \) for any \( i \in \mathbb{N} \).

3. Successors are reachable by a transition, in the sense that \( (\sigma_w,(x_{w0},\ldots,x_{w|\sigma|-1})) \in \delta(x_w) \) holds, where \( \rho(w) \) is labeled with \( (\sigma_w,x_w) \) such that \( \sigma_w \neq \ast \), and \( \rho(wi) \) is labeled with \( (\sigma_{wi},x_{wi}) \) for any \( 0 \leq i < |\sigma| \).

A partial run of a PPTA is defined similarly, except that Cond. 3 in the above is not required.

A partial run is thought of as an interim result of running an automaton \( \mathcal{A} \), after only finitely many steps. The properness requirement embodies the intuition that, in one-step execution of an automaton from some state, all of the successors of the state (together with the \( \Sigma \)-label for the state) are created at once. See Fig. 3 each of the five trees there are examples of partial runs.

Definition A.7. For \( \tau \in \text{Tree}_\Sigma \) and \( w \in \text{Dom}(\tau) \), the \( w \)-th subtree of \( \tau \) is a tree \( \tau_w \in \text{Tree}_\Sigma \) that is defined by \( \text{Dom}(\tau_w) = \{ w' \in \mathbb{N}^* | ww' \in \text{Dom}(\tau) \} \) and \( \tau_w(w') = \tau(ww') \).

A subtree of a run is called a subrun.

### B Supplementary Materials on Equational Systems

The intuitions in Def. 2.9 are put in the following precise terms.

Definition B.1 (solution). The solution of an equational system 6 is defined as follows, provided that all the necessary greatest and least fixed points exist. For each \( i \in [1,n] \) and \( j \in [1,\bar{i}] \), we define monotone functions

\[
\begin{align*}
  f_i^j : L_i \times L_{i+1} \times \cdots \times L_n &\rightarrow L_i & \text{and} & \quad f_j^i : L_{i+1} \times L_{i+2} \times \cdots \times L_n &\rightarrow L_j
\end{align*}
\]

**Figure 3** Execution of a generative probabilistic tree automaton. Here \( \delta(x_0)(\sigma_1^{(2)},(x_1,x_2)) = 2/3, \delta(x_1)(\sigma_1^{(1)},x_3) = 1/2, \) and so on.
as follows, inductively on \(i\). For the base case \(i = 1\):

\[
f_1^i(l_1, \ldots, l_n) := f_1(l_1, \ldots, l_n) \quad \text{and} \quad l_1^{(1)}(l_2, \ldots, l_n) := \eta_1 \left[ f_1^1(l_2, \ldots, l_n) : L_1 \to L_1 \right].
\]

In the last line we take the lfp or gfp (according to \(\eta_1 \in \{\mu, \nu\}\)) of the (monotone) function \(f_1^1(l_2, \ldots, l_n) : L_1 \to L_1\).

For the step case, the function \(f_{i+1}^i\) makes use of the \(i\)-th interim solutions \(l_1^{(i)}, \ldots, l_i^{(i)}\) for the variables \(u_1, \ldots, u_i\) obtained so far:

\[
f_{i+1}^i(l_{i+1}, \ldots, l_n) := f_{i+1}(l_1^{(i)}(l_{i+1}, \ldots, l_n), \ldots, l_i^{(i)}(l_{i+1}, \ldots, l_n), l_{i+1}, \ldots, l_n).
\]

We then let

\[
l_{i+1}^{(i+1)}(l_{i+2}, \ldots, l_n) := \eta_{i+1} \left[ f_{i+1}^i(l_{i+2}, \ldots, l_n) : L_{i+1} \to L_{i+1} \right]
\]

and use it to obtain the \((i + 1)\)-th interim solutions \(l_1^{(i+1)}, \ldots, l_i^{(i+1)}\). That is, for each \(j \in [1, i],\)

\[
l_j^{(i+1)}(l_{i+2}, \ldots, l_n) := l_j^{(i)}(l_{i+1}^{(i+1)}(l_{i+2}, \ldots, l_n), l_{i+2}, \ldots, l_n).
\]

Finally, the solution \((l_1^{(n)}, \ldots, l_n^{(n)}) \in L_1 \times \cdots \times L_n\) of the equational system \(\Gamma\) is defined by \((l_1^{(n)}, \ldots, l_n^{(n)}) := (l_1^{(n)}, \ldots, l_n^{(n)})\), where we identify a function \(l_j^{(n)} : 1 \to L_j\) with an element of \(L_j\). It is easy to see that all the functions \(f_1^i\) and \(l_j^{(i)}\) involved here are monotone.

**Remark B.2** \((\mathcal{K}(\mathcal{G})(X, 1)\) is not a complete lattice). Since \(\mathcal{G}\) is the sub-Giry monad we have that \(\mathcal{G}1\) is isomorphic to the unit interval \([0, 1]\), and they are complete lattices. The homset \(\mathcal{K}(\mathcal{G})(X, 1) = \text{Meas}(X, \mathcal{G}1)\) is however not a complete lattice in general, because of the measurability requirement.

For a counterexample let \(X = [0, 1] \) and \(X_0 \subseteq X\) be a non-measurable subset (it is well-known that such \(X_0\) exists). For each measurable subset \(P \subseteq X\) consider its characteristic function \(\chi_P : X \to \mathcal{G}1 \cong [0, 1], \chi_P(x) = 1\) if \(x \in P\) and \(\chi_P(x) = 0\) otherwise. Then \(\chi_P\) is measurable and hence an element of \(\mathcal{K}(\mathcal{G})(X, 1)\). Now assume that the supremum \(f := \bigcup_{P \subseteq X_0} \chi_P\) exists in \(\mathcal{K}(\mathcal{G})(X, 1)\).

- For each \(x \in X_0\) we have \(f(x) = 1\) since \(\{x\} \subseteq X_0\) is measurable.
- For each \(x \in X \setminus X_0\) we have \(f(x) = 0\). Assume otherwise: then the function \(f[x \mapsto 0]\), defined by \(y \mapsto f(y)\) (if \(y \neq x\)) and \(x \mapsto 0\), is greater than \(\chi_P\) (for each measurable \(P \subseteq X_0\)) and measurable (since for every measurable \(Q\), the sets \(Q \cup \{x\}\) and \(Q \setminus \{x\}\) are measurable). This contradicts with the minimality of the supremum \(f\).

Therefore we conclude \(f = \chi_{X_0}\). This is a contradiction, since \(\chi_{X_0}\) is not a measurable function.

The following results are about notions of homomorphism of equational systems and preservation of solutions: they are inspired by a similar result in domain theory (about preservation of least fixed points). Lem. B.4 is a rather straightforward generalization of the domain theory result. The condition we require in Lem. B.5 is rather restrictive—especially Cond. 2—but they are satisfied by our applications.

**Lemma B.3.** Let \(E\) and \(E'\) be the following equational systems, over posets \(L_1, \ldots, L_n\) and \(L'_1, \ldots, L'_n\), respectively. Note that the “polarities” \(\eta_1, \ldots, \eta_n\) are the same.

\[
E := \begin{bmatrix}
  u_1 & =_{\eta_1} & f_1(u_1, \ldots, u_n) \\
  \vdots & & \vdots \\
  u_n & =_{\eta_n} & f_n(u_1, \ldots, u_n)
\end{bmatrix}
\]

\[
E' := \begin{bmatrix}
  u'_1 & =_{\eta_1} & f'_1(u'_1, \ldots, u'_n) \\
  \vdots & & \vdots \\
  u'_n & =_{\eta_n} & f'_n(u'_1, \ldots, u'_n)
\end{bmatrix}
\]
Let
\[ \varphi_1 : L_1 \to L'_1, \ldots, \varphi_n : L_n \to L'_n \]
be a family of monotone functions, subject to the following conditions.

1. \( \varphi_i(f_i(l_1, \ldots, l_n)) = f'_i(\varphi_1(l_1), \ldots, \varphi_n(l_n)) \) for each \( i \in [1, n] \) and \( l_i \in L_i \). That is,
\[ L_1 \times \cdots \times L_n \xrightarrow{\varphi_1 \times \cdots \times \varphi_n} L'_1 \times \cdots \times L'_n \]
commutes for each \( i \in [1, n] \).

2. Let \( i \in [1, n] \), and \( l_{i+1} \in L_{i+1}, \ldots, l_n \in L_n \). Let us define the following posets of “interim fixed points under parameters \( l_{i+1}, \ldots, l_n \).”
\[ L^{(1, \ldots, i)} := \{ (l_1, \ldots, l_i) \mid \forall j \in [1, i]. l_j = f_j(l_1, \ldots, l_i, l_{i+1}, \ldots, l_n) \} \]
\[ L'^{(1, \ldots, i)} := \{ (l'_1, \ldots, l'_i) \mid \forall j \in [1, i]. l'_j = f'_j(l'_1, \ldots, l'_i, \varphi_1(l_{i+1}), \ldots, \varphi_n(l_n)) \} \]

Let us further define a function \( \varphi^{(1, \ldots, i)} : L^{(1, \ldots, i)} \to L'^{(1, \ldots, i)} \) by:
\[ \varphi^{(1, \ldots, i)}(l_1, \ldots, l_i) := (\varphi_1(l_1), \ldots, \varphi_i(l_i)) \]
where its well-definedness—i.e. that \( (\varphi_1(l_1), \ldots, \varphi_i(l_i)) \) indeed belongs to \( L'^{(1, \ldots, i)} \)—is readily verified from Cond. 7.

We require that \( \varphi^{(1, \ldots, i)} \) is an order isomorphism, for each \( i \) and \( l_{i+1}, \ldots, l_n \), with its inverse denoted by \( \psi^{(1, \ldots, i)} \).

Under these assumptions, if the system \( E' \) has a solution \( l_1^{(1)} = \ldots = l_n^{(1)} \), the other system \( E \) also has a solution \( l_1^{(0)} = \ldots = l_n^{(0)} \). Moreover \( \varphi_1(l_1^{(1)}) = l_1^{(0)} \), \( \ldots \), \( \varphi_n(l_n^{(1)}) = l_n^{(0)} \).

