CONVERGENCE OF THE EMPIRICAL TWO-SAMPLE U-STATISTICS WITH β-MIXING DATA

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Abstract. We consider the empirical two-sample U-statistic with strictly β-mixing strictly stationary data and investigate its convergence in Skorohod spaces. We then provide an application of such convergence.

1. Introduction and main results

In this paper, we investigate the large sample behavior of the empirical distribution function of the two-sample U-statistic, given by

\[ e_n(s, t) := \frac{1}{n^{3/2}} \sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} (\mathbb{1}\{g(X_i, X_j) \leq s\} - \mathbb{P}\{g(X_i, X_j) \leq s\}), \quad 0 \leq t \leq 1, s \in \mathbb{R}, \quad (1.1) \]

and viewed as a two-parameter process indexed by the parameters \((s, t) \in \mathbb{R} \times [0, 1]\). We call the process \((e_n(s, t))_{(s,t) \in [0,1]}\) a two-sample empirical U-process, and we will show that this process converges, as \(n \to \infty\), to a Gaussian limit process, if the underlying process \((X_i)_{i \geq 1}\) is short range dependent.

Date: June 16, 2020.

Key words and phrases. Two-sample U-statistics, empirical process, functional central limit theorem, mixing processes, short-range dependence.
Related processes have been studied in the literature. Dehling, Fried, Garcia and Wendler [8] have analyzed the large sample behavior of the two-sample $U$-process, defined by

$$
\frac{1}{n^{3/2}} \sum_{i=1}^{[nt]} \sum_{j=\lfloor nt \rfloor + 1}^{n} h(X_i, X_j), \quad 0 \leq t \leq 1,
$$

where $h: \mathbb{R}^2 \to \mathbb{R}$ is a measurable function, also called kernel of the $U$-statistic. Note that, for fixed value of $s \in \mathbb{R}$, the two-sample empirical $U$-process is a two-sample $U$-process, with the indicator kernel $h(x, y) = 1 \{g(x, y) \leq s\}$. Dehling et al. [8] proved convergence of this process to a Gaussian limit process, in case the underlying data are functionals of an absolutely regular process, thus extending earlier results by Csörgő and Horváth [4] for i.i.d. data.

Along another line, various authors have investigated the large sample behavior of the one-sample empirical $U$-process, defined as

$$
\frac{1}{n^{3/2}} \sum_{1 \leq i < j \leq n} (1 \{g(X_i, X_j) \leq s\} - \mathbb{P}(g(X_1, X_2) \leq s)), \ s \in \mathbb{R},
$$

where $g: \mathbb{R}^2 \to \mathbb{R}$ is a symmetric kernel. Serfling [15] initiated the study of these processes for i.i.d. data in connection with so-called generalized $L$-, $M$-, and $R$-statistics. Dehling, Denker and Philipp [7] proved an almost sure invariance principle, again for i.i.d. data. Arcones and Yu [1] investigated the one-sample empirical $U$-process for absolutely regular data. Motivated by applications to estimation of the correlation dimension of chaotic dynamical systems, Borovkova, Burton and Dehling [3] showed weak convergence of the one-sample empirical $U$-process to a Gaussian limit process, when the underlying data are functionals of absolutely regular processes. Lévy-Leduc, Boistard, Moulines, Reisen and Taqqu [12] studied the empirical $U$-process under long-range dependence.

In the present paper, we focus on the two-parameter empirical $U$-process, defined in (1.1), when the underlying data are generated by a stationary mixing stochastic process $(X_i)_{i \geq 1}$. From the results of Dehling et al. [8], we obtain convergence of the process $(e_n(s_0, t))_{0 \leq t \leq 1}$ for a fixed $s_0$. Here, we will extend this result to the two-parameter process $(e_n(s, t))_{(s, t) \in \mathbb{R} \times [0, 1]}$.

In this paper, we focus on the case of mixing sequences. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The $\alpha$-mixing and $\beta$-mixing coefficients between two sub-$\sigma$-algebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{F}$ are defined defined respectively by

$$
\alpha(\mathcal{A}, \mathcal{B}) = \sup \{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, A \in \mathcal{A}, B \in \mathcal{B}\};
$$

$$
\beta(\mathcal{A}, \mathcal{B}) = \frac{1}{2} \sup \left\{ \sum_{i=1}^{l} \sum_{j=1}^{r} |\mathbb{P}(A_i \cap B_j) - \mathbb{P}(A_i)\mathbb{P}(B_j)| \right\},
$$

where the supremum runs over all the partitions $(A_i)_{i=1}^{l}$ and $(B_j)_{j=1}^{r}$ of $\Omega$ of elements of $\mathcal{A}$ and $\mathcal{B}$ respectively.

Given a sequence $(X_i)_{i \geq 1}$, we associate its sequences of $\alpha$ and $\beta$-mixing coefficients by letting

$$
\alpha(k) := \sup_{l \geq 1} \alpha(\mathcal{F}_1^l, \mathcal{F}_{l+k}^\infty),
$$

$$
\beta(k) := \sup_{l \geq 1} \beta(\mathcal{F}_1^l, \mathcal{F}_{l+k}^\infty),
$$

where $\mathcal{F}_v, 1 \leq u \leq v \leq +\infty$ is the $\sigma$-algebra generated by the random variables $X_i$, $u \leq i \leq v$ ($u \leq i$ for $v = +\infty$).
1.1. Convergence of the two-sample $U$-statistic in Skorohod spaces $D([−R, R] \times [0, 1])$. Let us state one of the two main results of the paper.

**Theorem 1.1.** Let $(X_i)_{i \in \mathbb{Z}}$ be a strictly stationary sequence. Let $e_n$ be the two-sample $U$-statistics empirical process with kernel $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined for $n \geq 1$, $0 \leq t \leq 1$ and $s \in \mathbb{R}$ by

$$e_n (s, t) := \frac{1}{n^{3/2}} \sum_{i=1}^{[nt]} \sum_{j=\lceil nt \rceil + 1}^{n} \left(1 \{g(X_i, X_j) \leq s\} - P \{g(X_i, X_j) \leq s\}\right).$$

(1.6)

Suppose that the following four conditions holds.

(A.1) For all $u \in \mathbb{R}$, the random variable $g(u, X_1)$ has a density $f_{1,u}$ and $\sup_{u \in \mathbb{R}} f_{1,u}(x) < +\infty$.

(A.2) For all $v \in \mathbb{R}$, the random variable $g(X_1, v)$ has a density $f_{2,v}$ and $\sup_{v \in \mathbb{R}} f_{2,v}(x) < +\infty$.

(A.3) There exists a $p > 2$ such that $\sum_{k \geq 1} k^p \alpha(k)$ converges.

(A.4) The series $\sum_{k \geq 1} k^\beta(k)$ converges.

Then for all $R$,

$$e_n (s, t) \to W (s, t) \text{ in distribution in } D([−R, R] \times [0, 1]),$$

(1.7)

where $(W (s, t), s \in \mathbb{R}, t \in [0, 1])$ is a centered Gaussian process, with covariance given for $0 \leq t \leq t' \leq 1$ and $s, s' \in \mathbb{R}$ by the following formula:

$$\text{Cov} \left(W (s, t), W \left(s', t'\right)\right) = t (1 - t) \left(1 - t'\right) C_{1,1} (s, s') + t \left(1 - t\right) \left(t' - t\right) C_{2,1} (s, s') + t' \left(1 - t\right) C_{2,2} (s, s'),$$

(1.8)

where for $i, j = 1; 2$ and $s, s' \in \mathbb{R}$,

$$C_{i,j} (s, s') = \sum_{k \in \mathbb{Z}} \text{Cov} \left(h_{i,s} (X_k), h_{j,s'} (X_k)\right),$$

(1.9)

$$h_{1,s} (u) = P \{g(u, X_1) \leq s\},$$

(1.10)

$$h_{2,s} (v) = P \{g(X_1, v) \leq s\}.$$  

(1.11)

**Remark 1.2.** We did not make a symmetry assumption on $g$. When $g$ is symmetric, in the sense that $g(u, v) = g(v, u)$ for all $u$ and $v \in \mathbb{R}$, the covariance of the limiting process $W$ reads

$$\text{Cov} \left(W (s, t), W \left(s', t'\right)\right) = t (1 - t) \left(1 + 2t' - 2t\right) C_{1,1} (s, s').$$

(1.12)

In practical cases, the probability $P \{g(X_i, X_j) \leq s\}$ is unknown, and we only have the values of $1 \{g(X_i, X_j) \leq s\}, 1 \leq i < j \leq n$, at our disposal. This leads to an analogous result as Theorem 1.1, where the quantity $P \{g(X_i, X_j) \leq s\}$ is replaced by its empirical estimator $\left(\frac{n}{2}\right)^{-1} \sum_{1 \leq i < j \leq n} 1 \{g(X_i, X_j) \leq s\}$.

**Theorem 1.3.** Let $(X_i)_{i \in \mathbb{Z}}$ be a strictly stationary sequence. Let $e'_n$ be the two-sample $U$-statistics empirical process with kernel $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined for $n \geq 1$, $0 \leq t \leq 1$ and $s \in \mathbb{R}$ by

$$e'_n (s, t) := \frac{1}{n^{3/2}} \sum_{i=1}^{[nt]} \sum_{j=\lceil nt \rceil + 1}^{n} \left(1 \{g(X_i, X_j) \leq s\} - \frac{1}{\binom{n}{2}} \sum_{1 \leq i' < j' \leq n} 1 \{g(X_{i'}, X_{j'}) \leq s\}\right).$$

(1.13)

Suppose that the assumptions (A.1), (A.2), (A.3) and (A.4) hold. Then for all $R$,

$$e'_n (s, t) \to W (s, t) \text{ in distribution in } D([−R, R] \times [0, 1]),$$

(1.14)
where \((W(s, t), s \in \mathbb{R}, t \in [0, 1])\) is a centered Gaussian process, with covariance given for \(0 \leq t \leq t' \leq 1\) and \(s, s' \in \mathbb{R}\) by the following formula:

\[
\text{Cov} \left( W(s, t), W(s', t') \right) = t \left( 1 - t \right) \left( 1 - t' \right) C_{1,1} \left( s, s' \right) + \left( t' - t \right) t \left( 1 - t' \right) C_{2,1} \left( s, s' \right) + \left( 1 - t' \right) t' C_{2,2} \left( s, s' \right) - 2t \left( 1 - t \right) \left( 1 - t' \right) \left( 2 + t' - 2t - t' \right) C_1^2 \left( s, s' \right) - 2t \left( 1 - t \right) \left( 1 - t' \right) C_2^2 \left( s, s' \right) + 4t \left( 1 - t \right) t' \left( 1 - t' \right) \left( t'^2 + t + \frac{10}{3} \right) C^n \left( s, s' \right),
\]

where

\[
h_{1,s} (u) = \mathbb{P} \{ g(u, X_1) \leq s \},
\]
\[
h_{2,s} (v) = \mathbb{P} \{ g(X_1, v) \leq s \},
\]
\[
a_s = h_{1,s} - h_{2,s},
\]
for \(i, j = 1, 2\),

\[
C_{i,j} \left( s, s' \right) = \sum_{k \in \mathbb{Z}} \text{Cov} \left( h_{i,s}(X_0), h_{j,s'}(X_k) \right),
\]
\[
C_1^a \left( s, s' \right) = \sum_{k \in \mathbb{Z}} \text{Cov} \left( h_{1,s}(X_0), a_{s'}(X_k) \right)
\]
\[
C_2^a \left( s, s' \right) = \sum_{k \in \mathbb{Z}} \text{Cov} \left( a_s(X_0), a_{s'}(X_k) \right).
\]

**Remark 1.4.** When \(g\) is symmetric, \(h_{1,s} = h_{2,s}\) and \(a_s = 0\) hence the covariance admits the simpler form

\[
\text{Cov} \left( W(s, t), W(s', t') \right) = t \left( 1 - t' \right) \left( 1 - 2t + 2t' \right) C_{1,1} \left( s, s' \right), t \leq t', s \in \mathbb{R},
\]

in particular, we get the same limiting process as in the centering of the indicator by their expectation (see Remark 1.2).

