Solving congruence equations using Bernstein forms

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Abstract

We present a subdivision method to solve systems of congruence equations. This method is inspired in a subdivision method, based on Bernstein forms, to solve systems of polynomial inequalities in several variables and arbitrary degrees. The proposed method is exponential in the number of variables.

Keywords: Congruence equations, Bernstein form, Integer polynomial programming, Solver

2010 MSC: 11D79, 11Y50, 45G15, 68W30, 14Q20

Introduction

Overview of the problem

An important aspect within the study of algebraic varieties over characteristic $p$ is to know the number of points that a variety has. This led to the statement of Weil’s conjectures proved by Deligne thanks to the analysis made by Grothendieck and others. In this article we are interested in computing, not only the number of points in an algebraic variety modulo $p$ but also, the coordinates of these points. We propose a computational method to find solutions to the following system of congruence equations

\[
\begin{align*}
\begin{cases}
 h_1(x_1, \ldots, x_n) & \equiv 0 \pmod{(p_1)} \\
 \vdots \\
 h_r(x_1, \ldots, x_n) & \equiv 0 \pmod{(p_r)}
\end{cases}
\end{align*}
\]

where $h_1, \ldots, h_r \in \mathbb{Z}[x_1, \ldots, x_n]$ and $p_1, \ldots, p_r \in \mathbb{Z}_{\geq 2}$. To achieve this goal we first analyze a known method in nonlinear programming and then adapt it to our needs.

Polynomial programming is the process of solving a system of equalities and inequalities between polynomial functions along with an objective function to be maximized (or minimized). Typically, such a system can be described as:

\[
\begin{align*}
\text{maximize } & q_1(x) \\
\text{subject to } & q_i(x) \geq 0, \quad 2 \leq i \leq r.
\end{align*}
\]
The polynomials $q_i$ are assumed to have real coefficients but in this article we study integer polynomial programming assuming $q_1, \ldots, q_r \in \mathbb{Q}[x_1, \ldots, x_n]$ and $x \in \mathbb{Z}^n$.

Since the resolution of Hilbert’s Tenth problem by Matiyasevich, [19], several results appeared proving that there cannot exist any general algorithm to solve integer polynomial programming (it is non computable). For example, in [16], the author proves that the problem of minimizing a linear form over quadratic constraints in integer variables is not computable by a recursive function. Hence, in order to avoid this phenomena, it is necessary to add some constraints to the original problem. Given that we are interested in applications to congruence equations we bound the variables to some region $D = [a_1, b_1] \times \ldots \times [a_n, b_n] \subseteq \mathbb{R}^n$ restricting the problem of adapting an integer polynomial programming

$$\begin{align*}
\text{maximize } & q_1(x) \\
\text{subject to } & q_i(x) \geq 0, \quad 2 \leq i \leq r \\
& x \in D \cap \mathbb{Z}^n.
\end{align*}$$

to a solver of congruence equations

$$\begin{cases}
    h_1(x) \equiv 0 \pmod {p_1} \\
    \vdots \\
    h_r(x) \equiv 0 \pmod {p_r}
\end{cases}$$

where $x \in D \cap \mathbb{Z}^n$, $D = [0, k]^n$ and $k$ is the least common multiple of $p_1, \ldots, p_r$, $k = \text{lcm}(p_1, \ldots, p_r)$.

**Existing work**

There are many local and global methods to solve a Diophantine equation. The first thing to do is to check whether the problem has a solution locally. If it has none, the problem is solved since our equation has no solution, but if it has, the local information that we obtain may already completely solve the problem, or may not so it is necessary to study the equation globally.

The global methods include sophisticated tools from number theory, for example factorization over a number field, computations of class groups and unit groups, Diophantine approximation techniques, linear forms in logarithms, elliptic curves, modular forms and Galois representations and more.

Regarding congruence equations we can say that there are general methods for solving both linear and quadratic congruence equations however solutions to the general polynomial congruence is intractable. Standard algorithms can be found at [27] and also at [4, 5]. Several software packages implement these techniques, see for example MagmaSoft, SageMath, Kant/Kash.

Integer polynomial programming is a subclass of the more general class *Mixed integer non-linear programming* (MINLP). Approaches to solve these problems may be classified as either stochastic or deterministic. Stochastic techniques employ some element of randomness in their search for the global optimum and consequently rely on a statistical argument to prove convergence to the global optimum, see [26]. Deterministic methods have the advantage that they provide a rigorous guarantee of global optimality of any solution produced within a specified tolerance. Examples of deterministic techniques are cutting plane methods, interval methods, primal-dual methods, outer approximation, generating functions, etc. For instance using generating functions of convex sets, it is proven in [1] that there exists a polynomial-time algorithm...
to find the maximum value of the objective function without knowing where this maximum is achieved. Also, a very successful methodology is the spatial branch and bound algorithm [18, 28]. The aim of branch and bound techniques, is to divide the problem into subclasses to be solved with convex or linear approximations, which are powerful and flexible for modeling and solving decision problems. Despite their widespread use, few available software packages provide any guarantee of correct answers or certification of results. Possible inaccuracy is caused by the use of floating-point numbers. Floating-point calculations require the use of built-in tolerances for testing feasibility and optimality, and can lead to calculation errors in the solution of linear programming relaxations and in the methods used for creating cutting planes to improve these relaxations. See in [6] where the author gave a method to guarantee the result avoiding numerical errors.

There is a large variety of commercial and academic noncommercial software, but only a few of those software packages solve general non-convex problems to global optimality. For a review see [3], [2], [17, §1] and/or [8]. It is worth mention that there is a large literature reviewing different methods (and software) developed to solve convex and non-convex MINLP problems, furthermore, every two years, Robert Fourer publishes a list of currently available codes in the field of linear and integer programming, the latest edition can be found at the web page of OR/MS-Today, Software Survey section, [12].

In this article we use Bernstein forms. The advantage of using Bernstein forms is that they produce robust and reliable algorithms. They are used to solve systems of polynomial equations and also integer polynomial programming, see for example [13, 22, 23]. We review an algorithm using Bernstein forms to solve integer polynomial programming and then we adapt it to solve congruence problems.

Before recalling the next theorem, let us introduce some notations. Let \((d_1, \ldots, d_n)\) be a multi-degree, \(s = (d_1 + 1) \ldots (d_n + 1)\) and let \(D = [a_1, a_1 + 2^k_1] \times \ldots \times [a_n, a_n + 2^k_n], (a_1, \ldots, a_n) \in \mathbb{Z}^n, (k_1, \ldots, k_n) \in \mathbb{N}_0^n\). Choose some index \(j, 1 \leq j \leq n\), and then divide into two halves the \(j\)-side of \(D\),

\[
D_j^l = \prod_{i=1}^{j-1} [a_i, a_i + 2^k_i] \times [a_j, a_j + 2^{k_j-1}] \times \prod_{i=j+1}^n [a_i, a_i + 2^k_i],
\]

\[
D_j^r = \prod_{i=1}^{j-1} [a_i, a_i + 2^k_i] \times [a_j + 2^{k_j-1}, a_j + 2^k_j] \times \prod_{i=j+1}^n [a_i, a_i + 2^k_i].
\]

**Theorem.** For every \(q_1, \ldots, q_t \in \mathbb{Z}[x_1, \ldots, x_n]_{\leq (d_1, \ldots, d_n)}\), there exists a rectangular matrix \(v \in \mathbb{Q}^{t \times r}\) such that the values of \(q_i\) over \(D_j^l\) (resp. \(D_j^r\)) are bounded by the minimum and the maximum of \(\{w^L_{i,k}\}_{k=1}^r\) (resp. \(\{w^R_{i,k}\}_{k=1}^r\)), where \(w^L_i = M_j^L v, \ u^R = M_j^R v, 1 \leq i \leq r\). The matrices \(\{M_j^L, M_j^R\}_{j=1}^n\) depend on \((d_1, \ldots, d_n)\).

Also, the value of \(q_i\) at \((a_1, \ldots, a_n)\) is equal to \(w^L_i\), and the value of \(q_i\) at \((a_1, \ldots, a_{j-1}, a_j + 2^{k_j-1}, a_{j+1}, \ldots, a_n)\) is equal to \(w^R_i, 1 \leq i \leq r\).

