HILBERT 90 FOR BIQUADRATIC EXTENSIONS

ROMAN DWILEWICZ, JÁN MINÁČ, ANDREW SCHULTZ, AND JOHN SWALLOW

1. INTRODUCTION: “COMMERCE WITH THE GREAT UNIVERSE”.

Here something stubborn comes,
Dislodging the earth crumbs
And making crusty rubble.
It comes up bending double,
And looks like a green staple.
It could be seedling maple,
Or artichoke, or bean.
That remains to be seen.
—Richard Wilbur, “Seed Leaves”

We linger over the simplest theorems, the exquisite ones, succinct and stately. Something quickens: dare we hope that such a theorem, a result of already uncommon brevity, might grow into a result of uncommon grandeur? We fancy it a seedling, a tiny seed-case somehow containing within itself its own generalization. We imagine that its hidden truths, unknown to us now, await only fertile soil and close observation. In the quadratic case of Hilbert’s Theorem 90 we find just such a seedling.

Turning it over in our hands, we naturally wonder what we might one day behold. Friends tell us to expect what is affectionately known as “Hilbert 90,” a result in the theory of cyclic extensions. But quadratic Hilbert 90, as we shall now call the quadratic case, does not lead only to Hilbert 90. Those who have ventured into Galois cohomology know that Andreas Speiser sensed something different, a cohomological result in a noncohomological era. Emmy Noether recognized the power of Speiser’s result and used it to great effect: the result then became known as Noether’s theorem. As we shall see, even tiny quadratic Hilbert 90 contains suggestions of cohomology.
More recently, others have felt fresh intimations, imagining something different still. The spectacular announcement by Vladimir Voevodsky of the validity of the Bloch-Kato conjecture contains, in fact, a generalization of Hilbert’s Theorem 90 to Milnor $K$-theory.\footnote{The interested reader should consult \textcite{6} and \textcite{3} for some initial background on Hilbert’s Theorem 90 and \textcite{13, p. 30} for its cohomological generalization. To observe the use of Hilbert 90-type theorems in the partially published work of Markus Rost and Voevodsky on the Bloch-Kato conjecture, see \textcite{11} and \textcite{12}. For further original sources on Hilbert 90 and its cohomological generalization see Ernst Kummer’s early discovery of a special case \textcite{4}, followed by Speiser’s result \textcite{9} and Noether’s application \textcite{7}.}

In this article we propose to linger just long enough to investigate the very first shoots of the generalization from quadratic to biquadratic field extensions. The quadratic case is well known and already significant. The biquadratic case, particularly the difference between it and the quadratic case, appears new. Observing the shoots break the surface and bend back upon themselves, we encounter the possibility that the unfurling leaves aim in an uncharted direction, somewhere between the traditional Hilbert 90 result on cyclic extensions and Speiser’s and Noether’s cohomological result. Exploring this new possibility, we discern connections among multiplicative groups of fields, values of binary quadratic forms, a bit of module theory over group rings, and even Galois cohomology.

2. “ARTICHOKE, OR BEAN”? THE QUADRATIC SEED-CASE. The classical form of Hilbert 90 states that the kernel of the norm operator on a cyclic extension is as small as possible. We begin by making this statement precise in the context of a quadratic extension.

Let $F$ be a field of characteristic different from two. Then any quadratic extension $L$ of $F$ takes the form $L = F(\sqrt{a})$ for a suitable element $a$ of $F$ that is not contained in the set $F^2$ of squares of elements of $F$. The set $\{1, \sqrt{a}\}$ is a basis for the vector space $L$ over the field $F$, so each element $\ell$ in $F(\sqrt{a})$ can be expressed uniquely as $\ell = f_1 + f_2 \sqrt{a}$ with $f_1$ and $f_2$ in $F$.

The field extension $L/F$ has a special property: if an irreducible polynomial $g(x)$ with coefficients in $F$ has a root in $L$, then $L$ contains a complete set of distinct roots of $g$. Put more precisely, if for some element $\ell_1$ of $L$ and some polynomial $h(x)$ with coefficients in $L$ we can factor $g(x) = (x - \ell_1) \cdot h(x)$, then there exists a constant $c$ in $L$ and a collection of distinct elements $\ell_i$ in $L$ such that $g(x) = c \cdot \prod_i (x - \ell_i)$.
Field extensions $L/F$ enjoying this property are called Galois extensions. As a set, the Galois group $\text{Gal}(L/F)$ of a Galois extension $L/F$ consists of all field automorphisms of $L$ leaving $F$ pointwise fixed. The group operation is function composition.

The great discovery of Galois theory is that some fundamental properties of Galois extensions $L$ of $F$ are already captured by the corresponding Galois groups $\text{Gal}(L/F)$. For instance, if the Galois group has precisely $n$ subgroups, then Galois theory tells us that there exist precisely $n$ fields $M$ satisfying $F \subseteq M \subseteq L$. To emphasize the importance of certain group-theoretic properties of the Galois group, we use the same adjectives to modify the corresponding Galois field extension. For instance, a field extension $L/F$ is called cyclic if $L/F$ is a Galois extension and the associated Galois group $\text{Gal}(L/F)$ is a cyclic group. An important result in Galois theory—and one that we will have frequent occasion to use—is the fact that the subset of $L$ consisting of elements fixed by every element of the Galois group $\text{Gal}(L/F)$ is precisely $F$.

