PEQcheck: Localized and Context-aware Checking of Functional Equivalence
(Technical Report)

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Abstract. A refactoring must preserve the program’s functionality. However, not all refactorings are correct. Preservation of the functionality must be checked. Since programs are rarely formally specified, we use the original program as functional specification and check whether the original and refactored program are functionally equivalent. More concretely, our PEQcheck technique follows a common approach and reduces equivalence checking to program verification. To increase efficiency, PEQcheck generates several verification tasks, namely one per refactored code segment and not one per function as typically done by prior work. Additionally, PEQcheck takes the context of the segments into account. For example, only modified, live variables need to be equivalent and read-only variables can be shared between original and refactored code segments. We proved soundness of our PEQcheck technique and implemented it in a prototype tool. Our evaluation shows that the localized checking of PEQcheck can indeed be beneficial.

Keywords: Functional equivalence · Equivalence checking · Functional Equivalence Checking · Software Verification · Program Generation.

1 Introduction

Refactoring [22,13] is applied to improve the software’s qualities. For example, execution hot spots can be parallelized to improve the software’s performance—the motivation for our work is code parallelization with OpenMP [25]. While a refactoring aims at improving the software’s quality, it simultaneously must preserve the software’s functionality. To avoid that a refactoring inadvertently alters the functionality, the refactored software must be verified.

Different approaches exist to guarantee that a refactored program preserves the functionality. For example, one could prove the correctness of the applied refactoring rules [14,39,19]. However, this does not work for arbitrary, manual refactorings. A common technique used in industry is regression testing [14], but testing typically fails to examine all program behaviors and, thus, may miss regressions. An alternative to testing is formal software verification [11]. Incremental and regression verification techniques [42,17,32,7,29,15] aim at efficient
re-verification of changed programs. Many of those techniques require a specification of the functional behavior, which often does not exist. In contrast, regression verification techniques that check the functional equivalence of the original and refactored software do not suffer from this problem.

To show equivalence of two programs or functions, one can apply relational program verification [54], use a (bi)simulation relation [46,33,10,9], transform the programs into models and prove model equivalence [35,38,34,3], compute symbolic summaries and check if they are equivalent [28,2], convert the equivalence problem into a Horn constraint problem [12], or combine program generation with verification [15,16,31,20,41,8,36,1]. The latter solution encodes the equivalence problem as a verification task, which is then verified by a standard verifier. This solution is particularly appealing because it does not require specific proof techniques. Furthermore, it can directly profit from and easily switch between existing verification technologies.

We are interested in checking equivalence of a sequential program and its OpenMP parallelization. Our goal is to reduce equivalence checking to a program verification problem. Unfortunately, many existing approaches [15,16,31,20,41] focus on sequential programs and are unsound for parallel programs. For example, they assume that a function called with the same inputs (including global variables) returns the same result and, thus, replace function calls by uninterpreted functions. This assumption is, however, not necessarily true when another thread interferes with the function execution. While CIVL [36] and RVT [8] support parallel programs, they perform equivalence checking on program or function level. In order to reduce the state space that needs to be considered during verification, we want to check equivalence on the level of (parallelized) code segments. Among the approaches that reduce equivalence checking to program verification, only the approach of Abadi et al. [1] supports (parallelized) code segments. However, their approach requires a bijection between inputs of the code segments as well as between their outputs. Especially, a bijection between inputs is an unnecessarily strict assumption, which e.g., may prohibit the use of strategy patterns in only one of the code segments.

In this paper, we present PEQCHECK, a sound approach that generates verification tasks to check equivalence of code segments. While our motivation for PEQCHECK is to check the equivalence of pairs of a sequential and a parallelized code segment, PEQCHECK can also be used to check equivalence of sequential code segments as well as to check equivalence of parallel code segments.

PEQCHECK uses dataflow analyses to determine the context of the code segments, i.e., how variables are used in and after the code segments. This information allows PEQCHECK to develop a fine-grained differentiation of variables, which is used to reduce the complexity of the generated verification task. For example, non-modified variables are shared, inputs are only initialized equally when they are used before they are defined in the code segment, and equivalence checking is limited to modified variables that are used after the code segments. Existing approaches partially consider some of these optimizations. For example, SymDiff [20] restricts equivalence checking to modified variables and RVT [15,16]...
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Listing 1.1. Sequential program

```c
int sum2_par(unsigned char N) {
    int j, sum;
    sum = N;
    for (j=N-1; j>=0; j--)
    {
        sum += j;
    }
    return sum + 2;
}
```

Listing 1.2. Parallelized program

```c
int sum2_par(unsigned char N) {
    int i, sum;
    sum = 0;
    #pragma omp parallel for reduction(+:sum)
    for (i=1; i <= N; i++)
    {
        sum += i;
    }
    return sum + 2;
}
```

Listing 1.3. Verification task

```c
int main() {

    unsigned char N;
    int i, j, sum, sum_s;

    N = random_uchar();
    sum_s = N;
    for (j=N-1; j>=0; j--)
    {
        sum_s += j;
    }
    sum = 0;
    #pragma omp parallel for reduction(+:sum)
    for (i=1; i <= N; i++)
    {
        sum += i;
    }
    assert (sum_s == sum);
    return 0;
}
```

**Fig. 1.** Example sequential program, its parallelization, and the derived verification task for equivalence checking.

only initializes global variables that are written to by at least one of the programs. However, to the best of our knowledge, PEQcheck is the first regression verification approach that uses such a fine-grained differentiation of variables.

We proved soundness of our PEQcheck approach. As a proof-of-concept, we also implemented a prototype for PEQcheck and applied it to several examples. Our evaluation shows that non-equivalence is detected and that focusing on code segments in equivalence checking can be beneficial.

### 1.1 Illustration

We demonstrate our PEQcheck approach with the sequential and parallelized program on the left-hand side of Fig. 1. Both programs compute the sum of the first $N$ numbers plus 2. To show that the two functions are equivalent, we prove the equivalence of the two highlighted code segments. The resulting verification task is shown on the right-hand side of Fig. 1. We explain its construction next.

First, we find out which variables are used in the code segments and how. We will use this information to determine which variables (1) can be shared, (2) need to be declared, (3) need to be equivalent, and (4) whether and how to initialize
the variables. To this end, we identify the variables $V$ used in the code segments, which of them are modified ($M$) in the code segment, which of them are used in the code segment before they are defined ($UB$), and which are live after the code segment ($L$).

We get $V_{seq} = \{j, N, sum\}$, $V_{par} = \{i, N, sum\}$, $M_{seq} = \{j, sum\}$, $M_{par} = \{i, sum\}$, $UB_{seq} = UB_{par} = \{N\}$, and $L_{seq} = L_{par} = \{sum\}$.

We use the variable names to relate the variables of the two segments. Thus, our approach fails if there exist variables with the same name, but different types. Furthermore, we allow both code segments to use additional (input) variables, e.g., the parallelized code segment uses additional variable $i$.

After we determined the different sets, we continue with the identification of shared variables, which is important for code generation. Variables that are used in both code segments (i.e., $V_{seq} \cap V_{par}$) and are not modified can be safely shared. In contrast, modified variables that occur in both programs need to be duplicated and the sequential code segment will use the duplicated variables instead. The code segments in our example share common variable $N$ and variable $sum$ needs to be duplicated. Here, we use $sum_*$ for the duplicated variable.

Finally, we build the verification task. A verification task starts with the declaration of the variables, which is the union of $V_{seq}$, $V_{par}$, and the duplicated variables. For our example, we must declare variables $\{i, j, N, sum, sum_\}$.

After the declarations, variables are non-deterministically initialized whenever necessary. All variables that are used before they are defined by the code segment need to be initialized. Hence, we initialize all variables $UB^*_{seq} \cup UB^*_{par}$, where $UB^*_{seq}$ is derived from $UB_{seq}$ by replacing duplicated variables by their duplicate. If a variable and its duplicate are initialized, the duplicate will be initialized with the same value as the original variable. Our example verification task calls function `random ushort` to non-deterministically initialize variable $N$. Thereafter, it executes the sequential and parallelized code segment. The sequential segment uses the duplicated variables where necessary. At last, the task checks the equivalence of all shared, modified variables that may be live afterward, i.e., variables from $(M_{seq} \cup M_{par}) \cap (L_{seq} \cup L_{par})$. It uses one assert statement for each variable that checks the equivalence of the variable and its duplicate. Our example checks the equivalence of variables $sum$ and $sum_*$.

2 Programs

For the sake of representation, we restrict ourselves to imperative programs on integer variables and exclude synchronization primitives because we do not study synchronization issues. The following grammar describes our programs:

\[
S ::= E \mid v := e \mid assert \mid if \langle e \rangle \ then \ S_1 \ else \ S_2 \mid while \langle e \rangle \ do \ S \mid S_1 ; S_2 \mid [S_1 ; \cdots ; S_n]
\]

1. Note that we can safely ignore variables that are only defined in the scope of the code segment. They do not require declaration nor initialization. Neither can they be accessed after the code segment.
2. Providing the relation of the variables, one can overcome this problem.
3. Our implementation supports C programs with OpenMP pragmas for parallelization.
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We let $E$ denote the empty program and expect that arithmetic expressions $aexpr$ and boolean expressions $bexpr$ are constructed by applying standard operators on integers. More complex programs $S$ are composed of subprograms $S_i$. To identify subprograms unambiguously, we associate each basic statement with a label $\ell$ that is unique in the program.

The set $\mathcal{V}$ denotes all program variables and $\mathcal{V}(S) \subseteq \mathcal{V}$ describes the variables of (sub)program $S$, i.e., all variables that occur in an arithmetic or boolean expression of $S$ or on the left-hand side of an assignment in $S$. Similarly, subset $\mathcal{V}(expr) \subseteq \mathcal{V}$ refers to the variables occurring in expression $expr$.

To build proper verification tasks, PEQCHECK needs to rename certain program variables. To this end, we use a bijective, renaming function $\rho : \mathcal{V} \mapsto \mathcal{V}$ and replace all occurrences of any variable $v$ by $\rho(v)$. The result is the renamed program $R(S, \rho)$. Similarly, $R(expr, \rho)$ denotes the renaming of $expr$. In Fig. 1 we apply the renaming function $\rho_{\text{sum2}}$ to the sequential code segment, where $\rho_{\text{sum2}}(\text{sum}) = \text{sum}, \rho_{\text{sum2}}(\text{sum}, s) = \text{sum}$, and $\rho_{\text{sum2}}(v) = v$ otherwise. For the program semantics, we rely on an operational semantics, which defines executions as transitions between execution states. An execution state is a pair of a program plus a data state. A data state $\sigma : \mathcal{V} \mapsto \mathbb{Z}$ assigns to each variable an integer value. We use $\Sigma$ to denote the set of all data states and define $\rho(\sigma) \in \Sigma$ such that for all $v \in \mathcal{V}$ : $\rho(\sigma)(v) = \sigma(\rho^{-1}(v))$. Furthermore, for any $\sigma, \sigma' \in \Sigma$ and any subset $V \subseteq \mathcal{V}$, we write $\sigma \rightharpoonup_V \sigma'$ if for all $v \in V : \sigma(v) = \sigma'(v)$.

An execution step is defined by the rules shown in Fig. 2. The state update $\sigma[v := \sigma(aexpr)]$ returns a new data state $\sigma'$ with $\sigma(w) = \sigma'(w)$ for all $w \in \mathcal{V}$ with $w \neq v$ and $\sigma'(v) = \sigma(aexpr)$. Since we do not specify the expression syntax, we also do not specify their evaluation. However, we require that expression evaluation is (a) deterministic, (b) only depends on the variables used in the expression, i.e., $\forall \sigma, \sigma' \in \Sigma : \sigma \rightharpoonup_V \sigma' \Rightarrow \sigma(\text{expr}) = \sigma'(\text{expr})$, and (c) is consistent with renaming, i.e., $\sigma(\text{expr}) = \rho(\sigma)(R(\text{expr}, \rho))$. In addition, we assume that the evaluation of a variable is the look up of its value and the equivalence of two variables checks that their values are identical, i.e., $\forall v, v' \in \mathcal{V} : \sigma(v = v') \implies \sigma(v) = \sigma(v')$.

\[ \begin{array}{c}
(v := aexpr, \sigma) \xrightarrow{aexpr} (E, \sigma[v := \sigma(aexpr)]) \\
\sigma(\text{bexpr}) = \text{true} \xrightarrow{\text{assert} \_ \_ \_ \text{bexpr}, \sigma} (E, \sigma) \\
\sigma(\text{bexpr}) = \text{false} \xrightarrow{\text{assert} \_ \_ \_ \text{bexpr}, \sigma} (E, \sigma) \\
(\text{if} \_ \_ \_ \text{bexpr} \text{ then } S_1 \text{ else } S_2, \sigma) \xrightarrow{\text{bexpr}} (S_1, \sigma) \\
(\text{while} \_ \_ \_ \text{bexpr} \text{ do } S, \sigma) \xrightarrow{\text{bexpr}} (S, \sigma) \\
(\text{sum2}, \sigma) \xrightarrow{\text{sum2}} (S_1, \sigma') \\
(\text{sum2}, \sigma) \xrightarrow{\text{sum2}} (S_1 \| S_2, \sigma') \\
\end{array} \]
Next, we inductively define the executions $ex(S)$ of a program $S$.

$\sigma \in \Sigma \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
In this section, we present how our approach encodes the partial equivalence of two subprograms into a verification task and show that PEQCHECK is sound. Furthermore, we discuss limitations of PEQCHECK.

To encode partial equivalence of two subprograms, we need to ensure that both get equal inputs. Two approaches exist in the literature: (1) store the state before executing the first subprogram, save the results, and restore the state [20,31] or (2) assign equal values to the inputs of the two subprograms [15,41]. To store and restore the state, one can either use dedicated methods [31] to write and read them from (persistent) memory or copy the variable values to and from additional variables that do not occur in the program. The first option requires
the verifier to understand the dedicated methods, which cannot be assumed for arbitrary verifiers, and we, therefore, exclude it. Similar to the store and restore approach with additional variables, also the approach (2) needs to duplicate (shared, modified) variables. While the first solution has the advantage that it does not need to rename variables, we think that assigning equal values before executing the subprograms allows the verifier to more easily learn about the relation of the variables in the two subprograms. Thus, our encoding, which we discuss next, follows approach (2).

Remember from Section 1.1 that the PEQcheck encoding consists of three parts: (1) the (equal) initialization of variables, (2) the execution of the two subprograms, and (3) checking equivalence of output variables. We start to describe the generic construction of the individual parts, deferring the constraints on the inputs for the moment and forgoing to label program statements.

The goal of the initialization part is to ensure that the common inputs are equal. While equalization of inputs is not necessary for soundness, we use it to increase the success of equivalence checking. Without initialization, the verification of an equivalence task only succeeds if the two subprograms are equal independently of the input. The initialization aims at making duplicated input variables equal and assumes that non-duplicated (input) variables will not be modified by any of the two subprograms. PEQcheck will ensure this assumption. There exist two alternatives to ensure equivalent inputs: (1) assign the duplicated variable the value of the variable or (2) assign the variable the value of the duplicated variable. Since the PEQcheck encoding starts with the initialization, at the beginning of a program all variables are unconstrained, and both variables are part of $V$, it does not matter which alternative one chooses. We chose alternative (2), which reflects that we initialize the variables of the parallelized code segment with the values of the counterparts in the sequential code segment. Given a renaming function $\rho$ and a sequence $V$ of inputs that must be equal and are duplicated, we add one assignment per such variable that assigns to the variable the value of the duplicated variable.

\[
\text{init}(\rho, V) := \begin{cases} 
E & \text{if } V = \langle \rangle \\
v := \rho(v) & \text{if } V = \langle v \rangle \\
v := \rho(v); \text{init}(\rho, V') & \text{if } V = \langle v \rangle \circ V'
\end{cases}
\]

Next, we discuss the equalization part, which should check that output variables are equal. As for initialization, we only check equivalence of duplicated variables. We use assert statements to check equivalence. To learn which output variables threaten partial equivalence, we use one assert statement per output variable that compares the value of the original and duplicated variable. Given a renaming function $\rho$ and a sequence $V$ of outputs that must be equal and are duplicated,
the following definition implements our idea of the equalization part.

\[
equal(\rho, V) := \begin{cases} 
E & \text{if } V = \emptyset \\
\text{assert } \rho(v) == v; & \text{if } V = \{v\} \\
\text{assert } \rho(v) == v; equal(\rho, V') & \text{if } V = \{v\} \circ V'
\end{cases}
\]

Now, we can put everything together into a task for equivalence checking. As shown by the following definition, we need a renaming function and two sets of variables. Set \( I \) describes the variables that should be equally initialized and set \( C \) those that should be checked for equivalence. Given these information, we sequentially compose the initialization, the renamed subprogram \( S_1 \), the subprogram \( S_2 \), and the equalization. We use the function \( \text{toSeq} \) to transform a set of variables into a sequence, e.g., using their lexical ordering, and, thus, make them available for \( \text{init} \) and \( \text{equal} \).

\[
eq \text{task}(S_1, S_2, \rho, I, C) := \text{init}(\rho, \text{toSeq}(I)); \mathcal{R}(S_1, \rho); S_2; equal(\rho, \text{toSeq}(C))
\]

So far, we generically described how to encode an equivalence task. To be sound, we need to carefully choose the inputs, especially the renaming function \( \rho \) and variables \( I \) for the initialization.

