Pluricomplex Green’s functions and Fano manifolds

Nicholas McCleerey and Valentino Tosatti

Abstract. We show that if a Fano manifold does not admit Kähler-Einstein metrics then the Kähler potentials along the continuity method subconverge to a function with analytic singularities along a subvariety which solves the homogeneous complex Monge-Ampère equation on its complement, confirming an expectation of Tian-Yau.

Keywords. Fano manifold; pluricomplex Green function; algebraic Kähler potentials

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[Français]

Titre. Fonctions de Green pluricomplexes et variétés de Fano

Résumé. Nous montrons que si une variété de Fano n’admet aucune métrique de Kähler-Einstein alors, suivant la méthode de continuité, les potentiels kählériens sous-convergent vers une fonction à singularités analytiques le long d’une sous-variété, sur le complémentaire de laquelle la fonction est solution de l’équation de Monge-Ampère complexe homogène. Cela confirme une attente de Tian-Yau.

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1. Introduction

Let $X^n$ be a Fano manifold, i.e. a compact complex manifold with $c_1(X) > 0$. A Kähler-Einstein metric on $X$ is a Kähler metric $\omega$ which satisfies

$$\text{Ric}(\omega) = \omega.$$ 

This implies that $[\omega] = c_1(X)$. We assume throughout this paper that $X$ does not admit a Kähler-Einstein metric. This is known to be equivalent to K-unstability by [13] (see also [40]), but we will not use this fact.

We fix a Kähler metric $\omega$ with $[\omega] = c_1(X)$, with Ricci potential $F$ defined by $\text{Ric}(\omega) = \omega + \sqrt{-1} \partial \bar{\partial} F$ (normalized by $\int_X (e^F - 1) \omega^n = 0$). We consider Kähler metrics $\omega_t$ with $[\omega_t] = c_1(X)$ which satisfy

$$\text{Ric}(\omega_t) = t \omega_t + (1-t) \omega.$$ 

We can write $\omega_t = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t$ and the functions $\varphi_t$ solve the complex Monge-Ampère equation [46]

$$\omega^n_t = e^{F - t \varphi_t} \omega^n.$$  \hspace{1cm} (1.1)

A solution $\varphi_t$ exists on $[0, R(X))$ where $R(X) \leq 1$ is the greatest lower bound for the Ricci curvature of Kähler metrics in $c_1(X)$ [36]. It is known [35, 38] that since $X$ does not admit Kähler-Einstein metrics, we must have that $\lim_{t \to R(X)} \sup_X \varphi_t = +\infty$. We fix a sequence $t_i \to R(X)$ and write $\varphi_t := \varphi_{t_i}$ and $\omega_t := \omega_{t_i}$. Using this result, together with multiplier ideal sheaves, Nadel [29, Proposition 4.1] proved that (up to passing to a subsequence) the measures $\omega^n_t$ converge to zero (as measures) on compact sets of $X \setminus V$ for some proper analytic subvariety $V \subset X$, and in [44] the second-named author improved this to uniform convergence.

By weak compactness of closed positive currents in a fixed cohomology class, up to subsequences we can extract a limit $\rho$ of $\varphi_t - \sup_X \varphi_t$ (which may depend on the subsequence), which is an unbounded $\omega$-psh function, and the convergence happens in the $L^1$ topology.

In their work [41, p.178], Tian-Yau expressed the expectation that $\rho$ should have logarithmic poles along a proper analytic subvariety $V \subset X$, and that it should satisfy $(\omega + \sqrt{-1} \partial \bar{\partial} \rho)^n = 0$ on $X \setminus V$, so that $\rho$ could be thought of as a kind of pluricomplex Green’s function (see also [38, p.238] and [39, p.109]).

In this note we confirm Tian-Yau’s expectation:

**Theorem 1.1.** Let $X$ be a Fano manifold without a Kähler-Einstein metric, and let $\omega_t = \omega + \sqrt{-1} \partial \bar{\partial} \varphi_t$ be the solutions of the continuity method (1.1). Given any sequence $t_i \in [0, R(X))$ with $t_i \to R(X)$, choose a subsequence such that $\varphi_{t_i} - \sup_X \varphi_{t_i}$ converge in $L^1(X)$ to an $\omega$-psh function $\rho$. Then we can find $m \geq 1$ and an $\omega$-psh function $\psi$ on $X$ with analytic singularitites

$$\psi = \frac{1}{m} \log \sum_{j=1}^{p} \lambda_j^2 |S_j|^2_{h^m_i},$$  \hspace{1cm} (1.2)
for some $\lambda_j \in (0, 1]$ and some sections $S_j \in H^0(X, K_X^{-m})$, with nonempty common zero locus $V \subset X$ such that $\rho - \psi$ is bounded on $X$, and on $X \setminus V$ we have
\[
(\omega + \sqrt{-1} \partial \overline{\partial} \rho)^n = 0,
\]
where the Monge-Ampère product is in the sense of Bedford-Taylor [6].

In particular, Theorem 1.1 implies that the non-pluripolar Monge-Ampère operator of $\rho$ (defined in [12]) vanishes identically on $X$. On the other hand, there is another meaningful Monge-Ampère operator that can be applied to $\rho$. Indeed, the fact that $\rho - \psi \in L^\infty(X)$ implies that $\rho$ itself has analytic singularities. In [3] Andersson-Blocki-Wulcan defined a Monge-Ampère operator for $\omega$-psh functions with analytic singularities (generalizing earlier work of Andersson-Wulcan [4] in the local setting). In general, applying this Monge-Ampère operator to $\rho$ will produce a Radon measure $\mu$ on $X$ (which may be identically zero in some cases), which by Theorem 1.1 is supported on the analytic set $V$, thus providing geometrically interesting examples of unbounded quasi-psh functions on compact Kähler manifolds with Monge-Ampère operator concentrated on a subvariety (see also [1, 2] for related results in the local setting). In particular, this answers [11, Question 1 (c)], an open problem raised at the AIM workshop “The complex Monge-Ampère equation” in August 2016 (cf. the related [23, Question 12]).

