Weak solutions of the three-dimensional hypoviscous elastodynamics with finite kinetic energy

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Abstract

We construct weak solutions to the 3D hypoviscous incompressible elastodynamics with finite kinetic energy which was unknown in literatures. Our result holds for fractional hypoviscosity $(-\Delta)^\theta$, where $0 \leq \theta < 1$. The proof consists of a convex integration scheme with new building blocks of 2D intermittency and suitable temporal correctors, which are motivated by the inherent geometric structure of the viscoelastic equations.

Keywords: Viscoelastic Fluid; Convex Integration; Weak Solution

1 Introduction

We consider the following Oldroyd-B system describing incompressible viscoelastic fluids posed on the periodic box $\mathbb{T}^3$:

$$
\begin{align*}
\partial_t v + v \cdot \nabla v + (-\Delta)^\theta v + \nabla p &= \text{div}(FF^T), \\
\partial_t F + v \cdot \nabla F &= \nabla v F, \\
\text{div } v &= 0, \\
\text{div } F^T &= 0.
\end{align*}
$$

(1.1)

Here $\theta \in [0, 1)$ is a given constant, $v(t, x)$ represents the velocity field of fluids, $p(t, x)$ the scalar pressure, and $F(t, x)$ the deformation gradient tensor. We focus on solutions with the following normalization:

$$
\int_{\mathbb{T}^3} v dx = 0, \quad \int_{\mathbb{T}^3} (F - \text{Id}) dx = 0,
$$

where $\text{Id}$ is the identity matrix. Throughout the paper we will adopt the notations of

$$(\nabla v)_{ij} = \frac{\partial v_i}{\partial x_j}, \quad (\nabla \cdot F)_i = \partial_j F_{ij}. $$

Here and in what follows, we use the summation convention over repeated indices.

Our goal is to construct weak solutions of (1.1). We first introduce the notion of weak solutions that we consider in this paper as follows:

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Definition 1.1. Let $T > 0$, we say $(v, F) \in C([0, T]; L^2(T^3))$ is a weak solution of system (1.1) if $v$ and $F^T$ are weakly divergence free, and they satisfy (1.1) in the sense of distributions, that’s to say,
\begin{align*}
&\int_0^T \int_{T^3} \partial_t \psi \cdot v + \nabla \psi : (v \otimes v - FF^T) - (-\Delta)^\theta \psi \cdot v dx dt = 0, \\
&\int_0^T \int_{T^3} \partial_t \psi \cdot F^k + \nabla \psi : (F^k \otimes v - v \otimes F^k) dx dt = 0, \quad k = 1, 2, 3,
\end{align*}
hold for any divergence free function $\psi \in C_0^\infty((0, T) \times T^3)$, where $F^k(k = 1, 2, 3)$ is the $k$-th column vector of $F$.

Before proceeding, we mention some famous works in the elastic and viscoelastic fluid equations. In [34], Sideris and Thomases proved the global well-posedness of classical solutions to the 3D incompressible isotropic elastodynamics with small initial displacements. For 2D incompressible isotropic elastodynamics, the almost global existence was proved in [26] for small initial data. By observing an improved null structure for the system in Lagrangian coordinates, Lei proved the global well-posedness of 2D incompressible elastodynamics in [27]. As for viscoelastic fluid, Chemin and Masmoudi [8] proved the existence of local and global small solutions in critical Besov spaces. The global well-posedness near equilibrium was first obtained in [28] for the two-dimensional case. Lei and Zhou [24] obtained similar results through an incompressible limit process working directly on the deformation tensor $F$. More recently, Hu and Lin [20] gave a global weak solution with discontinuous initial data in 2D. Zhu [35] proved the global existence of small solutions without any physical structure in 3D. We also refer to [9, 18, 21, 25, 29] for related results. To our knowledge, there is no result for existence of global weak solutions when the initial data is not small. In this paper, we construct infinitely many weak solutions with finite energy, while the initial data can be arbitrarily large. Theorem 1.2 also demonstrates the nonuniqueness of $C_t L^2_x$ weak solutions to the Cauchy problem of (1.1), where the initial data is satisfied in $L^2$ space.

The main result is stated as follows:

Theorem 1.1. Suppose $\theta \in [0, 1)$ is a given constant. For any $T > 0$, there exist non-trivial weak solutions $(v, F) \in C([0, T]; H^3(T^3))$ of (1.1) in the sense of Definition 1.1, where the constant $\beta > 0$ depends only on $\theta$.

More precisely, we have the following result of h-principle type:

Theorem 1.2. For any given $\theta \in [0, 1)$ and $T > 0$, suppose $u$ is a vector field and $G$ is a tensor which are smooth on $[0, T] \times T^3$ and satisfy
\begin{align*}
\text{div } u &= 0, \quad \text{div } G^T = 0, \\
\int_{T^3} u dx &= 0, \quad \int_{T^3} (G - \text{Id}) dx = 0.
\end{align*}
Then, for any fixed $\varepsilon > 0$, there exists a weak solution $(v, F) \in C([0, T]; H^3(T^3))$ of (1.1) in the sense of Definition 1.1, satisfying
\begin{align*}
&\|v - u\|_{L^\infty_t L^1_x} + \|F - G\|_{L^\infty_t L^1_x} \leq \varepsilon, \quad (1.2) \\
&\text{supp}_t(v, F) \subseteq O_\varepsilon(\text{supp}_t(u, G)), \quad (1.3)
\end{align*}
where the constant $\beta > 0$ depends only on $\theta$, and we denote $\text{supp}_t(v,F) = \text{supp}_t v \cup \text{supp}_t F$ and

$$O_\varepsilon(S) := \{ t \in [0,T] : \exists s \in S, \text{ such that } |s-t| \leq \varepsilon \}.$$ 

**Remark 1.3.** By Theorem 1.2, system (1.1) admits nontrivial weak solutions with compact temporal supports. This implies the non-uniqueness of weak solutions to (1.1) in the sense of Definition 1.1.

When $\theta = 0$, the system (1.1) is referred as incompressible elastodynamics, which has similar structure with the following 3D ideal magnetohydrodynamic (MHD) equations:

$$\begin{align*}
\partial_t u + \nabla(u \otimes u - B \otimes B) + \nabla p &= 0, \\
\partial_t B + \nabla(B \otimes u - u \otimes B) &= 0, \\
\text{div } u &= \text{div } B = 0,
\end{align*}$$

where $u$ is the velocity field and $B$ the magnetic field. Actually, let $F^k (k = 1, 2, 3)$ denote the $k$-th column vector of $F$, then we can reformulate (1.1) as

$$\begin{align*}
\partial_t v + \nabla(v \otimes v - F^k \otimes F^k) + (-\Delta)^\theta v + \nabla p &= 0, \\
\partial_t F^i + \nabla(F^i \otimes v - v \otimes F^i) &= 0, \quad i = 1, 2, 3, \\
\text{div } v &= \text{div } F^i = 0. 
\end{align*}$$

(1.4)

We find that even though the deformation tensor is a matrix, each column vector $F^i$ obeys similar equations with the magnetic field $B$ in MHD equations. In [1], Beekie et al. applied the convex integration scheme to construct $C^1 L^2$ weak solutions to the ideal MHD equations, using intermittent shear flows as building blocks. The shear flows have 1D spatial intermittency and thus only permits a viscosity term $(-\Delta)^\theta u$ with $\theta < \frac{3}{4}$. In this paper, our scheme works for $\theta < 1$ by constructing new building blocks with 2D spatial intermittency and introducing suitable temporal correctors. Specially, our method can also be applied to MHD equations. Recently, after our work, Li et al. [30] extended this to the full Lions exponent $\theta < \frac{5}{4}$ for viscous and resistive MHD equations by introducing an extra temporal intermittency.

The proof of Theorem 1.2 builds on the scheme of convex integration, which can be traced back to the work of Nash [33]. In [33], Nash used an iteration scheme to construct $C^1$ isometric embeddings. Then, De Lellis and Székelyhidi Jr. applied Nash’s idea to fluid dynamics and devised in [14,16] a “convex integration” scheme leading to continuous dissipative solutions of the Euler equation, which was a significant step to solve the flexible part of Onsager’s conjecture. Subsequently, after a series of advancements [2,3,17,22], the Onsager’s conjecture was finally resolved by Isett in [23], using a key ingredient by Daneri and Székelyhidi Jr. [13]. Then, Buckmaster et al. [5] proved the same results for dissipative solutions. For the rigid part of Onsager’s conjecture, one can refer to [11,19]. The scheme has also been extended and adapted to various problems in mathematical physics, see [1,7,10,12,15] and references therein.

Furthermore, the convex integration techniques have fundamental analogies with the theory of turbulence, and features of turbulent flows (such as intermittency) have inspired researchers to develop and extend the convex integration constructions. Recently, the method was extended to the Navier-Stokes equations in [6], by constructing 3D intermittent Beltrami flows to treat the dissipative term. And more intermittent building blocks were adapted in the convex integration scheme, such as intermittent jets [4] and intermittent Mikado flows [32].
In this paper, we construct a new building block with 2D intermittency to adapt to the inherent geometric structure of viscoelastic equations \([1.4]\). More precisely, we introduce the intermittent velocity and deformation flows of the following forms:

\[
W_\xi = \phi_\xi \varphi_{\xi_1} \xi, \quad E_\xi = \phi_\xi \varphi_{\xi_1} \xi_2,
\]

where \((\xi, \xi_1, \xi_2) \subset \mathbb{Q}^3\) is an orthonormal basis, and \(\phi_\xi, \varphi_{\xi_1}\) are spatial concentration functions with one oscillation direction \(\xi, \xi_1\) respectively, see Section 2.3 for details. As mentioned in [1], the previous building blocks (intermittent Beltrami flows, intermittent jets, respectively viscous eddies), are not applicable to system \((1.4)\). The geometry of the nonlinear terms of system \((1.4)\) requires the building blocks’ direction of oscillation to be orthogonal to two direction vectors, only permitting the usage of 1D intermittency. To overcome this difficulty, we introduce two new types of temporal correctors to cancel the high spatial oscillations of nonlinear terms.

The remainder of this paper is organized as follows. In Section 2 we introduce the convex integration scheme, present the main iteration scheme and give a proof of the Theorem 1.2. In Section 3 we define the perturbations and show the main estimates. Finally, we estimate the new stresses parts by parts and complete the iteration in Section 4.

Notations: Throughout the paper we use the following notations:

\[
\|f\|_{C^{N, x,t}} = \sum_{0 \leq \kappa \leq N} \|\partial_{\kappa} f\|_{L^\infty},
\]

and

\[
\|f\|_{W^{s, p}} = \|f\|_{L^\infty, W^{s, p}} = \sum_{0 \leq \kappa \leq s} \|D^\kappa f\|_{L^\infty, L^p},
\]

where \(\kappa = (\kappa_1, \kappa_2, \kappa_3)\) is the multi-index and \(D^\kappa = \partial_{x_1}^{\kappa_1} \partial_{x_2}^{\kappa_2} \partial_{x_3}^{\kappa_3}\).

For \(f \in L^2_2(\mathbb{T}^3)\), we define the average integral operator

\[
\int_{\mathbb{T}^3} f dx := \frac{1}{(2\pi)^3} \int f dx,
\]

and the projection operator

\[
P_{\neq 0} f = f - \int_{\mathbb{T}^3} f,
\]

which projects a function onto its nonzero frequencies. We will write \(A \lesssim B\) to denote that there exists a constant \(C > 0\) such that \(A \leq CB\).

