Derived Representation Schemes and Noncommutative Geometry

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Abstract. Some 15 years ago M. Kontsevich and A. Rosenberg [KR] proposed a heuristic principle according to which the family of schemes \{\text{Rep}_n(A)\} parametrizing the finite-dimensional representations of a noncommutative algebra \( A \) should be thought of as a substitute or ‘approximation’ for ‘Spec(\( A \))’. The idea is that every property or noncommutative geometric structure on \( A \) should induce a corresponding geometric property or structure on \( \text{Rep}_n(A) \) for all \( n \). In recent years, many interesting structures in noncommutative geometry have originated from this idea. In practice, however, if an associative algebra \( A \) possesses a property of geometric nature (e.g., \( A \) is a NC complete intersection, Cohen-Macaulay, Calabi-Yau, etc.), it often happens that, for some \( n \), the scheme \( \text{Rep}_n(A) \) fails to have the corresponding property in the usual algebro-geometric sense. The reason for this seems to be that the representation functor \( \text{Rep}_n \) is not ‘exact’ and should be replaced by its derived functor \( D\text{Rep}_n \) (in the sense of non-abelian homological algebra). The higher homology of \( D\text{Rep}_n(A) \), which we call representation homology, obstructs \( \text{Rep}_n(A) \) from having the desired property and thus measures the failure of the Kontsevich-Rosenberg ‘approximation.’ In this paper, which is mostly a survey, we prove several results confirming this intuition. We also give a number of examples and explicit computations illustrating the theory developed in [BKR] and [BR].

1. Introduction

If \( k \) is a field, the set of all representations of an associative \( k \)-algebra \( A \) in a finite-dimensional vector space \( V \) can be given the structure of an affine \( k \)-scheme, called the representation scheme \( \text{Rep}_V(A) \). The group \( \text{GL}_k(V) \) acts naturally on \( \text{Rep}_V(A) \), with orbits corresponding to the isomorphism classes of representations. If \( k \) is algebraically closed and \( A \) is finitely generated, the equivariant geometry of \( \text{Rep}_V(A) \) is closely related to the representation theory of \( A \). This relation has been extensively studied (especially in the case of finite-dimensional algebras) since the late 70’s, and the schemes \( \text{Rep}_V(A) \) have become a standard tool in representation theory of algebras (see, for example, [Ga], [Bo], [Ge] and references therein).

More recently, representation schemes have come to play an important role in noncommutative geometry. Let us recall that in classical (commutative) algebraic geometry, there is a natural way to associate to a commutative algebra \( A \) a geometric object — the Grothendieck prime spectrum \( \text{Spec}(A) \). This defines a contravariant functor from commutative algebras to affine schemes, which is an (anti)equivalence of categories. Attempts to extend this functor to the category of all associative algebras have been largely unsuccessful. In [KR] M. Kontsevich and A. Rosenberg proposed a heuristic principle according to which the family of schemes \( \{\text{Rep}_V(A)\} \) for a given algebra \( A \) should be thought of as a substitute (or “approximation”) for “Spec(\( A \))”. The idea is that every property or noncommutative geometric structure on \( A \) should naturally induce a corresponding geometric property or structure on \( \text{Rep}_V(A) \) for...
all $V$. This viewpoint provides a litmus test for proposed definitions of noncommutative analogues of classical geometric notions. In recent years, many interesting structures in noncommutative geometry have originated from this idea: NC smooth spaces [CQ, KR, LeB], formal structures and noncommutative thickenings of schemes [Ka], LBW, noncommutative symplectic and bisymplectic geometry [G2, LeB1, CEG, Bo, BC], double Poisson brackets and noncommutative quasi-Hamiltonian spaces [VdB, VdB1, CB, MT]. In practice, however, the Kontsevich-Rosenberg principle works well only when $A$ is a (formally) smooth algebra, since in that case $\text{Rep}_V(A)$ are smooth schemes for all $V$. To extend this principle to arbitrary algebras we proposed in [BKR] to replace $\text{Rep}_V(A)$ by a DG scheme $\text{DRep}_V(A)$, which is obtained by deriving the classical representation functor in the sense of Quillen’s homotopy category of DG algebras. When applied to $\text{Rep}_V(A)$ to $\text{DRep}_V(A)$ amounts, in a sense, to desingularizing $\text{Rep}_V(A)$, so one should expect that $\text{DRep}_V(A)$ will play a role similar to the role of $\text{Rep}_V(A)$ in the geometry of smooth algebras.

To explain this idea in more detail let us recall that the representation scheme $\text{Rep}_V(A)$ is defined as a functor on the category of commutative $k$-algebras:

$$(1) \quad \text{Rep}_V(A) : \text{Comm Alg}_k \to \text{Sets}, \quad B \mapsto \text{Hom}_{\text{alg}_k}(A, \text{End}_V \otimes_k B).$$

It is well known that (1) is (co)representable, and we denote the corresponding commutative algebra by $A_V = k[\text{Rep}_V(A)]$. Now, varying $A$ (while keeping $V$ fixed) we can regard (1) as a functor on the category $\text{Alg}_k$ of associative algebras; more precisely, we define the representation functor in $V$ by

$$(2) \quad (-)_V : \text{Alg}_k \to \text{Comm Alg}_k, \quad A \mapsto k[\text{Rep}_V(A)].$$

The representation functor can be extended to the category of differential graded (DG) algebras, $\text{DGA}_k$, which has a natural model structure in the sense of [Q1, Q2]. It turns out that $(-)_V$ defines a left Quillen functor on $\text{DGA}_k$, and hence it has a total derived functor $\mathcal{L}(-)_V : \text{Ho}(\text{DGA}_k) \to \text{Ho}(\text{CDGA}_k)$ on the homotopy category of DG algebras. When applied to $A$, this derived functor is represented by a commutative DG algebra $\text{DRep}_V(A)$. The homology of $\text{DRep}_V(A)$ depends only on $A$ and $V$, with $H_0[\text{DRep}_V(A)]$ being isomorphic to $k[\text{Rep}_V(A)]$. Following [BKR], we will refer to $H_\bullet[\text{DRep}_V(A)]$ as the representation homology of $A$ and denote it by $H_\bullet(A, V)$. The action of $\text{GL}(V)$ on $k[\text{Rep}_V(A)]$ extends naturally to $\text{DRep}_V(A)$, and we have an isomorphism of graded algebras

$$H_\bullet[\text{DRep}_V(A)^{\text{GL}(V)}] \cong H_\bullet(A, V)^{\text{GL}(V)}.$$  

Now, let $HC(A)$ denote the cyclic homology of the algebra $A$. There is a canonical trace map

$$(3) \quad \text{Tr}_V(A)_0 : HC_0(A) \to k[\text{Rep}_V(A)]^{\text{GL}(V)}$$

defined by taking characters of representations. One of the key results of [BKR] is the construction of the higher trace maps

$$(4) \quad \text{Tr}_V(A)_n : HC_n(A) \to H_n(A, V)^{\text{GL}(V)}, \quad \forall n \geq 0,$$

extending (3) to the full cyclic homology. It is natural to think of (3) as derived (or higher) characters of finite-dimensional representations of $A$. In accordance with Kontsevich-Rosenberg principle, various standard structures on cyclic and Hochschild homology (e.g., Bott periodicity, the Connes differential, the Gerstenhaber bracket, etc.) induce via (3) new interesting structures on representation homology. We illustrate this in Section 5.3, where we construct an analogue of Connes’ periodicity exact sequence for $H_\bullet(A, V)$. We should mention that the idea of deriving representation schemes is certainly not new: the first construction of this kind was proposed in [CK] (cf. Section 5.3 below), and there are nowadays several different approaches (see, e.g., [Ka, BCHR, TV]). However, the trace maps (3) seem to be new, and the relation to cyclic homology has not appeared in the earlier literature.

The aim of this paper is threefold. First, we give a detailed overview of [BKR] and [BR] leaving out most technical proofs but adding motivation and necessary background on homotopical algebra and model categories. Second, we prove several new results on derived representation schemes refining and extending [BKR]. Third, we give a number of explicit examples and computations illustrating the theory.
We would like to conclude this introduction with a general remark that clarifies the meaning of representation homology from the point of view of noncommutative geometry. If an associative algebra $A$ possesses a property of geometric nature (for example, $A$ is a NC complete intersection, Cohen-Macaulay, Calabi-Yau, etc.), it may happen that, for some $V$, the scheme $\text{Rep}_V(A)$ fails to have a corresponding property in the usual algebro-geometric sense. The reason for this seems to be that the representation functor $\text{Rep}_V$ is not exact, and it is precisely the higher homology $H_n(A,V)$, $n \geq 1$, that obstructs $\text{Rep}_V(A)$ from having the desired property. In other words, representation homology measures the failure of the Kontsevich-Rosenberg “approximation.” In Section 6, we prove two results confirming this.

In Section 4, after reviewing the Feigin-Tsygan construction of relative cyclic homology $HC_*$ (see Theorem 21), we define canonical trace maps $\text{Tr}_V(S(A)_*) : HC_{*-1}(S(A)) \to H_*(S\setminus A,V)$ relating the cyclic homology of an $S$-algebra $A$ to its representation homology. In particular, for $S = k$, we get the derived character maps (4). The main result of this section, Theorem 15, describes an explicit chain map $T : CC(A) \to D\text{Rep}_V(A)$ that induces on homology the trace maps (4). We also draw reader’s attention to Theorem 17 which summarizes the main results of our forthcoming paper [BR]. In Section 5, we define and construct the abelianization of the representation functor. The main result of this section, Theorem 18, shows that the abelianized representation functor is precisely (the DG extension of) Van den Bergh’s functor introduced in [VdB]. This is a new result that has not appeared in [BKR]. As a consequence, we give a simpler and more conceptual proof of Theorem 5 which was one of the main results of [BKR]. Theorem 18 also leads to an interesting spectral sequence that clarifies the relation between representation homology and Andr`e-Quillen homology (see Section 5.5). Finally, in Section 6 we give some examples. One notable result is Theorem 27 which says that

$$H_n(k[x,y], V) = 0 , \forall n > \dim V ,$$

where $k[x,y]$ is the polynomial algebra in two variables. We originally conjectured studying the homology of $k[x,y]$ with the help of Macaulay2. It came as a surprise that this vanishing result is a simple consequence of a known theorem of Knutson [Kn].

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Notation and Conventions

Throughout this paper, $k$ denotes a base field of characteristic zero. An unadorned tensor product $\otimes$ stands for the tensor product $\otimes_k$ over $k$. An algebra means an associative $k$-algebra with 1; the category of such algebras is denoted $\text{Alg}_k$. Unless stated otherwise, all differential graded (DG) objects (complexes, algebras, modules, ...) are equipped with differentials of degree $-1$. The Koszul sign rule is systematically used. For a graded vector space $V$, we denote by $\Lambda(V)$ its graded symmetric algebra over $k$: i.e., $\Lambda(V) := \text{Sym}_k(V_{ev}) \otimes_k \Lambda_k(V_{odd})$, where $V_{ev}$ and $V_{odd}$ are the even and the odd components of $V$.

2. Model categories

A model category is a category with a certain structure that allows one to do non-abelian homological algebra (see [Q1], [Q2]). Fundamental examples are the categories of topological spaces and simplicial sets. However, the theory also applies to algebraic categories, including chain complexes, differential
graded algebras and differential graded modules. In this section, we briefly recall the definition of model categories and review the results needed for the present paper. Most of these results are well known; apart from the original works of Quillen, proofs can be found in [Hir] and [Ho]. For an excellent introduction we recommend the Dwyer-Spalinski article [DS]; for examples and applications of model categories in algebraic topology see [GS] and [He]; for spectacular recent applications in algebra we refer to the survey papers [K] and [S].

2.1. Axioms. A (closed) model category is a category $C$ equipped with three distinguished classes of morphisms: weak equivalences ($\sim$), fibrations ($\to$) and cofibrations ($\hookrightarrow$). Each of these classes is closed under composition and contains all identity maps. Morphisms that are both fibrations and weak equivalences are called acyclic fibrations and denoted $\sim\to$. Morphisms that are both cofibrations and weak equivalences are called acyclic cofibrations and denoted $\sim\hookrightarrow$. The following five axioms are required.

MC1. $C$ has all finite limits and colimits. In particular, $C$ has initial and terminal objects, which we denote ‘$e$’ and ‘$*$’, respectively.

MC2. Two-out-of-three axiom: If $f : X \to Y$ and $g : Y \to Z$ are maps in $C$ and any two of the three maps $f$, $g$, and $gf$ are weak equivalences, then so is the third.

MC3. Retract axiom: Each of the three distinguished classes of maps is closed under taking retracts; by definition, $f$ is a retract of $g$ if there is a commutative diagram

$$
\begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow & \downarrow & \downarrow \\
Y & \xrightarrow{g} & Y'
\end{array}
$$

such that the composition of the top and bottom rows is the identity.

MC4. Lifting axiom: Suppose that

$$
\begin{array}{ccc}
A & \xrightarrow{X} & X \\
\downarrow & & \downarrow \\
B & \xrightarrow{Y} & Y
\end{array}
$$

is a square in which $A \to B$ is a cofibration and $X \to Y$ is a fibration. Then, if either of the two vertical maps is a weak equivalence, there is a lifting $B \to X$ making the diagram commute. We say that $A \to B$ has the left-lifting property with respect to $X \to Y$, and $X \to Y$ has a right-lifting property with respect to $A \to B$.

MC5. Factorization axiom: Any map $A \to X$ in $C$ may be factored in two ways:

(i) $A \xrightarrow{\sim} B \xrightarrow{\sim} X$,

(ii) $A \xrightarrow{\sim} Y \xrightarrow{\sim} X$.

An object $A \in \text{Ob}(C)$ is called fibrant if the unique morphism $A \to *$ is a fibration in $C$. Similarly, $A \in \text{Ob}(C)$ is cofibrant if the unique morphism $e \to A$ is a cofibration in $C$. A model category $C$ is called fibrant (resp., cofibrant) if all objects of $C$ are fibrant (resp., cofibrant).

Remark. The notion of a model category was introduced by Quillen in [Q1]. He called such a category closed whenever any two of the three distinguished classes of morphisms determined the third. In [Q2], Quillen characterized the closed model categories by the five axioms stated above. Nowadays, it seems generally agreed to refer to a closed model category just as a model category. Also, in the current literature (see, e.g., [Hir] and [Ho]), the first and the last axioms in Quillen’s list are often stated in the stronger form: in MC1, one usually requires the existence of small (not only finite) limits and colimits, while MC5 assumes the existence of functorial factorizations.

Example. Let $A$ be an algebra, and let $\text{Com}^\oplus(A)$ denote the category of complexes of $A$-modules that have zero terms in negative degrees. This category has a standard (projective) model structure, where the weak equivalences are the quasi-isomorphisms, the fibrations are the maps that are surjective in all positive degrees and the cofibrations are the monomorphisms whose cokernels are complexes with
2.2. Natural constructions. There are natural ways to build a new model category from a given one:

2.2.1. The axioms of a model category are self-dual: if \( \mathcal{C} \) is a model category, then so is its opposite \( \mathcal{C}^{opp} \). The classes of weak equivalences in \( \mathcal{C} \) and \( \mathcal{C}^{opp} \) are the same, while the classes of fibrations and cofibrations are interchanged.

2.2.2. If \( S \in \text{Ob}(\mathcal{C}) \) is a fixed object in a model category \( \mathcal{C} \), then the category \( \mathcal{C}_S \) of arrows \( \{S \to A\} \) starting at \( S \) has a natural model structure, with a morphism \( f : A \to B \) being in a distinguished class in \( \mathcal{C}_S \) if and only if \( f \) is in the corresponding class in \( \mathcal{C} \). Dually, there is a similar model structure on the category of arrows \( \{A \to S\} \) with target at \( S \).

2.2.3. The category \( \text{Mor}(\mathcal{C}) \) of all morphisms in a model category \( \mathcal{C} \) has a natural model structure, in which a morphism \( (\alpha, \beta) : f \to f' \) given by the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
\downarrow{f} & & \downarrow{f'} \\
B & \xrightarrow{\beta} & B'
\end{array}
\]

is a weak equivalence (resp., a fibration) iff \( \alpha \) and \( \beta \) are weak equivalences (resp., fibrations) in \( \mathcal{C} \). The morphism \( (\alpha, \beta) \) is a cofibration in \( \text{Mor}(\mathcal{C}) \) iff \( \alpha \) is a cofibration and also the induced morphism \( B \amalg_A A' \to B' \) is cofibration in \( \mathcal{C} \) (cf. [11]).

2.2.4. Let \( \mathcal{D} := \{a \leftarrow b \to c\} \) be the category with three objects \( \{a, b, c\} \) and the two indicated non-identity morphisms. Given a category \( \mathcal{C} \), let \( \mathcal{C}^{\mathcal{D}} \) denotes the category of functors \( \mathcal{D} \to \mathcal{C} \). An object in \( \mathcal{C}^{\mathcal{D}} \) is pushout data in \( \mathcal{C} \):

\[
X(a) \leftarrow X(b) \to X(c),
\]

and a morphism \( \varphi : X \to Y \) is a commutative diagram

\[
\begin{array}{ccc}
X(a) & \xleftarrow{\varphi_a} & X(b) & \xrightarrow{\varphi_b} & X(c) \\
\downarrow{\varphi} & & \downarrow{\varphi} & & \downarrow{\varphi} \\
Y(a) & \xleftarrow{} & Y(b) & \xrightarrow{} & Y(c)
\end{array}
\]

If \( \mathcal{C} \) is a model category, then there is a (unique) model structure on \( \mathcal{C}^{\mathcal{D}} \), where \( \varphi \) is a weak equivalence (resp., fibration) if \( \varphi_a, \varphi_b, \varphi_c \) are weak equivalences (resp., fibrations) in \( \mathcal{C} \). The cofibrations in \( \mathcal{C}^{\mathcal{D}} \) are described as the morphisms \( \varphi = (\varphi_a, \varphi_b, \varphi_c) \), with \( \varphi_b \) being a cofibration and also the two induced maps \( X(a) \amalg_{X(b)} Y(b) \to Y(a) \), \( X(c) \amalg_{X(b)} Y(b) \to Y(c) \) being cofibrations in \( \mathcal{C} \). Dually, there is a (unique) model structure on the category of pullback data \( \mathcal{C}^{\mathcal{D}} \), where \( \mathcal{D} := \{a \to b \leftarrow c\} \).

2.3. Homotopy category. In an arbitrary model category, there are two different ways to define a homotopy equivalence relation. For simplicity of exposition, we will assume that \( \mathcal{C} \) is a fibrant model category, in which case we can use only one definition (‘left’ homotopy) based on the cylinder objects.

If \( A \in \text{Ob}(\mathcal{C}) \), a cylinder on \( A \) is an object \( \text{Cy}l(A) \in \text{Ob}(\mathcal{C}) \) given together with a diagram

\[
A \amalg A \xrightarrow{i} \text{Cy}l(A) \xrightarrow{\sim} A,
\]

factoring the natural map \( (\text{id}, \text{id}) : A \amalg A \to A \). By MC5(ii), such an object exists for all \( A \) and comes together with two morphisms \( i_0 : A \to \text{Cy}l(A) \) and \( i_1 : A \to \text{Cy}l(A) \), which are the restrictions of \( i \) to
the two canonical copies of $A$ in $A \amalg A$. In the category of topological spaces, there are natural cylinders:

$$\text{Cyl}(A) = A \times [0, 1],$$

with $i_0 : A \to A \times [0, 1]$ and $i_1 : A \to A \times [0, 1]$ being the obvious embeddings. However, in general, the cylinder objects $\text{Cyl}(A)$ are neither unique nor functorial in $A$.

Dually, if $X \in \text{Ob}(\mathcal{C})$, a path object on $X$ is an object $\text{Path}(X)$ together with a diagram

$$X \xrightarrow{i_0} \text{Cyl}(X) \xrightarrow{i_1} X \times X$$

factoring the natural map $(\text{id}, \text{id}) : X \to X \times X$.

If $f, g : A \to X$ are two morphisms in $\mathcal{C}$, a homotopy from $f$ to $g$ is a map $H : \text{Cyl}(A) \to X$ from a cylinder object on $A$ to $X$ such that the diagram

$$
\begin{array}{ccc}
A & \xrightarrow{i_0} & \text{Cyl}(A) & \xrightarrow{i_1} & A \\
\downarrow f & & \uparrow H & & \downarrow g \\
X & & & & X
\end{array}
$$

commutes. If such a map exists, we say that $f$ is homotopic to $g$ and write $f \sim g$.

If $A$ is cofibrant, the homotopy relation between morphisms $f, g : A \to X$ can be described in terms of path objects: precisely, $f \sim g$ iff there exists a map $H : A \to \text{Path}(X)$ for some path object on $X$ such that

$$
\begin{array}{ccc}
A & \xrightarrow{f} & \text{Path}(X) & \xrightarrow{g} & X \\
\downarrow H & & \downarrow H & & \downarrow H \\
X & & & & X
\end{array}
$$

commutes. Also, if $A$ is cofibrant and $f \sim g$, then for any path object on $X$, there is a map $H : A \to \text{Path}(X)$ such that the above diagram commutes.