This theorem is related to the resolution of integer polynomial programming as follows. The subdivision process tests if a given region may have solutions or definitely not. If it may, then the region is divided and the process starts again, otherwise it is rejected. At the end, the process produces candidates of the form \(\prod_{i=1}^n [z_i, z_i + 1]\) and test if \((z_1, \ldots, z_n)\) is a solution of the system. Both stages of this algorithm depend on the previous theorem.
Main result

In this article we present a subdivision method based on Bernstein forms to compute the solutions of a system of congruence equations. The idea of using Bernstein forms to make a “solver” is not new, see for example [13, 22, 23] and [11, 20, 21]. One of our contribution is the application of Bernstein forms to congruence equations. The key ingredient is that we managed to make the subdivision process and the process of rewriting the system in Bernstein form as a matrix multiplication.

An advantage of Bernstein forms is that the algorithm can test the existence of a solution in a large region by testing bounds on the coordinates of a matrix. An important feature of the algorithm is that it does not rely on numerical computations and as a result, it gives a certified answer, see Theorem 4.2. The algorithm works by performing a matrix multiplication at each iteration with triangular rational matrices \( \{ M_L^i, M_R^i \}_{i=1}^n \). These matrices depend on the multi-degree of the system \((d_1, \ldots, d_n)\) and we prove that it is convenient to have these 2\(^n\) matrices previously computed for a large multi-degree \((d_1, \ldots, d_n)\), see Theorem 4.4. It is worth mentioning that the algorithm can work with any system of congruence equations, independently if the numbers \(p_1, \ldots, p_r\) are primes or not.

The first contribution that we present is Theorem 3.3 where we give the expected number of matrix multiplications that the algorithm for integer polynomial programming requires. It is easily seen that this algorithm has an exponential complexity, but in practice, we prove that the complexity depends on the “size” of the real solution of the system, see Proposition 3.1. This “size” can be estimated using tools from algebraic geometry (dimension, degree) and in particular, Theorem 3.3 implies that the algorithm is faster if we add more equations, that is, if there are less solutions to the system. We used concepts and results from the theory of Branching processes to prove Theorem 3.3.

**Theorem.** Let \( \lambda \) be the complexity number (Definition 3.2) of \( \{ q_1, \ldots, q_r \} \) over \( D = \prod_{i=1}^n [a_i, a_i + 2^k] \) and let \( K = k_1 + \ldots + k_n \). Then, the expected number of matrix multiplications produced by the algorithm is

\[
\sum_{i=0}^K 2^{\sum_{j=0}^{i-1} (1 - (1 - \lambda)^{2^j})} - 1.
\]

Regarding systems of congruence equations we first give Lemma 4.1 that characterizes existence of solutions in a real interval. Then, we give the pseudocode of \texttt{SolveCE} and prove Theorem 4.2.

**Theorem.** The output of \texttt{SolveCE}((\(p_1, \ldots, p_r\), \(v\), (\(k, \ldots, k\)), (0, \ldots, 0)) is the set

\[\{ x \in D \cap \mathbb{Z}^n : p_1|h_1(x), \ldots, p_r|h_r(x) \},\]

where \( D = [0, 2^k] \times \ldots \times [0, 2^k], k = \lceil \log_2(\text{lcm}(p_1, \ldots, p_r)) \rceil \) and \( v \) is the matrix associated to \( h_1, \ldots, h_r \).

Finally, we compute the expected number of matrix multiplications produced by \texttt{SolveCE}. This number depends on a parameter \( \lambda \) (similar to the complexity number).
Theorem. Let \( h_1, \ldots, h_r \in \mathbb{Z}[x_1, \ldots, x_n] \), \( p_1, \ldots, p_r \in \mathbb{Z}_{\geq 2} \), \( k = \lceil \log_2(\text{lcm}(p_1, \ldots, p_r)) \rceil \). Then, the expected number of matrix multiplications produced by \( \text{SolveCE} \) in \( D = \prod_{i=1}^{n} [0, 2^k] \) is

\[
\sum_{i=0}^{n^k} 2^i \prod_{j=0}^{i-1} \left( 1 - (1 - \lambda)^{2^{i-j}} \right) - 1.
\]

This number is bounded between a polynomial and an exponential expression in \( \lambda \).

Summary

In Section 1 we give some preliminary results; we recall the concept of Bernstein Bases and degree elevation, fixing a multi-degree \((d_1, \ldots, d_n)\) we associate to every region \( D \) a square matrix and to every system \( \{q_1, \ldots, q_r\} \) a rectangular matrix. Finally, with these results, we prove in Theorem 1.13 our main tool. In Section 2 we present the algorithm to solve integer polynomial programming \( \text{SolveIPP} \) and in Section 3 we study its expected complexity and give several examples computed using our implementation in SageMath [30]. Finally in Section 4 we present the pseudocode \( \text{SolveCE} \) to solve congruence equations and study its expected complexity. We compare in a table \( \text{SolveCE} \) with the brute-force algorithm (Definition 4.5).

1. Preliminaries

In this and the next section we present a known algorithm to solve integer polynomial programming. Later we adapt it to solve congruence equations.

We are interested in the integral points of semi-algebraic sets defined by a system of multivariate rational polynomial equations in \( D = [a_1, b_1] \times \cdots \times [a_n, b_n] \subseteq \mathbb{R}^n \). Specifically, let \( q_1 \in \mathbb{Q}[x_1, \ldots, x_n] \) be some multivariate polynomial and let \( S' \) be the intersection between \( D \cap \mathbb{Z}^n \) and some semi-algebraic set defined by some polynomials \( q_2, \ldots, q_r \in \mathbb{Q}[x_1, \ldots, x_n] \). We want to find the maximum \( \theta \) of \( q_1 \) over \( S' \) and more general, to describe the set \( S \).

\[
S = \{ z \in S' : q_1(z) = \theta \}.
\]

Notation 1.1. First of all, we want to give a unifying description of the set \( S' \). It is defined as the intersection of equalities and inequalities. The first simplification to make is to assume that the polynomials have integer coefficients and then, since we are working over the integers, we can always assume that \( S' \) is defined only with inequalities of the form \( q \geq 0 \). Concretely, we can make the following changes for \( q, q' \in \mathbb{Z}[x_1, \ldots, x_n] \):

- Replace the inequality \( q \leq q' \) with the inequality \( q' - q \geq 0 \).
- Replace the equality \( q = q' \) with the inequalities \( q - q' \geq 0 \) and \( q' - q \geq 0 \).
- Replace the inequality \( q > q' \) with the inequality \( q - q' - 1 \geq 0 \).
- Replace the inequality \( q < q' \) with the inequality \( q' - q - 1 \geq 0 \).

Then, without loss of generality, we can assume that \( S' \) is given as,

\[
S' = \{ z \in D \cap \mathbb{Z}^n : q_2(z) \geq 0, \ldots, q_r(z) \geq 0 \},
\]

where \( q_2, \ldots, q_r \in \mathbb{Z}[x_1, \ldots, x_n] \) are some multivariate polynomials. Also, we can always assume that \( q_1 \) has integer coefficients and that we are searching for the maximum of \( q_1 \) over \( S' \) making the following changes,
• Replace \( \min(q_1) \) with \( \max(-q_1) \).

• If there is no \( q_1 \) to maximize, then maximize \( q_1 \equiv 0 \).

Then, the solutions of the system is the set \( S \),

\[
S = \left\{ z \in S' : q_1(z) = \max_{S'}(q_1) \right\}.
\]

Now that we have a unifying way to present the system, let us define (or recall) Bernstein bases. To simplify the notation, we use standard multi-index notation. Letters in boldface represent multi-indexes. For example \( d = (d_1, \ldots, d_n) \).

**Definition 1.2 (Bernstein Basis).** According to [21, Proposition 2], any polynomial \( p \in \mathbb{R}[x_1, \ldots, x_n] \) of multi-degree \( (d_1, \ldots, d_n) \) can be written in Bernstein form,

\[
p(x) = p(x_1, \ldots, x_n) = \sum_{k=0}^{d_1} \ldots \sum_{k_n=0}^{d_n} \beta_{k_1 \ldots k_n} B_{d_1\ldots d_n,k_1\ldots k_n}(x_1, \ldots, x_n) = \sum_{k=0}^{d} \beta_k B_{d,k}(x),
\]

where \( \beta_k = \beta_{k_1 \ldots k_n} \in \mathbb{R} \) and

\[
B_{d,k}(x) = \binom{d_1}{k_1} x_1^{k_1}(1 - x_1)^{d_1 - k_1} \ldots \binom{d_n}{k_n} x_n^{k_n}(1 - x_n)^{d_n - k_n} = \binom{d}{k} x^k (1 - x)^{d-k}.
\]

The polynomials \( B_{d,k} \) have rational coefficients and multi-degree \( d \) for every \( k \). The ordered set \( \{ B_{d,k} : 0 \leq k \leq d \} \) is called the **Bernstein basis** of \( \mathbb{R}[x]_{d,d} \) or the **d-Bernstein basis** to emphasize the multi-degree. The order is the lexicographical order. The numbers \( \beta_k \) are called Bernstein coefficients.

