The Gaussian entropy of fermionic systems

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We consider the entropy and decoherence in fermionic quantum systems. By making a Gaussian Ansatz for the density operator of a collection of fermions we study statistical 2-point correlators and express the entropy of a system fermion in terms of these correlators. In a simple case when a set of \( N \) thermalised environmental fermionic oscillators interacts bi-linearly with the system fermion we can study its time dependent entropy, which also represents a quantitative measure for decoherence and classicalization. We then consider a relativistic fermionic quantum field theory and take a mass mixing term as a simple model for the Yukawa interaction. It turns out that even in this Gaussian approximation, the fermionic system decoheres quite effectively, such that in a large coupling and high temperature regime the system field approaches the temperature of the environmental fields.

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I. INTRODUCTION

The density operator contains complete information about quantum statistical systems, and hence it can be used to study various properties of such systems, such as correlators, particle numbers, entropy and decoherence. However, the evolution of realistic physical systems is governed by interacting field theories, and only rarely the density operator is known beyond a perturbative approximation, which can be, nevertheless, very useful for weakly coupled regimes.

Even the Gaussian part of the density operator contains important information about the entropy and decoherence of the system, which can be neatly encoded in the statistical two point function, as was firstly pointed out independently by two groups of authors \[1, 2\]. This correlator approach to entropy, decoherence and classicalization has been extensively used in the context of weakly interacting bosonic systems \[1\]–\[10\]. However very little is known about the entropy and decoherence in fermionic systems, and the corresponding literature is scarce \[11\]–\[13\]. In this paper we present a first study of entropy and decoherence in relativistic fermionic field theories. For simplicity, we consider here only simple bilinear interactions, which are in field theory known as mass mixing. Since our Hamiltonian is quadratic in the fields, an initial Gaussian density operator will remain Gaussian as the system evolves, and a complete information about the density operator can be given in terms of equal-time 2-point correlators, which is the strategy we use in this work. For pedagogical reasons, we begin by considering coupled fermionic quantum oscillators, and only then move on to field theory.

If the density operator of a system \( \hat{\rho} \) is known, the (information) entropy \( S_{VN} \) can be calculated by the von Neumann formula,

\[
S_{VN} = -\langle \ln(\hat{\rho}) \rangle = -\text{Tr}[\hat{\rho}\ln(\hat{\rho})].
\]

Now, by making use of the Heisenberg evolution equation for the density operator, one can easily show that the von Neumann entropy is conserved for closed systems. In practice however no observer \( O \) will have access to a complete information of any nontrivial system \( S \) (with many interacting degrees of freedom), making the system open. Such systems will interact with an environment \( E \) which is, by definition, inaccessible to \( O \). The loss of information associated with this inaccessibility generically leads to decoherence \[14\]–\[18\], a rather qualitative concept that describes how a system evolves into a state which most closely resembles a classical state. However, at the same time the (reduced) von Neumann entropy \[1\] of the system alone \(^1\) is no longer conserved due to this loss of information. Entropy generation thus provides a quantitative measure of decoherence and classicalization.

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\(^1\) The von Neumann entropy of the reduced system density operator is in literature also known as the entanglement entropy \[19, 20\], as the information about the entanglement between the system and environment is lost in the reduced density matrix.
This statement requires clarification, because decoherence is an observer-dependent concept, whereas entropy can be defined without introducing an observer, but of course it does depend on the system – environment split. To make our work as general as possible, we kept the observer implicit throughout. Yet, a natural observer \( \mathcal{O} \) is the one which entangles with those states that diagonalise the density operator. In this work we identify those states as the Fock states for which statistical (average) particle number is defined. Decoherence is now induced by tracing over the inaccessible environmental degrees of freedom, and is perfect from the viewpoint of the observer entangled with the Fock states that does not see the entanglement between system and environment. As these Fock states get more occupied, and the system’s entropy increases, the system gets more classical. This observer plays a special role in the class of all observers sensitive to Gaussian properties of the density operator, and that is that this observer sees the most quantum properties of the system. All other observers, such as the momentum or position operator, will perceive the system as more classical. We can make this more concrete by considering the example of a highly squeezed pure state (that describes e.g. linearized cosmological perturbations at the end of inflation). That state will be perceived as classical by the observer that measures the position spread of the state (in the sense that \( \Delta x \) will be much greater than in the pure vacuum state), but it will be perceived as a pure quantum state by the observer that entangles with the Fock states that diagonalize the system’s density operator.

But, what can be the inaccessible information that can be justifiably called an environment? The most common example is a thermal bath of particles interacting weakly with the system that is observed. In this case, the system–environment correlations – also known as entanglement – are not observable. These correlations can be in 2-point \( SE \) correlators (such as considered in this work) or in higher order \( n \)-point functions. Higher order correlators are always suppressed by some power of the coupling constant, and hence they are typically small in a weakly coupled regime. For example, in Refs. \([2, 9]\) the leading \( SE \) correlator that is neglected corresponds to a 3-point \( SE \) correlator.

In this work we take the interaction to be of a Yukawa type, schematically

\[
\mathcal{L}_{\text{int}} = -y_{ij} \hat{\psi}_i \hat{\phi} \hat{\psi}_j , \tag{2}
\]

where \( \hat{\phi} \) and \( \hat{\psi}_j \) denote a scalar and fermionic quantum field, respectively. The simplest approximation in which one can treat this interaction is to neglect quantum fluctuations of the scalar field, i.e. to replace the scalar field by its expectation value,

\[
\hat{\phi} \rightarrow \phi \equiv \text{Tr}[\rho \hat{\phi}] .
\]

Within this Gaussian approximation the Yukawa interaction reduces to a mass mixing term,

\[
\mathcal{L}_{\text{int}} \rightarrow -\hat{\psi}_i M_{ij} \hat{\psi}_j , \quad M_{ij} = y_{ij} \phi , \tag{3}
\]

the Hamiltonian becomes quadratic in the fields, and the problem becomes exactly soluble. We shall use numerical techniques to obtain an exact solution to this simplified problem, and we shall express the Gaussian density operator as the Hamiltonian becomes quadratic in the fields, and the problem becomes exactly soluble. We shall use numerical techniques to obtain an exact solution to this simplified problem, and we shall express the Gaussian density operator in terms of equal-time 2-point correlators. The information we consider inaccessible to \( O \) is in the \( SE \) and \( EE \) 2-point correlators, and hence entropy gets generated. The so-called Gaussian von Neumann entropy for the system field, which is derived from the Gaussian density matrix alone, yields a good quantitative measure of decoherence for nearly Gaussian systems. However, for highly non-Gaussian systems, one has to modify the entropy definition to incorporate the relevant non-Gaussian features of the state [8]. In the simplified problem (3), the Gaussian von Neumann entropy can be analytically calculated in terms of 2-point (statistical) correlators of the system degrees of freedom. This has also been done for various bosonic systems in Refs. [2, 7, 8]. In this work we set out to derive the Gaussian (von Neumann) entropy for fermionic systems in terms of correlators of the system.

The Gaussian entropy is generally not conserved in the presence of interactions, which could be either environmental interactions, or self-interactions. Several case studies for bosonic systems [11, 2, 7, 8, 10] have indeed shown that the Gaussian entropy increases for interacting systems, thereby quantitatively describing decoherence and classicalization. As far as we know, a quantitative description of decoherence for fermionic systems is still lacking. A better understanding of decoherence in fermionic systems can be applicable in many situations of physical interest. For instance, one species of fermions could mix with others through mass-type terms, such as quarks through the CKM matrix [21, 22] or neutrinos through the PMNS [23, 24] matrix. Other examples include Yukawa interactions, or condensed matter systems with interacting fermions. Here a framework is provided for calculating the growth of entropy for a system of fermions interacting \( \text{via} \) a fermionic mass matrix [8]. As explained above, this model represents the simplest (Gaussian) approximation to the more realistic Yukawa interaction (2).

The outline of this work is as follows: in section [11] the simplest example of a one-dimensional fermionic harmonic oscillator is discussed. A general Ansatz is made for the density operator, after which the particle number and entropy are derived in terms of the statistical correlators of the system. In section [14] we make a connection with some existing literature by working with the density operator in the coherent state basis. Next, in section [1B] the simplest possible
interactions are added to the fermionic system: $N$ environmental fermionic oscillators coupled bilinearly to the system oscillator. Though not completely realistic, this example provides an insight into how a loss of information leads to an increase in the entropy of a fermionic system, and some specific examples are shown. In section III we switch our attention to the more realistic fermionic quantum field theory. After discussing diagonalisation of the Dirac Hamiltonian in III A an Ansatz is made for the density operator in terms of mixing particle and antiparticle states in section III B. The Gaussian entropy is derived in terms of the statistical correlators. In section III C the fermionic entropy is generalised in the presence of $N$ fermionic degrees of freedom. Finally, in section III D the first realistic example of entropy generation in fermionic quantum field theory is discussed, which is the simple case of one fermionic species mixing with other species through mass terms.

II. ENTROPY GENERATION IN FERMIONIC QUANTUM MECHANICS

The most general Ansatz for the density operator of a free fermionic quantum mechanical system (fermionic harmonic oscillator) with the Lagrangian,

$$L_0 = \psi^\dagger (i\partial_t - \omega(t)) \psi,$$

(4)

can be written as $^2$,

$$\hat{\rho}(t) = \frac{1}{Z} \exp(-\hat{a} \hat{\psi}^\dagger \hat{\psi}),$$

(5)

where $a(t)$ is a (complex valued) function of time, and $1/Z$ is the normalisation constant determined by the usual trace condition,

$$\text{Tr}[\hat{\rho}(t)] = 1,$$

(6)

and $\hat{\psi}$ is the (Grassmannian) fermionic operator (here expressed in the Schrödinger picture) satisfying the usual canonical anticommutation relation,

$$\{\hat{\psi}, \hat{\psi}^\dagger\} = 1.$$

(7)

Now making use of the Grassmannian nature of the operators $\psi^\dagger$ and $\psi$ with $(\hat{\psi})^2 = 0, (\hat{\psi}^\dagger)^2 = 0$, and of (7), we can expand (5) as,

$$\hat{\rho}(t) = \frac{1}{Z} \left( 1 + \left[ e^{-a} - 1 \right] \hat{N} \right),$$

(8)

where we introduced the fermionic number operator $\hat{N} = \hat{\psi}^\dagger \hat{\psi}$ with $\hat{N}^n = \hat{N} (n = 1, 2, \ldots)$. The Hilbert space of this theory is two dimensional, and can be conveniently represented in terms of the Fock space basis vectors $\{|0\rangle, |1\rangle\}$, defined by,

$$\hat{N}|n\rangle = n|n\rangle.$$

(9)

The trace of the density operator (6) is easily evaluated in this basis,

$$\text{Tr}[\hat{\rho}(t)] = \frac{1}{Z} \sum_{n=0,1} \langle n | \left( 1 + \left[ e^{-a} - 1 \right] \hat{N} \right) |n\rangle,$$

(10)

such that the general Gaussian fermionic density operator is properly normalised according to (6) by $Z = 1 + \exp(-a)$. It is also convenient to express the density operator as

$$\hat{\rho}(t) = (1 - \bar{n}(t)) + (2\bar{n}(t) - 1) \hat{N},$$

(11)

$^2$ The anticommutation relation (7) implies that the other possible Gaussian term $\exp(-b\hat{\psi}^\dagger \hat{\psi})$ can be expressed in terms of $\exp(-a\hat{\psi}^\dagger \hat{\psi})$ plus an appropriate change in the normalisation constant, and hence does not constitute a new term. In presence of interactions, the lagrangian $L_\psi$ can be written as $L_\psi = L_0 + L_{int}$, where $L_{int} = -j_0^\dagger \hat{\psi} - \hat{\psi}^\dagger j_0$. In this case the density operator can still be written as in (5), where now $\hat{\psi}$ and $\hat{\psi}^\dagger$ denote the suitably shifted fields, as shown in appendix B.
where the average particle number $\bar{n}$ is defined as,

$$\langle \bar{N} \rangle = \text{Tr}[\hat{\rho}(t)\bar{N}] = \frac{1}{e^\beta + 1} \equiv \bar{n}(t).$$

(12)

The (von Neumann) entropy is then simply,

$$S = -\text{Tr}[\hat{\rho} \ln(\hat{\rho})] = -\sum_{n=0,1} \langle n \rangle \left\{ [(1 - \bar{n}) + (2\bar{n} - 1)\bar{N}] \ln [(1 - \bar{n}) + (2\bar{n} - 1)\bar{N}] \right\} |n\rangle.$$

This evaluates to

$$S = -(1 - \bar{n}) \ln(1 - \bar{n}) - \bar{n} \ln(\bar{n}),$$

(13)

which is the standard expression for the entropy of $\bar{n}$ free (non-interacting) fermions, where $\bar{n}$ is the average number of fermions in the system defined in (12).

