Noether Symmetries of Two-Field Cosmological Models

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Abstract. We summarize our work on “hidden” Noether symmetries of multifield cosmological models and the classification of those two-field cosmological models which admit such symmetries.

INTRODUCTION

Cosmological models with at least two real scalar fields are of increasing interest in cosmology (for example, as models of quintessence as well as in descriptions of inflation which may allow for contact with string theory). Careful study shows that certain such models possess “hidden” Noether symmetries [1, 2], which – when present – allow one to simplify various problems and sometimes to construct exact solutions which are not accessible through other means. We summarize our recent work aimed at the classification of $n$-field cosmological models admitting such symmetries. For the case of two real scalar fields, our results show that all such models are of generalized $\alpha$-attractor type [3] with scalar manifold a disk, a punctured disk or an annulus [4], endowed with a complete metric with fixed value $K = -3/8$ of the Gaussian curvature.

FLRW COSMOLOGY WITH SCALAR MATTER

Definition 1 A scalar triple is an ordered system $(\mathcal{M}, \mathcal{G}, V)$, where:

- $(\mathcal{M}, \mathcal{G})$ is a Riemannian $n$-manifold (called scalar manifold)
- $V \in C^\infty(\mathcal{M}, \mathbb{R})$ is a smooth real-valued function defined on $\mathcal{M}$ (called scalar potential).

We shall assume throughout the paper that $\mathcal{M}$ is connected, oriented and of dimension $n \geq 2$ and that $\mathcal{G}$ is complete. Moreover, we assume for simplicity that $V > 0$ on $\mathcal{M}$. In most applications to cosmology, the scalar manifold is non-compact; however, notice that it need not be simply-connected.

Each scalar triple defines a scalar cosmological model, which describes Einstein gravity on a space-time with topology $\mathbb{R}^4$, coupled to $n = \dim \mathcal{M}$ scalar fields described by a smooth map $\phi \in C^\infty(\mathbb{R}^4, \mathcal{M})$ from the space-time to the scalar manifold $\mathcal{M}$; the scalar potential describes self-interaction of the scalar fields. The cosmological model has action:

$$ S_{\mathcal{M}, \mathcal{G}, V}[g, \phi] = \int_{\mathbb{R}^4} \left[ \frac{R(g)}{2} - \frac{1}{2} \text{Tr}_g \phi^\ast(\mathcal{G}) - V \circ \phi \right] \text{vol}_g, $$

where the space-time metric $g$ has ‘mostly plus’ signature. Locally, we have $\text{Tr}_g \phi^\ast(\mathcal{G}) = g^{ij} \partial_i \phi^i \partial_j \phi^j$, where we use the Einstein summation convention. Completeness of the scalar manifold metric $\mathcal{G}$ ensures conservation of energy in such models.

The metric $g$ of simply connected and spatially flat FLRW universe $(\mathbb{R}^4, g)$ has the following squared line element in global Cartesian coordinates $(t = x^0, x^1, x^2, x^3)$, where the strictly positive smooth function $a \in C^\infty(\mathbb{R}, R_{>0})$ is called the scale factor:

$$ ds^2_g := -dt^2 + a(t)^2 d\vec{x}^2, \quad \text{where } \vec{x} = (x^1, x^2, x^3). $$
This implies \( \text{vol}_k = \sqrt{\det g} d^3 \bar{x} dt = a(t)^3 d^3 \bar{x} dt \). Spatially homogeneous scalar field configurations in an FLRW universe are described by maps \( \varphi \) which depend only on the cosmological time \( t \):

\[
\varphi(t, \bar{x}) = \varphi(t) .
\]

This gives \( \text{Tr}_g \varphi^* (\mathcal{G}) = ||\dot{\varphi}||^2_g \), where \( \cdot^\text{def.} \frac{d}{dt} \).

