RELATIVE STATE MEASURES OF CORRELATIONS IN BIPARTITE QUANTUM SYSTEMS

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Received (received date)
Revised (revised date)

Everett’s concept of relative state can be viewed as a map that contains information about correlations between measurement outcomes on two quantum systems. We demonstrate how geometric properties of the relative state map can be used to develop operationally well-defined measures of the total correlation in bipartite quantum systems of arbitrary state space dimension. These measures are invariant under local unitary transformations and non-increasing under local operations. We show that some known correlation measures have a natural interpretation in terms of relative states.

Keywords: Correlations; relative states

Communicated by: to be filled by the Editorial

1 Introduction

Ever since the formulation of the EPR argument [1], the predicted correlations between outcomes of localized quantum tests have been considered a distinctive and important feature of quantum mechanics, with bearings on both interpretative issues [2,3,4] and possible applications [5,6]. There are several open problems related to correlations within the quantum mechanical framework. The most important one is probably the qualitative question whether a given \(n\)-partite state is separable or entangled, i.e., if the correlations between the subsystems can be prepared by local operations and classical communication, or if global unitary evolution (or a source of shared entanglement) is required. Related to this question is its quantitative counterpart: how much classical correlation and entanglement does a quantum state contain? This question has resulted in proposed measures of correlation and entanglement, which can be divided roughly into two categories: one that focuses on the violation of Bell-CHSH type inequalities [7,8], and another that quantifies the ability of states to serve as a resource in some communication task, e.g., entanglement of formation [9] and distillable entanglement [10].
One of the contexts in which quantum correlations play a significant role is quantum measurement theory. The measurement process may be analyzed in terms of the correlations in a closed composite system consisting of a system of interest $S$ and an apparatus $A$. If we denote the basis states of $S$ and $A$ by $|s_i\rangle$ and $|a_i\rangle$, respectively, and if the former initially is in the superposition $\alpha|s_0\rangle + \beta|s_1\rangle$, then the measurement can be described in terms of a unitary evolution resulting in the transformation $(\alpha|s_0\rangle + \beta|s_1\rangle) \otimes |a_0\rangle \mapsto \alpha|s_0\rangle \otimes |a_0\rangle + \beta|s_1\rangle \otimes |a_1\rangle$. The entangled state of $S$ and $A$ corresponds to a superposition of the possible apparatus states, which seems to be in contradiction with the definite outcome presented by the apparatus.

Everett’s “relative state formulation of quantum mechanics” [11] provides a framework to deal with the $S+A$ correlation and circumvent the “measurement problem”. It introduces a natural “if - then” perspective, equivalent to that of conditional probabilities: if we observe the outcome $a_0$ ($a_1$) then the state of $S$ is $s_0$ ($s_1$), and, according to Everett, that is all there is to know. Mathematically, this may be understood as a map from the space of apparatus states to that of the system of interest. In this framework, the entangled state is a representation of the relation between the possible outcomes in one measurement to those of another. This makes the relative state formalism and the notion of conditional states potentially useful to study correlations encoded in quantum states, as shown in e.g. the context of entanglement [12] and steerability [13].

The purpose of this paper is to develop operationally well-defined correlation measures for arbitrary bipartite states by using certain geometric properties of the corresponding relative state map. For pure states, these measures coincide with known entanglement measures such as concurrence hierarchies [14] and $I$ concurrence [15]. We extend these pure state measures to arbitrary mixed bipartite systems for which we obtain measures that are invariant under local unitary transformations as well as non-increasing under local operations. On the other hand, these measures may increase under local operations and classical communication (LOCC), a feature that reflects the fact that they quantify the total correlation in mixed quantum states.

This paper is organized as follows. In the next section, we introduce the concept of relative states in Hilbert space and operator formalisms. While the former framework is restricted to pure bipartite states, the latter allows for an extension of the relative state description to arbitrary mixtures of bipartite states. In section 3 we demonstrate how to exploit the relative state concept to quantify correlations in bipartite quantum systems of arbitrary Hilbert space dimension. The correlation measures are illustrated in section 4. The paper ends with the conclusions.

2 Relative states

2.1 Hilbert space formalism

Let a bipartite system $S$ consisting of subsystems $A$ and $B$ be in a pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. Let $\dim \mathcal{H}_A = \dim \mathcal{H}_B = d$ and $|\psi\rangle = \sum_{ij}^{d} \alpha_{ij} |ij\rangle$, where $\{|ij\rangle\}$ is a product basis of the joint state space. Following Refs. [12, 16, 17], $|\psi\rangle$ defines the relative state map $L_\psi : \mathcal{H}_A \mapsto \mathcal{H}_B$. The relative state operator $L_\psi$, taking a state $|\phi\rangle \in \mathcal{H}_A$ to a state $|\phi\rangle \in \mathcal{H}_B$ according to

$$L_\psi |\phi\rangle = \langle \phi |\psi\rangle |\phi\rangle.$$

If $d_A = \dim \mathcal{H}_A < d_B = \dim \mathcal{H}_B$ there exists a Schmidt decomposition with a maximum of $d_A$ components, hence the state is effectively a $d_A \otimes d_A$. 


may be viewed as a partial scalar product, i.e., $\langle \varphi | \psi \rangle \equiv \langle (\varphi | \otimes \hat{1}_B) | \psi \rangle$. The relative state operator can be expressed as $L_\psi = \hat{\alpha} T$, with $\hat{\alpha} = \sum_{ij} \alpha_{ij} |i\rangle \langle j|$ and $T$ denotes complex conjugation in the $\{|k\}\rangle$ basis. The map $L_\psi$ is anti-linear, i.e.,

$$L_\psi(a|\varphi_1\rangle + b|\varphi_2\rangle) = a^*L_\psi|\varphi_1\rangle + b^*L_\psi|\varphi_2\rangle,$$

and becomes anti-unitary in the case of a maximally entangled $|\psi\rangle$. Furthermore, $L_\psi^\dagger : \mathcal{H}_B \rightarrow \mathcal{H}_A$ such that $L_\psi L_\psi^\dagger = \text{Tr}_A |\psi\rangle \langle \psi| = \rho_B$ and $L_\psi^\dagger L_\psi = \text{Tr}_B |\psi\rangle \langle \psi| = \rho_A$. The state $|\phi\rangle$ is subnormalized $\langle \phi | \phi \rangle = \langle \varphi | \rho_A | \varphi \rangle \leq 1$.

In the following, we refer to an argument $|\varphi\rangle$ of the relative state map as a \textit{hypo-state}, which can be understood as an actual post-measurement state of one of the subsystems. The conditional state $|\phi\rangle$ we call \textit{re-state}, short for a state relative to a hypo-state $|\varphi\rangle$.

The relative state map is a convenient way to express the fact that if Alice and Bob share the above pure bipartite state $|\psi\rangle$ and Alice chooses to measure an observable $Q$ with eigenstates $|\varphi_k\rangle$, then she can, when an outcome $k$ is obtained, predict the result of a specific projective measurement at Bob’s site. If the shared state is entangled with $d$ non-zero Schmidt coefficients, there will be a one-to-one correspondence between states of $A$ and $B$, and each such pair is understood as a \textit{relative state}.

2.2 \textit{Operator formalism}

A more general approach to relative states can be developed in terms of linear maps of operators acting on Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. This framework allows for mixed hypo-states that may arise in non-projective measurements on one of the parties of bipartite states.