Let us now make a connection with the familiar expressions for a thermal fermionic density matrix [25]. According to the Fermi-Dirac distribution, the average occupancy of a state with energy $E$ is given by,

$$\bar{n}_{\text{FD}} = \frac{1}{e^{\beta E} + 1},$$

(14)

where $\beta = 1/(k_B T)$ is the inverse temperature. Of course,

$$\bar{n}_{\text{FD}} = \langle \bar{N} \rangle = \text{Tr}[\hat{\rho}_{\text{th}}\bar{N}],$$

where $\hat{\rho}_{\text{th}}$ denotes a thermal density operator. By comparing with the general expression for $\hat{\rho}$ (11) and with (12), it is now easily seen that the thermal density matrix is obtained upon identification, $a \to \beta E$, such that,

$$\hat{\rho}_{\text{th}} = \frac{1}{e^{\beta E} + 1} \left( e^{\beta E} + \left[ 1 - e^{\beta E} \right] \bar{N} \right),$$

(15)

or, equivalently,

$$\hat{\rho}_{\text{th}} = (1 - \bar{n}_{\text{th}}) + (2\bar{n}_{\text{th}} - 1)\bar{N},$$

(16)

which is, as expected, of the same form as the general entropy (13). The thermal density operator (16) implies the following well known expression for the entropy of a thermal Fermi gas,

$$S_{\text{th}} = -(1 - \bar{n}_{\text{th}}) \ln(1 - \bar{n}_{\text{th}}) - \bar{n}_{\text{th}} \ln(\bar{n}_{\text{th}}).$$

(17)

Let us now consider a bit more closely Eq. (13). In the spirit of the Schwinger-Keldysh out-of-equilibrium formalism, it is convenient to introduce the following 2-point functions,

$$iS^{++}(t; t') = \langle T[\hat{\psi}(t)\hat{\psi}^\dagger(t')] \rangle$$

$$iS^{+-}(t; t') = -\langle \hat{\psi}(t')\hat{\psi}(t) \rangle$$

$$iS^{-+}(t; t') = \langle \hat{\psi}(t)\hat{\psi}^\dagger(t') \rangle$$

$$iS^{--}(t; t') = \langle T[\hat{\psi}(t)\hat{\psi}^\dagger(t')] \rangle,$$

(18)

where here $\hat{\psi}(t)$ and $\hat{\psi}^\dagger(t)$ are the Heisenberg picture operators, and $T$ and $\bar{T}$ denote time ordering and anti-time ordering operations, defined as,

$$iS^{++}(t; t') = \theta(t - t')iS^{-+}(t; t') + \theta(t' - t)iS^{+-}(t; t')$$

$$iS^{--}(t; t') = \theta(t - t')iS^{+-}(t; t') + \theta(t' - t)iS^{-+}(t; t'),$$

(19)

such that,

$$iS^{++} + iS^{--} = iS^{+-} + iS^{-+}.$$
The retarded and advanced Green functions are then,\[ iS^r = iS^{++} - iS^{+-} = -(iS^{--} - iS^{-+}), \quad iS^a = iS^{++} - iS^{-+} = -(iS^{--} - iS^{+-}). \]

Notice that \( iS^r \) and \( iS^a \) can be also written as
\[ iS^r(t; t') = \theta(t - t')\langle \hat{\psi}(t), \hat{\psi}^\dagger(t') \rangle, \quad iS^a(t; t') = -\theta(t' - t)\langle \hat{\psi}(t), \hat{\psi}^\dagger(t') \rangle. \] (20)

The statistical and causal (spectral) two point functions are defined as
\[ F_\psi(t; t') = \frac{1}{2} (iS^{--}(t, t') + iS^{+-}(t, t')) = \frac{1}{2} \langle [\hat{\psi}(t), \hat{\psi}^\dagger(t')] \rangle, \]
\[ \rho_\psi(t; t') = \frac{1}{2i} S^c(t; t') = \frac{1}{2} (iS^{--}(t, t') - iS^{+-}(t, t')) = \frac{1}{2} \langle [\hat{\psi}(t), \hat{\psi}^\dagger(t')] \rangle. \] (21)

such that \( \rho_\psi(t; t) = 1/2 \) \((S^c(t; t) = i)\). By making use of the identity, \( \hat{\psi}^\dagger \hat{\psi} = (1/2)\{\hat{\psi}^\dagger, \hat{\psi}\} + (1/2)\{\hat{\psi}^\dagger, \hat{\psi}\}, \) one can obtain a simple relation between \( \bar{n}(t) \) and \( F_\psi(t; t)\):
\[ \bar{n}(t) = \langle \hat{\psi}^\dagger(t) \hat{\psi}(t) \rangle = \frac{1}{2} - F_\psi(t; t) = 1 - 2\bar{n}(t) = \tanh \left( \frac{a}{2} \right), \] (22)

where in the last step we defined \(^3\)
\[ \Delta_\psi(t) \equiv 2F_\psi(t; t) = 1 - 2\bar{n}(t) = \tanh \left( \frac{a}{2} \right). \] (23)

Note that in Eq. (22) the average particle number is computed using time dependent operators \( \psi(t), \psi^\dagger(t) \), although \( \bar{n} \) in Eq. (12) was computed using Schrödinger picture operators \( \hat{\psi}, \hat{\psi}^\dagger \) and the density operator \( \hat{\rho} \). The time dependence of the operators can be absorbed under the trace into the density operator, which then takes the form \(^5\) for equal-time operators.

Eq. (22) represents a relation between the invariant of the correlators and the Gaussian invariant of the density matrix, which are in this single fermion case simply \( F_\psi(t; t) \) and \( \alpha(t) \), respectively. In different systems with multiple correlators and a more complicated Gaussian density matrix such a relation can still be found. An example is the bosonic case, discussed in footnote \(^9\) and in appendix \( \text{A} \) of equal-time operators.

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Note that \( \bar{n} \in [0, 1] \) and \( \Delta_\psi \in [-1, 1] \), which can be appreciated from Eq. (23), making the interpretation of \( \Delta_\psi \) as the invariant phase space area of the state for fermions dubious. It is hence better to think about \( \Delta_\psi \) as the Gaussian invariant of a fermionic state, while \( \bar{n} = (1 - \Delta_\psi)/2 \in [0, 1] \) is more like the phase space area.

For thermal states, for which \( \bar{n}_{\text{th}} \in [0, 1/2], \Delta_{\psi\text{th}} \) acquires natural values, \( \Delta_{\psi\text{th}} \in [0, 1], \) and hence there is no problem. In fact, \( \Delta_\psi \) becomes negative only when higher energy states are overpopulated, \( i.e. \) when \( \bar{n} > 1/2 \). Relation (23) allows us to relate the fermionic entropy \(^13\) to the Gaussian invariant \( \Delta_\psi(t) \),
\[ S_\psi = -\frac{1 + \Delta_\psi}{2} \ln \left( \frac{1 + \Delta_\psi}{2} \right) - \frac{1 - \Delta_\psi}{2} \ln \left( \frac{1 - \Delta_\psi}{2} \right), \] (24)

which is to be compared with the analogous expression for bosons in Eq. (A10) of appendix \( \text{A} \).

A. Coherent states

In order to make a connection to the existing literature \(^11\)-\(^13\) \(^26\), here we rephrase our results in terms of fermionic coherent states \( |\theta\rangle \), defined by,
\[ \hat{\psi}|\theta\rangle = \theta|\theta\rangle. \] (25)

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\(^3\) The definition \(^23\) is the fermionic equivalent of the invariant (phase space area) \( \Delta \) of a bosonic Gaussian state in Eq. (A8) of appendix \( \text{A} \) where we present an analogous derivation of the entropy for bosons. The form of the Gaussian invariant \( \Delta_\psi \) for fermions is so simple because the fermionic density operator is diagonal in \( \hat{N} \equiv \hat{\psi}^\dagger \hat{\psi} \), implying that the fermionic density matrix is diagonal in the fermionic particle state basis.
When expressed in terms of Fock (particle number) states \(|\theta\rangle\), the coherent \textit{ket} and \textit{bra} states are given by,

\[ |\theta\rangle = |0\rangle - \theta |1\rangle, \quad \langle\theta| = \langle 0| - \langle 1|\theta, \]

where \(\theta = \theta^*\) and Grassmann variables obey a Grassmann algebra, \(\theta_i\theta_j = -\theta_j\theta_i\) (recall that complex conjugation for Grassmann variables is reminiscent of a hermitian conjugation, \((\theta_i\theta_j)^* = \theta_j^*\theta_i^* = -\theta_i^*\theta_j^*)\). By making use of the well known relations,

\[
\hat{\psi}|0\rangle = 0, \quad \hat{\psi}^\dagger|0\rangle = |1\rangle, \quad \hat{\psi}|1\rangle = |0\rangle, \quad \hat{\psi}^\dagger|1\rangle = 0,
\]

one sees that \(26\) is indeed an eigenstate of the operator \(\hat{\psi}\) with the eigenvalue \(\theta\). Note that the Fock space element \(|0\rangle\) commutes with Grassmann variables, while \(|1\rangle = \hat{\psi}^\dagger|0\rangle\) anticommutes, such that the coherent states \(|\theta\rangle\) commute with Grassmann variables. The coherent state \(|\theta\rangle\) is fixed uniquely by the requirement \(25\) up to a normalisation constant \(N = 1 + \bar{\theta}\theta/2\), where \(b\) is a complex number. Our choice of normalisation corresponds to

\[
\langle\theta|\theta\rangle = 1 + \bar{\theta}\theta, \tag{27}
\]

which is Grassmann valued. One may attempt to normalise to unity by choosing \(N = 1 - \bar{\theta}\theta/2\). The problem with this is that then the operator \(\hat{\psi}^\dagger\) does not act on \(|\theta\rangle\) in a desired manner. In fact, one can show that the normalisation \(27\) is uniquely fixed by the requirement,

\[
\hat{\psi}^\dagger|\theta\rangle = - \frac{d}{d\theta}|\theta\rangle. \tag{28}
\]

Indeed, when \(\hat{\psi}^\dagger\) acts on \(|\theta\rangle\) defined in \(26\) one gets \(|1\rangle\), and when the derivative \(-d/d\theta\) acts on the same state, one again gets \(|1\rangle\). A different normalisation would not give this result. Finally, as a final check of consistency we consider how the anticommutator acts on \(|\theta\rangle\),

\[
\{\hat{\psi}, \hat{\psi}^\dagger\}|\theta\rangle = \left(-\hat{\psi}\frac{d}{d\theta} + \hat{\psi}^\dagger\theta\right)|\theta\rangle = \left(\frac{d}{d\theta}\hat{\psi} - \theta\hat{\psi}^\dagger\right)|\theta\rangle = \left(\frac{d}{d\theta}\theta + \theta\frac{d}{d\theta}\right)|\theta\rangle = |\theta\rangle,
\]

as it should be from \(7\). When projected on a coherent state basis, the elements of the density matrix \(11\) become of the form,

\[
\rho(\bar{\theta}', \theta; t) \equiv \langle\theta'|\hat{\rho}(t)|\theta\rangle = (1 - \bar{n}) + \bar{n}\bar{\theta}'\theta = (1 - \bar{n}) \exp\left(\frac{\bar{n}'}{1 - \bar{n}}\bar{\theta}'\theta\right), \tag{29}
\]

which is not diagonal. This is to be contrasted with a diagonal \textit{Ansatz} used \textit{e.g.} in Ref. \[13\].

Let us now consider properties of the coherent state basis in more detail. Taking a trace of the density operator in the coherent state representation yields,\(^4\)

\[
\text{Tr} [\hat{\rho}] = \int d\theta \int d\bar{\theta} \exp(\bar{\theta}\theta)\langle\theta|\hat{\rho}|\theta\rangle = \int d\theta \int d\bar{\theta} (1 + \bar{\theta}\theta) [(1 - \bar{n}) + \bar{n}\bar{\theta}'\theta] = 1, \tag{30}
\]

where in the last step we used the usual integration rules, \(\int d\theta = 0, \int d\theta\theta = 1\). The integration measure factor \(\exp(\bar{\theta}\theta)\) in \(30\) is necessary to get the traces correctly.

One can now use a decomposition of unity \(^5\),

\[
I_\theta = \int d\bar{\theta} \int d\theta \exp(-\bar{\theta}\theta)|\theta\rangle\langle\theta|, \tag{31}
\]

\(^4\) The expression for the trace in Eq. \(30\) can be derived as follows: the trace of an operator \(\mathcal{O}\) is in the Fock basis defined as \(\text{Tr}[\mathcal{O}] = \langle 0|\mathcal{O}|0\rangle + \langle 1|\mathcal{O}|1\rangle\), where the Fock space elements can be expressed in terms of coherent states as \(|0\rangle = \int d\theta \theta |\theta\rangle, \langle 1| = - \int d\theta |\theta\rangle, \langle 0| = \int d\bar{\theta} \theta |\theta\rangle, \langle 1| = \int d\bar{\theta} |\theta\rangle\).

\(^5\) Note that \(\text{Tr}[I_\theta] = 2\), as it should be, where \(I_\theta\) is given in Eq. \(31\).
to recast $\hat{\rho}$ as,

$$\hat{\rho} = \int d\theta d\bar{\theta} e^{-\bar{\theta} \theta} |\theta\rangle \langle \theta| \int d\theta d\bar{\theta} e^{-\bar{\theta} \theta} |\theta\rangle \langle \theta| \int d\theta d\bar{\theta} e^{-\bar{\theta} \theta} |\theta\rangle \equiv \int d\theta d\bar{\theta} e^{-\bar{\theta} \theta} \int d\theta d\bar{\theta} e^{-\bar{\theta} \theta} \hat{\rho}(\theta, \bar{\theta}, t),$$

(32)

where $\hat{\rho}(\theta, \bar{\theta}, t)$ are elements of the density operator in the coherent state representation (see Eq. (29)),

$$\hat{\rho}(\theta, \bar{\theta}, t) = |\theta\rangle \langle \theta| \equiv \int d\theta d\bar{\theta} e^{-\bar{\theta} \theta} \hat{\rho}(\theta, \bar{\theta}, t),$$

(33)

with $\rho(\theta, \bar{\theta}, t) = Z^{-1} \exp(M \theta \bar{\theta})$ given in Eq. (29). $\hat{\rho}(\theta, \bar{\theta}, t)$ of Eq. (33) is obviously non-diagonal. However $\hat{\rho}$ of Eq. (32) can be cast in a diagonal basis by inserting the Ansatz

$$\rho(\theta, \bar{\theta}, t) = \int d\tilde{\zeta} d\zeta e^{\bar{\theta} \zeta - \bar{\zeta} \theta} P(\zeta),$$

(34)

such that

$$\hat{\rho} = \int d\tilde{\zeta} d\zeta P(\zeta) |\zeta\rangle \langle \zeta|.$$  

(35)

This is the so-called Glauber $P$ representation [27] for fermions. Inverting (34), the function $P(\zeta)$ is related to the density matrix in the diagonal elements of the coherent state basis as

$$P(\zeta) = \int d\theta d\bar{\theta} e^{-\bar{\zeta} \theta - \zeta \bar{\theta}} |\zeta\rangle \langle \zeta|.$$  

(36)

The elements of the density matrix in the coherent state basis have been found in [29], and by integrating over $\theta, \bar{\theta}$ in (36) one finds:

$$P(\zeta) = \bar{n} + (1 - \bar{n}) \zeta \bar{\zeta}.$$  

(37)

It is possible to return to the Fock basis via the $P$ representation (35) using Eqs. (26) and (37) and integrating over $\bar{\zeta}, \zeta$,

$$\hat{\rho}(t) = \sum_{n=0}^{1} |n\rangle \langle n| = \sum_{n=0}^{1} |1 - \bar{n} + (2\bar{n} - 1)n\rangle \langle n| = |0\rangle \langle 0| + |1\rangle \langle 1|.$$  

(38)

For one degree of freedom the (diagonal) Fock number basis is by far superior to the coherent state basis [35] for studying properties of the fermionic density operator, an important example being the von Neumann entropy defined in [1] and calculated in [13]. The reason is that the Fock states are orthogonal, contrary to the coherent states which satisfy $\langle \zeta | \theta \rangle = e^{\zeta \theta}$, see Eq. (27). Still, the von Neumann entropy can be derived from the density operator in the coherent state basis by using the replica trick, which is demonstrated in appendix D. There we also generalise to $N$ fermionic degrees of freedom, a case which is discussed in more detail in section III C.

### B. Fermionic interactions in quantum mechanics

Interactions can be included in the quantum mechanical fermionic theory [1] by introducing general current terms into the Lagrangian

$$L_{\psi} \equiv L_0 + L_{\text{int}}; \quad L_{\text{int}} = -\frac{1}{2} \hat{j}_{\psi} \hat{j}_{\psi} + \hat{j}_{\psi} \hat{j}_{\psi}.$$  

(39)

Formally the linear current terms can be absorbed into the free field theory [1] by shifting the fermionic fields, see appendix B. For these free shifted fermionic fields the von Neumann entropy is conserved.