Let us give examples where the assumptions (A.1) and (A.2) are satisfied. Let \(g_1\) and \(g_2\) be functions defined from \(\mathbb{R}\) to itself. Assume that \(g_1(X_1)\) has a density \(f_1\) and \(g_2(X_1)\) has a density \(f_2\), where \(f_1\) and \(f_2\) are bounded.

(1) Let \(g_1(u) \mapsto g_1(u) + g_2(v)\). Then \(g(u, X_1)\) has density \(f_{1,u}\) where \(f_{1,u}(x) = f_2(x - g_1(u))\), hence \(\sup_{x,u \in \mathbb{R}} f_{1,u}(x) = \sup_{x \in \mathbb{R}} f_2(x) < +\infty\) and similarly, \(\sup_{x,u \in \mathbb{R}} f_{2,u}(x) = \sup_{x \in \mathbb{R}} f_1(x) < +\infty\).

(2) Let \(g_1(u) \mapsto |g_1(u) - g_2(v)|\). Then \(f_{1,u}(x) = 0\) for \(x < 0\) and for \(x \geq 0\),

\[
f_{1,u}(x) = f_2(g_1(u) + x) + f_2(g_1(u) - x)
\]

hence

\[
\sup_{x,u \in \mathbb{R}} f_{1,u}(x) \leq \sup_{x,u \in \mathbb{R}} f_3(x) + \sup_{x,u \in \mathbb{R}} f_2(g_1(u) - x) \leq 2 \sup_{x \in \mathbb{R}} f_2(x) < +\infty
\]

and by a similar reasoning, we also derive that

\[
\sup_{x,u \in \mathbb{R}} f_{2,u}(x) \leq 2 \sup_{x \in \mathbb{R}} f_1(x) < +\infty.
\]
1.2. Application. The following corollaries are a consequence of Theorems 1.1 and 1.3.

**Corollary 1.5.** Under the conditions of Theorems 1.1 and 1.3 the following convergences in distribution take place for all positive $R$:

\[
\sup_{0 \leq t \leq 1} \sup_{-R \leq s \leq R} |e_n(s, t)| \rightarrow \sup_{0 \leq t \leq 1} \sup_{-R \leq s \leq R} |W(s, t)|; \tag{1.26}
\]

\[
\sup_{0 \leq t \leq 1} \sup_{-R \leq s \leq R} |e'_n(s, t)| \rightarrow \sup_{0 \leq t \leq 1} \sup_{-R \leq s \leq R} |W'(s, t)|. \tag{1.27}
\]

**Corollary 1.6.** Let $\mu$ be a finite measure on the Borel subsets of $\mathbb{R}$. Then under the conditions of Theorems 1.1 and 1.3 the following convergences in distribution take place

\[
\sup_{0 \leq t \leq 1} \int_{-R}^{R} e_n(s, t)^2 d\mu(s) \rightarrow \sup_{0 \leq t \leq 1} \int_{-R}^{R} W(s, t)^2 d\mu(s) \tag{1.28}
\]

\[
\sup_{0 \leq t \leq 1} \int_{-R}^{R} e'_n(s, t)^2 d\mu(s) \rightarrow \sup_{0 \leq t \leq 1} \int_{-R}^{R} W'(s, t)^2 d\mu(s). \tag{1.29}
\]

2. Proof

The proof of Theorems 1.1 and 1.3 will be done according to the following steps.

1. Let $h_s: \mathbb{R}^2 \rightarrow \mathbb{R}$ be the kernel defined by $h_s(u, v) = \mathbbm{1}\{g(u, v) \leq s\}$. The Hoeffding’s decomposition of this kernel gives a splitting of the empirical two-sample $U$-statistics into a linear part and a degenerated part.

2. We prove the convergence of the finite dimensional distributions of the linear part to the corresponding ones of the process $W$.

3. Then we prove that the process associated to the linear part converges to $W$ in $D([\mathbb{R} \times [0, 1]])$ for all $R > 0$.

4. Finally, we show the negligibility of the contribution of the degenerated part.

We do it first in the context of Theorem 1.1. The proof of Theorem 1.3 is closely related. Consequently, we will only mention the required modifications.

2.1. Proof of Theorem 1.1.

2.1.1. Hoeffding’s decomposition. Let $h_s: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by $h_s(u, v) = \mathbbm{1}\{g(u, v) \leq s\}$. Let us do the Hoeffding’s decomposition of $h_s$ for each fixed $s$. Let $\theta_s := \mathbb{P}\{g(X_1, X'_1) \leq s\}$, where $X'_1$ is an independent copy of $X_1$,

\[
h_{1,s}(u) = \mathbb{P}\{g(u, X_1) \leq s\} - \theta_s, \quad h_{2,s}(v) = \mathbb{P}\{g(X_1, v) \leq s\} - \theta_s, \tag{2.1}
\]

and

\[
h_{3,s}(u, v) = h_s(u, v) - h_{1,s}(u) - h_{2,s}(v) - \theta_s. \tag{2.2}
\]
Then
\[
e_n (s, t) = \frac{1}{n^{3/2}} \sum_{i=1}^{[nt]} \sum_{j=\lfloor nt \rfloor +1}^{n} (h_s (X_i, X_j) - \mathbb{E} [h_s (X_i, X_j)])
\]
\[
= \frac{1}{n^{3/2}} \sum_{i=1}^{[nt]} \sum_{j=\lfloor nt \rfloor +1}^{n} (h_{3,s} (X_i, X_j) - \mathbb{E} [h_{3,s} (X_i, X_j)]) + \frac{1}{n^{3/2}} \sum_{i=1}^{[nt]} \sum_{j=\lfloor nt \rfloor +1}^{n} (h_{1,s} (X_i) - \mathbb{E} [h_{1,s} (X_i)])
\]
\[
+ \frac{1}{n^{1/2}} \sum_{i=1}^{[nt]} \sum_{j=\lfloor nt \rfloor +1}^{n} (h_{2,s} (X_j) - \mathbb{E} [h_{2,s} (X_j)])
\]
or in other words,
\[
e_n (s, t) = R_n (s, t) + W_n (s, t),
\]
where
\[
R_n (s, t) = \frac{1}{n^{3/2}} \sum_{i=1}^{[nt]} \sum_{j=\lfloor nt \rfloor +1}^{n} (h_{3,s} (X_i, X_j) - \mathbb{E} [h_{3,s} (X_i, X_j)])
\]
\[
W_n (s, t) = \frac{n - [nt]}{n^{1/2}} \sum_{i=1}^{[nt]} (h_{1,s} (X_i) - \mathbb{E} [h_{1,s} (X_i)]) + \frac{[nt]}{n^{1/2}} \sum_{j=\lfloor nt \rfloor +1}^{n} (h_{2,s} (X_j) - \mathbb{E} [h_{2,s} (X_j)]).
\]

Moreover, observe that by the assumptions (A.1) and (A.2), there exists a constant $M$ such that for all $i \in \{1, 2, 3\}$,
\[
\sup_{x \in \mathbb{R}} \sup_{s \leq s'} \frac{|h_{1,i'} (x) - h_{1,s} (x)|}{s' - s} \leq M.
\]

2.1.2. Convergence of the finite dimensional distributions of the linear part.

**Proposition 2.1.** For all $d \geq 1$ and all $s_1 < \cdots < s_d$ and $0 \leq t_1 < \cdots < t_d \leq 1$, the vector $(W_n (s_\ell, t_\ell))_{\ell=1}^{d}$ converges in distribution to $(W (s_\ell, t_\ell))_{\ell=1}^{d}$, where $W_n$ is defined by (2.5) and $W$ is like in Theorem 1.1.

**Proof.** We will use the Cramer-Wold device. Let $(a_{k,\ell})_{k,\ell=1}^{d}$ be a family of real numbers. We have to prove that
\[
\sum_{k,\ell=1}^{d} a_{k,\ell} W_n (s_\ell, t_k) \to \sum_{k,\ell=1}^{d} a_{k,\ell} W (s_\ell, t_k) \text{ in distribution.}
\]
To this aim, we will express $\sum_{k,\ell=1}^{d} a_{k,\ell} W_n (s_\ell, t_k)$ as a sum of linear combinations of a mixing sequence random variables, and then apply a central limit theorem, namely, Theorem A.1. Let
\[
I_{n,u} = \{i \in \mathbb{N} \mid [nt_{u-1} + 1 \leq i \leq [nt_u], 2 \leq u \leq d,
\]
Indeed, by Proposition A.2, for $i \in I_{n,u}$ and $i' \in I_{n,u'}$

$$|\text{Cov}(Y_{n,i}, Y_{n,i'})| \leq \alpha (i'-i) \left(2 \sum_{k,\ell=1}^{d} |a_{k,\ell}| \right)^2 =: \alpha (i'-i) K$$

hence

$$\frac{1}{n} \left| \text{Cov} \left( \sum_{i \in I_{n,u}} Y_{n,i}, \sum_{i' \in I_{n,u'}} Y_{n,i'} \right) \right| \leq \frac{K}{n} \sum_{i=[nt_u]}^{[nt_u]+1} \sum_{i'=[nt_{u'}]+1}^{[nt_{u'}]+1} \alpha (i'-i).$$

Doing the change of index $j = i' - i$ in the inner sum, it follows that

$$\frac{1}{n} \left| \text{Cov} \left( \sum_{i \in I_{n,u}} Y_{n,i}, \sum_{i' \in I_{n,u'}} Y_{n,i'} \right) \right| \leq \frac{K}{n} \sum_{i=[nt_u]}^{[nt_u]+1} \sum_{j=[nt_{u'}]+1}^{[nt_{u'}]+1} \alpha (j)$$

or in other words,

$$\sum_{k,\ell=1}^{d} a_{k,\ell} W_n (s_{\ell}, t_k) = \sum_{u=1}^{d+1} \sum_{u \in I_{n,u}} \sum_{k,\ell=1}^{d} a_{k,\ell} \frac{n - [nt_k]}{n^{3/2}} \sum_{h_{1,\ell}} (X_i) - \mathbb{E} [h_{1,\ell} (X_i)]) + \sum_{u=1}^{d+1} \sum_{u \in I_{n,u}} \sum_{k,\ell=1}^{d} a_{k,\ell} \frac{[nt_k]}{n^{3/2}} \sum_{h_{2,\ell}} (X_i) - \mathbb{E} [h_{2,\ell} (X_i)]$$

