Automorphisms of certain affine complements in projective space

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Abstract. We prove that every biregular automorphism of the affine algebraic variety \( \mathbb{P}^M \setminus S \), \( M \geq 3 \), where \( S \subset \mathbb{P}^M \) is a hypersurface of degree \( m \geq M + 1 \) with a unique singular point of multiplicity \( (m - 1) \), resolved by one blow up, is a restriction of some automorphism of the projective space \( \mathbb{P}^M \) preserving the hypersurface \( S \); in particular, for a general hypersurface \( S \) the group \( \text{Aut}(\mathbb{P}^M \setminus S) \) is trivial.

Bibliography: 24 titles.

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§ 1. Introduction

1.1. Statement of the main result. Let \( \mathbb{P} = \mathbb{P}^M \) be the complex projective space of dimension \( M \geq 3 \) and \( S \subset \mathbb{P} \) be a hypersurface of degree \( m \geq M + 1 \) with a unique singular point \( o \in S \) of multiplicity \( m - 1 \) that can be resolved by one blow up. More precisely, let \( \sigma: \mathbb{P}^+ \rightarrow \mathbb{P} \) be the blow up of the point \( o \) with the exceptional divisor \( E = \sigma^{-1}(o) \cong \mathbb{P}^{M-1} \). We assume that the strict transform \( S^+ \subset \mathbb{P}^+ \) is a nonsingular hypersurface and the projectivised tangent cone \( S^+ \cap E \) is a nonsingular hypersurface of degree \( m - 1 \) in \( E \cong \mathbb{P}^{M-1} \). The main result of the present paper is the following claim.

Theorem 1. Every automorphism \( \chi \) of the affine algebraic set \( \mathbb{P} \setminus S \) is the restriction of some projective automorphism \( \chi_P \in \text{Aut} \mathbb{P} \) preserving the hypersurface \( S \). In particular, for a Zariski general hypersurface \( S \) the group \( \text{Aut}(\mathbb{P} \setminus S) \) is finite and trivial.

Due to certain well-known facts about automorphisms of projective hypersurfaces (see, for instance, [1]) Theorem 1 is easily implied (see §5) by a somewhat more general fact. Let \( S' \subset \mathbb{P} \) be another hypersurface of degree \( m \) with a unique singular point \( o' \in S' \) of multiplicity \( m - 1 \) that is resolved by one blow up (in the sense specified above). Then the following claim is true.
Theorem 2. Every isomorphism of affine algebraic varieties
\[ \chi : \mathbb{P} \setminus S \to \mathbb{P} \setminus S' \]
is the restriction of some projective automorphism \( \chi_{\mathbb{P}} \in \text{Aut} \mathbb{P} \) transforming the hypersurface \( S \) into \( S' \).

Obviously, \( \chi_{\mathbb{P}}(o) = o' \). It is Theorem 2 that we prove below.

If \( z_1, \ldots, z_M \) is a system of affine coordinates on \( \mathbb{P} \) with the origin at the point \( o \), then the hypersurface \( S \) is defined by
\[ f(z_*) = q_{m-1}(z_*) + q_m(z_*) = 0, \] (1.1)
where \( q_i(z_*) \) are homogeneous polynomials of degree \( i \) in the coordinates \( z_* \). An irreducible hypersurface of that type is rational and it is this property that makes the problem of describing the group of automorphisms \( \text{Aut}(\mathbb{P} \setminus S) \) meaningful; see the discussion in §2 below.

1.2. Structure of the paper. The paper is organized in the following way. In §2 we discuss the general problem of describing the automorphisms of affine complements and what little is known in that direction (in nontrivial cases), and also some well-known conjectures and incomplete projects. In §3 we start the proof of Theorem 2: for an arbitrary isomorphism of affine varieties
\[ \chi : \mathbb{P} \setminus S \to \mathbb{P} \setminus S' \]
we define the key numerical characteristics (such as the ‘degree’) and obtain the standard relations between them (for instance, an analogue of the ‘Noether-Fano inequality’ for the affine case). In §4 we construct the resolution of the maximal singularity of the map \( \chi \), which is now considered a birational map (Cremona transformation) \( \chi_{\mathbb{P}} : \mathbb{P} \dashrightarrow \mathbb{P} \), the restriction of which onto the affine complement \( \mathbb{P} \setminus S \) is an isomorphism onto \( \mathbb{P} \setminus S' \). Finally, in §5 we exclude the maximal singularity, which completes the proof of Theorem 2.

§2. Automorphisms of affine complements

Let \( X \) be a nonsingular projective rationally connected variety and \( Y \) and \( Y' \) be irreducible ample divisors, so that their complements \( X \setminus Y \) and \( X \setminus Y' \) are affine varieties. Two natural questions can be asked:

1) are the affine varieties \( X \setminus Y \) and \( X \setminus Y' \) isomorphic?
2) if \( Y' = Y \), what is the group of biregular automorphisms \( \text{Aut}(X \setminus Y) \)?

