Global solutions to planar magnetohydrodynamic equations with radiation and large initial data

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Abstract

The existence and uniqueness of the global classical solution for the planar magnetohydrodynamic equations are proved for large initial data. The model equations are coupled with the thermal radiation and are supplemented with free boundary and initial conditions. The existence proof relies on the classical Leray–Schauder fixed theorem together with some new a priori estimates in Lagrangian coordinates. The result holds for more general heat conductivity, which is plausible and interesting in physics. Particularly, it is also valid for the case of constant transport coefficients.

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1. Introduction

Magnetohydrodynamics (MHD) is the study of the complex macroscopic interaction between the magnetic fields and electrically conducting fluids. It is widely applied to astrophysics, geophysics and plasma physics in practice. Due to the existence of magnetic fields, which exert forces on currents, the dynamics of conducting fluids is considerably influenced. On the other hand, the development of electric currents simultaneously affects changes in the magnetic fields. Moreover, the thermal radiation is another important physical quantity in the high temperature regime, which significantly affects the dynamics of fluids. Indeed, the dynamics of many astrophysical models are often shaped and controlled by magnetic fields and radiative fields. The mathematical model of 3D MHD is governed by a set of equations in Eulerian coordinates [2, 12, 13, 19, 28] given by

\[ \partial_t \rho + \text{div}(\rho u) = 0, \]  
\[ \partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p = \text{div} S + (\nabla \times H) \times H, \]  
\[ H_t - \nabla \times (u \times H) = -\nabla \times (\nu \nabla \times H), \quad \text{div} H = 0, \]  

(1.1a) \hspace{1cm} (1.1b) \hspace{1cm} (1.1c)
\[ \begin{align*}
\partial_t\mathcal{E} + \text{div}((\mathcal{E}' + p)\mathbf{u}) + \text{div}Q &= \text{div}(\mathbf{S}\mathbf{u} + \nu\mathbf{H} \times (\nabla \times \mathbf{H})) \\
+ \text{div}(u(\mathbf{u} \times \mathbf{H}) \times \mathbf{H}),
\end{align*} \]

(1.1d)

Here \( \rho \) is the density of fluid flows, \( p \) is the pressure and \( \nu \) denotes the magnetic diffusion coefficient. \( \mathbf{u} \in \mathbb{R}^3 \) is the velocity and \( \mathbf{H} \in \mathbb{R}^3 \) is the magnetic field. The symbol \( \otimes \) denotes the kronecker tensor product. \( \mathcal{E} \) stands for the total energy expressed by

\[ \mathcal{E} = \rho \left( e + \frac{1}{2} |\mathbf{u}|^2 \right) + \frac{1}{2} |\mathbf{H}|^2 \]

with \( e \) the internal energy, \( \frac{1}{2} |\mathbf{u}|^2 \) is the kinetic energy and \( \frac{1}{2} |\mathbf{H}|^2 \) is the magnetic energy and \( \mathcal{E}' \) is defined as

\[ \mathcal{E}' = \rho \left( e + \frac{1}{2} |\mathbf{u}|^2 \right). \]

The viscous stress tensor

\[ \mathbf{S} = \lambda' (\text{div} \mathbf{u}) I + \mu (\nabla \mathbf{u} + (\nabla \mathbf{u})^\top), \]

where \( \lambda' \) and \( \mu \) are the viscosity coefficients of the flows with \( \lambda' + 2\mu > 0 \). \( I \) is the \( 3 \times 3 \) identity matrix. \( \nabla \mathbf{u}^\top \) is the transpose of the matrix \( \nabla \mathbf{u} \). The pressure \( p \) of the gas obeys the equation of state given by

\[ p = p_G + p_R = R\rho \theta + \frac{a}{3} \theta^4, \]

(1.2)

where \( p_G = R\rho \theta \) is the gaseous (elastic and thermal) pressure with \( R \) being a constant depending on the material properties of the gas and \( \theta \) the temperature of fluid flows, and \( p_R = \frac{a}{3} \theta^4 \) in (1.2) denotes the radiative pressure with \( a > 0 \) being the Stefan–Boltzmann constant. Accordingly, the internal energy \( e \) is

\[ e = e_G + e_R = C_v \theta + \frac{a}{\rho} \theta^4 \]

(1.3)

with \( C_v \) being the heat capacity of the gas at constant volume, and similarly the heat flux \( Q \) is

\[ Q = -\kappa \nabla \theta. \]

The heat conductivity \( \kappa \), motivated by physical experience for some important regimes [32], is taken as that in \([2, 12, 13, 28, 31, 33]\)

\[ \kappa_1 (1 + \theta^q) \leq \kappa = \kappa (\rho, \theta) \leq \kappa_2 (1 + \theta^q), \quad q \geq 0, \]

(1.4)

where \( \kappa_1 \) and \( \kappa_2 \) are positive constants. For a more detailed physical explanation and mathematical derivation of (1.1a)–(1.1d) see the appendix of [2].

In this paper, we will establish the existence and uniqueness of global solution for planar MHD fluid flows in Hölder space. The model equations are coupled with the thermal radiation and are supplemented with free boundary and initial conditions in the setting of (1.4). More precisely, the motion of flows is supposed to be in the \( x \)-direction and uniform in the transverse direction \( (y, z) \), i.e.

\[ \rho = \rho(x, \tau), \quad \theta = \theta(x, \tau), \quad \mathbf{u} = (u, w)(x, \tau), \quad \mathbf{w} = (u_2, u_3), \quad \mathbf{H} = (b_1, b)(x, \tau), \quad \mathbf{b} = (b_2, b_3). \]

Thus, the model equations (1.1a)–(1.1d) are reduced to a system of equations in the Eulerian coordinates as in \([2, 3, 9–11, 25, 28]\), given by

\[ \rho_x + (\rho u)x = 0, \quad x \in \Omega_\tau := (a(\tau), b(\tau)), \quad \tau > 0, \]

(1.5a)

\[ (\rho u)_x + (\rho u^2 + p + \frac{1}{2} |\mathbf{b}|^2)_x = (\lambda u)_x, \]

(1.5b)
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\[(\rho w)_t + (\rho uw - b)_x = (\mu w)_x, \quad (1.5c)\]
\[b_t + (ub - w)_x = (v b)_x, \quad (1.5d)\]
\[(\rho e)_t + (\rho eu)_x - (\kappa \theta)_x = \lambda u^2 + \mu |w|^2 + v |b|^2 - pu, \quad (1.5e)\]

where \(\lambda = \lambda' + 2\mu\) and \(b_1 = 1\).

To supplement system (1.5a)-(1.5e), we impose the initial conditions,

\[(\rho, u, \theta, w, b)(x, 0) = (\rho_0, u_0, \theta_0, w_0, b_0)(x), \quad (1.6)\]

and the free boundary conditions

\[(\lambda u - p, w, b, \theta_0)(f'(\tau), \tau) = 0, \quad f(\tau) = a(\tau), \quad b(\tau), \quad (1.7)\]

where \(f(\tau)\) denotes the free boundary defined by \(f'(\tau) = u(f(\tau), \tau)\).

The planar MHD equations (1.5a)-(1.5e) have been studied by many authors under the various growth constraints on the heat conductivity (1.4). In particular, the existence of global smooth solution for the Dirichlet boundary problem was addressed by Zhang and Xie [33] for \(q > \frac{5}{4}\) in (1.4). In [21], Qin et al proved the existence of global solutions and exponential stability of solutions in \(H^2(i = 1, 2, 4)\) for \(q > 2\) and references cited therein. Similar results were obtained by Ducomet and Zlotnik [5, 6] for one-dimensional compressible Navier–Stokes equations \((w = b = 0)\). More recently, Umehara and Tani showed the global existence of a unique classical solution to a free boundary problem of the one-dimensional radiative and reactive gases. Subsequently, the result was extended by Qin and Yao [20] and Wang and Xie [31].

Another relevant question is that of the existence and uniqueness of global solution to MHD equations (1.5a)-(1.5e) with \(a = 0\) in (1.2) and (1.3). The problem was first discussed for constant transport coefficients by Kawashima and Okada [15] where the smooth global solution near the constant state was investigated for (1.5a)-(1.5e) and (1.6). Later, Hoff and Tsyganov [11] showed the existence of global weak solution for small energy, and they also considered the uniqueness and continuous dependence on initial data of weak solution of the Cauchy problem for (1.5a)-(1.5e). However, it is well known that the corresponding Navier–Stokes equations were solved for large initial data in [17] long time ago. The main reason is due to the presence of magnetic fields and its interaction with hydrodynamic motion. On the other hand, Fan et al [9, 10] showed the global existence of weak solutions to planar MHD compressible flows and vanishing shear limit problem when \(q \geq 1\) in (1.4) or \(\kappa(\rho, \theta) \geq C/\rho\). Similar results can also be found in [2, 3] for \(q \geq 2\). The existence and continuous dependence of large solution is discussed in [3, 10] for the initial boundary problem. One can see from those papers that condition (1.4) plays an important role in the proof of existence.

The aim of this paper is to show the existence and uniqueness of global solution to the initial boundary problem (1.5a)-(1.5e) to (1.7) in the context of (1.2)-(1.4) and Hölder space. In particular, our result is also valid in the case of the transport coefficients \(\lambda, \mu, v, \kappa\) are positive constant, which is interesting and plausible in physics. The existence result is proved by applying the classical Leray–Schauder fixed theorem and the theory of the linear parabolic equations in the framework of Lagrangian coordinates, where it is convenient and equivalent to make various operation. To fulfill the conditions of the Leray–Schauder fixed theorem, some a priori estimates on the solution \((\rho, u, w, b, \theta)\) are necessary. To achieve this, we need to overcome the difficulties caused by magnetic fields and its interaction with the hydrodynamic motion of MHD fluid flows. On the other hand, the higher order nonlinearity in \(\theta\) of \(p, e\) and \(\kappa\) makes the a priori upper bound of temperature more sensitive than those works mentioned above. In our context, a new kind of a priori estimates will be provided with the integrability of temperature and the gradient of density (see lemmas 3.9 and 3.10) dominated by transverse velocity and the magnetic fields while an a priori bound of the second derivatives.
of transverse velocity and magnetic field can be obtained in other ways. This in turn implies the \textit{a priori} estimates of lemmas 3.9 and 3.10. Moreover, we also make use of the theory of the linear parabolic equations, embedding theorem and a delicate interpolation technique to reduce the minimal assumption on heat conductivity to $q \geq 0$ in (1.4) in the process of \textit{a priori} estimates. Therefore our results can be viewed as an extension of the previous works of [2, 5, 6, 26, 31, 33, 34].

Last but not at least, there are also extensively studies on MHD hydrodynamics, which are beyond our ability to reference comprehensively. See for instance [7, 8, 12, 13, 22–24, 29, 30, 35] and references cited therein.

We organize the rest of this paper as follows. In the next section, we begin with introducing the Lagrangian coordinates to transform problem (1.5a)–(1.5e) to (1.7) into an equivalent one (2.2a)–(2.2e) to (2.4). Here, the function spaces and an embedding theorem are also introduced. In section 3, we first formulate our main result and then divide the proof into three subsections. Finally, we summarize the paper in section 4.

