Sign equidistribution of Legendre polynomials

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Abstract
We prove sign equidistribution of Legendre polynomials: the ratio between the lengths of the regions in the interval $[-1, 1]$ where the Legendre polynomial assumes positive versus negative values, converges to one as the degree grows. The proof method also has application to the symmetry conjecture for a basis of eigenfunctions in the sphere.

1 Introduction

The importance of Legendre polynomials, from classical potential theory to modern computational methods, stems from the method of separation of variables in mathematical physics. They appear naturally in the spherical harmonic decompositions of functions in spherical coordinates.

The zeros of these polynomials have been extensively studied in the past two centuries. They are known to be simple and belong to the interval $[-1, 1]$. A classical result due to Bruns affirms that the roots $\theta_j$ of Legendre polynomials $P_n(\cos(\theta))$ equidistribute in $[0, \pi]$ as the degree $n$ grows (cf. [1]). More concretely, if we denote the increasing sequence of zeroes by

$$\theta_1 < \theta_2 < \cdots < \theta_n$$

the following inequalities hold

$$\frac{j - \frac{1}{2}}{n + \frac{1}{2}} \pi \leq \theta_j \leq \frac{j}{n + \frac{1}{2}} \pi$$
for \( j = 1, \ldots, n \) (cf. equation 6.6.2 in [8]). Markoff and Stieljes improved this to

\[
\frac{j - \frac{1}{2}}{n} \pi \leq \theta_j \leq \frac{j}{n + 1} \pi
\]

for \( j = 1, \ldots, [n/2] \) that extends by symmetry considerations to inequalities for all the zeros (cf. equation 6.6.4 in [8]; or the original articles [4, 7, 9]). This was finally improved by Szegö in 1936 who showed

\[
\frac{j - \frac{1}{4}}{n + \frac{1}{2}} \pi \leq \theta_j \leq \frac{j}{n + 1} \pi
\]

(cf. [9] and equation 6.6.5 in [8]).

The main result of this paper explores yet another equidistribution property of Legendre polynomials, that we call sign equidistribution: we would say that a sequence of real polynomials \( P_n \), or its zeros, sign equidistribute in an interval \( I \) if the length of the set where the polynomial \( P_n \) is positive equals the length of the set where the polynomial is negative in the limit \( n \to \infty \). This notion was introduced in [5] in connection with the symmetry conjecture for the semiclassical limit of eigenfunctions on compact Riemannian manifolds.

**Theorem 1.1** (Sign equidistribution) Let \( \{\theta_j\}_{j=1}^n \) be the increasing sequence of zeros corresponding to the \( n \)th Legendre polynomial \( P_n(\cos(\theta)) \). For any closed interval \( I \subseteq (0, \pi) \) containing an even number of roots we have

\[
\left| \sum_{\theta_j \in I} (-1)^j \theta_j \right| = \frac{\text{length}(I)}{2} + O(n^{-1})
\]

where the constant is independent on \( n \) but might depend on \( I \).

Unfortunately, the bounds of Bruns–Szegö are not enough to obtain this, and improving Szegö’s result seems a difficult task. We follow a different route; our method of proof provides a general result that is of independent interest, from which Theorem 1.1 follows straightforwardly. Indeed,

**Theorem 1.2** Let \( \{\theta_j\}_{j=1}^n \) be the increasing sequence of zeros corresponding to the \( n \)th Legendre polynomial \( P_n(\cos(\theta)) \). Let \( I \subseteq (0, \pi) \) be a fixed closed interval. Let \( f \) be any analytic function in a neighbourhood \( U \subseteq \mathbb{C} \) of the interval \( I \). The following holds:

\[
\sum_{\theta_j \in I} (-1)^j f(\theta_j) = \sum_j (-1)^j f \left( \frac{2\pi j - \pi/2}{2n + 1} \right) + O(n^{-1})\|f\|_{L^\infty(U)}
\]

where the second sum extends over those \( j \) such that \( \theta_j \in I \) and the constant is independent on \( n \) and \( f \) but might depend on \( I \) and \( U \).
This result is intimately related to the so-called symmetry conjecture on the semi-classical limit of eigenfunctions. The symmetry conjecture asserts that on a given Riemannian manifold \((M, g)\) the area of positiveness of a Laplace-Beltrami eigenfunction tends to equal its area of negativeness as the eigenvalue grows. The conjecture has been disproved by the authors in [5]. The counterexamples are explicit but the proof is a computer assisted argument for the three dimensional flat torus. It is nevertheless easy to observe that the conjecture is true in the particular case of the two dimensional flat torus (loc. cit.). This might suggest its truth in the case of surfaces.

