Game Description Logic with Integers:  
A GDL Numerical Extension

Munyque Mittelmann and Laurent Perrussel

Université de Toulouse - IRIT, Toulouse, France
{munyque.mittelmann, laurent.perrussel}@irit.fr

Abstract. Many problems can be viewed as games, where one or more agents try to ensure that certain objectives hold no matter the behavior from the environment and other agents. In recent years, a number of logical formalisms have been proposed for specifying games among which the Game Description Language (GDL) was established as the official language for General Game Playing. Although numbers are recurring in games, the description of games with numerical features in GDL requires the enumeration from all possible numeric values and the relation among them. Thereby, in this paper, we introduce the Game Description Logic with Integers (GDLZ) to describe games with numerical variables, numerical parameters, as well as to perform numerical comparisons. We compare our approach with GDL and show that when describing the same game, GDLZ is more compact.

Keywords: Game Description Language · Knowledge Representation · General Game Playing.

1 Introduction

Many problems, as multiagent planning or process synchronization, can be viewed as games, where one or more agents try to ensure that certain objectives hold no matter the behavior from the environment and other agents [4]. Thereby, a number of logical formalisms have been proposed for specifying game structures and its properties, such as the Game Logic [10, 11], the Dynamic Game Logic for sequential [16] and simultaneous games [17], the GameGolog language [4] and so on. Among this formalisms, the Game Description Language (GDL) [1, 7] has been established as the official language for the General Game Playing (GGP) Competition. Due to the GDL limitations, such as its restriction to deterministic games with complete state information, several works investigate GDL extensions to improve its expressiveness. Zhang and Thielscher (2014) [18] provide a GDL extension using a modality for linear time and state transition structures. They also propose two dual connectives to express preferences in strategies.

Another extension is called GDL with Incomplete Information (GDL-II) and it was proposed to describe nondeterministic games with randomness and incomplete state knowledge [12][13]. A different approach to deal with this problem is
the Epistemic GDL, that allows to represent imperfect information games and provides a semantical model that can be used for reasoning about game information and players’s epistemic status [6]. GDL with Imperfect Information and Introspection (GDL-III) is an extension of GDL-II to include epistemic games, which are characterized by rules that depend on the knowledge of players [14,15]. In order to model how agents can cooperate to achieve a desirable goal, Jiang et al. (2014) present a framework to combine GDL with the coalition operators from Alternating-time Temporal Logic and prioritized strategy connectives [5].

Although numbers are recurring in game descriptions (e.g. Monopoly, Nim game), neither GDL or its extensions incorporate numerical features. In these approaches, numbers can be designed as index in propositions or actions but not directly used as state variables. Thereby, describing games with numerical features can lead to an exhaustive enumeration of all possible numeric values and the relation between them. In the context of planning problems, numerical features have been introduced in Planning Domain Description Language (PDDL) by its first versions [3,9] and improved by PDDL 2.1 [2,8]. In PDDL 2.1, a world state contains an assignment of values to a set of numerical variables. These variables can be modified by action effects and used in expressions to describe actions’ preconditions and planning goals.

Similarly to the approach of PDDL 2.1, in this paper, we introduce the GDL extension Game Description Logic with Integers (GDLZ) that incorporates numerical variables, parameters and comparisons. Regarding that board games are mainly described with discrete values, our approach only considers the integer set. We compare our approach with GDL and show that a game description in GDLZ is more compact than the corresponding description in GDL.

This paper is organized as follows. In Section 2 we introduce the framework by means of state transition structures and we present the language syntax and semantics. In Section 3 we define the translation between GDLZ and GDL and we compare both languages. Section 4 concludes the paper, bringing final considerations.

2 Game Description Logic with Integers

In this section, we introduce a logical framework for game specification with integer numbers. The framework is an extension from the GDL state transition model and language [18], such that it defines numerical variables and parameters. We call the framework Game Description Logic with Integers, denoted GDLZ.

To describe a game, we first define a game signature, that specifies who are the players (the agents), what are the possible actions for each player and what are the aspects that describe each state in the game (the propositions and numerical variables). We define a game signature as follows:

Definition 1. A game signature \( S \) is a tuple \((N, A, \Phi, X)\), where:

\[- N = \{r_1, r_2, \ldots, r_k\} \text{ is a nonempty finite set of agents;} \]
- \( A = \bigcup_{r \in N} A^r \) where \( A^r = \{ a_1^r(\bar{z}_1), \ldots, a_m^r(\bar{z}_m) \} \) consists of a nonempty set of actions performed by agent \( r \in N \), where \( \bar{z}_i \in \mathbb{Z}^l \) is a possibly empty tuple of \( l \) integer values representing the parameters for the action \( a_i^r \), \( i \leq m \) and \( l \in \mathbb{N} \). For convenience, we occasionally write \( a_i^r \) for denoting an action \( a_i^r(\bar{z}_i) \in A \);  
- \( \Phi = \{ p, q, \cdots \} \) is a finite set of atomic propositions for specifying individual features of a game state;  
- \( X = (x_1, x_2, \cdots, x_n) \) is a tuple of numerical variables for specifying numerical features of a game state.

Given a game signature, we define a state transition model, that allows us to represent the key aspects of a game, such as the winning states for each agent, the legal actions in each state and the transitions between game states.

**Definition 2.** Given a game signature \( S = (N, A, \Phi, X) \), a state transition \( ST \) model \( M \) is a tuple \((W, \bar{w}, T, L, U, g, \pi_\phi, \pi_Z)\), where:

- \( W \) is a nonempty set of states;  
- \( \bar{w} \in W \) is the initial state;  
- \( T \subseteq W \) is a set of terminal states;  
- \( L \subseteq W \times A \) is a legality relation, describing the legal actions at each state;  
- \( U : W \times D \rightarrow W \) is an update function, where \( D = \prod_{r \in N} A^r \) denote the set of joint actions, specifying the transitions for each joint state;  
- \( g : N \rightarrow 2^W \) is a goal function, specifying the winning states for each agent;  
- \( \pi_\phi : W \rightarrow 2^\Phi \) is the valuation function for the state propositions;  
- \( \pi_Z : W \rightarrow \mathbb{Z}^n \) is the valuation function for the state numerical variables, such that \( \pi_Z(w) \) is a tuple of integer values assigned to the variables \( X \) at state \( w \in W \). Let \( \pi_Z^i(w) \) denote the \( i \)-th value of \( \pi_Z(w) \).

Given \( d \in D \), let \( d(r) \) be the individual action for agent \( r \) in the joint action \( d \). Let \( L(w) = \{ a \in A \mid (w, a) \in L \} \) be the set of all legal actions at state \( w \).

**Definition 3.** Given an \( ST \)-model \( M = (W, \bar{w}, T, L, U, g, \pi_\phi, \pi_Z) \), a path is a finite sequence of states \( \bar{w} \xrightarrow{d_1} w_1 \xrightarrow{d_2} \cdots \xrightarrow{d_e} w_e \) such that \( e \geq 0 \) and for any \( j \in \{ 1, \cdots, e \} \): (i) \( \{ w_0, \cdots, w_{e-1} \} \cap T = \emptyset \), where \( w_0 = \bar{w} \); (ii) \( d_j(r) \in L(w_{j-1}) \) for any \( r \in N \); and (iii) \( w_j = U(w_{j-1}, d_j) \).

A path \( \delta \) is complete if \( w_e \in T \). Given \( \delta \in \mathcal{P} \), let \( \delta[j] \) denote the \( j \)-th reachable state of \( \delta \), \( \theta(\delta, j) \) denotes the joint action taken at stage \( j \) of \( \delta \); and \( \theta_r(\delta, j) \) denotes the action of agent \( r \) taken at stage \( j \) of \( \delta \). Finally, the length of a path \( \lambda \), written \( |\lambda| \), is defined as the number of joint actions.

Describing a game with the \( ST \)-model is not practical, especially when modeling large games. Hereby, given a game signature \( S = (N, A, \Phi, X) \), we introduce a variant of the language for \( GDL \) (\( \mathcal{L}_{GDL} \) for short) to describe a \( GDLZ \) game in a more compact way by encoding its rules.
2.1 Syntax

The *language* is denoted by $L_{GDLZ}$ and a *formula* $\varphi$ in $L_{GDLZ}$ is defined by the following Backus-Naur Form (BNF) grammar:

$$\varphi ::= p \mid initial \mid terminal \mid legal(a'(\bar{z})) \mid wins(r) \mid does(a'(\bar{z})) \mid \neg \varphi \mid \varphi \land \varphi \mid \bigcirc \varphi \mid z > z \mid z < z \mid z = z \mid \langle \bar{z} \rangle$$

where, $p \in \Phi, r \in N, a' \in A^r, \bar{z}$ is a number list and $z$ is a numerical term.

Let $\varepsilon$ denote the empty word. A number list $\bar{z}$ is defined as:

$$\bar{z} ::= z \mid z, \bar{z} \mid \varepsilon.$$ 

Finally, a numerical term $z$ is defined by $L_z$, which is generated by the following BNF:

$$z ::= z' \mid x' \mid add(z,z) \mid sub(z,z) \mid min(z,z) \mid max(z,z)$$

where $z' \in \mathbb{Z}$ and $x' \in X$.

Other connectives $\lor, \to, \leftrightarrow, \top$ and $\bot$ are defined by $\neg$ and $\land$ in the standard way. The comparison operators $\leq, \geq$ and $\neq$ are defined by $\lor, \geq, <$ and $=,$ respectively, as follows: (i) $z_1 < z_2 \lor z_1 = z_2$, (ii) $z_1 > z_2 \lor z_1 = z_2$ and (iii) $z_1 > z_2 \lor z_1 < z_2$.

Intuitively, *initial* and *terminal* specify the initial state and the terminal state, respectively; $\text{does}(a'(\bar{z}))$ asserts that agent $r$ takes action $a$ with the parameters $\bar{z}$ at the current state; $\text{legal}(a'(\bar{z}))$ asserts that agent $r$ is allowed to take action $a$ with the parameters $\bar{z}$ at the current state; and $\text{wins}(r)$ asserts that agent $r$ wins at the current state. The formula $\bigcirc \varphi$ means "$\varphi$ holds at the next state". The formulas $z_1 > z_2, z_1 < z_2, z_1 = z_2$ means that a numerical term $z_1$ is greater, less and equal to a numerical term $z_2$, respectively. Finally, $\langle \bar{z} \rangle$ asserts the current values for the numerical variables, i.e. the $i$-th variable in $X$ has the $i$-th value in $\bar{z}$, for $0 \leq i \leq |X|$. Notice that $\langle \bar{z} \rangle$ could be represented by a conjunction over each $x_i \in X$ of formulas $x_i = z_i$, where $z_i \in L_z$ is the current value of the variable $x_i$. However, $\langle \bar{z} \rangle$ provides a short cut and it is more meaningful, in the sense that it is strictly related to the valuation of the numerical variables in a given state.

For numerical terms, $\text{add}(z_1, z_2)$ and $\text{sub}(z_1, z_2)$ specify the value obtained by adding and subtracting $z_2$ from $z_1$, respectively. The formulas $\text{min}(z_1, z_2)$ and $\text{max}(z_1, z_2)$ specify the minimum and maximum value between $z_1$ and $z_2$, respectively. The extension of the comparison operators $>, <, =, \leq, \geq$ and $\neq$ to multiple arguments is straightforward.

If $\varphi$ is not in the form $\neg \varphi', \bigcirc \varphi'$ or $\varphi' \land \varphi''$, for any $\varphi', \varphi'' \in L_{GDLZ}$, then $\varphi$ is called an atomic formula. We say that a numerical variable occurs in an atomic formula $\varphi$ if (i) $\varphi$ is either in the form $\text{legal}(a'(\bar{z}))$, $\text{does}(a'(\bar{z}))$ or $\langle \bar{z} \rangle$ and there is a $x \in X$ in the numerical list $\bar{z}$; (ii) $\varphi$ is either in the form $z_1 < z_2, z_1 > z_2$ or $z_1 = z_2$ and $z_1 \in X$ or $z_2 \in X$. 
2.2 Semantics

The semantics for the GDLZ language is given in two steps. First, we define function \( v \) to assign the meaning of numerical terms \( z \in \mathcal{L}_z \) in a specified state (Definition 4). Next, a formula \( \varphi \in \mathcal{L}_{GDLZ} \) is interpreted with respect to a stage in a path (Definition 5).

