Mathematical theory of physical vector fields

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We study the dynamics and statistics of real vector fields in flat \((n + 1)\)-dimensional space-time with an emphasis on the field topology and stochasticity in physical applications such as stochastic magnetic and velocity fields in cosmological systems. Vector field topology change plays an important role in a wide range of physical systems, e.g., magnetic topology change strongly affects the dynamics in a plasma and is responsible for phenomena such as solar flares and magnetic field generation in cosmological objects. We show that the natural field topology defined by the metric, induced by the Euclidean vector norm, is not generally preserved in time even for smooth fields. Moreover, this topology is defined in the vector space using open balls as the bases, which contain nearby vectors in vector space but not necessarily in real space. Hence, the topology change for physical fields such as magnetic fields, which occurs in a finite volume in Euclidean space, cannot be formulated properly using this topology. However, any vector field corresponds to a dynamical system with a topology in the corresponding phase space. This phase space topology, unlike the natural topology in Euclidean space, is defined using open balls that contain nearby vectors in both vector space and real space and hence is more physical. In addition, it is preserved under time translation if certain conditions including time reversal invariance are satisfied by the field. If these mathematical conditions are not satisfied, therefore, the field’s topology can spontaneously change as the field evolves in time. In this context, similar to topological entropy, which measures the complexity of a dynamical system in the phase space, a simple quantity is defined for a vector field which measures its spatial complexity in real space. For stochastic fields, this spatial complexity can be taken as a measure of the field’s stochasticity level. Generalizing a previous work based on renormalization group invariance, we show that corresponding to any arbitrary vector field, there exists a scalar field whose properties provide a means to quantify the vector field’s spatial complexity, stochasticity level and dissipation rate.

I. INTRODUCTION

The important role that the concept of a vector field, mathematical object that assigns a vector to each point in space, and its generalization to tensor fields play in different branches of physics is almost obvious. An intuitive notion of topology is also associated with vector fields which is usually visualized in terms of field lines (aka integral curves or streamlines). Yet, this intuition is not very aligned with the mathematical definition of topology in the context of topological spaces. Topology and topology change are seldom given a precise operational meaning in physical applications, e.g., in plasma physics and astrophysics literature. But in fact, as it turns out, the notion of field line and also that of field’s topology are not trivial at all, in particular in the context of stochastic fields and diffusive media. To give an example, the magnetic field threading an electrically conducting fluid such as a plasma is said to have a certain topology that may change spontaneously; a process referred to as magnetic reconnection which may strongly affect the system’s dynamics, e.g., it may be the underlying process producing solar flares. However, it is not clear, from a mathematical point of view, what is exactly meant by magnetic topology change in such dynamic, diffusive and chaotic environments. In mathematics, homeomorphisms are continuous functions between topological spaces that keep the topological properties intact; if a time dependent vector field has a preserved topology, its time translation must be a homeomorphism, i.e., it should continuously take the field at time \(t_0\) and map it to another field with the same topology at a later time \(t_1 > t_0\). How do we define the field’s topology in terms of open sets and a topological space in the first place? How do we ensure that the field’s time translation is a homeomorphism? Under what conditions, the topology may change and, for example, magnetic reconnection can occur? In fact, for time dependent fields, it turns out that the field lines, i.e., integral curves, are not in general continuously deforming curves in space, i.e., the field lines are not necessarily continuous in time even if the field itself is and stronger conditions should be satisfied. Thus we have no trivial way to visualize the field’s time evolution in terms of its integral curves. In order to address such fundamental questions, therefore, one needs a precise mathematical approach applicable to a variety of physical situations. We will define the topology corresponding to a general vector field in an attempt to show that the appropriate consideration of this problem naturally involves the theory of dynamical systems.

The other related concept is the stochasticity associated with a vector field, which can for example arise because of turbulence. Here as well, we seem to have a reasonable and intuitive phenomenology that can be even easily visualized, for example, in terms of random field lines in space which evolve in an indeterministic manner. But, this is a vague language which encounters mathematical difficulties when examined closely. Yet, it seems to play a fundamental role in many problems in physics, for instance velocity and magnetic fields often show stochastic behavior in cosmological objects \([1]; [2]; [3]; [4]; [5], [6]\). The concept of stochasticity has a clear meaning for a scalar variable—a variable whose numerical values are outcomes of a random process such as tossing
a fair coin. In fact, many phenomena in nature involve random processes, a quantitative understanding of which usually requires a statistical approach. In such processes, one deals with a random scalar, which might depend on other variables. This dependence is usually implied by using the term field; a stochastic (scalar) field is a stochastic variable whose possible numerical values depend on other variables. As a straightforward generalization, one can also define a stochastic vector as a vector whose components are stochastic variables. With this terminology, it is easy to define a stochastic vector field as a mathematical object that assigns to each point in space a stochastic vector. By definition, this means an infinite number of vectors, one vector defined at each point in space. Although this seems a straightforward generalization, however, care must be taken to avoid the misuse of commonplace concepts of stochastic field lines and topology of such stochastic fields.

In order to associate an intuitive notion to topology of a stochastic vector field, one needs to establish relationships between different vectors defined at different points in space (or on a manifold) for example in terms of a metric (although this is not necessary; a topology can be defined even if the space lacks a metric). In other words, in general for a stochastic vector field, not only we wish to study the randomness of vectors defined at any point in space but also we are interested to see how vector field stochasticity manifests itself globally in space. Even if a vector field comes with a well-defined and known probability measure, which is unlikely in real situations, its topology, once it is defined, may still completely be obscured by such a measure as it only assigns a probability to the field at every point in space with no emphasis on global relationships between different points in space or time.

Electric field topology ([7]; [8]), magnetic field topology ([9]; [10]) and velocity field topology (see e.g., [1]; [11]) are commonly used terminologies in different fields of research. In many problems, for example magnetic reconnection, the field topology plays a crucial role (see e.g., [12]; [13]; [14]; [6]). The concept of vector field stochasticity as well is commonly used in different contexts. For instance, recent models of magnetic reconnection rely on the effects of turbulence to explain the fast reconnection rates observed in astrophysical systems, therefore, such models inevitably deal with indeterministic behavior in magnetic and velocity fields ([15]; [3]; [14]; [16]; [4]). Spontaneous stochasticity of particle trajectories and the corresponding velocity fields in turbulent flows ([17]; [18]; [2]) as well as stochastic magnetic fields ([19]; [15]; [3]; [4]) have been studied in different contexts, however, a simple statistical measure for the stochasticity level of such vector fields has only recently been developed [20]. Based on this statistical approach, more recent work has linked magnetic stochasticity to magnetic diffusion [21] and kinetic stochasticity [22]. In the present paper, we attempt to define and formulate the stochasticity and topology of time dependent vector fields in a rigorous manner. We also briefly revisit few commonplace concepts such as stochastic field lines in the hope that it helps clarify their mathematical definitions which in turn might lead to gaining a deeper physical insight.

The plan of this paper is as follows: in §II, we first briefly review the fundamental properties of time dependent vector fields in real Euclidean space and illustrate the difficulties encountered in using integral curves in a quantitative consideration. Also, the time translation operator, H"older singular fields and renormalization using distributions are discussed. In §III, we define vector field topology using the metric induced by the vector norm and show that this natural topology is of little interest in physical applications. A phase space is defined in the context of dynamical systems theory with a built-in topology which is shown to be the standard topology appropriate for physical applications. Vector field stochasticity is defined and briefly discussed, which generalizes a previous work on magnetic field stochasticity. In §IV, we summarize and discuss our results and their physical implications.

II. TIME EVOLUTION

In general, a vector field $\mathbf{F}$ can be defined as a map from a manifold $\mathcal{M}$ to its tangent bundle $T\mathcal{M}$:

$$\mathbf{F} : \mathcal{M} \rightarrow T\mathcal{M},$$

$$\mathbf{x} \rightarrow \mathbf{F}(\mathbf{x}),$$

such that the image of $\mathbf{x} \in \mathcal{M}$, i.e., $\mathbf{F}(\mathbf{x})$, lies in the tangent space at $\mathbf{x}$, i.e., $T_{\mathbf{x}}\mathcal{M}$. In this paper, we will assume $\mathcal{M} = \mathbb{R}^n$, i.e., the real $n$-dimensional Euclidean space, unless stated otherwise. We will consider a real, time-dependent vector field $\mathbf{F}(\mathbf{x}, t)$ such that

$$\mathbf{F} : \mathbb{R}^n \times I_t \rightarrow \mathbb{R}^n,$$

where $t \in I_t \subseteq \mathbb{R}$, with the notation $\mathbf{F}(\mathbf{x}, t) = \mathbf{F}(\tilde{x}) = (F_1(\tilde{x}), ..., F_n(\tilde{x}))$, where $\tilde{x} = (x_1, ..., x_n; t)$. If the field is defined for all times, i.e., $I_t = \mathbb{R}$, we have a flow, otherwise a semi-flow. It goes without saying that a straightforward generalization of multivariate calculus can be applied to vector fields, e.g., $\mathbf{F}(\mathbf{x})$ is continuous if $\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{a})$.

