Can the Aharonov-Bohm effect be used to detect or refute Superseparability?

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Abstract

In 1988, Reeh showed that the representation of the canonical commutation relations that corresponds to the Aharonov-Bohm effect depends on the magnetic flux \( \Phi \). It can be integrated to a representation of the Weyl group only if the flux is quantized. It follows from Reeh’s analysis that representations for \( \Phi_1 \neq \Phi_2 \) are generally inequivalent. As a result, two identical charged bosons may be found in inequivalent representations of the CCR. If unitary inequivalence is a restriction on superposability, then these two particles should not feel each other’s presence even when they are in close physical proximity! If they do feel each other’s presence, then unitary inequivalence is not a restriction on superposability, and the question arises: what does unitary inequivalence mean? This paper suggests an experiment that can distinguish between these two possibilities and provides a brief account of the theory behind it, which depends upon the subtle notion of self-adjointness of unbounded operators.
1 Introduction

Superseparability may be defined as the polar opposite of entanglement: two identical charged bosons, with state vectors that have considerable spatio-temporal overlap, are unable to feel each other’s presence because the state vectors lie in disjoint Hilbert spaces and cannot be superposed. Mathematically, this possibility appears to be contained in von Neumann’s Hilbert space formulation of quantum mechanics, but whether or not it is realized in nature can only be ascertained by experiment. This note describes the principle of a possible experiment based on the magnetic Aharonov-Bohm effect.

The mathematical phenomena that suggest the experiment are subtleties hidden in the notion of self-adjointness for unbounded operators. Experience shows that these subtleties can generally be disregarded in practical applications of quantum mechanics; among physicists, only the mathematically minded are likely to be familiar with them. For this reason, a brief review of the basic definitions and results, due mostly to von Neumann, is provided in Section 4. It follows an essential review of the historical background in Section 3. The paper begins with a description of the experimental scheme in Section 2, the theory of the experiment is given, after the historical and mathematical excursions, in Section 5. If the experiment turns out to be feasible, its results – be they positive or negative – will be consequential, and implications of the possible results are discussed briefly, in a non-speculative manner, in Section 6.

2 Scheme of the experiment

The purpose of the experiment is to determine whether or not two beams of identical bosons, prepared in inequivalent representations of the CCR, can interfere with each other. A possible scheme for such an experiment is shown in figure 1.

A coherent beam of charged bosons (e.g., $\alpha$-particles or deuterons) from a source $S$ is split into two by a beam-splitter $P$. One beam goes through the chamber $A$, the other through the chamber $B$. Neither chamber contains a
magnetic field, and the two are electromagnetically isolated from each other. They contain the magnetic flux lines $\Phi_{A,B}$ respectively (perpendicular to the plane of the paper); at least one of these fluxes is continuously variable over a certain range. The experiment consists of observing changes in the interference pattern at the detector $D$ as $\Delta \Phi = \Phi_B - \Phi_A$ is varied.

![Figure 1: Scheme for a noninterferometer](image)

As will be shown in Section 5, von Neumann’s Hilbert space formulation of quantum mechanics [16] suggests that the two beams should interfere only when $\Delta \alpha = q \Delta \Phi / 2\pi$ is an integer, where $q$ is the charge of the boson, which will be $-e$ for deuterons and $-2e$ for $\alpha$-particles. (We use units in which $\hbar = c = 1$.) For non-integral values of $\Delta \alpha$, the two beams will belong to disjoint Hilbert spaces and should not – if the phrase disjoint Hilbert spaces has physical meaning – be able to interfere with each other. For this reason, the scheme of the figure is called a noninterferometer. It is also possible that the above interpretation of quantum mechanics is invalid, and that both beams belong to the same Hilbert space. In that case the interference pattern should merely shift, as $\Delta \alpha$ is varied, returning to the original state when $\Delta \alpha$ has changed exactly by unity; the fringe shift should be periodic, with period 1. If the experiment is realizable, the case $\Phi_A = \Phi_B \neq 0$ will correspond to the standard Aharonov-Bohm effect, so that this effect may be used to test the electromagnetic isolation of the chambers $A$ and $B$. The reader is referred to the monograph by Peshkin and Tonomura [5] for a historical account of the Aharonov-Bohm effect, and to the review article [13] and monograph [12] for the decisive experiments.

