Feasibility of a stationary micro-SQUID

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Abstract

The standard operation of a dc SQUID leads to oscillatory electric fields that emit electromagnetic radiation and can change the state of the measured sample. A stationary SQUID could be advantageous when back action on the measured sample has to be avoided. We study a superconducting loop that encloses a magnetic flux, connected to a superconducting and to a normal electrode, when a fixed electric current between the electrodes flows through the loop. The considered circuit does not contain Josephson junctions. We find that in a very broad range of parameters the current flow converges to a stationary regime. The potential difference between the electrodes depends on the magnetic flux, so that measuring this voltage would provide information on the enclosed flux. The influence of thermal noise was estimated. The sizes of the voltage and of the power dissipation could be appropriate for the design of a practical fluxmeter. We found narrow ranges of flux at which the voltage varies sharply with the flux.

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I. INTRODUCTION

Superconducting quantum interference devices (SQUIDs) are the most sensitive available fluxmeters. SQUIDs are classified into rf SQUIDs, which consist of a loop interrupted by a single Josephson junction and are monitored at radio frequency, and dc SQUIDs, with two Josephson junctions. The dc SQUID can really be used as a “dc” fluxmeter by gradually increasing the current through it, until its flux dependent critical current is reached; however, in the more practical procedure, the current through the SQUID is larger than its critical value, giving rise to a flux dependent average voltage.

More in detail, the voltage in the dc SQUID oscillates at a frequency that follows from the Josephson relation. As a consequence the SQUID emits electromagnetic radiation, which exerts a back action on the measured sample. In the case of nanoscopic samples, this back action can alter the state that the SQUID was intended to measure. Therefore, there are situations in which a stationary instrument can be advantageous over the standard SQUID, even if less precise.

Stationary fluxmeters do exist, such as the ballistic Hall magnetometer, the normal-metal-insulator-superconductor (NIS) interferometer, or the superconducting quantum interference proximity transistor. Here we propose a simple circuit, which under appropriate circumstances could be the fluxmeter of choice.

II. PROPOSAL

The circuit that we propose is as follows. A loop of superconducting material encloses the magnetic flux $\Phi$ that we intend to measure. Short wires at the left and the right connect the loop to “banks” that serve as electrodes. The connectors and the left bank are made of the same material as the loop, whereas the right bank is a normal metal. A fixed current is driven from the superconducting to the normal bank, and the potential difference between them is measured. If this potential difference is a function of $\Phi$, then this circuit provides a measurement of $\Phi$. Our proposal is sketched in Fig. 1.

Additional ideas for SQUIDs without Josephson junctions have been considered in Refs. 8–10. The difference between the circuit in Fig. 1 and the NIS interferometer is that the insulating barrier has been replaced with the connector to the normal bank. Another
FIG. 1: Circuit designed to measure the flux $\Phi$. The circuit consists of a superconducting loop that encloses the flux, with connections to two electrodes. The electrode at the right is made of a normal metal. The cross section of the connector at the left is $w_S$ and the cross section at the right is $w_N$. Both branches of the loop have the same length. The cross section in each branch is the same function of the arc length, and varies monotonically from $w_S$ to $w_N$.

circuit that can operate as a SQUID and involves normal components was investigated in Ref. 11, but the authors did not address the question of whether a stationary regime is attained.

Heuristic arguments Due to fluxoid quantization, current will not only flow from one connector to the other, but a circulating current will also be present. It follows than the current in one of the branches is expected to exceed half of the total current, increasing the amount of current that has to be transported as normal current. We may therefore expect an increase in the voltage between the banks as the enclosed flux deviates from an integer number of quantum fluxes. This is true also for a standard SQUID above its critical current, but in the standard case the phase of the order parameter changes at a different rate in each of the banks, forcing an oscillatory behavior. In our case, the order parameter vanishes at the normal bank, so that the phase becomes meaningless, and oscillations will not necessarily occur.

The current between the banks should not be too large, since in this case a considerable segment of the loop becomes essentially normal, making the circuit insensitive to the flux. On the other hand, the current ought to be sufficiently large to cause the order parameter to vanish at a point in the loop when the flux becomes half a quantum. This feature enables
a continuous passage between consecutive winding numbers, thus avoiding hysteresis.