Suppose the field $\mathbf{F}$ is governed by a general evolution equation of the following form:

$$\frac{\partial}{\partial t} \mathbf{F}(\mathbf{x}, t) = \mathbf{f}(\mathbf{F}, \partial^\kappa \mathbf{F}, \mathbf{x}, t),$$

(1)

where the notation $\partial^\kappa \mathbf{F}$ is used to imply that $\mathbf{f}$ may involve spatial derivatives of order $\kappa \in \mathbb{N}$. For the sake of simplicity, we will use the notation $\mathbf{f}(\mathbf{x}, t)$ throughout this paper keeping in mind that $\mathbf{f}$ may contain $\mathbf{F}$ and its spatial derivatives of any order $\kappa \in \mathbb{N}$. In physical problems, the field is often studied

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1 A general governing equation can take the following form in terms of the field components $F_j$:  

$$\frac{\partial^{n_i}F_i}{\partial t^{n_i}} = f_i(t, x_1, ..., x_n, F_1, ..., F_n, ..., \frac{\partial^{k_j}F_j}{\partial t^{k_j}x_1^{k_1}...x_n^{k_n}}),$$

with $i, j = 1, 2, ..., n; k = k_0 + k_1 + ... + k_n \leq n_j; k_0 < n_j$. The initial
in a region of space with some boundaries, thus the problem becomes a boundary value problem which has a unique solution provided that appropriate boundary conditions are applied. We will assume, throughout this paper, the existence of such a unique solution in a spatial volume $V$ without directly referring to any boundary condition.

The differential equation governing the time evolution of $\mathbf{F}$ can also be used to derive an evolution equation for the direction vector $\hat{\mathbf{F}} = \frac{\mathbf{F}}{|\mathbf{F}|}$ as well as the magnitude of $\mathbf{F}$, which is $F = |\mathbf{F}| = \sqrt{F_x^2 + \ldots + F_n^2}$; the Euclidean norm. The derivative of the unit vector $\hat{\mathbf{F}}$ is given by

$$
\partial_t \hat{\mathbf{F}} = \frac{1}{F} (\mathbf{F} \partial_t \mathbf{F} - \mathbf{F} \partial_t F).
$$

Noting that $\partial_t F \equiv \partial_t (F^2)^{1/2} = (\mathbf{F} \cdot \partial_t \mathbf{F})(F^2)^{-1/2}$ which is $(\mathbf{F} / F) \cdot \partial_t \mathbf{F} = (\partial_t \mathbf{F}) \cdot \mathbf{F}$, we find

$$
\partial_t \hat{\mathbf{F}} = \frac{1}{F} \left[ \partial_t \mathbf{F} - \hat{\mathbf{F}} \left( \partial_t \mathbf{F} \right) \right] = \frac{1}{F} \left( \partial_t \mathbf{F} - (\partial_t \mathbf{F}) \cdot \mathbf{F} \right).
$$

Obviously, the terms inside the brackets are the perpendicular component (with respect to $\mathbf{F}$) of the evolution equation of $\mathbf{F}$, eq. (1), that is

$$
\partial_t \mathbf{F} = \left( \partial_t \mathbf{F} \right) \perp = \frac{\mathbf{f}_l}{F}.
$$

Thus, the direction of $\mathbf{F}$ is determined solely by the perpendicular (with respect to $\mathbf{F}$) component of $\mathbf{f}$. Similarly, it is easy to show that the magnitude of $\mathbf{F}$ is determined by the parallel (with respect to $\mathbf{F}$) component of $\mathbf{f}$. We have

$$
\partial_t \mathbf{F} = \left( \partial_t \mathbf{F} \right) \parallel = f_\parallel, \text{ or } \partial_t \left( F^2 / 2 \right) = F f_\parallel.
$$

It follows that

$$
\begin{align*}
\partial_t \hat{\mathbf{F}} = 0 & \iff \mathbf{f}_\perp = 0 \iff \mathbf{f} \times \mathbf{F} = 0, \\
\partial_t \mathbf{F} = 0 & \iff f_\parallel = 0 \iff \mathbf{f} \cdot \mathbf{F} = 0.
\end{align*}
$$

Therefore, pointwise, $\mathbf{f} \times \mathbf{F} = 0$ and $\mathbf{f} \cdot \mathbf{F} = 0$ constrain, respectively, the topology and magnitude of the field. These simple observations play an important role in statistical considerations related to the field’s stochasticity level and topology change discussed in a later section.

### A. Singular Fields and Distributions

In any measurement, what one measures for a vector field at a given point is in fact an average field over a region in spacetime (or the corresponding manifold), that is, the measured field at point $(x, t)$ is an average field over a small spatial volume around $x$ during a time interval $\Delta t$. Mathematically, this average or renormalized field can be represented in terms of distributions. Roughly speaking, such coarse-graining or renormalization simply translates into multiplying the field, and its governing equations, by a smooth (infinitely differentiable) weight function and integrating. In order to renormalize a given field $\mathbf{F}(x, t)$ at a spatial scale $l > 0$, therefore, one can write

$$
\mathbf{F}_l(x, t) = \int_V G_l(r) \mathbf{F}(x + r, t) d^n r,
$$

where $G_l(r) = l^{-n} G(r/l)$ with $G(r)$ being a smooth and rapidly decaying kernel, i.e., $G \in C^\infty_c(\mathbb{R}^n)$; the space of infinitely-differentiable functions with compact support. We will call $\mathbf{F}$ the bare field whereas $\mathbf{F}_l$ is the renormalized, or coarse-grained, field. The high-pass filtered field is also given by

$$
\mathbf{F}(x, t) = \mathbf{F}_l(x, t) + \delta \mathbf{F}_l(x, t).
$$

Note that we could similarly coarse-grain the field with respect to time using a test function $g_\tau(x) = l^{-1} g(\tau)$; $g \in C^\infty_c(\mathbb{R}^n)$. The spatiotemporal renormalization at scale $l$ can be written as

$$
\mathbf{F}_l(x, t) = \int_V d^n r G_l(\tau) \int d\tau g_\tau(x) \mathbf{F}(x + r, t + \tau),
$$

where both test functions $g_\tau(x)$ and $G_l(\tau)$ are assumed to be rapidly decaying. Throughout this paper, only spatial renormalization will be considered unless stated otherwise. Without loss of generality, we will also assume the following mathematical properties for the test function $G$:

$$
G(r) \geq 0,
$$

2A function $g$ is said to have a compact support (set of its arguments for which $g \neq 0$) if $g = 0$ outside of a compact set (equivalent to closed and bounded sets in $\mathbb{R}^m$).

3In this paper, we assume that equations are non-dimensionalized or multiplied by appropriate coefficients to avoid inconsistencies regarding physical dimensions of different quantities such as $x$ and $t$. 

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conditions are given at time $t = t^0$ on a hyper-surface (called Cauchy data) in the following form:

$$
\frac{\partial^k}{\partial t^k} F_i(t^0, x_1, \ldots, x_n) = g_i^{(k)}(x_1, \ldots, x_n),
$$

with $k = 0, 1, 2, \ldots, n_i - 1$. The Cauchy-Kowalevski theorem guarantees a unique solution for the above initial value problem providing that the functions $f_j$ and $g_j$ are analytic. More precisely, if all $f_j$ functions are analytic in a neighborhood of the point $(t^0, x_1, \ldots, x_n, F_0, F_1, \ldots, F_n, \ldots, \frac{\partial^k F_j}{\partial t^0 \partial x_1 \ldots \partial x_n} |_{t=t^0})$ and all functions $g_j^{(k)}$ are analytic in some neighborhood of $(x_1, \ldots, x_n)$, then the above Cauchy problem has a unique, analytic solution in some neighborhood of $(t^0, x_1, \ldots, x_n)$. 

