1. Introduction

Whether or not a given system has positive ground state energy is a widely studied problem with significant repercussions in physics, particularly in quantum mechanics. It follows from the classical Hardy inequality that the bottom of the spectrum of the Dirichlet Laplacian on a domain in $\mathbb{R}^n$ that satisfies the outer cone condition is positive if and only if its inradius is finite (see [D95]). Whereas spectral behavior of the Dirichlet Laplacian is insensitive to boundary geometry, the story for the $\bar{\partial}$-Neumann Laplacian is different. Since the work of Kohn [Ko63, Ko64] and Hörmander [H65], it has been known that existence and regularity of the $\bar{\partial}$-Neumann Laplacian closely depend on the underlying geometry (see the surveys [BSt99, Ch99, DK99, FS01] and the monographs [CS99, St09]).

Let $\Omega$ be a domain in $\mathbb{C}^n$. It follows from the classical Theorem B of Cartan that if $\Omega$ is pseudoconvex, then the Dolbeault cohomology groups $H^{0,q}(\Omega)$ vanish for all $q \geq 1$. (More generally, for any coherent analytic sheaf $\mathcal{F}$ over a Stein manifold, the sheaf cohomology groups $H^q(X, \mathcal{F})$ vanish for all $q \geq 1$.) The converse is also true ([Se53], p. 65). Cartan’s Theorem B and its converse were generalized by Laufer [L66] and Siu [Siu67] to a Riemann domain over a Stein manifold. When $\Omega$ is bounded, it follows from Hörmander’s $L^2$-existence theorem for the $\bar{\partial}$-operator that if $\Omega$ is in addition pseudoconvex, then the $L^2$-cohomology groups $\tilde{H}^{0,q}(\Omega)$ vanish for $q \geq 1$. The converse of Hörmander’s theorem also holds, under the assumption that the interior of the closure of $\Omega$ is the domain itself. Sheaf theoretic arguments for the Dolbeault cohomology groups can be modified to give a proof of this fact (cf. [Se53, L66, Siu67, Br83, O88]; see also [Fu05] and Section 3 below).

In this expository paper, we study positivity of the $\bar{\partial}$-Neumann Laplacian, in connection with the above-mentioned classical results, through the lens of spectral theory. Our emphasis is on the interplay between spectral behavior of the $\bar{\partial}$-Neumann Laplacian and the geometry of the domains. This is evidently motivated by Marc Kac’s famous question “Can one hear the shape of a drum?” [Ka66]. Here we are interested in determining the geometry of a domain in $\mathbb{C}^n$ from the spectrum of the $\bar{\partial}$-Neumann Laplacian. (See [Fu05, Fu08] for related results.) We make an effort to present a more accessible and self-contained
treatment, using extensively spectral theoretic language but bypassing sheaf cohomology theory.

2. Preliminaries

In this section, we review the spectral theoretic setup for the ∂-Neumann Laplacian. The emphasis here is slightly different from the one in the extant literature (cf. [PK72, CS99]). The ∂-Neumann Laplacian is defined through its associated quadratic form. As such, the self-adjoint property and the domain of its square root come out directly from the definition.

Let \( Q \) be a non-negative, densely defined, and closed sesquilinear form on a complex Hilbert space \( H \) with domain \( \mathcal{D}(Q) \). Then \( Q \) uniquely determines a non-negative and self-adjoint operator \( S \) such that \( \mathcal{D}(S^{1/2}) = \mathcal{D}(Q) \) and

\[
Q(u, v) = \langle S^{1/2}u, S^{1/2}v \rangle
\]

for all \( u, v \in \mathcal{D}(Q) \). (See Theorem 4.4.2 in [D95], to which we refer the reader for the necessary spectral theoretic background used in this paper.) For any subspace \( L \subset \mathcal{D}(Q) \), let \( \lambda(L) = \sup\{Q(u, u) \mid u \in L, \|u\| = 1\} \). For any positive integer \( j \), let

\[
\lambda_j(Q) = \inf\{\lambda(L) \mid L \subset \mathcal{D}(Q), \dim(L) = j\}.
\]

The resolvent set \( \rho(S) \) of \( S \) consists of all \( \lambda \in \mathbb{C} \) such that the operator \( S - \lambda I : \mathcal{D}(S) \to H \) is both one-to-one and onto (and hence has a bounded inverse by the closed graph theorem).

The spectrum \( \sigma(S) \), the complement of \( \rho(S) \) in \( \mathbb{C} \), is a non-empty closed subset of \([0, \infty)\). Its bottom \( \inf \sigma(S) \) is given by \( \lambda_1(Q) \). The essential spectrum \( \sigma_e(S) \) is a closed subset of \( \sigma(S) \) that consists of isolated eigenvalues of infinite multiplicity and accumulation points of the spectrum. It is empty if and only if \( \lambda_j(Q) \to \infty \) as \( j \to \infty \). In this case, \( \lambda_j(Q) \) is the \( j \)th eigenvalue of \( S \), arranged in increasing order and repeated according to multiplicity.

The bottom of the essential spectrum \( \inf \sigma_e(T) \) is the limit of \( \lambda_j(Q) \) as \( j \to \infty \). (When \( \sigma_e(S) = \emptyset \), we set \( \inf \sigma_e(S) = \infty \).