It should be recalled that if a flux $\Phi$ is quantized, then $2e\Phi = 2\pi n$, where

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3The terms superseparability and noninterferometer were introduced in [10].
$n$ is an integer, so that $\Delta \alpha = (n_B - n_A)/2$ for deuterons and $\Delta \alpha = n_B - n_A$ for $\alpha$-particles. Therefore superseparability will never be observed with $\alpha$-particles if both fluxes are quantized.

We shall end this section with a few reservations and a remark. The configuration described above is an ideal which may be hard to realize in the laboratory; that is why we have called it the ‘scheme’ of an experiment. It is unlikely that the chambers $A$ and $B$ can be perfectly isolated from each other. It may not be possible to prepare a state with a sharp value of $\alpha$ if the flux is not quantized. Finally, the theory of Section 5 would be applicable only if the interiors of chambers $A$ and $B$ are, for purposes of the experiment, reasonable approximations to the punctured plane. On the other hand, we know – if only by hindsight – that the Aharonov-Bohm effect can be observed under conditions that are less than ideal. Therefore the possibility that superseparability may also be testable under less than ideal conditions should not be ruled out of hand.

3 Historical background

We begin by recalling two basic facts. (i) The Born-Jordan commutation relation $[p, q] = -iI$ cannot be represented by finite-dimensional matrices if $I$ is required to be the identity matrix. (ii) If it is represented on an infinite-dimensional Hilbert space $\mathcal{H}$ with $I$ as the identity operator, then at least one of $p$ and $q$ must be represented by an unbounded operator (see, for example, [11]).

Unbounded operators are not defined everywhere on a Hilbert space, and are discontinuous wherever they are defined. They give rise to mathematical phenomena that are not encountered in the theory of finite dimensional matrices, and it requires considerable effort to invest with meaning even the simplest of assertions, such as $[A, B] = 0$, if $A$ and $B$ are unbounded. The basic structures of quantum mechanics, namely matrix mechanics, wave mechanics and transformation theory were laid down in 1925–27 [3], but unbounded operators began to be explored only in 1929–1930 [14, 11]. The ‘first quantum revolution’ (this term is due to Aspect [11]) was completed while unbounded...
operators were still terra incognita even to mathematicians. In retrospect, one is struck by the fact that transformation theory could be developed with scant understanding of the operators that were to be transformed. By what magic was this achieved?

The answer lies in an ansatz due to Hermann Weyl and a theorem proven by von Neumann.

In 1928, Weyl published his book *Gruppentheorie und Quantenmechanik* [17]. In this book he replaced the canonical commutation relations (CCR) for $N$ degrees of freedom by a $2N$-parameter Lie group, which had the CCR as its Lie algebra. This group has become known as the *Weyl group*, and we shall denote it by $W_N$. We shall give the argument for $N = 1$; the general case merely requires a cumbersome modification of the notation (see [17], pp. 272–276).

Let $a, b \in \mathbb{R}$ and define, formally,

$$u(a) = \exp(iap), \quad v(b) = \exp(ibq). \tag{1}$$

From the properties of the exponential function, it follows that

$$u(a)u(a') = u(a + a'), \quad v(b)v(b') = v(b + b'). \tag{2}$$

Write $u(-a) = u(a)^{-1}$, $v(-b) = v(b)^{-1}$ and $u(0) = v(0) = 1$. Formal computation yields the result

$$u(a)v(b)u(a)^{-1}v(b)^{-1} = e^{iab}1. \tag{3}$$

By definition, the Weyl group $W_1$ consists of the set of elements (1), with multiplication defined by (2) and (3). The element 1 is the identity of the group. The group $W_1$ is nonabelian and noncompact, with $\mathbb{R}^2$ as the group manifold, and is a Lie group. The same is true of the Weyl group $W_N$ for $N$ degrees of freedom, except that its group manifold is $\mathbb{R}^{2N}$.

Being noncompact, the Weyl groups have no finite dimensional unitary representations. In a unitary representation, the elements $u(a)$ and $v(b)$ of $W_1$ are represented by unitary operators $U(a)$ and $V(b)$ on the Hilbert space $\mathcal{H}$, and similar statements hold for $W_N$. A result known as Stone’s theorem asserts that a one-parameter group of unitaries $\{U(t)\}$ on a Hilbert space

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4The definition of an infinite-dimensional unitary representation includes a continuity condition that we have not specified. The same condition is used in the definition of one-parameter groups of unitaries.
has an infinitesimal generator $H$: $U(t) = \exp(\i Ht)$, where $H$ is self-adjoint. It is bounded if \{U(t)\} is compact ($t \in S_1$, the circle) and unbounded if \{U(t)\} is not compact ($t \in \mathbb{R}$). A representation of $\mathcal{W}_N$ defines, uniquely, a representation of its Lie algebra – the CCR – by self-adjoint operators. In the representation so defined, at least one member of any canonical pair $p, q$ is represented by an unbounded operator.

