ON METALLIC SEMI-SYMMETRIC METRIC $F$–CONNECTIONS

CAGRI KARAMAN

ABSTRACT. In this article, we generate a metallic semi-symmetric metric $F$-connection on a locally decomposable metallic Riemann manifold. Also, we examine some features of torsion and curvature tensor fields of this connection.

1. INTRODUCTION

The topic of connection with torsion on a Riemann manifold has been studied with great interest in literature. Firstly, Hayden defined the concept of metric connection with torsion [3]. For a linear connection $\nabla$ with torsion on a Riemann manifold $(M, g)$, if $\nabla g = 0$, then linear connection $\nabla$ is called a metric connection. Then, Yano constructed a connection whose torsion tensor has the form: $S(X, Y) = \omega(Y)X - \omega(X)Y$, where $\omega$ is a 1–form, [15] and named this connection as semi-symmetric connection.

In [11], Prvanovic has defined a product semi-symmetric $F$–connection on locally decomposable Riemann manifold and worked its curvature properties. A locally decomposable Riemann manifold is expressed by the triple $(M, g, F)$ and the conditions $\nabla F = 0$ and $g(FX, Y) = g(X, FY)$ are provided, where $F, g$ and $\nabla$ are product structure, metric tensor and Riemann connection (or Levi-Civita connection) of $g$ on manifold respectively. For further references, see [8, 9, 10, 12].

The positive root of the equation $x^2 - x - 1 = 0$ is the number $x_1 = \frac{1 + \sqrt{5}}{2}$, which is called golden ratio. The golden ratio has many applications and has played an important role in mathematics. One of them is a golden Riemann manifold $(M, g, \varphi)$ endowed with golden structure $\varphi$ and Riemann metric tensor $g$. The golden structure $\varphi$ created by Crasmareanu and Hretcanu is actually root of the equality $\varphi^2 - \varphi - 1 = 0$ [5]. In [2], the authors have defined golden semi-symmetric metric $F$–connections on a locally decomposable golden Riemann manifold and examined torsion, projective curvature, conharmonic curvature and curvature tensors of this connection. Also, the golden ratio has many important generalizations. One
of the them is metallic proportions or metallic means family which was introduced by de Spinadel in [6, 7]. The positive root of the equation \( x^2 - px - q = 0 \) is called the metallic means family, where \( p \) and \( q \) are two positive integer. Also, the solution of the metallic means family is as follows

\[
\sigma_{p,q} = p + \sqrt{p^2 + 4q}.
\]

These numbers \( \sigma_{p,q} \) are also named \((p, q)\) metallic numbers. In the last equation,

- if \( p = q = 1 \), then the number \( \sigma_{1,1} = \frac{1 + \sqrt{5}}{2} \) is golden ratio;
- if \( p = 2 \) and \( q = 1 \), then the number \( \sigma_{2,1} = 1 + \sqrt{2} \) is silver ratio, which is used for fractal and Cantorian geometry;
- if \( p = 3 \) and \( q = 1 \), then the number \( \sigma_{3,1} = \frac{3 + \sqrt{13}}{2} \) is bronze ratio, which plays an important role in dynamical systems and quasicrystals and so on.

Inspired by the metallic number family, Hretcanu and Crasmareanu was introduced metallic Riemann structure [4]. Indeed, a metallic structure is polynomial structure such that

\[
F^2 p F - q I = 0,
\]

where \( F \) is \((1,1)\)-tensor …eld on manifold.

Given a Riemann manifold \((M, g)\) endowed with the metallic structure \( F \), if

\[
g(FX, Y) = g(X, FY)
\]

or equivalently

\[
g(FX, FY) = pg(FX, Y) + qg(X, Y)
\]

for all vector fields \( X \) and \( Y \) on \( M \), then the triple \((M, g, F)\) is called a metallic Riemann manifold.

In [1], For almost product structures \( J \) and the Tachibana operator \( \phi_F \), the authors proved that the manifold \((M, g, F)\) is a locally decomposable metallic Riemannian manifold iff \( \phi_J g = 0 \). In this article, we made a semi-symmetric metric \( F \)-connection with metallic structure \( F \) on a locally decomposable metallic Riemann manifold. Then we examine some properties related to its torsion and curvature tensors.

