Limit Theorems for Multi-Group Curie-Weiss Models via the Method of Moments

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Abstract

We study a multi-group version of the mean-field or Curie-Weiss spin model. For this model, we show how, analogously to the classical (single-group) model, the three temperature regimes are defined. Then we use the method of moments to determine for each regime how the vector of the group magnetisations behaves asymptotically. Some possible applications to social or political sciences are discussed.

Keywords: Curie-Weiss model, mean-field model, limit theorem, method of moments

2020 Mathematics Subject Classification: 60F05, 82B20

1 Definition of the Model

The Curie-Weiss model (CWM) is usually defined for a single set of spins or binary random variables. We have a fixed number of spins, which assume one of two values, ±1. So the state space of the model is \{-1, 1\}^N, where \(N \in \mathbb{N}\) is the number of spins. The spins have a tendency to align with each other. If the majority of other spins is positive, then the conditional probability of a given spin being positive is greater than 1/2. The probability measure that describes the single-group CWM for every spin configuration \((x_1, \ldots, x_N) \in \{-1, 1\}^N\) is given by

\[
\mathbb{P}(X_1 = x_1, \ldots, X_N = x_N) = Z^{-1} \exp\left(\frac{\beta}{2N} \left(\sum_{i=1}^{N} x_i\right)^2\right),
\]

where \(\beta \geq 0\) is the inverse temperature parameter.

The CWM has been extensively studied. It is named after Pierre Curie and Pierre Weiss, who first used the mean-field approach to study phase transitions in spin models. The CWM is also called the Husimi-Temperley model, since it was first introduced by Husimi [12] and Temperley [26]. Subsequently, it was discussed by Kac [14], Ellis-Newman [7], and many other authors. For an overview and more references see Thompson [27] and Ellis [6]. More recently, the CWM has been used in the context of social and political interactions. The idea of using models from statistical mechanics to study social interactions goes back to Föllmer [10]. The Curie-Weiss model specifically was first employed in [2]. See e.g. [5, 11, 17, 18, 28, 25, 24, 21] for other applications. Multi-group versions of this model were introduced in [4] and [1], and analysed by other authors as well [5, 9, 8, 3, 23, 22]. The authors of the present article have studied special cases of the models considered in this article with only two groups in [19, 20]. See the PhD thesis [28] of one of the authors for a detailed exposition of not only the CWM but other models of voting behaviour.

As mentioned, this multi-group model has been analysed by several different authors. Among the articles published, [9, 22] contain many of the same limit theorems to be found in this article. The methods employed in the earlier articles are mainly analytical in nature. We present proofs using the method of moments which has a more combinatorial flavour. We also prove strong laws of large numbers and a conditional central limit theorem for the low temperature regime, which to our knowledge have not been published in other articles.

The limit theorems that hold for this multi-group model are similar to the theorems which hold for the single-group model (see Chapters IV and V of [6] for an exposition). There are laws of large numbers and central...
limit theorems for the vector of group magnetisations, and these are multivariate versions of the results for the single-group model.

Multi-group models of this type have been used to study social dynamics and political decision making. For example, consider a population of voters belonging to different groups or constituencies such as a federal republic with a number of partially autonomous states. Typical examples are the Council of the European Union and to a large extent the Electoral College of the USA. Each group elects a representative to vote on its behalf in a council. The voters within each group influence each other in their decisions, more so than voters belonging to different groups do. In fact, it may even be the case that voters belonging to different groups tend to make contrary decisions.

To study such a situation, we define a model with different groups of spins that potentially interact with each other in different ways. Let

\[
(X_{11}, X_{12}, \ldots, X_{1N_1}, \ldots, X_{M1}, X_{M2}, \ldots, X_{MN_M}) \in \prod_{\lambda=1}^{M} \{-1,1\}^{N_\lambda} = \{-1,1\}^N
\]

be a random spin configuration of the entire population. \(X_{\lambda i}\) is the \(i\)-th spin of \(N_{\lambda}\) in group \(\lambda \in \{1, \ldots, M\}\). Each group \(\lambda\) consists of \(N_{\lambda}\) spins, hence \(\sum_{\lambda=1}^{M} N_{\lambda} = N\). Instead of a single inverse temperature parameter, there is a coupling matrix that describes the interactions. We will call this matrix \(J := (J_{\lambda \mu})_{\lambda,\mu=1,\ldots,M}\). Throughout this paper we will always assume that \(J\) is positive semi-definite. Just as in the single-group model, there is a Hamiltonian function that assigns each configuration a certain energy level. This energy level can also be interpreted as how costly a certain situation is in terms of the conflict between different voters.

\[
\mathbb{H}(x_{11}, \ldots, x_{MN_M}) := -\frac{1}{2N} \sum_{\lambda,\mu=1}^{M} J_{\lambda \mu} \sum_{i=1}^{N_{\lambda}} \sum_{j=1}^{N_{\mu}} x_{\lambda i} x_{\mu j}.
\]

Instead of each spin interacting with each other spin in exactly the same way, spins in different groups \(\lambda, \mu\) are coupled by a coupling constant \(J_{\lambda \mu}\). These coupling constants subsume the inverse temperature parameter \(\beta\) found in the single-group model. We note that, depending on the signs of the coupling parameters \(J_{\lambda \mu}\), different configurations have different energy levels assigned to them by \(\mathbb{H}\). If all coupling parameters are positive, there are two configurations that have the lowest possible energy levels: \((-1, \ldots, -1)\) and \((1, \ldots, 1)\). All other configurations receive higher energy levels. The highest levels are those where the spins are evenly split (or very close to it in case of odd group sizes).

We want to emphasise that the Hamiltonian \(\mathbb{H}\) – similarly to that of the single-group model defined in 1 – does not contain a term reflecting an external magnetic field. The reason for this is that applications to social sciences discussed above usually presuppose a symmetric probability measure on the space of spin configurations \(\{-1,1\}^N\).

**Definition 1.** A collection \((X_{11}, \ldots, X_{1N_1}, \ldots, X_{M1}, \ldots, X_{MN_M})\) of \(\{-1,1\}\)-valued random variables is called an \(M\)-group Curie-Weiss model with coupling matrix \(J\) if the probability of each of the \(2^N\) spin configurations is given by

\[
P(X_{11} = x_{11}, \ldots, X_{MN_M} = x_{MN_M}) := Z^{-1} e^{-\mathbb{H}(x_{11}, \ldots, x_{MN_M})},
\]

where each \(x_{\lambda i}\) is in \(\{-1,1\}\) and \(Z\) is a normalisation constant which depends on \(N\) and \(J\). The measure \(P\) is called the ‘canonical ensemble’ associated to the energy \(\mathbb{H}\).

As there is no external magnetic field, the measure \(P\) satisfies the symmetry condition

\[
P(X_{11} = x_{11}, \ldots, X_{MN_M} = x_{MN_M}) = P(X_{11} = -x_{11}, \ldots, X_{MN_M} = -x_{MN_M})
\]

for all configurations.

In the definition of homogeneous coupling matrices below we will use the notation

**Definition 2.** Let \((c)_{\lambda,\mu=1,\ldots,M}\) stand for an \(M \times M\) matrix with each entry equal to the constant \(c\).
We will deal with two classes of coupling matrices in this article:

1. Homogeneous coupling matrices \( J = (\beta)_{\lambda,\mu=1,\ldots,M} \), where all entries are equal to the same constant \( \beta \geq 0 \). In this case, \( J \) is positive semi-definite, but not positive definite.

2. Heterogeneous coupling matrices \( J = (J_{\lambda,\mu})_{\lambda,\mu=1,\ldots,M} \), which we assume to be positive definite.

We are interested in the asymptotic behaviour of the so called magnetisations. These are the sums of all spins belonging to each group:

\[
S := (S_1, \ldots, S_M) := \left( \sum_{i_1=1}^{N_1} X_{1i_1}, \ldots, \sum_{i_M=1}^{N_M} X_{Mi_M} \right).
\]

Since these magnetisations grow without bound as \( N \to \infty \), we need to normalise them by dividing each component \( S_\lambda \) by a suitable power \( \gamma \) of \( N_\lambda \). The power will turn out to depend on the regime the model is in, which we will see is determined by \( J \) and the asymptotic relative group sizes \( \alpha_1, \ldots, \alpha_M \):

\[
\alpha_\lambda := \lim_{N \to \infty} \frac{N_\lambda}{N}, \quad N_\lambda \to \infty \text{ as } N \to \infty.
\]

Note that we assume that as the overall population goes to infinity, so does each group. The \( \alpha_\lambda \)'s sum up to 1. We do not assume that all \( \alpha_\lambda \) are necessarily positive. If \( \alpha_\lambda = 0 \), we will say that group \( \lambda \) is ‘small’.

Throughout this article, we use Greek letters \( \lambda, \mu, \nu \) to index groups and Latin letters \( i, j, k \) to index the individual spins.

This article consists of seven sections. After this introduction, we define the three regimes of the model in Section 2. Then, we present and discuss the results of this paper in Section 3. The remainder of the article is dedicated to the proof of these results. We introduce some combinatorial concepts in Section 4 that are necessary for the application of the method of moments. Next, we calculate correlations of the spin variables of the form \( E(X_{11}\cdots X_{1k_1}\cdots X_{Mi_1}\cdots X_{Mk_M}) \) in Section 5. The moments of \( S \) are then calculated using these correlations in Section 6. Finally, we prove a strong law of large numbers for the magnetisations in Section 7.

Acknowledgement: We thank the referees for their careful reading of the manuscript. Their suggestions helped to improve the paper considerably.

## 2 The Three Regimes of the Model

We call the regimes of the model ‘temperature regimes’ because in the single-group model the parameter \( \beta \) can be interpreted as the inverse temperature. There, \( \beta < 1 \) is called the ‘high temperature regime’, \( \beta = 1 \) the ‘critical regime’, and \( \beta > 1 \) the ‘low temperature regime’. We use the same definition for homogeneous coupling matrices, as it turns out that \( S \) behaves differently in each of these three regimes.

Note that the Gibbs measure (3) for the homogeneous model is identical to the Gibbs measure of the single-group model (1). However, the magnetisations (4) form a random vector not a scalar as in the single-group model. Also, the entries of this random vector are correlated for \( \beta > 0 \). Thus, the analysis of the homogeneous model goes beyond the classical results concerning the classical CWM. We remark that the coupling matrix \( (\beta)_{\lambda,\mu} \) of the homogeneous model has determinant 0, so it is not a special case but rather a limit case of the heterogeneous model which has strictly positive determinant.

**Definition 3.** For homogeneous coupling matrices, we define the high temperature regime to be \( \beta < 1 \), the critical regime to be \( \beta = 1 \), and the low temperature regime to be \( \beta > 1 \).

For heterogeneous coupling matrices, the situation is somewhat more complicated. The parameter space is

\[
\Phi := \left\{ (\alpha_1, \ldots, \alpha_M) \mid \alpha_1, \ldots, \alpha_M \geq 0, \sum_{\lambda=1}^{M} \alpha_\lambda = 1 \right\} \times \{ J \mid J \text{ is an } M \times M \text{ positive definite matrix} \}.
\]
We define $\alpha := \text{diag}(\alpha_1, \ldots, \alpha_M)$, where ‘diag’ stands for a diagonal matrix with the entries given between parentheses, and

$$H := J^{-1} - \alpha. \quad (5)$$

We call the above matrix $H$ because, as we shall later see, it is the Hessian matrix of a function whose minima we have to find. We define the inverse of the coupling matrix $L := J^{-1}$. For future reference, we also define the ‘square root’ of the diagonal matrix $\alpha$:

$$\sqrt{\alpha} := \text{diag}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_M}). \quad (6)$$

**Definition 4.** For heterogeneous coupling matrices, the ‘high temperature regime’ is the set of parameters

$$\Phi_h := \{ \phi \in \Phi \mid H \text{ is positive definite} \}.$$

**Remark 5.** High temperature means weak interaction between voters. They influence each other’s decisions weakly, but polarisation of votes is still possible and happens frequently.

Now we define the critical regime for heterogeneous coupling matrices.

**Definition 6.** For heterogeneous coupling matrices, the ‘critical regime’ is the set of parameters

$$\Phi_c := \{ \phi \in \Phi \mid H \text{ is positive semi-definite but not positive definite} \}.$$ 

Having defined the high temperature and critical regimes, we can now define the low temperature regime as the complement of the union of these two sets in the parameter space.

**Definition 7.** For heterogeneous coupling matrices, the ‘low temperature regime’ is the set of parameters

$$\Phi_l := \Phi \setminus (\Phi_h \cup \Phi_c).$$

**Remark 8.** The coupling constants in the low temperature regime are high in relation to the inverse group sizes. Low temperature means very strong interactions between voters.

### 3 Results

We discuss the results for homogeneous and heterogeneous coupling matrices together, as they are qualitatively similar. We comment on any differences.

#### 3.1 High Temperature Results

Let for all $x \in \mathbb{R}^M$ the symbol $\delta_x$ stand for the Dirac measure at the point $x$, and $N(0, C)$ for the multivariate normal distribution with mean 0 and covariance matrix $C$. We shall write $' \xrightarrow{N \to \infty} ' \text{ for weak convergence as } N \to \infty$.

In the high temperature regime, we have a Law of Large Numbers (LLN) and a Central Limit Theorem (CLT). First the LLN:

**Theorem 9.** For homogeneous and heterogeneous coupling matrices, in their respective high temperature regimes, we have

$$\left( \frac{S_1}{N_1}, \ldots, \frac{S_M}{N_M} \right) \xrightarrow{N \to \infty} \delta_0.$$ 

**Remark 10.** The above LLN holds even in a strong sense, i.e. in terms of almost sure convergence rather than convergence in distribution. See Section 7.

Next we state the CLT. In the following, let $I$ be the identity matrix. Its dimensions should be clear from the context, such as below, where $I$ is an $M \times M$ matrix.
Theorem 11. For homogeneous and heterogeneous coupling matrices, in their respective high temperature regimes, we have
\[
\left( \frac{S_1}{\sqrt{N_1}}, \ldots, \frac{S_M}{\sqrt{N_M}} \right) \overset{N \to \infty}{\Longrightarrow} N(0, C),
\]
where the covariance matrix \(C\) is given by
\[
C = I + \sqrt{\alpha} \Sigma \sqrt{\alpha},
\]
and the matrix \(\Sigma\) is
\[
\left( \beta_{1 \lambda} - \beta_{1 \mu} \right)_{\lambda, \mu = 1, \ldots, M}
\]
if \(J\) is homogeneous and \(H^{-1}\) if \(J\) is heterogeneous.

Remark 12. If a group \(\nu\) is small, i.e. \(\alpha_{\nu} = 0\), then the marginal distribution of \(\frac{S_\nu}{\sqrt{N_\nu}}\) asymptotically follows a standard normal distribution and is asymptotically independent of all other \(\frac{S_\lambda}{\sqrt{N_\lambda}}\).

3.2 Critical Regime Results

The LLN holds for the critical regime as well.

