Semi-implicit Euler schemes for ordinary
differential inclusions

Janosch Rieger
Institut für Mathematik, Universität Frankfurt
Postfach 111932, D-60054 Frankfurt a.M., Germany
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Abstract
Two semi-implicit Euler schemes for differential inclusions are pro-
posed and analyzed in depth. An error analysis shows that both semi-
implicit schemes inherit favorable stability properties from the differential
inclusion. Their performance is considerably better than that of the im-
plicit Euler scheme, because instead of implicit inclusions only implicit
equations have to be solved for computing their images. In addition, they
are more robust with respect to spatial discretization than the implicit
Euler scheme.

Key words. Semi-implicit Euler schemes, differential inclusions, initial value
problems
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1 Introduction
Consider the ordinary differential inclusion
\[ \dot{x}(t) \in F(t, x(t)), \quad t \in [0, T], \quad x(0) = x_0 \in \mathbb{R}^d. \]  
A solution of \( \mathbb{P} \) is an absolutely continuous function \( x : [0, T] \to \mathbb{R}^d \) which satisfies the inclusion almost everywhere. An approximation of the set \( S([0, T], x_0) \) of all solutions of \( \mathbb{P} \) is possible, but often too complex to be of practical value. For many problems, however, it suffices to compute the reachable sets
\[ \mathcal{R}(t, x_0) = \{ x(t) : x(\cdot) \in S([0, T], x_0) \} \]
for \( t \in [0, T] \), which are the sets of all states that can be reached at time \( t \) by an arbitrary solution starting from \( x_0 \) at time zero. Basic properties of the reachable set regarded as a set-valued mapping depending on initial value and end time are well-known (see \( \mathbb{I} \)).
One of the first attempts to approximate the set of solutions and the reachable set of (1) by an Euler-like scheme was presented in [9]. A broad overview over classical literature with a focus on Runge-Kutta schemes is presented in [12]. The numerical method proposed in [10] only uses extremal points of the right-hand side, so that a fully discretized scheme is obtained for right-hand sides with finitely many extremal points. A detailed analysis of spatial discretization effects is presented in [5]. Recently, error estimates for the set-valued Euler scheme were given for differential inclusions with state constraints (see [2]) and non-convex differential inclusions (see [14]). First numerical methods for the approximation of solution sets of elliptic partial differential inclusions have been proposed in [3] and [13].

A set-valued implicit Euler scheme has been analyzed in [6]. It has very good analytical properties, and it is based on an implicit function theorem that is given in [4]. If applied to stiff differential inclusions, it is considerably more efficient than the explicit Euler scheme, because it senses the correct asymptotic behavior, while the reachable sets of the explicit Euler scheme do not only oscillate, but grow exponentially in diameter if the temporal step-size is not small enough. The construction of the spatial discretization of the implicit Euler scheme, however, requires explicit knowledge of the one-sided Lipschitz constant and the modulus of continuity of the right-hand side and is very sensitive to ill-estimated constants. This is the main motivation for the development of the semi-implicit Euler schemes (4) and (20a, 20b) analyzed in the present paper. In addition, their performance is considerably better than that of the implicit Euler scheme, because instead of implicit inclusions only implicit equations have to be solved for computing their images.

Both semi-implicit schemes are not Runge-Kutta schemes in the sense of [12].

## 2 Preliminaries

Let the power set and the spaces of nonempty compact and nonempty convex and compact subsets of \( \mathbb{R}^d \) be denoted by \( \mathcal{P}(\mathbb{R}^d) \), \( \mathcal{C}(\mathbb{R}^d) \) and \( \mathcal{CC}(\mathbb{R}^d) \). The Euclidean norm is denoted by \( | \cdot | \), while \( |A| := \sup_{a \in A} |a| \) denotes the maximal norm of the elements of a set \( A \in \mathcal{C}(\mathbb{R}^d) \). For \( A, B \in \mathcal{C}(\mathbb{R}^d) \), the one-sided and the symmetric Hausdorff distance are given by

\[
\text{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} |a - b|
\]

\[
\text{dist}_H(A, B) := \max\{\text{dist}(A, B), \text{dist}(B, A)\}.
\]

For \( A \in \mathcal{C}(\mathbb{R}^d) \) and \( r > 0 \) denote \( B_r(A) := \{x \in \mathbb{R}^d : \text{dist}(x, A) \leq r\} \). If \( A = \{x\} \) is a point, \( B_r(x) \) is simply the closed ball with radius \( r \) centered at \( x \).

For a set-valued mapping \( F : \mathbb{R}^d \to \mathcal{C}(\mathbb{R}^d) \) and \( A \subset \mathbb{R}^d \) define \( F(A) := \bigcup_{a \in A} F(a) \). Thus the composition \( F \circ G \) of two set-valued mappings \( F \) and \( G \) is given by \((F \circ G)(x) := \bigcup_{y \in G(x)} F(y)\), and \( F^k(x) := (F \circ \ldots \circ F)(x) \) is defined by induction. The projection \( \text{Proj} : \mathbb{R}^d \times \mathcal{C}(\mathbb{R}^d) \to \mathcal{C}(\mathbb{R}^d) \) is defined by
Proj(\(x, A\)) := \{a \in A : |x - a| \leq |x - a'| \text{ for all } a' \in A\}. If \(A \in CC(\mathbb{R}^d)\), then Proj(\(x, A\)) is a singleton.

A set-valued mapping \(F : \mathbb{R}^d \to C(\mathbb{R}^d)\) is called continuous if it is continuous w.r.t. the Euclidean metric and the Hausdorff distance and \(L\)-Lipschitz continuous if
\[
\text{dist}_H(F(x), F(x')) \leq L|x - x'| \text{ for all } x, x' \in \mathbb{R}^d.
\]

The notion of relaxed one-sided Lipschitz set-valued mappings generalizes the concepts of Lipschitz continuity and the (strong) one-sided Lipschitz property. A detailed analysis of this property can be found in [7] and several other works of this author.

**Definition 1.** A mapping \(F : \mathbb{R}^m \to CC(\mathbb{R}^d)\) is called relaxed one-sided Lipschitz with constant \(l \in \mathbb{R}\) (or \(l\)-ROSL) if for every \(x, x' \in \mathbb{R}^d\) and \(y \in F(x)\) there exists some \(y' \in F(x')\) such that
\[
\langle y - y', x - x' \rangle \leq l|x - x'|^2.
\]

A single-valued function \(f : \mathbb{R}^d \to \mathbb{R}^d\) is \(l\)-ROSL if and only if it is one-sided Lipschitz with constant \(l\) (or \(l\)-OSL) in the classical sense, i.e. if it satisfies
\[
\langle f(x) - f(x'), x - x' \rangle \leq l|x - x'|^2
\]
for all \(x, x' \in \mathbb{R}^d\).

**Lemma 2.** If \((\Omega, \mathcal{A}, \mu)\) is a measurable space with \(\mu(\Omega) = 1\) and \(g : \Omega \to \mathbb{R}^d\) is a \(\mu\)-integrable function, then for every \(A \in CC(\mathbb{R}^d)\), the estimate
\[
\text{dist}(\int_{\Omega} g(\omega)d\mu(\omega), A) \leq \int_{\Omega} \text{dist}(g(\omega), A)d\mu(\omega)
\]
holds.

**Proof.** Apply Jensen’s inequality to the convex and Lipschitz continuous function \(\text{dist}(\cdot, A) : \mathbb{R}^d \to \mathbb{R}\).

Throughout this paper, it will be assumed that there exists a splitting
\[
F(t, x) = f(t, x) + M(t, x),
\]
where \(f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d\) and \(M : [0, T] \times \mathbb{R}^d \to CC(\mathbb{R}^d)\) satisfy the following properties.

\(A1\) The function \((t, x) \mapsto f(t, x)\) is continuous and there exists a continuous integrable function \(l_f : [0, T] \to \mathbb{R}\) such that the mapping \(x \mapsto f(t, x)\) is OSL with constant \(l_f(t)\) for any \(t\).

\(A2\) The mapping \((t, x) \mapsto M(t, x)\) is continuous w.r.t. the Hausdorff metric and there exists a continuous integrable function \(L_M : [0, T] \to \mathbb{R}_+\) such that the multimap \(x \mapsto M(t, x)\) is Lipschitz with constant \(L_M(t)\) for every \(t\).
As a consequence, the set-valued mapping $F$ is jointly continuous, and $x \mapsto F(t, x)$ is $l(t)$-ROSL with $l(t) = l_f(t) + L_M(t)$.

In view of the following remark, the above assumptions are not unreasonable.

