Parallelization of high-dimensional single-photon quantum gates

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Local quantum gates represent basic building blocks of quantum computers. In photonic domain, these gates act on individual photons and are usually implemented as single optical devices. In cases when the same gate is to be applied simultaneously to multiple photons, the corresponding physical implementation consists of a stack of identical devices. We present a systematic approach that allows one to replace this stack with a single device supplemented by pre- and post-processing stages. As a result, considerable savings in resources are obtained. We exemplify our scheme for the case of the orbital angular momentum (OAM) of single photons and demonstrate the scheme for high-dimensional Pauli gates and their arbitrary powers. As a byproduct, we also present a simple argument that explicitly shows how to construct an arbitrary quantum gate acting on OAM of single photons, where only conventional optical elements are utilized.

I. INTRO

The field of quantum computation and quantum communication has seen marvelous progress in recent years, which is partly due to the maturity of technologies that enable the manipulation of single quantum systems [1,2]. With the increasing quality of quantum computational platforms, the need for large scale deployment will rise accordingly. Since the resulting physical infrastructure may be formidable complex, it is vital to investigate how the corresponding experimental setups can be simplified in a systematic way. When the same operation is to be applied to multiple quantum systems, one can use a single device and apply it successively to each of them. This approach requires an active control of the device, which may not be optimal in cases when the stability and high performance of the whole infrastructure are of concern. The passive approach is thus preferable. When implemented in a naive way, the passive approach requires as many devices as there are systems—one device is dedicated to each system. Such a scenario is clearly not optimal and the question arises whether one can find a more efficient implementation.

In this paper, we answer this question affirmatively. We present a passive scheme that requires significantly fewer resources than the naive approach. For concreteness, we consider the situation when the quantum information is carried by the orbital angular momentum (OAM) of single photons. The OAM of a photon amounts to quantized twists of the photon’s wave function [3,4] and has served in a multitude of experiments as a high-dimensional quantum carrier of information—for given d one can always consider a d-dimensional subspace of OAM spanned by eigenstates $|0\rangle, \ldots, |d-1\rangle$, where the information is encoded in a superposition of these eigenstates. The OAM of photons has been experimentally utilized in quantum teleportation [5,6], high-dimensional quantum key distribution [7], generation of high-dimensionally entangled quantum states [8,9], as well as in more fundamental experiments that study the correspondence principle for a very high number of OAM quanta [10,11].

We begin our discussion in Sec. II where a general parallelization scheme of local quantum gates is presented. In this scheme, many instances of the same device are replaced by its single instance that effectively operates on many parties. To quantify the advantage of the scheme in comparison with the naive approach, we study the case of gates operating on OAM. For them explicit formulas are given that determine the number of optical elements needed. At first, we show in Sec. III how to construct an arbitrary unitary in OAM of single photons while making use of only conventional optical elements, such as beam splitters, holograms, and Dove prisms. The stack of such gates is henceforth referred to as the naive approach. The parallelized scheme for OAM is introduced in Sec. IV and compared with the naive approach in Sec. V in terms of the number of optical elements required by the two approaches. The schemes can be simplified using the polarization of photons as shown in Sec. VI. To demonstrate a specific example of an operation that can be parallelized, in Sec. VII we construct the setups for the high-dimensional Pauli gates and their integer powers, together with their parallelized versions. Such gates are used for example in the construction of Heisenberg-Weyl observables, which in turn find applications in the quantum state tomography [12,13]. An analytical derivation is presented that gives us the exact number of beam splitters required by both the non-parallelized and parallelized schemes for Pauli gates. We summarize our results in Sec. VIII.

II. PARALLELIZATION

As has been already mentioned in the introduction, there are cases when a quantum algorithm or a communication protocol require multiple applications of the same unitary $U$. For example, a photonic quantum algorithm may require several photons to be prepared in the same initial (internal) state $|\psi\rangle$, but a physical process generates them in a different state $|\varphi\rangle$. A unitary $U$, whose
FIG. 1. Some computational tasks require the application of the same unitary operation to multiple path modes. (a) In the preprocessing stage of a quantum computer, individual high-dimensional quantum systems, on which the calculation is to be performed, have to be initialized into a given state $|\psi\rangle$. When the physical source $S_i$ produces a system in a different state $|\phi\rangle$, a unitary $U$ (implemented by a device $U_O$) can turn this state into $|\psi\rangle$ for each system. All these local unitaries can be grouped into a single operation, represented by a dashed box. (b) A qualitatively different scenario is that of a single quantum system that propagates along a superposition of multiple paths $p_i$ and is subjected to an operation $U$ that acts only on its internal degree of freedom (such as OAM). In the real physical implementation this one operation corresponds to a set of $n$ identical devices $U_O$, enclosed in the dashed box, each applied to a different path. (c) The series of identical devices $U_O$ acting on the internal degree of freedom can be replaced by a parallelized scheme. This scheme consists of two swap operators and only a single operation $U_P$, which acts on paths rather than the internal degree of freedom.

physical implementation we denote by $U_O$, can then be applied to each photon to turn $|\phi\rangle$ into $|\psi\rangle$, as shown in Fig. 1(a). The resulting collection of unitaries $U_O$ can be viewed as a single parallelized operation $U^{(par)}_O$ acting on many photons. Another, qualitatively different, scenario is when a single photon propagates in a superposition of multiple paths and one wants to apply operation $U$ only to its internal degree of freedom, such as OAM. In the experimental realization, $U$ is implemented as a series of identical setups with one setup $U_O$ in each path, see Fig. 1(b). Both aforementioned scenarios can be implemented by the same physical setup, whose action on single photons can be represented by the transformation

$$a^\dagger_{m,p} \mapsto \sum_{k=0}^{d-1} U_{km} a^\dagger_{k,p},$$

where $a^\dagger_{m,p}$ is the creation operator associated with an internal mode $m$ in path $p$. The implementation of the series of identical high-dimensional unitaries $U_O$ may require a lot of resources. In the naive approach, $U^{(par)}_O$ requires a number of elements that scales linearly with the number of systems.

