Closed formulas for total 2-domination number of Cartesian product of complete graphs

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Abstract

Let $G = (V, E)$ be a finite undirected graph. A set $S$ of vertices in $V$ is said to be total $k$-dominating if every vertex in $V$ is adjacent to at least $k$ vertices in $S$. The total $k$-domination number, $\gamma_k(G)$, is the minimum cardinality of a total $k$-dominating set in $G$. In this work we study the total $k$-domination number of Cartesian product of two complete graphs which is a natural lower bound of the total $k$-domination number of Cartesian product of two graphs. We obtain new lower and upper bounds for the total $k$-domination number of Cartesian product of two complete graphs. Some asymptotic behavior are obtained as a consequence of these bounds we found. Furthermore, we obtain closed formulas for the total 2-domination number of Cartesian product of two complete graphs, which has been an open problem implicit included in previous works.

Keywords: Total dominating sets, total domination numbers, Cartesian product

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1. Introduction

We begin by stating the terminology. Throughout this paper, $G = (V, E)$ denotes a simple graph of order $|V| = n$. We denote two adjacent vertices $u$ and $v$ by $u \sim v$. For a nonempty set $X \subseteq V$ and a vertex $v \in V$ the degree of $v$ in $X$ will be denoted by $d_X(v) = |\{u \in X : u \sim v\}|$. The subgraph induced by $S \subseteq V$ will be denoted by $(S)$.

Let $G = (V, E)$ be a graph. A set $S$ of vertices in $V$ is said to be $k$-dominating if every vertex $v \in V \setminus S$ satisfies $d_S(v) \geq k$. The $k$-domination number, $\gamma_k(G)$, is the minimum cardinality of a $k$-dominating set in $G$. A set $S \subseteq V$ is said to be total $k$-dominating if every vertex in $V$ is adjacent to at least $k$ vertices in $S$. The total $k$-domination number, $\gamma_{kt}(G)$, is the minimum cardinality of a total $k$-dominating set in $G$. The notion of total domination in a graph was introduced by Cockayne, Dawes and Hedetniemi in [7]. The total domination number ($k = 1$) has also been studied in Cartesian product of graphs in [5, 11]. This conjecture states that the dominating number of the Cartesian product of two graphs is greater than or equal the product of the dominating number of both factor graphs. Domination and some well-known variations have continuously been studied, see e.g. [1, 8, 9, 10, 14] and the references therein. We recall that the Cartesian product of two graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ is the graph $G \square H = (V, E)$, such that $V = \{(u, v) : u \in V(G), v \in V(H)\}$ and two vertices $(u_1, v_1), (u_2, v_2) \in V$ are adjacent in $G \square H$ if and only if, either $u_1 = u_2$ and $v_1 \sim v_2$, or $v_1 = v_2$ and $u_1 \sim u_2$. From this definition, it follows that the Cartesian product of two graphs is commutative. When we refer to the Cartesian product of complete graphs $K_n \square K_m$, we denote $V(K_n) := \{v_1, \ldots, v_n\}$ and $V(K_m) := \{w_1, \ldots, w_m\}$.

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The first approach to domination in graph appears within the problem of the five queens, i.e., place five queens on a chessboard so that every free square is dominated by at least one queen. Note that the solutions to this problem are dominating sets in the graph whose vertices are the 64 squares of the chessboard and vertices $a, b$ are adjacent if a queen may move from $a$ to $b$ in one move or queen occupies vertex $a$. More recently, a problem on total domination appeared as Questions 3 of the 40th International Mathematical Olympiad, which was equivalent to determining the total domination number of the Cartesian product of two path graphs with same even order, i.e., $\gamma_t(P_2n \square P_2n)$. Recently, several authors have studied the total domination of product of graphs like Cartesian, strong and lexicographic, see e.g., [2, 3, 8, 12].

In this paper we discuss the total $k$-domination number of Cartesian product of two complete graphs. We obtain lower and upper bounds for $\gamma_{kt}(K_n \square K_m)$ for $k \geq 2$ that improve the bounds in [1], see, e.g., Theorems 2.6 and 3.14. In addition, we obtain a double recurrence formula given in Theorems 3.6 and 3.7 with initial conditions given in Theorems 3.9 and 3.11 which easily calculates $\gamma_{kt}(K_n \square K_m)$ for every $n, m \geq 2$, see Theorem 3.12. Besides, these results deduce asymptotic behavior for $\gamma_{kt}(K_n \square K_m)$, see Theorems 3.15.

Sometimes, throughout this work, we refer to the following equivalent problem in an $n \times m$ board which could be a convenient tool to visualize and obtain $\gamma_{kt}(K_n \square K_m)$ for $k \geq 2$.

**Problem 1.** Determine the minimum number of chess-rooks placed at distinct squares of an $n \times m$ board such that each cell is dominated by at least $k$ rooks considering that no rook dominated the square where it is placed.

Clearly, the solution of Problem 1 is $\gamma_{kt}(K_n \square K_m)$, and consequently, each rook configuration that gives a solution of the problem is a minimum total $k$-dominating set of $K_n \square K_m$. Furthermore, a total $k$-dominating set of $K_n \square K_m$ provides a configuration of rooks that satisfies the Problem 1 too. Note that for every two graphs $G, H$ with orders $n, m$, respectively, we have that for every $k \geq 1$, $\gamma_{kt}(K_n \square K_m)$ is a natural lower bound of $\gamma_{kt}(G \square H)$ since $G \square H \subseteq K_n \square K_m$, i.e., $\gamma_{kt}(K_n \square K_m) \leq \gamma_{kt}(G \square H)$. Indeed, every closed formula obtained in this work is a lower bound for the Cartesian product of two graphs with respective orders.

