Accurate estimates of 3D Ising critical exponents using the Coherent-Anomaly Method

Miroslav Kolesik* and Masuo Suzuki
Department of Physics, University of Tokyo,
Bunkyo-ku, Tokyo 113, Japan
March 23, 2022

cond-mat/9411109

Abstract

An analysis of the critical behavior of the three-dimensional Ising model using the coherent-anomaly method (CAM) is presented. Various sources of errors in CAM estimates of critical exponents are discussed, and an improved scheme for the CAM data analysis is tested. Using a set of mean-field type approximations based on the variational series expansion approach, accuracy comparable to the most precise conventional methods has been achieved. Our results for the critical exponents are given by $\alpha = 0.108(5)$, $\beta = 0.327(4)$, $\gamma = 1.237(4)$ and $\delta = 4.77(5)$.

Key words: Ising model, critical exponents, coherent-anomaly method, series expansion

Running title: 3D Ising critical exponents using CAM

1 Introduction

This article is devoted to a study of critical properties of the 3D Ising model by the coherent-anomaly method (CAM) [1, 2, 3]. The Ising model has been investigated intensively by various techniques, including the CAM, for many years. The long-standing interest in this model is caused by the fact that it is an ideal system for testing new methods of studying critical phenomena as well as by its great theoretical importance itself.

A great deal of effort made for understanding of this model has been aimed at the calculation of critical exponents. However, there are still some uncertainty and scatter in the results available in the literature on critical phenomena. The most accurate estimates are usually provided by extensive Monte Carlo simulations, series expansions and field theoretic methods. The CAM, while proving its versatility and very good accuracy in many physical systems (see e.g. [4] for a recent review), seems to exhibit slightly less precise results for 3D-Ising critical exponents.

*Permanent address: Institute of Physics, SAS, Dúbravská cesta 9, Bratislava 842 28, Slovakia
The purpose of the present work is twofold. Firstly, we intend to demonstrate that within the coherent-anomaly approach it is possible to calculate critical indices with accuracy comparable to the one reached by the most precise conventional methods. Secondly, we want to discuss various sources of errors of the CAM estimates in some detail in order to show a possible way for further improvement of the results.

The article is organized as follows. We give a brief summary of the coherent-anomaly method in the next section for convenience. Then, we discuss the main sources of errors in the CAM estimates of critical exponents from a general point of view, and propose a new scheme to improve CAM results. In the third section we describe the generation of the series of mean-field type approximations used in this study, which is based on the variational series expansion (VSE). Analysis and our results together with a brief review of the available data for critical exponents are encompassed in the fourth section. Finally, we present some concluding remarks in Section 5.

2 Coherent-anomaly method

2.1 Brief summary of the CAM

The coherent-anomaly method is a general approach for extracting the true critical behavior of the system under investigation from the systematic behavior of the classical criticality of a series of mean-field type approximations \[1, 2, 3\]. It is based on the coherent-anomaly scaling exhibited by certain quantities in such a series. The starting point for each CAM analysis is thus a set of approximation for the given model. For each member of this set (labeled by \(L\), the order of approximation), one calculates its critical temperature \(T^c_L\) together with the so-called critical mean-field coefficient \(\bar{Q}_L\) for each singular quantity \(Q\). The coefficient \(\bar{Q}_L\) characterizes the classical singular behavior of \(Q\) in the vicinity of the critical point:

\[ Q_L \sim \bar{Q}_L(T/T^c_L - 1)^{\omega_{\text{class}}} \]  

where \(\omega_{\text{class}}\) stands for the mean-field value of the corresponding critical index. If the exact critical behavior is characterized by the exponent \(\omega\)

\[ Q \sim (T/T^* - 1)^\omega \]  

with \(T^*\) being the exact critical temperature, the mean-field critical coefficients \(\{\bar{Q}_L\}\) can be shown to obey the following scaling formula \[1, 2, 3\]

\[ \bar{Q}_L \sim (T^c_L - T^*)^{\omega - \omega_{\text{class}}} \]  

Thus, making use of the data \(\{T^c_L, \bar{Q}_L\}\), we can obtain an estimate for the exact critical index \(\omega\). For example, the values of the critical indices \(\alpha, \beta, \gamma\) and \(\delta\) can be estimated from

\[ \bar{c}_L \sim (T^c_L - T^*)^{-\alpha} \]  

\[ \bar{m}_L \sim (T^c_L - T^*)^{\beta - 1/2} \]  

\[ \bar{\chi}_L \sim (T^c_L - T^*)^{1 - \gamma} \]  

\[ \bar{m}_L^c \sim (T^c_L - T^*)^{-\psi}, \quad \psi = \gamma(\delta - 3)/3(\delta - 1) \]
where $c_L$, $m_L$, $\chi_L$ and $m^c_L$ are the mean-field critical coefficients of the specific heat, magnetization, susceptibility and of the critical magnetization, respectively. They fulfill the relations \[1\]

\[
\bar{\chi}_L \bar{c}_L/(\bar{m}_L)^2 = \text{const} \quad \text{and} \quad (\bar{\chi}_L)^2 \bar{c}_L/(\bar{m}^c_L)^3 = \text{const},
\]

independently of the used series of approximations. As a consequence, it is an inherent property of the CAM that the scaling relations $\alpha + 2\beta + \gamma = 2$ and $\gamma = \beta(\delta - 1)$ are always satisfied by resulting exponents.

