Thermodynamic geometry and phase transition of spinning AdS black holes

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Employing the thermodynamic geometry approach, we explore phase transition of four dimensional spinning black holes in an anti-de Sitter (AdS) spaces and found the following novel results. (i) Contrary to the charged AdS black hole, thermodynamic curvature of the spinning AdS black hole diverges at the critical point, without needing normalization. (ii) There is a certain region with small entropy in the space of parameters for which the thermodynamic curvature is positive and the repulsive interaction dominates. Such behavior exists even when the pressure is extremely large. (iii) The dominant interactions in the microstructure of extremal spinning AdS black holes are strongly repulsive, which is similar to an ideal gas of fermions at zero temperature. (iv) The maximum of thermodynamic curvature, $|R|$, is equal to $C_p$, maximum values for the Van der Waals fluid in the supercritical region. While for the black hole, they are close to each other near the critical point.

I. INTRODUCTION

Thermodynamic fluctuation provides a unique frame for the geometrical description of thermodynamical systems in equilibrium. Particular interest goes to the covariant version, known as Ruppeiner geometry [1], which consists of a metric that measures the probability of a fluctuation between two thermodynamic equilibrium states. The Riemannian scalar curvature, known as thermodynamic curvature, arises from such a metric is a fundamental object in the Ruppeiner geometry which contains information about inter-particles interaction. More specifically, a negative (positive) sign of the thermodynamic curvature determines an attractive (repulsive) interaction between particles. While zero value for the thermodynamic curvature means there is no interaction between particles [2–4]. The absolute value of the thermodynamic curvature in the asymptotic critical region is related to correlation length in fluids [3].

Since the discovery of entropy and temperature of black holes [5, 6], it has been well established that one can regard black hole as a thermodynamic system characterized by a set of thermodynamic variables. During the past decades, various thermodynamic properties of black holes, especially the phase transition and critical behavior, have been widely studied in the literatures [7–10]. In recent years, considerable attentions have been arisen to investigate thermodynamic phase transition of anti-de Sitter (AdS) black holes in an extended phase space, where the first law of black hole thermodynamics is extended by treating the cosmological constant as a thermodynamic variable [11–17]. The investigations on thermodynamic phase transition of black holes in the extended phase space have disclosed some interesting phenomena, such as Van der Waals liquid-vapor phase transition [17], zeroth-order phase transition [18], reentrant phase transition [19, 20], triple critical point [21], superfluid like phase transition [22] and many others.

In the context of black hole thermodynamics, thermodynamic curvature in the Ruppeiner geometry provides a powerful tool to explore microscopic behavior of black holes. The obtained results can also be compared with accessible experimental systems. Thermodynamic curvature has been investigated for various types of black holes (see e.g. [23–28] and references therein). It has been disclosed that thermodynamic curvature does not diverge at the critical point, contrary to the fluid systems. Recently, two new normalized thermodynamic curvature for a charged AdS black hole have been proposed, which diverge at the critical point of phase transition [29–31]. These thermodynamic curvature are constructed via the heat capacity at constant volume [29, 30] and adiabatic compressibility [31] and have the same behavior for the large black hole. In [31] it was shown that the normalized thermodynamic curvature diverges to positive infinity for the extremal black holes. More recently, the behavior of these two normalized thermodynamic curvature was studied for several different black holes [32–37].

In this paper, we explore thermodynamic phase structure of four-dimensional rotating AdS black hole. We consider an extended phase space in the pressure ($P$) and entropy ($S$) plane, in which the small-like and large-like black holes are separated by the maximum of the specific heat at constant pressure in the supercritical region. Besides, we provide simple analytical expressions for critical quantities. From the thermodynamic fluctuation metric in the entropy representation, we obtain a Ruppeiner line element of rotating-AdS black holes in the pressure-entropy coordinates, where it is also valid for the ordinary thermodynamic systems, such as the simple Van der Waals fluid. Then, by using the thermodynamic curvature, we explore the microscopic properties of the system and compare it with the one of the Van der Waals fluid system. In particular, we investigate the behavior of the maximum of the specific heat at constant pressure and minimum of thermodynamic curvature for these systems in the supercritical region. We find that, for both cases, the thermodynamic curvature diverges at the critical point and it goes to positive infinity for the extremal

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black holes. Finally, the critical behavior of thermodynamic curvature for the characteristic curves is studied and their critical exponents are calculated.

