Brownian motion with killing and reflection and the “hot–spots” problem

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Abstract

We investigate the “hot–spots” property for the survival time probability of Brownian motion with killing and reflection in planar convex domains whose boundary consists of two curves, one of which is an arc of a circle, intersecting at acute angles. This leads to the “hot–spots” property for the mixed Dirichlet–Neumann eigenvalue problem in the domain with Neumann conditions on one of the curves and Dirichlet conditions on the other.

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1 Introduction

The “hot spots” conjecture, formulated by J. Rauch in 1974, asserts that the maximum and the minimum of the first nonconstant Neumann eigenfunction for a smooth bounded domain in $\mathbb{R}^n$ are attained on the boundary and only on the boundary (see [4] for more precise formulation). The conjecture has received a lot of attention in recent years and partial results have been obtained in [12], [4], [11], [1], [2], [13]. Counterexamples for (nonconvex) domains in the plane and on surfaces have been given in [8], [7] and [10]. We refer the reader to [5] where a different proof of the result in [13] is given and for more details on the above literature.

The conjecture is widely believed to be true for arbitrary convex domains in the plane but surprisingly even this remains open. For planar convex domains (and indeed for any simply connected domain) the conjecture can be formulated in terms of a mixed Dirichlet–Neumann eigenvalue problem as discussed in [5]. The purpose of this note is to explore this mixed boundary value problem further and in particular to extend the results in [13] and [5].

We assume for the rest of the paper that $D$ is a planar convex domain for which the Laplacian with Neumann boundary conditions has discrete spectrum. The eigenvalues of the Laplacian are a sequence of nonnegative numbers tending to infinity and 0 is always an eigenvalue with eigenfunction 1. Let $\mu_1$ be the first nonzero eigenvalue. Under various conditions on $D$, it is shown in [7] that $\mu_1$ is simple. In general the multiplicity of $\mu_1$ is at most 2 (see [7]). Let $\varphi_1$ be any Neumann eigenfunction corresponding to $\mu_1$. The strongest form of the “hot–spots” conjecture (see [4] for other weaker forms) asserts that $\varphi_1$ attains its maximum on, and only, $\partial D$.

The set $\gamma = \{x \in D : \varphi_1(x) = 0\}$ is called the nodal line for $\varphi_1$. It follows from Pólya’s comparisons of Dirichlet and Neumann eigenvalues that $\varphi_1$ does not have closed nodal lines. That is, $\gamma$ a smooth simple curve intersecting the boundary at exactly two points and divides the domain into two simply connected domains $D_1$ and $D_2$, called nodal domains. We can take $\varphi_1 > 0$ on $D_1$ and $\varphi_1 < 0$ on $D_2$. The function $\varphi_1$ is an eigenfunction corresponding to the smallest eigenvalue for the Laplacian in $D_1$ with Dirichlet boundary conditions on $\gamma$ and Neumann boundary conditions on $\partial D_1 \setminus \gamma$. The “hot–spots” conjecture is equivalent to the assertion that this function takes its maximum on, and only on, $\partial D_1 \setminus \gamma$.

The results in [13] and [5] can be stated in terms of the above mixed Dirichlet-Neumann boundary value problem as follows. Suppose that $D$ is planar convex domain whose boundary consists of the curve $\gamma_1$ and the line segment $\gamma_2$. Let $\mu_1$ be the lowest eigenvalue for the Laplacian in $D$ with
Neumann boundary conditions on $\gamma_1$ and Dirichlet boundary conditions on $\gamma_2$. Let $\psi_1 : \overline{D} \to [0, \infty)$ be the ground state eigenfunction (unique up to a multiplicative constant) corresponding to $\mu_1$. Then $\psi_1$ attains its maximum on, and only on, $\gamma_1$. In fact, the results in [13], [5] prove more. Let $B_t$ be a reflecting Brownian motion in $D$ starting at $z \in \overline{D}$ which is killed on $\gamma_2$, and let $\tau$ denote its lifetime (the first time $B_t$ hits $\gamma_2$). Then, for an arbitrarily fixed $t > 0$, the function $u(z) = P_z^{\tau} \{ \tau > t \}$ attains is maximum, as a function of $z \in \overline{D}$, on, and only on, $\gamma_1$. Furthermore, both function $u(z)$ and $\psi_1(z)$ are strictly increasing as $z$ moves toward the boundary $\gamma_1$ of $D$ along hyperbolic line segments. (See [13] and [5] for the precise definitions of hyperbolic line segments and for the details of how the result for $u$ implies the result for $\psi_1$.) The following question, first raised in [5], naturally arises:

**Question.** Given a bounded simply connected planar domain whose boundary consists of two smooth curves, what conditions must one impose on these two curves in order for the ground state eigenfunction of the mixed boundary value problem (Dirichlet conditions on one curve and Neumann on the other) to attain its maximum on the boundary and only on the boundary?

In this paper we prove the following theorem which extends the results in [13] and [5] by replacing the hypothesis that $\gamma_2$ is a line segment by the hypothesis that $\gamma_2$ is an arc of a circle.

