Lattice Analogues of $N = 2$ Superconformal Models via Quantum Group Truncation

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We obtain lattice models whose continuum limits correspond to $N = 2$ superconformal coset models. This is done by taking the well known vertex model whose continuum limit is the $G \times G/G$ conformal field theory, and twisting the transfer matrix and modifying the quantum group truncation. We find that the natural order parameters of the new models are precisely the chiral primary fields. The integrable perturbations of the conformal field theory limit also have natural counterparts in the lattice formulation, and these can be incorporated into an affine quantum group structure. The topological, twisted $N = 2$ superconformal models also have lattice analogues, and these emerge as an intermediate part of our analysis.

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1. Introduction

In this paper we describe some exactly solvable lattice models whose continuum limit yields the $N = 2$ superconformal coset models based on hermitian symmetric spaces. Our approach is, to some degree, a natural extension of the lattice analogues of the $N = 2$ supersymmetric minimal series via fusions of the six vertex model. We do not use a manifestly supersymmetric formulation: instead the lattice formulation employed here parallels the free bosonic and para-toda formulation of the $N = 2$ superconformal models. The starting point of our construction is the well known vertex model, whose continuum limit is a $G_k \times G_1/G_{k+1}$ conformal coset model. By passing to a topological model and “untwisting” one can get a class of $N = 2$ superconformal coset models. We translate this prescription into the construction of lattice model analogues of these $N = 2$ superconformal theories. It is unclear whether the lattice model itself has any form of supersymmetry even though the continuum limit does. We do, however, suspect that there is some hidden supersymmetry since the lattice models we describe can be “twisted” back into topological lattice models that are the direct analogues of the topological, twisted $N = 2$ superconformal theories of. We suspect that it would not be so simple to obtain such topological lattice models unless there were some kind of hidden supersymmetry generators that provide the truncation to the topological lattice models.

Once the lattice analogue of an $N = 2$ superconformal coset is obtained, one finds that it comes equipped with a set of natural operators. These operators appear as an affine extension of the underlying quantum group structure of the vertex model, and we identify these operators as lattice analogues of the most relevant supersymmetric perturbations of the $N = 2$ superconformal model. In the continuum limit such perturbations lead to a massive integrable field theory. It also turns out to be straightforward to identify the natural lattice order parameters, and we find that the corresponding operators renormalize to Landau-Ginzburg fields of the $N = 2$ superconformal coset model in the continuum limit.

In this paper we will concentrate on the so-called SLOHSS models, but our results can readily be generalized via fusion procedure to arbitrary $N = 2$ supersymmetric models based upon hermitean symmetric spaces. We begin in section 2 by reviewing relationships between coset models $G_{k,1} \equiv G_k \times G_1/G_{k+1}$ and the SLOHSS models. We then exploit this in section 3 to construct the new lattice models in both the vertex and restricted

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1 This miserable acronym stands for Simply Laced, Level One, Hermitian Symmetric Space models.
height formulations. The lattice analogues of the grassmannian models are described in some detail and the Boltzmann weights are computed in the restricted height formulation. In section 4 we discuss various lattice operators, and in section 5 we discuss some of the questions that are raised by this work.

2. Coulomb gas formulation of $SLOHSS$ models

We consider the $N = 2$ superconformal coset models of the form

$$
\frac{G_1 \times SO(dim(G/H))}{H},
$$

where $G$ is simply laced and of level one, and $G/H$ is a hermitean symmetric space. The free field formulation has been discussed extensively elsewhere \[3\]\[4\]\[5\]\[6\], but to summarize, and fix notation, we consider $\ell$ free bosons, $\phi$, (where $\ell$ is the rank of $G$). The stress tensor, $T(z)$, has the form

$$
T(z) = -\frac{1}{2} (\partial \phi(z))^2 + i (\beta - \frac{1}{\beta}) \rho_H \cdot \partial^2 \phi(z),
$$

where

$$
\beta = \sqrt{\frac{g}{g+1}} ,
$$

$\rho_H$ is the Weyl vector of the subgroup $H$ of $G$, and $g$ is the dual Coxeter number of $G$. The central charge of the model is $c = 3M_{g+1}$, where $M$ is the complex dimension of $G/H$. The screening currents are

$$
e^{i\beta \alpha_j \cdot \phi(z)} \text{ and } e^{-i\frac{1}{\beta} \alpha_j \cdot \phi(z)} , \quad j = 1, \ldots, \ell - 1 ,
$$

where the $\alpha_j$ are the simple roots of $H$. The two supercurrents, $G^+(z)$ and $G^-(z)$, can be represented by vertex operators,

$$
G^+(z) = e^{-i\frac{1}{\beta} \gamma \cdot \phi(z)} \quad G^-(z) = e^{i\frac{1}{\beta} \psi \cdot \phi(z)} ,
$$

where $\gamma$ is the simple root of $G$ that extends the simple root system of $H$ to a simple root system for $G$, and $\psi$ is the highest root of $G$. The $U(1)$ current of the $N = 2$ superconformal algebra is given by,

$$
J(z) = 2i(\beta - \frac{1}{\beta}) (\rho_G - \rho_H) \cdot \partial \phi ,
$$

where $\rho_G$ is the Weyl vector of $G$. 

The chiral primary fields are easily identified. They can all be represented by vertex operators,

\[ e^{-i(\beta - \frac{1}{\beta}) \Lambda \cdot \phi(z)} \quad \text{with} \quad \Lambda = w(\rho_G) - \rho_G , \tag{2.7} \]

where \( w \) an element of the Weyl group, \( W(G) \), of \( G \) is chosen so that \( w(\rho_G) \) is a dominant weight of \( H \). The vertex operator (2.7) has \( N = 2 \), \( U(1) \) charge:

\[ Q = \frac{1}{g(g + 1)} 2(\rho_G - \rho_H) \cdot [\rho_G - w(\rho_G)] . \tag{2.8} \]

The topologically twisted \( N = 2 \) superconformal model is obtained from the \( N = 2 \) superconformal model by taking:

\[ T_{top}(z) = T(z) + \frac{1}{2} \partial J(z) = -\frac{1}{2}(\partial \phi)^2 + i(\beta - \frac{1}{\beta}) \rho_G \cdot \partial^2 \phi , \tag{2.9} \]

and using \( G^+(z) \) (which now has conformal weight equal to 1) as a screening current. The physical states of the topological model precisely the chiral primary fields (2.7).

The partition functions of the original (untwisted) \( N = 2 \) superconformal field theory can easily be computed using the Coulomb gas formulation. One can also obtain them from a direct computation of the branching functions. Since the SLOHSS models do not have any fixed points under the field identifications generated by spectral flow, the branching functions can be identified with the characters of the model. Following, we introduce \( M \) complex fermions, \( \lambda^\alpha(z) \) and \( \lambda^{-\bar{\alpha}}(z) \), to describe \( SO_1(dim(G/H)) \). The labels, \( \bar{\alpha} \), are the positive roots of \( G \) that are not positive roots of \( H \). In order to calculate the branching functions of the model we need to analyze the conformal embedding of \( H \) into \( SO(dim(G/H)) \). The Cartan subalgebra of \( H \) is realized via

\[ h^i(z) = \sum_\alpha \bar{\alpha}^i \lambda^{-\bar{\alpha}}(z) \lambda^\alpha(z) . \tag{2.10} \]