2 Convex Integration Scheme

For every integer \(q > 0\), we will construct a solution \((v_q, R^i_q, F^i_q, R^1_q, F^1_q, R^2_q, F^2_q, R^3_q, F^3_q)\) to the approximation system

\[
\begin{align*}
\partial_t v_q + \text{div}(v_q \otimes v_q - F^k_q \otimes F^k_q) + (-\Delta)^{\theta} v_q + \nabla p_q &= \text{div} R^o_q, \\
\partial_t F^i_q + \text{div}(F^i_q \otimes v_q - v_q \otimes F^i_q) &= \text{div} R^i_q, \quad i = 1, 2, 3, \\
\text{div} v_q &= \text{div} F^i_q = 0,
\end{align*}
\]
where the Reynolds stress $R^v_q$ is assumed to be a trace-free symmetric matrix, and $R^i_q$ is assumed to be skew-symmetric. Besides, $\|R^v_q\|_{L^1}$ and $\|R^i_q\|_{L^1}$ go to zero as $q \to \infty$, and $(v_q, F^1_q, F^2_q, F^3_q)$ will converge to a weak solution of (1.4).

For $\theta \in (0,1)$ given in system (1.1), we denote

$$
\theta_* = \begin{cases}
2\theta - 1, & \frac{1}{2} < \theta < 1; \\
0, & 0 \leq \theta \leq \frac{1}{2}.
\end{cases}
$$

Then, we shall fix a parameter $\alpha \in \mathbb{Q}$ satisfying

$$
0 < \alpha \leq \frac{1 - \theta_*}{8} \in \left(0, \min \left\{ \frac{1 - \theta}{4}, \frac{1}{8} \right\} \right).
$$

In order to quantify the convergence of the stresses, we introduce a frequency parameter $\lambda_q$ and an amplitude parameter $\delta_q$ defined as follows:

$$
\lambda_q = a b^\beta, \quad \delta_q = \lambda_q^{3\theta} \lambda_q^{-2\beta},
$$

where a large parameter $b \in \mathbb{N}$ and a small parameter $\beta > 0$ would be fixed such that

$$
800 < ab \in \mathbb{N}, \quad \beta b^2 \leq \frac{1}{16},
$$

and $a \gg 1$ will be chosen as a sufficiently large multiple of a geometric constant $N_A \in \mathbb{N}$ (which is fixed in Remark 2.4).

By induction, we will assume the following estimates on the solution of (2.1) at level $q$:

$$
\sum_{i=1}^3 \|R^i_q\|_{L^1} \leq \delta_{q+1}, \quad \|R^v_q\|_{L^1} \leq \delta_{q+1},
$$

$$
\|R^i_q\|_{C^1_{x,t}} + \|R^v_q\|_{C^1_{x,t}} \leq \lambda_q^{10},
$$

$$
\sum_{i=1}^3 \|F^i_q\|_{C^1_{x,t}} + \|v_q\|_{C^1_{x,t}} \leq \lambda_q^4.
$$

**Proposition 2.1 (Main Iteration).** There exist a universal constant $M$ and a sufficiently large parameter $a = a(\theta, b, \beta)$ such that the following holds: let $(v_q, R^v_q, F^1_q, R^i_q)$ be a solution of (2.1) satisfying the inductive estimates (2.6) - (2.8), then there exist functions $(v_{q+1}, R^v_{q+1}, F^1_{q+1}, R^i_{q+1})$ solving (2.1) and satisfying (2.6) - (2.8) with $q$ replaced by $q + 1$. Furthermore, we have

$$
\|v_{q+1} - v_q\|_{L^2} + \sum_{i=1}^3 \|F^i_{q+1} - F^i_q\|_{L^2} \leq M \delta_{q+1}^4,
$$

$$
\|v_{q+1} - v_q\|_{L^1} + \sum_{i=1}^3 \|F^i_{q+1} - F^i_q\|_{L^1} \leq \delta_{q+2},
$$

$$
\bigcup_{i=1}^3 \text{supp}_t(v_{q+1}, R^v_{q+1}, F^i_{q+1}, R^i_{q+1}) \subseteq \bigcup_{i=1}^3 O_{\delta_{q+2}} \left(\text{supp}_t(v_q, R^v_q, F^i_q, R^i_q)\right).
$$
2.1 Proof of Theorem 1.2

Following [1], we introduce the symmetric inverse divergence operator \( R \) and skew-symmetric inverse divergence operator \( R^F \) by

\[
(Rv)_{kl} = \partial_k \Delta^{-1} v^l + \partial_l \Delta^{-1} v^k - \frac{1}{2} (\delta_{kl} + \partial_k \partial_l \Delta^{-1}) \text{div} \Delta^{-1} v,
\]

\[
(R^F f)_{ij} := \epsilon_{ijk}(-\Delta)^{-1}(\text{curl} f)_k,
\]

where \( i, j, k, l \in \{1, 2, 3\} \), \( \epsilon_{ijk} \) is the Levi-Civita tensor, and functions \( v, f : \mathbb{R}^3 \to \mathbb{R}^3 \) satisfies \( \int_{\mathbb{R}^3} v = 0 \) and \( \text{div} f = 0 \) respectively. By a direct calculation we see that \( \text{div} R(v) = v \) and \( \text{div} R^F(f) = f \). The operator \( R \) returns a symmetric, trace-free matrix, and \( R^F \) returns a skew-symmetric matrix. By standard Calderon-Zygmund and Schauder estimates we have

\[
\|R\|_{L^p \to W^{1,p}} \lesssim 1, \quad \|R\|_{C^0 \to C^0} \lesssim 1, \quad \|R \neq 0 u\|_{L^p} \lesssim \|\nabla \omega^{-1} \neq 0 u\|_{L^p}, \tag{2.12}
\]

for \( 1 < p < \infty \). The above estimates also hold for \( R^F \).

Now we turn to the proof of Theorem 1.2. For \( q = 0 \), take \( v_0 = u, F^0_0 = G^0 \), and define \( p_0, R^0_0, R^0_0 \) as

\[
p_0 = -\frac{1}{3} |v_0|^2 + \frac{1}{3} \sum_{i=1}^3 |F^0_0|^2, \quad R^0_0 = R (\partial_t v_0 + (-\Delta)^{\delta} v_0) + v_0 \otimes v_0 - F^0_0 \otimes F^0_0 + p_0 \text{Id}, \quad R^0_0 = R^F (\partial_t F^0_0) + F^0_0 \otimes v_0 - v_0 \otimes F^0_0.
\]

Then for \( a \) large enough, \( (v_0, R^0_0, F^0_1, R^0_0, F^0_0, R^0_0, F^0_0, R^0_0) \) satisfies system (2.1) and obeys the estimates (2.6)-(2.8). For \( q \geq 1 \), we inductively apply Proposition 2.1 to get a sequence of solutions \( (v_q, R^0_q, F^1_q, R^1_q, F^2_q, R^2_q, F^3_q, R^3_q) \) satisfying the inductive estimates (2.6)-(2.8). Using the bound (2.9) and interpolation, we obtain

\[
\sum_{q \geq 0} \|v_{q+1} - v_q\|_{H^{\beta'}} + \sum_{q \geq 0} \sum_{i=1}^3 \|F^i_{q+1} - F^i_q\|_{H^{\beta'}}
\]

\[
\leq \sum_{q \geq 0} \|v_{q+1} - v_q\|_{L^2} \|v_{q+1} - v_q\|_{H^{\beta'}} + \sum_{q \geq 0} \sum_{i=1}^3 \|F^i_{q+1} - F^i_q\|_{L^2} \|F^i_{q+1} - F^i_q\|_{H^{\beta'}}
\]

\[
\leq \sum_{q \geq 0} \sum_{i=1}^3 \delta_{q+1} \lambda_{q+1}^{\beta'} \lambda_{q+1}^{1-\beta'} = \lambda_1 \sum_{q \geq 0} \lambda_{q+1}^{(1-\beta')2} \sum_{q \geq 0} \lambda_{q+1}^{-\beta} + 4 \beta' \lesssim 1,
\]

for \( \beta' < \frac{\beta}{1+\beta} \). Hence the solution sequence is Cauchy and there exists a limit \( (v, F^1, F^2, F^3) = \lim_{q \to \infty} (v_q, F^1_q, F^2_q, F^3_q) \), which is a weak solution of (1.4) because \( \lim_{q \to \infty} R^0_q = \lim_{q \to \infty} R^0_q = 0 \) in \( C([0, 1]; L^1(\mathbb{T}^3)) \). Finally, the estimate (1.2), (1.3) are direct results of (2.10) and (2.11). This completes the proof of Theorem 1.2.

2.2 Geometric Lemmas

The idea of the construction of perturbations mainly comes from the following two lemmas, with which we can cancel the previous stress by the low frequency of quadratic terms. One may refer to [1] for the proof of the lemmas.
Lemma 2.2. (Representation of symmetric matrices) There exists a constant $\varepsilon_v > 0$ and a finite set $\Lambda_v \subset S^2 \cap \mathbb{Q}^3$ consisting of vectors $\xi$ with associated orthonormal basis $(\xi, \xi_1, \xi_2)$, such that for each symmetric trace-free matrix $R \in B_{\varepsilon_v}(\text{Id})$ we have the identity

$$R = \frac{1}{2} \sum_{\xi \in \Lambda_v} (\gamma_\xi(R))^2 (\xi \otimes \xi),$$

where $\gamma_\xi \in C^\infty(B_{\varepsilon_v}(\text{Id}))$, $\xi \in \Lambda_v$.

Lemma 2.3. (Representation of skew-symmetric matrices) For every $N \in \mathbb{N}$, there exists a constant $\varepsilon_F > 0$ and pairwise disjoint sets $\Lambda_i \subset S^2 \cap \mathbb{Q}^3$, $i = 1, 2, \ldots, N$, consisting of vectors $\xi$ with associated orthonormal basis $(\xi, \xi_1, \xi_2)$, such that for each skew-symmetric matrix $R \in B_{\varepsilon_F}(0)$ we have the identity

$$R = \sum_{\xi \in \Lambda_i} (\gamma_\xi(R))^2 (\xi_2 \otimes \xi - \xi \otimes \xi_2),$$

for each $i = 1, 2, \ldots, N$. Here $\gamma_\xi \in C^\infty(B_{\varepsilon_F}(0))$, $\xi \in \Lambda_i$.

Remark 2.4. We choose $\Lambda_v$ and $\Lambda_i (i = 1, 2, 3)$ such that there do not exist two parallel vectors in two different sets, and for two vectors $\xi \neq \xi'$, the associated orthonormal bases satisfy $\xi_1 \neq \xi'_1$. For convenience, we denote $\Lambda = \bigcup_{i=1}^3 \Lambda_i \cup \Lambda_v$. By our choice, there exists a constant $N_\Lambda \in \mathbb{N}$ such that 

$$\{N_\Lambda \xi, N_\Lambda \xi_1, N_\Lambda \xi_2\} \subset \mathbb{Z}^3, \xi \in \Lambda.$$

2.3 Intermittent Flow

Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a smooth cutoff function supported on the interval $[-1, 1]$. Assume it is normalized in such a way that $\phi := -\frac{d^2}{dx^2} \Psi$ satisfies

$$\int_{\mathbb{R}} \phi^2(x) dx = 2\pi.$$

For parameters $0 < \sigma \ll r \ll 1$, we define the rescaled functions

$$\phi_r(x) := \frac{1}{r^{1/2}} \phi \left( \frac{x}{r} \right), \quad \phi_\sigma(x) := \frac{1}{\sigma^{1/2}} \phi \left( \frac{x}{\sigma} \right), \quad \Psi_\sigma(x) := \frac{1}{\sigma^{1/2}} \Psi \left( \frac{x}{\sigma} \right).$$

