Diffusion-induced dissipation and mode coupling in nanomechanical resonators

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(Dated: June 6, 2014)

We study a system consisting of a particle adsorbed on a carbon nanotube resonator. The particle is allowed to diffuse along the resonator, in order to enable study of e.g. room temperature mass sensing devices. The system is initialized in a state where only the fundamental vibration mode is excited, and the ring-down of the system is studied by numerically and analytically solving the stochastic equations of motion. We find two mechanisms of dissipation, induced by the diffusing adsorbate. First, short-time correlations between particle and resonator motions means that the net effect of the former on the latter does not average out, but instead causes dissipation of vibrational energy. For vibrational amplitudes that are much larger than the thermal energy this dissipation is linear; for small amplitudes the decay takes the same form as that of a nonlinearly damped oscillator. Second, the particle diffusion mediates a coupling between vibration modes, enabling energy transfer from the fundamental mode to excited modes, which rapidly reach thermal equilibrium.

I. INTRODUCTION

Nanoelectromechanical (NEM) resonators hold great promise for applications in inertial mass sensing. Carbon nanotubes (CNTs) in particular are suited when striving for high sensitivity, due to their extremely low mass. Recently, using CNT resonators, yoctogram sensitivity was achieved in experiments. In mass sensing applications, it is commonly assumed that an adsorbate, once attached to the surface, remains in the same positions during the time of measurement. However, at elevated temperatures thermal fluctuations can cause the adsorbate to change its position along the tube via diffusion. As the resonant frequency of the system depends on the position of the adsorbate, this gives rise to frequency fluctuations with accompanying phase noise.

For driven resonators, the effect of such frequency fluctuations has recently been studied both theoretically as well as experimentally. The effect manifests in a broadening and/or changed shape of the resonant response. However, broadening also arises from dissipation of mechanical energy. Dissipation and the origin of Q-factor limitations in nanoelectromechanical systems has been a long standing research topic where there are still unresolved issues. Recently, the connection between dissipation and nonlinear phenomena in NEM-resonators has begun to attract attention. This is partly because of the presence of nonlinear damping in carbon nanoresonators, and partly due to the recognition that geometric nonlinearites themselves give rise to dissipation and spectral broadening. While it was shown in Refs. and that a diffusing particle on an otherwise linear resonator induces both spectral broadening as well as nonlinear response to driving, we here adress the question whether the same mechanism also gives rise to dissipation and mode coupling.

For concreteness, we study in this paper the dissipation of mechanical energy induced by a diffusing adsorbate along a carbon nanotube-resonator. Due to the inertial back-action from the resonator on the diffusing particle, short-time correlations between resonator and particle motions cause energy dissipation; an effect also seen in molecular dynamics studies on graphene resonators. In order to isolate the dissipative contribution to the resonance broadening, we simulate ring-down measurements such as the ones performed in Refs. and Our model is a doubly clamped one-dimensional resonator constrained to move in the plane. A small mass is adsorbed on the resonator, and allowed to diffuse along it as shown in Figure. The resonator is excited in its lowest flexural vibration mode, by means of e.g. a nearby gate, and the subsequent free evolution of the system is studied.

We find that the inertial force acting on the particle causes its motion to have frequency components twice that of the fundamental mode. Because of retardation, this short-time correlation between adsorbate motion and resonator motion causes dissipation, leading to a non-exponential decay of mechanical energy; Eventually thermal equilibrium is reached. In addition, as the particle changes position, a mode coupling is induced that transfers energy to higher-lying modes, causing them as well to equilibriize. An example of this process is shown in Figure.
II. EQUATIONS OF MOTION FOR
A RESONATOR WITH A DIFFUSING PARTICLE

As shown in Figure 1 we consider a resonator of length $L$ with mass $M = L_0 \rho$ and bending rigidity $\kappa$. Neglecting longitudinal displacement, the Lagrangian density for the unperturbed resonator is

$$L_0 = \frac{1}{2} \rho \dot{w}^2 - \frac{1}{2} \sigma \dot{w}_X^2 - \frac{1}{2} \kappa \dot{w}_X^2.$$  \hfill (1)