Denote by $\mathcal{B}(\mathcal{H})$ the space of Hermitian operators with unit trace, and $\mathcal{S}(\mathcal{H})$ the non-negative cone of subnormalized density operators. A bipartite state $\varrho \in \mathcal{S}(\mathcal{H}_A \otimes \mathcal{H}_B)$ defines a map $\mathcal{L}_\varrho : \mathcal{B}(\mathcal{H}_A) \rightarrow \mathcal{B}(\mathcal{H}_B)$. If $Y \in \mathcal{B}(\mathcal{H}_A)$, then the map is given by

$$\mathcal{L}_\varrho(Y) = \text{Tr}_A \left[ (Y \otimes \hat{1}_B) \varrho \right] \in \mathcal{B}(\mathcal{H}_B).$$

For $Y$ being states $\tau$ of system $A$ (i.e., $\tau \in \mathcal{S}(\mathcal{H}_A)$), then $\mathcal{L}_\varrho : \mathcal{S}(\mathcal{H}_A) \rightarrow \mathcal{S}(\mathcal{H}_B)$ is the relative state map that takes hypo-states $\tau \geq 0$ on $\mathcal{H}_A$ to (subnormalized) re-states $\pi \geq 0$ on $\mathcal{H}_B$. The map $\mathcal{L}_\varrho$ is linear in the space of density operators, i.e., if $a, a'$ are real numbers and $\tau, \tau' \geq 0$, then

$$\mathcal{L}_\varrho(a\tau + a'\tau') = a\mathcal{L}_\varrho(\tau) + a'\mathcal{L}_\varrho(\tau').$$

The norm of the re-state is the probability of finding the hypo-state in the global state.

A mixed hypo-state $\tau$ can be understood as the post-measurement state resulting from a (non-unique) set of projections obtained with certain probabilities. Alternatively, one may interpret the relative state map $\mathcal{L}_\varrho : \mathcal{S}(\mathcal{H}_A) \rightarrow \mathcal{S}(\mathcal{H}_B)$ in terms of an outcome $E = V\sqrt{\tau}$ ($\tau \geq 0$ and $V$ unitary) of a local generalized measurements on the $A$ system, resulting in the post-measurement state $\tau = \text{Tr}_A (E \otimes \hat{1}_B E^\dagger \otimes \hat{1}_B) = \text{Tr}_A (\tau \otimes \hat{1}_B)$ of the $B$ system.

We can represent density operators and observables as elements of a real vector space $V$ and the relative state map can be expressed as a linear map of vectors. The corresponding vector elements can be interpreted as the expectation values of measured observables. Let
A bipartite state can be expressed as
\[ \varrho = \sum_{kl} M_{kl} K_A^k \otimes K_B^l, \] (6)
where
\[ M_{kl} = \text{Tr}[K_A^k \otimes K_B^l \varrho]. \] (7)

The matrix \( M \) is a representation of \( \varrho \) with respect to the chosen basis. Local states \( \tau, \pi \) are represented by real-valued vectors \( a \in \mathcal{V}(H_A) \) and \( b \in \mathcal{V}(H_B) \) with elements
\[ a_k = \text{Tr}[K_A^k \tau], \quad b_l = \text{Tr}[K_B^l \pi]. \] (8)

We can express the map in Eq. (3) for \( Y = \tau \) as
\[ \tau = \sum_j a_j K_A^j \mapsto \pi = M_{\varrho} \left( \sum_j a_j K_A^j \right), \]
\[ = \sum_{jkl} a_j M_{kl} \text{Tr}[K_A^k K_A^j] K_B^l = \sum_{kl} a_k M_{kl} K_B^l. \] (9)
Hence, \( b_l = \sum_k a_k M_{kl} \) or equivalently \( b = M^T a \). The relative state map is represented by
\[ M^T : \mathcal{V}(H_A) \mapsto \mathcal{V}(H_B) \] (10)
and conversely
\[ M : \mathcal{V}(H_B) \mapsto \mathcal{V}(H_A). \] (11)

3 Correlations
A probability distribution \( P(X,Y) \) over two random variables \( X,Y \) taking values \( x_i, y_j \) is correlated if \( P(X,Y) \neq P(X)P(Y) \), where \( P(X) \) and \( P(Y) \) are the marginal distributions of \( P(X,Y) \). The above condition can also be stated in terms of conditional probabilities: if there exist a pair \( (i, j \neq i) \) such that
\[ P(X|Y = y_i) \neq P(X|Y = y_j), \] (12)
where \( P(X|Y = y_i) \) denotes the probability distribution over \( X \) given the outcome \( Y = y_i \), then \( P(X,Y) \) is correlated. The condition Eq. (12) states that a probability distribution is correlated if information about an outcome of \( Y \) alters the prediction about the outcome of \( X \) (and vice versa). Thus, a way to characterize the correlation in a probability distribution is to compare the set of conditional probabilities given an exhaustive set of mutually exclusive conditionals \( \{y_i\} \), since the set of conditional probabilities contains information about how the
random variables are correlated, i.e., which outcomes $x_i$ are correlated with which outcomes $y_j$.

A quantum state $\rho \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$ is correlated if $\rho \neq \rho_A \otimes \rho_B$, where $\rho_A$ and $\rho_B$ are the reduced states of the $A$ and $B$ subsystems, respectively. The relative state formalism allows us to employ Eq. (12) in a quantum context: a bipartite state $\rho$ is correlated if there exists a pair of Hermitean operators $Y' \neq Y$ such that

$$\text{Tr}_A(Y' \otimes \hat{1}_B \rho) \neq \lambda \text{Tr}_A(Y \otimes \hat{1}_B \rho)$$

for any real number $\lambda$. Here, the conditional probabilities are replaced by relative states. The basic idea of the following analysis is to measure correlations in terms of how much the re-states differ for different choices of hypo-states. Specifically, the aim is to quantify correlations in terms of the difference of the conditional predictions contained in the re-states. In this way, we demonstrate how the geometrical properties of the relative state map can be used to develop correlation measures in arbitrary bipartite quantum systems.

### 3.1 Pure state correlation measures

Let us consider the bipartite pure product state $|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle$ and the corresponding relative state map $L_\psi$. For any hypo-state $|\varphi\rangle \in \mathcal{H}_A$, the re-state is

$$|\phi\rangle = \langle \varphi | \psi \rangle = \langle \varphi | \psi_A \rangle |\psi_B\rangle,$$

i.e., $L_\psi$ maps the whole Hilbert space $\mathcal{H}_A$ to the same ray in $\mathcal{H}_B$, that is, to the same state. This expresses the fact that for an uncorrelated state $|\psi_A\rangle \otimes |\psi_B\rangle$, a measurement outcome at site $A$ does not change the predictions about measurements at site $B$. Now, consider instead an entangled two-qubit state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$. If we choose two hypo-states $|\varphi\rangle, |\varphi'\rangle$ such that the corresponding re-states $|\phi\rangle, |\phi'\rangle$ are non-zero, then $|\varphi\rangle \neq |\varphi'\rangle$ implies that $|\phi\rangle \neq z|\phi'\rangle$, i.e., a measurement outcome at site $A$ does change the predictions regarding measurements at site $B$, as illustrated in Fig. 1.

![Fig. 1. An illustration of the correlation measure for a 2⊗2 system in pure state $|\psi\rangle = c_0|00\rangle + c_1|11\rangle$. The picture shows the real planes in $\mathcal{H}_A$ and $\mathcal{H}_B$ spanned by the local Schmidt bases. Two different choices of orthonormal hypo-states in $\mathcal{H}_A$, $|\varphi_i\rangle$ and $|\tilde{\varphi}_i\rangle$, maps to their respective re-states $|\phi_i\rangle$ and $|\tilde{\phi}_i\rangle$ in $\mathcal{H}_B$. The areas spanned by the restates are shown in gray.](image-url)

By using these properties of the relative state map we may develop measures that quantify bipartite correlations. These measures are based upon geometric properties of the wedge.
product, \( \wedge \) defined as follows. Let \( \{ |j\rangle \}_{j=1}^d \) be an orthonormal basis of a \( d \) dimensional Hilbert space \( \mathcal{H} \). Consider the vectors \( |\xi_i\rangle = \sum_{j=1}^d \eta_j^{(i)} |j\rangle \), \( i = 1, \ldots, k \leq d \). We define the \( k \)-product of these vectors as

\[
|\xi_1 \wedge \cdots \wedge \xi_k\rangle \sim |\xi_1\rangle \wedge \cdots \wedge |\xi_k\rangle \equiv \sum_{1 \leq \mu_1 < \cdots < \mu_k \leq d} \sum_{j_1, \ldots, j_k} \epsilon^{\mu_1 \cdots \mu_k}_{j_1 \cdots j_k} \eta_{j_1}^{(1)} \cdots \eta_{j_k}^{(k)} |\mu_1 \cdots \mu_k\rangle,
\]