With these definitions, we return to the quadratic extension $L = F(\sqrt{a})$ and find that $L/F$ is a cyclic extension whose Galois group $G = \text{Gal}(L/F)$ contains precisely two elements: the identity automorphism, denoted by id, and $\sigma$, which sends $\ell = f_1 + f_2\sqrt{a}$ to $\sigma(\ell) = f_1 - f_2\sqrt{a}$. We can verify that the elements of $L$ fixed by $\text{Gal}(L/F)$ are, in fact, those lying in $F$ by noticing that if $\sigma(\ell) = \ell$, then $f_2 = 0$, placing $\ell = f_1$ in $F$.

The norm function $N_{L/F} : L \to F$ for such a quadratic extension is defined by the formula $N_{L/F}(\ell) = \ell \cdot \sigma(\ell) = f_1^2 - af_2^2$, and, omitting zero from the domain, we find that $N_{L/F}$ restricts to a homomorphism from the multiplicative group $L^\times := L \setminus \{0\}$ to the multiplicative group $F^\times$ of $F$. Since there is no chance for confusion, we write $N_{L/F}$ for this restricted function.

It is natural to determine the kernel of this homomorphism. Since $N(\ell) = \ell \cdot \sigma(\ell) = N(\sigma(\ell))$ for each $\ell$ in $L^\times$, we see that elements of the form $\ell/\sigma(\ell)$ certainly lie in $\ker N_{L/F}$. Hilbert’s Theorem 90 says that $\ker N_{L/F}$ contains no other elements.

**Theorem 1 (Hilbert 90 for $F(\sqrt{a})/F$).** For a quadratic extension $F(\sqrt{a})$ of $F$ it is the case that

\[
\ker \left( N_{F(\sqrt{a})/F} : F(\sqrt{a})^\times \to F^\times \right) = \left\{ \frac{\ell}{\sigma(\ell)} : \ell \in F(\sqrt{a})^\times \right\}.
\]
Not surprisingly, this simple statement is already important. For example, the classical parameterization of Pythagorean triples is a beautiful consequence of this statement. The idea of the proof can be traced back to Olga Taussky [10, pp. 808–809] (who was in fact a coeditor of Hilbert’s collected works), and Noam Elkies independently discovered the proof in a short, attractive note [2]. See also Takashi Ono’s book [8, pp. 4–5].

Proof of Theorem 1. Let \( t \) belong to \( \ker N_{L/F} \). If \( t = -1 \), then \( -1 = t = \sqrt{a}/\sigma(\sqrt{a}) \). Assume therefore that \( t \neq -1 \). Set \( \ell = 1 + t \). Then

\[
\sigma(\ell) = 1 + \sigma(t) = \sigma(t) \cdot t + \sigma(t) = \sigma(t) \cdot (t + 1) = \sigma(t) \cdot \ell.
\]

Hence \( \sigma(\ell)/\ell = \sigma(t) \). Applying \( \sigma \) to both sides and remembering that \( \sigma^2 = \text{id} \) we obtain \( \ell/\sigma(\ell) = t \), as required. Therefore

\[
\ker N_{L/F} \subseteq \left\{ \frac{\ell}{\sigma(\ell)} : \ell \in L^\times \right\}.
\]

We have already observed the reverse inclusion. \( \square \)

The generalization of this result to an arbitrary cyclic extension is well known as “Hilbert 90.” Many of us have come through precisely such underbrush while ambling across our own fields. Speiser saw, however, that in the right environment the seedling that is quadratic Hilbert 90 produces something similar but different—something that our naturalist friends might call “homologous.”

Let us review the example in Theorem 1 and observe how the seed might develop into an elegant result in Galois cohomology. Consider a map \( f : G \to L^\times \) satisfying

\[
f(\gamma_1 \cdot \gamma_2) = (\gamma_1(f(\gamma_2))) \cdot f(\gamma_1) \quad (\gamma_1, \gamma_2 \in G).
\]

Because of its similarity to a homomorphism, such a map is called a crossed homomorphism. From the equality \( f(\text{id}) = (\text{id}(f(1))) \cdot f(\text{id}) = f(\text{id})^2 \) we deduce that \( f(\text{id}) = 1 \). As \( G \) is especially simple in our context, it follows that a crossed homomorphism is determined by \( f(\sigma) \). From \( 1 = f(\sigma^2) = (\sigma(f(\sigma))) \cdot f(\sigma) \) we see that \( f(\sigma) \) lies in \( \ker N_{L/F} \).

Conversely, choosing any element \( t \) from \( \ker N_{L/F} \), we can define a crossed homomorphism \( f : G \to L^\times \) by \( f(1) = 1, f(\sigma) = t \). Appealing to Theorem 1 we have \( f(\sigma) = \ell/\sigma(\ell) \) for some \( \ell \in L^\times \). In particular we find that for any crossed homomorphism \( f \) there exists \( \ell \) in \( L^\times \) such that \( f(g) = \ell/g(\ell) \) for all \( g \) in \( G \).
For an arbitrary Galois group \( G = \text{Gal}(L/F) \) the set of all crossed homomorphisms \( f : G \to L^\times \) is, in fact, an Abelian group under the standard operation of multiplying functions: for \( f_1 \) and \( f_2 \) crossed homomorphisms from \( G \) to \( L^\times \), we define \( f_1 \cdot f_2 \) by \( (f_1 \cdot f_2)(g) = f_1(g)\cdot f_2(g) \) for \( g \in G \). An important subgroup of the group of crossed homomorphisms is the subgroup of coboundaries. A map \( f : G \to L^\times \) given by the formula \( f(g) = \lambda / g(\lambda) \) for some fixed \( \lambda \) in \( L^\times \) is called a coboundary. Such a map is easily seen to be a crossed homomorphism. The first cohomology group \( H^1(G, L^\times) \) of \( G \) with coefficients in \( L^\times \) is defined to be the group of crossed homomorphisms modulo the subgroup of coboundaries:

\[
H^1(G, L^\times) = \frac{\text{crossed homomorphisms}}{\text{coboundaries}}.
\]

In this language, Theorem 1 translates to \( H^1(G, L^\times) = \{1\} \). Such an elegant outcome is what Speiser sensed in the seedling: he proved that \( H^1(G, L^\times) = \{1\} \) for all Galois extensions \( L/F \) with group \( G \).

