A GENERALIZATION OF KATO’S LOCAL $\varepsilon$-CONJECTURE 
FOR $(\varphi, \Gamma)$-MODULES OVER THE ROBBA RING.

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Abstract. The aim of this article is to generalize Kato’s (commutative) $p$-adic local $\varepsilon$-conjecture [Ka93b] for families of $(\varphi, \Gamma)$-modules over the Robba ring. In particular, we prove the generalized local $\varepsilon$-conjecture for rank one $(\varphi, \Gamma)$-modules, which is a generalization of Kato’s theorem [Ka93b] for rank one Galois representations. The key ingredients are the recent results of Kedlaya-Pottharst-Xiao [KPX12] on the finiteness of cohomology of $(\varphi, \Gamma)$-modules and the theory of Bloch-Kato’s exponential map for $(\varphi, \Gamma)$-modules developed in [Na13].

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1. Introduction.

1.1. Introduction. Since the works of Kisin [Ki03], Colmez [Co08], and Belläiche-Chenevier [BelCh09],···, the theory of \((\varphi, \Gamma)\)-modules over the (relative) Robba ring becomes one of the main streams in the theory of \(p\)-adic Galois representations. In particular, the recent works of Pottharst [Po13a] and Kedlaya-Pottharst-Xiao [KPX12] established the fundamental theorems (comparison with the Galois cohomology, finiteness, base change property, Tate duality, Euler-Poincaré formula···) in the theory of the cohomology of \((\varphi, \Gamma)\)-modules over the relative Robba ring over \(\mathbb{Q}_p\)-affinoid algebras. As is suggested and actually given in [KPX12], [Po13b], their results are expected to have many applications in number theory (e.g. eigenvarieties, non-ordinary case of Iwasawa theory). On the other hand, in [Na13], the author of this article generalized the theory of Bloch-Kato’s exponential map and Perrin-Riou’s exponential map in the framework of \((\varphi, \Gamma)\)-modules over the Robba ring. Since these maps are very important tools in Iwasawa theory, he expects that the results of [Na13] also have many applications.

As an application of the both theories, the purpose of this article is to generalize Kato’s \(p\)-adic local \(\varepsilon\)-conjecture [Ka93b] in the framework of \((\varphi, \Gamma)\)-modules over the relative Robba ring over \(\mathbb{Q}_p\)-affinoid algebras, which we briefly explain in this introduction: see §3 for the precise definitions. Let \(G_{\mathbb{Q}_p}\) be the absolute Galois group of \(\mathbb{Q}_p\). Let \(\Lambda\) be a semi-local ring such that \(\Lambda/\mathfrak{m}_\Lambda\) is a finite ring of the order a power of \(p\), where \(\mathfrak{m}_\Lambda\) is the Jacobson radical of \(\Lambda\). Let \(T\) be a \(\Lambda\)-representation of \(G_{\mathbb{Q}_p}\), i.e. a finite projective \(\Lambda\)-module with a continuous \(\Lambda\)-linear \(G_{\mathbb{Q}_p}\)-action. Let \(C_{\text{cont}}^*(G_{\mathbb{Q}_p}, T)\) be the complex of continuous cochains of \(G_{\mathbb{Q}_p}\) with the values in \(T\). By the classical theory of Galois cohomology of \(G_{\mathbb{Q}_p}\), this complex is a perfect complex of \(\Lambda\)-module which satisfies the base change property, Tate duality,···. This fact enables us to define the determinant \(\text{Det}_\Lambda(C^*(G_{\mathbb{Q}_p}, T))\) which is a (graded) invertible \(\Lambda\)-module. Modifying this module by multiplying a kind of \(\text{det}_\Lambda(T)\), one can canonically attach an invertible module

\[\Delta_\Lambda(T)\]

called the fundamental line of the pair \((\Lambda, T)\), which is compatible with base change and Tate duality. Our main objects are the pairs \((A, M)\) where \(A\) is a \(\mathbb{Q}_p\) affinoid and \(M\) is a \((\varphi, \Gamma)\)-module over the relative Robba ring \(\mathcal{R}_A\) over \(A\). By the results of [KPX12], then we can similarly attach a graded invertible \(A\)-module

\[\Delta_A(M)\]

such that, for a pair \((\Lambda, T)\) and a continuous homomorphism \(f : \Lambda \to A\), there exists a canonical comparison isomorphism \(\Delta_A(T) \otimes_\Lambda A \cong \Delta_A(D_{\text{rig}}(T \otimes_\Lambda A))\) by the result of [Po13a]. The following conjecture is Kato’s conjecture if \((B, N) = (\Lambda, T)\), and our conjecture if \((B, N) = (A, M)\). See Conjecture 3.9 for the precise formulation.
**Conjecture 1.1.** (Conjecture 3.9) We can uniquely define a $B$-linear isomorphism

$$
\varepsilon_{B,\zeta}(N) : \Delta_B(N) \simto 1_B
$$

for each pair $(B,N)$ of type $(\Lambda,T)$ or $(A,M)$ and each $\mathbb{Z}_p$-basis $\zeta$ of $\mathbb{Z}_p(1)$, which is compatible with any base changes $B \to B'$, exact sequences $0 \to N_1 \to N_2 \to N_3 \to 0$, and Tate duality, and satisfies the following:

1. For any $f : \Lambda \to A$ as above, we have
   $$
   \varepsilon_{\Lambda,\zeta}(T) \otimes \text{id}_A = \varepsilon_{A,\zeta}(\text{D}_{\text{rig}}(T \otimes_{\Lambda} A))
   $$
   under the canonical isomorphism $\Delta_{\Lambda}(T) \otimes_{\Lambda} A \simto \Delta_A(\text{D}_{\text{rig}}(T \otimes_{\Lambda} A))$.

2. Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let $N$ be a de Rham representation of $G_{\mathbb{Q}_p}$ or de Rham $(\varphi, \Gamma)$-module over $\mathcal{R}_L$. Then we have
   $$
   \varepsilon_{L,\zeta}(N) = \varepsilon_{L,\zeta}^\text{dR}(N),
   $$
   where the isomorphism
   $$
   \varepsilon_{L,\zeta}^\text{dR}(N) : \Delta_L(N) \simto 1_L
   $$
   which is called the de Rham $\varepsilon$-isomorphism is defined using the Bloch-Kato exponential and the dual exponential of $N$ and the local factors ($L$-factor, $\varepsilon$-constant, “gamma factor”) associated to $\text{D}_{\text{pst}}(N)$ and $\text{D}_{\text{pst}}(N^*)$.

**Remark 1.2.** To define the condition (vi) for de Rham $(\varphi, \Gamma)$-modules, we need the results of [Na13].

Roughly speaking, this conjecture says that the local factor which appears in the functional equation of the $L$-functions of motif $p$-adically interpolate to all the families of $p$-adic Galois representations and also rigid analytically interpolate to all the families of $(\varphi, \Gamma)$-modules in a compatible way. In fact, in [Ka93a], Kato formulated a conjecture called the generalized Iwasawa main conjecture which asserts the existence of a compatible family of “zeta”-isomorphisms $z_\Lambda(\mathbb{Z}[1/S], T) : \Delta_\Lambda^{\text{global}}(T) \simto 1_\Lambda$ for any $\Lambda$-representation $T$ of $G_{\mathbb{Q},S}$ ($S$ is a finite set of primes) which interpolate the special values of $L$-functions of motif, and also in [Ka93b] formulated another conjecture called the global $\varepsilon$-conjecture which asserts the functional equation between $z_\Lambda(\mathbb{Z}[1/S], T)$ and $z_\Lambda(\mathbb{Z}[1/S], T^*)$ with the $(p$-part of ) local factor $\varepsilon_{\Lambda,\zeta}(T|_{G_{\mathbb{Q}_p}})$.

In [Ka93b] (see also [Ve13]), Kato proved the local (and even the global) $\varepsilon$-conjecture for the rank one case. As a generalization of his theorem, our main theorem of this article is the following.

**Theorem 1.3.** (Theorem 3.12) The conjecture 1.1 is true for the rank case.

Finally in this introduction, we give the following two remarks concerning the future directions of our research.
Remark 1.4. From this theorem, we can immediately obtain some results in the trianguline case. We say that a \((\varphi, \Gamma)\)-module \(M\) over \(\mathcal{R}_A\) is trianguline if \(M\) has a filtration \(\mathcal{F}: 0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M\) whose graded quotients \(\text{gr}_{\mathcal{F}, i}(M)\) are rank one \((\varphi, \Gamma)\)-modules over \(\mathcal{R}_A\). Then, we can construct an \(\varepsilon\)-isomorphism \(\varepsilon_{\mathcal{F}, A, \zeta}(M): \Delta_A(M) \xrightarrow{\sim} 1_A\) as the products of \(\varepsilon_{A, \zeta}(\text{gr}_{\mathcal{F}, i}(M))\) defined in the above theorem. Then, it is easy to show that this isomorphism satisfies the de Rham condition (vi) in the conjecture (but we need to show this isomorphism is independent of the choice of \(\mathcal{F}\)). In particular, because the crystalline \((\varphi, \Gamma)\)-modules and even the twists of these by universal cyclotomic character are trianguline, we can compare our result with the previous known results on Kato’s local \(\varepsilon\)-conjecture for some twists of crystalline representations [BB08], [LVZ13]. Because this paper becomes enough long and we may hope to have some global applications in the triangle case, we will study the trianguline case in another paper [Na].

Remark 1.5. The non trianguline case is much more difficult but is much more interesting because this case corresponds to the supercuspidal representations of \(\text{GL}_2(\mathbb{Q}_p)\) via local Langlands, whose \(\varepsilon\)-constants are very difficult to explicitly describe. In a project with Seidai Yasuda, we are now trying to attack the conjecture in this (and the rank two) case using some results of [Co10] and the result of [Na13] on the Perrin-Riou’s exponential map of de Rham \((\varphi, \Gamma)\)-modules. In particular, we have an idea of how to construct the \(\varepsilon\)-isomorphism in the rank two case and have a very rough idea of how to relate this isomorphism with the Bloch-Kato exponential map. The \(\varepsilon\)-conjecture in this case is extremely interesting because we expect that this problem is intimately related with the compatibility of \(p\)-adic and classical local Langlands correspondence for \(\text{GL}_2(\mathbb{Q}_p)\), which has been proven only by a global method [Em].

1.2. Structure of the paper. In §2, we recall the results of [KPX12], [Po13a], and [Na13]. After recalling the definition of \((\varphi, \Gamma)\)-modules over the relative Robba ring, we recall the main results of [KPX12], [Po13a] on the cohomology of \((\varphi, \Gamma)\)-modules, i.e., comparison with Galois cohomology, finiteness, base change property, Euler-Poincaré formula, Tate duality, and the classification of rank one objects, all of which are essential for the formulation of our conjecture. We next recall the result of [Na13] on the theory of Bloch-Kato’s exponential map of \((\varphi, \Gamma)\)-modules. Because the result of [Na13] is not sufficient for our purpose, we slightly generalize the result of this paper. In particular, we show the existence of “Bloch-Kato’s fundamental exact sequences” involving \(D_{\text{cris}}(M)\) (Proposition 2.20), establishing the Bloch-Kato’s duality for the finite cohomology of \((\varphi, \Gamma)\)-modules (Proposition 2.23). The explicit formulae of our Bloch-Kato’s exponential maps (Proposition 2.22) are frequently used in later sections.

In §3, using the preliminaries recalled in §2, we formulate our \(\varepsilon\)-conjecture and state our main theorem of this paper. Because the conjecture is formulated by using the notion of determinant, we first recall this notion in §3.1. In §3.2, using
the determinant of cohomology of \((\varphi, \Gamma)\)-modules, we define a graded invertible \(A\)-module \(\Delta_1(M)\) called the fundamental line for any \((\varphi, \Gamma)\)-module \(M\) over \(\mathcal{R}_A\). In §3.3, for any de Rham \((\varphi, \Gamma)\)-module \(M\), we define a trivialization (called de Rham \(\varepsilon\)-isomorphism) of the fundamental line using the Bloch-Kato fundamental exact sequence, Deligne-Langlands-Fontaine-Perrin-Riou's \(\varepsilon\)-constants and the “gamma -factor” associated to \(D_{\text{psl}}(M)\). In §3.4, we formulate our conjecture and compare our conjecture with Kato’s conjecture, and state our main theorem of this article, which solves the conjecture for all rank one \((\varphi, \Gamma)\)-modules.

§4 is the main part of this paper, where we prove the conjecture for the rank one case. In §4.1, using the theory of analytic Iwasawa cohomology \[KPX12, \text{Po13b},\] and using the standard technique of \(p\)-adic Fourier transform, we construct our \(\varepsilon\)-isomorphism for all rank one \((\varphi, \Gamma)\)-modules. In §4.2, we show that our \(\varepsilon\)-isomorphism defined in §4.1 specializes to the de Rham \(\varepsilon\)-isomorphism defined in §3.2 at each de Rham points. In §4.2.1, we first verify this condition (which we call the de Rham condition) for the “generic” rank one de Rham \((\varphi, \Gamma)\)-modules by establishing a kind of explicit reciprocity law (Proposition 4.12, 4.18). In the process of proving this, we prove a proposition (Proposition 4.14.4) on the compatibility of our \(\varepsilon\)-isomorphism with a natural differential operator. Using the result in the generic case and the density argument, we prove the compatibility of our \(\varepsilon\)-isomorphism with Tate duality and compare with Kato’s \(\varepsilon\)-isomorphism. In §4.2.2, we verify the de Rham condition via explicit calculations for the exceptional case which includes the case of \(\mathcal{R}, \mathcal{R}(1)\) (the \((\varphi, \Gamma)\)-modules corresponding to \(\mathbb{Q}_p, \mathbb{Q}_p(1)\) respectively).

In the appendix, we explicitly calculate the cohomologies \(H^1_{\varphi, \gamma}(\mathcal{R}(1))\) and \(H^i_{\varphi, \gamma}(\mathcal{R})\), which will be used in §4.2.2.

1.3. Notation. Throughout this paper, we fix a prime number \(p\). The letter \(A\) will always denote a \(\mathbb{Q}_p\)-affinoid algebra; we use \(\text{Max}(A)\) to denote the associated rigid analytic space. Fix an algebraic closure \(\overline{\mathbb{Q}}_p\) of \(\mathbb{Q}_p\), and we consider any finite extension \(K\) of \(\mathbb{Q}_p\) inside \(\overline{\mathbb{Q}}_p\). Let \(|-|: \overline{\mathbb{Q}}_p \to \mathbb{Q}_{>0}\) to be the absolute value such that \(|p| = p^{-1}\). For \(n \geq 0\), let denote \(\mu_{p^n}\) for the set of \(p^n\)-th power roots of unity in \(\overline{\mathbb{Q}}_p\), and put \(\mu_{p^n}^\infty := \cup_{n \geq 1} \mu_{p^n}\). For a finite extension \(K\) of \(\mathbb{Q}_p\), put \(K_n := K(\mu_{p^n})\) for \(\infty \geq n \geq 0\). Let denote \(\chi: \Gamma_{\mathbb{Q}_p} := \text{Gal}(\mathbb{Q}_{p, \infty}/\mathbb{Q}_p) \to \mathbb{Z}_p^\times\) for the cyclotomic character given by \(\gamma(\zeta) = \zeta^{\chi(\gamma)}\) for \(\gamma \in \Gamma\) and \(\zeta \in \mu_{p^n}\). Set \(G_K := \text{Gal}(\overline{\mathbb{Q}}_p/K), H_K := \text{Gal}(\overline{\mathbb{Q}}_p/K_{\infty})\), and \(\Gamma_K := \text{Gal}(K_{\infty}/K)\). We let \(k\) be the residue field of \(K\), with \(F := W(k)[1/p]\). For \(k \in \mathbb{Z}\), put \(\mathbb{Z}_p(k) := \mathbb{Z}_p(1)^{\otimes k}\) equipped with a natural action of \(\Gamma_K\). For a \(\mathbb{Z}_p[G_K]\)-module \(N\), let denote \(N(k) := N \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(k)\). When we fix a generator \(\zeta = \{\zeta^n\}_{n \geq 0} \in \mathbb{Z}_p(1)\), we put \(e_k := \zeta^k\) and \(e_k := \bar{e}_k^k \in \mathbb{Z}_p(k)\) for \(k \in \mathbb{Z}\). For a continuous \(G_K\)-module \(N\), let denote by \(C^\bullet_{\text{cont}}(G_K, N)\) the complex of continuous cochains of \(G_K\) with values in \(N\). Denote by \(H^i(K, N) := H^i(C^\bullet_{\text{cont}}(G_K, N))\).
For a group $G$, denote $G_{\text{tor}}$ for the subgroup of $G$ consisting of all torsion elements in $G$. If $G$ is finite group, denote $|G|$ for the order of $G$.

For a commutative ring $R$, let denote by $P_{fg}(R)$ the category of finitely generated, projective $R$-modules. For $N \in P_{fg}(R)$, denote by $\text{rk}_RN$ the rank of $N$ and by $N^\vee := \text{Hom}_R(N,R)$. Let $[\_]: M_1 \times M_2 \to R$ be a perfect pairing. Then we always identify by $M_2 \xrightarrow{\sim} M_1^\vee$ by $x \mapsto (y \mapsto [y,x])$ Let denote by $\text{D}^-(R)$ the derived category of bounded below complexes of $R$-mouldes. Let denote by $\text{D}^b_{\text{perf}}(R)$ (respectively $\text{D}^b_{\text{per}}(R)$) the full subcategory of $\text{D}^-(R)$ consisting of the complexes of $R$-modules which are quasi-isomorphic to a complex $P^\bullet$ of $P_{fg}(R)$ concentrated in degrees in $[a,b]$ (respectively bounded degree). Define a duality functor $\text{RHom}_R(-,R): \text{D}^b_{\text{perf}}(R) \to \text{D}^b_{\text{per}}(R)$ given by sending a bound complex $P^\bullet$ of $P_{fg}(R)$ to $\text{Hom}_R(P^\bullet,R)$. This is well defined and $\text{RHom}_R(-,R) \circ \text{RHom}_R(-,R)$ is the identity. Define the notion $\chi_R(-)$ of Euler characteristic for any objects of $\text{D}^b_{\text{perf}}(R)$ such that, for a bound complex $P^\bullet$ of $P_{fg}(R), \chi_R(P^\bullet) := \sum_{i \in \mathbb{Z}} (-1)^i \text{rk}_R P^i \in \text{Map}(\text{Spec}(R),\mathbb{Z})$. For a complex $(P^\bullet,d_\bullet)$ of $R$-modules, the differential $d_\bullet$ on the dual complex $\text{Hom}_R(P^\bullet,R)$ is defined by $(P^\bullet)^\vee := (P^\bullet)^\vee : f \mapsto (-1)^{n-1} f \circ d_{n-1}$. If $P^i$ is finite projective, then the the canonical isomorphism $P^\bullet \xrightarrow{\sim} ((P^\bullet)^\vee)^\vee : x \mapsto (f \mapsto f(x))$ induces an isomorphism

\[(1) \quad (P^\bullet,-d_\bullet) \xrightarrow{\sim} (\text{Hom}_R(\text{Hom}_R(P^\bullet,R),R),(d_\bullet^\vee)^\vee).\]

2. Cohomology and Bloch-Kato exponential of $(\varphi, \Gamma)$-modules

2.1. Cohomology of $(\varphi, \Gamma)$-modules. In this subsection, we recall the definition of $(\varphi, \Gamma)$-modules and the definition of their cohomologies following [KPX12], and then recall the results of their article on the finiteness of the cohomology.

Put $\omega := p^{-1/(p-1)} \in \mathbb{R}_{>0}$. For $r \in \mathbb{Q}_{>0}$, define the $r$-Gauss norm $|-|_r$ on $\mathbb{Q}_p[T^\pm]$ by the formula $|\sum_i a_i T^i|_r := \max_i \{|a_i| \omega^n\}$. For $0 < s \leq r \in \mathbb{Q}_{>0}$, we write $A_1^{[s,r]}$ for the rigid analytic annulus defined over $\mathbb{Q}_p$ in the variable $T$ with radii $|T| \in [\omega^r, \omega^s]$; its ring of analytic functions, denoted by $\mathcal{R}^{[s,r]}_A$, is the completion of $\mathbb{Q}_p[T^\pm]$ with respect to the norm $|\cdot|_{[s,r]} := \max\{|a|_r| \cdot |_s\}$. Also allow $r$ (but not $s$) to be $\infty$, in which case $A_1^{[s,r]}$ is interpreted as the rigid analytic disc in the variable $T$ with radii $|T| \leq \omega^s$; its ring of analytic functions $\mathcal{R}^{[s,\infty]}_A = \mathcal{R}^{[s,\infty]}_A$ is the completion of $\mathbb{Q}_p[T]$ with respect to $|\cdot|_A$. Let $A$ be a $\mathbb{Q}_p$-affinoid algebra. Let $\mathcal{R}^{[s,r]}_A$ denote the ring of rigid analytic functions on the relative annulus (or disc if $r = \infty$) $\text{Max}(A) \times A_1^{[s,r]}$; its ring of analytic functions is $\mathcal{R}^{[s,r]}_A := \mathcal{R}^{[s,r]}_A \otimes_{\mathbb{Q}_p,A}$. Put $\mathcal{R}^{[s,r]}_A := \bigcap_{0 \leq r \leq s} \mathcal{R}^{[s,r]}_A$ and $\mathcal{R}_A := \cup_{0 \leq r} \mathcal{R}^{[s,r]}_A$.

Let $k'$ be the residue field of $K_\infty$, with $F' := W(k')[1/p]$. Put $\bar{e}_K := [K_\infty : F'^\bullet]$. 

\[6\]
For $0 < s \leq r$, we set $\mathcal{R}^{[s,r]}(\pi_K)$ to be the formal substitution of $T$ by $\pi_K$ in the ring $\mathcal{R}^{[s/r,k/r,k]}$. We set $\mathcal{R}^{[s,r]}_{A}(\pi_K) := \mathcal{R}^{[s,r]}(\pi_K) \otimes_{\mathbb{Q}_p} A$. We define $\mathcal{R}^{A}_{A}(\pi_K), \mathcal{R}_{A}(\pi_K)$ similarly; the latter is referred to as the relative Robba ring over $A$ for $K$.

By the theory of fields of norms, there exists a constant $C(K) > 0$ and, for any $0 < r \leq C(K)$, we can equip $\mathcal{R}^{A}_{A}(\pi_K)$ with a finite étale algebra free of rank $[K_\infty : \mathbb{Q}_p, K]$, where $H_{Q_p}/H_K$. More generally, for any finite extensions $L \supseteq K \supseteq \mathbb{Q}_p$, we can naturally equip $\mathcal{R}^{A}_{A}(\pi_L)$ with a structure of finite étale algebra free of rank $[L_\infty : K_\infty]$ with the Galois group $H_K/H_L$ for any $0 < r \leq \min\{C(K), C(L)\}$.

There are commuting $A$-linear actions of $\Gamma_K$ on $\mathcal{R}^{[s,r]}(\pi_K)$ and of an operator $\varphi : \mathcal{R}^{[s,r]}(\pi_K) \rightarrow \mathcal{R}^{[s/p,r/p]}(\pi_K)$ for $0 < s \leq r \leq C(K)$. The actions on the coefficients $F'$ are the natural ones, i.e. $\Gamma_K$ through its quotient $\text{Gal}(F'/F)$ and $\varphi$ by the canonical lift of $p$-th Frobenius on $k'$. For $0 < s \leq r \leq C(K)$, $\varphi$ makes $\mathcal{R}^{[s/p,r/p]}(\pi_K)$ into a free $\mathcal{R}^{[s,r]}(\pi_K)$-module of rank $p$, and we obtain a $\Gamma_K$-equivariant left inverse map $\psi : \mathcal{R}^{[s/p,r/p]}(\pi_K) \rightarrow \mathcal{R}^{[s,r]}(\pi_K)$ by formula $\frac{1}{p} \varphi - 1 \circ \text{Tr}_{\mathcal{R}^{[s/p,r/p]}(\pi_K) \otimes \mathcal{R}^{[s,r]}(\pi_K)}$. The map $\psi$ naturally extends to the maps $\mathcal{R}^{[r]}(\pi_K) \rightarrow \mathcal{R}^{A}_{A}(\pi_K)$ and $\mathcal{R}_{A}(\pi_K) \rightarrow \mathcal{R}_{A}(\pi_K)$.

**Remark 2.1.** In fact, these rings are constructed using Fontaine’s rings of $p$-adic periods. We don’t have any canonical choice of the parameter $\pi_K$ for general $K$, but the ring $\mathcal{R}_{A}(\pi_K)$ and the actions of $\varphi, \Gamma_K$ don’t depend on the choice of $\pi_K$.

More precisely, $\mathcal{R}(\pi_K)$ is defined as a subring of the ring $\mathcal{B}_{\text{rig}}$ of $p$-adic periods defined in [Ber02], and this subring does not depend on the choice of $\pi_K$, and the actions of $\varphi, \Gamma_K$ is induced by the natural actions of $\varphi, G_K$ on $\mathcal{B}_{\text{rig}}$.

However, for unramified $K$, once we fix a $\mathbb{Z}_p$-basis $\zeta := \{\zeta_n\}_{n \geq 0}$ of $\mathbb{Z}_p(1) := \lim_{\leftarrow n \geq 0} \mu_{p^n}$, we have a natural choice of $\pi_K$ as follows. Let $\overline{\mathbb{Z}_p}$ be the integral closure of $\overline{\mathbb{Q}_p}$, and let $\overline{E}^+ := \bigcap_{n \geq 0} \overline{\mathbb{Z}_p}/p\overline{\mathbb{Z}_p}$ be the projective limit with respect to $p$-th power map, and let $[-] : \overline{E}^+ \rightarrow W(\overline{E}^+)$ be the Teichmüller lift to the ring $W(E^+)$ of Witt vectors. Under the fixed $\zeta$, we can choose $\pi_K = \pi_{Q_p} = \pi_\zeta := (\pi_{Q_p})_{n \geq 0} - 1 \in W(\overline{E}^+) \subseteq \mathcal{B}_{\text{rig}}^\dagger$, and then $\varphi$ and $\Gamma_{Q_p}$ act by $\varphi(\pi_\zeta) = (1 + \pi_\zeta)^p - 1$ and $\chi(\pi_\zeta) = (1 + \pi_\zeta)^{\chi(\gamma)} - 1$ for $\gamma \in \Gamma_{Q_p}$.

**Notation 2.2.** From §3, we will concentrate on the case where $K = \mathbb{Q}_p$ and fix $\zeta := \{\zeta_n\}_{n \geq 0}$ as above. Then, we use the notation $\Gamma := \Gamma_{Q_p}$, $\pi := \pi_{Q_p}$, and omit $(\pi_{Q_p})$ from the notation of Robba rings by writing, for example, $\mathcal{R}^{[s,r]}$ instead of $\mathcal{R}^{[s,r]}(\pi_{Q_p})$. In this case, $\mathcal{R}^{[s/p,r/p]} = \oplus_{0 \leq i \leq p-1}(1 + \pi)^i \varphi(\mathcal{R}^{[s,r]})$, so if $f = \sum_{i=0}^{p-1}(1 + \pi)^i \varphi(f_i)$ then $\psi(f) = f_0$. We define the special element $t = \log(1 + \pi) \in \mathcal{R}^{[s,r]}$. We have $\varphi(t) = pt$ and $\gamma(t) = \chi(\gamma)t$ for $\gamma \in \Gamma$. 7
We first recall the definitions of \( \varphi \)-modules over \( \mathcal{R}_A(\pi_K) \) following Definition 2.2.12 of [KPX12].

**Definition 2.3.** Choose \( 0 < r_0 \leq C(K) \). A \( \varphi \)-module over \( \mathcal{R}_A^{r_0}(\pi_K) \) is a finite projective \( \mathcal{R}_A^{r_0}(\pi_K) \)-module equipped with a \( \mathcal{R}_A^{r_0/p}(\pi_K) \)-linear isomorphism \( \varphi^* M^{r_0} \overset{\sim}{\rightarrow} M^{r_0} \otimes \mathcal{R}_A^{r_0}(\pi_K) \mathcal{R}_A^{r_0/p}(\pi_K) \). A \( \varphi \)-module \( M \) over \( \mathcal{R}_A(\pi_K) \) is a base change to \( \mathcal{R}_A(\pi_K) \) of a \( \varphi \)-module over some \( \mathcal{R}_A^{r_0}(\pi_K) \).

For a \( \varphi \)-module \( M^{r_0} \) over \( \mathcal{R}_A^{r_0}(\pi_K) \) and for \( 0 < s \leq r \leq r_0 \), we set \( M^{[s,r]} = M^{r_0} \otimes \mathcal{R}_A^{r_0}(\pi_K) \mathcal{R}_A^{[s,r]}(\pi_K) \) and \( M^s = M^{r_0} \otimes \mathcal{R}_A^{r_0}(\pi_K) \mathcal{R}_A^s(\pi_K) \). For \( 0 < s \leq r \), the given isomorphism \( \varphi^* (M^{r_0}) \overset{\sim}{\rightarrow} M^{r_0/p} \) induces a \( \varphi \)-semilinear map \( \varphi : M^s \rightarrow \varphi^* M^s \overset{\sim}{\rightarrow} \varphi^* M^{r_0} \otimes \mathcal{R}_A^{r_0/p}(\pi_K) \mathcal{R}_A^{s/p}(\pi_K) \overset{\sim}{\rightarrow} M^{r_0/p} \otimes \mathcal{R}_A^{r_0/p}(\pi_K) \mathcal{R}_A^{s/p}(\pi_K) = M^{s/p} \), where the first map \( M^s \rightarrow \varphi^* M^s \) is given by \( x \rightarrow x \otimes 1 \in M^s \otimes \mathcal{R}_A^{s}(\pi_K) \). We also define \( \varphi^* M^s \otimes \mathcal{R}_A^{s/p}(\pi_K) \mathcal{R}_A^{s/p}(\pi_K) =: \varphi^* M^s \) and the second isomorphism is the base change of the given isomorphism \( \varphi^* M^{r_0} \overset{\sim}{\rightarrow} M^{r_0/p} \). This map \( \varphi \) also induces an \( A \)-linear homomorphism \( \psi : M^{s/p} = \varphi(M^s) \otimes_{\varphi^*(\mathcal{R}_A^{s}(\pi_K))} \mathcal{R}_A^{s/p}(\pi_K) \rightarrow M^s \) given by \( \psi(m \otimes f) = m \otimes \psi(f) \) for \( m \in M^s \) and \( f \in \mathcal{R}_A^{s/p}(\pi_K) \). For a \( \varphi \)-module \( M \) over \( \mathcal{R}_A(\pi_K) \), the maps \( \varphi : M^s \rightarrow M^{s/p} \) and \( \psi : M^{s/p} \rightarrow M^s \) naturally extend to \( \varphi : M \rightarrow M \) and \( \psi : M \rightarrow M \).

We recall the definition of \( (\varphi, \Gamma) \)-modules over \( \mathcal{R}_A(\pi_K) \) following Definition 6.1.1 of [KPX12].

**Definition 2.4.** Choose \( 0 < r_0 \leq C(K) \). A \( (\varphi, \Gamma) \)-module over \( \mathcal{R}_A^{r_0}(\pi_K) \) is a \( \varphi \)-module over \( \mathcal{R}_A^{r_0}(\pi_K) \) equipped with a commuting semilinear continuous action of \( \Gamma_K \). A \( (\varphi, \Gamma) \)-modules over \( \mathcal{R}_A(\pi_K) \) is a base change of a \( (\varphi, \Gamma) \)-module over \( \mathcal{R}_A^{r_0}(\pi_K) \) for some \( 0 < r_0 \leq C(K) \).

We can generalize these notions for general rigid analytic space as in Definition 2.2.12 of [KPX12].

**Definition 2.5.** Let \( X \) be a rigid analytic space over \( \mathbb{Q}_p \). A \( (\varphi, \Gamma) \)-module over \( \mathcal{R}_X(\pi_K) \) is a compatible family of \( (\varphi, \Gamma) \)-modules over \( \mathcal{R}_A(\pi_K) \) for each affinoid \( \text{Max}(A) \) of \( X \).

For \( (\varphi, \Gamma) \)-modules \( M, N \) over \( \mathcal{R}_X(\pi_K) \). We define \( M \otimes N := M \otimes_{\mathcal{R}_X(\pi_K)} N \) for the tensor product equipped with the diagonal action of \( (\varphi, \Gamma_K) \). We also define \( M^\vee := \text{Hom}_{\mathcal{R}_X(\pi_K)}(M, \mathcal{R}_X(\pi_K)) \) for the dual \( (\varphi, \Gamma) \)-module.

For a \( (\varphi, \Gamma) \)-module \( M \) over \( \mathcal{R}_A(\pi_K) \). We denote \( r_M := \text{rank}_{\mathcal{R}_A(\pi_K)} M \in \text{Map}(\text{Spec}(\mathcal{R}_A(\pi_K)), \mathbb{Z}_{\geq 0}) \) for the rank of \( M \), where \( \text{Map}(-,-) \) is the set of continuous maps and \( \mathbb{Z}_{\geq 0} \) is equipped with the discrete topology. We will see later (in Remark 2.15) that \( r_M \) is in fact in \( \text{Map}(\text{Spec}(A), \mathbb{Z}_{\geq 0}) \), i.e. we have \( r_M = \text{pr} \circ f_M \) for unique \( f_M \in \text{Map}(\text{Spec}(A), \mathbb{Z}_{\geq 0}) \), where \( \text{pr} : \text{Spec}(\mathcal{R}_A(\pi_K)) \rightarrow \text{Spec}(A) \) is the natural projection. We also denote by \( r_M := f_M \).

The importance of \( (\varphi, \Gamma) \)-module follows from the following theorem.
Theorem 2.6. (Theorem 3.11 of [KL10]) Let $V$ be a vector bundle over $X$ equipped with a continuous $\mathcal{O}_X$-linear action of $G_K$. Then there is functorially associated to $V$ a $(\varphi, \Gamma)$-module $D_{\text{rig}}(V)$ over $\mathcal{R}_X(\pi_K)$. The rule $V \mapsto D_{\text{rig}}(V)$ is fully faithful and exact, and it commutes with base change in $X$.

For example, we have a canonical isomorphism $D_{\text{rig}}(A(k)) = \mathcal{R}_A(\pi_K)(k)$ for $k \in \mathbb{Z}$.

From §3, we concentrate on $K = \mathbb{Q}_p$ and rank one $(\varphi, \Gamma)$-modules over $\mathcal{R}_X$. Here, we recall the result of [KPX12] concerning the classification of rank one modules. Actually, they obtained a similar result for general $K$, but we don’t recall it because we don’t use it.

Definition 2.7. For a continuous homomorphism $\delta : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$, we define $\mathcal{R}_X(\delta)$ to be the rank one $(\varphi, \Gamma)$-module $\mathcal{R}_X \cdot e_\delta$ over $\mathcal{R}_X$ with $\varphi(e_\delta) = \delta(p)e_\delta$ and $\gamma(e_\delta) = \delta(\chi(\gamma))e_\delta$ for $\gamma \in \Gamma$.

Theorem 2.8. (Theorem 6.1.10 of [KPX12]) Let $M$ be a rank 1 $(\varphi, \Gamma)$-module over $\mathcal{R}_X$. Then, there exist a continuous homomorphism $\delta : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$ and an invertible sheaf $\mathcal{L}$ on $X$, the pair of which is unique up to isomorphism, such that $M \sim \mathcal{R}_X(\delta) \otimes \mathcal{O}_X \mathcal{L}$.

Notation 2.9. i) For $\delta, \delta' : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times$, we fix an isomorphism $\mathcal{R}_X(\delta) \otimes \mathcal{R}_X(\delta') \sim \mathcal{R}_X(\delta \delta')$ by $e_\delta \otimes e_{\delta'} \mapsto e_{\delta \delta'}$, and fix $\mathcal{R}_X(\delta)^\vee \sim \mathcal{R}_X(\delta^{-1})$ by $e_\delta \mapsto e_{\delta^{-1}}$.

ii) For $k \in \mathbb{Z}$, we define a continuous homomorphism $x^k : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times : y \mapsto y^k$. Define $[x] : \mathbb{Q}_p^\times \to \Gamma(X, \mathcal{O}_X)^\times : p \mapsto p^{-1}, a \mapsto 1$ for $a \in \mathbb{Z}_p^\times$. Then, the homomorphism $x[x]$ corresponds to Tate twist, i.e. we have an isomorphism $\mathcal{R}_X(1) \sim \mathcal{R}_X(x[x])$. When we fix a generator $\zeta \in \mathbb{Z}_p(1)$, we identify $\mathcal{R}_X(1) = \mathcal{R}_X(x[x])$ by $e_1 \mapsto e_{x[x]}$.

We next recall some cohomology theories concerning $(\varphi, \Gamma)$-modules. Denote by $\Delta$ for the largest $p$-power torsion subgroup of $\Gamma_K$. Fix $\gamma \in \Gamma_K$ whose image in $\Gamma_K/\Delta$ is a topological generator. For a $\Delta$-module $M$, put $M^\Delta = \{m \in M | \sigma(m) = m \text{ for all } \sigma \in \Delta\}$.