Note that in general a formula for the total mass of $\mu$ is proved in [3, Theorem 1.2], and it satisfies
\[
\int_X \mu \leq \int_X \omega^n,
\]
with strict inequality in general (but it is not hard to see that if $\dim X = 2$ and $V$ is a finite set then equality holds). Therefore, the measure $\mu$ is in general different from the measures that one obtains as weak limits of $(\omega + \sqrt{-1} \partial \overline{\partial} \psi_i)^n$ (up to subsequences), whose total mass is always equal to $\int_X \omega^n$.

**Remark 1.2.** [Remark added in proof] After this work was posted on the arXiv, and partly prompted by it, Blocki [10] modified the definition of the Monge-Ampère operator for $\omega$-psh functions with analytic singularities of [4, 3], and with his definition the total mass is always equal to $\int_X \omega^n$. It is an interesting question to determine whether this Monge-Ampère operator equals the weak limit of $(\omega + \sqrt{-1} \partial \overline{\partial} \psi_i)^n$.

**Remark 1.3.** As in the second-named author’s previous work [44], Theorem 1.1 has a direct counterpart for solutions of the normalized Kähler-Ricci flow, instead of the continuity method (1.1). The statement is identical to Theorem 1.1, except that now the sequence $t_i$ goes to $+\infty$. The proof is also almost verbatim the same, and the partial $C^0$ estimate along the flow is proved in [14, 15] (see also [5]). All other ingredients used also have well-known counterparts for the flow (see [44]). We leave the simple details to the interested reader.

**Remark 1.4.** The behavior of the solutions $\omega_t$ of (1.1) as $t \to R(X)$ has been investigated in the past. If the manifold is K-stable, [20] show that $\omega_t$ converge smoothly to a Kähler-Einstein metric. If on the other hand no such metric exists, the blowup behavior of $\omega_t$ has been investigated in [29, 44] in the setting of this paper, and also in [20, 26, 33] by allowing reparametrizations of the metrics by diffeomorphisms.

The proof of Theorem 1.1 relies on the partial $C^0$ estimate for solutions of (1.1) which was established by Székelyhidi [37]. We recall this in section 2, together with a well-known reformulation of this estimate (Proposition 2.1). In section 3 we observe that this gives us the singularity model function $\psi$ in (1.2), and it also implies that $\rho$ has the same singularity type as $\psi$. In section 4 we show the general fact that every $\omega$-psh function on $X$ with the same singularity type as $\psi$ has vanishing Monge-Ampère operator outside $V$, thus proving Theorem 1.1. This relies on a geometric understanding of the rational map defined by the sections $\{S_j\}$ as in Theorem 1.1. Lastly, in section 5 we discuss the pluricomplex Green’s function with the same singularity type as $\psi$. 
2. The partial $C^0$ estimate

To start we fix some notation. We choose a Hermitian metric $h$ on $K_X^{-1}$ with curvature $R_h = \omega$ (such $h$ is unique up to scaling), and let $h^m$ be the induced metric on $K_X^{-m}$, for all $m \geq 1$. Let $N_m = \dim H^0(X, K_X^{-m})$, and for any $m \geq 1$ define the density of states function

$$\rho_m(\omega) = \sum_{j=1}^{N_m} |S_j|_{h^m}^2,$$

where $S_1, \ldots, S_{N_m}$ are a basis of $H^0(X, K_X^{-m})$ which is orthonormal with respect to the $L^2$ inner product $\langle S_1, S_2 \rangle_{h^m} = \omega$. Clearly $\rho_m(\omega)$ is independent of the choice of basis, and is also unchanged if we scale $h$ by a constant. The integral $\int_X \rho_m(\omega)\omega^m$ equals $N_m$, and if $m$ is sufficiently large so that $K_X^{-m}$ is very ample, then $\rho_m(\omega)$ is strictly positive on $X$. If we apply this same construction to the metrics $\omega_t$ and Hermitian metrics $h_t = h e^{-\psi}$ we get a density of states function $\rho_m(\omega_t)$. Following [39], we say that a “partial $C^0$ estimate” holds if there exist $m \geq 1$ and a constant $C > 0$ such that

$$\inf_X \rho_m(\omega_t) \geq C^{-1},$$

holds for all $t \in [0, R(X))$. The reason for this name is explained by the following proposition, which is essentially well-known (see [39, Lemma 2.2] and [43, Proposition 5.1]), but we provide the details for convenience:

**Proposition 2.1.** If a partial $C^0$ estimate holds then there exists $m \geq 1$, such that for all $\varepsilon > 0$ we can find a constant $C > 0$ so that for all $t \in [\varepsilon, R(X))$ we can find real numbers $1 = \lambda_1(t) \geq \ldots \geq \lambda_{N_m}(t) > 0$ and a basis $\{S_j(t)\}_{1 \leq j \leq N_m}$ of $H^0(X, K_X^{-m})$, orthonormal with respect to the $L^2$ inner product of $\omega, h^m$, such that for all $t \in [0, R(X))$ we have

$$\sup_X |q_t - \sup_X q_t - \frac{1}{m} \log \sum_{j=1}^{N_m} \lambda_j(t)^2 |S_j(t)|_{h^m}^2 | \leq C.$$  

(2.2)

In the rest of the paper we will fix a value of $\varepsilon > 0$ once and for all, for example $\varepsilon = R(X)/2$. The precise choice is irrelevant, since we are only interested in the behavior as $t \to R(X)$.