In realistic situations it is very hard to have a complete information about the current operator $\hat{j}_{\psi}(t)$ however, which makes the diagonalisation procedure [17] impracticable, or even impossible. Namely, in condensed matter systems, the coupling current is often given by a superposition of many (fermionic) degrees of freedom, whose precise time evolution is not known. In a quantum field theoretical setting one can have for example a Yukawa coupling term,
\[ -y\dot{\phi}(x)\dot{\psi}(x)\dot{\psi}(x), \text{ such that the coupling current } \dot{j}_\phi \text{ corresponds to a composite operator, } \dot{j}_\phi(x) = y\dot{\phi}(x)\dot{\psi}(x), \text{ making the diagonalisation procedure very hard, if not impossible. For that reason we adopt here the point of view that no (useful) information is known about the evolution of the current } \dot{j}_\psi. \text{ This } \textit{loss of information} \text{ leads to entropy generation, which is what we study next.}

The simplest nontrivial example is the quantum mechanical case when the current consists of \( N \) environmental oscillators in thermal equilibrium. In this case,

\[ \dot{j}_\psi(t) = \sum_{i=1}^{N} \lambda_i \hat{\psi}_i(t), \]  

(40)

where the \( \hat{\psi}_i \) represent the environmental fermionic oscillators. The form of the current (40) is motivated by mass mixing, which can be considered as an approximation to the Yukawa coupling, cf. Eqs. (2–3). The system is represented by a single fermionic oscillator \( \psi_x \) which is coupled bilinearly to the environmental oscillators through couplings \( \lambda_i \). The interaction between the environmental oscillators is assumed to be zero in our toy model. The \textit{loss of information} in this case is that we cannot observe (correlations of) the environmental oscillators, nor its interaction with the system. The complete action of system, environment and interactions in our toy model is

\[ S[\hat{\psi}_x, \{ \hat{\psi}_i \}] = \int dt \left\{ L_S[\hat{\psi}_x] + L_E[\{ \hat{\psi}_i \}] + L_{\text{int}}[\hat{\psi}_x, \{ \hat{\psi}_i \}] \right\}, \]  

(41)

with

\[ L_S[\hat{\psi}_x] = \hat{\psi}_x^\dagger (i\partial_t - \omega_0) \hat{\psi}_x \]

\[ L_E[\{ \hat{\psi}_i \}] = \sum_{i=1}^{N} \hat{\psi}_i^\dagger (i\partial_t - \omega_i) \hat{\psi}_i, \]

\[ L_{\text{int}}[\hat{\psi}_x, \{ \hat{\psi}_i \}] = - \sum_{i=1}^{N} \lambda_i \left( \hat{\psi}_x^\dagger \hat{\psi}_i + \hat{\psi}_i^\dagger \hat{\psi}_x \right). \]

(42)

Note that by the hermiticity of \( L_{\text{int}} \), all \( \lambda_i^* = \lambda_i \) are real. The fermionic oscillators only depend on time, \( \text{i.e. } \hat{\psi}_x = \hat{\psi}_x(t) \) and \( \hat{\psi}_i = \hat{\psi}_i(t) \). The anticommutation relations satisfied by \( \hat{\psi}_x \) and \( \hat{\psi}_i \) are

\[ \{ \hat{\psi}_x(t), \hat{\psi}_x^\dagger(t') \} = 1 \]

\[ \{ \hat{\psi}_i(t), \hat{\psi}_j^\dagger(t') \} = \delta_{ij}, \]

with all others being zero. Of our interest are the statistical correlators (21), which are for our model defined as

\[ F_{xx}(t; t') = \frac{1}{2} \langle [\hat{\psi}_x(t), \hat{\psi}_x^\dagger(t')] \rangle \]

\[ F_{qi,qj}(t; t') = \frac{1}{2} \langle [\hat{\psi}_q(t), \hat{\psi}_q^\dagger(t')] \rangle \]

\[ F_{xxq}(t; t') = \frac{1}{2} \langle [\hat{\psi}_x(t), \hat{\psi}_q^\dagger(t')] \rangle \]

\[ F_{qi,x}(t; t') = \frac{1}{2} \langle [\hat{\psi}_q(t), \hat{\psi}_x^\dagger(t')] \rangle, \]

(43)

where \( \hat{\psi}_x(t) \) and \( \hat{\psi}_q(t) \) are here Heisenberg picture operators. Our goal is to calculate the entropy for the system. For a free fermionic theory the entropy is given by Eqs. (23) and (24). Without interactions, the Gaussian invariant \( \Delta \) is constant and the entropy is conserved. If we switch on interactions the Gaussian invariant \( \Delta \) is time-dependent. The entropy of the system with the Gaussian \textit{Ansatz} for \( \hat{\rho} \) with time dependent \( a(t) \) is related to \( \Delta_{xx} \) as in Eq. (24):

\[ S_x(t) = - \frac{1 + \Delta_{xx}(t)}{2} \ln \left( \frac{1 + \Delta_{xx}(t)}{2} \right) - \frac{1 - \Delta_{xx}(t)}{2} \ln \left( \frac{1 - \Delta_{xx}(t)}{2} \right), \]  

(44)

with

\[ \Delta_{xx}(t) = 2F_{xx}(t; t) = 1 - 2\bar{n}_{xx}(t), \]  

(45)
and $n_{xx}(t)$ the average particle number for the system fermions. The proper way to derive the entropy of the system is to trace over the environmental degrees of freedom in the density operator, and calculate the entropy from this reduced density operator. The corresponding reduced von Neumann entropy is the same as Eq. (44), i.e. $S^\text{reduced}_N(t) = S_x(t)$.\footnote{The proof goes as follows. The reduced density matrix is defined as $\rho^\text{red} = \text{Tr}_E[\hat{\rho}]$, where the subscript $E$ denotes the environment, which in this example is the group of oscillators $\{\hat{\psi}_q\}$. The reduced von Neumann entropy is the usual $S^\text{reduced}_N(t) = -\text{Tr}[\rho^\text{red} \ln \rho^\text{red}]$. Now most importantly, if we calculate correlators of the system, we have

$$\langle 1 - \Delta_{xx} \rangle/2 = \langle \hat{\psi}^\dagger_x \hat{\psi}_x \rangle = \text{Tr}[\hat{\rho} \hat{\psi}^\dagger_x \hat{\psi}_x] = \sum_{n_x} \langle n_x | n_{x_1} ... n_{x_N} | \hat{\rho} \hat{\psi}^\dagger_x \hat{\psi}_x | n_{x_1} ... n_{x_N} | n_x \rangle = \text{Tr}[\rho^\text{red} \hat{\psi}^\dagger_x \hat{\psi}_x].$$

Here $\hat{\psi}_x$, $\hat{\psi}^\dagger_x$ and $\hat{\rho}$ are taken to be in the Schrödinger picture, which ensures that $\hat{\psi}^\dagger_x \hat{\psi}_x$ does not evolve with time, whereas $\hat{\rho}$ controls the evolution. The trace is taken over a complete set of orthogonal, time independent Fock states. Eq. (46) shows that the correlators, and thus the invariant area (45) and entropy (44) for the system are the same whether you first trace over the environment in the density matrix or you consider the full density matrix.

Thus, in order to investigate the growth of entropy for the system fermionic oscillator $\hat{\psi}_x$, we should find the statistical correlator $F_{xx}(t,t)$ defined in Eq. (43). The equations of motion for the fermionic operators follow from the action (41).

\begin{equation}
\begin{aligned}
(i\partial_t - \omega_0) \hat{\psi}_x(t) &= \sum_{i=1}^N \lambda_i \hat{\psi}_q_i(t) \\
(i\partial_t - \omega_i) \hat{\psi}_q_i(t) &= \lambda_i \hat{\psi}_x(t) \\
(-i\partial_t - \omega_0) \hat{\psi}^\dagger_x(t) &= \sum_{i=1}^N \lambda_i \hat{\psi}^\dagger_q_i(t) \\
(-i\partial_t - \omega_i) \hat{\psi}^\dagger_q_i(t) &= \lambda_i \hat{\psi}^\dagger_x(t).
\end{aligned}
\end{equation}

From these equations of motion we can derive coupled differential equations for the statistical equal-time correlators

\begin{equation}
\begin{aligned}
(i\partial_t - (\omega_0 - \omega_j))F_{xx}(t,t) &= -\lambda_j F_{xx}(t,t) + \sum_{i=1}^N \lambda_i F_{q_i,x}(t,t) \\
(i\partial_t - (\omega_j - \omega_0))F_{xq_j}(t,t) &= \lambda_j F_{xx}(t,t) - \sum_{i=1}^N \lambda_i F_{q_i,x}(t,t) \\
(i\partial_t - (\omega_i - \omega_j))F_{q_i,x}(t,t) &= \lambda_i F_{xq_j}(t,t) - \lambda_j F_{q_i,x}(t,t).
\end{aligned}
\end{equation}

These conditions can be solved with suitable initial conditions. We take the system fermionic oscillator to be initially in a state with average particle number zero. The environmental fermionic oscillators are assumed to be in a thermal state according to the Fermi-Dirac distribution with energy $E_i = \omega_i$, see Eq. (44). Thus

\begin{equation}
\begin{aligned}
F_{xx}(t_0;t_0) &= \frac{1}{2} \\
F_{q_i,x}(t_0;t_0) &= \delta_{ij} \frac{1}{2} \tanh \left( \frac{\beta \omega_i}{2} \right) \\
F_{xq_j}(t_0;t_0) &= 0.
\end{aligned}
\end{equation}

With these initial conditions Eqs. (48) can be solved. We first treat the simple case of two coupled oscillators, then the general case of $N$ coupled oscillators.

1. Two coupled fermionic oscillators

For two coupled fermionic oscillators we consider the case $N = 1$ in the action (41). Thus there is one environmental oscillator $\hat{\psi}_q \equiv \hat{\psi}_{q_1}$ coupled to the system oscillator $\hat{\psi}_x$ through a coupling $\lambda \equiv \lambda_1$. This simple example can be solved...
analytically. The procedure is explained in appendix C. As a final result we find an explicit expression for the Gaussian invariant of the system represented by the fermions $\psi_x$,

$$\Delta_{xx}(t) = 1 - 2\bar{n}_E \left(\frac{2\lambda}{\bar{\omega}}\right)^2 \sin^2 \left(\frac{\bar{\omega}}{2}(t - t_0)\right),$$

where

$$\bar{n}_E = \frac{1}{e^{\omega_0} + 1},$$

$$\bar{\omega} = \sqrt{(\omega_0 - \omega_1)^2 + 4\lambda^2},$$

with the frequencies $\omega_0$ and $\omega_1$ of the system and environment oscillators, respectively. The entropy of the system $S_x$ is subsequently found using Eq. (44). Figures 1 and 2 show the evolution of entropy for a system coupled to one environmental oscillator with $\omega_1 = 1.5\omega_0$ and $\lambda = 0.5\omega_0$ at different values of $\beta$. The dashed line indicates the entropy in the case when the system is completely thermalised (17) and the dotted line is the maximum entropy $S_{\text{max}} = \ln(2)$. The Gaussian invariant (50) satisfies the correct properties: initially $\Delta_{xx}(t_0) = 1$ and the entropy (44) is zero. For zero coupling ($\lambda = 0$), the two oscillators do not interact and the Gaussian invariant remains conserved, leaving zero entropy. For general coupling the Gaussian invariant oscillates between 1 and some value $> 0$ with an angular frequency $\bar{\omega}$. The corresponding entropy then oscillates between 0 and the thermal entropy. Only in the limit when $\beta \to 0$ and $\omega_0 \to \omega_1$ (resonant regime) the maximum entropy $S_{\text{max}} = \ln(2)$ is reached (for $\Delta_{xx} = 0$).

2. $N + 1$ coupled fermionic oscillators

In this section we consider the more general case of one system oscillator bilinearly coupled to $N$ environmental fermions. In order to find the growth of entropy of the system Eqs. (48) must be solved for the statistical correlators with initial conditions (49). This can be done numerically. We have used the $N = 1$ case treated analytically above as the test case for our numerical studies. For simplicity the system oscillator couples equally to all the environmental oscillators, i.e. $\lambda_i \equiv \lambda$. If the frequencies of the environmental oscillators are taken in a narrow range away from $\omega_0$, they will effectively behave as a single oscillator, leading to similar plots as figures 1 and 2.

In figures 3-6 the system entropy is calculated by taking 50 environmental oscillators with frequencies in the range of $[0 - 5] \times \omega_0$. The equal couplings to the system oscillator are $\lambda_i \equiv \lambda = 0.15\omega_0$. In figures 3-4 the environmental frequencies are equally spaced, i.e. $\omega_i = 0.1i\omega_0$, $i = 1, \ldots, 50$, with $\beta = 1$ and $\beta = 0.1$, respectively. The system entropy rapidly increases to the value of the entropy in case the system is completely thermalised, see Eq. (17). For higher environmental temperature (lower $\beta$) the late time entropy gets closer to the maximum entropy $S_{\text{max}} = \ln(2)$, which is only reached for $\beta \to 0$. In general, there will be fluctuations in the entropy due to constructive and destructive
δn/n is consistent with the expected behaviour, figure 7 we have shown the amplitude of fluctuations in late time statistical particle number for N and N entropy contains some random fluctuations. As the entropy increases to the same values as in figures 3–4, but due to the random choice of frequencies the late-time interference of the environmental oscillators. Due to the specific distribution of environmental oscillators in Figs. 5–6 the environmental frequencies of the 50 oscillators are randomly selected in the interval \([0, 50] \times \omega_0\). The entropy again rapidly reaches the thermal entropy \(S_{\text{th}}\) (dashed line), which at such high temperatures (low \(\beta\)) almost coincides with the maximum entropy \(S_{\text{max}} = \ln(2)\) (dotted line).

We shall now consider the entropy of a fermionic field theory whose action is given by,

\[
S[\psi] = \int d^4x \mathcal{L}_\psi, \quad \mathcal{L}_\psi = \bar{\psi}(x) \gamma^\mu \partial_\mu \psi(x) - m \bar{\psi}(x) \psi(x) + \mathcal{L}_{\text{int}}, \quad \mathcal{L}_{\text{int}} = -\bar{\psi}(x) j_\psi(x) \psi(x) - \bar{\psi}(x) j_\psi(x),
\]

where \(\psi(x)\) is a space-time spinor, \(\bar{\psi}(x) = \psi^\dagger \gamma^0\), \(j_\psi(x)\) is a spinorial current, \(j_\psi(x) = j_\psi^\dagger \gamma^0\) and \(\gamma^\mu\) are Dirac’s matrices obeying a Clifford algebra with an anticommutation relation,

\[
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \quad \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1).
\]

III. ENTROPY GENERATION IN A FERMIONIC QUANTUM FIELD THEORY

We shall now consider the entropy of a fermionic field theory whose action is given by,
Figure 5: Same plot as Fig. 3 but here the environmental frequencies have been randomly selected in the same interval, i.e. \( \omega_i \in [0-5] \times \omega_0 \). Due to the random choice of the environmental frequencies, the entropy randomly fluctuates around the thermal value.