Let \( T_k : H_k \to H_{k+1} , \ k = 1, 2 \), be densely defined and closed operators on Hilbert spaces. Assume that \( \mathcal{R}(T_1) \subset \mathcal{N}(T_2) \), where \( \mathcal{R} \) and \( \mathcal{N} \) denote the range and kernel of the operators. Let \( T_k^* \) be the Hilbert space adjoint of \( T_k \), defined in the sense of Von Neumann by

\[
\mathcal{D}(T_k^*) = \{u \in H_{k+1} \mid \exists C > 0, |\langle u, T_kv \rangle| \leq C\|v\|, \forall v \in \mathcal{D}(T_k)\}
\]

and

\[
\langle T_k^*u, v \rangle = \langle u, T_kv \rangle, \quad \text{for all } u \in \mathcal{D}(T_k^*) \text{ and } v \in \mathcal{D}(T_k).
\]

Then \( T_k^* \) is also densely defined and closed. Let

\[
Q(u, v) = \langle T_k^*u, T_k^*v \rangle + \langle T_2u, T_2v \rangle
\]

with its domain given by \( \mathcal{D}(Q) = \mathcal{D}(T_k^*) \cap \mathcal{D}(T_2) \). The following proposition elucidates the above approach to the ∂-Neumann Laplacian.

**Proposition 2.1.** \( Q(u, v) \) is a densely defined, closed, non-negative sesquilinear form. The associated self-adjoint operator \( \square \) is given by

\[
\mathcal{D} (\square) = \{f \in H_2 \mid f \in \mathcal{D}(Q), T_2f \in \mathcal{D}(T_2^*), T_1^*f \in \mathcal{D}(T_1)\}, \quad \square = T_1T_1^* + T_2^*T_2.
\]

**Proof.** The closedness of \( Q \) follows easily from that of \( T_1 \) and \( T_2 \). The non-negativity is evident. We now prove that \( \mathcal{D}(Q) \) is dense in \( H_2 \). Since \( \mathcal{N}(T_2) = \mathcal{R}(T_2^* \cap \mathcal{D}(T_2) \cap \mathcal{N}(T_2^*) \cap \mathcal{D}(T_2) = \mathcal{N}(T_2) \oplus \mathcal{D}(T_2) \cap \mathcal{N}(T_2)\).
we have
\[ \mathcal{D}(Q) = \mathcal{D}(T_1^*) \cap \mathcal{D}(T_2) = (\mathcal{N}(T_2) \cap \mathcal{D}(T_1^*)) \oplus (\mathcal{D}(T_2) \cap \mathcal{N}(T_2)^\perp) \]

Since \( \mathcal{D}(T_1^*) \) and \( \mathcal{D}(T_2) \) are dense in \( H_2 \), \( \mathcal{D}(Q) \) is dense in \( \mathcal{N}(T_2) \oplus \mathcal{N}(T_2)^\perp = H_2 \).

It follows from the above definition of \( \square \) that \( f \in \mathcal{D}(\square) \) if and only if \( f \in \mathcal{D}(Q) \) and there exists a \( g \in H_2 \) such that
\[
Q(u, f) = \langle u, g \rangle, \quad \text{for all } u \in \mathcal{D}(Q)
\]
(c.f. Lemma 4.4.1 in [D95]). Thus
\[ \mathcal{D}(\square) \supset \{ f \in H_2 \mid f \in \mathcal{D}(Q), T_2 f \in \mathcal{D}(T_2^*), T_1^* f \in \mathcal{D}(T_1) \}. \]

We now prove the opposite containment. Suppose \( f \in \mathcal{D}(\square) \). For any \( u \in \mathcal{D}(T_2) \), we write \( u = u_1 + u_2 \in (\mathcal{N}(T_1^*) \cap \mathcal{D}(T_2)) \oplus \mathcal{N}(T_1^*)^\perp \). Note that \( \mathcal{N}(T_2)^\perp \subset \mathcal{R}(T_2^*)^\perp = \mathcal{N}(T_2) \). It follows from (2.3) that
\[
\langle T_2 u, T_2 f \rangle = \langle T_2 u_1, T_2 f \rangle = Q(u_1, f) = \langle u_1, g \rangle \leq \|u\| \cdot \|g\|.
\]
Hence \( T_2 f \in \mathcal{D}(T_2^*) \). The proof of \( T_1^* f \in \mathcal{D}(T_1) \) is similar. For any \( w \in \mathcal{D}(T_1^*) \), write \( w = w_1 + w_2 \in (\mathcal{N}(T_2) \cap \mathcal{D}(T_1^*)) \oplus \mathcal{N}(T_1^*)^\perp \). Note that \( \mathcal{N}(T_2)^\perp = \mathcal{R}(T_2^*) \subset \mathcal{N}(T_1^*) \). Therefore, by (2.3),
\[
\langle T_1^* w, T_1^* f \rangle = \langle T_1^* w_1, T_1^* f \rangle = Q(w_1, f) = \langle w_1, g \rangle \leq \|w\| \cdot \|g\|.
\]
Hence \( T_1^* f \in \mathcal{D}(T_1^*) = \mathcal{D}(T_1) \). It follows from the definition of \( \square \) that for any \( f \in \mathcal{D}(\square) \) and \( u \in \mathcal{D}(Q) \),
\[
\langle \square f, u \rangle = \langle \square^1 f, \square^1 u \rangle = Q(f, u) = \langle T_1^* f, T_1^* u \rangle + \langle T_2 f, T_2 u \rangle = \langle (T_1^* f + T_2^* f) u, f \rangle.
\]
Hence \( \square = T_1^* + T_2^* T_2 \).

The following proposition is well-known (compare [165], Theorem 1.1.2 and Theorem 1.1.4; [83], Proposition 3; and [Sh92], Proposition 2.3). We provide a proof here for completeness.

**Proposition 2.2.** \( \inf \sigma(\square) > 0 \) if and only if \( \mathcal{R}(T_1) = \mathcal{N}(T_2) \) and \( \mathcal{R}(T_2) \) is closed.