**Proof.** First, it is well-known that for all $m \geq 1$ and $\varepsilon > 0$ there is a constant $C$ such that for all $t \in [\varepsilon, R(X))$ we have

$$\rho_m(\omega_t) \leq C.$$  

(2.3)

To see this, first observe for every $S \in H^0(X, K_X^{-m})$ we have

$$\Delta_{\omega_t} |S|_{h^m}^2 = |\nabla S|^2_{h_t} - 2m|S|_{h^m}^2 \geq -2m|S|_{h^m}^2,$$

(2.4)

and that since $\text{Ric}(\omega_t) \geq 2t\omega_t \geq \varepsilon \omega_t$, Myers’ Theorem gives a uniform upper bound for $\text{diam}(X, \omega_t)$ and then Croke [19] and Li [27] show that the Sobolev constant of $(X, \omega_t)$ has a uniform upper bound. We can then apply Moser iteration to (2.4) to get

$$\sup_X |S|_{h^m}^2 \leq C \int_X |S|_{h^m}^2 \omega_t^m \leq C,$$

(2.5)
provided we assume that \( \int_X |S_{1j}|^2 \omega^n = 1 \). Taking now an orthonormal basis of sections and summing we obtain (2.3).

Thanks to (2.3) we know that for \( t \in [\epsilon, R(X)] \) a partial \( C^0 \) estimate is equivalent to

\[
\sup_X |\log \rho_m(\omega_t)| \leq C. \tag{2.6}
\]

We now take a basis \( \{\bar{S}_j(t)\}_{1 \leq j \leq N_m} \) of \( H^0(X, K_X^{-m}) \) orthonormal with respect to the \( L^2 \) inner product of \( \omega_t, h_t^m \) and notice that since \( h_t^m = e^{-m\varphi_t} h^m \) we clearly have

\[
q_t = \frac{1}{m} \log \left( \sum_{j=1}^{N_m} |\bar{S}_j(t)|^2_{h_t^m} \right),
\]

which is equivalent to

\[
q_t - \frac{1}{m} \log \sum_{j=1}^{N_m} |\bar{S}_j(t)|^2_{h_t^m} = -\frac{1}{m} \log \rho_m(\omega_t). \tag{2.7}
\]

It follows from (2.6) and (2.7) that that for \( t \in [\epsilon, R(X)] \) a partial \( C^0 \) estimate is equivalent to an estimate

\[
\sup_X \left| q_t - \frac{1}{m} \log \sum_{j=1}^{N_m} |\bar{S}_j(t)|^2_{h_t^m} \right| \leq C.
\]

We now choose another basis \( \{S_j\}_{1 \leq j \leq N_m} \) of \( H^0(X, K_X^{-m}) \) orthonormal with respect to the \( L^2 \) inner product of \( \omega_t, h_t^m \). After modifying \( S_j \) and \( \bar{S}_j(t) \) by \( t \)-dependent unitary transformations, we obtain orthonormal bases \( \{S_j(t)\}_{1 \leq j \leq N_m} \) with respect to \( \omega_t, h_t^m \), and \( \{\bar{S}_j(t)\}_{1 \leq j \leq N_m} \) with respect to \( \omega_t, h_t^m \) such that

\[
\bar{S}_j(t) = \mu_j(t) S_j(t),
\]

for some positive real numbers \( \mu_j(t) \), with \( \mu_1(t) \geq \ldots \geq \mu_{N_m}(t) > 0 \). We then let \( \lambda_j(t) = \mu_j(t)/\mu_1(t) \) and we see that a partial \( C^0 \) estimate is equivalent to

\[
\sup_X \left| q_t - \frac{1}{m} \log \mu_1(t) - \frac{1}{m} \log \sum_{j=1}^{N_m} \lambda_j(t)^2 |S_j(t)|^2_{h_t^m} \right| \leq C. \tag{2.8}
\]

We now claim that if a partial \( C^0 \) estimate holds, then for all \( t \in [\epsilon, R(X)] \) we also have

\[
\left| \frac{1}{m} \log \mu_1(t) - \sup_X \varphi_t \right| \leq C. \tag{2.9}
\]

Once this is proved, combining (2.8) and (2.9) we get (2.2). To prove (2.9), first use (2.5) to get

\[
C \geq \sup_X |S_1(t)|^2_{h_t^m} \geq \mu_1(t)^2 \sup_X |S_1(t)|^2_{h_t^m} e^{-m \sup_X \varphi},
\]

and the fact \( \int_X |S_1(t)|^2_{h_t^m} \omega^n = 1 \) implies that \( \sup_X |S_1(t)|^2_{h_t^m} \geq 1/\text{Vol}(X, \omega) \), and so

\[
\left( \frac{1}{m} \log \mu_1(t) - \sup_X \varphi_t \right) \leq C.
\]

On the other hand the partial \( C^0 \) estimate (2.1) implies that

\[
C^{-1} \leq \rho_m(\omega_t) = \sum_{j=1}^{N_m} |\bar{S}_j(t)|^2_{h_t^m} \leq \mu_1(t)^2 \sum_{j=1}^{N_m} |S_j(t)|^2_{h_t^m} e^{-m \varphi}, \tag{2.10}
\]
and we clearly have that
\[ \sup_j \sup_X |S_j(t)|_{\Omega_m}^2 \leq C, \quad (2.11) \]
since the sections \( \{S_j(t)\} \) are just varying in a compact unitary group (or one can also repeat the Moser iteration argument of (2.3) for the fixed metric \( \omega \)). This together with (2.10), evaluated at the point where \( \varphi_t \) achieves its maximum, gives the reverse inequality
\[ \left( \sup_X \varphi_t - \frac{2}{m} \log \mu_1(t) \right) \leq C, \]
which completes the proof of (2.9). \( \square \)

3. The singularity model function

The next goal is to use the partial \( C^0 \) estimate in Proposition 2.1 to construct a singular \( \omega \)-psh function \( \psi \) which will have the same singularity type of any weak limit of the normalized solutions \( \varphi_t - \sup_X \varphi_t \) of the continuity method.

