Singularity degree of structured random matrices

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Abstract

We consider the density of states of structured Hermitian random matrices with a variance profile. As the dimension tends to infinity the associated eigenvalue density can develop a singularity at the origin. The severity of this singularity depends on the relative positions of the zero submatrices. We provide a classification of all possible singularities and determine the exponent in the density blow-up, which we label the singularity degree.

1 Introduction

Traditionally the theory of random matrices has focussed on models with a high degree of symmetry. The most prominent examples are the complex Hermitian Gaussian unitary (GUE) and real symmetric Gaussian orthogonal (GOE) ensembles with independent and identically distributed (i.i.d.) Gaussian entries above the diagonal. Their distributions are invariant under action of the unitary and orthogonal group, respectively, and their joint eigenvalue distributions admit closed formulas [5, 11, 12].

Already for a Wigner matrix with i.i.d. non-Gaussian entries, up to the Hermitian symmetry constraint, no such formula is available. Nevertheless, its distribution is still invariant under index permutation and, as its dimension tends to infinity, the empirical eigenvalue distribution still converges to the celebrated semicircle law [21].

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To account for applications with more complex underlying geometries, additional structure is imposed on the matrix entries and the invariance of the index space under all permutation dropped, resulting in structured random matrix models. This is achieved e.g. by assuming that the entries have different distributions and, thus, different variances. The eigenvalue density of such general Wigner-type matrices, $H$, deviates from the semicircle and depends on these variances \cite{14, 19, 3}. Examples of such ensembles include block band matrices and adjacency matrices of inhomogeneous Erdős-Rényi graphs. In both cases the index space is partitioned into $K$ equally sized sets with $n$ elements each, that encode the inhomogeneity of the model. More precisely, the entry variances $s_{lk} = \mathbb{E}[|h_{ij}^{lk}|^2]$ of the Hermitian matrix $H = (h_{ij}^{lk})$ depend only on the block indices $l,k = 1,...,K$ and are independent of the internal indices $i,j = 1,...,n$. We call $S = (s_{lk})_{l,k=1}^K$ the variance profile of $H$.

For fixed $K$, as $n$ tends to infinity, the empirical eigenvalue distribution of $H$ converges weakly, in probability, to the self-consistent density of states, $\rho$, which depends on the variance profile $S$. For dense matrices with sufficiently strong moment assumptions on the entry distribution this is well established (see e.g. \cite{14}), for inhomogeneous Erdős-Rényi graphs in the regime of diverging mean degree it was shown in \cite{10}. The self-consistent density of states, $\rho$, is the probability measure on $\mathbb{R}$ whose Stieltjes transform is \[ \langle m(z) \rangle = \frac{1}{K} \sum_k m_k(z), \] where $m_k = m_k(z)$ is the unique solution to
\begin{equation}
-\frac{1}{m_l(z)} = z + \sum_{k=1}^K s_{lk} m_k(z), \quad \text{Im} z > 0
\end{equation}
satisfying $\text{Im} m_k(z) > 0$ for all $k$ when $\text{Im} z > 0$. Another interpretation of this equation and its relation to the self-consistent density of states stems from free probability theory. In this context $\rho$ is interpreted as the distribution of the operator valued semicircular element $\sum l,k \sqrt{s_{lk}} E_{lk} \otimes c_{lk} \in \mathbb{C}^{K \times K} \otimes \mathcal{A}$, where $c_{lk} = c_{kl}$ are free semicircular elements in a non-commutative probability space $\mathcal{A}$ for $l \leq k$ and $(E_{lk})$ denotes the canonical basis of $\mathbb{C}^{K \times K}$, a model that was studied in \cite{6}.

As was shown in \cite{1}, the measure $\rho$ is symmetric around the origin and has a bounded density away from it. The behavior of this density has been studied in detail in \cite{1, 2}. Existence of a bounded density at the origin is ensured only under the assumption that the number and location of vanishing entries of $S$ is controlled, i.e. boundedness depends on the zero pattern of $S$. This assumption stems from the fact that too many zero entries in $S$ force certain row and columns in $H$ to be linearly dependent and, thus, $H$ to be singular. In Proposition 2.1 we provide necessary and sufficient conditions on $S$ to give rise to an asymptotic eigenvalue distribution $\rho$ with bounded density at the origin.

The behavior of $\rho$ at the origin is closely related to the dependence of the condition number $\text{Cond}(H) = \|H\|\|H^{-1}\|$ of $H$ on its dimension. For $N \times N$ - Wigner type
matrices, $H$, with $N = nK$ and uniform lower bound on the entries of $S$ the condition number satisfies $\text{Cond}(H) \sim \|H^{-1}\| \sim N$, which is a consequence of fixed energy universality at the origin [16]. This result was first established for Wigner matrices in [7]. A more geometric proof for the asymptotics of the smallest singular value of Wigner matrices was given in [20]. Such behavior is expected because, at the origin the associated density $\rho$ is bounded from above and below by a positive constant and, thus, the $N^{-1}$-quantile $\gamma = \gamma(N)$, defined through $\int_0^\gamma \rho(\tau)d\tau = N^{-1}$, satisfies $\gamma \sim N^{-1}$. The strong rigidity of eigenvalues of classical random matrix ensembles, i.e. their tendency to concentrate strongly around their expected location, motivates the conjecture $\text{Cond}(H) \sim \gamma^{-1}$ for a wide class of ensembles. Assuming $\lim_{\tau \to 0}|\tau|^{\sigma}\rho(\tau) > 0$ exists for some exponent $\sigma \in [0,1)$, this translates to $\log\text{Cond}(H) \sim \log\|H^{-1}\| \sim \frac{1}{1-\sigma}\log N$. We provide numerical evidence for this conjecture in Appendix B.

In this work we determine the singularity degree $\sigma$ and show that the self-consistent density of states either has an atom at the origin or a $|\tau|^{-\sigma}$ singularity. Furthermore, we determine the value of $\sigma$, depending on the zero pattern of $S$, for all possible variance profiles in Theorem 2.8. To this end we develop a solution and stability theory in Section 5 for a discrete averaging problem on a directed graph that is induced on the index space by the zero pattern of $S$. The solution to this problem determines the singular behavior of each component $m_k$ of the solution to (1) in a neighborhood of $z=0$. The most singular component, in turn, determines the singularity degree.

In the final stages of writing this manuscript we became aware that, independently of our work, related results have been obtained by O. Kolupaev. In [15] the singularity degree is determined for a variance profile $S=(s_{lk})$ with vanishing entries below and non-zero entries on and right above the anti-diagonal, i.e. when $s_{lk} = 0$ for $l+k > K+1$, as well as $s_{lk} > 0$ for $l+k = K+1$ and $l+k = K$. For this setting our Theorem 2.8 shows that $\sigma = \frac{K-1}{K+1}$, in agreement with [15]. In [15] the non-zero variances $s_{lk}$ on the individual blocks are allowed to be non-constant with uniform bounds from above and away from zero, a direction we do not pursue here.

2 Main results

Let $S=(s_{lk})_{l,k=1}^K$ be a matrix with non-negative entries. The self-consistent density of states associated to this variance profile is the probability measure $\rho$ on the real line whose Stieltjes transform is $\frac{1}{K} \sum_{k=1}^K m_k(z)$, with $m(z) = (m_1(z), \ldots, m_K(z))$ the unique solution to the vector Dyson equation (1) such that $\text{Im} m_k(z) > 0$. According to [1, Lemma 4.5] all $m_k(z)$ are bounded as long as $z$ with $\text{Im} z > 0$ is bounded away from zero. In particular, the measure $\rho$ has a Lebesgue-density away from zero. Thus, it has the form

$$\rho(d\tau) = \rho(\tau)d\tau + \kappa \delta_0(d\tau),$$

(2)
with $\kappa \in [0,1]$ and $\tau \mapsto \rho(\tau)$ is a bounded function.

To provide a classification of the singular behavior of $\rho$ in terms of $S$ we recall a few basic notions for matrices $R \in \mathbb{R}^{K \times K}$ with non-negative entries. For any permutation $\sigma$ of $[K] = \{1,\ldots,K\}$, the vector $(r_{i\sigma(j)})_{i=1}^{K}$ is called a diagonal of $R$. The matrix $R$ is said to have support if it has a diagonal with strictly positive entries. It is said to have total support if every non-negative entry lies on some positive diagonal.

**Proposition 2.1 (Singularity at zero).** Depending on the support properties of $S$, the self-consistent density of states (2) has a point mass at zero, a density blow-up or a bounded density. More precisely

(i) If $S$ has total support, then $\kappa = 0$ and the density $\rho(\tau)$ is bounded.

(ii) If $S$ has support, but not total support, then $\kappa = 0$ and $\lim_{\tau \to 0} |\tau|^{\sigma} \rho(\tau)$ exists and is a positive, finite number for some $\sigma \in (0,1)$.

(iii) If $S$ does not have support, then $\kappa = \frac{|I|+|J|-K}{K} > 0$, where $I,J \subset [K]$ are such that $(S_{ij})_{i \in I, j \in J}$ is a zero matrix with maximal perimeter, $2(|I|+|J|)$.

The proof of Proposition 2.1 is found at the end of Section 4. Although we provide a complete proof, the case (i) of $S$ having total support can also be inferred from a combination of [4, Proposition 3.10] and [1, Theorem 2.10]. Our main result, Theorem 2.8 below, identifies the exponent $\sigma$ from (ii) of Proposition 2.1 in terms of the entries of $S$ that vanish identically. To state it we introduce a relation on the index set $[K]$ that only depends on this zero pattern. The exponent $\sigma$ is then determined by the length of the longest increasing sequence compatible with this relation. Before we can define the relation we need a few preparations.

A non-negative matrix $R \in \mathbb{R}^{K \times K}$ is called fully indecomposable (FID) (see e.g. [8] for equivalent characterizations) if for any $I,J \subset \{1,\ldots,K\}$ such that $|I|+|J| \geq K$ the submatrix $R_{IJ} := (r_{ij})_{i \in I, j \in J}$ is not identically zero. Using this notion we now give a normal form for $S$ that is achieved by permuting the indices. We are not aware of this normal form previously appearing in the literature.

**Lemma 2.2 (Normal form of symmetric non-negative matrix).** Let $R \in \mathbb{R}^{K \times K}$ be a symmetric matrix with non-negative entries that has support. Then there is a permutation matrix $P = (\delta_{i\sigma(j)})_{i,j=1}^{K}$ of the indices with permutation $\sigma$ acting on
such that $R$ can be brought into the normal form

$$PRP^t = \begin{pmatrix} \ast & \ast & \cdots & \ast & \widetilde{R}_1 \\ \ast & \ast & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots \\ \ast & 0 & \cdots & \ast \\ \widetilde{R}_M & 0 & \cdots & 0 \end{pmatrix},$$

(3)

where all $R_i = R_i^t \in \mathbb{R}^{k_{M+i} \times k_{M+i}}$, and $\widetilde{R}_j \in \mathbb{R}^{k_j \times k_j}$ are FID. The normal form (3) has a $3 \times 3$-block structure that is subdivided into a $(L+2M) \times (L+2M)$-block structure with blocks of dimensions $(k_1, \ldots, k_{2M+L})$ such that $k_{2M+L+1-j} = k_j$ for $j \in [M]$ and $\sum_{i=1}^L k_{M+i} + 2 \sum_{j=1}^M k_j = K$. Referring to the $3 \times 3$ block structure from (3) the bold zeros in the $(2,3), (3,2)$ and $(3,3)$ blocks indicate that these blocks are zero matrices. The bold zeros in the $(2,2)$ block indicate that this block is itself block diagonal with $R_1, \ldots, R_M$ along the diagonal containing the only nonzero entries. Furthermore, the bold zeros in the $(1,3)$ and $(3,1)$ blocks indicate that these blocks have zero entries below the inverse block diagonal containing the matrices $\widetilde{R}_j$ and $\widetilde{R}_j^t$, respectively. The $\ast$-symbols in the $(1,1), (1,2)$ and $(2,1)$ blocks, as well as above the inverse block diagonals of the $(1,3)$ and $(3,1)$ blocks indicate arbitrary non-negative entries.

The normal form is not unique. In particular, permutation of the internal indices within each of the $L+2M$ blocks and permutation of the block indices corresponding to $R_1, \ldots, R_L$ result in different normal forms. The proof of Lemma 2.2 is presented in Appendix C.

We now introduce some definitions, based on the normal form.

**Definition 2.3 (0-1 mask).** To the $(L+2M) \times (L+2M)$-block structure induced by the normal form (3) of $R$ we associate a symmetric zero-one matrix $T = T^t \in \{0,1\}^{(L+2M) \times (L+2M)}$, called the 0-1 mask, whose entries are zero if and only if the corresponding block in $R$ is a zero matrix.

The 0-1 mask $T$ induces a natural pairing between indices, as well as, a relation on its index set $[L+2M]$, given by the following definitions.

**Definition 2.4 (Complement index).** For an index $i \in \mathbb{N} \cup (M+L+\mathbb{N})$ we define its complement index $\hat{i} := L+2M-i+1$ and for an index $i \in M+\mathbb{N}$ we
set $\hat{i} := i$.

**Definition 2.5 (Order).** For two distinct indices $i, j \in [L+2M]$ we write $i \triangleleft j$ if $t_{ij} = 1$.

We remark that the relation $\triangleleft$ between indices can be extended to a partial ordering on the index set, but we will not need this extension. In Section 5, we will introduce the directed graph induced by $\triangleleft$. From this directed graph we have the following natural notion of length.

**Definition 2.6 (Length).** We call $i_0, i_1, \ldots, i_k$ an increasing sequence of indices if $i_0 \triangleleft i_1 \triangleleft \cdots \triangleleft i_k$ and $k$ its length. We denote the length of the longest such increasing sequence by $\ell_\triangleleft (R)$.

The notation $\ell_\triangleleft (R)$ is justified by the following lemma, which we prove in Appendix C.

**Lemma 2.7 (Well-definedness of $\ell_\triangleleft (R)$).** The length of the longest increasing sequence in Definition 2.6 does not depend on the choice of normal form.

In Appendix A we present an example that illustrates the relationship between the variance profile $S$, its 0-1 mask $T$ and the induced relation $\triangleleft$ on the index set of $T$. We also show how these relations can be graphically depicted. Now we state our main result that expresses the singularity degree of the self-consistent density of states in terms of $\ell_\triangleleft (S)$. Its proof is presented at the end of Section 4.

**Theorem 2.8 (Classification of singularities).** Given a symmetric $K \times K$-matrix $S$ with non-negative entries that has support, let $\rho$ be the Lebesgue-density of the corresponding self-consistent density of states associated to $S$, i.e. the probability measure on $\mathbb{R}$ whose Stieltjes transform is $\frac{1}{K} \sum_k m_k(z)$ with $m_k(z)$ the unique solution to (1) with positive imaginary parts. Then $\rho$ has a singularity at the origin of degree $\sigma = \frac{\ell_\triangleleft (S)}{\ell_\triangleleft (S)+2}$, i.e. the limit

$$\lim_{\tau \to 0} |\tau|^{\sigma} \rho(\tau) > 0$$

exists as a finite positive number.

**Remark 2.9.** In the case that $S$ has total support, $\ell_\triangleleft (S) = 0$, which means the density $\rho(\tau)$ remains bounded in a neighborhood of the origin by (4).

We conclude this Section with an outline of the of the remainder of the paper. In Section 3, we determine the power law scaling of the solution to the Dyson equation (1) and introduce an averaging property that the scaling exponents satisfy. In Section 4, we introduce a rescaling of the Dyson equation, using the scaling exponents determined in Section 3. We then show the rescaled equations are stable.
at the singularity. In Section 5, we show that the averaging property, introduced in Section 3, has a unique solution in a generalized setting, and prove some properties of this solution when applied to the analysis of (1). In the Appendix we present an example, numerics for the condition numbers of certain structured random matrices, as well as list and prove properties of non-negative matrices.

**Notation**

We now introduce a few notations that are used throughout this work. We use the comparison relation $\varphi \lesssim \psi$ (or $\psi \gtrsim \varphi$) between two positive quantities $\varphi, \psi > 0$ if there is a constant $C = C(S) > 0$, only depending on the variance profile $S$, such that $\varphi \leq C \psi$. We write $\varphi \sim \psi$ in case $\varphi \lesssim \psi$ and $\varphi \gtrsim \psi$ both hold. When a vector is compared with a scalar, it is meant that relation holds in each component of the vector. For vectors $f = (f_i), g = (g_i)$ we interpret $f \lesssim g$ entrywise, i.e. $f_i \lesssim g_i$ holds for all $i$. In general, we consider $C^d$ as an algebra with entrywise operations, i.e. we write $\phi(f) := (\phi(f_i))_i$ for a function $\phi: C^d \to C$ applied to a vector $f$ and $f g := (f_i g_i)_i$ for the product of two vectors. Our scalar products are normalized, meaning that for $f, g \in C^d$, $\langle f, g \rangle := \frac{1}{d} \sum_i f_i g_i$ and we use the short hand $\langle f \rangle := \langle 1, f \rangle = \frac{1}{d} \sum_i f_i$ for the average of a vector. We denote by $e_i$ the $i^{th}$ standard basis vector.

