Differential Privacy for Sequential Algorithms

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Abstract—We study the differential privacy of sequential statistical inference and learning algorithms that are characterized by random termination time. Using the two examples: sequential probability ratio test and sequential empirical risk minimization, we show that the number of steps such algorithms execute before termination can jeopardize the differential privacy of the input data in a similar fashion as their outputs, and it is impossible to use the usual Laplace mechanism to achieve standard differentially private in these examples. To remedy this, we propose a notion of weak differential privacy and demonstrate its equivalence to the standard case for large i.i.d. samples. We show that using the Laplace mechanism, weak differential privacy can be achieved for both the sequential probability ratio test and the sequential empirical risk minimization with proper performance guarantees. Finally, we provide preliminary experimental results on the Breast Cancer Wisconsin (Diagnostic) and Landsat Satellite Data Sets from the UCI repository.[1]

I. INTRODUCTION

Privacy is a major concern in utilities involving statistical inference from collected personal data, such as personalized medicine, social media analysis, recommendation systems, to name a few. Among various mathematical measures of privacy, the research on differential privacy has flourished during the past decade [1] and has been applied extensively to various statistical inference, learning and optimization problems [2]–[6]. Differentially private probably approximately correct (PAC) learning is based on a body of literature established during the past decade; see for instance [7], [8] and references therein.

Generally speaking, differential privacy requires the probability distribution of the output of an algorithm to not change significantly when small changes are made to the inputs. The extent to which the output of an algorithm varies as its input varies is called sensitivity. A common approach to make statistical inference and learning algorithms differentially private is to first upper bound the sensitivity and then add proportional random noise to the output [9]. This usually results in a trade-off between the privacy of the data and the accuracy of the learning.

Previously, differential privacy is mostly studied in terms of statistical inference and learning algorithms that use fixed numbers of samples. This is partly because, differential privacy, at its inception, aimed at handling static databases with known sizes [1], [10], and there is a natural connection between randomized query of databases and statistical inference and learning from a given set of samples. In these cases, the sensitivity of the algorithms is usually provably bounded, therefore the sensitivity based randomization mechanism can ensure the differential privacy of the data.

In many cases, sequential algorithms are more efficient and easier to implement, especially when there is a given requirement on the accuracy of the result [11]–[15]. Generally, these sequential algorithms take one or a small number of samples at each round and evaluate the quality of samples gathered so far and the accuracy of prospective results using some stopping condition. Then, they decide to draw more samples if the evaluation fails, or give the final result otherwise. Since these algorithms stop once the given accuracy requirement is satisfied, they are usually more efficient than their counterpart algorithms that use fixed sample numbers.

Recently, there is an effort to make sequential learning and optimization algorithms differentially private [16]–[18]. This current work differs by treating the executed steps before termination as observable to the outside world. The varying termination time presents challenges to preserving differential privacy, as it may change dramatically under the alteration of a single datum value and thus reveal information about its existence. Thus, achieving differential privacy for these algorithms in the standard way is very difficult if not impossible. This motivates us to propose a notion of weak differential privacy, that is achievable for most sequential algorithms.

Our notion of weak differential privacy is based on the condition that the data used by the sequential algorithm are drawn independently from some known distribution. On average, the empirical distribution of these data is close to this known distribution. We define weak differential privacy so that the privacy of data is preserved in this average case. A similar idea appears in [19], but there, the average is only taken for unknown data.

Our notion of differential privacy is weaker than the standard differential privacy, since the latter 1) does not assume knowledge of the underlying distribution of the data and 2) preserves the privacy of data in all cases including the aforementioned average case. However, our weak differential privacy implies standard differential privacy almost surely, when the size of data is large.

For two classic sequential algorithms: sequential probability ratio test (SPRT) [20] and sequential empirical risk minimization (SERM) [12]–[15], we show that weak differential privacy can be achieved by the common exponential mechanism [9] with provably bounded loss of performance. We also show this experimentally on two real-world data
sets: the Breast Cancer Wisconsin (Diagnostic) and the Landsat Satellite Data Sets, from the UCI repository.

