The Density of Rational Points on Cayley’s Cubic Surface

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Abstract

The Cayley cubic surface is given by the equation $\sum_{i=1}^{4} X_i^{-1} = 0$. We show that the number of non-trivial primitive integer points of size at most $B$ is of exact order $B(\log B)^6$, as predicted by Manin’s conjecture.

1 Introduction

The Cayley cubic surface is defined in $\mathbb{P}^3$ by the equation

$$\frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} + \frac{1}{X_4} = 0$$

or equivalently by

$$C : C(X_1, X_2, X_3, X_4) = X_2X_3X_4 + X_1X_3X_4 + X_1X_2X_4 + X_1X_2X_3 = 0.$$ 

It has four singularities, at the points $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Moreover there are exactly 9 lines in the surface, and all of these are defined over the rationals. Three of the lines have the form $X_i + X_j = X_k + X_l = 0$, while the remaining six have the shape $X_i = X_j = 0$. We shall write $U$ for the complement of these lines in the surface $C$.

The aim of this paper is to consider the density of rational points on the surface $C$. It transpires that “most” of the rational points lie on one of the lines described above. We shall think of such points as being “trivial” and exclude them from our counting function. We therefore define

$$N^*(B) = \# \{ x \in \mathbb{Z}^4 : x \in U, \max |x_i| \leq B \},$$

where $x = (x_1, x_2, x_3, x_4)$. Indeed, since vectors $x$ which are scalar multiples of each other represent the same projective point, it is natural to
consider only primitive vectors $x$. (A vector $x$ is said to be primitive if $\text{h.c.f.}(x_1, x_2, x_3, x_4) = 1$.) With this in mind we set

$$N(B) = \# \{ x \in \mathbb{Z}^4 : x \in U, \max |x_i| \leq B, \text{ x primitive} \}.$$ 

The corresponding number of rational points in $\mathbb{P}^3$ is $\frac{1}{2} N(B)$, since $x$ and $-x$ represent the same point. Our two counting functions are closely related, since

$$N^*(B) = \sum_{h \leq B} N(B/h),$$

as one readily verifies.

Manin (see Batyrev and Manin [1]) has given a very general conjecture which would predict in our case that

$$N(B) \sim cB(\log B)^6,$$

for a suitable positive constant $c$. For an arbitrary cubic surface one expects something of this type, with the exponent of the logarithm being one less that the rank of the Picard group of the surface, and in our case this rank is 7. Unfortunately the conjecture has only been established for a small number of extremely simple cubic surfaces, all of which are singular. For example, several authors have considered the surface $X_1X_2X_3 = X_4^3$, see de la Bretèche [2], Fouvry [3], Heath-Brown and Moroz [8] and Salberger [9]. The Cayley surface, while still singular, is considerably more intricate than any previous example. The goal of the present paper is to establish the following estimates.

**Theorem** We have

$$B(\log B)^6 \ll N(B) \ll B(\log B)^6.$$

Of the two inequalities here, the lower bound is relatively easy to prove. Indeed Slater and Swinnerton-Dyer [10] have established the lower bound corresponding to Manin’s conjecture for any non-singular cubic surface defined over $\mathbb{Q}$, providing that it contains two skew lines defined over $\mathbb{Q}$. Although our surface is singular, it does contain several pairs of skew lines, and these are crucial to our argument. It would have been somewhat easier to have established upper bounds of order $B^{1+\varepsilon}$, with an arbitrary positive constant $\varepsilon$, or indeed of order $B(\log B)^A$ for some large constant $A$. However to achieve the correct exponent 6 requires more work.

It is natural to ask how close we come to establishing an asymptotic formula for $N(B)$. An analysis of the argument in §6 shows that the difficulty arises through our use of Lemma 6, which gives an upper bound for the
number of primitive lattice points in \( \mathbb{Z}^3 \), lying in a box, and which satisfy a given linear equation. It is not obvious how one could formulate a useful version of this which replaced the upper bound by an asymptotic formula.

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2 The Universal Torsor

Our goal in this section is to use factorization information to analyze the equation \( C(x) = 0 \), introducing further variables which will be of smaller size than the original variables \( x_1, \ldots, x_4 \), and which will satisfy additional equations. Although we shall not make any use of the fact, we note that these new variables describe the ‘Universal Torsor’ for the Cayley cubic. For the purposes of this analysis it will be convenient to introduce the convention that the letters \( i, j, k, l \) will denote generic distinct indices from the set \( \{1, 2, 3, 4\} \).

It is useful to begin by observing that none of the variables \( x_i \) can vanish. For if \( x_i = 0 \), then the equation \( C(x) = 0 \) implies that \( x_j x_k x_l = 0 \), so that the point \( x \) must lie on one of the excluded lines \( X_i = X_j = 0 \). We now set

\[
y_i = \text{h.c.f.}(x_j, x_k, x_l),
\]

According to our convention this should be taken to mean that \( y_i \) and \( y_j \) are coprime whenever \( i \) and \( j \) are distinct. Since \( y_j, y_k, y_l \) are pairwise coprime and all divide \( x_i \), their product divides \( x_i \), and similarly for the other indices. We may therefore set

\[
x_i = y_j y_k y_l z_i.
\]

The definition (2.1) now reduces to

\[
\text{h.c.f.}(y_k y_l z_j, y_j y_l z_k, y_j y_k z_l) = 1,
\]

In view of (2.2), this is equivalent to the two conditions

\[
\text{h.c.f.}(y_i, z_i) = 1,
\]
and
\[ h.c.f.(z_i, z_j, z_k) = 1. \tag{2.4} \]
Moreover the equation \( C(x) = 0 \) becomes
\[ z_2z_3z_4y_1 + z_1z_3z_4y_2 + z_1z_2z_4y_3 + z_1z_2z_3y_4 = 0 \tag{2.5} \]
on recalling that none of \( x_1, \ldots, x_4 \) can vanish.

Our problem is therefore reduced to counting solutions of the equation \( \tag{2.5} \), lying in the region \(|y_jy_ky_i| \leq B, \) and subject to the constraints \( \tag{2.2}, \tag{2.3} \) and \( \tag{2.4} \). Moreover solutions in which any of the variables is zero are to be discounted, since they produce points \( x \) on one of the lines in the surface \( C \). Similarly solutions with
\[ z_jz_kz_ilt + z_i z_iky_j = 0 \]
are to be discounted.

We now perform a second reduction. We begin by defining
\[ z_{ij} = z_{ji} = h.c.f.(z_i, z_j). \tag{2.6} \]
In view of \( \tag{2.4} \) we have
\[ h.c.f.(z_{ij}, z_{ik}) = 1. \]
Since \( z_{ij}, z_{ik}, z_{il} \) all divide \( z_i \), and are coprime in pairs, it follows that their product divides \( z_i \). We may therefore write
\[ z_i = B_iw_i \]
where
\[ B_i = z_{ij}z_{ik}z_{il}. \tag{2.7} \]
The definition \( \tag{2.6} \) then reduces to
\[ h.c.f.(z_{ik}z_{il}w_i, z_{jk}z_{ji}w_j) = 1, \]
or equivalently
\[ h.c.f.(w_i, w_j) = 1, \tag{2.8} \]
\[ h.c.f.(w_i, z_{jk}) = 1, \tag{2.9} \]
and
\[ h.c.f.(z_{ab}, z_{cd}) = 1, \quad (\{a, b\}, \{c, d\} \text{ distinct}). \tag{2.10} \]
The equation (2.7) now becomes
\[ A_1 w_1 w_3 w_3 y_1 + A_2 w_1 w_3 w_4 y_2 + A_3 w_1 w_2 w_4 y_3 + A_4 w_1 w_2 w_3 y_4 = 0, \quad (2.11) \]
where
\[ A_i = z_{jk} z_{jl} z_{kl}. \quad (2.12) \]
We therefore see that \( w_i | A_i w_j w_k w_l y_i \). In view of (2.8) and (2.9) this implies that \( w_i | z_i \). Since \( w_i | z_i \) we conclude from (2.3) that \( w_i = \pm 1 \).

We now have
\[ x_i = B_i y_j y_k y_l w_i, \]
with \( w_i = \pm 1 \). However, in making the definitions (2.1) and (2.6), the highest common factors are only defined up to sign. Let us assume, temporarily, that we chose the variables \( y_i \) and \( z_{ij} \) to be positive. We proceed to replace each \( y_i \) by \( w_i y_i \), whence
\[ x_i = \varepsilon B_i y_j y_k y_l, \quad (2.13) \]
where the variables \( z_{ij} \) are positive but \( y_i \) may be of either sign. After these changes the equation (2.11) reduces to
\[ A_1 y_1 + A_2 y_2 + A_3 y_3 + A_4 y_4 = 0. \quad (2.14) \]
Moreover the condition (2.4) is implied by (2.10), while (2.3) is equivalent to
\[ \text{h.c.f.}(y_i, z_{ij}) = 1. \quad (2.15) \]
We may therefore summarize our conclusions as follows.

