ON SUPPORT $\tau$-TILTING GRAPHS OF GENTLE ALGEBRAS

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Abstract. Let $A$ be a finite-dimensional gentle algebra over an algebraically closed field. We investigate the combinatorial properties of support $\tau$-tilting graph of $A$. In particular, it is proved that the support $\tau$-tilting graph of $A$ is connected and has the so-called reachable-in-face property. This property was conjectured by Fomin and Zelevinsky for exchange graphs of cluster algebras which was recently confirmed by Cao and Li.

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1. INTRODUCTION AND MAIN RESULTS

1.1. Background. Support $\tau$-tilting module is the central notion of $\tau$-tilting theory [AIR], which generalizes the classical tilting module. This class of modules has been investigated in various contexts [CDT, DF] and has been found to be deeply connected with other content of representation theory, such as torsion classes, silting objects, $t$-structures (cf. [AIR, BY] for instance). In contrast to tilting modules, the support $\tau$-tilting module can always be mutated at an arbitrary indecomposable direct summand to obtain a new support $\tau$-tilting module. Therefore, the support $\tau$-tilting modules may have a richer combinatorial structure than tilting modules. In particular, various cluster phenomenon in cluster algebras have been found in representation theory of finite-dimensional algebras via support $\tau$-tilting modules.

Let $A$ be a finite-dimensional algebra over an algebraically closed field $k$. Denote by $s\tau$-tilt $A$ the set of isomorphism classes of basic support $\tau$-tilting $A$-modules. The support $\tau$-tilting graph $\mathcal{H}(s\tau$-tilt $A)$ is the exchange graph of support $\tau$-tilting $A$-modules, which encodes the information of mutations. In particular, the support $\tau$-tilting graph $\mathcal{H}(s\tau$-tilt $A)$ is regular and can be regarded as a counterpart of the exchange graph of the

Key words and phrases. Gentle algebras; Support $\tau$-tilting modules; Marked surface.
cluster algebra. One of the basic questions in $\tau$-tilting theory is to determine the number of connected components of $\mathcal{H}(\mathcal{s}\tau\text{-tilt } A)$. We refer to [AIR, DJJ, BMRRT, FG, QZ, Y] for recent progress on this question for certain classes of finite-dimensional algebras. All the known results suggest the following folklore conjecture, which strengthens the additive reachability conjecture in cluster algebras (cf. [Q, Remark 5.9]).

**Conjecture 1.1.** Let $A$ be a connected finite-dimensional $k$-algebra. If there is a path between $A$ and $0$ in $\mathcal{H}(\mathcal{s}\tau\text{-tilt } A)$, then $\mathcal{H}(\mathcal{s}\tau\text{-tilt } A)$ has precisely one connected component.

The main purpose of this note is to study combinatorial properties of support $\tau$-tilting graphs for gentle algebras. Gentle algebras were introduced by Assem and Skowroński [AS], as an important class of representation-tame finite-dimensional algebras. The class of gentle algebras is closed under derived equivalence [S, SZ] and has attracted a lot of attention due to its occurrence in various contexts, such as Fukaya categories [HKK], dimer models [B] and cluster theory [ABCP, BZ].

### 1.2. Main results

Let $k$ be an algebraically closed field and $A$ a finite-dimensional $k$-algebra. Denote by $\text{mod } A$ the category of finitely generated right $A$-modules. Let $\tau$ be the Auslander-Reiten translation of $\text{mod } A$. For any $M \in \text{mod } A$, denote by $|M|$ the number of pairwise non-isomorphic indecomposable summands of $M$; $M$ is called *basic* if the number of indecomposable summands of $M$ equals $|M|$; $M$ is called *$\tau$-rigid* if $\text{Hom}_A(M, \tau M) = 0$; $M$ is called *$\tau$-tilting* if it is $\tau$-rigid and $|M| = |A|$. A *$\tau$-rigid pair* is a pair $(M, P)$ with $M \in \text{mod } A$ and $P$ a finitely generated projective $A$-module, such that $M$ is $\tau$-rigid and $\text{Hom}_A(P, M) = 0$. A $\tau$-rigid pair is called a *support $\tau$-tilting pair* if $|M| + |P| = |A|$. In this case, $M$ is a support $\tau$-tilting $A$-module and $P$ is uniquely determined by $M$ up to isomorphism provided that $P$ is basic. In the following we always identify support $\tau$-tilting modules with support $\tau$-tilting pairs.

Let $(M, P)$ and $(N, Q)$ be two $\tau$-rigid pairs. We say that $(N, Q)$ is a direct summand of $(M, P)$ if $N$ and $Q$ are direct summands of $M$ and $P$ respectively. A $\tau$-rigid pair $(M, P)$ is *indecomposable* if $|M| + |P| = 1$. In particular, each basic support $\tau$-tilting pair has $|A|$ indecomposable direct summands. A basic $\tau$-rigid pair $(M, P)$ is *almost complete support $\tau$-tilting* if $|M| + |P| = |A| - 1$. It has been proved in [AIR] that there exist exactly two non-isomorphic basic support $\tau$-tilting pairs $(M_i, P_i)$ such that $(M, P)$ is a direct summand of $(M_i, P_i)$ for $i = 1, 2$. In this case, $(M_1, P_1)$ and $(M_2, P_2)$ are *mutation* of each other. Clearly, $(M_1, P_1)$ and $(M_2, P_2)$ are only different in one indecomposable direct summand.

**Definition 1.2** (support $\tau$-tilting graph). The *support $\tau$-tilting graph* $\mathcal{H}(\mathcal{s}\tau\text{-tilt } A)$ has vertex set indexed by the isomorphism classes of basic support $\tau$-tilting $A$-modules, and two basic support $\tau$-tilting modules are connected by an edge if and only if they are mutations of each other.
Our first main result is about the number of the connected components of \( \mathcal{H}(s\tau\text{-tilt} A) \) of a gentle algebra \( A \), which provides new evidences for Conjecture 1.1.

**Theorem 1.3.** Let \( A \) be a finite-dimensional gentle algebra over an algebraically closed field \( k \). The support \( \tau \)-tilting graph \( \mathcal{H}(s\tau\text{-tilt} A) \) of \( A \) has precisely one connected component.

It is known that the graph \( \mathcal{H}(s\tau\text{-tilt} A) \) is isomorphic to the full subgraph of the silting graph of \( A \) consisting of 2-term silting complexes (see [AIR]). Note that the silting quiver of a gentle algebra possibly have infinitely many connected components (see [Du]).

In [DIJ], Demonet, Iyama and Jasso introduced a simplicial complex \( \Delta(A) \) for a finite-dimensional algebra \( A \) via \( \tau \)-tilting theory. In particular, there is a one-to-one correspondence between the \( d \)-simplexes of \( \Delta(A) \) and the basic \( \tau \)-rigid pairs of \( A \) which have exactly \( d + 1 \) indecomposable summands. The simplicial complex \( \Delta(A) \) has pure dimension of \( |A| − 1 \). The support \( \tau \)-tilting graph \( \mathcal{H}(s\tau\text{-tilt} A) \) can be identified with the dual graph of \( \Delta(A) \). Under this identification, each basic \( \tau \)-rigid pair \((M, P)\) determines a face \( F_{(M,P)} \) of \( \mathcal{H}(s\tau\text{-tilt} A) \). Namely, the face \( F_{(M,P)} \) is the full subgraph of \( \mathcal{H}(s\tau\text{-tilt} A) \) consisting of basic support \( \tau \)-tilting pairs which admit \((M, P)\) as a direct summand.

Let \((M, P)\) and \((N, Q)\) be basic support \( \tau \)-tilting pairs. We say \((N, Q)\) is mutation-reachable by \((M, P)\) if one can obtain \((N, Q)\) from \((M, P)\) by a finite sequence of mutations. Equivalently, there is a path from \((M, P)\) to \((N, Q)\) in \( \mathcal{H}(s\tau\text{-tilt} A) \). The following definition is inspired by [FZ, Conjecture 4.14(3)] for exchange graphs of cluster algebras, which was recently confirmed in [CL]. We also remark that the definition can be formulated for an abstract exchange graph in [BY].