**Remark 3.** A control system with affine linear controls has the form

$$
\dot{x}(t) = f(t, x(t)) + A(t, x(t))u(t), \quad u(t) \in U
$$

with a function $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$, a matrix-valued mapping $A : [0, T] \times \mathbb{R}^d \to \mathbb{R}^{d \times m}$ and a convex and compact control set $U \subset \mathbb{R}^m$. If $f$ satisfies A1) and $A$ is continuous and $L_A$-Lipschitz in the second argument, then the right-hand side of the corresponding differential inclusion

$$
\dot{x}(t) \in f(t, x(t)) + A(t, x(t))U
$$

exhibits a splitting of type [3], where $M(t, x) := A(t, x)U$ is jointly continuous and $L_A\|U\|$-Lipschitz (and hence $L_A\|U\|$-ROSL) in the second argument. In [12, Theorem 4.1], Runge-Kutta schemes are analyzed in this setting. The results there focus on the order of convergence of these schemes and not on stiffness and asymptotic behavior.

Differential equations with uncertainty are a special case of the above control system with $m = d$, $A(t, x) = r(t, x) \cdot \text{id}$, where $r : [0, T] \times \mathbb{R}^d \to \mathbb{R}^+$ is a non-negative real-valued function, and $U = B_1(0)$.

A set-valued semi-implicit Euler scheme has been considered, but not thoroughly analyzed in [11], where attractors of the partial differential inclusion

$$
\frac{d}{dt} u^{(N)}(t) \in \Delta u^{(N)} + F^{(N)}(u^{(N)})
$$

with Dirichlet boundary condition and multivalued nonlinearity $F$ and its Galerkin approximations

$$
\frac{d}{dt} u^{(N)}_n(t) \in \Delta^{(N)} u^{(N)}_n(t) + F(N)(u^{(N)}_n(t))
$$

are investigated. As all eigenvalues of the discretized Laplace operator $\Delta^{(N)}$ are negative, the Galerkin differential inclusion exhibits a splitting of the right-hand side into a OSL component $\Delta^{(N)} u^{(N)}_n(t)$ with negative Lipschitz constant and a Lipschitz continuous nonlinearity $F^{(N)}(u^{(N)}_n(t))$. Hence the semi-implicit Euler scheme

$$
u^{(N)}_{n+1} \in v^{(N)}_n + h\Delta^{(N)} u^{(N)}_{n+1} + hF^{(N)}(u^{(N)}_n(t))
$$

is solvable for any step-size $h > 0$ with solution

$$
u^{(N)}_{n+1} \in (I - h\Delta^{(N)})^{-1}[v^{(N)}_n + hF^{(N)}(u^{(N)}_n(t))].$$
3 The parameterized semi-implicit Euler scheme

In this section, the parameterized version of the semi-implicit Euler scheme will be analyzed. One step of this method is a multivalued mapping \( \Phi: D \to \mathcal{P}(\mathbb{R}^d) \) given by

\[
\Phi(t, x; h) = \{ z \in \mathbb{R}^d : z \in x + hf(t + h, z) + hM(t, x) \},
\]

where \( D \subset [0, T] \times \mathbb{R}^d \times \mathbb{R}_+ \) is a suitable domain of definition (see Section 3.1). The scheme \( \Phi \) is implicit in the single-valued part \( f \) of \( F \) and explicit in the set-valued component \( M \). It is called the parameterized semi-implicit Euler scheme, because its solution set can be parameterized over the set \( M(t, x) \) as shown in Lemma 9.

Approximations of the reachable sets \( \mathcal{R}(t, x_0) \), \( t \in [0, T] \), of (1) are obtained by iterating the scheme \( \Phi \) on a temporal grid. Define \( \Delta h = \{ t_0, t_1, \ldots, t_N \} \), where \( N \in \mathbb{N}, 0 = t_0 < t_1 < \ldots < t_N = T, h_n := t_{n+1} - t_n \) for \( n = 0, \ldots, N - 1 \) and \( h := (h_0, \ldots, h_{N-1}) \). Throughout this paper, the following condition will be imposed on \( \Delta h \).

A3) For any \( n \in \{0, \ldots, N-1\} \), the step-size \( h_n \) satisfies \( l_f(t_{n+1})h_n < 1 \).

In view of Theorem 7, this assumption guarantees that the iterates of the numerical scheme are well-defined.

**Definition 4.** A sequence \( \{y_n\}_{n=0}^N \subset \mathbb{R}^d \) is called a trajectory of the parameterized semi-implicit Euler scheme \( \Phi \) associated with (1) if

\[
y_{n+1} \in \Phi(t_n, y_n; h_n) \text{ for } n = 0, \ldots, N - 1, \quad y_0 = x_0.
\]

The set of all such trajectories is denoted by \( \mathcal{S}_{\Phi}(\Delta h, x_0) \).

As in continuous time, the reachable set \( \mathcal{R}_\Phi(t_n, x_0) \) is the set of all states that can be reached at time \( t_n \) by a discrete trajectory starting from \( x_0 \) at time zero.

**Definition 5.** For any \( t_n \in \Delta h \), the reachable set \( \mathcal{R}_\Phi \) of the parameterized semi-implicit Euler scheme \( \Phi \) is given by

\[
\mathcal{R}_\Phi(t_n, x_0) = \{ y_n : \{y_k\}_{k=0}^N \in \mathcal{S}_{\Phi}(\Delta h, x_0) \}.
\]

According to Theorem 7, the reachable set \( \mathcal{R}_\Phi(t_n, x_0) \) depends continuously on \((h, x_0)\) and is Lipschitz continuous in the initial value, because it is a composition of multimaps with this property.

The properties of \( \Phi \) regarded as a set-valued mapping depending on time, space, and step-size are investigated in Section 3.1, while Section 3.2 deals with the dynamics and convergence of the solution sets. In Section 3.3, the spatial discretization of the parameterized semi-implicit Euler scheme is analyzed, which is much easier and more robust than that of the fully implicit Euler scheme presented in [1]. All estimates in Sections 3.2 and 3.3 are given in terms of the solution sets, because these imply the same estimates for the reachable sets.
3.1 Properties of the scheme as a multivalued mapping

Define \( D := \{(t, x, h) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}_+ : l_f(t + h)h < 1\} \).

**Lemma 6.** The set \( D \) is relatively open in \([0, T] \times \mathbb{R}^d \times \mathbb{R}_+\).

**Proof.** Fix \((t, x, h) \in D\). Since \( l_f(t + h)h < 1\) and \((t', x', h') \rightarrow l_f(t' + h')h'\) is continuous, there exists some \( R > 0\) such that \( l_f(t' + h')h' < 1\) for all \((t', x', h') \in B_R(t, x, h) \cap ([0, T] \times \mathbb{R}^d \times \mathbb{R}_+)\), which is a relatively open neighborhood of \((t, x, h)\). \(\square\)

The set-valued mapping \( G : \mathbb{R}^d \rightarrow \mathbb{C}(\mathbb{R}^d) \) defined by
\[
G_{t,x,h}(z) := x + hf(t, h, z) - z + hM(t, x)
\]
is continuous and ROSL with constant \( l_{G_{t,x,h}} := -(1 - l_f(t + h)h) \). The parameterized semi-implicit Euler scheme can be represented as
\[
\Phi(t, x; h) = \{ z \in \mathbb{R}^d : z \in x + hf(t, h, z) + hM(t, x) \} = S_{G_{t,x,h}}(0),
\]
where \( S_{G_{t,x,h}}(0) := \{ z \in \mathbb{R}^d : 0 \in G_{t,x,h}(z) \} \).

**Theorem 7.** For any \((t, x, h) \in D\), the set \( \Phi(t, x; h) \) is nonempty and compact for each \( x \in \mathbb{R}^d \) and satisfies
\[
diam \Phi(t, x; h) \leq \frac{h}{1 - l_f(t + h)h} \sup_{\xi \in \Phi(t, x; h)} diam M(t, \xi). \tag{7}
\]

For any \( y \in \mathbb{R}^d \), the scheme satisfies the estimate
\[
dist(y, \Phi(t, x; h)) \leq \frac{1}{1 - l_f(t + h)h} \dist(y, x + hf(t, h, y) + hM(t, x)), \tag{8}
\]
\[
dist(\Phi(t, x; h), y) \leq \frac{1}{1 - l_f(t + h)h} \dist(x + hf(t, h, y) + hM(t, x), y). \tag{9}
\]

**Proof.** By Theorem 3.1 \( \Phi(t, x; h) = S_{G_{t,x,h}}(0) \) is nonempty and compact and satisfies (7). Moreover, (8) and (9) are implied by Theorem 3.1 because for any \( y \in \mathbb{R}^d \),
\[
\begin{align*}
\dist(y, \Phi(t, x; h)) &= \dist(y, S_{G_{t,x,h}}(0)) \leq \frac{1}{1 - l_f(t + h)h} \dist(0, G_{t,x,h}(y)) \\
&\leq \frac{1}{1 - l_f(t + h)h} \dist(y, x + hf(t, h, y) + hM(t, x)), \\
\dist(\Phi(t, x; h), y) &= \dist(S_{G_{t,x,h}}(0), y) \leq \frac{1}{1 - l_f(t + h)h} \dist(G_{t,x,h}(y), 0) \\
&\leq \frac{1}{1 - l_f(t + h)h} \dist(x + hf(t, h, y) + hM(t, x), y).
\end{align*}
\]
\(\square\)
Corollary 8. The iterates of the parameterized semi-implicit Euler scheme given by (5) and (6) are well-defined.

