Normalization of a nonlinear representation of a Lie algebra, regular on an abelian ideal

Mabrouk BEN AMMAR

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Abstract

We consider a nonlinear representation of a Lie algebra which is regular on an abelian ideal, we define a normal form which generalizes that defined in [2].

1 Introduction

The study of a vector fields in a neighborhood of a point on a complex manifold is, of course, reduced to that of a vector fields $T$ in a neighborhood of the origin of $E = \mathbb{C}^n$.

If $T$ is regular at the origin of $E$, we know that there exists a coordinate system $(x_1, \ldots, x_n)$ of $E$ in which the vector fields can be expressed as

$$T = \frac{\partial}{\partial x_1},$$

that is, there exists a local diffeomorphism $\phi$ of $E$ such that

$$\phi(0) = 0 \quad \text{and} \quad \phi^*(T) = \frac{\partial}{\partial x_1}.$$

If $T$ is analytic then $\phi$ can be chosen to be analytic [9].

If $T$ is singular at the origin of $E$, then the situation is not so simple; generically, $T$ is linearizable, that is, there exists a local diffeomorphism $\phi$ of $E$ such that

$$\phi(0) = 0 \quad \text{and} \quad \phi^*(T) = \sum_{j,k} a_{jk} x_j \frac{\partial}{\partial x_k}, \quad a_{jk} \in \mathbb{C}.$$

But, the linearization is not always possible, so, we introduce the notion of normal form of a vector fields. Such a normal form, by construction, must enjoy the following properties:

(a) If $T$ is any vector fields then there exists a local diffeomorphism $\phi$ of $E$ such that $\phi(0) = 0$ and $\phi^*(T)$ is in normal form.

(b) This normal form is unique, that is, if $\phi_1^*(T)$ and $\phi_2^*(T)$ are in normal form then $\phi_1^*(T) = \phi_2^*(T)$. 
This problem has been studied extensively since Poincaré in formal, analytical or $C^\infty$ contexts. The normal form of $T$ is then:

$$T' = S^1 + N$$

where $S^1$ is semisimple and $N$ is nilpotent satisfying

$$[S^1, N] = 0.$$  

Let us now consider the case, not of a single vector fields, but of a finite dimensional Lie algebra $\mathfrak{g}$ of vector fields, or, if we prefer, not the case of a germ of local actions on $E$, but rather the case of germs of local actions of a Lie group $G$ on $E$.

Let us consider the formal or analytic setting. If all vector fields are singular at 0, then we are in the presence of a nonlinear representation $T$ of a Lie algebra $\mathfrak{g}$ in the sense of Flato, Pinczon and Simon [5]:

$$T : \mathfrak{g} \longrightarrow \mathcal{X}_0(E), \ X \longmapsto T_X \text{ such that } [T_X, T_Y] = T_{[X,Y]},$$

where $\mathcal{X}_0(E)$ is the space of vector fields which are singular at 0. In this case, we use the structure of the Lie algebra $\mathfrak{g}$ to precise the convenient normal forms. The notion of normal form given in [2] generalizes those known for the nilpotent and semisimple cases ([1], [3], [6], [7]). Let $\mathfrak{r}$ be the solvable radical of $\mathfrak{g}$, we know that, by the Levi-Malcev decomposition theorem, $\mathfrak{g}$ can be decomposed

$$(\mathcal{D}) \quad \mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$$

where $\mathfrak{s}$ is semi-simple. The nonlinear representation $T$ of the complex Lie algebra $\mathfrak{g}$ is said to be normal with respect the Levi-Malcev decomposition $(\mathcal{D})$ of $\mathfrak{g}$, if its restriction $T|_\mathfrak{s}$ is linear and its restriction $T|_\mathfrak{r}$ to $\mathfrak{r}$ is in the following form

$$T|_\mathfrak{r} = D^1 + N^1 + \sum_{k \geq 2} T^k|_\mathfrak{r},$$

where the linear part $D^1 + N^1$ of $T|_\mathfrak{r}$ is such that $D^1$ is diagonal with coefficients $\mu_1, \ldots, \mu_n$, (the $\mu_i$ are elements of $\mathfrak{r}^*$) and $N^1$ is strictly upper triangular. Moreover, $T^k$ has the following form in the coordinates $x_i$ of $E$