2. Preliminary results

In this section, the Lagrangian coordinates as well as some function space and elementary inequalities will be presented.

We first introduce the Lagrangian coordinates to translate the free boundary problem (1.5a)–(1.5e) to (1.7) to fixed one for convenience, which is equivalent under consideration. Let

$$y = \int_{a(t)}^{x} \rho(\xi, t) \, d\xi, \quad t = \tau. \quad (2.1)$$

Then $0 \leq y \leq 1 = \int_{0}^{1} \rho(\xi, t) \, d\xi$ which is the total mass of fluid flows without loss of generality. Then system (1.5a)–(1.5e) canonically becomes

$$v_t = uy, \quad (2.2a)$$

$$u_t = \left( -p - \frac{1}{2} |b|^2 + \frac{\lambda uy}{v} \right) y, \quad (2.2b)$$

$$w_t = \left( b + \frac{\mu w}{v} \right) y, \quad (2.2c)$$

$$(v b)_t = \left( w + \frac{\nu b}{v} \right) y, \quad (2.2d)$$

$$e_t = \left( \frac{\kappa}{v} \theta \right) y + \left( -p + \frac{\lambda}{v} u_y \right) u_y + \frac{\mu |w|^2}{v} + \frac{\nu |b|^2}{v}, \quad (2.2e)$$

where $v = 1/\rho$ is the specific volume.

The initial boundary conditions are

$$(v, u, \theta, w, b)(y, 0) = (v_0(y), u_0(y), \theta_0(y), w_0(y), b_0(y)), \quad (2.3)$$

and boundary conditions

$$\left( \lambda \frac{u_t}{v} - p, w, \theta \right)(d, t) = 0, \quad \quad d = 0, 1. \quad (2.4)$$

In the following, some function spaces (see for instance [18]) will be listed out. Assume that $\Omega := (0, 1)$ and $m \geq 0$ be an integer. Then the symbol $C^{m+s}(\Omega)$ represents the spaces of functions which are H"older continuous with exponent $s$ up to order $m$ and its norm is defined by

$$|u|_{m+s} := \sum_{k=0}^{m} |D^k u(x)| + \sup_{x, y \in \Omega, x \neq y} \frac{|D^m u(x) - D^m u(y)|}{|x - y|^s},$$
where \( D = \frac{d}{dx} \). Likewise, we can also define

\[
|u|^{(0)} := \sup_{(x, t) \in Q_T} |u(x, t)|,
\]

\[
|u|^{(\sigma)}_{1} := \sup_{(x, t), (y, \tau) \in Q_T, x \neq y} \frac{|u(x, t) - u(y, \tau)|}{|x - y|^\sigma},
\]

\[
|u|^{(\sigma)}_{T} := \sup_{(x, t), (y, s) \in Q_T, t \neq s} \frac{|u(x, t) - u(x, s)|}{|t - s|^\sigma},
\]

where \( Q_T = (0, 1) \times (0, T) \) and \( 0 < \sigma, \sigma' \leq 1 \). Then the spaces \( C^{\sigma, \sigma'} \) denotes the Banach space of functions on \( Q_T \) which are uniformly Hölder continuous with exponent \( \sigma \) in \( x \) and \( \sigma' \) in \( t \) while its norm is

\[
|u|_{\sigma, \sigma'} := |u|^{(0)} + |u|^{(\sigma)}_{1} + |u|^{(\sigma')}_{T}.
\]

Particularly, the space \( C^{2\alpha, 1+\alpha/2}_{x,t} \) for \( 0 < \alpha < 1 \) means that \( (u_{xx}, u_{t}) \in (C^{\alpha, \alpha/2}_{x,t}(Q_T))^2 \), and its norm is defined by

\[
|u|_{2\alpha, 1+\alpha/2} := |u|^{(0)} + |u|^{(\alpha)}_{1} + |u_{xx}|_{\alpha, \alpha/2} + |u_{t}|_{\alpha, \alpha/2}.
\]

For brevity, the norm of \( C^{\alpha}[0, 1] \) and \( C^{\alpha, \alpha/2}(Q_T) \) will be denoted by \( \| \cdot \|_\alpha \) and \( \| \cdot \|_{\alpha, \alpha/2} \).

In the proof of existence, we will use repeatedly the embedding \( W^{1, 1}(0, 1) \hookrightarrow L^\infty(0, 1) \) (see [1] for details):

\[
|u(x)|^{(1)} \leq \int_0^1 (|u| + |u_x|) \, dx, \quad \forall u \in W^{1, 1}(0, 1).
\]

Moreover, the form of interpolation inequality employed in our context is the same as lemma 3.3 in [4].

**Lemma 2.1.** Let \( \phi(x, t) \) be a function on \( Q_T \) satisfying the following conditions:

(a) \( \phi \) is uniformly Hölder continuous in \( t \) with exponent \( \gamma \in (0, 1) \), i.e. \( |\phi(x, t) - \phi(x, \tau)| \leq C_1 |t - \tau|^{\gamma}, 0 \leq x \leq 1, 0 \leq \tau \leq t \leq T, \) where \( C_1 \) is a positive constant independent of \( t \).

(b) \( \phi_1(x, t) \) exists, and is uniformly Hölder continuous in \( x \) with exponent \( \delta \in (0, 1) \), i.e. \( |\phi_1(x, \tau) - \phi_1(x, t)| \leq C_2 |z - \tau|^{\delta}, 0 \leq x \leq T, \) where \( C_2 \) is a positive constant independent of \( t \).

Then \( \phi_1(x, t) \) is also uniformly Hölder continuous in \( t \) with exponent \( \xi = \frac{\delta \gamma}{1 + \gamma - \delta}, \) i.e. \( |\phi_1(x, t) - \phi_1(x, \tau)| \leq C_3 |t - \tau|^{\xi}, 0 \leq x \leq 1, 0 \leq \tau \leq t \leq T, \) where \( C_3 \) is a positive constant independent of \( t \).

Its proof can be found in [18, chapter II, lemma 3.1].

With these preliminaries, we are now ready to state the main result for system (2.2a)–(2.2e) to (2.4).

**3. Main result**

Let us first formulate our main theorem.

**Theorem 3.1.** Let \( q \geq 0 \) in (1.4), and let \( \lambda, \mu, \nu \) be positive constants. Assume that the initial data \( v_0(y), u_0(y), w_0(y), b_0(y), \theta_0(y) \) satisfy

\[
C_0^{-1} \leq v_0(y), \quad \theta_0(y) \leq C_0,
\]

for some constant \( C_0 \geq 1 \) and

\[
(v_0(y), u_0(y), w_0(y), b_0(y), \theta_0(y)) \in C^{1+\alpha}(\Omega) \times C^{2\alpha}(\Omega)^6.
\]
for $\alpha \in (0, 1)$. Then there exists a unique solution $(v, u, w, b, \theta)$ of the initial boundary value problem (2.2a)–(2.2e) to (2.4) such that

$$(v, v_\gamma, v_t) \in C^{\alpha,\alpha/2}(Q_T)^3$$

and

$$(u, w, b, \theta) \in C^{2+\alpha, 1+2/\alpha}(Q_T)^6,$$

where $\Omega = (0, 1)$ and $Q_T = (0, 1) \times (0, T)$ for any fixed $T > 0$.

**Remark 3.1.** In particular, the result of global existence is also valid when $\lambda, \mu, \nu, \kappa$ are positive constants, which has not been studied in previous works [21, 33]. Moreover, the heat conductivity can also be taken as $\kappa = \kappa_1 + \kappa_2 \theta^\gamma$ instead of the form of (1.4), where $\kappa_1$ and $\kappa_2$ are positive constants. Therefore, our results can be viewed as an extension of [2, 21, 26, 28, 33, 34].

The existence proof of theorem 3.1 is based on the classical Leray–Schauder fixed theorem (see for instance theorem 3.2), which is also employed by Dafermos and Hsiao [4] and Kawohl [16]. Following their strategy, the complete proof is divided into three subsections. Global a priori estimates on solutions to system (2.2a)–(2.2e) to (2.4) will first be shown so as to satisfy the conditions of theorem 3.2. Then, the Hölder estimates of solutions will be obtained in view of the theory of parabolic equations. Finally, the existence and uniqueness of solutions will proved completely.

### 3.1. Global a priori estimates

In this section, we will establish complete global a priori estimates of the solution $(v, u, w, b, \theta)$ on $[0, 1] \times [0, T]$ with $T > 0$. We first obtain the upper and lower bounds on density based on energy estimates. Then, we will obtain the upper and lower bounds on temperature, which are essential to guarantee all a priori estimates satisfying $q \geq 0$ in (1.4). At last, the evaluation on the derivatives of $(v, u, w, b, \theta)$ are also achieved by delicate interpolation techniques and embedding theorem. In the following, we will denote $C(C(T))$ generic positive constant, which may depend on the initial data (fixed time $T$).

#### 3.1.1. Upper and lower bounds of density

The following lemma give the basic energy estimates (see [3]).

**Lemma 3.1.** We have

$$\int_0^1 \left( e + \frac{1}{2} (u^2 + |w|^2 + v|b|^2) \right) \, dy \leq C. \quad (3.1)$$

**Proof.** Using the boundary conditions (2.4), we derive from (2.2a) to (2.2e) that

$$\frac{d}{dr} \int_0^1 \left( e + \frac{1}{2} (u^2 + |w|^2 + v|b|^2) \right) \, dy = 0,$$

which implies (3.1). \qed

And then, we immediately arrive at the uniform upper bound of density for all time.

**Lemma 3.2.** Suppose the hypotheses of theorem 3.1 are valid, then

$$v(y, t) \geq C, \quad \forall (y, t) \in (0, 1) \times (0, T),$$

(3.2)
\[
\rho(y, t) \leq C^{-1}, \quad \forall (y, t) \in (0, 1) \times (0, T)
\] (3.3)
and
\[
\int_0^1 (\theta^4 + |b|^2) \, dy \leq C, \quad \forall t \in (0, T).
\] (3.4)

**Proof.** Equation (2.2b) can be rewritten as
\[
u_t = (\lambda(\ln v)_t - p - \frac{1}{2}|b|^2)_y.
\]
Integrating it over \([0, y] \times [0, t]\), we obtain
\[
\lambda \ln v = \lambda \ln v_0(y) + \int_0^t (p + \frac{1}{2}|b|^2) \, ds + \int_0^y (u - u_0) \, dx.
\] (3.5)
By [lemma 3.1](#), we have
\[
\left| \int_0^y (u - u_0) \, dx \right| \leq C \int_0^1 u^2 \, dy + C \int_0^1 u_0^2 \, dy \leq C,
\]
which along with (3.5) imply that
\[
\ln v \geq -C,
\]
or equivalently
\[
\rho(y, t) \leq e^C.
\]
On the other hand, it follow from (3.1) and (3.3) that
\[
\int_0^1 \theta^4 \, dy = \int_0^1 \rho v \theta^4 \, dy \leq C \int_0^1 v \theta^4 \, dy \leq C \int_0^1 e \, dy \leq C.
\]
Similarly, one has
\[
\int_0^1 |b|^2 \, dy = \int_0^1 \rho v |b|^2 \, dy \leq C \int_0^1 v |b|^2 \, dy \leq C.
\]
This ends the proof. \(\Box\)

