Black Hole Shadows: How to Fix the Extended Gravity Theory

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Abstract. The first images of black hole shadows open new possibilities to develop modern extended gravity theories. We discuss the shadow calculations in non-rotating case both when $g_{11} = -\tilde{g}_{00}$ and $g_{11} \neq -\tilde{g}_{00}$. We demonstrate the application to few different models: Horndesky theory with Gauss-Bonnet invariant, loop quantum gravity and conformal gravity. The difference of these theories from shadow models with the theory of general relativity is shown. In addition we show that when the rotation is taken into account the requirements to the observational accuracy decrease.

Keywords: extended gravity, black hole, black hole shadow, Event Horizon Telescope
1 Introduction

More than 100 years ago first solutions of black hole type were obtained theoretically and nowadays the existence of these objects was proven by a lot of observational tests. The dynamics of binary systems [1], the results of gravitational wave astronomy [2], direct images of black hole shadows from Event Horizon Telescope [3] are the most well-known examples. The General Theory of Relativity (GR) describes almost all the astronomical data with great accuracy. Meanwhile some problems as dark matter, dark energy, the early Universe evolution, quantum gravity and so on are not resolved yet. Therefore new gravity models are developed to include the explanations of these phenomena. These new gravity models, extending GR, are, for example, $f(R)$ gravity [4], scalar-tensor theories including the generic case with second order field equations: Horndesky theory [5–8], teleparallel models [9], gravity models with conformal symmetry [10, 11], loop quantum gravity [12–14] and many other ones. It seems important to constraint these approaches in maximally wide parameter range and the study of black holes neighbourhoods provides such a possibility thanks to the achievements in shadow imaging.

As the physical equipment has the limiting accuracy each experimental result cause alternative explanations from different theories [15]. At the first step the most simple model is usually chosen. Further more detailed data (with some values treated as characteristic ones) allows to choose the underlying theory. To proceed this one has to take into account the effects from further perturbative orders (for example, as during the old discussion on possibility to distinguish different types of black holes at Large Hadron Collider [16]). So the shadow size being the first value obtaining at observations is used to estimate the model predictions at our work. We take the well-known GR space-times: Schwarzschild one, Kerr one, ... as the basic approximations.

Previously we discussed the shadow modelling, last stable orbit and strong gravitational lensing calculations in a case when the third approximation in spherically-symmetric space-time is taken into account. Such metrics represent the continuation of Reissner-Nordstrom
one by the next $r^{-1}$ expansion order[17]. In[18] the rotation was included. It was shown that the restriction on maximal shadow size equal to $4M$ in Reissner-Nordstrom case[17] is not actual. The consideration could be extended to next perturbation orders $r^{-n}$. Note that the observational results of shadow size, last stable orbit, strong gravitational lensing could be used to correct the metric expansions. To obtain the black hole (BH) parameters with third correction in its metrics two different probes are required. Increasing amount of probes allows to calculate more expansion orders. Also in rotating case each point of shadow border after extracting serves as an independent probe. The increasing of the rotational speed expands the range of potential probes simplifying the calculation of expansion orders. Therefore in our paper we calculate the distribution of background intensity for BH shadow. We also extend the consideration to the case $g_{11} \neq -g_{00}^{-1}$ and demonstrate the applications to the BH metrics in Horndesky theory with Gauss-Bonnet invariant [8], BH in modified Hayward metric[13, 14] and BH in new massive conformal gravity [11].

The rest part of the paper is organized as follows. Section 2 is devoted to the Taylor expansions of the metrics, Section 3 describes the calculation of the background intensity distribution in BH shadow, Section 4 extends the consideration to the case $g_{11} \neq -g_{00}^{-1}$ and discusses the mentioned examples, Section 5 contains the discussion and conclusions.