In 1930 von Neumann proved that, for finite $N$, the Weyl group $\mathcal{W}_N$ has only one irreducible unitary representation \[15\]. He gave the name Schrödinger operators to the representatives of the canonical variables $p_j, q_j$, $j = 1, \ldots, N$, and titled his paper ‘Die Eindeutigkeit Schrödingersche Operatoren’. His result has become known as ‘von Neumann’s uniqueness theorem’. If the CCR were equivalent to the Weyl group, it would explain why quantum mechanics could be developed ahead of the theory of unbounded operators without falling into gross error.

A Lie group defines a unique Lie algebra, but the converse is not true. The simplest examples are the covering groups of compact non-simply-connected Lie groups. Examples of this phenomenon that are relevant to elementary particle physics were unearthed by Michel as early as 1962 \[4\]. The canonical commutation relations are not abstractly equivalent to the Weyl group; as we shall see below, the $p_j, q_k$ will not even generate a Lie group unless they are represented by self-adjoint operators. However, the requirement of self-adjointness cannot be met in some simple and realizable physical situations.

## 4 Self-adjointness

Let $\mathcal{H}$ be a Hilbert space and $A$ an operator on it. If there exists a positive number $K$ such that $||A\psi|| \leq K||\psi||$ for all $\psi \in \mathcal{H}$, then $A$ is said to be bounded. If no such $K$ exists, then $A$ is said to be unbounded. An unbounded operator $A$ is not defined everywhere on $\mathcal{H}$; the subset $\mathcal{D}(A) \subseteq \mathcal{H}$ on which it is defined is called the domain of $A$. If $\mathcal{D}(A)$ is not dense in $\mathcal{H}$ then $A$ is not (yet) mathematically manageable, and one generally assumes that $A$ is densely defined, i.e., $\mathcal{D}(A)$ is dense in $\mathcal{H}$.

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5Self-adjoint operators on infinite-dimensional Hilbert spaces will be defined precisely in Section 4. The exponential $\exp(\i At)$, $t \in \mathbb{R}$ of the unbounded self-adjoint operator $A$ needs definition, but we shall content ourselves with the statement that it turns out to have the expected properties.
In the rest of this section we shall deal only with unbounded operators, and therefore the adjective ‘unbounded’ will be omitted.

An operator $A$ is called closed if the set of ordered pairs $\{(\psi, A\psi) | \psi \in \mathcal{D}(A)\}$ is a closed subset of $\mathfrak{H} \times \mathfrak{H}$. An operator $A_1$ is an extension of $A$ if $\mathcal{D}(A) \subset \mathcal{D}(A_1)$ and $A_1\psi = A\psi$ for $\psi \in \mathcal{D}(A)$; one writes $A \subset A_1$. An operator is called closable if it has a closed extension. Every closable operator $A$ has a smallest closed extension, which is denoted by $\bar{A}$.

In matrix theory, the adjoint is defined by $(Tx, y) = (x, T^*y)$. In operator theory, one has to take domains into consideration. Let $\varphi, \xi \in \mathfrak{H}$ such that $(A\psi, \varphi) = (\psi, \xi)$ for all $\psi \in \mathcal{D}(A)$, and define $A^*$ by $A^*\varphi = \xi$. Then $\mathcal{D}(A^*)$ is precisely the set of these $\varphi$. One can show that if $A$ is densely defined, then $A^*$ is closed. Furthermore, $A^*$ is densely defined if and only if $A$ is closable, and if it is, then $(\bar{A})^* = A^*$.

If $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $A\varphi = A^*\varphi$ for all $\varphi \in \mathcal{D}(A)$, then $A$ is called symmetric. If $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $A\varphi = A^*\varphi$ for all $\varphi \in \mathcal{D}(A)$, then $A$ is called self-adjoint. Self-adjoint operators form a subclass of symmetric operators.

A symmetric operator may have no self-adjoint extension, it may have many self-adjoint extensions, or it may have only one. In the last case, it is called essentially self-adjoint. One can show that if $A$ is essentially self-adjoint, then its closure $\bar{A}$ is self-adjoint, i.e., $\bar{A}$ is the unique self-adjoint extension of $A$.