**3 Block singularity degree**

Throughout this section we assume that the variance profile $S$ has support and we determine the degree of singularity for each individual entry of the solution $m(z)$ to (1) as $z \to 0$. Here and in the following sections, we will always assume without loss of generality that $S$ is already in normal form, i.e. that (3) holds with $R = S$ and $P = 1$. This is achieved by simply permuting the indices in (1). Correspondingly, we index $S$ and $m$ by block indices corresponding to the $(L+2M) \times (L+2M)$-block structure in (3), i.e. we write $S = (S_{ij})_{i,j=1}^{L+2M}$ and $m = (m_i)_{i=1}^{L+2M}$, where $S_{ij} \in \mathbb{R}^{k_i \times k_j}$ and $m_i \in \mathbb{C}^{k_i}$ with $(k_1, \ldots, k_{L+2M})$ such that $\sum_{i=1}^{L} k_{M+i} + 2 \sum_{j=1}^{M} k_j = K$. In particular, $k_i = k_i$, where we recall the definition of the complement index $\tilde{i} = L + 2M - i + 1$ from Definition 2.4. In accordance with the notation in (3) we denote $S_{ii} = S_{i-M}$ for $i \in M + [L]$ and $S_{i} = S_i$ for $i \in [M]$.

We identify the power law asymptotics of $m$ as $z \to 0$ from its restriction to the imaginary line, $m(i\eta) = iv(\eta)$, $\eta > 0$. (5)

The main advantage of this representation is that $v(\eta) > 0$ for all $\eta$. Expressed in
terms of $v_i \in \mathbb{R}^{k_i}$ the Dyson equation (1) takes the form

$$\frac{1}{v_i} = \eta + \sum_{j=1}^{L+2M} S_{ij} v_j.$$  \hfill (6)

As in (6) we will often omit the dependence of $v$ on $\eta$ from our notation. For later use we record the a priori bound

$$\min \{ \eta, \frac{1}{\eta} \} \leq v(\eta) \leq \frac{1}{\eta}, \quad \eta > 0.$$  \hfill (7)

The upper bound holds because the right hand side of (6) is bounded from below by $\eta$. The lower bound in (7) then follows by using $v \leq 1/\eta$ on the right hand side of (6).

The following lemma makes use of the relation $i \downarrow j$ on the block indices $i,j \in [L+2M]$ of $S$, introduced in Definition 2.5, for the normal form of a general square matrix $R$ with non-negative entries. Here and in what follows, we expand this relation to $[L+2M] \cup \{0, \infty\}$ by setting $0 \downarrow i$ (resp. $i \downarrow \infty$) if there does not exist a $j \in [L+2M]$ such that $j \downarrow i$ (resp. $i \downarrow j$). We also define

$$v_0(\eta) := \eta, \quad v_\infty(\eta) := \frac{1}{\eta}.$$  \hfill (8)

Lemma 3.1, below, identifies the exponents, $f_i$, for the power law behavior of $v_i(\eta)$ as $\eta \to 0$ and states stability of the defining equation, i.e. stability of the Dyson equation on the power law scale. It is a consequence of the more general Theorem 5.4 and Lemma 5.5 that show existence, uniqueness and stability of the solution to a general min-max averaging problem with boundary condition, of which the following is a special case. Its proof is postponed to the end of Section 5.

**Lemma 3.1 (Min-max-averaging of indices).** Let $f = (f_i)$ be a real-valued vector with index set $[L+2M] \cup \{0, \infty\}$, such that $f_0 = -1$, $f_\infty = 1$ and $f_i = 0$ for all $i \in M+[L]$. There is a unique choice of numbers $f_i \in (-1,1)$ for all remaining indices $i \in [M] \cup (M+[M])$ such that

$$f_i = \frac{1}{2} \left( \min_{j \downarrow i} f_j + \max_{j \uparrow i} f_j \right).$$  \hfill (9)

All $f_i$ with $i \in [L+2M]$ are rational numbers that satisfy $f_i < f_j$ for $i \downarrow j$ and the largest and smallest among them are

$$\sigma := \max_{i \in [L+2M]} f_i = \frac{\ell_\infty(S)}{\ell_\infty(S) + 2}, \quad \min_{i \in [L+2M]} f_i = -\sigma.$$  \hfill (10)

Additionally, the vector $(f_i)_{i=1}^{L+2M}$ is antisymmetric with respect to $i \mapsto \hat{i}$, i.e.

$$f_i = -f_{\hat{i}}.$$  \hfill (11)

Furthermore, there are constants $c, C > 0$ such that for all real-valued vectors $g = (g_i)$ with index set $[L+2M] \cup \{0, \infty\}$, the following implication holds true:

$$\|g - f\|_\infty \leq c \quad \text{implies} \quad |g - f| \leq C \|d\|_\infty,$$  \hfill (12)
where

\[
d_i := \begin{cases} 
  g_i - f_i, & \text{for } i \in (M + [L]) \cup \{0, \infty\} \\
  g_i - \frac{1}{2} (\min_{j \in \mathcal{J}} g_j + \max_{j \in \mathcal{J}} g_j), & \text{for } i \in [M] \cup (M + L + [M]).
\end{cases} 
\]

The main result of this section is the following proposition that identifies the singularity degree for each block index.

**Proposition 3.2** (Block singularity degree). Let \( f = (f_i) \) be the unique real-valued vector from Lemma 3.1 with index set \([L+2M] \cup \{0, \infty\} \). Then

\[
v_i(\eta) \sim \eta^{-f_i}, \quad \eta \in (0,1], i \in [L+2M].
\]

**Proof.** The proof proceeds in three steps. In the first step we show that \( v_i \sim \langle v_i \rangle \) for all block indices \( i \), i.e., the solution has a uniform asymptotic behavior within each block. Note that the bound \( v_i \leq \eta \langle v_i \rangle \leq \langle v_i \rangle \) is always satisfied.

In the second step we prove that the exponents \( f_i \) satisfy the min-max-averaging condition (9). In the final, third step we use the stability of the Dyson equation on the power law scale from Lemma 3.1 to establish (14).

**Step 1:** Here we show the following uniform comparison relations on the blocks:

\[
v_i \sim 1, \quad i \in M + [L] \quad \text{and} \quad v_i \sim \langle v_i \rangle \sim \frac{1}{\langle v_i \rangle} \sim \frac{1}{v_i}, \quad i \in [M].
\]

To prove (15) we use a reformulation of (6) as a variational principle. By [1, Section 6.2] the solution \( \psi(\eta) = (\psi_i)_{i=1}^{L+2M} \) is the unique minimizer of the functional

\[
J_\eta(x) := \frac{1}{2} \langle xSx \rangle + \langle \log x \rangle + \eta \langle x \rangle, \quad x \in (0, \infty)^K.
\]

In particular, the value of the functional at \( x = v \) is bounded from above by its value on the constant vector \( e = (1,...,1) \in \mathbb{R}^K \), i.e.

\[
J_0(v) \leq J_\eta(v) \leq J_\eta(e) \leq 1.
\]

Since all matrices \( S_i, \tilde{S}_j \), from the normal form (3) with \( R = S \) and \( P = 1 \), are fully indecomposable, there exist permutation matrices \( P_i, \tilde{P}_j \) such that \( S_i^P := S_i P_i \) and \( \tilde{S}_j^P := \tilde{S}_j \tilde{P}_j \) are primitive with positive main diagonal (cf. Lemma C.2 in Section C). Inserting these permutation matrices and using (16) yields the bound

\[
1 \geq J_0(v) \geq \sum_{i \in M + [L]} \langle v_i, S_{i-M}^P v_i^P \rangle + \sum_{i \in [M]} \langle v_i, \tilde{S}_i^P v_i^P \rangle - \sum_{i \in [L+2M]} \langle \log v_i \rangle \geq \sum_{i \in [L+2M]} \langle \varphi(v_i v_i^P) \rangle,
\]

where \( v_i^P = P_i^P v_i \) for \( i \in M + [L] \), \( v_i^P = \tilde{P}_i^P v_i \) for \( i \in [M] \) and \( \varphi(x) := c x - \log x \) for some \( \eta \)-independent constant \( c > 0 \). Since \( \varphi \) is coercive on the positive half line we conclude

\[
v_i v_i^P \sim 1, \quad i \in [L+2M].
\]
From (6) we see that the symmetric block matrix $F=(D(v_i)S_{ij}D(v_j))_{i,j=1}^{L+2M}$ in $\mathbb{R}^{K\times K}$ with non-negative entries satisfies $F \leq e$. Here $D(v_i) \in \mathbb{R}^{k_i\times k_i}$ denotes the diagonal matrix with $v_i$ along its main diagonal. By the Perron-Frobenius theorem there is a vector $f$ with non-negative entries such that $Ff = \|F\|f$. Taking the inner product with $e$ and using the symmetry of $F$ we find
\[
\langle e, f \rangle \geq \langle Fe, f \rangle = \langle e, Ff \rangle = \|F\| \langle e, f \rangle.
\]
Since $\langle e, f \rangle > 0$, we infer that $\|F\| \leq 1$. This argument was also used in [1, Proof of Lemma 6.10] in similar context. From $\|F\| \leq 1$ we also conclude that $\|F_i\| \leq 1$ and $\|\tilde{F}_i\| \leq 1$, where we set $F_i := D(v_i)S_{i,M}^P D(v_i^P) = D(v_i)S_{i,M}^P D(v_i^P)$ for $i \in M + [L]$ (recalling that in this case $i = \hat{i}$ by Definition 2.4) and $\tilde{F}_i := D(v_i)\tilde{S}_i^P D(v_i^P)$ for $i \in [M]$. Thus,
\[
1 \geq \|F^k_i\| \geq \|D(v_i)(S_{i,M}^P)^k D(v_i^P)\|, \quad 1 \geq \|\tilde{F}^k_i\| \geq \|D(v_i)(\tilde{S}_i^P)^k D(v_i^P)\|, \quad (18)
\]
holds for any $k \in \mathbb{N}$. In both cases we used the lower bound on $v_i v_i^P$ from (17) in the second inequality. Since $S_i^P$ and $\tilde{S}_i^P$ are primitive we can choose $k$ large enough so that the entries of their $k$-th powers are all positive. Then (18) together with the lower bound from (17) implies $\langle v_i \rangle \langle v_i \rangle \sim 1$. Again by (17) the claim (15) follows because $1 \sim v_i v_i^P \lesssim v_i \langle v_i \rangle$ and $v_i \langle v_i \rangle \lesssim \langle v_i \rangle \langle v_i \rangle \sim 1$.

Step 2: Here, we show that for every $i \in [L+2M]$ the relation
\[
\langle v_i \rangle^2 \sim \left(\max_{j \neq \hat{i}} \langle v_j \rangle \right) \left(\min_{j \neq \hat{i}} \langle v_j \rangle \right),
\]
holds true, where the maximum and minimum are taken with $j \in [L+2M] \cup \{0, \infty\}$.

We multiply equation (6) on both sides by $v_i$ and take its average. Then we subtract the resulting equation from the one where $i$ is replaced by the complement index $\hat{i}$. Due to the symmetry of $S$ and that the dimensions $k_i$ and $k_\hat{i}$ are equal, we have that the term $\langle v_i, S_{\hat{i}i}v_i \rangle = \langle v_\hat{i}, S_{\hat{i}i}v_\hat{i} \rangle$ and the constant term cancels from both sides. We are therefore left with:
\[
\sum_{j \neq \hat{i}} \langle v_i, S_{\hat{i}j}v_j \rangle + \eta \langle v_i \rangle = \sum_{j \neq \hat{i}} \langle v_i, S_{\hat{i}j}v_j \rangle + \eta \langle v_i \rangle,
\]
(20)
where $j \in [L+2M]$. We use the convention that empty sums are 0. Then using that $v_i \sim \langle v_i \rangle$, from (15), and the definition of $T = (t_{ij})$, in Definition 2.3, we have the comparison relations
\[
\langle v_i, S_{\hat{i}j}v_j \rangle \sim t_{ij} \langle v_i \rangle \langle v_j \rangle, \quad \langle v_i, S_{\hat{i}j}v_j \rangle \sim t_{ij} \langle v_i \rangle \langle v_j \rangle \sim \eta \langle v_i \rangle \langle v_j \rangle
\]
(21)
for any block indices $i,j \in [L+2M]$. We conclude
\[
\sum_{j \neq \hat{i}} \langle v_j \rangle \sim \langle v_i \rangle \sum_{j \neq \hat{i}} \langle v_j \rangle \sim \langle v_i \rangle \langle v_j \rangle \sim \langle v_i \rangle \langle v_j \rangle \sim \langle v_i \rangle \langle v_j \rangle \sim \frac{1}{\langle v_i \rangle},
\]
(21)
where the sum is over \( j \in [2M+L] \cup \{0,\infty\} \), i.e. we include \( v_0 = \eta \) and \( v_\infty = 1/\eta \). In the first and third relation we used \( \langle v_i \rangle v_i \sim 1 \) from (15) and for the second relation we used (20) along with (21). The claim (19) then follows.

Step 3: Note that (14) is trivial for \( \eta \in [c',1] \) with any constant \( c' > 0 \) because of (7) and recall the definition of \( f = (f_i) \) from Lemma 3.1, as well as the definition of \( v_0 = \eta \) and \( v_\infty = \frac{1}{\eta} \) from (8). Thus we assume \( \eta \in (0,c') \) with \( c' \) chosen sufficiently small. We conclude the proof of the proposition by taking the logarithm on both sides of (19) and dividing by \( \log \eta \). With
\[
g_i(\eta) := -\frac{\log(v_i(\eta))}{\log \eta}, \quad i \in [2M+L]
\]
and \( g_0(\eta) := -1, \ g_\infty(\eta) := 1 \) we find \( |d_i| \lesssim |\log \eta|^{-1} \) with \( d_i = d_i(\eta) \) defined as in (13). Furthermore, \( g_i(\eta) \in [-1,1] \) because of the a priori bound (7). Now let \( \eta_n \downarrow 0 \) be a sequence such that \( \tilde{g} := \lim_{n \to \infty} g(\eta_n) \) exists. Since \( d_i \to 0 \) as \( \eta \downarrow 0 \) this limit \( \tilde{g} \) solves the same min-max averaging problem (9) as \( f \) with identical boundary conditions (cf. first relation in (15) and (8)). By the uniqueness of the solution to (9) with given boundary conditions, we conclude \( f = \tilde{g} \). Since this is true for any convergent sequence \( g(\eta_n) \), we conclude that \( \lim_{n \to \infty} g(\eta_n) = f \). In particular, \( g \) can be continuously extended to \( \eta = 0 \) and the local stability (12) for this min-max averaging problem, as well as \( \|d\|_\infty \lesssim |\log \eta|^{-1} \) implies
\[
|g_i(\eta) - f_i| \lesssim \frac{1}{|\log \eta|}
\]
for \( \eta \in (0,c') \) and \( c' > 0 \) small enough. Altogether (14) is proven.

\[\square\]

4 Singular stability

In this section, we show the Dyson equation (6) on the imaginary line can be rescaled at the \( \eta = 0 \) singularity so that the solution of this rescaled equation has a limit when \( \eta \downarrow 0 \). Furthermore, the rescaled equation is stable and therefore its solution is smooth in a neighborhood of \( \eta = 0 \). In particular, we will see that the solution to the Dyson equation admits an expansion in fractional powers of \( z \).

Within this section we will often identify vectors \( a \in \mathbb{C}^{L+2M} \) with vectors in \( \mathbb{C}^K \) via the embedding \( a = (a_1,\ldots,a_{L+2M}) \in \mathbb{C}^K \), where \( a_i \in \mathbb{C}^{k_i} \) is a constant vector. In particular, for \( a \in \mathbb{C}^{L+2M} \) and \( b = (b_1,\ldots,b_{L+2M}) \in \mathbb{C}^K \) with \( b_i \in \mathbb{C}^{k_i} \), we have \( ab = (a_1b_1,\ldots,a_{L+2M}b_{L+2M}) \in \mathbb{C}^K \). We also define for \( \gamma \in \mathbb{R} \) the \( \gamma \)-powers of complex numbers as a holomorphic function \( z \mapsto z^\gamma \) with branch cut on the negative half line such that \( 1^\gamma := 1 \). Furthermore, we introduce the notation \( E_{ij}(A) \in \mathbb{C}^{K \times K} \) for the block matrix with the matrix \( A \in \mathbb{C}^{k_i \times k_j} \) in \( (i,j) \)-block and zeros everywhere else.
Proposition 4.1. Let \( f = (f_i)_{i \in [L + 2M]} \) be the rational numbers from Lemma 3.1 with the indices 0 and \( \infty \) removed. There exists an \( \varepsilon > 0 \) and a holomorphic function \( \tilde{v} : \mathbb{D}_\varepsilon \to \mathbb{C}^K \) such that \( \tilde{v}(\omega) > 0 \) entrywise for all \( \omega \in (-\varepsilon, \varepsilon) \). With this \( \tilde{v} \), the solution, \( m \), to the Dyson equation (1) satisfies
\[
 m(z) = i(-i z)^{-f} \tilde{v}((-i z)^{1/Q}),
\]
for all \( z \in \mathbb{D}_\varepsilon \cap \mathbb{H} \), where \( Q = \text{LCD}(f) \) is the least common denominator of \( f \).