3. “THE STALK IN TIME UNBENDS”: Biquadratic Extensions. Replacing \( G \) with the Klein 4-group \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), we shall observe the shoots of our seed “bending double” in this new context. It will appear, though, as if the new growth points toward something unexpected, something akin to both the traditional Hilbert 90 result and Speiser’s cohomological theorem on \( H^1(G, L^\times) \).

Let \( E \) be a Galois extension of \( F \) with \( G = \text{Gal}(E/F) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \). Since we assume that \( F \) has characteristic other than two, there exist \( a_1 \) and \( a_2 \) in \( F^\times \setminus (F^\times)^2 \) such that \( E = F(\sqrt{a_1}, \sqrt{a_2}) \), and the group \( G \) is generated by two automorphisms \( \sigma_1 \) and \( \sigma_2 \) such that

\[
\frac{\sigma_i(\sqrt{a_j})}{\sqrt{a_j}} = (-1)^{\delta_{ij}},
\]

where \( \delta_{ij} \) is Kronecker’s delta function: \( \delta_{ij} = 0 \) if \( i = j \) and \( \delta_{ij} = 1 \) if \( i \neq j \). The lattice of subfields of \( E \) then takes the form

\[
\begin{array}{c}
\text{E} \\
\text{E}_1 & \text{E}_3 & \text{E}_2 \\
\text{F}
\end{array}
\]
where \( E_1 = F(\sqrt{a_1}) \), \( E_2 = F(\sqrt{a_2}) \), and \( E_3 = F(\sqrt{a_1a_2}) \). We say that an extension \( E/F \) satisfying these properties is a biquadratic extension.

Since \( E = E_1(\sqrt{a_2}) \), \( E = E_2(\sqrt{a_1}) \), and \( E = E_3(\sqrt{a_1a_2}) \) are all quadratic extensions of fields of characteristic different from two, the field extensions \( E/E_1 \), \( E/E_2 \), and \( E/E_3 \) are also Galois extensions, with respective Galois groups \( \text{Gal}(E/E_1) = \{\text{id}, \sigma_1\} \), \( \text{Gal}(E/E_2) = \{\text{id}, \sigma_2\} \), and \( \text{Gal}(E/E_3) = \{\text{id}, \sigma_1\sigma_2\} \).

Given quadratic Hilbert 90 for the two extensions \( E/E_1 \) and \( E/E_2 \), we demonstrate that \( H^1(G, E^\times) = \{1\} \) is equivalent to another condition, itself very much in the spirit of the original Hilbert 90 statement. In fact, we show that this condition is the “difference” between Speiser’s result for biquadratic extensions and quadratic Hilbert 90, in the following sense: by adding the new result to quadratic Hilbert 90, we recover Speiser’s result in the biquadratic case.

Some corollaries of this condition have been rediscovered several times, but the connection with Hilbert 90 and the natural proof that we present seem new. To establish this equivalence, we rephrase the condition \( H^1(G, E^\times) = \{1\} \) in the language of elements of the multiplicative group \( E^\times \).

Consider a crossed homomorphism \( f : G \rightarrow E^\times \). Since \( f(\text{id}) = 1 \) and \( f(\sigma_1 \cdot \sigma_2) = (\sigma_1(f(\sigma_2))) \cdot f(\sigma_1) \) we see that \( f \) is determined by the values \( \alpha_i = f(\sigma_i) \). We further observe that

\[
1 = f(\sigma_i^2) = (\sigma_i(\alpha_i)) \cdot \alpha_i = N_{E/E_i}(\alpha_i) \quad (i = 1, 2)
\]

and that

\[
(\sigma_1(\alpha_2)) \cdot \alpha_1 = f(\sigma_1 \cdot \sigma_2) = f(\sigma_2 \cdot \sigma_1) = (\sigma_2(\alpha_1)) \cdot \alpha_2.
\]

Conversely, it is easy to check that for any given elements \( \alpha_1 \) and \( \alpha_2 \) in \( E^\times \) such that \( N_{E/E_i}(\alpha_i) = 1 \) and \( (\sigma_1(\alpha_2)) \cdot \alpha_1 = (\sigma_2(\alpha_1)) \cdot \alpha_2 \) there exists a unique crossed homomorphism \( f : G \rightarrow E^\times \) such that \( f(\sigma_i) = \alpha_i \).

Because \( H^1(G, E^\times) = \{1\} \), we also know that for a given crossed homomorphism \( f \) there exists \( \beta \) in \( E^\times \) such that \( f(g) = \beta / g(\beta) \) for each \( g \) in \( G \). In particular,

\[
\alpha_i = f(\sigma_i) = \frac{\beta}{\sigma_i(\beta)}.
\]

Therefore the cohomological identity \( H^1(G, E^\times) = \{1\} \) can be reformulated as follows:
**Theorem 2.** Let $E = F(\sqrt{a_1}, \sqrt{a_2})$ be a Galois extension of $F$ with $G = \text{Gal}(E/F) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, and let $\sigma_1$ and $\sigma_2$ be generators of $G$ that satisfy relation (1).