### 2.2 Correction terms for the CAM data analysis

In general, it is difficult to estimate the accuracy of the CAM results. This is mainly due to the fact that we do not know a priori whether or not we are already in the asymptotic scaling region. Usually, even very simple mean-field type approximations show a quite good coherent-anomaly, but one can always observe deviations from the ideal CAM scaling.

From our experience with the CAM, we know that these deviations depend very much on the specific series of approximations used and are mainly of two kinds. Sometimes one can detect a deviation which changes smoothly with the increasing order of the approximation. In such a case it is possible to weaken its effect by incorporating suitable corrections into the CAM scaling formulas. However, as a rule, the CAM data contain also departures from the ideal scaling which seem to be erratic, without any clear dependence on the order of the approximation. Such a “noise” is usually negligible in comparison with other sources of errors in our estimates, but as will be seen, it can be the limiting factor in the accuracy of the present method even with a very well behaved series of approximations. Namely, if there are no pronounced “smooth” deviations, the noise becomes discernible and, because of its random character, it is hardly possible to improve the results by introducing further corrections.

Before being limited by this subtle effect, however, one has to tackle one more problem. Namely, the results of the CAM analysis generally depend on the choice of an independent variable. For instance,

\[
\bar{\chi}_L \sim \left(\frac{T^c_L - T^*}{T^*}\right)^{1-\gamma} \quad \text{corresponds to} \quad \bar{\chi}_L \sim \left(\frac{\beta^c_L}{\beta^*}\right)^{\gamma-1} \left(\frac{\beta^* - \beta^c_L}{\beta^*}\right)^{1-\gamma},
\]

in the inverse parameter $\beta = 1/T$. The first factor on the right-hand side produces a background contribution in the log-log CAM plot when going from the temperature $T$ to the inverse temperature $\beta$. This is why one obtains different estimates for critical exponents using different variables. This effect should vanish for extremely good approximations, but this is not the case in practice. Naturally, there is no distinguished “right” variable to be preferred. Then, how to make our results insensitive to the choice of an independent variable? The idea is to introduce a correction factor which cancels the background terms induced by transformation of variables:

\[
\bar{\chi}_L \sim \left(\frac{x^c_L}{x^*}\right)^{\phi} \left(\frac{|x^* - x^c_L|}{x^*}\right)^{1-\gamma},
\]
where $x$ is a variable playing the role of the temperature or of its inverse. The exponent $\phi$ is determined from the requirement that this formula is invariant under the exchange $x \leftrightarrow x^{-1}$. It is easy to see that the appropriate value is given by

$$\phi = (\gamma - 1)/2$$  \hspace{1cm} (11)

The critical exponent estimated in terms of the CAM formula (10) with (11) is the same, whether we use $T$ or the inverse temperature $\beta$. Naturally, the following formulas similar to the one for the susceptibility exponent can be used for other critical indices:

$$\bar{c}_L \sim \left(\frac{x^c_L}{x^*}\right)^{\alpha/2} \left(\frac{|x^* - x^c_L|}{x^*}\right)^{-\alpha} = (|\Delta_L|)^{-\alpha},$$  \hspace{1cm} (12)

$$\bar{m}_L \sim \left(\frac{x^c_L}{x^*}\right)^{(1/2-\beta)/2} \left(\frac{|x^* - x^c_L|}{x^*}\right)^{\beta-1/2} = (|\Delta_L|)^{\beta-1/2},$$  \hspace{1cm} (13)

$$\bar{\chi}_L \sim \left(\frac{x^c_L}{x^*}\right)^{(\gamma-1)/2} \left(\frac{|x^* - x^c_L|}{x^*}\right)^{1-\gamma} = (|\Delta_L|)^{1-\gamma},$$  \hspace{1cm} (14)

$$\bar{m}_c^L \sim \left(\frac{x^c_L}{x^*}\right)^{\psi/2} \left(\frac{|x^* - x^c_L|}{x^*}\right)^{-\psi} = (|\Delta_L|)^{-\psi}, \psi = \gamma(\delta - 3)/3(\delta - 1)$$  \hspace{1cm} (15)