We recall that (22) is interpreted as
\[
 m_i = i(-i z)^{-f} \tilde{v}_i((-i z)^{1/Q}),
\]
where \( m_i, \tilde{v}_i \in \mathbb{C}^{k_i} \) and \( f_i > 0 \). We will prove Proposition 4.1 at the end of this section.

Instead of directly analyzing the stability of (1) and (6) we now rescale these equations and their solution by the block singularity degrees \( f_i \) given in Proposition 3.2. We begin by defining
\[
 \omega := \eta^{1/Q} > 0, \quad \tilde{v}(\omega) := v(\omega^Q) \omega^{Qf}.
\]
We develop a system of equations which \( \tilde{v} = \tilde{v}(\omega) \) satisfies (see (32) and (33) below), which also admit a non-degenerate limit as \( \eta \to 0 \). For this purpose we multiply (6) by \( \tilde{v} \) to arrive at
\[
 1 = \tilde{v} \tilde{S} \tilde{v} + \eta^{-f} \tilde{v},
\]
where we defined the rescaled variance profile
\[
 \tilde{S}(\omega) := D(\omega^{-Qf})SD(\omega^{-Qf}) = D(\eta^{-f})SD(\eta^{-f}).
\]
Here, \( D(\eta^{-f}) \in \mathbb{C}^{K \times K} \) denotes the diagonal matrix with the block constant vector \( \eta^{-f} \in \mathbb{C}^K \) along the main diagonal. In the following, we view \( \tilde{v}, \tilde{S}, \) and \( \eta \) as functions of \( \omega \), as in (23). Because we assume \( S \) to be in the normal form, (3), and \( f_i < f_j \) for \( i \neq j \), all entries of \( \tilde{S} \) remain bounded as \( \eta \to 0 \).

As it stands the naive limit of equation (24), as \( \eta \to 0 \), does not have a unique solution. To circumvent this issue we separate its leading order and sub-leading order terms. These, \( \omega \) independent, contributions to \( \tilde{S} \) in the \( \eta \to 0 \) are denoted
\[
 S^0 := \sum_{i=1}^{L+2M} E_{ii}^0(S_{ii}), \quad S^\omega := \sum_{i=1}^{L+2M} \sum_{j \in I_i} E_{ij}^\omega(S_{ij}),
\]
where we recall the notation for the block matrices \( E_{ij}^0(\cdot) \), introduced at the beginning of the section, and define the set \( I_i \) of successor indices, \( l \), of \( i \) for which the \( f_l \)-value is minimal, namely
\[
 I_i := \{ l \in [L+2M] : i \downarrow l \text{ and } f_i = \min_{j : i \downarrow j} f_j \}.
\]

If \( i \) does not have any successors \( i \downarrow j \), then the corresponding sum in (26) is empty and, thus, equal to zero. We remark that \( S^0 \) coincides with the FID skeleton \( S_{\text{FID}} \) of \( S \) that will be introduced later in Definition C.3.
The following lemma expresses the \( \omega \)-expansion of \( \tilde{S} \) in terms of \( S^0 \) and \( S^\ell \). For its statement we define
\[
h_i := \frac{1}{2} \left( \min_{j \in \Lambda} f_j - \max_{j \in \Lambda} f_j \right), \quad i \in [L+2M].
\] (28)
Here, we interpret \( \min_{j \in \Lambda} f_j := 1 \) and \( \max_{j \in \Lambda} f_j := -1 \) in case the set of \( j \in [L+2M] \) over which the minimum and maximum, respectively, are taken is empty.

**Lemma 4.2.** The numbers \( h_i \) defined in (28) are rational, positive and satisfy the following properties:
1. The number \( h_i \) quantifies the smallest change between \( f_i \) and its predecessor or its successor values, i.e.
\[
h_i = f_i - \max_{j \in \Lambda} f_j = \min_{j \in \Lambda} f_j - f_i.
\]
   Once again we interpret the case where the maximum/minimum sets are empty as in the definition of \( h \).
2. The vector \( (h_i)_{i=1}^{L+2M} \) is symmetric under the exchange \( i \leftrightarrow b_i \), i.e.
\[
h_i = h_{b_i}.
\]
3. The rescaled variance profile from (25) has the expansion
\[
\tilde{S}(\omega) = S^0 + D(\omega^{Qh})S^\ell + D(\omega^{1+Qh})E(\omega),
\] (29)
where \( E(\omega) \) is a polynomial in \( \omega \) with coefficients in \( \mathbb{C}^{K \times K} \).

**Proof.** Part 1 of the lemma is an immediate consequence of Lemma 3.1. In particular, the positivity of \( h_i \) holds because due to this lemma \( j \leftrightarrow i \) implies the inequality \( f_j < f_i \). To prove Part 2, we first recall from (11), that \( f_i = -f_{\tilde{i}} \). Additionally, by the symmetry of \( T = T^t \) from Definition 2.3 and its connection to the relation \( \bullet \) from Definition 2.5 we have that \( i \bullet j \) implies \( \tilde{j} \bullet \tilde{i} \). These two facts imply that if \( f_i = \min_{j \in \Lambda} f_j \) then \( f_{\tilde{i}} = \max_{j \in \Lambda} f_j \). Substituting this relationship into Part 1 gives
\[
h_i = \min_{j \in \Lambda} f_j - f_i = -\max_{j \in \Lambda} f_j + f_i = h_{\tilde{i}}.
\]

To verify Part 3, we begin by considering \( \tilde{S}_{it} \). Since \( f_i = -f_{\tilde{i}} \), we have
\[
\tilde{S}_{it} = \tilde{S}_{i\tilde{t}}.
\]
We now consider the remaining blocks for which \( S \) is non-zero, which, by the definition of \( \bullet \), are of the form \( S_{ij} \) for some \( i \bullet j \). Considering such a pair we have
\[
\tilde{S}_{ij} = \eta^{-k - f_j} S_{ij} = \eta^{-f_i + f_j} S_{i\tilde{j}}.
\]
Since \( i \bullet j \) we have \( f_i < f_j \) here. By \( \eta' = \omega^{Qf} \), we then see that every entry of \( \tilde{S}_{ij} \) is a polynomial in \( \omega \). Additionally, the leading order of \( S \) is given by \( S^0 \).

We now show the next order term scales like \( \eta^h \) by considering the off-diagonal block matrix with the largest Frobenius norm (denoted \( \| \cdot \|_F \)). For an index \( i \) with
\[ i \mapsto \infty \text{ we have } \| \tilde{S}_i \|_F = 0 \text{ for all } j \text{ with } i \neq j \text{ by the structure of the normal form (3). Otherwise we find}
\]
\[
\max_{j,j \neq i} \| \tilde{S}_{ij} \|_F = \max \eta^{-f_i+f_j} \| \tilde{S}_{ij} \|_F = \max \eta^{h_i} \| \tilde{S}_{ij} \|_F \sim \eta^{h_i},
\]
where the second equality uses the maximum is achieved when \(-f_i+f_j\) is minimized, which, by the final equality of Part 1, is \(h_i\). Furthermore, the indices \(j\), for which \(\| \tilde{S}_{ij} \|_F \sim \eta^{h_i}\) holds, are exactly \(j \in I_i\). Thus, the order \(\eta^{h_i}\) terms are given by \(S^>\). Since all entries of \(\tilde{S}\) are polynomials in \(\omega\) the expansion (29) follows. \(\square\)

We rewrite (20) in terms of \(\tilde{v}\) and \(\tilde{S}\) to find
\[
\sum_{k: k \neq l} \langle \tilde{v}_k, \tilde{S}_{ik} \tilde{v}_k \rangle + \eta^{1-f_i} \langle \tilde{v}_i \rangle - \sum_{k: k \neq l} \langle \tilde{v}_k, \tilde{S}_{ik} \tilde{v}_k \rangle - \eta^{1-f_i} \langle \tilde{v}_k \rangle = 0 \tag{30}
\]
for \(l \in [M]\). Comparing this expression to (29) we see that multiplying (30) by \(\eta^{-h_i}\) will ensure these equations have a non-trivial limit when \(\eta \to 0\), that just depends on \(S^>\), because the \(S^0\) has been canceled.

The limit of (24) at \(\eta \downarrow 0\) along with (30) gives too many equations, i.e. we have an overdetermined system with superfluous equations. On the other hand, the \(\eta \downarrow 0\) limit of (24) alone does not uniquely fix the solution \(\tilde{v}\) because only the leading order \(\tilde{S}(0) = S^0\) enters in the equation. In order to resolve the issue, we project (24) onto the orthogonal complement of the subspace \(E_-\), which we introduce now. Let
\[
e_{l} : = e_l - e_l, \quad E_- : = \text{span}_{l \in [L+2M]} \{ e_l \}. \tag{31}
\]
We identify \(E_- \subset \mathbb{C}^{L+2M}\) via the embedding of its spanning block constant vectors in \(\mathbb{C}^K\) with a \(2M\)-dimensional subspace of \(\mathbb{C}^K\). Let \(P_- : \mathbb{C}^K \to E_-\) be the orthogonal projection onto the orthogonal complement of this subspace. Then we define \(\mathcal{F} : \mathbb{C}^K \times \mathbb{C} \to E_+ \times \mathbb{C}^M \cong \mathbb{C}^K\) as \(\mathcal{F} := (F_0, F)\) with
\[
F_0(x, \omega) := P_- (x, \tilde{S}(\omega) x + \eta(\omega)^{-f} x - 1), \tag{32}
\]
and \(F = (F_1, \ldots, F_M)\) given by
\[
F_l(x, \omega) := \eta(\omega)^{-h_l} \left( \sum_{k: k \neq l} \langle x_l, \tilde{S}(\omega)_{ik} x_k \rangle + \eta(\omega)^{-f_l} \langle x_l \rangle - \sum_{k: k \neq l} \langle x_l, \tilde{S}(\omega)_{ik} x_k \rangle - \eta(\omega)^{-f_l} \langle x_l \rangle \right), \tag{33}
\]
for \(\omega \neq 0\). Recalling that we treat \(\eta(\omega) = \omega_Q\) as a function of \(\omega\), we have \(\mathcal{F}(\tilde{v}(\omega), \omega) = 0\) from (30) and applying \(P_-\) to (24). This reformulation of the Dyson equation (6) removes the singularity at \(\omega = 0\) and still determines the solution uniquely, as we establish below. In fact, all entries of \(\mathcal{F}\) are polynomials in \(\omega\) and the entries of \(x\). For \(F_0\) in (32) this is obvious. For \(F_l\) in (33) it is a consequence of (29). Therefore, we can analytically extend \(\mathcal{F}\) to \(\mathcal{F}(x, 0)\) at \(\omega = 0\). The form of the extensions of the components \(F_l(x, 0)\) of \(\mathcal{F}(w, 0)\) from (33) to \(\omega = 0\) differs depending on whether \(l\) is such that \(0 \leq l\) or \(j \leq l\) for some \(j \neq 0\). When \(l\) satisfies
0 \llcorner l$, the $\hat{l}$-th block row of $S^\omega$ is zero because the corresponding set $\mathcal{I}_l$ in (26) is empty. Thus, we find
\begin{align*}
F_0(x,0) &= P^\perp(x,S^0x-1), \\
F_l(x,0) &= \sum_{k: k \neq l} (x_l, S_{lk}^\omega x_k) - (x_l).
\end{align*}
(34)
for all indices $l \in \llbracket M \rrbracket$ with $0 \llcorner l$ and the expression
\begin{align*}
F_l(x,0) &= \sum_{k: k \neq l} (x_l, S_{lk}^\omega x_k) - \sum_{k: k \neq l} (x_l, S_{lk}^\omega x_k)
\end{align*}
for indices $l \in \llbracket M \rrbracket$ such that there exists a $j \neq 0$ with $j \llcorner l$.

To analyze the limit of the equation $F(x,\omega) = 0$ as $\omega \to 0$ we define the natural candidate for its solution at $\omega = 0$ as
\begin{align*}
w := \limsup_{\eta \to 0} F(\eta) = \limsup_{\eta \to 0} \tilde{v}(\omega).
\end{align*}
(36)
Recalling Proposition 3.2, we see that
\begin{align*}
w \sim 1.
\end{align*}
(37)
due to (3.2). Furthermore, by Proposition 3.2, $\eta^l f \tilde{v} = \eta v \to 0$ as $\eta \to 0$ since, due to Lemma 3.1, $f_i < 1$ for all $i \in \llbracket L + 2M \rrbracket$. Thus, we infer $F(w,0) = 0$. The following lemma shows that the implicit function theorem can be applied to $F$ at $(x,\omega) = (w,0)$, which then provides existence of a unique analytic function $\tilde{v}(\omega)$ that satisfies $F(\tilde{v}(\omega), \omega) = 0$ for $\omega \in \mathbb{C}$ in a neighborhood of zero and coincides with the originally defined $\tilde{v}$ for $\omega > 0$. In particular this implies, that $\lim_{\eta \to 0} \tilde{v}(\eta)$ exists and equals $w$, i.e. the limsup in (36) can be replaced with a limit.

**Lemma 4.3.** The derivative of $F$ with respect to the $x$-variable, when evaluated at $(x,\omega) = (w,0)$, is invertible.

The proof of Proposition 4.1 follows immediately from Lemma 4.3.

**Proof of Proposition 4.1.** We first show that there exists an open neighborhood $U \subset \mathbb{C}$ of $0$ and a unique analytic function $\tilde{v}(\omega)$ on $U$ that coincides with the originally defined $\tilde{v}(\omega)$ from (23) for $\omega \in U \cap (0,1)$ such that $\tilde{v}(0) = w$ and $F(\tilde{v}(\omega), \omega) = 0$ for all $\omega \in U$.

To see this we use the implicit function theorem and Lemma 4.3. Since $F$ is a polynomial it is an analytic function of both arguments $x$ and $\omega$. Thus, Lemma 4.3 implies the existence of a real analytic function $\hat{v} : (-\varepsilon, \varepsilon) \to \mathbb{R}$ with $F(\hat{v}(\omega), \omega) = 0$ for all $\omega \in (-\varepsilon, \varepsilon)$ and some $\varepsilon > 0$. Since $\hat{v}(0) = w > 0$ by (37) we can choose $\varepsilon$ small enough to ensure $\inf_{|\omega| < \varepsilon} \hat{v}(\omega) > 0$. Furthermore, $F(\hat{v}(\omega), \omega) = 0$ is equivalent to (24) at $\eta = \omega^Q$ and $\tilde{v} = \hat{v}$. Since (24) has a unique positive solution for every $\omega > 0$, we conclude $\hat{v}(\omega) = \tilde{v}(\omega)$ for $\omega > 0$.

To verify (22) we see that by (23) and (5) we have
\begin{align*}
m(\eta) &= i v(\eta) = i\omega^{-Q} \tilde{v}(\omega)
\end{align*}

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for sufficiently small $\omega = \eta^{1/\theta} > 0$. By analyticity this extension coincides with $m$ in the sense of (22), finishing the proof.

**Proof of Lemma 4.3.** Applying (36) and (29) to (24) we conclude

$$1 = w S^0 w$$

by taking the limit $\eta \to 0$ of (24) along an appropriately chosen subsequence. We have that the right side of (38) is finite from (36). The structure of $S^0$ from (26) together with (38) implies that

$$D(w)S^0D(w)e_i^- = -e_i^-,$$

where $e_i^- \in \mathbb{C}^K$ is the block constant vector from (31). In particular, $P_{\perp}$ and $D(w)S^0D(w)$ commute.

For $l \in [L+M]$, let

$$e^+_l := e_l + e_i, \quad E_l := \{ ae^+_l : a \in \mathbb{C}^K \},$$

be subspaces of dimension $k_l$ in case $l \in M + [L]$ and of dimension $2k_l$ in case $l \in [M]$, respectively.

We now determine some properties of the spectrum of $D(w)S^0D(w)$. From (38) we also see that

$$D(w)S^0D(w)e^+_l = e^+_l,$$

for each $l \in [L+M]$. Since $e^+ := \sum_l e^+_l$ is an eigenvector with positive entries and eigenvalue 1, the Perron-Frobenius theorem, or its direct consequence [18, Theorem 1.6], implies that the spectrum of $D(w)S^0D(w)$ is contained in the interval $[-1,1]$. We now determine the multiplicity of the eigenvalues at $-1$ and $1$, by studying invariant subspaces.