In order to put the conjecture in context let us recall the following result contained in the seminal work of Donnelly and Fefferman

**Theorem 1.3** (Corollary 7.10 in [2]) Let \((M, g)\) be a real analytic Riemannian manifold. There exists a constant \(C\) such that, for any eigenfunction \(\psi\) of the Laplace-Beltrami operator:

\[
\frac{1}{C} \leq \frac{\text{vol}(\{x \in M : \psi(x) > 0\})}{\text{vol}(\{x \in M : \psi(x) < 0\})} \leq C.
\]

We emphasize that the constant \(C\) depends on the manifold, but not on the eigenvalue. This was improved to general smooth metrics on surfaces by Nadirashvili in [6]. In the case of the classical spherical harmonics \(Y_{nm}\) (i.e the standard basis of eigenfunctions on the sphere, see below) it can be proved as a consequence of the Bruns–Szegö inequalities.

As a rather straightforward application of Theorem 1.2 in the case of \(f(z) = \cos(z)\) we will provide a partial result towards the symmetry conjecture in the two dimensional sphere:

**Conjecture 1.4** (Symmetry) Let \(\{\psi_\lambda\}\) be a sequence of spherical harmonics. The limit

\[
\frac{\text{vol}(\{x \in M : \psi_\lambda(x) > 0\})}{\text{vol}(\{x \in M : \psi_\lambda(x) < 0\})} \to 1
\]

holds as \(\lambda\) grows to infinity.

Before stating it let us introduce the set \(B\) of eigenfunctions on \(S^2\) that consists of the Legendre polynomials \(P_n(\cos(\theta))\), and the eigenfunctions \(P_n^m(\cos(\theta))\cos(m\phi)\) and \(P_n^m(\cos(\theta))\sin(m\phi)\), where \(P_n^m\) denotes the associated Legendre polynomials, \(1 \leq m \leq n\), \(\phi\) is the azimuthal angle variable and \(\theta\) the polar angle variable. We emphasize that the linear combinations of these functions do not belong to \(B\) (otherwise it would simply contain all the spherical harmonics of degree \(n\)).

**Theorem 1.5** (Symmetry for a basis of eigenfunctions of \(S^2\)) For any sequence of eigenfunctions \(\psi_n \in B\) with increasing eigenvalue \(n(n + 1)\):

\[
\lim_{n \to \infty} \frac{\text{vol}([x \in S^2 : \psi_n(x) > 0])}{\text{vol}([x \in S^2 : \psi_n(x) < 0])} = 1
\]
We remark that in the case of tori of any dimension, the existence of a basis of eigenfunctions with the above property is trivial.

The paper is organized as follows. In Sect. 2 we present the proof of Theorem 1.2, and explain how Theorem 1.1 follows from it. Section 3 is devoted to the proof of Theorem 1.5 as an application of these results.

2 Proof of Theorem 1.2

Before proceeding to the proof let us state a technical result we shall need later.

**Theorem 2.1** (Laplace’s formula) For any $\epsilon > 0$, the asymptotic

$$P_n(\cos(\theta)) = \sqrt{\frac{2}{n\pi \sin(\theta)}} \cos \left( \left( n + \frac{1}{2} \right) \theta - \frac{1}{4} \pi \right) + E_n(\theta),$$

with $E_n(\theta) = O(n^{-\frac{3}{2}})$, holds uniformly for $\theta \in (\epsilon, \frac{\pi}{2} - \epsilon)$. The first derivative satisfies

$$\frac{\partial}{\partial \theta} P_n(\cos(\theta)) = \frac{\partial}{\partial \theta} \left( \sqrt{\frac{2}{n\pi \sin(\theta)}} \cos \left( \left( n + \frac{1}{2} \right) \theta - \frac{1}{4} \pi \right) \right) + E'_n(\theta),$$

with $E'_n(\theta) = O(n^{-\frac{1}{2}})$. The constants involved are independent on $n$ but do depend on the fixed $\epsilon > 0$.