Theorem 13. For homogeneous coupling matrices, in the critical regime, we have
\[
\left( \frac{S_1}{N_1^{3/4}}, \ldots, \frac{S_M}{N_M^{3/4}} \right) \overset{N \to \infty}{\Longrightarrow} \delta_0.
\]
If \(M = 2\) and \(J\) is heterogeneous, we have
\[
\left( \frac{S_1}{N_1}, \frac{S_2}{N_2} \right) \overset{N \to \infty}{\Longrightarrow} \delta_0.
\]

Remark 14. The above LLN holds even in a strong sense. See Section 7.

Similarly to the Central Limit Theorem 11, we can normalise with a power \(\gamma < 1\) in order to obtain a limiting distribution which is not concentrated in the origin. In the critical regime, the appropriate normalising power is \(\gamma = 3/4\).

For homogeneous coupling matrices, we have

Theorem 15. For homogeneous coupling matrices, in the critical regime, we have
\[
\left( \frac{S_1}{N_1^{3/4}}, \ldots, \frac{S_M}{N_M^{3/4}} \right) \overset{N \to \infty}{\Longrightarrow} \mu.
\]
The probability measure \(\mu\) on \(\mathbb{R}^M\) has moments of order \((K_1, \ldots, K_M), K := \sum_{\nu=1}^M K_\nu, with K even,
\[
m_{K_1,\ldots,K_M}(\mu) := 12^n \Gamma \left( \frac{K+1}{4} \right) \prod \frac{K_\lambda}{\Gamma \left( \frac{K_\lambda + 1}{4} \right)} \alpha_1^{\frac{K_1}{4}} \cdots \alpha_M^{\frac{K_M}{4}},
\]
and \(m_{K_1,\ldots,K_M}(\mu) = 0\) if \(K\) is odd.

For heterogeneous coupling matrices, we have a result for two groups. We define \(L := J^{-1}\). For \(M = 2\), we will call the entries
\[
L = \left( \begin{array}{cc} L_1 & -L \\ -L & L_2 \end{array} \right).
\]

Theorem 16. Let \(M = 2\). For heterogeneous coupling matrices, in the critical regime, we have
\[
\left( \frac{S_1}{N_1^{3/4}}, \frac{S_2}{N_2^{3/4}} \right) \overset{N \to \infty}{\Longrightarrow} \mu.
\]
The probability measure \(\mu\) on \(\mathbb{R}^2\) has moments of order \((K, Q) m_{K,Q}(\mu) :=
\[
\left[ \frac{12}{\alpha_1 (L_2 - \alpha_2)^2 + \alpha_2 (L_1 - \alpha_1)^2} \right]^{K+Q} (L_1 - \alpha_1)^{Q/4} (L_2 - \alpha_2)^{Q/4} \prod \frac{\Gamma \left( \frac{K+Q+1}{4} \right)}{\Gamma \left( \frac{Q}{4} \right)} \alpha_1^{Q/4} \alpha_2^{Q/4}
\]
if \(K + Q\) is even and 0 otherwise.
Remark 17. For the special case $J_{11} = J_{22} = J$, $\alpha_1 = \alpha_2 = 1/2$, and $J + J_{12} = 2$, the moments of the limiting distribution are

$$12 \frac{K+Q}{\Gamma \left( \frac{K+Q+1}{4} \right)} \frac{\alpha_1}{\Gamma \left( \frac{1}{4} \right)} \alpha_2 = \frac{4}{\lambda},$$

and hence identical to those for the model with a homogeneous coupling matrix and $\beta = 1$.

Remark 18. If a group $\nu$ is small, then the marginal distribution of $\frac{S_0}{N^V}$ is asymptotically the Dirac measure $\delta_0$, and therefore asymptotically independent of all other $S_\lambda/N^3$. It is unclear what the joint distribution $\mu$ is. However, we can deduce the limiting distribution of two linear transformations of $\left( \frac{S_{\lambda_1}}{N^1}, \frac{S_{\lambda_2}}{N^2} \right)$. Let $\nu_\eta$ be the probability measure on $\mathbb{R}$ given by the density function proportional to $\exp(-\eta x^4)$, $x \in \mathbb{R}$.

**Theorem 19.** Let $M = 2$ and $\alpha_1, \alpha_2 \neq 0$. In the critical regime, these results hold:

For homogeneous coupling matrices, we have

$$\sqrt{\frac{\alpha_1}{\alpha_2}} S_1 / \sqrt{N} \xrightarrow{N \to \infty} N(0,1),$$

and

$$\frac{S_1}{2 (\alpha_1 N^3)^{1/4}} + \frac{S_2}{2 (\alpha_2 N^3)^{1/4}} \xrightarrow{N \to \infty} \nu_\eta$$

with $\eta = \frac{1}{12}$.

For heterogeneous coupling matrices,

$$\sqrt{\frac{L_1 - \alpha_1}{\alpha_1 N^3}} S_1 - \sqrt{\frac{L_2 - \alpha_2}{\alpha_2 N^3}} S_2 \xrightarrow{N \to \infty} N \left( 0, 1 + \frac{L_1 - \alpha_1}{\alpha_1} + \frac{L_2 - \alpha_2}{\alpha_2} \right),$$

and

$$\sqrt{\frac{L_1 - \alpha_1}{(\alpha_1 N^3)^{1/4}}} S_1 + \sqrt{\frac{L_2 - \alpha_2}{(\alpha_2 N^3)^{1/4}}} S_2 \xrightarrow{N \to \infty} \nu_\eta$$

with $\eta = \frac{1}{2^{3/4}} \left( \frac{\alpha_1}{(L_1 - \alpha_1)^{1/4}} + \frac{\alpha_2}{(L_2 - \alpha_2)^{1/4}} \right)$.

The result in Theorem 19 for homogeneous coupling matrices shows that the sequence of random variables

$$\frac{S_1}{2 (\alpha_1 N^3)^{1/4}} + \frac{S_2}{2 (\alpha_2 N^3)^{1/4}}$$

converges to the same limiting distribution $\nu_{1/2}$ as the sequence $(S_1 + S_2) / N^{3/4}$. The latter statement is a well-known result (see e.g. Theorem V.9.5 in [6]), and Theorem 19 says that the same limiting distribution is obtained by summing the two suitably scaled group magnetisations $S_1$ and $S_2$.

### 3.3 Low Temperature Results

For homogeneous coupling matrices, we have a limit theorem similar to the single-group case. As in the single-group model, we need to solve the so-called Curie-Weiss equation

$$\tanh(\beta t) = t.$$  \hspace{1cm} (7)

For $\beta \leq 1$, this equation has a single solution which is $t = 0$. For $\beta > 1$, there are three different solutions: $-t_1, 0, t_1$ with $t_1 > 0$.

**Definition 20.** We define $m(\beta)$ as 0 if $\beta \leq 1$ and as $t_1$ if $\beta > 1$. 

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Theorem 21. For homogeneous coupling matrices, in all regimes, we have
\[
\left(\frac{S_1}{N_1}, \ldots, \frac{S_M}{N_M}\right) \quad N \to \infty \quad \frac{1}{2} \left(\delta(-m(\beta), \ldots, -m(\beta)) + \delta(m(\beta), \ldots, m(\beta))\right).
\]

As far as we know, there is no limit theorem for the low temperature regime for heterogeneous coupling matrices in the most general case. Instead, there are results for some special cases, such as \(M = 2\) (see [20]) and \(M > 2\) with groups of equal size (see [22]). The main difficulty in the low temperature regime is the analysis of the properties of the function \(F\) defined in (12). If the coupling matrix \(J\) has strictly positive entries, then it is clear that the global minima of \(F\) are located in the positive and negative orthant. What is not known is the number of minima of \(F\) in each orthant. If we assume that there is at most a single local minimum in each orthant – which holds for special cases \(M = 2\) and groups of equal size mentioned above – then we can state a limit theorem.

Theorem 22. For heterogeneous coupling matrices with positive entries, assume the function \(F\) defined in (12) has at most one local minimum in each orthant, and let \(\bar{m} \in \mathbb{R}^M\) be the local minimum found in the positive orthant. We set \(m := \tanh \bar{m}\), applying the \(\tanh\) function componentwise. Then, in the low temperature regime, we have
\[
\left(\frac{S_1}{N_1}, \ldots, \frac{S_M}{N_M}\right) \quad N \to \infty \quad \frac{1}{2} \left(\delta - m + \delta m\right).
\]

Note that above \(m\) is a point in \(\mathbb{R}^M\), as opposed to the previous theorem concerning homogeneous coupling matrices.

Other limit theorems of the same type hold for different assumptions on the signs of the non-diagonal entries of \(J\).

The previous theorem says that in the low temperature regime the magnetisations are significant in the sense that the Law of Large Numbers as in Theorems 9 and 13 does not hold here. We can investigate the fluctuations around the two points of concentration \(\pm m\) which turn out to be normally distributed under suitable scaling. This is the subject of the next theorem, which is a conditional CLT for the low temperature regime.

Theorem 23. Assume one of the following assumptions hold:

1. Let \(J\) be a homogeneous coupling matrix.

2. Let \(J\) be a heterogeneous coupling matrix with positive entries and let \(M = 2\).

3. Let \(J\) be a heterogeneous coupling matrix with positive entries, and assume the function \(F\) defined in (12) has at most one local minimum in each orthant of \(\mathbb{R}^M\).

We set \(m := (m(\beta), \ldots, m(\beta))\) if \(J\) is homogeneous and \(m := \tanh \bar{m}\) if \(J\) is heterogeneous and \(\bar{m}\) the minimum of \(F\) in the positive orthant.

Then, conditioning on \(S_\nu > 0\) for all groups \(\nu\), we have
\[
\left(\frac{1}{\sqrt{N_1}} \sum_{i_1=1}^{N_1} (X_{1i_1} - m_1), \ldots, \frac{1}{\sqrt{N_M}} \sum_{i_M=1}^{N_M} (X_{M1_M} - m_M)\right) \quad N \to \infty \quad \mathcal{N}(0, E).
\]

Similarly, under the condition \(S_\nu < 0\) for all groups \(\nu\), we have
\[
\left(\frac{1}{\sqrt{N_1}} \sum_{i_1=1}^{N_1} (X_{1i_1} + m_1), \ldots, \frac{1}{\sqrt{N_M}} \sum_{i_M=1}^{N_M} (X_{M1_M} + m_M)\right) \quad N \to \infty \quad \mathcal{N}(0, E).
\]

The covariance matrix \(E\) is the same in both limiting distributions, and
\[
E = \text{diag} \left(1 - m^2\right) + \sqrt{\alpha} H^{-1}(\bar{m}) \sqrt{\alpha},
\]
where \(H^{-1}(\bar{m})\) is the inverse of the Hessian matrix of \(F\) at \(\pm \bar{m}\).
In the remainder of this paper, we will prove the above results by the method of moments. The structure of most of these proofs is the following:

1. By expanding the moments

\[
E \left[ \left( \sum_{i_1=1}^{N_1} X_{1i_1} \right)^{K_1} \left( \sum_{i_2=1}^{N_2} X_{2i_2} \right)^{K_2} \cdots \left( \sum_{i_M=1}^{N_M} X_{M_iM} \right)^{K_M} \right],
\]

we obtain a huge sum with correlations of the form

\[
E \left( \prod_{\nu=1}^{M} \prod_{k_{\nu}=1}^{K_{\nu}} X_{\nu i_{\nu k_{\nu}}} \right) = E \left( X_{1i_{11}} \cdots X_{1i_{1K_1}} \cdots X_{Mi_{M1}} \cdots X_{M_{i_{M K_M}}} \right).
\]

2. We calculate these correlations asymptotically for large \( N \):
   
   (a) First we use a Hubbard-Stratonovich transformation to express probabilities given by the Curie-Weiss measure \( P \) as an integral: for the single-group model, this transformation consists of
   
   \[
   \exp \left( \frac{\beta}{2N} \sum_{i=1}^{N} X_i^2 \right) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp \left( -\frac{y^2}{2} \right) \exp \left( \frac{\beta}{N} \sum_{i=1}^{N} X_i \right) dy.
   \]

   (b) Then we employ Laplace’s method to estimate the above integral for large \( N \).

3. Finally, we calculate the moments of the normalised \( S \), and prove they converge to the claimed limit.

4 Combinatorial Concepts

Whenever we use the method of moments, we will have to evaluate sums of the form

\[
E \left[ \left( \sum_{i_1=1}^{N_1} X_{1i_1} \right)^{K_1} \left( \sum_{i_2=1}^{N_2} X_{2i_2} \right)^{K_2} \cdots \left( \sum_{i_M=1}^{N_M} X_{M_iM} \right)^{K_M} \right],
\]

we obtain a huge sum with correlations of the form

\[
E \left( \prod_{\nu=1}^{M} \prod_{k_{\nu}=1}^{K_{\nu}} X_{\nu i_{\nu k_{\nu}}} \right) = E \left( X_{1i_{11}} \cdots X_{1i_{1K_1}} \cdots X_{Mi_{M1}} \cdots X_{M_{i_{M K_M}}} \right).
\]

To do the book-keeping for these huge sums we introduce a few combinatorial concepts taken from [15]. Let \( |A| \) stand for the cardinality of the set \( A \).

**Definition 24.** Let \( L \) be a natural number. We define a multiindex \( \underline{i} = (i_1, i_2, \ldots, i_L) \in \{1, 2, \ldots, N\}^L \).

1. For \( j \in \{1, 2, \ldots, N\} \), we set
   
   \[ \nu_j(\underline{i}) := |\{k \in \{1, 2, \ldots, L\} \mid i_k = j\}|. \]

2. For \( \ell = 0, 1, \ldots, L \), we define
   
   \[ \rho_\ell(\underline{i}) := |\{j \mid \nu_j(\underline{i}) = \ell\}| \]
   
   and

   \[ \rho(\underline{i}) := (\rho_1(\underline{i}), \ldots, \rho_L(\underline{i})). \]

The numbers \( \nu_j(\underline{i}) \) represent the multiplicity of each index \( j \in \{1, 2, \ldots, N\} \) in the multiindex \( \underline{i} \), and \( \rho_\ell(\underline{i}) \) represents the number of indices in \( \underline{i} \) that occur exactly \( \ell \) times. We shall call \( \rho(\underline{i}) \) the profile of the multiindex \( \underline{i} \).

**Lemma 25.** For all \( \underline{i} = (i_1, i_2, \ldots, i_L) \in \{1, 2, \ldots, N\}^L \), we have \( \sum_{\ell=1}^{L} \ell \rho_\ell(\underline{i}) = L \).
We use this property of profiles to define

**Definition 26.** Let $\underline{x} = (r_1, \ldots, r_L)$ be such that $\sum_{\ell=1}^L \ell r_\ell = L$ hold. We call $\underline{x}$ a profile vector. We define

$$w_L(\underline{x}) = \left| \{ \underline{i} \in \{1, \ldots, N\}^L \mid \underline{\rho}(\underline{i}) = \underline{x} \} \right|$$

to represent the number of multiindices $\underline{i}$ that have a given profile vector $\underline{x}$.

We now define the set of all profile vectors for a given $L \in \mathbb{N}$.