**Theorem 1.1.** Suppose $D$ is a bounded convex planar domain whose boundary consists of two curves $\{\gamma_1(t)\}_{t \in [0,1]}$ and $\{\gamma_2(t)\}_{t \in [0,1]}$ one of which is an arc of a circle, and suppose that the angle between the curves $\gamma_1$ and $\gamma_2$ is less than or equal to $\frac{\pi}{2}$. That is, the angle formed by the two half-tangents at $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$ is less than or equal to $\frac{\pi}{2}$. Let $B_t$ be a reflecting Brownian motion in $D$ killed on $\gamma_2$ and let $\tau_D$ denote its lifetime. Then, for each $t > 0$ arbitrarily fixed, the function $u(z) = P_z^{\tau} \{ \tau_D > t \}$ attains it maximum on, and only on, $\gamma_1$.

**Corollary 1.2.** (“Hot-spots” for the mixed boundary value problem.) Let $D$ be as in Theorem 1.1. Let $\psi_1$ be a first mixed Dirichlet-Neumann eigenfunction for the Laplacian in $D$, with Neumann boundary conditions on $\gamma_1$ and Dirichlet boundary conditions on $\gamma_2$. Then $\psi_1(z), \, z \in \overline{D}$, attains its maximum on, and only on, $\gamma_1$.

As in [13] and [5], the functions $u(z)$ and $\psi_1(z)$ are increasing along hyperbolic line segments in $D$, in the case when $\gamma_2$ is an arc of a circle and along Euclidean radii contained in $D$ in the case when $\gamma_1$ is an arc of a circle. We shall make this precise later. The proof of Theorem 1.1...
is presented in the next section. The idea for the case when $\gamma_2$ is an arc of a circle is to construct a convex domain starting from $D$, by symmetry with respect to a circle (the circle which contains the arc $\gamma_2$), and then use the stochastic inequality for potentials proved in \cite{5}. This inequality also follows from the coupling arguments in \cite{13} which have the advantage that they work in several dimensions. Hence we will discuss this inequality in several dimensions. While at this point we have no applications for this more general inequality, we believe the inequality is of independent interest.

The case when $\gamma_1$ is an arc of a circle is treated by a coupling argument right in the domain itself.

2 Preliminary Results

The proof of Theorem \ref{thm:main} is different depending on which one of curves $\gamma_1$ or $\gamma_2$ is an arc of a circle. For the proof of the case when $\gamma_2$ is an arc of a circle, we need several preliminary results.

**Proposition 2.1.** Let $D$ be as in Theorem \ref{thm:main} and suppose that $\gamma_2$ is an arc of a circle $C = \partial B(z_0, R)$. Let $D_s$ be the domain which is symmetric to the domain $D$ with respect to the circle $C$, that is

$$D_s = \{z_0 + \frac{R^2}{z - z_0} : z \in D\}.$$

Then $D^* = D \cup \gamma_2 \cup D_s$ is a convex domain.

**Proof.** For a complex number $z$ we will use $\Re z$ and $\Im z$ to denote the real, respectively the imaginary part of the complex number $z \in \mathbb{C}$. Without loss of generality we can assume that $C = \partial B(0, 1)$ is the circle centered at the origin of radius 1 and that $\gamma_1(0)$ and $\gamma_1(1)$ are symmetric with respect to the vertical axis, that is $\Im \gamma_1(0) = \Im \gamma_1(1)$. Further, we may assume that $\gamma_2$ contains the point $-i$.

We will first show that $\Im \gamma_1(0) \leq 0$. To see this, note that since the domain $D$ is convex, it lies below its half-tangent at the point $\gamma_1(0)$, and by the angle restriction this half-line lies below the line passing through $\gamma_1(0)$ and the origin. If $\Im \gamma_1(0) > 0$ then also $\Im \gamma_1(1) = \Im \gamma_1(0) > 0$, and therefore the point $\gamma_1(1) \in \partial D$ does not lie below (or on) the line determined by $\gamma_1(0)$ and 0, a contradiction. We must therefore have $\Im \gamma_1(0) = \Im \gamma_1(1) \leq 0$.

If $\Im \gamma_1(0) = \Im \gamma_1(1) = 0$, by the angle restriction at these points, together with the fact that $D$ is a convex domain (and hence $\gamma_1$ is a concave down curve), it follows that the curve $\gamma_1$ is in this case the line segment $[-1, 1]$,
and therefore $D = \{ z \in \mathbb{C} : \Im z < 0, |z| < 1 \}$. The proof is trivial in this case since $D_s = \{ z \in \mathbb{C} : \Im z < 0, |z| > 1 \}$, and therefore $D^* = D \cup \gamma_2 \cup D_s = \{ z \in \mathbb{C} : \Im z < 0 \}$ which is a convex domain.

A similar argument shows that if $0 \in \gamma_1 \subset \partial D$, then the curve $\gamma_1$ consists of the union of the two line segments from $\gamma_1(0)$ to 0, respectively from 0 to $\gamma_1(1)$, hence $D$ is a sector of the unit disk. It follows that $D^* = D \cup \gamma_2 \cup D_s = \{ z \in \mathbb{C} : \Im z < 0, |z| > 1 \}$, and therefore $D^* = D \cup \gamma_2 \cup D_s$ is contained in $\{ z \in \mathbb{C} : \Im z < 0 \}$, which is again a convex set.