The first formula corresponds to the classical Laplace’s formula for which a number of proofs and refinements can be found in Szegö’s treatise [8]. As for the second part, one can adapt the arguments there to provide a proof, we provide details on the Appendix that complement the arguments within Szegö’s treatise. The authors are grateful to the anonymous referee who pointed out that the second formula can also be found as a particular case of equation 8.8.1 in [8].

The basic idea is to employ the argument principle of complex analysis for a specific choice of contour integration inside a strip containing $I \subset (\epsilon, \frac{\pi}{2} - \epsilon)$, which provides the identity

$$\frac{1}{2\pi i} \oint_{\Gamma} f(z) \frac{\partial P_n(\cos(z))}{\partial z} dz = \sum_{\theta_j \in I} (-1)^j f(\theta_j)$$

where the contour has the form of a braid alternating winding number around consecutive zeros as in the figure (cf. [3]). Notice that we can restrict our analysis to the $(0, \frac{\pi}{2})$, as the Legendre polynomials satisfy $P_n(-x) = (-1)^n P_n(x)$. For the sake of clarity, let us focus now on the particular case $I := (\epsilon, \frac{\pi}{2} - \epsilon)$. 

In what follows we will write $E$ instead of $E_n$ and define

$$A(\theta) := \sqrt{\frac{2}{n\pi \sin(\theta)}} \cos \left( \left( n + \frac{1}{2} \right) \theta - \frac{1}{4} \pi \right)$$

to ease the notation, the dependence on $n$ is understood. Theorem 2.1 implies that the integral can be written as

$$\frac{1}{2\pi i} \oint_{\Gamma} \left( \frac{f(z)A'(z)}{A(z)} + \frac{f(z)E'(z)}{A(z)} - f(z)(A'(z) + E'(z)) \frac{E(z)}{A(z)(A(z) + E(z))} \right) dz$$

where we have used that

$$\frac{1}{A(z) + E(z)} - \frac{1}{A(z)} = -\frac{E(z)}{A(z)(A(z) + E(z))}.$$

The main term clearly satisfies

$$\frac{1}{2\pi i} \oint_{\Gamma} \frac{f(z)A'(z)}{A(z)} dz = \sum_{\theta_j \in I} (-1)^j f(\theta_j^0).$$

where

$$\theta_j^0 = \frac{j - \frac{1}{4}}{n + \frac{1}{2}} \pi,$$

denote the zeroes of $A$ i.e those of $\cos((n + 1/2)\theta - \pi/4)$.

To bound the remaining terms, the idea is to show that they consist of a gradient, which integrates to zero, plus some extra terms that go to zero as $n$ grows to infinity.

More precisely, observe that, on the one hand

$$f(z)(A' + E') \frac{E}{A(A + E)} = f(z)(A' + E') \frac{E}{A^2} - f(z)(A' + E') \frac{E^2}{A^2(A + E)}.$$
and on the other hand,
\[ \frac{f(z)E'}{A} - f(z)A' \frac{E}{A^2} = \frac{\partial}{\partial z} \left( f(z) \frac{E}{A} \right) - \frac{f'(z)E}{A}. \]

Putting both expressions together we see that the remaining term is equal to
\[ \frac{1}{2\pi i} \oint_{\Gamma} \frac{\partial}{\partial z} \left( f(z) \frac{E}{A} \right) dz - \frac{1}{2\pi i} \oint_{\Gamma} \left( f'(z)E - f(z)(A' + E') \frac{E^2}{A^2(A + E)} + f(z)E' \frac{E}{A^2} \right) dz. \]