The fermionic partition function, with an insertion of \( e^{-2\pi i \nu, h^i} \), are

\[ \chi_R^{\pm}(\tau; \nu) = q^{M/12} \prod_\alpha \left[ (e^{-i\pi \bar{\alpha} \cdot \nu} \pm e^{i\pi \bar{\alpha} \cdot \nu}) \prod_{n=1}^{\infty} (1 \pm q^n e^{-2\pi i \bar{\alpha} \cdot \nu})(1 \pm q^n e^{2\pi i \bar{\alpha} \cdot \nu}) \right] \]

\[ \chi_{N.S.}^{\pm}(\tau; \nu) = q^{-M/24} \prod_\alpha \prod_{n=1}^{\infty} (1 \pm q^{n-\frac{1}{2}} e^{-2\pi i \bar{\alpha} \cdot \nu})(1 \pm q^{n-\frac{1}{2}} e^{2\pi i \bar{\alpha} \cdot \nu}) . \tag{2.10} \]
These characters are interrelated by spectral flow:

\[
\begin{align*}
\chi_{N.S.}^\pm(\tau; \nu + \frac{1}{g}(\rho_G - \rho_H)) &= \chi_{N.S.}^\pm(\tau; \nu) \\
\chi_R^\pm(\tau; \nu + \frac{1}{g}(\rho_G - \rho_H)) &= (-i)^M \chi_R^\pm(\tau; \nu) \\
\chi_{N.S.}^\pm(\tau; \nu + \frac{1}{g}(\rho_G - \rho_H)\tau) &= (\pm 1)^M q^{-M/8} e^{-2\pi i(\rho_G - \rho_H) \cdot \nu} \chi_{N.S.}^\pm(\tau; \nu) \\
\chi_R^\pm(\tau; \nu + \frac{1}{g}(\rho_G - \rho_H)\tau) &= q^{-M/8} e^{-2\pi i(\rho_G - \rho_H) \cdot \nu} \chi_R^\pm(\tau; \nu).
\end{align*}
\]

More significantly, if \(\Delta(G)\) is the Weyl-Kac denominator of \(G\):

\[
\Delta(G) = \prod_{\alpha \in \Delta^+(G)} \left[ (e^{-i\pi\alpha \cdot \nu} - e^{i\pi\alpha \cdot \nu}) \right] \times \prod_{n=1}^{\infty} (1 - q^n)\ell(1 - q^n e^{-2\pi i \alpha \cdot \nu}),
\]

then one can write:

\[
\chi_R^\pm(\tau; \nu) = \frac{\Delta(G)}{\Delta(H)}.
\]

Using the Weyl-Kac denominator formula for \(G\) one immediately obtains:

\[
\chi_R^\pm(\tau; \nu) = q^{M/12} \frac{1}{\Delta(H)} \sum_{w \in W(G)} \sum_{\alpha \in M(G)} \epsilon^\pm(w, \alpha) q^{\frac{1}{2g}[w(\rho_G) + g\alpha]^2 - \frac{1}{2g}\rho_G^2 e^{-2\pi i \nu \cdot [w(\rho_G) + g\alpha]}}.
\]

where \(M(G)\) is the root lattice of \(G\), and

\[
\epsilon^-(w, \alpha) = \epsilon(w), \quad \epsilon^+(w, \alpha) = \epsilon(w)e^{-2\pi i \cdot [w(\rho_G) - \rho_G + g\alpha]}.
\]

Using (2.11) one then has

\[
\chi_{N.S.}^\pm(\tau; \nu) = q^{M/12} \frac{1}{\Delta(H)} \sum_{w \in W(G)} \sum_{\alpha \in M(G)} \epsilon^\pm(w, \alpha) q^{\frac{1}{2g}[w(\rho_G) + g\alpha - (\rho_G - \rho_H)]^2 - \frac{1}{2g}\rho_G^2} \times e^{-2\pi i \nu \cdot [w(\rho_G) + g\alpha - (\rho_G - \rho_H)]}.
\]

From this one can reduce the highest weight representations of affine \(SO_1(dim(G/H))\) into finitely many such representations of \(H_{g-h}\). (The number \(h\) is the dual Coxeter number
of $H$, and should be thought of as a vector if $H$ is semi-simple. The dual Coxeter number of the $U(1)$ factor is defined to be zero.) In the Ramond sector we have

$$\chi_R^\pm(\tau; \nu) = \sum_{w \in \mathbb{W}(G)} \sum_{\alpha \in M(G)} \chi_{\lambda(\alpha, w)}^H(\tau; \nu),$$

(2.17)

where

$$\lambda(\alpha, w) = w(\rho_G) - \rho_H + g\alpha$$

(2.18)

and $\chi_{\lambda}^H$ is the character of $H$ at level $g - h$ with highest weight $\lambda$. The Weyl element, $w \in \mathbb{W}(G)$, is chosen so that $\lambda(\alpha, w)$ is a highest weight of $H$.

Now introduce the level one characters of $G$

$$\chi_A^G(\tau; \nu) = \frac{1}{\eta(\tau)} \sum_{\alpha \in M(G)} q^{\frac{1}{2}(\Lambda + \alpha)^2} e^{-2\pi i \nu (\Lambda + \alpha)},$$

(2.19)

and multiply this by the expressions (2.14) and (2.16) for $\chi_R^\pm$ or $\chi_{N.S.}^\pm$. By rearranging the sum it is elementary to factor out the branching functions. At this point it is also valuable to keep track of the $N = 2$, $U(1)$ current. One then obtains the branching functions for the $N = 2$ Hilbert space with an insertion of $e^{-2\pi i \zeta J_0}$ into the trace:

$$b_{\lambda, R_\pm} = \frac{1}{\eta(\tau)} \sum_{w \in \mathbb{W}(G)} \sum_{\alpha \in M(G)} e^{\pm(w, \alpha)} q^{\frac{1}{4} [w(\rho_G) - \beta(\Lambda_+ - \rho_H) + \sqrt{g(g+1)}\alpha]^2}$$

$$\times e^{\frac{4\pi i}{\sqrt{g(g+1)}} \zeta(\rho_G - \rho_H) \cdot [\frac{1}{2} w(\rho_G) - \beta(\Lambda_+ - \rho_H) + \sqrt{g(g+1)}\alpha]}$$

(2.20)

$$b_{\lambda, N.S._\pm} = \frac{1}{\eta(\tau)} \sum_{w \in \mathbb{W}(G)} \sum_{\alpha \in M(G)} e^{\pm(w, \alpha)} q^{\frac{1}{4} [w(\rho_G) - \beta(\Lambda_+ - \rho_H) + \sqrt{g(g+1)}\alpha]^2}$$

$$\times e^{\frac{4\pi i}{\sqrt{g(g+1)}} \zeta(\rho_G - \rho_H) \cdot [\frac{1}{2} w(\rho_G) - \beta(\Lambda_+ - \rho_H) + \sqrt{g(g+1)}\alpha]}$$

(2.21)

where $\beta$ is given by (2.3). Taking the usual $A$-type modular invariant we obtain the modified gaussian partition function

$$Z = \frac{1}{2|Z(G)|} \sum_{\lambda \in \lambda_{\lambda_-, \pm}} |b_{\lambda, u}|^2$$

$$= \frac{1}{2|W(H)||Z(G)| |\eta(\tau)|^2} \sum_{w \in \mathbb{W}(G)} \sum_{\alpha \in M(G)} \sum_{u \in \frac{M(G)^*}{\Gamma}} \sum_{v_1, v_2 \in \Gamma} \sum_{\xi = 0, 1} \sum_{\eta = 0, 1} \epsilon(w)$$

$$\times q^{\frac{1}{2}(\nu_L + \eta s)^2} q^{\frac{1}{2}(\nu_R + \eta s)^2} e^{-4\pi i s \cdot (\zeta_L v_L - \zeta_R v_R)} e^{-2\pi i \xi s \cdot (v_L - v_R)},$$

