Adiabatic approach to large-amplitude collective motion with the higher-order collective-coordinate operator

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We propose a new set of equations to determine the collective Hamiltonian including the second-order collective-coordinate operator on the basis of the adiabatic self-consistent collective-coordinate (ASCC) theory. We illustrate, with the two-level Lipkin model, that the collective operators including the second-order one are self-consistently determined. We compare the results of the calculations with and without the second-order operator and show that, without the second-order operator, the agreement with the exact solution becomes worse as the excitation energy increases, but that, with the second-order operator included, the exact solution is well reproduced even for highly excited states. We also reconsider which equations one should adopt as the basic equations in the case where only the first-order operator is taken into account, and suggest an alternative set of fundamental equations instead of the conventional ASCC equations. Moreover, we briefly discuss the gauge symmetry of the new basic equations we propose in this paper.

1. Introduction

In recent papers, we elucidated the relation among the higher-order collective operators, the \( a^\dagger a \) terms, and the gauge symmetry and its breaking in the adiabatic self-consistent collective-coordinate (ASCC) theory [1–3]. The ASCC method [4] is a practical method of describing the large-amplitude collective motion, which is an adiabatic approximation to the SCC method [5] and can be regarded as an advanced version of the adiabatic time-dependent Hartree-Fock(-Bogoliubov)[ATDHF(B)] theory.

Although several versions of the ATDHF theory had been proposed so far, they encountered difficulties such as non-uniqueness of the solution (See Refs. [6–8] for a review). As for the ATDHF(B) theory, which is an extension of the ATDHF theory including the pairing correlation, Dobaczewski and Skalski attempted to develop an ATDHF(B) theory assuming the axial quadrupole deformation parameter \( \beta \) as a collective coordinate [9]. Recently, Li et al also attempted to construct the five-dimensional collective Hamiltonian on the basis of the ATDHF(B) theory [10]. However, the extension of the ATDHF theory to the ATDHF(B) theory is not straightforward, because one needs to decouple the pair-rotational mode from the collective mode of interest.

In the ASCC method, Matsuo et al [4] first assumed the gauge angle dependence of the state vector in the form

\[
|\phi(q,p,\varphi,n)\rangle = e^{-i\varphi N} e^{i\tilde{G}(q,p,n)} |\phi(q)\rangle.
\]

(1.1)
Here, \((q, p)\) are collective coordinate and conjugate momentum. \(n = N - N_0\) is the particle number measured from a reference value \(N_0\), and \(\varphi\) is the gauge angle conjugate to \(n\). With this form of the state vector, the collective Hamiltonian is independent of the gauge angle \(\varphi\), and it is guaranteed that the expectation value of the particle number is conserved. Furthermore, there appears no gauge-angle degree of freedom in the equations of motion. In Ref. [4], Matsuo et al considered the expansion of \(\hat{G}(q, p, n)\) up to the first order \(\hat{G}(q, p, n) = p\hat{Q}(q) + n\hat{\Theta}(q)\) and derived the moving-frame HFB (Hartree–Fock–Bogoliubov) & QRPA (quasiparticle random phase approximation) equations, which are equations of motion in the ASCC theory.

On the basis of Ref. [4], Hinohara et al performed a numerical calculation and encountered a difficulty of finding the solution due to the numerical instability [11]. They found that this instability was caused by the symmetry under the following transformation under which the basic equations are invariant if the collective-coordinate and particle-number operators commute, i.e. \([\hat{Q}, \hat{N}] = 0\).

\[
\hat{Q} \to \hat{Q} + \alpha \hat{N},
\]

\[
\hat{\Theta} \to \hat{\Theta} + \alpha \hat{P},
\]

\[
\lambda \to \lambda - \alpha \partial_q V,
\]

\[
\partial_q \lambda \to \partial_q \lambda - \alpha C
\]

As this transformation changes the phase of the state vector, they called it the "gauge" transformation and proposed a gauge-fixing prescription to remove the redundancy associated with the gauge symmetry. Their prescription for the gauge fixing is as follows. While the moving-frame HFB & QRPA equations are invariant under the above transformation at the HF equilibrium point \(\partial_q V = 0\), it is not the case at non-equilibrium points unless \([\hat{Q}, \hat{N}] = 0\). Therefore, they first require the commutativity of the collective-coordinate and particle-number operators \([\hat{Q}, \hat{N}] = 0\) for the gauge symmetry of the moving-frame HFB & QRPA equations, and then fix the gauge. With this prescription, they succeeded in obtaining the solution and applied the one-dimensional (1D) ASCC method to the multi-\(O(4)\) model [11] and the oblate-prolate shape-coexistence phenomena in the proton-rich Se and Kr isotopes [12, 13] (Here, we mean by the \(D\)-dimensional ASCC method that the dimension of the collective coordinate \(q\) is \(D\).)

After the successful application of the 1D ASCC method, an approximate version of the 2D ASCC method, the constrained HFB plus local QRPA method, was proposed and applied to large-amplitude quadrupole collective dynamics [14–21]. Moreover, the 1D ASCC method without the pairing correlation was also applied to nuclear reaction [22, 23] (In these calculations, the so-called curvature term was neglected.) However, little progress had been made in understanding of the gauge symmetry, until very recently we analyzed the gauge symmetry and its breaking under more general gauge transformations in the ASCC theory, on the basis of the Dirac-Bergmann theory of the constrained systems [1, 2]. There, it was shown that the gauge symmetry in the ASCC method is broken by the adiabatic approximation, and that the gauge symmetry is partially retained by containing the higher-order collective operators in the adiabatic expansion.

According to the generalized Thouless theorem (Refs. [5, 24–27]), \(\hat{G}\) in Eq. (1.1) can be written in terms of only \(a^\dagger a^\dagger\) and \(aa\) terms. We shall call \(a^\dagger a^\dagger\) and \(aa\) terms A-terms and
\( a^\dagger a \) and \( aa^\dagger \) (or equivalently \( a^\dagger a \) and constant terms) B-terms, respectively. In Hinohara’s prescription, the commutativity \([\hat{Q}, \hat{N}] = 0\) is required, which implies that one needs to introduce B-terms in \( \hat{Q} \) in contrast with the theorem. Thus, there are two approaches to conserve the gauge symmetry: one is the approach with higher-order operators consisting of only A-terms (Approach A), and the other with only the first-order operator containing B-terms as well as A-terms (Approach B). The relation between the two approaches was investigated in Ref. [3], and it was shown that the inclusion of the B-terms in the collective-coordinate operator is equivalent to that of a certain kind of the higher-order operators.

What should be emphasized is that these higher-order collective operators and B-terms contribute to the equations of motion and the inertial mass directly. As shown in Ref. [1], the gauge symmetry is associated with the constraint on the particle number and appears when the pairing correlation is taken into account. However, the higher-order operators and B-terms can affect dynamics through the equations of motion and the inertial mass even if there is no pairing correlation. In fact, in the ATDHF theories proposed in the 1970’s, similar things had been recognized. In his paper on the ATDHF in 1977 (Ref. [28]), Villars mentioned the extension of his ATDHF theory including the higher-order operator (more strictly, the extension with the first- and third-order operators and no second-order operator) and preannounced a publication on it: “Ref. 17) A. Toukan and F. Villars, to be published” in Ref. [28]. However, as far as the author knows, it was not published after all. It is noteworthy that the second-order operator was not included in his extension. In this paper, we include the second-order operator because it does contribute to the equations of motion and the inertial mass. On the other hand, in the ATDHF theory by Baranger and Vénéri (Ref. [29]), they proposed the density matrix in the form of \( \rho(t) = e^{i\chi(t)\rho_0(t)e^{-i\chi(t)}} \) with Hermitian and time-even \( \rho_0(t) \) and \( \chi(t) \). They emphasized that \( \chi(t) \) can be written in terms of A-terms only, but included B-terms as well as A-terms in the treatment of the translational motion. Thus, although the necessity of the inclusion of the higher-order operators or B-terms was already recognized in the ATDHF theories in the 1970’s, there has been no general theory to determine them.

Now that the relation between the higher-order operators and B-terms has been revealed in Ref. [3], we propose in this paper a new set of basic equations to determine the higher-order operator, which is different from Hinohara’s prescription. In Ref. [3], it is shown that, with Hinohara’s prescription, one can take into account the contribution from the second- and third-order collective operators in an effective way by including the B-part of the collective-coordinate operator \( \hat{Q}_B \). That is an advantage of Hinohara’s prescription. In the case without the pairing correlation, however, one cannot determine the higher-order collective operators through Hinohara’s prescription because \([\hat{Q}, \hat{N}] = 0\) is automatically satisfied without \( \hat{Q}_B \).

(In the first place, there exists no gauge symmetry when the pairing correlation is not taken into account, and there is no reason to require \([\hat{Q}, \hat{N}] = 0\).) Without the pairing correlation, no way has been known to include the higher-order contribution. In this sense, the cases with and without the pairing correlation have not been treated on an equal basis so far. Hinohara’s prescription is based on Approach B, in which the B-part of the collective-coordinate operator \( \hat{Q}_B \) is introduced. It is also worth mentioning that the Hilbert space in Approach A is larger than that in Approach B because only the higher-order collective
operator which are written in terms of (multiple) commutators of \( \hat{Q}_A \) and \( \hat{Q}_B \) can be taken into account in Approach B. In this paper, we employ Approach A.

In the conventional ASCC theory without the pairing correlation, only the first-order operators consisting of A-terms are taken into account, and the contributions from the higher-order operators to the equations of motion and the inertial mass are missing. However, it cannot be neglected from a simple order counting in the adiabatic expansion. In Refs. [22, 23], Wen and Nakatsukasa successfully reproduced the inertial mass against the translational motion without including the higher-order collective operators. Actually, for the mode with \( \partial_q V = 0 \), the higher-order operators and B-terms do not contribute to the inertial mass. The reason for this is explained in Ref. [3] from the viewpoint of Approach B, but it also applies to Approach A straightforwardly. For general collective modes of interest, however, \( \partial_q V = 0 \) is not satisfied, and a theory with which one can correctly evaluate the contribution from the higher-order operators is necessary. In Ref. [23], Wen and Nakatsukasa failed to find the solution at a certain point on the collective path and pointed out a possibility to solve this problem by including the pairing correlation. Depending on the particle number, or on the collective coordinate even in one nucleus, the pairing gap changes and can vanish. Therefore, a theory is favorable with which one can treat the cases with and without the pairing correlation on an equal footing. In this paper, we first consider the case without the pairing correlation on the basis of Approach A including the second-order collective-coordinate operator \( \hat{Q}^{(2)} \) and propose a new set of fundamental equations to determine the collective operators. The set of fundamental equations for the case with the pairing correlation included is derived in a straightforward way.

The paper is organized as follows. Sect. 2 describes the formulation. In Sect. 2.1, we propose a set of the basic equations including the second-order collective-coordinate operator \( \hat{Q}^{(2)} \). In Sect. 2.3, we introduce the two-level Lipkin model, and it is shown in Sect. 2.3 that the basic equations proposed in Sect. 2.1 are reduced to one differential equation in the case of the two-level Lipkin model. We give the collective Schrödinger equation in Sect. 2.4, and the solution to the conventional ASCC equations without \( \hat{Q}^{(2)} \) is given in Sect. 2.5. The numerical results are shown in Sect. 3. We compare the calculations with and without \( \hat{Q}^{(2)} \) employing the Lipkin model. The numerical results show that, for low-energy states, both of the calculations reproduce the exact solution well, but that, with increasing the excitation energy, the agreement with the exact solution becomes worse when the second-order collective operator \( \hat{Q}^{(2)} \) is neglected. With \( \hat{Q}^{(2)} \), the exact solution is well reproduced even for higher excited states. In Sect. 4, we consider which equations should be adopted as the basic equations of motion when only the first-order operator \( \hat{Q}^{(1)} \) is included, taking the Lipkin model as a simple example. We propose an alternative set of the basic equations including an equation introduced in Sect. 2 instead of the conventional moving-frame RPA equation of \( O(p^2) \). In Sect. 5, we briefly discuss the gauge symmetry of the basic equations derived in Sect. 2 with the pairing correlation included. Concluding remarks are given in Sect. 6. In Appendix A, the expressions of the derivatives of the collective operators are given in terms of quasispin operators in the Lipkin model. In Appendix B, the potential curvature \( C \) in the \( q \) space with \( B(q) = 1 \) is given in the case where only the first-order operator is taken into account.
2. Formulation

