Schwarz–Pick Lemma for Harmonic and Hyperbolic Harmonic Functions

Adel Khalfallah, Miodrag Mateljević, and Bojana Purtić

Abstract. We establish some inequalities of Schwarz–Pick type for harmonic and hyperbolic harmonic functions on the unit ball of $\mathbb{R}^n$ and we disprove a recent conjecture of Liu (Int Math Res Not, 2021).

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1. Introduction

By $\omega_n$ or $V(B^n)$ we denote the $n$-volume of the unit ball $B^n$ in $\mathbb{R}^n$, and by $\sigma_n$ the $(n-1)$-volume of the unit sphere $S^{n-1}$; note that $\sigma_n = n\omega_n$. Next, $\sigma$ denotes the rotation invariant Borel measure on $S^{n-1}$, $\sigma^0 = \sigma/\sigma_n$ and $|.|$ is the Euclidean norm. Thus $\sigma^0$ is the unique rotation invariant normalized Borel measure on $S^{n-1}$ such that $\sigma^0(S^{n-1}) = 1$. In this paper, the expressions $\omega_{n-1}/\omega_n$ and $\sigma_{n-1}/\sigma_n$ often appear so it is convenient to denote them by $\omega_*(n)$ and $\sigma_*(n)$ respectively, that is,

$$\omega_*(n) := \frac{\omega_{n-1}}{\omega_n}, \quad \text{and} \quad \sigma_*(n) := \frac{\sigma_{n-1}}{\sigma_n}.$$

Recall that a mapping $u \in C^2(B^n, \mathbb{R})$ is said to be hyperbolic harmonic if $\Delta_h u = 0$, where $\Delta_h$ is the hyperbolic Laplacian operator defined by

$$\Delta_h u(x) = (1 - |x|^2)^2 \Delta u + 2(n - 2)(1 - |x|^2) \sum_{i=1}^n x_i \frac{\partial u}{\partial x_i}(x),$$

here $\Delta$ denotes the Laplacian on $\mathbb{R}^n$. Clearly for $n = 2$, hyperbolic harmonic and harmonic functions coincide.
In [1], Liu proved the Khavinson conjecture, which says for bounded harmonic functions on the unit ball of $\mathbb{R}^n$ the sharp constants in the estimates for their radial derivatives and for their gradients coincide.

**Theorem A** [1]. For $n \geq 3$, if $u$ is a bounded harmonic function on $\mathbb{B}^n$ into $\mathbb{R}$, then we have the following sharp inequality

$$|\nabla u(x)| \leq \frac{c_n}{1 - |x|^2} \Phi_n(|x|)|u|_\infty, \quad x \in \mathbb{B}^n,$$

with $c_n = (n - 1)\omega_*(n)$, and

$$\Phi_n(r) = \int_{-1}^{1} \frac{|t - \frac{n-2}{2}r| (1 - t^2)^{n-3}}{(1 - 2tr + r^2)^{n-2}} dt.$$

For more details and development regarding the Khavinson conjecture for harmonic functions, see [2–8].

In [9], the author further proved that, when $n \geq 4$, the function $\Phi_n$ is decreasing on $[0,1]$, thus

$$\max_{r \in [0,1]} \Phi_n(r) = \Phi_n(0) = \frac{2}{n - 1}.$$

In contrast, if $n = 3$, then

$$\Phi_3(r) = \frac{2}{3} \frac{(1 + \frac{1}{3}r^2)^{3/2} - 1 + r^2}{r^2}$$

is strictly increasing on $[0,1]$ and attains its maximum at $r = 1$, thus

$$\max_{r \in [0,1]} \Phi_3(r) = \Phi_3(1) = \frac{16}{9\sqrt{3}}.$$

**Theorem 2** ([9], Schwarz–Pick lemma for harmonic functions). Let $u$ be a real-valued bounded harmonic function on the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$.

(1) When $n = 2$ or $n \geq 4$, the following sharp inequality holds:

$$|\nabla u(x)| \leq 2\omega_*(n) \frac{|u|_\infty}{1 - |x|^2}, \quad x \in \mathbb{B}^n.$$  \hspace{1cm} (1.1)

Equality holds if and only if $x = 0$ and $u = U \circ T$ for some orthogonal transformation $T$, where $U$ is the Poisson integral of the function that equals 1 on a hemisphere and $-1$ on the remaining hemisphere.