III. MINIMAL MODEL

A. Formalism

For simplicity, we consider a quasi-1D circuit, and the position dependence of all physical quantities will be completely determined by the arc length. In this section we assume that thermal contact is sufficiently effective to take heating away, so that the entire circuit is kept at a uniform temperature $T$; the expected influence of Joule heating and of supercurrent heat transport will be examined in Sec. [IVB]. At this stage we will also ignore thermal fluctuations.

In Refs. 6 and 7 it is assumed that the entire voltage drop occurs across the insulating barrier, so that the superconducting circuit has a uniform potential and a static description is possible. In contrast, in our case the electrochemical potential varies along the superconducting circuit, so that a time dependent formalism is necessary, and convergence to a stationary regime cannot be assumed a priori.

This section will rely on time-dependent Ginzburg–Landau model (TDGL), which is the simplest model for the description of the proposed circuit. More realistic models will be considered in Sec. [IV].

The proximity effect is built-in into the Ginzburg–Landau model: near the superconducting electrode, superconductivity will be strong, even above the critical current, and near the normal electrode superconductivity will be weak, even for very small currents. As a consequence, part of the current along the circuit will be normal current, giving rise to a potential difference between the electrodes.

We choose a gauge with no tangential vector potential along the connectors and uniform tangential vector potential $A$ along the loop. We write $\epsilon = 1 - T/T_c$, where $T_c$ is the critical temperature of the superconductor, and denote by $\varphi$ the electrochemical potential. The unit of length will be denoted by $x_0$, by $t_0$ the unit of time, by $\varphi_0$ the unit of voltage, by $A_0$ the unit of vector potential, and by $j_0$ the unit of the current density. We take

$$x_0 = \xi(0) , \quad t_0 = \frac{\pi \hbar}{8k_BT_c} , \quad A_0 = \frac{hc}{2e\xi(0)} , \quad \varphi_0 = \frac{4k_BT_c}{\pi e} , \quad j_0 = \frac{4\sigma k_BT_c}{\pi e\xi(0)} .$$

(1)
Here $\xi(0)$ is the coherence length at $T = 0$, $k_B$ is Boltzmann’s constant, $e$ is the electron charge, and $\sigma$ is the normal conductivity.

Let us first consider the case of uniform cross section, $w_S = w_N$; in this case the current density is the same at both connectors (assuming electroneutrality), and will be denoted by $j$. With the units in Eq. (1), the TDGL equation and Ohm’s law read

$$\partial_t \psi = D^2 \psi + \left( \epsilon - |\psi|^2 - i\varphi \right) \psi \quad (2)$$

and

$$\partial_x \varphi = u \text{Im} (\psi^* D \psi) - j \quad (3).$$

Here $\psi$ is the order parameter, with normalization imposed by Eq. (2), $\partial_t$ and $\partial_x$ denote partial differentiation with respect to the time $t$ and to the arc length $x$ along the wire, $D$ is the operator $D = \partial_x - IA$ and $u$ is the ratio between the relaxation times of $\psi$ and $j$. Along the branches of the loop, $j$ has to be set as the current density that enters the corresponding branch. Following Ref. 12, we denote by $4L$ the perimeter of the loop, and therefore the flux that it encloses is $4LA$. Invoking the units in Eq. (1) we obtain $|A| = \pi |\Phi|/2L\Phi_0$, where $\Phi_0 = \pi \hbar c/e$ is the quantum of flux. Accordingly, along one of the branches of the loop we set $A \to \pi \Phi/2L\Phi_0$, along the other branch we set $A \to -\pi \Phi/2L\Phi_0$ (for our purpose it makes no difference which branch is assigned the positive sign), and set $A \to 0$ along the connectors.