2. Preliminaries

Let \( M \) be an \( n \)-dimensional manifold. Throughout this paper, tensor fields, connections and all manifolds are always assumed to be differentiable of class \( C^\infty \)

For a \((1,1)\)-tensor \( F \) and a \((r,s)\)-tensor \( K \), The tensor \( K \) is named as a pure tensor with regard to the tensor \( F \), if the following condition is holds:

\[
K^{j_1\ldots j_r}_{m_1\ldots m_s} F^{m}\ = \ K^{j_1\ldots j_r}_{i_1\ldots i_s} F^{m} = \ldots = K^{j_1\ldots j_r}_{i_1\ldots i_s} F^{i_1 m} = \ldots = K^{j_1\ldots j_r}_{i_1\ldots i_s} F^{i_1 m} \]

where \( K^{j_1j_2\ldots j_r}_{i_1i_2\ldots i_s} \) and \( F \) is the components the tensor \( K \) and \((1,1)\)-tensor \( F \) respectively. Also, the Tachibana operator applied to a pure \((r, s)\)-tensor \( K \) is given
by

$$(\phi_F K)^{j_1 \ldots j_r}_{i_1 \ldots i_s} = F^s_k \partial_{m_1}^{j_1 \ldots j_r} F^s_j \partial_{m_s}^{i_1 \ldots i_s} - \partial_k (K \circ F)^{j_1 \ldots j_r}_{i_1 \ldots i_s}$$

$$+ \sum_{\lambda=1}^r (\partial_{i_\lambda} F^m_k) K^{j_1 \ldots j_r}_{i_1 \ldots i_s}$$

$$+ \sum_{\mu=1}^r \left( \partial_{k} F^\mu_j - \partial_{m} F^\mu_k \right) K^{j_1 \ldots j_r}_{i_1 \ldots i_s},$$

where

$$(K \circ F)^{j_1 \ldots j_r}_{i_1 \ldots i_s} = K^{j_1 \ldots j_r}_{m_1 \ldots i_s} F^m_i = \ldots = K^{j_1 \ldots j_r}_{i_1 \ldots i_s} F^m_i$$

$$= K^{j_1 \ldots j_r}_{i_1 \ldots i_s} \circ F = \ldots = K^{j_1 \ldots j_r}_{i_1 \ldots i_s} \circ F.$$

The equation (2.1) firstly defined by Tachibana [14] and the applications of this operator have been made by many authors [13, 16]. For the pure tensor $K$, if the condition $\phi_F K = 0$ holds, then $K$ is called as a \( \phi \)-tensor. Specially, if the $(1,1)$-tensor $F$ is a product structure, then $K$ is a decomposable tensor [14].

A metallic Riemannian manifold is a manifold $M$ equipped with a $(1,1)$-tensor field $F$ and a Riemannian metric $g$ which satisfy the following conditions:

$$F^2 - pF - qI = 0$$

and

$$g(FX, Y) = g(X, FY)$$

Also, the equation (2.3) equal to $g(FX, FY) = pg(FX, Y) + qg(X, Y)$, where $p, q$ are positive integers. The last two equations in local coordinates are as follows:

$$F^j_i F^i_k = p F^j_k + q \delta^j_i$$

and

$$F^k_i g_{kj} = F^k_j g_{ik},$$

It is obvious that $F^j_i F^i_k = p F^j_i + q g_{ij}$ and $F^i_j = F^j_i$ (symmetry) from (2.4) and (2.5). The almost product structure $J$ and metallic structure $F$ on $M$ are related to each other as follows [4],

$$J_{\pm} = \frac{p}{2} I \pm \left( \frac{2\sigma_{p,q} - p}{2} \right) F$$

or conversely

$$F_{\pm} = \pm \left( \frac{2}{2\sigma_{p,q} - p} J - \frac{p}{2\sigma_{p,q} - p} I \right),$$

where $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ which is the root of the (2.2). Also, it is obvious from (2.7) that a Riemann metric $g$ is pure with regard to a metallic structure $F$ if and only
if the Riemann metric $g$ is pure with regard to the almost product structure $J$. By using (2.7) and (2.1), we have
\[ \phi_p K = \pm \frac{2}{2\sigma_p - p} \phi_J K \] (2.8)
for any $(r, s)$—tensor $K$. We note that a metallic Riemann manifold $(M, g, F)$ is a locally decomposable metallic Riemann manifold if and only if the Riemann metric $g$ is a decomposable tensor, i.e., $(\phi_J g)_{ki} = 0$ and the condition $(\phi_J g)_{ki} = 0$ is equivalent to $\nabla_k J_i^j = 0$ [1].