**Lemma 3.3.** It holds that
\[
0 < \int_0^1 v \, dy \leq C(1 + t).
\] (3.6)

**Proof.** Integrating \([0, y]\) over (2.2b) yields
\[
\int_0^y u_t \, dx = \frac{\lambda u_y}{v} - (p + \frac{1}{2}|b|^2),
\]
which implies by multiplying \(v\) and integrating with respect to \(y\) and \(t\)
\[
\lambda \int_0^1 v \, dy = \lambda \int_0^1 v_0(y) \, dy + \int_0^t \int_0^1 \left( p + \frac{1}{2}|b|^2 \right) \, dy \, ds + \int_0^t \int_0^1 v \left( \int_0^y u_t \, dx \right) \, dy \, ds.
\] (3.7)
For the second term on the right-hand side of (3.7), it is deduced that
\[
\int_0^t \int_0^1 \left( p + \frac{1}{2}|b|^2 \right) \, dy \, ds \leq C \int_0^t \int_0^1 (e + v|b|^2) \, dy \, ds \leq C(1 + t).
\] (3.8)
On the other hand, we obtain from (2.2a) by integration
\[ v(y, t) = v_0(y) + \int_0^t u_y(y, s) \, ds. \]
Thus, it leads the third term on the right-hand side of (3.7) to
\[
\int_0^t \int_0^1 v \left( \int_0^y u_y \, dx \right) \, dy \, ds
= \int_0^1 \left( v_0(y) + \int_0^t u_y(y, s) \, ds \right) \left( \int_0^y u \, dx \right) \, dy
+ \int_0^t \int_0^1 u^2 \, dy \, ds
- \int_0^1 v_0(y) \left( \int_0^y u_0(x) \, dx \right) \, dy
= \int_0^1 v_0(y) \left( \int_0^y (u - u_0(x)) \, dx \right) \, dy
+ \int_0^t \int_0^1 u^2 \, dy \, ds
- \int_0^1 u(y, t) \left( \int_0^t u(y, s) \, ds \right) \, dy,
\]
which along with (3.7) and (3.8) ends the proof.

Lemma 3.4. One has
\[ U(t) + \int_0^t V(\tau) \, d\tau \leq C(T), \quad \forall t \in (0, T), \tag{3.9} \]
where
\[ U(t) = \int_0^1 \left( C_v(\theta - 1 - \log \theta) + R(v - 1 - \log v) \right) \, dy, \]
and
\[ V(t) = \int_0^1 \left( \frac{\lambda u_y^2}{v\theta} + \frac{\mu|w_y|^2}{v\theta} + \frac{v|b_y|^2}{v\theta} + \frac{\kappa\theta_y^2}{v\theta^2} \right) \, dy. \]

**Proof.** According to the expression of \( p \) and \( e \), we thereby compute
\[ e_0 \theta_t + \theta p_0 u_y = \frac{\lambda u_y^2}{v} + \frac{\mu|w_y|^2}{v} + \frac{v|b_y|^2}{v} + \left( \frac{\kappa}{v\theta} \right)_y, \]
which implies that
\[
\frac{d}{dt} \int_0^1 \left( C_v \log \theta + R \log v + \frac{4}{5} \alpha v\theta^3 \right) \, dy
= \int_0^1 \left( \frac{\lambda u_y^2}{v\theta} + \frac{\mu|w_y|^2}{v\theta} + \frac{v|b_y|^2}{v\theta} + \frac{\kappa\theta_y^2}{v\theta^2} \right) \, dy.
\]
Integrating it over \((0, 1) \times (0, t)\) yields
\[ U(t) + \int_0^t V(\tau) \, d\tau \leq C \left( 1 + \int_0^1 v\theta^3 \, dy \right). \]
On the other hand, one has by (3.6)
\[ \int_0^1 v\theta^3 \, dy \leq \left( \left( \int_0^1 v\theta^4 \, dy \right)^\frac{1}{2} \left( \int_0^1 v \, dy \right)^\frac{1}{2} \right)^2 \leq C(T), \]
which ends the proof in conjunction with (3.1).

Furthermore, we can deduce some a priori estimates for \( \theta \) and \( b \).
Lemma 3.5. For all $0 < t < T$, we have
\[ \int_0^t \max_{[0,1]} \theta^{q+4}(y, s) \, ds \leq C(T), \quad q \geq 0 \] (3.10)
and
\[ \int_0^t |b|_{L^\infty([0,1])}^2 \, ds \leq C(T). \] (3.11)

Proof. There exists $y(t)$ by the mean value theorem such that
\[ \theta(y, t) = \int_0^1 \theta \, dy, \]
where $y(t) \in [0, 1]$ for each $t \in [0, T]$. Furthermore, it yields by Hölder inequality
\begin{align*}
\theta(y, t)^{\frac{q+4}{q+2}} &= \left( \int_0^1 \theta \, dy \right)^{\frac{q+4}{q+2}} + \frac{q+4}{2} \int_{y(t)}^1 \theta(\xi, t)^{\frac{q+4}{q+2} - \frac{1}{2}} b_\xi(\xi, t) \, d\xi \\
&\leq C \left( 1 + \int_0^1 \frac{\kappa_1^2 \theta_{\xi]}{v^2 \theta} \, dy \right) \leq C \left[ 1 + \left( \int_0^1 \frac{v\theta^{q+4}}{\kappa} \, dy \right)^{\frac{2}{q+4}} V(t)^{\frac{q+4}{q+2}} \right] \\
&\leq C \left[ 1 + \left( \int_0^1 \frac{v\theta^{q+4}}{1 + \theta^{q+4}} \, dy \right)^{\frac{2}{q+4}} V(t)^{\frac{q+4}{q+2}} \right] \leq C \left[ 1 + \left( \int_0^1 v\theta^{q+4} \, dy \right)^{\frac{2}{q+4}} V(t)^{\frac{q+4}{q+2}} \right] \\
&\leq C (1 + V(t)^{\frac{q+4}{q+2}}),
\end{align*}
where $V(t)$ is the same as that in lemma 3.4. Then taking square on both sides of the above inequality, we can obtain (3.10) by integrating it over $[0, t]$ with the help of (3.1) and (3.9).

The proof of (3.11) is directly deduced from (3.1) and (3.10). Indeed, recalling the boundary conditions (2.4), we find that
\begin{align*}
|b|^2 &= 2 \int_0^1 b \cdot b_\xi(\xi, t) \, d\xi \\
&\leq C \int_0^1 \frac{v|b_\xi|^2}{\theta} \, dy + C \int_0^1 v\theta|b|^2 \, dy,
\end{align*}
hence
\[ \int_0^t |b|_{L^\infty([0,1])}^2 \, ds \leq C \int_0^t \int_0^1 \frac{v|b_\xi|^2}{\theta} \, dy \, ds + C \int_0^t \max_{[0,1]} \theta \left( \int_0^1 v|b|^2 \, dy \right) \, ds \leq C(T). \]
This completes the proof. \qed

Given these preliminary processes, the lower bound on the density can be inferred in the following lemma.

Lemma 3.6. One has
\[ \rho(y, t) \geq C(T). \] (3.12)
Lemma 3.7. Recalling (3.5) and using (3.10) and (3.11), we find that
\[ \lambda \ln v = \lambda \ln v_0(y) + \int_0^t \left( p + \frac{1}{2} |b|^2 \right) \, dt + \int_0^1 (u - u_0) \, dx \]
\[ \leq C(T) + C \int_0^t \max_{[0,1]} \theta^4 \, ds + C \int_0^t \|b\|^2_{L^\infty([0,1])} \, ds + C \int_0^1 u^2 \, dy \]
\[ \leq C(T), \]
which ends the proof.

3.1.2. Upper and lower bounds on temperature. In this section, the upper and lower bounds on temperature are derived. First, we will deduce a priori estimates of the derivatives of \((u, w, b)\) in advance.

Lemma 3.8. One has
\[ \int_0^t \int_0^1 (u_x^2 + |w_x|^2 + |b_x|^2) \, dy \, ds \leq C(T). \] (3.13)

Proof. Equality (2.2e) can be rewritten as
\[ e_t + pu_y = \frac{\lambda u_y^2}{v} + \frac{\mu |w|^2}{v} + \frac{v |b|^2}{v} + \left( \frac{k}{v} \theta_y \right) y. \] (3.14)
Then integrating (3.14) over \([0, 1] \times [0, t]\), we obtain from (3.4) and (3.10)
\[ \frac{1}{2} \int_0^t \int_0^1 \lambda u_y^2 \, dy \, ds \leq \int_0^t (e - e_0) \, dy + \int_0^t \int_0^1 pu_y \, dy \, ds \]
\[ \leq C + \frac{1}{2} \int_0^t \int_0^1 \lambda u_y^2 \, dy \, ds + C \int_0^t \int_0^1 p^2 \, dy \, ds \]
\[ \leq C + \frac{1}{2} \int_0^t \int_0^1 \lambda u_y^2 \, dy \, ds + C \int_0^t \max_{[0,1]} \theta^4 \left( \int_0^1 \theta^4 \, dy \right) \, ds \]
\[ \leq C(T) + \frac{1}{2} \int_0^t \int_0^1 \lambda u_y^2 \, dy \, ds, \]
which ends the proof according to (3.10) and (3.12).

Lemma 3.8. For all \(t \in [0, T]\), the following inequalities hold
\[ \int_0^t \int_0^1 |b \cdot b_x|^2 \, dy \, ds \leq C(T) \] (3.15)
and
\[ \int_0^t \int_0^1 |b|^8 \, dy \, ds \leq C(T). \] (3.16)

Proof. By (3.4) and using the embedding \(W^{1,1}(0, 1) \hookrightarrow L^\infty(0, 1)\), one has
\[ \int_0^t \int_0^1 |b|^8 \, dy \, ds \leq \int_0^t \max_{[0,1]} |b|^6 \left( \int_0^1 |b|^2 \, dy \right) \, ds \]
\[ \leq \int_0^t \int_0^1 |b|^6 \, dy \, ds + C \int_0^t \int_0^1 |b|^2 |b \cdot b_x| \, dy \, ds \]
\[ \leq \frac{1}{2} \int_0^t \int_0^1 |b|^8 \, dy \, ds + C \int_0^t \int_0^1 \frac{|b \cdot b_x|^2}{v} \, dy \, ds + C(T), \]
which implies that
\[
\int_0^t \int_0^1 |b| \frac{8}{dy} \, ds \leq C \int_0^t \int_0^1 \frac{|b|^2 |b_y|^2}{v} \, dy \, ds + C(T).
\] (3.17)

Multiplying by \(4|b|^2b\) on both sides of (2.2d), one has
\[
(v|b|^8)_y = \left(\frac{v^2 b_y + w}{v}\right) \cdot 4|b|^2b - 3v \varepsilon |b|^4,
\]
which follows from (3.11), (3.13) and Young inequality with \(\varepsilon\) that
\[
\int_0^1 v|b|^4 \, dy + 12v \int_0^1 \int_0^1 \frac{|b|^2 |b_y|^2}{v} \, dy \, ds
\]
\[
= \int_0^1 (v|b|^4(y,0) \, dy - 12 \int_0^1 \int_0^1 |b|^2 w \cdot b_y \, dy \, ds - 3 \int_0^t \int_0^1 u_y |b|^4 \, dy \, ds
\]
\[
\leq \varepsilon \int_0^1 \int_0^1 \frac{|b|^2 |b_y|^2}{v} \, dy \, ds + C(\varepsilon) \int_0^t \int_0^1 \frac{|b|^2 |b|^2}{v} \, dy \, ds + C(T)
\]
\[
+ \varepsilon \int_0^1 \int_0^1 |b|^8 \, dy \, ds + C(\varepsilon) \int_0^1 \int_0^1 u_y^2 \, dy \, ds + C(T)
\]
\[
\leq \varepsilon \int_0^1 \int_0^1 \frac{|b|^2 |b_y|^2}{v} \, dy \, ds + \varepsilon \int_0^1 \int_0^1 |b|^8 \, dy \, ds + C(T),
\]
and then we deduce both (3.15) and (3.16) by taking sufficiently small \(\varepsilon\) if (3.17) is substituted into the above inequality. \(\square\)

In the following, we will verify that the temperature is dominated by transverse velocity and magnetic fields.