**Proof.** By induction on \( i \) we shall prove existence of \( l_1^{(i)}, \ldots, l_i^{(i)} \) such that \( \varphi_j(l_j^{(i)}(l_{i+1}, \ldots, l_n)) = l_j^{(i+1)}(\varphi_1(l_{i+1}), \ldots, \varphi_n(l_n)) \), for each \( j \in [1, i] \) and for any “parameters” \( l_{i+1} \in L_{i+1}, \ldots, l_n \in L_n \). Let us fix \( i \) and assume that the claim holds up to \( i - 1 \).

There is no need of distinguishing the base case \( i = 1 \) from the step case: it is easy to take proper care of the occurrences of \( i - 1 \) in the proof below. We also assume \( \eta_i = \mu \); the case when \( \eta_i = \nu \) is symmetric.

First we shall describe a construction that turns a fixed point of \( f_i^{(i)}(\ldots, \varphi(l_{i+1}), \ldots, \varphi(l_n)) \) (in \( L'_i \)) into that of \( f'_i(\ldots, \varphi_1(l_{i+1}), \ldots, \varphi_n(l_n)) \) (in \( L_i \)). Recall that we have assumed existence of a solution of \( E' \); according to Def. B.1 this requires that \( f_i^{(i)}(\ldots, \varphi(l_{i+1}), \ldots, \varphi_n(l_n)) \) has a least point; let it be denoted by \( \tilde{l}_i^{(i)} \). Then the following fixed point equality about \( l_j^{(i-1)}(\tilde{l}_i, \varphi_1(l_{i+1}), \ldots, \varphi_n(l_n)) \) also holds for each \( j \in [1, i-1] \), by the definition of \( l_j^{(i-1)} \).

\[ l_j^{(i-1)}(\tilde{l}_i, \varphi_1(l_{i+1}), \ldots, \varphi_n(l_n)) = f'_j \left( \begin{array}{c} l_j^{(i-1)}(\tilde{l}_i, \varphi_1(l_{i+1}), \ldots, \varphi_n(l_n)), \ldots, \n l_j^{(i-1)}(\tilde{l}_i, \varphi_1(l_{i+1}), \ldots, \varphi_n(l_n)) \end{array} \right) \]

This means that the following tuple belongs to \( L^{(1, \ldots, i-1)} \subseteq L_1 \times \cdots \times L'_i \), the domain of \( \psi^{(1, \ldots, i-1)} \).
\[ \left( l_1^{(i-1)}(\tilde{l}_1, \varphi_1(l_{i+1}), \ldots, \varphi_n(l_n)), \ldots, l_i^{(i-1)}(\tilde{l}_i, \varphi_1(l_{i+1}), \ldots, \varphi_n(l_n)), \tilde{l}_i \right) \]
We use a (somewhat confusing) notation of letting $\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i)$ denote the $i$-th coprojection of the applied result. That is,

$$
\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i) := (\kappa_i \circ \psi_i^{(l_1, \ldots, l_n)})(\tilde{l}_i, l_i, \ldots, l_n),
$$

We shall see that $\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i)$ is indeed a fixed point of $f_j^{\pi_i}(\tilde{l}_i, l_i+1, \ldots, l_n)$. We have $\varphi_i(\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i)) = \tilde{l}_i$, since $\varphi^{(l_1, \ldots, l_n)}$ is assumed to be the inverse of $\psi^{(l_1, \ldots, l_n)}$. Therefore for each $i \in \{1, i-1\}$ the following holds.

$$
\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i, \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)) = \psi_i^{(l_1, \ldots, l_n)}(\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))
$$

Now we use the induction hypothesis $\varphi_j(\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i, l_{i+1}, \ldots, l_n)) = \psi_i^{(l_1, \ldots, l_n)}(\varphi_i(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)))$; substituting $\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i)$ for $l_i$ in it we have the following.

$$
\varphi_j(\psi_j^{(l_1, \ldots, l_n)}(\tilde{l}_j, l_{i+1}, \ldots, l_n)) = \psi_j^{(l_1, \ldots, l_n)}(\varphi_j(\psi_j^{(l_1, \ldots, l_n)}(\tilde{l}_j), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)))
$$

By (16) and (15) we have

$$
\varphi_j(\psi_j^{(l_1, \ldots, l_n)}(\tilde{l}_j, l_{i+1}, \ldots, l_n)) = \psi_j^{(l_1, \ldots, l_n)}(\tilde{l}_j, \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)),
$$

which implies the following equalities.

$$
\tilde{l}_i = f_i^{\pi_i}(\tilde{l}_i, l_{i+1}, \ldots, l_n),
$$

$$
= (\varphi_i \circ f_i)(\tilde{l}_i, l_{i+1}, \ldots, l_n),
$$

$$
= (\varphi_i \circ f_i)(\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i), l_{i+1}, \ldots, l_n),
$$

$$
\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i) = f_i^{\pi_i}(\tilde{l}_i, l_{i+1}, \ldots, l_n),
$$

which means $\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i)$ is a fixed point of $f_i^{\pi_i}(\tilde{l}_i, l_{i+1}, \ldots, l_n)$.

Then we focus on the special case $\tilde{l}_i = f_i^{\pi_i}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))$ (i.e. when we specifically choose the least fixed point as $\tilde{l}_i$); we shall show that $\psi_i^{(l_1, \ldots, l_n)}(\tilde{l}_i) = (\varphi_i(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)))$ is the least fixed point. (Recall that $\eta_i$ is assumed to be $\mu$.) Let $\tilde{l}_i$ be an arbitrary fixed point, i.e. $\tilde{l}_i = f_i^{\pi_i}(\tilde{l}_i, l_{i+1}, \ldots, l_n)$. By applying $\varphi_i$, we
have
\[
\varphi_i(\tilde{i}) = \varphi_i(f_i^1(\tilde{i}, l_{i+1}, \ldots, l_n))
\]
\[
= f'_i \left( \begin{array}{c}
\varphi_1(l_i^{-1}(\tilde{i}, l_{i+1}, \ldots, l_n)), \\
\varphi_{i-1}(l_{i-1}^{-1}(\tilde{i}, l_{i+1}, \ldots, l_n)), \\
\varphi_i(\tilde{i}), \\
\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)
\end{array} \right)
\]
\[
= f'_i \left( \begin{array}{c}
l_i^{-1}(\varphi_i(\tilde{i}), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)), \\
\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)
\end{array} \right)
\]
\[
= f'_i (\varphi_i(\tilde{i}), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)), \quad \text{by the induction hypothesis}
\]
\[
= f'_i (\varphi_i(\tilde{i}), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))
\]
\[
= f'_i (\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))
\]

Since \(l_i^{(i)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))\) is the least fixed point of \(f'_i(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))\),
\(l_i^{(i)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)) \subseteq \varphi_i(\tilde{i})\) holds. Now by applying \(\psi_i^{(l_{i+1}, \ldots, l_n)}\), we obtain
\[
\psi_i^{(l_{i+1}, \ldots, l_n)}(f_i^{(i)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))) \subseteq \tilde{i} ;
\]
thus \(\psi_i^{(l_{i+1}, \ldots, l_n)}(l_i^{(i)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)))\).

Now we have shown \(l_i^{(1)}(l_{i+1}, \ldots, l_n) = \psi_i^{(l_{i+1}, \ldots, l_n)}(l_i^{(1)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)))\), from which \(\varphi_i(l_i^{(1)}(l_{i+1}, \ldots, l_n)) = l_i^{(1)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))\) easily follows by applying \(\varphi_i\). Furthermore, for each \(j\) such that \(j < i\), we have
\[
\varphi_j(l_j^{(1)}(l_{i+1}, \ldots, l_n)) = \varphi_j(l_j^{(1)}(l_j^{(i)}(l_{i+1}, \ldots, l_n), l_{i+1}, \ldots, l_n))
\]
\[
= l_j^{(i-1)}(\varphi_j(l_j^{(i)}(l_{i+1}, \ldots, l_n), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)))
\]
\[
= l_j^{(i-1)}(l_j^{(i)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))
\]
\[
= l_j^{(i)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)) .
\]

**Lemma B.4.** Let \(E\) and \(E'\) be equational systems, as in Lem. B.3, over \(L_1, \ldots, L_n\) and \(L'_1, \ldots, L'_n\), respectively. We assume the same conditions as in Lem. 2.11 for both \(E\) and \(E'\), that is: all \(L_i\) and \(L'_i\) are \(\omega/\omega^{op}\)-cpo’s; and all \(f_i\) and \(f'_i\) are \(\omega/\omega^{op}\)-continuous.

Let
\[
\varphi_i: L_i \rightarrow L'_i, \quad \ldots, \quad \varphi_n: L_n \rightarrow L'_n
\]
be monotone functions such that:
1. each \(\varphi_i\) is both \(\omega\)-continuous and \(\omega^{op}\)-continuous;
2. each \(\varphi_i\) preserves greatest and least elements (\(\varphi_i(T) = T\) and \(\varphi_i(\bot) = \bot\)); and
3. the following diagram commutes for each \(i \in [1, n]\\)

\[
\begin{array}{ccc}
L_1 \times \cdots \times L_n & \xrightarrow{\varphi_1 \times \cdots \times \varphi_n} & L'_1 \times \cdots \times L'_n \\
\downarrow \phi_i & & \downarrow \phi' \\\nL_i & \xrightarrow{\varphi_i} & L'_i
\end{array}
\]
Then $\varphi_i(l^{\text{sol}}_i(l_{i+1}, \ldots, l_n)) = l^{\text{sol}}_i(l_{i+1}, \ldots, \varphi_n(l_n))$ holds for each $i \in [1, n]$.

Proof. By induction on $i$, we shall show $\varphi_i(l^{i}(l_{i+1}, \ldots, l_n)) = l^{i}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))$ and $\varphi_i(f^i(l_{i+1}, \ldots, l_n)) = f^i(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))$. As in the proof of Lem. 2.11, we do not distinguish the case $i = 1$, and assume $\eta_i = \mu$.

