Classification of integrable discrete equations of octahedron type

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Abstract

We use the consistency approach to classify discrete integrable 3D equations of the octahedron type. They are naturally treated on the root lattice $Q(A_3)$ and are consistent on the multidimensional lattice $Q(A_N)$. Our list includes the most prominent representatives of this class, the discrete KP equation and its Schwarzian (multi-ratio) version, as well as three further equations. The combinatorics and geometry of the octahedron type equations are explained. In particular the consistency on the 4-dimensional Delaunay cells has its origin in the classical Desargues theorem of projective geometry. The main technical tool used for the classification is the so called tripodal form of the octahedron type equations.

Key words: integrability, multidimensional consistency, Hirota equation, dKP equation, multi-ratio equation, Desargues configuration, root lattice, tripodal form

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1 Introduction

This paper is devoted to the study of certain integrable discrete three-dimensional equations. Prototypic examples of such equations are the discrete KP (dKP), or the discrete bilinear Hirota equation, and its Schwarzian version. Both equations play a pivotal role in modern mathematics [16, 18, 21, 22, 24, 25, 26, 27, 31, 36, 39].

The dKP equation for a complex lattice function (field) $x : \mathbb{Z}^3 \rightarrow \mathbb{C}$ was introduced in [20] in the form

$$\alpha x(m + e_1)x(m - e_1) + \beta x(m + e_2)x(m - e_2) + \gamma x(m + e_3)x(m - e_3) = 0$$

(1)

(here $m \in \mathbb{Z}^3$, $e_i$ is the unit vector of the $i$-th coordinate direction, and $\alpha, \beta, \gamma$ are three arbitrary complex coefficients). It relates six fields assigned to any elementary octahedron of the even sublattice $\mathbb{Z}^3_{even}$ of $\mathbb{Z}^3$, centered at the odd point $m \in \mathbb{Z}^3$. The even sublattice $\mathbb{Z}^3_{even}$ is also known as the face centered cubic lattice (fcc lattice). Its 3-cells are octahedra and tetrahedra.

The fcc lattice is isomorphic to the root lattice of the type $A_3$,

$$Q(A_3) = \{ n = (n_0, \ldots, n_3) \in \mathbb{Z}^3 : n_0 + n_1 + n_2 + n_3 = 0 \}.$$  

(2)

The six vertices of every octahedron cell in $Q(A_3)$ can be written as $n + e_i + e_j$, $i, j \in \{0, 1, 2, 3\}$ (with $n$ from the hyperplane $n_0 + n_1 + n_2 + n_3 = -2$). The dKP equation relates the fields $x$ at these vertices and reads:

$$\alpha x(n + e_0 + e_1)x(n + e_2 + e_3) + \beta x(n + e_0 + e_2)x(n + e_1 + e_3) + \gamma x(n + e_0 + e_3)x(n + e_1 + e_2) = 0.$$  

(3)

This representation of the dKP equation turns out to be more conceptual and plays the key role in the integrability analysis of this paper.
Since on the hyperplane $n_0 + n_1 + n_2 + n_3 = 0$ any coordinate is determined by other three, by forgetting one coordinate we obtain a one-to-one correspondence of the lattice $Q(A_3)$ with $\mathbb{Z}^3$, $(n_0, n_1, n_2, n_3) \leftrightarrow (n_1, n_2, n_3)$. In this realization, the dKP equation becomes an equation on $\mathbb{Z}^3$:

$$\alpha x(n + e_1)x(n + e_2 + e_3) + \beta x(n + e_2)x(n + e_3 + e_1) + \gamma x(n + e_3)x(n + e_1 + e_2) = 0. \quad (4)$$

In this form, introduced in [28], it relates fields assigned to six out of eight vertices of any elementary cube of $\mathbb{Z}^3$ (the fields at one pair of the opposite vertices, $n$ and $n + e_1 + e_2 + e_3$, do not participate in the equation).

Throughout the present paper, we will use the following abbreviation for lattice functions: $x$ for $x(n)$, $x_i$ for $x(n + e_i)$, $x_{-i}$ for $x(n - e_i)$, $x_{ij}$ for $x(n + e_i + e_j)$, etc. In this notation, the dKP equation in the $Q(A_3)$-form (3) can be put as

$$\alpha x_0x_{23} + \beta x_0x_{13} + \gamma x_0x_{12} = 0, \quad (5)$$

while in the $\mathbb{Z}^3$-form (4) it can be put as

$$\alpha x_1x_{23} + \beta x_2x_{13} + \gamma x_3x_{12} = 0. \quad (6)$$

We will call equations like (6) the octahedron type equations, as opposed to the cube type equations, whose best known representative is the dBKP equation introduced in [28]:

$$\alpha x_1x_{23} + \beta x_2x_{13} + \gamma x_3x_{12} + \delta xx_{123} = 0. \quad (7)$$

This latter equation relates the fields at all eight vertices of any elementary cube of $\mathbb{Z}^3$. The octahedron type equations could be considered as a subclass of the cube type equations, however they have different properties and require for a different analysis.

**Definition 1.** Equation of octahedron type is the relation

$$F(x_{01}, x_{02}, x_{03}, x_{12}, x_{13}, x_{23}) = 0$$

for the unknown function $x : Q(A_3) \to \mathbb{C}$, or, equivalently, the relation

$$F(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}) = 0$$

for the unknown function $x : \mathbb{Z}^3 \to \mathbb{C}$.

Our only assumption concerning the function $F$ is that it should be locally analytic and satisfy

**irreducibility condition:** equation $F = 0$ can be locally solved with respect to any variable, the result depending on all other variables, i.e., on any solution of $F = 0$ we have $F_x \neq 0$, where $x$ denotes any argument of the equation.

This condition forbids, in particular, equations with $F$ of the form $F = AB$, where $A$ or $B$ does not depend on some variable (for example, if $A_x = 0$, then $F_x = AB_x$, and this vanishes on the solution $A = 0$). Also the so called ultradiscrete equations with piecewise constant functions $F$ are excluded from consideration. On the other hand, the
irreducibility condition does not forbid the case when the solution of $F = 0$ is multivalued, for instance, $F = AB$ with both $A$ and $B$ depending on all variables. We handle such situations by working always in the neighborhood of some solution branch, where the theorem on implicit function applies.

It is important to observe that irreducible equations of the octahedron type are 3D systems in the following sense: a generic solution of such an equation on $\mathbb{Z}^3$ can be defined by a well posed Cauchy problem with generic 2D initial data. For instance, for the dKP equation in the form (1) such initial data are constituted by prescribing the values of $x$ on the planes $n_3 = 0$ and $n_3 = -1$.

In the work [3] we pushed forward the idea that integrability of discrete equations is synonymous with their multidimensional consistency. For a certain class of 2D equations, this notion was put in the basis of a classification of integrable cases. In the present work, we classify multidimensionally consistent (integrable) equations of the octahedron type. A general idea of consistency leads to the following formulation.

**Definition 2.** Consider a system on $\mathbb{Z}^N$ consisting of (possibly different) octahedron type equations

$$F(x_i, x_j, x_k, x_{ij}, x_{ik}, x_{jk}) = 0$$

on all affine 3D sublattices $c + \mathbb{Z}e_i + \mathbb{Z}e_j + \mathbb{Z}e_k$. It is called multidimensionally consistent if it has a solution whose restrictions on all 3D sublattices are generic solutions of corresponding equations.

This definition is a literal repetition of the corresponding notion for cubic type equations and does not take into account the specific feature of the octahedron type situation. If the lattice $\mathbb{Z}^N$ lattice in this definition is treated as a realization of the root lattice $Q(A_N) = \{n = (n_0, \ldots, n_N) \in \mathbb{Z}^{N+1} : n_0 + n_1 + \ldots + n_N = 0\}$, by forgetting the coordinate $n_0$, i.e., $Q(A_N) \ni (n_0, n_1, \ldots, n_N) \leftrightarrow (n_1, \ldots, n_N) \in \mathbb{Z}^N$, then the corresponding octahedron type equations (8) live on the sublattices

$$Q(A_3) = \{(n_0, n_i, n_j, n_k) : n_0 + n_i + n_j + n_k = \text{const}\}.$$

However, these are by far not all $Q(A_3)$ sublattices of $Q(A_N)$, but only those involving the coordinate $n_0$, which becomes distinguished in this formulation. A more symmetric notion would be:

**Definition 3.** Consider a system on $Q(A_N)$ consisting of (possibly different) octahedron type equations

$$F(x_{im}, x_{jm}, x_{km}, x_{ij}, x_{ik}, x_{jk}) = 0$$

on all affine sublattices $Q(A_3) = \{(n_i, n_j, n_k, n_m) : n_i + n_j + n_k + n_m = \text{const}\}$. It is called multidimensionally consistent if it has a solution whose restrictions on all $Q(A_3)$ sublattices are generic solutions of corresponding equations.

This version is more natural but looks much more restrictive, since there are $\binom{N+1}{4}$ sublattices $Q(A_3)$ of $Q(A_N)$, and only $\binom{N}{4}$ sublattices $\mathbb{Z}^3$ of $\mathbb{Z}^N$. For instance, in the case $N = 4$ the lattice $Q(A_4)$ has 5 sublattices $Q(A_3)$, of which only four involve the distinguished coordinate axis $n_0$. Nevertheless, in Section 3.5 we prove the following result (see Proposition 4).
Theorem 1. Definitions 2 and 3 of the multidimensional consistency of octahedron type equations are equivalent.

Now we formulate the main result of this work. We classify multidimensionally consistent systems of octahedron type equations (in notation of Definition 3) modulo the group of admissible transformations. This group consists of changes of independent variables $n$ generated by the affine Weyl group of $Q(A_N)$ (permutations of indices and translations by lattice vectors) extended by the simultaneous inversion of all coordinates $n \mapsto -n$, as well as of non-autonomous point transformations of dependent variables, $x(n) \mapsto f(x(n), n)$.

Theorem 2. Any multidimensionally consistent system of octahedron type equations on $Q(A_N)$ can be reduced by an admissible transformation to a system whose restriction to any sublattice $Q(A_3)$ has one of the following forms:

$$x_{ij}x_{km} - x_{ik}x_{jm} + x_{jk}x_{im} = 0,$$

(χ₁)

$$\frac{(x_{im} - x_{ij})(x_{jm} - x_{jk})(x_{km} - x_{ik})}{(x_{ij} - x_{jm})(x_{jk} - x_{km})(x_{ik} - x_{im})} = -1,$$

(χ₂)

$$\frac{x_{ik} - x_{ij}}{x_{im}} + \frac{x_{ij} - x_{jk}}{x_{jm}} + \frac{x_{jk} - x_{ik}}{x_{km}} = 0,$$

(χ₃)

$$\frac{x_{ik} - x_{jk}}{x_{km}} = x_{ij} \left( \frac{1}{x_{jm}} - \frac{1}{x_{im}} \right),$$

(χ₅)

Moreover, each of equations (χ₁), (χ₂) is multidimensionally consistent with itself (that is, with the set of the same equations for all $Q(A_3)$ sublattices), while multidimensional sets involving any of the equations (χ₃), (χ₄) include with necessity (χ₂) on some of the $Q(A_3)$ sublattices, and multidimensional sets involving (χ₅) include with necessity (χ₄) on some of the $Q(A_3)$ sublattices. The detailed description of consistent sets is given in Theorem 21.

Remark 1. All these equations in $\mathbb{Z}^3$-form already appeared in the literature. They were derived by the direct linearization method in [9, 32, 31, 30, 14, 8]. First steps towards consistency of the dKP equation on a root lattice of type $A$ were made in [35, 34].

Remark 2. It is instructive to look at the (somewhat more traditional) $\mathbb{Z}^3$-form of equations from Theorem 2. Recall that they are obtained by forgetting one of the indices. The high symmetry grade of equations (χ₁), (χ₂) yields that forgetting any one of the indices leads to the same equation on $\mathbb{Z}^3$, namely

$$x_{ij}x_{k} - x_{ik}x_{j} + x_{jk}x_{i} = 0,$$

resp.

$$\frac{(x_i - x_{ij})(x_j - x_{jk})(x_k - x_{ik})}{(x_{ij} - x_j)(x_{jk} - x_k)(x_{ik} - x_i)} = -1.$$
As for equations \((\chi_3)-(\chi_5)\), one gets for each of them several seemingly different equations on \(\mathbb{Z}^3\), like
\[
\frac{x_{ik} - x_{ij}}{x_i} + \frac{x_{ij} - x_{jk}}{x_j} + \frac{x_{jk} - x_{ik}}{x_k} = 0
\]
and
\[
\frac{x_i - x_j}{x_k} + \frac{x_j - x_{ij}}{x_{jk}} + \frac{x_{ij} - x_i}{x_{ik}} = 0,
\]
both of which follow from \((\chi_4)\) by forgetting one of the indices. Without the unifying and symmetric \(Q(A_3)\) notation, it is not easy to recognize the equivalence of the latter two equations. For instance, the (non-commutative versions of) these equations are listed in \([30]\) as different equations (1.6) and (1.9).

**Remark 3.** One can get equations \((\chi_4), (\chi_5)\) from \((\chi_2)\) via simple limiting transitions. Performing in \((\chi_2)\) a non-autonomous point transformation \(x(n) \mapsto \epsilon^{-\alpha} x(n)\), which amounts to the replacement of \(x_{im}, x_{jm}, x_{km}\) by \(\epsilon^{-1} x_{im}, \epsilon^{-1} x_{jm}, \epsilon^{-1} x_{km}\), and then sending \(\epsilon \to 0\), we arrive at equation \((\chi_4)\). Analogously, performing in \((\chi_4)\) the point transformation \(x(n) \mapsto \delta^n x(n)\), which amounts to the replacement of \(x_{ik}, x_{jk}, x_{km}\) by \(\delta x_{im}, \delta x_{jm}, \delta x_{km}\), and then sending \(\delta \to 0\), we arrive at equation \((\chi_5)\). Of course, these limiting transitions do not belong to our group of admissible transformations.

The contents of the paper is as follows.

In Section 2 we give, for the sake of completeness, a simple and well known derivation of the dKP equation and the related octahedron type equations from the compatibility of auxiliary linear problems. The main result of the present work, Theorem 2 (or its detailed version Theorem 21), says that equations derived in this section exhaust the list of multidimensionally consistent equations of this type.

Section 3 is devoted to the definition of multidimensional consistency for the octahedron type equations. This definition is not quite straightforward, since the underlying lattice \(Q(A_3)\) is more sophisticated than the standard cubic one. The main problem is to find a suitable multidimensional lattice containing several copies of the \(Q(A_3)\) lattice which can simultaneously support generic solutions of the discrete octahedron type equations. The fact that this problem is not trivial is illustrated in Section 3.2 by a failure of one possible definition. A successful definition is then illustrated by the example of the dKP equation in Section 3.3 and formulated in full generality in Section 3.5. An elementary combinatorial cell (a 4-cell of the root lattice \(Q(A_4)\) with five octahedral faces) is best illustrated by the Desargues configuration, see Section 3.4.

Section 4 contains the central technical observation: each octahedron equation of the multidimensionally consistent system can be written in eight ways in the so called tripodal form, and the tripodal forms of equations on the adjacent octahedra must combine themselves in a very special way. Actually, the necessary conditions for consistency established in this section are the basis for the subsequent solution of the classification problem.

This solution starts in Section 5 where we classify all octahedron type equations admitting eight tripodal forms. This conditions turns out to be stringent enough to produces a finite list of equations, given in Theorem 7.

Finally, in Section 6 we combine tripodal equations into consistent systems, a complete list of which is given in Theorem 21.
2 Equations of octahedron type through linear problems

All equations from the list are related to each other via difference substitutions, and can be easily derived from simple linear problems like

\[ f_2 - f_1 = af, \quad f_3 - f_1 = bf. \]  \hspace{1cm} (11)

It should be noticed that all four faces of the tetrahedron with the vertices \( f, f_1, f_2, f_3 \) are on the same footing, because, due to (11), there hold also the further linear equations:

\[ f_3 - f_2 = cf, \quad f_2 - f_1 = d(f_3 - f_1), \]  \hspace{1cm} (12)

where \( c = b - a, \; d = a/b \). Any two out of the four equations (11), (12) are equivalent to (11).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1}
\caption{Linear equations on triangles.}
\end{figure}

It is easy to compute the compatibility conditions (which express the equality of two alternative expressions for \( f_{23} \) through the initial values \( f, f_1, f_{11} \), see Figure 1):

\[ a_3 + b_1 = b_2 + a_1, \quad a_3b = b_2a. \]  \hspace{1cm} (13)

These formulas should be interpreted as a map \((a, a_1, b, b_1) \mapsto (a_3, b_2)\). This map defines the evolution of the initial data consisting, for instance, of the values of \( a \) prescribed on the coordinate plane 12 and the values of \( b \) prescribed on the coordinate plane 13. Expressing in (13) \( a \) and \( b \) through \( f \) according to (11), we obtain an equation connecting 6 out of 8 values of \( f \) at the vertices of an elementary cube, of the type \( (\chi_4) \):

\[ \frac{f_{13} - f_{12}}{f_1} + \frac{f_{12} - f_{23}}{f_2} + \frac{f_{23} - f_{13}}{f_3} = 0. \]  \hspace{1cm} (14)

Thus, there naturally appears the decomposition of the cubic lattice into tetrahedra (which support the linear problem) and octahedra (which support the nonlinear equations).