(2.22)
where
\[ \Gamma = \sqrt{g(g+1)} M(G) \]  
(2.23)

\[ v_L = v + u + v_1 ; \quad v_R = w(v) + u + v_2 \]  
(2.24)

and
\[ s = \frac{1}{\sqrt{g(g+1)}} (\rho_G - \rho_H) . \]  
(2.25)

The partition function (2.22) represents \( \operatorname{Tr}[q^{L_0-c/24} \bar{q}^{\bar{L}_0-c/24} e^{-2\pi i (\zeta_L \bar{J}_0 - \zeta_R \bar{J}_0)}] \) taken over the entire Hilbert space, and the factor of \( |Z(G)|^{-1} \) is a division by the order of the center of \( G \) and factors out the field identifications induced by the spectral flow by the center \[ [15] [16] [9] \]. In (2.22), the sum over \( \eta = 0 \) and \( \eta = 1 \) corresponds to the sum over the Neveu-Schwarz and Ramond sectors respectively. The sum over \( \xi = 0 \) and \( \xi = 1 \) corresponds, respectively, to the insertion or absence of \( (-1)^F \) in the trace. It is, perhaps, also, amusing to note that for fixed \( w \in W(G) \) the sum over \( (v_L; v_R) \) in (2.22) is a sum over a Lorentzian self-dual lattice and thus (2.22) is modular invariant even before one sums over \( W(G) \).

The sum over \( \eta \) and \( \xi \) is a simple example of the shifted lattice construction that maps one modular invariant to another \[ [18] \].

It is now highly instructive to compare the \( N = 2 \) superconformal model (2.1) to the coset model \( G_{k,1} \equiv G_k \times G_1 / G_{k+1} \). We first note that the branching functions and partition functions can be computed in the same manner as for the \( N = 2 \) supersymmetric models\[ [14] \]. The diagonal modular invariant partition function for the \( G_{k,1} \) model is given by a very similar formula to (2.22):
\[ Z = \frac{1}{|W(G)||Z(G)|} \frac{1}{|\eta(\tau)|^{2\ell}} \sum_{w \in W(G)} \sum_{v_1 \in \frac{1}{M(G)^*}} \sum_{u \in \frac{1}{\delta M(G)^*}} \sum_{v_2 \in \Gamma} \epsilon(w) q^{\frac{1}{2} (v_L)^2} q^{\frac{1}{2} (v_R)^2} \]  
(2.26)

where one now has
\[ \beta = \sqrt{\frac{k+g}{k+g+1}} , \]  
(2.27)

\[ \Gamma = \sqrt{(k+g)(k+g+1)} M(G) , \]  
(2.28)

but \( v_L \) and \( v_R \) are still given in terms of \( u, v, v_1 \) and \( v_2 \) by (2.24).

This form of the partition function can also be obtained from the well known free bosonic formulation of the \( G_{k,1} \) coset models \[ [19] \]. In this formulation, the energy momentum tensor is
\[ T_G(z) = -\frac{1}{2} (\partial \phi)^2 + i(\beta - \frac{1}{\beta}) \rho_G \cdot \partial^2 \phi , \]  
(2.29)
where $\beta$ is given by (2.27). The screening currents of the $G_{k,1}$ model are $e^{i\beta \alpha_j \cdot \phi}$ and $e^{-i\beta \alpha_j \cdot \phi}$ for $j = 1, \ldots, \ell$, where $\alpha_j$ are the simple roots of $G$. (In the notation following equation (2.3) we are taking $\alpha_{\ell} \equiv \gamma$.) The primary fields can be represented by

$$
\Phi_{\Lambda_1, \Lambda_2} = e^{(-i(\beta \Lambda_2 - \frac{1}{2} \Lambda_1) \cdot \phi(z))},
$$

(2.30)

where $\Lambda_1$ is an affine highest weight of $G_k$ and $\Lambda_2$ is an affine highest weight of $G_{k+1}$. The labels $\Lambda_1$ and $\Lambda_2$ also directly correspond, in the obvious way, to the branching functions of $G_{k,1}$. (The label of $G_1$ in the numerator of the coset is equal to $\Lambda_2 - \Lambda_1$ modulo the root lattice of $G$).

One should note that the topologically twisted $N = 2$ superconformal model is almost the same as the a $G_{k,1}$ model with $k = 0$ [20][4], but with an important modification - the $G_{0,1}$ model is a very particular, supersymmetry preserving perturbation of the twisted $N = 2$ model [3]. This perturbation is generated by the use of the screening current $e^{-i\beta \gamma \cdot \phi(z)}$. We will remark further upon this perturbation in section 4.

To make contact with the lattice model one can Poisson re-sum (2.22) and (2.26) over the lattice $u + v_1 \in \beta M(G)$. For the $N = 2$ superconformal partition function one finds

$$
Z = \frac{A}{(2\text{Im}(\tau))^\frac{1}{2} |\eta(\tau)|^{2\ell}} \sum_{x \in \frac{1}{\beta} M(G)} \sum_{v \in \frac{1}{\beta} \mathcal{M}^*(G)} \sum_{\eta(\tau) = 0,1} \sum_{\eta_0(\tau) \in \Gamma} \sum_{s} \frac{e^{-\frac{\pi}{\beta \text{Im}(\tau)} |x - \tau (v-w(v)+v_0) - 2(\zeta_L - \zeta_R)s|^2}}{x \cdot (v+w(v)-v_0+2\eta s)} e^{-2\pi i (\zeta_L + \zeta_R + \xi) s \cdot (v-w(v)+v_0)}
$$

(2.31)

where $A = (2|W(H)||Z(G)|\frac{1}{2}|\beta^\ell|^{-1}$ is an irrelevant normalization constant. The result for $G_{k,1} \equiv G_k \times G_1/G_{k+1}$ is identical (up to another normalization constant ), but with $\xi = \eta = \zeta_L = \zeta_R = 0$ and $\beta$ and $\Gamma$ given by (2.27) and (2.28).

It has been convincingly argued [21][22][23] that (at least for $G = SU(n)$) the Poisson re-summed form of the partition function (2.24) corresponds to that obtained from the IRF model based on the representations of $G$. (This will be described more fully in the next section). The key observation that we wish to make here is that the SLOHS model is a very simple modification of the $G \times G/G$ model. To go from $G_{k,1}$ to (2.1), one sets $k = 0$ and then untwists the resulting topological energy-momentum tensor (2.8). In the continuum theory this means subtracting the total derivative $\frac{1}{2} \partial J(z)$ from (2.29) (with $k = 0$). In the next section we will see that this amounts to choosing a particular value of $q$ and modifying a boundary term in the transfer matrix of the lattice model. The only other changes that are necessary in order to get the model (2.1) are some appropriate
phase insertions, and these can be read off from (2.31) and its counterpart in the \( G_{k,1} \) model. Specifically, in the partition function (2.31), each term in the soliton sum is given an additional phase that depends upon the \( SO(dim(G/H)) \), or fermionic, sector of (2.1). This phase is:

\[
\exp\left[ 2\pi i \eta x \cdot s - \xi y \cdot s \right], \quad y = v - w(v) + v_0,
\]  

(2.32)

where \( s \) is defined in (2.25) and the significance of \( \eta \) and \( \xi \) is described below (2.25). In terms of the lattice model, the momentum vectors \( x \) and \( y \) in (2.31) and (2.32) represent the change in the height, \( \Delta \phi \), as one periodically identifies the lattice. The extra phases, (2.32), must therefore be properly incorporated in order to obtain the desired lattice model from the lattice model corresponding to \( G_{k,1} \).