The boundary conditions require continuity of the order parameter and assume equilibrium at the electrodes. The potential at the superconducting electrode is taken as zero, and the order parameter as $\psi_S = \epsilon^{1/2}$. At the normal electrode, the order parameter is required to vanish. At the contacts between the ring and the connectors, we require continuity of the potential and of the order parameter. We also impose the constraint that the sum of total currents along the branches of the loop has to equal the current along the connectors.

Equations (2)-(3) are invariant under the gauge transformation $A \to A + C$, $\psi \to \exp(iCx)\psi$ where $C$ is constant. If $C$ is an integer multiple of $\pi/2L$, also the continuity conditions remain invariant. It follows that any physical property of the circuit is periodic in $\Phi$, with periodicity $\Phi_0$. In addition, switching the sign of $\Phi$ amounts to exchanging the lower and upper branches of the circuit, so that the potential difference between the electrodes has to be a symmetric function of $\Phi$.

TDGL is not expected to be quantitatively correct, but one of the features that makes it a valuable tool is the scaling with length. Equations (2)-(3) are invariant under the
transformation $x \rightarrow L'x$, $L \rightarrow L'L$, $A \rightarrow L'^{-1}A$, $t \rightarrow L'^2t$, $\psi \rightarrow L'^{-1}\psi$, $\varphi \rightarrow L'^{-2}\varphi$, $\epsilon \rightarrow L'^{-2}\epsilon$ and $j \rightarrow L'^{-3}j$, since each of the terms in Eqs. (2) and (3) is multiplied by $L'^{-3}$. The boundary conditions are also invariant under this transformation, provided that the ratios among the lengths of the connectors and the perimeter of the loop are kept unchanged. Therefore, for this simple model we can limit our study to a single value of $L$, and the solutions for any other value are obtained by scaling. We note that $L^2\varphi$, $\Phi$ and $L\epsilon^1/\xi(0) = L/\xi(T)$ are invariant under this scaling. This scaling breaks down if $L$ is not large in comparison to the average diffusion distance between inelastic collisions.\textsuperscript{14,15}

**B. Results**

Equations (2)–(3) were solved numerically, as described in the Appendix. In the case $w_S = w_N$ we studied the range $L^2/\xi^2 \lesssim 10$. The solutions that we found in this range always converged to a stationary regime. We are interested in potential differences that are continuous functions of $\Phi$, but the winding number of $\psi$ around the loop cannot change continuously unless $\psi$ vanishes at some point. For this goal, sufficiently large currents are required. Currents that lead to appropriate behavior were found empirically.

The lower curve in Fig. 2 shows the voltage between the electrodes as a function of $\Phi$ for $L^2\epsilon = 10$ and $L^3j = 120$. The considered current density is about twice the nominal “critical current density” $2u(\epsilon/3)^{3/2} \approx 70L^{-3}$. Due to periodicity and symmetry of $\varphi_N(\Phi)$, the interval $0 \leq \Phi \leq 0.5\Phi_0$ would suffice to describe this function for arbitrary flux, but a larger interval is presented in order to show continuity. For $\Phi = 0.5\Phi_0$, the numeric value of $|\psi|$ at the right extreme of the loop is of the order of $3 \times 10^{-3}\psi_S$; this value decreases if a denser computational grid is used. These and the following results were obtained for connectors of length 0.08$L$.

Let us now consider the case $w_S \neq w_N$ and denote the cross section within the loop by $w(x)$. The TDGL equation and Ohm’s law become\textsuperscript{16}

$$\partial_t \psi = D^2\psi + (\epsilon - |\psi|^2 - i\varphi)\psi + w^{-1}w'D\psi$$

(4)

and

$$\partial_x \varphi = u \text{Im}(\psi^*D\psi) - jw_s/w,$$

(5)

where $w' = \partial_x w$ and $j$ denotes either the current density along the connector to the super-
FIG. 2: Potential difference between the electrodes as a function of the flux for temperature
\[(1 - 10\xi^2(0)/L^2)T_c\]. \(\varphi_N\) is the electrochemical potential at the normal electrode. The blue line is for \(w_S = w_N\) and the red line for \(w_S = 3w_N\). In all cases the perimeter of the loop is \(4L\) and the length of each connector is \(0.08L\). For \(w_S = w_N\) the current density along the connectors is \(120j_0\xi^3(0)/L^3\); for \(w_S = 3w_N\), the current density along the left connector is \(75j_0\xi^3(0)/L^3\). For the purpose of comparison between the two curves, the curve for \(w_S = 3w_N\) was lowered by 15 units.

conducting electrode, or the current density at the left extreme of the appropriate branch of the loop. For simplicity, we take \(w(x)\) as the linear interpolation between \(w_S\) and \(w_N\).