And we periodize the above functions so that we can view the resulting functions as functions defined on $\mathbb{T}$. We fix a large parameter $\lambda$ such that $\lambda \sigma \in \mathbb{N}$, and a large time oscillation parameter $\mu \gg \sigma^{-1}$. For every $\xi \in \Lambda$, define

$$\phi_\xi(t, x) := \phi_{\xi, r, \sigma, \lambda, \mu}(t, x) = \phi_r(\lambda \sigma N_\Lambda (\xi \cdot x + \mu t)), \quad \phi_{\xi_1}(x) := \phi_{\xi_1, \sigma, \lambda}(x) = \phi_\sigma(\lambda \sigma N_\Lambda (\xi_1 \cdot x)), \quad \Psi_{\xi_1}(x) := \Psi_{\xi_1, \sigma, \lambda}(x) = \Psi_\sigma(\lambda \sigma N_\Lambda (\xi_1 \cdot x)),$$

which are $\left( \frac{\pi}{\lambda \sigma} \right)^3$ periodic. By definition, we have

$$-\Delta \Psi_{\xi_1}(x) = \lambda^2 N_\Lambda^2 \phi_{\xi_1}(x), \quad \text{and} \quad \xi \cdot \nabla \phi_\xi = \frac{1}{\mu} \partial_t \phi_\xi. \quad (2.13)$$
The intermittent velocity flows are defined by

\[ W_\xi(t, x) := \phi_\xi(t, x) \varphi_{\xi_1}(x) \xi, \quad \xi \in \Lambda, \]

and the intermittent deformation flows are defined by

\[ E_\xi(t, x) := \phi_\xi(t, x) \varphi_{\xi_1}(x) \xi_2, \quad \xi \in \bigcup_{i=1}^{3} \Lambda_i, \]

Since the map \( x \mapsto \lambda \sigma N \Lambda (\xi \cdot x + \mu t, \xi_1 \cdot x, \xi_2 \cdot x) \) is the composition of a rotation by a rational orthogonal matrix mapping \( \{e_1, e_2, e_3\} \) to \( \{\xi, \xi_1, \xi_2\} \), a translation, and a rescaling by integers, we have

\[ \int_{T^3} \phi_\xi \varphi_{\xi_1} \, dx = 0, \quad \int_{T^3} \phi_\xi' \varphi_{\xi_1'} \, dx = \frac{1}{(2\pi)^3} \int_{T^3} \phi_\xi' \varphi_{\xi_1'} \, dx = 1. \] (2.14)

Moreover, the following estimates hold:

**Lemma 2.5.** For any \( 1 \leq p \leq \infty, M, N \in \mathbb{N} \), and \( \xi \neq \xi' \) we have the following estimates

\[ \|\nabla^M \partial^N \phi_\xi\|_{L^p} \lesssim (\lambda \sigma)^{M+N} \mu^{\frac{1}{2} - \frac{2}{p} - M - N} \mu^N, \] (2.15)

\[ \|\nabla^M \varphi_{\xi_1}\|_{L^p} + \|\nabla^M \Psi_{\xi_1}\|_{L^p} \lesssim \lambda^M \sigma^{\frac{1}{2} - \frac{1}{p}}. \] (2.16)

\[ \|\nabla^M (\phi_\xi \varphi_{\xi_1})\|_{L^p} + \|\nabla^M (\phi_\xi \Psi_{\xi_1})\|_{L^p} \lesssim \lambda^M \sigma^{\frac{1}{2} - \frac{1}{p} - \frac{1}{2}}. \] (2.17)

\[ \|\phi_\xi \varphi_{\xi_1} \phi_\xi' \varphi_{\xi_1'}\|_{L^p} \lesssim \sigma^{\frac{1}{p} - 1} r^{-1}, \] (2.18)

where the implicit constants only depend on \( p, N \) and \( M \).

**Proof.** The first three inequalities follow from [1]. Here we only prove the last inequality. Recall that \( \phi_\xi, \varphi_{\xi_1} \), are \( \left(\frac{\pi}{\lambda \sigma}\right)^3 \) periodic. In each box with side length \( \frac{2\pi}{\lambda \sigma} \), the support of \( \phi_\xi \varphi_{\xi_1} \) consists of at most \( 2N \lambda \) parallel thin cubes with length \( \sim \lambda^{-1} \) (in the direction of \( \xi_1 \)), width \( \sim (\lambda \sigma)^{-1} r \) (in the direction of \( \xi \)) and height \( \sim (\lambda \sigma)^{-1} \) (in the direction of \( \xi_2 \)). And in each box the support of \( \phi_\xi' \varphi_{\xi_1'} \) consists of another family of parallel cubes with the same size. Due to our specific choice of the set \( \Lambda \) (see Remark 2.4), the two families of support cubes are unparallel and the angel between any two different vectors \( \xi \neq \xi' \) are larger than a universal constant \( \theta_\lambda > 0 \). In view of this, the intersections of two different support cubes are contained in much smaller cubes, with the length and width bounded by \( \sim \lambda^{-1} \), and the height \( \sim (\lambda \sigma)^{-1} \). Moreover, since the number of such cubes are bounded by \( 4N^2 \lambda^3 \), the total size of the support of \( \phi_\xi \varphi_{\xi_1} \phi_\xi' \varphi_{\xi_1'} \) in each box with side length \( \frac{2\pi}{\lambda \sigma} \) is bounded by \( \sim \frac{1}{\lambda^3 \sigma^3} \). We multiply the estimate by \( (\lambda \sigma)^3 \) to derive

\[ |\text{supp} \phi_\xi \varphi_{\xi_1} \phi_\xi' \varphi_{\xi_1'}| \lesssim \sigma^2, \]

\[ \|\phi_\xi \varphi_{\xi_1} \phi_\xi' \varphi_{\xi_1'}\|_{L^1} \lesssim |\text{supp} \phi_\xi \varphi_{\xi_1} \phi_\xi' \varphi_{\xi_1'}| \|\phi_\xi \varphi_{\xi_1} \phi_\xi' \varphi_{\xi_1'}\|_{L^\infty} \leq \frac{\sigma}{r}. \]

Interpolating between \( L^1 \) and \( L^\infty \) yields the estimate (2.18). \( \square \)

Now we fix the values of the parameters \( \lambda, \sigma, r \) and \( \mu \) as

\[ \lambda := \lambda_{q+1}, \quad \sigma := \lambda_{q+1}^{-(1-2\alpha)}, \quad r := \lambda_{q+1}^{-(1-6\alpha)}, \quad \mu := \lambda_{q+1}^{1-\alpha}. \] (2.19)
3 The Perturbation

3.1 Mollification

In order to avoid the loss of derivatives, we mollify the velocity and the deformation fields. Let
\[
\psi_\epsilon(t) = \epsilon^{-1} \psi \left( \frac{t}{\epsilon} \right), \quad \tilde{\psi}_\epsilon(x) = \epsilon^{-3} \tilde{\psi} \left( \frac{x}{\epsilon} \right),
\]
be the standard 1D and 3D Friedrichs mollifier sequences respectively, with
\[
supp \psi \subseteq (-1, 1), \quad supp \tilde{\psi} \subseteq B_1(0).
\]
Define a mollification of \(v_q, F^i_q, R^w_q, \) and \(R^i_q \) (\(i = 1, 2, 3\)) in space and time at length scale \(\ell\) by
\[
v_\ell := \left( v_q * \tilde{\psi}_\ell \right) *_t \psi_\ell, \quad R^w_\ell := \left( R^w_q * x \tilde{\psi}_\ell \right) *_t \psi_\ell, \\
F^i_\ell := \left( F^i_q * x \tilde{\psi}_\ell \right) *_t \psi_\ell, \quad R^i_\ell := \left( R^i_q * x \tilde{\psi}_\ell \right) *_t \psi_\ell.
\]
Then one has
\[
\bigcup_{i=1}^{3} \text{supp}_t (v_\ell, R^w_\ell, F^i_\ell, R^i_\ell) \subset \bigcup_{i=1}^{3} O_\ell \left( \text{supp}_t (v_q, R^w_q, F^i_q, R^i_q) \right).
\]
By (2.1) we obtain that (3.1)
\[
\begin{aligned}
& \partial_t v_\ell + \text{div}(v_\ell \otimes v_\ell - F^k_\ell \otimes F^k_\ell) + (-\Delta)^q v_\ell + \nabla p_\ell = \text{div} (R^w_\ell + R^w_{\text{comm}}), \\
& \partial_t F^i_\ell + \text{div}(F^i_\ell \otimes v_\ell - v_\ell \otimes F^i_\ell) = \text{div} (R^i_\ell + R^i_{\text{comm}}), \quad i = 1, 2, 3,
\end{aligned}
\]
where the traceless symmetric commutator stress \(R^w_{\text{comm}}\) and the skew-symmetric stress \(R^i_{\text{comm}}\) are given by
\[
R^w_{\text{comm}} = (v_q \otimes v_q) - (F^i_q \otimes F^i_q) - \left( (v_q \otimes v_q - F^i_q \otimes F^i_q) * x \tilde{\psi}_\ell \right) *_t \psi_\ell,
\]
\[
R^i_{\text{comm}} = F^i_q \otimes v_q - v_q \otimes F^i_q - \left( F^i_q \otimes v_q - v_q \otimes F^i_q \right) * x \tilde{\psi}_\ell *_t \psi_\ell.
\]
Here we use \(a \otimes b\) to denote the traceless part of tensor \(a \otimes b\). The new pressure \(p_\ell\) is defined as
\[
p_\ell = \left( p_q * x \tilde{\psi}_\ell \right) *_t \psi_\ell - \frac{1}{3} |v_\ell|^2 + \frac{1}{3} \sum_{i=1}^{3} |F^i_\ell|^2 + \frac{1}{3} \left( \left( |v_q|^2 - \sum_{i=1}^{3} |F^i_q|^2 \right) * x \tilde{\psi}_\ell \right) *_t \psi_\ell.
\]
In view of (2.6), (2.7) and (2.8) we have for \(M \in \mathbb{N}, N \in \mathbb{N}_+\),
\[
\left\| \nabla^M R^w_\ell \right\|_{L^1} + \sum_{i=1}^{3} \left\| \nabla^M R^i_\ell \right\|_{L^1} \lesssim \ell^{-M} \delta_{q+1},
\]
\[
\left\| R^w_\ell \right\|_{C^N_{t,x}} + \sum_{i=1}^{3} \left\| R^i_\ell \right\|_{C^N_{t,x}} \lesssim \lambda_{q}^{10} \ell^{-N+1},
\]
\[
\left\| v_\ell \right\|_{C^N_{t,x}} + \sum_{i=1}^{3} \left\| F^i_\ell \right\|_{C^N_{t,x}} \lesssim \lambda_{q}^{4} \ell^{-N+1}.
\]
For commutator stresses, we follow the estimate in [11] and obtain
\[ \| R^i_{\text{comm}} \|_{L^1} \lesssim \| R^i_{\text{comm}} \|_{C^0_{t,x}} \lesssim \ell^2 \| v_q \|_{C^1_{t,x}} \| F^i_q \|_{C^1_{t,x}} \lesssim \ell^2 \lambda_q^4, \]
\[ \| R^w_{\text{comm}} \|_{L^1} \lesssim \| R^w_{\text{comm}} \|_{C^0_{t,x}} \lesssim \ell^2 (\| v_q \|_{C^1_{t,x}}^2 + \sum_{i=1}^3 \| F^i_q \|_{C^1_{t,x}}^2 ) \lesssim \ell^2 \lambda_q^8. \]

Now we fix the parameter \( \ell \) as
\[ \ell := \lambda_q^{-20}. \] (3.7)
Recall that \( \beta b^2 \leq \frac{1}{10} \), hence
\[ \| R^w_{\text{comm}} \|_{L^1} + \sum_{i=1}^3 \| R^i_{\text{comm}} \|_{L^1} \lesssim \delta_{q+2}. \] (3.8)