We can therefore assume that $\Im \gamma_1(0) = \Im \gamma_1(1) < 0$ and $0 \notin D \cup \partial D$. It follows that for any points $w_1, w_2 \in D^* = D \cup \gamma_2 \cup D_s$, the line segment $[w_1, w_2]$ may intersect the circle $C$ only on the arc $\gamma_2$ (and not on $C - \gamma_2$). Since $D$ is convex domain, it follows that $D^* = D \cup \gamma_2 \cup D_s$ is a convex domain if and only if

$$w_1 \in D_s, w_2 \in \gamma_2 \cup D_s \text{ s.t. } [w_1, w_2] \cap \gamma_2 \in \{ \emptyset, \{w_2\} \} \Rightarrow [w_1, w_2] \subset D^*,$$

where $[w_1, w_2]$ denotes the line segment with endpoints $w_1$ and $w_2$.

Since the set is symmetric to a line with respect to $C$ is a circle passing through the origin, by letting $z_1, z_2$ be the symmetric points of $w_1$, respec-
tively $w_2$ with respect to $C$, \textbf{(2.1)} can be rewritten equivalently as

\begin{equation}
(2.2) \quad z_1 \in D, z_2 \in \gamma_2 \cup D \text{ s.t. } \tilde{z}_1 \tilde{z}_2 \cap \gamma_2 \in \{\emptyset, \{z_2\}\} \Rightarrow \tilde{z}_1 \tilde{z}_2 \subset \gamma_2 \cup D,
\end{equation}

where $\tilde{z}_1 \tilde{z}_2$ denotes the arc of the circle $C(0, z_1, z_2)$ passing through $z_1, z_2$ and 0, between (and including) $z_1$ and $z_2$, and not containing 0. If the points $z_1, z_2$ and 0 are collinear, the arc $\tilde{z}_1 \tilde{z}_2$ becomes the line segment $[z_1, z_2]$.

To show the claim, we will prove \textbf{(2.2)}. Let $z_1 \in D, z_2 \in \gamma_2 \cup D$ such that $\tilde{z}_1 \tilde{z}_2 \cap \gamma_2 \in \{\emptyset, \{z_2\}\}$. If the points 0, $z_1$ and $z_2$ are collinear, $\tilde{z}_1 \tilde{z}_2 = [z_1, z_2] \subset \gamma_2 \cup D$, so we may assume that 0, $z_1$ and $z_2$ are not collinear.

Assume first that the circle $C(0, z_1, z_2)$ does not intersect $C$. Since $\gamma_1$ bounds the convex domain $D$, the intersection $\gamma_1 \cap C(0, z_1, z_2)$ consists of exactly two points $u_1$ and $u_2$ (see Figure 1). It follows that the intersection between $D$ and $C(0, z_1, z_2)$ is the arc $\tilde{u}_1 \tilde{u}_2$, and therefore we have $\tilde{z}_1 \tilde{z}_2 \subset \tilde{u}_1 \tilde{u}_2 \subset D$ in this case.

If the circle $C(0, z_1, z_2)$ intersects $C$, the intersection $C(0, z_1, z_2) \cap D$ is either one or two (connected) arcs $c_1$ and $c_2$. Note that $z_1$ and $z_2$ must lie on the same connected arc $c_i$ ($i = 1$ or $i = 2$), for otherwise the intersection $\tilde{z}_1 \tilde{z}_2 \cap \gamma_2$ would consist of two distinct points (the two endpoints of $c_1$ and $c_2$ lying on $\gamma_2$). If $z_1, z_2 \in c_1$, since $c_1$ is a connected arc lying in $D$, we have $\tilde{z}_1 \tilde{z}_2 \subset c_1 \cup \gamma_2 \subset D \cup \gamma_2$ and the claim follows. This completes the proof of the Proposition.

Using the Schwarz reflection principle and the above lemma, we can prove the following

**Corollary 2.1.** Let $D$ be as in Theorem \textbf{1.1} and suppose that $\gamma_2$ is an arc of a circle. Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk and $U^+ = \{z \in U : \Im z > 0\}$ be the upper half-disk. Let $f : U^+ \rightarrow \mathcal{D}$ be a conformal map such that $f([-1, 1]) = \gamma_2$. Then $f$ extends to a conformal map from $U$ onto the convex domain $D^*$.

**Proof.** Assume $\gamma_2$ is an arc of a circle $\partial B(z_0, r)$ of radius $r$ centered at $z_0$.

Consider the function $\tilde{f} : U \rightarrow \mathbb{C}$ defined by

\[ \tilde{f}(z) = \begin{cases} f(z), & z \in U^+ \\ z_0 + \frac{r^2}{f(z) - z_0}, & z \in U \setminus U^+ \end{cases}. \]

Since $f$ maps the line segment $[-1, 1]$ onto the arc $\gamma_2$ of the circle $\partial B(z_0, r)$, by the Schwarz symmetry principle it follows that $\tilde{f}$ is a conformal extension of $f$, from the unit disk $U$ onto the domain $D^* = D \cup \gamma_2 \cup D_s$, which by Proposition \textbf{2.1} is a convex domain. \qed
Corollary 2.2. If \( f \) is as in Corollary 2.1, then for any \( \theta \in [0, 2\pi) \) arbitrarily fixed, \( r \left| f'(re^{i\theta}) \right| \) is an increasing function of \( r \in (0, 1) \).