Here, $w = w(X, t)$ is the transverse displacement (see Figure 1), $X$ is the coordinate measured along the resonator, and $w_X = \partial w / \partial X$. In the limit of small vibration amplitude and/or large prestrain, the built-in tension $\sigma$ can be assumed independent of $w$. The unperturbed Eigenfrequencies $\omega_n$ and Eigenmodes $\phi_n(X)$ are found from solving the corresponding equation of motion, see appendix A. For convenience, we will work with Eigenmodes normalized so that $\int dx \phi_n(x) \phi_m(x) = \delta_{nm}$, and with boundary conditions corresponding to a doubly clamped beam: $w(0) = w(L) = w_X(0) = w_X(L) = 0$.

Including the adsorbate of mass $m = \epsilon M$ at $X = x(t)$, the total Lagrangian is

$$\mathcal{L} = L_0 + \frac{1}{2} m \dot{d}(x - X) \left( \ddot{d} + (\dot{w} + \dot{w}_X \dot{d})^2 \right).$$  \hfill (2)

We expand the displacement in Eigenmodes, $w(X,t) = \sum_n q_n(t) \phi_n(X)$, and find the equations of motions

$$\ddot{q}_n + \omega_n^2 q_n - \epsilon \phi_n(x) \sum_k \omega_k^2 q_k \phi_k(x) = 0,$$  \hfill (3)

$$\frac{\ddot{X}}{\gamma} \sum_{k,l} \omega_k^2 q_k \phi_k \phi'_l = \sqrt{D} \eta(t),$$  \hfill (4)

where $\eta(t)$ is a delta-correlated Gaussian noise, i.e. $\langle \eta(t) \eta(t') \rangle = \delta(t - t')$, and $D = 2k_B T / m \gamma$. Throughout, we assume reflecting boundary conditions for the diffusing particle. The equations are derived using the approximation $\dot{d}^2 w(x(t), t) \approx - \sum \omega_k^2 q_n(t) \phi_k(x)$, which is equivalent to claiming that the effect of the added mass is a small correction to the unperturbed motion.

The non-linear system of Equations (3-4) is numerically integrated using a second-order algorithm. Some results are shown in Figures 2 and 3 using parameters $D = D / \omega_0 L^2 = 2.85 \times 10^{-4}$, $\epsilon = 1.82 \times 10^{-2}$, and $\gamma = 0.24 \omega_0$. These results are further discussed in Section IV.

The resonator dimensions are chosen to be experimentally realistic: length 1 µm, diameter 5 nm, and fundamental resonant frequency $\omega_0 = 2 \pi \times 108$ MHz.

III. SINGLE MODE DYNAMICS

To understand the observed energy decay we first focus on a single flexural mode. This simplification of the equations of motion is motivated by simulations, which have not shown any qualitative dependence on the number of included modes. That is, even when only the fundamental mode of the resonator is included, the two regimes identified in Figure 3 are evident. Below, we will along the length, and the eventual thermalization of all vibration modes. Also note that the higher-lying modes reach an internal equilibrium very rapidly compared to the slow decay of the fundamental mode energy. As illustrated in Figure 3, one can identify two distinct limiting cases. The first, high-amplitude limit is characterized by the particle being trapped at the antinode of the vibration around $x \approx L/2$. In this regime, the energy of the resonator decays linearly in time. As the amplitude of the resonator vibrations decreases, thermal fluctuations overcome the inertial trapping potential, the particle starts to diffuse along the entire length of the nanotube and the decay rate is no longer linear.