**Lemma 3.9.** One has
\[
\int_0^1 \theta^8 \, dy + \int_0^1 \int_0^1 \kappa \theta^8 \theta_y^2 \, dy \, ds
\]
\[
\leq C(T) + C(T) \left( \int_0^1 \int_0^1 w_{yy}^2 \, dy \, ds \right)^{\frac{1}{2}} + C(T) \left( \int_0^1 \int_0^1 b_{yy}^2 \, dy \, ds \right)^{\frac{1}{2}}.
\]

**Proof.** Multiplying by \(\theta^4\) on both sides of (2.2e), and then integrating it over \([0,1] \times [0,t]\) leads to
\[
\int_0^1 \theta^8 \, dy + \int_0^1 \int_0^1 \kappa \theta^8 \theta_y^2 \, dy \, ds
\]
\[
\leq C \int_0^1 \int_0^1 \left( \theta^8|u_y| + \theta^4 u_y^2 \right) \, dy \, ds + C \int_0^1 \int_0^1 \left( \frac{\mu w_x^2}{v} + \frac{v b_x^2}{v} \right) \cdot \theta^4 \, dy \, ds
\]
\[
\leq C + C \int_0^1 \int_0^1 \left( \theta^{12} + |u_y|^3 \right) \, dy \, ds + C \int_0^1 \left( ||w_x||^2_{L^\infty} + ||b_x||^2_{L^\infty} \right) \int_0^1 \theta^4 \, dy \, ds
\]
\[
\leq C + C \int_0^1 \int_0^1 \left( \theta^{12} + |u_y|^3 \right) \, dy \, ds
\]
\[
+ C \left( \int_0^1 \int_0^1 w_{yy}^2 \, dy \, ds \right)^{\frac{1}{2}} + C \left( \int_0^1 \int_0^1 b_{yy}^2 \, dy \, ds \right)^{\frac{1}{2}}.
\] (3.18)
On the other hand, if let $h = \int_0^y u \, d\xi$, it satisfies by $(2.2b)$

$$h_t = \frac{\lambda}{v} h_{yy} - \left( p + \frac{1}{2} |b|^2 \right),$$

$$h|_{t=0} = h_0(y),$$

$$h|_{y=0,1} = 0.$$  \hfill (3.19)

The standard $L^p$ estimates of solutions to the linear parabolic problem yield

$$\int_0^1 \int_0^1 |u_y|^3 \, dy \, dx = \int_0^1 \int_0^1 |h_{yy}|^3 \, dy \, dx \leq C \left( 1 + \int_0^1 \int_0^1 (p^3 + |b|^6) \, dy \, dx \right)^{\frac{1}{2}},$$

and then, we obtain by (3.18)

$$\int_0^1 \theta^8 \, dy + \int_0^1 \int_0^1 \kappa \theta^4 \theta^2 \, dy \, dx \leq C + C \left( \int_0^1 \int_0^1 w^2_{yy} \, dy \, dx \right)^{\frac{1}{2}} + C \left( \int_0^1 \int_0^1 b^2_{yy} \, dy \, dx \right)^{\frac{1}{2}}$$

$$+ C \left( \int_0^1 \int_0^1 w^2_{yy} \, dy \, dx \right)^{\frac{1}{2}} + C \left( \int_0^1 \int_0^1 b^2_{yy} \, dy \, dx \right)^{\frac{1}{2}} \leq C(T) + C \int_0^1 \max_{[0,1]} \theta^2 \int_0^1 \theta^3 \, dy \, ds$$

$$+ C \left( \int_0^1 \int_0^1 w^2_{yy} \, dy \, dx \right)^{\frac{1}{2}} + C \left( \int_0^1 \int_0^1 b^2_{yy} \, dy \, dx \right)^{\frac{1}{2}}.$$

The proof is completed using Gronwall’s inequality and lemma 3.5. \hfill □

The following lemma also declare a relationship of density and velocity, as well as magnetic fields.

**Lemma 3.10.** For any $\varepsilon > 0$, it satisfies that

$$\int_0^1 v_y^2 \, dy + \int_0^1 \theta v_y^2 \, dy \, ds \leq C(T) + \varepsilon \left( \int_0^1 \int_0^1 w^2_{yy} \, dy \, dx \right)^{\frac{1}{2}} + \varepsilon \left( \int_0^1 \int_0^1 b^2_{yy} \, dy \, dx \right)^{\frac{1}{2}}.$$

**Proof.** We can rewrite equation $(2.2b)$ as follows:

$$\left( u - \frac{\lambda v_y}{v} \right)_t = - \left( p + \frac{1}{2} |b|^2 \right)_y.$$  \hfill (3.20)
Multiplying by \((u - \frac{\lambda}{v}v_y)\) on both sides of (3.20) and integrating it over \((0, 1) \times (0, t)\), we find that
\[
\frac{1}{2} \int_0^1 \left(u - \frac{\lambda}{v}v_y\right)^2 \, dy + \int_0^t \int_0^1 \frac{\lambda R \theta}{v^3} v_y^2 \, dy \, ds
\]
\[
= \frac{1}{2} \int_0^1 \left(u - \frac{\lambda}{v}v_y\right)^2 \, dy + \int_0^t \int_0^1 \frac{R \theta uv_y}{v^2} \, dy \, ds
\]
\[
- \int_0^t \int_0^1 \left[R\left(\frac{v + 4a}{3}\right) \theta_y + b \cdot b_y\right] \left(u - \frac{\lambda}{v}v_y\right) \, dy \, ds. \tag{3.21}
\]
To complete the proof, it only remains to evaluate all terms on the right-hand side of (3.21).

By lemma 3.2 and using Young inequality with \(\varepsilon\), we obtain
\[
\int_0^t \int_0^1 \frac{R \theta uv_y}{v^2} \, dy \, ds \leq \varepsilon \int_0^t \int_0^1 \theta v_y^2 \, dy \, ds + C \int_0^t \int_0^1 \max_{[0,1]} \theta \cdot \left(\int_0^1 u^2 \, dy\right) \, ds
\]
\[
\leq \varepsilon \int_0^t \int_0^1 \theta v_y^2 \, dy \, ds + C \int_0^t \int_0^1 \max_{[0,1]} \theta \, dy \, ds + C(T).
\tag{3.22}
\]

By lemma 3.4, we have
\[
\left| \int_0^1 \int_0^1 \left[R\left(\frac{v + 4a}{3}\right) \theta_y\right] \left(u - \frac{\lambda}{v}v_y\right) \, dy \, ds \right| \leq C \int_0^t \int_0^1 \frac{\kappa \theta^2}{\theta^2} \, dy \, ds
\]
\[
+ \varepsilon \int_0^t \int_0^1 \kappa \theta^3 \theta_y^2 \, dy \, ds + \int_0^t \int_0^1 \frac{\theta^2 + \theta^3}{\kappa} \left(u - \frac{\lambda}{v}v_y\right)^2 \, dy \, ds
\]
\[
\leq C + \varepsilon \int_0^t \int_0^1 \kappa \theta^3 \theta_y^2 \, dy \, ds + \int_0^t \int_0^1 \max_{[0,1]} \theta^2 + \theta^3 \cdot \int_0^1 \left(u - \frac{\lambda}{v}v_y\right)^2 \, dy \, ds. \tag{3.23}
\]

In addition, by (3.15), we have
\[
\left| \int_0^t \int_0^1 b \cdot b_y \cdot \left(u - \frac{\lambda}{v}v_y\right) \, dy \, ds \right| \leq C \int_0^t \int_0^1 |b \cdot b_y|^2 \, dy \, ds + C \int_0^t \int_0^1 \left(u - \frac{\lambda}{v}v_y\right)^2 \, dy \, ds
\]
\[
\leq C(T) + C \int_0^t \int_0^1 \left(u - \frac{\lambda}{v}v_y\right)^2 \, dy \, ds. \tag{3.24}
\]

Finally, substituting (3.22)–(3.24) into (3.21) leads to
\[
\int_0^1 \left(u - \frac{\lambda}{v}v_y\right)^2 \, dy + \int_0^t \int_0^1 \theta v_y^2 \, dy \, ds \leq C(T) + \varepsilon \int_0^t \int_0^1 \kappa \theta^3 \theta_y^2 \, dy \, ds
\]
\[
+ C \int_0^t \left(\int_0^1 \frac{\theta^2 + \theta^3}{\kappa} + 1\right) \cdot \int_0^1 \left(u - \frac{\lambda}{v}v_y\right)^2 \, dy \, ds,
\]
and the proof is completed by invoking Gronwall’s inequality and lemma 3.9.

Combining lemmas 3.9 and 3.10, we can give a priori estimates of velocity and magnetic fields, which play an important role in the case of the constant heat conductivity.