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and its inverse is defined as
\[ \hat{T}^{-1}(\epsilon) = \hat{T}(-\epsilon), \]
with
\[ \hat{T}(0) = \mathbb{I}, \]
where \( \mathbb{I} \) is the identity operator. Because \( \epsilon \in \mathbb{R} \) can be positive or negative, the inverse map, \( \hat{T}^{-1} \), is well-defined if the governing equation, eq. (1), is invariant under the time reversal operator \( \hat{\Theta} : t \to -t; \)
\[ -\partial_t F(x, -t) = f(x, -t). \]

With this condition satisfied, there are two possibilities: either we have
\[
\begin{align*}
F(x, -t) &= +F(x, t), \\
f(x, -t) &= -f(x, t),
\end{align*}
\]
which indicates an even field \( F \) and an odd source field \( f \), or else we have
\[
\begin{align*}
F(x, -t) &= -F(x, t), \\
f(x, -t) &= +f(x, t),
\end{align*}
\]
which indicates an odd \( F \) and an even \( f \).

In passing, we also note that the time translation operator can be written as
\[ e^{-i\hat{H}\Delta t}, \]
where \( \hat{H} = \frac{\partial T}{\partial t} \) and thus, formally, the quantum translation operator \( e^{-i\hat{H}\Delta t} \) takes the form \( e^{\Delta t \frac{\partial T}{\partial t}} \), i.e., \( \Delta t = \epsilon = 0 \) in our notation.

4In some applications, it is convenient to decompose any arbitrary field \( F(x, t) \) in \( \mathbb{R}^n \), into a homogeneous \( \mathbf{F}(t) \) and an inhomogeneous \( \mathbf{F}'(x, t) \) part: \( \mathbf{F}(x, t) = \mathbf{F}(t) + \mathbf{F}'(x, t) \) where \( \mathbf{F}(t) = \int_V \mathbf{F}(x, t) d^n x / V \) and \( \int_V \mathbf{F}'(x, t) d^n x = 0. \)

B. Time Translation and Field Line Evolution

In order to study the time evolution of \( F \), assuming \( F(x, t_0) \) is given at time \( t_0 \), we can solve eq. (1) to obtain \( F(x, t_0 + \epsilon) \) at a different time \( t_1 = t_0 + \epsilon \) for an infinitesimal \( \epsilon \in \mathbb{R} \) (besides the trivial case of \( \epsilon = 0 \), corresponding to the identity operator, if \( \epsilon > 0 \) we move forward in time, otherwise backward). We can represent this as a linear time translation operator (shift operator, or a lag operator in time series analysis), \( \hat{T} : \mathbb{R}^n \to \mathbb{R}^n \):
\[ \hat{T}(\epsilon) F(x, t_0) = F(x, t_0 + \epsilon), \]
which is linear
\[ \hat{T}(\epsilon) \left( \alpha F(x_0, t_0) + \beta F(x_1, t_0) \right) \]
\[ = \alpha F(x_0, t_0 + \epsilon) + \beta F(x_1, t_0 + \epsilon), \forall \alpha, \beta \in \mathbb{R}, \]
and its inverse is defined as
\[ \hat{T}^{-1}(\epsilon) = \hat{T}(-\epsilon), \]
with
\[ \hat{T}(0) = \mathbb{I}, \]
where \( \mathbb{I} \) is the identity operator. Because \( \epsilon \in \mathbb{R} \) can be positive or negative, the inverse map, \( \hat{T}^{-1} \), is well-defined if the governing equation, eq. (1), is invariant under the time reversal operator \( \hat{\Theta} : t \to -t; \)
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\]
which indicates an even field \( F \) and an odd source field \( f \), or else we have
\[
\begin{align*}
F(x, -t) &= -F(x, t), \\
f(x, -t) &= +f(x, t),
\end{align*}
\]
which indicates an odd \( F \) and an even \( f \).

In passing, we also note that the time translation operator can be written as
The real function \( \hat{\epsilon}(e) := e^{\hat{\epsilon} \hat{\beta}} \),

which is defined operationally in terms of a Taylor series in \( e \). Hence,

\[
e^{\hat{\epsilon} \hat{\beta}} F(x, t) = F(x, t + e),
\]

which, incidentally, leads to

\[
e^{\hat{\epsilon} \hat{\beta}(s) \hat{\beta}} G(x, s) = G(x, t^{-1}(t(s) + e)),
\]

where \( t = t(s) \) is taken such that\(^6\)

\[
\begin{align*}
\frac{dt}{ds} & := \frac{1}{\hat{\beta}(s)}, \\
G(x, s) & = F(x, t(s)).
\end{align*}
\]

Next let us consider the field lines of \( F \) which can be considered as parametric curves whose tangent vector at any point \( x \) is parallel to \( F \) at that point. At a given time \( t_0 \), therefore, we can parametrize these curves using the arc-length \( s \)

\[
\begin{align*}
\frac{d\xi(s, t_0)}{ds} & = \hat{F}(\xi(s, t_0), t_0), \label{eq:18} \\
\xi(0, t_0) & = x,
\end{align*}
\]

where \( \hat{F} = F / ||F|| \) is the direction (unit) vector. If the unit vector field \( \hat{F} \) is Lipschitz continuous\(^7\) in \( x \), then the above differential equation has a unique solution\(^8\) and hence there exists a unique field-line \( \xi_x(s, t_0) \) passing through \( x \) at time \( t_0 \). Nevertheless, if the field is Hölder singular, then there may exist infinitely many such integral curves (solutions) satisfying the above differential equation\(^9\). Thus for Hölder-singular fields, the concept of field line is not generally well-defined. On the other hand, for a renormalized field \( F_\ell \), integration by parts shows that any spatial derivative \( \nabla_x \) acting on \( F_\ell \) can be made to act on \( G \) instead of \( F \) inside the integral, which implies that \( F_\ell \) is Lipschitz-continuous even if \( F \) is not. Thus the notion of field line is well-defined for the renormalized field \( F_\ell \) even if \( F \) is Hölder singular.

In order to understand how the field lines evolve in time, we write the equation of the integral curves, given by eq.\((18)\), at another time \( t_0 + \epsilon \) for a real \( \epsilon \);

\[
\begin{align*}
\frac{d\xi(s, t_0 + \epsilon)}{ds} & = \hat{F}(\xi_x(s, t_0 + \epsilon), t_0 + \epsilon), \\
\xi(0, t_0 + \epsilon) & = x.
\end{align*}
\]

The condition for \( \xi_x \) to be continuous in \( t \) is

\[
\lim_{\epsilon \to 0} ||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)|| \to 0 \text{ for any } s.
\]

We write

\[
\begin{align*}
||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)|| & = ||\int_0^\epsilon ds' \left[ \hat{F}(\xi_x(s', t_0 + \epsilon), t_0 + \epsilon) - \hat{F}(\xi_x(s', t_0), t_0) \right]|| \\
& \leq \int_0^\epsilon ds' \left[ ||\hat{F}(\xi_x(s', t_0 + \epsilon), t_0 + \epsilon) - \hat{F}(\xi_x(s', t_0), t_0)|| \right].
\end{align*}
\]

Assuming that \( \hat{F} \) is Lipschitz in spacetime\(^10\) position vector \( \vec{x} = (x, t) \), i.e.,

\[
||\hat{F}(\vec{x}_2) - \hat{F}(\vec{x}_1)|| \leq K_0 ||\vec{x}_2 - \vec{x}_1||^2 + |t_2 - t_1|^2,
\]

for some \( K_0 > 0 \), we can write

\[
\begin{align*}
||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)|| & \leq K_0 \int_0^\epsilon ds' \sqrt{||\xi_x(s', t_0 + \epsilon) - \xi_x(s', t_0)||^2 + \epsilon^2}.
\end{align*}
\]

Therefore, we find

\[
\begin{align*}
\lim_{\epsilon \to 0} ||\xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0)|| \to 0 \text{ for any } s.
\end{align*}
\]

\(9\)The Peano theorem can still be used here to infer the existence of solutions, however, the uniqueness of a solution is guaranteed by the Picard-Lindelöf theorem which requires Lipschitz continuity of \( F \) as discussed before.