3. The Metallic Semi-Symmetric metric $F$—connection

Let $(M, g, F)$ be a locally decomposable metallic Riemann manifold. We consider an affine connection $\nabla$ on $M$. If the affine connection $\nabla$ holds
\[ i) \quad \nabla_h g_{ij} = 0 \]
\[ ii) \quad \nabla_h F_{ij} = 0, \]
then it is called a metric $F$—connection. In the special case, when the torsion tensor $\tilde{S}_{ij}^k$ of $\nabla$ is as following shape
\[ \tilde{S}_{ij}^k = \omega_j \delta_i^k - \omega_i \delta_j^k + \frac{1}{q} \left( \omega_t F_j^t F_i^k - \omega_t F_i^t F_j^k \right), \] (3.2)
where $\omega_i$ are local ingredients of an $1$—form, we say that the affine connection $\nabla$ is a metallic semi-symmetric metric connection.

Let $\bar{\Gamma}_{ij}^k$ be the ingredients of the metallic semi-symmetric metric connection $\nabla$. If we put
\[ \bar{\Gamma}_{ij}^k = \Gamma_{ij}^k + T_{ij}^k, \] (3.3)
where $\Gamma_{ij}^k$ and $T_{ij}^k$ are the ingredients of the Riemann connection $\nabla$ of $g$ and $(1, 2)$—tensor field $T$ on $M$ respectively, then the torsion tensor $\tilde{S}_{ij}^k$ of $\nabla$ is as following form
\[ \tilde{S}_{ij}^k = \bar{\Gamma}_{ij}^k - \bar{\Gamma}_{ji}^k = T_{ij}^k - T_{ji}^k. \]
When the connection (3.3) provides the condition (i) of (3.1), by applying the method in [3], we get
\[ T_{ij}^k = \omega_j \delta_i^k - \omega_i \delta_j^k + \frac{1}{q} \left( \omega_t F_j^t F_i^k - \omega_t F_i^t F_j^k \right), \]
where $\omega^k = \omega_i g^i$ and $F^k = F_i^t g^i$ and $F_{ij} = F_j^k g_{ik}$. Hence the connection (3.3) becomes the following form
\[ \bar{\Gamma}_{ij}^k = \Gamma_{ij}^k + \omega_j \delta_i^k - \omega_i \delta_j^k + \frac{1}{q} \left( \omega_t F_j^t F_i^k - \omega_t F_i^t F_j^k \right). \] (3.4)
Also, by using the connection (3.4), we obtain the following equation with a simple calculation:

\[ \nabla_k F_i^j = g_{ki} (\omega^t F_i^j - \omega_i F^{jt}) = 0. \]

Therefore, the connection \( \nabla \) given by (3.4) is named metallic semi-symmetric metric \( F \)-connection.

4. CURVATURE AND TORSION PROPERTIES OF THE METALLIC SEMI-SYMMETRIC METRIC \( F \)-CONNECTION

In this section, we examine some properties associated with the torsion and curvature tensor of the connection (3.4).

Let \((M, g, F)\) be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). We say easily that the torsion tensor \( \tilde{\Sigma} \) of the connection (3.4) is pure. Indeed, by using (2.4) and (3.2), we get

\[ \tilde{\Sigma}_{im}^k F_j^m = \tilde{\Sigma}_{mj}^k F_i^m = \tilde{\Sigma}_{ij}^m F_k^m. \]

In [13], the author prove that a \( F \)-connection is pure \( \iff \) torsion tensor of that connection is pure. Thus, the connection (3.4) provides the following condition:

\[ \tilde{\Gamma}_{mj}^k F_i^m = \tilde{\Gamma}_{im}^k F_j^m = \tilde{\Gamma}_{ij}^m F_k^m. \]