The same task of simultaneous application of $U$ on $n$ paths can be nevertheless achieved with just a single device, when the internal modes (e.g., OAM eigenstates) are transformed into the path encoding, as shown in Fig. 1(c). The key role in this approach is played by the swap operator, henceforth referred to simply as the swap, whose action on input states reads

$$\text{SWAP}(|m\rangle_O |p\rangle_P) = |p\rangle_O |m\rangle_P ,$$

where $|m\rangle_O$ denotes internal mode $m$ and $|p\rangle_P$ stands for the $p$-th propagation mode. For convenience, let us explain the scheme of Fig. 1(c) by considering the multiple-system scenario with one system in each path. Before the application of $U^{(par)}_O$, each path mode carries a single system and the internal modes of each system’s state contain the information that is to be processed. In the first stage of the scheme in Fig. 1(c), a swap exchanges the roles of internal and path modes. The information is thus encoded in the path degree of freedom, while at the same time the internal modes play the role of identification labels for individual systems. In the second stage, the path-encoded implementation $U_P$ of the desired unitary $U$ is applied. This operation transforms the path modes according to $U$ and leaves the internal modes unaffected. Even though this property may not be satisfied in general, in many cases this is indeed the case as $U_P$ can be constructed only with beam splitters and phase shifters \cite{[14]}, which leave e.g. OAM, polarization, or frequency of photons unaffected. In the third stage, a swap is applied again in order to give the internal and path modes their original meaning. As a result, the internal modes in each path are transformed according to $U$ and path modes label different systems. This procedure is summarized by the formula

$$U^{(par)}_O = \text{SWAP}^{-1} \cdot U_P \cdot \text{SWAP},$$

which represents the main result of this paper and which we henceforth refer to as the parallelized scheme. In the
following, we consider a special case of unitaries that act on OAM of single photons. For these the number of optical elements is discussed that are required by the naive and parallelized schemes. To that end, we first present the naive scheme in the next section.

III. ARBITRARY UNITARY IN OAM

One of the core results of the quantum computation theory is that there exist universal sets of operations, out of which any other unitary operation can be constructed. For the case of operations acting on the OAM of a single photon, such universal sets have been presented in Refs. [15,16]. Here we put forward a very simple argument that shows explicitly that such a universal set can be constructed only from conventional optical elements. Let us denote by $U$ the abstract $d$-dimensional unitary operation and let $U_O$ and $U_P$ be its implementations for the OAM and the path degrees of freedom, respectively. The idea underlying our argument is a special case of the approach introduced in the preceding section: to build $U_O$ one first transforms the incoming OAM eigenstates into the path encoding and then applies $U_P$, for which general implementation schemes are known [13,17,15] that use only beam splitters and phase shifters. At the end, the propagation modes are transformed back into the OAM eigenstates. The transition between the OAM and path encodings is performed by a $d$-dimensional OAM sorter $S$. The sorter turns an OAM eigenstate $|m\rangle_O$ with $m$ quanta of OAM and propagating along the zeroth path $|0\rangle_P$, into the fundamental mode $|0\rangle_O$ that propagates along the $m$-th path $|m\rangle_P$, such that

$$S(|m\rangle_O |0\rangle_P) = |0\rangle_O |m\rangle_P.$$  

(4)

The sorter can be understood as a special case of the swap operator in Eq. (2) for $p = 0$. The OAM sorter can be implemented in multiple ways [19–22]. Here we employ the interferometric implementation that consists merely of beam splitters, Dove prisms, and holograms [23,24] and whose structure is shown in section VII A. The whole scheme is then compactly represented by the formula

$$U_O = S^{-1} \cdot U_P \cdot S.$$  

(5)

In the scheme, the use is made of $O(d^2)$ beam splitters, $O(d^2)$ phase shifters, $O(d)$ Dove prisms, and $O(d)$ holograms as can be deduced from the structure of the OAM sorter [25,27] and the Reck et al. scheme [14].

The path-encoded unitary $U_P$, as well as the OAM sorter, are implemented as networks of many interferometers. Beam splitters thus play an important role in their construction. In the following, we will estimate the complexity of the OAM implementation $U_O$ by counting the beam splitters used in its construction. A similar discussion can also be done for other optical elements. Let $N_O(d)$ be the number of beam splitters required in the scheme of Eq. (5). If $N_P(d)$ is the number of beam splitters that are necessary to implement $U_P$, then

$$N_O(d) = N_P(d) + 2N_S(d),$$  

(6)

where

$$N_S(d) = 2(d - 1)$$  

(7)

is the number of beam splitters that implement the OAM sorter in dimension $d$ [25]. Later on, we compare this number with the number of beam splitters used in the parallelized version of the scheme represented by Eq. (5).

IV. PARALLELIZATION IN OAM

Even though the relation (3) holds for any internal degree of freedom, it is not obvious how to implement efficiently the SWAP operator in a general case. For the case of OAM and path we can utilize the efficient design of Ref. [26] whose detailed structure is presented in section VII A. The path-encoded unitary $U_P$ can be implemented using Reck et al. scheme [14]. As a result, only conventional optical elements are used in the parallelized setup of Eq. (5). Namely, beam splitters, phase shifters, holograms, and Dove prisms. Moreover, each beam splitter in the Reck et al. scheme has to be supplemented by two extra mirrors, such that the sign of OAM eigenstates is unaffected by the reflection off the beam splitter’s interface [26]. Let us denote by $N_O^{(par)}(n,d)$ the number of beam splitters employed in this parallelized scheme. Analogously to Eq. (6) we obtain

$$N_O^{(par)}(n,d) = N_P(d) + 2N_{SWAP}(n,d),$$  

(8)

where $N_{SWAP}(n,d)$ is the number of beam splitters used to implement the swap operator with $n$ input and $d$ output paths. In Appendix A the exact values of $N_{SWAP}(n,d)$ are presented. A similar discussion can also be done for other optical elements.