**Proof.** Assume \( \inf \sigma(\square) > 0 \). Then 0 is in the resolvent set of \( \square \) and hence \( \square \) has a bounded inverse \( G: H_2 \rightarrow \mathcal{D}(\square) \). For any \( u \in H_2 \), write \( u = T_1^* T_1^* u + T_2^* T_2 u \). If \( u \in \mathcal{N}(T_2) \), then \( 0 = (T_2 u, T_2 Gu) = (T_2 T_2^* Gu, T_2 Gu) = (T_2 T_2^* Gu, T_2 T_2 Gu) = 0 \) and \( u = T_1^* T_1^* Gu \). Therefore, \( \mathcal{R}(T_1) = \mathcal{N}(T_2) \). Similarly, \( \mathcal{R}(T_2^*) = \mathcal{N}(T_1^*) \). Therefore \( T_2^* \) and hence \( T_2 \) have closed range. To prove the opposite implication, we write \( u = u_1 + u_2 \in \mathcal{N}(T_2) \oplus \mathcal{N}(T_2)^\perp \), for any \( u \in \mathcal{D}(Q) \). Note that \( u_1, u_2 \in \mathcal{D}(Q) \). It follows from \( \mathcal{N}(T_2) = \mathcal{R}(T_1) \) and the closed range property of \( T_2 \) that there exists a positive constant \( c \) such that \( c\|u_1\|^2 \leq \|T_1^* u_1\|^2 \) and \( c\|u_2\|^2 \leq \|T_2 u_2\|^2 \). Thus
\[
c\|u\|^2 = c(\|u_1\|^2 + \|u_2\|^2) \leq \|T_1^* u_1\|^2 + \|T_2 u_2\|^2 = Q(u, u).
\]
Hence \( \inf \sigma(\square) \geq c > 0 \) (cf. Theorem 4.3.1 in [D95]).

Let \( \mathcal{N}(Q) = \mathcal{N}(T_1^*) \cap \mathcal{N}(T_2) \). Note that when it is non-trivial, \( \mathcal{N}(Q) \) is the eigenspace of the zero eigenvalue of \( \square \). When \( \mathcal{R}(T_1) \) is closed, \( \mathcal{N}(T_2) = \mathcal{V}(T_1) \cap \mathcal{N}(Q) \). For a subspace \( L \subset H_2 \), denote by \( P_L \) the orthogonal projection onto \( L^\perp \) and \( T_2|_{L^\perp} \) the restriction of \( T_2 \) to \( L^\perp \). The next proposition clarifies and strengthens the second part of Lemma 2.1 in [Fu05].

**Proposition 2.3.** The following statements are equivalent:
(1) \( \inf \sigma_e(\square) > 0. \)
(2) \( \mathcal{R}(T_1) \) and \( \mathcal{R}(T_2) \) are closed and \( \mathcal{N}(Q) \) is finite dimensional.
(3) There exists a finite dimensional subspace \( L \subset \mathcal{D}(T_1^*) \cap \mathcal{N}(T_2) \) such that \( \mathcal{N}(T_2) \cap L \perp = P_{L \perp}(\mathcal{R}(T_1)) \) and \( \mathcal{R}(T_2 |_{L \perp}) \) is closed.

**Proof.** We first prove (1) implies (2). Suppose \( a = \inf \sigma_e(\square) > 0. \) If \( \inf \sigma(\square) > 0, \) then \( \mathcal{N}(Q) \) is trivial and (2) follows from Proposition 2.2. Suppose \( \inf \sigma(\square) = 0. \) Then \( \sigma(\square) \cap [0, a) \) consists only of isolated points, all of which are eigenvalues of finite multiplicity of \( \square \) (cf. Theorem 4.5.2 in [D95]). Hence \( \mathcal{N}(Q), \) the eigenspace of the eigenvalue 0, is finite dimensional. Choose a sufficiently small \( c > 0 \) so that \( \sigma(\square) \cap [0, c) = \{0\}. \) By the spectral theorem for self-adjoint operators (cf. Theorem 2.5.1 in [D95]), there exists a finite regular Borel measure \( \mu \) on \( \sigma(\square) \times \mathbb{N} \) and a unitary transformation \( U : H_2 \to L^2(\sigma(\square) \times \mathbb{N}, d\mu) \) such that \( U\square U^{-1} = M_\varepsilon, \) where \( M_\varepsilon \varphi(x, n) = \varepsilon \varphi(x, n) \) is the multiplication operator by \( \varepsilon \) on \( L^2(\sigma(\square) \times \mathbb{N}, d\mu). \) Let \( P_{\mathcal{N}(Q)} \) be the orthogonal projection onto \( \mathcal{N}(Q). \) For any \( f \in \mathcal{D}(Q) \cap \mathcal{N}(Q)^\perp, \)
\[
UP_{\mathcal{N}(Q)}f = \chi_{[0,c)}Uf = 0,
\]
where \( \chi_{[0,c)} \) is the characteristic function of \( [0, c) \). Hence \( Uf \) is supported on \( [c, \infty) \). Therefore,
\[
Q(f, f) = \int_{\sigma(\square) \times \mathbb{N}} x|Uf|^2 d\mu \geq c\|Uf\|^2 = c\|f\|^2.
\]
It then follows from Theorem 1.1.2 in [H65] that both \( T_1 \) and \( T_2 \) have closed range.

To prove (2) implies (1), we use Theorem 1.1.2 in [H65] in the opposite direction: There exists a positive constant \( c \) such that
\[
(2.4) \quad c\|f\|^2 \leq Q(f, f), \quad \text{for all } f \in \mathcal{D}(Q) \cap \mathcal{N}(Q)^\perp.
\]
Proving by contradiction, we assume \( \inf \sigma_e(\square) = 0. \) Let \( \varepsilon \) be any positive number less than \( c. \) Since \( L_{[0,\varepsilon]} = \mathcal{R}(\chi_{[0,\varepsilon]}(\square)) \) is infinite dimensional (cf. Lemma 4.1.4 in [D95]), there exists a non-zero \( g \in L_{[0,\varepsilon]} \) such that \( g \perp \mathcal{N}(Q). \) However,
\[
Q(g, g) = \int_{\sigma(\square) \times \mathbb{N}} x\chi_{[0,\varepsilon)}|Ug|^2 d\mu \leq \varepsilon\|Ug\|^2 = \varepsilon\|g\|^2,
\]
contradicting (2.4).