It is shown in Appendix B that the parameter that must be small if perturbation theory is to be valid is $\omega_0 \epsilon / \gamma D = \frac{1}{2} m \omega_0^2 q_0^2 / k_B T$. However, this parameter is useful beyond perturbation; it is the ratio between vibrational and thermal energy of the adsorbate. The inertial trapping potential is proportional to the vibrational energy, so the particle remains confined as long as $\omega_0 \epsilon / \gamma D \gg 1$, and diffuses freely when $\omega_0 \epsilon / \gamma D < 1$. This parameter is shown as a function of time in the center panel of Figure 3 which is in agreement between the value of $\omega_0 \epsilon / \gamma D$ and the behavior of the adsorbate.
We measure time in units of $\omega_0^{-1}$, and change to action angle variables ($E(t), \theta(t)$) via the transformations $q_0(t) = L\sqrt{E} \cos(\omega_0 t + \theta)$ and $\dot{q}_0(t) = -\omega_0 L \sqrt{E} \sin(\omega_0 t + \theta)$. Then, the equations take the form

$$
\begin{align*}
\partial_t E &= -\epsilon \phi_0^2 \sin 2\nu, \quad (\nu = \theta + \tau), \\
\partial_t \theta &= -\epsilon \phi_0^2 \cos^2 \nu, \\
\partial_t \xi &= \frac{\omega_0}{2\gamma} \cos \nu \partial_\xi \phi_0^2 + \sqrt{D} \eta(t),
\end{align*}
$$

where $\xi = x/L$, $\tau = \omega_0 t$, and $D = D/\omega_0 L^2$. From Equation (7) it is quite clear that performing an RWA-approximation here leads to $\partial_t E = 0$ and that the effect of the particle diffusion is only to cause fluctuations in resonant frequency. Hence, in order for $\partial_t E \neq 0$, $\xi$ must contain a frequency component $\sin 2\nu$ which arises from the first term in Equation (7), so the observed decay in energy stems from short-time correlations with frequency $2\omega_0$ between particle and resonator motions.

A. Large amplitude vibrations, confined particle

When $\omega_0 E/\gamma D \gg 1$, the thermal fluctuations cannot overcome the inertial trapping potential and the adsorbate fluctuates around the antinode of the flexural mode. In this regime, the phase noise is typically small and can be neglected when estimating the decay rate. Furthermore, as the particle is at all times in the vicinity of the antinode we can make the approximation $\partial_\xi \phi_0^2 \approx 2\phi_0(0)\phi_0'(0)\xi = -k\xi$, which renders the diffusion equation (7) linear. Solving for the particle motion yields

$$
\xi(\tau) = \sqrt{D} \int_{\tau'}^\tau d\tau' \eta(\tau') e^{-\omega_0 \tau'/2\gamma} I(\tau, \tau') \cos ^2 \nu' .
$$

Inserting back into the equation for $E$, omitting the term vanishing upon averaging over fast fluctuations, and assuming $E$ to be slow, one finds

$$
\partial_t E \approx \frac{1}{2} \epsilon D k E \sin 2\pi \int_{-\infty}^\tau d\tau' \, e^{-\omega_0 \pi(\tau')/2\gamma} I(\tau', \tau') \cos^2 \nu' .
$$

Averaging over fast oscillations,

$$
\partial_t E \approx \epsilon D k E I \left( \frac{k \omega_0 \mathcal{E}(\tau)}{\gamma} \right)
$$

where the integral $I$ is defined as

$$
I(x) = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \sin 2\pi \int_{-\infty}^\tau d\tau' \, e^{-x} I(\tau', \tau') \cos^2 \nu' .
$$

The integral is well approximated by the expression $I(x) \approx -\left(4 + \sqrt{\pi}(x/2) \coth(x/2) - 1 \right)$. Hence, for large amplitudes such that $k \omega_0 E(\tau)/\gamma \gg 1$, the decay becomes linear, i.e.

$$
\partial_t \mathcal{E} \approx -\frac{2D\gamma}{\sqrt{\pi} \omega_0}.
$$

FIG. 3: (color online) Top: fundamental mode energy as a function of time, together with a linear fit to initial decay. Center: the parameter $\omega_0 \mathcal{E}/\gamma D$, shown below to govern the qualitative behavior of the system, as a function of time. The horizontal dashed line indicates where $\omega_0 \mathcal{E} = \gamma D$. Bottom: evolution of the probability distribution $p(\xi, t)$, where $\xi = x/L$, for the particle to be at a certain position $0 < \xi < 1$ along the resonator as a function of time. For large initial amplitudes, the particle remains inertially trapped at the antinode of the vibration around $\xi \approx 0.5$. The corresponding energy decay is linear in time. As the thermal fluctuations overcome the inertial trapping potential, the particle diffuses freely, and the energy decays algebraically towards equilibrium.

treat the two regimes separately, beginning with the large amplitude-case.