### 3.2 The Principal Part of Perturbations

Define the principal part of the perturbations as
\[ w^p_{q+1} = \sum_{\xi \in \Lambda} a_{\xi} W_{\xi} = \sum_{\xi \in \Lambda} a_{\xi} \phi_{\xi} \varphi_{\xi} \xi, \quad \text{and} \quad \epsilon^p_{q+1} = \sum_{\xi \in \Lambda} a_{\xi} E_{\xi} = \sum_{\xi \in \Lambda} a_{\xi} \phi_{\xi} \varphi_{\xi} \xi, \]
where the amplitude functions \( a_{\xi} \) are to be determined such that \( R^w_q \) and \( R^i_q \) can be cancelled by applying Lemma 2.2 and Lemma 2.3. In order to cancel all the stresses, the velocity perturbation \( w_{q+1} = v_{q+1} - v_t \) should contain wavevectors from \( \Lambda = \bigcup_{i=1}^{3} \Lambda_i \cup \Lambda_v \). (Wavevectors from \( \Lambda_v \) are used to cancel \( R^w_q \), and wavevectors from \( \Lambda_i \) take part in the cancellation of \( R^i_q \).) The deformation perturbation \( \epsilon^p_{q+1} = F^p_{q+1} - F^i_q \) should contain wavevectors from \( \Lambda_i \). To achieve this, we first introduce a smooth increasing function \( \chi \) satisfying
\[ \chi(z) = \begin{cases} 1, & 0 \leq z \leq 1; \\ z, & z \geq 2. \end{cases} \]

And we define a temporal cut-off as in [31]: let \( \Phi_q(t) : [0, T] \to [0, 1] \) be a smooth cut-off function that satisfies
\[ \Phi_q(t) = 1 \text{ on } \supp \Phi_q \subset O_{\ell} \left( \supp \chi \right), \quad \| \Phi_q \|_{C^k} \leq \ell^{-k}, \quad k = 1, 2, 3. \] (3.9)

Then we set
\[ \rho_i(t, x) := 2 \delta_{q+1} \varepsilon_{\rho}^{-1} \chi \left( \delta_{q+1}^{-1} | R^i_q(t, x) | \right), \quad i = 1, 2, 3, \]
where \( \varepsilon_{\rho} \) is the small radius in the geometric Lemma 2.3. It is easy to verify that pointwise we have
\[ \left| \frac{R^i_q(t, x)}{\rho_i(t, x)} \right| \leq \frac{R^i_q(t, x)}{2 \delta_{q+1} \varepsilon_{\rho}^{-1} \chi \left( \delta_{q+1}^{-1} | R^i_q(t, x) | \right)} \leq \varepsilon_{\rho}. \]

By definition and (3.3) we have
\[ \| \rho_i \|_{L^1} \lesssim \delta_{q+1} + \| R^i_q \|_{L^1} \lesssim \delta_{q+1}, \] (3.10)
where the implicit constant only depends on $\varepsilon_F$. Furthermore, by standard Hölder estimates (see Proposition C.1 in [2]), (3.15), and the parameter inequality $\ell \ll \delta_{q+1}$, we obtain
\[
\|\rho_i\|_{C_{t,x}^0} \lesssim \ell^{-1}, \quad \|\rho_i^+\|_{C_{t,x}^1} \lesssim \ell^{-1}, \quad \|\rho_i^-\|_{C_{t,x}^1} \lesssim \delta_{q+1},
\]
(3.11)
\[
\|\rho_i^+\|_{C_{t,x}^N} \lesssim \delta_{q+1}^{\frac{1}{2}} \left( \|\delta_{q+1}^{-1} R_{i,\ell}^+\|_{C_{t,x}^N} + \|\delta_{q+1}^{-1} R_{i,\ell}^\pm\|_{C_{t,x}^N} \right) \lesssim \delta_{q+1}^{\frac{1}{2}} \ell^{-N} + \delta_{q+1}^{\frac{1}{2}} \ell^{-N} \lesssim \ell^{-2N}
\]
(3.12)
\[
\|\rho_i^-\|_{C_{t,x}^N} \lesssim \ell^{-N} \delta_{q+1}^{-N-1},
\]
(3.13)
for $N = 1, 2, 3$. Then for each $i = 1, 2, 3$, we define the deformation amplitude functions
\[
a_{\xi}(t, x) := a_{\xi,j}(t, x) = \rho_i^+ \Phi_\xi(t) \gamma_\xi \left( \frac{-R_i^j}{\rho_i} \right), \quad \xi \in \Lambda_i.
\]

By Lemma 2.2 and (2.14), we have
\[
-R_i^j = \sum_{\xi \in \Lambda_i} a_\xi^2 (\xi_2 \otimes \xi - \xi \otimes \xi_2) = \sum_{\xi \in \Lambda_i} a_\xi^2 \int \phi_\xi^2 \varphi_{\xi_1}^2 (\xi_2 \otimes \xi - \xi \otimes \xi_2).
\]
Thus, we get
\[
\sum_{\xi \in \Lambda_i} a_\xi^2 \phi_\xi^2 \varphi_{\xi_1}^2 (\xi_2 \otimes \xi - \xi \otimes \xi_2) = R_i^j = \sum_{\xi \in \Lambda_i} a_\xi^2 \varphi_{\xi_1}^2 (\xi_2 \otimes \xi - \xi \otimes \xi_2) \tag{3.14}
\]
Before giving the definition of the velocity amplitude functions, we first emphasize some factors that need to be considered. In order to cancel all the stresses, the velocity perturbation should have wavevectors from both $\Lambda_u$ and $\bigcup_{i=1}^3 \Lambda_i$. The wavevectors in $\Lambda_u$ will be used to cancel the previous Reynolds stress $R_i^j$, as well as a new matrix $R_F$ which is related to wavevectors from $\bigcup_{i=1}^3 \Lambda_i$:
\[
R_F = \sum_{i=1}^3 \sum_{\xi \in \Lambda_i} a_\xi^2 (\xi \otimes \xi - \xi_2 \otimes \xi_2),
\]
(3.15)
see section 4.3 for details. Note that the estimate of $L^1$ norm of $R_F$ is comparable to that of $R_q^v$ (see Lemma 3.1 below). It is too large to go into the new Reynolds stress so must be canceled together with $R_q^v$. This motivates us to define $\rho_v$ and the associated velocity amplitudes as
\[
\rho_v(t, x) := 2 \delta_{q+1}\varepsilon_v^{-1} \chi \left( \delta_{q+1}^{-1} R_i^v(t, x) + R_F \right),
\]
\[
a_{\xi}(t, x) := a_{\xi,v}(t, x) = \rho_v^+ \Phi_\xi(t) \gamma_\xi \left( \frac{R_i^v}{\rho_v} \right), \quad \xi \in \Lambda_v.
\]
Analogously we can verify that $\text{Id} - \frac{R_i^v + R_F}{\rho_v}$ is in the domain of functions $\gamma_\xi$ in Lemma 2.20 and the following equality holds:
\[
\sum_{\xi \in \Lambda_v} a_\xi^2 \phi_\xi^2 \varphi_{\xi_1}^2 (\xi \otimes \xi) + R_i^v + R_F = \rho_v \text{Id} + \sum_{\xi \in \Lambda_v} a_\xi^2 \varphi_{\xi_1}^2 (\phi_\xi^2 \varphi_{\xi_1}^2) (\xi \otimes \xi). \tag{3.16}
\]
We note that the definitions of $a_\xi$ for $\xi \in \Lambda_u$, respectively for $\xi \in \bigcup_{i=1}^3 \Lambda_i$, are different. Throughout the paper we abuse this notation and write $a_\xi = a_{\xi,i}$ for $\xi \in \Lambda_i$ and $a_\xi = a_{\xi,v}$ for $\xi \in \Lambda_v$.

Now we present the following estimates for the amplitude functions:
Lemma 3.1. (Estimate of the amplitude functions) For any $\xi \in \Lambda = \bigcup_{i=1}^{3} \Lambda_i \cup \Lambda_v$ and $N = 1, 2, 3$, we have the following estimates

$$
\|a_\xi\|_{C_{t,x}^N} \lesssim \epsilon^{-2N},
$$
(3.17)

$$
\|a_\xi\|_{L^2} \lesssim \delta_{q+1}^\frac{1}{q}.
$$
(3.18)

Proof. We first consider $\xi \in \bigcup_{i=1}^{3} \Lambda_i$. By (3.5) and (3.11)-(3.13), one gets that for $i = 1, 2, 3$,

$$
\|a_\xi\|_{C_{t,x}^N} \lesssim \|\phi_i\|_{C_{t,x}^N} + \|\rho_i\|_{C_{t,x}^N} \Big( \|\Phi_q\|_{C_{t,x}^N} + \|R_q\|_{C_{t,x}^N} + \|R_q - R_q\|_{C_{t,x}^N} \Big) \lesssim \epsilon^{-2N}, \quad \xi \in \Lambda_i.
$$

Moreover, (3.10) yields that

$$
\|a_\xi\|_{L^2} \lesssim \|\phi_i\|_{L^2}^{\frac{1}{q}} \|\gamma_\xi\|_{C^0} \lesssim \delta_{q+1}^\frac{1}{q}, \quad \xi \in \Lambda_i.
$$

It remains to prove the estimates for $\xi \in \Lambda_v$. Observe that

$$
\|R_q\|_{C_{t,x}^N} \lesssim \sum_{i=1}^{3} \sum_{\xi \in \Lambda_i} \|a_\xi\|_{C_{t,x}^N} \lesssim \epsilon^{-2N}, \quad \|R_q\|_{L^1} \lesssim \sum_{i=1}^{3} \sum_{\xi \in \Lambda_i} \|a_\xi\|_{L^2} \lesssim \delta_{q+1}.
$$

Using the same techniques used to derive (3.10) and (3.12), we obtain

$$
\|\rho_v\|_{L^1} \lesssim \delta_{q+1}, \quad \|\rho_v\|_{C_{t,x}^N} \lesssim \epsilon^{-\frac{1}{2}N}.
$$

Hence we get

$$
\|a_\xi\|_{C_{t,x}^N} \lesssim \|\phi_i\|_{C_{t,x}^N} + \|\rho_v\|_{C_{t,x}^N} \Big( \|\Phi_q\|_{C_{t,x}^N} + \|R_q\|_{C_{t,x}^N} + \|R_q - R_q\|_{C_{t,x}^N} \Big) \lesssim \epsilon^{-3N},
$$

$$
\|a_\xi\|_{L^2} \lesssim \|\rho_v\|_{L^2}^{\frac{1}{q}} \|\gamma_\xi\|_{C^0} \lesssim \delta_{q+1}^\frac{1}{q}, \quad \xi \in \Lambda_v.
$$

This completes the proof. \qed

3.3 Incompressibility Correctors

Notice that $w^{p}_{q+1}$ and $e^{i,p}_{q+1}$ are not divergence free, in order to fix it, we introduce the incompressibility correctors

$$
w^{c}_{q+1} := \frac{1}{N^2 \Lambda_q} \sum_{\xi \in \Lambda} (\nabla a_\xi \times (\phi_\xi \Psi_\xi, \xi)) + \nabla a_\xi \times (\phi_\xi \Psi_\xi, \xi) + a_\xi \nabla \phi_\xi \times (\nabla \Psi_\xi, \xi),
$$

$$
e^{i,c}_{q+1} := \frac{1}{N^2 \Lambda_q} \sum_{\xi \in \Lambda} (\nabla a_\xi \times (\phi_\xi \Psi_\xi, \xi_2)) + \nabla a_\xi \times (\phi_\xi \Psi_\xi, \xi_2) - a_\xi \Delta \phi_\xi \Psi_\xi, \xi_2.
$$

By definition one has

$$
w^{p}_{q+1} + w^{c}_{q+1} = \frac{1}{N^2 \Lambda_q} \nabla \nabla \sum_{\xi \in \Lambda} a_\xi \phi_\xi \Psi_\xi, \xi,
$$

$$
e^{i,p}_{q+1} + e^{i,c}_{q+1} = \frac{1}{N^2 \Lambda_q} \nabla \nabla \sum_{\xi \in \Lambda} a_\xi \phi_\xi \Psi_\xi, \xi_2.
$$

Hence $\text{div}(w^{p}_{q+1} + w^{c}_{q+1}) = \text{div}(e^{i,p}_{q+1} + e^{i,c}_{q+1}) = 0$. 