It is natural to consider a biregular isomorphism \( \chi : X \setminus Y \to X \setminus Y' \) (if it exists) as a birational automorphism \( \chi_X \in \text{Bir} X \) which is regular on the affine open set \( X \setminus Y \) and maps it isomorphically onto \( X \setminus Y' \). The case when \( \chi_X \in \text{Aut} X \) is a biregular automorphism of the variety \( X \) and the corresponding isomorphism \( \chi \) of affine complements itself will be said to be trivial. We therefore consider the following problem: are there any nontrivial isomorphisms \( \chi : X \setminus Y \to X \setminus Y' \), when
\[ \chi_X \in \text{Bir} X \setminus \text{Aut} X, \]
and, correspondingly, are the groups
\[ \text{Aut}(X \setminus Y) \quad \text{and} \quad \text{Aut}(X)_Y \]
the same (the second symbol means the stabilizer of the divisor \( Y \) in the group \( \text{Aut}(X) \))? In particular, Theorem 1 claims that
\[ \text{Aut}(\mathbb{P} \setminus S) = \text{Aut}(\mathbb{P})_S \]
for hypersurfaces \( S \subset \mathbb{P} \), described in §1.

**Proposition 1.** Let \( \chi \) be a nontrivial isomorphism of affine complements \( X \setminus Y \) and \( X \setminus Y' \). Then \( Y \) and \( Y' \) are birationally ruled varieties, that is to say, for some irreducible varieties \( Z \) and \( Z' \) of dimension \( \dim X - 2 \) the varieties \( Y \) and \( Y' \) are birational to the direct products \( Z \times \mathbb{P}^1 \) and \( Z' \times \mathbb{P}^1 \), respectively.

**Proof.** The birational map \( \chi^{-1}_X \) is regular at the generic point of the divisor \( Y' \), and its image cannot be the generic point of the divisor \( Y \): in such a case \( \chi_X \) would be an isomorphism in codimension 1 and for that reason a biregular automorphism, contrary to our assumption. Therefore, \( (\chi^{-1}_X)_* Y' \subset X \) is an irreducible subvariety of codimension at least 2 (which is, of course, contained in \( Y' \)). Now consider a resolution of singularities \( \varphi: \widetilde{X} \to X \) of the map \( \chi_X \). By what we said above, there is an exceptional divisor \( E \subset \widetilde{X} \) of this resolution such that
\[ (\chi \circ \varphi)|_E: E \to Y' \]
is a birational map. Therefore, \( Y' \) is a birationally ruled variety. For \( Y \) we argue in a symmetric way, which completes the proof.

**Remark 1.** Assume in addition that \( Y' \) is a rationally connected variety. Then in the notation of the proof of Proposition 1 we conclude that the centre of the exceptional divisor \( E \) on \( X \), that is, the irreducible subvariety \( \varphi(E) \), is a rationally connected variety. This is also true for the centres of the divisor \( E \) on the ‘lower’ storeys of the resolution of \( \varphi \).

**Example 1.** There are no nontrivial isomorphisms of affine complements \( \mathbb{P} \setminus Y \) and \( \mathbb{P} \setminus Y' \) if \( Y \subset \mathbb{P} \) is a nonsingular hypersurface of degree at least \( M + 1 \). Indeed, the hypersurface \( Y \) is not a birationally ruled variety.

Assume now that \( Y \subset X \) is a Fano variety. It is well known (see, for instance, [2], Ch. 2), that birationally rigid Fano varieties are not birationally ruled. Therefore, if \( Y \) is a birationally rigid variety, then every isomorphism of affine complements \( X \setminus Y \) and \( X \setminus Y' \) (where \( Y' \subset X \) is an irreducible ample divisor) is trivial, that is, it extends to an automorphism of the variety \( X \). In particular, the group \( \text{Aut}(X \setminus Y) \) is \( \text{Aut}(X)_Y \). This makes it possible to construct numerous examples of affine complements with no nontrivial isomorphisms and automorphisms. Below we give some of them.

**Example 2.** Let \( V \subset \mathbb{P}^4 \) be a smooth three-dimensional quartic. Because of its birational superrigidity (see [3]) the affine complement \( \mathbb{P}^4 \setminus V \) has no nontrivial automorphisms and isomorphisms. The same is true for quartics with at most isolated double points, provided that the variety \( V \) is factorial and its singularities are terminal; see [4]–[6].
Example 3. Let $V \subset \mathbb{P}^M$ be a general smooth hypersurface of degree $M$, where $M \geq 5$. Because of its birational superrigidity (see [7]) the affine complement $\mathbb{P} \setminus V$ has only trivial automorphisms and isomorphisms. The same is true if we allow $V$ to have quadratic singularities of rank at least 5 (see [8]). This example generalizes naturally for Fano complete intersections. Let $k \geq 2$,

$$Y = F_1 \cap \cdots \cap F_k \subset \mathbb{P}^{M+k}$$

be a nonsingular complete intersection of codimension $k$, where $F_i$ is a hypersurface of degree $d_i$ and let