**Definition 27.** Let $\Pi^{(L)} = \left\{ \underline{x} \in \{0, 1, \ldots, L\}^L \mid \sum_{\ell=1}^L \ell r_\ell = L \right\}$. Some important subsets of $\Pi^{(L)}$ are $\Pi_k^{(L)} = \left\{ \underline{x} \in \Pi^{(L)} \mid r_\ell = k \right\}$, $\Pi_0^{(L)} = \left\{ \underline{x} \in \Pi^{(L)} \mid r_\ell = 0 \text{ for all } \ell \geq 3 \right\}$, and $\Pi^+(L) = \left\{ \underline{x} \in \Pi^{(L)} \mid r_\ell > 0 \text{ for some } \ell \geq 3 \right\}$. We can also combine superscripts and subscripts. Then we have, e.g., $\Pi_0^{(L)} = \left\{ \underline{x} \in \Pi^{(L)} \mid r_\ell = 0 \text{ for all } \ell \neq 2 \right\}$.

We shall write for any $\underline{i} \in \{1, 2, \ldots, N\}^L$ $X_{\underline{i}} = X_{i_1} \cdots X_{i_L}$. For any $\underline{x} \in \Pi^{(L)}$, let $\underline{j} \in \{1, 2, \ldots, N\}^L$ be such that $\underline{\rho}(\underline{j}) = \underline{x}$. Then we let $X_{\underline{x}}$ stand for $X_{\underline{j}}$. This definition is not problematic if we are only interested in the expectation

$$E \left( X_{\underline{x}} \right) = E \left( X_{\underline{j}} \right),$$

and the random variables $X_1, \ldots, X_N$ are exchangeable.

If there are $M$ sets $\{1, 2, \ldots, N\}^L$, and for each $\nu k \in \{1, 2, \ldots, N\}^L$, we then set $\underline{i} := (i_{\nu k})$ and write $X_{\underline{i}}$ for $X_{i_1} \cdots X_{i_M}$. Similarly, if we have profile vectors $\underline{r}_\nu \in \Pi^{(L)}$, and $\underline{j}_\nu \in \{1, 2, \ldots, N\}^L$ such that $\underline{\rho}(\underline{j}_\nu) = \underline{r}_\nu$, then we write $X_{\underline{r}_\nu}$ for $X_{\underline{j}_\nu} \cdots X_{\underline{j}_M}$.

**Proposition 28.** For $\underline{r} \in \Pi^{(L)}$, set $r_0 := N - \sum_{\ell=1}^L \ell r_\ell$. Then

$$w_L(\underline{r}) = \frac{N! L!}{r_1! r_2! \ldots r_L! r_0! \prod_{\ell=1}^L 1! r_\ell! 2! r_\ell! \ldots L! r_\ell!}.$$  

If we let $N$ go to infinity, then we have

$$w_L(\underline{r}) \approx \frac{N^{\sum_{\ell=1}^L r_\ell} L!}{r_1! r_2! \ldots r_L! \prod_{\ell=1}^L 1! r_\ell! 2! r_\ell! \ldots L! r_\ell!}.$$  

This proposition is based on Theorem 3.14 and Corollary 3.18 in [15].

## 5 Correlations

In this section, we compute correlations of the form

$$E \left( \prod_{\nu=1}^M \prod_{k=1}^{K_\nu} X_{\nu k_\nu} \right) = E \left( X_{i_1} \cdots X_{i_{1k_1}} \cdots X_{M i_M} \cdots X_{M i_{MK_M}} \right),$$

where for $\nu = 1, \ldots, M$ the sequence $i_{\nu k_\nu}$ addresses $K_\nu$ spins in the group $\nu$.

Since $X_{12}^2 = 1$, it is enough to compute (8) for pairwise distinct index pairs $(\nu, \nu') \nu = 1, \ldots, M, \nu' = 1, \ldots, K_\nu$.

From the definition in (2), it is clear that the $X_{\nu t}$ are exchangeable within groups, i.e. for pairwise distinct index pairs, the random vectors

$$\left(X_{1_{i_{11}}, \ldots, i_{1k_1}}, \ldots, X_{M i_{M1}}, \ldots, X_{M i_{MK_M}} \right) \quad \text{and} \quad \left(X_{1_1}, \ldots, X_{1_{K_1}}, \ldots, X_{M 1}, \ldots, X_{M K_M} \right)$$

have the same distributions. Consequently, for these indices

$$m_{K_1, \ldots, K_M} := E \left( \prod_{\nu=1}^M \prod_{k=1}^{K_\nu} X_{\nu k_\nu} \right) = E \left( \prod_{\nu=1}^M \prod_{i_{\nu k_\nu}=1}^{i_{\nu k_\nu}} X_{\nu k_\nu} \right).$$

So the identity of the specific indices chosen from each group does not affect the correlation; only the number of different indices matters. The correlations differ according to the class of coupling matrix and the regime. We first show how to calculate the correlation for homogeneous coupling. These results have been known for a while and can also be used to calculate moments in the single-group model.
5.1 Homogeneous Coupling

When there is homogeneous coupling, all random variables are exchangeable and the (joint) distribution of the $X_{\nu}$ is actually the same as the distribution of a single-group CWM with $N = \sum_\nu N_\nu$.

To calculate the correlations in this case, we may therefore use estimates known for the single-group model.

**Definition 29.** Real-valued sequences $f_N, g_N$ are called *asymptotically equal* (as $N \to \infty$), in short $f_N \approx g_N$, if

$$\lim_{N \to \infty} \frac{f_N}{g_N} = 1.$$ 

The next theorem gives asymptotic expressions for the expectations.

**Theorem 30.** For homogeneous coupling matrices, the expectations

$$m_{k_1,\ldots,k_M} = \mathbb{E}(X_{11} \cdots X_{k_11} \cdots X_{M1} \cdots X_{Mk_M})$$

are asymptotically equal to:

1. if $\beta < 1$,

$$(k_1 + \cdots + k_M - 1)! \left(\frac{\beta}{1 - \beta}\right)^{k_1 + \cdots + k_M} \frac{1}{N^{k_1 + \cdots + k_M}};$$

2. if $\beta = 1$,

$$12^{k_1 + \cdots + k_M} \frac{\Gamma(k_1 + \cdots + k_M + 1)}{\Gamma(\frac{1}{4})} \frac{1}{N^{k_1 + \cdots + k_M}};$$

3. if $\beta > 1$,

$$m(\beta)^{k_1 + \cdots + k_M},$$

where $m(\beta)$ is the unique strictly positive solution of (7);

provided $k_1 + \cdots + k_M$ is even. If $k_1 + \cdots + k_M$ is odd, then $m_{k_1,\ldots,k_M} = 0$ for all $\beta \geq 0$.

**Proof.** This is Theorem 4.3 in [16].

5.2 Heterogeneous Coupling

When there is heterogeneous coupling, only the random variables within each group $X_{\lambda_i}$ are exchangeable. The case $M = 2$ and high temperature was analysed in a similar fashion in [20].

In the following, we set

$$x := (x_{11}, \ldots, x_{1N_1}, \ldots, x_{MN_M}) \in \{-1,1\}^N, \quad s_\lambda(x) := \sum_{i=1}^{N_\lambda} x_{\lambda i}, \quad \text{and} \quad s(x) := (s_1, \ldots, s_M)^T.$$ 

We calculate the correlations by adapting Laplace’s method. The assumption that the coupling matrix is positive definite is of key importance for the Hubbard-Stratonovich transformation.

**Proposition 31** (Multidimensional Hubbard-Stratonovich). Let $J$ be a heterogeneous coupling matrix, and $x = (x_{11}, \ldots, x_{MN_M}) \in \{-1,1\}^N$ a spin configuration. Then

$$e^{-\mathbb{E}(x)} = e^{\mathbb{E}(x)} = \sqrt{\det J} \left(\frac{2}{2\pi N^{1/4}}\right)^{2N} \int_{\mathbb{R}^M} e^{-\frac{u^T J u}{2N}} e^{u^T J s(x)} du.$$ 

**Proof.** This is a straightforward calculation.
We change variables $y := N^{-1} Ju$ in (10). Recall that we write $L$ for $J^{-1}$, and we obtain

$$e^{-H(x)} = \frac{N^{\frac{M}{2}}}{(2\pi)^{\frac{M}{2}} \sqrt{\det J}} \int_{\mathbb{R}^M} e^{-\frac{N}{2} (y^T L y)} e^{y^T s(x)} \, dy.$$  \hfill(11)

Let us define

$$Z_{k_1,\ldots,k_M} := \sum_{x \in \{-1,1\}^N} \left[ \prod_{i=1}^{k_1} x_{1i} \cdots \prod_{i=k_{M+1}}^{k_M} x_{MiM} \right] e^{-\mathcal{H}(x)}.$$ 

Observe that

$$\frac{Z_{k_1,\ldots,k_M}}{Z_{0,\ldots,0}} = \mathbb{E} (X_{11} \cdots X_{1k_1} \cdots X_{M1} \cdots X_{Mk_M}) = m_{k_1,\ldots,k_M}.$$ 

Using (11) and letting $c_N$ stand for the factor multiplying the integral in (11), we compute

$$Z_{k_1,\ldots,k_M} = c_N \int_{\mathbb{R}^M} e^{-\frac{N}{2} y^T L y} \sum_{x \in \{-1,1\}^N} \left[ \prod_{\lambda=1}^{M} \left( \sum_{x \in \{-1,1\}} e^{x_{\lambda x_{\lambda}}} \right)^{k_{\lambda}} \prod_{\lambda=1}^{M} e^{-y_{\lambda x_{\lambda}}} \right] \, dy$$

$$= c_N \int_{\mathbb{R}^M} e^{-\frac{N}{2} y^T L y} \prod_{\lambda=1}^{M} \left( \sum_{x \in \{-1,1\}} e^{y_{\lambda x_{\lambda}}} \right)^{k_{\lambda}} \prod_{\lambda=1}^{M} e^{-y_{\lambda x_{\lambda}}} \, dy$$

$$= 2^N c_N \int_{\mathbb{R}^M} \left( \prod_{\lambda=1}^{M} \tanh(y_{\lambda})^{k_{\lambda}} \right) e^{\left( -\frac{N}{2} y^T L y - \sum_{\lambda=1}^{M} N_{\lambda} \ln \cosh(y_{\lambda}) \right)} \, dy.$$ 

We summarise:

**Theorem 32.** Define the functions $F, F_N : \mathbb{R}^M \to \mathbb{R}$ by:

$$F_N(y) := \frac{1}{2} \sum_{\lambda, \nu=1}^{M} J_{\lambda \nu}^{-1} y_{\lambda} y_{\nu} - \sum_{\lambda=1}^{M} \frac{N_{\lambda}}{N} \ln \cosh(y_{\lambda}), \quad F(y) := \frac{1}{2} \sum_{\lambda, \nu=1}^{M} J_{\lambda \nu}^{-1} y_{\lambda} y_{\nu} - \sum_{\lambda=1}^{M} \alpha_{\lambda} \ln \cosh(y_{\lambda}).$$  \hfill(12)

Then

$$m_{k_1,\ldots,k_M} = \mathbb{E} (X_{11} \cdots X_{1k_1} \cdots X_{M1} \cdots X_{Mk_M})$$

$$= \int_{\mathbb{R}^M} \left( \prod_{\lambda=1}^{M} \tanh(y_{\lambda})^{k_{\lambda}} \right) e^{-N F_N(y)} \, dy \approx \int_{\mathbb{R}^M} \left( \prod_{\lambda=1}^{M} \tanh(y_{\lambda})^{k_{\lambda}} \right) e^{-N F(y)} \, dy.$$  \hfill(13)

Let us define the probability measure $P_y$ on $\{-1,1\}$ by $P_y(1) = \frac{1}{2}(1 + \tanh y)$ and its $k$-fold product by $P_y^\otimes k$. The corresponding expectation is denoted by $E_y$ and $E_y^\otimes k$, respectively.

From Theorem 32, we easily get a de-Finetti-type representation for the Curie-Weiss measure $\mathbb{P}$ defined in (3):

**Corollary 33.** We have

$$\mathbb{P} (X_{11} = x_{11} \cdots X_{1N_1} = x_{1N_1} \cdots X_{MN_M} = x_{MN_M})$$

$$= \frac{1}{\int_{\mathbb{R}^M} e^{-NF_N(y)} \, dy} \int_{\mathbb{R}^M} P_y^{\otimes N_1} (x_{11},\ldots,x_{1N_1}) \cdots P_y^{\otimes N_M} (x_{M1},\ldots,x_{MN_M}) e^{-N F_N(y)} \, dy$$

$$\approx \frac{1}{\int_{\mathbb{R}^M} e^{-NF(y)} \, dy} \int_{\mathbb{R}^M} P_y^{\otimes N_1} (x_{11},\ldots,x_{1N_1}) \cdots P_y^{\otimes N_M} (x_{M1},\ldots,x_{MN_M}) e^{-N F(y)} \, dy.$$  \hfill(14)
and

\[ m_{k_1, \ldots, k_M} = \int_{\mathbb{R}^M} \frac{1}{e^{-N F(y)}} \, dy \int_{\mathbb{R}^M} \left[ E_{y_1}^{N_1} \left( \prod_{i=1}^{k_1} x_{1i} \right) \right] \cdots \left[ E_{y_M}^{N_M} \left( \prod_{i=1}^{k_M} x_{Mi} \right) \right] e^{-N F(y)} \, dy \]

\[ \approx \frac{1}{\int_{\mathbb{R}^M} e^{-N F(y)} \, dy} \int_{\mathbb{R}^M} \left[ E_{y_1}^{N_1} \left( \prod_{i=1}^{k_1} x_{1i} \right) \right] \cdots \left[ E_{y_M}^{N_M} \left( \prod_{i=1}^{k_M} x_{Mi} \right) \right] e^{-N F(y)} \, dy. \] (15)

**Proof.** Equation (13) is just a rewriting of (13). Equation (14) follows since the probability measures involved are determined by their moments. \( \square \)

Our goal is to apply Laplace’s method to evaluate the integrals in (13) asymptotically. In order to do so, we have to analyse the critical points of the function \( F \). We start our analysis with the high temperature regime.

### 5.2.1 High Temperature Regime

**Proposition 34.** In the high temperature regime, i.e. if the matrix \( H = J^{-1} - \alpha = L - \alpha \) is positive definite, the function \( F \) defined in (12) has a unique global minimum at the origin. The matrix \( H \) is the Hessian of \( F \) at 0.

**Proof.** The Hessian of \( F \) is given by

\[ H(y) := \left( \frac{\partial^2 F}{\partial x_{\lambda} \partial x_{\nu}}(y) \right) = J^{-1}_{\lambda \nu} - \delta_{\lambda \nu} \alpha_{\nu} \left( 1 - \tanh (y_{\nu})^2 \right), \]

where the symbol \( \delta_{\lambda \nu} \) stands for the Kronecker delta. Since \( H = H(0) \) is assumed to be positive definite, by monotonicity of the eigenvalues of \( H(y) \) these matrices are positive definite for all \( y \). Thus, the function \( F \) is strictly convex.