12
3.4 Temporal Correctors

In addition to the incompressibility correctors, we introduce the temporal correctors, which are defined by

\[ w^i_{q+1} = \frac{1}{\mu} \sum_{\xi \in \Lambda} P_H P_{\neq 0} (a^2 \phi \partial^2_x \phi \xi_i) \xi, \quad \text{and} \quad e^{i,t}_{q+1} = \frac{1}{\mu} \sum_{\xi \in \Lambda} P_H P_{\neq 0} (a^2 \phi \partial^2_x \phi \xi_i) \xi. \]  

(3.19)

Here \( P_H = \text{Id} - \nabla (\Delta^{-1} \text{div}) \) denote the usual Helmholtz projector onto divergence free fields. It is easy to verify that \( w^i_{q+1} \) and \( e^{i,t}_{q+1} \) are both divergence free and have zero mean. By definition we have

\[ \partial_t w^i_{q+1} = -\frac{1}{\mu} \sum_{\xi \in \Lambda} P_{\neq 0} \partial_t (a^2 \phi \partial^2_x \phi \xi_i) + \frac{1}{\mu} \sum_{\xi \in \Lambda} \nabla (\Delta^{-1} \text{div} \partial_t (a^2 \phi \partial^2_x \phi \xi_i)) \].

(3.20)

\[ \partial_t e^{i,t}_{q+1} = -\frac{1}{\mu} \sum_{\xi \in \Lambda} P_{\neq 0} \partial_t (a^2 \phi \partial^2_x \phi \xi_i) + \frac{1}{\mu} \sum_{\xi \in \Lambda} \nabla (\Delta^{-1} \text{div} \partial_t (a^2 \phi \partial^2_x \phi \xi_i)). \]

(3.21)

We define the perturbations:

\[ w_{q+1} := w_{q+1}^p + w_{q+1}^c + w_{q+1}^t, \quad e^{i}_{q+1} = e^{i,p}_{q+1} + e^{i,c}_{q+1} + e^{i,t}_{q+1}, \]

(3.22)

which are mean zero and divergence-free. The new velocity field and deformation vectors are defined as

\[ v_{q+1} := v + w_{q+1}, \quad F^i_{q+1} = F^i + e^i_{q+1}. \]

(3.23)

3.5 Verification of Inductive Estimates for Perturbations

To estimate the \( L^2 \) norm of \( w_{q+1}^p \) and \( e^{i,p}_{q+1} \), we introduce the following \( L^p \) Decorrelation Lemma from [31]:

**Lemma 3.2.** Let \( f, g \in C^\infty(T^3) \), and \( g \) is \((\frac{\kappa}{\mu})^3\) periodic, \( \kappa \in \mathbb{N} \). Then for \( 1 \leq p \leq \infty \),

\[ \|fg\|_{L^p} \lesssim \|f\|_{L^p} \|g\|_{L^p} + \kappa^{-\frac{3}{2}} \|f\|_{C^1} \|g\|_{L^p}. \]

Now we prove the inductive estimates of the perturbations \( w_{q+1} \) and \( e^{i}_{q+1} \).

**Proposition 3.3.** The perturbations \( w_{q+1} \) and \( e^{i}_{q+1} \) obey the following bounds

\[ \|w_{q+1}\|_{L^2} + \sum_{i=1}^{3} \|e^{i}_{q+1}\|_{L^2} \leq M \delta_{q+1}^{\frac{1}{4}}, \]

(3.24)

\[ \|w_{q+1}\|_{L^p} + \sum_{i=1}^{3} \|e^{i}_{q+1}\|_{L^p} \lesssim \ell^{-3} (\sigma \rho)^{\frac{1}{2}} \left( \frac{\kappa^2}{\mu^2} \right), \]

(3.25)

\[ \|w_{q+1}\|_{W^{1,p}} + \sum_{i=1}^{3} \|e^{i}_{q+1}\|_{W^{1,p}} \lesssim \ell^{-3} \lambda_{q+1} (\sigma \rho)^{\frac{1}{2}} \left( \frac{\kappa^2}{\mu^2} \right), \]

(3.26)

\[ \|w_{q+1}\|_{C^0_{\ell,x}} + \sum_{i=1}^{3} \|e^{i}_{q+1}\|_{C^0_{\ell,x}} \leq \frac{1}{10} \lambda_{q+1}^{4}, \]

(3.27)

for \( p \in [1, +\infty] \). Here the constant \( M \) depends only on \( \varepsilon_v \) and \( \varepsilon_F \).
Proof. Applying Lemma 3.2 with \( f = a_\xi, \ g = \phi_\xi \varphi_{\xi_1}, \ \kappa = \lambda_{q+1} \sigma \), we get the following estimate

\[
\|w_{q+1}^p\|_{L^2} + \sum_{i=1}^3 \|e_{q+1}^{i,p}\|_{L^2} \lesssim \sum_{\xi \in \Lambda} \|a_\xi\|_{L^2} \|\phi_\xi \varphi_{\xi_1}\|_{L^2} + (\lambda_{q+1} \sigma)^{-\frac{1}{2}} \|a_\xi\|_{C^1} \|\phi_\xi \varphi_{\xi_1}\|_{L^2} \\
\lesssim \delta_{q+1}^{\frac{1}{2}} + \epsilon^{-3}(\lambda_{q+1} \sigma)^{-\frac{1}{2}}.
\]

In view of Lemma 2.6 and Lemma 3.1, we obtain for any \( p \in [1, \infty) \)

\[
\|w_{q+1}^p\|_{L^p} + \|e_{q+1}^{i,p}\|_{L^p} \lesssim \|a_\xi\|_{C^1} \|\phi_\xi \Psi_{\xi_1}\|_{L^p} \lesssim \epsilon^{-3}(\sigma r)^{-\frac{1}{2}} - \frac{1}{4},
\]  

(3.28)

and

\[
\|w_{q+1}^p\|_{L^p} \lesssim \frac{1}{\lambda_{q+1}^2} \left( \|\nabla^2 a_\xi \phi_\xi \Psi_{\xi_1}\|_{L^p} + \|\nabla a_\xi \Psi_{\xi_1}\|_{L^p} + \|a_\xi \nabla \phi_\xi \times \Psi_{\xi_1}\|_{L^p} + \|a_\xi \nabla \phi_\xi \times \Psi_{\xi_1}\|_{L^p} \right) \\
\lesssim \frac{1}{\lambda_{q+1}^2} \left( \|a_\xi\|_{C^1} \|\phi_\xi \Psi_{\xi_1}\|_{L^p} + \|a_\xi\|_{C^1} \|\phi_\xi \Psi_{\xi_1}\|_{W^{1,p}} + \|a_\xi\|_{C^1} \|\phi_\xi \Psi_{\xi_1}\|_{W^{1,p}} \right) \\
\lesssim \lambda_{q+1}^{-2} \left( \epsilon^{-6}(\sigma r)^{-\frac{1}{2}} + \epsilon^{-3}(\sigma r)^{-\frac{1}{2}} + \epsilon^{-1}(\lambda_{q+1} \sigma)^{-\frac{1}{2}} \right) \\
\lesssim \lambda_{q+1}^{-2} \epsilon^{-6}(\sigma r)^{-\frac{1}{2}} + (1 + \lambda_{q+1}^2 + \lambda_{q+1} \sigma r^{-1}) \lesssim \epsilon^{-6}(\sigma r)^{-\frac{1}{2}} - \frac{1}{2}.
\]

Analogously, for \( e_{q+1}^{i,c} \) we also have

\[
\|e_{q+1}^{i,c}\|_{L^p} \lesssim \epsilon^{-6}(\sigma r)^{-\frac{1}{2}} - \frac{1}{2}.
\]  

(3.29)

By the definition \( (3.19) \) of \( u_{q+1}^i \) we get

\[
\|w_{q+1}^i\|_{L^p} \lesssim \frac{1}{\mu} \|a_\xi\|_{C^1} \|\phi_\xi \varphi_{\xi_1}\|_{L^{2p}} \lesssim \epsilon^{-6}(\sigma r)^{-\frac{1}{2}} - 1.
\]

For \( e_{q+1}^{i,t} \) we also arrive at

\[
\|e_{q+1}^{i,t}\|_{L^p} \lesssim \frac{1}{\mu} \|a_\xi\|_{C^1} \|\phi_\xi \varphi_{\xi_1}\|_{L^{2p}} \lesssim \epsilon^{-6}(\sigma r)^{-\frac{1}{2}} - 1.
\]  

(3.30)

The above estimates yield

\[
\|w_{q+1}\|_{L^2} + \sum_{i=1}^3 \|e_{q+1}^{i}\|_{L^2} \lesssim \delta_{q+1}^{\frac{1}{2}} + \epsilon^{-3}(\lambda_{q+1} \sigma)^{-\frac{1}{2}} + \epsilon^{-6}(\sigma r)^{-\frac{1}{2}} - 1 \lesssim \delta_{q+1}^{\frac{1}{2}}
\]

\[
\|w_{q+1}\|_{L^p} + \sum_{i=1}^3 \|e_{q+1}^{i}\|_{L^p} \lesssim \epsilon^{-3}(\sigma r)^{-\frac{1}{2}} + \epsilon^{-6}(\sigma r)^{-\frac{1}{2}} + \epsilon^{-6}(\sigma r)^{-\frac{1}{2}} \lesssim \epsilon^{-3}(\sigma r)^{-\frac{1}{2}} - \frac{1}{2},
\]

where we used Definition \( (2.19) \) and parameter inequality \( (2.3) \). The implicit constants in the above estimates only depend on \( \epsilon_{2v} \) and \( \epsilon_{F} \). Hence we can choose a large \( M \) such that \( (3.24) \) holds.

Now we consider \( (3.20) \). By Lemma 2.6 and Lemma 3.1, we have

\[
\|w_{q+1}^p\|_{W^{1,p}} + \sum_{i=1}^3 \|e_{q+1}^{i,p}\|_{W^{1,p}} \lesssim \sum_{\xi \in \Lambda} \|a_\xi\|_{C^1} \|\phi_\xi \varphi_{\xi_1}\|_{W^{1,p}} \lesssim \epsilon^{-3}(\sigma r)^{-\frac{1}{2}} - \frac{1}{2}.
\]  

(3.31)

14
Moreover, we obtain from (2.15) and (2.16) that $\|\phi_\xi\|_{W^{k, p}} \lesssim (\lambda_{q+1} \sigma)^{\frac{k}{p} - \frac{1}{2} - k}$ and $\|\Psi_\xi\|_{W^{k, p}} \lesssim \lambda_{q+1}^k \sigma^{\frac{1}{2} - k}$, $k = 0, 1, 2, 3$. Then for $w^c_{q+1}$ and $e^{i, c}_{q+1}$ we have

$$
\|w_{q+1}^c\|_{W^{1, p}} + \sum_1^3 \|e_{q+1}^{i, c}\|_{W^{1, p}} \lesssim \sum_{\xi \in \Lambda} \lambda_{q+1}^{-2} \|a_\xi\|_{C^3} \|\phi_\xi\|_{W^{2, p}} \|\Psi_\xi\|_{W^{2, p}} + \|\phi_\xi\|_{W^{2, p}} \|\Psi_\xi\|_{W^{1, p}} + \|\phi_\xi\|_{W^{3, p}} \|\Psi_\xi\|_{L^p} \lesssim \lambda_{q+1}^{-2} \varepsilon^{-9} \left((\lambda_{q+1} \sigma) r^\frac{1}{p} - \frac{3}{2} \lambda_{q+1}^2 + (\lambda_{q+1} \sigma)^2 r^\frac{1}{p} - \frac{3}{2} \lambda_{q+1} + (\lambda_{q+1} \sigma)^3 r^\frac{1}{p} - \frac{3}{2} \lambda_{q+1} \right) \sigma^\frac{1}{2} - \frac{1}{2}
$$

where we used the fact $\sigma \ll r$. Similarly, in order to estimate $w^t_{q+1}$ and $e^{i, t}_{q+1}$, using (2.17) and Lemma 3.1 one obtains

$$
\|w^t_{q+1}\|_{W^{1, p}} + \|e^{i, t}_{q+1}\|_{W^{1, p}} \lesssim \sum_{\xi \in \Lambda} \frac{1}{\mu} \|a_\xi\|_{C^3} \|\phi_\xi\|_{W^{1, p}} \|\Psi_\xi\|_{W^{1, p}} \lesssim \varepsilon^{-6} \mu^{-1} \lambda_{q+1}^\sigma^\frac{1}{p} - \frac{1}{2} r^\frac{1}{p} - 1. \ (3.32)
$$

We conclude from (3.31)-(3.32) that

$$
\|w_{q+1}\|_{W^{1, p}} + \|e_{q+1}\|_{W^{1, p}} \lesssim \varepsilon^{-3} \lambda_{q+1} \sigma^\frac{1}{p} - \frac{1}{2} r^\frac{1}{p} - \frac{1}{2} + \varepsilon^{-9} \lambda_{q+1} \sigma^\frac{1}{p} - \frac{1}{2} r^\frac{1}{p} - \frac{1}{2} + \varepsilon^{-6} \mu^{-1} \lambda_{q+1}^\sigma^\frac{1}{p} - \frac{1}{2} r^\frac{1}{p} - 1
$$

which yields (3.26).