The parallelized scheme of Eq. (5) was applied in Ref. [25] in the special case of the high-dimensional Fourier transform acting on OAM. The interferometric implementation of this transform was presented for the first time in Ref. [26], where the resulting scheme requires $O(d)$ optical elements. One of its components is the lower-dimensional Fourier transform, which is applied only to the OAM degree of freedom and leaves the path degree of freedom unaffected. This scenario corresponds exactly to the case of Fig. 1(b). When the series of lower-dimensional Fourier transforms is replaced by a single path-encoded transform, the scaling improves into $O(\sqrt{d} \log(d))$, as was recently shown in Ref. [28]. This special case motivates the study of the scaling of the general parallelized case, as discussed in the following section.

V. SCALING OF RESOURCES

To quantify the improvement brought by the use of the parallelized scheme, we introduce the ratio of the number
From formula (10) it immediately follows that for large $n$, the asymptotic behavior (10) are depicted in Fig. 2. The orange surface is given by formula (10) that expresses the approximate scaling of ratio $r_{\text{Reck}}$. The violet surface is likewise given by formula (12) derived for permutations. The bottom right corner of the plot corresponds to $n = d = 2048$, for which $r_{\text{Reck}} \sim 10^{-4}$ for general unitaries and $r_{\text{perm}} \sim 10^{-3}$ for permutations. The gray plane divides the cases with $n > d$ and $n \leq d$. For specific classes of unitaries one obtains different scaling estimates. The extreme case is represented by unitaries that correspond to mere permutations of modes. For those, we get $N_P(d) = 0$ as no beam splitters are necessary to permute paths. The ratio for both $n > d$ and $n \leq d$ cases then scales as

$$r_{\text{perm}}(n,d) \sim \frac{\log_2(d)}{2n} + \frac{\log_2(n)}{4d}.$$  \hspace{1cm} (12)

When the dimension of the OAM space greatly exceeds the number of paths, the ratio reduces to $r_{\text{perm}}(n,d \gg n) \sim \log_2(d)/(2n)$. Unless the dimension grows exponentially faster than $n$, the parallelized scheme is also in this extreme case more efficient than the stack of $n$ independent setups. A sample of exact values of $r_{\text{perm}}$ as well as the asymptotic behavior (12) are depicted in Fig. 2.

VI. POLARIZATION-ENHANCED SCHEME

Additional savings in resources are possible when a polarization is utilized as an auxiliary degree of freedom. The setups of Eqs. 3 and 6 have a symmetric structure, where the sorter (or the swap) is applied both before and after the path-encoded operation $U_P$. One can get rid of the second sorter or swap by utilizing polarization, as demonstrated in Fig. 3(a), and (b). Let us describe the working principle of the non-parallelized scheme; the parallelized scheme of Fig. 3(b) works analogously. The polarization effectively controls whether the photons propagate through the setup forward or backward. Suppose the photons enter the setup of Fig. 3(a) in $H$ polarization. They evolve through the setup as in the original scheme until they reach a series of half-wave plates, which rotate $H$ polarization into $V$ polarization. The photons
late how large savings in resources one obtains when one gates and their powers can be implemented and calculational basis. In the following, we discuss how these \( \omega \) splitters is for the non-parallelized scheme equal to beam splitter. back into the sorter, but now backward. At the end, the are then reflected by a series of polarizing beam splitters back into the sorter, but now backward. At the end, the photons leave via the lower port of the initial polarizing beam splitter. The sum of both polarizing and non-polarizing beam splitters is for the non-parallelized scheme equal to \( N_O^{(pol)}(d) = N_P(d) + N_S(d) + d + 1 \) It holds that \( N_O(d) - N_O^{(pol)}(d) = d - 3 \), which is approximately equal to one half of the number of beam splitters that make up the removed OAM sorter. This minor reduction in the number of beam splitters can be of importance in real experimental setups. Analogously, for the parallelized scheme one gets \( N_O^{(pol-par)}(n,d) = N_P(d) + N_S^{(par)}(n,d) + d + n \), for which \( N_O^{(par)}(n,d) - N_O^{(pol-par)}(n,d) = (n/2) \log_2(n) + d \log_2(n) + d - 4n + 1 \) when \( n \leq d \) (for \( n > d \) a similar formula is obtained). The number of beam splitters is thus reduced approximately by the number of beam splitters required to construct the removed swap operator.

VII. HIGH-DIMENSIONAL PAULI GATES

The brute-force scheme of Eq. (5) for a single system as well as the parallelized scheme of Eq. (3) for the multiple-system case can be simplified considerably for specific unitary operations. One class of such operations are the Pauli operators, prominent examples of local quantum gates. The \( d \)-dimensional Pauli \( X \) gate and \( Z \) gate are defined by \( 12 \)

\[
X_d(|q\rangle) = |(q + 1) \mod d\rangle, \tag{13}
\]

\[
Z_d(|q\rangle) = \omega^k |q\rangle, \tag{14}
\]

where \( \omega = \exp(2\pi i/d) \), and where \( \{|q\rangle\}_{q=0}^{d-1} \) form the computational basis. In the following, we discuss how these gates and their powers can be implemented and calculate how large savings in resources one obtains when one uses the parallelized scheme. The setups for \( X^k \) gates are first constructed for the single-system scenario in subsection VIIA, the corresponding parallelized schemes are then presented in subsection VIIB. In subsection VII C we briefly discuss the powers of \( Z \) gate and their parallelization.