**2. On the total $k$-domination number of $K_n \square K_m$**

In order to obtain the main results, we collect some results through technical lemmas that will prove useful.

**Lemma 2.1.** For every $2 \leq k \leq n \leq m$,

$$\left\lfloor \frac{k(n+1)}{2} \right\rfloor + 1 \leq \gamma_{kt}(K_n \square K_m) \leq kn. \quad (2.1)$$

**Proof.** Without loss of generality, we can assume that $m \geq n$ as the Cartesian product of graphs commute. On the one hand, consider $S \subseteq V(K_n \square K_m)$ a total $k$-dominating set of $K_n \square K_m$. Since $d_{K_n \square K_m}(v) = n + m - 2$ for every $v \in S$ and $d_S(u) \geq k$ for every $u \in V(K_n \square K_m)$, we have $|S|(n + m - 2) \geq knm$. Thus, we have

$$|S| \geq \frac{knm}{n + m - 2} = \frac{k}{2} (n + 1) + \frac{k(m - n)(n - 1) + 2k}{2(n + m - 2)} \geq \frac{k(n + 1)}{2} + \frac{k}{n + m - 2}.$$

In order to obtain the second inequality, it suffices to choice $S := V(K_n) \times \{w_1, \ldots, w_k\}$ for distinct vertices $w_1, \ldots, w_k$ in $K_m$, since $S$ is a total $k$-dominating set of $K_n \square K_m$. \hfill $\square$

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1 One sample solution is queens on d4, e7, f5, g8, h6.

2 Reminder that rooks, in the game of chess, are major pieces which may move horizontally or vertically to any other square in their rank (row) or file (column).
Lemma 2.2. In every rook configuration of a Problem 1 solution for 2 ≤ k ≤ n ≤ m, there is a row or a column with at least $\left\lceil \frac{k+2}{2} \right\rceil$ rooks.

Furthermore, if $\gamma_{kt}(K_n \sqcap K_m) < k \min\{n, m\}$ then there is at least one rook in each row and column.

Proof. Consider a rook placed in a square on board. The square must be dominated by at least k other rooks, thus the number of rooks in its row plus the number of rooks in its column must be at least $k + 2$, so the result follows.

On the other hand, if there is a row (column, resp.) without a rook then the squares in that row (column, resp.) must be dominated by at least k distinct rooks. \qed

The following result states that in every configuration of rooks satisfying the condition of Problem 1 contains at least $\gamma_{kt}(K_r \sqcap K_s)$ rooks within every r rows and s columns for $r, s \geq k$.

Lemma 2.3. Let $A, B$ be an r-set of $V(K_n)$ and an s-set of $V(K_m)$, respectively, with $2 \leq k < r \leq n$ and $k < s \leq m$. If $S$ is a total k-dominating set of $K_r \sqcap K_m$, then $|S \cap [(A \times V(K_m)) \cup (V(K_n) \times B)]| \geq \gamma_{kt}(K_r \sqcap K_s)$.

Proof. Without loss of generality, we can assume that $A = \{v_1, \ldots, v_r\}$ and $B = \{w_1, \ldots, w_s\}$. Note that if $r = n$ and or $s = m$ the result is obvious. Thus we can assume that $r < n$ and $s < m$. Let us consider $V_1 := \{v_1, \ldots, v_r\} \times \{w_1, \ldots, w_s\}, V_2 := \{v_1, \ldots, v_r\} \times \{w_{s+1}, \ldots, w_m\}, V_3 := \{w_{s+1}, \ldots, w_m\} \times \{w_1, \ldots, w_s\}$ and $V := V_1 \cup V_2 \cup V_3$. Without loss of generality, we can assume that $r \leq s$.

On the one hand, assume that $S \cap V_1 = \emptyset$. Hence, since $d_{S \cap [(V_2 \cup V_3) \times \{v_i, w_i\}]} = d_S((v_i, w_i)) \geq k$ for every $1 \leq i \leq r$ and $\{N((v_i, w_i)) \cap (V_2 \cup V_3)\}_{i=1}^r$ is a set of pairwise disjoint subsets of $V_2 \cup V_3$, we have $|S \cap (V_2 \cup V_3)| \geq kr \geq \gamma_{kr}(K_r \sqcap K_s)$.

On the other hand, assume that $S \cap V_1 \neq \emptyset$. Define $f_i := |S \cap \{v_i\} \times \{w_1, \ldots, w_m\}|$ for $1 \leq i \leq r$ and $c_j := |S \cap \{v_1, \ldots, v_r\} \times \{w_j\}|$ for $1 \leq j \leq s$. Without loss of generality we can assume that $f_1 \leq f_2 \leq \ldots \leq f_r$ and $c_1 \leq c_2 \leq \ldots \leq c_s$. Note that $f_i + c_j \geq k$ for every $1 \leq i \leq r$ and $1 \leq j \leq s$; moreover, if $(v_i, w_j) \in S$ we have $f_i + c_j \geq k + 2$. Besides, if $f_1 + c_1 > k$, then $f_i + c_j > k$ for every $1 \leq i \leq r$ and every $1 \leq j \leq s$, and consequently, we can swap every element $(v_x, w_y)$ in $S \cap V_2$ ($S \cap V_3$, resp.) with an element in $S \cap V_1$ remaining the total k-dominating in $V_1$.