THE MINISUPERSPACE LAGRANGIAN AND THE FRIEDMANN CONSTRAINT

Substituting (2) and (3) in (1) and ignoring integration over \( \bar{x} \) produces the minisuperspace action:

\[
S_{\mathcal{H}, \mathcal{G}, \mathcal{V}}[a, \varphi] = \int_{-\infty}^{\infty} dt L_{\mathcal{H}, \mathcal{G}, \mathcal{V}}(a(t), \dot{a}(t), \varphi(t), \dot{\varphi}(t)) ,
\]

where the minisuperspace Lagrangian takes the form:

\[
L_{\mathcal{H}, \mathcal{G}, \mathcal{V}}(a, \dot{a}, \varphi, \dot{\varphi}) = a^3 \left[-3H^2 + \frac{1}{2} ||\dot{\varphi}||^2_g - V(\varphi) \right] .
\]

Here \( H \overset{\text{def.}}{=} \frac{\dot{a}}{a} \) is the Hubble parameter. This autonomous Lagrangian describes a mechanical system with \( n + 1 \) degrees of freedom and configuration space \( \mathcal{N} = \mathbb{R}_{>0} \times \mathcal{M} \). The \( g^{ij} \) component of the Einstein equations gives the Friedmann constraint:

\[
\frac{1}{2} ||\dot{\varphi}||^2_g + V(\varphi) = 3H^2 ,
\]

which is equivalent with the zero energy shell condition. Indeed, the canonical momenta of \( L \) are:

\[
p_a = \frac{\partial L}{\partial \dot{a}} = -6a \ddot{a} = -6a^2 H , \quad p_i = \frac{\partial L}{\partial \varphi^i} = G_{ij}(\varphi) a^3 \dot{\varphi}^j = \mathcal{G} p^j
\]

while the phase space is the cotangent bundle \( \mathcal{P} \overset{\text{def.}}{=} T^* \mathcal{N} \simeq \mathbb{R}_{>0} \times \mathbb{R} \times T^* \mathcal{M} \) to the configuration space, endowed with its canonical symplectic structure with local form:

\[
\omega = dp_a \wedge da + dp_i \wedge d\varphi^i .
\]

The minisuperspace Hamiltonian is:

\[
\mathcal{H} = p_a \dot{a} + p_i \dot{\varphi}^i - L = -\frac{1}{12a^2} p_a^2 + \frac{1}{2a^3} ||p||^2_g + a^3 V \circ \varphi = a^3 \left[-3H^2 + \frac{1}{2} ||\dot{\varphi}||^2_g + V \circ \varphi \right]
\]

and hence the Friedmann constraint amounts to requiring \( \mathcal{H} = 0 \). Since the Hamiltonian is conserved, the Friedmann constraint (6) amounts to an algebraic relation between the integration constants of the solutions to the Euler-Lagrange equations of (5).

The Euler-Lagrange and cosmological equations

The Euler-Lagrange equations of (5) are equivalent with:

\[
3H^2 + 2H + \frac{1}{2} ||\dot{\varphi}||^2_g - V(\varphi) = 0 \quad (8)
\]

\[
(\nabla_i + 3H) \dot{\varphi}^i + (\text{grad}_g V)(\varphi) = 0
\]

where:

\[
\nabla_i \varphi^i = \varphi^i + \Gamma^i_{jk} \varphi^j \varphi^k
\]

\[
\text{grad}_g V = (\text{grad}_g V)^i \partial_i = [\mathcal{G}^{ij} \partial_j] V \partial_i
\]

and \( \partial_j := \frac{\partial}{\partial \varphi^j} \).
Proposition 2 When supplemented with the Friedmann constraint, the Euler-Lagrange equations (8) are equivalent with the cosmological equations:

\[ 6H^2 - \| \dot{\phi} \|_{\mathcal{g}}^2 - 2V(\phi) = 0 \]
\[ \nabla_t \phi + 3H \phi + (\text{grad}_\mathcal{g} V)(\phi) = 0 \]  
(9)

Remark 3 When \( \dot{a} > 0 \), one can eliminate \( H \) algebraically from the first cosmological equation as:

\[ H(t) = \frac{1}{\sqrt{6}} \sqrt{\| \phi(t) \|_{\mathcal{g}}^2 + 2V(\phi(t))} \]  

Then (9) gives the reduced cosmological equation for \( \phi(t) \):

\[ \nabla_t \phi(t) + \sqrt{\frac{3}{2}} \sqrt{\| \phi(t) \|_{\mathcal{g}}^2 + 2V(\phi(t))} \phi(t) + (\text{grad}_\mathcal{g} V)(\phi(t)) = 0 \]  
(10)

which defines a (dissipative) geometric dynamical system on \( T.\mathcal{M} \).