A. \( X \) gate and its powers

The implementation scheme of the \( d \)-dimensional \( X \) gate acting on OAM was given in Ref. \( 27 \). That implementation consists of OAM exchangers, which are passive two-input two-output optical devices \( 20 \). They are composed of a Leach interferometer \( 23 \) augmented with two holograms, as shown in Fig. (4a). The order \( k \) of an OAM exchanger \( \text{EX}_k \) determines its sorting properties. The inverse \( \text{EX}_k^{-1} \) of an exchanger is implemented in an almost identical way to that of \( \text{EX}_k \), only the rotation of one Dove prism is reversed, see Fig. (4b).

The \( X \) gate can be built out of OAM exchangers in an arbitrary dimension, but here we focus only on dimensions of the form \( d = 2^M \). The modification of the original scheme for such dimensions is presented in Fig. (4c). The number of beam splitters required in such a scheme is equal to

\[
N_X(d) = 4 \log_2(d). \tag{15}
\]

This scheme can be obtained by starting from the naive implementation in Eq. (5), where the OAM sorters are constructed as binary-tree networks of OAM exchangers \( 23, 24 \). This case is explicitly shown in Fig. (4d), where the path-encoded implementation \( U_P \) of the \( X \) gate corresponds to the path permutation that connects the output ports of the sorter \( S \) on the left with the input ports of the inverted sorter \( S^{-1} \) on the right. Due to the structure of the path permutation, many exchangers \( \text{EX}_k \) from \( S \) are followed by their inverses \( \text{EX}_k^{-1} \) from \( S^{-1} \). All these exchangers can be obviously removed without any effect on the final state. The resulting optical network is identical to that in Fig. (4c).

The \( X \) gate is a specific example of a cyclic permutation of the basis states. We can obtain all other cyclic permutations by taking powers of the \( X \) gate. Specifically

\[
X^k_d(|q\rangle) = |(q + k) \mod d\rangle, \tag{16}
\]

where \( k \in \mathbb{N} \). In the following, we present the interferometric setups of \( X^k \) gates as well as the number of beam splitters that it requires. Due to the cyclic property of the \( X \) gate, it holds that \( X^{d-k} = (X^k)^{-1} \). Consequently, it suffices to study only powers \( k \leq d/2 \) as the implementation of \( X^k \) for \( k > d/2 \) is obtained as the implementation for \( X^{d-k} \) operated backwards. We proceed analogously to the case of the \( X \) gate. We again start from the general scheme in Eq. (5) and remove all the exchangers that do not affect the final state. In Figs. (4 e), (f), and (g) the explicit form of \( X^k \) gate is shown for \( d = 8 \) and \( k = 2, 3, 4 \), respectively.
FIG. 4. The interferometric implementation of the $X$ gate together with its integer powers. (a) The OAM exchanger $EX_k$ of order $k$ is built from two holograms and a Leach interferometer [23] with a Dove prism rotated through $\pi/(2k)$. Optical elements: holo—hologram, Dove—Dove prism, BS—50:50 beam splitter. (b) The inverse of the OAM exchanger, $EX_k^{-1}$, has almost the identical structure to that of $EX_k$, only the Dove prism is rotated through $-\pi/(2k)$. For convenience, we use two slightly different symbols to denote the inverse of the OAM exchanger, as shown in the figure. (c) The $X$ gate in dimension $d = 2^M$ of OAM can be constructed as a series of OAM exchangers of orders $2^k$ for $k = 0,\ldots,M-1$, followed by the reversed series of the same structure. The depicted setup corresponds to $d = 8$. Generalizations for higher dimensions follow analogously the same structure. (d) The $X$ gate is constructed from two OAM sorters, marked by shaded rectangles in the figure, from which redundant exchangers are removed. These exchangers can be grouped into blocks of increasing size, which are enclosed in dashed-line rectangles. The unused paths, as well as redundant exchangers, are drawn in faded color. The remaining exchangers can be reordered in order to get rid of the path permutations. The resulting setup is exactly the one shown in (c). (e) The same principles are also used when constructing the integer powers $X^k$ of the $X$ gate. When the exponent $k$ is a power of two, i.e., $k = 2^m$, the path permutation has a repetitive structure and the whole setup is effectively split into $k$ identical subsetups. (f) For a general exponent $k$ the path permutation has a more complicated structure. (g) The number of exchangers that have to be retained in the final setup increases with the exponent $k$ until it attains the form $k = d/2$ when no exchangers can be removed.

In order to understand how the scheme in Eq. (5) can be simplified for general dimension $d = 2^M$ and general power $1 \leq k \leq d/2$ one notes that the cyclic permutations of propagation paths can be expressed as a series of path crossings of increasing size, see the middle part of Fig. 4(d). When $k$ is a power of two, i.e., $k = 2^m$, the path permutation has a repetitive structure, cf. Fig. 4(e) and (g). In such cases, the setup effectively decomposes into $k$ subsetups of the same structure and smaller size. For example, the setup of $X^2$ in Fig. 4(e) in dimension $d = 8$ can be viewed as two smaller setups for $X$ in dimension $d = 4$. In general, the setup of $X^k$ for power $k = 2^m$ in dimension $d = 2^M$ can be seen as $k$ subsetups implementing $X$ gate in dimension $d' = d/k = 2^{M-m}$.
These setups are sandwiched between two OAM sorters with \(k\) output paths. This way, we obtain the simplified scheme for general dimensions \(d\) and powers \(k\) that are both powers of two. The number of beam splitters necessary to implement \(X^k\) in such cases is then

\[
N_X(d, k = 2^m) = k N_X(d/k) + 2 N_S(k) = 4 \left( k \log_2 \left( \frac{d}{k} \right) + k - 1 \right),
\]

(17)

where \(m\) is an integer such that \(2^m \leq k < 2^{m+1}\). For \(k = 2^m\) we recover formula (18).