The upper curve in Fig. 2 shows the voltage between the electrodes as a function of \(\Phi\) for \(L^2\epsilon = 10\) and \(L^3j = 75\) (the current density at the connector to the normal electrode is three times larger). In order to enhance visibility, this curve has been lowered by 15 units. Again, no discontinuity is visible in this curve.

In the case \(w_S > w_N\) we have also looked into larger values of \(L^2\epsilon\). Figure 3 shows the voltage between the electrodes for \(L^2\epsilon = 30\), for \(w_S = 3w_N\) and for \(w_S = 6w_N\). To enhance visibility, the curve for \(w_S = 6w_N\) was lowered by 40 units.

Close to \(\Phi \approx 0.14\Phi_0\) the curve for \(w_S = 3w_N\) has a steep slope, but is continuous and reversible. By means of an appropriate bias, this steep slope can serve to attain a high flux sensitivity.

The trend suggested by Figs. 2-3 is that the flux-modulation of the voltage increases with \(L^2\epsilon\), and the slope \(d\varphi_N/d\Phi\) becomes more uniform when \(w_S/w_N\) increases.
FIG. 3: Potential difference as a function of the flux for temperature \((1 - 30\xi^2(0)/L^2)T_c\). The red line stands for \(w_S = 3w_N\) and \(j = 320j_0\xi^3(0)/L^3\); the black line, for \(w_S = 6w_N\) and \(j = 230j_0\xi^3(0)/L^3\). For the purpose of comparison between the two cases, the curve for \(w_S = 6w_N\) was lowered by \(40\xi^2(0)\phi_0\). The length ratios are the same as in Fig. [2]. The inset shows the region near the inflection point for \(w_S = 3w_N\) in an expanded scale.

IV. MORE REALISTIC MODELS

In this section we will successively refine TDGL into a realistic model.

A. The Kramer–Watts-Tobin (KWT) model

A model that can be justified as long as there is local equilibrium is the Kramer–Watts-Tobin model,\(^{14,15}\) in which the difference in the relaxation times of the absolute value and of the phase of the order parameter is taken into account. In this case Eq. (1) has to be replaced with

\[
\hbar \left( \partial_t + i\varphi + \frac{w\tau_{\text{in}}^2}{2} \frac{\partial|\psi|^2}{\partial t} \right) \psi = \mathcal{D}^2\psi + \frac{w'}{w} \mathcal{D}\psi + (\epsilon - |\psi|^2) \psi ,
\]

where \(\tau_{\text{in}}\) is the inelastic collision time and

\[
h = (1 + w\tau_{\text{in}}^2|\psi|^2)^{-1/2} .
\]
FIG. 4: Potential difference as a function of the flux according to the Kramer–Watts-Tobin model, for a circuit with perimeter $400\xi(0)$ and $w_S = 3w_N$, for surrounding temperature $0.997T_c$, current $j = 3.2 \times 10^{-4}j_0$ and $u\tau_{in}^2 = 10^4t_0^2$. The solid red line was evaluated assuming that the current does not produce heating, whereas the black line assumes a local temperature described by Eq. (10) with $\eta = 3 \times 10^3$. The dashed line is a horizontal expansion of the solid red line, in the range where it has a steep slope; its vertical scale is common to the other lines, but its horizontal scale is given in the upper axis.

With some algebra, this equation can be cast in a form that is appropriate for Euler iterations:

$$\partial_t \psi = h^{-1}[D^2 + (w'/w)D]\psi - i\varphi \psi$$
$$+h\psi(\epsilon - |\psi|^2 - u\tau_{in}^2\text{Re}(\psi^*[D^2 + (w'/w)D]\psi)) \ .$$

(8)

In the limit $\tau_{in} \to 0$, KWT reduces to TDGL.

