Negative Index Lens Aberrations

D. Schurig and D.R. Smith
Physics Department, University of California, San Diego, La Jolla, CA, 92093
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We examine the Seidel aberrations of thin spherical lenses composed of media with refractive index not restricted to be positive. We find that consideration of this expanded parameter space allows reduction or elimination of more aberrations than is possible with only positive index media. In particular we find that spherical lenses possessing real aplanatic focal points are possible only with negative index. We perform ray tracing, using custom code that relies only on Maxwell’s equations and conservation of energy, that confirms the results of the aberration calculations.

In 1968, V. G. Veselago proposed the idea that a material could have a negative index of refraction, and described how this would impact many basic electromagnetic phenomena. In recent years, there has been great interest in this subject due to experimental demonstrations of negative index artificial materials, and the introduction of the perfect lens concept. A perfect lens relies only on the permittivity, \( \epsilon = \mu \) of the material, and it is not restricted to be positive. We find that consideration of this expanded parameter space allows reduction or elimination of more aberrations than is possible with only positive index media. In particular we find that spherical lenses possessing real aplanatic focal points are possible only with negative index. We perform ray tracing, using custom code that relies only on Maxwell’s equations and conservation of energy, that confirms the results of the aberration calculations.

The monochromatic imaging quality of a lens can be characterized by the five Seidel aberrations: spherical, coma, astigmatism, field curvature and distortion. These well known corrections to the simple Gaussian optical formulas are calculated from a fourth order expansion of the deviation of a wave front from spherical. (A spherical wave front converges to an ideal point focus in ray optics). The coefficients in this expansion quantify the non-ideal focusing properties of an optical element for a given object and image position. We find that there is an asymmetry of several of the Seidel aberrations with respect to index about zero. Considering that an interface with a relative index of +1 is inert and one of relative index -1 is strongly refractive, this asymmetry is not surprising. However, our conclusion that the asymmetry can yield superior focusing properties for negative index lenses is not obvious.

We note that negative index media are necessarily frequency dispersive, which implies increased chromatic aberration and reduced bandwidth. However, diffractive optics, which possess a similar limitation, have found utility in narrow band applications.

To confirm the analytical aberration results, we developed custom ray tracing code that does not rely on the sign of the index to determine the path of the ray, but relies only on the permittivity, \( \epsilon \), the permeability, \( \mu \), Maxwell’s equations and conservation of energy.

Between interfaces, in homogenous media, the ray propagates in a straight line following the direction of the Poynting vector. Refraction across an interface, from a region labeled 1 into a region labeled 2, is handled as follows. Wave solutions are sought that satisfy the dispersion relation (obtained from Maxwell’s equations) in region 2,

\[
\frac{c^2}{n_2^2} \mathbf{k}_2 \cdot \mathbf{k}_2 = \varepsilon_2 \mu_2,
\]

where \( \mathbf{k}_2 \) is the wave vector in region 2. The solutions must also satisfy a boundary match to the incident wave, requiring

\[
\mathbf{n} \times (\mathbf{k}_2 - \mathbf{k}_1) = 0,
\]

where \( \mathbf{n} \) is the unit normal to the interface. The outgoing, refracted, wave must carry energy away from the surface if the incident wave carried energy in,

\[
(P_2 \cdot \mathbf{n})(P_1 \cdot \mathbf{n}) > 0,
\]

where \( P = \frac{1}{2} \text{Re} (E \times H^*) \) is the time averaged Poynting vector. Finally, the wave must not be exponentially growing or decaying, \( \text{Im} (\mathbf{k}_2) = 0 \), since the media are assumed passive and lossless. If a solution exists that satisfies all the above criteria, the ray is continued with the new found wave vector and Poynting vector. Furthermore, since we consider only isotropic media the solution will be unique.