The gradient of \( F \) vanishes in the origin. Hence, the origin is a local minimum. By strict convexity, 0 is the unique minimum. \( \square \)

**Remark 35.** The above proof actually shows that the origin is the unique minimum of \( F \) even if \( H \) is merely positive semi-definite. In fact, for positive semi-definite \( H \), the Hessian \( H(y) \) of \( F \) is positive definite for all \( y \neq 0 \).

**Definition 36.** If \( A \) is a positive definite \( M \times M \)-matrix and \( k \in \mathbb{N}_0^M \) is a multiindex, we denote by \( m_k(A) \) the \( k \)-th moment of the multivariate normal distribution \( N(0, A) \), i.e.

\[ m_k(A) := \frac{1}{(2 \pi \det A)^{M/2}} \int_{\mathbb{R}^M} x_1^{k_1} \cdots x_M^{k_M} e^{-\frac{1}{2} (x^T A^{-1} x)} \, dx. \]

**Theorem 37.** Let \( X_{11}, \ldots, X_{1N_1}, \ldots, X_{M1}, \ldots, X_{MN_M} \) be a sequence of random variables which are \( M \)-group Curie-Weiss distributed with positive definite coupling matrix \( J \) and assume that \( \lim_{N \to \infty} \frac{N}{N+1} = \alpha_{\nu} \).

If the model is in the high temperature regime, i.e. if the matrix \( H = J^{-1} - \alpha \) is positive definite, then

\[ m_{k_1, \ldots, k_M} := \mathbb{E} (X_{11} \cdots X_{1k_1} \cdots X_{M1} \cdots X_{Mk_M}) \approx \frac{1}{N^{k_1 + \cdots + k_M}} m_k(H^{-1}) \quad \text{as} \quad N \to \infty. \]

Theorem 37 follows from the following proposition which can be considered a special case of Laplace’s method in higher dimension (for background on Laplace’s method see for example [29]).

**Proposition 38.** Suppose \( H = J^{-1} - \alpha \) is positive definite. For \( k = (k_1, \ldots, k_M) \in \mathbb{N}_0^M \), \( k = \sum k_{\nu} \), and for \( F \) as in (12), we have

\[ \int_{\mathbb{R}^M} \left( \prod_{\lambda=1}^{M} \tanh(y_{\lambda})^{k_{\lambda}} \right) e^{-N F(y)} \, dy \approx \frac{1}{N^{k_1 + \cdots + k_M}} \int_{\mathbb{R}^M} \left( \prod_{\nu=1}^{M} y_{\nu}^{k_{\nu}} \right) e^{-\frac{1}{2} (y^T H y)} \, dy. \]
Proof (Theorem 37). We assume Proposition 38 for the moment. Recall that

\[ m_{k_1, \ldots, k_M} = \frac{Z_{k_1, \ldots, k_M}}{Z_{0, \ldots, 0}} \approx \frac{\int_{\mathbb{R}^M} \left( \prod_{\lambda=1}^M \tanh(y_\lambda)^{k_\lambda} \right) e^{-NF(y)} \, dy}{\int_{\mathbb{R}^M} e^{-NF(y)} \, dy}. \]

Applying Proposition 38 to both the numerator and the denominator in the above expression and noting that

\[
\int_{\mathbb{R}^M} e^{-\frac{1}{2}y^T H y} \, dy = (2\pi \det(H^{-1}))^{M/2}
\]

proves the theorem. \(\square\)

We next prove Proposition 38.

Proof (Proposition 38). Let us use the shorthand notation

\[ \tanh^k(y) = \prod_{\lambda=1}^M \tanh(y_\lambda)^{k_\lambda} \quad \text{and} \quad y^k = \prod_{\nu=1}^M y_\nu^{k_\nu} \]

in this proof as well as the remainder of this article.

We compute, by a change of variables,

\[
N^{k+M} \int_{\mathbb{R}^M} \tanh^k(y) \, e^{-N\left(\frac{1}{2}y^T J^{-1} y - \sum_\nu \ln \cosh(y_\nu)\right)} \, dy
= N^{k} \int_{\mathbb{R}^M} \tanh^k \left( \frac{x}{\sqrt{N}} \right) e^{-\left(\frac{1}{2}x^T J^{-1} x - \sum_\nu \ln \cosh \left( \frac{x_\nu}{\sqrt{N}} \right) \right)} \, dx. \quad (16)
\]

Now, we expand \(\ln \cosh(x)\) and \(\tanh(x)\) around \(x_0 = 0\) using Taylor’s theorem (in Lagrange form):

\[
\ln \cosh(x) = \frac{1}{2} x^2 - \frac{1}{2} \tanh^2(\tilde{x}) \, x^2 \quad \text{and} \quad \tanh(x) = x - \frac{\tanh(\tilde{x})}{\cosh^2(\tilde{x})} \, x^2,
\]

where \(\tilde{x}\) and \(\tilde{x}\) are points between 0 and \(x\).

Inserting this in (16) gives

\[
\int_{\mathbb{R}^M} \prod_{\nu=1}^M \left( x_\nu - \frac{1}{\sqrt{N} \cosh^2(\tilde{x})} \, x_\nu^2 \right)^{k_\nu} e^{-\frac{1}{2}(y^T (J^{-1} - \alpha) y) - \frac{1}{2N} \sum_\nu \alpha_\nu \tanh^2(\tilde{x}) \, x_\nu^2} \, dx. \quad (17)
\]

The integrand in (17) converges to \(x^k e^{-\frac{1}{2}(x^T H x)}\). Moreover, the integrand is bounded in absolute value by an expression of the form \(C(1+x^2)^k e^{-\gamma x^2}\) which is integrable. We may therefore apply the dominated convergence theorem and arrive at the assertion of the proposition. \(\square\)

5.2.2 Critical Regime

Now we turn to the critical regime. We already know from Proposition 34 that \(F\) has a unique global minimum at the origin in the critical regime as well as in the high temperature regime. For the proof of this result, see [20]. We remind the reader that we set

\[
L := J^{-1} = \begin{pmatrix} L_1 & -\tilde{L} \\ -\tilde{L} & L_2 \end{pmatrix}.
\]
Theorem 39. Let $L_\nu - \alpha_\nu > 0$ for both groups and $(L_1 - \alpha_1)(L_2 - \alpha_2) = \bar{L}^2$. Then, for all $K, Q \in \mathbb{N}_0$, the expected value $E(X_{11} \cdots X_{1K}X_{21} \cdots X_{2Q})$ is asymptotically given by the expression

$$
\left[ \frac{12}{\alpha_1(L_2 - \alpha_2)^2 + \alpha_2(L_1 - \alpha_1)^2} \right]^{K+Q} \frac{(L_1 - \alpha_1)\bar{L}^2(L_2 - \alpha_2)\Gamma(K+Q+1)}{\Gamma(\frac{1}{4})} \frac{1}{N^{K+Q}}
$$

if $K + Q$ is even and 0 otherwise.

Remark 40. For the special case $M = 2$, $J = (\bar{J}_1 \bar{J}_2 J_1 J_2)$, $J_1 = J_2 = J$, $\alpha_1 = \alpha_2 = 1/2$, and $J + \bar{J} = 2$, the correlations are asymptotically equal to

$$
12 \frac{K+L}{4} \frac{\Gamma(K+L+1)}{\Gamma(\frac{1}{4})} \frac{1}{N^{K+L}}.
$$

These correlations are identical to those for the model with homogeneous coupling matrix and $\beta = 1$.

5.2.3 Low Temperature Regime

By Theorem 32,

$$
m_{k_1,\ldots,k_M} \approx \frac{\int_{\mathbb{R}^M} \tanh^k(y) e^{-NF(y)} dy}{\int_{\mathbb{R}^M} e^{-NF(y)} dy},
$$

where we used the notation introduced in the proof of Proposition 38. To estimate these integrals for large $N$, we can use a standard version of Laplace’s method, since contrary to the situation in Proposition 38, the tanh function is not 0 at $\bar{m}$, the minimum of $F$ in the low temperature regime. Let $H(\bar{m})$ stand for the Hessian of $F$ at the minimum. We apply Theorem 3 on p. 495 of [29] to the numerator and the denominator, which yields

$$
m_{k_1,\ldots,k_M} \approx \left( \frac{2\pi}{N} \right)^{M/2} m^k (\det H(\bar{m}))^{-1/2} \exp(-NF(\bar{m})) = m^k.
$$

Note that in the numerator the function $g_0$ given in Theorem 3 in [29] is $g_0(y) = \tanh^k y$ for all $y \in \mathbb{R}^M$, whereas in the denominator we chose the constant function $g_0 = 1$.

This concludes the calculation of the correlations. We will now use these correlations to calculate the moments of the normalised sums $S$.

6 Moments of $S$

We divide this section into three subsections according to the regime of the model.

6.1 High Temperature Regime

We first prove the Law of Large Numbers. In this case, the normalised sums are

$$
S := \left( \frac{S_1}{N_1}, \ldots, \frac{S_M}{N_M} \right).
$$

Later on in this proof, we will distinguish between sum vectors of different dimensions, in which case we will use a superindex in $S^M$, indicating that the vector is of dimension $M \in \mathbb{N}$.

We define $M_{K_1,\ldots,K_M}$ to be the moments of order $K = (K_1,\ldots,K_M)$ of this random vector, and set $K := \sum_{i=1}^M K_i$. Our task consists of calculating the large $N$ limit of these moments.
We can express the moments in the following fashion:

\[ M_{K_1,\ldots,K_M} = \frac{1}{N_1^{K_1}} \cdots \frac{1}{N_M^{K_M}} \sum_{\mathbf{i}} \mathbb{E} \left( X_{i_{11}} \cdots X_{i_{MK_M}} \right), \tag{18} \]

where the sum in the last line is over all \( i_{\nu1}, \ldots, i_{\nu K_{\nu}} \in \{1, \ldots, N_{\nu}\} \) for each group \( \nu \). This is where we use the correlations calculated previously. The above sum has \( N^K \) summands in it. We have to find a way to reduce this to a more manageable number. We use Proposition 28 to deal with the sum. The number of multiindices \( \mathbf{i} = (i_{11}, \ldots, i_{MK_M}) \) with a given profile \( \mathbf{r} = (r_1, \ldots, r_M) \) is asymptotically

\[ w_{\mathbf{r}}(\mathbf{\ell}) \approx \prod_{\lambda=1}^{M} \frac{N^{\sum_{l=1}^{L} r_{\lambda l}}}{r_{\lambda 1}! r_{\lambda 2}! \cdots r_{\lambda K_{\lambda}}!} \left( \frac{K_{\lambda}!}{\prod_{l=1}^{r_{\lambda}} 2^l \cdot \cdots \cdot r_{\lambda K_{\lambda}}!} \right). \tag{19} \]

Thus we can state

\[ \sum_{\mathbf{r}} w_{\mathbf{r}}(\mathbf{\ell}) \mathbb{E} \left( X_{\mathbf{\ell}} \right), \tag{20} \]

where the sum is over all possible profile vectors \( \mathbf{\ell} \).

The correlations are dependent on the coupling matrix. Since each \( X_{\lambda i} \) raised to an even power equals 1, we only focus on the indices which occur an odd number of times.

By Theorems 30 and 37, the correlations include powers of \( N \) that depend on the number of indices occurring an odd number of times. More precisely, up to a multiplicative constant, the correlations are asymptotically

\[ N^{-\frac{1}{2} \left( \sum_{m_1=0}^{K_{1}/2} r_{1,2m_1+1} + \cdots + \sum_{m_M=0}^{K_{M}/2} r_{M,2m_M+1} \right)}, \]

Therefore, in the sum above, each summand has the following power of \( N \):

\[ \frac{1}{N_{1}^{K_{1}} \cdots N_{M}^{K_{M}}} N_{1}^{\sum_{l=1}^{L} r_{1 l}} \cdots N_{M}^{\sum_{l=1}^{L} r_{M l}} N^{-\frac{1}{2} \left( \sum_{m_1=0}^{K_{1}/2} r_{1,2m_1+1} + \cdots + \sum_{m_M=0}^{K_{M}/2} r_{M,2m_M+1} \right)}. \]

Hence, the exponent of \( N \) in each summand is given by

\[ -K + \sum_{\lambda=1}^{M} \left( \frac{|K_{\lambda}/2|}{m_{\lambda}} + \frac{1}{2} \sum_{m_{\lambda}=0}^{K_{\lambda}/2} r_{\lambda,2m_{\lambda}+1} \right). \tag{21} \]

Lemma 41. For all \( \mathbf{\ell} \), the inequality \( (21) < 0 \) holds.

Proof. We omit the straightforward proof. \( \square \)

Since there are only finitely many (their number being independent of \( N \)) summands in \( (20) \), each of which converges to 0, the above lemma proves Theorem 9.

Now we turn to the CLT 11. In the proof of the LLN above, none of the summands contribute asymptotically to the moments, which therefore converge to 0. If we normalise the sums \( \mathbf{S} \) by a power of \( N \) equal to \( \frac{1}{2} \), we will see that the summands with profile vectors belonging to \( \Pi_{0}(K_{\lambda}) \) (see Definition 27) for each group \( \lambda \) contribute positive quantities to the moments. We proceed similarly to the above by defining \( M_{K_1,\ldots,K_M} \) to be the moments of order \( K = (K_1, \ldots, K_M) \) of the random vector

\[ \mathbf{S} := \left( \frac{S_1}{\sqrt{N_1}}, \ldots, \frac{S_M}{\sqrt{N_M}} \right), \]

and setting \( K := \sum_{i=1}^{M} K_i \). The moments can be expressed as

\[ M_{K_1,\ldots,K_M} = \frac{1}{N_{1}^{K_{1}}} \cdots \frac{1}{N_{M}^{K_{M}}} \sum_{\mathbf{\ell}} w_{\mathbf{r}}(\mathbf{\ell}) \mathbb{E} \left( X_{\mathbf{\ell}} \right). \tag{22} \]
Asymptotically, the powers of $N$ in (22) are
\[
\frac{1}{N_1^{\lambda_1}} \cdots \frac{1}{N_M^{\lambda_M}} N_{\sum_{i=1}^{K_1} r_{\lambda_1} + \cdots + \sum_{m=1}^{K_M} r_{\lambda_M}} N^{-\frac{1}{2} \sum_{\lambda} |\lambda|/2} N_{\sum_{m=0}^{r_{\lambda_1}} r_{\lambda_1,2m+1} + \cdots + \sum_{m=0}^{r_{\lambda_M}} r_{\lambda_M,2m+1}}.
\]

We separate the sum in (22) into two summands: $A_1$, where the sum runs over $\mathbf{r}$ such that for all $\lambda, r_{\lambda} \in \Pi^0(\lambda)$, and $A_2$ for all other $\mathbf{r}$. We claim

**Proposition 42.** The limit of $A_2$ as $N \to \infty$ is 0.

**Proof.** Let $\mathbf{r}$ be such that $r_{\nu} \in \Pi^+({K_\nu})$ for some group $\nu$.

We have for any group $\lambda$ the factor
\[
\frac{1}{N_1^{\lambda_1}} \cdots \frac{1}{N_M^{\lambda_M}} N_{\sum_{i=1}^{K_1} r_{\lambda_1} + \cdots + \sum_{m=1}^{K_M} r_{\lambda_M}} N^{-\frac{1}{2} \sum_{\lambda} |\lambda|/2} N_{\sum_{m=0}^{r_{\lambda_1}} r_{\lambda_1,2m+1} + \cdots + \sum_{m=0}^{r_{\lambda_M}} r_{\lambda_M,2m+1}},
\]
which has a power of $N$ equal to
\[
-\frac{K_\lambda}{2} + \sum_{i=1}^{K_\lambda} r_{\lambda_i} - \frac{1}{2} \sum_{m=0}^{r_{\lambda_1,2m+1} + \cdots + \sum_{m=0}^{r_{\lambda_M,2m+1}}} = \frac{1}{2} \left( -K_\lambda + \sum_{m=0}^{r_{\lambda_1,2m+1} + \cdots + \sum_{m=0}^{r_{\lambda_M,2m+1}}} \right) \leq 0,
\]
with equality if and only if
\[
\sum_{m=0}^{r_{\lambda_1,2m+1} + \cdots + \sum_{m=0}^{r_{\lambda_M,2m+1}}} = \sum_{i=1}^{K_\lambda} r_{\lambda_i} = K_\lambda.
\]
The first equality above holds if and only if $r_{\lambda,2m+1} = 0$ for all $\ell > 2$.