Equations (6)–(8) do not obey the scalings that we found for TDGL, forcing us to fix the length of our circuit.

The lower curve in Fig. 4 shows the voltage as a function of the flux for parameters that correspond to the lower curve in Fig. 3 but this time the evolution of the order parameter followed Eq. (8) with $u\tau_{in}^2 = 10^4t_0^2$, which is a typical value for low-$T_c$ superconductors with strong coupling. We note that the main effect of KWT in comparison to TDGL is that the range of fluxes for which the voltage rises steeply is shifted towards $\Phi = 0.5\Phi_0$. We remark
that the flow pattern converges to a stationary situation.

At the inflection point, $d|\varphi_N|/d(\Phi/\Phi_0) \approx 0.35$, i.e., the flux sensitivity is $0.35\varphi_0/\Phi_0$. For $T_c \sim 10$ K, the sensitivity is $\sim 4 \times 10^{-4} V/\Phi_0$. Assuming that the trend is the same as in Fig. 3, higher sensitivities would be found for smaller ratios $w_S/w_N$ and for smaller lengths.

B. Heating due to current flow

We assume that inelastic scattering lengths are short with respect to $L$, so that a local temperature, common to electrons and phonons, can be defined. Using the units in Eq. (1), the power deposited at a given position can be written as:

$$\text{density of power input} = \sigma^{-1}\{j_N^2 + u|\partial_t + i\varphi\psi|^2 + uw^{-1}\partial_x \text{Re}[w(D\psi)^*(\partial_t + i\varphi)\psi]\}. \quad (9)$$

Here $j_N = jw_s/w - u\text{Im}(\psi^*D\psi)$ is the normal current density, so that the first term in Eq. (9) stands for the Joule heating. The second term arises from the relaxation of the order parameter, and the third term from the heat carried by the supercurrent.

Assuming that a stationary situation is achieved, this power has to diffuse away. Heat can either diffuse along the wire or to the substrate, but assuming that the wire is thin and the substrate has good thermal conductivity, heat will mainly diffuse to the substrate, and its flow rate will be proportional to the difference between the local temperature and that of the substrate; therefore, as a crude model, we estimate that the local temperature increment will be proportional to the density of power dissipation, i.e.

$$\epsilon = \epsilon_0 - \eta\{j_N^2 + u|\partial_t + i\varphi\psi|^2 + uw^{-1}\partial_x \text{Re}[w(D\psi)^*(\partial_t + i\varphi)\psi]\}, \quad (10)$$

where $\epsilon_0$ corresponds to the temperature of the substrate and $\eta$ is a constant, expected to decrease when the thickness of the circuit decreases and when the thermal conductivity of the substrate increases.

The upper curve in Fig. 4 shows the voltage as a function of the flux for the same parameters as for the lower curve, but now current heating was taken into account, assuming Eq. (10), with $\eta = 3 \times 10^3 j_0^{-2}$. We see that the effect of current heating is similar to that of decreasing the ratio $w_N/w_S$: the steep rise becomes gradual. The similarity between the two cases may be attributed to the reduction of the ability to carry current in the vicinity of the normal electrode.
As expected, the hottest region in the circuit is found in the connector at the right, close to the branching point. Along the branches of the loop, the temperature gradually increases as the right connector is approached; this temperature is larger in the branch where the total current is smaller. Surprisingly, higher temperatures are encountered for $\Phi \approx 0$ than for $\Phi \approx 0.5\Phi_0$, but in the former case the hot regions along the branches of the loop are shorter.

For values of $\eta$ that are larger (respectively smaller) than $\eta = 3 \times 10^3 j_0^{-2}$, we may expect to obtain a $\varphi_N(\Phi)$ curve that is similar to the upper curve in Fig. 4 provided that the surrounding temperature is lowered (respectively raised), so as to yield the same temperature at the right extreme of the loop.

