Masses and decay constants of pions and kaons in mixed-action staggered chiral perturbation theory

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Lattice QCD calculations with different staggered valence and sea quarks can be used to improve determinations of quark masses, Gasser-Leutwyler couplings, and other parameters relevant to phenomenology. We calculate the masses and decay constants of flavored pions and kaons through next-to-leading order in staggered-valence, staggered-sea mixed-action chiral perturbation theory. We present the results in the valence-valence and valence-sea sectors, for all tastes. As in unmixed theories, the taste-pseudoscalar, valence-valence mesons are exact Goldstone bosons in the chiral limit, at non-zero lattice spacing. The results reduce correctly when the valence and sea quark actions are identical, connect smoothly to the continuum limit, and provide a way to control light quark and gluon discretization errors in lattice calculations performed with different staggered actions for the valence and sea quarks.

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I. INTRODUCTION

The quark masses and CKM matrix elements are fundamental parameters of the Standard Model. To understand their values in terms of the underlying physics and probe the limits of the Standard Model, they must be extracted from experiment with greater precision. In addition, the low-energy couplings (LECs) of chiral perturbation theory (ChPT) parametrize the strong interactions at energies small compared to the scale of chiral symmetry breaking. Improving knowledge of the Standard Model and chiral effective theory parameters requires improved calculations of strong force contributions to the relevant hadronic matrix elements.

Mixed-action lattice QCD calculations can be used to calculate hadronic matrix elements while exploiting the advantages of different discretizations of the fermion action. In practice, computationally more expensive fermions with smaller discretization effects or other desirable features are used for the valence quarks, while computationally less expensive fermions are used for the sea quarks, to include the effects of vacuum polarization. The construction of chiral effective theories for lattice QCD incorporates discretization effects, thereby relating the chiral and continuum extrapolations and providing greater control of the continuum limit.

Staggered ChPT (SChPT) was developed to analyze results of lattice calculations with staggered fermions [1, 2]; it has been used extensively to control extrapolations to physical light-quark masses and to remove dominant light-quark and gluon discretization errors [3]. Mixed-action ChPT was developed for lattice calculations performed with Ginsparg-Wilson valence quarks and Wilson sea quarks [4, 5]. The formalism for staggered sea quarks and Ginsparg-Wilson valence quarks was developed in Ref. [6]. Mixed-action ChPT for differently improved staggered fermions was introduced for calculations of the $K^0 - \bar{K}^0$ bag parameters entering $\xi_K$ in and beyond the Standard Model [7, 8] and the $K \rightarrow \pi\nu$ vector form factor [9].

We have calculated the pion and kaon masses and axial-current decay constants in all taste representations at next-to-leading order (NLO) in mixed-action SChPT. The results generalize those of Refs. [2, 11, 14] to the mixed-action case; the results could be used to improve determinations of LECs poorly determined by existing analyses and to improve determinations of light-quark masses, the Gasser-Leutwyler couplings, and the pion and kaon decay constants.

In Sec. II we review the formulation of mixed-action SChPT. Results for the masses are presented in Sec. III and for the decay constants, in Sec. IV. In Sec. V we conclude.
II. MIXED-ACTION STAGGERED CHIRAL PERTURBATION THEORY

As for ordinary, unmixed SChPT, the theory is constructed in two steps. First one builds the Symanzik effective continuum theory (SET) for the lattice theory. Then one maps the operators of the SET into those of ChPT [11, 12, 7, 14].

A. Symanzik effective theory

Through NLO the SET may be written

\[ S_{\text{eff}} = S_{\text{QCD}} + a^2 S_6 + \ldots, \]

where \( S_{\text{QCD}} \) has the form of the QCD action, but possesses taste degrees of freedom and respects the continuum taste SU(4) symmetry. To account for differences in the masses of valence and sea quarks in lattice calculations, the SET can be formulated with bosonic ghost quarks and fermionic valence and sea quarks [15]. We use the replica method [16] and so include in the action only (fermionic) valence and sea quarks.

The operators in \( S_6 \) have mass-dimension six, and they break the continuum symmetries to those of the mixed-action lattice theory. In valence and sea sectors, they break the continuum symmetries to those of the unmixed theory. To account for differences in the masses of valence and sea quarks and fermionic valence and sea quarks [15], we use a four-fermion operators respecting the remnant taste degrees of freedom and respects the continuum symmetry \( \Gamma \).