The structure of the proof also resembles to that of Lem. 2.11. We can easily check the claim for the function $f^i$, by induction hypothesis. For the solution $l^{i}(l_{i+1}, \ldots, l_n)$, recall that the proof of Lem. 2.11 asserts that $l^{i}(l_{i+1}, \ldots, l_n)$ is equal to $\bigcup_{j<\omega}[f^i(l_{i+1}, \ldots, l_n)]j(\bot)$, with continuity of $f^i$. Thus we have $\varphi_i(l^{i}(l_{i+1}, \ldots, l_n)) = l^{i}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))$ by straightforward induction. Then the claim easily follows also for each $l^{i}(l_{i+1}, \ldots, l_n)$. ◀

C Generative Probabilistic Parity Tree Automata

C.1 Generative Systems and Reactive Systems

The notion of probabilistic tree automaton we study in this paper as an example is a generative one. This is in contrast to reactive probabilistic systems (studied e.g. in [6]): a generative system generates a (possibly infinite) tree (Fig. 3 is a step-by-step illustration of a generation process)—hence the probability with which each single tree is generated is zero except for some singular cases—whereas a reactive system takes a tree as input and assigns a probability to it. The difference can be technically formulated in the types of transition functions:

$$X \rightarrow \mathcal{G}(\prod_{\sigma \in \Sigma} X^{[\sigma]}) \text{ for generative;}$$

$$X \rightarrow \prod_{\sigma \in \Sigma} \mathcal{G}(X^{[\sigma]}) \text{ for reactive.}$$

The difference has been discussed extensively for word (instead of tree) automata. See e.g. [33, 28, 11].

In the current generative (as opposed to reactive) setting, it does not make much sense to talk about the probability with which each single tree is generated. For example let $\Sigma = \{\text{hd}, \text{tl}\}$ and assume that each operation is unary. The generative automaton in Fig. 4 is then a model of a fair coin; and it generates any single infinite sequence with probability 0. This is a prototypical one that motivates the need for measure theory in the context of probabilistic systems (like in [23]); note that the set of $\Sigma$-trees (that are $\Sigma$-words if all symbols are unary) is uncountable.

C.2 Languages of Generative Probabilistic Tree Automata

Definition C.1 (measurable structures of $\text{Tree}_{\Sigma}$ and $\text{Run}_X$). Let $\lambda$ be a partial $\Sigma$-tree. The cylinder set associated to $\lambda$, denoted by $\text{Cyl}_{\Sigma}(\lambda)$, is the set of (proper, non-partial) $\Sigma$-trees
that have \( \lambda \) as their “prefix.” That is,
\[
\text{Cyl}_\Sigma(\lambda) := \{ \tau \in \text{Tree}_\Sigma \mid \forall w \in \text{Dom}(\lambda). \ (\lambda(w) = \tau(w) \text{ or } \lambda(w) = *) \}.
\]
The (smallest) \( \sigma \)-algebra generated by the family \( \{\text{Cyl}_\Sigma(\lambda) \mid \lambda \text{ is a partial } \Sigma\text{-tree} \} \) will be denoted by \( \mathfrak{F}_\Sigma \).

For a partial run \( \xi \) of \( \mathcal{X} \), the cylinder set \( \text{Cyl}_\mathcal{X}(\xi) \subseteq \text{Run}_\mathcal{X} \) associated to \( \xi \) is defined similarly but slightly differently. Precisely:
\[
\text{Cyl}_\mathcal{X}(\xi) := \left\{ \rho \in \text{Run}_\mathcal{X} \mid \forall w \in \text{Dom}(\xi). \ \left( \begin{array}{l}
\pi_1(\xi(w)) = \pi_1(\rho(w)) \text{ or } \\
\pi_2(\xi(w)) = \pi_2(\rho(w))
\end{array} \right) \right\}.
\]
Here \( \pi_1(\sigma, x) = \sigma \) and \( \pi_2(\sigma, x) = x \). These cylinder sets generate a \( \sigma \)-algebra over \( \text{Run}_\mathcal{X} \), which shall be denoted by \( \mathfrak{F}_\mathcal{X} \).

▶ **Lemma C.2.** The function \( \text{NoDiv}_\mathcal{X} : \mathcal{X} \to [0, 1] \) is the greatest fixed point of the function \( \Psi'_{\mathcal{X}} : [0, 1]^\mathcal{X} \to [0, 1]^\mathcal{X} \), defined by
\[
\Psi'_{\mathcal{X}}(f)(x) := \sum_{(\sigma, (x_1, \ldots, x_{|\sigma|})) \in \prod_{\sigma \in \Sigma} \mathcal{X}^{\sigma}} \delta(x) \cdot \left( \pi_1(\sigma, x_1, \ldots, x_{|\sigma|}) \right) \cdot \prod_{i \in [1..|\sigma|]} \text{NoDiv}_\mathcal{X,k}(x_i).
\]

**Proof.** The proof is essentially by Kleene’s fixed point theorem. Consider the sequence \( \text{NoDiv}_\mathcal{X,0}, \text{NoDiv}_\mathcal{X,1}, \ldots : \mathcal{X} \to [0, 1] \) in Def. 5.9. Then \( \text{NoDiv}_\mathcal{X,0} \) is the greatest element in \([0, 1]^\mathcal{X}\) (with respect to the pointwise order) and the sequence is obviously decreasing. Moreover, the function \( \Psi'_{\mathcal{X}} \) is easily seen to be “continuous” in the sense that \( \Psi'_{\mathcal{X}}(\bigwedge_{k \in \mathbb{N}} \text{NoDiv}_\mathcal{X,k}) = \bigwedge_{k \in \mathbb{N}} \Psi'_{\mathcal{X}}(\text{NoDiv}_\mathcal{X,k}) \). Therefore by an argument similar to the one for Kleene’s theorem, \( \bigwedge_{k \in \mathbb{N}} \text{NoDiv}_\mathcal{X,k} = \text{NoDiv}_\mathcal{X} \) is the greatest fixed point of \( \Psi'_{\mathcal{X}} \).

▶ **Lemma C.3.** The subprobability pre-measure \( \mu_{\mathcal{X}}^{\text{Run}} \) over cylinder sets defined in (13) of Def. 5.4 determines uniquely a subprobability measure over the whole \( \sigma \)-algebra \( \mathfrak{F}_\mathcal{X} \).

**Proof.** We rely on Carathéodory’s extension theorem [13] here. For using the theorem, since we have
\[
\text{Cyl}_\mathcal{X}(\xi) = \prod_{(\sigma, (x_1, \ldots, x_{|\sigma|})) \in \prod_{\sigma \in \Sigma} \mathcal{X}^{\sigma}} \text{Cyl}_\mathcal{X}(\xi_w, \sigma, x_1, \ldots, x_{|\sigma|}),
\]
it suffices to show what follows.

Let \( \xi \) be a partial run of \( \mathcal{X} \), \( w \in \mathbb{N}^* \) be such that \( w \in \text{Dom}(\xi) \) and \( \xi(w) = (\ast, x) \) (hence \( w \) is a leaf of \( \xi \)). For each \( \sigma \in \Sigma \) and \( x_0, \ldots, x_{|\sigma|-1} \in \mathcal{X} \), let \( \xi_w, x_1, \ldots, x_{|\sigma|} \) be the partial run that “extends” the leaf \( w \) with \( (\sigma, (x_1, \ldots, x_{|\sigma|})) \). Precisely:
\[
\text{Dom}(\xi_w, x_1, \ldots, x_{|\sigma|}) := \text{Dom}(\xi) \cup \{w1, \ldots, w|\sigma|\},
\]
\[
\xi_w, \sigma, x_1, \ldots, x_{|\sigma|}(w') := \begin{cases} 
(\sigma, \xi(w)) & \text{if } w' = w \\
(\ast, x_i) & \text{if } w' = wi \\
\xi(w') & \text{otherwise.}
\end{cases}
\]

Then
\[
\mu_{\mathcal{X}}^{\text{Run}}(\text{Cyl}_\mathcal{X}(\xi)) = \sum_{(\sigma, (x_1, \ldots, x_{|\sigma|})) \in \prod_{\sigma \in \Sigma} \mathcal{X}^{\sigma}} \mu_{\mathcal{X}}^{\text{Run}}(\text{Cyl}_\mathcal{X}(\xi_w, \sigma, x_1, \ldots, x_{|\sigma|})).
\]
To show this claim, by the bottom-up way of the definition of $P_X$ (Def. 5.9), it suffices to show that

$$\text{NoDiv}_X(x) = \sum_{(\sigma, (x_1, \ldots, x_{|\sigma|})) \in \prod_{\sigma \in \Sigma} X^{\sigma|\sigma|}} \delta(x) (\sigma, (x_1, \ldots, x_{|\sigma|})) \cdot \prod_{i \in [1, |\sigma|]} \text{NoDiv}_X(x_i).$$

This just means that NoDiv$_X$ is a fixed point of $\Psi'_X$, a fact proved in Lem. C.2.

\section{Omitted Proofs}

\subsection{Proof of Lem. 2.11}

Proof. By induction on $i$, we shall show $\omega$- and $\omega^{op}$-continuity of $f_i^j$ and $l_i^j$ (here $j \leq i$), and existence of the solution $l_i^0$. (Monotonicity of those is almost clear.) Let us fix $i$ and assume that the claim holds up to $i - 1$. There is no need of distinguishing the base case ($i = 1$) from the step case: it is easy to take proper care of the occurrences of $i - 1$ in the proof below. We also assume $\eta_i = \mu$; the case when $\eta_i = \nu$ is symmetric.