C. Thermal noise

In this section we take thermal noise into account, and investigate to what extent it limits the flux resolution of the proposed device. Thermal noise affects the evolutions of the electrochemical potential and of the order parameter. We will add its influence using a formalism in which length and time are discretized.

In the case of the electrochemical potential, we add the Johnson noise. If $\varphi_k$ and $\varphi_{k+1}$ are the electrochemical potentials in two consecutive cells, with a distance $\ell$ between their centers, at periods of time $\tau$ we have to add to $\varphi_{k+1} - \varphi_k$ a fluctuating term with gaussian distribution, zero average, and variance

$$\langle [\Delta(\varphi_{k+1} - \varphi_k)]^2 \rangle = \varphi_0^2 \Gamma \varphi \frac{T}{T_c \tau},$$

(11)

where $\Gamma \varphi = \pi e^2 \ell / h w \sigma$, $w$ stands for the cross section between the two cells, and $\Delta$ stands for the deviation from the value that would be obtained in the absence of fluctuations.

The fluctuating additions to the order parameter have been discussed in previous studies. If $\psi_k = |\psi_k| \exp(i\chi_k)$ is the order parameter in cell $k$, then at intervals of time $\tau$ we add to $|\psi_k|$ a fluctuating term with gaussian distribution, average

$$\langle \Delta|\psi_k| \rangle = \psi \frac{h_k^2}{2|\psi_k|} T \tau,$$

(12)

and variance

$$\langle (\Delta|\psi_k|)^2 \rangle - \langle \Delta|\psi_k| \rangle^2 = \psi h_k T \tau,$$

(13)
where $\Gamma_{\psi} = \Gamma_\psi \xi^2(0) / u \ell^2$, $h_k$ is obtained from Eq. (7) by setting $\psi = \psi_k$, and the cross section $w$ has to be taken as the average in cell $k$. Finally, the addition to the argument is a fluctuating term with gaussian distribution, zero average, and variance

$$
\langle (\Delta \chi_k)^2 \rangle = \frac{\Gamma_\psi}{h_k |\psi_k|^2} \frac{T \tau}{T_c t_0}.
$$

(14)

Obviously, the influence of thermal fluctuations becomes negligible for sufficiently low temperature. Therefore, a more relevant question is whether the signatures encountered in the previous sections are still encountered when the temperature is lowered away from $T_c$. With this in mind, we have studied the influence of thermal noise for $T = 0.9 T_c$. Accordingly, we reduced the perimeter of the circuit and increased the current density.

Figure 5 shows typical temporal variations of the voltage across the circuit, for some particular runs. The colored curves correspond to $j = 0.095$. The curve for $\Phi = 0.225 \Phi_0$ is typical of the case $0 \leq \Phi \lesssim 0.4 \Phi_0$; in this case the voltage fluctuates around some average value. On the other hand, for $0.4 \Phi_0 \lesssim \Phi \lesssim 0.6 \Phi_0$, as seen in the curve for $\Phi = 0.55 \Phi_0$, it is clear that fluctuations are not normally distributed. Instead, there are large fluctuations that persist for long periods of time, suggesting that the circuit attains metastable regimes. These large fluctuations can be associated with changes in the winding number of the order parameter around the loop; these changes are also detected in the fraction of the total current that flows through a given branch of the circuit. The curves for $\Phi = 0.4 \Phi_0$ and for $\Phi = 0.5 \Phi_0$ have a shorter time span, and depict special situations for which a large fluctuation was present during most of the sampled time.

When the current density is raised to $j = 0.1$, large fluctuations become quite frequent and the flux range where they are present becomes broader. The black curve in Fig. 5 shows that these fluctuations are not rare for $\Phi = 0.25 \Phi_0$.

It might be objected that if the system wanders between two states, it is not strictly in a “stationary” regime. Note however that thermal noise is present also in a standard SQUID, in addition to its inherent oscillatory behavior.