B. Leading order chiral Lagrangian

The first three terms are identical to the kinetic energy, mass, and anomaly operators of the unmixed theory, respectively. To construct the potential \( \mathcal{V} \), the projection operators are conveniently included in spurions. The result can be written

\[ \mathcal{V} = \mathcal{U} + \mathcal{U}' - C_{\text{mix}} \text{Tr}(\tau_3 \Sigma \Sigma^\dagger), \]

where the last term is a taste-singlet potential new in the mixed-action theory, with \( \tau_3 \equiv P_\sigma - P_\bar{\sigma} \). The potentials \( \mathcal{U} \) and \( \mathcal{U}' \) contain single- and double-trace operators, respectively, that are direct generalizations of those in unmixed SChPT. The operators in \( \mathcal{U}^{(i)} \) have independent LECs for the valence-valence, sea-sea, and valence-sea sectors. We write

\[ \mathcal{U} = \mathcal{U}_{v\bar{v}} + \mathcal{U}_{\sigma\bar{\sigma}} + \mathcal{U}_{v\sigma}, \]

\[ \mathcal{U}' = \mathcal{U}_{v\bar{v}}' + \mathcal{U}_{\sigma\bar{\sigma}}' + \mathcal{U}_{v\sigma}', \]

where

\[ -\mathcal{U}_{v\bar{v}} = C_{v\bar{v}}^4 \text{Tr}(\xi_5 P_v \Sigma \xi_5 P_v \Sigma^\dagger) \]

\[ + C_{v\bar{v}}^6 \sum_{\mu < \nu} \text{Tr}(\xi_\mu P_v \Sigma \xi_\nu P_v \Sigma^\dagger) \]

\[ + \frac{C_{v\bar{v}}^3}{2} [\text{Tr}(\xi_\nu P_v \Sigma \xi_\nu P_v \Sigma) + \text{p.c.}] \]

\[ + \frac{C_{v\bar{v}}^4}{2} [\text{Tr}(\xi_5 P_v \Sigma \xi_5 P_v \Sigma) + \text{p.c.}], \]

\[ -\mathcal{U}_{\sigma\bar{\sigma}} = C_{\sigma\bar{\sigma}}^4 \text{Tr}(\xi_5 P_\sigma \Sigma \xi_5 P_\sigma \Sigma^\dagger) \]

\[ + C_{\sigma\bar{\sigma}}^6 \sum_{\mu < \nu} \text{Tr}(\xi_\mu P_\sigma \Sigma \xi_\nu P_\sigma \Sigma^\dagger) \]

\[ + \frac{C_{\sigma\bar{\sigma}}^3}{2} [\text{Tr}(\xi_\nu P_\sigma \Sigma \xi_\nu P_\sigma \Sigma) + \text{p.c.}] \]

\[ + \frac{C_{\sigma\bar{\sigma}}^4}{2} [\text{Tr}(\xi_5 P_\sigma \Sigma \xi_5 P_\sigma \Sigma) + \text{p.c.}], \]

\[ -\mathcal{U}_{v\sigma} = C_{v\sigma}^4 \text{Tr}(\xi_5 P_v \Sigma \xi_5 P_\sigma \Sigma^\dagger) + \text{p.c.} \]

\[ + C_{v\sigma}^6 \sum_{\mu < \nu} [\text{Tr}(\xi_\mu P_v \Sigma \xi_\nu P_\sigma \Sigma^\dagger) + \text{p.c.}] \]

\[ + C_{v\sigma}^3 [\text{Tr}(\xi_\nu P_v \Sigma \xi_5 P_\sigma \Sigma) + \text{p.c.}] \]

\[ + C_{v\sigma}^4 [\text{Tr}(\xi_5 P_v \Sigma \xi_5 P_\sigma \Sigma) + \text{p.c.}], \]

\[ -\mathcal{U}'_{v\bar{v}} = C_{v\bar{v}}^{4(1)} \text{Tr}(\xi_\nu P_v \Sigma) \text{Tr}(\xi_\nu P_v \Sigma) + \text{p.c.} \]

\[ + C_{v\bar{v}}^{4(2)} [\text{Tr}(\xi_\nu P_v \Sigma) \text{Tr}(\xi_\nu P_v \Sigma) + \text{p.c.}] \]

\[ + C_{v\bar{v}}^{4(3)} [\text{Tr}(\xi_\nu P_v \Sigma) \text{Tr}(\xi_\nu P_v \Sigma^\dagger)], \]
\begin{align}
-\mathcal{W}'_{\sigma \sigma} &= \frac{C_{2\sigma}}{4} \left[ \text{Tr}(\xi_{\sigma} P_{\sigma} \Sigma) \text{Tr}(\xi_{\sigma} P_{\sigma} \Sigma) + p.c. \right] \\
&+ \frac{\xi_{2\sigma}}{2} \left[ \text{Tr}(\xi_{\sigma} P_{\sigma} \Sigma) \text{Tr}(\xi_{\sigma} P_{\sigma} \Sigma) + p.c. \right] \\
&+ \frac{C_{5\sigma}}{2} \left[ \text{Tr}(\xi_{\sigma} P_{\sigma} \Sigma) \text{Tr}(\xi_{\sigma} P_{\sigma} \Sigma) \right],
\end{align}

where \( p.c. \) indicates the parity conjugate. In the unmixed case, \( C_{\text{mix}} = 0 \), \( C_{uu} = C_{\sigma \sigma} = C_{\sigma \sigma} = C \), and the potential \( \mathcal{V} \) reduces to that of ordinary SChPT. Restricting attention to two-point correlators of sea-sea particles yields results of the unmixed theory, as expected [15].