We find that the form of the expressions for the Seidel aberrations of thin spherical lenses found in the optics literature are unchanged by the consideration of negative index media. We reached this conclusion by re-deriving these expressions, from first principles, using only the definition of optical path length and Fermat’s Principle. We interpret the optical path length, \( \text{OPL} = \int_C n(s) ds \), to be the phase change (in units of free space wavelength) that a wave would undergo along the path \( C \), if \( C \) is oriented parallel to the Poynting vector. The optical path may have contributions that are negative where the Poynting vector and the wave vector are antiparallel, i.e. where the index is negative. These aberration formulae are further corroborated by agreement with the results of our ray tracing.

The wave aberration, \( \Delta \text{OPL} \), is the difference in optical path length of a general ray and a reference ray,
where the reference ray passes through the optic axis in the aperture stop and the general ray is parameterized by its coordinate in the aperture stop, and its coordinate in the image plane, \( h \) (Fig. 1). To be in the Gaussian optic limit, where spherical interfaces yield perfect imaging, \( r \) and \( h \) must be near zero. A series expansion of the wave aberration in these parameters

\[
\Delta OPL = \sum_{l,m,n=0}^{\infty} C_{lmn} (r \cdot r)^l (r \cdot h)^m (h \cdot h)^n
\]

yields corrections to Gaussian optics of any desired order. The lowest order corrections for a thin spherical lens with aperture stop in the plane of the lens are given by

\[
C_{200} = \frac{1}{32 f^3 n (n-1)^2} \times \left[ n^3 + (n-1)^2 (3n+2) p^2 + 4 (n+1) pq + (n+2) q^2 \right]
\]

\[
C_{110} = \frac{-1 - p}{8 f^3 n (n-1)} \left[ (2n+1) (n-1) p + (n+1) q \right]
\]

\[
C_{020} = \frac{(1 - p)^2}{8 f^3},
\]

\[
C_{101} = \frac{(1 - p)^2}{16 f^3 n} (n+1),
\]

\[
C_{011} = 0.
\]

These coefficients are the Seidel aberrations: spherical, coma, astigmatism, field curvature and distortion respectively. Also appearing in these expressions are \( p \), the position factor, and \( q \), the shape factor, where we follow the definitions of Mahajan. The position factor is given by

\[
p \equiv 1 - \frac{2 f'}{S'},
\]

where \( f' \) is the focal length referred to the image side and \( S' \) is the image position. Through the thin spherical lens imaging equation,

\[
\frac{1}{S'} - \frac{1}{S} = \frac{1}{f'} = (n-1) \left( \frac{1}{R_1} - \frac{1}{R_2} \right),
\]

where \( S \) is the object position and \( R_1 \) and \( R_2 \) are the lens radii of curvature, the position factor is directly related to the magnification,

\[
M = \frac{S'}{S} = \frac{p + 1}{p - 1}
\]

The shape factor is given by

\[
q \equiv \frac{R_2 + R_3}{R_2 - R_1}
\]

A lens with a shape factor of 0 is symmetric, and \( \pm 1 \) is a plano-convex lens. Using the shape and position factor, all thin spherical lens configurations are described.

We will first examine the very important case of a source object at infinite distance. This is a position factor of \(-1\). We are left with two parameters that can be used to reduce aberrations, \( n \) and \( q \). We will set the value of \( q \) to eliminate one of the aberrations and compare the remaining aberrations as a function of index. We will restrict our attention to moderate values of index. At large absolute values of index, the aberrations approach the same value independent of sign, but dielectric lenses with high index have significant reflection coefficients due to the impedance mismatch to free space.