$$T^k_X = \sum_{i=1}^n \sum_{|\alpha| = k} \Lambda_i^\alpha(X)x_1^{\alpha_1} \cdots x_n^{\alpha_n} \frac{\partial}{\partial x_i}, \quad X \in \mathfrak{r},$$

where any coefficient $\Lambda_i^\alpha$ can be nonzero only if it is resonant, that means, the corresponding linear form $\mu_i^\alpha \in \mathfrak{r}^*$ defined by

$$\mu_i^\alpha = \sum_{r=1}^n \alpha_r \mu_r - \mu_i,$$

is a root of $\mathfrak{r}$, (eigenvalue of the adjoint representation of $\mathfrak{r}$). The representation $T$ is said to be normalizable with respect $(\mathcal{D})$ if it is equivalent to a normal one with respect $(\mathcal{D})$.  

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In this paper, we study the situation where the $T_X$ are not all in $X_0(E)$. More precisely, we introduce the notion of nonlinear representation $(T,E)$ of $g$ in $E$ such that

$$T_X = T_X^0 + \sum_{k \geq 1} T_X^k \in \mathcal{X}(E), \quad X \in g,$$

where $\mathcal{X}(E)$ is the space of formal vector fields on $E$ (not necessarily vanishing at 0), in particular, $T_X^0 \in E$. This situation seems to be more delicate in its generality, but here we treat the case where the Lie algebra $g$ can be written

$$g = g_0 + m,$$

where $m$ is an abelian ideal of $g$ and $g_0$ is a subalgebra of $g$. The Lie algebras of groups of symmetries of simple physical systems are often of this type (Galilée group, Poincaré group, ...). Moreover, we assume that $T_X^0 = 0$ if $X \in g_0$ and $T_X^0 \neq 0$ if $X \in m \setminus \{0\}$.

We also assume that $T$ is analytic and we will prove in this case that we can always put the representation $T$ in the following normal form: there exists a coordinate system $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ of $E$ such that:

(a) For all $X \in m$, $T_X$ can be expressed as

$$T_X = \sum_{i=1}^{p} a_i \frac{\partial}{\partial x_i},$$

where $a_i \in m^*$.

(b) $T|_{g_0}$ is normal au sense de [2], moreover, for all $X \in g_0$ and $k \geq 2$, $T_X^k$ has the following form

$$T_X^k = \sum_{i=1}^{p} \sum_{|\alpha|=k} \Lambda_i^\alpha(X) y_1^{\alpha_1} \cdots y_q^{\alpha_q} \frac{\partial}{\partial x_i} + \sum_{i=1}^{q} \sum_{|\alpha|=k} \Gamma_i^\alpha(X) y_1^{\alpha_1} \cdots y_q^{\alpha_q} \frac{\partial}{\partial y_i}.$$

2 Notations and definitions

Let $E$ be a complex vector space with dimension $n$. The space of symmetric $k$-linear applications from $E \times \cdots \times E$ to $E$ is identified with the space $L(\otimes_s^k E, E)$ of linear maps from $\otimes_s^k E$ to $E$, where $\otimes_s^k E$ is the space of symmetric $k$-tensors on $E$. Denote by $L(E)$ the space $L(E, E)$. Let $(e_1, \ldots, e_n)$ be a basis of $E$. For $\alpha \in \mathbb{N}^n$ such that $|\alpha| = k$ and $i \in \{1, \ldots, n\}$ we define $e_\alpha^i \in L(\otimes_s^k E, E)$ by:

$$e_\alpha^i \circ \sigma_k(\otimes \beta_1 e_1 \cdots \otimes \beta_n e_n) = \delta_{\alpha_1} \cdots \delta_{\alpha_n} e_i,$$

where $\delta$ is the Kronecker symbol, and $\sigma_k$ is the symmetrization operator from $\otimes E$ to $\otimes_s^k E$ defined by