**C. Tree-level masses and propagators**

As in the unmixed theory, the potential \( \mathcal{V} \) contributes to the tree-level masses of the pions and kaons, which fall into irreducible representations (irreps) of \( \Gamma_4 \times SO(4) \). For a taste \( t \) pseudo-Goldstone boson (PGB) \( \phi_{xy}^t \) composed of quarks with flavors \( x, y \), \( x \neq y \),

\[
m_{xy,t}^2 = \mu(m_x + m_y) + a^2 \Delta_{xy}^t,
\]

where \( F \) labels the taste \( \Gamma_4 \times SO(4) \) irrep (pseudoscalar, axial, tensor, vector, or scalar). The mass splitting \( \Delta_{xy}^t \) depends on the LEC of the taste-singlet potential \( (C_{\text{mix}}) \), the LECs in the single-trace potential \( (\mathcal{V}) \), and the sector (valence or sea) of the quark flavors \( x \) and \( y \). Expanding the LO Lagrangian through \( \mathcal{O}(\phi^2) \), we have

\[
\Delta_{xy}^{uu} = \frac{8}{f^2} \sum_{b \neq t} C_{b}^{uu} (1 - \theta^{tb} \theta^{bb}),
\]

\[
\Delta_{xy}^{\sigma \sigma} = \frac{8}{f^2} \sum_{b \neq t} C_{b}^{\sigma \sigma} (1 - \theta^{tb} \theta^{bb}),
\]

\[
\Delta_{xy}^{\sigma} = \frac{16C_{\text{mix}}}{f^2} + \frac{8}{f^2} \sum_{b \neq t} \left[ \frac{1}{2} (C_{b}^{uu} + C_{b}^{\sigma \sigma}) - C_{b}^{\sigma \sigma} \theta^{tb} \theta^{bb} \right],
\]

where the splitting is \( \Delta_{xy}^{uu} \) if both quarks are valence quarks \( (xy \in uu) \), \( \Delta_{xy}^{\sigma \sigma} \) if both quarks are sea quarks \( (xy \in \sigma \sigma) \), and \( \Delta_{xy}^{\sigma} \) otherwise. The sub(super)script \( b \) and taste \( t \) are indices labeling the generators of the fundamental irrep of \( U(4) \). The numerical constant \( \theta^{tb} = +1 \) if the generators for \( t \) and \( b \) commute and \(-1 \) if they anti-commute. The LEC \( C_b = C_1, C_6, C_3, \) or \( C_4 \) if \( b \) labels a generator corresponding to the \( P, T, V, \) or \( A \) irrep of \( \Gamma_4 \times SO(4) \), respectively. The residual chiral symmetry in the valence-valence sector, as for the unmixed theory, implies \( F = P \) particles are Goldstone bosons for \( a \neq 0, m_q = 0 \), and therefore \( \Delta_{xy}^{uu} = 0 \). The same is not true for the taste pseudoscalar, valence-sea PGBs, and generically, \( \Delta_{xy}^{\sigma \sigma} \neq 0 \). In the flavor-neutral sector, \( x = y \), the PGBs mix in the taste singlet, vector, and axial irreps. The Lagrangian mixing terms (hairpins) are

\[
\frac{1}{2} \delta_{il} \phi_{ij}^l \phi_{lj}^j + \frac{1}{2} \delta_{V}^{\sigma \sigma} \phi_{ij}^l \phi_{lj}^l + \frac{1}{2} \delta_{V}^{\sigma} \phi_{ij}^l \phi_{lj}^j + \delta_{V}^{\sigma \sigma} \phi_{ij}^l \phi_{lj}^l \]

\[(V \rightarrow A, \mu \rightarrow \mu_5),\]

where \( i, j \) are flavor indices; \( \mu (\mu_5) \) is a taste index in the vector (axial) irrep; and we use an overbar (underbar) to restrict summation to the valence (sea) sector. The \( \delta \)-term accounts for the anomaly. In continuum ChPT, taking \( \delta \rightarrow \infty \) at the end of the calculation decouples the \( \eta' \) [17]. In SChPT, taking \( \delta \rightarrow \infty \) decouples the \( \eta' \). The flavor-singlets in other taste irreps are PGBs and do not decouple [2]. The \( \delta_{V_A}^{\sigma \sigma, \sigma \sigma} \)-terms are lattice artifacts from the double-trace potential \( a^2 \mathcal{W}' \), and the couplings \( \delta_{V_A}^{\sigma \sigma, \sigma \sigma} \) depend linearly on its LECs,

\[
\delta_{V}^{\sigma} = \frac{16a^2}{f^2} (C_{2A}^{\sigma} - C_{5A}^{\sigma}) \quad \delta_{V}^{\sigma \sigma} = \frac{16a^2}{f^2} (C_{2A}^{\sigma \sigma} - C_{5A}^{\sigma \sigma}) \quad \delta_{V}^{\sigma \sigma} = \frac{16a^2}{f^2} (C_{2A}^{\sigma \sigma} - C_{5A}^{\sigma \sigma}).
\]