$$d_1 + \cdots + d_k = M + k,$$

$M \geq 4$, that is, $Y$ is a nonsingular $M$-dimensional Fano variety of index 1. For $i \in \{1, \ldots, k\}$ set

$$X_i = \bigcap_{j \neq i} F_j$$

and assume that $X_i$ is also nonsingular. Then $X_i \subset \mathbb{P}^{M+k}$ is an $(M+1)$-dimensional Fano variety of index $d_i + 1$, which contains $Y$ as a very ample divisor, so that the complement $X_i \setminus Y$ is an affine variety. If the set of integers $(d_1, \ldots, d_k)$ satisfies the conditions in any of the papers [9]–[11], and the variety $Y$ is sufficiently general in its family, then due to its birational superrigidity

$$\text{Aut}(X_i \setminus Y) = \text{Aut}(X_i)_Y$$

(and a similar claim holds for automorphisms). Of course, these arguments are nontrivial only for those cases when $\text{Aut}X_i \neq \text{Bir}X_i$: for instance, for $k = 2$ and $(d_1, d_2) = (2, M)$ the variety $X_2$ is a $(M + 1)$-dimensional quadric and its group of birational automorphisms is the Cremona group of rank $M + 1$. We get another nontrivial example for $k = 2$ and $(d_1, d_2) = (3, M - 1)$, where $X_2$ is an $(M + 1)$-dimensional cubic hypersurface which has a huge group of birational automorphisms. Using other families of birationally superrigid or rigid Fano varieties, one can construct more nontrivial examples of affine complements all of whose automorphisms are trivial.

Example 4. In [12] it was shown that a very general hypersurface $V_d \subset \mathbb{P}$ for $d \geq \frac{2}{3}M$ is not birationally ruled. Therefore, for such hypersurfaces their affine complements $\mathbb{P} \setminus V_d$ have no nontrivial isomorphisms or automorphisms.

Example 5. In [13] it was shown that a Zariski general hypersurface $V_{M-1} \subset \mathbb{P}$ for $M \geq 16$ has no other structures of a rationally connected fibre space apart from pencils of hyperplane sections. In particular, it has no structures of a conic bundle and for that reason is not birationally ruled. It follows that for those hypersurfaces the affine complements $\mathbb{P} \setminus V_{M-1}$ have no nontrivial isomorphisms or automorphisms.

Unfortunately, if the variety $Y$ is birationally ruled, then the problem of describing the isomorphisms of the affine complement $X \setminus Y$ becomes very hard (except in trivial cases, when, for instance, the variety $X$ itself satisfies $\text{Bir}X = \text{Aut}X$). The only complete result here is Theorem 2 in the present paper. As for the main objects of study today, they are particular classes of three-dimensional affine complements, such as the complement $\mathbb{P}^3 \setminus S$ to a cubic surface (nonsingular or with
prescribed singularities) or the affine space \( \mathbb{A}^3 \) and certain similar affine varieties. In respect of complements to cubic surfaces there is a classical conjecture, stated by Gizatullin in [14], p.6: if a cubic surface \( S \) is nonsingular, then its complement \( \mathbb{P}^3 \setminus S \) has no nontrivial automorphisms. However, if the cubic surface has a double point, then nontrivial automorphisms do exist — they were discovered by Lamy and Blanc (as far as the author knows, those examples were not published). A similar conjecture was stated by Dubouloz in the case when \( X \) is a Fano double cover of index 2, branched over a surface \( W \subset \mathbb{P}^3 \) of degree 4 and \( S \) is the inverse image of a plane in \( \mathbb{P}^3 \).

In respect of the groups of automorphisms of affine varieties a huge amount of material has been accumulated; there are a lot of results about special groups of automorphisms, dynamical properties of particular automorphisms etc. We only point out the three recent papers [15]–[17]; see also the bibliographies in those papers.

The groups of automorphisms of affine algebraic surfaces are much better understood: here we have such fundamental results as the complete description of the groups of automorphisms of the plane \( \text{Aut} \mathbb{A}^2 \); see [18] and [19]. This direction is still being actively explored [20]–[23].

§ 3. Start of the proof of Theorem 2

Let

\[
\chi : \mathbb{P} \setminus S \to \mathbb{P} \setminus S'
\]

be an isomorphism of affine varieties. Assume that \( \chi \) is nontrivial, that is, the corresponding birational map \( \chi_\mathbb{P} : \mathbb{P} \dasharrow \mathbb{P} \) is not a biregular isomorphism. Let

\[
\varphi : \widetilde{\mathbb{P}} \to \mathbb{P}
\]

be its resolution (a sequence of blow ups with nonsingular centres), so that \( \psi = \chi_\mathbb{P} \circ \varphi : \widetilde{\mathbb{P}} \to \mathbb{P} \) is a regular map. Furthermore, set \( \mathcal{E}_\varphi \) to be the set of prime \( \varphi \)-exceptional divisors. By assumption, for the strict transform of \( S' \) we have

\[
T = (\chi_\mathbb{P} \circ \varphi)^{-1}_{\ast} S' \in \mathcal{E}_\varphi.
\]