Proof. As in [13], we have:

\[
\frac{\partial}{\partial r} \ln r \left| f'(re^{i\theta}) \right| = \frac{1}{r} + \Re \frac{\partial}{\partial r} \ln f'(re^{i\theta}) \]
\[
= \frac{1}{r} + \Re \frac{\partial}{\partial r} \ln f'(re^{i\theta}) \]
\[
= \frac{1}{r} + \Re \frac{e^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} \]
\[
= \frac{1}{r} \Re(1 + re^{i\theta} f''(re^{i\theta})) \]

for any \( r \in (0, 1) \) and \( \theta \in (0, 2\pi) \).

By the above proposition, \( f \) extends to a convex map \( f : U \rightarrow D^* \); it is known (see [9]) that any convex map \( f : U \rightarrow \mathbb{C} \) satisfies the inequality

\[
\Re(1 + zf''(z)f'(z)) > 0, \quad z \in U,
\]

which shows that the quantity on the right side of (2.3) is strictly positive, and therefore \( \ln r \left| f'(re^{i\theta}) \right| \) is a strictly increasing function of \( r \in (0, 1) \) for any \( \theta \in [0, 2\pi) \) arbitrarily fixed, which proves the claim.

To complete the proof of Theorem 1.1 in the case when \( \gamma_2 \) is an arc of a circle we will use the following theorem which may be of independent interest.

Theorem 2.3. Let \( U_d = \{ \zeta \in \mathbb{R}^d : \| \zeta \| < 1 \} \) be the unit ball in \( \mathbb{R}^d \), \( d \geq 2 \), and let \( U_d^+ = \{ \zeta = (\zeta_1, \ldots, \zeta_d) \in U_d : \zeta_n > 0 \} \) be the upper hemisphere in \( \mathbb{R}^d \). Suppose that \( V : U_d^+ \rightarrow (0, \infty) \) is a continuous potential for which \( r^2 V(r\zeta) \) is a nondecreasing function of \( r \in (0, \frac{1}{\| \zeta \|}) \) for any \( \zeta \in U_d^+ \) arbitrarily fixed.

That is, suppose that

\[
r_1^2 V(r_1 \zeta) \leq r_2^2 V(r_2 \zeta),
\]

for all \( \zeta \in U_d^+ \), \( 0 < r_1 < r_2 < \frac{1}{\| \zeta \|} \). Let \( B_t \) be a reflecting Brownian motion in \( U_d^+ \) killed on the hyperplane \( H = \{ \zeta = (\zeta_1, \ldots, \zeta_d) \in \mathbb{R}^d : \zeta_d = 0 \} \), and let \( \tau_{U_d^+} \) denote its lifetime. Then for any arbitrarily fixed \( t > 0 \) and \( \zeta \in U_d^+ \),
\( P^{\tau_1 \zeta} \left\{ \int_0^{\tau_d^+} V(B_s)ds > t \right\} \) is a nondecreasing function of \( r \in (0, \frac{1}{\|\zeta\|}) \). That is,

\[
(2.5) \quad P^{r_1 \zeta} \left\{ \int_0^{\tau_d^+} V(B_s)ds > t \right\} \leq P^{r_2 \zeta} \left\{ \int_0^{\tau_d^+} V(B_s)ds > t \right\},
\]

for all \( t > 0, \zeta \in U_d^+ \) and

\[
0 < r_1 < r_2 < \frac{1}{\|\zeta\|}.
\]

Moreover, if the inequality in (2.4) is a strict inequality, so is the one in (2.5).

**Remark 2.1.** For \( d = 2 \), the Proposition as stated is proved in [5]. It also follows from the arguments in [13]. However, the proof in [13] can be made to work for all \( d \geq 2 \) and this is the argument we follow here.

**Proof.** Fix \( t > 0, \zeta \in U_d^+ \) and \( 0 < r_1 < r_2 < \frac{1}{\|\zeta\|} \). Following [13], we consider a scaling coupling of reflecting Brownian motions \( (B_t, \tilde{B}_t) \) in the unit ball \( U_d \) starting at \( (r_1 \zeta, r_2 \zeta) \). More precisely, let \( B_t \) be reflecting Brownian motion in \( U_d \) starting at \( r_1 \zeta \in U_d \), with its natural filtration \( \mathcal{F}_t \), and consider

\[
(2.6) \quad \tilde{B}_t = \frac{1}{M_{\alpha_t}} B_{\alpha_t}, \quad t \geq 0,
\]

where

\[
(2.7) \quad M_t = \frac{r_1}{r_2} \vee \sup_{s \leq t} \|B_s\|,
\]

\[
(2.8) \quad A_t = \int_0^t \frac{1}{M_s^2} ds,
\]

and

\[
(2.9) \quad \alpha_t = \inf \{ s > 0 : A_s \geq t \}.
\]

