FLUCTUATIONS, CORRELATIONS AND FINITE VOLUME EFFECTS IN HEAVY ION COLLISION

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Finite volume corrections to higher moments are important observable quantities. They make possible to differentiate between different statistical ensembles even in the thermodynamic limit. It is shown that this property is a universal one. The classical grand canonical distribution is compared to the canonical distribution in the rigorous procedure of approaching the thermodynamic limit.

I. INTRODUCTION

Fluctuations and correlations measured in heavy ion collision processes give better insight into dynamical and kinematical properties of the dense hadronic medium created in ultrarelativistic heavy ion collisions. Particle production yields are astonishingly well reproduced by thermal models, based on the assumption of noninteracting gas of hadronic resonances [1]. Systems under considerations are in fact so close to the thermodynamic limit that final volume effects can be neglected — at least when productions yields are considered.

The aim of the paper is to show that finite volume effects become more and more important when higher moments, e.g. correlations and fluctuations are considered. The basic physical characterization of the system described by means of the thermal model are underlying probability densities that given physical observables of the system have specified values. The only way to reproduce those probability distribution is by means of higher and higher probability moments. Those moments are in fact the only quantities which are phenomenologically available and can be used for the verification of theoretical predictions. Finite volume effects are also important for the lattice QCD calculations.

Particle yields in heavy ion collision are the first moments, so they lead to rather crude comparisons with the model. Fluctuations and correlations are second moments so they allow for the better understanding of physical processes in the thermal equilibrium.

A preliminary analysis of the increasing volume effects was given in [2, 3]. It has been rigorously shown an influence of $O(1/V)$ terms for a new class physical observables — semi-intensive quantities [3]. Those results completely explained also ambiguities noted in [4], related to "spurious non-equivalence" of different statistical ensembles used in the description of heavy ion collision processes.

This paper is devoted to a further analysis of $O(1/V)$ terms. It is shown that those terms are not specific for systems with subsidiary internal symmetries but appear also in the simplest "classical" problems of statistical physics.

II. CHOICE OF VARIABLES

In the thermodynamical limit the relevant probabilities distributions are those related to densities. These distributions are expressed by moments calculated for densities — not for particles. In the practice, however, we measure particles — not densities as we do not know related volumes. Fortunately, volumes can be omitted by taking corresponding ratios.

Let us consider e.g. the density variance $\Delta n^2$. This can be written as

$$\Delta n^2 = \langle n^2 \rangle - \langle n \rangle^2 = \frac{\langle N^2 \rangle - \langle N \rangle^2}{V^2}.$$
By taking the relative variance

$$\frac{\Delta n^2}{\langle n \rangle^2} = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2},$$

volume-dependence vanishes.

A. Semi-intensive variables

A special care should be taken for calculations of ratios of particles moments. Although moments are extensive variables their ratios can be finite in the thermodynamic limit. These ratios are examples of semi-intensive variables. They are finite in the thermodynamic limit but those limits depend on volume terms in density probability distributions. One can say that semi-intensive variables "keep memory" where the thermodynamic limit is realized from.

Let consider as an example the scaled particle variance

$$\frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle^2} = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle}.$$

The term

$$\frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle}.$$

tends to zero in the thermodynamic limit as $O(V^{-1})$. So a behavior of the scaled particle variance depends on the $O(V^{-1})$ term in the scaled density variance. A more detailed analysis of semi-intensive variables is given in [3].

To clarify this approach let us consider a well known classical problem of Poisson distribution but taken in the thermodynamic limit.

III. GRAND CANONICAL AND CANONICAL ENSEMBLES

A. Poisson distribution in the thermodynamic limit

Let us consider the grand canonical ensemble of noninteracting gas. A corresponding statistical operator is

$$\hat{D} = \frac{e^{-\beta H + \gamma N}}{\text{Tr} e^{-\beta H + \gamma N}}. \quad (1)$$

This leads to the partition function

$$\mathcal{Z}(V, T, \gamma) = e^{z e^\gamma}. \quad (2)$$

where $z$ is one-particle partition function

$$z(T, V) = \frac{V}{(2\pi)^3} \int d^3 p \ e^{-\beta E(p)} \equiv V z_0(T), \quad (3)$$

A $\gamma$ parameter ($= \beta \mu$) is such to provide the given value of the average particle number $\langle N \rangle = V \langle n \rangle$. This means that

$$e^\gamma = \frac{\langle n \rangle}{z_0}. \quad (4)$$

Particle moments can be written as

$$\langle N^k \rangle = \frac{1}{\mathcal{Z}} \frac{\partial^k \mathcal{Z}}{\partial \gamma^k}. \quad (5)$$
The parameter $\gamma$ is taken in final formulæ as a function $\gamma((n), z_0)$ from Eq (4).