where $\Delta_L = (x^*/x^c_L)^{1/2} - (x^c_L/x^*)^{1/2}$. Thus, we propose to fit the critical exponents to the CAM data using the variable $\Delta_L$ instead of the usual temperature or its inverse. It is worth to note that these formulas are nearly invariant with respect to more general transformation $x \rightarrow x^a$. Namely, the change $\Delta_L \rightarrow (x^*/x^c_L)^{a/2} - (x^c_L/x^*)^{a/2}$ does not produce corrections linear in $x^c_L - x^*$ in the log-log CAM plot. In the present case, we have obtained practically the same results with $x = \beta^\pm a$ for the values $a \in (0.1, 2)$; the differences appear only at the sixth decimal place of the estimated exponents, which is far beyond the accuracy of our estimates. This means that we can choose the variable $x$ in (12) - (14) quite arbitrarily without changing the estimates of critical indices.

From the practical point of view it is better to treat the exponent of the correction term $\phi$ as a free parameter, and first to try to fit it together with the usual exponent $\omega - \omega_{\text{class}}$ (and possibly with $T^*$) to the CAM data. If the resulting parameters are stable with respect to different subsets of the available data $\{T^*_L, Q^*_L\}$, then it means that some kind of the above-mentioned smooth background is observed. In such a case, the fitted values of correction-term exponents should be preferred to those listed in formulas (12) - (14). However, if we cannot detect the background corrections reliably, we have to resort to our invariant scheme (12) - (14) fixing the correction-term exponents. Testing this scheme is one of the aims of the present work.

### 3 Generation of mean-field approximations for CAM – VSE method

The crucial ingredient in the CAM is a series of approximations for, let’s say, the free energy of the system under investigation. Such a series is required to converge to its exact limit, and each approximation should give a mean-field type solution of the model, i.e. it should exhibit a classical singularity at its critical point. Many types of approximations
have been used within the CAM approach, and it seems that the best choice depends on the model studied. In this work we employ the so-called variational series expansion (VSE) method for calculating our approximations for the CAM.

3.1 Mapping the Ising model onto a 256-vertex model

Let us consider the simple-cubic Ising model described by the Hamiltonian

$$\mathcal{H} = - \sum_{<i,j>} s_i s_j - H \sum_i s_i$$

(16)

Instead of an external field \(H\) we prefer to use the dimensionless field \(h = \beta H\) below. The partition function

$$Z = \sum_{\{s_i\}} \exp(-\beta \mathcal{H})$$

(17)

can be rewritten as follows:

$$\sum_{\{s_{abc}\}} \prod_{(xyz)} w(s_{xyz}, s_{x+1yz}, s_{x+1y+z}, s_{xy+1z}, s_{xyz+1}, s_{x+1y+1z+1}, s_{xy+1z+1})$$

(18)

where the product runs only over the triples \((xyz)\) in which all entries are either even or odd, and

$$w(s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8) = \exp\left[\beta \left( s_1 s_2 + s_2 s_3 + s_3 s_4 + s_4 s_1 + s_5 s_6 + s_6 s_7 + s_7 s_8 + s_8 s_5 + s_1 s_5 + s_2 s_6 + s_3 s_7 + s_4 s_8 \right) \right] \exp\left( h/2 \sum_i s_i \right)$$

(19)

is the contribution to the statistical weight coming from the eight spins located in the corners of a cube. These weights \(\{w\}\) can be considered as the vertex weights of a 256-vertex model defined on a bcc lattice, in which the original spins represent the edges of the bcc lattice. The summation over all spin configurations is now understood as the summation over all configurations of the edge states.

Vertex models are known to be gauge-invariant (see e.g [8] [9]). Namely, their partition function remains unchanged when one replaces the given vertex weights by the transformed ones. We use the following parameterization of the gauge transformation:

$$\tilde{w}(s_1, \ldots, s_8) = \sum_{r_i = \pm 1} V_{s_1r_1} \cdots V_{s_8r_8} w(r_1, \ldots, r_8)$$

(20)

with \(V\) being an orthogonal matrix dependent on the gauge parameter \(y\):

$$V(y) = \frac{1}{\sqrt{1 + y^2}} \left( \begin{array}{cc} 1 & y \\ -y & 1 \end{array} \right)$$

(21)

Thus, the vertex weights can be considered as functions of the gauge parameter \(y\), and we do not distinguish between the original and the transformed weights below. It is the gauge parameter which is used for generating mean-field approximations based on the formal series expansion for the vertex model.
Naturally, the sc Ising model can be transformed also onto a two-state vertex model defined on the original lattice. Our choice for the bcc lattice has the following motivation. Firstly, we expect to obtain better approximations with this representation. For instance, our lowest-order approximation (described below) is already better than the Bethe approximation, and relatively small graphs embedded in the coarse-grained lattice typically encompass many original spins. Effectively, a single vertex in the bcc lattice represents four spins. Secondly, in this way we arrive at a new classification scheme for graphs contributing to our series expansions - very roughly speaking, clusters are counted in an order different from that in the formulation on the original lattice, and it turns out that the resulting coherent anomaly is essentially free of pronounced corrections to the CAM scaling. From this point of view, we find that the behavior of a series generated for the sc lattice is worse.