Considering only the fundamental mode, i.e. $w(X, t) = q_0(t)\phi_0(X)$, one finds

$$
\begin{align*}
\dot{q}_0 + \omega_0^2 \left[ 1 - \epsilon \phi_0^2(x) \right] q_0 &= 0, \\
\dot{x} &= \frac{\omega_0^2}{2\gamma} \partial_x \phi_0^2(x) + \sqrt{D} \eta(t),
\end{align*}
$$

where the integral $I$ is defined as

$$
I(x) = \frac{1}{2\pi} \int_0^{2\pi} d\sigma \sin 2\pi \int_{-\infty}^\tau d\tau' \, e^{-x} I(\tau', \tau') \cos^2 \nu' .
$$

The integral is well approximated by the expression $I(x) \approx -\left(4 + \sqrt{\pi}(x/2) \coth(x/2) - 1 \right)$. Hence, for large amplitudes such that $k \omega_0 E(\tau)/\gamma \gg 1$, the decay becomes linear, i.e.
Comparisons with simulation shows that this result is correct within an order of magnitude, even when excited modes are included: see Figure 4. However, as discussed in Section IV, adding more channels of decay increases the decay rate.

B. Small amplitude vibrations, unconfined particle

If \( \omega_0 \mathcal{E} / (\gamma D) \ll 1 \), the system (12) can be solved by means of perturbation theory. This limit can be seen to be equivalent to the assumption that the vibration amplitude be small enough that the particle is not inertially trapped at an antinode, and falls in the typical parameter regime encountered in most experimental situations. As an example, for a single Kr-atom on a 100 MHz CNT-resonator vibrating with an amplitude of \( q_0 = 3 \) nm at \( T = 1 \) K, one has \( \omega_0 \mathcal{E}_0 / \gamma D = \mathcal{E}_{vib} / k_B T \approx 0.03 \).

The corresponding Fokker-Planck equation (FPE) for the distribution function \( p(\xi, E, \nu, \tau) \) reads:

\[
[\partial_\tau + \partial_\nu]p(\xi, E, \nu, \tau) = \epsilon \phi_0^2 \mathcal{E} \sin(2\nu) \partial_\xi p + \epsilon \phi_0^2 \mathcal{E} \frac{1 + \cos(2\nu)}{4\nu} \partial_\nu p - \frac{\omega_0 \mathcal{E}}{4\nu} \frac{1 + \cos(2\nu)}{2} \partial_\xi^2 p + \frac{D}{2} \partial_\xi^2 p. \tag{12}
\]

As noted above, the dissipation of mechanical motion stems from the correlation between the motion of the particle and the resonating beam, induced by the last term in Eq. (12). These correlations occur on a time-scale \( \omega_0^{-1} \) which is much shorter than the scale of the rate of change of energy. Hence, we can find the dissipation rate by making a separation Ansatz for fast and slow time scales by the approximation \( p \approx p_0(\xi, \tau)p_1(\xi, \nu, \tau) \) (a more formal derivation is found in Appendix B). This decouples Equation (12) into one equation for slow time-scales and one for fast time-scales, where particle position is described by the latter:

\[
\partial_\tau p_1 = -\frac{\omega_0 \mathcal{E}}{4\nu} \left[ 1 + \cos(2\nu) \right] \partial_\xi [p_1 \partial_\xi \phi^2] + \frac{D}{2} \partial_\xi^2 p_1. \tag{13}
\]