Proof. Assumption A3) implies that \((t_n, x, h_n) \in D\) for any \(x \in \mathbb{R}^d\), and hence the defining implicit inclusion (4) is solvable in every step according to Theorem 7.

The following lemma will be useful for the discussion of connectedness and convexity of the images of \(\Phi\).

Lemma 9. For fixed \((t, x, h) \in D\), the image of the parameterized semi-implicit Euler scheme can be represented as

\[
\Phi(t, x; h) = \bigcup_{m \in M(t, x)} z(m),
\]

where \(z(m)\) is the unique solution \(z\) of the implicit equation

\[
z = x + hf(t + h, z) + hm.
\]

Moreover, the mapping \(z : M(t, x) \to \mathbb{R}^d\) is \(h^{-1}l_f(t + h; n)\)-Lipschitz.

Proof. Theorem 31 applied to the equation

\[
0 = x + hf(t + h, z) - z + hm
\]
guarantees the existence of a solution of (11). Assume that (11) has two solutions \(z\) and \(z'\). Then

\[
|z - z'| = h(f(t + h, z) - f(t + h, z'), z - z') \leq l_f(t + h)h|z - z'|^2,
\]

which forces \(|z - z'| = 0\), because \(l_f(t + h)h < 1\).

If \(z, z' \in \Phi(t, x; h)\), then there exist \(m, m' \in M(t, x)\) such that \(z\) and \(z'\) satisfy (11) with \(m\) and \(m'\). Theorem 31 applied to (12) ensures that

\[
|z(m) - z(m')| \\
\leq -l_{G_{t,x,h}}^{-1}|x + hf(t + h, z(m')) - z(m') + hm| \\
\leq -l_{G_{t,x,h}}^{-1}|x + hf(t + h, z(m')) - z(m') + hm'| + l_{G_{t,x,h}}^{-1}h|m - m'| \\
= -l_{G_{t,x,h}}^{-1}h|m - m'|.
\]

Corollary 10. For fixed \((t, x, h) \in D\), the set \(\Phi(t, x; h)\) is path-connected.

Proof. The set \(\Phi(t, x; h) = z(M(t, x))\) is path-connected, because \(M(t, x)\) is path-connected and \(z : M(t, x) \to \mathbb{R}^d\) is continuous.

Corollary 11. If \(d = 1\), then the set \(\Phi(t, x; h)\) is a convex interval.
Proof. Every compact connected subset of $\mathbb{R}^1$ is a convex interval. \hfill $\square$

The images of the parameterized semi-implicit Euler scheme are not necessarily convex for $d > 1$.

**Example 12.** Consider the mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$F(x_1, x_2) = f(x_1, x_2) + M(x_1, x_2) = \frac{1}{2} \left( -x_1 - x_2 \right) + \left( \left[ 0, \frac{11}{2} \right] \right).$$

Because of

$$\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, f(x_1, x_2) \rangle = \frac{1}{2} \langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} -x_1 - x_2 \\ x_1 - x_2 - x_3^2 \end{pmatrix} \rangle = -\frac{1}{2} (x_1^2 + x_2^2 + x_3^2) \leq -\frac{1}{2} (x_1^2 + x_2^2),$$

the mappings $f$ and $F$ are $-\frac{1}{2}$-ROSL. According to Lemma 9, the image $\Phi(0, 0, 1)$ is a Lipschitz continuous curve parameterized over $M(0, 0) = \left( \left[ 0, \frac{11}{2} \right] \right)$ and hence over the interval $\left[ 0, \frac{11}{2} \right]$. Solving (11) for $m_0 = 0$, $m_1 = \frac{43}{16}$ and $m_2 = \frac{13}{2}$ yields $z(m_0) = (0, 0)^T$, $z(m_1) = \left( \frac{11}{2}, \frac{1}{2} \right)^T$ and $z(m_2) = (4, 1)^T$. As these three points are not on a straight line (see Figure 1), the image $\Phi(0, 0, 1)$ is not convex.

**Theorem 13.** The set-valued mapping $(t, x, h) \mapsto \Phi(t, x; h)$ is jointly continuous on $D$, and $x \mapsto \Phi(t, x; h)$ is $\frac{1 + h M(t)}{1 - t_i(t + h)x}$-Lipschitz. Moreover, $\Phi(t, x; 0) = x$ for any $t \in \mathbb{R}$.
Proof. Fix any \((t, x, h) \in D\). By Lemma \[\text{Lemma} \] there exists some \(R > 0\) such that \(D_R := B_R(t, x, h) \cap \([0, T] \times \mathbb{R}^d \times \mathbb{R}_+\) \subset D\) and hence \(D_R = B_R(t, x, h) \cap D\). By Theorem \[\text{Theorem} \]
\[
\|\Phi(t', x'; h')\| \leq \frac{1}{1-lf(t' + h')h'}\|x' + h'f(t' + h', 0) + h'M(t', x')\| \tag{13}
\]
for any \((t', x', h') \in D_R\). Since \(D_R\) is compact and the right-hand side of \[\text{13} \]
is continuous in \((t', x', h')\), there exists some \(C > 0\) such that
\[
\|\Phi(t', x'; h')\| \leq C \text{ for all } (t', x', h') \in D_R,
\]
so that \(\Phi(D_R) \subset B_C(0)\). For any \(z' \in \Phi(t', x'; h')\) with \((t', x', h') \in D_R\),
\[
\text{dist}(z', \Phi(t, x; h)) \leq -l^{-1}_{r', x, h} \text{dist}(0, G_{t, x, h}(z'))
\leq -l^{-1}_{r', x, h} \big( \text{dist}(0, G_{t', x', h'}(z')) + \text{dist}(G_{t', x', h'}(z'), G_{t, x, h}(z')) \big)
\leq -l^{-1}_{r', x, h}(\|x' - x\| + \text{dist}(h'M(t', x'), hM(t, x)))
- l^{-1}_{r', x, h}[h'f(t' + h', z') - hf(t + h, z')] =: \varphi(t', x', h', z')
\]
according to Theorem \[\text{Theorem} \]. The function \(\varphi : D_R \times B_C(0) \to \mathbb{R}_+\) is continuous, and for any \(z' \in B_C(0)\), \(\varphi(t', x', h', z') \to 0\) as \((t', x', h') \to (t, x, h)\). Assume that
\[
\sup_{z' \in B_C(0)} \varphi(t', x', h', z') \to 0 \text{ as } (t', x', h') \to (t, x, h).
\]
Then there exists some \(\varepsilon > 0\) and a sequence \((t'_n, x'_n, h'_n, z'_n)_{n \in \mathbb{N}} \subset D_R \times B_C(0)\), such that \(\varphi(t'_n, x'_n, h'_n, z'_n) > \varepsilon\) and \((t'_n, x'_n, h'_n) \in B_{1/n}(t, x, h)\). As \(B_C(0)\) is compact, there exists a subsequence \((t''_n, x''_n, h''_n, z''_n)_{n \in \mathbb{N}'}, \ \mathbb{N}' \subset \mathbb{N}\), and some \(z_0 \in B_C(0)\) such that \(z''_n \to z_0\) as \(n' \to \infty\). But then
\[
\varphi(t''_n, x''_n, h''_n, z''_n) \to \varphi(t, x, h, z_0) = 0
\]
is a contradiction, so that
\[
\sup_{z' \in B_C(0)} \varphi(t', x', h', z') \to 0 \text{ as } (t', x', h') \to (t, x, h).
\]
Hence
\[
\text{dist}(\Phi(t', x'; h'), \Phi(t, x; h)) = \sup_{z' \in \Phi(t', x'; h')} \text{dist}(z', \Phi(t, x; h)) \to 0.
\]
The proof of the statement \(\text{dist}(\Phi(t, x; h), \Phi(t', x'; h')) \to 0\) is almost identical, because the set \(\Phi(t, x; h)\) is compact according to Theorem \[\text{Theorem} \]. Lipschitz continuity of \(\Phi\) in \(x\) follows from \[\text{14} \] with \(t = t'\) and \(h = h'\).
3.2 Dynamic properties and convergence analysis

The solutions of the differential inclusion and the parameterized semi-implicit Euler scheme are uniformly bounded.

