Janus-Facedness of the Pion: Analytic Instantaneous Bethe–Salpeter Models

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Abstract. Inversion enables the construction of interaction potentials underlying — under fortunate circumstances even analytic — instantaneous Bethe–Salpeter descriptions of all lightest pseudoscalar mesons as quark–antiquark bound states of Goldstone-boson nature.

1 Introduction: quark–antiquark bound states of Goldstone-boson identity

Within quantum chromodynamics, the pions or, as a matter of fact, all light pseudoscalar mesons must be interpretable as both quark–antiquark bound states and almost massless (pseudo) Goldstone bosons related to the spontaneously (and, to a minor extent, also explicitly) broken chiral symmetries of QCD.

Relativistic quantum field theory describes bound states by their Bethe–Salpeter amplitudes, $\Phi(p)$, controlled by the homogeneous Bethe–Salpeter equation defined (for two bound particles of individual and relative momenta $p_1, p_2$) by their full propagators $S_{1,2}(p_{1,2})$ and the integral kernel $K(p, q)$ that encompasses their interactions (notationally suppressing dependences on the total momentum $p_1 + p_2$):

$$
\Phi(p) = \frac{i}{(2\pi)^4} \int d^4q K(p, q) \Phi(q) S_2(-p_2) .
$$

The application of suitably adapted inversion techniques\cite{1} allows us to retrieve all the underlying interactions — rooted, of course, in QCD — analytically in the form of a (configuration-space) central potential $V(r), r = |x|$, from presumed solutions to the Bethe–Salpeter equation\cite{2}. By that, we are put in a position to construct exact analytic Bethe–Salpeter solutions for all massless pseudoscalar mesons\cite{3} in the sense of establishing in a rigorous manner the analytic relationships between interactions and resulting solutions: all analytic findings\cite{4} can be confronted with associated numerical outcomes\cite{5}.

2 Sequence of simplifying assumptions crucial for the inversion formalism

By a few steps, we cast the Bethe–Salpeter equation into a shape that allows us to talk about potentials.

1. Assuming, for each involved quark, both instantaneous interactions and free propagation, with a mass dubbed as constituent, simplifies the Bethe–Salpeter equation to a bound-state equation for the Salpeter amplitude $\phi(p)$, obtained from the Bethe–Salpeter amplitude by integration over $p_0$:

$$
\phi(p) \propto \int dp_0 \Phi(p) .
$$

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Generically, for a spin-$\frac{1}{2}$ fermion and a spin-$\frac{1}{2}$ antifermion of equal constituent masses $m$, bound to a spin-singlet state (which, for instance, clearly is the case for any such pseudoscalar state), its three-dimensional wave function involves just two independent components, here called $\varphi_{1,2}(p)$:

$$\phi(p) = \left[ \varphi_1(p) \frac{\gamma_0 (\mathbf{p} \cdot \mathbf{p} + m)}{E(p)} + \varphi_2(p) \right] \gamma_5 , \quad E(p) \equiv \sqrt{p^2 + m^2} , \quad p \equiv |p| .$$

2. Upon supposing that the quark interactions in the kernel respect spherical and Fierz symmetries, our bound-state equation for $\phi(p)$ collapses to the system of coupled radial eigenvalue equations

$$2 E(p) \varphi_2(p) + 2 \int_0^{\infty} \frac{dq q^2}{(2\pi)^2} V(p,q) \varphi_2(q) = \tilde{M} \varphi_1(p) \; , \quad 2 E(p) \varphi_1(p) = \tilde{M} \varphi_2(p) \; , \quad q \equiv |q| \; ,$$

for the bound-state mass eigenvalue $\tilde{M}$. Therein, $V(r)$ enters via its Fourier–Bessel transform

$$V(p, q) = \frac{8\pi}{pq} \int_0^{\infty} dr \sin(p r) \sin(q r) V(r) .$$

3. In the strictly massless (Goldstone) case $\tilde{M} = 0$, the system decouples: one Salpeter component, $\varphi_1(p)$, is doomed to vanish, $\varphi_1(p) \equiv 0$, whereas the surviving Salpeter component $\varphi_2(p)$ satisfies

$$E(p) \varphi_2(p) + \int_0^{\infty} \frac{dq q^2}{(2\pi)^2} V(p,q) \varphi_2(q) = 0 .$$