From the structure of the OAM exchanger, Fig. 4(a), and the fact that the resulting setup for any \(X^k\) gate consists only of the OAM exchangers, it is clear that the exact same formula (19) applies also to the number of employed Dove prisms and holograms. For the number of mirrors we get twice as large number and there is no need for phase shifters.

### B. Parallelized scheme for \(X^k\) gates

One can parallelize the \(X\) gate by starting from the setup in Eq. (3), where \(U_P\) is the path-encoded \(X\) gate, and then removing all the redundant optical elements. To see which elements are not necessary, let us take a close look at the internal structure of the swap operator demonstrated in Fig. 4(a). The swap consists of two functionally different parts—one \(E\) block and a series of \(H\) blocks of increasing size. The \(E\) block is a network of exchangers \(E_{X_k}\) of increasing orders of the form \(k^{\delta}\) and is shown explicitly for each swap in Fig. 4. The structure of \(H\) blocks is not of interest in our discussion and can be found in Ref. [20].

The removal of redundant exchangers in the case of parallelized \(X\) gate is depicted in Fig. 4(a) explicitly for the special example of \(n = d = 8\). Analogously to the procedure of the previous section, there are many instances where an OAM exchanger \(E_{X_k}\) is followed by its inverse \(E_{X_{k^{-1}}}^{-1}\). These exchangers can be removed without affecting the final state. The resulting number of beam splitters required to implement the parallelized version of the \(X\) gate in dimension \(d = 2^M\) for \(n = 2^K\) paths is equal to

\[
N_X^{(\text{par})}(n, d) = n \log_2(n) + 2n - 2,
\]

(20)

provided that \(n \geq d\), see Appendix [C]. This formula does not depend on the dimension \(d\), only on the number of paths. The naive approach consisting in stacking \(n\) non-parallelized schemes would require \(N_X(d) = 4n \log_2(d)\) beam splitters. The saving in resources is thus approximately equal to

\[
r_X(n, d) \approx \frac{\log_2(n)}{4 \log_2(d)}
\]

(21)

for large enough dimensions \(d\) and number of paths \(n \geq d\). When the number of paths is approximately equal to the dimension, the ratio above approaches a constant factor of \(1/4\) and the parallelization of Eq. (3) provides a moderate improvement over the naive approach. When \(n \leq d\), the formula (21) is modified, but even then the improvement resulting from the parallelized scheme is rather moderate.

Let us turn our attention to the parallelized version of the \(X^k\) gate, where we focus on the case with \(n \geq d\). A similar discussion can also be done for the case \(n \leq d\), which is studied in Appendix [C]. One again starts from the scheme in Eq. (3). This step for \(d = n = 8\) is shown in Figs. 5(b), (c), and (d) for all \(X^k\) gates with \(k \leq d/2\). The second step consists in removing the redundant exchangers and has been studied in some detail for the non-parallelized version in the previous section. As shown in Appendix [C] the number of beam splitters retained in the final implementation for an arbitrary \(k \leq d/2\) reads

\[
N_X^{(\text{par})}(n, d, k) = n \log_2(n) + 2n - 4k + 2 + 2d \left( \frac{k}{2^m} + m - 1 \right),
\]

(22)

where \(m\) is an integer such that \(2^m \leq k < 2^{m+1}\). This expression simplifies for \(k = 1\) into the formula (20) derived for the parallelized version of the \(X\) gate. A similar discussion can also be done for other optical elements with similar results.

The number of beam splitters in the parallelized scheme is shown in Fig. 6 for \(n = 16\) propagation paths and dimensions \(d \leq n\). For comparison, the naive approach that involves \(n\) copies of the non-parallelized \(X^k\) gate is also shown. To put these numbers into context, we note that in Ref. [2] an experiment has been reported recently, where 50 polarizing beam splitters and a bulk interferometer representing 300 beam splitters were used. Even though the naive approach exceeds these numbers already for \(d = 8\), the parallelized scheme allows for the construction of an arbitrary power of the \(X\) gate even for \(d = 16\). At most 162 beam splitters are required in such a case.

When we compare the scaling for the naive approach utilizing \(n\) identical copies of the \(X^k\) gate and the pa-
allelization of Eq. (3), we obtain a scaling ratio that approaches
\[ r_X(n, d, k) \sim \frac{1}{4k} \log_2(n) \log_2(d). \]  

For high powers \( k \), we, therefore, save more resources by making use of the parallelized version. In this formula, we assumed \( k \) to be constant. We can, however, also consider \( k \) that scales with the dimension \( d \). For instance,
FIG. 6. The number of beam splitters $N$ in the naive and parallelized implementations of $d$-dimensional $X^k$ gates. The gates act on $n = 16$ propagations paths in dimensions $d = 2, 4, 8, 16$ and $1 \leq k \leq d - 1$. The naive implementation, represented by dark bars, requires as many as 960 beam splitters for $d = 16$ and $k = 8$. On the contrary, the parallelized implementation, represented by bright bars, needs only 162 beam splitters in such a case.

the most resource-demanding scenario is when $k = d/2$. In such a case one obtains

$$r_X(n, d, d/2) \lesssim \frac{3 \log_2(n)}{4d}. \quad (24)$$

Unless the number of paths exceeds exponentially the dimension, the parallelized scheme of Eq. (3) offers in this scenario substantial savings in resources when compared to the naive approach.