(15)

where \( \epsilon^{\mu_1 \cdots \mu_k}_{j_1 \cdots j_k} \) is the Levi-Civita tensor, defined as \( \epsilon^{\mu_1 \cdots \mu_k}_{j_1 \cdots j_k} = +1 \) \((-1)\) if \( j_1 \cdots j_k \) is an even \(\) odd permutation of \( \mu_1 \ldots \mu_k \) \ and zero otherwise. Note in particular that the \( k \)-product vanishes if \( \xi_i \) are linearly dependent. \( |\xi_1 \wedge \cdots \wedge \xi_k\rangle \) is an element of the exterior space \( \Omega^k(\mathcal{H}) \) with norm

\[
|\xi_1 \wedge \cdots \wedge \xi_k\rangle^2 \equiv \sum_{1 \leq \mu_1 < \cdots < \mu_k \leq d} \sum_{j_1, \ldots, j_k} \epsilon^{\mu_1 \cdots \mu_k}_{j_1 \cdots j_k} \eta_{j_1}^{(1)} \cdots \eta_{j_k}^{(k)}|^2.
\]

(16)

For a set of vectors \( \{ \psi_i \} \) in a real three dimensional vector space \( \mathcal{V}^3 \), the two-fold wedge product \( \psi_{ij} = \psi_i \wedge \psi_j \in \Omega^2(\mathcal{V}^3) \) can be identified with the directed surface element spanned by the two vectors, with area \( |\psi_{ij}| \). Correspondingly, the three-fold wedge product \( \psi_{ijk} = \psi_i \wedge \psi_j \wedge \psi_k \in \Omega^3(\mathcal{V}^3) \) represents a directed volume element, with volume \( |\psi_{ijk}| \) (see Fig. 2 for an illustration). This geometric intuition carries over to complex higher dimensional spaces; the \( k \)-fold wedge product \( \xi_{i_1 \cdots i_k} = \xi_{i_1} \wedge \cdots \wedge \xi_{i_k} \) can be seen as the oriented \( k \)-dimensional rhomboid spanned by the vectors, with \( k \)-volume \( |\xi_{i_1 \cdots i_k}| \).

Given a general bipartite system prepared in the pure state \( |\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B \), where we assume that \( \dim \mathcal{H}_A = \dim \mathcal{H}_B = d \), a set of hypo-states \( \{ |\phi_i\rangle \}_{i=1}^d \), \( |\phi_i\rangle \in \mathcal{H}_A \), is chosen such that \( \text{Sp}\{ |\phi_i\rangle \} \cong \mathcal{H}_A \). We obtain a set of re-states \( \{ |\phi_i\rangle \}_{i=1}^d \), \( |\phi_i\rangle \in \mathcal{H}_B \), via \( |\phi_i\rangle = L_\psi |\phi_i\rangle = \langle \phi_i |\psi\rangle \). Our basic measure of correlation with respect to any \( k \)-tuple of hypo-states \( \{ |\varphi_{i_1}\rangle, \ldots, |\varphi_{i_k}\rangle \} \subseteq \{ |\phi_i\rangle \}_{i=1}^d \) is given by

\[
\lambda_{i_1 \cdots i_k} = \frac{|\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}|}{|\phi_{i_1} \wedge \cdots \wedge \varphi_{i_k}|}.
\]

(17)

We may interpret \( \lambda_{i_1 \cdots i_k} \) as follows. Each \( k \)-tuple \( \{ |\varphi_{i_1}\rangle, \ldots, |\varphi_{i_k}\rangle \} \subseteq \{ |\varphi_i\rangle \}_{i=1}^d \) of hypo-states is a basis of a \( k \)-dimensional subspace \( \mathcal{H}_{A^{i_1 \cdots i_k}} \subseteq \mathcal{H}_A \), and we will call \( \lambda_{i_1 \cdots i_k} \) a measure of the \( k \)-level correlation between that subspace and subsystem \( B \).

To make this notion clearer, consider a \( 3 \times 3 \) system in the state \( |\psi\rangle = \sum_{i=1}^3 \sqrt{p_i} |i\rangle \), and a choice of hypo-states as \( |\varphi_i\rangle = |i\rangle \), with the corresponding re-states given by \( |\phi_i\rangle = L_\psi |\phi_i\rangle = \sqrt{p_i} |i\rangle \). By Eq. (17), we have three quantities for the two-level correlations \( \lambda_{ij} = |\phi_i \wedge \phi_j| = \sqrt{p_i p_j} \), \( i < j \), and one for the three-level correlation \( \lambda_{123} = |\phi_1 \wedge \phi_2 \wedge \phi_3| = \sqrt{p_1 p_2 p_3} \), see Fig. 2 where a similar example with a different choice of hypo-states is illustrated. The quantity \( \lambda_{13} = \sqrt{p_1 p_3} \) quantifies the difference between the restates \( |\phi_1\rangle, |\phi_3\rangle \), and hence corresponds to how much our predictions about measurements on system \( B \) differs with the two post-measurement states \( |\varphi_1\rangle, |\varphi_3\rangle \) of \( A \), i.e., when the outcome corresponding to \( |\phi_3\rangle \) is discarded. Equivalently, \( \lambda_{13} \) measures the effective \( 2 \otimes 2 \) entanglement in the state \( |\psi_1\rangle = (|1\rangle |1\rangle + |3\rangle |3\rangle) \otimes (|1\rangle |1\rangle + |3\rangle |3\rangle) |\psi\rangle \) resulting from a projection onto the subspace \( \mathcal{H}_{A}^3 \otimes \mathcal{H}_{B}^3 \). The three-level quantity \( \lambda_{123} \) measures the volume spanned by the re-states, i.e., how much the predictions differ when all three post-measurement states \( |\varphi_i\rangle \) are taken into
account. On the other hand, if \( p_3 = 0 \), then \( \lambda_{13} = \lambda_{23} = \lambda_{123} = 0 \), where \( \lambda_{13} = \lambda_{23} = 0 \) reflects that \( |\psi\rangle = \sqrt{\frac{2}{3}} |11\rangle \) is a product state (the subspace \( \mathcal{H}_{123} \) is not correlated with \( B \)), and \( \lambda_{123} = 0 \) means that there exist no correlations that is not two-level. As is shown in Fig.

**Fig. 2.** An illustration of the correlation measures \( \lambda_{ij} , \lambda_{123} \) for a \( 3 \times 3 \) system in two pure states with different Schmidt-number, \( |\psi\rangle = \sum_{k=1}^{2} \tilde{c}_k |kk\rangle \) and \( |\tilde{\psi}\rangle = \sum_{k=1}^{3} \tilde{c}_k |kk\rangle \). In a) a choice of hypostates \( |\psi_i\rangle \in \mathcal{H}_A \) is depicted, and b) shows the re-states \( L_\psi |\phi_i\rangle = |\phi_i\rangle \in \mathcal{H}_B \), which span the volume \( \lambda_3 = |\phi_1 \wedge \phi_2 \wedge \phi_3| \) taken as the measure of three-level correlations. The areas of the faces of the rhomboid are given by \( \lambda_12 = |\phi_1 \wedge \phi_2| \), \( \lambda_13 = |\phi_1 \wedge \phi_3| \) and \( \lambda_{23} = |\phi_2 \wedge \phi_3| \), and they are measures of the two-level correlations betwen the respective two-dimensional subspaces. In c) the re-states \( L_\tilde{\psi} |\varphi_1\rangle = |\tilde{\phi}_1\rangle \) are shown, which lies in the subspace (shown in gray) spanned by the Schmidt vectors \( |1\rangle, |2\rangle \). Consequently, for \( |\tilde{\psi}\rangle \), the three-level correlations are \( \lambda_{123} = |\tilde{\phi}_1 \wedge \tilde{\phi}_2 \wedge \tilde{\phi}_3| = 0 \), whereas \( \lambda_{ij} \neq 0 \), i.e., \( |\tilde{\psi}\rangle \) only contains two-level correlations.