2 Black hole Solutions in Extended Gravity

The generic description of asymptotically flat static spherically-symmetric space-time in modified gravity represents the extension of Schwarzschild metric in the form:

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

(2.1)

where $A(r)$ and $B(r)$ are metric functions depending upon radial coordinate $r$. The standard Schwarzschild metrics in Planckian units $G = c = \hbar = 1$ corresponds to

$$A(r) = B^{-1}(r) = 1 - \frac{2M}{r}. $$

(2.2)

Note that Schwarzschild metric, Reissner-Nordstrom one and their extensions represent the first terms of Taylor expansion at $r >> 2M$, where $M$ is a mass of central object. It is possible to consider the Schwarzschild metrics as a way to describe the star’s trajectories around central BH. Reissner-Nordstrom metrics as next expansion order describes the influence of electrical or tidal charges [19] and sometimes could drastically change shadow properties [17, 18, 20, 21].

Here we would like to point out that in extended gravity models a degenerated case of “symmetrical” metric functions $A(r) = B(r)^{-1}$ could occur. So the event horizon position is defined as $A(r) = 0$. When this equation more than one solution one has to consider the external one.

Now we restrict the series of $A(r)$ by the third expansion order as:

$$A(r) = 1 - \frac{2M}{r} + \frac{Q}{r^2} + \frac{C_3}{r^3}. $$

(2.3)

We normalize all the values on BH mas: $\hat{r} = r/M$, $q = Q/M^2$, $c_3 = C_3/M^3$, therefore in Planckian units $M$, $Q$, $C_3$ and $r$ become unit free. Hence the configuration space becomes two dimensional and:

$$A(\hat{r}) = 1 - \frac{2}{\hat{r}} + \frac{q}{\hat{r}^2} + \frac{c_3}{\hat{r}^3}. $$

(2.4)
3 Background of Black Hole Shadow in Extended Gravity

The set of unstable photon orbits forms a photon sphere defining the BH shadow boundary. Photons from a source situated at big distance with sighting parameter \( b \) greater than its critical value \( b_{\text{ph}} \) pass outside this sphere and reach the distance observer. Photons with \( b < b_{\text{ph}} \) interact with the BH forming a spot at the image: the shadow. So the visible shadow image from non-rotating (or slowly rotating) BH has a form of a disk. The radius of this disk is defined by critical sighting parameter (\( b_{\text{ph}} = 3\sqrt{3}M \) for Schwarzschild BH [22]). The shadow image form is corrected by the strong gravitational lensing.

Consider the optically thin accretion disk surrounding the compact object [23]. We follow [22] and modify his approach for the symmetric case \( A(r) = B(r) \) (2.1). Note that the radiation is emitted from all the surface outside the horizon including the regions inside the photon sphere. The specific intensity \( I_{\nu_0} \) that could be registered (usually measuring in \( \text{ergs}^{-1}\text{cm}^{-2}\text{str}^{-2}\text{Hz}^{-1} \)) at visible photon frequency \( \nu_0 \) and the position \((X, Y)\) (coordinates at image plane) on the sky sphere is equal to [22]:

\[
I_{\nu_0} = \int_\gamma z^3 j(\nu_e) dl_{\text{prop}} , \tag{3.1}
\]

where \( \nu_e \) is emitted frequency, \( z = \nu_0/\nu_e \) is redshift, \( j(\nu_e) \) is volume unit emitting potential of resting source, \( dl_{\text{prop}} = -k_\alpha u_\alpha d\lambda \) is the differential of length unit in the source frame, \( k^\mu \) is 4-speed of the photon, \( u_\mu^e \) is 4-speed of the BH and \( \lambda \) is the affine parameter along the photon \( \gamma \) trajectory. Index \( \gamma \) means the integration along isotropic geodesics. Red-shift \( z \) is defined as [22]:

\[
z = \frac{k_\alpha u_\alpha^0}{k_\beta u_\beta^0} , \tag{3.2}
\]

where \( u_\alpha^0 = (1, 0, 0, 0) \) is 4-speed of a distance observer.