**Remark 1.3.** If a polynomial is given in monomial form \( p(x) = \sum_{i=0}^{d} c_i x^i \), it is easy to make a change of basis from monomial basis to Bernstein basis. Specifically, the Bernstein coefficients are computed as follows:

\[
\beta_k = \sum_{i=0}^{k} c_i \binom{k}{i}
\]

Note that \( \{ \beta_k \} \subseteq \mathbb{Q} \) if \( \{ c_i \} \subseteq \mathbb{Q} \). Recall from [21, Corollary 1] that if \( m_1 \) (resp. \( m_2 \)) is the minimum (resp. maximum) of the Bernstein coefficients \( \beta_k \), then we have the **fundamental inequality**

\[
m_1 \leq p(x) \leq m_2, \quad \forall x \in [0, 1]^n.
\]

**Lemma 1.4 (Degree elevation).** Let \( p \in \mathbb{Z}[x]_{d,d} \) be a polynomial defined over \( [0, 1]^n \) and let \( m_1, m_2 \in \mathbb{R} \) be such that \( m_1 < p(x) < m_2 \) for all \( x \in [0, 1]^n \). Then, there exists \( d' \) such that for every \( d'' \geq d' \) the Bernstein coefficients \( \beta_k \) of \( p \) in \( d'' \)-Bernstein basis satisfy,

\[
m_1 < \beta_k < m_2, \quad \forall k \leq d''.
\]
Proof. Given that \( m_1 < p(x) < m_2 \), there exists \( \delta > 0 \) such that \( m_1 + \delta < p(x) < m_2 - \delta \) for all \( x \in [0, 1]^{n} \). Also, from [24, Eq. 3.1] there exists \( m \in \mathbb{N} \) such that \( |\beta_k - p(k/m)| \leq \delta \) for all \( k \leq d' := (m, \ldots, m) \). Then,

\[
\beta_k = (\beta_k - p(k/m)) + p(k/m) < \delta + (m_2 - \delta) = m_2
\]

and

\[
\beta_k = (\beta_k - p(k/m)) + p(k/m) > -\delta + (m_1 + \delta) = m_1.
\]

Example 1.5. Consider the following polynomial over \([0, 1]^{n}\),

\[
p(x) = -\left( x - \frac{1}{2} \right)^2 - \frac{1}{10}.
\]

Then \( \max_{[0,1]}(p) = -1/10 \). The Bernstein coefficients in degree 2 of \( p \) are \((-7/20, 3/20, -7/20)\). Taking the Bernstein coefficients in degree \( d' = 3 \) we get \((-7/20, -1/60, -1/60, -7/20)\). In particular, for every \( d'' \geq 3 \),

\[
\frac{1}{10} \leq \max_{k \leq d''}(\beta_k) \leq \frac{1}{60}.
\]

Let us start analyzing the regions \( D \). Our goal is to define subdivision matrices for \([0, 1]^{n}\) and then, to treat any \( D \) in a uniform way.

Definition 1.6 (Matrix associated to a box and to a multi-degree). We say that \( D \subseteq \mathbb{R}^{n} \) is a box if there exist \( a, b \in \mathbb{R}^{n} \) such that

\[
D = [a_1, b_1] \times \ldots \times [a_n, b_n].
\]

Let \( \varphi : \mathbb{R}^{n} \rightarrow \mathbb{R}^{n} \) be \( \varphi(x) = (b - a)x + a \). Clearly \( \varphi \) maps bijectively \([0, 1]^{n}\) to \( D \) and defines, via pull-back, a linear map \( \mathbb{R}[x] \rightarrow \mathbb{R}[x] \) given by \( q \mapsto \varphi^{*}(q) := q(b - a)x + a \).

Fix a multi-degree \( d \). We say that the matrix \( M \) of the map \( \varphi^{*} \) in the \( d \)-Bernstein basis is the matrix associated to the box \( D \) and to the multi-degree \( d \). The matrix \( M \) is a \( s \times s \)-square matrix, where \( s = (d_1 + 1) \ldots (d_n + 1) \) and also, if \( [a_i, b_i]_{i=1}^{n} \subseteq \mathbb{Q} \), then \( M \in \mathbb{Q}^{s \times s} \).

Notation 1.7. Fix a multi-degree \( d \). Let us denote \( M_{ij}^{L} \) or \( M_{ij}^{U}(d) \) (resp. \( M_{ij}^{R} \) or \( M_{ij}^{B}(d) \)) to the matrix associated to the box \([0, 1]^{n-1} \times [0, 1/2] \times [0, 1]^{n-i} \) (resp. \([0, 1]^{n-1} \times [1/2, 1] \times [0, 1]^{n-i} \)) for \( 1 \leq i \leq n \) and to \( d \) (see Definition 1.6). Recall that if \( s = (d_1 + 1) \ldots (d_n + 1) \), then \( M_{ij}^{L}, M_{ij}^{R} \in \mathbb{Q}^{s \times s} \).

Proposition 1.8. The matrix \( M_{ij}^{L}(d) \) is lower triangular and \( M_{ij}^{R}(d) \) is upper triangular, \( 1 \leq i \leq n \). If \( i \neq j \), \( M_{ij}^{L} \) commutes with \( M_{ij}^{U} \) and with \( M_{ij}^{R} \), but \( M_{ij}^{L} M_{ij}^{R} \neq M_{ij}^{R} M_{ij}^{L} \).

Proof. The shape of the matrices \( M_{ij}^{U} \) and \( M_{ij}^{R} \) follows from [21, Proposition 5]. Let us prove the commutativity.

Let \( \varphi_{i}^{L} \) (resp. \( \varphi_{i}^{R} \)) be the unique affine isomorphism sending \([0, 1]^{n} \) to \([0, 1]^{n-1} \times [0, 1/2] \times [0, 1]^{n-i} \) (resp. \([0, 1]^{n-1} \times [1/2, 1] \times [0, 1]^{n-i} \)). The matrix representation of \( \varphi_{i}^{L} \) in \( d \)-Bernstein basis is \( M_{ij}^{L} \) (same for \( \varphi_{i}^{R} \) and \( M_{ij}^{R} \)). Then, we need to prove that if \( i \neq j \), then \( \varphi_{i}^{L} \) commutes
with \( \varphi_i \) and with \( \varphi_i^k \), but \( \varphi_i^k \varphi_i^R \neq \varphi_i^R \varphi_i^k \). Clearly \( \varphi_i^k \varphi_i^R \) and \( \varphi_i^R \varphi_i^k \) (resp. \( \varphi_i^k \varphi_i^k \) and \( \varphi_i^R \varphi_i^R \)) are affine isomorphisms sending \([0, 1]^n\) to the same box. By uniqueness, \( \varphi_i^k \varphi_i^R = \varphi_i^R \varphi_i^k \) (resp. \( \varphi_i^k \varphi_i^k = \varphi_i^R \varphi_i^R \)).

Given that the box associated to \( \varphi_i^k \varphi_i^R \) is different from the box associated to \( \varphi_i^R \varphi_i^k \), we obtain

\[
[0, 1)^{i-1} \times [1/2, 3/4] \times [0, 1]^{n-i} \neq [0, 1)^{i-1} \times [1/4, 1/2] \times [0, 1]^{n-i}.
\]

\( \square \)

**Proposition 1.9.** Fix a multi-degree \( \mathbf{d} \). Let \( k_i, l_i \in \mathbb{N}_0 \) be such that \( l_i < 2^{k_i} \), \( 1 \leq i \leq n \). Let \( D \) be the box defined as

\[
D = \left[ \frac{l_1}{2^{k_1}}, \frac{l_1+1}{2^{k_1}} \right] \times \ldots \times \left[ \frac{l_n}{2^{k_n}}, \frac{l_n+1}{2^{k_n}} \right]
\]

and let \( M \) be the matrix associated to \( D \) and \( \mathbf{d} \). Then, there exists a factorization of \( M \) into a product of matrices \( M^1_i(\mathbf{d}) \) and \( M^R_i(\mathbf{d}) \). The sum of the multiplicities of the matrices \( M^1_i(\mathbf{d}) \) and \( M^R_i(\mathbf{d}) \) in this product is \( k_i \), \( 1 \leq i \leq n \).