First, let us look at the renaming function. To guarantee that the initialization part equalizes \( v \) and its duplicate \( \rho(v) \) for all variables \( v \) in the set \( I \), we require that (a) renaming does not mess up the initialization, i.e., \( \forall v \in I : \rho(v) = v \lor \rho(v) \notin I \). To ensure that the execution of subprograms \( \mathcal{R}(S_1, \rho) \) and \( S_2 \) do not interfere with each other, the renaming function must (b) prohibit interfering, i.e., \( \forall v \in \mathcal{V}(S_1) \cup \mathcal{M}(S_2) : \rho(v) \notin \mathcal{M}(S_2) \) and \( \forall v \in \mathcal{M}(S_1) : \rho(v) \notin \mathcal{V}(S_2) \cup \mathcal{M}(S_1) \). We call renaming functions fulfilling the latter constraints appropriate for renaming. The renaming function \( \rho_{\text{sum2}} \) introduced in the previous section is appropriate for renaming.

To ensure that both subprograms \( S_1 \) and \( S_2 \) get the same input, the initialization part needs to consider all variables that are duplicated and used before definition in \( S_1 \) or \( S_2 \).

In practice, we consider overapproximations \( \text{UB}(S_1) \subseteq U_1 \subseteq \mathcal{V}(S_1) \) and \( \text{UB}(S_2) \subseteq U_2 \subseteq \mathcal{V}(S_2) \) of the variables used before definition and overapproximations \( \mathcal{M}(S_1) \subseteq M_1 \subseteq \mathcal{V}(S_1) \) and \( \mathcal{M}(S_2) \subseteq M_2 \subseteq \mathcal{V}(S_2) \) of the modified variables. We use this information to restrict the equivalence check to a subset \( C \subseteq M_1 \cup M_1 \) of the possibly modified variables. The concrete choice of \( C \) depends on variable liveness and, thus, the subset of output variables. Furthermore, we build a bijective renaming function \( \rho_{\text{switch}} \) that switches all modified variables \( (M_1 \cup M_2) \) with a non-program variable and keeps all other variables. Given an injective function \( \text{switch} : M_1 \cup M_2 \to \mathcal{V} \setminus (\mathcal{V}(S_1) \cup \mathcal{V}(S_2)) \), which describes how to switch, we define \( \rho_{\text{switch}} \) as follows: for all variables \( v \in \mathcal{V} \) the renamed variable is \( \rho_{\text{switch}}(v) = \text{switch}(v) \) if \( v \in M_1 \cup M_2 \), \( \rho_{\text{switch}}(v) = v_m \) if there exists \( v_m \in M_1 \cup M_2 \) and \( \text{switch}(v_m) = v \), and \( \rho_{\text{switch}}(v) = v \) in all other cases.\(^6\)

\(^6\) While the initialization part is not required for soundness, if we include initialization, it must work properly for PEQCHECK to be sound.

\(^7\) We rename the variables in \( \text{img}(\text{switch}) \) to guarantee bijectivity.
that \( \rho_{\text{switch}} \) is renaming function appropriate for renaming and fulfills condition (a) for \( I = (U_1 \cap U_2) \cap (M_1 \cup M_2) \), which we will use during task construction. In our example, we use \( \text{switch}_{\text{sum2}} : \text{sum} \mapsto \text{sum}_{\text{s}} \) to generate \( \rho_{\text{sum2}} \).

Next, we discuss soundness of our encoding. Our encoding should ensure that if the verification of the encoded equivalence task succeeds, i.e., none of its executions violates an assertion, then the two subprogram \( S_1 \) and \( S_2 \) will be partially equivalent with respect to the unmodified variables and the variables \( C \), which are checked for equivalence. The following theorem ensures this property when creating the equivalence task with the inputs discussed above.

**Theorem 1.** Let \( S_1 \) and \( S_2 \) be two (sub)programs, \( UB(S_1) \subseteq U_1 \subseteq V(S_1) \), \( UB(S_2) \subseteq U_2 \subseteq V(S_2) \), \( M(S_1) \subseteq M_1 \subseteq V(S_1) \), \( M(S_2) \subseteq M_2 \subseteq V(S_2) \), \( \rho_{\text{switch}} \) a renaming function, and \( C \subseteq M_1 \cup M_2 \). Define the equivalence task to be \( S = eq_{\text{task}}(S_1, S_2, \rho_{\text{switch}}, (U_1 \cap U_2) \cap (M_1 \cup M_2), C) \).

If all execution \((S, \sigma) \rightarrow^* (S', \sigma') \in \text{ex}(S)\) do not violate an assertion, then \( S_1 \equiv_{\forall \left((M(S_1) \cup M(S_2)) \setminus C\right)} S_2 \).

**Proof.** See appendix A.3

The theorem above tells us how to use PEQCHECK when applying it to the complete program. However, our goal is to split equivalence checking of two programs into equivalence checking of pairs of subprograms.

Given two programs \( S \) and \( S' \), we assume that there exists a partial, injective replacement function \( \gamma \) such that \( S' \) can be derived from \( S \) by replacing all subprogram \( S_1 \) with \( S_1 \in \text{dom}(\gamma) \) by \( \gamma(S_1) \). We write \( \Gamma(S, \gamma) \) to describe the result of this replacement. Furthermore, we assume the following: Programs \( E, |E| \ldots |E| \notin \text{dom}(\gamma) \). The domain \( \text{dom}(\gamma) \) only contains subprograms of \( S_1 \) and all subprograms in the domain \( \text{dom}(\gamma) \) do not occur in a parallel statement of \( S \). Similarly, we assume that all subprograms in the image \( im(\gamma) \) of \( \gamma \) do not occur in a parallel statement of \( S' \). Thus, we e.g. ensure that thread interference cannot invalidate the result of PEQCHECK’s equivalence checking. Note that such a replacement function always exists. One can always use \( \gamma = \{(S, S')\} \).

Given a replacement function \( \gamma \), PEQCHECK builds one equivalence task per pair \( (S_1, S_2) \in \gamma \). For the task generation, we consider overapproximations \( UB(S_1) \subseteq U_1 \subseteq V(S_1) \) and \( UB(S_2) \subseteq U_2 \subseteq V(S_2) \) of the variables used before definition, overapproximations \( M(S_1) \subseteq M_1 \subseteq V(S_1) \) and \( M(S_2) \subseteq M_2 \subseteq V(S_2) \) of the modified variables, and overapproximations \( L(S_1, S, V) \subseteq L_1 \subseteq V \) and \( L(S_2, S', V) \subseteq L_2 \subseteq V \) of the variables live after \( S_1 \) and \( S_2 \), where \( V \) denotes the output variables. Given these sets, we generate the equivalence task \( eq_{\text{task}}(S_1, S_2, \rho_{\text{switch}}, (U_1 \cap U_2) \cap (M_1 \cup M_2), (M_1 \cup M_2) \cap (L_1 \cup L_2)) \).

After generation of the verification tasks, PEQCHECK verifies each verification task and declares \( S \) and \( S' \) to be equivalent if none of the tasks violates an assertion. The following theorem shows that the PEQCHECK approach is sound when the variables used before definition are not overapproximated.

\(^8\) Proved by Lemma 12 in the appendix.
Theorem 2. Let $S$ and $S'$ be two programs, $\gamma$ be a replacement function such that $S' = \Gamma(S, \gamma)$, and $V \subseteq \mathcal{V}$ be a set of outputs. If for all $(S_1, S_2) \in \gamma$ there exists $\mathcal{M}(S_1) \subseteq M_1 \subseteq \mathcal{V}(S_1)$, $\mathcal{M}(S_2) \subseteq M_2 \subseteq \mathcal{V}(S_2)$, $\mathcal{L}(S_1, S, V) \subseteq L_1 \subseteq \mathcal{V}$, $\mathcal{L}(S_2, S', V) \subseteq L_2 \subseteq \mathcal{V}$, and renaming function $\rho_{\text{switch}}$ such that the equivalence task $\text{eq}_{\text{task}}(S_1, S_2, \rho_{\text{switch}}, (\mathcal{U}B(S_1) \cap \mathcal{U}B(S_2)) \cap (M_1 \cup M_2), (M_1 \cup M_2) \cap (L_1 \cup L_2))$ does not violate an assertion, then $S \equiv_V S'$.

Proof. See appendix A.4

In practice, one typically overapproximates the variables used before definition. However, our proof attempts taught us that not all overapproximations are sound. The problem is that modifications are defined semantically while live variables are defined syntactically. If the initialization step equalizes a variable whose value is not identical before, its not modified in variables are defined syntactically. If the initialization step equalizes a variable that is live, is not modified in $S_2$, and becomes live in $S_2$ afterwards, then the comparison in the equalization task will use the wrong value for the variable in $S_2$. One can circumvent this problem if the overapproximation of the modified variables for $S_2$ always considers all assignments in $S_2$.

Theorem 3. Let $S$ and $S'$ be two programs, $\gamma$ be a replacement function such that $S' = \Gamma(S, \gamma)$, and $V \subseteq \mathcal{V}$ be a set of outputs. If for all $(S_1, S_2) \in \gamma$ there exists overapproximations $\mathcal{U}B(S_1) \subseteq U_1 \subseteq \mathcal{V}(S_1)$, $\mathcal{U}B(S_2) \subseteq U_2 \subseteq \mathcal{V}(S_2)$, $\mathcal{M}(S_1) \subseteq M_1 \subseteq \mathcal{V}(S_1)$, $\mathcal{M}(S_2) \cup \{v \in \mathcal{V} | \exists s_2 \rightarrow^* s_k \vdash_{\text{expr}} s_r \in \text{syn}_P(S_2)\} \subseteq M_2 \subseteq \mathcal{V}(S_2)$, $\mathcal{L}(S_1, S, V) \subseteq L_1 \subseteq \mathcal{V}$, $\mathcal{L}(S_2, S', V) \subseteq L_2 \subseteq \mathcal{V}$, and renaming $\rho_{\text{switch}}$ s.t. $\text{eq}_{\text{task}}(S_1, S_2, \rho_{\text{switch}}, (U_1 \cap U_2) \cap (M_1 \cup M_2), (M_1 \cup M_2) \cap (L_1 \cup L_2))$ does not violate an assertion, then $S \equiv_V S'$.

Proof. See appendix A.4

3.1 Discussion

While we proved soundness of our PEQCHECK approach, PEQCHECK is not complete, which could have been expected because functional equivalence of two programs is undecidable [15]. For example, our encoding misses some equivalences. There exist equivalent code segments like in Fig. 8 for which the verification task violates an assertion. The problem in Fig. 8 is that variable $y$ is live, is not used before definition, but is not defined on all program paths. One could circumvent this problem by adding variables that are only modified on some paths to the set of initialized variables. A similar problem occurs when one of the two code segments violates an assertion. To deal with assertions $\text{assert bexpr}$, in the program, one could replace them by $\text{while bexpr do} E$. Another problem is that an equivalence task considers all input values, while during program execution the code segments are likely only executed with a subset of them. One could try to determine (an overapproximation of) the input ranges for code segments and enhance the initialization with code that restricts the inputs to these values. However, it is unlikely that one can determine the precise ranges in all cases. Another aspect is the choice of code segments. For example, code segments must
Listing 1.4. Original program

```c
int foo_orig(int x, int y)
{
    if (x>0) y=1;
    return y;
}
```

Listing 1.5. Modified program

```c
int foo_mod(int x, int y)
{
    if (!(x<=0)) y=1;
    return y;
}
```

Fig. 3. Behaviorally equivalent original and modified subprograms for which our PEQ-CHECK approach fails to show equivalence

Listing 1.6. Verification task

```c
int main()
{
    int x, y,s, y;
    x = random_int();
    if (x>0) y_s=1;
    if (!(x<=0))
        y=1;
    assert (y_s == y);
    return 0;
}
```

Listing 1.7. Sequential program

```c
int sum_seq(int N)
{
    int i, sum = 0, a[N];
    for (i = 0; i < N; i++)
        a[i] = N - i;
    for (i = 0; i < N; i++)
        sum += a[i];
    return sum;
}
```

Listing 1.8. Parallelized program

```c
int sum_par(int N)
{
    int i, sum = 0, a[N];
    #pragma omp parallel for
    for (i = 0; i < N; i++)
        a[i] = i + 1;
    #pragma omp parallel for reduction(+: sum)
    for (i = 0; i < N; i++)
        sum += a[i];
    return sum;
}
```

Fig. 4. Behaviorally equivalent sequential and parallelized program whose first code segments (highlighted in orange) are not identical

be subprograms and they must not occur in a parallel statement. Note that this only limits the granularity of code segments, but not the applicability of the approach. More importantly, we may miss equivalent programs when using a wrong choice of code segments. For example, consider the sequential and the parallelized program shown in Fig. 4 whose first for loops are not identical. If we use two code segments, one per for loop, then equivalence checking fails. In contrast, it succeeds if we choose the code segment to contain both for loops.

4 Experiments

Our goal is to demonstrate the PEQCHECK approach and evaluate whether localized equivalence checking is beneficial. To demonstrate the generality of
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PEQcheck, we apply it to parallelized programs and to different versions of sequential programs. Note that we did not compare PEQcheck with existing approaches that encode equivalence into program verification tasks because they are not available [31] (for C programs [20]) or the compilation failed [15].

**PEQcheck Implementation.** To perform equivalence checking with PEQcheck, one must (1) determine the code segments, (2) generate the verification tasks for the code segments, and (3) verify the tasks. So far, we have tool support for steps (2) and (3) and performed step (1) manually, i.e., in our examples, we inserted pragma statements `#pragma scope i` and `#pragma epochs i` to define the start and end of code segment i.

To generate the verification tasks, we implemented a prototype tool for the approach from Section 3. Our prototype builds on the ROSE compiler framework [30] (v0.9.13.0). It applies ROSE’s live analysis to detect the variables live afterwards and runs ROSE’s reaching definition analysis to identify which variables are modified and which are used before definition. Since these analyses are intraprocedural, we overapproximate the behavior of global variables and parameters passed. For example, we assume that global variables and non-scalar parameters are always live. Similarly, non-scalar parameters passed to and global variables used by another function are always modified and used before definition. However, note that our prototype does not support all C features. For instance, we do not properly handle pointers and do not support all types, especially, we do not support type definitions. Furthermore, we do not support inputs from files or standard input. Similarly, we do check outputs to files, etc.

For the verification of the generated tasks, we use the two existing verifiers CIVL [36] (version 1.20] with theorem prover Z3 [23] (version 4.8.8)) and CPAchecker [6] (version 1.9.1). To verify a task task.c with OpenMP constructs, we run CIVL using the following command line.

```
civl verify -input omp_thread_max=2 -checkDivisionByZero=false -checkMemoryLeak=false -timeout=300 nondet_funs.c task.c
```

Hence, we limit CIVL’s verification to 5 min and two threads. File `nondet_funs.c` implements the random input functions, which are restricted to elements from [-5;5]. For the verification of sequential tasks `task.c`, we run CPAchecker’s default analysis using the following command line.

```
scripts/cpa.sh -default -spec config/specification/Assertion.spc -timelimit 300s -preprocess task.c
```

We use scripts to automatically execute steps (2) and (3) on our examples.