**Definition 4.** Given an ST-model \( M \), a state \( w \) and the functions minimum and maximum let us define function \( v : W \times \mathcal{L}_z \to \mathbb{Z} \), associating any \( z_i \in \mathcal{L}_z \) in a state \( w \in W \) to a number in \( \mathbb{Z} \):

\[
v(z_i, w) = \begin{cases}
  z_i & \text{if } z_i \in \mathbb{Z} \\
  \pi_z(w) & \text{if } z_i = x_i \in X \\
  v(z'_i, w) + v(z''_i, w) & \text{if } z_i = \text{add}(z'_i, z''_i) \\
  v(z'_i, w) - v(z''_i, w) & \text{if } z_i = \text{sub}(z'_i, z''_i) \\
  \text{minimum}(v(z'_i, w), v(z''_i, w)) & \text{if } z_i = \text{min}(z'_i, z''_i) \\
  \text{maximum}(v(z'_i, w), v(z''_i, w)) & \text{if } z_i = \text{min}(z'_i, z''_i)
\end{cases}
\]

**Definition 5.** Let \( M \) be an ST-Model. Given a complete path \( \delta \) of \( M \), a stage \( j \) on \( \delta \), a formula \( \varphi \in \mathcal{L}_{GDLZ} \) and function \( v \), we say \( \varphi \) is true (or satisfied) at \( j \) of \( \delta \) under \( M \), denoted by \( M, \delta, j \models \varphi \), according to the following definition:

\[
\begin{align*}
M, \delta, j \models p & \iff p \in \pi_\varphi(\delta[j]) \\
M, \delta, j \models \neg \varphi & \iff M, \delta, j \not\models \varphi \\
M, \delta, j \models \varphi_1 \land \varphi_2 & \iff M, \delta, j \models \varphi_1 \text{ and } M, \delta, j \models \varphi_2 \\
M, \delta, j \models \text{initial} & \iff \delta[j] = \bar{w} \\
M, \delta, j \models \text{terminal} & \iff \delta[j] \in T \\
M, \delta, j \models \text{wins}(r) & \iff \delta[j] \in g(r) \\
M, \delta, j \models \text{legal}(a^r(z)) & \iff a^r(v(z, \delta[j]) : z \in z) \in L(\delta[j]) \\
M, \delta, j \models \text{does}(a^r(z)) & \iff \theta_r(\delta, j) = a^r(v(z, \delta[j]) : z \in z) \\
M, \delta, j \models \circ \varphi & \iff \text{if } j < |\delta|, \text{ then } M, \delta, j+1 \models \varphi \\
M, \delta, j \models z_1 > z_2 & \iff v(z_1, \delta[j]) > v(z_2, \delta[j]) \\
M, \delta, j \models z_1 < z_2 & \iff v(z_1, \delta[j]) < v(z_2, \delta[j]) \\
M, \delta, j \models z_1 = z_2 & \iff v(z_1, \delta[j]) = v(z_2, \delta[j]) \\
M, \delta, j \models \langle z \rangle & \iff \langle v(z, \delta[j]) : z \in z \rangle = \pi_\varphi(\delta[j])
\end{align*}
\]

A formula \( \varphi \) is globally true through \( \delta \), denoted by \( M, \delta \models \varphi \), if \( M, \delta, j \models \varphi \) for any stage \( j \) of \( \delta \). A formula \( \varphi \) is globally true in an ST-Model \( M \), written \( M \models \varphi \), if \( M, \delta \models \varphi \) for all complete paths \( \delta \) in \( M \), that is, \( \varphi \) is true at every reachable state. A formula \( \varphi \) is valid, denoted by \( \models \varphi \), if it is globally true in

\(^1\) Through the rest of this paper, the functions minimum\((a, b)\) and maximum\((a, b)\) respectively return the minimum and maximum value between \(a, b \in \mathbb{Z}\).
every ST-model of an appropriate signature. Finally, let $\Sigma$ be a set of formulas in $\mathcal{L}_{GDLZ}$, then $M$ is a model of $\Sigma$ if $M \models \varphi$ for all $\varphi \in \Sigma$.

Whenever $j \geq |\delta|$, the validity of $M, \delta, j \models \Box \varphi$ is irrelevant, since $\delta[j]$ is the last state reachable in $\delta$. A formula $\langle \vec{z} \rangle$ is valid at a stage $j$ in a path $\delta$ under $M$ only when it corresponds to the valuation of the numerical variables at $\delta[j]$.

The following propositions show that if a player does an action at a stage in a path, then (i) he does not any other action in the same stage and (ii) the action taken is legal.

**Proposition 1.** $\models \text{does}(a^r(\vec{z})) \rightarrow \bigwedge_{b^r \neq a^r \in A^r} \bigwedge_{z^r \neq \vec{z} \in \mathbb{Z}^n} \neg \text{does}(b^r(\vec{z}'))$.

**Proof.** Assume $M, \delta, j \models \text{does}(a^r(\vec{z}))$ for some $\vec{z}$ iff $\theta_r(\delta, j) = \text{does}(a^r(\vec{z}))$. Then, for any $\vec{z}' \neq \vec{z}, b^r \neq a^r, \theta_r(\delta, j) \neq \text{does}(b^r(\vec{z}'))$. Thereby, $M, \delta, j \not\models \bigwedge_{b^r \neq a^r \in A^r} \bigwedge_{z^r \neq \vec{z} \in \mathbb{Z}^n} \text{does}(b^r(\vec{z}'))$ and $M, \delta, j \models \bigwedge_{b^r \neq a^r \in A^r} \bigwedge_{z^r \neq \vec{z} \in \mathbb{Z}^n} \neg \text{does}(b^r(\vec{z}'))$.

**Proposition 2.** $\models \text{does}(a^r(\vec{z})) \rightarrow \text{legal}(a^r(\vec{z}))$.

**Proof.** Assume $M, \delta, j \models \text{does}(a^r(\vec{z}))$, then $a^r = \theta^r(\delta, j)$. And by the definition of $\delta$, $a^r(\vec{z}) \in L(\delta[j])$, so $M, \delta, j \models \text{legal}(a^r(\vec{z}))$.

Next, we illustrate the representation of a game with numerical features in GDLZ. First, we define the game signature and the game description in $\mathcal{L}_{GDLZ}$. Next, we define the ST-model by which it is possible to evaluate the $\mathcal{L}_{GDLZ}$ semantics. Finally, we illustrate a path in the game.

**Example 1.** $\langle \gamma_1, \cdots, \gamma_k \rangle$-Nim Game) A $\langle \gamma_1, \cdots, \gamma_k \rangle$-Nim Game consists in $k$ heaps. Each heap starts with $\gamma_i$ sticks, where $1 \leq i \leq k$. Two players take turns in removing sticks from one heap. The game ends when all heaps are empty. A player wins if it is not his turn when the game ends.

To represent a $\langle \gamma_1, \cdots, \gamma_k \rangle$-Nim Game in terms in GDLZ, we first specify the agents, the actions, the propositions, and the numerical variables involved in the game. Thus, the game signature, written $\mathcal{S}_{\text{k-nim}}$, is described as follows:

- $N_{\text{k-nim}} = \{ \text{Player}_1, \text{Player}_2 \}$;
- $A_{\text{k-nim}} = \{ \text{reduce}^r(m, s) \mid s \in \mathbb{N}, 1 \leq m \leq k \} \cup \{ \text{noop}^r \}$, where reduce$^r(m, s)$ denotes the action that player $r$ removes $s$ sticks from the $m$-th heap and noop$^r$ denotes that player $r$ does action noop;
- $\Phi_{\text{k-nim}} = \{ \text{turn}(r) \mid r \in \{ \text{Player}_1, \text{Player}_2 \} \}$, where turn$(r)$ says that it is player $r$’s turn now;
- $X_{\text{k-nim}} = \{ \text{heap}_i \mid 1 \leq i \leq k \}$, where heap$_i$ represents the amount of sticks in the $i$-th heap.

Given a player $r \in N_{\text{k-nim}}$, we denote $-r$ as the opponent of $r$, i.e. $-r = \text{Player}_2$ if $r = \text{Player}_1$ and $-r = \text{Player}_1$ otherwise. The rules of the $\langle \gamma_1, \cdots, \gamma_k \rangle$-Nim Game can be expressed by GDLZ- formulas as shown Figure 1.

Statement 1 says that the Player$_1$ has the first turn and that the $k$ heaps starts with $\gamma_1, \cdots, \gamma_k$ sticks, respectively. Statement 2 and 3 specify the winning states for each player and the terminal states of the game, respectively.
Let $\langle m, h_1, \ldots, h_k \rangle$ denote the $m$-th heap. The semantics for the language is based on the state transition model, written $\Sigma_{k\text{-nim}}$, as follows:

- $W_{k\text{-nim}} = \{ (t_1, t_2, \langle x_1, \ldots, x_k \rangle) : t_1 \in \{ turn(\text{Player}_1), \neg turn(\text{Player}_1) \} \land t_2 \in \{ turn(\text{Player}_2), \neg turn(\text{Player}_2) \} \land x_i \in \mathbb{N}, 1 \leq i \leq k \}$ is the set of states, where $t_1, t_2$ specify the turn taking and $x_i$ represents the amount of sticks in the $i$-th heap, i.e. the integer value assigned to $heap_i$;
- $\bar{w}_{k\text{-nim}} = \{ turn(\text{Player}_1), \neg turn(\text{Player}_2), \langle \gamma_1, \ldots, \gamma_k \rangle \}$;
- $T_{k\text{-nim}} = \{ (turn(\text{Player}_1), \neg turn(\text{Player}_2), (0, \ldots, 0)), \neg turn(\text{Player}_1), turn(\text{Player}_2), (0, \ldots, 0)) \}, i.e. all heaps are empty;
- $L_{k\text{-nim}} = \{ \{ (t_1, t_2, \langle x_1, \ldots, x_k \rangle), reduce^r(m, s) : t_r = turn(r) \land 1 \leq s \leq heap_m \} \cup \{ (t_1, t_2, \langle x_1, \ldots, x_k \rangle), \text{noop}^r : t_r = \neg turn(r) \} : \forall (t_1, t_2, \langle x_1, \ldots, x_k \rangle) \in W_{k\text{-nim}} \land r \in N_{k\text{-nim}} ;$;
- $U_{k\text{-nim}} : W_{k\text{-nim}} \times D_{k\text{-nim}} \rightarrow W_{k\text{-nim}}$ is defined as follows: for all $\langle t_1, t_2, \langle x_1, \ldots, x_k \rangle \rangle \in W_{k\text{-nim}}$ and all $(reduce^r(m, s), noop^r) \in D_{k\text{-nim}}$, let $U_{k\text{-nim}}(\langle t_1, t_2, \langle x_1, \ldots, x_k \rangle \rangle, (reduce^r(m, s), noop^r)) = \langle t_1', t_2', \langle x_1', \ldots, x_k' \rangle \rangle$, such that $\{ t_1', t_2', \langle x_1', \ldots, x_k' \rangle \}$ are the same as $\{ t_1, t_2, \langle x_1, \ldots, x_k \rangle \}$, except by its components $t'_1, t'_2$ and $x'_i$ are updated as follows: $t'_1 = turn(\text{Player}_1)$ iff $t_1 = turn(\text{Player}_2)$, otherwise $t'_1 = \neg turn(\text{Player}_1); t'_2 = turn(\text{Player}_2)$ iff $t_1 = turn(\text{Player}_1)$, otherwise, $t'_2 = \neg turn(\text{Player}_2)$; and for $1 \leq i \leq k$:

\[
x'_i = \begin{cases} x_i - s & \text{if } reduce^r(i, s) \text{ and } 1 \leq s \leq x_i \\ x_i & \text{otherwise} \end{cases}
\]
For all \( \langle t_1, t_2, (x_1, \cdots, x_k) \rangle \in W_{k-nim} \) and all \( (a^r, a^{-r}) \neq (\text{reduce}^r(m, s), \text{noop}^r) \) in \( D_{k-nim} \), let \( U_{k-nim}(\langle t_1, t_2, (x_1, \cdots, x_k) \rangle, (a^r, a^{-r})) = \langle t_1, t_2, (x_1, \cdots, x_k) \rangle \).