Figure 6: Same plot as Fig. 4 but here the environmental frequencies have been randomly selected in the same interval, i.e. \( \omega_i \in [0-5] \times \omega_0 \). Again the entropy fluctuates around the thermal value much more frequently due to the random choice of environmental frequencies.

Figure 7: Amplitude of fluctuations in statistical particle number at late time for \( N=25 \) (purple, dashed), \( N=50 \) (red, solid) and \( N=100 \) (blue, thick solid) environmental oscillators. \( \delta n/n \) is defined as \( \delta n/n \equiv (\bar{n}_{xx} - n_{th})/n_{th} \), where \( \bar{n}_{xx} \) is the statistical particle number of the system \((45)\) and \( n_{th} = (\exp(\beta \omega_0) + 1)^{-1} \). For all lines \( \lambda = 0.1, \beta = 0.5 \) and the environmental frequencies have been chosen randomly in the same range, \( \omega_i \in [0-5] \times \omega_0 \). In general, there are fluctuations in the late time entropy due to the finite amount of environmental oscillators that constructively and destructively interfere. The plot clearly shows that the amplitude of these late time fluctuations decreases as \( N \) increases.

The action \((52)\) is a general Ansatz describing many realistic systems. Examples include: a fermionic field in a heat bath of many fermionic degrees of freedom (similar to the quantum mechanical case in section \([1, B]\)); one quark flavour coupled to other quark flavours through the CKM matrix \([21, 22]\); one neutrino flavour coupled to other neutrino flavours through the PMNS matrix \([23, 24]\); but also many systems with true interactions, such as Yukawa type \( j_\psi \rightarrow y_{ij} \bar{\psi}_j \phi \) with \( \phi \) a scalar field.

In the correlator approach to decoherence \([1, 2, 5, 10]\) the Gaussian von Neumann entropy of a system is expressed in terms of the Gaussian (statistical) correlators of the degrees of freedom of the system. These correlators are derived from the density matrix and they are commonly expressed in terms of those fields that diagonalise the free Hamiltonian. The reason is that the time evolution of off-diagonal correlators is zero for non-interacting fields, which leads to a simple form of the density matrix. Simple means here that the density matrix can be written as a direct product of the density matrices for single degrees of freedom. For interacting theories such as the examples mentioned above, the off-diagonal correlators (between the different components of the diagonalised Hamiltonian) are in general nonzero and the density matrix has a more complicated form. We will discuss this more thoroughly in the coming sections.

In our trivial example, the quantum mechanical case for free fermions, the Hamiltonian is diagonal because there is
only one degree of freedom. Thus the Ansatz (5) for the Gaussian density matrix is simple (and already the most general one), and so is the expression for the entropy in terms of the statistical correlator (24). For a more complicated system such as fermionic fields in 3 + 1 dimensions, which we discuss now, the system is described by a spinor with four components for every wavenumber. In order to find the entropy, we first find the fields that diagonalise the free part of the Hamiltonian, then use an Ansatz for the density matrix in terms of those fields, and diagonalise it by transforming it into an appropriate Fock basis.

A. Diagonalisation of the Hamiltonian

As usual, fermions are quantised by employing an anticommutation relation,

$$\{ \tilde{\psi}_\alpha(\vec{x}, t), \tilde{\psi}^\dagger_\beta(\vec{y}, t) \} = \delta_{\alpha,\beta} \delta^3(\vec{x} - \vec{y}) ,$$  \hspace{1cm} (54)

(here $\alpha, \beta \in \{1, 2, 3, 4\}$ are spinor indices, which are in other equations suppressed) and thus – just as in the case of the fermionic quantum mechanics discussed in section II – they are Grassmannian operators.

Upon varying the action (52) with respect to $\tilde{\psi}(x)$ and $\psi(x)$, one gets the following operator equations for $\tilde{\psi}(x)$ and $\psi(x)$,

$$\gamma^\mu \partial_\mu \tilde{\psi}(x) - m \tilde{\psi}(x) = \hat{j}_\psi(x) , \quad -i \partial_\mu \psi(x) \gamma^\mu - m \psi(x) = \hat{j}_\psi(x) ,$$  \hspace{1cm} (55)

related by hermitian conjugation for real $m$. The simplest nontrivial case is when the current is generated by a mixing mass term. In this case $\hat{j}_\psi(x) = \sum_{n=1}^N m_0 \tilde{\psi}_j(x)$ and $\psi_0(x) \equiv \psi(x)$, $m_{00} \equiv m$, cf. Eqs. (24).

Here we shall study only time dependent problems, and we shall work in a spatial cube of volume $V$, such that it is convenient to transform these equations into a spatial momentum space, defined by,

$$\hat{\psi}(x) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \tilde{\psi}(\vec{k}, t) e^{i \vec{k} \cdot \vec{x}} ; \quad \hat{\psi}(\vec{k}, t) = \frac{1}{\sqrt{V}} \int d^3 x \tilde{\psi}(x) e^{-i \vec{k} \cdot \vec{x}} ,$$  \hspace{1cm} (56)

where $\vec{k} = (2\pi/L) \vec{n}$, $L = V^{1/3}$ is the linear size of the cube $V$, $\vec{n} = (n_1, n_2, n_3)$, $n_i \in \mathbb{Z}$ and $\mathbb{Z}$ is the set of integers. With these definitions we then get Eqs. (54) and (55) in momentum space,

$$\{ \tilde{\psi}_\alpha(\vec{k}, t), \tilde{\psi}^\dagger_\beta(\vec{k'}, t) \} = \delta_{\alpha,\beta} \delta_{\vec{k},\vec{k}'} ,$$  \hspace{1cm} (57)

and

$$(\gamma^0 \partial_t - \gamma^i \cdot \vec{k} - m) \tilde{\psi}(\vec{k}, t) = \hat{j}_\psi(\vec{k}, t) , \quad -i \partial_\mu \psi(\vec{k}, t) \gamma^\mu + \tilde{\psi}(\vec{k}, t) (\vec{k} \cdot \gamma - m) = \hat{j}_\psi(\vec{k}, t) ,$$  \hspace{1cm} (58)

where

$$\hat{j}_\psi(\vec{k}, t) = \frac{1}{\sqrt{V}} \int d^3 x \tilde{j}_\psi(x) e^{-i \vec{k} \cdot \vec{x}} ; \quad \hat{j}_\psi(\vec{k}, t) = \frac{1}{\sqrt{V}} \int d^3 x \tilde{j}_\psi(x) e^{i \vec{k} \cdot \vec{x}} .$$  \hspace{1cm} (59)

Because the problem at hand is linear, there is no momentum mixing. Since we are interested in time evolution, we can work in the helicity eigenbasis, in which

$$\hat{\psi}(\vec{k}, t) = \sum_{h=\pm} \hat{\psi}_h(\vec{k}, t) \otimes \xi_h(\vec{k}) , \quad \hat{j}_\psi(\vec{k}, t) = \sum_{h=\pm} \hat{j}_h(\vec{k}, t) \otimes \xi_h(\vec{k}) ,$$  \hspace{1cm} (60)

where $\xi_h(\vec{k})$ are the two-component helicity eigenspinors, satisfying

$$\hat{h} \xi_h \equiv \vec{k} \cdot \vec{\sigma} \xi_h = h \xi_h ,$$  \hspace{1cm} (61)

where $\hat{h}$ is the helicity operator in the two-by-two (Bloch) representation of Clifford algebra, which can be defined in terms of the helicity operator $\hat{H}$ as follows, $\hat{H} = \vec{k} \cdot \vec{\sigma} = \text{diag}(\hat{h}, \hat{h}) = I_2 \otimes \hat{h}$, $\hat{\Sigma} = \gamma^0 \vec{\gamma} \gamma^5$. By making use of the Bloch decomposition of the Clifford algebra in the Weyl/chiral representation,

$$\gamma^0 \rightarrow \rho^1 \otimes I , \quad \gamma^i \rightarrow \rho^2 \otimes \sigma^i , \quad \gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3 \rightarrow -\rho^3 \otimes I ,$$  \hspace{1cm} (62)
where \( \sigma^i, \rho^i \ (i = 1, 2, 3) \) are the Pauli matrices, equation \(^{58}\) (when multiplied by \( \gamma^0 \)) can be rewritten as
\[
(i\partial_t + h k \rho^3 - m \rho^1)\hat{\psi}_h(\vec{k}, t) = \rho^1 \hat{j}_h(\vec{k}, t),
\]
(63)
where \( \hat{\psi}_h(\vec{k}, t) \) is the two spinor whose components describe the two chiralities \( \hat{L}_h \) and \( \hat{R}_h \) and \( k = \| \vec{k} \| \). Here we shall consider the simpler case when the mass matrix is time independent. \(^7\) In this case a further (orthogonal) rotation,
\[
R = c_\phi - i \rho^3 s_\phi, \quad R^T = c_\phi + i \rho^3 s_\phi, \quad R : R^T = I = R^T : R, \quad \tan(2\phi) = \frac{m}{hk}, \quad \sin(2\phi) = \frac{m}{\omega}, \quad \cos(2\phi) = \frac{hk}{\omega},
\]
(64)
diagonalises equation \(^{63}\), where \( \omega = \sqrt{k^2 + m^2}, \ c_\phi \equiv \cos(\phi) \) and \( s_\phi \equiv \sin(\phi) \). The resulting (diagonalised) equation \(^{63}\) is of the form,
\[
(i\partial_t + \omega \rho^3)\hat{\Psi}_h(\vec{k}, t) = (c_{2\phi} \rho^1 - s_{2\phi} \rho^3)\hat{j}_h(\vec{k}, t),
\]
(65)
where
\[
\hat{\Psi}_h = R\hat{\psi}_h = \begin{pmatrix} \hat{\psi}_{h1} \\ \hat{\psi}_{h2} \end{pmatrix}, \quad \hat{J}_h = R\hat{j}_h = \begin{pmatrix} \hat{j}_{h1} \\ \hat{j}_{h2} \end{pmatrix},
\]
(66)
and we made use of,
\[
R\rho^3 R^T = c_{2\phi} \rho^1 + s_{2\phi} \rho^1, \quad R\rho^1 R^T = c_{2\phi} \rho^3 - s_{2\phi} \rho^3.
\]
When Eq. \(^{65}\) is rewritten in components, we get that the positive and negative frequency modes (particles and antiparticles) obey
\[
(i\partial_t + \omega)\hat{\psi}_{h1} = -\frac{m}{\omega}\hat{j}_{h1} + \frac{hk}{\omega}\hat{j}_{h2},
\]
\[
(i\partial_t - \omega)\hat{\psi}_{h2} = \frac{hk}{\omega}\hat{j}_{h1} + \frac{m}{\omega}\hat{j}_{h2}.
\]
(67)
In the absence of currents, the problem is reduced to the diagonal one, and there is no mixing between different states. We can define a Fock basis \( | \vec{k} \rangle \) (\( \vec{k} \) is the 3-momentum), which – in the absence of interactions – diagonalises the Hamiltonian. However, in general this procedure does not diagonalise the density matrix. Only when the source currents vanish and there is no initial entanglement between \( \hat{\psi}_{h1} \) and \( \hat{\psi}_{h2} \) (i.e., the Fock basis simultaneously diagonalises the Hamiltonian and density operator. In that case we can define the density operator as a direct product (\( \text{cf. Eqs. (12) and (11)} \)) of the density operators for the different fermionic components. In general, however, fermionic interactions (modeled by the currents \( \hat{j}_{h1,2} \)) generate mixing between the \( \hat{\psi}_{h1,2} \) and \( \hat{j}_{h1,2} \) fields, as can be seen from Eq. \(^{67}\). Therefore, this mixing should also be included in the Gaussian density matrix for a fermionic field.

**B. Density operator and fermionic entropy**

Following the previous discussion, a more general \( \textit{Ansatz} \) for the Gaussian density operator for an interacting quantum field is
\[
\hat{\rho}(t) = \frac{1}{Z} \exp \left( -\sum_{\vec{k}, h} \hat{\Psi}_h^\dagger(\vec{k}, t) \Lambda_h(k, t) \hat{\Psi}_h(\vec{k}, t) \right),
\]
(68)
where \( \hat{\Psi} \) is defined in \(^{66}\), \( Z \) is the normalisation constant determined by \( \text{Tr}[\hat{\rho}] = 1 \) and
\[
\Lambda_h(k, t) = \begin{pmatrix} \lambda_{h,11} & \lambda_{h,12} \\ \lambda_{h,21} & \lambda_{h,22} \end{pmatrix}, \quad \lambda_{h,ij} \equiv \lambda_{h,ij}(k, t).
\]
(69)

\(^7\) When the mass matrix is time dependent, \( m = m(t) \) and \( \omega = \omega(t) \), which can occur \( e.g. \) during a phase transition in the early Universe, then a unitary matrix is needed to diagonalise the \( 2 \times 2 \) problem \(^{63}\), where now \( \theta = \theta(t) \) and \( \phi = \phi(t) \).
From now on we suppress the momentum labels and time dependence for the matrix elements $\lambda_{i,j}$. Note that the parameters only depend on $k = ||\vec{k}||$ due to the assumed spatial homogeneity and isotropy of the state. In our Ansatz (68) the positive and negative helicity states, as well as the different momentum states, do not mix, but the $\hat{\psi}_{1,2}$ and $\bar{\psi}_{1,2}$ states do mix through the parameter $\lambda_{12}$. One could consider adding linear terms in $\hat{\Psi}, \hat{\Psi}^\dagger$ in (68) that would correspond to a nonzero value of $\langle \hat{\Psi}, \langle \hat{\Psi}^\dagger \rangle$. However, if the expectation value is initially zero, it will remain zero when the system evolves and Eq. (68) is the most general Ansatz for $\hat{\rho}$. The density operator can be diagonalised by a unitary transformation $U$

$$U = \begin{pmatrix} \cos \theta & e^{-i\phi} \sin \theta \\ e^{i\phi} \sin \theta & \cos \theta \end{pmatrix},$$

with $\theta = \theta(t)$ and

$$\cos 2\theta = \frac{\lambda_{11} - \lambda_{22}}{\sqrt{(\lambda_{11} - \lambda_{22})^2 + 4|\lambda_{12}|^2}}, \quad \sin 2\theta = \frac{-2|\lambda_{12}|}{\sqrt{(\lambda_{11} - \lambda_{22})^2 + 4|\lambda_{12}|^2}}, \quad e^{i\phi} = \frac{\lambda_{12}}{|\lambda_{12}|},$$

such that the diagonalised density operator becomes

$$\hat{\rho}(t) = \frac{1}{Z} \exp \left( - \sum_{\vec{k},h} \hat{\psi}_{h}^d(\vec{k}) \Lambda_{h}^d(k, t) \hat{\psi}_{h}^\dagger(\vec{k}) \right),$$

where

$$\hat{\psi}_{h}^d(\vec{k}) = U \hat{\Psi}_{h}(\vec{k}, t) = \begin{pmatrix} \hat{\psi}_{h+}(\vec{k}, t) \\ \hat{\psi}_{h-}(\vec{k}, t) \end{pmatrix}, \quad \Lambda_{h}^d(k, t) = U \Lambda_{h}(k, t) U^\dagger = \begin{pmatrix} \lambda_{h+} & 0 \\ 0 & \lambda_{h-} \end{pmatrix}, \quad \lambda_{h\pm} \equiv \lambda_{h\pm}(k, t).$$