We can easily show that the function

$$f_i^j(l_1, \ldots, l_n) := f_i(l_i^{j-1}(l_1, \ldots, l_n), \ldots, l_i^{0}(l_1, \ldots, l_n), l_1, \ldots, l_n)$$

is $\omega$- and $\omega^{op}$-continuous, by continuity of $l_i^{j-1}$ on induction hypothesis, and continuity of $f_i$ in the assumption. By Kleene’s fixed point theorem we can construct $l_i^j(l_1, \ldots, l_n)$— which is defined to be the least fixed point of $f_i^j(l_1, \ldots, l_n)$— together with the above $\omega$-continuity of $f_i^j$. We let

$$l_i^j(l_{i+1}, \ldots, l_n) = \bigcup_{j < \omega} [f_i^j(l_1, l_{i+1}, \ldots, l_n)]^j(\perp).$$

Since $\perp$ is the least element in $L_i$, we are ensured to obtain an $\omega$ chain of $\bigcup_{j < \omega} [f_i^j(l_1, l_{i+1}, \ldots, l_n)]^j(\perp)$. The supremum is a fixed point because we have

$$[f_i^j(l_1, l_{i+1}, \ldots, l_n)]^j(\perp) = [f_i^j(l_{i+1}, \ldots, l_n)]^j(\perp),$$

by induction hypothesis.

The obtained fixed point is readily verified to be the least, thanks to the minimality of $\perp$.

Now we show $\omega$-continuity of $l_i^j$. To this end we use the following easy observation: it is shown for each $j < \omega$ by induction.

$$[f_i^j(l_1, \ldots, l_{i+k}, \ldots, l_n)]^j(\perp) = [f_i^j(l_1, \ldots, l_{i+k}, \ldots, l_n)]^j(\perp),$$

By taking supremum of the above for $j < \omega$, the $\omega$-continuity of $l_i^j$ is shown as follows.

$$l_i^j(l_{i+1}, \ldots, l_n) = \bigcup_{k < \omega} [f_i^j(l_1, l_{i+1}, \ldots, l_n)]^j(\perp)$$

by (18)

$$= \bigcup_{j < \omega} [f_i^j(l_1, l_{i+1}, \ldots, l_n)]^j(\perp)$$

by (19)

$$= \bigcup_{k < \omega} [f_i^j(l_1, l_{i+1}, \ldots, l_n)]^j(\perp)$$

by (18)

$$= \bigcup_{k < \omega} l_i^j(l_{i+1}, \ldots, l_n).$$
Next we show $\omega^{op}$-continuity of $l^{(i)}$. It suffices to show that $\bigcap_{k, \omega} l^{(i)}_k(l_{i+1,k}, \ldots, l_{n,k})$ is the least (pre-) fixed point of $f_i^1(\bigcap_{k, \omega} l_{i+1,k}, \ldots, l_{n,k})$, since $l^{(i)}_k(l_{i+1}, \ldots, l_n)$ is defined to be $\text{lfp}[f_i^1(\bigcap_{k, \omega} l_{i+1,k}, \ldots, l_{n,k})]$. Let us take an arbitrary pre-fixed point $\bar{l}_i \supseteq f_i^1(\bigcap_{k, \omega} l_{i+1,k}, \ldots, l_{n,k})$. Then we have

$$\bar{l}_i \supseteq f_i^1(\bigcap_{k, \omega} l_{i+1,k}, \ldots, l_{n,k})$$

$$= \bigcap_{k, \omega} f_i^1(l_{i+1,k}, \ldots, l_{n,k})$$

by induction hypothesis

$$\supseteq \bigcap_{k, \omega} f_i^1(l^{(i)}_k(l_{i+1,k}, \ldots, l_{n,k}), l_{i+1,k}, \ldots, l_{n,k})$$

$$= \bigcap_{k, \omega} l^{(i)}_k(l_{i+1,k}, \ldots, l_{n,k})$$

thus $\bigcap_{k, \omega} l^{(i)}_k(l_{i+1,k}, \ldots, l_{n,k})$ is the least (pre-) fixed point.

We have shown $\omega$- and $\omega^{op}$-continuity of $l^{(i)}_k$; for the other interim solutions

$$l^{(i)}_{j}(l_{i+1}, \ldots, l_n) := l^{(i-1)}_{j}(l^{(i)}_k(l_{i+1}, \ldots, l_n), l_{i+1}, \ldots, l_n),$$

for $j < i$, continuity is also shown, in the same manner as in (17).

### D.2 Proof of Lem. 4.4

**Proof.** For a branch $\pi = (x_1, \sigma_1) \cdot (x_2, \sigma_2) \cdots$ of a run $\rho \in \text{Run}_X$, let $|\pi|$ be the length of $\pi$, and $|\pi|_{=j} := |\{k \mid x_k \in X_j\}|$. Note that $|\pi|$ and $|\pi|_{=j}$ can be $\omega$. For $m \in |\pi|$, let $\rho_{\pi,m}$ be the subrun of $\rho$ that follows after $(x_1, \sigma_1) \cdot (x_2, \sigma_2) \cdots (x_m, \sigma_m)$. Moreover, we write $X_{\leq j}$ and $X_{> j}$ for $\bigcup_{i \leq j} X_i$ and $\bigcup_{i > j} X_i$ respectively. Recall that $l^{(i)} : \mathcal{P}(\text{Run}_X \times X_{i+1}) \times \cdots \times \mathcal{P}(\text{Run}_X \times X_n) \rightarrow \mathcal{P}(\text{Run}_X \times X_i)$ denotes the $j$-th interim solution (Def. 3.1).

We first prove that: for each $j \in [1, n]$, sets $l_{j+1} \in \mathcal{P}(\text{Run}_X \times X_{j+1}), \ldots, l_n \in \mathcal{P}(\text{Run}_X \times X_n)$ of runs, a priority $i \in [1, j]$, a run $\rho \in l^{(i)}_{j+1}(l_{j+1}, \ldots, l_n) \cap \text{Run}_X$, and a (possibly-infinitive) branch $\pi = (x_1, \sigma_1) \cdot (x_2, \sigma_2) \cdots$ of $\rho$, we have either of the following conditions.

- We have $x_m \in X_{\leq j}$ for each $m \in |\pi|$. Moreover, $\max\{j' \mid |\pi|_{=j'} = \omega\}$ is even when $|\pi| = \omega$.
- There exists $m \in |\pi|$ such that $x_m \in X_{> j}$. Moreover, if we choose the minimum $m$ among such (i.e. $x_m' \in X_{\leq j}$ for every $m' < m$), then $\rho_{\pi,m} \in \bigcup_{i > j} l_{j'}$.

We prove this by induction on $j$. Note that there is no need of distinguishing the base case ($j = 1$) from the step case.

**Case: $j$ is odd ($u_j$ is $\mu$-variable).** It is not hard to see, for each $k \in \omega$, that

$$\rho \in \left[\bigcup_{\phi X} (l^{(i-1)}_{j', l_{j'+1}, \ldots, l_n}) \cup \cdots \cup (l^{(i-1)}_{j+1, l_{j+1}, \ldots, l_n}) \cup \bigcup_{\phi X} \cup l_{j+1} \cup \cdots \cup l_n) \cap \text{Run}_X\right]^k(\emptyset)$$

(20)

if and only if, for every branch $\pi = (x_1, \sigma_1) \cdot (x_2, \sigma_2) \cdots$ of $\rho$, either of the following conditions is satisfied.

- We have $x_m \in X_{\leq j}$ for each $m \in |\pi|$. Moreover, $|\pi|_{=j} \leq k$ and $\max\{j' \mid |\pi|_{=j'} = \omega\}$ is even when $|\pi| = \omega$.
- There exists $m \in |\pi|$ such that $x_m \in X_{> j}$. Moreover, if we choose the minimum $m$ among such (i.e. $x_m' \in X_{\leq j}$ for every $m' < m$), then $|(\sigma_1, x_1) \cdots (\sigma_{m-1}, x_{m-1})|_{=j} \leq k$ and $\rho_{\pi,m} \in \bigcup_{i > j} l_{j'}$. 
It is easy to see that the interim solution $l^{(j)}_1(l_{j+1}, \ldots, l_n)$ is obtained by taking the supremum of \( \rho \), for $k \in \omega$. Therefore, for $i = j$, the claim is discharged. The proof for $i < j$ is easy.

**Case: $j$ is even ($u_j$ is a $\nu$-variable).** As in the former case, we can see that

$$\rho \in \left( \bigcup_{l_{j+1}, \ldots, l_n = \varnothing} \bigcup_{u_1, \ldots, u_n} \bigcup_{l_{j+1}, \ldots, l_n} \bigcup_{l_{j+1}, \ldots, l_n} \bigcup_{l_{j+1}, \ldots, l_n} \right) \bigcap \text{Run}_{X_n, X_j} \right)^k \text{Run}_{X_n, X_j}$$

(21)

if and only if, for every branch $\pi = (x_1, \sigma_1)(x_2, \sigma_2) \ldots$ of $\rho$, either of the following conditions is satisfied.

- We have $x_m \in X_{\leq j}$ for each $m \in |\pi|$. Moreover, $|\pi| = |j|$ or $\max\{i' \mid |\pi| = j' = \omega\}$ is even when $|\pi| = \omega$.
- There exists $m \in |\pi|$ and such that $x_m \in X_{> j}$. Moreover, if we choose the minimum $m$ among such (i.e. $x_m \in X_{\leq j}$ for every $m' < m$), then $|(\sigma_1, x_1) \cdots (\sigma_m, x_m)| = k$ or $\rho_{\pi, m} \in \bigcup_{j > j} l_{j'}$.

It is easy to see that the interim solution $l^{(j)}_1(l_{j+1}, \ldots, l_n)$ is obtained by taking the infimum of (21), for $k \in \omega$. Therefore, for $i = j$, the claim is discharged and the proof for $i < j$ is easy.

Hence we can prove the claim for all $j \in [1, n]$. Letting $j = n$, Lem. 4.4 follows.