$$\sigma_k(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} (v_{\sigma(1)}, \ldots, v_{\sigma(k)}).$$
Obviously, \((e^\alpha_i)\) is a basis of \(L(\otimes^k_s E, E)\). Let \(\mathcal{X}(E)\) (respectively \(\mathcal{X}_0(E)\)) be the set of formal power series (or formal vector fields)

\[
T = \sum_{k=1}^{\infty} T^k, \quad \text{(respectively} \quad T = \sum_{k=0}^{\infty} T^k) \]

where \(T^k \in L(\otimes^k_s E, E)\) (with \(L(\otimes^0_s E, E) = E\)). Therefore, we can write

\[
T^k = \sum_{|\alpha|=k} \sum_{i=1}^{n} e^\alpha_i. \]

Of course, any analytical vector fields \(X\) on \(E\) which is singular in 0 admits a Taylor expansion:

\[
\sum_{i=1}^{n} \sum_{k=1}^{\infty} \sum_{|\alpha|=k} \Lambda^\alpha_{i_1} \cdots x_{i_n}^{\alpha_n} \frac{\partial}{\partial x_{i_1}} = \sum_{|\alpha|=i} \Lambda^\alpha_{i_1} x^{\alpha_1} \partial_{i_1}. \]

It can be identified with the formal vector fields:

\[
\sum_{i=1}^{n} \sum_{k=1}^{\infty} \sum_{|\alpha|=k} \Lambda^\alpha_{i} e^\alpha_{i}. \]

If \(|\alpha| = 0\) we agree that \(e^\alpha_i = e_i\).

Of course, \(\mathcal{X}(E)\) can be endowed with a Lie algebra structure defined, for \(e^\alpha_i \in L(\otimes^{|\alpha|}_s E, E)\) and \(e^\beta_j \in L(\otimes^{|\beta|}_s E, E)\), by

\[
[e^\alpha_i, e^\beta_j] = e^\alpha_i \ast e^\beta_j - e^\beta_j \ast e^\alpha_i \in L(\otimes^{|\alpha|+|\beta|-1}_s E, E), \]

where

\[
e^\alpha_i \ast e^\beta_j = b_{ij} e^\alpha_{i+j} - e^\beta_{j} e^\alpha_{i}, \quad 1_i = (0, \ldots, 0, 1, 0, \ldots, 0), (1 \text{ in } i\text{th place}). \]

**Definition 2.1** A formal vector fields \(T\) in \(\mathcal{X}(E)\) is said to be analytic al if the power series

\[
\sum_{k \geq 1} T^k(\otimes^k v) = T_X(v), \quad v \in E, \]

converges in a neighborhood of the origin of \(E\).

**Definition 2.2** Two formal vector fields \(T\) and \(T'\) are said to be equivalent if there exists an element \(\phi = \sum_{k \geq 1} \phi^k\) of \(\mathcal{X}_0(E)\) such that \(\phi^1\) is invertible and

\[
\psi \ast T = T' \circ \psi \]

where

\[
T' \circ \phi = \sum_{k \geq 1} \sum_{j=1}^{k} T'^{ij} \circ \left( \sum_{i_1 + \cdots + i_j = k} \phi^{i_1} \otimes \cdots \otimes \phi^{i_j} \right) \circ \sigma_k. \]

\(T\) and \(T'\) are said to be analytically equivalent if \(\phi\) is analytic.
Remark 2.3 For the composition law $\circ$, the map $\phi$ is invertible if and only if $\phi^1$ is an automorphism of $E$.

Definition 2.4 Let $\mathfrak{g}$ be a Lie algebra with finite dimension over $\mathbb{C}$. A nonlinear (formal) representation $(T, E)$ of $\mathfrak{g}$ in $E$ is a linear map:

$$T : \mathfrak{g} \to \mathcal{X}(E), \quad X \mapsto T_X,$$

such that

$$[T_X, T_Y] = T_{[X,Y]}, \quad X, Y \in \mathfrak{g}.$$

Definition 2.5 We say that two representations $(T, E)$ and $(T', E)$ are equivalent if there exists $\phi \in \mathcal{X}^0(E)$ such that $\phi \circ T_X = T'_X \circ \phi$, $X \in \mathfrak{g}$.