Finally, we estimate the $C^1$ norm of $w_{q+1}$ and $e^i_{q+1}$. Applying Lemma 2.5 we obtain

$$
\|w_{q+1}^c\|_{C^{1, \alpha}} + \sum_{i=1}^3 \|e^i_{q+1}\|_{C^{1, \alpha}} \lesssim \lambda_{q+1}^{-2} \|\phi_\xi\|_{C^{1, \alpha}} \|\Psi_\xi\|_{C^{1, \alpha}} \lesssim \lambda_{q+1}^{-2} \lambda_{q+1} \sigma^\frac{1}{p} - \frac{1}{2} \lambda_{q+1}^\sigma^\frac{1}{p} - \frac{1}{2} \ll \lambda_{q+1}^4,
$$

and

$$
\|w_{q+1}^t\|_{C^{1, \alpha}} + \sum_{i=1}^3 \|e^i_{q+1}\|_{C^{1, \alpha}} \lesssim \mu^{-1} \|\phi_\xi\|_{C^{1, \alpha}} \|\Psi_\xi\|_{C^{1, \alpha}} \lesssim \mu^{-1} \lambda_{q+1}^2 \sigma^{-1} r^{-1} \ll \lambda_{q+1}^4.
$$

This completes the proof of (3.22).}

Applying standard mollification estimates we have

$$
\|v_q - v_\ell\|_{L^2} \lesssim \|v_q - v_\ell\|_{C^0} \lesssim \ell \|v_q\|_{C^1} \lesssim \ell \lambda_q^4 \ll \delta_{q+1}^4, \quad (3.33)
$$

$$
\|F^i_q - F^i_\ell\|_{L^2} \lesssim \|F^i_q - F^i_\ell\|_{C^0} \lesssim \ell \|F^i_q\|_{C^1} \lesssim \ell \lambda_q^4 \ll \delta_{q+1}^4, \quad (3.34)
$$

where we used the fact that $\beta \sigma^2 \ll \frac{1}{16}$. Combining (3.24), (3.33), and (3.34) we obtain

$$
\|v_{q+1} - v_\ell\|_{L^2} \lesssim \|w_{q+1}\|_{L^2} + \|v_q - v_\ell\|_{L^2} \lesssim \delta_{q+1}^4,
$$

$$
\|F^i_{q+1} - F^i_\ell\|_{L^2} \lesssim \|e^i_{q+1}\|_{L^2} + \|F^i_q - F^i_\ell\|_{L^2} \lesssim \delta_{q+1}^4.
$$
Hence the solutions we construct satisfy the inductive estimate (2.9). Similarly, one can check that

$$\|v_q - v_{q+1}\|_{L^1} + \|F^i_{q+1} - F^i_q\|_{L^1} \lesssim \ell \lambda^4_q \ll \delta_{q+2}.$$  

Combining this with (3.25) to obtain

$$\|v_{q+1} - v_q\|_{L^1} + \|F^i_{q+1} - F^i_q\|_{L^1} \lesssim \|w_{q+1}\|_{L^1} + \|c^i_{q+1}\|_{L^1} + \|v_q - v_{q+1}\|_{L^1} + \|F^i_{q+1} - F^i_q\|_{L^1} \ll \delta_{q+2},$$

which yields the inductive estimate (2.10). We then check the $C^1$ estimate for $v_{q+1}$ and $F^i_{q+1}$. Using (3.6) and (3.27) to obtain that

$$\|v_{q+1}\|_{C^1_{t,x}} \leq \|v_{t}\|_{C^1_{t,x}} + \|w_{q+1}\|_{C^1_{t,x}} \leq \frac{1}{5} \lambda^4_{q+1},$$

$$\|F^i_{q+1}\|_{C^1_{t,x}} \leq \|F^i_{t}\|_{C^1_{t,x}} + \|c^i_{q+1}\|_{C^1_{t,x}} \leq \frac{1}{5} \lambda^4_{q+1}.$$  

This proves the inductive estimates we have claimed in (2.8). Finally, we estimate $\text{supp}_t(v_{q+1}, F^i_{q+1}, F^2_{q+1}, F^3_{q+1})$. By definition, it is easy to check that the amplitude functions satisfy

$$\bigcup_{\xi \in \Lambda} \text{supp}_t a_\xi \subseteq \text{supp}_t \Phi_q \subset O_t \left( \text{supp}_t (R^v_q, R^1_q, R^2_q, R^3_q) \right) \subseteq O_{2t} \left( \text{supp}_t (R^v_q, R^1_q, R^2_q, R^3_q) \right).$$  

(3.35)

Recall that $\ell = \lambda^{-20}_q$ and $\delta_{q+2} = \lambda^{3\beta}_q \lambda^{-2\beta^2}_q$, by (2.5) one gets $10\ell \leq \delta_{q+2}$. We obtain that

$$\text{supp}_t (w_{q+1}, e^1_{q+1}, e^2_{q+1}, e^3_{q+1}) \subseteq \bigcup_{\xi \in \Lambda} \text{supp}_t a_\xi \subseteq O_{\delta_{q+2}} \left( \text{supp}_t (R^v_q, R^1_q, R^2_q, R^3_q) \right).$$

Combining this with (3.1) yields that

$$\text{supp}_t (v_{q+1}, F^i_{q+1}, F^2_{q+1}, F^3_{q+1}) \subseteq \text{supp}_t (w_{q+1}, e^1_{q+1}, e^2_{q+1}, e^3_{q+1}) \cup \text{supp}_t (v_{t}, F^i_{t}, F^2_{t}, F^3_{t})$$

$$\subseteq \bigcup_{i=1}^3 O_{\delta_{q+2}} \left( \text{supp}_t (v_{q}, R^v_q, F^i_q, R^1_q) \right).$$  

(3.36)

4 The Estimate of Stresses

4.1 Decomposition of the Stresses

Our goal is to show that $R^v_{q+1}$ and $R^i_{q+1}$ satisfy the inductive estimates (2.6), (2.7) and (2.11). Recall that $v_{q+1} = v_t + w_{q+1}$, and $F^i_{q+1} = F^i_t + e^i_{q+1}$. Using (5.2), (5.22) and (5.23) we obtain

$$\text{div} R^i_{q+1} = \partial_t e^i_{q+1} + \text{div} \left( e^i_{q+1} \otimes v_t + F^i_t \otimes w_{q+1} - w_{q+1} \otimes F^i_t - v_t \otimes e^i_{q+1} \right)$$

$$+ \text{div} \left( e^i_{q+1} \otimes (w^e_{q+1} + w^l_{q+1}) - (w^c_{q+1} + w^l_{q+1}) \otimes e^i_{q+1} \right)$$

$$+ \left( e^{i,c}_{q+1} + e^{i,l}_{q+1} \right) \otimes w^p_{q+1} - w^p_{q+1} \otimes \left( e^{i,c}_{q+1} + e^{i,l}_{q+1} \right)$$

$$+ \text{div} \left( e^{i,p}_{q+1} \otimes w^p_{q+1} - w^p_{q+1} \otimes e^{i,p}_{q+1} + R^i_t \right) + \partial_t e^{i,t}_{q+1} + \text{div} R^c_{q+1} + \text{div} R^c_{q+1}.$$
and
\[
\text{div } R_{q+1}^\text{e} - \nabla p_{q+1}
= (-\Delta)^{\ell} w_{q+1}^p + \partial_t (w_{q+1}^p + w_{q+1}^c) + \text{div} \left( v_t \otimes w_{q+1} + w_{q+1}^p \otimes v_t - F_t^i \otimes e_{q+1}^i - e_{q+1}^i \otimes F_t^i \right) \\
+ \text{div} \left( w_{q+1} \otimes (w_{q+1}^c + w_{q+1}^e) + (w_{q+1}^c + w_{q+1}^e) \otimes w_{q+1}^p - e_{q+1}^i \otimes (e_{q+1}^i + e_{q+1}^t) - (e_{q+1}^i + e_{q+1}^t) \otimes e_{q+1}^i \right) \\
+ \text{div} \left( w_{q+1}^p \otimes w_{q+1}^p - e_{q+1}^i \otimes e_{q+1}^i - R_t^p \right) + \partial_t w_{q+1}^c + \text{div} R_{\text{corr}}^c - \nabla p_t.
\]

4.2 Linear and Corrector Errors

The symmetric and skew-symmetric inverse divergence operators allow us to define different parts of the Reynolds stresses as follows:

\[
R_{\text{lin}}^i = R^F \left( \partial_t (e_{q+1}^i + e_{q+1}^c) \right) + (e_{q+1}^i \otimes v_t + F_t^i \otimes w_{q+1} - w_{q+1} \otimes F_t^i - v_t \otimes e_{q+1}^i),
\]
\[
R_{\text{corr}}^i = e_{q+1}^i \otimes (w_{q+1}^c + w_{q+1}^e) - (w_{q+1}^c + w_{q+1}^e) \otimes e_{q+1}^i + (e_{q+1}^c + e_{q+1}^t) \otimes w_{q+1}^p - w_{q+1}^p \otimes (e_{q+1}^c + e_{q+1}^t),
\]
\[
R_{\text{lin}}^v = R(-\Delta)^{\ell} w_{q+1} + R \partial_t (w_{q+1}^p + w_{q+1}^c) + v_t \otimes w_{q+1} + w_{q+1} \otimes v_t - F_t^i \otimes e_{q+1}^i - e_{q+1}^i \otimes F_t^i,
\]
\[
R_{\text{corr}}^v = w_{q+1} \otimes (w_{q+1}^c + w_{q+1}^e) + (w_{q+1}^c + w_{q+1}^e) \otimes w_{q+1}^p - e_{q+1}^i \otimes (e_{q+1}^i + e_{q+1}^t) - (e_{q+1}^i + e_{q+1}^t) \otimes e_{q+1}^i.
\]

We first estimate the linear errors. Recall that

\[
w_{q+1}^p + w_{q+1}^c = \frac{1}{N^2 \Lambda^q} \text{curl} \text{curl} \sum_{\xi \in \Lambda} a_\xi \phi_\xi \Psi_\xi \xi,
\]
\[
e_{q+1}^i + e_{q+1}^c = \frac{1}{N^2 \Lambda^q} \text{curl} \text{curl} \sum_{\xi \in \Lambda} a_\xi \phi_\xi \Psi_\xi \xi_2.
\]