VARIATIONAL SYMMETRIES

The tangent space to the configuration space \( \mathcal{N} = \mathbb{R}_{>0} \times \mathcal{M} \) has the decomposition \( T\mathcal{N} = T_1\mathcal{N} \oplus T_2\mathcal{N} \), where \( T_1 \) and \( T_2 \) are the pullbacks of the vector bundles \( T\mathbb{R}_{>0} \) and \( T\mathcal{M} \) through the canonical projections. Accordingly, any vector field \( X \in \mathcal{T}(\mathcal{N}) \) decomposes as:

\[ X = X_1 + X_2 \]

In local coordinates \( (U, a, \varphi^i) \) on \( \mathcal{N} \), we have:

\[ X_1(a, \varphi) = X^a(a, \varphi) \frac{\partial}{\partial a} \]  
\[ X_2(a, \varphi) = X^i(a, \varphi) \frac{\partial}{\partial \varphi^i} \]

Definition 4 A (strong) variational symmetry of \( L \) is a vector field \( X \in \mathcal{T}(\mathcal{N}) \) which satisfies the Noether symmetry condition:

\[ \mathcal{L}_X^1(L) = 0 \]  
(11)

Here \( X^1 \in \mathcal{T}(T\mathcal{N}) \) is the first jet prolongation of \( X \), with local form:

\[ X^1 = X(a, \varphi) + X^a(a, \varphi) \frac{\partial}{\partial \dot{a}} + X^i(a, \varphi) \frac{\partial}{\partial \dot{\varphi}^i} \]

The Noether symmetry condition takes the local form:

\[ X^a \frac{\partial L}{\partial \dot{a}} + X^i \frac{\partial L}{\partial \dot{\varphi}^i} + \dot{X}^a \frac{\partial L}{\partial a} + \dot{X}^i \frac{\partial L}{\partial \varphi^i} = 0 \]  
(12)

where the total time derivative \( \dot{\lambda} \) of a locally-defined function \( \lambda \) on the configuration space has the local expression:

\[ \dot{\lambda}(a, \varphi) := \frac{\partial \lambda}{\partial t} + \frac{\partial \lambda}{\partial a} \dot{a} + \frac{\partial \lambda}{\partial \varphi^i} \dot{\varphi}^i \]

The characteristic system for variational symmetries

Theorem 5 For the Lagrangian (5), the Noether symmetry condition amounts to the requirement that \( X_1 \) and \( X_2 \) have the following forms:

\[ X_1(a, \varphi) = \frac{\Lambda(\varphi)}{\sqrt{a}} \frac{\partial}{\partial a} \]  
\[ X_2(a, \varphi) = \left[ Y^i(\varphi) - \frac{4}{a^{3/2}} (\text{grad}_\mathcal{g} \Lambda)(\varphi) \right] \frac{\partial}{\partial \varphi^i} \]
where $\Lambda \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ and $Y \in \mathcal{X}(\mathcal{M})$ satisfy the characteristic system of the scalar triple $(\mathcal{M}, \mathcal{G}, V)$:

\[
\text{Hess}_\mathcal{G}(\Lambda) = \frac{3}{8} \mathcal{G} \Lambda, \quad \mathcal{X}_\mathcal{G}(Y) = 0
\]

\[
\langle dV, d\Lambda \rangle_\mathcal{G} = \frac{3}{4} V \Lambda, \quad Y(V) = 0,
\]

which can also be written in the index-full form:

\[
\left( \partial_i \partial_j - \Gamma^k_{ij} \partial_k \right) \Lambda = \frac{3}{8} \mathcal{G}_{ij} \Lambda, \quad \nabla_i Y_j + \nabla_j Y_i = 0
\]

\[
\mathcal{G}^{ij} \partial_i V \partial_j \Lambda = \frac{3}{4} V \Lambda, \quad Y^i \partial V = 0.
\]

Here $\text{Hess}_\mathcal{G}(\Lambda) \overset{\text{def}}{=} \nabla \Lambda$ (the Hesse tensor of $\Lambda$) and $\mathcal{X}_\mathcal{G}(Y) \overset{\text{def}}{=} \text{Sym}^2(\nabla Y)$ (the Killing tensor of $Y$).