Let the notation be as in Proposition 2.1, and in particular we fix once and for all a value of \( m \geq 1 \) given there. We can find a sequence \( t_i \to R(X) \) and an \( \omega \)-psh function \( \rho \) with \( \sup_X \rho = 0 \) such that \( \varphi_t - \sup_X \varphi_t \to \rho \) in \( L^1(X) \), and pointwise a.e. Passing to a subsequence, we can find a basis \( \{S_j\}_{1 \leq j \leq N_m} \) of \( H^0(X, K_X^{-m}) \) orthonormal with respect to the \( L^2 \) inner product of \( \omega, h^m \), such that \( S_j(t_i) \to S_j \) smoothly as \( i \to \infty \), for all \( 1 \leq j \leq N_m \). The change of basis matrix from \( \{S_j\}_{1 \leq j \leq N_m} \) to \( \{S_j(t_i)\}_{1 \leq j \leq N_m} \) induces an automorphism \( \sigma(t) \) of \( \mathbb{C}P^{N_m-1} \), such that \( \sigma(t_i) \to \text{Id} \) smoothly as \( i \to \infty \).

For ease of notation, write
\[ \psi_t = \frac{1}{m} \log \sum_{j=1}^{N_m} \lambda_j(t)^2 |S_j(t)|_{\Omega_m}^2. \]
These functions are Kähler potentials for \( \omega \) since
\[ \omega + \sqrt{-1} \partial \bar{\partial} \psi_t = \frac{t^* \sigma(t)^* \tau(t)^* \omega_{FS}}{m} > 0, \quad (3.1) \]
where \( t : X \to \mathbb{C}P^{N_m-1} \) is the Kodaira embedding map given by the sections \( \{S_j\}_{1 \leq j \leq N_m} \), the map \( \tau(t) \) is the automorphism of \( \mathbb{C}P^{N_m-1} \) induced by the diagonal matrix with entries \( \{\lambda_j(t)\}_{1 \leq j \leq N_m} \), and \( \omega_{FS} \) is the Fubini-Study metric on \( \mathbb{C}P^{N_m-1} \). The identity in (3.1) follows directly from the definition of the Fubini-Study metric \( \omega_{FS} \) on \( \mathbb{C}P^{N_m-1} \), which on \( \mathbb{C}^{N_m} \setminus \{0\} \) is given explicitly by \( \omega_{FS} = \sqrt{-1} \partial \bar{\partial} \log \sum_{j=1}^{N_m} |z_j|^2 \), and from the fact that the curvature of \( h \) is \( \omega \).

Up to passing to a subsequence of \( t_i \), we may assume that \( \lambda_j(t_i) \to \lambda_j \) as \( i \to \infty \) for all \( j \), and we have
\[ 1 = \lambda_1 > \cdots > \lambda_p > 0 = \lambda_{p+1} = \cdots = \lambda_{N_m}, \]
for some \( 1 \leq p < N_m \). The case \( p = N_m \) is impossible because by (2.2) it would imply a uniform \( L^\infty \) bound for \( \varphi_t \), and so \( X \) would admit a Kähler-Einstein metric. For the same reason, the set \( V := \{S_1 = \cdots = S_p = 0\} \) must be a nonempty proper analytic subvariety of \( X \).

Note that thanks to (2.2) we can write
\[ \omega^n_t = e^{F(t)}(\varphi_t - \sup_X \varphi_t) e^{-t} \sup_X \varphi_t \omega^n \leq C e^{t \varphi_t} e^{-t} \sup_X \varphi_t \omega^n, \]
and since the term \( e^{t \varphi_t} \) is uniformly bounded on compact sets of \( X \setminus V \), we see immediately that
\[ \omega^n_t \to 0, \quad (3.2) \]
uniformly on compact sets of \(X \setminus V\) (this result was proved in [44] without using the partial \(C^0\) estimate, which was not available at the time, with weaker results established earlier in [29]).

Let then

\[
\psi = \frac{1}{m} \log \sum_{j=1}^{p} \lambda_j^2 |S_j|^2, 
\]

which is a smooth function on \(X \setminus V\) which approaches \(-\infty\) uniformly on \(V\). Since \(e^{m\psi_i} \to e^{m\psi}\) smoothly on \(X\), and since \(\psi_i\) are smooth and \(\omega_i\)-psh, it follows that \(\psi\) is \(\omega\)-psh. This will be our singularity model function in the rest of the argument, as we now explain:

**Lemma 3.1.** Define the class

\[ \mathcal{C} = \{ \eta \in PSH(X, \omega) \mid \eta - \psi \in L^\infty(X) \}, \]

of \(\omega\)-psh functions with the same singularity type as \(\psi\). Then we have that \(\rho \in \mathcal{C}\).

**Proof.** Recall that we have \(\varphi_i - \sup_X \varphi_i \to \rho\) a.e. on \(X\). Thanks to (2.2), the function \(\rho\) satisfies

\[
|\rho - \psi| \leq C, \tag{3.3}
\]
a.e. on \(X\), which implies the same inequality on all of \(X\) by elementary properties of psh functions (cf. [25, Theorem K.15]), thus showing that \(\rho \in \mathcal{C}\). \(\square\)

## 4. Understanding the class \(\mathcal{C}\)

We now exploit the geometry of our setting to gain a better understanding of the class of functions \(\mathcal{C}\).

The sections \(\{\lambda_j S_j\}_{1 \leq j \leq p}\) define a rational map \(\Phi : X \to \mathbb{CP}^p\), with indeterminacy locus \(Z \subset V\) (this inclusion is in general strict, since \(\text{codim} Z > 2\) while \(V\) may contain divisorial components). Let \(Y\) be the image of \(\Phi\), i.e. the closure of \(\Phi(X \setminus Z)\) in \(\mathbb{CP}^p\), which is an irreducible projective variety. By resolving the indeterminacies of \(\Phi\) we get a modification \(\mu : \tilde{X} \to X\), obtained as a sequence of blowups with smooth centers, and a holomorphic map \(\Psi : \tilde{X} \to Y\) such that \(\Psi = \Phi \circ \mu\) holds on \(\tilde{X} \setminus \mu^{-1}(Z)\). We may also assume without loss of generality that \(\mu\) principalizes the ideal sheaf generated by \(\{S_j\}_{1 \leq j \leq p}\), so that we have