(2) When $n = 3$, we have

$$|\nabla u(x)| \leq \frac{8}{3\sqrt{3}} \frac{|u|_\infty}{1 - |x|^2}, \quad x \in \mathbb{B}^3.$$  \hspace{1cm} (1.2)

The constant $\frac{8}{3\sqrt{3}}$ here is the best possible.

**Remark 1.1.** (1) Note that the inequality (1.1) holds when $n = 3$ at $x = 0$. Curiously, the inequality (1.1) fails when $n = 3$ in general. Note that $\frac{8}{3\sqrt{3}} \approx 1.5396$, while the constant $\frac{2\sqrt{n-1}}{V(\mathbb{B}^n)}$ in (1.1) equals to $\frac{3}{2}$ when $n = 3$. 
The inequality (1.1) at \( x = 0 \) was previously proved in [10, Theorem 6.26] and in [11, Corollary 1] for harmonic functions fixing the origin.

The classical Schwarz–Pick lemma states that an analytic function of \( \mathbb{D} \) into itself satisfies

\[
|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D},
\]

where \( \mathbb{D} \) denotes the unit disc of the complex plane \( \mathbb{C} \). For complex-valued harmonic function of \( \mathbb{D} \) into itself, Colonna [12] proved the following sharp Schwarz–Pick lemma:

\[
|Df(z)| \leq \frac{4}{\pi} \frac{1 - |u(z)|^2}{1 - |z|^2} - \frac{1}{2}, \quad z \in \mathbb{D},
\]

where \( |Df(z)| = |\frac{\partial f(z)}{\partial z}| + |\frac{\partial f(z)}{\partial \bar{z}}| \).

In the planar case, Kalaj and Vuorinen [13] obtained the following inequality for real harmonic functions with values in \((-1, 1)\).

\[
|\nabla u(z)| \leq \frac{4}{\pi} \frac{1 - |u(z)|^2}{1 - |z|^2}, \quad |z| < 1.
\]

(1.3)

Based on (1.1) and (1.3), Liu suggested the following conjecture.

**Conjecture 1.** [9] If \( n \geq 4 \) and \( u : \mathbb{B}^n \to (-1, 1) \) is a harmonic function, then

\[
|\nabla u(x)| \leq 2\omega_*(n) \frac{1 - u^2(x)}{1 - |x|^2}, \quad |x| < 1.
\]

(1.4)

First, by providing a counter-example, we disprove Conjecture 1 for \( n \geq 4 \). Our main tool is Theorem 3 giving a sharp estimate of the norm of the gradient at zero of functions having generalized Poisson transformations; such estimate is based on the Burgeth’s method, see [11,14].

Let us introduce some notations. If \( x \in \mathbb{B}^n \setminus \{0\} \), define \( \hat{x} = \frac{x}{|x|} \in \mathbb{S}^{n-1} \) and \( \hat{0} = e_n = (0, \ldots, 1) \), the north pole. \( S(\hat{x}, \gamma) \) denotes the hyperspherical cap with center \( \hat{x} \) and contact angle \( \gamma \in [0, \pi] \):

\[
S(\hat{x}, \gamma) = \{ y \in \mathbb{S}^{n-1} : \langle \hat{x}, y \rangle > \cos \gamma \}.
\]

For \( \alpha, \beta \in \mathbb{R} \), \( \beta > 0 \), the generalized Poisson kernel is defined by

\[
P_{\alpha,\beta}(x, y) = \frac{(1 - |x|^2)^\alpha}{|x - y|^{2\beta}} , \quad x \in \mathbb{B}^n \text{ and } y \in \mathbb{S}^{n-1}.
\]

For \( f \in L^1(\mathbb{S}^{n-1}) \), set

\[
P_{\alpha,\beta}[f](x) = \int_{\mathbb{S}^{n-1}} P_{\alpha,\beta}(x, y) f(y) \, d\sigma^0(y).
\]