Definition 2.6 A nonlinear formal representation $T$ of $\mathfrak{g}$ is said to be analytic if the power series

$$\sum_{k \geq 1} T^k_X(v^k e) = T_X(v), \quad v \in E$$

converges in a neighborhood of 0 in $E$, for all $X \in \mathfrak{g}$.

In the following, we consider a nonlinear analytic representation $(T, E)$ of a complex finite dimensional Lie algebra $\mathfrak{g}$ in $E = \mathbb{C}^n$, such that

$$\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{m},$$

where $\mathfrak{m}$ is an abelian ideal of $\mathfrak{g}$ and $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$. Moreover, we assume that

$$T^0_X = 0 \quad \text{if} \quad X \in \mathfrak{g}_0 \quad \text{and} \quad T^0_X \neq 0 \quad \text{if} \quad X \in \mathfrak{m} \setminus \{0\},$$

and then we normalize $T$ step by step.

3 Normalization of $T|_\mathfrak{m}$

Proposition 3.1 There exists a coordinate system $(x_1, \ldots, x_p, y_1, \ldots, y_q)$, $p + q = n$, of $E$ such that, for all $X \in \mathfrak{m}$, $T_X$ can be expressed as

$$T_X = \sum_{i=1}^p a_i \frac{\partial}{\partial x_i},$$

where $a_i \in \mathfrak{m}^*$. 

Proof. Since the representation $(T, E)$ is analytic, then, there exists a neighborhood $\mathcal{U}$ of the origin of $E$ such that

$$T_x(v) \in E,$$

for any $v$ in $\mathcal{U}$ and $X$ in $\mathfrak{g}$. So, for any $v$ in $\mathcal{U}$, we consider the subspace $E_v$ spanned by the vectors $T_X(v)$, where $X$ browse $\mathfrak{m}$. If $(X_1, \ldots, X_p)$ is a basis of $\mathfrak{m}$, then the $T_{X_i}(v)$ are
generators of $E_v$, thus, $\dim E_v \leq p$. But, the system $(T_{X_1}(0), \ldots, T_{X_p}(0))$ is independent, indeed, the condition

$$\alpha_1 T_{X_1}(0) + \cdots + \alpha_p T_{X_p}(0) = T^0_{\alpha_1 X_1(0) + \cdots + \alpha_p X_p} = 0$$

implies that $\alpha_1 X_1(0) + \cdots + \alpha_p X_p = 0$ since $T^0$ does not vanish on $m \setminus \{0\}$. Thus, $\dim E_0 = p$.

The determinant map is a continuous map, then there exists a neighborhood $V \subset \mathcal{U}$ such that $\dim E_v \geq p$, for all $v \in V$. Therefore, $\dim E_v = p$, for all $v \in V$.

Thus, the map

$$v \mapsto E_v$$

is an involutive integrable distribution, therefore, the Frobenius theorem ensures the existence of an analytic coordinate system $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ of $E$ such that the $\frac{\partial}{\partial x_i}$, $i = 1, \ldots, p$, generate this distribution.

\[ \square \]

4 Normalization of $T|_{\mathfrak{g}_0}$

Now, we begin the second step to normalize the representation $(T, E)$. We know that, for any $X$ in $\mathfrak{g}_0$, we have

$$T_X = \sum_{k \geq 1} T^k_X.$$ 

We consider the coordinate system $(x_1, \ldots, x_p, y_1, \ldots, y_q)$ of $E$ defined in the previous section and we prove the following results

**Proposition 4.1** For any $X$ in $\mathfrak{g}_0$, $T^1_X$ has the following form

$$T^1_X = \sum_{i,j} a_{ij}(X)y_j \frac{\partial}{\partial x_i} + \sum_{i,j} b_{ij}(X)y_j \frac{\partial}{\partial y_i} + \sum_{i,j} c_{ij}(X)x_j \frac{\partial}{\partial x_i}$$

and for $k \geq 2$, $T^k_X$ has the following form

$$T^k_X = \sum_{i,|\alpha|=k} A_i^\alpha(X)y^\alpha \frac{\partial}{\partial x_i} + \sum_{i,|\alpha|=k} B_i^\alpha(X)y^\alpha \frac{\partial}{\partial y_i}.$$ 