**Lemma 1** Let \( x \in U \) be a primitive integral solution of \( C(x) = 0 \). Then either \( x \) or \( -x \) takes the form (2.13), with non-zero integer variables \( y_i \) and positive integer variables \( z_{ij} \) constrained by the conditions (2.2), (2.10) and (2.15), and satisfying the equation (2.14). Moreover none of \( A_1 y_1 + A_2 y_2 \), \( A_1 y_1 + A_3 y_3 \) or \( A_1 y_1 + A_4 y_4 \) may vanish.

Conversely, if \( y_i \) and \( z_{ij} \) are as above, then the vector \( x \) given by (2.13) will be a primitive integral solution of \( C(x) = 0 \) lying in \( U \).

To proceed further, we note that the equation (2.14) implies that
\[ z_{ij} | z_{kl} (z_{ik} z_{il} y_j + z_{jk} z_{jl} y_i), \]
whence (2.11) yields
\[ z_{ij} z_{ik} y_j + z_{jk} z_{ji} y_i. \]

We therefore write
\[ z_{ik} z_{ii} y_j + z_{jk} z_{ji} y_i = z_{ij} v_{ij}, \] (2.16)
so that equation (2.14) is equivalent to each of the relations
\[ v_{ij} + v_{kl} = 0. \]

Note that \( v_{ij} = v_{ji} \), since \( z_{ij} \) is also symmetric in the indices \( ij \). We now calculate that
\[
z_{ij} v_{ij} z_{ik} v_{ik} = (z_{ik} z_{ii} y_j + z_{jk} z_{ji} y_i) (z_{ij} z_{ii} y_k + z_{jk} z_{kl} y_i)
= z_{ii}^2 z_{ij} z_{ik} y_j y_k + z_{jk} y_i (z_{ik} z_{ii} z_{kl} y_j + z_{jl} z_{ij} z_{kl} y_k + z_{jk} z_{ji} z_{kl} y_i),
\]
whence (2.14) yields
\[ z_{ij} v_{ij} z_{ik} v_{ik} = z_{ii}^2 z_{ij} z_{ik} y_j y_k - z_{jk} y_i z_{ij} z_{ik} z_{jk} y_i. \]

We therefore conclude that
\[ v_{ij} v_{ik} = z_{ii}^2 y_j y_k - z_{jk} y_i y_l. \] (2.17)

3 The Lower Bound

To tackle the lower bound problem in our theorem we begin by considering solutions for which the variables \( z_{ij} \) are fixed, and relatively small, while the variables \( y_i \) are comparatively large, and lie in the dyadic ranges
\[ Y_i < |y_i| \leq 2 Y_i. \] (3.1)

In the notation given by (2.7) and (2.12) we observe that the condition \( \max |x_i| \leq B \) is equivalent to
\[ A_i A_j A_k |y_i y_j y_k| \leq BP, \] (3.2)
where
\[ P = z_{12} z_{13} z_{14} z_{23} z_{24} z_{34} = A_i B_i. \] (3.3)
We shall choose
\[ Y_i = \left[ \frac{(BP)^{1/3}}{2 A_i} \right], \] (3.4)
We will then have (3.2) whenever the $y_i$ lie in the ranges (3.1). We shall assume moreover that
\[ P \leq B^\delta. \] (3.5)
where $\delta$ is a small positive constant to be specified later, see (3.14). We shall write
\[ N = N(z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34}) \]
for the number of solutions $(y_1, y_2, y_3, y_4)$ of (2.14), in the ranges (3.1), subject to the constraints (2.2) and (2.15), and not on any of the lines $A_1y_1 + A_iy_i = 0$.

The main difficulty in establishing our lower bound comes from the co-primality conditions (2.2) and (2.15). To handle these we begin by setting
\[ Q = P \prod_{p \leq \sqrt{\log B}} p \]
and writing $N_1$ for the number of solutions in which (2.2) is replaced by the weaker condition
\[ (y_i, y_j, Q) = 1, \text{ for all } i \neq j. \]
We take $N_2$ to be the number of solutions in which some pair $y_i, y_j$ has a prime factor $p | y_i, y_j$ with $p \nmid Q$. Clearly we then have
\[ N \geq N_1 - N_2. \] (3.6)

We begin by estimating $N_1$, and first note that there can be at most
\[ Y_1Y_j \ll (BP)^{2/3} \ll B^{2/3+28/3} \] (3.7)
solutions on one of the lines $A_1y_1 + A_iy_i = A_jy_j + A_ky_k = 0$, by (3.4) and (3.5). This bound will turn out to be of negligible size. We can therefore ignore the condition that solutions may not lie on such a line. We now proceed by picking out the coprimality conditions with the Möbius function. Let
\[ N_3 = N_3(d_1, \ldots, d_4; d_{12}, \ldots, d_{34}) \]
denote the number of solutions of the equation (2.14), with $y_i$ in the ranges (3.1), and such that $d_i | y_i$ and $d_{ij} | y_i, y_j$ for every choice of indices. Then
\[ N_1 = \sum_{d_i | B_i} \mu(d_1) \ldots \mu(d_4) \sum_{d_{ij} | Q} \mu(d_{12}) \ldots \mu(d_{34}) N_3, \] (3.8)
since the condition (2.15) is equivalent to $\text{h.c.f.}(y_i, B_i) = 1$. We must now estimate $N_3$. We shall write $h_1$ for the lowest common multiple of $d_{12}, d_{13}, d_{14}$ and $d_1$, and similarly for $h_2, h_3$ and $h_4$. We may then re-interpret $N_3$ as the
number of integer triples \((n_1, n_2, n_3)\) for which \(A_i h_i | n_i\) and \(A_4 h_4 | n_1 + n_2 + n_3\), and which lie in the region

\[
\mathcal{R} : A_i Y_i < |n_i| \leq 2 A_i Y_i \quad (i \leq 3), \quad A_4 Y_4 < |n_1 + n_2 + n_3| \leq 2 A_4 Y_4. \quad (3.9)
\]

The divisibility conditions define an integer sublattice \(\Lambda \leq \mathbb{Z}^3\), such that

\[
N_3 = \#(\Lambda \cap \mathcal{R}).
\]

We shall need to compute the determinant of \(\Lambda\), or, what is the same thing, its index in \(\mathbb{Z}^3\). This is most easily done locally. Write

\[
A_i h_i = \prod_p p^{\nu(p, i)}
\]

and let \(\Lambda_p \leq \mathbb{Z}^3\) be the lattice for which \(p^{\nu(p, i)} | n_i\) and \(p^{\nu(p, 4)} | n_1 + n_2 + n_3\).

Let \(\nu(p, 0) = \min \nu(p, i)\) and let \(\Lambda_p^{(0)} \leq \mathbb{Z}^3\) be defined by the conditions \(p^{\nu(p, i)-\nu(p, 0)} | n_i\) and \(p^{\nu(p, 4)-\nu(p, 0)} | n_1 + n_2 + n_3\). Then a moment’s thought reveals that

\[
\det(\Lambda_p^{(0)}) = p^{\nu(p, 1)-\nu(p, 0)} \cdot p^{\nu(p, 2)-\nu(p, 0)} \cdot p^{\nu(p, 3)-\nu(p, 0)} \cdot p^{\nu(p, 4)-\nu(p, 0)},
\]

and

\[
\det(\Lambda_p) = p^{3\nu(p, 0)} \det(\Lambda_p^{(0)}),
\]

whence

\[
\det(\Lambda_p) = \frac{\prod_p p^{\nu(p, i)}}{\text{h.c.f.}(p^{\nu(p, i)})}.
\]

We now observe that \(\Lambda\) is the intersection of the various \(\Lambda_p\), which have pairwise coprime indices in \(\mathbb{Z}^3\). It therefore follows that

\[
\det(\Lambda) = \prod_p \frac{\prod_p p^{\nu(p, i)}}{\text{h.c.f.}(p^{\nu(p, i)})} = \prod_i A_i h_i.
\]

We may choose a basis \(b_1, b_2, b_3\) of \(\Lambda\) with \(|b_i| \ll \det(\Lambda)\). Taking \(M\) to be the \(3 \times 3\) integer matrix formed from the vectors \(b_i\) we see that \(\Lambda = M \mathbb{Z}^3\), and that \(\det(M) = \det(\Lambda)\). If \(\mathcal{R}\) is the region \((3.9)\) then

\[
\#(\Lambda \cap \mathcal{R}) = \#(\mathbb{Z}^3 \cap M^{-1} \mathcal{R}).
\]