**Environmental Set Up.** We run our experiments on a machine with an Intel i7-8565U CPU (frequency of 1.8 GHz), 32 GB RAM, and an Ubuntu 18.04. To count the lines of codes, we use cloc v1.7. [10]

**Benchmark.** To check equivalence of sequential and parallelized programs, we consider four own examples and parallelized the *spec.c files from the functional equivalence suite (FEVS) [37]. We exclude programs diffusion1d-gd,
Table 1. Results of applying PEQcheck to a sequential program and its parallelization

| Benchmark tasks | #segments | LOC | Pseq | Ppar | Pseg | Pall | Cseq | Cpar | Cseg | Call | status |
|-----------------|-----------|-----|------|------|------|------|------|------|------|------|--------|
| adder-s2        | 1         | 10  | 11   | 21   | 44   | 4    | 5    | 7    | 4    | ✓    | ERR    |
| adder-s-e       | 1         | 10  | 11   | 21   | 44   | 5    | 5    | 6    | 4    | ×    | ERR    |
| adder-s         | 1         | 10  | 11   | 21   | 44   | 5    | 5    | 8    | 4    | ✓    | ERR    |
| ex              | 1         | 12  | 13   | 25   | 25   | 5    | 5    | 5    | 5    | ✓    | ✓      |
| adder2-nd       | 1         | 15  | 16   | 30   | 54   | 5    | 6    | 8    | 4    | ✓    | ERR    |
| adder2          | 2         | 17  | 20   | 28   | 28   | 8    | 5    | 6    | 5    | ✓    | ERR    |
| adder-e         | 2         | 17  | 19   | 26   | 26   | 8    | 5    | 6    | 5    | ✓    | ERR    |
| adder2-nd-e     | 1         | 15  | 16   | 30   | 54   | 6    | 6    | 20   | 4    | ×    | ERR    |
| adder2-nd       | 1         | 15  | 16   | 31   | 54   | 6    | 5    | 83   | 4    | ✓    | ✓      |
| adder           | 2         | 17  | 19   | 26   | 26   | 8    | 5    | 6    | 5    | ✓    | ERR    |
| diffusion1d-nd  | 3         | 44  | 51   | 61   | 197  | 11   | 6    | 52   | 6    | ✓    | ✓      |
| diffusion1d     | 2         | 43  | 45   | 42   | 153  | 8    | 6    | 13   | 48   | ✓    | ×      |
| diffusion2d-nd  | 2         | 50  | 59   | 91   | 237  | 9    | 6    | 13   | 7    | ERR  | ERR    |
| diffusion2d     | 2         | 63  | 67   | 106  | 242  | 9    | 6    | 19   | 306  | ×    | mem    |
| factorial2      | 1         | 11  | 15   | 28   | 26   | 6    | 6    | 8    | 5    | ✓    | ✓      |
| factorial-e     | 1         | 11  | 14   | 27   | 25   | 5    | 5    | 6    | 5    | ✓    | ×      |
| factorial       | 1         | 11  | 12   | 25   | 23   | 6    | 6    | 8    | 5    | ✓    | ✓      |
| gauceslim-e     | 2         | 98  | 108  | 64   | 228  | 9    | 6    | 23   | 5    | ×    | mem    |
| gauceslim       | 3         | 100 | 111  | 64   | 229  | 11   | 6    | 28   | 5    | ×    | mem    |
| integrate       | 1         | 59  | 60   | 150  | 163  | 7    | 8    | 1    | 1    | EX   | EX     |
| integrate       | 1         | 59  | 60   | 150  | 163  | 7    | 8    | 1    | 1    | EX   | mem    |
| matmat          | 1         | 33  | 37   | 78   | 133  | 6    | 6    | 304  | 5    | TO   | ERR    |
| mean-e          | 1         | 17  | 18   | 31   | 54   | 6    | 5    | 7    | 6    | ×    | ERR    |
| mean            | 1         | 17  | 18   | 31   | 54   | 6    | 6    | 81   | 7    | ✓    | ERR    |
| wave1d-nd       | 2         | 99  | 101  | 61   | 315  | 9    | 7    | 13   | 12   | ×    | ERR    |
| wave1d          | 2         | 87  | 89   | 136  | 295  | 9    | 7    | 17   | 309  | ×    | mem    |
| diffusion2d-gd  | 2         | 99  | 101  | 61   | 315  | 9    | 7    | 13   | 12   | ×    | ERR    |
| nbody           | 2         | 99  | 101  | 61   | 315  | 9    | 7    | 13   | 12   | ×    | ERR    |
| diffusion2d     | 2         | 87  | 89   | 136  | 295  | 9    | 7    | 17   | 309  | ×    | mem    |

diffusion2d-gd, and nbody due to missing header files. We also exclude fib, which we failed to parallelize, and use the iterative instead of the recursive factorial implementation. To deal with I/O inputs, we replaced them by calls to random functions. We also replaced the assert statements. As local code segments, we use the parallel code units.

To check equivalence of two sequential program versions, we consider the non-recursive programs used in Rêve [12] (except for loop4 and loop5, which were not available). As local code segments, we selected the smallest subprogram influenced by a change.

To evaluate whether localized equivalence checking is beneficial, we consider a second set of code segments, named all. This set contains one code segment per program that covers the complete program.

4.1 PEQcheck on Parallelized Programs

Table I shows the results of applying PEQcheck to the benchmark tasks with the parallelized programs. The first section of the table shows our own examples (ex is the example from Fig. I) and the second shows the FEVS examples. Benchmark tasks ending on -e are incorrect parallelizations.
For each benchmark task, we report the number of local code segments\textsuperscript{11}, the lines of code of the sequential and parallelized program as well as the lines of code of the verification tasks for both configurations of code segments. If more than one verification task exists in the local configuration \textit{seg}, we report the largest number of lines of code among all tasks. For both configurations of code segments, we also provide the total time spent on generating the verification tasks plus the total time spent on verification and the verification results.

First, let us look at the results using \textsc{PEQcheck} with local code segments (configuration \textit{seg}). Studying the lines of code (LOC), we observe that an encoded verification task is often larger than the sequential and parallelized program. One reason for the larger size is that the programs are often rather small, i.e., the code segments dominate the program code and the tasks contain the code of the sequential and the parallelized code segment. Additionally, the verification task requires statements for initialization and for inspection of the equivalence of output variables. Inspecting column $t_{\text{seg}}^V$, which reports the time for generating all verification tasks of a benchmark task, we notice that task generation is rather fast. This may change if the input programs get significantly larger. Finally, let us consider the verification of the generated equivalence tasks (columns $t_{\text{seg}}^V$ and $s_{\text{seg}}$). We observe that \textsc{CIVL} fails to verify some equivalence tasks due to an error (ERR), an exception (EX), or a time out (TO). In case of status $\times_{\text{mem}}$, \textsc{CIVL} detects an out of bounds access or an invalid dereference. These violations exist because in the generated tasks the size of pointer-based arrays and their size assumed by the program mismatch. Note that this is not a problem of the \textsc{PEQcheck} approach in general. On the one hand, it is C specific issue with our \textsc{PEQcheck} implementation, which we plan to fix in future. On the other hand, executions causing an out of bounds access or an invalid dereference do not terminate normally and, thus, do not need to be considered for partial equivalence. Considering status $\checkmark$ and $\times$, we notice that equivalence ($\checkmark$) is only reported for equivalent tasks and that the inequivalence ($\times$) of some tasks is detected. Thus, localized equivalence checking with \textsc{PEQcheck} can correctly detect (in)equivalence.

Next, let us compare \textsc{PEQcheck} with localized equivalence checking (configuration \textit{seg}) against all at once checking (configuration all). Although the encoding times are similar in both configurations, we observe that the tasks for configuration all are rarely smaller and often large than the local tasks. This can be an indication that localized checking reduces complexity. Moreover, the verification in configuration all fails more often. One reason for more errors are the handling of (random) input functions. The encoding uses static local variables to ensure that the sequential and parallelized code get the same identical value for their ith call to a random function. \textsc{CIVL} seems to have problems with such variables. Comparing the tasks for which both configurations reported either $\checkmark$ or $\times$, we observe little difference between the two configurations. Summing up, localized equivalence checking is beneficial for our examples.

\textsuperscript{11} By construction, there exists one code segment per task in the set all.
Our second experiment illustrates that the PEQcheck approach is not limited to parallelization. Table 2 shows our evaluation results for the pairs of sequential programs from our benchmark. The structure of Tab. 2 is similar to Tab. 1. Again, let us look at the results using PEQcheck with local code segments (configuration seg). Looking at Tab. 2, we observe that also the sequential verification tasks are typically larger than the two input programs. Additionally, task generation is fast. Considering the verification, we observe that the verifier CPAchecker times out (status TO) for some tasks. Additionally, CPAchecker reports an inequivalence for barthe and loop3, although these two benchmark tasks represent behaviorally equivalent modifications. Our inspection reveals that the equivalence tasks are indeed inequivalent, but the two programs execute the two segments with a restricted set of inputs and are therefore equivalent. Nevertheless, equivalence (status ✓) is reported for some of the equivalent tasks. Moreover, for all pairs of programs that are inequivalent (i.e., benchmark task with suffix -e), inequivalence is detected (i.e., status × is reported). Thus, localized equivalence checking with PEQcheck can also correctly detect (in)equivalence for pairs of sequential programs.

Next, let us compare PEQcheck with localized equivalence checking (configuration seg) against all at once checking (configuration all). In this set of benchmarks, configuration all is identical to checking equivalence of functions, the approach typically taken in the related work. Again, the times for the encoding are similar. Additionally, the sizes of the verification tasks do not differ significantly. The reasons are that even the localized segments enclose most of the functions’ code, often only excluding variable declaration and initialization. Nevertheless, the verification in configuration all times out more often and ex-

### Table 2. Results of applying PEQcheck to pairs of sequential programs

| Benchmark tasks | LOC | timeenc (s) | timenc (max) | timeCPA (s) | status |
|-----------------|-----|-------------|--------------|-------------|--------|
| barthe          | 1   | 16          | 16           | 39          | 35     | 6      | 5      | 6      | 304    | ×      | TO     |
| barthe-e        | 1   | 19          | 22           | 44          | 44     | 6      | 6      | 7      | 326    | ×      | TO     |
| barthe2         | 1   | 14          | 14           | 31          | 29     | 5      | 5      | 379    | 327    | TO     | TO     |
| barthe2-big     | 1   | 19          | 19           | 33          | 39     | 5      | 5      | 349    | 346    | TO     | TO     |
| barthe2-big2    | 1   | 24          | 24           | 33          | 49     | 6      | 5      | 351    | 340    | TO     | TO     |
| bugs15          | 2   | 13          | 13           | 24          | 25     | 6      | 5      | 62     | 6      | ✓      | ✓      |
| digits10        | 1   | 29          | 32           | 75          | 75     | 5      | 5      | 23     | 27     | ✓      | ✓      |
| digits10-e      | 1   | 26          | 29           | 62          | 62     | 5      | 5      | 4      | 5      | ×      | ×      |
| loop            | 1   | 11          | 11           | 24          | 24     | 6      | 5      | 343    | 368    | TO     | TO     |
| loop2           | 1   | 11          | 11           | 26          | 24     | 5      | 5      | 348    | 368    | TO     | TO     |
| loop3           | 1   | 14          | 14           | 26          | 32     | 5      | 6      | 5      | 420    | ×      | TO     |
| loop5-e         | 1   | 14          | 14           | 29          | 29     | 6      | 6      | 6      | 7      | ×      | ×      |
| nested-while    | 2   | 21          | 19           | 26          | 43     | 7      | 5      | 9      | 377    | ✓      | TO     |
| nested-while-e  | 2   | 20          | 18           | 26          | 41     | 7      | 5      | 10     | 8      | ×      | ×      |
| simple-loop     | 1   | 9           | 9            | 17          | 17     | 6      | 6      | 6      | 5      | ✓      | ✓      |
| simple-loop-e   | 1   | 10          | 10           | 18          | 18     | 5      | 5      | 5      | 5      | ×      | ×      |
| while-if        | 1   | 20          | 20           | 55          | 53     | 5      | 5      | 8      | 5      | ✓      | ✓      |
| while-if-e      | 1   | 17          | 17           | 42          | 40     | 5      | 6      | 9      | 6      | ×      | ×      |
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Except for bug15 is not significantly faster. Only in case of bug15, the verification in configuration all profits from restricted input values. Hence, also for the sequential examples localized equivalence checking can be beneficial.

5 Related Work

Checking functional equivalence is a special case of relational program verification [5,43]. To verify relational properties between two programs, Barthe et al. suggest to merge them into a product program [4] and verify the product program. A product program integrates the two programs tighter than a sequential composition because it executes synchronous steps in lockstep.

Still, lots of approaches directly focus on function equivalence. Some approaches [35,38,3,40,34] transform the two programs, which should be proven equivalent, into models and check model equivalence. Other approaches, e.g., [46,33,10,9], aim at finding a (bi)simulation relation between the two programs. Pathg [45] checks equivalence of an OpenMP program and its sequential version (the program without the OpenMP directives). It assumes that only race conditions can cause inequivalence and, therefore, apply symbolic simulation on segments with race conditions to check whether the races affect the output. Fractal symbolic analysis [21] transforms the two programs, which should be proven equivalent, into two simpler programs whose equivalence implies the equivalence of the original programs. Then, it compares the guarded symbolic expressions (descriptions of the effect of a program on a variable) of all variables that are modified and live. Rève [12] translates the equivalence of two deterministic functions into verification conditions, more concretely, Horn constraints with uninterpreted symbols. DSE [28] and ARDiff [2] use symbolic execution to compute the summaries of two functions and check the logical equivalence of the summaries to decide the functional equivalence of the two functions. DSE and ARDiff abstract certain common code regions by uninterpreted functions. In contrast, we analyze the equivalence of code regions that differ.

A common approach, which we also follow, is to encode the equivalence check as a program [15,16,31,20,41,8,36,11]. Often, the encoded program equally, but non-deterministically initializes the inputs (e.g., global variables, parameters), sequentially executes the two functions, and finally checks the equivalence of the outputs (e.g., global variables, return values). Regression verification [15,16], as implemented in RVT, SymDiff [20], and RIE [41] apply this idea for each matched pair of sequential functions. RVT and SymDiff use uninterpreted functions for function call [12] and RIE uses summaries. While RVT [15,16] and RIE [41] rename variables and ensure that matching variables get the same input values, SymDiff [20] and UC-Klee [31] store and restore the initial state and save the state after each function execution. Moreover, RIE [41] considers heap equivalence instead of equivalence of output variables. Also, approaches for parallel programs exist. Chaki et al. [8] show how to encode equivalence of multi-threaded programs

\footnote{RVT only replaces calls that are recursive or already proven equivalent.}
into one sequential verification task per function pair. CIVL [36] can encode and check functional equivalence for concurrent programs using pthreads, OpenMP, MPI, etc. It builds a single composite program and equalizes the inputs. However, CIVL requires that input and output variables are manually specified. Abadi et al. [1] describe a method to encode functional equivalence of a sequential and a parallelized code segment. Their method wraps the two code segments into two separate functions, which get the same input variables, but different output variables. Input and output variable detection uses dataflow analyses, but the detection is not further specified. To be functionally equivalent, the input and output variables of the two segments must be identical.

Some approaches [26,27,28,18] go beyond the detection of equivalence and output under which conditions the two programs are equivalent.

6 Conclusion

Program refactoring is a common task in software development and ensuring the correctness of a refactoring, i.e., ensuring that the refactoring is behavior preserving, has obtained lots of attention. Since programs are rarely formally specified, one typically uses the original program as behavior specification and shows that original and refactored program are functionally equivalent.

We propose PEQCHECK to check functional equivalence of original and refactored program and proved its soundness. PEQCHECK reduces equivalence checking to program verification. Motivated by rather local OpenMP parallelizations, PEQCHECK generates one verification task per modified code segment, e.g., a parallelized code segment. PEQCHECK’s other novelty is that it takes the context of the code segments, i.e., how variables are used in and after the code segments, into account when creating the verification tasks. Together, context-awareness and localized checking aim at reducing the complexity of the verification task.

Our experiments show that such a reduction is indeed achieved for some examples. Furthermore, we demonstrate that PEQCHECK is not limited to equivalence checking of sequential programs and their parallelization, but that it can be used with pairs of sequential programs, too. While our experiments also testify one of PEQCHECK’s incompleteness issues (overapproximation of the input space may lead to a missed equivalence), this problem is common in many modular verification approaches. All in all, our evaluation shows that the PEQCHECK approach is feasible and can be beneficial.

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A Proofs

This appendix contains the proofs of our theorems.

A.1 Auxiliary Lemmas on Program Executions

Lemma 1. Let S be a program.

∀p = (S₀, σ₀) \rightarrow^{op} \ldots \rightarrow^{op} (Sₙ, σₙ) ∈ ex(S) : ∀σ′₀ ∈ Σ : σ₀ = |_{i \in bexpr(p)} σ₀' :
∃(S₀, σ₀') \rightarrow^{op} \ldots \rightarrow^{op} (Sₙ, σₙ') ∈ ex(S) : σₙ = |_{i \in V | σ₀(v) = σ₀'(v) \land 1 ≤ i ≤ n : op_i \equiv v := expr} σₙ'

Proof. Prove by induction on the length of the executions. Show for all programs S that for all executions of length n the following holds:

∀p = (S₀, σ₀) \rightarrow^{op} \ldots \rightarrow^{op} (Sₙ, σₙ) ∈ ex(S) : ∀σ₀' ∈ Σ : σ₀ = |_{i \in bexpr(p)} σ₀' :
∃(S₀, σ₀') \rightarrow^{op} \ldots \rightarrow^{op} (Sₙ, σₙ') ∈ ex(S) : σₙ = |_{i \in V | σ₀(v) = σ₀'(v) \land 1 ≤ i ≤ n : op_i \equiv v := expr} σₙ'

Base case (n=0): Let S and σ₀' be arbitrary. By definition, (S₀, σ₀') ∈ ex(S).
Since n = 0, we have \{ v ∈ V | σ₀(v) = σ₀'(v) \} and \{ v ∈ V | σ₀(v) = σ₀'(v) \}.