**Lemma 14.** There exists some constant $C > 0$ such that every solution $x(t)$ of the differential inclusion and every trajectory $\{y_n\}_n$ of the parameterized semi-implicit Euler scheme with $y_0 = x_0$ are contained in $B_C(0) \subset \mathbb{R}^d$.

**Proof.** Consider an arbitrary solution $x(t)$ of (1). By the ROSL property, for every $t \in [0, T]$ there exists an element $v(t) \in F(t, 0)$ such that

$$\langle \dot{x}(t) - v(t), x(t) \rangle \leq l(t)|x(t)|^2,$$

and hence

$$|x(t)| \frac{d}{dt} |x(t)| = \langle \dot{x}(t) - v(t), x(t) \rangle + \langle v(t), x(t) \rangle \leq l(t)|x(t)|^2 + |v(t)||x(t)|,$$

so that

$$\frac{d}{dt} |x(t)| \leq l(t)|x(t)| + |v(t)| \leq l(t)|x(t)| + \|F(t, 0)\|$$

whenever $x(t) \neq 0$. Applying the Gronwall Lemma on every subinterval $[t_k, t_{k+1}] \subset [0, T]$ with $|x(t)| \geq 1$ for any $t \in [t_k, t_{k+1}]$ yields the desired bound.

Boundedness of the discrete trajectories is shown by induction. By assumption, $y_0 = x_0 \in B_{l_0}(0)$. Assume that there exists some $R_k > 0$ such that $y_k \in B_{R_k}(0)$ for all discrete trajectories $\{y_n\}_n$. Then estimate 10 from Theorem 9 yields that

$$|y_{k+1}| \leq \|\Phi(t_k, y_k; h_k)\| \leq \frac{1}{1 - l_f(t_{k+1})h_k} |y_k + h_k f(t_{k+1}, 0) + h_k M(t_k, y_k)|$$

$$\leq \frac{1}{1 - l_f(t_{k+1})h_k} \sup_{y \in B_{R_k}(0)} \|y + h_k f(t_{k+1}, 0) + h_k M(t_k, y)\| =: R_{k+1} < \infty,$$

because $B_{R_k}(0)$ is compact and the continuous function $y \rightarrow y + h_k M(t_k, y)$ is bounded on $B_{R_k}(0)$. An iteration of this argument shows that any trajectory $\{y_n\}_n$ is contained in $B_R(0)$ with $R := \max_{k=0, \ldots, N} R_k$.

As a consequence, the ODI (1) and the parameterized semi-implicit Euler scheme can be considered on the compact set $[0, T] \times B_C(0)$, where $f$ and $M$ are uniformly continuous so that the moduli of continuity $\tau_f, \chi_f, \tau_M : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ given by

$$\tau_f(\delta) := \sup\{|f(t, x) - f(t', x)| : t, t' \in [0, T], |t - t'| \leq \delta, x \in B_C(0)\},$$

$$\chi_f(\delta) := \sup\{|f(t, x) - f(t, x')| : t \in [0, T], x, x' \in B_C(0), |x - x'| \leq \delta\},$$

$$\tau_M(\delta) := \sup\{\text{dist}(M(t, x), M(t', x)) : t, t' \in [0, T], |t - t'| \leq \delta, x \in B_C(0)\}$$

are well-defined. The error term

$$\Gamma(h, t) := \tau_f(h) + \chi_f(Ph) + \tau_M(h) + L_M(h)Ph$$

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with \( P := \max_{x \in B_C(0)} |f(x)| + \max_{x \in B_C(0)} \|M(x)\| \) will appear frequently in the following discussion. Note that
\[
\|\Gamma(h, \cdot)\|_\infty \to 0 \text{ as } h \to 0, \tag{15}
\]
because \( L_M \) is bounded.

Proposition 15 is an existence and stability result for solutions of the semi-implicit Euler scheme (4) and therefore a discrete counterpart of Theorem 32.

**Proposition 15.** For any sequence \( \{x_n\}_n \subset \mathbb{R}^d \), there exists a solution \( \{y_n\}_n \) of the parameterized semi-implicit Euler scheme (4) satisfying
\[
|x_n - y_n| \leq e^{\sum_{k=0}^{n-1} \frac{1}{1 - \frac{1}{l_f(t_{k+1})h_k}} L_M(t_k)h_k} |x_0 - y_0| + n-1 \sum_{k=0}^{n-1} e^{\sum_{j=k}^{n-1} \frac{1}{1 - \frac{1}{l_f(t_{j+1})h_{j+1}} + \sum_{j=k+1}^{n-1} L_M(t_j)h_j} h_k g_k, \tag{16}
\]
where \( g_n := \text{dist}(\frac{1}{h_n} (x_{n+1} - x_n), f(t_{n+1}, x_{n+1}) + M(t_n, x_n)) \).

**Proof.** Let the sequence \( \{x_n\}_{n=0}^N \) be given, and assume that a trajectory \( \{y_n\}_{n=0}^{\tilde{n}} \) of (4) has already been constructed for some \( \tilde{n} < N \). By (8), there exists some \( y_{\tilde{n}+1} \in \Phi(t_{\tilde{n}}, y_{\tilde{n}}; h) \) such that
\[
|y_{\tilde{n}+1} - y_{\tilde{n}+1}| \\
\leq \frac{1}{1 - l_f(t_{\tilde{n}+1})h_{\tilde{n}}} \text{dist}(x_{\tilde{n}+1}, y_{\tilde{n}} + h_{\tilde{n}}f(t_{\tilde{n}+1}, x_{\tilde{n}+1}) + h_{\tilde{n}}M(t_{\tilde{n}}, y_{\tilde{n}})) \\
= \frac{1}{1 - l_f(t_{\tilde{n}+1})h_{\tilde{n}}} \text{dist}(x_{\tilde{n}+1} - x_{\tilde{n}} + x_{\tilde{n}} - y_{\tilde{n}} + h_{\tilde{n}}f(t_{\tilde{n}+1}, x_{\tilde{n}+1}) + h_{\tilde{n}}M(t_{\tilde{n}}, y_{\tilde{n}})) \\
\leq \frac{1}{1 - l_f(t_{\tilde{n}+1})h_{\tilde{n}}} |x_{\tilde{n}} - y_{\tilde{n}}| + \frac{h_{\tilde{n}}}{1 - l_f(t_{\tilde{n}+1})h_{\tilde{n}}} \text{dist}(\frac{1}{h_{\tilde{n}}} (x_{\tilde{n}+1} - x_{\tilde{n}}), f(t_{\tilde{n}+1}, x_{\tilde{n}+1}) + M(t_{\tilde{n}}, x_{\tilde{n}})) \\
+ \frac{h_{\tilde{n}}}{1 - l_f(t_{\tilde{n}+1})h_{\tilde{n}}} \text{dist}(M(t_{\tilde{n}}, x_{\tilde{n}}), M(t_{\tilde{n}}, y_{\tilde{n}})) \\
\leq \frac{1 + L_M(t_{\tilde{n}})h_{\tilde{n}}}{1 - l_f(t_{\tilde{n}+1})h_{\tilde{n}}} |x_{\tilde{n}} - y_{\tilde{n}}| + \frac{h_{\tilde{n}}}{1 - l_f(t_{\tilde{n}+1})h_{\tilde{n}}} g_{\tilde{n}},
\]
so that by induction
\[
|x_n - y_n| \\
\leq \prod_{k=0}^{n-1} \frac{1 + L_M(t_{k})h_{k}}{1 - l_f(t_{k+1})h_{k}} |x_0 - y_0| + \sum_{k=0}^{n-1} \left( \prod_{j=k+1}^{n-1} \frac{1 + L_M(t_{j})h_{j}}{1 - l_f(t_{j+1})h_{j}} \right) \frac{h_k}{1 - l_f(t_{k+1})h_{k}} g_k
\]
for all \( n \in \{0, \ldots, N\} \). The well-known formulas \( \frac{1}{1 - lh} = 1 + \frac{lh}{1 - lh} \) for \( lh < 1 \) and \( 1 + t \leq e^t \) for \( t \in \mathbb{R} \) yield the desired statement. \( \square \)
One-sided error estimates are a simple consequence of Proposition 15.

**Corollary 16.** For every solution $x(\cdot)$ of the differential inclusion (1), there exists a trajectory $\{y_n\}_n$ of the semi-implicit Euler scheme (4) such that

$$|x(t_n) - y_n| \leq \sum_{k=0}^{n-1} e^{\int_{t_k}^{t_{k+1}} \frac{1}{2} h_k \Gamma(h_k,t_k)} + \sum_{j=k+1}^{n-1} L_{M(t_j)} h_j \Gamma(h_k,t_k)$$

(17)

for $n = 0, \ldots, N$.