2.1. Basic equations with the second-order operator $\hat{Q}^{(2)}$

First, we consider the equations of motion without pairing correlation. In preceding papers [2, 3], we considered the ASCC theory with the second- and third-order collective-coordinate operators. Here we adopt the approach with the higher-order operators consisting of only A-terms, i.e., Approach A. The state vector in the ASCC theory is given by

$$|\phi(q,p)\rangle = e^{i\hat{G}(q,p)}|\phi(q)\rangle. \quad (2.1)$$

Here, $\hat{G}$ is expanded as

$$\hat{G}(q,p) = p\hat{Q}^{(1)}(q) + \frac{1}{2}p^2\hat{Q}^{(2)}(q) + \frac{1}{3!}p^3\hat{Q}^{(3)}(q). \quad (2.2)$$

with

$$\hat{Q}^{(i)}(q) = \sum_{\alpha\beta} Q^{(i)}_{\alpha\beta} a^\dagger_\alpha a^\dagger_\beta + Q^{(i)\ast}_{\alpha\beta} a_\beta a_\alpha \quad (i = 1, 2, 3). \quad (2.3)$$

The equations of motion in the ASCC theory is derived from the invariance principle of the Schrödinger equation

$$\delta\langle\phi(q,p)|i\partial_t - \hat{H}|\phi(q,p)\rangle = 0, \quad (2.4)$$

which can be rewritten into the equation of collective submanifold (CS):

$$\delta\langle\phi(q,p)|\hat{H} - \frac{\partial H}{\partial p} \hat{P} - \frac{\partial H}{\partial q} \hat{Q}|\phi(q,p)\rangle = 0, \quad (2.5)$$

with $(\hat{P}, \hat{Q}) := (i\partial_q, -i\partial_p)$. The collective Hamiltonian is given by

$$\mathcal{H}(q,p) = V(q) + \frac{1}{2}B(q)p^2 \quad (2.6)$$

with

$$V(q) = \langle\phi(q)|\hat{H}|\phi(q)\rangle, \quad (2.7)$$

$$B(q) = \langle\phi(q)||[\hat{H}, i\hat{Q}^{(2)}]|\phi(q)\rangle - \langle\phi(q)||[\hat{H}, \hat{Q}^{(1)}], \hat{Q}^{(1)}]|\phi(q)\rangle. \quad (2.8)$$

One can see that the second-order operator $\hat{Q}^{(2)}$ contributes to the inertial mass, and that one cannot neglect the term involving $\hat{Q}^{(2)}$ in $B(q)$ from a simple order counting, as both of the first and second terms of the right-hand side of Eq. (2.8) are $O(p^2)$ terms.

By substituting the state vector (2.1) into Eq. (2.5) and expanding in powers of $p$, the moving-frame HF & RPA equations, which are the equations of motion in the ASCC theory (without the pairing), are obtained as
Moving-frame HF equation

\[ \delta \langle \phi(q) | \hat{H} - \partial_q V \hat{Q}^{(1)} | \phi(q) \rangle = 0, \tag{2.9} \]

Moving-frame RPA equations

\[ \delta \langle \phi(q) | [\hat{H}, \hat{Q}^{(1)}] - \frac{1}{i} B(q) \hat{P} - \frac{1}{i} \partial_q V \hat{Q}^{(2)} | \phi(q) \rangle = 0, \tag{2.10} \]

\[ \delta \langle \phi(q) | [\hat{H} - \partial_q V \hat{Q}^{(1)}, \frac{1}{i} B \hat{P}] - B(q) C(q) \hat{Q}^{(1)} \]

\[ - \frac{1}{2} \partial_q V \left\{ [\hat{H}, \hat{Q}^{(1)}], \hat{Q}^{(1)} \right\} + [\hat{H}, \frac{1}{i} \hat{Q}^{(2)}] \]

\[ + \partial_q V \left( \hat{Q}^{(3)} + \frac{1}{2} \hat{Q}^{(1)}, \frac{1}{i} \hat{Q}^{(2)} \right) \}

\[ \delta \langle \phi(q) | \phi(q) \rangle = 0, \tag{2.11} \]

where

\[ C := \partial_q^2 V(q) - \Gamma(q) \partial_q V := \partial_q^2 V(q) + \frac{1}{2B} \partial_q B \partial_q V. \tag{2.12} \]

The moving-frame HF & RPA equations are derived from the \( O(1), O(p) \) and \( O(p^2) \) expansions of the equation of CS (2.5). While the moving-frame HF equation (2.9) is the zeroth-order equation of CS with respect to \( p \), i.e., the \( O(1) \) terms of Eq. (2.5), the moving-frame RPA equation of \( O(p) \) (2.10) is the first-order equation of CS. On the other hand, the moving-frame RPA equation of \( O(p^2) \) (2.11) is not the \( O(p^2) \) expansion of the equation of CS (2.5) itself, which is given by Eq. (2.16), but is derived from the second-order equation of CS (2.16) and the \( q \)-derivative of the zeroth-order equation of CS, i.e., the moving-frame HF equation (2.9). (See Ref. [2] for the detailed derivation.) The fourth- and higher-order operators \( \hat{Q}^{(i)} (i \geq 4) \) do not contribute to the equations of motion up to this order.

Here we expand \( \hat{G} \) up to \( \hat{Q}^{(2)} \) and omit \( \hat{Q}^{(i)} (i \geq 3) \). \( \hat{Q}^{(3)} \) contributes to the moving-frame RPA equation of \( O(p^2) \) (2.11) but does not contribute to the equations of motion of \( O(1) \) and \( O(p) \) and the inertial mass. When the expansion up to \( \hat{Q}^{(2)} \) is taken, there are one more unknown operators than in the conventional ASCC equations taking up to \( \hat{Q}^{(1)} \). Thus, one more equation is necessary to determine the collective operators and the state vector self-consistently. The moving-frame equations are derived by expanding the equation of CS (2.5) in powers of \( p \). The \( O(p^3) \) equation of CS is a possible candidate for such an equation to add to the three moving-frame equations. However, as easily confirmed, there appears \( \hat{Q}^{(4)} \) as well as \( \hat{Q}^{(3)} \) in the \( O(p^3) \) equation of CS, so one needs to make an approximation neglecting \( \hat{Q}^{(3)} \) and \( \hat{Q}^{(4)} \) to close the set of the equations.

Therefore, we adopt another equation, that is, the \( q \)-derivative of the moving-frame HF equation. As mentioned above, the moving-frame RPA equations of \( O(p^2) \) (2.11) is derived from the second-order equation of CS and the \( q \)-derivative of the moving-frame HF equation [2, 4]. Here we consider the following equations as a set of the basic equations in the case where \( \hat{Q}^{(2)} \) is included.

\[ \delta \langle \phi(q) | \hat{H} - \partial_q V \hat{Q}^{(1)} | \phi(q) \rangle = 0, \tag{2.13} \]

\[ \delta \langle \phi(q) | [\hat{H}, \hat{Q}^{(1)}] - \frac{1}{i} B(q) \hat{P} - \frac{1}{i} \partial_q V \hat{Q}^{(2)} | \phi(q) \rangle = 0, \tag{2.14} \]
\[
\delta \langle \phi(q) | \hat{H} - \partial_q V \hat{Q}^{(1)} , \frac{1}{i} B(q) \hat{P} - B(q) C(q) \hat{Q}^{(1)} - \partial_q V B(q) D_q \hat{Q}^{(1)} | \phi(q) \rangle = 0, \quad (2.15)
\]

\[
\delta \langle \phi(q) | \frac{1}{2} [\hat{H}, \hat{Q}^{(1)}] , \hat{Q}^{(1)} - B(q) D_q \hat{Q}^{(1)} - \frac{i}{2} \hat{H}, \hat{Q}^{(2)} \rangle - \frac{i}{4} \partial_q V [\hat{Q}^{(1)}, \hat{Q}^{(2)}] | \phi(q) \rangle = 0, \quad (2.16)
\]

where the covariant derivative of \( \hat{Q}^{(1)} \) is given by

\[
D_q \hat{Q}^{(1)} = \partial_q \hat{Q}^{(1)} + \Gamma(q) \hat{Q}^{(1)}. \quad (2.17)
\]

Eq. (2.16) is the second-order equation of collective submanifold with \( \hat{Q}^{(3)} \) omitted. The term \( [\hat{Q}^{(1)}, \hat{Q}^{(2)}] \) is kept in Eq. (2.16) but does not contribute because it is a B-term (See Ref. [3]). The moving-frame RPA equation of \( O(p^2) \) (2.11) is obtained by eliminating \( D_q \hat{Q}^{(1)} \) from Eq. (2.15) using Eq. (2.16) multiplied by \( \partial_q V \). Hence, for \( \partial_q V \neq 0 \), it is equivalent to adopt Eqs. (2.14) and (2.15) as basic equations to adopting Eqs. (2.14) and (2.11) with \( Q^{(3)} \) omitted. In Ref. [22, 23] by Wen and Nakatsukasa, Eq. (2.15) without the \( D_q \hat{Q}^{(1)} \) term is called “moving” RPA equation of \( O(p^2) \). It is somewhat misleading, however, because this equation does not contain \( O(p^2) \) terms. Eq. (2.15) is derived from the first derivative of the equation of CS with respect to \( \phi \), and thus it would rather be called the equation of “\( O(q) \)”. The other moving-frame RPA equation (2.14) is of \( O(p) \). At the HF equilibrium point \( \partial_q V = 0 \), they reduce to the ordinary RPA equations, in which the collective coordinate and momentum operators are involved in a symmetric form. At the equilibrium point \( \partial_q V = 0 \), the moving-frame HF equation (2.9) reduces to the ordinary HF equation. Eq. (2.16) gives the relation between \( \partial_q \hat{Q}^{(1)} \) and \( \hat{Q}^{(2)} \).

In the above equations, there appears the covariant derivative of \( \hat{Q}^{(1)} \). In actual calculations, one can choose a coordinate system with \( B(q) = 1 \) by the scale transformation of \( q \). Here we mean by the symbol \( \equiv 1 \) that the inverse inertial mass \( B(q) \) is unity everywhere along the collective path. Then, the covariant derivative reduces to the ordinary partial derivative, and the equations may be solved by approximating \( \partial_q \hat{Q}^{(1)}(q) \) by finite difference. The simplest scheme of the finite difference is \( \partial_q \hat{Q}^{(1)}(q) = \left[ \hat{Q}^{(1)}(q) - \hat{Q}^{(1)}(q - \delta q) \right] / \delta q \). At the HF(B) equilibrium point \( (\partial_q V = 0) \), Eqs. (2.13)-(2.15) reduces to the HF & RPA equations, and \( (\hat{Q}^{(1)}, \hat{P}) \) can be obtained. Starting from the equilibrium point and approximating the derivative of the collective-coordinate operator \( \partial_q \hat{Q}^{(1)} \) with the finite difference, one can solve the above set of the equations.

Above we have considered the case without pairing correlation, the basic equations in the case with the pairing correlation can be obtained in a straightforward way as follows.

\[
\delta \langle \phi(q) | \hat{H}_M | \phi(q) \rangle = 0, \quad (2.18)
\]

\[
\delta \langle \phi(q) | [\hat{H}_M, \hat{Q}^{(1)}] - \frac{1}{i} B(q) \hat{P} - \frac{1}{i} \partial_q V \hat{Q}^{(2)} | \phi(q) \rangle = 0, \quad (2.19)
\]

\[
\delta \langle \phi(q) | [\hat{H}_M, \frac{1}{i} B(q) \hat{P}] - B(q) C(q) \hat{Q}^{(1)} - \partial_q V B(q) D_q \hat{Q}^{(1)} - \partial_q \lambda \hat{N} | \phi(q) \rangle = 0, \quad (2.20)
\]

\[
\delta \langle \phi(q) | \frac{1}{2} [\hat{H}_M, \hat{Q}^{(1)}], \hat{Q}^{(1)} - B(q) D_q \hat{Q}^{(1)} - \frac{i}{2} [\hat{H} - \lambda \hat{N}, \hat{Q}^{(2)}] - \frac{i}{4} \partial_q V [\hat{Q}^{(1)}, \hat{Q}^{(2)}] | \phi(q) \rangle = 0, \quad (2.21)
\]
where the moving-frame Hamiltonian is given by

\[ \hat{H}_M = \hat{H} - \lambda \tilde{N} - \partial_q \hat{Q}^{(1)}, \]  

(2.22)

with \( \tilde{N} \) is the particle-number operator measuring from \( N_0 \), \( \tilde{N} = \hat{N} - N_0 \). In Sect. 5, we shall give a brief consideration on the gauge symmetry of these equations.

2.2. The Lipkin model

In this section, we introduce the Lipkin model with two \( N \)-fold degenerate levels [30]. We follow the formulation of Ref. [31] by Holzwarth. We label the states in the upper and lower level by \( p = 1, 2, \ldots, N \) and \( -p \), respectively.