Defining for $i \in I_{n,u}$, $1 \leq u \leq d+1$ the random variable $Y_{n,i}$ by

$$Y_{n,i} := \sum_{k,\ell=1}^{d} a_{k,\ell} \frac{n - [nt_k]}{n^{3/2}} 1 \{u \leq k\} (h_{1,\ell} (X_i) - \mathbb{E} [h_{1,\ell} (X_i)])$$

$$+ \sum_{u=1}^{d+1} \sum_{u \in I_{n,u}} \sum_{k,\ell=1}^{d} a_{k,\ell} \frac{[nt_k]}{n^{3/2}} 1 \{u \geq k\} (h_{2,\ell} (X_i) - \mathbb{E} [h_{2,\ell} (X_i)])$$

it follows that $\sum_{k,\ell=1}^{d} a_{k,\ell} W_n (s_{\ell}, t_k) = n^{-1/2} \sum_{n=1}^{n} Y_{n,i}$. Observe also that $\mathbb{E} [Y_{n,i}] = 0$. We want to check the conditions of Theorem A.1. The first condition follows from

$$|Y_{n,i}| \leq \sum_{k,\ell=1}^{d} |a_{k,\ell}| (1 \{u \leq k\} |h_{1,\ell} (X_i) - \mathbb{E} [h_{1,\ell} (X_i)])$$

$$+ \sum_{u=1}^{d+1} \sum_{u \in I_{n,u}} \sum_{k,\ell=1}^{d} a_{k,\ell} \frac{[nt_k]}{n^{3/2}} \{u \geq k\} |h_{2,\ell} (X_i) - \mathbb{E} [h_{2,\ell} (X_i)]] \leq 2 \sum_{k,\ell=1}^{d} |a_{k,\ell}|.$$
and since \([nt_u] \leq [nt_{u'} - 1] + 1\) for \(n\) large enough, we get

\[
\frac{1}{n} \left| \text{Cov} \left( \sum_{i \in I_{n,u}} Y_{n,i}, \sum_{i' \in I_{n,u'}} Y_{n,i'} \right) \right| \leq \frac{K}{n} \sum_{i=1}^{[nt_{u'} - 1]} \sum_{j=0}^{\infty} \alpha(j); \quad (2.17)
\]

doing the change \(k = [nt_{u'} - 1] - i\) gives

\[
\frac{1}{n} \left| \text{Cov} \left( \sum_{i \in I_{n,u}} Y_{n,i}, \sum_{i' \in I_{n,u'}} Y_{n,i'} \right) \right| \leq \frac{K}{n} \sum_{k=0}^{[nt_{u'} - 1] - 1} \sum_{j=k+1}^{\infty} \alpha(j); \quad (2.18)
\]

which goes to zero in view of assumption (A.3). Therefore,

\[
\lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \left[ \left( \sum_{i=1}^{n} Y_{n,i} \right)^2 \right] = \sum_{u=1}^{d+1} \lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \left[ \left( \sum_{i \in I_{n,u}} Y_{n,i} \right)^2 \right]. \quad (2.19)
\]

By stationarity,

\[
\mathbb{E} \left[ \left( \sum_{i \in I_{n,u}} Y_{n,i} \right)^2 \right] = \mathbb{E} \left[ \left( \sum_{i=1}^{[nt_u] - [nt_{u-1}]} Y_{n,i} \right)^2 \right] \quad (2.20)
\]

and defining for \(1 \leq u \leq d + 1\) the random variable

\[
Z_{i,u} := \sum_{k,l=1}^{d} a_{k,l} (1 - t_k) \mathbf{1} \{u \leq k\} (h_{1,s_k} (X_i) - \mathbb{E}[h_{1,s_k} (X_i)])
\]

\[
+ \sum_{k,l=1}^{d} a_{k,l} t_k \mathbf{1} \{u \geq k + 1\} (h_{2,s_k} (X_i) - \mathbb{E}[h_{2,s_k} (X_i)]), \quad (2.21)
\]

we have \(|Y_{n,i} - Z_{i,u}| \leq K/n\) for a constant \(K\) independent of \(n\) and \(i\) hence

\[
\lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \left[ \left( \sum_{i=1}^{n} Y_{n,i} \right)^2 \right] = \sum_{u=1}^{d+1} \lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \left[ \left( \sum_{i=1}^{[nt_u] - [nt_{u-1}]} Z_{i,u} \right)^2 \right]. \quad (2.22)
\]

By expanding the square and using stationarity, it follows that

\[
\lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \left[ \left( \sum_{i=1}^{n} Y_{n,i} \right)^2 \right] = \sigma^2 \quad (2.23)
\]

where

\[
\sigma^2 = \sum_{u=1}^{d+1} (t_u - t_{u-1}) \sum_{i \in \mathbb{Z}} \text{Cov}(Z_{0,u}, Z_{i,u}). \quad (2.24)
\]

This is the variance of \(\sum_{k,l=1}^{d} a_{k,l} N_{k,l}\), where \((N_{k,l})_{k,l=1}^{d}\) is a centered Gaussian vector having covariance

\[
\text{Cov}(N_{k,l}, N_{k',l'}) = \sum_{u=1}^{d+1} (t_u - t_{u-1}) \sum_{i \in \mathbb{Z}} Z_{k,l,0, i}^{(u)} Z_{k',l', i}^{(u)}, \quad (2.25)
\]

where

\[
Z_{k,l,0, i}^{(u)} = (1 - t_k) \mathbf{1} \{u \leq k\} (h_{1,s_k} (X_i) - \mathbb{E}[h_{1,s_k} (X_i)])
\]

\[
+ t_k \mathbf{1} \{u \geq k + 1\} (h_{2,s_k} (X_i) - \mathbb{E}[h_{2,s_k} (X_i)]). \quad (2.26)
\]
We can also check by splitting the sum over \( u \) that for \( k \leq k' \),
\[
\text{Cov} (N_{k,t}, N_{k', t'}) = \text{Cov} (W (s_t, t_k), W (s_{t'}, t_{k'})).
\] (2.27)

Now, it remains to check the third condition of Theorem A.1. We take \( a_k := \alpha (k) \) and since \( Y_{n,t} \) is a function of \( X_i \), the inequality \( \alpha_n (k) \leq \alpha (k) \) holds.

This ends the proof of Proposition 2.1.

\[ \square \]

2.1.3. Convergence of the linear part.

**Proposition 2.2.** For all \( R > 0 \), the sequence \( \left(W_n (s, t), s \in [-R, R], t \in [0, 1]\right)_{n \geq 1} \) converges in distribution in \( D ([ -R, R] \times [0, 1]) \) to \( (W (s, t), s \in [-R, R], t \in [0, 1]) \).

In order to prove Proposition 2.2, we will use the following convergence criterion in \( D ([0, 1] \times [0, 1]) \), which is Corollary 1 in [5].

**Theorem 2.3.** Let \( \xi_n, n \geq 1 \) be stochastic processes defined on \( [0, 1]^2 \), taking values in \( \mathbb{R} \), and whose paths are in the space \( D ([0, 1]^2) \) almost surely. We make the following assumptions:

1. the finite-dimensional distributions of \( \xi_n \) converges to the corresponding ones of a process \( \xi \) having continuous paths;
2. the process \( \xi_n \) can be written as the difference to two coordinate-wise non decreasing processes \( \xi_n^1 \) and \( \xi_n^2 \);
3. the exists constants \( \gamma > \beta > 2 \), \( c \in (0, \infty) \) such that for all \( n \geq 1 \), \( \text{E} [\xi_n (0, 0)^\gamma] \leq c \) and 
\[
\text{E} \left[ \frac{\xi_n (s, t) - \xi_n (s', t')}{\gamma} \right] \leq c \left\| (s, t) - (s', t') \right\|_\infty^\beta \text{ whenever } \left\| (s, t) - (s', t') \right\|_\infty \geq n^{-1}.
\] (2.28)
4. the following convergence in probability holds:
\[
\max_{1 \leq j_1, j_2 \leq n} \left| \xi_n^1 \left( \frac{j_1}{n}, \frac{j_2}{n} \right) - \xi_n^1 \left( \frac{j_1 - 1}{n}, \frac{j_2}{n} \right) \right| + \left| \xi_n^2 \left( \frac{j_1}{n}, \frac{j_2}{n} \right) - \xi_n^2 \left( \frac{j_1}{n}, \frac{j_2 - 1}{n} \right) \right| \rightarrow 0,
\] (2.29)

Then \( \xi_n \) converges weakly to \( \xi \) in \( D ([0, 1]^2) \).

In order to prove Proposition 2.1, we will use Theorem 2.3 in the following setting:
\[
\xi_n (s, t) := W_n (-R + 2R s, t), s, t \in [0, 1].
\] (2.30)

The convergence of the finite dimensional distributions to those of \( (W (-R + 2R s, t), s, t \in [0, 1]) \) is guaranted by Proposition 2.1. The covariance function of this process is Lipschitz continuous hence this process has continuous paths.

We will now express \( \xi_n \) as the difference of two coordinatewise non-decreasing process. To ease the notations, we will write \( F_{1,i} (u) := \Pr \{ g (u, X_1) \leq -R + 2R s \} \) and \( F_{2,i} (u) := \Pr \{ g (X_1, v) \leq -R + 2R s \} \) and \( m_{i,s} := \text{E} [F_{1,i} (X_0)], i \in \{1, 2\} \). These four functions are non-negative and non-decreasing in \( s \). The following equalities take place:
\[
\xi_n (s, t) = \frac{n - [nt]}{n^{3/2}} \sum_{i=1}^{[nt]} (F_{1,i} (X_i) - m_{1,i}) + \frac{[nt]}{n^{3/2}} \sum_{j=[nt]+1}^{n} (F_{2,i} (X_j) - m_{2,i})
\]
\[
= \frac{n - [nt]}{n^{3/2}} \sum_{i=1}^{[nt]} F_{1,i} (X_i) + \frac{[nt]}{n^{3/2}} \sum_{j=[nt]+1}^{n} F_{2,i} (X_j) - m_{1,i} \frac{n - [nt]}{n^{3/2}} [nt] - m_{2,i} \frac{[nt]}{n^{3/2}} (n - [nt]),
\]
and continuing the decomposition, we obtain

\[
\xi_n(s, t) = \frac{1}{n^{1/2}} \sum_{i=1}^{[nt]} F_{1,s} (X_i) - \frac{[nt]}{n^{3/2}} \sum_{i=1}^{[nt]} F_{1,s} (X_i) + \frac{[nt]}{n^{3/2}} \sum_{j=1}^{n} F_{2,s} (X_j) - \frac{[nt]}{n^{3/2}} \sum_{j=1}^{n} F_{2,s} (X_j)
\]

\[- m_{1,s} \frac{[nt]}{n^{1/2}} - m_{2,s} \frac{[nt]}{n^{1/2}} + m_{1,s} \frac{[nt]^2}{n^{3/2}} + m_{2,s} \frac{[nt]^2}{n^{3/2}}. \quad (2.31)
\]