The usual ordering of the aberrations is from highest to lowest in the order of \( r \), the aperture coordinate. This is the ordering of most image degradation to least if one is forming images with significant lens aperture, but small to moderate image size, which is a common occurrence in applications. Thus, spherical aberration is an obvious target for elimination. However, there are no roots of \( C_{200} \) for values of index greater than one, which is why this aberration is referred to as spherical aberration, since it appears to be inherent to spherical lenses. The usual practice is to eliminate coma (the next in line), and it so happens that the resulting lens has a value for the spherical aberration that is very near the minimum obtainable. Adjusting the shape factor, \( q \), is often called lens bending. If we bend the lens for zero coma, that is find the roots of \( C_{110} \) with respect to \( q \) we obtain

\[
q_c = \frac{(2n+1)(n-1)}{n+1}.
\]

We plug this value for \( q \) and \( p = -1 \) into \( 5a \) and plot the remaining three non-zero aberration coefficients as well as \( q_c \) in Fig. 2. We note that there are two values of index where \( q = 1 \), which represent a plano-concave/convex lens. Setting \( 10 \) equal to one we obtain,

\[
n^2 - n - 1 = 0.
\]

the roots of which are the ubiquitous golden ratios, \( n = \phi \approx 1.62 \) and \( n = 1 - \phi \approx -0.62 \). We also note
that there is a window of index values near \( n = -0.7 \) where both the spherical aberration and field curvature are small. There is no equivalent window in positive index.

Several ray tracing diagrams with both meridional rays and ray spot diagrams are shown for specific values of index in Fig. 2. The reference lens has index \( \phi \), which is close to typical values used in visible optical lenses and near enough to \( n = 1 \) for reasonably low reflection. The lenses of negative index shown are in fact closer to \( n = -1 \), which is the other index which permits perfect transmission, so this is a fair comparison. The negative index lenses all show significantly tighter foci than the positive index lens.

These expressions have real values only for \( n \leq 1/4 \), so an implementation of such a lens (embedded in free space) is not possible with normal materials. It is a surprising and significant result that negative index permits an entire family of spherical aberration free spherical lenses that can focus a distant object to a real focus, Fig. 3. The solution with the negative sign in the expression for \( q_s \) (solid curves) has less coma for moderate negative values of index, so ray tracing diagrams are shown for that solution. We note that at \( n = -1 \), the field curvature is also zero, thus this lens has only two of the five Seidel aberrations, coma and astigmatism. For a positive index reference we use the zero coma, \( n = \phi \) lens from above. Here again, negative index lenses achieve a tighter focus.

\[
q_s = \frac{2(n^2 - 1) \pm n\sqrt{1 - 4n}}{n + 2}. \tag{12}
\]
than a comparable positive index lens.

Now we examine the case of \(|p| < 1\), that is a real object and real image both at finite position. Since \(p\) and \(q\) are both free parameters, we can conceivably eliminate two aberrations. If we eliminate spherical aberration and coma the resulting lens is called aplanatic. It is a well known, though incorrect, result that a spherical lens can only have virtual aplanatic focal pairs. The correct statement is that only negative index spherical lenses can have real aplanatic focal pairs.

If we set \(C_{200}\) and \(C_{110}\) to zero and solve for \(p\) and \(q\), we obtain four solutions, the two non-trivial ones are given by

\[
p_{sc} = \pm \frac{n + 1}{n - 1}, \tag{13a}
\]

\[
q_{sc} = \pm (2n + 1). \tag{13b}
\]

We will focus on the solution with a minus sign for \(p\) and the plus sign for \(q\). This solution has smaller aberrations for lens configurations that magnify an image. The other solution is better for image reduction. Inserting the expressions (13a) into (13b) we have plotted the two remaining non-zero coefficient as well as the values of \(p_{sc}\) and \(q_{sc}\) (Fig. 4). Ray diagrams are shown for lenses with magnifications of \(-1\), \(-2\) and \(-3\). Also shown is a reference reference lens for each. The reference lenses (which cannot be aplanatic) are of moderate index and \(f/\#\) as the lenses they are compared to. They are bent for zero coma but also have spherical aberration near the minimum possible for the configuration. Again the negative index lenses produce superior foci.