In this model, the Hamiltonian is given by

\[ H = \frac{1}{2} \epsilon \sum_{p>0} (c_p^\dagger c_p - c_{-p}^\dagger c_{-p}) + \frac{1}{2} V \sum_{p,p'>0} (c_p^\dagger c_{p'}^\dagger c_{-p} c_{-p'} + c_{-p}^\dagger c_{-p'}^\dagger c_p c_{p'}) \]

\[ = \epsilon \hat{K}_0 + \frac{1}{2} V (\hat{K}_+ \hat{K}_+ + \hat{K}_- \hat{K}_-), \]  

(2.23)

with

\[ \hat{K}_0 = \frac{1}{2} \sum_{p>0} (c_p^\dagger c_p - c_{-p}^\dagger c_{-p}), \]  

(2.24)

\[ \hat{K}_+ = \sum_{p>0} c_p^\dagger c_{-p}, \quad \hat{K}_- = \hat{K}_+^\dagger. \]  

(2.25)

The quasispin operators satisfy

\[ [\hat{K}_+, \hat{K}_-] = 2 \hat{K}_0, \quad [\hat{K}_0, \hat{K}_\pm] = \pm \hat{K}_\pm. \]  

(2.26)

We assume that the system contains \( N \) particles. If there is no interaction \( V = 0 \), the lower levels are fully occupied and the upper levels are completely empty in the ground state. We denote the ground state in the non-interacting case by \( |0\rangle \). Then, we have

\[ c_p |0\rangle = 0, \quad c_{-p}^\dagger |0\rangle = 0, \quad \hat{K}_- |0\rangle = 0. \]  

(2.27)

We define the particle/hole creation and annihilation operators \( a_{\pm p} \) as follows:

\[ a_p = c_p, \quad a_p^\dagger = c_p^\dagger, \]  

(2.28)

\[ a_{-p} = c_{-p}^\dagger, \quad a_{-p}^\dagger = c_{-p}. \]  

(2.29)

Since \( a_{\pm p} |0\rangle = 0 \), \( |0\rangle \) is the vacuum with respect to \( a_{\pm p} \). \( a_{\pm p} \) satisfies the canonical commutation relations of fermions. With \( a_{\pm p} \), the quasispin operators are written as

\[ \hat{K}_0 = \frac{1}{2} \sum_{p>0} (a_p^\dagger a_p - a_{-p} a_{-p}^\dagger) = \frac{1}{2} \sum_p (a_p^\dagger a_p + a_{-p} a_{-p}) - \frac{N}{2}, \]  

(2.30)

\[ \hat{K}_+ = \sum_{p>0} a_p^\dagger a_{-p}, \quad \hat{K}_- = \sum_{p>0} a_{-p} a_p. \]  

(2.31)

Following Ref. [31], we introduce the "deformed" state as

\[ |a\rangle = \exp(a \hat{K}_+) |0\rangle, \]  

(2.32)
and the “deformed” operators as
\[
\begin{pmatrix}
\alpha_p^+ \\
\alpha_{-p}^-
\end{pmatrix} =
\begin{pmatrix}
\cos \alpha & -\sin \alpha e^{-i\psi} \\
\sin \alpha e^{i\psi} & \cos \alpha
\end{pmatrix}
\begin{pmatrix}
\alpha_p^- \\
\alpha_{-p}^+
\end{pmatrix}. \tag{2.33}
\]

With the deformed operators, we shall define the new quasispin operators:
\[
\hat{I}_0 = \frac{1}{2} \sum_{p>0} (\alpha_p^+ \alpha_p - \alpha_{-p}^+ \alpha_{-p}),
\]
\[
\hat{I}_+ = \sum_{p>0} \alpha_p^+ \alpha_{-p}, \quad \hat{I}_- = \hat{I}_+ = \sum_{p>0} \alpha_{-p}^+ \alpha_p,
\]
which are connected with the undeformed quasispin operators by
\[
\begin{pmatrix}
\hat{I}_0 \\
\hat{I}_+ \\
\hat{I}_-
\end{pmatrix} =
\begin{pmatrix}
\cos \phi & -\frac{1}{2} e^{-i\psi} \sin \phi & -\frac{1}{2} e^{-i\psi} \sin \phi \\
\frac{1}{2} e^{-i\psi} \sin \phi & \cos^2 \frac{1}{2} \phi & -e^{-2i\psi} \sin^2 \frac{1}{2} \phi \\
\frac{1}{2} e^{i\psi} \sin \phi & -e^{2i\psi} \sin^2 \frac{1}{2} \phi & \cos^2 \frac{1}{2} \phi
\end{pmatrix}
\begin{pmatrix}
\hat{K}_0 \\
\hat{K}_+ \\
\hat{K}_-
\end{pmatrix}. \tag{2.36}
\]

The inverse transformation is given by
\[
\begin{pmatrix}
\hat{K}_0 \\
\hat{K}_+ \\
\hat{K}_-
\end{pmatrix} =
\begin{pmatrix}
\cos \phi & \frac{1}{2} e^{i\psi} \sin \phi & \frac{1}{2} e^{-i\psi} \sin \phi \\
-e^{-i\psi} \sin \phi & \cos^2 \frac{1}{2} \phi & e^{-2i\psi} \sin^2 \frac{1}{2} \phi \\
e^{-i\psi} \sin \phi & -e^{2i\psi} \sin^2 \frac{1}{2} \phi & \cos^2 \frac{1}{2} \phi
\end{pmatrix}
\begin{pmatrix}
\hat{I}_0 \\
\hat{I}_+ \\
\hat{I}_-
\end{pmatrix}. \tag{2.37}
\]

As shown in Ref. [31], \(|a\rangle\) is not normalized, and the normalized deformed state is given by
\[
|\phi\rangle = \left(\cos \frac{1}{2} \phi\right)^N \langle a\rangle. \tag{2.38}
\]

As in Ref. [31], we set \(\psi = 0\) and denote the quasispin operators as \(\hat{J}_i (= \hat{I}_i|_{\psi=0})\). Then the Hamiltonian can be written as
\[
\hat{H} = 2E\hat{J}_0 + \frac{1}{2} \sin \phi \left(1 - \chi \cos \phi\right)(\hat{J}_+ + \hat{J}_-)
\]
\[
+ \frac{1}{4} V \sin^2 \phi \left(4\hat{J}_0^2 + (4N - 2)\hat{J}_0\right) - \frac{1}{2} V \sin^2 \phi \hat{J}_+ \hat{J}_-
\]
\[
- V \sin \phi \cos \phi \left[\hat{J}_+ \hat{J}_0 + \hat{J}_0 \hat{J}_+ + \frac{N}{2} \left(\hat{J}_+ + \hat{J}_-\right)\right] + \frac{1}{2} V \left(1 - \frac{1}{2} \sin^2 \phi \right)(\hat{J}_+ \hat{J}_+ + \hat{J}_- \hat{J}_-), \tag{2.39}
\]

where
\[
2E = \epsilon \left(\cos \phi + \chi \sin^2 \phi\right),
\]
\[
\chi = (1 - N)V/\epsilon. \tag{2.40}
\]

We shall define
\[
\tilde{J}_0 = \hat{J}_0 + \frac{N}{2}, \tag{2.41}
\]
and rewrite the Hamiltonian as follows.
\[
\hat{H} = V(\phi) + 2E\tilde{J}_0 + \frac{1}{2} \epsilon \sin \phi \left(1 - \chi \cos \phi\right)(\tilde{J}_+ + \tilde{J}_-)
\]
\[
+ \frac{1}{4} V \sin^2 \phi \left(4\tilde{J}_0^2 - 2\tilde{J}_0\right) - \frac{1}{2} V \sin^2 \phi \tilde{J}_+ \tilde{J}_-
\]
\[
- V \sin \phi \cos \phi [\tilde{J}_+ \tilde{J}_0 + \tilde{J}_0 \tilde{J}_+] + \frac{1}{2} V \left(1 - \frac{1}{2} \sin^2 \phi \right)(\tilde{J}_+ \tilde{J}_+ + \tilde{J}_- \tilde{J}_-). \tag{2.42}
\]
As easily confirmed, the last four terms in Eq. (2.42) are normally-ordered quartic terms of \((\alpha, \alpha^\dagger)\). Here, \(V(\phi)\) corresponds to the collective potential, and is given by

\[
V(\phi) = \langle \phi | \hat{H} | \phi \rangle = -\frac{N}{2} \epsilon (\cos \phi + \frac{1}{2} \chi \sin^2 \phi).
\] (2.43)

The reader should not confuse \(V(\phi)\) with the interaction parameter \(V\).

The Hartree–Fock equations is given by the condition for \(V(\phi)\) to take an extremum,

\[
\frac{\partial V}{\partial \phi} = \frac{N}{2} \epsilon \sin \phi (1 - \chi \cos \phi) = 0,
\] (2.44)

which is equivalent to the condition for the third term in Eq. (2.42) to vanish. Examples of the collective potential for \(\chi \leq 1\) are depicted in Figs. 1 and 2. In all the numerical calculations in this paper, we set the energy splitting between the two levels \(\epsilon = 1\). In other words, we measure energy in units of \(\epsilon\).

2.3. Moving-frame equations with the Lipkin model

We shall consider the basic equations (2.13)-(2.16) employing the Lipkin model. In general cases, the basic equations may be solved by approximating \(\partial_q \hat{Q}^{(1)}(q)\) with the finite difference. With this simple model, the basic equations reduce to one first-order differential equation of \(Q^{(1)}(\phi)\) as shown below. Following Ref. [31], we assume that any collective operator is written in terms of the quasispin operators \(\hat{J}_\pm\) and \(\hat{J}_0\). In Approach A, the collective operators are written in terms of only A-terms, which implies that they are written in terms of \(\hat{J}_\pm\). Noting the Hermiticity and the time-reversal symmetry of the collective operators, one finds

\[
\hat{Q}^{(1)}(q) = Q^{(1)}(q)(\hat{J}_+ + \hat{J}_-),
\] (2.45)

\[
\hat{B}\hat{P}(q) = iBP(q)(\hat{J}_+ + \hat{J}_-),
\] (2.46)

\[
\hat{Q}^{(2)}(q) = iQ^{(2)}(q)(\hat{J}_+ + \hat{J}_-).
\] (2.47)

Here, \(Q^{(1)}(q), BP(q)\) and \(Q^{(2)}(q)\) are real numbers. While \(\hat{Q}^{(1)}\) is time-even, \(\hat{P}\) and \(\hat{Q}^{(2)}\) are time-odd. The canonical-variable condition [4] determines the normalization of
\((Q^{(1)}(q), P(q))\) as follows.

\[
\langle \phi(q)|[\hat{Q}^{(1)}, \hat{P}]|\phi(q)\rangle = i \leftrightarrow Q^{(1)}P = \frac{1}{2N}. \tag{2.48}
\]

As shown in Appendix A, the derivatives of \((\hat{Q}^{(1)}, \hat{P})\) are written as

\[
\partial_q \hat{Q}^{(1)}(q) = \partial_q Q^{(1)}(q)(\dot{J}_+(q) + \dot{J}(q)) + \frac{2}{N}\dot{J}_0(q), \tag{2.49}
\]

\[
\partial_q \hat{P}(q) = i\partial_q P(q)(\dot{J}_+(q) - \dot{J}_0(q)). \tag{2.50}
\]

In the ASCC method, as only the variation of A-type, \(a^\dagger a|\phi(q)\rangle\), is taken, so here we take the variation of the form \(\dot{J}_+|\phi(q)\rangle\).

We shall give the expression of the inertial mass \(B(q)\) first. It is given by Eq. (2.8), and using Eqs. (2.42), (2.45), and (2.46), we have

\[
B(q) = -N\varepsilon \sin \phi (1 - \chi \cos \phi) Q^{(2)} + 2N \left[ \varepsilon \cos \phi + \varepsilon \chi (1 + \sin^2 \phi) \right] \left( Q^{(1)} \right)^2. \tag{2.51}
\]

Although we choose the coordinate system with \(B(q) \equiv 1\), we keep \(B(q)\) explicitly in the expressions below. One can always choose such a coordinate system with \(B = 1\) by the scale transformation of the collective coordinate \(q\). Note that \(Q^{(1)}\) and \(Q^{(2)}\) are rank-1 and rank-2 contravariant tensors, respectively. \(B(q)\) is a rank-2 contravariant tensor, and \(P\) is a rank-1 covariant tensor (vector).

Let us move on to the equations of motion. The moving-frame HF equation (2.13) reads

\[
\frac{1}{2} \sin \phi (1 - \chi \cos \phi) - \partial_q V Q^{(1)} = 0. \tag{2.52}
\]

From Eq. (2.44), we have

\[
\frac{\partial V}{\partial \phi} = \frac{\partial V}{\partial q} \frac{\partial q}{\partial \phi} = \frac{1}{2} N\varepsilon \sin \phi (1 - \chi \cos \phi). \tag{2.53}
\]

By comparing this with Eq. (2.52), we find

\[
\frac{\partial \phi}{\partial q} = \frac{1}{NQ^{(1)}(q)} = 2P(q). \tag{2.54}
\]

The last equality follows from the canonical-variable conditions (2.48).