Figure 1: Auxiliary subdivision for a total k-dominating set in $K_n \sqcap K_m$.

Now we assume that $f_1 + c_1 = k$. Since $S \cap V_1 \neq \emptyset$ there exists $1 < p \leq r$ such that $f_1 = f_{p-1} < f_p$, or $1 < q \leq s$ such that $c_1 = c_{q-1} < c_q$. Thus, we have a configuration like the one in Figure 1. Consider that both $p$ and $q$ exist, and define

$$\begin{array}{c|c|c}
q-1 & s-q+1 \\
p-1 & & \\
r-p+1 & & \\
\end{array}$$
\[
V_{11} := \{v_1, \ldots, v_{p-1}\} \times \{w_1, \ldots, w_{q-1}\}, \quad V_{12} := \{v_1, \ldots, v_{p-1}\} \times \{w_q, \ldots, w_s\},
\]

\[
V_{21} := \{v_p, \ldots, v_r\} \times \{w_1, \ldots, w_{q-1}\}, \quad \text{and} \quad V_{22} := \{v_p, \ldots, v_r\} \times \{w_q, \ldots, w_s\}.
\]

Perhaps, some of the set \(V_{12}, V_{21}, V_{22}\) can be empty-set, but not all since at least \(p\) or \(q\) exists. Note that \(S \cap V_{11} = \emptyset\) since \(f_i + c_j = k\) for every \(1 \leq i < p\) and every \(1 \leq j < q\). Therefore, we can swap elements \((v_x, w_y)\) in \(S \cap V_2\) \((S \cap V_3, \text{resp.)}\) with elements in \(S \cap (V_{12} \cup V_{22})\) \((S \cap (V_{21} \cup V_{22}), \text{resp.)}\) remaining the total \(k\)-domination in \(V_1\). Notice that if the swapping described above empties both \(V_2\) and \(V_3\), then the result follows. Define \(S'\) as the new total \(k\)-total dominating set \((at \text{ least on } V_1)\) obtained after the swapping described above. Assume the swapping left a remainder in \(V_2\) \((V_3, \text{resp.)}\), then \(V_{22} \subseteq S\) and \(V_{12} \subseteq S\) \((V_{21} \subseteq S, \text{resp.)}\). Note that if \(p = 2\) or \(q = 2\), the result follows since \(r, s > k\) and \(V_{11}\) is included in a row or a column. Assume now \(p, q > 2\). Note that if \(S' \cap (\{v_{r+1}, \ldots, v_n\} \times \{w_1, \ldots, w_{q-1}\}) \neq \emptyset\), then \(S' \cap (\{v_{r+1}, \ldots, v_n\} \times \{w_{j}\})\) has the same cardinality, \(x\), for every \(1 \leq j \leq q - 1\), so we can swap those elements in \(S'\) with the elements in \(x\) rows in \(V_{11}\) remaining the total \(k\)-domination in \(V_1\). Define the new total \(k\)-dominating set \((at \text{ least on } V_1)\) by \(S''\). Notice that the case when \(S'' \cap (\{v_{r+1}, \ldots, v_n\} \times \{w_1, \ldots, w_{q-1}\}) \neq \emptyset\), is analogous. Therefore, \(S'' \cap V_1\) is a total \(k\)-dominating set in \(V_1\). □

The follow result is a consequence of Lemmas \ref{2.1} and \ref{2.3}.  

**Proposition 2.4.** For every \(2 \leq k \leq n \leq m\),  
\[
\gamma_{kt}(K_{n+1} \boxtimes K_{m+\lfloor \frac{k}{2}\rfloor + 1}) \geq \gamma_{kt}(K_n \boxtimes K_m) + \left\lfloor \frac{k}{2} \right\rfloor + 1.
\]

**Proof.** Let \(S\) be a minimum total \(k\)-dominating set of \(K_{n+1} \boxtimes K_{m+\lfloor \frac{k}{2}\rfloor + 1}\), so \(|S| = \gamma_{kt}(K_{n+1} \boxtimes K_{m+\lfloor \frac{k}{2}\rfloor + 1})\). By Lemma \ref{2.1} there is a rows in a configuration given by \(S\) with at least \(\left\lfloor \frac{k}{2} \right\rfloor + 1\) rooks. Without loss of generality we can assume that \((v_{n+1}, w_{m+1}),(v_{n+1}, w_{m+2}), \ldots, (v_{n+1}, w_{m+\lfloor \frac{k}{2}\rfloor + 1}) \in S\). Then, by applying Lemma \ref{2.3} to the first \(n\) rows and first \(m\) columns, we obtain the result. □

Using a similar reasoning as above in Lemma \ref{2.3} we obtain the following result.  

**Theorem 2.5.** For every \(2 \leq k < n \leq m\), we have  
\[
\gamma_{(k+1)t}(K_n \boxtimes K_m) \leq \gamma_{kt}(K_n \boxtimes K_m) + k.
\]  
Furthermore, the inequality is sharp.  