Set \( B = \varphi(T) \) to be the centre of the exceptional divisor \( T \) on \( \mathbb{P} \), which is an irreducible subvariety of codimension at least 2, and moreover \( B \subset S \). Therefore, we get the positive integers \( a = a(T, \mathbb{P}) \) (the discrepancy of the divisor \( T \) with respect to \( \mathbb{P} \)) and

\[
b = \text{ord}_T S = \text{ord}_T \varphi^* S.
\]

Furthermore, let \( \Sigma \) be the strict transform of the linear system of hyperplanes with respect to \( \chi_\mathbb{P} \). This is a mobile linear system \( \Sigma \subset |nH| \), where \( H \) is a hyperplane in \( \mathbb{P} \), and \( n \geq 2 \). Set

\[
\nu = \text{ord}_T \varphi^* \Sigma.
\]

Proposition 2. The following equalities are true:

\[
bn = \nu m + 1 \quad \text{and} \quad (M + 1)b = am + (M + 1).
\] (3.1)
Proof. We write down $\mathcal{E} = \mathcal{E}_\varphi \setminus \{T\}$, so that $\mathcal{E}_\varphi = \mathcal{E} \coprod \{T\}$. Let $D \in \Sigma$ be a general divisor, $\tilde{D} \in \tilde{\Sigma}$ be its strict transform on $\tilde{\mathbb{P}}$, where $\tilde{\Sigma}$ is the strict transform of the linear system $\Sigma$ on $\tilde{\mathbb{P}}$ with respect to $\varphi$. Let $\tilde{S} \subset \tilde{\mathbb{P}}$ be the strict transform of the hypersurface $S$. By the symbol $\tilde{K}$ we denote the canonical class of the variety $\tilde{\mathbb{P}}$.

We obtain the following presentations:

$$
\tilde{D} \sim nH - \nu T - \sum_{E \in \mathcal{E}} \nu_E E,
$$

$$
\tilde{K} = -(M + 1)H + aT + \sum_{E \in \mathcal{E}} a_E E,
$$

$$
\tilde{S} \sim mH - bT - \sum_{E \in \mathcal{E}} b_E E,
$$

where the coefficients $\nu_E$, $a_E$ and $b_E$ have the obvious meaning (in order to simplify the formulae we write $H$ instead of $\varphi^* H$). Consider the family of lines $\mathcal{L}$ on $\mathbb{P}$. Obviously, a general line $L \in \mathcal{L}$ does not meet the set

$$
\bigcup_{E \in \mathcal{E}} \psi(E) \cup \psi(\tilde{S}),
$$

since the latter has codimension at least 2 (recall that $\tilde{S} \subset \tilde{\mathbb{P}}$ is a $\psi$-exceptional divisor). Therefore, the strict transform $\tilde{L} \subset \tilde{\mathbb{P}}$ satisfies

$$
(\tilde{L} \cdot \tilde{D}) = 1, \quad (\tilde{L} \cdot \tilde{K}) = -(M + 1) \quad \text{and} \quad (\tilde{L} \cdot \tilde{S}) = 0. \quad (3.2)
$$

Moreover, $(\tilde{L} \cdot T) = (L \cdot S') = m$. Set

$$
d = (\tilde{L} \cdot H).
$$

Obviously, $d$ is the degree of the curve $\varphi(\tilde{L}) \subset \tilde{\mathbb{P}}$ in the usual sense. Finally, $(\tilde{L} \cdot E) = 0$ for every exceptional divisor $E \in \mathcal{E}$. Therefore, (3.2) implies the relations

$$
dn - \nu m = 1, \quad -(M + 1)d + am = -(M + 1) \quad \text{and} \quad dm - bm = 0.
$$

The last equality implies that $d = b$. Now equalities (3.1) follow in a straightforward way and the proof is complete.

Remark 2. Relations (3.1) imply the equality

$$
\nu = \frac{a}{M + 1} n + \frac{n - 1}{m}.
$$

Since $n \geq 2$, we obtain

$$
\nu > \frac{a}{M + 1} n.
$$

This is the usual Noether-Fano inequality for the birational map $\chi_\mathbb{P}$. Therefore, the prime divisor $T$ (the strict transform of the hypersurface $S'$ on $\tilde{\mathbb{P}}$) is a maximal singularity of the linear system $\Sigma$ (see, for instance, Definition 1.4 in [2], Ch. 2).
Although relations (3.1) are sufficient for the proof of Theorem 2, we will show similar relations for every infinitely near divisor $E \in \mathcal{E}$. Recall that we have defined the integers $a_E = a(E, \mathbb{P})$ and $b_E = \text{ord}_E \varphi^* S,$ where the discrepancy is understood with respect to the birational morphism $\varphi$. Let $a'_E$ be the discrepancy of the divisor $E$ with respect to the birational morphism $\psi$ and $b'_E = \text{ord}_E \psi^* S'$, so that we get the equality

$$e_K = \psi^* K_P + a'_E e_S + \sum_{E \in \mathcal{E}} a_E E$$

(3.3)

and the presentation

$$e_{S'} = T \sim \psi^* (mH) - b'E - \sum_{E \in \mathcal{E}} b'E,$$

where $a' > 0$ and $b' > 0$ have the same sense in respect of the image of the map $\chi$ as $a$ and $b$ have in respect of the original projective space $\mathbb{P}$.