3. Lattice models

The relationship between the conformal coset models

\[
\mathcal{G}_{k,\ell} = \frac{G_k \times G_\ell}{G_{k+\ell}},
\]  

(3.1)

and lattice models is fairly well established \[24\] \[25\] \[26\]. The conformal model only appears in the continuum limit and at the critical point of the lattice model. For simplicity we will only consider the critical transfer matrices here. For \( \ell = 1 \), the vertex description of the lattice model is obtained by building the transfer matrix from the \( \tilde{R} \)-matrix for the fundamental representation of \( G \), and then performing a quantum group truncation \[25\] \[27\] \[28\]. One considers evolution from left to right on the usual 45\(^0\) lattice (see figure 1). A constant time-slice is a vertical zig-zag that runs from the top to the bottom of the lattice. Suppose this zig-zag has 2\(L\) edges. To each edge one associates a copy of the fundamental representation, \( V \), of \( G \) and the Hilbert space of the time slice is \( V = V \otimes 2L \).

\[\text{We will be considering the vertex model transfer matrix with free boundary conditions. To discuss the vertex model transfer matrices with periodic boundary conditions one has to do some unpleasant technical modifications. See, for example, \[27\].}\]
The transfer matrix is given by

\[
T(u, q) = \left[ \prod_{p=1}^{L} X_{2p-1}(u, q) \right] \left[ \prod_{p=1}^{L-1} X_{2p}(u, q) \right],
\]

where \( X_p(u, q) = \frac{1}{2i} \tilde{R}(u, q) \), and this matrix acts on the tensor product of the \( p^{th} \) and \((p + 1)^{th}\) copy of \( V \). The matrix, \( \tilde{R}(u, q) \), is the \( \tilde{R} \)-matrix for quantum \( G \), and maps \( V \otimes V \rightarrow V \otimes V \). For \( SU(n) \), \( V \) is the \( n \)-dimensional representation, and one has [29]

\[
\tilde{R}(u, q) = (xq - x^{-1}q^{-1}) \sum_{\alpha=1}^{n} E_{\alpha\alpha} \otimes E_{\alpha\alpha} + (x - x^{-1}) \sum_{\alpha \neq \beta}^{n} E_{\alpha\beta} \otimes E_{\beta\alpha}
\]

\[
+ (q - q^{-1}) \left[ x \sum_{\alpha > \beta} E_{\alpha\alpha} \otimes E_{\beta\beta} + x^{-1} \sum_{\alpha < \beta} E_{\alpha\alpha} \otimes E_{\beta\beta} \right],
\]

where \( x = e^{iu} \), \( E_{\alpha\beta} \) is an \( n \times n \) matrix whose entries \((E_{\alpha\beta})_{ij}\) are equal to \( \delta_{\alpha i} \delta_{\beta j} \). Literally, by construction, the transfer matrix (3.2) commutes with the action of \( U_q(G) \) on
Furthermore, the lattice model is integrable since $\tilde{R}(u,q)$ satisfies the Yang-Baxter equation

$$(\tilde{R}(u; q) \otimes I)(I \otimes \tilde{R}(u+v; q))(\tilde{R}(v; q) \otimes I) = (I \otimes \tilde{R}(v; q))(\tilde{R}(u+v; q) \otimes I)(I \otimes \tilde{R}(u; q)).$$

where $I$ is the identity matrix on one copy of $V$.

By going to the extreme anisotropic limit ($u \to 0$), one can extract a “spin-chain” Hamiltonian from $T$. This is defined by

$$H = T^{-1}(u) \frac{\partial}{\partial u} T(u)|_{u=0}.$$  

An important property of $H$ is that it has a non-trivial boundary-term:

$$H_{bdry} = -\frac{2}{g} \rho_G \cdot (h_1 - h_{2L}),$$

where $h_j$ is the Cartan sub-algebra (C.S.A.) generator of $U_q(G)$ acting on the $j^{th}$ edge of the lattice. This boundary term is essential for $H$ to commute with the quantum group $\mathbb{U}_q(G)$. Moreover, this term has the effect of shifting all the ground state energies exactly parallel to introducing a boundary charge proportional to $\rho_G$ into a gaussian model.

There are two equivalent height model or IRF descriptions of the foregoing vertex model, and both these descriptions can be viewed as arising from a change of basis in $V$. The first height model, sometimes called the unrestricted height model (or BCSOS model), is a completely trivial basis change: the heights take values on the entire weight lattice of $G$ and are assigned to vertices on the lattice so that the difference in height between two adjacent vertices is precisely the weight of $V$ that is assigned to the interconnecting edge in the vertex description. The second height description is much better adapted to the quantum group. One breaks the lattice into time slices, each of which consists of a vertical zig-zag of vertices and edges (see figure 1). One starts at the top of the lattice with some fixed representation of $G$ assigned to the first vertex in each zig-zag. Representations, $W_p$, of $G$ are then associated to each vertex, $p$, down a zig-zag so that $W_{p+1} \subseteq V \otimes W_p$. One also keeps track of the total C.S.A. eigenvalue (weight) of the state in the Hilbert space $V$. A list of highest weights of representations $W_p$ and the total C.S.A. eigenvalue along a zig-zag determines a unique state in $V$. The set of such states forms a basis for $V$. One thus has an association of representations of $G$ to vertices on the lattice, and the IRF Boltzmann weights can be obtained by performing this basis change on the transfer matrix $T(u)$. The fact that the transfer matrix of the vertex model
commutes with $U_q(G)$ means that the Boltzmann weights only care about the highest weight labelling of the vertices and are independent of the overall C.S.A. charge. 

Let $k \in \mathbb{Z}$, $k \geq 0$, and suppose that

$$q^{k+g+1} = \pm 1$$

where $k + g + 1$ is the smallest such integer. One can now perform a quantum group truncation that leads to the lattice analogue of the $G_{k,1}$ models. In the vertex model formulation the partition functions and the correlation functions of the truncated model can easily be written down in terms of the untruncated model. One simply uses the modified trace \[ \text{Tr} \mathcal{O} = \text{tr} \mathcal{O} \mu \otimes \cdots \otimes \mu, \] 

where

$$\mu = q^{2\rho_G \cdot h}.$$ 

Once again, the insertion of the factor of $\mu$ can be thought of as a modification of the charge at infinity in the conformal model.

If one converts the foregoing into the height description that uses the representations $W_p$, one finds that because of the modified trace, only those representations with non-vanishing $q$-dimension contribute to partition and correlation functions. Therefore one can simply restrict the heights to those that correspond to the type $II$ representations of $U_q(G)$. Equivalently one can restrict to those weights that are highest weights of the affine $G$ at level $k+1$. The transfer matrix, of course, preserves such a truncation. For obvious reasons this is called the restricted height or $IRF$ model.

One of the easiest methods of getting at the conformal field theoretic limit of these lattice models is to consider the unrestricted height model whose heights lie on the weight lattice of $G$. It can be argued that this model renormalizes to a continuum limit consisting of $\ell$ free bosons (where $\ell$ is the rank of $G$). The modified trace of the truncated model, along with appropriately chosen spatial boundary terms, can be argued to give the winding modes, or solitons of the gaussian model certain phases that depend upon the winding numbers. In this way, one can arrive at the Poisson re-summed version of the partition function with the correct phase factors. One can also reverse this procedure and try to use the weight factors of a non-trivial modular invariant,
gaussian partition functions in order to determine the corresponding modifications to the lattice theory.