The resulting probability distribution to obtain $N$ particles under condition that the average number of particles is $\langle N \rangle$ is equal to Poisson distribution

$$P_{(N)}(N) = \frac{(N)^N}{N!} e^{-N}.$$

We introduce corresponding probability distribution $P$ for the particle number density $n = N/V$

$$P_{(n)}(n; V) = V P_{(n)}(V n) = V \frac{(V(n))^{Vn}}{\Gamma(Vn + 1)} e^{-V(n)}.$$  

(6)

For large $V n$ we are using an asymptotic form of Gamma function

$$\Gamma(V n + 1) \sim \sqrt{2\pi(V n)}^{Vn-1/2} e^{-V n} \left\{ 1 + \frac{1}{12Vn} + \mathcal{O}(V^{-2}) \right\}.$$

This gives

$$P_{(n)}(n; V) \sim V^{1/2} \frac{1}{\sqrt{2\pi n}} \left( \frac{n}{n} \right)^{Vn} e^{V(n-n)} \left\{ 1 - \frac{1}{12Vn} + \mathcal{O}(V^{-2}) \right\}.$$  

(7)

This expression in singular in the $V \to \infty$ limit. To estimate a large volume behavior of the probability distribution (6) one should take into account a generalized function limit. So we are going to calculate an expression

$$\langle G \rangle_V = \int dn \ G(n) P_{(n)}(n; V),$$

where $P_{(n)}(n; V)$ is replaced by the asymptotic form from Eq (7). In the next to leading order in $1/V$ one should calculate

$$V^{1/2} \frac{1}{\sqrt{2\pi}} \int dn \ \frac{G(n)}{n^{1/2}} e^{V S(n)} - V^{-1/2} \frac{1}{12\sqrt{2\pi}} \int dn \ \frac{G(n)}{n^{3/2}} e^{V S(n)}.  

(8)

where

$$S(n) = n \ln(n) - n \ln n + n - \langle n \rangle.$$ 

An asymptotic expansion of the function $\langle G \rangle_V$ is given by the classical Watson-Laplace theorem

**Theorem 1** Let $I = [a, b]$ be the finite interval such that

1. $\max_{x \in I} S(x)$ is reached in the single point $x = x_0$, $a < x_0 < b$.

2. $f(x), S(x) \in C(I)$.

3. $f(x), S(x) \in C^\infty$ in the vicinity of $x_0$, and $S^{(k)}(x_0) \neq 0$.

Then, for $\lambda \to \infty$, $\lambda \in S_\epsilon$, there is an asymptotic expansion

$$F(\lambda) \sim e^{\lambda S(x_0)} \sum_{k=0}^{\infty} c_k \lambda^{-k-1/2},$$  

(9a)

$$c_k = \frac{\Gamma(k + 1/2)}{(2k)!} \left( \frac{d}{dx} \right)^{2k} \left. f(x) \left( \frac{S(x_0) - S(x)}{(x - x_0)^2} \right)^{-k-1/2} \right|_{x=x_0}.$$  

(9b)

$S_\epsilon$ is here a segment $|\arg z| \leq \frac{\pi}{2} - \epsilon < \frac{\pi}{2}$ in the complex $z$-plane.
To obtain $O(1/V)$ formula the first term in (8) should be calculated till the next to leading order term in $1/V$. For the second term it is enough to perform calculations in the leading order only.

The first term gives the contribution

$$
V^{1/2} \frac{1}{\sqrt{2\pi}} \int \frac{dn}{n^{3/2}} G(n) e^{S(n)} = G(\langle n \rangle) + \frac{1}{12\langle n \rangle V} G(\langle n \rangle) + \frac{\langle n \rangle}{2V} G''(\langle n \rangle),
$$

(10a)

and the second term gives

$$
V^{-1/2} \frac{1}{12\sqrt{2\pi}} \int \frac{dn}{n^{3/2}} G(n) e^{S(n)} = \frac{1}{12\langle n \rangle V} G(\langle n \rangle),
$$

(10b)

So we have eventually

$$
\langle G \rangle_V = G(\langle n \rangle) + \frac{\langle n \rangle}{2V} G''(\langle n \rangle) + O(V^{-2}),
$$

(11)

for any function $G$.

This gives us the exact expression for the density distribution in the large volume limit

$$
\mathcal{P}(n; V) \sim \delta(n - \langle n \rangle) + \frac{\langle n \rangle}{2V} \delta''(n - \langle n \rangle) + O(V^{-2}).
$$

(12)

We are now able to obtain arbitrary density moments up to $O(V^{-2})$ terms.

$$
\langle n^k \rangle_V = \int dn n^k \mathcal{P}(n; V) = \langle n \rangle^k + \frac{k(k-1)}{2V} \langle n \rangle^{k-1} + O(V^{-2}).
$$

(13)

We have for the second moment (intensive variable!)

$$
\langle n^2 \rangle_V = \langle n \rangle^2 + \frac{\langle n \rangle}{V} + O(V^{-2}).
$$

This means

$$
\Delta n^2 = \frac{\langle n \rangle}{V} \to 0.
$$

(14)

as expected in the thermodynamic limit.

The particle number and its density are fixed in the canonical ensemble so corresponding variances are always equal to zero. The result can be seen as an example of the equivalence of the canonical and grand canonical distribution in the thermodynamic limit. This equivalence is clearly visible from the Eq (12) where the delta function in the first term can be considered as the particle number density distribution in the canonical ensemble.