Therefore, defining

\[
\xi_n^o (s, t) := \frac{1}{n^{1/2}} \sum_{i=1}^{[nt]} F_{1,s} (X_i) + \frac{[nt]}{n^{3/2}} \sum_{j=1}^{n} F_{2,s} (X_j) + m_{1,s} \frac{[nt]^2}{n^{3/2}} + m_{2,s} \frac{[nt]^2}{n^{3/2}} \text{ and } \quad (2.32)
\]

\[
\xi_n^* (s, t) := \frac{[nt]}{n^{3/2}} \sum_{i=1}^{[nt]} F_{1,s} (X_i) + \frac{[nt]}{n^{3/2}} \sum_{j=1}^{[nt]} F_{2,s} (X_j) + m_{1,s} \frac{[nt]}{n^{1/2}} + m_{2,s} \frac{[nt]}{n^{1/2}}. \quad (2.33)
\]

the processes $\xi_n^o$ and $\xi_n^*$ are both coordinatewise non-decreasing and $\xi_n = \xi_n^o - \xi_n^*$. Now, let $p$ be such that Assumption (A.3) is satisfied and let $\gamma := 2p$ and $\beta := p$ (since $p > 2$, the condition $\beta > 2$ is fulfilled). In view of Theorem 2.3, the proof of Proposition 2.2 will be complete once we show the following:

\[
\sup_{n \geq 1} \sup_{s \in [0,1]} \sup_{|t'| \geq n^{-1}} \sup_{|\xi - s| \geq n^{-1}} \sum_{i=1}^{[nt']} \left| \xi_n^o (s, t) - \xi_n^* (s, t') \right|^2 \leq C |t - t'|^p; \quad (2.34)
\]

\[
\sup_{n \geq 1} \sup_{s, s' \in [0,1]} \sup_{|t| \geq n^{-1}} \sup_{|\xi - s| \geq n^{-1}} \sum_{i=1}^{[nt]} \left| \xi_n^o (s, t) - \xi_n^* (s', t) \right|^2 \leq C |s - s'|^p; \quad (2.35)
\]

\[
\max_{0 \leq j_1, j_2 \leq n} \xi_n^o \left( \frac{j_1}{n} + \frac{1}{n}, \frac{j_1}{n}, \frac{j_2}{n} \right) - \xi_n^o \left( \frac{j_1}{n}, \frac{j_2}{n} \right) \to 0 \text{ in probability}; \quad (2.36)
\]

\[
\max_{0 \leq j_1, j_2 \leq n} \xi_n^* \left( \frac{j_1}{n}, \frac{j_2 + 1}{n} \right) - \xi_n^* \left( \frac{j_1}{n}, \frac{j_2}{n} \right) \to 0 \text{ in probability}. \quad (2.37)
\]

In order to prove (2.34) and (2.35), we need to control the differences in $s$ and $t$. Let us show (2.34). We first control $|\xi_n^o (s, t) - \xi_n^* (s, t')|$ for $t' - t \geq 1/n$. By definition of $\xi_n$,

\[
|\xi_n^o (s, t) - \xi_n^* (s, t')| \leq \left| \frac{n - [nt]}{n^{3/2}} \sum_{i=1}^{[nt']} (F_{1,s} (X_i) - m_{1,s}) \right| + \left| \frac{[nt']}{n^{3/2}} \sum_{i=[nt]+1}^{n} (F_{2,s} (X_i) - m_{2,s}) \right| \leq \frac{n}{n^{3/2}} \sum_{i=1}^{[nt']} (F_{1,s} (X_i) - m_{1,s}) + \frac{[nt']}{n^{3/2}} \sum_{i=[nt]+1}^{n} (F_{2,s} (X_i) - m_{2,s}) \quad (2.38)
\]

Decomposing the sum $\sum_{i=1}^{[nt']} \sum_{i=[nt]+1}^{n}$ as $\sum_{i=1}^{[nt']} + \sum_{i=[nt]+1}^{n}$ and the sum $\sum_{i=1}^{n} \sum_{i=[nt]+1}^{n}$ as $\sum_{i=1}^{[nt']} + \sum_{i=[nt]+1}^{n}$, we derive that

\[
|\xi_n^o (s, t) - \xi_n^* (s, t')| \leq \left| \frac{[nt'] - [nt]}{n^{3/2}} \sum_{i=1}^{[nt']} (F_{1,s} (X_i) - m_{1,s}) \right| + \left| \frac{[nt']}{n^{3/2}} \sum_{i=[nt]+1}^{n} (F_{2,s} (X_i) - m_{2,s}) \right| \leq \frac{[nt'] - [nt]}{n^{3/2}} \sum_{i=1}^{[nt']} (F_{1,s} (X_i) - m_{1,s}) + \frac{[nt'] - [nt]}{n^{3/2}} \sum_{i=[nt]+1}^{n} (F_{2,s} (X_i) - m_{2,s}) + \frac{[nt']}{n^{3/2}} \sum_{i=[nt]+1}^{n} (F_{2,s} (X_i) - m_{2,s}) \quad (2.39)
\]

Taking into account that $t' - t \geq 1/n$, it follows that

\[
[n]t' - [nt] \leq nt' - nt + 1 \leq 2n (t' - t) \quad (2.40)
\]
hence

$$
\begin{align*}
\left| \xi_n(s,t) - \xi_n(s,t') \right| & \leq \frac{t' - t}{n^{1/2}} \left| \sum_{i=1}^{[nt]} (F_{1,i} (X_i) - m_{1,s}) \right| + \frac{1}{n^{1/2}} \sum_{i=[nt]+1}^{[nt']} (F_{1,i} (X_i) - m_{1,s}) \\
+ 2 \frac{t' - t}{n^{1/2}} \sum_{i=[nt]+1}^{n} (F_{2,i} (X_i) - m_{2,s}) & + \frac{1}{n^{1/2}} \sum_{i=[nt]+1}^{[nt']} (F_{2,i} (X_i) - m_{2,s}) \cdot (2.41)
\end{align*}
$$

By stationarity,

$$
\begin{align*}
\mathbb{E} \left[ \left| \xi_n(s,t) - \xi_n(s,t') \right|^{2p} \right] & \leq C_p \left( \frac{t' - t}{n^{1/2}} \right)^{2p} \mathbb{E} \left[ \left| \sum_{i=1}^{[nt]} (F_{1,i} (X_i) - m_{1,s}) \right|^{2p} \right] \\
+ C_p n^{-p} \mathbb{E} \left[ \left| \sum_{i=1}^{[nt']-[nt]} (F_{1,i} (X_i) - m_{1,s}) \right|^{2p} \right] + C_p \left( \frac{t' - t}{n^{1/2}} \right)^{2p} \mathbb{E} \left[ \left| \sum_{i=1}^{[nt']-[nt]} (F_{2,i} (X_i) - m_{2,s}) \right|^{2p} \right] \\
+ C_p n^{-p} \mathbb{E} \left[ \left| \sum_{i=1}^{[nt']-[nt]} (F_{2,i} (X_i) - m_{2,s}) \right|^{2p} \right].
\end{align*}
$$

(2.42)

By application of Proposition A.4 to each of the four terms of the right hand side of (2.42) and using boundedness of $F_{1,i} (X_i) - m_{1,s}$ and $F_{2,i} (X_i) - m_{2,s}$ by 2, we derive that for all $s, t, t' \in [0, 1]$ such that $t' - t \geq 1/n$,

$$
\begin{align*}
\mathbb{E} \left[ \left| \xi_n(s,t) - \xi_n(s,t') \right|^{2p} \right] & \leq C \left( p, (\alpha (k))_{k \geq 1} \right) (t' - t)^{2p} n^n \\
& + C \left( p, (\alpha (k))_{k \geq 1} \right) \left( \frac{[nt']-[nt]}{n} \right)^{2p} + C \left( p, (\alpha (k))_{k \geq 1} \right) (t' - t)^{2p}. \quad (2.43)
\end{align*}
$$

and using again (2.40), we derive (2.34).

Let us show (2.35). This time, we have to control the increments in the first variable. For $s, s', t \in [0, 1]$ such that $s' - s \geq 1/n$,

$$
\begin{align*}
\left| \xi_n(s', t) - \xi_n(s, t) \right| & \leq \frac{n - [nt]}{n^{3/2}} \sum_{i=1}^{[nt]} (F_{1,i} (X_i) - F_{1,i'} (X_i) + m_{1,s'} - m_{1,s}) \\
+ \frac{[nt]}{n^{3/2}} \sum_{i=[nt]+1}^{n} (F_{2,i} (X_i) - F_{2,i'} (X_i) + m_{2,s'} - m_{2,s}) \cdot (2.44)
\end{align*}
$$

Bounding $n-[nt]$ and $[nt]$ by $n$, we derive by stationarity that

$$
\begin{align*}
\mathbb{E} \left[ \left| \xi_n(s', t) - \xi_n(s, t) \right|^{2p} \right] & \leq C_p n^{-p} \mathbb{E} \left[ \left| \sum_{i=1}^{[nt]} (F_{1,i} (X_i) - F_{1,i'} (X_i) + m_{s'} - m_{s}) \right|^{2p} \right] \\
+ C_p n^{-p} \mathbb{E} \left[ \left| \sum_{i=1}^{[nt]} (F_{2,i} (X_i) - F_{2,i'} (X_i) + m_{2,s'} - m_{2,s}) \right|^{2p} \right].
\end{align*}
$$

(2.45)
and taking into account the inequalities $|F_{1,s}(x) - F_{2,s'}(x)| \leq 2RM|s - s'|$ and $|F_{2,s}(x) - F_{2,s'}(x)| \leq 2RM|s - s'|$, we derive that

$$
E \left[ |\xi_n(s',t) - \xi_n(s,t)|^{2p} \right] \leq C(s' - s)^p,
$$

showing (2.35).

Now we show (2.36). Going back to the expression of $\xi_n^*$ given by (2.33), we derive that

$$
|\xi_n^*(s,t) - \xi_n^*(s',t')| \leq \frac{1}{n^{3/2}} \sum_{i=1}^{n} \left( |F_{1,s}(X_i) - F_{1,s'}(X_i)| + |F_{2,s}(X_i) - F_{2,s'}(X_i)| \right)
\leq \sqrt{n|m_{1,s} - m_{1,s'}| + \sqrt{n|m_{2,s} - m_{2,s'}|} \leq 8\sqrt{n}|s' - s|
$$

hence

$$
\max_{0 \leq j_1, j_2 \leq n} |\xi_n^* \left( \frac{j_1}{n} + \frac{j_2}{n} \right) - \xi_n^* \left( \frac{j_1'}{n} + \frac{j_2'}{n} \right) | \leq \frac{8R}{\sqrt{n}}
$$

giving (2.36).

Finally, we show (2.37). For $0 \leq t \leq t' \leq 1$ and $s \in [0,1]$, denoting $c_s := F_{1,s} + F_{2,s}$ and $m_s = m_{1,s} + m_{2,s}$, the following inequalities hold:

$$
|\xi_n^*(s,t) - \xi_n^*(s,t')| \leq \frac{[nt'] - [nt]}{n^{3/2}} \sum_{i=1}^{[nt]} c_s(X_i) - \frac{[nt'] - [nt]}{n^{3/2}} \sum_{i=1}^{[nt]} c_s(X_i) + m_s \frac{[nt'] - [nt]}{n^{1/2}}
\leq \frac{4}{n^{3/2}} \left( [nt'] - [nt] \right) \left( [nt] + [nt'] \right) + 8 \frac{[nt'] - [nt]}{n^{1/2}}
\leq \frac{8}{n^{3/2}} \left( n(t' - t) + 1 \right) + 8 \frac{n(t' - t) + 1}{n^{1/2}}
\leq 16n^{1/2}(t' - t) + 16n^{-1/2}
$$

As a consequence, we derive that

$$
\max_{0 \leq j_1, j_2 \leq n} |\xi_n^* \left( \frac{j_1}{n} + \frac{j_2 + 1}{n} \right) - \xi_n^* \left( \frac{j_1}{n} + \frac{j_2}{n} \right) | \leq 32n^{-1/2},
$$

and (2.37) follows. This ends the proof of Proposition 2.2.