\[
\mu^* (\omega + \sqrt{-1} \partial \overline{\partial} \psi) = \theta + [E],
\]

where \(E\) is an effective \(\mathbb{R}\)-divisor with \(\mu(E) \subset V\), and \(\theta\) is a smooth closed semipositive \((1,1)\) form on \(\tilde{X}\). We will denote by \(\omega_{FS,\rho}\) the Fubini-Study metric on \(\mathbb{CP}^p\). To identify \(\theta\), note that on \(X \setminus V\) we have by definition \(\omega + \sqrt{-1} \partial \overline{\partial} \psi = \frac{\Psi^* \omega_{FS,\rho}}{m}\), and so on \(\tilde{X} \setminus \mu^{-1}(V)\) we have

\[
\mu^* (\omega + \sqrt{-1} \partial \overline{\partial} \psi) = \frac{\mu^* \Phi^* \omega_{FS,\rho}}{m} = \frac{\Psi^* \omega_{FS,\rho}}{m},
\]

and so \(\theta = \frac{\Psi^* \omega_{FS,\rho}}{m}\) on \(\tilde{X} \setminus \mu^{-1}(V)\), and hence everywhere since both sides of this equality are smooth forms on all of \(\tilde{X}\). This proves the key relation

\[
\mu^* (\omega + \sqrt{-1} \partial \overline{\partial} \psi) = \frac{\Psi^* \omega_{FS,\rho}}{m} + [E]. \tag{4.1}
\]

Let \(\tilde{X} \twoheadrightarrow \hat{Y} \xrightarrow{q} Y\) be the Stein factorization of \(\Psi\), where \(\hat{Y}\) is an irreducible projective variety, the map \(\nu\) has connected fibers, and \(q\) is a finite morphism.

We have that \(q^* \omega_{FS,\rho}\) is a smooth semipositive \((1,1)\) form on \(\hat{Y}\), in the sense of analytic spaces. Since \(\nu\) has compact connected fibers, a standard argument shows that the set of \(\frac{\Psi^* \omega_{FS,\rho}}{m}\)-psh functions on \(\tilde{X}\)
can be identified with the set of (weakly) $\frac{d^* \omega_{FS,p}}{m}$-psh functions on $\tilde{Y}$ via $\nu^*$ (indeed the restriction of every $\Psi^* \omega_{FS,p}$-psh function to any fiber of $\nu$ is plurisubharmonic and hence constant on that fiber). We will use this standard argument several other times in the following.

Here and in the following, as in [21], a weakly quasi-psh function on a compact analytic space means a quasi-psh function on its regular part which is locally bounded above near the singular set. As shown in [21, §1], weakly quasi-psh functions are the same as usual quasi-psh functions if the analytic space is normal, and otherwise they can be identified with quasi-psh functions on its normalization.

**Proposition 4.1.** Given any function $\eta \in C$, there is a unique bounded weakly $\frac{d^* \omega_{FS,p}}{m}$-psh function $u$ on $\tilde{Y}$ such that

$$\mu^* \eta = \mu^* \psi + \nu^* u.$$  

(4.2)

Conversely, given any bounded weakly $\frac{d^* \omega_{FS,p}}{m}$-psh function $u$ on $\tilde{Y}$ there is a unique function $\eta \in C$ such that (4.2) holds.

The relation in (4.2) thus allows us to identify the class $C$ with the class of bounded weakly $\frac{d^* \omega_{FS,p}}{m}$-psh functions on $\tilde{Y}$.

Next, we observe that

**Proposition 4.2.** We have that

$$\dim Y < \dim X.$$

This is a consequence of our assumption that $X$ does not admit a Kähler-Einstein metric.

Lastly, every function $\eta \in C$ belongs to $L^\infty_{loc}(X \setminus V)$, and so its Monge-Ampère operator $(\omega + \sqrt{-1} \partial \bar{\partial} \eta)^n$ is well-defined on $X \setminus V$ thanks to Bedford-Taylor [6]. Combining the results in Propositions 4.1 and 4.2 we will obtain:

**Theorem 4.3.** For every $\eta \in C$ we have that

$$(\omega + \sqrt{-1} \partial \bar{\partial} \eta)^n = 0,$$

on $X \setminus V$.

In particular, this holds for the function $\rho$, thanks to Lemma 3.1, and Theorem 1.1 thus follows from these.

**Proof of Proposition 4.1.** If $\eta$ is an $\omega$-psh function on $X$ with $\eta - \psi \in L^\infty(X)$, i.e. $\eta$ is an element of $C$, then using (4.1) we can write

$$\mu^*(\omega + \sqrt{-1} \partial \bar{\partial} \eta) = \frac{\Psi^* \omega_{FS,p}}{m} + \sqrt{-1} \partial \bar{\partial} \mu^*(\eta - \psi) + [E],$$

where $E$ is as in (4.1) and $\mu^*(\eta - \psi) \in L^\infty(\tilde{X})$. Applying the Siu decomposition, we see that

$$\frac{\Psi^* \omega_{FS,p}}{m} + \sqrt{-1} \partial \bar{\partial} \mu^*(\eta - \psi) \geq 0,$$

weakly, and so

$$\mu^*(\eta - \psi) = \nu^* u_\eta,$$

for a bounded weakly $\frac{d^* \omega_{FS,p}}{m}$-psh functions $u_\eta$ on $\tilde{Y}$, which is uniquely determined by $\eta$ (and $\psi$, which we view as fixed).
Conversely, given a bounded weakly \( \frac{q^i\omega_{FS,p}}{m} \)-psh function \( u \) on \( \tilde{Y} \), we have that \( \nu^* u \) is \( \frac{\Psi^i\omega_{FS,p}}{m} \)-psh and bounded on \( \check{X} \) and so
\[
0 \leq \frac{\Psi^i\omega_{FS,p}}{m} + [E] + \sqrt{-1} \partial \bar{\partial} \nu^* u = \mu^* \omega + \sqrt{-1} \partial \bar{\partial} (\mu^* \psi + \nu^* u),
\]
and so \( \mu^* \psi + \nu^* u \) descends to an \( \omega \)-psh function \( \eta_u \) on \( X \) with \( \eta_u - \psi \in L^\infty(X) \), i.e. \( \eta_u \in \mathcal{C} \).