**Proof.** Without loss of generality, we may assume that \( n = 1 \) and \( D = [l/2^k, (l+1)/2^k] \). Let us call \( M^1_i = L \) and \( M^R_i = R \).

Induction in \( k \). If \( k = 1 \), then \( l = 0 \) or \( l = 1 \). Hence, \( M = L \) or \( M = R \). Assume now the result for \( k - 1 \) and let us prove it for \( k \). We have two possibilities, \( l \) is even or \( l + 1 \) is even. If \( l = 2l' \) is even, then \( D \) is included in \( D' = [l/2^{k-1}, (l+2)/2^{k-1}] = [l'/2^{k-1}, (l'+1)/2^{k-1}] \). By the inductive hypothesis, \( D' \) has associated a matrix \( M' \) that can be factorized as \( k - 1 \) products of the matrices \( L \) and \( R \), \( M' = L^{l'}R^{l-1} \ldots \). Given that \( D \) is equal to the first half of \( D' \), \( M = LM' \). The case in which \( l + 1 \) is even is similar. \( \square \)

**Definition 1.10.** Let \( F(k) \) be the set of functions from \([1, \ldots, k]\) to \([L, R]\). The cardinal of \( F(k) \) is \( 2^k \).

Fix a multi-degree \( \mathbf{d} \) and \( k \in \mathbb{N}_0^n \). The set of \((\mathbf{d}, k)\)-subdivision matrices is defined as

\[
\left\{ M^{f_1(1)}_1 \ldots M^{f_k(1)}_1 M^{f_1(2)}_2 \ldots M^{f_k(2)}_2 \ldots M^{f_1(n)}_n \ldots M^{f_k(n)}_n : (f_1, \ldots, f_n) \in F(k_1) \times \ldots \times F(k_n) \right\}.
\]

If \( k = 0 \), we use the convention \( F(k) = \{0\} \) and \( M^0_1 \) is the identity matrix. According to Proposition 1.9, the cardinal of this set is equal to the number of boxes obtained by subdividing \( k_i \) times in direction \( i \), the box \([0, 1]^n \). Hence, the cardinal is equal to \( 2^{k_1 + \ldots + k_n} \). \( \square \)

Now that we know how to treat different boxes \( D \) as matrices, let us give some results on how to treat polynomials as vectors.

**Proposition 1.11.** Let \( \mathbf{d} \in \mathbb{N}_0^n \) \( s = (d_1 + 1) \ldots (d_n + 1) \), \( D = [a_1, b_1] \times \ldots \times [a_n, b_n] \) and let \( q \in \mathbb{R}[\mathbf{x}]_d \) be a multivariate polynomial of multi-degree \( \leq \mathbf{d} \). Then, there exist a vector \( v = v(q) \in \mathbb{R}^s \) such that,

- **The first coefficient** \( v_1 \) is equal to \( q(a_1, \ldots, a_n) \).

\[ v_1 = q(a_1, \ldots, a_n). \]

- **The values of** \( q \) **over** \( D \) ***are bounded by*** \( \min(v) \) ***and*** \( \max(v) \).

\[
\min([v_i]_{i=1}^s) \leq q(\mathbf{x}) \leq \max([v_i]_{i=1}^s), \quad \forall \mathbf{x} \in D.
\]
Theorem 1.13. Let \( \varphi : \mathbb{R}^n \to \mathbb{R}^n \) be the unique affine isomorphism sending \([0, 1]^n\) to \( D, \varphi(x) = (b-a)x + a \). Then, the pull-back of \( q \) over \( D \), defines a polynomial \( \varphi'(q) = q((b-a)x + a) \) over \([0, 1]^n\). Let \( v \) be the vector representation of \( \varphi'(q) \) in \( d \)-Bernstein basis. Then,
\[
\min(v) \leq q(x) \leq \max(v), \quad \forall x \in D
\]
and from [21, Proposition 4],
\[
v_1 = b_0 = \varphi'(q)(0) = q(\varphi(0)) = q(a_1, \ldots, a_n).
\]

\[\Box\]

Corollary 1.12. Let \( d \in \mathbb{N}_0^n \), \( s = (d_1 + 1) \ldots (d_n + 1) \), \( D = [a_1, b_1] \times \ldots \times [a_n, b_n] \) and let \( q_1, \ldots, q_r \in \mathbb{R}[x]_{d} \) be polynomials of multi-degree \( \leq d \). Then, there exist a rectangular matrix \( v \in \mathbb{R}^{s \times r} \) with columns \( v_i = v_i(q_i) \) such that,
- The coefficient \( v_{i1} \) is equal to \( q_i(a_1, \ldots, a_n) \),
\[
v_{i1} = q_i(a_1, \ldots, a_n), \quad 1 \leq i \leq r.
\]
- The values of \( q_i \) over \( D \) are bounded by the minimum and the maximum of \( \{v_{i1}, \ldots, v_{ir}\} \).
\[
\min(v_{j1}) \leq q_i(x) \leq \max(v_{j1}), \quad \forall x \in D, \quad 1 \leq i \leq r.
\]

\[\Box\]

Proof. Define the matrix \( v = (v_1, \ldots, v_r) \in \mathbb{R}^{s \times r} \), where \( v_i = v_i(q_i) \) is defined as in Proposition 1.11. Combining the previous results on boxes and polynomials, we obtain the following important result.

Theorem 1.13. Let \( d = (d_1, \ldots, d_n) \) be a multi-degree, \( s = (d_1 + 1) \ldots (d_n + 1) \) and let \( D = [a_1, a_1 + 2^{k_1}] \times \ldots \times [a_n, a_n + 2^{k_n}] \) be a box, where \( k = (k_1, \ldots, k_n) \in \mathbb{N}_0^n \) is fixed.

For every \( q_1, \ldots, q_r \in \mathbb{Z}[x]_{d} \) there exists a rectangular matrix \( v \in \mathbb{Q}^{s \times r} \) and for every box \( D' \subseteq D \) of the form
\[
D' = [z_1, z_1 + 2^{k_1}] \times \ldots \times [z_n, z_n + 2^{k_n}],
\]
\[
z_i = a_i + l_i 2^{k_i}, \quad 0 \leq k_i' \leq k_i, \quad 0 \leq l_i < 2^{k_i-k_i'}, \quad l_i \in \mathbb{N}_0, \quad 0 \leq i \leq n,
\]
there exists a \((d, k)\)-subdivision matrix \( M \in \mathbb{Q}^{s \times s} \) such that \( w = Mv \) satisfies
- The coefficient \( w_{i1} \) is equal to \( q_i(z_1, \ldots, z_n) \),
\[
w_{i1} = q_i(z_1, \ldots, z_n), \quad 1 \leq i \leq r.
\]
- The values of \( q_i \) over \( D' \) are bounded by the minimum and the maximum of \( \{w_{i1}, \ldots, w_{ri}\} \).
\[
\min(w_{j1}) \leq q_i(x) \leq \max(w_{j1}), \quad 1 \leq i \leq r, \quad x \in D'.
\]
Proof. Let \( \phi(x_1, \ldots, x_n) = (a_1 + 2^{k_1}x_1, \ldots, a_n + 2^{k_n}x_n) \) and consider the system \( \varphi(q_1), \ldots, \varphi(q_r) \) over \([0, 1]^n\). Let \( v \in \mathbb{Q}^{\times n} \) be the rectangular matrix such that its \( i \)-th column is the vector representing \( \varphi(q_i) \) in the Bernstein basis, \( 1 \leq i \leq r \).

The idea now is to compare the system \( \{q_i\}_{i=1}^n \) over \( D \) and the system \( \{\varphi(q_i)\}_{i=1}^n \) over \([0, 1]^n\).