Let us consider the rest of the basic equations. First, we shall see the case where \(\partial_q V = 0\). Eqs. (2.14) and (2.15) reduce to the RPA equations:

\[
\delta \langle \phi|[\hat{H}, \hat{Q}^{(1)}] - \frac{1}{i}B\hat{P}|\phi\rangle = 0, \tag{2.55}
\]

\[
\delta \langle \phi|[\hat{H}, \frac{1}{i}B\hat{P}] - BC\hat{Q}^{(1)}|\phi\rangle = 0. \tag{2.56}
\]

Using Eqs. (2.42), (2.45), and (2.46), we have

\[
\delta \langle \phi|[\hat{H}, -\frac{1}{i}B\hat{P}] = 0 \leftrightarrow \langle \phi|\hat{J}_- \left( [\hat{H}, \hat{Q}^{(1)}] - \frac{1}{i}B\hat{P} \right) |\phi\rangle = 0 \leftrightarrow \left[ \varepsilon \cos \phi + \varepsilon \chi (1 + \sin^2 \phi) \right] Q^{(1)} - BP = 0. \tag{2.57}
\]

\[
\delta \langle \phi|[\hat{H}, \frac{1}{i}B\hat{P}] - \hat{Q}^{(1)}|\phi\rangle = 0 \leftrightarrow \langle \phi|\hat{J}_- \left( [\hat{H}, \frac{1}{i}B\hat{P}] - \hat{Q}^{(1)} \right) |\phi\rangle = 0, \tag{2.58}
\]

\[
\leftrightarrow \left[ \varepsilon \cos \phi + \varepsilon \chi \sin^2 \phi - \varepsilon \chi \cos^2 \phi \right] BP - BCQ^{(1)} = 0.
\]
The RPA equations can be written as

\[
\begin{pmatrix}
\epsilon \cos \phi + \epsilon \chi \sin^2 \phi + \epsilon \\
-BC \\
- \epsilon \cos \phi + \epsilon \chi (\sin^2 \phi - \cos^2 \phi)
\end{pmatrix}
\begin{pmatrix}
Q^{(1)} \\
BP
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}.
\tag{2.59}
\]

Here, \( BC \) plays a role of the eigenfrequency squared \( \omega^2 \) and is given by

\[
BC = \omega^2 = [\epsilon \cos \phi + \epsilon \chi \sin^2 \phi + \epsilon \chi] [\epsilon \cos \phi + \epsilon \chi (\sin^2 \phi - \cos^2 \phi)].
\tag{2.60}
\]

The RPA solution reads

\[
\begin{pmatrix}
Q^{(1)} \\
BP
\end{pmatrix}
= \sqrt{\frac{B}{2N}} \begin{pmatrix}
[\epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi)]^{-\frac{1}{2}} \\
[\epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi)]^{\frac{1}{2}}
\end{pmatrix}.
\tag{2.61}
\]

We have used the canonical-variable condition (2.48) for the normalization. Here \( \phi \) in the above expression is a solution to the HF equation where \( \partial_q V = 0 \). One easily sees that \( \phi = 0 \) and \( \phi = \pi \) are always solutions to the HF equation (2.44), and that for \( \chi > 1 \) there is another solution \( \phi_0 := \cos^{-1}(1/\chi) \). For \( \chi > 1 \), the RPA solution at \( \phi = \phi_0 \) is given by

\[
\begin{pmatrix}
Q^{(1)} \\
BP
\end{pmatrix}
= \sqrt{\frac{B}{2N}} \begin{pmatrix}
[2\epsilon \chi]^{-\frac{1}{2}} \\
[2\epsilon \chi]^{\frac{1}{2}}
\end{pmatrix}.
\tag{2.62}
\]

The solutions at \( \phi = 0 \) and \( \phi = \pi \) are

\[
\begin{pmatrix}
Q^{(1)} \\
BP
\end{pmatrix}
= \sqrt{\frac{B}{2N}} \begin{pmatrix}
[\epsilon (\chi \pm 1)]^{-\frac{1}{2}} \\
[\epsilon (\chi \pm 1)]^{\frac{1}{2}}
\end{pmatrix},
\tag{2.63}
\]

where + and – correspond to \( \phi = 0 \) and \( \phi = \pi \), respectively.

Next, noting that \([\hat{Q}^{(1)}, \hat{Q}^{(2)}] \) is a B-term, one finds that Eq. (2.16) is independent of \( \partial_q V \). From Eq. (2.16), we obtain

\[
\epsilon \chi \cos \phi \sin \phi \left( Q^{(1)} \right)^2 + BD_q Q^{(1)} - \frac{1}{2} \left[ \epsilon \cos \phi + \epsilon \chi (\sin^2 \phi - \cos^2 \phi) \right] Q^{(2)} = 0,
\tag{2.64}
\]

where \( D_q Q^{(1)} = \partial_q Q^{(1)} + \Gamma(q) Q^{(1)} \). This equation gives the relation between \( \partial_q Q^{(1)} \) and \( Q^{(2)} \). (At this point, the value of \( \partial_q Q^{(1)} \) at the HF equilibrium point is unknown, so is \( Q^{(2)} \).) The partial derivative \( \partial_q Q^{(1)} \) can be rewritten as

\[
\partial_q Q^{(1)} = \frac{\partial Q^{(1)}}{\partial \phi} \frac{\partial \phi}{\partial q} = 2P \frac{\partial Q^{(1)}}{\partial \phi} = \frac{1}{N Q^{(1)}} \frac{\partial Q^{(1)}}{\partial \phi},
\tag{2.65}
\]

using Eq. (2.54), so if we choose the coordinate system with \( B(q) \equiv 1 \) and can express \( Q^{(2)} \) in terms of \( Q^{(1)} \), then Eq. (2.64) reduces to a differential equation of \( Q^{(1)} \) with respect to \( \phi \).

We shall move on to the case where \( \partial_q V \neq 0 \). Then, as long as Eq. (2.16) is adopted as one of the basic equations, the set of Eqs. (2.14) and (2.15) are equivalent to the moving-frame
RPA equations and leads to

\[
\left[ \epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi) \right] Q^{(1)} - BP - \partial_q V Q^{(2)} = 0, \tag{2.66}
\]

\[
\left[ \epsilon \cos \phi + \epsilon \chi (\sin^2 \phi - \cos^2 \phi) \right] BP - BC Q^{(1)}
\]

\[-\frac{1}{2} \left[ \epsilon \cos \phi + \epsilon \chi (\sin^2 \phi - \cos^2 \phi) \right] \partial_q V Q^{(2)} + \epsilon \chi \cos \phi \sin \phi \partial_q V \left( Q^{(1)} \right)^2 = 0. \tag{2.67}
\]

We shall express \(BC = \omega^2\) as a function of \(Q^{(1)}\) or equivalently \(P\) [See Eq. (2.48)]. By eliminating \(\partial_q V Q^{(2)}\) from Eq. (2.67) with use of Eq. (2.66), we find

\[
BC = 3N \left[ \epsilon \cos \phi + \epsilon \chi (\sin^2 \phi - \cos^2 \phi) \right] BP^2
\]

\[-\frac{1}{2} \left[ \epsilon \cos \phi + \epsilon \chi (\sin^2 \phi - \cos^2 \phi) \right] \left[ \epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi) \right]
\]

\[+ \frac{1}{2} \epsilon^2 \chi \cos \phi \sin^2 \phi (1 - \chi \cos \phi) \tag{2.68}
\]

Here we have used the moving-frame HF equation (2.52) and the canonical-variable condition (2.48). Now \(BC\) is written in terms of \(P\) and is readily rewritten in terms of \(Q^{(1)}\). Eq. (2.68) can be also obtained by calculating the product of \(B\) and \(C\) through Eqs. (2.76), (2.70), and (2.12) in the \(\phi\) space.

Let us remove \(Q^{(2)}\) from Eq. (2.64). With the moving-frame RPA equation (2.66)×\(Q^{(1)}\) and the moving-frame HF equation (2.52), we obtain

\[
Q^{(2)} = \frac{2N \left[ \epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi) \right] \left( Q^{(1)} \right)^2 - B}{N \epsilon \sin \phi (1 - \chi \cos \phi)}. \tag{2.69}
\]

Note that this is the indeterminate form of 0/0 when \(\partial_q V = 0\). Eq. (2.69) can be also obtained from the expression of the inertial mass (2.51).

By substituting Eq. (2.69) into Eq. (2.64), and setting \(B = 1\), we obtain the differential equation of \(Q^{(1)}\) below.

\[
\partial_\phi Q^{(1)} = \frac{1}{2} \left[ \cos \phi + \chi (\sin^2 \phi - \cos^2 \phi) \right] \left\{ 2N \left[ \epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi) \right] \left( Q^{(1)} \right)^2 - 1 \right\} \frac{Q^{(1)}}{\sin \phi (1 - \chi \cos \phi)}
\]

\[-N \epsilon \chi \cos \phi \sin \phi \left( Q^{(1)} \right)^3. \tag{2.70}
\]

Now all one has to do is to integrate the above differential equation as an initial-value problem. We choose \(\phi = 0\) as the initial point. From the symmetry of the system, we can assume that \(Q^{(1)}(\phi)\) is symmetric under the reflection about \(\phi = 0\) and differentiable at \(\phi = 0\), and then \(\partial_\phi Q^{(1)}(\phi = 0) = 0\). From the \(O(p^2)\) equation of CS (2.64) and \(\partial_\phi Q^{(1)}(\phi = 0) = 0\), it follows that \(Q^{(2)}(\phi)\) vanishes at \(\phi = 0\).

The relation between \(q\) and \(\phi\) can be also obtained from Eq. (2.54) and we find

\[
q = N \int_{\phi_0}^{\phi} Q^{(1)}(\phi') d\phi'. \tag{2.71}
\]

We have chosen the origin \(q = 0\) such that \(\phi = \phi_0 = \cos^{-1}(1/\chi)\) at \(q = 0\).

Before ending this subsection, we shall give a remark. We have seen that the \(q\)-derivative of the \(O(1)\) equation of CS can be used as one of the basic equations independent of the
moving-frame HF & RPA equations. One may wonder if also the $q$-derivative of the $O(p)$ equation of CS may be used as a basic equation. However, there appears no independent equation, if one differentiates the $O(p)$ equation of CS with respect to $q$. In the case of the Lipkin model, the $q$-derivative of the $O(p)$ equation of CS (2.14) leads to

\[
2\epsilon\chi \cos \phi \sin \phi Q^{(1)} - \frac{1}{N} \epsilon \sin \phi (1 - \chi \cos \phi)
- \left( B \partial_q P + \partial_q V \partial_q Q^{(2)} \right) - CQ^{(2)} + \partial_q B \left( \frac{1}{2B} \partial_q V Q^{(2)} - \partial_q B P \right) = 0,
\]

(2.72)

which can be simplified with $B \equiv 1$ and the canonical-variable condition (2.48) as below.

\[
\left[ \epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi) + \frac{1}{2N(Q^{(1)})^2} \right] \partial_q Q^{(1)}
+ \frac{1}{N} \epsilon \sin \phi (2\cos \phi - 1) - CQ^{(2)} - \partial_q V \partial_q Q^{(2)} = 0.
\]

(2.73)

The above equation (2.73) is also directly obtained by differentiating Eq. (2.66) with respect to $q$ and using Eqs. (2.54) and (2.48), so it is not independent of those equations. Thus, Eq. (2.73) gives no new condition to the set of the $O(1)$ and $O(p)$ equations of CS. Which equations are independent of one another depends on the form of the state vector.

2.4. Collective mass and Schrödinger equation

When we choose the coordinate system such that $B(q) = 1$, the collective Hamiltonian reads

\[
\mathcal{H} = \frac{1}{2} \dot{\phi}^2 + V(\phi).
\]

(2.74)

We shall rewrite the kinetic energy in terms of $\phi$.

\[
T = \frac{1}{2} \dot{\phi}^2 = \frac{1}{2} \left( \frac{\partial \phi}{\partial \phi} \right)^2 \dot{\phi}^2 := \frac{1}{2} M(\phi) \dot{\phi}^2.
\]

(2.75)

Here, the inertial mass $M(\phi)$ can be written as

\[
M(\phi) = \left( \frac{\partial \phi}{\partial \phi} \right)^2 = N^2(Q^{(1)})^2 = \frac{1}{4P^2},
\]

(2.76)

where we have used Eq. (2.54).