We do some preparations before proving the equivalence of (3) with (1) and (2). Let \( L \) be any finite dimensional subspace of \( \mathcal{D}(T_1^*) \cap \mathcal{N}(T_2). \) Let \( H_2' = H_2 \ominus L. \) Let \( T_2' = T_2 |_{H_2'} \) and let \( T_1'' : T_1' |_{H_2'} \). Then \( T_2' : H_2' \to H_3 \) and \( T_1'' : H_2' \to H_1 \) are densely defined, closed operators. Let \( T_1' : H_1 \to H_2' \) be the adjoint of \( T_1''. \) It follows from the definitions that \( \mathcal{D}(T_1') \subset \mathcal{D}(T_1^*). \) The finite dimensionality of \( L \) implies the opposite containment. Thus, \( \mathcal{D}(T_1) = \mathcal{D}(T_1') \). For any \( f \in \mathcal{D}(T_1) \) and \( g \in \mathcal{D}(T_1^*) = \mathcal{D}(T_1') \cap L^\perp, \)
\[
\langle T_1'f, g \rangle = \langle f, T_1''g \rangle = \langle f, T_1^*g \rangle = \langle T_1f, g \rangle.
\]
Hence \( T_1' = P_{L^\perp} \circ T_1 \) and \( \mathcal{R}(T_1') = P_{L^\perp}(\mathcal{R}(T_1)) \subset \mathcal{N}(T_2'). \) Let
\[
Q'(f, g) = \langle T_1'^*f, T_1'^*g \rangle + \langle T_2'f, T_2'g \rangle
\]
be the associated sesquilinear form on \( H_2' \) with \( \mathcal{D}(Q') = \mathcal{D}(Q) \cap L^\perp. \)

We are now in position to prove that (2) implies (3). In this case, we can take \( L = \mathcal{N}(Q). \) By Theorem 1.1.2 in [H65], there exists a positive constant \( c \) such that
\[
Q(f, f) = Q'(f, f) \geq c\|f\|^2, \quad \text{for all } f \in \mathcal{D}(Q').
\]
We then obtain (3) by applying Proposition 2.2 to $T'_1$, $T'_2$, and $Q'(f, g)$.

Finally, we prove (3) implies (1). Applying Proposition 2.2 in the opposite direction, we know that there exists a positive constant $c$ such that

$$Q(f, f) \geq c\|f\|^2, \quad \text{for all } f \in D(Q) \cap L^\perp.$$  

The rest of the proof follows the same lines of the above proof of the implication (2) $\Rightarrow$ (1), with $N(Q)$ there replaced by $L$.

We now recall the definition of the $\overline{\partial}$-Neumann Laplacian on a complex manifold. Let $X$ be a complex hermitian manifold of dimension $n$. Let $C^\infty_{(0,q)}(X) = C^\infty(X, \Lambda^{0,q}T^*X)$ be the space of smooth $(0, q)$-forms on $X$. Let $\overline{\partial}_q: C^\infty_{(0,q)}(X) \to C^\infty_{(0,q+1)}(X)$ be the composition of the exterior differential operator and the projection onto $C^\infty_{(0,q+1)}(X)$.

Let $\Omega$ be a domain in $X$. For $u, v \in C^\infty_{(0,q)}(X)$, let $\langle u, v \rangle$ be the point-wise inner product of $u$ and $v$, and let

$$\langle\langle u, v \rangle\rangle = \int_\Omega \langle u, v \rangle dV$$

be the inner product of $u$ and $v$ over $\Omega$. Let $L^2_{(0,q)}(\Omega)$ be the completion of the space of compactly supported forms in $C^\infty_{(0,q)}(\Omega)$ with respect to the above inner product. The operator $\overline{\partial}_q$ has a closed extension on $L^2_{(0,q)}(\Omega)$. We also denote the closure by $\overline{\partial}_q$. Thus $\overline{\partial}_q: L^2_{(0,q)}(\Omega) \to L^2_{(0,q+1)}(\Omega)$ is densely defined and closed. Let $\overline{\partial}_q$ be its adjoint. For $1 \leq q \leq n-1$, let

$$Q_q(u, v) = \langle\langle \overline{\partial}_q u, \overline{\partial}_q v \rangle\rangle + \langle\langle \overline{\partial}_q - 1 u, \overline{\partial}_q - 1 v \rangle\rangle$$

be the sesquilinear form on $L^2_{(0,q)}(\Omega)$ with domain $D(Q_q) = D(\overline{\partial}_q) \cap D(\overline{\partial}_q - 1)$. The self-adjoint operator $\square_q$ associated with $Q_q$ is called the $\overline{\partial}$-Neumann Laplacian on $L^2_{(0,q)}(\Omega)$. It is an elliptic operator with non-coercive boundary conditions [ KN65 ] .

The Dolbeault and $L^2$-cohomology groups on $\Omega$ are defined respectively by

$$H^{0,q}(\Omega) = \{ f \in C^\infty_{(0,q)}(\Omega) \mid \overline{\partial}_q f = 0 \} \quad \text{and} \quad \overline{H}^{0,q}(\Omega) = \{ f \in L^2_{(0,q)}(\Omega) \mid \overline{\partial}_q f = 0 \}.$$  

These cohomology groups are in general not isomorphic. For example, when a complex variety is deleted from $\Omega$, the $L^2$-cohomology group remains the same but the Dolbeault cohomology group could change from trivial to infinite dimensional. As noted in the paragraph preceding Proposition 2.3 when $R(\overline{\partial}_{q-1})$ is closed in $L^2_{(0,q)}(\Omega)$, $\overline{H}^{0,q}(\Omega) \cong N(\square_q$).