Each equations in (13) can be interpreted as a conservation law and allows for an introduction of a potential. For instance, due to the first equation in (13) one can introduce a function \( \rho \) on the vertices of the cubic lattice, such that

\[ a = \rho_2 - \rho_1, \quad b = \rho_3 - \rho_1, \]
which brings the second equation in (13) into the form
\[(\rho_{23} - \rho_{13})(\rho_3 - \rho_1) = (\rho_{23} - \rho_{12})(\rho_2 - \rho_1),\]
which is equivalent to (\chi_3). Analogously, the second equation allows us to introduce a vertex function \(q\) such that
\[a = \frac{q_2}{q}, \quad b = \frac{q_3}{q},\]
which brings the first equation to the form
\[\frac{q_{12} - q_{13}}{q_1} = q_{23}\left(\frac{1}{q_3} - \frac{1}{q_2}\right),\]
which is nothing but (\chi_5). Finally, setting
\[q = \frac{\tau_1}{\tau} \Rightarrow a = \frac{\tau_{12} \tau}{\tau_1 \tau_2}, \quad b = \frac{\tau_{13} \tau}{\tau_1 \tau_3},\]
we can rewrite equation (16) as
\[(T_1 - 1) \left(\frac{\tau_{23} \tau_{13}}{\tau_1 \tau_{23}} - \frac{\tau_{32} \tau_{13}}{\tau_1 \tau_{23}}\right) = 0,
\]
which yields the Hirota equation (\chi_1)
\[\tau_{23} \tau_{13} - \tau_{32} \tau_{12} = c \tau_1 \tau_{23}.\]

In order to obtain equation (\chi_2), one has to start with the linear problems
\[\psi_2 - \psi = u(\psi_1 - \psi), \quad \psi_3 - \psi = v(\psi_1 - \psi).\]
These linear problems are not essentially different from (11): on one hand, they are gauge equivalent, and on the other hand, they have already appeared in (12), so that on lattices of a sufficiently large dimension both types of linear problems coexist. The condition of compatibility of (18) leads to the nonlinear equations
\[(u_3 - 1)(v - 1) = (v_2 - 1)(u - 1), \quad u_3 v_1 = v_2 u_1,\]
which are interpreted, as before, as a map \((u, u_1, v, v_1) \mapsto (u_3, v_2)\). Variables \(\psi\) satisfy equation of the type (\chi_2):
\[\frac{(\psi_1 - \psi_1)(\psi_2 - \psi_2)(\psi_3 - \psi_3)}{(\psi_2 - \psi_2)(\psi_3 - \psi_3)(\psi_3 - \psi_1)} = -1.\]
Like in the previous example, both equations (19) can be interpreted as conservation laws and allow for an introduction of a potential. However, this time these two equations are similar and lead both to the same equations. For instance, one can resolve the second equation in (19) by introducing a vertex function \(f\) according to
\[u = \frac{f_1}{f_2}, \quad v = \frac{f_1}{f_3},\]
and then the first equation in (19) leads again to (14).

One can add one more equation which appears via the substitution

\[ h = \frac{q_1}{q_3} = \frac{\tau_1 \tau_3}{\tau_1 \tau_2}, \]

which leads to

\[ (h_{12} - 1)(h_3 - 1) = h_2 h_{13}(1 - h_1^{-1})(1 - h_{23}^{-1}). \quad (21) \]

This equation will be discussed in Section 7 under the name \((Y)\). One can express the variables \(h\) directly through \(\psi\), by composing all intermediate transformations:

\[
\begin{align*}
\psi - \psi_1 & = \frac{1}{u - 1} \\
\psi - \psi_2 & = \frac{1}{v - 1} \\
\psi - \psi_3 & = \frac{\psi_2 - \psi_1}{\psi_3 - \psi_1} \\
\psi - \psi & = (\psi_2 - \psi_1)(\psi_3 - \psi)
\end{align*}
\]

It can be checked directly that this substitution turns equation (21) into a product of four copies of equation (20), at the original vertex and at its shifts in three coordinate directions.

Thus, all equations \((\chi_1)–(\chi_5)\) and (21) are related to one another in a rather simple manner:

\[
\begin{align*}
(15) & = (\chi_5) \\
(21) & = (Y) \\
(20) & = (\chi_2) \\
(14) & = (\chi_4) \\
(16) & = (\chi_5) \\
(17) & = (\chi_1)
\end{align*}
\]

3 Multidimensional consistency

3.1 Consistency for equations of the cube type

The idea behind the definition of multidimensional consistency given in Section 1 (Definitions 2, 3) is rather general and can be implemented for various types of discrete systems. However, it refers to the properties of equations on the whole multidimensional lattice, which is difficult to verify. It would be preferable to have some sufficient conditions for multidimensional consistency which refer to some local (finite) piece of the lattice.

For 3D equations of the cube type, like dBKp equation (7), such a local sufficient condition refers to one 4D cube, as follows (see, e.g., [3]). A solution of a non-degenerate equation of the cube type which is generic on each 3D sublattice can be defined by the initial data consisting of the values of \(x\) on all two-dimensional coordinate planes. Indeed, these data are obviously independent, and the inductive application of the equation allows one to extend the solution from the coordinate planes to the whole of \(\mathbb{Z}^N\) (compute \(x_{ijk}\) from the known \(x, x_i\) and \(x_{ij}\)), provided one does not encounter contradictions in this inductive process. It is sufficient to verify the lack of contradictions within one 4D cube, which is done as follows. Initial data within one 4D cube are:

\[ x, x_i, x_{ij} \ (1 \leq i < j \leq 4). \]
Application of equations on the four cubic faces adjacent to the vertex \( x \) yields the values \( x_{ijk} \) (\( 1 \leq i < j < k \leq 4 \)), and then application of equations on the four cubic faces adjacent to the vertex \( x_{1234} \) yields four \textit{a priori} different values for this last field. The 4D consistency takes place (and implies multidimensional consistency) if these four values identically coincide (thus, one has three conditions in terms of 11 initial data to be fulfilled). A more detailed discussion can be found in [3], where it was shown that the dBKP equation satisfies this criterium.

### 3.2 Lack of consistency for the dKP equation on the face centered cubic lattice

The type of initial value problem discussed for the cube type equations in the previous section is not applicable to 3D equations of the octahedron type, because of the absence of the fields \( x_{ijk} \) from the equations (10). Hence, the very notion of the multidimensional consistency in this concrete situation has to be modified.

Here we analyze a possible definition of 4D consistency for the dKP equation suggested by its original form (1). This definition turns out to be unsuccessful, so that this section is not necessary for the further reading, however we hope that it will clearly demonstrate the non-triviality of the problem of finding the suitable notion.

Equation (1) decomposes into two independent systems, one for the fields \( x : \mathbb{Z}^3_{\text{even}} \rightarrow \mathbb{C} \) on the so called even (or black) sublattice

\[
\mathbb{Z}^3_{\text{even}} = \left\{ m = (m_1, m_2, m_3) \in \mathbb{Z}^3 : m_1 + m_2 + m_3 \equiv 0 \pmod{2} \right\},
\]

and another one for the fields \( x : \mathbb{Z}^3_{\text{odd}} \rightarrow \mathbb{C} \) on the odd (or white) sublattice

\[
\mathbb{Z}^3_{\text{odd}} = \left\{ m = (m_1, m_2, m_3) \in \mathbb{Z}^3 : m_1 + m_2 + m_3 \equiv 1 \pmod{2} \right\} ;
\]

it will be enough to consider the half defined on \( \mathbb{Z}^3_{\text{even}} \) (say). The latter is known as the face centered cubic (fcc) lattice. Its set of vertices is clearly in a one-to-one correspondence to \( \mathbb{Z}^3 \), but its (Delaunay) cell structure is quite different. Its 2-cells are equilateral triangles, while its 3-cells are octahedra and tetrahedra. Equation (1) relates six fields assigned to any elementary octahedron of \( \mathbb{Z}^3_{\text{even}} \), centered at \( m \in \mathbb{Z}^3_{\text{odd}} \). Two octahedra with a nonempty intersection either share an edge (those whose centers are neighbors in \( \mathbb{Z}^3_{\text{odd}} \)) or either share a vertex (those whose centers are diagonal neighbors in \( \mathbb{Z}^3_{\text{odd}} \), i.e., are at distance 2 from one another).

One could attempt to define the multidimensional consistency of octahedron type equations by imposing them on all \( \mathbb{Z}^3_{\text{even}} \) sublattices in

\[
\mathbb{Z}^N_{\text{even}} = \left\{ m = (m_1, \ldots, m_N) \in \mathbb{Z}^N : m_1 + \ldots + m_N \equiv 0 \pmod{2} \right\}.
\]

However, this idea turns out to be invalid. Indeed, one can show the inconsistency of two copies of (1), corresponding to the sublattices (123) and (124):

\[
x(m + e_3)x(m - e_3) - ax(m + e_1)x(m - e_1) + bx(m + e_2)x(m - e_2) = 0, \quad (22)
\]

\[
x(m + e_4)x(m - e_4) - cx(m + e_1)x(m - e_1) + dx(m + e_2)x(m - e_2) = 0, \quad (23)
\]

10
where \(a, b, c, d\) are arbitrary constants. As initial data for these two equations one can take the values of \(x\) at four two-dimensional planes parallel to the coordinate plane 12:

\[
x(m_1, m_2, 0, 0), \ x(m_1, m_2, -1, 0), \ x(m_1, m_2, 0, -1), \ x(m_1, m_2, -1, -1).
\]

These data are free, in the sense that they are not subject to any equation among (22), (23), or remaining two equations corresponding to the sublattices 134 and 234, since any of these equations contains at least one pair of points differing by \(2\varepsilon_3\) or by \(2\varepsilon_4\). In particular, consider the following points in these two planes:

(\(m_1, m_2\) = \(\pm 2, 0\), \(1, \pm 1\), \(0, 0\), \(-1, \pm 1\), \(0, \pm 2\) with \((m_3, m_4) = (0, 0)\);

(\(m_1, m_2\) = \(\pm 1, 0\), \(0, \pm 1\) with \((m_3, m_4) = (-1, 0), (0, -1)\);

(\(m_1, m_2\) = \(0, 0\) with \((m_3, m_4) = (-1, -1)\).

These are round points on Fig. 2. Now equations (22), (23) allow us to compute \(x\) at the triangular points

(\(0, 0, 1, -1\), \(0, 0, -1, 1\), \(\pm 1, 0, 1, 0\), \(0, \pm 1, 1, 0\), \(\pm 1, 0, 0, 1\), \(0, \pm 1, 0, 1\),

and then two different values at the square point \((0, 0, 1, 1)\). It can be checked that, for any choice of the non-vanishing coefficients \(a, b, c, d\), these two values of \(x(0, 0, 1, 1)\) do not coincide identically (as functions of the initial data).

**Figure 2.** Consistency check for equations (22), (23) leads to a contradiction at the uppermost vertex.

Note that two neighboring octahedra in different three-dimensional lattices \(\mathbb{Z}^3\) with two common coordinate directions (as those underlying equations (22), (23)) do not share triangular faces; rather, they share two pairs of antipodal points, or an 2D equatorial square-formed section spanned by these points.
3.3 Consistency of the dKP equation on a 4D cube

Here we prove a local statement about the 4D consistency for the $\mathbb{Z}^3$ version (6) of the dKP equation, which deals with the octahedra contained within one elementary 4D cube of $\mathbb{Z}^4$. Each of the eight 3D cubic faces of the 4D cube contains such an octahedron, so that we have the eight octahedra

$$[jkm] = \{x_j, x_k, x_m, x_{jk}, x_{jm}, x_{km}\} \quad \text{and} \quad T_i[jkm] = \{x_{ij}, x_{ik}, x_{im}, x_{ijk}, x_{ijm}, x_{ikm}\}$$

(where \{i, j, k, m\} = \{1, 2, 3, 4\}). Neither of the equations on these octahedra involves the fields $x$ and $x_{1234}$, so that only 14 out of 16 vertices are involved. It turns out that one can take 9 of them as independent initial data, for instance, $x_1$, $x_2$, $x_3$, $x_4$, $x_{14}$, $x_{24}$, $x_{34}$, $x_{134}$, $x_{234}$.

and to find the remaining 5 fields using only three of the equations, namely those for the octahedra [124], [134], [234]:

$$x_{12}x_4 - x_{14}x_2 + x_{24}x_1 = 0,$$

$$x_{13}x_4 - x_{14}x_3 + x_{34}x_1 = 0,$$

$$x_{23}x_4 - x_{24}x_3 + x_{34}x_2 = 0,$$

(24)

and for their shifted copies $T_3[124]$, $T_2[134]$, $T_1[234]$:

$$x_{123}x_{34} - x_{134}x_{23} + x_{234}x_{13} = 0,$$

$$x_{123}x_{24} - x_{124}x_{23} + x_{234}x_{12} = 0,$$

$$x_{123}x_{14} - x_{124}x_{13} + x_{134}x_{12} = 0.$$

(25)

We use equations (24) to determine $x_{12}$, $x_{13}$, $x_{23}$. Then, we use the first equation in (25) to determine $x_{123}$. Finally, we have two alternative answers for $x_{124}$ which come from the last two equations in (25). We now show that these two values coincide as functions of the initial data, so that the 4D consistency takes place. The following computations might make an impression of a repeated application of a certain skillful trick, but we will show later that this trick is actually a key structural feature common for all consistent octahedron type equations, namely the so called tripodal form of these equations.

First of all, we show that the values of $x_{12}$, $x_{13}$, $x_{23}$ determined from (24), together with the initial data $x_1$, $x_2$, $x_3$, automatically satisfy the dKP equation on the octahedron [123]. To this end, we rewrite (24) as

$$\frac{x_{12}}{x_1x_2} - \frac{x_{14}}{x_1x_4} + \frac{x_{24}}{x_2x_4} = 0,$$

$$\frac{x_{13}}{x_1x_3} - \frac{x_{14}}{x_1x_4} + \frac{x_{34}}{x_3x_4} = 0,$$

$$\frac{x_{23}}{x_2x_3} - \frac{x_{24}}{x_2x_4} + \frac{x_{34}}{x_3x_4} = 0.$$

(26)

An obvious linear combination of these equations immediately leads to

$$\frac{x_{12}}{x_1x_2} - \frac{x_{13}}{x_1x_3} + \frac{x_{23}}{x_2x_3} = 0,$$

12
which is equivalent to
\[ x_{12}x_3 - x_{13}x_2 + x_{23}x_1 = 0. \] (27)

Second, we show that the values of \( x_{12}, x_{13}, x_{23} \) determined from (24), together with \( x_{14}, x_{24}, x_{34} \), automatically satisfy an equation which literally coincides with dKP. For this aim, we rewrite equations (24) in another equivalent form:

\[
\begin{align*}
\frac{x_{12}}{x_{14}x_{24}} - \frac{x_2}{x_{4}x_{24}} + \frac{x_1}{x_{4}x_{14}} &= 0, \\
\frac{x_{13}}{x_{14}x_{34}} - \frac{x_3}{x_{4}x_{34}} + \frac{x_1}{x_{4}x_{14}} &= 0, \\
\frac{x_{14}x_3}{x_{23}} - \frac{x_{4}x_3}{x_{3}} + \frac{x_2}{x_3} &= 0.
\end{align*}
\] (28)

A suitable linear combination of these equations leads to

\[
\begin{align*}
\frac{x_{12}}{x_{14}x_{24}} - \frac{x_1}{x_{14}x_{34}} + \frac{x_3}{x_{24}x_{34}} &= 0,
\end{align*}
\]

which is equivalent to
\[ x_{12}x_{34} - x_{13}x_{24} + x_{23}x_{14} = 0. \] (29)

Finally, we show the 4D consistency. The first equation in (25) used to determine \( x_{123} \) is equivalent to

\[
\begin{align*}
\frac{x_{34}}{x_{13}x_{23}} - \frac{x_{134}}{x_{13}x_{123}} + \frac{x_{234}}{x_{23}x_{123}} &= 0.
\end{align*}
\] (30)

The last two equations in (25) used to determine \( x_{124} \) are equivalent to

\[
\begin{align*}
\frac{x_{24}}{x_{12}x_{23}} - \frac{x_{124}}{x_{12}x_{123}} + \frac{x_{234}}{x_{13}x_{123}} &= 0, \\
\frac{x_{14}}{x_{12}x_{13}} - \frac{x_{124}}{x_{12}x_{123}} + \frac{x_{34}}{x_{13}x_{123}} &= 0.
\end{align*}
\] (31)

They give the same value of \( x_{124} \) if and only if

\[
\frac{x_{14}}{x_{12}x_{13}} - \frac{x_{24}}{x_{12}x_{23}} + \frac{x_{34}}{x_{13}x_{23}} - \frac{x_{234}}{x_{23}x_{123}} = 0.
\]

Combining the latter equation with (30), we see that the 4D consistency condition is reduced to

\[
\begin{align*}
\frac{x_{14}}{x_{12}x_{13}} - \frac{x_{24}}{x_{12}x_{23}} + \frac{x_{34}}{x_{13}x_{23}} &= 0,
\end{align*}
\]

which is equivalent to the already proven equation (29).

### 3.4 Cell structure of the lattice \( Q(A_N) \)

Considerations of Section 3.3 give a hint towards a valid formulation of the notion of multidimensional consistency for the octahedron type equations. In particular, equation (29) for the even shifts calls for an interpretation as an equation on a further octahedron present in the 4D lattice. Such interpretation is enabled by an embedding of the lattice \( Q(A_3) \) into the root lattice \( Q(A_N) \), given in (9). We start with a short description of the Delaunay cell structure of the four-dimensional lattice \( Q(A_4) \) \([10, 29]\). We do not go into detail and give an elementary description appropriate for our purposes. For each \( N \) there are \( N \) sorts of \( N \)-cells of \( Q(A_N) \) denoted by \( P(k, N), k = 1, \ldots, N \).
Two sorts of 2-cells:

\(P(1, 2)\): black triangles \(\{x_i, x_j, x_k\}\),

\(P(2, 2)\): white triangles \(\{x_{ij}, x_{ik}, x_{jk}\}\);

Three sorts of 3-cells:

\(P(1, 3)\): black tetrahedra \(\{x_i, x_j, x_k, x_\ell\}\), with all four facets being black triangles,

\(P(2, 3)\): octahedra \(\langle i \rangle = \{x_{ij}, x_{ik}, x_{i\ell}, x_{jk}, x_{j\ell}, x_{k\ell}\}\), the eight triangular facets of each octahedron being bi-colored, consult Fig. 3;

\(P(3, 3)\): white tetrahedra \(\{x_{ijk}, x_{ij\ell}, x_{i\ell k}, x_{jk\ell}\}\), with all four facets being white triangles;

Four sorts of 4-cells:

\(P(1, 4)\): black 4-simplices \(\{x_i, x_j, x_k, x_\ell, x_m\}\), with all five facets being black tetrahedra,

\(P(2, 4)\): 4-ambo-simplices, in the terminology of [10], with the ten vertices \(x_{ij}\), \(i, j \in \{0, 1, \ldots, 4\}\); each such polytope has five octahedral facets \(\langle i \rangle = \{jk\ell m\}\) and five black tetrahedral facets \(T_i\{x_j, x_k, x_\ell, x_m\}\);

\(P(3, 4)\): 4-ambo-simplices with the ten vertices \(x_{ijk}\), \(i, j, k \in \{0, 1, 2, 3, 4\}\); each of these polytopes has five octahedral facets \(T_i\{jk\ell m\}\) and five white tetrahedral facets;

\(P(4, 4)\): white 4-simplices \(\{x_{ij\ell k}, x_{ijkm}, x_{ij\ell m}, x_{i\ell km}, x_{jk\ell m}\}\), with all five facets being white tetrahedra.