We solve the collective Schrödinger equation in the $\phi$ space rather than the $q$ space because it is more convenient for the comparison between the calculations with and without $\hat{Q}^{(2)}$. The classical collective Hamiltonian is written in terms of $\phi$ as

\[
\mathcal{H} = \frac{1}{2} M(\phi) \dot{\phi}^2 + V(\phi).
\]

(2.77)

The quantized Hamiltonian is given by

\[
\hat{H}_{\text{coll}} = \hat{T} + V = -\frac{1}{2} \Delta + V(\phi),
\]

(2.78)

with

\[
\Delta = \frac{1}{\sqrt{M}} \partial_\phi \sqrt{M} \frac{1}{M} \partial_\phi
= \frac{1}{M} \partial_\phi^2 - \frac{1}{2} \frac{1}{M^2} (\partial_\phi M) \partial_\phi = B \partial_\phi^2 + \frac{1}{2} \partial_\phi B \partial_\phi,
\]

(2.79)

with $B(\phi) = M^{-1}(\phi)$. For this Hamiltonian, we solve the collective Schrödinger equation under the boundary condition explained in the next subsection.
2.5. Solution in the case without $\hat{Q}^{(2)}$

In the ASCC theory without $\hat{Q}^{(2)}$, the equations of motion are given by the moving-frame HF equation (2.52) and the moving-frame RPA equations with $Q^{(2)}$ omitted,

$$[\epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi)] Q^{(1)} - BP = 0, \quad (2.80)$$

$$[\epsilon \cos \phi + \epsilon \chi (\sin^2 \phi - \cos^2 \phi)] BP - BCQ^{(1)} + \epsilon \chi \cos \phi \sin \phi \partial_q V \left( Q^{(1)} \right)^2 = 0. \quad (2.81)$$

From Eq. (2.80) and the canonical-variable condition (2.48), we obtain

$$\left( \frac{Q^{(1)}}{BP} \right) = \sqrt{\frac{B}{2N}} \left[ \frac{[\epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi)]^{-\frac{1}{2}}}{[\epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi)]^{\frac{1}{2}}} \right]. \quad (2.82)$$

We shall call the moving-frame RPA equations without the curvature term [the last term in Eq. (2.81)] the local RPA (LRPA) equations. As $(Q^{(1)}, BP)$ are determined from Eq. (2.80) and the canonical-variable condition (2.48), the solution $(Q^{(1)}, BP)$ of the moving-frame RPA equations coincides with that to the LRPA solution. Then, the inverse collective mass is given by

$$B(\phi) = M^{-1}(\phi) = \frac{2}{N} \epsilon \left[ \cos \phi + \chi (1 + \sin^2 \phi) \right], \quad (2.83)$$

which coincides with the ATDHF mass (41) and the GCM mass (72) in Ref. [31]. When $\chi < 1$, $B$ is positive for sufficiently small $\phi \in [0, \pi]$, but $B$ becomes 0 at a certain point, beyond which $(\hat{Q}^{(1)}, \hat{P})$ is no longer Hermitian and $B < 0$. We shall denote the point at which $B = 0$ by $\phi_{\text{max}} \in (0, \pi)$. We solve the collective Schrödinger equation with the boundary condition for the collective wave functions to vanish outside the region $(-\phi_{\text{max}}, \phi_{\text{max}})$, because the potential energy is sufficiently high there. When $\chi > 1$, $B$ is always positive. Then, we solve the collective Schrödinger equation with the periodic boundary condition. We employ these boundary conditions similarly in the case with $Q^{(2)}$ included.

The eigenfrequency $\omega^2 = BC$ of the moving-frame RPA equations without $\hat{Q}^{(2)}$ is obtained from Eq. (2.81) as

$$BC = [\epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi)] \left[ \epsilon \cos \phi + \epsilon \chi (\sin^2 \phi - \cos^2 \phi) \right]$$

$$+ \frac{1}{2} \epsilon^2 \chi \sin^2 \phi \cos \phi (1 - \chi \cos \phi), \quad (2.84)$$

and $\partial_q V$ is also obtained analytically

$$\partial_q V = \sqrt{\frac{N}{2B}} \epsilon \sin \phi (1 - \chi \cos \phi) \left[ \epsilon \cos \phi + \epsilon \chi (1 + \sin^2 \phi) \right]^{\frac{1}{2}}. \quad (2.85)$$

3. Numerical results

We compare the numerical results obtained by solving Eq. (2.70) with those obtained from the conventional ASCC equations without $\hat{Q}^{(2)}$. For the symmetry of the system, we solve Eq. (2.70) in the region $[0, 180^\circ]$ as an initial-value problem, starting from $\phi = 0$.

As an example of the calculations for $\chi > 1$, we show the calculated results for $\chi = 1.8, N = 10, \epsilon = 1$ in Fig. 3. We depict (a) $Q^{(1)}$, (b) $Q^{(2)}$, (c) $\partial_q V$, (d) $\partial_q VQ^{(2)}$, (e) $\omega^2$ and (f) $C$ as functions of $\phi$. When $Q^{(2)}$ is not included, $Q^{(1)}$, shown in Fig. 3(a), is obtained analytically
Fig. 3: Numerical results obtained by solving Eq. (2.70) for $N = 10$, $\chi = 1.8$, $\epsilon = 1.0$ plotted as function of $\phi$: (a) $Q^{(1)}$, (b) $Q^{(2)}$, (c) $\partial_q V$, (d) $\partial_q V Q^{(2)}$, (e) $\omega^2$, and (f) $C(\phi)$. For (a) $Q^{(1)}$, (c) $\partial_q V$, (e) $\omega^2$ and (f) $C(\phi)$, the results obtained by solving the ASCC equations without $Q^{(2)}$ are shown with the solid line for comparison. The dashed line in the panel (a) indicates the value of the RPA solution (2.61) at $\phi = \phi_0$. (See also Fig. 4.)

Fig. 4: Same as Fig. 3(a), but magnified around $\phi = \phi_0$. The dashed line indicates the value of the RPA solution (2.61) at $\phi = \phi_0$. At $\phi = \phi_0$, $\hat{Q}^{(1)}$ coincides the RPA solution.
as seen in the previous section. It coincides with the RPA solutions at the potential extrema \( \phi = 0, \phi_0 \) and \( \pi \). One can see that, also with \( \hat{Q}^{(2)} \), the calculated result coincides with the RPA solutions at the potential extrema. (In Fig. 4, the magnified figure around \( \phi = \phi_0 = 56.3^\circ \) is shown). It is also noteworthy that, for \( \phi \lesssim \phi_0 \), both of the calculations give similar results, but for \( \phi \gtrsim \phi_0 \), the deviation between the two becomes larger. [We shall give a minor remark. At \( \phi = 180^\circ \), the differentiability of \( Q^{(1)} \) seems to be broken if we assume \( B(\phi) \) is periodic. However, this is not a serious problem because the potential energy is sufficiently high and the collective wave function almost vanishes there. We also integrated Eq. (2.70) from \( \phi = 0^\circ \) to \( \phi = 180^\circ \) with the boundary condition \( \partial_\phi Q^{(1)} = 0 \) as in the case from \( \phi = 0 \). The obtained solution coincides with the RPA solutions at the potential extrema, but in turn the differentiability at \( \phi = 0 \) is broken. Therefore, we adopt here the solution obtained by the integration from \( \phi = 0 \).]

In Fig. 3(b), \( Q^{(2)} \) is plotted. It vanishes at \( \phi = 0 \) and rapidly decreases as \( \phi \) approaches to \( 180^\circ \). However, it does not diverge there but converges to a finite value. \( Q^{(2)} \) is involved in the equations of motion in the form of \( \partial_q V Q^{(2)} \), so we plot \( \partial_q V Q^{(2)} \) in Fig. 3(d). As \( \partial_q V = 0 \) at the potential extrema \( \phi = 0^\circ, \phi_0 \), and \( 180^\circ \) and \( Q^{(2)} \) is always finite, their product vanishes at \( \phi = 0, \phi_0 \), and \( 180^\circ \). Thus, the equations of motion reduce to the HF & RPA equations there. [Note that, \( \phi = 0, 180^\circ \) are (local) potential maxima, although the RPA equations are usually solved at the potential minimum.]

Figs. 3(e) and 3(f) display the eigenfrequency squared \( \omega^2 \) and the potential curvature \( C \) as functions of \( \phi \), respectively. In the calculation with \( \hat{Q}^{(2)} \) included, \( \omega^2 \), which is calculated through Eq. (2.68), coincides with the product of the curvature \( C \) and the inverse inertial mass \( B \), i.e., \( \omega^2 = BC \). However, it is not the case when \( \hat{Q}^{(2)} \) is ignored. (It is seen that the position of the zero of \( \omega^2 \) around \( 120^\circ \) is different from that of \( C \).) This will be investigated in the next section.

The inverse inertial mass \( B(\phi) \) calculated with \( Q^{(2)} \) is shown in Fig. 5, in comparison with that calculated without \( Q^{(2)} \). In Fig. 6, the ratio of the inverse inertial mass calculated without \( \hat{Q}^{(2)} \) to that with \( \hat{Q}^{(2)} \),

\[
\frac{B(\phi) \text{ w/o } \hat{Q}^{(2)}}{B(\phi) \text{ w/ } \hat{Q}^{(2)}} = \frac{M(\phi) \text{ w/ } \hat{Q}^{(2)}}{M(\phi) \text{ w/o } \hat{Q}^{(2)}},
\]

is plotted. Reflecting the difference in \( Q^{(1)} \), while for \( \phi \lesssim \phi_0 \), the difference between the two is not so large, it becomes more significant for \( \phi \gtrsim \phi_0 \). There, the inverse inertial mass with \( \hat{Q}^{(2)} \) is larger than that without \( \hat{Q}^{(2)} \), and depending on \( \phi \) or equivalently on the collective coordinate \( q \), the difference can be by a factor of 3 approximately. The difference of the inertial masses becomes more important as the excitation energy increases, because the component of the collective wave function increases in the region with \( \phi \gtrsim \phi_0 \), where the potential energy is high and the mass difference becomes larger.

Table 1 displays the comparison of the excitation energies calculated including \( \hat{Q}^{(2)} \) with those calculated without \( \hat{Q}^{(2)} \) and the exact solution for \( N = 10, \chi = 1.8 \). For a first few excited states, the difference between the two calculated results is small (by several percents), and both of the calculations are in good agreement with the exact solution, although the excitation energy of the first excited state is somewhat overestimated. With increasing the excitation energy, the difference between the two calculations becomes larger and amounts to \( 10 - 20\% \). Without \( \hat{Q}^{(2)} \), the deviation from the exact solution become larger with increasing
Fig. 5: Inverse inertial mass $B(\phi)$ obtained by solving (2.70) for $N = 10, \chi = 1.8, \epsilon = 1.0$. The inverse inertial mass obtained without $\hat{Q}^{(2)}$ is also shown for comparison.

Fig. 6: Ratio of the inverse inertial mass $B(\phi)$ calculated without $Q^{(2)}$ to that with $Q^{(2)}$ for $N = 10, \chi = 1.8, \epsilon = 1.0$.

the excitation energy, while, with $\hat{Q}^{(2)}$, the deviation stays relatively small. The fourth and fifth columns in Table 1 show the deviations of the calculated excitation energies from the exact solution. Although the calculation without $\hat{Q}^{(2)}$ gives a better agreement with the exact solution for the first excited state, for all the other states, the calculation with $\hat{Q}^{(2)}$ included gives a better agreement.

Fig. 7 shows the excitation energies of the first three excited states as functions of $\chi$ with $N = 10$. Although the calculation with $\hat{Q}^{(2)}$ does not always give a better agreement with the exact solution than that without $\hat{Q}^{(2)}$, as a whole, the agreement with the exact solution is rather good for both of the calculations with and without $\hat{Q}^{(2)}$.

Fig. 8 displays the results for the next three, the fourth, fifth and sixth excited states. Without $\hat{Q}^{(2)}$, as excitation energy increases, the calculated excitation energies start to deviate from the exact solution, and the excitation energies are systematically underestimated. On the other hand, the excitation energies calculated with $\hat{Q}^{(2)}$ are still in good agreement with the exact solution. These results suggest that the role of the inertial mass becomes more important as the excitation energy increases.
Table 1: Excitation energies calculated with $Q^{(2)}$ for $\epsilon = 1, \chi = 1.8, N = 10$ in comparison with those calculated without $Q^{(2)}$ and the exact solution. The first, second, and third columns show the excitation energies of the exact solution, the calculation without $\hat{Q}^{(2)}$, and with $\hat{Q}^{(2)}$, respectively. The fourth and fifth columns show the deviation from the exact solution for the excitation energies calculated without and with $Q^{(2)}$, respectively. All the energies are in units of $\epsilon(= 1)$. The rightmost column shows the percentage of the excitation-energy increase by $\hat{Q}^{(2)}$: \[
\frac{E(w/ Q^{(2)}) - E(w/o Q^{(2)})}{E(w/o Q^{(2)})} \times 100.
\]

We performed the calculation with a larger particle number $N = 40$ similarly, and the results are shown in Figs. 9 and 10. As seen in the previous section, without $\hat{Q}^{(2)}$, $B(\phi)$ is obtained analytically and is proportional to $N^{-1}$, whereas the collective potential energy is proportional to $N$. If the $N$ dependence of the inertial mass in the case with $\hat{Q}^{(2)}$ is similar to that without $\hat{Q}^{(2)}$, with increasing $N$, the potential energy becomes larger relative to the kinetic energy, and the difference of the inertial mass between the two calculations becomes less important. Actually, for $N = 40$, the two calculations give similar results and both are in good agreement with the exact solution.