For the group $\nu$, we have by assumption some $k > 2$ with $r_{\nu,k} > 0$, and hence $r_{\nu,k} < K_{\nu,k}$. That implies
\[
-\frac{K_\nu}{2} + \sum_{\nu=1}^{K_\nu} r_{\nu,\lambda} - \frac{1}{2} \sum_{m=0}^{r_{\nu,2m+1}} < K_\nu.
\]
So the overall power of $N$ in each summand is negative and it converges to 0 as $N \to \infty$. There are only finitely many such summands in $A_2$, and we conclude that $A_2$ converges to 0 as $N \to \infty$.

It follows from this proposition that
\[
M_{K_1,\ldots,K_M} \approx A_1 = \frac{1}{N_1^{\sum_{i=1}^{K_1} r_{\lambda_1}}} \cdots \frac{1}{N_M^{\sum_{m=1}^{K_M} r_{\lambda_M}}} \sum_{\mathbf{r}: \mathbf{r} \in \Pi^0(\mathbf{K})} w_{\mathbf{K}}(\mathbf{r}) \xi(\mathbf{X}_\mathbf{r})
\]
\[
= \frac{1}{N_1^{\lambda_1}} \cdots \frac{1}{N_M^{\lambda_M}} \sum_{\mathbf{r}: \mathbf{r} \in \Pi^0(\mathbf{K})} \prod_{\lambda=1}^{M} N_{\sum_{m=1}^{r_{\lambda_1}} r_{\lambda_1} + \cdots + \sum_{m=0}^{r_{\lambda_M}} r_{\lambda_M}} \frac{K_\lambda!}{2^{r_{\lambda_1}} r_{\lambda_1}!} \xi(\mathbf{X}_\mathbf{r})
\]
\[
= \frac{1}{N_1^{\lambda_1}} \cdots \frac{1}{N_M^{\lambda_M}} \sum_{k_1=0}^{K_1} \cdots \sum_{k_M=0}^{K_M} \prod_{\lambda=1}^{M} N_{\sum_{m=0}^{r_{\lambda_1}} r_{\lambda_1,2m+1} + \cdots + \sum_{m=0}^{r_{\lambda_M}} r_{\lambda_M,2m+1}} \frac{K_\lambda!}{2^{r_{\lambda_1}} r_{\lambda_1}!} \xi(\mathbf{X}_\mathbf{r}). \quad (23)
\]

In the last line, each sum is over those $k_\lambda$ which have the same parity as $K_\lambda$ and $\mathbf{r}$ is the profile vector where $r_{\lambda} = (k_\lambda, (K_\lambda - k_\lambda)/2)$ for all groups $\lambda$.

Of course, the expectation $\xi(\mathbf{X}_\mathbf{r})$ depends on the class of coupling matrix. As a reminder, if $J$ is homogeneous, then
\[
\xi(\mathbf{X}_\mathbf{r}) \approx (k_1 + \cdots + k_M - 1)! \left( \frac{\beta}{1 - \beta} \right)^{k_1 + \cdots + k_M} \frac{1}{N^{\sum_{i=1}^{K_1} r_{\lambda_1}}}.
\]

(24)
If, on the other hand, $J$ is heterogeneous, then
\[ \mathbb{E} (X_J) \approx m_{k_1, \ldots, k_M} \left( \frac{1}{N^{\frac{\ell_1 + \cdots + \ell_n}{2}}} \right), \quad (25) \]

Recall that in the last line above $m_{k_1, \ldots, k_M} \left( \frac{1}{N^{\frac{\ell_1 + \cdots + \ell_n}{2}}} \right)$ stands for the moment of centred multivariate normal distributions with covariance matrix $H^{-1}$.

We now show that the correlation in (24) can be expressed in a similar fashion as in (25).

**Definition 43.** Let for all $n \in \mathbb{N}$ and all $\ell_1, \ldots, \ell_n \in \mathbb{N}_0$ the notation $m_{\ell_1, \ldots, \ell_n}^n(C)$ stand for an $n$-dimensional multivariate moment of an $n$-dimensional multivariate normal distribution with mean 0 and covariance matrix $C$. The superindex $n$ indicates the dimension of the distribution.

We state Isserlis’s Theorem which we will use in the proof of the next proposition.

**Theorem 44 (Isserlis’s Theorem).** Let $\ell_1, \ldots, \ell_n \in \mathbb{N}_0$. Assume $(Z_1, \ldots, Z_n)$ follows $\mathcal{N}(0, C)$, $C = (c_{ij})_{i,j=1,\ldots,n}$. Define a set $\mathbf{Z} = \bigcup_{i=1}^{M+1} \{ Z_1, \ldots, Z_{\ell_i} \}$ with $\ell_i$ copies of $Z_i$ for each $i = 1, \ldots, n$. If the sum $\ell_1 + \cdots + \ell_n$ is odd, the moment $m_{\ell_1, \ldots, \ell_n}^n(C)$ is 0 due to the symmetry of the normal distribution. Otherwise we let $\mathcal{P}$ be the set of all possible pair partitions of the set $\mathbf{Z}$. Let for any $\pi \in \mathcal{P}$ $\pi(k) = (i, j)$ if the $k$-th pair in $\pi$ has one copy of $Z_i$ and one copy of $Z_j$. Then
\[ m_{\ell_1, \ldots, \ell_n}^n(C) = \sum_{\pi \in \mathcal{P}} \prod_{k=1}^{\ell_1 + \cdots + \ell_n} c_{\pi(k)}. \]

**Proof.** For a proof see the original publication [13]. \( \square \)

**Proposition 45.** Let $\ell_1, \ldots, \ell_n \in \mathbb{N}_0$. Let $C = (c)_{\lambda,\mu=1,\ldots,n}$ be a homogeneous matrix (recall Definition 2). Then
\[ m_{\ell_1, \ldots, \ell_n}^n(C) = m_{1^{\ell_1} \cdots + 1^{\ell_n}}^1(c). \]

**Proof.** By Isserlis’s Theorem,
\[ m_{\ell_1, \ldots, \ell_n}^n(C) = \sum_{\pi \in \mathcal{P}} \prod_{k=1}^{\ell_1 + \cdots + \ell_n} c_{\pi(k)} = \sum_{\pi \in \mathcal{P}} c_{\pi(k)}^{\ell_1 + \cdots + \ell_n} = (\ell_1 + \cdots + \ell_n - 1)!! c_{\ell_1 + \cdots + \ell_n}^{\ell_1 + \cdots + \ell_n} = m_{1^{\ell_1} \cdots + 1^{\ell_n}}^1(c). \]

**Corollary 46.** The homogeneous coupling correlation (24) is equal to
\[ \mathbb{E} (X_J) \approx m_{k_1, \ldots, k_M}^M \left( (\bar{\beta})_{\lambda,\mu=1,\ldots,M} \right) \frac{1}{N^{\frac{\ell_1 + \cdots + \ell_n}{2}}}, \]
where $\bar{\beta} = \frac{\beta}{1 - \beta}$.

We can therefore treat the two classes of coupling matrix $J$ simultaneously by setting
\[ \Sigma := (\sigma_{\lambda,\mu})_{\lambda,\mu=1,\ldots,M} := \begin{cases} \bar{\beta}, & \text{if } J \text{ is homogeneous,} \\ (H^{-1})_{\lambda,\mu}, & \text{if } J \text{ is heterogeneous.} \end{cases} \quad (26) \]

Next, we present a recursive formulation of Isserlis’s Theorem.

**Proposition 47.** Let $M \in \mathbb{N}, l_1, \ldots, l_{M+1} \in \mathbb{N}_0$, and $m_{l_1, \ldots, l_{M+1}}^{M+1}$ be the moment of order $(l_1, \ldots, l_{M+1})$ of an $(M+1)$-dimensional multivariate normal distribution with covariance matrix
\[ m_{1, \ldots, 1, l_{M+1}}^{M+1} = (\sigma_{i,j})_{i,j=1,\ldots,M+1}. \]
Then the moment \( m^{M+1}_{l_1, \ldots, l_{M+1}} \) can be calculated in terms of the covariances and the \( M \)-dimensional moments \( m^M_{i_1, \ldots, i_M} \) of the \( M \)-dimensional multivariate normal distribution of the first \( M \) entries with covariance matrix

\[
\Sigma^M = (\sigma_{i,j})_{i,j=1,\ldots,M}
\]
as follows:

\[
\begin{align*}
m^{M+1}_{l_1, \ldots, l_{M+1}} &= \sum_{(r_1, \ldots, r_M) \in A} \frac{l_1! \cdots l_{M+1}!}{2^{\frac{M(M-1)}{2}} r_1! \cdots r_M!(l_1 - r_1)!(\cdots(l_M - r_M)!)} \\
&\quad \cdot \sigma^r_{1,M+1} \cdots \sigma^r_{M,M+1} \Sigma_{M+1,M+1} \cdot m^{M}_{l_1-1, \ldots, l_{M+1}-1, M - r_M},
\end{align*}
\]

where \( A = \{ (r_1, \ldots, r_M) \mid \forall i = 1, \ldots, M : 0 \leq r_i \leq l_i \wedge l_{M+1} \) and \( l_i - r_i, l_{M+1} - \sum_{i=1}^M r_i \) are even\}.}

Proof. Let \( Z_1, \ldots, Z_{M+1} \) follow \( \mathcal{N}(0, \Sigma^{M+1}) \). By Isserlis’s Theorem, the higher order moments of an \( (M+1) \)-dimensional multivariate normal distribution can be calculated as the sum over all possible pair partitions of the component variables. To calculate the moment

\[
m^{M+1}_{l_1, \ldots, l_{M+1}} := m^{M+1}_{l_1, \ldots, l_{M+1}} (\mathcal{N}(0, \Sigma^{M+1}))
\]
of order \((l_1, \ldots, l_{M+1})\), we define a set \( Z = \bigcup_{i=1}^{M+1} \{ Z_1, \ldots, Z_{l_i} \} \) with \( l_i \) copies of \( Z_i \) for each \( i = 1, \ldots, M+1 \). If the sum \( l_1 + \cdots + l_{M+1} \) is odd, the moment \( m^{M+1}_{l_1, \ldots, l_{M+1}} \) is 0 due to the symmetry of the normal distribution. Otherwise we let \( \mathcal{P} \) be the set of all possible pair partitions of the set \( Z \). Let for any \( \pi \in \mathcal{P} \) \( \pi(k) = (i, j) \) if the \( k \)-th pair in \( \pi \) has one copy \( Z_i \) and one copy \( Z_j \). Then, by Isserlis,

\[
m^{M+1}_{l_1, \ldots, l_{M+1}} = \sum_{\pi \in \mathcal{P}} \prod_{k=1}^{l_1 + \cdots + l_{M+1}} \sigma(\pi(k))
\]

holds. We express the \((M+1)\)-dimensional moment \( m^{M+1}_{l_1, \ldots, l_{M+1}} \) in terms of the \( M \)-dimensional moments \( m^M_{l_1, \ldots, l_{M+1}} \). Each of the \( l_{M+1} \) copies of \( Z_{M+1} \) can be paired up with a copy of \( Z_1, \ldots, Z_M \) or another copy of \( Z_{M+1} \). Let the number of copies of \( Z_{M+1} \) paired up with a \( Z_i \) for \( i = 1, \ldots, M \) be \( r_i \). Then there are \( r := \sum_{i=1}^M r_i \) remaining copies of \( Z_{M+1} \) that must be paired, hence \( l_{M+1} - r \) must be even. Applying Isserlis to the \( M \)-dimensional moment, a single pair partition \( \pi \) contributes

\[
\sigma^r_{1,M+1} \cdots \sigma^r_{M,M+1} \Sigma_{M+1,M+1} m^M_{l_1-1, \ldots, l_{M+1}-1, M - r_M}
\]
to (28). How many of these pair partitions \( \pi \) for a given profile \((r_1, \ldots, r_M)\) are there?

For each \( i = 1, \ldots, M \), we have to pick \( r_i \) from the \( l_{M+1} \) copies of \( Z_{M+1} \) to be paired with a \( Z_i \) and \( l_{M+1} - r \) to be paired with another \( Z_{M+1} \):

\[
\binom{l_{M+1}}{l_i, r_i, l_{M+1} - r}
\]

We have to select for each \( i = 1, \ldots, M \) the corresponding \( r_i \) copies of \( Z_i \):

\[
\binom{l_{M+1}}{l_i}
\]

For each \( i = 1, \ldots, M \), the selected \( r_i Z_i \) and \( r_i Z_{M+1} \) can be paired in \( r_i \) ways.

The remaining \((l_{M+1} - r) Z_{M+1}\) can be ordered into \((l_{M+1} - r - 1)!! = \frac{(l_{M+1} - r)!}{2^{\frac{l_{M+1} - r}{2}} \cdot \frac{l_{M+1} - r}{2}}\) pairs. Multiplying all of these terms together and simplifying, we obtain

\[
\frac{l_1! \cdots l_{M+1}!}{\left(\frac{l_{M+1} - \sum_{i=1}^M r_i}{2}\right)! \left(\frac{l_{M+1} - \sum_{i=1}^M r_i}{2}\right)! r_1! \cdots r_M! (l_1 - r_1)! \cdots (l_M - r_M)!}
\]

Then we sum over all profiles \((r_1, \ldots, r_M)\) to obtain the desired result. \(\square\)
Given the covariance matrix $\Sigma$ defined in (26), we set

$$C := I + \sqrt{\alpha} \Sigma \sqrt{\alpha}.$$  

Before proceeding with the proof of the CLT, we state an auxiliary lemma.