Applying MC5(ii) to the canonical morphism $e \to A$, we obtain a cofibrant object $QA$ with an acyclic fibration $QA \xrightarrow{\sim} A$. This is called a cofibrant resolution of $A$. As usual, a cofibrant resolution is not unique, but it is unique up to homotopy equivalence: for any pair of cofibrant resolutions $QA, Q'A$, there exist morphisms

$$QA \xrightarrow{f} Q'A$$

such that $fg \sim \text{Id}$ and $gf \sim \text{Id}$. By MC4, for any morphism $f : A \to X$ and any cofibrant resolutions $QA \xrightarrow{\sim} A$ and $QX \xrightarrow{\sim} A$ there is a map $\tilde{f} : QA \to QX$ making the following diagram commute:

$$
\begin{array}{ccc}
QA & \xrightarrow{f} & QX \\
\downarrow & & \downarrow \\
A & \xrightarrow{f} & X
\end{array}
$$

(6)

We call this map a cofibrant lifting of $f$; it is uniquely determined by $f$ up to homotopy.

When $A$ and $X$ are both cofibrant objects in $\mathcal{C}$, homotopy defines an equivalence relation on $\text{Hom}_\mathcal{C}(A, X)$. In this case, we write

$$\pi(A, X) := \text{Hom}_\mathcal{C}(A, X) / \sim .$$

The homotopy category of $\mathcal{C}$ is now defined to be a category $\text{Ho}(\mathcal{C})$ with $\text{Ob}(\text{Ho}(\mathcal{C})) = \text{Ob}(\mathcal{C})$ and

$$\text{Hom}_{\text{Ho}(\mathcal{C})}(A, X) := \pi(QA, QX) ,$$

where $QA$ and $QX$ are cofibrant resolutions of $A$ and $X$. For $A$ and $A'$ both cofibrant objects in $\mathcal{C}$, it is easy to check that

$$f \sim h : A \to A', \quad g \sim k : A' \to X \quad \Rightarrow \quad gf \sim hk : A \to X .$$

This ensures that the composition of morphisms in $\text{Ho}(\mathcal{C})$ is well defined.
There is a canonical functor $\gamma : \mathcal{C} \to \text{Ho}(\mathcal{C})$ acting as the identity on objects while sending each morphism $f \in \mathcal{C}$ to the homotopy class of its lifting $\tilde{f} \in \text{Ho}(\mathcal{C})$, see [I].

**Theorem 1.** Let $\mathcal{C}$ be a model category, and $\mathcal{D}$ any category. Given a functor $F : \mathcal{C} \to \mathcal{D}$ sending weak equivalences to isomorphisms, there is a unique functor $\bar{F} : \text{Ho}(\mathcal{C}) \to \mathcal{D}$ such that $\bar{F} \circ \gamma = F$.

Theorem [I] shows that the category $\text{Ho}(\mathcal{C})$ is the abstract (universal) localization of the category $\mathcal{C}$ at the class $W$ of weak equivalences. Thus $\text{Ho}(\mathcal{C})$ depends only on $\mathcal{C}$ and $W$. On the other hand, the model structure on $\mathcal{C}$ is not determined by $\mathcal{C}$ and $W$: it does depend the choice of fibrations and cofibrations in $\mathcal{C}$ (see [Q1], § I.1.17, Example 3). The fibrations and cofibrations are needed to control the morphisms in $\text{Ho}(\mathcal{C})$.

**2.4. Derived functors.** Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between model categories. A (total) left derived functor of $F$ is a functor $L F : \text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{D})$ given together with a natural transformation $L F \circ \gamma \to \gamma D \circ F$ which are universal with respect to the following property: for any pair $G : \text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{D})$, $s : G \circ \gamma \to \gamma D \circ F$ there is a unique natural transformation $s' : G \to L F$ such that

There is a dual notion of a right derived functor $R F$ obtained by reversing the arrows in the above definition (cf. 2.2.1). If they exist, the functors $L F$ and $R F$ are unique up to canonical natural equivalence. If $F$ sends weak equivalences to weak equivalence, then both $L F$ and $R F$ exist and, by Theorem [I] $L F = \gamma \bar{F} = R F$,

where $\bar{F} : \text{Ho}(\mathcal{C}) \to \mathcal{D}$ is the extension of $F$ to $\text{Ho}(\mathcal{C})$. In general, the functor $F$ does not extend to $\text{Ho}(\mathcal{C})$, and $L F$ and $R F$ should be viewed as the best possible approximations to such an extension ‘from the left’ and ‘from the right’, respectively.

**2.5. The Adjunction Theorem.** One of the main results in the theory of model categories is Quillen’s Adjunction Theorem. This theorem consists of two parts: part one provides sufficient conditions for the existence of derived functors for a pair of adjoint functors between model categories and part two establishes a criterion for these functors to induce an equivalence at the level of homotopy categories. We will state these two parts as separate theorems. We begin with the following observation which is a direct consequence of basic axioms.

**Lemma 1.** Let $\mathcal{C}$ and $\mathcal{D}$ be model categories. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a pair of adjoint functors. Then the following conditions are equivalent:

(a) $F$ preserves cofibrations and acyclic cofibrations,
(b) $G$ preserves fibrations and acyclic fibrations,
(c) $F$ preserves cofibrations and $G$ preserves fibrations.

A pair of functors $(F, G)$ satisfying the conditions of Lemma [I] is called a Quillen pair; it should be thought of as a ‘map’ (or morphism) of model categories from $\mathcal{C}$ to $\mathcal{D}$. The next theorem justifies this point of view.
**Theorem 2.** Let \( F: \mathcal{C} \rightleftarrows \mathcal{D}: G \) be a Quillen pair. Then the total derived functors \( LF \) and \( RG \) exist and form an adjoint pair

\[
LF: \text{Ho}(\mathcal{C}) \rightleftarrows \text{Ho}(\mathcal{D}): RG.
\]

The functor \( LF \) is defined by

\[
LF(A) = \gamma F(QA), \quad LF(f) = \gamma F(\tilde{f}),
\]

where \( QA \sim \rightarrow A \) is a cofibrant resolution in \( \mathcal{C} \) and \( \tilde{f} \) is a lifting of \( f \), see \([\text{DS}]\).

For a detailed proof of Theorem 2 we refer to \([\text{DS}]\), Sect. 9; here, we only mention one useful result on which this proof is based.

**Lemma 2 (K. Brown).** If \( F: \mathcal{C} \rightarrow \mathcal{D} \) carries acyclic cofibrations between cofibrant objects in \( \mathcal{C} \) to weak equivalences in \( \mathcal{D} \), then \( LF \) exists and is given by formula \((8)\).

**Remark.** In the situation of Theorem 2 if \( \mathcal{D} \) is a fibrant category, then \( RG = G \). This follows from the fact that the derived functor \( RG \) is defined by applying \( G \) to a fibrant resolution similar to \((8)\).

**Example.** Let \( \mathcal{C}^D \) be the category of pushout data in a model category \( \mathcal{C} \) (see [22A]). The colimit construction gives a functor \( \text{colim}: \mathcal{C}^D \rightarrow \mathcal{C} \) which is left adjoint to the diagonal (‘constant diagram’) functor

\[
\Delta: \mathcal{C} \rightarrow \mathcal{C}^D, \quad A \mapsto \{ A \xleftarrow{\text{Id}} A \xrightarrow{\text{Id}} A \}.
\]

Theorem 2 applies in this situation giving the adjoint pair

\[
L\text{colim}: \text{Ho}(\mathcal{C}^D) \rightleftarrows \text{Ho}(\mathcal{C}): R\Delta.
\]

The functor \( L\text{colim} \) is called the homotopy pushout functor. Similarly one defines the homotopy pullback functor \( R\lim \) which is right adjoint to \( L\Delta \) (see [DS], Sect. 10).

Now, we state the second part of Quillen’s Theorem.

**Theorem 3.** The derived functors \((7)\) associated to a Quillen pair \((F,G)\) are (mutually inverse) equivalences of categories if and only if the following condition holds: for each cofibrant object \( A \in \text{Ob}(\mathcal{C}) \) and fibrant object \( B \in \text{Ob}(\mathcal{D}) \) a morphism \( f: A \rightarrow G(B) \) is a weak equivalence in \( \mathcal{C} \) if and only if the adjoint morphism \( f^* : F(A) \rightarrow B \) is a weak equivalence in \( \mathcal{D} \).

A Quillen pair \((F,G)\) satisfying the condition of Theorem 3 is called a Quillen equivalence. The fundamental example of a Quillen equivalence is the geometric realization and the singular set functors relating the categories of simplicial sets and topological spaces (see [Q1]):

\[
|-|: \text{SSets} \rightleftarrows \text{Top}: \text{Sing}(-).
\]

We give another well-known example coming from algebra. Recall that if \( A \) is a DG algebra, the category \( \text{DGMod}(A) \) of DG modules over \( A \) is abelian and has a natural model structure, with weak equivalences being the quasi-isomorphisms.

**Proposition 1.** Let \( f: R \rightarrow A \) be a morphism of DG algebras. The corresponding induction and restriction functors form a Quillen pair

\[
f^*: \text{DGMod}(R) \rightleftarrows \text{DGMod}(A): f_*
\]

If \( f \) is a quasi-isomorphism, \((f^*, f_*)\) is a Quillen equivalence.

Proposition 1 is a special case of a general result about module categories in monoidal model categories proved in [SS1] (see loc. cit, Theorem 4.3).
2.6. Quillen homology. For a category \( \mathcal{C} \), let \( \mathcal{C}^{ab} \) denote the category of abelian objects in \( \mathcal{C} \). Recall that \( A \in \text{Ob}(\mathcal{C}) \) is an abelian object if the functor \( \text{Hom}_{\mathcal{C}}(-, A) \) is naturally an abelian group. Assuming that \( \mathcal{C} \) has enough limits, this is known to be equivalent to the ‘diagrammatic’ definition of an abelian group structure on \( A \): i.e., the existence of multiplication \( (m : A \times A \to A) \), inverse \( (i : A \to A) \) and unit \( (s \to A) \) morphisms in \( \mathcal{C} \), satisfying the usual axioms of an abelian group (see, e.g., [GM], Ch. II, Sect. 3.10). Note that the forgetful functor \( i : \mathcal{C}^{ab} \to \mathcal{C} \) is faithful but not necessarily full. For example, the abelian objects in the categories \( \text{Sets} \) and \( \text{Groups} \) are the same: namely, the abelian groups; however, \( i : \mathcal{C}^{ab} \to \mathcal{C} \) is a full embedding only for \( \mathcal{C} = \text{Groups} \).

Now, let \( \mathcal{C} \) be a model category. Following Quillen (see [Q1], § II.5), we assume that the forgetful functor \( i : \mathcal{C}^{ab} \to \mathcal{C} \) has a left adjoint \( Ab : \mathcal{C} \to \mathcal{C}^{ab} \) called abelianization, and there is a model structure on \( \mathcal{C}^{ab} \) such that

\[
(9) \quad Ab : \mathcal{C} \rightleftarrows \mathcal{C}^{ab} : i
\]
is a Quillen pair. Then, by Theorem 2, \( Ab \) has a total left derived functor \( LAb : \text{Ho}(\mathcal{C}) \to \text{Ho}(\mathcal{C}^{ab}) \), which is called the Quillen homology of \( \mathcal{C} \). Assume, in addition, that the model structure on \( \mathcal{C}^{ab} \) is stable, i.e. there is an invertible suspension functor \( \Sigma : \text{Ho}(\mathcal{C}^{ab}) \to \text{Ho}(\mathcal{C}^{ab}) \) making \( \text{Ho}(\mathcal{C}^{ab}) \) a triangulated category (cf. [HO], Sect. 7.1). Then, for any \( X \in \text{Ob}(\mathcal{C}) \) and \( A \in \text{Ob}(\mathcal{C}^{ab}) \), we can define the Quillen cohomology of \( X \) with coefficients in \( A \) by

\[
H^n_\mathcal{C}(X, A) = \text{Hom}_{\mathcal{C}^{ab}}(LAb(X), \Sigma^{-n} A)
\]
This construction unifies basic (co-)homology theories of spaces, groups and algebras (see [Q1], § II.5). We briefly discuss only three well-known examples related to algebras (see [Q4]).

**Example 1.** Let \( \mathcal{C} = \text{DGLA}_k \) be the category of DG Lie algebras over \( k \). This category has a natural model structure, with weak equivalences being quasi-isomorphisms (see [Q2], Part II, Sect. 5). The abelian objects in \( \mathcal{C} \) are just the abelian Lie algebras (i.e., the DG Lie algebras with zero bracket). The category \( \mathcal{C}^{ab} \) can thus be identified with \( \text{Com}(k) \). The abelianization functor \( Ab : \text{DGLA}_k \to \text{Com}(k) \) is given by \( g \mapsto g/[g, g] \). If \( g \) is an ordinary Lie algebra, and \( \mathcal{L} \sim g \) is a cofibrant resolution of \( g \) in \( \text{DGLA}_k \), then

\[
(10) \quad H_n(\mathcal{L}/[\mathcal{L}, \mathcal{L}]) \cong H_{n+1}(g, k), \quad \forall n \geq 0.
\]
Thus, the Quillen homology of \( g \) agrees with the usual Lie algebra homology with trivial coefficients.

**Example 2.** Let \( \text{DGA}_k \) be the category of associative DG algebras. Unlike in \( \text{DGLA}_k \), the only abelian object in \( \text{DGA}_k \) is the zero algebra. To get more interesting examples, we fix an algebra \( A \in \text{Ob}(\text{DGA}_k) \) and consider the category \( \mathcal{C} := \text{DGA}_k/A \) of algebras over \( A \). (So an object in \( \mathcal{C} \) is a DG algebra \( B \) given together with a DG algebra map \( B \to A \).) In this case, it is easy to show that \( \mathcal{C}^{ab} \) is equivalent to the (abelian) category \( \text{DG Bimod}(A) \) of DG bimodules over \( A \). The equivalence is given by the semi-direct product construction

\[
(11) \quad A \ltimes (-) : \text{DG Bimod}(A) \to \text{DGA}_k/A,
\]
assigning to a bimodule \( M \) the DG algebra \( A \ltimes M \) together with the canonical projection \( A \ltimes M \to A \). Note that \( A \ltimes M \) is an abelian object in \( \mathcal{C} \) because \( \text{Hom}_{\mathcal{C}}(B, A \ltimes M) \cong \text{Der}_k(B, M) \), where \( \text{Der}_k(B, M) \) is an abelian group (in fact, a vector space) of \( k \)-linear derivations \( \partial : B \to M \). On the other hand, for any \( A \)-bimodule \( M \), there is a natural isomorphism

\[
\text{Der}_k(B, M) \cong \text{Hom}_{\text{DG Bimod}(A)}(\Omega^1_k(B/A), M),
\]
where \( \Omega^1_k(B/A) := A \otimes B \Omega^1_k(B) \otimes B A \) and \( \Omega^1_k B \) denotes the kernel of the multiplication map \( B \otimes B \to B \). Thus, for \( \mathcal{C} = \text{DGA}_k/A \), the Quillen pair \( \Omega^1_k(-/A) \) can be identified with

\[
(12) \quad \Omega^1_k(-/A) : \text{DGA}_k/A \cong \text{DG Bimod}(A) : A \ltimes (-).
\]

---

1. We will discuss the properties of this category as well as its commutative counterpart in Section 2.7 below.
If \( A \) is an ordinary \( k \)-algebra, the Quillen homology of \( \mathcal{C} \) essentially coincides with Hochschild homology: precisely, we have

\[
H_n[\mathcal{L}^{-1}(B/A)] = \begin{cases} 
\Omega^1_k(B/A) & \text{if } n = 0 \\
\text{Tor}_{n+1}^R(A, A) & \text{if } n \geq 1
\end{cases}
\]

The derived abelianization functor \( \mathcal{L}^{-1} \) of \( A \) is called the noncommutative cotangent complex of \( A \) by \( \mathcal{L}^{-1}(B/A) \). The Quillen homology of \( A \) with coefficients in a bimodule \( M \) can be identified with Hochschild cohomology of \( M \) (see [Q4], Proposition 3.6).

**Example 3.** Let \( \text{CDGA}_k \) be the category of commutative DG \( k \)-algebras. As in the case of associative algebras, for any \( A \in \text{Ob}(\text{CDGA}_k) \), the semi-direct product construction defines a fully faithful functor

\[
A \ltimes (-) : \text{DG Mod}(A) \rightarrow \text{CDGA}_k/A,
\]

whose image is the subcategory of abelian objects in \( \text{CDGA}_k/A \). The Quillen pair \( (\mathcal{L}, \mathcal{R}) \) is then identified with

\[
\Omega^1_{\text{com}}(-)/A : \text{CDGA}_k/A \rightleftarrows \text{DG Mod}(A) : A \ltimes (-).
\]

Here the abelianization functor is given by \( \Omega^1_{\text{com}}(B/A) := A \otimes B \Omega^1_{\text{com}}(B) \), where \( \Omega^1_{\text{com}}(B) \) is the module of Kähler differentials of the commutative \( k \)-algebra \( B \). The corresponding derived functor \( \mathcal{L}^{-1}_{\text{com}}(A) \) evaluated at the identity morphism of \( A \) is usually denoted \( L_{k/A} \) and called the cotangent complex of \( A \). By definition, this is an object in the homotopy category \( \text{Ho}(\text{CDGA}_k) \), which can be computed by the formula \( L_{k/A} \cong A \otimes_R \Omega^1_{\text{com}}(R) \), where \( R \xrightarrow{\sim} A \) is a cofibrant resolution of \( A \). The homology of the cotangent complex

\[
D_q(k/A) := H_q(L_{k/A}) \cong H_q(A \otimes_R \Omega^1_{\text{com}}(R))
\]

is called the André-Quillen homology of \( A \). More generally, the André-Quillen homology with coefficients in an arbitrary module \( M \in \text{DG Mod}(A) \) is defined by

\[
D_q(k/A, M) := H_q(L_{k/A} \otimes_A M).
\]

Taking the Hom complex with \( L_{k/A} \) instead of tensor product defines the corresponding cohomology. The construction of André-Quillen (co-)homology theory was historically the first real application of model categories. The original paper of Quillen [Q4] seems still to be the best exposition of foundations of this theory. Many interesting examples and applications can be found in the survey paper [I].

**2.7. Differential graded algebras.** By a DG algebra we mean a \( \mathbb{Z} \)-graded unital associative \( k \)-algebra equipped with a differential of degree \(-1\). We write \( \text{DGA}_k \) for the category of all such algebras and denote by \( \text{CDGA}_k \) the full subcategory of \( \text{DGA}_k \) consisting of commutative DG algebras. On these categories, there are standard model structures which we describe in the next theorem.

**Theorem 4.** The categories \( \text{DGA}_k \) and \( \text{CDGA}_k \) have model structures in which

(i) the weak equivalences are the quasi-isomorphisms,
(ii) the fibrations are the maps which are surjective in all degrees,
(iii) the cofibrations are the morphisms having the left-lifting property with respect to acyclic fibrations (cf. MC4).

Both categories \( \text{DGA}_k \) and \( \text{CDGA}_k \) are fibrant, with the initial object \( k \) and the terminal 0.

Theorem [4] is a special case of a general result of Hinich on model structure on categories of algebras over an operad (see [H], Theorem 4.1.1 and Remark 4.2). For \( \text{DGA}_k \), a detailed proof can be found in [I]. Note that the model structure on \( \text{DGA}_k \) is compatible with the projective model structure on the category \( \text{Com}_k \) of complexes. Since a DG algebra is just an algebra object (monoid) in \( \text{Com}_k \), Theorem [4] follows also from [SST1] (see op. cit., Sect. 5).

It is often convenient to work with non-negatively graded DG algebras. We denote the full subcategory of such DG algebras by \( \text{DGA}_k^+ \) and the corresponding subcategory of commutative DG algebras by \( \text{CDGA}_k^+ \). We recall that a DG algebra \( R \in \text{DGA}_k^+ \) is called semi-free if its underlying graded algebra \( R \cdot \) is free (i.e.,
$R_#$ is isomorphic to the tensor algebra $T_k V$ of a graded $k$-vector space $V$). More generally, we say that a DG algebra map $f : A \to B$ in $\text{DGA}_k^+$ is a semi-free extension if there is an isomorphism $B_# \cong A_# \coprod T_k V$ of underlying graded algebras such that the composition of $f_#$ with this isomorphism is the canonical map $A_# \to A_# \coprod T_k V$. Here, $\coprod$ denotes the coproduct (free product) in the category of graded algebras over $k$.

Similarly, a commutative DG algebra $S \in \text{CDGA}_k^+$ is called semi-free if $S_# \cong \Lambda_k V$ for some graded vector space $V$. A morphism $f : A \to B$ in $\text{CDGA}_k^+$ is an (semi) free extension if $f_#$ is isomorphic to the canonical map $A_# \to A_# \otimes \Lambda_k V$.

**Theorem 5.** The categories $\text{DGA}_k^+$ and $\text{CDGA}_k^+$ have model structures in which
(i) the weak equivalences are the quasi-isomorphisms,
(ii) the fibrations are the maps which are surjective in all positive degrees,
(iii) the cofibrations are the retracts of semi-free algebras (cf. MC3).

Both categories $\text{DGA}_k^+$ and $\text{CDGA}_k^+$ are fibrant, with the initial object $k$ and the terminal 0.