We refer the reader to [ De ] for an extensive treatise on the subject and to [ H65 ] and [ O82 ] for results relating these cohomology groups.

### 3. Positivity of the spectrum and essential spectrum

Laufer proved in [ L75 ] that for any open subset of a Stein manifold, if a Dolbeault cohomology group is finite dimensional, then it is trivial. In this section, we establish the following $L^2$-analogue of this result on a bounded domain in a Stein manifold:

**Theorem 3.1.** Let $\Omega \subset X$ be a domain in a Stein manifold $X$ with $C^1$ boundary. Let $\square_q$, $1 \leq q \leq n-1$, be the $\overline{\partial}$-Neumann Laplacian on $L^2_{(0,q)}(\Omega)$. Assume that $N(\square_q) \subset W^1(\Omega)$. Then $\inf \sigma(\square_q) > 0$ if and only if $\inf \sigma_e(\square_q) > 0$.
The proof of Theorem 3.1 follows the same line of arguments as Laufer’s. We provide the details below.

Let \( H^\infty(\Omega) \) be the space of bounded holomorphic functions on \( \Omega \). For any \( f \in H^\infty(\Omega) \), let \( M_f \) be the multiplication operator by \( f \):
\[
M_f: L^2_{(0,q)}(\Omega) \rightarrow L^2_{(0,q)}(\Omega), \quad M_f(u) = fu.
\]
Then \( M_f \) induces an endomorphism on \( \widetilde{H}^{0,q}(\Omega) \). Let \( I \) be set of all holomorphic functions \( f \in H^\infty(\Omega) \) such that \( M_f = 0 \) on \( \widetilde{H}^{0,q}(\Omega) \). Evidently, \( I \) is an ideal of \( H^\infty(\Omega) \). Assume \( \inf \sigma_e(\Box_q) > 0 \). To show that \( \widetilde{H}^{0,q}(\Omega) \) is trivial, it suffices to show that \( 1 \in I \).

**Lemma 3.2.** Let \( \xi \) be a holomorphic vector field on \( X \) and let \( f \in I \). Then \( \xi(f) \in I \).

**Proof.** Let \( D = \xi \cdot \partial: C^\infty_{(0,q)}(\Omega) \rightarrow C^\infty_{(0,q)}(\Omega) \), where \( \cdot \) denotes the contraction operator. It is easy to check that \( D \) commute with the \( \Box \) operator. Therefore, \( D \) induces an endomorphism on \( \widetilde{H}^{0,q}(\Omega) \). (Recall that under the assumption, \( \widetilde{H}^{0,q}(\Omega) \cong \mathcal{N}(\Box_q) \subset W^1(\Omega) \).) For any \( u \in \mathcal{N}(\Box_q) \),
\[
D(fu) - fu = \xi \cdot \partial(fu) - f \xi \cdot \partial u = \xi(f)u.
\]
Notice that \( \Omega \) is locally starlike near the boundary. Using partition of unity and the Friedrichs Lemma, we obtain \( [D(fu)] = 0 \). Therefore, \( [\xi(f)u] = [D(fu)] - [fD(u)] = [0] \).

We now return to the proof of the theorem. Let \( F = (f_1, \ldots, f_{n+1}): X \rightarrow \mathbb{C}^{2n+1} \) be a proper embedding of \( X \) into \( \mathbb{C}^{2n+1} \) (cf. Theorem 5.3.9 in [H91]). Since \( \Omega \) is relatively compact in \( X \), \( f_j \in H^\infty(\Omega) \). For any \( f_j \), let \( P_j(\lambda) \) be the characteristic polynomial of \( M_{f_j}: \widehat{H}^{0,q}(\Omega) \rightarrow \widehat{H}^{0,q}(\Omega) \). By the Cayley-Hamilton theorem, \( P_j(M_{f_j}) = 0 \) (cf. Theorem 2.4.2 in [HJ85]). Thus \( P_j(f_j) \in I \).

The number of points in the set \( \{(\lambda_1, \lambda_2, \ldots, \lambda_{2n+1}) \in \mathbb{C}^{2n+1} \mid P_j(\lambda_j) = 0, 1 \leq j \leq 2n+1 \} \) is finite. Since \( F: X \rightarrow \mathbb{C}^{2n+1} \) is one-to-one, the number of common zeroes of \( P_j(f_j(z)) \), \( 1 \leq j \leq 2n+1 \), on \( X \) is also finite. Denote these zeroes by \( z^k \), \( 1 \leq k \leq N \). For each \( z^k \), let \( g_k \) be a function in \( I \) whose vanishing order at \( z^k \) is minimal. (Since \( P_j(f_j) \in I \), \( g_k \neq 0 \).) We claim that \( g_k(z^k) \neq 0 \). Suppose otherwise \( g_k(z^k) = 0 \). Since there exists a holomorphic vector field \( \xi \) on \( X \) with any prescribed holomorphic tangent vector at any given point (cf. Corollary 5.6.3 in [H91]), one can find an appropriate choice of \( \xi \) so that \( \xi(g_j) \) vanishes to lower order at \( z^k \). According to Lemma 3.2, \( \xi(g_j) \in I \). We thus arrive at a contradiction.

Now we know that there are holomorphic functions, \( P_j(f_j), 1 \leq j \leq 2n+1 \), and \( g_k, 1 \leq k \leq N \), that have no common zeroes on \( X \). It then follows that there exist holomorphic functions \( h_j \) on \( X \) such that
\[
\sum P_j(f_j)h_j + \sum g_kh_k = 1.
\]
(See, for example, Corollary 16 on p. 244 in [GR65], Theorem 7.2.9 in [H91], and Theorem 7.2.5 in [Kr01]. Compare also Theorem 2 in [Sk72].) Since \( P_j(f_j) \in I, g_k \in I, \) and \( h_j \in H^\infty(\Omega) \), we have \( 1 \in I \). We thus conclude the proof of Theorem 3.1.