To summarize, one can think of the vertex formulation (or the equivalent description in terms of unrestricted heights) as being the direct counterpart of the continuum free bosonic theory. The quantum group truncation is the counterpart of the Feigen-Fuchs-Felder screening prescriptions. The restricted height model then emerges as the lattice counterpart of the conformal model $G_{k,1}$, and it requires no further modifications or truncation. There are some important subtleties about spatial boundary conditions, and we will comment upon these at the end of this section. Our strategy will therefore be the following: we will use the vertex/unrestricted height description and its relation to the free bosonic theory to determine the how to modify the $\hat{R}$ matrix and how to further modify the traces (3.9). We will then change the basis to the height model and pass to the restricted height model and thus obtain the Boltzmann weights of a lattice model whose continuum limit is (2.1).

The first step to getting the lattice analogue of the model (2.1) is to take the vertex description of the $G_{k,1}$ models, set $k = 0$ and untwist the transfer matrix. This can be done by “conjugating” $\hat{R}$ the $\check{R}$-matrix of $G$ \cite{34,35}. Define:

$$\hat{R}'(x,q) = \left[1 \otimes x^{-\frac{2}{\hbar}(\rho_G - \rho_H)h}\right] \hat{R}(x,q) \left[x^{\frac{2}{\hbar}(\rho_G - \rho_H)h} \otimes 1\right].$$

(3.11)

This $\hat{R}'$-matrix has several important properties. It, of course, still satisfies the Yang-Baxter equations (3.5).

It commutes with $U_q(H')$, where $H'$ is the semi-simple factor of $H$ (i.e. $H = H' \times U(1)$), because $(\rho_G - \rho_H)h$ defines the $U(1)$ that is orthogonal to $H'$. If one employs $\hat{R}'$ to construct the transfer matrix, one can easily verify that the net effect of “conjugation” is to simply add boundary terms to the transfer matrix $T(u)$. In particular, the analogue of (3.6) has a boundary term of the form (3.7), but with $\rho_G$ replaced by $\rho_H$. This yields precisely the required shift in the ground state energy to go from the central charge, $c = 0$, of the topological $G_{0,1}$ theory to the correct value of $c$ for the $N = 2$ supersymmetric coset model.

The only other step that is required to obtain the lattice analogue of (2.1) is to re-examine the modified trace (3.9) in the light of our earlier comments. From the comparison

\footnote{Note, this is not really a true conjugation of the $\hat{R}$-matrix as the left hand factor in (3.11) is not the inverse of the right hand factor in (3.11).}
of the \( N = 2 \) superconformal partition function with that of the \( G_{k,1} \) model, we see that
the untwisted \( G_{0,1} \) partition function gives the \( N = 2 \) superconformal partition function
in the Ramond sector with an insertion of \((-1)^F\). In the sum over solitons in (2.31) the
operator \((-1)^F\) corresponds to weighting a soliton by a phase:

\[
e^{-\frac{2\pi i}{g+1} (\rho_G - \rho_H) \cdot x},
\]

where \( x \) is the winding vector in the timelike direction. Therefore, to obtain lattice partition
functions and correlation functions without such an insertion of \((-1)^F\), we must remove
this phase from the modified trace of (3.9). On the lattice Hilbert space, \( V \), the operator
\((-1)^F\) is simply:

\[
\Delta(q^{-2(\rho_G - \rho_H) \cdot h}),
\]

where \( \Delta \) is the co-product. Making a further insertion of (3.13) into (3.9) merely amounts
to replacing the modified trace (3.9) by the \( H \)-modified trace:

\[
Tr_H(O) = tr[O \mu_H \otimes \cdots \otimes \mu_H],
\]

where

\[
\mu_H = q^{2\rho_H \cdot h}.
\]

If one converts this to the height formulation, this new \( H \)-modified trace means that the
only those heights that correspond to type II representations of \( U_q(H') \) will contribute.

Thus the essential idea is that we are using a vertex model based upon \( G \), but only
performing a quantum group truncation with respect to \( H' \). Since one has \( q = e^{\frac{\pi}{g+1}} \)
this means that one truncates \( H' \) highest weights to those weights that are highest weight
labels of affine \( H' \) at level \( g - h + 1 \). The states that lie purely in the Hilbert space of the
\( U(1) \) factor of \( H \) are completely unaffected by the truncation and the boundary charge.
This direction still corresponds to an unrestricted, free \( U(1) \).

Perhaps the most convincing argument as to why the foregoing construction is the
correct one is obtained by reconsidering, and expanding upon, our discussion of the con-
formal field theory defined by (2.1). Because the factor of \( G \) in (2.1) is of level one, it
can be replaced by a factor of \( H \) at level one (because the rank of \( G \) is equal to the rank

\[\footnote{Given the form of the phase \((-1)^F\), this operator should correspond to something of the form
\( \Delta(q^{a(\rho_G - \rho_H) \cdot h}) \), where \( a \) is a constant. It will soon become evident as to why one has \( a = -2 \).} \]
of $H$). In addition, the fact that the embedding of $H$ into $SO_1(dim(G/H))$ is con-
formal means that one can replace the factor of $SO(dim(G/H))$ by $H_{g-h}$. The im-
portant proviso is that one can make these replacements provided that one restricts the repre-
sentations of $H_1$ and $H_{g-h}$ to those combinations that make up the representations of $G_1$ and
$SO_1(dim(G/H))$. Hence, modulo this statement about represen-
tations, one is dealing with a coset: $H_1 \times H_{g-h}/H_{g-h+1}$. Now recall that $H = H' \times U(1)$, where $H'$ is semi-simple, and once again, modulo the careful treatment of the radii of $U(1)$ factors and the correct association of $U(1)$ charges with representations of $H'$, one can cancel the $U(1)$ factors be-
tween the numerator and denominator and see that one is really working with a conformal coset model:

$$\frac{H_1 \times H'_{g-h}}{H'_{g-h+1}} \times U(1).$$

(3.16)

The $U(1)$ factor in (3.16) is precisely the $N = 2$, $U(1)$ current (2.6). Much of the
labour in section 2 was spent upon deriving exactly which Hilbert spaces of the $H'$-coset model (3.16) were to be employed, and determining the associated $N = 2$, $U(1)$ charges so that one would obtain (2.1). The result of this labour may be summarized as follows.

One uses (2.18) in $H_{g-h}$ for the Ramond sector, while one uses

$$\lambda(\alpha, w) = w(\rho_G) - \rho_G + g\alpha$$

(3.17)

in $H_{g-h}$ in the Neveu-Schwarz sector. In the $H_1$ factor one simply employs the weights of $H$ that are, in fact, weights of $G$. Now observe that vectors in (3.17) are actually roots of $G$, and that the vectors in (2.18) are roots of $G$ shifted by $\rho_G - \rho_H$. Consequently one can roughly think of (2.18) as (3.16) with a selection rule on the $U(1)$ charge that amounts to requiring that the $U(1)$ charges are added so that the weights of $H'$ extend to weights of $G$, or at least weights of $G$ up to a possible shift by $\rho_G - \rho_H$. This selection rule is also evident in the partition function (2.31), where the summation is over winding modes on $\frac{1}{\beta} M(G)$.

The new vertex model introduced above has been constructed in such a manner that it also exhibits the foregoing $H'$-coset structure. First, the corresponding unrestricted height model has a continuum limit that is described by $\ell$ free bosons. We have ensured that the transfer matrix commutes with $U_q(H')$, and that the spin-chain hamiltonian only has non-hermitian boundary terms in the $H'$ direction. We have arranged the modified trace so as to only perform the $H'$ quantum group truncation, and the value $q$:

$$q \equiv e^{\frac{ig}{\beta}}$$

(3.18)
is precisely the correct one to obtain (3.16) at the right level. As remarked above, the states that lie in the Hilbert space of the $U(1)$ factor are completely unaffected by the truncation and boundary charge, and this direction thus corresponds to an unrestricted, free $U(1)$. The fact that we have built the model starting with the $\hat{R}$ matrices of $G$, and heights that are weights of $G$ means that the continuum limit will have $U(1)$ charges associated with $H'$ representations in accordance with the selection rule described above. These facts all give us considerable confidence that we have indeed given a lattice description of a model whose continuum limit is (2.1).

