QUADRATIC CHABAUTY FOR MODULAR CURVES

SAMIR SIKSEK

Abstract. Let $X/\mathbb{Q}$ be a curve of genus $g \geq 2$ with Jacobian $J$ and let $\ell$ be a prime of good reduction. Using Selmer varieties, Kim defines a decreasing sequence

$$X(\mathbb{Q}_\ell) \supseteq X(\mathbb{Q}_\ell)_1 \supseteq X(\mathbb{Q}_\ell)_2 \supseteq \cdots$$

all containing $X(\mathbb{Q})$. Thanks to the work of Coleman, the ‘Chabauty set’ $X(\mathbb{Q}_\ell)_1$ is known to be finite provided the ‘Chabauty condition’ rank $J(\mathbb{Q}) < g$ holds. In this case one has a practical strategy that often succeeds in computing the set of rational points $X(\mathbb{Q})$. Balakrishnan and Dogra have recently shown that the ‘quadratic Chabauty set’ $X(\mathbb{Q}_\ell)_2$ is finite provided

$$\text{rank } J(\mathbb{Q}) < g + \text{rank NS}(J) - 1,$$

where NS$(J)$ is the Néron-Severi group of $J/\mathbb{Q}$. In view of this it is interesting to give families of curves where rank NS$(J) \geq 2$ and where therefore quadratic Chabauty is more likely to succeed than classical Chabauty. In this note we show that this is indeed the case for all modular curves of genus $\geq 3$.

1. Introduction

Let $X$ be a smooth algebraic curve of genus $g \geq 2$ defined over $\mathbb{Q}$ and write $J$ for its Jacobian. Let $\ell$ be a prime of good reduction for $X$. The method of Chabauty and Coleman [3] yields a subset $X(\mathbb{Q}_\ell)_1 \subseteq X(\mathbb{Q}_\ell)$ (which we may term the ‘Chabauty set’) containing $X(\mathbb{Q})$ that can be explicitly described in terms of Coleman integrals. Coleman [3] showed that $X(\mathbb{Q}_\ell)_1$ is finite provided the ‘Chabauty condition’ holds:

$$\text{rank } J(\mathbb{Q}) < g.$$

This implies that $X(\mathbb{Q})$ is finite. Of course Faltings’ Theorem [4] asserts the finiteness of $X(\mathbb{Q})$ without assumptions beyond $g \geq 2$. However, Faltings’ Theorem is ineffective. In contrast (see for example [7] or [10]) the method of Chabauty and Coleman often allows for the computation of $X(\mathbb{Q})$ provided the Chabauty condition (1) is satisfied and one knows a Mordell–Weil basis for $J(\mathbb{Q})$ (or even for a subgroup of $J(\mathbb{Q})$ of finite index).

In [6], Kim defined a family of Selmer varieties $\text{Sel}(U_n)$ giving a decreasing sequence of subsets

$$X(\mathbb{Q}_\ell) \supseteq X(\mathbb{Q}_\ell)_1 \supseteq X(\mathbb{Q}_\ell)_2 \supseteq \cdots \supseteq X(\mathbb{Q}),$$

which can be computed in terms of iterated Coleman integrals. This offers hope of a strategy to compute $X(\mathbb{Q})$ even if the Chabauty condition (1) fails. Indeed, the conjectures of Bloch and Kato imply that $X(\mathbb{Q}_\ell)_n$ is finite for $n$ sufficiently

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The following theorem of Balakrishnan and Dogra \cite{1, Lemma 3} gives a criterion for the finiteness of \(X(\mathbb{Q}_\ell)_2\).

**Theorem 1** (Balakrishnan & Dogra \cite{1}). Suppose
\[
\text{rank } J(\mathbb{Q}) < g + \text{rank } \text{NS}(J) - 1,
\]
where \(\text{NS}(J)\) is the Néron–Severi rank of \(J\) over \(\mathbb{Q}\). Then \(X(\mathbb{Q}_\ell)_2\) is finite.

Indeed, Balakrishnan & Dogra explain how, under condition (2), a function can be constructed that cuts out \(X(\mathbb{Q}_\ell)_2\) explicitly (following \cite{2} this method is termed Quadratic Chabauty). They then apply this method to determine the rational points on several genus 2 curves that have the form
\[
X : y^2 = a_6 x^6 + a_4 x^4 + a_2 x^2 + a_0,
\]
where \(\text{rank } J(\mathbb{Q}) = 2\).