C. $Z$ gate and its powers

The implementation of the $d$-dimensional Pauli $Z$ gate is especially simple—a single Dove prism rotated through the angle of $\pi/d$ will do. The $k$-th power of the $Z$ gate then corresponds to an application of the Dove prism rotated through $k\pi/d$. The simplicity of the $Z$ gate implementation implies that in this case the parallelized scheme of Eq. (5) is not necessary. The stack of $n$ independent Dove prisms is much easier to build than the optical network consisting of beam splitters, Dove prisms, holograms, phase shifters, and mirrors as required by the parallelized scheme. The $Z$ gate in OAM is thus an example of a gate, where the parallelization does not actually bring any advantage.

VIII. CONCLUSION

We present a scheme that allows for the efficient implementation of a series of identical unitaries, each of which acts on an internal degree of freedom of a high-dimensional system. This parallelized scheme applies the same unitary on many propagation paths. To illustrate the advantages of the scheme, we focus on the orbital angular momentum of single photons and calculate the number of corresponding optical elements needed in the construction of the naive and parallelized schemes. Such a scheme can find applications e.g. in multiplexed quantum and classical communication channels, where the internal modes of photons are used as carriers of information.

Moreover, we present a simple argument that demonstrates how to implement with conventional optical elements an arbitrary unitary operation that acts on the orbital angular momentum of a single photon. The parallelized scheme can be understood as a generalization of this argument. We analyze how many optical elements are saved by utilizing the parallelized scheme and present estimates of their number for several general classes of unitaries. The savings for a general unitary scale approximately linearly with the number $n$ of involved communication parties, i.e., the naive approach requires roughly $n$ times more elements than our approach. Moreover, precise analytical formulas are derived for the case of $X$ gate and its integer powers $X^k$. The setups for $X^k$ gates are constructed explicitly for dimensions of the form $d = 2^M$.

The need to minimize the complexity of an experimental setup is driven primarily by the technical difficulties that accompany sophisticated optical arrangements. Advantages brought by the parallelized scheme presented in this work can render feasible those operations that would be otherwise out of reach of the present-day technology. Our approach relies on the utilization of the swap operator that exchanges the state of the OAM and path degrees of freedom. The very same method can be used also in other degrees of freedom, provided that the corresponding swap operator can be constructed. As available resources differ for different physical systems, the complexity estimates in those cases might differ from the estimates presented in this paper.

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[1] F. Arute, K. Arya, R. Babbush, D. Bacon, J. C. Bardin, R. Barends, R. Biswas, S. Boixo, F. G. S. L. Brandao, D. A. Buell, B. Burkett, Y. Chen, Z. Chen, B. Chiaro,
Appendix A: Swap operator

The interferometric implementation of the OAM-path swap operator was given in Ref. [26]. Here we present the number of beam splitters necessary for its implementation. Due to the structure of the swap operator, we have to discuss the case with $n \leq d$ and that with $n > d$ separately. When both $n$ and $d$ are powers of two and $n \leq d$, the number of beam splitters that implement the swap operator is given by

\[ N_{\text{SWAP}}(n, d) = \frac{n}{2} \log_2(n) + d \log_2(d) + 3n + 2d + 1. \quad (A1) \]

In the opposite case with $n \geq d$ one obtains

\[ N_{\text{SWAP}}(n, d) = \frac{n}{2} \log_2(n) + d \log_2(d) + n - 2d + 1. \quad (A2) \]

When $n$ or $d$ are not powers of two, we construct the swap with $2^r$ input and $2^s$ output paths, where $r$ and $s$ are such that $2^{s-1} < n \leq 2^r$ and $2^{s-1} < d \leq 2^s$. 

[References omitted for brevity]
Formulas (A1) and (A2) then represent upper bounds on the number of utilized beam splitters.

From the implementation of the swap operator it follows that whenever the number $n$ of input paths exceeds the number $d$ of output paths, only a specific class of incoming OAM eigenstates gets swapped correctly. Specifically, only eigenstates of the form $|0\rangle, |n/d\rangle, |2n/d\rangle, \ldots, |kn/d\rangle$ for $k \in \mathbb{Z}$ can then be used in our parallelized scheme.

**Appendix B: Construction of non-parallelized $X^k$ gates**

In this section we derive the formulas presented in the main text that quantify the number of beam splitters required in the interferometric implementation of the $X^k$ gates. We prove the most general case of Eq. (19), from where all the other cases follow.

First, we determine how many exchangers can be removed from the non-simplified setup for the non-parallelized $X^k$ gate. We calculate the resulting number of beam splitters $N_{X}(d, k)$ by subtracting the number of redundant beam splitters from the number of beam splitters constituting the two OAM sorters like

$$N_{X}(d, k) = 2N_{S}(d) - 2R_{S}(d, k), \quad (B1)$$

where $N_{S}(d)$ is given by (7) and $R_{S}(d, k)$ is the number of beam splitters removed from one OAM sorter. The latter quantity can be calculated by taking a close look at the structure of the path permutation that corresponds to the $k$-th power of the $X$ gate, see Fig. 4. Let us focus first on the $X$ gate itself. The corresponding path permutation consists of path crossings that are grouped in blocks of increasing size, the specific example of which for $d = 8$ is shown in Fig. 4(d). Due to the hierarchical structure of the OAM sorter implementation, the 8-dimensional sorter can be understood as two 4-dimensional sorters connected by a single OAM exchanger. In Fig. 4(d), the first 4-dimensional sorter is enclosed in a dashed box. The largest path crossings connect this sorter with the 4-dimensional inverted sorter. The 4-dimensional sorter and its inverse annihilate each other resulting in removing $N_{S}(4)$ beam splitters from the 8-dimensional sorter on the left and the same number of beam splitters from the inverted sorter on the right. This same observation can also be made for all the smaller groups of path crossings. As a result, from the setup of the 4-dimensional $X$ gate, where $d = 2^M$, one can remove

$$R_{S}(d, 1) = \sum_{j=1}^{M-1} N_{S}(2^j) = 2(d - \log_2(d) - 1) \quad (B3)$$

beam splitters with no effect on the operation of the gate. Using (B3) to calculate $N_{X}(d, 1)$ (B1) we recover formula (15) for the $X$ gate derived in Ref. [27].