Definition 2.10. For a $(\varphi, \Gamma)$-module $M$ over $\mathcal{R}_A(\pi_K)$, we define the complexes $C_{\varphi, \gamma}(M)$ and $C_{\psi, \gamma}(M)$ of $A$-modules concentrated in degree $[0, 2]$, and define a morphism $\Psi_M$ between them as follows:

\[
C_{\varphi, \gamma}(M) = \left[ M^\Delta \xrightarrow{(\gamma-1, \varphi-1)} M^\Delta \oplus M^\Delta \xrightarrow{(\varphi-1, \psi-1)} M^\Delta \right]
\]

\[
C_{\psi, \gamma}(M) = \left[ M^\Delta \xrightarrow{(\gamma-1, \psi)} M^\Delta \oplus M^\Delta \xrightarrow{(\psi-1)} M^\Delta \right].
\]

The map $\Psi_M$ is quasi-isomorphism by Proposition 2.3.4 of [KPX12].
For \( i \in \mathbb{Z}_{\geq 0} \), define \( H^i_{\phi,\gamma}(M) \) for the \( i \)-th cohomology of \( C^\bullet_{\phi,\gamma}(M) \), called the \((\phi, \Gamma)\)-cohomology of \( M \). We similarly define \( H^i_{\psi,\gamma}(M) \) to be the \( i \)-th cohomology of \( C^\bullet_{\psi,\gamma}(M) \), called the \((\psi, \Gamma)\)-cohomology of \( M \). In this article, we freely identify \( C^\bullet_{\phi,\gamma}(M) \) (respectively \( H^i_{\phi,\gamma}(M) \)) with \( C^\bullet_{\psi,\gamma}(M) \) (respectively \( H^i_{\psi,\gamma}(M) \)) via the quasi-isomorphism \( \Psi_M \).

More generally, for \( h = \phi, \psi \) and any module \( N \) with a commuting actions of \( h \) and \( \Gamma \), we similarly define the complexes \( C^\bullet_{h,\gamma}(N) \)and denote the resulting cohomology by \( H^i_{h,\gamma}(N) \). We denote \([x, y] \in H^i_{h,\gamma}(N) \) (respectively \([z] \in H^2_{h,\gamma}(N) \)) for the element represented by a one cocycle \((x, y) \in N^\Delta \oplus N^\Delta \) (respectively by \( z \in N^\Delta \)). The functor \( N \mapsto C^\bullet_{h,\gamma}(N) \) from the category of topological \( A \)-modules which are Hausdorff with commuting continuous actions of \( h, \Gamma_K \) to the category of complexes of \( A \)-modules is independent of the choice of \( \gamma \) up to canonical isomorphism, i.e. for another choice \( \gamma' \in \Gamma_K \), we have a canonical isomorphism

\[
C^\bullet_{h,\gamma'}(N) = [N^\Delta \xrightarrow{\gamma-1h-1} N^\Delta \oplus N^\Delta \xrightarrow{(h-1)\oplus(1-\gamma)} N^\Delta] \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
C^\bullet_{h,\gamma}(N) = [N^\Delta \xrightarrow{(\gamma'-1h-1)} N^\Delta \oplus N^\Delta \xrightarrow{(h-1)\oplus(1-\gamma')} N^\Delta].
\]

(3)

For a commutative ring \( R \), let denote \( D^-(R) \) for the derived category of bounded below complexes of \( R \)-modules. We use the same notation \( C^\bullet_{h,\gamma}(N) \in D^-(A) \) for the object represented by this complex.

Let \( V \) be a finite projective \( A \)-module with a continuous \( A \)-linear action of \( G_K \). Let denote \( C^\bullet_{\cont}(G_K, V) \) for the complex of continuous \( G_K \)-cochains with values in \( V \), and denote \( H^i(G_K, V) \) for the cohomology. By Theorem 2.8 of [Po13a], we have a functorial isomorphism

\[
C^\bullet_{\cont}(G_K, V) \xrightarrow{\sim} C^\bullet_{\phi,\gamma}(D_{\rig}(V))
\]

in \( D^-(A) \) and a functorial \( A \)-linear isomorphism

\[
H^i(G_K, V) \xrightarrow{\sim} H^i_{\phi,\gamma}(D_{\rig}(V)).
\]

**Definition 2.11.** For \((\phi, \Gamma)\)-modules \( M, N \) over \( \mathcal{R}_A(\pi_K) \), we have a natural \( A \)-bilinear cup product morphism

\[
C^\bullet_{\phi,\gamma}(M) \times C^\bullet_{\phi,\gamma}(N) \to C^\bullet_{\phi,\gamma}(M \otimes N),
\]

see Definition 2.3.11 of [KPX12] for the definition. This induces an \( A \)-bilinear graded commutative cup product pairing

\[
\cup : H^i_{\phi,\gamma}(M) \times H^j_{\phi,\gamma}(N) \to H^{i+j}_{\phi,\gamma}(M \otimes N).
\]

For example, this is defined by the formulae

\[
x \cup [y] := [x \otimes y] \text{ for } i = 0, j = 2, \\
[x_1, y_1] \cup [x_2, y_2] := [y_1 \otimes \phi(x_2) - x_1 \otimes \gamma(y_2)] \text{ for } i = j = 1.
\]
Definition 2.12. Let denote by $M^* := M^\vee(1)$ for the Tate dual of $M$. Using the cup product, the evaluation map $ev : M^* \otimes M \to \mathcal{R}_A(\pi_K)(1) : f \otimes x \mapsto f(x)$, the comparison isomorphism $H^2(K, A(1)) \tilde{\to} H^2_{\varphi, \gamma}(\mathcal{R}_A(\pi_K)(1))$ and the Tate’s trace map $H^2(K, A(1)) \to A$, one gets Tate duality pairings

$$
C^\bullet_{\varphi, \gamma}(M^*) \times C^\bullet_{\varphi, \gamma}(M) \to C^\bullet_{\varphi, \gamma}(M^* \otimes M) \to C^\bullet_{\varphi, \gamma}(\mathcal{R}_A(\pi_K)(1)) \to H^2_{\varphi, \gamma}(\mathcal{R}_A(\pi_K)(1))[-2] \\
\sim H^2(K, A(1))[-2] \sim A[-2]
$$

and

$$
< -, - > : H^i_{\varphi, \gamma}(M^*) \times H^{2-i}_{\varphi, \gamma}(M) \to A.
$$

Remark 2.13. In the appendix, we explicitly describe the isomorphism $H^2_{\varphi, \gamma}(\mathcal{R}_A(1)) \to H^2(G_{\mathbb{Q}_p}, A(1)) \to A$ using the residue map; see Proposition 4.26.

One of the main results of [KPX12] which is crucial to formulate our conjecture is the following.

Theorem 2.14. (Theorem 4.4.3, Theorem 4.4.4 of [KPX12]) Let $M$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_A(\pi_K)$.

1. $C^\bullet_{\varphi, \gamma}(M) \in D_{\text{perf}}^{0,2}(A)$. In particular, the cohomology groups $H^i_{\varphi, \gamma}(M)$ are finite $A$-modules.
2. Let $A \to A'$ be a continuous morphism of $\mathbb{Q}_p$-affinoid algebras. Then, the canonical morphism $C^\bullet_{\varphi, \gamma}(M) \otimes_A A' \to C^\bullet_{\varphi, \gamma}(M \otimes_A A')$ is a quasi-isomorphism.

In particular, if $A'$ is flat over $A$, we have $H^i_{\varphi, \gamma}(M) \otimes_A A' \tilde{\to} H^i_{\varphi, \gamma}(M \otimes_A A')$.
3. (Euler characteristic formula) We have $\chi_A(C^\bullet_{\varphi, \gamma}(M)) = -[K : \mathbb{Q}_p] \cdot r_M$.
4. (Tate duality) Tate duality pairing defined in Definition 2.12 induces a quasi-isomorphism

$$
C^\bullet_{\varphi, \gamma}(M) \tilde{\to} \text{RHom}_A(C^\bullet_{\varphi, \gamma}(M^*), A)[-2].
$$

Remark 2.15. By the equality of (3) of the above theorem, the rank $r_M \in \text{Map}(\text{Spec}(\mathcal{R}_A(\pi_K)), \mathbb{Z}_{\geq 0})$ is contained in $\text{Map}(\text{Spec}(A), \mathbb{Z}_{\geq 0})$.

Let $X$ be a rigid analytic space over $\mathbb{Q}_p$ and let $M$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_X(\pi_K)$. By (1) and (2) of the above theorem, the correspondence $U \mapsto H^i_{\varphi, \gamma}(M|_U)$ for each affinoid open $U$ in $X$ defines a coherent $\mathcal{O}_X$-module for each $i \in [0, 2]$, for which we also denote by $H^i_{\varphi, \gamma}(M)$.

2.2. Bloch-Kato exponential for $(\varphi, \Gamma)$-modules. In §2 of [Na13], we developed the theory of Bloch-Kato exponential purely in terms of $(\varphi, \Gamma)$-modules over Robba ring. This subsection is a complement of §2 of [Na13]; we recall the definitions of Bloch-Kato exponential and the dual exponential for $(\varphi, \Gamma)$-modules over the (relative) Robba ring, and we slightly generalize the results of [Na13], which are needed to formulate our conjecture in the next section.
Define $n(K) \geq 1$ to be the minimal integer $n$ such that $1/p^{n-1} \leq \ell_K C(K)$, and put $\mathcal{R}_A^{(n)}(\pi_K) = \mathcal{R}_A^{1/(p^{n-1}\ell_K)}(\pi_K)$ for $n \geq n(K)$. For $n \geq n(K)$, one has a $\Gamma_K$-equivariant $A$-algebra homomorphism

$$
t_n : \mathcal{R}_A^{(n)}(\pi_K) \to (K_n \otimes_{Q_p} A)[[t]]$$

such that

$$t_n(\pi) = \zeta_p^n \exp \left(\frac{t}{p^n}\right) - 1 \quad \text{and} \quad t_n(a) = \varphi^{-n}(a) \quad (a \in F') \, .$$

For $n \geq n(K)$, we have the following commutative diagrams:

$$
\begin{array}{ccc}
\mathcal{R}_A^{(n)}(\pi_K) & \xrightarrow{t_n} & (K_n \otimes_{Q_p} A)[[t]] \\
\bigg\uparrow \varphi & & \bigg\uparrow \text{can} \\
\mathcal{R}_A^{(n+1)}(\pi_K) & \xrightarrow{t_{n+1}} & (K_{n+1} \otimes_{Q_p} A)[[t]]
\end{array}
$$

where $\text{can}$ is the canonical injection and $\frac{1}{p} \text{Tr}_{K_{n+1}/K_n}$ is defined by

$$\sum_{k \geq 0} a_k t^k \mapsto \sum_{k \geq 0} \frac{1}{p} \text{Tr}_{K_{n+1}/K_n}(a_k)t^k .$$

Let $M$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_A(\pi_K)$ obtained as a base change of a $(\varphi, \Gamma)$-module $M^{(r)}$ over $\mathcal{R}_A^{(r)}(\pi_K)$ for some $0 < r_0 \leq c(K)$. Define $n(M) \in \mathbb{Z}_{\geq n(K)}$ to be the minimal integer such that $1/p^{n-1} \leq \ell_K r_0$. Put $M^{(n)} = M^{1/(p^{n-1}\ell_K)}$ for $n \geq n(M)$, then $\varphi$ and $\psi$ induce $\varphi : M^{(n)} \to M^{(n+1)}$ and $\psi : M^{(n+1)} \to M^{(n)}$ respectively. Define

$$D^{+}_n(M) = M^{(n)} \otimes_{\mathcal{R}_A^{(n)}(\pi_K), t_n} (K_n \otimes_{Q_p} A)[[t]] \quad (\text{respectively } D^{+}_n(M) = D^{+}_n(M)[1/t]) ,$$

which is a finite projective $(K_n \otimes_{Q_p} A)[[t]]$-module (respectively $(K_n \otimes_{Q_p} A)(t)$-module ) with a semilinear action of $\Gamma_K$. We also denote $t_n : M^{(n)} \to D^{+}_n(M)$ for the map defined by $x \mapsto x \otimes 1$.

Using the base change of the Frobenius structure $\varphi^*(M^{(n)}) \sim M^{(n+1)}$ by the map $t_{n+1}$, we obtain a $\Gamma_K$-equivariant $(K_{n+1} \otimes_{Q_p} A)[[t]]$-linear isomorphism

$$D^{+}_n(M) \otimes_{(K_n \otimes_{Q_p} A)[[t]]} (K_{n+1} \otimes_{Q_p} A)[[t]] \xrightarrow{\sim} \varphi^*(M^{(n)}) \otimes_{\mathcal{R}_A^{(n+1)}(\pi_K), t_{n+1}} (K_{n+1} \otimes_{Q_p} A)[[t]] \xrightarrow{\sim} M^{(n+1)} \otimes_{\mathcal{R}_A^{(n+1)}(\pi_K), t_{n+1}} (K_{n+1} \otimes_{Q_p} A)[[t]] = D^{+}_n(M) .$$

Using this isomorphism, we obtain $\Gamma_K$-equivariant $(K_n \otimes_{Q_p} A)[[t]]$-linear morphisms

$$\text{can} : D^{+}_n(M) \xrightarrow{x \mapsto x \otimes 1} D^{+}_n(M) \otimes_{(K_n \otimes_{Q_p} A)[[t]]} (K_{n+1} \otimes_{Q_p} A)[[t]] \xrightarrow{\sim} D^{+}_n(M)$$

and
\[
\frac{1}{p} \text{Tr}_{K_{n+1}/K_n} : D_{\text{diff},n+1}^+ (M) \xrightarrow{\sim} D_{\text{diff},n}^+(M) \otimes (K_n \otimes \mathbb{Q}_p A)[[t]] (K_{n+1} \otimes \mathbb{Q}_p A)[[t]]
\]

\[
x \otimes f \rightarrow \frac{1}{p} \text{Tr}_{K_{n+1}/K_n} (f) x \quad \rightarrow \quad D_{\text{diff},n}^+(M).
\]

These naturally induce \( \text{can} : D_{\text{diff},n}^+(M) \rightarrow D_{\text{diff},n+1}^+(M) \) and \( \frac{1}{p} \text{Tr}_{K_{n+1}/K_n} : D_{\text{diff},n+1}^+(M) \rightarrow D_{\text{diff},n}^+(M) \), and we have the following commutative diagrams:

\[
\begin{array}{ccc}
M^{(n)} & \xrightarrow{\iota_n} & D_{\text{diff},n}^+(M) \\
\downarrow \varphi & & \downarrow \text{can} \\
M^{(n+1)} & \xrightarrow{\iota_n+1} & D_{\text{diff},n+1}^+(M) \\
\end{array}
\]

\[
\begin{array}{ccc}
M^{(n+1)} & \xrightarrow{\iota_n+1} & D_{\text{diff},n+1}^+(M) \\
\downarrow \psi & & \downarrow \frac{1}{p} \text{Tr}_{K_{n+1}/K_n} \\
M^{(n)} & \xrightarrow{\iota_n} & D_{\text{diff},n}^+(M).
\end{array}
\]

Put \( D_{\text{diff}}^{(+)} (M) := \lim_{n \geq n(M)} D_{\text{diff},n}^{(+)} (M) \), where the transition map is \( \text{can} : D_{\text{diff},n}^{(+)} (M) \rightarrow D_{\text{diff},n+1}^{(+)} (M) \). Then, we have \( D_{\text{diff}}^{(+)} (M) = D_{\text{diff},n}^{(+)} (M) \otimes (K_n \otimes \mathbb{Q}_p A)[[t]] (K_{n+1} \otimes \mathbb{Q}_p A)[[t]] \) for any \( n \geq n(M) \), where we define \( (K_{\infty} \otimes \mathbb{Q}_p A)[[t]] = \bigcup_{m \geq 1} (K_m \otimes \mathbb{Q}_p A)[[t]] \).

For an \( A[\Gamma_K] \)-module \( N \), we define a complex of \( A \)-modules concentrated in degree \([0,1]\)

\[
C^\gamma_\varphi(N) = [N^\Delta \xrightarrow{\gamma-1} N^\Delta]
\]

and denote \( H_\gamma^\varphi(N) \) for the cohomology of \( C^\gamma_\varphi(N) \). If \( N \) is a topological Hausdorff \( A \)-module with a continuous action of \( \Gamma_K \), the complex \( C^\gamma_\varphi(N) \) is also independent of the choice of \( \gamma \) up to canonical isomorphism.

Let \( M \) be a \((\varphi, \Gamma)\)-modules over \( R_A(\pi_K) \). For \( n \geq n(M) \) and \( M_0 = M, M[1/t] \), we define a complex \( \tilde{C}^\varphi_\gamma(M_0^{(n)}) \) concentrated in degree \([0,2]\) by

\[
\tilde{C}^\varphi_\gamma(M_0^{(n)}) := [M_0^{(n)}, \Delta \xrightarrow{(\gamma-1)(\varphi-1)} M_0^{(n)} \oplus M_0^{(n+1), \Delta} \xrightarrow{(\varphi-1)(1-\gamma)} M_0^{(n+1), \Delta}].
\]

Of course, we have \( \lim_{n \rightarrow n(M)} \tilde{C}^\varphi_\gamma(M_0^{(n)}) = C^\varphi_\gamma(M_0) \), where the transition map is the natural one induced by the canonical inclusion \( M_0^{(n)} \hookrightarrow M_0^{(n+1)} \). We define another complex

\[
C^\varphi_\gamma(M_0) := \lim_{n \rightarrow n(M)} \tilde{C}^\varphi_\gamma(M_0^{(n)}),
\]

where the transition map is the natural one induced by \( \varphi : M_0^{(n)} \rightarrow M_0^{(n+1)} \). We similarly define \( C^\varphi_\gamma(M_0) \) and denote \( H^\varphi_\gamma(M_0) \) for the cohomology of \( C^\varphi_\gamma(M_0) \) (respectively \( H^\varphi_\gamma(M_0) \)). For \( n \geq n(M) \), we equip \( C^\gamma_\varphi(M_0^{(n)}) \) with a structure of a complex of \( F \)-vector spaces by \( ax := \varphi^n(a)x \) for \( a \in F, x \in C^\gamma_\varphi(M_0^{(n)}) \). Then, \( C^\varphi_\gamma(M_0) \) (respectively \( H^\varphi_\gamma(M_0) \)) is also naturally equipped with a structure of a complex of \( F \)-vector spaces (respectively a \( F \)-vector space).
By the compatibility of \( \varphi : M^{(n)} \hookrightarrow M^{(n+1)} \) and can : \( \mathbf{D}_{\text{diff},n}^{+}(M) \to \mathbf{D}_{\text{diff},n+1}^{+}(M) \) with respect to the map \( \iota_n : M^{(n)} \to \mathbf{D}_{\text{diff},n}^{+}(M) \), the map \( \iota_n \) induces canonical maps

\[
\iota : C_{\gamma}^{(\varphi)}(M) \to C_{\gamma}^{(\varphi)}(\mathbf{D}_{\text{diff}}^{+}(M)) \quad \text{and} \quad \iota : C_{\gamma}^{(\varphi)}(M[1/t]) \to C_{\gamma}^{(\varphi)}(\mathbf{D}_{\text{diff}}(M))
\]

which are \( F \otimes_{Q_p} A \)-linear.

**Lemma 2.16.** For \( n \geq n(M) \), the natural maps

\[
C_{\gamma}^{*}(\mathbf{D}_{\text{diff},n}^{+}(M)) \to C_{\gamma}^{*}(\mathbf{D}_{\text{diff},n+1}^{+}(M)), \quad C_{\gamma}^{*}(M^{(n)}) \to C_{\gamma}^{*}(M_{0}^{(n+1)})
\]

and

\[
\tilde{C}_{\varphi,\gamma}^{*}(M_{0}^{(n)}) \to \tilde{C}_{\varphi,\gamma}^{*}(M_{0}^{(n+1)})
\]

for \( M_{0} = M, M[1/t] \) which are induced by \( \varphi \) are quasi-isomorphism. Similarly, the maps

\[
C_{\gamma}^{*}(\mathbf{D}_{\text{diff},n}(M)) \to C_{\gamma}^{*}(\mathbf{D}_{\text{diff}}^{+}(M)), \quad C_{\gamma}^{*}(M_{0}^{(n)}) \to C_{\gamma}^{(\varphi)}(M_{0})
\]

and

\[
\tilde{C}_{\varphi,\gamma}^{*}(M_{0}^{(n)}) \to C_{\gamma}^{(\varphi)}(M_{0})
\]

for \( M_{0} = M, M[1/t] \) are quasi-isomorphism.

**Proof.** The latter statement is trivial if we can prove the first statement. Let’s prove the first statement. We first note that \( \gamma - 1 : (M_{0}^{(n)})^{\psi=0} \to (M_{0}^{(n)})^{\psi=0} \) is isomorphism for \( n \geq n(M) + 1 \) by Theorem 3.1.1 of [KPX12] (precisely, this fact for \( M_{0} = M[1/t] \) follows from the proof of this theorem). Taking the base change of this isomorphism by the map \( \iota_n : \mathcal{R}_{A}^{(\pi_K)}(K_{n} \otimes_{Q_p} A)[[t]] \), we also have that \( \gamma - 1 : (\mathbf{D}_{\text{diff},n}^{+}(M))^{1_{\text{Tr}K_{n/K_{n-1}}=0}} \to (\mathbf{D}_{\text{diff},n}^{+}(M))^{1_{\text{Tr}K_{n/K_{n-1}}=0}} \) is isomorphism for \( n \geq n(M) + 1 \). Using these facts, we prove the lemma as follows. Here, we only prove that the map \( C_{\gamma}^{*}(M_{0}^{(n)}) \to C_{\gamma}^{*}(M_{0}^{(n+1)}) \) induced by \( \varphi : M_{0}^{(n)} \to M_{0}^{(n+1)} \) is quasi-isomorphism for \( n \geq n(M) \) because the other cases can be proven in the same way. Because we have a \( \Gamma_{K} \)-equivariant decomposition \( M_{0}^{(n+1)} = \varphi(M_{0}^{(n)}) \oplus (M_{0}^{(n+1)})^{\psi=0} \), we obtain a decomposition \( C_{\gamma}^{*}(M_{0}^{(n+1)}) = \varphi(C_{\gamma}^{*}(M_{0}^{(n)})) \oplus C_{\gamma}^{*}(M_{0}^{(n+1)})^{\psi=0} \). Because the complex \( C_{\gamma}^{*}(M_{0}^{(n+1)})^{\psi=0} \) is acyclic by the above remark and \( \varphi : M_{0}^{(n)} \to M_{0}^{(n+1)} \) is injection, the map \( \varphi : C_{\gamma}^{*}(M_{0}^{(n)}) \to C_{\gamma}^{*}(M_{0}^{(n+1)}) \) is quasi-isomorphism.

\[\square\]

For another canonical map \( C_{\gamma}^{*}(M_{0}^{(n)}) \to C_{\gamma}^{*}(M_{0}) \) which is induced by the canonical inclusion \( M^{(n)} \hookrightarrow M \), we can show the following lemma.

**Lemma 2.17.** For \( n \geq n(M) \) and \( M_{0} = M, M[1/t] \), the inclusion

\[
H^{0}_{\varphi}(M_{0}^{(n)}) \hookrightarrow H^{0}_{\varphi}(M_{0})
\]

induced by the canonical inclusion \( M_{0}^{(n)} \hookrightarrow M_{0} \) is isomorphism.
Proof. It suffices to show that $H^0_\gamma(M_0^{(n)}) \hookrightarrow H^0_\gamma(M_0^{(n+1)})$ is an isomorphism for each $n \geq n(M)$. We first prove this claim when $A$ is a finite $\mathbb{Q}_p$-algebra. In this case, we may assume that $A = \mathbb{Q}_p$. Because we have an inclusion $\iota_n : H^0_\gamma(M_0^{(n)}) \hookrightarrow H^0_\gamma(D_{dif}(M))$ and the latter is a finite dimensional $\mathbb{Q}_p$-vector space, $H^0_\gamma(M_0^{(n)})$ is also finite dimensional. Because $\varphi : C^*_\gamma(M_0^{(n)}) \to C^*_\gamma(M_0^{(n+1)})$ is quasi-isomorphism for $n \geq n(M)$ by the above lemma, we obtain an isomorphism $\varphi : H^0_\gamma(M_0^{(n)}) \cong H^0_\gamma(M_0^{(n+1)})$. In particular, the dimension of $H^0_\gamma(M_0^{(n)})$ is independent of $n \geq n(M)$.

Hence, the canonical inclusion $H^0_\gamma(M_0^{(n)}) \hookrightarrow H^0_\gamma(M_0^{(n+1)})$ is isomorphism.

We next prove the claim for general $A$. By Lemma 6.4 of [KL10], there exists a strict inclusion $A \hookrightarrow \prod_{i=1}^k A_i$ of topological rings, in which each $A_i$ is a finite algebra over a completely discretely valued field. If we similarly define the rings $\mathcal{R}_{A_i}(\pi_K)$, $\mathcal{R}_{A_i}(\pi_K)$, we can generalize the notions concerning $(\varphi, \Gamma)$-modules for $\mathcal{R}_{A_i}(\pi_K)$. In particular, the above claim holds for $M_{0,i} := M_0 \otimes_A A_i$ for each $i$. Consider the following canonical diagram of exact sequences,

$$
\begin{array}{cccccc}
0 & \longrightarrow & M_0^{(n)} & \longrightarrow & M_0^{(n+1)} & \longrightarrow & M_0^{(n+1)}/M_0^{(n)} \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \prod_{i=1}^k M_{0,i}^{(n)} & \longrightarrow & \prod_{i=1}^k M_{0,i}^{(n+1)} & \longrightarrow & \prod_{i=1}^k M_{0,i}^{(n+1)}/M_{0,i}^{(n)} \longrightarrow 0.
\end{array}
$$

If we can show that the right vertical arrow is injection, then the claim for $A$ follows from the claim for each $A_i$ by a simple diagram chase. To show that the right vertical arrow is injection, we may assume that $M = \mathcal{R}_A(\pi_K)$ because $M^{(n)}$ is finite projective over $\mathcal{R}_A^{(n)}(\pi_K)$ for each $n$. Then, the natural map $\mathcal{R}_A^{(n+1)}(\pi_K)[1/t]/\mathcal{R}_A^{(n)}(\pi_K)[1/t] \to \prod_{i=1}^k \mathcal{R}_{A_i}^{(n+1)}(\pi_K)[1/t]/\mathcal{R}_{A_i}^{(n)}(\pi_K)[1/t]$ is injection because the inclusion $A \hookrightarrow \prod_{i=1}^k A_i$ is strict, which proves the claim for general $A$, hence proves the lemma.

\[ \square \]

Remark 2.18. The author doesn’t know whether the natural map $H^1_\gamma(M_0^{(n)}) \to H^1_\gamma(M_0)$ induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0$ is isomorphism or not.

For the $(\varphi, \Gamma)$-cohomology, we can prove the following lemma.

Lemma 2.19. (1) For $n \geq n(M)$ and for $M_0 = M, M[1/t]$, the map

$$
\tilde{C}^{\bullet, \gamma}_\varphi(M_0^{(n)}) \to C^{\bullet, \gamma}_\varphi(M_0)
$$

induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0$ is quasi-isomorphism.

(2) In $D^{-}(A)$, the isomorphism

$$
C^{\bullet, \gamma}_\varphi(M_0) \cong C^{(\varphi, \gamma)}_\varphi(M_0)
$$
which is obtained as the composition of the inverse of the isomorphism $\tilde{C}_{φ,γ}^\bullet(M_0^{(n)}) \xrightarrow{\sim} C_{φ,γ}^\bullet(M_0^{(n)})$ in (1) with the isomorphism $\tilde{C}_{φ,γ}^\bullet(M_0^{(n)}) \xrightarrow{\sim} C_{φ,γ}^\bullet(M_0)$ in Lemma 2.16 is independent of the choice of $n \geq n(M)$.

**Proof.** For $n \geq n(M)$, we define a map $f : \tilde{C}_{φ,γ}^\bullet(M_0^{(n)}) \to \tilde{C}_{φ,γ}^\bullet(M_0^{(n+1)})^{[1]}$ by $f_1 : M_0^{(n),\Delta} \oplus M_0^{(n+1),\Delta} \to M_0^{(n+1),\Delta} : (x, y) \mapsto y$, and $f_2 : M_0^{(n+1),\Delta} \to M_0^{(n),\Delta} \oplus M_0^{(n+2),\Delta} : x \mapsto (x, 0)$. This gives a homotopy between $φ : \tilde{C}_{φ,γ}^\bullet(M_0^{(n)}) \to \tilde{C}_{φ,γ}^\bullet(M_0^{(n+1)})$ and can : $\tilde{C}_{φ,γ}^\bullet(M_0^{(n)}) \to \tilde{C}_{φ,γ}^\bullet(M_0^{(n+1)})$ induced by the canonical inclusion $M_0^{(n)} \hookrightarrow M_0^{(n+1)}$. Hence, can : $\tilde{C}_{φ,γ}^\bullet(M_0^{(n)}) \to \tilde{C}_{φ,γ}^\bullet(M_0^{(n+1)})$ is also isomorphism by Lemma 2.16 and the map $\tilde{C}_{φ,γ}^\bullet(M_0^{(n)}) \to C_{φ,γ}^\bullet(M_0)$ is also isomorphism by taking the limit, which proves (1).

In a similar way, we can show that the map can : $C_{φ,γ}^\bullet(M_0^{(n)}) \to C_{φ,γ}^\bullet(M_0)$ induced by the canonical inclusions can : $M_0^{(n)} \to M_0^{(n+1)}$ for any $n \geq n(M)$ is homotopic to the identity map. Hence, we obtain the following commutative diagram in $D^-(A)$ for any $n \geq n(M)$,

\[
\begin{array}{ccc}
\tilde{C}_{φ,γ}^\bullet(M_0^{(n)}) & \longrightarrow & C_{φ,γ}^\bullet(M_0) \\
\downarrow \text{can} & & \downarrow \text{id} \\
\tilde{C}_{φ,γ}^\bullet(M_0^{(n+1)}) & \longrightarrow & C_{φ,γ}^\bullet(M_0),
\end{array}
\]

from which we obtain the second statement in the lemma.

We define a morphism $f : C_{φ,γ}^\bullet(M_0) \to C_{φ,γ}^\bullet(M_0)$ in $D^-(A)$ as the composition of the isomorphism $C_{φ,γ}^\bullet(M_0) \xrightarrow{\sim} C_{φ,γ}^\bullet(M_0)$ in the above lemma (2) with the map $C_{φ,γ}^\bullet(M_0) \to C_{φ,γ}^\bullet(M_0)$ which is induced by $\tilde{C}_{φ,γ}^\bullet(M_0^{(n)}) = [M_0^{(n),\Delta}] \xrightarrow{(γ-1)\oplus(φ-1)} M_0^{(n),\Delta} \oplus M_0^{(n+1),\Delta} \xrightarrow{(φ-1)\oplus(1-γ)} M_0^{(n+1),\Delta}$

We define $g : C_{φ,γ}^\bullet(M) \xrightarrow{f} C_{φ,γ}^\bullet(M) \xrightarrow{\delta} C_{φ,γ}^\bullet(D^\bullet_{\text{id}}(M)).$

We denote can : $C_{φ,γ}^\bullet(M_0) \to C_{φ,γ}^\bullet(M_0)$ for the map induced by the canonical inclusion can : $M_0^{(n)} \to M_0^{(n+1)}$ for each $n \geq n(M)$. Under these notations, we prove the following proposition, which is a modified version of Theorem 2.8 of [Na13].
Proposition 2.20. We have functorial two distinguished triangles (horizontal ones) and the map from the above to the below:

\[
C^\bullet_{\varphi, \gamma}(M) \xrightarrow{d_1} C^\bullet_{\varphi, \gamma}(M[1/t]) \oplus C^\bullet_\gamma(D^+_\text{diff}(M)) \xrightarrow{d_2} C^\bullet_\gamma(D^+_\text{diff}(M)) \xrightarrow{\leftarrow} (5) 0
\]

with

\[
d_1(x) = (x, g(x)), \quad d_2(x, y) = g(x) - y,
\]

and

\[
d_3(x) = (f(x), g(x)), \quad d_4(x, y) = ((\text{can} - 1)x, g(x) - y).
\]

Remark 2.21. In §2 of [Na13], we essentially proved that the above horizontal line in this proposition is distinguished triangle. For the application to the local \(\varepsilon\)-conjecture, we also need the below triangle, which involves \(D^K_{\text{cris}}(M) := H^0(M[1/t])\).

Proof. ( of Proposition 2.20) We first show that the above horizontal line in the proposition is a distinguished triangle. Actually, this is the content of Theorem 2.8 of [Na13], but we briefly recall the proof because we also use it to prove that the below line is a distinguished triangle. In this proof, we assume \(\Delta = \{1\}\) for simplicity; the general case just follows by taking the \(\Delta\)-fixed parts.