**Remark.** (1) Unlike the above-mentioned result of Laufer on the Dolbeault cohomology groups [L75], Theorem 3.1 is not expected to hold if the boundedness condition on \( \Omega \) is removed (compare [W83]). It would be interesting to know whether Theorem 3.1 remains true if the assumption \( \mathcal{N}(\Box_q) \subset W^1(\Omega) \) is dropped and whether it remains true for unbounded pseudoconvex domains.
(2) Notice that in the above proof, we use the fact that $\mathcal{R}(\partial_{q-1})$ is closed, as a consequence of the assumption $\inf \sigma_e(\Box_q) > 0$ by Proposition 2.3. It is well known that for any infinite dimensional Hilbert space $H$, there exists a subspace $R$ of $H$ such that $H/R$ is finite dimensional but $R$ is not closed. However, the construction of such a subspace usually involves Zorn’s lemma (equivalently, the axiom of choice). It would be of interest to know whether there exists a domain $\Omega$ in a Stein manifold such that $\widetilde{H}^{0,q}(\Omega)$ is finite dimensional but $\mathcal{R}(\partial_{q-1})$ is not closed.

(3) We refer the reader to [Sh09] for related results on the relationship between triviality and finite dimensionality of the $L^2$-cohomology groups using the $\overline{\partial}$-Cauchy problem. We also refer the reader to [B02] for a related result on embedded CR manifolds.

4. Hearing pseudoconvexity

The following theorem illustrates that one can easily determine pseudoconvexity from the spectrum of the $\overline{\partial}$-Neumann Laplacian.

**Theorem 4.1.** Let $\Omega$ be a bounded domain in $\mathbb{C}^n$ such that $\text{int}\,(\text{cl}\,(\Omega)) = \Omega$. Then the following statements are equivalent:

1. $\Omega$ is pseudoconvex.
2. $\inf \sigma(\Box_q) > 0$, for all $1 \leq q \leq n-1$.
3. $\inf \sigma_e(\Box_q) > 0$, for all $1 \leq q \leq n-1$.

The implication $(1) \Rightarrow (2)$ is a consequence of Hörmander’s fundamental $L^2$-estimates of the $\overline{\partial}$-operator [H65], in light of Proposition 2.2, and it holds without the assumption $\text{int}\,(\text{cl}\,(\Omega)) = \Omega$. The implications $(2) \Rightarrow (1)$ and $(3) \Rightarrow (1)$ are consequences of the sheaf cohomology theory dated back to Oka and Cartan (cf. [Se53, L66, Siu67, Br83, O88]). A elementary proof of $(2)$ implying $(1)$, as explained in [Fu05], is given below. The proof uses sheaf cohomology arguments in [L66]. When adapting Laufer’s method to study the $L^2$-cohomology groups, one encounters a difficulty: While the restriction to the complex hyperplane of the smooth function resulting from the sheaf cohomology arguments for the Dolbeault cohomology groups is well-defined, the restriction of the corresponding $L^2$ function is not. This difficulty was overcome in [Fu05] by appropriately modifying the construction of auxiliary $(0,q)$-forms (see the remark after the proof for more elaborations on this point).

We now show that $(2)$ implies $(1)$. Proving by contradiction, we assume that $\Omega$ is not pseudoconvex. Then there exists a domain $\Omega' \supset \Omega$ such that every holomorphic function on $\Omega$ extends to $\Omega'$. Since $\text{int}\,(\text{cl}\,(\Omega)) = \Omega$, $\Omega' \setminus \text{cl}\,(\Omega)$ is non-empty. After a translation and a unitary transformation, we may assume that the origin is in $\Omega' \setminus \text{cl}\,(\Omega)$ and there is a point $z^0$ in the intersection of the $z_n$-plane with $\Omega'$ that is in the same connected component of the intersection of the $z_n$-plane with $\Omega$.

Let $m$ be a positive integer (to be specified later). Let $k_q = n$. For any $\{k_1, \ldots, k_{q-1}\} \subset \{1, 2, \ldots, n-1\}$, we define

$$u(k_1, \ldots, k_q) = \frac{(q - 1)! (\partial \cdots \partial \bar{\partial})^{m-1}}{r_m^q} \sum_{j=1}^q (-1)^j (\partial \cdots \partial \bar{\partial} \hat{\partial}_{k_j} \cdots \hat{\partial}_{k_1} \cdots \partial \bar{\partial}_{k_{q-j}} \cdots \partial \bar{\partial}_{k_q}),$$

where $r_m = |z_1|^{2m} + \ldots + |z_n|^{2m}$. As usual, $\hat{\partial}_{k_j}$ indicates the deletion of $\partial \bar{\partial}_{k_j}$ from the wedge product. Evidently, $u(k_1, \ldots, k_q) \in L^2_{(0,q-1)}(\Omega)$ is a smooth form on $\mathbb{C}^n \setminus \{0\}$. Moreover,
u(k_1, \ldots, k_q) is skew-symmetric with respect to the indices \((k_1, \ldots, k_{q-1})\). In particular, \(u(k_1, \ldots, k_q) = 0\) when two \(k_j\)'s are identical.