3.2 Formal series expansion for the vertex model

The second step in the VSE method is to calculate a formal series expansion for the general 256-vertex model on the bcc lattice. Thus, for the purpose of the generation of the expansion we need to consider all the allowed weights as independent variables. Because of the symmetry of the lattice, the complete set of 256 vertex weights \( w(s_1, s_2, \ldots, s_8) \) consists of 22 equivalence classes \( \omega_i \) \((i = 0, \ldots, 21)\) represented e.g. by

\[
\begin{align*}
\omega_0 &= w(+++++++) \\
\omega_1 &= w(+++++++),
\end{align*}
\]

\[
\vdots
\]

\[
\begin{align*}
\omega_18 &= w(--+-+-) \\
\omega_19 &= w(--+-+-),
\end{align*}
\]

\[
\omega_20 = w(-----+) \quad \omega_21 = w(-------)
\]

As a starting point for generating the expansion, we need a ground-state. At this stage, the convergence properties of the formal expansion are irrelevant, and we can choose the configuration with all edges in the state + as our formal ground-state. (Note that this ground-state has nothing to do with the configuration with all spin aligned up, because we have not specified the gauge yet.) Each vertex in this configuration has the weight \( \omega_0 \) (see (22)), and, consequently, the ground-state contribution to the dimensionless free energy per site is \(-\log(\omega_0)\).

Next, we have to take into account excitations above the ground-state. They are represented by graphs weakly embedded in the lattice. The bonds connecting nodes of these graphs correspond to the edges in the state −. According to the states on its incident edges, each vertex belonging to such a graph falls into one of the vertex classes \( \omega_i \), and the weight of the graph is given by \( \prod_i (\omega_i/\omega_0)^{n_i} \) where \( n_i \) stands for the number of those nodes which belong to the class \( i \).

We use the no-free-ends method described in Ref. [10] which allows us to eliminate the so-called free-ends graphs from the calculation. The free-ends graphs are those which contain at least one node with a single incident edge in the state −, i.e., the node of the class \( \omega_1 \). Thus, for calculating our series we have to generate all possible no-free-ends graphs (up to a certain size \( L \)) embedded in the bcc lattice. Because the minimal number of nodes of a no-free-ends graph is four, it is sufficient to consider only graphs with two connected components for calculating the expansion up to the order 11. We calculated separately the contributions of graphs with single and two connected components. In
this part of the calculation we discriminated between the two sublattices in order to
make it possible to use the so-called code balance (i.e. the symmetry with respect to the
sublattice exchange) for checking our series. We implemented a kind of a shadow method
which treats the two sublattices in a completely different way, such that the sublattice
symmetry is restored only in the final expansion, while both the one- and two-component
contributions lack this symmetry. This turns to be a useful check to our calculations.

Having calculated the no-free-ends part of the series we have generated the free-ends
part by a simple algebraic procedure explained in detail in Ref. [10]. After summing all
the contributions, the final series expansion has the form

\[ F_L = \log(\omega_0) + \sum_{n=2}^{L} f_n \left( \left\{ \frac{\omega_i}{\omega_0} \right\}_{i=1}^{21} \right) \]  

(23)

Here (we use the logarithm of the statistical sum rather than the free energy.), \( f_n \) is a
homogeneous polynomial of order \( n \) representing the contributions of all graphs (connected
or not) with \( n \) vertices, and \( L \) denotes the maximal order included in the expansion. We
have generated the expansion up to the order \( L = 10 \). The final series consists of about
\( 3 \times 10^4 \) terms, and it is therefore impossible to present it here.

3.3 From series expansions to mean-field approximations

To extract physical information from the formal series expansion (23), we return to the
weights (19) - (21) describing our original model. They depend on the inverse temperature
\( \beta \), external field \( h \) and on the gauge parameter \( y \). The dependence of the free energy on
the gauge should vanish in the limit \( L \to \infty \), provided the limit exists. However, our
truncated expansion \( F_L(\beta, h, y) \) depends on \( y \) for arbitrary finite order \( L \). In order to
restore the gauge invariance of the free energy, at least locally, we impose a minimal-
sensitivity condition

\[ \frac{\partial F_L(\beta, h, y)}{\partial y} = 0 \]  

(24)

This stationarity condition is a self-consistency equation for determining the value of the
gauge parameter within the VSE scheme [1]. If there are more solutions to the stationarity
condition, we choose the one which corresponds to the minimal free energy.