The fundamental differences between symmetric and self-adjoint operators are:

1. The spectrum of a self-adjoint operator is a subset of the real line, whereas the spectrum of a symmetric operator is a subset of the complex plane; a symmetric operator is self-adjoint if and only if its spectrum is a subset of the real line.

2. A self-adjoint operator can be exponentiated, i.e., if $A$ is self-adjoint then $\exp(itA)$ is defined for all $t \in \mathbb{R}$; a symmetric operator which is not self-adjoint cannot be exponentiated.

The representation problem for the CCR (one degree of freedom) may now be formulated as follows: Find all pairs of essentially self-adjoint operators.

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\[\text{6}\text{Von Neumann used the term Hermitian, but current usage seems to limit this term to operators on finite-dimensional vector spaces.}\]

\[\text{7}\text{Von Neumann used the term Hermitian hypermaximal.}\]
$P, Q$, densely defined on a common domain $\mathfrak{D}$, such that $[P, Q]\phi = -i\phi$ for every $\phi \in \mathfrak{D}$. There are infinitely many such (inequivalent) pairs; the interested reader is referred to Schmudgen \cite{9} for details, and for references to earlier works.

If $A$ and $B$ are self-adjoint, are defined on a common dense domain $\mathfrak{D}$ and commute on $\mathfrak{D}$, then $\exp(iaA)$ and $\exp(ibB)$ are defined for all $a, b \in \mathbb{R}$ and commute. However, if $A$ and $B$ are merely essentially self-adjoint, are defined on $\mathfrak{D}$ and commute on $\mathfrak{D}$, then $\exp(ia\bar{A})$ and $\exp(ib\bar{B})$ do not necessarily commute. This fact, which at first seems highly counterintuitive, was unearthed by Nelson in 1958; for details and references, see Reed and Simon \cite{6, 7}.

We shall conclude this section with an example. The group of isometries of $\mathbb{R}^2$ consists of translations and rotations. The group of isometries of the punctured plane $\mathbb{R}^2 \setminus \{0\}$ is the group of rotations about the origin $O$. What happens to the translation operators on $\mathbb{R}^2$, namely $\exp(iax)$ and $\exp(iby)$, $a, b \in \mathbb{R}$ (where $p_x = -i\partial/\partial x$, $p_y = -i\partial/\partial y$), when the origin is excised?

The operators $\partial/\partial x, \partial/\partial y$ are defined on sets of differentiable functions. A function which is differentiable on $\mathbb{R}^2$ is necessarily differentiable on $\mathbb{R}^2 \setminus \{0\}$, but the latter has a richer supply of differentiable functions than $\mathbb{R}^2$, e.g., the function $r^{-1}\exp(-r^2/2)$ (which is also square-integrable). Restricting the domain enlarges the set of differentiable functions on which $p_x$ and $p_y$ are defined. This enlargement changes the spectra of these operators, which in turn leads to the failure of self-adjointness and exponentiability.

## 5 Theory of the experiment

In 1988, Helmut Reeh showed that that the ‘Nelson phenomenon’ could be found in the Aharonov-Bohm effect \cite{8}. For brevity, let us call a spinless particle of charge $q$ moving in a plane perpendicular to a trapped magnetic flux – the classical Aharonov-Bohm example – an $AB$-particle. Owing to cylindrical symmetry, the motion of an AB-particle is essentially two-dimensional. Its canonical operators may be written, formally, as

$$p = -i\frac{\partial}{\partial x} + qA, \quad q = \text{multiplication by } x.$$  \hspace{1cm} (4)
Boldface symbols denote 2-vectors in the $XY$-plane. The vector potential $A$ (up to a gauge) can be written, in terms of the magnetic flux $\Phi$, as

$$A = \frac{\Phi}{2\pi r} e,$$

(5)

where $r = (x^2 + y^2)^{1/2}$ and $e$ is the unit vector at $(x, y)$ tangent to the circle $r = \text{const}$:

$$e = \left(-\frac{y}{r}, \frac{x}{r}\right).$$

We shall set $\alpha = q\Phi/2\pi$ and use (5) to rewrite the quantities $p$ of (4) as

$$p^\alpha = -i \frac{\partial}{\partial x} + \alpha e,$$

(6)