It is now relatively simple to summarize the restricted height formulation of the $N = 2$ supersymmetric models. The IRF formulation of the $\mathcal{G}_{k,1}$ models can be described in terms of a graph generated by the fusion rules of $G$ at level $k$. In particular, one generates a directed graph by considering which representation can be connected by fusion with the fundamental representation, $V$. For the $N = 2$ models the situation is very similar, except, it works on the graph defined by all the weights of $G$ that are highest weights of $H'$ at level $g - h + 1$. (The graph may be infinite in one direction because the $U(1)$ charge is arbitrary). One can make it a directed graph by considering which vertices can be connected by fusions with $V$. It should, of course be remembered that $V$ will be decomposable into at least two $H$ representations, and one must consider all the irreducible pieces in describing the fusion graph. The Boltzmann weights in the height description can be computed using the basis change that underlies the IRF $\leftrightarrow$ vertex correspondence, but almost all of the hard work can be circumvented by using the known solution for the $\mathcal{G}_{k,1}$ models.

As an example, consider the grassmannian models, which are of the form (2.1) with $G = SU(m+n)$ and $H = SU(m) \times SU(n) \times U(1)$. The $\hat{R}$ matrix is given by (3.4), but with $\alpha, \beta$ running from 1 to $m+n$. Let the indices $a, b$ and $i, j$ run from 1 to $m$ and from $m+1$ to $m+n$ respectively. Let $\hat{R}_{(1)}(u, q)$ and $\hat{R}_{(2)}(u, q)$ be the diagonal $m \times m$ and $n \times n$ blocks in $\hat{R}$. Note that the sub-matrices $\hat{R}_{(1)}$ and $\hat{R}_{(2)}$ are simply the $\hat{R}$-matrices for $SU(m)$ and $SU(n)$ respectively. Under the “conjugation” operation (3.11), these sub-matrices are not modified. One can also easily verify that the only part of (3.4) that is modified is the third term for $\alpha > m, \beta \leq m$ or $\alpha \leq m, \beta > m$. Indeed, one finds that in $\hat{R}'$, with $\alpha$ and $\beta$ in the foregoing index ranges, this third term reduces to:

$$(q - q^{-1}) \left[ \sum_{\alpha > m, \beta \leq m} + \sum_{\alpha \leq m, \beta > m} \right] E_{\alpha \alpha} \otimes E_{\beta \beta} ,$$
It follows immediately that
\[ \tilde{R}'(u,q)_{ai,bj} = \tilde{R}'(u,q)_{ia,jb} = (q - q^{-1}) \delta_{ab} \delta_{ij} \]
\[ \tilde{R}'(u,q)_{ia,bj} = \tilde{R}'(u,q)_{ai,jb} = (x - x^{-1}) \delta_{ab} \delta_{ij}. \] (3.19)

Converting the face transfer matrix (3.3) into the IRF language is now elementary. Consider the face with assigned heights \((\Lambda_{p-1}, \Lambda_p, \Lambda_{p+1}, \Lambda'_p)\), as shown in figure 2. Decompose each height according to \(\Lambda \equiv (\lambda, \nu; q)\), where \(\lambda\) is a highest weight of \(SU_n(m)\), \(\nu\) is a highest weight of \(SU_m(n)\), and \(q\) is the \(U(1)\) charge: \(q = 2(\rho_G - \rho_H) \cdot \Lambda\). Because of the foregoing block decomposition of \(\tilde{R}'\), the height model transfer matrix has three ways in which it can act:

(i) The \(\lambda\) label can evolve exactly as it does in the \(SU_1(m) \times SU_n(m)/SU_{n+1}(m)\) height model, with \(q'_p = q_p\) and \(\nu'_p = \nu_p\).

(ii) The \(\nu\) label can evolve exactly as it does in the \(SU_1(n) \times SU_m(n)/SU_{m+1}(n)\) height model, with \(q'_p = q_p\) and \(\lambda'_p = \lambda_p\).

(iii) The off-diagonal terms of (3.19) act.

![Figure 2](image_url)  

Figure 2. In the IRF formulation the heights, \(\Lambda\), are associated to vertices as shown.

The expression for \(X_p(u,q)\) in situations (i) and (ii) is well known (see, for example, [36][26]). One can write
\[ X_p(u,q) = \sin(\gamma + u)I_p - \sin(u)U_p, \] (3.20)
where \( q = e^{i\gamma} \) and \( x = e^{iu} \). Introduce the vectors \( e_1 \equiv \nu_1, e_n \equiv -\nu_{n-1} \) and \( e_j \equiv \nu_j - \nu_{j-1} \) for \( j = 2, \ldots, n-1 \), where \( \nu_j \) is the \( j^{th} \) fundamental weight of \( SU(n) \), and define a function

\[
s_{jk}(\nu) \equiv \sin \left( \frac{\pi}{m + n + 1} (e_j - e_k) \cdot \nu \right).
\]

The operator, \( U_p \), then has the form:

\[
U_p \equiv (1 - \delta_{jl}) \left( \frac{s_{jl}(\nu + e_j)s_{jl}(\nu + e_k)}{s_{jl}(\nu)} \right)^{\frac{1}{2}},
\]

where \( \Lambda_{p-1} \equiv (\lambda, \nu; q) \), \( \Lambda_p \equiv (\lambda, \nu + e_j; q - m) \), \( \Lambda_{p+1} \equiv (\lambda, \nu + e_j + e_l; q - 2m) \) and \( \Lambda'_p \equiv (\lambda, \nu + e_k; q - m) \). The evolution in the \( SU(m) \) factor is similar. One should also note that if \( n = 1 \), the evolution in the “\( SU(1) \)” direction merely involves a shift of the \( U(1) \) charge and, as can be seen from \([3,4]\), the transfer matrix is simply a multiplicative factor of \( \sin(\gamma + u) \) (i.e. the matrix \( U_p \) is zero).

In the foregoing components of the transfer matrix one either had \( \lambda_{p-1} = \lambda_p = \lambda_{p+1} \) or \( \nu_{p-1} = \nu_p = \nu_{p+1} \). The off-diagonal parts of \( \tilde{R}' \) deal with the evolution when neither of these equalities hold. However, for \( X_p(u, q) \) to be non-zero, one must still have either (i) \( \lambda_p = \lambda_{p-1} \) and \( \nu_{p+1} = \nu_p \) or (ii) \( \nu_p = \nu_{p-1} \) and \( \lambda_{p+1} = \lambda_p \). In which case one has

\[
X_p(u, q) = \sin(\gamma) I + \sin(u) E,
\]

where \( E \) is an operator that is equal to 1 if (i) \( \lambda'_p = \lambda_{p+1} \) and \( \nu'_p = \nu_{p-1} \), or (ii) \( \lambda'_p = \lambda_{p-1} \) and \( \nu'_p = \nu_{p+1} \), and \( E \) vanishes otherwise. Putting it more directly: If \( \Lambda_{p-1} = (\lambda_{p-1}, \nu_{p-1}; q) \), \( \Lambda_p = (\lambda_{p-1}, \nu_p; q - m) \) and \( \Lambda_{p+1} = (\lambda_{p+1}, \nu_p; q + n - m) \) then \( X_p(u, q) \) is \( \sin(\gamma) \) or \( \sin(u) \) depending upon whether \( \Lambda'_p = (\lambda_{p-1}, \nu_p; q - m) \) or \( \Lambda'_p = (\lambda_{p+1}, \nu_{p-1}; q + n) \) respectively. If \( \Lambda_{p-1} = (\lambda_{p-1}, \nu_{p-1}; q) \), \( \Lambda_p = (\lambda_p, \nu_{p-1}; q + n) \) and \( \Lambda_{p+1} = (\lambda_p, \nu_{p+1}; q + n - m) \) then \( X_p(u, q) \) is \( \sin(\gamma) \) or \( \sin(u) \) depending upon whether \( \Lambda'_p \equiv (\lambda_p, \nu_{p-1}; q + n) \) or \( \Lambda'_p \equiv (\lambda_{p-1}, \nu_{p+1}; q - m) \) respectively.