Consider the simple model of accretion gas in spherically-symmetric space-time. Following [22] we suppose that the gas freely falls in radial direction to the center of BH with the following 4-speed:

\[
u_e^t = \frac{1}{A(\hat{r})}, u_e^r = -\sqrt{1 - A(\hat{r})}, u_e^\theta = u_e^\phi = 0 . \tag{3.3}
\]

\( k^\mu = \dot{x}^\mu \) was calculated yet in [22], using these results:

\[
k^r/k^t = \pm \sqrt{\frac{1}{A(\hat{r})} \left( \frac{1}{A(\hat{r})} - \frac{b^2}{\hat{r}^2} \right)} , \tag{3.4}
\]

where the sign “+(-)” denotes the moving from (to) BH. So the redshift transfers to [24]:

\[
z = \frac{1}{\frac{1}{A(\hat{r})} - \frac{k_\alpha}{k_\beta} \sqrt{1 - A(\hat{r})}} . \tag{3.5}
\]

Continue following [22] for the shadow profile we consider the model where the frequency of resting source is \( \nu_s \), radiation is monochromatic and has the radial profile \( 1/\hat{r}^2 \):

\[
j(\nu_e) \propto \frac{\delta(\nu_e - \nu_s)}{\hat{r}^2} , \tag{3.6}
\]
where $\delta$ is Dirac delta-function. The differential of length unit in the resting frame is defined as:

$$dl_{\text{propo}} = -k_\alpha u^\alpha e_d\lambda = -\frac{k^4}{zkr^2} dr.$$  \hfill \text{(3.7)}

Integrating the Eq. (3.1) over all the observing frequencies one obtains the observable intensity of photons at the position $(X, Y)$ on sky sphere $[22]$:

$$I_{\text{obs}}(X, Y) \propto \int \frac{z^3k^4}{r^2} dr.$$  \hfill \text{(3.8)}

The unique value of sighting parameter $b$ corresponds to each position at the image plane $(X, Y)$ and is equal to $b^2 \propto X^2 + Y^2$. After the numerical integration we obtain the intensity profile of BH shadow.

The dependence of shadow size upon $q$ and $\alpha$ was calculated earlier in $[17]$. It was shown that if the shadow size is greater than $4M$ it could be described only with the help of addition degree of freedom, namely $q$. Hence in the first expansion order such shadow has to be parametrized by the Reissner-Nordstrom metrics. When the intensity profile begins to differ one has to incorporate next perturbation orders.

When the third and further expansions are considered the shadow description stops to be unique and allows a set of different parameter combinations. This occurs because the increasing of equation order leads to the addition solutions appearing. Therefore it requires more observational data to constraint the theoretical model. To proceed in addition to the shadow size one can use such values as the last stable orbit radius, strong gravitational lensing of the bright object close to BH and the distribution of background intensity. As previously the consideration starts from the Schwarzschild space-time.

The figure 1 demonstrates the intensity of the shadow profile for BH shadow with $q = 0.2519$, $\alpha = -0.7515$. The key moment is that its size is equal to the corresponding Schwarzschild one. Extracting the difference from the Schwarzschild BH normalized on maximal intensity leads to: $I_{\text{max}} \approx 0.6$ (Fig. 2). As one can conclude from Fig. 2, this difference grows when additional parameters increase. The maximal difference takes place near the shadow boundary, then it vanishes while going to infinity. The difference inside shadow is constant. Further, from Fig. 2 one concludes that for fixing this difference in the observations the intensity resolution greater than $0.1\%$ from maximal intensity is required. Note that each point of the profile could serve as an addition probe of BH potential.

4 Shadow model at $A(r) \neq B^{-1}(r)$. 

In general spherically-symmetric case $A(r) \neq B^{-1}(r)$ (see Eq. (2.1)). To extend our consideration we start from equations of motion in the form:

$$\left(\frac{d\hat{r}}{d\tau}\right)^2 + \frac{L^2}{B(\hat{r})\hat{r}^2} = \frac{E^2}{A(\hat{r})B(\hat{r})},$$  \hfill \text{(4.1)}

$$\frac{d\phi}{d\tau} = \frac{L}{\hat{r}^2},$$  \hfill \text{(4.2)}

where $E$ is photon energy, $L$ is the angular momentum of the photon beam and $\tau$ is affine parameter. After substitution Eq. (4.2) to Eq. (4.1) these equations transform to:

$$u(\hat{r}) = \left(\frac{d\hat{r}}{d\phi}\right)^2 = \frac{\hat{r}^4}{D^2A(\hat{r})B(\hat{r})} - \frac{\hat{r}^2}{B(\hat{r})}.$$  \hfill \text{(4.3)}
where $D = L/E$ is sighting parameter of the photon beam. Analogously to the symmetric case the end of the shadow corresponds to the place where the photon trajectory becomes unstable. The corresponding equation is:

$$u(r) = 0, \quad \frac{du(r)}{dr} = 0, \quad \frac{d^2u(r)}{dr^2} > 0.$$  

(4.4)