The model structure on $\text{CDGA}_k^+$ described in Theorem 5 is a ‘chain’ version of a well-known model structure on the category of commutative cochain DG algebras. This last structure plays a prominent role in rational homotopy theory and the verification of axioms for $\text{CDGA}_k^+$ can be found in many places (see, e.g., [BG] or [CM], Chap. V). The model structure on $\text{DGA}_k^+$ is also well known: a detailed proof of Theorem 5 for $\text{DGA}_k^+$ can be found in [M]. The assumption that $k$ has characteristic 0 is essential in the commutative case: without this assumption, $\text{CDGA}_k^+$ is not (Quillen) equivalent to the model category of simplicial commutative $k$-algebras. On the other hand, it is known that the model category $\text{DGA}_k^+$ is Quillen equivalent to the model category of simplicial associative $k$-algebras over an arbitrary commutative ring $k$ (see [SS2], Theorem 1.1).

### 2.8. DG schemes.

Working with commutative DG algebras it is often convenient to use the dual geometric language of DG schemes. In this section, we briefly recall basic definitions and facts about DG schemes needed for the present paper. For more details, we refer to [CK], Section 2. We warn the reader that, unlike [CK], we use the homological notation: all our complexes and DG algebras have differentials of degree $-1$.

A DG scheme $X = (X_0, \mathcal{O}_{X_\bullet})$ is an ordinary $k$-scheme $X_0$ equipped with a quasicoherent sheaf $\mathcal{O}_{X_\bullet}$ of non-negatively graded commutative DG algebras such that $\mathcal{O}_{X_0} = \mathcal{O}_{X_0}$. A DG scheme is called affine if $X_0$ is affine: the category of affine DG schemes is (anti-)equivalent to $\text{CDGA}_k^+$. Since $\mathcal{O}_{X_\bullet}$ is non-negatively graded, the differential $d$ on $\mathcal{O}_{X_\bullet}$ is linear over $\mathcal{O}_{X_0}$, and $H_0(\mathcal{O}_{X_\bullet}) = \mathcal{O}_{X_0}/d\mathcal{O}_{X_1}$ is the quotient of $\mathcal{O}_{X_0}$. We write $\pi_0(X) := \text{Spec} H_0(\mathcal{O}_{X_\bullet})$ and identify $\pi_0(X)$ with a closed subscheme of $X_0$.

A DG scheme $X$ is called smooth (or a DG manifold) if $X_0$ is a smooth variety, and $\mathcal{O}_{X_\bullet}$ is locally isomorphic (as a sheaf of graded $\mathcal{O}_{X_0}$-algebras) to the graded symmetric algebra

$$\mathcal{O}_{X_\bullet} = \Lambda_{\mathcal{O}_{X_0}}(E_\#)$$

where $E_\# = \oplus_{i\geq 1} E_i$ is a graded $\mathcal{O}_{X_0}$-module whose components $E_i$ are finite rank locally free sheaves on $X_0$. (Note that we do not require $E_\#$ to be bounded, i.e. $E_i$ may be nonzero for infinitely many $i$’s.)

Now, given a DG scheme $X$ and a closed $k$-point $x \in X_0$, we define the DG tangent space $(T_x X)_\bullet$ at $x$ to be the derivation complex

$$\mathbf{(16)} \quad (T_x X)_\bullet := \operatorname{Der}(\mathcal{O}_{X_\bullet}, k_x) ,$$

where $k_x$ is the 1-dimensional DG $\mathcal{O}_{X_\bullet}$-module corresponding to $x$. The homology groups of this complex are denoted

$$\mathbf{(17)} \quad \pi_i(X, x) := H_i(T_x X) , \quad i \geq 0 ,$$

and called the derived tangent spaces of $X$ at $x$. A morphism $f : X \to Y$ of DG schemes induces a morphism of complexes $(d_x f)_\bullet : (T_x X)_\bullet \to (T_y Y)_\bullet$, and hence linear maps

$$\mathbf{(18)} \quad (d_x f)_i : \pi_i(X, x) \to \pi_i(Y, y) ,$$
where \( y = f(x) \). Dually, the DG cotangent space \((T_x^*X)_*\) at a point \( x \in X_0 \) is defined by taking the complex of Kähler differentials:
\[
(T_x^*X)_* := \Omega^1_{\text{com}}(\mathcal{O}_X)_x = m_x/m_x^2,
\]
where \( m_x \subset \mathcal{O}_x \) is the maximal DG ideal corresponding to \( x \). The topological notation \((\Gamma)\) for the derived tangent spaces is justified by the following proposition, which is analogous to the Whitehead Theorem in classical topology.

**Proposition 2** ([Ka], Proposition 1.3). Let \( f : X \to Y \) be a morphism of smooth DG schemes. Then \( f \) is a quasi-isomorphism if and only if

1. \( \pi_0(f) : \pi_0(X) \to \pi_0(Y) \) is an isomorphism of schemes,
2. for every closed point \( x \in X_0 \), the differential \( d_xf \) induces linear isomorphisms
   \[ \pi_i(X, x) \to \pi_i(Y, f(x)), \quad \forall i \geq 0. \]

The proof of Proposition 2 is based on the next lemma which is of independent interest (see [CK2], Sect. 2.2.3).

**Lemma 3.** Let \( X = (X_0, O_{X,*}) \) be a smooth DG scheme, and let \( \hat{O}_{X,x} := O_{X,*} \otimes_{O_{X_0}} \hat{O}_{X_0,x} \) denote the complete local DG ring corresponding to a closed \( k \)-point \( x \in \pi_0(X) \). Then there is a convergent spectral sequence
\[
E^2 = A^* [H_*(T_x^*X)] \Rightarrow H_*(\hat{O}_{X,x})
\]
for \( x \in \pi_0(X) \).

Crucial to the proof of Lemma 3 is the fact that \( \hat{O}_{X,x} \) coincides with the completion of \( O_{X,x} \) with respect to the \( m_x \)-adic topology. If \( f : X \to Y \) satisfies the conditions (1) and (2) of Proposition 2 for any \( x \in \pi_0(X) \), the map \( \hat{f}_x : \hat{O}_{Y,y} \to \hat{O}_{X,x} \) induces a quasi-isomorphism between \( E^2 \)-terms of the spectral sequences \((19)\) associated to the local rings \( \hat{O}_{X,x} \) and \( \hat{O}_{Y,y} \). Since these local rings are complete, the Eilenberg-Moore Comparison Theorem implies that \( \hat{f}_x \) is a quasi-isomorphism. By Krull’s Theorem, the map \( f \) itself is then a quasi-isomorphism.

### 3. Representation Schemes

In this section, we extend the representation functor (2) to the category of DG algebras. We show that such an extension defines a representable functor which is actually a left Quillen functor in the sense of Lemma 1. A key technical tool is the universal construction of ‘matrix reduction’, which (in the case of ordinary associative algebras) was introduced and studied in [B] and [C]. The advantage of this construction is that it produces the representing object for (2) in a canonical form as a result of application of some basic functors on the category of algebras.

**3.1. DG representation functors.** Let \( S \in \text{DGA}_k \) be a DG algebra, and let \( \text{DGA}_S \) denote the category of DG algebras over \( S \). By definition, the objects of \( \text{DGA}_S \) are the DG algebra maps \( S \to A \) in \( \text{DGA}_k \) and the morphisms are given by the commutative triangles
\[
\begin{array}{ccc}
S & \xrightarrow{f} & B \\
\downarrow & & \downarrow \\
A & \xrightarrow{} & \mathcal{O}_A
\end{array}
\]
We will write a map \( S \to A \) as \( S \setminus A \), or simply \( A \), when we regard it as an object in \( \text{DGA}_S \). For \( S \in \text{Alg}_k \), we also introduce the category \( \text{Alg}_S \) of ordinary \( S \)-algebras (i.e., the category of morphisms \( S \to A \) in \( \text{Alg}_k \)) and identify it with a full subcategory of \( \text{DGA}_S \) in the natural way.

Let \((V, d_V)\) be a complex of \( k \)-vector spaces of finite (total) dimension, and let \( \text{End} V \) denote its graded endomorphism ring with differential \( df = d_V f - (-1)^i f d_V \), where \( f \in \text{End}(V)_i \). Fix on \( V \) a DG \( S \)-module structure, or equivalently, a DG representation \( S \to \text{End} V \). This makes \( \text{End} V \) a DG algebra over \( S \), i.e., an object of \( \text{DGA}_S \). Now, given a DG algebra \( A \in \text{DGA}_S \), an \( S \)-representation of \( A \) in \( V \) is, by
definition, a morphism $A \to \text{End} V$ in $\text{DGA}_S$. Such representations form an affine DG scheme which is defined as the functor on the category of commutative DG algebras:

\[(20) \quad \text{Rep}_V(S\backslash A) : \text{CDGA}_k \to \text{Sets}, \quad C \mapsto \text{Hom}_{\text{DGA}_S}(A, \text{End} V \otimes C).\]

Our proof of representability of $\text{(20)}$ is based on the following simple observation. Denote by $\text{DGA}_{\text{End}(V)}$ the category of DG algebras over $\text{End} V$ and consider the natural functor

\[(21) \quad G : \text{DGA}_k \to \text{DGA}_{\text{End}(V)}, \quad B \mapsto \text{End} V \otimes B,\]

where $\text{End} V \otimes B$ is viewed as an object in $\text{DGA}_{\text{End}(V)}$ via the canonical map $\text{End} V \to \text{End} V \otimes B$.

**Lemma 4.** The functor $\text{(21)}$ is an equivalence of categories.

For a detailed proof we refer to [BKR], Lemma 2.1. Here we only note that the inverse functor to $\text{(21)}$ is given by

\[(22) \quad G^{-1} : \text{DGA}_{\text{End}(V)} \to \text{DGA}_k, \quad (\text{End} V \to A) \mapsto A_{\text{End}(V)},\]

where $A_{\text{End}(V)}$ is the (graded) centralizer of the image of $\text{End} V$ in $A$.

Next, we introduce the following functors

\[(23) \quad \sqrt{\cdot} : \text{DGA}_S \to \text{CDGA}_k, \quad S\backslash A \mapsto (\text{End} V \Pi_S A)^{\text{End}(V)},\]

\[(24) \quad (-)_{\sqrt{\cdot}} : \text{DGA}_S \to \text{CDGA}_k, \quad S\backslash A \mapsto (\sqrt{S\backslash A})_{\text{End}(V)},\]

where $\Pi_S$ denotes the coproduct in the category $\text{DGA}_S$ and $(-)_{\sqrt{\cdot}} : \text{DGA}_k \to \text{CDGA}_k$ stands for ‘commutativization’, i.e. taking the quotient of a DG algebra $R$ by its two-sided commutator ideal: $R_{\sqrt{\cdot}} := R/[[R, R]]$. The following proposition is an easy consequence of Lemma 4.

**Proposition 3.** For any $S\backslash A \in \text{DGA}_S$, $B \in \text{DGA}_k$ and $C \in \text{CDGA}_k$, there are natural bijections

\[(a) \quad \text{Hom}_{\text{DGA}_k}(\sqrt{S\backslash A}, B) \cong \text{Hom}_{\text{DGA}_S}(A, \text{End} V \otimes B),\]

\[(b) \quad \text{Hom}_{\text{CDGA}_k}((S\backslash A)_{\sqrt{\cdot}}, C) \cong \text{Hom}_{\text{DGA}_S}(A, \text{End} V \otimes C).\]

**Proof.** The tensor functor $B \mapsto \text{End} V \otimes B$ in (a) can be formally written as the composition

\[(25) \quad \text{DGA}_k \xrightarrow{G} \text{DGA}_{\text{End}(V)} \xrightarrow{F} \text{DGA}_S,\]

where $G$ is defined by $\text{(21)}$ and $F$ is the restriction functor via the given DG algebra map $S \to \text{End} V$. Both $F$ and $G$ have natural left adjoint functors: the left adjoint of $F$ is obviously the coproduct $A \mapsto \text{End} V \Pi_S A$, while the left adjoint of $G$ is $G^{-1}$, since $G$ is an equivalence of categories (Lemma 4). Now, by definition, the functor $\sqrt{\cdot}$ is the composition of these left adjoint functors and hence the left adjoint to the composition $\text{(25)}$. This proves part (a). Part (b) follows from (a) and the obvious fact that the commutativization functor $(-)_{\sqrt{\cdot}} : \text{DGA}_k \to \text{CDGA}_k$ is left adjoint to the inclusion $\iota : \text{CDGA}_k \hookrightarrow \text{DGA}_k$. \hfill \Box

Part (b) of Proposition 3 can be restated in the following way, which shows that $\text{Rep}_V(S\backslash A)$ is indeed an affine DG scheme in the sense of Section 2.8

**Theorem 6.** For any $S\backslash A \in \text{DGA}_S$, the commutative DG algebra $(S\backslash A)_{\sqrt{\cdot}}$ represents the functor $\text{(21)}$.

The algebras $\sqrt{S\backslash A}$, $(S\backslash A)_{\sqrt{\cdot}}$ and the isomorphisms of Proposition 3 can be described explicitly. To this end, we choose a linear basis $\{v_i\}$ in $V$ consisting of homogeneous elements, and define the elementary endomorphisms $\{e_{ij}\}$ in $\text{End} V$ by $e_{ij}(v_k) = \delta_{jk} v_i$. These endomorphisms are homogeneous, the degree of $e_{ij}$ being $|v_i| - |v_j|$, and satisfy the obvious relations

\[(26) \quad \sum_{i=1}^{d} e_{ii} = 1, \quad e_{ij} e_{kl} = \delta_{jk} e_{il},\]
where \( d := \dim_k V \). Now, for each homogeneous element \( a \in \End V \otimes_S A \), we define its ‘matrix’ elements by

\[
(27) \quad a_{ij} := \sum_{k=1}^d (-1)^{(|a|+|e_{kl}|)|e_{kl}|} e_{ki} a e_{kj}, \quad i, j = 1, 2, \ldots, d.
\]

A straightforward calculation using (26) shows that \( [a_{ij}, e_{lk}] = 0 \) for all \( i, j, k, l = 1, 2, \ldots, d \). Since \( \{e_{ij}\} \) spans \( \End V \), this means that \( a_{ij} \in \sqrt{S \setminus A} \), and in fact, it is easy to see that every homogeneous element of \( \sqrt{S \setminus A} \) can be written in the form (27). By Lemma 4, the map

\[
(28) \quad \psi : \End V \otimes \sqrt{S \setminus A} \to \End V \otimes (S \setminus A), \quad a \mapsto \sum_{i,j=1}^d e_{ij} \otimes a_{ij}
\]

is a DG algebra isomorphism which is inverse to the canonical (multiplication) map

\[
\End V \otimes \sqrt{S \setminus A} \to \End V \otimes_S A.
\]

Using (28), we can now write the bijection of Proposition 3(b):

\[
\Hom_{\text{DG}}(\sqrt{S \setminus A}, B) \to \Hom_{\text{DG}}(A, \End V \otimes B), \quad f \mapsto (\text{Id} \otimes f) \circ \psi|_A,
\]

where \( \psi|_A \) is the composition of (28) with the canonical map \( A \to \End V \otimes_S A \). As the algebra \( (S \setminus A)_V \) is, by definition, the maximal commutative quotient of \( \sqrt{S \setminus A} \), it is also spanned by the elements (27) taken modulo the commutator ideal.

**Remark.** For ordinary \( k \)-algebras, Proposition 3 and Theorem 6 were originally proven in [11] (Sect. 7) and [10] (Sect. 6). In these papers, the functor (23) was called the ‘matrix reduction’ and a different notation was used. Our notation \( \sqrt{\phantom{S \setminus A}} \) is borrowed from [LBW], where (23) is used for constructing noncommutative thickenings of classical representation schemes.

### 3.2. Deriving the representation functor.

As explained in Section 2.2, the categories \( \text{DGA}_k \) and \( \text{CDGA}_k \) have natural model structures, with weak equivalences being the quasi-isomorphisms. Furthermore, for a fixed DG algebra \( S \), the category of \( S \)-algebras, \( \text{DGA}_S \), inherits a model structure from \( \text{DGA}_k \) (cf. 2.2.2). Every DG algebra \( S \setminus A \in \text{DGA}_S \) has a cofibrant resolution \( Q(S \setminus A) \to S \setminus A \) in \( \text{DGA}_S \), which is given by a factorization \( S \to Q \to A \) of the morphism \( S \to A \in \text{DGA}_k \). By Theorem 4 the homotopy category \( \Ho(\text{DGA}_S) \) is equivalent to the localization of \( \text{DGA}_S \) at the class of weak equivalences in \( \text{DGA}_S \). We denote the corresponding localization functor by \( \gamma : \text{DGA}_S \to \Ho(\text{DGA}_S) \); this functor acts as identity on objects while maps each morphism \( f : S \setminus A \to S \setminus B \) to the homotopy class of its cofibrant lifting \( \tilde{f} : Q(S \setminus A) \to Q(S \setminus B) \) in \( \text{DGA}_S \), see (6). The next theorem is one of the main results of [BK1] (see loc. cit., Theorem 2.2).

**Theorem 7.** (a) The functors \((-)_V : \text{DGA}_S \rightleftarrows \text{CDGA}_k : \End V \otimes -\) form a Quillen pair.

(b) \((-)_V \) has a total left derived functor defined by

\[
L(-)_V : \Ho(\text{DGA}_S) \to \Ho(\text{CDGA}_k), \quad S \setminus A \mapsto Q(S \setminus A)_V, \quad \gamma f \mapsto \gamma(\tilde{f}_V).
\]

(c) For any \( S \setminus A \in \text{DGA}_S \) and \( B \in \text{CDGA}_k \), there is a canonical isomorphism

\[
\Hom_{\Ho(\text{CDGA}_k)}(L(S \setminus A)_V, B) \cong \Hom_{\Ho(\text{CDGA}_k)}(A, \End V \otimes B).
\]

**Proof.** By Proposition 3(b), the functor \((-)_V \) is left adjoint to the composition

\[
\text{CDGA}_k \leftarrow \text{DGA}_k \xrightarrow{\End V \otimes -} \text{DGA}_S,
\]

which we still denote \( \End V \otimes -\). Both the forgetful functor \( - \) and the tensoring with \( \End V \) over a field are exact functors on \( \text{Com}_k \); hence, they map fibrations (the surjective morphisms in \( \text{DGA}_k \)) to fibrations and also preserve the class of weak equivalences (the quasi-isomorphisms). It follows that \( \End V \otimes -\) preserves fibrations as well as acyclic fibrations. Thus, by Lemma 1, \((-)_V : \text{DGA}_S \rightleftarrows \text{CDGA}_k : \End V \otimes -\) is a Quillen pair. This proves part (a). Part (b) and (c) now follow directly from Quillen’s Adjunction
Theorem (see Theorem 2). For part (c), we need only to note that $G := \text{End} V \otimes -$ is an exact functor in Quillen’s sense, i.e. $RG = G$, since $\text{CDGA}_k$ is a fibrant model category. \qed

**Definition.** By Theorem 7 the assignment $S \backslash A \mapsto Q(S \backslash A)_V$ defines a functor

$$\text{DRep}_V : \text{Alg}_S \to \text{Ho}(\text{CDGA}_k)$$

which is independent of the choice of resolution $Q(S \backslash A) \sim S \backslash A$ in $\text{DGA}_S$. Abusing terminology, we call $\text{DRep}_V(S \backslash A)$ a relative derived representation scheme of $A$. The homology of $\text{DRep}_V(S \backslash A)$ is a graded commutative algebra, which depends only on $S \backslash A$ and $V$. We write

$$H_\bullet(S \backslash A, V) := H_\bullet[\text{DRep}_V(S \backslash A)]$$

and refer to (29) as representation homology of $S \backslash A$ with coefficients in $V$. In the absolute case when $S = k$, we simplify the notation writing $\text{DRep}_V(A) := \text{DRep}_V(k \backslash A)$ and $H_\bullet(A, V) := H_\bullet(k \backslash A, V)$.

We now make a few remarks related to Theorem 7

3.2.1. For any cofibrant resolutions $p : Q(S \backslash A) \sim S \backslash A$ and $p' : Q'(S \backslash A) \sim S \backslash A$ of a given $S \backslash A \in \text{DGA}_S$, there is a quasi-isomorphism $f_V : Q(S \backslash A)_V \sim Q'(S \backslash A)_V$ in $\text{CDGA}_k$. Indeed, by the identity map on $A$ lifts to a morphism $f : Q(S \backslash A) \sim Q'(S \backslash A)$ such that $p' f = p$. This morphism is automatically a weak equivalence in $\text{DGA}_S$, so $f$ is an isomorphism in $\text{Ho}(\text{DGA}_S)$. It follows that $L(\gamma f)_V$ is an isomorphism in $\text{Ho}(\text{CDGA}_k)$. But $Q(S \backslash A)_V$ and $Q'(S \backslash A)_V$ are both cofibrant objects, so $L(\gamma f)_V = \gamma (f_V)$ in $\text{Ho}(\text{CDGA}_k)$. Thus $f_V$ is a quasi-isomorphism in $\text{CDGA}_k$.