Step case (n=1→n): Let S, p = (S₀, σ₀) \rightarrow^{op} \ldots \rightarrow^{op} (Sₙ, σₙ) ∈ ex(S) and
σ₀' ∈ Σ with σ₀ = |_{i \in bexpr(p)} σ₀' be arbitrary. By definition, p' = (S₁, σ₁) \rightarrow^{op} (Sₙ, σₙ) ∈ ex(S₁). Consider three cases:

Case op₁ \equiv \text{nop} Due to semantics, S₁ starts with E or \{E\} \ldots \{E\}, σ₀ = σ₁, and (S₀, σ₀') \rightarrow^{op} (S₁, σ₀'). Furthermore, \{ v ∈ V | σ₀(v) = σ₀'(v) \land 1 ≤ i ≤ n : op_i \equiv v := expr \} = \{ v ∈ V | σ₀(v) = σ₀'(v) \land 2 ≤ i ≤ n : op_i \equiv v := expr \}.

By definition UB_{ex}(p') ⊆ UB_{ex}(p). By induction hypothesis, (S₁, σ₁') \rightarrow^{op} (Sₙ, σₙ') ∈ ex(S₁) : σ₁' = σₙ' ∧ σₙ = |_{i \in V | σ₀(v) = σ₀'(v) \land 2 ≤ i ≤ n : op_i \equiv v := expr} σₙ'.

The induction hypothesis follows.

Case op₁ \equiv \text{bexpr} Due to semantics, S₀ starts with E or \{E\} \ldots \{E\}, σ₀ = σ₁, and (S₀, σ₀') \rightarrow^{bexpr} (S₁, σ₀'). Furthermore, \{ v ∈ V | σ₀(v) = σ₀'(v) \land 1 ≤ i ≤ n : op_i \equiv v := expr \} = \{ v ∈ V | σ₀(v) = σ₀'(v) \land 2 ≤ i ≤ n : op_i \equiv v := expr \}.

By definition UB_{ex}(p') ⊆ UB_{ex}(p). By induction hypothesis, (S₁, σ₁') \rightarrow^{op} (Sₙ, σₙ') ∈ ex(S₁) : σ₁' = σₙ' ∧ σₙ = |_{i \in V | σ₀(v) = σ₀'(v) \land 2 ≤ i ≤ n : op_i \equiv v := expr} σₙ'.

The induction hypothesis follows.

Case op₁ \equiv \text{aexpr} Due to definition, V(aexpr) ⊆ UB_{ex}(p). Thus, σ₀(aexpr) = σ₀'(aexpr). Due to semantics, (S₀, σ₀) \rightarrow^{aexpr} (S₁, σ₀) and (S₀, σ₀') \rightarrow^{aexpr} (S₁, σ₁) implies σ₀ = σ₁. Furthermore, \{ v ∈ V | σ₀(v) = σ₀'(v) \land 1 ≤ i ≤ n : op_i \equiv v := expr \} = \{ v ∈ V | σ₀(v) = σ₀'(v) \land 2 ≤ i ≤ n : op_i \equiv v := expr \}.

By definition UB_{ex}(p') ⊆ UB_{ex}(p). By induction hypothesis, (S₁, σ₁') \rightarrow^{op} \ldots \rightarrow^{op} (Sₙ, σₙ') ∈ ex(S₁) : σ₁' = σₙ' ∧ σₙ = |_{i \in V | σ₀(v) = σ₀'(v) \land 2 ≤ i ≤ n : op_i \equiv v := expr} σₙ'.

The induction hypothesis follows.

\footnote{Since \text{-bexpr} is boolean expression, too, it is covered by this case.}
Lemma 2. Let $S$ be a program.

$$\forall \sigma \in \Sigma : (S,\sigma) \rightarrow^* (E,\sigma') \in ex(S) \implies \sigma = |_{V \setminus M(S)} \sigma'$$

Proof. Prove by induction on the length of the executions. Show for all programs $S$ that $\forall \sigma \in \Sigma : (S,\sigma) \rightarrow^n (E,\sigma') \in ex(S) \implies \sigma = |_{V \setminus M(S)} \sigma'$.

Base case ($n=0$): Then, $S = E$ and $\sigma = \sigma_0$. The hypothesis follows.

Step case ($n=1$): Let $S$ be arbitrary and $p = (S_0,\sigma_0) \rightarrow_{op_1} \ldots \rightarrow_{op_n} (E,\sigma_n) \in ex(S)$. By definition $p' = (S_1,\sigma_1) \rightarrow_{op_1} \ldots \rightarrow_{op_n} (E,\sigma_n) \in ex(S_1)$ and $M(S_1) \subseteq M(S)$. Consider two cases:

Case $op_1 \equiv \text{bexpr}$ or $op_1 = \text{nop}$: Due to semantics, $\sigma_0 = \sigma_1$. By induction, $\sigma_1 = |_{V \setminus M(S_1)} \sigma_n$. Hence, $\sigma_0 = |_{V \setminus M(S)} \sigma_n$.

Case $op_1 \equiv v := \text{aexpr}$: Due to semantics, $\sigma_1 = \sigma_0[v := \sigma_0(\text{aexpr})]$ and, hence, $\sigma_0 = |_{V \setminus \{v\}} \sigma_1$. By induction, $\sigma_1 = |_{V \setminus M(S_1)} \sigma_n$. Since $M(S_1) \cup \{v\} \subseteq M(S)$, we conclude $\sigma_0 = |_{V \setminus M(S)} \sigma_n$.

Lemma 3. Let $S$ be a program and $\rho$ be a renaming function.

$$\forall (S_0,\sigma_0) \rightarrow_{op_1} (S_1,\sigma_1) \in ex(S) :$$

$$\exists (R(S_0,\rho),\rho(\sigma_0)) \rightarrow_{R(\sigma_0)} (S_1,\rho(\sigma_1)) \in ex(R(S,\rho))$$

Proof. Prove by induction over the length $n$ of the derivation of $(S_0,\sigma_0) \rightarrow_{op_1} (S_1,\sigma_1)$ that there exists $(R(S_0,\rho),\rho(\sigma_0)) \rightarrow_{R(\sigma_0)} (S_1,\rho(\sigma_1))$.

Base case ($n=1$) Due to semantics, $S_0$ is an assignment, assert statement, if- or while-statement, empty parallel statement, or a sequence starting with an empty program. Consider eight cases.

Case 1 ($S_0 \equiv v := \text{aexpr}$): Then, $R(S_0,\rho) = (E,\sigma_0[v := \sigma_0(\text{aexpr})])$. Due to the semantics, we conclude that $(S_0,\sigma_0) \rightarrow_{v:=\text{aexpr}} (E,\sigma_0[v := \sigma_0(\text{aexpr})])$ and $(R(S,\rho),\rho(\sigma_0)) \rightarrow_{R(\sigma_0)} (E,\rho(\sigma_0)[v := \sigma_0(\text{aexpr})]) \in ex(R(S,\rho))$.

By definition of $\rho(\sigma_0)$, $\rho(\sigma_0)(R(\text{aexpr},\rho)) = \sigma_0(\text{aexpr})$. We can conclude that $\rho(\sigma_0)[v := \sigma_0(\text{aexpr})] = \rho(\sigma_0)[v := \sigma_0(\text{aexpr})]$.

The hypothesis follows.

Case 2 ($S_0 \equiv \text{assert}_E \text{bexpr}$): Then, $R(S_0,\rho) = \text{assert}_E R(\text{bexpr},\rho)$. Due to the semantics, $(S_0,\sigma_0) \rightarrow_{\text{bexpr}} (S_1,\sigma_1)$ implies $\sigma_1 = \sigma_0, \sigma_0(\text{bexpr}) = true$, and $S_1 = E$. Since $true = \sigma_0(\text{bexpr}) = \rho(\sigma_0)(R(\text{bexpr},\rho))$, we conclude from the semantics that $(R(S,\rho),\rho(\sigma_0)) \rightarrow_{R(\text{bexpr},\rho)} (E,\sigma_1') \in ex(R(S,\rho))$ and $\sigma_1' = \rho(\sigma_0)$. Since $R(E,\rho) = E$, the hypothesis follows.

Case 3 ($S_0 \equiv \text{if} \text{bexpr then } S' \text{ else } S'' \land \sigma_0(\text{bexpr})$): Then, $R(S,\rho) = \text{if } R(\text{bexpr},\rho) \text{ then } R(S',\rho)$ else $R(S'',\rho)$. Due to the semantics, $(S_0,\sigma_0) \rightarrow_{\text{bexpr}}
\((S_1, \sigma_1)\) with \(\sigma_1 = \sigma_0\), and \(S_1 = S'\). Since \(true = \sigma_0(\text{expr}) = \rho(\sigma_0)(R(\text{expr}, \rho))\), we conclude from the semantics that \((R(S_0, \rho), \rho(\sigma_0)) \stackrel{R(\text{expr}, \rho)}{\rightarrow} (R(S', \rho), \rho(\sigma_0))\). Since \(\sigma_0 = \sigma_1\), the induction hypothesis follows.

**Case 4** \((S_0 \equiv \text{if}_{\ell} \ \text{expr} \ \text{then} \ S' \ \text{else} \ S'' \ \text{\&} \ \neg \sigma_0(\text{expr}))\) Analogously to case 3.

**Case 5** \((S_0 \equiv \text{while}_{\ell} \ \text{expr} \ \text{do} \ S' \ \text{\&} \ \sigma_0(\text{expr}))\) We infer that \(R(S, \rho) = \text{while}_{\ell} \ \rho(\text{expr}) \ \text{do} \ R(S', \rho)\). Due to the semantics, \((S_0, \sigma_0) \stackrel{\text{expr}}{\rightarrow} (S_1, \sigma_1)\) with \(\sigma_1 = \sigma_0\), and \(S_1 = S'; S_0\). By definition of \(\rho(\sigma_0)\), \(\rho(\sigma_0)(R(\text{expr}, \rho)) = \sigma_0(\text{expr}) = \text{true}\). Thus, \((R(S_0, \rho), \rho(\sigma_0)) \stackrel{R(\text{expr}, \rho)}{\rightarrow} (R(S', \rho); R(S_0, \rho), \rho(\sigma_0))\). Since \(R(S_1, \rho) = R(S'; S_0, \rho) = R(S', \rho); R(S_0, \rho)\), the induction hypothesis follows.

**Case 6** \((S_0 \equiv \text{while}_{\ell} \ \text{expr} \ \text{do} \ S' \ \text{\&} \ \neg \sigma_0(\text{expr}))\) We infer that \(R(S, \rho) = \text{while}_{\ell} \ \rho(\text{expr}, \rho) \ \text{do} \ R(S', \rho)\). Due to the semantics, \((S_0, \sigma_0) \stackrel{\neg \text{expr}}{\rightarrow} (E, \sigma_0)\). By definition of \(\rho(\sigma_0)\), \(\rho(\sigma_0)(R(\text{expr}, \rho)) = \sigma_0(\text{expr}) = \text{false}\). Due to the semantics, \((R(S, \rho), \rho(\sigma_0)) \stackrel{R(\text{expr}, \rho)}{\rightarrow} (E, \rho(\sigma_0))\). Since \(R(E, \rho) = E\) and \(\neg R(\text{expr}, \rho) = R(\neg \text{expr}, \rho)\), the hypothesis follows.

**Case 7** \((S_0 \equiv [E] \ldots [E])\) Then, \(R(S_0, \rho) = [E] \ldots [E]\). Due to the semantics, \((S_0, \sigma_0) \stackrel{\text{nop}}{\rightarrow} (S_1, \sigma_1)\) implies \(\sigma_1 = \sigma_0\) and \(S_1 = E\). Due to \(R(\text{nop}, \rho) = \text{nop}\), also \((R(S_0, \rho), \rho(\sigma_0)) \stackrel{R(\text{nop}, \rho)}{\rightarrow} (E, \rho(\sigma_0))\) \(\in ex(R(S_0, \rho))\). Since \(R(E, \rho) = E\), the hypothesis follows.

**Case 8** \((S_0 \equiv E; S)\) Then, \(R(S_0, \rho) = E; R(S, \rho)\). Due to the semantics, \((S_0, \sigma_0) \stackrel{\text{nop}}{\rightarrow} (S_1, \sigma_1)\) implies \(\sigma_1 = \sigma_0\) and \(S_1 = S\). Due to \(R(\text{nop}, \rho) = \text{nop}\), also \((R(S_0, \rho), \rho(\sigma_0)) \stackrel{R(\text{nop}, \rho)}{\rightarrow} (R(S, \rho), \rho(\sigma_0))\) \(\in ex(R(S_0, \rho))\). The hypothesis follows.

**Step case** \((n \rightarrow n+1):\) Consider a ordered sequence of the derivation steps, which are derived from a derivation tree for \((S_0, \sigma_0) \stackrel{\text{op}_{\ell}}{\rightarrow} (S_1, \sigma_1)\) such that a step required by another step in the tree occurs earlier in the sequence. Since \(n+1 > 1\), the last step in the sequence is a computational sequential composition steps or a parallel composition step. Consider two cases.

**Case 1** \((S_0 \equiv S; S' \land S \neq E)\) Then, \(R(S_0, \rho) = R(S, \rho); R(S', \rho)\). Due to semantics, there exists \(S, \sigma_0 \stackrel{\text{op}_n}{\rightarrow} S''\) that can be derived in less than \(n + 1\) steps and \(S_1 = S''; S'\). By induction, there exists \((R(S, \rho), \rho(\sigma_0)) \stackrel{R(\text{op}_n, \rho)}{\rightarrow} (S'', \rho, \rho(\sigma_1))\). We conclude \((R(S; S', \rho), \rho(\sigma_0)) \stackrel{R(\text{op}_n, \rho)}{\rightarrow} (S'', \rho, \rho(\sigma_1))\). Since \(R(S', \rho); R(S'', \rho) = R(S'; S''', \rho)\), the induction hypothesis follows.

**Case 2** \((S_0 \equiv [S_0^1] \ldots [S_0^n] \land \exists j \in [1, n]: S_j \neq E)\) Then, \(R(S_0, \rho) = [R(S_1, \rho)] \ldots [R(S_j, \rho)] \ldots [R(S_n, \rho)]\). Due to semantics, there exists \(j \in [1, n]\) such that \(S_j, \sigma_0 \stackrel{\text{op}_{\ell}}{\rightarrow} S''\), which can be derived in less than \(n + 1\) steps, and \(S_1 = [S_0^1] \ldots [S_j] \ldots [S_n']\). By induction, there exists \((R(S', \rho), \rho(\sigma_0)) \stackrel{R(\text{op}_{\ell}, \rho)}{\rightarrow} (S'', \rho, \rho(\sigma_1))\). Hence, \((R([S_1^1] \ldots [S_j^1] \ldots [S_n^1], \rho), \rho(\sigma_0)) \stackrel{R(\text{op}_{\ell}, \rho)}{\rightarrow} ([R(S_1^1, \rho)] \ldots [R(S_j^1, \rho)] \ldots [R(S_n^1, \rho)], \rho(\sigma_1))\). Finally, taking into account that \([R(S_1, \rho)] \ldots [R(S_j, \rho)] \ldots [R(S_n, \rho)] = R([S_0^1] \ldots [S_j] \ldots [S_n'], \rho)\), the induction hypothesis follows.
Lemma 4. Let $S$ be a program and $\rho$ be a renaming function.
\[
\forall (S_0, \sigma_0) \xrightarrow{op_1} \ldots \xrightarrow{op_n} (S_n, \sigma_n) \in ex(S) : \\
\exists (\mathcal{R}(S_0, \rho, \rho(\sigma_0)) \xrightarrow{op_1, \rho} \ldots \xrightarrow{op_n, \rho} (\mathcal{R}(S_n, \rho), \rho(\sigma_n)) \in ex(\mathcal{R}(S, \rho))
\]

Proof. Prove by induction on the length of the executions. Show for all programs $S$ that $\forall (S_0, \sigma_0) \xrightarrow{op_1} \ldots \xrightarrow{op_n} (S_n, \sigma_n) \in ex(S) : \exists (\mathcal{R}(S_0, \rho, \rho(\sigma_0)) \xrightarrow{op_1, \rho} \ldots \xrightarrow{op_n, \rho} (\mathcal{R}(S_n, \rho), \rho(\sigma_n)) \in ex(\mathcal{R}(S, \rho))$.

Base case ($n=0$): Let $S$ and $\sigma \in \Sigma$ be arbitrary. By definition, there exists $(\mathcal{R}(S, \rho), \rho(\sigma)) \in ex(\mathcal{R}(S, \rho))$. The induction hypothesis follows.

Step case ($n \rightarrow n+1$): By definition of executions, $(S_0, \sigma_0) \xrightarrow{op_1} \ldots \xrightarrow{op_n} (S_n, \sigma_n) \in ex(S)$ implies $(S_0, \sigma_0) \xrightarrow{op_1} \ldots \xrightarrow{op_{n-1}} (S_{n-1}, \sigma_{n-1}) \in ex(S)$ and $(S_{n-1}, \sigma_{n-1}) \xrightarrow{op_n} (S_n, \sigma_n)$. Furthermore, $(S_{n-1}, \sigma_{n-1}) \xrightarrow{op_n} (S_n, \sigma_n) \in ex(S)$.