The lens of index \(-1\) and magnification \(-1\) is particularly interesting. At this index value the field curvature is also zero. This remarkable lens configuration has only one of the five Seidel aberrations, astigmatism. This is confirmed by ray tracing which shows a one dimensional "spot" at the image plane. This is perfect focusing in the sagittal plane. Perfect focusing also occurs in the meridional plane, in front of sagittal focus.

One may ask why this asymmetric lens, \(q = -1\), performs so well in a symmetric configuration, \(p = 0\). This lens can be equivalently viewed as a biconcave doublet with one component having index \(-1\) and the other having index 1, i.e. free space. Driven by this observation, we found that all biconcave doublets with arbitrary indices of \(\pm n\) have identical focusing properties. The only observable difference is in the internal rays, which are always symmetric about the planer interface, but make more extreme angles at higher index magnitude.

Fabrication of any of these negative index lenses is quite feasible using periodically structured artificial materials. Current artificial material designs can operate at frequencies from megahertz through terahertz\[8\], where there are numerous communication and imaging applications. For example, lens antennas could benefit both from increased gain, and by a reduction of mass, afforded by low density artificial materials. Furthermore, these lenses are even easier to implement than a perfect lens, since they lack its severe structure period per wavelength requirements and are more tolerant to losses\[9\]. Negative index lenses at visible light frequencies may also be possible, by using photonic crystals, which have shown potential for negative refraction\[10\, 11\].

Using the current optical system design paradigm, aberrations are minimized by combining elements with coefficients of opposite sign\[12\]. However, more elements mean greater complexity and cost. Taking advantage of an expanded parameter space that includes negative

\[
\begin{array}{c|c|c|c|c}
 n & p & q & r_{\text{rms}} & \text{zoom} \\
-1 & 0 & -1 & 0.08 & 5x \\
-0.5 & 0.3 & -0.3 & 0.38 & 1x \\
-0.5 & 1.5 & -0.3 & 0.38 & 1x \\
-1 & 1/2 & 0.33 & 0.07 & 10x100x \\
\end{array}
\]
index can reduce the number of required elements—possibly even to one.

[1] V. G. Veselago, Sov. Phys. Usp. 10, 509 (1968).
[2] D. R. Smith, W. Padilla, D. C. Vier, S. C. Nemat-Nasser, and S. Schultz, Phys. Rev. Lett. 84, 4184 (2000).
[3] J. B. Pendry, Phys. Rev. Lett. 85, 3966 (2000).
[4] C. G. Parazzoli, K. Li, S. J. McLean, R. B. Gregor, and M. H. Tanielian, Applied Physics Letters (2004).
[5] V. N. Mahajan, Optical Imaging and Aberrations, vol. 1 (SPIE, Bellingham, Washington, 1998), 1st ed.
[6] B. Kress and P. Meyrueis, Digital Diffractive Optics: An Introduction to Planar Diffractive Optics and Related Technology (Wiley, Hoboken, New Jersey, 2000), 1st ed.
[7] M. Livio, The Golden Ratio: The Story of PHI, the World’s Most Astonishing Number (Broadway, 2003), 1st ed.
[8] T. J. Yen, W. J. Padilla, N. Fang, D. C. Vier, D. R. Smith, J. B. Pendry, D. N. Basov, and X. Zhang, Science 303, 1494 (2004).
[9] D. R. Smith, D. Schurig, M. Rosenbluth, S. Schultz, S. A. Ramakrishnan, and J. B. Pendry, Applied Physics Letters 82, 1506 (2003).
[10] P. V. Parimi, W. T. Lu, P. Vodo, and S. Sridhar, Nature 426, 404 (2004).
[11] E. Cubukcu, K. Aydin, E. Ozbay, S. Foteinopoulou, and C. M. Soukoulis, Nature 423, 604 (2004).
[12] E. Hecht, Optics (Addison-Wesley, Massachusetts, 1998), 3rd ed.