From the structure of $S^0$ in (26) we infer that $D(w)S^0D(w)$ leaves each of the $E_l$ invariant. Furthermore, since $S_0$ is FID for all $l$, its restriction to $E_l$ is irreducible, with period two if $l \in [M]$ and aperiodic if $l \in M + [L]$. In the aperiodic case, from the Perron-Frobenius theorem, 1 is a non-degenerate eigenvalue of $D(w)S^0D(w)|_{E_l} = D(w_l)S_0D(w_l)$ and $\text{Spec}(D(w_l)S_0D(w_l)) \setminus \{1\} \subset (-1,1)$. Here, $A|_E$ denotes the restriction of $A$ to an invariant subspace $E$. Due to (38) the Perron-Frobenius eigenvector is $e^+_i$ in this case. In the period two case, $-1$ and 1 are the only eigenvalues of $D(w)S^0D(w)|_{E_l}$ with magnitude 1. These eigenvalues are non-degenerate and due to (38) and (39), the corresponding eigenvectors are $e^-_i$ and $e^+_i$, respectively. Together we then conclude that $E_-$ is the eigenspace of $D(w)S^0D(w)$ corresponding to eigenvalue $-1$.

We now turn to showing the derivative of $F$ with respect to $x$ is invertible at $(x, \omega) = (w,0)$. We write $\nabla_h F_i$ for the derivative of $F_i$ with respect to the $x$-coordinate, in the direction $h \in \mathbb{C}^K$. Using (38) and that $D(w)S_0D(w)$ commutes...
with $P_\perp$ by (39) we get
\[
\nabla_h F_0(w,0) = P_\perp (h S^0 w + w S^0 h) = (1 + D(w) S^0 D(w)) P_\perp (h/w).
\]
From the information that all eigenvectors of $D(w) S^0 D(w)$ with corresponding eigenvalue $-1$ belong to $E_-$ we obtain that $\nabla_h F_0(w,0) = 0$ if and only if $h/w \in E_-.$

We now verify that the derivative of the remaining equations does not vanish when $h = we_-$, for any non-zero $e_- \in E_-$. If $l \in [M]$ is an index with $0 \perp l$, then (cf. (34))
\[
\nabla_h F_l(w,0) = \sum_{k:k \neq l} \langle h_l, S^>_w w_k \rangle + \sum_{k:k \neq l} \langle w_l, S^>_w h_k \rangle - \langle h_l \rangle
\]
and otherwise we have (cf. (35))
\[
\nabla_h F_l(w,0) = \sum_{k:k \neq l} \langle h_l, S^>_w w_k \rangle + \sum_{k:k \neq l} \langle w_l, S^>_w h_k \rangle - \sum_{k:k \neq l} \langle h_l, S^>_w w_k \rangle - \sum_{k:k \neq l} \langle w_l, S^>_w h_k \rangle.
\]

We evaluate (41) and (42) at $h = \sum_{k=1}^M \alpha_k (w_k e_k - w_k e_k) = w \sum \alpha_k e_k$ with $\alpha = (\alpha_k)_{k=1}^M \in \mathbb{C}^M$. For the $\langle w_l, S^>_w h_k \rangle$ terms in (42), we see that if $S^>_w$ is non-zero, then $k \in [M]$, and therefore $h_k = \alpha_k w_k$. On the other hand, the $h_k$ terms contributing to $\sum_{k:k \neq l} \langle w_l, S^>_w h_k \rangle$ with non-zero $S^>_w$ in (41) and (42) are either of the form $h_k = -\alpha_k w_k$ if $k \leq M$ or $h_k = \alpha_k w_k$ if $k > M$. To differentiate these cases, let
\[
s_l := \begin{cases} 1 & \text{if } S^>_w \text{ is non-zero for some } j > M \\ 0 & \text{otherwise.} \end{cases}
\]

We now show which of the cases that $h_k$ is realized depends only on $l$, meaning we have that the $k$ indices for which $S^>_w$ is non-zero are either all greater than $M$ or all less than or equal to $M$. First note that $f_j$ is negative for all $j \in [M]$, and nonnegative otherwise, so it suffices to show $f_j$ has the same sign for all non-negative $S^>_w$. We then recall that $S^>_w$ is non-negative if and only if $j \in \mathcal{I}_1$, and can conclude that $f_j$ is constant for $j \in \mathcal{I}_1$, so in particular is sign definite on $\mathcal{I}_1$.

Evaluating the derivatives gives
\[
\nabla_h F_l(w,0) = \alpha_l \sum_{k:k \neq l} \langle w_l, S^>_w w_k \rangle - (-1)^{s_l} \alpha_l \sum_{k:k \neq l} \langle w_l, S^>_w w_k \rangle + \alpha_l \langle w_l \rangle
\]
in the first case (41) and
\[
\nabla_h F_l(w,0) = \alpha_l \sum_{k:k \neq l} \langle w_l, S^>_w w_k \rangle + \alpha_l \sum_{k:k \neq l} \langle w_l, S^>_w w_k \rangle - (-1)^{s_l} \alpha_l \sum_{k:k \neq l} \langle w_l, S^>_w w_k \rangle - \alpha_l \langle w_l, S^>_w w_k \rangle
\]
in the second case (42).

Altogether we see that, $\nabla_h F(w,0) = 0$ for some $0 \neq h \in \mathbb{C}^K$ is equivalent to $A \alpha = 0$, for some $0 \neq \alpha \in \mathbb{C}^M$, where the matrix $A = (a_{ij})_{i,j=1}^M \in \mathbb{C}^{M \times M}$ is given by
\[
a_{ij} := (-1)^{s_i+1} \langle w_i, S^>_w w_j \rangle - \langle w_i, S^>_w w_j \rangle.
\]
for all $i,j \in [M]$ with $i \neq j$, and
\[
    a_{ii} := \sum_{j : j \neq i} \langle w_i, S_{ij}^{\geq} w_j \rangle + \sum_{j : j \neq i} \langle w_i, S_{ij}^{\leq} w_j \rangle + \mathbb{1}(0 \triangleright i) \langle w_i \rangle,
\]
(43)
for indices $i \in [M]$. Note that the second sum is empty if $0 \triangleright i$ and that the diagonal entries of $A$ are all positive. In the case $0 \triangleright i$, the term $\langle w_i \rangle$ in (43) is positive due to (37) and otherwise the second sum contains at least one positive term since $\mathcal{I}_i$, defined in (27), is non-empty. The off-diagonal elements satisfy
\[
\sum_{j \in [M] \setminus i} |a_{ij}| \leq a_{ii}
\]
(44)
with strict inequality in the case $0 \triangleright i$.

In order to show $A$ is invertible, we decompose $A := D + O$ into its diagonal part $D$ and off-diagonal part $O$. Since the diagonal entries are non-zero we can rewrite $A = D(1 + D^{-1}O)$. It then suffices to show that the inverse of $1 + D^{-1}O$ exists. To see this, we define the matrix $|D^{-1}O|$ by taking entrywise absolute values, i.e. $|D^{-1}O|_{ij} := |(D^{-1}O)_{ij}|$. The following lemma, whose proof we postpone until after we have finished proving Lemma 4.3, will allow us to show $1 + D^{-1}O$ is invertible.

**Lemma 4.4.** There is $p \in \mathbb{N}$ such that the row sums of $|D^{-1}O|^p$ satisfy
\[
\max_{l \in [M]} \sum_{k=1}^{M} (|D^{-1}O|^p)_{lk} < 1.
\]
(45)

Because the Perron-Frobenius eigenvalue of $|D^{-1}O|^p$ is bounded from above by the maximum row sum, (45) implies that all eigenvalues of $|D^{-1}O|^p$ have magnitude strictly less than 1. Since Wielandt’s theorem (see for instance, [13, Lemma 3.2]) states that the spectral radius of any square matrix $A$ is bounded by the spectral radius of $|A|$, we conclude that all the eigenvalues of $D^{-1}O$ have magnitude strictly less than 1 and $1 + D^{-1}O$ is invertible. Thus, the matrix $A$ is invertible and the only solution to $Aa = 0$ is the trivial solution, $a = 0$. We conclude $\nabla F(w, 0)$ is invertible, finishing the proof of the lemma.

**Proof of Lemma 4.4.** First we show that for each $l \in [M]$, there exists a finite sequence of indices $i_0, i_1, \ldots, i_p \in [M]$ with $i_0 = l$ and $0 \triangleright i_p$ such that $|D^{-1}_{i_k i_{k+1}} O_{i_k i_{k+1}}| > 0$.

We construct the sequence inductively. If $0 \triangleright l$, there is nothing to show since (44) becomes a strict inequality and we can chose $p_l = 1$ and $i_l = l$. In all other cases, assuming $i_0, \ldots, i_k$ have been chosen, we pick an index $j \in \mathcal{I}_{i_k}$, i.e. $S_{i_k j}^{\geq} \neq 0$, and let $i_{k+1} := j$. We stop the procedure once $0 \triangleright i_{k+1}$. In particular, here we have $\mathcal{I}_{i_k} \neq \emptyset$. We see this sequence satisfies the desired property as
\[
|D^{-1}_{i_k i_{k+1}} O_{i_k i_{k+1}}| = D^{-1}_{i_k i_{k+1}} \langle w_{i_k}, S_{i_k i_{k+1}}^{\geq} w_{i_{k+1}} \rangle > 0,
\]
for each \( k \). In the above inequality, we have used that \( \langle w_{i_k}, S_{i_{k+1}} w_{i_{k+1}} \rangle = 0 \), as \( i_{k+1} \neq i_k \). Positivity of these coefficients then implies that \( (|D^{-1}O|^k)_{i_{k+1}} > 0 \) for all \( k < p_l \).

Now that we have constructed a sequence as stated above, we show that
\[
\sum_k (|D^{-1}O|^n)_{lk} < 1. \tag{46}
\]

From (44), we have
\[
\sum_j |D^{-1}O|_{ij} \leq 1 \tag{47}
\]
for all \( i \), with strict inequality when \( 0 \neq i \). For any positive \( q \), we bound the row sums of \( |D^{-1}O|^q \) by writing them as products of row sums of \( |D^{-1}O| \), namely
\[
\sum_k (|D^{-1}O|^q)_{lk} = \sum_{i_1} |D^{-1}O|_{i_1} \sum_{i_2} |D^{-1}O|_{i_1i_2} \ldots \sum_k |D^{-1}O|_{i_{q-1}k} \leq 1
\]
Choosing \( p_l \) and \( i_p \) as defined in the beginning of the proof we get:
\[
\sum_k (|D^{-1}O|^{p_l})_{li_k} < \sum_k (|D^{-1}O|^{p_l-1})_{li_k}< 1, \tag{48}
\]
as \( 0 \neq i_0 \). Combining this bound with the general bound (47) we find
\[
\sum_k (|D^{-1}O|^{p})_{lk} \leq \sum_i (|D^{-1}O|^{p_l-1})_{li} \leq 1,
\]
where the strict inequality holds because it holds for the summand with index \( i = i_{p_l} \) due to (48) and we use (47) once again to bound the summands indexed by \( i \neq i_{p_l} \).

Finally (45) holds because (46), when combined with (47) also implies that (46) holds for \( p > p_l \). Thus, with the choice \( p := \max(p_l), \) the inequality (46) holds uniformly in \( l \).

\[ \square \]

**Proof of Theorem 2.8.** Since (1) has a unique solution with positive imaginary part and by taking the complex conjugate on both sides of (1) we see that \( m(-\bar{z}) = -m(z) \). In particular, \( \rho(\tau) = \rho(-\tau) \) and it suffices to show (4) for \( \tau > 0 \). By Proposition 4.1 the function \( z \mapsto m(z) \) the function has a holomorphic extension to \( U \setminus \{0, \infty\} \), where \( U \) is a neighborhood of the origin in the complex plane. We denote the extension again by \( m(z) \). The claim (4) follows from
\[
\pi \rho(\tau) = (\text{Im} m(\tau)) = \tau^{-\theta} \theta(\tau)
\]
for small enough \( \tau > 0 \), where \( \theta \) is continuous with \( \theta(0) > 0 \). Indeed, by Proposition 4.1 we have
\[
\text{Im} m(\tau) = \tau^{-f} (\text{Re} \phi_1) \text{Re} \tilde{v} - \text{Im} \phi_1 \text{Im} \tilde{v} = \tau^{-f} (\tilde{v}(0)g + O(\tau^{1/Q})) = \tau^{-f} (\tilde{v}(0) + O(\tau^{1/Q})),
\]
where \( g := \text{Re} \tilde{v} \) and we used that the analytic function \( \tilde{v} = \tilde{v}((-i \tau)^{1/Q}) \) satisfies \( \tilde{v} = \tilde{v}(0) + O(\tau^{1/Q}) \) and \( \tilde{v}(0) > 0 \). Since \( f_l \in (-1,1) \) by Lemma 3.1, the
vector \( g \) has strictly positive entries. Thus, we find
\[
\theta(\tau) = \langle \tau^{\sigma-I} (\tilde{v}(0)g + \mathcal{O}(\tau^{1/\mathcal{Q}})) \rangle = \text{Re}([-i]^{-\sigma} \sum_{i=1}^{2M+L} 1(f_i = \sigma) \langle \tilde{v}(0) e_i \rangle + \mathcal{O}(\tau^{1/\mathcal{Q}}),
\]
where in the second equality we used (10) and \( f_j \leq \sigma - \frac{1}{\mathcal{Q}} \) for all indices \( j \) with \( f_j < \sigma \).

**Proof of Proposition 2.1.** In case (i), since \( S \) has total support its FID skeleton is either one large block or of the form
\[
\begin{pmatrix}
0 & S_{12} \\
S_{21} & 0
\end{pmatrix}.
\]
In either case, \( \ell_{\infty}(S) = 0 \). By Theorem 2.8, we then have that the self-consistent density of states is bounded. Case (ii), follows directly from Theorem 2.8.

We divide the proof of Case (iii) into several steps. From now on we assume \( S \) does not have support. Additionally, we assume that \( S \) has no zero rows, because if row \( i \) was a zero row then the index \( i \) in (1) would decouple from the rest of the Dyson equation, with the associated solution given by \( m_i(z) = -1/z \), i.e. implying a contribution to the atom of \( \rho \) at the origin of size \( 1/K \). In Step 1, we write \( S \) in a normal form, based on its largest zero block. This block structure naturally splits the solution to (1) into three components. In Step 2, we give a lower bound comparable to \( \eta^{-1} \) on the third of these components and therefore also on the averaged solution to (1) along the imaginary line \( z = i\eta \). This is consistent with an atom at \( z = 0 \).

In Step 3 we show that the first out of the three components of the solution decays proportional to \( \eta \) as \( \eta \to 0 \). In Step 4 we determine the precise weight of the atom at the origin in the self-consistent density by establishing that the second out of the three components of \( v(\eta) \) is much smaller than \( \eta^{-1} \) in the \( \eta \to 0 \) limit.

**Step 1:** We begin writing \( S \) in a normal form based on its largest zero block.

**Lemma 4.5 (Normal Form for matrices without support).** Let \( S \) be a symmetric matrix with non-negative entries and without support. There exist \( I,J \subseteq [K] \) such that \( |I| + |J| > K \) and a permutation matrix \( P \) such that
\[
P^t S P = \begin{pmatrix}
S_{11}^{12} & S_{12}^{12} & S_{13}^{12} \\
S_{21}^{12} & S_{22}^{12} & 0 \\
S_{31}^{12} & 0 & 0
\end{pmatrix}
\]
where \( S_{11}^{12} \in \mathbb{R}^{(K-|J|) \times (K-|J|)} \), \( S_{12}^{12} \in \mathbb{R}^{(K-|J|) \times (|J|-|I|)} \), and \( S_{13}^{12} \in \mathbb{R}^{(K-|J|) \times |I|} \). The above form is chosen so that \( S_{22}^{12} \) has support and that for each \( k = 1,...,(K-|J|) \) there is no set of \( k \) rows of \( S_{13}^{12} \) such that all the non-zero entries of these rows lie in \( k \) or fewer columns.
The above decomposition creates an \(|I| \times |J|\) submatrix of zeros such that \(|I|+|J|\) is maximized and that additionally, among all such choices, \(|J|\) is as large as possible. If \(S^{13}\) were chosen with a \(k \times k\) submatrix that contained all the non-zero entries of the corresponding rows, then these entries (and the corresponding entries of \(S^{31}\)) could be permuted into the bottom left (top right) corner and absorbed into \(S^{22}\). This process would leave \(S^{22}\) with support and strictly increase \(|J|\). We defer the proof until Appendix C. We now assume \(S\) is in this normal form, i.e. \(P\) is the identity matrix.

Step 2: We now partition the solution of (6) along the blocks of \(S\). Let \(v(z) = (a,b,c)\) where \(a \in \mathbb{R}^{K-|J|}\), \(b \in \mathbb{R}^{|J|-|I|}\), and \(c \in \mathbb{R}^{|I|}\).