First of all, it is easy to check that \( \varphi \) maps bijectively the box \( A \subseteq [0, 1]^n \) to \( D \subseteq D' \),

\[
A = \prod_{i=1}^n [z_i - a_i, z_i + 2^{k_i} - 2^{k_i}], \quad D' = \prod_{i=1}^n [z_i, z_i + 2^{k_i}].
\]

Let \( \phi_A \) be the affine isomorphism sending \([0, 1]^n \) to \( A \) and let \( \phi_D \) be the one sending \([0, 1]^n \) to \( D' \), that is, \( \phi_A(x_1, \ldots, x_n) = ((z_1 - a_1 + x_1 2^{k_1})/2^{k_1}, \ldots, (z_n - a_n + x_n 2^{k_n})/2^{k_n}) \) and \( \phi_D(x_1, \ldots, x_n) = (z_1, z_2, \ldots, z_n) \). Clearly \( \phi_D = \phi \circ \phi_A \), hence the Bernstein coefficients of \( \phi_D(q_i) \) are the same as the Bernstein coefficients of \( \phi_A(q_i) \). Then, the values of \( \{q_i\}_{i=1}^n \) over \( A \) are the same as the values of \( \{\varphi(q_i)\}_{i=1}^n \) over \( A \). The result follows from Proposition 1.9 by noting that \( A = \prod_{i=1}^n [l_i/2^{k_i}, (l_i + 1)/2^{k_i}] \) and letting \( M \) be the matrix associated to \( A \) and \( d \).

**Definition 1.14.** Boxes appearing in the previous theorem can be parameterized by two multi-indexes \((I, k')\). Let \( a \in \mathbb{Z}^n \), \( k \in \mathbb{N}^n_0 \) and let \( D = \prod_{i=1}^n [a_i, a_i + 2^{k_i}] \). For every \((I, k')\) such that \( 0 \leq k' \leq k \) and \( 0 \leq i < 2^{k_i-k'_i}, l_i \in \mathbb{N}_0, 1 \leq i \leq n \) define \( D_{I,k'} \) as

\[
D_{I,k'} = \prod_{i=1}^n [a_i + l_i 2^{k'_i}, a_i + (l_i + 1) 2^{k'_i}].
\]

Otherwise, define \( D_{I,k'} = \emptyset \). The vectors \( a \) and \( k \) are omitted from the notation. Notice that \( D_{0,k} = D \) and that the volume of \( D_{I,k'} \) is \( 2^{k_1+\cdots+k_{n-I}} \) (or 0). The index \( I \) represents the position and the index \( k' \) the volume of the box \( D_{I,k'} \).

\[\square\]

2. Subdivision algorithm

Let \( d \in \mathbb{N}^n_0 \) be a multi-degree, \( s = (d_1 + 1) \ldots (d_n + 1) \) and let \( \{M^k_{I_1}(d), M^k_{I_0}(d)\}_{k=1}^s \subseteq \mathbb{Q}^{\times n} \). Given that these matrices depend only on \( d \), it is possible to have them previously computed. For example, depending on the application, we can compute these matrices to solve systems in \( \mathbb{Z}[x]_{<d} \) for some large \( d >> 0 \).

Let \( a \in \mathbb{Z}^n \), \( k \in \mathbb{N}^n_0 \), \( D = [a_1, a_1 + 2^{k_1}] \times \cdots \times [a_n, a_n + 2^{k_n}] \) and let \( q_1, \ldots, q_r \in \mathbb{Z}[x]_{<d} \). We want to describe the following sets,

\[
S' = \{ z \in D \cap \mathbb{Z}^n \colon q_2(z) \geq 0, \ldots, q_r(z) \geq 0 \}, \quad S = \left\{ z \in S' \colon q_1(z) = \max_{S'}(q_1) \right\}.
\]

Let \( v \in \mathbb{Q}^{\times n} \) denote the rectangular matrix associated to \( q_1, \ldots, q_r \) (see Theorem 1.13). Initialize
the global variables \( S := \emptyset \) and \( \theta := -\infty \) and run the function \( \text{SolveIPP}(v, 0, k) \).

**Function SolveIPP\((w, l, k')\)**

**Data:** Global variable \( \theta \), a global set \( S \) and matrices \( \{M^l_i(d), M^R_i(d)\}_{i=1}^{n} \).

1. if \( \max(w_1) \geq \theta \) and \( \max(w_2) \geq 0 \) and \( \ldots \) and \( \max(w_r) \geq 0 \) then

2. \[ k'_j := \max(k') \]

3. if \( k'_j > 0 \) then

4. \( \text{SolveIPP}(M^l_i(w, 1 + l_j e_j, k' - e_j)) \)

5. \( \text{SolveIPP}(M^R_i(w, 1 + l_j e_j + e_j, k' - e_j)) \)

6. else if \( w_{11} \geq \theta \) and \( w_{12} \geq 0 \) and \( \ldots \) and \( w_{1r} \geq 0 \) then

7. if \( w_{11} > \theta \) then

8. \( S := \{\} \)

9. \( \theta := w_{11} \)

10. \( S := S \cup \{a + 1\} \)

Let us explain the subdivision process of \( \text{SolveIPP} \) (Solve Integer Polynomial Programming). The first thing to say is that it is a recursive algorithm that makes two matrix multiplications in each call, \( M^l_i \) and \( M^R_i \). The rationality of the matrices makes it robust and reliable and the resulting set \( S \) is equal to its mathematical counterpart \( S \) (see Theorem 2.1 below).

According to Theorem 1.13 the values in \( (w, l, k') \) give information about the polynomials \( \{q_1, \ldots, q_r\} \) over the box \( D_{lk} \). The condition in Step 1 is given to test if there exist solutions on \( D_{lk} \), in other words, if we keep \( D_{lk} \) or we reject it. Assuming that we do not reject \( D_{lk} \), we test in Step 3 if \( D_{lk} \) can be divided.

If \( D_{lk} \) can be divided, we define \( D_L = D_{vl,e_j,k-e_j} \) and \( D_R = D_{vl,e_j+e_j,k-e_j} \) and we analyze the values of the system over these boxes by defining \( w^l = M^l_i v \) and \( w^R = M^R_i v \). The notation \( e_j \) indicates the \( j \)-th canonical vector,

\[
(e_j)_i = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}, \quad 1 \leq i \leq n.
\]

If \( D_{lk} \) cannot be divided, that is \( k' = 0 \), we test in Step 6 if \( a + 1 \) is actually a solution of the system. If this is the case, we save \( \alpha = a + 1 \) in \( S \). If \( w_{11} > \theta \), we reset \( S \) and set \( \theta = q_1(\alpha) \). This last process guarantees that \( q_1(\alpha) = \theta \) for every \( \alpha \in S \).

**Theorem 2.1.** Let \( n \in \mathbb{N}, d, k \in \mathbb{N}_0, a \in \mathbb{Z}^n, q_1, \ldots, q_r \in \mathbb{Z}[x_1, \ldots, x_n]_{\leq k} \), let \( D = \prod_{i=1}^{n}[a_i, a_i + 2k] \) be a box and let \( v \) be the matrix associated to \( q_1, \ldots, q_r \) as in Theorem 1.13. Then, the output of the function \( \text{SolveIPP}(v, 0, k) \) is the set \( S \),

\[
S' = \{z \in D \cap \mathbb{Z}^r : q_2(z) \geq 0, \ldots, q_r(z) \geq 0\}, \quad S = \left\{z \in S' : q_1(z) = \max(q_1)\right\}.
\]

**Proof.** The compactness of \( D \) implies that the set \( S' \) is finite. If \( S' \neq \emptyset \), the set \( q_1(S') \) has a maximum value \( \theta \) and the space of solutions \( S \) is equal to \( q_1^{-1}(\theta) \cap S' \).

Let us prove that every \( z \in S \) is in the output of \( \text{SolveIPP} \) (the reciprocal is straightforward). Let \( D' \) be a box such that \( z \in D' \cap S \). Then \( \max_D(q_1) \geq \theta \) and \( \max_D(q_i) \geq 0, 2 \leq i \leq r \). In particular, the box \( D' \) is not rejected by the algorithm and is subdivided. Then, without loss of generality, we may suppose that \( D' = \prod_{i=1}^{n}[z_i, z_i + 1] \) and clearly, \( z \) is in the output of \( \text{SolveIPP} \). Note that after some steps, the value of the global variable \( \theta \) is equal to the maximum of \( q_1 \) over \( S' \). \( \square \)
3. Expected complexity

Given that the number of matrix multiplications can be computed as the number of times the function SolveIPP is called (minus one), the complexity of the algorithm can be computed in terms of boxes. The worst case scenario is when every box is subdivided (exponential case). The best case occurs when almost every box is unnecessary (linear case). Thus, the complexity of the algorithm is related to the number of unnecessary boxes, and it can be described in probabilistic terms. The following proposition gives a geometric representation of the set of unnecessary boxes.