The rest of the paper is organized as follows. We give the preliminaries and problem setups for sequential algorithms in Section III. In Section IV, we explain why achieving differential privacy in the strict sense is difficult, and propose the notion of weak differential privacy. In Section IV, we achieve weak differential privacy for two classic sequential algorithms: SPRT and SERM. In Section V, we show promising experimental results on the Breast Cancer Wisconsin (Diagnostic) and the Landsat Satellite Data Sets from the UCI repository. We conclude in Section VI.

II. PROBLEM FORMULATION

We consider the differential privacy of a general sequential algorithm $\mathcal{A}$. The input algorithm is an (infinite) sequence of independent and identically distributed (i.i.d.) sample data $X = X_1, X_2, \ldots$ referred to as a data stream, from some probability space $(\Omega, \Sigma, P)$. Let $\Omega' = \Omega \cup (\Omega \times \Omega) \cup \ldots$ be the set of finite sequences taken from $\Omega$. Then, a sequential algorithm is defined as a tuple $\mathcal{A} = (T_{\mathcal{A}}, f_{\mathcal{A}})$, where

- $T_{\mathcal{A}} : \Omega^* \to \{0, 1\}$ is a stopping condition, where 0 and 1 stand for false and true, respectively;
- $f_{\mathcal{A}} : \Omega^* \to \mathcal{\Theta}$ is a return function, where $\mathcal{\Theta}$ is the set of outputs of the algorithm $\mathcal{A}$. The set $\mathcal{\Theta}$ can be of general type, such as binary or real numbers $\mathbb{R}$.

The sequential algorithm $\mathcal{A}$ executes on a data stream $X$ as follows. Iteratively for $n = 1, 2, \ldots$ if $\tau(X_1, X_2, \ldots) = 1$, then $\mathcal{A}$ stops and returns $T_{\mathcal{A}}(X_1, X_2, \ldots)$, otherwise, it continues. The number of iterations $\mathcal{A}$ executed on the data stream $X$ is denoted by $\tau_{\mathcal{A}}$ and is referred to as the stopping step. Mathematically, $\tau_{\mathcal{A}}$ is a stopping time [21] defined on the random process $X$, so we may also write it as $\tau_{\mathcal{A}}(X)$. Below, we introduce two examples of sequential algorithms.

A. Sequential Hypothesis Testing

A simple sequential algorithm is the sequential probability ratio test (SPRT) [22]. It judges the correctness between two hypothesis with bounded statistical error. We consider an SPRT algorithm $\mathcal{A}$ for two hypotheses on the parameter $p$ of a Bernoulli distribution

$$H_0 : p = p_0 \quad H_1 : p = p_1, \quad p_0 < p_1$$

(1)

with false positive (FP) ratio $\alpha$ and false negative (FN) ratio $\beta$. Iteratively for $n = 1, 2, \ldots$ the algorithm $\mathcal{A}$ takes the sample $X_n$ from the Bernoulli distribution and computes

$$\Lambda_n = \sum_{i=1}^{n} \ln \left( \frac{p_1^n (1 - p_1)^{1 - X_i}}{p_0^n (1 - p_0)^{1 - X_i}} \right).$$

(2)

The stopping condition is given by

$$T_{\mathcal{A}} = \begin{cases} 0, & \text{if } \ln \left( \frac{\beta}{1 - \alpha} \right) < \Lambda_n < \left( \frac{1 - \beta}{\alpha} \right), \\ 1, & \text{otherwise.} \end{cases}$$

(3)

And the return function is given by

$$f_{\mathcal{A}} = \begin{cases} 0, & \text{if } \Lambda_n \leq \ln \left( \frac{\beta}{1 - \alpha} \right), \\ 1, & \text{if } \Lambda_n \geq \ln \left( \frac{1 - \beta}{\alpha} \right). \end{cases}$$

(4)

Finally, the stopping step $\tau_{\mathcal{A}}$ is finite with probability 1.

B. Sequential Empirical Risk Minimization

Another widely-used sequential algorithm is the sequential empirical risk minimization (SERM) [23]. Consider a supervised learning problem of finding the minimizer $f_{\text{min}}$ from a class of functions $F$ with finite VC-dimension on a probability space $(\Omega, \Sigma, P)$:

$$f_{\text{min}} = \arg\min_{f \in F} P(f) = \arg\min_{f \in F} \int_{\Omega} f dP.$$ 

(5)

This problem can be solved in the probably approximately correct (PAC) sense with the usual empirical risk minimization algorithm using samples of fixed sizes, computed from the accuracy requirement and the VC-dimension of $F$. However, it can be solved more efficiently via a sequential empirical risk minimization algorithm proposed in [12]–[15].

