GENUS-ONE STABLE MAPS, LOCAL EQUATIONS, AND VAKIL-ZINGER’S DESINGULARIZATION

YI HU AND JUN LI

Abstract. We describe an algebro-geometric approach to Vakil-Zinger’s desingularization of the main component of the moduli of genus one stable maps to $\mathbb{P}^n$ [6, 7]. The new approach provides complete local structural results for this moduli space as well as for the desingularization of the entire moduli space and should fully extend to higher genera.

1. Introduction

Let $\overline{M}_1(\mathbb{P}^n, d)$ be the moduli of degree $d$ genus one stable maps [2] to $\mathbb{P}^n$ and let $\overline{M}_1(\mathbb{P}^n, d)_0 \subset \overline{M}_1(\mathbb{P}^n, d)$ be the primary component that is the closure of the open subset of all stable morphisms $[u, C] \in \overline{M}_1(\mathbb{P}^n, d)$ with smooth domains. In [6, 7], Vakil-Zinger found a canonical desingularization $\tilde{\overline{M}}_1(\mathbb{P}^n, d)_0$ of $\overline{M}_1(\mathbb{P}^n, d)_0$ by performing “virtual” blowing-ups on $\overline{M}_1(\mathbb{P}^n, d)$. They also showed that for $\tilde{f}_0 : \tilde{X} \to \mathbb{P}^n$ and $\tilde{\pi} : \tilde{X} \to \tilde{\overline{M}}_1(\mathbb{P}^n, d)_0$, the pull-back universal family over $\tilde{\overline{M}}_1(\mathbb{P}^n, d)_0$, the direct image sheaf

$$\tilde{\pi}_* \tilde{f}_0^* \mathcal{O}_{\mathbb{P}^n}(5)$$

is locally free. Those desingularizations are useful for applying Atiyah-Bott localization formula to the hyperplane relation proved by Li-Zinger in [4]. While the desingularization result in [7] is algebro-geometric, its proof is analytic in nature.

This paper provides an algebro-geometric approach to these desingularization results. It will be a part of an algebro-geometric approach of the relation between ordinary and reduced genus one GW-invariants of complete intersection in products of projective spaces [1]. It will also serve as our first step to generalize the structure results on moduli spaces of genus one stable maps to higher genera.

Our proof consists of two stages. At first, we use the classical method of studying special linear series on curves to give an algebro-geometric proof of the local equations of $\overline{M}_1(\mathbb{P}^n, d)$, obtained by Zinger in [8, §2.3]. After that, we “modular” blow up $\overline{M}_1(\mathbb{P}^n, d)$ and prove that the resulting stack has smooth irreducible components. The blowup construction used in this paper follows that in [7].
We now briefly outline our approach. Since a stable map \([u, C]\) to a projective space \(\mathbb{P}^n\) is given by
\[ u = [u_0, \cdots, u_n] : C \longrightarrow \mathbb{P}^n, \quad u_i \in H^0(u^*\mathcal{O}(1)), \]
its deformation is determined by the combined deformation of the curve \(C\) and the sections \(\{u_i\}\). Since moduli spaces of curves is smooth, the singularity of \(\overline{M}_1(\mathbb{P}^n, d)\) is caused by the non-locally freeness of the direct image sheaf \(\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(1)\) of the universal family
\[ \pi : \mathcal{X} \longrightarrow \overline{M}_1(\mathbb{P}^n, d) \quad \text{and} \quad f : \mathcal{X} \longrightarrow \mathbb{P}^n \]
of \(\overline{M}_1(\mathbb{P}^n, d)\).

To study the non-locally freeness of the direct image sheaf, by assigning to each stable map \([u, C]\) the divisor \(u^{-1}(0) \subset C\), locally we can view \(\overline{M}_1(\mathbb{P}^n, d)\) as a stack over the Artin stack \(\mathcal{D}_1\) of pairs \((C, D)\) of genus one nodal curves \(C\) and effective divisors \(D \subset C\). Over each chart \(V \subset \mathcal{D}_1\), by picking an auxiliary section of the universal curve \(\pi : C \longrightarrow V\) with \(D \subset C\) the tautological divisor, we construct explicitly a complex \(R \cdot \pi_* = [\mathcal{O} \oplus (d+1)\mathcal{V} \phi \longrightarrow \mathcal{O}_V]\) whose sheaf cohomology gives the cohomology \(R \cdot \pi_*\mathcal{O}_C(D)\).

We then apply the deformation theory of nodal curves to derive a simple explicit form of the homomorphism \(\phi\) in \(R \cdot \pi_*\). Under a suitable trivialization,
\[ \phi = (0, \zeta_1, \cdots, \zeta_d), \]
where each \(\zeta_i\) is a suitable product of the pull back of regular functions whose vanishing loci are irreducible components of nodal curves in \(\mathcal{D}_1\). This description enables us to derive explicit local equations of \(\overline{M}_1(\mathbb{P}^n, d)\) (see Theorem 5.7). They are analogous to the equations described by Zinger [8, §2.3].

To construct the desingularization of the moduli space, we introduce the Artin stack \(\overline{M}_1^{wt}\) of pairs \((C, w)\) of genus one nodal curves \(C\) with non-negative weights \(w \in H^2(C, \mathbb{Z})\), meaning that \(w(\Sigma) \geq 0\) for all irreducible \(\Sigma \subset C\). The stack \(\overline{M}_1^{wt}\) is smooth and contains closed substacks \(\Theta_k\) whose general point is a pair \((C, w)\) such that \(C\) consists of a smooth elliptic curve \(C_e\) with \(k\) rational curves attached to \(C_e\) and the restriction of \(w\) to \(C_e\) is zero, while its restriction to the other components of \(C\) is non-zero. We then blow up \(\overline{M}_1^{wt}\) successively along \(\Theta_1, \Theta_2, \cdots\), etc., to obtain \(\widetilde{\overline{M}}_1^{wt}\). The desingularization of \(\overline{M}_1(\mathbb{P}^n, d)\) is
\[ \widetilde{\overline{M}}_1(\mathbb{P}^n, d) = \overline{M}_1(\mathbb{P}^n, d) \times_{\overline{M}_1^{wt}} \widetilde{\overline{M}}_1^{wt}. \]

After a detailed study of the lifting of the local equations of \(\overline{M}_1(\mathbb{P}^n, d)\) mentioned earlier, we prove that the irreducible components of \(\widetilde{\overline{M}}_1(\mathbb{P}^n, d)\) are smooth and intersect transversally. We also prove that for each irreducible component \(\widetilde{\overline{M}}_1(\mathbb{P}^n, d)_\mu \subset \widetilde{\overline{M}}_1(\mathbb{P}^n, d)\) with \((\widetilde{\pi}_\mu, \widetilde{f}_\mu)\) the pull-back universal family on \(\widetilde{\overline{M}}_1(\mathbb{P}^n, d)_\mu\), the direct image sheaf
\[ \widetilde{\pi}_\mu^* \widetilde{f}_\mu^* \mathcal{O}_{\mathbb{P}^n}(r) \]
is locally free. It is of rank $dr$ over the desingularization of the primary component and of rank $dr + 1$ elsewhere. We comment that the results for the primary component were proved by Vakil-Zinger [7]. Weighted graph was also used to study stable maps to $\mathbb{P}^2$ by Pandharipande [5].

This paper is organized as follows. In §2, we outline our approach, stating the main desingularization theorems 2.8 and 2.10 and the main local structure theorems 2.16 and 2.18. In §3, we introduce the notion of weighted rooted trees of weighted nodal curves. In the following section, we state and prove the main structural result of the direct image sheaf $\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$.

Finally, in §5 we prove the theorems stated in §2.

We'd like to thank Zinger for numerous suggestions on improving the presentation of the paper and for pointing out several oversights. The second author also thanks him for valuable discussion during their collaboration. The first author was partially supported by NSA; the second author was partially supported by NSF DMS-0601002.

Throughout the paper, we fix an arbitrary algebraically closed base field $k$. All schemes in this paper are assumed to be noetherian over $k$.

2. Canonical desingularization of $\overline{M}_1(\mathbb{P}^n, d)$

In this section, we state our main results.

2.1. The Artin stack of weighted nodal curves.

Definition 2.1. Let $C$ be a curve. We set

$$H^2(C, \mathbb{Z})^+ = \{w \in H^2(C, \mathbb{Z}) \mid w(\Sigma) \geq 0 \text{ for } C \subset \Sigma \text{ irreducible}\}.$$ 

A weighted nodal curve is a pair $(C, w)$ consisting of a connected nodal curve $C$ and a weight assignment $w \in H^2(C, \mathbb{Z})^+$.

2.2. For any scheme $S$ and flat family of nodal curves $C \rightarrow S$, a weight assignment of $C/S$ is a section $w$ of the sheaf $R^2\pi_*\mathcal{O}_C^+$, where $R^2\pi_*\mathcal{O}_C^+ \subset R^2\pi_*\mathcal{O}_C$ is the subsheaf consisting of all sections $w$ whose restrictions $w(s)$ to any closed point $s \in S$ lies in $H^2(C_s, \mathbb{Z})^+$. Here $w(s)$ is the image of $w$ under the pullback homomorphism $R^2\pi_*\mathcal{O}_C \rightarrow H^2(C_s, \mathbb{Z})$.

A flat family of weighted nodal curves over $S$ is a pair $(C/S, w)$ of a flat family of nodal curves over $S$ together with a weight assignment over $C/S$. We say that two families of weighted nodal curves $(C/S, w)$ and $(C'/S', w')$ are equivalent if there is an isomorphism $h : C/S \rightarrow C'/S$ such that $w = h^*w'$.

A weighted nodal curve is called stable if every smooth ghost (weight 0) rational curve $B \subset C$, $B$ contains at least three nodes of $C$. Fixing a genus $g > 0$, we form a groupoid

$$\mathcal{M}_g^{\text{wt}} : (\text{Schemes}) \rightarrow (\text{Sets}),$$

that sends any scheme $S$ to the set of all equivalence classes of flat families of stable weighted nodal curves of genus $g$ over $S$. 
Mimicking the proof that the stack $\mathcal{M}_g$ of nodal curves of genus $g$ is an Artin stack, we obtain

**Proposition 2.3.** The groupoid $\mathcal{M}^{wt}_g$ is a smooth Artin stack of dimension $3g-3$. The projection $\mathcal{M}_g^{wt} \to \mathcal{M}_g$ by forgetting the weight assignment is étale.

Note that $\mathcal{M}^{wt}_g$ is the disjoint union of infinitely many smooth components, each of which is indexed by the total weight of the weighted curves. Because of the stability requirement, each component is of finite type.

### 2.2. Blowups of $\overline{M}_1(\mathbb{P}^n, d)$.

To describe the desingularization of $\overline{M}_1(\mathbb{P}^n, d)$, the notion of core curve is pivotal.

**Definition 2.4.** The core of a connected genus-one curve $C$ is the unique smallest (by inclusion) subcurve of arithmetic genus one. The core of a weighted curve $(C, w) \in \mathcal{M}_1^{wt}$ is called ghost if the induced weight on the core is zero.

### 2.5. The stack $\mathcal{M}_1^{wt}$ contains an open substack $\hat{\Theta}_0$ consisting of weighted curves with non-ghost cores. The complement $\mathcal{M}_1^{wt} \setminus \hat{\Theta}_0$ admits a natural partition according to the number of rational trees attached to the ghost core curves: $\hat{\Theta}_k$ is the subset of pairs $(C, w)$ such that $C$ can be obtained from the ghost core $C_e \subset C$ by attaching $k$ (connected) trees of rational curves to the core $C_e$ at $k$ distinct smooth points of $C_e$. Then $\mathcal{M}_1^{wt} = \bigsqcup_{k \geq 0} \hat{\Theta}_k$. We let $\Theta_k$ be the closure of $\hat{\Theta}_k$.

### 2.6. We can successively blow up $\mathcal{M}_1^{wt}$ along the loci $\Theta_k$. The locus $\Theta_1$ is a Cartier divisor; blowing up along $\Theta_1$ does nothing. We start by blowing up $\mathcal{M}_1^{wt}$ along the locus $\Theta_2$, which is a smooth codimension 2 closed substack of $\mathcal{M}_1^{wt}$; we denote the resulting stack by $\mathcal{M}_1^{wt}_{[2]}$, which is smooth. Inductively, after obtaining $\mathcal{M}_1^{wt}_{[k-1]}$, we blow it up along the proper transform $\Theta_{[k-1], k} \subset \mathcal{M}_1^{wt}_{[k-1]}$ of the closed substack $\Theta_k \subset \mathcal{M}_1^{wt}$. Since $\Theta_{[k-1], k}$ is a smooth closed substack of $\mathcal{M}_1^{wt}_{[k-1]}$ of codimension $k$, the new stack is smooth. We continue this process for all $k = 2, 3, \cdots$. Since each connected component of $\mathcal{M}_1^{wt}$ is of finite type, the blowup process on this component will terminate after finitely many steps. Therefore, the limit stack, which is the blowup of $\mathcal{M}_1^{wt}$ along the proper transforms of $\Theta_k$ in $\mathcal{M}_1^{wt}_{[k-1]}$ for all $k \geq 2$, is a well-defined smooth Artin stack; we denote this stack by $\tilde{\mathcal{M}}_1^{wt}$.

### 2.7. To induce a blowup on $\overline{M}_1(\mathbb{P}^n, d)$, we form the fiber product

$$\overline{M}_1(\mathbb{P}^n, d) = \overline{M}_1(\mathbb{P}^n, d) \times_{\mathcal{M}_1^{wt}} \tilde{\mathcal{M}}_1^{wt}.$$ 

Here the morphism $\overline{M}_1(\mathbb{P}^n, d) \to \mathcal{M}_1^{wt}$ is defined as follows. The first Chern class $c_1(f^* \mathcal{O}_{\mathbb{P}^n}(1))$ gives a weight assignment to the domain curves of the
universal family $f: \mathcal{X} \to \mathbb{P}^n$ of $\overline{M}_1(\mathbb{P}^n, d)$, making $\mathcal{X}$ a family of weighted nodal elliptic curves. This family then defines a tautological morphism

$$\overline{M}_1(\mathbb{P}^n, d) \to \mathcal{M}_1^{\text{wt}}$$

that is a lift of the tautological morphism $\overline{M}_1(\mathbb{P}^n, d) \to \mathcal{M}_1$. Note that since $c_1(f^*\mathcal{O}_{\mathbb{P}^n}(1))$ has degree $d$, defining $\tilde{M}_1(\mathbb{P}^n, d)$ requires blowing up $\mathcal{M}_1^{\text{wt}}$ along the proper transforms of $\Theta_k$ from $k = 2$ to $k = d$. That is,

$$\tilde{M}_1(\mathbb{P}^n, d) = \overline{M}_1(\mathbb{P}^n, d) \times_{\mathcal{M}_1^{\text{wt}}} \mathcal{M}_1^{\text{wt}}.$$ 

We can now succinctly reformulate the end result of the virtual blowup construction of [7, §4.3].