To determine the number of beam splitters that can be removed from the general setup of $X^k$ gate we study first exponents that are powers of two. The path permutation for $X^k$ gate with $k = 2^n$ is equivalent to a series of permutations for $k$ different $(d/k)$-dimensional $X$ gates, see $X^4$ in Fig. 7(a). The number of beam splitters we can remove is thus equal to $R_{S}(d, 2^n) = 2^n R_{S}(d/2^n, 1)$. To determine this number for a general exponent $k$, we first express it in its binary form as $k = \sum_{j=0}^{m} b_j 2^j$, where $b_j \in \{0, 1\}$. In other words, $k$ is a sum of powers of two and the corresponding path permutation is a composition of permutations for exponents of the form $k_j = 2^j$. In Fig. 7(a) an example of the path permutation for $X^5$ in dimension $d = 16$ is shown, which is a composition of the path permutations for $k = 1$ and $k = 4$. As is apparent from the figure, it is the larger power $k = 4$ that imposes more constraints on the number of OAM exchangers that can be removed. In general, let us have a path permutation for the exponent of the form $k = k_m + k'$, where $k_m = 2^m$ and $0 \leq k' < 2^m$. The crucial observation here is that the path permutation for $k'$ consists of crossings that can be divided into two layers. The first layer contains large crossings that connect paths that are more then $d/k_m$ positions apart, see the dashed box in Fig. 7(a). This layer plays no role in the number of redundant exchangers. The second layer is a series of blocks, each of which permutes $d/k_m$ paths. These blocks are of two types. The first type corresponds to $(d/k_m)$-dimensional $X$ gate and is depicted by shaded rectangles in Fig. 7. The second type is just the identity on $d/k_m$ paths and is represented by empty rectangles in Fig. 7. When composing the permutation for $k'$ and that for $k_m$ the situation differs for the two types of blocks. The empty blocks are just replaced with permutations for $(d/k_m)$-dimensional $X$ gate, as demonstrated in Fig. 7(b). In contrast, the blocks of the first type are $X$ gates of size $d/k_m$ and so are also blocks of the path permutation for $k_m$. The two-fold application of the same path-encoded $d'$-dimensional $X$ gate is equivalent to two parallel permutations corresponding to $d'/2$-dimensional $X$ gate, as demonstrated in Fig. 7(c). After the decomposition we thus end up with $k = k_m + k'$ blocks that correspond to $(d/(2k_m))$-dimensional $X$ gates and additional $2^{m+1} - k$ empty blocks. For the number of beam splitters that we can remove we, therefore, obtain

$$R_{S}(d, k) = k R_{S}(d/2^{m+1}, 1) + (2^{m+1} - k) N_{S}(d/2^{m+1}) \quad (B4)$$

When written explicitly, this quantity is equal to

$$R_{S}(d, k) = 2(d - 2^{m+1} - k \log_2(d/2^{m+1})) \quad (B5)$$

from where we obtain Eq. (19) as was our goal to show.

**Appendix C: Construction of parallelized $X^k$ gates**

In this section we derive the formulas presented in the main text that quantify the number of beam splitters required in the interferometric implementation of the parallelized $X^k$ gates. We prove the most general case of
FIG. 7. Composition of the cyclic path permutations. (a) A specific example of the composition of the path permutations for $X^1$ gate and $X^4$ gate. As a result, the path permutation for $X^5$ gate is obtained. The path permutation for the power $k$ that is not a power of two can be decomposed into two layers as shown for $X^1$ gate. The first layer enclosed in a dashed box consists of large path crossings that do not affect the number of beam splitters that can be removed. The second layer consists of two types of blocks. The first type, depicted as a solid box, corresponds to the lower-dimensional $X$ gate. The second type, depicted as an empty box, is just the identity. When composing the path permutation for a general $k$ with a permutation for $k_m$ that is a power of two, i.e., $k_m = 2^m$, the two types of blocks are composed differently. (b) The empty box of size $d'$ is composed with a block corresponding to the $d'$-dimensional $X$ gate. The resulting permutation is thus just the $X$ gate, which itself can be decomposed into a $(d'/2)$-dimensional $X$ gate and an empty box of size $d'/2$. (c) The solid box is composed with another solid box. This composition corresponds to the two-fold application of $d'$-dimensional $X$ gate and results in a path permutation corresponding to two $(d'/2)$-dimensional $X$ gates.

Eq. (22), from where all the other cases follow. We proceed analogously to the previous section.

We treat only cases when both $n$ and $d$ are powers of two. Let us assume first that $n \geq d$. The resulting number of beam splitters is equal to

$$N_X^{(\text{par})}(n,d,k) = 2N_{\text{SWAP}}(n,d) - 2R_{\text{SWAP}}(n,d,k), \quad (C1)$$

where $N_{\text{SWAP}}(n,d)$ is given by $[\text{A2]}$ and where $R_{\text{SWAP}}(n,d,k)$ is the number of beam splitters that can be removed from one swap operator. Let us treat the $k = 1$ case first, cf. Fig. 6(a). From the figure we see that we can remove a series of blocks of increasing size, where each block, henceforth referred to as an $E$ sub-block, is a complete network of OAM exchangers. The largest removable $E$ sub-block resides on $d' = d/2$ paths. It consists of $\log_2(d')$ layers of exchangers $EX_l$, where in each layer there are $d'/2$ exchangers with the same order $l$. This totals $(d'/2) \log_2(d')$ exchangers or, equivalently

$$N_E(d') = d' \log_2(d') \quad (C2)$$

beam splitters. From there we calculate that the total number of beam splitters that we can remove from the setup of $d$-dimensional parallelized $X$ gate equals

$$R_{\text{SWAP}}(n,d,1) = \sum_{j=1}^{M-1} N_E(2^j) = d \log_2(d) - 2d + 2, \quad (C4)$$

where $d = 2^M$. Applying the same line of thought for higher powers $k$ as in the case of the non-parallelized $X$ gate, we conclude that the following relation holds true

$$R_{\text{SWAP}}(n,d,k) = k R_{\text{SWAP}}(n,d/2^{m+1},1) + (2^{m+1} - k) N_E(d/2^{m+1}). \quad (C5)$$