The quantity \( \lambda_{i_1...i_k} \) is independent of the choice of hypo-states as long as \( \text{Sp} \{ |\varphi_{i_1}\rangle, \ldots, |\varphi_{i_k}\rangle \} \cong \mathcal{H}_{i_1...i_k} \). To see this, let \( \{ |\varphi_{i_1}\rangle \}_{k-1} \) form an orthonormal basis of \( \mathcal{H}_{i_1...i_k} \) and define another set \( \{ |\varphi'_{i_1}\rangle \}_{i=1}^{K} \) of arbitrary basis vectors via

\[
|\varphi'_{i_1}\rangle = \sum_m c_{1m} |\varphi_{i_m}\rangle ,
\]

where \( c_{1m} \) are elements of a complex-valued invertible \( k \times k \) matrix. Define the corresponding set of re-states \( \{ |\varphi'_{i_1}\rangle \}_{i=1}^{k} \) as \( |\varphi'_{i_1}\rangle = L_\psi |\varphi_{i_1}\rangle \). By the anti-linearity of the relative state map, we obtain

\[
|\varphi'_{i_1}\rangle = L_\psi \left( \sum_m c_{1m} |\varphi_{i_m}\rangle \right) = \sum_m c_{1m} |\varphi_{i_m}\rangle .
\]

Explicit evaluation of the wedge product of the non-orthogonal basis elements yields

\[
\varphi'_{i_1} \wedge \cdots \wedge \varphi'_{i_k} = \left( \sum_{j_1...j_k} \epsilon_{j_1...j_k} e_{1j_1} \cdots e_{kj_k} \right) \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} ,
\]

where \( \epsilon_{i_1...i_d} \) denotes the Levi-Civita tensor. By performing the corresponding expansion of
the set of primed re-states $\phi'_i$, we obtain

$$
\phi'_i \wedge \cdots \wedge \phi'_k = \left( \sum_{j_1 \cdots j_k} \epsilon_{j_1 \cdots j_k} c_{i_1 j_1} \cdots c_{i_k j_k} \right) \phi_i \wedge \cdots \wedge \phi_k,
$$

which is essentially the same expression as in Eq. (20) up to a complex conjugation of the coefficient $c_{lm}$. Thus, we conclude that

$$
|\phi_i \wedge \cdots \wedge \phi_k| = |\phi'_i \wedge \cdots \wedge \phi'_k|.
$$

(22)

To simplify the notation, we henceforth assume that a set of hypo-states form an orthogonal basis and thus omit the denominator in Eq. (17).

The quantities $\lambda_{i_1 \cdots i_k}$ have the following properties. If the re-states $(\phi_1, \ldots, \phi_k)$ are linearly dependent, then $\lambda_{i_1 \cdots i_k} = 0$, which reflects that one can find two hypo-states in $\mathcal{H}_{i_1 \cdots i_k}$ that maps to the same ray in $\mathcal{H}_B$. Furthermore, $\lambda_{i_1 \cdots i_k}$ vanishes if $|\psi\rangle$ lacks support in some part of the subspace spanned by the hypo-states (one or several re-states will have zero norm). On the other hand, max $\lambda_{i_1 \cdots i_k} = (1/\sqrt{k})^k$ and this value is saturated if $|\psi\rangle$ is maximally entangled on $\mathcal{H}_{i_1 \cdots i_k}$ and $|\varphi_i\rangle$, $i \in (i_1 \ldots i_k)$ span this subspace.

The quantities $\lambda_{i_1 \cdots i_k}$ are in general not invariant under local unitaries. To see this, consider the local unitary transformation $|\psi\rangle \mapsto |\psi'\rangle = U_A \otimes 1_B |\psi\rangle$, which implies that $|\phi_i\rangle \mapsto |\phi'_i\rangle = \langle \varphi_i | U_A \otimes 1_B |\psi\rangle$. In other words, the transformed re-states would correspond to a set of hypo-states $|\varphi'\rangle = U_A^\dagger |\varphi_i\rangle$, defining a different subspace decomposition of $\mathcal{H}_A$ leading to that $\lambda_{i_1 \cdots i_k}$ may change. (The exception is $\lambda_d = \phi_1 \wedge \cdots \wedge \phi_d$ that contains all re-states.) However, we have seen that the different choices of orthonormal bases of hypo-states are equivalent with local unitary transformations (on subsystem A) of the global state, and hence the question of invariance under change of hypo-states are equivalent to that of invariance under local unitary transformations. We now define

$$
\Lambda_k^2 = d^k \binom{d}{k}^{-1} \sum_{i_1 < \cdots < i_k} \lambda_{i_1 \cdots i_k}^2,
$$

(23)

where the sum is over all unique $k$-tuples of re-states and the normalization factor on the right-hand side is chosen so that $\Lambda_k = 1$ for all $k$ if the global state is maximally entangled.

**Theorem.** For a $d \times d$-dimensional bipartite system, the members of the set $\{\Lambda_k\}_{k=1}^d$ are invariant under local unitary transformations.

**Proof.** Let $|\psi\rangle = \sum_{i=1}^d \sqrt{p_i} |ii\rangle$ be the bipartite state on Schmidt form. We can make a choice of hypo-states such that $|\varphi_i\rangle = |i\rangle$, with the corresponding re-states $|\phi_i\rangle = L_\psi |\varphi_i\rangle = \sqrt{p_i} |i\rangle$. First, we consider a local unitary on subsystem B, i.e.,

$$
|\psi\rangle \mapsto |\tilde{\psi}\rangle = \tilde{1}_A \otimes U_B |\psi\rangle = \sum_{i=1}^d \sqrt{p_i} |i\rangle \otimes U_B |i\rangle,
$$

(24)

from which we see that the re-states transform according to $|\phi_i\rangle \mapsto |\tilde{\phi}_i\rangle = U_B |\phi_i\rangle$. The corresponding transformation of the $k$-vectors then reads

$$
\phi_i \wedge \cdots \wedge \phi_i \mapsto \tilde{\phi}_i \wedge \cdots \wedge \tilde{\phi}_k = U_B^\otimes k (\phi_i \wedge \cdots \wedge \phi_k),
$$

(25)
which means that the unitary $U_B$ on $\mathcal{H}_B$ induces a unitary $U_B^{\otimes k}$ on the exterior space $\Omega^k(\mathcal{H}_B)$. Clearly, this cannot change the norm of the $k$-vector, since

$$\tilde{\Lambda}_{i_1...i_k} = (\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}) (U_B^{\otimes k})\dag (U_B^{\otimes k}) (\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}) = |\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}|^2 = \lambda_{i_1...i_k},$$

and thus we have that $\tilde{\Lambda}_k = \Lambda_k$ under $U_B$.

Consider now a local unitary on subsystem $A$, i.e., let $|\psi\rangle$ be defined as before but let $|\psi\rangle \mapsto |\tilde{\psi}\rangle = U_A \otimes 1_B |\psi\rangle$. In this case, the re-states transform as

$$|\phi_i\rangle \mapsto |\tilde{\phi}_i\rangle = L_\psi|\varphi_i\rangle = \langle \varphi_i|U_A \otimes 1_B|\psi\rangle = L_\psi U_A^\dag|\varphi_i\rangle = L_\psi|\tilde{\varphi}_i\rangle,$$

i.e., a local unitary on subsystem $A$ is equivalent to the inverse transformation of the hypo-states. If we denote $(U_A^\dag)_{ij} = u_{ij}$, the transformed hypo-states are related to the original ones according to $|\tilde{\varphi}_j\rangle = \sum_i u_{ij}|\varphi_i\rangle$, and we have that $|\tilde{\phi}_i\rangle = L_\psi|\tilde{\varphi}_i\rangle = \sum_j u_{ij}|\varphi_j\rangle$.