**Theorem 4.1.** Let \((M, g, F)\) be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). If the 1-form \( \omega \) is a \( \phi \)-tensor, then the torsion tensor \( \tilde{S} \) of the connection (3.4) is a \( \phi \)-tensor and holds following equation:

\[ F_k^m (\nabla_m \tilde{S}_{ij}^l) = F_i^m (\nabla_k \tilde{S}_{mj}^l) = F_j^m (\nabla_i \tilde{S}_{km}^l). \quad (4.1) \]

**Proof.** Let \((M, g, F)\) be a locally decomposable metallic Riemann manifold. Since a zero tensor is pure, a \( F \)-connection with torsion-free is always pure. Hence, we can say that the Levi-Civita connection \( \nabla \) of \( g \) on \( M \) is always pure with respect to \( F \).

If we implement the Tachibana operator \( \phi_F \) to the torsion tensor \( \tilde{S} \) of the connection (3.4), then we have

\[ (\phi_F \tilde{S})_{kij}^l = F_k^m (\partial_m \tilde{S}_{ij}^l) - \partial_k (\tilde{S}_{mj}^l F_i^m) = F_k^m (\nabla_m \tilde{S}_{ij}^l + \Gamma^s_{mi} \tilde{S}_{kj}^l + \Gamma^s_{mj} \tilde{S}_{is}^l - \Gamma^l_{ms} \tilde{S}_{ij}^l) - F_i^m (\nabla_k \tilde{S}_{mj}^l + \Gamma^s_{km} \tilde{S}_{sj}^l + \Gamma^s_{kj} \tilde{S}_{ms}^l - \Gamma^l_{ks} \tilde{S}_{mj}^l). \]

When the torsion tensor \( \tilde{S} \) and Levi-Civita connection \( \nabla \) are pure, the above relation reduces to

\[ (\phi_F \tilde{S})_{kij}^l = F_k^m (\nabla_m \tilde{S}_{ij}^l) - F_i^m (\nabla_k \tilde{S}_{mj}^l). \quad (4.2) \]
Substituting (3.2) into (4.2), we get

\begin{equation}
(\phi_F \tilde{S})_{kij}^l = \left[ (\nabla_m \omega_j) F_k^m - (\nabla_k \omega_m) F_j^m \right] \delta_i^l \\
- \left[ (\nabla_m \omega_i) F_k^m - (\nabla_k \omega_m) F_i^m \right] \delta_j^l \\
+ \frac{1}{q} (\nabla_m \omega_s) F_k^m F_s^j \delta_i^l - \frac{p}{q} (\nabla_k \omega_s) F_j^s \delta_i^l \\
- \frac{1}{q} (\nabla_m \omega_s) F_k^m F_i^s \delta_j^l - \frac{p}{q} (\nabla_k \omega_s) F_i^s \delta_j^l.
\end{equation}

Also, for the 1–form \( \omega \), we calculate

\begin{equation}
(\phi_F \omega)_{kj} = F_k^m (\partial_m \omega_j) - \partial_k (F_j^m \omega_m) \\
= F_k^m (\nabla_m \omega_j + \Gamma^s_{mj} \omega_s) - F_j^m (\nabla_k \omega_m + \Gamma^s_{km} \omega_s) \\
= F_k^m (\nabla_m \omega_j) - F_j^m (\nabla_k \omega_m).
\end{equation}

From last equation, we can say that the 1–form \( \omega \) is a \( \phi \)–tensor iff

\begin{equation}
F_k^m (\nabla_m p_j) = F_j^m (\nabla_k p_m).
\end{equation}

Assuming that the 1–form \( \omega \) is a \( \phi \)–tensor, thanks to (2.4) the relation (4.3) becomes \( (\phi_F \tilde{S})_{kij}^l = 0 \), i.e., the torsion tensor \( \tilde{S} \) is a \( \phi \)–tensor. Also, from the equation (4.2) we get

\begin{equation}
F_k^m (\nabla_m \tilde{s}_{ij}) = F_i^m (\nabla_k \tilde{s}_{mj}) = F_j^m (\nabla_k \tilde{s}_{im}).
\end{equation}

The proof is complete.