However \(M^{-1} \mathcal{R}\) has volume \(\text{meas}(\mathcal{R})/\det(M)\), is bounded by \(O(1)\) planar sides, and lies in a sphere of radius \(r\), say, where \(r \ll \|M^{-1}\| \max Y_i\). Here
$\|M^{-1}\|$ is the modulus of the largest entry in $M^{-1}$, so that $\|M^{-1}\| \ll \det(\Lambda)$. It follows that

$$\mathcal{N}_3 = \#(\Lambda \cap \mathcal{R}) = \#(\mathbb{Z}^3 \cap M^{-1} \mathcal{R}) = \frac{\text{meas}(\mathcal{R})}{\det(M)} + O(r^2).$$

Since $d_i | B_i$ and $d_{ij} | Q$ we have $A_i h_i \leq A_i B_i Q = PQ$, by (3.3). Thus (3.3) yields $\det(\Lambda) \leq P^4 Q^4 \ll B^{12\delta}$, since

$$Q \ll P \exp(O(\sqrt{\log B})) \ll B^{2\delta}.$$

We therefore deduce that

$$\mathcal{N}_3 = \text{meas}(\mathcal{R}) \frac{\text{h.c.f.}(A_i h_i)}{\prod_{i} A_i h_i} + O(B^{2/3 + 25\delta}).$$

We now insert this into (3.8), so that

$$\mathcal{N}_1 = \text{meas}(\mathcal{R}) \sum_{d_i, d_{ij}} \mu(d_1) \cdots \mu(d_4) \mu(d_{12}) \cdots \mu(d_{34}) \frac{\text{h.c.f.}(A_i h_i)}{\prod_{i} A_i h_i}$$

$$+ O(B^{2/3 + 26\delta}),$$

(3.10)

since the usual estimate for the divisor function shows that there are $O(B^\delta)$ divisors $d_i, d_{ij}$ in total.

It remains to consider the sum

$$\sum_{d_i, d_{ij}} \mu(d_1) \cdots \mu(d_4) \mu(d_{12}) \cdots \mu(d_{34}) \frac{\text{h.c.f.}(A_i h_i)}{\prod_{i} A_i h_i}.$$

By multiplicativity we see that this is a product of local factors $e_p$, say. For primes $p \nmid P$ we define the integer $N$, temporarily, as the number of quadruples $(x_1, x_2, x_3, x_4) \pmod{p}$ satisfying $\text{h.c.f.}(x_i, x_j, p) = 1$ for $i \neq j$, and such that $x_1 + x_2 + x_3 + x_4 \equiv 0 \pmod{p}$. We then find, again using the Möbius function, that $N = p^3 e_p$. An easy computation then yields

$$e_p = 1 - \frac{6}{p^2} + \frac{5}{p^3}. \quad (3.11)$$

For the remaining primes $p$ we note that $p$ will divide exactly one $z_{ij}$, by (2.10), and we suppose without loss of generality that $z_{12}$ contains $p$ with exponent $e \geq 1$, say. We then let $A'_i = A_i$ if $i = 1$ or 2, and $A'_i = p^{1-e} A_i$ for $i = 3$ or 4. Since $p^2 \nmid A_i h_i$ for $i = 1$ or 2 we have

$$\text{h.c.f.}(A_i h_i) = \text{h.c.f.}(A'_i h_i),$$
and hence

\[ \frac{\text{h.c.f.}(A_i h_i)}{\prod_i A_i h_i} = p^{2-2e} \frac{\text{h.c.f.}(A'_i h_i)}{\prod_i A'_i h_i}. \]

Now let \( N \) denote, temporarily, the number of quadruples \((x_1, x_2, x_3, x_4) \in \mathbb{N}^4\) satisfying the conditions

\[
\begin{align*}
  &x_1, x_2 \leq p^2, \quad x_3, x_4 \leq p, \\
  &\text{h.c.f.}(x_i, x_j, p) = 1 \quad \text{for} \quad i \neq j, \\
  &\text{h.c.f.}(x_1, p) = \text{h.c.f.}(x_2, p) = 1,
\end{align*}
\]

and

\[
x_1 + x_2 + px_3 + px_4 \equiv 0 \pmod{p^2}.
\]

We then find, using the Möbius function once more, that

\[ N = p^{2e+4}e_p, \]

and another easy computation then produces

\[ e_p = (1 - \frac{1}{p})(1 - \frac{1}{p^2})p^{-2e}. \tag{3.12} \]

The formulae (3.11) and (3.12) show that

\[
\sum_{d_i, d_{ij}} \mu(d_1) \ldots \mu(d_4) \mu(d_{12}) \ldots \mu(d_{34}) \frac{\text{h.c.f.}(A_i h_i)}{\prod_i A_i h_i} \gg P^{-2\phi(P)}.
\]

and since we clearly have \( \text{meas}(\mathcal{R}) \gg BP \), from (3.4) and (3.9), we deduce from (3.5), (3.7) and (3.10) that

\[ \mathcal{N}_1 \gg \frac{B \phi(P)}{P} \tag{3.13} \]

providing that we take

\[ \delta = \frac{1}{84}. \tag{3.14} \]

We turn now to \( \mathcal{N}_2 \), which we must estimate from above. We start by considering the contribution from solutions in which \( p \mid y_1, y_2 \), say, with \( p \nmid Q \) and \( R < p \leq 2R \). We begin with the following preliminary observations. Clearly there are no solutions with \( R \gg Y_1 \), and so we may suppose that \( R \ll Y_1 \). Moreover, if \( \delta \leq 1/7 \) then (3.3), (3.4) and (3.5) yield

\[
z_{12} \leq \frac{P}{A_1} \leq \frac{(BP)^{1/6}}{A_1} \ll Y_1^{1/2}.
\]

It therefore follows that

\[
1 \ll \frac{(Y_1)^{1/2}Y_1^{1/2}}{z_{12}} = \frac{Y_1}{R^{1/2}z_{12}}.
\]
Thus

\[ 1 + \frac{Y_1}{R_{z_{12}}} \ll \frac{Y_1}{R^{1/2}z_{12}}. \] (3.15)

Since we are seeking an upper bound for \( N_2 \), the coprimality conditions can be dropped. If we set \( y_1 = pt_1, y_2 = pt_2 \) then we have \( Y_i/R \ll |t_i| \ll Y_i/R \) for \( i = 1, 2 \). Moreover, since \( z_{12}|A_3, A_4 \) we have \( A_1t_1 \equiv -A_2t_2 (\mod z_{12}) \). However \( z_{12} \) is coprime to \( A_1 \), so that each admissible value of \( t_2 \) determines \( O(1 + Y_1/(Rz_{12})) \) values of \( t_1 \). It therefore follows from (3.15) that there are \( O(Y_1Y_2R^{-3/2}z_{12}^{-1}) \) possible pairs \( t_1, t_2 \).

For each such pair we now estimate how many triples \( p, y_3, y_4 \) there might be. We put \( A_1t_1 + A_2t_2 = z_{12}s \). Then

\[ ps + \frac{A_3}{z_{12}}y_3 + \frac{A_4}{z_{12}}y_4 = 0, \]

whence

\[ ps + \frac{A_3}{z_{12}}y_3 \equiv 0 (\mod \frac{A_4}{z_{12}}). \]

Since \( A_3/z_{12} \) and \( A_4/z_{12} \) are coprime, by (2.10), we see that each value of \( p \) determines \( y_3 \) modulo \( A_4/z_{12} \), producing \( O(1 + Y_3z_{12}/A_4) \) values. However if \( \delta \leq 1/7 \) then (3.4) and (3.3) suffice to show that \( 1 \ll Y_3/A_4 \), so we will have \( O(RY_3z_{12}/A_4) \) possible pairs \( p, y_3 \), each of which determines at most one admissible \( y_4 \).

These bounds show that the range \( R < p \leq 2R \) contributes

\[ \frac{Y_1Y_2}{R^{3/2}z_{12}} \frac{RY_3z_{12}}{A_4} = \frac{A_1Y_1A_2Y_2A_3Y_3}{P^2R^{1/2}} \ll \frac{B}{P^{3/2}} \]

to \( N_2 \). If we now sum \( R \gg \sqrt{\log B} \) over powers of two we deduce that

\[ N_2 \ll \frac{B}{P}(\log B)^{-1/4}. \]

Since \( \phi(P)/P \gg (\log \log P)^{-1} \) we deduce from (3.13) that \( N_2 = o(N_1) \) and hence, via (3.6), that

\[ N \gg \frac{B \phi(P)}{P}. \]

We summarize our conclusions thus far as follows.