4. Basic equations in the case without higher-order operators

In the previous section, we determined $\hat{Q}^{(2)}$ adopting the $q$-derivative of the $O(1)$ equation of CS as well as the moving-frame HF & RPA equations. In other words, we employed the $O(1), O(p)$ and $O(p^2)$ equations of CS and the $q$-derivative of the $O(1)$ equation of CS as independent basic equations. It was shown that, with these equations, one can determine $\hat{Q}^{(2)}$, with which the agreement with the exact solution was improved in the Lipkin model.

One question may arise here. In the conventional ASCC method with only the first-order operator $Q^{(1)}$ included, the moving-frame RPA equation of $O(p^2)$ is adopted as one of the basic equations, which is derived from the $O(p^2)$ equation of CS and $q$-derivative of the $O(1)$ equation of CS (the moving-frame HF equation). What if one adopts the $O(p^2)$ equation of CS or the $q$-derivative of the $O(1)$ equation of CS instead of the moving-frame RPA equation of $O(p^2)$? In the ASCC method without the higher-order operators, three equations of motion are necessary, two of which are the moving-frame HF equation and moving-frame RPA equation of $O(p)$. To elucidate the role of $O(p^2)$ terms in the equation of CS, below we
investigate three cases where, as the last one of the equations of motion, we adopt (i) the $O(p^2)$ equation of CS, (ii) the conventional moving-frame RPA equation of $O(p^2)$, or (iii) the $q$-derivative of the $O(1)$ equation of CS.

4.1. The $O(p^2)$ equation of collective submanifold

When $\hat{Q}^{(2)}$ and $\hat{Q}^{(3)}$ are neglected, the $O(p^2)$ equation of CS (2.16) reads

$$\delta \langle \phi(q) \rangle \frac{1}{2} [[\hat{H}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - B \partial_q \hat{Q}^{(1)} + \frac{1}{2} \partial^2_q B \hat{Q}^{(1)} | \phi(q) \rangle = 0. \quad (4.1)$$

In the case of the Lipkin model, using Eqs. (2.42), (2.45), and (2.49), we obtain

$$-\epsilon \chi \cos \phi \sin \phi \left( Q^{(1)} \right)^2 - B \partial_q Q^{(1)} + \frac{1}{2} \partial^2_q B Q^{(1)} = 0. \quad (4.2)$$
Let us move to the coordinate system with $B \equiv 1$. Using Eq. (2.65) and multiplying both sides by $Q^{(1)}$, we find

$$
\epsilon \chi \cos \phi \sin \phi \left( Q^{(1)} \right)^3 + \frac{1}{N} \partial_\phi Q^{(1)} = 0 \iff - \frac{1}{(Q^{(1)})^3} \partial_\phi Q^{(1)} = N \epsilon \chi \cos \phi \sin \phi. \tag{4.3}
$$

Here we have used the canonical-variable condition (2.48). The above equation is easily integrated with the initial condition (RPA solution) at $\phi = 0$ shown in Eq. (2.63) as

$$
Q^{(1)} = \frac{1}{\sqrt{2N}} \left[ \epsilon + \epsilon \chi \left( 1 - \frac{1}{2} \sin^2 \phi \right) \right]^{-\frac{1}{2}}. \tag{4.4}
$$

Obviously, this is inconsistent with $Q^{(1)}$ which is determined from the $O(p)$ equation of CS (2.82). This inconsistency is thought to be caused by the neglect of $\hat{Q}^{(2)}$ and $\hat{Q}^{(3)}$. Thus, one cannot adopt the set of the $O(p)$ and $O(p^2)$ equations of CS in this case.
4.2. The moving-frame RPA equation of $O(p^2)$

Next, we shall adopt the moving-frame RPA equation of $O(p^2)$. We have already investigated this case in Sect. 2.5. In the case of the two-level Lipkin model, $(Q^{(1)}, P)$ are determined from the moving-frame RPA equation of $O(p)$ and the canonical-variable condition only, as shown in Eq. (2.82). Therefore, there is no difference in the solution $(Q^{(1)}, P)$ between the two cases where the moving-frame RPA equation of $O(p^2)$ is adopted and where the $q$-derivative of the $O(1)$ equation of CS is adopted. In both cases, $BC$ plays a role of the squared eigenfrequency of the eigenvalue equations, and it is in the eigenfrequency that there appears a difference between the two cases. We have already seen the eigenfrequency squared for the moving-frame RPA equations without the higher-order operators in Sect. 2.5, and it is given by Eq. (2.84). If the set of the basic equations are self-consistent, the eigenfrequency squared $\omega^2$ should coincide with the product of the inverse inertial mass $B$ and the potential curvature $C$.

We shall calculate the product of the inverse inertial mass $B$ and the potential curvature $C$ in the conventional moving-frame HF & RPA equations without the higher-order operators. The inverse inertial mass $B$ has been already obtained in Eq. (2.83). The Christoffel symbol of the second kind is given by

$$\Gamma = -\frac{1}{2B} \frac{dB}{d\phi} = -\frac{1}{2P^2} \frac{d}{d\phi} P^2 = \frac{1}{2M} \frac{dM}{d\phi} = \frac{1}{2(Q^{(1)})^2} \frac{d}{d\phi} (Q^{(1)})^2 = \frac{1}{Q^{(1)}} \frac{dQ^{(1)}}{d\phi},$$

from which we have

$$\Gamma = -\frac{1}{2(2E + \epsilon \chi)} \frac{d}{d\phi} (2E + \epsilon \chi) = \frac{1}{2(2E + \epsilon \chi)} (\epsilon \sin \phi - 2\epsilon \chi \sin \phi \cos \phi).$$

Then, $C$ is calculated as

$$C = \frac{d^2V}{d\phi^2} - \Gamma \frac{dV}{d\phi} = \frac{1}{2} N \left(2E - \epsilon \chi \cos^2 \phi \right) - \frac{N}{4(2E + \epsilon \chi)} \epsilon^2 \sin^2 \phi \left(1 - 2\chi \cos \phi \right) \left(1 - \chi \cos \phi \right).$$

Thus, we obtain

$$BC = (2E - \epsilon \chi \cos^2 \phi) (2E + \epsilon \chi) + \frac{1}{2} \epsilon^2 \sin^2 \phi \left(1 - \chi \cos \phi \right) \left(2 \chi \cos \phi - 1\right).$$

The first term coincides with the squared eigenfrequency of the LRPA equations $\omega_{LRPA}$, in which the curvature term [the third term in Eq. (2.11)] is omitted. The second term is the contribution from the connection term, and the curvature term in the moving-frame RPA equations should give this contribution. As $BC$ is a scalar, the same result is obtained with the calculation in the $q$ space (See Appendix B).

Clearly, the squared eigenfrequency of the moving-frame RPA equation without $\hat{Q}^{(2)}$ (2.84) and the product of the potential curvature and inertial mass parameter $BC$ (4.8) are different by

$$-\frac{1}{2} \epsilon^2 \sin^2 \phi (1 - \chi \cos \phi)^2 = -2 \left(\partial_q V Q^{(1)} \right)^2 = -\frac{2}{N^2} \left(\frac{dV}{d\phi} \right)^2,$$

and in this sense the self-consistency is broken. This can be attributed to the fact that the higher-order operators are neglected in the conventional moving-frame RPA equations.
Fig. 11 shows the comparison of $BC$ with the squared eigenfrequency of the moving-frame RPA equations $\omega^2$ and that of the LRPA equations $\omega^2_{LRPA}$ without the higher-order operators. One can see that the squared eigenfrequencies almost coincide with $BC$ for $\phi \lesssim \phi_0$, but their deviations from $BC$ become large for $\phi \gtrsim \phi_0$.

4.3. The $q$-derivative of the $O(1)$ equation of collective submanifold

Last, we consider the case where the $q$-derivative of the $O(1)$ equation of CS (2.15) is adopted. We shall obtain the eigenfrequency squared $BC = \omega^2$ in the eigenvalue problem given by the $O(p)$ equation of CS and the $q$-derivative of the $O(1)$ equation of CS, and compare with the product of $B$ and $C$ we obtained in the previous subsection.

By substituting Eqs. (2.42), (2.45), (2.46), and (2.49) into Eq. (2.15), we obtain

$$[\varepsilon \cos \phi + \varepsilon \chi (\sin^2 \phi - \cos^2 \phi)] BP - BCQ^{(1)} + \frac{1}{2} \partial_q B \partial_q VQ^{(1)} - B \partial_q VQ^{(1)} = 0. \quad (4.10)$$

The third term vanishes when we employ the coordinate system with $B \equiv 1$. By rewriting $\partial_q V$ and $\partial_q Q^{(1)}$ with use of Eqs. (2.52) and (2.65), respectively, $BC$ can be rewritten in terms of $(Q^{(1)}, P)$. Then, with Eq. (2.82), we readily obtain

$$BC = [\varepsilon \cos \phi + \varepsilon \chi (\sin^2 \phi - \cos^2 \phi)] [\varepsilon \cos \phi + \varepsilon \chi (1 + \sin^2 \phi)]$$

$$+ \frac{1}{2} \varepsilon^2 \sin^2 \phi (1 - \chi \cos \phi)(2 \chi \cos \phi - 1), \quad (4.11)$$

Unlike the conventional moving-frame RPA equations without $\hat{Q}^{(2)}$, the eigenfrequency squared (4.11) coincides with $BC$ (4.8) obtained from the potential curvature and the inverse inertial mass, so they are self-consistent in this case. While the higher-order operators are involved in the moving-frame RPA equation of $O(p^2)$, which were neglected in the previous subsection, the $q$-derivative of the $O(1)$ equation of CS does not contain the higher-order operators, and no approximation is made. In this sense, the $q$-derivative of the $O(1)$ equation of CS is better than the moving-frame RPA equation of $O(p^2)$ with the higher-order operators ignored. It is noteworthy that $\hat{Q}^{(2)}$ is omitted in the $O(p)$ equation of CS in both of the two cases.
5. Discussion

So far we have considered a method of determining $\hat{Q}^{(2)}$ in the case without the pairing correlation. As shown in Refs. [2, 3], the higher-order operators have much to do with the gauge symmetry. In this section, we briefly discuss the gauge symmetry of the basic equations when the pairing correlation is included.

First we shall reconsider the gauge symmetry that Hinohara et al found [11]. As mentioned in Introduction, Hinohara et al encountered the numerical instability caused by the gauge symmetry of the basic equations, and needed to introduce a gauge-fixing prescription for successful calculation. This is a "numerical" problem in the sense explained below. The moving-frame HFB & QRPA equations including up to the first-order operator $\hat{Q}^{(1)}$ are given by,

$$\delta \langle \phi(q) | \hat{H}_M | \phi(q) \rangle = 0,$$  \hspace{1cm} (5.1)

$$\delta \langle \phi(q) | [\hat{H}_M, \hat{Q}^{(1)}] - \frac{1}{i} B(q) \hat{P} | \phi(q) \rangle = 0,$$  \hspace{1cm} (5.2)

$$\delta \langle \phi(q) | [\hat{H}_M, \frac{1}{i} B(q) \hat{P}] - B(q) C(q) \hat{Q}^{(1)} - \partial_q \lambda \hat{N} - \frac{1}{2} \partial_q V[[\hat{H}_M, \hat{Q}^{(1)}], \hat{Q}^{(1)}] | \phi(q) \rangle = 0.$$  \hspace{1cm} (5.3)

with $\hat{H}_M = \hat{H} - \lambda \hat{N} - \partial_q V \hat{Q}^{(1)}$. (Assume that $\hat{Q}^{(1)}$ contains only A-terms.) When $\partial_q V = 0$, these equations are invariant under the transformation (1.2)-(1.5). When $\partial_q V \neq 0$, the basic equations are not invariant under this transformation. Nevertheless, there occurred the numerical instability. This may be understood as follows.