2.1.4. Negligibility of the degenerated part.

**Proposition 2.4.** Let $R_n$ be defined by (2.4). Under the conditions of Theorem 1.1, the following convergence holds:

$$
\sup_{-R \leq s \leq R} \sup_{0 \leq t \leq 1} |R_n(s,t)| \rightarrow 0 \text{ in probability.}
$$

As a first step, we can look to the control of the supremum on $t$ for a fixed $s$. Then

$$
\sup_{0 \leq t \leq 1} |R_n(s,t)| = \frac{1}{n^{3/2}} \max_{1 \leq \ell \leq n-1} \left| \sum_{i=1}^{\ell} \sum_{j=\ell+1}^{n} (h_{3,s}(X_i, X_j) - E[h_{3,s}(X_i, X_j)]) \right|
$$

The problem is that the index $\ell$ over which we take the maximum appear in both sums and we cannot directly apply maximal inequality in Lemma A.5.
Here is a way to overcome this issue. Denote $h_{i,j} := h_{3,s} (X_i, X_j) - E [h_{3,s} (X_i, X_j)]$ and for a fixed $n$, $S_t := \sum_{i=1}^{\ell} \sum_{j=\ell+1}^{n} h_{i,j}, 1 \leq \ell \leq n - 1$ and $S_0 = 0$. Then for $2 \leq \ell \leq n$,

$$S_{\ell} - S_{\ell-1} = \sum_{i=1}^{\ell-1} \sum_{j=\ell+1}^{n} h_{i,j} - \sum_{i=1}^{\ell-1} \sum_{j=\ell}^{n} h_{i,j}$$

$$= \sum_{i=1}^{\ell-1} \sum_{j=\ell+1}^{n} h_{i,j} + \sum_{j=\ell+1}^{n} h_{\ell,j} - \sum_{i=1}^{\ell-1} \sum_{j=\ell+1}^{n} h_{i,j} - \sum_{i=1}^{\ell-1} h_{i,\ell} \tag{2.57}$$

$$= \sum_{j=\ell+1}^{n} h_{\ell,j} - \sum_{i=1}^{\ell-1} h_{i,\ell} \tag{2.58}$$

hence

$$S_k = \sum_{i=1}^{k} \sum_{j=k+1}^{n} h_{i,j} = \sum_{i=1}^{k} (S_i - S_{i-1}) = \sum_{i=1}^{k} \sum_{j=i+1}^{n} h_{i,j} - \sum_{1 \leq i \leq \ell \leq k} h_{i,\ell}. \tag{2.59}$$

Therefore, the maximum over $k$ can be treated by using the maximum of degenerated $U$-statistic for the term $\sum_{1 \leq i \leq \ell \leq k} h_{i,\ell}$. The other one can be reduced to a similar contribution by using this time the data and $(X_n, \ldots, X_1)$ instead of $(X_1, \ldots, X_n)$.

Using Lemma A.5, we derive that for each fixed $s$,

$$\mathbb{P} \left\{ \sup_{0 \leq t \leq 1} |R_n (s, t)| > \varepsilon \right\} \leq C \varepsilon^{-2} \log \frac{n}{n}, \tag{2.60}$$

where the constant $C$ depends only on $(\beta (k))_{k \geq 1}$.

We divide the interval $[-R, R]$ in intervals of length $\delta_n$, where $\delta_n$ with be specified later. Let $a_k := -R + 2Rk\delta_n$ and the interval

$$I_k := [a_{k-1}, a_k], 1 \leq k \leq \lfloor 1/\delta_n \rfloor + 1 =: B_n \tag{2.61}$$

(here for simplicity, we omit the dependence in $n$ for $a_k$ and $I_k$).

Then

$$\sup_{-R \leq s \leq R} \sup_{0 \leq t \leq 1} |R_n (s, t)| \leq \max_{1 \leq k \leq B_n} \sup_{s \in I_k} \sup_{0 \leq t \leq 1} |R_n (s, t)|. \tag{2.62}$$

In order to handle the supremum on $I_k$, we need the following lemma:

**Lemma 2.5.** Let $a$ and $b$ be two real numbers such that $a < b$ and let $f : \mathbb{R} \to \mathbb{R}$ be a function which can be expressed as a difference of two non-decreasing functions $f_1$ and $f_2$. Then

$$\sup_{s \in [a, b]} |f (s)| \leq |f (a)| + |f (b)| + |f_2 (b) - f_2 (a)|. \tag{2.63}$$

**Proof.** Let $s \in [a, b]$. Then by non-decreasingness of $f_1$ and $f_2$,

$$f (s) = f_1 (s) - f_2 (s) \leq f_1 (b) - f_2 (a) = f (b) + f_2 (b) - f_2 (a) \leq |f (a)| + |f (b)| + |f_2 (b) - f_2 (a)|. \tag{2.64}$$

Moreover,

$$f (s) = f_1 (s) - f_2 (s) \geq f_1 (a) - f_2 (b) = f (a) + f_2 (a) - f_2 (b) \geq - |f (a)| - |f_2 (b) - f_2 (a)|,$$

which allows to conclude. $\square$

In the expression $\sup_{s \in I_k} \sup_{0 \leq t \leq 1} |R_n (s, t)|$, the supremum over $t$ is actually a maximum; for the supremum over $s$, we will apply Lemma 2.5 in the following setting: for a fixed $t \in [0, 1]$,

$$f_1 (s) = \sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} \left\{ 1 \{ g (X_i, X_j) \leq s \} + E [h_{1,s} (X_i)] + E [h_{1,s} (X_j)] \right\}; \tag{2.65}$$
\[
f_2(s) = \sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} \left( \tilde{h}_{1,i}(X_i) + \tilde{h}_{2,i}(X_j) + \mathbb{P}\{g(X_i, X_j) \leq s\} \right),
\]

where \( \tilde{h}_{1,i}(u) = \mathbb{P}\{g(u, X_2) \leq s\} \) and \( \tilde{h}_{2,i}(v) = \mathbb{P}\{g(X_1, v) \leq s\} \). In view of (2.63), we derive that

\[
\sup_{-R \leq s \leq R} \sup_{0 \leq t \leq 1} |R_n(s, t)| \leq Z_n + c_n
\]

where

\[
Z_n := 2 \max_{1 \leq k \leq B_n} \sup_{0 \leq t \leq 1} |R_n(a_k, t)|
\]

\[
+ \frac{1}{n^{3/2}} \max_{0 \leq t \leq 1} \max_{1 \leq k \leq B_n} \sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} \left| \tilde{h}_{1,a_k}(X_i) - \tilde{h}_{1,a_k-1}(X_i) + \tilde{h}_{2,a_k}(X_j) - \tilde{h}_{2,a_k-1}(X_j) \right|
\]

and

\[
c_n := \frac{1}{n^{3/2}} \max_{0 \leq t \leq 1} \max_{1 \leq k \leq B_n} \sum_{i=1}^{[nt]} \sum_{j=[nt]+1}^{n} \{\mathbb{P}\{g(X_i, X_j) \leq a_k\} - \mathbb{P}\{g(X_i, X_j) \leq a_{k-1}\}\},
\]

We first show that \( \{Z_n\}_{n \geq 1} \) goes to zero in probability. After having rearranged the second term of the right hand side of (2.67), we end up with the estimate

\[
P\{Z_n > 4\varepsilon\} \leq P\left\{\max_{1 \leq k \leq B_n} \sup_{0 \leq t \leq 1} |R_n(a_k, t)| > \varepsilon \right\}
\]

\[
+ P\left\{\frac{1}{\sqrt{n}} \max_{1 \leq k \leq B_n} \sum_{i=1}^{n} \left( \tilde{h}_{1,a_k}(X_i) - \tilde{h}_{1,a_k-1}(X_i) \right) > \varepsilon \right\}
\]

\[
+ P\left\{\frac{1}{\sqrt{n}} \max_{1 \leq k \leq B_n} \sum_{j=1}^{n} \left( \tilde{h}_{2,a_k}(X_j) - \tilde{h}_{2,a_k-1}(X_j) \right) > \varepsilon \right\}.
\]

Let us estimate the first term of the right hand side of (2.70). The use of (2.61) and a union bound yields

\[
P\left\{\max_{1 \leq k \leq B_n} \sup_{0 \leq t \leq 1} |R_n(a_k, t)| > \varepsilon \right\} \leq CB_n\varepsilon^{-2} \frac{\log n}{n}.
\]

Now, using the assumptions (A.1) and assumption (A.2), we derive that

\[
\max_{1 \leq k \leq B_n} \sum_{i=1}^{n} \left( \tilde{h}_{q,a_k}(X_i) - \tilde{h}_{q,a_k-1}(X_i) \right) \leq n \max_{1 \leq k \leq B_n} M_q(a_k - a_{k-1}) \leq M_q n \delta_n, q \in \{1, 2\}
\]

hence the condition

\[
\lim_{n \to +\infty} \sqrt{n} \delta_n = 0
\]

guarantees the convergence in probability of the last two terms of (2.70).

We thus need

\[
\lim_{n \to +\infty} \sqrt{n} \delta_n = 0; \quad \lim_{n \to +\infty} \frac{\log n}{n \delta_n} = 0
\]

which can be done by choosing \( \delta_n = n^{-3/4} \).

It remains to show the convergence to zero of the sequence \( \{c_n\}_{n \geq 1} \) defined by (2.69). To this aim, we estimate for all \( i < j \) the probability \( p_{i,j} := P\{g(X_i, X_j) \in (a_{k-1}, a_k]\} \) by using coupling. We can find a random variable \( X'_j \) independent of \( X_i \) and having the same distribution as \( X_j \) such that
\[ \mathbb{P} \{ X_j \neq X'_j \} \leq \beta (j - i). \] Therefore, \( p_{i,j} \leq \beta (j - i) + \mathbb{P} \{ g (X_i, X'_i) \in (a_{k-1}, a_k) \} \) and \( c_n \leq c'_n + c''_n \), where

\[
c'_n = \frac{1}{n^{3/2}} \max_{1 \leq \ell \leq n} \sum_{i=1}^{\ell} \sum_{j=\ell+1}^n \beta (j - i) \tag{2.75}
\]

\[
c''_n = \frac{1}{n^{3/2}} \max_{1 \leq \ell \leq n} \max_{1 \leq k \leq \ell} \sum_{i=1}^{\ell} \sum_{j=\ell+1}^n \mathbb{P} \{ g (X_i, X'_j) \in (a_{k-1}, a_k) \}. \tag{2.76}
\]

Let us show that \( c'_n \to 0 \). For a fixed \( \ell \in \{ 1, \ldots, n \} \),

\[
\sum_{i=1}^{\ell} \sum_{j=\ell+1}^n \beta (j - i) \leq \sum_{i=1}^{\ell} \sum_{k=\ell-i}^n \beta (k) \leq n \sum_{k=0}^n \beta (k) \tag{2.77}
\]

hence

\[
c'_n \leq n^{-1/2} \sum_{k=0}^n \beta (k) \tag{2.78}
\]

and by assumption (A.4), we derive that \( c'_n \to 0 \).