Then arbitrary elements $\alpha_1$ and $\alpha_2$ of $E^\times$ satisfy conditions

1. $N_{E/E_1}(\alpha_1) = 1 = N_{E/E_2}(\alpha_2)$

and

2. $\alpha_1 \cdot \sigma_1(\alpha_2) = \alpha_2 \cdot \sigma_2(\alpha_1)$

if and only if

3. there exists $\beta$ in $E^\times$ such that $\alpha_i = \beta/\sigma_i(\beta)$.

This result can be found in [1] p. 756.

Seen as results about kernels of norm functions, Theorem 2 and Hilbert 90 are similar: each shows that a kernel is suitably minimal. However, since Theorem 2 describes the simultaneous vanishing of norms from different quadratic extensions under the additional compatibility condition (2), it is best to think of Theorem 2 as a version of Hilbert 90 with a compatibility condition.

We want to see just how far this new growth reaches above the seed-case. We find that, assuming quadratic Hilbert 90, the conditions of Theorem 2 are equivalent to a statement that the kernel of a particular operator is minimal.

In order to formulate this result, it is convenient to view $E^\times$ as a $\mathbb{Z}[G]$-module. The group ring $\mathbb{Z}[G]$ is defined to be the set of formal sums of integer multiples of the elements of $G$, 

$$\mathbb{Z}[G] = \{c_0\text{id} + c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_1\sigma_2 : c_i \in \mathbb{Z}\},$$

Together with addition and multiplication operations reminiscent of the operations of polynomial addition and multiplication, as follows. We first treat $\text{id}$, $\sigma_1$, $\sigma_2$, and $\sigma_1\sigma_2$ as indeterminates and add or multiply two elements of $\mathbb{Z}[G]$ formally. Then we replace products of these “indeterminates” with the corresponding product in the group $G$. We derive the following formulas:

$$(c_0\text{id} + c_1\sigma_1 + c_2\sigma_2 + c_3\sigma_1\sigma_2) + (d_0\text{id} + d_1\sigma_1 + d_2\sigma_2 + d_3\sigma_1\sigma_2) :=
(c_0 + d_0)\text{id} + (c_1 + d_1)\sigma_1 + (c_2 + d_2)\sigma_2 + (c_3 + d_3)\sigma_1\sigma_2$$
and
\[(c_0 \text{id} + c_1 \sigma_1 + c_2 \sigma_2 + c_3 \sigma_1 \sigma_2) \cdot (d_0 \text{id} + d_1 \sigma_1 + d_2 \sigma_2 + d_3 \sigma_1 \sigma_2) :=
(c_0 d_0 + c_1 d_1 + c_2 d_2 + c_3 d_3) \text{id} + (c_0 d_1 + c_1 d_0 + c_2 d_3 + c_3 d_2) \sigma_1 +
(c_0 d_2 + c_1 d_3 + c_2 d_0 + c_3 d_1) \sigma_2 + (c_0 d_3 + c_1 d_2 + c_2 d_1 + c_3 d_0) \sigma_1 \sigma_2.
\]

A \(\mathbb{Z}[G]\)-module \(M\) is defined to be an Abelian group \(M\) together with a given action of \(\mathbb{Z}[G]\) on \(M\).

In particular, the Abelian group \(E^\times\) is a \(\mathbb{Z}[G]\)-module. We specify the action of \(\mathbb{Z}[G]\) on \(E^\times\) by extending the action of \(\sigma_i\) on \(E^\times\) in a natural way: for \(\gamma\) in \(E^\times\) and \(c_0, c_1, c_2, c_3\) in \(\mathbb{Z}\) we set
\[(c_0 \text{id} + c_1 \sigma_1 + c_2 \sigma_2 + c_3 \sigma_1 \sigma_2) \cdot \gamma = \gamma^{c_0} \cdot \sigma_1(\gamma)^{c_1} \cdot \sigma_2(\gamma)^{c_2} \cdot \sigma_1 \sigma_2(\gamma)^{c_3}.
\]
Since all of the \(\mathbb{Z}[G]\)-modules we consider are, like \(E^\times\), multiplicative groups, we continue to write the module operation multiplicatively. For example, \((\text{id} + \sigma_1) \cdot m\) can be also be written \(m \cdot \sigma_1(m)\). Similarly, we can write \((\text{id} - \sigma_1) \cdot m\) as \(m/\sigma_1(m)\). For ease of notation, from now on we shall abbreviate \(\text{id}\) by 1. Thus 1, considered as an element of our group ring \(\mathbb{Z}[G]\), will denote the identity element. From the context it will be clear when 1 denotes an element of \(\mathbb{Z}[G]\). For instance, \((1 + \sigma_i) \cdot m\) will denote \((\text{id} + \sigma_i) \cdot m\) for an element \(m\) of a \(\mathbb{Z}[G]\)-module \(M\).

Our approach to the new growth begins with a general theorem about \(\mathbb{Z}[G]\)-modules for groups \(G\) isomorphic to the Klein 4-group. A corollary that we deduce from the theorem will bring the new growth into sharp relief. In order to extend the property of quadratic Hilbert 90 from \(E^\times\) to general \(M\), we say that \((M, \sigma_1, \sigma_2)\) satisfies QH90 if for a given element \(m\) of \(M\) the condition \((1 + \sigma_i) \cdot m = 1\) implies that there exists \(n\) in \(M\) such that \(m = (1 - \sigma_i) \cdot n\).