For the present model, it can be shown that \( y = 1 \) is a solution to (24) for arbitrary
temperature, provided the external field vanishes. In the low temperature region there
exist two more solutions which correspond to the ordered phase (these two solutions differ
in the sign of the magnetization). The critical point for a given order \( L \) can be located
easily as the temperature at which this couple of low-temperature solutions appear. Analyzing
the formal expansion \( F_L \) in the vicinity of the critical point, we can derive the
following formulas for the critical mean-field coefficients:

\[ \bar{c}_L = \partial_{\beta \beta} F - 3(\beta_L^2)^2 (\partial_{yy} F)^2 / \partial_{yyyy} F \]  

(25)

\[ \bar{m}_L = \partial_{hy} F (\bar{c}_L^2 (\partial_{yy} F)^2 / \partial_{yyyy} F)^{1/2} \]  

(26)

\[ \bar{\chi}_L = (\partial_{hy} F)^2 / \partial_{yyyy} F \]  

(27)

\[ \bar{m}_L^c = \partial_{hy} F (\bar{c}_L^2 (\partial_{yy} F)^2 / \partial_{yyyy} F)^{1/3} \]  

(28)

\[ 1 \text{ The series is available upon request at the e-mail address fyzikomi@shpa.phys.s.u-tokyo.ac.jp (till June 1995) and/or fyzikomi@savba.savba.sk (after June 1995) } \]
where all derivatives are taken at the critical point \((\beta^c_L, h = 0, y = 1)\).

Among the critical coefficients, the specific-heat coefficient \(\bar{c}_L\) deserves special attention. The first term on right-hand side of (25) expresses the critical specific heat in the high-temperature phase, and the complete formula corresponds to the ordered phase. In principle, the CAM scaling should be observed in each of these terms. Nevertheless, the high-temperature term contains also the contribution from the regular part of the specific heat. This is why it does not exhibit a good CAM scaling. On the other hand, the second term represents the difference between the high- and low-temperature specific heats and is therefore free of the regular part. For this reason we restrict ourselves to the latter one. Moreover, in order to satisfy the relations (8) strictly in each order \(L\), we have changed one of the \(\beta^c_L\) factors into exact critical value \(\beta^*\). Thus, we use

\[
\bar{c}_L = -3\beta^c_L \beta^* (\partial_{yy\beta} F)^2 / \partial_{yyyy} F
\]

instead of (25). As we shall see in the next Section, we can arrive, in this way, at the estimate for the specific heat exponent in very good agreement with the most precise results available. On the other hand we cannot extract any reliable information from the specific heat in the disordered phase.

4 CAM analysis

We have calculated the mean-field critical coefficients (25) - (28) for approximations with \(L = 0, \ldots, 10\). The obtained values together with the corresponding critical inverse temperatures are listed in Table 1. The CAM plots are depicted in Fig. 1. One can see that our series of approximation really exhibits a very good coherent-anomaly. Let us describe our analysis in some detail.

For accurate estimation of the critical exponents within the CAM, one needs a very precise value for the exact critical temperature. Fortunately, for the Ising model we know the critical temperature with high accuracy from MC studies. In what follows we use the value \(\beta^* = 0.221652\) from the MCRG simulation [11] for MC studies.

First, we tried to find out whether or not there is some smooth correction to the CAM scaling discernible in our data. We fitted our critical coefficients according to formulas (12) - (15) without correction terms but with added corrections to scaling as in Ref. [3]. Then we included the correction terms but considered their exponents as free parameters to be fitted. In both cases we were not able to fix the corrections to the CAM scaling, because the resulting parameters as well as the estimated exponents depended strongly on chosen subsets of our CAM data; omitting just a single point often led to a completely different output. This means that the deviations of the critical coefficients from the expected CAM scaling do not show any systematic tendency but have a random character. Consequently, we have to use our invariant scheme (12) - (15).

Thus, we have fitted our critical coefficients, using formulas (12) - (15), to various subsets of the available data in order to compare the results. The obtained estimates of the critical exponents are listed in Table 2 together with specification of the used data.

Having the critical coefficients without pronounced correction to scaling, we tend to trust more the estimates based on the most complete data set. However, we check here for the stability of the resulting critical exponents with respect to omitting some points.
The most reliable results are listed in the first part of Table 2. One can see that the agreement between them is satisfactory.

Then we have calculated a list of estimates using only pairs of points (see the second part of Table 2) in order to get feeling about the precision of our estimates. Again, we see that the agreement is quite good. (Naturally, exponents obtained from near points, e.g., neighbors in the CAM plot, are often rather different but there is no good reason to take such “estimates” seriously).