Although the mass splittings and hairpin couplings are different in the three sectors, the tree-level propagator can be written in the same form as in the unmixed case. We have \( (k, l) \) are flavor indices

\[
G_{ij,kl}^{th}(p^2) = \delta_{il} \left( \frac{\delta_{kl} \delta_{jk}}{p^2 + m_{ij,t}^2} + \delta_{ij} \delta_{kl} D_{il}^{kl} \right),
\]

where the disconnected propagators vanish (by definition) in the pseudoscalar and tensor irreps, and for the singlet, vector, and axial irreps,

\[
D_{ij}^l = \frac{-1}{I_i J_i} \left( \frac{\delta_{ij}^l}{I_i J_i} + \delta_{ij}^l \delta_{ij}^{kl} \right) \quad \text{for } ij \notin \nu \nu,
\]

\[
D_{ij}^l = \frac{-1}{I_i J_i} \left( \frac{(\delta_{ij}^l)^2}{I_i J_i} + \delta_{ij}^l + \delta_{ij}^l \delta_{ij}^{kl} \right) \quad \text{for } ij \in \nu \nu,
\]

where \( \delta_{ij}^l \equiv \delta, I_i \equiv p^2 + m_{ii,t}^2, J_i \equiv p^2 + m_{ij,t}^2 \), and we use the replica method to quench the valence quarks [10].
and root the sea quarks \[2\], so that
\[
\sigma_t \equiv \sum \frac{1}{p^2 + m_{ii,t}^2} = \frac{1}{4} \sum \frac{1}{p^2 + m_{ii',t}^2},
\]
(24)
\[
\bar{\sigma}_t = \sum \frac{1}{p^2 + m_{ii,t}^2} = 0.
\]
(25)

The index \(i'\) is summed over the physical sea quark flavors. As for the continuum, partially quenched case \[18\], the factors arising from iterating sea quark loops can be reduced to a form convenient for doing loop integrations. For three nondegenerate, physical sea quarks \(u, d, s\), we have
\[
\frac{1}{1 + \delta_F \sigma_t} = \frac{(p^2 + m_{uu,t}^2)(p^2 + m_{dd,t}^2)(p^2 + m_{ss,t}^2)}{(p^2 + (m_{\pi t}^2)(p^2 + m_{\eta t}^2)(p^2 + m_{\eta' t}^2)},
\]
(26)
where \(m_{\pi t}^2, m_{\eta t}^2,\) and \(m_{\eta' t}^2\) are the eigenvalues of the matrices (for tastes \(F = I, V, A\))
\[
\begin{pmatrix}
m_{uu,t}^2 + \frac{\delta_F \sigma_t}{4} & \frac{\delta_F \sigma_t}{4} & \frac{\delta_F \sigma_t}{4} \\
\frac{\delta_F \sigma_t}{4} & m_{dd,t}^2 + \frac{\delta_F \sigma_t}{4} & \frac{\delta_F \sigma_t}{4} \\
\frac{\delta_F \sigma_t}{4} & \frac{\delta_F \sigma_t}{4} & m_{ss,t}^2 + \frac{\delta_F \sigma_t}{4}
\end{pmatrix}.
\]
(27)

In the disconnected propagator \(D_{ij}^t\), an additional piece appears in the valence-valence sector (Eq. (23)). As noted in Refs. \[7, 14\], this piece has the form of a quenched disconnected propagator, for which \(\sigma_t = 0\), and the assumption of factorization leads us to expect its suppression. In the unmixed case, the mass splittings and hairpin couplings in the valence and sea sectors are degenerate, and the propagator reduces.

\section*{III. NEXT-TO-LEADING ORDER CORRECTIONS TO MASSES}

For a taste \(t\) PGB \(\phi_{xy}^t\) composed of quarks with flavors \(x, y, x \neq y\), the mass is defined in terms of the self-energy, as in continuum ChPT. The NLO mass can be obtained by adding the NLO self-energy to the tree-level mass,
\[
M_{xy,t}^2 = m_{xy,t}^2 + \Sigma_{xy,t}(-m_{xy,t}^2).
\]
(28)
\(\Sigma_{xy,t}\) consists of connected and disconnected tadpole loops with vertices from the LO Lagrangian at \(O(\phi^4)\) and tree-level graphs with vertices from the NLO Lagrangian at \(O(\phi^2)\). The tadpole graphs contribute the leading chiral logarithms, while the tree-level terms are analytic in the quark masses and the square of the lattice spacing.

We have not attempted to enumerate all terms in the NLO Lagrangian. It consists of generalizations of the Gasser-Leutwyler terms \[19\], as in ordinary, unmixed SCHPT, as well as generalizations of the Sharpe-Van de Water Lagrangian \[20\] to the mixed action case. There also exist additional operators including traces over taste-singlets; such operators vanish in the unmixed theory.

Given the different kinds of operators in the NLO Lagrangian, the analytic terms at NNLO have the same form as those in the unmixed theory, but with distinct LECs for valence-valence, sea-sea, and valence-sea PGBs. We have calculated the tadpole graphs for the sea-sea PGBs and find them identical to the results in the unmixed theory, as expected \[15\]. Below we consider the tadpole graphs for the valence-valence and valence-sea PGBs.