3.2.2. The analogue of Theorem 7 holds for the pair of functors $\sqrt{-} : \text{DGA}_S \rightleftarrows \text{DGA}_k : \text{End} V \otimes -$, which are adjoint to each other by Proposition 3(a). Thus, $\sqrt{-}$ has the left derived functor

$$L\sqrt{-} : \text{Ho}(\text{DGA}_S) \to \text{Ho}(\text{DGA}_k), \quad L\sqrt{Q(S \backslash A)} := \sqrt{Q(S \backslash A)},$$

which is left adjoint to $\text{End} V \otimes -$ on the homotopy category $\text{Ho}(\text{CDGA}_k)$.

3.2.3. If $V$ is a complex concentrated in degree 0, the functors $(-)_V$ and $\text{End} V \otimes -$ restrict to the category of non-negatively graded DG algebras and still form the adjoint pair

$$(-)_V : \text{DGA}_S^+ \rightleftarrows \text{CDGA}_k^+ : \text{End} V \otimes -.$$ 

The categories $\text{DGA}_S^+$ and $\text{CDGA}_k^+$ have natural model structures (see Theorem 5), for which all the above results, including Theorem 7 hold, with proofs being identical to the unbounded case.

3.2.4. The representation functor $\text{DRep}_V$ naturally extends to the category $\text{Alg}_k$ of simplicial $k$-algebras, and one can also use the model structure on this last category to construct the derived functors of $\text{DRep}_V$. However, for any commutative ring $k$, the model category $\text{Alg}_k$ is known to be is Quillen equivalent to the model category $\text{DGA}_k^+$ (see SS2, Theorem 1.1). Also, if $k$ is a field of characteristic zero (as we always assume in this paper), the corresponding categories of commutative algebras $\text{SComm Alg}_k$ and $\text{CDGA}_k^+$ are Quillen equivalent (see Q2, Remark on p. 223). Thus, at least when $V$ is a complex concentrated in degree 0, the derived representation functors $\text{DRep}_V$ constructed using simplicial and DG resolutions are naturally equivalent.

3.3. Basic properties of $\text{DRep}_V(S \backslash A)$.

3.3.1. We begin by clarifying how the functor $\text{DRep}_V$ depends on $V$. Let $\text{DGMod}(S)$ be the category of DG modules over $S$, and let $V$ and $W$ be two modules in $\text{DGMod}(S)$ each of which has finite dimension over $k$.

**Proposition 4** (BKR, Proposition 2.3). If $V$ and $W$ are quasi-isomorphic in $\text{DGMod}(S)$, the corresponding derived functors $L(-)_V$ and $L(-)_W : \text{Ho}(\text{DGA}_S) \to \text{Ho}(\text{CDGA}_k)$ are naturally equivalent.

The proof of this proposition is based on the following lemma, which is probably known to the experts.

**Lemma 5.** Let $V$ and $W$ be two bounded DG modules over $S$, and assume that there is a quasi-isomorphism $f : V \sim W$ in $\text{DGMod}(S)$. Then the DG algebras $\text{End} V$ and $\text{End} W$ are weakly equivalent in $\text{DGA}_S$, i.e. isomorphic in $\text{Ho}(\text{DGA}_S)$. 

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As an immediate consequence of Proposition 3, we get

**Corollary 1.** If $V$ and $W$ are quasi-isomorphic $S$-modules, then $\text{DRep}_V(S\backslash A) \cong \text{DRep}_W(S\backslash A)$ for any algebra $S\backslash A \in \text{Alg}_S$. In particular, $H_*(S\backslash A, V) \cong H_*(S\backslash A, W)$ as graded algebras.

3.3.2. **Base change.** Let $R \xrightarrow{\varphi} S \xrightarrow{f} A$ be morphisms in $\text{DGA}_k$. Fix a DG representation $\rho : S \rightarrow \text{End} V$, and let $\rho_R := \varphi \circ \rho$. Using $\rho$ and $\rho_R$, define the representation functors $(S\backslash -)_V : \text{DGA}_S \rightarrow \text{CDGA}_k$, and $(R\backslash -)_V : \text{DGA}_R \rightarrow \text{CDGA}_k$ and consider the corresponding derived functors $L(S\backslash -)_V$ and $L(R\backslash -)_V$.

**Theorem 8** ([BKR], Theorem 2.3). (a) The commutative diagram

$$
(R\backslash S)_V \xrightarrow{(R\backslash f)_V} (R\backslash A)_V
$$

is a cocartesian square in $\text{CDGA}_k$.

(b) There is a homotopy commutative diagram

$$
L(R\backslash S)_V \xrightarrow{L(f)_V} L(R\backslash A)_V
$$

which is a cocartesian square in $\text{Ho}(\text{CDGA}_k)$.

Let us state the main corollary of Theorem 8 which may be viewed as an alternative definition of $\text{DRep}_V(S\backslash A)$. It shows that our construction of relative $\text{DRep}_V$ is a ‘correct’ one from homotopical point of view (cf. [Q2], Part I, 2.8).

**Corollary 2.** For any $S\backslash A \in \text{Alg}_S$, $\text{DRep}_V(S\backslash A)$ is a homotopy cofibre of the natural map $\text{DRep}_V(S) \rightarrow \text{DRep}_V(A)$, i.e.

$$
\text{DRep}_V(S) \xrightarrow{k} \text{DRep}_V(A) \xrightarrow{} \text{DRep}_V(S\backslash A)
$$

is a pushout diagram in $\text{Ho}(\text{CDGA}_k^+)$.

The above result suggests that the homology of $\text{DRep}_V(S\backslash A)$ should be related to the homology of $\text{DRep}_V(S)$ and $\text{DRep}_V(A)$ through a standard spectral sequence associated to a fibration. To simplify matters we will assume that $V$ is a 0-complex and work in the category $\text{DGA}_k^+$ of non-negatively graded DG algebras (cf. Remark 3.2.3).

**Corollary 3.** Given $R \rightarrow S \rightarrow A$ in $\text{DGA}_k^+$ and a representation $S \rightarrow \text{End}(V)$, there is an Eilenberg-Moore spectral sequence with

$$E_2^{p, q} = \text{Tor}_p^{H_*(R, S)_V}(k, H_*(R\backslash A, V))$$

converging to $H_*(S\backslash A, V)$.

3.3.3. The next result shows that $\text{DRep}_V(S\backslash A)$ is indeed the ‘higher’ derived functor of the classical representation scheme $\text{Rep}_V(S\backslash A)$ in the sense of homological algebra.

**Theorem 9.** Let $S \in \text{Alg}_k$ and $V$ concentrated in degree 0. Then, for any $S\backslash A \in \text{Alg}_S$,

$$H_0(S\backslash A, V) \cong (S\backslash A)_V$$

where $(S\backslash A)_V$ is a commutative algebra representing $\text{Rep}_V(S\backslash A)$. 

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Theorem 7 implies, in particular, that DRep_V(S\A) is trivial whenever Rep_V(S\A) is trivial. Indeed, if Rep_V(S\A) is empty, then (S\A)_V = 0. By Theorem 9 this means that 1 = 0 in H_1[DRep_V(S\A)], hence H_d[DRep_V(S\A)] is the zero algebra. This, in turn, means that DRep_V(S\A) is acyclic and hence DRep_V(S\A) = 0 in Ho(CDGA^+_k) as well.

**Example.** Take the first Weyl algebra A_1(k) := k⟨x,y⟩/(xy − yx − 1). Since k has characteristic zero, A_1(k) has no (nonzero) finite-dimensional modules. So Rep_V[A_1(k)] is empty and DRep_V[A_1(k)] = 0 for all V ≠ 0. Note that, even if we allow V to be a chain complex, we still get DRep_V[A_1(k)] = 0, by Proposition 4.

3.4. The invariant subfunctor. We will keep the assumption that V is a 0-complex and assume, in addition, that S = k. Let GL(V) ⊂ End(V) denote, as usual, the group of invertible endomorphisms of V. Consider the right action of GL(V) on End(V) by conjugation, α → g^(-1)αg, and extend it naturally to the functor End(−) : CDGA_k → CDGA_k. Through the adjunction of Proposition 3(b), this right action induces a (left) action on the representation functor (−)_V : CDGA_k → CDGA_k, so we can define its invariant subfunctor

\[ (−)^{GL}_V : CDGA_k → CDGA_k, \quad A → A^{GL}_V. \]

Unlike (−)_V, the functor (32) does not seem to have a right adjoint, so it is not a left Quillen functor. The Quillen Adjunction Theorem does not apply in this case. Still, using Brown’s Lemma 2 we prove

**Theorem 10** ([BKR], Theorem 2.6). (a) The functor (32) has a total left derived functor

\[ L(−)^{GL}_V : Ho(CDGA_k) → Ho(CDGA_k). \]

(b) For any A ∈ CDGA_k, there is a natural isomorphism of graded algebras

\[ H_*(L(A)^{GL}_V) \cong H_*(A, V)^{GL}(V). \]

If A ∈ Alg_k, abusing notation we will sometimes write DRep_V(A)^{GL} instead of L(A)^{GL}_V.

3.5. The Ciocan-Fontanine-Kapranov construction. For an ordinary k-algebra A and a k-vector space V, Ciocan-Fontanine and Kapranov introduced a derived affine scheme, RAct(A, V), which they called the derived space of actions of A (see [CK], Sect. 3.3). Although the construction of RAct(A, V) is quite different from our construction of DRep_V(A), Proposition 3.5.2 of [CK] shows that, for a specific resolution of A, the DG algebra k[RAct(A, V)] satisfies the adjunction of Proposition 3(b). Since k[RAct(A, V)] and DRep_V(A) are independent of the choice of resolution, we conclude

**Theorem 11.** If A ∈ Alg_k and V is a 0-complex, then k[RAct(A, V)] ≅ DRep_V(A) in Ho(CDGA^+_k).

The fact that k[RAct(A, V)] is independent of resolutions was proved in [CK] by a fairly involved calculation using spectral sequences. Strictly speaking, this calculation does not show that RAct(−, V) is a Quillen derived functor. In combination with Theorem 11 our main Theorem 7 can thus be viewed as a strengthening of [CK] — it implies that RAct(A, V) is indeed a (right) Quillen derived functor on the category of DG schemes.

3.6. Explicit presentation. Let A ∈ Alg_k. Given an semi-free resolution \( R \) \( \sim \) A in DG^+_k, the DG algebra R_V can be described explicitly. To this end, we extend a construction of Le Bruyn and van de Weyer (see [LBW], Theorem 4.1) to the case of DG algebras. Assume, for simplicity, that V = k^d. Let \( \{x^{α}\}_{α ∈ I} \) be a set of homogeneous generators of a semi-free DG algebra R, and let \( d_R \) be its differential. Consider a free graded algebra \( \tilde{R} \) on generators \( \{x^{α}_{ij} : 1 ≤ i, j ≤ d, \ α ∈ I\} \), where \( |x^{α}_{ij}| = |x^{α}| \) for all i, j. Forming matrices \( X^{α} := ||x^{α}_{ij}|| \) from these generators, we define the algebra map

\[ π : R → M_d(\tilde{R}), \quad x^{α} → X^{α}, \]

where \( M_d(\tilde{R}) \) denotes the ring of \( (d × d) \)-matrices with entries in \( \tilde{R} \). Then, letting \( d'(x^{α}_{ij}) := ||π(dx^{α})||_{ij} \), we define a differential \( \tilde{d} \) on generators of \( \tilde{R} \) and extend it to the whole of \( \tilde{R} \) by linearity and the Leibniz
rule. This makes \( \hat{R} \) a DG algebra. The commutativization of \( \hat{R} \) is a free (graded) commutative algebra generated by (the images of) \( x^n_{ij} \) and the differential on \( \hat{R}_{2n} \) is induced by \( d \).

**Theorem 12 ([BKR], Theorem 2.8).** There is an isomorphism of DG algebras \( \sqrt{R} \cong \hat{R} \). Consequently, \( R_V \cong \hat{R}_{2n} \).

Using Theorem 12, one can construct a finite presentation for \( R_V \) (and hence an explicit model for \( \text{DRep}_V(A) \)) whenever a finite semi-free resolution \( R \to A \) is available. We will apply this theorem in Section 6 where we study representation homology for three classes of algebras: noncommutative complete intersections, Koszul and Calabi-Yau algebras, which have canonical ‘small’ resolutions.

### 4. Cyclic Homology and Higher Trace Maps

In this section, we construct canonical trace maps \( \text{Tr}_V(S\setminus A)_n : \text{HC}_{n-1}(S\setminus A) \to H_n(S\setminus A, V) \) relating the cyclic homology of an \( S \)-algebra \( A \in \mathfrak{Alg}_S \) to its representation homology. In the case when \( S = k \) and \( V \) is concentrated in degree 0, these maps can be viewed as derived characters of finite-dimensional representations of \( A \).

#### 4.1. Relative cyclic homology

We begin by recalling the Feigin-Tsygan construction of cyclic homology as a non-abelian derived functor on the category of algebras (see \([FT_1, FT_2]\)). To the best of our knowledge, this construction does not appear in standard textbooks on cyclic homology (like, e.g., \([L]\) or \([W]\)). One reason for this is perhaps that while the idea of Feigin and Tsygan is very simple and natural, the proofs in \([FT_1, FT_2]\) are obtained by means of spectral sequences and are fairly indirect. In \([BKR]\), we develop a more conceptual (and in fact, slightly more general) approach and give proofs using simple model-categorical arguments. What follows is a brief summary of this approach: for details, we refer to \([BKR]\), Section 3.

If \( A \) is a DG algebra, we write \( A_2 := A/[A, A] \), where \([A, A] \) is the commutator subspace of \( A \). The assignment \( A \mapsto A_2 \) is obviously a functor from \( \mathfrak{DGA}_k \) to the category of complexes \( \mathfrak{Com}(k) \): thus, a morphism of DG algebras \( f : S \to A \) induces a morphism of complexes \( f_2 : S_2 \to A_2 \). Fixing \( S \in \mathfrak{DGA}_k \), we now define the functor

\[
C : \mathfrak{DGA}_{S} \to \mathfrak{Com}(k) , \quad (S \xrightarrow{f} A) \mapsto \text{cone}(f_2) ,
\]

where ‘cone’ refers to the mapping cone in \( \mathfrak{Com}(k) \).

The category \( \mathfrak{Com}(k) \) has a natural model structure with quasi-isomorphisms being the weak equivalence and the epimorphisms being the fibrations. The corresponding homotopy category \( \text{Ho}(\mathfrak{Com}(k)) \) is isomorphic to the (unbounded) derived category \( \mathcal{D}(k) := \mathcal{D}(\mathfrak{Com}) \) (cf. Theorem 11).

**Theorem 13.** The functor \([S] \) has a total left derived functor \( \text{LC} : \text{Ho}(\mathfrak{DGA}_S) \to \mathcal{D}(k) \) given by

\[
\text{LC}(S\setminus A) = \text{cone}(S_2 \to Q(S\setminus A)_2) ,
\]

where \( S \to Q(S\setminus A) \) is a cofibrant resolution of \( S \to A \) in \( \mathfrak{DGA}_S \).

Theorem 13 implies that the homology of \( \text{LC}(S\setminus A) \) depends only on the morphism \( S \to A \). Thus, we may give the following

**Definition.** The (relative) cyclic homology of \( S\setminus A \in \mathfrak{DGA}_S \) is defined by

\[
\text{HC}_{n-1}(S\setminus A) := \text{H}_n[\text{LC}(S\setminus A)] = \text{H}_n[\text{cone}(S_2 \to Q(S\setminus A)_2)] .
\]

If \( S \to A \) is a map of ordinary algebras and \( S \xrightarrow{i} QA \xrightarrow{\sim} A \) is a cofibrant resolution of \( S \to A \) such that \( i \) is a semi-free extension in \( \mathfrak{DGA}_S \), then the induced map \( i_2 : S_2 \xleftarrow{} (QA)_2 \) is injective, and

\[
C(S\setminus QA) = \text{cone}(i_2) \cong (QA)_2/S_2 \cong QA/[QA, QA] + i(S) .
\]

In this special form, the functor \( C \) was originally introduced by Feigin and Tsygan in \([FT]\) (see also \([FT_1]\) ); they proved that the homology groups \([S] \) are independent of the choice of resolution using spectral sequences. Theorem 13 is not explicitly stated in \([FT, FT_1]\), although it is implicit in several
calculated. We emphasize that, in the case when $S$ and $A$ are ordinary algebras, our definition of relative cyclic homology \((34)\) agrees with the Feigin-Tsygan one.

One of the key properties of relative cyclic homology is the existence of a long exact sequence for composition of algebra maps. Precisely,

**Theorem 14** ([FT], Theorem 2). Given DG algebra maps $R \to S \to A$, there is an exact sequence in cyclic homology

\[ \ldots \to \text{HC}_n(R \setminus S) \to \text{HC}_n(R \setminus A) \to \text{HC}_n(S \setminus A) \to \text{HC}_{n-1}(R \setminus S) \to \ldots \]

In fact, the long exact sequence \((36)\) arises from the distinguished triangle in $\mathcal{D}(k)$:

\[ \text{LC}(R \setminus S) \to \text{LC}(R \setminus A) \to \text{LC}(S \setminus A) \to \text{LC}(R \setminus S)[1], \]

the construction of \((37)\) is given in [BKR], Theorem 3.3.

If $A$ is an ordinary algebra over a field of characteristic zero, its cyclic homology $\text{HC}_*(A)$ is usually defined as the homology of the cyclic complex $\text{CC}(A)$ (cf. [L], Sect. 2.1.4):

\[ \text{CC}_n(A) := A^{\otimes (n+1)}/\text{Im}(\text{Id} - t_n), \quad b_n : \text{CC}_n(A) \to \text{CC}_{n-1}(A), \]

where $b_n$ is induced by the standard Hochschild differential and $t_n$ is the cyclic operator defining an action of $\mathbb{Z}/(n+1)$ on $A^{\otimes (n+1)}$:

\[ t_n : A^{\otimes (n+1)} \to A^{\otimes (n+1)}, \quad (a_0, a_1, \ldots, a_n) \mapsto (-1)^n(a_n, a_0, \ldots, a_{n-1}). \]

The complex $\text{CC}(A)$ contains the canonical subcomplex $\text{CC}(k)$: the homology of the corresponding quotient complex $\overline{\text{HC}}_*(A) := H_*(\text{CC}(A)/\text{CC}(k))$ is called the reduced cyclic homology of $A$. Both $\text{HC}(A)$ and $\overline{\text{HC}}(A)$ are special cases of relative cyclic homology in the sense of Definition \((34)\). Precisely, we have the following result (due to Feigin and Tsygan [FT]).

**Proposition 5.** For any $k$-algebra $A$, there are canonical isomorphisms

(a) $\text{HC}_n(A) \cong \text{HC}_n(A \setminus 0)$ for all $n \geq 0$,

(b) $\overline{\text{HC}}_n(A) \cong \text{HC}_{n-1}(k \setminus A)$ for all $n \geq 1$.

**Proof.** (a) For any (unital) algebra $A$, the DG algebra $A \langle x \rangle := A \ast k \langle x \rangle$ coincides with the bar construction of $A$ and hence is acyclic. The canonical morphism $A \to A \langle x \rangle$ provides then a cofibrant resolution of $A \to 0$ in $\text{DGA}_A$. In this case, we can identify

\[ \text{LC}(A \setminus 0) \cong \text{cone}(A_k \to A \langle x \rangle_k) \cong A \langle x \rangle/(A + [A \langle x \rangle, A \langle x \rangle]) \cong \text{CC}(A)[1], \]

where the last isomorphism (in degree $n > 0$) is given by $a_1 x a_2 x \ldots a_n x \leftrightarrow a_1 \otimes a_2 \otimes \ldots \otimes a_n$. On the level of homology, this induces isomorphisms $\text{HC}_{n-1}(A \setminus 0) \cong H_n(\text{CC}(A)[1]) \cong \text{HC}_{n-1}(A)$.

(b) With above identification, the triangle \((37)\) associated to the canonical maps $k \to A \to 0$ yields

\[ \text{LC}(k \setminus A) \cong \text{cone}(\text{LC}(k \setminus 0) \to \text{LC}(A \setminus 0))[1] \cong \text{cone}(\text{CC}(k) \to \text{CC}(A)). \]

Whence $\text{HC}_{n-1}(k \setminus A) \cong \overline{\text{HC}}_n(A)$ for all $n \geq 1$. \(\square\)

As a consequence of Theorem \([14]\) and Proposition \([5](a)\), we get the fundamental exact sequence associated to an algebra map $S \to A$:

\[ \ldots \to \text{HC}_n(S \setminus A) \to \text{HC}_n(S) \to \text{HC}_n(A) \to \text{HC}_{n-1}(S \setminus A) \to \ldots \to \text{HC}_0(S) \to \text{HC}_0(A) \to 0. \]

In particular, if we take $S = k$ and use the isomorphism of Proposition \([5](b)\), then \((14)\) becomes

\[ \ldots \to \text{HC}_n(k) \to \text{HC}_n(A) \to \overline{\text{HC}}_n(A) \to \text{HC}_{n-1}(k) \to \ldots \to \text{HC}_0(A) \to \overline{\text{HC}}_0(A) \to 0. \]

**Remark.** The isomorphism of Proposition \([5](a)\) justifies the shift of indexing in our definition \((34)\) of relative cyclic homology. In [FT], cyclic homology is referred to as an additive $K$-theory, and a different notation is used. The relation between the Feigin-Tsygan notation and our notation is $\text{K}_n^+(A, S) = \text{HC}_{n-1}(S \setminus A)$ for all $n \geq 1$. 19
4.2. Trace maps. Let $V$ be a complex of $k$-vector spaces of total dimension $d$. The natural map $k \hookrightarrow \text{End}(V) \rightarrow \text{End}(V)_k$ is an isomorphism of complexes, which we can use to identify $\text{End}(V)_k = k$. This defines a canonical (super) trace map $\operatorname{Tr}_V : \text{End} V \rightarrow k$ on the DG algebra $\text{End} V$. Explicitly, $\operatorname{Tr}_V$ is given by $$\operatorname{Tr}_V(f) = \sum_{i=1}^d (-1)^{|v_i|} f_{ii},$$ where $\{v_i\}$ is a homogeneous basis in $V$ and $\|f_{ij}\|$ is the matrix representing $f \in \text{End} V$ in this basis.