The first integral is clearly zero, since the integrand is the derivative of \( f(z)E/A \), which is meromorphic in a domain containing the curve \( \Gamma \) and no pole lies on the curve. As for the other one, it can be bounded as
\[ O \left( n^{-1} + n^{-3/2} \| A' \|_\infty + \frac{n^{-1}}{\alpha^2} \ell(\Gamma) \sup_{z \in \Gamma} (|f(z)| + |f'(z)|) \right). \]

Here \( \ell(\Gamma) \) is the length of the contour \( \Gamma \), and we have used the fact that \( E = O(n^{-\frac{3}{2}}) \), \( E' = O(n^{-\frac{3}{2}}) \) and, for the denominators, we claim \( A = \Omega(n^{-\frac{1}{2}}) \), i.e. the fact that the contour can be chosen so that on it, \( |A| \geq n^{-\frac{1}{2}} \alpha \) for some \( \alpha > 0 \) depending on \( \epsilon \) but independent of \( n \). The bound \( \| A' \|_\infty = \tilde{O}(n^{1/2}) \) together with the above shows that
\[ \sum_{\theta_j \in I} (-1)^j f(\theta_j) = \sum_{\theta_j \in I} (-1)^j f(\theta_j^0) + O(n^{-1}) \sup_{z \in \Gamma} (|f(z)| + |f'(z)|) \]
and using Cauchy’s integral formula we get
\[ \sum_{\theta_j \in I} (-1)^j f(\theta_j) = \sum_{\theta_j \in I} (-1)^j f(\theta_j^0) + O(n^{-1}) ||f||_{L^\infty(U)} \]
as claimed.

To justify our claim that \( A = \Omega(n^{-\frac{1}{2}}) \) let us consider the contour as in Fig. 1 that stays \( \frac{1}{2n+1} \) away from the zeroes \( \theta_j^0 \) of \( A(\theta) \). By Bruns–Szegö inequality, the set of balls just described also contain the zeroes \( \theta_j \) of \( P_n \). On the other hand
\[ \alpha = \inf \left| \cos \left( \left( n + \frac{1}{2} \right) z - \frac{1}{4} \pi \right) \right| > 0 \]
where the infimum is taken on the complement to the union of balls, and it is independent of \( n \).

The argument works, mutatis mutandis, for the zeroes contained in any other interval \( I \subset [0, \pi] \), by simply adapting the contour \( \Gamma \). Finally, Theorem 1.1 follows as a
corollary taking \( f(z) = z \) and using the fact that the zeroes \( \theta^0_j \) clearly equidistribute which implies that the main term

\[
\sum_{\theta^0_j \in I} (-1)^j \theta^0_j = \left| I \right| \frac{1}{2} + O(n^{-1}).
\]

3 Application to the symmetry conjecture

Let us observe first that any eigenfunction in the form of an associated Legendre polynomial already satisfies the conjecture, in fact (because of the symmetries of the \( \cos(m\varphi) \) and \( \sin(m\varphi) \) factors) the quotient is exactly one half for any degree! Furthermore, Legendre polynomials of odd degree \( k \) also satisfy the conjecture, since they verify \( P_k(-z) = -P_k(z) \). Thus we can focus our attention to the Legendre polynomials of even degree \( n \).

The surface area of the part of the two dimensional sphere \( C \) contained between two parallel planes \( z = a \) and \( z = b \) is \( 2\pi(a - b) \).

Thus, if \( z_j \) are the roots of an even degree Legendre polynomial \( P_n(z) \), either its area of positiveness or its area of negativeness is \( 2\pi \) times the absolute value of the alternating sum

\[
\sum_{j=1}^{n} (-1)^j z_j = \sum_{j=1}^{n} (-1)^j \cos(\theta_j).
\]
Applying Theorem 1.2 we have that, for any \( \epsilon > 0 \), the above series is equal to

\[
\sum_{\theta_j \in (\epsilon, \pi - \epsilon)} (-1)^j \cos \left( \frac{2\pi j - \pi/2}{2n + 1} \right) + O_\epsilon(n^{-1}) + O(\epsilon)
\]

where the constants on the first error term might depend on \( \epsilon \) but not those on the second (which is purely geometrical in nature, i.e. to compensate for the end points).