**Proof.** The statement follows from Proposition 15 with $x(t_0) = x_0 = y_0$ and

$$g_n = \text{dist}(\frac{1}{h_n} (x(t_{n+1}) - x(t_n)), f(t_{n+1}, x(t_{n+1})) + M(t_n, x(t_n)))$$

$$= \text{dist}(\frac{1}{h_n} \int_{t_n}^{t_{n+1}} \dot{x}(t) dt, f(t_{n+1}, x(t_{n+1})) + M(t_n, x(t_n)))$$

$$\leq \frac{1}{h_n} \int_{t_n}^{t_{n+1}} \text{dist}(\dot{x}(t) dt, f(t_{n+1}, x(t_{n+1})) + M(t_n, x(t_n))) dt$$

$$\leq \frac{1}{h_n} \int_{t_n}^{t_{n+1}} \text{dist}(f(t, x(t)) + M(t, x(t)), f(t_{n+1}, x(t_{n+1})) + M(t_n, x(t_n))) dt$$

$$\leq \Gamma(h_n, t_n),$$

because

$$|x(t) - x(t')| \leq \int_t^{t'} |\dot{x}(s)| ds \leq \Phi h_n \text{ for all } t, t' \in [t_n, t_{n+1}].$$

The estimate for the other semi-distance follows from the generalized Filippov Theorem 32.

**Corollary 17.** For every trajectory $\{y_n\}_n$ of the semi-implicit Euler scheme (4), there exists a solution $x(\cdot)$ of the ordinary differential inclusion (1) such that

$$|x(t_n) - y_n| \leq \int_{t_n}^{t_{n+1}} e^{\int_{t_n}^{t} L_M(s) ds} \Gamma(||h||, t) dt$$

(18)

for $n = 0, \ldots, N$.

**Proof.** Consider a trajectory $\{y_n\}_n$ of the parameterized semi-implicit Euler scheme and its linear interpolation

$$y(t) = y_n + (t - t_n) f(t_{n+1}, y_{n+1}) + (t - t_n) m, \quad t \in [t_n, t_{n+1}],$$

where $m \in M(t_n, y_n)$. Then

$$g(t) = \text{dist}(y(t), F(t, y(t)))$$

$$= \text{dist}(f(t_{n+1}, y_{n+1}) + m, f(t, y(t)) + M(t, y(t)))$$

$$\leq |f(t_{n+1}, y_{n+1}) - f(t, y(t))| + \text{dist}(M(t_n, y_n), M(t, y(t)))$$

$$\leq \Gamma(h_n, t)$$
for all $t \in [t_n, t_{n+1}]$. As the mapping $x \mapsto F(t, x)$ is $(l_f(t) + L_M(t))$-ROSL, Theorem 32 implies piecewise and thus globally the existence of a solution $x(\cdot)$ of the differential inclusion (1) satisfying (18).

Remark 18. a) In view of (15) and Corollaries 16 and 17,

$$\text{dist}_H(S([0, T], x_0), S_{\Phi}(\Delta h, x_0)) \to 0 \text{ as } |h|_\infty \to 0,$$

which implies convergence

$$\text{dist}_H(R(T, x_0), R_{\Phi}(T, x_0)) \to 0 \text{ as } |h|_\infty \to 0.$$

Moreover, error estimates (16) and (18) allow to exploit negative ROSL-constants of the single-valued part $f$, but not of $M$, which is due to the semi-implicit construction of the scheme.

b) If $l_f(\cdot) \equiv l_f$, $L_M(\cdot) \equiv L_M$, and $h_n \equiv h$ are constant and $(t, x) \mapsto f(t, x)$ and $(t, x) \mapsto M(t, x)$ are $L$-Lipschitz, then estimate (17) simplifies to

$$|x(t_n) - y_n| \leq 2L(1 + P)he^{(l_f + L_M)t_n}$$

and (18) becomes

$$|x(t_n) - y_n| \leq \frac{2L(1 + P)h}{l_f + L_M}(e^{(l_f + L_M)t_n} - 1)$$

for every $n \in \{0, \ldots, N\}$.

3.3 Spatial discretization

The parameterized semi-implicit Euler scheme analyzed in Sections 3.1 and 3.2 is discrete in time, but not in space. For a practical implementation, it is inevitable to introduce a spatial discretization, which is simple for explicit numerical schemes (see [5]) but causes problems in the case of the fully implicit Euler scheme (see [6]). The aim of this Section is to show that the parameterized semi-implicit Euler scheme can be discretized in a straight-forward way without losing its favorable properties.

To this end, consider the grid $\Delta_\rho := \rho \mathbb{Z}^d \subset \mathbb{R}^d$ for some $\rho > 0$. The projection $P_\rho : \mathcal{P}(\mathbb{R}^d) \to \mathcal{P}(\Delta_\rho)$ from the subsets of $\mathbb{R}^d$ to the subsets of $\Delta_\rho$ is defined by $P_\rho(A) := B_{\Delta_\rho}(A) \cap \Delta_\rho$.

Lemma 19. If $A \subset \mathbb{R}^d$ is nonempty, then

$$\text{dist}_H(A, P_\rho(A)) \leq \frac{\sqrt{d}}{2} \rho.$$

In particular, the set $P_\rho(A)$ is nonempty.
Proof. Let \( \text{round} : \mathbb{R} \to \mathbb{Z} \) be the usual rounding function. If \( x \in A \), then the element \( x^\rho \in \Delta_\rho \) specified by

\[
x_n^\rho := \rho \cdot \text{round}(\frac{x_n}{\rho}), \quad n = 1, \ldots, d,
\]
satisfies

\[
|x - x^\rho| \leq \sqrt{(\frac{\rho}{2})^2 + \ldots + (\frac{\rho}{2})^2} = \frac{\sqrt{d}}{2} \rho,
\]
so that \( x^\rho \in B_{\frac{\sqrt{d}}{2} \rho}(A) \cap \Delta_\rho = P_\rho(A) \). As \( x \in A \) was arbitrary,

\[
\text{dist}(A, P_\rho(A)) \leq \frac{\sqrt{d}}{2} \rho.
\]

Since \( P_\rho(A) \subset B_{\frac{\sqrt{d}}{2} \rho}(A) \), it is clear that \( \text{dist}(P_\rho(A), A) \leq \frac{\sqrt{d}}{2} \rho \).

According to Lemma [2] the image \( \Phi(t, x; h) \) of the parameterized semi-implicit Euler scheme is a union of solutions of implicit equations parameterized over \( M(t, x) \). It is therefore natural to discretize the parameter set \( M(t, x) \), solve the corresponding implicit equations, and map the results to the spatial grid, which leads to the definition

\[
\hat{\Phi}(t, x; h, \rho, \varepsilon) := P_\rho(\{z \in x + hf(t + h, z) + hP_\varepsilon(M(t, x))\}),
\]
where \( \rho > 0 \) and \( \varepsilon > 0 \) are the mesh sizes of grids in state and velocity space. Set \( \rho := (\rho_0, \ldots, \rho_N) \) and \( \varepsilon := (\varepsilon_0, \ldots, \varepsilon_{N-1}) \).

**Definition 20.** A sequence \( \{\hat{y}_n\}_{n=0}^N \subset \mathbb{R}^d \) is called a trajectory of the fully discretized parameterized semi-implicit Euler scheme \( \hat{\Phi} \) associated with (1) if

\[
\hat{y}_{n+1} \in \hat{\Phi}(t_n, \hat{y}_n; h_n, \rho_{n+1}, \varepsilon_n) \quad \text{for} \quad n = 0, \ldots, N-1, \quad \hat{y}_0 \in P_{\rho_0}(\{x_0\}).
\]

The set of all such trajectories is denoted \( S_\hat{\Phi}(\Delta_h, x_0) \). Its dependence on \( \rho \) and \( \varepsilon \) is suppressed for the sake of readability.