\subsection*{A. Valence-valence sector}

For any \(\Gamma_4 \times SO(4)\) irrep, the calculation of the valence-valence PGB self-energies proceeds as for the unmixed
case \[2 \text{[12]}. \] Quark flow diagrams corresponding to the tadpole graphs are shown in diagrams (a-f) of Fig. 1. The kinetic energy, mass, and \( \mathcal{Z} \) vertices yield graphs of types (a), (c), and (d), and the taste-singlet potential vertices (\( \propto C_{\text{mix}} \)) yield graphs of type (a),

\[
\frac{a^2 C_{\text{mix}}}{3f^2(4\pi f)^2} \sum_{i'q} \ell(m_{i'q}, b),
\]

where \( i' \) is summed over \( \mathcal{Z}, \bar{y}, i' \) is summed over the physical sea quarks \( u, d, s \) and \( \ell(m^2) = m^2 \ln(m^2/\Lambda^2) + \delta_1(mL) \) is the chiral logarithm, with \( \Lambda \) the scale of dimensional regularization and \( \delta_1 \) the correction for finite spatial volume \[21]. (\( L \) is the spatial extent of the lattice.)

Vertices from \( \mathcal{Z}' \) yield graphs of types (b), (e), and (f). The hairpin vertex graphs are of types (e) and (f). As in the unmixed case, they can be combined and eliminated in favor of a contribution of type (d). In the mixed-action case, the necessary identity is \( (t \in V, A) \)

\[
\frac{\delta_{\mathcal{F}}^{\text{F}}}{p^2 + m_{\mathcal{F}}^2, t} + \frac{\delta_{\mathcal{F}}^{\text{F}}}{4} \sum_{i'} D_{\mathcal{F}}^L = -(p^2 + m_{\mathcal{F}}^2, t)D_{\mathcal{F}}^L.
\]

This relation follows from Eqs. \[22 \text{ and } 23\].

As in the unmixed theory, graphs of type (b) come from vertices \( \propto \omega_{\mathcal{F}}^{\text{F}} \equiv 16(C_{\text{mix}}^{\text{F}} + C_{\text{mix}}^{\text{F}})/f^2 \) for \( \mathcal{F} = V, A \); they have the same form as those in the unmixed case \[12\].

Adding the various contributions and evaluating the result at \( p^2 = -m_{\mathcal{F}}^2, t \), we have the NLO, one-loop contributions to the self-energies of the valence-ghost PGBs,

\[
-\Sigma_{\mathcal{F}5}^{\text{NLO loop}}(-m_{\mathcal{F}}^2, t) = \frac{a^2}{48(4\pi f)^2} \times \sum_c \left[ \left( \Delta_{\mathcal{F}v5}^{\text{F}} - \Delta_{\mathcal{F}v5}^{\text{F}} + \frac{16c_{\text{mix}}^{\text{F}}}{f^2} \right) \sum_{i'q} \ell(m_{i'q}, c) 
\right.
\]

\[
+ \frac{3}{2} \left( \sum_{c \in V, A} \omega^{\text{F}} \tau_{cb} \tau_{cb}(1 + \theta^2) \ell(m_{i'q}, c) \right) 
\]

\[
+ \frac{1}{12(4\pi f)^2} \int \frac{d^4q}{\pi^2} \times \sum_c \left[ a^2 \left( \Delta_{\mathcal{F}v5}^{\text{F}} + \frac{16c_{\text{mix}}^{\text{F}}}{f^2} \right) \right.
\]

\[
+ 2 \left( 2(1 - \theta^2) + \rho^2 \right)q^2 + \left( 2(1 + 2\theta^2) + \rho^2 \right) \ell(m_{\mathcal{F}5}, 5)
\]

\[
+ 2a^2 \Delta_{\mathcal{F}v5}^{\text{F}} + a^2 \left( 2\theta^2 \Delta_{\mathcal{F}v5}^{\text{F}} + 2 + \rho^2 \right) \ell(m_{\mathcal{F}5}, 5) \right],
\]

which is the generalization of the results of Ref. \[2 \text{ to the mixed-action case.} \] As in ordinary SchPT, only graphs of type (d) contribute. To generalize to the mixed-action theory, one has only to replace the disconnected propagators \( D_{\mathcal{F}5}^{\text{F}} \) with their counterparts in the mixed-action theory.

### B. Valence-sea sector

We consider mesons \( \phi_{\mathcal{F}5} \) with one valence quark \( \mathcal{Z} \) and one sea quark \( y \). For tadpoles with vertices from the kinetic energy and mass terms of the LO Lagrangian (Eq. 8), we find graphs of types (a), (c), and (d).

\[
\frac{1}{48(4\pi f)^2} \sum_{c'i'} \left[ (p^2 + \mu(m_{\mathcal{F}} + m_{\mathcal{F}}) - a^2 \Delta_{\mathcal{F}c}^{\text{F}}) \ell(m_{i'q}, c) 
\right.
\]

\[
+ (p^2 + \mu(m_{\mathcal{F}} + m_{\mathcal{F}}) - a^2 \Delta_{\mathcal{F}c}^{\text{F}}) \ell(m_{y'i'}, c) \right]
\]
\[ + \frac{1}{12(4\pi f)^2} \sum_c \int \frac{d^4 q}{\pi^2} \left[ \left( p^2 + q^2 + \mu(m_\pi + m_\eta) \right) D_{\pi\pi}^c \right. \]
\[ + \left( p^2 + q^2 + \mu(m_\pi + m_\eta) \right) D_{\eta\eta}^c \]
\[ - 2g_{ct} \left( p^2 + q^2 - \mu(m_\pi + m_\eta) \right) D_{\nu\eta}^c \] \]

where \( \tilde{g}' \) is summed over the physical sea-quark flavors.