Theorem 2.3 and Remark 2.4 of [13] show that \( \tilde{B}_t \) is an \( (\mathcal{F}_{\alpha_t}) \)-adapted reflecting Brownian in \( U_n \).
Letting $\tau_{U_d}^{+}, \tilde{\tau}_{U_d}^{+}$ denote the killing times of $B_t$, respectively $\tilde{B}_t$, on the hyperplane $H = \{\zeta = (\zeta_1, ..., \zeta_d) \in \mathbb{R}^d : \zeta_d = 0\}$, we have almost surely

$$\tau_{U_d}^{+} = \inf\{s > 0 : B_s \in H\} = \inf\{\alpha_u > 0 : B_{\alpha_u} \in H\} = \inf\{\alpha_u > 0 : \tilde{B}_u \in H\} = \alpha_{\inf\{u > 0 : \tilde{B}_u \in H\}} = \alpha_{\tilde{\tau}_{U_d}^{+}},$$

and therefore we obtain

$$\int_{0}^{\tau_{U_d}^{+}} V(B_s)ds = \int_{0}^{\alpha_{\tilde{\tau}_{U_d}^{+}}} V(B_s)ds = \int_{0}^{\tilde{\tau}_{U_d}^{+}} V(B_{\alpha_u})d\alpha_u = \int_{0}^{\tilde{\tau}_{U_d}^{+}} V(B_{\alpha_u})M_{\alpha_u}^{2} du \leq \int_{0}^{\tilde{\tau}_{U_d}^{+}} V(\frac{1}{M_{\alpha_u}}B_{\alpha_u})du = \int_{0}^{\tilde{\tau}_{U_d}^{+}} V(\tilde{B}_u)du.$$ 

The inequality above follows from the assumption that $r^2 V(r\zeta)$ is a nondecreasing function of $r$ for $\zeta \in U_d^+$ arbitrarily fixed:

$$V(B_{\alpha_u}) = 1^2 V(1B_{\alpha_u}) \leq \frac{1}{M_{\alpha_u}^2} V(\frac{1}{M_{\alpha_u}}B_{\alpha_u}),$$

since by (2.7) we have $M_{\alpha_u} \leq 1$ for all $u \geq 0$.

By the construction above, $(B_t, \tilde{B}_t)$ is a pair of reflecting Brownian motions in $U_d$ starting at $(r_1\zeta, r_2\zeta)$, and the inequality (2.10) shows that we have in particular

$$Pr_{r_1\zeta}\left\{\int_{0}^{\tau_{U_d}^{+}} V(B_s)ds > t\right\} \leq Pr_{r_2\zeta}\left\{\int_{0}^{\tilde{\tau}_{U_d}^{+}} V(\tilde{B}_s)ds > t\right\},$$

which proves the first part of the Theorem (2.3).

To prove the strict increasing part of the theorem, we will use the following support lemma for the $n$-dimensional Brownian motion (see [13], page 374.)
Lemma 2.4. Given an $d$-dimensional Brownian motion $B_t$ starting at $x$ and a continuous function $f : [0, 1] \rightarrow \mathbb{R}^d$ with $f(0) = x$ and $\varepsilon > 0$, we have

$$P^x(\sup_{t \leq 1} \| B_t - f(t) \| < \varepsilon) > 0.$$  

Assume now that we have strict inequality in (2.4). By the continuity of the potential $V : \overline{U_d^+} \rightarrow (0, \infty)$ and the strict monotonicity of $r^2 V(r\zeta)$ for $0 < r < \frac{1}{\|\zeta\|}$, we have

$$\int_0^1 V((1 - u)r_1\zeta)du < \int_0^1 \left(\frac{r_2}{r_1}\right)^2 V\left(\frac{r_2}{r_1}(1 - u)r_1\zeta\right)du,$$

and therefore we can choose $T > 0$ such that

$$T \int_0^1 V((1 - u)r_1\zeta)du < t < T \int_0^1 \left(\frac{r_2}{r_1}\right)^2 V\left(\frac{r_2}{r_1}(1 - u)r_1\zeta\right)du,$$

and we may further choose $\varepsilon > 0$ and $\delta > 0$ small enough so that

$$\int_0^{1+\frac{\delta}{1+\delta}} V((1 - u)r_1\zeta)du < \frac{T}{1+\delta} \int_0^{1-\frac{\delta}{1+\delta}} \left(\frac{r_2}{r_1}\right)^2 V\left(\frac{r_2}{r_1}(1 - u)r_1\zeta\right)du.$$