For \(n \geq n(M)\), we have the following exact sequence of \(A\)-modules:

\[
0 \to M^{(n)} \xrightarrow{c_1} M^{(n)[1/t]} \oplus \prod_{m \geq n} D^+_{\text{diff}, m}(M) \xrightarrow{c_2} \bigcup_{k \geq 0} \prod_{m \geq n} \frac{1}{t^k} D^+_{\text{diff}, m}(M) \to 0
\]

with

\[
c_1(x) = (x, (t_m(x))_{m \geq n}) \quad \text{and} \quad c_2(x, (y_m)_{m \geq n}) = (t_m(x) - y_m)_{m \geq n}
\]

by Lemma 2.9 of [Na13] (precisely, we proved it when \(A\) is a finite \(\mathbb{Q}_p\)-algebra, but we can prove it for general \(A\) in the same way). For \(n \geq n(M)\) and \(k \geq 0\), we define a complex \(\tilde{C}^\bullet_{\varphi, \gamma, k}(D^+_{\text{diff}, n}(M))\) concentrated in degree in \([0, 2]\) by

\[
\prod_{m \geq n} \frac{1}{t^k} D^+_{\text{diff}, m}(M) \xrightarrow{b_0} \prod_{m \geq n} \frac{1}{t^k} D^+_{\text{diff}, m}(M) \oplus \prod_{m \geq n+1} \frac{1}{t^k} D^+_{\text{diff}, m}(M) \xrightarrow{b_1} \prod_{m \geq n+1} \frac{1}{t^k} D^+_{\text{diff}, m}(M)
\]

with

\[
b_0((x_m)_{m \geq n}) = (((\gamma - 1)x_m)_{m \geq n}, (x_{m-1} - x_m)_{m \geq n+1})
\]

and

\[
b_1((x_m)_{m \geq n}, (y_m)_{m \geq n+1}) = ((x_{m-1} - x_m) - (\gamma - 1)y_m)_{m \geq n+1}.
\]
Put $\tilde{C}^\bullet_{\varphi,\gamma}(D_{\text{diff},n}(M)) = \bigcup_{k\geq 0} \tilde{C}^k_{\varphi,\gamma}([\mathbb{D}_{\text{diff},n}^+(M))]$. By the above exact sequence (7), we obtain the following exact sequence of complexes of $A$-modules:

$$0 \to \tilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}) \to \tilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus \tilde{C}^\bullet_{\varphi,\gamma}(D_{\text{diff},n}^+(M)) \to \tilde{C}^\bullet_{\varphi,\gamma}(D_{\text{diff},n}(M)) \to 0.$$ 

Moreover, the map $C^\bullet(\mathbb{D}_{\text{diff},n}^+(M)) \to \tilde{C}^\bullet_{\varphi,\gamma}(\mathbb{D}_{\text{diff},n}^+(M))$ which is defined by

$$\prod_{m\geq n} D_{\text{diff},m}^+(M) \longrightarrow \prod_{m\geq n} D_{\text{diff},m}^+(M) \oplus \prod_{m\geq n+1} D_{\text{diff},m}^+(M) \longrightarrow \prod_{m\geq n+1} D_{\text{diff},m}^+(M)$$

and the similar map $C^\bullet_{\gamma}(\mathbb{D}_{\text{diff},n}(M)) \to \tilde{C}^\bullet_{\varphi,\gamma}(\mathbb{D}_{\text{diff},n}(M))$ are easily seen to be quasi-isomorphism because we have the following exact sequence

$$0 \to D_{\text{diff},n}^{(+)}(M) \xrightarrow{x \to (x)(m\geq n)} \prod_{m\geq n} D_{\text{diff},m}^{(+)}(M) \xrightarrow{(x_m)_{m\geq n} \to (x_{m-1} - x_m)_{m\geq n+1}} \prod_{m\geq n+1} D_{\text{diff},m}^{(+)}(M) \to 0.$$ 

Put $\tilde{C}^\bullet_{\varphi,\gamma}(\mathbb{D}_{\text{diff},n}^{(+)}(M)) := \lim_{n \to n, a} \tilde{C}^\bullet_{\varphi,\gamma}(\mathbb{D}_{\text{diff},n}^{(+)}(M))$ where the transition map $a^\bullet : \tilde{C}^\bullet_{\varphi,\gamma}(\mathbb{D}_{\text{diff},n}^{(+)}(M)) \to \tilde{C}^\bullet_{\varphi,\gamma}(\mathbb{D}_{\text{diff},n}^{(+)}(M))$ is defined by

$$a^0((x_m)_{m\geq n}) = (x_m)_{m\geq n+1}, \quad a^1((x_m)_{m\geq n}, (y_m)_{m\geq n+1}) = ((x_m)_{m\geq n+1}, (y_m)_{m\geq n+2}),$$

and

$$a^2((x_m)_{m\geq n+1}) = (x_m)_{m\geq n+2}.$$ 

We also define $\tilde{C}_{\varphi,\gamma}^{(q)}(\mathbb{D}_{\text{diff},n}^{(+)}(M)) := \lim_{n \to n, a^q} \tilde{C}^\bullet_{\varphi,\gamma}(\mathbb{D}_{\text{diff},n}^{(+)}(M))$ where the transition map $a^{q\bullet}$ is defined by

$$a^{q0}((x_m)_{m\geq n}) = (x_{m-1})_{m\geq n+1}, \quad a^{q1}((x_m)_{m\geq n}, (y_m)_{m\geq n+1}) = ((x_{m-1})_{m\geq n+1}, (y_{m-1})_{m\geq n+2}),$$

and

$$a^{q2}((x_m)_{m\geq n+1}) = (x_{m-1})_{m\geq n+2}.$$ 

Then, it is easy to see that quasi-isomorphism $C^\bullet_{\gamma}(\mathbb{D}_{\text{diff},n}^{(+)}(M)) \simeq \tilde{C}^\bullet_{\varphi,\gamma}(\mathbb{D}_{\text{diff},n}^{(+)}(M))$ defined in (8) is compatible with the transition maps $C^\bullet_{\gamma}(\mathbb{D}_{\text{diff},n}^{(+)}(M)) \leftrightarrow C^\bullet_{\gamma}(\mathbb{D}_{\text{diff},n+1}^{(+)}(M))$, $a^\bullet$ and $a^{q\bullet}$, hence induces quasi-isomorphisms

$$C^\bullet_{\gamma}(\mathbb{D}_{\text{diff}}^{(+)}(M)) \simeq \tilde{C}^\bullet_{\varphi,\gamma}(\mathbb{D}_{\text{diff}}^{(+)}(M)) \quad \text{and} \quad \tilde{C}^\bullet_{\varphi,\gamma}(\mathbb{D}_{\text{diff}}^{(+)}(M)) \simeq \tilde{C}^{(q)}_{\varphi,\gamma}(\mathbb{D}_{\text{diff}}^{(+)}(M)).$$

For $\tilde{C}^{(q)}_{\varphi,\gamma}(\mathbb{D}_{\text{diff}}^{(+)}(M))$, we also have a left inverse

$$\tilde{C}^{(q)}_{\varphi,\gamma}(\mathbb{D}_{\text{diff}}^{(+)}(M)) \to C^\bullet_{\gamma}(\mathbb{D}_{\text{diff}}^{(+)}(M))$$
of the above quasi-isomorphism \( C^\ast_n(D_{\text{diff}}^+(M)) \to \tilde{C}_{\varphi,\gamma}^\ast(D_{\text{diff}}^+(M)) \) which is obtained as the limit of the map

\[
\prod_{m \geq n} D_{\text{diff},m}^+(M) \rightarrow \prod_{m \geq n} D_{\text{diff},m}^+(M) \oplus \prod_{m \geq n+1} D_{\text{diff},m}^+(M) \rightarrow \prod_{m \geq n+1} D_{\text{diff},m}^+(M)
\]

Taking the limits of the map \( \tilde{C}_{\varphi,\gamma}^\ast(M^{(n)}) \to \tilde{C}_{\varphi,\gamma}^\ast(D_{\text{diff},m}^+(M)) : x \mapsto (t_m(x))_{m \geq n_0} \) (\( n_0 = n, n + 1 \)), we obtain the following maps

\[
C_{\varphi,\gamma}^\ast(M) \to \tilde{C}_{\varphi,\gamma}^\ast(D_{\text{diff}}^+(M)) \text{ and } C_{\varphi,\gamma}^\ast(M) \to \tilde{C}_{\varphi,\gamma}^\ast(D_{\text{diff}}^+(M)).
\]

Taking the limit of the exact sequence (17) with respect to the transition map induced by the canonical inclusion \( M^{(n)}_0 \to M^{(n+1)}_0 \) and \( a_\ast \), and taking the quasi-isomorphism \( C_{\gamma}^\ast(D_{\text{diff}}^+(M)) \to \tilde{C}_{\varphi,\gamma}^\ast(D_{\text{diff}}^+(M)) \) in (11), we obtain the following exact triangle which is the top horizontal line in the proposition

\[
C_{\varphi,\gamma}^\ast(M) \rightarrow C_{\varphi,\gamma}^\ast(M[1/t]) \oplus C_{\gamma}^\ast(D_{\text{diff}}^+(M)) \rightarrow C_{\gamma}^\ast(D_{\text{diff}}^+(M)) \rightarrow 1.
\]

On the other hands, because we have

\[
C_{\varphi,\gamma}^\ast(M^{(n)}[1/t]) = \text{Cone}(1 - \phi : C_{\gamma}^\ast(M^{(n)})[1/t]) \to C_{\gamma}^\ast(M^{(n+1)}[1/t])[1]
\]

for \( n \geq n(M) \) (where we define \( \text{Cone}(f : M^\ast \to N^\ast)[-1]^n = M^n \oplus N^{n-1} \) and \( d : M^n \oplus N^{n-1} \to M^{n+1} \oplus N^n : (x, y) \mapsto (d_M(x), -f(x) - d_N(y)) \)), taking the limit of the exact sequence (17) with respect to the transition map induced by \( \phi : M^{(n)}_0 \to M^{(n+1)}_0 \) and \( a_\ast \), and taking the left inverse \( \tilde{C}_{\varphi,\gamma}^\ast(D_{\text{diff}}^+(M)) \to C_{\gamma}^\ast(D_{\text{diff}}^+(M)) \) in (11), and identifying \( C_{\varphi,\gamma}^\ast(M) \to \tilde{C}_{\varphi,\gamma}^\ast(M) \) by Lemma 2.19 (2), we obtain the following exact triangle which is the bottom horizontal arrow in the proposition

\[
C_{\varphi,\gamma}^\ast(M) \rightarrow C_{\varphi,\gamma}^\ast(M[1/t]) \oplus C_{\gamma}^\ast(D_{\text{diff}}^+(M)) \rightarrow C_{\gamma}^\ast(D_{\text{diff}}^+(M)) \rightarrow 1.
\]

with \( d_3(x) = (f(x), g(x)) \) and \( d_4(x, y) = ((\text{can} - 1)(x), g(x) - y) \), which proves the proposition.

We next recall some notions concerning the \( p \)-adic Hodge theory for \((\varphi, \Gamma)\)-modules over the Robba ring. For a \((\varphi, \Gamma)\)-module \( M \) over \( \mathcal{R}_A(\pi_K) \), let denote

\[
D_{\text{diff}}^K(M) := H^0_0(D_{\text{diff}}^+(M)) \quad \text{and} \quad D_{\text{diff}}^K(M)^i := D_{\text{diff}}^K(M) \cap t^i D_{\text{diff}}^+(M)
\]

for \( i \in \mathbb{Z} \). Let denote

\[
D_{\text{cris}}^K(M) := H^0_0(M[1/t]).
\]

By Lemma 2.15 \( \varphi : C_{\gamma}^\ast(M[1/t]) \to C_{\gamma}^\ast(M[1/t]) \) induces a \( \varphi \)-semilinear automorphism

\[
\varphi : D_{\text{cris}}^K(M) \to D_{\text{cris}}^K(M).
\]
More precisely, by Lemma 2.17, we have $D_{\text{cris}}(M) = H^0_n(M^{(n)}[1/t])$ and $\varphi$ induces an automorphism $\varphi : H^0_n(M^{(n)}[1/t]) \xrightarrow{\varphi} H^0_n(M^{n+1}[1/t]) = H^0_n(M^{(n)}[1/t])$ for $n \geq n(M)$. Using these facts, we define an isomorphism

$$j_1 : D_{\text{cris}}^K(M) = H^0_n(M^{(n)}[1/t]) \xrightarrow{\varphi^n} H^0_n(M^{(n)}[1/t]) \xrightarrow{\sim} H_{\gamma}(\varphi,0)(M[1/t]),$$

which does not depend on the choice of $n$. Then, the map $\iota : C_{\gamma}^*(M[1/t]) \rightarrow C_{\gamma}^*(D_{\text{diff}}(M))$ induces a $F \otimes_{\mathbb{Q}_p} A$-linear injection

$$\iota : D_{\text{cris}}^K(M) \xrightarrow{j_1} H_{\gamma}(\varphi,0)(M[1/t]) \xrightarrow{\iota} D_{\text{diff}}^K(M).$$

We define another isomorphism

$$j_2 : D_{\text{cris}}^K(M) \xrightarrow{j_1} H_{\gamma}(\varphi,0)(M[1/t]) \xrightarrow{\text{can}} H_{\gamma}(\varphi,0)(M[1/t]),$$

where $H_{\gamma}(\varphi,0)(M[1/t]) \xrightarrow{\text{can}} H_{\gamma}(\varphi,0)(M[1/t])$ is the map induced by $\text{can} : C_{\gamma}^*(M[1/t]) \rightarrow C_{\gamma}^*(M[1/t])$, which is isomorphism by Lemma 2.19. Then, we obtain the following commutative diagram

$$\begin{array}{ccc}
D_{\text{cris}}^K(M) & \xrightarrow{1-\varphi} & D_{\text{cris}}^K(M) \\
\downarrow{j_1} & & \downarrow{j_2} \\
H_{\gamma}(\varphi,0)(M[1/t]) & \xrightarrow{\text{can-id}} & H_{\gamma}(\varphi,0)(M[1/t]).
\end{array}$$

Let denote

$$\exp_M : D_{\text{diff}}^K(M) \rightarrow H^1_{\varphi,\gamma}(M), \quad \exp_f, M : D_{\text{cris}}^K(M) \xrightarrow{j_2} H_{\gamma}(\varphi,0)(M[1/t]) \rightarrow H^1_{\varphi,\gamma}(M)$$

for the boundary maps obtained by taking the cohomology of the exact triangles in Proposition 2.20. We define

$$H^1_{\varphi,\gamma}(M) = \text{Im}(D_{\text{diff}}^K(M) \xrightarrow{\exp_M} H^1_{\varphi,\gamma}(M))$$

and

$$H^1_{\varphi,\gamma}(M) = \text{Im}(D_{\text{cris}}^K(M) \oplus D_{\text{diff}}^K(M) \xrightarrow{\exp_f, M \oplus \exp_M} H^1_{\varphi,\gamma}(M)).$$

We call the latter group the finite cohomology. Put $t_M(K) := D_{\text{diff}}^K(M) / D_{\text{diff}}^K(M)^0$. By Proposition 2.20, we obtain the following diagram of exact sequences (13)

$$\begin{array}{ccc}
0 & \rightarrow & H^0_{\varphi,\gamma}(M) \xrightarrow{x \mapsto x} D_{\text{cris}}^K(M)^{\varphi=1} \xrightarrow{x \mapsto x(x)} t_M(K) \xrightarrow{\exp_M} H^1_{\varphi,\gamma}(M) \rightarrow 0 \\
\text{id} & \downarrow & x \mapsto x \downarrow & \xrightarrow{\varphi=0(x)} & \xrightarrow{x \mapsto x} \\
0 & \rightarrow & H^0_{\varphi,\gamma}(M) \xrightarrow{x \mapsto x} D_{\text{cris}}^K(M) \xrightarrow{d_5} D_{\text{cris}}^K(M) \oplus t_M(K) \xrightarrow{d_6} H^1_{\varphi,\gamma}(M) \rightarrow 0
\end{array}$$

with

$$d_5(x, y) = ((1 - \varphi)x, \iota(x)) \quad \text{and} \quad d_6 = \exp_f, M \oplus \exp_M,$$
where we also denote \( \exp_M : t_M(K) \to H^1_{\varphi,\gamma}(M) \) which is naturally induced by \( \exp_M : D^+_{\text{diff}}(M) \to H^1_{\varphi,\gamma}(M) \).

By the proof of Proposition 2.20 we obtain the following explicit formulae of \( \exp_M \) and \( \exp_{f,M} \), which are very useful in applications.

**Proposition 2.22.** We have the following formulae.

1. For \( x \in D^+_\text{diff}(M) \), take \( \tilde{x} \in M^{(n)}[1/t] \Delta (n \geq n(M)) \) such that
   \[ t_m(\tilde{x}) - x \in D^+_{\text{diff},m}(M) \]
   for any \( m \geq n \) (such \( \tilde{x} \) exists by the exact sequence (5) in the proof of Proposition 2.20). Then we have
   \[ \exp_M(x) = [(\gamma - 1)\tilde{x}, (\varphi - 1)\tilde{x}] \in H^1_{\varphi,\gamma}(M). \]
2. For \( x \in D^+_\text{cria}(M) \), take \( \tilde{x} \in M^{(n)}[1/t] \Delta (n \geq n(M)) \) such that
   \[ t_n(\tilde{x}) \in D^+_{\text{diff},n}(M) \]
   and
   \[ t_{n+k}(\tilde{x}) - \sum_{i=1}^{k} t_{n+i}(\varphi^n(x)) \in D^+_{\text{diff},n+k}(M) \]
   for any \( k \geq 1 \) (we remark that we have \( \varphi^n(x) \in M^{(n)}[1/t] \) by Lemma 2.17 and that such \( \tilde{x} \) exists by the exact sequence (5)). Then we have
   \[ \exp_{f,M}(x) = [(\gamma - 1)\tilde{x}, (\varphi - 1)\tilde{x} + \varphi^n(x)] \in H^1_{\varphi,\gamma}(M). \]

**Proof.** These formulae directly follow from simple but a little bit long diagram chases in the proof of Proposition 2.20. For the convenience of the reader, we give a proof of these formulae.

We first prove the formula (1). By the proof of Proposition 2.20 the above exact triangle in this proposition is obtained by taking the limit of the composition of the quasi-isomorphism
\[
\tilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}) \Rightarrow \text{Cone}(\tilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus \tilde{C}^\bullet_{\varphi,\gamma}(D^+_{\text{diff},n}(M)) \to \tilde{C}^\bullet_{\varphi,\gamma}(D^+_{\text{diff},n}(M)))[-1] := C^\bullet_1
\]
(which is obtained by the exact sequence (7)) with the inverse of the quasi-isomorphism
\[
C^\bullet_2 := \text{Cone}(\tilde{C}^\bullet_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus C^\bullet_1(D^+_{\text{diff},n}(M)) \to C^\bullet_1(D^+_{\text{diff},n}(M)))[−1] \Rightarrow C^\bullet_1
\]
induced by the quasi-isomorphism \( C^\bullet_1(D^+_{\text{diff},n}(M)) \to \tilde{C}^\bullet_{\varphi,\gamma}(D^+_{\text{diff},n}(M)) : x \mapsto (x)_{m \geq n} \) of (10).

By definition of \( \exp_M(−) \), for \( x \in H^0_{\gamma}(D^+_{\text{diff},n}(M)) \), these quasi-isomorphisms send \( \exp_M(x) \) (which we see as an element of \( H^1(\tilde{\tilde{C}}^\bullet_{\varphi,\gamma}(M^{(n)})) \)) to the element \( [0, 0, x] \in H^1(C^\bullet_2) \) represented by \( (0, 0, x) \in \tilde{C}^1_{\varphi,\gamma}(M^{(n)}[1/t]) \oplus C^1_1(D^+_{\text{diff},n}(M)) \oplus C^1_1(D^+_{\text{diff},n}(M)) \).

Take \( \tilde{x} \in M^{(n)}[1/t] \Delta \) satisfying the condition in (1), then it suffices to show that
\[(\gamma - 1)\tilde{x}, (\varphi - 1)\tilde{x}\] in \(H^1(\tilde{C}_{\varphi, \gamma}(M^{(n)}))\) and \([0, 0, x] \in H^1(C^*_1)\) are the same element in \(H^1(C^*_1)\). By definition, \([(\gamma - 1)\tilde{x}, (\varphi - 1)\tilde{x}]\) is sent to

\[([(\gamma - 1)\tilde{x}, (\varphi - 1)\tilde{x}], (t_m((\gamma - 1)\tilde{x})), (t_m((\varphi - 1)\tilde{x})), m_{\geq n}, 0, 0, x] \in H^1(C^*_1)\]

and \([0, 0, x] \in H^1(C^*_1)\) is sent to

\[[0, 0, (-x)]_{m \geq n} \in H^1(C^*_1)\]

in \(H^1(C^*_1)\), both are represented by elements of \(\tilde{C}_{\varphi, \gamma}^1(M^{(n)}[1/t]) \oplus \tilde{C}_{\varphi, \gamma}^1(D_{\text{diff}, n}(M)) \oplus \tilde{C}_{\varphi, \gamma}^0(D_{\text{diff}, n}(M))\) (we remark the sign; for \(f : C^* \to D^*\), we define \(D^{*-1} \to \text{Cone}(C^* \to D^*)[-1]\) by \(x \mapsto (-x, 0)\) and \(\text{Cone}(C^* \to D^*)[-1] \to C^*\) is defined by \((x, y) \mapsto y\). Then, it is easy to check that the difference of these two elements is the coboundary of the element

\[(\tilde{x}, (t_m(\tilde{x}) - x)_{m \geq n}) \in C^0_1 = M^{(n)}[1/t] \oplus \prod_{m \geq n} D_{\text{diff}, m}(M),\]

which proves (1).

We next prove (2). The below exact triangle in Proposition \([22,20]\) is obtained by taking the limit of the composition of the quasi-isomorphism \(C_{\varphi, \gamma}^*(M^{(n)}) \isomorph C^*_1\) defined above with the quasi-isomorphism

\[C^*_1 \isomorph \text{Cone}(\tilde{C}_{\varphi, \gamma}^*(M^{(n)}[1/t]) \oplus C^*_1(D_{\text{diff}, n}(M)) \to C^*_1(D_{\text{diff}, n}(M)))[-1] := C^*_3\]

induced by the map \(\prod_{m \geq n} D_{\text{diff}, m}(M) \to D_{\text{diff}, n}(M) : (x_m)_{m \geq n} \mapsto x_n\), with the inverse of the quasi-isomorphism

\[C^*_3 \isomorph \text{Cone}(C^*_1(M^{(n)}[1/t]) \oplus C^*_1(D_{\text{diff}, n}(M)) \to C^*_1(M^{(n+1)}[1/t]) \oplus C^*_1(D_{\text{diff}, n}(M)))[-1] := C^*_4\]

which is naturally obtained by the identity

\[\tilde{C}_{\varphi, \gamma}^*(M^{(n)}[1/t])) = \text{Cone}(C^*_1(M^{(n)}[1/t])) \xrightarrow{1 - \tilde{\tau}_n} C^*_1(M^{(n+1)}[1/t]))[-1].\]

For \(x' \in H^0_\gamma(M^{(n+1)}[1/t])\), the image of \(x'\) by the first boundary map of the cone \(C^*_1\) is equal to \([0, 0, x', 0] \in H^1_\gamma(C^*_1)\) which is represented by the element \((0, 0, x', 0) \in C^1_\gamma(M^{(n)}[1/t]) \oplus C^1_\gamma(D_{\text{diff}, n}(M)) \oplus C^0_\gamma(M^{(n+1)}[1/t]) \oplus C^0_\gamma(D_{\text{diff}, n}(M))\). Take \(x' \in M^{(n)}[1/t] \Delta\) such that \(\tau_n(x') \in D_{\text{diff}, n}(M)\) and that \(\tau_{n+k}(x') - \sum_{l=1}^k \tau_{n+l}(x') \in D_{\text{diff}, n+k}(M)\) for any \(k \geq 1\), then, by definition of the map \(j_2 : D_{\text{cris}, n}(M) \isomorph H^1_\gamma(M^{(n)}[1/t]) \xrightarrow{\exp_{f,M}}\), it suffices to show that the element \([(\gamma - 1)x', (\varphi - 1)x', x')] \in H^1_\gamma(\tilde{C}_{\varphi, \gamma}(M^{(n)}))\) is sent to \([0, 0, x', 0] \in H^1_\gamma(C^*_1)\) by the above quasi-isomorphisms. By definition, the element \([(\gamma - 1)x', (\varphi - 1)x', x'] \in H^1_\gamma(C^*_1)\)
by the above quasi-isomorphism. Then, it is easy to check that the difference of this element with \([0, 0, x', 0]\) is the coboundary of the element

\[(x', t_n(x')) \in C_4^0 = M^{(n)}[1/t]^{\Delta} \oplus D_{\text{diff}, n}(M)^{\Delta},\]

which proves the formula (2).

We next generalize Bloch-Kato duality concerning the finite cohomology for \((\varphi, \Gamma)\)-modules. Let \(L = A\) be a finite extension of \(\mathbb{Q}_p\), and let \(M\) be a \((\varphi, \Gamma)\)-module over \(\mathcal{R}_L(\pi_K)\). We say that \(M\) is de Rham if the equality \(\dim_K D_{\text{dr}}^K(M) = [L : \mathbb{Q}_p] \cdot r_M\) holds. When \(M\) is de Rham, we have a natural \(L\)-bilinear perfect pairing

\[(M^*)_f \times D_{\text{dr}}^K(M) \xrightarrow{(f_x) \mapsto f(x)} D_{\text{dr}}^K(\mathcal{R}_L(1)) = L \otimes_{\mathbb{Q}_p} K \frac{1}{e_1} \xrightarrow{\varphi, \gamma} L,\]

which induces a natural isomorphisms

\[D_{\text{dr}}^K(M) \xrightarrow{\sim} D_{\text{dr}}^K(M^*)^\vee \quad \text{and} \quad D_{\text{dr}}^K(M)^0 \xrightarrow{\sim} t_{M^*}(K)^\vee.\]

**Proposition 2.23.** Let \(L = A\) be a finite extension of \(\mathbb{Q}_p\), and let \(M\) be a de Rham \((\varphi, \Gamma)\)-module over \(\mathcal{R}_L(\pi_K)\). Then, \(H_{\varphi, \Gamma}^1(M^*_f)\) is the orthogonal complement of \(H_{\varphi, \Gamma}^1(M_f)\) with respect to Tate duality pairing \(<, >:\ H_{\varphi, \Gamma}^1(M^*) \times H_{\varphi, \Gamma}^1(M) \to L.\)

**Proof.** We remark that we have \(\dim_L H_{\varphi, \Gamma}^1(M_f) = \dim_L (t_M(K)) + \dim_L H_{\varphi, \Gamma}^0(M)\) by the below exact sequence of \([13]\). Using this formula for \(M, M^*\), it is easy to check that we have \(\dim_L H_{\varphi, \Gamma}^1(M_f) + \dim_L H_{\varphi, \Gamma}^1(M^*_f) = \dim_L H_{\varphi, \Gamma}^1(M)\) under the assumption that \(M\) is de Rham. Hence, it suffices to show that we have \(<x, y> = 0\) for any \(x \in H_{\varphi, \Gamma}^1(M^*_f)\) and \(y \in H_{\varphi, \Gamma}^1(M_f)\) by comparing the dimensions. By definition of \(H_{\varphi, \Gamma}^1(M^*)_f\), this claim follows from the following lemma \([2.24]\).

\[\square\]

Let \(M\) be a \((\varphi, \Gamma)\)-module over \(\mathcal{R}_A(\pi_K)\) (we don’t need to assume that \(M\) is de Rham). Using the isomorphism \(j_2 : D_{\text{cris}}^K(M^*) \xrightarrow{\sim} H_{\varphi}^1(M^*[1/l]),\) define an \(A\)-bilinear pairing

\[h(-, -) : (D_{\text{cris}}^K(M^*) \oplus D_{\text{dr}}^K(M^*)) \times (H_{\varphi}^1(M[1/l]) \oplus H_{\gamma}^1(D_{\text{diff}}^K(M))) \to H_{\varphi}^1(M^* \otimes M[1/l]) \oplus H_{\gamma}^1(D_{\text{diff}}^K(M^* \otimes M))\]

by

\[h((x, y), ([z], [w])) := ([j_2(x) \otimes z], [y \otimes w]).\]

**Lemma 2.24.** For \((x, y) \in D_{\text{cris}}^K(M^*) \oplus D_{\text{dr}}^K(M^*)\) and \(z \in H_{\varphi, \Gamma}^1(M)\), we have

\[f_2(h((x, y), g(z))) = (\exp_{f, M^*}(x) + \exp_{M^*}(y)) \cup z \in H_{\varphi, \Gamma}^2(M^* \otimes M),\]
where
\[ g : H^1_{\varphi,\gamma}(M) \to H^{(\varphi),1}_\gamma(M) \oplus H^1_{\text{dR}}(M) \]
is induced by \( d_3 \) and
\[ f_2 : H^{(\varphi),1}_\gamma(M^* \otimes M[1/\ell]) \oplus H^1_{\text{dR}}(M^* \otimes M) \to H^{\varphi,\gamma}_2(M^* \otimes M) \]
is the second boundary map of the below exact triangle of Proposition \[2.29\].

**Proof.** The equality \( \exp_M(y) \cup z = f_2(h((0,y), g(z))) \) for \( y \in D^K_{\text{dR}}(M^*) \), \( z \in H^1_{\varphi,\gamma}(M) \) is proven in Lemma 2.13 of \[Na13\]. Hence, it suffices to show the equality
\[ \exp_{f,M}(x) \cup z = f_2(h((x,0), g(z))) \]
for \( x \in D^K_{\text{cris}}(M^*) \), whose proof is also just a diagram chase similar to the proof of Proposition \[2.22\] hence we omit the proof.

Finally, we compare our exponential map with Bloch-Kato exponential map for \( p \)-adic representations. Here, we assume that \( A = \mathbb{Q}_p \) for simplicity, we can do the same things for any \( L = A \) a finite \( \mathbb{Q}_p \)-algebra.

For an \( \mathbb{Q}_p \)-representation \( V \) of \( G_K \), by Bloch-Kato \[BK90\], we have the following diagram of exact sequences
\[ (15) \]
\[ 0 \to H^0(K,V) \xrightarrow{x \to x} D^K_{\text{cris}}(V) \xrightarrow{x \to x} \gamma \xrightarrow{t_V(K)} H^1_e(K,V) \to 0 \]
\[ 0 \to H^0(K,V) \xrightarrow{x \to x} D^K_{\text{cris}}(V) \xrightarrow{d_5} D^K_{\text{cris}}(V) \oplus t_V(K) \xrightarrow{d_6} H^1_f(K,V) \to 0 \]
with
\[ d_5(x,y) = ((1 - \varphi)x, \gamma) \text{ and } d_6 = \exp_{f,V} \oplus \exp_V, \]
which is associated to the exact sequences which is the tensor product of \( V \) ( over \( \mathbb{Q}_p \)) with the Bloch-Kato fundamental exact sequences
\[ 0 \to \mathbb{Q}_p \xrightarrow{x \to x} B^{x=1}_{\text{cris}} \oplus B^+_{\text{dR}} \xrightarrow{(x,y) \to x-y} B_{\text{dR}} \to 0 \]
\[ 0 \to \mathbb{Q}_p \xrightarrow{x \to x} B^{x=1}_{\text{cris}} \oplus B^+_{\text{dR}} \xrightarrow{(x,y) \to (x,1-x-y)} B_{\text{cris}} \oplus B_{\text{dR}} \to 0. \]
We want to compare the diagram \((15)\) with the diagram \((13)\) for \( M = D_{\text{rig}}(V) \). More generally, as in §2.4 of \[Na13\], we compare the similar diagram defined below for a \( B \)-pair \( W = (W,e,W^+_{\text{dR}}) \) with the diagram \((13)\) for the associated \((\varphi,\Gamma)\)-module \( D_{\text{rig}}(W) \). For the definitions of \( B \)-pairs and the definition of the functor \( W \mapsto D_{\text{rig}}(W) \) which gives an equivalence between the category of \( B \)-pairs and that of \((\varphi,\Gamma)\)-modules over \( \mathcal{R}(\pi_K) \) which we don’t recall here, see §2.5 \[Na13\] and \[Ber08a\].
Let $W = (W_e, W_{\text{dR}}^+)$ be a $B$-pair for $K$. Put $W_{\text{cris}} := B_{\text{cris}} \otimes_{B_e} W_e$, which is naturally equipped with an action of $\varphi$. Because we have an exact sequence $0 \to B_{\text{cris}}^{\varphi=1} \to B_{\text{cris}} \xrightarrow{1-\varphi} B_{\text{cris}} \to 0$, we have a natural quasi-isomorphism (the vertical arrows) between the following two complexes of $G_K$-modules concentrated in degree $[0, 1]$

$$\begin{align*}
W_e \oplus W_{\text{dR}}^+ & \xrightarrow{(x,y)\mapsto x-y} W_{\text{dR}} \\
\downarrow (x,y)\mapsto(x,y) & \quad \downarrow x\mapsto(0,x) \\
W_{\text{cris}} \oplus W_{\text{dR}}^+ & \xrightarrow{(x,y)+(1-\varphi)x,x-y} W_{\text{cris}} \oplus W_{\text{dR}}^+.
\end{align*}$$

Put

$$C_{\text{cont}}^\bullet(G_K, W) := \text{Cone}(C_{\text{cont}}^\bullet(G_K, W_e) \oplus C_{\text{cont}}^\bullet(G_K, W_{\text{dR}}^+) \to C_{\text{cont}}^\bullet(G_K, W_{\text{dR}}))[−1]$$

and

$$C_{\text{cont}}^\bullet(G_K, W) := C_{\text{cont}}^\bullet(G_K, W_{\text{cris}}) \oplus C_{\text{cont}}^\bullet(G_K, W_{\text{dR}}^+) \to C_{\text{cont}}^\bullet(G_K, W_{\text{dR}}^+)[−1].$$

We identify

$$H^i(K, W) := H^i(C_{\text{cont}}^\bullet(G_K, W)) = H^i(C_{\text{cont}}^\bullet(G_K, W))$$

by the above quasi-isomorphism. Put $D_{\text{cris}}^K(W) := H^0(K, W_{\text{cris}})$, $D_{\text{dR}}^K(W) := H^0(K, W_{\text{dR}})$, and $D_{\text{dR}}^K(W)^j := D_{\text{dR}}^K(W) \cap t^j W_{\text{dR}}^+$ for $i \in \mathbb{Z}$. Taking the cohomology of the mapping cones, then we obtain the similar diagram of exact sequences as in (15) for $W$. By definition, it is clear that the diagram (15) for the associated $B$-pair $W(V) := (B_e \otimes_{Q_p} V, B_{\text{dR}}^+ \otimes_{Q_p} V)$ is canonically isomorphic to the diagram (15) for $V$ defined by Bloch-Kato.

Our comparison result is the following.

**Proposition 2.25.**

1. We have the following functorial isomorphisms

   (i) $H^i(K, W) \xrightarrow{\sim} H^i_{\varphi=1}(D_{\text{rig}}(W))$,
   (ii) $D_{\text{dR}}^K(W)^j \xrightarrow{\sim} D_{\text{dR}}^K(D_{\text{rig}}(W))^j$ for $j \in \mathbb{Z}$,
   (iii) $D_{\text{cris}}^K(W) \xrightarrow{\sim} D_{\text{cris}}^K(D_{\text{rig}}(W))$.

2. The isomorphisms in (1) induces an isomorphism from the diagram (15) for $W$ to the diagram (13) for $D_{\text{rig}}(W)$.

**Proof.** We already proved (i), (ii) of (1) and the comparison of the above exact sequence in (15) for $W$ with that in (13) for $D_{\text{rig}}(W)$, see Theorem 2.21 of [Na13] or the references in the proof of this theorem.

Moreover, the isomorphism (iii) may be well known to the experts, but we give a proof of this because we couldn’t find suitable references. In this proof, we freely use the notations in §2.5 [Na13] or in [Ber08a]; please see these references. We first note that the inclusion $(\text{B}_{\text{rig}}[1/t] \otimes_{B_e} W_e)_{G_K}^+ \hookrightarrow D_{\text{cris}}^K(W)$ induced by the natural
such that to exp\(D_{\text{rig}}(W)\) is a finite dimensional \(\mathbb{Q}_p\)-vector space on which \(\varphi\) acts as an automorphism. Moreover, in the same way as the proof of Proposition 3.4 of [Ber02], we can show that the natural inclusion
\[(\widetilde{B}_{\text{rig}}^+[1/t] \otimes_{B_{\text{rig}}} W)_{K^\text{cris}} \hookrightarrow (\widetilde{B}_{\text{rig}}^+[1/t] \otimes_{B_{\text{rig}}} W)_{G_K}\]
is also isomorphism. Because we have
\[
\widetilde{B}_{\text{rig}}^+[1/t] \otimes_{B_{\text{rig}}} W = \widetilde{B}_{\text{rig}}^+[1/t] \otimes_{\mathcal{R}((\pi_K)[1/t])} D_{\text{rig}}(W)[1/t]
\]
by definition of \(D_{\text{rig}}(W)\), it suffices to show that the natural inclusion
\[
D_{\text{cris}}^K(D_{\text{rig}}(W)) \hookrightarrow (\widetilde{B}_{\text{rig}}^+[1/t] \otimes_{\mathcal{R}((\pi_K)[1/t])} D_{\text{rig}}(W)[1/t])_{G_K} := D_0
\]
is isomorphism. Moreover, it suffices to show that \(D_0\) is contained in \(D_{\text{rig}}(W)[1/t]\). This claim is proven as follows. Define \(\mathcal{R}((\pi_K) \otimes_F D_0 \subseteq \widetilde{B}_{\text{rig}}^+[1/t] \otimes_{\mathcal{R}((\pi_K)[1/t])} D_{\text{rig}}(W)[1/t]\), which are \((\varphi, \Gamma)\)-module over \(\mathcal{R}((\pi_K)\) (respectively \((\varphi, G_K)\)-module over \(\widetilde{B}_{\text{rig}}^+[1/t]\)). Then, by Théorème 1.2 of [Ber09], the natural map
\[
\mathcal{R}((\pi_K) \otimes_F D_0 \to \widetilde{B}_{\text{rig}}^+[1/t] \otimes_{\mathcal{R}((\pi_K)[1/t])} D_{\text{rig}}(W)[1/t] : a \otimes x \mapsto a \cdot x
\]
(which is actually an inclusion) of \((\varphi, G_K)\)-modules factors through \(\mathcal{R}((\pi_K) \otimes_F D_0 \to D_{\text{rig}}(W)[1/t]\), in particular we have \(D_0 \subseteq D_{\text{rig}}(W)[1/t]\), which proves the claim.