By \( A_n^{\text{cap}}(\gamma) \) we denote the normalized \((n - 1)\)-dimensional volume of the spherical cap with contact angle \( \gamma \).
Theorem 3 [15]. Let $h^* : S^{n-1} \to [-1, 1]$ be a bounded function on $S^{n-1}$ with values in $[-1, 1]$ and $h = P_{\alpha, \beta}[h^*]$. Then,

$$|\nabla h(0)| \leq D_n(\gamma, \beta) := \frac{4\beta}{n} \omega_{n-1} (\sin \gamma)^{n-1},$$

(1.5)

where $\gamma$ is the unique angle in $[0, \pi]$ such that

$$A = A_n^{cap}(\gamma) = \frac{1 + h(0)}{2}. \quad (1.6)$$

The estimate (1.5) is sharp and

$$h_{\alpha, \beta}^0 := P_{\alpha, \beta}[\mathbb{1}_{S(e_n, \gamma)} - \mathbb{1}_{S^c(e_n, \gamma)}]$$

(1.7)

is an extremal function, where $S^c(e_n, \gamma) = S^{n-1} \setminus S(e_n, \gamma)$.

It is readable that

$$D_n(\gamma, \beta) = \sup |\nabla h(0)|,$$

where the supremum is taken over all functions $h$ which satisfy the assumptions of Theorem 3 with the constraint $h(0) = a$, where $\gamma$ is determined by $A_n^{cap}(\gamma) = \frac{1+\alpha}{2}$.

As a corollary, we obtain an estimate of the gradient at zero for harmonic functions obtained for $(\alpha, \beta) = (1, \frac{n}{2})$ and hyperbolic-harmonic functions obtained for $(\alpha, \beta) = (n-1, n-1)$ in terms of their values at the origin.

Corollary 1 [15]. Let $h : \mathbb{B}^n \to (-1, 1)$ be a harmonic or hyperbolic harmonic function. Then the following sharp estimates hold:

$$|\nabla h(0)| \leq \begin{cases} 2\omega_*(n)(\sin \gamma)^{n-1} & \text{if } h \text{ is harmonic,} \\ 4\sigma_*(n)(\sin \gamma)^{n-1} & \text{if } h \text{ is hyperbolic-harmonic,} \end{cases}$$

where $\gamma$ is the unique angle in $[0, \pi]$ such that

$$A_n^{cap}(\gamma) = \frac{1 + h(0)}{2}. \quad (1.7)$$

These estimates are sharp and $h^0_{1, \frac{n}{2}}$ (resp., $h^0_{n-1, n-1}$) is an extremal harmonic (resp., hyperbolic-harmonic) function on $\mathbb{B}^n$, see (1.7).

2. Main Results

We are now in a position to establish our main results. Our first result is the following.

Theorem 2.1. For $n \geq 4$, the harmonic function $h^0_{1, \frac{n}{2}}$ defined in (1.7) provides a counter-example to Conjecture 1.
The proof is based on Corollary 1 and some basic properties of the volume of the unit ball in $\mathbb{R}^n$. Recently, several authors presented interesting monotonicity properties of the $\omega_n$, the volume of the unit ball in $\mathbb{R}^n$. The sequence itself is not monotonic and attains its maximum at $n = 5$. It is worth noting that $\omega_n(\gamma) = \omega_{n-1}/\omega_n$ has very interesting properties and there are remarkable upper and lower bounds, see for instance Borgwardt [16, p. 253] and Alzer [17]. Using these estimates, we prove the following.

**Theorem 2.2.** For $n \geq 4$ and $a \in [-1, 1]$. Then the following inequalities hold

\[ (\sin \gamma_a)^{n-1} \geq 1 - a^2. \]  

(2.1)

Moreover, the equality holds for $a = -1, 0, 1$.

\[ (\sin \gamma_a)^{n-1} \leq \frac{n-1}{4\pi}(1 - a^2). \]  

(2.2)

Moreover, the equality holds for $a = -1, 1$. $\gamma_a$ is the unique angle in $[0, \pi]$ such that $A_{n,\gamma_a}^\cap = \frac{1+a}{2}$.

Thus combining Corollary 1 and the inequality (2.1), we disprove Liu’s conjecture.