These two constructions are inverses to each other, and so we obtain the desired bijective correspondence between functions in \( \mathcal{C} \) and bounded weakly \( \frac{q^i\omega_{FS}}{m} \)-psh functions on \( \check{Y} \).

**Proof of Proposition 4.2.** On \( X \) we have the estimate
\[
\omega_i \geq C^{-1} \frac{\tau^i \sigma(t)^* \tau(t)^* \omega_{FS}}{m},
\]
which is a direct consequence of the partial \( C^0 \) estimate (see e.g. [24, Lemma 4.2]). We can also give a direct proof by calculating
\[
\Delta_{\omega_i} \left( \log \frac{\tau^i \sigma(t)^* \tau(t)^* \omega_{FS}}{m} - A(\varphi_i - \sup_{X} \varphi_i - \psi_i) \right) \geq \frac{\tau^i \sigma(t)^* \tau(t)^* \omega_{FS}}{m} - C,
\]
if \( A \) is sufficiently large, and applying the maximum principle together with the partial \( C^0 \) estimate (2.2) (for this calculation we used that the bisectional curvature of the metrics \( \frac{\tau^i \sigma(t)^* \tau(t)^* \omega_{FS}}{m} \) have a uniform upper bound independent of \( t \).

If we had \( \dim Y = \dim X \) then the rational map \( \Phi \) would be generically finite, so there would be a nonempty open subset \( U \subset X \backslash V \) such that \( \Phi|_U \) is a biholomorphism with its image. Recall that \( \Phi \) is the rational map defined by the sections \( \{1 : j \}_{1 \leq j \leq p} \), while \( \iota : X \hookrightarrow \mathbb{C}P^{N-1} \) is the embedding defined by the sections \( \{ S_j \}_{1 \leq j \leq N} \), and so \( \Phi = \tilde{\tau} \circ P \circ \iota \) where \( P : \mathbb{C}P^{N-1} \rightarrow \mathbb{C}P^{p-1} \) is the linear projection given by \( [z_1 : \ldots : z_N] \mapsto [z_1 : \ldots : z_p] \) and \( \tilde{\tau} : \mathbb{C}P^{p-1} \rightarrow \mathbb{C}P^{p-1} \) is the automorphism given by
\[
[z_1 : \ldots : z_p] \mapsto [\lambda_1 z_1 : \ldots : \lambda_p z_p].
\]
In particular, on the embedded open \( n \)-fold \( \iota(U) \), we have that \( P|_{\iota(U)} \) is also a biholomorphism with its image. The automorphisms \( \tau(t_i) \) descend to automorphisms \( \tilde{\tau}(t_i) \) on \( \mathbb{C}P^{p-1} \), and now as \( i \rightarrow \infty \) these converge smoothly to the automorphism \( \tilde{\tau} \). Thus \( P \circ \tau(t_i) \circ \sigma(t_i) \circ \iota = \tilde{\tau}(t_i) \circ P \circ \sigma(t_i) \circ \iota \), which converge smoothly as maps to \( \tilde{\tau} \circ P \circ \iota = \Phi \) on \( U \) as \( i \rightarrow \infty \).

Since \( \Phi \) is an isomorphism on \( U \), smooth convergence gives us that \( P \circ \tau(t_i) \circ \sigma(t_i) \circ \iota \) is a local isomorphism. Thus, after possibly shrinking \( U \),
\[
P : (\tau(t_i) \circ \sigma(t_i) \circ \iota)(U) \rightarrow (P \circ \tau(t_i) \circ \sigma(t_i) \circ \iota)(U) = (\tilde{\tau}(t_i) \circ P \circ \sigma(t_i) \circ \iota)(U)
\]
is an isomorphism, and for \( i \) large the open sets \( (\tilde{\tau}(t_i) \circ P \circ \sigma(t_i) \circ \iota)(U) \subset \mathbb{C}P^{p-1} \) converge to the open set \( (\tilde{\tau} \circ P \circ \iota)(U) \) in the Hausdorff sense. Up to shrinking \( U \), there is an open subset \( V \subset \mathbb{C}P^{p-1} \) that contains \( (\tilde{\tau}(t_i) \circ P \circ \sigma(t_i) \circ \iota)(U) \) for all \( i \) large, and still \( P^{-1} \) is well-defined on \( V \) (and \( P : P^{-1}(V) \rightarrow V \) is a biholomorphism), so that \( P^{-1}(V) \) contains \( (\tau(t_i) \circ \sigma(t_i) \circ \iota)(U) \) for all \( i \) large, and on \( P^{-1}(V) \) we have
\[
P^* \omega_{FS,p} \leq C \omega_{FS},
\]
On \( U \) we also have that \( \frac{\tau^i \sigma(t_i)^* \tau(t_i)^* P^* \omega_{FS,p}}{m} \) converges smoothly to \( \frac{\Phi^i \omega_{FS,p}}{m} \), which is a Kähler metric on \( U \). Thanks to (4.3) and (4.4), on \( U \) we have
\[
\omega_i \geq C^{-1} \frac{\tau^i \sigma(t_i)^* \tau(t_i)^* \omega_{FS}}{m} \geq C^{-1} \frac{\tau^i \sigma(t_i)^* \tau(t_i)^* \omega_{FS,p}}{m} \geq C^{-1} \frac{\Phi^i \omega_{FS,p}}{m},
\]
for all \( i \) large, which implies that \( \int_U \omega_i^n \geq C^{-1} \), which is absurd thanks to (3.2).
Remark 4.4. In particular we see that if $\dim Y = 0$ (i.e. $Y$ is a point) then we have $\mathcal{C} = \{\psi + s\}_{s \in \mathbb{R}}$. On the other hand as long as $\dim Y > 0$ the class $\mathcal{C}$ is always rather large.