Assuming \( \mathcal{E}_{vib} / k_B T \ll 1 \) one finds to first order in \( \mathcal{E} \) the steady state solution

\[
p_1 = 1 + \frac{\omega_0 \mathcal{E}}{4\nu} \sum_n \lambda_n f_n \frac{\sin(2\nu)}{\lambda_n^2 + 16} \cos(n\pi\xi),
\]

where \( \lambda_n = DN^2 \pi^2 \) and \( f_n = \int_0^1 d\xi \cos(n\pi\xi) \phi^2(\xi) \). Inserting this solution into the FPE equation (12), and integrating over position \( \xi \) and the fast variable \( \nu \) yields

\[
\partial_\tau p_0(\xi, \tau) = \epsilon \frac{2\omega_0}{D\gamma} \sum_n \frac{\lambda_n f_n^2}{\lambda_n^2 + 16} \partial_\xi \mathcal{E}^2 p_0. \tag{14}
\]

The solution to this equation is \( p(\xi, \tau) = \mathcal{E}^{-2} f \left( 1 - \frac{\alpha \mathcal{E}}{\mathcal{E}} \right) \), where

\[
\alpha = \epsilon \frac{\omega_0^2 L^2}{k_B T} \sum_n f_n^2 \frac{\lambda_n}{16 + \lambda_n^2}. \tag{15}
\]

FIG. 4: (color online) Damping parameter \( \alpha \), as calculated by Eq. (15) (solid line) and by a numerical fit (dots). The same simulation parameters as in Fig. 2 were used, with the exception of the initial amplitude, here \( \mathcal{E}_0(0) = 10^{-4} \). The ratio \( \omega_0 \mathcal{E}_0 / \gamma D \) was varied by changing the simulation temperature. We see that the perturbative approach is indeed valid for \( \omega_0 \mathcal{E}_0 / \gamma D \ll 1 \). In the intermediate region \( \omega_0 \mathcal{E}_0 / \gamma D \lesssim 1 \), Eq. (15) overestimates the magnitude of the damping, but captures the overall shape of the curve and the location of the maximal damping.

If \( p(\mathcal{E}, 0) = \delta(\mathcal{E} - \mathcal{E}_0) \) the ensemble averaged energy \( \langle \mathcal{E} \rangle \) decays without dispersion and one obtains the characteristic ringdown of a nonlinearly damped oscillator,

\[
\langle \mathcal{E}(\tau) \rangle = \frac{\mathcal{E}_0}{1 + \alpha \mathcal{E}_0 \tau}. \tag{16}
\]

This expression does indeed agree well with simulation in the parameter space where perturbation theory is valid; see Fig. 4.

To include dispersion, and to reach a proper thermal equilibrium state, fluctuation corrections must be included. As shown in Appendix B, this leads to the following Fokker-Planck equation for the reduced probability density,

\[
\partial_\tau p_0 = \alpha \partial_\mathcal{E} \mathcal{E}^2 [p_0 + (\epsilon D/(\omega_0)) \partial_\mathcal{E} p_0].
\]

Noting that \( \epsilon D/(\omega_0) = 2k_B T/(M \omega_0^2 L^2) \), we see that this FPE also gives the proper thermal equilibrium stationary solution \( p_0(\tau \to \infty) \propto \exp(-M \omega_0^2 q_0^2 / 2k_B T) \).

IV. MULTIMODE DYNAMICS, THERMALIZATION

While the qualitative behavior of the dynamics remain unchanged by incorporating more flexural modes, a quantitative change takes place. If the higher-lying modes are initially at rest, we find that exciting the system in only the fundamental mode rapidly causes the higher modes
to be thermalized. Once in thermal equilibrium, they provide additional channels for energy dissipation from the fundamental mode.

The thermalization of the higher modes stem from the stochastic additive noise terms in Eq. (3). As initially $\mathcal{E}_0 \gg \mathcal{E}_{n>0}$, the transient behavior is described by

$$\dot{q}_n + \omega_n^2 q_n \approx \epsilon \omega_n^2 q_0 \phi_n \phi_0 \propto \epsilon \sqrt{\mathcal{E}_0}, \; n > 0.$$ 

To a first approximation, we would thus expect the energy of the higher-lying modes to have a transient behavior $\langle E_n \rangle \sim \epsilon^2 \mathcal{E}_0 t$. In order to investigate this, we define the thermalization time $\tau_{\text{therm}}$ as the time when the energy of an excited mode first exceeds the thermal energy. Simulations were made at a constant temperature but for values of $\epsilon$ and $\mathcal{E}_0(0)$ ranging over several orders of magnitude; the resulting values for $\tau_{\text{therm}}$ are shown in Fig. 5.