As was already mentioned, it is difficult to determine the accuracy of the CAM estimates of exponents in general, because one can never exclude a possibility of systematic deviations. Nevertheless, we believe that the effect of possible corrections to scaling causing systematic deviations is buried in the noise of the CAM data in the present case. Consequently, the comparison between the results obtained from different subsets of the available data could give a good estimate for errors of our critical exponents. In this way we arrive at our final estimates as

$$\begin{align*}
\alpha & = 0.108(5), \\
\beta & = 0.327(4), \\
\gamma & = 1.237(4), \\
\delta & = 4.77(5), \\
\psi & = 1.94(5),
\end{align*}$$

where the values come from our most reliable fit (1-7) (see Table 2) and the error bars are chosen such that they essentially cover all the most significant estimates in Table 2. Alternatively, we have calculated mean values from the estimates listed in the second part of Table 2 and obtained the results

$$\begin{align*}
\alpha & = 0.1092(40), \\
\beta & = 0.3267(28), \\
\gamma & = 1.2373(23), \\
\delta & = 4.791(39), \\
\psi & = 1.945(25)
\end{align*}$$

where errors quoted correspond just to the standard deviations. It should be remarked that the estimated values have smaller error bars but taking mean values is not necessarily justified. We, therefore, prefer the values. We would like to stress again that our results are consistent with the scaling relations $\alpha + 2\beta + \gamma = 2$ and $\gamma = \beta(\delta - 1)$. The values of the critical exponents $\nu$ and $\eta$ calculated from and the scaling relations $d\nu = 2 - \alpha$, $\eta = 2 - \gamma/\nu$ are $\nu = 0.631$ and $\eta = 0.039$.

Finally, we present some recent estimates of 3D Ising critical exponents in Table 3 for comparison with our results. We can conclude that our present estimates are in agreement with results of other methods within the error bars and that our accuracy is comparable to the one of the most accurate previous results. In comparison with other CAM-based works, on the other hand, the present results show distinct improvement.

5 Concluding remarks

In the present work we have shown that the coherent-anomaly method can compete with the most accurate methods based on series expansions and/or large-scale computer simulations. It turns out that the choice of the series of classical approximations used in the CAM analysis is very important. The present improvement in comparison with the previous CAM-based studies was achieved mainly owing to the new type of the series expansion used here, which exhibits a coherent-anomaly essentially free of systematic corrections.

The second important point in the present approach is the use of the invariant scheme. It is useful whenever we have the mean-field critical coefficients for which it
is not possible to extract the corrections to the CAM scaling reliably. In such a case it represents the most natural way for resolving the ambiguity related to the choice of the independent variable. Without using the correction terms we would obtain, for instance, the values $\gamma = 1.2395(25)$ and $\gamma = 1.2350(23)$ (compare them to (31)) from fitting in the inverse temperature $\beta$ and in the temperature $T$, respectively. Similarly, also other indices would be systematically different, depending on the variable used.

The bottleneck of the whole calculation is the generation of the series expansion. We calculated it up to the order $L = 10$ using only about three hours of CPU time of a small HP workstation. Thus, the computer demands of the present method are essentially smaller than in the methods based on series expansions as well as on MC simulations. This immediately rises the question whether it is possible to improve our results by using a faster machine, more computer time and more memory space in order to extend our series. The answer is, unfortunately, no. That is because the uncertainty in our estimates is dominated by the chaotic deviations from the ideal CAM scaling, and we would need many new terms to make this effect smaller. We have estimated that we would need more than 20 days with the same computer to extend the series up to order 13, but that would not be sufficient to reduce error bars significantly. From this follows that we had better try to reduce the noise in the CAM data rather than to use brute force.

One possible way could be perhaps to extend the variational series expansion scheme in the sense of correlated mean-field theories, or towards a (quasi)continuous family of classical approximations, similarly as in Ref. [29]. We hope to report on progress in this direction in the future.
Table 1. Critical mean-field coefficients for the specific heat, $\tilde{c}_L$, magnetization, $\bar{m}_L$, susceptibility, $\tilde{\chi}_L$, and for the critical magnetization, $\bar{m}^c_L$. (Approximations of the order $L = 0, 1, 2, 3$ have the same classical critical behavior within the VSE method and were excluded from the analysis from the very beginning.)