Denoting the Fourier–Bessel transform of the kinetic term $E(p) \varphi_2(p)$ by $T(r)$, the potential $V(r)$ may be simply read off from the configuration-space representation of this bound-state equation:

$$T(r) + V(r) \varphi_2(r) = 0 \quad \Rightarrow \quad V(r) = - \frac{T(r)}{\varphi_2(r)} .$$

### 3 Constraints on lightest-pseudoscalar-meson Bethe–Salpeter amplitudes

Information on the input Salpeter component $\varphi_2(p)$ can be gained from the full quark propagator $S(p)$, which is determined by its mass function $M(p^2)$ and its wave-function renormalization function $Z(p^2)$:

$$S(p) = \frac{i Z(p^2)}{p - M(p^2) + i \epsilon} , \quad p \equiv \gamma^\mu p^\mu , \quad \epsilon \downarrow 0 .$$

Studies of $S(p)$ within the Dyson–Schwinger framework, preferably done in Euclidean space signalled by underlined quantities, allow for pivotal insights. In the chiral limit, a Ward–Takahashi identity links [7] this quark propagator to the flavour-nonsinglet pseudoscalar-meson Bethe–Salpeter amplitude [3]:

$$\Phi(k) \approx \frac{M(k^2)}{k^2 + M^2(k^2)} \gamma^5 + \text{subleading contributions} .$$

First, in order to devise analytically accessible scenarios, we exploit two crucial pieces of information:
1. In the chiral limit, phenomenologically sound Dyson–Schwinger studies \[8\] imply, for the quark mass function \( M(x^2) \), at large Euclidean momenta \( x^2 \) a decrease essentially proportional to \( 1/x^2 \).

2. From axiomatic quantum field theory, we may deduce \[9\] that the presence of an inflection point at finite space-like momenta \( x^2 > 0 \) in the quark mass function \( M(x^2) \) entails colour confinement. Of course, any imposition of such kind of requirements on \( M(x^2) \) has to be reflected by \( \Phi(x) \). An ansatz for \( \Phi(x) \) compatible with both constraints, involving a mass parameter, \( \mu \), and a mixing parameter, \( \eta \), is

\[
\Phi(x) = \left[ \frac{1}{(x^2 + \mu^2)^2} + \frac{\eta x^2}{(x^2 + \mu^2)^3} \right] \gamma_\mu, \quad \mu > 0, \quad \eta \in \mathbb{R}.
\]

An integration of this \( \Phi(x) \) with respect to the time component of the Euclidean momentum \( x \) results in

\[
\varphi_2(p) \propto \frac{1}{(p^2 + \mu^2)^3/2} + \frac{\eta p^2 + \mu^2/4}{(p^2 + \mu^2)^3/2}, \quad p \equiv |p|,
\]

in configuration space expressible in terms of modified Bessel functions of the second kind \( K_\nu(z) \) \[10\]:

\[
\varphi_2(r) \propto 4(1 + \eta) K_0(\mu r) - \eta \mu r K_1(\mu r).
\]

(1)

For \( \eta \) values satisfying \( \eta < -1 \) or \( \eta > 0 \), \( \varphi_2(r) \) has one zero, which clearly induces a singularity in \( V(r) \).

### 4 Analytic outcomes \[3, 4\] for interquark potentials exhibiting confinement

For a few particular values of the dimensionless ratio \( m/\mu \), the analytic expression of \( V(r) \) can be found \[3, 4\]. (Throughout this section, any quantity has to be understood in units of the adequate power of \( \mu \).) As a consequence of our ansatz for \( \Phi(x) \), giving rise to the particular form \[11\] of \( \varphi_2(r) \), for \( \eta \neq -1 \) each extracted \( V(r) \) will develop, at the spatial origin \( r = 0 \), a logarithmically softened Coulomb singularity:

\[
V(r) \xrightarrow{r \to 0} \begin{cases} \text{const} & \text{for } \eta < -1, \\ r \ln r & \text{for } \eta > 0. \end{cases}
\]