**Theorem 3.** Let \(G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}\), and let \(\sigma_1\) and \(\sigma_2\) be generators of \(G\). Let \(M\) be a \(\mathbb{Z}[G]\)-module, written multiplicatively, and suppose that \((M, \sigma_1, \sigma_2)\) satisfies QH90. Then
\[\ker(1 - \sigma_1)(1 - \sigma_2) = \ker(1 - \sigma_1) \cdot \ker(1 - \sigma_2)\quad \text{(II)}\]
if and only if for \(m_1\) and \(m_2\) in \(M\) the two conditions

\[(1) \quad (1 + \sigma_1) \cdot m_1 = 1 = (1 + \sigma_2) \cdot m_2\]

and

\[(2) \quad m_1 \cdot \sigma_1(m_2) = m_2 \cdot \sigma_2(m_1)\]

imply the truth of

(3) there exists \( n \) in \( M \) such that \( m_i = (1 - \sigma_i) \cdot n \) \( (i = 1, 2) \).

In relation (11) the expression \( \ker(1 - \sigma_1) \cdot \ker(1 - \sigma_2) \) denotes the set of all products \( k_1 \cdot k_2 \) of elements \( k_1 \) in \( \ker(1 - \sigma_1) \) and \( k_2 \) in \( \ker(1 - \sigma_2) \).

**Proof.** Suppose first that

\[
\ker(1 - \sigma_1)(1 - \sigma_2) = \ker(1 - \sigma_1) \cdot \ker(1 - \sigma_2)
\]

and that elements \( m_1 \) and \( m_2 \) of \( M \) satisfy conditions (11) and (12). Since \( (1 + \sigma_i) \cdot m_i = 1 \), the QH90 property gives us elements \( n_i \) in \( M \) with \( m_i = (1 - \sigma_i) \cdot n_i \). Using (12) we obtain

\[
1 = \frac{(1 - \sigma_2) \cdot m_1}{(1 - \sigma_1) \cdot m_2} = (1 - \sigma_2)(1 - \sigma_1) \cdot (n_1/n_2).
\]

Hence \( n_1/n_2 \) lies in \( \ker(1 - \sigma_1)(1 - \sigma_2) = \ker(1 - \sigma_1) \cdot \ker(1 - \sigma_2) \). Recall that for each \( i \) the set \( \ker(1 - \sigma_i) \) is a group. Therefore, if \( e \) lies in \( \ker(1 - \sigma_i) \), then so does \( 1/e \). Thus there exists \( e_i \) in \( \ker(1 - \sigma_i) \) such that \( n_1/n_2 = e_2/e_1 \), and we set \( n = n_1 \cdot e_1 = n_2 \cdot e_2 \). We then see that

\[
(1 - \sigma_i) \cdot n = m_i \quad (i = 1, 2),
\]

establishing (3).

Now suppose that conditions (11) and (12) imply condition (3). If \( x = x_1 \cdot x_2 \), where \( x_i \) belongs to \( \ker(1 - \sigma_i) \), then

\[
(1 - \sigma_1)(1 - \sigma_2) \cdot x = ((1 - \sigma_2)(1 - \sigma_1) \cdot x_1) \cdot ((1 - \sigma_1)(1 - \sigma_2) \cdot x_2)
= 1,
\]

providing the inclusion \( \ker(1 - \sigma_1) \cdot \ker(1 - \sigma_2) \subseteq \ker(1 - \sigma_1)(1 - \sigma_2) \).

To verify the more interesting, opposite inclusion, consider an arbitrary \( x \) in \( \ker(1 - \sigma_1)(1 - \sigma_2) \). If \( m_1 = (1 - \sigma_1) \cdot x \) and \( m_2 = 1 \), then

\[
(1 + \sigma_1) \cdot m_1 = ((1 + \sigma_1)(1 - \sigma_1)) \cdot x = 1 = (1 + \sigma_2) \cdot m_2.
\]

Also

\[
\frac{(1 - \sigma_2) \cdot m_1}{(1 - \sigma_1) \cdot m_2} = (1 - \sigma_2)(1 - \sigma_1) \cdot x = 1.
\]

Since we have verified (11) and (12) for \( m_1 \) and \( m_2 \), (3) tells us that there exists \( n \) in \( M \) such that \( m_1 = (1 - \sigma_1) \cdot x = (1 - \sigma_1) \cdot n \) and \( 1 = m_2 = (1 - \sigma_2) \cdot n \). Therefore \( x/n \) belongs to \( \ker(1 - \sigma_1) \) and \( n \) to \( \ker(1 - \sigma_2) \). We conclude that \( x = (x/n) \cdot n \) lies in \( \ker(1 - \sigma_1) \cdot \ker(1 - \sigma_2) \). \( \square \)
Remark. Since \( \ker(1 - \sigma_1) \cdot \ker(1 - \sigma_2) \subseteq \ker(1 - \sigma_1)(1 - \sigma_2) \) for all \( \mathbb{Z}[G] \)-modules, relation (III) says that \( \ker(1 - \sigma_1)(1 - \sigma_2) \) is as small as possible.

Returning to the \( \mathbb{Z}[G] \)-module \( E^\times \), we observe that \( (E^\times, \sigma_1, \sigma_2) \) satisfies QH90. Indeed, for each \( i \) and for \( \gamma \) in \( E^\times \) the condition \( \gamma \cdot \sigma_i(\gamma) = 1 \) implies that there exists \( \delta \) in \( E^\times \) such that \( \gamma = \delta/\sigma_i(\delta) \). Therefore Theorems 2 and 3 applied to \( M = E^\times \) immediately give the following corollary, which makes the new growth visible:

**Corollary 1.** Let \( E = F(\sqrt{a_1}, \sqrt{a_2}) \) be a Galois extension of \( F \) with \( G = \text{Gal}(E/F) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), and let \( \sigma_1 \) and \( \sigma_2 \) be generators of \( G \) that satisfy relation (I). Then

\[
\ker(1 - \sigma_1)(1 - \sigma_2) = \ker(1 - \sigma_1) \cdot \ker(1 - \sigma_2).
\]

Remark. By requiring elements \( \sigma_1 \) and \( \sigma_2 \) of \( G \) to satisfy relation (I), we are viewing this result from only one particular direction. The equivalence in Theorem 2 implies the truth of every version of relation (III) obtained by substituting two arbitrary generators of \( G \) in place of \( \sigma_1 \) and \( \sigma_2 \), while the conjunction of any version of relation (III) with quadratic Hilbert 90 gives the equivalence in Theorem 2. Hence the particular “difference” we find in the corollary depends only on our vantage point.