Proof of Theorem 4.3. Thanks to Proposition 4.1, every $\eta \in \mathcal{C}$ satisfies $\mu^* \eta = \mu^* \psi + \nu^* u$ for some bounded weakly $\frac{q^* \omega_{FS,p}}{m}$-psh function $u$ on $\tilde{Y}$. Then using (4.1) we have

$$\mu^*(\omega + \sqrt{-1} \partial \bar{\partial} \eta) = \frac{\Psi^* \omega_{FS,p}}{m} + \sqrt{-1} \partial \bar{\partial} \nu^* u + [E],$$

and so if $K$ is any compact subset of $X \setminus V$, since $\mu$ is an isomorphism on $\mu^{-1}(K)$, we get

$$\int_K (\omega + \sqrt{-1} \partial \bar{\partial} \eta)^n = \int_{\mu^{-1}(K)} \mu^*(\omega + \sqrt{-1} \partial \bar{\partial} \eta)^n$$

$$= \int_{\mu^{-1}(K)} \nu^* \left( \frac{q^* \omega_{FS,p}}{m} + \sqrt{-1} \partial \bar{\partial} u \right)^n = 0,$$

since $\dim \tilde{Y} = \dim Y < \dim X$ by Proposition 4.2.

\[\square\]

5. The pluricomplex Green’s function

We can also consider the pluricomplex Green’s function with singularity type determined by $\psi$, namely

$$G = \sup \{ u \mid u \in \text{PSH}(X, \omega), u \leq 0, u \leq \psi + O(1)^* \}, \quad (5.1)$$

which is the compact manifold analog of the construction in [31], and has been studied in detail in [18, 30, 32] and references therein. In particular, since $\psi$ has analytic singularities, it follows from [31, 32] that $G \in \mathcal{C}$.

Thanks to Proposition 4.1 we can write

$$\mu^* G = \mu^* \psi + \nu^* F, \quad (5.2)$$

for a bounded weakly $\frac{q^* \omega_{FS,p}}{m}$-psh function $F$ on $\tilde{Y}$. The function $F$ is itself given by a suitable envelope.

Proposition 5.1. The pluricomplex Green’s function $G$ satisfies (5.2) where $F$ is the envelope on $\tilde{Y}$ given by

$$F = \sup \{ w \mid w \in \text{PSH}(\tilde{Y}, q^* \omega_{FS,p}/m), w \leq -\nu, \mu^* \psi \}^*, \quad (5.3)$$

and where we are writing

$$\nu(f)(y) = \sup_{x \in \nu^{-1}(y)} f(x),$$

for any function $f$ on $\tilde{X}, y \in \tilde{Y}$.

In other words, $F$ is given by a quasi-psh envelope with obstacle $-\nu, \mu^* \psi$ on $\tilde{Y}$.

Proof. Write $E = \sum_i \lambda_i E_i$ for $E_i$ prime divisors and $\lambda_i \in \mathbb{R}_{>0}$, and for each $i$ fix a defining section $s_i$ of $\mathcal{O}(E_i)$ and a smooth metric $h_i$ on $\mathcal{O}(E_i)$ with curvature form $R_i$. For brevity, we will write $|s|^2 = \prod_i |s|^2_{h_i}$ and $R_h = \sum \lambda_i R_i$. Then the Poincaré-Lelong formula gives

$$[E] = \sqrt{-1} \partial \bar{\partial} \log |s|^2 + R_h,$$
and we obtain that $\mu^* \omega - R_h$ is cohomologous to $\frac{\Psi^* \omega_{FS, p}}{m}$ and

$$
\mu^* \omega - R_h = \frac{\Psi^* \omega_{FS, p}}{m} + \sqrt{-1} \partial \bar{\partial} (\log |s|^2_h - \mu^* \psi),
$$

and $\mu^* \psi - \log |s|^2_h$ is smooth on all of $\tilde{X}$. Note that if we denote by

$$
\tilde{G} = \sup \{ u \mid u \in PSH(\tilde{X}, \mu^* \omega), u \leq 0, u \leq \log |s|^2_h + O(1) \}^*,
$$

then we have that $\tilde{G} = \mu^* G$ (this is again because every $\mu^* \omega$-psh function on $\tilde{X}$ is in fact the pullback of an $\omega$-psh function on $X$).

As in [28], we use a trick from [8, Section 4] (see also [31]), to show that

$$
\tilde{G} = \log |s|^2_h + \sup \{ v \mid v \in PSH(\tilde{X}, \mu^* \omega - R_h), v \leq -\log |s|^2_h \}^*.
$$

For the reader’s convenience, we supply the simple proof. Denote the right hand side by $\hat{G}$. For one direction, if $v$ is $(\mu^* \omega - R_h)$-psh and satisfies $v \leq -\log |s|^2_h$, then $u := v + \log |s|^2_h$ satisfies $u \leq 0$ but also since $v \leq C$ on $\tilde{X}$, we see that $u \leq \log |s|^2_h + C$, and also

$$
\mu^* \omega + \sqrt{-1} \partial \bar{\partial} u = \mu^* \omega + \sqrt{-1} \partial \bar{\partial} \log |s|^2_h + \sqrt{-1} \partial \bar{\partial} v
$$

$$
= \mu^* \omega - R_h + [E] + \sqrt{-1} \partial \bar{\partial} v
$$

$$
\geq \mu^* \omega - R_h + \sqrt{-1} \partial \bar{\partial} v \geq 0,
$$

and so $\hat{G} \leq \tilde{G}$. Conversely, if $u$ is $\mu^* \omega$-psh and satisfies $u \leq 0$ and $u \leq \log |s|^2_h + C$ for some $C$, then the Siu decomposition of $\mu^* \omega + \sqrt{-1} \partial \bar{\partial} u$ contains $[E]$ and so

$$
0 \leq \mu^* \omega + \sqrt{-1} \partial \bar{\partial} u - [E] = \mu^* \omega - R_h + \sqrt{-1} \partial \bar{\partial} (u - \log |s|^2_h),
$$

and so $v := u - \log |s|^2_h$ is $(\mu^* \omega - R_h)$-psh and satisfies $v \leq -\log |s|^2_h$, and it follows that $\hat{G} \leq \tilde{G}$, which proves our claim.