Specifically, iteratively for $n = 1, 2, \ldots$, the SERM algorithm $\mathcal{A}$ draws the samples $X_n$ from the probability space and compute the empirical Rademacher average

$$r_{\mathcal{A}}(n) = \frac{1}{n} \sup_{f \in F} \left| \frac{1}{n} \sum_{i=1}^{n} \sigma_i f(X_i) \right|.$$ 

(6)

where $\sigma_1, \ldots, \sigma_n$ are i.i.d. Rademacher random variables, taking values in $\{-1, 1\}$ with probability 1/2. The stopping condition is defined by

$$T_{\mathcal{A}} = \begin{cases} 1, & \text{if } r_{\mathcal{A}}(n) < \alpha \text{ and } n > N_{\alpha, \beta}, \\ 0, & \text{otherwise.} \end{cases}$$

(7)

where

$$N_{\alpha, \beta} = \frac{2}{\alpha^2} \ln \frac{2}{\beta (1 - e^{-\frac{\alpha^2}{2}})}.$$ 

(8)

And the return function is given by

$$f_{\mathcal{A}} = \arg\min_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} f(X_i).$$

(9)

It is guaranteed that the minimizer is probably approximately correct (PAC)

$$\mathbb{P} \left[ |P(f_{\text{ERM}}) - P(f^*)| > \alpha \right] < \beta.$$ 

(10)

III. DIFFERENTIAL PRIVACY IN SEQUENTIAL ALGORITHMS

Now, we discuss the differential privacy of the sequential algorithm $\mathcal{A} = (\tau_{\mathcal{A}}, f_{\mathcal{A}})$ defined in Section III. To begin with, we recall that the input of $\mathcal{A}$ is a data stream $X = X_1, X_2, \ldots$, i.e., an (infinite) sequence of i.i.d. sample data from a probability space. And the public observation of $\mathcal{A}$ is twofold: the stopping step $\tau_{\mathcal{A}}(X)$ (the iteration at which $\mathcal{A}$ stops) and the value of the return function $f_{\mathcal{A}}(X_1, \ldots, X_{\tau_{\mathcal{A}}})$.

Following the seminal work [10], we start to discuss the differential privacy for the sequential algorithm $\mathcal{A}$ as follows. Generally, the algorithm $\mathcal{A}$ is differentially private, if the value of the data stream $X$ cannot be easily inferred from the random public observations $(\tau_{\mathcal{A}}(X), f_{\mathcal{A}}(X_1, \ldots, X_{\tau_{\mathcal{A}}})).$
This requires that the probability distributions of the (random) public observations changes mildly with the value of the data stream.

To formally capture this, we consider two data streams that are only different in one entry [1].

**Definition 1 (Adjacency):** Two data streams $X = X_1, X_2, \ldots$ and $X' = X'_1, X'_2, \ldots$ are adjacent, if there exists $K \in \mathbb{N}$ such that for any $n \in \mathbb{N} \setminus \{K\}$, it holds that $X_n = X'_n$.

Differential privacy requires for any two adjacent data streams $X$ and $X'$, the probability distributions of the two (random) public observations $(\tau_{\mathcal{A}}(X), f_{\mathcal{A}}(X_1, \ldots, X_{\tau_{\mathcal{A}}}))$ and $(\tau_{\mathcal{A}}(X'), f_{\mathcal{A}}(X'_1, \ldots, X'_{\tau_{\mathcal{A}}}))$ should be similar enough to satisfy the following inequality.