**Definition 21.** For any \( t_n \in \Delta_h \), the reachable set \( \mathcal{R}_\hat{\Phi}(t_n, x_0) \) of the fully discretized semi-implicit Euler scheme is given by

\[
\mathcal{R}_\hat{\Phi}(t_n, x_0) := \{\hat{y}_n \in \mathbb{R}^d : \{\hat{y}_n\}_{n=0}^N \in S_\hat{\Phi}(\Delta_h, x_0)\}.
\]

**Proposition 22.** For every trajectory \( \{y_n\}_{n=0}^N \subset S_\Phi(\Delta_h, x_0) \), there exists a trajectory \( \{\hat{y}_n\}_{n=0}^N \in S_\hat{\Phi}(\Delta_h, x_0) \) such that

\[
|y_n - \hat{y}_n| \leq \frac{\sqrt{d}}{2} \rho_0 \prod_{k=0}^{n-1} \frac{1 + L_M(t_k)h_k}{1 - l_f(t_k+1)h_k} + \frac{\sqrt{d}}{2} \sum_{k=0}^{n-1} \left( \prod_{j=k+1}^{n-1} \frac{1 + L_M(t_j)h_j}{1 - l_f(t_{j+1})h_j} \right) (\rho_{k+1} + \frac{\varepsilon_{k+1}h_k}{2}) \tag{19}
\]

for all \( n = 0, \ldots, N \), and for every trajectory \( \{\hat{y}_n\}_{n=0}^N \in S_\hat{\Phi}(\Delta_h, x_0) \), there exists a trajectory \( \{y_n\}_{n=0}^N \in S_\Phi(\Delta_h, x_0) \) satisfying estimate (19).
Proof. The statement of the proposition follows by induction. Let \( \{y_n\}_{n=0}^{N} \in S_{\Phi}(\Delta_n) \) be an arbitrary trajectory. For \( n = 0 \), take \( y_0 \in P_{\rho_0}(y_0) \) such that

\[
|y_0 - \hat{y}_0| = \text{dist}(\{x_0\}, P_{\rho_0}(\{x_0\})) \leq \frac{\sqrt{d}}{2} \rho_0.
\]

Now assume that a trajectory \( \{\hat{y}_k\}_{k=0}^n \), \( n \in \{0, \ldots, N-1\} \), of (19) has been constructed that satisfies (19). By Theorem 7 there exists some \( y \in \Phi(t_n, \hat{y}_n; h_n) \) with

\[
|y_{n+1} - y| \leq \frac{1+L_M(t_n)h_n}{1-l_f(t_{n+1})h_n}|y_n - \hat{y}_n|.
\]

By definition, there exists some \( m \in M(t_n, \hat{y}_n) \) such that

\[
y \in \hat{y}_n + h_n f(t_{n+1}, y) + h_n m,
\]

and by the properties of the projection \( P_{\epsilon_n} \), there exists a vector \( \hat{m} \in \mathbb{P}_{\epsilon_n}(M(t_n, \hat{y}_n)) \) with \(|m - \hat{m}| \leq \frac{\sqrt{d}}{2} \epsilon_n\). Let \( z \in \mathbb{R}^d \) be the unique solution of the implicit equation

\[z = \hat{y}_n + h_n f(t_{n+1}, z) + h_n \hat{m},\]

and select some arbitrary \( \hat{y}_{n+1} \in P_{\rho_{n+1}}(z) \). Then \( \hat{y}_{n+1} \in \Phi(t_n, \hat{y}_n; h_n, \rho_{n+1}, \epsilon_n) \), and it follows from Lemma 8 that

\[
|y_{n+1} + \hat{y}_{n+1}| \leq |y_{n+1} - y| + |y - z| + |z - \hat{y}_{n+1}|
\]

\[
\leq \frac{1+L_M(t_n)h_n}{1-l_f(t_{n+1})h_n}|y_n - \hat{y}_n| + \frac{\sqrt{d} \epsilon_n h_n}{1-l_f(t_{n+1})h_n} + \frac{\sqrt{d}}{2} \rho_{n+1}
\]

\[\leq \frac{\sqrt{d}}{2} \rho_0 \prod_{k=0}^{n} \frac{1+L_M(t_k)h_k}{1-l_f(t_{k+1})h_k} + \frac{\sqrt{d}}{2} \sum_{k=0}^{n} \left( \prod_{j=k+1}^{n} \frac{1+L_M(t_j)h_j}{1-l_f(t_{j+1})h_j} \right) (\rho_{k+1} + \frac{\epsilon_n h_k}{1-l_f(t_{k+1})h_k}),\]

so that the sequence \( \{\hat{y}_k\}_{k=0}^{n+1} \) satisfies (19).

The other direction can be shown with the same arguments. \( \square \)

Remark 23. a) In view of Proposition 22

\[
\text{dist}_H(S_{\Phi}(\Delta_n, x_0), S_{\hat{\Phi}}(\Delta_n, x_0)) \to 0
\]

when \( \epsilon \to 0 \) and \( \rho = o(h) \). All error estimates allow to exploit negative OSL-constants of \( f \).

b) If \( l_f(\cdot) \equiv l_f, \ L_M(\cdot) \equiv L_M \), \( h \) and \( \rho \) are constant, then estimate (19) changes to

\[
\text{dist}_H(S_{\Phi}(0, x_0, t_n), S_{\hat{\Phi}}(0, P_{\rho}(\{x_0\}), t_n))
\]

\[
\leq \frac{\sqrt{d}}{2} \rho e^{\int_{t_n}^{t_{n+1}} L_M(t) + \frac{l_f(t)}{h}} + \frac{\sqrt{d}}{2} \frac{\epsilon M L + \frac{l_f(t_n)}{h} - 1}{L_M + \frac{l_f(t_n)}{h}} \left( \frac{\rho + \epsilon}{1-l_f h} \right)
\]

\[15\]
for \( n = 0, \ldots, N \).

The term \( \sqrt{2} \rho \exp\left(\frac{tN}{1-t/h} + L_M t_n\right) \) originates from the projection of the initial value \( x_0 \) to the grid \( \Delta_{\rho} \). If the grid is centered in \( x_0 \) instead of the origin, this error term vanishes. This is impossible if the initial value is in fact an initial set that must be discretized. As discussed in [5], it is reasonable to choose \( \rho = h^2 \) and \( \varepsilon = h \) in order to obtain first-order convergence.

c) An implementation of the fully discretized scheme \( \hat{\Phi} \) is much easier than that of the implicit Euler scheme presented in [6], where explicit knowledge of the one-sided Lipschitz constant and the moduli of continuity of the right-hand side are required not only for the error estimates, but for the practical computations. Underestimation of these parameters can lead to divergence or failure of the implicit Euler scheme, whereas overestimation implies pessimistic estimates and smaller step-sizes than necessary as usual. The semi-implicit scheme \( \hat{\Phi} \) does not have any such flaws.

4 The semi-implicit split scheme

In this section, a simpler version of the semi-implicit Euler scheme will be investigated. One step of this method is the multivalued mapping \( \Psi: D \to P(\mathbb{R}^d) \) given by

\[
\Psi(t, x; h) := z + hM(t, x),
\]

where \( D \) is the domain defined in Section 3.1 and \( z \in \mathbb{R}^d \) is the unique solution of the implicit equation

\[
z = x + hf(t + h, z).
\]

The obvious advantage of this scheme is that only one implicit equation must be solved in every step instead of a whole family of such equations. This is achieved by fully separating the problem of solving implicit equations and treating the set-valued part \( M \).

The discussion of the semi-implicit split scheme will be concise. On one hand, the techniques are similar to those used in Section 3. On the other hand, useful statements about the solution of implicit equation (20b), which is in fact the defining equation of the classical implicit Euler scheme, can be deduced from the results in Section 3 applied to the right-hand side \( F(t, x) = \{ f(t, x) \} \).

By Lemma 9, equation (20b) admits a unique solution \( z \) when \( (t, x, h) \in D \), and by Theorem 13 and Assumption A2), the mapping \( (t, x, h) \mapsto \Psi(t, x; h) \) is jointly continuous and \( x \mapsto \Psi(t, x; h) \) is Lipschitz continuous with constant \( \frac{1}{1-\frac{t}{1+h}} + L_M h \) on \( D \). Convexity of the images \( \Psi(t, x; h) \) is evident.

Assumption A3) guarantees that the iterates of the numerical scheme are well-defined.
Definition 24. A sequence \( \{y_n\}_{n=0}^{N} \subset \mathbb{R}^d \) is called a trajectory of the semi-implicit split scheme \( \Psi \) associated with (1) if
\[
y_{n+1} \in \Psi(t_n, y_n; h_n) \quad \text{for} \quad n = 0, \ldots, N - 1, \quad y_0 = x_0.
\] (21)
The set of all such trajectories is denoted by \( \mathcal{S}_\Psi(\Delta h, x_0) \).

Definition 25. For any \( t_n \in \Delta h \), the reachable set \( \mathcal{R}_\Psi \) of the semi-implicit split scheme \( \Psi \) is given by
\[
\mathcal{R}_\Psi(t_n, x_0) = \{ y_n : \{ y_k \}_{k=0}^{N} \in \mathcal{S}_\Psi(\Delta h, x_0) \}. \tag{22}
\]

By the above, the reachable set \( \mathcal{R}_\Psi(t_n, x_0) \) depends continuously on \( (h, x_0) \) and is Lipschitz continuous in the initial value, because it is a composition of multimaps with this property.

Boundedness of trajectories can be shown as in Lemma (14) so that the moduli of continuity \( \tau_f, \chi_f, \tau_M \) and the constant \( P \) are well-defined.

