Stratified Two-Phase Ranked Set Sampling

Raghunath Arnab
Department of Statistics, University of Botswana and Department of Statistics, University of KwaZulu-Natal, South Africa
Email: arnabr@mopipi.ub.bw

George Anderson
Department of Computer Science, University of Botswana, Botswana
Email: andersong@mopipi.ub.bw

John. O. Olaomi
Department of Statistics, University of South Africa, South Africa
Email: olaomjo@unisa.ac.za

B. C. Rodríguez
Department of Statistics and Operations Research, Universidad de Granada, Spain
Email: beacr@ugr.es

Abstract

We propose an alternative two-phase stratified ranked set sampling. A comparison of the performances of the proposed estimators made by simulation studies using both real and simulated data sets. It is found that the proposed two-phase stratified regression estimator beats its competitors in literature.

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1. Introduction

Ranked set sampling (RSS) introduced by McIntyre (1952), was used to estimate the mean pasture and forage yield. The RSS is employed when precise measurement of the variable of interest is difficult or expensive, but one can easily rank the variable without measuring the variable by an inexpensive method such as visual perception, judgment and auxiliary information. For example, in the problem of estimating the mean height of trees in a forest, one can rank the heights of a small sample of two or three trees standing nearby easily by visual inspection without measuring them. In estimating the number of bacterial cells per unit volume, we can rearrange two or three test tubes easily in order of concentration using optical instruments without measuring exact values. In the RSS, instead of selecting a single sample of size \( m \), we select \( m \)-sets of samples each of size \( m \). In each set, we rank all the elements but we only measure one of them. Finally, the average of the \( m \)-measured units is taken as an estimate of the population mean.

Takahashi and Wakimoto (1968) and Takahasi (1970) provided the theoretical justification for using the ranked set sampling. They proved that when the ranking is perfect, the sample mean of the RSS is an unbiased estimator of the population mean and the variance of the RSS mean is smaller than that of the sample mean of the simple random sampling with
replacement (SRSWR) of the same size. Dell and Clutter (1972) proved that the sample mean based on the RSS is unbiased for the population mean regardless of the ranking error and it is at least as precise as the SRSWR sample mean of the same size. Stokes (1977) considered the performance of the Dell and Clutter estimator when the regression of the study variable (y) and the ranking variable (x) is linear, and y and x follow certain model. Yu and Lam (1997) proposed regression estimator when x and y follow a bivariate normal distribution and found on the basis of simulation studies that their proposed regression estimator performs better than the naive estimator, unless the correlation between x and y is low (|ρ|<0.4). Kadilar et al., (2006) and Arnab and Olaomi (2015) proposed an improved estimator of mean μ_y, the population mean of the study variable y using the ranking variable as an auxiliary variable x when the population mean μ_x of x is unknown. Zamanzade and Al-Omari (2016) developed a new ranked set sampling for estimating the population mean and variance, called neoteric ranked set sampling (NRSS) under perfect and imperfect ranking conditions while Mahdizadeh and Zamanzade (2018) introduced stratified pair ranked set sampling (SPRSS) and utilized it in estimating the population mean, with some theoretical results.

In this paper, we propose two alternative estimators for two-phase sampling where in the first phase; information only on the ranking variable x is collected. Based on the observed x - values, the population is divided into a number of homogeneous strata. From each of the stratum so formed, one selects ranked set samples independently using proportional allocation. The performances of the proposed estimators are compared by simulation studies using both real tree data collected by Platt et al. (1988) and generated bivariate normal data. We found that the proposed two-phase stratified regression estimator performs better in respect of relative bias (RB) and mean-square error (MSE) than those of naïve and Yu and Lam (1997) estimators for the tree data and it behaves better in most situations for the simulated bivariate data.

1.1. Rank set sampling by SRSWR method
First, we choose a small number m (set size) such that one can easily rank the m elements of the population with sufficient accuracy. Then the selection of RSS is as follows: Select a sample of m^2 units from a population U by SRSWR method. Allocate these m^2 units at random into m sets each of size m. Rank all the units in a set with respect to the values of the variable of interest y from 1(minimum) to m(maximum) by a very inexpensive method such as eye inspection. At this stage, no actual measurement is done. After the ranking has been completed, the unit holding rank i (i=1,...,m) in the i th set is actually measured. This completes a cycle of the sampling. One repeats the process for r cycles to obtain the desired sample of size n=mr. Thus, in a RSS, a total of m^2r units are drawn from the population but only mr of them are measured and the rest mr(m−1) are discarded. We call these measured mr observations “ranked set sample”. Since the ordering of a large number of observations is difficult, increase of sample size n=mr is done by increasing the number of cycles r. It is well known that μ RSS (m,r), the sample mean the RSS of size n=mr is unbiased for the population mean μ_y.

1.2. Judgment ranking
Sometimes, perfect ranking (no error in ranking) is not possible. In such cases, we use judgment ranking where each of the selected samples is ranked by an approximate method
such as visual inspection, expert opinion or use of concomitant variable. It should be noted
that some tests have been developed in the literature to assess the assumption of perfect
ranking in RSS. Some of these include Frey, et al (2007), Zamanzade, et al (2012) and
Zamanzade and Vock (2018).
Let \( y_{i(j)k} \) be the smallest \( j \) th “judgment order statistic” corresponding to order statistic
\( x_{i(j)k} \) of the concomitant variable \( x \) in the \( i \) th set of the cycle \( k \). In case the judgment
ranking is perfect \( y_{i(j)k} \) becomes equal to the \( j \) th order statistic \( y_{i(j)k} \), otherwise if the
judgment process is imperfect, we find \( y_{i(j)k} \neq y_{i(j)k} \).