4. “AND STARTS TO RAMIFY”: A FORM REVEALED.

Now we establish relation (III) directly and record several of its consequences. One corollary in particular has been rediscovered several times under a certain attractive disguise as a statement on binary quadratic forms. Although several ingenious and beautiful proofs of this statement have been obtained in the literature, the proof we offer may be the most transparent.

**Theorem 4.** Let \( E = F(\sqrt{a_1}, \sqrt{a_2}) \) be a Galois extension of \( F \) with \( G = \text{Gal}(E/F) = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \), and let \( \sigma_1 \) and \( \sigma_2 \) be generators of \( G \) that satisfy relation (I). Then the following sets are identical:

\[
K_1 = \ker(1 - \sigma_1)(1 - \sigma_2),
K_2 = \ker(1 - \sigma_1) \cdot \ker(1 - \sigma_2),
K_3 = \langle E_1^\times, E_2^\times \rangle,
K_4 = \{ e \in E^\times : N_{E/E_3}(e) \in F^\times \},
K_5 = \{ e \in E^\times : N_{E/E_3}(e) \in N_{E_1/F}(E_1^\times) \cdot N_{E_2/F}(E_2^\times) \}.
\]
In the definition of $K_3$, $\langle E_1^\times, E_2^\times \rangle$ denotes the smallest subgroup of $E^\times$ containing $E_1^\times$ and $E_2^\times$, where $E_i = F(\sqrt{a_i})$. In the definitions of $K_4$ and $K_5$, we recall that $E_3 = F(\sqrt{a_1a_2})$. Finally, in the definition of $K_5$, the product of norm groups denotes the set of all products of elements of the first set with the second.

Taking $N_{E/E_3}$ of the sets $K_4$ and $K_5$ we obtain

$$N_{E/E_3}(E^\times) \cap F^\times \subseteq N_{E_1/F}(E_1^\times) \cdot N_{E_2/F}(E_2^\times).$$

The reverse inequality follows from the proof of Theorem 4, since for arbitrary $\gamma_i$ in $E_i^\times$ and $\gamma = \gamma_1 \cdot \gamma_2$ we have $N_{E/E_3}(\gamma) = N_{E_1/F}(\gamma_1) \cdot \gamma_2)$. Thus we see that

$$N_{E/E_3}(E^\times) \cap F^\times = N_{E_1/F}(E_1^\times) \cdot N_{E_2/F}(E_2^\times). \quad (III)$$

This equality gives us, in the form of Corollary 2, the equality of binary quadratic forms mentioned at the beginning of the section. The proof of the corollary, which we present after the proof of Theorem 4, is a routine translation of equation (III) into the language of binary quadratic forms.

**Corollary 2.** Suppose that $a$ and $b$ belong to $F^\times$ and $x$ and $y$ to $F(\sqrt{b})$. Then $x^2 - ay^2$ is in $F$ if and only if there exist $x_i$ and $y_i$ in $F$ ($i = 1, 2$) with the property that

$$x^2 - ay^2 = (x_1^2 - ay_1^2)(x_2^2 - aby_2^2).$$

For a nice proof of and further references to this corollary see [5, Proposition 1.5] and the comments preceding it.

**Proof of Theorem 4** $K_2 = K_3$: First notice that $E = E_1(\sqrt{a_2}) = E_2(\sqrt{a_1})$. Since the elements of $E$ fixed by the elements of $\text{Gal}(E/E_i)$ are precisely those in $E_i$, we infer that $(1 - \sigma_i) \cdot \ell = \ell / \sigma_i(\ell) = 1$ if and only if $\ell$ lies in $E_i$. Since the domain of $\text{ker}(1 - \sigma_i)$ is $E^\times$, we obtain $\text{ker}(1 - \sigma_i) = E_i^\times$.

$K_1 \subseteq K_4$: Recall that

$$\text{ker}(1 - \sigma_1)(1 - \sigma_2) = \{ e \in E^\times : e \cdot \sigma_1\sigma_2(e) = \sigma_1(e) \cdot \sigma_2(e) \}.$$ 

Also $e \cdot \sigma_1\sigma_2(e) = N_{E/E_3}(e)$, a member of $E_3^\times$. Now assume that

$$f = e \cdot \sigma_1\sigma_2(e) = \sigma_1(e) \cdot \sigma_2(e).$$
Then $\sigma_1(e \cdot \sigma_1 \sigma_2(e)) = \sigma_1(e) \cdot \sigma_2(e) = f$. Because $\sigma_1$ and $\sigma_2$ generate $G$, we see that $f$ is fixed by $G$. Since the elements of $E$ fixed by $G$ lie in $F$, we deduce that $f$ lies in $F^\times$. Hence
\[
\ker(1 - \sigma_1)(1 - \sigma_2) \subseteq \{ e \in E^\times : N_{E/E_3}(e) \in F^\times \}.
\]

$K_4 \subseteq K_3$: Let $e$ in $E^\times$ have $f = N_{E/E_3}(e)$ in $F^\times$. Because the set \( \{1, \sqrt{a_1}, \sqrt{a_2}, \sqrt{a_1a_2}\} \) is basis for the vector space $E$ over the field $F$, we can write $e$ as a linear combination of the elements of this basis: $e = f_0 + f_1\sqrt{a_1} + f_2\sqrt{a_2} + f_3\sqrt{a_1a_2}$. We calculate $N_{E/E_3}(e)$:
\[
f = N_{E/E_3}(e) = e \cdot \sigma_1 \sigma_2(e) = e \cdot (f_0 - f_1\sqrt{a_1} - f_2\sqrt{a_2} + f_3\sqrt{a_1a_2}).
\]
Multiplying out the product and determining the coefficient of the basis element $\sqrt{a_1a_2}$, we deduce that $f_0f_3 - f_1f_2 = 0$.