A more involved situation appears for particle number moments (extensive variable!). Eq (13) translated to the particle number gives

$$
\langle N^k \rangle = V^k \langle n \rangle^k + V^{k-1} \frac{k(k-1)}{2} \langle n \rangle^{k-1} + O(V^{k-2}),
$$

(15)

One gets for the scaled variance (semi-intensive variable!)

$$
\frac{\Delta N^2}{\langle N \rangle} = 1,
$$

(16)

what should be compared with zero obtained for the canonical distribution.

The mechanism for such a seemingly unexpected behavior is quite obvious. The grand canonical and the canonical density probability distributions tend to the same thermodynamic limit. There are different however for any finite volume. Semi-intensive variables depend on coefficients at those finite volume terms so they are different also in the thermodynamic limit.
B. Energy distribution

It is interesting to perform similar calculation for the energy distribution in both ensembles. Energy moments and an average energy density can be written as

\[ \langle E^k \rangle = (-1)^k \frac{1}{Z} \frac{\partial^k Z}{\partial^k \beta^k}; \quad \langle \epsilon \rangle = -\frac{d z_0}{d \beta} e^\gamma. \]  

(17)

One gets from Eq (17)

\[ \langle E^k \rangle = V^k \langle \epsilon \rangle^k + V^{k-1} \frac{k(k-1)}{2} \langle \epsilon \rangle^{k-2} \frac{d^2 z_0}{d \beta^2} + O(V^{k-2}). \]  

(18)

The grand canonical energy density distribution follows

\[ P(\epsilon|\langle n \rangle, \langle \epsilon \rangle) = \delta (\epsilon - \langle \epsilon \rangle) + \frac{\langle n \rangle}{2V} \mathcal{R}^{GC} \left( \frac{\langle \epsilon \rangle}{\langle n \rangle} \right) \delta'' (\epsilon - \langle \epsilon \rangle) + O(V^{-2}). \]  

(19)

\[ \mathcal{R}^{GC} \left( \frac{\langle \epsilon \rangle}{\langle n \rangle} \right) = \frac{1}{z_0} \left. \frac{d^2 z_0}{d \beta^2} \right|_{\beta = \beta(\langle \epsilon \rangle / \langle n \rangle)}. \]

For the canonical distribution a corresponding statistical operator is

\[ \hat{D} = \frac{e^{-\beta \hat{H}}}{\text{Tr} e^{-\beta \hat{H}}}. \]  

(20)

This leads to the partition function

\[ Z(V, T) = \frac{z^N}{N!} = \frac{e^{V n \log z}}{N!}. \]  

(21)

Internal energy moments are given by Eq (17). In particular

\[ \langle \epsilon \rangle = -\frac{n}{z_0} \frac{d z_0}{d \beta}. \]  

(22)

For the energy moments one gets now

\[ \langle E^k \rangle = V^k \langle \epsilon \rangle^k + V^{k-1} \frac{k(k-1)}{2} \langle \epsilon \rangle^{k-2} n \frac{\partial}{\partial \beta} \left( \frac{1}{z_0} \frac{\partial z_0}{\partial \beta} \right) + O(V^{k-2}). \]  

(23)

A corresponding probability distribution is

\[ P(\epsilon|n, \langle \epsilon \rangle) = \delta (\epsilon - \langle \epsilon \rangle) + \frac{n}{2V} \mathcal{R}^C \left( \frac{\langle \epsilon \rangle}{n} \right) \delta'' (\epsilon - \langle \epsilon \rangle) + O(V^{-2}), \]  

(24)

where \( \mathcal{R}^C \) is given here as

\[ \mathcal{R}^C \left( \frac{\langle \epsilon \rangle}{n} \right) = \left. \frac{\partial}{\partial \beta} \left( \frac{1}{z_0} \frac{\partial z_0}{\partial \beta} \right) \right|_{\beta = \beta(\langle \epsilon \rangle / n)}. \]

\[ \text{[1]} \text{ For a review see, e.g., P. Braun-Munzinger, K. Redlich and J. Stachel: Quark Gluon Plasma 3 eds. R. C. Hwa and X. N. Wang (World Scientific, Singapore 2004) 491-599; A. Andronic and P. Braun-Munzinger: Lect. Notes Phys. 652 35 (2004)} \]

\[ \text{[2]} \text{ J. Cleymans, K. Redlich and L. Turko: Phys. Rev. C 71 047902 (2005) } \]

\[ \text{[3]} \text{ J. Cleymans, K. Redlich and L. Turko: J. Phys. G 31 1421 (2005) } \]

\[ \text{[4]} \text{ V. V. Begun, M. Gazdzicki, M. I. Gorenstein and O. S. Zozulya: Phys. Rev. C 70 034901 (2004); V. V. Begun, M. I. Gorenstein, A. P. Kostyuk and O. S. Zozulya: Phys. Rev. C 71 054904 (2005); V. V. Begun, M. I. Gorenstein and O. S. Zozula: Phys. Rev. C 72 014902 (2005); A. Keränne, F. Becattini, V.V. Begun, M.I. Gorenstein, O.S. Zozulya, J. Phys. G 31 S1095 (2005) } \]