**Lemma 48.** Let $(L_1, \ldots, L_{M+1})$ be a fixed $(M+1)$-tuple of non-negative integers. Let $\{(r, r_1, \ldots, r_M, l_1, \ldots, l_{M+1})\}$ be a set of tuples of non-negative integers such that $r = \sum_{i=1}^{M} r_i$. We define the following conditions:

\begin{align*}
0 & \leq l_i \leq L_i \quad (i = 1, \ldots, M + 1) \\
0 & \leq r \leq l_{M+1} \\
0 & \leq r_i \leq l_i \wedge l_{M+1} \quad (i = 1, \ldots, M) \\
0 & \leq r_i \leq L_i \quad (i = 1, \ldots, M) \\
0 & \leq r \leq L_{M+1} \\
r_i & \leq l_i \leq L_i \quad (i = 1, \ldots, M) \\
r & \leq l_{M+1} \leq L_{M+1}.
\end{align*}

Let

\begin{align*}
A & := \{(r, r_1, \ldots, r_M, l_1, \ldots, l_{M+1}) \mid (29), (30), (31) \text{ hold}\}, \\
B & := \{(r, r_1, \ldots, r_M, l_1, \ldots, l_{M+1}) \mid (32), (33), (34), (35) \text{ hold}\}.
\end{align*}

Then we have $A = B$.

**Proof.** We omit the proof which is a straightforward verification. \qed

Now we are prepared to prove the CLT.

**Proof of Theorem 11.** The proof proceeds by induction on $M$. For $M = 1$, we show that $\frac{\overline{S}}{\sqrt{N_1}}$ is asymptotically univariate normally distributed with variance $1 + \alpha_1 \sigma_{11} = C$, where $\sigma_{11}$ is the sole entry of the covariance matrix $\Sigma$ defined in equation (26), by the method of moments.

Let $K \in \mathbb{N}_0$. Then the moment $M_K^1$ is $0$ if $K$ is odd. Otherwise it is asymptotically equal to

$$m_{2k}^1 = \frac{K!}{(2k)!} \sum_{k=0}^{K/2} \frac{a_1^k}{(k - k)!} 2^{k/2 - k} k!^2 \sigma_{11}^k \sigma_{11}^k = \frac{K!}{(2k)!^2} \sum_{k=0}^{K/2} \frac{(K/2)!(a_1 \sigma_{11})^k}{k!(k - k)!} = (K - 1)!!(1 + a_1 \sigma_{11})^{K/2}.$$

Therefore, the claim holds for $M = 1$. Let for the rest of this proof

$$m_{M+1}^{L_1, \ldots, L_{M+1}} := m_{L_1, \ldots, L_{M+1}}^1(\Sigma).$$

Assume the claim holds for some $M \in \mathbb{N}$ and we show that it also holds for $M + 1$ by proving that for all $L_1, \ldots, L_{M+1} \in \mathbb{N}_0$ the moments of order $(L_1, \ldots, L_{M+1})$ satisfy the recursive relation

$$M_{L_1, \ldots, L_{M+1}}^{M+1} = \sum_{(r_1, \ldots, r_M) \in A} \frac{L_1! \cdots L_{M+1}!}{(L_{M+1} - \frac{\sum_{i=1}^{M} r_i}{2})! (r_1! \cdots r_{M-1}! \cdots r_{M+1}! (L_1 - r_1)! \cdots (L_M - r_M))} \cdot (\sqrt{\alpha_1} \sigma_{11}^{r_1, M+1})^{r_1} \cdots (\sqrt{\alpha} \sigma_{M+1}^{r_M, M+1})^{r_M} \cdot (1 + \alpha_M \sigma_{M+1, M+1})^{L_{M+1} - \frac{\sum_{i=1}^{M} r_i}{2}} M_{L_1 - r_1, \ldots, L_M - r_M}^{M+1}. \quad (36)$$
where $A$ is given in Lemma [48]. If the sum $L_1 + \cdots + L_{M+1}$ is odd, the moment $M_{L_1,\ldots,L_{M+1}}^{M+1}$ is 0. If the sum is even, $M_{L_1,\ldots,L_{M+1}}^{M+1}$ is asymptotically equal to

$$
\sum_{l_1=0}^{L_{M+1}} \cdots \sum_{l_{M+1}=0}^{L_{M+1}} \frac{L_1! \cdot \alpha_{1,1}^{L_1}}{l_1!(\frac{l_1}{2})! \cdot \frac{L_1+1}{2}!} \cdots \frac{L_{M+1}! \cdot \alpha_{M+1,1}^{L_{M+1}}}{l_{M+1}!(\frac{l_{M+1}}{2})! \cdot \frac{L_{M+1}+1}{2}!} m_{L_1,\ldots,L_{M+1}}^{M+1},
$$

where each sum goes over $l_i$ with values with the same parity as $L_i$. We next express $m_{L_1,\ldots,L_{M+1}}^{M+1}$, which is a moment of an $(M+1)$-dimensional normal distribution recursively according to (27). Then we switch summation indices from $A$ to $B$ as defined by Lemma [48].

$$
= \sum_{r=0}^{L_{M+1}} \sum_{(r_1,\ldots,r_M)} \frac{L_1! \cdot \alpha_{1,1}^{L_1}}{(r_1-1)! \cdot \cdots \cdot (r_{M+1}-1)!} \sigma_{r_1,\ldots,r_M}^{L_{M+1}+1} \sigma_{r_1,\ldots,r_M}^{M+1} \frac{L_{M+1}! \cdot \alpha_{M+1,1}^{L_{M+1}}}{(L_{M+1}-r_1)! \cdot \cdots \cdot (L_{M+1}-r_M)!} \\
\cdot \left(\frac{(L_{M+1}-r)}{2}\right)^{L_{M+1}-1} r_1! \cdot \cdots \cdot r_M! \cdot (l_1-1)! \cdot \cdots \cdot (l_M-r)! \\
\cdot \left(\frac{(L_{M+1}-r)}{2}\right)^{L_{M+1}-1} r_1! \cdot \cdots \cdot r_M! \cdot (l_1-1)! \cdot \cdots \cdot (l_M-r)!.
$$

Now we switch indices once more: define for all $i = 1,\ldots,M$ $s_i := l_i - r_i$ and $s := l_{M+1} - r$. After some lengthy calculations, we see that the sum is

$$
= \sum_{r=0}^{L_{M+1}} \sum_{(r_1,\ldots,r_M)} \frac{L_1! \cdot \alpha_{1,1}^{L_1}}{(r_1-1)! \cdot \cdots \cdot (r_{M+1}-1)!} \sigma_{r_1,\ldots,r_M}^{L_{M+1}+1} \sigma_{r_1,\ldots,r_M}^{M+1} \frac{L_{M+1}! \cdot \alpha_{M+1,1}^{L_{M+1}}}{(L_{M+1}-r_1)! \cdot \cdots \cdot (L_{M+1}-r_M)!} \\
\cdot \left(\sqrt{\alpha_{r_1,\ldots,r_M}^{L_{M+1}+1} \sigma_{r_1,\ldots,r_M}^{M+1}}\right)^{s_1} \cdot \left(\sqrt{\alpha_{r_1,\ldots,r_M}^{L_{M+1}+1} \sigma_{r_1,\ldots,r_M}^{M+1}}\right)^{s_2} \cdot \left(1 + \alpha_{M+1}^{L_{M+1}+1} \sigma_{L_{M+1}+1}^{M+1}\right)^{L_{M+1}-r_1} \cdots \cdot \left(1 + \alpha_{M+1}^{L_{M+1}+1} \sigma_{L_{M+1}+1}^{M+1}\right)^{L_{M+1}-r_M}.
$$

As we can see, this last expression is equal to (36), and therefore the recursive relation holds for the moments $M^{M+1}$ in terms of the moments $M^M$ holds. This implies that if for some $M \in \mathbb{N}$ $S^M$ is asymptotically normally distributed, then so is $S^{M+1}$. This concludes the proof by induction.

We have thus shown all results for the high temperature regime.

### 6.2 Critical Regime

In the critical regime, we proceed similarly to the above analysis of the high temperature regime. We first prove the Law of Large Numbers [13] and then the fluctuations results Theorems [10] and [16].

We use the previously calculated correlations presented in Theorems [30] and [39]. For the Law of Large Numbers, the moments can be expressed analogously to Section 6.1 and the proof that these moments converge to 0 as $N \rightarrow \infty$ is identical to the proof found there.

For the fluctuations results, we normalise the sums $S$ by a power of $N$ equal to $\frac{4}{3}$. Then, we will see that the summands with profile vectors belonging to $\Pi^{(K)}$ for each group $\lambda$ contribute positive quantities to the moments. We proceed similarly to the above by defining $M_{K_1,\ldots,K_M}$ to be the moments of order $K = (K_1,\ldots,K_M)$ of the random vector

$$
S := \left(\frac{S_1}{N_1^{\frac{4}{3}}},\ldots,\frac{S_M}{N_M^{\frac{4}{3}}}\right),
$$

and setting $K := \sum_{i=1}^{M} K_i$. The moments are

$$
M_{K_1,\ldots,K_{M-1}} = \sum_{i=1}^{M} \frac{1}{N_i^{\frac{4}{3}}} \cdots \frac{1}{N_{M-1}^{\frac{4}{3}}} \sum_{\xi} u_\lambda(\xi) \mathbb{E} \left( X_\xi \right).
$$

We separate the sum in (37) into two summands: $A_1$, where $\xi$ satisfies for all $\lambda r_\lambda \in \Pi^{(K)}$, and $A_2$ all other $\xi$. This means $A_1$ contains all those profile vectors that describe multiindices with no repeating indices at all, and $A_2$ all other profile vectors.

We claim
Proposition 49. The limit of $A_2$ as $N \to \infty$ is 0.

Proof. We omit the proof, which proceeds along the same lines as before.

It follows from this proposition that
\[ M_{K_1, \ldots, K_M} \approx A_1 = \frac{1}{N_1^{K_1}} \cdots \frac{1}{N_M^{K_M}} \exp \left( \sum_{\lambda} \lambda_1 \gamma_{11} + \lambda_2 \gamma_{22} \right), \tag{38} \]
where the profile vector $\mathbf{r}$ in the last expression is the one that has $r_{\lambda 1} = K_\lambda$ for all groups $\lambda$. The correlations $\text{E} \left( X_1^2 \right)$ have the same powers of $N$ but different constants that depend on $K_\lambda$ in the homogeneous and the heterogeneous case. We set the constants $c_{K_1, \ldots, K_M}$ equal to
\[
\left\{ \begin{array}{ll}
\frac{\gamma_{11}}{r_{\lambda 1}^{K_1}} & \text{if } J \text{ is homogeneous}, \\
(\gamma_{11} + \gamma_{22})^{\frac{1}{2}} & \text{if } M = 2 \text{ and } J \text{ is heterogeneous}.
\end{array} \right. \tag{39}
\]
With this definition,
\[ A_1 \approx c_{K_1, \ldots, K_M} \frac{1}{N_1^{K_1}} \cdots \frac{1}{N_M^{K_M}}, \]
which proves Theorems 15 and 16.

6.2.1 Proof of Theorem 19

We prove Theorem 19. In a departure from the method of moments, we use the de Finetti representation to show the convergence of the sequence of characteristic functions of the transformations given in Theorem 19.

We prove the statement
\[ T_N := \frac{\sqrt{L_1 - \alpha_1}}{\sqrt{\alpha_1 N_1}} S_1 - \frac{\sqrt{L_2 - \alpha_2}}{\sqrt{\alpha_2 N_2}} S_2 \Rightarrow \mathcal{N} \left( 0, 1 \right) \]
for heterogeneous coupling. The other statements can be shown analogously.

We set $\gamma_\lambda := \sqrt{L_\lambda - \alpha_\lambda}$ for both groups. The characteristic function of $T_N$ is defined as
\[ \varphi_N(t) := \text{E} \left( \exp \left( \frac{\gamma_1}{\sqrt{\alpha_1 N_1}} S_1 - \frac{\gamma_2}{\sqrt{\alpha_2 N_2}} S_2 \right) \right). \]

We have to prove that $\varphi_N$ converges pointwise to the characteristic function $\varphi(t) = \exp \left( - \left( 1 + \frac{\alpha_1}{\alpha_2} \right) \frac{t^2}{2} \right)$ of a centred normal distribution with variance equal to $1 + \frac{\alpha_1^2}{\alpha_2} + \frac{\alpha_2^2}{\alpha_2}$.

By Theorem 32, the characteristic function $\varphi_N$ can be expressed as
\[ \varphi_N(t) = Z^{-1} \int_{\mathbb{R}^{2\gamma}} E_y \left( \exp \left( \frac{\gamma_1}{\sqrt{\alpha_1 N_1}} S_1 - \frac{\gamma_2}{\sqrt{\alpha_2 N_2}} S_2 \right) \right) \cdot e^{-N \left( \frac{1}{2} y^T L y - \sum_{\lambda} \frac{\gamma_\lambda}{N_\lambda} \ln \cosh y_\lambda \right)} \, dy. \]

The normalisation constant $Z^{-1}$ is equal to $\int_{\mathbb{R}^{2\gamma}} e^{-N \left( \frac{1}{2} y^T L y - \sum_{\lambda} \frac{\gamma_\lambda}{N_\lambda} \ln \cosh y_\lambda \right)} \, dy$. The transformations we will be doing to the integral above are equally applied to $Z^{-1}$. Hence, we will be omitting multiplicative constants that stem from variable switches in the integral.