Figure 3. Vertex enumeration of the octahedron \(\langle i \rangle = \{jk\ell m\}\), where \(i \in \{0, 1, 2, 3, 4\}\) and \(\{j, k, \ell, m\} = \{0, 1, 2, 3, 4\} \setminus \{i\}\). Opposite vertices carry complementary pairs of indices. Facets are bi-colored; there are four white triangles like \(\{jk, j\ell, k\ell\}\) missing one of the indices (\(m\) in this case), and four black triangles like \(\{jk, j\ell, jm\}\) = \(T_j\{k, \ell, m\}\) sharing one common index (\(j\) in this case).
The combinatorial description of higher dimensional cells in the root lattices $Q(A_N)$ with bigger $N$ is given analogously.

The affine Weyl group is generated by permutations of indices and translations. It permutes the cells of the same sort.

We finish this section with a suggestive description of the admittedly somewhat complicated combinatorics of a 4-ambo-simplex $P(2, 4)$ whose five octahedral faces $⟨0⟩$, $⟨1⟩$, $⟨2⟩$, $⟨3⟩$, $⟨4⟩$ carry a quintuple of consistent octahedron type equations. Remarkably, this description not only has a combinatorial meaning, but also has a direct relation to the multidimensional consistency of equation $(\chi_2)$, explained in [7, p. 285]. Consider a map $x : Q(A_N) \to \mathbb{R}P^n$ satisfying the following condition: the image of any white triangle $\{x_{ij}, x_{ik}, x_{jk}\}$ is a collinear triple of points. Such maps were introduced in the three-dimensional situation by Schief [33] under the name “Laplace-Darboux lattices”. He also observed the relation of their four-dimensional consistency to the Desargues theorem (private communication; see [2, 6]). These maps (under the name “Desargues maps”) are studied in detail in the recent work by Doliwa [12, 13]. A connection of the Desargues theorem to equation $(\chi_2)$ appeared in [1].

It is easy to realize that the image of an octahedron $[ijkm]$ is a complete quadrilateral, as on Fig. 4. It contains four lines which are images of the white triangular faces $\{x_{ij}, x_{ik}, x_{jk}\}$, and four triangles which are images of the black triangular faces $T_i\{x_j, x_k, x_m\}$. An analytic description of the complete quadrilateral is given, according to the classical Menelaus theorem, by equation $(\chi_2)$.

![Figure 4](image_url)

Figure 4. Geometric interpretation of equation $(\chi_2)$: a Menelaus configuration, or a complete quadrilateral, is an image of the octahedron $[0123]$; four lines correspond to the white triangles $\{jk, j\ell, k\ell\}$, while four triangles correspond to the black triangles $\{jk, j\ell, jm\}$.

The image of the 4-ambo-simplex $P(2, 4)$ is then a configuration like the one on Fig. 5. One can recognize here five complete quadrilaterals, corresponding to the octahedra $⟨i⟩ = [jk\ell m]$, as well as the images of the five black tetrahedra $T_i\{x_j, x_k, x_\ell, x_m\}$. The ten lines are the images of the ten white triangular faces of the octahedra. The configuration on Fig. 5 illustrates one of the most important incidence theorems of the classical projective geometry – the Desargues theorem. Its five-fold symmetry is not obvious from the first glance (and from the original formulation of the theorem), but it was well known to the
classics, see, e.g., [19].

Figure 5. The image of a cell $P(2,4)$ – Desargues configuration 103. Ten lines correspond to the the white triangles $\{jk, j\ell, k\ell\}$. One can clearly recognize the images of the five black tetrahedra $\{ij, ik, i\ell, im\} = T_i \{j, k, \ell, m\}$, and five complete quadrilaterals, which are images of the octahedral faces $\langle i \rangle = [jk\ell m]$, each one has six vertices missing one of the indices ($i$ in this case).

Remark 4. The image of the 4-ambo-simplex $P(3,4)$ turns out to be less interesting. It is a configuration 10254 in the projective plane, i.e., five lines in general position with their ten pairwise intersection points, see Fig. 6. This figure does not support any non-trivial incidence theorem.

3.5 Consistent triples of octahedron type equations

A local definition of the 4D consistency of octahedron type equations should deal with the 4-cells of $Q(A_4)$ possessing octahedral faces, i.e., with 4-ambo-simplices. We consider in more detail an ambo-simplex whose tetrahedral faces are black. Its five octahedral faces

$$\langle 0 \rangle, \langle 1 \rangle, \langle 2 \rangle, \langle 3 \rangle, \langle 4 \rangle,$$

are characterized by the property that any two of them share a triangular face. Two 4-ambo-simplices are said to be adjacent if they share a common octahedral face. For instance, for any fixed $i = 0, 1, \ldots, 4$, the octahedra

$$T_0 T_i^{-1} \langle 0 \rangle, T_1 T_i^{-1} \langle 1 \rangle, T_2 T_i^{-1} \langle 2 \rangle, T_3 T_i^{-1} \langle 3 \rangle, T_4 T_i^{-1} \langle 4 \rangle$$

serve as the faces of a 4-ambo-simplex adjacent to the previous one, with the common octahedron $\langle i \rangle$.

In the rest of this section, we will work with the representation of the lattice $Q(A_4)$ as $\mathbb{Z}^4$, which corresponds to the representation of the octahedron type equations in the form (8). This representation appears through a projection from $\mathbb{Z}^5$ to $\mathbb{Z}^4$ along the 0-th coordinate axis (that is, by forgetting the index $n_0$). It is useful to remember that the
Figure 6. The image of a cell $P(3, 4)$ – a configuration $10_25_4$. Five lines correspond to the five white tetrahedra $\{ijk, ij\ell, i\ell k, jk\ell\}$. One can also recognize five complete quadrilaterals, which are images of the octahedral faces $\mathcal{T}_i(i)$, each one has six vertices sharing one of the indices ($i$ in this case).

Shift operators $T_i$ in the $\mathbb{Z}^4$ representation of $Q(A_4)$ stand for the operators $T_iT_0^{-1}$ in the standard representation (9) of $Q(A_4)$. As already mentioned, we will consider triples of octahedron type equations corresponding to three 3D sublattices of $\mathbb{Z}^4$, say the sublattices (124), (134), (234):

$$
\begin{align*}
F(x_1, x_2, x_3, x_{12}, x_{14}, x_{24}) &= 0, \\
G(x_1, x_3, x_4, x_{13}, x_{14}, x_{34}) &= 0, \\
H(x_2, x_3, x_4, x_{23}, x_{24}, x_{34}) &= 0.
\end{align*}
$$

(32)

An elementary combinatorial structure supporting these equations is a triple of octahedra

$$(3), (2), (1),$$

(33)

which are just three out of five octahedral faces of a 4-ambo-simplex (in particular, any two octahedra of a triple share a triangular face). We will say that another triple of octahedra of the same coordinate directions is adjacent to the triple (33) if they are faces of an adjacent 4-ambo-simplex, and the common octahedral face of the 4-ambo-simplices belongs to neither of the triples. Thus, for two adjacent triples there is a further (seventh) octahedron sharing a face with any octahedron of the both triples. We will say that the both triples of octahedra are adjacent along the seventh one. For instance, the triple

$$
T_3(3), T_2(2), T_1(1),
$$

(34)

is adjacent to triple (33) along the octahedron (0), and the triple

$$
T_3T_4^{-1}(3), T_2T_4^{-1}(2), T_1T_4^{-1}(1),
$$

(35)

is adjacent to triple (33) along the octahedron (4). We are now in a position to formulate a local definition of 4D consistency of the octahedron type equations.
**Definition 4.** A triple of the octahedron type equations (32) is called 4D consistent if they can be imposed (admit generic solutions) on any two adjacent triples of octahedra.

Thus, Definition 4 requires to consider equations (32) on the triple of octahedra (33) and on its two adjacent triples, (34) and (35).

To verify the first requirement of Definition 4, one can consider the same set of 9 initial data as used in Section 3.3 for the dKP equation, namely

\[ x_1, x_2, x_3, x_4, x_{14}, x_{24}, x_{34}, x_{134}, x_{234}. \]  

Equations for the three octahedra (33) determine the fields \( x_{12}, x_{13}, x_{23} \):

\[ x_{12} = f(x_1, x_2, x_4, x_{14}, x_{24}), \]
\[ x_{13} = g(x_1, x_3, x_4, x_{14}, x_{34}), \]
\[ x_{23} = h(x_2, x_3, x_4, x_{24}, x_{34}), \]

then equation for the octahedron \( T_3(3) \) determines the field \( x_{123} \):

\[ x_{123} = \hat{f}(x_1, x_{13}, x_{23}, x_{134}, x_{234}). \]

and finally two equations for the octahedra \( T_2(2) \) and \( T_1(1) \) deliver two a priori different values for \( x_{124} \):

\[ x_{124} = \hat{g}(x_{12}, x_{23}, x_{24}, x_{123}, x_{234}) = \hat{h}(x_{12}, x_{13}, x_{14}, x_{123}, x_{134}). \]

The first requirement in Definition 4 is that these two values of \( x_{124} \) identically coincide as functions of the 9 initial data (36).

One can proceed similarly to verify the second requirement of Definition 4: consider the set of 9 initial data

\[ x_1, x_2, x_3, x_4, x_{14}, x_{24}, x_{34}, x_{134}, x_{234}, x_{134}, x_{234}. \]  

One starts with equations (37) for the three octahedra (33), solves equation for the octahedron \( T_3^{-1}(3) \) to determine the field \( x_{123,4} \):

\[ x_{123,4} = \hat{f}(x_3, x_{13}, x_{23}, x_{134}, x_{234}), \]

and finally has two equations for the octahedra \( T_2T_4^{-1}(2) \) and \( T_1T_4^{-1}(1) \) which give two a priori different values for \( x_{12,4} \):

\[ x_{12,4} = \hat{g}(x_2, x_{12}, x_{23}, x_{23,4}, x_{123,-4}) = \hat{h}(x_1, x_{12}, x_{13}, x_{13,-4}, x_{123,-4}). \]

The second requirement in Definition 4 is that the two values of \( x_{12,4} \) coincide as functions of the 9 initial data (40).

For the formulation of the following statements we use the following convention: the derivatives are denoted by the lower indices:

\[ f_1 = \frac{\partial f}{\partial x_1}, \ldots, \quad h_{34} = \frac{\partial h}{\partial x_{34}} \]

(since we will not need higher order derivatives, the multiple indices should cause no misunderstanding).
Proposition 3. If the triple of the octahedron type equations (37) is 4D consistent, then the following holds:

\[
\begin{align*}
\text{rk} \begin{pmatrix} f_1 & f_2 & 0 & f_4 \\ g_1 & 0 & g_3 & g_4 \\ 0 & h_2 & h_3 & h_4 \end{pmatrix} & \leq 2, \\
\text{rk} \begin{pmatrix} f_4 & f_{14} & f_{24} & 0 \\ g_4 & g_{14} & 0 & g_{34} \\ 0 & h_4 & 0 & h_{24} \end{pmatrix} & \leq 2.
\end{align*}
\] (43)

Proof. The first claim follows from the consistency of equations for the triples (33), (34), while the second follows from the consistency of equations for the triples (33), (35). Each of the matrices in (43) contains the derivatives with respect to the fields not belonging to the octahedron along which the corresponding triple is adjacent. Both statements are verified similarly, therefore we only prove the first one. We differentiate condition \( \hat{g} = \hat{h} \) in (39) with respect to the initial data:

\[
\begin{align*}
\partial_{x_1} : & \quad (\hat{g}_{12} f_1 + \hat{g}_{123} \hat{f}_{13} g_1 = \hat{h}_{12} f_1 + \hat{h}_{13} g_1 + \hat{h}_{123} \hat{f}_{13} g_1, \\
\partial_{x_2} : & \quad (\hat{g}_{12} f_2 + \hat{g}_{23} h_2 + \hat{g}_{123} \hat{f}_{23} h_2 = \hat{h}_{12} f_2 + \hat{h}_{123} \hat{f}_{23} h_2, \\
\partial_{x_3} : & \quad (\hat{g}_{23} h_3 + \hat{g}_{123} (f_{13} g_3 + f_{23} h_3) = \hat{h}_{123} (f_{13} g_3 + f_{23} h_3), \\
\partial_{x_4} : & \quad (\hat{g}_{12} f_4 + \hat{g}_{23} h_4 + \hat{g}_{123} (f_{13} g_4 + f_{23} h_4) = \hat{h}_{12} f_4 + \hat{h}_{13} g_4 + \hat{h}_{123} (f_{13} g_4 + f_{23} h_4).
\end{align*}
\]

These equations say that the row vector

\[
(\hat{g}_{12} - \hat{h}_{12}, \quad \hat{h}_{13} + (\hat{g}_{123} - \hat{h}_{123}) \hat{f}_{13}, \quad \hat{g}_{23} + (\hat{g}_{123} - \hat{h}_{123}) \hat{f}_{23})
\]

belongs to the left kernel of the matrix

\[
\begin{pmatrix} f_1 & f_2 & 0 & f_4 \\ g_1 & 0 & g_3 & g_4 \\ 0 & h_2 & h_3 & h_4 \end{pmatrix},
\]

so that the latter matrix has rank \( \leq 2 \).

Proposition 4. If the triple of the octahedron type equations (37) is consistent, then some octahedron type equations are automatically fulfilled on the sublattices \( \langle 4 \rangle = (123) \) and \( \langle 0 \rangle = (1234) \):

\[
K(x_1, x_2, x_3, x_{12}, x_{13}, x_{23}) = 0
\]

(44)

and

\[
L(x_{12}, x_{13}, x_{23}, x_{14}, x_{24}, x_{34}) = 0.
\]

(45)

Proof. The fact that

\[
\text{rk} \begin{pmatrix} f_1 & f_2 & 0 & f_4 \\ g_1 & 0 & g_3 & g_4 \\ 0 & h_2 & h_3 & h_4 \end{pmatrix} \leq 2
\]

can be reformulated as follows: if we consider (37) as a system for the unknowns \( x_1, x_2, x_3, x_4 \) and solve the first two equations of this system for \( x_1, x_4 \), then the substitution of the result into the last equation will cancel out \( x_2, x_3 \) identically. This is precisely statement (45). Relation (44) follows similarly. Thus, consistency of a triple of equations on two adjacent triples of octahedra yields that the vertices of the connecting octahedron also satisfy certain equation.
Thus, the 4D consistency of a triple of the octahedron type equations yields that actually all coordinate directions are on the same footing: each of the five root lattices $Q(A_3)$ contained in $Q(A_4)$ carries its own equation of the octahedron type.

**Remark 5.** It is not clear how to extend the initial data (36) or (40) in order to get a set of independent initial data for the whole 4D lattice $\mathbb{Z}^4$. A possible formulation applicable to the whole of $\mathbb{Z}^4$ is the following: the initial data consist of the values of $x$ on the coordinate 2D planes $(i4)$ for all $i = 1, 2, 3$ (they intersect along the coordinate axis 4). Within two neighboring 4D cubes, one has the following set of 11 independent initial data:

$$x_1, x_2, x_3, x_4, x_{14}, x_{24}, x_{34}, x_{144}, x_{244}, x_{344}$$

(46)

(the last four data coming from the second 4D cube). One can determine $x_{12}, x_{13}, x_{23}$ from equations (37), and then $x_{124}, x_{134}, x_{234}$ from equations (37) shifted in the 4-th coordinate direction: $x_{124} = f(x_{14}, x_{24}, x_{34}, x_{144}, x_{244}), T_4(f)$, etc. The 4D consistency conditions appear from the comparison of the three values of $x_{123}$ from (38) that must coincide identically as functions of the initial data:

$$x_{123} = f(g, h, x_{34}, T_4(g), T_4(h)) = g(f, h, x_{24}, T_4(f), T_4(h)) = h(f, g, x_{14}, T_4(f), T_4(g)).$$

(47)

We will show that this notion of 4D consistency quickly leads to the same necessary conditions formulated in Proposition 3. Denote the exterior functions $f, g, h$ in (47), as well as their derivatives, by the bar, so that, for instance,

$$\bar{f} = T_3(f) = f(g, h, x_{34}, T_4(g), T_4(h)), \quad \bar{f}_1 = T_3(f_1).$$

The following equations are obtained by differentiating (47):

$$\begin{align*}
\partial x_1 &: \quad \bar{f}_1g_1 = \bar{g}_1f_1 = \bar{h}_2f_1 + \bar{h}_3g_1, \\
\partial x_2 &: \quad \bar{f}_2h_2 = \bar{g}_1f_2 + \bar{g}_3h_2 = \bar{h}_2f_2, \\
\partial x_3 &: \quad \bar{f}_3g_3 + \bar{f}_2h_3 = \bar{g}_3h_3 = \bar{h}_3g_3, \\
\partial x_4 &: \quad \bar{f}_1g_4 + \bar{f}_2h_4 = \bar{g}_1f_4 + \bar{g}_3h_4 = \bar{h}_2f_4 + \bar{h}_3g_4, \\
\partial x_{14} &: \quad \bar{f}_1T_4(g_4) + \bar{f}_2T_4(h_4) = \bar{g}_1T_4(f_4) + \bar{g}_3T_4(h_4) = \bar{h}_2T_4(f_4) + \bar{h}_3T_4(g_4), \\
\partial x_{144} &: \quad \bar{f}_1T_4(g_{14}) = \bar{g}_1T_4(f_{14}) = \bar{h}_2T_4(f_{14}) + \bar{h}_3T_4(g_{14}), \\
\partial x_{24} &: \quad \bar{f}_2T_4(h_{24}) = \bar{g}_1T_4(f_{24}) + \bar{g}_3T_4(h_{24}) = \bar{h}_2T_4(f_{24}), \\
\partial x_{34} &: \quad \bar{f}_1T_4(g_{34}) + \bar{f}_2T_4(h_{34}) = \bar{g}_3T_4(h_{34}) = \bar{h}_3T_4(g_{34}).
\end{align*}$$

(49)

Now equations (48) say that the row vectors

$$(-\bar{g}_1, \quad \bar{f}_1, \quad \bar{f}_2 - \bar{g}_3) \quad \text{and} \quad (\bar{g}_1 - \bar{h}_2, \quad -\bar{h}_3, \quad \bar{g}_3)$$

belong to the left kernel of the matrix

$$\begin{pmatrix}
\bar{f}_1 & \bar{f}_2 & 0 & \bar{f}_4 \\
\bar{g}_1 & 0 & \bar{g}_3 & \bar{g}_4 \\
0 & \bar{h}_2 & \bar{h}_3 & \bar{h}_4
\end{pmatrix}. $$

20
Analogously, equations (49) say that the row vectors
\((-\bar{g}_{14}, \ f_{14}, \ f_{24} - \bar{g}_{34})\) and \((\bar{g}_{14} - \bar{h}_{24}, \ -\bar{h}_{34}, \ \bar{g}_{34})\)
belong to the left kernel of the \((T_{4}\text{-shifted})\) matrix
\[
\begin{pmatrix}
  f_4 & f_{14} & f_{24} & 0 \\
  g_4 & g_{14} & 0 & g_{34} \\
  h_4 & 0 & h_{24} & h_{34}
\end{pmatrix}.
\]
Thus, both matrices have rank \(\leq 2\).