**Proof.** Let \(S\) be a minimum total \(k\)-dominating set of \(K_n \boxtimes K_m\). Define \(f_i := |S \cap \{(v_i) \times \{w_1, \ldots, w_m\}\}|\) for \(1 \leq i \leq n\) and \(c_j := |S \cap \{(v_1, \ldots, v_n) \times \{w_j\}\}|\) for \(1 \leq j \leq m\). Note that \(f_i + c_j \geq k\) for every \(1 \leq i \leq n\) and \(1 \leq j \leq m\); moreover, if \((v_i, w_j) \in S\), \(f_i + c_j \geq k + 2\); and if \(f_i + c_j = k + 1\), then \((v_i, w_j) \notin S\). Without loss of generality, we can assume that there exist \(p, q \geq 1\) and \(r, s, t, u \geq 0\) such that \(f_1 = \ldots = f_p\), \(f_p + 1 = f_{p+1} = \ldots = f_{p+r}\), \(f_p + r + 1 = f_{p+r+1} = \ldots = f_{p+r+t}\), \(f_{p+r+t} < f_{p+r+t+1} = \ldots \leq f_n\), and \(c_1 = \ldots = c_q\), \(c_q + 1 = c_{q+1} = \ldots = c_{q+s}\), \(c_q + s + 1 = c_{q+s+1} = \ldots \leq c_{q+s+u}\), \(c_{q+s+u} < c_{q+s+u+1} = \ldots \leq c_m\). We define the following regions.
Some of the sets $V_{xy}$ with $1 \leq x, y \leq 4$ can be an empty-set. Figure 2 shows an auxiliary view of such an arrangement. Besides, $|S \cap V_{11}| = |S \cap V_{13}| = |S \cap V_{21}| = 0$ and all vertices in $V_{14}, V_{23}, V_{32}, V_{33}, V_{43}, V_{44}$, $V_{42}, V_{43}, V_{44}$ are total dominated by at least $k+1$ vertices in $S$. Thus, $S' := S \cup \{(v_1) \times \{w_1, w_2, \ldots, w_{q+s+u}\}\}$ dominates with at least $k+1$ vertices to all vertices in $K_n \Box K_m$ except to $S \cap \{(v_1) \times \{w_{q+s+u+1}, \ldots, w_{m}\}\}$ which $S$ dominates with exactly $k$ vertices. Therefore, we can obtain $S''$ by adding to $S'$ as many elements as in $S \cap \{(v_1) \times \{w_{q+s+1}, \ldots, w_{q+s+u}\}\}$ within its corresponding column. It is easy to check that $S''$ is a total $(k+1)$-dominating set of $K_n \Box K_m$.

![Figure 3: Minimum total $k$-dominating configurations for $K_4 \Box K_4$ from $k = 2$ to $k = 3$.](image)

The inequality in (2.2) is sharp, for instance, $\gamma_{2l}(K_4 \Box K_4) = 6$ and $\gamma_{3l}(K_4 \Box K_4) = 10$. Figure 3 shows a sharp case of the inequality (2.2).

**Theorem 2.6.** For every $2 \leq k < n \leq m$, we have

$$\gamma_{kl}(K_n \Box K_m) \leq \left\lfloor \frac{k}{2} \right\rfloor \left( n + m - 2 \left\lfloor \frac{k}{2} \right\rfloor + 2 \right) - 1.$$

**Proof.** Start with the grid representation of $K_n \Box K_m$ and rooks populating the first $\left\lfloor \frac{k}{2} \right\rfloor$ rows and $\left\lfloor \frac{k}{2} \right\rfloor$ columns, except squares $(i, j)$ with $1 \leq i, j < \left\lfloor \frac{k}{2} \right\rfloor$ or $i, j > \left\lceil \frac{k}{2} \right\rceil$: like appear in Figure 4. Therefore, we can easily verify that every square is $k$ dominated. \qed
What we notice from the figure above is that, in most cases, the configuration can be improved. This prompts the following corollary.

**Corollary 2.7.** For every $2 \leq k$ and $n, m \geq k + 2$, we have

$$
\gamma_{kt}(K_n \square K_m) \leq \left(\left\lceil \frac{k}{2} \right\rceil - 1\right) \left(n + m - 2 \left\lceil \frac{k}{2} \right\rceil + 2\right).
$$

3. Closed formulas for $\gamma_{2t}(K_n \square K_m)$

We have the following consequence from Lemma 2.1 when $k = 2$.

**Proposition 3.1.** For every $m \geq 2$ we have $\gamma_{2t}(K_2 \square K_m) = 4$.

The following result is a particular version of Lemma 2.2 for $k = 2$ with a little improvement.

**Lemma 3.2.** For every $n, m \geq 2$, if $\gamma_{2t}(K_n \square K_m) < 2 \min\{n, m\}$ then the following statements hold in every rook configuration of a Problem 1 solution

(i) there is a rook in each row (column, resp.),

(ii) there is a row (column, resp.) with at least three rooks.

**Proof.** (of Part ii) Since $\gamma_{2t}(K_n \square K_m) < 2 \min\{n, m\}$ there is a row (column, resp.) with just one rook, the square with that rook must be dominated by others two rooks located in the same column (row, resp.).