\(10\)Note that one may also use the Minkowski metric here, which is

\[
||\vec{x}_2 - \vec{x}_1|| = \sqrt{||x_2 - x_1||^2 + |t_2 - t_1|^2}.
\]

In any case, the continuity of the integral curves in time requires continuity of \( F \) in spacetime and not just space.
\[
\frac{\partial}{\partial s} \left\| \xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0) \right\| \\
\leq K_0 \sqrt{\left\| \xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0) \right\|^2 + \epsilon^2} \\
\leq K_0 \left( \left\| \xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0) \right\| + |\epsilon| \right),
\]
which implies
\[
\left\| \xi_x(s, t_0 + \epsilon) - \xi_x(s, t_0) \right\| \leq |\epsilon| \left( \frac{K_0 \epsilon}{K_0} \right).
\]

For any finite but arbitrarily large \( s > 0 \), we can take \( |\epsilon| \) small enough to make the RHS of (20) arbitrarily small, which indicates that \( \xi_x \) is uniformly continuous in time. Consequently, \( \xi_x(s, t) \) is uniformly continuous in \( t \), provided that \( \hat{F}(x, t) \) is Lipschitz in \( x = (x, t) \). Lipschitz continuity of \( \hat{F} \) in \( x = (x, t) \) indicates that \( \hat{F} \) is Lipschitz in both \( x \) and \( t \) which can be seen from the last line of (19). Also note that if \( \hat{F} \) is uniformly Lipschitz, i.e., \( \| F(x) - F(y) \| \leq K' \| x - y \| \) for some \( K' > 0 \), and \( \hat{F} \) has an upper bound, i.e., \( \exists M' > 0 \) s.t. \( \| F \| \leq M' \), then \( \hat{F} \) will be uniformly Lipschitz continuous;
\[
\| F(x) - F(y) \| \leq M' \left| \hat{F}(x) - \hat{F}(y) \right| \leq M' K' \| x - y \|. 
\]

On the other hand, if \( F \) is uniformly Lipschitz, i.e., \( \| F(x) - F(y) \| \leq K \| x - y \| \) for some \( K > 0 \) and has a lower bound, i.e., \( \exists M > 0 \) s.t. \( \| F \| \geq M \), then \( \hat{F} \) will be uniformly Lipschitz;
\[
\left| \hat{F}(x) - \hat{F}(y) \right| \leq \frac{1}{M} \| F(x) - F(y) \| \leq \frac{K}{M} \| x - y \|. 
\]

It should be emphasized that in order to have well-defined field lines, the Lipschitz-continuity of \( \hat{F} \) in spatial position vector \( x \) is required, otherwise the Picard-Lindelöf theorem cannot be applied to ensure a unique solution for \( \partial \xi_x / \partial s = \hat{F} \). In order to have continuously deforming field lines in time, the stronger condition of the Lipschitz continuity of \( \hat{F} \) in space-time position vector \( \tilde{x} = (x, t) \) should hold. If the field’s magnitude \( \| F \| \) (or energy density \( \| F \|^2 / 2 \)) is bounded from below and above, i.e., there are positive numbers \( M \) and \( M' \) such that \( M \leq \| F \| \leq M' \), the Lipschitz continuity of \( \hat{F} \) follows from the Lipschitz continuity of \( \hat{F} \) and vice versa.

### III. TOPOLOGY

Any set equipped with a metric, i.e., a notion of distance between its elements, is a metric space, and any metric space is a topological space. On the other hand, any vector field \( F(x) \) comes with a natural metric induced by the vector norm, i.e., the distance between \( F(x) \) and \( F(y) \) can be defined as \( \| F(x) - F(y) \| \), which is non-negative and satisfies the triangle inequality. Hence, a vector field as a set of vectors in \( \mathbb{R}^n \) is a topological space.

For a given vector field \( F \), let us define an open ball \( B_{\mathcal{R}}(F(x_0, t_0)) \), with radius \( \mathcal{R} > 0 \), around the vector \( F(x_0, t_0) \) as
\[
B_{\mathcal{R}}(F(x_0, t_0)) = \{ F(x, t_0); \| F(x, t_0) - F(x_0, t_0) \| < \mathcal{R} \}. 
\]

It should be emphasized that, in general, this ball is not localized in real Euclidean space \( \mathbb{R}^n \), i.e., it doesn’t imply \( \| x - x_0 \| < c \) for some \( c > 0 \), unless \( F \) is biLipschitz, that is to say there is an \( F_0 > 0 \) such that \( \frac{1}{F_0} \| x - x_0 \| \leq \| F(x, t_0) - F(x_0, t_0) \| \leq F_0 \| x - x_0 \| \). Therefore, for a general, non biLipschitz field, the ball defined by eq.(21) is a set of vectors \( F(x, t_0) \) which are close to \( F(x, t_0) \) as measured by the metric induced by the vector norm. If we visualize the field in space, e.g., a three-dimensional magnetic field around a magnet, then we will see that the ball defined above is a set of vectors which are scattered in space. If we define open sets using these open balls as the bases, we will have a metric topology for the vector field \( F \). We denote this topology by \( \hat{F} \), hence the vector field \( F \) naturally defines a topological space \( (F, F) \).

Let us consider the conditions under which the field \( F \) keeps its topology as it evolves in time. In other words, we look for conditions under which the time translation operator \( \hat{T}(\epsilon) \) acting on a given vector field \( F \) is a homeomorphism, i.e., a continuous, bijective map with continuous inverse. Continuous topological deformations do not change the topology; one can deform (i.e., map) an object such as a donut (i.e., a topological space) without cutting and gluing it (i.e., a continuous map which takes nearby points to nearby points) to a coffee mug (i.e., another topological space). Conversely, one should be able to recover the donut by deforming the coffee mug (i.e., the continuous map should have a continuous inverse). In addition, all points on the donut should be mapped; each point to only one point (i.e., an onto and one-to-one map). Instead of two topological spaces \( X_0 \) and \( X_1 \), e.g., a donut and a coffee mug, one may consider a vector field at two different times \( t_0 \) and \( t_1 = t_0 + \epsilon \). With our definition for vector field topology given above, the time translation operator \( \hat{T} \) should be a homeomorphism to preserve the topological properties of the field as it evolves in time.

In order for \( \hat{T}(\epsilon) \) to be one-to-one, it should satisfy the following condition:
\[
\hat{T}(\epsilon) F(x_1, t_0) = \hat{T}(\epsilon) F(x_2, t_0) \rightarrow F(x_1, t_0) = F(x_2, t_0), 
\]
that is
\[
F(x_1, t_0 + \epsilon) = F(x_2, t_0 + \epsilon) \rightarrow F(x_1, t_0) = F(x_2, t_0). 
\]

Obviously, this condition will not be satisfied in general. In other words, even for smooth and well-defined vector fields, the time evolution operator \( \hat{T} \) is not in general a homeomorphism, hence the natural topology of the vector field can change over time no matter how smooth the field \( F \) is. The other issue is that the open balls, the bases defining open
sets for the field’s natural topology, are not necessarily (except for bi-Lipschitz fields) localized in real space as discussed above. This is extremely restricting in many physical applications. It is more desirable to define a topology such that, roughly speaking, open balls are associated with close points in both real space and vector space. Such an open ball around $F(x_0, t_0)$, at a given time $t_0$, is defined as

$$B_r(F(x_0, t_0)) = \{(F(x, t_0)) : \sqrt{\|x - x_0\|^2 + \|F(x, t_0) - F(x_0, t_0)\|^2} < r\}. \quad (22)$$