4 Tripodal forms of the octahedron type equations

We now derive further analytic consequences of the necessary conditions of 4D consistency formulated in Proposition 3, which can be formulated as
\[
f_1 g_3 h_2 + f_2 g_1 h_3 = 0, \quad f_2 g_3 h_4 = f_4 g_3 h_2 + f_2 g h_3, \quad (50)
\]
and
\[
f_{14} g_{34} h_{24} + f_{24} g_{14} h_{34} = 0, \quad f_{24} g_{34} h_4 = f_4 g_{34} h_{24} + f_{24} g h_{34}, \quad (51)
\]
respectively.

**Proposition 5.** If the triple of octahedron type equations (37) is 4D consistent, then it can be cast into the form
\[
\begin{align*}
  a(x_1, x_4, x_{14}) - b(x_2, x_4, x_{24}) &= p(x_{12}, x_{14}, x_{24}), \\
  c(x_3, x_4, x_{34}) - a(x_1, x_4, x_{14}) &= q(x_{13}, x_{14}, x_{34}), \\
  b(x_2, x_4, x_{24}) - c(x_3, x_4, x_{34}) &= r(x_{23}, x_{24}, x_{34}),
\end{align*} \quad (52)
\]
and simultaneously into the form
\[
\begin{align*}
  A(x_1, x_4, x_{14}) - B(x_2, x_4, x_{24}) &= P(x_{12}, x_{14}, x_{24}), \\
  C(x_3, x_4, x_{34}) - A(x_1, x_4, x_{14}) &= Q(x_{13}, x_{14}, x_{34}), \\
  B(x_2, x_4, x_{24}) - C(x_3, x_4, x_{34}) &= R(x_{23}, x_{24}, x_{34}).
\end{align*} \quad (53)
\]

**Proof.** We prove that the general solution of equations (50) is of the form
\[
\begin{align*}
  f &= \phi(a(x_1, x_4, x_{14}) - b(x_2, x_4, x_{24}), x_{14}, x_{24}), \\
  g &= \psi(c(x_3, x_4, x_{34}) - a(x_1, x_4, x_{14}), x_{14}, x_{34}), \\
  h &= \chi(b(x_2, x_4, x_{24}) - c(x_3, x_4, x_{34}), x_{24}, x_{34}),
\end{align*}
\]
which is obviously equivalent to (52). First equation in (50) implies
\[
\begin{align*}
  \frac{f_1}{f_2}(x_1, x_2, x_4, x_{14}; x_{24}) &= -\frac{g_1 / g_3 (x_1, x_3, x_4, x_{14}, x_{34})}{h_2 / h_3 (x_2, x_3, x_{24}, x_{34})} = -\frac{\alpha(x_1, x_4, x_{14})}{\beta(x_2, x_4, x_{24})}
\end{align*}
\]
(it is sufficient to choose some fixed values of $x_3$ and $x_{34}$ in the middle expression). Now,

\[
\frac{\alpha g_3}{g_1} = \frac{\beta h_3}{h_2} = -\gamma(x_3, x_4, x_{34}),
\]

where $\gamma$ denotes the common value of both ratios. Clearly, $\alpha, \beta, \gamma$ are defined up to a common factor, possibly depending on $x_4$. We arrive at

\[
\beta f_1 + \alpha f_2 = 0, \quad \alpha g_3 + \gamma g_1 = 0, \quad \gamma h_2 + \beta h_3 = 0.
\]

Choosing some $a = a(x_1, x_4, x_{14}), \ b = b(x_2, x_4, x_{24}), \ c = c(x_3, x_4, x_{34})$ so that $\alpha = a_1, \ \beta = b_2, \ \gamma = c_3$, we come to the conclusion that the functions $f, g, h$ can be represented as

\[
f = \phi(a(x_1, x_4, x_{14}) - b(x_2, x_4, x_{24}), x_4, x_{14}, x_{24}),
\]

\[
g = \psi(c(x_3, x_4, x_{34}) - a(x_1, x_4, x_{14}), x_4, x_{14}, x_{34}),
\]

\[
h = \chi(b(x_2, x_4, x_{24}) - c(x_3, x_4, x_{34}), x_4, x_{24}, x_{34}).
\]

Substituting this into the second equation in (50), one finds:

\[
\frac{\phi_4}{\phi'} + \frac{\psi_4}{\psi'} + \frac{\chi_4}{\chi'} = 0, \tag{54}
\]

where prime denotes the derivatives with respect to the first arguments of the functions. Differentiating this equation with respect to $x_1$ and $x_2$ yields:

\[
\left(\frac{\phi_4}{\phi'}\right)' = \left(\frac{\psi_4}{\psi'}\right)' = \left(\frac{\chi_4}{\chi'}\right)' = \delta(x_4),
\]

where $\delta$ is the common value of all expressions. Now, integration yields:

\[
\frac{\phi_4}{\phi'} = \delta(a - b) + \lambda(x_4, x_{14}) - \mu(x_4, x_{24}), \quad \frac{\psi_4}{\psi'} = \delta(c - a) + \nu(x_4, x_{34}) - \lambda(x_4, x_{14}),
\]

\[
\frac{\chi_4}{\chi'} = \delta(b - c) + \mu(x_4, x_{24}) - \nu(x_4, x_{34})
\]

(the form of the integration constants follows from (54)). At this point we use the remaining freedom in the definition of the functions $a, b, c$, which can be changed to

\[
\tilde{a} = k(x_4)a + \ell(x_4, x_{14}), \quad \tilde{b} = k(x_4)b + m(x_4, x_{24}), \quad \tilde{c} = k(x_4)c + n(x_4, x_{34}),
\]

with arbitrary $k, \ell, m, n$. Denoting

\[
\phi(a - b, x_4, x_{14}, x_{24}) = \tilde{\phi}(\tilde{a} - \tilde{b}, x_4, x_{14}, x_{24}),
\]

\[
\psi(c - a, x_4, x_{14}, x_{34}) = \tilde{\psi}(\tilde{c} - \tilde{a}, x_4, x_{14}, x_{34}),
\]

\[
\chi(b - c, x_4, x_{24}, x_{34}) = \tilde{\chi}(\tilde{b} - \tilde{c}, x_4, x_{24}, x_{34}),
\]

it is easy to see that

\[
\phi' = k\tilde{\phi}', \quad \phi_4 = \tilde{\phi}_4 + (k'(a - b) + \ell_4 - m_4)\tilde{\phi}',
\]

\[
\psi' = k\tilde{\psi}', \quad \psi_4 = \tilde{\psi}_4 + (k'(c - a) + \ell_4 - n_4)\tilde{\psi}',
\]

\[
\chi' = k\tilde{\chi}', \quad \chi_4 = \tilde{\chi}_4 + (k'(b - c) + \ell_4 - m_4)\tilde{\chi}'.
\]
so that

$$\phi_4 = \phi_4 + k' a - b + \ell_4 - m_4,$$

and analogously for $\psi, \chi$. The choice $k'/k = \delta$, $\ell_4 = k\lambda$, $m_4 = k\mu$, $n_4 = k\nu_4$ leads to

$$\tilde{\phi}_4 = \tilde{\psi}_4 = \tilde{\chi}_4 = 0,$$

and this gives the desired representation. The solution of the second pair of equations (51) is of the same structure and leads to representation (53).

\[\text{Figure 7. } \text{Tripodal forms (52) of equations (37) on the octahedra } \langle 3 \rangle, \langle 2 \rangle, \langle 1 \rangle, \text{ and the tripodal form (55) of equation (45) on the octahedron } \langle 0 \rangle \text{ sum up to zero}\]

We call equations (52) and (53) tripodal forms of equations (37). For instance, the three terms in the first equation in (52) correspond to the three (triangular) legs (1,4,14), (2,4,14), and (12,14,24) of the tripod with the head (4,14,24), see Fig. 7. Adding the three tripodal forms (52) of equations on the octahedra (3), (2), (1) leads to the equation

$$p(x_{12}, x_{14}, x_{24}) + q(x_{13}, x_{14}, x_{34}) + r(x_{23}, x_{24}, x_{34}) = 0,$$

which is nothing but the tripodal form of equation (45) on the octahedron (0), with the head (14,24,34).

Similarly, adding the three tripodal forms (53) of equations on the octahedra (3), (2), (1) leads to the equation

$$P(x_1, x_2, x_{12}) + Q(x_1, x_3, x_{13}) + R(x_2, x_3, x_{23}) = 0,$$
Figure 8. Tripodal forms (53) of equations (37) on the octahedra \langle 3 \rangle, \langle 2 \rangle, \langle 1 \rangle, and the tripodal form (56) of equation (44) on the octahedron \langle 4 \rangle sum up to zero

which is the tripodal form of equation (44) on the octahedron \langle 4 \rangle, with the head (1,2,3). Thus, Proposition 5 subsumes (and is actually much stronger than) Proposition 4.

Moreover, there exist further tripodal representations of equations (37) and (44), (45), due to the symmetry of all coordinates. Each equation under consideration admits eight tripodal representations, since each face of the octahedron can be chosen as the head of the tripod. In total we have 40 such representations. In order to put them in a unified form, we will use the realization of the $Q(A_4)$ lattice as a hyperplane (9) in $\mathbb{Z}^5$, and will denote the leg functions in the tripodal form by the letter $a$ with three superscripts which are the indices of the three arguments, omitting the arguments themselves. Our convention will be that this notation is symmetric with respect to the first and the third argument (the base of the leg), while the second argument (the spike of the leg) enters the equation only once. In this notation, formulas of Proposition 5 can be written so: for the quadruple of octahedra \langle i \rangle, \langle j \rangle, \langle k \rangle, \langle m \rangle, the tripodal forms

$$
\begin{align*}
\langle i \rangle & : \quad a^{jn,jm,mn} - a^{kn,km,mn} = a^{jn,jk,km}, \\
\langle j \rangle & : \quad a^{kn,km,mn} - a^{in,im,mn} = a^{kn,ik,in}, \\
\langle k \rangle & : \quad a^{in,im,mn} - a^{jn,jm,mn} = a^{in,ij,jn} 
\end{align*}
$$

(57)

sum up to the tripodal form

$$
\langle m \rangle : \quad a^{jn,jk,kn} + a^{kn,ik,in} + a^{in,ij,jn} = 0.
$$
The following theorem summarizes the results of the present section.

**Theorem 6.** A quintuple of the octahedron type equations is 4D consistent if and only if, in any quadruple of octahedra such that any two of them share a triangular face, the tripodal forms of any three equations sum up to the tripodal form of the fourth one.

*Proof.* The necessity follows from Proposition 5, the sufficiency is proved by literally the same argument as used in Section 3.3 for the proof of the 4D consistency of the dKP equation.

## 5 Tripodal equations

### 5.1 Definitions and notation

We have shown that each equation of a 4D consistent system can be written in 8 ways in a tripodal form. In the present section, we temporarily forget about consistency and analyze just this property of a single octahedron type equation. The enumeration of the vertices of an octahedron and of the corresponding variables will be different from the previously used: it will be convenient to enumerate them just from 1 to 6 as on Fig. 9.

Our task here will be to describe all equations

$$
\Phi(x_1, x_2, x_3, x_4, x_5, x_6) = 0,
$$

which are locally equivalent to equations of the form

$$
a(x_i, x_{7-k}, x_j) + b(x_j, x_{7-i}, x_k) + c(x_k, x_{7-j}, x_i) = 0
$$

for each of the eight triples \((i, j, k)\), corresponding to the faces of an octahedron. Such equations will be called *tripodal*. 

---

**Figure 9.** Left: enumeration of the vertices of an octahedron: \((1,2,3)\) is a face, and opposite vertices carry complementary indices which sum up to 7. Right: the tripodal form with the head \((i,j,k)\).
As usual, it is supposed that equation (58) is irreducible, in particular, neither of the partial derivatives $\Phi_i$ vanishes identically. The answer is determined up to point transformations
\[ \tilde{x}_i = X_i(x_i), \quad X'_i \neq 0. \] (59)
Functions $a, b, c$ may be different for different faces. It will be convenient to denote them by superscripts pointing to the arguments, for instance, $a^{123} = a^{123}(x_1, x_2, x_3)$. Here the second index is distinguished, and it is supposed that the notation is symmetric with respect to the first and the third index: $a^{123} = a^{321}$. This means that $a^{123}(x_1, x_2, x_3) = a^{321}(x_3, x_2, x_1)$ (but of course it is not supposed that $a^{123}(x_1, x_2, x_3) = a^{321}(x_1, x_2, x_3)$). Opposite vertices of an octahedron will be denoted by the same letter in the lower and the upper cases, e.g., $I = 7 - i$. Thus, the tripodal form of equation with the head $(ijk)$ takes the form
\[ a^{iKj} + a^{ilk} + a^{kji} = 0, \]
see Fig. 9. The set of all eight tripodal forms looks like this:

| head:          | tripodal form:                   |
|----------------|----------------------------------|
| $(1, 2, 3)$    | $a^{142} + a^{263} + a^{351} = 0$, |
| $(1, 2, 4)$    | $a^{132} + a^{264} + a^{451} = 0$, |
| $(1, 3, 5)$    | $a^{123} + a^{365} + a^{541} = 0$, |
| $(1, 4, 5)$    | $a^{124} + a^{465} + a^{531} = 0$, |
| $(2, 3, 6)$    | $a^{213} + a^{356} + a^{642} = 0$, |
| $(2, 4, 6)$    | $a^{214} + a^{456} + a^{632} = 0$, |
| $(3, 5, 6)$    | $a^{315} + a^{546} + a^{623} = 0$, |
| $(4, 5, 6)$    | $a^{415} + a^{536} + a^{624} = 0$. |

Clearly, this structure is rather restrictive. It turns out to be possible to find a complete list of tripodal equations. It is given and discussed in the following subsection, while the rest of the section is devoted to the proof that the list is exhaustive, indeed.

5.2 Classification of tripodal equations

**Theorem 7.** All tripodal equations (58), up to the point transformations (59) and transpositions $i \leftrightarrow I, (i, j) \leftrightarrow (J, I)$, are given by the following list:

\[
\begin{align*}
  x_1x_6 + x_2x_5 + x_3x_4 &= 0, \\
  (x_1 - x_4)(x_2 - x_6)(x_3 - x_5) + (x_4 - x_2)(x_6 - x_3)(x_5 - x_1) &= 0, \\
  (x_1 - x_2)x_4 + (x_2 - x_3)x_6 + (x_3 - x_1)x_5 &= 0, \\
  x_1x_6 &= (x_2 + x_3)^{-1}(x_4 + x_5), \\
  x_1x_6 &= x_2 + x_3 + x_4 + x_5, \\
  x_1x_2x_3x_4 &= x_5 + x_6, \\
  x_1 + x_2 + x_3 + x_4 + x_5 + x_6 &= 0.
\end{align*}
\] (T1, T2, T3, T4, T5, T6, T7)

It is easy to see that the first three equations are equivalent to $(\chi_1), (\chi_3), (\chi_2)$, upon re-naming $(x_{12}, x_{13}, x_{23}) \rightarrow (x_4, x_5, x_6)$. The proof of the theorem will be given in the
following subsections. Here we will just check that all equations of the list are tripodal, indeed, that is, admit all eight representations (60).

The results of this check are summarized in Table 10, which lists the functions $a(x, y, z)$ acting as the legs $a_{ijk} = a(x_i, x_j, x_k)$ for equations of the list. Recall that the variables $x, z$ are on the equal footing, while $y$ plays a distinguished role. In particular, the functions $a$ always depend on $y$ but is sometimes independent on $x$ or $z$. Functions $a$ are defined up to point transformations (59). Besides, they can be multiplied by an arbitrary constant, one can flip $x, z$ and add arbitrary combinations of the type $\mu(x) + \nu(z)$ (as long as one such function is considered, and not all three legs of the tripodal form). At this stage, we are not concerned with adjusting all these transformations for consistent equations, therefore the table contains arbitrary (possibly simple) representatives for the leg functions. This arbitrariness notwithstanding, we will see in Section 6 that this table is very useful for putting separate tripodal equations into consistent quintuples. The third column of the table contains the class according to the subdivision at the end of Section 5.3.

**Equation (T_1).** The tripodal form with the head $(1, 2, 3)$ is

$$(1, 2, 3) : \quad \frac{x_4}{x_1x_2} + \frac{x_6}{x_2x_3} + \frac{x_5}{x_3x_1} = 0,$$

other ones are obtained by reflections $x_i \leftrightarrow x_I$ and $(x_i, x_j) \leftrightarrow (x_J, x_I)$, which generate the symmetry group of an octahedron and leave the equation invariant.

**Equation (T_2).** The multiplicative tripodal form $(1, 2, 3)$ can be seen directly from the equation. All other ones follow again by reflections $x_i \leftrightarrow x_I$ and $(x_i, x_j) \leftrightarrow (x_J, x_I)$, although the invariance of the equation under these reflections is less obvious than in the case of equation $T_1$.