To show that $\Lambda_k = \Lambda_k$, we first note that the $U_A$ induces a corresponding transformation of the $k$-vectors

$$\phi_{j_1} \wedge \cdots \wedge \phi_{j_k} \mapsto \tilde{\phi}_{j_1} \wedge \cdots \wedge \tilde{\phi}_{j_k} = \left(\sum_i u_{i_1j_1}^* \phi_i\right) \wedge \cdots \wedge \left(\sum_i u_{i_kj_k}^* \phi_i\right),$$

and summing the squared norms of the new $k$-vectors, we get

$$\sum_{1 \leq j_1 < \cdots < j_k \leq d} |\tilde{\phi}_{j_1} \wedge \cdots \wedge \tilde{\phi}_{j_k}|^2 = \sum_{1 \leq j_1 < \cdots < j_k \leq d} u_{i_1j_1}^* u_{m_1j_1} \cdots u_{i_kj_k}^* u_{m_kj_k} |\phi_{\mu_1} \wedge \cdots \wedge \phi_{\mu_k}|^2.$$

Here, we have used that the set $\{\phi_{\mu_1} \wedge \cdots \wedge \phi_{\mu_k}\}_{1 \leq \mu_1 < \cdots < \mu_k \leq d}$ is an orthogonal basis of $\Omega^k(\mathcal{H}_B)$, which follows from the orthogonality of the Schmidt-basis and that $|\varphi_i\rangle = \sqrt{P_i}|i\rangle$.

Now, to see that the factors labeled by $\mu_1, \mu_2$, each sum up to one as required, first note that the rows of a unitary is an orthonormal set of vectors, i.e., we have that $\sum_j u_{\mu_jj}^* u_{\mu_jj} = \delta_{im}$.

The determinant of the identity can then be expanded according to

$$1 = \det \delta_{im} = \frac{1}{k!} \sum_{j_1...j_k} \epsilon_{j_1...j_k} \epsilon_{m_1...m_k} u_{\mu_1j_1}^* u_{\mu_mj_1} \cdots u_{\mu_kj_k}^* u_{\mu_mj_k},$$

where we have used the definition of the Levi-Civita tensor and that we can restrict the sums over $j_1...j_k$.

The $k = 1$ invariant is just normalization and does not provide any information about the correlation between the subsystems. Therefore, we take the correlation measures to consist of the set $\{\Lambda_k\}_{k=2}^d$. □
Note that the $k$-level invariants are not independent since $\Lambda_k \neq 0$ implies that $\Lambda_l \neq 0$ for all $l < k$. Geometrically, this expresses the fact that a non-zero volume must be bounded by non-zero areas. More explicitly, the lower order invariants are related to $\Lambda_d$ as

$$\Lambda_k \geq \left( \frac{d}{k} \right)^{1/2} \left( \Lambda_d \right)^{k/d}, \quad d > k,$$

which gives a lower bound for $k$th order invariant.

We may relate the $\Lambda_k$’s to known entanglement measures by using the Schmidt form $|\psi\rangle = \sum_{k=1}^{d} \sqrt{p_k} |\varphi_k\rangle \otimes |\phi_k\rangle$, where $\langle \varphi_k | \varphi_l \rangle = \delta_{kl}$ and $\langle \phi_k | \phi_l \rangle = \delta_{kl}$, such that $L_\psi |\varphi_k\rangle = \sqrt{p_k} |\phi_k\rangle$.

Since the re-states are subnormalized, mutually orthogonal vectors, it follows that

$$\Lambda_k^2 = \sum_{i_1 < \ldots < i_k} |\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}|^2 = \sum_{i_1 < \ldots < i_k} p_{i_1} \cdots p_{i_k}. \quad (32)$$

Hence, the invariants are equivalent to the symmetric polynomials in the Schmidt coefficients, i.e., the concurrence hierarchies proposed in Ref. [14]. The pure state invariant $\Lambda_2$ is recognized as the $I$ concurrence [15]

$$C_1^2 = 4 \sum_{i < j} p_i p_j, \quad (33)$$

up to a factor. For $d = 2$ (qubit) systems, $\Lambda_2$ is the only non-trivial invariant and equals half the pure state concurrence [18].

The relative state approach may further be used to give the following alternative geometric interpretation of pure state concurrence for qubit systems. Let $|\Psi\rangle = \sqrt{p_0} |00\rangle + \sqrt{p_1} |11\rangle$ and consider the orthonormal hypo-states $|\varphi_0\rangle = \alpha |0\rangle + \beta |1\rangle$ and $|\varphi_1\rangle = -\beta^* |0\rangle + \alpha^* |1\rangle$ with complex-valued $\alpha$ and $\beta$ such that $|\alpha|^2 + |\beta|^2 = 1$. The corresponding re-states read $|\phi_0\rangle = \sqrt{p_0 \alpha^*} |0\rangle + \sqrt{p_1 \beta^*} |1\rangle \sim a = (\sqrt{p_0 \alpha^*}, \sqrt{p_1 \beta^*})$ and $|\phi_1\rangle = \sqrt{p_0 \beta^*} |0\rangle + \sqrt{p_1 \alpha^*} |1\rangle \sim b = (\sqrt{p_0 \beta^*}, \sqrt{p_1 \alpha^*})$. The area $A$ spanned by $a$ and $b$ is

$$A = \sqrt{|a|^2 |b|^2 - |a^* \cdot b|^2} = \sqrt{p_0 p_1}, \quad (34)$$

which is half the pure state concurrence of the two-qubit state $\psi$. Thus, concurrence is essentially the area spanned by two re-states, as is shown in Fig. [1].

### 3.2 Mixed state correlation measures

Let $\varrho \in S(\mathcal{H}_A \otimes \mathcal{H}_B)$ be a bipartite state and assume that $d = \dim \mathcal{H}_A \leq \dim \mathcal{H}_B$. Let $\{\tau_i\}_{i=1}^{d^2}$, $\tau_i \in B(\mathcal{H}_A)$, be a set of Hermitian operators on $\mathcal{H}_A$ such that $\text{Sp} \{\tau_i\} \cong B(\mathcal{H}_A)$ and define the corresponding set of operators $\{\pi_i\}_{i=1}^{d^2}$, $\pi_i \in S'(\mathcal{H}_B)$, as

$$\pi_i = L_\varrho (\tau_i) = \text{Tr}_A \left[ \tau_i \otimes \hat{1}_B \right]. \quad (35)$$

The basic correlation measures now read

$$v_{i_1 \ldots i_k} = \frac{|\pi_{i_1} \wedge \cdots \wedge \pi_{i_k}|}{|\tau_{i_1} \wedge \cdots \wedge \tau_{i_k}|}. \quad (36)$$

In analogy with the pure state case, these measures are independent of choice of $\{\tau_i\}$. In particular, if $\{\tau_i\}$ is an orthogonal set the denominator in Eq. (36) can be omitted. Note,
however, the operational interpretation of \( \{ \tau_i \} \) and \( \{ \pi_i \} \) as states cannot be maintained for such a choice, since the space of density operators cannot be equipped with a complete orthogonal basis of positive operators. Nonetheless, due to the independence of the choice of \( \{ \tau_i \} \), we refer to \( \{ \tau_i \} \) and \( \{ \pi_i \} \) as hypo-states and re-states in the following, regardless of whether all members of the sets represent valid states or not.

To evaluate the wedge product, it is convenient to move to the Hilbert-Schmidt representation of states and observables as real-valued vectors and matrices. Thus, we make the substitutions \( \rho \rightarrow M, \{ \tau_i \} \rightarrow \{ a_i \}, \) and \( \{ \pi_i \} \rightarrow \{ b_i \} \), where \( a_i \) and \( b_i \) are related via the linear map \( a_i \rightarrow b_i = M^t a_i \). Then

\[
\rho_{i_1} \wedge \cdots \wedge \rho_{i_k} \rightarrow b_{i_1} \wedge \cdots \wedge b_{i_k}.
\]

(37)

To illustrate this substitution, let us consider the case of a product state \( \rho = \rho_A \otimes \rho_B \). We find \( M = r_A r_B^T \), where \( r_{A;i} = \text{Tr} [K^A_i \rho_A] \) and \( r_{B;i} = \text{Tr} [K^B_i \rho_B] \) for some local operator bases \( \{ K^A_i \} \) and \( \{ K^B_i \} \). Hence, for a product state, the relative state map takes any \( a \in \mathcal{V}_A \) to a vector proportional to \( r_B \): \( M^t a = r_B (r_A^T \cdot a) \), which implies \( (M^t a_1) \wedge \cdots \wedge (M^t a_k) = 0 \) for any \( k \)-tuple \( (a_1, \ldots, a_k) \in \mathcal{V}_A \).