**Proof.** Consider $(X_1, \ldots, X_p) \in m^p$ such that

$$T_{X_i} = \frac{\partial}{\partial x_i}.$$ 

Since $m$ is an ideal of $m$, then $[m, \mathfrak{g}_0] \subset m$, and therefore, for any $X$ in $\mathfrak{g}_0$ and $j = 1, \ldots, p$, we have

$$T_{[X_j, X]} = \left[ \frac{\partial}{\partial x_j}, T_X \right] = \left[ \frac{\partial}{\partial x_j}, T^k_X \right] = \sum_{i=1}^p \alpha_i \frac{\partial}{\partial x_i},$$

for some $\alpha_i \in \mathbb{C}$. In particular, for $k \geq 2$, we have

$$\left[ \frac{\partial}{\partial x_j}, T^k_X \right] = 0,$$
and then the result follows. □

Now, for any \( X \) in \( g_0 \), we define
\[
A_X = \sum_{i,\alpha} A^\alpha_i(X) y^\alpha \frac{\partial}{\partial x_i} = \sum_{k \geq 1} A^k_X, \quad (A^1_i = \alpha_{ij}),
\]
\[
B_X = \sum_{i,\alpha} B^\alpha_i(X) y^\alpha \frac{\partial}{\partial x_i} + \sum_{i,j} c_{ij}(X)x_j \frac{\partial}{\partial x_i} = \sum_{k \geq 1} B^k_X,
\]
\[
H_X = \sum_{i,j} b_{ij}(X)y_j \frac{\partial}{\partial y_i},
\]
\[
K_X = \sum_{i,j} c_{ij}(X)x_j \frac{\partial}{\partial x_i},
\]
and we consider the subspaces \( E_1 \) and \( E_2 \) corresponding, respectively, to coordinate systems \((x_1, \ldots, x_p)\) and \((y_1, \ldots, y_q)\).

**Proposition 4.2**

i) \((H, E_1)\) and \((K, E_2)\) are two linear representations of \( g_0 \).

ii) For any \( X \) in \( g_0 \) we have
\[
A_{[X,Y]} = [A_X, B_Y] + [B_X, A_Y],
\]
\[
B_{[X,Y]} = [B_X, B_Y].
\]

Thus, \((B, E)\) is a nonlinear representation of \( g_0 \) in \( E \).

**Proof.**

i) Obvious.

ii) For any \( X \) in \( g_0 \), we have
\[
T_X = A_X + B_X,
\]
then we have
\[
T_{[X,Y]} = A_{[X,Y]} + B_{[X,Y]} = [A_X + B_X, A_Y + B_Y] = [A_X, B_Y] + [B_X, A_Y] + [B_X, B_Y].
\]
We easily check that
\[
A_{[X,Y]} = [A_X, B_Y] + [B_X, A_Y],
\]
\[
B_{[X,Y]} = [B_X, B_Y].
\]

□

Now, we consider a Levi-Malcev decomposition of \( g_0 \):
\[
(D) \quad g = \mathfrak{t} \oplus \mathfrak{s}
\]
where \( \mathfrak{s} \) is semi-simple and \( \mathfrak{t} \) is the solvable radical of \( g \). We triangularize simultaneously the \( H_X \) and \( K_X \), where \( X \) browses \( \mathfrak{t} \) (Lie theorem [8]). Thus, we define the linear forms: \( \mu_1, \ldots, \mu_p \in \mathfrak{t}^* \) and \( \nu_1, \ldots, \nu_q \in \mathfrak{t}^* \). The \( \mu_1(X), \ldots, \mu_p(X), \nu_1(X), \ldots, \nu_q(X) \) are the eigenvalues of \( B^1(X) \), indeed, \( B^1_X = H_X + K_X \). They are also the eigenvalues of \( T^1(X) \), since \( A^1_X \in \text{L}(E_2, E_1) \). Thus, the simultaneous triangulation of \( H \) and \( K \) leads to that of \( T^1 \) by retaining the first \( p \) components in \( E_1 \) and the \( q \) other in \( E_2 \).