We next prove that the below exact sequences in (15) for \(W\) is isomorphic to that in (13) for \(D_{\text{rig}}(W)\) by the isomorphisms in (1) of this proposition. Because the other commutativities are clear, or were already proven in Theorem 2.21 [Na13], it suffices to show that the following diagram commutes

\[
\begin{array}{ccc}
D_{\text{cris}}^K(D_{\text{rig}}(W)) & \xrightarrow{\exp_{f,D_{\text{rig}}(W)}} & H^1_{\varphi,\gamma}(D_{\text{rig}}(W)) \\
\downarrow \cong & & \downarrow \cong \\
D_{\text{cris}}^K(W) & \xrightarrow{\exp_{f,W}} & H^1(K,W).
\end{array}
\]

(16)

In the same way as the proof of Theorem 2.21 [Na13], we assume that \(\Delta = \{1\}\), and using the canonical identifications
\[
H^1(K,W) \cong \text{Ext}^1(B,W), \quad H^1_{\varphi,\gamma}(D_{\text{rig}}(W)) \cong \text{Ext}^1(\mathcal{R}((\pi_K), D_{\text{rig}}(W))
\]
(where we denote by \(B = (B_{\text{cris}}, B_{\text{cris}}^+)\) for the trivial \(B\)-pair), it suffices to show that, for \(a \in D_{\text{cris}}^K(D_{\text{rig}}(W))\), the extension corresponding to \(\exp_{f,D_{\text{rig}}(W)}(a)\) is sent to the extension corresponding to \(\exp_{f,W}(a)\) by the inverse functor \(W(-)\) of \(D_{\text{rig}}(-)\). We prove this claim as follows. Take \(n \geq 1\) sufficiently large such that \(a \in (D_{\text{rig}}(W)[1/t])_{\Gamma^\text{cris}}\). Take \(\tilde{a} \in D_{\text{rig}}(W)[1/t]\) satisfying the condition in (2) of Proposition 2.22. By (2) of this proposition, then the extension \(D_a\) corresponding to \(\exp_{f,D_{\text{rig}}(W)}(a)\) is written by

\[
[0 \to D_{\text{rig}}(W) \xrightarrow{\varphi((x,y) \mapsto 0)} D_{\text{rig}}(W) \oplus \mathcal{R}(\pi_K)e \xrightarrow{(x,y) \mapsto y} \mathcal{R}(\pi_K) \to 0]
\]
such that
\[
\varphi((x,y)) = (\varphi(x) + \varphi(y)((\varphi - 1)\tilde{a} + \varphi^n(a)), \varphi(y)e)
\]
and
\[
\gamma((x,y)) = (\gamma(x) + \gamma(y)(\gamma - 1)\tilde{a}, \gamma(y)e)
\]
and we define $W_{2.21}$ of $\text{Na13}$;
and
$G_W$ surjection $\text{D}$

Then, by definition of the functor $f$, $W$

On the other hand, by definition of $\text{exp}_f$, the extension $W_{\alpha} := (W_{e,\alpha}, W_{\text{dr},\alpha} := W_{\text{dr}} \oplus B_{\text{dr}}^+ e_{\text{dr}})$
corresponding to $\text{exp}_f(a)$ is defined by

for $x \in W_{\text{dr}}^+$, $y \in B_{\text{dr}}^+$, $g \in G_K$, and $W_{e,\alpha}$ is defined as the kernel of the following surjection

$W_{\text{cris},\alpha} := W_{\text{cris}} \oplus B_{\text{cris}} e_{\text{cris}} \to W_{\text{cris},\alpha} : (x, ye_{\text{cris}}) \mapsto ((\varphi - 1)x + \varphi(y)a, (\varphi - 1)ye_{\text{cris}})$
on which $G_K$ acts by $g(e_{\text{cris}}) = e_{\text{cris}}$ (actually, this is equal to the kernel of the surjection

$W_{\text{rig},\alpha} := W_{\text{rig}} \oplus \widetilde{B}_{\text{rig}}^+[1/t] e_{\text{cris}} \to W_{\text{rig},\alpha} : (x, ye_{\text{cris}}) \mapsto ((\varphi - 1)x + \varphi(y)a, (\varphi - 1)ye_{\text{cris}})$

where we define $W_{\text{rig}} := \widetilde{B}_{\text{rig}}^+[1/t] \otimes B_e W_e$ and the isomorphism $B_{\text{dr}} \otimes B_e W_{e,\alpha} \sim B_{\text{dr}} \otimes B_{e,\alpha} W_{e,\alpha}^+$ is defined by

Then, by definition of the functor $D_{\text{rig}}(-)$ in §2.2 $\text{Ber08a}$ (where the notation $D(-)$ is used), $\widetilde{B}_{\text{rig}}^{[1/r]} \otimes_R (\pi_K) D_{\text{rig}}^{[n]}(W_a)$ is equal to

Then, by definition of the functor $D_{\text{rig}}(-)$ in §2.2 $\text{Ber08a}$ (where the notation $D(-)$ is used), $\widetilde{B}_{\text{rig}}^{[1/r]} \otimes R(\pi_K) D_{\text{rig}}^{[n]}(W_a)$ is equal to

Because we have $\widetilde{B}_{\text{rig}}^{[1/r]} \otimes B_e W_{e,\alpha} = \widetilde{B}_{\text{rig}}^{[1/r]} \otimes B_{\text{rig}}^{[1/t]} W_{\text{rig},\alpha}$, $\varphi^{-m}(e_{\text{cris}}) = e_{\text{cris}} - \sum_{k=1}^{m-1} \varphi^{-k}(\alpha)$ for $m \geq 1$ and we have $\iota_{n+k} \circ \varphi^n = \varphi^{-k}$, it is easy to see that the group (17) is equal to

which is easily seen to be isomorphic to $\widetilde{B}_{\text{rig}}^{[1/r]} \otimes R(\pi_K) D_{\text{rig}}^{(n)}(\varphi, G_K)$-module.
Therefore, we obtain the isomorphism $D_{\text{rig}}(W_a) \sim D_a$
as an extension by Théorème 1.2 of $\text{Ber09}$, which proves the proposition.$\square$
3. Local $\varepsilon$-conjecture for $(\varphi, \Gamma)$-modules over the Robba ring

From now on, we assume that $K = \mathbb{Q}_p$, and we freely omit the notation $\mathbb{Q}_p$, i.e.,
we use the notation $\Gamma$, $\mathcal{R}_A$, $\mathcal{D}_{\text{dR}}(M)$, $\mathcal{D}_{\text{cris}}(M)$, $t_M$ etc instead of
$\mathcal{R}_{\mathbb{Q}_p}$, $\mathcal{R}_A(\pi_{\mathbb{Q}_p})$, $\mathcal{D}^\mathbb{Q}_p(M)$, $\mathcal{D}^{\mathbb{Q}_p}_{\text{cris}}(M)$, $t_M(\mathbb{Q}_p) \cdots$.
Moreover, because Kato’s and our conjectures are formulated after fixing a $\mathbb{Z}_p$-basis $\zeta = \{\zeta_{lp}\}_{n \geq 0}$ of $\mathbb{Z}_p(1)$, we also fix a parameter $\pi := \pi_{\zeta}$ of $\mathcal{R}_A$ and denote $t = \log(1 + \pi)$ as in Notation 2.2.

In this section, we formulate a conjecture which is a natural generalization of
Kato’s $(p$-adic $)$ local $\varepsilon$-conjecture, where the main objects were $p$-adic or torsion representations of $G_{\mathbb{Q}_p}$, for $(\varphi, \Gamma)$-modules over the relative Robba ring $\mathcal{R}_A$. Because the article [Ka93b] in which the conjecture was stated has been unpublished until now, and because the compatibility of our conjecture with his conjecture is an important part of our conjecture, here we also recall his original conjecture.

3.1. Determinant functor. Kato’s and our conjectures are formulated using the
theory of the determinant functor ([KM76]). In this subsection, we briefly recall
this theory following [KM76], §2.1 of [Ka93a].

Let $R$ be a commutative ring. We define a category $\mathcal{P}_R$ such that whose objects
are pairs $(L, r)$ where $L$ is an invertible $R$-module and $r : \text{Spec}(R) \to \mathbb{Z}$ is a
locally constant function, whose morphisms are defined by $\text{Mor}_{\mathcal{P}_R}((L, r), (M, s)) :=
\text{Isom}_R(L, M)$ if $r = s$, or empty otherwise. We call the objects of this category
graded invertible $R$-modules. The category $\mathcal{P}_R$ is equipped with the structure of a
(tensor) product defined by $(L, r) \cdot (M, s) := (L \otimes_R M, r + s)$ with commutativity
constraint $(L, r) \cdot (M, s) \sim (M, s) \cdot (L, r) : l \otimes m \mapsto (-1)^{rm} m \otimes l$ (locally). The unit
object for the product is $1 := (R, 0)$. For each $(L, r)$, let $L^{-1} := \text{Hom}_R(L, R)$,
then $(L, r)^{-1} := (L^{-1}, -r)$ is an inverse of $(L, r)$ and one has an isomorphism $e : $(L, r) \cdot (L^{-1}, -r) \sim 1$ induced by the usual evaluation map $L \otimes_R \text{Hom}_R(L, R) \sim R : x \otimes f \mapsto f(x)$. For a ring homomorphism $f : R \to R'$, one has a base change functor $(-) \otimes_R R' : \mathcal{P}_R \to \mathcal{P}_{R'}$ defined by $(L, r) \mapsto (L, r) \otimes_R R' := (L \otimes_R R', r \circ f^\ast)$ where $f^\ast : \text{Spec}(R') \to \text{Spec}(R)$.

For a category $\mathcal{C}$, denote by $(\mathcal{C}, \text{is})$ for the category such that whose objects are
the same as $\mathcal{C}$ and the morphisms are all isomorphisms in $\mathcal{C}$. Define a functor

$$\text{Det}_R : (\mathcal{P}_{fg}(R), \text{is}) \to \mathcal{P}_R : P \mapsto (\text{det}_R P, \text{rk}_R P)$$

where $\text{rk}_R : \mathcal{P}_{fg}(R) \to \mathbb{Z}_{\geq 0}$ is the $R$-rank of $P$ and $\text{det}_R P := \wedge^{\text{rk}_R P} P$. Note that $\text{Det}_R(0) = 1$ is the unit object and one has for a short exact sequence
$0 \to P_1 \to P_2 \to P_3 \to 0$ of objects in $\mathcal{P}_{fg}(R)$ a functorial isomorphism (put
$r_i := \text{rk}_R P_i$)

$$\text{Det}_R P_2 \sim \text{Det}_R P_1 \cdot \text{Det}_R P_3$$

induced by

$$x_1 \wedge \cdots \wedge x_{r_1} \wedge x_{r_1+1} \wedge \cdots \wedge x_{r_2} \mapsto (x_1 \wedge \cdots \wedge x_{r_1}) \otimes (x_{r_1+1} \wedge \cdots \wedge x_{r_2})$$

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where \(x_1, \ldots, x_{r_1}\) are local sections of \(P_1\) and \(\overline{f}_i \in P_3\) is the image of \(x_i \in P_2\).

For a bounded complex \(P^\bullet\) in \(\mathbf{P}_{\mathbb{Q}}(R)\), define \(\text{Det}_R(P^\bullet) \in \mathcal{P}_R\) by

\[
\text{Det}_R(P^\bullet) := \prod_{i \in \mathbb{Z}} \text{Det}_R(P_i)^{(-1)^i}.
\]

More generally, the theory of determinants of [KM76] enables us to uniquely (up to canonical isomorphism) extend this definition to a functor

\[
\text{Det}_R : (\mathbf{D}_{\text{perf}}^b(R), \text{is}) \to \mathcal{P}_R
\]

such that the isomorphism (18) extends to the following situation: for any exact sequence of complexes of \(R\)-modules \(0 \to P_1^\bullet \to P_2^\bullet \to P_3^\bullet \to 0\) such that \(P_i^\bullet\) are quasi-isomorphic to bounded complexes in \(\mathbf{P}_{\mathbb{Q}}(R)\), there exists a functorial isomorphism

\[
\text{Det}_R(P_1^\bullet) \xrightarrow{\sim} \text{Det}_R(P_2^\bullet) \cdot \text{Det}_R(P_3^\bullet).
\]

Throughout this paper, we fix such a \(\text{Det}_R\). The functor \(\text{Det}_R\) also satisfies the following: for \(P^\bullet \in \mathbf{D}_{\text{perf}}^b(R)\), if \(H^i(P^\bullet)[0] \in \mathbf{D}_{\text{perf}}^b(R)\) for any \(i\), then there exists a canonical isomorphism

\[
\text{Det}_R(P^\bullet) \xrightarrow{\sim} \prod_{i \in \mathbb{Z}} \text{Det}_R(H^i(P^\bullet)[0])^{(-1)^i}.
\]

For \((L, r) \in \mathcal{P}_R\), define \((L, r)^\vee := (L^\vee, r) \in \mathcal{P}_R\) and define an evaluation isomorphism \((L, r) \cdot (L, r)^\vee \xrightarrow{\sim} (R, 2r) : x \otimes f \mapsto f(x)\). For \(P \in \mathbf{P}_{\mathbb{Q}}(R)\), then we have a canonical isomorphism \(\text{Det}_R(P^\vee) \xrightarrow{\sim} \text{Det}_R(P)^\vee\) defined by the isomorphism

\[
det_R(P^\vee) \xrightarrow{\sim} (\text{det}_R P)^\vee : f_1 \wedge \cdots \wedge f_r \mapsto (x_1 \wedge \cdots \wedge x_r) \mapsto \sum_{\sigma \in S_r} \text{sgn}(\sigma) f_1(x_{\sigma(1)}) \cdots f_r(x_{\sigma(r)}).
\]

This naturally extends to \((\mathbf{D}_{\text{perf}}^b(R), \text{is})\), i.e. for any \(P^\bullet \in \mathbf{D}_{\text{perf}}^b(R)\), there exists a functorial isomorphism \(\text{Det}_R(\mathbf{R}\text{Hom}_R(P^\bullet, R)) \xrightarrow{\sim} \text{Det}_R(P^\bullet)^\vee\). Using this isomorphism and the evaluation isomorphism \(\text{Det}_R(P^\bullet) \cdot \text{Det}_R(P^\bullet)^\vee \xrightarrow{\sim} (R, 2\chi_R(P^\bullet))\), we obtain an isomorphism

\[
\text{Det}_R(P^\bullet) \cdot \text{Det}_R(\mathbf{R}\text{Hom}_R(P^\bullet, R)) \xrightarrow{\sim} (R, 2\chi_R(P^\bullet)).
\]

For a map \(f : P^\bullet \to P^\bullet\) of complexes of \(R\)-modules such that \(P^\bullet \in \mathbf{D}_{\text{perf}}^b(R)\), we define a canonical trivialization

\[
\text{Det}_R(\text{Cone}(P^\bullet \xrightarrow{f} P^\bullet)[-1]) \xrightarrow{\sim} \text{Det}_R(P^\bullet) \cdot \text{Det}_R(P^\bullet)^{-1} \xrightarrow{\sim} 1_R
\]

where the first isomorphism is induced by the natural exact sequence

\[
0 \to P^\bullet \xrightarrow{x \mapsto (x, 0)} \text{Cone}(P^\bullet \xrightarrow{f} P^\bullet)[-1] \xrightarrow{(x, y) \mapsto y} P^\bullet[-1] \to 0
\]

and the second isomorphism is the canonical evaluation isomorphism.
3.2. **Fundamental lines.** Both Kato’s conjecture and ours concern with the existence of a compatible family of canonical trivialization of some graded invertible modules defined by using the determinants of the Galois cohomologies of Galois representations or \((\varphi, \Gamma)\)-modules. We call these graded invertible modules the fundamental lines, of which we explain in this subsection.

Kato’s conjecture concerns with pairs \((\Lambda, T)\) such that

(i) \(\Lambda\) is a noetherian semi-local ring which is complete with respect to the \(m_{\Lambda}\)-adic topology (where \(m_{\Lambda}\) is the Jacobson radical of \(\Lambda\)) such that \(\Lambda/m_{\Lambda}\) is a finite ring with the order a power of \(p\),

(ii) \(T\) is a \(\Lambda\)-representation of \(G_{\mathbb{Q}_p}\), i.e. a finite projective \(\Lambda\)-module equipped with a continuous \(\Lambda\)-linear action of \(G_{\mathbb{Q}_p}\).

Our conjecture concerns with pairs \((A, M)\) such that

(i) \(A\) is a \(\mathbb{Q}_p\)-affinoid algebra,

(ii) \(M\) is a \((\varphi, \Gamma)\)-module over \(R_A\).

For each pair \((B, N) = (\Lambda, T)\) or \((A, M)\) as above, we’ll define graded invertible \(\Lambda\)-modules \(\Delta_{B, i}(N) \in P_B\) for \(i = 1, 2\) as below, and the fundamental line will be defined as \(\Delta_B(N) = \Delta_{B, 1}(N) \cdot \Delta_{B, 2}(N) \in P_B\).

We first define \(\Delta_{\Lambda, i}(T)\) for \((\Lambda, T)\). Denote by \(C_{\text{cont}}^\bullet(G_{\mathbb{Q}_p}, T)\) for the complex of continuous cochains of \(G_{\mathbb{Q}_p}\) with values in \(T\). It is known that \(C_{\text{cont}}^\bullet(G_{\mathbb{Q}_p}, T) \in \mathbf{D}^-_{\text{perf}}(\Lambda)\) is contained in \(\mathbf{D}_{\text{perf}}(\Lambda)\) and that satisfies the similar properties (1), (2), (3), (4) in Theorem 2.14. In particular, we can define a graded invertible \(\Lambda\)-module

\[\Delta_{\Lambda, 1}(T) := \text{Det}_\Lambda(C_{\text{cont}}^\bullet(G_{\mathbb{Q}_p}, T)),\]

(whose degree is \(-r_T := -\text{rk}_\Lambda T\) by the Euler-Poincaré formula) which satisfies the following properties:

(i) For each continuous homomorphism \(f : \Lambda \to \Lambda'\), there exists a canonical \(\Lambda'\)-linear isomorphism

\[\Delta_{\Lambda, 1}(T) \otimes_\Lambda \Lambda' \sim \Delta_{\Lambda', 1}(T \otimes_\Lambda \Lambda'),\]

(ii) For each exact sequence \(0 \to T_1 \to T_2 \to T_3 \to 0\) of \(\Lambda\)-representations of \(G_{\mathbb{Q}_p}\), there exists a canonical \(\Lambda\)-linear isomorphism

\[\Delta_{\Lambda, 1}(T_2) \sim \Delta_{\Lambda, 1}(T_1) \cdot \Delta_{\Lambda, 1}(T_3),\]

(iii) Tate duality \(C_{\text{cont}}^\bullet(G_{\mathbb{Q}_p}, T) \sim \mathbf{R}\text{Hom}_\Lambda(C_{\text{cont}}^\bullet(G_{\mathbb{Q}_p}, T^*), \Lambda)[-2]\) and the isomorphism \([20]\) induce a canonical \(\Lambda\)-linear isomorphism

\[\Delta_{\Lambda, 1}(T^*) \cdot \Delta_{\Lambda, 1}(T) \sim (\Lambda, -2 \cdot r_T).\]

We next define \(\Delta_{\Lambda, 2}(T)\) as follows. For \(a \in \Lambda^\times\), we define

\[\Lambda_a := \{ x \in W(\mathbb{F}_p) \otimes_{\mathbb{Z}_p} \Lambda | (\varphi \otimes \text{id}_\Lambda)(x) = (1 \otimes a)x \},\]

which is an invertible \(\Lambda\)-module. In the same way as Theorem 2.8, for any rank one \(\Lambda\)-representation \(T_0\), there exists unique (up to isomorphism) pair \((\delta_{T_0}, \mathcal{L}_{T_0})\)
where $\delta_{T_0} : Q_p^\times \rightarrow \Lambda^x$ is a continuous homomorphism and $\mathcal{L}_{T_0}$ is an invertible $\Lambda$-module such that $T_0 \sim \Lambda(\delta_{T_0}) \otimes_\Lambda \mathcal{L}_{T_0}$, where we denote by $\delta_{T_0} : G_{Q_p} \rightarrow \Lambda^x$ for the continuous character which satisfies $\delta_{T_0} \circ \text{rec}_{Q_p} = \delta_{T_0}$. Under these definitions, we define $a(T) := \delta_{\det T}(p) \in \Lambda^x$, and define an invertible $\Lambda$-module

$$\mathcal{L}_\Lambda(T) := \Lambda_{a(T)} \otimes_\Lambda \det \Lambda T$$

and define a graded invertible $\Lambda$-module

$$\Delta_{\Lambda,2}(T) := (\mathcal{L}_\Lambda(T), r_T).$$

Because we have a canonical isomorphism $\Lambda_{a_1} \otimes_\Lambda \Lambda_{a_2} \sim \Lambda_{a_1a_2} : x \otimes y \mapsto xy$ for any $a_1, a_2 \in \Lambda$, $\Delta_{\Lambda,2}(T)$ also satisfies the similar properties:

(i) For $f : \Lambda \rightarrow \Lambda'$, there exists a canonical isomorphism $\Delta_{\Lambda,2}(T) \otimes_\Lambda \Lambda' \sim \Delta_{\Lambda,2}(T \otimes_\Lambda \Lambda')$,

(ii) For $0 \rightarrow T_1 \rightarrow T_2 \rightarrow T_3 \rightarrow 0$, there exists a canonical isomorphism $\Delta_{\Lambda,2}(T_2) \sim \Delta_{\Lambda,2}(T_1) \cdot \Delta_{\Lambda,2}(T_3)$,

(iii) Let $r_T$ be the rank of $T$, then there exists a canonical isomorphism

$$\Delta_{\Lambda,2}(T^*) \cdot \Delta_{\Lambda,2}(T) \sim (\Lambda(r_T), 2 \cdot r_T).$$

For example, (iii) is obtained using the fact that $\delta_{\det T^*}(p) = \delta_{\det T}(p)^{-1}$ and that $\Lambda \otimes \Lambda^{-1} \sim \Lambda_1 = \Lambda$, and using the the canonical evaluation pairing

$$\det T^* \otimes_\Lambda \det T \sim \det T^* \otimes_\Lambda \text{Hom}_\Lambda(\det T^*, \Lambda(r_T)) \sim \Lambda(r_T) : x \otimes f \mapsto f(x).$$

Finally, we define

$$\Delta_\Lambda(T) := \Delta_{\Lambda,1}(T) \cdot \Delta_{\Lambda,2}(T) \in \mathcal{P}_B,$$

then $\Delta_\Lambda(T)$ also satisfies the similar properties (i), (ii) as those for $\Delta_{\Lambda,i}(T)$ and

(iii) There exists a canonical isomorphism

$$\Delta_\Lambda(T^*) \cdot \Delta_\Lambda(T) \sim (\Lambda(r_T), 0).$$

Next, we define the fundamental line $\Delta_\Lambda(M)$ for $(\varphi, \Gamma)$-modules $M$ over $\mathcal{R}_A$. Let $A$ be a $Q_p$-affinoid algebra, and let $M$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_A$. By Theorem 2.14 of Kedlaya-Pottharst-Xiao, we can define a graded invertible $A$-module

$$\Delta_{\Lambda,1}(M) := \text{Det}_A C_{\varphi, \gamma}^* (M) \in \mathcal{P}_A$$

which satisfies the similar properties (i), (ii), (iii) as those for $\Delta_{\Lambda,1}(T)$. We next define $\Lambda_{\Lambda,2}(M)$ as follows. By Theorem 2.8 of Kedlaya-Pottharst-Xiao, there exists unique ( up to isomorphism ) pair $(\delta_{\det M} : \mathcal{L}_{\det M})$ where $\delta_{\det M} : Q_p^\times \rightarrow A^x$ is a continuous homomorphism and $\mathcal{L}_{\det M}$ is an invertible $A$-module such that $\det M \sim \mathcal{R}_A(\delta_{\det M}) \otimes_\mathcal{R}_A \mathcal{L}_{\det M}$. Then, we define an $A$-module

$$\mathcal{L}_A(M) := \{ x \in \det M | \varphi(x) = \delta_{\det M}(p)x, \gamma(x) = \delta_{\det M}(\chi(\gamma))x \ (\gamma \in \Gamma) \}$$
which is an invertible \( A \) module because it is isomorphic to \( \mathcal{L}_{\text{det} R_A M} \), and define a graded invertible \( A \)-module
\[
\Delta_{A,2}(M) := (\mathcal{L}_A(M), r_M) \in \mathcal{P}_A.
\]
By definition, it is easy to check that \( \Delta_{A,2}(M) \) satisfies the similar properties (i), (ii), (iii) as those for \( \Delta_{A}(T) \). Finally, we define a graded invertible \( A \)-module \( \Delta_A(M) \) which we call the fundamental line by
\[
\Delta_A(M) := \Delta_{A,1}(M) \cdot \Delta_{A,2}(M) \in \mathcal{P}_A,
\]
which also satisfies the similar properties (i), (ii), (iii) as those for \( \Delta_A(T) \).

More generally, let \( X \) be a rigid analytic space over \( \mathbb{Q}_p \), and let \( M \) be a \((\varphi, \Gamma)\)-module over \( R_X \). By the base change property (i) of \( \Delta_A(M) \), we can also functorially define a graded invertible \( O_X \)-module \( \Delta_X(M) \in \mathcal{P}_{O_X} \) on \( X \) (we can naturally generalize the notion of graded invertible modules in this setting) such that there exists a canonical isomorphism
\[
\Gamma(\text{Max}(A), \Delta_X(M)) \xrightarrow{\sim} \Delta_A(M|_{\text{Max}(A)})
\]
for any affinoid open \( \text{Max}(A) \subseteq X \).

We next compare the Kato’s fundamental line \( \Delta_{\Lambda}(T) \) with our fundamental line \( \Delta_A(M) \). Let \( f : \Lambda \to A \) be a continuous ring homomorphism, where \( \Lambda \) is equipped with \( m_{\Lambda} \)-adic topology and \( A \) is equipped with \( p \)-adic topology. Let \( T \) be a \( \Lambda \)-representation of \( G_{\mathbb{Q}_p} \). Let denote by \( M := D_{\text{rig}}(T \otimes_{\Lambda} A) \) for the \((\varphi, \Gamma)\)-module over \( R_A \) associated to the \( A \)-representation \( T \otimes_{\Lambda} A \) of \( G_{\mathbb{Q}_p} \). By Theorem 2.8 of [Po13a], there exists a canonical quasi-isomorphism
\[
C_{\text{cont}}(G_{\mathbb{Q}_p}, T) \otimes_{\Lambda} A \xrightarrow{\sim} C_{\varphi, \gamma}(M),
\]
and this induces an \( A \)-linear isomorphism
\[
\Delta_{\Lambda,1}(T) \otimes_{\Lambda} A \xrightarrow{\sim} \Delta_{A,1}(M).
\]
We also have the following isomorphism.

**Lemma 3.1.** In the above situation, there exists a canonical \( A \)-linear isomorphism
\[
\Delta_{A,2} \otimes_{\Lambda} A \xrightarrow{\sim} \Delta_{A,2}(M).
\]

**Proof.** By definition, it suffices to show the lemma when \( T \) is rank one. Hence, we may assume that \( T = \Lambda(\hat{\delta}) \otimes_{\Lambda} \mathcal{L} \) for a continuous homomorphism \( \hat{\delta} : \mathbb{Q}_p^x \to \Lambda^x \) and an invertible \( \Lambda \)-module \( \mathcal{L} \) (where \( \hat{\delta} \) is the character of \( G_{\mathbb{Q}_p}^{ab} \) such that \( \hat{\delta} \circ \text{rec}_{\mathbb{Q}_p} = \delta \)). Moreover, because we have a canonical isomorphism
\[
D_{\text{rig}}((\Lambda(\hat{\delta}) \otimes_{\Lambda} \mathcal{L}) \otimes_{\Lambda} A) \xrightarrow{\sim} D_{\text{rig}}(\Lambda(\hat{\delta}) \otimes_{\Lambda} A) \otimes_{A}(\mathcal{L} \otimes_{\Lambda} A)
\]
by the exactness of \( D_{\text{rig}}(-) \), it suffices to show the lemma when \( \mathcal{L} = \Lambda \).

Because the image of \( H_{\mathbb{Q}} := \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_{p,\infty}) \) in \( G_{\mathbb{Q}_p}^{ab} \) is the closed subgroup which is topologically generated by \( \text{rec}_{\mathbb{Q}_p}(p) \), we have
\[
D_{\text{rig}}(\Lambda(\hat{\delta}) \otimes_{\Lambda} A) = (W(\overline{\mathbb{F}}_p) \hat{\otimes}_{\mathbb{Z}_p} A(\hat{\delta}))^{\text{rec}_{\mathbb{Q}_p}(p) = 1} \otimes_{A} \mathcal{R}_A.
\]
by definition of $D_{\text{rig}}(-)$, and the right hand side is isomorphic to $\mathcal{R}_A(f \circ \delta)$. Hence, we obtain
\[
\mathcal{L}_A(M) = (W(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} \Lambda(\delta))^{\text{rec}_{\mathbb{Q}_p}(p)=1} \otimes_A \mathcal{R}_A)_{\varphi=f(\delta(p)), \Gamma=f \circ \delta} X
= (W(\overline{\mathbb{F}}_p) \otimes_{\mathbb{Z}_p} \Lambda(\delta))^{\text{rec}_{\mathbb{Q}_p}(p)=1} \otimes_A A = \mathcal{L}_A(T) \otimes_A A,
\]
which proves the lemma. \hfill \Box

Taking the products of these two canonical isomorphisms, we obtain the following corollary.

**Corollary 3.2.** In the above situation, there exists a canonical isomorphism
\[
\Delta_A(T) \otimes_A A \sim \Delta_A(M).
\]

**Example 3.3.** The typical example of the above base change property is the following. For $A$ as above, let denote by $X$ for the associated rigid analytic space. More precisely, $X$ is the union of affinoids $\text{Max}(A_n)$ for $n \geq 1$, where $A_n$ is the $\mathbb{Q}_p$-affinoid algebra defined by $A_n := A[\frac{1}{p}]^\wedge[1/p]$ (for a ring $R$, denote by $R^\wedge$ for the $p$-adic completion). Let $T$ be a $A$-representation of $G_{\mathbb{Q}_p}$, and let denote by $M_n := D_{\text{rig}}(T \otimes_A A_n)$. Because $M_n$ is compatible with the base change with respect to the canonical map $A_n \to A_{n+1}$ for any $n$, $\{M_n\}_{n \geq 1}$ defines a $(\varphi, \Gamma)$-module $M$ over $\mathcal{R}_X$. Then, the canonical isomorphism $\Delta_A(T) \otimes_A A_n \sim \Delta_{A_n}(M_n)$ defined in the above corollary glues to an isomorphism
\[
\Delta_A(T) \otimes_A \mathcal{O}_X \sim \Delta_X(M).
\]
Moreover, using the terminology of coadmissible modules (ST03), we can define this comparison isomorphism without using sheaves. Let define $A_{\infty} := \Gamma(X, \mathcal{O}_X)$ and $\Delta_{A_\infty}(M_{\infty}) := \varprojlim_n \Delta_{A_n}(M_n)$. Taking the limit of the isomorphism $\Delta_A(T) \otimes_A A_n \sim \Delta_{A_n}(M_n)$ we obtain an $A_{\infty}$-linear isomorphism
\[
\Delta_A(T) \otimes_A A_{\infty} \sim \Delta_{A_\infty}(M_{\infty}).
\]
Then, the theory of coadmissible modules (Corollary 3.3 of ST03) says that to consider the isomorphism $\Delta_A(T) \otimes_A \mathcal{O}_X \sim \Delta_X(M)$ is the same as to consider the isomorphism $\Delta_A(T) \otimes_A A_{\infty} \sim \Delta_{A_\infty}(M_{\infty})$. Actually, we will frequently use the latter object like $\Delta_{A_\infty}(M_{\infty})$ in §4.

**3.3. de Rham $\varepsilon$-isomorphism.** In this subsection, we assume that $L = A$ is a finite extension of $\mathbb{Q}_p$. We functorially define a trivialization
\[
\varepsilon^{\text{dR}}_L(M) : \Delta_L(M) \sim (L, 0)
\]
which we call the de Rham $\varepsilon$-isomorphism for each de Rham $(\varphi, \Gamma)$-module $M$ over $\mathcal{R}_L$ and for each $\mathbb{Z}_p$-basis $\zeta = \{\zeta_n\}_{n \geq 0}$ of $\mathbb{Z}_p(1)$.

Let $M$ be a de Rham $(\varphi, \Gamma)$-module over $\mathcal{R}_L$. We first recall the definition of Deligne-Langlands' and Fontaine-Perrin-Riou’s $\varepsilon$-constant associated to $M$ (Dei3,[FP94]).
We first briefly recall the theory of \( \varepsilon \)-constants of Deligne and Langlands \( [De73] \). Let \( W_{Q_p} \subseteq G_{Q_p} \) be the Weil group of \( Q_p \). Let \( E \) be a field of characteristic zero, and let \( V \) be a finite dimensional \( E \)-vector space equipped with a continuous \( E \)-linear action of \( W_{Q_p} \) (where \( V \) is equipped with the discrete topology). Let denote by \( V^\vee \) for the dual \( \text{Hom}_E(V, E) \) of \( V \). Denote by \( E(|x|) \) the rank one \( E \)-representation of \( W_{Q_p} \) corresponding to the continuos homomorphism \( |x| : Q_p^\times \to E^\times : p \mapsto 1/p, a \mapsto 1(a \in Z_p^\times) \) via the local class field theory. Put \( V^\vee(|x|) := V^\vee \otimes_E E(|x|) \). In this article, we assume that \( E \) is a field which contains \( Q_p(\zeta_p\infty) \). The definition of the \( \varepsilon \)-constants depends on the choice of an additive character of \( Q_p \) and a Haar measure on \( Q_p \). In this article, we fix the Haar measure \( dx \) on \( Q_p \) for which \( Z_p \) has measure 1. For each \( Z_p \)-basis \( \zeta = \{ \zeta_p^n \}_{n \geq 0} \) of \( Z_p(1) \), we define an additive character \( \psi : Q_p \to E^\times \) such that \( \psi(1/p^n) := \zeta_p^n \) for \( n \geq 1 \). In this article, we don’t recall the precise definition of \( \varepsilon \)-constants, but we recall here some of their basis properties for the additive character \( \psi \) and the fixed Haar measure \( dx \). Under these fixed datum, we can attach a constant \( \varepsilon(V, \psi, dx) \in E^\times \) for each \( V \) as above which satisfies the following properties (we denote \( \varepsilon(V, \zeta) := \varepsilon(V, \psi, dx) \) for simplicity):

1. For each exact sequence \( 0 \to V_1 \to V_2 \to V_3 \to 0 \) of finite dimensional \( E \)-vector spaces with continuous actions of \( W_{Q_p} \), we have
   \[
   \varepsilon(V_2, \zeta) = \varepsilon(V_1, \zeta)\varepsilon(V_3, \zeta).
   \]

2. For each \( a \in Z_p^\times \), we define \( \zeta^a := \{ \zeta_p^n \}_{n \geq 1} \). Then, we have
   \[
   \varepsilon(V, \zeta^a) = \text{det}_E V(\text{rec}_{Q_p}(a))\varepsilon(V, \zeta).
   \]

3. \( \varepsilon(V, \zeta)\varepsilon(V^\vee(|x|), \zeta^{-1}) = 1 \).

4. \( \varepsilon(V, \zeta) = 1 \) if \( V \) is unramified.

5. If \( \dim_E V = 1 \) and corresponds to a locally constant homomorphism \( \delta : Q_p^\times \to E^\times \) via the local class field theory, then
   \[
   \varepsilon(V, \zeta) = \delta(p)^{n(\delta)} \sum_{i \in (Z/p^n(\delta)Z)^\times} \delta(i)^{-1}\zeta_p^{n(i)},
   \]
   where \( n(\delta) \geq 0 \) is the conductor of \( \delta \), i.e. the minimal integer \( n \geq 0 \) such that \( \delta|_{1+p^nZ_p} = 1 \) (then \( \delta|_{Z_p^\times} \) factors through \( (Z/p^n(\delta)Z)^\times \)).

Next, we define the \( \varepsilon \)-constant for each de Rham \( (\varphi, \Gamma) \)-module over \( \mathcal{R}_L \) following Fontaine-Perrin-Riou \( [FP94] \). Let \( M \) be a de Rham \( (\varphi, \Gamma) \)-module over \( \mathcal{R}_L \). Then \( M \) is potentially semi-stable by the result of Berger (for example, see Théorème III.2.4 of \( [Ber08b] \)) which is based on the Crew’s conjecture proven by Andrée Mebkhout, Kedlaya. Hence, we can define a filtered \( (\varphi, N, G_{Q_p}) \)-module \( D_{\text{pst}}(M) := \bigcup_{K \subseteq G_{Q_p}} D_{\text{st}}^K(M|_K) \) which is a free \( Q_p^{ur} \otimes Q_p \)-module whose rank is \( r_M \), where \( K \) run through all the finite extensions of \( Q_p \) and we define \( D_{\text{st}}^K(M|_K) := (\mathcal{R}_L(\pi_K)|[\log(\pi), 1/t] \otimes_{\mathcal{R}_L} M)^{\Gamma_K = 1} \). Following Fontaine, we define a structure of a \( Q_p^{ur} \otimes Q_p \)-representation of \( W_{Q_p} \) on \( D_{\text{pst}}(M) \) by \( g(x) := \varphi^{v(g)}(g \cdot x) \) for \( g \in W_{Q_p} \) and...
$x \in D_{pst}(M)$, where we denote by $g \cdot x$ for the natural action of $G_{\mathbb{Q}_p}$ on $D_{pst}(M)$ and $v : W_{\mathbb{Q}_p} \to W_{\mathbb{Q}_p}^{ab} \xrightarrow{\text{rec}^{-1}} \mathbb{Q}_p^\times \xrightarrow{v} \mathbb{Z}$. Taking the base change of $D_{pst}(M)$ by the natural inclusion $\mathbb{Q}_p^{ur} \otimes_{\mathbb{Q}_p} L \hookrightarrow \mathbb{Q}_p^{ab} \otimes_{\mathbb{Q}_p} L$, and decomposing $\mathbb{Q}_p^{ab} \otimes_{\mathbb{Q}_p} L \to \prod_{\tau} L_{\tau}$ into a finite product of fields $L_{\tau}$. We obtain a $L_{\tau}$-representation of $W_{\mathbb{Q}_p}$ which we denote by $D_{pst}(M)_{\tau}$ for each $\tau$. Hence, we can define the $\varepsilon$-constant $\varepsilon(D_{pst}(M)_{\tau}, \tau(\zeta)) \in L_{\tau}^\times$, where $\tau(\zeta)$ is the image of $\zeta$ in $L_{\tau}$. Then,

$$
\varepsilon_L(D_{pst}(M), \zeta) := (\varepsilon(D_{pst}(M)_{\tau}, \tau(\zeta)))_{\tau} \in \prod_{\tau} L_{\tau}^\times
$$

is contained in $L_{\tau}^\times := (\mathbb{Q}_p(\zeta_{\infty}) \otimes_{\mathbb{Q}_p} L)^\times \subseteq (\mathbb{Q}_p(\zeta_{\infty}) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p^{ur} \otimes_{\mathbb{Q}_p} L)^\times$ because it is easy to check that $\varepsilon_L(D_{pst}(M), \zeta)$ is fixed by $1 \otimes \varphi \otimes 1$.