As for the sea-sea and valence-valence sectors, the \( q^2 D_{\pi\pi}^c \) and \( q^2 D_{\eta\eta}^c \) terms can be eliminated in favor of a \( q^2 D_{\nu\eta}^c \) term. But for the valence-sea mesons, an additional term arises, with the form of a connected contribution [graph (e) of Fig. 1]. The necessary identities are

\[ (q^2 + 2\mu m_\pi)D_{\pi\pi}^c = \frac{\delta^\nu_\pi}{\delta^\nu_\eta}(q^2 + m_\pi^2, t)D_{\pi\pi}^t \] \[ - a^2 \Delta^\nu_\pi \] \[ (q^2 + 2\mu m_\eta)D_{\eta\eta}^c = \frac{\delta^\nu_\eta}{\delta^\nu_\pi}(q^2 + m_\eta^2, t)D_{\eta\eta}^t \] \[ - a^2 \Delta^\nu_\eta \]

which hold for \( t \in F = V, A, I \). Applying these identities to the above result gives

\[ \frac{1}{48(4\pi f)^2} \sum_{c \in V, A} \left[ \left( p^2 + \mu(m_\pi + m_\eta) - a^2 \Delta^\nu_\pi \right) \ell(m_\pi^2, c) \right] \]
\[ + \left( p^2 + \mu(m_\pi + m_\eta) - a^2 \Delta^\nu_\eta \right) \ell(m_\eta^2, c) \]
\[ - \frac{1}{12(4\pi f)^2} \sum_{c \in V, A} \left( \delta^\nu_\pi - \frac{\delta^\nu_\pi^2}{\delta^\nu_\eta^2} \right) \ell(m_\pi^2, c) \]
\[ + \frac{1}{12(4\pi f)^2} \sum_c \int \frac{d^4 q}{\pi^2} \times \]
\[ \left[ \left( p^2 + \mu(m_\pi + m_\eta) - a^2 \Delta^\nu_\pi \right) D_{\pi\pi}^c \right. \]
\[ + \left( p^2 + \mu(m_\pi + m_\eta) - a^2 \Delta^\nu_\eta \right) D_{\eta\eta}^c \]
\[ + \left( -2g_{ct}p^2 + \left( \delta^\nu_\pi \delta^\nu_\eta + \delta^\nu_\eta \delta^\nu_\pi \right) - 2g_{ct}^2 \right) q^2 \]
\[ + \left( \delta^\nu_\pi \delta^\nu_\eta + \theta_{ct} \right) (2\mu m_\pi) + \left( \delta^\nu_\pi \delta^\nu_\eta + \theta_{ct} \right) (2\mu m_\eta) \]
\[ + a^2 \left( \delta^\nu_\pi \delta^\nu_\eta \Delta^\nu_\pi + \delta^\nu_\eta \delta^\nu_\pi \Delta^\nu_\eta \right) D_{\nu\eta}^c \] \]

From the taste-singlet potential, we find contributions not only from graphs of type (a), as in the valence-valence sector, but also from graphs of types (c) and (d),

\[ \frac{a^2 C_{mix}}{3f^2(4\pi f)^2} \sum_b \left[ \left( 8\ell(m_\pi^2, b) + \ell(m_\eta^2, b) \right) \right. \]
\[ + 4 \int \frac{d^4 q}{\pi^2} \left( D_{\pi\pi}^b + D_{\eta\eta}^b - 2g_{ct} D_{\nu\eta}^b \right) \] \]

From the single-trace potential \( \mathcal{U} \), we have graphs of types (a), (c), and (d),

\[ \frac{a^2 C_{mix}}{48(4\pi f)^2} \sum_b \left[ \left( \Delta^\nu_{ct} \ell(m_\pi^2, b) \right) \right. \]
\[ + \left( \Delta^\nu_{ct} \ell(m_\eta^2, b) \right) \]
\[ + 4 \int \frac{d^4 q}{\pi^2} \left( \Delta^\nu_{ct} D_{\pi\pi}^b + \Delta^\nu_{ct} D_{\eta\eta}^b \right. \]
\[ + 2\Delta^\nu_{ct} D_{\nu\eta}^b \] \]

where

\[ \Delta^\nu_{ct} = \frac{8}{f^2} \sum_{b \neq \ell} \left[ 4C_{\nu\pi} + C_{\nu\eta} \right] \]
\[ + \frac{8}{f^2} \sum_{b \neq \ell} \left[ C_{\nu\pi} \left( \frac{9}{2} - 4\theta_{ct} \theta_{bt} \right) \right. \]
\[ + C_{\nu\eta} \left( 3\theta_{ct} \theta_{bt} - 4\theta_{ct} \theta_{bt} \right) \]
\[ + \frac{8}{f^2} \sum_{b \neq \ell} \left[ C_{\nu\pi} \left( \frac{9}{2} - 4\theta_{ct} \theta_{bt} \right) \right. \]
\[ + C_{\nu\eta} \left( 3\theta_{ct} \theta_{bt} - 4\theta_{ct} \theta_{bt} \right) \]
\[ + \frac{8}{f^2} \sum_{b \neq \ell} \left[ C_{\nu\pi} \left( \frac{9}{2} - 4\theta_{ct} \theta_{bt} \right) \right. \]
\[ + C_{\nu\eta} \left( 3\theta_{ct} \theta_{bt} - 4\theta_{ct} \theta_{bt} \right) \]