Let us denote by \( \lambda_1, \ldots, \lambda_n \) the roots of \( T^1|_e \), and consider the elements \( \lambda^\alpha_j \) of \( \mathfrak{t}^* \) defined by
\[
\lambda^\alpha_j(X) = \sum_{i=1}^n \alpha_i \lambda_i(X) - \lambda_j(X),
\]
where \( j = 1, \ldots, n \) and \( \alpha \in \mathbb{N} \).
Definition 4.3 We say that \((\alpha, j)\) is resonant if \(\lambda_{j}^{\alpha}\) is a root of \(\mathfrak{r}\).

Definition 4.4 An element \(X_0\) in \(\mathfrak{r}\) is said to be vector resonance if, for any \(\alpha\), for any \(i\) and for any \(j\),
\[
\lambda_{j}^{\alpha}(X_0) = v_i(X_0) \Rightarrow \lambda_{j}^{\alpha} = v_i,
\]
where the \(v_i\) are the roots of \(\mathfrak{r}\).

For the two following lemmas see, for instance, [1] and [2].

Lemma 4.5 The set of vector resonance is dense in \(\mathfrak{r}\).

Lemma 4.6 There exists a resonance vector in \(\mathfrak{r}\) such that \([X_0, s] = 0\).

The resonant pairs \((\alpha, j)\) appearing here are of two types:

1. \(\sum_{i=1}^{p} \alpha_i \nu_i - \mu_j\) is a root of \(\mathfrak{r}\).
2. \(\sum_{i=1}^{p} \alpha_i \nu_i - \nu_j\) is a root of \(\mathfrak{r}\).

Let us denote by
\[
R = \{ (\alpha, j) \mid (\alpha, j) \text{ resonant of type (1)} \}
\]
and by
\[
R' = \{ (\alpha, j) \mid (\alpha, j) \text{ resonant of type (2)} \}
\]

Theorem 4.7 \(T|_{\mathfrak{r}}\) is normalizable in the sense of [2], that means, there exists an analytic operator \(\phi = \sum \phi^{k}\) in \(X_0(E)\), such that \(\phi^{1}\) is invertible and, for any \(X\) in \(\mathfrak{r}\), \(\phi \ast T_X \circ \phi^{-1}\) is in the form:
\[
\sum_{(\alpha, i) \in R} \Lambda_{i}^{\alpha}(X) y_{\alpha}^{i} \frac{\partial}{\partial x_{i}} + \sum_{(\alpha, i) \in R'} \Gamma_{i}^{\alpha}(X) y_{\alpha}^{i} \frac{\partial}{\partial y_{i}}.
\]

Proof. Let \(X_0\) be a resonance vector of \(\mathfrak{r}\) such that \([X_0, s] = 0\). It is well known that \(T_{X_0}\) is analytically normalizable: there exists an analytic operator \(\phi = \sum \phi^{k}\) in \(X_0(E)\), such that \(\phi^{1}\) is invertible and, for any \(X\) in \(\mathfrak{r}\),
\[
\phi \ast T_{X_0} \circ \phi^{-1} = T'_{X_0},
\]
where \(T'_{X_0} = S^{1}_{X_0} + N_{X_0}\) with
\[
S^{1}_{X_0} = \sum_{i=1}^{p} \mu_i(X_0) x_i \frac{\partial}{\partial x_i} + \sum_{i=1}^{q} \nu_i(X_0) y_i \frac{\partial}{\partial y_i},
\]
and
\[
[S^{1}_{X_0}, N_{X_0}] = 0.
\]
Therefore, \(N_{X_0}\) has the following form
\[
N_{X_0} = \sum_{(\alpha, i) \in R_0} \Lambda_{i}^{\alpha}(X_0) y_{\alpha}^{i} \frac{\partial}{\partial x_{i}} + \sum_{(\alpha, i) \in R'_0} \Gamma_{i}^{\alpha}(X_0) y_{\alpha}^{i} \frac{\partial}{\partial y_{i}},
\]
where
\[ R_0 = \{ (\alpha, i) \mid \sum_{j=1}^{q} \alpha_j \nu_j - \mu_i = 0 \} \quad \text{and} \quad R_0' = \{ (\alpha, i) \mid \sum_{j=1}^{q} \alpha_j \nu_j - \nu_i = 0 \}. \]

The operator \( \phi \) normalizes \( T|_r \) (see [2]).