The term $e^{-N \left( \frac{1}{2} y^T L y - \sum_{\lambda} \frac{\gamma_\lambda}{N_\lambda} \ln \cosh y_\lambda \right)}$ can be approximated by a Taylor series. For simplicity's sake, we call the coordinates $\left( x, y \right)$ instead of $\left( y_1, y_2 \right)$. We calculate the derivatives of $F$ up to order 2 in order to approximate $F$ around $(0, 0)$:
\[ F(x, y) = \frac{1}{2} \left( (\gamma_1 x - \gamma_2 y)^2 + \frac{\alpha_1}{6} x^4 + \frac{\alpha_2}{6} y^4 \right). \]
Recall that conditionally on \((x, y) \in \mathbb{R}^2\), the spin variables are all independent, within each group also identically distributed. Therefore,

\[
E_{(x,y)} \left( \exp \left( it \left( \frac{\gamma_1}{\sqrt{\alpha_1 N_1}} S_1 - \frac{\gamma_2}{\sqrt{\alpha_2 N_2}} S_2 \right) \right) \right) = \frac{E_x \left( \exp \left( it \frac{\gamma_1}{\sqrt{\alpha_1 N_1}} S_1 \right) \right)}{E_y \left( \exp \left( it \frac{\gamma_2}{\sqrt{\alpha_2 N_2}} S_2 \right) \right)}, \tag{40}
\]

and

\[
E_x \left( \exp \left( it \frac{\gamma_1}{\sqrt{\alpha_1 N_1}} S_1 \right) \right) = \exp \left( it \frac{\gamma_1}{\sqrt{\alpha_1 N_1}} X_{11} \right) E_y \left( \exp \left( it \frac{\gamma_1}{\sqrt{\alpha_1 N_1}} (X_{11} - x) \right) \right) \approx \exp \left( it \gamma_1 x \sqrt{N} \right) \left( 1 - (1 - x^2) \frac{\gamma_1^2 t^2}{2 \alpha_1 N_1} + O \left( \frac{1}{N_1^{3/2}} \right) \right) \left( 1 - (1 - y^2) \frac{\gamma_1^2 t^2}{2 \alpha_1 N_1} + O \left( \frac{1}{N_1^{3/2}} \right) \right),
\]

where we used a Taylor series for the exponential function. This last term is asymptotically equal to

\[
\exp \left( it \gamma_1 x \sqrt{N} \right) \cdot \exp \left( - (1 - x^2) \frac{\gamma_1^2 t^2}{2 \alpha_1} \right).
\]

We calculate an asymptotic expression similarly for group two and obtain for the conditional expectation (40)

\[
\exp \left( it (\gamma_1 x - \gamma_2 y) \sqrt{N} \right) \cdot \exp \left( - \left( (1 - x^2) \frac{\gamma_1^2}{\alpha_1} + (1 - y^2) \frac{\gamma_1^2}{\alpha_1} \right) \frac{t^2}{2} \right).
\]

When we apply Laplace’s method to approximate the value of the integral, the factor

\[
\exp \left( - \left( (1 - x^2) \frac{\gamma_1^2}{\alpha_1} + (1 - y^2) \frac{\gamma_1^2}{\alpha_1} \right) \frac{t^2}{2} \right)
\]

contributes only with its value at \((x, y) = (0, 0)\). Proceeding from the de Finetti representation of the characteristic function \(\varphi_N\) given above, we substitute

\[
u := (\gamma_1 x - \gamma_2 y) \sqrt{N} \quad \text{and} \quad v := (\gamma_1 x + \gamma_2 y) N^{1/4},
\]

and the integral becomes (up to a multiplicative constant)

\[
\int_{\mathbb{R}^2} \exp (i \nu u) \exp \left( - \left( \frac{\gamma_1^2}{\alpha_1} + \frac{\gamma_2^2}{\alpha_2} \right) \frac{t^2}{2} \right) e^{-\frac{1}{4} \left[ u^2 + \frac{\alpha_1}{\gamma_1^2} (\frac{\gamma_1}{\gamma_1})^4 + \frac{\alpha_2}{\gamma_2^2} (\frac{\gamma_2}{\gamma_2})^4 \right]} du dv \\
\approx \exp \left( - \left( \frac{\gamma_1^2}{\alpha_1} + \frac{\gamma_2^2}{\alpha_2} \right) \frac{t^2}{2} \right) \int_{\mathbb{R}^2} \exp (i \nu u) e^{-\frac{1}{4} \left[ u^2 + \frac{\alpha_1}{\gamma_1^2} (\frac{\gamma_1}{\gamma_1})^4 + \frac{\alpha_2}{\gamma_2^2} (\frac{\gamma_2}{\gamma_2})^4 \right]} du dv
\]

by dominated convergence. Now note that the integral above can be separated into two factors. The factor

\[
\int_{\mathbb{R}} e^{-\frac{1}{2 \alpha_1} \left( \frac{\alpha_1}{\gamma_1^2} + \frac{\alpha_2}{\gamma_2^2} \right) v} dv \text{ cancels out with the corresponding term in the normalisation constant } Z^{-1}.
\]

The only term left in \(Z^{-1}\) is \(\int_{\mathbb{R}^2} e^{-\frac{\gamma_1^2}{2 \alpha_1} u^2} du\). Thus we get

\[
\varphi_N \left( t \right) \approx \exp \left( - \left( \frac{\gamma_1^2}{\alpha_1} + \frac{\gamma_2^2}{\alpha_2} \right) \frac{t^2}{2} \right) \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \exp (i \nu u) e^{-\frac{1}{2} u^2} du \\
= \exp \left( - \left( 1 + \frac{\gamma_1^2}{\alpha_1} + \frac{\gamma_2^2}{\alpha_2} \right) \frac{t^2}{2} \right).
\]

This concludes the proofs of the critical regime results.
6.3 Low Temperature Regime

The limit Theorem 22 for the magnetisations in the low temperature regime easily follows from the correlations. The normalised sums are

\[ S := \left( \frac{S_1}{N_1}, \ldots, \frac{S_M}{N_M} \right), \]

and we define \( M_{K_1, \ldots, K_M} \) to be the moments of order \( \mathbf{K} = (K_1, \ldots, K_M) \) of this random vector, and set \( K := \sum_{i=1}^{M} K_i \). These moments equal

\[ M_{K_1, \ldots, K_M} = \mathbb{E} \left( \left( \frac{S_1}{N_1} \right)^{K_1} \cdots \left( \frac{S_M}{N_M} \right)^{K_M} \right) = \frac{1}{N_1} \cdots \frac{1}{N_M} \sum_{\mathbf{r}} w_{\mathbf{K}}(\mathbf{r}) \mathbb{E} \left( X_{\mathbf{r}} \right). \] (41)

The correlations \( \mathbb{E} \left( X_{\mathbf{r}} \right) \) are constant with respect to \( N \) for such \( \mathbf{r} \) where each group \( \lambda \) has a profile vector \( r_{\lambda} \) with \( r_{\lambda 1} = K_\lambda \). For all other types of profiles \( \mathbf{r} \), the summands converge to 0.

We therefore separate the sum in (41) into two summands:

\[ A_1 = \frac{1}{N_1^{K_1} \cdots N_M^{K_M}} \sum_{\mathbf{r} : \mathbf{r}_\lambda \in \Pi_{K_\lambda}} w_{\mathbf{K}}(\mathbf{r}) \mathbb{E} \left( X_{\mathbf{r}} \right), \]

\[ A_2 = \frac{1}{N_1^{K_1} \cdots N_M^{K_M}} \sum_{\mathbf{r} : \mathbf{r}_\lambda \notin \Pi_{K_\lambda}} w_{\mathbf{K}}(\mathbf{r}) \mathbb{E} \left( X_{\mathbf{r}} \right). \]

We claim

**Proposition 50.** The limit of \( A_2 \) as \( N \to \infty \) is 0.

**Proof.** The proof is very similar to that of Proposition 42. We choose to omit it.

It follows from this proposition that

\[ M_{K_1, \ldots, K_M} \approx A_1 \approx m^K, \]

where \( m \) stands for \((m(\beta), \ldots, m(\beta))\) if \( J \) is homogeneous and \( m := \tanh \bar{m} \), the componentwise tanh of \( \bar{m} \), if \( J \) is heterogeneous and \( \bar{m} \) the minimum of \( F \) in the positive orthant. This concludes the proof of Theorems 21 and 22.

6.3.1 Proof of Theorem 23

Now we turn to the proof of the conditional CLT presented in Theorem 23. We prove the statement for heterogeneous coupling matrices conditional on \( S_\nu > 0 \) for each group \( \nu \). The other statements can be shown analogously. Let \( \bar{m} \) and \( m \) be defined as in Theorem 23. We define the function \( \bar{F} \) such that its value at the global minimum is 0:

\[ \bar{F}(y) := F(y) - F(\bar{m}). \]

We define the sequence of normalised random vectors

\[ \Sigma := \left( \frac{1}{\sqrt{N_1}} \sum_{i_1=1}^{N_1} (X_{1i_1} - m_1), \ldots, \frac{1}{\sqrt{N_M}} \sum_{i_M=1}^{N_M} (X_{Mi_M} - m_M) \right), \]
as well as the sequence of random variables
\[ \chi^+ := \begin{cases} 1, & \text{if for all groups } \nu \ S_\nu > 0, \\ 0, & \text{otherwise}. \end{cases} \]

We have to calculate the moments of all orders \( \mathbf{K} = (K_1, \ldots, K_M) \in \mathbb{N}_0^M \) of the random vectors \( (\Sigma \chi^+)^{\mathbf{K}} \). Note that \( (\Sigma \chi^+)^{\mathbf{K}} = \Sigma^{\mathbf{K}} \chi^+ \) holds for all \( \mathbf{K} \neq \mathbf{0} \). We define the sequence of integrals
\[ Z_{\mathbf{K}} := \int_{\mathbb{R}^M} e^{-N \bar{F}(y)} E_y \otimes N (\Sigma^{\mathbf{K}} \chi^+) \, dy. \]
The moments can thus be calculated by the formula
\[ \mathbb{E} \left( (\Sigma \chi^+)^{\mathbf{K}} \right) = \frac{Z_{\mathbf{K}}}{Z_0}. \]

We state a result concerning the speed of convergence of sums of independent random variables which we will use in this proof.

**Lemma 51.** Let \((X_n)\) be a sequence of independent random variables on a probability space \((\Omega, \mathcal{F}, P)\). Assume that for each \(m \in \mathbb{N}_0\) there is a constant \(T_m\) such that
\[ \sup_{n \in \mathbb{N}} \mathbb{E} (X_n^m) \leq T_m < \infty. \]
Then, for each \(V \in \mathbb{N}\), there is a constant \(C_V\) such that for any \(a > 0\)
\[ P \left( \left| \frac{1}{N} \sum_{n=1}^N (X_n - E(X_n)) \right| > a \right) \leq \frac{C_V}{a^{2V} N^V}. \]

**Proof.** This is Theorem 3.24 in [15]. \(\square\)

We first show

**Lemma 52.** \(Z_{\mathbf{K}}^-\) converges to 0 faster than any power of \(N\).

**Proof.** Let \(Q \in \mathbb{N}\). We prove that \(E_y \otimes N (\Sigma^{\mathbf{K}} \chi^+)\) goes to 0 faster than \(1/N^Q\).
Given a \(y \in A_-\), the Cauchy-Schwarz inequality states that
\[ E_y \otimes N (\Sigma^{\mathbf{K}} \chi^+) \leq E_y \otimes N (\Sigma^{2\mathbf{K}} \chi^+) \leq P_y \otimes N (\chi^+ = 1)^{\frac{1}{2}}. \]
We calculate upper bounds for both expressions on the right hand side of the above inequality. Due to the conditional independence of the random variables $X_{\lambda i}$, we have

$$E_y^{\otimes N} (\Sigma^{2K}) = E_y^{\otimes N} \left( \left( \frac{1}{\sqrt{N_1}} \sum_{i_1=1}^{N_1} (X_{i_11} - m_1) \right)^{2K_1} \right) \cdots E_y^{\otimes N} \left( \left( \frac{1}{\sqrt{N_M}} \sum_{i_M=1}^{N_M} (X_{i_M1} - m_M) \right)^{2K_M} \right).$$

(42)

The conditionally i.i.d. random variables $X_{\nu i}$ for fixed $\nu$ and $i = 1, \ldots, N_\nu$ each have a conditional expectation of $y_\nu \leq -\frac{m_\nu}{2}$, so a lower bound on the distance between $E_y (X_{\nu i})$ and $m_\nu$ is given by

$$|E_y (X_{\nu i}) - m_\nu| = |y_\nu - m_\nu| \geq \frac{3}{2} m_\nu > 0.$$ 

Thus, we cannot expect these expectations to converge to 0. However, we have the following upper bound for each factor in (42):

$$\left( \frac{1}{\sqrt{N_\nu}} \sum_{i_\nu=1}^{N_\nu} (X_{\nu i_\nu} - m_\nu) \right)^{2K_\nu} \leq \frac{1}{N_\nu} \left( \sum_{i_\nu=1}^{N_\nu} (|X_{\nu i_\nu}| + |m_\nu|) \right)^{2K_\nu} \leq \frac{1}{N_\nu} (2N_\nu)^{2K_\nu} = 2^{2K_\nu} N_\nu^{K_\nu}.$$ 

Thus, we obtain the upper bound

$$E_y^{\otimes N} (\Sigma^{2K}) \leq \left( \prod_{\nu=1}^{M} 2^{2K_\nu} N_\nu^{K_\nu} \right)^{\frac{1}{2}} \approx 2^K \prod_{\nu=1}^{M} \frac{N_\nu^{K_\nu}}{N_\nu^{K_\nu}} N^{\frac{K}{2}},$$

(43)

where we set for the rest of this section $K := \sum_{\nu=1}^{M} K_\nu$. We use Lemma 51 in the third step below to obtain an upper bound for $P_y^{\otimes N} (\chi^+ = 1)$:

$$P_y^{\otimes N} (\chi^+ = 1) = \prod_{\nu=1}^{M} P_y^{\otimes N} \left( \frac{1}{N_\nu} \sum_{i_\nu=1}^{N_\nu} (X_{\nu i_\nu} - y_\nu) + y_\nu > 0 \right) \leq \prod_{\nu=1}^{M} P_y^{\otimes N} \left( \frac{1}{N_\nu} \sum_{i_\nu=1}^{N_\nu} (X_{\nu i_\nu} - t_\nu) \right) \leq \prod_{\nu=1}^{M} \frac{C_{2V} (-y_\nu)}{N_\nu^{2V}} \leq \prod_{\nu=1}^{M} \frac{C_{2V} (-\frac{m_\nu}{2})}{N_\nu^{2V}} = \prod_{\nu=1}^{M} \frac{2^{4V} C_{2V}}{(m_\nu)^{4V} \alpha_\nu^{2V}} \cdot \frac{1}{N^{2MV}}.$$

(44)

For a given $K$, choose $V(K) := \frac{K}{2} + Q + 1$. Then, using the upper bounds in (43) and (44), we obtain

$$E_y^{\otimes N} (\Sigma^K \chi^+) \leq c N^{K \frac{1}{N^{MV(K)}}} = c N^{-(M-1)\frac{K}{2} - M(Q+1)},$$

which goes to 0 faster than $1/N^{Q}$ for any number of groups $M \in \mathbb{N}$. Set $Q' := (M-1)\frac{K}{2} + M(Q+1)$. The constant $c$ depends only on $K$ and $Q$ but not on $N$.

Thus, we see that for any $Q \in \mathbb{N}$

$$Z_K \leq c \frac{1}{N^{Q'}} \int_{A^{-}} e^{-N F(y)}dy \rightarrow 0 \text{ as } N \rightarrow \infty.$$ 

Note that the sequence of integrals above converges. This concludes the proof of the lemma. \qed

We next show
Lemma 53. $Z^K_0$ converges to 0 exponentially fast in $N$.