**Proposition 3.** For every divisor $E \in \mathcal{E}$ the following equalities hold:

$$ (M + 1)b_E + ma'_E = (M + 1)b'_E + ma_E $$

(3.4)

and

$$ b_E n = mn + b'_E. $$

(3.5)

**Proof.** Consider a mobile family of curves $\mathcal{C}$ on $\mathbb{P}$ with the following properties:

1) every $C \in \mathcal{C}$ is an irreducible rational curve of degree $l \geq 2$;

2) the strict transform $\widetilde{C}$ of a general curve $C \in \mathcal{C}$ on $\widetilde{\mathbb{P}}$ with respect to the birational morphism $\psi$ meets $E$ transversally at a unique point $p_C$ of general position on $E$ and does not meet other $\psi$-exceptional divisors; in particular $\widetilde{C} \cap \widetilde{S} = \emptyset$;

3) the curves in the family $\mathcal{C}$ sweep out a Zariski open subset of the space $\mathbb{P}$.

Such a family of rational curves is easy to construct using the methods of elementary algebraic geometry; see [2], Ch. 2, §3. Let $p \in E$ be a point of general position, $q = \psi(p) \in \mathbb{P}$ be its image on $\mathbb{P}$ and $(v_1, \ldots, v_M)$ be a system of affine coordinates on $\mathbb{P}$ with the origin at that point. We construct the curve $C$ in the parametric form:

$$ v_1 = \alpha_{1,1} t + \cdots + \alpha_{1,l} t^l, $$

$$ \ldots \ldots \ldots \ldots \ldots \ldots $$

$$ v_M = \alpha_{M,1} t + \cdots + \alpha_{M,l} t^l, $$

where $l$ is sufficiently large. In [24] (see also [2], Ch. 2, Theorem 3.1) it was shown that there is a set of coefficients $\alpha_{i,j}$, $i = 1, \ldots, M$, $j = 1, \ldots, a'_E$ (in fact, instead of $a'_E$ we can take an essentially smaller number, but we do not need that), such that for any coefficients $\alpha_{i,j}$ with $j \geq a'_E + 1$ the strict transform of such a curve meets $E$ transversally at the point $p$ when $t = 0$. Varying the coefficients $\alpha_{i,j}$
for \( j \geq a'_E + 1 \), one can ensure that the curve \( C \) goes through \( q \) only when \( t = 0 \) and intersects the closed subset of codimension \( \geq 2 \)

\[
\bigcup_{E \in \mathcal{E}} \psi(E) \cup \psi(\tilde{S})
\]

only at \( q \). Such curves have properties 1)–3).

Now we argue in exactly the same way as in the proof of Proposition 2. We have the equality

\[
(C \cdot \tilde{D}) = l = dn - \nu(lm - b'_E) - \nu_E,
\]

where \( d = \deg \varphi(C) \). Multiplying \( C \) by the canonical class \( \tilde{K} \) and using the presentation (3.3) we obtain

\[
-(M + 1)d + a(lm - b'_E) + a_E = -(M + 1)l + a'_E.
\]

Finally, multiplying the curve \( \tilde{C} \) by \( \tilde{S} \) we get

\[
dm - b(lm - b'_E) - b_E = 0
\]

(the expression \( lm - b'_E \) in brackets is the ‘residual intersection’ \( (\tilde{C} \cdot T) \)). From here, using (3.1), we get (3.4) and (3.5) by means of easy computations. This completes the proof of Proposition 3.

§4. The resolution of the maximal singularity

Let

\[
\varphi_{i,i-1} : X_i \to X_{i-1},
\]

\( i = 1, \ldots, N \), be the resolution of the maximal singularity \( T \) of the linear system \( \Sigma \) in the sense of [2], Ch. 2, that is, \( X_0 = \mathbb{P} \), and each map \( \varphi_{i,i-1} \) is the blow up of the (possibly singular) irreducible subvariety \( B_{i-1} \subset X_{i-1} \), which is the centre of the exceptional divisor \( T \) on \( X_{i-1} \). Set \( E_i = \varphi_{i,i-1}^{-1}(B_{i-1}) \). The strict transform of the subvariety \( R \subset X_i \) on a higher storey of the resolution \( X_j \), where \( j > i \), is denoted by adding the upper index \( j \): we write \( R^j \).

For \( j > i \) we set

\[
\varphi_{j,i} = \varphi_{i+1,i} \circ \cdots \circ \varphi_{j,j-1} : X_j \to X_i.
\]

The exceptional divisor \( E_N \subset X_N \) of the last blow up realizes the maximal singularity \( T \): the birational map

\[
\varphi^{-1} \circ \varphi_{N,0} : X_N \dashrightarrow \mathbb{P}^\circ
\]

is regular at the general point of the divisor \( E_N \) and maps it onto \( T \).