**Definition 1.4 (reachable-in-face).** The support \( \tau \)-tilting graph \( \mathcal{H}(s\tau\text{-tilt} A) \) has the reachable-in-face property if for any mutation-reachable basic support \( \tau \)-tilting pairs \((M, P)\) and \((N, Q)\) such that \((M, P)\) and \((N, Q)\) have a common direct summand \((L, R)\), there is a path from \((M, P)\) to \((N, Q)\) lying in the face \( F_{(L,R)} \) determined by \((L, R)\). In this case, we also say that \( A \) has the reachable-in-face property.

The definition of reachable-in-face property is related to the non-leaving-face property, which was introduced in [CP] for polytopes. Here we generalize it to the support \( \tau \)-tilting graph of an arbitrary finite-dimensional \( k \)-algebra \( A \). We say that the support \( \tau \)-tilting graph \( \mathcal{H}(s\tau\text{-tilt} A) \) has the non-leaving-face property provided that for any mutation-reachable basic support \( \tau \)-tilting pairs \((M, P)\) and \((N, Q)\) with maximal common direct summand \((L, R)\), every path with minimal length linked \((M, P)\) and \((N, Q)\) lies in the face \( F_{(L,R)} \) determined by \((L, R)\). Clearly, the non-leaving-face property implies the reachable-in-face property. It is worth mentioning that for a 2-Calabi-Yau tilted gentle algebra \( A \) arising from a marked surface without punctures, Brüstle and Zhang [BZ2] have proved the non-leaving-face property for the support \( \tau \)-tilting graph of \( A \).
For an arbitrary finite-dimensional $k$-algebra, we do not know whether its support $\tau$-tilting graph has the reachable-in-face property. We do know that the reachable-in-face property holds for the following class of algebras:

1. $\tau$-tilting finite algebras [DIJ], i.e., algebras with finitely many pairwise non-isomorphic indecomposable $\tau$-rigid modules;
2. Cluster-tilted algebras arising from hereditary abelian categories [FG];
3. More generally, 2-Calabi-Yau tilted algebras [C].

Our second result shows that the reachable-in-face property holds for gentle algebras.

**Theorem 1.5.** Let $A$ be a finite-dimensional gentle algebra over an algebraically closed field $k$. The support $\tau$-tilting graph $\mathcal{H}(s\tau\text{-tilt } A)$ has the reachable-in-face property.

Our proofs of Theorem 1.3 and Theorem 1.5 are inspired by the reduction approach in [FG], where the Iyama-Yoshino’s reduction was applied to study the connectedness of cluster-tilting graph of a hereditary category. In present paper, the Iyama-Yoshino’s reduction has been replaced by $\tau$-reduction in the sense of Jasso [J]. We introduce the notion of $\tau$-reachable and totally $\tau$-reachable for an arbitrary finite-dimensional $k$-algebra (cf. Definition 3.2 and Definition 4.1) and prove Theorem 1.3 and Theorem 1.5 by showing that every finite-dimensional gentle $k$-algebra has the totally $\tau$-reachable property. In particular, we have the following general result.

**Theorem 1.6** (Theorem 4.5). Let $A$ be a finite-dimensional $k$-algebra. Then $A$ is totally $\tau$-reachable if and only if $A$ has the reachable-in-face property and the support $\tau$-tilting graph $\mathcal{H}(s\tau\text{-tilt } A)$ is connected.

The paper is organized as follows. In Section 2, we recall basic properties of gentle algebras and their geometric models. In Section 3, we introduce the definition of $\tau$-reachability for indecomposable $\tau$-rigid pairs and establish the $\tau$-reachable property for gentle algebras. In Section 4, we first give the proof of Theorem 1.6 and then deduce Theorem 1.3 and Theorem 1.5. The reduction theory of $\tau$-rigid pairs is presented in Appendix A.

**Acknowledgments.** The authors thank Professor Wen Chang and Professor Thomas Brüstle for their interest and helpful comments. They are grateful to Håvard Utne Terland for comments on the converse of Proposition 3.4. This work is partially supported by the National Natural Science Foundation of China (Grant No. 11801297, 11971326, 12071315).

2. Gentle algebras and their geometric models

Let $Q$ be a quiver. Denote by $Q_0$ the set of vertices and $Q_1$ the set of arrows. For an arrow $\alpha \in Q_1$, we denote by $s(\alpha)$ and $t(\alpha) \in Q_0$ the source and target of $\alpha$ respectively.
2.1. **Gentle algebras.** A finite-dimensional \( k \)-algebra \( A \) is *gentle* if it admits a presentation \( A = kQ/I \) satisfying the following conditions:

(G1) Each vertex of \( Q \) is the source of at most two arrows and the target of at most two arrows;

(G2) For each arrow \( \alpha \), there is at most one arrow \( \beta \) (resp. \( \gamma \)) such that \( \beta \alpha \in I \) (resp. \( \gamma \alpha \notin I \));

(G3) For each arrow \( \alpha \), there is at most one arrow \( \beta \) (resp. \( \gamma \)) such that \( \alpha \beta \in I \) (resp. \( \alpha \gamma \notin I \));

(G4) \( I \) is generated by paths of length 2.

The following is a direct consequence of definition of gentle algebras.

**Lemma 2.1.** Let \( A \) be a finite-dimensional gentle algebra over \( k \) and \( e \) an idempotent of \( A \). The factor algebra \( A/\langle e \rangle \) is also gentle.

The following result plays a fundamental role in our reduction approach.

**Lemma 2.2** ([S, Theorem 1.1]). Let \( A \) be a finite-dimensional gentle algebra over \( k \) and \( M \) a rigid \( A \)-module (i.e., \( \text{Ext}_A^1(M, M) = 0 \)), the endomorphism algebra \( \text{End}_A(M) \) is a gentle algebra.

2.2. **Tilings and permissible curves.** In this subsection, we mainly follow [BS] and [HZZ].

A marked surface is a pair \( (S, M) \), where \( S \) is a connected oriented Riemann surface with non-empty boundary \( \partial S \), and \( M \subset \partial S \) is a finite set of marked points on the boundary. A connected component of \( \partial S \) is called a boundary component of \( S \). A boundary component \( B \) of \( S \) is unmarked if \( M \cap B = \emptyset \). We allow unmarked boundary components of \( S \). A boundary segment is the closure of a component of \( \partial S \setminus M \).

An arc on \( (S, M) \) is a continuous map \( \gamma : [0, 1] \to S \) such that

- \( \gamma(0), \gamma(1) \in M \) and \( \gamma(t) \in S \setminus M \) for \( 0 < t < 1 \);
- \( \gamma \) is neither null-homotopic nor homotopic to a boundary segment.

We always consider arcs on \( S \) up to homotopy relative to their endpoints and up to inverse, where the inverse of an arc \( \gamma \) on \( S \) is defined as \( \gamma^{-1}(t) = \gamma(1 - t) \) for \( t \in [0, 1] \). Denote by \( C(S) \) the set of arcs on \( S \). An arc \( \gamma \) is called a loop if \( \gamma(0) = \gamma(1) \).

For any arcs \( \gamma_1, \gamma_2 \in C(S) \) which are in a minimal position, the intersection number between them is defined to be

\[
\text{Int}(\gamma_1, \gamma_2) := |\{(t_1, t_2) \mid 0 < t_1, t_2 < 1, \gamma_1(t_1) = \gamma_2(t_2)\}|.
\]

An arc \( \gamma \in C(S) \) is said to be without self-intersections if \( \text{Int}(\gamma, \gamma) = 0 \).

A partial triangulation \( T \) of \( (S, M) \) is a collection of arcs in \( C(S) \) without self-intersections and such that \( \text{Int}(\gamma_1, \gamma_2) = 0 \) for any \( \gamma_1, \gamma_2 \in T \). A triple \( (S, M, T) \) is called a tiling provided that \( (S, M) \) is a marked surface and \( T \) is a partial triangulation.
of \((S, M)\) such that \(S\) is divided by \(T\) into a collection of regions (also called tiles) of the following types:

(I) monogons with exactly one unmarked boundary component in their interior (see the left picture in Figure 1);

(II) digons with exactly one unmarked boundary component in their interior (see the right picture in Figure 1);

(III) \(m\)-gons, with \(m \geq 3\) and whose edges are arcs of \(T\) and at most one boundary segment, and whose interior contains no unmarked boundary component of \(S\);

(IV) 3-gons bounded by two boundary segments and one arc of \(T\), and whose interior contains no unmarked boundary component of \(S\).