Using this definition, for each de Rham $(\varphi, \Gamma)$-module $M$ over $\mathcal{R}_L$, we construct a trivialization $\varepsilon^{\text{dR}}_L(M) : \Delta_L(M) \xrightarrow{\sim} 1_L$ as follows. Decomposing $\Delta_L(M)$ into the product

$$(\Delta_L,1(M) \cdot \text{Det}_L(D_{\text{dR}}(M))) \cdot (\text{Det}_L(D_{\text{dR}}(M))^{-1} \cdot \Delta_L,2(M)),$$

this trivialization is defined as the product of trivializations

$$
\theta_L(M) : \Delta_L,1(M) \cdot \text{Det}_L(D_{\text{dR}}(M)) \xrightarrow{\sim} 1_L
$$

and

$$
\theta_{\text{dR},L}(M, \zeta) : \text{Det}_L(D_{\text{dR}}(M))^{-1} \cdot \Delta_L,2(M) \xrightarrow{\sim} 1_L
$$

which are defined as follows (we remark that, as you can guess from the notation, only $\theta_{\text{dR},L}(M, \zeta)$ depends on the choice of $\zeta$).

We first define $\theta_L(M) : \Delta_L,1(M) \cdot \text{Det}_L(D_{\text{dR}}(M)) \xrightarrow{\sim} 1_L$. By the result of §2.2, we have a canonical exact sequence of $L$-vector spaces

$$
0 \to H^0_{\varphi, \gamma}(M_0) \to D_{\text{cris}}(M_0) \xrightarrow{\varphi \mapsto [(1-\varphi)x, x]} D_{\text{cris}}(M_0) \otimes t_{M_0} \xrightarrow{\exp_f \cdot M_0 \otimes \exp_{\text{dR}} f} H^1_{\varphi, \gamma}(M_0)_f \to 0
$$

for $M_0 = M, M^*$. This exact sequence for $M_0 = M$ naturally induces an isomorphism

$$
\theta^1_L(M) : \text{Det}_L(H^0_{\varphi, \gamma}(M)) \cdot \text{Det}_L(H^1_{\varphi, \gamma}(M)_f)^{-1} \cdot \text{Det}_L(t_M) \xrightarrow{\sim} 1_L,
$$

For $M_0 = M^*$, using Tate duality and the canonical de Rham duality

$$
D_{\text{dR}}(M)^0 \xrightarrow{\sim} t_{M^*}^\vee : x \mapsto ([y], y)_{\text{dR}},
$$

and Proposition 2.23, we define a map

$$
\exp^*_{M} : H^1_{\varphi, \gamma}(M)_f := H^1_{\varphi, \gamma}(M)/H^1_{\varphi, \gamma}(M)_0 \xrightarrow{\varphi \mapsto [(1-\varphi)x, x]} H^1_{\varphi, \gamma}(M^*)^\vee \\
\xrightarrow{\exp^*_{M}} t_{M^*}^\vee \xrightarrow{\sim} D_{\text{dR}}(M)^0
$$

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which is called the dual exponential map and was studied in §2.4 of [Na13]. Using this map, as the the dual of the exact sequence (22) for $M_0 = M^*$, we obtain an exact sequence

$$0 \to H^1_{\varphi, \gamma}(M)_f \xrightarrow{\exp_{\Gamma} \otimes \exp_{\varphi}} D_{\text{cris}}(M^*)^\vee \oplus D_{\text{dR}}(M)^0 \to D_{\text{cris}}(M^*)^\vee \to H^2_{\varphi, \gamma}(M) \to 0.$$  

This exact sequence induces a canonical isomorphism

$$\theta^2_L(M) : \text{Det}_L(H^1_{\varphi, \gamma}(M)/f)^{-1} : \text{Det}_L(H^2_{\varphi, \gamma}(M)) \cdot \text{Det}_L(D_{\text{dR}}(M)^0) \sim 1_L.$$  

Taking the product, we obtain an isomorphism

$$\theta'_L(M) := \theta^1_L(M) \cdot \theta^2_L(M) : \Delta_{L,1}(M) \cdot \text{Det}_L(D_{\text{dR}}(M)) \sim 1_L.$$  

Using $\theta'_L(M)$ and the “gamma” constant $\gamma_{L,M} \in \mathbb{Q}^\times$ which is defined by

$$\gamma_{L,M} := \prod_{r \leq 0} \left\{ \frac{(-1)^r}{(-r)!} \right\}^{-\dim_L(\text{gr}^{-r} D_{\text{dR}}(M))} \prod_{r \geq 1} \left\{ -(r - 1)! \right\}^{-\dim_L(\text{gr}^{-r} D_{\text{dR}}(M))},$$

we finally define

$$\theta_L(M) := \gamma_{L,M} \cdot \theta'_L(M) : \Delta_{L,1}(M) \cdot \text{Det}_L(D_{\text{dR}}(M)) \sim 1_L.$$  

Next, we define an isomorphism $\theta_{\text{dR}}(L, M, \zeta) : \text{Det}_L(D_{\text{dR}}(M))^{-1} : \Delta_{L,2}(M) \sim 1_L$. To define this, we show the following lemma.

**Lemma 3.4.** Let $\{h_1, h_2, \ldots, h_{r_M}\}$ be the set of Hodge-Tate weights of $M$ (with multiplicity). Put $h_M := \sum_{i=1}^{r_M} h_i$. For any $n \geq n(M)$ such that $\varepsilon_L(D_{\text{pst}}(M), \zeta) \in L_n := \mathbb{Q}_p(\zeta_{p^n}) \otimes_{\mathbb{Q}_p} L$, the map

$$L(M) \to D_{\text{dR}, n}(\det R_L M) = L_n((t)) \otimes_{\mathbb{C}_L} (\det R_L M)^{(n)} : x \mapsto \frac{\delta_{\text{det} R_L M}(x)}{\varepsilon_L(D_{\text{pst}}(M), \zeta) t^{h_M} \otimes \varphi^n(x)}$$

induces an isomorphism

$$f_{M, \zeta} : L(L(M)) \sim D_{\text{dR}}(\det R_L M),$$

and doesn’t depend on the choice of $n$.

**Proof.** The independence of $n$ follows from the definition of the transition map $D_{\text{dR}, n}(-) \to D_{\text{dR}, n+1}(-)$.

We show that $f_{M, \zeta}$ is isomorphism. Comparing the dimensions, it suffices to show that the image of the map in the lemma is contained in $D_{\text{dR}}(\det R_L M)$, i.e. is fixed by the action of $\Gamma$. Because we have $\varepsilon_L(D_{\text{pst}}(M), \zeta) / \varepsilon_L(D_{\text{dR}}(R_L M), \zeta) \in L^\times (\subseteq L^\times)$, it suffices to show the claim when $M$ is of rank one. We assume that $M$ is of rank one. By the classification of rank one de Rham $(\varphi, \Gamma)$-modules, there exists a locally constant homomorphism $\delta : \mathbb{Q}_p^\times \to L^\times$ such that $M \sim R_L(\delta \cdot x^{h_M})$.  

\[36\]
The corresponding representation $D_{\text{pst}}(M)$ of $W_{Q_p}$ is given by the homomorphism $\overline{\delta} \cdot |x|^h_{\text{mt}} : Q_p^\times \to L^\times$ via the local class field theory. By the property (2) of $\varepsilon$-constants, then we have $\gamma(\varepsilon_L(D_{\text{pst}}(M), \zeta)) = \overline{\delta}(\chi(\gamma))\varepsilon_L(D_{\text{pst}}(M), \zeta)$ for $\gamma \in \Gamma$, which proves the claim because we have $\gamma(f^n(x)) = \overline{\delta}(\chi(\gamma))\gamma^h_{\text{mt}} f^n(x)$ for $x \in L_L(M), \gamma \in \Gamma$ by definition.

Because we have a canonical isomorphism $D_{\text{dr}}(\det_{RL}M) \sim \det_D(D_{\text{dr}}(M))$, the isomorphism $f_{M, \zeta}$ induces an isomorphism $f_{M, \zeta} : \Delta_{L,2}(M) \sim \Delta_{L,2}(M)$. Multiplying by $\det_D(D_{\text{dr}}(M))^{-1}$, we obtain an isomorphism

$$\theta_{DR, L}(M, \zeta) : \det_D(D_{\text{dr}}(M))^{-1} \cdot \Delta_{L,2}(M) \sim 1_L.$$

**Remark 3.5.** The isomorphism $f_{M, \zeta}$, and hence the isomorphism $\theta_{DR, L}(M, \zeta)$ depend on the choice of $\zeta$. If we choose another $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(1)$ which can be written as $\zeta^a := \{\zeta_p^a\}_{n \geq 0}$ for unique $a \in \mathbb{Z}_p^\times$, then $f_{M, \zeta^a}$ is defined using $\varepsilon_L(D_{\text{pst}}(M), \zeta^a)$ and the parameter $\pi_{\zeta^a}$ (see remark 2.1) and $t_a := \log(1 + \pi_{\zeta^a})$. Because we have $\varepsilon_L(D_{\text{pst}}(M), \zeta^a) = \det D_{\text{pst}}(M)(\text{rec}_{Q_p}(a))\varepsilon_L(D_{\text{pst}}(M), \zeta)$ and $\pi_{\zeta^a} = (1 + \pi)^a - 1$ and $t_a = at$, we have $f_{M, \zeta^a} = \frac{1}{\det_{RL}\Delta_{L,2}(M)} f_{M, \zeta}$, and hence we also have

$$\theta_{DR, L}(M, \zeta^a) = \frac{1}{\det_{RL}\Delta_{L,2}(M)} \cdot \theta_{DR, L}(M, \zeta).$$

Finally, we define a trivialization $\varepsilon_{L, L, \zeta}^{\text{dr}}(M) : \Delta_L(M) \sim 1_L$ as follows.

**Definition 3.6.** We define an isomorphism $\varepsilon_{L, \zeta}^{\text{dr}}(M) : \Delta_L(M) \sim 1_L$ which is called the de Rham $\varepsilon$-isomorphism by

$$\varepsilon_{L, \zeta}^{\text{dr}}(M) := \theta_L(M) \cdot \theta_{DR, L}(M, \zeta),$$

and similarly define $\varepsilon_{L, \zeta^a}^{\text{dr}}(M) := \theta_L(M) \cdot \theta_{DR, L}(M, \zeta^a)$ for $a \in \mathbb{Z}_p^\times$, which is equal to $\frac{1}{\det_{RL}\Delta_{L,2}(M)} \cdot \varepsilon_{L, \zeta}^{\text{dr}}(M)$ by the above remark.

**Remark 3.7.** In [Ka93a], Kato defined his de Rham $\varepsilon$-isomorphism $\varepsilon_{L, \zeta}^{\text{dr}}(V) : \Delta_L(V) \sim 1_L$ (using a different notation) for each de Rham $L$-representation $V$ of $G_{Q_p}$ using the original Bloch-Kato exponential map. Using Proposition 2.25 it is easy to check that Kato’s isomorphism $\varepsilon_{L, \zeta}^{\text{dr}}(V)$ can be identified with our isomorphism $\varepsilon_{L, \zeta}^{\text{dr}}(D_{\text{rig}}(V))$ under the comparison isomorphism $\Delta_L(V) \sim \Delta_L(D_{\text{rig}}(V))$ defined in Corollary 3.2 Therefore, in this article, we don’t recall the definition of Kato’s de Rham $\varepsilon$-isomorphism $\varepsilon_{L, \zeta}^{\text{dr}}(V)$, and formulate Kato’s and our conjecture (Conjecture 3.9) using only our de Rham $\varepsilon$-isomorphism $\varepsilon_{L, \zeta}^{\text{dr}}(M)$.

Finally in this subsection, we prove a lemma on the compatibility of de Rham $\varepsilon$-isomorphism with Tate duality.
Lemma 3.8. Under the canonical isomorphism
\[ \Delta_L(M) \cdot \Delta_L(M) \sim (L(r_M), 0), \]
the isomorphism \( \epsilon_{L, \zeta}^{\text{DR}}(M) \cdot \epsilon_{L, \zeta}^{\text{DR}}(M) : \Delta_L(M) \cdot \Delta_L(M) \sim 1_L \) is equal to the isomorphism \( (L(r_M), 0) \sim 1_L : e_{r_M} \mapsto (-1)^{r_M}. \)

Proof. By the canonical isomorphism \( D_{\text{DR}}(M) \sim D_{\text{DR}}(M)^\vee : x \mapsto (y \mapsto [x, y]_{\text{DR}}) \)
and the isomorphism \( 20 \), we have an isomorphism
\[ (24) \quad \text{Det}_L(D_{\text{DR}}(M)) \cdot \text{Det}_L(D_{\text{DR}}(M)) \sim (L, 2r_M) \]
By definition of \( \epsilon_{L, \zeta}^{\text{DR}}(M) = \theta_L(M) \cdot \theta_{\text{DR}, L}(M, \zeta) \), it suffices to show the following two equalities:
(i) We have
\[ \theta_L(M) \cdot \theta_L(M) = (-1)^{h_M + r_M} \cdot \text{id}_L \]
under the canonical isomorphism
\[ \Delta_{L,1}(M) \cdot \text{Det}_L(D_{\text{DR}}(M)) \cdot \Delta_{L,1}(M) \cdot \text{Det}_L(D_{\text{DR}}(M)) \sim (L, -2r_M) \cdot (L, 2r_M) \sim 1_L, \]
where the first isomorphism is the product of the canonical isomorphism \( \Delta_{L,1}(M) \cdot \Delta_{L,1}(M) \sim (L, -2r_M) \) with the above isomorphism \( 24 \) and the last one is induced by \( L \otimes_L L \rightarrow L : a \otimes b \mapsto ab. \)
(ii) We have
\[ \theta_{\text{DR}, L}(M, \zeta^{-1}) \cdot \theta_{\text{DR}, L}(M, \zeta) = (e_{r_M} \mapsto (-1)^{h_M}) \]
under the canonical isomorphism
\[ \Delta_{L,2}(M) \cdot \text{Det}_L(D_{\text{DR}}(M)^{-1}) \cdot \Delta_{L,2}(M) \cdot \text{Det}_L(D_{\text{DR}}(M)^{-1}) \sim (L(r_M), 2r_M) \cdot (L, 2r_M)^{-1} \sim (L(r_M), 0). \]
where the first isomorphism is the product of the canonical isomorphism \( \Delta_{L,2}(M) \cdot \Delta_{L,2}(M) \sim (L(r_M), 2r_M) \) with the isomorphism \( \text{Det}_L(D_{\text{DR}}(M)^{-1}) \). \( \text{Det}_L(D_{\text{DR}}(M)^{-1}) \) is induced by \( (a \cdot e_{r_M}) \otimes f \mapsto f(a) \cdot e_{r_M}. \)
We first show the equality (ii). We may assume that \( \det_{\mathcal{R}_L} M = \mathcal{R}_L(\delta_{x^M}), \) then we naturally have \( \det_{\mathcal{R}_L} M^* = \mathcal{R}_L(\delta_{-1x^{-M}})e_{r_M}. \) By definition, it suffices to show that the isomorphism
\[ L(r_M) \xrightarrow{e_{r_M} \mapsto e_{r_M} - \delta_{x^{-1}} e_{r_M}} \mathcal{L}(M^*) \otimes_L \mathcal{L}(M) \]
\[ \xrightarrow{f_{M^*, \zeta^{-1}} \otimes f_{M, \zeta}} \text{Det}_{\mathcal{R}_L}(M^*) \otimes_L \text{Det}_{\mathcal{R}_L}(M) \sim L \]
sends $e_{r_M}$ to $(-1)^{h_M}$. Because we have

$$f_{M^*,\zeta^{-1}} \otimes f_{M,\zeta}(e_{\delta^{-1}x^{-h_M}} \cdot e_{r_M} \otimes e_{\delta x^{h_M}})$$

and have $D_{\text{pst}}(M^*) = D_{\text{pst}}(M)^\vee([x])$, the claim follows from the formula

$$\varepsilon_L(D_{\text{pst}}(M), \xi)\varepsilon_L(D_{\text{pst}}(M^*), \xi^{-1}) = 1.$$

We next prove the equality (i). Put $s_M := \dim_L t(M)$. By definition, it is easy to check that we have

$$\gamma_{L,M^*} \cdot \gamma_{L,M} = (-1)^{h_M+r_M-s_M}.$$  

Define a subcomplex $C_{\varphi,\gamma}(M)_f$ of $C_{\varphi,\gamma}(M)$ by $C_{\varphi,\gamma}(M) = C_{\varphi,\gamma}(M)$,

$$C_{\varphi,\gamma}(M)_f := \{ x \in \text{Ker}(C^1_{\varphi,\gamma}(M) \to C^2_{\varphi,\gamma}(M)) \mid [x] \in H^1_{\varphi,\gamma}(M)_f \}$$

and $C^2_{\varphi,\gamma}(M)_f = 0$. Put $C_{\varphi,\gamma}(M)_f := C_{\varphi,\gamma}(M)/C_{\varphi,\gamma}(M)_f$. By Proposition 2.23, the isomorphism $\Delta_{L,1}(M^*) \cdot \Delta_{L,1}(M) \sim (L, -2r_M)$ is written as the product of

$$\text{Det}_L(C_{\varphi,\gamma}(M^*)_f) \cdot \text{Det}_L(C_{\varphi,\gamma}(M)_f) \sim (L, -2s_M)$$

with

$$\text{Det}_L(C_{\varphi,\gamma}(M^*)_f) \cdot \text{Det}_L(C_{\varphi,\gamma}(M)_f) \sim (L, -2r_M + 2s_M).$$

By definition of $\theta^0_L(M)$, we have

$$\theta^0_L(M) \cdot \theta^0_L(M) = \text{id}_{1_L}$$

under the product of the isomorphism (27) with the canonical isomorphism $\text{Det}_L(t_{M^*}) \cdot \text{Det}_L(D_{\text{dr}}(M)^0) \sim (L, 2r_M - 2s_M)$. On the other hand, we have

$$\theta^2_L(M^*) \cdot \theta^2_L(M) = (-1)^{h_M} \cdot \text{id}_{1_L}$$

under the similar isomorphism by the equality (1) in the notation. Because we have $\theta_L(M) = \gamma_{L,M} \cdot \theta^0_L(M) \cdot \theta^2_L(M)$, the claim (ii) follows from the equalities (25), (28), (29).

\[\square\]

3.4. Formulation of the local $\varepsilon$-conjecture. In this subsection, using the definitions in the previous subsections, we formulate the following conjecture which we call local $\varepsilon$-conjecture. This conjecture is a combination of Kato’s original $\varepsilon$-conjecture for $(A, T)$ with our conjecture for $(A, M)$. To state both situations in a same time, we use notation $(B, N)$ for $(A, T)$ or $(A, M)$, and $f : B \to B'$ for $f : A \to A'$. 

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Conjecture 3.9. We can uniquely define a $B$-linear isomorphism

$$\varepsilon_{B,\zeta}(N) : \Delta_B(N) \rightarrow 1_B$$

for each pair $(B, N)$ as above and for each $\mathbb{Z}_p$-basis $\zeta$ of $\mathbb{Z}_p(1)$ satisfying the following conditions.

(i) Let $f : B \rightarrow B'$ be a continuous homomorphism. Then, we have

$$\varepsilon_{B,\zeta}(N) \otimes \text{id}_{B'} = \varepsilon_{B',\zeta}(N \otimes_B B')$$

under the canonical isomorphism $\Delta_B(N) \otimes_B B' \rightarrow \Delta_{B'}(N \otimes_B B')$.

(ii) Let $0 \rightarrow N_1 \rightarrow N_2 \rightarrow N_3 \rightarrow 0$ be an exact sequence. Then, we have

$$\varepsilon_{B,\zeta}(N_2) = \varepsilon_{B,\zeta}(N_1) \cdot \varepsilon_{B,\zeta}(N_3)$$

under the canonical isomorphism $\Delta_B(N_2) \rightarrow \Delta_B(N_1) \cdot \Delta_B(N_3)$.

(iii) For $a \in \mathbb{Z}_p^\times$, we have

$$\varepsilon_{B,\zeta a}(N) = \frac{1}{\delta_{\det(B)(a)}} \cdot \varepsilon_{B,\zeta}(N).$$

(iv) The isomorphism

$$\varepsilon_{B,\zeta^{-1}}(N^*) \cdot \varepsilon_{B,\zeta}(N) : \Delta_B(N^*) \cdot \Delta_B(N) \rightarrow 1_B$$

is equal to the isomorphism $(B(r_N), 0) \rightarrow (B(r_N), 0)$ under the canonical isomorphism $\Delta_B(N^*) \cdot \Delta_B(N) \rightarrow (B(r_N), 0)$.

(v) Let $f : \Lambda \rightarrow A$ be a continuous homomorphism, and let $M := D_{\text{rig}}(T \otimes \Lambda A)$ be the associated $(\varphi, \Gamma)$-module obtained by the base change of $T$ with respect to $f$. Then, we have

$$\varepsilon_{A,\zeta}(T) \otimes \text{id}_A = \varepsilon_{A,\zeta}(M)$$

under the canonical isomorphism $\Delta_{\Lambda}(T) \otimes \Lambda A \rightarrow \Delta_A(M)$ defined in Corollary 3.2.

(vi) Let $L = A$ be a finite extension of $\mathbb{Q}_p$, and let $M$ be a de Rham $(\varphi, \Gamma)$-module over $\mathcal{R}_L$. Then we have

$$\varepsilon_{L,\zeta}(M) = \varepsilon_{L,\zeta}^{\text{de R}}(M).$$

Remark 3.10. The Kato’s original conjecture ([Ka93b]) is just the restriction of the above conjecture to the pairs $(\Lambda, T)$. As we explained in Remark 3.7, the condition (v) was stated using his de Rham $\varepsilon$-isomorphism $\varepsilon_{L,\zeta}^{\text{de R}}(V)$ for each de Rham $L$-representation $V$ of $G_{\mathbb{Q}_p}$.

Remark 3.11. In the Kato’s conjecture, the uniqueness of $\varepsilon$-isomorphism was not explicitly predicted. Recently, it is known that the de Rham points (even crystalline points) are Zariski dense in “universal” families of $p$-adic representations, or $(\varphi, \Gamma)$-modules in many cases ([Co08], [Kis10] for two dimensional case, [Ch13], [Na11] for general case), hence we add the uniqueness assertion in our conjecture.
In [Ka93b], Kato proved his conjecture for the rank one case. As a generalization of his theorem, our main theorem of this article is the following, whose proof is given by the next section.

**Theorem 3.12.** The conjecture 3.9 is true for the rank case. More precisely, we can uniquely define a $B$-linear isomorphism $\varepsilon_{B,\zeta}(N) : \Delta_B(N) \xrightarrow{\sim} 1_B$ of for each pair $(B, N)$ such that $N$ is of rank one and for each $\mathbb{Z}_p$-basis $\zeta$ of $\mathbb{Z}_p(1)$ satisfying the conditions (i), (iii), (iv), (v), (vi).

4. Rank one case

In [Ka93b], Kato proved his $\varepsilon$-conjecture using the theory of Coleman homomorphism which interpolates the exponential maps and the dual exponential maps of rank one de Rham $p$-adic representations of $G_{\mathbb{Q}_p}$. In particular, so called the explicit reciprocity law, which is the explicit formula of its interpolation properties, was very important in his proof.

In this final section, we first construct the $\varepsilon$-isomorphism $\varepsilon_A,\zeta(M) : \Delta_A(M) \xrightarrow{\sim} 1_A$ for any rank one $(\phi, \Gamma)$-module $M$ by interpreting the theory of Coleman homomorphism in terms of $p$-adic Fourier transform (e.g. Amice transform, Colmez transform), which seems to be standard for the experts of the theory of $(\phi, \Gamma)$-modules. Then, we prove that this isomorphism satisfies the de Rham condition (vi) by establishing the “explicit reciprocity law” of our “Coleman homomorphism” using our theory of Bloch-Kato’s exponential map developed in §2.2, which seems to be a new contribution of our article in the theory of $(\phi, \Gamma)$-modules.

4.1. Construction of the $\varepsilon$-isomorphism. We first recall the theory of analytic Iwasawa cohomology of $(\phi, \Gamma)$-modules over the Robba ring after [Po93b] and [KPX12]. Let $A(\Gamma) := \mathbb{Z}_p[[\Gamma]]$ be the Iwasawa algebra of $\Gamma$ with coefficients in $\mathbb{Z}_p$ and let $m$ be the Jacobson radical of $A(\Gamma)$. For each $n \geq 1$, define a $\mathbb{Q}_p$-affinoid algebra $R^{[1/p^n, \infty]}(\Gamma) := (A(\Gamma)[[\frac{m}{p}]]^{\wedge})[1/p]$ where, for any ring $R$, let denote by $R^{\wedge}$ for the $p$-adic completion. Let $X_n := \text{Max}(R^{[1/p^n, \infty]}(\Gamma))$ be the associated affinoid. Define $X := \bigcup_{n \geq 1} X_n$, which is a disjoint union of open unit discs. For $n \geq 1$, consider the rank one $(\phi, \Gamma)$-module

$$Dfm_n := R^{[1/p^n, \infty]}(\Gamma) \hat{\otimes} \mathbb{Q}_p R e = R_{R^{[1/p^n, \infty]}(\Gamma)} e$$

with

$$\phi(1 \hat{\otimes} e) = 1 \hat{\otimes} e \text{ and } \gamma(1 \hat{\otimes} e) = [\gamma]^{-1} \hat{\otimes} e \text{ for } \gamma \in \Gamma.$$ 

Put $Dfm := \lim_{n \to \infty} Dfm_n$; this is a $(\phi, \Gamma)$-module over the relative Robba ring over $X$. For $M$ a $(\phi, \Gamma)$-module over $R_A$, we define the cyclotomic deformation of $M$ to be

$$Dfm(M) := \lim_{n \to \infty} Dfm_n(M).$$
with
\[
Dfm_n(M) := M \otimes_R Dfm_n \sim M \otimes \mathcal{R}_A^{[1/p^n, \infty]}(\Gamma)e,
\]
which is a \((\varphi, \Gamma)\)-module over the relative Robba ring over Max\((A) \times X\). This \((\varphi, \Gamma)\)-module is the universal cyclotomic deformation of \(M\) in the sense that, for each continuous homomorphism \(\delta_0 : \Gamma \to A^\times\), we have a natural isomorphism
\[
Dfm(M) \otimes_{R^\infty_A(\Gamma), f_{\delta_0}} A \sim M(\delta_0) : (x \otimes e) \otimes a \mapsto f_{\delta_0}(\eta)axe_{\delta_0}
\]
for \(x \in M\) and \(e \in R^\infty_A(\Gamma)e\) and \(a \in A\), where
\[
f_{\delta_0} : R^\infty_A(\Gamma) \to A
\]
is the continuous \(A\)-algebra homomorphism defined by
\[
f_{\delta_0}([\gamma]) := \delta_0(\gamma)^{-1}
\]
for \(\gamma \in \Gamma\) \((M(\delta_0) := M \otimes_A Ae_{\delta_0} = Me_{\delta_0}\) is defined by \(\gamma(xe_{\delta_0}) := \delta_0(\gamma)x e_{\delta_0}\) for \(x \in M\) and \(\gamma \in \Gamma\).

By Theorem 4.4.8 of [KPX12], we have a natural quasi-isomorphism of \(R^\infty_A(\Gamma)\)-modules
\[
g_\gamma : C^\bullet_{\psi, \gamma}(Dfm(M)) \sim C^\bullet_{\psi}(M).
\]
This quasi-isomorphism is obtained as a composition of a (system of) quasi-isomorphisms
\[
C^\bullet_{\psi, \gamma}(Dfm_n(M)) \sim C^\bullet_{\psi}(M) \hat{\otimes}_{R^\infty_A(\Gamma)} \mathcal{R}_A^{[1/p^n, \infty]}(\Gamma)
\]
which are naturally induced by the following diagrams of short exact sequences of \(\mathcal{R}_A^{[1/p^n, \infty]}(\Gamma)\)-modules for \(n \geq 1\) (put \(\log_0(a) := \frac{\log(a)}{p^{p(a)}}\) for \(a \in \mathbb{Z}_p^\times\),
\[
(30)\quad
\begin{array}{cccccc}
0 & \longrightarrow & Dfm_n(M) & \xrightarrow{\gamma^{-1}} & Dfm_n(M) & \xrightarrow{f} \mathcal{R}_A^{[1/p^n, \infty]}(\Gamma) & \longrightarrow & 0 \\
\longrightarrow & \longrightarrow & \psi^{-1} & \psi^{-1} & \psi^{-1} & & \\
0 & \longrightarrow & Dfm_n(M) & \xrightarrow{\gamma^{-1}} & Dfm_n(M) & \xrightarrow{f} \mathcal{R}_A^{[1/p^n, \infty]}(\Gamma) & \longrightarrow & 0,
\end{array}
\]
where
\[
f_\gamma \left( \sum_i x_i \hat{\otimes} \eta_i e \right) := \frac{1}{|\Gamma_{\text{tor}}|\log_0(\chi(\gamma))} \sum_i x_i \hat{\otimes} \eta_i
\]
for \(x_i \in M\), \(\eta_i e \in \mathcal{R}_A^{[1/p^n, \infty]}(\Gamma)e\) with the inverse of the natural quasi-isomorphism
\[
C^\bullet_{\psi}(M) \sim \lim_{\longleftarrow} C^\bullet_{\psi}(M \otimes \mathcal{R}_A^\infty(\Gamma) \mathcal{R}_A^{[1/p^n, \infty]}(\Gamma)) \sim \lim_{\longleftarrow} C^\bullet_{\psi}(M \otimes \mathcal{R}_A^\infty(\Gamma) \mathcal{R}_A^{[1/p^n, \infty]}(\Gamma)),
\]
(see Theorem 4.4.8 of [KPX12] and Theorem 2.8 (3) of [Po13b] for the proof). This quasi-isomorphism is canonical in the sense that, for another \(\gamma' \in \Gamma\) whose image
in \( \Gamma/\Delta \) is a topological generator, we have the following commutative diagram

\[
\begin{array}{ccc}
C_{\psi,\gamma}(\text{Dfm}(M)) & \xrightarrow{g_\gamma} & C_{\psi}(M) \\
\downarrow^{\iota_{\gamma,\gamma}} & & \downarrow^{\text{id}} \\
\n & & \\
C_{\psi,\gamma}(\text{Dfm}(M)) & \xrightarrow{g_\gamma} & C_{\psi}(M).
\end{array}
\] (31)

Moreover, using the natural isomorphism \( \text{Dfm}(M) \otimes_{\mathcal{R}_A}(\Gamma), f_{\delta_0} A \sim M(\delta_0) \) and the quasi-isomorphism \( g_\gamma \), we obtain a natural quasi-isomorphism

\[
g_{\gamma,\delta_0} : C_{\psi,\gamma}(M(\delta_0)) \sim C_{\psi,\gamma}(\text{Dfm}(M) \otimes_{\mathcal{R}_A}(\Gamma), f_{\delta_0} A) \sim C_{\psi,\gamma}(\text{Dfm}(M)) \otimes_{\mathcal{R}_A}(\Gamma), f_{\delta_0} A \sim C_{\psi}(M) \otimes_{\mathcal{R}_A}(\Gamma), f_{\delta_0} A,
\] (32)

where the second isomorphism follows from the fact that any \((\varphi, \Gamma)\)-module \( M_0 \) over \( \mathcal{R}_{A_0} \) is flat over \( A_0 \) for any \( A_0 \) (see Corollary 2.1.7 of [KPX12]). This quasi-isomorphism can be written in a more explicit way as follows. To recall this, we see \( A \) as a \( \mathcal{R}_A(\Gamma) \)-module by the map \( f_{\delta_0} \). Then, we can take the following projective resolution of \( A \),

\[
0 \to \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{1,\gamma}} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{2,\gamma}} A \to 0,
\]

where \( p_{\delta_0} := \frac{1}{|\Delta|} \sum_{\sigma \in \Delta} \delta_0^{-1}(\sigma)\sigma \in \mathcal{R}_A^\infty(\Gamma) \) (this is an idempotent ) and

\[
d_{1,\gamma}(\eta) := (\delta_0(\gamma)[\gamma] - 1)\eta, \text{ and } d_{2,\gamma}(\eta) := \frac{1}{|\Gamma| \log_0(\chi(\gamma))} f_{\delta_0}(\eta).
\]

This resolution induces a natural quasi-isomorphism

\[
C_{\psi}(M) \otimes_{\mathcal{R}_A^\infty(\Gamma)} [\mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{1,\gamma}} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0}] \sim C_{\psi}(M) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0} A} A.
\]

Moreover, using the natural isomorphism

\[
M \otimes_{\mathcal{R}_A(\Gamma)} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \sim M(\delta) : m \otimes \lambda p_{\delta_0} \mapsto \lambda p_{\delta_0}(me_{\delta_0}),
\]

we obtain a natural quasi-isomorphism

\[
C_{\psi}(M) \otimes_{\mathcal{R}_A(\Gamma)} [\mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0} \xrightarrow{d_{1,\gamma}} \mathcal{R}_A^\infty(\Gamma) \cdot p_{\delta_0}] \sim C_{\psi,\gamma}(M(\delta_0)).
\]

Composing both, we obtain a natural quasi-isomorphism

\[
C_{\psi}(M) \otimes_{\mathcal{R}_A^\infty(\Gamma), f_{\delta_0} A} A \sim C_{\psi,\gamma}(M(\delta_0)),
\]

which is easily to see that this is equal to \( g_{\gamma,\delta_0} \).

Using the theory of analytic Iwasawa cohomology, we can describe the fundamental line \( \Delta_{\mathcal{R}_A^\infty(\Gamma)}(\text{Dfm}(M)) \) as follows. The quasi-isomorphism \( g_\gamma \) and the quasi-isomorphism \( C_{\psi,\gamma}(\text{Dfm}(M)) \sim C_{\psi}(\text{Dfm}(M)) \) induce a natural isomorphism in \( \mathcal{P}_{\mathcal{R}_A^\infty(\Gamma)} \)

\[
\Delta_{\mathcal{R}_A^\infty(\Gamma),1}(\text{Dfm}(M)) \sim \text{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_{\psi,\gamma}(\text{Dfm}(M))) \sim \text{Det}_{\mathcal{R}_A^\infty(\Gamma)}(C_{\psi}(M)).
\]
Moreover, because we have
\[
\Delta_{\mathcal{R}_A(\Gamma)}(\text{Dfm}(M)) = \lim_{\to} \Delta_{A,2}(M) \otimes_A \mathcal{R}_A^{1/p,\infty}(\Gamma) \mathcal{e}_A^{\otimes_\mathbb{R}}(\Gamma)
\]

\[
\cong \Delta_{A,2}(M) \otimes_A \mathcal{R}_A^{\infty}(\Gamma),
\]

where the last isomorphism is just the division by $\mathcal{e}_A^{\otimes_\mathbb{R}}$, we obtain a canonical isomorphism
\[
(\Delta_{\mathcal{R}_A(\Gamma)}(\text{Dfm}(M))) \cong \mathcal{R}_A^{\infty}(\Gamma).
\]

Therefore, taking a product with $(\Delta_{A,2}(M) \otimes_A \mathcal{R}_A^{\infty}(\Gamma))^{-1}$, defining an isomorphism
\[
\Delta_{\mathcal{R}_A(\Gamma)}(\text{Dfm}(M)) \cong 1_{\mathcal{R}_A^{\infty}(\Gamma)}
\]

by the argument after (33), it suffices to define an isomorphism
\[
\epsilon_{\mathcal{R}_A(\Gamma)}(\text{Dfm}(\mathcal{R}_A(\delta_\lambda))) : \Delta_{\mathcal{R}_A(\Gamma)}(\text{Dfm}(\mathcal{R}_A(\delta_\lambda))) \cong 1_{\mathcal{R}_A^{\infty}(\Gamma)}
\]

because we have $\Delta_{A,2}(\mathcal{R}_A(\delta_\lambda)) = \Delta_{A,2}(Ae^\lambda)$. We construct this isomorphism $\epsilon_5$ by the following steps, which are based on the re-interpretation of the theory of the Coleman homomorphism in terms of the $p$-adic Fourier transform.