**Theorem 2.8.** $\tilde{M}_1(\mathbb{P}^n, d)$ is a DM-stack with normal crossing singularities.

For a refined version of this theorem with explicit local equations, see Theorem 5.22.

**2.9.** Desingularizations of the sheaves $\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k)$ over $\tilde{M}_1(\mathbb{P}^n, d)$ are essential ingredients in computing genus one GW-invariants of complete intersections [4, 9]. The blowup $\tilde{M}_1(\mathbb{P}^n, d)$ contains a primary irreducible component whose generic points are stable maps with non-ghost elliptic core. We denote this component by $\tilde{M}_1(\mathbb{P}^n, d)_0$. The other irreducible components are indexed by the set of all partitions of $d$. For $\mu$ either 0 or a partition of $d$, let

$$\tilde{\pi}_\mu: \tilde{\mathcal{X}}_\mu \to \tilde{M}_1(\mathbb{P}^n, d)_\mu \quad \text{and} \quad \tilde{f}_\mu: \tilde{\mathcal{X}}_\mu \to \mathbb{P}^n$$

be the pull back of the universal map over $\overline{M}_1(\mathbb{P}^n, d)$.

The following theorem is due to Vakil-Zinger [7].

**Theorem 2.10.** For every $k \geq 0$, the direct image sheaf $\tilde{\pi}_*\tilde{f}_*\mathcal{O}_{\mathbb{P}^n}(k)$ is a locally free sheaf over $\tilde{M}_1(\mathbb{P}^n, d)_\mu$ with $\mu$ either 0 or a partition of $d$. It is of rank $kd$ when $\mu = 0$ and of rank $kd + 1$ otherwise.

We will treat the two theorems in the reverse order from what was done in [7]. Specifically, we will first prove a structure result for the direct image sheaf $\pi_*f^*\mathcal{O}_{\mathbb{P}^n}(k)$ for all positive integers $k$. We then derive local defining equations for $\overline{M}_1(\mathbb{P}^n, d)$ and $\tilde{M}_1(\mathbb{P}^n, d)$, and obtain theorems 2.8 and 2.10 as corollaries.

**2.3. Local defining equations for $\overline{M}_1(\mathbb{P}^n, d)$.**

**2.11.** For later use, we form the Artin stack of stable pairs $(C, D)$ of (connected) nodal elliptic curves $C$ with effective divisors $D \subset C$ supported on the smooth loci of $C$. Here $(C, D)$ is stable if any smooth rational curve in $C$ disjoint from $D$ contains at least three nodes of $C$. We denote this stack by $\mathcal{D}_1$. It maps to $\mathcal{M}_1^{\text{wt}}$ by sending $(C, D)$ to $(C, c_1(D))$. The morphism $\mathcal{D}_1 \to \mathcal{M}_1^{\text{wt}}$ is smooth and has connected fibers. In particular, the connected components of $\mathcal{D}_1$ are indexed by the degree of the effective divisors.
2.12. For any closed point \([u, C] \in \overline{M}_1(\mathbb{P}^n, d)\), we let
\[ u = [u_0, \cdots, u_n] : C \rightarrow \mathbb{P}^n, \quad u_i \in \Gamma(C, u^*\mathcal{O}_{\mathbb{P}^n}(1)), \]
be the associated stable morphism. By a change of homogeneous coordinates on \(\mathbb{P}^n\), we can assume that \(u_0^{-1}(0) \subset C\) is a smooth simple divisor \(D\) of degree \(d\). Once \(D \subset C\) is fixed, we can choose \(u_0\) to be the constant section \(1 \in \Gamma(C, \mathcal{O}_C) \subset \Gamma(C, \mathcal{O}_C(D))\). Under this convention, the remaining sections \(u_1, \cdots, u_n\) are uniquely determined by the morphism \(u : C \rightarrow \mathbb{P}^n\). Consequently, deforming \(u : C \rightarrow \mathbb{P}^n\) is equivalent to deforming the pair \((C, D)\) and the sections \(u_1, \cdots, u_n\).

2.13. We next choose a small open neighborhood \([u, C] \in U \subset \overline{M}_1(\mathbb{P}^n, d)\). Let
\[ \pi : \mathcal{X} \rightarrow U \quad \text{and} \quad f : \mathcal{X} \rightarrow \mathbb{P}^n \]
be the the universal family over \(U\) with \(\mathcal{X} \cong C\) the fiber over \([u, C]\). Let \(S = f^*[x_0 = 0]\) (where \([x_0, \cdots, x_n]\) are the homogeneous coordinates of \(\mathbb{P}^n\)). By shrinking \(U\) if necessary, we can assume that \(S\) is away from the singular points of the fibers \(\mathcal{X}/U\). This way, we obtain a morphism
\[ U \rightarrow \mathcal{D}_1, \quad [u', C'] \in U \mapsto (C', C' \cap S). \]

2.14. We now construct the deformation space of the data \((C, D, u_1, \cdots, u_n)\). We let \(\mathcal{V} \rightarrow \mathcal{D}_1\) be a smooth chart of \((C, D) \in \mathcal{D}_1\) that contains the image of \(U \subset \mathcal{D}_1\); let \((\mathcal{C}_0, \mathcal{D}_0) = (C, D)\) for some point \(0 \in \mathcal{V}\), and let
\[ \rho : C \rightarrow \mathcal{V} \]
be the projection. We set \(\mathcal{L} = \mathcal{O}_C(D)\). Set theoretically, our deformation space is the union \(\bigcup_{v \in \mathcal{V}} H^0(\mathcal{C}_v, \mathcal{L}|_{\mathcal{C}_v})^\oplus n\). The deformation is singular at points where the core curve of \(\mathcal{C}_v\) is ghost due to \(H^1(\mathcal{C}_v, \mathcal{L}_v) \neq 0\).

The algebraic construction of the deformation space is via a standard trick.

2.15. By shrinking \(\mathcal{V}\) if necessary, we can find a section \(\mathcal{A} \subset C\) of \(\mathcal{C}/\mathcal{V}\), away from \(\mathcal{D}_1\), such that it passes through smooth parts of the core curves of all fibers of \(\mathcal{C}/\mathcal{V}\). Because \(\mathcal{L}(\mathcal{A})\) is effective and has positive degree on the core curve of every fiber of \(\mathcal{C}/\mathcal{V}\), we have
\[ R^1\rho_*\mathcal{L}(\mathcal{A}) = 0 \quad \text{and} \quad \rho_*\mathcal{L}(\mathcal{A}) \text{ is locally free}. \]
We let \(\mathcal{E}_\mathcal{V}\) be the total space of the vector bundle \(\rho_*\mathcal{L}(\mathcal{A})^\oplus n\) and let \(p : \mathcal{E}_\mathcal{V} \rightarrow \mathcal{V}\) be the projection. Then the tautological restriction homomorphism
\[ \text{rest} : \rho_*\mathcal{L}(\mathcal{A})^\oplus n \rightarrow \rho_*\mathcal{L}(\mathcal{A})^\oplus n|_{\mathcal{A}} \]
lifts to a section
\[ F \in \Gamma(\mathcal{E}_\mathcal{V}, p^*\rho_*\mathcal{L}(\mathcal{A})^\oplus n|_{\mathcal{A}}). \]

**Theorem 2.16.** Let \(\mathcal{U} = \mathcal{V} \times_{\mathcal{D}_1} U\). Then there is a canonical open immersion \(\mathcal{U} \rightarrow (F = 0) \subset \mathcal{E}_\mathcal{V}\).
These local equations are made explicit in Theorem 5.7.

2.17. We let $\widetilde{M}_1^{wt} \to M_1^{wt}$ be the blowup described in 2.16. We form the fiber product

$$\widetilde{D}_1 = D_1 \times_{\widetilde{M}_1^{wt}} \widetilde{M}_1^{wt}, \quad \widetilde{V} = V \times_{D_1} \widetilde{D}_1 \quad \text{and} \quad \widetilde{U} = \widetilde{V} \times_{D_1} U.$$ 

Let $\eta : \widetilde{V} \to V$ be the projection. Then

$$E_{\widetilde{V}} = E_V \times_V \widetilde{V}$$

is the total space of the pull back bundle $\eta^*p_*\mathcal{L}(A)^{\otimes n}$. The immersion $U \to E_V$ of Theorem 2.16 induces an immersion $\widetilde{U} \to E_{\widetilde{V}}$.

**Theorem 2.18.** Let $\xi : E_{\widetilde{V}} \to E_V$ be the projection and let $\widetilde{F} = \xi^*(F)$ be the pull back section in $\xi^*p^*\rho_*\mathcal{L}(A)^{\otimes n}|_A$. After shrinking $V$ if necessary and fixing a suitable trivialization $\xi^*p^*\rho_*\mathcal{L}(A)^{\otimes n}|_A \cong \mathcal{E}_{\widetilde{V}}^{\otimes n}$, we can find $(d + n)$ regular functions $w_1, \ldots, w_n, \xi_1, \ldots, \xi_d$ over $E_{\widetilde{V}}$ such that

$$\widetilde{F} = (w_1\xi_1 \cdots \xi_d, \ldots, w_n\xi_1 \cdots \xi_d).$$

Further, each $w_i$ and $\xi_j$ has smooth vanishing locus and the vanishing locus of their product $(w_1 \cdots w_n \cdot \xi_1 \cdots \xi_d = 0)$ has normal crossing singularities.

Note that some $\xi_1, \ldots, \xi_d$ may be invertible. For more explicit local equations for $\widetilde{M}_1(\mathbb{P}^n, d)$; see Theorem 5.22.

### 3. Combinatorics of the Dual Graphs of Nodal Curves

In this section, we discuss the combinatorics of the dual graphs of nodal curves and introduce terminally weighted trees for weighted elliptic nodal curves. This combinatorics is not strictly necessary for our presentation, but they do make our exposition more intuitive and formulas more elegant.

#### 3.1. Terminally weighted rooted trees

**3.1.** Let $\gamma$ be a connected rooted tree with $o$ its root\(^1\). The root $o$ defines a unique partial ordering on the set of all vertices of $\gamma$ according to the descendant relation so that the root is the unique minimal element; all non-root vertices are descendants of the root $o$. We call a vertex terminal if it has no descendants; in the combinatorial world, this is also called a leaf. Equivalently, a non-root terminal vertex is a non-root vertex that has exactly one edge connecting to it. Following this convention, the root is terminal only when $\gamma$ consists of a single vertex. In the following, given a rooted tree $\gamma$ we denote by $\text{Ver}(\gamma)$ the set of its vertices, by $\text{Ver}(\gamma)^*$ its non-root vertices, and by $\text{Ver}(\gamma)^*$ its terminal vertices. Note that for any vertex $v$ in $\gamma$, there is a unique (directed) path between $o$ and $v$; this is the maximal chain of vertices $o = v_0 \prec v_1 \prec \cdots \prec v_r = v$.

---

\(^1\)All trees are connected in this article.
3.2. Next we consider weighted rooted trees. A weighted rooted tree is a pair \((\gamma, w)\) consisting of a rooted tree \(\gamma\) together with a function on its vertices \(w: \text{Ver}(\gamma) \rightarrow \mathbb{Z}_\geq 0\), called the weight function. A vertex is positive if its weight is positive; a ghost vertex is a vertex with zero weight. The total weight of the tree is the sum of all its weights.

In this paper, we will consider only terminal weighted rooted trees.

**Definition 3.3.** A weighted rooted tree \((\gamma, w)\) is terminal if all terminal vertices are positive and all positive vertices are terminal; it is called stable (resp. semistable) if every ghost non-root vertex has at least three (resp. two) edges attached to it.

When the weight \(w\) is understood, we will use \(\gamma\) to denote the weighted rooted tree \((\gamma, w)\) as well as its underlying rooted tree \(\gamma\) with weights removed.

3.4. For positive \(d\), we let \(\Lambda_d\) (resp. \(\Lambda_d^{ss}\)) be the set of all stable (resp. semistable) terminal weighted rooted trees of total weight \(d\). The set \(\Lambda_d\) is finite, while \(\Lambda_d^{ss}\) is infinite. The set \(\Lambda_d^{ss}\) admits the following geometrical operations that will be useful for our discussion.

3.5. The first operation is pruning a tree. Given \(\gamma \in \Lambda_d^{ss}\), to prune \(\gamma\) from a vertex \(v\), we simply remove all descendants (i.e. those \(u\) with \(u \succ v\)) and the edges connecting the removed vertices. After pruning \(\gamma\) from \(v\), the new graph has \(v\) as its terminal vertex. If \(\gamma\) is a weighted tree, we define the weight of \(v\) in the pruned tree to be the sum of the original weight of \(v\) and the weight of the removed vertices; we keep the weights of the other vertices unchanged. Note that the resulting pruned tree is terminally weighted as well.

The second operation is collapsing a vertex. Collapsing a vertex \(v\) in \(\gamma\) is a two-step process: first merge \(v\) with its unique ascendent, removing the edge between them and assigning the sum of their weights to the merged vertex,
and then prune the resulting tree along all positively weighted non-terminal vertices, repeating the process as long as possible.

The third operation is *specialization*: it is the inverse operation of a collapsing.

The fourth operation is *advancing* a vertex. Let $v$ be a vertex in $\gamma$ and let $\bar{v}$ be its direct ascendant. To advance $v$, replace every edge connecting $\bar{v}$ to a direct descendant $v_i$ other than $v$ by an edge connected $v_i$ to $v$ and then prune the resulting tree along all positively weighted non-terminal vertices, repeating the process as long as possible.