**Definition 2 (Strong Differential Privacy):** A sequential algorithm $\mathcal{A} = (\tau_{\mathcal{A}}, f_{\mathcal{A}})$ is $(\varepsilon, \delta)$-strongly differentially private, if

$$\mathbb{P}\left[ (\tau_{\mathcal{A}}(X), f_{\mathcal{A}}(X_1, \ldots, X_{\tau_{\mathcal{A}}})) \in O \times \{K\} \right] \leq e^\varepsilon \mathbb{P}\left[ (\tau_{\mathcal{A}}(X'), f_{\mathcal{A}}(X'_1, \ldots, X'_{\tau_{\mathcal{A}}})) \in O \times \{K\} \right] + \delta$$

holds for any two adjacent data streams $X = X_1, X_2, \ldots$ and $X' = X'_1, X'_2, \ldots$, $K \in \mathbb{N}$, and $O \subseteq \Omega$, where $\Omega$ is the range of the return function $f_{\mathcal{A}}$. Specially, $(\varepsilon, 0)$-strongly differentially private is referred to as $\varepsilon$-strongly differentially private.

We refer to Definition 2 as **strong differential privacy**, as the condition (11) should hold for any two adjacent data streams.

**A. Difficulties for Strong Differential Privacy**

Following Sections II-A and II-B it is easy to check that neither of the two example sequential algorithms is strongly differentially private. The common approach to imposing differential privacy on an algorithm is by randomizing the public observation [9]. For simplicity, for the public observation of a sequential algorithm $\mathcal{A}$, we focus on the stopping step $\tau_{\mathcal{A}}$ of a sequential algorithm $\mathcal{A}$ and ignore the return function $f_{\mathcal{A}}$ for now. If for two given adjacent data streams $X$ and $X'$, the two stopping steps $\tau_{\mathcal{A}}(X)$ and $\tau_{\mathcal{A}}(X')$ are nonrandom, and $|\tau_{\mathcal{A}}(X) - \tau_{\mathcal{A}}(X')| = D(X, X')$ for some $D \in \mathbb{N}$, then we should add a mean-zero Laplace noise with parameter $\varepsilon$ to $\tau_{\mathcal{A}}$. When $D > \varepsilon \max_{X, X'} D(X, X')$ for some $\varepsilon > 0$, then the sequential algorithm $\mathcal{A}$ achieves $\varepsilon$-strong differential privacy.

The above technique depends critically on the boundedness of $\max_{X, X'} D(X, X')$, which is generally true for nonsequential algorithms [1]. However, this condition is violated for both sequential probability ratio test and sequential empirical risk minimization from Sections II-A and II-B

**Sequential probability ratio test:** Following Section II-A let $N_0 = |\{i \in [n]|X_i = 0\}|$ and $N_1 = |\{i \in [n]|X_i = 1\}|$. Then, the stopping condition (3) can be expressed by

$$T_{\mathcal{A}} = \begin{cases} 0, & \text{if } N_0 - N_1 < \frac{\ln \frac{\varepsilon}{\delta}}{\ln \frac{\varepsilon}{\nu}} \text{ and } N_1 - N_0 < \frac{\ln \frac{\varepsilon}{\delta}}{\ln \frac{\varepsilon}{\nu}} \\ 1, & \text{otherwise.} \end{cases}$$

Now consider two adjacent data streams

$$X = (1, \ldots, 1, -1, -1, -1, \ldots)$$

$$X' = (1, \ldots, 1, -1, -1, -1, \ldots).$$

For $X$, the algorithm $\mathcal{A}$ stops at $\tau_{\mathcal{A}}(X) = K_1 + 1$, while for $X'$, the algorithm never stops. Thus, $D(X, X') = |\tau_{\mathcal{A}}(X) - \tau_{\mathcal{A}}(X')| = +\infty$.

**Sequential empirical risk minimization:** Following Section II-B consider the probability space of uniform distribution on $(0, 1, 2)$ and the function class $\mathcal{F} = \{f_1, f_2\}$ with $f_1(\{0, 1, 2\}) = \{1, 0, 1\}$ and $f_2(\{0, 1, 2\}) = \{0, 1, 0\}$. Let $1/\varepsilon < N_{\alpha, \beta}$. Now, consider two adjacent data streams

$$X = (0, \ldots, 0, 1, 0, 0, \ldots)$$

$$X' = (0, \ldots, 0, 0, 0, 0, \ldots).$$

For $X$, the algorithm $\mathcal{A}$ stops at $\tau_{\mathcal{A}}(X) = N_{\alpha, \beta} + 1$, while for $X'$, the algorithm never stops. Thus, $D(X, X') = |\tau_{\mathcal{A}}(X) - \tau_{\mathcal{A}}(X')| = +\infty$.