- \( g_{k-nim}(r) = \{ (t_1, t_2, (0, \cdots, 0)) \} \), where \( t_r = -\text{turn}(r) \) and \( t_{-r} = \text{turn}(r) \).

Finally, for each state \( w = (t_1, t_2, (x_1, \cdots, x_k)) \in W_{k-nim} \), let

- \( \pi_{\Phi, k-nim}(w) = \{ \text{turn}(r) : t_r = \text{turn}(r) \} \);
- \( \pi_{Z, k-nim}(w) = (x_1, \cdots, x_k) \).

Let \( M_{k-nim} = (W_{k-nim}, \pi_{k-nim}, T_{k-nim}, L_{k-nim}, U_{k-nim}, g_{k-nim}, \pi_{\Phi, k-nim}, \pi_{Z, k-nim}) \) be the ST-model for the \( k \)-Nim Game.

Consider, for instance, \( k = 2 \) and \( \langle \gamma_1, \gamma_2 \rangle = (5, 3) \), i.e. there are only two heaps and their starting values are 5 and 3, respectively. Figure 2 illustrates a path in \( M_{k-nim} \). The state \( w_0 \) represents the initial state. In \( w_0 \), it is the turn of Player_1 and he removes 5 sticks from the first heap. In the state \( w_1 \), the first heap is empty and players can only remove sticks from the second heap. It is now Player_2’s turn and he reduces 2 sticks from the second heap. In the state \( w_2 \), Player_2 removes the last stick from the second heap. Finally, in the state \( w_3 \), there is no stick remaining in any heap, thereby it is a terminal state. Since it is Player_2’s turn, Player_1 wins the game.

The next proposition shows that soundness does hold, i.e. the framework provides a sound description for the \( k \)-Nim Game. Notice that as \( M_{k-nim} \) is not the unique model for \( \Sigma_{k-nim} \), thereby, the completeness does not hold.

**Proposition 3.** \( M_{k-nim} \) is an ST-model and it is a model of \( \Sigma_{k-nim} \).

**Proof.** It is routine to check that \( M_{k-nim} \) is actually an ST-model. Given any complete path \( \delta \), any stage \( t \) on \( \delta \) in \( M_{k-nim} \), we need to verify that each rule is true at \( t \) of \( \delta \) under \( M_{k-nim} \).

Let us consider Rule 4. Assume \( M_{k-nim}, \delta, t \models \bigwedge_{m \in \{1, \ldots, k\}} \bigwedge_{s \in \{1, \ldots, \gamma_m\}} \text{legal} (\text{reduce}^r(m, s)) \) iff \( \text{reduce}^r(m, s) \in L_{k-nim}(\delta[t]) \) iff \( t_r = \text{turn}(r) \) and \( 1 \leq s \leq \text{heap}_m \) (by the definition of \( L_{k-nim} \)) iff \( \text{turn}(r) \in \pi_{\Phi, k-nim}(\delta[t]) \) (by the definition of \( \pi_{\Phi, k-nim} \)) and \( 1 \leq s \leq x_m \), where \( x_m \) is the value assigned to \( \text{heap}_m \) at stage \( \delta[t] \) iff \( M_{k-nim}, \delta, t \models 1 \leq s \leq \text{heap}_m \land \text{turn}(r) \).

Let us verify Rule 7. Assume \( M_{k-nim}, \delta, t \models \bigwedge_{i \in \{1, \ldots, k\}} \bigwedge_{\gamma_i} \neg \text{terminal} \land (h_1, \cdots, h_k) \land \bigvee_{r \in \mathbb{N}} \bigvee_{m \in \{1, \ldots, k\}} \bigvee_{s \in \{1, \ldots, \gamma_m\}} \text{does}(\text{reduce}^r(m, s)) \). Since \( \neg \text{terminal} \), by the path definition, we have \( t < |\delta| \). For some \( r \in \mathbb{N}, 1 \leq m \leq k \) and \( 1 \leq s \leq \gamma_m \), it is true that \( \text{does}(\text{reduce}^r(m, s)) \), then \( \theta_r(\delta, t) = \text{reduce}^r(m, s) \in L_{k-nim}(\delta[t]) \). Since \( M_{k-nim}, \delta, t \models (h_1, \cdots, h_k) \), for some \( h_i \in \{1, \cdots, \gamma_i\} \) and any \( i \in \{1, \cdots, k\} \), by the definition of \( U_{k-nim} \), we have \( M_{k-nim}, \delta, t + 1 \models (x_1', \cdots, x_k') \), where \( x_i' = \text{heap}_m - s \) if \( \text{reduce}^r(m, s) \) and for any \( j \neq m \) and \( 1 \leq j \leq k \), \( x_j' = \text{heap}_j \). By function \( v \), we know that \( v(\text{sub}(h_m, s)) = \text{heap}_m - s \). Therefore, \( M_{k-nim}, \delta, t + 1 \models (h_1, \cdots, \text{sub}(h_m, s), \cdots, h_k) \) and so \( M_{k-nim}, \delta, t \models \bigcirc (h_1, \cdots, \text{sub}(h_m, s), \cdots, h_k) \).

The remaining rules are proved in a similar way.
Thus, \( \phi \) corresponds to the formula \( \psi \) induction on \( M \) following way: first it gets all subformulas of \( \phi \).

2.3 Model Checking

Verifying the validity of a formula at a stage of a path in a model. The addition of numerical features in GDL does not increase the complexity at which is the same complexity then the model checking for GDL. In other words, the model checking problem for GDLZ is the following: Given a GDLZ-formula \( \varphi \), an ST-model \( M \), a path \( \delta \) of \( M \) and a stage \( j \) on \( \delta \), determine whether \( M, \delta, j \models \varphi \) or not.

Let \( \text{Sub}(\varphi) \) be the set of all subformulas\(^2\) of \( \varphi \). Algorithm \( \text{alg-1} \) works in the following way: first it gets all subformulas of \( \varphi \) and orders them in \( S \) by its ascending length. Thus, \( S(\vert \varphi \vert) = \varphi \), i.e. the position \( \vert \varphi \vert \) in the vector \( S \) corresponds to the formula \( \varphi \) itself, and if \( \phi_i \) is a subformula of \( \phi_j \), then \( i < j \). An induction on \( S \) label each subformula \( \phi_i \) depending on whether or not \( \phi_i \) is true in \( M \) at \( \delta[j] \). If \( \phi_i \) does not have any subformula, its truth value is obtained.

---

\(^2\) We say that \( \psi \) is a subformula of \( \varphi \in L_{GDLZ} \) if either (i) \( \psi = \varphi \); (ii) \( \varphi \) is of the form \( \neg \varphi' \) or \( \bigcirc \varphi' \) and \( \psi \) is a subformula of \( \varphi' \); or (iii) \( \varphi \) is of the form \( \varphi' \land \varphi'' \) and \( \psi \) is a subformula of either \( \varphi' \) or \( \varphi'' \).
directly from the semantics. Since $S$ is ordered by the formulas length, if $\phi_i$ is either in the form $\phi' \land \phi''$ or $\neg \phi'$ the algorithm labels $\phi_i$ according to the label assigned to $\phi'$ and/or $\phi''$. If $\phi_i$ is in the form $\Box \phi'$, its label will be recursively defined according to $\phi'$ truth value in $\delta[j + 1]$. As Algorithm 1 visits each node at most once, and the number of nodes in the tree is not greater than the size of $\varphi$, it can be clearly implemented in a polynomial-time deterministic Turing machine with PTIME.

Algorithm 1 $isTrue(M, \delta, j, \varphi)$

**Input:** an ST-model $M$, a path $\delta$ of $M$, a stage $j$ and a formula $\varphi \in L_{GDLZ}$.

**Output:** true if $M, \delta, j \models \varphi$, and false otherwise

1: $S \leftarrow \text{Sub}(\varphi)$ ordered by ascending length
2: Let $\text{reg}[1 \ldots \text{size}(S)]$ be a boolean array
3: for $i \leftarrow 1$ to $\text{size}(S)$ do
4:   $\phi \leftarrow S[i]$
5:   if $(\phi = \phi' \land \phi'')$ then
6:     $\text{reg}[i] \leftarrow \text{reg}[\text{getIndex}(S, \phi')] \land \text{reg}[\text{getIndex}(S, \phi'')]$
7:   else if $(\phi = \Box \phi')$ then
8:     $\text{reg}[i] \leftarrow isTrue(M, \delta, j + 1, \phi')$
9:   else if $(\phi = \neg \phi')$ then
10:    $\text{reg}[i] \leftarrow \neg \text{reg}[\text{getIndex}(S, \phi')]$
11: else $\text{reg}[i] \leftarrow M, \delta, j \models \phi$
12: return $\text{reg}[\text{size}(S)]$

In Section 3.3 we show that $L_{GDL} \subseteq L_{GDLZ}$, i.e. any formula in GDL is also a formula in GDLZ. Thereby, Algorithm 1 can also be used in the model checking problem for GDL.

3 Translation Between GDLZ and GDL

In this section, we investigate translation maps among GDLZ and GDL models and descriptions. We first consider the general case where the GDLZ ST-model can have infinite components. Next, we restrict to the case where a GDLZ ST-model is finite. Finally, we compare both languages in order to show the succinctness of GDLZ descriptions over GDL descriptions.

Given a GDLZ ST-model $M$, a complete path $\delta$ in $M$ and a formula $\varphi \in L_{GDLZ}$, in the Sections 3.1 and 3.2 our goal is to construct a GDL ST-model $M'$, a path $\delta'$ in $M'$ and a formula $\varphi' \in L_{GDL}$ such that, for any stage $j$ on $\delta$, if $M, \delta, j \models \varphi$ then $M', \delta', j \models \varphi'$.

3.1 From GDLZ Paths and Models to GDL Models

In a GDL ST-model, the sets of states, actions and atomic propositions are finite. Since it does not hold for GDLZ ST-models, it is not possible to define a
complete translation from every GDLZ model to a GDL model. However, since any GDLZ path is a finite sequence of states and joint actions, we can define a partial translation from GDLZ ST-models to GDL ST-models based on the reached states and joint actions performed in a complete path. In other words, we can translate a run in a GDLZ model into a GDL model. Let us formally describe the translation.

Through the rest of this section, we fix the GDLZ ST-model $M = (W, w, T, L, U, g, \pi_\Phi, \pi_\Sigma)$ with a game signature $S = (N, A, X, \Phi)$ and the complete path $\delta = w \xrightarrow{d_1} w_1 \xrightarrow{d_2} \cdots \xrightarrow{d_e} w_e$ in $M$.

Given the path $\delta$ in $M$, we next define a shortcut to refer to the smallest and biggest integers occurring in $\delta$ and the set of all actions performed in $\delta$.

**Definition 6.** Given $M$ and $\delta$, we denote $\delta_{\text{min}}$ and $\delta_{\text{max}}$ as the smallest and biggest integer, respectively, occurring in any parameter list $z$ from any action $a \in \{d_1, d_2, \ldots, d_e\}$ and in any $\pi_\Sigma(w)$, for $w \in \{w, w_1, \ldots, d_e\}$.

**Definition 7.** Given $M$ and $\delta$, let $A^\delta = \{d_j(r) : r \in N \& 1 \leq j \leq e\}$ denote the set of all actions performed in $\delta$.

Since we are aware of the path numerical range, we are able to construct a partial model translation. The translation is restricted to the states and actions involved in a given path.

**Definition 8.** Given a GDLZ ST-model $M$ and $\delta$, we construct an associated GDL ST-model $M' = (W', w', T', L', U', g', \pi')$ with a game signature $S' = (N', A', \Phi')$. The components $w$ and $N = \{r_1, \ldots, r_N\}$ are the same for $M$ and $M'$.