The method of Balakrishnan & Dogra at first seems rather special. For a ‘generic’ curve \(X\) it is known that the Néron–Severi group has rank 1 (and is in fact spanned by the class of the principal polarization). Hence the inequality (2) is equivalent to the inequality (1), and the method of Balakrishnan & Dogra does not yield any more than the classical Chabauty. Arguably modular curves are the most interesting family of curves. In this note, we give an explicit sufficiency criterion, based on the theorem of Balakrishnan & Dogra, for the finiteness of \(X(\mathbb{Q}_\ell)_2\) when \(X\) is a modular curve. In fact we show that for any modular curve \(X/\mathbb{Q}\) of genus \(\geq 3\) we have \(\text{rank } \text{NS}(J) \geq 2\), thus quadratic Chabauty is more likely to succeed than classical Chabauty.

**2. The Balakrishnan–Dogra Criterion and Modular Curves**

Let \(G\) be a congruence subgroup of \(\text{SL}_2(\mathbb{Z})\) and let \(X = X_G\) be the corresponding modular curve which we shall suppose is defined over \(\mathbb{Q}\) (sufficient conditions on \(G\) for this to hold are given in \cite{3} Section 1.2). Assume that \(X\) has genus \(g \geq 2\) and write \(J\) for the Jacobian of \(X\). Let \(f_1, \ldots, f_n\) be representatives of the Galois orbits of weight 2 cuspidal eigenforms for \(G\) (these may be computed via the modular symbols algorithm as in \cite{11}). The theory of Eichler–Shimura attaches to each \(f_i\) an abelian variety of \(\text{GL}_2\)-type, which we shall denote by \(A_i\). Then
\[
J \approx A_1 \times \cdots \times A_n,
\]
where \(\approx\) denotes isogeny over \(\mathbb{Q}\). As \(A_i\) is of \(\text{GL}_2\)-type, we have \(\text{End}(A_i) \otimes \mathbb{Q}\) is a number field \(F_i\) of degree \(\dim A_i\), which is either totally real or CM. The fields \(F_i\) are generated by the Hecke eigenvalues of \(f_i\). We arrange the \(f_i\) so that \(F_1, \ldots, F_m\) are totally real and \(F_{m+1}, \ldots, F_n\) are CM.

**Corollary 2.1.** With the above notation and assumptions, suppose
\[
\text{rank } J(\mathbb{Q}) < -1 + 2 \sum_{i=1}^m [F_i : \mathbb{Q}] + \frac{3}{2} \sum_{i=m+1}^n [F_i : \mathbb{Q}].
\]
Then, for any prime \(\ell\) of good reduction for \(X\), the set \(X(\mathbb{Q}_\ell)_2\) is finite.

**Proof.** Note that
\[
g = \dim(J) = \sum_{i=1}^n \dim A_i = \sum_{i=1}^n [F_i : \mathbb{Q}].
\]
Thus (5) maybe rewritten as

\[
\text{rank } J(\mathbb{Q}) < (g - 1) + \sum_{i=1}^{m} [F_i : \mathbb{Q}] + \frac{1}{2} \sum_{i=m+1}^{n} [F_i : \mathbb{Q}].
\]

It is sufficient to show, thanks to the theorem of Balakrishnan & Dogra, that

\[
\text{rank } \text{NS}(J) \geq \sum_{i=1}^{m} [F_i : \mathbb{Q}] + \frac{1}{2} \sum_{i=m+1}^{n} [F_i : \mathbb{Q}].
\]

We now invoke the isomorphism of $\mathbb{Q}$-vector spaces

\[
\text{NS}(J) \otimes \mathbb{Q} \cong \text{End}^{(s)}(J) \otimes \mathbb{Q},
\]

where $\text{End}^{(s)}(J)$ is the subring of $\text{End}(J)$ left invariant by the Rosati involution (see [5, Section 1] for example). However, the Rosati involution restricted to $\text{End}(A_i)$ induces complex conjugation on $F_i$ ([8, page 196]). Thus we see that $\text{End}^{(s)}(J) \otimes \mathbb{Q}$ contains the algebra

\[
F_1 \times \cdots \times F_m \times E_{m+1} \times \cdots \times E_n,
\]

where, for $m + 1 \leq i \leq n$, we denote the maximal totally real subfield of $F_i$ by $E_i$. As $F_i$ is CM, we have $[E_i : \mathbb{Q}] = [F_i : \mathbb{Q}]/2$. The corollary follows. \qed

It is evident that (6) (and hence (5)) is weaker than Chabauty’s condition (11) whenever $g \geq 3$. If $g = 2$ then (6) is weaker if either $J$ has real multiplication, or is isogenous to a product of two elliptic curves.

**Remark.** If $J$ has multiplicity 1 (meaning that the simple factors $A_i$ are pairwise non-isogenous) then $\text{End}(J) \cong \text{End}(A_1) \times \cdots \times \text{End}(A_n)$ and the above argument in fact shows that $\text{NS}(J) \otimes \mathbb{Q}$ is isomorphic to the algebra (7).

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Mathematics Institute, University of Warwick, Coventry, CV4 7AL, United Kingdom
E-mail address: s.siksek@warwick.ac.uk