**Equation (T_3)** is already written in the tripodal form (123). Arranging the terms in a different order leads to the tripodal form for the opposite face:

$$(4, 5, 6) : \quad x_1(x_4 - x_5) + x_3(x_5 - x_6) + x_2(x_6 - x_4) = 0.$$
Thus, the equation is invariant under the involution

\[ P : (x_1, x_2, x_3) \leftrightarrow (x_6, x_5, x_4). \]

Further, one finds the multiplicative tripodal form

\[ (1, 2, 4) : \frac{x_1 - x_3}{x_3 - x_2} \frac{x_4 - x_5}{x_6 - x_4} = 1. \]

All other tripodal forms are obtained with the help of the involution \( P \) and the cyclic permutation

\[ Z : x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_1, \quad x_6 \rightarrow x_6 \rightarrow x_4 \rightarrow x_6. \]

\textit{Equation (T_4)} is invariant under the transformations

\[ x_1 \leftrightarrow x_6; \quad (x_2, x_5) \leftrightarrow (x_3, x_4); \quad (x_1, x_2, x_3, \gamma) \leftrightarrow (x_1^{-1/\gamma}, x_5, x_4, -1/\gamma), \]

which allows one to obtain all the tripodal forms from the following two:

\[ (1, 2, 3) : \frac{x_4}{x_1} - x_6(x_2 + x_3)^\gamma + \frac{x_5}{x_1} = 0, \]

\[ (1, 2, 4) : (\log x_1 + \gamma \log(x_2 + x_3)) + \log x_6 - \log(x_4 + x_5) = 0. \]

\textit{Equation (T_5)}. In this case the symmetry group is generated by the transformations

\[ x_i \leftrightarrow x_I; \quad (x_2, x_5) \leftrightarrow (x_3, x_4), \]

which allows one to obtain all the tripodal forms from one of them, for instance, from

\[ (1, 2, 3) : \frac{x_4 + x_2}{x_1} - x_6 + \frac{x_3 + x_5}{x_1} = 0. \]

It is worth mentioning that, although here the first and the third terms are similar to the legs encountered for equations \((T_3)\) and \((T_4)\) with \(\gamma = 1\), this is a different type of legs, since the role of the distinguished variable \(y\) is different.

\textit{Equation (T_6)}. Here there are transformations

\[ (x_1, x_6) \leftrightarrow (x_2, x_5); \quad x_3 \leftrightarrow x_4, \]

which make it sufficient to give the following three tripodal forms:

\[ (1, 2, 3) : x_1x_4x_2 - \frac{x_6}{x_3} - \frac{x_5}{x_3} = 0, \]

\[ (1, 3, 5) : \log(x_1x_2x_3) - \log(x_5 + x_6) + \log x_4 = 0, \]

\[ (3, 5, 6) : (\log x_3 + \log x_1) + (\log x_4 - \log(x_5 + x_6)) + \log x_2 = 0. \]

\textit{Equation (T_7)}. This case is trivial.
5.3 Reduction to functions of two variables

Let us prove some useful relations which follow from the (local) equivalence of the two tripodal forms

\[(i, j, K) : a^{ikj} + a^{jIK} + a^{KJI} = 0, \quad (i, j, k) : a^{iKj} + a^{jIk} + a^{kJI} = 0, \ (61)\]

which correspond to two faces sharing a common edge \(ij\). We will use the subscripts \(i\) to denote the partial derivatives \(\partial x_i\).

**Proposition 8.** The following identities hold true:

\[
\frac{a^{ikj}}{a^{Kji}} = a^{iKj} + a^{kJi}, \quad \frac{a^{jKj}}{a^{jIK}} = a^{iKj} + a^{jIk}, \ (62)
\]

\[
\left(\frac{1}{a^{Kji}_j}ight)_K = \left(\frac{1}{a^{Kji}_j}ight)_i, \quad \left(\frac{1}{a^{Kji}_j}ight)_k = \left(\frac{1}{a^{Kji}_j}ight)_i \left(\frac{1}{a^{jIK}_K}ight)_K. \ (63)
\]

Since each one contains five variables only, they are satisfied identically and not by virtue of equations \((61)\).

**Proof.** The numerators do not vanish identically because of the local solvability of equations with respect to each variable. We shall prove only the first identities in each pair, since the second ones follow by the permutation \((i, I) \leftrightarrow (j, J)\). Note also that all identities are invariant under the flip \(k \leftrightarrow K\). Solving equations \((61)\) for \(x_I\), we obtain an equation of the following form:

\[f(x_j, x_K, a^{ikj} + a^{KJI}) = g(x_j, x_k, a^{iKj} + a^{kJi}),\]

and by differentiation there follow equations of the form

\[f_K + f' a^{Kji}_K = g' a^{Kji}_K, \quad f'(a^{ikj} + a^{Kji}) = g'(a^{ikj} + a^{kJi}), \quad f' a^{Kji}_j = g' a^{Kji}_j,\]

where the prime denotes the derivatives of the functions \(f, g\) with respect to their last arguments. Now \((62)\) follows immediately from the second and the third equations, while from the first and the third we derive

\[\frac{f_K}{f'} = \frac{a^{Kji}_j}{a^{Kji}_j}, \quad a^{Kji}_j = \phi = \phi a^{Kji}_j,\]

which yields

\[\left(\frac{a^{Kji}_j}{a^{Kji}_j} a^{iKj} - a^{KJI}_K\right) a^{ikj}_j = (a^{Kji}_j) a^{Kji}_j \left(\frac{1}{a^{Kji}_j}\right).\]

Dividing by \(a^{iKj}_j a^{ikj}_j (a^{Kji}_j)^2\) and differentiating with respect to \(x_j\), we arrive at \((63)\).

Equations \((62)\) will play the main role in the subsequent analysis. Equations \((63)\) are needed only for the proof of the following property, which, in its turn, will only be used in the proof of Proposition 18. Nevertheless, it is convenient to do the job right now, since all necessary formulas are at hand.
Proposition 9. If at least one of the functions $a_{IK}^{iK}$, $a_{KJ}^{iJ}$, $a_{lK}^{iK}$, $a_{kl}^{jK}$ does not vanish identically, then

$$\frac{a_{ik}^{ik} a_{kj}^{kj}}{(a_k^{ik})^2} = \frac{a_{Kj}^{iK} a_{Kj}^{iK}}{(a_K^{ik})^2}. \tag{64}$$

Proof. Differentiating (62) with respect to $x^k$ and $x^K$, we obtain equations

$$a_{ik}^{ik} \frac{1}{a_{KJ}^{iK}} = a_{iK}^{iK} \frac{1}{a_{KJ}^{iK}}, \quad a_{kj}^{kj} \frac{1}{a_{jI}^{iK}} = a_{Kj}^{iK} \frac{1}{a_{Kj}^{iK}}.$$

Along with (63) they build a pair of linear homogeneous systems, one for $(1/a_{KJ}^{iK})_K$, $(1/a_{KJ}^{iK})_k$ and another for $(1/a_{jI}^{iK})_K$, $(1/a_{jI}^{iK})_k$. Determinants of these systems coincide.

The statement of the proposition says that if at least one of the systems has a non-trivial solution, then their common determinant vanishes. \hfill \square

The following representation is a direct consequence of identities (62).

Proposition 10. The functions $a_{ik}^{ik}$ and $a_{iK}^{iK}$ are of the form

$$a_{ik}^{ik} = a_{ij}^{ij} b^k + p_{ik} + p_{kj}, \quad a_{iK}^{iK} = a_{ij}^{ij} b^K + p_{iK} + p_{Kj}, \quad b^k b^K \neq 0. \tag{65}$$

Proof. Differentiating equations (62), we find:

$$a_{ij}^{ik} a_{jI}^{iK} = a_{ij}^{iK} a_{Kj}^{iK}, \quad a_{ij}^{ik} a_{lK}^{iK} = a_{ij}^{iK} a_{lK}^{iK}. \tag{66}$$

Setting the variables $x_I, x_J, x_K$ to arbitrary constants, we find:

$$a_{ij}^{ik} = a_{ij}^{ij} b^{ij} = a_{ij}^{ij} b^{ij} \Rightarrow a_{ij}^{ik} = a_{ij}^{ij} b^{ij},$$

and an integration leads to formula (65). Here the function $a^{ij}$ is defined up to addition of the terms $\mu^i + \nu^j$ and multiplication by a constant, which can be taken into account by a re-definition of $b$ and $p$. It is easy to see from (66) that this function can be chosen the same in the both formulas, and that the factors $b$ can be taken non-vanishing without restriction of generality. \hfill \square

Remark 6. Taking into account all the tripodal forms (60), we come to the conclusion that each function $a_{lmn}^{lmn}$ has a representation of the type (65), so that the form of the equation is already found up to functions of two variables. However it should be understood that the notation in formulas (65) refer to some fixed pair of the tripodal forms (61). If we would like to apply them to all possible index sets simultaneously then we would be forced to use more complicated enumeration for the functions $b$ and $p$. However, there will be no need to do this, since we will only perform a pairwise comparison of the tripodal forms.

Proposition 11. If $a_{ij}^{ij} \neq 0$ then either the both functions $b^k$ and $b^K$ are different from constants or the both are constants, the tripodal forms (61) being either

$$a_{ij}^{ij} b^k + c_{ij}^{ij} b^K = 0 \iff a_{ij}^{ij} b_K + c_{ij}^{ij} b_K = 0 \tag{67}$$
or

\[(a^{ij} + p^{ik} + p^{kj}) + (c^{jI} + p^{Kj}) + (c^{JI} + p^{iK}) = 0\]

\[
\iff (a^{ij} + p^{iK} + p^{Kj}) + (c^{jI} + p^{kj}) + (c^{JI} + p^{ik}) = 0,
\]  \hspace{1cm} (68)

respectively.

**Proof.** We still did not exhaust the content of identities (66). If \(a^{ij}_{ikj} \neq 0\) then they are reduced to

\[b^k a^k_{ij} - b^K a^K_{ij} = 0, \quad b^k a^k_{ikj} - b^K a^K_{ikj} = 0,\]  \hspace{1cm} (69)

which give, upon integration,

\[a^K_{ij} = \frac{c^{JI}}{b^k} + d^{ki}, \quad a^{jI}_{ik} = \frac{c^{jI}}{b^k} + d^{jk},\]  \hspace{1cm} (70)

\[a^K_{Kj} = \frac{c^{jI}}{b^k} + d^{Kj}, \quad a^{jK}_{iK} = \frac{c^{ji}}{b^k} + d^{ki} + d^{Kj}.\]  \hspace{1cm} (71)

Thus, the tripodal forms (61) can be put as

\[(a^{ij} b^k + p^{ik} + p^{kj}) + \left(\frac{c^{jI}}{b^k} + d^{ji}\right) + \left(\frac{c^{ji}}{b^k} + d^{ki}\right) = 0,\]

\[(a^{ij} b^K + p^{iK} + p^{Kj}) + \left(\frac{c^{ji}}{b^k} + d^{Kj}\right) + \left(\frac{c^{jI}}{b^k} + d^{ki}\right) = 0.\]

It is easy to see that they are equivalent if and only if

\[b^K (p^{ik} + p^{kj} + d^{Kj}) = b^k (p^{iK} + p^{kj} + d^{Kj} + d^{ki}),\]

from which it follows \(b^K (p^{ik} + p^{kj}) = b^k (p^{iK} + p^{Kj}).\) Now if \(b^k = 0,\) then \(p^{ik} + p^{kj} \neq 0,\) since otherwise the first tripodal form would not contain \(x_k,\) but then also \(b^K = 0,\) and, setting \(b^k = b^K = 1,\) we find formulas (68).

If, on the contrary, \(b^k \neq 0,\) then

\[p^{ik} + p^{kj} = b^k (p^i + p^j) + q^i + q^j, \quad p^{iK} + p^{Kj} = b^K (p^i + p^j) + \tilde{q}^i + \tilde{q}^j.\]

It is not difficult to see that these terms can be assumed to vanish (this can be achieved by a re-definition of other terms). But then

\[d^{iK} + d^{Kj} = \frac{d^i + d^j}{b^k}, \quad d^{jk} + d^{ki} = \frac{d^j + d^k}{b^k},\]

and these terms can be again absorbed by the functions \(c^{jI}, c^{ji}.\) As a result, the tripodal forms (65) can be put as (67).

\[\square\]

The further analysis does not require for complicated computations, but rather for a detailed bookkeeping of different cases. Taking into account the possibilities listed in Proposition 11, one can take care of this by separating the following classes of equations:

I. \(a^{ikj}_{ikj} \neq 0\) for at least one triple \(i, k, j;\)
II. \(a^{ikj}_{ikj} = 0\) for all \(i, k, j,\) but \(a^{ikj}_{ikj} \neq 0\) for at least one triple;
III. \(a^{ikj}_{ikj} = 0\) for all \(i, k, j.\)

Clearly, this exhausts all logical possibilities. A complete description of these cases will be sufficient for a proof of Theorem 7.
5.4 Class I

Setting, without restriction of generality, \( b^k = x_k \), \( b^K = x_K \), we put the tripodal forms (67) as
\[
\frac{a^{ij} x_k}{x_K} + \frac{c^{ji}}{x_K} = 0, \quad \iff \quad \frac{a^{ij} x_K}{x_k} + \frac{c^{ji}}{x_k} = 0. \tag{72}
\]
Now we have to compare them with the other tripodal forms. Since functions \( a^{jIk} \), \( a^{kJi} \) have to be of the form (65), as well, we immediately find that
\[
c^{ij} = c^i b^j + d^i, \quad c^{ji} = c^j b^i + d^j.
\]
Assume first that \( c^{ij}_j \neq 0 \) or \( c^{ji}_i \neq 0 \). For definiteness, let \( c^{ij}_j \neq 0 \). Compare (72) with the tripodal form \( (I,j,k) \): \( a^{IKj} + a^{iik} + a^{kJi} = 0 \). Apply formulas (70), from which there follows:
\[
a^{ijk} = \frac{c^{ij}}{x_k} b^i + q^{ji} + q^{ik};
\]
while the function \( a^{iKj} \) should admit the both representations
\[
a^{iKj} = a^{ij} x_K = \frac{c^{Kj}}{b^i} + d^j \quad \Rightarrow \quad a^{iKj} = \frac{x_K}{b^i}. \tag{73}
\]
Since, by assumption, \( a^{ij}_j \neq 0 \), there follows \( b^i \neq 0 \), but then there holds also \( b^j \neq 0 \), and the equation is reduced to the form
\[
\frac{c^{Kj}}{b^i} + a^{jk} b^j + c^{kJ}_j = 0.
\]
Subtracting this from (72), we come to
\[
\frac{c^{ij}}{x_k} + \frac{c^{ji}}{x_k} = a^{jk} b^j + \frac{c^{kJ}}{b^i},
\]
which yields that the equation is reduced by the point transformations (59) to the form \( (T_1) \).

Now let \( c^{ij}_j = c^{ji}_i = 0 \), then equation (72) takes the form
\[
\log a^{ij} + \log x_K - \log(c_I + c_J) + \log x_K = 0.
\]
This case is covered (after an obvious change of enumeration) by the following proposition which refers to somewhat more general equations. This is done in order to avoid unnecessary repetitions for the cases II and III, which also lead to equations of this type. In this proposition it is not assumed that the equation falls into one of the three classes separated above. Actually, the analysis of results performed in Section 5.2 shows that equation \( (T_6) \) belongs to the class I, equation \( (T_7) \) belongs to the class III, and equation \( (T_4) \) belongs to the class I if \( \lambda \neq 1 \) and to the class III if \( \lambda = 1 \).

**Proposition 12.** Let one of the tripodal forms of equation (58) be as follows:
\[
p^{IK} + p^{Ki} + p^{ik} + q^i + q^J = 0. \tag{73}
\]
Then it is reduced to one of the equations \( (T_4) \), \( (T_6) \), or \( (T_7) \).
Proof. Setting, without restriction of generality, \( q^j = x_j, \) \( q^J = x_J, \) we put the equation as
\[
p^I K + p^K i + p^i k + x_j + x_J = 0. \tag{74}
\]
Let us show that if \( p^K i_k \neq 0, \) then \( p^I K = 0 \) and \( p^i k = 0. \)

In order to prove the first of these equations, compare two tripodal forms (61),
\[
(i, j, K) : a^{ikj} + a^{jIK} + a^{KJi} = 0, \quad (i, j, k) : a^{IKj} + a^{JI k} + a^{kJi} = 0,
\]
assuming that (74) is the first one of them, so that \( a^{ikj} = p^i k + x_j, \) \( a^{jIK} = p^I K \) and \( a^{KJi} = p^K i + x_J. \) From the first identity (62) we find:
\[
p^i k + p^K i = \frac{a^{iKj}}{a^{kJi}} a^i_k + a^i_j \Rightarrow p^K i = \frac{a^{iKj}}{a^{kJi}}. \tag{75}
\]
Suppose that \( p^K i_k \neq 0. \) Then, differentiating the latter equation with respect to \( x_k, \) \( x_J, \)
we find \( a^{kJi} = a^{kJi} J = 0. \) Therefore, \( a^{kJi} = \mu x_j + r^kJ, \) so the tripodal form \( (i, j, k) \) becomes
\[
a^{IKj} + a^{jIK} + \mu x_j + r^kJ = 0, \quad \mu \neq 0.
\]

Eliminating \( x_J \) with the help of (74), we obtain the identity
\[
a^{IKj} + a^{jIK} - \mu (p^I K + p^K i + p^i k + x_j) = 0 \Rightarrow p^I K = 0.
\]
The second equation is shown analogously, by comparing (74) with the tripodal form \( (I, j, K) \) (it is enough to use the symmetry \( i \leftrightarrow K, k \leftrightarrow I). \)

Thus, we have shown that (74) takes one of the following two forms:
\[
p^I + p^K i + p^i k + x_j + x_J = 0 \quad \text{or} \quad p^I K + p^i k + x_j + x_J = 0.
\]
But it is clear that the first case is included into the second one, upon the re-labeling \( i \to k \to I \to K \to i. \) Therefore we can assume that in equation (74) there holds \( p^K i = 0. \)
Note that this makes the equation invariant under the re-labeling \( i \leftrightarrow k, \) \( I \leftrightarrow K, \) which does not affect the tripodal form \( (i, j, k). \) Now, returning to identity (75), we find:
\[
p^i k = \frac{a^{iKj}}{a^{kJi}} \Rightarrow a^{iKj} = a^i_j = 0
\]
and, using the symmetry just pointed out,
\[
p^i k = \frac{a^{jIK}}{a^{kJi}} \Rightarrow a^{jIK} = a^i_j = 0.
\]
As a result, we put the tripodal forms \( (i, j, K) \) and \( (i, j, k) \) as
\[
p^I K + p^i k + x_j + x_J = 0, \quad a^{Kj} + a^{jI} + a^{kJi} = 0,
\]
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and, solving for $x_J$, we arrive at the identity

$$p^{IK} + x_J = \varphi(x_K, x_I, a^{Ki} + a^{Ij}) \Rightarrow p^{IK}_J = \frac{a^{Kj}}{a^{Ji}} = \frac{a^K}{a^I},$$

where the last equality is obtained by setting $x_J = \text{const}$. Upon suitable point transformations of $x_I$ and $x_K$, we can put $p^{IK}$ as $p^{IK} = P(x_I + x_K)$, while $a^{Kj} = a^{Ij} = r^j$. There follows easily

$$\frac{1}{P'p^{IK}} = \frac{a^{Kj}}{a^{Ji}} + \frac{a^{Ij}}{r^j},$$

and, differentiating twice with respect to $x_I$ or to $x_K$, we find $(1/P')'' = 0$, or

$$P(y) = \alpha \log(y + \beta) + \gamma \quad \text{or} \quad P(y) = \alpha y + \beta.$$  

Of course, for $p^{ik}$ analogous formulas are found (by comparison with the tripodal form $(I,j,K)$ and upon suitable point transformations of $x_i, x_k$), so $p^{ik} = Q(x_i + x_k)$ with

$$Q(y) = \lambda \log(y + \mu) + \nu \quad \text{or} \quad Q(y) = \lambda y + \mu.$$  

Different combinations lead, after further obvious point transformations, to the cases $(T_4)$, $(T_6)$, $(T_7)$.