**Proposition 3.1.** Let \( n \in \mathbb{N} \), \( d, k \in \mathbb{N}_0 \), \( a \in \mathbb{Z}^n \), \( q_1, \ldots, q_r \in \mathbb{Z}[x_1, \ldots, x_n]_{\leq d} \), \( D = \prod_{i=1}^n [a_i, a_i + 2^k] \) and let \( \theta \) be the maximum of \( q_1 \) over \( S' = \{ q_i \geq 0 \}_{i=2}^r \cap D' \cap \mathbb{Z}^n \). Let \( R \) be the set of real solutions of the system,

\[
R = \{ z \in D : q_1(z) \geq \theta, q_2(z) \geq 0, \ldots, q_r(z) \geq 0 \}.
\]

Then, there exists \( d' \in \mathbb{N}_0 \) such that for every \( d'' \geq d' \) the following sentences are equivalent for every non-empty box \( D_{l,k} \):

1. The matrix associated to the system over \( D_{l,k} \) satisfies the condition in Step 1.
2. \( D_{l,k} \cap R \neq \emptyset \).
3. There exists \( D_{r,0} \subseteq D_{l,k} \) such that \( D_{r,0} \cap R \neq \emptyset \).

**Proof.** Let us prove the existence of \( d' \). First, replace \( q_1 \) by \( q_1 - \theta \), and write condition in Step 1 as \( \{ \max(w_i) \geq 0 \}_{i=1}^r \). In each box \( D_{l,k} \) take some \( d_{l,k} \) satisfying

\[
\max(q_i) < 0 \implies \max(w_i) < 0, \quad 1 \leq i \leq r,
\]

where the rectangular matrix \( w = (w_1, w_2, \ldots, w_r) \) is the one associated to the system over the box \( D_{l,k} \) and multi-degree \( d_{l,k} \), see Lemma 1.4. Let \( d' \) be the maximum of \( \{ d_{l,k} \} \).

Assume that we have a box \( D_{l,k} \) without real solutions in it. Let us see that the algorithm applied in multi-degree \( d' \) rejects this box immediately. Given that \( D_{l,k} \) does not contain real solutions, there exists some \( i, 1 \leq i \leq r \), such that \( \min(q_i) < 0 \). Then, \( \min(w_i) < 0 \) and the box \( D_{l,k} \) is rejected.

The equivalence between the last two sentences follows from the fact that it is possible to cover \( D_{l,k} \) with boxes of the form \( D_{l,0} \).

**Definition 3.2.** Let \( n \in \mathbb{N} \), \( d, k \in \mathbb{N}_0 \), \( a \in \mathbb{Z}^n \), \( q_1, \ldots, q_r \in \mathbb{Z}[x_1, \ldots, x_n]_{\leq d} \) and \( D = \prod_{i=1}^n [a_i, a_i + 2^k] \), let \( \theta \) be the maximum of \( q_1 \) over \( S' = \{ q_i \geq 0 \}_{i=2}^r \cap D' \cap \mathbb{Z}^n \) and let \( R \) be the set of real solutions, \( R = \{ z \in D : q_1(z) \geq \theta, q_2(z) \geq 0, \ldots, q_r(z) \geq 0 \} \).

The complexity number of the system \( \{ q_1, \ldots, q_r \} \) over \( D \) is defined as

\[
\lambda(R) = \frac{|\{ \theta : D_{l,0} \cap R \neq \emptyset \}|}{2^{k_1 + \ldots + k_n}}, \quad \lambda(R) \in \mathbb{Q}, \; 0 \leq \lambda(R) \leq 1.
\]

The complexity number can be described in probabilistic terms, it is the probability of a box \( D_{l,0} \) to have real solutions.

**Theorem 3.3.** Let \( \lambda \) be the complexity number of \( \{ q_1, \ldots, q_r \} \) over \( D = \prod_{i=1}^n [a_i, a_i + 2^k] \) and let \( K = k_1 + \ldots + k_n \). Let \( d' \) be the multi-degree from Proposition 3.1. Then, the expected number of
multiplications between a square triangular matrix in \( \mathbb{Q}^{s' \times s'} \) and a rectangular matrix in \( \mathbb{Q}^{s' \times r} \), \( s' = (d'_1 + 1) \ldots (d'_n + 1) \) produced by \texttt{SolveIPP} is

\[
\sum_{i=0}^{K} 2^i \prod_{j=0}^{i-1} (1 - (1 - \lambda)^{2^{j+1}}) - 1.
\]

This number is bounded between \((2(2K)^{2k+1} - 1)/(2K - 1)\) (or \(K\) if \(\lambda = 1/2\)) and \(2kK - 1\).

**Proof.** Each time a box is subdivided, the function \texttt{SolveIPP} produces two matrix multiplications. Hence, the expected number of matrix multiplications is equal to the expected number of divided boxes. Equivalently, it is the expected number of boxes processed by \texttt{SolveIPP} minus 1 (\(D\) is not a divided box). According to Proposition 3.1, the probability of a box \(D_{lk}\) to be subdivided depends on the existence of some \(D_{l'}g\) such that \(D_{l'}g \subseteq D_{lk}\) and \(D_{l'}g \cap R \neq \emptyset\). There are a total of \(2^{k' - j'_l + k'}\) possible boxes \(D_{l'}g\) inside \(D_{lk}\) hence the probability is equal to \(1 - (1 - \lambda)^{2^{j' + k'}}\).

We can model the problem of finding the expected number of boxes as a branching process, see [15]. We say that \(D_{lk}\) is in generation \(i\) if \(k'_l + \ldots + k'_i = K - i\). In other words, if the volume of \(D_{lk}\) is equal to \(2^{K - i}\). Let \(Z_k\) be the number of boxes in each generation. Boxes in generation \(i\) have probability \(\lambda_i\) (resp. \(1 - \lambda_i\)) to add two (resp. 0) boxes to the next generation,

\[
\lambda_i = 1 - (1 - \lambda)^{2^{i+1}}, \quad 0 \leq i \leq K.
\]

These boxes can be interpreted as independent identically distributed random variables \(D_i\) with common generating function \(1 - \lambda_i + \lambda_i \lambda^2\).

Thus \(Z_{i+1} = D_i + \ldots + D_{Z_i}\) and from [14, §5.1 Th.25] we obtain \(E(Z_{i+1}) = 2\lambda_i E(Z_i)\), hence

\[
E(Z_0) = 1, \quad E(Z_{i+1}) = 2\lambda_i E(Z_i) = 2^{i+1} \lambda_i \ldots \lambda_0, \quad 0 \leq i \leq K - 1.
\]

The number of expected boxes is \(E(Z_0) + \ldots + E(Z_K)\).

**Example.** Consider the equation \(y = x\) in \([0, 2] \times [0, 2]\). Clearly, it has two boxes with real solutions, \([0, 1] \times [0, 1]\) and \([1, 2] \times [1, 2]\). Then, \(\lambda = 2/4\) and the expected number of boxes processed by \texttt{SolveIPP} for \(d'\) as in Proposition 3.1 is \(91/16\), a number between \(2 + 1 = 3\) and \(2^3 - 1 = 7\). The function applied to this example passes Step 1, 7 times.

**Example.** Consider the system \(y - x^2 = 0\) in \(D = [0, 2^2] \times [0, 2^2]\). The system has three integral solutions, \((0, 0), (1, 1)\) and \((2, 4)\). The real curve pass through 8 boxes of the form \(D_{l'g}\). Then,

\[
\lambda = 8/64, \quad (\lambda_0, \ldots, \lambda_7) \approx (.99, .98, .88, .65, .41, .23, .12).
\]

Hence, the expected number of boxes processed by \texttt{SolveIPP} for \(d'\) as in Proposition 3.1 is approximately 34. In this example Step 1 occurs 33 times.

**Remark.** In [23] the authors manage to to improve (a variant of) \texttt{SolveIPP} to produce better performance. Also, it is possible to adapt \texttt{SolveIPP} to treat systems of continuous functions by using Stone-Weierstrass Theorem [29, Cor.1] and interpolation techniques [7, 9].