**D.3 Proof of Lem. 4.5**

Proof. In what follows we shall work with the semantic domains $L_i := \prod_{x \in X_i} P(\text{Run}_X)$ and $L'_i := \prod_{x \in X_i} P(\text{Tree}_X)$, which are easily seen to be equivalent to the formulation in Lem. 4.3. We write

$$\varphi_i := \prod_{x \in X_i} P(\text{DelSt}) : L_i \rightarrow L'_i$$

for each $i \in [1, n]$. Here $P(\text{DelSt}) : P(\text{Run}_X) \rightarrow P(\text{Tree}_X)$ is defined by direct images. Furthermore we write $f_i, f'_i$ for the following functions (that occur on the right-hand sides of the relevant equational systems), for each $i \in [1, n]$.

$$f_i : L_1 \times \cdots \times L_n \rightarrow L_i, \quad f_i(u_1, \ldots, u_n) := \big( \bigcup_{\delta} \langle u_1, \ldots, u_n \rangle \big) \upharpoonright X_n$$,

$$f'_i : L'_1 \times \cdots \times L'_n \rightarrow L'_i, \quad f'_i(u'_1, \ldots, u'_n) := \big( \bigcup_{\delta} \langle u'_1, \ldots, u'_n \rangle \big) \upharpoonright X_n$$.

It is straightforward to see that the following diagram commutes, for each $i \in [1, n]$.

$$\begin{array}{ccc}
L_1 \times \cdots \times L_n & \xrightarrow{\varphi_1 \times \cdots \times \varphi_n} & L'_1 \times \cdots \times L'_n \\
\downarrow f & & \downarrow f' \\
L_i & \xrightarrow{\varphi_i} & L'_i
\end{array}$$

(22)

In view of Lem. 4.4 it suffices to show that, on the solution $l^1_n, \ldots, l^1_n$ of the equational system $E$ in [9] and the solution $l^1_n, \ldots, l^1_n$ of the equational system $E'$ in [10], we have $\varphi_i(l^1_n) = l^1_n$ for each $i \in [1, n]$.

Towards this end we shall prove the following by induction on $i \in [1, n]$.

For each $l_{i+1} \in L_{i+1}, \ldots, l_n \in L_n$:

- We have $\varphi_i(l^{(i)}_1(l_{i+1}, \ldots, l_n)) = l^{(i)}_1(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n))$, where $l^{(i)}_1 : L_{i+1} \times \cdots \times L_n \rightarrow L_i$ is the $i$-th interim solution of $E$ for $u_i$ (Def. B.1); $l^{(i)}_1$ is the same for $E'$. 
On the other \( i \)-th interim solutions, too, we have \( \varphi_j(l_j(l_{i+1}, \ldots, l_n)) = l_j^i(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)) \), for each \( j \in [1, i - 1] \).

By showing the above we will obtain \( \varphi_i(l_{i}^{\text{sol}}) = l_{i}^{\text{sol}} \), as a special case, for each \( i \in [1, n] \).

The main technical difficulty lies in the first item; the second is easy. Let us first assume that \( i \) is odd, that is, \( \eta_i = \mu \). In this case, by the Cousot-Cousot construction of least fixed points (that is via transfinite induction), we have some ordinal \( \alpha \) where the increasing approximation sequence

\[
\bot \leq \left( f^i_{\bot}(l_{i+1}, \ldots, l_n) \right) \leq \left( f^i_{\bot}(l_{i+1}, \ldots, l_n) \right)^2 \leq \cdots
\]

stabilizes, yielding

\[
l^{i}(l_{i+1}, \ldots, l_n) = \mu \left[ f^{\alpha}_{\bot}(l_{i+1}, \ldots, l_n) \right] \text{ by def. of } l^{i}(l_{i+1}, \ldots, l_n)
\]

for \( E \). For \( E' \) the situation is similar, and \( l^{i}_{+}(l_{i+1}, \ldots, l_n) \) is given as a suitable limit of a (transfinite) increasing sequence.

Let us note the following.

\[
(\varphi_i \circ f^{i}_{\bot})(l_{i+1}, \ldots, l_n)
\]

\[
= (\varphi_i \circ f^{i}_{\bot}) \left( l^{i-1}_{\bot}(l_{i+1}, \ldots, l_n), \ldots, l^{i-1}_{\bot}(l_{i+1}, \ldots, l_n), \ldots, l^{i-1}_{\bot}(l_{i+1}, \ldots, l_n), \right)
\]

\[
= f^{i}_{\bot}\left( \varphi_1(l^{i-1}_{\bot}(l_{i+1}, \ldots, l_n), \ldots, \varphi_{i-1}(l^{i-1}_{\bot}(l_{i+1}, \ldots, l_n)), \varphi_i(l_{i+1}, \ldots, l_n) \right) \text{ by def. of } f^{i}_{\bot}
\]

\[
= f^{i}_{\bot}\left( \varphi_1(l^{i-1}_{\bot}(l_{i+1}, \ldots, l_n), \ldots, \varphi_{i-1}(l^{i-1}_{\bot}(l_{i+1}, \ldots, l_n)), \varphi_i(l_{i+1}, \ldots, l_n) \right) \text{ by } (22)
\]

\[
= f^{i}_{\bot}\left( \varphi_i(l_{i+1}, \ldots, l_n) \right) \text{ by def. of } f^{i}_{\bot}.
\]

We shall use this in showing that, for each ordinal \( \beta \), we have the following. Here \( \bot \) is the least element of \( L_i \).

\[
\left( \varphi_i \circ f^{i}_{\bot}(l_{i+1}, \ldots, l_n) \right)^{\beta}(\bot) = \left( f^{i}_{\bot}(l_{i+1}, \ldots, l_n) \right)^{\beta}(\bot) \in L_{i}^i .
\]

Indeed: the base case (\( \beta = 0 \)) is obvious; the step case follows from (23); and for the limit case (\( \beta \) is a limit ordinal), we use the fact that \( \varphi_i = \bigwedge_{x \in X} \mathcal{P}(\text{DelSt}) \) —defined by direct images— preserves suprema (i.e. unions). Together with the fact that \( \varphi_i \) preserves least elements, we see that \( \varphi_i \) carries the Cousot-Cousot sequence in \( L_i \) (for computing \( l^{i}_{i}(l_{i+1}, \ldots, l_n) \)) to the one in \( L_i \) (for computing \( l^{i}_{i}(l_{i+1}, \ldots, l_n) \)). This proves

\[
\varphi_i(l^{i}_{i}(l_{i+1}, \ldots, l_n)) = l^{i}_{i}(l_{i+1}, \ldots, l_n).
\]

Let us now assume that \( i \) is even, that is, \( \eta_i = \nu \). We shall again prove the claim by scrutinizing the Cousot-Cousot sequences for \( l^{i}_{i}(l_{i+1}, \ldots, l_n) \) and \( l^{i}_{i}(l_{i+1}, \ldots, l_n) \).
Writing
\[\Phi := f_i^1(\varphi_{i+1}(l_{i+1}), \ldots, l_n) \quad \text{by def.} \quad f_i \left( l_{1}^{(i-1)}(\varphi_{i+1}(l_{i+1}), \ldots, l_n), \ldots, l_{i-1}^{(i-1)}(\varphi_{i+1}(l_{i+1}), \ldots, l_n), \ldots, l_{n}^{(i-1)}(\varphi_{i+1}(l_{i+1}), \ldots, l_n) \right) \]
and
\[\Phi' := f_i^1(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)) \quad \text{by def.} \quad f_i' \left( l_{1}^{(i-1)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)), \ldots, l_{i-1}^{(i-1)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)), \ldots, l_{n}^{(i-1)}(\varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)) \right), \]
the relevant Cousot-Cousot sequences are as follows.

\[T \geq \Phi(T) \geq \cdots \geq \Phi^n(T) \geq \cdots \quad \text{in } L_i, \quad \text{and} \quad T \geq \Phi'(T) \geq \cdots \geq \Phi'^n(T) \geq \cdots \quad \text{in } L'_i. \tag{24}\]

Unlike the previous case where \(\eta_i = \mu\), it is not the case that the first sequence is carried \emph{exactly} to the second by \(\varphi_i\). Instead we shall show the following two claims.

1. For each ordinal \(\alpha\) we have \(\varphi_i(\Phi^n(T)) \leq \Phi'^n(T)\).

2. We have \(R \in L_i\) such that: \(R\) is a \(\Phi\)-postfixed point (i.e. \(R \leq \Phi(R)\)); and \(\varphi_i(R) = \nu\Phi'\).

Showing these items 1-2 proves the claim (namely \(\varphi_i(\nu\Phi) = \nu\Phi'\)). Indeed: taking \(\alpha_0\) such that \(\nu\Phi' = \Phi'^\alpha_0(T)\), we have

\[\nu\Phi' = \Phi'^\alpha_0(T) \geq \varphi_i(\Phi'^\alpha_0) \geq \varphi_i(\nu\Phi) \quad \text{where we used monotonicity of } \varphi_i;\]

conversely, for \(R\) in the item 2 we have \(R \leq \nu\Phi\)—because \(\nu\Phi\) is the greatest \(\Phi\)-postfixed point (the Knaster-Tarski theorem)—hence

\[\nu\Phi' = \varphi_i(R) \leq \varphi_i(\nu\Phi).\]

The item 1 is shown by (transfinite) induction on \(\alpha\). The base case is obvious. For the step case,

\[\Phi'^{\alpha+1}(T) \geq \Phi\left(\varphi_i(\Phi^n(T))\right) \quad \text{by ind. hyp. (for } \alpha\text{), and that } \Phi' \text{ is monotone}\]

\[= f_i' \left( l_{1}^{(i-1)}(\varphi_i(\Phi^n(T)), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)), \ldots, l_{i-1}^{(i-1)}(\varphi_i(\Phi^n(T)), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)), \ldots, l_{n}^{(i-1)}(\varphi_i(\Phi^n(T)), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)) \right) \quad \text{by def. of } \Phi'\]

\[= f_i' \left( (\varphi_i \circ l_{1}^{(i-1)})(\Phi^n(T), l_{i+1}, \ldots, l_n), \ldots, (\varphi_i \circ l_{i-1}^{(i-1)})(\Phi^n(T), l_{i+1}, \ldots, l_n), \varphi_i(\Phi^n(T)), \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n) \right) \quad \text{by ind. hyp. (for } i-1\text{)} \tag{25}\]

\[= (\varphi_i \circ f_i) \left( l_{1}^{(i-1)}(\Phi^n(T), l_{i+1}, \ldots, l_n), \ldots, l_{i-1}^{(i-1)}(\Phi^n(T), l_{i+1}, \ldots, l_n), \Phi^n(T), l_{i+1}, \ldots, l_n \right) \quad \text{by } 22\]

\[= \varphi_i(\Phi^n(T)) = \varphi_i(\Phi'^{\alpha+1}(T)) \quad \text{by def. of } \Phi.\]

For the limit case, we have

\[\varphi_i(\Phi^n(T)) = \varphi_i\left(\bigwedge_{\alpha' < \alpha} \Phi'^{\alpha'}(T)\right) \leq \bigwedge_{\alpha' < \alpha} \varphi_i(\Phi'^{\alpha'}(T)) \leq \bigwedge_{\alpha' < \alpha} \Phi'^{\alpha'}(T) = \Phi'^{\alpha}(T),\]
where the first inequality is due to monotone of \( \varphi_i \) and the second is by the induction hypothesis (on \( \alpha' \)). This proves the item 1.