The data has been fitted to a model $\tau_{\text{therm}} \propto (\epsilon^0 \mathcal{E}_0)^b$, and the exponent $a = 1.8615$ was determined by minimizing the sum of squared residuals of the linear fit shown as a black line in Fig. 5. The slight deviation from the theoretical value of $a = 2$ is likely due to the present definition of $\tau_{\text{therm}}$, which will always be smaller than the time taken for all higher-lying modes to reach $k_B T$. Finally, we note that for $\mathcal{E}_0(0) \gg k_B T$, the thermalization of excited modes occur on a time scale much shorter than the decay of the fundamental mode. Hence, on the time-scale relevant for studying the ring-down of the resonator, it is a good approximation to assume that all excited modes are in thermal equilibrium.

The fact that the mode coupling strength and energy transfer between modes are determined only by the energy in the modes is further corroborated by considering the decay rate of the fundamental mode energy once the higher modes have thermalized. As shown in Fig. 6, for each additional mode we include in the simulation, an additional channel for energy transfer away from the fundamental mode is made available and the decay rate (initially) increases linearly with the number of added modes. As each individual mode has the same energy $\sim k_B T$, each mode contributes an equal amount to the fundamental mode dissipation.

Clearly, there must be a cut-off at which this is no longer true. Such a cutoff can be estimated by noting that due to the influence of the fundamental mode, the dissipation of energy from the fundamental mode is associated with adsorbate dynamics occurring on a timescale $\omega_0^{-1}$ corresponding to a diffusion length scale of $\sqrt{2\pi D/\omega_0}$. If this length is larger than half the wavelength of the $n^{th}$ mode, the effective coupling to this mode will average to zero and not contribute to dissipation. The wavelength is $\lambda_n = 2L/(n+1)$, giving the cut-off condition that only modes with $n \lesssim n_{\text{max}} \approx \sqrt{\frac{2}{\omega_0}}$ will contribute to the dissipation of the fundamental mode. For the parameter values used in Fig. 5, we find $n_{\text{max}} \approx 20$.

V. CONCLUSIONS

We have studied the effect of a diffusing adsorbed particle on a vibrating 1D nanomechanical resonator, initially excited in its fundamental flexural mode. Studying the free ring-down of the mode, focusing on the energy
transfer induced by the diffusion we find that there are two effects that cause vibrational energy to dissipate.

First, the inertial force exerted on the adsorbed particle causes short-time correlations between adsorbant motion and flexural vibrations. For large initial amplitudes such that $m_0^2a^2/kyT \gg 1$, the particle is trapped at the antinode of vibration, and the decay of vibrational energy is linear in time, approximately given by expression (11). For lower amplitudes, when the particle diffuses freely along the resonator, the decay takes the same form as that of a nonlinearly damped oscillator according to Equation (10).

Second, the diffusing particle also provides a stochastic coupling between different flexural vibration modes. This stochastic coupling provides an additional channel for energy transfer from the fundamental mode for each added flexural mode. This second mechanism can further cause short-time correlations between adsorbant molecules.

We find that the Eigenmodes satisfy the equation

$$\kappa n^2 - \sigma k^2 - \rho \omega_n^2 = 0. \quad (A3)$$

with roots $\pm k_n^+ = \pm ik_n^-$, where

$$k_n^\pm = \sqrt{\frac{\sigma^2}{4\kappa^2} + \frac{\rho}{\kappa} + \frac{\sigma}{2\kappa}}, \quad n = 0, 1, 2, \ldots (A4)$$