| No. | $L$ | $\beta^c_L$ | $\tilde{c}_L$ | $\bar{m}_L$ | $\tilde{\chi}_L$ | $\bar{m}^c_L$ |
|-----|-----|-----------|-------------|-------------|----------------|-------------|
| 0   | 0-3 | 0.206633  | 6.848577    | 8.181219    | 1.083122       | 4.169696    |
| 1   | 4   | 0.214738  | 7.599206    | 9.563006    | 1.333713       | 4.959262    |
| 2   | 5   | 0.214844  | 7.650375    | 9.622822    | 1.341417       | 4.989491    |
| 3   | 6   | 0.216791  | 7.944006    | 10.24680    | 1.464803       | 5.357788    |
| 4   | 8   | 0.217938  | 8.251377    | 10.74478    | 1.550638       | 5.635969    |
| 5   | 7   | 0.218671  | 8.316348    | 11.07017    | 1.633118       | 5.849366    |
| 6   | 9   | 0.219200  | 8.521609    | 11.47035    | 1.711093       | 6.083345    |
| 7   | 10  | 0.219384  | 8.643945    | 11.67254    | 1.746871       | 6.197225    |
Table 2. In the first part of this table we have listed our most reliable estimates for critical exponents using different subsets of our CAM-data. The notation $x - y$ in the first column means that all the points from $x$ up to $y$ (numbered as in Table 1) were used for calculating the exponents. In the second part of Table 2 we have listed estimates as obtained using only two points indicated in the first column.

| data | $\alpha$     | $\beta$     | $\gamma$     | $-\psi$     |
|------|---------------|--------------|---------------|--------------|
| 1 - 4| 0.12598       | 0.31645      | 1.24111       | 0.20273      |
| 1 - 5| 0.10991       | 0.32650      | 1.23709       | 0.19470      |
| 1 - 6| 0.10721       | 0.32839      | 1.23602       | 0.19308      |
| 1 - 7| 0.10783       | 0.32736      | 1.23655       | 0.19394      |
| 2 - 4| 0.12279       | 0.31996      | 1.23728       | 0.19912      |
| 2 - 5| 0.10514       | 0.33050      | 1.23386       | 0.19095      |
| 2 - 6| 0.10321       | 0.33167      | 1.23345       | 0.19004      |
| 2 - 7| 0.10563       | 0.32994      | 1.23448       | 0.19153      |

| data | $\alpha$     | $\beta$     | $\gamma$     | $-\psi$     |
|------|---------------|--------------|---------------|--------------|
| 1 , 3| 0.12424       | 0.30662      | 1.26251       | 0.21642      |
| 1 , 4| 0.13092       | 0.31473      | 1.23962       | 0.20339      |
| 1 , 5| 0.10604       | 0.32791      | 1.23814       | 0.19411      |
| 1 , 6| 0.10942       | 0.32630      | 1.23798       | 0.19513      |
| 1 , 7| 0.11443       | 0.32292      | 1.23973       | 0.19796      |
| 2 , 4| 0.12334       | 0.32014      | 1.23638       | 0.19870      |
| 2 , 5| 0.10000       | 0.33213      | 1.23574       | 0.19049      |
| 2 , 6| 0.10458       | 0.32969      | 1.23603       | 0.19222      |
| 2 , 7| 0.11001       | 0.32602      | 1.23794       | 0.19530      |
| 3 , 5| 0.09286       | 0.34332      | 1.22050       | 0.17795      |
| 3 , 6| 0.10174       | 0.33649      | 1.22528       | 0.18410      |
| 3 , 7| 0.10987       | 0.33049      | 1.22914       | 0.18939      |
| 4 , 6| 0.07708       | 0.34370      | 1.23552       | 0.18270      |
| 4 , 7| 0.09356       | 0.33329      | 1.23987       | 0.19110      |
| 5 , 7| 0.14037       | 0.30750      | 1.24464       | 0.20988      |
| Author  | year | Ref. | α      | β      | γ       | δ       |
|---------|------|------|--------|--------|---------|---------|
| Adler   | 1983 | [12] | 1.239(3) |        |         |         |
| Adler   | 1982 | [13] |        |        |         | 1.238(3) |
| Bhanot  | 1994 | [14] | 0.104(4) |        |         |         |
| Bhanot  | 1992 | [15] | 0.207(4) | 0.308(5) |         |         |
| Ferenberg | 1991 | [16] |        | 0.3258 |         | 1.239(7) |
| Guttmann | 1994 | [17] | 0.101(4) |        |         |         |
| Guttmann | 1993 | [18] | 0.110(5) | 0.329(9) |         |         |
| Guttmann | 1987 | [19, 20] | 0.104(6) |        |         | 1.239(3) |
| Guttmann | 1986 | [21] |        |         |         | 1.240(6) |
| Ito     | 1991 | [22] |        | 0.324(4) |        |         |
| Le Guillou | 1987 | [23] |        | 0.3270(15) | 1.2390(25) |         |
| Le Guillou | 1980 | [24] |        | 0.3250(15) | 1.2341(20) |         |
| Nickel  | 1991 | [25] |        |        |         | 1.238    |
| Nickel  | 1990 | [26] |        | 0.11(2)  |        | 1.237(2) |
| Oitmaa  | 1991 | [27] |        | 0.096(6) | 0.318(4) | 1.25(2)  |
| Ruge    | 1994 | [28] |        |         | 0.319(5) | 1.237    |
| Cenedese | 1994 | [29, *] | 0.105 | 0.318 | 1.258 | 4.957 |
| Fujiki  | 1990 | [30, *] | 0.075 | 0.297 | 1.329 | 5.471 |
| Katori  | 1994 | [31, *] |        |        |        | 1.296    |
| Katori  | 1987 | [32] |        |        |        | 1.258    |
| Kinosita | 1992 | [33] |        |        |        | 1.246    |
| Kolesik | 1993 | [34] |        | 0.331 |        | 1.238    |
| Monroe  | 1988 | [35, *] |        | 0.35  |        | 1.32     |
| this work |        | * | 0.108(5) | 0.327(4) | 1.237(4) | 4.77(5) |