Notice that Proposition 3.1 implies that the strategy of dividing a side of a box in half is not optimal. For example, consider the equation \(x - 2\) over \([0, 4]\). The function \texttt{SolveIPP} divides \([0, 4]\) in two halves making two matrix multiplications, but given that \texttt{SolveIPP} do not reject these two halves, it makes four multiplications more.

Instead of dividing \([0, 4]\) into \([0, 2]\) and \([2, 4]\) it is better (from a performance point of view) to divide it into \([0, 2]\) and \([2, 4]\) or even better into \([0, 1]\) and \([2, 4]\).\[\square\]
Remark. Given \( d \in \mathbb{N}^n \), we need to compute the triangular matrices \( \begin{bmatrix} M^L_i \end{bmatrix}_{i=1}^{s} \subseteq \mathbb{Q}^{n \times s} \), \( s = \prod_{i=1}^{n} (d_i + 1) \) and save them into the computer memory. If \( ns(s+1) \) is greater than the total amount of available memory it is possible to use techniques from computer science such as Parallel computing or Map-Reduce \([10, 25]\) in order to increase the available space of memory. An alternative (or complementary) solution is to use toric Bernstein polynomials to minimize the number of monomials, \([31, \text{Cor.2}]\). \( \square \)

4. Solving systems of congruence equations

In this section we apply the subdivision method presented in Section 2 to solve congruence equations (SolveCE). Let \( h_1, \ldots, h_r \in \mathbb{Z}[x_1, \ldots, x_n]_{\leq 0} \) and let \( p_1, \ldots, p_r \in \mathbb{Z}_{\geq 2} \) be natural numbers (possibly coprimes). Here we propose a method to find solutions \( z \in \mathbb{Z}^n \) to the system

\[
\begin{align*}
\begin{cases}
h_1(z) \equiv 0 \mod(p_1), \\
\vdots \\
h_r(z) \equiv 0 \mod(p_r).
\end{cases}
\end{align*}
\]

In order to do so, we need to adapt some of the propositions presented so far. Notice that any solution \( z \in \mathbb{Z}^n \) can be represented in the box \([0, 2^1] \times \ldots \times [0, 2^k]\), where \( k = \left\lceil \log_2(\text{lcm}(p_1, \ldots, p_r)) \right\rceil \in \mathbb{Z}_{\geq 2} \).

\( \text{lcm}(x, y) \) is the least common multiple of \( x \) and \( y \) and \( \lceil x \rceil \) is the least integer greater than or equal to \( x \). The following simple Lemma is the key ingredient of our method.

Lemma 4.1. Let \( p \in \mathbb{Z}_{\geq 2} \), let \( m_1 \leq m_2 \) be two integers and let \( r_1 \) (resp. \( r_2 \)) be the remainder of \( m_1 \) (resp. \( m_2 \)) divided by \( p \). The following sentences are equivalent,

1. \( m_2 - m_1 < p - 1 \) and \( 0 < r_1 \leq r_2 \),  
2. There is no multiple of \( p \) between \( m_1 \) and \( m_2 \).

Proof. Let \( q_1, q_2 \in \mathbb{Z} \) be such that \( m_1 = q_1 p + r_1 \) and \( m_2 = q_2 p + r_2 \). Given that \( m_1 \leq m_2 < m_1 + p - 1 \), we have the inequality,

\[
p - 1 > m_2 - m_1 = (q_2 - q_1) p + (r_2 - r_1) \geq 0.
\]

1\( \Rightarrow \)2) Given that \( 0 \leq r_2 - r_1 \leq p - 1 \), we have \((q_2 - q_1) p \geq 0 \) and from \( p - 1 > (q_2 - q_1) p \geq 0 \) it follows \( q_2 = q_1 \). Then, there exists \( q \in \mathbb{Z} \) such that \( m_1 = q p + r_1 \) and \( m_2 = q p + r_2 \). Assume now, that there exists \( a \in \mathbb{Z} \) such that \( m_1 \leq ap \leq m_2 \). Then, \( m_2 - ap \) is also \( < p - 1 \) and it follows \( a = q \). Hence,

\[
m_1 \leq qp \leq m_2 \iff qp + r_1 \leq qp \leq qp + r_2 \iff r_1 \leq 0 \leq r_2.
\]

A contradiction, because \( r_1 > 0 \).

2\( \Rightarrow \)1) It is clear that if \( m_2 - m_1 \geq p - 1 \), then there exists a multiple of \( p \) between \( m_1 \) and \( m_2 \). Same if \( r_1 = 0 \) or \( r_2 = 0 \). Assume \( m_2 - m_1 < p - 1 \) and \( r_1 > r_2 \) and let us prove that there exists a multiple of \( p \) between \( m_1 \) and \( m_2 \).

Using the inequality \( p - 1 > (q_2 - q_1) p + (r_2 - r_1) \geq 0 \) and \( 0 < r_1 - r_2 \leq p - 1 \) it follows that \( q_2 = q_1 + 1 \). Then, \( q_1 p \) is in between \( m_1 \) and \( m_2 \). \( \square \)
A direct application of Lemma 4.1 gives the following function.

\textbf{Function hasSol}(m_1,m_2, p)

\textbf{Data:} \( m_1 \leq m_2 \in \mathbb{Q} \) and \( p \in \mathbb{Z}_{\geq 2} \).

\textbf{Output:} True if there is a multiple of \( p \) between \( m_1 \) and \( m_2 \). False otherwise.

1. \((m_1, m_2) \leftarrow ([m_1], [m_2])\)
2. \((r_1, r_2) \leftarrow (m_1 \% p, m_2 \% p)\)
3. \textbf{if} \( m_2 - m_1 < p - 1 \) \textbf{and} \( 0 < r_1 \leq r_2 \) \textbf{then}
4. \quad \textbf{return} \ True
5. \textbf{return} \ False

Step 1 in hasSol is based on the following fact. There exists an integer \( i \) such that \( m_1 \leq i \leq m_2 \) if only if \([m_1] \leq i \leq [m_2]\), where \([m_1]\) is the least integer greater than or equal to \( m_1 \) and \([m_2]\) is the greatest integer less than or equal to \( m_2 \). The notation \( m_1 \% p \) in Step 2 means the remainder of dividing \( m_1 \) by \( p \). Same for \( m_2 \% p \).

Now, we can adapt the function SolveIPP. Fix some \( d \in \mathbb{N}_0 \) and let \( \{M^i_j(d), M^r_j(d)\}_{i=1}^n \) be the matrices defined in 1.7. Let \( h_1, \ldots, h_r \in \mathbb{Z}[x]_{\leq d} \) and let \( p_1, \ldots, p_r \in \mathbb{Z}_{\geq 2} \) be natural numbers. Let \( k := \lceil \log_2(\text{lcm}(p_1, \ldots, p_r)) \rceil \in \mathbb{Z}_{\geq 2} \) and \( D = [0, 2^k] \times \ldots \times [0, 2^k] \). Denote \( p = (p_1, \ldots, p_r) \), \( \mathbf{0} = (0, \ldots, 0) \) and \( \mathbf{k} = (k, \ldots, k) \).

We want to describe the following set,

\[ S' = \{z \in D \cap \mathbb{Z}^n : h_1(z) \equiv 0 \mod (p_1), \ldots, h_r(z) \equiv 0 \mod (p_r)\} \]

Let \( v \) be the rectangular matrix associated to \( h_1, \ldots, h_r \) as in Theorem 1.13. Initialize the global set \( S' := \emptyset \) and run SolveCE\((p, v, \mathbf{k}, \mathbf{0})\).

\textbf{Function SolveCE}(p,v,k', l)

\textbf{Data:} A global set \( S' \) and matrices \( \{M^i_j(d), M^r_j(d)\}_{i=1}^n \).