Assume first that both $f_2$ and $f_3$ are nonzero. Thus $f_0 = f_2t$ and $f_1 = f_3t$ for some $t$ in $F^\times$. Substituting these values into our expression for $e$ gives
\[
e = (f_2 + f_3\sqrt{a_1}) \cdot (t + \sqrt{a_2}) \in \langle E_1^\times, E_2^\times \rangle,
\]
the subgroup of $E^\times$ generated by $E_1^\times$ and $E_2^\times$. If $f_2 = 0$, then $f_0 = 0$ or $f_3 = 0$. In the first case
\[
e = \sqrt{a_1} \cdot (f_1 + f_3\sqrt{a_2}) \in \langle E_1^\times, E_2^\times \rangle,
\]
and in the second case
\[
e = f_0 + f_1\sqrt{a_1} \in E_1^\times \subseteq \langle E_1^\times, E_2^\times \rangle.
\]
The case $f_3 = 0$ is handled in the same way. Thus we see that
\[
\{ e \in E^\times : N_{E/E_3}(e) \in F^\times \} \subseteq \langle E_1^\times, E_2^\times \rangle.
\]

$K_3 \subseteq K_1$: Since the elements of $E$ fixed by Gal($E/E_i$) lie in $E_i$, we find that $E_i^\times \subseteq \ker(1 - \sigma_i)$ ($i = 1, 2$). We therefore obtain
\[
E_i^\times \subseteq \ker(1 - \sigma_1)(1 - \sigma_2),
\]
from which the stated inclusion follows.

We have established that the sets $K_1$, $K_2$, $K_3$, and $K_4$ coincide, so it remains to show that set $K_5$ is identical to the others. The inclusion $K_5 \subseteq K_4$ is easy to see:
\[
W := \{ e \in E^\times : N_{E/E_3}(e) \in N_{E_1/F}(E_1^\times) \cdot N_{E_2/F}(E_2^\times) \} \subseteq \{ e \in E^\times : N_{E/E_3}(e) \in F^\times \}.
\]
We argue that $K_3 \subseteq K_5$, as follows. Let $\gamma = \gamma_1 \cdot \gamma_2$, where $\gamma_i$ is in $K_i^\times$. Note that $N_{E_i/E_3}(\gamma_i) = \gamma_i \cdot \sigma_1 \sigma_2(\gamma_i) = N_{E_i/F}(\gamma_i)$ because $\sigma_1 \sigma_2$ acts on both $E_1$ and $E_2$ nontrivially. It follows that 

$$
N_{E_i/E_3}(\gamma) \in N_{E_1/F}(E_1^\times) \cdot N_{E_2/F}(E_2^\times).
$$

Therefore $\gamma$ belongs to $W$. Hence the sets $K_1$, $K_2$, $K_3$, $K_4$, and $K_5$ are identical. □

Proof of Corollary 2. Recall that the nonzero values of a binary quadratic form $x^2 - dy^2$ for $d$ in $F$ form a group under multiplication. Hence if $b$ is a square in $F$ our statement follows. Therefore assume that $b$ lies in $F \times \setminus (F \times)^2$.

Next observe that the set of values assumed by a quadratic form $x^2 - c^2 y^2$ ($c \neq 0$) as $x$ and $y$ range over $F$ is $F$ itself:

$$
f = \left( \frac{f + 1}{2} \right)^2 - c^2 \left( \frac{f - 1}{2c} \right)^2 \quad (f \in F).
$$

Therefore if either $a$ or $ab$ is a square in $F$, the statement is true as well. (Notice that if $ab$ is a square in $F$, then $a$ is a square in $F(\sqrt{b})$ and $x^2 - ay^2$ assumes all elements of $F(\sqrt{b})$.)

Finally, assume that none of the elements $a$, $b$, or $ab$ is a square in $F$. Let $a_1 = a$ and $b = a_2/a_1$. Since $E = E_3(\sqrt{a_1}) = E_3(\sqrt{a})$, we find that

$$
N_{E_i/E_3}(E_i^\times) \cap F^\times = \{ x^2 - ay^2 : x, y \in E_3, x^2 - ay^2 \in F^\times \}.
$$

Also note that $E_3 = F(\sqrt{a_1 a_2}) = F(a_1 \cdot \sqrt{a_2/a_1}) = F(\sqrt{b})$. From $N_{E_i/F}(E_i^\times) = \{ x_i^2 - a_i y_i^2 : x_i, y_i \in F, x_i^2 - a_i y_i^2 \neq 0 \}$ the result follows. □

5. “BE VAGUELY VAST, AND CLIMB”: FURTHER DIRECTIONS. We have observed the first shoots of the generalization from quadratic to biquadratic extensions. To understand what truly lies cached within the seed-case, we must watch closely as $E/F$ is replaced with bicyclic extensions, tricyclic extensions, and even general Abelian extensions. In particular, it is natural to ask what the “difference” is between bicyclic Hilbert 90 and biquadratic Hilbert 90, and, more generally, between Hilbert 90 for an Abelian extension and Hilbert 90 for a bicyclic extension. Perhaps a generalization of Theorem 1 similarly connects a notion of cohomological vanishing with minimal kernels of related operators.
Like seedlings ourselves, however, we have been “forced to make choice of ends,” and have accepted—not unlike this article itself—“the doom of taking shape.” But what you, the reader, may yet heave aloft—“that remains,” as the poet says, “to be seen.”