**Figure 1.** Tiles of type I and II

An arc segment in a tile \(D\) is a curve \(\eta : [0, 1] \to D\) such that \(\eta(0)\) and \(\eta(1)\) are in the edges of \(D\), and \(\eta(t), 0 < t < 1\) are in the interior of \(D\). An arc segment in \(D\) is called permissible with respect to \(T\) if it satisfies one of the following conditions (cf. [HZZ, Definition 2.1] and compare [BS, Definition 3.1]).

(P1) One endpoint \(P\) of \(\eta\) is in \(M\) and the other \(Q\) is in the interior of a non-boundary edge, say \(\gamma\) of \(D\), such that \(\eta\) is not isotopic to a segment of an edge of \(D\) relative to their endpoints, and after moving \(P\) along the edges of \(D\) in anticlockwise order to the next marked point, say \(P'\), the new arc segment obtained from \(\eta\) is isotopic to a segment of \(\gamma\) relative their endpoints. See Figure 2 for all the possible cases of permissible arc segments satisfying (P1).

**Figure 2.** Condition (P1)
The endpoints of $\eta$ are in the interiors of non-boundary edges $x, y$ (which are possibly not distinct) of $D$ such that $\eta$ has no self-intersections, $x, y$ have a common endpoint $p_\eta \in M$ and $\eta$ cuts out an angle from $D$ as shown in Figure 3. We denote by $\Delta(\eta)$ the local triangle cut out by $\eta$.

Figure 3. Condition (P2)

For any arc $\gamma \in C(S)$, we always assume that $\gamma$ is in a minimal position with $T$.

**Definition 2.3** ([BS, Definition 3.1] and [HZZ, Definition 2.6]). An arc $\gamma$ on $S$ is called permissible (with respect to $T$) if each arc segment of $\gamma$ divided by $T$ is permissible.

We denote by $P(T) \subset C(S)$ the set of permissible arcs on $S$. For any $\gamma \in P(T)$ and any $a \in T$, we define $l_a(\gamma) = \text{int}(\gamma, a)$. The length of $\gamma$ (with respect to $T$) is defined to be

$$l(\gamma) = \sum_{a \in T} l_a(\gamma).$$

2.3. **Geometric interpretation of $\tau$-tilting theory.** For any finite-dimensional $k$-algebra $A$, denote by $\tau$-rigid $A$ the set of isomorphism classes of basic $\tau$-rigid modules in $\text{mod } A$, and by $\text{ind}$-$\tau$-rigid $A$ the subset of $\tau$-rigid $A$ consisting of indecomposable ones.

The following result is useful to study $\tau$-rigid modules of gentle algebras.

**Lemma 2.4.** Let $A = kQ/I$ be a finite-dimensional gentle algebra. Then there exists a tiling $(S, M, T)$ such that

- we have a complete set of primitive orthogonal idempotents $\{e_a \mid a \in T\}$ of $A$ indexed by $T$, and
- there is a bijection

$$M : \{\gamma \in P(T) \mid \text{int}(\gamma, \gamma) = 0\} \to \text{ind}$-$\tau$-rigid $A,$

satisfying $l_a(\gamma) = \dim \text{Hom}_A(e_a A, M(\gamma))$ for any $a \in T$ and $\gamma \in P(T)$.

In particular, $\dim M(\gamma) = l(\gamma)$ for any $\gamma \in P(T)$. Moreover, the bijection $M$ induces a bijection

$$\{R \subset P(T) \mid \text{int}(\gamma_1, \gamma_2) = 0, \ \forall \gamma_1, \gamma_2 \in R\} \to \tau$-rigid $A,$
mapping $R$ to $\oplus_{\gamma \in R} M(\gamma).$
Proof. By [BS, Theorems 2.10 and 3.8], there is a tiling \((S, M, T)\) with \(|T| = |Q_0|\) and such that to any \(\gamma \in P(T)\), there is associated indecomposable module \(M(\gamma) \in \text{mod} A\) satisfying \(\text{Int}(\gamma, a) = \dim \text{Hom}_A(e_a A, M(\gamma))\) for any \(a \in T\). The two bijections in the lemma then follows from [HZZ, Propositions 5.3 and 5.6]. \(\square\)

Remark 2.5. In [BS, Theorem 2.10], any finite-dimensional gentle algebra is realized as a tiling algebra; in [BS, Theorem 3.8], \(M\) is a bijection from \(P(T)\) to the set of non-zero string modules in \(\text{mod} A\); the bijections in [HZZ, Propositions 5.3 and 5.6] also exist for skew-gentle algebras.

We fix some notations for a permissible arc in a tiling.

Notations 2.6. Let \((S, M, T)\) be a tiling and \(\gamma \in P(T)\). Let \(m = l(\gamma)\). Fix an orientation of \(\gamma\). We denote by \(P_1, \ldots, P_m\) the intersections of \(\gamma\) and \(T\) in order, which are in \(a_1, \ldots, a_m \in T\) respectively, and which divide \(\gamma\) into arc segments \(\gamma_1, \ldots, \gamma_{m+1}\) in order. Note that we may have \(a_i = a_j\) for different \(i\) and \(j\).

For any \(1 \leq j \leq m\), we denote by \(\delta_j\) (resp. \(\delta_j'\)) the left half (resp. the right half) segment of \(a_j\) divided by \(P_j\), where left/right is w.r.t. the orientation of \(\gamma\). See Figure 4. We fix the orientations of each \(\delta_j, \delta_j'\) such that \(P_j = \delta_j(1) = \delta_j'(1)\).

Using the notations in Notations 2.6, we call \(P_j\) is left most if \(\delta_j\) does not contain any other \(P_i\).

We simply denoted by \(\gamma_1 \circ \gamma_2\) the concatenation of \(\gamma_1 : [0, 1] \to S\) and \(\gamma_2 : [0, 1] \to S\) with \(\gamma_1(1) = \gamma_2(0)\). The following lemma is useful to construct new permissible arcs from a given one, which will be used in the next section.

Lemma 2.7. Let \(\gamma \in P(T)\) without self-intersections. Using the notations in Notations 2.6, for any \(1 < j < m+1\), either \(\delta_{j-1}, \delta_j\) and \(\gamma_j\), or \(\delta_{j-1}', \delta_j'\) and \(\gamma_j\), are the three edges of the triangle \(\triangle(\gamma_j)\).

(a) In the former case, if \(P_{j-1}\) is left most, then so is \(P_j\).

(b) In the later case, the concatenation \(\delta_{j-1}' \circ \gamma_j\) is isotopic to a permissible arc segment satisfying condition \((P1)\).
Proof. Since $\gamma_j$ is a permissible arc segment satisfying condition (P2), it cuts out an angle formed by either $\delta_{j-1}$ and $\delta_j$, or $\delta'_{j-1}$ and $\delta'_j$. So the first assertion holds, see Figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure5.png}
\caption{Cases in Lemma 2.7}
\end{figure}

In the former case, since $\delta_{j-1}$, $\delta_j$ and $\gamma_j$ are the edges of $\Delta(\gamma_j)$ in the clockwise order, $\delta_{j-1} \circ \gamma_j$ is isotopic to $\delta_j$. Suppose that $P_{j-1}$ is left most. If the interior of $\delta_j$ contains a $P_i$, then there is an arc segment $\eta$ of $\gamma$ in the interior of $\Delta(\gamma_j)$ which has $P_i$ as an endpoint. Since $\gamma$ does not have self-intersection, $\eta$ does not cross $\gamma_j$. Then the other endpoint of $\eta$ has to be in the interior of $\delta_{j-1}$, a contradiction. So $P_j$ is also left most.