**Remark 2.3.** (i) For $n = 3$, we have $(\sin \gamma_a)^{n-1} = 1 - a^2$.

(ii) For $n = 2$, $(\sin \gamma_a)^{n-1} \leq 1 - a^2$.

Indeed, in the planar case, $A_{2,\gamma_a}^\cap = \frac{\gamma_a}{\pi} = \frac{1+a}{2}$, thus $\gamma_a = \frac{\pi}{2} + \frac{\pi a}{2}$ and $\sin \gamma_a = \cos(\frac{\pi a}{2}) \leq 1 - a^2$.

As for $n = 3$, we have $\frac{4}{n} \omega_{n-1}/\omega_n = 1$ and combining Theorem 3 and the inequality (2.2) in Theorem 2.2, it yields the following.

**Theorem 2.4.** Let $n \geq 3$ and $h^* : S^{n-1} \rightarrow [-1, 1]$ be a function on $S^{n-1}$ with values in $[-1, 1]$ and $h = P_{\alpha,\beta}[h^*]$. Then,

\[ |\nabla h(0)| \leq \beta(1 - |h(0)|^2). \]  

(2.3)

The constant $\beta$ is sharp in (2.3).

In particular, we get the following estimate of the gradient at zero for harmonic and hyperbolic harmonic functions.

**Corollary 2.** Let $n \geq 3$ and $h : \mathbb{B}^n \rightarrow (-1, 1)$ be a harmonic or hyperbolic harmonic function. Then the following sharp estimates hold:

\[ |\nabla h(0)| \leq \begin{cases} \frac{n}{2} (1 - |h(0)|^2) & \text{if } h \text{ is harmonic}, \\ (n-1)(1 - |h(0)|^2) & \text{if } h \text{ is hyperbolic-harmonic}. \end{cases} \]

Furthermore, this inequality is strict for $n \geq 4$. 
Using the ball of center $x$ and radius $1 - |x|$ in the harmonic case and Möbius transformations in the hyperbolic-harmonic case, we obtain the following.

**Theorem 2.5.** Let $n \geq 3$ and $h : \mathbb{B}^n \to (-1,1)$ be a harmonic function, then

$$|\nabla h(x)| \leq \frac{n}{2} \frac{1 - |h(x)|^2}{1 - |x|}.$$  

In addition, this inequality is strict for $n \geq 4$.

**Theorem 2.6.** Let $n \geq 3$ and let $h : \mathbb{B}^n \to (-1,1)$ be a hyperbolic harmonic function, then

$$|\nabla h(x)| \leq (n - 1) \frac{1 - |h(x)|^2}{1 - |x|^2}.$$  

Therefore,

$$d_{h_2}(h(x_1), h(x_2)) \leq (n - 1) d_{h_n}(x_1, x_2), \quad x_1, x_2 \in \mathbb{B}^n,$$

where $d_{h_n}$ denotes the hyperbolic distance of the unit ball $\mathbb{B}^n$. For $n = 2$, $d_{h_2}$ is simply the Poincaré distance of the unit disc.

This result is connected to the Khavinson conjecture for hyperbolic harmonic functions, see [18].

**Theorem 4.** [18, Theorem 2] Let $n \geq 3$ and let $h : \mathbb{B}^n \to (-1,1)$ be a hyperbolic harmonic function, then

$$|\nabla h(x)| \leq 4\sigma^*(n) \frac{1}{1 - |x|^2}.$$  

For vector valued functions, we prove the following.

**Theorem 2.7.** Let $n \geq 3$, $m \geq 1$ and $h : \mathbb{B}^n \to \mathbb{B}^m$ be a harmonic or hyperbolic harmonic function. Then the following estimates hold.

$$|Dh(x)| \leq \begin{cases}  
\frac{n}{2} \frac{1}{1 - |x|} & \text{if } h \text{ is harmonic}, \\
(n - 1) \frac{1}{1 - |x|^2} & \text{if } h \text{ is hyperbolic-harmonic},  
\end{cases}$$

where

$$|Dh(x)| = \sup_{|v|=1} |Dh(x)v|.$$  

In addition, it yields the following.