From the equation (2.8), it is obvious that the torsion tensor \( \tilde{S} \) of the connection (3.4) and the 1–form \( \omega \) are hold following equality

\[ \phi_j \tilde{S} = 0 \quad \text{and} \quad \phi_j \omega = 0, \]

i.e., they are decomposable tensors, where \( J \) is the product structure associated with the metallic structure \( F \). From on now, we shall consider 1–form \( \omega \) is a \( \phi \)–tensor (or decomposable tensor), i.e., the following conditions are provided:

\[ F_k^m (\nabla_m \omega_j) = F_j^m (\nabla_k \omega_m) \]

and

\[ J_k^m (\nabla_m \omega_j) = J_j^m (\nabla_k \omega_m). \]

It is well known that the curvature tensor \( \tilde{R}_{ijk}^l \) of the connection (3.4) is as follows:

\[ \tilde{R}_{ijk}^l = \partial_l \tilde{\Gamma}_{jk}^l - \partial_j \tilde{\Gamma}_{ik}^l + \tilde{\Gamma}_{im}^l \tilde{\Gamma}_{jk}^m - \tilde{\Gamma}_{jm}^l \tilde{\Gamma}_{ik}^m. \]
Then, the curvature tensor $\mathcal{R}_{ijk \ell}$ can be expressed

$$
\mathcal{R}_{ijk \ell} = R_{ijk \ell} + \delta_{j}^{\ell} \mathcal{A}_{ik} - \delta_{i}^{\ell} \mathcal{A}_{jk} + g_{ik} \mathcal{A}_{j \ell} - g_{jk} \mathcal{A}_{i \ell}
$$

(4.5)

where $R_{ijk \ell}$ are the ingredients of the Riemann curvature tensor of the Riemann connection $\nabla$ and

$$
\mathcal{A}_{jk} = \nabla_{j} \omega_{k} - \nabla_{k} \omega_{j} + \frac{1}{2} \omega^{m} \omega_{m} g_{kj} - \frac{1}{q} \omega_{m} \omega_{t} F_{k}^{t} F_{j}^{m} + \frac{1}{2q} \omega^{m} \omega_{t} F_{m}^{t} F_{jk}.
$$

(4.6)

It is clear that the tensor $A$ provide

$$
\mathcal{A}_{jk} - \mathcal{A}_{kj} = 0
$$

if and only if 1-form $\omega$ is closed.

Also, from the equation (4.5), we obtain

$$
\overline{\mathcal{R}}_{ijkl} = R_{ij} + (4 - n) \mathcal{A}_{ij} - trace \mathcal{A} g_{ij}
$$

(4.8)

where $R_{ij}$ is Ricci tensors of the Riemann connection $\nabla$ of $g$ and

$$
trace \mathcal{A} = \mathcal{A}^{i}_{i} = \nabla_{i} \omega^{i} + \left( \frac{n - 4}{2} \right) \omega_{i} \omega^{i} - \frac{1}{q} \omega_{t} \omega^{m} F_{m}^{t} (p - \frac{1}{2} F_{i}^{j} i).
$$

Contracting the last equation with $g^{ik}$, for the scalar curvature $\tau$ of the connections (3.4), we get

$$
\tau = \tau + 2 (2 - n) trace \mathcal{A} + \frac{2}{q} \left( p - F_{t}^{j} \right) F_{j}^{t} A^{i}_{i},
$$

(4.9)

where $\tau$ is scalar curvature of Levi-Civita connection $\nabla$ of $g$. From the equation (4.8), we can have

$$
\overline{\mathcal{R}}_{jk} - \overline{\mathcal{R}}_{kj} = (n - 4) \left( \mathcal{A}_{jk} - \mathcal{A}_{kj} \right) + \frac{1}{q} \left( 2p - F_{t}^{j} \right) F_{k}^{t} \left( \mathcal{A}_{jt} - \mathcal{A}_{tj} \right).
$$

(4.10)