Here, we dropped the time dependence in $\hat{\psi}_{h}^d(\vec{k})$, since these are operators in the Schrödinger picture. The eigenvalues are

$$\lambda_{\pm} = \frac{1}{2} (\lambda_{h11} + \lambda_{h22}) \pm \frac{1}{2} \sqrt{(\lambda_{h11} - \lambda_{h22})^2 + 4|\lambda_{h12}|^2} = \frac{1}{2} \text{Tr}[^h \Lambda_{h}] \pm \frac{1}{2} \sqrt{(\text{Tr}[^h \Lambda_{h}])^2 - 4\text{Det}[^h \Lambda_{h}]}.$$ 

In the second line the eigenvalues $\lambda_{h\pm}$ are expressed in terms of the Gaussian invariants of the exponent of the density matrix, that is, in terms of the trace and determinant of $\Lambda_{h} = \Lambda_{h}(k, t)$. As in the quantum mechanical case, we can identify the statistical particle number $\hat{N}_{h\pm}(\vec{k}) = \hat{\psi}_{h\pm}^\dagger(\vec{k}) \hat{\psi}_{h\pm}(\vec{k})$ and introduce a Fock basis $|n_{h\pm}(\vec{k})\rangle$ (not to be confused with the Fock basis $|n^d_{h\pm}(\vec{k})\rangle$ that diagonalises the Hamiltonian, discussed above). Of course

$$\hat{N}_{h\pm}(\vec{k})|n_{h\pm}(\vec{k})\rangle = n_{h\pm}(k)|n_{h\pm}(\vec{k})\rangle, \quad \hat{N}_{h\pm}(\vec{k}) = \hat{\psi}_{h\pm}^\dagger(\vec{k}) \hat{\psi}_{h\pm}(\vec{k}).$$

The trace of the density operator can now be taken easily, and by demanding $\text{Tr}[\hat{\rho}] = 1$ the normalisation is

$$Z = \prod_{\vec{k},h\pm} \left( 1 + e^{-\lambda_{h\pm}(k, t)} \right).$$

Thus the density operator can be written as a direct product,

$$\hat{\rho}(t) = \prod_{\vec{k},h\pm} \hat{\rho}_{h\pm}(\vec{k}, t), \quad \hat{\rho}_{h\pm}(\vec{k}, t) = (1 - \bar{n}_{h\pm}(k, t)) + (2\bar{n}_{h\pm}(k, t) - 1)\bar{N}_{h\pm}(\vec{k}),$$

where the average particle number is

$$\bar{n}_{h\pm}(k, t) = \langle \bar{N}_{h\pm}(\vec{k}) \rangle = \text{Tr}[\hat{\rho}(t)\bar{N}_{h\pm}(\vec{k})] = [1 + \exp(\lambda_{h\pm}(k, t))]^{-1}.$$ 

The entropy is then (cf. Eq. (13)),

$$S = \sum_{\vec{k}h\pm} s_{h\pm}(k, t), \quad s_{h\pm}(k, t) = -(1 - \bar{n}_{h\pm}(k, t)) \ln (1 - \bar{n}_{h\pm}(k, t)) - \bar{n}_{h\pm}(k, t) \ln (\bar{n}_{h\pm}(k, t)).$$
The main difference between the quantum fermionic oscillator studied in section II B and the (free) quantum field theoretic oscillator is that for each fermionic mode there are four distinct states (particles), and hence four contributions to the entropy for each mode \( k \): two from the two helicity states and two from the particle/antiparticle states. Furthermore, Eq. (67) implies that both environmental particles and antiparticles will in general contribute to the entropy of the system (anti-)particle. In the non-relativistic limit when \( k \to 0 \) particle-antiparticle mixing is absent. This is a property of the particular form of the interaction (52), which for the purpose of this paper we take to be an operator valued scalar fermionic current density, which can be generated, for example, by a mass mixing term. Other possible interaction Lagrangians that occur in nature include: pseudo-scalar, vector and pseudo-vector fermionic currents. For simplicity we shall only consider here the scalar fermionic current.

Just like the quantum mechanical case the average particle numbers \( \bar{n}_{h,\pm}(k, t) \) (and the entropy) can be expressed in terms of statistical correlators. Using Eqs. (68), (74) and (76) one finds

\[
\langle \hat{\psi}_{h1}^\dagger \hat{\psi}_{h1} \rangle = -\frac{\partial}{\partial \lambda_{h,11}} \ln Z = \frac{1}{2} \left( \bar{n}_{h,+} + \bar{n}_{h,-} \right) + \frac{1}{2} \left( \bar{n}_{h,+} - \bar{n}_{h,-} \right) \frac{\lambda_{h,11} - \lambda_{h,22}}{\sqrt{(\lambda_{h,11} - \lambda_{h,22})^2 + 4|\lambda_{h,12}|^2}} \\
\langle \hat{\psi}_{h2}^\dagger \hat{\psi}_{h2} \rangle = -\frac{\partial}{\partial \lambda_{h,22}} \ln Z = \frac{1}{2} \left( \bar{n}_{h,+} + \bar{n}_{h,-} \right) - \frac{1}{2} \left( \bar{n}_{h,+} - \bar{n}_{h,-} \right) \frac{\lambda_{h,11} - \lambda_{h,22}}{\sqrt{(\lambda_{h,11} - \lambda_{h,22})^2 + 4|\lambda_{h,12}|^2}} \\
\langle \hat{\psi}_{h1}^\dagger \hat{\psi}_{h2} \rangle = -\frac{\partial}{\partial \lambda_{h,12}} \ln Z = \frac{1}{2} \left( \bar{n}_{h,+} - \bar{n}_{h,-} \right) - \frac{2\lambda_{h,12}}{\sqrt{(\lambda_{h,11} - \lambda_{h,22})^2 + 4|\lambda_{h,12}|^2}} \\
\langle \hat{\psi}_{h2}^\dagger \hat{\psi}_{h1} \rangle = -\frac{\partial}{\partial \lambda_{h,12}^*} \ln Z = \frac{1}{2} \left( \bar{n}_{h,+} - \bar{n}_{h,-} \right) - \frac{2\lambda_{h,12}}{\sqrt{(\lambda_{h,11} - \lambda_{h,22})^2 + 4|\lambda_{h,12}|^2}}.
\]

(80)

Here \( \hat{\psi}_{hi} = \hat{\psi}_{hi}(k, t) \), \( \bar{n}_{h,\pm} = \bar{n}_{h,\pm}(k, t) \) and \( \lambda_{ij} = \lambda_{h,ij}(k, t) \). We can easily relate the correlators above to the equal time statistical correlators for the \( \hat{\psi}_{hi} \) fields

\[
\langle \hat{\psi}_{h1}^\dagger \hat{\psi}_{hj} \rangle = \frac{1}{2} \left\{ \langle \hat{\psi}_{h1}^\dagger \hat{\psi}_{hj} \rangle \right\} = -\frac{1}{2} \left\{ \langle \hat{\psi}_{hj} \hat{\psi}_{h1}^\dagger \rangle \right\} = \frac{1}{2} \delta_{ij} - F_{h,ji},
\]

(81)

where \( F_{h,ij} = F_{h,ij}(k; t; t) \). The average particle number expressed in terms of the statistical correlators is then

\[
\bar{n}_{h,\pm} = \frac{1}{2} - \frac{1}{2} (F_{h,11} + F_{h,22}) \pm \text{sgn}(F_{h,22} - F_{h,11}) \frac{1}{2} \sqrt{(F_{h,22} - F_{h,11})^2 + 4F_{h,12}^2F_{h,21}}.
\]

(82)

The correctness of this expression can be checked when going to the single quantum mechanical fermion case, thus only keeping for example the \( \hat{\psi}_{1} \) fields and setting \( \hat{\psi}_{2} \) fields to zero. For specific \( h, \tilde{k} \) there is only one remaining particle number, and it agrees with Eq. (22). Moreover, in the absence of interactions and with zero initial mixing, the Ansatz for the density matrix (68) becomes diagonal, i.e. \( \lambda_{12} = 0 \). As we stated earlier in the introduction to this section, the density matrix then becomes a direct product of different single fermion density matrices. Indeed, the non-interacting case gives \( \bar{n}_{h,+} = \frac{1}{2} - F_{h,11}, \bar{n}_{h,-} = \frac{1}{2} - F_{h,22} \), which agrees with the average particle number for a single fermionic oscillator (22). Thus the entropy for a non-interacting fermionic field is simply given by the sum of the entropies of the components of the diagonalised Hamiltonian, what was to be expected.

In principle we could have also made an Ansatz for the density operator (68) in terms of rotated fermion fields, for example in terms of left- and right-handed fields. Of course, the resulting entropy should not depend on the basis in which the Ansatz is made, but one basis may be more convenient than the other. In order to clarify this, we can define (just as in the single fermion case) Gaussian invariants of the correlators

\[
\Delta_{h,\pm} = 1 - 2\bar{n}_{h,\pm} = \tanh \left( \frac{\lambda_{h,\pm}}{2} \right).
\]

(83)

The fact that the \( \Delta_{h,\pm} \) are Gaussian invariants of the correlators becomes more clear when introducing a \( 2 \times 2 \) matrix of statistical correlators

\[
F_h = \begin{pmatrix} F_{h,11} & F_{h,12} \\ F_{h,21} & F_{h,22} \end{pmatrix},
\]

(84)

such that

\[
\Delta_{h,\pm} = \text{Tr}[F_h] \mp \sqrt{(\text{Tr}[F_h])^2 - 4\text{Det}[F_h]}.
\]

(85)

Both the trace and the determinant are invariant under a change of basis, thus also the expressions for the Gaussian invariant are indeed invariant, as are the particle number (82) and the entropy (79). Moreover, because with (74)
the eigenvalues $\lambda_{\pm}$ can be expressed in terms of Gaussian invariants of the density operator, Eq. (83) presents the relation between the Gaussian invariants of the correlators and those of the density matrix.

As a final comment, note that the entropy (79) only gives a limited amount of information. The complete density operator (68) contains more information and we can separate it into two parts. The "mostly classical" information is stored in the spectrum, which we define as

$$\text{Spec}[\hat{\rho}] \equiv \{\lambda_i\}, \quad (86)$$

where $\lambda_i$ $(i = 1, \ldots, N)$ are the eigenvalues of $-\ln(Z\hat{\rho})$ and can be read off from the diagonalised form of the density operator. To be more precise, for an $N$-state system the components of the spectrum are

$$\lambda_i = \langle 0|\ldots|i\ldots|0_N|(-\ln[Z\hat{\rho}]|0_N\rangle\ldots|1_i\rangle\ldots|0\rangle,$$

where here $\hat{\rho}$ is assumed to be written in diagonal form and the $|n_i\rangle$ are the Fock states used to diagonalise the density operator. An example of the spectrum for a two-state system can be read off from Eq. (72), where the two eigenvalues $\lambda_{h\pm}$ (4) contain the "mostly classical information". They are related to the averaged particle numbers $\bar{n}_{h\pm} = (e^{\lambda_{h\pm}} + 1)^{-1}$, which is what a late time observer entangled with the Fock states of the system would identify with a thermal distribution of fermionic particles. On the other hand, the "mostly quantum" information is stored in the off-diagonal components of $\hat{\rho}$, which describe mixing (entanglement) between different states in the original (non-diagonal) basis. In section III D below we show both $\{\lambda_i\}$ and $\{\bar{n}_i\}$.

The Fock states (75) are a natural candidate for pointer states [15], which are selected by the environment, and in which the system becomes classical, explaining the above term "mostly classical". Hence, these Fock states are particularly useful when considering the process of classicalization of a quantum system, and in fact we use them to define it. The rate of statistical particle number increase we identify as the rate of classicalization $^8$. The rotation matrix that brings the density operator to a diagonal form has no classical analogue, thereby justifying the name "mostly quantum". Of course, even though the Fock states (75) do exist at early times, the system is then not yet classical. One therefore needs a more precise definition of when the system becomes classical.

Zurek states [18] that pointer states are stable under the influence of the environment, but provides no deeper insight into why this is so. We believe that the stability of pointer states can be explained by entropic considerations. Namely, in the limit when the number of environmental oscillators $N$ becomes large, elements of the reduced density matrix in the diagonal Fock basis become stable (up to small statistical fluctuations) because most of the volume of the total Hilbert space (of the system + environment) corresponds to an almost constant average occupation values of the Fock states (75). This represents a quantum generalisation of the ergodic hypothesis.

As with regards to classicality, there is a notable difference between bosonic and fermionic systems. While, in the case of bosonic systems, in the high temperature limit, one can speak of large occupation numbers of certain oscillators, yielding a well defined classical field theoretic limit, no classical field theoretic limit exists for fermions (simply because fermionic occupation numbers must lie in between 0 and 1). However, that does not mean that there is no classical limit for fermionic systems. In the case when the fermionic mass $m$ is very big and the spatial size of the fermionic system is large (e.g. when there are many available system states and they are dense in energy), one speaks of a classical particle limit, even when the occupation number of each of the states is much less than 1. In this case, the number of fermions is well-defined (particle number fluctuations are suppressed) and the Pauli blocking is not important. An important example of the classical particle limit is the classical fermionic thermal case, in which $mc^2 \gg k_B T$, such that fermions get distributed according to the Maxwell distribution, which is of course classical. An analogous classical particle limit exists for bosonic systems as well.