**Lemma 3.11.** It holds that
\[
|b|_L^{\infty((0,1) \times (0,T))} + \int_0^1 \left(|b_y|^2 + |w_y|^2\right) \, dy
\]
\[
+ \int_0^t \int_0^1 \left(|b|^4 + |w|^4 + |b_{yy}|^2 + |w_{yy}|^2\right) \, dy \, ds \leq C(T). \tag{3.25}
\]
Proof. Multiplying (2.2c) by $w_{yy}$ and integrating it over $[0, 1] \times [0, t]$, one has

$$\frac{1}{2} \int_0^1 |w_y|^2 \, dy = \frac{1}{2} \int_0^1 |w_y(y, 0)|^2 \, dy - \int_0^t \int_0^1 \left( b + \frac{\mu w_y}{v} \right)_y \cdot w_{yy} \, dy \, ds \leq C - C(T) \int_0^t \int_0^1 |w_{yy}|^2 \, dy \, ds + C \int_0^t \int_0^1 \left( |b_y| + |v_y| |w_y| \right) |w_{yy}| \, dy \, ds$$

$$\leq C - \frac{3C(T)}{4} \int_0^t \int_0^1 |w_{yy}|^2 \, dy \, ds + C(T) \int_0^t \int_0^1 |b_y|^2 \, dy \, ds$$

$$+ C(T) \int_0^t \max_{[0,1]} |w_y|^2 \left( \int_0^1 v_y^2 \, dy \right) \, ds$$

$$\leq C - \frac{3C(T)}{4} \int_0^t \int_0^1 |w_{yy}|^2 \, dy \, ds + \varepsilon \int_0^t \int_0^1 b_{yy}^2 \, dy \, ds.$$
which implies that
\[
\int_0^1 |b_y|^2 \, dy + \int_0^t \int_0^1 |b_{yy}|^2 \, dy \, ds \leq C(T) + \varepsilon \int_0^t \int_0^1 w_{xy}^2 \, dy \, ds.
\]
This together with (3.26) leads to
\[
\int_0^1 \left( |w_y|^2 + |b_y|^2 \right) \, dy + \int_0^t \int_0^1 \left( |w_{yy}|^2 + |b_{yy}|^2 \right) \, dy \, ds \leq C(T).
\]
By the above estimates and using the embedding theorem, we obtain
\[
\int_0^t \int_0^1 |w_y|^4 \, dy \, ds \leq \int_0^t \max_{[0,1]} \left| w_y \right|^2 \left( \int_0^1 |w_y|^2 \, dy \right) \, ds
\leq C(T) \int_0^t \int_0^1 \left( |w_y|^2 + |w_{yy}|^2 \right) \, dy \, ds
\leq C(T).
\]
Similarly, we have
\[
\int_0^t \int_0^1 |b_y|^4 \, dy \, ds \leq \int_0^t \max_{[0,1]} |b_y|^2 \left( \int_0^1 |b_y|^2 \, dy \right) \, ds
\leq C(T) \int_0^t \int_0^1 \left( |b_y|^2 + |b_{yy}|^2 \right) \, dy \, ds
\leq C(T).
\]
This completes the proof of the lemma.\qed

With (3.25) in hand, we can reexamine lemmas 3.9 and 3.10 to obtain the following corollary.

**Corollary 3.12.** We have
\[
\int_0^1 v_y^2 \, dy + \int_0^t \int_0^1 \theta v_y^2 \, dy \, ds \leq C(T) \tag{3.27}
\]
and
\[
\int_0^t \max_{[0,1]} \theta^{q+13} \, ds + \int_0^1 \theta^8 \, dy + \int_0^t \int_0^1 \kappa \theta^3 \theta_y^2 \, dy \, ds \leq C(T). \tag{3.28}
\]

Similarly, using $L^p$ estimates of solutions to the linear parabolic problem again, we obtain from (3.16) and (3.19)
\[
\int_0^t \int_0^1 |u_y|^4 \, dy \, ds = \int_0^t \int_0^1 |h_{yy}|^4 \, dy \, ds
\leq C \left( 1 + \int_0^t \int_0^1 \left( p^4 + |b|^8 \right) \, dy \, ds \right)
\leq C \left( 1 + \int_0^t \int_0^1 \theta^{16} \, dy \, ds \right)
\leq C \left( 1 + \int_0^t \max_{[0,1]} \theta^8 \int_0^1 \theta^8 \, dy \, ds \right)
\leq C(T). \tag{3.29}
\]
In order to obtain the upper bound on temperature, we also need to establish higher order a priori estimates of \((v, u, \theta, w, b)\). Thus, we introduce some auxiliary new variables motivated by [14, 16].

\[
X := \int_0^t \int_0^1 (1 + \theta^q) \theta^2 \, dy \, ds,
\]
\[
Y := \max_{0 \leq t \leq T} \int_0^1 (1 + \theta^2q) \theta^2 \, dy,
\]
\[
Z := \max_{0 \leq t \leq T} \int_0^1 u_v^2 \, dy.
\]

By interpolation and embedding theorem, it follows from (3.32) that
\[
|uy|_0 \leq C(1 + Z^{q+5}).
\]

**Lemma 3.13.** Under the assumptions of theorem 3.1, we have
\[
\max_{[0,1] \times [0,t]} \theta(y,s) \leq C \frac{Y}{\sqrt{\theta}} + CY^{1/2},
\]
for \((y, t) \in (0, 1) \times (0, T)\).

**Proof.** By the embedding theorem, we deduce that
\[
\max_{[0,1]} \theta^q(y, t) \leq C \int_0^1 \theta^q \, dy + C \int_0^1 (1 + \theta)^q \theta^q \, dy.
\]

By (3.28), the Hölder inequality and the Young inequality, we have
\[
\max_{[0,1]} \theta^q(y, t) \leq C \max_{[0,1]} \theta^{q+1} \int_0^1 \theta^2 \, dy + C \int_0^1 (1 + \theta)^q \theta^q \, dy
\]
\[
\leq \frac{1}{2} \max_{[0,1]} \theta^q + C \left( \int_0^1 (1 + \theta)^q \theta^q \, dy \right)^{\frac{1}{2}} \left( \int_0^1 (1 + \theta)^q \theta^q \, dy \right)^{\frac{1}{2}} + C
\]
\[
\leq \frac{1}{2} \max_{[0,1]} \theta^q + C(T) \left( \int_0^1 (1 + \theta)^q \theta^q \, dy \right)^{\frac{1}{2}} + C(T)
\]
\[
\leq \frac{1}{2} \max_{[0,1]} \theta^q + CY^{1/2} + C(T).
\]
The proof is complete.

**Lemma 3.14.** We have
\[
X + Y \leq C(T) \left( 1 + Z^{q+5} \right).
\]

**Proof.** We introduce the function as in [16, 26]
\[
K(v, \theta) := \int_0^\theta \frac{\kappa(v, \xi)}{v} \, d\xi.
\]
The simple calculation leads to
\[
K_v = \frac{\kappa}{v} \theta_v + K_v u_y,
\]
\[
K_{yy} = \left( \frac{\kappa}{v} \theta_y \right)_v + K_v v y u_y + K_v u_{yy} + \left( \frac{\kappa}{v} \right)_v v y \theta_v,
\]
\[
|K_v|, |K_{yy}| \leq C(1 + \theta^{q+1}).
\]
Multiplying (2.2) by $K_t$ and integrating it over $(0, 1) \times (0, t)$, we obtain

$$
\int_0^t \int_0^1 \left( \kappa e \theta_t + \theta p_0 u_y - \frac{\lambda u_y^2}{v} - \frac{\mu |w|^2}{v} - \frac{v |b|^2}{v} \right) K_t \ dy \ ds + \int_0^t \int_0^1 \frac{\kappa}{v} \theta_t K_{t,x} \ dy \ ds = 0,
$$

or equivalently

$$
\int_0^t \int_0^1 \frac{\kappa e \theta_t^2}{v} \ dy \ ds + \int_0^t \int_0^1 \frac{\kappa}{v} \theta_t \left( \frac{\kappa}{v} \theta_y \right)_t \ dy \ ds = \sum_{i=1}^{6} I_i,
$$

where $I_i$ is defined for $i = 1, \ldots, 6$ by

$$
I_1 = - \int_0^t \int_0^1 \frac{\kappa}{v} e \theta_t K_{t,x} u_y \ dy \ ds,
$$

$$
I_2 = - \int_0^t \int_0^1 \left( \theta p_0 u_y - \frac{\lambda u_y^2}{v} - \frac{\mu |w|^2}{v} - \frac{v |b|^2}{v} \right) \frac{\kappa}{v} \theta_t \ dy \ ds,
$$

$$
I_3 = - \int_0^t \int_0^1 \left( \theta p_0 u_y - \frac{\lambda u_y^2}{v} - \frac{\mu |w|^2}{v} - \frac{v |b|^2}{v} \right) K_{t,x} u_y \ dy \ ds,
$$

$$
I_4 = - \int_0^t \int_0^1 \frac{\kappa}{v} \theta_t K_{t,x} v_y u_y \ dy \ ds,
$$

$$
I_5 = - \int_0^t \int_0^1 \frac{\kappa}{v} \theta_t K_{t,x} v_y \ dy \ ds,
$$

$$
I_6 = - \int_0^t \int_0^1 \frac{\kappa}{v} \theta_t \left( \frac{\kappa}{v} \theta_y \right)_t v_y \ dy \ ds.
$$

In the following, we will evaluate all terms in (3.39) according to (3.36)–(3.38).

First, one has by (3.30) and (3.31)

$$
\int_0^t \int_0^1 \frac{\kappa e \theta_t^2}{v} \ dy \ ds \geq C \int_0^t \int_0^1 (1 + \theta^3)(1 + \theta^q) \theta_t^2 \ dy \ ds \geq C X,
$$

and

$$
\int_0^t \int_0^1 \frac{\kappa}{v} \theta_t \left( \frac{\kappa}{v} \theta_y \right)_t \ dy \ ds
$$

$$
\geq \frac{1}{2} \int_0^1 \left( \frac{\kappa}{v} \theta_y \right)^2 \ dy - \frac{1}{2} \int_0^1 \left( \frac{\kappa}{v} \theta_y \right)^2 (y, 0) \ dy
$$

$$
\geq C \int_0^1 (1 + \theta^q)^2 \theta_y^2 \ dy - C \geq CY - C.
$$

Second, we have by Cauchy–Schwarz inequality and (3.34)

$$
|I_1| = \left| \int_0^t \int_0^1 e \theta_t K_{t,x} u_y \ dy \ ds \right|
$$

$$
\leq C \int_0^t \int_0^1 (1 + \theta^q)^2 |\theta_t||u_y| \ dy \ ds
$$

$$
\leq \varepsilon X + C \varepsilon^2 \int_0^t \int_0^1 (1 + \theta^q)^2 u_y^2 \ dy \ ds
$$

$$
\leq \varepsilon (X + Y) + C,
$$

(3.42)
for any fixed sufficiently small $\epsilon > 0$, and similarly, by recalling (3.25) and (3.29), one has
\[
|I_2| = \left| \int_0^t \left( \frac{\lambda u_2^2 - \mu |w_{2z}|^2 - v|b_z|^2}{v} \right) \frac{\kappa}{v} \theta_t \, dy \, ds \right|
\leq C \int_0^t \int_0^1 \left[ (1 + \theta)^{\eta + 4} |u_z \theta_t| + (1 + \theta)^{\eta} |\theta_t| (u_2^2 + |w_z|^2 + |b_z|^2) \right] \, dy \, ds
\leq \epsilon X + C_\epsilon |(1 + \theta)^{\eta + 4}| \int_0^1 \int_0^1 u_2^2 \, dy \, ds
+ C_\epsilon |(1 + \theta)^{\eta + 4}| \int_0^1 \int_0^1 (|u_z|^4 + |w_z|^4 + |b_z|^4) \, dy \, ds
\leq \epsilon (X + Y) + C,
\] (3.43)
and
\[
|I_3| = \left| \int_0^t \left( \frac{\lambda u_3^2 - \mu |w_{3z}|^2 - v|b_z|^2}{v} \right) \frac{\kappa}{v} \theta_{z} \, dy \, ds \right|
\leq C \int_0^t \int_0^1 \left[ (1 + \theta)^{\eta + 5} u_3^2 \right] \, dy \, ds + \int_0^t \int_0^1 \left[ (1 + \theta)^{\eta + 1} |u_z|^3 \right] \, dy \, ds
+ C \int_0^t \int_0^1 \left[ (1 + \theta)^{\eta + 1} |u_z|^3 \right] \, dy \, ds
\leq C \frac{\epsilon Y}{v^{1/4}} + C \frac{\epsilon Y}{v^{1/4}} \int_0^1 \int_0^1 (u_2^2 + |w_z|^4 + |b_z|^4) \, dy \, ds + C
\leq \epsilon Y + C.
\] (3.44)