![Operations on weighted trees](image)

**Figure 2. Operations on weighted trees**

### 3.6

Instead of drawing a picture, a weighted rooted tree can also be described compactly as follows. Here is an example. Let

$$\gamma = o[a(2), b[c(1), d(1)]] .$$

This is a weighted rooted tree whose root is $o$; the other vertices are labelled by $a, b, c, d$. The vertices inside a square bracket are the descendants of the vertex immediately proceeding the bracket; the weights of the terminal vertices $a, c,$ and $d$ are indicated in the following parenthesis. Collapsing $b$, we obtain $o[a(2), c(1), d(1)]$. Advancing $b$, we get $o[b[a(2), c(1), d(1)]]$. Advancing $a$, we obtain $o[a(4)]$. 
When the weight function is irrelevant to the discussion, we will drop any reference to weights. For example, the above tree $\gamma$ would then be written as $\gamma = o[a, b[c, d]]$

3.7. In this compact representation of a tree, pruning a tree from a vertex $v$ means removing the bracket immediately after $v$ and assigning it the total weight of the vertices inside the bracket. Collapsing a ghost vertex $v$ is removing the vertex $v$ as well as the closest brackets “[,]” associated to it. Advancing a ghost vertex $v$ is moving all the other vertices located inside of the same square bracket as $v$ into the square bracket following $v$.

3.8. Observe that advancing can make a stable tree semistable. For example, consider the stable tree $o[a[b[d, e], c]]$. If we advance $b$, we obtain $o[a[b[c, d, e]]]$, which is semistable but not stable, since the vertex $a$ has only two edges attached to it.

3.3. Monoidal transformations of weighted trees.

To keep track of the changes of the strata of $\overline{M}_1(\mathbb{P}^n, d)$ after blowups, we need the notion of monoidal transformations of weighted trees. We begin with the following.

**Definition 3.9.** Let $\gamma$ be a semi-stable terminally weighted tree with root $o$ and at least one non-root vertex. The trunk of $\gamma$ is the maximal chain $o = v_0 \prec \ldots \prec v_r$ of vertices in $\gamma$ such that each vertex $v_i$ with $i < r$ has exactly one immediate descendant.

In this case, we abbreviate the trunk by $ov_r$; we call $v_r$ the branch vertex of $\gamma$ if it is not a terminal vertex. Otherwise we call $\gamma$ a path tree. When $v_r = o$, we say the tree has no trunk. Figure 1 shows two trees: the first one has no trunk; the second one has a trunk.

Let $\gamma$ be a tree with trunk $ov_r$. Then $\gamma$ can be obtained by attaching $\ell > 1$ rooted trees, called branches, $\gamma'_1, \ldots, \gamma'_\ell$, to the trunk so that the roots of $\gamma'_i$ are direct descendants of $v_r$. According to our convention, $\gamma$ can be expressed as

$$\gamma = \overline{ov_r}[]\gamma'_1, \ldots, \gamma'_\ell := o[v_1]\cdots[v_r[\gamma'_1, \ldots, \gamma'_\ell]]].$$

The tree $\gamma$ has no branches if and only if it is a path-tree.

**Definition 3.10.** Let $br(\gamma)$ denote the number of branches of $\gamma$. We call $\gamma$ simple if all of the branches are stable.

**Definition 3.11.** A monoidal transform of a simple terminally weighted tree $\gamma$ is a tree obtained by advancing one of the immediate descendants of the branch vertex of $\gamma$, if $\gamma$ has a branch vertex. We denote the set of monoidal transforms of $\gamma$ by $\text{Mon}(\gamma)$.

Note that every tree in $\Lambda_d$ is simple, and if $\gamma$ is a simple tree, so is every monoidal transform of $\gamma$. 
Lemma 3.12. Let $\gamma$ be a simple terminally weighted tree and $\tilde{\gamma} \in \text{Mon}(\gamma)$. Then either $\text{br}(\tilde{\gamma}) = 0$, which is when $\tilde{\gamma}$ is a path-tree, or $\text{br}(\tilde{\gamma}) \geq \text{br}(\gamma) + 1$. The same conclusion holds when $\tilde{\gamma}$ is a collapsing of $\gamma$ at a direct descendant of the branch vertex.

Proof. If $\tilde{\gamma}$ is the result of advancing a direct descendant $v$ of the branch vertex $\bar{v}$ of $\gamma$ and $v$ is not terminal, then the direct descendants of $v$ in $\tilde{\gamma}$ are the direct descendants of $v$ in $\gamma$ and the direct descendants of $\bar{v}$ in $\gamma$ other than $v$. Furthermore, $v$ is the branch vertex of $\tilde{\gamma}$ in this case; thus, $\text{br}(\tilde{\gamma}) \geq \text{br}(\gamma) + 1$. On the other hand, if $v$ is terminal in $\gamma$, then $\tilde{\gamma}$ is the path from $o$ to $v$ in $\gamma$. The proof of the second statement is similar. □

3.13. To index the strata of the various blowups of $\overline{M}_1(\mathbb{P}^n, d)$, we introduce the following. We set $\Lambda_{d,[1]} = \Lambda_d$ and define $\Lambda_{d,[k]}$ inductively for $k \geq 2$:

$$\Lambda_{d,[k]} = \{\gamma \in \Lambda_{d,[k-1]} | \text{br}(\gamma) \geq k+1\} \cup \{\gamma \in \text{Mon}(\gamma') | \gamma' \in \Lambda_{d,[k-1]}, \text{br}(\gamma') = k\}.$$

Lemma 3.14. For any $\gamma \in \Lambda_{d,[k]}$ with $1 \leq k \leq d$, either $\text{br}(\gamma) = 0$ or $k + 1 \leq \text{br}(\gamma) \leq d$. In particular, $\Lambda_{d,[d]}$ consists of path trees only.

Proof. First, the fact the total weight of a terminally weighted tree $\gamma$ is $d$ implies that $\text{br}(\gamma) \leq d$. The assertion of the lemma holds for $k = 1$. On the other hand, Lemma 3.12 implies that if the assertion holds for $k$, then it also holds for $k + 1$. □

3.4. Terminally weighted trees of weighted nodal curves.

3.15. Let $C$ be a connected nodal genus one curve. We associate to $C$ the dual $\gamma'_C$, with vertices corresponding to the irreducible components of $C$ and the edges to the nodes of $C$. Since the arithmetic genus of $C$ is 1, either $\gamma'_C$ has a unique vertex corresponding to the genus one irreducible component of $C$ or $\gamma'_C$ contains a unique loop. In the first case, we designate that vertex the root of $\gamma'_C$; in the latter case we shall contract the whole loop to a single vertex and designate it the root of the resulting tree. We will call the resulting rooted tree the reduced dual tree of $C$ and denoted it by $\gamma_C$.

3.16. Next we consider a weighted nodal genus one curve $(C, w)$. The weight $w$ induces weights on the vertices of the dual graph $\gamma'_C$ of $C$. If $\gamma'_C$ contains a loop, then upon contracting the loop, we assign the resulting root the total weight of the loop. This way, we obtain a natural weighted tree $(\gamma_C, w')$ associated to the weighted curve $(C, w)$.

In general, the $\gamma_C$ with this weight function may not be terminally weighted. If not, we can prune $\gamma_C$ along all positively weighted non-terminal vertices to obtain a terminally weighted rooted tree. We call the result the terminally weighted rooted tree of $(C, w)$ and denoted it by $(\gamma_C, w)$. In the case that $(C, w)$ is understood, we shall write it simply as $\gamma$.

Following this construction, each terminal vertex $v$ of $\gamma$ is a positively weighted vertex of $\gamma'_C$. We let $C_v$ be the union of all irreducible components of $C$ whose associated vertices $v'$ (in $\gamma'_C$) satisfy $v \preceq v'$. Then, $C_v$ is a tree
of rational curves. Further, if $\gamma$ has zero weight root and $v_1, \cdots, v_k$ are its terminal vertices, then $C - \bigcup_{i=1}^{k} C_{v_i}$ is the maximal weight zero connected subcurve of $C$ that contains the core elliptic curve.

3.5. A stratification of $\overline{M}_1(\mathbb{P}^n, d)$. Any stable map $[u, C] \in \overline{M}_1(\mathbb{P}^n, d)$ naturally gives rise to a weighted nodal genus one curve $(C, w)$, where the weight of an irreducible component of $C$ is the degree of the map $u$ on that component. We will then call the associated terminally weighted rooted tree of $(C, w)$ the terminally weighted rooted tree of the stable map $[u]$. We denote it by $(\gamma[u], w)$. It is stable.

**Definition 3.17.** For any $\gamma \in \Lambda_d$, we define $\overline{M}_1(\mathbb{P}^n, d)_\gamma$ be the subset of all $[u] \in \overline{M}_1(\mathbb{P}^n, d)$ whose associated terminally weighted rooted trees is $\gamma$.

**Lemma 3.18.** Each $\overline{M}_1(\mathbb{P}^n, d)_\gamma$ is a smooth, locally closed substack of $\overline{M}_1(\mathbb{P}^n, d)$; together they form a stratification of $\overline{M}_1(\mathbb{P}^n, d)$.

**Proof.** Suppose $\gamma \in \Lambda_d$ has $\ell$ terminal vertices, indexed by $1, \cdots, \ell$ and of weights $d_1, \cdots, d_\ell > 0$ with $\sum_{i=1}^{\ell} d_i = d$. Without the weight, $\gamma$ is the reduced dual graph of some genus 1 nodal curve. We denote by $M_\gamma$ the stratum in $M_{1, \ell}$ consisting of stable genus 1 curves whose reduced dual graph is $\gamma$ with terminal vertices replaced by the corresponding marked points. Then $\overline{M}_1(\mathbb{P}^n, d)_\gamma$ is (up to equivalence by automorphisms)

$$\{([C_0, p_1, \cdots, p_\ell], [u_i, C_i, q_i]|_{\ell}) \in M_\gamma \times \prod_{i=1}^{\ell} \overline{M}_{0, 1}(\mathbb{P}^n, d_i) \mid u_1(p_1) = \cdots = u_\ell(p_\ell)\}.$$

Since $\mathbb{P}^n$ has ample tangent bundle, $\overline{M}_{0, 1}(\mathbb{P}^n, d_i)$ smooth and the evaluation morphisms $u_i$ are submersions. Hence $\overline{M}_1(\mathbb{P}^n, d)_\gamma$ is smooth. $\square$

4. The Structure of the Direct Image Sheaf

In this section, we state and prove structure results of the direct image sheaf $\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$.

4.1. Terminology.

**Definition 4.1.** Let $C$ be a proper nodal curve with arithmetic genus $g > 0$. We call a node $q$ of $C$ a separating node if $C - q$ is disconnected. Similarly, we call an irreducible component $\Sigma \subset C$ a separating component if $C - \Sigma$ is disconnected.

Along the same line, we introduce

**Definition 4.2.** An inseparable curve is a connected curve with no separating node; an inseparable component of $C$ is an inseparable subcurve of $C$ that is not a proper subcurve of another inseparable subcurve of $C$. 
4.3. We say that a (separating) node $q$ separates $x$ and $y \in C$ if $x$ and $y$ lie in different connected components of $C - q$; we say that the node $q$ lies between $x$ and $y$ in this case. We let $N_{[x,y]}$ be the collection of all nodes that lie between $x$ and $y$. This notion extends beyond nodes: for any smooth point $t \in C$, we denote by $C_t$ the inseparable component of $C$ that contains $t$; we say $t$ (or $C_t$) separates or lies between $x$ and $y \in C$ if $x$ and $y$ lie in different connected components of $C - C_t$.

4.4. For a nodal elliptic curve $C$ and two distinct smooth points $a$ and $b$ on the core of $C$, we have

\[(4.1) \quad h^0(C, \mathcal{O}_C(a - b)) = 0 \quad \text{and} \quad h^1(C, \mathcal{O}_C(a - b)) = 0; \]

for any point $\delta$ of $C$ distinct from $a, b$, we have

\[(4.2) \quad h^0(C, \mathcal{O}_C(\delta + a - b)) = 1 \quad \text{and} \quad h^1(C, \mathcal{O}_C(\delta + a - b)) = 0. \]

4.5. Let $X$ be a scheme, $D$ a Cartier divisor of $X$, and $Z$ a closed subscheme of $X$. We will write $\mathcal{O}_Z(D)$ for the restriction $\mathcal{O}_X(D)|_Z$.

4.6. Our aim is to describe the structure of $\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(k)$; we let $m = dk$ in the rest of this section. When we investigate the structure of $\overline{M}_1(\mathbb{P}^n, d)$ via $\pi_* f^* \mathcal{O}_{\mathbb{P}^n}(1)$, we will specialize to $m = d$.

4.7. Consider the substack $\mathcal{D}_1^m$ of the Artin stack $\mathcal{D}_1$ of pairs $(C, D)$ of nodal elliptic curves with effective degree $d$ divisors $D \subset C$. Let $(C, D) \in \mathcal{D}_1^m$ be a point with the divisor $D$ simple and supported on the smooth locus of $C$. We let $\mathcal{V} \rightarrow \mathcal{D}_1^m$ be a smooth chart containing $(C, D)$. Again, let $(C, D)$ be the tautological family over $\mathcal{V}$ with $(C_0, D_0) = (C, D)$ for some point $0 \in \mathcal{V}$ and $\rho : C \rightarrow \mathcal{V}$ be the projection; set $\mathcal{L} = \mathcal{O}_C(D)$. As in 2.15, we choose a general section $\mathcal{A}$ of $\mathcal{C}/\mathcal{V}$ and this time around also an additional general section $\mathcal{B}$ of $\mathcal{C}/\mathcal{V}$ such that they are disjoint from $\mathcal{D}$ and pass through the core of every fiber of $\mathcal{C}/\mathcal{V}$. This is possible after shrinking $\mathcal{V}$ if necessary. By the Mittag-Leffler exact sequence, the sheaf $\rho_* \mathcal{L}$ over $\mathcal{V}$ is the kernel sheaf of

\[(4.3) \quad \psi : \rho_* \mathcal{L}(\mathcal{A}) \longrightarrow \rho_* \mathcal{O}_\mathcal{A}(\mathcal{A}). \]

4.8. The complex of locally free sheaves of $\mathcal{O}_\mathcal{V}$-modules

$$[R^*] = [\rho_* \mathcal{L}(\mathcal{A}) \longrightarrow \rho_* \mathcal{O}_\mathcal{A}(\mathcal{A})]$$

has sheaf cohomology $[R^* \rho_* \mathcal{L}]$. Further, for any scheme $g : T \rightarrow \mathcal{V}$ with the induced family

$$\rho_T : C_T = C \times_{\mathcal{V}} T \longrightarrow T \quad \text{and} \quad D_T = D \times_{\mathcal{V}} T,$$

since $R^1 \rho_* \mathcal{L}(\mathcal{A}) = R^1 \rho_* \mathcal{O}_\mathcal{A}(\mathcal{A}) = 0$, by cohomology and base change,

$$R^1 \rho_T \mathcal{O}_{C_T}(D_T) \equiv h^i(g^*[R^*]).$$
4.9. To get hold of the sheaf $\pi^*f^* \mathcal{O}_p^*(k)$, we shall study the local structure of the homomorphism $(4.3)$. As in 2.12-2.14 we only need to consider the case that $D$ is a smooth simple divisor on $C$; we will assume that this holds.