**C. Generalisation to $N$ degrees of freedom**

Up to now a density operator $\hat{\rho}$ for two degrees of freedom (at fixed helicity $h$ and momentum $\vec{k}$) was considered. Explicit diagonalisation led to a formula for the "phase space" $\Delta$'s and the entropy (79). The diagonal elements were represented by the invariants $\text{Tr}[F_{h}]$ and $\text{Det}[F_{h}]$ of the statistical Greens function matrix. Here we consider a more general setting with $N$ degrees of freedom. The (Gaussian) Ansatz for $\hat{\rho}$ is now

$$\hat{\rho} = \frac{1}{Z} \exp(-\frac{1}{2} a_{ij}^\dagger \bar{a}_{ij}), \quad (i, j = 1, \ldots, N). \quad (87)$$

$^8$ Of course, the classicalization rate is observer dependent and different observers will measure different classicalization rates. For example, the position operator $\langle \Delta \hat{x}^2 \rangle$ will perceive a different (typically larger) rate of classicalization.
Indeed this is equivalent to a $\rho(\theta', \theta)$ in the coherent state representation:

$$\rho(\theta', \theta) = \langle \theta' \mid \rho \mid \theta \rangle = \frac{1}{Z} \exp(\theta'_i M_{ij} \theta_j),$$

with

$$M_{ij} = (e^{-a})_{ij}$$

$$Z = \text{Det}[I + e^{-a}],$$

as one can easily see by diagonalising the hermitian (real, symmetric) matrices $a, M$ simultaneously (as in the $2 \times 2$ case considered before). The 2-correlators $\langle \hat{\psi}_{k}^{+} \hat{\psi}_{l} \rangle$ are related to the statistical "matrix" $F_{kl}$ and to the $a_{ik}$ introduced above:

$$\bar{n}_{kl} \equiv \langle \hat{\psi}_{k}^{+} \hat{\psi}_{l} \rangle = \frac{1}{2} \delta_{kl} - F_{lk}(t; t) = \frac{1}{2}(I - \Delta)_{kl},$$

which is a suitable generalisation to many degrees of freedom of the one degree of freedom result \(22\). $\langle \hat{\psi}_{k}^{+} \hat{\psi}_{l} \rangle$ can be also obtained by differentiating $\text{Tr}[\hat{\rho}]$ of Eq. \(87\) with respect to $-a_{kl}$:

$$- \frac{\partial}{\partial a_{kl}} \text{Tr}[\hat{\rho}] = 0 = \text{Tr} \left[ \frac{\exp(-\hat{\psi}_{k}^{+} a_{ij} \hat{\psi}_{l})}{Z} \right] - \text{Tr} \left[ \exp(-\hat{\psi}_{k}^{+} a_{ij} \hat{\psi}_{l}) \right] \frac{\partial}{\partial a_{kl}} \left( \frac{1}{Z} \right),$$

where the first term is just $\langle \hat{\psi}_{k}^{+} \hat{\psi}_{l} \rangle$ and the second term is evaluated as

$$-Z \frac{\partial}{\partial a_{kl}} \left( \frac{1}{Z} \right) = \frac{1}{Z} \frac{\partial}{\partial a_{kl}} \text{Det}[I + e^{-a}]$$

$$= \frac{\partial}{\partial a_{kl}} \text{Tr} \ln (I + e^{-a})$$

$$= - \left( e^{-a} \right)_{lk} = - \left( \frac{I}{I + e^{a}} \right)_{lk}.$$

We have used here the identity $\text{Det}[A] = \exp\{\text{tr}\ln(A)\}$, and that $(\partial/\partial a_{ij})\text{Tr}(a) = (f'(a))_{ji}$, where $f$ is some function of the matrix $a_{ij}$. We thus obtain the generalisation of Eqs. \(22\),

$$\bar{n}_{kl} = \frac{1}{2}(I - \Delta)_{kl} = \left[(I + e^{a})^{-1}\right]_{lk}.$$

Diagonalising $a, \Delta, F$ with the same rotation we can use a sum of terms of type Eq. \(24\) for the entropy. This is a trace and rotating back inside the trace we obtain

$$S = -\text{Tr} \left[ \frac{I + \Delta}{2} \ln \left( \frac{I + \Delta}{2} \right) + \frac{I - \Delta}{2} \ln \left( \frac{I - \Delta}{2} \right) \right] = -\text{Tr} \left[ (I - \bar{n}) \ln(I - \bar{n}) + \bar{n} \ln \bar{n} \right].$$

For this formula to work the eigenvalues of $\bar{n}$ must lie in the interval $[0, 1]$, which is indeed the case for fermionic systems. The result \(94\) we have also obtained using the replica trick for the density operator $\hat{\rho}$ in the coherent representation \(88\) without a diagonalisation procedure. Appendix \(D\) contains the calculational details. Of course in order to evaluate \(94\), diagonalisation of $\Delta, a$ is again the fastest method to obtain the entropy. In the next section we discuss the growth of entropy for Dirac fermions mixing through a mass matrix, where we will also show the spectrum of $\hat{\rho} (86)$ as well as the average particle number.

## D. Fermion mass mixing

A simple model for interacting fermions is a model of different fermion species mixing through a mass matrix. Similar to the quantum mechanical action of bilinearly coupled fermions \(41\), the action for fermion mass mixing

$$S[\hat{\psi}_x, \{\hat{\psi}_y\}] = \int d^4x \left\{ \mathcal{L}_{S}[\hat{\psi}_x] + \mathcal{L}_{E}[\{\hat{\psi}_y\}] + \mathcal{L}_{\text{int}}[\hat{\psi}_x, \{\hat{\psi}_y\}] \right\},$$

\(95\).
with
\[
\mathcal{L}_S[\hat{\psi}_x] = \hat{\psi}_x^\dagger(x)(i\gamma^\mu \partial_\mu - m_0)\hat{\psi}_x(x)
\]
\[
\mathcal{L}_E[\hat{\psi}_q] = \sum_{i=1}^{N} \hat{\psi}_q^\dagger_i(x)(i\gamma^\mu \partial_\mu - m_i)\hat{\psi}_q_i(x)
\]
\[
\mathcal{L}_{\text{int}}[\hat{\psi}_x, \{ \hat{\psi}_q_i \}] = -\sum_{i=1}^{N} \left( m_{0i} \hat{\psi}_x^\dagger \hat{\psi}_q_i + m_{i0} \hat{\psi}_q_i^\dagger \hat{\psi}_x \right).
\]  
(96)

From now on we assume that the mass mixing parameters are real, \( m_{0i} = m_{i0} \). Next, we follow the same steps as in section III A: we first transform all fields to momentum space as in (56), then go to the helicity eigenbasis using (60), and finally rotate the fields as in Eq. (66) (with a different rotation matrix \( R_i \) for different species). The resulting action is
\[
S[\hat{\psi}_x, \{ \hat{\psi}_q_i \}] = \int dt \sum_{k, h} \left\{ \hat{\psi}_x^\dagger(k, t)(i\partial_t + \omega_0 \rho^3)\hat{\psi}_x(k, t) + \sum_{i=1}^{N} \hat{\psi}_q_i^\dagger(k, t)(i\partial_t + \omega_i \rho^3)\hat{\psi}_q_i(k, t) \right. \\
- \sum_{i=1}^{N} m_{0i} \left( \hat{\psi}_x^\dagger R_0 \rho^1 R_i^T \hat{\psi}_q_i + \hat{\psi}_q_i^\dagger R_0 \rho^1 R_i^T \hat{\psi}_x \right) \right\}.
\]  
(97)

where \( \omega_0 = \sqrt{m_0^2 + \|k\|^2} \) and \( \omega_i = \sqrt{m_i^2 + \|k\|^2} \). \( R = R(\theta) \) is the rotation matrix that diagonalises the free part of the action for \( \hat{\psi}_x \), whereas \( R_i = R(\theta_i) \) diagonalises the action for \( \hat{\psi}_q_i \). When the masses of different species are the same, \( m_0 = m_i \), the rotation matrices will be the same, \( R = R_i \). In general the interaction term is
\[
R_0 \rho^1 R_i^T R_i \rho^3 R_i^T = \cos(\theta + \theta_i) \rho^1 - \sin(\theta + \theta_i) \rho^3,
\]  
(98)

with the \( \theta, \theta_i \) defined as in Eq. (64). The equations of motions follow directly from the action (97)
\[
(i\partial_t + \omega_0 \rho^3)\hat{\psi}_x(k, t) = \sum_{i=1}^{N} m_{0i} R_0 \rho^1 R_i^T \hat{\psi}_q_i(k, t)
\]
\[
(i\partial_t + \omega_i \rho^3)\hat{\psi}_q_i(k, t) = m_{0i} R_0 \rho^1 R_i^T \hat{\psi}_x(k, t)
\]
\[
(-i\partial_t \hat{\psi}_x^\dagger(k, t) + \omega_0 \hat{\psi}_x^\dagger(k, t) \rho^3) = \sum_{i=1}^{N} m_{0i} \hat{\psi}_q_i^\dagger(k, t) R_0 \rho^1 R_i^T
\]
\[
(-i\partial_t \hat{\psi}_q_i^\dagger(k, t) + \omega_i \hat{\psi}_q_i^\dagger(k, t) \rho^3) = m_{0i} \hat{\psi}_x^\dagger(k, t) R_0 \rho^1 R_i^T.
\]  
(99)

Note that each line consists of two equations for the two components of the spinors \( \hat{\psi}_x \) and \( \hat{\psi}_q_i \). Differential equations for the statistical equal-time correlators can be derived from the equations of motions (99). It is only necessary to derive the statistical correlators for fields with the same helicity. Remember that there is no helicity mixing in the action. For the system alone, the statistical correlators obey,
\[
i\partial_t F_{xx,11} = \sum_{i=1}^{N} m_{0i} \left[ \cos(\theta + \theta_i) \left( F_{q,x,21} - F_{q,x,12} \right) - \sin(\theta + \theta_i) \left( F_{q,x,11} - F_{q,x,12} \right) \right]
\]
\[
i\partial_t F_{xx,22} = \sum_{i=1}^{N} m_{0i} \left[ \cos(\theta + \theta_i) \left( F_{q,x,12} - F_{q,x,21} \right) + \sin(\theta + \theta_i) \left( F_{q,x,12} - F_{q,x,11} \right) \right]
\]
\[
(i\partial_t + 2\omega_0) F_{xx,12} = \sum_{i=1}^{N} m_{0i} \left[ \cos(\theta + \theta_i) \left( F_{q,x,22} - F_{q,x,11} \right) - \sin(\theta + \theta_i) \left( F_{q,x,12} + F_{q,x,11} \right) \right].
\]  
(100)

Here we have used a shorthand notation, with
\[
F_{xx,hmn} \equiv F_{xx,hmn}(k; t) = \frac{1}{2} \langle [\hat{\psi}_x^\dagger h m(k, t), \hat{\psi}_x^\dagger h n(k, t)] \rangle, \quad m, n = 1, 2, \quad \text{etc.}
\]  
(101)
Note that \( F_{xx,hmn}^* = F_{xx,hnm} \), \( F_{xq,hmn}^* = F_{q;x,hnm} \) and \( F_{q,q,hmn}^* = F_{q,q,hnm} \). With these relations the remaining equation for \( F_{xx,h21} \) in \([100]\) can be found easily by taking the complex conjugate. Similarly, the equations of motion for the environmental correlators are:

\[
\begin{align*}
(\partial_t + (\omega_i - \omega_j)) F_{q,q,h11} &= m_{0i} \left[ \cos(\theta + \theta_j) F_{q,q,h21} - \sin(\theta + \theta_i) F_{q,q,h11} \right] \\
&\quad - m_{0j} \left[ \cos(\theta + \theta_j) F_{q,x,h12} - \sin(\theta + \theta_j) F_{q,x,h11} \right] \\
(\partial_t - (\omega_i - \omega_j)) F_{xq,h22} &= m_{0i} \left[ \cos(\theta + \theta_i) F_{xq,h12} + \sin(\theta + \theta_i) F_{xq,h22} \right] \\
&\quad - m_{0j} \left[ \cos(\theta + \theta_j) F_{xq,h21} + \sin(\theta + \theta_j) F_{xq,h22} \right] \\
(\partial_t + (\omega_i + \omega_j)) F_{q,q,h12} &= m_{0i} \left[ \cos(\theta + \theta_i) F_{q,q,h22} - \sin(\theta + \theta_i) F_{q,q,h12} \right] \\
&\quad - m_{0j} \left[ \cos(\theta + \theta_j) F_{q,x,h11} + \sin(\theta + \theta_j) F_{q,x,h12} \right].
\end{align*}
\] (102)

Again, the remaining equation for \( F_{q,q,h21} \) can be obtained by complex conjugation of the third line above. Finally, the system-environment correlators obey the equations

\[
\begin{align*}
(\partial_t + (\omega_0 - \omega_i)) F_{xq,h11} &= \sum_{j=1}^{N} m_{0j} \left[ \cos(\theta + \theta_j) F_{q,q,h21} - \sin(\theta + \theta_j) F_{q,q,h11} \right] \\
&\quad - m_{0i} \left[ \cos(\theta + \theta_i) F_{xx,h12} - \sin(\theta + \theta_i) F_{xx,h11} \right] \\
(\partial_t - (\omega_0 - \omega_i)) F_{xq,h22} &= \sum_{j=1}^{N} m_{0j} \left[ \cos(\theta + \theta_j) F_{q,q,h12} + \sin(\theta + \theta_j) F_{q,q,h22} \right] \\
&\quad - m_{0i} \left[ \cos(\theta + \theta_i) F_{xx,h21} + \sin(\theta + \theta_i) F_{xx,h22} \right] \\
(\partial_t + (\omega_0 + \omega_i)) F_{xq,h12} &= \sum_{j=1}^{N} m_{0j} \left[ \cos(\theta + \theta_j) F_{q,q,h22} - \sin(\theta + \theta_j) F_{q,q,h12} \right] \\
&\quad - m_{0i} \left[ \cos(\theta + \theta_i) F_{xx,h11} + \sin(\theta + \theta_i) F_{xx,h12} \right] \\
(\partial_t - (\omega_0 + \omega_i)) F_{xq,h21} &= \sum_{j=1}^{N} m_{0j} \left[ \cos(\theta + \theta_j) F_{q,q,h11} + \sin(\theta + \theta_j) F_{q,q,h21} \right] \\
&\quad - m_{0i} \left[ \cos(\theta + \theta_i) F_{xq,h22} - \sin(\theta + \theta_i) F_{xq,h21} \right].
\end{align*}
\] (103)

Taking the complex conjugate of these equations gives the final equations of motion for the environment-system correlators. This results in a closed system of \((N + 1)^2 \times 2^2 \times 2\) equations for the correlators of the components of \((N + 1)\) coupled 2-spinors at different helicities. These coupled first order differential equations can be solved (numerically) with initial conditions corresponding to environmental oscillators in chemical equilibrium:

\[
\begin{align*}
F_{xx,h11}(k; t_0; t_0) &= F_{xx,h22}(k; t_0; t_0) = \frac{1}{2} \\
F_{q,q,h11}(k; t_0; t_0) &= \delta_{ij} \frac{1}{2} \tanh \left( \frac{\beta(\omega_i - \mu_i)}{2} \right) \\
F_{q,q,h22}(k; t_0; t_0) &= \delta_{ij} \frac{1}{2} \tanh \left( \frac{\beta(\omega_i + \mu_i)}{2} \right),
\end{align*}
\] (104)

and all others are initially equal to zero. According to Eqs. \([104]\) at \(t = t_0\) there is no mixing between the different components of the 2-spinors in the helicity eigenbasis. Remember that these '1,2' components are the fields that diagonalize the Hamiltonian in a non-interacting theory; they are the positive and negative frequency states, or particles and antiparticles. Note that the initial state \([104]\) allows for nonvanishing chemical potentials \(\mu_i\) for the environmental fields. The chemical potentials have an opposite sign for particles and antiparticles. Moreover, we have assumed that initially there is no mixing between the system and the environment. The physical picture is therefore that initially there is no mass-mixing between the different fermion species, but at \(t = t_0\) the coupling is switched on and the entropy of the system can grow. That is, we consider only the entropy of the system

\[
S_x(t) = \sum_{k,h,\pm} s_{xx,h,\pm}(k, t)
= \sum_{k,h,\pm} \left[ \frac{1 + \Delta_{xx,h,\pm}(k, t)}{2} \ln \left( \frac{1 + \Delta_{xx,h,\pm}(k, t)}{2} \right) - \frac{1 - \Delta_{xx,h,\pm}(k, t)}{2} \ln \left( \frac{1 - \Delta_{xx,h,\pm}(k, t)}{2} \right) \right],
\] (105)
After some time it fluctuates around an equilibrium value. This late time entropy should scale as \( \beta \) Gaussian invariant. In Figs. 8 and 9 the total system entropy increases due to interactions with the environment.