**Theorem 2.8.** Let $n \geq 3$, $m \geq 1$ and $h : \mathbb{B}^n \to \mathbb{B}^m$ be a harmonic or hyperbolic harmonic function. Then the following estimates hold:
\[ |\nabla |h||x|| \leq \begin{cases} \frac{n}{2} \frac{1-|h(x)|^2}{1-|x|^2} & \text{if } h \text{ is harmonic,} \\ \frac{1-(n-1)}{1-|x|^2} \frac{1-|h(x)|^2}{1-|x|^2} & \text{if } h \text{ is hyperbolic-harmonic.} \end{cases} \]

Furthermore, this inequality is strict for \( n \geq 4 \).

Recall that
\[ |\nabla |h||(x)| = \sup_{\beta \in S^{n-1}} \lim_{t \to 0^+} \frac{|h(x+t\beta)|-|h(x)|}{t}. \]

Thus \( |\nabla |h||(x)| \) coincides with the gradient of \(|h|\) at \( x \), if \( h(x) \neq 0 \). Moreover, if \( h(x) = 0 \), then \( |\nabla |h||(x)| = |Dh(x)| \). Therefore, Theorem 2.8 can be seen as an extension of Theorem 2.7.

We mention that in [13], the authors considered the corresponding theorem for vector harmonic functions defined on the unit disc, see [13, Theorem 1.10]. A Schwarz lemma for the modulus of a vector-valued analytic functions was previously considered in [19].

In [14], Burgeth introduced \( R_{\alpha,\beta}(|x|,h(x)) \) and provides estimates for the radial derivative in terms of \(|x|\) and \( h(x)\). In the three-dimensional case, we get explicit formula for \( R_{\alpha,\beta}(|x|,h(x)) \). In particular, we can modify [14, Corollary 3] in the following way.

**Theorem 2.9.** If \( u : \mathbb{B}^3 \to (-1, 1) \) is a harmonic function, then for each \( x \in \mathbb{B}^3 \),
\[
-3 - \frac{|x|u(x)}{1-|x|^2}(1-u^2(x))/2 \leq Dr\,u(x) \leq \frac{3-|x|u(x)}{1-|x|^2}(1-u^2(x))/2. \tag{2.4}
\]

Hence
\[
|\nabla u(x)| \leq 2 \frac{1-u^2(x)}{1-|x|^2}, \tag{2.5}
\]

and therefore
\[
d_{h_{2}}(u(x_1), u(x_2)) \leq 2d_{h_{3}}(x_1, x_2), \quad x_1, x_2 \in \mathbb{B}^3. \tag{2.6}
\]

We should mention that, recently, several versions of Schwarz lemma for harmonic and pluriharmonic mappings were established, see [20–25].

We close our paper, by the following question as an adjustment to Liu’s conjecture.

**Question 1.** Let \( n \geq 3 \) and \( h : \mathbb{B}^n \to (-1, 1) \) be a harmonic function. Is it true that
\[
|\nabla h(x)| \leq \frac{n}{2} \frac{1-|h(x)|^2}{1-|x|^2}, \quad x \in \mathbb{B}^n? \]
3. Proofs of the Main Results

Proof of Theorem 2.1

To simplify notations, let us denote

\[ A = A_n^{\text{cap}}(\gamma). \]

Using spherical coordinates, the formula for the normalized area of the spherical cap with contact angle \( \gamma \in [0, \pi] \) is given by

\[ A = \sigma_*(n)A_0(\gamma), \quad (3.1) \]

where

\[ A_0(\gamma) = \int_0^\gamma (\sin \theta)^{n-2}d\theta, \]

see for example [26].

Lemma 3.1. Using the notations of Theorem 3, for \( \gamma \in (0, \pi) \), we have

\[ D_n(\gamma, \beta) = \beta C_n(\gamma)(1 - |h(0)|^2), \quad (3.2) \]

where

\[ C_n(\gamma) = \frac{1}{(n-1)} A_0(\gamma)(1 - A_n^{\text{cap}}(\gamma)), \quad (3.3) \]

and

\[ C_n(\gamma) \to 1 \quad \text{as} \quad \gamma \to 0^+. \]

Proof. By the constraint condition (1.6), we have \( 1 + h(0) = 2A \) and \( 1 - h(0) = 2(1 - A) \). Therefore

\[ 1 - |h(0)|^2 = 4A(1 - A). \quad (3.4) \]