Due to Lemma 4 there exists $(\mathcal{R}(S_{n-1}, \rho), \rho(\sigma_{n-1})) \xrightarrow{op_{n-1}, \rho} (\mathcal{R}(S_n, \rho), \rho(\sigma_n)) \in ex(\mathcal{R}(S_n, \rho))$. Hence, $(\mathcal{R}(S_0, \rho), \rho(\sigma_0)) \xrightarrow{op_1, \rho} \ldots \xrightarrow{op_n, \rho} (\mathcal{R}(S_n, \rho), \rho(\sigma_n)) \in ex(\mathcal{R}(S_n, \rho))$.

Corollary 1. Let $S$ be a program and $\rho$ be a renaming function.
\[
\forall (\mathcal{R}(S_0, \rho, \rho(\sigma_0)) \xrightarrow{op_1, \rho} \ldots \xrightarrow{op_n, \rho} (\mathcal{R}(S_n, \rho), \rho(\sigma_n)) \in ex(S) : \\
\exists (S_0, \sigma_0) \xrightarrow{op_1} \ldots \xrightarrow{op_n} (S_n, \sigma_n) \in ex(S) :
\]

Proof. By construction, $\mathcal{R}(S, \rho))$ is a program and $\rho^{-1}$ is a bijective function.

Due to Lemma 4 \forall $\sigma \in \Sigma$, $\exists (\mathcal{R}(S_0, \rho, \rho(\sigma_0)) \xrightarrow{op_1, \rho} \ldots \xrightarrow{op_n, \rho} (\mathcal{R}(S_n, \rho), \rho(\sigma_n)) \in ex(\mathcal{R}(S, \rho)) \exists (\mathcal{R}(S_1, \rho), \rho^{-1}(\rho(\sigma_0)))) \xrightarrow{op_{n-1}, \rho^{-1}} \ldots \xrightarrow{op_1, \rho^{-1}} (\mathcal{R}(S_0, \rho), \rho^{-1}(\rho(\sigma_n)) \in ex(\mathcal{R}(S, \rho))$.

Since $\rho^{-1} \circ \rho = id$, there exists $(S_0, \sigma_0) \xrightarrow{op_1} \ldots \xrightarrow{op_n} (S_n, \sigma_n) \in ex(S)$.

Lemma 5. Let $S_1$ and $S_2$ be two programs and $\rho$ a renaming function that is appropriate for renaming. $\forall \sigma \in \Sigma : (\mathcal{R}(S_1, \rho), \sigma) \rightarrow^* (E, \sigma_1) \in ex(\mathcal{R}(S_1, \rho)) \wedge (S_2, \sigma) \rightarrow^* (E, \sigma_2) \in ex(S_2) \Rightarrow \exists (\mathcal{R}(S_1, \rho), S_2, \sigma) \rightarrow^* (E, \sigma') \in ex(\mathcal{R}(S_1, \rho); S_2)$:

Example: $\sigma' = \{ v \in \mathcal{M}(S_1) : \rho(v) \} \wedge \sigma' = \{ v \in \mathcal{M}(S_2) : \rho(v) \} \wedge \sigma_1 \wedge \sigma_2 \wedge \rho \in \mathcal{M}(S_1)$

Proof. First, show $\sigma = \{ v \in \mathcal{M}(S_2) : \rho(v) \} \wedge \mathcal{M}(\mathcal{R}(S_1, \rho))$. Due to Lemma 4 $\sigma = \{ v \in \mathcal{M}(S_2) : \rho(v) \} \wedge \mathcal{M}(\mathcal{R}(S_1, \rho))$. Since $\rho$ is appropriate for renaming, we conclude $(\mathcal{R}(S_1, \rho)) \cap \mathcal{M}(\mathcal{R}(S_1, \rho)) = \emptyset$. Hence, $\sigma = \{ v \in \mathcal{M}(S_2) : \rho(v) \} \wedge \mathcal{M}(\mathcal{R}(S_1, \rho))$.

Due to Lemma 4 \exists $(S_2, \sigma) \rightarrow^* (E, \sigma') \in ex(S_2)$. Due to semantics, $(\mathcal{R}(S_1, \rho); S_2, \sigma) \rightarrow^* (S_2, \sigma_1) \rightarrow^* (E, \sigma') \in ex(\mathcal{R}(S_1, \rho); S_2)$. From Lemma 4 we conclude $\sigma_1 = \{ v \in \mathcal{M}(S_2) : \rho(v) \} \wedge \mathcal{M}(\mathcal{R}(S_1, \rho)) = \emptyset$. The claim follows.
Lemma 6. Let $S$ and $S'$ be two programs, $\gamma$ be a replacement function such that $S' = \Gamma(S, \gamma)$, and $V \subseteq \mathcal{V}$ be a set of outputs. If $\exists S_1^1, S_2^1 : S_1^1 \in \text{dom}(\gamma) \land (S = S_1^1; S_2^1 \lor S = S_1^1)$, then $\exists \sigma, \sigma' \in \Sigma : (S, \sigma) \xrightarrow{\text{op}_1} (S_1, \sigma_1) \in \text{ex}(S) \land \sigma = \langle \varepsilon(S, V) \cup \varepsilon(S', V) \rangle \sigma'$.

Proof. Let $S$ and $S'$ be two programs, $\gamma$ be a replacement function such that $S' = \Gamma(S, \gamma)$ and $\exists S_1^1, S_2^1 : S_1^1 \in \text{dom}(\gamma) \land (S = S_1^1; S_2^1 \lor S = S_1^1)$, $V \subseteq \mathcal{V}$ be a set of outputs and $\sigma, \sigma' \in \Sigma$ with $\gamma = \langle \varepsilon(S, V) \cup \varepsilon(S', V) \rangle \sigma'$.

Assume $p = (S, \sigma) \xrightarrow{\text{op}_1} (S_1, \sigma_1) \in \text{ex}(S)$. By definition, $\varepsilon(S) \cup \varepsilon(S') \subseteq \mathcal{L}(S, V) \cup \mathcal{L}(S', V)$, Consider two cases.

Case 1 ($S_1 = E$): Due to the semantics, we conclude that either $S = E; E$, or $S$ is not a sequential composition, but a statement. If $S$ is an assignment, an assertion or a parallel statement, we conclude from $S \notin \text{dom}(\gamma)$, $S' = \Gamma(S, \gamma)$, and $\gamma$ does not replace statements in parallel statements that $S' = S$. Similarly, if $S = E; E$, also $S = S'$. Due to Lemma 1, $\exists (S', \sigma') \xrightarrow{\text{op}_1} (E, \sigma')$ and $\sigma_1 = \langle \varepsilon(S, V) \cup \varepsilon(S', V) \rangle \sigma'$. Due to the definition of live variables, we conclude that $\sigma_1 = \langle \varepsilon(S, V) \cup \varepsilon(E, V) \rangle \sigma'$.

Case 2 ($S_1 \neq E$): Since replacements do not occur in parallel statements and $\exists S_1^1, S_2^1 : S_1^1 \in \text{dom}(\gamma) \land (S = S_1^1; S_2^1 \lor S = S_1^1)$, we conclude that $\exists S_1^1, S_2^1, S_3^1, S_4^1 : S = S_1^1; S_2^1 \lor S' = S_3^1; S_4^1 \lor S = \Gamma(S_1^1, \gamma) \land S = \Gamma(S_2^1, \gamma) \lor S = S_1^1 \land S = S_3^1; S_4^1 \land S' = \Gamma(S_1^1, \gamma)$ and either $S_1^1 = S_3^1$ or $S_1^1$ and $S_3^1$ are either both if- or both-while statements with the same condition and the if/else-body, the loop body of $S_2^1$ is a replacement of the body of $S_1^1$. First, consider the first case ($S_1^1 = S_3^1$). Due to semantics, either (1) $S_1 = E; S_2^1$ and $(S_1^1, \sigma) \xrightarrow{\text{op}_1} (E, \sigma_1)$, (2) $S_1 = S_2^1 \land S_1^1 = E$, $\sigma = \sigma_1$, and $(S, \sigma) \xrightarrow{\text{nop}} (S_1, \sigma_1)$, or (3) $S_1 = S_3^1; S_2^1$ and $(S_1^1, \sigma) \xrightarrow{\text{op}_1} (S_3^1, \sigma_1)$. Due to Lemma 1 in case (1) $\exists (S_1^1, \sigma') \xrightarrow{\text{op}_1} (E, \sigma_1')$ and $\sigma' = \sigma_1'$, and in case (3) $\exists (S_1^1, \sigma') \xrightarrow{\text{op}_1} (S_3^1, \sigma_1')$. Furthermore, $\sigma_1 = \langle \varepsilon(S, V) \cup \varepsilon(S_1, V) \rangle \sigma'_1$. Due to semantics, in case (1) $\exists (S', \sigma') \xrightarrow{\text{op}_1} (E; S_2^1, \sigma'_1)$, in case (2) $(S', \sigma') \xrightarrow{\text{nop}} (S_1^1, \sigma')$, and in case (3) $\exists (S', \sigma') \xrightarrow{\text{op}_1} (S_3^1; S_2^1, \sigma'_1)$. Since $S_1^1 = S_3^1 = \Gamma(S_3^1, \gamma)$, $\gamma$ is only defined for subprograms of $S$ and statements (thus, subprograms) can be uniquely identified via labels, we get $\Gamma(S_3^1, \gamma) = S_3^1$. Hence, $\Gamma(S_2^1; S_2^1, \gamma) = S_5^1; S_4^1 \lor S_5^1 \lor S_4^1 \lor S_5^1$. Similarly, $\Gamma(E; S_2^1, \gamma) = E; \Gamma(S_2^1, \gamma) = S_5^1; S_4^1$. Moreover, $\sigma_1 = \langle \varepsilon(S_1, V) \cup \varepsilon(S_1, V) \rangle \sigma'_1$, and the definition of live variable analyses let us conclude that $\sigma_1 = \langle \varepsilon(S_1, V) \cup \varepsilon(S_1, V) \rangle \sigma'_1$.

Second, consider that $(S_1^1 \neq S_3^1)$. We know that $S_1^1$ and $S_3^1$ are either both if- or both-while statements with the same condition and the if/else-body, the loop body of $S_2^1$ is a replacement of the body of $S_1^1$. Due to semantics, definition of live variable analysis, $\sigma = \langle \varepsilon(S, V) \cup \varepsilon(S_1, V) \rangle \sigma'$, and the replacement function, we
Lemma 8. Let $\sigma = \sigma_1$ and $\sigma' = \sigma'_1$, and either $S = S'_1$ and exists $(S', \sigma') \xrightarrow{op_1} (S'_1, \sigma_1)$ in $ex(S')$ with $S'_1 = \Gamma(S_1, \gamma)$ (due to $S$; while expr do $S$ is no sub-program of $S$) or $S = S'_1; S'_2$ and $(S', \sigma') \xrightarrow{op_2} (S'_1, \sigma_1) \in ex(S')$ with $S'_1 = \Gamma(S_1, \gamma)$. Due to the definition of live variables, we conclude that $\sigma_1 = |_{\Gamma(S_1, \gamma)} \cup \Gamma(S'_1, \gamma)} \sigma'_1$.

Proof. Proof by induction on the cardinality of $V$.

Corollary 2. Let $S$ and $S'$ be two programs, $\gamma$ be a replacement function such that $S' = \Gamma(S, \gamma)$, and $V \subseteq V$ be a set of outputs. For all $(S_0, \sigma_0) \xrightarrow{op_1} \ldots \xrightarrow{op_n} (S_n, \sigma_n) \in ex(S)$, if for all $0 \leq i < n$ not exists $S_i^{a_1}, S_i^{a_2}$ such that $(S_i = S_i^{a_1}; S_i^{a_2} \lor S_i = S_i^{a_1})$ and $S_i^{a_1} \in \text{dom}(\gamma)$, then $\forall \sigma'_0 \in \Sigma : \sigma_0 = |_{\Gamma(S, \gamma)} \cup \Gamma(S', \gamma)} \sigma'_0 \Rightarrow \exists (\Gamma(S_0, \gamma), \sigma'_0) \xrightarrow{op_1} \ldots \xrightarrow{op_n} (\Gamma(S_n, \gamma), \sigma'_n) \in ex(S') : \forall 0 \leq i \leq n : \sigma_i = |_{\Gamma(S_i, \gamma)} \cup \Gamma(S_i, \gamma)} \sigma'_i$.

Proof. Proof by induction.

Base case (i=0): By definition $(S', \sigma) = (\Gamma(S, \gamma), \sigma) \in ex(S')$ for arbitrary $\sigma \in \Sigma$ (including all $\sigma'_0$ with $\sigma_0 = |_{\Gamma(S, \gamma)} \cup \Gamma(S', \gamma)} \sigma'_0$).

Step case $(n - 1 \rightarrow n)$: Due to Lemma 8 there exists $(\Gamma(S, \gamma), \sigma') \xrightarrow{op_1} (\Gamma(S_1, \gamma), \sigma'_1)$ with $\sigma_1 = |_{\Gamma(S, \gamma)} \cup \Gamma(S', \gamma)} \sigma'_1$. By induction, $(\Gamma(S_1, \gamma), \sigma'_1) \xrightarrow{op_2} \ldots \xrightarrow{op_n} (\Gamma(S_n, \gamma), \sigma'_n) \in ex(S') \land \forall 1 \leq i \leq n : \sigma_i = |_{\Gamma(S_i, \gamma)} \cup \Gamma(S_i, \gamma)} \sigma'_i$. By definition, the induction hypothesis follows.

A.2 Auxiliary Lemmas for Soundness of Initialization and Equalization Part

Lemma 7. Let $\rho$ be a renaming function and $V \subseteq V$ a subset of variables such that $\forall v \in V : \rho(v) = v \lor \rho(v) \notin V$. Then, $\forall \sigma \in \Sigma : (\text{init}(\rho, \text{toSeq}(V)), \sigma) \rightarrow^* (E, \sigma') \Rightarrow \forall v \in V : \sigma'(v) = \rho'(v) = \sigma(\rho(v))$.

Proof. Proof by induction on the cardinality of $V$.

Base case ($|V| = 0$) $|V| = 0$ implies $V = \emptyset$, the hypothesis trivially holds.

Step case ($|V| = n, n > 0$) Let $\text{toSeq}(V) = v_1, \ldots, v_n$ and $\sigma \in \Sigma$ be arbitrary. Then, $\text{init}(\rho, \text{toSeq}(V)) = \text{init}(\rho, v_1, \ldots, v_n) = v := \rho(v); \text{init}(\rho, v_2, \ldots, v_n) = v := \rho(v); \text{init}(\rho, \text{toSeq}(V \setminus \{v_1\}))$. Due to semantics, $(\text{init}(\rho, \text{toSeq}(V)), \sigma) \rightarrow^* (E, \sigma')$ implies that $(\text{init}(\rho, \text{toSeq}(V)), \sigma) \xrightarrow{v_1 := \rho(v_1)} (\text{init}(\rho, \text{toSeq}(V \setminus \{v_1\}), \sigma[v_1 := \rho(v_1)]) \rightarrow^* (E, \sigma')$. Thus, we get $\text{init}(\rho, \text{toSeq}(V \setminus \{v_1\}), \sigma[v_1 := \rho(v_1)]) = \sigma(v_1)$. By induction, $\forall v \in V \setminus \{v_1\} : \sigma'(v) = \sigma'(\rho(v)) = \sigma(\rho(v))$. Since $\forall v \in V : \rho(v) = v \lor \rho(v) \notin V$ and $\text{M}(\rho, \text{toSeq}(V \setminus \{v_1\})) \subseteq \{v_2, \ldots, v_n\}$, we get $\text{M}(\rho, \text{toSeq}(V \setminus \{v_1\}) \cap \{v_1, \rho(v_1)\} = \emptyset$. Due to Lemma 2 $\sigma[v := \rho(v)](v_1) = \sigma'(v_1)$ and $\sigma[v := \rho(v)](\rho(v_1)) = \sigma'(\rho(v_1))$. Hence, $\sigma'(v_1) = \sigma'(\rho(v)) = \sigma(\rho(v))$. The induction hypothesis follows.

Lemma 8. Let $\rho$ be a renaming function and $V \subseteq V$ a subset of variables. Then, $\forall \sigma \in \Sigma : (\text{equal}(\rho, \text{toSeq}(V)), \sigma) \rightarrow^* (E, \sigma') \Rightarrow \forall v \in V : \sigma(v) = \sigma(\rho(v))$.

Proof. Proof by induction on the cardinality of $V$.

Base case ($|V| = 0$) $|V| = 0$ implies $V = \emptyset$, the hypothesis trivially holds.
Step case \((|V| = n, n > 0)\) Let \(\text{toSeq}(V) = v_1, \ldots, v_n\) and \(\sigma \in \Sigma\) be arbitrary. From definition, we conclude that \(\text{equal}(\rho, \text{toSeq}(V)) = \text{equal}(\rho, v_1, \ldots, v_n) = \text{assert } (\rho(v_1) = v_1; \text{equal}(\rho, v_2, \ldots, v_n) = \text{assert } (\rho(v_1) = v_1; \text{equal}(\rho, \text{toSeq}(V \setminus \{v_1\})))\). Due to semantics and \(\text{equal}(\rho, \text{toSeq}(V), \sigma) \rightarrow^* (E, \sigma')\), we infer that \(\text{equal}(\rho, \text{toSeq}(V), \sigma) \rightarrow^* \text{equal}(\rho, \text{toSeq}(V \setminus \{v_1\})), \sigma) \rightarrow^* (E, \sigma')\) and \(\sigma(\rho(v_1)) = v_1 = \text{true}\). Thus, \(\sigma(\rho(v_1)) = \sigma(v_1)\). By induction, \(\forall v \in V \setminus \{v_1\} : \sigma(v) = \sigma(\rho(v))\). The induction hypothesis follows.