**Visible and Hessian symmetries**

The characteristic system separates into the $\Lambda$-system:

\[
\left( \partial_i \partial_j - \Gamma^k_{ij} \partial_k \right) \Lambda = \frac{3}{8} \mathcal{G}_{ij} \Lambda, \quad \mathcal{G}^{ij} \partial_i V \partial_j \Lambda = 2 V \Lambda
\]

and the $Y$-system:

\[
\nabla_i Y_j + \nabla_j Y_i = 0, \quad Y^i \partial V = 0,
\]

where we use the rescaled scalar manifold metric $\mathcal{G} \overset{\text{def}}{=} \frac{3}{8} \mathcal{G}$. By Theorem\[5\] strong infinitesimal Noether symmetries have the form:

\[
X = X_\Lambda + Y,
\]

where:

\[
X_\Lambda \overset{\text{def}}{=} \frac{\Lambda}{\sqrt{\mathcal{G}}} \partial_\mu - \frac{4}{a^2} \text{grad}_\mathcal{G} \Lambda,
\]

with $\Lambda \in \mathcal{C}^\infty(\mathcal{M}, \mathbb{R})$ and $Y \in \mathcal{X}(\mathcal{M})$.

**Definition 6**

- A non-trivial vector field $X_\Lambda$ depending on a given solution $\Lambda$ of the $\Lambda$-system is called a **Hessian symmetry** of $(\mathcal{M}, \mathcal{G}, V)$. We say that the scalar cosmological model is **Hessian** if it admits a non-trivial Hessian symmetry.

- A non-trivial solution $Y$ of the $Y$-system is called a **visible symmetry** of $(\mathcal{M}, \mathcal{G}, V)$. We say that the scalar cosmological model is **visibly symmetric** if it admits a non-trivial visible symmetry.

Let $N(\mathcal{M}, \mathcal{G}, V)$ denote the vector space of solutions to the characteristic system and $N_H(\mathcal{M}, \mathcal{G}, V)$ and $N_V(\mathcal{M}, \mathcal{G}, V)$ denote the vector spaces of solutions of the $\Lambda$- and $Y$-systems respectively.

**Proposition 7** There exists a linear isomorphism:

\[
N(\mathcal{M}, \mathcal{G}, V) \simeq \mathbb{R} N_H(\mathcal{M}, \mathcal{G}, V) \oplus N_V(\mathcal{M}, \mathcal{G}, V).
\]

In particular, a scalar triple $(\mathcal{M}, \mathcal{G}, V)$ admits strong Noether symmetries if it is Hessian or visibly symmetric (or both).

Since existence of visible symmetries is a well-studied problem, we focus on characterizing Hessian symmetries.

**Remark 8** As shown in\[2\], the integral of motion of a Hessian symmetry allows one to simplify the computation of the number of e-folds along cosmological trajectories, for which it gives the non-integral formula:

\[
N_{\theta_0}(t) = \frac{2}{3} \log \left[ \frac{\Lambda(\phi(t_0)) + \frac{1}{2} H(t_0) \Lambda(\phi(t_0)) + (d\Lambda)(\phi(t_0))(\phi(t_0))(t - t_0)}{\Lambda(\phi(t))} \right]
\]

in terms of a solution $\phi(t)$ of the reduced cosmological equation\[10\].
**Hesse functions and Hesse manifolds**

**Definition 9** A Hesse function of $\mathcal{M}, G$ is a smooth global solution $\Lambda \in C^\infty(\mathcal{M}, \mathbb{R})$ of the Hesse equation:

$$\text{Hess}_G(\Lambda) = GA \iff \left( \partial_i \partial_j - \Gamma^k_{ij} \partial_k \right) \Lambda = G_{ij} \Lambda .$$

We denote by $\mathcal{H}(\mathcal{M}, G)$ the linear space of Hesse functions of $(\mathcal{M}, G)$. When this space is non-trivial, we say that $(\mathcal{M}, G)$ is a Hesse manifold. (Notice that a Hesse manifold need not be a Hessian manifold, i.e. its metric need not be given locally by the Hessian of a function!)