On the set \( \{1, \ldots, N\} \) there is a natural structure of an oriented graph: \( i \to j \), if and only if \( i > j \) and the inclusion

\[
B_{i-1} \subset E_j^{i-1}
\]

holds. If the vertices \( i \) and \( j \) are not joined by an oriented edge, we write \( i \not\to j \).
For \( i \neq j \) we denote the number of paths in that graph from the vertex \( i \) to the vertex \( j \) by the symbol \( p_{ij} \) (so that \( p_{ij} = 0 \) for \( i < j \) and \( p_{ij} \geq 1 \) for \( i > j \)). For convenience we set \( p_{ii} = 1 \) for \( i = 1, \ldots, N \). Finally, to simplify our notations, we write \( p_i \) instead of \( p_{Ni} \). Let 

\[
\delta_i = \text{codim } B_{i-1} - 1
\]

be the elementary discrepancies. Then the following equality holds:

\[
a = \sum_{i=1}^{N} p_i \delta_i.
\]

Let us also introduce the elementary multiplicities

\[
\nu_i = \text{mult}_{B_{i-1}} \Sigma^{i-1},
\]

\( i = 1, \ldots, N \) (where, in accordance with the general principle of notations, \( \Sigma^{i-1} \) means the strict transform of the mobile linear system \( \Sigma \) on \( X_{i-1} \)) and

\[
\mu_i = \text{mult}_{B_{i-1}} S^{i-1},
\]

\( i = 1, \ldots, N \). Obviously,

\[
\nu = \sum_{i=1}^{N} p_i \nu_i \quad \text{and} \quad b = \sum_{i=1}^{N} p_i \mu_i.
\]

Note that for some \( k \leq N \) the strict transform \( S^{k-1} \) contains \( B_{k-1} \), but \( S^k \) no longer contains \( B_k \), so that \( \mu_{k+1} = \cdots = \mu_N = 0 \), and for that reason

\[
b = \sum_{i=1}^{k} p_i \mu_i.
\]

If \( B_0 \neq o \) is not the unique singular point of the hypersurface \( S \), then it is obvious

\[
\mu_1 = \cdots = \mu_k = 1,
\]

and so \( b = p_1 + \cdots + p_k \). If \( B_0 = o \), then \( \mu_1 = m - 1 \) and by the assumption about the singularities of the divisor \( S \) the strict transform \( S^1 \) is smooth, so that \( \mu_i = 1 \) for \( 2 \leq i \leq k \). Therefore, if \( B_0 = o \), then

\[
b = (m - 1)p_1 + p_2 + \cdots + p_k.
\]

Finally, we point out one property of the numbers \( p_i \). Since \( \varphi_{i,i-1}(B_i) = B_{i-1} \) by construction (\( B_i \) is the centre of the exceptional divisor \( T \) on \( X_i \), and \( B_{i-1} \) is its centre on \( X_{i-1} \)), the dimensions \( \text{dim } B_i \) do not decrease as \( i \) is growing. Accordingly, the codimensions \( \text{codim } B_i \) do not increase as \( i \) is growing, so that \( \delta_1 \geq \delta_2 \geq \cdots \geq \delta_N \). Assume that for some \( k_1 < k \) the centres of the blow ups \( B_{i-1}, i = k_1 + 1, \ldots, k \), have the maximum dimension \( M - 2 \).
Proposition 4. Under the assumptions above for $k - k_1 \geq 3$ the subgraph with the vertices $k_1 + 1, \ldots, k$ is a chain:

$$k_1 + 1 \leftarrow k_1 + 2 \leftarrow \cdots \leftarrow k,$$

that is, between the vertices of the subgraph there are no other arrows apart from the consecutive ones $i \leftarrow i + 1$. Moreover,

$$p_{k_1+1} = \cdots = p_{k-1}.$$

**Proof.** By the definition of $k$, for $i \leq k$ we have $B_{i-1} \subset S^{i-1}$, where the divisor $S^{i-1}$ is nonsingular at the general point $B_{i-1}$ for $i \geq k_1 + 1$. Therefore, for $k_1 + 1 \leq i \leq k - 2$ we have

$$B_i = E_i \cap S^i \quad \text{and} \quad B_{i+1} = E_{i+1} \cap S^{i+1}$$

(since $B_i$ is contained in both $E_i$ and $S^i$ and has codimension 2, and the same is true for $B_{i+1}$), and $E_i$ and $S^i$ ($E_{i+1}$ and $S^{i+1}$) meet transversally at the general point of $B_i$ (of $B_{i+1}$, respectively), so that $E_{i+1}^i$ and $S^{i+1}$ do not meet over a point of general position in $B_i$. Therefore, $B_{i+1} \not\subset E_{i+1}^i$ and the first claim of the proposition is shown.

In particular, $k \rightarrow k-2$. But then for any vertex $e \geq k+1$ we also have $e \rightarrow k-2$, so that every path from the vertex $N$ to the vertex $i \leq k - 2$ must go through the vertex $k - 1$. This proves the second claim of Proposition 4.