For the item 2 we first observe the fixed-point property of \( \nu \Phi' \), expanding the definition of \( \Phi' \) and furthermore that of \( f' \):

\[
(\nu \Phi')_x = \left\{ (\sigma, (\tau_1, \ldots, \tau_{\sigma}) | \exists x_1, \ldots, x_{|\sigma|}. (\sigma, (x_1, \ldots, x_{|\sigma|})) \in \delta(x), \quad \forall k \in [1, |\sigma|]. \\
x_k \in X_1 \Rightarrow \tau_k \in \left( (\nu \Phi', \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)) \right)_{x_k}, \\
x_k \in X_{i-1} \Rightarrow \tau_k \in \left( (\nu \Phi', \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)) \right)_{x_k}, \\
x_k \in X_i \Rightarrow \tau_k \in (\nu \Phi')_{x_k}, \\
x_k \in X_{i+1} \Rightarrow \tau_k \in (\varphi_{i+1}(l_{i+1}))_{x_k}, \\
\ldots, \\
x_k \in X_n, \Rightarrow \tau_k \in (\varphi_n(l_n))_{x_k}. \right\}
\]

(26)

for each \( x \in X_1 \). It is then not hard to see that, for each \( \Sigma \)-tree \( \tau \) that belongs to \( (\nu \Phi')_x \), we can find at least one run \( \rho \) of \( X \) so that DelSt(\( \rho \)) = \( \tau \). This fact is proved by decorating each node of \( \tau \) with an \( X \)-label, coinductively from top to bottom, starting with \( x \). Concretely, once an \( X \)-label \( x' \) is assigned to a certain node, we operate as follows.

- If \( x' \in X_k \) with \( k \in [i + 1, n] \), then the subtree \( \tau' \) starting at the current node belongs to the set \( (\varphi_k(l_k))_{x'} \). Recalling that \( \varphi_k = \mathcal{P}(\text{DelSt}) \), we can find a run \( \rho' \in l_k \) such that DelSt(\( \rho' \)) = \( \tau' \); we decorate \( \tau' \) according to \( \rho' \).

- If \( x' \in X_i \) then the subtree \( \tau' \) starting at the current node belongs to \( (\nu \Phi')_{x'} \). We invoke the fixed-point property (26) to find the \( X \)-labels \( x_1, \ldots, x_{|\sigma|} \) for the children of the current node.

- If \( x' \in X_k \) with \( k \in [1, i - 1] \), we note that the set \( \left( (\nu \Phi', \varphi_{i+1}(l_{i+1}), \ldots, \varphi_n(l_n)) \right)_{x'} \) to which the subtree \( \tau' \) starting at the current node should belong to—consists of those trees \( \tau \) with the following property: \( \tau \) has a prefix \( \tau_0 \) that is the image under DelSt of a prefix \( \rho_0 \) of some run of \( X \) starting from \( x' \); \( \rho_0 \) has \( X \)-labels from \( X_i \cup X_{i+1} \cup \cdots \cup X_n \) only at those nodes where \( \tau_0 \) ends but \( \tau \) continues; and, at each such node \( x'' \),
  - \( x'' \in X_i \) implies that the subtree of \( \tau \) starting there belongs to \( (\nu \Phi')_{x''} \), and
  - \( x'' \in X_j \) (for \( j \in [i + 1, n] \)) implies that the subtree of \( \tau \) starting there belongs to \( (\varphi_j(l_j))_{x''} \).

This fact is shown in the current induction on \( i \). We can then decorate the prefix \( \tau_0' \) of \( \tau' \) according to \( \rho'_0 \) (in the above notations); once we hit \( X \)-labels from \( X_i \cup X_{i+1} \cup \cdots \cup X_n \) we continue according to the above other cases.

For each \( \tau \in (\nu \Phi')_x \) we collect its decorations \( \rho \); and we let \( R \in L_i = \bigcup_{x \in X} \mathcal{P}(\text{Run}_X) \) defined by its closure under subtrees. It is then obvious that \( R \leq \Phi(R) \) (since \( R \) is closed under subtrees) and \( \varphi_i(R) = \nu \Phi' \) (since for each \( \tau \in (\nu \Phi')_x \) we included its decoration). This proves the item 2 and proves the claim.

\( \blacksquare \)

\textbf{Remark D.1.} The sequences (24) do not match step-by-step, already in the following simple example. Assume that \( F = \{ *, \} \times \{ \} \), every edge below is labeled with \( * \), and every state
is accepting.

Let the top node denoted by $x$. Then after $\omega$ steps in the first Cousot-Cousot sequence every potential run from $x$ is eliminated (one with length $n$ is eliminated after $n$ steps). However in the second Cousot-Cousot sequence, the word $\ast \omega = \ast \ast \ldots$ is eliminated only after $\omega + 1$ steps: $\ast \omega \in \bigcup_{n<\omega} \Phi^m(\top)$ because, for each $n$, $x$ has a run of length $n$.

D.4 Proof of Lem. 5.2
The following fact, which gives an explicit construction of the final coalgebra $\zeta : Z \Rightarrow FX$, is standard.

**Sublemma D.2 (26).** Let $F : \text{Meas} \rightarrow \text{Meas}$ be a (standard Borel) polynomial functor. Let $Z$ be a limit of its final sequence (up to $\omega$)—the measurable structure of $Z$ is the weakest one such that all projections $\pi_i$ are measurable. In this case the functor $F$ preserves the limit $Z$ and we have the following mediating isomorphism $\zeta$.

\[
\begin{array}{c}
\pi_0 \quad \pi_1 \quad \pi_2 \\
\downarrow \quad \downarrow \quad \downarrow \\
F^1 \quad F^2 \ldots \quad F^\omega
\end{array}
\xrightarrow{\zeta} Z \quad \text{(limit)}
\]

By a standard argument like in 23, $\zeta : Z \Rightarrow FZ$ is a final coalgebra in $\text{Meas}$. □

We also use the fact that the Kleisli inclusion functor $J$ lifts the limit to $2$-limit in $\mathcal{K}(\mathcal{G})$.

**Sublemma D.3 (29).** The Kleisli inclusion functor $J : \text{Meas} \rightarrow \mathcal{K}(\mathcal{G})$ for the sub-Giry monad $\mathcal{G}$ preserves the limits in 27. This yields, in particular, the following limit.

\[
\begin{array}{c}
\pi_0 \quad \pi_1 \quad \pi_2 \\
\downarrow \quad \downarrow \quad \downarrow \\
F^1 \quad F^2 \ldots \quad F^\omega
\end{array}
\xrightarrow{\zeta} Z \quad \text{(limit)}
\]

Moreover $Z$ here is in fact a $2$-limit: if two cones $(\gamma_k : X \Rightarrow F^k1)_{k \in \omega}$ and $(\gamma'_k : X \Rightarrow F^k1)_{k \in \omega}$ satisfy $\gamma_k \leq \gamma'_k$ for each $k \in \omega$, then the mediating arrows $(\gamma_k)_{k \in \omega}, (\gamma'_k)_{k \in \omega} : X \Rightarrow Z$ satisfy $(\gamma_k)_{k \in \omega} \leq (\gamma'_k)_{k \in \omega}$.

**Proof.** The claim follows from the result in 26 that: the sub-Giry monad $\mathcal{G}$ preserves limits over an $\omega^{\text{op}}$-sequence, provided that the latter consists of standard Borel spaces and surjective measurable functions. This is indeed the setting in 27, and the result yields the following limit.

\[
\begin{array}{c}
\gamma_0 \quad \gamma_1 \quad \gamma_2 \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{G}^1 \quad \mathcal{G}F^1 \ldots \quad \mathcal{G}F^\omega
\end{array}
\xrightarrow{\zeta} \mathcal{G}Z \quad \text{(limit)}
\]
It is straightforward to see that: cones over the sequence in $\mathcal{K}^\text{F}$ are precisely those over the sequence in $\mathcal{K}^\text{G}$; and the correspondence carries over to mediating arrows. Here the following easy observation plays a crucial role: for any $f: Y \to X$, $g: Z \to GX$ and $h: Z \to GY$,

$$X \xrightarrow{f} Y \xrightarrow{g} Z \text{ in } \mathcal{K}^\text{G} \quad \text{if and only if} \quad GX \xleftarrow{g} GY \xleftarrow{h} Z \text{ in } \text{Meas.} \quad (30)$$

The last “monotonicity” condition is easy, too, exploiting the fact that the measurable structure of $Z$ is the weakest one such that all projections $\pi_i$ are measurable. ▷

Now we shall prove Lem. 3.7.2.

**Proof.** We first define $\Delta^\omega: \mathfrak{G}^\mathfrak{G} \to \mathfrak{G}^\mathfrak{G}$. Let $(h_A: X \to 1) \in \mathfrak{G}^\mathfrak{G}$.

For each $k \in \omega$, we define an arrow $\gamma^A_k: X_A \to \mathcal{F}^k 1$ by induction on $k$ as follows:

$$\gamma^A_0 := h_A$$
$$\gamma^A_k := \gamma^A_{k+1} \circ \rho \circ g_b \circ c_A$$

Here $c_{[A,B]} = c \circ k_{[A,B]}$; and $k_{[A,B]}: X_{[A,B]} \to X_A + X_B$ denotes the canonical coprojection.