The propositional set $\Phi'$ is constructed over both $\Phi$ and $X$ as follows: $\Phi' = \{p, \text{smaller}(z_1, z_2), \text{bigger}(z_1, z_2), \text{equal}(z_1, z_2), \text{succ}(z_1, z_2), \text{prec}(z_1, z_2), x(q) : p \in \Phi, x \in X, \delta_{\text{min}} \leq q_1, z_1, z_2 \leq \delta_{\text{max}}\}$. The notation $x(q)$ represents the proposition “variable $x$ has the value $q$”.

For integrating the GDLZ comparison operators $<, >$ and $=$ in GDL, we need to define the order between the numerical terms in the translated model. Let $\pi_\Phi \subseteq \Phi'$ denote a set of propositions describing the numerical order, such as: $\pi_\Phi = \{\text{succ}(z, z + 1), \text{prec}(z + 1, z), \text{equal}(z_1, z_2) : \delta_{\text{min}} \leq z < \delta_{\text{max}} \& \delta_{\text{min}} \leq z_1 \leq \delta_{\text{max}}\} \cup \{\text{smaller}(z_1, z_2) : \delta_{\text{min}} \leq z_1, z_2 \leq \delta_{\text{max}} \& \delta_{\text{min}} < z_1 \leq \delta_{\text{max}}\} \cup \{\text{bigger}(z_1, z_2) : \delta_{\text{min}} \leq z_1, z_2 \leq \delta_{\text{max}} \& \delta_{\text{min}} < z_1 \leq \delta_{\text{max}}\}$. For any $a^r(z_1, \ldots, z_l) \in A^\delta$, $a^r_2z_2, \ldots, z_l \in A'$. We define an action translation $Tr^r : A^\delta \rightarrow A'$ associating every action in $A^\delta$ with an action in $A'$:

$Tr^r(a^r(z_1, \ldots, z_l)) = a^r_{z_1, \ldots, z_l}$

where $a^r(z_1, \ldots, z_l) \in A^\delta$.

The $M'$ components $W', T', L', U', g'$ and $\pi'$ are defined as follows:

- $W' = \{w, w_1, \ldots, w_e\}$
- $T' = \{w_e\}$
Given a path set \( N = \{r_1, \ldots, r_k\} \), define a path translation \( \text{Tr}^\lambda : \delta \to \delta' \) associating a path \( \delta = \vec{w} \xrightarrow{d_1} w_1 \xrightarrow{d_2} \cdots \xrightarrow{d_e} w_e \) in \( M \) with a path \( \delta' \) in \( \text{Tr}^m(M, \delta) \): \( \text{Tr}^\lambda(\delta) = \vec{w} \xrightarrow{d'_1} w_1 \xrightarrow{d'_2} \cdots \xrightarrow{d'_e} w_e \) where \( d'_i = (\text{Tr}^a(d_i(r_1)), \ldots, \text{Tr}^a(d_i(r_k))) \). For any state sequence, if \( w = (\vec{w}, T') \) in \( M \), we have that \( \text{Tr}^\lambda(\delta) = \vec{w} \xrightarrow{d'_1} w_1 \xrightarrow{d'_2} \cdots \xrightarrow{d'_e} w_e \), where \( (\vec{w}, T') = (\text{Tr}^a(w, T), \text{Tr}^a(T')) \).

As shown next propositions, given a path in a GDLZ model, the translation of the GDLZ model is a GDL model. Moreover, the translation of a path in a GDLZ model is a path in the translation of the GDLZ model.

**Proposition 4.** If \( M \) is a GDL model and \( \delta \) a complete path in \( M \), then \( \text{Tr}^m(M, \delta) \) is a GDL ST-model.

**Proof.** Given \( M \) and \( \delta \), let \( \text{Tr}^m(M, \delta) = (W', \vec{w}', T', U', g', \pi') \), with \( S' = (N, \mathcal{A}', \Phi') \). Since \( \delta \) is a finite sequence of states and joint actions, we have that \( W', \mathcal{A}' \) and \( \Phi' \) are ensured to be finite sets.

Since \( \text{Tr}^a : \mathcal{A}^\delta \to \mathcal{A}' \) is an injective function, each \( a \in \mathcal{A}^\delta \) will be assigned to a unique \( a' \in \mathcal{A}' \). By the path definition, we know that \( d_j(r) \in L(w_{j-1}) \), for every \( r \in N, 1 \leq j \leq e \). Then, it is easy to see that \( L' \subseteq W' \times \mathcal{A}' \). By \( \text{Tr}^m \) definition, we know that \( \vec{w} \in W', T' \subseteq W' \) and \( g'(r) \subseteq \{\{w_e\}, \emptyset\} \), thereby \( g'(r) \subseteq 2^W' \), for \( r \in N \). Furthermore, for every stage \( 1 \leq j \leq e \), we have \( U'(w_{j-1}, (\text{Tr}^a(d_i(r_1)), \ldots, \text{Tr}^a(d_i(r_k)))) = U(w_{j-1}, (d_j(r_1), \ldots, d_j(r_k))) \), thus \( U'(w_{j-1}, (\text{Tr}^a(d_i(r_1)), \ldots, \text{Tr}^a(d_i(r_k)))) \in W' \). Finally, since \( \text{Tr}^m \) defines \( \Phi' = \{\Phi, \lambda X, \delta_{min} \leq q, z_1, z_2, \delta_{max}\} \), then for every \( w \in W' \), we have that \( \pi'(w) \in \{\pi \phi(w) \cup \pi z \cup \{q \in \pi z(w), x \in X\}\} \) and thus \( \pi'(w) \subseteq 2^{\mathbb{W'}} \). Therefore, \( \text{Tr}^m(M, \delta) \) is a GDL ST-model.

**Proposition 5.** If \( \delta \) is a path in a GDLZ model \( M \) then \( \text{Tr}^\lambda(\delta) \) is a path in \( \text{Tr}^m(M, \delta) \).

**Proof.** Given \( M \), \( \delta \) and \( \text{Tr}^m(M, \delta) = (W', \vec{w}', T', U', g', \pi') \) with \( S = (N, \mathcal{A}', \Phi') \). Then \( \text{Tr}^\lambda(\delta) = \vec{w} \xrightarrow{d'_1} w_1 \xrightarrow{d'_2} \cdots \xrightarrow{d'_e} w_e \). By the GDLZ path definition, for \( e \geq 0 \) and for any \( j \in \{1, \ldots, e\} \), we have \( \{w_0, \cdots, w_{e-1}\} \cap T = \emptyset \), where \( w_0 = \vec{w} \).

For every \( r \in N \), we have that \( d_j(r) \in L(w_{j-1}) \). Since the action translation \( \text{Tr}^a \) assigns each action in \( \mathcal{A}^\delta = \{d_j(r) : r \in N, 1 \leq j \leq e\} \) to an
unique action in $A'$, then the translation from the action of agent $r$ in the joint action $d_j$ will be in the set of the translated legal actions in state $w_{j-1}$, i.e. $Tr_{\alpha}(d_j(r)) \in L'(w_{j-1})$, where $L'(w_{j-1}) = \{ Tr_{\alpha}(a) \in A^\delta | (w_{j-1}, Tr_{\alpha}(a)) \in L' \}$. Thus, $Tr_{\alpha}(d_j(r)) \in L'(w_{j-1})$. Finally, since $\delta$ is path, $w_j = U(w_{j-1}, d_j) = U(w_{j-1}, (d_j(r_1), \ldots, d_j(r_k)))$. Then, $w_j = U'(w_{j-1}, (Tr_{\alpha}(d_j(r_1)), \ldots, Tr_{\alpha}(d_j(r_k))))$, that is, $w_j = U'(w_{j-1}, (a^{r_1'}, \ldots, a^{r_k'})) = U'(w_{j-1}, d'_{j})$.

Thus, we have that $Tr_{\lambda}(\delta) = \bar{w} \overset{d_1'}{\rightarrow} \overset{d_2'}{\rightarrow} \cdots \overset{d'_{k}}{\rightarrow} w_c$ is a path in the GDL ST-model $Tr^m(M, \delta)$. Furthermore, if $\delta$ a complete path in $M$, then $w_c \in T$ and $Tr_{\lambda}(\delta)$ is also a complete path in $Tr^m(M, \delta)$.

Next, we show how to translate GDLZ formulas to GDL. Likewise to the model translation, the translation is restricted to a path.

**From GDLZ Paths and Formulas to GDL Formulas.** Let us briefly recall GDL grammar. Given a GDL game signature $S' = (N, A', \Phi')$, a formula $\varphi' \in \mathcal{L}_{GDL}$ is defined by the following BNF:

$$\varphi' ::= p \mid initial \mid terminal \mid legal(a') \mid wins(r) \mid does(a') \mid \neg \varphi \mid \varphi \land \varphi \mid \bigcirc \varphi$$

where $p \in \Phi'$, $r \in N$ and $a' \in A'$.

Given a path $\delta$ in a GDLZ ST-model $M$, we next define a translation for formulas in $\mathcal{L}_{GDLZ}$ to $\mathcal{L}_{GDL}$. Each numerical term $z \in L$ occurring in a formula $\varphi \in \mathcal{L}_{GDLZ}$ is translated by its semantic interpretation through function $v$ (see Definition 4).

**Definition 10.** Given a GDLZ ST-model $M = (N, A, X, \Phi)$, a path $\delta$ in $M$, a stage $j$ in $\delta$ and function $v$ (see Definition 4). A translation $Tr_{\varphi}$ from a formula $\varphi \in \mathcal{L}_{GDLZ}$ in a state $\delta[j]$ to a formula $\varphi' \in \mathcal{L}_{GDL}$ is defined as follows:

- $Tr_{\varphi}(\varphi, \delta[j]) = \varphi$ for all $\varphi \in \Phi \cup \{ initial, terminal, wins(r) \}$;
- $Tr_{\varphi}(\neg \varphi, \delta[j]) = \neg Tr_{\varphi}(\varphi, \delta[j])$;
- $Tr_{\varphi}(\varphi_1 \land \varphi_2, \delta[j]) = Tr_{\varphi_1}(\varphi_1, \delta[j]) \land Tr_{\varphi_2}(\varphi_2, \delta[j])$;
- $Tr_{\varphi}(\bigcirc \varphi, \delta[j]) = \bigcirc Tr_{\varphi}(\varphi, \delta[j + 1])$;
- $Tr_{\varphi}(\text{legal}(\alpha', \delta[j]), \delta[j]) = \text{legal}(Tr_{\alpha}(\alpha', v(z : z \in \bar{z})))$ iff $\text{legal}(\alpha', v(z, \delta[j]) : z \in \bar{z}) = \delta, \delta[j]$; otherwise $Tr_{\varphi}(\text{legal}(\alpha', \delta[j]), \delta[j]) = \neg \text{legal}(Tr_{\alpha}(\alpha', v(z : z \in \bar{z})))$;
- $Tr_{\varphi}(\text{does}(\alpha', \delta[j]), \delta[j]) = \text{does}(Tr_{\alpha}(\alpha', v(z : z \in \bar{z})))$;
- $Tr_{\varphi}(\langle \bar{z}, \delta[j] \rangle) = \bigwedge_{i=1}^{\bar{z}} x_i(v(q_i, \delta[j]))$;
- $Tr_{\varphi}(z_1 < z_2, \delta[j]) = \text{smaller}(v(z_1, \delta[j]), v(z_2, \delta[j]))$;
- $Tr_{\varphi}(z_1 > z_2, \delta[j]) = \text{bigger}(v(z_1, \delta[j]), v(z_2, \delta[j]))$;
- $Tr_{\varphi}(z_1 = z_2, \delta[j]) = \text{equal}(v(z_1, \delta[j]), v(z_2, \delta[j]))$.

Where $r \in N, x_i \in X, q_i$ is the $i$-th value in $\bar{z}$ and $0 \leq i \leq |\bar{z}|$.