**B. Weak Differential Privacy**

Noting the difficulties in deriving strongly differentially privacy sequential algorithms, we propose a weak notion of differential privacy. This is based on the fact that the data stream $X$ is drawn i.i.d. from a probability space $(\Omega, \Sigma, \mu)$. Thus, encountering extreme cases like in Section III-A is very rare. Even though achieving differential privacy for all data streams is virtually impossible, it is possible for most data streams from that probability space.

Unlike strong differential privacy from Definition 2 weak differential privacy focuses on the average probability distributions of the (random) public observations, when the $n$th entry of a data stream $X$ changes. This average is derived by taking the expected value of the other entries, denoted by $X_{-n}$, with respect to their probability distribution $P$. Thus, the average public observation is given by $E_{X_{-n}}[\tau_{\mathcal{A}}(X), f_{\mathcal{A}}(X_1, \ldots, X_{\tau_{\mathcal{A}}})]$. By requiring that for any entry $n \in \mathbb{N}$, this average public observation only changes slightly, we introduce the weak differential privacy below.

**Definition 3 (Weak Differential Privacy):** A sequential algorithm $\mathcal{A} = (\tau_{\mathcal{A}}, f_{\mathcal{A}})$ is $(\varepsilon, \delta)$-weakly differentially private, if

$$\mathbb{P}\left[ E_{X_{-n}}[\tau_{\mathcal{A}}(X), f_{\mathcal{A}}(X_1, \ldots, X_{\tau_{\mathcal{A}}})] \in O \times \{K\} \right]$$

$$\leq e^\varepsilon \mathbb{P}\left[ E_{X'_{-n}}[\tau_{\mathcal{A}}(X'), f_{\mathcal{A}}(X'_1, \ldots, X'_{\tau_{\mathcal{A}}})] \in O \times \{K\} \right] + \delta$$

(13)
holds for any two adjacent data streams \( X = X_1, X_2, \ldots \) and \( X' = X'_1, X'_2, \ldots, K \in \mathbb{N} \), and \( O \subseteq \Theta' \), where \( \Theta' \) is the range of the return function \( f_{\delta} \). Specially, \((\varepsilon, 0)\)-weakly differentially private is referred to as \( \varepsilon \)-weakly differentially private.

Definition 3 is indeed weaker than Definition 2 since the condition (13) can be derived from the condition (11) by taking conditional expectation on \( X_{-u} \). Finally, we note that weak differential privacy is almost equivalent to strong differential privacy, when \( \tau_d \) is large, because the empirical distribution of the samples converges to \( P \) almost surely. Thus, we have the following relation between weak and strong differential privacy.

Proposition 1: If a sequential algorithm \( \mathcal{A} \) is \((\varepsilon, \delta)\)-weakly differentially private, then it is \((\varepsilon, \delta)\)-strongly differentially private almost surely for a random data stream, as \( \tau_d(X) \to \infty \).

IV. WEAK DIFFERENTIAL PRIVACY FOR SEQUENTIAL ALGORITHMS

In this section, we apply the idea of weak differential privacy to the sequential probability ratio test and the sequentially empirical risk minimization from Sections II-A and II-B. We show that by randomizing a proper set of parameters, weak differential privacy can be achieved on both of them.

A. Sequential Probability Ratio Test

Following Section II-A, we design a weakly differentially private sequential probability ratio test by first randomizing the stopping condition. In (2), when a single sample is altered, the log-likelihood ratio \( \Lambda_n \) changes its value by

\[
d = \ln \frac{p_1(1 - p_0)}{p_0(1 - p_1)}.
\]

(14)

So, to achieve \( \varepsilon \)-differential privacy, the stopping condition should be randomized by

\[
\tau_{\delta} = \begin{cases} 
0, & \text{if } l + \Delta < \Lambda_n < u + \Delta, \\
1, & \text{otherwise}.
\end{cases}
\]

(15)

where

\[
u = \ln((1 - \beta)/\alpha), \quad l = \ln(\beta/(1 - \alpha)),
\]

(16)

and

\[
\Delta \sim \text{Laplace}(\frac{1}{\varepsilon d} \mathbb{1}_{[u, \infty]}).
\]

(17)

obey the Laplace distribution confined to the interval \([u, \infty]\). Now, \((\varepsilon, \exp(\varepsilon \min(-\ln \alpha, -\ln \beta))/2)\)-weak differential privacy is achieved for solely observing the stopping step \( \tau_{\delta} \).