But on the other hand, it is easy to see that

\[
I = \sum_{j=1}^{n} (-1)^j \cos \left( \frac{2\pi j - \pi/2}{2n + 1} \right) = -1 + o(1)
\]

where \( o(1) \) is a function that tends to zero as \( n \) grows to infinity. To prove it we shall use the identity

\[
\cos \left( \frac{2\pi (j + 1) - \pi/2}{2n + 1} \right) - \cos \left( \frac{2\pi j - \pi/2}{2n + 1} \right) = -\frac{2\pi}{2n + 1} \sin \left( \frac{2\pi j - \pi/2}{2n + 1} \right) + O(n^{-2})
\]

which comes from the Taylor approximation \( \cos(x) = \cos(x) - \sin(x)\epsilon + O(\epsilon^2) \) valid uniformly in \([\frac{-\pi}{2}, \frac{\pi}{2}]\). Using this we can rewrite the series as

\[
\sum_{j=1}^{n} (-1)^j \cos \left( \frac{2\pi j - \pi/2}{2n + 1} \right) = -\frac{1}{2} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{4\pi}{2n + 1} \sin \left( \frac{2\pi (2k + 1) - \pi/2}{2n + 1} \right) + O(n^{-1}) = -\frac{1}{2} \int_{0}^{\pi} \sin(\theta) d\theta + o(1) = -1 + o(1),
\]

where in the last step we have recognized the sum as a Riemann integral approximation.

Summing up, we get

\[
\sum_{j=1}^{n} (-1)^j z_j = -1 + o(1) + O_\epsilon(n^{-1}) + O(\epsilon).
\]

Therefore, the difference

\[
\text{vol}\{x \in S^2 : P_n(x) > 0\} - \text{vol}\{x \in S^2 : P_n(x) < 0\} = o(1) + O_\epsilon(n^{-1}) + O(\epsilon).
\]

One can divide by \( \text{vol}\{x \in S^2 : P_n(x) < 0\} \), which is clearly bounded below away from zero, and then take the limit as \( n \) grows, obtaining that

\[
\lim_{n \to \infty} \frac{\text{vol}\{x \in S^2 : P_n(x) > 0\}}{\text{vol}\{x \in S^2 : P_n(x) < 0\}} = 1 + O(\epsilon).
\]
Since this is true for any $\epsilon > 0$, this concludes the proof.

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**Declarations**

**Conflict of interest** The authors have no conflict of interest.

### 4. Appendix

As already mentioned the first asymptotic

$$P_n(\cos(\theta)) = \sqrt{\frac{2}{n\pi \sin(\theta)}} \cos \left( \left( n + \frac{1}{2} \right) \theta - \frac{1}{4} \pi \right) + E$$

with $E = O(n^{-3/2})$ is known as Laplace formula. The first part of the statement corresponds to Theorem 8.21.2 from [8]. We refer the reader to this reference for further details. The second part of Theorem 2.1 can be found as a particular case of equation 8.8.1 in [8] (we thank the referee for pointing this out to us). One can as well follow the approach of Stieltjes (cf. Theorem 8.21.5 loc. cit.) which provides an error that can be explicitly written as

$$E(\theta) = \frac{2}{\pi} \text{Im} \left( \frac{e^{i(n+1)\theta} e^{i(\pi/4-\theta/2)}}{(2 \sin \theta)^{1/2}} \int_0^1 t^n (1-t)^{-1/2} \frac{1}{\pi} \int_0^\pi \frac{z \sin^2(\varphi)}{1-z \sin^2(\varphi)} d\varphi dt \right)$$

where

$$z = (1-t) \frac{e^{i(\theta-\pi/2)}}{2 \sin \theta}$$

cf. section 8.5 in [8], specifically equation 8.5.1 considering $R_p(\theta)$ for $p = 1$. Taking derivatives in the identity above and using the fact that $\theta \in (\epsilon, \pi - \epsilon)$, so that $\sin(\theta)$ is bounded away from zero, one concludes the proof as in the original (cf. bound 8.5.5 loc. cit.).

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