Third, using (3.10) and (3.33) yields
\[
|I_4| = \left| \int_0^t \left( \frac{\lambda}{v} \theta_{vy} v u_y \right) \, dy \, ds \right|
\leq C \left( \int_0^t \int_0^1 \frac{\kappa \theta^2}{\theta^2} \, dy \, ds \right)^{1/2} \times \left( \int_0^t \int_0^1 (1 + \theta)^{\eta + 4} u_2^2 \, dy \, ds \right)^{1/2}
\leq CZ\frac{\epsilon Y}{v^{1/4}}
\leq \epsilon Y + CZ\frac{\epsilon Y}{v^{1/4}},
\] (3.45)
and
\[
|I_5| = \left| \int_0^t \left( \frac{\lambda}{v} \theta_{vy} u_{yy} \right) \, dy \, ds \right|
\leq C \left( \int_0^t \int_0^1 \frac{\kappa \theta^2}{\theta^2} \, dy \, ds \right)^{1/2} \times \left( \int_0^t \int_0^1 (1 + \theta)^{\eta + 4} u_2^2 \, dy \, ds \right)^{1/2}
\]
Finally, one has

\[ |I_0| = \left| \int_0^t \int_0^1 \frac{\kappa}{v} \left( \frac{\kappa}{v} \right)_y v_y \theta_y \, dy \, ds \right| \]

\[ \leq C \int_0^t \int_0^1 \frac{\kappa}{v} \left(1 + \theta \right)^{\gamma(0)} \left| v_y \theta_y \right| \, dy \, ds \]

\[ \leq \varepsilon X + C_y \int_0^t \max_{[0, 1]} \left( \frac{\kappa}{v} \right)_y^2 \left( \int_0^1 v_y^2 \, dy \right) \, ds \]

\[ \leq \varepsilon X + C_y Y_{x_5} \int_0^t \max_{[0, 1]} \left( \frac{\kappa}{v} \right)_y^2 \left( \int_0^1 v_y^2 \, dy \right) \, ds. \]  

(3.46)

By the embedding theorem, one has

\[ \int_0^t \max_{[0, 1]} \left( \frac{\kappa}{v} \right)_y^2 \, dy \, ds \leq C \int_0^t \int_0^1 \left( \frac{\kappa}{v} \right)_y^2 \, dy \, ds + C \int_0^t \int_0^1 \left| \frac{\kappa}{v} \right| \, dy \, ds \]

\[ \leq C \left| \frac{\kappa}{v} \right|_{(0)} \int_0^t \int_0^1 \kappa \theta_y^2 \, dy \, ds \]

\[ + C \left( \int_0^t \int_0^1 \left(1 + \theta \right)^{\gamma(0)} \left( \int_0^1 \frac{\kappa}{v} \theta_y^2 \, dy \right) \, ds \right)^{\frac{1}{2}} \]

\[ \leq C \left\{ \left(1 + \theta \right)^{\gamma(2)} \left( \int_0^t \int_0^1 \left( \frac{\kappa}{v} \right)_y^2 \, dy \, ds \right)^{\frac{1}{2}} \right\}. \]

In particular, we also have

\[ \int_0^t \int_0^1 \left(1 + \theta \right)^{\gamma(2)} \varepsilon_0 \theta_y^2 \, dy \, ds \leq C \int_0^t \int_0^1 \left(1 + \theta \right)^{\gamma(0)} \, dy \, ds \]

\[ \leq C \left( X + Y_{x_5} \right) \]

and

\[ \int_0^t \int_0^1 \left(1 + \theta \right)^{\gamma(0)} \rho_0 \theta_y^2 \, dy \, ds \leq C \int_0^t \int_0^1 \left(1 + \theta \right)^{\gamma(10)} \theta_y^2 \, dy \, ds \]

\[ \leq C \left(1 + Y_{x_5}^{e(10)} \right), \]

and

\[ \int_0^t \int_0^1 \left(1 + \theta \right)^{\gamma(2)} \left( u_y^4 + |w_y|^{12} + |b_y|^4 \right) \, dy \, ds \]

\[ \leq C \left(1 + \theta \right)^{\gamma(2)} \int_0^t \int_0^1 \left( u_y^4 + |w_y|^{12} + |b_y|^4 \right) \, dy \, ds \leq C \left( 1 + Y_{x_5}^{e(2)} \right), \]
Lemma 3.15. The following a priori estimates hold:
\[ |\theta|^{(0)} + |u_\gamma|^{(0)} + |u|^{(0)} \leq C(T) \]
and
\[ \int_0^t (\theta_x^2 + u_\gamma^2 + u_r^2) \, dy + \int_0^t \int_0^1 (\theta_t^2 + |b_t|^2 + u_r^2) \, dy \, ds \leq C(T). \]

**Proof.** Differentiate (2.2b) with respect to \( t \), multiply it by \( u_r \), and then integrate to obtain
\[
\frac{d}{dr} \int_0^1 u_r^2 \, dy + \lambda \int_0^1 \frac{u_r^2}{v^2} \, dy = \int_0^1 \left( \left( p + \frac{1}{2} |b_t|^2 \right) t \right) u_r \, dy.
\]
This implies that
\[
\int_0^1 u_r^2 \, dy + \int_0^t \int_0^1 u_r^2 \, dy \, ds \leq C + C \int_0^t \int_0^1 u_r^2 |u_r| \, dy \, ds
\]
\[ + C \int_0^t \int_0^1 |p_t u_r| \, dy \, ds + C \int_0^t \int_0^1 |b_t u_r| \, dy \, ds \]
\[ \leq \frac{3}{4} \int_0^t \int_0^1 u_r^2 \, dy \, ds + C \int_0^t \int_0^1 u_r^2 \, dy \, ds
\]
\[ + C \int_0^t \int_0^1 p_t^2 \, dy \, ds + C \int_0^t \int_0^1 |b_t|^2 \, dy \, ds + C(T) \]
\[ \leq C(T) + \frac{3}{4} \int_0^t \int_0^1 u_r^2 \, dy \, ds + C \int_0^t \int_0^1 p_t^2 \, dy \, ds + C \int_0^t \int_0^1 |b_t|^2 \, dy \, ds,
\]
which leads to
\[
\int_0^1 u_r^2 \, dy + \int_0^t \int_0^1 u_r^2 \, dy \, ds \leq C(T) + C \int_0^t \int_0^1 p_t^2 \, dy \, ds + C \int_0^t \int_0^1 |b_t|^2 \, dy \, ds.
\]
We notice that
\[
\int_0^t \int_0^1 p_t^2 \, dy \, ds = \int_0^t \int_0^1 (p_v v + p_\theta t)^2 \, dy \, ds
\]
\[ = \int_0^t \int_0^1 \left( -\frac{R \theta}{v^2} u_r + \left( R \rho + \frac{4\theta}{3} \right) \theta_t \right)^2 \, dy \, ds
\]
\[ \leq C \int_0^t \int_0^1 \theta^2 u_r^2 \, dy \, ds + C \int_0^t \int_0^1 (1 + \theta^2) \theta_t^2 \, dy \, ds.
\]
\[ \leq C(T) |\theta^2|^{(0)} \int_0^t \int_0^1 u_y^2 \, dy \, ds + C \int_0^t \int_0^1 (1 + \theta^6) \theta_y^2 \, dy \, ds \]
\[ \leq C(T) Y^{\frac{4+x}{3(1-x)}} + X + X \eta^{\frac{4+x}{3(1-x)}} \]
\[ \leq C(T) \left(1 + Z^{\frac{4+x}{3(1-x)}} \right). \]  

From (3.25) and (3.27) it follows that
\[ \int_0^t \int_0^1 \frac{|b|}{y} \, dy \, ds \leq \int_0^t \int_0^1 \left( |b|^2 u_x + |w_y|^2 + |b_{yy}|^2 + |b_y|^2 u_y^2 \right) \, dy \, ds \]
\[ \leq C \int_0^t \int_0^1 u_y^2 \, dy \, ds + C \int_0^t \int_0^1 (|w_y|^2 + |b_y|^2) \, dy \, ds \]
\[ + C \int_0^t \max_{[0,1]} |b_y|^2 \left( \int_0^1 u_y^2 \, dy \right) \, ds \]
\[ \leq C + C \int_0^t \int_0^1 \left( |b_y|^2 + |b_{yy}|^2 \right) \, dy \, ds \]
\[ \leq C(T). \]  

Substituting (3.52) and (3.53) into (3.51) yields that
\[ \int_0^1 u_y^2 \, dy + \int_0^1 u_{yy}^2 \, dy \, ds \leq C \left(1 + Z^{\frac{4+x}{3(1-x)}} \right). \]

On the other hand, it is also deduced from (2.2b) that
\[ u_{yy} = \frac{v}{\lambda} \left( u_t + \left( p + \frac{|b|^2}{2} \right)_{yy} + \frac{\lambda v_x u_x}{y^2} \right), \]
so that integrating (3.55) with respect to \( y \) and using (3.54) yields
\[ \int_0^1 u_{yy} \, dy \leq C \int_0^1 \left( u_t^2 + p_x^2 + |b|^2 \cdot |b_y|^2 + v_y^2 u_x^2 \right) \, dy \]
\[ \leq C \int_0^1 u_y^2 \, dy + C \int_0^1 \left( p_y^2 + v_{xy}^2 u_x^2 \, dy + |b_x|^2 \right) \, dy \]
\[ \leq C \left(1 + Z^{\frac{4+x}{3(1-x)}} \right) \int_0^1 \left(1 + \theta^6 \right) \theta_y^2 \, dy + \left( |\theta^2|^{(0)} + |u_x^2|^{(0)} \right) \int_0^1 u_y^2 \, dy \]
\[ \leq C \left(1 + Z^{\frac{4+x}{3(1-x)}} + Y^{\frac{4+x}{3(1-x)}} + Z^4 \right) \]
\[ \leq C(T) \left(1 + Z^{\frac{4+x}{3(1-x)}} \right). \]

Thus we have \( Z \leq C(T) \) due to \( 0 < \frac{4+x}{3(1-x)} < 1 \), and then \( X \) and \( Y \) are also bounded. Subsequently, \( |\theta|^{(0)}, |u_x|^{(0)}, \int_0^1 \left( u_t^2 + \theta_x^2 + u_y^2 \right) \, dy \) and \( \int_0^t \int_0^1 \left( u_y^2 + \theta_y^2 \right) \, dy \, ds \) are also bounded. \( \square \)

In the following, the lower bound of temperature is obtained.