Substituting $N_E$ from Eq. (C2) and $R_{\text{SWAP}}$ from Eq. (C4), the formula above assumes the form

$$R_{\text{SWAP}}(n,d,k) = d \log_2 \left( \frac{d}{2^{m+1}} \right) + 2k \left( 1 - \frac{d}{2^{m+1}} \right), \quad (C6)$$

which is independent of $n$ and from which we obtain Eq. (22), as we wanted to show. The swap operator consists of a series of $H$ blocks and a large $E$ block of OAM exchangers that has $n$ input and $d$ output paths.
Note that whenever \( n \neq d \), this large \( E \) is not a complete network of exchangers. Nevertheless, all the smaller \( E \) subblocks, which we remove, already are.

This is no longer the case when the dimension exceeds the number of paths, i.e., \( n \leq d \), which makes the discussion a little bit more involved. All the \( E \) subblocks with at most \( n \) paths are again complete networks of exchangers, but the incomplete structure of the \( E \) block is reflected in the subblocks with more than \( n \) paths. It can be shown that the number of beam splitters employed in the construction of an \( E \) block with \( n' \) input and \( d' \) output paths, where \( n' \leq d' \), is equal to

\[
N_E(n', d') = d' \log_2(n') + 2(d' - n'). \tag{C7}
\]

Note that \( \text{(C7)} \) reduces both to \( \text{(7)} \) and \( \text{(C2)} \) for \( n' = 1 \) and \( n' = d' \), respectively. For the number of beam splitters that can be removed from the \( X \) gate we obtain

\[
R_{\text{SWAP}}(n, d, 1) = \sum_{j=1}^{K} N_E(2^j, 2^j) + \sum_{j=K+1}^{M-1} N_E(n, 2^j)
\]

\[
= d \log_2(n) + 2n \log_2(n) - 2n \log_2(d) + 2d - 4n + 2, \tag{C8}
\]

where \( d = 2^m \) and \( n = 2^K \). Formula \( \text{(C5)} \) has to be modified into

\[
R_{\text{SWAP}}(n, d, k) = k R_{\text{SWAP}}(n', d/2^{m+1}, 1) + (2^{m+1} - k) N_E(n', d/2^{m+1}), \tag{C9}
\]

where

\[
n' = \min(n, d/2^{m+1}). \tag{C10}
\]

Due to the dependency \( \text{(C10)} \), the discussion has to treat two different scenarios—either \( k < d/n \), for which \( n' = n \), or \( k \geq d/n \), for which \( n' = d/2^{m+1} \). From \( \text{(C9)} \) we can calculate the explicit forms for \( R_{\text{SWAP}}(n, d, k) \) and plug those into \( \text{(C1)} \). For the case of \( n \leq d \) we obtain

\[
N_X^{(\text{par})}(n, d, k) = n \log_2(n) - 4k + 2 - 6n
\]

\[
+ 4n(k + 2^{m+1}) + 4kn \log_2 \left( \frac{d}{n 2^{m+1}} \right), \tag{C11}
\]

when \( k < d/n \), and

\[
N_X^{(\text{par})}(n, d, k) = n \log_2(n) - 4k + 2 - 6n
\]

\[
+ 4 \frac{d}{2^{m+1}} (k + 2^{m+1}) - 2d \log_2 \left( \frac{d}{n 2^{m+1}} \right), \tag{C12}
\]

when \( k \geq d/n \). As for the savings in the number of beam splitters, we mention four special cases for \( n < d \):

- When \( k = 1 \), the ratio \( \text{(9)} \) is approximately equal to

\[
r_X(n, d, 1) \approx 1 - \frac{3 \log_2(n)}{4 \log_2(d)}. \tag{C13}
\]

For \( n = d \) this ratio reduces to \( r_X(n, d, 1) \approx 1/4 \), which is consistent with formula \( \text{(21)} \).

- The case of \( k < d/n \) is a representative of “small powers”. Especially for \( n \ll d \) almost all the powers \( k \) satisfy \( k \leq d/n \). The ratio then scales like

\[
r_X(n, d, k) \sim 1 - \frac{\log_2(n)}{\log_2(d/k)}. \tag{C14}
\]

- The case of \( k = d/n \) represents the “medium powers”. The ratio \( \text{(9)} \) behaves approximately like

\[
r_X(n, d, d/n) \approx \frac{n}{4d} + \frac{2}{\log_2(n)}. \tag{C15}
\]

For \( d \approx n \approx 1 \) this ratio reduces to \( r_X(n, d, d/n) \approx 1/4 \), which is again consistent with \( \text{(21)} \). For \( d \gg n \) it reduces to \( r_X(n, d, d/n) \approx 2/ \log_2(n) \).

- As we treat only powers in the range \( 1 \leq k \leq d/2 \), see the main text for the explanation, we choose \( k = d/2 \) as a representative of “large powers”. For that case we obtain

\[
r_X(n, d, d/2) \approx \frac{5 \log_2(n)}{8n}. \tag{C16}
\]