**Lemma 2** For a given admissible set of values \( z_{12}, \ldots, z_{34} \) satisfying

\[ P = \prod z_{ij} \leq B^{1/84} \]

there are

\[ \gg \frac{B \phi(P)}{P} \]

corresponding values of \( y_1, y_2, y_3, y_4 \).
To complete the proof of the lower bound part of our theorem, we observe that any square-free value of $P$ will factorize into values $z_1, \ldots, z_{34}$ satisfying (2.10) and (3.3) in exactly $d_6(P)$ ways. (Here $d_6(\ldots)$ is the generalized divisor function.) Thus

$$N(B) \gg \sum_{P \leq B^{1/84}} \mu(P)^2 d_6(P) \frac{B \phi(P)}{P}$$

and a standard estimation using Perron’s formula then produces the required bound

$$N(B) \gg B(\log B)^6.$$

4 The Upper Bound—Basic Estimates

In contrast to the work of the previous section, in giving an upper bound for $N(B)$ we can ignore questions of coprimality whenever we wish to do so. Instead our principal technical problem will be to control precisely the number of logarithms appearing in our estimates.

We shall need to understand the equations (2.16) and (2.17), and our results are summarized as follows.

**Lemma 3** Let real numbers $K_1, \ldots, K_7 > 0$ be given, and let $N_1$ denote the number of solutions $n_i \in \mathbb{N}$ to the equation

$$n_1n_2n_3 + n_4n_5n_6 = n_7n_8 \quad (K_i < n_i \leq 2K_i, \ 1 \leq i \leq 7)\tag{4.1}$$

subject to the condition

$$\text{h.c.f.}(n_1n_2n_3, n_4n_5n_6) = 1.$$

Then

$$N_1 \ll K_1K_2K_3K_4K_5K_6.\tag{4.2}$$

Similarly, if $N_2$ is the number of solutions of

$$n_1n_2n_3 = n_4n_5n_6 + n_7n_8, \tag{4.3}$$

under the same conditions, then

$$N_2 \ll K_1K_2K_3K_4K_5K_6.$$
Lemma 4 Let real numbers $K_1, \ldots, K_7 > 0$ be given, and let $N_3$ denote the number of solutions $n_i \in \mathbb{N}$ of the equation

\[ n_1^2 n_2 n_3 + n_4^2 n_5 n_6 = n_7 n_8 \quad (K_i < n_i \leq 2K_i, \ 1 \leq i \leq 7) \quad (4.4) \]

subject to the condition (4.1). Then

\[ N_3 \ll K_1 K_2 K_3 K_4 K_5 K_6 \max\{\left(\frac{K_1^2 K_2 K_3}{K_4^2 K_5 K_6}\right)^{1/4}, \left(\frac{K_4^2 K_5 K_6}{K_1^2 K_2 K_3}\right)^{1/4}\}. \quad (4.5) \]

If $N_4$ is the corresponding number of solutions for the equation

\[ n_1^2 n_2 n_3 - n_4^2 n_5 n_6 = n_7 n_8 \quad (K_i < n_i \leq 2K_i, \ 1 \leq i \leq 7) \quad (4.6) \]

we have

\[ N_4 \ll \{1 + \frac{\log K_1 K_4}{(K_2 K_3 K_5 K_6)^{1/3}}\} K_1 K_2 K_3 K_4 K_5 K_6 \]

\[ \times \max\{\left(\frac{K_1^2 K_2 K_3}{K_4^2 K_5 K_6}\right)^{1/4}, \left(\frac{K_4^2 K_5 K_6}{K_1^2 K_2 K_3}\right)^{1/4}\}. \quad (4.7) \]

We may think of the bound for $N_1$, for example, as describing the number of divisors of $n_1 n_2 n_3 + n_4 n_5 n_6$ which lie in specified dyadic ranges. Note that we do not impose a condition on the size of $n_8$. We may remark that in both lemmas we can use the standard bound for the divisor function to show that each 6-tuple $(n_1, \ldots, n_6)$ determines $O((\max K_i)^\varepsilon)$ pairs of divisors $n_7, n_8$, for any fixed $\varepsilon > 0$. This immediately yields the bounds

\[ N_1, N_2, N_3, N_4 \ll (K_1 K_2 K_3 K_4 K_5 K_6)^{1+\varepsilon}, \]

so that the important aspect of Lemma 3 is the removal of the exponent $\varepsilon$. It would be relatively easy to replace the $\varepsilon$ power by a power of a logarithm, but this would be insufficient for our purposes. In relation to Lemma 4 we conjecture that the factor

\[ \max\{\left(\frac{K_1^2 K_2 K_3}{K_4^2 K_5 K_6}\right)^{1/4}, \left(\frac{K_4^2 K_5 K_6}{K_1^2 K_2 K_3}\right)^{1/4}\} \]

may be removed in both cases. However it is not possible to delete the term

\[ 1 + \frac{\log K_1 K_4}{(K_2 K_3 K_5 K_6)^{1/3}} \]

in our estimate for $N_4$. Indeed, when $K_1 = K_4 = K_7$ and $K_2 = K_3 = K_5 = K_6 = 1/2$ we easily find that $N_4 \gg K_1 K_4 \log(K_1 K_4)$. Thus our bounds are
not as sharp as we would like, but they are optimal in the critical case in which $K_1^2K_2K_3$ and $K_4^2K_5K_6$ have the same order of magnitude.

Before beginning the proofs of these results we observe that the condition (4.1) implies that the three terms $n_1n_2n_3$, $n_4n_5n_6$ and $n_7n_8$ are coprime in pairs. We shall use this fact repeatedly without further comment, in relation to both lemmas.

In this section we shall prove Lemma 3. The treatment of Lemma 4, which we defer to the next section, uses some of the same principles, but is much more involved. We begin by considering $N_1$. By the symmetry we may assume that

$$K_1K_2K_3 \gg K_4K_5K_6. \quad (4.8)$$

It is then clear that $N_1 = 0$ unless

$$\frac{K_1K_2K_3}{K_7} \ll n_8 \ll \frac{K_1K_2K_3}{K_7}, \quad (4.9)$$

as we shall now assume. We write this condition as $K_8 \ll n_8 \ll K_8$. We may then suppose, by symmetry, that $K_7 \geq K_8$, whence (4.9) implies that $K_8 \ll (K_1K_2K_3)^{1/2}$. We then apply the following estimate.

**Lemma 5** Let $K_1, K_2, K_3 > 0$ and let $q \ll (K_1K_2K_3)^{1/2}$. Then for any integer $a$ coprime to $q$, we have

$$\# \{(n_1, n_2, n_3) \in \mathbb{N}^3 : K_i < n_i \leq 2K_i, n_1n_2n_3 \equiv a \pmod{q}\} \ll K_1K_2K_3/\phi(q).$$

We shall prove this in a moment. However if we apply it to the current situation we see, on taking $q = n_8$ and summing over $n_4, n_5, n_6$ and $n_8$, that

$$N_1 \ll K_1K_2K_3K_4K_5K_6 \sum_{K_8 \ll n_8 \ll K_8} \phi(n_8)^{-1}.$$ 

The required bound (4.2) now follows, since

$$\sum_{K_8 \ll n_8 \ll K_8} \phi(n_8)^{-1} \ll 1.$$

To handle $N_2$ we note that we automatically have (4.8) if there are to be any solutions. We can then proceed exactly as before providing that

$$K_1K_2K_3 \geq 16K_4K_5K_6,$$
since this is enough to ensure that (4.9) holds. It therefore remains to consider the case in which

\[ K_1 K_2 K_3 \ll K_4 K_5 K_6 \ll K_1 K_2 K_3. \]  
(4.10)

In this case we shall assume that

\[ K_1 \geq K_2 \geq K_3, \quad \text{and} \quad K_4 \geq K_5 \geq K_6, \]

as we may, by the symmetry. It follows in particular that

\[ K_2 K_3 K_5 K_6 \leq (K_1 K_2 K_3 K_4 K_5 K_6)^{2/3}. \]  
(4.11)

We now write \( N_{2,a}(q) \) for the number of solutions \((n_1, \ldots, n_6)\) corresponding to each value \( n_7 = q \), so that

\[ N_2 \ll \sum_{K_7 < q \leq 2K_7} N_{2,a}(q). \]  
(4.12)

Moreover, if we set

\[ K_8 = K_1 K_2 K_3 / K_7, \]  
(4.13)

then it is apparent that we must have \( n_8 \ll K_8 \) in any solution of (4.3). Thus

\[ N_2 \ll \sum_{q \ll K_8} N_{2,b}(q), \]  
(4.14)

where \( N_{2,b}(q) \) counts the solutions corresponding to a given value \( n_8 = q \). We plan to use (4.12) when \( K_7 \ll (K_1 K_2 K_3)^{1/2} \). If this condition fails to hold we must have \( K_8 \ll (K_1 K_2 K_3)^{1/2} \), in which case we shall employ (4.14).