To calculate the shadow size one has to find the maximal solution of Eqs (4.4). We proceed this numerically for the set of extended gravity models.

5 Shadow in Different Theories

5.1 Horndesky Theory

Horndesky model [25] is the most general case of scalar-tensor gravity with the second order field equations [26]. After GW170817 event Horndesky theory was constrained and now it is used in the extended form of DHOST (degenerated higher-order scalar-tensor) theories with a lot of interesting consequences [27]. Therefore we consider the BH type solution in Horndesky

$q=0.2519 \ c_3=-0.7515$
Figure 2. The dependence of the absolute value of the intensity difference $|I - I_{sh}|/I_{max}$ between BH with additional parameters $q$ and $\alpha$ and Schwarzschild one upon the distance from BH center on image plane $X$ in the units of $M$.

theory with the linear coupling with Gauss-Bonnet invariant [8]. Metric functions are:

$$A(r) = 1 - \frac{2M}{r} - \frac{2C_7}{r^7}$$  \hspace{1cm} (5.1)

$$B(r)^{-1} = 1 - \frac{2M}{r} - \frac{C_7}{r^4}$$  \hspace{1cm} (5.2)

where $C_7$ is the specific combination of model constants. In [8] only positive values of $C_7$ were considered. Otherwise the requirement that the position of the horizon $A(r_h) = 0$ must be situated outside the surface $B(r) = 0$ fulfills only if $C_7 < 0$. When $C_7 > 0$ object does not represent a BH. So it is reasonable to suppose that the metric (5.1) is valid outside the photon sphere but near the horizon a more accurate expansion compatible with the BH definition is required. The numerical results of shadow size value depending upon metric functions from Eq. (5.1) are presented at Fig. 3. The shadow size for metric functions (5.1) differs from corresponding Schwarzschild values less than 0.01% (at $|C_7| < 0.5$) even when $C_7$ value is compatible with $M$ one. This means that observational data compatible with simple model also does not forbid Horndesky theory.
Figure 3. The dependence of shadow size \((D)\) against the combination of model constants \(C_7\) for Horndesky theory coupled with Gauss-Bonnet invariant (in the units of \(M, M = 1\)).

5.2 Loop Quantum Gravity

As the next application we consider loop quantum gravity as a perspective direction for quantifying gravity [28]. We apply the modified Hayward metric [13, 14] as BH solution. The important property of this BH is the absence of a central singularity that is why it is called regular BH. Its modification includes a time delay and the 1-loop quantum correction. The metric functions in this case are:

\[
A(r) = (1 - \frac{2Mr^2}{r^3 + 2Ml^2})(1 - \frac{\alpha \beta M}{\alpha r^3 + \beta M})
\]

\[
B(r)^{-1} = 1 - \frac{2Mr^2}{r^3 + 2Ml^2}
\]

where \(l\) is a convenient encoding of the central energy density \(3/8\pi l^2\), the constant \(\alpha\) describes the time delay between the center and infinity and \(\beta\) is related to the 1-loop quantum corrections of the Newtonian potential. These parameters were constrained in [13, 14] as: \(0 \leq \alpha < 1, \beta_{\text{max}} = 41/(10\pi)\). When \(l > \sqrt{16/27} M\) the object has no horizon. After solving the Eqs. (5.1) one obtains the shadow size for different \(l, \alpha\) and \(\beta\). Fig.4 shows the characteristic dependencies of the shadow size against \(l, \alpha\) and \(\beta\). During increasing of \(l\) the shadow size decreases. In opposite the increasing of \(\alpha\) and \(\beta\) leads to shadow size increasing. If \(\beta \geq 0\) the minimal shadow size is reached at \(l = \sqrt{16/27} M, \beta = \alpha = 0\) and is equal to \(4.92M\). The maximal shadow size occurs when \(l = 0, \beta = 41/(10\pi), \alpha = 1\) and is equal to \(5.32M\). Note that the shadows of such size could be described also by the Reissner-Nordstrom
space-time. So using only the shadow size it is impossible to find all these parameters and new observational data is required.