### 5.5 Class II

Now let $a^{ikj}_{ij} = 0$ for all $i, k, j$, but $a^{ikj}_{ij} \neq 0$ for at least one triple. For this triple we find two equivalent tripodal forms (68)

$$(a^{ij} + p^{ik} + p^{kj}) + (c^{iI} + p^{Kj}) + (c^{jI} + p^{IK}) = 0$$

$\Leftrightarrow$  

$$(a^{ij} + p^{ik} + p^{Kj}) + (c^{iI} + p^{kj}) + (c^{jI} + p^{ik}) = 0$$

which have to be compared with other ones. Here it is convenient to distinguish subcases depending on which mixed derivatives of the pairs of functions $p^{ik}, p^{kj}$ and $p^{ik}, p^{Kj}$ vanish.

First assume that for at least one pair the mixed derivatives of the both functions do not vanish. For the sake of definiteness, let $p^{ik} \neq 0$, $p^{kj} \neq 0$. Note that formula (69) with constant $b^K, b^k$ yields that

$$a^{ikj}_{ij} \neq 0 \Rightarrow a^{iK}_{1K} = a^{Kj}_{1i} = a^{ik}_{1k} = a^{kj}_{k} = 0.$$  

Applying this implication to the second of the tripodal forms, we find:

$$a^{jK}_{jK} = 0, \quad a^{jK}_{jk} = p^{kj}_{kj} \neq 0 \Rightarrow a^{jK}_{j} = a^{iK}_{i} = 0 \Rightarrow c^{jK}_{j} = p^{jK}_{j} = 0;$$

$$a^{kK}_{ki} = 0, \quad a^{kK}_{ki} = p^{ik}_{ik} \neq 0 \Rightarrow a^{kK}_{j} = a^{iK}_{i} = 0 \Rightarrow c^{jK}_{j} = p^{Kj}_{j} = 0,$$

and as a consequence the equation takes the form

$$a^{ij} + p^{ik} + p^{kj} + q^I + q^J + q^K = 0,$$
and after the change \( q^n \to x_n \) we come to equation
\[
p^{ij} + p^{jk} + p^{ki} + x_I + x_J + x_K = 0. \tag{76}
\]

If the previous subcase does not take place, then we assume first that one of the pairs \( p^{ik}, p^{kj} \) or \( p^{iK}, p^{Kj} \) contains exactly one function with vanishing mixed derivatives. For definiteness, let \( p^{ik} \neq 0 \) and \( p^{kj} = 0 \). Then, as before, \( c^{Ij}_j = p^{Kj}_K = 0 \), and we come to an equation of the form
\[
a^{ij} + p^{ik} + q^I + c^{ji} + p^{iK} = 0,
\]

or, upon a point change of variables, to equation
\[
p^{ij} + p^{jk} + p^{ik} + p^{iJ} + p^{iK} + x_I = 0. \tag{77}
\]

Finally, if all mixed derivatives \( p^{ik}, p^{kj}, p^{iK}, p^{Kj} \) vanish, then our equation belongs to the special type already dealt with in Proposition 12:
\[
a^{ij} + q^k + c^{jI} + q^K + c^{iI} = 0;
\]
we have seen that such equations cannot belong to the case II.

As a result, case II is reduced to equations of two special types. They are dealt with in the following two statements which result in equation \( \text{(T5)} \) as the only possible one.

**Proposition 13.** There exist no equations of the class II with one of the tripodal forms as in \( \text{(76)} \).

**Proof.** We compare \( \text{(76)} \) with the tripodal form
\[(I, J, K) : \quad a^{IkJ} + a^{JiK} + a^{KjI} = 0.\]

Solving for \( x_k \), we obtain an identity
\[
f(x_I, x_J, a^{JiK} + a^{KjI}) = g(x_i, x_j, x_I + x_J + x_K),
\]
whence
\[
\log a^{IkJ}_i - \log a^{KjI}_j = \log g_i - \log g_j = h(x_i, x_j, x_I + x_J + x_K).
\]
Differentiating with respect to \( x_I, x_J, \) and \( x_K \), and using the symmetry of the equation under permutations of \( i, j, k \), we find:
\[
\begin{align*}
- \frac{a^{KjI}}{a^{KjI}_j} & = \frac{a^{JiK}}{a^{JiK}_j} = \frac{a^{JiK}}{a^{JiK}_i} - \frac{a^{KjI}}{a^{KjI}_j} = \nu^K, \\
- \frac{a^{IkJ}}{a^{IkJ}_k} & = \frac{a^{KjI}}{a^{KjI}_k} = \frac{a^{IkJ}}{a^{IkJ}_j} - \frac{a^{IkJ}}{a^{IkJ}_j} = \lambda^I, \\
- \frac{a^{JiK}}{a^{JiK}_i} & = \frac{a^{IkJ}}{a^{IkJ}_i} = \frac{a^{IkJ}}{a^{IkJ}_k} - \frac{a^{JiK}}{a^{JiK}_i} = \mu^J,
\end{align*}
\]
where $\lambda^I$, $\mu^J$, and $\nu^K$ denote the common values of the corresponding expressions. Since, by assumption, the equation belongs to the class II, we have $a_{jiK}^{j} = 0$, and, differentiating the last equality in the first equation with respect to $x_J$, we find:

$$\mu^J \nu^K = -\frac{a_{jiK}^{j} a_{ji}^i}{(a_{jiK}^{j})^2} = 0,$$

and analogously $\nu^K \lambda^I = \lambda^I \mu^J = 0$; since there holds additionally $\lambda^I + \mu^J + \nu^K = 0$, all three functions actually vanish. This means that the tripodal form $(IJK)$ is reduced, upon a point transformation of $x_i, x_j, x_k$, to the form analogous to (76):

$$a^{IJ} + a^{JK} + a^{KI} + x_i + x_j + x_k = 0.$$

Resolving once again the both tripodal forms for $x_k$, we find an identity

$$a^{IJ} + a^{JK} + a^{KI} + x_i + x_j = g(x_i, x_j, x_I + x_J + x_K),$$

whence

$$a^{IJ} + a^{JK} = a^I + a^J = a^K + a^K.$$

These equations are easy to solve, and we arrive at the tripodal form

$$\lambda(x_I + x_J + x_K)^2 + \mu(x_I + x_J + x_K) + \nu + x_i + x_j + x_k = 0.$$

The case $\lambda = 0$ leads to equation (77) which does not belong to the class II, while for $\lambda \neq 0$, as easily shown, this tripodal form cannot be equivalent to equation (76).

**Proposition 14.** Equations of the class II with one of the tripodal forms as in (77) can be reduced to (75).

**Proof.** Comparing with the tripodal form

$$(I, J, K) : \quad a^{lkj} + a^{iK} + a^{Klj} = 0,$$

we find:

$$p^{ij}_j = a^{ijkl} + a^{ijkl}, \quad p^{IK}_K = a^{iKlj} + a^{iKlj}.$$

There follows $a^{ijk}_{JK} = 0$. Further, assume that $a^{ijk}_{I} \neq 0$. Then, differentiating the first equation with respect to $x_I$ and $x_i$, we find $a^{ikj}_{I} = a^{ikj}_{I} = 0$, so that the tripodal form is

$$a^{kJ} + a^{iKj} + a^{Klj} = 0.$$

Solving for $x_I$, we find:

$$p^{ij}_j + p^{ik} + p^{iK} = f(x_K, x_J, a^{kJ} + a^{iK}),$$

and applying $\partial_j \partial_k$ leads to $0 = f'_j$, where the prime stands for the derivative of $f$ with respect to the third argument. But then $p^{ij}_j = 0$. 

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Exchanging the roles of \( j \) and \( k \), we prove that the implications hold:

\[
a_{ji}^k \neq 0 \Rightarrow a_{jk}^i = a_{kj}^i = p_{ij}^j = 0; \quad a_{ji}^k \neq 0 \Rightarrow a_{ki}^j = a_{ik}^j = p_{ik}^k = 0.
\]

If \( a_{ji}^k \neq 0 \) and \( a_{ik}^J \neq 0 \) simultaneously, then \( p_{ij}^j = p_{ik}^k = 0 \), and our equation reduces to

\[
p^j + p^k + p^J + p^K + x_I = 0,
\]

but this is, up to notation, a particular case of the previous special equation (76). Let, for example, \( a_{ji}^k \neq 0 \) and \( a_{ik}^k = 0 \). Then (79) is fulfilled, and, applying \( \partial_J \partial_K \), we come to

\[
0 = f_K' \quad \text{(recall that } a_{ji}^J = 0), \quad \text{but then also } p_{ik}^k = 0, \quad \text{and we come again to an equation of the type (76)}:
\]

\[
p^j + p^k + p^J + p^K + x_I = 0.
\]

There remains the only possibility \( a_{ji}^k = a_{ik}^k = 0 \). In this case there follows from (78) that

\[
\begin{pmatrix}
  p_{ij}^j \\
  p_{ik}^k
\end{pmatrix}_i = \begin{pmatrix}
  p_{ik}^k \\
  p_{ij}^j
\end{pmatrix}_i = 0,
\]

and, using the symmetry of our equation under \( k \leftrightarrow K \) (that is, comparing (77) with the tripodal form \( (IJk) \) instead of \( (IJK) \)), we find also:

\[
\begin{pmatrix}
  p_{ij}^j \\
  p_{ik}^k
\end{pmatrix}_i = \begin{pmatrix}
  p_{ik}^k \\
  p_{ij}^j
\end{pmatrix}_i = 0.
\]

All these equations yield:

\[
p^j + p^k + p^J + p^K = (q^j + q^k + q^J + q^K)r^i + s^i,
\]

moreover \( r_i \neq \text{const} \), since otherwise we would come to equation (\( T_7 \)) which does not belong to the class II. Up to changes of variables, equation (77) is reduced to

\[
x_j + x_k + x_J + x_K = x_I x_i + s^i.
\]

The tripodal form \( (I, J, K) \) in this case is \( a_{1k}^J + a^i + a_{kj}^I = 0 \), and, solving for \( x_j \), we find:

\[
x_I x_i + s^i - x_k - x_J - x_K = f(x_K, x_I, a_{1k}^J + a^i) \quad \Rightarrow \quad x_I + s_i = -\frac{a_i^J}{a_{1k}^J}.
\]

Differentiating the last equation with respect to \( x_i \) and \( x_I \), we obtain: \( a_{ii}^i = 0 \), but then \( s_{II}^i = 0 \), and an additional linear change of variables leads to equation (\( T_5 \)).
5.6 Class III

In this case it turns out to be convenient to change notation and to re-write the tripodal forms (60) as

\[\begin{align*}
\text{faces:} & \quad (1, 2, 3), (4, 5, 6) \\
& \quad (1, 2, 4), (3, 5, 6) \\
& \quad (1, 3, 5), (2, 4, 6) \\
& \quad (1, 4, 5), (2, 3, 6)
\end{align*}\]

\[\begin{align*}
\text{equations:} & \quad p^{14} + p^{42} + p^{26} + p^{63} + p^{35} + p^{51} = 0, \\
& \quad q^{13} + q^{32} + q^{26} + q^{44} + q^{45} + q^{51} = 0, \\
& \quad r^{63} + r^{32} + r^{21} + r^{14} + r^{45} + r^{56} = 0, \\
& \quad s^{64} + s^{42} + s^{21} + s^{13} + s^{35} + s^{56} = 0
\end{align*}\]

(through a re-ordering of terms each of these equations gives two tripodal forms, corresponding to opposite faces).

**Proposition 15.** Each term in equations (80), as a function \(f(x, y)\) of its arguments, is of the following type:

\[f = a(x)b(y) + c(x) + d(y) \quad \text{or} \quad f = \rho \log(a(x) + b(y)) + c(x) + d(y).\]  

**Proof.** First we show that \(f = p^{14}, \ x = x_1, \ y = x_4\). We use the first identity (62), with \((i, j, k) = (1, 2, 3)\) and \((4, 5, 6)\):

\[
\frac{a_1^{13} + a_1^{451}}{a_5^{351}} = \frac{a_4^{142} + a_4^{351}}{a_5^{351}}, \quad \frac{a_4^{465} + a_4^{124}}{a_2^{124}} = \frac{a_4^{415} + a_4^{624}}{a_2^{624}}.
\]

This yields:

\[
\frac{q_1^{13} + q_1^{51}}{q_5^{351}} = \frac{p_1^{14} + p_1^{51}}{p_5^{351}}, \quad \frac{s_4^{64} + s_4^{42}}{s_2^{42}} = \frac{p_4^{14} + p_4^{42}}{p_2^{42}} + \frac{p_4^{42}}{p_2^{42}} + \frac{p_4^{42}}{p_2^{42}}.
\]

and, setting all variables except for \(x_1, x_4\) to constants, we come to (82).

Further, computing the mixed derivatives, we find:

\[-f_{xy} = \frac{\alpha(x)\delta'(y)}{(\beta(x) + \delta(y))^2} = \frac{\lambda(y)\kappa'(x)}{(\mu(y) + \kappa(x))^2}.
\]

If at least one of the functions \(\alpha, \delta', \lambda, \kappa'\) vanishes, then \(f = c(x) + d(y)\). Otherwise, taking logarithms and applying \(\partial_x \partial_y\), we come to

\[
\frac{\beta'(x)\delta'(y)}{(\beta(x) + \delta(y))^2} = \frac{\mu'(y)\kappa'(x)}{(\mu(y) + \kappa(x))^2}.
\]

If \(\beta' = \mu' = 0\), then equations (82) are easily integrated and lead to \(f = a(x)b(y) + c(x) + d(y)\). Finally, if \(\beta'\) and \(\mu'\) do not vanish, then the two latter equations yield \(\alpha/\beta' = \lambda/\mu' = \text{const}\), and an integration finishes the proof. \(\square\)
The proposition just proven is not easy to use directly, since it is not yet known how are the representations \(81\) corresponding to different terms of the tripodal forms \(80\) related to one another. We start with filtering away the case when these forms contain too many terms of the kind \(a(x) + b(y)\).

**Proposition 16.** If each of the forms \(80\) contains terms with vanishing mixed derivatives, then one of the forms contains at least four such terms, and it is a particular case of the equation \(73\).

**Proof.** By differentiation we derive from \(83\):

\[
 s_{21}^{21} = 0 \iff p_{14}^{14} = 0 \iff q_{45}^{45} = 0,
\]

that is, the presence of one term with vanishing mixed derivatives in one of the forms yields the presence of such terms in further two forms. This chain can be continued to a closed cycle:

\[
 p_{14}^{14} = 0 \Rightarrow q_{45}^{45} = 0 \Rightarrow s_{56}^{56} = 0 \Rightarrow p_{63}^{63} = 0 \Rightarrow q_{32}^{32} = 0 \Rightarrow s_{21}^{21} = 0 \Rightarrow p_{14}^{14} = 0,
\]

not containing \(r^{ij}\). Of course, all forms are on the same footing, so that starting, e.g., from \(r_{14}\), we would get the cycle

\[
 r_{14}^{14} = 0 \Rightarrow p_{42}^{42} = 0 \Rightarrow q_{26}^{26} = 0 \Rightarrow r_{63}^{63} = 0 \Rightarrow p_{35}^{35} = 0 \Rightarrow q_{51}^{51} = 0 \Rightarrow r_{14}^{14} = 0,
\]

not containing \(s^{ij}\). Therefore, if the terms with vanishing mixed derivatives are present in each of the forms, then one of the forms accumulates at least four such terms. For instance, if both cycles above take place then \(p_{14}^{14} = p_{63}^{63} = p_{42}^{42} = p_{35}^{35} = 0\), and the first equation in \(80\) becomes \(p^4 + p^{20} + p^3 + p^{51} = 0\).

To compare the general terms \(81\) we will use the following lemma. Note that the function \(F\) in this lemma defines a general solution of the Liouville equation \((\log F)_{xy} = -2F\).