**Lemma 3.16.** One has
\[ \theta(y, t) \geq C(T). \]
Proof. Let $\Theta = \frac{1}{p}$. Then (3.14) is written as
\[
e_0\Theta_t = \left(\frac{C}{v}\Theta_y\right)_y - \left(\frac{2\kappa^2}{v\Theta} + \frac{\mu|w|_2^2\Theta^2}{v} + \frac{v|b|_2^2\Theta^2}{v}\right)
- \frac{\lambda\Theta^2}{v} \left(u_y - \frac{vp_0}{2\lambda\Theta}\right)^2 + \frac{vp_0^2}{4\lambda}.
\]
By (3.3) and (3.49), we have
\[
\Theta_t \leq \frac{1}{e_0} \left(\frac{C}{v}\Theta_y\right)_y + C(T),
\]
for some positive constant $C(T)$. Define the operator $\mathcal{L} := -\frac{\partial}{\partial t} + \frac{1}{e_0} \frac{\partial}{\partial y} \left(\frac{C}{v}\frac{\partial}{\partial y}\right)$ and then
\[
\begin{cases}
\mathcal{L}\tilde{\Theta} < 0, & \text{in } Q_T = (0,1) \times (0,T), \\
\tilde{\Theta}_{|y=0} \geq 0, & \text{on } [0,1], \\
\tilde{\Theta}_{|y=1} = 0, & \text{on } [0,T].
\end{cases}
\]
where $\tilde{\Theta}(y,t) = C(T)t + \max_{[0,1]} \frac{1}{\Theta_0(y)} - \Theta(y,t)$, and by the comparison theorem (see [18]), one has
\[
\min_{(y,t) \in Q_T} \tilde{\Theta}(y,t) \geq 0,
\]
which is inferred
\[
\Theta(y,t) \geq \left(Ct + \max_{[0,1]} \frac{1}{\Theta_0(y)}\right)^{-1}
\]
for any $(y,t) \in Q_T$. □

3.1.3. A priori estimates of derivatives for $(v, u, w, b, \theta)$. In this section, we continue to show some higher order a priori estimates on derivatives of $(v, u, w, b, \theta)$ which is necessary to establish the existence of the classical global solutions.

Lemma 3.17. One has
\[
|w_y, b_y, v_y|(0) + \int_0^1 \left(|w|^2 + |v^4| + |w_{yy}|^2 + |b_{yy}|^2\right) dy + \int_0^t \int_0^1 \left(|w|^2 + |b|^2 + \Theta^2_{yy}\right) dy ds \leq C(T).
\]
(3.56)

Proof. First, we differentiate (2.2c) with respect to $t$, multiply it with $w_t$ and then integrate over $(0,1) \times (0,t)$
\[
\frac{1}{2} \int_0^1 |w_t|^2 dy + \int_0^t \int_0^1 \frac{\mu}{v} |w_{y_j}|^2 dy ds
= \frac{1}{2} \int_0^1 |w_t|^2(y,0) dy + \int_0^t \int_0^1 \frac{\mu}{v} u_y w_y \cdot w_t dy ds - \int_0^t \int_0^1 b_y \cdot w_y dy ds
\leq C + \frac{1}{2} \int_0^t \int_0^1 \frac{\mu}{v} |w_{y_j}|^2 dy ds + C(T) \int_0^1 \int_0^1 (u_t^4 + |w_t|^2 + |b|^2) dy ds,
\]
which follows from (3.25), (3.49) and (3.50)
\[
\frac{1}{2} \int_0^1 |w_t|^2 dy + \int_0^t \int_0^1 |w_{y_j}|^2 dy ds \leq C(T).
\]
(3.57)
Similarly, we obtain from (2.2c)
\[ w_{yy} = \frac{v}{\mu} \left( w_y - b_y + \frac{\mu}{v^2} v_y w_y \right), \] (3.58)
which leads to
\[
\int_0^1 |w_{yy}|^2 \, dy \leq C \int_0^1 \left( |w_y| + |b_y| + v_y^2 |w_y|^2 \right) \, dy \\
\leq C + C \max_{[0,1]} |w_y|^2 \int_0^1 v_y^2 \, dy \\
\leq C(T) + C \int_0^1 |w_y|^2 \, dy + \frac{1}{2} \int_0^1 |w_{yy}|^2 \, dy \\
\leq C(T) + \frac{1}{2} \int_0^1 |w_{yy}|^2 \, dy,
\]
i.e.
\[
\int_0^1 |w_{yy}|^2 \, dy \leq C(T).
\]
Second, we further deduce that
\[
\int_0^t \int_0^1 \frac{k^2}{v^2} \theta_y^2 \, dy \, ds \leq C \int_0^t \int_0^1 \left( \theta_y^2 + u_y^2 + u_{yy}^2 + |w_y|^4 + |b_y|^4 + v_y^2 \theta_y^2 + \theta_y^4 \right) \, dy \, ds \\
\leq C + C \int_0^t \int_0^1 \left( v_y^2 + \theta_y^2 \right) \, dy \, ds \\
\leq C + C \int_0^t \int_0^1 \theta_y^2 \, dy \, ds + \frac{1}{2} \int_0^t \int_0^1 \frac{k^2}{v^2} \theta_y^2 \, dy \, ds,
\]
and then
\[
\int_0^t \int_0^1 \theta_y^2 \, dy \, ds \leq C(T) + C \int_0^t \int_0^1 \theta_y^2 \, dy \, ds \leq C(T).
\] (3.59)
In addition, by (3.5), one has
\[
\lambda (\ln v)_y = \lambda (\ln v_0(y))_y + \int_0^t \left( p_y + b \cdot b_y \right) \, ds + (u - u_0)
\]
or equivalently
\[
v_y^2 \leq C + C \int_0^t \left( |b_y|^2 + p_y v_y^2 + p_y^2 \theta_y^2 \right) \, ds \\
\leq C + C \int_0^t \left( |b_y|^2 + |b_{yy}|^2 + \theta_y^2 + \theta_{yy}^2 \right) \, dy \, ds + C \int_0^t v_y^2 \, ds \\
\leq C + C \int_0^t v_y^2 \, ds.
\]
Thus, we deduce that \(v_y\) is bounded by Gronwall’s inequality. Lastly, differentiate (2.2d) with respect to \(t\) and then multiply by \((v b)\), and integrate to obtain
\[
\frac{1}{2} \frac{d}{dt} \int_0^1 (v b)_y^2 \, dy + \int_0^1 v |b_{yy}|^2 \, dy = - \int_0^1 \frac{1}{v} b_{yy} \cdot (b u_{yy} + b_y u_x + b_x v_y) \, dy \\
+ \int_0^1 \frac{1}{v^2} v_y b_y \cdot (b u_{yy} + b_y u_x + b_x v_y + b_y v) \, dy
\]
\[
- \int_0^1 w_t \cdot (bu_{yy} + b_y u_y + b_y v + b_y v) \, dy \leq \varepsilon \int_0^1 \nu |b_{y t}|^2 \, dy \\
+ C \int_0^1 (b^2_y + b^2_t + w^2) \, dy + C \int_0^1 (u^2_{yy} + |w_{yy}|^2) \, dy.
\]

(3.60)

So we obtain from (3.25), (3.50) and (3.57)

\[
\int_0^1 (\nu b^2_t) \, dy + \int_0^t \int_0^1 |b_{y t}|^2 \, dy \, ds \leq C,
\]

(3.61)

by choosing a sufficiently small \(\varepsilon\) in (3.60). Furthermore, by (2.2d), one has

\[
\int_0^1 |b_{y y}|^2 \, dy \leq C \int_0^1 ((v b^2_t) + |w_{y y}|^2 + u^2_{y y} |b_{y y}|^2) \, dy \\
\leq C(T) + C \int_0^1 |b_{y y}|^2 \, dy \leq C(T),
\]

(3.62)

and similarly

\[
|b_y|^2 \leq C \int_0^1 |b_y|^2 \, dy + C \int_0^1 |b_{y y y}|^2 \, dy \leq C(T).
\]

This completes the proof. \(\square\)

**Lemma 3.18.** We have

\[
|\theta_{y y}(t)| + \int_0^1 (\theta_t^2 + \theta_{y y}^2) \, dy + \int_0^1 \int_0^1 \theta_{y y}^2 \, dy \, ds \leq C(T).
\]

(3.63)

**Proof.** Differentiate (2.2e) with respect to \(t\), multiply it by \(e_0 \theta_t\) and then integrate it over \([0, 1]\)

\[
\frac{d}{dt} \int_0^1 \frac{(e_0 \theta_t)^2}{2} \, dy + \int_0^1 \frac{\kappa}{v} e_0 \theta_{y y t}^2 \, dy \\
= \int_0^1 \left[ - p_0 e_0 u_y \theta_t^2 - \theta p_{00} e_0 u_y \theta_t^2 - \theta p_{00} e_0 u_y \theta_t^2 - \theta p_0 e_0 u_y \theta_t^2 \\
+ \frac{2 \lambda}{v} e_0 u_y \theta_{y y} \theta_t - \frac{\lambda}{v} e_0 u_y \theta_{y y} \theta_t + \frac{2 \mu}{v} e_0 v_y \cdot \theta_{y y t} \theta_t - \frac{\mu}{v} e_0 v_y |w_{y y}|^2 \theta_t \\
+ \frac{2 \nu}{v} e_0 \theta_t b_y \cdot b_{y y} - \frac{\nu}{v^2} e_0 |b_y|^2 \theta_t - \left( \frac{\kappa}{v} \right) e_0 v_y \theta_{y y} \theta_t - \left( \frac{\kappa}{\theta} \right) e_0 \theta_{y y}^2 \theta_t \\
- \left( \frac{\kappa}{\theta} \right) \theta_{y y} \theta_{y y t} \theta_t + \left( \kappa \right) \theta_{y y t} \theta_{y y t} - \left( \frac{\kappa}{\theta} \right) \theta_{y y} \theta_{y y t} - \left( \frac{\kappa}{\theta} \right) \theta_{y y t} \theta_{y y t} \right] \, dy,
\]

which implies by integration

\[
\int_0^1 \theta_t^2 \, dy + \int_0^t \int_0^1 \theta_{y y t}^2 \, dy \, ds \leq C \left( 1 + \int_0^t \int_0^1 (v^2_t + \theta_{xx}^2) \theta_t \, dy \, ds \right) \\
\leq C \left( 1 + \varepsilon \int_0^t \int_0^1 \theta_{y y t}^2 \, dy \, ds + C \varepsilon \int_0^t \int_0^1 \theta_t^2 \, dy \, ds \right).
\]
Thus \( \int_0^1 \theta^2_y \, dy \) and \( \int_0^T \int_0^1 \theta^2_{yy} \, dy \, ds \) are also bounded. Taking the same operation used for \( b_{yy} \) and \( w_{yy} \) in (3.58) and (3.62), we deduce from (2.2e)

\[
\int_0^1 \theta^2_{yy} \, dy \leq \int_0^1 \frac{1}{\lambda^2} (1 + \theta^2 + u^2_{yy} + u^2 + v^2_\gamma + \theta^2 + \theta^2_\gamma) \, dy \leq C \left( 1 + \max_{[0,1]} \theta^2 + \int_0^1 (v^2_\gamma + \theta^2_\gamma) \, dy \right)
\]

\[
\leq \varepsilon \int_0^1 \theta^2_{yy} \, dy + C(T),
\]

which implies the boundedness of \( \int_0^1 \theta^2_{yy} \, dy \), and so is \( |\theta_y|^{(0)} \) by employing the embedding theorem.

\[
\square
\]

3.2. Hölder estimates on the \((v, u, w, b, \theta)\)

In this section, the Hölder estimates on the solution to system (2.2a)–(2.2e) to (2.4) are established.