The return function \( f_{\delta} \) of the algorithm should also be randomized to achieve differential privacy. Here, we adopt the exponential algorithm from [9] to guarantee \( \epsilon' \)-differential privacy for solely observing the return function. Combining the public observation of the stopping step \( \tau_{\delta} \) and the return function \( f_{\delta} \) in the usual way [1], we derive the the overall privacy level by

\[
\exp(\min(-\ln \alpha, -\ln \beta))/2)
\]

(15)

For the stopping step condition (13), we observe that the stopping condition is random, and the actual false positive (FP) and false negative (FN) ratios \(\alpha_F\) and \(\alpha_N\) of the proposed algorithm is random, and their expectations satisfies

\[
\alpha_F \leq \int_{\mathbb{R}} \frac{e^{-(u+x)}}{2} \exp(\varepsilon x) \mathbb{1}_{[-\ln \alpha, -\ln \beta]}(u) du \leq \frac{\varepsilon \alpha}{1 - \varepsilon},
\]

(18)

and similarly

\[
\alpha_N \leq \frac{\varepsilon \beta}{1 - \varepsilon}.
\]

(19)

Algorithm 1 Weakly differentially private SPRT

Require: FP, FN ratios \(\alpha, \beta\), privacy levels \(\varepsilon, \epsilon'\)

1. Draw \(\Delta\) from (17) with \(d, u, l\) from (14) (16)
2. Run SPRT with stopping condition (15)
3. Randomize the return by exponential algorithm by \(\epsilon'\).

Finally, we check the statistical accuracy of Algorithm 1.

Theorem 1: Algorithm 1 is \((\max\{\varepsilon, \varepsilon'\}, \exp(\varepsilon \min(-\ln \alpha, -\ln \beta))/2\)-weakly differentially private with the expected false positive and false negative ratios less than \(\varepsilon \alpha/(1 - \varepsilon) + 1/\exp \varepsilon' + 1\) and \(\varepsilon \beta/(1 - \varepsilon) + 1/\exp \varepsilon' + 1\), respectively.

B. Sequential Empirical Risk Minimization

Following Section II-B, we design a weakly differentially private sequential probability ratio test by first randomizing the stopping condition. The sequential empirical risk minimization algorithm is already random due to Rademacher random variables. This automatically gives a degree of privacy.

To begin with, we consider the stopping step. For any \(f \in \mathcal{F}\), we note that \(f(X_i)\) obeys Bernoulli distribution with parameter \(P(f)\). For Rademacher random variables \(\sigma_i\), we have

\[
\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \sigma_i f(X_i) \to \text{Gaussian}(0, P(f)), \quad n \to \infty
\]

(20)

by Central Limit Theorem. Consequently, we have

\[
\frac{1}{\sqrt{n}} \sup_{f \in \mathcal{F}} \left| \sum_{i=1}^{n} \sigma_i f(X_i) \right| \to \text{Gaussian}(0, \lambda), \quad n \to \infty\]

(21)

where \(\lambda = \sup_{f \in \mathcal{F}} P(f)\), as the process with the largest \(P(f)\) will dominate as \(n \to \infty\).

The probability of terminating at \(N_{\alpha, \beta}\) is

\[
\mathbb{P}[\tau = N_{\alpha, \beta}] = \int_{0}^{\alpha N_{\alpha, \beta}} \sqrt{\frac{2}{\lambda \pi N_{\alpha, \beta}}} e^{-\frac{x^2}{\lambda N_{\alpha, \beta}}} dx \approx \sqrt{\frac{2\lambda}{\alpha^2 \pi N_{\alpha, \beta}}} e^{-\frac{\alpha^2 N_{\alpha, \beta}}{2\lambda}}
\]