Given a path in a GDLZ model, we show that the translation of a GDLZ formula is a GDL formula. Furthermore, if the GDLZ formula is valid at a stage in the path, its translation will be valid at the same stage in the translated path in the translated model.
Proposition 6. Given a GDLZ ST-model $M$, a path $\delta$ in $M$, a stage $j$ in $\delta$ and function $v$, if $\varphi$ is a formula in $\mathcal{L}_{GDLZ}$ then $\text{Tr}^\varphi(\varphi, [j])$ is a formula in $\mathcal{L}_{GDL}$.

Proof. Given a GDLZ model $M = (W, \bar{w}, T, L, U, g, \pi_\Phi, \pi_Z)$, with a game signature $S = (N, A, \Phi, X)$, a path $\delta$ in $M$, a stage $j$ in $\delta$ and function $v$. Let $M' = \text{Tr}^m(M, \delta)$, with $S' = (N, A', \Phi')$. Assume that $\varphi \in \mathcal{L}_{GDLZ}$, we show that $\text{Tr}^\varphi(\varphi, [j]) \in \mathcal{L}_{GDL}$ for each form of $\varphi$:

- If $\varphi$ is of the form $p, \text{initial}, \text{terminal}, \text{wins}(r), \neg \varphi, \varphi \land \varphi$ or $\bigcirc \varphi$, where $p \in \Phi$ and $r \in N$, then $\text{Tr}^\varphi(\varphi, [j])$ assigns $\varphi$ to the exactly corresponding $\varphi' \in \mathcal{L}_{GDL}$. Thus, $\text{Tr}^\varphi(\varphi, [j]) \in \mathcal{L}_{GDL}$.

- If $\varphi$ is of the form $\text{legal}(a^r(\bar{z}))$ or $\text{does}(a^r(\bar{z}))$, where $r \in N$, then $\text{Tr}^\varphi(\varphi, [j]) = \text{legal}(\text{Tr}^\varphi(a^r(v(z) : z \in \bar{z})))$ or $\text{Tr}^\varphi(\varphi, [j]) = \text{does}(\text{Tr}^\varphi(a^r(v(z) : z \in \bar{z})))$, respectively. Since $\text{Tr}^\varphi$ is an injective function from $\mathcal{A}^\delta$ to $\mathcal{A}'$, we have that $\text{Tr}^\varphi(a^r) = a'^r \in \mathcal{A}'$. Therefore, $\text{legal}(a'^r), \text{does}(a'^r) \in \mathcal{L}_{GDL}$ and $\text{Tr}^\varphi(\varphi, [j]) \in \mathcal{L}_{GDL}$.

- If $\varphi$ is of the form $z_1 < z_2, z_1 > z_2$, or $z_1 = z_2$, we show that each $\text{smaller}(z_1, z_2), \text{bigger}(z_1, z_2), \text{equal}(z_1, z_2), \delta_{\min} \leq q_1, z_1, z_2 \leq \delta_{\max}$ belongs to $\mathcal{L}_\Phi$ and $\varphi_1 \land \varphi_2 \in \mathcal{L}_{GDL}$. To show that $\varphi \in \mathcal{L}_{GDL}$.

- Finally, if $\varphi$ is of the form $(\bar{z})$, then $\text{Tr}^\varphi(\varphi, [j]) = \text{legal}(\text{Tr}^\varphi(x_i(v(z) : z \in \bar{z})))$, where $x_i \in X, q_i$ is the $i$-th value of $\bar{z}$ and $0 \leq i \leq |\bar{z}|$. We have that $\{x_i(q) : x \in X, \delta_{\min} \leq q \leq \delta_{\max}\} \subseteq \Phi'$. Since that for each $p \in \Phi', p \in \mathcal{L}_{GDL}$, we have that each $x_i(q_i) \in \mathcal{L}_{GDL}$. Moreover, for any $\varphi_1, \varphi_2 \in \mathcal{L}_{GDL}$, we also have $\varphi_1 \land \varphi_2 \in \mathcal{L}_{GDL}$, then $(x_1(q_1) \land (x_2(q_2)) \cdot \cdot \cdot \land x_||q||) = x_1(q_1) \cdot \cdot \cdot \land x_||q||) = \text{Tr}^\varphi(\varphi, [j]) \in \mathcal{L}_{GDL}$.

Theorem 1. If $M, \delta, j \models \varphi$ then $\text{Tr}^m(M, \delta), \text{Tr}^\lambda(\delta), j \models \text{Tr}^\varphi(\varphi, [j])$.

Proof. Given a GDLZ model $M = (W, \bar{w}, T, L, U, g, \pi_\Phi, \pi_Z)$, with the game signature $S = (N, A, \Phi, X)$, a complete path $\delta$, a stage $j$ on $\delta$, a formula $\varphi \in \mathcal{L}_{GDLZ}$ and the function $v$. Let $M' = (W', \bar{w}, T', L', U', g', \pi')$, with $S' = (N, A', \Phi')$, be the GDL translation of $M$, i.e. $M' = \text{Tr}^m(M, \delta), \delta' = \text{Tr}^\lambda(\delta)$ and $\delta_{\min}, \delta_{\max} \in \mathbb{Z}$ denote the integer bounds in $\delta$.

For any integers $\delta_{\min} \leq z_1, z_2 < \delta_{\max}, \pi_\varphi \subseteq \pi'(\delta[j])$ enumerated its predecessor and successor and define all the cases were bigger$(z_1, z_2)$, smaller$(z_1, z_2)$ and equal$(z_1, z_2)$ are true. Let $\varphi' = \text{Tr}^\varphi(\varphi, [j])$. We assume that $M, \delta, j \models \varphi$ and show that then we have $M', \delta', j \models \varphi'$ for every $\varphi$.

- If $\varphi$ is on the form $p \in \Phi$, we have $\text{Tr}^\varphi(p, [j]) = p$. By $\mathcal{L}_{GDLZ}$ semantics, we know that $p \in \pi_\varphi(\delta[j])$. In the ST-model translation, we have the valuation function constructed such that $\pi'(\delta[j]) = \{\pi_\varphi(\delta[j])\} \cup \{\pi_\varphi \cup \{x(q) : q \in \pi_Z(\delta[j]), x \in X\}\}. Then, $p \in \pi'(\delta[j])$ and $M', \delta', j \models p$.

- If $\varphi$ is on the form $\neg \psi, \varphi_1 \land \varphi_2, \text{initial}, \text{terminal}, \text{wins}(r), \text{legal}(a^r(\bar{z}))$, $\text{does}(a^r(\bar{z}))$, or $\bigcirc \varphi$, since $\text{Tr}^m$ and $\text{Tr}^\varphi$ assigns each GDLZ action and formula to an unique GDL state, action and formula, respectively, due to both languages semantics it is easy to see that $M', \delta', j \models \text{Tr}^\varphi(\varphi, [j])$, whenever $M, \delta, j \models \varphi$.  

If \( \phi \) is on the form \( z_1 > z_2 \), we have \( Tr^\phi(z_1 > z_2, \delta[j]) = \text{bigger}(v(z_1, \delta[j]), v(z_2, \delta[j])) \). By \( \mathcal{L}_{GDLZ} \) semantics, we know that \( v(z_1, \delta[j]) > v(z_2, \delta[j]) \), i.e. \( v(z_1, \delta[j]) \) is bigger then \( v(z_2, \delta[j]) \), then \( \text{bigger}(v(z_1, \delta[j]), v(z_2, \delta[j])) \in \pi_z \).

3.2 From Finite GDLZ Model to GDL Model

Let us consider the case where the GDLZ ST-model has finite components. In this case, we are able to define a complete model translation, instead of partial based on a path. In other words, all possible runs over the finite GDLZ ST-model can be translated. Next, we characterize a finite GDLZ ST-model.

**Definition 11.** Given two arbitrary bounds \( z_{\min} \leq z_{\max} \in \mathbb{Z} \), a finite GDLZ ST-model \( M_f = (W_f, \bar{w}_f, T_f, L_f, U_f, g_f, \pi_f, \pi_{\Sigma_f}) \), with the game signature \( \mathcal{S}_f = (N_f, \mathcal{A}_f, \mathcal{X}_f, \Phi_f) \) is a subset of GDLZ ST-models that have the following aspects: (i) \( z_{\min} \leq z_{\max} \) for any \( a^r(z_1, \cdots, z_i) \in \mathcal{A}_f, 1 \leq i \leq o \) and \( r \in N_f \); (ii) \( W_f \) and \( \mathcal{A}_f \) are finite sets; and (iii) \( z_{\min} \leq q_i \leq z_{\max} \), for any \( (q_1 \cdots q_n) = \pi_{\Sigma}(w) \), \( 1 \leq i \leq n \) and \( w \in W_f \).

Through the rest of this section, we fix the bounds \( z_{\min} \) and \( z_{\max} \) as well as the finite GDLZ ST-model \( M_f = (W_f, \bar{w}_f, T_f, L_f, U_f, g_f, \pi_f, \pi_{\Sigma_f}) \) with a game signature \( \mathcal{S}_f = (N_f, \mathcal{A}_f, \mathcal{X}_f, \Phi_f) \) and \( N_f = \{ r_1, \cdots, r_k \} \). Let us show how any finite GDLZ ST-model can be translated into a GDL ST-model.

**Definition 12.** Given the finite GDLZ ST-model \( M_f \) and its signature \( \mathcal{S}_f \), we define the GDL ST-model \( M'_f = (W_f, \bar{w}_f, T_f, L'_f, U'_f, g_f, \pi'_f) \) with a game signature \( \mathcal{S}'_f = (N_f, \mathcal{A}_f', \Phi'_f) \). The components \( W_f, \bar{w}_f, T_f, g_f \) and \( N_f \) are the same for \( M_f \) and \( M'_f \).

We construct \( \Phi'_f \) over both \( \Phi_f, X_f \) and its values. Although \( X_f \) is a finite set, each one of its components has an integer value in each state \( w \in W_f \). As \( \Phi'_f \) is finite, we construct it with the bounds \( z_{\min} \) and \( z_{\max} \). Since \( \mathbb{Z} \) is a countable set, for any \( z_{\min} \) and \( z_{\max} \), we can define a finite enumeration of integer values.
The set of atomic propositions is defined as follows: \( \Phi^a_f = \{p, \text{smaller}(z_1, z_2), \text{bigger}(z_1, z_2), \text{equal}(z_1, z_2), \text{prec}(z_1, z_2), x(q) : p \in \Phi, x \in X_f, z_{\text{min}} \leq q, z_1, z_2 \leq z_{\text{max}}\} \).

We define an action translation \( \text{Tr}^a_f : \mathcal{A}_f \rightarrow \mathcal{A}'_f \) associating every action in \( \mathcal{A}_f \) with an action in \( \mathcal{A}'_f \) as follows:

\[
\text{Tr}^a_f(a^r(z_1, \cdots, z_i)) = a^r_{2i, \cdots, z_i}
\]

where \( a^r(z_1, \cdots, z_i) \in \mathcal{A}_f, z_{\text{min}} \leq z_i \leq z_{\text{max}} \) and \( 0 \leq i \leq l \).

Note that \( \text{Tr}^a_f \) is an injective function. Thereby, we can define the GDL components \( \mathcal{A}'_f \) and \( L'_f \) based on \( \text{Tr}^a_f \), as follows: (i) \( \mathcal{A}'_f = \{\text{Tr}^a_f(a^r(z_1, \cdots, z_i)) : a^r(z_1, \cdots, z_i) \in \mathcal{A}_f\}\); and (ii) \( L'_f = \{(w, \text{Tr}^a_f(a)) : (w, a) \in L_f\}\). For each \( w \in W_f \), each \( r \in N_f \) and each joint action \( (a^{r_1}, \ldots, a^{r_n}) \in \prod_{r \in N_f} \mathcal{A}'_f \), where \( \mathcal{A}'_f \in \mathcal{A}_f \), the update function is defined as: \( U_f(w, (\text{Tr}^a_f(a^{r_1}), \ldots, \text{Tr}^a_f(a^{r_n}))) = U_f(w, (a^{r_1}, \ldots, a^{r_n})) \).