The red lines in Fig. 6 show average voltages as a function of the flux for $j = 0.095$, obtained by monitoring the voltage during a lapse of time during which the winding number changed a handful of times. In order to assess the error, the sampling time was divided into 49 intervals, the average voltage was evaluated for each interval, and then the statistical error was estimated as the standard deviation of the voltage values for these intervals, divided by
FIG. 5: Potential difference as a function of time for miscellaneous runs. The colored curves are for \( j = 0.095 j_0 \), and the black curve, for \( j = 0.1 j_0 \). The number next to each colored curve is the flux in units of \( \Phi_0 \). For \( \Phi = 0.225 \Phi_0 \), for \( \Phi = 0.55 \Phi_0 \), and for the black curve, time is shown in the lower horizontal axis and the binning is 2400 \( t_0 \); for \( \Phi = 0.4 \Phi_0 \) and for \( \Phi = 0.5 \Phi_0 \), time is shown in the upper axis and the binning is 120 \( t_0 \). Other parameters: \( T = 0.9 T_c \), \( L = 20 \xi(0) \), \( w_S = 3 w_N \), \( w_{\text{in}}^2 = 10^4 t_0^2 \), \( \eta = 0 \), \( \xi(0) = 10 \text{ nm} \), \( \sigma w_S = 10^{-6} \Omega^{-1} \text{cm} \). For visibility, the curve for \( \Phi = 0.225 \Phi_0 \) has been lowered by 0.1 \( \varphi_0 \), those for \( \Phi = 0.4 \Phi_0 \) and \( \Phi = 0.5 \Phi_0 \) have been raised by 0.1 \( \varphi_0 \), and the curve for \( j = 0.1 j_0 \) has been raised by 0.3 \( \varphi_0 \).

7. As a rough picture of the flux dependence of the voltage, we also evaluated the voltage for several fluxes, for much shorter sampling times. The results are shown as blue dots in Fig. 6.

The purple lines in Fig. 6 show average voltages for \( j = 0.1 \). It appears that in this case our samplings involved a sufficient number of large fluctuations and the results are statistically meaningful.

We can estimate the flux resolution of our circuit as \( 0.5 \Phi_0 \Delta \varphi \sqrt{\Delta t/|\varphi(0.5 \Phi_0) - \varphi(0)|} \), where \( \Delta \varphi \) is the largest statistical error of \( \varphi \) and \( \Delta t \) is the sampling time, and the power dissipated by the circuit is \( w_S |\varphi_N| \). For the upper curve in Fig. 6, the resolution is of the order of \( 20 \Phi_0 t_0^{1/2} \).
FIG. 6: Average of the potential difference with thermal fluctuations, as a function of the flux. The red lines and the blue dots refer to $j = 0.095j_0$; the purple lines refer to $j = 0.1j_0$. The lines were obtained from sampling times that extended during $1.176 \times 10^5 t_0$, and represent error bars. The dots were obtained from samplings that lasted for $5.88 \times 10^3 t_0$, and do not show error bars. The gray lines are guides for the eye. The other parameters are as in Fig. 5.

V. DISCUSSION

We have looked for a superconducting circuit that attains a stationary regime and gives rise to a flux dependent voltage. We have found that this situation is indeed met within a very broad range of parameters. We have not conducted a systematic study to obtain the optimal parameters, but have rather limited ourselves to show a proof of concept for a proposed fluxmeter. Since the working state of this fluxmeter is not periodic, it is expected to exert less back action on the measured system.

The circuit considered in Fig. 6 has a perimeter of $80\xi(0)$, which for $\xi(0) \sim 10\text{nm}$ would be of the order of a micrometer. For flux changes of $0.5\Phi_0$, the voltage changes by about $0.4k_B T_c/e$. For $T_c \sim 10\text{K}$ this voltage is of the order of $0.3\text{mV}$, which should be readily measurable. For $T = 0.9T_c$, the flux resolution of this circuit is $\sim 20\Phi_0^{1/2}$, which for $T_c \sim 10\text{K}$ is of the order of $10^{-5}\Phi_0/\text{Hz}^{1/2}$. Taking $\sigma w_s \sim 10^{-6}\Omega^{-1}\text{cm}$, the power dissipated would be of the order of $10^{-7}\text{W}$.