Now, fix $S \in \text{DGA}_k$ and a DG algebra map $\varrho : S \rightarrow \text{End} V$ making $V$ a DG module over $S$. For an $S$-algebra $A \in \text{DGA}_S$, consider the (relative) DG representation scheme $\text{Rep}_V(S\backslash A)$, and let $\pi_V : A \rightarrow \text{End} V \otimes (S\backslash A)_V$ denote the universal representation of $A$ corresponding to the identity map in the adjunction of Proposition 3(b). Consider the morphism of complexes $$\pi_V : A \xrightarrow{\pi_V} \text{End} V \otimes (S\backslash A)_V \xrightarrow{\operatorname{Tr}_V \otimes \text{Id}} (S\backslash A)_V.$$

Since $\pi_V$ is a map of $S$-algebras, and the $S$-algebra structure on $\text{End} V \otimes (S\backslash A)_V$ is of the form $\varrho \otimes \text{Id}$, (41) induces a map $\operatorname{Tr}_V \circ \pi_V : A \rightarrow (S\backslash A)_V$, which fits in the commutative diagram

$$\begin{array}{ccc}
S_2 & \xrightarrow{\pi_V \circ \varrho} & A_2 \\
\downarrow{\operatorname{Tr}_V \circ \varrho} & & \downarrow{\operatorname{Tr}_V \circ \pi_V} \\
k & \xrightarrow{\pi_V} & (S\backslash A)_V
\end{array}$$

This, in turn, induces a morphism of complexes

$$\text{cone}(S_2 \rightarrow A_2) \rightarrow (S\backslash A)_V,$$

where we write $\bar{R} = R/k \cdot 1_R$ for a unital DG algebra $R$. The family of morphisms (43) defines a natural transformation of functors from $\text{DGA}_S$ to $\text{Com}(k)$:

$$\operatorname{Tr}_V : \mathcal{C} \rightarrow (-)_V.$$

The next lemma is a formal consequence of Theorem 7 and Theorem 13.

**Lemma 6.** $\operatorname{Tr}_V$ induces a natural transformation $\mathcal{L} \rightarrow \mathcal{L}(-)_V$ of functors $\text{Ho}(\text{DGA}_S) \rightarrow \mathcal{D}(k)$.

For any (non-acyclic) unital DG algebra $R$, we have

$$H_n(R) \cong \begin{cases} 
H_0(R) & , \quad n = 0 \\
H_n(R) & , \quad n \neq 0
\end{cases}$$

This is immediate from the long homology sequence arising from $0 \rightarrow k \rightarrow R \rightarrow \bar{R} \rightarrow 0$. Hence, if $A \in \text{Alg}_S$ is an ordinary algebra, applying the natural transformation of Lemma 6 to $S\backslash A$ and using (45), we can define

$$\operatorname{Tr}_V(S\backslash A)_n : \text{HC}_{n-1}(S\backslash A) \rightarrow H_n(S\backslash A, V) , \quad n \geq 1.$$

Assembled together, these trace maps define a homomorphism of graded commutative algebras

$$\Delta \operatorname{Tr}_V(S\backslash A)_* : \Delta(\text{HC}(S\backslash A)[1]) \rightarrow H_*(S\backslash A, V),$$

where $\Delta$ denotes the graded symmetric algebra of a graded $k$-vector space $W$.

We examine the trace maps (46) and (47) in the special case when $S = k$ and $V$ is a single vector space concentrated in degree 0. In this case, by Proposition 5, the maps (46) relate the reduced cyclic homology of $A$ to the (absolute) representation homology:

$$\operatorname{Tr}_V(A)_n : \text{HHC}_n(A) \rightarrow H_n(A, V) , \quad n \geq 1.$$
Now, for each $n$, there is a natural map $\text{HC}_n(A) \to \overline{\text{HC}}_n(A)$ induced by the projection of complexes $\text{CC}(A) \to \overline{\text{CC}}(A)$, cf. (48). Combining this map with (48), we get

$$\text{Tr}_V(A)_n : \text{HC}_n(A) \to H_n(A, V) , \ n \geq 0$$.

Notice that (49) is defined for all $n$, including $n = 0$. In the latter case, $H_0(A, V) \cong A_V$ (by Theorem 4.2), and $\text{Tr}_V(A)_0 : A \to A_V$ is the usual trace induced by $A \xrightarrow{\pi} A_V \otimes \text{End}_k V \xrightarrow{\text{Id} \otimes \text{Tr}} A_V$.

The linear maps (49) define an algebra homomorphism

$$\Lambda \text{Tr}_V(A)_* : \Lambda[\text{HC}(A)] \to H_* (A, V)$$.

Since, for $n \geq 1$, (49) factor through reduced cyclic homology, (50) induces

$$\overline{\text{Tr}}_V(A)_* : \text{Sym} (\text{HC}_0(A)) \otimes \Lambda (\text{HC}_{\geq 1}(A)) \to H_* (A, V)$$,

where $\overline{\text{HC}}_{\geq 1}(A) := \bigoplus_{n \geq 1} \text{HC}_n(A)$.

**Proposition 6.** The image of the maps (50) and (51) is contained in $H_* (A, V)^{\text{GL}(V)}$.

Our next goal is to construct an explicit morphism of complexes $T : \text{CC}(A) \to R_V$ that induces the trace maps (49). Recall that if $R \in \text{DGA}_k$ is a DG algebra, its (reduced) bar construction $B(R)$ is a (noncounital) DG coalgebra, which is a universal model for twisting cochains with values in $R$ (see HMS, Chap. II). Explicitly, $B(R)$ can be identified with the tensor coalgebra $B(R) = \bigoplus_{n \geq 1} R[1] \otimes^n$, the universal twisting cochain being the canonical map $\hat{\theta} : B(R) \to R$ of degree $-1$.

Now, let $\pi : R \overset{\sim}{\to} A$ be a semi-free resolution of an algebra $A$ in $\text{DGA}_k^+$. By functoriality of the bar construction, the map $\pi$ extends to a surjective quasi-isomorphism of DG coalgebras $B(R) \simplyto B(A)$, which we still denote by $\pi$. This quasi-isomorphism has a section $f : B(A) \to B(R)$ in the category of DG coalgebras, that is uniquely determined by the twisting cochain $\theta_\pi := \hat{\theta} f : B(A) \to R$. The components $f_n : A \otimes^n \to R_{n-1}$, $n \geq 1$, of $\theta_\pi$ satisfy the Maurer-Cartan equations

$$\pi f_1 = \text{Id}_A$$

$$d_R f_2 = f_1 m_A - m_R (f_1 \otimes f_1)$$

$$d_R f_n = \sum_{i=1}^{n-1} (-1)^{i-1} f_{n-i}(\text{Id}_A \otimes (\text{Id}_A \otimes \text{Id}_A)^{(n-1-i)}) + \sum_{i=1}^{n-1} (-1)^{i} m_R (f_i \otimes f_{n-i}) , \ n \geq 2$$,

where $m_A$ and $m_R$ denote the multiplication maps of $A$ and $R$, respectively. Giving the maps $f_n : A \otimes^n \to R_{n-1}$ is equivalent to giving a quasi-isomorphism of $A_{\infty}$-algebras $f : A \to R$, which induces the inverse of $\pi$ on the level of homology. The existence of such a quasi-isomorphism is a well-known result in the theory of $A_{\infty}$-algebras (see [K1], Theorem 3.3). Since $\pi : R \to A$ is a homomorphism of *unital* algebras, we may assume that $f$ is a (strictly) unital homomorphism of $A_{\infty}$-algebras: this means that, in addition to (52) - (54), we have the relations (cf. [K1], Sect. 3.3)

$$f_1(1) = 1 \quad f_n(a_1, a_2, \ldots, a_n) = 0 \quad n \geq 2$$,

whenever one of the $a_i$’s equals 1.

To state the main result of this section, we fix a $k$-vector space $V$ of (finite) dimension $d$ and, for the given semi-free resolution $R \to A$, consider the DG algebra $R_V = (\sqrt{R})_2$. As explained in Remark following Proposition 3, the elements of $R_V$ can be written in the ‘matrix’ form as the images of $a_{ij} \in \sqrt{R}$, see (27), under the commutativization map $\sqrt{R} \to R_V$. With this notation, we have

**Theorem 15** ([BK], Theorem 4.2). *The trace maps (49) are induced by the morphism of complexes $T : \text{CC}(A) \to R_V$, whose $n$-th graded component $T_n : A \otimes^{(n+1)}/\text{Im}(1 - t_n) \to (R_V)_n$ is given by*

$$T_n(a_1, a_2, \ldots, a_{n+1}) = \sum_{i=1}^{d} \sum_{k \in [n+1]} (-1)^{nk} f_{n+1}(a_{1+k}, a_{2+k}, \ldots, a_{n+1+k})_{ii} ,$$

*where $(f_1, f_2, \ldots)$ are defined by the relations (52) - (54) and (55).*
For $n = 0$, it is easy to see that (53) induces

$$\text{Tr}_V(A)_0 : A \to H_0(A, V) = A_V, \quad a \mapsto \sum_{i=1}^d a_{ii},$$

which is the usual trace map on $\text{Rep}_V(A)$.

We can also write an explicit formula for the first trace $\text{Tr}_V(A)_1 : \text{HC}_1(A) \to H_1(A, V)$. For this, we fix a section $f_1 : A \to R_0$ satisfying \([\xi]\), and let $\omega : A \otimes A \to R_0$ denote its ‘curvature’:

$$\omega(a, b) := f_1(ab) - f_1(a)f_1(b), \quad a, b \in A.$$

Notice that, by \([\xi]\), $\text{Im} \omega \subseteq \text{Ker} \pi$. On the other hand, $\text{Ker} \pi \cong \text{R}_1 \cong \text{R}_1/\text{d} \text{R}_2$, since $\text{R}$ is acyclic in positive degrees. Thus, identifying $\text{Ker} \pi = \text{R}_1/\text{d} \text{R}_2$ via the differential on $\text{R}$, we get a map $\tilde{\omega} : A \otimes A \to \text{R}_1/\text{d} \text{R}_2$ such that $\text{d} \tilde{\omega} = \omega$. Using this map, we define

$$\text{ch}_2 : \text{CC}_1(A) \to \text{R}_1/\text{d} \text{R}_2, \quad (a, b) \mapsto [\tilde{\omega}(a, b) - \tilde{\omega}(b, a)] \text{ mod } \text{d} \text{R}_2.$$

Since $\tilde{\omega} \equiv f_2 (\text{mod } \text{d} \text{R}_2)$, cf. \([\xi]\), it follows from \([\xi]\) that $\text{Tr}_V(A)_1$ is induced by the map

$$\text{Tr}_V(A)_1 : (a, b) \mapsto \sum_{i=1}^d \text{ch}_2(a, b)_{ii}.$$

**Remark.** The notation ‘\(\text{ch}_2\)’ for \((57)\) is justified by the fact that this map coincides with the second Chern character in Quillen’s Chern-Weil theory of algebra cochains (see \([Q2]\)). It would be interesting to see whether the higher traces $\text{Tr}_V(A)_n$ can be expressed in terms of higher Quillen-Chern characters.

### 4.3. Relation to Lie algebra homology

There is a close relation between representation homology and the homology of matrix Lie algebras. To describe this relation we first recall a celebrated result of Loday-Quillen \([LQ]\) and Tsygan \([T]\) which was historically at the origin of cyclic homology theory.

For a fixed $k$-algebra $A$ and finite-dimensional vector space $W$, let $H_*(\text{gl}_W(A); k)$ denote the homology of the Lie algebra $\text{gl}_W(A) := \text{Lie}(\text{End} W \otimes A)$. The Loday-Quillen-Tsygan Theorem states that there are natural maps

$$H_{n+1}(\text{gl}_W(A); k) \to \text{HC}_n(A), \quad \forall n \geq 0,$$

which, in the limit $\text{gl}_W(A) \to \text{gl}_\infty(A)$, induce an isomorphism of graded Hopf algebras

$$H_*(\text{gl}_\infty(A); k) \sim A(\text{HC}(A)[1]).$$

Explicitly, the maps \([\xi]\) are induced by the morphisms of complexes

$$A^{n+1} \text{gl}_W(A) \xrightarrow{\vartheta_*} \text{CC}_*(\text{End} W \otimes A) \xrightarrow{\text{tr}_*} \text{CC}_*(A),$$

where $\vartheta_*$ is defined by

$$\vartheta_n(\xi_0 \wedge \xi_1 \wedge \ldots \wedge \xi_n) = \sum_{\sigma \in S_n} \text{sgn}(\sigma)(\xi_{\sigma(1)}, \ldots, \xi_{\sigma(n)}),$$

and $\text{tr}_*$ is given by the generalized trace maps $\text{tr}_n : (\text{End} W \otimes A)^{\otimes(n+1)} \to A^{\otimes(n+1)}$ (see \([L]\), 10.2.3).

It turns out that there is a natural map relating the Lie algebra homology $H_*(\text{gl}_W(A); k)$ to representation homology of $A$. To construct this map we will realize the Lie algebra homology as Quillen homology of the category $\text{DGLA}_k^+$ of non-negatively graded DG Lie algebras (cf. Example 1 in Section \([2.0]\)). This category has a natural model structure, which is compatible with the model structure on $\text{DGA}_k$ via the forgetful functor $\text{Lie} : \text{DGLA}_k^+ \to \text{DGA}_k^+$ (see \([Q2]\), Part II, Sect. 5). Fix a cofibrant resolution $\alpha : R \xrightarrow{\sim} A$ of $A$ in $\text{DGA}_k^+$. Then, for each finite-dimensional vector space $W$, tensoring $\alpha$ by $\text{End}(W)$ yields an acyclic fibration in $\text{DGA}_k^+$ which, in turn, yields (via the forgetful functor) an acyclic fibration in $\text{DGLA}_k^+$:

$$\bar{\alpha} : \text{gl}_W(R) \xrightarrow{\sim} \text{gl}_W(A).$$
Now, let $\beta : \mathcal{L}_W \to g\mathfrak{l}_W(A)$ be a cofibrant resolution of $g\mathfrak{l}_W(A)$ in $\text{DGLA}_k^+$. Since $\tilde{\alpha}$ is an acyclic fibration, $\beta$ lifts through $\tilde{\alpha}$ giving a quasi-isomorphism $\tilde{\beta} : \mathcal{L}_W \to g\mathfrak{l}_W(R)$. Combining this quasi-isomorphism with traces induces the map of complexes

$$\mathcal{L}_W/[\mathcal{L}_W, \mathcal{L}_W] \tilde{\beta} g\mathfrak{l}_W(R)/[g\mathfrak{l}_W(R), g\mathfrak{l}_W(R)] = (\text{End} W \otimes R)^{\nu} \to R^\nu.$$  

Now, for any cofibrant resolution $\mathcal{L} \to g$ of in $\text{DGLA}_k^+$, the complex $\mathcal{L}/[\mathcal{L}, \mathcal{L}]$ computes the Lie algebra homology of $g$ with trivial coefficients, see (10). Thus, for any $V$ and $W$, (61) induces the maps

$$H_{n+1}(gL_A(A); k) \to H_n(A, V), \quad n \geq 0.$$  

Letting $W = k^d$ and taking the inductive limit (as $d \to \infty$), we identify

$$\lim \downarrow H_n(g\mathfrak{l}_W(A); k) \cong H_*(\lim \downarrow g\mathfrak{l}_W(A); k) = H_*(g\mathfrak{l}_\infty(A); k).$$

With this identification, (62) induces the maps

$$H_{n+1}(g\mathfrak{l}_\infty(A); k) \to H_n(A, V), \quad \forall n \geq 0.$$  

**Theorem 16** ([BR], Theorem 4.3). For each $n \geq 0$, the maps (62) and (63) factor through the Loday-Quillen-Tsygan map (59). The induced maps are precisely the trace maps (49).

Note that for $n = 0$, the map (62) is simply the composition of obvious traces

$$H_1(g\mathfrak{l}_W(A); k) \cong g\mathfrak{l}_W(A)/[g\mathfrak{l}_W(A), g\mathfrak{l}_W(A)] = (\text{End} W \otimes A)^{\nu} \to A^{\nu} \to A \to A_V,$$

so the claim of Theorem 16 is immediate in this case.

**Remark.** The homology of a Lie algebra with trivial coefficients has a natural coalgebra structure (cf. [L 10.1.3]). One can show that the degree $(-1)$ map $\tau : H_*(g\mathfrak{l}_W(A); k) \to H_*(A, V)^{GL(V)}$ defined by (62) is a twisting cochain with respect to the coalgebra structure on $H_*(g\mathfrak{l}_W(A); k)$. In the stable limit (see Section 4.4 below), $\tau$ becomes an acyclic twisting cochain, which means that the Lie algebra homology of $g\mathfrak{l}_\infty(A)$ is Koszul dual to the stable representation homology of $A$. For a precise statement of this result and its implications we refer the reader to [BR] Section 5 (see, in particular, op. cit., Theorem 5.2).

### 4.4. Stabilization theorem

If $A$ is an ordinary algebra, a fundamental theorem of Procesi ([P]) implies that the traces of elements of $A$ generate the algebra $A_V^{GL(V)}$; in other words, the algebra map

$$\text{Sym}[\text{Tr}_V(A) \otimes] : \text{Sym}[\text{HC}_0(A)] \to A_V^{GL(V)}$$

is surjective for all $V$. A natural question is whether this result extends to higher traces: namely, is the full trace map

$$A\text{Tr}_V(A) : A[\text{HC}(A)] \to H_*(A, V)^{GL(V)}$$

surjective? We address this question in the forthcoming paper [BR], where by analogy with matrix Lie algebras (see [T] LQ] we approach it in two steps. First, we ‘stabilize’ the family of maps (65) passing to an infinite-dimensional limit $\dim V \to \infty$ and prove that (65) becomes an isomorphism in that limit. Then, for a finite-dimensional $V$, we construct obstructions to $H_*(A, V)^{GL(V)}$ attaining its ‘stable limit’. These obstructions arise as homology of a complex that measures the failure of (65) being surjective. Thus, the answer to the above question is negative. A simple counterexample will be given in Section 6.3 below.

We conclude this section by briefly explaining the stabilization procedure of [BR]. We will work with unital DG algebras $A$ which are **augmented over** $k$. We recall that the category of such DG algebras is naturally equivalent to the category of non-unital DG algebras, with $A$ corresponding to its augmentation ideal $\bar{A}$. We identify these two categories and denote them by $\text{DGA}_{k/k}$. Further, to simplify the notation we take $V = k^d$ and identify $\text{End} V = \mathbb{M}_d(k)$, $GL(V) = GL_k(d)$; in addition, for $V = k^d$, we will write $A_V$ as $A_d$. Bounding a matrix in $\mathbb{M}_d(k)$ by 0's on the right and on the bottom gives an embedding $\mathbb{M}_d(k) \hookrightarrow \mathbb{M}_{d+1}(k)$ of non-unital algebras. As a result, for each $B \in \text{CDGA}_k$, we get a map of sets

$$\text{Hom}_{\text{DGA}_{k/k}}(\bar{A}, \mathbb{M}_d(B)) \to \text{Hom}_{\text{DGA}_{k/k}}(\bar{A}, \mathbb{M}_{d+1}(B)).$$
defining a natural transformation of functors from $\mathbf{CDGA}_k$ to $\mathbf{Sets}$. Since $B$’s are unital and $A$ is augmented, the restriction maps

\begin{equation}
\text{Hom}_{\mathbf{CDGA}}(A, M_d(B)) \xrightarrow{\sim} \text{Hom}_{\mathbf{CDGA}}(\bar{A}, M_d(B)) \ , \ \varphi \mapsto \varphi|_{\bar{A}}
\end{equation}

are isomorphisms for all $d \in \mathbb{N}$. Combining (66) and (67), we thus have natural transformations

\begin{equation}
\text{Hom}_{\mathbf{CDGA}}(A, M_d(-)) \to \text{Hom}_{\mathbf{CDGA}}(A, M_{d+1}(-)) .
\end{equation}

By standard adjunction, (68) yield an inverse system of morphisms \( \{ \mu_{d+1,d} : A_{d+1} \to A_d \} \) in $\mathbf{CDGA}_k$. Taking the limit of this system, we define

\[ A_\infty := \lim_{d \in \mathbb{N}} A_d . \]

Next, we recall that the group $\text{GL}(d)$ acts naturally on $A_d$, and it is easy to check that $\mu_{d+1,d} : A_{d+1} \to A_d$ maps the subalgebra $A_{d+1}^{\text{GL}}$ of $\text{GL}$-invariants in $A_{d+1}$ to the subalgebra $A_d^{\text{GL}}$ of $\text{GL}$-invariants in $A_d$. Defining $\text{GL}(\infty) := \lim_d \text{GL}(d)$ through the standard inclusions $\text{GL}(d) \to \text{GL}(d+1)$, we extend the actions of $\text{GL}(d)$ on $A_d$ to an action of $\text{GL}(\infty)$ on $A_\infty$ and let $A_{d}^{\text{GL}(\infty)}$ denote the corresponding invariant subalgebra. Then one can prove (see [TT])

\[ A_{\infty}^{\text{GL}(\infty)} \cong \lim_{d \in \mathbb{N}} A_d^{\text{GL}(d)} . \]

This isomorphism allows us to equip $A_{\infty}^{\text{GL}(\infty)}$ with a natural topology: namely, we put first the discrete topology on each $A_d^{\text{GL}(d)}$ and equip $\prod_{d \in \mathbb{N}} A_d^{\text{GL}(d)}$ with the product topology; then, identifying $A_{\infty}^{\text{GL}(\infty)}$ with a subspace in $\prod_{d \in \mathbb{N}} A_d^{\text{GL}(d)}$ via (69), we put on $A_{\infty}^{\text{GL}(\infty)}$ the induced topology. The corresponding topological DG algebra will be denoted $A_{\infty}^{\text{GL}}$.