In fact, we have the following result as a consequence of Lemma 3.2. This result also appears in [4, Proposition 3.2] when $k = 2$. 

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Figure 4: Total $k$-dominating configuration for $K_n \square K_m$. 

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Proposition 3.3. For every \( n \leq m \), we have \( \gamma_{2t}(K_n \sqcup K_m) \geq \min\{m+2, 2n\} \). Furthermore, \( \gamma_{2t}(K_n \sqcup K_m) = 2n \) for \( m \geq 2n - 2 \).

Note that Lemma 2.3 for \( k = 2 \) could be extended until \( r, s \geq 2 \) (instead of \( r, s > 2 \)). It is easily seen that \( |S \cap V| \geq 4 \) when \( r = 2 \). In order to obtain the exact value of \( \gamma_{2t}(K_n \sqcup K_n) \) we need the following interesting result will be useful to obtain some of the main results of this work.

Theorem 3.4. For every \( 6 \leq n \leq m \),

\[
\gamma_{2t}(K_n \sqcup K_m) \geq \min\{\gamma_{2t}(K_3 \sqcup K_3) + \gamma_{2t}(K_{n-3} \sqcup K_{m-3}), \gamma_{2t}(K_4 \sqcup K_4) + \gamma_{2t}(K_{n-4} \sqcup K_{m-4})\}.
\]

(3.3)

Proof. Note that if \( \gamma_{2t}(K_n \sqcup K_m) = 2n \), then the inequality holds. Hence, we can assume that \( \gamma_{2t}(K_n \sqcup K_m) < 2n \). Let \( S \) be a minimum total 2-dominating set of \( K_n \sqcup K_m \). By Lemma 3.2 there is a vertex \((v_1, w_1) \in V(K_n \sqcup K_m)\) such that \( |S \cap \{(v_i) \times V(K_m)\}| \geq 3 \) and \( |S \cap (V(K_n) \times \{w_j\})| \geq 3 \). Without loss of generality, we can assume that \( i = j = 1 \). Assume first that \((v_1, w_1) \notin S\). Without loss of generality, we can assume that \((v_1, w_2), (v_1, w_3), (v_2, w_1), (v_3, w_1), (w_4, w_1) \in S\), i.e., \( S \) has the configuration in Figure 5 right. Denote by \( A := S \cap V_1 \) where \( V_1 := \{v_1, v_2, v_3, v_4\} \times \{w_1, w_2, w_3, w_4\} \). Clearly, \(|A| \geq \gamma_{2t}(K_4 \sqcup K_4) = 6\). Hence, by Lemma 2.3 we have \(|S \setminus A| \geq \gamma_{2t}(K_{n-4} \sqcup K_{m-4})\), and consequently, \( \gamma_{2t}(K_n \sqcup K_m) \geq \gamma_{2t}(K_4 \sqcup K_4) + \gamma_{2t}(K_{n-4} \sqcup K_{m-4})\). The proof when \((v_1, w_1) \in S\) is analogous. Note that \( S \) has the configuration in Figure 5 left, and Lemma 2.3 gives \( \gamma_{2t}(K_n \sqcup K_m) \geq \gamma_{2t}(K_3 \sqcup K_3) + \gamma_{2t}(K_{n-3} \sqcup K_{m-3})\).

\[\square\]

![Figure 5: Auxiliar configurations for Theorem 3.4](image)

Lemma 3.5. For every \( n, m \geq 2 \), we have

\[
\gamma_{2t}(K_n \sqcup K_m) + 1 \leq \gamma_{2t}(K_{n+1} \sqcup K_{m+1}) \leq \gamma_{2t}(K_n \sqcup K_m) + 2.
\]

(3.4)

Proof. Let \( S' \) be a total 2-dominating set of \( K_{n+1} \sqcup K_{m+1} \) and consider \((v', w') \in S'\). By Lemma 2.3 we have \( \gamma_{2t}(K_{n+1} \sqcup K_{m+1}) - 1 = |S' \setminus \{(v', w')\}| \geq \gamma_{2t}(K_n \sqcup K_m)\), and so, the first inequality in (3.4) holds. Let \( S \) be a total 2-dominating set of \( K_n \sqcup K_m \). If \(|S| = \gamma_{2t}(K_n \sqcup K_m) = 2\min\{n, m\}\) then \( \gamma_{2t}(K_{n+1} \sqcup K_{m+1}) \leq 2\min\{n+1, m+1\} = |S| + 2 \). Assume that \(|S| < 2\min\{n, m\}\). Then by Lemma 3.2 there are vertices \( v \in V(K_n) \) and \( w \in V(K_m) \) such that \(|S \cap \{(v) \times V(K_m)\}| \geq 3\) and \(|S \cap (V(K_n) \times \{w\})| \geq 3\). Thus it is a simple matter to check that \( S \cup \{(v, w_{m+1}), (v_{n+1}, w)\} \) is a total 2-dominating set of \( K_{n+1} \sqcup K_{m+1}\). \[\square\]

Lemma 3.5 and Theorem 3.4 have the following direct consequence.