The rest of the paper is organized as follows. In Sec. II, we first give a brief review on thermodynamics of four-dimensional rotating AdS black hole in the extended phase space and then determine thermodynamic phase structure in the $P$-$S$ plane. Next, we obtain the Ruppeiner metric in $(P,S)$ coordinates, and using this, we study in details the microscopic properties of the black hole and Van der Waals system in Sec. III. Section IV is devoted to investigating the thermodynamic curvature near the critical region. In Sec. V, we present our summary and discussion. In Appendix we calculate the thermodynamic curvature of Van der Waals system using the Ruppeiner metric in $(P,S)$ coordinates.

II. THERMODYNAMIC PHASE STRUCTURE

Let us begin with a brief review of the thermodynamics of single spinning AdS black holes in four dimensions, based on Refs [12, 38]. The mass of the Kerr-AdS black hole with the pressure ($P$) is [12]

$$M(S, P, J) = \frac{1}{2} \sqrt{\frac{(1 + 8PS/3)[4\pi^2 J^2 + S^2(1 + 8PS/3)]}{\pi S}},$$

where $S$ and $J$ are the entropy and angular momentum, respectively. By identifying the black hole mass as the enthalpy, the first law of thermodynamics reads

$$dM = TdS + \Omega dJ + VdP,$$

where $T$ is the Hawking temperature, $\Omega$ the angular velocity, $V$ the thermodynamic volume, which are given by

$$\Omega = \frac{\pi J}{SM}(1 + 8PS/3),$$

$$V = \frac{2}{3\pi M} (S^2[1 + 8PS/3] + 2\pi^2 J^2),$$

$$T = \frac{1}{8\pi M} \left([1 + 8PS/3](1 + 8PS) - 4\pi^2 J^2/S^2) \right).$$

The internal energy $U$ is obtained from $M$ via the Legendre transformation, $U = M - PV$, and it is given by

$$U(S, V, J) = \frac{\pi}{S} \left\{ \left( \frac{3V}{4\pi} \right) \left( \frac{S^2}{2\pi^2} + J^2 \right) - J^2 \sqrt{\frac{3V}{4\pi}} - \left( \frac{S}{\pi} \right)^3 \right\}.$$  

In this representation, the first law of the black hole thermodynamics is written as

$$dU = TdS + \Omega dJ - PdV.$$  

Now, we turn to study the critical behavior of the rotating-AdS black hole by investigating the specific heat at constant pressure

$$C_p = T \frac{\partial S}{\partial T} \bigg|_P,$$

where we have also fixed $J$. For constant $J$ and $P = P_c$, the value of critical point can be determined by an inflection point

$$\frac{\partial T}{\partial S} \bigg|_{P_c} = 0, \quad \frac{\partial^2 T}{\partial S^2} \bigg|_{P_c} = 0.$$  

Using the temperature formula in Eq. (5), the critical quantities are obtained analytically as

$$S_c = \frac{24\pi J}{(73 + 6\sqrt{87})^{1/3} + (73 - 6\sqrt{87})^{1/3} - 5} \approx 28.719J,$$

$$P_c = \frac{(73 + 6\sqrt{87})^{1/3} + (73 - 6\sqrt{87})^{1/3} - 5}{768\pi J} \approx 0.003/J,$$

$$T_c^2 = \frac{(6137 + 768\sqrt{87})^{1/3} - 239}{384\pi^2 J} \approx 0.002/J.$$  

These quantities are the same, numerically, as ones found in Ref. [39]. Here we present their analytical expressions for the first time in a compact form. For $P > P_c$, the specific heat at constant pressure is positive, i.e., black hole is thermodynamically stable. However, below $P_c$, there exists a certain range of quantities, for which the specific heat at constant pressure is negative ($C_p < 0$). This corresponds to a thermodynamic instability of the black hole which is remedied by the Maxwell equal area construction, $\oint VdP = 0$, indicating a first order phase transition between small and large black holes. The region of the first order phase transition, which is obtained from the the Maxwell construction, is identified in the $P$-$S$ plane in Fig. 1. The small and large black hole phases are located at the left and right of the shaded region, respectively. In Fig. 1, the extremal black hole curve (corresponding to zero temperature) is denoted by the gray dashed line and the critical point is indicated by a black solid circle. The left region of the gray dashed curve is physically excluded because the temperature becomes negative.