In fact, this corresponds to an open ball in the phase space $(x, F)$: the desired topology we are after is basically the topology in the phase space defined by the corresponding dynamical system with the governing equations$^{11}$

$$\frac{dx}{dt} = F, \quad \frac{dF}{dt} = f. \quad (23)$$

Assuming that the second equation in (23) uniquely determines $F$ (e.g., with appropriate boundary conditions), the first equation requires an initial condition in the form

$$x(t_0) = x_0$$

to have a unique solution as a trajectory in the phase space (with uniqueness guaranteed by applying appropriate conditions on $F$, e.g., Lipschitz continuity). However, we are not interested in any particular trajectory here; we are interested in all possible trajectories that $F$ defines in the phase space. Each trajectory is determined by its corresponding initial condition and can be visualized as the trajectory (in the phase space) of a particle moving with the time dependent velocity $F$. The trajectories are solutions of the following non-autonomous differential equation (cf. (18) in the previous section):

$$\begin{cases}
\frac{dx(t)}{dt} = F(x(t), t), \\
x(t_0) = x_0,
\end{cases} \quad (24)$$

which has a unique solution if $F$ is uniformly Lipschitz continuous in $x$ and continuous in $t$ (cf. Lipschitz continuity of $F$ in $x$ for eq. (18) to define unique field lines). The time translation operator, acting at any point $(x, F(x, t))$ in the phase space, can be represented as

$$\hat{T}_\epsilon(x, F(x, t)) = (x, F(x, t + \epsilon)). \quad (25)$$

It is easy to see that $\hat{T}_\epsilon$ is an onto, one-to-one, and continuous map with continuous inverse. For its continuity, for example, we note that $\hat{T}_\epsilon$, for any $\epsilon \in \mathbb{R}$, is continuous (so is its inverse for $\hat{T}_\epsilon^{-1}(\epsilon) = \hat{T}_\epsilon(-\epsilon)$) if it is continuous at $\epsilon = 0$. In order to show this for any $t$, the following $L_1$-norm should vanish in the limit $\epsilon \to 0$,

$$\lim_{\epsilon \to 0} \int_{t \in \mathbb{R}} dt \left\| \hat{T}_\epsilon(x, F(x, t)) - \hat{T}_\epsilon(0)(x, F(x, t)) \right\| = 0,$$

Thus the condition for the continuity of $\hat{T}_\epsilon^{-1}(\epsilon)$ is

$$\lim_{\epsilon \to 0} \int_{t \in \mathbb{R}} dt \left\| F(x, t + \epsilon) - F(x, t) \right\| \to 0, \quad (26)$$

which follows if $F$ is uniformly continuous in $t$.\(^{12}\) In order to keep the phase space topology preserved in time, we need to ensure that the phase space at any given time $t_0$, as a topological space, is homeomorphic to the phase space at another time $t_1$. The condition of continuity for $\hat{T}_\epsilon^{-1}(\epsilon) = \hat{T}_\epsilon(-\epsilon)$, on the other hand, requires equations given by (23) to be time reversal invariant, which requires $F$ to be odd, i.e., $F(x, -t) = -F(x, t)$ and $f$ to be even, i.e., $f(x, -t) = +f(x, t)$.

In short, the topology associated with a given vector field $F$ is naturally defined in the phase space. Moreover, in order to ensure that time evolution keeps the topological properties of the phase space, the field $F(x, t)$ is required to be (i) Lipschitz continuous in $x$, (ii) uniformly continuous in $t$ and (iii) odd under time reversal, i.e., $F(x, -t) = -F(x, t)$, such that its governing equation, $\partial_t F = f$, eq.(1), is time reversal invariant.

### A. Spatial Complexity and Topological Deformation

The velocity field $u$ at a fixed point $x$ in a river has an almost well-defined and definite direction if observed from a distant point; e.g., if the river flows from east to west, the velocity field will point from east to west. One may call this the large scale velocity field. However, as we approach the river and look at smaller and smaller scales around the point $x$, we see more complex motions in different directions—the direction of the small scale velocity field will not generally be from east to west. The large scale velocity at point $x$ can be defined as the coarse-grained field at a large scale $L$, see eq.(5);

$$u_L(x, t) = \int_V G_L(r) u(x + r, t) d^3r,$$

which is the average velocity of a fluid parcel of size $L$ located at $x$ (because $G$ is a rapidly decaying function, hence

\(^{11}\)In the language of differential equations, $dx(t)/dt = F(x, t)$ is a non-autonomous equation because $F$ explicitly depends on time $t$. However, we can eliminate this time-dependence by introducing a new variable, $\tau(t)$ such that $d\tau(t)/dt = 1$. With this choice, the equation $dx(t)/dt = F(x(t), \tau(t))$ becomes autonomous. However, this does not simplify the task of solving the equations since we have increased the dimension of the problem by introducing a new function.

\(^{12}\)The shift operator defined by $\hat{O}_a f(x) = f(x + a)$ is continuous if $f$ has compact support and is continuous, which implies that $f$ is uniformly continuous.
the integral gets more contributions from points at a distance \( \leq L \) from \( x \) than distant points). The small scale field at \( x \) is similarly defined as

\[
u_l(x, t) = \int_V G_l(r) \mathbf{u}(x + r, t) d^3r,\]

which is the velocity of a fluid parcel of size \( l \) located at \( x \). How different are the directions of the small and large scale fields at point \( x' \)? Denote by \( \theta \) the angle between these vectors, which is a function of space and time, i.e., \( \theta = \theta(x, t) \) and can be obtained easily using the inner product \( \mathbf{u}_l \cdot \mathbf{u}_L = u_l u_L \cos \theta \). Hence we can use this angle to quantify the difference between the directions of the large and small scale velocity fields at point \( x \) and time \( t \). The larger is the deviation angle \( \theta \), the more complex the flow is at point \( (x, t) \). Moreover, if the flow is turbulent, it becomes a stochastic variable which measures the level of randomness in the velocity field at \( x, t \)(see \S III D). The quantity \( u_l, u_L \) not only tells us how strong the small and large scale velocity fields are at \( x \) (because it depends on the magnitudes \( u_l \) and \( u_L \)) but also it tells us how parallel they are (because of its dependence on \( \theta \)).

For a vector field \( \mathbf{F} \), we define

\[
\psi_{l,L}(x, t, t) = \frac{1}{2} \mathbf{F}_l(x, t) \cdot \mathbf{F}_L(x, t)
\]

\[
= \frac{1}{2} \int_V d^3r G_l(r) \int_V d^3r' G_L(r') \mathbf{F}(x + r, t) \cdot \mathbf{F}(x + r', t),
\]

as a generalized energy density at point \( x \). The function \( \psi_{l,L} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is in fact a scalar field\(^{13}\) which noting that \( \lim_{l \to 0} \mathbf{F}_l \to \mathbf{F} \) can be thought of as a generalization of the energy density;\(^{14}\) \( U(x, t) \);

\[
U(x, t) = \frac{1}{2} \mathbf{F}(x, t) \cdot \mathbf{F}(x, t) = \frac{1}{2} F^2(x, t).
\]

It is also more convenient to consider direction and magnitude of the field separately by writing \( \psi_{l,L}(x, t) = \phi_{l,L}(x, t) \chi_{l,L}(x, t) \) with two scalar fields

\[
\phi_{l,L}(x, t) = \begin{cases} 
\mathbf{F}_l(x, t) \cdot \mathbf{F}_L(x, t); & F_l \neq 0 & F_L \neq 0, \\
0; & otherwise,
\end{cases}
\]

and

\[
\chi_{l,L}(x, t) = \frac{1}{2} F_l(x, t) F_L(x, t).
\]

From equations (11) and (12), we realize that \( \phi_{l,L} \) is related to the field’s topology whereas \( \chi_{l,L} \) is associated with the field’s magnitude. In fact, for non-zero vectors \( \mathbf{F}_l \) and \( \mathbf{F}_L \), the scalar field \( \phi_{l,L} \) is the cosine of the angle between two coarsely-grained components of the vector field \( \mathbf{F}(x, t) \) at different scales \( l \) and \( L \) at point \( (x, t) \), i.e., \( \phi_{l,L} = \cos \alpha = \mathbf{F}_l \cdot \mathbf{F}_L \). At any given point \((x, t)\), this scalar field is simply what is known as the cosine similarity between two vectors.\(^{15}\) This scalar field is called the topology field to imply its relationship with the vector field topology [20]. On the other hand, \( \chi_{l,L} \) is in fact twice the geometric mean of the field energy densities at scales \( l \) and \( L \). In other words, we can write

\[
\chi_{l,L} = 2 \sqrt{U_l U_L}
\]

where \( U_l = F_l^2 \) and similarly \( U_L = F_L^2 \). For simplicity, we will drop the index \( l, L \) hereafter.