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**Theorem 3.6.** For every \( n, m \geq 2 \), if there is a minimum total 2-dominating set \( S \) of \( K_n \square K_m \) such that \( S \cap \{v\} \times V(K_m) \neq \emptyset \) and \( S \cap (V(K_n) \times \{w\}) \neq \emptyset \) for every \( v \in V(K_n) \) and \( w \in V(K_m) \) then

\[
\gamma_{2t}(K_{n+4} \square K_{m+4}) = \gamma_{2t}(K_n \square K_m) + 6. \tag{3.5}
\]

Figure 6 left shows a minimal configuration for a total 2-dominating set of \( K_6 \square K_7 \) which satisfies (3.5); however, it does not verify the condition of Theorem 3.6 since any total 2-dominating set \( S \) of \( K_2 \square K_3 \) with \( S \cap \{v\} \times V(K_3) \neq \emptyset \) and \( S \cap (V(K_2) \times \{w\}) \neq \emptyset \) for every \( v \in V(K_2) \) and \( w \in V(K_3) \) is non-minimal. Similarly, Figure 6 right shows a non-minimal configuration for a total 2-dominating set of \( K_6 \square K_8 \) which does not verify neither the condition of Theorem 3.6 nor the equality in (3.3) since \( \gamma_{2t}(K_6 \square K_8) = 11 \neq 10 = \min\{6 + 4, 5 + 6\} \).

![Figure 6: Minimal configuration for 6×7 (left) and non-minimal for 6×8 (right).](image-url)

**Theorem 3.7.** For every \( 3 \leq n \leq m \), if \( \gamma_{2t}(K_n \square K_m) < 2n \) then

\[
\gamma_{2t}(K_{n+1} \square K_{m+3}) = \gamma_{2t}(K_n \square K_m) + 3. \tag{3.6}
\]

**Proof.** Let \( S \) be a minimum total 2-dominating set of \( K_{n+1} \square K_{m+3} \). By Lemma 3.2 there is \( v \in V(K_{n+1}) \) such that \( |S \cap \{v\} \times V(K_{m+3})| \geq 3 \). Without loss of generality, we can assume that \( A = \{(v_{n+1}, w_{m+1}), (v_{n+1}, w_{m+2}), (v_{n+1}, w_{m+3})\} \subseteq S \). Then, by Lemma 2.3, we have \( \gamma_{2t}(K_{n+1} \square K_{m+3}) - 3 = |S \setminus A| \geq \gamma_{2t}(K_{n+1} \square K_m) \), and so, the inequality \( \gamma_{2t}(K_{n+1} \square K_{m+3}) \geq \gamma_{2t}(K_n \square K_m) + 3 \) holds. Let \( S' \) be a total 2-dominating set of \( K_n \square K_m \). Thus it is a simple matter to check that \( S' \cup A \) is a total 2-dominating set of \( K_{n+1} \square K_{m+3} \) obtaining \( \gamma_{2t}(K_{n+1} \square K_{m+3}) \leq \gamma_{2t}(K_n \square K_m) + 3 \). \( \square \)

Now, we approach the case \( n = m \), i.e., to compute \( \gamma_{2t}(K_n \square K_n) \). The proof of the following Proposition is recommended to the reader.

**Proposition 3.8.** We have \( \gamma_{2t}(K_2 \square K_2) = 4 \), \( \gamma_{2t}(K_3 \square K_3) = 5 \), \( \gamma_{2t}(K_4 \square K_4) = 6 \), \( \gamma_{2t}(K_5 \square K_5) = 8 \) and \( \gamma_{2t}(K_6 \times K_6) = 10 \).

**Theorem 3.9.** For every \( n \geq 2 \) we have

\[
\gamma_{2t}(K_n \square K_n) = \begin{cases} 
(3n)/2, & \text{if } n \equiv 0 \pmod{4}, \\
(3n + 1)/2, & \text{if } n \equiv 1 \pmod{2}, \\
(3n + 2)/2, & \text{if } n \equiv 2 \pmod{4}.
\end{cases} \tag{3.7}
\]

**Proof.** First we proceed by induction on \( n \) for obtaining

\[
\gamma_{2t}(K_n \square K_n) \geq \begin{cases} 
6k - 2, & \text{if } n = 4k - 2, \\
6k - 1, & \text{if } n = 4k - 1, \\
6k, & \text{if } n = 4k, \\
6k + 2, & \text{if } n = 4k + 1.
\end{cases} \tag{3.8}
\]
By Proposition 3.8, as well as other similar results. The proof of the following proposition is also recommended to the reader.

Theorem 3.11. By Proposition 3.8, \((3.8)\) holds for \(k = 1\). Assume that \((3.8)\) holds for \(k = r\). Hence, Theorem 3.4 gives

\[
\begin{align*}
\gamma_2(K_{4r+2} \sqcap K_{4r+2}) &\geq \min \{5 + \gamma_2(K_{4r-1} \sqcap K_{4r-1}), 6 + \gamma_2(K_{4r-2} \sqcap K_{4r-2})\} \geq 6r + 4, \\
\gamma_2(K_{4r+3} \sqcap K_{4r+3}) &\geq \min \{5 + \gamma_2(K_{4r} \sqcap K_{4r}), 6 + \gamma_2(K_{4r-1} \sqcap K_{4r-1})\} \geq 6r + 5, \\
\gamma_2(K_{4r+4} \sqcap K_{4r+4}) &\geq \min \{5 + \gamma_2(K_{4r+1} \sqcap K_{4r+1}), 6 + \gamma_2(K_{4r} \sqcap K_{4r})\} \geq 6r + 6, \\
\gamma_2(K_{4r+5} \sqcap K_{4r+5}) &\geq \min \{5 + \gamma_2(K_{4r+2} \sqcap K_{4r+2}), 6 + \gamma_2(K_{4r+1} \sqcap K_{4r+1})\} \geq 6r + 8.
\end{align*}
\]

We continue in this fashion obtaining a configuration for \(S\) that yields the equality by putting in diagonal matter \(k - 1\) configurations of \(4 \times 4\) blocks and another configuration with size congruent with \(n\) modulo 4.