For the supercritical region, which is at higher pressures and entropies than the critical point, we illustrate the local maximum of the specific heat at constant pressure ($C_p$) in Fig. 1 by the purple dotted line. The local maximum of $C_p$ commences from $(\tilde{P}, \tilde{S}) \approx (1.69, 1.45)$ and terminates at the critical point, where it goes to infinity and $\tilde{P} = P/P_c$ and $\tilde{S} = S/S_c$ are the reduced pressure and entropy, respectively. This curve can be viewed as an extension to the coexistence line, which divides the supercritical region into two phases [40, 41]. Here, the small-like and large-like black holes are separated by the local maximum of $C_p$ in the supercritical region beyond the critical point.
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vanishing (brown dot-
for rotating AdS black hole Eq.(
AdS black hole and the Van der Waals fluid.
behavior of the thermodynamic curvature for the rotating
extremal black hole (\(T = 0\)) and the local maximum of \(C_p\),
respectively; a black spot represents the critical point. The
several isothermal lines are denoted by the thin curves where,
top to bottom, the temperature is decreased. The region
on the left of the gray dashed curve is excluded since it leads
to negative temperature.

FIG. 1: Phase structure of the rotating AdS black hole in
\(P-S\) plane. The coexistence region of the first order phase
transition between the large and small black holes is identified.
The gray dashed and purple dotted curves correspond to the
external black hole (\(T = 0\)) and the local maximum of \(C_p\),
respectively; a black spot represents the critical point. The
domain regime. In this region, when approaching the gray

III. THERMODYNAMIC CURVATURE

To set up a thermodynamic Riemannian geometry, we
consider the rotating AdS black hole in the canonical
(fixed \(J\)) ensemble of extended phase space so that its
thermodynamic state is specified by the internal energy
\(U\) and volume \(V\). The line element of the geometry,
which characterizes the distance between thermodynamic
states, is given by [1]

\[
dl^2 = -\frac{\partial^2 S}{\partial x^\mu \partial x^\nu} dx^\mu dx^\nu, \quad (11)
\]

where \(S\) is entropy and \(x^\mu = (U, V)\). Using the first law
for rotating AdS black hole Eq.(7) and the Maxwell relation,
one can express the line element Eq.(11) as follows\(^1\)

\[
dl^2 = \frac{1}{T} \left( \frac{\partial T}{\partial S} \right)_P dS^2 - \frac{1}{T} \left( \frac{\partial V}{\partial P} \right)_S dP^2. \quad (12)
\]

By computing the Riemannian curvature scalar, \(R\), (thermodynamic curvature) from the metric, one can get some
information about the interparticle interaction in
the thermodynamic system. In particular, the positive (negative) sign of the thermodynamic curvature specifies that
the dominant interaction is repulsive (attractive) [2–4].
On the other hand, \(R = 0\) shows there is no interaction
in the system [42]. In what follows, we examine the behavior of the thermodynamic curvature for the rotating
AdS black hole and the Van der Waals fluid.

\(^1\) Although this line element is derived for the rotating AdS black
hole, it remains valid for an ordinary thermodynamic system [43].

FIG. 2: Behavior of the thermodynamic curvature \(R\) for the
rotating AdS black hole.

For the four-dimensional rotating AdS black hole, the
thermodynamic curvature is readily calculated as

\[
R = \frac{B(\tilde{S}, \tilde{P})}{JT[(\partial T/\partial S)_P]^2}. \quad (13)
\]

where \(B(\tilde{S}, \tilde{P})\) is a complicated function of the reduced
pressure \((\tilde{P})\) and entropy \((\tilde{S})\) and \(\tilde{T} = T/T_c\) is the reduced temperature. Note that \(R\) is proportional to
the inverse of angular momentum in the reduced parameter
space. The behaviour of \(R\) is depicted in Fig. 2 as a
function of \(P/P_c\) and \(S/S_c\). One can see from Fig. 2 that \(R\) is positive in some region of the parameter space.
From Eq.(13), \(R\) diverges on \(\tilde{T} = 0\) and \((\partial T/\partial S)_P = 0\)
corresponding to the extremal black holes and diverging
specific heat at constant pressure, respectively.