We now fix some notational conventions. Let \(K = (k_1, \ldots, k_q)\) and \(J\) a collection of indices from \(\{k_1, \ldots, k_q\}\). Write \(d\bar{z}_K = d\bar{z}_{k_1} \wedge \ldots \wedge d\bar{z}_{k_q}\), \(\bar{z}_K^{m-1} = (\bar{z}_{k_1} \cdots \bar{z}_{k_q})^{m-1}\), and \(\bar{d}\bar{z}_{k_j} = d\bar{z}_{k_1} \wedge \ldots \wedge d\bar{z}_{k_j} \wedge \ldots \wedge d\bar{z}_{k_q}\). Denote by \((k_1, \ldots, k_q \mid J)\) the tuple of remaining indices after deleting those in \(J\) from \((k_1, \ldots, k_q)\). For example, \((2, 5, 3, 1 \mid (4, 1, 6, 4, 6)) = (2, 5, 3)\).

It follows from a straightforward calculation that

\[
\overline{\partial}u(k_1, \ldots, k_q) = -\frac{q!mz_K^{m-1}}{r_{m+1}} (r_m d\bar{z}_K + \left( \sum_{\ell=1}^{n} z_{\ell}^{m-1} z_{\ell}^{m} d\bar{z}_{\ell} \right) \wedge \left( \sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} \bar{d}\bar{z}_{k_j} \right))
\]

\[
= -\frac{q!mz_K^{m-1}}{r_{m+1}} \sum_{\ell=1}^{n} z_{\ell}^{m-1} z_{\ell}^{m} \left( \bar{\bar{z}}_\ell d\bar{z}_K + d\bar{z}_\ell \wedge \sum_{j=1}^{q} (-1)^j \bar{z}_{k_j} d\bar{z}_{k_j} \right)
\]

\[
= m \sum_{\ell=1}^{n-1} z_{\ell}^{m} u(\ell, k_1, \ldots, k_q).
\]

(4.2)

It follows that \(u(1, \ldots, n)\) is a \(\overline{\partial}\)-closed \((0, n-1)\)-form.

By Proposition 2.2, we have \(R(\overline{\partial}u_{q-1}) = N(\overline{\partial}u)\) for all \(1 \leq q \leq n-1\). We now solve the \(\overline{\partial}\)-equations inductively, using \(u(1, \ldots, n)\) as initial data. Let \(v \in L^2_{(0, n-2)}(\Omega)\) be a solution to \(\overline{\partial}v = u(1, \ldots, n)\). For any \(k_1 \in \{1, \ldots, n-1\}\), define

\[
w(k_1) = -mz_{k_1}^m v + (-1)^{1+k_1} u(1, \ldots, n \mid k_1).
\]

Then it follows from (4.2) that \(\overline{\partial}w(k_1) = 0\). Let \(v(k_1) \in L^2_{(0, n-3)}(\Omega)\) be a solution of \(\overline{\partial}v(k_1) = w(k_1)\).

Suppose for any \((q-1)\)-tuple \(K' = (k_1, \ldots, k_{q-1})\) of integers from \(\{1, \ldots, n-1\}, \ q \geq 2\), we have constructed \(v(K') \in L^2_{(0, n-q-1)}(\Omega)\) such that it is skew-symmetric with respect to the indices and satisfies

\[
\overline{\partial}v(K') = m \sum_{j=1}^{q-1} (-1)^j z_{k_j}^m v(K' \mid k_j) + (-1)^{q+1} K' u(1, \ldots, n \mid K') \tag{4.3}
\]

where \(|K'| = k_1 + \ldots + k_{q-1}\) as usual. We now construct a \((0, n-q-2)\)-forms \(v(K)\) satisfying (4.3) for any \(q\)-tuple \(K = (k_1, \ldots, k_q)\) of integers from \(\{1, \ldots, n-1\}\) (with \(K'\) replaced by \(K\)). Let

\[
w(K) = m \sum_{j=1}^{q} (-1)^j z_{k_j}^m v(K \mid k_j) + (-1)^{q+1} K' u(1, \ldots, n \mid K).
\]
Then it follows from (4.2) that
\[
\overline{\partial} w(K) = m \sum_{j=1}^{q} (-1)^j z^m_k \overline{\partial} v(K | k_j) + (-1)^{q+|K|} \overline{\partial} u(1, \ldots, n | K)
\]
\[
= m \sum_{j=1}^{q} (-1)^j z^m_k \left( m \sum_{1 \leq i < j} (-1)^i z^m_k \overline{\partial} v(K | k_j, k_i) + m \sum_{j < i \leq q} (-1)^{i-1} z^m_k \overline{\partial} v(K | k_j, k_i) \right) \nonumber
\]
\[
- (-1)^{q+|K|} \overline{\partial} u(1, \ldots, n | (K | k_j)) + (-1)^{q+|K|} \overline{\partial} u(1, \ldots, n | K)
\]
\[
= (-1)^{q+|K|} \left( -m \sum_{j=1}^{q} (-1)^{j-1} z^m_k u(1, \ldots, n | (K | k_j)) + \overline{\partial} u(1, \ldots, n | K) \right)
\]
\[
= (-1)^{q+|K|} \left( -m \sum_{j=1}^{q} z^m_k u(k_j, (1, \ldots, n | K)) + \overline{\partial} u(1, \ldots, n | K) \right) = 0
\]

Therefore, by the hypothesis, there exists a \( v(K) \in L^2_{(0, n-q-2)}(\Omega) \) such that \( \overline{\partial} v(K) = w(K) \). Since \( w(K) \) is skew-symmetric with respect to indices \( K \), we may also choose a likewise \( v(K) \). This then concludes the inductive step.