We now introduce the following result, which is part of Lemma 3 of the author’s work [5].

**Lemma 6** Let \( v \in \mathbb{Z}^3 \) be a primitive vector, and let \( H_i > 0 \) for \( i = 1, 2, 3 \) be given. Then the number of primitive vectors \( x \in \mathbb{Z}^3 \) for which \( v \cdot x = 0 \), and which lie in the box \(|x_i| \leq H_i \) \((i = 1, 2, 3)\), is at most

\[ 4 + 12\pi \frac{H_1 H_2 H_3}{\max H_i |v_i|} \leq 4 + 12\pi \frac{H_1 H_2}{|v_3|}. \]

Recall that an integer vector is said to be primitive if its coordinates have no common factor. In our applications this condition will be a consequence of (2.2), (2.10) and (2.13).

To bound \( N_{2,a}(q) \) we write the condition (4.3) as \( v \cdot x = 0 \) where \( v = (n_2 n_3, -n_5 n_6, -q) \) and \( x = (n_1, n_4, n_8) \). We set \( H_1 = 2K_1, H_2 = 2K_4 \) and

\[ H_3 = 8K_1 K_2 K_3 / q. \]
Then Lemma 6 produces the bound $O(1 + K_1 K_4 / q)$ for the number of triples $(n_1, n_4, n_8)$ and it follows on summing over $n_2, n_3, n_5, n_6$ and $q$ that

$$N_2 \ll \sum_{K_7 < q \leq 2K_7} N_{2,a}(q) \ll K_2 K_3 K_5 K_6 K_7 + K_1 K_2 K_3 K_4 K_5 K_6. \quad (4.15)$$

Alternatively, we may use $N_{2,b}(q)$, and write (4.3) as $v \cdot x = 0$ with $v = (n_2 n_3, -n_5 n_6, -q)$ and $x = (n_1, n_4, n_7)$. We set $H_1 = 2K_1, H_2 = 2K_4$ as before, and $H_3 = 2K_7$. This time Lemma 6 produces a bound

$$\ll 1 + \frac{K_1 K_4 K_7}{\max H_i |v_i|} \ll 1 + \frac{K_1 K_4 K_7}{K_1 K_2 K_3}$$

for the number of triples $(n_1, n_4, n_7)$. On summing over $n_2, n_3, n_5, n_6$ and $q$ we then find that

$$N_2 \ll \sum_{q \ll K_8} N_{2,b}(q) \ll K_2 K_3 K_5 K_6 K_8 + K_4 K_5 K_6 K_7 K_8.$$

In view of (4.13) we may combine this with (4.15) to deduce that

$$N_2 \ll K_2 K_3 K_5 K_6 \min(K_7, K_8) + K_1 K_2 K_3 K_4 K_5 K_6.$$ 

Since we have

$$\min(K_7, K_8) \ll (K_1 K_2 K_3)^{1/2} \ll (K_1 K_2 K_3 K_4 K_5 K_6)^{1/4},$$

by (4.10), we then deduce from (4.11) that

$$N_2 \ll (K_1 K_2 K_3 K_4 K_5 K_6)^{2/3} (K_1 K_2 K_3 K_4 K_5 K_6)^{1/4}$$

$$+ K_1 K_2 K_3 K_4 K_5 K_6$$

$$\ll K_1 K_2 K_3 K_4 K_5 K_6,$$

which completes the proof of our bound for $N_2$.

We must now establish Lemma 5. To do this we refer to the author’s work [3] on the divisor function $d_3(n)$ in arithmetic progressions. If we write

$$N(K_1, K_2, K_3; a, q)$$

$$= \# \{(n_1, n_2, n_3) \in \mathbb{N}^3 : K_i < n_i \leq 2K_i, n_1 n_2 n_3 \equiv a \pmod{q}\}$$

then the analysis of [3, §7] suffices to show that

$$N(K_1, K_2, K_3; a, q) = C(K_1, K_2, K_3; q) + O(E),$$
with $C(K_1, K_2, K_3; q)$ independent of $a$, and an error term

$$E = (K_1 K_2 K_3)^{3/4+\varepsilon} q^{-13/24} + (K_1 K_2 K_3)^{7/9+\varepsilon} q^{-7/12} + (K_1 K_2 K_3)^{10/13+\varepsilon} q^{-15/26}$$

$$+ (K_1 K_2 K_3)^{46/57+\varepsilon} q^{-12/19} + (K_1 K_2 K_3)^{86/107+\varepsilon} q^{-66/107},$$

for any fixed $\varepsilon > 0$. Since $q \ll (K_1 K_2 K_3)^{1/2}$ we deduce that

$$N(K_1, K_2, K_3; a, q) = C(K_1, K_2, K_3; q) + O(K_1 K_2 K_3 q^{-1}).$$

We may now average over $a$ coprime to $q$ to find that

$$\phi(q)^{-1} \# \{(n_1, n_2, n_3) \in \mathbb{N}^3 : K_i < n_i \leq 2K_i, \text{h.c.f.}(n_1 n_2 n_3, q) = 1\}$$

$$= C(K_1, K_2, K_3; q) + O(K_1 K_2 K_3 q^{-1}).$$

We deduce that $C(K_1, K_2, K_3; q) \ll K_1 K_2 K_3 / \phi(q)$, and Lemma 5 follows.

The reader should note that the work of Friedlander and Iwaniec [4] could have been used equally effectively at this point.

## 5 The Proof of Lemma 4

By symmetry, we may suppose at the outset that

$$K_1^2 K_2 K_3 \gg K_2^2 K_3 K_5. \quad (5.1)$$

We shall write $N_{3,a}(q)$ for the number of solutions $(n_1, \ldots, n_6)$ corresponding to each value $n_7 = q$, so that

$$N_3 \ll \sum_{K_7 < q \leq 2K_7} N_{3,a}(q). \quad (5.2)$$

Moreover, if we set

$$K_8 = K_1^2 K_2 K_3 / K_7,$$

then it is apparent that we must have $K_8 \ll n_8 \ll K_8$ in any solution of (4.4). Thus

$$N_3 \ll \sum_{K_8 \ll q \ll K_8} N_{3,b}(q), \quad (5.3)$$

where $N_{3,a}(q)$ counts the solutions corresponding to a given value $n_8 = q$.

We plan to use (5.2) when $K_7 \ll K_1 (K_2 K_3)^{1/2}$. If this condition fails to hold we must have $K_8 \ll K_1 (K_2 K_3)^{1/2}$, in which case we shall employ (5.3).
To bound $N_{3,a}(q)$ we write the condition \((1.4)\) as \(v \cdot x = 0\) where \(v = (n_1^2 n_2, n_1^2 n_5, -q)\) and \(x = (n_3, n_4, n_8)\), say. We set $H_1 = 2K_3, H_2 = 2K_6$ and
\[H_3 = 32K_1^2 K_2 K_3/q.\]
Then Lemma 6 produces the bound $O(1 + K_3 K_6/q)$ for the number of triples $(n_3, n_6, n_8)$ and it follows on summing over $n_1, n_2, n_4$ and $n_5$ that
\[N_{3,a}(q) \ll K_1 K_2 K_4 K_5 + K_1 K_2 K_3 K_4 K_5 K_6/q.\] (5.4)

In a precisely analogous way we find that
\[N_{3,b}(q) \ll K_3 K_4 K_6 + K_1 K_2 K_3 K_4 K_5 K_6/q.\] (5.5)

We may also use a vector $x$ involving $n_1$ and $n_4$. To do this, we let $t \in [0, q)$ run over the solutions of the quadratic congruence
\[t^2 n_2 n_3 + n_5 n_6 \equiv 0 \pmod{q},\]
and we write $\rho(q; n_2 n_3, n_5 n_6)$ for the number of such solutions $t$. We then see that, for fixed $n_2, n_3, n_5, n_6$ and $q$, we must have $n_1 \equiv tn_4 (\text{mod } q)$ for some value of $t$. This leads to an equation $v \cdot x = 0$ with $v = (1, -t, q)$ and $x = (n_1, n_4, m)$, with size restrictions given by $H_1 = 2K_1, H_2 = 2K_4$ and $H_3 = K_1 + K_4$, say. Thus Lemma 6 produces a bound $O(1 + K_1 K_4/q)$ for the number of solutions $n_1, n_4$ corresponding to a given value of $t$. We therefore obtain an estimate
\[N_{3,a}(q) \ll (1 + K_1 K_4/q) \sum_{n_2, n_3, n_5, n_6} \rho(q; n_2 n_3, n_5 n_6).\] (5.6)