We show that $(X_A, (\gamma^A_k: X_A \to \mathcal{F}^k 1)_{k \in \omega})$ is a cone over the sequence $1 \xrightarrow{\rho} \mathcal{F}_1 \xrightarrow{\rho} \mathcal{F}_2 \xrightarrow{\rho} \mathcal{F}_3 \xrightarrow{\rho} \cdots$. To this end, we show that for each $k \in \omega$, $\mathcal{F}^k \rho \circ \gamma^A_{k+1} = \gamma^A_k$ by induction on $k$. If $k = 0$, then:

$$\mathcal{F}^k \rho \circ \gamma^A_{k+1} = \rho \circ g_b \circ c_A$$

For $k > 0$, we have:

$$\mathcal{F}^k \rho \circ \gamma^A_{k+1} = \mathcal{F}^k \rho \circ \gamma^A_k \circ \rho \circ g_b \circ c_A$$

Hence $(X_A, (\gamma^A_k: X_A \to \mathcal{F}^k 1)_{k \in \omega})$ is a cone over the sequence $1 \xrightarrow{\rho} \mathcal{F}_1 \xrightarrow{\rho} \mathcal{F}_2 \xrightarrow{\rho} \mathcal{F}_3 \xrightarrow{\rho} \cdots$, and this implies that there uniquely exists a mediating arrow $h^A_{\omega}: X_A \to Z$.

We show that $h^A_{\omega}$ belongs to $\mathfrak{G}^\mathfrak{G}$, that is, $h^A_{\omega} = \gamma^A_{\omega} \circ \rho \circ g_b \circ c_A$. To this end, by the definition of $h^A_{\omega}$, it suffices to show that for each $k \in \omega$ we have

$$\rho \circ g_b \circ c_A = \gamma^A_k.$$
If \( k = 0 \), then we have:

\[
\begin{align*}
J\pi_k \circ (J\zeta^{-1} \circ F[h^1_A, g_B] \circ c_A) \\
= J!Z \circ J\zeta^{-1} \circ F[h^1_A, g_B] \circ c_A \\
&= J!F_1 \circ J!F_Z \circ F[h^1_A, g_B] \circ c_A \\
&= J!F_1 \circ F[J\pi_0 \circ h^1_A, J\pi_Z \circ g_B] \circ c_A \\
&= J!F_1 \circ F[J\pi_0 \circ h^1_A, J\pi_Z \circ g_B] \circ c_A \\
&= J!F_1 \circ F[\gamma^A_k, J\pi_Z \circ g_B] \circ c_A \\
&= J!F_1 \circ F[\gamma^A_k, J\pi_Z \circ g_B] \circ c_A \\
&= J\pi_0 =!Z
\end{align*}
\]

If \( k > 0 \), then we have:

\[
\begin{align*}
J\pi_k \circ (J\zeta^{-1} \circ F[h^1_A, g_B] \circ c_A) \\
= JF\pi_{k-1} \circ F[h^1_A, g_B] \circ c_A \\
= F[J\pi_{k-1} \circ h^1_A, J\pi_{k-1} \circ g_B] \circ c_A \\
= F[\gamma^A_{k-1}, J\pi_{k-1} \circ g_B] \circ c_A \\
= \gamma^A_k
\end{align*}
\]

We shall define \( \Delta^g_n : \mathcal{G}^g_n \rightarrow \mathcal{G}^g_n \) by \( \Delta^g_n(h_A) := h^1_A \); and let us show the monotonicity of \( \Delta^g_n \) above. Assume that \( h_A \sqsubseteq h' : X \rightarrow 1 \). Let \( \{X, (\gamma^A_k : X \rightarrow F^1_k)_{k \in \omega}\} \) and \( \{X, (\gamma^A_k : X \rightarrow F^1_k)_{k \in \omega}\} \) be cones that are induced by \( h_A \) and \( h' \) as above, respectively. Then by induction on \( k \in \omega \), we can show that \( \gamma^A_k \sqsubseteq \gamma^A_k \) for each \( k \in \omega \). As \( \{Z, (J\pi_k : Z \rightarrow F^1_k)_{k \in \omega}\} \) is a 2-limit, it implies that the mediating arrow induced by \( \{X, (\gamma^A_k : X \rightarrow F^1_k)_{k \in \omega}\} \) is less than or equal to the one induced by \( \{X, (\gamma^A_k : X \rightarrow F^1_k)_{k \in \omega}\} \)—which means \( \Delta^g_n(h_A) \sqsubseteq \Delta^g_n(h'_A) \), by definition.

To conclude the proof, we show that \( \Delta \) and \( \Gamma \) indeed constitute an isomorphism, that is,

1. \( \Delta^g_n(\Gamma_A(g_A)) = g_A \) if \( g_A \in \mathcal{G}^g_n \); and
2. \( \Gamma_A(\Delta^g_n(h_A)) = h_A \) if \( h_A \in \mathcal{G}^g_n \).  

[1] Let \( g_A \in \mathcal{G}^g_n \). Let \( h_A = \Gamma_A(g_A) \) and define a cone \( \{X, (\gamma^A_k : X \rightarrow F^1_k)_{k \in \omega}\} \) as above. Note that by definition of \( \Delta^g_n \), \( \Delta^g_n(\Gamma(g_A)) = h^1_A \) where \( h^1_A : X \rightarrow Z \) is the unique mediating arrow from \( \{X, (\gamma^A_k : X \rightarrow F^1_k)_{k \in \omega}\} \) to \( \{Z, (J\pi_k : Z \rightarrow F^1_k)_{k \in \omega}\} \).

For each \( k \in \omega \), we prove \( J\pi_k \circ g_A = \gamma^A_k \) by induction on \( k \). If \( k = 0 \), then

\[
\begin{align*}
J\pi_k \circ g_A &= J!Z \circ g_A \\
&= h_A & (\pi_0 =!Z) \\
&= \gamma^A_k & (by \ definition)
\end{align*}
\]
If $k > 0$, we have:

\[
\begin{align*}
J\pi_k &\circ g_A \\
&= J\pi_k \circ J\zeta^{-1} \circ \mathcal{T}[g_A, g_B] \circ c_A \\
&= J\mathcal{F}\pi_{k-1} \circ \mathcal{T}[g_A, g_B] \circ c_A \\
&= \mathcal{T}[J\pi_{n-1} \circ g_A, J\pi_{n-1} \circ g_B] \circ c_A \\
&= \mathcal{T}[[\gamma^A]^{k-1} \circ J\pi_{k-1} \circ g_A] \circ c_A \\
&= \gamma^A_k
\end{align*}
\]

(by induction hypothesis)

Therefore by uniqueness of the mediating arrow, we have $g_A = h^1_A$, and this implies Cond. 1.

By definition, $\Delta^\omega(h_A) = h^1_A$ where each $h^1_A$ is the unique mediating arrow from a cone $(X, (\gamma^A_k : X \to \mathcal{T}^k_1)_{k \in \omega})$ to the limit $(Z, (J\pi_k : Z \to \mathcal{T}^k_1)_{k \in \omega})$ where the former is defined as above. Letting $k = 0$, we have:

\[
\begin{align*}
\Gamma_A(h^1_A) &= J\Pi^1 \circ h^1_A \\
&= J\pi_0 \circ h^1_A \\
&= \gamma^A_0 \\
&= h_A
\end{align*}
\]

(h$_A^1$ is a mediating arrow)

(by definition)

This implies Cond. 2.

D.5 Proof of Lem. 5.3

Proof. It is straightforward that $\mathcal{K}(\mathcal{G})(X, 1)$ is both a pointed $\omega$-cpo and a pointed $\omega^{op}$-cpo (here restriction to $\omega$ is crucial for compatibility with measurable structures). Moreover, Kleisli composition $\circ$ in $\mathcal{K}(\mathcal{G})$ is seen to be $\omega$- and $\omega^{op}$-continuous, similarly to the proof of [5 Prop. 4.20]—thus the equational system $E'$ in [12] indeed has a solution $l^{pol}_1, \ldots, l^{pol}_n$ by Lem. 2.11

Recall the similarity between $\Phi_X, \Psi_X$ and the diagrams in [11]. We can prove $\Gamma \circ \Phi_X = \Psi_X \circ \Gamma$ (where $\Gamma$ is from Lem. 5.2, as shown in the above diagram; indeed $(\Gamma \circ \Phi_X)(g) = J\Pi^1 \circ J\zeta^{-1} \circ \mathcal{T}g \circ c_A$, and $(\Psi_X \circ \Gamma)(g) = J\Pi^1 \circ \mathcal{T}J\pi \circ \mathcal{T}g \circ c_A$). This discharges Cond. 1 of Lem. 5.3 where $E$ and $E'$ are taken as in [12]; Cond. 2 is discharged by Lem. 5.2. Therefore by taking $\Gamma$ as $\varphi$ and $\Delta^{[l_1, \ldots, l_n]}_{\psi^{[l_1, \ldots, l_n]}}$ in Lem. 5.3, we conclude existence of a solution $l^{pol}_1, \ldots, l^{pol}_n$ of $E$, and that $\Gamma([l^{pol}_1, \ldots, l^{pol}_n]) = [l^{pol}_1, \ldots, l^{pol}_n]$.

Finally we realize that $E$ in [12] is the same one as $E_X$ in Def. 3.5 therefore $tr^p(X) = [l^{pol}_1, \ldots, l^{pol}_n]$.