Let $\mathcal{L}(\mathbb{Z}_p, A)$ be the set of $A$-valued locally analytic functions on $\mathbb{Z}_p$, and define the action of $(\varphi, \psi, \Gamma)$ on it by
\[
\varphi(f)(y) := f(\frac{y}{p})(y \in p\mathbb{Z}_p) \text{ and } \varphi(f)|_{\mathbb{Z}_p^\times} := 0,
\]
\[
\psi(f)(y) := f(py) \text{ and } \gamma(f)(y) := \frac{1}{\lambda(\gamma)}f(\frac{y}{\lambda(\gamma)}) (\gamma \in \Gamma).
\]

One has a $(\varphi, \psi, \Gamma)$-equivariant $A$-linear surjection which we call the Colmez transform
\[
\text{Col} : \mathcal{R}_A \rightarrow \mathcal{L}(\mathbb{Z}_p, A)
\]

defined by
\[
\text{Col}(f(\pi))(y) := \text{Res}_0((1 + \pi)^y f(\pi)) \frac{d\pi}{(1 + \pi)},
\]
where \( \text{Res}_0 : \mathcal{R}_A \to A \) is defined by \( \text{Res}_0(\sum_{n \in \mathbb{Z}} a_n \pi^n) := a_{-1} \) (we remark that \( \text{Col} \) depends on the choice of the parameter \( \pi \), i.e. of \( \zeta \)). By this map, we obtain the following short exact sequence

\[
0 \to \mathcal{R}_A^\infty \to \mathcal{R}_A \xrightarrow{\text{Col} \otimes e_{\delta}} \text{LA}(\mathbb{Z}_p, A) \to 0.
\]

Twisting the action of \((\varphi, \psi, \Gamma)\) by \(\delta_\lambda\), we obtain the following \((\varphi, \psi, \Gamma)\)-equivariant exact sequence

\[
0 \to \mathcal{R}_A^\infty(\delta_\lambda) \to \mathcal{R}_A(\delta_\lambda) \xrightarrow{\text{Col} \otimes e_{\delta}} \text{LA}(\mathbb{Z}_p, A)(\delta_\lambda) \to 0,
\]

from which we obtain the following exact sequences of complexes

\[
0 \to C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda)) \to C_\psi^\bullet(\mathcal{R}_A(\delta_\lambda)) \to C_\psi^\bullet(\text{LA}(\mathbb{Z}_p, A)(\delta_\lambda)) \to 0.
\]

For each \( k \geq 0 \), we define the algebraic function

\[
y^k : \mathbb{Z}_p \to A : a \mapsto a^k,
\]

then \( Ay^k e_{\delta_\lambda} \subseteq \text{LA}(\mathbb{Z}_p, A)(\delta_\lambda) \) is a \( \psi \)-stable sub \( \mathcal{R}_A^\infty(\Gamma) \)-module. The natural inclusion

\[
C_\psi^\bullet(\oplus_{0 \leq k \leq N} Ay^k e_{\delta_\lambda}) \hookrightarrow C_\psi^\bullet(\text{LA}(\mathbb{Z}_p, A)(\delta_\lambda))
\]

is quasi-isomorphism for sufficiently large \( N \) by Lemma 2.9 of [Ch13].

Because we have \( Ay^k e_{\delta_\lambda}[0] \in D^b_{\text{perf}}(\mathcal{R}_A^\infty(\Gamma)) \) for \( k \geq 0 \), we have

\[
C_\psi^\bullet(\oplus_{0 \leq k \leq N} Ay^k e_{\delta_\lambda}) \in D^b_{\text{perf}}(\mathcal{R}_A^\infty(\Gamma))
\]

and we have a natural trivialization

\[
\text{Det}(\mathcal{R}_A^\infty(\Gamma))(C_\psi^\bullet(\oplus_{0 \leq k \leq N} Ay^k e_{\delta_\lambda})) \sim 1_{\mathcal{R}_A^\infty(\Gamma)},
\]

defined by the isomorphism (21) in §3.1.

By (35), (36), (37), we have

\[
C_\psi^\bullet(\text{LA}(\mathbb{Z}_p, A)(\delta_\lambda)) \in D^b_{\text{perf}}(\mathcal{R}_A^\infty(\Gamma)),
\]

and we obtain an isomorphism

\[
\iota_0 : \text{Det}(\mathcal{R}_A^\infty(\Gamma))(C_\psi^\bullet(\text{LA}(\mathbb{Z}_p, A)(\delta_\lambda))) \sim 1_{\mathcal{R}_A^\infty(\Gamma)}.
\]

Because \( C_\psi^\bullet(\text{LA}(\mathbb{Z}_p, A)(\delta_\lambda)) \) and \( C_\psi^\bullet(\mathcal{R}_A(\delta_\lambda)) \) are both perfect complexes, we also have

\[
C_\psi^\bullet(\mathcal{R}_A^\infty(\Gamma)) \in D^b_{\text{perf}}(\mathcal{R}_A^\infty(\Gamma))
\]

by the exact sequence (34). Because \( \mathcal{R}_A^\infty(\delta_\lambda) \xrightarrow{1-\psi} \mathcal{R}_A^\infty(\delta_\lambda) \) is surjection by Lemma 2.9 (v) of [Ch13], we have a natural quasi-isomorphism

\[
\mathcal{R}_A^\infty(\delta_\lambda)^{\psi-1}[-1] \sim C_\psi^\bullet(\mathcal{R}_A^\infty(\delta_\lambda)).
\]
Using this quasi-isomorphism, the exact sequence (33) induces a canonical isomorphism

\[ \mu_1 : \text{Det}_A(\Gamma)(C^\bullet(\mathcal{R}_A(\delta_\lambda))) \xrightarrow{\sim} \text{Det}_A(\Gamma)(C^\bullet(\mathcal{R}_A^\infty(\delta_\lambda))) \cdot \text{Det}_A(\Gamma)(C^\bullet(\mathcal{LA}(\mathbb{Z}_p, A)(\delta_\lambda))) \xrightarrow{\sim} \text{Det}_A(\Gamma)(C^\bullet(\mathcal{R}_A(\delta_\lambda))) \xrightarrow{\sim} \text{Det}_A(\Gamma)(\mathcal{R}_A^\infty(\delta_\lambda)^{\psi=1}[0])^{-1}, \]

where the second isomorphism is induced by the isomorphism \( \iota_0 \).

We next consider \( C^\bullet(\mathcal{R}_A^\infty(\delta_\lambda)) \). For a \( \mathcal{R}_A^\infty(\Gamma) \)-module \( M \) with linear \( \varphi, \psi \)-actions, we define a complex

\[ C^\bullet_\psi(M) := [M \xrightarrow{\psi} M] \in D^{[1,2]}(\mathcal{R}_A^\infty(\Gamma)), \]

and define a map of complexes \( \alpha_M : C^\bullet_\psi(M) \to C^\bullet_\varphi(M) \) by

\[
\begin{align*}
C^\bullet_\psi(M) : [M & \xrightarrow{\psi} M] \\
\downarrow_{\alpha_M} & \downarrow_{1-\varphi} \downarrow_{\text{id}_M} \\
C^\bullet_\varphi(M) : [M & \xrightarrow{\psi} M].
\end{align*}
\]

For \( N \geq 0 \), put \( D_N := \oplus_{0 \leq k \leq N} A t^k e_{\delta_\lambda} \). Consider the following diagram of exact sequences

\[
\begin{array}{ccccccccc}
0 & \rightarrow & C^\bullet_\psi(D_N) & \rightarrow & C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)) & \rightarrow & C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)/D_N) & \rightarrow & 0 \\
\alpha(D_N) \downarrow & & \alpha_{\mathcal{R}_A^\infty(\delta_\lambda)} \downarrow & & \alpha_{\mathcal{R}_A^\infty(\delta_\lambda)/D_N} \downarrow & & \\
0 & \rightarrow & C^\bullet_\psi(D_N) & \rightarrow & C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)) & \rightarrow & C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)/D_N) & \rightarrow & 0.
\end{array}
\]

The complex \( C^\bullet_\psi(D_N) \) is acyclic because \( \psi : At^k e_{\delta_\lambda} \to At^k e_{\delta_\lambda} \) is isomorphism for any \( k \geq 0 \). Because the map \( 1 - \varphi : \mathcal{R}_A^\infty(\delta_\lambda)/D_N \to \mathcal{R}_A^\infty(\delta_\lambda)/D_N \) is isomorphism for sufficiently large \( N \) by Lemma 2.9 (ii) of [Ch13], the map \( \alpha_{\mathcal{R}_A^\infty(\delta_\lambda)/D_N} \) is also isomorphism for sufficiently large \( N \). From these two facts and from the above diagram (41), for sufficiently large \( N \), we obtain the following distinguished triangle

\[
C^\bullet_\psi(D_N) \rightarrow C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)) \xrightarrow{\alpha_{\mathcal{R}_A^\infty(\delta_\lambda)}} C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)) \xrightarrow{[+1]} \cdots.
\]

Moreover, because we have the following diagram

\[
\begin{array}{ccc}
\mathcal{R}_A^\infty(\delta_\lambda)^{\psi=1}[-1] & \xrightarrow{\sim} & C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda)) \\
\downarrow_{1-\varphi} & & \downarrow_{\alpha_{\mathcal{R}_A^\infty(\delta_\lambda)}} \\
\mathcal{R}_A^\infty(\delta_\lambda)^{\psi=0}[-1] & \xrightarrow{\sim} & C^\bullet_\psi(\mathcal{R}_A^\infty(\delta_\lambda))
\end{array}
\]

the above triangle becomes

\[
\begin{array}{ccccccccc}
C^\bullet_\psi(D_N) & \rightarrow & \mathcal{R}_A^\infty(\delta_\lambda)^{\psi=1}[-1] & \xrightarrow{1-\varphi} & \mathcal{R}_A^\infty(\delta_\lambda)^{\psi=0}[-1] & \xrightarrow{[+1]} & \cdots.
\end{array}
\]
Because we have \( At^k e_{\delta_i}[0] \in D^b_{\perf}(R^\infty_A(\Gamma)) \), we also have \( C^\bullet_{\psi}(D_N) \in D^b_{\perf}(R^\infty_A(\Gamma)) \) and we similarly obtain a canonical isomorphism

\[
\tau_2 : \text{Det}_{R^\infty_A(\Gamma)}(C^\bullet_{\psi}(D_N)) \rightarrow 1_{R^\infty_A(\Gamma)}.
\]

By these arguments, the triangle \([42]\) naturally induces the following isomorphism

\[
\tau_3 : \text{Det}_{R^\infty_A(\Gamma)}(R^\infty_A(\delta))^{-1} \rightarrow \text{Det}_{R^\infty_A(\Gamma)}(C^\bullet_{\psi}(D_N)) \cdot \text{Det}_{R^\infty_A(\Gamma)}(R^\infty_A(\delta))^{-1} \rightarrow \text{Det}_{R^\infty_A(\Gamma)}(R^\infty_A(\delta))^{-1},
\]

where the second isomorphism is induced by \( \tau_2 \).

Because one has a natural \( R^\infty_A(\Gamma) \)-linear isomorphism

\[
R^\infty_A(\Gamma) \rightarrow (R^\infty_A)^{\psi = 0} : \eta \mapsto \eta \cdot (1 + \pi)
\]

(we remark that this isomorphism depends on the choice of \( \pi \), i.e. the choice of \( \zeta \)), we can also define an isomorphism \( R^\infty_A(\Gamma)e_{\delta, \lambda} \rightarrow R^\infty_A(\delta)^{\psi = 0} : \eta \cdot e_{\delta, \lambda} \mapsto (\eta \cdot (1 + \pi)) \cdot e_{\delta, \lambda} \), and this induces an isomorphism

\[
\tau_4 : \text{Det}_{R^\infty_A(\Gamma)}(R^\infty_A(\delta))^{-1} \rightarrow \text{Det}_{R^\infty_A(\Gamma)}(R^\infty_A(\delta))^{-1}.
\]

Composing the isomorphisms \( \tau_1, \tau_3, \tau_4 \), we finally obtain an isomorphism

\[
\tau_5 := \tau_4 \circ \tau_3 \circ \tau_1 : \text{Det}_{R^\infty_A(\Gamma)}(C^\bullet_{\psi}(R_A(\delta))) \rightarrow \text{Det}_{R^\infty_A(\Gamma)}(R^\infty_A(\delta))^{-1}.
\]

**Definition 4.1.** Using the identification \([33]\), we define the \( \varepsilon \)-isomorphism by

\[
\varepsilon_{R^\infty_A(\Gamma), \zeta}(\text{Dfm}(R_A(\delta))) \boxtimes \text{id}_{A'} = \varepsilon_{R^\infty_A(\Gamma), \zeta}(\text{Dfm}(R_A(\delta)))
\]

under the canonical isomorphism

\[
\Delta_{R^\infty_A(\Gamma)}(\text{Dfm}(R_A(\delta))) \boxtimes A' \rightarrow \Delta_{R^\infty_A(\Gamma)}(\text{Dfm}(R_A(\delta) \boxtimes A'))
\]

\[
\rightarrow \Delta_{R^\infty_A(\Gamma)}(\text{Dfm}(R_A(\delta))),
\]

where the last isomorphism is induced by the isomorphism

\[
R_A(\delta) \boxtimes A' \rightarrow R_A(\delta) \rightarrow g(\pi)e_{\delta_{\lambda}} \hat{a} \mapsto ag^f(\pi)e_{\delta_{f(\lambda)}},
\]

where we define \( g^f(\pi) := \sum_{n \in \mathbb{Z}} f(a_n)\pi^n \in R_{A'} \) for \( g(\pi) = \sum_{n \in \mathbb{Z}} a_n\pi^n \in R_A \).

Next, we consider a rank one \((\varphi, \Gamma)\)-module over \( R_A \) of the form \( R_A(\delta) \) for a continuous homomorphism \( \delta : \mathbb{Q}_p^\times \rightarrow A^\times \). Set

\[
\lambda := \delta(p) \text{ and } \delta_0 := \delta|_{\mathbb{Q}_p^\times}.
\]
which we freely see as a homomorphism $\delta_0 : \Gamma \to A^\times$ by identifying $\chi : \Gamma \sim \mathbb{Z}_p^\times$.

We define a unique continuous $A$-algebra homomorphism

$$f_{\delta_0} : \mathcal{R}_A(\Gamma) \to A$$

such that $f_{\delta_0}(\gamma) = \delta_0(\gamma)^{-1}$ for $\gamma \in \Gamma$. Then, we have a canonical isomorphism

$$\text{Dfm}(\mathcal{R}_A(\delta_\lambda)) \otimes_{\mathcal{R}_A(\Gamma), f_{\delta_0}} A \sim \mathcal{R}_A(\delta)$$

defined by

$$(f(\pi)e_{\delta_0} \otimes e_\delta) \otimes a := af_{\delta_0}(\eta)f(\pi)e_\delta$$

for $f(\pi) \in \mathcal{R}_A$, $\eta \in \mathcal{R}_A(\Gamma)$, $a \in A$. Hence, we obtain a natural isomorphism

$$\Delta_{\mathcal{R}_A(\Gamma)}(\text{Dfm}(\mathcal{R}_A(\delta_\lambda))) \otimes_{\mathcal{R}_A(\Gamma), f_{\delta_0}} A \sim \Delta_A(\mathcal{R}_A(\delta)).$$

**Definition 4.2.** We define an isomorphism

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta)) : \Delta_A(\mathcal{R}_A(\delta)) \sim 1_A$$

by

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta)) := \varepsilon_{\mathcal{R}_A(\Gamma), \zeta}(\text{Dfm}(\mathcal{R}_A(\delta_\lambda))) \otimes 1$$

under the above isomorphism.

Next, we consider a rank one $(\varphi, \Gamma)$-module of the form $\mathcal{R}_A(\delta) \otimes A L$ for an invertible $A$-module $L$.

**Lemma 4.3.** Let $M$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_A$ (of any rank), and let $L$ be an invertible $A$-module. Then, there exist a natural $A$-linear isomorphism

$$\Delta_A(M \otimes A L) \sim \Delta_A(M).$$

**Proof.** The natural isomorphism $C_{\varphi,\gamma}^\bullet (M \otimes A L) \sim C_{\varphi,\gamma}^\bullet (M) \otimes A L$ induces an isomorphism

$$\Delta_{A,1}(M \otimes A L) \sim \Delta_{A,1}(M) \cdot (L^{\otimes -r_M}, 0).$$

Because we also have a natural isomorphism $L_A(M \otimes A L) \sim L_A(M) \otimes A L^{\otimes r_M}$, we obtain a natural isomorphism

$$\Delta_{A,2}(M \otimes A L) \sim \Delta_{A,2}(M) \cdot (L^{\otimes r_M}, 0).$$

Then, the isomorphism in the lemma is obtained by taking the products of these isomorphisms with the canonical isomorphism $(L^{\otimes r_M}, 0) \cdot (L^{\otimes -r_M}, 0) \sim 1_A$. \hfill $\square$

**Definition 4.4.** We define an isomorphism

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta) \otimes A L) : \Delta_A(\mathcal{R}_A(\delta) \otimes A L) \sim 1_A$$

by

$$\varepsilon_{A,\zeta}(\mathcal{R}_A(\delta) \otimes A L) := \varepsilon_{A,\zeta}(\mathcal{R}_A(\delta))$$

under the above isomorphism $\Delta_A(M \otimes A L) \sim \Delta_A(M)$. 

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Finally, let $M$ be a rank one $(\varphi, \Gamma)$-module over $\mathcal{R}_A$. By Theorem 2.8, there exists a unique pair $(\delta, L)$ such that $g : M \xrightarrow{\sim} \mathcal{R}(\delta) \otimes_A L$. This isomorphism induces an isomorphism $g_* : \Delta_A(M) \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta) \otimes_A L)$.

**Definition 4.5.** Under the above situation, we define

$$\varepsilon_{A, L}(M) := \varepsilon_{A, L}(\mathcal{R}_A(\delta) \otimes_A L) \circ g_* : \Delta_A(M) \xrightarrow{\sim} 1_A.$$ 

**Lemma 4.6.** The isomorphism $\varepsilon_{A, L}(M)$ is well-defined, i.e. does not depend on $g$.

**Proof.** Because we have $\text{Aut}(\mathcal{R}_A(\delta) \otimes_A L) = A^\times$ (where $\text{Aut}(M)$ is the group of automorphisms of $M$ as $(\varphi, \Gamma)$-module over $\mathcal{R}_A$), it suffices to show the following lemma. □

**Lemma 4.7.** Let $M$ be a $(\varphi, \Gamma)$-module over $\mathcal{R}_A$. For $a \in A^\times$, let denote by $g_a : M \xrightarrow{\sim} M : x \mapsto ax$. Then, we have

$$(g_a)_* = \text{id}_{\Delta_A(M)}.$$ 

**Proof.** This lemma immediately follows from the fact that $g_a$ induces $\Delta_{1, A}(M) \xrightarrow{\sim} \Delta_{A, 1}(M) : x \mapsto a^{-1}x$ and $\Delta_{A, 2}(M) : x \mapsto a^{\pi}x$ by definition. □

**Remark 4.8.** For another choice of a $\mathbb{Z}_p$-basis $\zeta^a$ of $\mathbb{Z}_p(1) (a \in \mathbb{Z}_p^\times)$, we similarly define an isomorphism $\varepsilon_{A, \zeta^a}(M) : \Delta_A(M) \xrightarrow{\sim} 1_A$ using the parameter $\pi_{\zeta^a}$ instead of $\pi$.

**Remark 4.9.** By definition, it is clear that $\varepsilon_{A, \zeta}(M)$ constructed above satisfies the condition (i) in Conjecture 3.9. Moreover, it seems to be possible to directly prove the condition (iii), (iv), (v) of Conjecture 3.9 (more precisely, (iii) seems to be very easy to prove, and (v) also seems to be easy if one knows the Kato’s construction of $\varepsilon_{A, L}(T)$ for rank one $T$ in [Kato93]), and (iv) seems to be a little bit difficult but interesting to directly prove because it seems to be related to a generalization of Perrin-Riou’s $\text{Rec}(V)$ in our setting. In the next subsection, we prove these conditions using density arguments in the process of verifying the condition (vi).

**Remark 4.10.** Define $\mathcal{O}_E := \{ \sum_{n \in \mathbb{Z}} a_n \pi^n | a_n \in \mathbb{Z}_p, a_{-n} \to 0 (n \to +\infty) \}$, $\mathcal{O}_{E^+} := \mathbb{Z}_p[[\pi]]$, and $\mathcal{O}_{E^+, A} := \mathcal{O}_{E^+} \hat{\otimes}_{\mathbb{Z}_p} A$. Define $\mathcal{C}^0(\mathbb{Z}_p, A)$ to be the $A$-module of $A$-valued continuous functions on $\mathbb{Z}_p$. Using the similar exact sequence

$$0 \to \mathcal{O}_{E^+, A} \to \mathcal{O}_{E, A} \xrightarrow{\text{Col}} \mathcal{C}^0(\mathbb{Z}_p, A) \to 0,$$

and using the equivalence between the category of $A$-representations of $G_{\mathbb{Q}_p}$ with that of étale $(\varphi, \Gamma)$-modules over $\mathcal{O}_{E, A}$ [DeC01], it seems to be possible to define an epsilon isomorphism $\varepsilon_{A, \zeta}(\Lambda(\delta))$ for any $\delta : G_{\mathbb{Q}_p}^{ab} \to \Lambda^\times$ in the same way as the definition of $\varepsilon_{A, L}(\mathcal{R}_A(\delta))$. Using this $\varepsilon$-isomorphism, it is clear that our epsilon isomorphism $\varepsilon_{A, L}(\mathcal{R}_A(\delta))$ satisfies the condition (v) in Conjecture 3.9. Moreover,
if one know the Kato’s construction of his epsilon isomorphism, one can easily compare the above $\varepsilon_{A,\xi}(\Lambda(\delta))$ with Kato’s one.

### 4.2. Verification of the conditions (iii), (iv), (v), (vi).

In this final subsection, we prove that our $\varepsilon$-isomorphism $\varepsilon_{A,\xi}(M)$ constructed in the previous subsection satisfies the conditions (iii), (iv), (v), (vi) of Conjecture 3.9. In fact, the main part is to prove the condition (vi); the other conditions follow from it using density arguments.

Therefore, we mainly concentrate on the case where $A = L$ is a finite extension of $\mathbb{Q}_p$ in this subsection. Before verifying the condition (vi), we describe the isomorphism $\varepsilon_{L,\xi}(\mathcal{R}_L(\delta)) : \Delta_L(\mathcal{R}_L(\delta)) \xrightarrow{} \mathbb{1}_L$ for any continuous homomorphism $\delta = \delta_1 \delta_0 : \mathbb{Q}_p^\times \rightarrow \mathcal{L}_x^\times$ in a more explicit way. To do so, we need to describe the isomorphisms $\iota_{i,\delta} := \iota_i \otimes_{\mathcal{R}_L^\infty(\Gamma), f_{\delta_0}} \text{id}_L$ in more explicit way as follows.

For a $\mathcal{R}_{\Lambda}^\infty(\Gamma)$-module $N$, define a $\Gamma$-module $N(\delta_0) := Ne_{\delta_0}$ such that $\gamma(xe_{\delta_0}) = \delta_0(\gamma)(\gamma \cdot x)e_{\delta_0}$. Then, we have a natural quasi-isomorphism

$$N[-1] \otimes_{\mathcal{R}_L^\infty(\Gamma), f_{\delta_0}} L \xrightarrow{} N \otimes_{\mathcal{R}_L^\infty(\Gamma)} [\mathcal{R}_L^\infty(\Gamma)]p_{\delta_0} \xrightarrow{} \mathcal{R}_L^\infty(\Gamma)p_{\delta_0} \xrightarrow{} C^\bullet(\mathcal{N}(\delta_0)).$$

Hence, if $N[0] \in \mathcal{D}_b^b(\mathcal{R}_L^\infty(\Gamma))$, then we obtain a natural isomorphism

$$\text{Det}_L(N[-1]) \otimes_{\mathcal{R}_L^\infty(\Gamma), f_{\delta_0}} L \xrightarrow{} \text{Det}_L(C^\bullet(\mathcal{N}(\delta_0))) \xrightarrow{} \prod_{i=0,1} \text{Det}_L(H^i(\mathcal{N}(\delta_0)))^{(-1)^i}.$$}

Moreover, if $N$ is also equipped with a commuting linear action of $\psi$ such that $C_\psi(\Lambda) \in \mathcal{D}_b^b(\mathcal{R}_L^\infty(\Gamma))$, then we obtain a natural isomorphism

$$\text{Det}_L(C_\psi^\bullet(N)) \otimes_{\mathcal{R}_L^\infty(\Gamma), f_{\delta_0}} L \xrightarrow{} \text{Det}_L(C_\psi^\bullet(\mathcal{N}(\delta_0))) \xrightarrow{} \prod_{i=0}^{2} \text{Det}_L(H^i_{\psi,\gamma}(\mathcal{N}(\delta_0)))^{(-1)^i}.$$}

In particular, the isomorphism $\iota_{5,\delta} := \iota_{5} \otimes_{\mathcal{R}_L^\infty(\Gamma), f_{\delta_0}} L$ can be written as

$$\iota_{5,\delta} : \prod_{i=0}^{2} \text{Det}_L(H^i_{\psi,\gamma}(\mathcal{R}_L(\delta)))^{(-1)^i} \xrightarrow{} \text{Det}_{\mathcal{R}_L^\infty(\Gamma)}(\mathcal{R}_L^\infty(\Gamma)e_{\delta_0})^{-1} \otimes_{\mathcal{R}_L^\infty(\Gamma), f_{\delta_0}} L \xrightarrow{} \text{Det}_L(Le_{\delta})^{-1},$$

where the last isomorphism is induced by the isomorphism

$$\mathcal{R}_L^\infty(\Gamma)e_{\delta_0} \otimes_{\mathcal{R}_L^\infty(\Gamma), f_{\delta_0}} L \xrightarrow{} Le_{\delta} : (\eta e_{\delta_0}) \otimes a \mapsto a f_{\delta_0}(\eta)e_{\delta}.$$}

Therefore, to verify the condition (vi) when $\mathcal{R}_L(\delta)$ is de Rham, we need to relate the map $\iota_{5,\delta}$ with the Bloch-Kato’s exponential map or the dual exponential map.

To do so, we divide into the following three cases:

1. $\delta \neq x^{-k}, x^{k+1}|x|$ for any $k \in \mathbb{Z}_{\geq 0}$ (which we call the generic case).
2. $\delta = x^{-k}$ for $k \geq 0$.
3. $\delta = x^{k+1}|x|$ for $k \geq 0$. 50
We first verify the condition (vi) in the generic case via establishing a kind of explicit reciprocity laws (see Proposition 4.12 and Proposition 4.18). Then, we prove the condition (iii), (iv) and (v) using the generic case by density argument. Finally, we prove (vi) in the case (2) via direct calculations, and reduce the case (3) to the case (2) using the duality condition (iv).

In the remaining parts, we freely use the results of Colmez and Chenevier concerning the calculations of cohomologies $H^i_\psi,\gamma(L(\delta))$ and $H^i_\psi,\gamma(\text{LA}(\mathbb{Z}_p, L)(\delta))$; see Proposition 2.1 and Théorème 2.9 of \cite{Colmez08} and Lemme 2.9 and Corollaire 2.11 of \cite{Ch13}.

4.2.1. Verification of the condition (vi) in the generic case. In this subsection, we assume that $\delta$ is generic. Then, we have

$$H^i_\psi,\gamma(Lt^k e_\delta) = H^i_\psi,\gamma(Ly^k e_\delta) = 0$$

for any $k \in \mathbb{Z}_{\geq 0}$ and $i \in \{0, 1, 2\}$, and

$$H^i_\psi,\gamma(R_L(\delta)) = H^i_\psi,\gamma(R^\infty_L(\delta)) = 0$$

for $i = 0, 2$, and

$$\dim_L H^1_\psi,\gamma(R_L(\delta)) = \dim_L H^1_\psi,\gamma(R^\infty_L(\delta)) = 1.$$

Then, $t_{i,\delta} := t_i \otimes_{R^\infty_L(\delta)} \varphi(\delta) \otimes \text{id}_L$ is the isomorphism

$$\text{Det}_L(H^1_\psi,\gamma(R_L(\delta)))^{-1} \sim \text{Det}_L(H^1_\psi,\gamma(R^\infty_L(\delta)))^{-1}$$

induced by the isomorphism

$$H^1_\psi,\gamma(R^\infty_L(\delta)) \sim H^1_\psi,\gamma(R_L(\delta)) : [x] \mapsto [x, 0].$$

The isomorphism $t_{3,\delta} \circ t_{4,\delta}$ is the isomorphism

$$\text{Det}_L(H^1_\psi,\gamma(R^\infty_L(\delta)))^{-1} \stackrel{t_{3,\delta}}{\sim} \text{Det}_L(H^1_\psi,\gamma(R^\infty_L(\delta)))^{-1} \stackrel{t_{4,\delta}}{\sim} \text{Det}_L(\varphi e_\delta)^{-1},$$

where induced by the isomorphism

$$H^1_\psi,\gamma(R^\infty_L(\delta)) \sim H^1_\psi,\gamma(R^\infty_L(\delta)) : [x] \mapsto [x, 0].$$

where the last isomorphism is explicitly defined as follows. For an explicit definition of this isomorphism, it is useful to use the Amice transform. Let $D(\mathbb{Z}_p, L) := \text{Hom}_L(\text{LA}(\mathbb{Z}_p, L), L)$ be the algebra of $L$-valued distributions on $\mathbb{Z}_p$. By the theorem of Amice, we have an isomorphism

$$D(\mathbb{Z}_p, L) \sim R^\infty_L : \mu \mapsto f_\mu(\pi) := \sum_{n \geq 0} \mu \left( \binom{y}{n} \right) \pi^n$$

( which depends on the choice of $\pi$, i.e. $\zeta$ where $\binom{y}{n} := \frac{y(y-1) \cdots (y-n+1)}{n!}$. Then, the action of $(\varphi, \Gamma, \psi)$ on $R^\infty_L$ induces the action on $D(\mathbb{Z}_p, L)$, i.e.

$$\int_{\mathbb{Z}_p} f(y) \varphi(\mu)(y) := \int_{\mathbb{Z}_p} f(py) \mu(y), \quad \int_{\mathbb{Z}_p} f(y) \psi(\mu)(y) := \int_{\mathbb{Z}_p} f\left( \frac{y}{p} \right) \mu(y)$$

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and
\[ \int_{\mathbb{Z}_p} f(y)\sigma_a(\mu)(y) := \int_{\mathbb{Z}_p} f(ay)\mu(y), \]
where, for \( a \in \mathbb{Z}_p^{\times} \), we define \( \sigma_a \in \Gamma \) such that \( \chi(\sigma_a) = a \).

Using this notion, it is easy to see that the above isomorphism is defined by
\[ H^1_\gamma(\mathcal{R}_L(\delta)^{\psi=0}) \xrightarrow{\sim} L\mathbf{e}_\delta : [f_\mu \mathbf{e}_\delta] \mapsto \frac{1}{|\Gamma_{\text{tor}}| \log_0(\chi(\gamma))} \cdot \int_{\mathbb{Z}_p} \delta^{-1}(y)\mu(y)\mathbf{e}_\delta, \]
where we remark that we have an isomorphism \( D(\mathbb{Z}_p^\times, L)\mathbf{e}_\delta \xrightarrow{\sim} \mathcal{R}_L(\delta)^{\psi=0} : \mu \mathbf{e}_\delta \mapsto f_\mu \mathbf{e}_\delta \).

For a \( \Gamma \)-module \( N \), we define \( H^1(\Gamma, N) := N/N_0 \), where \( N_0 \) is the submodule generated by all \( (\gamma - 1)n \) for \( \gamma \in \Gamma \), \( n \in N \). Then, we have the following canonical isomorphism
\[ H^1(\Gamma, \mathcal{R}_L^\times(\delta)^{\psi=1}) \xrightarrow{\sim} H^1_\gamma(\mathcal{R}_L^\times(\delta)^{\psi=1}) : [f_\mu \mathbf{e}_\delta] \mapsto [[\Gamma_{\text{tor}}| \log_0(\chi(\gamma))]p_\Delta(f_\mu \mathbf{e}_\delta)] \]
(46) where “canonical” means that this is independent of \( \gamma \), i.e., is compatible with \( \iota_{\gamma, \gamma'} \). Composing this with \( \iota_{4, \delta} \circ \iota_{3, \delta} \), we obtain an isomorphism
\[ \text{Det}_L(H^1(\Gamma, \mathcal{R}_L(\delta)^{\psi=1}))^{-1} \xrightarrow{\sim} \text{Det}_L(L\mathbf{e}_\delta)^{-1} \]
induced by the isomorphism
\[ \iota_\delta : H^1(\Gamma, \mathcal{R}_L^\times(\delta)^{\psi=1}) \xrightarrow{\sim} H^1_\gamma(\mathcal{R}_L^\times(\delta)^{\psi=1}) \xrightarrow{\sim} L\mathbf{e}_\delta. \]

By the above arguments, we obtain the following lemma.

**Lemma 4.11.** We have
\[ \iota_\delta([f_\mu \mathbf{e}_\delta]) = \int_{\mathbb{Z}_p} \delta^{-1}(y)\mu(y) \]
for \([f_\mu \mathbf{e}_\delta] \in H^1(\Gamma, \mathcal{R}_L(\delta)^{\psi=1}) \).

**Proof.** For \( f_\mu \mathbf{e}_\delta \in \mathcal{R}_L^\times(\delta)^{\psi=1} \), we have \((1 - \varphi)(f_\mu \mathbf{e}_\delta) = ((1 - \varphi \psi)f_\mu) \cdot \mathbf{e}_\delta \). Then, the lemma follows from the formula
\[ \int_{\mathbb{Z}_p} f(x)(1 - \varphi \psi)\mu(x) = \int_{\mathbb{Z}_p} f(x)\mu(x). \]
for \( \mu \in D(\mathbb{Z}_p, L) \).

Next, we furthermore assume that \( \mathcal{R}_L(\delta) \) is de Rham. By the classification, it is equivalent to \( \delta = \tilde{\delta}x^k \) for \( k \in \mathbb{Z} \) and for a locally constant homomorphism \( \tilde{\delta} : \mathbb{Q}_p^\times \to L^\times \). In the generic case, we have the following isomorphisms of one dimensional \( L \)-vector spaces,

1. \( \exp_{R_{\mathcal{R}_L}(\delta^{-1}x^{|x|})}^* : H^1_{\psi, \gamma}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} D_{\text{dR}}(\mathcal{R}_L(\delta)) \) if \( k \leq 0 \).
2. \( \exp_{R_{\mathcal{R}_L}(\delta)} : D_{\text{dR}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} H^1_{\psi, \gamma}(\mathcal{R}_L(\delta)) \) if \( k \geq 1 \).
Let define \( n(\delta) \in \mathbb{Z}_{\geq 0} \) as the minimal integer such that \( \tilde{\delta}|_{(1+p^n\mathbb{Z}_p)\cap \mathbb{Z}_p^*} \) is trivial. Then, we have the following facts:

1. \( n(\delta) = 0 \) if and only if \( R_L(\delta) \) is crystalline.
2. \( \varepsilon_L(D_{\text{pst}}(R_L(\delta)), \zeta) = 1 \) if \( n(\delta) = 0 \).
3. \( \varepsilon_L(D_{\text{pst}}(R_L(\delta)), \zeta) = \tilde{\delta}(p)^{n(\delta)} \sum_{i \in (\mathbb{Z}/p^n(\delta))} \tilde{\delta}(i)^{-1} \zeta_{p^n(i)} \) if \( n(\delta) \geq 1 \).
4. \( \varepsilon_L(D_{\text{pst}}(R_L(\delta)), \zeta) \cdot \varepsilon_L(D_{\text{pst}}(R_L(\delta^{-1}x|x))), \zeta) = \tilde{\delta}(-1) \).

By definition of \( \varepsilon_L, \zeta(R_L(\delta)) \), \( t_i,\delta \) \((i = 1, 3, 4, 5)\), \( \epsilon \) and \( \varepsilon_L \) by Proposition 4.12 \((k \leq 0)\) and 4.11 \((k \geq 1)\), which can be seen as a kind of explicit reciprocity laws.

**Proposition 4.12.** If \( k \leq 0 \), then the following map

\[
H^1(\Gamma, R_L^\infty(\delta)^{\psi=1}) \xrightarrow{\exp^*_{R_L(\delta^{-1}x|x)}} D_{\text{dr}}(R_L(\delta)) = \left( \frac{1}{n} L_{\infty} e_\delta \right)^\Gamma
\]

(where the first isomorphism is defined by \([f e_\delta] \mapsto [\Gamma_{\text{tor}} \log_0(\chi(\gamma)) \rho_\Delta(f e_\delta), 0] \)

sends each element \([f_\mu e_\delta] \in H^1(\Gamma, R_L^\infty(\delta)^{\psi=1}) \) to

\[
(1) \quad \frac{(-1)^k}{(-k)!} \varepsilon(\varepsilon_L, \zeta; \varepsilon) \int_{\mathbb{Q}_p} \delta^{-1}(y) \mu(y) e_\delta \quad \text{if} \quad n(\delta) \neq 0,
\]

\[
(2) \quad \frac{(-1)^k \det_L(1-x^{-1})^{-1} \delta(\varepsilon_L) \delta^{-1}}{(-k)!} \int_{\mathbb{Q}_p} \delta^{-1}(y) \mu(y) e_\delta \quad \text{if} \quad n(\delta) = 0.
\]

**Proof.** Here, we prove the proposition only when \( k = 0 \), i.e. \( \delta = \tilde{\delta} \) is locally constant. We will prove it for general \( k \leq 0 \) after some preparations on the differential operator \( \partial \).