**Remark 10** One can show that any Hesse manifold is non-compact and that $\dim \mathcal{H}(\mathcal{M}, G) \leq n + 1$, where $n = \dim \mathcal{M}$. The space $\mathcal{H}(\mathcal{M}, G)$ is endowed with the Hesse pairing $(\ ,\ )_G$, a (possibly degenerate) natural bilinear symmetric pairing which is invariant under the action of the isometry group of $(\mathcal{M}, G)$ on Hesse functions. Moreover, one can show that $\dim \mathcal{H}(\mathcal{M}, G) = n + 1$ iff $(\mathcal{M}, G)$ is an elementary hyperbolic space form. See [3] for these and other results on Hesse manifolds.

**Example** The $n$-dimensional Poincaré ball $\mathbb{D}^n \overset{\text{def}}{=} (\mathbb{D}^n, G)$ (with Poincaré metric $G$) has $\dim \mathcal{H}(\mathcal{M}, G) = n + 1$. Let $\mathbb{R}^{1,n} = (\mathbb{R}^{n+1}, (\ ,\ ))$ denote the $(n+1)$-dimensional Minkowski space, where $(\ ,\ )$ is the canonical Minkowski pairing and let $E_0, E_1, \ldots, E_n$ be the canonical orthonormal basis of $\mathbb{R}^{1,n}$. Let $\Xi = (\Xi^0, \Xi^1, \ldots, \Xi^n) : \mathbb{D}^n \to \mathbb{R}^{n+1}$ denote the Weierstrass map, where $\Xi^0, \ldots, \Xi^n$ are the Weierstrass coordinates.

**Theorem 11** [5] There exists a bijective isometry $\Lambda : \mathbb{R}^{1,n} \overset{\text{def}}{\sim} (\mathcal{H}(\mathbb{D}^n), (\ ,\ )_G)$ such that:

$$\Lambda_E(u) \overset{\text{def}}{=} \Lambda(E_{\mu})(u) = (E_{\mu}, \Xi(u)) , \ \forall u \in \mathbb{D}^n , \ \forall \mu \in \{0, \ldots, n\} .$$

## NOETHER SYMMETRIES OF TWO-FIELD COSMOLOGICAL MODELS

Let $\mathcal{M} := \Sigma$ be a connected and oriented surface ($\dim \Sigma = 2$), endowed with the complete metric $\mathcal{G}$. Let $G \overset{\text{def}}{=} \frac{1}{8} \mathcal{G}$ and $V \in C^\infty(\Sigma, \mathbb{R})$. Then:

- If the two-field model defined by the scalar triple $(\Sigma, \mathcal{G}, V)$ admits a non-trivial Hessian symmetry, then $\Sigma$ is oriented-diffeomorphic to a disk $D$, a punctured disk $\mathcal{D}$ or an annulus $A(R)$ of modulus $\mu = 2 \log R > 0$ and $G$ is a complete metric of Gaussian curvature $K(G) = -1$ (see [2, 5, 8]). In particular, any such model is a generalized two-field $\alpha$-attractor [3] of elementary type [4], with fixed Gaussian curvature $K(\mathcal{G}) = -3/8$ of the scalar manifold metric $\mathcal{G}$.

- For each of the three cases above, reference [2] gives a complete description of the space $\mathcal{H}(\Sigma, G)$ of Hesse functions of $(\Sigma, G)$ and the explicit general form of the scalar potential $V$ which is compatible with existence of a non-trivial Hessian symmetry generated by a given Hesse function $\Lambda \in \mathcal{H}(\Sigma, G)$. It further gives the special forms of such potentials for which the model also admits visible symmetries and describes the space of such symmetries in each case.