We may also assume that $V$ is affine. After shrinking $V$ and an étale base change if necessary, we may assume that $D = \sum_{i=1}^m D_i$, where $\{D_i\}$ are disjoint sections of the family $C/V$. To the sheaf $L = \mathcal{O}_C(D)$, the standard inclusion $\mathcal{O}_C \subset \mathcal{O}_C(D)$ provides us a section $1 \in \Gamma(\rho_* L)$, called the obvious section. To capture other sections, we consider the inclusion of sheaves $M_i = \mathcal{O}_C(D_i + A - B) \subset M = \mathcal{O}_C(D + A - B)$ and the induced inclusions $\eta_i : \rho_* M_i \subset \rho_* M$.

Both are locally free since $R^1 \rho_* M_i$ and $R^1 \rho_* M = 0$ by (4.2). By Riemann-Roch, $\rho_* M_i$ is invertible and the rank of $\rho_* M$ is $m$. We then let $\varphi : \rho_* M \longrightarrow \rho_*(\mathcal{O}_A(D + A - B)) = \rho_*(\mathcal{O}_A(A))$ and $\varphi_i : \rho_* M_i \longrightarrow \rho_*(\mathcal{O}_A(D_i + A - B)) = \rho_*(\mathcal{O}_A(A))$ be the evaluation homomorphisms. Obviously, $\varphi_i = \varphi \circ \eta_i$. Since $V$ is assumed affine, the sheaf $\rho_*(\mathcal{O}_A(A)) \cong \mathcal{O}_V$.

Lemma 4.10. We have

1. $\rho_* L \cong \mathcal{O}_V \oplus \rho_* L(-B)$;
2. $\rho_* L(-B) \cong \ker \varphi$;
3. $\oplus_{i=1}^m \eta_i : \bigoplus_{i=1}^m \rho_* M_i \longrightarrow \rho_* M$ is an isomorphism, and $\bigoplus_{i=1}^m \varphi_i = \varphi \circ \bigoplus_{i=1}^m \eta_i$.

Consequently, the sheaf $\rho_* L$ is a direct sum of $\mathcal{O}_V$ with the kernel of the homomorphism

\[(4.4) \quad \bigoplus_{i=1}^m \varphi_i : \rho_* M_i \longrightarrow \mathcal{O}_V.\]

Proof. Taking the direct image of the exact sequence

\[0 \longrightarrow \mathcal{O}_C(D - B) \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_B(D) \longrightarrow 0,\]

we obtain the exact sequence

\[0 \longrightarrow \rho_* \mathcal{O}_C(D - B) \longrightarrow \rho_* \mathcal{O}_C(D) \overset{\alpha}{\longrightarrow} \rho_* \mathcal{O}_B(D).\]

Clearly, $\rho_* \mathcal{O}_B(D) \cong \mathcal{O}_V$. Also, $\alpha$ is surjective because the obvious section $1 \in \Gamma(\rho_* \mathcal{O}_C(D))$ maps surjectively onto $\rho_* \mathcal{O}_B(D)$. Finally, because $V$ is affine, $\operatorname{Ext}^1(\mathcal{O}_V, \rho_* \mathcal{O}_C(D - B)) = 0$. Therefore, the sequence

\[0 \longrightarrow \rho_* \mathcal{O}_C(D - B) \longrightarrow \rho_* \mathcal{O}_C(D) \overset{\alpha}{\longrightarrow} \rho_* \mathcal{O}_B(D) \longrightarrow 0\]

is exact and splits. This proves (1).
The second is obvious. We now obtain the third. Since both sheaves $\rho_*\mathcal{M}_i$ and $\rho_*\mathcal{M}$ are locally free, we only need to show that for any closed $z \in V$,

$$
\bigoplus_{i=1}^m \rho_*\mathcal{M}_i \otimes k(z) \to \rho_*\mathcal{M} \otimes k(z)
$$

is an isomorphism. Because $R^1\rho_*$ of $\mathcal{M}_i$ and $\mathcal{M}$ are zero, by base change, this is equivalent to that the tautological homomorphism

$$
\bigoplus_{i=1}^m \eta_i(z) : \bigoplus_{i=1}^m H^0(C_z, \mathcal{O}_{C_z}(D_i + A - B)) \to H^0(C_z, \mathcal{O}_{C_z}(D + A - B))
$$

is an isomorphism. Because both sides are of equal dimensions, it suffices to show that it is injective. For this, we look at the composite of $\bigoplus_{i=1}^m \eta_i(z)$ with

$$
\Phi_j(z) : H^0(C_z, \mathcal{O}_{C_z}(D + A - B)) \to H^0(C_z \cap D_j, \mathcal{O}_{C_z \cap D_j}(D + A - B)).
$$

Obviously, $\Phi_j(z) \circ \eta_i(z) = 0$ for $i \neq j$ and is an isomorphism for $j = i$. This shows that $\bigoplus_{i=1}^m \eta_i(z)$ is injective. This proves the last claim of the lemma.

4.11. The homomorphism

$$
\varphi : \rho_*\mathcal{M} \to \rho_*\mathcal{O}_A(D + A - B) \cong \rho_*\mathcal{O}_A(A)
$$

then is completely determined by the homomorphism

$$
\bigoplus_i \varphi_i : \bigoplus_i \rho_*\mathcal{M}_i \to \rho_*\mathcal{O}_A(D + A - B) \cong \rho_*\mathcal{O}_A(A).
$$

The homomorphism

$$
\varphi_i : \rho_*\mathcal{M}_i \to \rho_*\mathcal{O}_A(A)
$$

will be our focus in the next subsection.

4.3. The homomorphism $\varphi_i$.

4.12. Our strategy is to find an explicit expression for $\varphi_i$ so that its vanishing locus has precise geometric meaning. For this, we need some regular functions associated to the smoothing of nodes. By the deformation theory of nodal curves, for each separating node $q \in C$ there is a regular function $\zeta_q \in \Gamma(\mathcal{O}_Y)$ so that $\Sigma_q = \{\zeta_q = 0\}$ is the locus where the node $q$ is not smoothed; the divisor $\Sigma_q$ is an irreducible smooth Cartier divisor.

For any $1 \leq i \leq m$, we introduce

$$
\delta_i = D_i \cap C \quad \text{and} \quad a = A \cap C.
$$

We then collect all the nodes $q$ that lie between $\delta_i$ and $a$ (cf. 4.3) and form the product of their associated functions $\zeta_q$:

$$
\zeta_{[\delta, a]} = \prod_{q \in N_{[\delta, a]}} \zeta_q.
$$

In case $N_{[\delta, a]} = \emptyset$, we set $\zeta_{[\delta, a]} = 1$. 
Proposition 4.13. There are trivializations $\rho_*\mathcal{M}_i \cong \mathcal{O}_{\mathcal{V}}$ and $\rho_*\mathcal{O}_A(A) \cong \mathcal{O}_{\mathcal{V}}$ such that the homomorphism $\varphi_i$ is given by
\begin{equation}
\varphi_i = \zeta_{[\delta_i,a]} : \rho_*\mathcal{M}_i \longrightarrow \rho_*\mathcal{O}_A(A).
\end{equation}

Proof. Since $\mathcal{V}$ is affine, we fix a trivialization $\rho_*\mathcal{M}_i \cong \mathcal{O}_{\mathcal{V}}$, and keep the trivialization $\rho_*\mathcal{O}_A(A) \cong \mathcal{O}_{\mathcal{V}}$ mentioned before. This way, $\varphi_i \in \Gamma((\rho_*\mathcal{M}_i)^{\vee} \otimes \rho_*\mathcal{O}_A(A)) \cong \Gamma(\mathcal{O}_R)$.

Then the proposition is equivalent to that as divisors,
\begin{equation}
\varphi_i^{-1}(0) = \sum_{q \in N[\delta_i,a]} \zeta_q^{-1}(0) = \sum_{q \in N[\delta_i,a]} \Sigma_q.
\end{equation}

We next let $\eta : R \to \mathcal{V}$ be either a point or a smooth affine curve, we let $\pi_R : C_R \to R$ be $C_R = C \times_{\mathcal{V}} R$ over $R$, and let

\[ D_{R,i} = \mathcal{D}_i \times_{\mathcal{V}} R, \quad \mathcal{A}_R = \mathcal{A} \times_{\mathcal{V}} R \quad \text{and} \quad \mathcal{B}_R = \mathcal{B} \times_{\mathcal{V}} R \]

be the corresponding pull back divisors. Since $R^1\rho_*\mathcal{M}_i = 0$, by cohomology and base change, the natural homomorphism

\[ \eta^*\rho_*\mathcal{M}_i = \eta^*\rho_*\mathcal{O}_C(\mathcal{D}_i - \mathcal{B} + A) \longrightarrow \pi_{R*}\mathcal{O}_{C_R}(\mathcal{D}_{R,i} - \mathcal{B}_R + \mathcal{A}_R) \cong \mathcal{O}_R \]

is an isomorphism. Finally, let $\varphi_{R,i}$ be the composite

\[ \varphi_{R,i} : \eta^*\rho_*\mathcal{M}_i \cong \pi_{R*}\mathcal{O}_{C_R}(\mathcal{D}_{R,i} - \mathcal{B}_R + \mathcal{A}_R) \longrightarrow \varphi_{R*}\mathcal{O}_{C_R}(\mathcal{D}_{R,i} - \mathcal{B}_R + \mathcal{A}_R). \]

Then, if $R$ is a smooth curve not contained in $\varphi_i^{-1}(0)$, as divisors,

\[ \eta^{-1}(\varphi_i^{-1}(0)) = \varphi_{R,i}^{-1}(0) \subset R. \]

We now prove the claim. First let $R$ be a smooth point away from $\bigcup_{q \in N[\delta_i,a]} \Sigma_q$. Then $\mathcal{A}_R$, $\mathcal{B}_R$, and $\mathcal{D}_{R,i}$ lie in the same inseparable component of $\mathcal{C}_R$. Therefore $\pi_{R*}(\mathcal{O}_{C_R}(\mathcal{D}_{R,i} - \mathcal{B}_R)) = 0$ and $\varphi_{R,i} \not\equiv 0$ because

\[ \ker(\varphi_{R,i}) = \pi_{R*}(\mathcal{O}_{C_R}(\mathcal{D}_{R,i} - \mathcal{B}_R)) = 0. \]

If $R$ is a smooth point in $\Sigma_q$ for some $q \in N[\delta_i,a]$, then $\mathcal{B}_R$ and $\mathcal{D}_{R,i}$ lie in different inseparable components of $\mathcal{C}_R$. This time, $\pi_{R*}(\mathcal{O}_{C_R}(\mathcal{D}_{R,i} - \mathcal{B}_R)) \cong \mathcal{O}_R$. Therefore, for the same reason as above, $\varphi_{R,i} = 0$. This proves that $\varphi_i$ vanishes exactly along $\bigcup_{q \in N[\delta_i,a]} \Sigma_q$.

It remains to show that $\varphi_i$ vanishes at first order only along $\Sigma_q, q \in N[\delta_i,a]$. To prove this we only need to study $\varphi_i$ near a general point $p \in \Sigma_q$. We let $R \subset \mathcal{V}$ be an affine curve passing through $p$ and transversal to $\Sigma_q$ at $p = R \cap \Sigma_q$. After shrinking $p \in R$ if necessary, the family $\mathcal{C}_R \to R$ is the blowup of a family of smooth elliptic curves $\pi_R : E_R \to R$ at a point $q \in E_p = E_R \times_R p$.

We let $\xi : \mathcal{C}_R \to E_R$ be the projection and let $\mathcal{F}_0 \subset \mathcal{C}_p$ be the rational component, which is also the exceptional divisor of $\mathcal{C}_R$. Let $A = \xi(\mathcal{A}_R)$, $B = \xi(\mathcal{B}_R)$, and $D_i = \xi(\mathcal{D}_{R,i})$ be the image divisors in $E_R$. Then

\[ \xi^{-1}(A) = \mathcal{A}_R, \quad \xi^{-1}(B) = \mathcal{B}_R, \quad \text{and} \quad \xi^{-1}(D_i) = \mathcal{F}_0 + \mathcal{D}_{R,i}. \]
Further, since $\mathcal{O}_{\mathcal{F}_\nu}(\mathcal{D}_{R,i} + \mathcal{F}_0) \cong \mathcal{O}_{\mathcal{F}_\nu}$, the cokernel of the inclusion

(4.10) $\pi_{R\ast}\mathcal{O}_{\mathcal{C}_R}(\mathcal{D}_{R,i} - \mathcal{B}_R + \mathcal{A}_R) \longrightarrow \pi_{R\ast}\mathcal{O}_{\mathcal{C}_R}(\mathcal{D}_{R,i} + \mathcal{F}_0 - \mathcal{B}_R + \mathcal{A}_R)$

is $\pi_{R\ast}\mathcal{O}_{\mathcal{F}_0}(\mathcal{D}_{R,i} + \mathcal{F}_0)$, which is isomorphic to $k(p)$. Therefore, since $\varphi_{R,i}|_p = 0$, $\varphi_{R,i}$ factors through a homomorphism $\phi$ as shown in the commutative diagram

$$
\begin{array}{ccc}
\pi_{R\ast}\mathcal{O}_{\mathcal{C}_R}(\mathcal{D}_{R,i} + \mathcal{F}_0 - \mathcal{B}_R + \mathcal{A}_R) & \longrightarrow & \pi_{R\ast}\mathcal{O}_{\mathcal{A}}(A) \cong \mathcal{O}_R \\
\pi_{R\ast}\mathcal{O}_{\mathcal{E}_R}(\mathcal{D}_i - \mathcal{B} + \mathcal{A}) & \longrightarrow & \pi_{R\ast}\mathcal{O}_{\mathcal{A}}(A) \cong \mathcal{O}_R.
\end{array}
$$

Since the lower horizontal arrow is an isomorphism, $\phi$ is an isomorphism. Combined with that the cokernel of (4.10) is $k(p)$, this proves that $\varphi_{R,i}$ has precisely order one vanishing at $p \in R$. $\Box$

For the convenience of reference, we record an immediate consequence of Proposition 4.13.