In the unmixed case, \( \Delta_{ct}^{\nu,\sigma,\sigma} = \Delta_{ct}^{\nu,\sigma,\nu} = \Delta_{ct}^{\nu,\nu,\nu} = \Delta_{ct}^{\nu,\nu,\nu} = \Delta_{ct}^{\nu,\nu,\nu} = \Delta_{ct}^{\nu,\nu,\nu} \), and the contribution from \( \mathcal{U} \) reduces to [12]. We note that \( \Delta_2^{\nu,\sigma,\sigma} \) appears in both connected and disconnected terms.

From the double-trace potential \( \mathcal{U}' \), we have, after combining graphs of types (e) and (f) to eliminate those
of type (f),

\[
\sum_c \left[ \frac{3a^2}{8} g_{\text{tot}} g_{\text{tot}} \left( \omega_{\eta} + \frac{\theta_{ct}}{2} (\omega_{\eta} + \omega_{\eta}^*) \right) \ell(m_{\pi_\mu}^2) \right.
\]
\[
+ \int \frac{d^4q}{(2\pi)^2} \rho_{ct} \left( q^2 + m_{\pi_\mu}^2 + a^2 (\Delta_{\nu\nu} + \Delta_{\sigma\sigma}) \right) D_{\pi_\mu}^2 \right]
\]
\[
+ \frac{1}{3(4\pi)^2} \sum_{c \in V_A} \left[ (\delta_{\nu\nu} - (\delta_{\nu\nu})^2) (\delta_{\sigma\sigma}) (\ell(m_{\pi_\mu}^2)) \right.
\]
\[
+ \int \frac{d^4q}{(2\pi)^2} \left( 2 - \delta_{\nu\nu}/(\delta_{\nu\nu} - \delta_{\sigma\sigma}/(\delta_{\sigma\sigma}) \right) q^2
\]
\[
+ \left( 1 - \delta_{\nu\nu}/(\delta_{\nu\nu}) \right) (2m_{\pi_\mu} + a^2 \Delta_{\nu\nu})
\]
\[
+ \left( 1 - \delta_{\sigma\sigma}/(\delta_{\sigma\sigma}) \right) (2m_{\pi_\mu} + a^2 \Delta_{\sigma\sigma}) \right] D_{\pi_\mu}^2 \right].
\]

The reduction of this expression in the unmixed case is immediate. In the valence-valence sector and unmixed cases, the graphs of types (c) and (f) can be combined into a graph of type (d). In the valence-sea sector, we eliminate graphs of type (f) in favor of those of type (d), but a contribution of type (e) remains.

Adding the various contributions and evaluating the sum at \( p^2 = -m_{\pi_\mu}^2 \) gives, for graphs with connected propagators,

\[
-\Sigma_{\pi_\mu, t}^{\text{NLO loop}} \left( -m_{\pi_\mu}^2 \right) = \frac{a^2}{48(4\pi)^2} \times \left( \sum_c \left[ \Delta_{ct}^{\nu\sigma} - \Delta_{ct}^{\nu\sigma} - \Delta_{ct}^{\nu\sigma} + \frac{128C_{\text{mix}}}{f^2} \right] \sum_{\ell} \ell(m_{\pi_\mu}^2) \right.
\]
\[
+ \left( \Delta_{ct}^{\nu\sigma} - \Delta_{ct}^{\nu\sigma} - \Delta_{ct}^{\nu\sigma} + \frac{16C_{\text{mix}}}{f^2} \right) \sum_{\ell} \ell(m_{\pi_\mu}^2) \right]
\]
\[
+ \frac{3}{2} \sum_{b \in V_A} \tau_{ctb} \tau_{ctb} \left( \omega_{b}^{\nu} + \frac{\theta_{ct}}{2} (\omega_{b}^{\nu} + \omega_{b}^{\nu*}) \ell(m_{\pi_\mu}^2) \right]
\]
\[
+ \frac{1}{4(4\pi)^2} \sum_{c \in V_A} \left( \delta_{\nu\nu} - (\delta_{\nu\nu})^2/\delta_{\sigma\sigma} \right) (\ell(m_{\pi_\mu}^2))
\]