Let us show that \( c''_n \to 0 \). First, we notice that for all \( i < j \), the vector \( (X_i, X'_j) \) has the same distribution as \( (X, Y) \), where \( X \) and \( Y \) are independent and have the same distribution as \( X_1 \). Consequently,

\[
c''_n \leq \sqrt{n} \max_{1 \leq k \leq B_n} \mathbb{P} \{ g (X, Y) \in (a_{k-1}, a_k) \} \tag{2.79}
\]

By assumption (A.1) and accounting \( \delta_n = n^{-3/4} \), we derive that

\[
c''_n \leq M_1 \sqrt{n} \max_{1 \leq k \leq B_n} (a_k - a_{k-1}) \leq M \sqrt{n} \delta_n \leq M n^{1/2 - 3/4} \to 0. \tag{2.80}
\]

This ends the proof of Theorem 1.1.

2.2. Proof of Theorem 1.3.

2.2.1. Hoeffding’s decomposition. Using the functions \( h_{1,s}, h_{2,s} \) and \( h_{3,s} \) defined by (2.1) and (2.2), we derive that

\[
c'_n (s, t) = W'_n (s, t) + R'_n (s, t), \tag{2.81}
\]

where

\[
W'_n (s, t) = \frac{1}{n^{3/2}} (n - [nt]) \sum_{i=1}^{[nt]} (h_{1,s} (X_i) - \mathbb{E} [h_{1,s} (X_i)]) + \frac{1}{n^{3/2}} [nt] \sum_{j=\lceil nt \rceil + 1}^n (h_{2,s} (X_j) - \mathbb{E} [h_{2,s} (X_j)])
\]

\[
- \frac{[nt] (n - [nt])}{n^{3/2}} \left( \sum_{i=1}^{n-1} (n - i) (h_{1,s} (X_i) - \mathbb{E} [h_{1,s} (X_i)]) + \sum_{j=2}^n (j - 1) (h_{2,s} (X_j) - \mathbb{E} [h_{2,s} (X_j)]) \right) \tag{2.82}
\]

and

\[
R'_n (s, t) := \frac{1}{n^{3/2}} \sum_{i=1}^{[nt]} \sum_{j=\lceil nt \rceil + 1}^n (h_{3,s} (X_i, X_j) - \mathbb{E} [h_{3,s} (X_i, X_j)])
\]

\[
- \frac{[nt] (n - [nt])}{n^{3/2}} \left( \sum_{1 \leq i < j \leq n} \left( h_{3,s} (X_i, X_j) - \mathbb{E} [h_{3,s} (X_i, X_j)] \right) \right). \tag{2.83}
\]
2.2.2. Convergence of the finite dimensional distributions. We treat the convergence of the finite dimensional distributions.

**Proposition 2.6.** For all $d \geq 1$ and all $s_1 < \cdots < s_d$ and $0 \leq t_1 < \cdots < t_d \leq 1$, the vector $(W_n'(s_t, t_k))_{k,t=1}^d$ converges in distribution to $(W'(s_t, t_k))_{k,t=1}^d$, where $W_n'$ is defined by (2.82) and $W$ is like in Theorem 1.3.

Here again, we will prove the convergence of linear combinations, that is, for all $(a_k, \ell)_{k,\ell=1}^d$, the convergence in distribution

$$\sum_{k,\ell=1}^d a_k, \ell W_n'(s_t, t_k) \to \sum_{k,\ell=1}^d a_k, \ell W'(s_t, t_k)$$

holds. To this aim, we will express $\sum_{k,\ell=1}^d a_k, \ell W_n'(s_t, t_k)$ as a linear combination of a sum of functions of $X_i$. Using the definition of $I_{n,u}$ given by (2.8) for $1 \leq u \leq d + 1$, we derive that

$$\sum_{k,\ell=1}^d a_k, \ell W_n'(s_t, t_k) = \frac{1}{\sqrt{n}} \sum_{i=1}^n A_{n,i},$$

where, for $i \in I_{n,u},$

$$A_{n,i} = \frac{1}{n} \sum_{k,\ell=1}^d a_k, \ell (n - [nt_k]) (h_{1,s_k} (X_i) - E[h_{1,s_k} (X_i)]) 1 \{u \leq k\}$$

$$- \frac{1}{n} \sum_{k,\ell=1}^d a_k, \ell [nt_k] (n - [nt_k]) (h_{1,s_k} (X_i) - E[h_{1,s_k} (X_i)])$$

$$+ \frac{1}{n} \sum_{k,\ell=1}^d a_k, \ell [nt_k] (h_{2,s_k} (X_i) - E[h_{2,s_k} (X_i)]) 1 \{u > k\}$$

$$- \frac{1}{n} \sum_{k,\ell=1}^d a_k, \ell [nt_k] (n - [nt_k]) (h_{2,s_k} (X_i) - E[h_{2,s_k} (X_i)]).$$

In order to get rid of terms of the form $[nt_k]$ and reduce the dependence in $n$, we will define for $i \in I_{n,u}$ the random variable

$$Y_{n,i} = \sum_{k,\ell=1}^d a_k, \ell (1 - t_k) (h_{1,s_k} (X_i) - E[h_{1,s_k} (X_i)]) 1 \{u \leq k\}$$

$$- \frac{2}{n} (n - i) \sum_{k,\ell=1}^d a_k, \ell t_k (1 - t_k) (h_{1,s_k} (X_i) - E[h_{1,s_k} (X_i)]) + \sum_{k,\ell=1}^d a_k, \ell t_k (h_{2,s_k} (X_i) - E[h_{2,s_k} (X_i)]) 1 \{u > k\}$$

$$- \frac{2}{n} \sum_{k,\ell=1}^d a_k, \ell t_k (1 - t_k) (h_{2,s_k} (X_i) - E[h_{2,s_k} (X_i)]).$$

Since $|A_{n,i} - Y_{n,i}| \leq 1/n$, it suffices to show that $n^{-1/2} \sum_{i=1}^n Y_{n,i} =: n^{-1/2} S_n \to \sum_{k,\ell=1}^d a_k, \ell W'(s_t, t_k)$ in distribution. Here again, we will use Theorem A.1. The first condition can be easily checked by bounding $|h_{u,s_k} (X_i)|$ by 2 and the terms $i/n$ by 1. For the third condition, we also use $a_k = o(k)$, since each random variable $Y_{n,i}$ is a function of $X_i$. It remains to compute the limit of the sequence $(n^{-1} \text{Var}(S_n))_{n \geq 1}$.
By similar argument as those who gave (2.13), this reduces to compute for all $1 \leq u \leq d + 1$ the limit

$$\lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \left[ \left( \sum_{t \in I_{n,u}} Y_{n,i} \right)^{2} \right].$$

(2.88)

For convenience, we write

$$n \sum_{k,t=1}^{d} a_{k,t} (1-t_k) (h_{1,x_k} (X_i) - \mathbb{E} [h_{1,x_k} (X_i)]) 1 \{u \leq k\}$$

$$+ \sum_{k,t=1}^{d} a_{k,t} t_k (h_{2,x_t} (X_i) - \mathbb{E} [h_{2,x_t} (X_i)]) 1 \{u > k\}$$

$$- 2 \sum_{k,t=1}^{d} a_{k,t} (1-t_k) (h_{1,x_k} (X_i) - \mathbb{E} [h_{1,x_k} (X_i)]) = 2 \sum_{k,t=1}^{d} a_{k,t} t_k (h_{2,x_t} (X_i) - \mathbb{E} [h_{2,x_t} (X_i)]),$$

(2.89)

Then $Y_{n,i} = Z_i + Z'_i$ and consequently

$$\mathbb{E} \left[ \left( \sum_{t \in I_{n,u}} Y_{n,i} \right)^{2} \right] = \mathbb{E} \left[ \left( \sum_{t \in I_{n,u}} Z_i \right)^{2} \right] + 2 \mathbb{E} \left[ \sum_{t \in I_{n,u}} Z_i \sum_{t' \in I_{n,u}} Z'_{n,i} \right] + \mathbb{E} \left[ \sum_{t \in I_{n,u}} Z'_{n,i} \right]^{2}$$

(2.92)

For the first term, we get, by stationarity, that

$$\lim_{n \to +\infty} \frac{1}{n} \mathbb{E} \left[ \left( \sum_{t \in I_{n,u}} Z_i \right)^{2} \right] = (t_u - t_{u-1}) \sum_{i \in \mathbb{Z}} \text{Cov} (Z_0, Z_i)$$

(2.93)

The second and third term cannot be treated similarly because of the terms $i/n$. Nevertheless, we can use the following lemma.

**Lemma 2.7.** Let $(c_j)_{j \in \mathbb{Z}}$ be an absolutely summable sequence of real numbers such that $c_j = c_{-j}$ for all $j$ and let $0 \leq a < b \leq 1$. Then

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i,j=\lfloor an \rfloor + 1}^{\lfloor bn \rfloor} c_{j-i} = \frac{b^2 - a^2}{2} \sum_{j \in \mathbb{Z}} c_j;$$

(2.94)

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{i,j=\lfloor an \rfloor + 1}^{\lfloor bn \rfloor} i j c_{j-i} = \frac{b^3 - a^3}{3} \sum_{j \in \mathbb{Z}} c_j.$$

(2.95)

**Proof.** For the first convergence, we split the sum according to the values of $j - i$ (between $N_n - 1$ and $-N_n + 1$ where $N_n = \lfloor bn \rfloor - \lfloor an \rfloor$):

$$\sum_{i,j=\lfloor an \rfloor + 1}^{\lfloor bn \rfloor} i c_{j-i} = \sum_{k=-N_n+1}^{N_n-1} c_k \sum_{i=\lfloor an \rfloor + 1}^{\lfloor bn \rfloor} i 1 \{j - i = k\}. $$

(2.96)
The sum $\sum_{j=|an|+1}^{[bn]} 1 \{j - i = k\}$ is 1 if $|an| + 1 \leq i + k \leq [bn]$ and zero otherwise; for $k \geq 0$, this constraint means $|an| + 1 \leq i \leq [bn] - k$ and for $k < 0$, this means $|an| + 1 - k \leq i \leq [bn]$ hence

$$\frac{1}{n} \sum_{i,j=[an]+1}^{[bn]} \frac{i}{n} c_{j-i} = \sum_{k \in \mathbb{Z}} \frac{1}{n} 1 \{0 \leq k \leq N_n - 1\} c_k \sum_{i=[an]+1}^{[bn]-k} \frac{i}{n} + \frac{1}{n} \sum_{k \in \mathbb{Z}} 1 \{-N_n + 1 \leq k \leq -1\} c_k \sum_{i=[an]+1-k}^{[bn]} \frac{i}{n} \quad (2.97)$$

For a fixed $k$, the convergences

$$\frac{1}{n} \sum_{i=[an]+1}^{[bn]-k} \frac{i}{n} \rightarrow c_k \sum_{i=[an]+1-k}^{[bn]} \frac{i}{n} \rightarrow c_k \frac{b^2 - a^2}{2} \quad (2.98)$$

holds and these sequences are bounded in absolute value by $|c_k|$ hence (2.94) follows by dominated convergence.