**Lemma 17.** All solutions of the functional equation

\[
 \frac{g_x(x, z)h_y(y, z)}{(g(x, z) - h(y, z))^2} = F(x, y) \neq 0
\]

are given by the formulas

\[
 g = \frac{\delta(z)}{\gamma(z) - \alpha(x)} + \varepsilon(z), \quad h = \frac{\delta(z)}{\gamma(z) - \beta(y)} + \varepsilon(z), \quad F = \frac{\alpha'(x)\beta'(y)}{(\alpha(x) - \beta(y))^2}.
\]

**Proof.** Integration with respect to \(x\) leads to

\[
 \frac{h_y(y, z)}{g(x, z) - h(y, z)} = \tilde{h}(y, z) - \int F(x, y)dx.
\]

Setting \(y = y_0 = \text{const}\), we come to

\[
 \frac{\delta(z)}{g(x, z) - \varepsilon(z)} = \gamma(z) - \alpha(x),
\]
and since \( h_y \neq 0 \), we can choose \( y_0 \) so that the numerator and the denominator do not vanish identically. This yields the expression for \( g \). Further, we can use the invariance of the equation under non-degenerate linear-fractional transformations

\[
g \to \frac{p(z)g + q(z)}{r(z)g + s(z)}, \quad h \to \frac{p(z)h + q(z)}{r(z)h + s(z)}, \quad F \to F
\]

to bring \( g \) to the form \( g = \alpha(x) \), \( \alpha' \neq 0 \). Then

\[
\frac{\alpha'(x)h_y(y, z)}{(\alpha(x) - h(y, z))^2} = F(x, y) \Rightarrow \frac{h_y}{h_y} + \frac{2h_z}{h - \alpha(x)} = 0 \Rightarrow \frac{h_z\alpha'}{(h - \alpha(x))^2} = 0,
\]

that is, \( h = \beta(y) \), and the inverse linear-fractional transformation finishes the proof. \( \square \)

Now we are in a position to analyze the case where at least one of the tripodal forms (80) contains no terms with the vanishing mixed derivatives (representable as a sum of functions of a single variable). For definiteness, let it be the first form in (80).

**Proposition 18.** If \( p_{ij} \neq 0 \) for all \( i, j \), the equation (58) is equivalent to (T2) or to (T3).

**Proof.** We apply Proposition 9. We have: \( a_{ikj} = p^{ik} + p^{kj}, a_{jIK} = p^{jI} + p^{IK}, a_{Kji} = p^{KJ} + p^{Ij} \), and, since by assumption \( a_{jIK} = p^{jI}_{IK} \neq 0 \), the following equation is fulfilled:

\[
\frac{p^{ik}p^{kj}}{(p^{j_1} + p^{j_2})^2} = \frac{a_{iKj}a_{iKj}}{(a_{ikj}^2)^2} = A_{ij}.
\]

Then, according to Lemma 17, we have:

\[
p_{ik} + p_{kj} = \frac{\delta^k}{\gamma^k - \alpha^i - \gamma^k - \beta^j},
\]

so that the representations (81) for \( p_{ik} \) and for \( p_{kj} \) are tied in the following sense:

- if \( p^{ik} = a^{ik}b^k + c^i + d^k \), then \( p^{kj} = a^{kj}b^j + c^j - d^k \);
- if \( p^{ik} = \rho \log(a^i + b^k) + c^i + d^k \), then \( p^{kj} = -\rho \log(a^j + b^k) + c^j - d^k \).

Applying this to the pair \( p^{kj}, p^{jI} \), and further cyclically, we obtain the representations

\[
\begin{align*}
p^{ik} &= a^{ik}b^k + c^i + d^k, & p^{ik} &= \rho \log(a^i + b^k) + c^i + d^k, \\
p^{kj} &= a^{kj}b^j + c^j - d^k, & p^{kj} &= -\rho \log(a^j + b^k) + c^j - d^k, \\
p^{jI} &= a^{jI}b^j - c^j + d^j, & p^{jI} &= \rho \log(a^j + b^j) - c^j + d^j, \\
p^{IK} &= a^{IK}b^I + c^K - d^I, & p^{IK} &= -\rho \log(a^K + b^I) + c^K - d^I, \\
p^{KJ} &= a^{KJ}b^J - c^K + d^J, & p^{KJ} &= \rho \log(a^K + b^J) - c^K + d^J, \\
p^{JI} &= a^{JI}b^I + c^i - d^j, & p^{JI} &= -\rho \log(a^i + b^j) + c^i - d^j.
\end{align*}
\]

A comparison of the first and the last terms shows that

\[
\hat{a}^i = \varepsilon a^i, \quad \hat{c}^i = -c^i + \delta, \quad \hat{a}^i = \varepsilon a^i + \delta, \quad \hat{c}^i = -c^i + \sigma.
\]

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Summing up all the forms, we obtain, after point transformations and re-enumeration, the following equations:

\[
x_1 x_4 + x_4 x_2 + x_2 x_6 + x_6 x_3 + x_3 x_5 + \varepsilon x_5 x_1 = \delta,
\]

\[
(x_1 + x_4)(x_2 + x_6)(x_3 + x_5) = \sigma(x_4 + x_2)(x_6 + x_3)(x_5 + \varepsilon x_1 + \delta).
\]

One can be more precise about the constants here. For this aim, we solve these equations for one of the variables, say for \(x_6 = f(x_1, x_2, x_3, x_4, x_5)\), and compare with the second tripodal form (80), which yields another expression for

\[
x_6 = \varphi(x_2, x_4, q^{13} + q^{32} + q^{45} + q^{51}).
\]

It follows that \((\log(f_3/f_5))_{24} = 0\), which can be checked in a straightforward way for a given function \(f\). This check shows that the constants are uniquely determined, and we come to the equations

\[
x_1 x_4 + x_4 x_2 + x_2 x_6 + x_6 x_3 + x_3 x_5 - x_5 x_1 = 0,
\]

\[
(x_1 + x_4)(x_2 + x_6)(x_3 + x_5) = (x_4 + x_2)(x_6 + x_3)(x_5 + x_1).
\]

The first of them is reduced to \((T_3)\) after an additional change \(x_2 \rightarrow -x_2, x_6 \rightarrow -x_6\), while the second one coincides with \((T_2)\) after changing the signs of \(x_4, x_5, x_6\).

This finishes the proof of Theorem 7.

6 Classification of compatible quintuples

6.1 Separating away non-compatible equations

Now we have to combine tripodal equations into compatible quintuples, taking into account that the legs on the faces shared by different octahedra must coincide, according to Proposition 5. Table 10 allows us to exclude some a priori non-compatible combinations. For instance, equation of the type \((T_1)\) can only match either an equation of the same type or an equation of the type \((T_6)\). Other combinations are impossible, as shown in the following proposition.

Proposition 19. The functions

\[
xyz, \ xy, \ y, \ (x + z)^k y \ (k \neq 0), \ (x + y)z,
\]

\[
y + \log(x + z), \ \log(x + y) - \log(y + z), \ \log(x + y),
\]

are pairwise non-equivalent modulo transformations

\[
\tilde{a}(x, y, z) = \gamma a(f(x), g(y), h(z)) + \mu(x) + \nu(z)
\]

with non-constant \(f, g, h\), along with the flip \(x \leftrightarrow z\). Functions \((x + z)^k y\) with different \(k\) are also non-equivalent.

Proof. All proofs are similar, therefore we consider several pairs of functions as examples.
• The equality
\[ \gamma g(y) + \gamma \log(f(x) + h(z)) + \mu(x) + \nu(z) = \log \frac{x + y}{y + z}, \]
is impossible, as follows immediately by differentiation with respect to \(x\) and \(z\).

• Suppose that
\[ \gamma f(x)g(y)h(z) + \mu(x) + \nu(z) = (x + y)z, \]
then the differentiation gives \(\gamma f'(x)g'(y)h'(z) = 0\).

• Suppose that
\[ \gamma (f(x) + h(z))^k g(y) + \mu(x) + \nu(z) = (x + z)^m y. \]
Then \(g(y) = \alpha y + \beta\), and further \(\alpha \gamma (f(x) + h(z))^k = (x + z)^m\), so \(f(x) + h(z) = \text{const}(x + z)^{m/k}\). Differentiation with respect to \(x, z\) yields \(m = k\).

The reader is invited to work out the other pairs of functions.

A more careful analysis shows that equations \((T_1)\) and \((T_6)\) cannot be matched, although they have a common leg. More precisely, the following statement holds true.

**Proposition 20.** There do not exist compatible triples of equations where one of the equations is of the type \((T_4)\) with \(\gamma \neq 1\), or \((T_5)\), or \((T_6)\).

**Proof.** The proof is based on the fact that each of the equations listed in the proposition possesses a unique leg, which does not appear by equations of other types \((y(x + z)^\gamma\), \((x + y)z\), and \(y + \log(x + z)\), respectively). By virtue of Proposition 5, there follows that at least one further equation of the compatible triple has to be of the same type. Since all coordinate directions \(Z^4\) are on the same footing, we can assume that the compatible triple is given by equations \((52)\), the unique leg being the function \(a(x_1, x_4, x_{14})\) in the first and the second equations. Moreover, since this leg is unique, the equations themselves are recovered up to transformations \((59)\), in other words, the functions \(b(x_2, x_4, x_{24})\) and \(c(x_3, x_4, x_{34})\) can be determined. It turns out that \(b - c\) does not depend on \(x_4\), so that the third equation in \((52)\) is reducible, in contradiction with our standing assumption. In the following detailed analysis \(X_i\) stands for an arbitrary non-constant function of \(x_i\).

**Equation \((T_4)\) with \(\gamma \neq 1\).** Consider equation
\[ X_1X_{24} = (X_2 + X_{12})(X_4 + X_{14})^{-\gamma}. \]
Its tripodal form with the head \((4, 14, 24)\) is
\[ X_1(X_4 + X_{14})^\gamma - X_2/X_{24} = X_{12}/X_{24}, \]
and, comparing with the first equation in \((52)\), we find:
\[ a(x_1, x_4, x_{14}) = X_1(X_4 + X_{14})^\gamma + \lambda(x_{14}) + \mu(x_4), \]
\[ b(x_2, x_4, x_{24}) = X_2/X_{24} + \mu(x_4) + \nu(x_{24}). \]
This corresponds to a unique leg \( y(x + z) \) in Table 10, therefore the second equation in (52)

\[
X_3/X_{34} - X_1(X_4 + X_{14}) = -X_{13}/X_{34}.
\]

Therefore \( c(x_3, x_4, x_{34}) = X_3/X_{34} + \mu(x_4) + \kappa(x_{34}) \), but then the terms \( \mu(x_4) \) in the third equation in (52) cancel.

**Equation \((T_5)\).** Consider equation

\[
X_2X_{14} = X_1 + X_4 + X_{24} + X_{12} \Rightarrow \frac{X_1 + X_4}{X_{14}} - X_2 = -\frac{X_{24} + X_{12}}{X_{14}}.
\]

We have

\[
a(x_1, x_4, x_{14}) = \frac{X_1 + X_4}{X_{14}} + \lambda(x_{14}) + \mu(x_4), \quad b(x_2, x_4, x_{24}) = X_2 + \mu(x_4) + \nu(x_{24}).
\]

Due to the uniqueness of the leg \( a \) the second equation in (52) has to be of the type \((T_5)\), as well, and to have the tripodal form

\[
X_3 - \frac{X_1 + X_4}{X_{14}} = \frac{X_{34} + X_{23}}{X_{14}},
\]

whence \( c(x_3, x_4, x_{34}) = X_3 + \mu(x_4) + \kappa(x_{34}) \), and the third equation in (52) does not contain \( x_4 \).

**Equation \((T_6)\).** Consider equation

\[
X_1X_2X_{12}X_{24} = X_4 + X_{14} \Rightarrow (\log(X_4 + X_{14}) - \log X_1) - \log X_2 = \log(X_{12}X_{24}).
\]

We have

\[
a(x_1, x_4, x_{14}) = \log(X_4 + X_{14}) - \log X_1 + \lambda(x_{14}) + \mu(x_4),
\]

\[
b(x_2, x_4, x_{24}) = \log X_2 + \mu(x_4) + \nu(x_{24}).
\]

Also this \( a \) is unique, therefore the second equation in (52) has to be of the type \((T_6)\) and to have the tripodal form

\[
\log X_3 - (\log(X_4 + X_{14}) - \log X_1) = -\log(X_{13}X_{34}),
\]

whence \( c(x_3, x_4, x_{34}) = \log X_3 + \mu(x_4) + \kappa(x_{34}) \), and, as in the previous case, the third equation in (52) does not contain \( x_4 \).

### 6.2 Completing the classification

Note that after removing equations \((T_5)\) and \((T_6)\) from Table 10 the leg \( xy \) becomes unique, however the argumentation like in the proof of Proposition 20 is not possible. To handle with the remaining cases, it is not enough to use formulas (52) alone, and one has to refer to all tripodal forms, (57) and their shifted versions.

Before we formulate the final result, let us describe more precisely what transformations are allowed to bring the equations to the canonical form. First of all, these are autonomous point transformations \( x \rightarrow X(x) \) (the same at all lattice points). However, if only these are...
allowed, then the answer will contain many arbitrary constant parameters. It turns out that all these parameters are inessential and can be killed by non-autonomous transformations, which depend on the lattice point. Generally speaking, such transformations result in non-autonomous equations, therefore not all them should be allowed but only those special ones which render the transformed equation still autonomous. Their existence is related to a certain symmetry of the equations. For instance, if the equation is invariant under the one-parameter group of translations \( x \to x + a \), then it admits non-autonomous transformations \( x(i, j, k) \to x(i, j, k) + \alpha i + \beta j + \gamma k \), with the transformed equation being dependent on the parameters \( \alpha, \beta, \gamma \), which can be used to simplify the result. Similar transformations exist in all cases under consideration, and we use them to eliminate all the constant parameters (the situation is similar for continuous 3D integrable systems).

**Theorem 21.** Any compatible quintuple of irreducible shift-invariant octahedron type equations on \( Q(A_4) \) is equivalent, modulo non-autonomous point transformations, to one of the following systems (different indices stand for the shifts in different coordinate directions; recall that in the \( \mathbb{Z}^4 \) realization of the lattice \( Q(A_4) \) the shift \( T_0 \) in the coordinate direction \( 0 \) can be simply omitted).

**Five equations of type \((T_1)\):**

\[
x_{ij}x_{km} - x_{ik}x_{jm} + x_{im}x_{jk} = 0, \quad 0 \leq i < j < k < m \leq 4; \quad (\chi_1^5)
\]

**Five equations of type \((T_2)\):**

\[
\frac{(x_{ij} - x_{ik})(x_{kj} - x_{km})(x_{jm} - x_{im})}{(x_{ik} - x_{kj})(x_{km} - x_{jm})(x_{im} - x_{ij})} = -1, \quad i, j, k, m \in \{0, 1, 2, 3, 4\}; \quad (\chi_2^5)
\]

**Two different quintuples consisting of four equations of type \((T_3)\) and one equation of type \((T_2)\):**

\[
\begin{cases}
(x_{ik} - x_{ij})x_{0} + (x_{ij} - x_{jk})x_{j} + (x_{jk} - x_{ik})x_{k} = 0, & i, j, k \in \{1, 2, 3, 4\}, \\
(x_{12} - x_{13})(x_{23} - x_{34})(x_{24} - x_{14}) = -1;
\end{cases} \quad (\chi_3^3 \chi_2)
\]

and

\[
\begin{cases}
\frac{x_{ik} - x_{ij}}{x_{0}} + \frac{x_{ij} - x_{jk}}{x_{j}} + \frac{x_{jk} - x_{ik}}{x_{k}} = 0, & i, j, k \in \{1, 2, 3, 4\}, \\
(x_{12} - x_{13})(x_{23} - x_{34})(x_{24} - x_{14}) = -1;
\end{cases} \quad (\chi_4^4 \chi_2)
\]

**Three equations of type \((T_4)\) and two equations of type \((T_3)\):**

\[
\begin{cases}
x_{ij} - x_{i4} = x_{ij} \left( \frac{1}{x_{j0}} - \frac{1}{x_{i0}} \right), & i, j \in \{1, 2, 3\}, \\
x_{13} - x_{12} + \frac{x_{12} - x_{23}}{x_{20}} + \frac{x_{23} - x_{13}}{x_{30}} = 0, \quad (\chi_5^3 \chi_4^2)\\
x_{14} - x_{24} + \frac{x_{24} - x_{34}}{x_{23}} + \frac{x_{34} - x_{14}}{x_{13}} = 0.
\end{cases}
\]

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Proof. General scheme. We start with one tripodal equation, replacing variables $x_{ij}$ by yet unknown non-constant functions $X_{ij} = X_{ij}(x_{ij})$. Comparing terms in the tripodal forms (57), we are able to completely determine the consistent quintuple, up to ten arbitrary functions $X_{12}, \ldots, X_{45}$. To specify the latter, we use the shifted tripodal forms. One can show then that all functions $X_{ij}$ are related to one another via some linear-fractional transformations. Point transformations allow us to assume that the functions $X_{ij}$ are linear-fractional, with coefficients of different functions being connected by certain relations. Resolving these relations, we come to a consistent system containing several free parameters. Finally, we get rid of them using non-autonomous transformations.