In order to examine the thermodynamic curvature
more closely, we plot in Fig. 3 the vanishing (brown dotted line) and diverging (gray dashed line) curves of \(R\) as
well as the transition curve (light blue solid line) of small
and large black holes and local maximum of \(C_p\) (purple
dotted line), which were shown already in Fig. 1. In
Fig. 3, the shaded regions represent positive values of \(R\),
where the dominant interaction is repulsive. In contrast,
\(R\) is negative everywhere outside the shaded regions, indicating the dominant attractive interaction. Remarkably,
the transition and diverging curves coincide at the critical
point which is highlighted by a black spot. This
situation also occurs for ordinary thermodynamic systems
[24]. The white area to the left of the gray dashed line on
the left side of the figure is excluded because of a negative
temperature. One can see from Fig. 3 that the associated
\(R\) for the large black hole phase is negative. However, for
the small black hole phase, there exists a certain region
with positive \(R\), which is also present in the higher
pressure regime. In this region, when approaching the gray
reveals the existence of a region with negative interaction becomes strongly repulsive. The inset dashed curve from above, \( R \) diverges to \(+\infty\) and dominant interaction becomes strongly repulsive. The inset in Fig. 3 reveals the existence of a region with negative \( R \) in the shaded region when \( \bar{P} \) is greater than \( \approx 242.78 \). Moreover, in Fig. 3 we also display the local minimum of \( R \) in the supercritical region by the thin green line, which begins from \( (\bar{P}, \bar{S}) \approx (1.41, 1.38) \) and ends at the critical point where \( R \) goes to negative infinity.

In Fig. 4, we depict the coexistence curve (light blue solid line) of the Van der Waals vapor-liquid phase transition and maximum of \( C_p \) (purple dotted line) as well as the diverging (gray dashed line) and minimum (thin green line) of \( R \), where the expression of \( R \) is given in Appendix. According to Eq. (A4) and Fig. 4, \( R \) has negative values everywhere, indicating the dominant attractive interaction among the molecules. The coexistence and diverging curves coincide at the critical point, which is marked by a black dot. Furthermore, as also seen in Fig. 4, the maximum of \( C_p \) and minimum of \( R \) curves match each other in the supercritical region. For the region below the coexistence curve, the Van der Waals model is inapplicable, so it is not considered here.

### IV. CRITICAL PROPERTIES

To further clarify the critical behavior of thermodynamic curvature for the rotating AdS black hole and associated critical exponent, we investigate the thermodynamic curvature of characteristic curves around the critical point. To do so, in Fig. 5, we illustrate \( R \) along its minimum and maximum of \( C_p \) curves as well as along the transition curve for small and large black holes in the neighborhood of the critical temperature. As evident from the figure, the large black hole is at higher \(|R|\) than the small black hole and upon approaching the critical point, \( R \) in both phases diverges as

\[
R \approx -\frac{41.2}{J}|t|^{-2},
\]

with a universal critical exponent of 2, where \( t = T/T_c - 1 \) is the deviation from the critical temperature. In the supercritical regime, the local minimum of \( R \) and maximum of \( C_p \) curves are close together in thermodynamic curvature and they diverge from above \( T_c \) as

\[
R \approx -\frac{165.3}{J}|t|^{-2},
\]

implying a critical exponent of 2.

For the Van der Waals fluid, the thermodynamic curvature of the vapor and liquid along the coexistence curve...
near the critical temperature has the following form

\[ R = -\frac{1}{12} |t|^{-2}. \]  

(16)

Moreover, upon approaching the critical point from above along the minimum of \( R \) and maximum of \( C_p \) curves, \( R \) diverges with the exponent 2 as

\[ R = -\frac{1}{3} t^{-2}. \]  

(17)