Stokes (1977) derived the following results:

Theorem 1.

(i) \( \hat{\mu}_{\text{rss}} = \frac{1}{r} \sum_{k=1}^{r} \bar{y}_{m(k)} \) is an unbiased estimator for \( \mu_y \).

(ii) The variance of \( \hat{\mu}_{\text{rss}} \) is
\[
V(\hat{\mu}_{\text{rss}}) = \frac{1}{n} \left[ \sigma_y^2 - \frac{1}{m} \sum_{j=1}^{m} \tau^2_{y(m)} \right]
\]

(iii) An unbiased estimator of the variance of \( V(\hat{\mu}_{\text{rss}}) \) is
\[
\hat{V}(\hat{\mu}_{\text{rss}}) = \frac{1}{r(r-1)} \sum_{k=1}^{r} \left( \bar{y}_{m(k)} - \hat{\mu}_{\text{rss}} \right)^2
\]

1.3. Use of auxiliary variable with known mean \( \mu_x \)

Stokes (1977) considered the linear regression of \( y \) on \( x \) as follows:
\[
y = \mu_y + B(x - \mu_x) + \epsilon \tag{1}
\]
where \( x \) and \( \epsilon \) are independent random variables, \( E(\epsilon|x) = 0 \), \( V(\epsilon|x) = \sigma_y^2(1 - \rho^2) \), \( B = \rho \frac{\sigma_y}{\sigma_x} \)
and \( \rho \) is the correlation coefficient between \( x \) and \( y \).

The equation (1) yields
\[
y_{i(j)k} = \mu_y + B(x_{i(j)k} - \mu_x) + \epsilon_{i(j)k} \tag{2}
\]
Stokes (1977) further assumed that \( \frac{y - \mu_y}{\sigma_y} \) and \( \frac{x - \mu_x}{\sigma_x} \) have the same marginal distribution,
which holds for bivariate normal and bivariate Pareto distributions. Stokes (1977) derived
the following Theorem:

Theorem 2.

(i) \( E(\hat{\mu}_{y|\text{rss}}) = \mu_y \)

(ii) \( V(\hat{\mu}_{y|\text{rss}}) = \frac{1}{mr} \left[ \sigma_y^2 - \frac{\rho^2}{m} \sum_{j=1}^{m} \tau^2_{y(j)m} \right] \)

where \( \hat{\mu}_{y|\text{rss}} = \frac{1}{mr} \sum_{k=1}^{r} \sum_{j=1}^{m} y_{i(j)k} \) and \( \tau_{y(j)m} = E(y_{i(j)k}) - \mu_y \)
Following Stokes (1977), Yu and Lam (1997) proposed the following estimator of the population mean $\mu_y$ of $y$ as

$$\hat{\mu}_{yreg} = \hat{\mu}_{y(rss)} - \hat{B}(\hat{\mu}_{x(rss)} - \bar{x})$$

(3)

where

$$\hat{\mu}_{x(rss)} = \frac{1}{mr} \sum_{k=1}^{r} \sum_{j=1}^{m} x(j|k)$$

and

$$\hat{B} = \frac{\sum_{k=1}^{r} \sum_{j=1}^{m} (y(j|k) - \hat{\mu}_{y(rss)}) (x(j|k) - \hat{\mu}_{x(rss)})}{\sum_{k=1}^{r} (x(j|k) - \hat{\mu}_{x(rss)})^2}$$

(4)

Yu and Lam (1997) derived the following theorem:

**Theorem 3.**

(i) $E(\hat{\mu}_{yreg}) = \mu_y$

(ii) The optimum value of $\hat{B}$ that minimizes the variance of $\hat{\mu}_{yreg}$ is $B$.

(iii) $\text{Var}(\hat{\mu}_{yreg}) = \frac{\sigma_y^2}{mr} \left[ 1 + E\left( \frac{Z_{(rss)}^2}{S_z} \right) \right] = \frac{\sigma_y^2}{mr} \left[ 1 + E\left( \frac{\bar{x}_{(rss)}^2 - \mu_x^2}{s_x^2(rss)} \right) \right]$

where

$$Z_{(rss)} = \frac{1}{mr} \sum_{k=1}^{r} \sum_{j=1}^{m} (x(j|k) - \bar{x}_{(rss)})$$

$$S_z^2 = \frac{1}{mr} \sum_{k=1}^{r} \sum_{j=1}^{m} (x(j|k) - Z_{(rss)})^2$$

and

$$s_x^2(rss) = \frac{1}{mr} \sum_{k=1}^{r} \sum_{j=1}^{m} (x(j|k) - \bar{x}_{(rss)})^2.$$

1.4. Population mean with unknown $\mu_x$

Since $\mu_x$ is unknown, Yu and Lam (1997) considered a two-phase sampling procedure where in the first-phase, a relatively large sample $\tilde{s}$ of size $\tilde{n}$ is selected by the simple random sampling without replacement (SRSWOR) method from a population of size $N$ and only information on the auxiliary variable $x$ is collected. On the second-phase, a sub-sample $s$ of size $n (= rm)$ is selected from $\tilde{s}$ using ranked set sampling with $r$ cycles and information of study variable $y$ is obtained using $x$ as ranking variable. The proposed estimator for the population mean $\mu_y$ is

$$\hat{\mu}_{YM} = \hat{\mu}_{y(rss)} - \hat{B}(\hat{\mu}_{x(rss)} - \bar{x})'$$

(5)

where $\bar{x}' = \frac{1}{\tilde{n}} \sum_{i \in \tilde{s}} x_i$ and $\hat{B}$ is defined in (4).