Table 3. Some recent estimates for critical exponents of the 3D Ising model. We have included also works based on the CAM (denoted by *) for comparison. Our final estimates are listed in the last row of the table.
6 Figure caption

Fig. 1. CAM scaling of the critical mean-field coefficients for the specific heat, $\tilde{c}_L$ (+), magnetization, $\tilde{m}_L$ (□), susceptibility, $\tilde{\chi}_L$ (◇) and for the critical magnetization, $\tilde{m}_L^c$ (×). The distance from the true critical point is measured in $\Delta_L = (\beta^*/\beta_L^c)^{1/2} - (\beta^*/\beta^c)^{1/2}$. Critical coefficients were rescaled in order to get them into the same plotting area. The straight lines correspond to our most reliable fit (1-7) (see Table 2).
References

[1] M. Suzuki, J. Phys. Soc. Jpn. 55 (1986) 4205.
[2] M. Suzuki, M. Katori and X. Hu, J. Phys. Soc. Jpn. 56 (1987) 3092.
[3] M. Katori and M. Suzuki, J. Phys. Soc. Jpn. 56 (1987) 3113.
[4] M. Suzuki, K. Minami and Y. Nonomura, Physica A 205 (1994) 80.
[5] M. Kolesik and L. Šamaj, J. Phys. I (France) 3 (1993) 93.
[6] M. Kolesik and L. Šamaj, J. Stat. Phys. 72 (1993) 1203.
[7] M. Kolesik and L. Šamaj, Phys. Lett. A 177 (1993) 87.
[8] F. J. Wegner, Physica 68 (1973) 570.
[9] A. Gaaf and J. Hijmans, Physica A 80 (1975) 149.
[10] M. Kolesik, Physica A 202 (1994) 529.
[11] C. F. Baillie, R. Gupta, K. A. Hawick and G. S. Pawley, Phys. Rev. B 45 (1992) 10438.
[12] J. Adler, J. Phys. A 16 (1983) 3585.
[13] J. Adler, Moshe M and Privman V, Phys. Rev. B 26 (1982) 3958.
[14] G. Bhanot, M. Creutz, U. Glässner and K. Schilling, Phys. Rev. B 49 (1994) 12909.
[15] G. Bhanot, M. Creutz and J. Lacki, Phys. Rev. Lett. 69 (1992) 1841.
[16] A. M. Ferrenberg and D. P. Landau, Phys. Rev. B 44 (1991) 5081.
[17] A. J. Guttmann and I. G. Enting, (COND-MAT/9411002).
[18] A. J. Guttmann and I. G. Enting, J. Phys. A 26 (1993) 807.
[19] A. J. Guttmann, J. Phys. A 20 (1987) 1855.
[20] A. J. Guttmann, J. Phys. A 20 (1987) 1839.
[21] A. J. Guttmann, Phys. Rev. B 33 (1986) 5089.
[22] N. Ito and M. Suzuki, J. Phys. Soc. Jpn. 60 (1991) 1978.
[23] J. C. Le Guillou and J. Zinn-Justin, J. Phys. (Paris) 48 (1987) 19.
[24] J. C. Le Guillou and J. Zinn-Justin, Phys. Rev. B 21 (1980) 3976.
[25] B. G. Nickel and J. J. Rehr, J. Stat. Phys. 61 (1990) 1.
[26] B. G. Nickel, Physica 177 (1991) 189.
[27] J. Oitmaa, C. J. Hamer and W. Zheng, J. Phys. A 24 (1991) 2863.
[28] C. Ruge, P. Zhu and F. Wagner, Physica A 209 (1994) 431.

[29] P. Cenedese, J. M. Sanchez and R. Kikuchi, Physica A 209 (1994) 257.

[30] S. Fujiki, M. Katori and M. Suzuki, J. Phys. Soc. Jpn. 59 (1990) 2681.

[31] M. Katori and M. Suzuki, Prog. Theor. Phys. Suppl. 115 (1994) 83.

[32] Y. Kinosita, N. Kawashima and M. Suzuki, J. Phys. Soc. Jpn. 61 (1992) 3887.

[33] J. L. Monroe, Phys. Lett. A 131 (1988) 427.