As shown in Ref. [2], for $\partial_q V \neq 0$, the moving-frame HFB equation (5.1) is invariant under this gauge transformation, but the moving-frame QRPA equations (5.2) and (5.3) are not. Under the transformation, $[\hat{H}_M, \hat{Q}^{(1)}]$ transforms as below,

$$[\hat{H}_M, \hat{Q}^{(1)}] \rightarrow [\hat{H}_M, \hat{Q}^{(1)}] - \alpha \partial_q V[\hat{N}, \hat{Q}^{(1)}],$$  \hspace{1cm} (5.4)

so the $O(p)$ moving-frame QRPA equation (5.2) is not gauge invariant. The second term of the right-hand side comes from $\partial_q V \hat{Q}^{(1)}$ in $\hat{H}_M$. As for the moving-frame QRPA equation of $O(p^2)$ (5.3), the curvature term

$$-\frac{1}{2} \partial_q V[[\hat{H}_M, \hat{Q}^{(1)}], \hat{Q}^{(1)}]$$  \hspace{1cm} (5.5)

breaks the gauge symmetry as easily confirmed. Hinohara et al started the calculation from the HFB equilibrium point ($q = 0$) where $\partial_q V(q = 0) = 0$, and at the next step $q = \pm \delta q$, $\partial_q V(\delta q)$ is still small. Although there are gauge-symmetry-breaking terms in the moving-frame QRPA equations as seen above, they are proportional to $\partial_q V$, and their contributions are small in the vicinity of the HFB equilibrium point $\partial_q V(q) = 0$. Thus, the gauge symmetry is approximately retained, which leads to the numerical instability. If the gauge-symmetry-breaking terms gave sufficiently large contributions, the numerical instability would not occur. Then, once the gauge is fixed at the HFB equilibrium point, the gauge-fixing prescription would not be necessary to solve the moving-frame equations at non-equilibrium points. The term $[[H_M, \hat{Q}^{(1)}], \hat{Q}^{(1)}]$ originates from the $O(p^2)$ equation of CS [See Eq. (2.21)] and breaks the gauge symmetry (in other words, fixes the gauge). It is multiplied by $\partial_q V$ to remove $D_q \hat{Q}^{(1)}$ from the $q$-derivative of the $O(1)$ equation of CS in the derivation of
the moving-frame QRPA equation of $O(p^2)$, which leads to the small contribution in the moving-frame QRPA equation.

Next, we investigate the gauge symmetry of the basic equations we have proposed in this paper. First, let us assume the case where only the operators up to the first order are taken, and Eq. (2.20) is adopted instead of the moving-frame QRPA equation of $O(p^2)$. Concerning the gauge symmetry of Eq. (2.20), the part

$$[\hat{H}_M, \frac{1}{i} B(q) \hat{P}] - B(q) C(q) \hat{Q}^{(1)}(0) - \hat{B} \partial_q \lambda \hat{N}$$  \hspace{1cm} (5.6)

is gauge invariant, then what we have to check is the gauge symmetry of $\partial_q V D_q \hat{Q}^{(1)}$. It is sufficient to investigate the gauge symmetry in the vicinity of the HFB equilibrium point $\partial_q V = 0$. We choose the coordinate system with $B(q) = 1$, and then $D_q \hat{Q}^{(1)} = \partial_q \hat{Q}^{(1)}$. In the previous section, due to the simplicity of the Lipkin model, the basic equations reduced to one differential equation. In general cases, however, $\partial_q \hat{Q}^{(1)}$ should be approximated by finite difference. With the simplest scheme,

$$B \partial_q V(\delta q) \partial_q \hat{Q}(\delta q) = B \partial_q V(\delta q) \frac{1}{\delta q} \left[ \hat{Q}^{(1)}(\delta q) - \hat{Q}^{(1)}(0) \right]$$

$$= B \left[ \partial_q V(0) + \partial_q^2 V(0) \delta q \right] \frac{1}{\delta q} \left[ \hat{Q}^{(1)}(\delta q) - \hat{Q}^{(1)}(0) \right]$$

$$= BC(0) \hat{Q}^{(1)}(\delta q) - BC(0) \hat{Q}^{(1)}(0)$$

$$\rightarrow BC(0) \hat{Q}^{(1)}(\delta q) - BC(0) \hat{Q}^{(1)}(0) + \alpha BC(0) \hat{N}.$$  \hspace{1cm} (5.7)

It is not gauge invariant except for the zero modes.

Then, we investigate the gauge symmetry in the case where we adopt Eq. (2.18)-(2.21) as the basic equations. As shown in Ref. [2], Eqs. (2.18)-(2.19) are gauge invariant. Eq. (2.20) does not contain $\hat{Q}^{(2)}$, so the consideration above holds as it is. Unless $C(0) = 0$, the gauge symmetry is significantly broken even in the vicinity of HFB equilibrium point $\partial_q V = 0$, and the gauge is fixed. The term $[\hat{Q}^{(1)}, \hat{Q}^{(2)}]$ in Eq. (2.21) is a B-term and does not contribute, so Eq. (2.21) with this term omitted,

$$\delta \langle \phi(q) \rangle \frac{1}{2} \left[ [\hat{H}_M, \hat{Q}^{(1)}], \hat{Q}^{(1)} \right] - \partial_q \hat{Q}^{(1)} - \frac{i}{2} [\hat{H} - \lambda \hat{N}, \hat{Q}^{(2)}] \rangle \langle \phi(q) \rangle = 0,$$  \hspace{1cm} (5.8)

would be solved in the actual calculation. It is worth mentioning that the moving-frame QRPA equation of $O(p^2)$ and the $O(p^3)$ equation of CS with the $\hat{Q}^{(3)}$ term, which is ignored here, are gauge-invariant if the $[\hat{Q}^{(1)}, \hat{Q}^{(2)}]$ term is included (See Ref. [2]).

Under the gauge transformation, the first and third terms ($\times 2$) transform as

$$[[H_M, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i [\hat{H} - \lambda \hat{N}, \hat{Q}^{(2)}]$$

$$\rightarrow [[H_M, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i [\hat{H} - \lambda \hat{N}, \hat{Q}^{(2)}] + \alpha \partial_q V \left( [[\hat{N}, \hat{Q}^{(1)}], \hat{Q}^{(1)}] - i [\hat{N}, \hat{Q}^{(2)}] \right) + O(\alpha^2 \partial_q V),$$  \hspace{1cm} (5.9)

so the gauge-symmetry-breaking term is proportional to $\partial_q V$. The rest transforms as

$$\partial_q \hat{Q}^{(1)}(\delta q) = \left[ \hat{Q}^{(1)}(\delta q) - \hat{Q}^{(1)}(0) \right] / \delta q \rightarrow \partial_q \hat{Q}^{(1)}(\delta q) + \frac{\alpha}{\delta q} \hat{N}.$$  \hspace{1cm} (5.10)

This term breaks the gauge symmetry, and the gauge can be regarded as fixed.
6. Concluding remarks

In this paper, we have considered the ASCC theory including the second-order collective-coordinate operator \( \hat{Q}^{(2)} \) which consists of only A-terms, proposed a new set of basic equations to determine the collective operators including \( \hat{Q}^{(2)} \), and applied it to the two-level Lipkin model. We have compared the ASCC calculations with and without \( \hat{Q}^{(2)} \) and found that, for a first few low-energy states, the difference between the results of the two calculations is not so significant and that both of the calculations reproduce the exact solution well. However, with increasing the excitation energy, the deviation from the exact solution becomes larger in the case without \( \hat{Q}^{(2)} \), while, with \( \hat{Q}^{(2)} \), the agreement with the exact solution is good even for the higher excited states. As discussed in Refs. [2, 3] and this paper, \( \hat{Q}^{(2)} \) does contribute to the inertial mass. As the excitation energy increases, the kinetic energy becomes important relatively to the collective potential energy. The results shown in this paper illustrates the importance of the correct evaluation of the inertial mass.

We have also reconsidered the basic equations to adopt in the case where no higher-order operator is included. It has been shown that, in the conventional moving-frame RPA equations, the self-consistency is broken in the sense that the eigenfrequency squared \( \omega^2 \) does not coincide with the product of the potential curvature and the inverse inertial mass \( BC \). In contrast, when we employ the \( q \)-derivative of the \( O(1) \) equation of CS, in which no approximation is made for the higher-order operators, the relation \( \omega^2 = BC \) holds, and the self-consistency is kept. In the case of the two-level Lipkin model we have used, \( (\hat{Q}^{(1)}, \hat{P}) \) are determined from the \( O(p) \) equation of CS and the canonical-variable condition, and the difference between the moving-frame RPA equation of \( O(p^2) \) and the \( q \)-derivative of the \( O(1) \) equation of CS appears only in \( \omega^2 \). It would be interesting to investigate how \( (\hat{Q}^{(1)}, \hat{P}) \) are affected depending on which of the two equations to adopt, using the three-level Lipkin or more realistic models. The observation above may lead to an intuitive understanding as follows. The \( O(1) \) equation of CS is an equation for the state vector \( |\phi(q)\rangle \) and the Lagrange multiplier, once \( \hat{Q}^{(1)} \) is given. The \( O(p) \) equation of CS and the \( q \)-derivative of the \( O(1) \) equation of CS can be viewed as equations to determine \( \hat{Q}^{(1)} \) and \( \hat{P} \), respectively, while \( \hat{Q}^{(2)} \) is determined from the \( O(p^2) \) equation of CS. Note that all the basic equations should be solved self-consistently.

Although we have mainly studied the ASCC theory without the pairing correlation in this paper, the basic equations with the pairing correlation are also derived in a straightforward way, and we have briefly discussed the gauge symmetry of the basic equations. The gauge transformation changes the gauge angle and the chemical potential as well as the collective operators, and it plays an important role in the treatment of superfluid systems.

As shown in Refs. [1, 2], the equation of CS before the adiabatic expansion is gauge invariant, but the gauge symmetry is (partially) broken by the adiabatic expansion. In Refs. [1, 2], we analyzed the gauge symmetry under the general gauge transformation, and found that four examples or types of the gauge transformations play an essential role in the analysis. One of the four is the gauge transformation Eqs. (1.2) - (1.5) (We call it Example 1 in Refs. [1, 2]). The gauge symmetry of Example 1 can be retained by including the higher-order collective-coordinate operators. However, the symmetry under the gauge transformation of Example 3 in Ref. [1, 2], which mixes \( \hat{P} \) with \( \tilde{N} \), cannot be retained even if the higher-order operators are introduced. This can be regarded as a gauge fixing by the adiabatic expansion.
In Hinohara’s prescription, they require only the gauge symmetry under the transformation which mixes $\hat{Q}^{(1)}$ with $\tilde{N}$ (Example 1), and the gauge symmetry under the transformation which mixes $\hat{P}$ with $\tilde{N}$ (Example 3) is left broken. In this sense, Hinohara’s prescription attaches more weight to Example 1 than Example 3. If the two transformations are of equal weight, the basic equations breaking the symmetry under the gauge transformation of Example 1 can be adopted. Then one may regarded it as a gauge fixing as in Example 3. In general, if there is gauge symmetry, one can choose a convenient gauge, so a set of the basic equations which are not gauge invariant may be possible.

As mentioned in Introduction, the extension of the ATDHF to the ATDHFB theory is not straightforward, because one has to decouple the number-fluctuating mode from the collective mode of interest. For the collision of two nuclei, the chemical potentials of the two nuclei are different unless it is a collision between the same nuclides. To describe such a phenomenon, it is necessary to construct a theory with which one can treat the gauge degrees of freedom correctly. In Ref. [23], Wen and Nakatsukasa could not find a collective path connecting the superdeformed state and the ground state in $^{32}$S, and pointed out a possible improvement of the problem by including the pairing correlation. For that purpose, a theory is required which can treat both cases with and without the pairing correlation on an equal footing. The basic equations we have proposed in this paper has such an advantage and treat the higher-order collective operator in both of the cases on an equal footing. It would be very interesting to apply the set of the basic equations proposed here to systems with the pairing correlation, and it will be reported in a future publication.

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A. Derivatives of the collective operators
We shall obtain the derivative of the operator $\hat{O}(q)$ which is written in terms of the quasispin operators by evaluating

$$\frac{\partial_q \hat{O}(q)}{} = \lim_{\delta q \to 0} \frac{\hat{O}(q) - \hat{O}(q - \delta q)}{\delta q}. \quad (A1)$$

Here we consider the following three types of operators corresponding to $\hat{Q}^{(1)}$, $\hat{P}$ and $\hat{Q}_B$.

$$\hat{E}_A(q - \delta q) = E_A(q - \delta q)(\hat{J}_+(q - \delta q) + \hat{J}_-(q - \delta q)), \quad (A2)$$

$$\hat{O}_A(q - \delta q) = iO_A(q - \delta q)(\hat{J}_+(q - \delta q) - \hat{J}_-(q - \delta q)), \quad (A3)$$

$$\hat{E}_B(q - \delta q) = E_B\tilde{J}_0(q - \delta q) = E_B \left( \tilde{J}_0(q - \delta q) + \frac{N}{2} \right). \quad (A4)$$

While $\hat{Q}_B = Q_B\tilde{J}_0$ is not necessary in this paper, we have added it for completeness.