Along the imaginary axis we find
\[
\begin{align*}
a^{-1} &= S^{11}a + S^{12}b + S^{13}c + \eta, \\
b^{-1} &= S^{21}a + S^{22}b + \eta, \\
c^{-1} &= S^{31}a + \eta.
\end{align*}
\]

Multiplying each equation by the inverse of the left hand side and averaging gives
\[
\begin{align*}
1 &= \langle a, S^{11}a \rangle + \langle a, S^{12}b \rangle + \langle a, S^{13}c \rangle + \eta \langle a \rangle, \\
1 &= \langle b, S^{21}a \rangle + \langle b, S^{22}b \rangle + \langle c \rangle \eta, \\
1 &= \langle c, S^{31}a \rangle + \langle c \rangle \eta.
\end{align*}
\]

Then multiplying the first equation by \((K-|J|)\) and the third equation by \(|I|\) and taking the differences we get
\[
|I| - (K-|J|) = \eta |I| - (K-|J|) \left( \langle a, S^{11}a \rangle + \langle a, S^{12}b \rangle + \eta \langle a \rangle \right) \leq \eta |I| \langle c \rangle. 
\]

Thus, for all \(\eta > 0\) the average of the third block component is bounded from below by
\[
\langle c \rangle \geq \frac{|I| + |J| - K}{|I| \eta},
\]
implying the lower bound
\[
\langle v \rangle \geq \frac{|I| + |J| - K}{K \eta}.
\]

To show that the leading order of \(\langle v \rangle\) is in fact given by the right hand side, we will show that the terms dropped in the equality in (55) vanish in the limit \(\eta \rightarrow 0\). To do this, we will first show that \(a \sim \eta\) in the following step.

Step 3: We now partition the set \([K-|J|]\) into pieces on which the solution to the Dyson equation can be separately studied. This partition is induced by the equivalence relation \(\sim_R\), where \(i \sim_R j\) if there exists a power \(l > 0\) such that \((S^{13}S^{31})^l_{ij} > 0\). We denote the elements of this partition by \(I_x, x=1,2,...,p\). For any set \(S \subseteq [d]\) and vector \(w \in \mathbb{R}^d\), we denote the restriction of \(w\) to \(S\) by \(w_S = (w_i)_{i \in S}\). Additionally, for each \(i \in [K-|J|]\) we define its neighbors to be \(N_i := \{j : s^{13}_{ij} > 0\}\), and let \(N_x := \cup_{i \in I_x} N_i\).
For fixed $x = 1, \ldots, p$ we will now show that $a_i \sim c_j^{-1} \sim \eta$ for each $i \in \mathcal{I}_x$, $j \in N_x$. We begin at an index $i$, and show that there is an index $j \in N_i$ such that $a_i \sim c_j^{-1}$, which in turn implies there is an element $k$ such that $j \in N_k$ such that $a_k \sim c_j^{-1}$. This process continues until we exhaust all entries that are in the same partition as $i$. Note that if the iteration starts at index $i$, then on the $l$th step of this iteration we consider indices such that $(S^{13}S^{31})_{ij}^l > 0$, motivating the definition of our partition of the index set.

From the definition of $\mathcal{I}_x$ we have the equality of the unnormalized sums

$$\sum_{i \in \mathcal{I}_x} a_i (S^{13}c)_i = \sum_{j \in N_{\mathcal{I}_x}} c_j (S^{31}a)_j.$$  \hfill (56)

Indeed, if $i \in \mathcal{I}_x$ and $s_{ij}^{13} > 0$, then $j \in N_{\mathcal{I}_x}$. Additionally, for such $j \in N_{\mathcal{I}_x}$, if $k$ is such that $s_{jk}^{31} > 0$, then $(S^{13}S^{31})_{ik} > 0$ and we see that $k \in \mathcal{I}_x$.

Averaging (52) and (54) over $\mathcal{I}_x$ and $N_{\mathcal{I}_x}$ instead of all indices and using (56) gives the refined lower bound

$$\langle c_{N_{\mathcal{I}_x}} \rangle \geq \frac{|N_{\mathcal{I}_x}|-|\mathcal{I}_x|}{|N_{\mathcal{I}_x}| \eta}.$$ \hfill (57)

via the same computation as in (55), but with restricted sums. By construction of $S^{13}$, every set of $k$ rows has more than $k$ non-zero columns and therefore we have $|N_{\mathcal{I}_x}|-|\mathcal{I}_x| > 0$.

In the next step, we fix $\mathcal{I}_x$ and for notational simplicity drop the subscript $x$, i.e. let $\mathcal{I} = \mathcal{I}_x$. We will now show that $a_\mathcal{I}$ scales like the inverse of $c_{N_{\mathcal{I}_x}}$. From this we deduce that $a_i \sim \eta$ for all $i \in \mathcal{I}$. In what follows we will tacitly use the trivial bound (7).

Our main tool is the following inequality. For any subset $\mathcal{I}' \subset \mathcal{I}$ we have

$$\sum_{j \in N_{\mathcal{I}'}} \left( \sum_{k \in \mathcal{I} \setminus \mathcal{I}'} c_j s_{jk}^{31} a_k + \eta c_j \right) \geq |N_{\mathcal{I}'}| - |\mathcal{I}'|.$$ \hfill (58)

To verify (58) we note that the sum of the coordinates of $\mathcal{I}'$ in (49), after multiplying both sides by $a$, gives

$$|\mathcal{I}'| = \sum_{k \in \mathcal{I} \setminus \mathcal{I}'} a_k ((S^{11}a)_k + (S^{12}b)_k + (S^{13}c)_k + \eta) \geq \sum_{k \in \mathcal{I} \setminus \mathcal{I}'} a_k (S^{13}c)_k.$$  \hfill (59)

Similarly for the coordinates of $N_{\mathcal{I}'}$ in (51) we have

$$|N_{\mathcal{I}'}| = \sum_{j \in N_{\mathcal{I}'}} c_j ((S^{31}a)_j + \eta).$$  \hfill (59)

Now (58) follows from taking the difference, and noticing that each of the sums only contains indices in $\mathcal{I}$ and $N_{\mathcal{I}'}$.

Once again, by the construction of $S^{13}$, there must be more than $|\mathcal{I}'|$ columns with non-zero entries and therefore $|N_{\mathcal{I}'}| - |\mathcal{I}'| > 0$, making the lower bound in (58) non-trivial.

We will show now that for any subset $\mathcal{I}' \subset \mathcal{I}$ at least one of the following possibilities occurs: 1) $a_i \sim \eta$ for some $i \in \mathcal{I}'$ or 2) there exists an index $k \in \mathcal{I} \setminus \mathcal{I}'$.
such that \( a_k \geq a_i \) for some \( i \in I' \) (or both). Before verifying this fact, we show that this implies the desired result \( a_I \sim \eta \). We begin by choosing \( I'_0 \) the set of all indices \( i \in I \) such that \( a_i \geq \max_{j \in I} a_j \). If case 1) holds, then we are done, as \( i \in I'_0 \) implies \( a_i \geq a_j \) for all \( j \in I \) and therefore \( a_I \sim \eta \) for all \( j \in I \). If case 2) holds, then there is an index \( k \in I \setminus I'_0 \) such that \( a_k \) scales like the largest component \( a_i \) of \( a \) with index \( i \in I \). We then let \( I_1 = I'_0 \cup \{ k \} \) and repeat the argument, replacing \( I'_0 \) with \( I_1 \). If at any point case 1) holds, the lemma is proven. If case 2) holds, the argument is inductively repeated until \( I \) is exhausted. Once \( I \) is exhausted, case 2) can no longer hold and we get that there is an index \( i \in I \) such that \( a_i \sim \eta \) but by the inductive argument we have that \( a_i \sim a_j \sim \eta \) for all \( i, j \in I \), as desired.

We now verify that one of the two cases 1) or 2) from above must hold. From (58), for at least one \( j \in N_{I'} \) we have either \( \sum_{k \in I \setminus I'} c_j s_{jk} a_k \geq 1 \) or \( \eta c_j \geq 1 \). In the latter case the comparison relation \( c_j \sim \eta^{-1} \) holds because \( c_j \lesssim \eta^{-1} \) is trivially satisfied. Additionally, from (49) we have that

\[
a^{-1}_i \gtrsim \max_{j \in N_i} c_j.
\]  

(60)

If \( c_j \sim \eta^{-1} \), then the relation (60) implies that \( a_i \sim \eta \) for some \( i \in I' \) and therefore case 1) holds.

On the other hand, if \( \sum_{k \in I \setminus I'} c_j s_{jk} a_k \geq 1 \), then there exists a \( k \in I \setminus I' \) such that \( a_k \gtrsim c_j^{-1} \) for some \( j \in N_{I'} \). Additionally, from (60), we have for \( i \in I' \), that \( c_j^{-1} \gtrsim a_i \) for \( j \in N_{I'} \). Combining these two relations gives that \( a_k \gtrsim a_i \), so case 2) holds.

Having verified that at least one of the two cases 1) or 2) must hold, we then have that \( a_i \sim \eta \) for all \( i \in I \). Since the index \( x \) of \( I = I_x \) was chosen arbitrarily, we conclude \( a \sim \eta \). Having verified that at least one of the two cases 1) or 2) must hold, we then have that \( a_i \sim \eta \) for all \( i \in I \). Since the index \( x \) of \( I = I_x \) was chosen arbitrarily, we conclude \( a \sim \eta \).

Step 4: We are now left with showing that \( \langle a, S^{12} b \rangle \) converges to zero as \( \eta \to 0 \) since this implies that the inequality in (55) is asymptotically sharp. This follows by noting that the restriction (50) of (1) to the \( b \) coordinates is similar to a Dyson equation itself, as we will explain now. In fact, making the substitution \( b = b \eta^{-1}(\eta + S^{21} a) \) in (50) yields

\[
\tilde{b}^{-1} = \tilde{S}^{22} \tilde{b} + \eta,
\]

\[
\tilde{s}^{22}_{ij} := \frac{s^{22}_{ij}}{(1 + \eta^{-1}(S^{21} a)) (1 + \eta^{-1}(S^{21} a)_j)}.
\]

Thus, we see that \( \tilde{b} \) satisfies a Dyson equation with \( \tilde{S}^{22} = (\tilde{s}^{22}_{ij})_{i,j=|I|-|I|+1} \) as its variance profile. From Proposition 3.2, we conclude that \( \tilde{b} \) grows slower than \( \eta^{-1} \) since \( \tilde{S}^{22} \) has support by construction (cf. Lemma 4.5). We note that although the matrix \( \tilde{S}^{22} = S^{22}(\eta) \) is non-constant, its entries are uniformly bounded from above and away from zero, for all small enough \( \eta \). Thus, the comparison relations
in Proposition 3.2 remain valid. We infer \( (a, S^{12}b) \rightarrow 0 \), as desired. Combining this with the first equality of (55), we see that
\[
\lim_{\eta \downarrow 0} (v(\eta))\eta = \frac{|I| + |J| - K}{K}.
\]
From Stieltjes inversion, we infer that \( \rho \) has an atom with mass \( \frac{|I| + |J| - K}{K} \) at the origin. \( \Box \)

5 Min-max averaging problem

In this section, we solve a general version of the min-max averaging problem (9) for the exponents, \( f_i \), describing the asymptotic power law behavior of \( v_i(\eta) \) as \( \eta \rightarrow 0 \). Motivated by the relation from Definition 2.5 this general version is formulated on a directed graph without loops and allows for general boundary conditions. In particular, within this section, we use the same symbol \( \Rightarrow \) for the general relation on the directed graph. We will conclude this section with the proof of Lemma 3.1 by applying the general theory we now develop. For an example that illustrates the connection between the directed graphs studied in this section and the relation \( \Rightarrow \) on the index set of the 0-1 mask from Definition 2.5 we refer to Appendix A.

Let \( (X, E_X) \) be a non-empty finite directed graph with directed edges \( E_X \subset X^2 \). We write \( x \Rightarrow_X y \) if \( (x, y) \in E_X \) and say \( x \) is a predecessor of \( y \) and \( y \) is a successor of \( x \). If the set of underlying edges are clear from the context we simply write \( X \) instead of \( (X, E_X) \) and \( x \Rightarrow y \) instead of \( x \Rightarrow_X y \). For \( n \in \mathbb{N} \), a map \( \gamma: [0, n] \rightarrow X, i \mapsto \gamma_i \) with \( \gamma_i \Rightarrow \gamma_{i+1} \) is a path (from \( \gamma_0 \) to \( \gamma_n \)) of length \( \ell(\gamma) := n \). If a fixed path \( \gamma \) is chosen we will often use the notation \( [i] = [\gamma]_i = \gamma_i \) and we write \( x \Rightarrow y \) if \( \gamma \) is a path from \( x \) to \( y \). A directed graph \( (Y, E_Y) \) is a subgraph of \( (X, E_X) \) if \( Y \subset X \) and \( E_Y \subset E_X \). In this case we write \( (Y, E_Y) \subset (X, E_X) \) and \( (Y, E_Y) \subseteq (X, E_X) \) if equality does not hold for both inclusions. In the following we always consider relations \( \Rightarrow \) on directed graphs \( X \) without loops, i.e. there are no closed paths \( x \Rightarrow x \). In particular, no element of \( X \) is its own predecessor.

**Definition 5.1 (Past and future).** For any \( x \in X \) in a directed graph \( (X, E_X) \) we set
\[
\mathcal{P}^X_x := \{ y \in X : \exists \gamma \text{ such that } y \Rightarrow x \}, \quad \mathcal{F}^X_x := \{ y \in X : \exists \gamma \text{ such that } x \Rightarrow y \}.
\]
We call \( \mathcal{P}^X_x \) the past and \( \mathcal{F}^X_x \) the future of \( x \) (in \( X \)).

**Definition 5.2 (Min-max averaging).** Let \( f: X \rightarrow \mathbb{R} \) be a function on the directed graph \( X \). We say that \( f \) is increasing (on \( X \)) if \( f(x) \leq f(y) \) holds for all \( x, y \in X \) with \( x \Rightarrow y \). In case \( f(x) < f(y) \) for \( x, y \in X \) with \( x \Rightarrow y \), we say that \( f \) is strictly increasing (on \( X \)). For a subgraph \( (Y, E_Y) \subset (X, E_X) \) we say that \( f \) is min-max
averaging on $\mathcal{Y}$ inside $\mathcal{X}$ if it is increasing on $\mathcal{X}$ and
\[ f(y) = \frac{1}{2} \left( \min_{x \in \mathcal{X} \setminus \mathcal{Y}} f(x) + \max_{x \in \mathcal{X} \setminus \mathcal{Y}} f(x) \right) \] (61)
holds for all $y \in \mathcal{Y}$.

**Definition 5.3 (Boundary condition).** An increasing function $f : \mathcal{Y} \to \mathbb{R}$ on a subgraph $(\mathcal{Y}, E_\mathcal{Y}) \subset (\mathcal{X}, E_\mathcal{X})$ is called a boundary condition for $\mathcal{X}$ if $\mathcal{Y}$ contains all $x \in \mathcal{X}$ with an empty past or future in $\mathcal{X}$, the subgraph $\mathcal{Y}$ contains all edges in $\mathcal{X}$ between elements of $\mathcal{Y}$ (i.e. if $E_\mathcal{Y} = E_\mathcal{X} \cap \mathcal{Y}^2$), and $f(x) \leq f(y)$ for all $x, y \in \mathcal{Y}$ such that $y \in E_\mathcal{X} x$. If, additionally, $f$ satisfies the strict inequality $f(x) < f(y)$ for all $x, y \in \mathcal{Y}$ such that $y \in E_\mathcal{X} x$, then we say that $f$ is a strictly increasing boundary condition.

Note that the subgraph $\mathcal{Y}$ on which a boundary condition $f : \mathcal{Y} \to \mathbb{R}$ is defined is never empty. Indeed, since there are no loops in the finite graph $\mathcal{X}$, there always exists a maximal element without future and a minimal element without past.

**Theorem 5.4 (Solution of min-max averaging problem).** Let $\mathcal{X}$ be a finite directed graph without loops and $f : \mathcal{Y}_0 \to \mathbb{R}$ be a boundary condition. Then there is a unique extension $\hat{f} : \mathcal{X} \to \mathbb{R}$ of $f$ to $\mathcal{X}$, such that $\hat{f}$ is min-max averaging on $\mathcal{X} \setminus \mathcal{Y}_0$ inside $\mathcal{X}$. If $f$ is a strictly increasing boundary condition, then $\hat{f}$ is strictly increasing on $\mathcal{X}$.

**Proof.** We will iteratively define extensions $f_k : \mathcal{Y}_k \to \mathbb{R}$ for $k = 0, \ldots, L$ of $f$ and associated positive numbers $\delta_k$. We now give the important properties of $f_k$, $\mathcal{Y}_k$, and $\delta_k$. We will then verify that these properties hold. With a slight abuse of notation we identify $\gamma = (\gamma_i)_{i=0}^n$ with the set $\{\gamma_i : i \in \mathbb{N}\}$ in the following.