The fluctuations in late time entropy in Figs. 8 and 9 are rather large. The reason is that the bilinear coupling is not a true interaction term: each system field mode is only coupled to \( N \) of the environment field and zero chemical potential, \( \mu_1 = 0 \). The late time entropy is approximately 8 times higher than for \( \beta = (m_0)^{-1} \), which supports the scaling of late time entropy as \( T^3 \).

In the case of \( N = 1 \) environmental fields we have numerically solved the \( 16 \times 2 \) equations for the statistical correlators of particles and antiparticles of the system and environment at different helicities. In the numerical procedure a smooth selection of modes \( \vec{k} \) has been made, separated into spherical bins of size \( \Delta k \). The total system entropy \( S_{\text{tot}} \) is then calculated as

\[
S_{\text{tot}}(t) = \sum_{\vec{k},h,\pm} s_{xx,h,\pm}(k,t) = V \left( \frac{m_0}{2\pi} \right)^3 \sum_{k/\Delta k = 0}^{k = \infty} \sum_{h = \pm} 4\pi \left( \frac{k}{m_0} \right)^2 \left( \frac{\Delta k}{m_0} \right) s_{xx,h,\pm}(k,t),
\]

where \( s_{xx,h,\pm}(||\vec{k}||,t) \) is the system entropy per fermionic degree of freedom, i.e. for a state with quantum numbers \( \vec{k}, h, \pm \). The Gaussian invariants \( \Delta_{xx,h,\pm} \) are those defined in Eq. (85), with the subscript \( xx \) indicating that only the system correlators are used. Due to the loss of information (assuming environmental correlations are inaccessible) the system decoheres, leading to an increase in entropy for the system.

In the case of \( N = 1 \) environmental fields we have numerically solved the \( 16 \times 2 \) equations for the statistical correlators of particles and antiparticles of the system and environment at different helicities. In the numerical procedure a smooth selection of modes \( \vec{k} \) has been made, separated into spherical bins of size \( \Delta k \). The total system entropy \( S_{\text{tot}} \) is then calculated as

\[
S_{\text{tot}}(t) = \sum_{\vec{k},h,\pm} s_{xx,h,\pm}(k,t) = V \left( \frac{m_0}{2\pi} \right)^3 \sum_{k/\Delta k = 0}^{k = \infty} \sum_{h = \pm} 4\pi \left( \frac{k}{m_0} \right)^2 \left( \frac{\Delta k}{m_0} \right) s_{xx,h,\pm}(k,t),
\]

where \( V \) is the volume of the system and \( s_{xx,h,\pm}(k,t) \) is the average entropy per degree of freedom in a spherical bin with a momentum \( ||\vec{k}|| \) and a width \( \Delta k \ll k \). The maximum mode has been chosen such that \( \beta k_{\text{max}} \gg 1 \), since the inclusion of higher modes does not significantly change the total entropy. In Figs. 8 and 9 the entropy density for the system (in units of the inverse Compton wavelength cubed, \( \lambda_C^{-3} = (m_0/(2\pi))^{-3} \)) has been plotted for zero chemical potential, same mass and mass-mixing parameters, but different temperatures. In the absence of a chemical potential the particles and antiparticles evolve completely separately, i.e. \( F_{xx,h12} \) and \( F_{xx,h21} \) are zero. Because the initial conditions are identical the statistical correlators \( F_{xx,h11} \) and \( F_{xx,h22} \) behave equally, and so do the Gaussian invariants \( \Delta_{xx,h,\pm} \) of Eq. (85). Thus the total entropy is simply four times the entropy calculated from a single Gaussian invariant. In Figs. 8 and 9 the total system entropy increases due to interactions with the environment. After some time it fluctuates around an equilibrium value. This late time entropy should scale as \( \beta^{-3} = (k_B T)^{-3} \) in the relativistic limit where \( k_B T/m_0 = 1/(\beta m_0) \gg 1 \). Comparing Figs. 8 and 9 this appears to be the case. The fluctuations in late time entropy in Figs. 8 and 9 are rather large. The reason is that the bilinear coupling is not a true interaction term: each system field mode is only coupled to \( N = 1 \) environmental mode. Due to the unitary evolution, energy flows back and forth from the system to the environmental oscillator, resulting in large amplitude oscillations. As mentioned at the end of Sec. IIIA2 in the case of a true interaction each system mode couples effectively to infinitely many environmental modes, leading to an efficient thermalization of the system with an expected rate that is to a good approximation given by the perturbative rate, just as in the case of bosonic field theory [10].

Next, a distinction can be made between particles and antiparticles by introducing a nonzero chemical potential \( \mu_i \) for the environmental fermion species \( \psi_q \). In Figs. 10 and 11 the particle/antiparticle number densities \( n_q \) and the spectrum of the density operator \( \rho_q \) are shown at different values of \( \beta \). We have taken here the case of a system field interacting with 10 environmental fields, which have masses distributed around the system mass \( m_0 \). In general, the (anti)particle number oscillates between the initial value 0 and the value for perfect thermalisation, when the
Figure 10: Average (anti)particle number $\bar{n}_{h\pm}$ for the mode $k = m_0$ as a function of $m_0 t$ for $N = 10$ environmental fields. The environmental masses are distributed as $m_0 = (0.5i - 0.25) \times m_0$, or $m_1 = 0.25m_0, m_2 = 0.75m_0, ..., m_{10} = 4.75m_0$, and the couplings are all equal $m_0 = 0.2m_0$. The inverse temperature is $\beta = (m_0)^{-1}$ and the (equal) chemical potentials are $\mu_i = m_0$. Due to the chemical potential the antiparticle number density $\bar{n}_{h-}$ (solid blue) is suppressed with respect to the particle number density $\bar{n}_{h+}$ (solid red, thick). The dashed lines indicate the (anti)particle number densities for perfect thermalisation, $(\bar{n}_{th})_{h\pm} = (e^{\beta(\omega - \mu_i)} + 1)^{-1}$. Note that there is no distinction in particle number for $+ \text{ and } -$ helicity states because helicity mixing is absent. The dotted black line is the maximum fermionic particle number in the limit when $\beta \to 0, \bar{n}_{h\pm} \to 0.5$.

Figure 11: Average (anti)particle number $\bar{n}_{h\pm}$ for the mode $k = m_0$ as a function of $m_0 t$ for $N = 10$ environmental fields. The parameters are the same as those in Fig. 10, but the temperature is higher, $\beta = 0.5(m_0)^{-1}$. Both the particle (solid red, thick) and antiparticle (solid blue) number densities are larger than in Fig. 10, but the relative increase of the antiparticle number density is bigger.

Figure 12: Spectrum of the density operator for the mode $k = m_0$ as a function of $m_0 t$ for $N = 10$ environmental fields. The spectrum, defined in Eq. (86), are the $\lambda_{h\pm}$ of Eq. (83). The parameters are the same as those in Fig. 10. Due to the chemical potential $\lambda_{h-}$ (solid blue) is greater than $\lambda_{h+}$ (solid red, thick). When the system is completely thermalised the spectrum is $\lambda_{h\pm} = \beta(\omega_0 \mp \mu_1)$, indicated by the red and blue dashed lines for particles and antiparticles, respectively. Note that the initial value of $\lambda_{h\pm}$ is infinite as the initial particle number is zero.

Figure 13: Spectrum of the density operator for the mode $k = m_0$ as a function of $m_0 t$ for $N = 10$ environmental fields. The parameters are the same as those in Fig. 12, but the temperature is higher, $\beta = 0.5(m_0)^{-1}$. As temperature increases, the difference between $\lambda_{h+}$ and $\lambda_{h-}$ becomes smaller.

System fermions have the same temperature as the environmental fermions, approximately the initial temperature of the environment. Moreover, Figs. 10 and 11 clearly show that, for positive particle environmental chemical potentials,
the system antiparticle number density is suppressed with respect to the particle number density due to the nonzero chemical potentials. As usual, for higher temperatures (lower $\beta$) the particle numbers are closer to the maximum fermionic particle number $\bar{n}_{\text{max}}$ for $\beta \to 0$, but the relative increase of the antiparticle number with respect to particle number is greater.

The spectrum of the density operator (86) is shown in Figs. 12 and 13 and is related to the particle number as in Eq. (75). The eigenvalues $\lambda_{h,\pm}$ of the exponent of the density operator are initially infinite (corresponding to zero (anti)particle number), but oscillate on top of its thermal value of $\lambda_{h,\pm} = \beta (\omega_0 \mp \mu_1)$ at later time. When more environmental fields are added, the oscillations are damped and the eigenvalues, and thus the (anti)particle numbers move closer to a constant. This is similar to what happened in the quantum mechanical case, see Figs. 3–6.

IV. DISCUSSION

In this work we provide a quantitative description of the entropy of quantum mechanical and quantum field theoretic fermionic systems, which here consists of one system oscillator (or field) and $N$ environmental oscillators (or fields). In our correlator approach to decoherence the observer is assumed to be sensitive only to the 2-point correlators of the fermionic system oscillator (or field). In that case the reduced density operator of the system is Gaussian and the corresponding Gaussian entropy can be explicitly calculated in terms of the correlators. We have done this for a one-dimensional fermionic harmonic oscillator (24), for a fermionic quantum field theory (79) and for the general case of $N$ fermionic degrees of freedom (94). We have demonstrated that the Gaussian density matrix singles out a Fock basis in which it becomes diagonal. The Fock basis defines statistical particle number, whose dynamics can be used to define how a system evolves from quantum to classical. In that sense this Fock basis defines pointer states. An observer which measures statistical particle number, when it gets entangled with these pointer states, will experience complete decoherence. The decoherence rate associated with that observer can be defined to be the classicalization rate. In Sec. II B we make a rough estimate of this rate for the quantum mechanical case studied here, but we leave a more detailed study of the classicalization rate in realistic fermionic quantum field theories for future work.

For simplicity in this work we have considered fermionic problems where the system couples bilinearly to the environment. This problem has the advantage that it can be solved exactly by numerical methods. We have demonstrated that the dynamics in general leads to an increase in entropy of the system. When the system couples strongly to the environment in a thermal state at temperature $T$, at late times the system’s entropy reaches its thermal value at the same temperature $T$. Furthermore, in the field theoretic case we have shown that, when environmental fermionic fields are in a chemical equilibrium with the common particle chemical potential, the system field will eventually reach chemical equilibrium with the environment.

While in this work we focus our attention on the study of exactly soluble Gaussian systems with bilinear couplings, fermionic systems occurring in Nature are usually not of that type. An important example is a relativistic quantum field theory with Yukawa interactions (2), where the scalar field is the Standard Model Higgs, a candidate for which has recently been discovered [28–29], or the inflaton in inflationary models. A more sophisticated treatment of the scalar field in the Yukawa interactions is desirable, and one can foresee solving nonlinear, perturbative, Kadanoff-Baym equations for the fermionic and scalar fields, whereby scalar thermal fluctuations are also taken into account. Analogous equations have already been tackled within a certain approximation scheme for the bosonic case in [1, 2, 10]. We intend to address the analogous problem for fermions in future work.

Another interesting extension of this work would be to study the effects of CP violation by adding coupling to a pseudo-scalar fermionic current with time (or space) dependent mass mixing terms (thus emulating phase transitions in the early Universe). In this case, the CP violation would induce a difference between the particle and antiparticle numbers, which in the massless limit becomes the axial vector current. Studying how this axial current depends on the environmental temperature in the presence of a non-adiabatically changing mass would allow for a better understanding of baryogenesis and leptogenesis sources [30–36].

V. ACKNOWLEDGEMENTS

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Appendix A: Bosonic density operator and entropy

In this appendix we calculate the invariant (phase space) area and entropy for a quantum mechanical system of one bosonic degree of freedom with position operator $\hat{\phi}$ and momentum operator $\hat{\pi}$, based on the approach in Ref. [4]. The Ansatz for the bosonic density operator is

$$\hat{\rho}_B(t) = \frac{1}{Z} \exp \left[ -\frac{1}{2} \left( \alpha \hat{\pi}^2 + \beta \{\hat{\phi}, \hat{\pi}\} + \gamma \hat{\phi}^2 \right) \right], \quad \text{(A1)}$$

where $\{., .\}$ is the anticommutator and $\alpha, \beta, \gamma$ are real time dependent parameters. By defining bosonic creation and annihilation operators

$$\hat{a} = \sqrt{\frac{\sigma}{2\alpha}} \left( 1 + \frac{i\beta}{\sigma} \right) \hat{\phi} + i\frac{\alpha}{\sigma} \hat{\pi}, \quad \hat{a}^\dagger = \sqrt{\frac{\sigma}{2\alpha}} \left( 1 - \frac{i\beta}{\sigma} \right) \hat{\phi} - i\frac{\alpha}{\sigma} \hat{\pi}, \quad \sigma \equiv \sqrt{\alpha \gamma - \beta^2}, \quad \text{(A2)}$$

the density operator can be written in diagonalised form

$$\hat{\rho}_B(t) = \frac{1}{Z'} \exp(-\sigma \hat{N}_B), \quad Z' = \frac{Z}{e^{-\sigma/2}}. \quad \text{(A3)}$$

The bosonic particle number is defined in the usual way $\hat{N}_B = \hat{a}^{\dagger}_B \hat{a}_B$, with a corresponding Fock basis $|n_B\rangle$ defined through $\hat{N}_B|n_B\rangle = n_B|n_B\rangle$. Using this basis to take the trace and demanding that $\text{Tr}[\hat{\rho}_B] = 1$ we find

$$Z' = \text{Tr}[\exp(-\sigma \hat{N}_B)] = \sum_{n_B=0}^\infty \langle n_B|\exp(-\sigma \hat{N}_B)|n_B\rangle = \sum_{n_B=0}^\infty e^{-n_B\sigma} = \frac{1}{1 - e^{-\sigma}}. \quad \text{(A4)}$$