Consider the function $f : \mathbb{R} \rightarrow \mathbb{R}^d$ defined by

$$f(s) = (1 - \frac{(1 + \delta)}{T}s)r_1\zeta.$$

With the change of variable $u = \frac{1+\delta}{T}s$, the double inequality in (2.11) can be rewritten as

$$\frac{T}{1+\delta} \int_0^{1+\frac{\delta}{1+\delta}} V((1 - u)r_1\zeta)du < t < \frac{T}{1+\delta} \int_0^{1-\frac{\delta}{1+\delta}} \left(\frac{r_2}{r_1}\right)^2 V\left(\frac{r_2}{r_1}(1 - u)r_1\zeta\right)du.$$

By eventually choosing a smaller $\varepsilon > 0$, and by the uniform continuity of $V$ on $\overline{U^+}$, we also have

$$\int_0^{\frac{T}{1+\delta}} V(b(s))ds < t < \int_0^{\frac{T}{1+\delta}} \left(\frac{r_2}{r_1}\right)^2 V\left(\frac{r_2}{r_1}b(s)\right)ds,$$

for any continuous functions $b : [0, \frac{T}{1+\delta}] \rightarrow \mathbb{R}^d$ such that

$$\sup_{s \geq \frac{T}{1+\delta}} \| b(s) - f(s) \| < \varepsilon.$$
Let $B_t$ and $\tilde{B}_t$ be the reflecting Brownian motions in $U_d$ starting at $r_1 \zeta$, respectively $r_2 \zeta$, as constructed above. By Lemma 2.4, $B_t$ lies in the $\varepsilon$-tube about $f(t)$ for $0 < t < T$ with positive probability. That is,

$$P(\sup_{s \leq T} |B_s - f(s)| < \varepsilon) > 0.$$ 

We may assume that $\varepsilon > 0$ is chosen small enough so that this tube does not intersect $\partial U$, and therefore on a set $Q$ of positive probability, the coupled Brownian motion $\tilde{B}_s$ does not reach $\partial U_d$, hence the process $M_s$ is constant on this set. Thus, on $Q$ we have

(2.13) \[ M_s = \frac{r_1}{r_2}. \]

(2.14) \[ A_s = \int_0^s \frac{1}{M_0^2} du = \left( \frac{r_2}{r_1} \right)^2 s. \]

(2.15) \[ \alpha_s = A_s^{-1} = \left( \frac{r_1}{r_2} \right)^2 s. \]

and $\tau_{U_d}^+ = A_{\tau_{U_d}^+} = \left( \frac{r_2}{r_1} \right)^2 \tau_{U_d}^+$. Therefore on $Q$ we have

(2.16) \[ \int_0^{\tau_{U_d}^+} V(\tilde{B}_s) ds = \int_0^{\left( \frac{r_2}{r_1} \right)^2 \tau_{U_d}^+} V\left( \frac{1}{M_{\alpha_s}} B_{\alpha_s} \right) ds = \int_0^{\tau_{U_d}^+} \left( \frac{r_2}{r_1} \right)^2 V\left( \frac{r_2}{r_1} B_u \right) du > \int_0^{\tau_{U_d}^+} V(B_s) ds. \]

Also, by the construction of the set $Q$ we have $\frac{1-\varepsilon}{1+\delta} T < \tau_{U_d}^+ < \frac{1+\varepsilon}{1+\delta} T$ on $Q$, and combining with (2.14) and (2.16), we obtain the strict inequality

(2.17) \[ \int_0^{\tau_{U_d}^+} V(B_s) ds \leq \int_0^{\tau_{U_d}^+} V(B_s) ds \leq t < \int_0^{\tau_{U_d}^+} V(B_s) ds. \]
almost surely on \( Q \).

Therefore we have:

\[
P^r_1 \mathbb{1} \left\{ \int_0^{\tau^+_d} V(B_s)ds > t \right\} = P^r_1 \mathbb{1} \left\{ \int_0^{\tau^+_d} V(B_s)ds > t, Q \right\} + P^r_1 \mathbb{1} \left\{ \int_0^{\tau^+_d} V(B_s)ds > t, Q^c \right\} \\
\leq P^r_2 \mathbb{1} \left\{ \int_0^{\tilde{\tau}^+_d} V(\tilde{B}_s)ds > t, Q^c \right\} \\
< P^r_2 \mathbb{1} \{ Q \} + P^r_2 \mathbb{1} \left\{ \int_0^{\tilde{\tau}^+_d} V(\tilde{B}_s)ds > t, Q^c \right\} \\
= P^r_2 \mathbb{1} \left\{ \int_0^{\tilde{\tau}^+_d} V(\tilde{B}_s)ds > t, Q \right\} + P^r_2 \mathbb{1} \left\{ \int_0^{\tilde{\tau}^+_d} V(\tilde{B}_s)ds > t, Q^c \right\} \\
= P^r_2 \mathbb{1} \left\{ \int_0^{\tilde{\tau}^+_d} V(\tilde{B}_s)ds > t \right\},
\]

which proves the strict inequality in (2.5) in the case when the \( r^2 V(r \zeta) \) is a strictly increasing function of \( r \), ending the proof of Theorem 2.3. \( \Box \)

## 3 Proof of Theorem 1.1 and Corollary 1.2

For the proof of Theorem 1.1 we will distinguish the two cases.