**Example: Classification of Hessian two-field models whose rescaled scalar manifold is the Poincaré disk**

The following results are proved in [2] (Section 5), to which we refer the reader for further details and more explicit formulas written in adapted coordinates on the scalar manifold. By Theorem [11] the general Hesse function on the Poincaré disk $\mathbb{D} = (D, G)$ has the form:

$$\Lambda_B(u) = (B, \Xi(u)) = B_{\mu} \Xi^\mu(u) = B_{\mu} \left[ \frac{1+|u|^2}{1-|u|^2} B^1 - \frac{\Re u}{1-|u|^2} - \frac{2B^2}{1-|u|^2} \frac{\Im u}{1-|u|^2} \right] (u \in D \subset \mathbb{C}^2) \right) , \quad (18)$$


where $B = (B^0, B^1, B^2) \in \mathbb{R}^3$ is a non-vanishing 3-vector parameter, $(\cdot, \cdot)$ is the Minkowski pairing of signature $(1, 2)$ on $\mathbb{R}^3$ and $\Xi = (\Xi^0, \Xi^1, \Xi^2) : D \to \mathbb{R}^3$ is the Weierstrass map:

$$\Xi(u) \equiv \frac{1 + |u|^2}{1 - |u|^2} \begin{pmatrix} 1 & 2 \text{Re} u & 2 \text{Im} u \\ 1 - |u|^2 & 1 - |u|^2 & 1 - |u|^2 \end{pmatrix}.$$ 

The following statements hold for the two-field cosmological model whose scalar manifold is the Poincaré disk $\mathbb{D} = (D, G)$:

- **When $B \neq 0$ is timelike**, i.e. $(B, B) = -(B^0)^2 + (B^1)^2 + (B^2)^2 < 0$, the two-field model admits the Hessian symmetry generated by (18) iff the scalar potential has the form:

  $$V_B(u) = \omega(n_B(u)) \left[ \frac{\Lambda_B(u)^2}{(B, B)} - 1 \right],$$

  where $\omega$ is an arbitrary smooth function defined on the unit circle and $n_B(u)$ is the 3-vector given by:

  $$n_B(u) = \frac{(B, B) \Xi(u) - (B, \Xi(u)) B}{\sqrt{(B, B)(B, \Xi(u))^2 - (B, B)^2}} = \frac{(B, B) \Xi(u) - B \Lambda_B(u)}{\sqrt{(B, B)\Lambda_B(u)^2 - (B, B)^2}}.$$

- **When $B \neq 0$ is spacelike**, i.e. $(B, B) > 0$, the two-field model with scalar manifold $\mathbb{D}$ admits the Hessian symmetry generated by (18) iff its scalar potential $V$ has the form:

  $$V_B(u) = \omega(n_B(u)) \left[ \frac{\Lambda_B(u)^2}{(B, B)} + 1 \right],$$

  where $\omega \in \mathcal{C}^\infty(\mathbb{R})$ is an arbitrary smooth function defined on the real line and $n_B(u)$ is the timelike 3-vector given by:

  $$n_B(u) = \frac{((B, B)|\Xi(u) + (B, \Xi(u)) B}{\sqrt{(B, B)^2 + ((B, B)(B, \Xi(u))^2}}} = \frac{(B, B)|\Xi(u) + \Lambda_B(u) B}{\sqrt{(B, B)^2 + ((B, B)(\Lambda_B(u))^2}}}.$$

- **When $B \neq 0$ is lightlike**, i.e. $(B, B) = 0$, the two-field model with scalar manifold $\mathbb{D}$ admits the Hessian symmetry generated by (18) iff its scalar potential $V$ has the form:

  $$V_B(u) = \omega(B_0 n_B(u)) \frac{\Lambda_B(u)^2}{B_0^2},$$

  where $\omega \in \mathcal{C}^\infty(\mathbb{R})$ is an arbitrary smooth function defined on the real line and $n_B(u)$ is the lightlike 3-vector given by:

  $$n_B(u) = \frac{2(B, \Xi(u)) \Xi(u) - B}{2(B, \Xi(u))^2} = \frac{2\Lambda_B(u) \Xi(u) - B}{2\Lambda_B(u)^2}.$$

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