Recall that \( D_n(\beta, \gamma) = \frac{4\beta\omega_*(n)}{n}(\sin \gamma)^{n-1} = \frac{4\beta\sigma_*(n)}{n-1}(\sin \gamma)^{n-1}. \)

By (3.4), we obtain \( D_n(\beta, \gamma) = \frac{\beta}{n-1} \frac{\sigma_*(n) \sin \gamma)^{n-1}}{A(1 - A)} (1 - |h(0)|^2). \) Therefore, by (3.1), we get (3.3). The limit of \( C_n(\gamma) \) as \( \gamma \) goes to 0 equals to 1 is a direct consequence of the L’Hopital’s rule. \( \square \)

First, we disprove Conjecture 1. Assume that the inequality (1.4) is true. Then, in particular for \( x = 0 \), we have

\[ |\nabla u(0)| \leq 2\omega_*(n)(1 - a^2), \quad (3.5) \]

where \( a = u(0) \). Since the estimate (1.5) in Theorem 3 under the constrains \( a = u(0) \) is sharp, then there is extremal function \( u^0 \), such that \( D_n(\gamma, \beta) = |\nabla u^0(0)|. \)

Thus

\[ D_n(\gamma, \beta) = \beta C_n(\gamma)(1 - a^2) \leq 2\omega_*(n)(1 - a^2). \quad (3.6) \]
Therefore $\beta C_n(\gamma) \leq 2\omega_*(n)$. In particular, in harmonic case where $\beta = n/2$, we have

$$nC_n(\gamma) \leq 4\omega_*(n).$$

As the limit of $C_n(\gamma)$ is 1 as $\gamma \to 0$, it yields

$$n \leq 4\omega_*(n).\quad (3.7)$$

Since $\omega_n$ assumes its maximal value when $n = 5$, $\omega_*(5) < 1$, we disprove Liu’s conjecture.

We leave the reader to disprove Conjecture 1 for $n \geq 4$ using Alzer’s estimate below.

### 3.1. Proof of Theorem 2.2

A remarkable upper and lower bounds for the ratio $\frac{\omega_{n-1}}{\omega_n}$ are proved by Borgwardt[7, p.253]. He showed that for $n \geq 2$

$$\sqrt{\frac{n}{2\pi}} \leq \omega_*(n) \leq \sqrt{\frac{n + 1}{2\pi}}.\quad (3.8)$$

More refinements of these estimates are established by Alzer [17].

**Theorem 3.1** [17]. For $n \geq 2$, we have

$$\sqrt{\frac{n + A}{2\pi}} \leq \omega_*(n) \leq \sqrt{\frac{n + B}{2\pi}},\quad (3.9)$$

with the best possible constants

$$A = \frac{1}{2} \quad \text{and} \quad B = \frac{\pi}{2} - 1.$$

For more properties of the volume of the unit ball in $\mathbb{R}^n$, see [27,28].

As $\sigma_*(n) = \frac{n-1}{n} \omega_*(n)$ and using (3.8) or (3.9), one can easily check the following lemma.

**Lemma 3.2.** For $n \geq 4$, we have

$$\frac{1}{2} < \sqrt{\frac{n-1}{8}} < \sigma_*(n) < \frac{n-1}{4}.\quad (3.10)$$

Recall that the area of the spherical cap of contact angle $\gamma \in [0, \pi]$ is given by

$$A(\gamma) = \sigma_*(n) \int_0^\gamma \sin^{n-2}\theta d\theta.\quad (3.11)$$

Let $a \in [-1, 1]$, then there exists a unique angle $\gamma(a) \in [0, \pi]$ such that

$$A(\gamma(a)) = \frac{1 + a}{2}.\quad (3.12)$$

Clearly, the mapping $a \mapsto \gamma(a)$ is strictly increasing from $[-1, 1]$ to $[0, \pi]$.
Differentiating the Eq. (3.12) with respect to $a$, we obtain
\[ \sigma_*(n) \gamma'(a) \sin^{n-2} \gamma(a) = \frac{1}{2}, \quad \text{for } a \in (-1, 1). \] (3.13)