For \( p \in (1, 2) \), by (2.12), (2.13), (2.16) and (6.17) we have

\[
\| R \left( \partial_t (w_{q+1}^p + w_{q+1}^c) \right) \|_{L^p} \lesssim \lambda_{q+1}^{-2} \sum_{\xi \in \Lambda} \| R \text{ curl} \text{curl} \partial_t (a_\xi \phi_\xi \Psi_\xi \xi) \|_{L^p} \\
\lesssim \lambda_{q+1}^{-2} \left( \| a_\xi \|_{C^2} \| \partial_t \phi_\xi \|_{L^p} \| \Psi_\xi \|_{L^p} + \| a_\xi \|_{C^2} \| \partial_t \phi_\xi \|_{L^p} \| \nabla \Psi_\xi \|_{L^p} \right) \\
\lesssim \lambda_{q+1}^{-2} \left( 6^{-\mu} \left( \frac{\lambda_{q+1}}{r} \frac{1}{\sigma} \right)^2 \sigma^{p-\frac{1}{2}} \frac{1}{r} + 6^{-\mu} \frac{\lambda_{q+1}}{r} \sigma^{p-\frac{1}{2}} \frac{1}{r} \right) \\
\lesssim 6^{-\mu} \sigma^{\frac{1}{2}} \frac{1}{r} \frac{1}{r}.
\]
Similar result holds for \( R^F \left( \partial_t (e_{q+1}^p + e_{q+1}^c) \right) \) in \( R_{in}^i \). Next we estimate the high-low interaction terms in linear errors. In view of (3.6), (3.28), (3.29) and (3.30), one has
\[
\| v_t \otimes e_{q+1}^i \|_{L^1} \leq \| v_t \otimes e_{q+1}^i \|_{L^1} + \| v_t \otimes e_{q+1}^c \|_{L^1} + \| v_t \otimes e_{q+1}^t \|_{L^1} \\
\leq \| v_t \|_{C^0} \| e_{q+1}^i \|_{L^1} + \| v_t \|_{C^0} \| e_{q+1}^c \|_{L^1} + \| v_t \|_{C^0} \| e_{q+1}^t \|_{L^1} \\
\lesssim \lambda_q^4 \ell^{-3}(\sigma r)^1/2 + \lambda_q^4 \ell^{-6} \sigma^2 r^{-1/2} + \lambda_q^4 \ell^{-6} \mu^{-1} \\
\lesssim \lambda_q^4 \ell^{-6}(\sigma r)^{1/2}.
\]

Other high-low interaction terms can be handled similarly. For the dissipation term, combining (3.25) and (3.26) we obtain
\[
\| R(-\Delta)^\theta w_{q+1} \|_{L^p} \lesssim \| w_{q+1} \|_{W^{1,p}}^{1-\theta} \| w_{q+1} \|_{W^{1,p}}^\theta \lesssim \ell^{-\theta} \lambda_{q+1}^\theta (\sigma r)^{1/2},
\]
where \( \theta_* \) is defined by (2.2). Thus we obtain
\[
\| R_{in}^i \|_{L^1} \lesssim \| R(-\Delta)^\theta w_{q+1} \|_{L^p} + \| R \left( \partial_t (w_{q+1}^p + w_{q+1}^c) \right) \|_{L^p} \\
+ \| v_t \otimes w_{q+1} + w_{q+1} \otimes v_t - F_t^i \otimes e_{q+1}^i - e_{q+1}^i \otimes F_t^i \|_{L^1} \\
\lesssim \ell^{-\theta} \lambda_{q+1}^\theta (\sigma r)^{1/2} + \ell^{-\theta} \mu \sigma^{1/2} r^{1/2} + \lambda_q^4 \ell^{-6} (\sigma r)^{1/2} \\
\lesssim \ell^{-\theta} (\sigma r)^{1/2} (\lambda_{q+1}^\theta + \mu r^{-1}).
\]

Analogously we get
\[
\| R_{in}^i \|_{L^1} \lesssim \| R^F \partial_t \left( e_{q+1}^i + e_{q+1}^c \right) \|_{L^p} + \| e_{q+1}^i \otimes v_t + F_t^i \otimes w_{q+1} - w_{q+1} \otimes F_t^i - v_t \otimes e_{q+1}^i \|_{L^1} \\
\lesssim \ell^{-\theta} \mu \sigma^{1/2} r^{1/2} + \lambda_q^4 \ell^{-6} (\sigma r)^{1/2} \\
\lesssim \ell^{-\theta} \mu \sigma^{1/2} r^{1/2} + \lambda_q^4 \ell^{-6} (\sigma r)^{1/2}.
\]

Next we estimate the corrector errors. Due to the smallness of the corrector terms, we apply (3.21), (3.29), (3.30), and obtain
\[
\| R_{corr}^i \|_{L^1} \lesssim \| w_{q+1}^c + w_{q+1}^i \|_{L^2} \| e_{q+1} \|_{L^2} + \| w_{q+1} \|_{L^2} \| e_{q+1}^i + e_{q+1}^t \|_{L^2} \\
\lesssim (\ell^{-\theta} \sigma r^{-1} + \ell^{-\theta} \mu^{-1} (\sigma r)^{-1/2}) \delta_{q+1}^{1/2} \\
\lesssim \ell^{-\theta} \mu^{-1} (\sigma r)^{-1/2} \delta_{q+1}^{1/2}.
\]

And \( R_{corr}^v \) obeys the same estimate.
4.3 Oscillation Error

By (3.14) we have

\[
\text{div} \left( e_{q+1}^p \otimes w_{q+1}^p - w_{q+1}^p \otimes e_{q+1}^p + R^i_t \right)
\]

\[
= \text{div} \left( \sum_{\xi \in \Lambda_i} a_\xi^2 \phi_\xi^2 \varphi_{\xi_1}^2 (\xi_2 \otimes \xi - \xi \otimes \xi_2) + R^i_t \right) + \text{div} \left( \sum_{\xi \in \Lambda_i, \xi' \in \Lambda_i, \xi \neq \xi'} a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_1} \phi_{\xi'} \varphi_{\xi_1}' (\xi_2' \otimes \xi - \xi \otimes \xi_2') \right)
\]

\[
= \text{div} \left( \sum_{\xi \in \Lambda_i} a_\xi^2 \mathbb{P}_{\geq \lambda_{q+1}} (\phi_\xi^2 \varphi_{\xi_1}^2) (\xi_2 \otimes \xi - \xi \otimes \xi_2) \right) + \text{div} \left( \sum_{\xi \in \Lambda_i, \xi' \in \Lambda_i, \xi \neq \xi'} a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_1} \phi_{\xi'} \varphi_{\xi_1}' (\xi_2' \otimes \xi - \xi \otimes \xi_2') \right)
\]

:= \text{div}(E_{i,1} + E_{i,2}),
\]

where we also used that \( \phi_\xi \) and \( \varphi_{\xi_1} \) are \( \frac{2}{\lambda_{q+1} + \sigma} \) periodic, hence \( \mathbb{P}_{\geq 0} (\phi_\xi^2 \varphi_{\xi_1}^2) = \mathbb{P}_{\geq \lambda_{q+1}} (\phi_\xi^2 \varphi_{\xi_1}^2) \).

The oscillation stress is given by

\[
R^i_{\text{osc}} = \mathcal{R}^F \left( \text{div} E_{i,1} + \partial_t e_{q+1}^i \right) + E_{i,2}.
\]

It is easy to check that \( \text{div} \left( \text{div} E_{i,1} \right) = 0 \), hence the term \( \mathcal{R}^F \left( \text{div} E_{i,1} + \partial_t e_{q+1}^i \right) \) is well-defined. In view of (2.18), Lemma 5.1 and Lemma 3.2 the term \( E_{i,2} \) can be easily estimated as

\[
\| E_{i,2} \|_{L^1} \lesssim \sum_{\xi \in \Lambda_i, \xi' \in \Lambda_i, \xi \neq \xi'} \| a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_1} \phi_{\xi'} \varphi_{\xi_1}' (\xi_2' \otimes \xi - \xi \otimes \xi_2') \|_{L^1} \lesssim \sum_{\xi \in \Lambda_i, \xi' \in \Lambda_i, \xi \neq \xi'} \| a_\xi a_{\xi'} \|_{L^1} \| \phi_\xi \varphi_{\xi_1} \phi_{\xi'} \varphi_{\xi_1}' \|_{L^1} \lesssim \delta_{q+1}^{\sigma_1} \frac{1}{\sigma_1}.
\]

Then we focus on the term \( E_{i,1} \). By the definition of \( \phi_\xi \) and \( \varphi_{\xi_1} \), it is easy to check that \( \xi_2 \cdot \nabla (\phi_\xi^2 \varphi_{\xi_1}^2) = 0 \) and \( \xi \cdot \nabla \varphi_{\xi_1}^2 = 0 \), then one has

\[
\text{div} \left( (\phi_\xi^2 \varphi_{\xi_1}^2) (\xi_2 \otimes \xi - \xi \otimes \xi_2) \right) = \xi \cdot \nabla (\phi_\xi^2 \varphi_{\xi_1}^2) \xi_2 - \xi_2 \cdot \nabla (\phi_\xi^2 \varphi_{\xi_1}^2) \xi = (\xi \cdot \nabla \phi_\xi^2) \varphi_{\xi_1}^2 \xi_2.
\]

Using (2.13) we have

\[
\text{div} E_{i,1} = \sum_{\xi \in \Lambda_i} \mathbb{P}_{\geq \lambda_{q+1}} (\phi_\xi^2 \varphi_{\xi_1}^2) (\xi_2 \otimes \xi - \xi \otimes \xi_2) + \sum_{\xi \in \Lambda_i} \mathbb{P}_{\geq \lambda_{q+1}} (\phi_\xi^2 \varphi_{\xi_1}^2) (\xi_2 \otimes \xi - \xi \otimes \xi_2)
\]

\[
= \sum_{\xi \in \Lambda_i} \mathbb{P}_{\geq \lambda_{q+1}} (\phi_\xi^2 \varphi_{\xi_1}^2) (\xi_2 \otimes \xi - \xi \otimes \xi_2) + \frac{1}{\rho} \sum_{\xi \in \Lambda_i} \mathbb{P}_{\geq \lambda_{q+1}} (a_\xi^2 \partial_\xi^2 \phi_{\xi_1}^2 \xi_2).
\]

Notice that the amplitude term \( a_\xi^2 \) oscillates at a frequency much lower than \( \lambda_{q+1} \), so we can exploit the frequency separation between \( \nabla a_\xi^2 \) and \( \phi_\xi^2 \varphi_{\xi_1}^2 \) and gain a factor of \( \lambda_{q+1} \) from the inverse divergence operator. More precisely, we recall the following lemma, which is a variant of Lemma B.1 of [8]:

19
Lemma 4.1. Let \( a \in C^2(\mathbb{T}^3) \). For \( 1 < p < \infty \), and for any \( f \in L^p(\mathbb{T}^3) \), we have

\[
\|\nabla^{-1} P_{\neq 0} (a \mathcal{P}_{\geq k} f)\|_{L^p} \lesssim k^{-1} \|a\|_{C^2} \|f\|_{L^p}.
\]

Using (4.6) and (3.21), we have

\[
\text{div} E_{i_{t+1}} + \partial_t e_{i_{t+1}} = \sum_{\xi \in \Lambda_i} P_{\neq 0} \left( \nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}) (\xi_2 \otimes \xi - \xi \otimes \xi_2) \right) - \frac{1}{\mu} \sum_{\xi \in \Lambda_i} P_{\neq 0} \left( \partial_t a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}) \xi_2 \right)
\]

\[
+ \frac{1}{\mu} \sum_{\xi \in \Lambda_i} \nabla \Delta^{-1} \text{div} \partial_t (a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}) \xi_2) .
\]