Based on the quantities discussed above, we can now define the spatial complexity or self-entanglement (of order \( p \in \mathbb{N} \)) associated with the field \( \mathbf{F}(x, t) \) as\(^{16}\)

\[
s_p(t) = \frac{1}{2} || \phi - \phi \|_p. \]

For instance, taking \( p = 2 \), we find the second order self-entanglement:

\[
s_2(t) = \frac{1}{2} \left( \phi - \phi_{rms} \right)_{rms}. \]

This form resembles the definition of the conventional standard deviation corresponding to a random variable \( X \); \( \sigma_x = \langle (x - \langle x \rangle)^2 \rangle^{1/2} \) where \( \langle \cdot \rangle \equiv E[\cdot] \) denotes the expected value calculated in the usual way using a given probability measure. In fact, in case we have a probability measure, one can use the standard deviation defined in the conventional way, that is

\[
s(t) = \frac{1}{2} E \left[ \left( \phi - E[\phi] \right)^2 \right]. \]

Nevertheless, instead of probability measures, unlikely to be given in many real world applications, the definition (31) is based only on spatial volume averages, which are easy to calculate in practice. In any case, although its numerical value

\[^{13}\text{In general, for a complex field, one may consider} \psi_{l,L}(x, R, t) = 4 \mathbf{F}_l(x, t) \mathbf{F}^*_l(x + R, t), \text{where} \mathbf{F}^* \text{is the complex conjugate of} \mathbf{F}. \text{In this paper, we will only consider real fields and would take} R = 0 \text{ hence} \psi_{l,L}(x, R, 0, t) = \psi_{l,L}(x, t).\]

\[^{14}\text{For Lipschitz-continuous fields, we can consider an interesting limiting case as} \psi_{\infty}(x, t) := \lim_{l \to 0} \lim_{L \to \infty} \psi_{l,L}(x, t) = \frac{1}{2} \mathbf{F}(x, t) \cdot \mathbf{F}(x, t), \text{local field} \mathbf{F}(x, t) \text{global field} \mathbf{F}(x, t). \text{where} \mathbf{F}(x, t) = \int_V \mathbf{F}(x + r, t) d^3r / V \text{is the spatial volume average of} \mathbf{F}. \text{The above expression follows from the properties of the kernel} G_l(r).\]

\[^{15}\text{This is analogous to the Otsuka-Ochiai coefficient} \frac{X \cap Y}{|X \cup Y|}, \text{for two sets} X \text{and} Y, \text{where} |\cdot| \text{denotes the number of elements, as the similarity measure. This measure in fact reduces to the cosine similarity if} X \text{and} Y \text{are bit vectors (i.e., maps from a set of integer numbers to the interval} [0, 1].\}

\[^{16}\text{The} L_p \text{ norm of} \mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n \text{is the mapping} \mathbf{f} \to ||\mathbf{f}||_p = \left[ \int_V ||\mathbf{f}(x)||^p (d^nx/V)^{1/p} \right]^{1/p}. \text{For} p = 2, ||\mathbf{f}||_2 = f_{rms} \text{is the root-mean-square (rms) value of} \mathbf{f}. \text{For} p \leq q, ||\mathbf{f}||_p \leq ||\mathbf{f}||_q. \text{Also} ||\mathbf{f}||_\infty = \lim_{p \to \infty} ||\mathbf{f}||_p = ||\mathbf{f}||_{max}.\]
Its time derivative corresponds to the dissipation rate; 

\[ \frac{\partial \phi}{\partial t} = \left[ \frac{\partial \hat{F}_I}{\hat{F}_I} (I - \hat{F}_I \hat{F}_I) \right] \hat{F}_I + \left[ \frac{\partial \hat{F}_L}{\hat{F}_L} (I - \hat{F}_L \hat{F}_L) \right] \hat{F}_L. \]  

(41)

Here, \( I = I_{n \times n} \) is the identity tensor and \( (\cdot)_{\perp \cdot} \) represents the perpendicular component with respect to \( \cdot \). We find

\[ T_2(t) = \frac{1}{4 S_2} \int_V \left[ \hat{F}_I (I - \hat{F}_I \hat{F}_I) \right] \left[ \hat{F}_L (I - \hat{F}_L \hat{F}_L) \right] \frac{d^m \chi \cdot d^m \chi}{V}. \]  

(42)

The time evolution of the scalar field \( \chi(x, t) \) can be similarly obtained,

\[ \frac{\partial \chi}{\partial t} = \frac{1}{2} F_I F_I \left[ \left( \frac{\partial \chi}{\chi} \right)_{\parallel F_L} + \left( \frac{\partial \chi}{\parallel F_L} \right)_{\perp F_I} \right]. \]  

(43)

B. Scalar Fields

Although we are primarily interested in vector fields in this paper, however, it should be emphasized that a similar approach in terms of spatial complexity and energy, discussed in the previous section, can be applied to scalar fields. For any scalar field \( \Phi(x, t) \), with \( x \in \mathbb{R}^3 \) for simplicity, there exists a four-vector field, i.e., a \((3+1)\)-dimensional vector field in the Minkowski spacetime, which represents the spatiotemporal derivative;

\[ F^\mu = \partial^\mu \Phi = \left( \frac{\partial \Phi}{\partial t}, -\nabla \Phi \right), \quad \mu = 0, 1, 2, 3. \]  

(45)

The inner product of the vector field can be defined using the Minkowski metric \( \eta_{\mu \nu} = \text{diag}(+1, -1, -1, -1) \);

\[ \mathbf{F} \cdot \mathbf{E} \equiv F^\mu E_\mu = \eta_{\mu \nu} F^\mu E^\nu = F^0 E^0 - \sum_{i=1}^{3} F^i E^i. \]  

(46)
Therefore, the energy density associated with the field $F \equiv F^\mu$ is

$$U = \frac{1}{2} F^\mu F_\mu = \frac{1}{2} \dot{\Phi}^2 - \frac{1}{2} \nabla \Phi |^2, \quad (47)$$

which should look familiar as the energy terms in the Lagrangian densities corresponding to scalar fields such as Klein-Gordon field in quantum field theories. The corresponding scale split energy density, eq. (27), is defined as

$$\Psi = \frac{1}{2} F^L \dot{F}_L = \frac{\eta_{\mu\nu}}{2} F^\mu F^\nu = \frac{1}{2} \frac{\partial \Phi_l}{\partial t} \frac{\partial \Phi_l}{\partial t} - \frac{1}{2} \nabla \Phi_l \cdot \nabla \Phi_l. \quad (48)$$

The term $\frac{1}{2} \nabla \Phi_l \cdot \nabla \Phi_l$ is a measure of the “spatial patchiness” of the field $\Phi$ for $L \gg l$. If $\Phi$ represents the temperature on a two dimensional plate, for example, this term shows how the local temperature gradient is oriented relative to the global temperature gradient averaged over the whole plate. The temporal term can be interpreted in a similar manner.

The analogy between expressions such as eq. (48) and the kinetic energy terms in the Lagrangian densities commonly encountered in quantum field theory is motivating to employ the definitions given above in such contexts. However, care must be taken in coarse-graining the fields in relativistic theories. In a relativistic setup, spatial coarse-graining of $F$ must be taken in coarse-graining the fields in relativistic theories.

Now, it is easy to see that $\psi$ in fact defines an average, weighted auto-correlation for a real field $F$:

$$\psi(x, t) = \int d\tau \int d\tau' g_{T_l}(\tau)g_{T_l}(\tau') F(x, t + \tau)\cdot F(x, t + \tau'). \quad (50)$$

C. Topological Entropy

In a given phase space, we are interested to distinguish two distinct groups of points in a neighborhood of a given trajectory: those points whose distance grows over time as the system evolves from those points whose distance does not. This can be made precise in terms of a metric [23], although there are other equivalent ways to do so, e.g., in terms of covers in compact Hausdorff spaces [24]. The topological entropy is a way of counting the number of distinct trajectories which are generated as the dynamical system evolves in time\(^{18}\). For a dynamical system governed by a given iteration function, the topological entropy can be thought of as a measure of the exponential growth rate of the number of distinguishable orbits, which is an extended real number. In other words, topological entropy is a measure of the system’s complexity level.

