A Bayesian method for the analysis of deterministic and stochastic time series

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Time series modelling

- heteroscedastic, asymmetric noise on time and signal
- non-uniform time sampling

Measured data $D_j = (s_j, y_j)$ and uncertainties $\sigma_j = (\sigma_{s_j}, \sigma_{y_j})$

Model $M$ with parameters $\theta$

Likelihood of single data point: integrate over unknown true time ($t$) and signal ($z$)

$$P(D_j|\sigma_j, \theta, M) = \int_{t_j, z_j} \left[ P(D_j|t_j, z_j, \sigma_j) \right] \left[ P(t_j, z_j|\theta, M) \right] dt_j dz_j$$

Measurement model Time series model

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Model comparison

Likelihood of all data points is

\[ P(D|\sigma, \theta, M) = \prod_j P(D_j|\sigma_j, \theta, M) \]

Evidence is the likelihood marginalized over the parameter prior

\[ P(D|\sigma, M) = \int_\theta P(D|\sigma, \theta, M) P(\theta|M) \, d\theta \]

More robust alternative is the leave-one-out cross validation likelihood

\[ P(D_j|D_{-j}, \sigma, M) = \int_\theta P(D_j|\sigma_j, \theta, M) P(\theta|D_{-j}, \sigma_{-j}, M) \, d\theta \]

\[ L_{CV} = \prod_{j=1}^{J} P(D_j|D_{-j}, \sigma, M) \]

Calculate integrals by MCMC sampling of posterior

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Time series model

Deterministic mean plus stochastic variation of constant variance

\[ P(z_j|t_j, \theta, M) = \frac{1}{\sqrt{2\pi\omega}} e^{-\frac{(z_j - \eta(t_j))^2}{2\omega^2}} \quad \text{Gaussian} \]

\[ \eta(t_j) = \frac{a}{2} \cos[2\pi(\nu t + \phi)] + b \quad \text{sinusoidal} \]

- red solid: deterministic component
- red dashed: standard deviation of stochastic component
- black: true data
Time series model

Ornstein-Uhlenbeck process

A Stationary, Markov, Gaussian process

\[ dz(t) = -\frac{1}{\tau} z(t)dt + c^{1/2} \mathcal{N}(t; 0, dt) \]

\[ P(z_j \mid t_j, \theta, M) = \frac{1}{\sqrt{2\pi V_z}} e^{-\left( z_j - \mu_z \right)^2 / 2V_z} \]

\[ \mu_z = z_0 \nu \]

\[ V_z = \frac{c\tau}{2} (1 - \nu^2) \]

where \[ \nu = e^{-(t-t_0)/\tau} \] for \[ t > t_0 \]
Examples of OU process realizations

relaxation time, $\tau$

Different randomisations
Luminosity variations in ultra cool dwarf stars

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Luminosity variations in ultra cool dwarf stars

Models compared:

- constant (variability just due to measurement noise)
- constant with Gaussian stochastic component
- sinusoid with Gaussian stochastic component
- OU process
Luminosity variations in ultra cool dwarf stars

OU process

Sinusoid (8.3h, 13.3h)

Sinusoid + stochastic
Periodicity in biodiversity over past 550 Myr?

Rohde & Muller 2005

periodic model with additional fitted Gaussian noise

black = data
red = model fit

stochastic process (OU process)

CV likelihood is much higher for this model

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Summary

• a Bayesian method for modelling times series
  ‣ arbitrary time sampling and error models
  ‣ deterministic and stochastic times series
  ‣ use of cross-validation likelihood, a robust alternative to the evidence

• applications
  ‣ light curves of some very cool stars (and quasars) evolve stochastically
  ‣ no evidence for periodic variation of biodiversity over past 550 Myr

• more information and software: tinyurl.com/ctsmo
## Ultra cool dwarf model comparison results

Table 4. Log (base 10) LOO-CV likelihood of each model relative to that for the no-model for each light curve (log $L_{\text{LOO-CV}} - \log L_{\text{NM}}$).

| Light curve | OUprocess | Off+Stoch | Sin   | Sin+Stoch | Off+Sin+Stoch | No-model | p-value |
|-------------|------------|-----------|-------|-----------|---------------|----------|---------|
| 2m0345      | 3.26       | 2.07      | 0.15  | 2.06      | 2.66          | -13.60   | 4e-4    |
| 2m0913      | 0.44       | 0.72      | 0.23  | 0.97      | 0.10          | -53.39   | 7e-4    |
| 2m1145a     | 15.23      | 8.59      | 3.01  | 12.26     | 11.70         | -63.83   | <1e-9   |
| 2m1145b     | -0.73      | 1.96      | 2.00  | 2.69      | 2.95          | -39.71   | 1e-3    |
| 2m1146      | 0.67       | 0.56      | -0.08 | 0.21      | 1.17          | -26.83   | 3e-3    |
| 2m1334      | 14.95      | 12.82     | 4.06  | 16.86     | 16.12         | -65.88   | 1e-9    |
| sdss0539    | 5.50       | 1.99      | 4.93  | 4.48      | 4.67          | -19.62   | 3e-5    |
| calar3      | 3.60       | 1.43      | 5.65  | 5.11      | 4.28          | -28.06   | 6e-4    |
| sori31      | 2.04       | 2.12      | 1.02  | 2.59      | 1.90          | -11.16   | 4e-5    |
| sori33      | 1.49       | 0.66      | 2.14  | 1.85      | 2.12          | -8.39    | 2e-3    |
| sori45      | 6.70       | 4.32      | 5.08  | 6.23      | 6.32          | -29.93   | 5e-9    |

**Notes.** The penultimate column gives the value of the log likelihood for the no-model, log $L_{\text{NM}}$. The last column is the p-value for the hypothesis test from BJM.
Parameter posterior PDFs:

- Frequency, $\nu / \text{hr}^{-1}$
- Amplitude, $a / \text{mag}$
- Phase, $\phi$

black = posterior
red = prior
Parameter posterior PDFs: 2m1145a

black = posterior
red = prior
Parameter posterior PDFs: 2m1334

- Offset, $b$ / mag
- Frequency, $\nu$ / hr$^{-1}$
- Amplitude, $a$ / mag
- Phase, $\phi$
- Standard deviation, $\omega$ / mag

black = posterior
red = prior