Corollary 4.14. There are trivializations $\rho_{\ast}\mathcal{M}_i \cong \mathcal{O}_V$ and $\rho_{\ast}\mathcal{O}_A(A) \cong \mathcal{O}_V$ such that the homomorphism $\varphi$ is given by

(4.11) $\bigoplus_{i=1}^{m} \varphi_i : \bigoplus_{i=1}^{m} \rho_{\ast}\mathcal{M}_i \longrightarrow \rho_{\ast}\mathcal{O}_A(A), \quad \varphi_i = \zeta_{[\delta_i, a]}$.

4.15. The homomorphism (4.11) can be further simplified. Recall that $V$ is a neighborhood of $(C, D) \in \mathcal{D}_m^1 \subset \mathcal{D}_1$. The pair $(C, D)$ induces a weighted curve $(C, w)$ with $w = c_1(D)$. We let $\gamma$ be the terminally weighted tree of $(C, w)$ with terminal vertices

$$
\text{Ver}(\gamma)^t = \{1, \cdots, \ell\}.
$$

According to our convention, each non-root vertex $v \in \text{Ver}(\gamma)$ corresponds to a connected subcurve $C_v \subset C$ (c.f. §3.4); on the subcurve $C_v$ there is a unique separating node $q$ of $C$ that separates $C_v$ and the remainder part $C - C_v$. We call this node $q$ the node associated to $v$. With such node identified, for each vertex $v$ we define

$$
\zeta_v = \zeta_q \in \Gamma(\mathcal{O}_V),
$$

where $q$ the associated node of $v$. For any terminal vertex $i \in \text{Ver}(\gamma)^t$, we let

$$
\zeta_{[i, o]} = \prod_{i \geq v > o} \zeta_v.
$$

Theorem 4.16. The direct image sheaf $\rho_{\ast}\mathcal{L}$ is a direct sum of $\mathcal{O}_V^{\oplus (m-\ell+1)}$ with the kernel sheaf of the homomorphism

(4.12) $\bigoplus_{i=1}^{\ell} \varphi_i : \mathcal{O}_V^{\oplus \ell} \longrightarrow \mathcal{O}_V, \quad \varphi_i = \zeta_{[i, o]}$. 
Proof. We express $D$ as $\sum_{j=1}^{m} \delta_j$, and continue to denote by $v_1, \ldots, v_\ell$ the terminal vertices of $\gamma$. By our construction of the terminaly weighted tree $\gamma$ of the weighted curve $(C,w)$ of the pair $(C,D)$ (cf. [3.4]), each $v_i$ is associated to a connected tree $C_{v_i}$ of rational curves; there is a unique irreducible component $D_{v_i}$ of $C_{v_i}$ closest to the core of $C$. Again, by the construction of $\gamma$, at least one of $\{\delta_j\}_{j=1}^{m}$ lies on $D_{v_i}$; we pick one and index it by $\delta_i$.

Further, for every $\delta_j$, $1 \leq j \leq m$, there is $1 \leq i \leq \ell$, such that $\delta_i$ is between the point $a$ (of the core curve of $C$) and $\delta_j$ ($\delta_i$ and $\delta_j$ can be on the same irreducible component). This shows that $\zeta[\delta_i,a] | \zeta[\delta_j,a]$; thus, every $\zeta[\delta_j,a]$ is divisible by one of $\zeta[\delta_1,a], \ldots, \zeta[\delta_\ell,a]$.

Thus in the expression (4.11), we can choose a new basis for $\bigoplus \rho_\ast \mathcal{M}_j$ so that with respect to the new trivialization $\bigoplus_{j=1}^{m} \rho_\ast \mathcal{M}_j \cong \mathcal{O}_\mathcal{V}^{\oplus m}$, the homomorphism $\varphi = \bigoplus_{i=1}^{\ell} \varphi_i$ has the form

$$\varphi = \bigoplus_{i=1}^{\ell} \varphi_i \oplus 0 : \mathcal{O}_\mathcal{V}^{\oplus m} \rightarrow \mathcal{O}_\mathcal{V}.$$ 

Together with Lemma 4.10 and Corollary 4.14, this proves the theorem. \(\square\)

5. Local Equations of $\overline{M}_1(P^n,d)$ and its Desingularization

In this section, we prove the theorems stated in §2. In the meantime, we describe local defining equations for $\overline{M}_1(P^n,d)$ in terms of weighted trees.

5.1. Proof of Theorem 2.16

Recall that in Theorem 2.16, for any $[u,C] \in \overline{M}_1(P^n,d)$ we first pick a small open subset $[u,C] \in U \subset \overline{M}_1(P^n,d)$ and a homogeneous coordinate $[x_0, \ldots, x_n]$ of $P^n$ so that the pull back divisor $S = f^{-1}[x_0 = 0]$ is a family of simple divisors on the domain family $X$ of the universal family $f : X \rightarrow P^n$ of $U$; the family $X$ coupled with the divisor $f^{-1}[x_0 = 0]$ defines a tautological morphism $U \rightarrow \mathcal{D}_1$ of $\mathcal{D}_1$. We next pick a smooth chart $\mathcal{V} \rightarrow \mathcal{D}_1$ so that its image contains the image of $U \rightarrow \mathcal{D}_1$. Let $\mathcal{U} = \mathcal{V} \times_{\mathcal{D}_1} U$.

**Theorem 2.16** There is a canonical open immersion $\mathcal{U} \rightarrow (F = 0) \subset \mathcal{E}_\mathcal{V}$.

**Proof.** We continue with the notation introduced in §2. For instance, $(\mathcal{C}, \mathcal{D})$ is the tautological family over $\mathcal{V}$. We set

$$X' = X \times_{\mathcal{U}} \mathcal{U}, \quad D' = S \times_{\mathcal{U}} \mathcal{U}.$$ 

By the universality of $\mathcal{D}_1$,

$$X' = \mathcal{C} \times_{\mathcal{V}} \mathcal{U}, \quad D' = \mathcal{D} \times_{\mathcal{V}} \mathcal{U}.$$
We use $\alpha$ and $\tilde{\alpha}$ to denote the induced horizontal maps in the square

\[
\begin{array}{ccc}
X' & \xrightarrow{\tilde{\alpha}} & C \\
\pi' & \downarrow & \rho \\
U & \xrightarrow{\alpha} & V.
\end{array}
\]

Likewise, we denote $\mathcal{L}' = \mathcal{O}_{X'}(D')$ and $\mathcal{L} = \mathcal{O}_C(D)$. Then $\mathcal{L}' = \tilde{\alpha}^* \mathcal{L}$.

We now construct the promised open immersion

\[
\mu : U \longrightarrow \mathcal{E}_V.
\]

We let $f'$ be the composition of the projection $X' \rightarrow X$ with $f : X \rightarrow \mathbb{P}^n$; let $s_i = f'^*(x_i)$, $0 \leq i \leq n$, be the pull back sections in $\Gamma(\pi'_* \mathcal{L}')$. According to our convention, $s_0 = f'^*(x_0)$ is the section 1 induced by the inclusion $\mathcal{O}_{X'} \subset \mathcal{L}' = \mathcal{O}_{X'}(D')$. This way, all other $s_i$, $i \geq 1$, are canonically defined.

As mentioned in [14.5] we have a section $A$ of $\mathcal{C}/\mathcal{V}$. Its pull back section in $X'$ is $A' = A \times_V U$. Because $R^1 \rho_* \mathcal{L}(A) = 0$, by the cohomology and base change theorem, we have canonical isomorphism

\[
\alpha^* \rho_* \mathcal{L}(A) \cong \pi'_* \mathcal{L}'(A').
\]

We let $\iota : \pi'_* \mathcal{L}' \longrightarrow \pi'_* \mathcal{L}'(A') \cong \alpha^* \rho_* \mathcal{L}(A)$ be the inclusion. Then $\iota(s_i)$ is a section of $\alpha^* \rho_* \mathcal{L}(A)$. On the other hand, since $\mathcal{E}_V$ is the vector bundle $\rho_* \mathcal{L}(A)^{\oplus n}$, defining a $\mathcal{V}$-morphism $\mu : U \rightarrow \mathcal{E}_V$ is equivalent to giving a section of the pull back sheaf $\alpha^* \mathcal{E}_V = \alpha^* \rho_* \mathcal{L}(A)^{\oplus n}$.

We define the morphism $\mu$ in [5.2] to be the one induced by the homomorphism

\[
\mu(s) = (\iota(s_1), \cdots, \iota(s_n)) : \mathcal{O}_U \longrightarrow \alpha^* \rho_* \mathcal{L}(A)^{\oplus n}.
\]

To complete the proof, we need to show that $\mu$ factors through $(F = 0) \subset \mathcal{E}_V$ and the factored morphism $\mu' : U \rightarrow (F = 0)$ is an open immersion.

We first check that $\mu$ factors. By definition, $\mu$ factors if the pull back $\mu^*(F) \equiv 0$. Let $p : \mathcal{E}_V \rightarrow \mathcal{V}$ be the projection. By definition, $F$ is the composite

\[
F : \mathcal{O}_{\mathcal{E}_V} \xrightarrow{1} p^* \rho_* \mathcal{L}(A)^{\oplus n} \xrightarrow{r} p^* \rho_* (\mathcal{L}(A)^{\oplus n} |_A),
\]

where $1$ is the tautological section and $r$ is the restriction homomorphism. Therefore, $\mu^*(F)$ is the composite

\[
\mathcal{O}_U \xrightarrow{\mu^*(1)} \alpha^* \rho_* \mathcal{L}(A)^{\oplus n} \xrightarrow{\mu^*(r)} \alpha^* \rho_* (\mathcal{L}(A)^{\oplus n} |_A).
\]

Since $\alpha = p \circ \mu$, $\mu^*(1)$ is the composite

\[
\mathcal{O}_U \xrightarrow{(s)} \alpha^* \rho_* \mathcal{L}^{\oplus n} \xrightarrow{\iota} \alpha^* \rho_* \mathcal{L}(A)^{\oplus n}.
\]

Therefore $\mu^*(F)$ is the composite

\[
\mathcal{O}_U \xrightarrow{(s)} \alpha^* \rho_* \mathcal{L}^{\oplus n} \xrightarrow{\iota} \alpha^* \rho_* \mathcal{L}(A)^{\oplus n} \xrightarrow{\mu^* r} \mu^* p^* \rho_* (\mathcal{L}(A)^{\oplus n} |_A).
\]
Since \( \mu^*(r) \circ \iota = 0 \), we get \( \mu^*(F) = 0 \). This proves that \( \mu \) factors through
\[
\mu' : \mathcal{U} \longrightarrow (F = 0) \subset \mathcal{E}_V.
\]

We next prove that \( \mu' \) is an open immersion. We let \( Z = (F = 0) \subset \mathcal{E}_V \) and let \( \tau : Z \longrightarrow \mathcal{E}_V \) be the tautological immersion. Because \( \mathcal{E}_V \) is the total space of the vector bundle \( \rho_* \mathcal{L}(\mathcal{A})^{\oplus n} \) on \( \mathcal{V} \), the morphism \( \tau \) is equivalent to giving a section (homomorphism)
\[
s_\tau : \mathcal{O}_Z \longrightarrow \tau^* p^* \rho_* \mathcal{L}(\mathcal{A})^{\oplus n}.
\]
At the same time, \( \tau^*(F) = 0 \) is equivalent to the vanishing of the composite of the homomorphisms:
\[
(5.5) \quad \mathcal{O}_Z \xrightarrow{s_\tau} \tau^* p^* \rho_* \mathcal{L}(\mathcal{A})^{\oplus n} \xrightarrow{\tau^*(r)} \tau^* p^* \rho_* (\mathcal{L}(\mathcal{A})^{\oplus n}|_\mathcal{A}).
\]
We remark that because \( \alpha = p \circ \tau \circ \mu' \), by the universality property of \( \mathcal{E}_V \),
\[
(5.6) \quad \mu^*(s_\tau) = \iota(s) : \mathcal{O}_U \longrightarrow \mu^* \tau^* p^* \rho_* \mathcal{L}(\mathcal{A})^{\oplus n} \equiv \alpha^* p_* \mathcal{L}(\mathcal{A})^{\oplus n}.
\]
To continue, we will show that the vanishing \([5.5]\) provides us a family of stable morphisms parameterized by an open subset of \( Z \) that contains \( \mu'(\mathcal{U}) \). We let
\[
\mathcal{C}_Z = \mathcal{C} \times_\mathcal{V} Z, \quad \mathcal{D}_Z = \mathcal{D} \times_\mathcal{V} Z, \quad \mathcal{A}_Z = \mathcal{A} \times_\mathcal{V} Z, \quad \text{and} \quad \mathcal{L}'' = \mathcal{O}_{\mathcal{C}_Z}(\mathcal{D}_Z).
\]
Because \( R^1 \rho_* \mathcal{L}(\mathcal{A}) = 0 \), by the cohomology and base change theorem, we have the canonical identity
\[
\Gamma(Z, \tau^* p^* \rho_* \mathcal{L}(\mathcal{A})^{\oplus n}) = \Gamma(\mathcal{C}_Z, \mathcal{L}''(\mathcal{A}_Z)^{\oplus n}).
\]
This identity transforms \([5.5]\) into
\[
(5.7) \quad \mathcal{O}_{\mathcal{C}_Z} \xrightarrow{s''} \mathcal{L}''(\mathcal{A}_Z)^{\oplus n} \xrightarrow{\rho''} \mathcal{L}''(\mathcal{A}_Z)^{\oplus n}|_{\mathcal{A}_Z}.
\]
Because \( \rho'' \circ s'' \) is zero and the kernel of the second arrow is \( \mathcal{L}''^{\oplus n} \), \( s'' \) factors through a unique homomorphism
\[
(5.8) \quad s'' = (s''_1, \ldots, s''_n) : \mathcal{O}_{\mathcal{C}_Z} \longrightarrow \mathcal{L}''^{\oplus n}.
\]
Let \( s''_0 \) be the section 1 of \( \mathcal{O}_{\mathcal{C}_Z} \subset \mathcal{L}''^n = \mathcal{O}_{\mathcal{C}_Z}(\mathcal{D}_Z) \). The \( (n+1) \) sections \( s''_0, \ldots, s''_n \) considered as sections of \( \mathcal{L}'' \) on \( \mathcal{C}_Z \) define a morphism
\[
(5.9) \quad \left[ s''_0, \ldots, s''_n \right] : \mathcal{C}_Z \setminus \{ s''_0 = \cdots = s''_n = 0 \} \longrightarrow \mathbb{P}^n.
\]
To analyze the domain of this morphism, we notice that due to \([5.6]\), the morphism
\[
(5.10) \quad \left[ s_0, \ldots, s_n \right] = \left[ s''_0, \ldots, s''_n \right] \circ \tilde{\mu}' : \mathcal{X}' = \mathcal{X} \times_\mathcal{U} \mathcal{U} = \mathcal{C} \times_\mathcal{V} \mathcal{U} \longrightarrow \mathbb{P}^n,
\]
where \( \tilde{\mu}' \) is the lift of \( \mu' \) to \( \mathcal{X}' \). Therefore
\[
\mathcal{C}_Z \times_\mathcal{Z} \mu'(\mathcal{U}) \subset \mathcal{C}_Z \setminus \{ s''_0 = \cdots = s''_n = 0 \}.
\]
Because \( \mathcal{C}_Z \rightarrow Z \) is proper, there is an open \( W \subset Z \) containing \( \mu'(\mathcal{U}) \) so that
\[
\mathcal{C}_W = \mathcal{C}_Z \times_\mathcal{Z} W \subset \mathcal{C}_Z \setminus \{ s''_0 = \cdots = s''_n = 0 \}.
\]
We let
\[ f_W : C_W \rightarrow \mathbb{P}^n \]
be the restriction of \((5.9)\) to \(C_W\). Finally, because restricting to \(\mu'(U)\) this morphism is a family of stable morphisms, possibly after shrinking \(W \supset \mu'(U)\) if necessary, \(f_W\) is a family of stable morphisms.