Proposition 26. For any solution \( x(\cdot) \in \mathcal{S}([0, T], x_0) \), there exists a trajectory \( \{y_n\}_{n=0}^{N} \in \mathcal{S}_\Psi(\Delta h, x_0) \) such that
\[
|x(t_n) - y_n| \leq \sum_{k=0}^{n-1} \left( \prod_{j=k+1}^{n-1} \frac{1}{1 - l_f(t_{j+1})h_j} + L_M(t_j)h_j \right) h_k \tilde{\Gamma}(h_k, t_k) \tag{23}
\]
for \( n = 0, \ldots, N \), where
\[
\tilde{\Gamma}(h_k, t_k) := \frac{1}{1 - l_f(t_{k+1})h_k} (\tau_f(h_k) + \chi_f(Ph_k))
\]
\[
+ \chi_f \left( \int_{t_k}^{t_{k+1}} e^{\int_{u}^{t_{k+1}} l(u)du} P \, ds \right) + \tau_M(h_k) + L_M(t_k)Ph_k.
\]

Proof. The statement is obtained by induction. It is trivially satisfied for \( n = 0 \).

Let \( x(\cdot) \) be an arbitrary solution of (1), and assume that a trajectory \( \{y_k\}_{k=0}^{n} \) satisfying (23) has been constructed. By definition of a solution,
\[
\dot{x}(t) = f(t, x(t)) + m(t, x(t)), \quad x(0) = x_0
\]
with \( m(t, x(t)) \in M(t, x(t)) \). Consider the solution \( e(\cdot) \) of the auxiliary problem
\[
\dot{e}(t) = f(t, e(t)), \quad e(t_n) = x(t_n).
\]
The estimate given in the proof of Theorem 3.2 in [5] with \( F(t, x) := f(t, x) \) and \( s(t) := |x(t) - e(t)| \) yields
\[
|x(t_{n+1}) - e(t_{n+1})| \leq \int_{t_n}^{t_{n+1}} e^{\int_{s}^{t_{n+1}} l(u)du} |m(s, x(s))|ds
\]
\[
\leq \int_{t_n}^{t_{n+1}} e^{\int_{s}^{t_{n+1}} l(u)du} Pds.
\]
Define \( m := \text{Proj}(\frac{1}{t_n} \int_{t_n}^{t_{n+1}} m(s, x(s))ds, M(t_n, x(t_n))) \). Then \( y := z + h_n m \in \Psi(t_n, x(t_n), h_n) \), where \( z \) is the unique solution of

\[
0 = x(t_n) + h_n f(t_{n+1}, x(t_n)) - z.
\]  

(24)

By Theorem 31 applied to (24) with initial guess \( e(t_{n+1}) \),

\[
|e(t_{n+1}) - z| \leq \frac{1}{1 - l_f(t_{n+1})h_n} |x(t_n) + h_n f(t_{n+1}, e(t_{n+1})) - e(t_{n+1})| \leq \frac{1}{1 - l_f(t_{n+1})h_n} \int_{t_n}^{t_{n+1}} |f(t_{n+1}, e(t_{n+1})) - f(s, e(s))|ds \leq \frac{h_n}{1 - l_f(t_{n+1})h_n} (\tau_f(h_n) + \chi_f(Ph_n)),
\]

and

\[
|\int_{t_n}^{t_{n+1}} m(s, x(s))ds - h_n m| = \text{dist}(\int_{t_n}^{t_{n+1}} m(s, x(s))ds, h_n M(t_n, x(t_n))) \leq \int_{t_n}^{t_{n+1}} \text{dist}(M(s, x(s)), M(t_n, x(t_n)))ds \leq h_n (\tau_M(h_n) + L_M(t_n)Ph_n),
\]

so that

\[
|x(t_{n+1}) - y| = |x(t_n) + \int_{t_n}^{t_{n+1}} f(s, x(s)) + m(s, x(s))ds - [z + h_n m]| \leq |x(t_n) + \int_{t_n}^{t_{n+1}} f(s, e(s))ds - z| + \int_{t_n}^{t_{n+1}} |f(s, x(s)) - f(s, e(s))|ds + |\int_{t_n}^{t_{n+1}} m(s, x(s))ds - h_n m| \leq \tilde{\Gamma}(h_n, t_n),
\]

As the scheme \( \Psi \) is Lipschitz, there exists an element \( y_{n+1} \in \Psi(t_n, y_n, h_n) \) such that

\[
|x(t_{n+1}) - y_{n+1}| \leq |x(t_{n+1}) - y| + |y - y_{n+1}| \leq \left( \frac{1}{1 - l_f(t_{n+1})h_n} + L_M(t_n)h_n \right) |x(t_n) - y_n| + \tilde{\Gamma}(h_n, t_n),
\]

and hence \( y_{n+1} \) satisfies (23).

The convergence proof for the other semi-distance proceeds along the same lines as that of Corollary 17.

**Proposition 27.** For every trajectory \( \{y_n\} \in \mathcal{S}_\Psi(\Delta_h, x_0) \), there exists a solution \( x(\cdot) \in \mathcal{S}([0, T], x_0) \) such that

\[
|x(t_n) - y_n| \leq \int_0^{t_n} e^{\int_0^t l_f(s) + L_M(s)ds} \Gamma(|h|_\infty, t) dt
\]

for \( n = 0, \ldots, N \).
Remark 28.

a) In view of Propositions 26 and 27, the estimates allow to exploit negative ROSL-constants.

The set of all such trajectories is denoted $S \equiv S([0, T], x_0)$. For any

$$|x(t_n) - y_n| \leq \exp\left(\frac{L(1 + P)h}{1 - l_fhM} + \frac{1}{l_f}(\exp(l_fh) - 1) + (L + L_M P)\right).$$

A straight-forward spatial discretization of the scheme $\Psi$ is given by

$$\hat{\Psi}(t, x; h) := P_n(\Psi(t, x; h)),$$

i.e. the solution $z$ of (26) is computed and the set $z + M(t, x)$ is projected to the spatial grid. The concrete implementation of this process depends on the implementation of the mapping $M$ and is similar to that of the explicit Euler scheme (see [5]). It is therefore significantly more simple than the implementation of the fully implicit and the parameterized semi-implicit Euler schemes.

Definition 29. A sequence $\{\hat{y}_n\}_{n=0}^N \subseteq \mathbb{R}^d$ is called a trajectory of the fully discretized semi-implicit split scheme $\hat{\Psi}$ associated with (1) if

$$\hat{y}_{n+1} \in \hat{\Psi}(t_n, \hat{y}_n; h, \rho_{n+1}) \text{ for } n = 0, \ldots, N - 1, \quad \hat{y}_0 \in P_{\rho_0}(\{x_0\}).$$

The set of all such trajectories is denoted $S_\Psi(\Delta_h, x_0)$.

Definition 30. For any $t_n \in \Delta_h$, the reachable set $R_\Psi(t_n, x_0)$ of the fully discretized semi-implicit split scheme is given by

$$R_\Psi(t_n, x_0) \equiv \{\hat{y}_n \in \mathbb{R}^d : \{\hat{y}_n\}_{n=0}^N \in S_\Psi(\Delta_h, x_0)\}.$$ 

It is easy to see that for any $\{y_n\}_{n=0}^N \in S_\Psi(\Delta_h, x_0)$, there exists some $\{\hat{y}_n\}_{n=0}^N \in S_\Psi(\Delta_h, x_0)$ satisfying

$$|y_n - \hat{y}_n| \leq \frac{\sqrt{d}}{2} \sum_{k=0}^n \left(\prod_{j=k}^{n-1}\frac{1}{1 - l_f(t_{j+1})h_j} + L_M(t_j)h_j\right)\rho_k,$$

and for every $\{\hat{y}_n\}_{n=0}^N \in S_\Psi(\Delta_h, x_0)$, there exists some $\{y_n\}_{n=0}^N \in S_\Psi(\Delta_h, x_0)$ such that (26) holds.

If $l_f(\cdot) \equiv l_f, L_M(\cdot) \equiv L_M$, $h$ and $\rho$ are constant, then (26) can be replaced with

$$|y_n - \hat{y}_n| \leq \frac{\sqrt{d}}{2} \frac{\exp\left(\frac{(n+1)l_fh}{1 - l_fhM} + (n + 1)L_Mh\right) - 1}{h} \frac{\rho}{h}.$$ 

The unusual factor $(n + 1)$ originates from the projection of the initial value to the spatial grid (cp. Remark 28).
5 Performance

A comparison of the explicit and implicit Euler schemes on a stiff Michaelis-
Menten system was given in [6]. As the simulations look very similar when
the implicit Euler scheme is replaced with one of the two semi-implicit schemes
under discussion, no such graphics are presented here. Instead, the performance
of both fully discretized semi-implicit schemes is investigated when applied to
the Dahlquist-like test inclusion

\[
\dot{x}(t) \in -x(t) + [-1, 1], \quad x(0) = x_0 \in \mathbb{R},
\]

which seems to be an appropriate setting for testing implicit schemes. Both
methods are so much faster than the fully implicit Euler scheme from [6], which
is in addition dangerously sensitive to ill-estimated constants, that a detailed
comparison with this method is inadequate.