Our next task is evidently to examine averages of the function $\rho$. Suppose that h.c.f. $(a, b) = 1$. Then for odd $q$ we have
\[\rho(q; a, b) = \sum_{d|q} \mu(d)^2 \left(\frac{-ab}{d}\right),\]
where $(ab/d)$ is the Jacobi symbol. We then see that
\[\rho(q; a, b) \leq 4 \sum_{d|q} \mu(d)^2 \left(\frac{-ab}{d}\right)\]
whether $q$ is even or odd, where we take the Jacobi symbol to vanish for even $d$. We also note that the sum on the right is non-negative when $ab$ and $q$ are not coprime. Our aim is to estimate
\[\sum_{q \leq Q} \sum_{n_2, n_3, n_5, n_6} \rho(q; n_2 n_3, n_5 n_6) = S,\]
say. It will facilitate our argument to average over all 4-tuples \((n_2, n_3, n_5, n_6)\) in the relevant ranges, and not just those satisfying the coprimality condition (4.1). In view of the above remarks we clearly have

\[
S \ll \sum_{e \leq Q} \sum_{d \leq Q/e} \mu(d)^2 \sum_{n_2, n_3, n_5, n_6} \left( \frac{-n_2 n_3 n_5 n_6}{d} \right)
\ll KQ + \sum_{e \leq Q} S(e),
\]

(5.7)

where

\[ K = K_2 K_3 K_5 K_6 \]

and

\[
S(e) = \sum_{1 \neq d \leq Q/e} \mu(d)^2 \sum_{n_2, n_3, n_5, n_6} \left( \frac{-n_2 n_3 n_5 n_6}{d} \right).
\]

An immediate application of the author’s large sieve inequality for real character sums [7, Corollary 4] shows that

\[
S(e) \ll (KQ/e)^\varepsilon \{K(Q/e)^{1/2} + K^{1/2}(Q/e)\}
\]

(5.8)

for any fixed \(\varepsilon > 0\). If we use the Pólya-Vinogradov inequality, we find that

\[
S(e) = \sum_{1 \neq d \leq Q/e} \mu(d)^2 \sum_{n_2, n_3, n_5} \left( \frac{-n_2 n_3 n_5}{d} \right) \sum_{n_6} \left( \frac{n_6}{d} \right)
\ll K_2 K_3 K_5 \sum_{1 \neq d \leq Q/e} d^{1/2+\varepsilon}
\ll K_2 K_3 K_5 (Q/e)^{3/2+\varepsilon}.
\]

In the same way we find that

\[
S(e) \ll \frac{K_2 K_3 K_5 K_6}{K_i} (Q/e)^{3/2+\varepsilon}
\]

for any index \(i = 2, 3, 5, 6\). It therefore follows on taking \(K_i\) as the maximum of \(K_2, K_3, K_5\) and \(K_6\), that

\[
S(e) \ll K^{3/4} (Q/e)^{3/2+\varepsilon}.
\]

(5.9)

Alternatively, if \(N\) is not a square, we may use the Pólya-Vinogradov to
derive the bound
\[
\sum_{d \leq Q/e} \mu(d)^2 \left( \frac{N}{d} \right) = \sum_{d \leq Q/e, h | d} \mu(h) \left( \frac{N}{h^2} \right) \sum_{k \leq Q/eh^2} \left( \frac{N}{k} \right) \\
\ll \sum_{h \leq (Q/e)^{1/2}} N^{1/2} \log N \\
\ll (QN/e)^{1/2} \log N.
\] (5.10)

We can use this estimate to find that
\[
S(e) \ll K^{3/2+\epsilon}(Q/e)^{1/2},
\] (5.11)
since \(-n_2n_3n_5n_6\) is never a square. Comparing this bound with (5.8) and (5.9) we find that
\[
S(e) \ll (KQ/e)^{\epsilon} \min,
\]
where
\[
\min = \min\{K(Q/e)^{1/2} + K^{1/2}(Q/e)^3/2, K^{3/4}(Q/e)^{3/2}, K^{3/2}(Q/e)^{1/2}\} \\
\ll \min\{K(Q/e)^{1/2}, K^{3/4}(Q/e)^{3/2}, K^{3/2}(Q/e)^{1/2}\} \\
+ \min\{K^{1/2}(Q/e), K^{3/4}(Q/e)^{3/2}, K^{3/2}(Q/e)^{1/2}\} \\
\ll \{K(Q/e)^{1/2}\}^{3/5} \{K^{3/4}(Q/e)^{3/2}\}^{2/5} \\
+ \{K^{1/2}(Q/e)\}^{2/3} \{K^{3/2}(Q/e)^{1/2}\}^{1/3} \\
\ll (KQ/e)^{9/10} + (KQ/e)^{5/6} \\
\ll (KQ/e)^{9/10}.
\]

It follows that
\[
S(e) \ll (KQ/e)^{10/11},
\] (5.12)
say. Finally we insert this into (5.7) to deduce that
\[
S \ll KQ + K^{10/11}Q^{10/11} \sum_{e \leq Q} e^{-10/11} \ll KQ + K^{10/11}Q \ll KQ.
\] (5.13)

The above bound allows us to conclude from (5.6) that
\[
\sum_{Q/2 < q \leq Q} N_{3,a}(q) \ll (1 + K_1K_4/Q)KQ = K_2K_3K_5K_6Q + K_1K_2K_3K_4K_5K_6.
\]
On the other hand, (5.4) and (5.5) yield
\[ \sum_{Q/2 < q \leq Q} N_{3,\alpha}(q) \ll K_1 K_2 K_4 K_5 Q + K_1 K_2 K_3 K_4 K_5 K_6 \]
and
\[ \sum_{Q/2 < q \leq Q} N_{3,\alpha}(q) \ll K_1 K_3 K_6 Q + K_1 K_2 K_3 K_5 K_6. \]
Taking the minimum of these, and assuming that \( Q \ll K_1 (K_2 K_3)^{1/2} \), we obtain an estimate
\[ \sum_{Q/2 < q \leq Q} N_{3,\alpha}(q) \ll Q \min\{K_2 K_3 K_5 K_6, K_1 K_2 K_4 K_5, K_1 K_3 K_4 K_6\} \]
\[ \quad + K_1 K_2 K_3 K_4 K_5 K_6 \]
\[ \ll Q \{K_2 K_3 K_5 K_6\}^{1/2} \{K_1 K_2 K_4 K_5\}^{1/4} \{K_1 K_3 K_4 K_6\}^{1/4} + K_1 K_2 K_3 K_4 K_5 K_6 \]
\[ \ll Q K_1^{1/2} K_2^{3/4} K_3^{3/4} K_4^{1/2} K_5^{3/4} K_6^{3/4} + K_1 K_2 K_3 K_4 K_5 K_6 \]
\[ \ll K_1 (K_2 K_3)^{1/2} K_1^{1/2} K_2^{3/4} K_3^{3/4} K_4^{1/2} K_5^{3/4} K_6^{3/4} + K_1 K_2 K_3 K_4 K_5 K_6 \]
\[ \ll K_1^{3/2} K_2^{5/4} K_3^{5/4} K_4^{1/2} K_5^{3/4} K_6^{3/4} \]
(5.14)
in view of our assumption (5.1). This gives a satisfactory bound for (5.2).

We may handle \( N_{3,\beta}(q) \) in a precisely analogous way, thereby completing our treatment of (4.5).

The equation (4.6) introduces a couple of further difficulties. Firstly, the bound (5.10) is only valid when \( N \) is not a square. Previously we took \( N = -n_2 n_3 n_5 n_6 \), which can never be a square. However, if \( N = n_2 n_3 n_5 n_6 \), we must allow for the case in which \( n_2 n_3 n_5 n_6 \) is a square. The effect of this is to change the estimate (5.11) into
\[ S(e) \ll K^{3/2+\varepsilon} (Q/e)^{1/2} + K^{1/2+\varepsilon} (Q/e). \]
The additional term contributes \( O(K^{3/2}Q/e) \), say, to (5.12), whence (5.13) becomes
\[ S \ll KQ \{1 + \frac{\log(K_1 K_2 K_3 K_4 K_5 K_6)}{K_1^{1/3}}\} \ll KQ \{1 + \frac{\log K_1 K_4}{K_1^{1/3}}\}. \]
(5.15)
This introduces the extra factor we see in (4.7).