Now we are ready to complete the proof of Theorem 2.

§ 5. Exclusion of the maximal singularity

We write down the second equality in (3.1) in terms of elementary multiplicities and discrepancies:

$$(M + 1) \sum_{i=1}^{k} p_i \mu_i = m \sum_{i=1}^{N} p_i \delta_i + (M + 1). \quad (5.1)$$

We conclude immediately that $B_0 = o$ is the singular point of the hypersurface $S$. Otherwise all multiplicities $\mu_i = 1$, so that from (5.1) we would have obtained

$$0 = \sum_{i=1}^{k} (m\delta_i - M - 1)p_i + m \sum_{i=k+1}^{N} p_i \delta_i + (M + 1),$$

which is impossible since $m \geq M + 1$ and $\delta_i \geq 1$, so that all the three components in the right-hand side of the last formula are nonnegative and at least one of them is positive.

Thus $B_0 = o$. Here $\mu_1 = m - 1$ and $\delta_1 = M - 1$, so we get the equality

$$(2m - M - 1)p_1 = \sum_{i=2}^{k} (m\delta_i - M - 1)p_i + m \sum_{i=k+1}^{N} p_i \delta_i + (M + 1), \quad (5.2)$$
all components in which both in the right- and left-hand sides are nonnegative. By Remark 1, all centres $B_i$ of the blow ups $\varphi_{i+1,i}$ are rationally connected varieties. In particular, $B_1 \neq S^1 \cap E_1$ since $S^1 \cap E_1 \subset E_1 \cong \mathbb{P}^{M-1}$ is a nonsingular hypersurface of degree $m - 1 \geq M$, which is not rationally connected. Thus if $k \geq 2$, then $B_1$ is a subvariety of codimension at least 3 in $X_1$, so that $\delta_2 \geq 2$ and the coefficient at $p_2$ is no smaller than $2m - M - 1$. If also $3 \rightarrow 1$, then $p_1 = p_2$ (every path from the vertex $N$ to the vertex 1 must go through the vertex 2) and we obtain a contradiction: in (5.2) the right-hand side is strictly higher than the left-hand side. If $N = 2$, then $p_1 = p_2 = 1$ and we obtain a contradiction again: in this case (5.2) takes the form

$$2m - M - 1 = \delta_2 m - M - 1 + (M + 1)$$

with $\delta_2 \geq 2$, which is also impossible. We conclude that for $k \geq 2$, with necessity $N \geq 3$ and $3 \rightarrow 1$.

**Proposition 5.** The case $k = 1$ is impossible.

**Proof.** Assume the converse: $k = 1$. Then $b = (m - 1)p_1$. Let $Q \subset \mathbb{P}$ be a general hypersurface of degree $m$ with the point $o$ as a singular point of multiplicity $m - 1$. Since it is general, $B_1 \not\subset Q^1$, so that

$$\text{ord}_T \varphi^* Q = \text{ord}_{E_N} \varphi^*_0 Q = (m - 1)p_1 = b,$n

and it follows that for the strict transform $\widetilde{Q} \subset \widetilde{\mathbb{P}}$ we get the presentation

$$\widetilde{Q} \sim mH - bT - \sum_{E \in \mathcal{E}} q_E E.$$

(5.3)

Therefore, $(\widetilde{L} \cdot \widetilde{Q}) = 0$, where $\widetilde{L}$ is the strict $\psi$-transform of a general line $L \subset \mathbb{P}$ (see the proof of Proposition 2). But the curves $\widetilde{L}$ sweep out a Zariski open subset of $\mathbb{P}$, and the hypersurfaces $Q$ sweep out $\mathbb{P}$. This contradiction proves Proposition 5.

Set $l = \max\{i \mid i \rightarrow 1, 1 \leq i \leq N\}$.

**Proposition 6.** The case $l \leq k$ is impossible.

**Proof.** Assume the converse: $l \leq k$. We could see above that $N \geq 3$ and $3 \rightarrow 1$, so that $l \geq 3$. For any $i \leq l$, $i \geq 2$, we have

$$B_{i-1} \subset S^{i-1} \cap E_1^{i-1} \cap E_{i-1},$$

so that $\text{codim} B_{i-1} \geq 3$ and $\delta_i \geq 2$. We re-write the right-hand side of (5.2) as

$$\sum_{i=2}^{l} (m\delta_i - M - 1)p_i + \sum_{i=l+1}^{k} (m\delta_i - M - 1)p_i + m \sum_{i=k+1}^{N} p_i\delta_i + (M + 1).$$

The first component in this sum is no smaller than

$$(2m - M - 1)(p_2 + \cdots + p_l) = (2m - M - 1)p_1.$$

For that reason (5.2) is impossible, which completes the proof.
The last step in the proof of Theorem 2 is the following proposition.