while for the graphs with disconnected propagators, we have

\[
-\Sigma_{\pi_\mu, t}^{\text{NLO loop}} \left( -m_{\pi_\mu}^2 \right) = \frac{1}{12(4\pi)^2} \int \frac{d^4q}{(2\pi)^2} \times \left( \sum_c \left[ a^2 \left( \Delta_{ct}^{\nu\sigma} - \Delta_{ct}^{\nu\sigma} - \Delta_{ct}^{\nu\sigma} + \frac{16C_{\text{mix}}}{f^2} \right) \right.
\]
\[
+ \left( 8 - 3 \left( \delta_{\nu\sigma}^{\nu\sigma} + \delta_{\nu\sigma}^{\nu\sigma} \right) - 2\theta_{ct} + \rho_{ct} \right) q^2
\]
\[
+ \left( 4 - 3 \delta_{\nu\nu}^{\nu\nu} + 2\theta_{ct} + \rho_{ct} \right) (2\mu m_\pi)
\]
\[
+ a^2 \left( 2\theta_{ct} \Delta_{ct}^{\nu\sigma} + \left( 4 - 3 \delta_{\nu\nu}^{\nu\nu} + \delta_{\nu\nu}^{\nu\nu} \right) \Delta_{ct}^{\nu\sigma} \right)
\]
\[
+ \left( 4 - 3 \delta_{\nu\nu}^{\nu\nu} + \rho_{ct} \right) \Delta_{ct}^{\nu\sigma} \right) \right] D_{\pi_\mu}^2 \right].
\]

The reduction in the unmixed case is straightforward. There is no symmetry under \( \pi \leftrightarrow y \); when using the replica method, the valence and sea sectors of the effective theory are distinguished by the operations of partial quenching (the valence quarks) and rooting (the sea quarks). The taste-pseudoscalars are not Goldstone bosons (in the chiral limit) at non-zero lattice spacing, and the self-energy does not vanish in the chiral limit. In the continuum limit, the symmetry is restored, and the masses vanish, in accord with Goldstone’s theorem.

### IV. NEXT-TO-LEADING ORDER CORRECTIONS TO DECAY CONSTANTS

As for continuum and ordinary SChPT, the decay constants are defined by matrix elements of the axial currents,

\[
-\Sigma_{\pi_\mu, t} \left( -m_{\pi_\mu}^2 \right) = \left\langle 0 \left| J^{\mu}_{\pi_\mu} \right| \phi_{\pi_\mu}(p) \right\rangle.
\]

The NLO corrections are the same types of diagrams that appear in continuum and unmixed SChPT. We have one-loop wave function renormalization contributions [graphs (a), (c), and (d) of Fig. 1], one-loop graphs from insertions of the \( \mathcal{O}(\phi^3) \)-terms of the LO current [graphs (g), (h), and (i) of Fig. 1], and terms analytic in the quark masses and squared lattice spacing, from the NLO Lagrangian [11]. As for the NLO analytic corrections to the masses, the NLO analytic corrections to the decay constants have the same form as in the unmixed theory, with distinct LECs for the valence-valence, sea-sea, and valence-sea sectors.

Turning to the one-loop corrections, we note that the LO current is determined by the kinetic energy vertices of the LO Lagrangian; these vertices are the same in mixed-action and unmixed SChPT. Therefore, the LO current
in the mixed-action case is the same as the LO current in unmixed SChPT. Likewise, the NLO wave function renormalization corrections are determined by self-energy contributions from tadpoles with kinetic energy vertices from the LO Lagrangian. Moreover, nothing in the calculation of the relevant part of the self-energies or the current-vertex loops is sensitive to the sector of the external quarks.

Therefore, to generalize the one-loop graphs of the unmixed case, we have only to replace the propagators with those of the mixed-action theory. The results hold for all sectors of the mixed-action theory (valence-valence, sea-sea, and valence-sea). We have

$$\frac{f_{xy,t}^{\text{NLO loop}}}{f} = 1 - \frac{1}{8(4\pi f)^2} \sum_i \times \left[ \frac{1}{4} \int \frac{d^4 q}{\pi^2} (D^c_{xx} + D^c_{yy} - 2\theta^c D^c_{xy}) \right].$$

(54)

The form of this result is the same as that in the unmixed theory [13], and the reduction in the unmixed case is immediate.

V. CONCLUSION

In mixed-action SChPT, we have calculated the NLO loop corrections to the masses and decay constants of pions and kaons in all taste irreps. We have cross-checked all results by performing two independent calculations and verifying the results reduce correctly when valence and sea quark actions are the same. In the valence-sea sector, the taste pseudoscalars are Goldstone bosons in the chiral limit, at non-zero lattice spacing, as in ordinary, unmixed SChPT. The NLO analytic corrections arise from tree-level contributions of the (NLO) Gasser-Leutwyler and generalized Sharpe-Van de Water Lagrangians. They have the same form as in the unmixed case, with independent LECs in the valence-valence, sea-sea, and valence-sea sectors. The NLO loop corrections to the self-energies of the valence-valence pions and kaons are given in Eq. (51); those for the valence-sea pions and kaons are given in Eqs. (51) and (52); and those for the decay constants are given in Eq. (54). As given above, the results for the decay constants and valence-valence masses have the same form as the results in ordinary, unmixed SChPT; the results for the valence-sea masses have additional corrections that vanish in the ordinary, unmixed case. The loop integrals are the same as those in the unmixed theory.