We assume that \( k = 0 \). For such \( \delta \), we define a map

\[
g_{R_L(\delta)} : D_{\text{dr}}(R_L(\delta)) \to H^1(\Gamma, D_{\text{diff}}(R_L(\delta))) : x \mapsto [\log(\chi(\gamma)) x],
\]

which is easy to see to be isomorphism. By Proposition 2.16 of [Na13], we have the following commutative diagram

\[
\begin{array}{ccc}
H^1_{\psi,\gamma}(R_L(\delta)) & \xrightarrow{\exp^*_{R_L(\delta^{-1}x|x)}} & D_{\text{dr}}(R_L(\delta)) \\
\downarrow {\text{id}} & & \downarrow {g_{R_L(\delta)}} \\
H^1_{\psi,\gamma}(R_L(\delta)) & \xrightarrow{\text{can}} & H^1(\Gamma, D_{\text{diff}}(R_L(\delta)))
\end{array}
\]

Set \( n_0 := \max\{n(\delta), 1\} \) if \( p \neq 2 \), and set \( n_0 := \max\{n(\delta), 2\} \) if \( p = 2 \). Then, the image of \([f_\mu e_\delta] \in H^1(\Gamma, R_L^\infty(\delta)^{\psi=1}) \) \( \xrightarrow{\exp^*_{R_L(\delta^{-1}x|x)}} H^1_{\psi,\gamma}(R_L(\delta)) \) by the canonical map can : \( H^1_{\psi,\gamma}(R_L(\delta)) \to H^1_{\psi,\gamma}(R_L(\delta)) \) is equal to

\[
[\Gamma_{\text{tor}} \log_0(\chi(\gamma)) \rho_\Delta(n_0(f_\mu e_\delta))] \in H^1_{\psi,\gamma}(D_{\text{diff}}(R_L(\delta))).
\]
Hence, it suffices to calculate \( g_{R_L(\delta)}^{-1}(\Gamma_{\log_\eta}(\chi(\gamma))p_\Delta(t_n_0(f_\mu e_\delta))) \). By definition of \( g_{R_L(\delta)} \), it is easy to check that we have

\[
g_{R_L(\delta)}^{-1}(\Gamma_{\log_\eta}(\chi(\gamma))p_\Delta(t_n_0(f_\mu e_\delta))) = \frac{\Gamma_{\log_\eta}(\chi(\gamma))}{\log(\chi(\gamma))} \frac{1}{\langle Q_p(\zeta_{p^0}) : Q_p \rangle} \sum_{i \in (\mathbb{Z}/p^n \mathbb{Z})^\times} \sigma_i(t_n_0(f_\mu e_\delta)|_{t=0}).
\]

Concerning the right hand side, when \( n(\delta) \geq 1 \) if \( p \neq 2 \), or \( n(\delta) \geq 2 \) if \( p = 2 \), we have the following equalities, from which we obtain the equality (1),

\[
(RHS) = \frac{\Gamma_{\log_\eta}(\chi(\gamma))}{\log(\chi(\gamma))} \frac{1}{\langle Q_p(\zeta_{p^0}) : Q_p \rangle} \sum_{i \in (\mathbb{Z}/p^n \mathbb{Z})^\times} \sigma_i(t_n(\delta)(f_\mu e_\delta)|_{t=0})
\]

where the second equality follows form \( \frac{\Gamma_{\log_\eta}(\chi(\gamma))}{\log(\chi(\gamma))} = 1 \) for any \( p \), and the sixth equality follows from the fact that \( \sum_{i \in (\mathbb{Z}/p^n \mathbb{Z})^\times} \delta(i) \zeta_{p^0}^{ij} = 0 \) if \( p|j \), and the seventh and eighth follow from the properties of \( \varepsilon \)-constants listed before this proposition.

When \( n(\delta) = 0 \), we have \( n_0 = 1 \) if \( p \neq 2 \) and \( n_0 = 2 \) if \( p = 2 \), then we have the following equalities

\[
(RHS) = \frac{1}{p^{n_0}} \sum_{i \in (\mathbb{Z}/p^n \mathbb{Z})^\times} \sigma_i(t_{n_0}(f_\mu e_\delta)|_{t=0})
\]

where the first equality follows form \( \frac{\Gamma_{\log_\eta}(\chi(\gamma))}{\log(\chi(\gamma))} = \frac{1}{p^{n_0}} \) for any \( p \).

When \( p \neq 2 \), the last term is equal to

\[
\frac{1}{p^{n_0}} ((p - 1) \int_{\mu_{p^n}} \mu(y) - \int_{\mu_{p^n}} \mu(y))e_\delta
\]

because we have \( \sum_{i \in (\mathbb{Z}/p^n \mathbb{Z})^\times} \zeta_{p^0}^{ij} = p - 1 \) if \( p|j \) and \( \sum_{i \in (\mathbb{Z}/p^n \mathbb{Z})^\times} \zeta_{p^0}^{ij} = -1 \) if \( p \nmid j \).
Because \( f_\mu e_\delta \in \mathcal{R}^\infty(\delta)^{\psi=1} \), we have \( \psi(f_\mu) = \delta(p)f_\mu \), hence we have
\[
\int_{\mathbb{Z}_p} \mu(y) = \int_{\mathbb{Z}_p} \psi(\mu)(y) = \delta(p) \int_{\mathbb{Z}_p} \mu(y) = \delta(p)(\int_{\mathbb{Z}_p^\times} \mu(y) + \int_{\mathbb{Z}_p} \mu(y)),
\]
and we have
\[
\int_{\mathbb{Z}_p} \mu(y) = \frac{\delta(p)}{1 - \delta(p)} \int_{\mathbb{Z}_p^\times} \mu(y)
\]
because we have \( \delta(p) \neq 1 \) by the generic assumption on \( \delta \).

Therefore, we have
\[
\frac{1}{p\delta(p)}((p - 1) \int_{\mathbb{Z}_p} \mu(y) - \int_{\mathbb{Z}_p^\times} \mu(y))e_\delta = \frac{1}{p\delta(p)}((p - 1) \sum_{j \delta(p) - 1} \int_{\mathbb{Z}_p^\times} \mu(y)e_\delta = \frac{1}{p\delta(p)} \sum_{j \neq 1} \int_{\mathbb{Z}_p^\times} \mu(y)e_\delta,
\]
from which we obtain the equality (2) for \( p \neq 2 \).

When \( p = 2 \), then the last term is equal to
\[
\frac{1}{p\delta(p)^2}((2 \int_{\mathbb{Z}_2} \mu(y) - \int_{\mathbb{Z}_2^\times} \mu(y))e_\delta = \frac{1}{p\delta(p)^2} \sum_{j \neq 1} \int_{\mathbb{Z}_2^\times} \mu(y)e_\delta
\]
because we have \( \sum_{j \neq 1} \zeta_4^j \) is equal to 2 if \( j \equiv 0 \pmod{4} \), is equal to 0 if \( j \equiv 1, 3 \pmod{4} \), and is equal to \(-2\) if \( j \equiv 2 \pmod{4} \). Because we have \( \psi(f_\mu) = \delta(p)f_\mu \), we have
\[
\int_{\mathbb{Z}_2} \mu(y) = \int_{\mathbb{Z}_2^\times} \mu(y) = \delta(p) \int_{\mathbb{Z}_2} \mu(y) = \delta(p) \sum_{j \neq 1} \int_{\mathbb{Z}_2^\times} \mu(y)
\]
where the last equality follows from the same argument for \( p \neq 2 \).

Therefore, we have
\[
\frac{1}{p\delta(p)^2}((2 \int_{\mathbb{Z}_2} \mu(y) - \int_{\mathbb{Z}_2} \mu(y))e_\delta = \frac{1}{p\delta(p)^2} \sum_{j \neq 1} \int_{\mathbb{Z}_2^\times} \mu(y)e_\delta
\]
and we have
\[
\frac{1}{p\delta(p)}((p - 1) \int_{\mathbb{Z}_2} \mu(y) - \int_{\mathbb{Z}_2^\times} \mu(y))e_\delta = \frac{1}{p\delta(p)} \sum_{j \neq 1} \int_{\mathbb{Z}_2^\times} \mu(y)e_\delta,
\]
from which we obtain the equality (2) for \( p = 2 \).

\[\square\]

To prove the above proposition for general \( k \leq 0 \), we need to recall and prove some facts on the differential operator \( \partial \) defined in §2.4 of [Co08], which will be used to reduce the verification of the condition (vi) for general \( k \) to that for \( k = 0, 1 \) (even for the non generic case).

Let \( A \) be a \( \mathbb{Q}_p \)-affinoid algebra. We define an \( A \)-linear differential operator \( \partial : \mathcal{R}_A \to \mathcal{R}_A : f(\pi) \mapsto (1 + \pi)^{\partial f(\pi)} \). Let \( \delta : \mathbb{Q}_p^\times \to A^\times \) be a continuous homomorphism, then \( \partial \) naturally induces an \( A \)-linear, \((\varphi, \Gamma)\)-equivariant morphism
\[
\partial : \mathcal{R}_A(\delta) \to \mathcal{R}_A(\delta x) : f(\pi)e_\delta \mapsto \partial(f(\pi))e_{\delta x},
\]
where \( f(\pi) \in \mathcal{R}_A(\delta) \) and \( \delta x = \delta + x \delta(\pi) \in \mathcal{R}_A(\delta x) \).
which is in the following exact sequence

\[ 0 \to A(\delta) \to \mathcal{R}_A(\delta) \to \mathcal{R}_A(\delta x) \xrightarrow{fe_\delta \to \text{Res}_0(\delta \bar{\omega})e_{\delta x}^{-1}} A(\delta |x|^{-1}) \to 0. \]

By this exact sequence, when \( A = L \) is a finite extension of \( \mathbb{Q}_p \), we immediately obtain the following lemma.

**Lemma 4.13.** \( \partial : C_{\varphi,\gamma}^*(\mathcal{R}_L(\delta)) \to C_{\varphi,\gamma}^*(\mathcal{R}_L(\delta x)) \) is quasi-isomorphism except when \( \delta = 1, |x|. \)

Using the above exact sequence, and using the canonical trivialization ((211) in §3.1)

\[ \text{Det}_A(C_{\varphi,\gamma}^*(A(\delta'))) = \text{Det}_A(C_{\gamma}^*(A(\delta'))) \xrightarrow{1-\varphi} C_{\gamma}^*(A(\delta'))[-1] \xrightarrow{\sim} 1_A \]

for \( \delta' = \delta, \delta|x|^{-1} \), we obtain the following isomorphism which we also denote by \( \partial \):

\[ \partial : \Delta_{A,1}(\mathcal{R}_A(\delta)) \xrightarrow{\sim} \Delta_{A,1}(\mathcal{R}_A(\delta x)). \]

Taking the product of this isomorphism with the isomorphism

\[ \Delta_{A,2}(\mathcal{R}_A(\delta)) \xrightarrow{\sim} \Delta_{A,2}(\mathcal{R}_A(\delta x)) : a e_\delta \mapsto a e_{\delta x}, \]

we obtain the isomorphism

\[ \partial : \Delta_A(\mathcal{R}_A(\delta)) \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta x)). \]

By definition, it is clear that this isomorphism is compatible with any base change \( A \to A' \).

Concerning this isomorphism, we prove the following proposition.

**Proposition 4.14.** In the above situation, we have

\[ \varepsilon_{A,\zeta}(\mathcal{R}_A(\delta)) = \varepsilon_{A,\zeta}(\mathcal{R}_A(\delta x)) \circ \partial. \]

**Proof.** The proof of this proposition is a typical density argument, which will be used several times later.

If we define a unramified homomorphism \( \delta_Y : \mathbb{Q}_p^\times \to \Gamma(\mathbb{G}_m^{an}, \mathcal{O}_{\mathbb{G}_m^{an}}) : p \mapsto Y \) (where \( Y \) is the parameter of \( \mathbb{G}_m^{an} \)), then \( \mathcal{R}_A(\delta) \) is obtained as a base change of the “universal” rank one \( (\varphi, \Gamma) \)-module \( \text{Dfm}(\mathcal{R}_{\mathbb{G}_m^{an}}(\delta_Y)) \) over \( \mathcal{R}_X \times \mathbb{G}_m^{an} \) (\( X \) is the rigid analytic space associated to \( \mathbb{Z}_p[[\Gamma]] \)). Because the isomorphism \( \partial : \Delta_A(\mathcal{R}_A(\delta)) \xrightarrow{\sim} \Delta_A(\mathcal{R}_A(\delta x)) \) is compatible with any base change, it suffices to show the proposition for \( \text{Dfm}(\mathcal{R}_{\mathbb{G}_m^{an}}(\delta_Y)) \). Because \( X \times \mathbb{G}_m^{an} \) is reduced, it suffices to show it for the Zariski dense subset \( S_0 \) of \( X \times \mathbb{G}_m^{an} \) defined by

\[ S_0 := \{ (\delta_0, \lambda) \in X(L) \times \mathbb{G}_m^{an}(L) | L \text{ is a finite extension of } \mathbb{Q}_p, \delta := \delta_0 \delta_0 \text{ is generic} \}. \]

For any \( (\delta_0, \lambda) \in S_0(L) \), \( \varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) \) essentially corresponds to the isomorphism

\[ i_\delta : H^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\sim} L e_\delta : [f_\delta e_\delta] \mapsto \int_{\mathbb{Z}_p} \delta^{-1}(y) \mu(y) e_\delta \]
by Lemma 4.11 and by the arguments before this lemma. Then, the equality
\[ \varepsilon_{L,\zeta}(\mathcal{R}_L(\delta)) = \varepsilon_{L,\zeta}(\mathcal{R}_L(\delta x)) \circ \partial \] is equivalent to the commutativity of the following diagram
\[
\begin{array}{ccc}
H^1(\Gamma, \mathcal{R}^\infty_L(\delta)^{\psi=1}) & \xrightarrow{\iota^L} & Le_\delta \\
\partial \downarrow & & \downarrow e_\delta \mapsto e_{\delta x} \\
H^1(\Gamma, \mathcal{R}^\infty_L(\delta x)^{\psi=1}) & \xrightarrow{\iota^L} & Le_{\delta x}.
\end{array}
\]

Finally, this commutativity follows from the formula
\[
\int_{\mathbb{Z}_p} f(y) \partial(\mu)(y) = \int_{\mathbb{Z}_p} yf(y)\mu(y)
\]
for any \( f(y) \in LA(\mathbb{Z}_p, L) \), which finishes to prove the proposition. \( \Box \)

We next prove the compatibility of \( \partial \) with the de Rham \( \varepsilon \)-isomorphism \( \varepsilon_{L,\zeta}^{dR}(\mathcal{R}_L(\delta)) \) for de Rham rank one \((\varphi, \Gamma)\)-modules \( \mathcal{R}_L(\delta) \) under a condition on the Hodge-Tate weight of \( \mathcal{R}_L(\delta) \) as below.

Lemma 4.15. Let \( \mathcal{R}_L(\delta) \) be a de Rham \((\varphi, \Gamma)\)-module (here we don’t assume that \( \delta \) is generic). If the Hodge-Tate weight of \( \mathcal{R}_L(\delta) \) is not zero, i.e. we have \( \delta = \delta x^k \) such that \( k \neq 0 \), then we have the equality
\[ \varepsilon_{L,\zeta}^{dR}(\mathcal{R}_L(\delta)) = \varepsilon_{L,\zeta}^{dR}(\mathcal{R}_L(\delta x)) \circ \partial. \]

Remark 4.16. Of course, after verifying the condition (vi), this compatibility is trivial by the above proposition 4.14, but the difficulty for directly proving this compatibility depends on the Hodge-Tate weight. The proof for \( k = 0 \) is more difficult than that of other cases; see the proof of Proposition 4.18.

Proof. Because we have \( D_{dR}(\mathcal{R}_L(\delta)) = (L_{\infty} e_\delta)^\Gamma \), under the condition that \( k \neq 0 \), the differential operator \( \partial \) naturally induces an isomorphism
\[ \partial : D_{dR}(\mathcal{R}_L(\delta)) \rightarrow D_{dR}(\mathcal{R}_L(\delta x)) : a \rightarrow (-k)\frac{a}{tk+1} e_{\delta x} \]
because we have \( \partial(g(t)) = \frac{dg(t)}{dt} \) for \( g(t) \in L_{\infty}((t)) \). Hence, by definition of of \( \varepsilon_{L,\zeta}^{dR}(M) = \theta_L(M) \cdot \theta_{dR,L}(M, \zeta) \) and \( \gamma_{L,M} \) in §3.2, it suffices to show the following two equalities:
1. \( \theta'_L(\mathcal{R}_L(\delta)) = \theta'_L(\mathcal{R}_L(\delta x)) \circ \partial, \)
2. \( \theta_{dR,L}(\mathcal{R}_L(\delta), \zeta) = \frac{1}{\delta} \cdot \theta_{dR,L}(\mathcal{R}_L(\delta x), \zeta) \circ \partial, \)
under the isomorphisms
\[ \partial := \partial \otimes \partial : \Delta_{L,1}(\mathcal{R}_L(\delta)) \cdot \text{Det}_L(D_{dR}(\mathcal{R}_L(\delta))) \cong \Delta_{L,1}(\mathcal{R}_L(\delta x)) \cdot D_{dR}(\mathcal{R}_L(\delta x)) \]
and
\[ \partial := \partial^{-1} \otimes \partial : \text{Det}_L(D_{dR}(\mathcal{R}_L(\delta)))^{-1} \cdot \Delta_{L,2}(\mathcal{R}_L(\delta)) \cong \text{Det}_L(D_{dR}(\mathcal{R}_L(\delta x)))^{-1} \cdot \Delta_{L,2}(\mathcal{R}_L(\delta x)). \]
We first prove the equality (2). By definition of \( \theta_{dR,L}(M, \zeta) \), it suffices to show that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{L}_L(\mathcal{R}_L(\delta)) = L e_\delta & \xrightarrow{f_{\mathcal{R}_L(\delta), \zeta}} & D_{dR}(\mathcal{R}_L(\delta)) \\
\downarrow e_\delta \mapsto e_{\delta x} & & \downarrow e_\delta \mapsto (-k) \frac{e_\delta}{t} e_{\delta x} \\
\mathcal{L}_L(\mathcal{R}_L(\delta x)) = L e_{\delta x} & \xrightarrow{f_{\mathcal{R}_L(\delta x), \zeta}} & D_{dR}(\mathcal{R}_L(\delta x)),
\end{array}
\]

where the map \( f_{\mathcal{R}_L(\delta'), \zeta} \) (for \( \delta' = \delta, \delta x \)) is defined in Lemma \[5.4\]. This commutativity is obvious from definition of \( f_{\mathcal{R}_L(\delta_0), \zeta} \), because we have a natural isomorphism

\[
D_{pst}(\mathcal{R}_L(\delta)) \sim D_{pst}(\mathcal{R}_L(\delta x)) : \frac{a}{t^k} e_\delta \mapsto \frac{a}{t^{k+1}} e_{\delta x},
\]

in particular we have \( \varepsilon_L(D_{pst}(\mathcal{R}_L(\delta)), \zeta) = \varepsilon_L(D_{pst}(\mathcal{R}_L(\delta x)), \zeta) \).

We next prove the equality (1). Under the assumption that \( k \neq 0 \), it is easy to see that \( \partial \) induces the isomorphisms

\[
D_{dR}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} D_{dR}(\mathcal{R}_L(\delta x)), \quad \Fil^0 D_{dR}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} \Fil^0 D_{dR}(\mathcal{R}_L(\delta x))
\]

and

\[
D_{\text{cris}}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} D_{\text{cris}}(\mathcal{R}_L(\delta x)).
\]

and

\[
H^i_{\varphi, \gamma}(\mathcal{R}_L(\delta)) \xrightarrow{\sim} H^i_{\varphi, \gamma}(\mathcal{R}_L(\delta x))
\]

for any \( i = 0, 1, 2 \) by Lemma \[4.13\]. Therefore, by definition of \( \theta'_{dR,L}(\mathcal{R}_L(\delta)) \), it suffices to show that the following two diagrams of exact sequences are commutative for \( M = \mathcal{R}_L(\delta) \):

\[
\begin{array}{cccc}
H^0_{\varphi, \gamma}(M) & \longrightarrow & D_{\text{cris}}(M) & \longrightarrow & D_{\text{cris}}(M) \oplus t_M & \longrightarrow & H^1_{\varphi, \gamma}(M) \\
\downarrow \partial & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial \\
H^0_{\varphi, \gamma}(M(x)) & \longrightarrow & D_{\text{cris}}(M(x)) & \longrightarrow & D_{\text{cris}}(M(x)) \oplus t_{M(x)} & \longrightarrow & H^1_{\varphi, \gamma}(M(x))
\end{array}
\]

and

\[
\begin{array}{cccc}
H^1_{\varphi, \gamma}(M) & \longrightarrow & D_{\text{cris}}(M^*) \vee \oplus D_{dR}(M)^0 & \longrightarrow & D_{\text{cris}}(M) \vee & \longrightarrow & H^2_{\varphi, \gamma}(M) \\
\downarrow \partial & & \downarrow (-\partial') \oplus \partial & & \downarrow -\partial' & & \downarrow \partial \\
H^1_{\varphi, \gamma}(M(x)) & \longrightarrow & D_{\text{cris}}(M(x)^*) \vee \oplus D_{dR}(M(x))^0 & \longrightarrow & D_{\text{cris}}(M(x)^*) \vee & \longrightarrow & H^2_{\varphi, \gamma}(M(x)),
\end{array}
\]

where \( \partial' \) is the dual of \( \partial : D_{\text{cris}}(M(x)^*) = D_{\text{cris}}(\mathcal{R}_L(\delta^1|x|)) \xrightarrow{\sim} D_{\text{cris}}(\mathcal{R}_L(\delta^{-1}x|x|)) = D_{\text{cris}}(M^*) \). For the commutativity of the diagram \[48\], the only non trivial part
is the commutativity of the diagram

\[
\begin{array}{ccc}
D_{\text{crys}}(M) \oplus t_M & \xrightarrow{\exp_{M,f} \oplus \exp_{M(x)}} & H^1_{\varphi,\gamma}(M) \\
\partial \downarrow & & \partial \\
D_{\text{crys}}(M(x)) \oplus t_{M(x)} & \xrightarrow{\exp_{M(x),f} \oplus \exp_{M(x)}} & H^1_{\varphi,\gamma}(M(x)),
\end{array}
\]

but this commutativity easily follows from Proposition 2.22. Using the commutativity of (13) for \( M = \mathcal{R}_L(\delta^{-1}|x|) \), to prove the commutativity of (19), it suffices to show the commutativities of the following diagrams:

\[
\begin{array}{ccc}
D_{\text{dR}}(M) & \longrightarrow & D_{\text{dR}}(M^*)^\vee \\
\partial \downarrow & & \partial^\vee \\
D_{\text{dR}}(M(x)) & \longrightarrow & D_{\text{dR}}(M(x)^*)^\vee
\end{array}
\]

(50)

and

\[
\begin{array}{ccc}
H^i_{\varphi,\gamma}(M) & \longrightarrow & H^{2-i}_{\varphi,\gamma}(M^*)^\vee \\
\partial \downarrow & & \partial^\vee \\
H^i_{\varphi,\gamma}(M(x)) & \longrightarrow & H^{2-i}_{\varphi,\gamma}(M(x)^*)^\vee
\end{array}
\]

(51)

where the horizontal arrows are isomorphisms obtained by the canonical duality \( D_{\text{dR}}(M_0^*) \times D_{\text{dR}}(M_0) \to D_{\text{dR}}(\mathcal{R}_L(1)) = \mathbb{L}_{\mathbb{Z}}\{1\} \to \mathbb{L} \) and the Tate duality \(< -, - > : H^{2-i}_{\varphi,\gamma}(M_0^*) \times H^i_{\varphi,\gamma}(M_0) \to \mathbb{L} \) for \( M_0 = M, M(x) \). Because the commutativity of (50) is easy to check, here we only prove the commutativity of (51). Moreover, we only prove it for \( i = 2 \) because other cases are proven in the same way. For \( i = 2 \), it suffices to show the equality

\[
[\partial(f)g e_1] = -[f \partial(g) e_1] \in H^2_{\varphi,\gamma}(\mathcal{R}_L(1))
\]

for any \([fe_1] \in H^2_{\varphi,\gamma}(\mathcal{R}_L(\delta)) \) and \( ge_{\delta^{-1}|x|} \in H^0_{\varphi,\gamma}(\mathcal{R}_L(\delta^{-1}|x|)) \). Because we have \( \partial(fg) = \partial(f)g + f \partial(g) \), the equality follows from the fact that we have \([\partial(h)e_1] = 0 \) in \( H^2_{\varphi,\gamma}(\mathcal{R}_L(1)) \) for any \( h \in \mathcal{R}_L \).

\[ \square \]

**Remark 4.17.** Proposition 4.14 and Lemma 4.15 and the following proof of Proposition 4.12 should be generalized to a more general setting. Let \( M \) be a de Rham \((\varphi, \Gamma)\)-module over \( \mathcal{R}_L \) of any rank. In §3 of [Na13], we developed the theory of Perrin-Riou’s big exponential map for de Rham \((\varphi, \Gamma)\)-modules, which is a \( \mathcal{R}_L^\infty(\Gamma) \)-linear map \( H^1_{\psi,\gamma}(\text{Dfm}(M)) \to H^1_{\psi,\gamma}(\text{Dfm}(N_{\text{rig}}(M))) \) where \( N_{\text{rig}}(M) \subseteq M[1/t] \) is a de Rham \((\varphi, \Gamma)\)-module equipped with a natural action of the differential operator \( \partial \) defined by Berger. This big exponential map is defined using the operator \( \partial \), and our generalization of Perrin-Riou’s \( \delta(V) \) theorem (Theorem 3.21 [Na13]) states
that this map gives an isomorphism

$$\partial_M : \Delta_{\mathcal{R}_L^\infty(\delta)}(\text{Dfm}(M)) \xrightarrow{\sim} \Delta(\text{Dfm}(\mathcal{N}_{\text{rig}}(M))).$$

Therefore, as a generalization of Proposition 4.14, it is natural to conjecture that the conjectural $\varepsilon$-isomorphisms should satisfy

$$\varepsilon_{\mathcal{R}_L^\infty(\delta)}(\text{Dfm}(M)) = \varepsilon_{\mathcal{R}_L^\infty(\delta)}(\text{Dfm}(\mathcal{N}_{\text{rig}}(M))) \circ \partial_M,$$

which will be more precisely studied in [Na].

Using these results, we prove Proposition 4.14 for general $k \leq 0$ as follows.

**Proof.** (of Proposition 4.12 for general $k \leq 0$)

Let $\delta = \tilde{\delta} \in \mathbb{x}$ be a generic homomorphism such that $k \leq 0$. By the arguments before Proposition 4.12, it suffices to show the equality $\varepsilon_{\mathcal{R}_L^\infty}(\mathcal{R}_L(\tilde{\delta})) = \varepsilon^{\text{dR}}_{\mathcal{R}_L^\infty}(\mathcal{R}_L(\tilde{\delta}))$ (i.e. the condition (vi)).

Because we have

$$\varepsilon_{\mathcal{R}_L^\infty}(\mathcal{R}_L(\tilde{\delta})) = \varepsilon^{\text{dR}}_{\mathcal{R}_L^\infty}(\mathcal{R}_L(\tilde{\delta}))$$

by Proposition 4.12 for $k = 0$, we also obtain the equality

$$\varepsilon_{\mathcal{R}_L^\infty}(\mathcal{R}_L(\tilde{\delta})) = \varepsilon^{\text{dR}}_{\mathcal{R}_L^\infty}(\mathcal{R}_L(\tilde{\delta}))$$

because we have

$$\varepsilon_{\mathcal{R}_L^\infty}(\mathcal{R}_L(\tilde{\delta})) = \varepsilon_{\mathcal{R}_L^\infty}(\mathcal{R}_L(\tilde{\delta})) \circ \partial^{-k} \quad \text{and} \quad \varepsilon^{\text{dR}}_{\mathcal{R}_L^\infty}(\mathcal{R}_L(\tilde{\delta})) = \varepsilon^{\text{dR}}_{\mathcal{R}_L^\infty}(\mathcal{R}_L(\tilde{\delta})) \circ \partial^{-k}$$

by Proposition 4.14 and Lemma 4.15.

We next consider the case where $k \geq 1$. To verify the condition (vi), it suffices to show the following proposition.

**Proposition 4.18.** If $k \geq 1$, then the following map

$$H^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\exp_{\mathcal{R}_L^\infty(\delta)}^{-1}} D_{\text{dR}}(\mathcal{R}_L(\delta))$$

sends each element $[f_{\psi} e_\delta] \in H^1(\Gamma, \mathcal{R}_L^\infty(\delta)^{\psi=1})$ to

1. $-(k-1)1! \frac{\delta E_{\mathcal{R}_L^\infty(\delta)}(\mathcal{R}_L(\delta)^{\psi=1})}{\det L} \int_{\mathcal{X}_0} \delta^{-1}(y) \mu(y) e_\delta$ when $n(\delta) \neq 0$,

2. $-(k-1)1! \frac{\delta E_{\mathcal{R}_L^\infty(\delta)}(\mathcal{R}_L(\delta)^{\psi=1})}{\det L} \int_{\mathcal{X}_0} \delta^{-1}(y) \mu(y) e_\delta$ when $n(\delta) = 0$.

**Proof.** In the same way as the proof of Proposition 4.12, it suffices to show the proposition for $k = 1$ (i.e. $\delta = \tilde{\delta} \in \mathbb{x}$) using Proposition 4.14 and Lemma 4.15.

Hence, we assume $k = 1$. Then, in the similar way as the proof of Proposition 4.12 (for $k = 0$), we have the following commutative diagrams

$$\begin{align*}
H^1(\mathcal{R}_L^\infty(\tilde{\delta})) & \xrightarrow{\partial} H^1(\mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\mathcal{L}_\delta e_\delta} L e_\delta \\
H^1(\mathcal{R}_L^\infty(\tilde{\delta})) & \xrightarrow{\partial} H^1(\mathcal{R}_L^\infty(\delta)^{\psi=1}) \xrightarrow{\mathcal{L}_\delta e_\delta} L e_\delta
\end{align*}$$

(52)
such that all the arrows are isomorphisms by Lemma 4.13. Hence, reducing to the case of \(k = 0\), it suffices to show that the following diagram is commutative,

\[
\begin{array}{cccc}
H^1(\Gamma, \mathcal{R}_L^\infty(\tilde{\delta})^\psi=1) & \longrightarrow & H^1_{\psi, \gamma}(\mathcal{R}_L(\tilde{\delta})) & \xrightarrow{\exp_{\mathcal{R}_L(\tilde{\delta})^{\psi=1}}} & D_{\text{dr}}(\mathcal{R}_L(\tilde{\delta})) = (L_{\infty}e_3)^\Gamma \\
\downarrow & & \downarrow & & \downarrow_{\text{ae}^3 \rightarrow \tilde{\gamma}e_3} \\
H^1(\Gamma, \mathcal{R}_L^\infty(\delta)^\psi=1) & \longrightarrow & H^1_{\psi, \gamma}(\mathcal{R}_L(\delta)) & \xleftarrow{\exp_{\mathcal{R}_L(\delta)}} & D_{\text{dr}}(\mathcal{R}_L(\delta)) = (L_{\infty}^1H_{\delta})^\Gamma.
\end{array}
\]

The following proof of this commutativity is very similar to that of Theorem 3.10 of [Na13]. Take \([f e_3] \in H^1(\Gamma, \mathcal{R}_L^\infty(\delta)^\psi=1)\). If we denote by

\[
\alpha e_3 := \exp_{\mathcal{R}_L(\delta=1-x|x|)}(\Gamma_{\text{tor}}|\log(\chi(\gamma))p_{\Delta}(f e_3), 0]) \in D_{\text{dr}}(\mathcal{R}_L(\tilde{\delta})) \subseteq D_{\text{dir}}(\mathcal{R}_L(\tilde{\delta})),
\]

then it suffices to show the equality

\[
\exp_{\mathcal{R}_L(\delta)}(\alpha e_3) = |\Gamma_{\text{tor}}|\log(\chi(\gamma))|p_{\Delta}(\partial(f) e_3), 0].
\]

We prove this equality as follows. First, we have an equality

\[
\frac{|\Gamma_{\text{tor}}|\log(\chi(\gamma))}{\log(\chi(\gamma))} [t_{n}(p_{\Delta}(f e_3))] = [\alpha e_3] \in H^1_{\psi, \gamma}(D^+_{\text{dir}}(\mathcal{R}_L(\tilde{\delta})))
\]

for a sufficiently large \(n \geq 1\) by the explicit definition of \(\exp_{\mathcal{R}_L(\delta=1-x|x|)}\) (Proposition 2.16 of [Na13]). This equality means that there exists \(y_n \in D^+_{\text{dir}, n}(\mathcal{R}_L(\tilde{\delta}))^\Delta\) such that

\[
\frac{|\Gamma_{\text{tor}}|\log(\chi(\gamma))}{\log(\chi(\gamma))} t_{n}(p_{\Delta}(f e_3)) - \alpha e_3 = (\gamma - 1)y_n.
\]

If we define \(\nabla_0 := \frac{\log(\gamma)}{\log(\chi(\gamma))} \in \mathcal{R}_L^\infty(\Gamma)\), and define

\[
\nabla_0 = \frac{1}{\log(\chi(\gamma))} \sum_{m \geq 1} \frac{(-1)^m - 1}{m} \in \mathcal{R}_L^\infty(\Gamma),
\]

then we obtain the following equality

\[
(54) \quad t_{n}(\frac{|\Gamma_{\text{tor}}|\log(\chi(\gamma))}{\log(\chi(\gamma))} \nabla_0 \nabla_0 \nabla_0) (p_{\Delta}(f e_3)) = \frac{1}{\log(\chi(\gamma))} \alpha e_3 + \nabla_0(y_n) \in \frac{1}{\log(\chi(\gamma))} \alpha e_3 + tD^+_{\text{dir}, n}(\mathcal{R}_L(\tilde{\delta})).
\]

Because we have \(f e_3 \in \mathcal{R}_L(\tilde{\delta})^{\psi=1}\), we have

\[
(1 - \varphi)(p_{\Delta}(f e_3)) \in \mathcal{R}_L(\tilde{\delta})^{\Delta, \psi=0}.
\]

Hence, there exists \(\beta \in \mathcal{R}_L(\tilde{\delta})^{\Delta, \psi=0}\) such that

\[
(1 - \varphi)(p_{\Delta}(f e_3)) = (\gamma - 1)\beta
\]
for any $m \geq n + 1$, we have

$$
\ell_m \left( \frac{\nabla_0}{\gamma - 1} (p \Delta (fe_\delta)) \right) - \ell_{m-1} \left( \frac{\nabla_0}{\gamma - 1} (p \Delta (fe_\delta)) \right) = \ell_m ((1 - \varphi) (\frac{\nabla_0}{\gamma - 1} (p \Delta (fe_\delta)))) = \ell_m (\frac{\nabla_0}{\gamma - 1} ((1 - \varphi) (p \Delta (fe_\delta))))
$$

$$= \ell_m (\frac{\nabla_0}{\gamma - 1} ((\gamma - 1) \beta)) = \ell_m (\nabla_0 (\beta)) \in tD_{\text{dif},m}^\times (R_L(\delta))
$$

because we have $\nabla_0 (R_L(\delta)) \subseteq tR_L(\delta)$. In particular, we have

$$
(55) \quad \ell_m \left( \frac{\nabla_0}{\gamma - 1} (p \Delta (fe_\delta)) \right) - \ell_n \left( \frac{\nabla_0}{\gamma - 1} (p \Delta (fe_\delta)) \right) \in tD_{\text{dif},m}^\times (R_L(\delta))
$$

for any $m \geq n + 1$ by induction.

Because the map $R_L(\delta) \xrightarrow{\sim} \frac{1}{\gamma} R_L(\delta) : g e_\delta \mapsto \frac{\psi}{\delta} e_\delta$ is an isomorphism of $(\varphi, \Gamma)$-modules, the facts (54) and (55), and the explicit definition of the exponential map (Proposition 2.22 (1)) induce the following equality

$$
\exp_{R_L(\delta)} (\frac{\psi}{\delta} e_\delta) = [\Gamma_{\text{tor}} \log_0 (\chi(\gamma)) \frac{\nabla_0}{\gamma - 1} (p \Delta (\frac{\psi}{\delta} e_\delta)), (\psi - 1) \frac{\nabla_0}{\gamma - 1} (p \Delta (\frac{\psi}{\delta} e_\delta))]
$$

$$= [\Gamma_{\text{tor}} \log_0 (\chi(\gamma)) \nabla_0 (p \Delta (\frac{\psi}{\delta} e_\delta)), 0]
$$

$$= [\Gamma_{\text{tor}} \log_0 (\chi(\gamma)) [p \Delta (\partial (f) e_\delta)], 0],
$$

where the last equality follows from the equality $\nabla_0 (\frac{\psi}{\delta} e_\delta) = \partial (f) e_\delta$ because we have $\nabla_0 (\frac{1}{\gamma} e_\delta) = 0$ by the assumption that $k = 1$, from which the commutativity of the diagram (53) follows.

As a corollary of Proposition 4.12, 4.18 we verify the conditions (iii), (iv), (v) by the density argument.

Corollary 4.19. Let $M$ be a rank one $(\varphi, \Gamma)$-module over $R_A$. Then the isomorphism $\varepsilon_{A,\zeta} (M) : \Delta_A (M) \xrightarrow{\sim} 1_A$ which is defined in §4.1 satisfies the conditions (iii), (iv), and (v) of Conjecture 3.9.