Yu and Lam (1997) derived the following results for large $N$ and assuming the model (2) holds.

**Theorem 4.**

(i) $E(\hat{\mu}_{YM}) = \mu_y$

(ii) The optimum value of $\hat{B}$ that minimizes the variance of $\hat{\mu}_{YM}$ is $B$.
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\[
(iii) \ Var(\hat{\mu}_y) = \frac{\sigma_y^2}{mr} \left[ 1 + E \left( \frac{\left( \bar{Z}_{\text{RSS}} - \bar{Z} \right)^2}{S_z^2} \right) \right] \times \frac{\rho^2 \sigma_y^2}{\hat{n}}
\]

where \( Z = (x - \mu_x) / \sigma_x \), \( \bar{Z}_{\text{RSS}} \), \( \bar{X}_{\text{RSS}} \), \( S_Z^2 \) and \( S_{X_{\text{RSS}}}^2 \) are defined as in Theorem 3.

2. Two-phase stratified ranked set sampling

Initially, a relatively large sample \( \tilde{s} \) of size \( \tilde{n} \) is selected from the entire population by SRSWOR method. From each of the selected units of \( \tilde{s} \), information only on the concomitant variable \( x \) is obtained similar to Yu and Lam (1977) in two-phase sampling. Here, we assume that the condition of two-phase sampling is valid i.e. the cost of collecting data on \( x \) is much cheaper than that of the study variable \( y \). Observing the values of \( x \), the sampled units are classified into a number of strata \( H \) so that each of the stratum becomes homogeneous with respect to the variable under study \( y \). The number of strata will certainly depend on the characteristics of the variable \( y \) and sample size \( \hat{n} \). For example, noting eye estimates of heights or date of plantation, one can classify the plants as small, medium or big. Similarly, noting the CD counts of HIV patients, we may classify the conditions of the HIV infected patients into bad, very bad and severe. Let \( \tilde{s}_h \) be the set of units of size \( \tilde{n}_h \) falling in the \( h \)th stratum. Here \( \tilde{n}_h \) is a random variable taking values from 0 to \( \min(\tilde{n}_h, N_h) \) where \( N_h \) is the total number of the units in the \( h \)th stratum of the population. From the sampled \( \tilde{n}_h \) units of the \( h \)th stratum, a sub-sample of \( n_h \) units is selected using a ranked-set sampling procedure with \( h \) cycles of set-size \( m \) each using the \( x \) as ranking variable where \( \gamma_h (0 < \gamma_h < 1) \) are pre-determined fractions (vide Rao, 1973). Here we assume that (i) \( \hat{n} \) is so large that \( P(\tilde{n}_h \geq 1) = 1 \) and (ii) \( n_h \) are integers. For proportional allocation with fixed sample size \( n = \sum \tilde{n}_h \), \( \gamma_h = \gamma = n / \hat{n} \) and \( n_h = \tilde{n}_h n / \hat{n} = m n_h \).

Let the ranked set data collected from the \( h \) stratum be denoted by

\[ d_{h} = \left\{ \left( y_{j(h)|\tilde{n}_h}, x_{j(h)|\tilde{n}_h} \right) ; j = 1, ..., m; t_h = 1, ..., r_h \right\}; \ h = 1, ..., H \]

where \( x_{j(h)|\tilde{n}_h} \) is the \( j \)th ordered statistics for the concomitant variable \( x \) of the \( j \)th set of \( t_h \) th cycle of the \( h \)th stratum and \( y_{j(h)|\tilde{n}_h} \) be the corresponding judgment order statistic for the study variable for \( y \).

2.1. Estimator of the population mean without using auxiliary variable at the estimation stage

We propose the estimator for the population mean \( \mu_y \) without assuming any auxiliary information at the estimation stage as follows:
where \( w_h = \frac{\bar{n}_h}{n} \) and \( \hat{\mu}_{y(rs)}^h = \frac{1}{m} \sum_{j=1}^{m} \frac{y_{j(j)}^h}{\bar{n}_h} \).

**Theorem 5.**

(i) \( \mu_{y(rs)}^{st} \) is an unbiased estimator of \( \mu_y \).

(ii) \( V\left( \hat{\mu}_{y(rs)}^h \right) = \frac{1}{\bar{n}_h} \left( \sigma_{y}^2 - \frac{1}{m} \sum_{j=1}^{m} \sigma_{\hat{y}(j)|m}^2 \right) + \left( \mu_{hy} - \mu_y \right)^2 \)

where \( \mu_{hy} \) and \( \sigma_{y}^2 \) are the population mean and variance of \( y \) for the stratum \( h \),
\( \tau_{h(j)|m} = \mu_{h(j)|m} - \mu_{hy} \), \( \mu_{h(j)|m} = E\left(y_{j(j)|h}^h\right) \) and \( w_h \) is the proportion of units in the population of the \( h \)th stratum.