**Case 1. Suppose \( \gamma_2 \) is an arc of a circle.**

Let \( f \) a the conformal mapping given by Corollary 2.1 and let \( B_t \) be a reflecting Brownian motion in \( U^+ \) killed on hitting \([-1, 1]\), and denote its lifetime by \( \tau_{U^+} \). By Corollary 2.2 the potential \( V : U^+ \to \mathbb{R} \) defined by \( V(z) = |f'(z)|^2 \) satisfies the hypothesis of Theorem 2.3 and therefore we have

\[
P^{z_1} \left\{ \int_0^{\tau^+_U} |f'(B_s)|^2 ds > t \right\} \leq P^{z_2} \left\{ \int_0^{\tau^+_U} |f'(B_s)|^2 ds > t \right\},
\]
for all $t > 0$ and $z_1 = r_1 e^{i\theta}$, $z_2 = r_2 e^{i\theta}$ with $0 < r_1 < r_2 < 1$ and $0 < \theta < \pi$. By Lévy’s conformal invariance of the Brownian motion, this is exactly the same as

$$P^{f(z_1)} \{ \tau_D > t \} \leq P^{f(z_2)} \{ \tau_D > t \},$$

where $\tau_D$ is as in Theorem 1.1. From this it follows that the function $u(z) = P^{z} \{ \tau_D > t \}$ is nondecreasing as $z$ moves toward $\gamma_1$ along the curve $\gamma_0 = f\{ re^{i\theta} : 0 < r < 1 \}$, for any $\theta \in (0, \pi)$ arbitrarily fixed. This together with the real analyticity of the function $u(z)$ implies that $u(z)$ is in fact strictly increasing along the family of curves $\{ \gamma_0 : 0 < \theta < \pi \}$, which completes the proof of Theorem 1.1 when $\gamma_2$ is an arc of a circle.

Case 2. Suppose $\gamma_1$ is an arc of a circle.

Without loss of generality we may assume that $\gamma_1$ is an arc of the unit circle centered at the origin. An argument similar to the one in Proposition 2.1 shows that $0 \notin D$, and if $0 \in \partial D$ then the domain $D$ is a sector of the unit disk. It either case, the origin belongs to $U \setminus D$.

We claim that $U \setminus D$ is starlike with respect to the origin. If $0 \in \partial D$, the set $D$ is a sector of the unit disk $U$ and the claim follows. We can assume therefore that $0 \notin \overline{D}$. By the angle restriction in the hypothesis of our theorem, together with the convexity of the domain, it follows that $D$ is contained in a sector of the unit disk $U$, which without loss of generality may be assumed to be symmetric with respect to the imaginary axis. That is, $D \subset \{ z \in U : \alpha < \arg z < \pi - \alpha \}$, where $\alpha = \min\{ \arg \gamma_1(0), \arg \gamma_1(1) \} \in (0, \pi/2)$. Let $z \in U \setminus D$ and $t \in [0, 1]$ be arbitrarily fixed. If $z \notin (\alpha, \pi - \alpha)$ then $tz \in U \setminus \{ z \in U : \alpha < \arg z < \pi - \alpha \} \subset U \setminus D$. Thus $tz \in U \setminus D$ in this case. If $z \in (\alpha, \pi - \alpha)$ and $tz \notin U \setminus D$, then, since $\frac{1}{|z|}z \in \gamma_1 \subset \overline{D}$, we obtain by the convexity of $D$ that the line segment with endpoints $tz$ and $\frac{1}{|z|}z$ is contained in $D$, and in particular it follows that $z \in D$, a contradiction. In both cases we obtained that $tz \in U \setminus D$, which proves that $U \setminus D$ is starlike with respect to the origin.

We now follow the proof of Theorem 2.3 in the case $d = 2$. For arbitrarily fixed $t > 0$ and $r_1 e^{i\theta}, r_2 e^{i\theta} \in D$ with $r_1 < r_2$, let $(B_t, \tilde{B}_t)$ be a scaling coupling of reflecting Brownian motions in the unit disk $U$ starting at $(r_1 e^{i\theta}, r_2 e^{i\theta})$, as in the case of Theorem 1.1. We note that that if for $s > 0$ we have $\frac{1}{M_s} B_s \in \gamma_2 \subset U \setminus D$, then by the starlikeness of the set $U \setminus D$ also $B_s \in U \setminus D$. That is,

$$\frac{1}{M_s} B_s \notin D \Rightarrow B_{s'} \notin D \text{ for some } 0 < s' \leq s.$$
Recalling that $\tilde{B}_s = \frac{1}{\mu_{\alpha_s}} B_{\alpha_s}$ and that $\alpha_s \leq s$ for all $s > 0$, we can rewrite (3.3) as follows

(3.4) $\tilde{B}_s \notin D \Rightarrow B_{s'} \notin D$ for some $0 < s' \leq \alpha_s \leq s$.