Observe that

\[
\nabla \Delta^{-1} \text{div} \partial_t (a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}) \xi_2)
\]

\[
= \nabla \Delta^{-1} \text{div} (\partial_t a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}) \xi_2) + \nabla \Delta^{-1} \text{div} (a_{i_{t+1}}^2 \partial_t \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}) \xi_2)
\]

\[
= \nabla \Delta^{-1} \text{div} (\partial_t a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}) \xi_2) + \mu \nabla \Delta^{-1} (\xi_2 \cdot \nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}))
\]

\[
= \nabla \Delta^{-1} \text{div} (\partial_t a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}) \xi_2) + \mu \nabla \Delta^{-1} (\xi \cdot \nabla (\xi_2 \cdot \nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi})))
\]

\[
- \mu \nabla \Delta^{-1} ((\xi \cdot \nabla (\xi_2 \cdot \nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}))).
\]

Hence we get

\[
\| R^F (\text{div} E_{i_{t+1}} + \partial_t e_{i_{t+1}}) \|_{L^p}
\]

\[
\lesssim \sum_{\xi \in \Lambda_i} \left\{ \|\nabla^{-1} (\nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi})) \|_{L^p} + \frac{1}{\mu} \|\nabla^{-1} (\partial_t a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi})) \|_{L^p}
\]

\[
+ \frac{1}{\mu} \|\nabla^{-1} \nabla \Delta^{-1} \text{div} (\partial_t a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi})) \|_{L^p}
\]

\[
+ \|\nabla^{-1} \nabla \Delta^{-1} (\xi \cdot \nabla (\xi_2 \cdot \nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}))) \|_{L^p}
\]

\[
+ \|\nabla^{-1} \nabla \Delta^{-1} ((\xi \cdot \nabla (\xi_2 \cdot \nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}))) \|_{L^p}\right\}
\]

\[
\lesssim \sum_{\xi \in \Lambda_i} \left\{ \|\nabla^{-1} (\nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi})) \|_{L^p} + \frac{1}{\mu} \|\nabla^{-1} (\partial_t a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi})) \|_{L^p}
\]

\[
+ \|\nabla^{-1} (\xi_2 \cdot \nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi})) \|_{L^p} + \|\nabla^{-1} ((\xi \cdot \nabla (\xi_2 \cdot \nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi}))) \|_{L^p}\right\},
\]

(4.7)

where we used the fact that \( \|\nabla \Delta^{-1} \text{div} \|_{L^p \rightarrow L^p} + \|\nabla \Delta^{-1} \xi \cdot \nabla \|_{L^p \rightarrow L^p} + \|\nabla \Delta^{-1} P_{\neq 0} \|_{L^p \rightarrow L^p} \lesssim 1 \).

For the first term, we can apply Lemma \([4.1]\) with \( a = \nabla a_{i_{t+1}}^2, f = \phi_{\xi}^2 \psi_{\xi}, k = \lambda_{t+1}\sigma \) and obtain

\[
\|\nabla^{-1} (\nabla a_{i_{t+1}}^2 \mathcal{P}_{\geq \lambda_{t+1}\sigma} (\phi_{\xi}^2 \psi_{\xi})) \|_{L^p} \lesssim (\lambda_{t+1}\sigma)^{-1} \epsilon^{-9}(\sigma r)^{\frac{3}{2} - 1}.
\]

(4.8)

Other terms in (4.7) can be estimated similarly as the first term.

In view of (4.5) and (4.8), we can conclude that

\[
\| R^F_{\text{osc}} \|_{L^1} \lesssim \delta_{t+1} \frac{\sigma}{r} + (\lambda_{t+1}\sigma)^{-1} \epsilon^{-9}(\sigma r)^{\frac{3}{2} - 1}.
\]

(4.9)

20
Next we estimate the Reynolds oscillation stress $R_{\text{osc}}^w$. By (3.15), (3.16) and the fact that $\xi_2 \cdot \nabla (\phi_2^2 \varphi_2^2) = 0$, we can write

$$
\text{div}(u_{q+1}^w \otimes u_{q+1}^w - c_{q+1}^w \otimes c_{q+1}^w + R_{\xi}^v)
$$

$$
= \text{div} \left( \sum_{\xi \in A_\alpha} a_2^2 \phi_2^2 \varphi_2^2 (\xi \otimes \xi) + \sum_{i} a_2^2 \phi_2^2 \varphi_{2i} (\xi \otimes \xi - \xi_2 \otimes \xi_2) + R_{\xi}^v \right)
$$

$$
+ \text{div} \left( \sum_{\xi \neq \xi'} a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_i} \phi_\xi' \varphi_{\xi'_i} (\xi \otimes \xi') - \sum_{i} a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_i} \phi_\xi' \varphi_{\xi'_i} (\xi_2 \otimes \xi_2') \right)
$$

$$
= \text{div} \left( \sum_{\xi \in A_\alpha} a_2^2 \phi_2^2 \varphi_2^2 (\xi \otimes \xi) + R_{\xi}^v + R_F \right) + \text{div} \left( \sum_{i} a_2^2 \phi_2^2 \varphi_{\xi_i} (\xi \otimes \xi - \xi_2 \otimes \xi_2) \right)
$$

$$
+ \text{div} \left( \sum_{\xi \neq \xi'} a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_i} \phi_\xi' \varphi_{\xi'_i} (\xi \otimes \xi') - \sum_{i} a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_i} \phi_\xi' \varphi_{\xi'_i} (\xi_2 \otimes \xi_2') \right)
$$

$$
= \text{div} \left( \sum_{\xi \in A} a_2^2 \phi_2^2 \varphi_{\xi_i} (\xi \otimes \xi) \right) + \nabla \rho_v - \sum_{i} \xi_2 \cdot \nabla a_2^2 \phi_2^2 \varphi_{\xi_i} (\xi_2)
$$

$$
+ \text{div} \left( \sum_{\xi \neq \xi'} a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_i} \phi_\xi' \varphi_{\xi'_i} (\xi \otimes \xi') - \sum_{i} a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_i} \phi_\xi' \varphi_{\xi'_i} (\xi_2 \otimes \xi_2') \right).
$$

We put $\nabla \rho_v$ in the pressure $\nabla p_{\text{osc}}$. The last two terms can be estimated similarly as (4.5) by Lemma 4.1. Now we consider the first term. In view of (3.20) we obtain

$$
\text{div} \left( \sum_{\xi \in A} a_2^2 \phi_2^2 \varphi_{\xi_i} (\xi \otimes \xi) + \partial \epsilon_{q+1}^w \right)
$$

$$
= \sum_{\xi \in A} \nabla a_2^2 \phi_2^2 \varphi_{\xi_i} (\xi \otimes \xi) - \frac{1}{\sigma} \sum_{\xi} \partial \epsilon_{q+1}^w a_2^2 \phi_2^2 \varphi_{\xi_i} (\xi) + \frac{1}{\sigma} \sum_{\xi} \nabla (\Delta^{-1} \text{div} \partial \epsilon_{q+1}^w a_2^2 \phi_2^2 \varphi_{\xi_i} (\xi)).
$$

We also apply Lemma 4.1 to estimate the first two terms, and classify the last term into $\nabla p_{\text{osc}}$. The oscillation error $R_{\text{osc}}^w$ is given by

$$
R_{\text{osc}}^w = \mathcal{R} \left( \sum_{\xi \in A} \nabla a_2^2 \phi_2^2 \varphi_{\xi_i} (\xi \otimes \xi) - \frac{1}{\sigma} \sum_{\xi} \partial \epsilon_{q+1}^w a_2^2 \phi_2^2 \varphi_{\xi_i} (\xi) \right)
$$

$$
- \mathcal{R} \left( \sum_{i} \xi_2 \cdot \nabla a_2^2 \phi_2^2 \varphi_{\xi_i} (\xi_2) \right) + \sum_{\xi \neq \xi'} a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_i} \phi_\xi' \varphi_{\xi'_i} (\xi \otimes \xi')
$$

$$
- \sum_{i} a_\xi a_{\xi'} \phi_\xi \varphi_{\xi_i} \phi_\xi' \varphi_{\xi'_i} (\xi_2 \otimes \xi_2').
$$
Hence we can conclude
\[ \| R_{\text{osc}}^v \|_{L^1} \lesssim \delta_{q+1} \sigma + (\lambda_{q+1} \sigma)^{-1} \ell^{-15} (\sigma r)^{\frac{1}{2}}. \] (4.11)

### 4.4 Verification of Inductive Estimates for Stresses

In this section, we verify inductive estimate (2.6) for the stresses. By (3.8), (4.2)-(4.3), and (4.9)-(4.11) we can conclude
\[ \| R_{i_q+1}^i \|_{L^1} + \| R_{v_q+1}^v \|_{L^1} \leq \| R_{i_{\text{lin}}}^i \|_{L^1} + \| R_{osc}^i \|_{L^1} + \| R_{corr}^i \|_{L^1} + \| R_{osc}^v \|_{L^1} + \| R_{corr}^v \|_{L^1} + \| R_{comm}^v \|_{L^1} \approx \ell^{-15} (\sigma r)^{\frac{1}{2}} + \delta_{q+1} \sigma + \ell^{-6} \mu^{-1} (\sigma r)^{\frac{1}{2}} + \ell^2 \lambda_q^6. \]

holds for \( p > 1 \). By (2.4), (2.19) and (3.7), we obtain
\[ \| R_{i_q+1}^i \|_{L^1} + \| R_{v_q+1}^v \|_{L^1} \leq \lambda_{q+1}^{\frac{300}{b} - (2 - 8 \alpha)(\frac{1}{2} - 1)} \left( \lambda_{q+1}^{\theta - 1 + 4 \alpha} + \lambda_{q+1}^{-3 \alpha - 3 \beta} + \lambda_{q+1}^{3 \beta} \right) \]
We choose \( p \) such that
\[ \left( \frac{1}{p} - 1 \right) (8 \alpha - 2) = \frac{\alpha}{2}, \]
namely,
\[ p = \frac{4 - 16 \alpha}{4 - 17 \alpha} \in (1, 2). \]
By (2.2)-(2.5), one has
\[ \theta_* - 1 + 4 \alpha \leq -4 \alpha, \]
and
\[ \| R_{i_q+1}^i \|_{L^1} + \| R_{v_q+1}^v \|_{L^1} \leq \lambda_{q+1}^{\frac{300}{b} - \alpha} + \lambda_{q+1}^{3 \beta} \lambda_{q+1}^{-4 \alpha - 2 \beta} + \lambda_{q+1}^{3 \beta} \lambda_{q+1}^{-3 \alpha - 3 \beta} + \lambda_{q+1}^{\frac{32}{b}} + \lambda_{q+1}^{\frac{120}{b}} - 3 \alpha - 3 \beta. \]
where we have used the fact
\[ \max \left\{ \frac{300}{b} - \frac{\alpha}{2}, -4 \alpha, \frac{120}{b} - 3 \alpha, \frac{-32}{b} \right\} = \frac{32}{b} \leq -2 \beta. \]
Moreover, by Lemma 3.1 and Lemma 2.5 we obtain
\[ \| R_{i_q+1}^i \|_{C_{l_x}^2} + \| R_{v_q+1}^v \|_{C_{l_x}^2} \leq \sum_{\xi \in \Lambda} |a_{\xi}^2| C_{l_x}^2 |\phi_{\xi}^2| C_{l_x}^2 |\phi_{\xi}^2| C_{l_x}^2 \leq \lambda_{q+1}^{10}. \]
This yields the inductive estimate (2.7). Finally, by (3.35) and the definition of the new stresses (see (3.3), (4.1), (4.4) and (4.10)), we have

\[
\text{supp}_t(R_{v+1}, R_{q+1}, R_{q+1}, R_{q+1}) \subseteq \left( \bigcup_{\xi \in \Lambda} \text{supp}_t a_\xi \right) \cup \text{supp}_t(v_\ell, F_1^\ell, F_2^\ell, F_3^\ell) \\
\subseteq \bigcup_{i=1}^3 O_{3+2}^i(\text{supp}_t(v_q, R_q^v, F_q^i, R_q^i)).
\]

Combining this with (3.36), we obtain (2.11). This completes the proof of Proposition 2.1.

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