**Proposition 4.8** The normalization operator \( \phi \) of \( T|_r \) leaves invariant \( T|_m \).

**Proof.** To prove that \( \phi \) leaves invariant \( T|_m \) we will prove that
\[
\phi = \cdots (I + W_k) \circ \cdots \circ (I + W_1),
\]
where \( I \) is the identity of \( E \) and \( W_k \in L(\otimes^k E, E) \).

(a) \( B^1_{X_0} = H_{X_0} + K_{X_0} \) being decomposed into a semisimple part \( S^1_{X_0} \) and a nilpotent part, therefore, to reduce \( T^1_{X_0} \) it suffices to reduce \( A^1_{X_0} \). We decompose \( A^1_{X_0} \):
\[
A^1_{X_0} = A^1_{0X_0} + A^1_{1X_0},
\]
where
\[
A^1_{0X_0} = \sum_{(i,j), \nu_j = \mu_i} a_{ij}(X_0) y_i \frac{\partial}{\partial x_i} \in \ker \text{ad} S^1_{X_0}
\]
and
\[
A^1_{1X_0} = \sum_{(i,j), \nu_j \neq \mu_i} a_{ij}(X_0) y_i \frac{\partial}{\partial x_i} \in \ker \text{ad} S^1_{X_0}.
\]
We reduce \( A^1_{X_0} \) by removing \( A^1_{1X_0} \). Indeed, for any \( X \) and \( Y \) in \( r \),
\[
A^1_{[X,Y]} = [A^1_X, B^1_Y] + [B^1_X, A^1_Y],
\]
therefore, there exists \( W^1 \) in \( L(E) \) and a 1-cocycle \( V^1 \) such that
\[
A^1_X = [B^1_X, W^1] + V^1.
\]
In particular, \( A^1_{X_0} = [B^1_{X_0}, W^1] + V^1 \), then we can choose, for instance,
\[
W^1 = (\text{ad} B^1_{X_0}(A^1_{X_0})) \quad \text{and} \quad V^1_{X_0} = A^1_{0X_0}.
\]
The stability of the eigenvector subspaces of \( \text{ad} S^1_{X_0} \) by \( B^1_{X_0} \) proves that \( W^1 \in L(E_2, E_1) \), therefore, \( (I + W^1) \) is invertible and
\[
(I + W^1) \star T^1_{X_0} \circ (I + W^1)^{-1} = B^1_{X_0} + A^1_{0X_0}, \quad (1)
\]
\[
(I + W^1) \star \frac{\partial}{\partial x_i} \circ (I + W^1)^{-1} = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, p. \quad (2)
\]
\( T|_m \) is then stable by \( (i + W^1) \).
b) Now, we assume that $T_{X_0}$ is normalized up to order $k - 1$, we write

$$T_{X_0} = S^1_{X_0} + \sum_{1 \leq j < k} N^j_{X_0} + T^k_{X_0} + \sum_{j > k} T^j_{X_0}$$

and we decompose $T^k_{X_0}$:

$$T^k_{X_0} = T^k_{0X_0} + T^k_{1X_0},$$

where

$$T^k_{0X_0} \in \ker \text{ad} S^1_{X_0} \quad \text{and} \quad T^k_{1X_0} \in \text{Im} \text{ad} S^1_{X_0}.$$ 

Therefore, $T^k_{0X_0}$ has the following form

$$T^k_{0X_0} = N^k_{X_0} = \sum_{(\alpha, i) \in R_0} \Lambda^\alpha_i(X_0) y^\alpha \frac{\partial}{\partial x_i} + \sum_{(\alpha, i) \in R'_0} \Gamma^\alpha_i(X_0) y^\alpha \frac{\partial}{\partial y_i}.$$ 