The average particle number is

$$\langle \hat{N}_B \rangle = \text{Tr}[\hat{\rho}_B \hat{N}_B] = \frac{1}{e^\sigma - 1} \equiv \bar{n}_B, \quad \text{(A5)}$$

which indeed agrees with the Bose-Einstein distribution for a thermal state if we identify $\sigma = E/(k_B T)$, where $k_B$ is the Stefan-Boltzmann constant. The Gaussian correlators are obtained from $\hat{\rho}_B$ as

$$\langle \hat{\pi}^2 \rangle = -2 \frac{\partial}{\partial \alpha} \ln Z = \left( \bar{n}_B + \frac{1}{2} \right) \frac{\gamma}{\sigma},$$
$$\langle \hat{\phi}^2 \rangle = \left( \bar{n}_B + \frac{1}{2} \right) \frac{\alpha}{\sigma},$$
$$\frac{1}{2} \langle \{\hat{\phi}, \hat{\pi}\} \rangle = \left( \bar{n}_B + \frac{1}{2} \right) \frac{-\beta}{\sigma}. \quad \text{(A6)}$$

The statistical correlator for bosons is

$$F_\phi(t; t') = \frac{1}{2} \langle \{\hat{\phi}(t), \hat{\phi}(t')\} \rangle, \quad \text{(A7)}$$

which we use to define a Gaussian invariant

$$\Delta_\phi(t) = 4 \left[ \langle \hat{\phi}^2 \rangle \langle \hat{\pi}^2 \rangle - \langle \frac{1}{2} \{\hat{\phi}, \hat{\pi}\} \rangle^2 \right] = 4 \left[ F_\phi(t; t') \partial_t \partial_{t'} F_\phi(t; t') - (\partial_t F_\phi(t; t'))^2 \right]|_{t=t'}. \quad \text{(A8)}$$

$\Delta_\phi/2$ is the phase space area occupied by a Gaussian state in units of $\hbar$ [2]. In a free theory $\Delta_\phi = 1$ and conserved, whereas it increases for interacting theories. Using the correlators (A6) we find

$$\Delta_\phi(t) = 1 + 2\bar{n}_B(t) = \frac{1}{\tanh(\sigma/2)} \quad \text{(A9)}$$

which presents a relation between the invariant phase space area and the Gaussian invariant of the density matrix $\sigma$, and should be compared to the expression for fermions [23]. Finally, the bosonic entropy is

$$S_\phi = -\text{Tr}[\ln \hat{\rho}_B] = \frac{1 + \Delta_\phi}{2} \ln \frac{1 + \Delta_\phi}{2} - \frac{1 - \Delta_\phi}{2} \ln \frac{1 - \Delta_\phi}{2} = (1 + \bar{n}_B) \ln (1 + \bar{n}_B) - \bar{n}_B \ln \bar{n}_B. \quad \text{(A10)}$$
Appendix B: Fermionic shift and diagonalisation

The lagrangian for an interacting fermionic oscillator is given by (see Eqs. (24) and (39)),

\[ L_\psi = \hat{\psi}^\dagger (i\partial_t - \omega(t))\hat{\psi} - \hat{j}_\psi^\dagger \hat{\psi} - \hat{\psi}^\dagger \hat{j}_\psi. \]  

(B1)

This implies the equations of motion,

\[ (i\partial_t - \omega(t))\hat{\psi} = \hat{j}_\psi, \quad (-i\partial_t - \omega(t))\hat{\psi}^\dagger = \hat{j}_\psi^\dagger. \]  

(B2)

One can easily construct the free field solution (in the absence of currents),

\[
\hat{\psi}_0(t) = \exp\left(-i\int_0^t \omega d\tau\right)\hat{\psi}_0(0), \quad \hat{\psi}_0^\dagger(t) = \exp\left(i\int_0^t \omega d\tau\right)\hat{\psi}_0^\dagger(0),
\]  

(B3)

in terms of which we can express the (free) retarded and advanced Green functions (20) as,

\[ iS_0^R(t; t') = -i\theta(t - t')e^{-i\int_0^t \omega d\tau}, \quad iS_0^A(t; t') = i\theta(t - t')e^{i\int_0^t \omega d\tau} \]  

(B4)

With a help of \( iS_0^\alpha \) we can solve the general fermionic operator equations \( \hat{\psi}_0(t) = e^{-i\int_0^t \omega d\tau} \hat{\psi}_0(0) \) and the related \( \hat{\psi}_0^\dagger(t) \) we have,

\[ \hat{N}_0(t) = \hat{\psi}_0^\dagger(t)\hat{\psi}_0(t) = \hat{\psi}_0^\dagger(0)\hat{\psi}_0(0) = \hat{N}_0(0). \]  

(B6)

which implies that in this (source-free) case the density matrix \( \{1\} \) does not evolve in time, such that \( \hat{\alpha} = \text{const.} \), \( \hat{n} = \text{const.} \) and also the entropy \( \{13\} S = \text{const.} \).  

In the case when there is a nonvanishing current source, \( \hat{j}_\psi(t) \neq 0 \), one can think of the full solutions \( \{B5\} \) as a suitably shifted \( \hat{\psi}_0(t) \) (analogous to the bosonic Glauber’s coherent states), and therefore one can write the density operator in terms of the shifted fields

\[ \hat{\psi}(t) + \hat{j}_\psi(t) \equiv \hat{\psi}_0(t). \]  

(B7)

Eq. \( \{B6\} \) then implies that for that density operator the (von Neumann) entropy is conserved, as it should be.

Appendix C: Exact entropy for two coupled fermions

For one environmental oscillator the equations of motion for the statistical correlators Eqs. \( \{48\} \) become

\[
\begin{align*} 
\partial_t F_{xx}(t; t) &= -\lambda \Delta F(t; t) \\
\partial_t \Delta F(t; t) &= (\omega_0 - \omega_1)F_{xx}(t; t) - 2\lambda (F_{xx}(t; t) - F_{qq}(t; t)) \\
\partial_t F_{xx}(t; t) &= (\omega_0 - \omega_1)\Delta F(t; t) \\
\partial_t F_{qq}(t; t) &= \lambda \Delta F(t; t),
\end{align*}
\]  

(C1)

where \( \Delta F(t; t) = F_{xx}(t; t) - F_{qq}(t; t) \) and \( F_{xx}(t; t) + F_{qq}(t; t) = F_{xy}(t; t) + F_{yx}(t; t) \). The statistical correlator can be solved from the first line

\[ F_{xx}(t; t) = F_{xx}(t_0; t_0) - \frac{\lambda}{\lambda^2} \int_{t_0}^t dt' \Delta F(t'; t') = \frac{1}{\lambda} - \frac{\lambda}{\lambda^2} \int_{t_0}^t dt' \Delta F(t'; t'), \]  

(C2)

where we have used the initial conditions \( \{49\} \). After acting with \( \partial_t \) on the second line of \( \{C1\} \) one finds

\[ (\partial_t^2 + (\omega_0 - \omega_1)^2 + 4\lambda^2)\Delta F(t; t) = 0. \]  

(C3)
This equation can be solved with initial conditions for the correlator $\Delta F(t; t)$ from Eqs. (49) and for the first time derivative of the correlator $\Delta F(t; t)$ from the second line of (C1). This gives

$$\Delta F(t; t) = \frac{2\lambda}{\omega} \tilde{n}_E \sin[\tilde{\omega}(t - t_0)],$$  \hspace{1cm} (C4)

where

$$\tilde{\omega} = \sqrt{(\omega_0 - \omega_1)^2 + 4\lambda^2}$$

$$\tilde{n}_E = \frac{e^{3\omega_1} + 1}{2} - \frac{1}{2} \tanh \left[ \frac{1}{2} \beta \omega_1 \right].$$  \hspace{1cm} (C5)

Inserting the solution (C4) in (C2) one obtains the Gaussian invariant from Eq. (45),

$$\Delta_{xx}(t) = 1 - 2\tilde{n}_E \left( \frac{2\lambda}{\omega} \right)^2 \sin^2 \left[ \frac{\tilde{\omega}}{2}(t - t_0) \right].$$  \hspace{1cm} (C6)

Thus, the entropy of the system can be analytically calculated using (44).

**Appendix D: Entropy via the replica trick in coherent state basis**

Here we calculate the Gaussian von Neumann entropy (1) using the density operator in the coherent state basis (33). For convenience we use the exponentiated form of the elements of the density operator (see Eq. (29)),

$$\rho(\tilde{\theta}', \theta; t; \bar{t}) = \langle \bar{\theta}'| \hat{\rho} |\theta\rangle = \frac{1}{Z} \exp(\tilde{\theta}' M \theta),$$  \hspace{1cm} (D1)

where $M = \frac{\tilde{n}}{1 - \tilde{n}} = e^{-a}$ and $Z = \frac{1}{1 - \tilde{n}} = 1 + e^{-a} = 1 + M$. By making use of the replica trick the entropy can be expressed as

$$S = -\text{Tr}[\hat{\rho} \ln(\hat{\rho})] = -\lim_{n \to 0} \frac{\text{Tr}[\hat{\rho}^{n+1} - \hat{\rho}]}{n}.$$  \hspace{1cm} (D2)

The trace is defined in (30). By inserting $n$ unity operators (31) in (D2) and using $\langle \theta'| \theta \rangle = \exp(\theta' \theta)$ and Eq. (D1) one finds

$$\text{Tr}[\hat{\rho}^{n+1}] = \int d\bar{\theta} d\theta \exp(\bar{\theta} \theta) \prod_{i=1}^{n} \left[ \int d\bar{\theta}^{(i)} d\theta^{(i)} \exp(-\bar{\theta}^{(i)} \theta^{(i)}) \right] \rho(\bar{\theta}, \theta^{(1)}; t) \times \rho(\bar{\theta}^{(1)}, \theta^{(2)}; t) \times \ldots \times \rho(\bar{\theta}^{(n)}, \theta; t)$$

$$= Z^{-n-1} \int d\bar{\theta} d\theta \exp(\bar{\theta} \theta) \prod_{i=1}^{n} \left[ \int d\bar{\theta}^{(i)} d\theta^{(i)} \exp(-\bar{\theta}^{(i)} \theta^{(i)}) \right] \exp(\bar{\theta} M \theta^{(1)}) \times \exp(\bar{\theta}^{(1)} M \theta^{(2)}) \times \ldots \times \exp(\bar{\theta}^{(n)} M \theta)$$

$$= Z^{-n-1} (1 + M^n).$$  \hspace{1cm} (D3)

The Grassmann integrations have been performed explicitly in the last step. The resulting entropy (D2) becomes

$$S = -\lim_{n \to 0} \left\{ \frac{1 + M^{n+1}}{(1 + M)^{n+1}} - 1 \right\} = -\frac{M}{1 + M} \ln M + \ln(1 + M) = -(1 - \tilde{n}) \ln(1 - \tilde{n}) - \tilde{n} \ln \tilde{n},$$  \hspace{1cm} (D4)

which is indeed the same as the entropy derived earlier using the Fock basis (13).

The previous derivation can be generalised for $N$ fermionic degrees of freedom. In that case (see Eq. (88))

$$\rho(\tilde{\theta}', \theta; t; \bar{t}) = \langle \bar{\theta}'| \hat{\rho} |\theta\rangle = \frac{1}{Z} \exp(\tilde{\theta}' M_{ij} \theta_j),$$  \hspace{1cm} (D5)

with

$$M_{ij} = (e^{-a})_{ij} = \left( \frac{\tilde{n}}{1 - \tilde{n}} \right)_{ij}$$

$$Z = \text{Det}[\mathbb{I} + e^{-a}] = \text{Det}[\mathbb{I} + M].$$  \hspace{1cm} (D6)
Again the replica trick \( \{D2\} \) is used to obtain the entropy. The trace is of course taken over all \( N \) fermionic degrees of freedom, which are also included in the unit operation. After inserting the unity operators in \( \{D2\} \) one finds
\[
\text{Tr}[\rho^{n+1}] = \prod_{a=1}^{N} \left[ \int d\theta_a d\bar{\theta}_a \exp(\bar{\theta}_a \theta_a) \right] \times \prod_{b=1}^{N} \left[ \int d\theta_b d\bar{\theta}_b \exp(-\bar{\theta}_b \theta_b) \right] \times \ldots \times \prod_{c=1}^{N} \left[ \int d\theta_c d\bar{\theta}_c \exp(-\bar{\theta}_c \theta_c) \right]
\]
\[
\times \rho(\bar{\theta}, \theta^{(1)}; t) \times \rho(\bar{\theta}^{(1)}, \theta^{(2)}; t) \times \ldots \times \rho(\bar{\theta}^{(n)}, \theta; t)
\]
\[
= Z^{-n-1} \prod_{a=1}^{N} \left[ \int d\theta_a d\bar{\theta}_a \exp(\bar{\theta}_a \theta_a) \right] \times \prod_{b=1}^{N} \left[ \int d\theta_b d\bar{\theta}_b \exp(-\bar{\theta}_b \theta_b) \right] \times \ldots \times \prod_{c=1}^{N} \left[ \int d\theta_c d\bar{\theta}_c \exp(-\bar{\theta}_c \theta_c) \right]
\]
\[
\times \exp \left( \sum_{i,j} \bar{\theta}_i M_{ij} \theta_j^{(1)} \right) \times \exp \left( \sum_{k,l} \bar{\theta}_k M_{kl} \theta_l^{(2)} \right) \times \ldots \times \exp \left( \sum_{r,s} \bar{\theta}_r^{(n)} M_{rs} \theta_s \right) .
\]
(D7)

To avoid confusion the summations have been written out explicitly. In order to perform the Grassmann integrations the following identities prove to be useful,
\[
\text{Det}[1 + M] = \prod_{a=1}^{N} \left[ \int d\theta_a d\bar{\theta}_a \exp(\bar{\theta}_a \theta_a) \right] \exp(\sum_{i,j} \bar{\theta}_i M_{ij} \theta_j)
\]
\[
\exp(\sum_{i,j} \bar{\theta}_i (M^2)_{ij} \theta_j) = \prod_{b=1}^{N} \left[ \int d\theta_b d\bar{\theta}_b \exp(-\bar{\theta}_b \theta_b) \right] \exp(\sum_{i,k} \bar{\theta}_i M_{ik} \theta_k^{(1)} \right) \exp(\sum_{i,j} \bar{\theta}_i^{(1)} M_{ij} \theta_j).
\]
(D8)
(D9)

Applying these to Eq. \( \{D7\} \) we find
\[
\text{Tr}[\rho^{n+1}] = \text{Det}[I + M^{n+1}],
\]
(D10)
The entropy \( \{D2\} \) becomes
\[
S = -\lim_{n \to 0} \left( \frac{\text{Det}[I + M^{n+1}]}{(\text{Det}[I + M])^{n+1}} - 1 \right) = \text{Tr} \left[ -\frac{M}{I + M} \ln M + \ln(I + M) \right] = \text{Tr} [-((1 - \bar{n}) \ln(1 - \bar{n}) - \bar{n} \ln \bar{n}] .
\]
(D11)

which is indeed the entropy derived in Eq. \( \{D4\} \). Thus, by using the density operator in the coherent state basis in combination with the replica trick, no diagonalisation of the density operator is required in order to find the entropy.

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