Let \( \pi_{\Phi_f} \) be a set of propositions describing the numerical order, such that \( \pi_{\Phi_f} = \{\text{succ}(z, z + 1), \text{prec}(z + 1, z), \text{equal}(z_1, z_1) : z_{\text{min}} \leq z \leq z_{\text{max}} \} \cup \{\text{smaller}(z_1, z_2) : z_{\text{min}} \leq z_1, z_2 \leq z_{\text{max}} \} \cup \{\text{bigger}(z_1, z_2) : z_{\text{min}} \leq z_1, z_2 \leq z_{\text{max}} \} \). Finally, for all \( w \in W_f \), we construct the valuation \( \pi'_f(w) = \{\pi_{\Phi_f}(w) \cup \pi_{\Phi_f} \cup \{x(q) : q \in \pi_{\Phi_f}(w), x \in X_f\}\} \).

We note that \( M'_f \) is a bounded ST-model translation of \( M_f \) and write \( \text{Tr}^m_f(M_f) \).

The path translation consists at assigning each action appearing on it to the appropriated GDL action through \( \text{Tr}^a_f \).

**Definition 13.** Define a path translation \( \text{Tr}^\lambda_f : \delta_f \rightarrow \delta'_f \) associating every path \( \delta_f = w_f \xrightarrow{d_1} w_1 \xrightarrow{d_2} \ldots \xrightarrow{d_e} w_e \) in \( M_f \) with a path \( \delta'_f \) in \( M'_f \); \( \text{Tr}^\lambda_f(\delta_f) = \tilde{w}_f \xrightarrow{d'_1} \tilde{w}_1 \xrightarrow{d'_2} \ldots \xrightarrow{d'_e} \tilde{w}_e \) where \( d'_i = (a^{r_1}, \ldots, a^{r_n}) \in D_f, D_f = \prod_{r \in N_f} \mathcal{A}'_f \), \( A'_f \in \mathcal{A}_f, w_i \in W_f, d'_i = (\text{Tr}^a_f(a^{r_1}), \ldots, \text{Tr}^a_f(a^{r_n})) \) and \( 1 \leq i \leq e \).

It follows that the translations of a finite GDLZ model and a path in a finite GDLZ model are a model and a path in GDL, respectively.

**Proposition 7.** If \( M_f \) is a finite GDLZ model then \( \text{Tr}^m_f(M_f) \) is a GDL model.

**Proof.** Given \( M_f, S_f, \mathcal{S}'_f, \text{Tr}^m_f(M_f) = (W_f, \tilde{w}_f, T_f, L'_f, U'_f, g'_f, \pi'_f) \), with \( S'_f = (N_f, \mathcal{A}'_f, \Phi'_f) \), the integer bounds \( z_{\text{min}} \) and \( z_{\text{max}} \) and the construction of \( \text{Tr}^m_f \), we have that both the \( W_f, A'_f \) and \( \Phi'_f \) are ensured to be finite. Since \( \text{Tr}^a_f \) is an injective function, the proof proceeds in a similar way to the proof for Proposition 4.

**Proposition 8.** If \( \delta_f \) is a path in a finite GDLZ model \( M_f \) then \( \text{Tr}^\lambda_f(\delta_f) \) is a path in \( \text{Tr}^m_f(M_f) \).

**Proof.** Given \( \text{Tr}^a_f \) and \( \text{Tr}^\lambda_f \), the proof proceeds as the proof for Proposition 5.

Next, we show a complete translate from GDLZ formulas to GDL formulas. Likewise to the model translation, we use arbitrary bounds to restrict the numerical range in the formulas.
From bounded GDLZ Formulas to GDL Formulas. Assuming a GDLZ game signature $\mathcal{S}_f = (N_f, A_f, \Phi_f, X_f)$, the semantics of a numerical variable $x \in X_f$ in a $L_{GDLZ}$ formula is evaluated depending on the current game state.

To translate the meaning of a numerical variable $x \in X_f$ occurring in an atomic formula $\varphi \in L_{GDLZ}$ in the form $\text{legal}(a''(\vec{z}))$, $\text{does}(a''(\vec{z}))$, $\langle \vec{z} \rangle$, $z_1 < z_2$, $z_1 > z_2$ or $z_1 = z_2$, Algorithm 2, denoted $\text{removeVar}(\varphi)$, defines an intermediate formula $\varphi_x$ as the disjunction from all possible values $z_{\text{min}} \leq q \leq z_{\text{max}}$ for $x$ in $\varphi$ and $x(q)$. Algorithm 2 stops when there is no more occurrence of numerical variables in the resulting formula.

**Algorithm 2** removeVar($\varphi$)

**Input:** a formula $\varphi \in L_{GDLZ}$. Assume the variable set $X_f$ and $z_{\text{min}} \leq z_{\text{max}}$.

**Output:** a partially translated formula.

1. $I \leftarrow \{z_{\text{min}}, \ldots, z_{\text{max}}\}$
2. if ($\varphi = \text{"legal}(a''(z_1, \ldots, z_m))"$) then
3.   for each $z_i \in (z_1, \ldots, z_m)$ do
4.     if $z_i \in X_f$ then return $\bigvee_{q_i \in I}(\text{removeVar}(\text{legal}(a''(z_1, \ldots, q_i, \ldots, z_m)) \land z_i(q_i)))$
5. else if ($\varphi = \text{"does}(a''(z_1, \ldots, z_m))"$) then Proceeds as the previous case.
6. else if ($\varphi = \langle z_1, \ldots, z_m \rangle$) then
7.   for each $z_i \in (z_1, \ldots, z_m)$ do
8.     if $z_i \in X_f$ then return $\bigvee_{q_i \in I}(\text{removeVar}(\langle z_1, \ldots, q_i, \ldots, z_m \rangle) \land z_i(q_i)))$
9. else if ($\varphi = \text{"z_1 < z_2"}$) then
10.    if $z_1 \in X_f$ then return $\bigvee_{q_1 \in I}(\text{removeVar}(q_1 < z_2) \land z_1(q_1))$
11.   if $z_2 \in X_f$ then return $\bigvee_{q_2 \in I}(\text{removeVar}(z_1 < q_2) \land z_2(q_2))$
12. else if ($\varphi = \text{"z_1 > z_2"}$ or $\varphi = \text{"z_1 = z_2"}$) then Proceeds as the previous case.

   return $\varphi$

A numerical simple term $z_f$ is defined by $L_{z_f}$, which is generated by the following BNF:

$$z_f ::= z' | \text{add}(z_f, z_f) | \text{sub}(z_f, z_f) | \text{min}(z_f, z_f) | \text{max}(z_f, z_f)$$

where $z' \in \mathbb{Z}$. Note that $L_{z_f} \subseteq L_z$. Each numerical term $z_f \in L_{z_f}$ occurring in a formula $\varphi \in L_{GDLZ}$ is translated by its semantic interpretation through function $v_f$, defined in a similar way to Definition 4.

**Definition 14.** Let us define function $v_f : L_{z_f} \rightarrow \mathbb{Z}$, associating any $z_f \in L_{z_f}$ to a number in $\mathbb{Z}$:
Let \( \phi \) be a formula in GDLZ.

**Definition 15.** \( \phi \in \mathcal{L}_{GDLZ} \) is a bounded formula if, for any numerical term \( z_f \) occurring in \( \phi \), we have \( z_f \in \mathcal{L}_{z_f} \) and \( z_{\min} \leq v_f(z) \leq z_{\max} \) or if there is no occurrence of numerical terms in \( \phi \).

We next define a translation map for bounded formulas in \( \mathcal{L}_{GDLZ} \) to formulas in \( \mathcal{L}_{GDL} \). Each numerical simple term \( z_f \in \mathcal{L}_{z_f} \) occurring in a formula \( \varphi \in \mathcal{L}_{GDLZ} \) is translated by its semantic interpretation through function \( v_f \) (see Definition 14).

**Definition 16.** Given the GDLZ game signature \( S_f = (N_f, A_f, X_f, \Phi_f) \) and function \( v_f \), a translation \( \text{Tr}_f^z \) from a bounded formula \( \varphi \in \mathcal{L}_{GDLZ} \) to a formula \( \varphi' \in \mathcal{L}_{GDL} \) is defined as \( \text{Tr}_f^z = \text{Tr}_f^z(\text{removeVar}(\varphi)) \), where \( \text{Tr}_f^z \) is specified as follows:

1. \( \text{Tr}_f^z(\varphi) = \varphi \) for all \( \varphi \in \Phi_f \cup \{\text{initial, terminal, wins}(r)\} \cup \{x(q) : x \in X_f, z_{\min} \leq q \leq z_{\max}\}; \)
2. \( \text{Tr}_f^z(\varphi) = \text{Tr}_f^z(\text{removeVar}(\varphi)); \)
3. \( \text{Tr}_f^z(\psi_1 \land \psi_2) = \text{Tr}_f^z(\text{removeVar}(\psi_1)) \land \text{Tr}_f^z(\text{removeVar}(\psi_2)); \)
4. \( \text{Tr}_f^z(\neg \psi) = \neg \text{Tr}_f^z(\text{removeVar}(\psi)); \)
5. \( \text{Tr}_f^z(\phi_1 \lor \phi_2) = \text{Tr}_f^z(\text{removeVar}(\phi_1)) \lor \text{Tr}_f^z(\text{removeVar}(\phi_2)); \)
6. \( \text{Tr}_f^z(\phi_1 \rightarrow \phi_2) = \text{Tr}_f^z(\text{removeVar}(\phi_1)) \rightarrow \text{Tr}_f^z(\text{removeVar}(\phi_2)); \)
7. \( \text{Tr}_f^z(\text{legal}(a^i(z))) = \text{legal}(\text{Tr}_f^z(a^i(v_f(z)) : z \in \mathcal{Z})); \)
8. \( \text{Tr}_f^z(\text{does}(a^i(z))) = \text{does}(\text{Tr}_f^z(a^i(v_f(z)) : z \in \mathcal{Z})); \)
9. \( \text{Tr}_f^z(\text{smaller}(z_1, z_2)) = \text{smaller}(\text{Tr}_f^z(z_1), \text{Tr}_f^z(z_2)); \)
10. \( \text{Tr}_f^z(\text{bigger}(z_1, z_2)) = \text{bigger}(\text{Tr}_f^z(z_1), \text{Tr}_f^z(z_2)); \)
11. \( \text{Tr}_f^z(\text{equal}(z_1, z_2)) = \text{equal}(\text{Tr}_f^z(z_1), \text{Tr}_f^z(z_2)); \)

Where \( r \in N_f, x_i \in X_f, q_i \) is the \( i \)-th value in \( \mathcal{Z} \) and \( 0 \leq i \leq |\mathcal{Z}| \).

Let us illustrate the translation of GDLZ formulas into GDL using \( \text{Tr}_f^z \).

**Example 2.** Let \( I = \{z_{\min}, \ldots, z_{\max}\} \) and \( \varphi_1 = \text{does}(\text{reduce}(\text{heap}_1, \text{add}(1, 2))); \)
where \( \text{heap}_1 \in X_f \), then \( \text{Tr}_f^z(\varphi_1) = \bigvee_{h_1 \in \{z_{\min}, \ldots, z_{\max}\}} (\text{does}(\text{reduce}(h_1, 3)) \land \text{heap}_1(h_1)). \)

The translation of a GDLZ formula is a GDL formula. Furthermore, if the GDLZ formula is valid at a stage in the path in a finite GDLZ model, then its translation will be valid at the same stage in the translated path in the translated model.
Proposition 9. If \( \varphi \in \mathcal{L}_{\text{GDLZ}} \) then \( \text{Tr}_f^\pi(\varphi) \in \mathcal{L}_{\text{GDL}} \).