Let us consider the function $h$ defined on $[-1, 1]$ by
\[ h(a) := \sin^{n-1} \gamma(a) - 1 + a^2. \] (3.14)

As $h$ is even, it is enough to study the function $h$ on $[0, 1]$. For each $a \in [0, 1)$, we have
\[ h'(a) = (n - 1) \gamma'(a) \sin^{n-2} \gamma(a) \cos \gamma(a) + 2a. \]

In view of the Eq. (3.13), we get
\[ h'(a) = \frac{(n - 1) \cos \gamma(a)}{2\sigma_*(n)} + 2a. \]

The second derivative of $h$ is given by
\[ h''(a) = -\frac{n - 1}{2\sigma_*(n)} \sin \gamma(a) \gamma'(a) + 2. \]

Again using (3.13), we deduce that
\[ h''(a) = -\frac{n - 1}{4\sigma_*(n)^2} \sin^{3-n} \gamma(a) + 2. \]

For $n \geq 4$, we conclude that $h''$ is strictly decreasing on $[0, 1]$ because $\gamma$ is increasing with values in $[\pi/2, \pi]$. Moreover, we have
\[ h''(0) = -\frac{n - 1}{4\sigma_*(n)^2} + 2, \quad \lim_{a \to 1} h''(a) = -\infty. \] (3.15)

Using the inequality (3.10), it yields
\[ h''(0) > 0, \quad \text{for } n \geq 4. \] (3.16)

Therefore, there exists $a_n \in (0, 1)$ such that $h''(a) > 0$ on $(0, a_n)$ and $h''(a) < 0$ on $(a_n, 1)$. Thus $h'$ is increasing on $[0, a_n)$ and decreasing on $(a_n, 1)$. Moreover,
\[ h'(0) = 0 \quad \text{and} \quad h'(1) = -\frac{n - 1}{2\sigma_*(n)} + 2. \]

Now, using (3.10), we deduce that
\[ h'(1) < 0. \]

Therefore there exists $b_n \in (0, 1)$ such that $h$ is increasing on $(0, b_n)$ and decreasing on $(b_n, 1)$. As $h(0) = h(1) = 0$, we conclude that the $\min_{a \in [0, 1]} h(a) = h(0) = 0$. Finally, we get $h(a) \geq 0$ for all $a \in [0, 1]$, that is, $\sin^{n-1} \gamma(a) \geq 1 - a^2$, for $n \geq 4$.

Next, in order to prove the second estimate in Theorem 2.2, we need the following lemma.
Lemma 3.3. Let \( a \in [0, 1] \) and \( n \geq 4 \). Then

\[- \cos \gamma(a) \leq a,\]

Moreover, the equality holds at \( a = 0 \) or \( a = 1 \).

Proof. Consider the function \( G \) on \([0, 1]\) defined by

\[ G(a) = a + \cos \gamma(a). \]

\( G \) is differentiable on \((0, 1)\) and

\[ G'(a) = 1 - \sin \gamma(a) \gamma'(a) = 1 - \frac{1}{2\sigma_*(n)} \sin^{3-n} \gamma(a). \]

The function \( G' \) is strictly decreasing as \( \gamma(a) \) belongs to \([\pi/2, \pi]\). Moreover, we have

\[ G'(0) = 1 - \frac{1}{2\sigma_*(n)} > 0 \text{ and } \lim_{a \to 1} G'(a) = -\infty. \]

Therefore, \( G \) is strictly increasing on \([0, c_n]\) and decreasing on \((c_n, 1)\), where \( c_n \in (0, 1) \) is the unique zero of \( G' \). As \( G(0) = G(1) = 0 \), we get \( G(a) > 0 \) on \((0, 1)\). \( \square \)

It remains to prove the following: \((\sin \gamma)^{n-1} \leq \frac{n-1}{4\sigma_*(n)}(1 - a^2), \ n \geq 4.\)

Consider

\[ g(a) = 1 - a^2 - \frac{4\sigma_*(n)}{n-1} \sin^{n-1} \gamma(a). \]

Then

\[ g'(a) = -2a - 2\cos \gamma_n(a). \]

By Lemma 3.3, we have \( g'(a) \leq 0 \) and \( g \) is decreasing. Hence \( g(a) \geq g(1) = 0 \) on \([0, 1]\) and the conclusion follows.