The second difficulty is that if
\[ K_4^2 K_5 K_6 \gg K_2^2 K_2 K_3 \gg K_4^2 K_5 K_6 \]
(5.16)
we may no longer have the lower bound \( q \gg K_8 \) to use in the estimate

\[
N_4 \ll \sum_{q \ll K_8} N_{4,b}(q). \tag{5.17}
\]

We therefore assume now that (5.16) holds, and investigate the quantity \( N_{4,b}(q) \) further. Since

\[
n_1^2 n_2 n_3 - n_4^2 n_5 n_6 = n_7 q
\]

in this context, with \( n_7 \leq 2K_7 \), we can apply Lemma 6 with

\[
v = (n_1^2 n_2, -n_4^2 n_5, q), \quad x = (n_3, n_6, -n_7),
\]

and with \( H_1 = 2K_3, H_2 = 2K_6 \) and \( H_3 = 2K_7 \). Thus there are

\[
\ll 1 + \frac{H_1 H_2 H_3}{\max H_{|v_i|}} \ll 1 + \frac{K_3 K_6 K_7}{K_1^2 K_2 K_3}
\]
solutions \( x \). Summing over \( n_1, n_2, n_4, n_5 \) yields

\[
N_{4,b}(q) \ll K_1 K_2 K_4 K_5 (1 + \frac{K_6 K_7}{K_1^2 K_2}). \tag{5.18}
\]

Similarly one can show that

\[
N_{4,b}(q) \ll K_1 K_3 K_4 K_6 (1 + \frac{K_5 K_7}{K_1^2 K_2}). \tag{5.19}
\]

As before we need also an estimate in which we treat \( n_1 \) and \( n_4 \) as variables. By the argument used before we can produce \( \rho(q; n_2 n_3, -n_5 n_6) \) congruence conditions \( n_1 \equiv tn_4 \pmod{q} \). Each of these defines a lattice \( \Lambda \subseteq \mathbb{Z}^2 \) of points \( (n_1, n_4) \). Moreover we will have \( \det(\Lambda) = q \). The points \( (n_1, n_4) \) satisfy \( n_1 \leq 2K_1 \) and \( n_4 \leq 2K_4 \). Additionally we have

\[
|1 - \frac{n_4^2 n_5 n_6}{n_1^2 n_2 n_3}| \leq \frac{2K_7 q}{K_2^2 K_3 K_2},
\]

whence

\[
|1 - \frac{n_4 \sqrt{n_5 n_6}}{n_1 \sqrt{n_2 n_3}}| \leq \frac{2K_7 q}{K_1^2 K_2 K_3}.
\]

In view of our assumption (5.16) this may be written as

\[
|n_1 - \alpha n_4| \leq C \frac{K_7 q}{K_1 K_2 K_3},
\]
for some $\alpha = \alpha(n_2, n_3, n_4, n_5)$ and some absolute constant $C$. The above inequality, along with the condition $|n_4| \leq 2K_4$, defines a parallelogram of area 

$$8C \frac{K_4K_7q}{K_1K_2K_3} = A,$$

say, centred on the origin. This parallelogram may be mapped to a square $S$, centred on the origin, and having the same area $A$, by a projective mapping $M$ say, of determinant $1$. Enclose $S$ by a disc $D$ of area $\pi A/2$, and consider the ellipse $E = M^{-1}D$. This also has area $\pi A/2$. Moreover it contains the original parallelogram, and is centred at the origin. We are therefore in a position to apply the following result, due to the author \cite[Lemma 2]{5}.

**Lemma 7** Let $\Lambda \subseteq \mathbb{R}^2$ be a lattice, and let $E$ be an ellipse, centred on the origin, together with its interior. Then

$$\#(\Lambda \cap E) \leq 4 \left(1 + \frac{\text{meas}(E)}{\det(\Lambda)}\right).$$

This lemma allows us to conclude that there are

$$\ll 1 + \frac{K_4K_7}{K_1K_2K_3}$$

pairs $(n_1, n_4)$ for each set of values $t, n_2, n_3, n_5, n_6, q$. We may now proceed as before, using (5.15) to deduce that

$$\sum_{q \ll K_8} N_{4,b}(q) \ll \tau K_2K_3K_5K_6(1 + \frac{K_4K_7}{K_1K_2K_3})K_8,$$  \hspace{1cm} (5.20)$$

where we have set

$$\tau = 1 + \frac{\log K_1K_4}{K^{1/3}}$$

for convenience. We now deduce from (5.17), (5.18), (5.19) and (5.20) that

$$N_4 \ll \sum_{q \ll K_8} N_{4,b}(q) \ll \tau \min(K_1K_2K_4K_5K_8, K_1K_3K_4K_6K_8, K_2K_3K_5K_6K_8) \frac{K_7K_8}{K_1^2K_2K_3}$$

$$\ll \tau K_1^1/2K_2^{3/4}K_3^{3/4}K_4^{1/2}K_5^{3/4}K_6^{3/4}K_8 + \tau K_1K_2K_3K_4K_5K_6$$

as in the proof of (5.14). Since we only need (5.17) for the case $K_8 \ll K_1(K_2K_3)^{1/2}$, the required bound (4.7) follows.
6 Proof of the Upper Bound

We shall specify dyadic ranges
\[ X_i < |x_i| \leq 2X_i \]
for the original variables \(x_1, x_2, x_3, x_4\), and
\[ Z_{ij} < z_{ij} \leq 2Z_{ij} \]
for the variables introduced in §2, and we write
\[ \mathcal{N}(X_1, \ldots, X_4; Z_{12}, \ldots, Z_{34}) = \mathcal{N} \]
for the corresponding contribution to \(N(B)\). We obviously have
\[ X_i \ll B, \quad (6.1) \]
and the relation (2.13) implies that
\[ Z_{ij}Z_{ik}Z_{il} \ll X_i. \quad (6.2) \]

We shall find it convenient to re-order the indices so that
\[ X_1 \geq X_2 \geq X_3 \geq X_4. \quad (6.3) \]
Since any solution will have
\[ \frac{1}{2X_4} \leq \frac{1}{|x_4|} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} \leq \left| \frac{1}{x_1} \right| + \left| \frac{1}{x_2} \right| + \left| \frac{1}{x_3} \right| \leq \frac{1}{X_1} + \frac{1}{X_2} + \frac{1}{X_3} \leq \frac{3}{X_3} \]
we deduce that \( \mathcal{N} = 0 \) unless
\[ 6X_4 \geq X_3 \geq X_4, \quad (6.4) \]
as we henceforth assume. Moreover we have
\[ y_i^3 = \frac{x_jx_kx_lB_i}{(x_iA_i)^2} = \frac{x_1x_2x_3x_4P}{(x_iA_i)^3}, \]
with the notations (2.7), (2.12) and (3.3), so that
\[ Y_i \ll |y_i| \ll Y_i \]
where
\[ Y_i = \frac{(X_F)^{1/3}}{X_iZ_{jk}Z_{jl}Z_{kl}}, \quad (6.5) \]
with
\[ X = X_1X_2X_3X_4 \quad \text{and} \quad F = Z_{12}Z_{13}Z_{14}Z_{23}Z_{24}Z_{34}. \]

We begin by applying Lemma 3 to the equation (2.16), to show that there are
\[ \ll Z_{ik}Z_{il}Z_{jk}Z_{jl}Y_iY_j \]
possible sets of values for \( z_{ik}, z_{il}, z_{jk}, z_{jl}, y_i, y_j, v_{ij} \). For each set of values we proceed to examine (2.17), which we write in the form \( \mathbf{v} \cdot \mathbf{x} = 0 \) with
\[ \mathbf{v} = (v_{ij}, -z_{il}^2y_j, z_{jk}^2y_i) \quad \text{and} \quad \mathbf{x} = (v_{ik}, y_k, y_l). \]

In view of (2.2), (2.10) and (2.15) both \( \mathbf{v} \) and \( \mathbf{x} \) will be primitive. Moreover, the equation
\[ z_{ij}z_{il}y_k + z_{jk}z_{kl}y_i = z_{ik}v_{ik}, \]
which is an example of (2.16), yields
\[ z_{ik}v_{ik} \ll \max\{Z_{ij}Z_{il}Y_k, Z_{jk}Z_{kl}Y_i\} = \frac{(XF)^{1/3}}{Z_{jl}} \max\left\{ \frac{1}{X_k}, \frac{1}{X_i} \right\}. \]