**Proof:**

Since \( \bar{n}_h \), the number of units in \( n \) that fall into \( h \)th stratum is a random variable, we have from Rao (1973)
\[
E \left[ w_h \right] = W_h, \ V \left[ w_h \right] = \frac{W_h}{n} - W_h, \ Cov \left[ w_h, w_{h'} \right] = -\frac{W_h W_{h'}}{n^2}, \ E\left( \hat{\mu}_{y(rs)}^h | \bar{n}_h \right) = \mu_{hy} \text{ and }
\]
\[
V\left( \hat{\mu}_{y(rs)}^h | \bar{n}_h \right) = \frac{1}{\bar{n}_h} \left( \sigma_{y}^2 - \frac{1}{m} \sum_{j=1}^{m} \sigma_{\hat{y}(j)|m}^2 \right)
\]

Using (9), we get the following:

(i) \( E\left( \hat{\mu}_{y(rs)}^{st} \right) = E\left[ \sum_{h=1}^{H} w_h E\left( \hat{\mu}_{y(rs)}^h | \bar{n}_h \right) \right] = E\left[ \sum_{h=1}^{H} w_h \mu_{hy} \right] = \sum_{h=1}^{H} W_h \mu_{hy} = \mu_y \)

(ii) \( V\left( \hat{\mu}_{y(rs)}^{st} \right) = V\left[ \sum_{h=1}^{H} w_h^2 V\left( \hat{\mu}_{y(rs)}^h | \bar{n}_h \right) \right] + V\left[ \sum_{h=1}^{H} w_h E\left( \hat{\mu}_{y(rs)}^h | \bar{n}_h \right) \right] \) (10)

Now using Theorem 1, we find
\[
E\left( \hat{\mu}_{y(rs)}^{st} | \bar{n}_h \right) = \mu_{hy} \text{ and } V\left( \hat{\mu}_{y(rs)}^{st} | \bar{n}_h \right) = \frac{1}{n_h} \left( \sigma_{y}^2 - \frac{1}{m} \sum_{j=1}^{m} \sigma_{\hat{y}(j)|m}^2 \right)
\]

The first component of (10) is obtained from (11) as
\[
E\left( \sum_{h=1}^{H} w_h^2 V\left( \hat{\mu}_{y(rs)}^h | \bar{n}_h \right) \right) = E\left[ \sum_{h=1}^{H} \left( \frac{\bar{n}_h}{n_h} \right)^2 \frac{1}{\gamma_h \bar{n}_h} \left( \sigma_{y}^2 - \frac{1}{m} \sum_{j=1}^{m} \sigma_{\hat{y}(j)|m}^2 \right) \right] \text{ (noting, } n_h = \gamma_h \bar{n}_h \)
\]
\[
= \frac{1}{\bar{n}_h} \sum_{h=1}^{H} w_h \left( \sigma_{y}^2 - \frac{1}{m} \sum_{j=1}^{m} \sigma_{\hat{y}(j)|m}^2 \right)
\]

(12)

The second component of (10) is
\[
V\left[ \sum_{h=1}^{H} w_h E\left( \hat{\mu}_{y(rs)}^{st} | \bar{n}_h \right) \right] = V\left[ \sum_{h=1}^{H} w_h \mu_{hy} \right] = \sum_{h=1}^{H} \mu_{hy}^2 V\left( w_h \right) + \sum_{h=1}^{H} \sum_{h'=1}^{H} \mu_{hy} \mu_{h'y} \text{ Cov}\left( w_h, w_{h'} \right) \) (13)

Now using (9), we get
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\[
V \left( \sum_{h=1}^{H} w_h E \left( \tilde{\mu}_{yreg} | \tilde{n}_h \right) \right) = \frac{1}{\tilde{n}} \left[ \sum_{h=1}^{H} W_h \left( \frac{H}{H} \sum_{h=1}^{H} W_h \mu_{hy} - \left( \frac{H}{H} \sum_{h=1}^{H} W_h \mu_{hy} \right)^2 \right) \right] = \frac{1}{\tilde{n}} \left[ \sum_{h=1}^{H} W_h \left( \mu_{hy} - \mu_y \right)^2 \right]
\]  

(14)

Part II of the theorem follows from (10), (12) and (14).

Corollary 1.

For proportional allocation with fixed sample size \( n \) at the second phase, \( V \left( \tilde{\mu}_{yreg} \right) \) reduces to

\[
V \left( \tilde{\mu}_{yreg} ^{st} \right) = \sum_{h=1}^{H} W_h \left[ \frac{1}{n} \left( \sigma_{hy}^2 - \frac{1}{m} \sum_{j=1}^{m} \sigma_{(j|\eta)}^2 \right) \right] + \frac{1}{\tilde{n}} \left( \mu_{hy} - \mu_y \right)^2
\]