**Lemma 3.19.** Supposed that \((\rho, u, w, b, \theta)\) is a solution to system (2.2a)–(2.2e) to (2.4) and the initial data satisfy the hypotheses of theorem 3.1. Let \( T \) be an arbitrary positive constant. Then there exists a constant \( C(T) \), such that

\[
|v, v_y, v_t|^{\alpha, \alpha/2} + |u, w, b, \theta|^{2+\alpha, 1+\alpha/2} \leq C(T).
\]  

(3.64)

**Proof.** In view of (3.49), (3.56) and (3.63), we have

\[
|u_y|^{(0)} + |w_y|^{(0)} + |b_y|^{(0)} + |\theta_y|^{(0)} \leq C(T)
\]

which implies that

\[
(u, w, b, \theta) \in C^{1,0}(Q_T)^6.
\]

Furthermore, using the Cauchy–Schwarz inequality and (3.50), we have

\[
|u(y_1, t) - u(y_2, t)| = \left| \int_{y_2}^{y_1} u_{xx}(\xi, t) \, d\xi \right| \\
\leq \left( \int_0^1 u^2_{xx} \, dy \right)^{1/2} |y_1 - y_2|^{1/2} \\
\leq C(T) |y_1 - y_2|^{1/2},
\]

and

\[
|u(y, t_1) - u(y, t_2)| = \left| \int_{t_2}^{t_1} u_t \, dt \right| \\
\leq \left( \int_0^T u^2_t \, dt \right)^{1/2} |t_1 - t_2|^{1/2} \\
\leq \left( \int_0^T \int_0^1 (u^2_t + u^2_{yt}) \, dy \, dt \right)^{1/2} |t_1 - t_2|^{1/2} \\
\leq C(T) |t_1 - t_2|^{1/2},
\]

which lead to

\[
u \in C^{0,1}(Q_T), \quad u_y \in C^{1,0}(Q_T).
\]
Similarly, we have, for \((w, b, \theta)\)

\[(u, w, b, \theta) \in C^{0, \frac{\beta}{2}}(QT)^6, \quad (u_y, w_y, b_y, \theta_y) \in C^{1, \frac{\beta}{2}}(QT)^6.\]

Then, by lemma 2.1, one obtains that

\[(u_y, w_y, b_y, \theta_y) \in C^{1, \frac{\beta}{2}}(QT)^6.\]

In addition, it follows from \(v_t = u_y\) and \(v|_{t=0} = v_0(y) \in C^{1+\alpha}(\Omega)\) that

\[v \in C^{1, \frac{\beta}{2}}(QT).\]

Moreover, the specific volume \(v(y, t)\) can be solved from (3.5)

\[v(y, t) = \frac{1}{BD}(v_0(y) + R \int_0^t \theta BD \, ds)\]

where

\[B(y, t) = \exp\left(-\frac{1}{\lambda} \int_0^t \left( |b|^2 + \frac{4a}{3} \theta^3 \right) \, ds\right)\]

and

\[D(y, t) = \exp\left(-\frac{1}{\lambda} \int_0^t (u - u_0) \, dx\right).\]

Thus, it is easily obtained that

\[v_y = \frac{1}{BD} \left( v_0 - A v_0 + \frac{R}{\lambda} \int_0^t B D \theta_0 \, ds - \frac{R}{\lambda} \int_0^t \theta BD (A(y, s) - A(y, t)) \, ds\right),\]

where

\[A(y, t) = -\frac{1}{\lambda} \int_0^t \left( 2b \cdot b_y + \frac{4a}{3} \theta^3 \theta_y \right) \, ds - \frac{1}{\lambda} (u - u_0).\]

This can be derived that \(v_y \in C^{\beta, \beta/2}(QT)\) for \(\beta = \min\{\alpha, \frac{1}{2}\}\) by recalling \(v_0' \in C^\alpha(\Omega)\).

Finally, the equations (2.2b–2.2e) can be viewed as the linear parabolic equations

\[\begin{align*}
u_t - \lambda \nu_y &= \nu v_t - \lambda \nu v y = -b \cdot b_y + \frac{R \theta}{v} v y - \left( \nu + \frac{4a}{3} \theta^3 \right) \theta_y, \\
b_t - \nu \nu y &= \nu v y = b_y, \\
\end{align*}\]

\[\begin{align*}
e_y (v, \theta) \theta_t - &\frac{\kappa (v, \theta)}{v} \theta y y - \frac{\left( \kappa (v, \theta) \right)}{v} v y + \frac{\kappa \theta (v, \theta)}{v} \theta_y = p_u u \theta. \\
\end{align*}\]  

Indeed, the coefficients of equations (3.65) and its right-hand sides are Hölder continuous in \(y\) with exponent \(\beta\) and in \(t\) with exponent \(\beta/2\), and then it follows from the classical Schauder estimates (see for instance [4, 18]) that

\[|v, w, b, \theta| \leq C(T),\]

which further implies that

\[(v, u_y, w_y, b_y, \theta_y) \in C^{1, \frac{\beta}{2}}(QT)^7.\]

Taking the same process again as above, we obtain

\[v_y \in C^{\alpha, \alpha/2}(QT), \quad |u, w, b, \theta| \leq C(T),\]

This completes the proof of the lemma.
3.3. Existence and uniqueness of solutions

In this section, we will show the existence of global solution to system (2.2a)–(2.2e) to (2.4) with the help of the classical Leray–Schauder fixed theorem and global a priori estimates in section 3.1. The proof of uniqueness of solution can follow directly in a routine way.

We first state the classical Leray–Schauder fixed theorem.

**Theorem 3.2.** Let \( \mathfrak{B} \) be Banach space and suppose that \( P: [0, 1] \times \mathfrak{B} \rightarrow \mathfrak{B} \) has the following properties:

(i) For any fixed \( \sigma \in [0, 1] \) the map \( P(\sigma, \cdot) : \mathfrak{B} \rightarrow \mathfrak{B} \) is completely continuous.

(ii) For every bounded subset \( S \subset \mathfrak{B} \) the family of maps \( P(\cdot, X) : [0, 1] \rightarrow \mathfrak{B}, X \in S \) is uniformly equicontinuous.

(iii) There is a bounded subset \( S \) of \( \mathfrak{B} \) such that any fixed point in \( \mathfrak{B} \) of \( P(\sigma, \cdot) \), \( \sigma \in [0, 1] \) is contained in \( S \).

(iv) \( P(0, \cdot) \) has precisely one fixed point in \( \mathfrak{B} \).

Then \( P(1, \cdot) \) has at least one fixed point in \( \mathfrak{B} \).

In our case, we suppose that \( \mathfrak{B} \) is the Banach space of functions \((v, u, w, b, \theta)\) defined on \( Q_T \) and simultaneously, \((v, u, u_y, w, w_y, b, b_y, \theta, \theta_y) \in C^{1+\frac{1}{2}}(\overline{Q_T})^1\), with the norm

\[
||(v, u, w, b, \theta)||_{\mathfrak{B}} = ||(v, u, u_y, w, w_y, b, b_y, \theta, \theta_y)||_1 \\
= ||v||_1 + ||u||_1 + ||w||_1 + ||b||_1 + ||\theta||_1 + \\
+ ||u_y||_1 + ||w_y||_1 + ||b_y||_1 + ||\theta_y||_1.
\]

For \( \sigma \in [0, 1] \), we define \( P(\sigma, \cdot) \) as the map which carries \([\vec{v}, \vec{u}, \vec{w}, \vec{b}, \vec{\theta}] \) into \([v, u, w, b, \theta] \) by solving the system

\[
v_y = u_y, \tag{3.66a}
\]

\[
u_t - \frac{\lambda u_y}{v} = -\vec{b} \cdot \vec{b}_y - \vec{p}_v(\vec{v}, \vec{\theta}) v_y - \vec{p}_b(\vec{v}, \vec{\theta}) \vec{\theta}_y, \tag{3.66b}
\]

\[
w_t - \frac{\mu w_{yy}}{v} = \vec{b}_y - \frac{\mu \vec{w}_y}{v}, \tag{3.66c}
\]

\[
b_t - \frac{\nu b_{yy}}{v} = -\vec{b} \vec{w}_y + \frac{1}{v} \vec{w}_y - \frac{\nu \vec{b}_y}{v}, \tag{3.66d}
\]

\[
\vec{\theta}_y(\vec{v}, \vec{\theta}) \vec{\theta}_y - \frac{\kappa(\vec{v}, \vec{\theta})}{v} \theta_{yy} = \left( \frac{\kappa(\vec{v}, \vec{\theta})}{v} \right) \vec{\theta}_y v_y + \frac{\kappa_0(\vec{v}, \vec{\theta})}{v} \vec{\theta}_y, \tag{3.66e}
\]

with boundary conditions

\[
(w, b, \theta_y)|_{(d, t)} = 0 \tag{3.67}
\]

and

\[
\left( -\vec{p}(\vec{v}, \vec{\theta}) + \frac{\lambda}{v} u_y \right) |_{(d, t)} = 0, \quad d = 0, 1. \tag{3.68}
\]

and initial data

\[
v(y, 0) = (1 - \sigma) + \sigma v_0(y), \quad u(y, 0) = (1 - \sigma) + \sigma u_0(y), \quad w(y, 0) = \sigma w_0(y), \quad b(y, 0) = \sigma b_0(y), \quad \theta(y, 0) = (1 - \sigma) + \sigma \theta_0(y). \tag{3.69}
\]
Kawohl [16] has found a fixed point \((\bar{v}, \bar{u}, \bar{\theta})\) in \(B\) for \(P(0, -)\) in the case of free boundary when \(w = b = 0\). Similarly, we can also obtain a fixed point \((\bar{v}, \bar{u}, 0, 0, \bar{\theta})\) for \(P(0, -)\) with our system \((3.66a)–(3.66e)\) to \((3.69)\). The remainder is to verify that \(P\) satisfies other assumptions of theorem 3.2 in the context of our \textit{a priori} estimates for a solution to \((2.2a)–(2.2e)\) to \((2.4)\), which is the same procedure as that in [4, 16]. The existence of a solution to system \((2.2a)–(2.2e)\) to \((2.4)\) can be obtained by a standard continuation argument which connects \((3.66a)\) to a system without the terms \(v_y\) in \((3.66b)–(3.66e)\) (see for instance section 3 in [4] for details). Finally, the uniqueness of a solution to \((2.2a)–(2.2e)\) to \((2.4)\) can be proved in a standard fashion (see [4]). This completes the proof of theorem 3.1.

4. Conclusion

A global solution to system \((2.2a)–(2.2e)\) to \((2.4)\) is uniquely justified by the strategy devised in [4, 16]. The existence is established by means of the Leray–Schauder fixed point theorem. To illustrate the requirements of the application of the Leray–Schauder fixed point theorem, some new \textit{a priori} estimates are obtained by employing the theory of the linear parabolic equations and other techniques.

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