### 4.1 Analytically manageable scenario of massless quarks, i.e., of constituent mass \( m = 0 \)

For our choice of \( \varphi_2(r) \), \( V(r) \) involves both modified Bessel \( (I_n) \) and Struve \( (L_n) \) functions \[10\] \((n \in \mathbb{N})\), and rises in a confinement-betraying manner to infinity either at the zero of \( \varphi_2(r) \) or for \( r \to \infty \) (Fig. 1):

\[
V(r) = \frac{\pi [4 + \eta (4 + r^2)] [I_0(r) - I_0(\mu r)] + \pi (4 + 5 \eta) r [L_1(r) - L_1(\mu r)] + 4 (2 + 3 \eta) r}{2 r [4 (1 + \eta) K_0(r) - \eta r K_1(r)]}.
\]

### 4.2 Analytically expressible observation for quarks with common constituent mass \( m = \mu \)

For \( m = \mu \), the kinetic term \( T(r) \) is a mixture of Yukawa and exponential behaviour, whence (cf. Fig. 2)

\[
V(r) = -\frac{\pi [8 + \eta (8 - 3 \eta)] \exp(-r)}{4 r [4 (1 + \eta) K_0(r) - \eta r K_1(r)]} \xrightarrow{r \to \infty} -\frac{\text{const}}{\sqrt{r}} \xrightarrow{r \to \infty} 0 \quad (\text{const} > 0).
\]
Figure 1. Configuration-space interquark potential $V(r)$ of the Fierz-symmetric kernel $K(p, q)$, for the constituent quark mass $m = 0$ and mixture $\eta = 0$ (black), $\eta = 1$ (red), $\eta = 2$ (magenta), $\eta = -0.5$ (blue), or $\eta = -1$ (violet).

Figure 2. Configuration-space interquark potential $V(r)$ of the Fierz-symmetric kernel $K(p, q)$, for the constituent quark mass $m = 1$ and mixture $\eta = 0$ (black), $\eta = 0.5$ (red), $\eta = 1$ (magenta), $\eta = 2$ (blue), and $\eta = -1$ (violet).

5 Reliability check of findings: numerical determination of the potential [5]

Our findings may be scrutinized by use of the chiral-limit quark mass function’s pointwise form $M(k^2)$, provided graphically in Ref. [8] and shown in Fig. 3 as $M(k)$ with $k \equiv (k^2)^{1/2}$, which we parametrize by

$$M(k) = 0.708 \text{ GeV} \exp\left(-\frac{k^2}{0.655 \text{ GeV}^2}\right) + \frac{0.0706 \text{ GeV}}{1 + \left(\frac{k^2}{0.487 \text{ GeV}^2}\right)^{1.48}^{0.752}}.$$  

Note that the product of the two exponents in the second term above yields $1.48 \times 0.752 \approx 1.1$, which is pretty close to unity, as demanded by the large-$k$ constraint. Feeding this $M(k)$ parametrization into our inversion procedure, we obtain potentials that are finite at $r = 0$ and, for sufficiently small $m$, rise with $r$ to infinity but, for large $m$, remain negative, as illustrated in Fig. 3 for selected constituent mass values.
6 Summary of results, observations, discussion, conclusion, perspectives

We constructed confining potentials $V(r)$ that in cooperation with a Fierz-symmetric interaction kernel describe massless pseudoscalar quark–antiquark bound-state solutions of the Bethe–Salpeter equation. This is possible even analytically if focusing to specific aspects of the quark mass function’s behaviour. Two obstacles call for a particularly careful treatment: Numerically, $M(p^2)$ is known for only a limited range of $p^2$. For large $r$, both $T(r)$ and $\varphi_2(r)$ approach zero; thus, pinning down $V(r)$ in the limit $r \to \infty$ boils down to a division of zero by zero. Dropping the free quark propagation constraint [11] allows us to thoroughly take into account the effects of $M(p^2)$ and the quark wave-function renormalization [12].

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