2.2. Using auxiliary information at the estimation stage

Noting \( \tilde{\mu}_{y(rss)}^h = \frac{1}{m} \sum_{i=1}^{k_h} y_i^{(h)} / \eta_h \) and \( \hat{\mu}_{x(rss)}^h = \frac{1}{m} \sum_{i=1}^{k_h} x_i^{(h)} / \eta_h = \bar{x}_h^{(rss)} \) are unbiased estimators of the population means \( \mu_{hy} \) and \( \mu_{hx} \) of \( y \) and \( x \) of the stratum \( h \) respectively, we propose the following regression estimator of the population mean \( \mu_y \) of \( y \) as

\[
\tilde{\mu}_{yreg} ^{st} = \sum_{h=1}^{H} w_h \tilde{\mu}_{yreg}^h
\]

where \( w_h = \tilde{n}_h / \tilde{n} \), \( \tilde{\mu}_{yreg}^h = \mu_{y(rss)}^h - \hat{B}^h \left( \tilde{\mu}_{x(rss)}^h - \bar{x}_h^{(rss)} \right) \), \( \bar{x}_h^{(rss)} = \frac{1}{\tilde{n}_h} \sum_{i \in \tilde{S}_h} x_i \) and

\[
\hat{B}^h = \frac{\sum_{i=1}^{k_h} \sum_{j=1}^{m} \left( y_i^{(h)} - \mu_{y(rss)}^h \right) \left( x_i^{(h)} - \mu_{x(rss)}^h \right)}{\sum_{i=1}^{k_h} \sum_{j=1}^{m} \left( x_i^{(h)} - \mu_{x(rss)}^h \right)^2}
\]

Assume that \( y_i^{(h)} \) (\( x_i^{(h)} \)), is the value of \( y(x) \) for the \( i \) th unit of the \( h=1,..,H \) th stratum follows the model

\[
y_i^{(h)} = \mu_{hy} + B_h \left( x_i^{(h)} - \mu_{hx} \right) + \epsilon_i^{(h)}
\]

(15)

where \( x_i^{(h)} \) and \( \epsilon_i^{(h)} \) are independent random variables; \( E \left( x_i^{(h)} \right) = \mu_{hx} \), \( V \left( x_i^{(h)} \right) = \sigma_{hx}^2 \), \( E \left( y_i^{(h)} \right) = \mu_{hy} \),

\%; \( V \left( y_i^{(h)} \right) = \sigma_{hy}^2 \), \( E \left( \epsilon_i^{(h)} \right) \left( x_i^{(h)} \right) = 0 \), \( V \left( \epsilon_i^{(h)} \right) \left( x_i^{(h)} \right) = \sigma_{hy}(1-\rho_h^2) \), \( B_h = \rho_h \sigma_{hy} / \sigma_{hx} \), and \( \rho_h \) is the correlation coefficient between \( x \) and \( y \) of the \( h \) th stratum.

Theorem 6.

Under the model (15)

(i) \( \tilde{\mu}_{yreg} ^{st} \) is an unbiased estimator of \( \mu_y \)

(ii) \( V \left( \tilde{\mu}_{yreg} ^{st} \right) = \frac{1}{\tilde{n}} \sum_{h=1}^{H} W_h \left[ \frac{\sigma_{hy}^2 \left( 1-\rho_h^2 \right)}{\gamma_h} \left( 1 + E \left( \frac{Z_{hrss} - \bar{Z}_h}{S_{hz}^2} \right)^2 \right) + \rho_h^2 \sigma_{hy} \right] \left( \sum_{h=1}^{H} W_h \left( \mu_{hy} - \mu_y \right)^2 \right) \]
The second component of (16) is obtained from (14) as

\[ \rho_h \sigma_{hy}^2 \left( \frac{1 - \rho_h^2}{\gamma_h} \right) \left( \frac{\bar{x}_{hy\{r\}} - \bar{x}_h}{S_{hy\{r\}}} \right)^2 + \frac{\gamma_h^2 \rho_h^2 \sigma_{hy}^2}{m^2 \gamma_h} \]

where \( \bar{z}_{hy\{r\}} = \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} (z_{j(k)h} - \bar{x}_{hy\{r\}})^2 \),

\[ s_{hy\{r\}}^2 = \frac{1}{m} \sum_{k=1}^{m} \sum_{j=1}^{n} (z_{j(k)h} - \bar{x}_{hy\{r\}})^2 \], \( \bar{x}_h = \frac{1}{n_h} \sum_{i=1}^{n_h} x_{hi} \).

**Proof:**

(i) \( E(\hat{\mu}_{yreg}^h) = E \left( \sum_{h=1}^{H} w_{hy} E(\hat{\mu}_{yreg}^h | \bar{y}_h) \right) = E \left( \sum_{h=1}^{H} W_{hy} \mu_{hy} \right) = \sum_{h=1}^{H} W_{hy} \mu_{hy} = \mu_y \)

(ii) \( \text{Var}(\hat{\mu}_{yreg}^h) = E \left( \sum_{h=1}^{H} w_{hy}^2 \text{Var}(\hat{\mu}_{yreg}^h | \bar{y}_h) \right) + V \left( \sum_{h=1}^{H} W_{hy} E(\hat{\mu}_{yreg}^h | \bar{y}_h) \right) \) (16)

Now using (6), we find

\[ V(\hat{\mu}_{yreg}^h | \bar{y}_h) = \frac{\sigma_{hy}^2}{m^2} \left[ 1 + \frac{\rho_h^2 \sigma_{hy}^2}{m^2} + \frac{\rho_h^2 \sigma_{hy}^2}{m^2} \right] \]