D.6 Proof of Lem. 5.10

Proof. Without loss of generality, we can assume that $n$ is even. We shall append a state $\blacklozenge$ and a unary letter $\diamond$, that represent divergence explicitly, by trapping every divergence into the non-accepting infinite loop $(\diamond, \blacklozenge)(\diamond, \blacklozenge) \cdots$.
More concretely, we define a new PPTA $X_\bullet = ((X_1, \ldots, X_n), \{\bullet\}), \Sigma + (o, \delta_\bullet, s)$, where $\delta_\bullet : (X + \{\bullet\}) \to G(\prod_{\sigma \in \Sigma + (o)} X^{[\sigma]})$ is defined as follows.

$$\delta_\bullet(x)(\sigma, (x_1, \ldots, x_n)) :=
\begin{cases}
\delta(x)(\sigma, (x_1, \ldots, x_n)) & (x, x_1, \ldots, x_n \in X, \sigma \in \Sigma) \\
1 - \sum_{(\sigma(x_1, \ldots, x_{[\sigma]})) \in \prod_{\sigma \in \Sigma} X^{[\sigma]}} \delta((\sigma, (x_1, \ldots, x_{[\sigma]}))) & (n = 1, x \in X, x_1 = \bullet, \sigma = o) \\
1 & (n = 1, x = x_1 = \bullet, \sigma = o) \\
0 & (otherwise).
\end{cases}$$

Notice that $\{\bullet\}$ has an odd priority $n + 1$ that is maximum. Let $\tilde{t}^{pol}_{n_1}, \ldots, \tilde{t}^{pol}_{n+1}$ be the solution of the following equational system over $[0, 1]^{X + \{\bullet\}}$.

$$u_1' = \mu \Psi'_{X}([u_1', \ldots, u_n', u_{n+1}']) \mid X_1$$
$$\vdots$$
$$u_n' = \mu \Psi'_{X}([u_1', \ldots, u_n', u_{n+1}']) \mid X_n$$
$$u_{n+1}' = \mu \Psi'_{X}([u_1', \ldots, u_n', u_{n+1}']) \mid \{\bullet\}$$

The $(n + 1)$-th solution $\tilde{t}^{pol}_{n+1}$ is $([\bullet] \mapsto 0]$, since it is defined by the least fixed point of the identity function. Thus we can ignore the last equation and obtain the following equational system, without changing the other part of the solution $\tilde{t}^{pol}_{1}, \ldots, \tilde{t}^{pol}_{n}$.

$$u_1' = \mu \Psi'_{X}([u_1', \ldots, u_n', [\bullet] \mapsto 0]) \mid X_1$$
$$\vdots$$
$$u_n' = \mu \Psi'_{X}([u_1', \ldots, u_n', [\bullet] \mapsto 0]) \mid X_n$$

It is easy to see that $\Psi'_{X}(l_1, \ldots, l_n, [\bullet] \mapsto 0] = \Psi'_{X}(l_1, \ldots, l_n)$. Thus the solution $\tilde{t}^{pol}_{1}, \ldots, \tilde{t}^{pol}_{n}$ coincides with $l^{pol}_{1}, \ldots, l^{pol}_{n}$.

We shall define $\text{Run}_{X}$, in the similar manner to $\text{Run}_{X}$ (Def. A.3), except that any $\rho \in \text{Run}_{X}$ that contains a label $(\sigma, \bullet)$ where $\sigma \in \Sigma$ does not belong to $\text{Run}_{X}$. (Recall that in the current probabilistic setting, $\text{Run}_{X}$ is defined to permit arbitrary transitions between the states.)

We augment the equational system (9) (in Lem. A.4), which characterizes the accepting runs, with $\bullet$. Though the system (9) is defined with respect to $\text{Run}_{X}$ of an NBTA $X$, its definition naturally extends to runs of PBTAs. The definition of this augmented equational system is as follows.

$$u_1 = \mu \mathcal{V}_{X}([u_1, \ldots, u_n, \{\bullet\}] \cap \text{Run}_{X})$$
$$\vdots$$
$$u_n = \mu \mathcal{V}_{X}([u_1, \ldots, u_n, \{\bullet\}] \cap \text{Run}_{X})$$
$$u_{n+1} = \mu \mathcal{V}_{X}([u_1, \ldots, u_n, \{\bullet\}] \cap \text{Run}_{X})$$

Much like in the last case of (31), we can easily see that the (non-last) solution of the equational system (32) coincides with one of (9), which is $\text{AccRun}_{X}$. Note that here the definition of $\text{Run}_{X}$, which excludes a run with a $(\sigma, \bullet)$-labeled node, is crucial.

Now we aim to apply Lem. A.4 sending the solution of (32) (accepting runs) to one of (31) (acceptance probabilities), by $\mu^{\text{Run}}_{X}$. Notice that first for the equational system (32), each interim solution can be defined as either the $\omega$-supremum or the $\omega$-infimum (as in the proof of Lem. A.4), essentially because $\mathcal{V}_{X}$ is both $\omega$-continuous and $\omega^{op}$-continuous; thus
\(\delta\) can be solved within measurable spaces. This observation is required, since \(\mu_{\text{Run}}^{\mathcal{X}}\) is defined only over measurable sets of runs. Preservation of \(\bot\), is almost trivial; and \(\Psi_{\mathcal{X}}\) and \(\mu_{\text{Run}}^{\mathcal{X}}\) are both \(\omega\)-continuous and \(\omega^{\text{op}}\)-continuous by measurability.

The other conditions required in Lem. B.4 are as follows.

- **Commutativity:** \(\mu_{\text{Run}}^{\mathcal{X}}(\otimes_{\mathcal{X}} R) = \Psi_{\mathcal{X}}(\mu_{\text{Run}}^{\mathcal{X}}(R))\) for \(R \in \mathcal{P}( \text{Run}_{\mathcal{X}} )\)
- **Preservation of \(\top\):** \(\mu_{\text{Run}}^{\mathcal{X}}(\text{Run}_{\mathcal{X}}) = 1\)

The commutativity condition is easily seen; and the preservation of \(\top\) is due to the definition of \(\delta\) in which the “missing” probability is filled by the transitions to \(\bullet\).

Then by applying Lem. B.4 we have

\[
\mu_{\mathcal{X},\bullet}^{\text{Run}}(\text{AccRun}_{\mathcal{X},i}) = I^i_{\text{sol}}.
\]

Since \(\text{AccProb}(x) = \mu_{\text{Run}}^{\mathcal{X}}(\text{AccRun}_{\mathcal{X}})\) by definition, it suffices to show, for any \(x \in X\),

\[
\mu_{\mathcal{X},\bullet}^{\text{Run}}(\text{AccRun}_{\mathcal{X}}) = \mu_{\mathcal{X},\bullet}^{\text{Run}}(\text{AccRun}_{\mathcal{X}}).
\]

In fact, thanks to measurability, we only need to show that for any partial run \(\xi\) of \(\mathcal{X}\):

\[
\mu_{\mathcal{X},\bullet}^{\text{Run}}(\text{Cyl}_{\mathcal{X}}(\xi)) = \mu_{\mathcal{X},\bullet}^{\text{Run}}(\text{Cyl}_{\mathcal{X}}(\xi)). \tag{33}
\]

We note that \(\text{Cyl}_{\mathcal{X}}(\xi)\) does not contain any of \(\circ\) or \(\bullet\), because \(\xi\) is a run of \(\mathcal{X}\) and is not a run of \(\mathcal{X}_{\bullet}\). Therefore, by the inductive definition of \(\mu_{\mathcal{X}}^{\text{Run}}\) in Def. 5.9, (33) can be straightforwardly confirmed. This concludes the proof.

**D.7 Proof of Thm. 5.11**

**Proof.** We identify \(\mathcal{X}\) with a \((\mathcal{G}, F_{\Sigma})\)-system \((\mathcal{X}_1, \ldots, \mathcal{X}_n), \delta: X \rightarrow F_{\Sigma} X, s: 1 \rightarrow X\), and let \(1 = \{\bullet\}\). We can easily see that \(\Psi_{\mathcal{X}}\) (in Lem. 5.3) and \(\Psi'_{\mathcal{X}}\) (in Lem. 5.10) define exactly the same function. Therefore, by the claim of these two lemmas, we have \(\Gamma(\text{tr}^p(\mathcal{X})) = \text{AccProb}_{\mathcal{X}}\).

Now we note the following:

\[\Gamma([x \mapsto \mu_{\mathcal{X},\bullet}^{\text{Tree}}]) = J^!_{\text{Tree}} \circ ([x \mapsto \mu_{\mathcal{X},\bullet}^{\text{Tree}}]) = \mu_{\mathcal{X},\bullet}^{\text{Tree}}(\text{Tree}_{\Sigma}) = \mu_{\mathcal{X},\bullet}^{\text{Tree}}(\text{Cyl}_{\Sigma}(\bullet)) \quad \text{Def. 5.7} \]

where \(\bullet\) denotes the partial tree consisting of one node labeled by \(\bullet\) ("continuation", Def. 5.7). As \(\text{DelSt}^{-1}(\text{Cyl}_{\Sigma}(\bullet))\) is nothing but the set of all runs \(\text{Run}_{\mathcal{X}}\), we have

\[\Gamma([x \mapsto \mu_{\mathcal{X},\bullet}^{\text{Tree}}]) = \mu_{\mathcal{X},\bullet}^{\text{Run}}(\text{AccRun}_{\mathcal{X}}) = \text{AccProb}_{\mathcal{X}}\]

by the definition of \(\text{AccProb}_{\mathcal{X}}\) (in Lem. 5.10).

Combining the above two facts we obtain \(\Gamma(\text{tr}^p(\mathcal{X})) = \Gamma([x \mapsto \mu_{\mathcal{X},\bullet}^{\text{Tree}}])\). Recall that there is an inverse of \(\Gamma\), namely \(\Delta\) in Lem. 5.2; this yields \(\text{tr}^p(x) = \mu_{\mathcal{X},\bullet}^{\text{Tree}}\).

Now the claim is immediate, as below, where we have only to consider cylinder sets \(\text{Cyl}_{\Sigma}(\lambda)\) that generate the relevant \(\sigma\)-algebra.

\[\text{tr}^p(\mathcal{X})(\bullet)(\text{Cyl}_{\Sigma}(\lambda)) = \sum_{x \in X} s(x) \mu_{\mathcal{X},\bullet}^{\text{Tree}}(\text{Cyl}_{\Sigma}(\lambda)) = \mu_{\mathcal{X}}^{\text{Tree}}(\text{Cyl}_{\Sigma}(\lambda)) = \text{Lang}(\mathcal{X})(\text{Cyl}_{\Sigma}(\lambda)).\]