The results are displayed in Figure 2. The convergence of both methods \( \hat{\Phi} \)
and \( \hat{\Psi} \) measured in terms of the errors

\[
\max_{n \in \{0, \ldots, N_h\}} \text{dist}_H(\mathcal{R}(t_n, 5), \mathcal{R}_{\hat{\Phi}}(t_n, 5)),
\]

\[
\max_{n \in \{0, \ldots, N_h\}} \text{dist}_H(\mathcal{R}(t_n, 5), \mathcal{R}_{\hat{\Psi}}(t_n, 5))
\]

is linear in the constant step-size \( |h|_{\infty} = 5/N_h \), but not in the consumed time,
which is typical for numerical methods for differential inclusions (cp. [5]). Due
to its favorable analytical properties, the parameterized scheme is better when
the numerical errors are compared to the overall step-size. When performance
is measured in computation time, however, the split scheme is far more efficient
because of its simple spatial discretization. The reasoning below shows that this
performance gap will grow dramatically with the dimension of the state space.

Assume that the step-size \( h \) and the grid-width \( \rho \) are constant and that \( \varepsilon = h \)
is fixed. The computational costs caused by one time step of the parameterized
and the split schemes can be roughly expressed as

\[
\text{time}_{\text{par}} \approx C_{\text{scan}} \frac{\text{vol}(\text{domain})}{\rho^d} + \left( C_{\text{Newton}}(d) + C_{\text{eval}} \frac{\text{vol}(\text{image}(F))}{h^d} \right) \frac{\text{vol}(\text{curr. state})}{\rho^d}
\]

\[
\text{time}_{\text{split}} \approx C_{\text{scan}} \frac{\text{vol}(\text{domain})}{\rho^d} + \left( C_{\text{Newton}}(d) + C_{\text{eval}} \frac{\text{vol}(\text{image}(F))}{h^d} \right) \frac{\text{vol}(\text{curr. state})}{\rho^d}
\]

with notation

- \( C_{\text{scan}} \) – time needed to check whether some grid point is an element of the
current state
- \( C_{\text{Newton}}(d) \) – time needed to compute an approximate solution using New-
ton’s method (depends on space dimension \( d \))
- \( C_{\text{eval}} \) – time needed to evaluate and project the final result of one individ-
ual computation to the spatial grid.

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• domain – the domain in $\mathbb{R}^d$ on which the algorithm computes the solution sets

• curr. state – the reachable set at present time.

Since $\rho = h^2$ and $C_{\text{scan}} < C_{\text{eval}} \ll C_{\text{Newton}}(d)$, it is evident that the consumed time grows exponentially in $d$ and that

$$\text{time}_{\text{par}} \approx \frac{\text{vol(image}(F))}{h^d} \text{time}_{\text{split}}.$$ 

This rule of thumb is verified by Figure 2. As the error estimates for both schemes are linear in $|h|_\infty$, the parameterized semi-implicit Euler scheme cannot compete with the split scheme. It has, however, one advantage that is illustrated in the subsequent analysis of the Dahlquist-like inclusion (27).

![Figure 2: Error analysis of both methods tested on the Dahlquist-like equation (27) with $x_0 = 5$ on the time interval $[0, 5]$. The convergence of both methods is linear in the constant step-size $h$, but not in the consumed time.](image)

By monotonicity, inclusion (27) admits upper and lower solutions

$$x^{(+)}(t) = e^{-t}(x_0 - 1) + 1, \quad x^{(-)}(t) = e^{-t}(x_0 + 1) - 1,$$

and it is easy to see that the interval $[-1, 1]$ is the global attractor for the multivalued flow induced by (27). The parameterized semi-implicit Euler scheme has upper and lower solutions given by the recursions

$$y^{(+)}_{n+1} = \frac{y^{(+)}_n + h}{1 + h}, \quad y^{(-)}_{n+1} = \frac{y^{(-)}_n - h}{1 + h},$$
and its attractor coincides with that of the original inclusion. The split scheme admits upper and lower solutions given by the recursions

\[
\tilde{y}_{n+1}^{(+)} = \frac{\tilde{y}_n^{(+)} + h}{1 + h}, \quad \tilde{y}_{n+1}^{(-)} = \frac{\tilde{y}_n^{(-)} - h}{1 + h},
\]

and simple computations show that its attractor is \([-1 - h, 1 + h]\). As a consequence, the parameterized semi-implicit Euler scheme seems to be superior with respect to correct asymptotic behavior (in time).

6 Conclusion

The semi-implicit split scheme is at present the fastest numerical method applicable to stiff differential inclusions. In some sense, this paper finishes the discussion of explicit and implicit first-order methods of classical type, because combined with \[5\] it provides a fairly clear picture of what can be achieved with such schemes. Grid-based methods are currently the best we have, and they are far better than trajectory-based schemes, because they reduce the complexity in every step by projecting to the spatial grid and thus identifying many individual solutions.

Nevertheless, the performance of these methods leaves much to be desired. This is partly due to the fact that they are only approximations of first order, but it is mainly because of redundant computations that arise from the overlap of computed images and account for an overwhelming majority of the computational costs.

Two worthwhile future challenges are therefore clearly defined: It is necessary to obtain a better understanding of the fine structure of the solution set of the differential inclusion \((1)\) in order to be able to develop higher order schemes (of classical type) systematically. Moreover, the invention of non-classical schemes that avoid redundant computations must be pushed forward. Numerical methods which discretize and track only the boundary of the reachable sets have been tested experimentally with very promising results. The dynamics of the boundary, however, are complicated, and for that reason it is very difficult to prove error estimates for this type of schemes.

A Solvability and stability theorems

The following solvability theorem is a modification of Corollary 3 in \[6\]. Every continuous set-valued mapping is upper semicontinuous.

**Theorem 31.** Let \(G : \mathbb{R}^d \to \mathcal{CC}(\mathbb{R}^d)\) be upper semicontinuous and \(l\)-ROSL with \(l < 0\). Then for any \(y \in \mathbb{R}^d\), the set \(S_G(y) := \{z \in \mathbb{R}^d : y \in G(z)\}\) is nonempty and compact and satisfies

\[
\text{diam } S_G(y) \leq -\frac{1}{l} \sup_{z \in S(y)} \text{diam } G(x). \tag{28}
\]
Moreover, the estimates
\[
\text{dist}(x, S_G(y)) \leq -\frac{1}{l} \text{dist}(y, G(x)),
\]
\[
\text{dist}(S_G(y), x) \leq -\frac{1}{l} \text{dist}(G(x), y)
\]
hold for arbitrary \( x \in \mathbb{R}^d \).

**Proof.** As \( G \) is usc, \( S_G(y) \) is closed for any \( y \in \mathbb{R}^d \). Estimate (29) is Corollary 3 in [6]. If \( z \in S_G(y) \), then \( y \in G(z) \), and by the ROSL property there exists some \( \eta \in G(x) \) such that
\[
-|y-\eta| \cdot |z-x| \leq \langle y-\eta, z-x \rangle \leq l|z-x|^2.
\]
Hence
\[
|z-x| \leq -\frac{1}{l}|y-\eta| \leq -\frac{1}{l} \text{dist}(G(x), y),
\]
which proves (30). If, in addition, \( x \in S_G(y) \), then \( y \in G(x) \) and (28) follows from (30). \( \square \)

The stability Theorem below is cited from [6, Theorem 13]

**Theorem 32.** Let \( l : [0, T] \to \mathbb{R} \) be continuous, and \( F : [0, T] \to \mathcal{CC}^u(\mathbb{R}^d) \) be a jointly continuous set-valued mapping which is \( l(t) \)-ROSL in the second argument. Then for any given \( y \in C^1([0, T], \mathbb{R}^d) \), there exists a solution \( x(\cdot) \) of the differential inclusion
\[
\dot{x}(t) \in F(t, x(t)), \quad t \in [0, T], \quad x(0) = x_0
\]
such that
\[
|x(t) - y(t)| \leq e^{\int_0^t l(s)ds}|x(0) - y(0)| + \int_0^t e^{\int_\tau^t l(s)ds} g(s)ds
\]
for all \( t \in [0, T] \), where \( g \in C([0, T], \mathbb{R}_+) \) is defined by
\[
g(t) := \text{dist}(\dot{y}(t), F(t, y(t))).
\]
If \( l, F \) and \( y \) are defined on \([0, \infty)\) and have the same properties as above, then the above statement holds on the interval \([0, \infty)\).

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