Proof. Given a finite GDLZ model \( M_f \) with the game signature \( \mathcal{S} = (\mathcal{N}, \mathcal{A}, \Phi, X) \) and \( M'_f = \text{Tr}_f^\pi(M_f) \), with \( \mathcal{S}'_f = (\mathcal{N}'_f, \mathcal{A}'_f, \Phi'_f) \). Assume that \( \varphi \in \mathcal{L}_{\text{GDLZ}} \), since \( \text{Tr}_f^\pi(\varphi) = \text{Tr}_f^\pi(\text{removeVar}(\varphi)) \), we need to show that \( \text{Tr}_f^\pi(\text{removeVar}(\varphi)) \in \mathcal{L}_{\text{GDL}} \) for each form of \( \varphi \). If there is a numerical variable \( x \) in an atomic formula \( \varphi \), the method \( \text{removeVar}(\varphi) \) constructs \( \varphi_x \) as a disjunction from \( \varphi \) with every possible value of \( x \) between \( z_{\text{min}} \) and \( z_{\text{max}} \) and the proposition \( x(q) \). By \( \text{Tr}_f^m \) definition, \( \{ x(q) : x \in X, z_{\text{min}} \leq q \leq z_{\text{max}} \} \subseteq \mathcal{F} \) and thereby \( x(q) \in \mathcal{L}_{\text{GDL}} \).

The translation \( \text{Tr}_f^\pi \) proceeds assigning each subformula of \( \text{removeVar}(\varphi) \) to a \( \mathcal{L}_{\text{GDL}} \) formula. The proof proceeds as the proof for Proposition 8.

Theorem 2. If \( M_f \) is a finite GDLZ ST-model, \( \varphi \in \mathcal{L}_{\text{GDLZ}} \) is a bounded formula and \( M_f, \delta_f, j \models \varphi \) then \( \text{Tr}_f^\pi(M_f), \text{Tr}_f^\pi(\delta_f), j \models \varphi \).

Proof. Given a finite GDLZ ST-model \( M_f = (W_f, \bar{w}_f, T_f, L_f, U_f, g_f, \pi_{\Phi}, \pi_{\pi_f}) \), with the game signature \( \mathcal{S} = (\mathcal{N}, \mathcal{A}, X, \Phi) \), a complete path \( \delta_f \), a stage \( j \) on \( \delta_f \), a formula \( \varphi \in \mathcal{L}_{\text{GDLZ}} \) and the function \( \text{Tr}_f^\pi \). Let \( \text{Tr}_f^\pi(M_f) = M'_f = (W_f, \bar{w}_f, T_f, L_f', U_f, g_f, \pi'_f) \), \( \delta'_f = \text{Tr}_f^\pi(\delta_f) \), \( \varphi' = \text{Tr}_f^\pi(\varphi) \) and \( z_{\text{min}} \leq z_{\text{max}} \in \mathbb{Z} \) denote the integer bounds in \( \text{Tr}_f^\pi \).

The proof is performed in a similar way that in the proof for Theorem 1 except in the case where there are numerical variables occurring in \( \varphi \). Let us consider the case where \( \varphi \) is in the form \( \text{legal}(a(z_1, \ldots, z_m)) \) and we have only one numerical variable \( z_i \in X_f \) occurring in the parameter list \((z_1, \ldots, z_m)\), where \( 1 \leq i \leq m \). By \( \text{Tr}_f^\pi \) and Algorithm 2 definition, \( \text{Tr}_f^\pi(\varphi) = \text{Tr}_f^\pi(\text{removeVar}(\varphi)) = \bigwedge_{q_i \in \{z_{\text{min}}, \ldots, z_{\text{max}}\}}(\text{legal}(\text{Tr}_f^\pi(a(z_1, \ldots, q_i, \ldots, z_m))) \land z_i(q_i)) \). For any \( w \in W_f' \), \( z_i \in X_f \) and \( z_{\text{min}} \leq q'_i \leq z_{\text{max}} \), we have that \( z_i(q'_i) \in \pi'_f(w) \) if \( q'_i \) is the \( i \)-th value of \( \pi_{\pi_f}(w) \), i.e., variable \( z_i \) has the value \( q'_i \) in state \( w \). Thereby, \((\text{legal}(\text{Tr}_f^\pi(a(z_1, \ldots, q_i, \ldots, z_m))) \land z_i(q_i)) \) will hold only in the case where \( q_i = q'_i \).

Thus, \( \text{Tr}_f^\pi(M_f, \delta'_f, j \models \bigwedge_{q_i \in \{z_{\text{min}}, \ldots, z_{\text{max}}\}}(\text{legal}(\text{Tr}_f^\pi(a(z_1, \ldots, q_i, \ldots, z_m))) \land z_i(q_i)) \) if \( M_f, \delta'_f, j \models \text{legal}(a(z_1, \ldots, z_m)) \). Since \( \text{removeVar}(\text{legal}(a(z_1, \ldots, z_m))) \) will be recursively applied to every \( z_i \in X_f \) occurring in \((z_1, \ldots, z_m)\), it is easy to see that the result holds when we have two or more numerical variables in \( \text{legal}(a(z_1, \ldots, z_m)) \). The proof proceeds in a similar way if \( \varphi \) is either in the form \( \text{legal}(a(z_1, \ldots, z_m)) \), \((z_1, \ldots, z_m)\), \( z_1 < z_2, z_1 > z_2 \) or \( z_1 = z_2 \).

In the next section, we briefly describe how to translate GDLZ ST-models into GDLZ ST-models. Besides that, we show that GDL is a sublanguage of GDLZ.

3.3 From GDL to GDLZ

Conversely, we show that any GDL ST-model can be transformed into a GDLZ ST-model. Given a GDL ST-model \( M' = (W, \bar{w}, T, L, U, g, \pi') \) with a game signature \( \mathcal{S}' = (\mathcal{N}, \mathcal{A}, \Phi) \), we define an associated GDLZ ST-model \( M = (W, \bar{w}, T, L, U, g, \pi_{\Phi}, \pi_{\pi}) \) with the game signature \( \mathcal{S} = (\mathcal{N}, \mathcal{A}, X, \Phi) \), such that all elements are the same, except by \( \pi_{\Phi}, \pi_{\pi} \) and \( X \) and \( X \). These GDLZ components are defined as follows: (i) \( \pi_{\Phi}(w) = \pi'(w) \); (ii) \( \pi_{\pi}(w) = \emptyset \); and (iii) \( X = \emptyset \).
It follows that any formula \( \varphi \in \mathcal{L}_{GDL} \) is also a formula in GDLZ, i.e. \( \varphi \in \mathcal{L}_{GDLZ} \).

**Proposition 10.** If \( S' = (N, A, \Phi') \) and \( S = (N, A, X, \Phi) \) are GDL and GDLZ game signatures, respectively, and \( \Phi' \subseteq \Phi \), then \( \mathcal{L}_{GDL} \subseteq \mathcal{L}_{GDLZ} \).

**Proof.** Assume the GDL and GDLZ signatures \( S' = (N, A, \Phi') \) and \( S = (N, A, X, \Phi) \), respectively, and \( \Phi' \subseteq \Phi \), we show that for any \( \varphi \in \mathcal{L}_{GDL} \), \( \varphi \in \mathcal{L}_{GDLZ} \).

Assume \( \varphi \in \mathcal{L}_{GDL} \), if \( \varphi \) is of the form \( p, initial, terminal, wins(r), \neg \varphi, \varphi \land \varphi \) or \( \bigcirc \varphi \), where \( p \in \Phi \) and \( r \in N \), by the grammar definition of GDL, since \( \Phi' \subseteq \Phi \), we can easily see that \( \varphi \in \mathcal{L}_{GDLZ} \). Otherwise, if \( \varphi \) is of the form legal(a’), or does(a’), where \( a' \in A, r \in N \), we have that legal(a’), does(a’) \( \in \mathcal{L}_{GDLZ} \). By the numerical list \( \bar{z} \) grammar, we know that \( \bar{z} \) can be empty. Therefore, legal(a’(\( \bar{z} \))), does(a’(\( \bar{z} \))) \( \in \mathcal{L}_{GDLZ} \) or simply legal(a’), does(a’) \( \in \mathcal{L}_{GDLZ} \).

Thus, \( \mathcal{L}_{GDL} \subseteq \mathcal{L}_{GDLZ} \).

### 3.4 Succinctness

Next, we compare \( \mathcal{L}_{GDLZ} \) and \( \mathcal{L}_{GDL} \) in order to show the succinctness of \( \mathcal{L}_{GDLZ} \) in describing the same game. The following definition specifies when two sets of formulas in GDLZ and GDL describe the same game.

**Definition 17.** Two sets of formulas \( \Sigma_{GDLZ} \subseteq \mathcal{L}_{GDLZ} \) and \( \Sigma_{GDL} \subseteq \mathcal{L}_{GDL} \) describe the same game either (i) if \( \Sigma_{GDLZ} = \{ \varphi : \varphi \in \Sigma_{GDL} \} \) and \( \mathcal{L}_{GDL} \) and \( \mathcal{L}_{GDLZ} \) have, respectively, \( S' = (N, A, \Phi') \) and \( S = (N, A, \emptyset, \Phi) \); (ii) if \( \Sigma_{GDL} = \{ Tr^\varphi(\varphi, \delta[j]) : \varphi \in \Sigma_{GDLZ} \} \), given a GDLZ ST-model \( M \), a path \( \delta \) in \( M \) and a stage \( j \) in \( \delta \) or (iii) if \( \Sigma_{GDL} = \{ Tr^\varphi(\varphi) : \varphi \in \Sigma_{GDLZ} \} \), where every \( \varphi \in \Sigma_{GDLZ} \) is a bounded formula.

The following theorem show that (i) a GDLZ description has less subformulas and (ii) if we compare with the path translation, the growth is linear, if we compare with the complete translation, the growth is exponential.

**Theorem 3.** If \( \Sigma_{GDLZ} \) and \( \Sigma_{GDL} \) are two sets of formulas in \( \mathcal{L}_{GDLZ} \) and \( \mathcal{L}_{GDL} \), respec., describing the same game, then \( |\text{Sub}(\Sigma_{GDLZ})| \leq |\text{Sub}(\Sigma_{GDL})| \).

**Proof.** Assume the GDL and GDLZ game signatures \( S' = (N, A', \Phi') \) and \( S = (N, A, X, \Phi) \), respectively. Since \( \Sigma_{GDLZ} \) and \( \Sigma_{GDL} \) describe the same game, by Definition 17, we have either: (i) \( \Sigma_{GDLZ} = \{ \varphi : \varphi \in \Sigma_{GDL} \} \), \( S' = (N, A, \emptyset, \Phi) \) and \( S = (N, A, \emptyset, \Phi) \); (ii) \( \Sigma_{GDL} = \{ Tr^\varphi(\varphi, \delta[j]) : \varphi \in \Sigma_{GDLZ} \} \), for a GDLZ ST-model \( M \), a path \( \delta \) in \( M \) and a stage \( j \) in \( \delta \), or (iii) if \( \Sigma_{GDL} = \{ Tr^\varphi(\varphi) : \varphi \in \Sigma_{GDLZ} \} \), where every \( \varphi \in \Sigma_{GDLZ} \) is a bounded formula. In the first case, \( A' = A, \Phi' = \Phi, X = \emptyset \) and \( \Sigma_{GDLZ} = \{ \varphi : \varphi \in \Sigma_{GDL} \} \), we clearly have \( |\Sigma_{GDLZ}| = |\Sigma_{GDL}| \) and \( |\text{Sub}(\Sigma_{GDLZ})| = |\text{Sub}(\Sigma_{GDL})| \).

Given a path \( \delta \) in a GDLZ ST-model \( M \) and a stage \( j \), let us now consider the case (ii) where \( \Sigma_{GDL} = \{ Tr^\varphi(\varphi, \delta[j]) : \varphi \in \Sigma_{GDLZ} \} \). From \( Tr^\varphi(\varphi, \delta[j]) \), we have that any translation assigns \( \varphi \) to a corresponding \( \varphi' \) where \( |\text{Sub}(\varphi)| = |\text{Sub}(\varphi')| \),
except in the case where \( \varphi \) is of the form \( \langle \bar{z} \rangle \). If \( \varphi \) is of the form \( \langle \bar{z} \rangle \), then \( \varphi' \) will be constructed as \( \bigwedge_{i=1}^{n} \psi_i(q_i, w_i) \), where \( x_i \in X \) and \( q_i \) is the \( i \)-th value of \( \bar{z} \). Thus, \( |\text{Sub}(\varphi')| = |\bar{z}| |\text{Sub}(\varphi)| \). Since \( |\text{Sub}(\varphi)| = 1 \), then \( |\text{Sub}(\varphi')| = |\bar{z}| \).