But finally note that for all $x \in \tilde{X}$ we have

$$
\log |s|^2_h(x) + \sup \{ v(x) \mid v \in PSH(\tilde{X}, \mu^* \omega - R_h), v \leq -\log |s|^2_h \}
$$

$$
= \mu^* \psi(x) + \sup \{ v(x) \mid v \in PSH(\tilde{X}, \Psi^* \omega_{FS, p}/m), v \leq -\mu^* \psi \}
$$

$$
= \mu^* \psi(x) + \sup \{ w(v(x)) \mid w \in PSH(\tilde{Y}, q^* \omega_{FS, p}/m), w \leq -v, \mu^* \psi \}
$$

and taking the upper-semicontinuous regularization and using the claim above gives $\mu^* G = \mu^* \psi + v^* F$, which completes the proof.

Using Proposition 5.1 we can see that $F$ is continuous on a Zariski open subset of $\tilde{Y}$, using the following argument. Let $g : Y' \to \tilde{Y}$ be a resolution of the singularities of $\tilde{Y}$, then we have:

$$
g^* F = \sup \{ w \mid w \in PSH(Y', g^* q^* \omega_{FS, p}/m), w \leq -g^* \nu, \mu^* \psi \}^*.
$$

Note that $g^* q^* \omega_{FS, p}/m$ is semi-positive and big, and that $-g^* \nu, \mu^* \psi$ is continuous off of $g^{-1}(\nu(\mu^{-1}(\psi^{-1}(-\infty))))$, where it is unbounded. Using the trick in [28], we can replace the obstacle $-g^* \nu, \mu^* \psi$ with a globally continuous obstacle $h$ without changing $g^* F$. Now, approximate $h$ uniformly by smooth functions $h_j$. It is easy to see that the envelopes:

$$
F_j := \sup \{ w \mid w \in PSH(Y', g^* q^* \omega_{FS, p}/m), w \leq h_j \}^*.
$$

converge uniformly to $g^* F$. But then by [8], the $F_j$ are continuous away from the non-Kähler locus of $g^* q^* \omega_{FS, p}/m$ (a proper Zariski closed subset, see e.g. [12]), so we are done.
Remark 5.2. One is naturally led to wonder about what the optimal regularity of \( G \) is. The sharp \( C^{1,1} \) regularity (on a Zariski open subset) of envelopes of the form (5.3) has been recently obtained in \([17, 45]\) in Kähler classes and in \([16]\) in nef and big classes (see also \([7, 8, 9]\)) when the obstacle is smooth (or at least \( C^{1,1} \)), but in our case the regularity of \(-\nu, \mu^* \psi\) does not seem to be very good, especially near the points where \( \nu \) is not a submersion.

On the other hand, the first-named author \([28]\) has very recently obtained \( C^{1,1} \) regularity (on a Zariski open subset) of envelopes with prescribed analytic singularities, which include those of the form (5.1), generalizing results in \([32]\) in the case of line bundles. In our situation, the results of \([28, 32]\) do not apply since in (5.1) the functions \( u \) and \( \psi \) are both \( \omega \)-psh (while for these results one would need them to be quasi-psh with respect to two different \((1,1)\)-forms such that the cohomology class of their difference is big). Moreover, the main result of \([28]\) also allows for \( u \) and \( \psi \) being both \( \omega \)-psh, but then needs the condition that the total mass of the non-pluripolar Monge-Ampère operator of \( \psi \) be strictly positive. This is obviously not the case in our situation however, by Theorem 4.3.

Remark 5.3. One possibly interesting approach to studying higher regularity of functions \( v \in \mathcal{C} \) which are already continuous on \( X \setminus V \) is the following. Suppose \( \sup_X v = 0 \). Fix an \( M > 0 \) and let \( \Omega \) be the open set \( \Omega := \{ v < -M \} \). Then one can easily show using the comparison principle and Theorem 4.3 that we have:

\[
\max\{v, -M\} = V_\Omega - M,
\]

where here \( V_\Omega \) is the global (Siciak) extremal function for \( \Omega \). In particular, one sees that \( \Omega \) is regular. There is then a well-developed theory about Hölder continuous regularity for such functions (the so called HCP property), see e.g. \([34]\). It may be possible to use this theory to study \( G \), if one can first show that it is continuous in at least a neighborhood of \( V \). Another possibility may be to study regularity of the boundary of \( \Omega \) – see the very end of \([28]\).

Remark 5.4. On can also naturally ask whether the function \( \rho \) (and therefore also its singularity type \( \psi \)) in Theorem 1.1 is actually independent of the choice of subsequence \( t_i \), and also how regular \( \rho \) is on \( X \setminus V \). Our guess is that \( \rho \) is indeed uniquely determined, and is smooth on \( X \setminus V \). These properties would both follow if one could show that the map \( \Phi : X \to \tilde{Y} \) is independent of the chosen subsequence, and that the corresponding function \( u \) on \( \tilde{Y} \) given by Lemma 3.1 and Proposition 4.1 which satisfies

\[
\mu^* \rho = \mu^* \psi + v^* u,
\]

actually solves a suitable complex Monge-Ampère equation on \( \tilde{Y} \). In a related setting of Calabi-Yau manifolds fibered over lower-dimensional spaces, such a limiting equation after collapsing the fibers was obtained by the second-named author in \([42, \text{Theorem 4.1}]\).

Remark 5.5. Lastly, we can also ask whether the limit \( \rho \) (if it is unique) is necessarily equal to the pluricomplex Green’s function \( G \) up to addition of a constant. By remark 4.4 this is the case if the rational map \( \Phi \) is constant, so that \( \tilde{Y} \) is a point. In general though this seems rather likely false.

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