1. Initially we start on $\mathcal{Y}_0$ with $f_0 := f$.
2. The extensions are strict, i.e. $(\mathcal{Y}_k, E_k) \subset (\mathcal{Y}_{k+1}, E_{k+1})$ with $E_k := E_{\mathcal{Y}_k}$ and $f_{k+1}(x) = f_k(x)$ holds for all $x \in \mathcal{Y}_k$ and for all $k = 0, \ldots, L - 1$.
3. The future and past within $\mathcal{Y}_k$ of all elements of $\mathcal{Y}_k \setminus \mathcal{Y}_0$ are not empty i.e. for all $x \in \mathcal{Y}_k \setminus \mathcal{Y}_0$ we have $\mathcal{F}_{x}^{\mathcal{Y}_k} \neq \emptyset$ and $\mathcal{P}_{x}^{\mathcal{Y}_k} \neq \emptyset$.
4. Associated to the extensions are strictly increasing non-negative numbers $\delta_0 < \cdots < \delta_L$ defined by
\[ \delta_k := \min_\gamma \frac{f_k(x) - f_k(y)}{\ell(\gamma)}, \quad k = 0, \ldots, L - 1, \] (62)
where the minimum is taken over all paths $y \xrightarrow{\gamma} x$ in $\mathcal{X}$ with endpoints $x, y \in \mathcal{Y}_k$ such that the path moves through $\mathcal{X} \setminus \mathcal{Y}_k$, i.e. $\gamma \setminus \{x, y\} \subset \mathcal{X} \setminus \mathcal{Y}_k$ if $\ell(\gamma) > 1$ and $(y, x) \in E_\mathcal{X} \setminus E_k$ if $\ell(\gamma) = 1$. The extension then satisfies $\mathcal{Y}_{k+1} = \mathcal{Y}_k \cup \bigcup_{\gamma \in \Gamma_k} \gamma$, where $\Gamma_k$ is the set of all minimizing paths in (62) and $E_{k+1}$ consists of all edges in $E_k$ and all edges that are traversed by paths in $\Gamma_k$.
Additionally, the numbers $\delta_k$ satisfy the identity
\[ f_{k+1}(x) - f_{k+1}(y) = \delta_k, \quad (y,x) \in E_{k+1} \setminus E_k. \] (63)

5. For $k=1, \ldots, L$ the extension $f_k$ is min-max averaging on $\mathcal{Y}_k \setminus \mathcal{Y}_0$ inside $\mathcal{Y}_k$.

6. Finally, we have $\mathcal{Y}_L = \mathcal{X}$.

From this construction existence of the extension $\hat{f}$ follows by choosing $\hat{f} := f_L$: $\mathcal{Y}_L \to \mathbb{R}$ since $\mathcal{Y}_L = \mathcal{X}$. Initially we set $f_0 := f$ as required, and $\delta_0$ as in (62). Now we construct the extensions inductively until $\mathcal{Y}_L = \mathcal{X}$, which happens eventually because of property 2 above and because $\mathcal{X}$ is finite. Suppose that $f_i$ has been constructed for all $l \leq k < L$ with associated numbers $\delta_0 < \cdots < \delta_{k-1}$ such that properties 1 to 5 above are satisfied for the already constructed extensions. To define $\mathcal{Y}_{k+1}$, $f_{k+1}$ and $\delta_k$, given $\mathcal{Y}_k$ and $f_k$, we follow the suggestion from property 4. We pick $x,y$ and $y \xrightarrow{\gamma} x$ such that $\gamma \cap \mathcal{Y}_k = \{x,y\}$ and $f_k(x) - f_k(y) = \ell(\gamma) \delta_k$, i.e. $\gamma$ is a minimizer in (62).

Such path always exists. Indeed, since $\mathcal{X}$ is finite it suffices to show that the set of paths through $\mathcal{X} \setminus \mathcal{Y}_k$ with endpoints in $\mathcal{Y}_k$ is not empty. Since $k < L$ there is an element $u \in \mathcal{X} \setminus \mathcal{Y}_k$ or we have $\mathcal{Y}_k = \mathcal{X}$ and $E_K \subseteq E_\mathcal{X}$. In the latter case we pick $(y,x) \in E_\mathcal{X} \setminus E_K$ and $y \xrightarrow{\gamma} x$ the path of length $\ell(\gamma) = 1$. In the former case any largest element of $\mathcal{F}_u^\mathcal{X}$ and any smallest element of $\mathcal{P}_u^\mathcal{X}$ are in the boundary $\mathcal{Y}_k \subset \mathcal{Y}_k$. We follow an arbitrary path $u \xrightarrow{\gamma} x$, starting from $u$, inside $\mathcal{F}_u^\mathcal{X}$ until the first instance the path hits some $x \in \mathcal{Y}_k$. Then we backtrack along an arbitrary path $y \xrightarrow{\gamma} u$, ending at $u$, inside $\mathcal{P}_u^\mathcal{X}$ until the first instance the path hits some $y \in \mathcal{Y}_k$. The composition of paths $y \xrightarrow{\gamma} u \xrightarrow{\gamma} x$ runs from $y \in \mathcal{Y}_k$ to $x \in \mathcal{Y}_k$ through $\mathcal{X} \setminus \mathcal{Y}_k$.

Given a path $\gamma$ that minimizes (62) we define
\[ f_{k+1}(u) := f_k(y) + j \delta_k. \] (64)

Let $\Gamma_k$ be the set of all such minimizing paths and $\tilde{E}_k \subset \mathcal{X} \setminus E_k$ the set of edges that all these paths $\gamma \in \Gamma_k$ traverse, i.e. $([j], [j+1], \gamma) \in \tilde{E}_k$. Then we set $\mathcal{Y}_{k+1} := \mathcal{Y}_k \cup \bigcup_{\gamma \in \Gamma_k} \mathcal{E}_k \cup \tilde{E}_k$ and use (64) to define $f_{k+1}$. By construction $(\mathcal{Y}_{k+1}, E_{k+1}) \supseteq (\mathcal{Y}_k, E_k)$, i.e. property 2 is satisfied. Property 3 also holds because every $u \in \mathcal{Y}_{k+1} \setminus \mathcal{Y}_k$ satisfies $u = [j], \gamma$ for some $\gamma$ that minimizes (62) and $0 < j < \ell(\gamma)$. Thus, $[j-1], \gamma \in \mathcal{F}_u^{\mathcal{Y}_{k+1}}$ and $[j+1], \gamma \in \mathcal{F}_u^{\mathcal{Y}_{k+1}}$, i.e. the future and past of $u$ within $\mathcal{Y}_{k+1}$ are not empty.

We now show the definitions (64) are consistent, i.e. they do not depend on the choice of minimizing path in the case $\mathcal{Y}_{k+1} \setminus \mathcal{Y}_k \neq \emptyset$. Indeed, let $y_1 \xrightarrow{\gamma_1} x_1$, $y_2 \xrightarrow{\gamma_2} x_2$ be two minimizing paths that cross at some $u := [j], \gamma = [j], \gamma_2$. We construct two paths $\tau_1$ and $\tau_2$. For $\tau_1$ we follow $\gamma_1$ from $y_1$ to $u$ and then follow $\gamma_2$ from $u$ to $x_2$. For $\tau_2$ we follow $\gamma_2$ from $y_2$ to $u$ and then follow $\gamma_1$ from $u$ to $x_1$. Since
\[ f_k([0]_{\tau_i}) - f_k([\ell_i]_{\tau_i}) \geq \delta_k \ell_i \text{ with } \ell_i := \ell(\tau_i) \] holds by the definition of \( \delta_k \) we get
\[ \frac{f(x_2) - f(y_2)}{\ell_2 + j_1 - j_2} \leq \delta_k \quad \frac{f(x_1) - f(y_2)}{\ell_1 + j_2 - j_1} \leq \delta_k. \] (65)

From the first inequality we conclude
\[ f(y_1) + j_1 \delta_k \geq f(x_2) - \ell_2 \delta_k + j_2 \delta_k = f(y_2) + j_2 \delta_k. \]

The analogous bound coming from the second inequality of (65) implies equality. Thus, from (64) we see that \( f_{k+1}(u) = f(y_1) + j_1 \delta_k = f(y_2) + j_2 \delta_k \) is independent of the choice of path.

Next we verify property 4. First we show that \( \delta_k > \delta_{k-1} \) via proof by contradiction. Suppose therefore that \( \delta_k = \delta_{k-1} \) and let \( y \xrightarrow{\gamma} x \) be one of the minimizing paths \( \gamma \in \Gamma_{k-1} \) that have been used in the construction (64) of the extension \( f_{k+1} \). We follow an arbitrary path \( x \xrightarrow{\gamma} u \) through \( \mathcal{P}_{x}^{\gamma_{k}} \) until the first instance it hits \( u \in \mathcal{Y}_{k-1} \). Similarly we backtrack along an arbitrary path \( v \xrightarrow{\gamma} y \) through \( \mathcal{P}_{y}^{\gamma_{k}} \) until the first instance it hits \( v \in \mathcal{Y}_{k-1} \). Both are possible because of property 3. Then the joint path \( v \xrightarrow{\gamma} y \xrightarrow{\gamma} x \xrightarrow{\gamma} u \) satisfies
\[ f_{k-1}(u) - f_{k-1}(v) = (f_{k-1}(u) - f_k(x)) + (f_k(x) - f_k(y)) + (f_k(y) - f_{k-1}(v)) \]
\[ \leq \delta_{k-1} \ell(\gamma_1) + \delta_k \ell(\gamma) + \delta_{k-1} \ell(\gamma_2) = \delta_{k-1} (\ell(\gamma_1) + \ell(\gamma) + \ell(\gamma_2)), \] (66)
i.e. it is a minimizing path of \( \delta_{k-1} \) in \( \Gamma_{k-1} \). This contradicts the fact that at least one edge that \( \gamma \) traverses has to be in \( E_{k+1} \setminus E_k \). For the inequality in (66) we used (63) with \( k \) replaced by \( k-1 \) and that \( \delta_l \leq \delta_{k-1} \) for \( l \leq k-1 \). We conclude \( \delta_k > \delta_{k-1} \). The claim (63) as it stands is clear by the construction of \( f_{k+1} \) in (64).

Now we verify property 5, the min-max averaging of \( f_{k+1} \) on \( \mathcal{Y}_{k+1} \setminus \mathcal{Y}_0 \) inside \( \mathcal{Y}_{k+1} \), i.e. we check that \( f_{k+1} \) is increasing on \( \mathcal{Y}_{k+1} \) and that for every \( u \in \mathcal{Y}_{k+1} \setminus \mathcal{Y}_0 \) the identity
\[ \min_{u \xleftarrow{v} \mathcal{Y}_{k+1}} (f_{k+1}(v) - f_{k+1}(u)) = \min_{v \xrightarrow{u} \mathcal{Y}_{k+1}} (f_{k+1}(v) - f_{k+1}(u)) \] (67)
holds, where the minima are taken over \( v \in \mathcal{Y}_{k+1} \) and we write \( y \xleftarrow{\mathcal{Y}_{k+1}} x \) for \( (y,x) \in E_{k+1} \).

We first verify that (67) remains valid for \( u \in \mathcal{Y}_{k} \setminus \mathcal{Y}_0 \). Using that by property 3 the future and past within \( \mathcal{Y}_k \) of \( u \) are each non-empty, as well as (63) with \( k \) replaced by \( k-1 \), we see that both sides of (67) are less than or equal to \( \delta_{k-1} \). On the other hand \( f_{k+1}(u) - f_{k+1}(v) = \delta_k \) for every \( v \in \mathcal{Y}_{k+1} \) with \( (v,u) \in E_{k+1} \setminus E_k \) and \( f_{k+1}(v) - f_{k+1}(u) = \delta_k \) for every \( v \in \mathcal{Y}_{k+1} \) with \( (u,v) \in E_{k+1} \setminus E_k \). Since \( \delta_k > \delta_{k-1} \) adding the elements \( v \in \mathcal{Y}_{k+1} \setminus \mathcal{Y}_k \) in the minima of (67) does not effect the min-max averaging property on \( \mathcal{Y}_k \).

Now we verify that (67) is true for \( u \in \mathcal{Y}_{k+1} \setminus \mathcal{Y}_k \) and that \( f_{k+1} \) is increasing on \( \mathcal{Y}_{k+1} \). Indeed, in the case when \( \mathcal{Y}_{k+1} \setminus \mathcal{Y}_k = \emptyset \) the set of edges \( E_k = E_{k+1} \setminus E_k \) contains only elements of the form \( (y,x) \in \mathcal{Y}_{k+1}^2 \) with \( y \xleftarrow{\mathcal{Y}_{k+1}} x \) and by construction \( f_{k+1}(x) - f_{k+1}(y) = \delta_k \geq 0 \). Together with the fact that \( f_k \) was increasing on \( \mathcal{Y}_k \).
we conclude that $f_{k+1}$ is increasing on $\mathcal{Y}_{k+1}$. In the situation $\mathcal{Y}_{k+1} \setminus \mathcal{Y}_k \neq \emptyset$ we check that $f_{k+1}(v) - f_{k+1}(u) \geq 0$ for $(u,v) \in \mathcal{E}_{k+1}$ and that $f_{k+1}(u) - f_{k+1}(v) \geq 0$ for $(v,u) \in \mathcal{E}_{k+1}$, as well as (67) for any $u \in \mathcal{Y}_k \setminus \mathcal{Y}_0$. Let $y \rightarrow x$ be a path used to define $f_{k+1}$ in (64) and $u = [j], e \in \gamma \setminus \{x,y\} \subset \mathcal{Y}_{k+1} \setminus \mathcal{Y}_k$. Then the edges $u \rightarrow v$ and $v \rightarrow w$ giving rise to the set of $v$ the minima in (67) are taken over all belong to $\mathcal{E}_{k+1} \setminus \mathcal{E}_k$. We conclude that both sides of (67) equal $\delta_k$ due to (63). This finishes the construction of an extension $\hat{f}$ as claimed in the theorem.

Note that if $f_k$ is strictly increasing, then so is $f_{k+1}$, because in this case $\delta_k > 0$. By induction this implies that $\hat{f}$ is strictly increasing if the boundary condition $f$ is strictly increasing.

Now we are left with proving uniqueness of the extension $\hat{f}$. For that purpose let $\tilde{f}$ be an extension of $f$ as stated in the theorem. We will show inductively that $\tilde{f}$ coincides on $\mathcal{Y}_k$ with the extension $f_k$ from the construction above. On $\mathcal{Y}_0 = \mathcal{Y}$ the two function $f_0$ and $\tilde{f}$ coincide by assumption. Suppose now that $\tilde{f}$ coincides on $\mathcal{Y}_k$ with $f_k$ for some $k < L$. We now show that $\tilde{f}(x) = f_{k+1}(x)$ for all $x \in \mathcal{Y}_{k+1} \setminus \mathcal{Y}_k$. We define

$$\tilde{\delta}_k := \min \{ \tilde{f}(u) - \tilde{f}(v) : (v,u) \in \mathcal{E}_{k+1} \}.$$  

(68)

and pick a pair $(v,u) \in \mathcal{E}_{k+1} \setminus \mathcal{E}_{k}$ for which the minimum is attained. We show that the edge $(v,u)$ is traversed by a path $\gamma \in \Gamma_k$, that $\tilde{\delta}_k = \delta_k$ and that $\tilde{f}([j+1], \gamma) = \tilde{f}([j], \gamma) = \delta_k$ for all $j = 0, \ldots, \ell(\gamma) - 1$. Then $\tilde{f} = f_{k+1}$ on $\mathcal{Y}_{k+1}$ by construction of $f_{k+1}$.

We start by constructing a path $y \gamma \rightarrow x$ from some $y \in \mathcal{Y}_k$ to some $x \in \mathcal{X}_k$ with $\gamma \cap \mathcal{Y}_k = \{x,y\}$ such that $\tilde{f}([j+1], \gamma) = \tilde{f}([j], \gamma) = \delta_k$ for all $j = 0, \ldots, \ell(\gamma) - 1$. First, we iteratively construct a path $y \gamma \rightarrow v$, following $v$ through its past until we hit $y \in \mathcal{Y}_k$. Since $\tilde{f}$ satisfies (61) either $v \in \mathcal{Y}_k$ or $v \notin \mathcal{Y}_k$ and there is a $w \in \mathcal{X}$ with $(w,v) \in \mathcal{E}_{k+1} \setminus \mathcal{E}_k$ such that $\tilde{f}(v) - \tilde{f}(w) = \delta_k$. In the former case we stop and $\gamma_p$ is empty. In the latter case we continue to extend to the past from $w$ until we have constructed the path $\gamma_p$ such that $\tilde{f}([j+1], \gamma_p) - \tilde{f}([j], \gamma_p) = \tilde{\delta}_k$ for $j = 0, \ldots, \ell(\gamma_p) - 1$. Now we use the same procedure to construct a path $u \gamma \rightarrow x$, following $u$ through its future until we hit $x \in \mathcal{Y}_k$ and such that $\tilde{f}([j+1], \gamma_f) - \tilde{f}([j], \gamma_f) = \tilde{\delta}_k$ for $j = 0, \ldots, \ell(\gamma_f) - 1$. We call $\gamma$ the joint path $y \gamma \rightarrow v \gamma \rightarrow x$ that has constant increases in the values of $\tilde{f}$ of size $\tilde{\delta}_k$ along its edges.

Now we realize that $\tilde{\delta}_k = \delta_k$ from (62). Indeed, for any path $y \gamma \rightarrow x$ that traverses only edges from $\mathcal{E}_{k+1} \setminus \mathcal{E}_k$ with $\gamma \cap \mathcal{Y}_k = \{x,y\}$ we have $\tilde{f}([j+1], \gamma) - \tilde{f}([j], \gamma) \geq \tilde{\delta}_k$ by definition of $\tilde{\delta}_k$. Thus, $\tilde{f}(x) - \tilde{f}(y) \geq \ell(\gamma) \tilde{\delta}_k$ and equality holds for $\gamma = \gamma$ as constructed above. In particular, $\gamma \in \Gamma_k$ is a valid choice for the path used in the construction of $f_{k+1}$, the extension of $f_k$ to $\mathcal{Y}_k \cup \gamma \subset \mathcal{Y}_{k+1}$. Thus, $\tilde{f}$ is unique.
and the theorem is proven. □

The min-max averaging problem (61) is locally stable under perturbation in the following sense.