In other words, build \(S\) for every \(n = 4k + \alpha\) with \(k \geq 1\) and \(\alpha = -2, -1, 0, 1\). Take \(S_i\) as a minimum \(2\)-total domination set of \(\langle \{v_{4(i-1)+1}, \ldots, v_{4i}\} \times \{w_{4(i-1)+1}, \ldots, w_{4i}\} \rangle\) for \(i \leq k - 1\) (if \(k > 1\)) and \(S_k\) as a minimum \(2\)-total domination set for \(\langle \{v_{4(k-1)+1}, \ldots, v_{4k+\alpha}\} \times \{w_{4(k-1)+1}, \ldots, w_{4k+\alpha}\} \rangle\). Finally, take \(S := \bigcup_{i=1}^{k} S_i\) which is a total \(2\)-dominating set of \(K_n \sqcap K_n\).

We can use Theorem 3.8 and mathematical induction to obtain close formulas for \(\{\gamma_2(K_n \sqcap K_{n+1})\}_{n=2}^{\infty}\) as well as other similar results. The proof of the following proposition is also recommended to the reader.

Proposition 3.10. We have \(\gamma_2(K_2 \sqcap K_3) = 4, \gamma_2(K_3 \sqcap K_4) = 6, \gamma_2(K_4 \sqcap K_5) = 7, \gamma_2(K_5 \sqcap K_6) = 9, \gamma_2(K_6 \sqcap K_7) = 10\).

Theorem 3.11. For every \(n \geq 2\) we have

\[
\gamma_2(K_n \sqcap K_{n+1}) = \begin{cases} 
(3n + 2)/2, & \text{if } n \equiv 0 \pmod{2}, \\
(3n + 3)/2, & \text{if } n \equiv 1 \pmod{2}.
\end{cases}
\]

The following result is a direct application of the the recursive formulas in Theorems 3.6 and 3.7 with initial conditions given in Theorems 3.9 and 3.11. Note that by Proposition 3.3 we have that \(\gamma_2(K_n \sqcap K_m) = 2n\) when \(m \geq 3n\). Hence, for numerical reason, we can assume that \(\gamma_2(K_x \sqcap K_y) = 0\) if \(x \leq 0\) or \(y \leq 0\).

Theorem 3.12. For every \(2 \leq n \leq m\) we have

\[
\gamma_2(K_n \sqcap K_m) = \min \left\{ \gamma_2\left( K_{n-\left\lfloor \frac{m}{2} \right\rfloor} \sqcap K_{n-\left\lfloor \frac{m}{2} \right\rfloor} + \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor \right) + 3 \left\lfloor \frac{m-n}{2} \right\rfloor, 2n \right\}.
\]

Hence, we can obtain a closed formula for the total \(2\)-domination number of \(K_n \sqcap K_m\) for every \(n, m \geq 2\) using Theorem 3.12.

\[
\gamma_2(K_n \sqcap K_m) = \min \left\{ \gamma_2\left( K_{n-\left\lfloor \frac{m}{2} \right\rfloor} \sqcap K_{n-\left\lfloor \frac{m}{2} \right\rfloor} + \left\lfloor \frac{m}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor \right) + 3 \left\lfloor \frac{m-n}{2} \right\rfloor, 2n \right\}.
\]
We call $a = n - \left\lfloor \frac{m-n}{2} \right\rfloor$ and $b = 3 \left\lfloor \frac{m-n}{2} \right\rfloor$. Now, we have two cases, the case where $n, m$ have the same parity, i.e., $n \equiv m \pmod{2}$, and the case where $n, m$ have opposing parity, i.e., $n \not\equiv m \pmod{2}$. To refine that idea we let consider the quotient and remainder of $n, m$ by dividing by 8, that is $n = 8q_n + r_n$ and $m = 8q_m + r_m$, for $0 \leq r_n, r_m < 8$. We check these in two cases.

Let $r_n \equiv r_m \pmod{2}$. Then, $a = 8q_n + r_n - \frac{8q_m + r_m - 8q_n - r_n}{2} = 12q_n - 4q_m + \frac{3r_m - r_n}{2}$ and $b = 12(q_m - q_n) + 3 \cdot \frac{r_m - r_n}{2}$. Now we find the quotient and remainder for the divisor of 4 to use previous theorems, so we get that $a = 4 \left(3q_n - q_m + \frac{(3r_n - r_m)}{4} \right) + r_a$ where $0 \leq r_a < 4$ and $r_a \equiv \frac{3r_m - r_n}{2} \pmod{4}$. Now we use Theorem 3.9 to find $\gamma_{2t}(K_a \Box K_a)$ to get, for $a = 4q_a + r_a$ where $q_a = \left(3q_n - q_m + \frac{(3r_n - r_m)}{4} \right)$

$$\gamma_{2t}(K_a \Box K_a) = \begin{cases} 
6q_a & r_a \equiv 0 \pmod{4} \\
6q_a + 2 & r_a \equiv 1 \pmod{4} \\
6q_a + 4 & r_a \equiv 2 \pmod{4} \\
6q_a + 6 & r_a \equiv 3 \pmod{4}
\end{cases}$$