ACKNOWLEDGMENTS. We thank Noam Elkies for bringing to our attention the reference [8], which in turn had been brought to his attention by Franz Lemmermeyer. We also thank both referees for thoughtful comments and valuable suggestions that have improved our exposition. Roman Dwilewicz acknowledges support from the University of Missouri Research Board and the Polish Committee for Scientific Research grant KBN2P03A04415. Ján Mináč is grateful for support from Natural Sciences and Engineering Research Council of Canada grant R0370A01 and a Distinguished Professorship for 2004–2005 at the University of Western Ontario. Andrew Schultz thanks Ravi Vakil for his encouragement and direction in this and all other projects. John Swallow thanks the Institut de Mathématiques de Bordeaux for its hospitality during part of the writing of this article.

References

[1] I. G. Connell, Elementary generalizations of Hilbert’s Theorem 90, Canad. Math. Bull. 8 (1965) 749–757.
[2] N. D. Elkies, Pythagorean triples and Hilbert’s Theorem 90, this MONTHLY 110 (2003) 678.
[3] D. Hilbert, The Theory of Algebraic Number Fields (trans. I. Adamson), Springer-Verlag, Berlin, 1998.
[4] E. Kummer, Über eine besondere Art aus complexen Einheiten gebildeter Ausdrücke, J. Reine Angew. Math. 50 (1855) 212–232.
[5] D. B. Leep and T. L. Smith, Multiquadratic extensions, rigid fields and Pythagorean fields, Bull. London Math. Soc. 34 (2002) 140–148.
[6] F. Lemmermeyer and N. Schappacher, Introduction, in D. Hilbert, The Theory of Algebraic Number Fields (trans. I. Adamson), Springer-Verlag, Berlin, 1998, pp. xxiii–xxxvi.
[7] E. Noether, Der Hauptgeschlechtssatz für relativ-galoissche Zahlkörper, Math. Ann. 108 (1933) 411–419.
[8] T. Ono, Variations on a Theme of Euler. Quadratic Forms, Elliptic Curves, and Hopf Maps, Plenum, New York, 1994.
[9] A. Speiser, Zahlentheoretische Sätze aus der Gruppentheorie, Math. Zeit. 5 (1919) 1–6.
[10] O. Taussky, Sums of squares, this MONTHLY 77 (1970) 805–830.
[11] V. Voevodsky, Motivic cohomology with $\mathbb{Z}/2$-coefficients, Publ. Inst. Hautes Études Sci. 98 (2003) 59–104.
[12] ———, On motivic cohomology with $\mathbb{Z}/l$ coefficients, K-Theory Preprint Archive 639, 2003; available at www.math.uiuc.edu/K-theory/0639/.
[13] E. Weiss, Cohomology of Groups, Academic Press, New York, 1969.
[14] R. Wilbur, “Seed Leaves,” New and Collected Poems, Harcourt Brace Jovanovich, San Diego, 1988, pp. 129–130.
ROMAN DWILEWICZ is indigenous to Poland. He earned his master’s, doctoral, and habilitation degrees at the University of Warsaw, and then for many years he taught at the same institution. He has been a member of the Polish Academy of Sciences, as well as scientific director of its Institute of Mathematics and the Banach Center. Once transplanted halfway around the world, he met Ján Mináč when Ján, absorbed in thought, quite literally crashed into him.

Department of Mathematics and Statistics, University of Missouri-Rolla, Rolla, MO 65409
romand@umr.edu

JÁN MINÁČ spent years blissfully unaware of the fact that the Fields Medal clock had already begun ticking from the moment of his birth. He happily wasted his time daydreaming, playing soccer, reading all kinds of literature, and toying with prime numbers. Although now past forty, he is as excited as ever to keep playing with mathematics, blessed with great collaborators and the support of his wonderful wife Leslie. Summing these up, he reaches but one result, his own sort of Fields Medal.

Department of Mathematics, Middlesex College, University of Western Ontario, London, ON N6A 5B7, Canada
minac@uwo.ca

ANDREW SCHULTZ arrived at Davidson College intending to study economics. What was to be a brief fling with mathematics in his freshman year, however, quickly blossomed into love, and he chose to abandon all practicality in its pursuit. He spent the summers of 2001 and 2002 in the REU run by Arie Bialostocki at the University of Idaho, first as a participant and then as assistant director. He is currently a graduate student at Stanford, studying algebraic geometry with Ravi Vakil.

Department of Mathematics, Stanford University, Stanford, CA 94305-2125
aschultz@stanford.edu

JOHN SWALLOW is Kimbrough Associate Professor at Davidson College. In moments of pure serendipity he met, in the same semester, a sophomore named Andrew Schultz and a fellow MSRI member named Ján Mináč. He is the author of Exploratory Galois Theory (Cambridge, 2004) and is the 2006–2007 section lecturer for the MAA’s Southeastern Section. He has been elected a trustee of his alma mater, Sewanee: The University of the South.

Department of Mathematics, Davidson College, Box 7046, Davidson, NC 28035
josswallow@davidson.edu