In the latter case, since the concatenation $a_{j-1} \circ (\delta_{j-1} \circ \gamma_j)$ is isotopic to $\delta'_j$, where the orientation of $a_{j-1}$ is taken to be the direction from $\delta'_{j-1}$ to $\delta_{j-1}$. By definition, $\delta_{j-1} \circ \gamma_j$ is a permissible arc segment satisfying condition (P1). \qed

3. $\tau$-reachable property for gentle algebras

3.1. Definition of $\tau$-reachability. Let $A$ be a finite-dimensional $k$-algebra. Denote by $\mod A$ the category of finitely generated right $A$-modules.

Definition 3.1. Let $M$ and $N$ be $\tau$-rigid $A$-modules. We say that $M$ is $\tau$-reachable from $N$, denoted by $M \sim^\tau N$, if there exists a sequence of $\tau$-rigid $A$-modules $M_1, \ldots, M_s$ such that $M \oplus M_1, M_1 \oplus M_2, \ldots, M_{s-1} \oplus M_s, M_s \oplus N$ are $\tau$-rigid in $\mod A$, where $s \geq 0$.

By definition, $M$ is $\tau$-reachable from $N$ if and only if $N$ is $\tau$-reachable from $M$. All direct summands of a $\tau$-rigid module are $\tau$-reachable from each other. We extend the definition to $\tau$-rigid pairs.

Definition 3.2. Let $(M, P), (N, Q)$ be two $\tau$-rigid pairs in $\mod A$. We say that $(M, P)$ is $\tau$-reachable from $(N, Q)$, denoted by $(M, P) \sim^\tau (N, Q)$, if there exists a sequence of $\tau$-rigid pairs $(M_1, P_1), \ldots, (M_s, P_s)$ such that $(M \oplus M_1, P \oplus P_1), (M_1 \oplus M_2, P_1 \oplus P_2), \ldots, (M_{s-1} \oplus M_s, P_{s-1} \oplus P_s), (M_s \oplus N, P_s \oplus Q)$ are $\tau$-rigid pairs in $\mod A$.

Let $(M, P)$ be a $\tau$-rigid pair. By definition, $(M, 0)$ is $\tau$-reachable from $(0, P)$. Let $M, N$ be $\tau$-rigid modules, then $M \sim^\tau N$ implies $(M, 0) \sim^\tau (N, 0)$. It is also clear that $\tau$-reachability is reflexive, symmetric and transitive.
Definition 3.3. The algebra $A$ has the $\tau$-reachable property if any two indecomposable $\tau$-rigid pairs of $\text{mod} \ A$ are $\tau$-reachable from each other.

The above definitions are inspired by [FG, Definition 4.1]. The $\tau$-reachable property has a close relation with the connectedness of support $\tau$-tilting graph. In particular, by definition, we have

Proposition 3.4. Let $A$ be a finite-dimensional $k$-algebra such that $|A| > 1$. If the support $\tau$-tilting graph of $A$ is connected, then $A$ has the $\tau$-reachable property.

In particular, every $\tau$-tilting finite algebra $A$ with $|A| > 1$ has the $\tau$-reachable property.

Remark 3.5. For connected algebra, we don’t know whether the converse of Proposition 3.4 is true, but for disconnected algebra, the converse is false. In fact, suppose that $A = A_1 \times A_2$, where $A_1$ and $A_2$ are finite dimensional $k$-algebras. For each indecomposable $\tau$-rigid $A_1$-module $M_1$, the pair $(M_1, 0)$ is $\tau$-reachable from $(0, A_2)$ since $(M_1, 0), (M_1, A_2), (0, A_2)$ is exactly the desired sequence. Similarly, for each indecomposable $\tau$-rigid $A_2$-module $M_2$, the pair $(M_2, 0)$ is $\tau$-reachable from $(0, A_1)$. Thus the disconnected finite-dimensional $k$-algebra $A$ has the $\tau$-reachable property. On the other hand, as stated in [T, Lemma 2.6], the support $\tau$-tilting graph of the disconnected algebra $A$ has $n_1 n_2$ connected components, where $n_1$ (resp. $n_2$) is the number of connected components of $\mathcal{H}(s\tau\text{-tilt } A_1)$ (resp. $\mathcal{H}(s\tau\text{-tilt } A_2)$).

In the next subsection, we will establish the $\tau$-reachable property for gentle algebras, which plays a key role in the proof of the connectedness of support $\tau$-tilting graphs of gentle algebras. We remark that there exist algebras which do not satisfy the $\tau$-reachable property.

Remark 3.6. If $|A| = 1$, then the $\tau$-rigid pair $(0, 0)$ is almost complete. By [AIR, Theorem 0.4], there are exactly two non-isomorphic support $\tau$-tilting modules $A$ and $0$, which are related by a mutation. Hence the support $\tau$-tilting graph is connected. However, by definition, $A$ does not have the $\tau$-reachable property.

3.2. $\tau$-reachability of gentle algebras. Throughout this subsection, let $A$ be a finite-dimensional gentle algebra over $k$. Recall that a module $M$ is sincere if $\text{Hom}(P, M) \neq 0$ for any indecomposable projective $A$-module $P$.

Lemma 3.7. Assume that $|A| > 1$. Let $M$ be an indecomposable sincere $\tau$-rigid $A$-module. Then there is an indecomposable $\tau$-rigid $A$-module $N$ with $\dim N < \dim M$ such that $(M, 0)$ is $\tau$-reachable from $(N, 0)$.

Proof. Let $(\mathbf{S}, \mathbf{M}, \mathbf{T})$ be the tiling for $A$ given in Lemma 2.4. Then there is a permissible arc $\gamma$ without self-intersections and such that $M \cong M(\gamma)$. Let $m = l(\gamma) = \dim M$. 

Since $M$ is sincere, we have $\text{Int}(\gamma, a) \neq 0$ for any $a \in T$. By the assumption $|A| > 1$, we have $|T| > 1$ and $m > 1$. We are going to construct a permissible arc $\gamma'$ without self-intersections such that $\text{Int}(\gamma, \gamma') = 0$ and $l(\gamma') < l(\gamma)$, or two permissible arcs $\gamma', \gamma''$ without self-intersections such that $\text{Int}(\gamma, \gamma') = 0$, $l(\gamma') = l(\gamma)$ and $\text{Int}(\gamma'', \gamma') = 0$, $l(\gamma'') < l(\gamma')$.

Fixing an orientation of $\gamma$ and using the notations in Notation 2.6, we divide the proof into several cases.

Case 1: there is an arc $a \in T$ which contains at least one of $P_1$ and $P_m$ and at least one of $P_j, 1 < j < m$.

Reversing the orientation of $\gamma$ if necessary, we may assume $a$ contains $P_1$ and $P_j$ for some $1 < j < m$, such that there is no $P_i$ in $a$ between $P_1$ and $P_j$. See Figure 6. Denote by $\delta$ the segment of $a$ from $P_1$ to $P_j$. Let $g$ be the segment of $\gamma$ from $P_j$ to one endpoint of $\gamma$ such that the arc segment of $g$ connecting $P_j$ is on the different side of $a$ from $\gamma_1$. Take $\gamma' = \gamma_1 \circ \delta \circ g$. Then $\gamma'$ is divided by $T$ into arc segments $\gamma_1$ and the arc segments of $g$, all of which are permissible. So $\gamma'$ is a permissible arc. Since $\gamma$ does not cross the interior of $\delta$, we have $\text{Int}(\gamma', \gamma') = 0 = \text{Int}(\gamma, \gamma')$. Moreover, we have $l(\gamma') \leq \max\{j, m - (j - 1)\} < m = l(\gamma)$ since $1 < j < m$.

![Figure 6. Case 1 in the proof of Lemma 3.7](image)

Case 2: $a_1 = a_m$ does not contain $P_j$ for any $1 < j < m$, and $\gamma_1$ and $\gamma_{m+1}$ are on the different sides of the arc $a_1 = a_m$.

Let $\delta$ be the segment of $a_1 = a_m$ from $P_1$ to $P_m$. See Figure 7. Take $\gamma' = \gamma_1 \circ \delta \circ \gamma_{m+1}$. Then the arc segments of $\gamma$ divided by $T$ are $\gamma_1$ and $\gamma_{m+1}$, both of which are permissible. So $\gamma'$ is a permissible arc. Since $\gamma$ does not cross the interior of $\delta$, we have $\text{Int}(\gamma', \gamma') = 0 = \text{Int}(\gamma, \gamma')$. Moreover, we have $l(\gamma') = 1 < m = l(\gamma)$.