Hence, the Eigenmodes can be written as

$$\phi_{2n} = A_{2n} \cosh k_{2n}^-(X - \frac{L}{2}) + A_{2n} \cos k_{2n}^-(X - \frac{L}{2})$$

$$\phi_{2n+1} = B_{2n+1} \sinh k_{2n+1}^+ (X - \frac{L}{2})$$

where the boundary conditions $\phi_n(0) = \phi_n(L) = \phi'_n(0) = \phi'_n(L) = 0$ have been used to divide the $\phi_n$ into even and odd sets of Eigenmodes. Similarly, we find that the Eigenfrequencies $\omega_n$ are determined from the equation

$$\frac{\kappa}{k_n^+} = \pm \frac{\tan k_n L}{\tanh k_n L}, \quad (A6)$$

where upper/lower signs correspond to odd/even $n$. A good approximation for the ratio $\omega_n/\omega_0$ is $(2n + 1)^2/9$. Finally, the integration constants $A_n$ and $B_n$ are determined. The boundary conditions demand that

$$A_n' = -\frac{\cosh k_n L}{\cos k_n L} A_n, \quad B_n' = -\frac{\sinh k_n L}{\sin k_n L} B_n \quad (A7)$$

whereas the normalization condition $\int dX \phi_n^\dagger \phi_n = L\delta_{mn}$ determines

$$|A_n|^2 = 2 \left[ 1 + \frac{\sinh^2 k_n L}{k_n L} + \frac{\cosh^2 k_n L}{\cos^2 k_n L} \left( 1 + \frac{\sin k_n L}{k_n L} \right)^2 \right]^{-\frac{1}{2}},$$

$$|B_n|^2 = 2 \left[ 1 - \frac{\sinh^2 k_n L}{k_n L} - \frac{\cosh^2 k_n L}{\sin^2 k_n L} \left( 1 - \frac{\sin k_n L}{k_n L} \right)^2 \right]^{-\frac{1}{2}}. \quad (A8)$$

The final undetermined phase is chosen so that the Eigenfunctions $\phi_n$ are real.

### Appendix A: Determining the $\omega_n$ and $\phi_n(X)$

Here, we derive the flexural Eigenmodes and Eigenfrequencies for the unperturbed resonator.

The equation of motion corresponding to the Lagrangian $I$ is

$$\rho \ddot{w} - \sigma \partial_x^2 w + \kappa \partial_x^4 w = 0. \quad (A1)$$

Defining $\phi_n(X)$ and $\omega_n$, through $w(X, t) = e^{-i\omega_n t} \phi_n(X)$, we find that the Eigenmodes satisfy the equation

$$-\rho \omega_n^2 \phi_n - \sigma \phi_n'' + \kappa \phi_n''' = 0. \quad (A2)$$

The corresponding characteristic equation is

$$\kappa k^4 - \sigma k^2 - \rho \omega_n^2 = 0 \quad (A3)$$

The perturbation theory sketched in section III B can be put on more formal grounds. In this appendix we derive the reduced FPE by means of the methods in Ref. [24]. Introducing the variable $\nu = \omega \hat{t} + \theta$, the FPE reads

$$\partial_{\nu} p = \epsilon \phi_n^\dagger \xi \sin(2\nu) \partial_{\xi} \phi_n$$

$$\left(\frac{\epsilon \phi_n^\dagger}{2}(1 + \cos(2\nu)) - 1\right) \partial_{\nu} p$$

$$-\frac{\omega_0}{2} \left[ 1 + \cos(2\nu) \right] \partial_{\nu} p + D \partial_{\xi}^2 p. \quad (B1)$$

Upon expanding $p = \sum p_n (\xi, \epsilon, \tau)^{2n\nu}$ and introducing the vector $p = [.., p_1, p_0, p_1, ..]^T$, the FPE-equation can be rewritten as

$$\partial_{\tau} p = \epsilon \hat{L}_1 p + \frac{D}{2} \hat{L}_2 p,$$
For time scales \( \tau > \eta \) perturbatively to first order in \( L \) given by
\[
\beta_{\eta} \equiv \eta \delta_m \eta^{-1} \delta_m n_{m,n} \approx L n, m \text{ Eigenvalue pair has composite indices } (n, m).\]