We seek an operator $W_k$ in $L(\otimes_s^k E, E)$ such that

$$(1 + W_k) \star T_{X_0} = (S^1_{X_0} + \sum_{1 \leq j \leq k} N^j_{X_0} + \sum_{j > k} T^j_{X_0}) \circ (1 + W_k).$$

We can choose

$$W_k = (\text{ad} T^1_{X_0})^{-1}(T^1_{1X_0}).$$

The eigenvector subspaces of ad $S^1_{X_0}$ are stable by ad $T^1_{X_0}$, therefore, $W_k$ has the following form:

$$W_k = \sum_{(\alpha, i) \in R_0} W^\alpha_i y^\alpha \frac{\partial}{\partial x_i} + \sum_{(\alpha, i) \in R'_0} W'^\alpha_i y^\alpha \frac{\partial}{\partial y_i}.$$ 

Thus,

$$(I + W_k) \star \frac{\partial}{\partial x_i} \circ (I + W_k)^{-1} = \frac{\partial}{\partial x_i}, \quad i = 1, \ldots, p.$$ 

We proceed by induction, so, we construct

$$\phi = \cdots (I + W_k) \circ \cdots \circ (I + W_1)$$

which normalizes $T|_s$ and leaves invariant $T|m$.

5 Linearization of $T|_s$

Let us now consider the representation $(T', E)$ of $\mathfrak{g}$ in $E$, defined on $\mathfrak{g}$ by

$$T'_X = \phi \star T_X \circ \phi^{-1}.$$ 

The representation being $T'|_s$, as in [2], we construct an analytic invertible operator $\psi \in \mathcal{A}_0(E)$, which linearizes $T'|_s$ and leaves $T'|_s$ in normal form. This is possible through the choice of $X_0$ switching with $\mathfrak{s}$: $[X_0, \mathfrak{s}] = 0$.

On the other hand, by construction, the components of $\psi$ are independent of the coordinates $x_1, \ldots, x_p$, then $\psi$ leaves invariant $T'|_m = T|m$. 

10
6 Recapitulation

Consider a nonlinear analytic representation \((T, E)\) of a complex finite dimensional Lie algebra \(g\) in \(E = \mathbb{C}^n\), such that

\[
(D_1) \quad g = g_0 + m,
\]

where \(m\) is a \(p\)-dimensional abelian ideal of \(g\) and \(g_0\) is a subalgebra of \(g\). Moreover, we assume that

\[
T_X^0 = 0 \quad \text{if} \quad X \in g_0 \quad \text{and} \quad T_X^0 \neq 0 \quad \text{if} \quad X \in m \setminus \{0\}.
\]

We consider a Levi-Malcev decomposition of \(g_0\):

\[
(D_2) \quad g = r \oplus s
\]

where \(s\) is semi-simple and \(r\) is the solvable radical of \(g\). Then, we have

**Theorem 6.1** The representation \((T, E)\) of \(g\) can be normalized with respect the decompositions \((D_1)\) and \((D_2)\). That means, there exists a coordinate system \((x_1, \ldots, x_p, y_1, \ldots, y_q)\) of \(E\) such that

a) For all \(X \in m\), \(T_X\) can be expressed as

\[
T_X = \sum_{i=1}^{p} a_i \frac{\partial}{\partial x_i},
\]

where \(a_i \in m^*\).

b) \(T|_s\) is a linear representation of \(s\) in \(E\).

c) For any \(X\) in \(r\), \(T_X\) is in the form:

\[
T_X = \sum_{(\alpha, i) \in R} A_i^\alpha(X)y_\alpha \frac{\partial}{\partial x_i} + \sum_{(\alpha, i) \in R'} \Gamma_i^\alpha(X)y_\alpha \frac{\partial}{\partial y_i},
\]

where \(R_0 = \{(\alpha, i) \mid \sum_{j=1}^{q} \alpha_j \nu_j - \mu_i = 0\}\) and \(R'_0 = \{(\alpha, i) \mid \sum_{j=1}^{q} \alpha_j \nu_j - \nu_i = 0\}\).

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