We now define the correlation measures

\[
\Upsilon_k^2 = d^2 \binom{d^2}{k}^{-1} \sum_{i_1 < \cdots < i_k} \nu_{i_1 \ldots i_k}^2,
\]

(38)

where the normalization factor is chosen such that \( \Upsilon_k = 1 \) for maximally entangled states.

To demonstrate that \( \Upsilon_k \) are invariant under local unitary operations, we first need to define the corresponding transformation in the Hilbert-Schmidt representation. Let \( \rho \) be a state and let \( r \) be the Hilbert-Schmidt representation of the state given by \( r_i = \text{Tr} [K_i \rho] \). Furthermore, define \( \rho' = U \rho U^t \) where \( U \) is an arbitrary unitary transformation. Then

\[
r'_i = \text{Tr} [K_i \rho'] = \text{Tr} [U^t K_i U \rho],
\]

(39)

and thus the transformation of \( \rho \) corresponds to the inverse transformation of the basis elements \( K'_i = U^t K_i U \). From the orthonormality of \( \{ K_i \} \) we see that \( \text{Tr} [K'_i K'_j] = \text{Tr} [U^t K_i U U^t K_j U] = \delta_{ij}, \) i.e., \( \{ K'_i \} \) is also an orthonormal basis. The transformation of \( r \) is given by the orthogonal transformation

\[
r' = Rr, \quad R_{ij} = \text{Tr} [K_i K'_j].
\]

(40)

Since \( U \) is continuously connected to the identity, \( R \) is too, and hence the transformation is a rotation. The transformation \( U \) is also trace-preserving, which implies that \( R \) is restricted to act on a \( d^2 - 1 \) dimensional subspace of \( \mathcal{V} \), namely the plane orthogonal to the identity vector \( v_I \) with elements \( (v_I)_i = \text{Tr} [K_i] \).

A local unitary transformation of a bipartite state

\[
\rho \rightarrow \rho' = U_A \otimes U_B \rho U_A^t \otimes U_B^t
\]

(41)

induces the transformation \( M \rightarrow M' = R_B M R_A^t \), where

\[
(R_A)_{ij} = \text{Tr} [U_A^t K_i^A U_A K_j^A], \quad (R_B)_{kl} = \text{Tr} [U_B^t K_k^B U_B K_l^B].
\]

(42)
To see that the $\Upsilon_k$’s are invariant under such transformations, it suffices to note that the above proof of the invariance of the pure state quantities $\Lambda_k$ under unitary transformations, immediately goes through for local rotations of the real vectors $a_i$ and $b_i$ representing the hypo- and re-states, respectively.

For any bipartite $\rho$, there exists a unique Schmidt form

$$\rho = \sum_i \kappa_i \tilde{K}_A^i \otimes \tilde{K}_B^i, \quad (43)$$

where the Hermitian $\{\tilde{K}_A^i\}$ and $\{\tilde{K}_B^i\}$ are the particular orthonormal bases of operators on $\mathcal{H}_A$ and $\mathcal{H}_B$ – corresponding to the pure state Schmidt-bases – and the real numbers $\{\kappa_i\}$ are singular values of $M_{kl} = \text{Tr}(K_A^k \otimes K_B^l \rho)$. In analogy with the pure state case, the invariants $\Upsilon_k^2$ can be seen to be equivalent to the symmetric polynomials in $\kappa_i^2$, a form of correlation measures similar to those proposed in Ref. [19].

Since classical communication can increase correlations, it follows that $\Upsilon_k$ may increase under LOCC. However, as the following theorem shows, $\Upsilon_k$ are non-increasing under local operations.

**Theorem.** Suppose $\Upsilon_k \mapsto \tilde{\Upsilon}_k$ under a local operation

$$\rho \mapsto \tilde{\rho} = \mathcal{E}_{\text{LO}}(\rho) = \sum_{ij} A_i \otimes B_j \rho A_i^\dagger \otimes B_j^\dagger. \quad (44)$$

Then $\tilde{\Upsilon}_k \leq \Upsilon_k$.

**Proof.** We first note that a local operation takes the form

$$M \mapsto \tilde{M} = S_A M S_B^T, \quad (45)$$

and that the $S$ matrices have a polar decomposition $S = R[S]$, where $R$ is a rotation and $|S| = \sum_i q_i f_i f_i^T$, $0 \leq q_i \leq 1$, $f_i^T \cdot f_j = \delta_{ij}$, is a positive matrix. Let us first consider the case where $\mathcal{E}_{\text{LO}} = \mathcal{E}_B$ corresponding to $M \mapsto \tilde{M} = M S_B^T = M |S_B| R_B^T$. Since we have already proved that $\Upsilon_k$ are invariant under local unitaries, we may absorb $R_B$ into the choice of hypo-states $b_i = \sum_j b_j^{(i)} f_j$. Thus, the action of $S_B$ becomes

$$b_i \mapsto \tilde{b}_i = S_B b_i = \sum_j q_j b_j^{(i)} f_j. \quad (46)$$

We further note that $|b_{i_1} \wedge \cdots \wedge b_{i_k}|^2$ is the norm of the vector $b_{i_1} \cdots b_{i_k}$ in the exterior space $\Omega^k(B)$ of $B$. The set $\{f_{\mu_1} \wedge \cdots \wedge f_{\mu_k}\}_{1 \leq \mu_1 < \cdots < \mu_k \leq d}$ is an orthonormal ordered basis of $\Omega^k(B)$. Thus,

$$b_{i_1} \wedge \cdots \wedge b_{i_k} = \sum_{1 \leq \mu_1 < \cdots < \mu_k \leq d} \lambda_{\mu_1 \cdots \mu_k} b_{\mu_1}^{(i_1)} \cdots b_{\mu_k}^{(i_k)}, \quad (47)$$

where

$$\lambda_{\mu_1 \cdots \mu_k} = \sum_{m_1 \ldots m_k} \phi_{m_1 \ldots m_k}^{\mu_1 \cdots \mu_k} \phi_{m_1}^{(i_1)} \cdots \phi_{m_k}^{(i_k)} \quad (48)$$
and we may write
\[
v_{i_1 \ldots i_k}^2 = |b_{i_1} \wedge \ldots \wedge b_{i_k}|^2 = \sum_{1 \leq \mu_1 < \ldots < \mu_k \leq d} |b^{(i_1 \ldots i_k)}_{\mu_1 \ldots \mu_k}|^2.
\] (49)

Now, under the local operation $\mathcal{E}_B$, the correlation measure transforms as $v_{i_1 \ldots i_k}^2 \mapsto \bar{v}_{i_1 \ldots i_k}^2 = |\bar{b}_{i_1} \wedge \ldots \wedge \bar{b}_{i_k}|^2$, which can be written as
\[
\bar{v}_{i_1 \ldots i_k}^2 = \left| \left( \sum_{j_1} q_{j_1} b^{(i_1)}_{j_1} f_{j_1} \right) \wedge \ldots \wedge \left( \sum_{j_k} q_{j_k} b^{(i_k)}_{j_k} f_{j_k} \right) \right|^2
\]
\[
= \sum_{1 \leq \mu_1 < \ldots < \mu_k \leq d} q_{\mu_1}^2 \cdots q_{\mu_k}^2 |b^{(i_1 \ldots i_k)}_{\mu_1 \ldots \mu_k}|^2.
\] (50)

Here, we have used that $q_\mu$ are independent of the indices $i_1, \ldots, i_k$. Since $0 \leq q_\mu \leq 1$, it follows that $v_{i_1 \ldots i_k}$ is non-increasing. Thus, $\bar{\Upsilon}_k \leq \Upsilon_k$.