From the equation (4.10), we easily say that if the 1-form $\omega$ is closed, then $\overline{\mathcal{R}}_{jk} - \overline{\mathcal{R}}_{kj} = 0$. 
Lemma 4.2. Let \((M, g, F)\) be a locally decomposable metallic Riemann manifold endowed with the connection \(\nabla\). Then the tensor \(A\) given by \((4.6)\) is a \(\phi\)-tensor (or decomposable tensor) and thus the following relation holds:
\[
(\nabla_m A_{ij}) F^m_k = (\nabla_k A_{mj}) F^m_i = (\nabla_k A_{im}) F^m_j.
\]

Proof. The tensor \(A\) is pure with regard to \(F\). Indeed
\[
F^t_k A_{it} - F^t_i A_{tk} = (\nabla_i \omega_t) F^t_k - (\nabla_t \omega_i) F^t_k = 0.
\]

If the Tachibana operator is applied to the tensor \(A\), then we get
\[
(\phi_F A)_{kij} = (\nabla_m A_{ij}) F^m_k - (\nabla_k A_{mj}) F^m_i.
\]
Substituting \((4.6)\) into \((4.11)\), standard calculations give
\[
(\phi_F A)_{kij} = (\nabla_m A_{ij}) F^m_k - (\nabla_k A_{mj}) F^m_i.
\]

When we apply the Ricci identity to the \(1\)-form \(\omega\), we get
\[
(\nabla_m \nabla \omega_j) F^m_k = (\nabla_i \nabla \omega_j) F^m_k - \frac{1}{2} \omega_s R_{mijs} F^m_k
\]
and
\[
(\nabla_k \nabla \omega_j) F^m_i = (\nabla_i \nabla \omega_k) F^m_j - \frac{1}{2} \omega_s R_{kimjs} F^m_j.
\]
With the help of the last two equation, from \((4.12)\), the equation \((4.12)\) becomes as follows,
\[
(\phi_F A)_{kij} = -\frac{1}{2} \omega_s (R_{mijs} F^m_k - R_{kimjs} F^m_j).
\]

In a locally decomposable metallic Riemann manifold \((M, g, F)\), the Riemann curvature tensor \(R\) is pure \([1]\). This instantly gives \((\phi_F A)_{kij} = 0\). Hence, from \((4.11)\) we can write
\[
(\nabla_m A_{ij}) F^m_k = (\nabla_k A_{mj}) F^m_i = (\nabla_k A_{im}) F^m_j.
\]

Also, with help of \((2.8)\), we can say that \(\phi J A = 0\), i.e., the tensor \(A\) is decomposable, where \(J\) is the product structure associated with the metallic structure \(F\). \(\square\)
By using the purity of the tensor $A$, standard calculations give
\[
\tilde{R}_{i m k} F^m_j = \tilde{R}_{i j m} F^m_k = \tilde{R}_{i j k} F^m_m = \tilde{R}_{m j k} F^m_i,
\]
i.e., the curvature tensor $\tilde{R}$ is pure with respect to metallic structure $F$.

If Tachibana operator $\phi_F$ is applied to the curvature tensor $\tilde{R}$, then we get
\[
(\phi_F \tilde{R})_{k i j l} = F^m_k (\partial_{m} \tilde{R}_{i j l}^t) - \partial_{k} (\tilde{R}_{m j l}^t F^m_i) \tag{4.13}
\]
\[
= F^m_k (\nabla_m \tilde{R}_{i j l}^t + \Gamma_{m i}^s \tilde{R}_{s j l}^t + \Gamma_{m j}^s \tilde{R}_{i s l}^t + \Gamma_{m l}^s \tilde{R}_{i j s}^t - \Gamma_{m s}^t \tilde{R}_{i j l}^m) - F^m_i (\nabla_k \tilde{R}_{m j l}^t + \Gamma_{k m}^s \tilde{R}_{s j l}^t + \Gamma_{k j}^s \tilde{R}_{m s l}^t + \Gamma_{k l}^s \tilde{R}_{m j s}^t - \Gamma_{k s}^t \tilde{R}_{m j l}^m)
\]
\[
= (\nabla_m \tilde{R}_{i j l}^t) F^m_k - (\nabla_k \tilde{R}_{m j l}^t) F^m_i
\]
from which, by (4.5), we find
\[
(\phi_F \tilde{R})_{k i j l} = (\phi_F \tilde{R})_{k i j l}^t + (\nabla_m A_{j m}^l) F^m_i - (\nabla_m A_{i m}^k) F^m_k) \delta_{l}^t
\]
\[
+ [(\nabla_m A_{i k}^l) F^m_k - (\nabla_m A_{k i}^m) F^m_l)] \delta_{j}^t
\]
\[
+ [(\nabla_m A_{k l}^i) F^m_l - (\nabla_m A_{l k}^m) F^m_i)] g_{il}
\]
\[
+ [(\nabla_m A_{l i}^k) F^m_k - (\nabla_m A_{k l}^m) F^m_i)] g_{jl}.
\]