Equation (17) yields the first component of (16) as

\[ E \left( \sum_{h=1}^{H} w_{hy}^2 \text{Var}(\hat{\mu}_{yreg}^h | \bar{y}_h) \right) = E \left( \sum_{h=1}^{H} \frac{n_h}{\bar{y}_h} \right)^2 \left[ \frac{\sigma_{hy}^2}{\gamma_h \bar{y}_h} \left( \frac{\bar{z}_{hy\{r\}} - \bar{x}_h}{S_{hy\{r\}}} \right)^2 + \frac{\gamma_h^2 \rho_h^2 \sigma_{hy}^2}{m^2 \gamma_h} \right] \]

\[ = \frac{1}{n} \sum_{h=1}^{H} W_{hy} \left[ \frac{\sigma_{hy}^2}{\gamma_h} \left( \frac{\bar{z}_{hy\{r\}} - \bar{x}_h}{S_{hy\{r\}}} \right)^2 + \frac{\gamma_h^2 \rho_h^2 \sigma_{hy}^2}{m^2 \gamma_h} \right] \] (18)

The second component of (16) is obtained from (14) as

\[ V \left( \sum_{h=1}^{H} w_{hy} E(\hat{\mu}_{yreg}^h | \bar{y}_h) \right) = V \left( \sum_{h=1}^{H} W_{hy} \mu_{hy} \right) = \frac{1}{n} \sum_{h=1}^{H} W_{hy} \left( \mu_{hy} - \mu_y \right)^2 \] (19)

Part II of the theorem follows from (16), (18) and (19).

**Corollary 2.**

For proportional allocation with fixed sample size \( n \), \( V(\hat{\mu}_{yreg}^h) \) reduces to
Stratified Two-Phase Ranked Set Sampling

\[
V(\hat{\mu}_{y,\text{reg}}) = \frac{1}{n} \sum_{h=1}^{H} W_h \left[ \sigma_{hy}^2 (1 - \rho_h^2) \left( 1 + E \left( \frac{(Z_{h,\text{rss}} - Z_h)^2}{s_{h,y}^2} \right) \right) + \frac{\rho_h^2 \sigma_{hy}^2}{m} \right] + \frac{1}{n} \sum_{h=1}^{H} W_h \left( \mu_{hy} - \mu_y \right)^2
\]

where \( Z_{h,\text{rss}} = \frac{1}{m_{h}} \sum_{k=1}^{m_{h}} \sum_{j=1}^{m} x_{(j)k}^h - \frac{1}{m_{h}} \sum_{k=1}^{m_{h}} x_{(j)k}^h - \mu_{hx} \) \( \sigma_{hx} \) and \( Z_{h} = \frac{\bar{x}_{h} - \mu_{hx}}{\sigma_{hx}} \).

3. Comparison of stratified and un-stratified ranked set sampling strategies

It is very difficult to compare the performances of the proposed estimators \( t_2 = \hat{\mu}_{y,\text{reg}} \) and \( t_3 = \hat{\mu}_{y,\text{rss}} \) theoretically with the existing estimators \( t_0 = \hat{\mu}_{y,\text{rss}} \) (Stokes, 1977) and \( t_1 = \hat{\mu}_{M} \) (Yu and Lam, 1997), since the expressions of the variances \( \text{Var}(t_j); j = 0,1,2 \) involve several unknown parameters. Hence, we will compare the performances using simulation studies. For the simulation studies, we have considered both real and simulated data.

The real tree data related to the diameter in centimetres at breast height (\( x \)) and entire height (\( y \)) in feet of 396 trees which was originally collected by Platt et al. (1988) and used later by Chen et al. (2003) and Zamanzade and Mahdizadeh (2018). The mean diameter and heights of 396 (= \( N \)) trees are \( \mu_x = 20.970 \) cm and \( \mu_y = 52.967 \) feet respectively. The tree were partitioned into two strata with diameter less than equal 13.6 cm (stratum 1) and more than 13.6 cm (stratum 2) respectively.

The number of trees belonging to the stratum 1 and stratum 2 are equal to 198. In the first-phase, a sample \( \tilde{s} \) of size \( \tilde{n} \) is selected from the entire population by the SRSWOR method and information only on the auxiliary variable \( x \) is collected. Let \( \tilde{s}_i \) be the sample of size \( \tilde{n}_i \) that falls in the stratum \( i = 1,2 \). From the sample \( \tilde{s}_i \), a sub-sample of size \( s_i \) of size \( n_i = [\tilde{n}_i / \tilde{n}] \) is collected using proportional allocation by RSS sampling with \( m(=2,3,4) \) as set-size and \( n \) as a predetermined number. For our simulation studies we take the following combinations of \( (\tilde{n},n,m): (250,160,2), (250,160,3), (250,160,4); (250,100,2), (250,100,3), (250,100,4); (200,125,2), (200,125,3), (200,125,4); (200,80,2), (150,80,3), (200,80,4); (150,80,2), (150,80,3), (150,80,4); (150,60,2), (150,60,3), (150,60,4). \)

The simulated data comprises with 5 bivariate normal populations of sizes 600 (= \( N \)) each of which have the same \( \mu_x = 10, \mu_y = 25, \sigma_x = 2.5, \sigma_y = 4.5 \) but different values of \( \rho (=0.5, 0.6, 0.7, 0.8, 0.9) \). We divide each population into two strata as the tree population. From each of the 5 populations, ranked set samples of parameters \( (\tilde{n},n,m): (400,125,2), (400,125,3), (400,125,4); (400,80,2), (400,80,3), (400,80,4); (250,80,2), (250,80,3), (250,80,4) \) are selected.