Now let
\[
F = w(1, \ldots, n - 1) = m \sum_{j=1}^{n-1} z^m_j v(1, \ldots, j, \ldots, n - 1) - (-1)^{n + \frac{n(n-1)}{2}} u(n),
\]
where \( u(n) = -z^m_n / r_m \), as given by (4.1). Then \( F(z) \in L^2(\Omega) \) and \( \overline{\partial} F(z) = 0 \). By the hypothesis, \( F(z) \) has a holomorphic extension to \( \overline{\Omega} \). We now restrict \( F(z) \) to the coordinate hyperplane \( z' = (z_1, \ldots, z_{n-1}) = 0 \). Notice that so far we only choose the \( v(K)'s \) and \( w(K)'s \) from \( L^2 \)-spaces. The restriction to the coordinate hyperplane \( z' = 0 \) is not well-defined. To overcome this difficulty, we choose \( m > 2(n-1) \). For sufficiently small \( \varepsilon > 0 \) and \( \delta > 0 \),
\[
\left\{ \int_{\{|z'|<\varepsilon\} \cap \Omega} |(F + (-1)^{n+\frac{n(n-1)}{2}} u(n))(\delta z', z_n)|^2 dV(z) \right\}^{1/2}
\]
\[
\leq m \delta^m \varepsilon^m \sum_{j=1}^{n-1} \left\{ \int_{\{|z'|<\varepsilon\} \cap \Omega} |v(1, \ldots, \hat{j}, \ldots, n-1)(\delta z', z_n)|^2 dV(z) \right\}^{1/2}
\]
\[
\leq m \delta^m - 2(2n-1) \varepsilon^m \sum_{j=1}^{n-1} \|v(1, \ldots, \hat{j}, \ldots, n-1)\|_{L^2(\Omega)}.
\]

Letting \( \delta \to 0 \), we then obtain
\[
F(0, z_n) = -(-1)^{n+\frac{n(n-1)}{2}} u(n)(0, z_n) = (-1)^{n+\frac{n(n-1)}{2}} z_n^{-m}.
\]

for \( z_n \) near \( z_0 \). (Recall that \( z_0 \in \Omega \) is in the same connected component of \( \{z' = 0\} \cap \overline{\Omega} \) as the origin.) This contradicts the analyticity of \( F \) near the origin. We therefore conclude the proof of Theorem (4.1).

Remark. (1) The above proof of the implication \((2) \Rightarrow (1)\) uses only the fact that the \( L^2 \)-cohomology groups \( H^0, q(\Omega) \) are trivial for all \( 1 \leq q \leq n-1 \). Under the (possibly) stronger assumption \( \inf \sigma(\Box_q) > 0 \), \( 1 \leq q \leq n-1 \), the difficulty regarding the restriction of the
$L^2$ function to the complex hyperplane in the proof becomes superficial. In this case, the $\overline{\partial}$-Neumannn Laplacian $\Box_q$ has a bounded inverse. The interior ellipticity of the $\overline{\partial}$-complex implies that one can in fact choose the forms $v(K)$ and $w(K)$ to be smooth inside $\Omega$, using the canonical solution operator to the $\overline{\partial}$-equation. Therefore, in this case, the restriction to $\{z' = 0\} \cap \Omega$ is well-defined. Hence one can choose $m = 1$. This was indeed the choice in [L66], where the forms involved are smooth and the restriction posts no problem. It is interesting to note that by having the freedom to choose $m$ sufficiently large, one can leave out the use of interior ellipticity. Also, the freedom to choose $m$ becomes crucial when one proves an analogue of Theorem 4.1 for the Kohn Laplacian because the $\overline{\partial}_b$-complex is no longer elliptic. The construction of $u(k_1, \ldots, k_q)$ in (4.1) with the exponent $m$ was introduced in [Fu05] to handle this difficulty.

(2) One can similarly give a proof of the implication (3) $\Rightarrow$ (1). Indeed, the above proof can be easily modified to show that the finite dimensionality of $\tilde{\mathcal{H}}^{0,q}(\Omega)$, $1 \leq q \leq n - 1$, implies the pseudoconvexity of $\Omega$. In this case, the $u(K)$’s are defined by

$$ u(k_1, \ldots, k_q) = \frac{(\alpha + q - 1)! \overline{z}_m (\overline{z}_{k_1} \cdots \overline{z}_{k_q})^{m-1}}{r^m} \sum_{j=1}^{q} (-1)^j \overline{z}_{k_j} d\overline{z}_{k_1} \wedge \cdots \wedge \hat{d}\overline{z}_{k_j} \wedge \cdots \wedge d\overline{z}_{k_q}, $$

where $\alpha$ is any non-negative integers. One now fixes a choice of $m > 2(n-1)$ and let $\alpha$ runs from 0 to $N$ for a sufficiently large $N$, depending on the dimensions of the $L^2$-cohomology groups. We refer the reader to [Fu05] for details.

(3) As noted in Sections 2 and 3, unlike the Dolbeault cohomology case, one cannot remove the assumption $\text{int}(\text{cl}(\Omega)) = \Omega$ or the boundedness condition on $\Omega$ from Theorem 4.1. For example, a bounded pseudoconvex domain in $\mathbb{C}^n$ with a complex analytic variety removed still satisfies condition (2) in Theorem 3.1.

(4) As in [L66], Theorem 4.1 remains true for a Stein manifold. More generally, as a consequence of Andreotti-Grauert’s theory [AG62], the $q$-convexity of a bounded domain $\Omega$ in a Stein manifold such that $\text{int}(\text{cl}(\Omega)) = \Omega$ is characterized by $\inf \sigma(\Box_k) > 0$ or $\inf \sigma_e(\Box_k) > 0$ for all $q \leq k \leq n - 1$.

(5) It follows from Theorem 3.1 in [H04] that for a domain $\Omega$ in a complex hermitian manifold of dimension $n$, if $\inf \sigma_e(\Box_q) > 0$ for some $q$ between 1 and $n-1$, then wherever the boundary is $C^3$-smooth, its Levi-form cannot have exactly $n-q-1$ positive and $q$ negative eigenvalues. A complete characterization of a domain in a complex hermitian manifold, in fact, even in $\mathbb{C}^n$, that has $\inf \sigma_e(\Box_q) > 0$ or $\inf \sigma(\Box_q) > 0$ is unknown.

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