Case 3: both $a_1$ and $a_m$ do not contain $P_j$ for any $1 < j < m$, and if $a_1 = a_m$ then $\gamma_1$ and $\gamma_{m+1}$ are on the same side of $a_1$.

Reversing the orientation of $\gamma$ if necessary, we may assume that if $a_1 = a_m$, then $P_m$ is in the interior of $\delta'_1$. So we have that $P_1$ is left most. By Lemma 2.7, this case can be divided into the following two subcases 3.1 and 3.2.
Subcase 3.1: there exists 1 < j < m + 1 such that δ′ j−1, δ′ j and γ j are the three edges of Δ(γ j).

Choose the smallest j satisfying this condition. See Figure 8. Since P 1 is left most, using Lemma 2.7 repeatedly, we have that P 1, · · · , P j−1 are left most and the concatenation δ j−1 ∘ γ j is a permissible arc segment satisfying condition (P1). Since γ has no self-intersections and the interior δ j−1 does not cross γ, the interior of δ j−1 ∘ γ j does not cross itself and γ. Take γ′ = (δ j−1 ∘ γ j) ∘ γ j+1 ∘ · · · ∘ γ m+1. Then γ′ is a permissible arc satisfying Int(γ′, γ′) = 0 = Int(γ′, γ). Moreover, we have l(γ′) = m − (j − 1) < m = l(γ) since j > 1.

Subcase 3.2: for any 1 < j < m + 1, δ j−1, δ j and γ j are the three edges of Δ(γ j).

Then by Lemma 2.7 (b), all P j, 1 ≤ j ≤ m, are left most. We denote by O the common endpoint of δ 1, · · · , δ m. See Figure 9. This implies that for any a ∈ T, we have Int(γ, a) ≤ 2.

Subcase 3.2.1: there exists 2 ≤ j ≤ m such that a j ̸= a i for any 1 ≤ i ≤ m and i ̸= j.

Take γ′ = γ 1 ∘ γ 2 ∘ · · · ∘ γ j−1 ∘ (γ j ∘ δ j−1), see Figure 9. Applying Lemma 2.7 (b) to the inverse of γ, we have that γ j ∘ δ j−1 is a permissible arc segment satisfying condition (P1). So γ′ is a permissible arc. Since the interior of δ j does not cross γ, a similar
discussion as in Subcase 3.1 shows that \( \text{int}(\gamma', \gamma') = 0 = \text{int}(\gamma', \gamma) \). Moreover, we have \( l(\gamma') = j - 1 < m = l(\gamma) \).

**Subcase 3.2.2:** for any \( 2 \leq j \leq m \), there is \( 1 \leq i \leq m \) and \( i \neq j \) such that \( a_i = a_j \).

Since \( a_m \) does not contain \( P_2, \ldots, P_{m-1} \), we have \( a_m = a_1 \). So for any \( 1 \leq j \leq m \), there is \( i \neq j \) such that \( a_i = a_j \). Recall that \( 0 < \text{int}(\gamma, a) \leq 2 \) holds for any \( a \in T \) in Subcase 3.2. So in Subcase 3.2.2, we have \( \text{int}(\gamma, a) = 2 \) for any \( a \in T \), and \( m = 2|T| \).

It follows that all arcs in \( T \) are loops sharing the same endpoint, say \( O \), and \( \gamma \) cuts out each angle formed by two arcs in \( T \) exactly once. Recall that \( \gamma_1 \) and \( \gamma_{m+1} \) are on the same side of \( a_m = a_1 \). It follows that the tile containing \( \gamma_1 \) is the same as the tile containing \( \gamma_{m+1} \), none of whose angles is formed by two (not necessarily distinct) arcs in \( T \). So this tile is of type (IV), i.e. a 3-gon whose three edges are \( a_1 = a_m \) and two boundary components with \( O \) a common endpoint and without any unmarked boundary component in its interior. Thus, the boundary component of \( S \) containing \( O \) has two marked points, say \( P \) the other marked point. Then we have \( \gamma(0) = \gamma(1) = P \). See Figure 10.

Since \( \text{int}(\gamma, a) = 2 \) for any \( a \in T \), there is \( 1 < j < m - 1 \) such that \( a_j = a_{m-1} \). Let \( \delta \) be the segment of \( a_j = a_{m-1} \) from \( P_j \) to \( P_{m-1} \). The concatenation \( \delta \circ \gamma_m \) is isotopic to \( \gamma_m \) relative to the interiors of arcs in \( T \). So it is permissible. Take \( \gamma' = \gamma_m^{-1} \circ \gamma_m^{-1} \circ \cdots \circ \gamma_{j+1}^{-1} \circ (\delta \circ \gamma_m) \circ \gamma_{m+1}^{-1} \). Then \( \gamma' \) is a permissible curve with \( l(\gamma') = m - j + 2 \).

Since \( P_j \) and \( P_{m-1} \) are the only intersections between \( \gamma \) and \( a_j = a_{m-1} \), the interior of \( \delta \) does not cross \( \gamma \). So we have \( \text{int}(\gamma', \gamma') = 0 \) and \( \text{int}(\gamma', \gamma) = 0 \).

If \( j > 2 \), then \( l(\gamma') = m - j + 2 < m = l(\gamma) \) and we are done. If \( j = 2 \), then \( l(\gamma') = m = l(\gamma) \). We need to find another permissible arc \( \gamma'' \) without self-intersections such that \( \text{int}(\gamma'', \gamma') = 0 \) and \( l(\gamma'') < l(\gamma') \). Note that in case \( j = 2 \), \( \gamma_2 \) and \( \gamma_m \) are in the same tile, where both of the angles incident to \( a_2 = a_{m-1} \) have \( a_1 = a_m \) as an edge. So this tile is of type II, whose edges are \( a_1 = a_m \) and \( a_2 = a_{m-1} \) and which contains an unmarked boundary component in its interior. See Figure 11. Let \( \gamma'' \) be the concatenation of \( \gamma' \) with the boundary segment edge of \( \triangle(\gamma_1) \) from \( P \) to \( O \). By definition, the last arc
\(a_1 = a_m\)

\(a_2 = a_{m-1}\)

\(\gamma_1\)

\(\gamma_2\)

\(\triangle(\gamma_1)\)

\(\triangle(\gamma_2)\)

\(P_1\)

\(P_2\)

\(O\)

\(\gamma_{j+1}\)

\(\gamma_j\)

\(\gamma_{m+1}\)

\(\gamma_m\)

\(\gamma''\)

\(\gamma'\)

\(\gamma''\) is permissible in a tile of type (II). So \(\gamma''\) is permissible. Moreover, we have \(\text{Int}(\gamma'', \gamma'') = 0, \text{Int}(\gamma'', \gamma') = 0\) and \(l(\gamma'') = m - 1 < m = l(\gamma')\) as required.

\(a_2 = a_{m-1}\) in Subcase 3.2.2
Theorem 3.8. Let $A$ be a finite-dimensional gentle algebra over $k$ such that $|A| > 1$. Then $A$ has the $\tau$-reachable property.

Proof. It suffices to show that every indecomposable $\tau$-rigid pair $(M,0)$ is $\tau$-reachable from $(0,P)$ for some indecomposable projective $A$-module $P$. This is clear whenever $M$ is non-sincere. Now assume that $M$ is sincere, applying Lemma 3.7 repeatedly, we know that $M$ is $\tau$-reachable from a non-sincere indecomposable $\tau$-rigid $A$-module $N$. Consequently, $(M,0)$ is $\tau$-reachable from $(0,P)$ for some indecomposable projective $A$-module $P$. □

4. PROOFS OF THE MAIN RESULTS

Let $A$ be a finite-dimensional $k$-algebra. Let $(U,R)$ be a basic $\tau$-rigid pair. We refer to Appendix A for the reduction theory of $(U,R)$. Denote by $(T_{U,R},R)$ the Bongartz completion of $(U,R)$. Let $A_{(U,R)} := \text{End}_A(T_{U,R})/\langle e_U \rangle$ be the factor algebra of $\text{End}_A(T_{U,R})$ by the ideal generated by $e_U$, where $e_U$ is the idempotent of $\text{End}_A(T_{U,R})$ associated to the direct summand $U$.