where the $\alpha$-dependence of $p$ has been rendered explicit on the left. The problem is to define the formal quantities $p^\alpha_x$ and $p^\alpha_y$ in (6) as operators on the Hilbert space $L^2(\mathbb{R}^2 \setminus O) = L^2(\mathbb{R}^2)$; excision of a single point, here the origin $O$, has no real effect on an $L^2$-space, but – as we have seen earlier – changing the domains of differentiation operators ever so slightly can have drastic consequences. Reeh chose, for the common domain of $p^\alpha_x, p^\alpha_y$, the space $\mathcal{D}(\mathbb{R}^2 \setminus O)$ of smooth functions with compact support on $\mathbb{R}^2 \setminus O$. The space $\mathcal{D}(\mathbb{R}^2 \setminus O)$ is dense in $L^2(\mathbb{R}^2)$, and $p^\alpha_x$ and $p^\alpha_y$ are distributions on it. If $\varphi \in \mathcal{D}(\mathbb{R}^2 \setminus O)$, then it follows from curl $A = 0$ that $[p^\alpha_x, p^\alpha_y]\varphi = 0$.

Consider now the equation

$$p^\alpha_x \varphi = \left(-i \frac{\partial}{\partial x} - \alpha \frac{y}{x^2 + y^2}\right) \varphi = \lambda \varphi.$$

(7)

It is a linear homogeneous differential equation of the first order which can be solved explicitly for any $\lambda \in \mathbb{C}$, and the same holds for the equation $p^\alpha_y \psi = \lambda \psi$. The solutions do not have compact support. By exploiting these solutions, Reeh established the following results [8]:

1. The operators $p^\alpha_x$ and $p^\alpha_y$ are not self-adjoint; they are essentially self-adjoint.

2. Let $\bar{p}^\alpha_x$ and $\bar{p}^\alpha_y$ be their self-adjoint extensions, and define

$$V^\alpha_x(a) = \exp (i a \bar{p}^\alpha_x), \quad V^\alpha_y(b) = \exp (i b \bar{p}^\alpha_y).$$
Then
\[ V_x^\alpha(a)V_y^\alpha(b)V_x^\alpha(a)^{-1}V_y^\alpha(b)^{-1} = e^{i(\pi\alpha/2)\left[\epsilon(x)-\epsilon(x+a)\right] \left[\epsilon(y)-\epsilon(y-b)\right]} I, \quad (8) \]
where \( I \) is the identity operator, and
\[ \epsilon(t) = \begin{cases} 
1 & t > 1 \\
-1 & t < 1.
\end{cases} \]

Note that the product \([\ldots][\ldots] \) in the exponent on the right-hand side of (8) can only assume the values 0, ±4, so that the entire right-hand side can only assume the values \( I, \exp(\pm 2\pi i\alpha)I \). It follows that if \( \alpha \) is an integer, then the right-hand side of (8) equals the identity operator \( I \) for all admissible \( x, y, a, b \), but not if \( \alpha \) is not an integer; in this case the group generated by the operators \( \{x, y, p_x^\alpha, p_y^\alpha\} \) is no longer isomorphic with the Weyl group \( W_2 \). Clearly, the groups generated by these operators for \( \alpha = \alpha_1, \alpha_2 \) are not isomorphic with each other if \( \alpha_1 - \alpha_2 \) is not an integer, and therefore the representations of the CCR (for two degrees of freedom) they define are not unitarily equivalent.

The experiment suggested in Section 2 is designed to determine whether this mathematical inequivalence has observable physical consequences.

6 Interpretation of possible results

1. If superseparability is observed with charged boson beams, then – irrespective of the psychological effect of the observation – it will confirm that the notion of inequivalent irreducible representations of the CCR (for a finite number of degrees of freedom) is physically meaningful; vectors from two inequivalent representations cannot be superposed upon each other. It should then prompt the investigation of other possible effects that arise from the existence of inequivalent irreducible representations of the CCR. Note that the discussion is at the level of the first quantization.

2. If, however, the phenomenon is not observed, we shall have to conclude that something basic is lacking in our understanding of the linear space that underlies quantum mechanics: the question that David Hilbert asked Rolf Nevanlinna in the late 1920’s – Tell me, Rolf, what is this
Hilbert Space that the young people are talking about? – may not yet have been answered to the physicist’s satisfaction.

3. The theoretical considerations of Section 5 do not apply to fermions. The creation-annihilation operators for a fermion are bounded, and the canonical anticommutation relations for a finite number of degrees of freedom have only one irreducible unitary representation. This was proved by Jordan and Wigner in their very first paper on anticommutation relations in 1928. Therefore one should not expect to find the phenomenon of superseparability among fermions. If the experiment is performed with electrons and superseparability is observed, it will pose new and unsettling problems for theoretical physics.

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