In a locally decomposable metallic Riemann manifold $(M, g, F)$, since the Riemann curvature tensor $\tilde{R}$ is a tensor, considering Lemma 4.2, the last relation becomes $\phi_F \tilde{R} = 0$. Also, from the equation (2.8), we can say that $J e_R = 0$, where $J$ is the product structure associated with the metallic structure $F$. Thus we obtain the following theorem:

**Theorem 4.3.** Let $(M, g, F)$ be a locally decomposable metallic Riemann manifold endowed with the connection (3.4). The curvature tensor $\tilde{R}$ of the connection (3.4) is a $\phi$–tensor (or decomposable tensor).

**References**

[1] Gezer A., Karaman C., On metallic Riemannian structures. *Turk J Math*, 39, (2015), 954-962.
[2] Gezer A., Karaman C., On golden semi-symmetric metric $F$-connections, *Turk J Math*, DOI: 10.3906/mat-1510-77.
[3] Hayden H. A., Sub-spaces of a space with torsion. *Proc. London Math. Soc*. S2-34 (1932), 27-50.
[4] Hretcanu C., Crasmareanu M., Metallic structures on Riemannian manifolds. *Rev Un Mat Argentina* 2013; 54: 15-27.
[5] Crasmareanu M., Hretcanu C. E., Golden differential geometry. *Chaos Solitons Fractals* 38 (2008), no. 5, 1229–1238.
[6] de Spinadel VW., The metallic means family and multifractal spectra. *Nonlinear Anal Ser B* 1999; 36: 721-745.
[7] de Spinadel VW., The family of metallic means. *Vis Math* 1999; 1: 3.
[8] Pusic N., On some connections on locally product Riemannian manifolds-part II. *Novi Sad J. Math*. 41 (2011), no. 2, 41-56.
[9] Pusic N., On some connections on locally product Riemannian manifolds-part I. *Novi Sad J. Math*. 41 (2011), no. 2, 29-40.
ON METALLIC SEMI-SYMMETRIC METRIC $F-$CONNECTIONS

[10] Prvanovic M., Locally decomposable Riemannian manifold endowed with some semi-symmetric $F-$connection. Bull. Cl. Sci. Math. Nat. Sci. Math. No. 22 (1997), 45-56.
[11] Prvanovic M., Some special product semi-symmetric and some special holomorphically semi-symmetric $F-$connections. Publ. Inst. Math. (Beograd) (N.S.) 35(49) (1984), 139-152.
[12] Prvanovic M., Product semi-symmetric connections of the locally decomposable Riemannian spaces. Bull. Acad. Serbe Sci. Arts Cl. Sci. Math. Natur. (N.S.) 10 (1979), 17-27.
[13] Salimov A., Tensor operators and their applications. Mathematics Research Developments Series. Nova Science Publishers, Inc., New York, 2013. xii+186 pp.
[14] Tachibana S., Analytic tensor and its generalization. Tohoku Math. J. 12 (1960), 208-221.
[15] Yano K., On semi-symmetric metric connection. Rev. Roumaine Math. Pures Appl. 15 (1970), 1579-1586.
[16] Yano K., M. Ako, On certain operators associated with tensor fields. Kodai Math. Sem. Rep. 20 (1968), 414-436.

Current address: Ataturk University, Oltu Faculty of Earth Science, Geomatics Engineering, 25240, Erzurum-Turkey.
E-mail address: cagri.karaman@atauni.edu.tr
ORCID Address: https://orcid.org/0000-0001-6532-6317