**Lemma 5.5 (Min-max averaging stability).** Let \( f: \mathcal{Y} \to \mathbb{R} \) be a boundary condition, \( \hat{f}: \mathcal{X} \to \mathbb{R} \) the extension of \( f \) that is min-max averaging on \( \mathcal{X} \setminus \mathcal{Y} \) from Theorem 5.4 and \( d: \mathcal{X} \to \mathbb{R} \) an arbitrary function. Set
\[
\delta := \min_{x \in \mathcal{X} \setminus \mathcal{Y}} \min \mathcal{A}_x, \quad \mathcal{A}_x := \{ [f(u) - f(v)] : u, v \in \mathcal{X} \text{ with } u \not\equiv x \text{ or } x \not\equiv u, v \} \setminus \{0\},
\]
where \( \min \emptyset := \infty \). Suppose \( g: \mathcal{X} \to \mathbb{R} \) satisfies the perturbed min-max averaging problem
\[
\begin{align*}
g(y) &= f(y) + d(y), & \text{for all } y \in \mathcal{Y} \\
g(x) &= \frac{1}{2} \left( \min_{y \in \mathcal{X} \setminus \mathcal{Y}} g(y) + \max_{y \in \mathcal{X} \setminus \mathcal{Y}} g(y) \right) + d(x), & \text{for all } x \in \mathcal{X} \setminus \mathcal{Y},
\end{align*}
\]
and \( \|g - \hat{f}\|_\infty = \max_{x \in \mathcal{X}} |g(x) - \hat{f}(x)| < \frac{1}{2} \delta \). Then
\[
\|g - \hat{f}\|_\infty \leq 2\ell \cdot d\|_\infty,
\]
where \( \ell \) the length of the longest path in \( \mathcal{X} \).

**Proof.** For every \( x \in \mathcal{X} \setminus \mathcal{Y} \) we pick \( y_2^x, y_2^x \) with \( x \not\equiv y_2^x \) and \( y_2^x \equiv x \) such that
\[
g(y_2^x) = \min_{y \in \mathcal{X} \setminus \mathcal{Y}} g(y), \quad g(y_2^x) = \max_{y \in \mathcal{X} \setminus \mathcal{Y}} g(y).
\]
By definition of \( \delta \) and \( \|g - \hat{f}\|_\infty < \frac{1}{2} \delta \) the same identities hold with \( g \) replaced by \( \hat{f} \). Now we set \( h := g - \hat{f} \).

\[
h(x) = \frac{1}{2} \left( h(y_2^x) + h(y_2^x) \right) + d(x).
\]

Let \( x_0 \in \mathcal{X} \setminus \mathcal{Y} \) be such that \( |h(x_0)| = \|h\|_\infty \). For definiteness, we consider the case \( h(x_0) > 0 \). For \( h(x_0) < 0 \) the proof is analogous. Now we construct a path \( u \rightarrow x_\gamma \rightarrow \gamma \rightarrow v \) with \( u, v \in \mathcal{Y} \) by the relations \([j + 1] \gamma_\gamma = y_{[j] \gamma_\gamma} \) and \([j] \gamma_\gamma = y_{[j] \gamma_\gamma + 1} \gamma_\gamma \) until we hit the boundary \( \mathcal{Y} \). We set \( \gamma := \gamma_1 \) the shorter of the two paths. In particular, \( \ell(\gamma) \leq \ell/2 \). Again for definiteness suppose \( \gamma = \gamma_2 \). The case \( \gamma = \gamma_1 \) follows the same argument. Now we show by induction that
\[
h([j] \gamma) \geq \|h||\infty - (3^j - 1)d\|_\infty - 2d\|_\infty.
\]
At the beginning \( j = 0 \) we have \( h(x_0) = \|h\|_\infty \). Now suppose that (72) holds at \( j \), we now show its validity with \( j \) replaced by \( j + 1 \). Indeed, by construction of \( \gamma \) and (71) we have that for some \( y \in \mathcal{X} \),
\[
h([j + 1] \gamma) = 2h([j] \gamma) - h(y) - 2d([j] \gamma) \geq \|h||\infty - 2 \cdot 3^j d\|_\infty \geq \|h||\infty - (3^j + 1 - 1)d\|_\infty,
\]
where we used (72) in the first inequality. We evaluate (72) at \( j = \ell(\gamma) \) to see
\[
d(v) = h(v) \geq \|h||\infty - (3^{\ell(\gamma)} - 1)d\|_\infty.
\]
Here we used that $v \in \mathcal{Y}$ and the boundary condition in (69). The claim (70) now follows from $\ell(\gamma) \leq \ell/2$. 

Now we apply the general theory we developed to the specific setting in Lemma 3.1.

**Proof of Lemma 3.1.** We consider the directed graph $\mathcal{X} = \mathbb{L} + \mathbb{M} + \mathbb{K}$ with edges $i \xrightarrow{} j$ given by the relation in Definition 2.5 and its extension to 0 and $\infty$, defined before (8). As boundary condition we choose $\mathcal{Y}_0 = \{0, \infty\} \cup (\mathbb{M} + [L])$ and the function $f(i) = f$, with $f_0 = -1$, $f_\infty = 1$, as well as $f_i = 0$ for all $i \in \mathbb{M} + [L]$. Then (9) as well as (12) follow immediately from an application of Theorem 5.4 and Lemma 5.5 to this setting.

We now verify the properties of $f$ in our specific setting. By monotonicity, the maximum value of $f$ on $\mathbb{L} + \mathbb{M} + \mathbb{K}$ will occur at an $i$ such that $i \xrightarrow{} \infty$. Additionally, from (64) we see that the largest possible value will occur with $i \in \mathcal{Y}_1$ with $f_i = 1 - \delta_0$. Finally, the path defining $\delta_0$ in (62) either has length $\ell(S) + 2$ and connects 0 to $\infty$ or has length $\ell(S)/2 + 1$ and connects some $i \in \mathbb{M} + [L]$ to either 0 or $\infty$ (by symmetry both paths exist). In either case $\delta_0 = 2/(\ell(S) + 2)$, and thus

$$\max_{i \in [L+2M]} f_i = 1 - \frac{2}{\ell(S) + 2} = \frac{\ell(S)}{\ell(S) + 2},$$

as desired.

Finally, the relationship $f_i = -f_i$ follows by noting that $-f$ satisfies the min-max averaging property on the graph formed by switching all the direction of the edges and the boundary conditions. More precisely, let $\tilde{\mathcal{X}}$ be the graph with vertex set $\mathbb{L} + \mathbb{M} + \mathbb{K}$ with edge set $\tilde{E}_\mathcal{X}$ given $(x,y) \in \tilde{E}_\mathcal{X}$ if $(y,x) \in E_\mathcal{X}$. We write $\tilde{x} \xrightarrow{} \tilde{y}$ if $(x,y) \in E_\mathcal{X}$. As boundary condition we choose $\tilde{\mathcal{Y}}_0 = \{0, \infty\} \cup (\mathbb{M} + [L])$ and the function $\tilde{f}(i) = \tilde{f}_i$ with $\tilde{f}_0 = 1$, $\tilde{f}_\infty = -1$, as well as $\tilde{f}_i = 0$ for all $i \in \mathbb{M} + [L]$. We then note if a function $f$ satisfies the min-max averaging property for $\xrightarrow{}$ then $-f$ satisfies the min-max averaging property for $\xrightarrow{}$, so $\tilde{f}_i = -f_i$. On the other hand, as $i \xrightarrow{} j$ implies $\tilde{j} \xrightarrow{} \tilde{i}$, the graph $\tilde{\mathcal{X}}$ is exactly the graph formed by switching the labels $i$ with $\tilde{i}$. Then by uniqueness we have $\tilde{f}_i = f_i$, and we conclude $f_i = -f_i$. 

**A. Example**

We consider a variance profile, $S \in \mathbb{R}^{10 \times 10}$, with the zero pattern below, which can be brought into normal form with the permutation matrix, $P$. 
$S = \begin{pmatrix}
0 & 0 & 0 & \star & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \star \\
0 & \star & 0 & 0 & 0 & \star & 0 & 0 \\
\star & 0 & \star & \star & \star & 0 & 0 & 0 \\
\star & 0 & \star & 0 & 0 & 0 & 0 & \star \\
0 & 0 & \star & \star & 0 & 0 & 0 & \star \\
0 & 0 & 0 & 0 & 0 & \star & 0 & 0 \\
0 & \star & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & 0 & 0 & 0 & 0 & \star & \star & \star & 0 & 0
\end{pmatrix}$, $P = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$.

Here, each $\star$ represents a non-zero entry of $S$.

This leads to the following normal form and associated 0-1 mask $T \in \mathbb{R}^{7 \times 7}$, as defined in Definition 2.3:

$P^tSP = \begin{pmatrix}
0 & 0 & \star & 0 & 0 & \star & 0 \\
\star & 0 & \star & 0 & 0 & \star & 0 \\
0 & \star & \star & \star & 0 & 0 & 0 \\
\star & 0 & \star & 0 & 0 & \star & 0 \\
0 & 0 & \star & \star & 0 & 0 & 0 \\
0 & 0 & \star & 0 & 0 & \star & 0 \\
0 & \star & 0 & 0 & 0 & 0 & 0 \\
\star & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$, $T = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}$.

Following the indicated downwards staircase, the red entries 1 in $T$ mark the longest path with respect to the relation $\blacktriangledown$ induced by $T$ on its index set $[7]$ through Definition 2.5. More generally, we see that for $i \blacktriangledown j$ we find a down-left path from the $(i, b_i)$ entry to the $(j, b_j)$ of $T$ through its entry $(i, b_j)$. The existence of such a down-left path in the matrix can be more transparently illustrated by the following directed graph, in which an arrow from index $i$ to index $j$ indicates $i \blacktriangledown j$. This is the interpretation used in Section 5 in a more abstract setting.
In particular, the longest path is $1 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 7$ and, thus, the singularity degree is $\sigma = \frac{2}{3}$ by Theorem 2.8 because $\ell(S) = 4$ according to Definition 2.6. In fact, any down-left path in the matrix corresponds to a path in the directed graph. Figure 1 shows the eigenvalues of a random matrix with variance profile given by $S$, with the the block sizes, $n=200$ and the non-zero entries of the random matrix having variance $1/n=1/200$. The blue curve represents the self-consistent density of states, generated by solving the Dyson equation (1) associated to $S$ at $\text{Im}z=0.01$.

Figure 1: Histogram of eigenvalues of $H$ and solution to the MDE
B  Numerics

In this section, we present numerics on the least singular value to support the conjecture that it is given by the $N^{-1}$-quantile of the self-consistent density of states.

![Log-log plots with the size $n$ of blocks of the random matrices along the horizontal axis and the average of the least singular value over 200 trials on the vertical axis](image)

Figure 2: Log-log plots with the size $n$ of blocks of the random matrices along the horizontal axis and the average of the least singular value over 200 trials on the vertical axis.

Figure 2 shows the log-log plot of the average, over 200 simulations, of the least singular value of block matrices with complex Gaussian entries and variance profile $S_{ij}=1$ if $i+j \leq K+1$ and 0 otherwise. Each block is an $n \times n$ matrix. In Figure 2a $K=2$ and in Figure 2b $K=3$. The slope of the best fit line in Figure 2a is $-1.51 \approx -3/2$ and the slope of the best fit line in Figure 2b is $-1.96 \approx -2$, as conjectured.

C  Non-negative matrices

Here we collect a few facts and definitions concerning matrices with nonnegative entries that we use in this work. We refer to [9] for details. At the end of the section we prove Lemma 2.2 and Lemma 2.7.
Definition C.1. Let $R = (r_{ij})_{i,j=1}^K$ be a matrix with non-negative entries. For any permutation $\pi$ of $[K]$ we call $(r_{\pi(\pi^{-1}(i))})_{i=1}^K$ a diagonal of $R$. The diagonal with $\pi = \text{id}$ is called main diagonal. The matrix $P = (\delta_{\pi(j)})_{i,j=1}^K$ is called a permutation matrix. The 0-1 matrix $Z_R := (1(r_{ij} > 0))_{i,j=1}^K$ is called the zero pattern of $R$.

1. $R$ is said to have support if it has a positive diagonal. Equivalently, $R$ has support if there is a positive constant $c > 0$ such that $R \geq cP$ holds entry wise for some permutation matrix $P$.

2. $R$ is said to have total support if every positive entry of $R$ lies on some positive diagonal, i.e. if its zero pattern coincides with that of a sum of permutation matrices.

3. $R$ is said to be fully indecomposable (FID) if for any index sets $I, J \subset [K]$ with $|I| + |J| \geq K$ the submatrix $(r_{ij})_{i \in I, j \in J}$ is not a zero-matrix.

The following facts about FID matrices are well known in the literature.

Lemma C.2. Let $R \in \mathbb{R}^{K \times K}$ be a matrix with non-negative entries.
1. If $R$ is FID and $P$ a permutation matrix, then $RP$ and $PR$ are FID.
2. A matrix $R$ with non-negative entries is FID if and only if there exist a permutation matrix $P$ such that $RP$ is irreducible and has positive main diagonal.
3. If $R$ is FID, then there is an integer $k \in \mathbb{N}$ such that $R^k$ has strictly positive entries, i.e. $R$ is primitive.

Definition C.3 (FID-skeleton). Let $R \in \mathbb{R}^{K \times K}$ be a matrix with non-negative entries. Set $I := \{(i,j): r_{ij} \neq 0 \text{ does not lie on a positive diagonal of } R\}$, and $R_{\text{FID}} := (r_{ij} \mathbb{1}(i,j) \notin I))_{i,j=1}^K$. We call $R_{\text{FID}}$ the FID-skeleton of $R$.

The following lemma is the first step in the construction of the normal form in Lemma 2.2.

Lemma C.4 (Normal form for FID-skeleton). Let $R \in \mathbb{R}^{K \times K}$ be a symmetric matrix with non-negative entries that has support. Then $R_{\text{FID}}$ has total support and there
is a permutation matrix $P$ such that

$$PR_{\text{FID}}P^t = \begin{pmatrix}
  \vdots & R_1 & \vdots \\
  R_1 & \ddots & \vdots \\
  \vdots & \ddots & \ddots \\
  \end{pmatrix}$$

Here all $R_i \in \mathbb{R}^{k_i \times k_i}$ and $\tilde{R}_i \in \mathbb{R}^{k_i \times k_i}$ are FID, where $\sum_{i=1}^L k_i + 2 \sum_{j=1}^M \tilde{k}_j = K$. All other entries in (73) are zero. The right hand side of (73) is unique up to all permutations of the matrices $R_1, \ldots, R_L$, simultaneous permutations of the matrices $\tilde{R}_1, \ldots, \tilde{R}_M$ and their transposes, exchanging $\tilde{R}_i$ and $\tilde{R}_i^t$, as well as reindexing the matrix $R_i$, i.e. replacing it by $QR_iQ^t$ with some permutation matrix $Q$, and independently reindexing the rows and columns of $\tilde{R}_i$, i.e. replacing it by $Q_1R_iQ_2$ with permutation matrices $Q_1, Q_2$.

**Proof.** First we show that $R_{\text{FID}}$ has total support. Certainly $R_{\text{FID}}$ has support, because $R$ has a positive diagonal and all elements on that diagonal do not belong to $I$. Let $(i,j) \notin I$ with $r_{ij} > 0$. Then $(i,j)$ lies on a positive diagonal $(r_{i\pi(i)})$ of $R$, but so does every other entry on this diagonal. Thus, $(i,\pi(i)) \notin I$ for all $i$ and therefore $R_{\text{FID}}$ has the same positive diagonal on which $(i,j)$ lies.