We let
\[ \eta : W \rightarrow \overline{M}_1(\mathbb{P}^n, d) \]
be the tautological morphism induced by the family \(f_W\). Because of the identity \((5.10)\), the composite of \(\mu' : U \rightarrow W\) with \(\eta : W \rightarrow \overline{M}_1(\mathbb{P}^n, d)\) is identical to the projection \(U = V \times D_1 U \rightarrow U \subset \overline{M}_1(\mathbb{P}^n, d)\). Therefore, if we let
\[ W_0 = \eta^{-1}(U), \]
\(W_0 \subset Z\) is open and \(\mu'\) factor through \(\mu'' : U \rightarrow W_0\).

We claim that, with \(W_0 \subset Z\) endowed with the open subscheme structure of \(Z\), the morphism \(\mu''\) is an isomorphism. To prove this, we will construct the inverse of \(\mu''\). Let \(\eta'' : W_0 \rightarrow U\) be the morphism induced by \(\eta\). Because the composite \(\eta'' : W_0 \rightarrow U\) with the tautological \(U \rightarrow D_1\) is identical to the composite of \(p \circ \tau : W_0 \rightarrow E_V \rightarrow V\) with \(V \rightarrow D_1\), the pair \((\eta'', p \circ \tau)\) lifts to a morphism
\[ \zeta'' : W_0 \rightarrow U = V \times D_1 U. \]
Because of the identity \((5.10)\), the composite \(\zeta'' \circ \mu''\) is identical to the projection \(U \rightarrow U\). This implies that \(\zeta'' \circ \mu'' = \text{id}_U\). On the other hand, \(\mu'' \circ \zeta'' : W_0 \rightarrow E_V\) is exactly the inclusion \(W_0 \rightarrow E_V\), again due to the identity \((5.10)\), therefore \(\mu'' \circ \zeta'' = \text{id}_{W_0}\). Thus \(\mu''\) is an isomorphism. This proves the theorem. \(\square\)

### 5.2. Local defining equations of \(\overline{M}_1(\mathbb{P}^n, d)\) restated.

#### 5.2.

The local equation \(F = 0\) of Theorem 2.16 admits an elegant form in terms of the terminally weighted tree \(\gamma \in \Lambda_d\) of the associated weighted curves which we now describe.

#### 5.3.

Given a terminally weighted tree \(\gamma\), there are three equivalent ways to describe the local equation near the stratum \(\overline{M}_1(\mathbb{P}^n, d)_{\gamma}\).

The first is in a direct form: to every non-root vertex \(a \in \text{Ver}(\gamma)^*\), we associate the coordinate function of \(A^1\) indexed by \(a\): \(z_a \in \Gamma(\mathcal{O}_{\gamma})\). To a terminal vertex \(b \in \text{Ver}(\gamma)^t\), we associate \(n\) coordinate functions \(w_{b,1}, \ldots, w_{b,n} \in \Gamma(\mathcal{O}_{\gamma})\). We then set
\[ \Phi_{\gamma} = (\Phi_{\gamma,1}, \ldots, \Phi_{\gamma,n}), \quad \Phi_{\gamma,e} = \sum_{b \in \text{Ver}(\gamma)^t} z_{[b,a]} w_{b,e}, \quad z_{[b,a]} = \prod_{b \geq a > 0} z_a. \]
We make a convention that if \(\gamma = o\), we define \(\Phi_{o,e} = w_e\) and hence \(\Phi_o = (w_1, \ldots, w_n)\).
The second is by induction on $\gamma \neq 0$. If $a$ is a terminal vertex, we set $\Phi_{a,e} = w_{a,e}$ with $w_{a,e}$ as before. If $\gamma = o[\gamma_1, \cdots, \gamma_j]$ with $\gamma_i \in \Lambda_d$, having roots $v_i$, then set

$$\Phi_{\gamma,e} = z_{v_1}\Phi_{\gamma_1,e} + \cdots + z_{v_k}\Phi_{\gamma_j,e}.$$ 

The third is in terms of the bracket representation of $\gamma$. Each $\Phi_{\gamma,e}$ is derived from $\gamma$ by dropping the root $o$, replacing each ghost vertex $a$ by its associated function $z_a$, replacing each terminal vertex $b$ by $z_b w_{b,e}$, replacing “,” by “+”, and replacing “$[\cdots]$” by “$\{\cdots\}$”. The resulting expression $\Phi_{\gamma,e}$ is identical to the one from the second method.

**Example 5.4.** Take $\gamma = o[a,b[c,d]]$ (see the first tree in Figure 1). The domain of a generic $f$ in $\overline{\mathcal{M}}_1(\mathbb{P}^n, d)_\gamma$ has a genus-1 ghost component labelled by the root $a$, a genus zero ghost component labelled by $b$, and three rational tails labelled by $a$, $c$ and $d$, respectively. The tails $a$ are attached to the genus 1 ghost component. Tails $c$ and $d$ are attached to the rational component $b$. By the first approach,

$$\Phi_{\gamma,e} = z_a w_{a,e} + z_b z_c w_{c,e} z_b + z_b z_d w_{d,e} z_b.$$ 

By the second,

$$\Phi_{\gamma,e} = z_a w_{a,e} + z_b \Phi_{b[c,d]} = z_a w_{a,e} + z_b (z_c w_{c,e} + z_d w_{d,e}).$$ 

By the third,

$$\Phi_{\gamma,e} = z_a w_{a,e} + z_b (z_c w_{c,e} + z_d w_{d,e}).$$ 

**5.5.** We now describe the local model of the singularity type of $\overline{\mathcal{M}}_1(\mathbb{P}^n, d)$ near $\overline{\mathcal{M}}_1(\mathbb{P}^n, d)_\gamma$. We let

$$V_\gamma = \prod_{a \in \text{Ver}(\gamma)^*} \mathbb{A}^1 \cong \mathbb{A}^h \quad \text{and} \quad E_\gamma = V_\gamma \times (\prod_{b \in \text{Ver}(\gamma)^t} \mathbb{A}^1)^\times n \cong \mathbb{A}^{h+nt},$$

where $h$ (resp. $t$) is the cardinality of $\text{Ver}(\gamma)^*$ (resp. $\text{Ver}(\gamma)^t$). The expressions $\Phi_{\gamma,e}$ then become regular functions on $E_\gamma$ after we identify $z_a$ with the coordinate function of the $a$-th copy of $\prod_{a \in \text{Ver}(\gamma)^*} \mathbb{A}^1$, and identity $w_{b,e}$ with the coordinate function of the $b$-th copy of $\prod_{b \in \text{Ver}(\gamma)^t} \mathbb{A}^1$ in the $e$-th component of the product $(\cdot)^\times n$.

We define

$$Z_\gamma = \{(z_a, w_{b,e}) \in E_\gamma \mid \Phi_{\gamma,e}(z, w) = 0, \ 1 \leq e \leq n\}.$$ 

We then define the type $\gamma$ loci in $Z_\gamma$ to be

$$Z_\gamma^0 = \{(z, w) \in Z_\gamma \mid z_a = 0 \text{ for all } a \in \text{Ver}(\gamma)^*\}.$$ 

**Definition 5.6.** We say a DM-stack $S$ has singularity type $\gamma$ at a closed point $s \in S$ if there is a scheme $y \in Y$ and two smooth morphisms $q_1 : Y \to S$ and $q_2 : Y \to Z_\gamma$ such that $q_1(y) = s$ and $q_2(y) \in Z_\gamma^0$.

We have
**Theorem 5.7.** The stack $\overline{M}_1(\mathbb{P}^n, d)$ has singularity type $\gamma$ along $\overline{M}_1(\mathbb{P}^n, d)_\gamma$.

**Proof.** Let $[u, C] \in \overline{M}_1(\mathbb{P}^n, d)$ be a closed point with associated terminally weighted rooted tree $\gamma$. We let $U = \mathcal{V} \times_{\mathcal{V}_1} U \to \mathcal{V}$ be as in Theorem 2.16.

Theorem 4.16 provides trivializations $\rho^* \mathcal{L}(A) \cong (\bigoplus_{b \in \text{Ver}(\gamma)} \mathcal{O}_V) \oplus \mathcal{O}_V^{(d-\ell+1)}$ so that the restriction homomorphism $r : \rho^* \mathcal{L}(A) \longrightarrow \rho^* (\mathcal{L}(A)|_A)$ is given by

$$
\bigoplus_{b \in \text{Ver}(\gamma)} \zeta_{[b,o]} \oplus 0 : \left( \bigoplus_{b \in \text{Ver}(\gamma)} \mathcal{O}_V \right) \oplus \mathcal{O}_V^{(d-\ell+1)} \longrightarrow \mathcal{O}_V.
$$

The composite homomorphism $\rho^* \mathcal{L}(A)^{\oplus n} \xrightarrow{\cong} \left( \left( \bigoplus_{b \in \text{Ver}(\gamma)} \mathcal{O}_V \right) \oplus \mathcal{O}_V^{(\ell+1)} \right)^{\oplus n} \xrightarrow{\text{pr}} \left( \bigoplus_{b \in \text{Ver}(\gamma)} \mathcal{O}_V \right)^{\oplus n}$ induces a morphism

$$
\mathcal{E}_V \longrightarrow \mathcal{V} \times \prod_{b \in \text{Ver}(\gamma)} \mathbb{A}^1;
$$

the regular functions $\zeta_a$ define a morphism

$$
\phi = \prod_{a \in \text{Ver}(\gamma)^*} \zeta_a : \mathcal{V} \longrightarrow \left( \prod_{a \in \text{Ver}(\gamma)^*} \mathbb{A}^1 \right) = V_\gamma.
$$

Together, they define a morphism

$$
\tilde{\phi} : \mathcal{E}_V \longrightarrow \mathcal{V} \times \prod_{b \in \text{Ver}(\gamma)^*} \mathbb{A}^1 \longrightarrow E_\gamma = V_\gamma \times \prod_{b \in \text{Ver}(\gamma)^*} \mathbb{A}^1.
$$

We comment that since deformations of nodal curves are unobstructed, the morphisms $\phi$ and $\tilde{\phi}$ are smooth.

By Theorem 4.16,

$$
\tilde{\phi}^*(\Phi_\gamma) = F.
$$

This proves that $\tilde{\phi}|_{(F=0)} : (F = 0) \longrightarrow Z_\gamma$ is smooth, since $\tilde{\phi}$ is smooth.

Finally, because $U \to \overline{M}_1(\mathbb{P}^n, d)$ is smooth and $U \to (F = 0) \subset \mathcal{E}_V$ is an open immersion, and thus smooth, the composite

$$
U \longrightarrow (F = 0) \subset \mathcal{E}_V \longrightarrow Z_\gamma \subset E_\gamma
$$

is smooth. Also, it is clear that a lift $\xi \in U$ of $[u] \in \overline{M}_1(\mathbb{P}^n, d)_\gamma$ is mapped to a point in $Z_\gamma$. This proves that $\overline{M}_1(\mathbb{P}^n, d)$ has singularity type $\gamma$ at $[u, C]$. □
5.3. A stratification of a blowing up of $Z_{\gamma}$.

5.8. For the purposes of keeping track of blowups of $\mathcal{M}_U(\mathbb{P}^n, d)$, we need to classify the singularity types of the blowups of $Z_{\gamma}$. Such types will be classified by simple weighted rooted trees that are monoidal transformations and collapsings of $\gamma$.

5.9. We first classify the singularity types of the space $Z_{\gamma}$ for a simple terminally weighted tree $\gamma$. The singularity type of a $x \in Z_{\gamma}$ is defined by its associated tree $\gamma_x$. Let $x \in Z_{\gamma}$ (resp. $\in V_{\gamma}$) and let $x = (z_a, w^j_0)$ (resp. $x = (z_a)$) be its coordinate representation. The non-vanishing of $z_a$ identifies a subset of $\text{Ver}(\gamma)^*$:

$$I_x = \{a \in \text{Ver}(\gamma)^* \mid z_a \neq 0\}.$$  

We let $\gamma_x$ be the collapsing of $\gamma$ at vertices in $I_x$.

**Lemma 5.10.** The scheme $Z_{\gamma}$ has singularity type $\gamma_x$ at $x \in Z_{\gamma}$.

**Proof.** This is a direct check. □

5.11. We now investigate the blowing up of $Z_{\gamma}$. Let

$$\Pi_{\gamma,e} = \{x \in V_{\gamma} \mid \text{the root of } \gamma_x \text{ is the branch point and } \text{br}(\gamma_x) = k\}.$$  

It is clear that $\Pi_{\gamma,e}$ is smooth and locally closed. In general, the closure $\Pi_{\gamma,k}$ of $\Pi_{\gamma,k}$ in $V_{\gamma}$ is quite complicated. However, in the case when $\text{br}(\gamma) \geq k$, $\Pi_{\gamma,k}$ is smooth and bears a simple description.