\textbf{Output:} \( S' = \{z \in D \cap \mathbb{Z}^n : h_1(z) \equiv 0 \mod (p_1), \ldots, h_r(z) \equiv 0 \mod (p_r)\} \)

1. \textbf{if} \ hasSol \((\text{min}(w_1), \text{max}(w_1), p_1)\) \textbf{and} \ldots \textbf{and} \ hasSol \((\text{min}(w_r), \text{max}(w_r), p_r)\) \textbf{then}
2. \quad \( k'_j := \max(k') \)
3. \quad \textbf{if} \( k'_j > 0 \) \textbf{then}
4. \quad \quad \text{SolveCE}(p, M^{j, w}_j, k' - e_j, l + 1 \cdot e_j)
5. \quad \quad \text{SolveCE}(p, M^{j, w}_j, k' - e_j, l + 1 \cdot e_j + e_j)
6. \quad \textbf{else} \( p_1|w_1 \) \textbf{and} \ldots \textbf{and} \( p_r|w_r \) \textbf{then}
7. \quad \quad \( S' := S' \cup \{l\} \)

\textbf{Theorem 4.2.} The output of \ SolveCE\((p, v, (k, \ldots, k),(0, \ldots, 0))\) is the set

\[ S' = \{z \in D \cap \mathbb{Z}^n : p_1|h_1(z), \ldots, p_r|h_r(z)\}, \]

where \( D = [0, 2^k] \times \ldots \times [0, 2^k] \), \( k = \lceil \log_2(\text{lcm}(p_1, \ldots, p_r)) \rceil \) and \( v \) is the matrix associated to \( h_1, \ldots, h_r \), as in Theorem 1.13.

\textbf{Proof.} If there is a solution in a box \( D' \), then the process goes from Step 1 to Step 2 and \( D' \) gets subdivided unless the volume of \( D' \) is equal to 1. If the volume of \( D' \) is equal to 1 and there exists a solution \( l \in D' \), then the condition in Step 6 is satisfied and \( l \) is saved in \( S' \). \hfill \square
As before, we can estimate the expected number of matrix multiplications that SolveCE do. Let \( h : \mathbb{R}^n \to \mathbb{R}^l \) be the polynomial function \( h = (h_1, \ldots, h_r) \) and consider the discrete set
\[
X = \{(x_1, \ldots, x_r) \in \mathbb{Z}^r : p_i|x_1, \ldots, p_i|x_r\}.
\]
Notice that the set \( h^{-1}(x) \subseteq \mathbb{R}^n \) is a real algebraic variety for each \( x \in X \).

**Proposition 4.3.** Let \( n \in \mathbb{N}, d \in \mathbb{N}^r_0, h_1, \ldots, h_r \in \mathbb{Z}[x_1, \ldots, x_l], p_1, \ldots, p_r \in \mathbb{Z}_{\geq 2} \) and let \( k = \lceil \log_2(\text{lcm}(p_1, \ldots, p_r)) \rceil \). Let \( X = \{x \in \mathbb{Z}^r : p_i|x_1, \ldots, p_i|x_r\} \) and let \( h : \mathbb{R}^n \to \mathbb{R}, h = (h_1, \ldots, h_r) \). Then, there exists \( d' \in \mathbb{N}^r \) such that for every \( d'' \geq d' \) and every non-empty box \( D_{1,k} \), the following sentences are equivalent:

1. \( h^{-1}(X) = \emptyset \).
2. There exists \( D_{1,0} \subseteq D_{1,k} \) such that \( D_{1,0} \cap h^{-1}(X) \neq \emptyset \).

**Proof.** Clearly sentence 2 is equivalent to sentence 3. Let us prove the equivalence between sentences 1 and 2. Fix some \( l \) and \( k' \). Let us define
\[
m'_i := \begin{cases} \lfloor \min_{D_{1,k}}(h_i) \rfloor & \text{if } \min_{D_{1,k}}(h_i) > \lfloor \min_{D_{1,k}}(h_i) \rfloor \\
\min_{D_{1,k}}(h_i) - \frac{1}{2} & \text{if not}
\end{cases}
\]
\[
m''_i := \begin{cases} \lceil \max_{D_{1,k}}(h_i) \rceil & \text{if } \max_{D_{1,k}}(h_i) < \lceil \max_{D_{1,k}}(h_i) \rceil \\
\max_{D_{1,k}}(h_i) + \frac{1}{2} & \text{if not}
\end{cases}
\]
Choose a multi-degree \( d_{1,k} \) as in Lemma 1.4 such that
\[
m'_i < \min(w_i) \leq \max(w_i) < m''_i, \quad 1 \leq i \leq r,
\]
where \( w = (w_1, \ldots, w_r) \) is the rectangular matrix associated to \( (h_1, \ldots, h_r) \) over \( D_{1,k} \). By construction, there exists a multiple of \( p_i \) between \( \min_{D_{1,k}}(h_i) \) and \( \max_{D_{1,k}}(h_i) \) if and only if it is between \( \min(w_i) \) and \( \max(w_i) \). The result follows by taking \( d' \) as the maximum of \( \{d_{1,k}\} \). \( \square \)

**Theorem 4.4.** Let \( h_1, \ldots, h_r \in \mathbb{Z}[x_1, \ldots, x_l], p_1, \ldots, p_r \in \mathbb{Z}_{\geq 2}, k = \lceil \log_2(\text{lcm}(p_1, \ldots, p_r)) \rceil \) and let \( d' \) be as in Proposition 4.3. Let \( X = \{x \in \mathbb{Z}^r : p_i|x_1, \ldots, p_i|x_r\} \) and let \( h : \mathbb{R}^n \to \mathbb{R}, h = (h_1, \ldots, h_r) \). Then, the expected number of matrix multiplications produced by SolveCE in \( D = \prod_{i=1}^r [0, 2^{d_i}] \) is
\[
\sum_{l=0}^{2k} 2^{l-1} \prod_{j=1}^{l-1} \left( 1 - (1 - \lambda)^{2^{d_j}} \right) - 1, \quad \lambda := \frac{\#\{l : D_{1,0} \cap h^{-1}(X) \neq \emptyset\}}{2^{2k}}.
\]
The number \( \lambda \) is the probability of a box \( D_{1,0} \) to intersect \( h^{-1}(X) \).

**Proof.** According to Proposition 4.3 the probability of a box \( D_{1,k} \) to be rejected in Step 1 is \((1 - \lambda)^{2^{d_1} \cdots 2^{d_k}}\). The proof continues as in Theorem 3.3. \( \square \)

**Definition 4.5.** We define the brute-force algorithm as the algorithm consisting in testing each \( z \in D \cap \mathbb{Z}^n \) to be in \( S' \),
\[
S' = \{z \in D \cap \mathbb{Z}^n : p_i|h_1(z), \ldots, p_i|h_r(z)\}.
\]
This method performs $r 2^{nk}$ evaluations to characterize $S'$. Notice that there is no need for the sides of $D$ to be a power of two and the same is true in SolveCE. We chose to have sides to be a power of two to make the presentation clearer and also to be able to compare this algorithm with SolveCE.

For example, if $p_1 = p_2 = 5$ and $h_1 = x + 1$, $h_2 = y$ we need to take the starting box to be $D = [0, 8]^2$. The brute-force algorithm with the box $D = [0, 5]^2$ performs $25 \times 2$ evaluations to find the solution $(4, 0)$. The function SolveCE finds two (apparently different) solutions $(4, 0)$ and $(4, 5)$ in the box $D = [0, 8]^2$.

In order to avoid benefiting either algorithm, we choose powers of two for the numbers $p_1, \ldots, p_r$. In the next table we compare SolveCE with the brute-force algorithm. All computations were made using our implementation in SageMath.

| $p$   | $h$                      | Step 1 | Step 6 | BF |
|--------|--------------------------|--------|--------|----|
| 512    | $2x - 3$                 | 19     | 2      | 512|
| 64     | $x^2 + 3x - 4$           | 111    | 51     | 64 |
| 128    | $x^2 + 5x^2 - 9$         | 254    | 127    | 128|
| (512, 512) | $(x + 1, y)$          | 40     | 4      | 256244|
| (8, 8) | $(xy + 2x + y^2, 3y + x^2)$ | 123   | 60     | 64 |
| (128, 128) | $(y(x - 2), y^3 + 1)$  | 29124  | 13675  | 16384|
| (16, 16, 16) | $(x + y - z, 2x + y - 3z, z - 3 + y)$ | 491   | 84     | 4096|
| (4, 8, 16) | $(x + 5yz - z^2, y^2, x + z + 2)$ | 1923  | 720    | 512|

The first two columns codify the system $h \equiv 0 \mod (p)$. The column Step 1 is the number of times SolveCE pass through the first step. The column Step 6 is the number of boxes $D_{1,0}$ tested to have a solution. Finally, the last column is the number of evaluations that the brute-force algorithm must do. We can infer from the table that SolveCE is faster for linear systems, but for linear systems there are much better algorithms, [4, 27]. We have prioritized presentation over performance.

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