Now, for each $d \in \mathbb{N}$, we have the commutative diagram

\[ \begin{array}{ccc}
C(A) & \xrightarrow{\text{Tr}_d(A)} & A_d^{\text{GL}} \\
\mu_{d+1,d} & \downarrow & \downarrow \mu_{d+1,d} \\
A_{d+1}^{\text{GL}} & \xrightarrow{\text{Tr}_{d+1}(A)} & A_{d+1}^{\text{GL}} 
\end{array} \]

where $C(A)$ is the cyclic functor restricted to $\mathbf{DGA}_{k/k}$ (cf. Section 2.1). Hence, by the universal property of inverse limits, there is a morphism of complexes $\text{Tr}_\infty(A)_\bullet : C(A) \to A_{\infty}^{\text{GL}}$ that factors $\text{Tr}_d(A)_\bullet$, for each $d \in \mathbb{N}$. We extend this morphism to a homomorphism of commutative DG algebras:

\[ \text{Tr}_\infty(A)_\bullet : A[C(A)] \to A_{\infty}^{\text{GL}} . \]

The following lemma is one of the key technical results of [BR] (see loc. cit., Lemma 3.1).

\begin{lemma}
The map (70) is topologically surjective: i.e., its image is dense in $A_{\infty}^{\text{GL}}$.
\end{lemma}

Letting $A_{\infty}^{\text{Tr}}$ denote the image of (70), we define the functor

\[ (-)_{\infty}^{\text{Tr}} : \mathbf{DGA}_{k/k} \to \mathbf{CDGA}_k \ , \ A \mapsto A_{\infty}^{\text{Tr}} . \]

The algebra maps (70) then give a morphism of functors

\[ \text{Tr}_\infty(-)_\bullet : A[C(-)] \to (-)_{\infty}^{\text{Tr}} . \]

Now, to state the main result of [BR] we recall that the category of augmented DG algebras $\mathbf{DGA}_{k/k}$ has a natural model structure induced from $\mathbf{DGA}_k$. We also recall the derived Feigin-Tsygan functor $L(-) : \text{Ho}(\mathbf{DGA}_{k/k}) \to \text{Ho}(\mathbf{CDGA}_k)$ inducing the isomorphism of Proposition (b).

\begin{theorem}[BR, Theorem 4.2]
(a) The functor (71) has a total left derived functor $L(-)_{\infty}^{\text{Tr}} : \text{Ho}(\mathbf{DGA}_{k/k}) \to \text{Ho}(\mathbf{CDGA}_k)$.

(b) The morphism (72) induces an isomorphism of functors

\[ \text{Tr}_\infty(-)_\bullet : A[L(-)] \xrightarrow{\sim} L(-)_{\infty}^{\text{Tr}} . \]

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By definition, \( L(-)^{Tr} \) is given by \( L(A)^{Tr} = (QA)^{Tr} \), where \( QA \) is a cofibrant resolution of \( A \) in \( DGA_{k/k} \).

For an ordinary augmented \( k \)-algebra \( A \in Alg_{k/k} \), we set

\[
DRep^\infty(A)^{Tr} := (QA)^{Tr}.
\]

By part \((a)\) of Theorem \(17\) \( DRep^\infty(A)^{Tr} \) is well defined. On the other hand, part \((b)\) implies

**Corollary 4.** For any \( A \in Alg_{k/k} \), \( Tr^\infty(A) \) induces an isomorphism of graded commutative algebras

\[
\Lambda[HC(A)] \cong H_*[DRep^\infty(A)^{Tr}] .
\]

In fact, one can show that \( H_*[DRep^\infty(A)^{Tr}] \) has a natural structure of a graded Hopf algebra, and the isomorphism of Corollary \(25\) is actually an isomorphism of Hopf algebras. This isomorphism is analogous to the Loday-Quillen-Tsygan isomorphism \(10\) computing the stable homology of matrix Lie algebras \( gl_n(A) \) in terms of cyclic homology. Heuristically, it implies that the cyclic homology of an augmented algebra is determined by its representation homology.

## 5. Abelianization of the Representation Functor

"If homotopical algebra is thought of as ‘nonlinear’ or ‘non-additive’ homological algebra, then it is natural to ask what is the ‘linearization’ or ‘abelianization’ of this situation” (Quillen, \(\text{[QT]}\), § II.5). In Section 2.6 following Quillen, we defined the abelianization of a model category \( \mathcal{C} \) as the category \( \mathcal{C}^{ab} \) of abelian group objects in \( \mathcal{C} \). As a next step, one should ask for abelianization of a functor \( F: \mathcal{C} \to \mathcal{D} \) between model categories. We formalize this notion in Section 5.1 below, and then apply it to our representation functor \( (-)^\bullet_v: DGA_k \to CDGA_k \). As a result, for a given algebra \( A \in DGA_k \), we get an additive left Quillen functor

\[
(-)^{ab}_v : \text{DG Bimod}(A) \to \text{DG Mod}(A^v) ,
\]

relating the category of DG bimodules over \( A \) to DG modules over \( A^v \). In the case of ordinary algebras, this functor was introduced by M. Van den Bergh \(\text{[VdB]}\). He found that \(74\) plays a special role in noncommutative geometry of smooth algebras, transforming noncommutative objects on \( A \) to classical geometric objects on \( \text{Rep}_v(A) \). Passing from \( \text{Rep}_v(A) \) to \( D\text{Rep}_v(A) \), we constructed in \(\text{[BKR]}\) the derived functor of \(74\) and showed that it plays a similar role in the geometry of arbitrary (not necessarily smooth) algebras. The original definition of \(74\) in \(\text{[VdB]}\) is given by an explicit but somewhat \textit{ad hoc} construction (cf. \(\text{[ML]}\) below). Characterizing Van den Bergh’s functor as abelianization of the representation functor provides a conceptual explanation of the results of \(\text{[VdB]}\) and \(\text{[BKR]}\). At the derived level, this also leads to a new spectral sequence relating representation homology to Andrè-Quillen homology (see Section 5.3 below).

### 5.1. Abelianization as a Kan extension

Let \( F: \mathcal{C} \to \mathcal{D} \) be a right exact (i.e., compatible with finite colimits) functor between model categories. As in Section 2.6 we assume that \( \mathcal{C}^{ab} \) and \( \mathcal{D}^{ab} \) are abelian categories with enough projectives and the abelianization functors \( Ab_c: \mathcal{C} \to \mathcal{C}^{ab} \) and \( Ab_d: \mathcal{D} \to \mathcal{D}^{ab} \) exist and form Quillen pairs, see \(\text{[H]}\). In general, \( F \) may not descend to an additive functor \( F^{ab}: \mathcal{C}^{ab} \to \mathcal{D}^{ab} \) that would complete the commutative diagram

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow Ab_c & & \downarrow Ab_d \\
\mathcal{C}^{ab} & \xrightarrow{F^{ab}} & \mathcal{D}^{ab}
\end{array}
\]

Following a standard categorical approach (see \(\text{[ML]}\) Chapter X), we remedy this problem in two steps. First, we define the ‘best left approximation’ to \( F^{ab} \) (which we call the left abelianization) as a right Kan extension of \( Ab_d \circ F \) along \( Ab_c \). Precisely, the left abelianization of \( F \) is a right exact additive functor \( F^{ab}_l: \mathcal{C}^{ab} \to \mathcal{D}^{ab} \) together with a natural transformation \( t: F^{ab}_l \circ Ab_c \to Ab_d \circ F \) satisfying the following universal property:
For any pair $(G, s)$ consisting of a right exact additive functor $G : \mathcal{C}^{\text{ab}} \to \mathcal{D}^{\text{ab}}$ and a natural transformation $s : G \circ \mathcal{A}_\mathcal{E} \to \mathcal{A}_\mathcal{B}_\mathcal{D} \circ F$, there is a unique natural transformation $s' : G \to F^{\text{ab}}_{\mathcal{I}}$ such that the following diagram commutes:

$$
G \circ \mathcal{A}_\mathcal{E} \quad \xrightarrow{s} \quad \mathcal{A}_\mathcal{B}_\mathcal{D} \circ F
$$

Next, we say that $F$ is abelianizable if $F^{\text{ab}}_{\mathcal{I}}$ exists, and the corresponding natural transformation $t : F^{\text{ab}}_{\mathcal{I}} \circ \mathcal{A}_\mathcal{E} \to \mathcal{A}_\mathcal{B}_\mathcal{D} \circ F$ is a natural equivalence. In this case, we drop the subscript in $F^{\text{ab}}_{\mathcal{I}}$ and call $F^{\text{ab}}$ the abelianization of $F$. As usual, the above universal property guarantees that when it exists, the functor $F^{\text{ab}} : \mathcal{C}^{\text{ab}} \to \mathcal{D}^{\text{ab}}$ is unique up to a canonical isomorphism.

### 5.2. The Van den Bergh functor

In this section, we assume for simplicity that $S = k$ and $V$ is concentrated in degree 0. Given $R \in \text{DGA}_k$, let $\pi_V : R \to \text{End} V \otimes \sqrt{R}$ denote the universal DG algebra homomorphism, see Proposition [B(a)]. The complex $\sqrt{R} \otimes V$ is naturally a left DG module over $\text{End} V \otimes \sqrt{R}$ and right DG module over $\sqrt{R}$, so restricting the left action via $\pi$ we can regard $\sqrt{R} \otimes V$ as a DG bimodule over $R$ and $\sqrt{V}$. Similarly, we can make $V^* \otimes \sqrt{R}$ a $\sqrt{R}$-$R$-bimodule. Using these bimodules, we define the functor

$$
\sqrt{\_} : \text{DGMod}(R) \to \text{DGMod}(\sqrt{R}) , \quad M \mapsto (V^* \otimes \sqrt{R}) \otimes_R M \otimes_R (\sqrt{R} \otimes V) .
$$

Now, recall that $R_V := (\sqrt{R})^{\otimes_2}$ is a commutative DGA. Using the natural projection $\sqrt{R} \to R_V$, we regard $R_V$ as a DG bimodule over $\sqrt{R}$ and define

$$
(\_)^{\otimes_2} : \text{DGMod}(\sqrt{R}) \to \text{DGMod}(R_V) , \quad M \mapsto M_{\otimes_2} := M \otimes_{\sqrt{R}} R_V .
$$

Combining (77) and (78), we get the functor

$$
(\_)^{\otimes_2} : \text{DGMod}(R) \to \text{DGMod}(R_V) , \quad M \mapsto M^{\otimes_2} := (\sqrt{M})_{\otimes_2} .
$$

It is easy to check that, for any $M \in \text{DGMod}(R)$, there is a canonical isomorphism of $R_V$-modules

$$
M^{\otimes_2} = M \otimes_{R^e} (\text{End} V \otimes R_V) .
$$

Thus, (79) is indeed a DG extension of Van den Bergh’s functor defined in [VdB], Section 3.3.

The next lemma is analogous to Proposition [B] for DG algebras. We recall that, if $R$ is a DG algebra and $M, N$ are DG modules over $R$, the morphism complex $\text{Hom}(M, N)$ is a complex of vector spaces with $n$-th graded component consisting of all $R$-linear maps $f : M \to N$ of degree $n$ and the $n$-th differential given by $d(f) = d_N \circ f - (-1)^n f \circ d_M$.

**Lemma 8** ([BKR], Lemma 5.1). For any $M \in \text{DGMod}(R)$, $N \in \text{DGMod}(\sqrt{R})$ and $L \in \text{DGMod}(R_V)$, there are canonical isomorphisms of complexes

1. $\text{Hom}_{\sqrt{\_}^{\text{op}}}(\sqrt{M}, N) \cong \text{Hom}_{R^{e}}(M, \text{End} V \otimes N)$,
2. $\text{Hom}_{R^{e}}(M^{\otimes_2}, L) \cong \text{Hom}_{R^{e}}(M, \text{End} V \otimes L)$.

**Example.** Let $\Omega^1 R$ denote the kernel of the multiplication map $R \otimes R \to R$ of a DG algebra $R$. This is naturally a DG bimodule over $R$, which, as in the case of ordinary algebras, represents the complex of graded derivations $\text{Der}(R, M)$, i.e., $\text{Der}(R, M) \cong \text{Hom}_{R^{e}}(\Omega^1 R, M)$ for any $M \in \text{DGMod}(R)$ (see, e.g., [Q3], Sect. 3). Lemma 8 then implies canonical isomorphisms

$$
\sqrt{\Omega^1 R} \cong \Omega^1(\sqrt{R}) , \quad (\Omega^1 R)^{\otimes_2} \cong \Omega^1_{\text{com}}(R_V) .
$$

To prove (81) it suffices to check that $\Omega^1(\sqrt{R})$ and $\Omega^1_{\text{com}}(R_V)$ satisfy the adjunctions of Lemma 8 and then appeal to Yoneda’s Lemma. We leave this as an exercise to the reader.

We are now in position to state the main theorem of this section. This theorem justifies, in particular, our notation for the functor (79).
Theorem 18. The functor \( -^{ab} \) is the abelianization of the representation functor \( -^{ab} \).

Proof. Given a DG algebra \( R \in \text{DGA}_k \), we set \( C := \text{DGA}_k / R \) and \( D := \text{CDGA}_k / R \). Then, as in Section 2.6 (see Example 2 and Example 3), we can identify \( C^{ab} = \text{DG Bimod}(R) \) and \( D^{ab} = \text{DG Mod}(R) \). Under this identification, the abelianization functors \( A_{\text{ab}} \) and \( \text{Ab}_2 \) become

\[
\Omega^1(-/R) : \text{DGA}_k / R \to \text{DG Bimod}(R), \quad B \mapsto R \otimes_B \Omega^1(B) \otimes_B R,
\]

\[
\Omega^1_{\text{com}}(-/R) : \text{CDGA}_k / R \to \text{DG Mod}(R), \quad B \mapsto R_V \otimes_B \Omega^1_{\text{com}}(B),
\]

where \( \Omega^1(B) \) and \( \Omega^1_{\text{com}}(B) \) are the modules of noncommutative and commutative (Kähler) differentials, respectively. We prove Theorem 18 in two steps. First, we show that for the functor \( -^{ab} \), there is a canonical natural equivalence

\[
(82) \quad t : (-)^{ab} \circ \Omega^1(-/R) \cong \Omega^1_{\text{com}}(-/R) \circ (-)_V
\]

which makes \( -^{ab} \) a commutative diagram. Then, we verify the universal property stated in Section 5.1.

To establish \( 82 \) we will use the Yoneda Lemma. For any \( B \in \text{DGA}_k / R \) and \( L \in \text{DG Mod}(R) \), Lemma 8 together with \( 81 \) gives natural isomorphisms:

\[
\text{Hom}_{R_V}(\Omega^1(B/R)^{ab}, L) \cong \text{Hom}_{R_V}(\Omega^1(B/R), \text{End}(V) \otimes L)
\]

\[
\cong \text{Hom}_{R_V}(\Omega^1(B), \text{End}(V) \otimes L)
\]

\[
\cong \text{Hom}_{R_V}(\sqrt[3]{\Omega^1(B)}, L)
\]

\[
\cong \text{Hom}_{R_V}(\sqrt[3]{\Omega^1(B)}_{/2}, L)
\]

\[
\cong \text{Hom}_{R_V}(\Omega^1_{\text{com}}(B_V), L)
\]

\[
\cong \text{Hom}_{R_V}(R_V \otimes_{B_V} \Omega^1_{\text{com}}(B_V), L).
\]

Hence, \( \Omega^1(B/R)^{ab} \) is canonically isomorphic to \( R_V \otimes_{B_V} \Omega^1_{\text{com}}(B_V) \), which is equivalent to \( 82 \).

To verify the universal property for abelianization we will use the functorial isomorphism

\[
(83) \quad M = \text{Cok}[\Omega^1(R) \to \Omega^1(T_R M / R)]
\]

where \( T_R M \) is the tensor algebra of \( M \) equipped with the canonical projection \( T_R M \to R \). This isomorphism follows from the standard cotangent sequence for the tensor algebra \( T = T_R M \)

\[
T \otimes_R \Omega^1(T) \to \Omega^1(T) \to T \otimes_R M \otimes_R T \to 0,
\]

which is proved, for example, in [CQ] (see loc. cit., Corollary 2.10).

Now, given a right exact additive functor \( G : \text{DG Bimod}(R) \to \text{DG Mod}(R_V) \) with natural transformation

\[
s : G \circ \Omega^1(-/R) \to \Omega^1_{\text{com}}(-/R) \circ (-)_V
\]

we compose \( s \) with the inverse of \( 82 \) and use \( 83 \) to define the \( B_V \)-module maps

\[
G(M) = \text{Cok}[G \circ \Omega^1(R) \to G \circ \Omega^1(T_R M / R)] \xrightarrow{s_M^V} \text{Cok}[G \circ \Omega^1(R) \to \Omega^1(T_R M / R)] = M^{ab}_V
\]

The maps \( s^V_M \) define a natural transformation \( s' : G \to \cdot^{ab} \) making \( 76 \) commutative. This proves the required universal property and finishes the proof of the theorem.

Now, as in the case of DG algebras (cf. Theorem 7), Lemma 8 easily implies

Theorem 19. (a) The functors \( (-)^{ab} \) : \( \text{DG Bimod}(R) \cong \text{DG Mod}(R_V) : \text{End} V \otimes - \) form a Quillen pair.

(b) \( (-)^{ab} \) has a total left derived functor

\[
L(-)^{ab} : \mathcal{D}(\text{DG Bimod} R) \to \mathcal{D}(\text{DG Mod} R_V)
\]

which is left adjoint to the exact functor \( \text{End} V \otimes - \).
Now, for ordinary algebras, the derived Van den Bergh functor can be defined using a standard procedure in differential homological algebra (cf. [HMS, FHT]). Given $A \in \mathbf{Alg}_k$ and a complex $M$ of bimodules over $A$, we first choose a semi-free resolution $\varepsilon : R \to A$ in $\mathbf{DGA}_k$ and consider $M$ as a DG bimodule over $R$ via $\varepsilon$. Then, we choose a semi-free resolution $F(R, M) \to M$ in the category $\mathbf{DG Bimod}(R)$ and apply to $F(R, M)$ the functor (79). Combining Theorem 19 with Proposition 1 in Section 2.4, we get

**Corollary 5.** Let $A \in \mathbf{Alg}_k$, and let $M$ be a complex of bimodules over $A$. The assignment $M \mapsto F(R, M)_V$ induces a well-defined functor

$$L(-)_V : \mathcal{D}(\text{Bimod} A) \to \mathcal{D}(\mathbf{DG Mod} RV),$$

which is independent of the choice of the resolutions $R \to A$ and $F \to M$ up to auto-equivalence of $\mathcal{D}(\mathbf{DG Mod} RV)$ inducing the identity on homology.

This result can be also verified directly, using polynomial homotopies (see [BKR]).

**Definition.** For $M \in \mathbf{DG Bimod}(A)$, we call $H_* (M, V) := H_* [L(-)_V M^{ab}]$ the representation homology of the bimodule $M$ with coefficients in $V$. If $M \in \mathbf{Bimod}(A)$ is an ordinary bimodule viewed as a complex in $\mathcal{D}(\text{Bimod} A)$ concentrated in degree 0, then $H_0 (M, V) \cong M^{ab}_V$.

We now give some applications of Theorem 19.