Now we split $R_i$ into a direct sum of irreducible components, i.e. we permute its indices through a permutation matrix $P_1$ to transform it into a block diagonal matrix $P_1R_{\text{FID}}P_1^t=\oplus_i \tilde{R}_i$ with symmetric irreducible matrices $\tilde{R}_i$ as the diagonal blocks. If $\tilde{R}_i$ is FID, then we set $R_i := \tilde{R}_i$. Without loss of generality we assume that the first $L$ matrices are of this type. If $\tilde{R}_i$ is not FID, then it still must have total support because $R_{\text{FID}}$ has total support. Thus, by [1, Lemma A.6] it has the form

$$\tilde{R}_{L+i} = Q_i \begin{pmatrix} 0 & \tilde{R}_i \\ \tilde{R}_i^t & 0 \end{pmatrix} Q_i^t$$

with some permutation matrix $Q_i$. We set $P_2 := \oplus_{i=1}^L \oplus_{i=1}^M Q_i$, where $M$ is the number of matrices $\tilde{R}_i$ that are not FID. We conclude

$$P_2P_1R_{\text{FID}}P_1^tP_2^t = \oplus_{i=1}^L \oplus_{i=1}^M \begin{pmatrix} 0 & \tilde{R}_i \\ \tilde{R}_i^t & 0 \end{pmatrix}$$

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which is easily brought into the form (73) by permuting the blocks containing $R_k, R_t, \tilde{R}_b^t$. We are left with showing that $\tilde{R}_t$ is FID. Indeed, $\tilde{R}_t$ is irreducible since $\tilde{R}_{L+t}$ is irreducible. Now we choose permutation matrices $Q_1, Q_2$ so that $Q_1 \tilde{R}_t Q_2^T$ is a direct sum of FID matrices. For the existence of such permutations see e.g. [8, Theorem 4.2.8]. If $Q_1 \tilde{R}_t Q_2^T$ is the direct sum of more than one FID matrix, then $\tilde{R}_{L+t}$ is reducible by permutation $Q_1 \oplus Q_2$ of its indices. Thus $Q_1 \tilde{R}_t Q_2^T$ is already FID and so is $\tilde{R}_t$.

The statement about uniqueness from the lemma is clear from the form of (73). \hfill \square

**Proof of Lemma 2.2.** By Lemma C.4 we assume without loss of generality that the FID-skeleton $R_{\text{FID}}$ of $R$ as defined in Lemma C.4 is given by the right hand side of (73) with FID matrices $R_k$ and $\tilde{R}_t$. In particular, $R_{\text{FID}}$ induces a $(L+2M) \times (L+2M)$-block structure on the entries of $R$, whose blocks have dimensions $(\tilde{k}_1, \ldots, \tilde{k}_M, k_1, \ldots, k_L, \tilde{K}_M, \ldots, \tilde{k}_1)$.

**Step 1:** In this step we show that it suffices to consider the case of a zero-one matrix $R$ where $R = 1, \tilde{R}_t = 1$, i.e. the dimensions of the blocks in the $(L+2M) \times (L+2M)$-block structure induced by (73) are $k_i = 1$.

We associate to the block structure a zero-one matrix $T = T' \in \{0, 1\}^{(L+2M) \times (L+2M)}$ by setting $t_{ij} := 0$ if and only if the corresponding $(i, j)$-block in $R$ is a zero matrix. In particular, $t_{ii} = 1$ for all $i$, where $\tilde{i} := i$ for $i \in [M]$ and $\tilde{i} := 2M + L + 1 - i$ for $i \in [M] \cup (M + 1 + [M])$ is the complement index of $i$. We will show that $T$ has FID-skeleton

$$T_{\text{FID}} = X, \quad X := \begin{pmatrix} 0 & 0 & A_M \\ 0 & \mathbb{1}_L & 0 \\ A_M & 0 & 0 \end{pmatrix}, \quad (74)$$

where $\mathbb{1}_L \in \mathbb{R}^{L \times L}$ is the identity matrix and $A_M = (\delta_{i, M+1-j})_{i,j=1}^M$ is the permutation matrix inverting the order of indices in $[M]$. Thus, the FID-skeleton of $T$ exactly corresponds to the FID-skeleton of $R$ on the right hand side of (73). Since taking the skeleton commutes with any permutation of rows or columns, i.e. $(R P)_{\text{FID}} = R_{\text{FID}} P$ and $(P R)_{\text{FID}} = P R_{\text{FID}}$ for any permutation matrix $P$, any permutation $\pi$ of the indices $[2M + L]$ that brings $T$ into normal form also brings $R$ into normal form when acting on the blocks. Therefore, it suffices to consider the case $k_i = \tilde{k}_i = 1$.

To prove (74) we show the equivalent statement that the only positive diagonal of $TX$ is its main diagonal, where $X$ is the permutation matrix from (74). Let $\hat{X} \in \mathbb{R}^{K \times K}$ be the permutation matrix induced by the permutation $X$ acting on the blocks of $R$. Then $R \hat{X}$ has the FID matrices $\hat{R}_1, \ldots, \hat{R}_M, R_1, \ldots, R_L, \hat{R}^e_L, \ldots, \hat{R}^e_1$ along its block diagonal because $R_{\text{FID}}$ equals the right hand side of (73). There
are permutation matrices $\tilde{P}_i, \tilde{Q}_i, P_i$ such that $\tilde{R}_i\tilde{P}_i, \tilde{R}_i\tilde{P}_i, R_iP_i$ all have positive main diagonal. We set $\tilde{R} := RX\tilde{P}$ with the block index preserving permutation $\tilde{P} := \oplus_{i=1}^{M} \tilde{Q}_i \oplus \oplus_{i=1}^{L} P_i \oplus \oplus_{i=1}^{M} \tilde{P}_{M+1-i}$.

Then $\tilde{R}$ has positive main diagonal and the non-zero entries of $TX$ still correspond to the non-zero blocks of $\tilde{R}$. Since $\tilde{R}_\text{FID} = R_{\text{FID}} X \tilde{P}$ is block diagonal, $\tilde{R}$ does not have a positive diagonal that contains an entry of an off-diagonal block. We show now that the same is true for its positive powers.

**Claim:** For any $k \in \mathbb{N}$ the FID skeleton of $\tilde{R}^k$ is block diagonal.

Let $I_1, \ldots, I_{2M+L} \subset \llbracket K \rrbracket$ be the indices within the blocks of $\tilde{R}$. In particular, $(\llbracket I_1 \rrbracket, \ldots, \llbracket I_{2M+L} \rrbracket) = (k_1, \ldots, k_M, k_{M+1}, \ldots, k_{2M}, k_{2M+1})$ and $\tilde{R}_{ij} = (\tilde{r}_{ij})_{\alpha \in I_i, \beta \in I_j}$ the $(i, j)$-block in $\tilde{R}$. We prove the claim by contradiction. Therefore, suppose that $\tilde{R}^k$ contains a positive diagonal associated to a permutation $\pi$ that contains an entry of the off-diagonal block $(i, j)$ with $i \neq j$, i.e. that $(\tilde{R}^k)_{\alpha\pi(\alpha)}>0$ for all $\alpha$ with $\alpha_0 \in I_i$ and $\pi(\alpha_0) \in I_j$. We restrict our attention to the orbit of $\pi$ containing $\alpha_0$ and see that

\[
(\tilde{R}^k)_{\alpha_0\pi_1}(\tilde{R}^k)_{\alpha_1\pi_2} \cdots (\tilde{R}^k)_{\alpha_{n-1}\pi_0} > 0,
\]

where $\alpha_i := \pi^i(\alpha_0)$ and $\pi^n(\alpha_0) = \alpha_0$. We say that $\gamma = (\beta_0, \ldots, \beta_l)$ is a path if $\tilde{r}_{\alpha_0\alpha_{p+1}}>0$ for all $p = 0, \ldots, l-1$ and write $\beta_0 \xrightarrow{\gamma} \beta_l$ to emphasise the start and end point of the path. Then (75) means that there is a closed path $\gamma$ starting from $\alpha_0 \in I_i$ and running through $\alpha_1 \in I_j$. We write this path as a composition of the two paths $\alpha_0 \xrightarrow{\gamma_1} \alpha_1 \xrightarrow{\gamma_2} \alpha_0$. By removing all loops from the two paths $\gamma_1$ and $\gamma_2$ we end up with a path $\gamma = \alpha_0 \xrightarrow{\gamma_2} \alpha_1 \xrightarrow{\gamma_2} \alpha_0$ such that $\gamma_1$ and $\gamma_2$ both do not contain an index twice.

Now let $\gamma_>$ be the the shortest closed subpath of $\gamma$ such that

\[
\alpha_0 \xrightarrow{\gamma} \alpha_0 = \alpha_0 \xrightarrow{\gamma_2} \alpha_s \xrightarrow{\gamma_2} \alpha_0,
\]

where $\alpha_s \in I_i$ and $\gamma_2(\alpha_0) \nsubseteq \gamma(\alpha_0) \in I_i$. Furthermore, let $\gamma_<$ be the longest closed subpath of $\gamma_>$ with the property

\[
\alpha_s \xrightarrow{\gamma} \alpha_s = \alpha_s \xrightarrow{\gamma_2} \alpha_# \xrightarrow{\gamma_2} \alpha_# \xrightarrow{\gamma_2} \alpha_s,
\]

and $\alpha_# \nsubseteq I_i$. Then the closed path $\gamma := \alpha_s \xrightarrow{\gamma_2} \alpha_# \xrightarrow{\gamma_2} \alpha_s$ starts from $\alpha_s \in I_i$, runs through some $\alpha_# \nsubseteq I_i$ and does not contain a loop. Writing $\gamma = (\beta_0, \ldots, \beta_l)$ we set $\pi$ the corresponding cyclic permutation $\pi := (\beta_0, \ldots, \beta_l)$. Since $\tilde{R}$ has positive main diagonal and by the definition of paths, $(\tilde{r}_{\alpha\pi(\alpha)})_{\alpha=1}^{d}$ is a positive diagonal of $\tilde{R}$. This diagonal contains the entry $\tilde{r}_{\alpha_#\pi(\alpha_#)}$ with $\alpha_# \nsubseteq I_i$ and $\pi(\alpha_#) \in I_i$. This contradicts that fact that $\tilde{R}_\text{FID}$ is block diagonal and finishes the proof of the claim.
Now we return to completing Step 1 of proof and show that there is a power $k \in \mathbb{N}$ such that all blocks $(\hat{R}^k)_{ij}$ of $\hat{R}^k$ with block indices $(i,j)$ for which $(TX)_{ij} = 1$ have strictly positive entries. Indeed, since the diagonal blocks of $\hat{R}$ are FID and, thus, primitive, there is a power $l \in \mathbb{N}$ such that $(\hat{R}^l)_{ii}$ has strictly positive entries. Then we find

$$(\hat{R}^{2l+1})_{ij} \geq (\hat{R}^l)_{ii}(\hat{R}^l)_{jj} > 0,$$

where the inequalities are meant entrywise and the positivity holds because $(\hat{R})_{ij}$ is not a zero-matrix by the assumption $(TX)_{ij} = 1$ and the matrices $(\hat{R}^l)_{ii}, (\hat{R}^l)_{jj}$ have strictly positive entries.

Altogether we have now seen that there is $k \in \mathbb{N}$ such that $(TX)_{ij} = 1$ implies $(\hat{R}^k)_{ij}$ has strictly positive entries and $(\hat{R}^k)_{\text{FID}}$ is block diagonal. To get a contradiction, suppose now that $TX$ has a positive diagonal associated to a permutation $\pi \neq \text{id}$ of $[2M+L]$ that contains a nontrivial cycle $(i,\pi(i),\ldots,\pi^{n-1}(i))$. Then we choose arbitrary indices $\alpha_i \in I_{\pi(i)}$, where $I_j$ again denotes the indices within block $j$. We conclude $(\hat{R}^k)_{\alpha_i\alpha_{i+1}} > 0$ and since $\hat{R}^k$ has positive main diagonal the cyclic permutation $\pi := (\alpha_0,\alpha_1,\ldots,\alpha_{n-1})$ generates a positive diagonal of $\hat{R}^k$ that contains entries from off-diagonal blocks, in contradiction to $\hat{R}^k$ having block diagonal FID skeleton. This finishes the proof of (74) and, thus, of Step 1.

Step 2: By Step 1 it suffices to consider the case when $R_{\text{FID}} = X$, where $X$ is defined in (74). In this step we prove the following claim:

Claim: $R$ has a row with index $i \in \left\lbrack M \right\rbrack \cup (M+L+\left\lbrack M \right\rbrack)$ that contains exactly one non-zero entry.

We set $\hat{R} := RX$ with $X$ as in (74). Then the claim is equivalent to finding a row of $\hat{R} := RX$ with exactly one non-zero entry. Furthermore, we have $\hat{R}_{\text{FID}} = 1_{2M+L}$. We pick any initial index $i_0$ and inductively construct a sequence as follows. If $i_0,\ldots,i_k$ have been constructed, then we choose $i_{k+1}$ such that $\tilde{r}_{i_ki_{k+1}} > 0$ and $i_k \neq i_{k+1}$. This sequence procedure either terminates at some point, in which case we found a row $i_{k+1}$ of $\hat{R}$ whose only non-zero element is $\tilde{r}_{i_{k+1}i_{k+1}}$ or it creates a cycle, i.e. $i_{k+1} = i_0$. The latter is impossible because the cyclic permutation $\pi := (i_0,\ldots,i_k)$ would induce the positive diagonal $(\tilde{r}_{j\pi(j)})$ of $\hat{R}$, contradicting that $\hat{R}_{\text{FID}} = 1_{2M+L}$. Finally, we observe that the above procedure must terminate at an $i \in \left\lbrack M \right\rbrack \cup (M+L+\left\lbrack M \right\rbrack)$ because for $j \in M+\left\lbrack L \right\rbrack$ and $\tilde{r}_{ij} > 0$ we have, by the symmetry of $\hat{R}$ and definition of $\hat{R}$, that $\tilde{r}_{ij} = r_{ij} = r_{ji} \neq \tilde{r}_{ji}$. In other words the given algorithm will cannot terminate at such a $j$.

Step 3: In this step we inductively construct a permutation matrix $P$ for $R$ such
that (3) holds. By Step 1 we still assume $R_{\text{FID}} = X_{L,M}$ with $X_{L,M} = X$ as in (74). We start by choosing a row $i$ such that $r_{ii}$ is its only non-zero entry, where we recall the definition of the complement index $\hat{i} := i$ for $i \in M + [L]$ and $\hat{i} := 2M + L + 1 - i$ for $i \in [M] \cup (M + L + [M])$. This is possible by Step 2. If $i = \hat{i} \in M + [L]$, then we define a matrix $\tilde{R}$ by removing the $i$-th row and column from $R$. Since $r_{ii}$ was the only non-zero element we removed $\tilde{R}_{\text{FID}} = X_{L-1,M}$, i.e. we reduced the dimension of the problem by 1. If $i \notin M + [L]$, then we choose the permutation matrix $P$ associated to 

\[ \pi := \begin{cases} 
\text{id} & \text{if } i = 2M + L, \\
(i, \hat{i}) & \text{if } i = 1, \\
(i, 2M + L)(\hat{i}, 1) & \text{otherwise}, 
\end{cases} \]

which commutes $i$ to the last index and $\hat{i}$ to the first. The last row and column of $PRP^t$ have $r_{\hat{i}\hat{i}} = r_{ii}$ as their only non-zero element. We define $\tilde{R}$ by removing the last row and column from $PRP^t$ and find $\tilde{R}_{\text{FID}} = X_{L,M-1}$. Thus, we reduced the dimension by 2. We repeat this procedure, reducing the dimension of the problem in each step, until it becomes trivial. This finishes Step 3 and, thus, the proof of the lemma.

PROOF OF LEMMA 2.7. The FID skeleton of the normal form on the right hand side of (3) coincides with the right hand side of (73). In particular, the uniqueness statement in Lemma C.4 implies that the normal form in Lemma 2.2 is unique up to permutations of the indices within the blocks of dimensions $(k_1, ..., k_{2M+L})$ and certain permutations of the blocks themselves. The definition of the matrix $T$ in Definition 2.3 is independent of the former and will be effected by the latter only through permutation of its indices, i.e. depending on the normal form $T = (t_{ij})^L_{i,j=1}$ may change to $(t_{\pi(i)\pi(j)})^L_{i,j=1}$ for some permutation $\pi$. Thus, also the relation $\ell(R)$ from Definition 2.5 is unique up to a potential permutation of the indices, which leaves the length $\ell(R)$ of the longest path unaffected.

We now consider the case when $S$ does not have support and begin by recalling the Frobenius-König theorem, a proof of which can be found e.g. in [17].

THEOREM C.5 (Frobenius-König theorem). A matrix $S \in \mathbb{R}^{n \times n}$ with non-negative entries does not have support if and only if $S$ contains an $r \times s$-submatrix of zeros with $r + s = n + 1$.

We note this theorem is often stated with the equivalent first condition, that the zero pattern of the matrix has permanent equal to 0.

PROOF OF LEMMA 4.5. Since $S$ does not have support, the Frobenius-König theorem implies that there exists an $|I| \times |J|$ submatrix of zeros, with $|I| + |J| > K$.
By the symmetry of $S$ there also exists a $|J| \times |I|$ zero submatrix, so without loss of generality we assume $|I| \leq |J|$.

We then consider all zero submatrices with maximal length plus height, i.e. maximizing $|I| + |J|$, and choose $I,J$ corresponding to the submatrix with the largest height $|J|$. With this choice of $I$ and $J$, there is no set of $k$ rows of $S_{13}$ such that all the non-zero entries in these rows lie in $k$ or fewer columns, otherwise we could choose a submatrix of zeros with a larger height, as explained below the statement of Lemma 4.5.

Then the submatrix $S_{22}$ has no submatrix of zeros whose height plus width is greater than $|J| - |I|$. Otherwise a larger zero submatrix would have been chosen in the first step. Thus, again by the Frobenius-König theorem, the matrix $S_{22}$ has support.

\begin{proof}

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