Figure 4. The shadow size $D$ dependence upon the time delay $\alpha$ when $l = 0.5M$ and $\beta = 0.5$ (left image), upon the 1-loop quantum corrections $\beta$ when $l = 0.5M$, $\alpha = 0.5$ (central image), upon the central energy density $l$ when $\alpha = 0.5$, $\beta = 0.5$ (right image) for BH in modified Hayward metric (in the units of $M$, $M = 1$).

5.3 Conformal Gravity

The next example is conformal gravity [29] as such symmetry introduced in the action allows to quantify the theory. There are many branches, for example, models with nonlinear symmetry realization [30, 31]. As an example for shadows we take BH metrics in new massive conformal gravity [11]:

$$A(r) = 1 - \frac{2M}{r} + \frac{Q_s^2}{r^2} + \frac{Q_s^2(-M^2 + Q_s^2 + \frac{6}{m_2^2})}{3r^4} + \ldots \tag{5.5}$$

$$B(r)^{-1} = 1 - \frac{2M}{r} + \frac{2Q_s^2}{r^2} + \frac{2Q_s^2(-M^2 + Q_s^2 + \frac{6}{m_2^2})}{3r^4} + \ldots \tag{5.6}$$

where $Q_s$ is a scalar charge and $m_2$ is a massive spin-2 mode. This asymptote is obtained far from the horizon. The key point for shadow size calculation is the transfer of photons to photon orbit. For such an analysis this asymptote appears to be applicable. Fig. 5 shows the dependence of the shadow size against scalar charge $Q_s$ for different values of $m_2$. The ends of the lines correspond to the effect described in [17, 20]: the big values of $Q_s$ and $1/m_2$ cause the absence of photon sphere. Decreasing of $m_2$ value causes the shadow size to become smaller. To constraint the model parameters additional observational data is also required.

6 Discussion and Conclusions

The resolution of the first BH images from Event Horizon Telescope was not very high being about half of the object size[3]. Further improving of ground-based equipment could increase the resolution only for few times[32] (not orders). The maximally possible size of the ground base radio-telescope system is already reached. As it was demonstrated earlier [17, 18] and in this paper the constraining of extended gravity models requires the accuracy improving for few orders (not times!) than it was reached[3]. Therefore it is necessary to make the next step and use satellites. Moreover the shadow size measuring by itself is enough only for the theories based on Reissner-Nordstrom metrics and easier. For the theories with more
Figure 5. The dependence of the shadow size $D$ upon the scalar charge $Q_s$ for a BH in new massive conformal gravity with different values of massive spin-2 mode $m_2$ (in the units of $M$, $M = 1$). Black line corresponds to $m_2 \rightarrow \infty$, red one corresponds to $m_2 = 2$, blue one corresponds to $m_2 = 1$, green one corresponds to $m_2 = 0.707$, orange one corresponds to $m_2 = 0.577$, purple one corresponds to $m_2 = 0.5$.

complicated black hole metrics the amount of observational points must be increased. In a case of measuring the last stable orbit, strong gravitational lensing of bright stars [17] and the distribution of shadow background intensity the minimal requiring resolution is equal to 0.001 of shadow size for the additional coefficients compatible with BH mass $M$. Therefore the study of the shadow from fast rotating object seems to be more perspective. With the same values of additional coefficients the requiring resolution is about 0.01 of shadow size [18].

Here it is importantly to emphasis that the approach without taking into account a black hole rotation is valid only when its rotation speed is small and can be neglected. The inclusion of the rotation by itself increases the amount of probes necessary to distinguish between different extended gravity models but decreases the requirements to the observational accuracy.

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