### 5.3. Derived tangent spaces.

First, we compute the derived tangent spaces $\pi_i (\text{DRep}_V (A), \varrho)$ for $\text{DRep}_V (A)$ viewed as an affine DG scheme (see Section 2.8 for notation and terminology). Let $\varrho : A \to \text{End} V$ be a fixed representation of $A$. Choose a cofibrant resolution $R \to A$ and let $\varrho_V : R_V \to k$ be the DG algebra homomorphism corresponding to the representation $\varrho : R \to A \to \text{End} V$. Now, for any DG bimodule $M$, there is a canonical map of complexes induced by the functor (79):

$$\text{Der} (R, M) \cong \text{Hom}_R (\Omega^1 M, R) \to \text{Hom}_{R_V} (\Omega^1 (R_V), M_V) \cong \text{Der} (R_V, M_V).$$

We claim that for $M = \text{End} V$ viewed as a DG bimodule via $\varrho$, this map is an isomorphism. Indeed,

$$\text{Der} (R_V, k) \cong \text{Hom}_{R_V} (\Omega^1 (R_V), k) \cong \text{Hom}_{R_V} (\Omega^1 R_V, k) \text{[see (81)]} \cong \text{Hom}_R (\Omega^1 R, \text{End} V) \text{[see Lemma 8(b)]} \cong \text{Der} (R, \text{End} V).$$

This implies

$$\pi_* (\text{DRep}_V (A), \varrho) := H_* [\text{Der} (R_V, k)] \cong H_* [\text{Der} (R, \text{End} V)].$$

The following proposition is now a direct consequence of [BP, Lemma 4.2.1 and Lemma 4.3.2.

**Proposition 7.** There are canonical isomorphisms

$$\pi_i (\text{DRep}_V (A), \varrho) \cong \begin{cases} \text{Der} (A, \text{End} V) & \text{if } i = 0 \\ \text{HH}^{i+1} (A, \text{End} V) & \text{if } i \geq 1 \end{cases}$$

where $\text{HH}^* (A, \text{End} V)$ denotes the Hochschild cohomology of the representation $\varrho : A \to \text{End} V$.

As explained in Section 3.5 in the case when $V$ is a single vector space concentrated in degree 0 $\text{Rep}_V (R)$ is isomorphic to the DG scheme $\text{RA}ct (R, V)$ constructed in [CK]. This implies that $\pi_* (\text{DRep}_V (A), \varrho)$ should be isomorphic to $\pi_* (\text{RA}ct (R, V), \varrho)$, which is indeed the case, as one can easily see by comparing our Proposition 7 to [CK, Proposition 3.5.4(b).
5.4. Periodicity and the Connes differential. One of the most fundamental properties of cyclic homology is Connes’ periodicity exact sequence (cf. [L], 2.2.13):

\[ \ldots \to \mathcal{HH}_n(A) \xrightarrow{L} \mathcal{HH}_n(A) \xrightarrow{S} \mathcal{HH}_{n-2}(A) \xrightarrow{R} \mathcal{HH}_{n-1}(A) \to \ldots \]

This sequence involves two important operations on cyclic homology: the periodicity operator \( S \) and the Connes differential \( B \). It turns out that \( S \) and \( B \) induce (via trace maps) some natural operations on representation homology, and there is a periodicity exact sequence for \( H_n(A) \) similar to \([S]\). We briefly describe this construction below referring the reader to [BKR], Section 5.4, for details and proofs.

We begin by constructing the abelianized version of the trace maps \([87]\). Recall, if \( M \) is a bimodule over a DG algebra \( A \), a trace on \( M \) is a map of complexes \( \tau : M \to N \) vanishing on the commutator subspace \([A,M] \subseteq M\). Every trace on \( M \) factors through the canonical projection \( M \to M_\mathbb{Z} := M/[A,M] \), which is thus the universal trace. Given a finite-dimensional vector space \( V \), let \( \pi_V(M) \) denote the canonical map corresponding to \( \text{Id}_M \), which is thus the universal trace. Given a finite-dimensional vector space \( V \), let \( \pi_V(M) \) denote the canonical map corresponding to \( \text{Id}_M \), which is thus the universal trace.

\[ \text{Tr}_V(M) : M \xrightarrow{\pi_V(M)} \text{End} V \otimes M_\mathbb{V}^{ab} \xrightarrow{\text{Tr}_V \otimes \text{Id}} M_\mathbb{V}^{ab}, \]

is then obviously a trace, which is functorial in \( M \). Thus \([88]\) defines a morphism of functors

\[ \text{Tr}_V : (-)_V \to (-)_V^{ab}. \]

As in the case of DG algebras, we have the following result.

**Lemma 9.** \([89]\) induces a morphism of functors \( \mathcal{D}^{\text{DG Bimod}} A \to \mathcal{D}(k) : \)

\[ \text{Tr}_V : L(-)_V \to L(-)_V^{ab}, \]

where \( L(-)_V^{ab} \) is the derived representation functor introduced in Theorem \([4]\).

To describe \([87]\) on \( M \in \text{DG Bimod}(A) \) explicitly we choose an semi-free resolution \( p : R \xrightarrow{\sim} A \), regard \( M \) as a bimodule over \( R \) via \( p \) and choose a semi-free resolution of \( F(R,M) \xrightarrow{\sim} M \) in \( \text{DG Bimod}(R) \). Then \([87]\) is induced by the map \([88]\) with \( M \) replaced by \( F(R,M) \):

\[ \text{Tr}_V(M) : F(R,M)_V \to F(R,M)_V^{ab}. \]

Note that, if \( A \in \text{Alg}_k \) and \( M \in \text{Bimod}(A) \), then \( H_n(F(R,M)_V) \cong \text{HH}_n(A,M) \) for all \( n \geq 0 \), so \([88]\) induces the trace maps on Hochschild homology:

\[ \text{Tr}_V(M)_n : \text{HH}_n(A,M) \to H_n(M,V), \quad \forall n \geq 0, \]

where \( H_n(M,V) := H_n[L(M)_V^{ab}] \) is the representation homology of \( M \) in sense of Definition \([5]\).

Now, given an algebra \( A \in \text{Alg}_k \), fix an semi-free resolution \( p : R \xrightarrow{\sim} A \) in \( \text{DG}_k \) and consider the commutative DG algebra \( R_V \). Let \( \Omega^1_{\text{com}}(R_V) \) be the DG module of Kähler differentials of \( R_V \), and let \( \partial_V : R_V \to \Omega^1_{\text{com}}(R_V) \) denote the universal derivation (the de Rham differential) on \( R_V \). By Theorem \([12]\) \( R_V \) is isomorphic to a (graded) polynomial algebra. Hence \( \text{Ker}(\partial_V) \cong k \) for all \( V \). On the the hand, the cokernel of \( \partial_V \) is a nontrivial complex which is not, in general, acyclic positive degrees. We denote this complex by \( \Omega^1_{\text{com}}(R_V)/\partial R_V \), and for each integer \( n \geq 0 \), define

\[ \text{HC}_n(A,V) := \tilde{H}_n(A,V) \oplus H_{n-1}[\Omega^1_{\text{com}}(R_V)/\partial R_V] \oplus H_{n-2}[\Omega^1_{\text{com}}(R_V)/\partial R_V] \oplus \ldots \]

Note that the (reduced) representation homology \( \tilde{H}_n(A,V) \) appears as a direct summand of \( \text{HC}_n(A,V) \). It turns out that there are canonical maps

\[ \text{Tr}_V(A)_n : \overline{\text{HC}}_n(A) \to \text{HC}_n(A,V), \quad \forall n \geq 0, \]

lifting the traces \([49]\) to \( \text{HC}_n(A,V) \). Moreover, for all \( n \geq 0 \), one can construct natural maps \( S_V : \text{HC}_n(A,V) \to \text{HC}_{n-2}(A,V) \) and \( B_V : \tilde{H}_n(A,V) \to H_n(\Omega^1 A,V) \) making commutative the following
Here, the first isomorphism follows from the fact which is related to the Connes periodicity sequence (84) by the trace maps in (91). It is suggestive to call DGA commutative algebra 1-converging the representation homology of Ω given in (81) and the last again follows from the fact that

\[ \ldots \xrightarrow{B_V} \tilde{H}_n(A, V) \xrightarrow{B_V} H_n(\Omega^1 A, V) \]

(The rightmost trace in the second diagram is defined as in (83) for \( M = \Omega^1 A \).) Finally, there exists a long exact sequence

\[ \ldots \xrightarrow{B_V} \tilde{H}_n(A, V) \oplus H_{n-1}(\Omega^1 A, V) \xrightarrow{B_V} H_n(\Omega^1 A, V) \]

which is related to the Connes periodicity sequence (84) by the trace maps in (91). It is suggestive to call \( HC_n(A, V) \) the cyclic representation homology of \( A \).

5.5. Relation to Andrè-Quillen homology. Recall that the Andrè-Quillen homology of a commutative algebra \( C \) with coefficients in a module \( M \) is denoted \( D_*(k[1], M) \) (see Section 2.3 Example 3).

Now, fix \( A \in \text{CDGA}_k \), and let \( \pi : R \xrightarrow{\sim} A \) be a semi-free resolution of \( A \). Assume that, for some \( V \), the canonical map induced by \( \pi \):

\[ \Omega^1_{\text{com}}(R_V) \xrightarrow{\sim} A_V \otimes_{R_V} \Omega^1_{\text{com}}(R_V) \] is a quasi-isomorphism.

Then, there is a homological spectral sequence

\[ E^2_{pq} = D_p(k[A_V], H_q(A, V)) \Rightarrow H_n(\Omega^1 A, V) \]

converging the representation homology of \( \Omega^1 A \). Indeed, applying \( L\Omega^1_{\text{com}}(-/A_V) \circ L(\vdash)_V \) to the DG algebra \( R \), we have isomorphisms in the derived category of DG \( R_V \)-modules:

\[ L\Omega^1_{\text{com}}(R_V/A_V) \cong \Omega^1_{\text{com}}(R_V/A_V) := A_V \otimes_{R_V} \Omega^1_{\text{com}}(R_V) \cong \Omega^1_{\text{com}}(R_V) \cong (\Omega^1 A)^{ab}_V \cong L(\Omega^1 A)^{ab}_V. \]

Here, the first isomorphism follows from the fact \( R_V \) is semi-free in \( \text{CDGA}_k \) whenever \( R \) is semi-free in \( \text{CDGA}_k \) (cf. Theorem 12), the second isomorphism is a consequence of (92) and the third isomorphism is given in (81) and the last again follows from the fact that \( R \) is semi-free so that \( \pi : R \xrightarrow{\sim} A \) induces a semi-free resolution \( \Omega^1 R \xrightarrow{\sim} \Omega^1 A \) in the category of \( R \)-bimodules. Hence, we have the Grothendieck spectral sequence

\[ L_{p+q}\Omega^1_{\text{com}}(-/A_V) \circ L_q(A)_V \Rightarrow L_{p+q}(\Omega^1 A)^{ab}_V \]

which is precisely (93). We conclude with the following

**Example.** Let \( A \) be a formally smooth algebra in \( \mathfrak{Alg}_k \) (see Section 6.1 below). Assume that \( A \) has a semi-free resolution \( R \xrightarrow{\sim} A \) that is finitely generated in each degree. Then, by the proof of Theorem 21 we have a quasi-isomorphism \( R_V \xrightarrow{\sim} A_V \) which implies (92). Hence, the spectral sequence (93) exists in this case. Now, we actually have \( H_0(A, V) = 0 \) for all \( q > 0 \), while \( H_0(A, V) = A_V \). On the other hand, if \( A \) is formally smooth in \( \mathfrak{Alg}_k \), then \( A_V \) is formally smooth in the category of commutative \( k \)-algebras. This implies that \( D_p(k[A_V, \vdash] = 0 \) for all \( p > 0 \) (see [L], Theorem 3.5.6). Thus, the spectral sequence (93) collapses, giving isomorphisms \( H_0(\Omega^1 A, V) \cong \Omega^1_{\text{com}}(A_V) \) and \( H_n(\Omega^1 A, V) = 0 \) for all \( n > 0 \).

6. Examples

In this section, we will give a number of examples and explicit computations. We will focus on two classes of algebras: noncommutative complete intersections and Koszul algebras for which there are known ‘small’ canonical resolutions. We begin with a particularly simple class of algebras that are models for smooth spaces in noncommutative geometry (see [KR]).
6.1. Smooth algebras. Recall that a \( k \)-algebra \( A \) is called formally smooth (or quasi-free) if either of the following equivalent conditions holds (see [CQ, KR]):

1. \( A \) has cohomological dimension \( \leq 1 \) with respect to Hochschild cohomology.
2. The universal bimodule \( \Omega^1_A \) of derivations is a projective bimodule.
3. \( A \) satisfies the lifting property with respect to nilpotent extensions in \( \text{Alg}_k \): i.e., for every algebra homomorphism \( f : A \to B/I \), where \( I \subset B \) is a nilpotent ideal, there is an algebra homomorphism \( f : A \to B \) inducing \( f \).

A formally smooth algebra is called smooth if it is finitely generated. It is easy to see that a formally smooth algebra is necessarily hereditary ([CQ], Proposition 6.1), but a hereditary algebra may not be formally smooth (e.g., the Weyl algebra \( A_1(k) \)). Here are some well-known examples of smooth algebras:

- Finite-dimensional separable algebras.
- Finitely generated free algebras.
- Path algebras of (finite) quivers.
- The coordinate rings of smooth affine curves.
- If \( G \) is a (f.g. discrete group, its group algebra \( kG \) is smooth iff \( G \) is virtually free (i.e., \( G \) contains a free subgroup of finite index), see [LeB].

The class of formally smooth algebras is closed under some natural constructions: for example, coproducts and (universal) localizations of formally smooth algebras are formally smooth.

The key property of (formally) smooth algebras is given by the following well-known theorem (see, e.g., [G], Proposition 19.1.4).

**Theorem 20.** If \( A \) is a (formally) smooth algebra, then \( \text{Rep}_V(A) \) is a (formally) smooth scheme for every finite-dimensional vector space \( V \).

In other words, Theorem 20 says that the representation functor \( \text{Rep}_V \) preserves (formal) smoothness. This can be explained by the following

**Theorem 21.** Let \( A \) be a formally smooth algebra. Assume that \( A \) has a semi-free resolution in \( \text{DGA}_k^+ \) that is finitely generated in each degree. Then, for any finite-dimensional vector space \( V \),

\[
H_n(A, V) = 0, \quad \forall n > 0.
\]

**Remark.** It is natural to ask whether the vanishing condition (94) characterizes formally smooth algebras: that is, does (94) imply that \( A \) is formally smooth? The answer to this question is ‘no.’ A counterexample will be given in Section 6.2.2.

**Proof.** The proof of Theorem 21 is based on Proposition 2 of Section 2.8. We will use the notation and terminology introduced in that section. Let \( \rho : R \to A \) a semifree resolution of \( A \) in \( \text{DGA}_k \) that is finitely generated in each degree. Then, \( R_V \) defines a smooth affine DG scheme which, abusing notation, we denote \( \text{DRep}_V(A) \). By Theorem 20, \( \pi_0(\text{DRep}_V(A)) \cong \text{Rep}_V(A) \). On the other hand, \( \pi_0(\text{Spec}(A_V)) = \text{Spec}(A_V) \) is indeed the same as \( \text{Rep}_V(A) \), and the latter scheme is smooth by Theorem 20. Furthermore, by Proposition 4, for any \( \rho \in \text{Rep}_V(A) \),

\[
\pi_i(\text{DRep}_V(A), \rho) \cong H^{i+1}(A, \text{End} V), \quad i \geq 1.
\]

Since \( A \) is formally smooth, it follows that \( \pi_i(\text{DRep}_V(A), \rho) = 0 \) for all \( i \geq 1 \). Thus, the differential \( dp_\rho \) is a quasi-isomorphism of tangent spaces \( T_\rho \text{DRep}_V(A) \) to \( T_\rho \text{Rep}_V(A) \) for each representation \( \rho \) of \( A \) in \( \text{End} V \). Now, from Proposition 2 it follows that \( R_V \) is quasi-isomorphic to \( A_V \) (via \( p_V \)). Since \( H_n(A, V) \cong H_n(R_V) \) for all \( n \), the desired result follows.

We call an algebra \( A \) representation cofibrant if \( H_n(A, V) \) vanishes for all positive \( n \) and for each finite-dimensional \( k \)-vector space \( V \). The following result is analogous to the fact that a resolution by acyclic sheaves suffices to compute sheaf cohomology.
**Proposition 8.** Let $B \in \text{DGA}^+_k$. Suppose $S \rightarrow B$ is a resolution of $B$ by a DG algebra $S$ that is an extension of a representation cofibrant algebra $A$ by an honest cofibration, then, for any finite-dimensional vector space $V$,

$$H_n(S_V) \cong H_n(B,V).$$

**Proof.** Let $A \rightarrow S$ be the given cofibration. Consider a cofibrant resolution $R \rightarrow A$ of $A$ and note that the composite map $R \rightarrow A \rightarrow S$ makes $S$ an object in $\text{DGA}_R$. Let $R' \rightarrow S$ be a cofibrant resolution of $R \rightarrow S$ in $\text{DGA}_R$. Consider the pushout $U := A \amalg_T S$ in $\text{DGA}^+_k$. We claim that $U$ is quasi-isomorphic to $S$ (via the natural map $U \rightarrow S$ arising out of the universal property of $(U)$, and hence, to $B$. Indeed, since the model category $\text{DGA}^+_k$ is proper (cf. $\text{BKR}$ Proposition B.3)], the morphism $T \rightarrow U$ (coming from the pushout diagram) is a quasi-isomorphism. Since the resolution $T \rightarrow S$ is equal to the composition $T \rightarrow U \rightarrow S$, $U \rightarrow S$ is indeed a quasi-isomorphism. Further, since $A \rightarrow U$ is the pushout of a cofibration, it is a cofibration. Thus, $p : U \rightarrow S$ is a quasi-isomorphism between cofibrant objects in $\text{DGA}^+_k$. Since $T \rightarrow S$ is a fibration, so is $U \rightarrow S$. Thus, one obtains a homotopy inverse $i : S \rightarrow U$ of $p$ in $\text{DGA}^+_k$. By $\text{BKR}$ Proposition B.2], $ip$ is homotopic to the identity via an $M$-homotopy (while $pi = \text{Id}_S$). Thus, $i_V$ and $p_V$ are quasi-isomorphisms. It therefore suffices to check that

$$H_n(U_V) \cong H_n(B,V).$$

By definition, $H_n(B,V) \cong H_n(T_V)$. Since the functor $(-)_V$ preserves cofibrations and pushout diagrams,

$$U_V \cong A_V \amalg_{R_V} T_V$$

in $\text{CDGA}^+_k$. Since $R_V \rightarrow A_V$ is a quasi-isomorphism (as $A$ is representation cofibrant), and since the model category $\text{CDGA}^+_k$ is proper, $T_V \rightarrow U_V$ is a quasi-isomorphism in $\text{CDGA}^+_k$. This proves the desired result. \[\square\]

**6.2. Noncommutative complete intersections.** Let $F \in \text{Alg}_k$ be a smooth algebra (e.g., the tensor algebra of a finite-dimensional vector space), and let $J$ be a finitely generated 2-sided ideal of $F$.

**Definition.** The algebra $A = F/J$ (or the pair $J \subseteq F$) is called a noncommutative complete intersection (for short, NCCI) if $J/J^2$ is a projective bimodule over $A$.

This class of algebras has been studied, under different names, by different authors (see, e.g., $[\text{AH}, \text{A}, \text{GSh}, \text{Go}, \text{EG}]$). In the present paper, we will use the notation and terminology of $\text{EG}$. As in $\text{EG}$, we will work with graded connected algebras equipped with a non-negative polynomial grading. Such an algebra $A$ can be presented as the quotient of a free algebra generated by a finite set of homogeneous variables by the two-sided ideal generated by a finite collection of homogeneous relations. In other words, we may write

$$(95) \quad A = T_k V/\langle j(L) \rangle$$

where $V$ is a positively graded $k$-vector space of finite total dimension and $L$ is a finite-dimensional positively graded $k$-vector space equipped with an homomorphism $j : L \rightarrow T_k V$ of graded $k$-vector spaces (which can be chosen to be an embedding). Following $\text{EG}$, we refer to the triple $(V,L,j)$ as presentation data for $A$. It is easy to show that an algebra $A$ of the form $(95)$ is NCCI if and only if it has cohomological dimension $\leq 2$ with respect to Hochschild cohomology (see $\text{EG}$ Theorem 3.1.1)].

The class of (graded) noncommutative complete intersections is thus a natural extension of the class of smooth algebras.

Associated to the data $(V,L,j)$ there is a non-negatively graded DG algebra defined as follows. Place $V$ in homological degree 0 and place $L$ in homological degree 1 to obtain the $k$-vector space $V \oplus L[1]$ (which is graded homologically as well as polynomially). Then define the bigraded algebra $T_k(V \oplus L[1])$ and put on it a (unique) differential $d$ such that

$$d(l) = j(l) \in T_k V$$

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for all \(l\) in \(L[1]\). The resulting DG algebra is denoted \(\text{Sh}(A,(V,L,j))\) and called the Shafarevich complex\(^2\) corresponding to \((V,L,j)\). Note that \(H_0(\text{Sh}(A,(V,L,j))) \cong A\).

**Theorem 22** (see [AG, Go, EC]). A (graded connected) algebra \(A\) is NCCI iff it has presentation data \((V,L,j)\) such that the associated Shafarevich complex \(\text{Sh}(A,(V,L,j))\) is acyclic in all positive degrees.

Using the Shafarevich complex, we can study the representation homology of NCCI algebras. To avoid confusion with the data \((V,L,j)\), we will consider representations of \(A\) on a vector space \(k^n\), \(n \geq 1\). The corresponding representation functors will then be denoted by \(\text{Rep}_n(A)\) and \(\text{DRep}_n(A)\) instead of \(\text{Rep}_{k^n}(A)\) and \(\text{DRep}_{k^n}(A)\).