V. SUMMARY AND DISCUSSION

Thermodynamic geometry of black holes provides a powerful tool to explore microscopic structure of these systems and disclose the nature of interaction between their ingredient particles. In this paper, we have presented simple exact analytical expressions for the critical quantities of the Kerr-AdS black holes and constructed the phase diagram in the pressure-entropy parameter space, where the small black hole and large black hole phases are separated by a first order phase transition region below the critical point. Based on the locus of the maxima of the specific heat at constant pressure, we divided the supercritical region into small-like and large-like black hole regions. Indeed, the line of maxima is used as the Widom line, which is characterized by the maximum of the correlation length. In addition, starting from the Ruppeiner geometry in an entropy representation, we have derived the thermodynamic metric for the Kerr-AdS black holes in the pressure-entropy coordinates that is also valid for any ordinary thermodynamic system. We have explicitly shown that, contrary to the charged AdS black hole [43], thermodynamic curvature of the Kerr AdS black hole diverges at the critical point, without needing normalization. Comparing to the simple Van der Waals fluid, which has negative thermodynamic curvature everywhere, we have found that there is a certain region for the spinning AdS black holes with small entropy in the space of parameters for which the thermodynamic curvature is positive and the repulsive interaction dominates. Such behavior exists even when the pressure is extremely large. Another distinction is that the dominant interactions in the microstructure of extremal Kerr AdS black holes are strongly repulsive, which is similar to an ideal gas of fermions at zero temperature [2].

Taking into account the fact that the magnitude of the thermodynamic curvature is related to the correlation length, we have used the locus of the maximum of \( |R| \) to characterize the Widom line. We have found the maximum of \( |R| \) is equal to \( C_p \) maximum values for the Van der Waals fluid in the supercritical region. While for the black hole, they are close to each other near the critical point. Finally, we determined the critical behavior of thermodynamic curvature of spinning AdS black hole and find out that governs by a universal critical exponent of 2, which is the same as the Van der Waals fluid.

It would be interesting to study reentrant phase transitions and universal properties of higher dimensional rotating AdS black holes by employing the thermodynamic Riemannian geometry based on the fluctuations of the entropy and pressure.

**Note Added:** When this work was completed, we learned that another article [45] had addressed the same issue where it was shown that the thermodynamic curvature has different behavior at small entropy. However, our results differ from [45] in that we find a region within repulsive interaction area in which the thermodynamic curvature has negative values.

**Appendix A: Van der Waals model**

In this Appendix, we calculate the thermodynamic curvature for Van der Waals fluid in the \( P-S \) plane. The specific Helmholtz free energy of the Van der Waals, which contains two parameters \((a,b)\) reflecting intermolecular interaction and molecular size effects, is given by [44]

\[ F = -\frac{a}{v} - T \left( \ln[v-b] + \frac{3}{2} \ln[T] + \ln[\zeta] + 1 \right), \]  

(A1)

where \( \zeta = (m/2\pi)^{3/2} \) and \( m \) is a mass of atom. Here, \( T \) and \( v \) are the temperature and specific volume, respectively. It is important to note that \( v > b \). Using Eq.(A1), the pressure and entropy are obtained as

\[ \bar{P} = \frac{8\bar{T}}{3\bar{v} - 1} - \frac{3}{\bar{v}^2}, \quad \bar{s} = \frac{\bar{T}(3\bar{v} - 1)^{2/3}}{2^{5/3}}, \]  

(A2)

which is expressed in terms of the reduced thermodynamic variables

\[ \bar{T} = \frac{T}{T_c}, \quad \bar{P} = \frac{P}{P_c}, \quad \bar{v} = \frac{v}{v_c}, \quad \bar{s} = \frac{s}{s_c}, \]

where \( s \equiv e^{(2S-3)/3}/\zeta^{2/3} \) and \( S \) is entropy. The critical quantities are

\[ P_c = \frac{a}{27b^2}, \quad v_c = 3b, \quad s_c = \frac{2^{11/3}a}{27b^{1/3}}, \quad T_c = \frac{8a}{27b}. \]  

(A3)

Using the line element in \((P,S)\) coordinates Eq.(12), the thermodynamic curvature is obtained as

\[ R = \frac{(3\bar{v} - 1)^{8/3}[3(3\bar{v} - 1)^{8/3} - 2^{11/3}s^{2/3}]}{3[3(3\bar{v} - 1)^{8/3} - 2^{5/3}s^{2/3}]^2}, \]  

(A4)

which is independent of \( a \) and \( b \).
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