Proof. We first verify the conditions (iii), (iv). By definition of $\varepsilon_{A,\zeta} (R_A (\delta) \otimes_A L)$, it suffices to verify these conditions for $(\varphi, \Gamma)$-modules of the form $R_A (\delta)$ because the general case immediately follows from this case. Then, in the same way as the proof of Proposition 4.14 it suffices to verify these conditions for any $\delta = \delta_0 : Q_p^\times \to L^\times$ such that the point $(\delta_0, \lambda) \in X \times \mathbb{G}_m$ is contained in a Zariski dense subset $S_1$ of $X \times \mathbb{G}_m$ defined by

$$
S_1 := \{ (\delta_0, \lambda) \in X(L) \times \mathbb{G}_m (L) | [L : Q_p] < \infty, \delta \text{ is generic }, R_L (\delta) \text{ is de Rham } \}.
$$

For such $\delta$, the conditions (iii), (iv) follow from Definition 3.6 and Lemma 3.8 because we have $\varepsilon_{L,\zeta} (R_L (\delta)) = \varepsilon_{L,\zeta}^\text{deR} (R_L (\delta))$ by Propositions 4.12, 4.18.
We next verify the condition (v). Let \( (\Lambda, T) \) be as in Conjecture 3.9 (v). We recall that we defined a canonical isomorphism
\[
\Delta_\Lambda(T) \otimes_\Lambda A_\infty \sim \Delta_{A_\infty}(M_\infty)
\]
(see Example 3.3 for definition and notation). Because any continuous map \( \Lambda \to A \) factors through \( \Lambda \to A_\infty \to A \), it suffices to show the equality
\[
(56) \quad \varepsilon_{\Lambda, \zeta}(T) \otimes \text{id}_{A_\infty} = \varepsilon_{A_\infty, \zeta}(M_\infty)(:= \lim_n \varepsilon_{A_n, \zeta}(M_n)).
\]

Because the condition (v) is local for \( \text{Spf}(\Lambda) \), it suffices to verify (v) for \( \Lambda \)-representations of the form \( \Lambda(\tilde{\delta}) \) for \( \tilde{\delta} : G_{ab}^{G_{an}} \otimes \mathbb{Q}_p \rightarrow \Lambda \times \mathbb{Q}_p \). Let decompose \( \tilde{\delta} = \delta \circ \text{rec}_{\mathbb{Q}_p} \). Then, we can define a continuous \( \mathbb{Z}_p \)-algebra homomorphism \( \Lambda_k := \lim_n \mathbb{Z}_p[Y]/(p, (Y^k - 1))^n \rightarrow \Lambda : Y \mapsto \lambda \). Hence, the \( \Lambda \)-representation \( \Lambda(\tilde{\delta}) \) is obtained by a base change of the “universal” \( \mathbb{Z}_p[[\Gamma]] \otimes \mathbb{Z}_p \Lambda_k \)-representation \( T_{k \text{univ}} \) which corresponds to the homomorphism \( \delta_k \text{univ} : Q_p^\times \rightarrow (\mathbb{Z}_p[[\Gamma]] \otimes \mathbb{Z}_p \Lambda_k) \times : p \mapsto 1 \otimes 1, a \mapsto [\sigma_a^{-1}] \otimes 1 \) for \( a \in \mathbb{Z}_p^\times \). Hence, it suffices to verify the equality (56) for this universal one. In this case, because the associated rigid space is an admissible open of \( X \times \mathbb{G}_{m, \mathbb{Z}} \) defined by
\[
Z_k := \{(\delta_0, \lambda) \in X \times \mathbb{G}_{m, \mathbb{Z}} ||\lambda^k - 1| < 1\},
\]
and the associated \((\varphi, \Gamma)\)-module is isomorphic to the restriction of the universal one \( \text{Dfm}(\mathcal{R}_{\mathbb{G}_{m, \mathbb{Z}}}(\delta_Y)) \) defined in the proof of Proposition 4.14, it suffices to show the equality
\[
\varepsilon_{\mathbb{Z}_p[[\Gamma]] \otimes \mathbb{Z}_p \Lambda_k, \zeta}(T_{k \text{univ}}) \otimes \text{id}_{(Z_k, O_{Z_k})} = \varepsilon_{(Z_k, O_{Z_k}), \zeta}(\text{Dfm}(\mathcal{R}_{\mathbb{G}_{m, \mathbb{Z}}}(\delta_Y))|_{Z_k}).
\]
Because both sides satisfy the condition (vi) for any point \( (\delta_0, \lambda) \in Z_k \cap S_1 \) by Kato’s theorem ([Ka93b]) and by Proposition 4.12, 4.18, and because the set \( Z_k \cap S_1 \) is Zariski dense in \( Z_k \), the equality (18) follows by the density argument.

4.2.2. Verification of the condition (vi): the exceptional case. Finally, we verify the condition (vi) in the exceptional case, i.e. \( \delta = x^{-k} \) or \( \delta = x^{k+1}|x| \) for \( k \in \mathbb{Z}_{\geq 0} \).

We first reduce all the exceptional cases to the case of \( \delta = x|x| \).

Lemma 4.20. We assume that the equality
\[
\varepsilon_{L, \zeta}(\mathcal{R}_L(x|x|)) = \varepsilon^{\text{DR}}_{L, \zeta}(\mathcal{R}_L(x|x|))
\]
holds. Then the other equalities
\[
\varepsilon_{L, \zeta}(\mathcal{R}_L(\delta)) = \varepsilon^{\text{DR}}_{L, \zeta}(\mathcal{R}_L(\delta))
\]
also hold for all \( \delta = x^{k+1}|x|, x^{-k} \) for \( k \geq 0 \).
Proof. The equality for $\delta = x^0$ follows from that for $\delta = x |x|$ by the compatibility of $\varepsilon^{d_{\mathbb{R}}}(-)$ and $\varepsilon_{L,\zeta}(-)$ with the Tate duality, which are proven in Lemma 3.8 and Proposition 4.19. Then, the equality for $\delta = x^{k+1} |x|$ (respectively, $\delta = x^{-k}$) follows from that for $\delta = x |x|$ (respectively, $\delta = x^0$) by the compatibility of $\varepsilon^{d_{\mathbb{R}}}(-)$ and $\varepsilon_{L,\zeta}(-)$ with $\partial$, which are proven in Lemma 4.15 and Proposition 4.14.

Finally, it remains to show the equality
$$
\varepsilon_{L,\zeta}(\mathcal{R}_L(1)) = \varepsilon^{d_{\mathbb{R}}}((\mathcal{R}_L(1))
$$
(we identify $\mathcal{R}_L(x|x|) = \mathcal{R}_L(1) : f e_{x|x|} \mapsto f e_1$). Because $\mathcal{R}_L(1)$ is étale, this equality immediately follows from the Kato’s result because we have $\varepsilon_{L,\zeta}(\mathcal{R}_L(1)) = \varepsilon_{O_L,\zeta}(O_L(1)) \otimes \mathbb{G}_L$ under the canonical isomorphism $\Delta_L(\mathcal{R}_L(1)) \cong \Delta_O_L(O_L(1)) \otimes \mathbb{G}_L
$ by Corollary 4.19. However, here we give another proof of this equality only using the framework of $(\varphi, \Gamma)$-modules.

In the remaining part of this section, we prove this equality by explicit calculations. First, it is easy to see that the inclusion
$$
C^*(\mathcal{R}_L^\infty(1)^{\psi=1}) \hookrightarrow C^*(\mathcal{R}_L^\infty(1)),
$$
induced by the natural inclusion $\mathcal{R}_L \hookrightarrow \mathcal{R}_L^\infty(1)$ (here, $1_{\mathbb{Z}}$ is the constant function on $\mathbb{Z}_p$ with the constant value $1$) is quasi-isomorphism. This quasi-isomorphism and the quasi-isomorphism
$$
\psi : C^*(\mathcal{R}_L^\infty(1)^{\psi=1}) \cong C^*(\mathcal{R}_L^\infty(1)),
$$
and the long exact sequence associated to the short exact sequence
$$
0 \to \mathcal{R}_L^\infty(1) \to \mathcal{R}_L(1) \to \mathbb{Z}_p, \mathcal{R}_L(1) \to 0
$$
induce the following isomorphisms
$$
\alpha_0 : H^0_{\psi,\gamma}(L \cdot 1_{\mathbb{Z}} \mathbf{e}_1) \cong H^1(\Gamma, \mathcal{R}_L^\infty(1)^{\psi=1}),
$$
$$
\alpha_1 : H^1_{\psi,\gamma}(\mathcal{R}_L(1)) \cong H^1_{\psi,\gamma}(L \cdot 1_{\mathbb{Z}} \mathbf{e}_1) : [f_1 \mathbf{e}_1, f_2 \mathbf{e}_2] \mapsto (\text{Res}_0(f_1 \frac{d\pi}{1+\pi}) \cdot 1_{\mathbb{Z}} \mathbf{e}_1, \text{Res}_0(f_2 \frac{d\pi}{1+\pi}) \cdot 1_{\mathbb{Z}} \mathbf{e}_1),
$$
$$
\alpha_2 : H^2_{\psi,\gamma}(\mathcal{R}_L(1)) \cong H^2_{\psi,\gamma}(L \cdot 1_{\mathbb{Z}} \mathbf{e}_1) : [f \mathbf{e}_1] \mapsto \text{Res}_0(f \frac{d\pi}{1+\pi}) \cdot 1_{\mathbb{Z}} \mathbf{e}_1.
$$
Therefore, the isomorphism $\iota_{\delta,\lambda}[-1] : \prod_{i=1}^2 \text{Det}_L(\mathcal{H}^0_{\psi,\gamma}(\mathcal{R}_L(1)))^{-1} \cong \text{Det}_L(\mathcal{R}_L(1))$ is the composition of the isomorphisms $\beta_0$, $\beta_1$ and $\iota_{x|x|}$.

$$
\prod_{i=1}^2 \text{Det}_L(\mathcal{H}^0_{\psi,\gamma}(\mathcal{R}_L(1)))^{-1} \xrightarrow{\beta_0} \prod_{i=0}^2 \text{Det}_L(\mathcal{H}^0_{\psi,\gamma}(L \cdot 1_{\mathbb{Z}} \mathbf{e}_1))^{-1} \xrightarrow{\beta_1} \text{Det}_L(\mathcal{R}_L^\infty(1)^{\psi=1}) \xrightarrow{\iota_{x|x|}} \text{Det}_L(\mathcal{R}_L(1)) \xrightarrow{\iota_{\delta,\lambda}[-1]} \text{Det}_L(\mathcal{R}_L(1)^{\psi=1}) \xrightarrow{\iota_{x|x|}} \text{Det}_L(\mathcal{R}_L(1)) \xrightarrow{\iota_{\delta,\lambda}[-1]} \text{Det}_L(\mathcal{R}_L(1)).
$$
where $\beta_0$ is induced by $\alpha_i$ ($i = 0, 1, 2$), and $\beta_1$ is the canonical trivialization
\[
\beta_1 : 2 \prod_{i=0}^{1} \text{Det}_L(H^i_{\psi,\gamma}(L \cdot 1Z_p e_1))^{(-1)^{i-1}} \sim 1_L
\]
induced by the natural trivialization.

\[
\text{Det}_L(C^*(\lambda \cdot 1Z_p e_1)) = \text{Det}_L(\text{Con}(C^*(\lambda \cdot 1Z_p e_1) \xrightarrow{1-\psi} C^*(\lambda \cdot 1Z_p e_1))[-1])
\sim \text{Det}_L(C^*(\lambda \cdot 1Z_p e_1)) \cdot \text{Det}_L(C^*(\lambda \cdot 1Z_p e_1))^{-1} \sim 1_L
\]

To describe this isomorphism explicitly, we give several lemmas. The following lemma is directly follows from the definition.

**Lemma 4.21.** If we denote $\tilde{f}_0 := (1Z_p e_1)$ (respectively $\tilde{f}_{1,1} := (1Z_p e_1, 0)$ and $\tilde{f}_{1,2} := (0, 1Z_p e_1)$, respectively $\tilde{f}_2 := (1Z_p e_1)$ for the basis of $H^0_{\psi,\gamma}(L \cdot 1Z_p e_1)$ (respectively $H^1_{\psi,\gamma}(L \cdot 1Z_p e_1)$, respectively $H^2_{\psi,\gamma}(L \cdot 1Z_p e_1)$), then the canonical trivialization
\[
\beta_1 : H^0_{\psi,\gamma}(L \cdot 1Z_p e_1)^\vee \otimes_L \det L H^0_{\psi,\gamma}(L \cdot 1Z_p e_1) \otimes_L H^2_{\psi,\gamma}(L \cdot 1Z_p e_1)^\vee \sim L
\]
satisfies the equality
\[
\beta_1(\tilde{f}_0^\vee \otimes (\tilde{f}_{1,1} \wedge \tilde{f}_{1,2}) \otimes \tilde{f}_2^\vee) = -1.
\]

**Lemma 4.22.** The isomorphism
\[
H^0_{\psi,\gamma}(L \cdot 1Z_p e_1) \xrightarrow{\alpha_0} H^1(\Gamma, \mathcal{R}_L^\infty(1)^{\psi=1}) \xrightarrow{\iota_{|x|}} Le_1
\]
sends the element $\tilde{f}_0$ to $e_1 \in L(1)$.

**Proof.** Because we have $\text{Col}(\frac{1+\pi}{\pi}) = 1Z_p$ and $\psi(\frac{1+\pi}{\pi} e_1) = \frac{1+\pi}{\pi} e_1$, we have
\[
\alpha_0(\tilde{f}_0) = [\frac{1}{|\Gamma_{\text{tor}}| \log_0(\chi(\gamma))} (\gamma - 1)(\frac{1+\pi}{\pi} e_1)]
\]
by definition of the boundary map.

Because we have
\[
(\gamma - 1)(\frac{1+\pi}{\pi} e_1) = \partial(\log(\frac{\gamma(\pi)}{\pi})) e_1 \text{ and } \log(\frac{\gamma(\pi)}{\pi}) e_{|x|} \in \mathcal{R}_L^\infty(|x|)^{\psi=1},
\]
and because we have the following commutative diagram
\[
H^1(\Gamma, \mathcal{R}_L^\infty(|x|)^{\psi=1}) \xrightarrow{\iota_{|x|}} Le_{|x|}
\]
(57)
\[
\downarrow \partial \quad \downarrow e_{|x|} \mapsto e_1
\]
\[
H^1(\Gamma, \mathcal{R}_L^\infty(1)^{\psi=1}) \xrightarrow{\iota_{|x|}} Le_1,
\]
we have an equality
\[
\iota_{|x|}(\alpha_0(\tilde{f}_0)) = \frac{1}{|\Gamma_{\text{tor}}| \log_0(\chi(\gamma))} \iota_{|x|}([\partial(\log(\frac{\gamma(\pi)}{\pi})) e_1]) = \frac{1}{|\Gamma_{\text{tor}}| \log_0(\chi(\gamma))} \int_{Z_p} \mu_\gamma(y) e_1
\]
by Lemma 4.11, where we define $\mu_\gamma \in \mathcal{D}(\mathbb{Z}_p, L)$ such that $f_{\mu_\gamma}(\pi) = \log(\frac{\gamma(\pi)}{\pi})$.

We calculate $\int_{\mathbb{Z}_p} \mu_\gamma(y)$ as follows. Because we have $\psi(\mu_\gamma) = \frac{1}{p}\mu_\gamma$, we have

$$\int_{\mathbb{Z}_p} \mu_\gamma(y) = \int_{\mathbb{Z}_p} \psi(\mu_\gamma)(y) = \frac{1}{p} \int_{\mathbb{Z}_p} \mu_\gamma(y).$$

Hence, we obtain

$$\int_{\mathbb{Z}_p} \mu_\gamma(y) = \int_{\mathbb{Z}_p} \mu_\gamma(y) - \int_{p\mathbb{Z}_p} \mu_\gamma(y) = \int_{\mathbb{Z}_p} \mu_\gamma(y) - \frac{1}{p} \int_{\mathbb{Z}_p} \mu_\gamma(y)$$

$$= \frac{p-1}{p} \int_{\mathbb{Z}_p} \mu_\gamma(y) = \frac{p-1}{p} \log(\frac{\gamma(\pi)}{\pi})|_{\pi=0} = \frac{p-1}{p} \log(\chi(\gamma)).$$

Hence, we obtain

$$\iota_{x|e}(\alpha_0(\tilde{f}_0)) = \frac{\log(\chi(\gamma))}{|\Gamma_{x_0}|\log(\chi(\gamma))} \frac{p-1}{p} e_1 = e_1$$

for any prime $p$, which proves the lemma.

In the appendix, we define canonical basis $\{f_{1,1}, f_{1,2}\}$ of $H^1_{\psi,\gamma}(\mathcal{R}_L(1))$, $f_2 \in H^2_{\psi,\gamma}(\mathcal{R}_L(1))$, $e_0 \in H^0_{\psi,\gamma}(\mathcal{R}_L)$ and $\{e_{1,1}, e_{1,2}\}$ of $H^1_{\psi,\gamma}(\mathcal{R}_L)$; see the appendix for the definition.

**Corollary 4.23.** The isomorphism

$$\det_L H^1_{\psi,\gamma}(\mathcal{R}_L(1)) \otimes_L H^2_{\psi,\gamma}(\mathcal{R}_L(1))^\vee \xrightarrow{\sim} \text{Le}_1$$

which corresponds to $\iota_{5,6}[-1]$ sends the element $(f_{1,1} \wedge f_{1,2}) \otimes f_2^\vee$ to $-\frac{p-1}{p} e_1$.

**Proof.** By definition, we have

$$\alpha_1(f_{1,1}) = \frac{p-1}{p} \log(\chi(\gamma))\tilde{f}_{1,1}, \quad \alpha_1(f_{1,2}) = \frac{p-1}{p} \tilde{f}_{1,2}$$

and $\alpha_2(f_2) = \frac{p-1}{p} \log(\chi(\gamma))\tilde{f}_2$.

Then, the corollary follows from the previous lemmas.

Finally, because we have $\gamma_L, \mathcal{R}_L(1) = -1$, and because $\theta_{\text{DR}, \mathcal{R}_L}(\mathcal{R}_L(1), \zeta)$ corresponds to the isomorphism

$$\mathcal{L}_L(\mathcal{R}_L(1)) = \text{Le}_1 \sim \text{D}_{\text{DR}}(\mathcal{R}_L(1)) = \frac{1}{t} \text{Le}_1 : ae_1 \mapsto -\frac{a}{t} e_1,$$

it suffices to show the following lemma.

**Lemma 4.24.** The isomorphism

$$\det_L H^1_{\psi,\gamma}(\mathcal{R}_L(1)) \otimes_L H^2_{\psi,\gamma}(\mathcal{R}_L(1))^\vee \xrightarrow{\sim} \text{D}_{\text{DR}}(\mathcal{R}_L(1)) = \frac{1}{t} e_1$$

which corresponds to $\theta'_L(\mathcal{R}_L(1))$ sends the element $(f_{1,1} \wedge f_{1,2}) \otimes f_2^\vee$ to $-\frac{p-1}{pt} e_1$. 

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Proof. By definition, the above isomorphism is the one which is naturally induced by the exact sequence
\[
0 \to D_{\text{cris}}(R_L(1)) \xrightarrow{(1-\varphi)\circ \text{can}} D_{\text{cris}}(R_L(1)) \oplus D_{\text{dR}}(R_L(1)) \xrightarrow{\exp_{f, R_L(1)} \circ \exp_{R_L(1)}} H^1_{\psi, \gamma}(R_L(1))_f \to 0
\]
and the isomorphisms
\[
\exp_{f, R_L}^\vee : H^1_{\psi, \gamma}(R_L(1))/H^1_{\psi, \gamma}(R_L(1))_f \xrightarrow{\sim} D_{\text{cris}}(R_L)^\vee
\]
and
\[
D_{\text{cris}}(R_L)^\vee \xrightarrow{\sim} H^2_{\psi, \gamma}(R_L(1)),
\]
which is the dual of the natural isomorphism \(H^0_{\psi, \gamma}(R_L) \xrightarrow{\sim} D_{\text{cris}}(R_L)\).

We have \(\exp_{R_L(1)}(\frac{1}{1}e_1) = f_{1,2}\) by the proof of Lemma 4.25. Because we have
\[
\exp_{f, R_L}^\vee(f_{1,1}) = -d_0^\vee \in D_{\text{cris}}(R)^\vee
\]
we obtain that
\[
D_{\text{cris}}(R_L)^\vee \to H^2_{\psi, \gamma}(R_L(1)) : d_0^\vee \mapsto f_2
\]
by Lemma 4.28. Using these calculations, the lemma follows from diagram chase. \(\square\)

Appendix: Explicit calculations of \(H^i_{\psi, \gamma}(R_L)\) and \(H^i_{\psi, \gamma}(R_L(1))\)

In this appendix, we compare \(H^i(Q_p, L(k))\) with \(H^i_{\psi, \gamma}(R_L(k))\) explicitly for \(k = 0, 1\), and define canonical basis of \(H^i_{\psi, \gamma}(R_L(k))\), which are used to compare \(\varepsilon_{L, \psi}(R_L(1))\) with \(\varepsilon_{L, \psi}(R_L(1))\) in Corollary 4.23 and Lemma 4.24. All the results in this appendix seems to be known (see for example [Ben00]), but here we give another proof of these results in the framework of \((\varphi, \Gamma)\)-modules over the Robba ring. Of course, we may assume that \(L = Q_p\) by base change.

We first consider \(H^i_{\psi, \gamma}(R_{Q_p})\). If we identify by
\[
H^1(Q_p, Q_p) = \text{Hom}_{\text{cont}}(G_{Q_p}, Q_p) \xrightarrow{\sim} \text{Hom}_{\text{cont}}(Q_p^\times, Q_p) : \tau \mapsto \tau \circ \text{rec}_{Q_p},
\]
then this has a basis \([\text{ord}_p, \text{log}]\) defined by
\[
\text{ord}_p : Q_p^\times \to Q_p : p \mapsto 1, a \mapsto 0 \quad \text{for} \quad a \in \mathbb{Z}_p^\times,
\]
\[
\text{log} : Q_p^\times \to Q_p : p \mapsto 0, a \mapsto \log(a) \quad \text{for} \quad a \in \mathbb{Z}_p^\times.
\]
We define a basis \( e_0 \) of \( H^0_{\varphi,\gamma}(\mathbb{R}_{\Omega p}) \) and \( \{ e_{1,1}, e_{1,2} \} \) of \( H^1_{\varphi,\gamma}(\mathbb{R}_{\Omega p}) \) by

\[
e_0 = 1 \in \mathbb{R}_{\Omega p}, \quad e_{1,1} := [\log(\chi(\gamma)), 0], \quad e_{1,2} := [0, 1],
\]

which is independent of the choice of \( \gamma \), i.e. is compatible with the comparison isomorphism \( \iota_{\gamma,\gamma'} \). We can easily check that the canonical isomorphism \( H^1(\mathbb{Q}_p, \mathbb{Q}_p) \sim \rightarrow H^1_{\varphi,\gamma}(\mathbb{R}_{\Omega p}) \) sends \([\log]\) to \( e_{1,1} \) and \([\text{ord}_p]\) to \( e_{1,2} \).

We next consider \( H^1_{\varphi,\gamma}(\mathbb{R}_{\Omega p}(1)) \). Let denote by \( \kappa : \mathbb{Q}_p^\times \rightarrow H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \) for the Kummer map. Composing this with the canonical isomorphism \( H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \sim \rightarrow H^1_{\varphi,\gamma}(\mathbb{R}_{\Omega p}(1)) \), we obtain a homomorphism

\[
\kappa_0 : \mathbb{Q}_p^\times \rightarrow H^1_{\varphi,\gamma}(\mathbb{R}_{\Omega p}(1)).
\]

We define a homomorphism

\[
H^1_{\varphi,\gamma}(\mathbb{R}_{\Omega p}(1)) \rightarrow \mathbb{Q}_p \oplus \mathbb{Q}_p : \quad [f_1 e_1, f_2 e_1] \mapsto \left( \frac{1}{p-1} \cdot \log(\chi(\gamma)) \cdot \text{Res}_0(f_1 \frac{d\pi}{1+\pi}), -\frac{1}{p-1} \cdot \text{Res}_0(f_2 \frac{d\pi}{1+\pi}) \right),
\]

(we note that \( \frac{p-1}{p} \cdot \log(\chi(\gamma)) = |\Gamma_{\text{tor}}| \cdot \log_0(\chi(\gamma))) \), which is also independent of the choice of \( \gamma \), and is isomorphism. Using this isomorphism, we define a basis \( \{ f_{1,1}, f_{1,2} \} \) of \( H^1_{\varphi,\gamma}(\mathbb{R}_{\Omega p}(1)) \) such that \( f_{1,1} \) (respectively \( f_{1,2} \)) corresponds to \((1,0) \in L \oplus L\) (respectively \((0, 1)) \) by this isomorphism. We want to explicitly describe the map \( \kappa_0 \) using this basis. For this, we first prove the following lemma.

**Lemma 4.25.** For each \( a \in \mathbb{Z}_p^\times \), we have \( \kappa_0(a) = \log(a) \cdot f_{1,2} \).

**Proof.** By the classical explicit calculation of exponential map, we have

\[
\kappa(a) = \exp_{\mathbb{Q}_p(1)}(\frac{\log(a)}{t} e_1).
\]

Because we have the following commutative diagram

\[
\begin{array}{ccc}
\text{D}_{\text{dR}}(\mathbb{Q}_p(1)) & \xrightarrow{\exp_{\mathbb{Q}_p(1)}} & H^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) \\
\downarrow \sim & & \downarrow \sim \\
\text{D}_{\text{dR}}(\mathbb{R}_{\Omega p}(1)) & \xrightarrow{\exp_{\mathbb{R}_{\Omega p}(1)}} & H^1_{\varphi,\gamma}(\mathbb{R}_{\Omega p}(1))
\end{array}
\]

by Proposition 2.25, it suffices to show that

\[
\exp_{\mathbb{R}_{\Omega p}(1)}(\frac{1}{t} e_1) = f_{1,2}.
\]

We show this equality as follows. We first take some \( f \in (\mathbb{R}_{\Omega p})^\Delta \) such that \( f(\zeta_{p^n} - 1) = \frac{1}{p^n} \) for any \( n \geq 0 \), which is possible because we have an isomorphism...
\( \mathcal{R}_{Q_p}^\infty/t \to \prod_{n \geq 0} Q_p(\zeta_p^n) : f \mapsto (f(\zeta_p^n - 1))_{n \geq 0} \) by Lazard’s theorem [La62]. Then, the element \( \frac{f}{t} e_1 \in (\frac{1}{t} \mathcal{R}_{Q_p}(1))^\Delta \) satisfies

\[
\iota_n(\frac{f}{t} e_1) - \frac{1}{t} e_1 \in D_{\text{diff},n}(\mathcal{R}_{Q_p}(1))
\]

for any \( n \geq 1 \), because we have

\[
\iota_n(\frac{f}{t} e_1) \equiv p^n \cdot \frac{f(\zeta_p^n - 1)}{t} e_1 = \frac{1}{t} e_1 \pmod{D_{\text{diff},n}(\mathcal{R}_{Q_p}(1))}.
\]

By the explicit definition of \( \text{exp}_{\mathcal{R}_{Q_p}(1)} \) (Proposition 2.22 (1)), then we have

\[
\text{exp}_{\mathcal{R}_{Q_p}(1)}(\frac{1}{t} e_1) = [(\gamma - 1)(\frac{f}{t} e_1), (\varphi - 1)(\frac{f}{t} e_1)] \in H_{\varphi,\gamma}^1(\mathcal{R}_{Q_p}(1)).
\]

Hence, it suffices to show that

\[
\text{Res}_0(\frac{\gamma(f) - f}{t} \cdot \frac{d\pi}{1 + \pi}) = 0
\]

and

\[
\text{Res}_0(\frac{\varphi(f) - f}{p} \cdot \frac{1}{t} \cdot \frac{d\pi}{1 + \pi}) = -\frac{p - 1}{p}.
\]

Here, we only calculate \( \text{Res}_0(\frac{\varphi(f) - f}{p} \cdot \frac{1}{t} \cdot \frac{d\pi}{1 + \pi}) \), because the calculation of \( \text{Res}_0(\frac{\varphi(f) - f}{p} \cdot \frac{d\pi}{1 + \pi}) \) is similar. By definition of \( f \), we have

\[
\frac{\varphi(f)(\zeta_p^n - 1)}{p} - f(\zeta_p^n - 1) = \frac{f(\zeta_p^{n-1} - 1)}{p} - f(\zeta_p^n - 1) = \frac{1}{p} \cdot \frac{1}{p^{n-1}} - \frac{1}{p^n} = 0
\]

for each \( n \geq 1 \). Hence, we have

\[
(\frac{\varphi(f)}{p} - f) \in (\prod_{n \geq 1} Q_n(\pi))_{n \geq 1} R_{Q_p}^\infty
\]

by the theorem of Lazard [La62], where we define \( Q_n(\pi) := \varphi^{n-1}(\frac{\varphi(\pi)}{\pi}) \) for each \( n \geq 1 \). Because we have \( t = \pi \prod_{n \geq 1} (\frac{Q_n(\pi)}{p}) \), we obtain the equality

\[
\text{Res}_0((\frac{\varphi(f)}{p} - f) \cdot \frac{1}{t} \cdot \frac{d\pi}{1 + \pi}) = ((\frac{\varphi(f)}{p} - f) \cdot \frac{1}{p} \prod_{n \geq 1} \frac{Q_n(\pi)}{p} \cdot \frac{1}{1 + \pi})|_{\pi = 0} = (\frac{\varphi(f)}{p} - f)|_{\pi = 0} = \frac{f(0)}{p} - f(0) = -\frac{p - 1}{p},
\]

where the second equality follows from the fact that \( \frac{Q_n(0)}{p} = 1 \) for \( n \geq 1 \), which proves the lemma.

\[\square\]
Before calculating $\kappa_0(p) \in H^1_{\varphi,\gamma}(\mathcal{R}_{Q_p}(1))$, we explicitly describe the Tate’s trace map in terms of $(\varphi, \Gamma)$-modules. We note that we normalize the Tate’s trace map $H^2(Q_p, Q_p(1)) \xrightarrow{\sim} Q_p$ such that the cup product pairing

$$\langle,\rangle : H^1(Q_p, Q_p(1)) \times H^1(Q_p, Q_p) \xrightarrow{\cup} H^2(Q_p, Q_p(1)) \xrightarrow{\sim} Q_p$$

satisfies that

$$\langle \kappa(a), [\tau] \rangle = \tau(a)$$

for $a \in Q_p^\times$ and $[\tau] \in \text{Hom}(Q_p^\times, Q_p) = H^1(Q_p, Q_p)$.

**Proposition 4.26.** The map $\iota_\gamma : H^2_{\varphi,\gamma}(\mathcal{R}_{Q_p}(1)) \xrightarrow{\sim} H^2(Q_p, Q_p(1))$ which is the composition of the canonical isomorphism $H^2_{\varphi,\gamma}(\mathcal{R}_{Q_p}(1)) \xrightarrow{\sim} H^2(Q_p, Q_p(1))$ with the Tate’s trace map is explicitly defined by

$$\iota_\gamma([fe_1]) = -\frac{p}{p-1} \cdot \frac{1}{\log(\chi(\gamma))} \text{Res}_0(f \frac{d\pi}{1+\pi}).$$

**Proof.** Because the map $\iota : H^2_{\varphi,\gamma}(\mathcal{R}_{Q_p}(1)) \xrightarrow{\sim} Q_p : [fe_1] \mapsto \text{Res}_0(f \frac{d\pi}{1+\pi})$ is a well-defined isomorphism, there exists unique $\alpha \in Q_p^\times$ such that $\iota_\gamma = \alpha \cdot \iota$. We calculate $\alpha$ as follows.

We recall that the element $[\log(\chi(\gamma)), 0] \in H^1_{\varphi,\gamma}(\mathcal{R}_{Q_p})$ is the image of $[\log] \in H^1(Q_p, Q_p)$ by the comparison isomorphism. By the proof of Lemma 4.25 for each $a \in \mathbb{Z}_p^\times$, we have

$$\kappa_0(a) = \log(a)((\varphi - 1)(\frac{f}{t}e_1), (\varphi - 1)(\frac{f}{t}e_1)) \in H^1_{\varphi,\gamma}(\mathcal{R}_{Q_p}(1)),$$

where $f \in \mathcal{R}_{Q_p}^\infty$ is an element defined in the proof of Lemma 4.25. Because the cup products are compatible with the comparison isomorphism, then we have

(58) $\iota_\gamma(\kappa_0(a) \cup [\log(\chi(\gamma)), 0]) = \langle \kappa(a), [\log] \rangle = \log(a)$.

By definition of the cup product, we have

$$\kappa_0(a) \cup [\log(\chi(\gamma)), 0] = \log(a)((\varphi - 1)(\frac{f}{t}e_1) \otimes \varphi(\log(\chi(\gamma))))$$

$$= \log(a)\log(\chi(\gamma))((\varphi - 1)(\frac{f}{t}e_1)) \in H^2_{\varphi,\gamma}(\mathcal{R}_{Q_p}(1)).$$

Because we have $\text{Res}_0((\varphi - 1)(\frac{d\pi}{1+\pi})) = -\frac{p-1}{p}$ by the proof of Lemma 4.25 we obtain
\[ \iota_\gamma(\kappa_0(a) \cup \log(\chi(\gamma)), 0) = \alpha \cdot \iota(\kappa_0(a) \cup \log(\chi(\gamma)), 0) \]
\[ = \alpha \cdot \log(\chi(\gamma)) \cdot \log(a) \cdot \left[ (\varphi - 1) \left( \frac{f}{t} \right) e_1 \right] = -\alpha \cdot \log(\chi(\gamma)) \cdot \log(a) \cdot \frac{p - 1}{p}. \]

Comparing this equality with the equality (58), we obtain
\[ \alpha = -\frac{p}{p - 1} \cdot \frac{1}{\log(\chi(\gamma))}, \]
which prove the proposition.

Finally, we prove the following lemma, which completes the calculation of the map \( \kappa_0 : \mathbb{Q}_p^\times \to \mathbb{Q}_p \oplus \mathbb{Q}_p. \)

Lemma 4.27.
\[ \kappa_0(p) = f_{1,1}. \]

Proof. Take \( f_{1,1} = [f_1 e_1, f_2 e_1] \in H^1_{\varphi,\gamma}(\mathcal{R}_{\mathbb{Q}_p}(1)) \) a representative of \( f_{1,1}. \) By definition of the cup product, we have
\[ \iota_\gamma(f_{1,1} \cup e_{1,1}) = \iota_\gamma(f_{1,1} \cup [\log(\chi(\gamma)), 0]) \]
\[ = \iota_\gamma([f_2 e_1 \otimes \varphi(\log(\chi(\gamma)))] = -\frac{p}{p - 1} \cdot \Res_0(f_2, \frac{d\pi}{1 + \pi}) = 0, \]
and
\[ \iota_\gamma(f_{1,1} \cup e_{1,2}) = \iota_\gamma(f_{1,1} \cup [0, 1]) \]
\[ = \iota_\gamma(-[f_1 e_1 \otimes \gamma(1)]) = \frac{p}{p - 1} \cdot \frac{1}{\log(\chi(\gamma))} \cdot \Res_0(f_1, \frac{d\pi}{1 + \pi}) = 1 \]
by Proposition 4.26. Because \( \kappa(p) \in H^1(Q_p, \mathbb{Q}_p(1)) \) satisfies the similar formulae
\[ <\kappa(p), [\text{ord}_p]> = 1, \quad <\kappa(p), [\log]> = 0, \]
we obtain the equality
\[ \kappa_0(p) = f_{1,1}. \]

Using these lemmas, we obtain the following lemma. We define the basis \( f_2 \) of \( H^2_{\varphi,\gamma}(\mathcal{R}_L(1)) \) by \( f_2 := \iota_\gamma^{-1}(1). \)

Lemma 4.28. Tate’s duality pairings
\[ <, > : H^1_{\varphi,\gamma}(\mathcal{R}_L(1)) \times H^1_{\varphi,\gamma}(\mathcal{R}_L) \mapsto H^2_{\varphi,\gamma}(\mathcal{R}_L(1)) \mapsto L \]
and
\[ <, > : H^2_{\varphi,\gamma}(\mathcal{R}_L(1)) \times H^0_{\varphi,\gamma}(\mathcal{R}_L) \mapsto H^2_{\varphi,\gamma}(\mathcal{R}_L(1)) \mapsto L. \]
satisfies the following:

\[ <f_{1,1}, e_{1,1}> = 0, \quad <f_{1,1}, e_{1,2}> = 1 \]

and

\[ <f_{1,2}, e_{1,1}> = 1, \quad <f_{1,2}, e_{1,2}> = 0. \]

\[ <f_2, e_0> = 1. \]

**Proof.** That we have \( <f_{1,1}, e_{1,1}> = 0 \) and \( <f_{1,1}, e_{1,2}> = 1 \) is proven in Lemma 4.27. We prove the formula for \( f_{1,2} \). By Lemma 4.23, we have an equality \( f_{1,2} = \frac{1}{\log(a)} \kappa_0(a) \) for any non-torsion \( a \in \mathbb{Z}_p^* \). Hence, we obtain

\[ <f_{1,2}, e_{1,1}> = \frac{1}{\log(a)} <\kappa(a), [\log]> = 1 \]

and

\[ <f_{1,2}, e_{1,2}> = \frac{1}{\log(a)} <\kappa(a), [\text{ord}_p]> = 0 \]

by the compatibility of the cup products. Finally, that \( <f_2, e_0> = 1 \) is trivial by definition.

\( \square \)

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