A.3 Proof of Theorem 1

Theorem 1. Let \(S_1\) and \(S_2\) be two (sub)programs. Given overapproximation \(\mathcal{UB}(S_1) \subseteq U_1 \subseteq V(S_1)\) and \(\mathcal{UB}(S_2) \subseteq U_2 \subseteq V(S_2)\) of the variables used before definition and overapproximations \(\mathcal{M}(S_1) \subseteq M_1 \subseteq V(S_1)\) and \(\mathcal{M}(S_2) \subseteq M_2 \subseteq V(S_2)\) of the modified variables, a renaming function \(\rho_{\text{switch}}\), and \(C \subseteq M_1 \cup M_2\).

If all \(\text{eqTask}(S_1, S_2, \rho_{\text{switch}}, (U_1 \cap U_2) \cap (M_1 \cup M_2), C), \sigma) \rightarrow^* (S', \sigma') \in \text{ex}(\text{eqTask}(S_1, S_2, \rho_{\text{switch}}, (U_1 \cap U_2) \cap (M_1 \cup M_2), C))\) do not violate an assertion, then \(S_1 = V_1((\mathcal{M}(S_1) \cup \mathcal{M}(S_2))) \cap \mathcal{S}_2\).

Proof. Let \(\sigma \in \Sigma, (S_1, \sigma) \rightarrow^* (E, \sigma') \in \text{ex}(S_1), (S_2, \sigma) \rightarrow^* (E, \sigma'') \in \text{ex}(S_2),\) and \(u \in \{\text{ex}((\mathcal{M}(S_1) \cup \mathcal{M}(S_2))) \cap C\} \) is arbitrary.

Consider two cases. First, consider \(u \in \{\text{ex}((\mathcal{M}(S_1)) \cup \mathcal{M}(S_2))) \cap C\}\). Due to Lemma 2, \(\sigma(u) = \sigma'(u)\) and \(\sigma(u) = \sigma''(u)\). Hence, \(\sigma'(u) = \sigma''(u)\).

Second, consider \(u \in \{\text{ex}((\mathcal{M}(S_1) \cup \mathcal{M}(S_2))) \cap C\}\). Let \(\sigma_r \in \Sigma\) with \(\sigma_r = \text{ex}(\text{init}(\mathcal{S}_1)) \cap \mathcal{S}_2\).

Due to definition of \(\rho_{\text{switch}}\) such a data state exists.

Due to semantics and Lemma 7, \(\exists \text{ex}(\text{init}(\rho_{\text{switch}}(U_1 \cap U_2) \cap (M_1 \cup M_2)), \sigma_r) \rightarrow^* (E, \sigma_{\text{init}})\) with \(\forall v \in (U_1 \cap U_2) \cap (M_1 \cup M_2) : \sigma_{\text{init}}(v) = \text{init}(\rho_{\text{switch}}(v)) = \sigma_r(\rho_{\text{switch}}(v))\). By construction of \(\sigma_r\) and \(\rho_{\text{switch}}\), we further get \(\forall v \in (U_1 \cap U_2) \cap (M_1 \cup M_2) : \sigma_r(\rho_{\text{switch}}(v)) = \rho_{\text{switch}}(\sigma_{\text{init}}(\rho_{\text{switch}}(v))) = \sigma(\rho_{\text{switch}}(v))\). By Lemma 8 and \(\mathcal{M}(\text{init}(\rho_{\text{switch}}(U_1 \cap U_2) \cap (M_1 \cup M_2)) \subseteq (U_1 \cap U_2) \cap (M_1 \cup M_2)\), we infer that \(\forall v \in \{\text{ex}((U_1 \cap U_2) \cap (M_1 \cup M_2)) \) : \text{init}(v) = \sigma_r(v)\).

Hence, \(\sigma_r = \text{init}\).

Due to Lemma 10 where there exists \(\mathcal{R}(S_1, \rho_{\text{switch}}) \rightarrow^* (E, \rho_{\text{switch}}(\sigma')) \in \text{ex}(\mathcal{R}(S_1, \rho_{\text{switch}}))\). By definition of \(\sigma_r\) and, Lemma 10 there exists \(\mathcal{S}_2, \sigma_r = \mathcal{S}_2, \sigma_r \rightarrow^* (E, \sigma'') \in \text{ex}(\mathcal{S}_2)\) with \(\sigma'' = \text{ex}(\mathcal{S}_2) \cup \mathcal{S}_2\) and \(\mathcal{S}_1 = \text{ex}(\mathcal{S}_1) \cup \mathcal{S}_1\) with \(\sigma_r = \text{ex}(\mathcal{S}_r) \cup \mathcal{S}_2\).

Due to semantics, \(\sigma_r = \text{ex}(\mathcal{S}_r) \cup \mathcal{S}_2\), and all \(\text{eqTask}(S_1, S_2, \rho_{\text{switch}}, (U_1 \cap U_2) \cap (M_1 \cup M_2), C), \sigma) \rightarrow^* (S', \sigma') \in \text{ex}(\text{eqTask}(S_1, S_2, \rho_{\text{switch}}, (U_1 \cap U_2) \cap (M_1 \cup M_2), C))\) do not violate assertions, there exists \(\text{ex}(\rho_{\text{switch}}, (U_1 \cap U_2) \cap (M_1 \cup M_2), C), \sigma) \rightarrow^* (E, \sigma')\). Due to Lemma 8, we infer for all \(v \in C\) that \(\sigma_r(v) = \text{ex}(\rho_{\text{switch}}(v))\). Since \(u \in (\mathcal{M}(S_1) \cup \mathcal{M}(S_2)) \cap C\), we conclude that \(\sigma_r(\rho_{\text{switch}}(u)) = \sigma_r(u) = \sigma''(u)\) and \(\sigma_r(\rho_{\text{switch}}(u)) = \sigma_r(\rho_{\text{switch}}(u)) = \rho_{\text{switch}}(\sigma')(\rho_{\text{switch}}(u)) = \sigma'(\rho_{\text{switch}}(\rho_{\text{switch}}(u)))) = \sigma'(u)\).
A.4 Proof of Theorem 2

Lemma 9. Let $S_1$ and $S_2$ be two (sub)programs, $\mathcal{M}(S_1) \subseteq M_1 \subseteq V(S_1)$ and $\mathcal{M}(S_2) \subseteq M_2 \subseteq V(S_2)$ overapproximations of the modified variables, $\rho_{\text{switch}}$ a renaming function, $I = (UB(S_1) \cap UB(S_2)) \cap (M_1 \cup M_2)$, and $C \subseteq M_1 \cup M_2$. If all $(\text{eq-task}(S_1, S_2, \rho_{\text{switch}}, I, C), \sigma) \rightarrow^* (S', \sigma') \in ex(\text{eq-task}(S_1, S_2, \rho_{\text{switch}}, I, C))$ do not violate an assertion, then $\forall \sigma_2 \in \Sigma, V \subseteq V, \nu \in V \setminus ((\mathcal{M}(S_1) \cup \mathcal{M}(S_2)) \cap C) : p_1 = (S_1, \sigma_1) \rightarrow^* (E, \sigma_1') \in ex(S_1) \land p_2 = (S_2, \sigma_2) \rightarrow^* (E, \sigma_2') \in ex(S_2) \land UB(p_1) \cup UB(p_2) \subseteq V \land \sigma_1 =_{\nu} \sigma_2 \implies \forall \nu \sigma_2 \rightarrow^* \sigma_1(v) = \sigma_2(v)$.

Proof. Let $UB(S_1) \cup UB(S_2) \subseteq V \subseteq V$ and $u \in V \setminus ((\mathcal{M}(S_1) \cup \mathcal{M}(S_2)) \cap C)$ be arbitrary. Furthermore, consider arbitrary $p_1 = (S_1, \sigma_1) \rightarrow^* (E, \sigma_1') \in ex(S_1)$ and $p_2 = (S_2, \sigma_2) \rightarrow^* (E, \sigma_2') \in ex(S_2)$ with $\sigma_1 =_{\nu} \sigma_2$.

Consider two cases. First, consider $u \in V \setminus (\mathcal{M}(S_1) \cup \mathcal{M}(S_2))$. Due to Lemma 2, $\sigma_1(u) = \sigma_1'(u)$ and $\sigma_2(u) = \sigma_2'(u)$. Hence, $\sigma_1'(u) = \sigma_2'(u)$.

Second, consider $u \in (\mathcal{M}(S_1) \cup \mathcal{M}(S_2)) \cap C$. Let us consider $\sigma_r \in \Sigma$ with $\sigma_r = |_{\mathcal{M}(S_1) \cup \mathcal{M}(S_2)} \cup UB(S_2) \cap (M_1 \cup M_2), \sigma_r \rightarrow^* (E, \sigma_{init})$ with $\forall v \in (UB(S_1) \cap UB(S_2)) \cap (M_1 \cup M_2) : \sigma_{init}(v) = \sigma_{init}(\rho_{\text{switch}}(v)) = \sigma_r(\rho_{\text{switch}}(v))$. By construction of $\sigma_r$ and $\rho_{\text{switch}}$, $UB(S_1) \cap UB(S_2) \subseteq V \subseteq V, \land \sigma_1 =_{\nu} \sigma_2$, we get $\forall v \in (UB(S_1) \cap UB(S_2)) \cap (M_1 \cup M_2) : \sigma_r(\rho_{\text{switch}}(v)) = \rho_{\text{switch}}(\sigma_1)(\rho_{\text{switch}}(v)) = \rho_{\text{switch}}(\sigma_2(\rho_{\text{switch}}(v))) = \sigma_2(v)$.

Due to Lemma 2 and $M((\text{init}(\rho_{\text{switch}}, UB(S_1)) \cap UB(S_2)) \cap (M_1 \cup M_2)) \subseteq (UB(S_1) \cup UB(S_2)) \cap (M_1 \cup M_2)$, we infer that $\forall v \in V \setminus ((\mathcal{M}(S_1) \cup \mathcal{M}(S_2)) \cap C), \sigma_r \rightarrow^* (E, \sigma(v)) \in \sigma_{init}(v)$. Hence, $\sigma_r = \sigma_{init}$.

Due to Lemma 3 there exists $(R(S_1, \rho_{\text{switch}}), \rho_{\text{switch}}(\sigma_1)) \rightarrow^* (E, \rho_{\text{switch}}(\sigma_1')) \in ex(R(S_1, \rho_{\text{switch}}))$. By definition of $\sigma_r$ and Lemma 3 there exists execution $(S_2, \sigma_r) \rightarrow^* (E, \sigma_r') \in ex(S_2)$ with $\sigma_r' = |_{\mathcal{M}(S_1) \cup \mathcal{M}(S_2)} \cup UB(S_2) \cap (M_1 \cup M_2, C), \sigma_r \rightarrow^* (S', \sigma_r') \in ex(\mathcal{M}(S_1) \cup \mathcal{M}(S_2))$ such that $\rho_{\text{switch}}(\sigma_1') = |_{\mathcal{M}(S_1) \cup \mathcal{M}(S_2)} \cup UB(S_2) \cap (M_1 \cup M_2) \cap UB(S_2) \cap UB(S_2) \cap UB(S_2) \cap UB(S_2) \cap UB(S_2) \cap UB(S_2)$, $\sigma_r'$. Due to Lemma 5 there exists execution $(R(S_1, \rho_{\text{switch}}), S_2, \sigma_r) \rightarrow^* (E, \sigma_c) \in ex(R(S_1, \rho_{\text{switch}}))$ with $\sigma_c = |_{\mathcal{M}(S_1) \cup \mathcal{M}(S_2)} \cup UB(S_2) \cap UB(S_2) \cap UB(S_2) \cap UB(S_2) \cap UB(S_2) \cap UB(S_2) \cap UB(S_2) \cap UB(S_2)$, $\sigma_r'$. Due to semantics, $\sigma_r = \sigma_{init}$, and all $(\text{eq-task}(S_1, S_2, \rho_{\text{switch}}, UB(S_1) \cap UB(S_2)) \cap (M_1 \cup M_2), C, \sigma) \rightarrow^* (S', \sigma') \in ex(\text{eq-task}(S_1, S_2, \rho_{\text{switch}}, UB(S_1) \cap UB(S_2)) \cap (M_1 \cup M_2))$ do not violate assertions, there exists $(equal(\rho_{\text{switch}}, C), \sigma_c) \rightarrow^* (E, \sigma_c)$. Due to Lemma 3 we infer for all $v \in C$ that $\sigma_c(v) = \sigma_c(\rho_{\text{switch}}(v))$. Since $u \in (\mathcal{M}(S_1) \cup \mathcal{M}(S_2)) \cap C$, we conclude that $\sigma_c(\rho_{\text{switch}}(u)) = \sigma_c(u) = \sigma_r'(u) = \sigma_r'(v) \land \sigma_r(\rho_{\text{switch}}(u)) = \rho_{\text{switch}}(\sigma_1'(\rho_{\text{switch}}(u))) = \sigma_1'(v)$.

Theorem 2. Let $S$ and $S'$ be two programs, $\gamma$ be a replacement function such that $S' = \Gamma(S, \gamma)$, and $V \subseteq V$ be a set of outputs. If for all $(S_1, S_2) \in \gamma$ there exists $\mathcal{M}(S_1) \subseteq M_1 \subseteq V(S_1), \mathcal{M}(S_2) \subseteq M_2 \subseteq V(S_2), \mathcal{L}(S, S', V) \subseteq L_1 \subseteq V, \mathcal{L}(S_2, S', V) \subseteq L_2 \subseteq V$, and renaming function $\rho_{\text{switch}}$ such that the equivalence

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task eq\(\text{task}(S_1, S_2, \rho_{\text{switch}}, (UB(S_1) \cap UB(S_2)) \cap (M_1 \cup M_2), (M_1 \cup M_2) \cap (L_1 \cup L_2))\)
does not violate an assertion, then \(S \equiv \nu S'\).

**Proof.** Consider \((S_p, \sigma) \rightarrow^* (E, \sigma') = (S_0, \sigma_0) \rightarrow_{\rho_0}^* \ldots \rightarrow_{\rho_n}^* (S_n, \sigma_n)\) be a path for an arbitrary program \(S_p\). We define the splitting of the path into \(m \geq 0\) segments such that each segment represents either a sequence in which each program of the sequence’s states except for the last one does not start with a replaced subprogram or the execution of the subprogram that will be replaced. In case that there exists multiple replacements (nesting of replaced subprograms), we use the largest replacement. Show by induction over the number of segments that for all programs \(S_p\) such that \(\exists \sigma, \sigma' \in \Sigma : (S, \sigma) \rightarrow^* (S_p, \sigma')\) if \((S_p, \sigma) \rightarrow^* (E, \sigma') \in ex(S_p),\) \(S''_p = \Gamma(S_p, \gamma)\), \(\exists \sigma, \sigma' \in \Sigma : (S', \sigma) \rightarrow^* (S''_p, \sigma'),\) \(\sigma'' \in \Sigma\) with \(\sigma = \in (\Sigma, \nu_{\Sigma, \nu_{\Sigma, \Sigma}})\) \(\sigma''\), and \((S''_p, \sigma'') \rightarrow^* (E, \sigma'''') \in ex(S''_p),\) then \(\sigma' = \in (\Sigma, \nu_{\Sigma, \Sigma})\) \(\sigma''\). Thus, the induction hypothesis follows.

**Base case** (\(m=0\)): Since \(m = 0\), we conclude that \(S_p = E\). Since \(S''_p = \Gamma(S_p, \gamma) = \Gamma(E, \gamma)\), we conclude that \(S''_p = E\). Hence, \(\sigma' = \sigma \land \sigma'' = \sigma'''\). By assumption \(\sigma = \in (\Sigma, \nu_{\Sigma, \Sigma})\) \(\sigma''\). Thus, the induction hypothesis follows.

**Step case** (\(m > 0\)): Let \((S_0, \sigma_0) \rightarrow_{\rho_0}^* \ldots \rightarrow_{\rho_{i-1}}^* (S_i, \sigma_i)\) be the first segment and \(\sigma'_0 \in \Sigma\) be arbitrary such that \(\sigma = \in (\Sigma, \nu_{\Sigma, \Sigma})\) \(\sigma'_0\) and assume \((S''_p, \sigma'') \rightarrow^* (E, \sigma'''') \in ex(S''_p)\). We know that \(S_p = S_0\) and \(\sigma_0 = \sigma\). Consider two cases.