Finally, we need to consider the bi-local operation $M' = S_A M S_B^T$. This can be written as $M \mapsto M' = R_A |S_A| M |S_B| R_B^T$ and from the consecutive application of the above argument it is clear that
\[
\Upsilon_k \geq \bar{\Upsilon}_k \geq \Upsilon_k',
\] (51)

which completes the proof. □

If we calculate the invariants $\{\Upsilon_k\}_{k=2}^{d^2}$ for a pure state, we expect them to contain redundant information, since the entanglement in the pure state is characterized by the set $\{\Lambda_k\}_{k=2}^{d^2}$ of pure state invariants. To see how this manifests, consider a pure state with the Schmidt form $|\psi\rangle = \sum_i \sqrt{p_i}|i\rangle$. We make particular choice of the local basis operators, defining them in terms of the local Schmidt bases as
\[
E_k = |k\rangle\langle k|, \ k = 1, \ldots, d,
\]
\[
F_{ll'} = \frac{1}{\sqrt{2}} (|l\rangle\langle l'| + |l'|\langle l|), \ 1 \leq l < l' \leq d,
\]
\[
G_{mm'} = \frac{i}{\sqrt{2}} (|m\rangle\langle m'| - |m'|\langle m|), \ 1 \leq m < m' \leq d.
\] (52)

The Hilbert-Schmidt representation $M_\psi$ of the state $|\psi\rangle\langle\psi|$ is diagonal in this basis, with the diagonal values given by
\[
e_k = p_k = \text{Tr} \left[ E_k^A \otimes E_k^B |\psi\rangle\langle\psi| \right],
\]
\[
f_{ll'} = \sqrt{p_lp_{l'}} = \text{Tr} \left[ F_{ll'}^A \otimes F_{ll'}^B |\psi\rangle\langle\psi| \right],
\]
\[
g_{mm'} = -\sqrt{p_mp_{m'}} = \text{Tr} \left[ G_{mm'}^A \otimes G_{mm'}^B |\psi\rangle\langle\psi| \right].
\] (53)

Note that this is essentially the mixed state Schmidt decomposition given in Eq. (13) with mixed state Schmidt coefficients $\{e_k, f_{ll'}, g_{mm'}\}$, i.e., we have that
\[
|\psi\rangle\langle\psi| = \sum_{k=1}^d p_k E_k^A \otimes E_k^B + \sum_{1 \leq l < l' \leq d} \sqrt{p_l p_{l'}} (F_{ll'}^A \otimes F_{ll'}^B - G_{ll'}^A \otimes G_{ll'}^B).
\] (54)
The measures $\Upsilon_k$ are functions of the mixed state Schmidt coefficients $\{e_k, f_{ll'}, g_{mm'}\}$, which for pure states are, in turn, simple functions of the pure state Schmidt coefficients, as can be seen from Eq. \([53]\). Hence $\Upsilon_k$ can be expressed in terms of the pure state measures $\Lambda_k$. For some $k$ this relation becomes simple, e.g., one can show that

$$\Upsilon_2^2 = 2 \left( \Lambda_2^2 - \Lambda_2^4 \right), \quad \Upsilon_3^2 = 2 \left( \Lambda_2^4 - \Lambda_2^6 \right), \quad \Upsilon_4^2 = \Lambda_4^2d.$$  

4 Application: Quantum dynamics

We illustrate the correlation measures $\Upsilon_k$ by looking at how the correlations of a maximally entangled state $|\psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |ii\rangle$ of a $d \times d$ dimensional system changes under two types of decoherence.

We first consider the depolarization channel $E$ defined as the map

$$\psi \mapsto \varrho_W = E(\psi) = p\psi + (1-p)\varrho_*$$  

of $\psi$. Here, $\psi = |\psi\rangle\langle\psi|$ and $\varrho_* = \frac{1}{d^2} \mathbb{1}_{AB}$, i.e., the output $\varrho$ is a Werner state that connects the maximally entangled state $\psi$ for $p=1$ and the random mixture $\varrho_*$ for $p=0$. The relative state map $\mathcal{E}_W$ induced by $\varrho$ acts on a hypo-state $\tau \in \mathcal{S}_A$ as

$$\tau \mapsto \pi = \mathcal{E}_W(\tau) = p\mathcal{E}_\psi(\tau) + (1-p)\mathcal{E}_* (\tau),$$

where $\mathcal{E}_\psi$ and $\mathcal{E}_*$ are the maps induced by $\psi$ and $\varrho_*$, respectively. In particular, for any $Y \in \mathcal{B}(\mathcal{H}_A)$, we find $\mathcal{E}_*(Y) = \frac{1}{d^2} \text{Tr}[Y] \mathbb{1}_B$, i.e., the relative state map defined by $\varrho_*$ maps any element of $\mathcal{B}(\mathcal{H}_A)$ to an operator proportional to the reduced state of subsystem $B$.

Now, let $\{K_i^A\}$ and $\{K_i^B\}$ be orthonormal bases of $\mathcal{B}(\mathcal{H}_A)$ and $\mathcal{B}(\mathcal{H}_B)$, respectively, with the additional property that $K_1^A = \frac{1}{\sqrt{d}} \mathbb{1}_A$ and $K_1^B = \frac{1}{\sqrt{d}} \mathbb{1}_B$. This implies $\text{Tr}[K_i^A] = \text{Tr}[K_i^B] = 0$ for $i > 1$. By choosing $\tau_i = K_i^A$, we obtain the re-states

$$\pi_1 = \mathcal{E}_W(\tau_1) = \frac{1}{d^2} \mathbb{1}_B,$$

$$\pi_i = \mathcal{E}_W(\tau_i) = p\mathcal{E}_\psi(\tau_i) + (1-p)\mathcal{E}_* (\tau_i), \quad i > 1.$$  

Let us use these expressions to evaluate first $\Upsilon_2^2$ explicitly. The individual terms are given by $\nu_{i_1 i_2} = |\pi_{i_1} \wedge \pi_{i_2}|$, which can take two values

$$\nu_{1 i_2} = |\pi_1 \wedge \pi_{i_2}| = \frac{p}{d^2}, \quad 1 < i_2,$$

$$\nu_{i_1 i_2} = |\pi_{i_1} \wedge \pi_{i_2}| = \frac{p^2}{d^2}, \quad 1 < i_1 < i_2,$$

where we have used that $|\pi_i^A \wedge \pi_{i_j}^A| = 1/d^2$ for all $i \neq j$. Hence, we have that

$$\Upsilon_2^2 = \binom{d^2}{k}^{-1} p^2 \left( d^2 - 1 + \binom{d^2 - 1}{2} p^2 \right) = \frac{p^2}{d^2} \left( 2 + (d^2 - 2)p^2 \right).$$

Generalizing to arbitrary $k = 2, \ldots, d^2$ yields

$$\Upsilon_k^2 = d^{2k} \binom{d^2}{k}^{-1} \sum_{1 \leq i_1 < \ldots < i_k \leq d^2} \nu_{i_1 \ldots i_k}^2 = p^{2(k-1)} \left[ \frac{k}{d^2} + \left( 1 - \frac{k}{d^2} \right) p^2 \right].$$
which vanishes when $p \to 0$. The $k = 2, 3, 4, 9$ invariants are shown in Fig. 3 for $d = 3$.

Secondly, we look at how the invariants change under product-basis decoherence of the maximally entangled state $\psi$. Let the product basis be composed of the local Schmidt-bases, i.e., $E_i^A \otimes E_j^B = |i\rangle \langle i| \otimes |j\rangle \langle j|$. The channel $\mathcal{F}$ can be represented as

$$
\psi \mapsto \varrho_D = \mathcal{F}(\psi) = p\psi + (1-p) \sum_{ij=1}^d E_i^A \otimes E_j^B \psi E_i^A \otimes E_j^B .
$$

If we define the maximally decohered state $\Xi = \sum_{ij} E_i^A \otimes E_j^B \psi E_i^A \otimes E_j^B = \frac{1}{d} \sum_i E_i^A \otimes E_i^B$, then

$$
\mathcal{L}_D(\tau_i) = p\mathcal{L}_\psi(\tau_i) + (1-p)\mathcal{L}_\Xi(\tau_i)
$$

We choose the local basis operators given in Eq. (52). With the identification $\{\tau_i\} = \{E_k,F_{ll}',G_{mm}'\}$, we have $\mathcal{L}_\psi(\tau_i) = \mathcal{L}_\Xi(\tau_i)$ for $1 \leq i \leq d$, and $\mathcal{L}_\Xi(\tau_i) = 0$ for $i > d$. $\mathcal{L}_D(\tau_i)$ takes two values in terms of $\mathcal{L}_\psi(\tau_i)$:

$$
\mathcal{L}_D(\tau_i) = \begin{cases} 
\mathcal{L}_\psi(\tau_i), & 1 \leq i \leq d \\
p\mathcal{L}_\psi(\tau_i), & d < i \leq d^2 
\end{cases} .
$$

It is then a matter of combinatorics to show that the invariants for general $k$ are given by

$$
\Upsilon_k^2 = \binom{d^2}{k}^{-1} \sum_{l=0}^k \binom{d}{k-l} \binom{d^2-d}{l} p^{2l} .
$$

The $k = 2, 3, 4, 9$ invariants are shown in Fig. 4 for $d = 3$.