From the selected sample \( s \), we compute estimates \( t_0, t_1, t_2 \) and \( t_3 \). We will call the process of selection of two-phase sample \( s \) and obtaining estimates from the selected sample as an iteration. The iteration is repeated \( R=100,000 \) times. Let the values of the \( t_0 \)
, \( t_1 \), \( t_2 \) and \( t_3 \) based on the \( q \)th iteration be denoted by \( t_0(q), t_1(q), t_2(q) \) and \( t_3(q) \) respectively. The relative biases (RB) and the mean square errors (MSE) of the estimators are computed using the following formula:

\[
RB(t_j) = \frac{1}{\mu_y} \left( \frac{1}{R} \sum_{q=1}^{R} (t_j(q) - \mu_y) \right) \quad \text{and} \quad MSE(t_j) = \frac{1}{R} \sum_{q=1}^{R} (t_j(q) - \mu_y)^2 \quad ; \ j = 0, 1, 2, 3 \quad (20)
\]

The percentage relative efficiency of the estimator \( t_j \) compared with the conventional estimator \( t_0 = \hat{\mu}_{y(x)} \) given by

\[
PRE(t_j) = 100 \times \frac{MSE(t_0)}{MSE(t_j)} \quad \% \quad (21)
\]

The values of \( RB(t_j) \) and \( PRE(t_j) \) are computed for the five populations with \( m = 2, 3, 4 \) for the live and simulated populations and are given in the Table 1 and Table 2.

**Table 1: Tree data: Relative bias (RB) and Relative Efficiency (RE)**

| RB \( \bar{y}(n) = (250,160) \) | PRE \( \bar{y}(n) = (200,125) \) | RB \( \bar{y}(n) = (150,80) \) |
|---|---|---|
| \( \hat{\mu}_{y(x)} \) | -0.0376 | -0.0038 |
| \( \hat{\mu}_{ys} \) | -0.0493 | -0.0874 |
| \( \hat{\mu}_{ysx} \) | -0.0545 | -0.0329 |
| \( \hat{\mu}_{yss} \) | -0.0209 | 113.7285 |
| \( \hat{\mu}_{ys} \) | -0.0009 | -0.0311 |
| \( \hat{\mu}_{ys} \) | -0.0495 | 155.1414 |
| \( \hat{\mu}_{ys} \) | -0.0252 | 160.9543 |
| \( \hat{\mu}_{ys} \) | -0.0233 | 105.9229 |
| \( \hat{\mu}_{ys} \) | 0.0253 | 100.0000 |
| \( \hat{\mu}_{ys} \) | -0.0715 | 137.5715 |
| \( \hat{\mu}_{ys} \) | -0.0166 | 140.6034 |
| \( \hat{\mu}_{ys} \) | -0.0058 | 98.9452 |

| RB \( \bar{y}(n) = (250,100) \) | PRE \( \bar{y}(n) = (200,80) \) | RB \( \bar{y}(n) = (150,60) \) |
|---|---|---|
| \( \hat{\mu}_{y(x)} \) | -0.0581 | 100.0000 |
| \( \hat{\mu}_{ys} \) | -0.1029 | 233.4535 |
| \( \hat{\mu}_{ys} \) | -0.0236 | 239.2576 |
| \( \hat{\mu}_{ys} \) | 0.0012 | 129.7157 |
| \( \hat{\mu}_{ys} \) | 0.0011 | 100.0000 |
| \( \hat{\mu}_{ys} \) | -0.0899 | 198.1569 |
| \( \hat{\mu}_{ys} \) | 0.0150 | 207.1916 |
| \( \hat{\mu}_{ys} \) | 0.0518 | 123.5520 |
| \( \hat{\mu}_{ys} \) | -0.0186 | 100.0000 |
| \( \hat{\mu}_{ys} \) | -0.0783 | 173.7545 |
| \( \hat{\mu}_{ys} \) | -0.0212 | 178.7264 |
| \( \hat{\mu}_{ys} \) | -0.0282 | 113.9226 |
### Table 2: Bivariate Normal data: Relative bias (RB) and Relative Efficiency (RE)