Recall that a given DG algebra \(R \in \text{DGA}_k^+\) has a universal DG representation \(\pi_n : R \to M_n(R_n)\) defined for each \(n \geq 1\). For a matrix \(M\) in a DG-algebra \(S\), we denote the entry in row \(i\) and column \(j\) by \(M_{ij}\). For notational brevity, we shall denote the vector space \(X \otimes M_n(k)\) by \(X_n\) for any \(k\)-vector space \(X\). Let \(j_n : L_n \to (T_kV)_n\) denote the map

\[
l_{pq} := l \otimes e_{pq} \mapsto (\pi_n(j(l)))_{pq}
\]

Further, recall that for a finitely generated (polynomially graded) commutative algebra \(B\), given finite dimensional (polynomially graded) vector spaces \(W,S\) and a homomorphism \(f : S \to \Lambda(W)\) such that \(B = \Lambda(W)/(f(S))\), one can construct the Koszul complex \(K(B,(W,S,f)) := \Lambda(W \oplus S[1])\) equipped with the homological differential mapping each \(s \in S\) to \(f(s)\). It turns out that the representation functor transforms Shafarevich complexes to Koszul complexes. Indeed, with our notation, the following lemma is an immediate consequence of Theorem 22.

**Lemma 10.** Let \((V,L,j)\) be presentation data for \(A\). Then,

\[
(\text{Sh}(A,(V,L,j)))_n \cong K(A_n,(V_n,L_n,j_n)).
\]

This lemma suggests that we should indeed view a Shafarevich complex as a noncommutative Koszul complex. The next theorem shows that the representation homology of NCCI algebras is rigid in the sense of Auslander-Buchsbaum (see [AB]).

**Theorem 23.** If \(A\) is a NCCI algebra, then \(H_q(A,k^n) = 0\) implies \(H_p(A,k^n) = 0\) for all \(p \geq q\).

**Proof.** This follows from Lemma 10 and the rigidity of the usual Koszul complexes (see [AB], Proposition 2.6).

The following theorem gives a natural interpretation for the 1-st representation homology of NCCI algebras: namely, \(H_1(A,k^n)\) is an obstruction for the classical representation scheme \(\text{Rep}_n(A)\) to be a complete intersection.

**Theorem 24.** Let \(A\) be a NCCI algebra. Assume that \(H_1(A,k^n) = 0\) for some \(n \geq 1\). Then \(\text{Rep}_n(A)\) is a complete intersection.

**Proof.** Suppose that \((V,L,j)\) is presentation data for \(A\) making \(\text{Sh}(A,(V,L,j))\) acyclic in positive degree. Then, the Koszul complex \(K := K(A_n,(V_n,L_n,j_n))\) represents \(\text{DRep}_n(A)\) in \(\text{Ho}(_\text{CDGA}_k)\) (by Lemma 10). Suppose that \(H_1(A_n,u) = 0\). Then, by [AB] Proposition 2.6, the Koszul complex \(K\) is acyclic in all positive degrees. Since \(k[\text{Rep}_n(A)] \cong H_0(K)\), \(\text{Rep}_n(A)\) is a complete intersection.

Under extra (mild) assumptions, the vanishing of \(H_1(A,k^n)\) is not only sufficient but also necessary for \(\text{Rep}_n(A)\) to be a complete intersection. More precisely, we have

**Theorem 25.** Let \(A\) be a NCCI algebra given with presentation data \((V,L,j)\) such that \(\text{Sh}(A,(V,L,j))\) is acyclic in positive degrees.

(a) If \(\text{Rep}_n(A)\) is a complete intersection in \(\text{Rep}_n(T_kV)\) of dimension \(n^2(\dim_k V - \dim_k L)\), then \(H_p(A,k^n) = 0\) , \(\forall p > 0\).

---

\(^2\)This complex was originally introduced in [GSSh] in connection with the famous Golod-Shafarevich Theorem. We recommend the survey paper [Pi], where this connection as well as many other interesting applications of the Shafarevich complex are discussed.
(b) More generally, if \( \text{Rep}_n(A) \) is a complete intersection in \( \text{Rep}_n(T_k V) \), then \( H_q(A, k^n) \) is a free module over \( H_0(A, k^n) \) of rank \( \binom{p}{q} \), where \( p := \dim \text{Rep}_n(A) - n^2 (\dim_k V - \dim_k L) \).

**Proof.** If \( \text{Rep}_n(A) \) is a complete intersection in \( \text{Rep}_n(T_k V) \) implies that the Koszul complex \( K \) is acyclic in positive degrees. Since \( H_q(A, k^n) \cong H_q(K) \), choose a homogenous basis of \( L_n \) and choose a minimal set \( S \) from this homogenous basis such that its image under \( j_n \) generates the ideal \( I_n \) defining \( \text{Rep}_n(A) \) in \( \text{Rep}_n(T_k V) \). The \( k \)-linear span of \( S \) is a graded subspace \( L^{(0)} \) of \( L_n \), and \( \dim_k \text{Rep}_n(A) = n^2 \dim_k V - \dim_k L \). For any complement \( L^{-1} \) of \( L^{(0)} \) in \( L \), \( j(L^{-1}) \) is contained in the ideal \( I_n \). It follows from [2] that \( K \) is quasi-isomorphic to \( K(A_n, (V_n, L^0, j_n)) \otimes \Lambda(L^{-1}[1]) \). Since \( \text{Rep}_n(A) \) is a complete intersection, \( K(A_n, (V_n, L^0, j_n)) \) is acyclic in positive degrees and has 0-th homology \( k[\text{Rep}_n(A)] \). Thus, \( H_q(A, k^n) \cong k[\text{Rep}_n(A)] \otimes \Lambda^q L^{-1} \) as (polynomially graded) vector spaces. Finally, note that the number \( p \) in the statement of (c) is precisely \( \dim_k L^{-1} \). This proves (b), of which (a) is a special case. \( \square \)

Let \( A \) be a NCCI algebra with presentation data \((V, L, j)\) as in Theorem [25]. Set \( R := \text{Sh}(A, (V, L, j)) \) and denote the summand of polynomial degree \( r \) in \( R_p \) by \( R_p \) (with square brackets being reserved for denoting shifts in homological degree). Since \( R \) is acyclic in homological degrees \( p > 0 \), the map \( j : L \to T_k V \) is injective (see [24], Theorem 2.4). Consider the graded subspace \( L_0 := j^{-1}([T_k V, T_k V]) \) of \( L \). The embedding \( L_0 \hookrightarrow R_1 \) induces a linear map \( \phi : L_0 \to H_1(\bar{R}_2) \cong \text{HTC}_1(A) \).

Consider the restriction of the map \( \text{Tr}_n : R \to R_n \) to \( L_0 \). Clearly, \( \text{Tr}_n \vert_{L_0} \) is injective. We may therefore, identify \( L_0 \) with its image under \( \text{Tr}_n \) and choose a direct sum decomposition

\[ L_n \cong L_0 \oplus L_0^{-1} \]

as graded \( k \)-vector spaces. The following proposition now follows from Lemma [10].

**Proposition 26.** With above notation, there is an isomorphism of DG algebras

\[ R_n \cong K(A_n, (V_n, L_0^{-1}, j_n)) \otimes \Lambda(L_0[1]) \]

Consequently,

\[ H_*(A, k^n) \cong H_{\text{Koszul}}(A_n, (V_n, L_0^{-1}, j_n)) \otimes \Lambda(L_0[1]) \]

When the graded vector space \( L \) is concentrated in a single degree and when \( n > 1 \), one can further show (using the 2nd Fundamental Theorem of Invariant Theory) that the images of any basis of \( \text{Rep}_n(A) \) is a free \( \Lambda \)-module. This proves (b), of which (a) is a special case. \( \square \)

**Theorem 27.** \( H_0(A, k^n) = 0 \) for all \( p > n \).

**Proof.** The obvious presentation \( A = k[x, y]/(xy - yx) \) with natural polynomial grading \( (\deg(x) = \deg(y) = 1) \) shows that \( A \) is actually a NCCI algebra (see [24], Proposition 2.20). Indeed, for \( V := k \cdot x \oplus k \cdot y \), \( L := k \cdot t \) (with \( t \) in polynomial degree 2) and \( j(t) := xy - yx \), the Shafarevich complex is isomorphic to the DG algebra \( R := k[x, y, t] / dt = xy - yx \) which is acyclic in positive degrees. Thus, \( R_n \cong k[x_{ij}, y_{ij}, t_{ij}] | 1 \leq i, j \leq n | \) with variables \( t_{ij} \) in degree 1 and differential determined by the formula

\[ dT = [X, Y] \]

where \( X := (x_{ij}), Y := (y_{ij}), T := (t_{ij}) \in \text{Mat}(n)(R_n) \). By [Kn] Theorem 1, the \( (n^2 - n) \) elements \( \{dt_{ij}, 1 \leq i \neq j \leq n\} \) form a regular sequence in \( k[x_{ij}, y_{ij}] \). It follows from [2] Corollary 17.12 that \( H_0(R_n) = 0 \) for all \( p > n \). \( \square \)
Example \((n = 1)\). It is easy to see that 
\[ H_\bullet(k[x, y], k) \cong k[x, y] \otimes k.t \]
where \(t\) has degree 1. Hence, \(H_\bullet(k[x, y], k) \cong k[x, y] \oplus k[x, y].t\) is a rank 2 free module over \(k[x, y]\).

This simple example shows that \(DRep_v(A)\) does depend on the algebra \(A\), and not only on the affine scheme \(\text{Rep}_v(A)\). Indeed, comparing \(k[x, y]\) to the free algebra \(k(H)\), we see that \(DRep(k[x, y]) = \text{Rep}_1(k[x, y])\) but \(H_1(k[x, y], k) \neq H_1(k(x, y), k)\) because \(H_1(k(x, y), k) = 0\), by Theorem \([21]\).

Example \((n = 2)\). The algebra \(H_\bullet(k[x, y], k^2)\) is more complicated. Let \(g := \text{span}_k\{\xi, \tau, \eta\}\), where the variables \(\xi, \tau, \eta\) are in homological degree 1. Then, there is an isomorphism of graded algebras
\[ H_\bullet(k[x, y], k^2) \cong (k[x, y]_2 \otimes \Lambda_k g)/I \]
where the ideal \(I\) is generated by the following relations
\[
\begin{align*}
(1) & \quad \quad x_{12} \eta - y_{12} \xi = (x_{12}y_{11} - y_{12}x_{11}) \tau \\
(2) & \quad \quad x_{21} \eta - y_{21} \xi = (x_{21}y_{22} - y_{21}x_{22}) \tau \\
(3) & \quad \quad (x_{11} - x_{22}) \eta - (y_{11} - y_{22}) \xi = (x_{11}y_{22} - y_{11}x_{22}) \tau \\
(4) & \quad \quad \xi \eta = y_{11}(\xi \tau) - x_{11}(\eta \tau) = y_{22}(\xi \tau) - x_{22}(\eta \tau)
\end{align*}
\]
Thus, as a \(H_0\)-module, \(H_\bullet(k[x, y], k^2) \cong H_0 \oplus H_1 \oplus H_2\), where
\[
\begin{align*}
H_0 & \cong k[x, y]/2 \\
H_1 & \cong (H_0 \cdot \xi + H_0 \cdot \tau + H_0 \cdot \eta)/(\text{relations (1)-(3)}) \\
H_2 & \cong (H_0 \cdot \xi \tau + H_0 \cdot \eta \tau)/(x_{12} \eta \tau - y_{12} \xi \tau, x_{21} \eta \tau - y_{21} \xi \tau, (x_{11} - x_{22}) \eta - (y_{11} - y_{22}) \xi)
\end{align*}
\]
The above presentation of \(H_\bullet(k[x, y], k^2)\) was obtained with an assistance of \texttt{Macaulay2}.

Recall that for \(A = k[x, y]\), the cyclic homology \(H^iC(A) = 0\) for \(i > 1\), while \(H^0C(A) = A\) and \(H^1C(A) = \Omega^1 A/dA\). With these identifications, for all \(n\), the 0-th trace \(\text{Tr}(A)_0: H^0C(A) \to H_0(A, k^n)\) is obviously given by the formula \(x^iy^m \mapsto \text{Tr}(X^iY^m)\), while the 1-st trace \(\text{Tr}(A)_1: H^1C(A) \to H_1(A, k^n)\) is expressed by (see \[\texttt{BKR}\] Example 4.1):
\[
\begin{align*}
\text{Tr}_1(x^iy^mdx) &= \sum_{i=0}^{m-1} \text{Tr}(X^iYT^{m-1-i}) , \\
\text{Tr}_1(x^iy^mdy) &= -\sum_{j=0}^{l-1} \text{Tr}(X^jTX^{l-1-j}Y^m) .
\end{align*}
\]
Now, for \(n = 2\), the generators \(\xi, \eta\) and \(\tau\) in \([96]\) correspond to the classes of the elements \(\text{Tr}(XT)\), \(\text{Tr}(YT)\) and \(\text{Tr}(T)\). Thus, we see that
\[
\tau = \text{Tr}_1(ydx) , \quad \xi = \text{Tr}_1(xydx) , \quad \eta = \text{Tr}_1(xydy) .
\]
It follows that \(H_\bullet(k[x, y], k^2)\) is generated (as an algebra over \(H_0\)) by invariant traces of degree 1.

6.2.2. \textit{Derived} \(q\)-commuting schemes. For a parameter \(q \in k^*\), define \(A := k(x, y)/(xy - qyx)\). By \([\texttt{PR}]\) Proposition 2.20, this algebra is a \(\text{NCCI}\) whose \(\text{Sha}\) resolution is given by \(k(x, y, t)/dt = xy - qyx\). In the case when \(q\) is not a root of unity, \([\texttt{EG}]\) Proposition 5.3.1 shows that \(\text{Rep}_q(A)\) is a complete intersection of dimension \(n^2\) for all \(n \geq 1\). By Theorem \([25]\), we conclude that \(H_\bullet(A, k^n) = 0\) for all \(p > 0\), i.e. \(A\) is a representation cofibrant algebra. However, by \([\texttt{DI}]\) Theorem 5.3, the global dimension of \(A\) is equal to 2. Hence, \(A\) is not formally smooth.

More generally, for parameters \(q_1, q_2, \ldots, q_{m-1} \in k^*\), we can define (cf. \[\texttt{EG}\] Example 5.3.3)
\[
A = k(x_1, \ldots, x_m)/\langle \text{ad}_{q_1}(x_1) \ldots \text{ad}_{q_{m-1}}(x_{m-1})x_m \rangle
\]
where \( \text{ad}_q(x)(y) := xy - qyx. \) If all \( q_i \)'s are not roots of unity, then \( \text{Rep}_n(A) \) is a complete intersection of dimension \( n^2(m-1) \). Again, Theorem \( 24 \) implies the vanishing of the higher representation homology of \( A \), while [D] Theorem 5.3] shows that \( A \) is not formally smooth.

### 6.3. Koszul algebras

For any Koszul algebra \( A \) with quadratic linear relations, there is a canonical semifree resolution given in terms of the cobar construction of the dual coalgebra \( (A^!)^* \) (see [LV Chapter 3]). We illustrate the use of this resolution in three examples.

#### 6.3.1. Dual numbers

Let \( A := k[x]/(x^2) \) be the ring of dual numbers. This is a quadratic algebra which is Koszul dual to the tensor algebra \( T_k V \) of a one-dimensional vector space \( V \). It has a minimal semi-free resolution of the form \( R := k(x,t_1,t_2,t_3,\ldots) \) where \( \deg(x) = 0 \) and \( \deg(t_p) = p \), and the differential is given by

\[
d_{R_p} = x t_{p-1} - t_1 t_{p-2} + \ldots + (-1)^{p-1} t_{p-1} x, \quad p \geq 1.
\]

By Theorem \([12] \) \( H_\bullet(A,k^n) \) is then the homology of the commutative DG algebra

\[
R_n := k[x_{ij},(t_1)_{ij},(t_2)_{ij},\ldots \mid 1 \leq i,j \leq n],
\]

whose differential in the matrix notation is given by

\[
d_{R_p} = X T_{p-1} - T_1 T_{p-2} + \ldots + (-1)^{p-1} T_{p-1} X.
\]

For \( n = 1 \), using Macaulay2, we find

\[
\begin{align*}
H_0(A,k) & \cong A \\
H_1(A,k) & = 0 \\
H_2(A,k) & \cong A \cdot t_2 \\
H_3(A,k) & \cong A \cdot (xt_3 - 2t_1 t_2) \\
H_4(A,k) & \cong A \cdot t_2^2 \oplus A \cdot t_4 \\
H_5(A,k) & \cong A \cdot (-2t_1 t_2^2 + xt_2 t_3) \oplus A \cdot (-t_3 t_2 - 4t_1 t_4 + 2xt_5) \\
H_6(A,k) & \cong A \cdot t_2 t_4 \oplus A \cdot t_6 \\
H_7(A,k) & \cong A \cdot (-t_2 t_3 - 4t_1 t_2 t_4 + 2xt_5) \oplus A \cdot (-t_3 t_4 - 2t_1 t_6 + xt_7) \\
H_8(A,k) & \cong A \cdot t_2 t_6 \oplus A \cdot t_4^2 \oplus A \cdot t_8
\end{align*}
\]

The (reduced) cyclic homology of \( A \) is given by (see, e.g., [LQ], Section 4.3):

\[
\text{HC}_n(A) = \begin{cases} 
0 & \text{if } n = 2p + 1 \\
\kappa x^{2p+1} & \text{if } n = 2p
\end{cases}
\]

The odd traces \( \text{Tr}(A)_{2p+1} \) thus vanish, while the even ones are given by

\[
\text{Tr}(A)_{2p} : \text{HC}_n(A) \rightarrow H_{2p}(A,k), \quad x^{2p+1} \mapsto t_{2p}
\]

This example shows that the algebra map \( A \text{Tr}(A)_\bullet : A[\text{HC}(A)] \rightarrow H_\bullet(A,V)^{GL(V)} \) is not surjective in general, i.e. the Procesi Theorem [P] fails for higher traces (cf. Section [LQ]). Note also that, unlike in the case of NCCI algebras, the representation homology of \( A \) is not rigid in the sense that \( H_1 = 0 \) does not force the vanishing of higher homology.
6.3.2. Polynomials in three variables. Let \( A = k[x, y, z] \) be the polynomial ring in three variables. It has a minimal Koszul resolution of the form \( R = k[x, y, z; \xi, \theta, \lambda; t] \), where the generators \( x, y, z \) have degree 0; \( \xi, \theta, \lambda \) have degree 1 and \( t \) has degree 2. The differential on \( R \) is defined by
\[
\begin{align*}
  d\xi &= [y, z] , & d\theta &= [z, x] , & d\lambda &= [x, y] , & dt &= [x, \xi] + [y, \theta] + [z, \lambda] .
\end{align*}
\]
For \( V = k^n \), Theorem \[12\] implies that
\[
R_n \cong k[x_{ij}, y_{ij}, z_{ij}; \xi_{ij}, \theta_{ij}, \lambda_{ij}; t_{ij}] ,
\]
where the generators \( x_{ij}, y_{ij}, z_{ij} \) have degree zero, \( \xi_{ij}, \theta_{ij}, \lambda_{ij} \) have degree 1, and \( t_{ij} \) have degree 2.

Using the matrix notation \( X = \|x_{ij}\| , Y = \|y_{ij}\| \), etc., we can write the differential on \( R_n \) in the form
\[
\begin{align*}
  d\Xi &= [Y, Z] , & d\Theta &= [Z, X] , & d\Lambda &= [X, Y] , & dT &= [X, \Xi] + [Y, \Theta] + [Z, \Lambda] .
\end{align*}
\]
For \( n = 1 \), it is easy to see that the homology of \( R_n \) is just a graded symmetric algebra generated by the classes of \( x, y, z, \xi, \theta, \lambda, t \). Thus,
\[
H_\ast(A, k) \cong \Lambda(x, y, z, \xi, \theta, \lambda, t)
\]
This example shows that, unlike in the case of two variables, the representation homology of the polynomial algebra \( k[x, y, z] \) is not bounded.

6.3.3. Universal enveloping algebras. Let \( A = U(\mathfrak{sl}_2) \) be the universal enveloping algebra of the Lie algebra \( \mathfrak{sl}_2(k) \). As in previous example, \( A \) has a minimal resolution of the form \( R = k[x, y, z; \xi, \theta, \lambda; t] \) with generators \( x, y, z \) of degree 0; \( \xi, \theta, \lambda \) of degree 1 and \( t \) of degree 2. The differential on \( R \) is defined by
\[
\begin{align*}
  d\xi &= [y, z] + x , & d\theta &= [z, x] + y , & d\lambda &= [x, y] + z , & dt &= [x, \xi] + [y, \theta] + [z, \lambda] .
\end{align*}
\]
For \( V = k^n \), the corresponding algebra \( R_n \) has differential
\[
\begin{align*}
  d\Xi &= [Y, Z] + X , & d\Theta &= [Z, X] + Y , & d\Lambda &= [X, Y] + Z , & dT &= [X, \Xi] + [Y, \Theta] + [Z, \Lambda] .
\end{align*}
\]
For \( n = 1 \), it is easy to see that the homology of \( R_n \) is just the polynomial algebra generated by one variable \( t \) of degree 2. Hence \( H_\ast(A, k) \cong k[t] \).

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