Definition 4.1. We call $A$ has the totally $\tau$-reachable property provided that for any basic $\tau$-rigid pair $(U,R)$, the algebra $A_{(U,R)}$ is $\tau$-reachable when $|A_{(U,R)}| > 1$.

By definition, if $A$ is totally $\tau$-reachable and $|A| > 1$, then $A$ is $\tau$-reachable; if $|A| = 1$, then $A$ is always totally $\tau$-reachable.

Lemma 4.2. If $A$ is totally $\tau$-reachable, then for any basic $\tau$-rigid pair $(U,R)$, the algebra $A_{(U,R)}$ is also totally $\tau$-reachable.

Proof. For any basic $\tau$-rigid pair $(V,Q)$ in $\text{mod} A_{(U,R)}$, by Corollary A.5, there is a basic $\tau$-rigid pair $(V',Q')$ in $\text{mod} A$ such that $(A_{(U,R)})_{(V,Q)} \cong A_{(V',Q')}$. Since $A$ is totally $\tau$-reachable, $A_{(V',Q')}$ is $\tau$-reachable if $|A_{(V',Q')}| > 1$. This implies that $A_{(U,R)}$ is totally $\tau$-reachable. □

Proposition 4.3. Let $A$ be a finite-dimensional $k$-algebra. If $A$ is totally $\tau$-reachable, then the support $\tau$-tilting graph $\mathcal{H}(s\tau\text{-tilt}A)$ of $A$ is connected.

Proof. Use induction on $|A|$. If $|A| = 1$, by Remark 3.6, the support $\tau$-tilting graph of $A$ is connected. Assume that the theorem holds for the case $|A| < n$. Now suppose $|A| = n$. Let $(M,P)$ and $(N,Q)$ be two basic support $\tau$-tilting pairs of $A$-modules. We separate the remaining proof by considering whether $(M,P)$ and $(N,Q)$ share a non-zero direct summand.

Case 1: $(M,P)$ and $(N,Q)$ have a common direct summand $(U,R)$. By Lemma 4.2, $A_{(U,R)}$ is totally $\tau$-reachable with $|A_{(U,R)}| < |A| = n$. By the inductive hypothesis, the support $\tau$-tilting graph $\mathcal{H}(s\tau\text{-tilt}A_{(U,R)})$ of $A_{(U,R)}$ is connected. By Corollary A.4, the
The support \( \tau \)-tilting pairs containing \((U, R)\) as a direct summand. So \((M, P)\) and \((N, Q)\) are in this subgraph. It follows that they are mutation-reachable to each other. **Case 2:** \((M, P)\) and \((N, Q)\) do not have any common nonzero direct summand. Let \((M_1, P_1)\) be an indecomposable direct summand of \((M, P)\) and \((N_1, Q_1)\) an indecomposable direct summand of \((N, Q)\). Since \(A\) is \(\tau\)-reachable, \((M_1, P_1)\) is \(\tau\)-reachable from \((N_1, Q_1)\). By definition, there exists a sequence of \(\tau\)-rigid pairs \((L_1, R_1), \ldots, (L_t, R_t)\) such that \((L_i \oplus L_{i+1}, R_i \oplus R_{i+1})\), \(0 \leq i \leq t\), are \(\tau\)-rigid pairs in \(\text{mod} A\), where \((L_0, R_0) = (M_1, P_1)\) and \((L_{t+1}, R_{t+1}) = (N_1, Q_1)\). According to [AIR, Theorem 2.10], each \(\tau\)-rigid pair \((L_i \oplus L_{i+1}, R_i \oplus R_{i+1})\) can be completed into a support \(\tau\)-tilting pair \((\tilde{L}_i, \tilde{R}_i)\). Denote \((\tilde{L}_{-1}, \tilde{R}_{-1}) = (M, P)\) and \((\tilde{L}_{t+1}, \tilde{R}_{t+1}) = (N, Q)\). Then \((\tilde{L}_i, \tilde{R}_i)\) and \((\tilde{L}_{i+1}, \tilde{R}_{i+1})\) have a common direct summand \((L_{i+1}, R_{i+1})\) for \(-1 \leq i \leq t\). We conclude that \((M, P)\) is mutation-reachable by \((N, Q)\) by using the result in Case 1 repeatedly. This completes the proof.

**Corollary 4.4.** Let \(A\) be a finite-dimensional \(k\)-algebra. If \(A\) is totally \(\tau\)-reachable, then \(A\) has the reachable-in-face property.

**Proof.** By Lemma 4.2, for any basic \(\tau\)-rigid pair \((U, R)\) in \(\text{mod} A\), the algebra \(A_{(U, R)}\) is totally \(\tau\)-reachable. Then by Proposition 4.3, the support \(\tau\)-tilting graph of \(A_{(U, R)}\) is connected. This graph is isomorphic to the face \(\mathcal{F}_{(U, R)}\). So any face of \(\mathcal{H}(s\tau\text{-tilt } A)\) is connected. In particular, \(A\) has the reachable-in-face property.

**Theorem 4.5.** Let \(A\) be a finite-dimensional \(k\)-algebra. Then \(A\) is totally \(\tau\)-reachable if and only if \(A\) has the reachable-in-face property and the support \(\tau\)-tilting graph \(\mathcal{H}(s\tau\text{-tilt } A)\) is connected.

**Proof.** The “only if” part follows from Proposition 4.3 and Corollary 4.4. Let us prove the “if” part. Assume that \(A\) has the reachable-in-face property and the support \(\tau\)-tilting graph \(\mathcal{H}(s\tau\text{-tilt } A)\) is connected. Consequently, each face of \(\mathcal{H}(s\tau\text{-tilt } A)\) is connected. Let \((U, R)\) be a basic \(\tau\)-rigid pair of \(A\) such that \(|A_{(U, R)}| > 1\), we have to show that \(A_{(U, R)}\) is \(\tau\)-reachable. According to Corollary A.4, there is a bijection \(E_{(U, R)} : s\tau\text{-tilt-pair } A_{(U, R)} \rightarrow s\tau\text{-tilt-pair } A_{(U, R)}\) which commutes with the mutation. If follows that the support \(\tau\)-tilting graph \(\mathcal{H}(s\tau\text{-tilt } A_{(U, R)})\) of \(A_{(U, R)}\) is isomorphic to the face \(\mathcal{F}_{(U, R)}\). In particular, \(\mathcal{H}(s\tau\text{-tilt } A_{(U, R)})\) of \(A_{(U, R)}\) is connected. Now the result follows from Proposition 3.4.

**Proposition 4.6.** Any finite-dimensional gentle \(k\)-algebra is totally \(\tau\)-reachable.

**Proof.** Let \(A\) be a finite-dimensional gentle algebra over \(k\). For any basic \(\tau\)-rigid pair \((U, R)\) in \(\text{mod } A\), by Lemma 2.2, the endomorphism algebra \(\text{End}_A(T_{(U, R)})\) is gentle. By Lemma 2.1, the factor algebra \(A_{(U, R)}\) of \(\text{End}_A(T_{(U, R)})\) by the ideal generated by an idempotent is also gentle. Hence by Theorem 3.8, \(A_{(U, R)}\) is \(\tau\)-reachable if \(|A_{(U, R)}| > 1\). Thus, \(A\) is totally \(\tau\)-reachable.
Combining Proposition 4.6 and Theorem 4.5, we get Theorem 1.3 and Theorem 1.5.

**Appendix A. Reduction**

Let $A$ be a finite-dimensional $k$-algebra and $\text{mod} \ A$ the category of finitely generated right $A$-modules. For any $M \in \text{mod} \ A$, denote by $\text{Fac} \ M$ the full subcategory of $\text{mod} \ A$ consisting of all factors modules of direct sums of copies of $M$; denote

$$M^\perp = \{ N \in \text{mod} \ A \mid \text{Hom}_A(M, N) = 0 \}, \quad ^\perp M = \{ N \in \text{mod} \ A \mid \text{Hom}_A(N, M) = 0 \}.$$

Denote by $\tau$-rigid-pair $A$ the set of isoclasses of basic $\tau$-rigid pairs in $\text{mod} \ A$ and by $\tau$-tilt-pair $A$ the set of isoclasses of basic support $\tau$-tilting pairs in $\text{mod} \ A$. There is a partial order on $\tau$-tilt-pair $A$ that $(M, P) \leq (N, Q)$ if and only if $\text{Fac} \ M \subseteq \text{Fac} \ N$. There is a bijection from $\tau$-tilt-pair $A$ to the set of functorially finite torsion pairs in $\text{mod} \ A$, sending $(M, P)$ to $(\mathcal{T}_M, \mathcal{F}_M) := (\text{Fac} \ M, M^\perp)$, see [AIR, Theorem 2.7].