Before concluding this section we think it important to make some remarks about spatial boundary conditions. In the lattice analogues of the \( G_{k,1} \) models the relation between lattice boundary conditions and sectors of the continuum limit is rather subtle. One should begin by noting that the vertex model must necessarily have free boundary conditions if it is to commute with \( U_q(G) \) \([27]\). If one now constructs the partition function by using the modified trace \([3,9]\) and taking \( Z_v = Tr(T^v) \), then the spatial and temporal boundary conditions are very different, and \( Z_v \) will not be modular invariant in the continuum limit.
It is of obvious interest to determine which combination of characters one gets from $Z_v$ in the continuum limit. A detailed analysis for $SU(2)$ can be found in [27]. The results suggest that in general $Z_v$ will be a particular combination of characters $\chi_{0,\Lambda}$, where the subscripts connote the affine labels of $G_k$ and $G_{k+1}$ in the $\mathcal{G}_{k,1}$ model. Intuitively one can understand this as follows. The choice of the value of $q$ in (3.8) suggests that the lattice quantum group should be identified with the screening charge $s$ obtained from the operators $e^{i\beta \alpha_j \cdot \phi}$, as opposed to the operators $e^{i\beta_{\Lambda} \alpha_j \cdot \phi}$. This is because the former operators have braiding relationships that involve the $(k+g+1)_{th}$ roots of unity whereas the latter have braiding relationships that involve the $(k+g)_{th}$ roots of unity. The physical states on the lattice are constructed from non-trivial representations of the screening algebra and so one should expect the same thing in the continuum limit. This suggests that the Hilbert spaces should be those built from the primary field $\Phi_{0,\Lambda}$ in (2.30). Putting it somewhat differently, if one considers the lattice states that remain after the quantum group truncation, the corresponding heights are precisely the affine highest weights of $G_{k+1}$ (and not $G_k$).

While $Z_v$ is a very particular combination of the abovementioned characters, one can also extract a particular one of these characters by restricting the modified trace to those states for which the last representation, $W_{2L}$, in the sequence described earlier is fixed to a given representation (with highest weight corresponding to an affine highest weights of $G_{k+1}$) [27]. One can also go beyond the restricted class of characters by imposing spatially periodic boundary conditions and making the appropriate modifications to the transfer matrix. The partition function then gets contributions from all sectors of the theory, but one loses the simple quantum group invariance of the transfer matrix. With a considerable amount of hard work, one can recover a more exotic form of the quantum group structure and use it to see how each sector is accounted for in the total partition function [27].

There are parallel situations in the IRF descriptions of these models. For example, if one fixes the heights on the top and bottom of the lattice then, in the continuum limit, one gets partition functions that enumerate the subclass of characters, $\chi_{0,\Lambda}$, mentioned above. On the other hand, if one merely wishes to construct a modular invariant partition function for the lattice model of interest it is elementary to accomplish this in the restricted height formulation. One does the obvious thing and uses the Boltzmann weights of the model on a toroidal lattice (i.e. one that is periodically identified in space and time). Since the restricted height model needs no further modifications or truncations, one need not introduce any factors or phases at the boundaries. In the continuum limit one will get the
diagonal modular invariant for $G_{k,1}$ or $G \times SO(dim(G/H))/H$ depending upon the choice of the restricted Boltzmann weights. The purpose of taking the circuitous route through vertex and unrestricted height models is that this approach gives us a computational method of deducing the Boltzmann weights in the restricted IRF model.

As a final comment on the subject of boundary conditions, we expect that the extraction of the separate sectors of (2.1) will be no more complicated than it is for the $G_{k,1}$ models. As was observed in the previous section, the only difference in the unrestricted height model is the insertion of an extremely simple phases (2.32) in the spatial direction of the soliton sum.

4. Lattice operators

Having seen how closely related the SLOHSS models are to the lattice models based upon $G$, it is easy to translate other lattice results to the $N = 2$ supersymmetric theory. Consider, for instance the order parameters [25]. For the $G_{k,1}$ models, the natural order parameters renormalize to vertex operators

$$\psi_{\Lambda}(z, \bar{z}) = e^{i\alpha_0 \Lambda \cdot \phi(z, \bar{z})}, \quad (4.1)$$

where

$$\alpha_0 = \frac{1}{\sqrt{(k + g)(k + g + 1)}} \quad (4.2)$$

is the charge at infinity, and $\Lambda$ is the highest weight label of $G$ at level $k + 1$. Since the $N = 2$ supersymmetric model can be written in the form (3.16), the order parameters of the lattice model will renormalize to a similar vertex operator to (4.1), but with $\Lambda$ constructed out of the correct combination of $H'$ weights and $U(1)$ charges. Now recall that in the Neveu-Schwarz sector, the $H$ labels that come from $SO(dim(G/H))$ are given by (3.17). One should recall that in equation (3.17) one has $w \in W(G)/W(H)$ and $\alpha \in M(G)/M(H)$. For operators of the form (1.1), the translation by $\alpha$ can be neglected since it represents a trivial automorphism generated by spectral flow [3]. It follows that the set of natural order parameters of the lattice model will renormalize to

$$e^{i w(\rho_G) - \rho_G \cdot \phi(z)}.$$

As was noted in section 2, these are precisely the chiral primary fields of the $N = 2$ theory.
There are also further operators of interest that arise directly from the quantum group structure. There are two generators, $X_{\pm \gamma}$, of the quantum group $G$ that are not generators of quantum $H'$. These generators obey the commutator

$$[X_{\gamma}, X_{-\gamma}] = \frac{q^{\gamma \cdot h} - q^{-\gamma \cdot h}}{q - q^{-1}},$$

(4.4)

where $\gamma$ is the simple root of $G$ that extends the simple root system of $H'$ to one for $G$. One should recall that the original $\hat{R}$ matrix not only commutes with $\Delta(X_{\pm \alpha_j})$ for all simple roots $\alpha_j$ of $G$, but also satisfies

$$\hat{R} (X_{\pm \psi} \otimes q^{\mp \frac{1}{2} \alpha \cdot h} + q^{-\frac{1}{2} \alpha \cdot h} \otimes (x_{\pm 2} X_{\pm \psi})) = ((x_{\pm 2} X_{\pm \psi}) \otimes q^{\mp \frac{1}{2} \alpha \cdot h} + q^{-\frac{1}{2} \alpha \cdot h} \otimes X_{\pm \psi}) \hat{R}.$$  

(4.5)

This means that the matrix $\hat{R}'$ satisfies a similar equation:

$$\hat{R}' (X_{\pm \alpha} \otimes q^{-\frac{1}{2} \alpha \cdot h} + q^{\frac{1}{2} \alpha \cdot h} \otimes (x_{\pm 1} X_{\pm \alpha})) = ((x_{\pm 1} X_{\pm \alpha}) \otimes q^{-\frac{1}{2} \alpha \cdot h} + q^{\frac{1}{2} \alpha \cdot h} \otimes X_{\pm \alpha}) \hat{R}'.$$  