| m | $\mu_{\text{Y}_{\text{RSS}}}^{\text{RB}}$ | $\mu_{\text{Y}_{\text{RSS}}}^{\text{PRE}}$ | $\mu_{\text{Y}_{\text{RSS}}}^{\text{RB}}$ | $\mu_{\text{Y}_{\text{RSS}}}^{\text{PRE}}$ | $\mu_{\text{Y}_{\text{RSS}}}^{\text{RB}}$ | $\mu_{\text{Y}_{\text{RSS}}}^{\text{PRE}}$ | $\mu_{\text{Y}_{\text{RSS}}}^{\text{RB}}$ | $\mu_{\text{Y}_{\text{RSS}}}^{\text{PRE}}$ | $\mu_{\text{Y}_{\text{RSS}}}^{\text{RB}}$ | $\mu_{\text{Y}_{\text{RSS}}}^{\text{PRE}}$ |
|----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 2  | -0.007                          | 100.000                        | 0.000                           | 100.000                        | 0.000                           | 100.000                        | -0.009                          | 100.000                        | 0.000                           | 100.000                        |
|    | $\mu_{\text{YM}}^{\text{RB}}$   | 0.001                           | 105.711                         | 0.012                           | 123.375                         | 0.002                           | 133.247                         | 0.001                           | 166.876                         | -0.002                         | 260.631                        |
|    | $\mu_{\text{YM}}^{\text{PRE}}$  | 0.002                           | 106.214                         | -0.001                          | 123.803                         | -0.016                          | 133.173                         | -0.004                          | 166.398                         | -0.002                         | 260.938                        |
|    | $\mu_{\text{Y}_{\text{RSS}}}^{\text{RB}}$ | -0.001                         | 97.338                          | 0.002                           | 109.578                         | -0.002                          | 109.906                         | 0.001                           | 122.497                         | -0.002                         | 153.668                        |
| 3  | -0.001                          | 100.000                        | 0.004                           | 100.000                        | -0.001                          | 100.000                        | 0.002                           | 100.000                        | 0.006                           | 100.000                        |
|    | $\mu_{\text{YM}}^{\text{RB}}$   | 0.002                           | 101.634                         | -0.001                          | 113.628                         | 0.001                           | 122.631                         | 0.000                           | 146.398                         | -0.002                         | 211.965                        |
|    | $\mu_{\text{YM}}^{\text{PRE}}$  | 0.003                           | 101.947                         | -0.005                          | 113.161                         | -0.006                          | 122.289                         | -0.008                          | 147.418                         | -0.004                         | 213.146                        |
|    | $\mu_{\text{Y}_{\text{RSS}}}^{\text{RB}}$ | 0.001                           | 95.157                          | -0.006                          | 102.884                         | 0.006                           | 103.991                         | -0.004                          | 113.718                         | -0.003                         | 138.658                        |
| 4  | 0.000                           | 100.000                        | -0.006                          | 100.000                        | -0.007                          | 100.000                        | 0.001                           | 100.000                        | 0.006                           | 100.000                        |
|    | $\mu_{\text{YM}}^{\text{RB}}$   | 0.002                           | 100.109                         | 0.003                           | 109.169                         | 0.002                           | 115.823                         | 0.007                           | 135.626                         | -0.003                         | 186.343                        |
|    | $\mu_{\text{YM}}^{\text{PRE}}$  | 0.007                           | 100.293                         | 0.001                           | 108.531                         | 0.000                           | 113.804                         | -0.002                          | 135.412                         | -0.001                         | 185.834                        |
|    | $\mu_{\text{Y}_{\text{RSS}}}^{\text{RB}}$ | 0.007                           | 93.206                          | 0.002                           | 100.129                         | 0.010                           | 100.961                         | 0.004                           | 109.483                         | 0.002                          | 127.097                        |

| $\rho = 0.5$ | $\rho = 0.6$ | $\rho = 0.7$ | $\rho = 0.8$ | $\rho = 0.9$ |
|-------------|-------------|-------------|-------------|-------------|

(\bar{n}, n) = (400, 125)
3.1. Simulation Results

Relative biases of all the estimators are very low in general. For the tree data it ranges from -0.2260 to 0.0518. The relative biases for the simulated bivariate normal data is much lower than that of the tree data and it varies from -0.014 to 0.023. The estimators \( \hat{\mu}_{YM} \) and \( \hat{\mu}_{y_{reg}}^{st} \) using auxiliary information possess higher relative efficiency than the naïve estimator \( \hat{\mu}_{y(s)} \) in almost all situations. The proposed two-phase stratified regression estimator \( \hat{\mu}_{y_{reg}}^{st} \) performs the best, the next place is occupied by two-phase un-stratified regression estimator \( \hat{\mu}_{y_{reg}}^{un} \). The estimator \( \hat{\mu}_{y_{rss}}^{st} \) performs the best in all situations for the tree data with a maximum PRE 239.2576. The stratified estimator \( \hat{\mu}_{y_{rss}}^{st} \) performs better than naïve estimator in general but in some isolated situations, it possesses lower efficiency (with the minimum PRE = 86.252) than the naïve estimator. For a given combination of \((\tilde{n}, n)\), PREs of all the estimators for both the tree data and simulated data decrease with \( m \). For a given \( \tilde{n} \) and \( m \), PRE of the estimators decreases with \( n \). The relative efficiencies of all the estimators for the simulated data increase with the correlation coefficient \( \rho \). The Yule-estimator \( \hat{\mu}_{y_{reg}}^{un} \) performs slightly better than the proposed two-phase estimator \( \hat{\mu}_{y_{reg}}^{st} \) in scantly occasions.

4. Conclusion

Stokes (1977) recommended regression estimator for the ranked set sampling when the population mean of the auxiliary variable is known. Yu and Lam (1997) proposed the regression estimator in two-phase sampling when the population mean of the auxiliary variable is unknown. We also propose an alternative two-phase stratified ranked set sampling. On the basis of real and simulated data, it is found that the proposed regression estimator outperforms the other estimators in most situations, especially for the real tree data. We suggest therefore that instead of using two-phase sampling, one should use two-phase stratified sampling for small strata for improving efficiency of the Yu-Lam estimator. Determination of the asymptotic distribution of the proposed estimators, coverage probabilities of the asymptotic confidence intervals and Bootstrap confidence intervals using Akgul et al. (2018) are subjects of our future research.

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