To evaluate the derivative, one has to rewrite the quasispin operators at $q - \delta q$, $\hat{J}_\pm(q - \delta q)$ and $\tilde{J}_0(q - \delta q)$ in terms of $\hat{J}_\pm(q)$ and $\tilde{J}_0(q)$ at $q$. Let us denote Eq. (2.36) by

$$\begin{pmatrix} \hat{I}_0(q) \\ \hat{I}_+(q) \\ \hat{I}_-(q) \end{pmatrix} = A(\phi(q), \psi(q)) \begin{pmatrix} \hat{K}_0 \\ \hat{K}_+ \\ \hat{K}_- \end{pmatrix}. \quad (A5)$$
We define \( A(\phi(q)) := A(\phi(q), \psi(q) = 0) \), and then
\[
\begin{pmatrix} \hat{J}_0(q - \delta q) \\ \hat{J}_+(q - \delta q) \\ \hat{J}_-(q - \delta q) \end{pmatrix} = A(\phi(q - \delta q)) \begin{pmatrix} \hat{K}_0 \\ \hat{K}_+ \\ \hat{K}_- \end{pmatrix} = A(\phi(q - \delta q))A(\phi(q))^{-1} \begin{pmatrix} \hat{J}_0(q) \\ \hat{J}_+(q) \\ \hat{J}_-(q) \end{pmatrix}. \tag{A6}
\]
Let us simplify the notation with \((\phi(q - \delta q), \phi(q)) = (\phi_o, \phi_n)\), and then
\[
A(\phi(q - \delta q))A(\phi(q))^{-1} = A(\phi_o)A(\phi_n)^{-1}
= \begin{pmatrix} \cos \phi_o & -\frac{1}{2} \sin \phi_o & -\frac{1}{2} \sin \phi_o \\ \sin \phi_o & \cos^2 \frac{1}{2} \phi_o - \sin^2 \frac{1}{2} \phi_o \\ \sin \phi_o & -\sin^2 \frac{1}{2} \phi_o & \cos^2 \frac{1}{2} \phi_o \end{pmatrix}
\begin{pmatrix} \cos \phi_n & \frac{1}{2} \sin \phi_n & \frac{1}{2} \sin \phi_n \\ -\sin \phi_n & \cos^2 \frac{1}{2} \phi_n & -\sin^2 \frac{1}{2} \phi_n \\ -\sin \phi_n & -\sin^2 \frac{1}{2} \phi_n & \cos^2 \frac{1}{2} \phi_n \end{pmatrix}
= \begin{pmatrix} \cos(\phi_n - \phi_o) & -\frac{1}{2} \sin(\phi_n - \phi_o) & -\frac{1}{2} \sin(\phi_n - \phi_o) \\ \sin(\phi_n - \phi_o) & \cos^2 \frac{1}{2}(\phi_n - \phi_o) & -\sin^2 \frac{1}{2}(\phi_n - \phi_o) \\ \sin(\phi_n - \phi_o) & -\sin^2 \frac{1}{2}(\phi_n - \phi_o) & \cos^2 \frac{1}{2}(\phi_n - \phi_o) \end{pmatrix}. \tag{A7}
\]

Now the operators at \( q - \delta q \) are readily rewritten in terms of the quasispins at \( q \).
\[
\hat{E}_A(q - \delta q) = (E_A(q) - \delta q \partial_q E_A) \left[ -2 \sin \delta \phi \hat{J}_0(q) + \cos \delta \phi \left( \hat{J}_+(q) + \hat{J}_-(q) \right) \right], \tag{A8}
\]
where \( \delta \phi = \phi_n - \phi_o = \phi(q) - \phi(q - \delta q) \), which leads to
\[
\frac{(\hat{E}_A(q) - \hat{E}_A(q - \delta q))}{\delta q} = \frac{2 \sin \delta \phi \delta \phi}{\delta q} E_A(q) \hat{J}_0(q) + \partial_q E_A \cos \delta \phi \left( \hat{J}_+(q) + \hat{J}_-(q) \right) + O(\delta q). \tag{A9}
\]
Taking the limit \( \delta q \to 0 \), we have
\[
\partial_q \hat{E}_A = 2 E_A(q) \frac{\partial \phi}{\partial q} \hat{J}_0(q) + \partial_q E_A \left( \hat{J}_+(q) + \hat{J}_-(q) \right). \tag{A10}
\]
The derivative of a time-even operator of A-type \( \hat{E}_A \) contains both of A-terms and B-terms. Similarly,
\[
\hat{O}_A(q - \delta q) = i (O_A(q) - \delta q \partial_q O_A) \left( \hat{J}_+(q) - \hat{J}_-(q) \right), \tag{A11}
\]
from which we obtain
\[
\partial_q \hat{O}_A(q) = i \partial_q O_A \left( \hat{J}_+(q) - \hat{J}_-(q) \right). \tag{A12}
\]
The derivative of the B-type operator can be calculated similarly.
\[
\hat{E}_B(q - \delta q) = (E_B(q) - \delta q \partial_q E_B) \left[ \cos \delta \phi \left( \hat{J}_0(q) + \frac{N}{2} \right) + \frac{1}{2} \sin \delta \phi \left( \hat{J}_+(q) + \hat{J}_-(q) \right) \right], \tag{A13}
\]
from which we have
\[
\partial_q \hat{E}_B = \partial_q E_B \hat{J}_0(q) - \frac{1}{2} E_B(q) \frac{\partial \phi}{\partial q} \left( \hat{J}_+(q) + \hat{J}_-(q) \right). \tag{A14}
\]
The derivatives of the collective operators now read
\[
\begin{align}
\partial_q \hat{Q}^{(1)}(q) &= \partial_q Q^{(1)}(q) \left( \hat{J}_+(q) + \hat{J}_-(q) \right) + \frac{2}{N} \hat{J}_0(q), \\
\partial_q \hat{P}(q) &= i \partial_q P(q) \left( \hat{J}_+(q) - \hat{J}_-(q) \right), \\
\partial_q \hat{Q}_B(q) &= \partial_q Q_B(q) \hat{J}_0(q) - Q_B(q) P(q) \left( \hat{J}_+(q) + \hat{J}_-(q) \right).
\end{align}
\]
(A15) (A16) (A17)

Here we have used Eq. (2.54). The second terms of \( \partial_q \hat{Q}^{(1)} \) and \( \partial_q \hat{Q}_B(q) \) coincide with \( \frac{1}{\hbar} [\hat{P}, \hat{Q}^{(1)}] \) and \( \frac{1}{\hbar} [\hat{P}, Q_B] \), respectively.

### B. Curvature in the \( q \) space

In this Appendix, we shall evaluate the potential curvature \( C(q) \) in the \( q \) space when only the first-order collective-coordinate operator is taken into account. The collective potential \( V(\phi) \) as a function of \( \phi \) is given by Eq. (2.43).

In the \( q \) space with \( B(q) \equiv 1 \), the curvature \( C(q) \) is given by
\[
C(q) = \frac{d^2V}{dq^2} = \frac{d}{dq} \left( \frac{dV}{d\phi} \frac{d\phi}{dq} \right) = \frac{d^2V}{d\phi^2} \left( \frac{d\phi}{dq} \right)^2 + \frac{dV}{d\phi} \left( \frac{d^2\phi}{dq^2} \right). 
\]
(B1)

From Eqs. (2.54) and (2.82), the first term of Eq. (B1) reads
\[
\frac{d^2V}{d\phi^2} \left( \frac{d\phi}{dq} \right)^2 = \frac{1}{2}N \left( 2E - \epsilon \chi \cos^2 \phi \right) \cdot 4P^2 = (2E - \epsilon \chi \cos^2 \phi) (2E + \epsilon \chi) = \omega_{LRPA}^2. 
\]
\( \omega_{LRPA} \) is the eigenfrequency of the LRPA equations, in which the curvature term is neglected. Similarly, from Eqs. (2.54) and (2.82), we have
\[
\frac{d^2\phi}{dq^2} = 2 \frac{dP}{dq} = 2 \frac{dP}{d\phi} \frac{d\phi}{dq} = 4P \frac{dP}{d\phi} = \frac{1}{N} \epsilon \sin \phi (2\chi \cos \phi - 1). 
\]
(B3)

Then, the second term of Eq. (B1) is
\[
\frac{dV}{d\phi} \left( \frac{d^2\phi}{dq^2} \right) = \frac{1}{2} \epsilon^2 \sin^2 \phi (1 - \chi \cos \phi) (2\chi \cos \phi - 1). 
\]
(B4)

Thus, the product of the curvature and the inverse mass \( [B(q) \equiv 1] \) is obtained as
\[
BC = C(q) = (2E - \epsilon \chi \cos^2 \phi) (2E + \epsilon \chi) + \frac{1}{2} \epsilon^2 \sin^2 \phi (1 - \chi \cos \phi) (2\chi \cos \phi - 1), 
\]
(B5)

which coincides with Eqs. (4.8) and (4.11). The second term vanishes when \( 2\chi \cos \phi - 1 = 0 \) as well as at the potential extrema. One can immediately find that \( B(\phi)C(\phi) = B(q)C(q) \) by noticing that from Eqs. (2.54) and (2.76),
\[
\left( \frac{d\phi}{dq} \right)^2 = B(\phi), \quad \frac{d^2\phi}{dq^2} = 4P \frac{dP}{d\phi} = \frac{1}{2} \frac{dB(\phi)}{d\phi} = -\Gamma(\phi)B(\phi). 
\]
(B6)

### References

[1] K. Sato, Prog. Theor. Exp. Phys. **2015**, 123D01 (2015).
[2] K. Sato, Prog. Theor. Exp. Phys. **2017**, 033D01 (2017).
[3] K. Sato, Prog. Theor. Exp. Phys. **2017**, 123D03 (2017).
[4] M. Matsuo, T. Nakatsukasa, and K. Matsuyanagi, Prog. Theor. Phys. **103**, 959 (2000).
[5] T. Marumori, T. Maskawa, F. Sakata, and A. Kuriyama, Prog. Theor. Phys. **64**, 1294 (1980).
[6] K. Matsuyanagi, M. Matsuo, T. Nakatsukasa, N. Hinohara, and K. Sato, J. Phys. G 37, 064018 (2010).
[7] T. Nakatsukasa, K. Matsuyanagi, M. Matsuo, K. Yabana, Rev. Mod. Phys. 88, 045004 (2016).
[8] A. Klein, N. R. Walet, and G. Do Dang, Ann. Phys. 208, 90 (1991).
[9] J. Dobaczewski, J. Skałski, Nucl. Phys. A 369, 123 (1981).
[10] Z. P. Li, T. Nikšić, P. Ring, D. Vretenar, J. M. Yao, J. Meng, Phys. Rev. C 86, 034334 (2012).
[11] N. Hinohara, T. Nakatsukasa, M. Matsuo, and K. Matsuyanagi, Prog. Theor. Phys. 117, 451 (2007).
[12] N. Hinohara, T. Nakatsukasa, M. Matsuo, and K. Matsuyanagi, Prog. Theor. Phys. 119, 59 (2008).
[13] N. Hinohara, T. Nakatsukasa, M. Matsuo, and K. Matsuyanagi, Phys. Rev. C 80, 043065 (2009).
[14] N. Hinohara, K. Sato, T. Nakatsukasa, M. Matsuo, and K. Matsuyanagi, Phys. Rev. C 82, 064313 (2010).
[15] K. Sato and N. Hinohara, Nucl. Phys. A 849, 53 (2011).
[16] H. Watanabe et al. Phys. Lett. B 704, 270 (2011).
[17] N. Hinohara and Y. Kanada-En’yo, Phys. Rev. C 83, 014321 (2011).
[18] N. Hinohara, K. Sato, K. Yoshida, T. Nakatsukasa, M. Matsuo, and K. Matsuyanagi, Phys. Rev. C 84, 061302 (2011).
[19] N. Hinohara, Z. P. Li, T. Nakatsukasa, T. Nikšić, and D. Vretenar, Phys. Rev. C 85, 024323 (2012).
[20] K. Yoshida and N. Hinohara, Phys. Rev. C 83, 1 (2011).
[21] K. Sato, N. Hinohara, K. Yoshida, T. Nakatsukasa, M. Matsuo, and K. Matsuyanagi, Phys. Rev. C 86, 024316 (2012).
[22] K. Wen, T. Nakatsukasa, Phys. Rev. C 94, 054618 (2016).
[23] K. Wen, T. Nakatsukasa, Phys. Rev. C 96, 014610 (2017).
[24] D. J. Thouless, Nucl. Phys. B 21, 225 (1960).
[25] D. J. Rowe, A. Rymann, G. Rosensteel, Phys. Rev. A 22, 2362 (1980).
[26] P. Ring, P. Schuck, Nucl. Phys. A 292, 20 (1977).
[27] T. Suzuki, Nucl. Phys. A 398, 557 (1983).
[28] F. Villars, Nucl. Phys. A 285, 269 (1977).
[29] M. Baranger, M. Vénéri, Ann. Phys. 114, 123 (1978).
[30] H. J. Lipkin, N. Meshkow, A. J. Glick, Nucl. Phys. 62, 188 (1965).
[31] G. Holzwarth, Nucl. Phys. A 207, 545 (1973).