Proof of Theorem 2.4

Theorem 2.4 is a direct consequence of Theorems 3 and 2.2. In particular, in Corollary 2, we obtain the corresponding inequality for harmonic and hyperbolic harmonic functions, by considering the corresponding values of \( \beta \).

Proof of Theorems 2.5 and 2.6

Let \( h : \mathbb{B}^n \to (-1, 1) \) be a harmonic function and let \( x \in \mathbb{B}^n \). Consider \( g \) the harmonic function defined on \( \mathbb{B}^n \) by

\[ g(y) = h(x + y(1 - |x|)). \]

Clearly, we have \( g(0) = h(x) \) and \( |\nabla g(0)| = (1 - |x|)|\nabla h(x)| \). By applying Corollary 2 to the function \( g \), we get the desired inequality.

In the hyperbolic harmonic case, we compose with a Möbius transformation sending \( 0 \) to \( x \). More precisely, let \( x \in \mathbb{B}^n \) be fixed. By the Möbius
invariance of $\Delta_h$, the function $h \circ \varphi_x$ is also a bounded hyperbolic harmonic function, where

$$\varphi_x(y) := \frac{|y - x|^2 x - (1 - |x|^2)(y - x)}{1 - 2\langle y, x \rangle + |y|^2|x|^2},$$

which is a Möbius transformation of $\mathbb{B}^n$. Theorem 2.6 follows by replacing $h$ by $h \circ \varphi_x$ in Corollary 2 and noting that $\nabla (h \circ \varphi_x)(0) = -(1 - |x|^2)\nabla h(x)$, see [29, p. 18].

3.2. Proof of Theorem 2.7
As the proofs for the harmonic and the hyperbolic harmonic case are similar, we will provide only the proof in the harmonic setting. Let $h : \mathbb{B}^n \to \mathbb{B}^m$ be a harmonic vector-function and $\theta$ be a unit vector in $\mathbb{R}^m$. Consider $h_\theta$ the function defined by

$$h_\theta(x) = \langle h(x), \theta \rangle.$$

Then $h_\theta$ is a harmonic function with values in $(-1, 1)$. Consequently, by Theorem 2.5, we get

$$|\langle Dh(x)v, \theta \rangle| = |Dh_\theta(x)v| \leq n \frac{1}{2 \left(1 - |x|\right)}, \quad \text{for all } v \in \mathbb{R}^n, \quad |v| = 1.$$

Therefore, $|Dh(x)| \leq \frac{n}{2 \left(1 - |x|\right)}$.

3.3. Proof of Theorem 2.8
Let $x_0 \in \mathbb{B}^n$ and $h : \mathbb{B}^n \to \mathbb{B}^m$ be a harmonic function. By Theorem 2.7, we have to consider only the case where $h(x_0) \neq 0$. Define

$$g(x) := \langle h(x), \frac{h(x_0)}{|h(x_0)|} \rangle.$$

Then $g$ is a harmonic function on $\mathbb{B}^n$ with codomain $(-1, 1)$ with $g(x_0) = |h(x_0)|$. It follows from Theorem 2.5, that

$$|\nabla g(x_0)| \leq \frac{n}{2} \frac{1 - |g(x_0)|}{1 - |x_0|}.$$

Indeed, easy computations show that $|\nabla g(x_0)| = |\nabla |h(x_0)|$ as $g_{x_i}(x_0) = |h|_{x_i}(x_0)$, where $g_{x_i}(x_0)$ denotes the partial derivative of $g$ with respect to the variable $x_i$ at $x_0$.

Declarations
Conflict of interest The authors have no conflicts of interest to declare that are relevant to the content of this article.
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Adel Khalfallah  
Department of Mathematics  
King Fahd University of Petroleum and Minerals  
Dhahran 31261  
Saudi Arabia  
e-mail: khelifa@kfupm.edu.sa

Miodrag Mateljević and Bojana Purtić  
Faculty of Mathematics  
University of Belgrade  
Studentski Trg 16  
Belgrade  
Republic of Serbia  
e-mail: miodrag@matf.bg.ac.rs; bojanaj@matf.bg.ac.rs

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