Denote \( \Sigma_i = \Sigma_{GDLZ} - \{ \langle \bar{z} \rangle : \langle \bar{z} \rangle \in \mathcal{L}_{GDLZ} \} \), i.e. \( \Sigma_i \) is the subset of \( \Sigma_{GDLZ} \) without any formula \( \langle \bar{z} \rangle \). Thereby \( |\text{Sub}(\Sigma_i)| = |\text{Sub}(\{ \text{Tr}^\varphi(\varphi, \delta[j]) : \varphi \in \Sigma_i \})| \). Assuming \( k \) as the amount of formulas in the form \( \langle \bar{z} \rangle \in \Sigma_{GDLZ} \), we have \( |\text{Sub}(\Sigma_{GDLZ})| = |\text{Sub}(\Sigma_i)| + |\bar{z}|k \). Thereby, in the second case, we have \( |\text{Sub}(\Sigma_{GDLZ})| \leq |\text{Sub}(\Sigma_{GDLZ})| \).

Let us consider case (iii), where \( \Sigma_{GDL} = \{ \text{Tr}^\varphi(\varphi) : \varphi \in \Sigma_{GDLZ} \} \) and every \( \varphi \in \Sigma_{GDLZ} \) is a bounded formula. Let \( \mu = z_{\text{max}} - z_{\text{min}} \). The proof for case (iii) proceeds in the same way that for case (ii), except in the situation where there are numerical variables occurring in any \( \varphi \in \Sigma_{GDLZ} \). If we have at least one numerical variable occurring in \( \varphi \), we know that \( \varphi \) is either in the form \( \text{legal}(a^+(\bar{z})) \), \( \text{does}(a^+(\bar{z})) \), \( \langle \bar{z} \rangle \), \( z_1 < z_2 \), \( z_1 > z_2 \) or \( z_1 = z_2 \). Thereby, \( |\varphi| = 1 \) and \( |\text{Tr}^\varphi(\text{removeVar}(\varphi))| = 2\mu^n \times |\varphi| \), where \( \eta \) is the amount of numerical variables occurring in \( \varphi \). Thereby, \( |\varphi| < |\text{Tr}^\varphi(\text{removeVar}(\varphi))| \) and \( |\text{Sub}(\Sigma_{GDL})| \leq |\text{Sub}(\Sigma_{GDLZ})| \).

Denote \( \Sigma'_i = \Sigma_{GDLZ} - \{ \langle \bar{z} \rangle : \langle \bar{z} \rangle \in \mathcal{L}_{GDLZ} \} - \{ \varphi \in \Sigma_{GDLZ} : \varphi \text{ is at least one numerical variable in } \varphi \} \). Assuming \( k \) as the amount of formulas in the form \( \langle \bar{z} \rangle \in \Sigma_{GDLZ} \) and \( \kappa \) as the amount of formulas where occurs \( \eta \) numerical variables, we have \( |\text{Sub}(\Sigma_{GDL})| = |\text{Sub}(\Sigma'_i)| + 2\mu^n + |\bar{z}|k \).

Theorem 4. Given \( \Sigma_{GDLZ} \subseteq \mathcal{L}_{GDLZ} \), a GDLZ ST-model \( M \) with the game signature \( S = (N, A, \Phi, X) \):

1. If \( \Sigma_{GDL} = \{ \text{Tr}^\varphi(\varphi, \delta[j]) : \varphi \in \Sigma_{GDLZ} \} \), given a path \( \delta \) in \( M \) and a stage \( j \) in \( \delta \), then \( |\text{Sub}(\Sigma_{GDL})| \) grows in the order \( \mathcal{O}(n) \), where \( n = |\text{Sub}(\Sigma_i)| + |X|k \), the value \( k \) represents the amount of formulas in the form \( \langle \bar{z} \rangle \in \Sigma_{GDLZ} \) and \( \Sigma_i = \Sigma_{GDLZ} - \{ \langle \bar{z} \rangle : \langle \bar{z} \rangle \in \mathcal{L}_{GDLZ} \} \), i.e. \( \Sigma_i \) is the subset of \( \Sigma_{GDLZ} \) without any formula \( \langle \bar{z} \rangle \);

2. If \( \Sigma_{GDL} = \{ \text{Tr}^\varphi(\varphi) : \varphi \in \Sigma_{GDLZ} \} \), where every \( \varphi \in \Sigma_{GDLZ} \) is a bounded formula, then \( |\text{Sub}(\Sigma_{GDL})| \) grows in the order \( \mathcal{O}(n + \kappa\mu^n) \), where \( n = |\text{Sub}(\Sigma_i)| + |X|k \), the value \( k \) represents the amount of formulas in the form \( \langle \bar{z} \rangle \in \Sigma_{GDLZ} \), \( \Sigma_i = \Sigma_{GDLZ} - \{ \langle \bar{z} \rangle : \langle \bar{z} \rangle \in \mathcal{L}_{GDLZ} \} \) and \( \kappa \) is the amount of numerical variables occurring in \( \kappa \) variables.

Proof. Theorem 4 follows directly from the proof of Theorem 3.

The partial translation \( \text{Tr}^\varphi \) only concerns a fragment of the GDLZ model, that is the part of the model involved in a specific path. The size of a formula translated through \( \text{Tr}^\varphi \) has a linear growth over the number of numerical variables in \( X \) and the number of formulas in the form \( \langle \bar{z} \rangle \). Conversely, \( \text{Tr}^\varphi \) is a complete translation over finite GDLZ models. To represent a GDL formula in a GDL formula regardless of a specific path, we should remove the occurrence of numerical variables as numerical terms (see Algorithm 2). This procedure exponentially increases the size of the translated formula, depending mainly on the occurrence of numerical variables in the original GDL formula.
4 Conclusion

In this paper, we have introduced a GDL extension to describe games with numerical aspects, called GDLZ. In GDLZ, states are evaluated with propositions and an assignment of integer values to numerical variables. This allows us to define the terminal and goal states in terms of the numerical conditions. Furthermore, we define actions with numerical parameters, such that these parameters can influence over the action legality and over the state update. The language was extended mainly to include the representation of numerical variables and integer values as well as to allow numerical comparison.

We defined translations between GDLZ and GDL game models and formulas. Since GDL models have finite components, we can not define a complete model translation for any GDLZ model. We first defined a partial translation from any GDLZ model restricted to a specified path, i.e. only a run in the game is represented. Second, we defined a complete translation from GDLZ models with finite components and bounded formulas. We show that, in both cases, a translated GDLZ model, path or formula is a GDL model, path or formula, respectively. Furthermore, we prove that if a formula is satisfied at a stage in a path under a GDLZ model, its translation will also be satisfied at the same stage in the translated path under the translated model.

Finally, we show that, if we have a GDLZ and a GDL description for the same (finite) game, the GDLZ description is more succinct or equal, in terms of the quantity of subformulas in the description. More precisely, if the GDL game description is based on the partial translation from a GDLZ description restricted to one path, it is linearly larger than the GDLZ description. When we consider the complete model translation, the GDL description is exponentially larger than the GDLZ description.

Future work may extend GDLZ to define numerical rewards to players, stating their achievement when the game ends. It means that numerical variables may not have values assigned in some state of the model. Our aim is to investigate this new kind of numerical models. In our framework, it is possible to define both concurrent and sequential games. However, the legality of an agent’s action is independent from the actions of other agents. Thereby, it may be inappropriate to describe concurrent games where the actions of two agents change the same numerical variable. To overcome this limitation, future work may explore the definition of the legality function over joint actions.

Acknowledgments. Munyque Mittelmann and Laurent Perrussel acknowledge the support of the ANR project AGAPE ANR-18-CE23-0013.

References

1. Genesereth, M., Love, N., Pell, B.: General game playing: Overview of the AAAI competition. AI magazine 26(1), 1–16 (2005),
   http://www.aaai.org/ojs/index.php/aimagazine/article/viewArticle/1813
2. Gerevini, A.E., Saetti, A., Serina, I.: An approach to efficient planning with numerical fluents and multi-criteria plan quality. Artificial Intelligence 172(8-9), 899–944 (2008). https://doi.org/10.1016/j.artint.2008.01.002
3. Ghallab, M., Howe, A., Knoblock, C., McDermott, D., Ram, A., Veloso, M., Weld, D., Wilkins, D.: PDDL - The Planning Domain Definition Language. Tech. rep., AIPS-98 Planning Competition Committee (1998), http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.37.212
4. Giacomo, G.D., Lesp, Y., Pearce, A.R.: Situation Calculus-Based Programs for Representing and Reasoning about Game Structures. In: Proc. of the Twelfth International Conference on the Principles of Knowledge Representation and Reasoning (KR 2010). pp. 445–455 (2010)
5. Jiang, G., Zhang, D., Perrussel, L.: GDL Meets ATL: A Logic for Game Description and Strategic Reasoning. In: Pham, D.N., Park, S.B. (eds.) PRICAI 2014: Trends in Artificial Intelligence. pp. 733–746. Springer Int. Publishing, Cham (2014)
6. Jiang, G., Zhang, D., Perrussel, L., Zhang, H.: Epistemic GDL: A logic for representing and reasoning about imperfect information games. IJCAI International Joint Conference on Artificial Intelligence 2016-Janua, 1138–1144 (2016)
7. Love, N., Genesereth, M., Hinrichs, T.: General Game Playing: Game Description Language Specification. Tech. Rep. LG-2006-01, Stanford University, Stanford, CA (2006). http://logic.stanford.edu/reports/LG-2006-01.pdf
8. Maria Fox, Derek Long: PDDL2.1: An extension to PDDL for expressing temporal planning domains. Journal of Artificial Intelligence Research 20, 1–48 (2003), http://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.68.1957
9. McDermott, D.M.: The 1998 AI Planning Systems Competition. AI Magazine 21(2), 35 (2000). https://doi.org/10.1609/aimag.V21I2.1506
10. Parikh, R.: The Logic of Games and its Applications. North-Holland Mathematics Studies 102(C), 111–139 (1985). https://doi.org/10.1016/S0304-0208(08)73078-0
11. Pauly, M., Parikh, R.: Game Logic - An Overview. Studia Logica 75(2), 165–182 (nov 2003). https://doi.org/10.1023/A:1027354826364
12. Schiffel, S., Thielischer, M.: Representing and reasoning about the rules of general games with imperfect information. Journal of Artificial Intelligence Research 49, 171–206 (2014)
13. Thielischer, M.: A general game description language for incomplete information games. Proceedings of the Twenty-Fourth AAAI Conference on Artificial Intelligence (AAAI-10) pp. 994–999 (2010), https://www.aaai.org/ocs/index.php/AAAI/AAAI10/paper/view/1727
14. Thielischer, M.: GDL-III: A proposal to extend the game description language to general epistemic games. In: Proceedings of the European Conference on Artificial Intelligence (ECAI). vol. 285, pp. 1630–1631. Hague (2016). https://doi.org/10.3233/978-1-61499-672-9-1630
15. Thielischer, M.: GDL-III: A description language for epistemic general game playing. IJCAI Int. Joint Conference on Artificial Intelligence pp. 1276–1282 (2017)
16. Van Benthem, J.: Games in dynamic-epistemic logic. Bulletin of Economic Research 53(4), 219–248 (2001). https://doi.org/10.1111/1467-8586.00133
17. Van Benthem, J., Ghosh, S., Liu, F.: Modelling simultaneous games in dynamic logic. Synthese 165(2), 247–268 (2008). https://doi.org/10.1007/s11229-008-9390-y
18. Zhang, D., Thielischer, M.: Representing and Reasoning about Game Strategies. Journal of Philosophical Logic 44(2), 203–236 (2014). https://doi.org/10.1007/s10992-014-9334-6