We have considered temperatures close to $T_c$ just for numerical convenience and in order
to use theoretically simple models. However, from the practical point of view, we expect that low temperatures will offer a better performance: lower thermal noise and reduced danger of uncontrolled hot spots. With all other parameters taken equal, comparison of Figs. 2 and 3 suggests that lower temperatures would yield larger signals and steeper inflection points.

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Appendix A: Numerical solution of the dynamic equations

The basic technique is straightforward: in each connector or branch of the loop, \( \varphi(x) \) was obtained by trapezoidal integration of Eq. (3) or (5), and the evolution of \( \psi \) was followed by Euler iterations. In the following we describe the operations that were not entirely standard.

1. **Evaluation of** \( \mathcal{D}^p \psi \) \((p = 1, 2)\)

   a. **Case with no fluctuations**

   Each branch of the loop was discretized into a grid of \( 2^7 \) vertices. A fast Fourier transformation was performed on \( \psi(x) \), and the Fourier transformed function was multiplied by \((ik \mp A)^p\), where \( k \) is the reciprocal variable of \( x \). We then transformed back to \( x \)-space and obtained \( \mathcal{D}^p \psi \).

   Fourier transformation implicitly assumes that \( \psi(x) \) is periodic, and this feature is not obeyed for a single branch. In order to obtain a continuous periodic function with continuous first derivative, we proceeded as follows. (i) Only the \( 2^7 - 4 \) vertices in the middle of the grid had physical significance and represented the function \( \psi(x) \) in the physical domain \(-L \leq x \leq L\); the two first and the two last values were used to interpolate between \( \partial_x \psi \) at \( x = -L \) and \( x = L \), and ensure its continuity in the extended grid. (ii) A linear function was subtracted from \( \psi \), so that we obtain a new function \( \psi'(x) \) that has the same value at the beginning and the end of the grid; we then evaluated \( \mathcal{D}^p \psi' \) and added to it \( \mathcal{D}^p (\psi - \psi') \), which
is easily found analytically. We could have obtained periodicity automatically by dealing with the order parameter in the entire loop, but this is not advantageous, because in this case $\partial_x \psi$ is not continuous at the joints between the branches.

b. Case with fluctuations

In this case $\psi(x)$ is not a smooth function, and the Fourier method is not efficient. $D^p \psi$ was evaluated by means of finite differences.

2. Continuity at the contacts between the ring and the connectors

At these points, three branches meet. Charge conservation implies that the total current flowing out from these points has to vanish. Moreover, for a dense computational grid, the amount of normal to supercurrent conversion in the region involved becomes negligible, and we require that the total supercurrent flowing out from the contact point has to vanish, i.e., $\text{Im}(\psi_0^* \sum_{n=1}^3 D\psi_n) = 0$, where $\psi_0$ is the order parameter at the contact point, $\psi_n$ is the order parameter along branch number $n$, and the operator $D$ has to be interpreted as going outwards from the contact point. Assuming that the material in contact with the circuit is insulating, this condition generalizes to $\sum_{n=1}^3 D\psi_n = \sum_{n=1}^3 \partial_x \psi_n - i \psi_0 \sum_{n=1}^3 A_n = 0$, where $A_n$ is the component of the magnetic potential that goes out from the contact point, in the direction of branch $n$.

In our situation $\sum_{n=1}^3 A_n = 0$, so that we are left with $\sum_{n=1}^3 \partial_x \psi_n = 0$. This condition is discretized as $\sum_{n=1}^3 (\psi_{n,\text{next}} - \psi_0)/\Delta x_n = 0$, where $\psi_{n,\text{next}}$ is the order parameter in the vertex along branch $n$ next to the contact point, and $\Delta x_n$ is the distance to this vertex. From here $\psi_0$ is obtained as a weighted average: $\psi_0 = (\sum_{n=1}^3 \psi_{n,\text{next}}/\Delta x_n)/\sum_{n=1}^3 \Delta x_n^{-1}$. Clearly, this procedure ensures continuity of the order parameter at the contact points.

Continuity of $\varphi$ at the contact points is imposed by adjusting the division of the current between the two branches of the loop.

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