We use the same strategy to show (2.95). We start from

$$\sum_{i,j=[an]+1}^{[bn]} \frac{i}{n} c_{j-i} = \sum_{k=-N_n+1}^{N_n-1} \frac{i}{n} c_k \sum_{i=[an]+1}^{[bn]} \frac{i}{n} \sum_{j=[an]+1}^{[bn]} \frac{j}{n} 1 \{j - i = k\}. \quad (2.100)$$

and $\sum_{j=[an]+1}^{[bn]} 1 \{j - i = k\}$ is $(i + k)/n$ if $|an| + 1 \leq i + k \leq [bn]$ and zero otherwise hence

$$\frac{1}{n} \sum_{i,j=[an]+1}^{[bn]} \frac{i}{n} c_{j-i} = \frac{1}{n} \sum_{k \in \mathbb{Z}} 1 \{0 \leq k \leq N_n - 1\} c_k \sum_{i=[an]+1}^{[bn]-k} \frac{i}{n} + \frac{1}{n} \sum_{k \in \mathbb{Z}} 1 \{-N_n + 1 \leq k \leq -1\} c_k \sum_{i=[an]+1-k}^{[bn]} \frac{i}{n} \frac{i + k}{n}. \quad (2.101)$$

By applying Lemma 2.7 and the convergences (2.94) to $c_i := \text{Cov} (Z_0, Z_i')$ and (2.95) to $c_i := \text{Cov} (Z_0', Z_i')$ we finally obtain that

$$\frac{1}{n} \sum_{i \in l_n,u} \left( \sum_{i \in l_n,u} Y_{n,i} \right)^2 = \sum_{u=1}^{d+1} (t_u - t_{u-1}) \sum_{i \in \mathbb{Z}} \text{Cov} (Z_0, Z_i) + \sum_{u=1}^{d+1} \frac{t_u^2 - t_{u-1}^2}{2} \sum_{i \in \mathbb{Z}} \text{Cov} (Z_0, Z_i') + \sum_{u=1}^{d+1} \frac{t_u^3 - t_{u-1}^3}{3} \sum_{i \in \mathbb{Z}} \text{Cov} (Z_0', Z_i') =: \sigma^2. \quad (2.102)$$

If $\sigma = 0$, we get a weak convergence to 0, otherwise by Theorem A.1, we get the weak convergence of $\sum_{k,\ell=1}^{d} a_{k,\ell}W_{\ell} (s_{\ell}, t_k)$ to $\sum_{k,\ell=1}^{d} a_{k,\ell}N_{k,\ell}$, where $(N_{k,\ell})_{k,\ell=1}^{d}$ is a Gaussian vector such that for $k \leq k'$, $\text{Cov} (N_{k,\ell}, N_{k',\ell'}) = \text{Cov} (W (s_{\ell}, t_k), W (s_{\ell'}, t_{k'}))$, where
\[
\text{Cov} \left( W(s, t), W(s', t') \right) = t (1 - t) \left( 1 - t' \right) C_{1,1} (s, s') - 2 t (1 - t) t' \left( 1 - t' \right) C_1^a (s, s') \\
- 2 t^2 (1 - t) (1 - t') C_1^a (s', s) + 4 t^2 (1 - t) t' (1 - t') C_1^a (s, s') \\
+ (t' - t) t (1 - t') C_{2,1} (s, s') - 2 (t' - t) t' (1 - t') C_2^a (s, s') \\
- 2 (t' - t) t (1 - t') C_2^a (s', s') + 4 (t' - t) t (1 - t') (1 - t') C_1^a (s, s') \\
+ (1 - t') t' (1 - t') C_{2,2} (s, s') - 2 (1 - t') t' (1 - t') C_2^a (s, s') \\
- 2 (1 - t') t' (1 - t') C_2^a (s', s') + 4 (1 - t') t (1 - t') C_1^a (s, s') \\
- 2 t^2 (1 - t) t (1 - t') C_1^a (s, s') - 2 (1 - t') t' t (1 - t') C_2^a (s, s') \\
+ 4 t (1 - t) t' \left( 1 - t' \right) C_1^a (s, s') + \frac{4}{3} t (1 - t) t' \left( 1 - t' \right) C_1^a (s', s'). 
\] 

(2.103)

Simplifying this expression leads to the covariance mentioned in (1.15).

2.2.3. Convergence of the linear part. We also use Theorem 2.3. The convergence of \( \{W_n^\ast\}_{n \geq 1} \) also hold to a process having continuous paths. Define

\[
\xi_n (s, t) := W_n^\ast (-R + 2R s, t), s, t \in [0, 1],
\] 

(2.104)

where \( W_n^\ast \) is defined by (2.82). Observe that

\[
\xi_n (s, t) = W_n (-R + 2R s, t) \\
= \frac{[nt] (n - [nt])}{n^{3/2}} \left( \frac{1}{2} \right)^{n-1} \sum_{i=1}^{n-1} (n - i) \left( h_{1,-R+2Rs} (X_i) - \mathbb{E} \left[ h_{1,-R+2Rs} (X_i) \right] \right) \\
- \frac{[nt] (n - [nt])}{n^{3/2}} \left( \frac{1}{2} \right)^n \sum_{j=2}^{n} (j - 1) \left( h_{2,-R+2Rs} (X_j) - \mathbb{E} \left[ h_{2,-R+2Rs} (X_j) \right] \right). 
\] 

(2.105)

Now, it suffices to prove that

\[
\xi_{n,1} (s, t) := \frac{[nt] (n - [nt])}{n^{3/2}} \left( \frac{1}{2} \right)^{n-1} \sum_{i=1}^{n-1} (n - i) \left( h_{1,-R+2Rs} (X_i) - \mathbb{E} \left[ h_{1,-R+2Rs} (X_i) \right] \right) 
\] 

(2.106)

and

\[
\xi_{n,2} (s, t) := \frac{[nt] (n - [nt])}{n^{3/2}} \left( \frac{1}{2} \right)^n \sum_{j=2}^{n} (j - 1) \left( h_{2,-R+2Rs} (X_j) - \mathbb{E} \left[ h_{2,-R+2Rs} (X_j) \right] \right) 
\] 

(2.107)

satisfy the conditions 2, 3 and 4 of Theorem 2.3. Since the treatment of \( \xi_{n,1} \) is completely analogous to that of \( \xi_{n,2} \), we will do it only for the latter. By writing \( [nt] (n - [nt]) \) and \( h_{2,-R+2Rs} (X_j) \) as a difference of non-decreasing function, we can see that condition 2 of Theorem 2.3 is satisfied with

\[
\xi_{n,2}^* (s, t) = \frac{[nt]^2}{n^{3/2}} \left( \frac{1}{2} \right)^n \sum_{j=2}^{n} (j - 1) F_{2,s} (X_j) + \frac{[nt]^2}{n^{3/2}} \left( \frac{1}{2} \right)^n \sum_{j=2}^{n} (j - 1) \mathbb{E} \left[ F_{2,s} (X_j) \right]; 
\] 

(2.108)

where

\[
F_{2,s} (v) = \mathbb{P} \left[ g (X_1, v) \leq -R + 2Rs \right]. 
\] 

(2.109)

To check condition 3 of Theorem 2.3, we rewrite \( \sum_{j=2}^{n} (j - 1) F_{2,s} (X_j) \) in terms of partial sums of \( F_{2,s} (X_j) \), use triangle inequality for the \( L^{2p} \)-norm, then we apply Proposition A.4.

Similar estimates as those who led to (2.54) give

\[
\left| \xi_{n,2}^* \left( \frac{j_1}{n} \frac{j_2 + 1}{n} \right) - \xi_{n,2}^* \left( \frac{j_1}{n} \frac{j_2}{n} \right) \right| \leq 32n^{-1/2}. 
\] 

(2.110)
2.2.4. Negligibility of the degenerated part. In view of (2.4) and (2.83), it suffices to prove that
\[
\sup_{s \in [-R, R]} \sup_{0 \leq t < 1} \frac{[nt]}{n^{3/2}} \left( \frac{n}{2} \right)^{-1} \sum_{1 \leq i < j \leq n} \left( h_{3,s}(X_i) - \mathbb{E}[h_{3,s}(X_i)] \right) \rightarrow 0 \text{ in probability.} \tag{2.111}
\]
Bounding \(\frac{[nt]}{n^{3/2}} \left( \frac{n}{2} \right)^{-1}\) by \(2n^{2-3/2-2}\), it suffices to show that for all positive \(\varepsilon\),
\[
P \left\{ \sup_{s \in [-R, R]} \frac{1}{n^{3/2}} \sum_{1 \leq i < j \leq n} \left( h_{3,s}(X_i) - \mathbb{E}[h_{3,s}(X_i)] \right) > \varepsilon \right\} \rightarrow 0. \tag{2.112}
\]
This is done in the same way as in the proof of Proposition 2.4: we cut the interval \([-R, R]\) into intervals of length \(\delta_n = n^{-3/4}\) and do the same estimate. This ends the proof of Theorem 1.3.

2.3. Proof of Corollary 1.6. We would like to use directly the continuous mapping theorem. However, Theorems 1.1 and 1.3 only give the convergence in distribution on \(D([-R, R] \times [0,1])\) which leads to the convergence in \((2.18)\) and \((1.29)\) where the integral over \(R\) are replaced by integrals over \([-R, R]\). Then we show that the contribution of the integrals on \(R \setminus [-R, R]\) is negligible.

More formally, we will use the Theorem 4.2 in [2].

Proposition 2.8. Let \(\left\{ Y_n^{(R)}(R) \right\}_{n \geq 1}^{R \geq 1}\) be a family of random variable and let \((Y_n)_{n \geq 1}\) and \((Z_R)_{R \geq 1}\) be a sequence of random variables such that
\[
\begin{align*}
(1) & \text{ for all } R \geq 1, \text{ the sequence } \left\{ Y_n^{(R)}(R) \right\}_{n \geq 1}^{R \geq 1} \text{ converges in distribution to a random variable } Z_R; \\
(2) & \text{ the sequence } (Z_R)_{R \geq 1} \text{ converges in distribution to a random variable } Z \text{ and } \\
(3) & \text{ for all positive } \varepsilon, \\
\text{there exists } R_0 & \text{ such that } \\
\lim_{R \to +\infty} \limsup_{n \to +\infty} P \left\{ \left| Y_n^{(R)}(R) - Y_n \right| > \varepsilon \right\} = 0. \tag{2.113}
\end{align*}
\]
Then the sequence \((Y_n)_{n \geq 1}\) converges in distribution to \(Z\).