In the case where $r_n \not\equiv r_m \pmod{2}$, we have $a = 8q_n + r_n - \frac{8q_m + r_m - 8q_n - r_n}{2} = 12q_n - 4q_m + \frac{3r_m - r_n - 1}{2}$ and $b = 12(q_m - q_n) + 3 \cdot \frac{r_m - r_n - 1}{2}$. We follow the same method above instead using Theorem 3.11 with the expanded expressions of

$$\gamma_{2t}(K_a \Box K_{a+1}) = \begin{cases} 
6q_a + 1 & r_a \equiv 0 \pmod{4} \\
6q_a + 3 & r_a \equiv 1 \pmod{4} \\
6q_a + 5 & r_a \equiv 2 \pmod{4} \\
6q_a + 7 & r_a \equiv 3 \pmod{4}
\end{cases}$$

Combining results above, we have for any given $n, m$, a look up Table 1 for the total 2-domination number of $K_n \Box K_m$.

So for $Q = 6q_n + 6q_m$

| 2t Domination Number | $m \equiv 0$ | $m \equiv 1$ | $m \equiv 2$ | $m \equiv 3$ | $m \equiv 4$ | $m \equiv 5$ | $m \equiv 6$ | $m \equiv 7$ |
|----------------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| $n \equiv 0 \pmod{8}$ | $Q + 0$     | $Q + 1$     | $Q + 2$     | $Q + 3$     | $Q + 4$     | $Q + 5$     | $Q + 6$     | $Q + 7$     |
| $n \equiv 1 \pmod{8}$ | $Q + 1$     | $Q + 2$     | $Q + 3$     | $Q + 4$     | $Q + 5$     | $Q + 6$     | $Q + 7$     | $Q + 8$     |
| $n \equiv 2 \pmod{8}$ | $Q + 2$     | $Q + 3$     | $Q + 4$     | $Q + 5$     | $Q + 6$     | $Q + 7$     | $Q + 8$     | $Q + 9$     |
| $n \equiv 3 \pmod{8}$ | $Q + 3$     | $Q + 4$     | $Q + 5$     | $Q + 6$     | $Q + 7$     | $Q + 8$     | $Q + 9$     | $Q + 10$    |
| $n \equiv 4 \pmod{8}$ | $Q + 4$     | $Q + 5$     | $Q + 6$     | $Q + 7$     | $Q + 8$     | $Q + 9$     | $Q + 10$    | $Q + 11$    |
| $n \equiv 5 \pmod{8}$ | $Q + 5$     | $Q + 6$     | $Q + 7$     | $Q + 8$     | $Q + 9$     | $Q + 10$    | $Q + 11$    |             |
| $n \equiv 6 \pmod{8}$ | $Q + 6$     | $Q + 7$     | $Q + 8$     | $Q + 9$     | $Q + 10$    | $Q + 11$    |             |             |
| $n \equiv 7 \pmod{8}$ | $Q + 7$     | $Q + 8$     | $Q + 9$     | $Q + 10$    | $Q + 11$    |             |             |             |

Table 1: Table with the minimum values of the total 2-domination number of $K_n \Box K_m$ for $n, m \geq 2$.

Thus, we can take the minimum of table entry above or $2n$ to get our value for $\gamma_{2t}(K_n \Box K_m)$.

Theorems 3.9, 3.11 and 3.12 have a direct consequence which is a general result which improves the result in [4, Proposition 3.3] for $k = 2$.

**Theorem 3.13.** Let $G, H$ be two graphs without isolate vertex and order $n$ and $m$ respectively. Then

$$\gamma_{2t}(G \Box H) \geq \frac{3}{2} \min \{n, m\}.$$ 

Furthermore, if $n \leq m$

$$\lim_{n \to \infty} \frac{\gamma_{2t}(G \Box H)}{n} = \frac{3}{2}.$$
Theorems 2.5 and 3.12 directly give the following upper bound and asymptotic behavior of $\gamma_{kt}(K_n \Box K_m)$ for $k \geq 2$.

**Theorem 3.14.** For every $2 \leq k \leq n \leq m$ we have

$$\gamma_{kt}(K_n \Box K_m) \leq \min \left\{ \gamma_{2n} \left( K_{n-\left\lfloor \frac{m-n}{2} \right\rfloor} \Box K_{n-2\left\lfloor \frac{m-n}{2} \right\rfloor} + \left\lceil \frac{m-n}{2} \right\rceil \right) + 3 \left\lceil \frac{m-n}{2} \right\rceil \cdot 2n \right\} + (k - 2)n. \quad (3.11)$$

**Theorem 3.15.** Let $G, H$ be two graphs without isolate vertex and order $n$ and $m$, respectively. If $2 \leq k \leq n \leq m$ then

$$\liminf_{n \to \infty} \frac{\gamma_{kt}(G \Box H)}{n} \leq \frac{2k - 1}{2}.$$

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