**Proposition 7.** The case \( l > k \) is impossible.

**Proof.** Assume the converse: \( l > k \). As in the proof of the previous proposition, for any \( i \leq k, i \geq 2 \), we have

\[
B_{i-1} \subset S_1^{i-1} \cap E_1^{i-1} \cap E_{i-1} = (S_1 \cap E_1)^{i-1} \cap E_{i-1}.
\]

Set \( \Delta = S_1 \cap E_1 \). Consider the hypersurface \( Q \subset \mathbb{P} \), containing the point \( o \), which in the affine coordinates \( z_1, \ldots, z_M \) is defined by

\[
\tilde{f}(z_*) = q_{m-1}(z_*) + \tilde{q}_m(z_*) = 0,
\]

where \( q_{m-1}(z_*) \) is the same polynomial as in equation (1.1) of the hypersurface \( S \), and \( \tilde{q}_m(z_*) \) is a generic homogeneous polynomial of degree \( m \). Obviously, \( Q^1 \cap E_1 = \Delta \), the hypersurface \( Q^1 \subset X_1 \) is nonsingular and the intersection of \( Q^1 \) with \( E_1 \) is everywhere transversal. Therefore for every \( i \leq k, i \geq 2 \), we have

\[
B_{i-1} \subset \Delta^{i-1} \cap E_{i-1} = Q^{i-1} \cap E_1^{i-1} \cap E_{i-1}.
\]

On the other hand, by the definition of the number \( k \) we have \( B_k \not\subset S^k \), so that, because of the polynomial \( \tilde{q}_m \) being general, we have \( B_k \not\subset Q^k \). Thus

\[
\text{ord}_T \varphi^*Q = \text{ord}_{EN} \varphi_{N,0}Q = (m-1)p_1 + p_2 + \cdots + p_k = b.
\]

Now we argue in the same way, word for word, as in the proof of Proposition 5: for the strict transform \( \overline{Q} \subset \mathbb{P} \) we get the presentation (5.3), which immediately implies that \( (L \cdot \overline{Q}) = 0 \) for a general line \( L \subset \mathbb{P} \), which is impossible since the polynomial \( \tilde{q}_m \) is a general one. The proposition is proved.

The proof of Theorem 2 is now complete.

Let us prove Theorem 1. By Theorem 2 we need to show only the claim that the group \( \text{Aut}(\mathbb{P} \setminus S) = \text{Aut}(\mathbb{P})_S \) is finite, generically trivial. First of all, every projective automorphism \( \chi_\mathbb{P} \) preserving the hypersurface \( S \) maps \( o \) to itself. Let \( \text{Aut}(\mathbb{P})_o \subset \text{Aut}(\mathbb{P}) \) be the stabilizer of the point \( o \) and

\[
\pi: \text{Aut}(\mathbb{P})_o \rightarrow \text{Aut}(E)
\]

be the natural projection sending a projective automorphism \( \xi \in \text{Aut}(\mathbb{P})_o \) to the corresponding automorphism of the projectivized tangent space \( \mathbb{P}(T_o \mathbb{P}) \cong E \). Obviously, for every \( \chi_\mathbb{P} \in \text{Aut}(\mathbb{P})_S \) its image \( \pi(\chi_\mathbb{P}) \) preserves the hypersurface \( S^+ \cap E \) (that is, the hypersurface \( \{p_{m-1} = 0\} \) in the sense of equation (1.1)). By [1] the group \( \pi(\text{Aut}(\mathbb{P})_S) \) is finite and, for a Zariski general hypersurface \( S \), trivial. Setting

\[
\pi_S = \pi|_{\text{Aut}(\mathbb{P})_S},
\]

we see that it is sufficient to show that the kernel \( \text{Ker} \pi_S \) is trivial. This is really easy.

In a system of homogeneous coordinates \( (x_0 : x_1 : \cdots : x_M) \) such that \( o = (1 : 0 : \cdots : 0) \) every projective automorphism \( \xi \in \text{Ker} \pi \) has the form

\[
\xi: (x_0 : x_1 : \cdots : x_M) \mapsto (a_0x_0 + a_1x_1 + \cdots + a_Mx_M : x_1 : \cdots : x_M),
\]

where \( a_0, a_1, \ldots, a_M \) are constants.
where \( a_0 \neq 0 \). In such a system of coordinates the hypersurface \( S \) is given by the equation \( \Phi(x^*) = 0 \), where

\[
\Phi(x_0, \ldots, x_M) = x_0 q_{m-1}(x_1, \ldots, x_M) + q_m(x_1, \ldots, x_M);
\]

see (1.1). If \( \xi \in \text{Ker} \pi_S \), then the homogeneous polynomial

\[
(a_0x_0 + a_1x_1 + \cdots + a_Mx_M)q_{m-1}(x_1, \ldots, x_M) + q_m(x_1, \ldots, x_M)
\]

is proportional to \( \Phi(x^*) \). It is easy to see that this is only possible in one case, when \( a_0 = 1 \) and \( a_1 = \cdots = a_M = 0 \), that is, \( \xi = \text{id}_\mathbb{P} \). This completes the proof of Theorem 1.

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