Equation (T_1) can only be compatible with equations of the same type, as follows from Table 10. Equations (57) with $(i, j, k, m, n) = (1, 2, 3, 4, 0)$ read:

\begin{align*}
(1) : & \quad \frac{X_{34}}{X_{03}X_{04}} - \frac{X_{24}}{X_{02}X_{04}} + \frac{X_{23}}{X_{02}X_{03}} = 0, \\
(2) : & \quad \frac{X_{34}}{X_{03}X_{04}} - \frac{X_{14}}{X_{01}X_{04}} + \frac{X_{13}}{X_{01}X_{03}} = 0, \\
(3) : & \quad \frac{X_{24}}{X_{02}X_{04}} - \frac{X_{14}}{X_{01}X_{04}} + \frac{X_{12}}{X_{01}X_{02}} = 0.
\end{align*}

Indeed, in the first equation the functions $X_{ij}$ can be freely chosen; this defines the first term in the second equation. But then in the second term in (2) the dependence on $x_{40}$ is known, while the variables $x_{14}$ and $x_{01}$ (which were absent from (1)) enter via arbitrary new functions. Continuing in this fashion, we express all terms through ten arbitrary functions $X_{ij}$. As a corollary we get

\begin{align*}
(4) : & \quad X_{01}X_{23} - X_{02}X_{13} + X_{03}X_{12} = 0, \\
(0) : & \quad X_{12}X_{34} - X_{13}X_{24} + X_{14}X_{23} = 0.
\end{align*}

Equations (57) are herewith exhausted. To determine the functions $X_{ij}$ we have to consider the shifted tripodal forms. We write them as

\begin{align*}
T_1(1) : \quad T_1 \left( \frac{X_{02}}{X_{23}X_{24}} - \frac{X_{03}}{X_{23}X_{34}} + \frac{X_{04}}{X_{24}X_{34}} = 0 \right), \\
T_2(2) : \quad T_2 \left( \frac{X_{01}}{X_{13}X_{14}} - \frac{X_{03}}{X_{13}X_{34}} + \frac{X_{04}}{X_{14}X_{34}} = 0 \right), \\
T_3(3) : \quad T_3 \left( \frac{X_{01}}{X_{12}X_{14}} - \frac{X_{02}}{X_{12}X_{24}} + \frac{X_{04}}{X_{14}X_{24}} = 0 \right),
\end{align*}

and the comparison of the legs on the left-hand sides leads to relations like

$$\frac{\alpha X_{30}(x_0)}{X_{23}(x_2)X_{34}(x_4)} = \frac{X_{10}(x_0)}{X_{12}(x_2)X_{14}(x_4)} + \mu(x_2) + \nu(x_4).$$

It is easy to realize that the terms $\mu + \nu$ can be neglected here, and that the functions $X_{12}, X_{13}, X_{23}$ coincide up to constant factors, and the same is true for the functions $X_{01}, X_{02}, X_{03}$. Because of the symmetry of all the indices, we come to the conclusion that all ten functions are proportional. Without loss of generality we can set $X_{ij} = \alpha_{ij}x_{ij}$,
and a direct check shows that the equations are consistent for arbitrary values of $\alpha_{ij}$. All these parameters can be set equal to 1 upon use of the non-autonomous transformation

$$\tilde{x}(n_0, n_1, n_2, n_3, n_4) = \prod_{i,j} a_{i,j}^{n_i n_j} x(n_0, n_1, n_2, n_3, n_4),$$

and the answer takes the form of the quintuple $(\chi_{i,j,k,m,n}^5)$.

**Equation (T4) with $\gamma = 1$.** We consider this equation in the form

$$X_{04}(X_{24} - X_{34}) = X_{23}(X_{02} - X_{03}).$$

Identifying it with the first equation in (57) with $(i, j, k, m, n) = (1, 2, 3, 4, 0)$, we have:

$$a^{02,24,04} = X_{24}X_{04}, \quad a^{03,34,04} = X_{34}X_{04}, \quad a^{02,23,03} = X_{23}(X_{03} - X_{02}),$$

and the last two equations in (57) read:

$$\langle 2 \rangle : \quad X_{34}X_{04} - a^{01,14,04} = a^{03,13,01}, \quad \langle 3 \rangle : \quad a^{01,14,04} - X_{24}X_{04} = a^{01,12,02}.$$

Since the leg $xy$ is unique, these equations are of the type $(T_4)$, as well. This determines them in the following form:

$$X_{34}X_{04} - X_{14}X_{04} = X_{13}(X_{01} - Y_{03}), \quad X_{14}X_{04} - X_{24}X_{04} = X_{12}(Y_{02} - Y_{01}),$$

where $Y$, similarly to $X$, stand for arbitrary non-constant functions of the corresponding variables: $Y_{ij} = Y_{ij}(x_{ij})$. Summing up, we find the fourth equation

$$\langle 4 \rangle : \quad X_{12}(Y_{02} - Y_{01}) + X_{13}(X_{01} - Y_{03}) + X_{23}(X_{03} - X_{02}) = 0,$$

which can be of the type $(T_3)$ only. But then the corresponding functions $X$ and $Y$ are linearly related, and it is easy to see that we can assume that they plainly coincide (re-defining $X_{12}, X_{13},$ and $X_{15},$ if necessary). As a result, we come to the following quintuple of equations:

$$\begin{align*}
\langle 1 \rangle : \quad (X_{24} - X_{34})X_{04} &= X_{23}(X_{03} - X_{02}), \\
\langle 2 \rangle : \quad (X_{34} - X_{14})X_{04} &= X_{13}(X_{01} - X_{03}), \\
\langle 3 \rangle : \quad (X_{14} - X_{24})X_{04} &= X_{12}(X_{02} - X_{01}), \\
\langle 4 \rangle : \quad X_{12}(X_{02} - X_{01}) + X_{13}(X_{01} - X_{03}) + X_{23}(X_{03} - X_{02}) &= 0, \\
\langle 0 \rangle : \quad \frac{X_{14} - X_{24}}{X_{12}} + \frac{X_{24} - X_{34}}{X_{23}} + \frac{X_{34} - X_{14}}{X_{13}} &= 0.
\end{align*}$$

Equations (57) are now exhausted, and the remaining functional freedom has to be reduced by using their shifted versions:

$$\begin{align*}
T_1(1) : & \quad T_1(X_{23}X_{03} - X_{23}X_{02} - X_{04}(X_{24} - X_{34}) = 0), \\
T_2(2) : & \quad T_3(X_{13}X_{01} - X_{13}X_{03} - X_{04}(X_{34} - X_{14}) = 0), \\
T_3(3) : & \quad T_3(X_{12}X_{02} - X_{12}X_{01} - X_{04}(X_{14} - X_{24}) = 0).
\end{align*}$$
The comparison of legs on the left-hand sides leads to relations like

\[ \alpha X_{23}(x_2)X_{03}(x_0) = X_{12}(x_2)X_{01}(x_0) + \mu(x_2) + \nu(x_4), \]

where the terms \( \mu + \nu \) account for the freedom in the definition of the legs. There follows that the functions \( X_{12}, X_{13}, X_{23} \) are the same up to constant factors, while the relations between the functions \( X_{15}, X_{25} \) and \( X_{35} \) are affine. Further, equation (4) takes the form

\[
\alpha X_{04}(x_0)(X_{24}(x_{12}) - X_{34}(x_{13})) + \beta X_{04}(x_0)(X_{34}(x_{23}) - X_{14}(x_{12}))
+ X_{04}(x_0)(X_{14}(x_{13}) - X_{24}(x_{23})) = 0,
\]

and the comparison with the previous form leads to the conclusion that \( X_{14}, X_{24}, \) and \( X_{34} \) are affinely related to \( X_{12}, \) and \( X_{04} \) is affinely related to \( X_{15}. \) Finally, for \( \langle 0 \rangle \) we obtain the representation

\[
\frac{X_{02}(x_{12}) - X_{03}(x_{13})}{X_{04}(x_{14})} + \gamma \frac{X_{03}(x_{23}) - X_{01}(x_{12})}{X_{04}(x_{24})} + \delta \frac{X_{01}(x_{13}) - X_{02}(x_{23})}{X_{04}(x_{34})} = 0
\]

(it is enough to observe that equations (85) are invariant under the flip \( 4 \leftrightarrow 0 \) and the inversion of \( X_{04}, X_{12}, X_{13}, X_{23} \)), and a comparison with the previous representation shows that \( 1/X_{04} \) is affinely related to \( X_{14}. \) As a result, all functions are expressed through one of them (say, \( X_{12} \)), and without loss of generality we can put them as functions of \( x \) as

\[
X_{12} = x, \quad X_{13} = px, \quad X_{23} = qx, \quad X_{04} = 1/x,
\]

\[
X_{i4} = a_i x + b_i, \quad X_{0i} = c_i/x + d_i, \quad i = 1, 2, 3.
\]

Substitution into the shifted tripodal forms allows us to show that \( b_i = d_i = 0 \) and leads to equations

\[
\frac{a_i x_{i4} - a_j x_{j4}}{x_{04}} = Ax_{ij} \left( \frac{a_i}{x_{0j}} - \frac{a_j}{x_{0i}} \right), \quad i, j = 1, 2, 3.
\]

They are consistent for all parameter values. The non-autonomous change \( \tilde{x}(i, j, k, m, n) = a_i^1 a_j^2 a_k^3 A_{mn}^x x(i, j, k, m, n) \) allows us to set all parameters to 1, and we come to the consistent triple \( (x_3, x_4) \).

**Equation** \( (T_3) \). Now we do not consider equations of the type \( (T_4) \) anymore, and the leg \( y(x+z) \) becomes unique for equations of the type \( (T_3) \). It is not difficult to see that in a consistent quintuple containing such an equation at least four equations are of the same type. Indeed, either the tripodal form does not contain legs \( y(x+z) \), or all three legs are of this sort. Therefore, if one starts with such a leg for one equation of the triple \( (57) \), then all the legs for the three equations and for their sum (which is the tripodal form of the fourth equation) will be of this sort. We easily come to the following system:

\[
\begin{align*}
\langle 1 \rangle : & \quad X_{24}(X_{02} - X_{04}) + X_{34}(X_{04} - X_{03}) + X_{23}(X_{03} - X_{02}) = 0, \\
\langle 2 \rangle : & \quad X_{34}(X_{03} - X_{04}) + X_{14}(X_{04} - X_{01}) + X_{13}(X_{01} - X_{03}) = 0, \\
\langle 3 \rangle : & \quad X_{14}(X_{01} - X_{04}) + X_{24}(X_{04} - X_{02}) + X_{12}(X_{02} - X_{01}) = 0, \\
\langle 4 \rangle : & \quad X_{12}(X_{02} - X_{01}) + X_{13}(X_{01} - X_{03}) + X_{23}(X_{03} - X_{02}) = 0, \\
\langle 0 \rangle : & \quad \frac{(X_{14} - X_{12})(X_{24} - X_{23})(X_{34} - X_{13})}{(X_{12} - X_{24})(X_{23} - X_{34})(X_{13} - X_{14})} = -1.
\end{align*}
\]
Here \((4)\) follows from the first three equations in an obvious way, while \((0)\) is derived using their multiplicative representation

\[
\frac{X_{i4} - X_{ij}}{X_{ij} - X_{j4}} = \frac{X_{04} - X_{0j}}{X_{0i} - X_{04}}.
\]

In order to determine the functions \(X\), we compare legs in the shifted tripodal forms. The first three equations are re-written as

\[
T_1(1) : \quad T_1(X_{02}(X_{24} - X_{23}) + X_{03}(X_{23} - X_{34}) + X_{04}(X_{34} - X_{24}) = 0),
T_2(2) : \quad T_2(X_{01}(X_{13} - X_{14}) + X_{03}(X_{34} - X_{13}) + X_{04}(X_{14} - X_{34}) = 0),
T_3(3) : \quad T_3(X_{01}(X_{14} - X_{12}) + X_{02}(X_{12} - X_{24}) + X_{04}(X_{24} - X_{14}) = 0),
\]

and the legs comparison leads to equations like

\[
\alpha X_{03}(x_0)(X_{23}(x_2) - X_{34}(x_4)) = X_{01}(x_0)(X_{12}(x_2) - X_{14}(x_4)) + \mu(x_2) + \nu(x_4).
\]

Because of the symmetry of the indices 1, 2, 3, 4, we find that the functions \(X_{12}, X_{13}, X_{23}, X_{14}, X_{24}, X_{34}\) are affinely related, and the same is true for the functions \(X_{01}, X_{02}, X_{03}, X_{04}\). To establish a relation between the two sets of functions, it is enough to compare the legs \(T_1(a^{03,04,02})\) of equation \((1)\) and \(T_0(a^{13,14,12})\) of equation \((5)\):

\[
\log\frac{X_{02}(x_2) - X_{04}(x_4)}{X_{04}(x_4) - X_{03}(x_3)} = \lambda\log\frac{X_{12}(x_2) - X_{14}(x_4)}{X_{14}(x_4) - X_{13}(x_3)} + \mu(x_2) + \nu(x_3).
\]

Differentiating with respect to \(x_2\), we find:

\[
\frac{X_{25}'(x_2)}{X_{25}(x_2) - X_{45}(x_4)} = \frac{\lambda X_{12}'(x_2)}{X_{12}(x_2) - X_{14}(x_4)} + \mu'(x_2),
\]

whence \(X_{04}\) and \(X_{14}\) are related by a linear-fractional transformation. Without loss of generality, we can restrict ourselves to two cases:

1) \(X_{ij}(x) = a_{ij}x + b_{ij}, \quad i, j = 0, 1, 2, 3, 4;\)
2) \(X_{ij}(x) = a_{ij}x + b_{ij}, \quad i, j = 1, 2, 3, 4, \quad X_{0i}(x) = c_i/x + d_i, \quad i = 1, 2, 3, 4.\)

A rather tiresome analysis of the shifted tripodal forms shows that in the case 1) there are two possibilities: \(a_{ij} = \alpha_i\alpha_j, \quad b_{ij} = b_{12}\) or \(a_{ij} = 1, \quad b_{ij} = \beta_i + \beta_j,\) and in the case 2) \(a_{ij} = c_i c_j, \quad b_{ij} = b_{12}, \quad d_i = d_1.\) Non-autonomous dilations and translations allow us to reduce these cases to the quintuples \((\chi_{12}^1\chi_2)\) and \((\chi_{12}^4\chi_2)\).

**Equation** \((T_2)\). Now we can exclude from consideration equations of the type \((T_3)\). Then equation of the type \((T_2)\) can be a member of a consistent quintuple with equations of the same type only. As above, the tripodal forms \((57)\) allow us to show, starting with one equation, that a consistent quintuple must be of the form

\[
(X_{ij} - X_{ik})(X_{kj} - X_{km})(X_{jm} - X_{jm})(X_{im} - X_{ij}) = -1.
\]
Comparison of the shifted tripodal forms leads to relations like (87) with all possible permutations of indices, whence all \( X_{ij} \) are related by linear-fractional transformations. When we encountered such a situation above, we used the method of undetermined coefficients, but, fortunately, now one can dispense with it. Indeed, the same arguments as in Lemma 17 allow us to deduce from (87) that the linear-fractional relation between \( X_{02} \) and \( X_{12} \) is the same as between \( X_{04} \) and \( X_{14} \) or between \( X_{03} \) and \( X_{13} \) (with \( \lambda = 1 \) and \( \mu + \nu = 0 \)). This is true for other index sets, as well, so that

\[
X_{ik} = M_{ij} X_{jk}, \quad i \neq j \neq k \neq i,
\]

where \( M_{ij} \) are some linear-fractional function. There follows \( M_{ij} M_{jk} = M_{ik} \), and, setting \( L_i = M_{0i} \), \( L_0 = \text{id} \), we obtain \( M_{ij} = L_i L_j^{-1} \). But then \( L_i^{-1} X_{ik} = L_j^{-1} X_{jk} \), in particular, \( L_i^{-1} X_{0i} = L_j^{-1} X_{0j} \), and on the other hand, \( X_{ij} = L_i X_{0j} = L_j X_{0i} \), whence \( L_i L_j = L_j L_i \). Thus, \( X_{ij}(x) = L_i L_j X(x) \), where all \( L_i \) pairwise commute, and \( X(x) \) is an arbitrary function. Setting without loss of generality \( X(x) = x \), and performing a non-autonomous transformation \( \tilde{x}(n_0, n_1, n_2, n_3, n_4) = L_1^{n_1} L_2^{n_2} L_3^{n_3} L_4^{n_4} (x(n_0, n_1, n_2, n_3, n_4)) \), the situation is reduced to the case when all \( L_i = \text{id} \).

Equation (T7). A simple analysis of the tripodal forms (57) and their shifted versions shows that consistent systems of this type are reduced to linear ones with constant coefficients, which is of course not very interesting. Note however that in this case it makes sense to relax the condition of the shift invariance, which might lead to auxiliary linear problems for hypothetical 4D equations. This interesting possibility remains outside of the scope of the present paper.

7 Remarks on the Y-system

We are aware of one integrable equation on \( \mathbb{Z}^3 \) which formally belongs to the octahedron type equations and is not in our list. The so called Y-system (of type \( A_N \))

\[
h(m + e_1)h(m - e_1) = \frac{(1 - h(m + e_2))(1 - h(m - e_2))}{(1 - h^{-1}(m + e_3))(1 - h^{-1}(m - e_3))}
\]

(Y)

plays a prominent role in the theory of integrable systems [40, 5, 11, 15, 17, 21, 26, 37, 38]. The Y-system is closely related to equations (\( \chi_1 \))–(\( \chi_5 \)). However, its properties are rather different from those of (\( \chi_1 \))–(\( \chi_5 \)). It has been already shown in Section 2 that on \( \mathbb{Z}^3 \) this equation can be derived, through suitable changes of variables, from both (\( \chi_1 \)) and (\( \chi_2 \)).

We discuss here in more detail its relation to equation (\( \chi_2 \)).

Start with a 3D Desargues (or Laplace-Darboux) map as defined in Section 3.4, i.e., with a map \( x : Q(A_3) \to \mathbb{R}^P \) such that the image of any white triangle is a collinear triple of points. In the representation of \( Q(A_3) \) as \( \mathbb{Z}^3_{\text{even}} \), this can be seen as a 2D Q-net (indexed by \( m_1, m_2 \)) and its iterated Laplace transforms (indexed by \( m_3 \)), see [33, 12]; one can find definitions of a Q-net and of its Laplace transforms in [7]. Thus, the vertices of any white tetrahedron are collinear, and therefore one can introduce their (real) cross-ratio, called \( h \) (or \( 1 - h \), or else \( 1 - h^{-1} \), depending on the ordering of the arguments of the cross-ratio). Any black tetrahedron has six white neighbors, adjacent to it along its
six edges. A straightforward computation shows that the variables $h$ for these six white tetrahedra are related by equation (Y) which basically says that the product of six cross-ratios is equal to 1. To obtain it, one has to consider four octahedra adjacent to the black tetrahedron along its black faces. They carry four multi-ratio equations ($\chi_2$) coming from the corresponding four complete quadrilaterals. These equations, being multiplied, give the Y-system (the product of four multi-ratios can be re-arranged as a product of six cross-ratios). To sum up: variables $h$ of equation (Y) live on white tetrahedra, while the equation itself is assigned to black tetrahedra of $Q(A_3)$.

One can try to make this picture multidimensional, by starting with a multidimensional Desargues map $x : Q(A_N) \to \mathbb{RP}^n$. Then any 3D sublattice $x : Q(A_3) \to \mathbb{RP}^n$ will support its own copy of equation (Y), with the variables $h$ which show up in this copy and in no other: they are defined on 3-cells, and each 3-cell belongs to only one 3D sublattice. The only relation between these co-existing copies of equation (Y) will be the condition that the product of $h$’s for all tetrahedral faces of a white simplex is equal to 1.

Thus, although equation (Y) is closely related to multidimensionally consistent octahedron type equations, there is no reason for itself to be multidimensionally consistent.

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