**Lemma 5.12.** Let $\gamma$ be a simple tree and $k \geq 2$. If $\text{br}(\gamma) = k$, then $\Pi_{\gamma,k}$ consists of all $x \in V_{\gamma}$ such that $\text{br}(\gamma_x) = k$. If $\text{br}(\gamma) = 0$ or $\text{br}(\gamma) > k$, then $\Pi_{\gamma,k} = \emptyset$.

**Proof.** Let $o < v_1 < \cdots < v_r$ be the trunk of $\gamma$ and let $a_1, \cdots, a_k$ be the direct-descendants of the branch point $v_r$. Then $\Pi_{\gamma,k}$ consists of those $x = (z_a)$ so that

$$z_{v_1} \neq 0, \cdots, z_{v_r} \neq 0; \quad z_{a_1} = 0, \cdots, z_{a_k} = 0.$$  

Its closure is given by the vanishing of all $z_{a_j}$:

$$\Pi_{\gamma,k} = \{z_{a_1} = \cdots = z_{a_k} = 0\}.$$  

By the definition of $\gamma_x$, $x \in \Pi_{\gamma,k}$ if and only if $\text{br}(\gamma_x) = k$. The second claim is clear. □

5.13. We now consider the blowup of $V_{\gamma}$ along $\Pi_{\gamma,k}$ in the case $\text{br}(\gamma) = k$. We let

$$\gamma = \overline{ovr[\gamma_1, \cdots, \gamma_k]} = o[v_1 \cdots [v_r[\gamma_1, \cdots, \gamma_k] \cdots]],$$  

and let $a_i$ be the root of $\gamma_i$; thus, $a_1, \cdots, a_k$ are the direct descendants of the branch point $v_r$ of $\gamma$. Accordingly,

$$\Phi_{\gamma,e}(z, w) = (z_{v_1} \cdots z_{v_r})(z_{a_1} \Phi_{\gamma_1,e} + \cdots + z_{a_k} \Phi_{\gamma_k,e}).$$
We denote the blowup of $V_{\gamma}$ along $\Pi_{\gamma,k}$ by $V_{\gamma,[k]}$ and define

$$Z_{\gamma,[k]} = Z_{\gamma} \times V_{\gamma,[k]}.$$  

The scheme $Z_{\gamma,[k]}$ is a subscheme of $Z_{\gamma} \times \mathbb{P}^{k-1}$ defined by the equations $z_{aj}u_i = z_{ai}u_j$ for $1 \leq i, j \leq k$. (Here $[u_1, \ldots, u_k]$ is the homogeneous coordinate on $\mathbb{P}^{k-1}$.) In the affine open subset $\{u_i = 1\}$, we have $z_{aj} = z_{ai}u_j$ for $j \neq i$. Thus, over this chart, $Z_{\gamma,[k]}$ is defined by

$$z_1 \cdots z_r \cdot z_{ai} \left( \Phi_{\gamma,e} + \sum_{j \neq i} u_j \Phi_{\gamma_j,e} \right) = 0, \quad 1 \leq e \leq n. \tag{5.18}$$

There are two cases. If $\gamma_i$ is a single-vertex tree, then (5.18) becomes

$$z_1 \cdots z_r \cdot z_{ai} \left( w_{ai,e} + \sum_{j \neq i} u_j \Phi_{\gamma_j,e} \right) = 0, \quad 1 \leq e \leq n. \tag{5.19}$$

After introducing $\tilde{w}_{ai,e} = w_{ai,e} + \sum_{j \neq i} u_j \Phi_{\gamma_j,e}$, (5.19) becomes

$$z_1 \cdots z_r \cdot \tilde{w}_{ai,e} = 0, \quad 1 \leq e \leq n. \tag{5.20}$$

This is the system associated to the tree $\gamma' = \sigma_{v_r}[a_i]$, a path tree containing vertices $o, v_1, \ldots, v_r, a_i$. It is also the advancing $a_i$ in $\gamma$ and thus is in $\text{Mon}(\gamma)$. This shows that $Z_{\gamma,[k]} \cap \{u_i = 1\} \cong Z_{\gamma'}$.

If $\gamma_i$ is a nontrivial rooted tree and is of the form $\gamma_i = a_i[\gamma'_1, \ldots, \gamma'_m]$, where $m > 1$ because $\gamma$ is simple by assumption, the system (5.18) becomes

$$z_1 \cdots z_r \cdot z_{ai} \left( \sum_{s=1}^{m} z_{bs} \Phi_{\gamma'_s,e} + \sum_{j \neq i} u_j \Phi_{\gamma_j,e} \right) = 0, \quad 1 \leq e \leq n, \tag{5.21}$$

where $b_s$ is the root of $\gamma'_s$. After replacing $u_j$ by $z_{bj}$, (noticing that the system (5.21) does not contain the variables $z_{ai}$) this is the system associated to the tree $\gamma'$ that is obtained from $\gamma$ by advancing the vertex $a_i$. This shows that $Z_{\gamma,[k]} \cap \{u_i = 1\} \cong Z_{\gamma'}$.

The above yields the following statement.

**Lemma 5.14.** If $\gamma$ is a simple terminally weighted tree such that $\text{br}(\gamma) = k$, then the blowup $Z_{\gamma,[k]}$ of $Z_{\gamma}$ can be covered by open subsets isomorphic to $Z_{\gamma'}$ with $\gamma \in \text{Mon}(\gamma)$.

**5.15.** We remark here that in the case of (5.20), the vanishing locus

$$\tilde{w}_{ai,1} \cdots \tilde{w}_{ai,n} \cdot z_{a_1} \cdots z_{a_r} = 0$$

has normal crossing singularities.

**Example 5.16.** Consider $\gamma = o[a, b[e, d]]$. Then we have

$$\Phi_{\gamma,e} = z_a w_{a,e} + z_b (z_c w_{c,e} + z_d w_{d,e}), \quad 1 \leq k \leq n.$$ We blow up $Z_{\gamma}$ along the locus $\{z_a = z_b = 0\}$. The blow-up is a subspace of $Z_{\gamma} \times \mathbb{P}^1$ defined by equations $z_a u_b = z_b u_a$, where $[u_a, u_b]$ are the homogeneous
coordinates of $\mathbb{P}^1$. In the affine open subset $u_a = 1$, the equation $\Phi_{\gamma,e} = 0$ becomes

$$z_a(w_{a,e} + u_b(z_c w_{c,e} + z_d w_{d,e})) = 0;$$

it has normal crossing singularities, and the resulting system is associated to the tree $o[a]$, the advancing of $a$ of $o[a,b|c,d]$. In the affine open subset $u_b = 1$, the equation $\Phi_{\gamma,e} = 0$ becomes

$$z_b(u_a w_{a,e} + z_c w_{c,e} + z_d w_{d,e}) = 0;$$

this is the system associated to the tree $o[b|a,c,d]$, the advancing of $b$ of $o[a,b|c,d]$.

5.4. Local equations of $\widetilde{M}_1(\mathbb{P}^n, d)$.

5.17. We begin with recalling the notations and facts about the blown-up $\widetilde{M}_1(\mathbb{P}^n, d)$. In [2.5] we introduced $\Theta_k$ that is a smooth locally closed substack of $\mathcal{M}_1^{\text{wt}}$ of ghost core elliptic curves attached on $k$ distinct smooth points with $k$-rational tails; $\Theta_k$ is the closure of $\Theta_k$. In [2.6] we successively blow up $\mathcal{M}_1^{\text{wt}}$ along proper transforms of $\Theta_k$. Inductively, after obtaining $\mathcal{M}_1^{\text{wt}}_{1,[k-1]}$, we blow it up along the proper transform $\Theta_k_{[k-1]} \subset \mathcal{M}_1^{\text{wt}}_{1,[k-1]}$ of $\Theta_k \subset \mathcal{M}_1^{\text{wt}}$. We denote by $\Theta_k \subset \mathcal{M}_1^{\text{wt}}_{1,[k]}$ the proper transform of $\Theta_k$. We let $\mathcal{E}_{[k]} \subset \mathcal{M}_1^{\text{wt}}_{1,[k]}$ be the exceptional divisor of the $\mathcal{M}_1^{\text{wt}}_{1,[k]} \to \mathcal{M}_1^{\text{wt}}_{1,[k-1]}$. Finally, we denote $\mathcal{M}_1^{\text{wt}}$ the resulting limit stack.

As the image of $\widetilde{M}_1(\mathbb{P}^n, d) \to \mathcal{M}_1^{\text{wt}}$ is disjoint from $\Theta_k$ for $k > d$, the fiber product $\widetilde{M}_1(\mathbb{P}^n, d)$ can be defined after $d$-th blowing up of $\mathcal{M}_1^{\text{wt}}$:

$$\widetilde{M}_1(\mathbb{P}^n, d) = \overline{\mathcal{M}}_1(\mathbb{P}^n, d) \times_{\mathcal{M}_1^{\text{wt}}} \mathcal{M}_1^{\text{wt}}_{1,[d]};$$

To pave a way for our proof, we also need to record the intermediate blowup spaces. For this, we introduce for $k \geq 1$

$$\widetilde{M}_1(\mathbb{P}^n, d)_{[k]} = \overline{\mathcal{M}}_1(\mathbb{P}^n, d) \times_{\mathcal{M}_1^{\text{wt}}} \mathcal{M}_1^{\text{wt}}_{1,[k]}.$$

5.18. Recall that $\Lambda_d$ is the index set for the canonical stratification $\overline{\mathcal{M}}_1(\mathbb{P}^n, d) = \bigcup_{\gamma \in \Lambda_d} \overline{\mathcal{M}}_1(\mathbb{P}^n, d)_\gamma$. We set $\Lambda_d, [1] = \Lambda_d$ and defined $\Lambda_d, [k]$ inductively for $k \geq 2$ in [3.3]

Lemma 5.19. To each closed point $s \in \widetilde{M}_1(\mathbb{P}^n, d)_{[k]}$ we can find a graph $\gamma \in \Lambda_{d,[k]}$ and a smooth morphism $q_{\gamma,1} : W_\gamma \rightarrow \widetilde{M}_1(\mathbb{P}^n, d)_{[k]}$ whose image contains $s$ of which the following holds:

(i). there are smooth chart $\mathcal{V}_\gamma \to \mathcal{M}_1^{\text{wt}}_{1,[k]}$ and smooth morphisms $q_{\gamma,2}, \phi_\gamma$, $\psi_\gamma$, and $p_\gamma$ shown below, making the diagram commutative

$$
\begin{array}{ccc}
Z_\gamma & \xleftarrow{q_{\gamma,2}} & W_\gamma & \xrightarrow{q_{\gamma,1}} & \widetilde{M}_1(\mathbb{P}^n, d)_{[k]} \\
\downarrow \psi_\gamma & & \downarrow p_\gamma & & \\
\mathcal{V}_\gamma & \xleftarrow{\phi_\gamma} & \mathcal{V}_\gamma & \longrightarrow & \mathcal{M}_1^{\text{wt}}_{1,[k]};
\end{array}
$$


(ii) for any \( i \geq k + 1 \),

\[
W_\gamma \times \mathfrak{M}_{1,[k]} \Theta_i = W_\gamma \times V_\gamma \Pi_{\gamma,i}.
\]

**Proof.** The proof is by induction on \( k \). In the case \( k = 1 \), the desired morphisms are provided in the proof of Theorem 5.7. Suppose \( k \geq 2 \) and the lemma holds for a closed point \( s \in \tilde{M}_1(\mathbb{P}^n, d)[k-1] \). Let \( W_\gamma \) etc., be the corresponding data provided by the statement of the lemma.

If \( \text{br}(\gamma) = 0 \) or \( \text{br}(\gamma) > k \), then \( \gamma \in \Lambda_{d,[k+1]} \), while by (ii) with \( k \) replaced by \( k - 1 \) and Lemma 5.12

\[
W_\gamma \times \mathfrak{M}_{1,[k-1]} \Theta_k = \emptyset.
\]

Thus, \( s \) does not lies over the blowing up center of \( \mathfrak{M}_{1,[k-1]} \), and is in \( \tilde{M}_1(\mathbb{P}^n, d)[k] \). By shrinking \( W_\gamma \) if necessary, the morphism \( q_{\gamma,1} \) (provided by the inductive assumption) lifts to \( W_\gamma \to \tilde{M}_1(\mathbb{P}^n, d)[k] \) and the \( k \)-version of the statements (i) and (ii) are identical to its \((k-1)\)-version.