(22)
Thus, the log-likelihood ration of two adjacent data streams with \( f(X_i) = 0 \) and \( f(X'_i) = 1 \) satisfies

\[
\frac{\ln \mathbb{P}[\tau \mid f(X_i) = 1]}{\ln \mathbb{P}[\tau \mid f(X_i) = 0]} = \frac{\int_0^{\alpha N_{\alpha, \beta}^{-1}} \frac{2}{\lambda \pi (N_{\alpha, \beta}^{-1})} e^{-\frac{x^2}{2 \lambda (N_{\alpha, \beta}^{-1})}} \, dx}{\int_0^{\alpha N_{\alpha, \beta}^{-1}} \frac{2}{\lambda \pi (N_{\alpha, \beta}^{-1})} e^{-\frac{x^2}{2 \lambda (N_{\alpha, \beta}^{-1})}} \, dx} = \ln \left( \frac{\alpha N_{\alpha, \beta}^{-1}}{\sqrt{\frac{2}{\lambda \pi (N_{\alpha, \beta}^{-1})} e^{-\frac{\alpha^2 N_{\alpha, \beta}^{-1}}{2 \lambda}}} \right) \tag{23}
\]

noting that \( \alpha N_{\alpha, \beta} \gg 1 \) for \( \alpha \ll 1 \).

The probability of terminating at step \( \tau + N_{\alpha, \beta} \) satisfies asymptotically,

\[
\mathbb{P}[\tau = N_{\alpha, \beta}] = \int_0^{\alpha N_{\alpha, \beta}} \frac{\pi x}{\lambda} \sqrt{\frac{1}{1 - \lambda}} \tau^N e^{-\frac{(\alpha N_{\alpha, \beta} + x)^2}{2 \lambda N_{\alpha, \beta}}} \, dx 
\]

where the second exponential comes from an inverse Gaussian distribution describing first hitting. Again, the log-likelihood ration of two adjacent data streams with \( f(X_i) = 0 \) and \( f(X'_i) = 1 \) satisfies

\[
\ln \frac{\mathbb{P}[\tau = N_{\alpha, \beta} \mid f(X_i) = 1]}{\mathbb{P}[\tau = N_{\alpha, \beta} \mid f(X_i) = 0]} \approx \sqrt{\frac{2\lambda}{\alpha^2 \pi N_{\alpha, \beta}}} e^{-\frac{\alpha^2 N_{\alpha, \beta}}{2 \lambda}} \tag{25}
\]

By (23) and (23), we know that the randomness in the sequential learning algorithm gives the algorithm some level of weak differential privacy.

The return function \( f_{a, \beta} \) can be randomized by the exponential randomization algorithm proposed [8], [9] with any \( \epsilon \). Then the overall privacy level will be max\( \{\epsilon, \sqrt{2 / \lambda \pi N_{\alpha, \beta}} e^{-\alpha^2 N_{\alpha, \beta} / 2 \lambda} \max\{1, \lambda / \alpha\}\} \). The above discussions are summarized by Algorithm 2 and Theorem 2.

**Algorithm 2** Weakly differentially private SERM

**Require:** \( \alpha > 0, \beta \in (0, 1), \epsilon > 0 \)

1. Perform SERM with \((\alpha, \beta)\).
2. Randomize return by exponential mechanism by \( \epsilon \).

**Theorem 2:** Algorithm 2 is max\( \{\epsilon, e^{-\alpha^2 N_{\alpha, \beta} / 2 \lambda} \max\{1, \lambda / \alpha\} \sqrt{2 / \lambda \pi N_{\alpha, \beta}}\}\)-weakly differentially private.

In Algorithm 2 only the return function is randomized, therefore its sample efficiency is still the same as the non-private version. The accuracy of the algorithm will decrease in the same way as non-sequential empirical risk minimization [24], so the discussion is omitted.

**V. Case Studies**

We implemented Algorithm 2 in Python and studied its performance on two datasets from the UCI repository [25]. The Wisconsin Diagnostic Breast Cancer (WDBC) Data Set consists of 569 instances with 32 attributes: 30 real-valued input features, the diagnosis (M = malignant, B = benign) and the patient ID. We excluded the IDs from the features. Hence, we have a classification problem where the input is 30-dimensional real vector and the output is one of two classes. In such scenario, an analyst would need to preserve the privacy of the patients while being able to provide a classifier for a third party to diagnose new patients.