Let $(U, R)$ be a basic $\tau$-rigid pair. Denote by $\tau$-rigid-pair$(U, R) A$ the set of isoclasses of $\tau$-rigid pairs containing $(U, R)$ as a direct summand, and by $\tau$-tilt-pair$(U, R) A$ the set of isoclasses of support $\tau$-tilting pairs containing $(U, R)$ as a direct summand.

**Lemma A.1** ([AIR, Theorem 2.10] and [DIRRT, Theorem 4.4]). Let $(U, R)$ be a basic $\tau$-rigid pair in $\text{mod} \ A$. There are $(S_{(U,R)}, L_{(U,R)}) \leq (T_{(U,R)}, R) \in \tau$-tilt-pair$(U, R) A$ such that $\tau$-tilt-pair$(U, R) A$ is the interval of $\tau$-tilt-pair $A$ between $(S_{(U,R)}, L_{(U,R)})$ and $(T_{(U,R)}, R)$. Moreover, the corresponding torsion pairs are

$$(\mathcal{T}_{T_{(U,R)}}, \mathcal{F}_{T_{(U,R)}}) = (\perp \tau U \cap R^\perp, T_{(U,R)}^\perp) \text{ and } (\mathcal{T}_{S_{(U,R)}}, \mathcal{F}_{S_{(U,R)}}) = (\text{Fac} \ U, U^\perp).$$

The pair $(T_{(U,R)}, R)$ (resp. $(S_{(U,R)}, L_{(U,R)})$) is called the Bongartz complement (resp. co-Bongartz complement) of $(U, R)$. Let $\text{End}_A(T_{(U,R)})$ be the endomorphism algebra of $T_{(U,R)}$ and $e_U$ the idempotent of $\text{End}_A(T_{(U,R)})$ associated to the direct summand $U$. We denote by $A_{(U,R)} := \text{End}_A(T_{(U,R)})/\langle e_U \rangle$ the factor algebra of $\text{End}_A(T_{(U,R)})$ by the ideal generated by $e_U$. We have $|A_{(U,R)}| = |A| - |U| - |R|$, which is called the co-rank of $(U, R)$, denoted by co-rank$(U, R)$. Denote by

$$\mathcal{W}(U, R) = \mathcal{T}_{T_{(U,R)}} \cap \mathcal{F}_{S_{(U,R)}} = \perp \tau U \cap R^\perp \cap U^\perp.$$

**Lemma A.2** ([J, Theorem 1.4] (for $R = 0$) and [DIRRT, Theorem 4.12 (a,b)]). The subcategory $\mathcal{W}(U, R)$ is a wide subcategory of $\text{mod} \ A$, and the functor

$$\text{Hom}_A(T_{(U,R)}, -) : \text{mod} A \to \text{mod} A_{(U,R)}$$

restricts to an equivalence

$$F_{(U,R)} : \mathcal{W}(U, R) \to \text{mod} A_{(U,R)}.$$

A pair $(M, P)$ of objects in $\mathcal{W}(U, R)$ is called a $\tau$-rigid pair if $(F_{(U,R)}(M), F_{(U,R)}(P))$ is a $\tau$-rigid pair in $\text{mod} A_{(U,R)}$. Denote by $\tau$-rigid-pair $\mathcal{W}(U, R)$ the set of isoclasses of basic $\tau$-rigid pairs in $\mathcal{W}(U, R)$. For any $(V, Q) \in \tau$-rigid-pair $\mathcal{W}(U, R)$, denote by
τ-rigid-pair\(_{(V,Q)}\) \(\mathcal{W}(U, R)\) the subset of \(\tau\)-rigid-pair \(\mathcal{W}(U, R)\) consisting of the pairs containing \((V, Q)\) as a direct summand.

For any \(M \in \mathcal{W}(U, R)\), we denote by \(\text{Fac}_{\mathcal{W}(U,R)} M\) the full subcategory of \(\mathcal{W}(U, R)\) consisting of objects \(N\) such that there is an epimorphism in \(\mathcal{W}(U, R)\) from a direct sum of copies of \(M\) to \(N\). The following result is essentially from [BM].

**Proposition A.3.** Let \((U, R)\) be a basic rigid pair in \(\text{mod } A\). Then there is a bijection

\[
\mathcal{E}_{(U,R)} : \text{\(\tau\)-rigid-pair}_{(U,R)} A \rightarrow \tau\text{-rigid-pair} \mathcal{W}(U, R)
\]

\[
(M, P) \rightarrow (\mathcal{E}'_{(U,R)}(M, P), \mathcal{E}''_{(U,R)}(M, P))
\]

which commutes with direct sums and such that

1. for any \((M, P) \in \tau\text{-rigid-pair}_{(U,R)} A\), we have

\[
\text{co-rank}(M, P) = \text{co-rank}(\mathcal{E}'_{(U,R)}(M, P), \mathcal{E}''_{(U,R)}(M, P)),
\]

2. for any \((M, P) \in \text{sr-tilt-pair}_{(U,R)} A\), we have

\[
\text{Fac } M \cap \mathcal{W}(U, R) = \text{Fac}_{\mathcal{W}(U,R)}(\mathcal{E}'_{(U,R)}(M, P)).
\]

**Proof.** Applying [BM, Proposition 5.11] to \((0, R)\) in \(\text{mod } A\), we have that there is a bijection

\[
\mathcal{E}_{(0,R)} : \quad \text{\(\tau\)-rigid-pair}_{(0,R)} A \rightarrow \tau\text{-rigid-pair} \mathcal{W}(0, R)
\]

\[
(M, P) \rightarrow (M, \mathcal{E}''_{(0,R)}(M, P))
\]

which commutes with direct sums and such that for any \((M, P) \in \tau\text{-rigid-pair}_{(0,R)} A\), we have \(|P| = |\mathcal{E}''_{(0,R)}(M, P)| + |R|\). In particular, \((U, 0) = \mathcal{E}_{(0,R)}((U, R))\) and \((\text{T}_{(U,R)}, 0) = \mathcal{E}_{(0,R)}((\text{T}_{(U,R)}, R))\) are basic \(\tau\)-rigid pairs in \(\mathcal{W}(0, R)\). Since the map \(\mathcal{E}_{(0,R)}\) commutes with direct sums, it restricts to a bijection

\[
\quad \text{\(\tau\)-rigid-pair}_{(U,R)} A \rightarrow \text{\(\tau\)-rigid-pair}_{(U,0)} \mathcal{W}(0, R).
\]

Note that \(\mathcal{W}(0, R) = R^\perp\) is a Serre subcategory of \(\text{mod } A\), i.e., a full subcategory of \(\text{mod } A\) closed under extensions, factor modules and submodules. So for any \(N \in \mathcal{W}(0, R)\), we have \(\text{Fac } N = \text{Fac}_{\mathcal{W}(0,R)} N\). It follows that \((\text{T}_{(U,R)}, 0)\) is the Bongartz completion of \((U, 0)\) in \(\mathcal{W}(0, R)\).