If \( \text{br}(\gamma) = k \), we define \( V_{\gamma,[k]} \) to be the fiber product using the right square of (5.22) (see below). By the property (ii) and the construction of the blowups \( \mathfrak{M}_{1,[k]} \to \mathfrak{M}_{1,[k-1]} \) and \( V_{\gamma,[k]} \to V_\gamma \), there is a unique \( \phi_{\gamma,[k]} \) making the left square a fiber product:

\[
\begin{array}{ccc}
V_{\gamma,[k]} & \xleftarrow{\phi_{\gamma,[k]}} & V_{\gamma,[k]} \\
\downarrow & & \downarrow \\
V_\gamma & \xleftarrow{\phi_{\gamma}} & V_\gamma
\end{array}
\]

(5.22)

This shows that

\[
W_\gamma \times \mathfrak{M}_{1,[k-1]} \mathfrak{M}_{1,[k]} = W_\gamma \times V_\gamma V_{\gamma,[k]},
\]

which is smooth over \( Z_{\gamma,[k]} = Z_\gamma \times V_\gamma V_{\gamma,[k]} \). Therefore, by Lemma 5.14

\[
W_\gamma \times \mathfrak{M}_{1,[k-1]} \mathfrak{M}_{1,[k]} \text{ is covered by smooth morphisms}
\]

\[
W_\gamma \times Z_\gamma Z_{\gamma'} \longrightarrow W_\gamma \times \mathfrak{M}_{1,[k-1]} \mathfrak{M}_{1,[k]}
\]

with \( \gamma' \in \text{Mon}(\gamma) \subset \Lambda_{d,[k]} \) that satisfy the statements of the lemma. Finally, because

\[
\tilde{M}_1(\mathbb{P}^n, d)[k] = \tilde{M}_1(\mathbb{P}^n, d)[k-1] \times \mathfrak{M}_{1,[k-1]} \mathfrak{M}_{1,[k]},
\]

the lemma follows. \( \square \)

**5.20.** We now describe a stratification of \( \tilde{M}_1(\mathbb{P}^n, d) \). We define an equivalence relation on \( \Lambda_{d,[k]} \) by demanding that \( \gamma \sim \gamma' \) if \( \gamma \) is isomorphic to \( \gamma \) as rooted but unweighted trees. Note that \( W_\gamma \) and \( W_{\gamma'} \) has isomorphic germs at their origin if and only if \( \gamma \sim \gamma' \). For \( \gamma \in \Lambda_{d,[k]} \), we denote by \([\gamma]\) the equivalence class of \( \gamma \) in \( c \).
For any point $\xi \in \tilde{M}_1(\mathbb{P}^n, d)_{[k]}$, we let

$$(\eta, W_\gamma) \rightarrow (\xi, \tilde{M}_1(\mathbb{P}^n, d)_{[k]}) \quad \text{and} \quad (\eta, W_\gamma) \rightarrow (x, Z_\gamma)$$

be the smooth morphisms provided by the previous lemma. We define $[\gamma_\xi] = [\gamma_\xi]$. By the comment in the previous paragraph, the equivalence class $[\gamma_\xi]$ is independent of the choice of the chart covering $\xi$. Therefore, we can define

$$\tilde{M}_1(\mathbb{P}^n, d)_{[\gamma], [k]} = \{\xi \in \tilde{M}_1(\mathbb{P}^n, d)_{[k]} \mid [\gamma_\xi] = [\gamma]\}.$$  

In general, $\tilde{M}_1(\mathbb{P}^n, d)_{[\gamma], [k]}$ is a disjoint union of closed substacks.

It follows immediately from the previous lemma that

**Lemma 5.21.** The stack $\tilde{M}_1(\mathbb{P}^n, d)_{[k]}$ has a stratification $\bigsqcup \tilde{M}_1(\mathbb{P}^n, d)_{[\gamma], [k]}$ indexed by $[\gamma] \in \Lambda_{d,[k]} / \sim$ such that it has singularity type $[\gamma]$ along stratum $\tilde{M}_1(\mathbb{P}^n, d)_{[\gamma], [k]}$.

By Lemma 3.14, $\Lambda_{d,[d]}$ consists of only path trees. Note that in this case, $\gamma \sim \gamma'$ if and only if $\gamma = \gamma'$. Hence we have

**Theorem 5.22.** $\tilde{M}_1(\mathbb{P}^n, d)$ has a stratification $\bigsqcup_{\gamma \in \Lambda_{d,[d]}} \tilde{M}_1(\mathbb{P}^n, d)_\gamma$ and has normal crossing singularity type $\gamma$ along $\tilde{M}_1(\mathbb{P}^n, d)_\gamma := \tilde{M}_1(\mathbb{P}^n, d)_{[\gamma], [d]}$.

**5.5. Proof of Theorem 2.18**

**Proof.** For each $\gamma \in \Lambda_{d,[d]}$, it follows from the proof of Lemma 5.19 that to each $\xi \in \tilde{M}_1(\mathbb{P}^n, d)_\gamma$, we can find an étale $\xi \in \tilde{U}_\gamma \rightarrow \tilde{M}_1(\mathbb{P}^n, d)$, a scheme $\tilde{U}_\gamma$ and a smooth morphism $q_1 : \tilde{U}_\gamma \rightarrow \tilde{U}_\gamma$ with an open embedding $q_2 : \tilde{U}_\gamma \rightarrow (\tilde{F} = 0) \subset \mathcal{E}_\gamma$. Let $(\tilde{F} = 0) \rightarrow Z_\gamma \subset E_\gamma$ be induced by (5.15). By the same reasoning as in the proof of Theorem 5.7, if we let

$$\Phi_\gamma = (\Phi_\gamma, 1, \cdots, \Phi_\gamma, n) \in \Gamma(\mathcal{E}_{E_\gamma}^{\oplus n}),$$

and $\bar{\phi} : \mathcal{E}_\gamma \rightarrow E_\gamma$ be the natural morphism (similarly defined as (5.13)), then

$$\bar{\phi}_* (\Phi_\gamma) = \bar{F}.$$  

Since $\gamma \in \Lambda_{d,[d]}$, $\gamma$ is of the form $o[v_1 \cdots [v_r]]$ with $r \leq d$. Hence each

$$\Phi_{\gamma,e} = w_e z_1 \cdots z_r.$$  

This proves the theorem. \qed

**5.23. Proof of Theorem 2.8**

**Proof.** Theorem 5.22 is a refined version of Theorem 2.8. \qed

Finally, since the primary component $\overline{M}_1(\mathbb{P}^n, d)_0$ of $\overline{M}_1(\mathbb{P}^n, d)$ is irreducible and of dimension $(n + 1)d$, thus the primary component $\overline{M}_1(\mathbb{P}^n, d)_0$ of $\tilde{M}_1(\mathbb{P}^n, d)$, which is the proper transform of $\overline{M}_1(\mathbb{P}^n, d)_0$, is smooth and is defined by $(w_1 = \cdots = w_n = 0)$ in each chart.
5.6. Proof of Theorem 2.10  
We now prove

**Theorem 2.10.**  For any \( r \geq 0 \), the direct image sheaf \( \tilde{\pi}_* \tilde{f}_* \mathcal{O}_{\mathbb{P}^n}(r) \) is locally free over every component \( \tilde{M}_1(\mathbb{P}^n, d)_\mu \) where \( \mu \) is either 0 or a partition of \( d \). It is of rank \( rd \) when \( \mu = 0 \) and of rank \( rd + 1 \) otherwise.

**Proof.**  For \( r \geq 1 \), we set \( m = dr \) and follow the notations introduced earlier. We let \( \tilde{\xi} \in \tilde{M}_1(\mathbb{P}^n, d)_\mu \) be any point and let \( \xi \in \tilde{M}_1(\mathbb{P}^n, d) \) its image. We pick a local chart \( \mathcal{U} = \mathcal{V} \times_{\mathcal{D}_1} \mathcal{U}_r \) of \( \mathcal{D} \), and introduce

\[
\eta : \tilde{\mathcal{V}} = \mathcal{V} \times_{\mathcal{D}_1} \tilde{\mathcal{D}}_1 \longrightarrow \mathcal{V}, \quad \lambda : \tilde{\mathcal{U}} = \tilde{\mathcal{V}} \times_{\mathcal{D}_1} \mathcal{U} \longrightarrow \mathcal{U}.
\]

Then \( \tilde{\mathcal{U}} \) is a chart of \( \tilde{M}_1(\mathbb{P}^n, d) \).

Continue to write \((\mathcal{C}, \mathcal{D})\) the tautological family over \( \mathcal{V} \), we set

\[
\widetilde{\mathcal{X}}' = \mathcal{X} \times_{\mathcal{U}} \tilde{\mathcal{U}} = \mathcal{C} \times_{\mathcal{V}} \tilde{\mathcal{U}}, \quad \widetilde{\mathcal{D}}' = \mathcal{D} \times_{\mathcal{V}} \tilde{\mathcal{U}} = \mathcal{D} \times_{\mathcal{V}} \tilde{\mathcal{U}}.
\]

Here the second equality in each set of identities follows from the universality of \( \mathcal{D}_1 \). We let \( \tilde{\mathcal{A}}' = \mathcal{A} \times_{\mathcal{V}} \tilde{\mathcal{U}} \) and \( \tilde{\mathcal{L}}' = \mathcal{O}_{\widetilde{\mathcal{X}}'}(\widetilde{\mathcal{D}}') \). The square

\[
\begin{array}{ccc}
\mathcal{X}' & \overset{\beta}{\longrightarrow} & \mathcal{C} \\
\tilde{\mathcal{X}}' & \downarrow \tilde{\pi}' & \downarrow \rho \\
\tilde{\mathcal{U}} & \overset{\beta}{\longrightarrow} & \mathcal{V}
\end{array}
\]

combined with the cohomology and base change theorem, gives the following commutative diagram

\[
\tilde{\pi}'_* \tilde{\mathcal{L}}'(\tilde{\mathcal{A}}') \overset{\beta^*}{\longrightarrow} \tilde{\pi}'_* \mathcal{O}_{\tilde{\mathcal{X}}'}(\tilde{\mathcal{A}}')
\]

Thus investigating \( \tilde{\varphi} \) is equivalent to investigating \( \beta^* \varphi \).

For \( \varphi \), by (4.11) it is the direct sum of the zero homomorphism on \( \mathcal{O}_V \) with

\[
\bigoplus_{i=1}^m \varphi_i : \bigoplus_{i=1}^m \rho_* \mathcal{M}_i \longrightarrow \rho_* \mathcal{O}_A(A), \quad \varphi_i = \zeta_{[\delta, a]}.
\]

Therefore, \( \tilde{\varphi} \) is given by a direct sum of the zero homomorphism with the pullbacks of \( \varphi_i = \zeta_{[\delta, a]} \).

We now let \( \tilde{\gamma} = o[v_1, \ldots, v_r] \in \Lambda_{d, [d]} \) be the tree associated to \( \tilde{\xi} \) and let \( \gamma \) be the tree associated to \( \xi \). By Lemma 5.19 to \( \tilde{\xi} \in \tilde{M}_1(\mathbb{P}^n, d)_[\tilde{\xi}] \) we can find an étale neighborhood \( \tilde{U}_{\gamma} \ni \tilde{\xi} \to \tilde{M}_1(\mathbb{P}^n, d) \) and a scheme \( W_{\gamma} \) with smooth morphisms

\[
W_{\gamma} \overset{\theta}{\longrightarrow} \tilde{U}_{\gamma} \quad \text{and} \quad W_{\gamma} \longrightarrow Z_{\gamma}.
\]

(We can assume \( \tilde{U} = W_{\gamma} \).) By the inductive construction of 5.13, we know that the tree \( \tilde{\gamma} \) is obtained from \( \gamma \) by successively advancing vertices until a
terminal vertex is advanced. Assume that $1 \leq i \leq \ell$ is the terminal vertex (in $\gamma$) that is advanced in the last step (here we adopt the indexing scheme as arranged in the proof of Theorem 4.16). Then by following the same inductive construction of 5.13 step by step, we obtain

$$\beta^* \varphi_i = z_{v_1} \cdots z_{v_r} \quad \text{and} \quad \beta^* \varphi_j \big|_{\beta^* \varphi_i} \quad \text{for all} \quad j \neq i.$$ 

Thus as in the proof of Theorem 4.16, using a new basis of $\tilde{\pi}' \tilde{L}(\tilde{A}') = \beta^* \rho_* L(A) = O_{W_5} \otimes W_{\tilde{\gamma}} \oplus \bigoplus_{i=1}^{m} \beta^* \rho_* M_i \sim O_{W_5} \oplus W_{\tilde{\gamma}}$, we see that the kernel of $\tilde{\varphi}$ is a direct sum of $O_{W_5} \oplus W_{\tilde{\gamma}}$ with the kernel of the homomorphism

$$z_{v_1} \cdots z_{v_r} : O_{W_5} \rightarrow O_{W_5}.$$ 

Now if $\xi \in \tilde{M}_1(\mathbb{P}^n, d)_0$, then $z_{v_1} \cdots z_{v_r}$ does not vanish at general points of $\theta^{-1}(\hat{U}_5 \cap \tilde{M}_1(\mathbb{P}^n, d)_0)$. Hence the kernel sheaf of $\tilde{\varphi}$ is locally free of rank $m$ over $\theta^{-1}(\hat{U}_5 \cap \tilde{M}_1(\mathbb{P}^n, d)_0)$. If $\xi \in \tilde{M}_1(\mathbb{P}^n, d)_\mu$ for $\mu$ a partition of $d$, then, one of $z_{v_1}, \ldots, z_{v_r}$ vanishes along $\theta^{-1}(\hat{U}_5 \cap \tilde{M}_1(\mathbb{P}^n, d)_\mu)$. Hence the kernel sheaf of $\tilde{\varphi}$ is locally free of rank $m + 1$ over $\theta^{-1}(\hat{U}_5 \cap \tilde{M}_1(\mathbb{P}^n, d)_\mu)$. This proves the theorem. \hfill \Box

### 5.7. Remarks on moduli spaces of stable maps with marked points.

Finally, we point out that all the main results in this paper generalize directly to moduli space of genus one stable maps with marked points. This extension requires introducing blowup loci involving the marked points analogous to those described in [7].

## References

[1] H-L. Chang and J. Li, in preparation

[2] M. Kontsevich, *Enumeration of rational curves via torus actions*. In: The Moduli Space of Curves, ed. by R. Dijkgraaf, C. Faber, G. van der Geer, Progress in Math. vol. 129, Birkhäuser, 1995, 335–368.

[3] J. Li, *A degeneration formula of GW-invariants*. J. of Differential Geom. 60 (2002), no. 2, 199–293. MR1938113 (2004k:14096)

[4] J. Li and A. Zinger, *On the Genus-One Gromov-Witten Invariants of Complete Intersections*. to appear in J. of Differential Geom., [math.AG/0507104](http://arxiv.org/abs/math.AG/0507104)

[5] R. Pandharipande, *A Note On Elliptic Plane Curves With Fixed j-Invariant*. Proc. Amer. Math. Soc. 125 (1997), no. 12, 3471–3479.

[6] R. Vakil and A. Zinger, *A natural smooth compactification of the space of elliptic curves in projective space*. Electron. Res. Announc. Amer. Math. Soc. 13 (2007), 53–59.

[7] R. Vakil and A. Zinger, *A Desingularization of the Main Component of the Moduli Space of Genus-One Stable Maps into $\mathbb{P}^n$*. Geom. Topol. 12 (2008), no. 1, 1–95.
[8] A. Zinger, *On the Structure of Certain Natural Cones over Moduli Spaces of Genus-One Holomorphic Maps*. math.SG/0406104

[9] A. Zinger, *The reduced genus-one Gromov-Witten invariants of Calabi-Yau hypersurfaces*, math/0705.2397, to appear in J. Amer. Math. Soc.

Department of Mathematics, University of Arizona, USA.
E-mail address: yhu@math.arizona.edu

Department of Mathematics, Stanford University, USA.
E-mail address: jli@math.stanford.edu