(4.6)

for both $\alpha = \gamma$ and for $\alpha = -\psi$. Given the close relationship between the quantum group generators of the lattice model and the screening charges of the continuum theory, one would expect that these four affine quantum group generators, $X_{\pm \gamma}$ and $X_{\pm \psi}$, should, in the continuum limit, be identified with a subset of the vertex operators: $e^{i \beta \gamma \cdot \phi(z)}$, $e^{-i \frac{1}{\beta} \gamma \cdot \phi(z)}$, $e^{-i \beta \psi \cdot \phi(z)}$, $e^{i \frac{\alpha}{\beta} \psi \cdot \phi(z)}$, and their anti-holomorphic counterparts. To make the proper identification one should note that $X_{\gamma}$ and $X_{\psi}$ have the same $N = 2$, $U(1)$ charge and this charge is equal and opposite to that of $X_{-\gamma}$ and $X_{-\psi}$. This means that the continuum vertex operators should either all involve the coupling constant $\beta$ or should all involve $\frac{1}{\beta}$. The fact that $q^{k+g+1} = 1$ in the lattice model suggests that we should look for the same root of unity in braiding relations of the vertex operators. This then leads us to relate $X_{\gamma}$ and $X_{-\gamma}$ to $e^{i \beta \gamma \cdot \phi(z)}$ and $e^{i \beta \gamma \cdot \phi(\bar{z})}$ respectively, and $X_{-\psi}$ and $X_{\psi}$ to $e^{-i \beta \psi \cdot \phi(z)}$ and $e^{-i \beta \psi \cdot \phi(\bar{z})}$ respectively \footnote{One should remember that for supersymmetric perturbations of the conformal field theory, the non-trivial, off-critical conserved $U(1)$ charge is $Q = J_0 - \bar{J}_0$. This means that holomorphic and anti-holomorphic operators have an extra relative sign for their $U(1)$ charges.}. In the continuum field theory, these four operators correspond to $(G_{-\frac{1}{2}}^{-1} \Phi)(z)$, $(\bar{G}_{-\frac{1}{2}}^{-1} \Phi)(\bar{z})$, $(G_{+\frac{1}{2}}^{+} \Phi)(z)$ and $(\bar{G}_{+\frac{1}{2}}^{+} \Phi)(\bar{z})$, where $\Phi$ is the most relevant chiral primary field and $\bar{\Phi}$ is its anti-chiral conjugate.
The foregoing operators have extremely special properties. When used to perturb the
conformal model, they yield massive, integrable field theories \[8\][4][1]. These operators
also generate the perturbation that is needed to take the topological, twisted \(N = 2\)
superconformal model to the topological \(G_0 \times G_1/G_1\) model in such a way that the topological
correlation functions of \[37\] yield the fusion rules. It is interesting to note that the foregoing
operators also become a component of the supercurrent in these massive off-critical models.

The fact that the lattice analogues of the operators that give rise to integrable field
theories are precisely the operators that extend \(U_q(H')\) to the affine quantum group \(\widehat{U}_q(G)\)
suggests that the same might be true in the continuum. That is, if one uses a free field
description of the conformal model, then the screening prescription should combine with
the conformal perturbation theory to produce a realization of \(\widehat{U}_q(G)\). This phenomenon
has already been observed in the integrable field theories that appear as perturbations of
the non-supersymmetric minimal models \[38\]. For the vertex model analogues of these
non-supersymmetric models, the generators of \(U_q(\widehat{SU}(2))\) can be directly related to the
vertex operators of a screening current and of the integrable perturbation. Consequently,
the lattice affine quantum group gives us some further understanding of the results of \[38\],
and also leads to an intriguing prediction for the structure of the conformal perturbation
expansion in the perturbed SLOHSS models.

5. Conclusions

It is evident that one can find simple lattice formulations of \(N = 2\) supersymmetric
SLOHSS models by making a straightforward modification of the \(G_{k,1}\) models. We also
find it satisfying that the order parameters of the model become the chiral primary fields
become the Landau-Ginzburg fields of the \(N = 2\) superconformal model in the continuum
limit. The role played by the operators that extend \(U_q(H')\) to \(U_q(G)\) also provides a new
perspective on the continuum integrable field theories.

The technique that we have used here is basically to formulate a model using a group
\(G\), and then quantum group truncate with respect to a subgroup \(H\). This procedure
obviously admits generalizations (see, for example \[39\]). As in \[39\] the resulting theory
will probably only be unitary when \(G/H\) is a symmetric space \[6\]. On the other hand,

\[6\] Note that \(G/H\) does not need to be a hermitian symmetric space in order to produce a
unitary theory. A non-hermitian symmetric space yields an \(N = 1\) supersymmetric theory.
lack of unitarity has never seemed to be an impediment to finding physically interesting statistical mechanics models, and so there may well be some interesting non-unitary, as well as unitary, generalizations.

There are also several other clear directions for further research. It would be valuable to perform some Bethe Ansatz calculations to confirm that the lattice models described here do indeed yield the correct conformal weights for primary fields. It is also very probable that the fused vertex models, and their IRF counterparts, can be modified to obtain a formulation of the general $N = 2$ supersymmetric coset models

$$\frac{G_k \times SO(dim(G/H))}{H}.$$  \hfill (5.1)

It would be interesting to examine the details of how this works. In this paper we have also only examined the critical model, with Boltzmann weights tuned to the critical temperature. We next plan to obtain the off-critical Boltzmann weights for the lattice analogues of the $N = 2$ supersymmetric models. Among our aims in doing this is to use the corner transfer matrix methods to understand the critical model more precisely. We are also extremely curious to see whether we can find the lattice antecedents of the supersymmetry generators.

Finally, there is the question of the lattice analogues of the topological matter models that can be obtained from the coset models (2.1), and more generally, (5.1). One should recall that the first step in our construction of the new vertex models was to take the vertex model constructed from the $\tilde{R}$ matrix of $G$, and set $q = e^{i\pi/3} \ (i.e. \ put \ k = 0)$. The $U_q(G)$ quantum group structure is supposedly trivial for this value of $q$. It is however one of the lessons of the topological field theory that trivial representations often combine to make physically interesting theories. (Or to paraphrase Stanislaw Lem “Everybody knows that non-trivial representations do not exist, but each one does it in a different way.”) One finds that with this value of $q$, the only $G$ representations with non-vanishing $q$-dimension are those that correspond to representations of affine $G$ at of level one. When one passes to the restricted height model, the directed fusion graph \cite{26} is trivial since each representation of $G_1$ fuses with $V$ to yield an unique result. This means that all the lattice heights are fixed once one has chosen one of them. This virtually trivial lattice model is what we will refer to as the topological lattice model because, in the continuum limit, it is precisely the analogue of the perturbed, topologically twisted $N = 2$ superconformal theory that was discussed in \cite{6}. For the $\ell$-fused vertex model, or its IRF equivalent, the topological model
has $q = e^{\frac{i\pi}{\ell}}$ (i.e. $k = 0$). Once again, all the lattice heights are fixed by the choice of a single height somewhere on the lattice. This is because the directed fusion graph of the restricted height model is defined using a particular level $\ell$ representation, $V$, of $G$, and one can easily verify that the highest weight of $V$ defines a simple current of $G_\ell$ [10] [11]. This means that each representation of $G_\ell$ fuses with $V$ to generate an unique result. This rigid structure of the fusion graph of the topological lattice model leads us to expect that the $N$-point correlation functions of such a model will all be constants, and that the three point functions will reproduce the fusion algebra of $G_\ell$. In spite of the fact that this is intuitively very reasonable, it is still necessary to check the details, and perhaps more interesting, to see how such a topological sector embeds in the physical model.

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