The matrices \( A, B, N \), have components \( A_{m,n} = (2i)^{-1}[\delta_{m,n+1} - \delta_{m,n-1}], B_{m,n} = \delta_{m,n} + 2^{-1}[\delta_{m,n+1} + \delta_{m,n-1}], \) and \( N_{m,n} = n \delta_{m,n}. \)

Expanding in Eigenmodes of the operator \( \hat{L}_2 \), as \( p = \sum_n \beta_n(\mathcal{E}, \tau) v_n(\xi, \mathcal{E}) \), where \( \hat{L}_2 v_n = -\mu_n(\mathcal{E}) v_n \) yields
\[
(\partial_\tau + \frac{D \mu_n}{2}) \beta_n = \epsilon \sum_m \langle w_n, \hat{L}_1 \beta_m v_m \rangle .
\]
The inner product is here defined as \( \langle u, v \rangle \equiv \int d\xi \, u^\dagger v \) and the right eigenvectors \( w_n \) satisfy the adjoint equation \( \hat{L}_2^2 w_n = -\mu_n^* w_n \) with
\[
\hat{L}_2^2 = 4i D^{-1} \hat{N} + \frac{\omega \gamma}{2 D \gamma} \hat{B}(\partial_\xi \phi_0^2) \partial_\xi + \partial^2_\xi .
\]

For time scales \( \tau > D^{-1} \), we can make the approximation
\[
\begin{aligned}
\partial_\tau \beta_0 &= \epsilon \sum_n \langle w_0, \hat{L}_1 \beta_n v_n \rangle , \\
\beta_{\eta} &\approx \frac{2 \epsilon}{D \mu_n} \sum_{n \geq 1} \langle w_0, \hat{L}_1 \beta_0 v_0 \rangle .
\end{aligned}
\]
Combining the two gives
\[
\begin{aligned}
&\quad \partial_\tau \beta_0 = \epsilon \langle w_0, \hat{L}_1 \beta_0 v_0 \rangle \\
&\quad + \frac{2 \epsilon^2}{D} \sum_{n \geq 1} \langle w_0, \hat{L}_1 \mu_n^{-1} \langle w_n, \hat{L}_1 \beta_0 v_0 \rangle \rangle v_n.
\end{aligned}
\]

1. Perturbation theory for Eigenvectors

The Eigenvectors of \( \hat{L}_2 \) cannot obtained exactly. However, if the parameter \( \eta \equiv \frac{\gamma}{2D} \ll 1 \) we can find them perturbatively to first order in \( \eta \). Each Eigenvector – Eigenvalue pair has composite indices \( (n, m) \) and is to first order given by
\[
\begin{aligned}
v_{nm} &\approx v_{nm}^{(0)} + \frac{\omega \gamma}{2 D \gamma} \sum_{pq \neq nm} \langle w_{pq}^{(0)}, \hat{B} \partial_\xi \phi_{nm}^{(0)} \partial_\xi \phi_{pq}^{(0)} \rangle v_{pq}^{(0)}, \\
w_{nm} &\approx w_{nm}^{(0)} - \frac{\omega \gamma}{2 D \gamma} \sum_{pq \neq nm} \langle w_{pq}^{(0)}, \hat{B} \partial_\xi \phi_{nm}^{(0)} \partial_\xi \phi_{pq}^{(0)} \rangle v_{pq}^{(0)}, \\
\mu_{nm} &\approx \mu_{nm}^{(0)} + \frac{\omega \gamma}{2 D \gamma} \langle w_{nm}^{(0)}, \hat{B} \partial_\xi \phi_{nm}^{(0)} \partial_\xi \phi_{nm}^{(0)} \rangle .
\end{aligned}
\]

The unperturbed eigenvalues are \( \mu_{nm} = n^2 \pi^2 + 4 i D^{-1} m \) and the corresponding eigenvector has the \( k \)th component
\[
[v_{nm}^{(0)}]_k = [w_{nm}^{(0)}]_k = \sqrt{2} \cos(n \pi \xi) + \delta_{n,0} (1 - \sqrt{2}) \delta_{k,m} .
\]
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