Thus Lemma 6 may be applied with
\[ H_1 = c \left( \frac{(XF)^{1/3}}{Z_{ik}Z_{jl}} \max\left\{ \frac{1}{X_k}, \frac{1}{X_i} \right\} \right), \quad H_2 = cY_k, \quad H_3 = cY_l, \]
for a suitable constant \( c \). Since the remaining value \( z_{kl} \) is determined by (2.14), there are
\[ \mathcal{N} \ll Z_{ik}Z_{il}Z_{jk}Z_{jl}Y_iY_j \{1 + \frac{H_1H_2H_3}{\max(H_2V_2, H_3V_3)}\} \]
solutions to (2.14) in total, where
\[ V_2 = Z_{il}^2Y_j \quad \text{and} \quad V_3 = Z_{jk}^2Y_i. \]

We may then calculate, using (6.3), that the above bound is
\[ \ll \frac{(XF)^{2/3}}{Z_{kl}^2X_iX_j} + \max\left( \frac{1}{X_k}, \frac{1}{X_l} \right) \min(X_iX_l, X_jX_k). \]

Since this estimate is valid for any choice of \( i, j, k, l \) we may interchange \( i \) with \( j \), and \( k \) with \( l \), to deduce that
\[ \mathcal{N} \ll \frac{(XF)^{2/3}}{Z_{kl}^2X_iX_j} + \max\left( \frac{1}{X_l}, \frac{1}{X_j} \right) \min(X_jX_k, X_iX_l). \]
We now observe that our assumption (6.3) implies that \( \min(X_iX_l, X_jX_k) \leq X_2X_3 \) and that either
\[
\max\left( \frac{1}{X_k}, \frac{1}{X_i} \right) \leq \frac{1}{X_2}
\]
or
\[
\max\left( \frac{1}{X_l}, \frac{1}{X_j} \right) \leq \frac{1}{X_2}.
\]
It follows that
\[
\mathcal{N}(X_1, \ldots, X_4; Z_{12}, \ldots, Z_{34}) \ll \frac{(XF)^{2/3}}{Z_{kl}^2X_iX_j} + X_3.
\]
We apply this with \( i = 1, j = 4, k = 2, l = 3 \), so that
\[
\mathcal{N}(X_1, \ldots, X_4; Z_{12}, \ldots, Z_{34}) \ll \frac{(XF)^{2/3}}{Z_{23}^2X_1X_4} + X_3,
\]
and again with \( i = 2, j = 3, k = 1, l = 4 \), so that
\[
\mathcal{N}(X_1, \ldots, X_4; Z_{12}, \ldots, Z_{34}) \ll \frac{(XF)^{2/3}}{Z_{14}^2X_2X_3} + X_3.
\]
Since
\[
\min(A, B) \leq (AB)^{1/2}
\]
this yields
\[
\mathcal{N}(X_1, \ldots, X_4; Z_{12}, \ldots, Z_{34}) \ll \min\left\{ \frac{(XF)^{2/3}}{Z_{23}^2X_1X_4}, \frac{(XF)^{2/3}}{Z_{14}^2X_2X_3} \right\} + X_3
\]
\[
\ll \frac{(XF)^{2/3}}{Z_{14}Z_{23}X^{1/2}} + X_3
\]
\[
= \frac{X^{1/6}F^{2/3}}{Z_{14}Z_{23}} + X_3.
\] (6.7)

For an alternative estimate we begin by applying Lemma 4 to the equation (2.17), to show that the number of possible sets of values for \( v_{ij}, v_{ik}, z_{il}, z_{jk}, y_i, y_j, y_k, y_l \) is
\[
\ll \left\{ 1 + \frac{\log X}{(Y_iY_jY_kY_l)^{1/3}} \right\} Z_{il}Z_{jk}Y_iY_jY_kY_l \max\left\{ \left( \frac{Y_i^2Y_jY_k}{Z_{jk}^2Y_iY_l} \right)^{1/4}, \left( \frac{Z_{il}^2Y_jY_k}{Z_{il}^2Y_jY_k} \right)^{1/4} \right\}
\]
\[
\ll \sigma Z_{il}Z_{jk}X^{1/3}F^{-2/3} \max\left\{ \left( \frac{X_iX_l}{X_jX_k} \right)^{1/4}, \left( \frac{X_jX_k}{X_iX_l} \right)^{1/4} \right\},
\]
where
\[ \sigma = 1 + \log X \left( X F^{-2} \right)^{1/9} \].

For each such set of values we write (2.16) in the form \( \mathbf{v} \cdot \mathbf{x} = 0 \) with
\[ \mathbf{v} = (z_{il}y_j, z_{jk}y_i, -v_{ij}) \] and \( \mathbf{x} = (z_{ik}, z_{jl}, z_{ij}). \)

In view of (2.2), (2.10) and (2.15) both \( \mathbf{v} \) and \( \mathbf{x} \) will be primitive. We can therefore apply Lemma 6 with
\[ H_1 = 2Z_{ik}, \quad H_2 = 2Z_{jl}, \quad H_3 = 2Z_{ij}, \]
to deduce that there are
\[ \ll 1 + \frac{H_1 H_2 H_3}{\max(H_1 |v_1|, H_2 |v_2|)} \ll 1 + \frac{Z_{ik}Z_{jl}Z_{ij}}{\max(Z_{ik}Z_{il}Z_{jl}, Z_{jl}Z_{jk}Y_1)} \ll 1 + \frac{Z_{ik}Z_{il}Z_{ij}Z_{kl}}{(XF)^{1/3}} \min(X_i, X_j) \]
corresponding solutions \( z_{ik}, z_{jl}, z_{ij} \).

We apply these estimates with \( i = 1, j = 3, k = 2, l = 4 \), so that
\[ \max\left\{ \left( \frac{X_iX_l}{X_jX_k} \right)^{1/4}, \left( \frac{X_jX_k}{X_iX_l} \right)^{1/4} \right\} \ll (X_1/X_2)^{1/4}, \]
and \( \min(X_i, X_j) = X_3 \), in view of (5.3) and (5.4). Since the final remaining value \( z_{kl} = z_{24} \) is now determined by (2.14) it follows that
\[ \mathcal{N} \ll \sigma Z_{14}Z_{23}X^{1/3}F^{-2/3}(X_1/X_2)^{1/4} \left( 1 + \frac{Z_{12}Z_{34}Z_{13}Z_{24}}{(XF)^{1/3}}X_3 \right) = \sigma Z_{14}Z_{23}X^{1/3}F^{-2/3}(X_1/X_2)^{1/4} + (X_1/X_2)^{1/4}X_3. \]
We now combine this with (5.7), using the inequality (5.3) again, to deduce that
\[ \mathcal{N} \ll \sigma \min\left\{ \frac{X^{1/6}F^{2/3}}{Z_{14}Z_{23}}, Z_{14}Z_{23}X^{1/3}F^{-2/3}(X_1/X_2)^{1/4} \right\} + (X_1/X_2)^{1/4}X_3 \ll \sigma X^{1/4}(X_1/X_2)^{1/8} + (X_1/X_2)^{1/4}X_3. \]

We are finally in a position to sum over the various dyadic ranges for the \( X_i \) and \( Z_{ij} \), subject to (6.1) and (6.2). We begin by considering the summation over \( Z_{ij} \). The values of \( Z_{ij} \) are powers of 2, subject to the constraints
These imply that $F^2 \ll X \ll B^4$. Thus there are $O((\log B)^6)$ possible sets of values for the various $Z_{ij}$. Moreover there are $O((\log B)^5)$ sets of values for each given value of $F$. Since $F$ runs over powers of 2, subject to $F \ll X^{1/2}$ we conclude that

$$
\sum_{Z_{ij}} F^{2/9} \ll X^{1/9}(\log B)^5.
$$

We therefore deduce that

$$
\sum_{Z_{ij}} \sigma \ll (\log B)^6. \tag{6.8}
$$

It remains to consider the summation over values of the $X_i$, which also run over powers of 2. Here we observe that

$$
X^{1/4}(X_1/X_2)^{1/8} = X_1^{3/8}X_2^{1/8}X_3^{1/4}X_4^{1/4},
$$

and that (5.1) yields

$$
\sum_{X_i} X_i^e \ll B^e
$$

if $e > 0$. It follows that

$$
\sum_{X_1,X_2,X_3,X_4} X_1^{3/8}X_2^{1/8}X_3^{1/4}X_4^{1/4} \ll B.
$$

Similarly we have

$$
(X_1/X_2)^{1/4}X_3 \ll X_1^{1/4}X_2^{1/4}X_3^{1/4}X_4^{1/4},
$$

by (5.3) and (6.4), so that

$$
\sum_{X_1,X_2,X_3,X_4} (X_1/X_2)^{1/4}X_3 \ll B.
$$

Combining this with (6.8) completes the proof of the upper bound in our theorem.

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