Let \(\mathcal{W}_{\mathcal{W}(0,R)}(U, 0)\) be the wide subcategory of \(\mathcal{W}(0, R)\) induced by the \(\tau\)-rigid pair \((U, 0)\). Since \(\text{Fac } \text{T}_{(U,R)} = \perp \text{T}_{(U,R)} \subseteq \mathcal{W}(0, R)\) and \(\text{Fac } \text{T}_{(U,R)} = \text{Fac}_{\mathcal{W}(0,R)} \text{T}_{(U,R)}\), we have

\[
\mathcal{W}_{\mathcal{W}(0,R)}(U, 0) = \text{Fac}_{\mathcal{W}(0,R)} \text{T}_{(U,R)} \cap \perp \mathcal{W}(0, R) = \text{Fac } \text{T}_{(U,R)} \cap \mathcal{U} = \mathcal{W}(U, R).
\]

Applying [BM, Proposition 5.8] to \((U, 0)\) in \(\mathcal{W}(0, R)\), we have that there is a bijection

\[
\mathcal{E}_{(U,0)} : \quad \text{\(\tau\)-rigid-pair}_{(U,0)} \mathcal{W}(0, R) \rightarrow \tau\text{-rigid-pair} \mathcal{W}_{\mathcal{W}(0,R)}(U, 0) = \tau\text{-rigid-pair} \mathcal{W}(U, R)
\]

\[
(M, P) \rightarrow (\mathcal{E}'_{(U,0)}(M, P), \mathcal{E}''_{(U,0)}(M, P))
\]

which commutes with direct sums and such that for any \((M, P) \in \tau\text{-rigid-pair}_{(U,0)} \mathcal{W}(0, R)\), we have \(|M| + |P| = |\mathcal{E}'_{(U,0)}(M, P)| + |\mathcal{E}''_{(U,0)}(M, P)| + |U|\) and the map \(\mathcal{E}'_{(U,0)}\) is the same
as the map \( f \) in [J, Theorem 3.15]. So for any \((M, P) \in \tau\text{-rigid-pair}_{(U,0)} \mathcal{W}(0, R)\) with \(\text{co-rank}(M, P) = 0\), we have \(\text{Fac} M \cap \mathcal{W}(U, R) = \text{Fac}_{\mathcal{W}(U,R)} E'_{(U,0)}(M, P)\).

Composing the two bijections \(E_{(0, R)}\) and \(E_{(U, R)}\), we get the required bijection. \(\square\)

Corollary A.4. Let \((U, R)\) be a basic rigid pair in \(\text{mod} A\). There is a bijection

\[
E_{(U, R)} : \tau\text{-rigid-pair}_{(U, R)} A \to \tau\text{-rigid-pair}_{(U, R)} A
\]

which commutes with direct sums and restricts to a bijection

\[
E_{(U, R)} : \text{sr\text{-}tilt\text{-}pair}_{(U, R)} A \to \text{sr\text{-}tilt\text{-}pair}_{(U, R)} A
\]

which preserves the order and commutes with the mutation, and such that for any \((M, P) \in \text{sr\text{-}tilt\text{-}pair}_{(U, R)} A\), we have

\[
F_{(U, R)} (\mathcal{T}_M \cap \mathcal{W}(U, R)) = \mathcal{T}_{M'} \quad \text{and} \quad F_{(U, R)} (\mathcal{F}_M \cap \mathcal{W}(U, R)) = \mathcal{F}_{M'}
\]

where \((M', P') = E_{(U, R)}(M, P)\).

Proof. Let \(E_{(U, R)} = F_{(U, R)} \circ E_{(U, R)}\). By Proposition A.3 and Lemma A.2, this is a bijection from \(\tau\text{-rigid-pair}_{(U, R)} A\) to \(\tau\text{-rigid-pair}_{(U, R)} A\), commuting with direct sums and for any \((M, P) \in \tau\text{-rigid-pair}_{(U, R)} A\), we have \(\text{co-rank}(M, P) = \text{co-rank} E_{(U, R)}(M, P)\). So \(E_{(U, R)}\) restricts to a bijection from \(\text{sr\text{-}tilt\text{-}pair}_{(U, R)} A\) to \(\text{sr\text{-}tilt\text{-}pair}_{(U, R)} A\).

By Proposition A.3 and Lemma A.2 again, for any \((M, P) \in \text{sr\text{-}tilt\text{-}pair}_{(U, R)} A\), we have \(F_{(U, R)} (\mathcal{T}_M \cap \mathcal{W}(U, R)) = \mathcal{T}_{M'}\), which implies that \(E_{(U, R)}\) preserves the order. Since \(\text{sr\text{-}tilt\text{-}pair}_{(U, R)} A\) is an interval of \(\tau\text{-tilting-pair}\) by Lemma A.1 and the order can be induced by the mutation, the map \(E_{(U, R)}\) commutes with the mutation.

For any \(X \in \mathcal{W}(U, R)\), since \((\mathcal{T}_M, \mathcal{F}_M)\) is a torsion pair in \(\text{mod} A\), there is an exact sequence

\[
0 \to t_M(X) \to X \to f_M(X) \to 0
\]

with \(t_M(X) \in \mathcal{T}_M\) and \(f_M(X) \in \mathcal{F}_M\). Since \(X \in U^\perp\) and \(U^\perp\) is closed under submodules, we have \(t_M(X) \in U^\perp\). On the other hand, \(t_M(X) \in \mathcal{T}_M \subseteq \mathcal{T}_{(U, R)}\). So \(t_M(X) \in \mathcal{T}_{(U, R)} \cap U^\perp = \mathcal{W}(U, R)\). Since \(\mathcal{W}(U, R)\) is a wide subcategory of \(\text{mod} A\) by Lemma A.2, we have \(f_M(X) \in \mathcal{W}(U, R)\) too. This shows that \((\mathcal{T}_M \cap \mathcal{W}(U, R), \mathcal{F}_M \cap \mathcal{W}(U, R))\) is a torsion pair in \(\mathcal{W}(U, R)\). Then \((F_{(U, R)} (\mathcal{T}_M \cap \mathcal{W}(U, R)), F_{(U, R)} (\mathcal{F}_M \cap \mathcal{W}(U, R)))\) is a torsion pair in \(\text{mod} A_{(U, R)}\), which together with \(F_{(U, R)} (\mathcal{T}_M \cap \mathcal{W}(U, R)) = \mathcal{T}_{M'}\), implies that \(F_{(U, R)} (\mathcal{F}_M \cap \mathcal{W}(U, R)) = \mathcal{F}_{M'}\). \(\square\)

The following result is useful (cf. [BH, Corollary 1.2]).

Corollary A.5. Let \((U, R)\) be a basic \(\tau\text{-rigid pair}\) in \(\text{mod} A\) and \((N, Q)\) a basic \(\tau\text{-rigid pair}\) in \(\text{mod} A_{(U, R)}\). Then there exists a basic \(\tau\text{-rigid pair}\) \((N', Q')\) in \(\text{mod} A\) such that \(A_{(N', Q')} \cong (A_{(U, R)})_{(N, Q)}\).
Proof. By Corollary A.4, there is \((N', Q') \in \tau\)\text{-}\text{rigid-pair}_{(U,R)} A\) such that \(E_{(U,R)}(N', Q') = (N, Q)\). Since \(E_{(U,R)}\) commutes with direct sums and preserves the order, we have \(E_{(U,R)}(T_{(N', Q')}, Q') = (T_{(N, Q)}, Q)\) and \(E_{(U,R)}(S_{(N', Q')}, L_{(N', Q')}) = (S_{(N, Q)}, L_{(N, Q)})\). Then we have

\[
F_{(U,R)} \left( T_{(N', Q')} \cap W(U, R) \right) = T_{(N, Q)} \quad \text{and} \quad F_{(U,R)} \left( F_{S_{(N', Q')}} \cap W(U, R) \right) = F_{S_{(N, Q)}},
\]

Since \(T_{(N', Q')} \subseteq T_{(U,R)}\) and \(F_{S_{(N', Q')}} \subseteq F_{S(U,R)}\), we have

\[
T_{(N', Q')} \cap F_{S_{(N', Q')}} \subseteq T_{(U,R)} \cap F_{S(U,R)} = W(U, R),
\]

which implies

\[
\left( T_{(N', Q')} \cap W(U, R) \right) \cap \left( F_{S_{(N', Q')}} \cap W(U, R) \right) = T_{(N', Q')} \cap F_{S_{(N', Q')}} = W(N', Q').
\]

Hence we have \(F_{(U,R)}(W(N', Q')) = W(N, Q)\). By Lemma A.2, we have \(A_{(N', Q')} \cong (A_{(U,R)})_{(N,Q)}\) as required. \(\square\)

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