One can also consider the vector field $dx/dt = F$ in the phase space $(x, F)$ and ask how the complexity of the dynamical system, measured by the entropy, manifests itself in terms of the vector field $F$. The field $F(x, t)$, as a function of $x$, is the vector tangent to the trajectory $x(t)$. As the system evolves, therefore, the vector field $F$ determines the direction of motion in the phase space. Consequently, two distinct points on two close trajectories in the phase space will remain close if the corresponding tangent vectors of their trajectories remain close (in the tangent bundle). This argument suggests that the topological entropy is associated with the evolution of spatial configuration of the corresponding vector field, i.e., the topological deformation defined by eq. (34). For example, the velocity field corresponding to a laminar flow retains its untangled and smooth configuration in time, unlike the entangled and complex velocity field corresponding to a fully turbulent flow. These are expected to be associated respectively with lower and higher entropies in the corresponding phases spaces.

\(^{18}\)For a compact metric space $(M, d)$ equipped with a continuous map $g : M \rightarrow M$, one can define for each $n \in \mathbb{N}$, the metric ((23)], $d_n : M \times M \rightarrow \mathbb{R}$ as

$$d_n(x, y) = \max \{d(g^n(x), g^n(y)) : 0 \leq k \leq n - 1\},$$

for any $x, y \in M$. For any real, positive $\epsilon$ and $n \geq 1$, two points of $M$ are said to be $\epsilon$-close if their first $n$ iterates are $\epsilon$-close. A subset $N \subseteq M$ is $(n, \epsilon)$-separated provided that the distance between every distinct points of the subset $N$ is larger than or equal to $\epsilon$ as measured by the above metric. Suppose $N$ is the maximum cardinality of such an $(n, \epsilon)$-separated set, which is a finite number because of the compactness of $M$. The topological entropy of the map $g$, as a measure of the complexity of the corresponding dynamical system, is a non-negative, real number defined as

$$h(\epsilon) = \lim_{\epsilon \to 0} \lim_{n \to \infty} \frac{-1}{n} \log N(\epsilon, n) \cdot (n).$$
D. Stochasticity

An $n$-dimensional vector $\mathbf{X} = (X_1, \ldots, X_n)^T$ is a random (or stochastic) vector if its components $X_i$, $1 \leq i \leq n$, are random variables, which are defined on a probability space $(\Omega, \mathcal{F}, P)$, with sample space $\Omega$, $\sigma$-algebra $\mathcal{F}$, and probability measure $P$. The vector’s expected value is defined as

$$E[\mathbf{X}] = (E[X_1], \ldots, E[X_n])^T,$$

and its variance is defined as

$$\text{Var}[\mathbf{X}] = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{X} - E[\mathbf{X}])^T].$$

As a straightforward generalization, the cross-covariance between two random vectors $\mathbf{X}$ and $\mathbf{Y}$ is given by

$$\text{Cov}[\mathbf{X}, \mathbf{Y}] = E[(\mathbf{X} - E[\mathbf{X}])(\mathbf{Y} - E[\mathbf{Y}])^T].$$

Other statistical quantities can be defined in a similar manner, which are all widely used in multi-variate statistics.

We can generalize this multi-variate approach to vector fields. Let us define a stochastic vector field $\mathbf{F}(\mathbf{x})$ (with $\mathbf{x}$ in an open set $U \subseteq \mathbb{R}^n$ or on a manifold in general) as a field that assigns a random vector to each point $\mathbf{x}$ in its domain. A time dependent field $\mathbf{F}(\mathbf{x}, t)$ is stochastic if it is a stochastic field for any given $t \in I \subseteq \mathbb{R}$. At a fixed time $t_0$, the cross-covariance between $\mathbf{F}(\mathbf{x}, t_0)$ and $\mathbf{F}(\mathbf{y}, t_0)$ associated with points $\mathbf{x}$ and $\mathbf{y}$ is defined as

$$\text{Cov}_{t_0}[\mathbf{F}(\mathbf{x}, t_0), \mathbf{F}(\mathbf{y}, t_0)] = E \left[ \left( \mathbf{F}(\mathbf{x}, t_0) - E[\mathbf{F}(\mathbf{x}, t_0)] \right) \left( \mathbf{F}(\mathbf{y}, t_0) - E[\mathbf{F}(\mathbf{y}, t_0)] \right)^T \right].$$

As mentioned before, physically speaking, what is measured as a field $\mathbf{F}$ at a point in space and time is an average value. In fact, $\mathbf{F}(\mathbf{x}, t)$ is a mathematical object and only its spatiotemporally renormalized version $\tilde{\mathbf{F}}(\mathbf{x}, t)$ at a non-zero resolution $l$, eq.(7), has a definite physical meaning. This is because any physical measurement taking place at $(\mathbf{x}, t)$ requires a spatial volume around $\mathbf{x}$ as well as a time interval $\Delta t$ around $t$. For simplicity, let us consider only the spatial renormalization, in which case eq.(6), $\mathbf{F}(\mathbf{x}, t) = \tilde{\mathbf{F}}(\mathbf{x}, t) + \delta \mathbf{F}(\mathbf{x}, t)$, will indicate a measurement error; $\delta \mathbf{F}(\mathbf{x}, t)$. In fact, a level of uncertainty may be present in the bare field $\mathbf{F}$ itself, which, e.g., can arise from the source field $f$ or some unknown extra terms in the evolution equation $\partial_t \mathbf{F} = \mathbf{f}$ which we have ignored. Suppose that the source field $\mathbf{f}$ is a function only of $\tilde{\mathbf{F}}$ and $t$ and let us add a noise term, e.g., a Gaussian white noise $\tilde{\eta}(t)$ to eq.(1), $\partial_t \mathbf{F} = \mathbf{f}$, at a fixed point $\mathbf{x}$ in space. This leads to the following stochastic differential equation (SDE):

$$\frac{\partial \mathbf{F}(t)}{\partial t} = \mathbf{f}(t) + \sqrt{2\kappa_0} \tilde{\eta}(t), \quad \mathbf{F}(t_0) = \mathbf{F}_0, \quad (51)$$

where $\kappa_0 > 0$ is a constant and we have suppressed, by abuse of notation, the dependence on $\mathbf{x}$ in order to emphasize the time dependence. The Euler-Maruyama scheme can be used to discretize this equation;

$$\mathbf{F}_m = \mathbf{F}_{m-1} + \mathbf{f}(\mathbf{F}_{m-1}, t_{m-1})\delta t + \sqrt{2\kappa_0}(\mathbf{W}_m - \mathbf{W}_{m-1}),$$

where the Wiener variables $\mathbf{W}_m$ for $0 \leq m \leq M$ ($\mathbf{W}_0 = 0$) have Gaussian density [2];

$$P(\mathbf{W}_1, \ldots, \mathbf{W}_M) \propto \exp \left( \frac{-1}{2\delta t} \sum_{m=1}^{M} \left| \mathbf{W}_m - \mathbf{W}_{m-1} \right|^2 \right).$$

This distribution can be used to obtain the probability distribution for the field

$$P(\mathbf{F}) = \frac{1}{\det (\frac{\partial \mathbf{F}}{\partial \mathbf{W}})} P(\mathbf{W}),$$

where the Jacobian is block lower triangular with diagonal blocks $\partial \mathbf{F}_m/\partial \mathbf{W}_m = \sqrt{2\kappa_0} \mathbf{I}_{n \times n}$ with $\mathbf{I}_{n \times n}$ representing the $n \times n$ identity matrix. Therefore, $\det (\frac{\partial \mathbf{F}}{\partial \mathbf{W}}) = \text{const.}$, which leads to

$$P(\mathbf{F}_1, \ldots, \mathbf{F}_M) \propto \left( \frac{-1}{4\kappa_0} \sum_{m=1}^{M} \frac{\delta t}{\delta t} \left| \mathbf{F}_m - \mathbf{F}_{m-1} \right|^2 \right).$$

Integration with respect to $\mathcal{D} \mathbf{F} = \Pi_m dF_m$, and taking the limit $\delta t \to 0$ gives us a path integral. The transition probability, from an initial configuration $(\mathbf{F}_0, t_0)$ to a final configuration $(\mathbf{F}_f, t_f)$, at the given point $\mathbf{x}$, is given by the following path integral:

$$G_{\mathbf{x}}^{\alpha}(\mathbf{F}_f, t_f|\mathbf{F}_0, t_0) = \int_{\mathbf{F}(t_0) = \mathbf{F}_0} \mathcal{D} \mathbf{F} \delta^\alpha [\mathbf{F}_f - \mathbf{F}(t_f)]$$

$$\times \exp \left( \frac{-1}{4\kappa_0} \int_{t_0}^{t_f} d\tau \left| \tilde{\mathbf{F}}(\tau) - \mathbf{f}(\mathbf{x}(\tau), \tau) \right|^2 \right),$$

where $\delta^\alpha$ is $n$-dimensional delta function and $\tilde{\mathbf{F}} = \partial \mathbf{F}/\partial t$. Note that the noise term in eq.(51) breaks the time reversal invariance, therefore, the field’s topology will not be preserved in general. The function $G_{\mathbf{x}}^{\alpha}$ provides us with the probability that at any given point $\mathbf{x}$, the field evolves from $\mathbf{F}_0$ at time $t_0$ to $\mathbf{F}_f$ at time $t_f$.