Proof. Since $\tilde{F}$ has exactly two global minima at $\pm \bar{m} \notin A_0$, there is a $\delta > 0$ such that $\tilde{F}(\pm \bar{m}) = 0 < \delta \leq \tilde{F}(y)$ for all $y \in A_0$. Similarly to the proof of the last lemma, the expectation $E^\otimes_N (\Sigma^{2K})$ does not converge to zero for $y \in A_0$, as the distance between $E_y(X_{\nu\nu})$ and $m_{\nu\nu}$ is at least

$$|E_y(X_{\nu\nu}) - m_{\nu\nu}| = |y_{\nu\nu} - m_{\nu\nu}| \geq \frac{m_{\nu\nu}}{2} > 0.$$ 

However, we can once again use the upper bound for $E^\otimes_N (\Sigma^{2K})^\frac{1}{2}$ given by (13). We calculate

$$Z^K_0 \leq \int_{A_0} e^{-N\delta} E^\otimes_N (\Sigma^{2K})^\frac{1}{2} P^\otimes_N (\chi^+ = 1)^\frac{1}{2} \, dy$$

$$\leq 2^K \prod_{\nu=1}^M \delta_{\nu\nu}^{K_{\nu\nu}} e^{-N\delta} \int_{A_0} \, dy.$$ 

This last expression converges to 0 exponentially fast as $N \to \infty$. \hfill \Box

We turn our attention to $Z^K_+$ using

$$E^\otimes_N (\Sigma^K \chi^+) = E^\otimes_N (\Sigma^K) - E^\otimes_N (\Sigma^K (1 - \chi^+)),$$

we divide $Z^K_+$ into two parts

$$Z^K_+ = \int_{A_+} e^{-N\tilde{F}(y)} E^\otimes_N (\Sigma^K) \, dy \quad \text{(45)}$$

$$- \int_{A_+} e^{-N\tilde{F}(y)} E^\otimes_N (\Sigma^K (1 - \chi^+)) \, dy. \quad \text{(46)}$$

By the same reasoning as in Lemma 52, the expression (46) goes to 0 faster than any power of $N$. We centre our attention on the expression (45). We calculate the conditional expectation

$$E^\otimes_N (\Sigma^K) = E^\otimes_N \left( \prod_{\nu=1}^M \frac{\Sigma_{i_{\nu}\nu} - m_{\nu\nu}}{\sqrt{N_{\nu\nu}}} \right)^{K_{\nu\nu}} = E^\otimes_N \left( \sum_{i_{\nu}\nu=1}^{N_{\nu\nu}} Y_{i_{\nu}\nu} \right)^{K_{\nu\nu}}$$

where we set for each $\nu$ and each $i_{\nu}$ $Y_{i_{\nu}\nu} := X_{i_{\nu}\nu} - m_{\nu\nu}$.

As usual in these types of proofs, we now have to see what profiles of multiindices contribute asymptotically to the moments:

$$E^\otimes_N \left( \sum_{i_{\nu}\nu=1}^{N_{\nu\nu}} Y_{i_{\nu}\nu} \right)^{K_{\nu\nu}} = \sum_{j_1, \ldots, j_{K_{\nu\nu}}=1}^{N_{\nu\nu}} E^\otimes_N (Y_{\nu j_1} \ldots Y_{\nu j_{K_{\nu\nu}}}) = \sum_{\nu} w_{K_{\nu}}(\nu) E^\otimes_N (Y_{\nu}^{K_{\nu}}).$$

We have by Proposition 28

$$w_{K_{\nu}}(\nu) \approx \frac{N_{\nu}^{\frac{1}{2}}}{{r_1!}^{K_{\nu}}} \frac{2^{r_1} \ldots K_{\nu}!}{{r_1!}^{r_1} \ldots {K_{\nu}!}^{K_{\nu}}}.$$

We first note that for a group $\nu$ and $\nu \in \Pi^{0(K_{\nu})}$, i.e. a profile vector with

$$\nu = \left( r_1, \frac{K_{\nu} - r_1}{2}, 0, \ldots, 0 \right),$$

and $r_1$ and $K_{\nu}$ have the same parity, the expression becomes

$$w_{K_{\nu}}(\nu) \approx \frac{N_{\nu}^{\frac{1}{2}}}{{r_1!}^{K_{\nu}}} \frac{2^{r_1} \ldots K_{\nu}!}{{r_1!}^{r_1} \ldots {K_{\nu}!}^{K_{\nu}}}.$$ 

(47)
Due to the conditional independence of the $Y_{\nu_i}$, we have

$$E_{y}^{\otimes N} (Y_L) = E_{y}^{\otimes N} (Y_{\lambda_1})^r \cdot E_{y}^{\otimes N} (Y_{\lambda_1}^2)^{K_{\nu_r}} = (\tanh y_{\nu_r} - m_{\nu_r})^{r_1} (1 - 2 \tanh y_{\nu_r} \cdot m_{\nu_r} + m_{\nu_r}^2)^{K_{\nu_r}-r_1},$$

Let $\mathbf{r} := (r_1, r_2, \ldots, r_M)$ be a profile vector for all groups such that for each group $\nu$ we have $r_{\nu} = (r_{\nu_1}, \ldots, r_{\nu K_{\nu}}) \in \Pi^{0(K_{\nu})}$. Using Laplace’s method in a similar setting as in Proposition 42, we obtain

$$E(\mathbf{r}) := \int_{A_+} e^{-N\bar{F}(\mathbf{r})} E_y^{\otimes N} (Y_L) \, dy$$

$$= \int_{A_+} e^{-N\bar{F}(\mathbf{r})} \prod_{\nu=1}^M (\tanh y_{\nu_r} - m_{\nu_r})^{r_\nu} (1 - 2 \tanh y_{\nu_r} \cdot m_{\nu_r} + m_{\nu_r}^2)^{K_{\nu_r}-r_\nu} \, dy$$

$$\approx (2\pi)^{\frac{M}{2}} \sqrt{\det H^{-1}(\bar{m}) \left( \prod_{\nu=1}^M (1 - m_{\nu_r}^2)^{K_{\nu_r}-r_\nu} \frac{1}{N^{\frac{K_{\nu_r}-r_\nu}} \nu_{r}} \right)} \frac{1}{N^{\frac{K_{\nu_r}-r_\nu}} \nu_{r}} m_{r_1,\ldots,r_M} (H^{-1}(\bar{m})),$$

where $H^{-1}(\bar{m})$ in the last line stands for the inverse of the Hessian matrix of $\bar{F}$ in the points $\pm \bar{m}$. We need to normalise $E(\mathbf{r})$ by dividing the above expression by

$$E(\mathbf{0}) := \int_{A_+} e^{-N\bar{F}(\mathbf{0})} \, dy \approx (2\pi)^{\frac{M}{2}} \sqrt{\det H^{-1} \left( \frac{1}{N^{\frac{K_{\nu_r}-r_\nu}} \nu_{r}} m_{0,\ldots,0} (H^{-1}) \right)}.$$

Then we divide these two expressions and obtain

$$\frac{E(\mathbf{r})}{E(\mathbf{0})} = \left( \prod_{\nu=1}^M (1 - m_{\nu_r}^2)^{\frac{K_{\nu_r}-r_\nu}{2}} \frac{1}{N^{\frac{K_{\nu_r}-r_\nu}}} \nu_{r} \right) m_{r_1,\ldots,r_M} (H^{-1}).$$

(48)

Suppose there is at least one group $\nu$ such that

$$r_{\nu} = (r_{\nu_1}, \ldots, r_{\nu K_{\nu}}) \in \Pi^{+}(K_{\nu}),$$

i.e. there is some index that repeats at least three times. By the same reasoning as in Proposition 42, we conclude that summands corresponding to these profile vectors do not contribute asymptotically, as they converge to 0, contrary to those summands where $r_{\nu} \in \Pi^{0(K_{\nu})}$ for each group.

To calculate the asymptotic moments, we collect the constants from the expressions (47) and (48) and sum over all profile vectors where each group $\nu$ belongs to $\Pi^{0(K_{\nu})}$:

$$M_K (\Sigma^{\chi^+}) \approx \sum_{k_1=0}^{K_1} \cdots \sum_{k_M=0}^{K_M} \prod_{\lambda=0}^{K_{\lambda}} \frac{k_{\lambda}}{\lambda !} \frac{K_{\lambda}!}{\left( \frac{K_{\lambda}}{2} \right)^{K_{\lambda}!} \! \nu_{r}} (1 - m_{\nu_r}^2)^{\frac{K_{\lambda}!}{2 \lambda !} \nu_{r}} m_{k_1,\ldots,k_M} (H^{-1})$$

$$= \prod_{\nu=1}^M (1 - m_{\nu_r}^2)^{\nu_{r}} \sum_{k_1=0}^{K_1} \cdots \sum_{k_M=0}^{K_M} \prod_{\lambda=1}^{K_{\lambda}} \left( \frac{\alpha_{k_\lambda}}{1 - m_{\nu_r}^2} \right)^{\nu_{r}} \frac{K_{\lambda}!}{\lambda ! \left( \frac{K_{\lambda}}{2} \right)^{K_{\lambda}!} \! \nu_{r}} m_{k_1,\ldots,k_M} (H^{-1}),$$

(49)

where each sum is over those $k_\lambda$ which have the same parity as $K_{\lambda}$.

The second factor above,

$$\sum_{k_1=0}^{K_1} \cdots \sum_{k_M=0}^{K_M} \prod_{\lambda=1}^{K_{\lambda}} \frac{\alpha_{k_\lambda}}{1 - m_{\nu_r}^2} \frac{K_{\lambda}!}{\lambda ! \left( \frac{K_{\lambda}}{2} \right)^{K_{\lambda}!} \! \nu_{r}} m_{k_1,\ldots,k_M} (H^{-1}),$$

has the same structure as the moment given in (23). By the proof of the CLT 11, these are the moments of a centred multivariate normal distribution. Since in (49) there is another factor present, namely

$$\prod_{\nu=1}^M (1 - m_{\nu_r}^2)^{\nu_{r}},$$

we need the following
Lemma 54. Let $\ell_1, \ldots, \ell_n \in \mathbb{N}_0$. Let $m_{\ell_1, \ldots, \ell_n} := m_{\ell_1, \ldots, \ell_n}(C)$ be the moment of the centred multivariate normal distribution with covariance matrix $C = (c_{ij})_{i,j=1,\ldots,n}$. Let $s_1, \ldots, s_n$ be positive numbers. Then
\[
e_{s_1^2 \cdots s_n^2} m_{\ell_1, \ldots, \ell_n}
\]
is the moment of order $(\ell_1, \ldots, \ell_n)$ of the centred multivariate distribution with covariance matrix
\[
D := (d_{ij})_{i,j=1,\ldots,n} := \left(\sqrt{s_is_j} c_{ij}\right)_{i,j=1,\ldots,n}.
\]

Proof. We omit the proof which consists of an application of Isserlis’s Theorem. □

This lemma shows that the expression (49) is the moment of order $K$ of a centred multivariate normal distribution.

To calculate the covariance matrix $E$ as a function of the Hessian matrix $H$ of $F$ at $\pm \tilde{m}$, we use the formula (49). The diagonal entries of $E$ are
\[
e_{\lambda \lambda} = (1 - m_\lambda^2) \sum_{k_\lambda=0,2} \frac{\alpha_\lambda}{1-(m_\lambda^2)^k} \frac{2^k}{k_\lambda!} \frac{m_{0,\ldots,0,k_\lambda,0,\ldots,0}(H^{-1})}{2^{-k_\lambda/2}}
= 1 - m_\lambda^2 + \alpha_\lambda (H^{-1})_{\lambda \lambda}.
\]
The off-diagonal entries are given by
\[
e_{\lambda \mu} = \sqrt{1 - m_\lambda^2} \sqrt{1 - m_\mu^2} \sum_{k_\lambda=1, \kappa_\mu=1} \frac{\alpha_\lambda}{1-(m_\lambda^2)^{k_\lambda}} \frac{1!}{k_\lambda!} \frac{1!}{2^{-k_\lambda/2}} \frac{m_{0,\ldots,0,k_\lambda,0,\ldots,0}(H^{-1})}{2^{-k_\lambda/2}}
= \sqrt{\alpha_\lambda \alpha_\mu} (H^{-1})_{\lambda \mu}.
\]

We can thus express the covariance matrix as
\[
E = \text{diag} \left((1 - m_\lambda^2) + \sqrt{\alpha}H^{-1}\sqrt{\alpha}\right).
\]

7 Strong Laws of Large Numbers

In this section, we prove the statements in Remarks 10 and 14. It is well known that a sequence of random variables which converges in distribution to a constant also converges in probability. Thus the Weak Law of Large Numbers holds for the LLNs 9 and 13.

We next state a proposition that will allow us to show almost sure convergence.

Proposition 55. Let $(X_n)$ be a sequence of real random variables defined on the probability space $(\Omega, \mathcal{A}, P)$. Set $A_{m,k} := \{ |X_m| > \frac{1}{k} \}$, $A := \{ \lim_{n \to \infty} X_n = 0 \}$. If $\sum_{m=1} P(A_{m,k}) < \infty$ for all $k \in \mathbb{N}$, then $P(A) = 1$.

Proof. This is Proposition 3.33 in [15]. □

Now we show that the normalised sums $S_{\lambda}/N_{\lambda}$ converge to 0 faster than any power of $N$ when we are either in the high temperature or the critical regime.

Theorem 56. Assume we are in the high temperature or the critical regime, and let $K \in \mathbb{N}$ and $a > 0$. Then there is a constant $d_K$ such that
\[
P\left(\left\| \left(\frac{S_1}{N_1}, \ldots, \frac{S_M}{N_M}\right) \right\| > a \right) \leq \frac{d_K}{a^4KN^K},
\]
where $\| \cdot \|$ stands for the max norm on $\mathbb{R}^M$. 

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Proof. Let $K \in \mathbb{N}$. We show for all $\lambda$:

$$
P\left(\left|\frac{S_{\lambda}}{N_{\lambda}}\right| > a\right) \leq \frac{d_K}{a^{4K} N^K}.
$$

We have

$$
P\left(\left|\frac{S_{\lambda}}{N_{\lambda}}\right| > a\right) \leq \frac{1}{a^{4K} N_{\lambda}^{K}} \mathbb{E}\left(\left|\frac{S_{\lambda}}{N_{\lambda}^{K}}\right|^{4K}\right),
$$

where we applied Markov’s inequality.

If we are in the high temperature regime, then $\mathbb{E}\left(\left|\frac{S_{\lambda}}{N_{\lambda}^{K}}\right|^{4K}\right)$ converges to 0. In the critical regime, we have $\mathbb{E}\left(\left|\frac{S_{\lambda}}{N_{\lambda}^{K}}\right|^{4K}\right) \approx b_{4K}$, where $b_{4K} = c_{0,\ldots,0,4K,0,\ldots,0}^{K}$ from (39) with the entry $4K$ at the position $\lambda$ and 0 everywhere else. We are done once we set $d_K$ equal to the maximum of these $b_{4K}$ over all $\lambda$.

Now we can prove that the Strong Law of Large Numbers holds in the high temperature and critical regimes.

**Theorem 57.** Assume we are in the high temperature or the critical regime. Then

$$
\lim_{N \to \infty} \left(\frac{S_1}{N_1}, \ldots, \frac{S_M}{N_M}\right) = (0, \ldots, 0) \text{ a.s.}
$$

*Proof.* Let $\lambda \in \{1, \ldots, M\}$, and let for all $N, k \in \mathbb{N}$

$$
A_{N,k} := \left\{\left|\frac{S_{\lambda}}{N_{\lambda}}\right| > \frac{1}{k}\right\}.
$$

By the previous theorem, we have

$$
P\left(A_{N,k}\right) \leq \frac{d_2 k^8}{N^2}
$$

for some constant $d_2$. Then $\sum_{N=1}^{\infty} P\left(A_{N,k}\right) < \infty$ holds and we can apply Proposition 55 to conclude that

$$
P\left(\lim_{N \to \infty} \frac{S_{\lambda}}{N_{\lambda}} = 0\right) = 1.
$$

\[\square\]

**Declarations Section**

**Conflict of Interests**

Not applicable.

**Availability of Data and Materials**

Not applicable.
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