In order to prove (1.28) (respectively (1.29)), we apply Proposition 2.8 to
\[
Y_n^{(R)} = \sup_{0 \leq t \leq 1} \int_{[-R, R]} \varepsilon_n(s,t)^2 d\mu(s), Y_n = \sup_{0 \leq t \leq 1} \int_{R} \varepsilon_n(s,t)^2 d\mu(s) \tag{2.114}
\]
\[
Z_R = \sup_{0 \leq t \leq 1} \int_{[-R, R]} W(s,t)^2 d\mu(s) \tag{2.115}
\]
(respectively
\[
Y_n^{(R)} = \sup_{0 \leq t \leq 1} \int_{[-R, R]} \varepsilon'_n(s,t)^2 d\mu(s), Y_n = \sup_{0 \leq t \leq 1} \int_{R} \varepsilon'_n(s,t)^2 d\mu(s), \tag{2.116}
\]
\[
Z_R = \sup_{0 \leq t \leq 1} \int_{[-R, R]} W'(s,t)^2 d\mu(s)). \tag{2.117}
\]
Assumption 1 follows from the continuous mapping theorem applied with the functional
\[
\Phi_R(f) = \sup_{0 \leq t \leq 1} \int_{[-R, R]} f(s,t)^2 d\mu(s). \tag{2.118}
\]
In order to show that assumptions 2 and 3 hold, it suffices to show that
\[
\lim_{R \to +\infty} \sup_{n \geq 1} \int_{R \setminus [-R, R]} \mathbb{E} \left[ \sup_{0 \leq t \leq 1} \varepsilon_n(s,t)^2 \right] d\mu(s) = 0 \tag{2.119}
\]
and
\[
\lim_{R \to +\infty} \sup_{n \geq 1} \int_{R \setminus [-R, R]} \mathbb{E} \left[ \sup_{0 \leq t \leq 1} \varepsilon'_n(s,t)^2 \right] d\mu(s) = 0. \tag{2.120}
\]
Indeed, for assumption 2,

\[ |Z_R - Z| \leq \int_{R\setminus[-R,R]} \sup_{0 \leq t \leq 1} W(s,t)^2 \, d\mu(s) \]  \hspace{1cm} (2.121)

hence

\[ E[|Z_R - Z|] \leq \int_{R\setminus[-R,R]} E\left[ \sup_{0 \leq t \leq 1} W(s,t)^2 \right] \, d\mu(s). \]  \hspace{1cm} (2.122)

Using the weak convergence of \((c_n(s,t))_{n \geq 1}\) to \(W(s,t)\) on \(D([0,1])\) for a fixed \(s\) and then Fatou’s lemma, we derive that

\[ E[|Z_R - Z|] \leq \liminf_{n \to +\infty} \int_{R\setminus[-R,R]} E\left[ \sup_{0 \leq t \leq 1} e_n(s,t)^2 \right] \, d\mu(s). \]  \hspace{1cm} (2.123)

Moreover,

\[ E\left[ |Y_n^{(R)} - Y_n| \right] \leq \int_{R\setminus[-R,R]} E\left[ \sup_{0 \leq t \leq 1} e_n(s,t)^2 \right] d\mu(s) \leq \sup_{n \geq 1} \int_{R\setminus[-R,R]} E\left[ \sup_{0 \leq t \leq 1} e_n(s,t)^2 \right] d\mu(s), \]  \hspace{1cm} (2.124)

and similar inequalities hold where \(e_n\) and \(W\) are replaced by \(e'_n\) and \(W\).

Now, using the fact that \(\mu\) is a finite measure, it suffices to find uniform bounds in \(n\) and \(s\) for \(E\left[ \sup_{0 \leq t \leq 1} e_n(s,t)^2 \right]\) and \(E\left[ \sup_{0 \leq t \leq 1} e'_n(s,t)^2 \right]\), namely,

\[ \sup_{n \geq 1} \sup_{s \in \mathbb{R}} \left( E\left[ \sup_{0 \leq t \leq 1} e_n(s,t)^2 \right] + E\left[ \sup_{0 \leq t \leq 1} e'_n(s,t)^2 \right] \right) < +\infty. \]  \hspace{1cm} (2.125)

In view of the decompositions (2.3) and (2.81) and due to the fact that the dependence in \(t\) lies in \([nt]\), which is an integer between 0 and \(n\), it suffices to show that

\[ \sup_{n \geq 1} \sup_{s \in \mathbb{R}} E\left[ \max_{1 \leq k \leq n} W_n\left( s, \frac{k}{n} \right)^2 \right] < +\infty, \]  \hspace{1cm} (2.126)

\[ \sup_{n \geq 1} \sup_{s \in \mathbb{R}} E\left[ \max_{1 \leq k \leq n} W'_n\left( s, \frac{k}{n} \right)^2 \right] < +\infty, \]  \hspace{1cm} (2.127)

\[ \sup_{n \geq 1} \sup_{s \in \mathbb{R}} E\left[ \max_{1 \leq k \leq n} R_n\left( s, \frac{k}{n} \right)^2 \right] < +\infty, \]  \hspace{1cm} (2.128)

\[ \sup_{n \geq 1} \sup_{s \in \mathbb{R}} E\left[ \max_{1 \leq k \leq n} R'_n\left( s, \frac{k}{n} \right)^2 \right] < +\infty, \]  \hspace{1cm} (2.129)

where \(W_n, W'_n, R_n\) and \(R'_n\) are defined respectively by (2.5), (2.82), (2.4) and (2.83).

Let us show (2.126). Observe that

\[ \max_{1 \leq k \leq n} W_n\left( s, \frac{k}{n} \right)^2 \leq 2 \left( \frac{1}{n^{1/2}} \max_{1 \leq k \leq n} \left| \sum_{i=1}^{k} (h_{1,s}(X_i) - E[h_{1,s}(X_i)]) \right| \right)^2 \]

\[ + 2 \left( \frac{1}{n^{1/2}} \max_{1 \leq k \leq n} \left| \sum_{j=k+1}^{n} (h_{2,s}(X_j) - E[h_{2,s}(X_j)]) \right| \right)^2 \]  \hspace{1cm} (2.130)

hence using Theorem A.3 two times gives (2.126).
Let us show (2.127). In view of (2.82), the equality
\[
W_n'(s, t) = W_n(s, t) - \frac{[nt]}{n^{3/2}} \left[ \frac{1}{2} \sum_{i=1}^{n-1} \left( n - i \right) (h_{1,s}(X_i) - \mathbb{E}[h_{1,s}(X_i)]) \right. \\
- \frac{[nt]}{n^{3/2}} \left. \left( \frac{1}{2} \sum_{j=2}^{n} \left( j - 1 \right) (h_{2,s}(X_j) - \mathbb{E}[h_{2,s}(X_j)]) \right) \right) 
\] (2.131)
holds hence it suffices to show that
\[
\sup_{n \geq 1} \sup_{s \in [0, 1]} \frac{1}{n^3} \mathbb{E} \left[ \left( \sum_{i=1}^{n-1} \left( n - i \right) (h_{1,s}(X_i) - \mathbb{E}[h_{1,s}(X_i)]) + \sum_{j=2}^{n} \left( j - 1 \right) (h_{2,s}(X_j) - \mathbb{E}[h_{2,s}(X_j)]) \right)^2 \right] < +\infty. 
\]
This follows from a rewriting of the sums in terms of partial sums of $h_{1,s}(X_i)$ and $h_{2,s}(X_i)$ and an application of Theorem A.3.

Let us show (2.128). Letting $h_{1,j} := h_{3,s}(X_i, X_j) - \mathbb{E}[h_{3,s}(X_i, X_j)]$, we get in view of (2.60) that
\[
\mathbb{E} \left[ \max_{1 \leq k \leq n} R_n \left( s, \frac{k}{n} \right)^2 \right] = \mathbb{E} \left[ \frac{2}{n^3} \max_{1 \leq k \leq n} \left( \sum_{1 \leq i \leq k} h_{i,1} \right) \right]^2 + \mathbb{E} \left[ \frac{2}{n^3} \max_{1 \leq k \leq n} \left( \sum_{1 \leq j \leq k} h_{1,j} \right) \right]^2. 
\] (2.132)
Boundedness follows from Lemma A.5.

Let us show (2.129). Noticing that
\[
R_n'(s, t) = R_n(s, t) - \frac{1}{n^{3/2}} \frac{[nt]}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} (h_{3,s}(X_i, X_j) - \mathbb{E}[h_{3,s}(X_i, X_j)]), 
\] (2.133)
it suffices to show, in view of (2.128), that
\[
\sup_{n \geq 1} \sup_{s \in [0, 1]} \frac{1}{n^3} \mathbb{E} \left[ \left( \sum_{1 \leq i < j \leq n} (h_{3,s}(X_i, X_j) - \mathbb{E}[h_{3,s}(X_i, X_j)]) \right)^2 \right] < +\infty. 
\] (2.134)
This can be seen by an other use of Lemma A.5.

Acknowledgement This research was supported by the grant DFG Collaborative Research Center SFB 823 ‘Statistical modelling of nonlinear dynamic processes’.

Appendix A. Facts on Mixing Sequences

In this section, we collect the facts on mixing sequences we need in the proof.

**Theorem A.1** (Central limit theorem for row-wise mixing arrays, see [10]). Let $(x_{n,j})_{n \geq 1, 1 \leq j \leq \ell}$ be a triangular array of centered random variables. For $n \geq 1$, let $\alpha_n(k)$ be the $k$ mixing coefficient of the sequence $(Y_\ell)_{\ell \geq 1}$, where $Y_\ell = 0$ if $\ell \leq 0$ or $\ell \geq n+1$ and $Y_\ell = x_{n,\ell}$ for $1 \leq \ell \leq n$. Let $S_n := \sum_{\ell=1}^{n} x_{n,\ell}$ and suppose that the following conditions hold:

1. there exists a constant $M$ such that $\sup_{n \geq 1} \max_{1 \leq j \leq \ell} |x_{n,j}| \leq M$ almost surely;
2. $\lim_{n \to +\infty} n^{-1} \mathbb{V}ar(S_n) = \sigma^2 > 0$;
3. there exists a sequence $(a_k)_{k \geq 1}$ such that $\alpha_n(k) \leq a_k$ for all $n$ and all $k$ and for some $r > 0$,
\[
\sum_{k=1}^{\infty} k^r a_k < \infty, 
\] (A.1)

Then $\left( n^{-1/2} S_n \right)_{n \geq 1}$ converges in distribution to a centered normal random variable with variance $\sigma^2$.

We will also need the following covariance inequality, due to Ibragimov [11].

**Proposition A.2.** Let $X$ and $Y$ be two bounded random variables. Then
\[
\text{Cov}(X,Y) \leq 2\alpha(\sigma(X), \sigma(Y)) \|X\|_\infty \|Y\|_\infty. 
\] (A.2)
In order to control partial sums of an $\alpha$-mixing sequence, we need the following maximal inequality (see Theorem 3.1 in [13]).

**Theorem A.3.** Let $(X_i)_{i \geq 1}$ be a centered sequence of random variables bounded by $M$. Then

$$E \left[ \max_{1 \leq k \leq n} \left( \sum_{i=1}^{k} X_i \right)^2 \right] \leq 16M^2 n \sum_{k \geq 0} \alpha(k).$$

(A.3)

We need the following moment inequality for mixing sequences, in the spirit of Rosenthal’s inequality [14].

**Proposition A.4** (Theorem 2.5 in [13]). Let $p > 1$ and let $(X_i)_{i \geq 1}$ be a strictly stationary sequence of real-valued centered random variables bounded by $M$. Then

$$E \left[ |S_n|^p \right] \leq (8np)^p \int_0^1 \left( \alpha^{-1}(u) \right)^p du \leq (8np)^p \sum_k k^p \alpha(k)$$

(A.4)

where

$$\alpha^{-1}(u) = \text{Card} \{ k \geq 1, \alpha(k) \leq u \}, u \in [0, 1].$$

(A.5)

The treatment of the degenerated part requires the following moment inequality for a degenerated $U$-statistic, which is Lemma 2.4 in [9]. It was done in the case of a symmetric kernel, but a careful reading of the proof shows that it also works in the non-symmetric case.

**Lemma A.5.** Let $(X_i)_{i \geq 1}$ be a strictly stationary sequence and let $h: \mathbb{R}^2 \to \mathbb{R}$ be a measurable function bounded by $M$ and such that for all $x \in \mathbb{R}$, $E[h(X_1, x)] = E[h(x, X_1)] = 0$. Suppose also that $\sum_{k \geq 1} k \beta(k)$ converges. Then for $n \geq 2$, the following inequality holds:

$$E \left[ \max_{2 \leq k \leq n} \left( \sum_{1 \leq i < j \leq k} h(X_i, X_j) \right)^2 \right] \leq CM^2 n^2 \log n,$$

(A.6)

where $C$ depends only on $(\beta(k))_{k \geq 1}$.

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