The Statlog (Landsat Satellite) Data Set consists of multi-spectral values of pixels in 3 × 3 neighborhoods in satellite images with the class of the center pixels. Each pixel has four integer values between 0 and 255 and its class can be one of seven: red soil, cotton crop, grey soil, damp grey soil, soil with vegetation stubble, mixture class (all types present), and very grey damp soil. Hence, each instance consists of 36 integers between 0 and 255 and an integer between 1 and 7. Given the multi-spectral values of the 3 × 3 neighborhood pixels of a pixel, the goal is to predicts its class. The data set is separated to two: one for training with 4435 instances and one for testing with 2000 instances. This data set is larger than WDBC.

For each of the two datasets, we trained a 2-hidden layers neural network classifier with 90 nodes in the first layer and 50 nodes in the second. We chose the activation functions at all hidden nodes to be the Sigmoid function \( (\beta(x) = \frac{1}{1 + e^{-x}}) \). The output layer is a Softmax function \( f(x) = \frac{e^x}{\sum e^x} \).

Moreover, we restricted the values of all parameters to \([-2, 2]\). Training the best classifier is not the aim of the experiments. They just aim to show the effect of different levels of privacy on the accuracy of a good classifier. We chose \( \alpha = 0.2, \beta = 0.2 \) for the experiments on the WDBC data set and \( \alpha = 0.1, \beta = 0.1 \) for the ones on the Statlog data set.

To implement the exponential algorithm in Algorithm 2 we impose a metric structure on the class of functions \( \mathcal{F} \). Here, the metric can be equivalently defined on the parameters of the neural network: and we take the \( \ell_1 \)-norm multiplied by 1000 on the vector of parameters, since the trained parameters are mostly of order 0.01 to 0.1. In this case, to achieve \( \{0.1, 0.2, 0.5\}\)-weak differential privacy, the exponential algorithm requires adding independent Laplace noise with parameter \( \{100, 200, 500\} \) to each parameter.

For each case, we repeated the following experiment 10 times: we randomly shuffled the training data set and then ran Algorithm 2. Since the WDBC data set does not have a separate testing data set, the remaining instances that were not chosen as part of the training set would be considered as a test set. If the test set is smaller than 50, the experiment would be discarded and repeated again. We then trained the neural net on the training set to get the empirical risk minimizer by 10000 steps of size 0.001 of the ADAM optimizer using cross entropy error. After that, we checked
the accuracy of the classifier on both the training and testing data sets. The accuracy is the number of times the predicted class was equal to the correct class divided by the size of the data set. Then, we added to each of the parameters of the neural net independent Laplace noise with mean zero and exponential decay equal to the reciprocal of the term in Theorem 2. Again, we computed the accuracy of the new classifier on both the training and testing data sets. Finally, we computed the average of the results over the 10 repetitions along with the average size of the training data set (equivalently, the average of the stopping time). The results are shown in Tables I and II.

From Tables I and II as $\epsilon$ increases the differences between the private and non-private accuracy values decrease. That is because the exponential decay value of the Laplace noise added to the parameters decreases as $\epsilon$ increases. However, the level of privacy decreases as $\epsilon$ increases. Therefore, these tables show the trade-off between the performance and the privacy level. The stopping time is not related to $\epsilon$ and its variation between the different columns in both tables is just due the inherent randomness of the stopping time.

VI. CONCLUSIONS

In this work, we studied the differential privacy of sequential statistical inference and learning algorithms that are characterized by random termination. First, we demonstrated that for sequential probability ratio test and sequential empirical risk minimization, the number of steps executed before termination can leak information about the input data. Furthermore, we showed that it is impossible to design strictly differentially private versions of the algorithms. Thus, we proposed a weaker notion of differential privacy and proved that they are approximately equivalent when the inputs are a large number of i.i.d. samples. Then, we designed weakly differentially private versions of both the sequential probability ratio test and sequential empirical risk minimization algorithms with proper performance guarantees. Finally, we showed that the performance loss of the weakly differential privacy algorithms are reasonably small on the Breast Cancer Wisconsin (Diagnostic) and Landsat Satellite Data Sets.

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