Let $\mathbf{F}^\alpha$ denote a stochastic field. It follows that the coarse-grained component $\mathbf{F}^\alpha$, with $l > 0$, will also be a stochastic field. Therefore, for a stochastic, time dependent field $\mathbf{F}^\alpha(\mathbf{x}, t)$,
\[ \psi^s(x, t) = \frac{1}{2} \mathbf{F}^s \mathbf{F}^s, \quad (52) \]

\[ \phi^s(x, t) = \mathbf{F}^s \mathbf{F}^s, \quad (53) \]

and

\[ \chi^s(x, t) = \frac{1}{2} ||\mathbf{F}^s||^2 \quad (54) \]

are stochastic scalar fields for any pair of scales \( l, L \). In particular, because \( \phi^s \) is a physically dimensionless scalar and \(-1 \leq \phi^s \leq 1\), its deviation from unity or its mean value can be used as a statistical measure of the global stochasticity level of \( \mathbf{F}^s \). Hence, we can define the stochasticity level (of order \( p \in \mathbb{N} \)) of the stochastic field \( \mathbf{F}^s \) as its spatial complexity or self-entanglement:

\[ S_p^s(t) = \frac{1}{p} \| \phi^s(x, t) - 1 \|_p. \quad (55) \]

The second order stochasticity \( S_2(t) \), topological deformation \( T_2(t) \), cross energy \( E_2(t) \), and dissipation \( D_2(t) \) are given by

\[ S_2(t) = \frac{1}{2} (\phi^s - 1)_{rms}, \quad (56) \]

\[ T_2(t) = \frac{1}{4S_2^s(t)} \int_V (\phi^s - 1) \frac{\partial \phi^s}{\partial t} \frac{d^nx}{V} \quad (57) \]

\[ = \frac{1}{2} \int_V (\phi^s - 1)_{rms} \frac{\partial \phi^s}{\partial t} \frac{d^nx}{V}, \]

\[ E_2^s(t) = \chi_{rms}, \quad (58) \]

\[ D_2(t) = \frac{1}{E_2^s(t)} \int_V \chi \frac{\partial \chi}{\partial t} \frac{d^nx}{V} \quad (59) \]

\[ = \int_V \chi_{rms} \frac{\partial \chi}{\partial t} \frac{d^nx}{V}. \]

These quantities can be used, for example, to formulate magnetic reconnection, magnetic diffusion and probably magnetic dynamo theories in turbulent fluids ([20]; [22]; [21]).

Another related and important concept is the notion of stochastic field lines, widely used in describing stochastic fields e.g., turbulent velocity and magnetic fields (see e.g., [3]; [14]; [20] and references therein). For a stochastic field \( \mathbf{F}^s \), the equation defining its field lines will be an SDE:

\[ \begin{cases} \frac{d\xi^s(x, t)}{ds} = \hat{\mathbf{F}}^s(\xi^s(x, t) + \delta \mathbf{F}, t) + \sigma \xi^s(x, t) dW_s, \\ \xi^s(x, t) = x. \end{cases} \quad (60) \]

If \( \hat{\mathbf{F}}^s \) is not Lipschitz continuous, we can still define its field lines in terms of distributions. In fact, even for a deterministic, Hölder singular field, we can use eq.(6), that is \( \mathbf{F}(x, t) = \mathbf{F}_i(x, t) + \delta \mathbf{F}_i(x, t) \) and treat \( \delta \mathbf{F}_i \) as a noise term. Then the equation defining the field lines, eq.(18), becomes an SDE at a given time \( t_0 \):

\[ \begin{aligned} d\xi^s(x, t_0) &= \hat{\mathbf{F}}^s(\xi^s(x, t_0), t_0) + \sigma \xi^s(x, t_0) dW_s, \\ \xi^s(x, t_0) &= x, \end{aligned} \quad (61) \]

where \( W_s \geq 0 \) is a standard Wiener process and \( \sigma \) a Borel measurable function. This SDE has a strong and unique solution provided that, at a given time \( t_0 \), both \( \hat{\mathbf{F}}_i \) and \( \sigma \) are uniformly Lipschitz. For a Gaussian white noise, \( \sigma = \text{const.} \) (cf. eq.(51) above), hence the SDE given by eq.(61) will have unique solutions at different times—stochastic field lines.

IV. SUMMARY AND CONCLUSIONS

Vector fields in physics are usually visualized in terms of their integral curves, or field lines, while the field topology is implicitly used, in fact, only as a synonym for the field configuration in terms of its field lines, which differs from the mathematical notion of topology in the context of topological spaces. In fact, to have well-defined field lines for a vector field \( \mathbf{F} \), the Lipschitz-continuity of \( \mathbf{F} \) in spatial position vector \( x \) is required. To have continuously deforming field lines in time, the stronger condition of the Lipschitz continuity of \( \mathbf{F} \) in spacetime position vector \( \vec{x} = (x, t) \) should also be satisfied. These requirements are not met in many applications, e.g., velocity and magnetic fields in turbulent plasmas are non-Lipschitz and also have stochastic behavior.

As for the field’s topology and topology change, these notions are usually employed in physical applications, e.g., in plasma physics literature in the context of magnetic fields threading highly conducting fluids, to indicate a spontaneous change in the field configuration in terms of its field lines. This is once again different from what a topology change means in mathematics, i.e., maps between topological spaces which fail to be a homeomorphism and therefore may change the topological properties. In order to study the topology change of a given vector field \( \mathbf{F} \), one needs to (i) define vector field topology using the conventional mathematical language e.g., in terms of open sets, (ii) define a time translation map which takes a time dependent vector field as a topological space at time \( t_0 \) and maps it to another vector field at another arbitrary time \( t_1 \) and finally (iii) find the conditions under which such a time translation map is a homeomorphism, i.e., a continuous, bijective map with continuous inverse (for only homeomorphisms, by definition, preserve topological properties).

The Euclidean vector norm defines a metric for the vector field \( \mathbf{F} \), as \( d(\mathbf{F}(x), \mathbf{F}(y)) = ||\mathbf{F}(x) - \mathbf{F}(y)|| \). With this metric, the vector field defines a metric space and hence a topological space since there is a natural notion of distance between any pair of vectors \( \mathbf{F}(x) \) and \( \mathbf{F}(y) \). However, in physical
applications, one is interested in vectors which are not only close in the above sense but also are located at nearby points in real space. For example, when the magnetic field threading a plasma undergoes reconnection, we are concerned with the magnetic vectors in a spatial volume, i.e., the reconnection region. Thus we are interested in the vectors $F(x)$ and $F(y)$ for which

$$\sqrt{||x - y||^2 + ||F(x) - F(y)||^2} < r$$

for some $r > 0$, which naturally defines an open ball in the phase space $(x, F(x))$. Any trajectory $x(t)$ in this phase space is a solution to an initial value problem $dx(t)/dt = F(x(t), t)$, $x(0) = x_0$. Therefore, the vector field topology should be defined as the phase space topology. In order to ensure that time evolution preserves the topological properties of the phase space, the field $F(x, t)$ is required to be (i) Lipschitz continuous in $x$, (ii) uniformly continuous in $t$, (iii) odd under time reversal $F(x, -t) = -F(x, t)$ and (iv) have a time reversal invariant governing equation $\partial_t F = f$.

Field lines, which are defined only for Lipschitz fields and do not continuously transform in time unless strong conditions are satisfied, are not very appropriate objects in the study of the field topology, time evolution and topology change. Instead the trajectories in the phase space $(x, F)$ provide better means for such considerations. Dynamics and statistics of physical vector fields, such as stochastic and Hölder singular magnetic and velocity fields in turbulent plasmas and astrophysical environments, should be studied in the context of mathematical theory of non-autonomous dynamical systems.

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