First, assume that the first segment represents a sequence in which each program of the first \(i\)-1 states does not start with a replaced subprogram. Due to Corollary 2 there exists execution \((\Gamma(S_0, \gamma), \sigma'_0) \rightarrow^* \ldots \rightarrow^* (\Gamma(S_i, \gamma), \sigma'_i)\) with \(\sigma_i = \in (\Sigma, \nu_{\Sigma, \Sigma})\) \(\sigma'_i\). By assumption \(S''_p = \Gamma(S_0, \gamma)\). By definition, \((S_0, \sigma_0) \rightarrow^* (E, \sigma') \in ex(S_0)\), which consists of \(m - 1\) segments and is reachable from \(S\). Due to semantics, semantics being deterministic, and \((S''_p, \sigma'') \rightarrow^* (E, \sigma'''') \in ex(S''_p)\), there exists \((\Gamma(S_0, \gamma), \sigma'_0) \rightarrow^* (E, \sigma'''') \in ex(\Gamma(S_0, \gamma))\). Furthermore, since \(S''_p\) reachable from \(S_0\), \(S''_p = \Gamma(S_0, \gamma), (\Gamma(S_0, \gamma), \sigma'_0) \rightarrow^* (\Gamma(S_0, \gamma), \sigma'_i)\), also \(\Gamma(S_0, \gamma)\) reachable from \(S_0\). By induction, \(\sigma' = \in (\Sigma, \nu_{\Sigma, \Sigma})\) \(\sigma''\). The induction hypothesis follows.

Second, assume that the first segment is the execution of a subprogram \(S^*_p\) that will be replaced, i.e., \(S^*_p \in dom(\gamma)\) and \(S_p = S_0 = S^*_p \land S_1 = E \lor S_p = S_0 = \Gamma(S^*_p, S^*_p)\). Furthermore, from \(S''_p = \Gamma(S_0, \gamma)\), we conclude that \(S''_p = \gamma(S^*_p) = \Gamma(S_p, \gamma)\) if \(S_p = S_0 = S^*_p\) and \(S''_p = \Gamma(S_p, \gamma) = \Gamma(S^*_p, \gamma)\); \(\Gamma(S^*_p, \gamma)\) otherwise. Due to semantics, semantics being deterministic, and \((S''_p, \sigma'') \rightarrow^* (E, \sigma'''') \in ex(S''_p)\), there exists \((S^*_p, \sigma''_0) \rightarrow^* (S^*_p, \sigma'_0) \in ex(S^*_p)\) with \(S^*_p = \Gamma(S^*_p, \gamma)\). Furthermore, there exists \(S^*_p \rightarrow^* (E, \sigma'''') \in ex(S^*_p)\). We conclude that \((\Gamma(S^*_p, \gamma), \sigma''_0) \rightarrow^* (E, \sigma'_0)\) (semantics). Due to the definitions of \(UB\) and \(UB(S^*_p, \gamma) \subseteq UB(S_p, \gamma) \subseteq UB(S^*_p) \subseteq UB(L(S_p), \nu_{\Sigma, \Sigma})\).

Due to Lemma 3 \(\sigma_1 = \in (\Sigma, \nu_{\Sigma, \Sigma})\) \(\sigma'_0\). By definition, \((S^*_p, \sigma_1) \rightarrow^* (E, \sigma'_0) \in ex(S^*_p)\), which consists of \(m - 1\) segments and is reachable from \(S\). Furthermore, we can
conclude from \( S_2' \) reachable from \( S' \), also \( \Gamma(S_i, \gamma) \) reachable from \( S' \). By induction, \( \sigma' =_{\Gamma(E, V) \cup \Gamma(S', V)} \sigma'' \). The induction hypothesis follows.

Let \( \sigma \in \Sigma, \sigma, \sigma' \rightarrow^* (E, \sigma') \in ex(S), (S', \sigma) \rightarrow^* (E, \sigma'') \in ex(S') \), and \( u \in V \) be arbitrary. Since \( \sigma =_{\Gamma(S, \gamma)} \sigma, S' = \Gamma(S, \gamma), \) and \( (S', \sigma) \rightarrow^* (E, \sigma'') \in ex(S') \), the induction hypothesis gives us \( \sigma' =_{\Gamma(E, V) \cup \Gamma(S', V)} \sigma'' \). By definition of live variable analysis, \( u \in L(E, V) \). Hence, \( \sigma'(u) = \sigma''(u) \).

### A.5 Proof of Theorem 3

**Lemma 10.** Let \( S \) and \( S' \) be two programs, \( \gamma \) be a replacement function such that \( S' = \Gamma(S, \gamma) \), and \( V \subseteq \gamma \) be a set of outputs. If \( \neg \exists S_1', S_2 : S_1' \in dom(\gamma) \land (S = S_1', S_2 \lor S = S_1') \), then \( \exists \sigma, \sigma' \in \Sigma : (S, \sigma) \overset{\text{op}}{\rightarrow} (S_1, \sigma_1) \in ex(S) \land \sigma =_{\Gamma(S', V)} \sigma' \rightarrow \exists (S', \sigma') \overset{\text{op}}{\rightarrow} (S_1', \sigma_1') \in ex(S) \).

**Proof.** Let \( S \) and \( S' \) be two programs, \( \gamma \) be a replacement function such that \( S' = \Gamma(S, \gamma) \) and \( \neg \exists S_1', S_2 : S_1' \in dom(\gamma) \land (S = S_1', S_2 \lor S = S_1') \), \( V \subseteq \gamma \) be a set of outputs and \( \sigma, \sigma' \in \Sigma \) with \( \sigma =_{\Gamma(S', V)} \sigma' \). Assume \( p = (S, \sigma) \overset{\text{op}}{\rightarrow} (S_1, \sigma_1) \in ex(S) \). By definition, \( \mathcal{UB}(S') \subseteq L(S', V) \).

Consider two cases.

Case 1 (\( S_1 = E \)): Due to the semantics, we conclude that either \( S = E; E \), or \( S \) is not a sequential composition, but a statement. If \( S \) is an assignment, an assertion or a parallel statement, we conclude from \( S \notin dom(\gamma) \), \( S' = \Gamma(S, \gamma) \), and \( \gamma \) does not replace statements in parallel statements that \( S' = S \). Similarly, if \( S = E; E \), also \( S = S' \). Due to Lemma 10, \( \exists (S', \sigma') \overset{\text{op}}{\rightarrow} (E, \sigma_1) \) and \( \sigma_1 =_{\Gamma(E, V)} \sigma'_1 \). Due to the definition of live variables, we conclude that \( \sigma_1 =_{\Gamma(E, V)} \sigma'_1 \). If \( S \) is an if- or while-statement, we conclude from \( S \notin dom(\gamma) \) and \( S' = \Gamma(S, \gamma) \) that \( S' \) is an if-/while-statement and the condition is the same. Due to semantics, definition of live variable analysis, and \( \sigma =_{\Gamma(S', V)} \sigma' \), we then conclude that \( (S', \sigma') \overset{\text{op}}{\rightarrow} (E, \sigma') \in ex(S) \) and \( \sigma_1 = \sigma \).

Due to the definition of live variables, we conclude that \( \sigma_1 =_{\Gamma(E, V)} \sigma' \). By definition, \( E = \Gamma(E, \gamma) \).

Case 2 (\( S_1 \neq E \)): Since replacements do not occur in parallel statements and \( \neg \exists S_1', S_2 : S_1' \in dom(\gamma) \land (S = S_1', S_2 \lor S = S_1') \), we conclude that \( \exists S_1', S_2', S_3 : S = S_1', S_2 \lor S = S_1' \land S = S_2', S_3 \land S_3 = \Gamma(S_1', \gamma) \land S_1' = \Gamma(S_2', \gamma) \lor S = S_1' \land S_2' = S_3 \land S_4 = \Gamma(S_1', \gamma) \) and either \( S_3 = S_2 ' \) or \( S_2 ' = S_4 \) and \( S_2 ' \) is a replacement of the body of \( S_2 ' \). First, consider the first case \( S_1' = S_3 \). Due to semantics, either (1) \( S_1 = E; S_2 \) and \( (S_1', \sigma) \overset{\text{op}}{\rightarrow} (E, \sigma_1) \), (2) \( S_1 = S_2 \land S_1 = E, \gamma = \gamma_1, \) and \( (S, \sigma) \overset{\text{nop}}{\rightarrow} (S_1, \sigma_1) \), or (3) \( S_1 = S_2; S_2 \) and \( (S_1', \sigma) \overset{\text{op}}{\rightarrow} (S_2, \sigma_1) \). Due to Lemma 10, in case (1) \( \exists (S_1, \sigma') \overset{\text{op}}{\rightarrow} (E, \sigma'_1) \) and \( \sigma' = \sigma'_1 \), and in case (2) \( \exists (S_2, \sigma') \overset{\text{nop}}{\rightarrow} (S_4, \sigma'_1) \). Furthermore, \( \sigma_1 =_{\Gamma(E, V) \cup \Gamma(S', V)} \sigma'_1 \). Due to semantics, in case (1) \( \exists (S', \sigma') \overset{\text{op}}{\rightarrow} (E; S_2, \sigma'_1) \), in case (2) \( (S_2, \sigma') \overset{\text{nop}}{\rightarrow} (S_2, \sigma'_1) \), and in case (3) \( \exists (S', \sigma') \overset{\text{op}}{\rightarrow} (S_4; S_2, \sigma'_1) \). Since \( S_1' = S_3 = \Gamma(S_3, \gamma) \), \( \gamma \)
is only defined for subprograms of $S$ and statements (thus, subprograms) can be uniquely identified via labels, we get $I(S_1^1, \gamma) = S_2^1$. Hence, $I(S_1^1; S_2^1, \gamma) = S_1^1; I(S_2^1, \gamma) = S_2^1; I(E, S_2^1, \gamma) = E; I(S_2^1, \gamma) = S_2^1; S_2^1$. Moreover, $\sigma_1 = \|_{e \in V(\sigma_0); v \in e. expr} \sigma'_i$, and the definition of live variable analyses let us conclude that $\sigma_1 = \|_{e \in V(\sigma_0); v \in e. expr} \sigma'_i$.

Second, consider that $(S_1^1 \neq S_3^2)$. We know that $S_1^1$ and $S_3^2$ are either both if- or both while-statements with the same condition and the if/else-body, the loop body of $S_3^2$ is a replacement of the body of $S_1^1$. Due to semantics, definition of live variable analysis, $\sigma = \|_{e \in V(\sigma_0); v \in e. expr} \sigma'_i$, and the replacement function, we then conclude that $\sigma = \sigma_1$ and $\sigma' = \sigma'_1$ and either $S = S_1^1$ and exists $(S', \sigma') \op_1 (S^1_i, \sigma_1) \in ex(S')$ with $S_1^1 = I(S_1^1, \gamma)$ (due to $S$; \textbf{while} expr \textbf{do} $S$ is no subprogram of $S$) or $S = S_2^1; S_2^1$ and $(S', \sigma') \op_2 (S^1_i, \sigma_1) \in ex(S')$ with $S_1^1 = I(S_1^1, \gamma)$. Due to the definition of live variables, we conclude that $\sigma_1 = \|_{e \in V(\gamma, \gamma); v \in e. expr} \sigma'_i$.

Corollary 3. Let $S$ and $S'$ be two programs, $\gamma$ be a replacement function such that $S' = I(S, \gamma)$, and $V \subseteq V$ be a set of outputs. For all $(S_0, \sigma_0) \op_1 \ldots \op_n (S_n, \sigma_n) \in ex(S)$, if for all $0 \leq i < n$ not exists $S_1^1, S_2^2$ such that $(S_1^1 = S_1^1; S_2^2 \vee S_1^1 = S_2^1)$ and $S_1^1 \in dom(\gamma)$, then $\forall \sigma_0 \in \Sigma : \sigma_0 = \|_{e \in V(\gamma, \gamma); v \in e. expr} \sigma'_0 \implies \exists (I(S_0, \gamma), \sigma'_0) \op_1 \ldots \op_n (I(S_n, \gamma), \sigma'_n) \in ex(S') : \forall 0 \leq i \leq n : \sigma_i = \|_{e \in V(\gamma, \gamma); v \in e. expr} \sigma'_i$.

Proof. Proof by induction.

Base case (i=0): By definition $(S', \sigma) = (I(S, \gamma), \sigma) \in ex(S')$ for arbitrary $\sigma \in \Sigma$ (including all $\sigma' \in \Sigma$).

Step case (i=n): Due to Lemma 11, there exists $(I(S, \gamma), \sigma') \op_1 (I(S_1, \gamma), \sigma'_1) \ldots \op_n (I(S_n, \gamma), \sigma'_n) \in ex(S')$ : $\forall 0 \leq i \leq n : \sigma_i = \|_{e \in V(\gamma, \gamma); v \in e. expr} \sigma'_i$. By definition, the induction hypothesis follows.

Lemma 11. Let $S_1$ and $S_2$ be two (sub)programs of programs $S$ and $S'$, respectively. Consider arbitrary $(S, \sigma) \rightarrow^* (S_1, \sigma_1) \rightarrow^* (S_j, \sigma_j) \in ex(S)$ and $(S', \sigma') \rightarrow^* (S'_i, \sigma'_i) \rightarrow^* (S'_j, \sigma'_j) \in ex(S')$ such that $S_j = S_1 \land S_j = E \lor S_2 \land S_2 = E \lor S_1 = S_1 \land S_j = S_2$; $S_2 \subseteq S'_1 \subseteq S_1$; $(S_1, \sigma_1) \rightarrow^* (E, \sigma_j) \in ex(S_1)$, and $(S_2, \sigma'_i) \rightarrow^* (E, \sigma'_i) \in ex(S_2)$. Given overapproximations $UB(S_1) \subseteq U_1 \subseteq V(S_1)$ and $UB(S_2) \subseteq U_2 \subseteq V(S_2)$ and overapproximations $M(S_1) \subseteq M_1 \subseteq V(S_1)$ and $M(S_2) \cup \{v \in V \mid \exists S_2 \rightarrow^* S'_k \rightarrow^{expr} S'_j \in \text{syn}(S_2)\} \subseteq M_2 \subseteq V(S_2)$ of the modified variables, a renaming function $\rho_{\text{switch}}$, and $(L(S_2, S', V)) \cap (M_1 \cup M_2) \subseteq C \subseteq M_1 \cup M_2$ If all executions (eq $\text{task}(S_1, S_2, \rho_{\text{switch}}, (U_1 \cup U_2) \cap (M_1 \cup M_2), C), \sigma) \rightarrow^* (S', \sigma') \in ex(eq_{\text{task}}(S_1, S_2, \rho_{\text{switch}}, (U_1 \cup U_2) \cap (M_1 \cup M_2), C))$ do not violate an assertion, $V \subseteq V$, and $\sigma_i = \|_{e \in V(S'_j, v) \in e. expr} \sigma'_i$, then $\sigma_j = \|_{e \in V(S'_j, v) \in e. expr} \sigma'_j$.

Proof. Let $u \in L(S'_j, V)$ be arbitrary. In the following, we write $v$ is assigned in $p = (S_0, \sigma_0) \op_1 \ldots \op_n (S_n, \sigma_n) \in ex(S)$ if $\exists i \in [1, n] : \rho_i = v := e_{\text{expr}}$.

First, consider $u \notin (M(S_1) \cup M(S_2))$ and $\sigma_i(u) = \sigma_j(u)$. Due to Lemma 2, $\sigma_i(u) = \sigma_j(u)$ and $\sigma'_i(u) = \sigma'_j(u)$. Hence, $\sigma_j(u) = \sigma'_j(u)$.
Second, consider $u \in (M(S_1) \cup M(S_2))$ or $\sigma(u) \neq \sigma_j(u)$. Let $\sigma_r \in \Sigma$ with $\sigma_r = |_{\mathcal{L}(S'_r,V) \cup M(S_1) \cup M(S_2)}$ $\sigma'_r$ and $\sigma_r = |_{\cup \in \mathcal{L}(S'_r,V) \cup M(S_1) \cup M(S_2)} \rho_{\mathit{switch}}(\sigma_i)$. Due to definition of $\rho_{\mathit{switch}}$ such a data state exists.

Due to semantics and Lemma 2, there exists $(\mathit{init}(\rho_{\mathit{switch}}, (U_1 \cap U_2) \cap (M_1 \cup M_2)), \sigma_r) \rightarrow^* (E, \sigma_{\mathit{init}})$ with $\forall v \in (U_1 \cap U_2) \cap (M_1 \cup M_2) : \sigma_{\mathit{init}}(v) = \mathit{init}(\rho_{\mathit{switch}}(v)) = \sigma_r(\rho_{\mathit{switch}}(v))$. By construction of $\sigma_r$ and $\rho_{\mathit{switch}}$, and $\sigma_r = |_{\mathcal{L}(S'_r,V)}$ $\sigma_r'$, we get $\forall v \in (\mathcal{L}(S'_r,V)) : \sigma_r(\rho_{\mathit{switch}}(v)) = \rho_{\mathit{switch}}(\sigma_r(\rho_{\mathit{switch}}(v))) = \sigma_r(\rho_{\mathit{switch}}(\rho_{\mathit{switch}}(v))) = \sigma_r(v) = \sigma_r'(v)$. Due to Lemma 2 and modifications $M((\mathit{init}(\rho_{\mathit{switch}}, (U_1 \cup U_2) \cap (M_1 \cup M_2)), (U_1 \cap U_2) \cap (M_1 \cup M_2)) \subseteq (U_1 \cap U_2) \cap (M_1 \cup M_2)$, we infer that $\forall v \in V \setminus ((U_1 \cap U_2) \cap (M_1 \cup M_2)) : \sigma_{\mathit{init}}(v) = \sigma_r(v)$. Hence, $\sigma_r' = |_{\mathcal{L}(S'_r,V)} \sigma_{\mathit{init}}$ and $\sigma_{\mathit{init}} = |_{\cup \in \mathcal{L}(S'_r,V) \cup M(S_1) \cup M(S_2)} \rho_{\mathit{switch}}(\sigma_i)$.