This in turn is equivalent to

(3.5) $\tau_{\gamma_2} \leq \alpha_{\tilde{\gamma}_2} \leq \tilde{\tau}_{\gamma_2}$,

where $\tau_{\gamma_2}$ and $\tilde{\tau}_{\gamma_2}$ denote the killing times of $B_t$, respectively $\tilde{B}_t$, on the curve $\gamma_2$. From this, it follows that we have

(3.6) $P^{r_1 e^{i\theta}} \{\tau_{\gamma_2} > t\} \leq P^{r_2 e^{i\theta}} \{\tilde{\tau}_{\gamma_2} > t\}$.

Thus the function $u(z) = P^z \{\tau_D > t\}$ is nondecreasing on the part of the radii $r_\theta = \{re^{i\theta}, 0 < r < 1\}$ which is contained in the domain $D$. As before, this together with the real analyticity of the function $u$ shows that it is in fact strictly increasing. This completes the proof of the theorem.

The Corollary 1.2 follows from Theorem 1.1 exactly as in [5]. Briefly, by Proposition (3.5) of [5],

(3.7) $P^z \{\tau_D > t\} = e^{-\mu t} \int_D \psi_1(z) dw + \int_D R_t(z,w) dw$,

where

$e^{\mu t} R_t(z,w) \to 0$,

as $t \to \infty$, uniformly in $z, w \in D$. From this it follows that if $\gamma_2$ is an arc of a circle, the function $\psi$ is nondecreasing on the hyperbolic radii $\gamma_\theta$ and that if $\gamma_1$ is an arc of a circle the function $\psi$ is nondecreasing along the part of the radii $r_\theta$ which are in the domain. The strict increasing follows from the real analyticity. This proves Corollary 1.2.

In our application of Theorem 2.3, the strict increasing was not really used as this was derived from the fact that quantities involved are solutions of “nice” partial differential equations and hence are real analytic. It may be that the strict increasing of the quantity

$P^{r_\zeta} \left\{\int_{0}^{\tau_{U_d^+}} V(B_s) ds > t\right\}$

can also be proved by relating it to an appropriate PDE.

We end with some other remarks related to Theorem 2.3. Consider the Schrödinger operator $\frac{1}{2} \Delta u - Vu$ in $U_d^+$ with Dirichlet boundary conditions
on the part of $\partial U_d^+$ lying in the hyperplane $H = \{(\zeta_1, \ldots, \zeta_d) \in \mathbb{R}^d : \zeta_n = 0\}$, and Neumann boundary conditions on the “top” portion of the sphere. If we let $P_t^V(\xi, \zeta)$, $\xi, \zeta \in U_d^+$ be the heat kernel for this problem, then

$$u(\xi) = E^{\xi}\left\{e^{-\int_0^{\tau_{U_d^+}} V(B_s) \, ds} ; \tau_{U_d^+} > t \right\} = \int_{U_d^+} P_t^V(\xi, \zeta) \, d\zeta.$$

It would be interesting to investigate (under suitable assumptions on $V$) the monotonicity properties for the function $u(\xi)$. This will lead to “hot-spots” results for the Schrödinger operator defined above. We also refer the reader to [6] where a related problem is studied for the Dirichlet Schrödinger semigroup (in that case one has that near the boundary, and for large values of $t > 0$, the function corresponding function $u(\xi)$ decreases).

References

[1] R. Atar, *Invariant wedges for a two–point reflecting Brownian motion and the “hot spots” problem*. Elect. J. of Probab. 6, 18(2001), 1–19.

[2] R. Atar and K. Burdzy, *On the Neumann eigenfunctions in lip domains*. (preprint).

[3] R. Atar and K. Burdzy, *On nodal lines of Neumann eigenfunction*. (preprint).

[4] R. Bañuelos and K. Burdzy, *On the “hot spots” conjecture of J. Rauch*. J. Funct. Anal. 164 (1999), 1–33.

[5] R. Bañuelos and M. Pang, *An inequality for potentials and the “hot–spots” conjecture*, Indiana Math. J. (to appear)

[6] R. Bañuelos and M. Pang, *Lower bound gradient estimates for solutions of Schrödinger operators and heat kernels*, Comm in PDE, 64 (1999), 499–543.

[7] R. Bass and K. Burdzy, *Fiber Brownian motion and the “hot spots’ problem*. Duke Math J. 105 (2000), 25–58.

[8] K. Burdzy and W. Werner, *A counterexample to the “hot spots” conjecture*. Ann. Math. 149 (1999), 309–317.

[9] P. Duren, Univalent Functions, Springer Verlag New York, 1983.
[10] P. Freitas, Closed nodal lines and interior hot spots of the second eigen-
function of the Laplacian on surfaces. Indiana University Mathematics
Journal. 51 (2002), 305–316.

[11] D. Jerison and N. Nadirashvili The “hot spots” conjecture for domains
with two axes of symmetry. J. Amer. Math. Soc. 13 (2000), 741–772.

[12] B. Kawohl, Rearrangements and Convexity of Level Sets in PDE, Lecture
Notes in Math., 1150, Springer, Berlin, 1985.

[13] M. Pascu, Scaling coupling of reflected Brownian motion and the hot
spots problem. Trans. Amer. Math. Soc. 354 (2001), 4681–4702.

[14] D. Stroock, Probability Theory, An Analytic View, Cambridge University
Press, 1993.