For the depolarization channel, all $\Upsilon_k \to 0$ when $p \to 0$, i.e., all $k$-level correlations are suppressed and vanish for the final product state. However, the Werner state for $p \neq 0$ inherits
the symmetry of the maximally entangled state: for any choice of orthonormal measurement basis \{\ket{a_i}\} (i.e., an observable) at site A, there is a corresponding orthonormal basis \{\ket{b_i}\} at site B, in which the correlations (as measured by, e.g., mutual information) exhibited by the resulting probability distribution will be non-zero. In terms of the invariants, this is due to that \(\Upsilon_k \neq 0\), or, in terms of the \(k\)-vectors, that for any choice \{\ket{a_i}\} the objects \(\mathcal{L}_W(\ket{a_1}\bra{a_1}) \wedge \cdots \wedge \mathcal{L}_W(\ket{a_d}\bra{a_d})\) and \(\mathcal{L}_W(\tau_1) \wedge \cdots \wedge \mathcal{L}_W(\tau_d)\) have a \(d\)-dimensional intersection.

For the product-basis decoherence channel, which is of interest as, e.g., a model for measurement einselection, we see that \(\Upsilon_k \rightarrow 0\) for \(k > d\) and \(\Upsilon_k \rightarrow \left(\frac{d^2}{k}\right)^{-1/2} \left(\frac{d}{k}\right)^{1/2}\) for \(k \leq d\), when \(p \rightarrow 0\) (see Fig. 4). This can be related to proposed measures of classical correlations in quantum states \([20, 21]\), in particular, quantum discord \([21]\) defined as

\[
D(A : B|\{\tau_i\}) = S(\rho_B) - S(\rho) + \sum_i p_i S(\pi_i/p_i),
\]

where \(p_i = \text{Tr}[\pi_i]\), \(S(\rho)\) denotes the von Neumann entropy of the state \(\rho\), and we have the restriction \(\tau_i \in \mathcal{S}(\mathcal{H}_A)\), which ensures that \(\mathcal{L}_e(\tau_i) = \pi_i \in \mathcal{S}'(\mathcal{H}_B)\) and that \(\pi_i/p_i \in \mathcal{S}(\mathcal{H}_B)\). This definition also utilizes a relative state construction in that they are derived from entropies over subsystem \(B\) that are conditioned on measurements on subsystem \(A\). The minimum discord of a state

\[
D_{\text{min}}(A : B) = D(A : B|\{\tau_i\}_{\text{min}}) = S(\rho_B) - S(\rho) + \min_{\{\tau_i\}} \left(\sum_i p_i S(\pi_i/p_i)\right)
\]

quantifies the amount of information lost in the optimal correlation measurement.

The final state \(\Xi\) is the maximally correlated separable state and likewise the maximally correlated zero-discord state. A zero-discord state is characterized by that \(D_{\text{min}} = 0\), which

Fig. 4. The product basis decoherence of a maximally entangled state. Note that \(\Upsilon_k \rightarrow 0\) when \(p \rightarrow 0\) for \(k > d\), while it goes to a finite value for \(k \leq d\). The final state defined by \(\mathcal{F}\) for \(p = 0\) is a maximally correlated separable state.
Fig. 5. An illustration of the mixed state relative states and the mixed state correlation measures for the $2 \otimes 2$ system. In a) we show a three dimensional subspace of $B(H_A)$, spanned by basis operators $K_A^1 = \hat{1}/\sqrt{2}$, $K_A^2 = \sigma_x/\sqrt{2}$ and $K_A^3 = \sigma_z/\sqrt{2}$, and b) and c) shows the corresponding subspace of $B(H_B)$. The disc enclosed by the circle orthogonal to $\hat{1}$ is a subspace of the state space $S$, i.e., the $xz$-plane of the Bloch-sphere, with the pure states on the boundary. The corresponding subspace of subnormalized states $S'$ is the cone with the circle as its base. Three (out of four) hypo-states $\tau_1 = |0\rangle\langle 0|$, $\tau_2 = |1\rangle\langle 1|$, $\tau_3 = K_2$ are shown in a). The re-states $\pi_i = \mathcal{L}_\psi(\tau_i)$, defined by the maximally entangled state $\psi = \frac{1}{2} \sum_{k,l=0}^1 |kk\rangle\langle ll|$, is shown in b), where the volume spanned is $\nu_{123} = \nu_{\pi_1 \wedge \pi_2 \wedge \pi_3} = (1/2)^3$. In c) the restates $\tilde{\pi}_i = \mathcal{L}_\Xi(\tau_i)$ of the maximally classically correlated state $\Xi = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$, is shown, where $\nu_{123} = |\tilde{\pi}_1 \wedge \tilde{\pi}_2 \wedge \tilde{\pi}_3| = 0$ since $\tilde{\pi}_3 = 0$. The quantity $\nu_{12} = 1/4$ is the only nonzero contribution to the invariants, and hence $Y_4 = Y_3 = 0$ and only $Y_2 \neq 0$, which characterizes a two-qubit 0-discord state.

means that all its correlations can be extracted by a single measurement setup, and consequently that the state is robust under this particular measurement, and that the classical mutual information over the probability distribution obtained by measuring in a product basis equals the quantum mutual information of the state. This can be related to the invariants in the following way: If $Y_k \neq 0$ for $k > d$, then $D_{\text{min}}(A : B) \neq 0$, and if $D_{\text{min}}(A : B) = 0$, then $Y_k = 0$ for $k > d$. The first implication we understand as that the re-states span a $k > d$-dimensional subspace of $S_B$, while the (complete) set of projectors constituting a measurement basis only span a $d$-dimensional subspace, thus the $k$-volume spanned by the re-states "collapses" into a $d$-volume upon measurement, which is what we see in the example of product basis decoherence. The restates of the pre- and post-measurement states $\psi$ and $\Xi$ are shown in Figs. 5b and 5c, respectively. Conversely, the second implication illustrates that the re-states of a zero-discord state, which is robust under some product basis measurement, can maximally span a $d$-dimensional subspace. It also follows that, contrary to the symmetric Werner state, that one can find a set of $d$-hypo-states $\{|a'_i\rangle\}$ (bases $\{|a'_i\rangle\}$ and $\{E_i\}$ are mutually unbiased) such that $\mathcal{L}_\Xi(\langle a'_i | a'_j \rangle) = \mathcal{L}_\Xi(\langle a'_j | a'_j \rangle)$, and none of the state’s correlations can be extracted.

5 Conclusions

The concept of relative state, originally developed by Everett [11] to deal with the measurement problem in quantum mechanics, has been used to construct measures of correlations in pure bipartite quantum states of arbitrary dimension. The basic idea is to quantify how much information one observer can obtain about measurements that can be performed by another observer, if they are allowed only to do local, projective measurements. These correlation measures have been shown in detail to be invariant under local unitary transformations of
Relative state measures of correlations in bipartite quantum systems

We have further shown that the present correlation measures coincide with those given by concurrence hierarchies [14] and I concurrence [15], providing an alternative operational interpretation of these measures.

We have extended the notion of relative state to generalized measurements. This allows for studies of the correlation structure of mixed bipartite states. The corresponding measures quantify the total correlation in the sense that they vanish for product states, and are non-increasing under local operations, but may increase under LOCC. We have illustrated the behavior of the mixed state correlation measures for bipartite systems of arbitrary dimension undergoing two different types of open system dynamics.

Acknowledgments

PR acknowledges financial support from the Göran Gustafsson Foundation. ES acknowledges support from the National Research Foundation and the Ministry of Education (Singapore).

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