Due to Lemma 2 there exists $(\mathcal{R}(S_1, \rho_{\mathit{switch}}), (\mathcal{R}(S_2, \rho_{\mathit{switch}}), \mathcal{R}(S_1, \rho_{\mathit{switch}})), \mathcal{R}(S_2, \rho_{\mathit{switch}}), \mathcal{R}(S_1, \rho_{\mathit{switch}}))) \rightarrow^* (E, \rho_{\mathit{switch}}(\sigma_i))$. Since $\sigma' = |_{\mathcal{L}(S'_r,V)} \sigma_{\mathit{init}}$, by definition $UB(S_2) \subseteq \mathcal{L}(S'_r,V)$, and $\sigma_{\mathit{init}} = |_{\cup \in \mathcal{L}(S'_r,V) \cup M(S_1) \cup M(S_2)} \rho_{\mathit{switch}}(\sigma_i)$, we infer from Lemma 2 that $\mathcal{R}(S_1, \rho_{\mathit{switch}}), \mathcal{R}(S_2, \rho_{\mathit{switch}}), \mathcal{R}(S_1, \rho_{\mathit{switch}})) \rightarrow^* (E, \sigma') \in \mathcal{R}(\mathcal{R}(S_1, \rho_{\mathit{switch}}), \mathcal{R}(S_1, \rho_{\mathit{switch}}), \mathcal{R}(S_2, \rho_{\mathit{switch}}))$ such that $\mathcal{R}(S_1, \rho_{\mathit{switch}}), \mathcal{R}(S_2, \rho_{\mathit{switch}}), \mathcal{R}(S_1, \rho_{\mathit{switch}})) \rightarrow^* (E, \sigma') \in \mathcal{R}(\mathcal{R}(S_1, \rho_{\mathit{switch}}), \mathcal{R}(S_1, \rho_{\mathit{switch}}), \mathcal{R}(S_2, \rho_{\mathit{switch}}))$ with $\sigma_c = |_{\mathcal{L}(S'_r,V) \cup M(S_1) \cup M(S_2)} \rho_{\mathit{switch}}(\sigma_i)'$ and $\sigma_{\mathit{init}} = |_{\mathcal{L}(S'_r,V) \cup M(S_1) \cup M(S_2)} \rho_{\mathit{switch}}(\sigma_i)''$. By definition of $\rho_{\mathit{switch}}$, all $(\mathit{eq}_{\mathit{task}}(S_1, S_2, \rho_{\mathit{switch}}, U_1 \cap U_2 \cap (M_1 \cup M_2), C), \sigma) \rightarrow^*(S', \sigma') \in \mathit{eq}_{\mathit{task}}(S_1, S_2, \rho_{\mathit{switch}}, U_1 \cap U_2 \cap (M_1 \cup M_2), C) \setminus \mathit{eq}_{\mathit{task}}(S_1, S_2, \rho_{\mathit{switch}}, U_1 \cap U_2 \cap (M_1 \cup M_2), C))$ do not violate assertions, there exists $(eq_{\mathit{task}}(S_1, S_2, \rho_{\mathit{switch}}, U_1 \cap U_2 \cap (M_1 \cup M_2), C), \sigma) \rightarrow^*(E, \sigma')$. Due to Lemma 2 we infer for all $v \in C$ that $\sigma_r(v) = \sigma_c(\rho_{\mathit{switch}}(v))$.

Distinguish two cases. First, consider $u \in (M(S_1) \cup M(S_2))$. We conclude that $u \in C$. Hence, $\sigma_c(u) = \sigma_c(\rho_{\mathit{switch}}(u))$. By definition of $\mathcal{L}$, we conclude that $u \in \mathcal{L}(S'_r,V)$ or $u$ is assigned on $(S_2, \sigma_i)' \rightarrow^* (E, \sigma_j') \in \mathcal{L}(S'_r,V)$. Due to Lemma 2 we infer that $u \in \mathcal{L}(S'_r,V)$ or $u$ is assigned on $p'$. We conclude that $\sigma_c(u) = \sigma_c'(u) = \sigma_c' |_{\mathcal{L}(S'_r,V)} \rho_{\mathit{switch}}(\sigma_i)'(u) = \sigma_c(\rho_{\mathit{switch}}(u)) = \rho_{\mathit{switch}}(\rho_{\mathit{switch}}(u)) = \sigma_j(\rho_{\mathit{switch}}(\rho_{\mathit{switch}}(u))) = \sigma_j(u)$. Since $\sigma_c(u) = \sigma_c(\rho_{\mathit{switch}}(u))$, we get $\sigma_j'(u) = \sigma_j(u)$.

Second, consider $\sigma_j(u) \neq \sigma_j'(u)$ and $u \notin (M(S_1) \cup M(S_2))$. Since $u \in \mathcal{L}(S'_r,V)$ and $\sigma(u) \neq \sigma_j(u)$, we conclude that $u \notin \mathcal{L}(S'_r,V)$. By definition of $\mathcal{L}$, we conclude that $u$ is assigned on $(S_2, \sigma_i)' \rightarrow^* (E, \sigma_j') \in \mathcal{L}(S'_r,V)$. Hence, $u \in M_2$ and, therefore, $u \in C$. Then, due to Lemma 2 $u$ is assigned on $p'$. Furthermore, we conclude that $\sigma_c(u) = \sigma_c'(u) = \sigma_c' |_{\mathcal{L}(S'_r,V)} \rho_{\mathit{switch}}(\sigma_i)'(u) = \sigma_c(\rho_{\mathit{switch}}(u)) = \rho_{\mathit{switch}}(\rho_{\mathit{switch}}(u)) = \sigma_j(\rho_{\mathit{switch}}(\rho_{\mathit{switch}}(u))) = \sigma_j(u)$. Since $u \in C$, we conclude that $\sigma_j'(u) = \sigma_c(u) = \sigma_c(\rho_{\mathit{switch}}(u)) = \sigma_j(u)$.

**Theorem 3.** Let $S$ and $S'$ be two programs, $\gamma$ be a replacement function such that $S' = T(S, \gamma)$, and $V \subseteq \mathcal{V}$ be a set of outputs. If for all $(S_1, S_2) \in \gamma$ there exists overapproximations $UB(S_1) \subseteq U_1 \subseteq V(S_1), UB(S_2) \subseteq U_2 \subseteq V(S_2), M(S_1) \subseteq M_1 \subseteq V(S_1), M(S_2) \subseteq M_2 \subseteq V(S_2), \mathcal{L}(S_1, S_2, V) \subseteq L_1 \subseteq V, \mathcal{L}(S_2, S', V) \subseteq L_2 \subseteq V$, and renaming
\( \rho_{\text{switch}} \) s.t. eq-\text{Lask}(S_1, S_2, \rho_{\text{switch}}, (U_1 \cup U_2) \cap (M_1 \cup M_2), (M_1 \cup M_2) \cap (L_1 \cup L_2)) \) does not violate an assertion, then \( S \equiv V \ S' \).

**Proof.** Consider \( (S_p, \sigma) \rightarrow^* (E, \sigma') = (S_0, \sigma_0) \overset{\text{op}_1}{\rightarrow} \ldots \overset{\text{op}_n}{\rightarrow} (S_n, \sigma_n) \) be a path for an arbitrary program \( S_p \). We define the splitting of the path into \( m \geq 0 \) segments such that each segment represents either a sequence in which each program of the sequence’s states except for the last one does not start with a replaced subprogram or the execution of the subprogram that will be replaced. In case that there exists multiple replacements (nesting of replaced subprograms), we use the largest replacement. Show by induction over the number of segments that for all programs \( S_p \) such that \( \exists \sigma, \sigma' \in \Sigma : (S, \sigma) \rightarrow^* (S, \sigma') \) if \( (S_p, \sigma) \rightarrow^* (E, \sigma') \) \( \in \text{ex}(S_p) \), \( S'_p = \Gamma(S_p, \gamma) \), \( \exists \sigma, \sigma' \in \Sigma : (S', \sigma) \rightarrow^* (S'_p, \sigma'), \sigma'' \in \Sigma \) with \( \sigma = \|_{\Sigma(S'_p, V)} \sigma' \), and \( (S'_p, \sigma'') \rightarrow^* (E, \sigma'\''') \in \text{ex}(S'_p) \), then \( \sigma' = \|_{\Sigma(E, V)} \sigma'\''\).}

**Base case** \((m=0): \) Since \( m = 0 \), we conclude that \( S_p = E \). Since \( S'_p = \Gamma(S_p, \gamma) = \Gamma(E, \gamma) \), we conclude that \( S'_p = E \). Hence, \( \sigma' = \sigma \land \sigma'' = \sigma''' \). By assumption \( \sigma = \|_{\Sigma(S'_p, V)} \sigma' \). Thus, the induction hypothesis follows.

**Step case** \((m > 0): \) Let \( (S_0, \sigma_0) \overset{\text{op}_1}{\rightarrow} \ldots \overset{\text{op}_n}{\rightarrow} (S_i, \sigma_i) \) be the first segment and \( \sigma_i' \in \Sigma \) be arbitrary such that \( \sigma_0 = \|_{\Sigma(S_i, V)} \sigma_i' \) and assume \( (S'_p, \sigma'') \rightarrow^* (E, \sigma'''') \in \text{ex}(S'_p) \). We know that \( S_p = S_0 \) and \( \sigma_0 = \sigma \). Consider two cases.

First, assume that the first segment represents a sequence in which each program of the first i-1 states does not start with a replaced subprogram. Due to Corollary \( \ref{corollary} \) there exists execution \((\Gamma(S_0, \gamma), \sigma_0') \overset{\text{op}_2}{\rightarrow} \ldots \overset{\text{op}_n}{\rightarrow} (\Gamma(S_i, \gamma), \sigma_i') \) with \( \sigma_i' = \|_{\Sigma(\Gamma(S_i, \gamma), V)} \sigma_i' \). By assumption \( S'_p = \Gamma(S_0, \gamma) \). By definition, \((S_i, \sigma_i) \rightarrow^* (E, \sigma') \in \text{ex}(S_i) \), which consists of \( m - 1 \) segments and is reachable from \( S \). Due to semantics, semantics being deterministic, and \((S'_p, \sigma_0) \rightarrow^* (E, \sigma'''') \in \text{ex}(S'_p) \), there exists \((\Gamma(S_i, \gamma), \sigma_i') \rightarrow^* (E, \sigma'''') \in \text{ex}(\Gamma(S_i, \gamma)) \). Furthermore, since \( S_p \) reachable from \( S' \), \( S'_p = \Gamma(S_0, \gamma) \), and \((\Gamma(S_0, \gamma), \sigma_0) \rightarrow^* (\Gamma(S_i, \gamma), \sigma_i') \), also \( \Gamma(S_i, \gamma) \) reachable from \( S' \). By induction, \( \sigma' = \|_{\Sigma(E, V)} \sigma''' \). The induction hypothesis follows.

Second, assume that the first segment is the execution of a subprogram \( S^*_p \) that will be replaced, i.e., \( S^*_p \in \text{dom}(\gamma) \) and \( S_p = S_0 = S^*_p \land S_i = E \lor S_p = S_0 = S^*_p, S_i \land (S^*_p, \sigma_0) \rightarrow^* (E, \sigma_i) \in \text{ex}(S^*_p) \). Furthermore, from \( S'_p = \Gamma(S_p, \gamma) \), we conclude that \( \Gamma(S^*_p, \gamma) = \Gamma(S_p, \gamma) \) if \( S_p = S_0 = S^*_p \) and \( \Gamma(S_p, \gamma) = \Gamma(S^*_p, \gamma) \). By semantics, semantics being deterministic, and \( (S'_p, \sigma_0) \rightarrow^* (E, \sigma'''') \in \text{ex}(S'_p) \), there exists \((S'_i, \sigma_i') \in \text{ex}(S'_p) \) with \( S'_i = \Gamma(S_i, \gamma) \) and \((\Gamma(S^*_p, \gamma), \sigma_0) \rightarrow^* (E, \sigma_i') \). Furthermore, there exists \((S'_i, \sigma''''\) \in \text{ex}(S^*_p) \), we conclude that \((\Gamma(S^*_p, \gamma), \sigma_0) \rightarrow^* (E, \sigma_i') \) (semantics). Due to Lemma \( \ref{lemma} \), \( \sigma_i = \|_{\Sigma(S_i, V)} \sigma_i' \). By definition, \((S_i, \sigma_i) \rightarrow^* (E, \sigma) \in \text{ex}(S_i) \), which consists of \( m - 1 \) segments and is reachable from \( S \). Furthermore, we can conclude from \( S'_p \) reachable from \( S'_i \), also \( \Gamma(S_i, \gamma) \) reachable from \( S' \). By induction, \( \sigma' = \|_{\Sigma(E, V)} \sigma''' \). The induction hypothesis follows.

Let \( \sigma \in \Sigma, (S, \sigma) \rightarrow^* (E, \sigma') \in \text{ex}(S), (S', \sigma) \rightarrow^* (E, \sigma'') \in \text{ex}(S') \), and \( u \in V \) be arbitrary. Since \( \sigma = \|_{\Sigma(S', V)} \sigma \), \( S' = \Gamma(S, \gamma) \), and \( (S', \sigma) \rightarrow^* (E, \sigma''') \in \text{ex}(S') \), we conclude that \( \sigma' = \|_{\Sigma(E, V)} \sigma''' \). The induction hypothesis follows.
ex(S'), the induction hypothesis gives us \( \sigma' =_{L(E,V)} \sigma'' \). By definition of live variable analysis, \( u \in L(E,V) \). Hence, \( \sigma'(v) = \sigma''(v) \).

### A.6 Correctness of \( \rho_{\text{switch}} \)

**Lemma 12.** Let \( S_1 \) and \( S_2 \) be two (sub)programs. Given overapproximation \( UB(S_1) \subseteq U_1 \subseteq V(S_1) \) and \( UB(S_2) \subseteq U_2 \subseteq V(S_2) \) of the variables used before definition and overapproximations \( M(S_1) \subseteq M_1 \subseteq V(S_1) \) and \( M(S_2) \subseteq M_2 \subseteq V(S_2) \) of the modified variables. Any function \( \rho_{\text{switch}} \) is appropriate for renaming and ensures \( \forall v \in (U_1 \cap U_2) \cap (M_1 \cup M_2) : \rho_{\text{switch}}(v) = v \lor \rho_{\text{switch}}(v) \notin (U_1 \cap U_2) \cap (M_1 \cup M_2) \).

**Proof.** Let \( \text{switch} : M_1 \cup M_2 \to V \setminus (V(S_1) \cup V(S_2)) \) be an arbitrary injective function. Due to injectivity of function \( \text{switch} \) and the construction of \( \rho_{\text{switch}} \), function \( \rho_{\text{switch}} \) is bijective.

Since \( M(S_2) \subseteq M_2 \subseteq V(S_2) \) and by definition of \( \rho_{\text{switch}} \) for all \( v \in V(S_1) \) either \( \rho_{\text{switch}}(v) = v \) and \( v \notin M_1 \cup M_2 \) or \( \rho_{\text{switch}}(v) \notin V(S_1) \cup V(S_2) \), we infer \( \forall v \in V(S_1) \cup M(S_2) : \rho(v) \notin M(S_2) \).

Since \( M(S_1) \subseteq M_1 \) and for all \( v \in M_1 \cup M_2 \) renamed variable \( \rho_{\text{switch}}(v) \in V \setminus (V(S_1) \cup V(S_2)) \), we infer \( \forall v \in M(S_1) : \rho(v) \notin V(S_2) \cup M(S_1) \).

We conclude that \( \rho_{\text{switch}} \) is appropriate for renaming.

By definition, \( (U_1 \cap U_2) \cap (M_1 \cup M_2) \subseteq U_1 \cap U_2 \subseteq V(S_1) \cup V(S_2) \). By construction of \( \rho_{\text{switch}} \), for all \( v \in V(S_1) \cup V(S_2) \) either \( \rho_{\text{switch}}(v) = v \) or \( \rho_{\text{switch}}(v) \notin V(S_1) \cup V(S_2) \). Hence, \( \forall v \in (U_1 \cap U_2) \cap (M_1 \cup M_2) : \rho_{\text{switch}}(v) = v \lor \rho_{\text{switch}}(v) \notin (U_1 \cap U_2) \cap (M_1 \cup M_2) \).