On the Initial Algebra and Final Co-algebra of some Endofunctors on Categories of Pointed Metric Spaces

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Abstract

We consider two endofunctors of the form $F: X \rightarrow M \otimes X$, where $M$ is a non degenerate module, related to the unit interval and the Sierpinski gasket, and their final co-algebras. The functors are defined on the categories of bi-pointed and tri-pointed metric spaces, with continuous maps, short maps or Lipschitz maps as the choice of morphisms.

First we demonstrate that the final co-algebra for these endofunctors on the respective category of pointed metric spaces with the choice of continuous maps is the final co-algebra of that with short maps and after forgetting the metric structure is of that in the set setting. We use the fact that the final co-algebra can be obtained by a Cauchy completion process, to construct the mediating morphism from a co-algebra by means of the limit of a sequence obtained by iterating the co-algebra. We also show that the Sierpinski gasket $(S, \sigma)$ is not the final co-algebra for these endofunctors when the morphism is restricted to being Lipschitz maps.

1. Introduction

This paper considers the initial algebra and final co-algebra for two particular endofunctors $F_i: C \rightarrow C$, where $C$ is the category of $i$-pointed metric spaces with continuous maps (\textit{MS}_i^C), short maps (\textit{MS}_i^S) or Lipschitz maps (\textit{MS}_i^L), and $i = 2, 3$. The two functors are based on the unit
interval $[0, 1]$ and the Sierpinski gasket. These definitions are motivated by [6] and have been considered previously in [4] and [5]. We need some definitions to start.

A bi-pointed set is a set having two distinguished elements. We denote such a set by a triple $(X, \top, \bot)$, where $\top$ and $\bot$ are the two distinguished elements. In a similar manner, a tri-pointed set, denoted by a quadruple $(X, T, L, R)$, consists of a set and three distinguished elements. We will often omit the distinguished points from the description of the set and simply write “Let $X$ be a tri-pointed set...”. To differentiate distinguished points of two (or more) $i$-pointed sets $X$ and $Y$, we will use subscripts, such as $\bot_X$ and $\top_Y$ for $\bot$ of $X$ and $\top$ of $Y$ respectively (in the case of $i = 2$).

One can similarly define an $i$-pointed set for any $i \in \mathbb{N}$. There is a category of $i$-pointed sets, $\text{Set}_i$, whose objects are $i$-pointed sets and morphisms are functions which preserve the distinguished elements. An $i$-pointed metric space $(X, d)$ is an $i$-pointed set $X = (X, x_1, \ldots, x_i)$ equipped with a one-bounded metric ($d(x, y) \leq 1$ $\forall x, y \in X$) such that the distance between any pair of distinguished elements is 1. The class of $i$-pointed metric spaces can be raised to the categories $\text{MS}_i^S$, $\text{MS}_i^L$ and $\text{MS}_i^C$, where the morphisms are respectively short maps, Lipschitz maps and continuous maps that preserves the distinguished elements. Note that $\text{MS}_i^L$ and $\text{MS}_i^S$ are subcategories of $\text{MS}_i^C$, $\text{MS}_i^S$ is a subcategory of $\text{MS}_i^L$ and all these three categories ($\text{MS}_i^S$, $\text{MS}_i^L$, $\text{MS}_i^C$) are subcategories of $\text{Set}_i$. In this paper we will consider only the cases $i = 2$ and $i = 3$.

The functors $F_i$ were defined, for example, in [4], and we invite the reader to refer it for details. The functors $F_i$ are defined at the level of $\text{Set}_i$ by $F_i X_i = M_i \otimes X_i$, for a particular set $M_i$. Here $M_i \otimes X_i$ is the set of equivalence classes of a particular equivalence relation defined on $M_i \times X_i$. We will denote the equivalence class of an element $(m, x) \in M_i \times X_i$ by $m \otimes x$. Details for the two cases $i = 2$ and $i = 3$ are given below.

For the bi-pointed case, we take $M_2 = \{l, r\}$ and consider the equivalence relation on $M_2 \times X$ generated by the relation $(l, \top) \sim (r, \bot)$. The set $M_2 \otimes X$ is lifted to a bi-pointed set by choosing $l \otimes \bot$ and $r \otimes \top$ as $\bot_{M_2 \otimes X}$ and $\top_{M_2 \otimes X}$ respectively. This description is based on Freyd’s description of the unit interval $[0, 1]$ as a final co-algebra (see [4]).
For the tri-pointed case, we take $M_3 = \{a, b, c\}$ and consider the equivalence relation on $M_3 \times X$ generated by the relations $(b, T) \sim (a, L)$, $(a, R) \sim (c, T)$ and $(c, L) \sim (b, R)$. The set $M_3 \otimes X$ is lifted to a tri-pointed set by choosing $a \otimes T$, $b \otimes L$ and $c \otimes R$ as $T_{M_3 \otimes X}$, $L_{M_3 \otimes X}$ and $R_{M_3 \otimes X}$ respectively. This description glues three copies of $X$ as shown in the diagram and is based on the Sierpinski gasket (see [4]).

Given a morphism $f : X \to Y$ of $i$-pointed sets, $F_i f : M_i \otimes X \to M_i \otimes Y$ is given by $F_i f(m \otimes x) = m \otimes f(x)$. This definition makes $F_i$ an endofunctor on $\textbf{Set}_i$. One can easily verify that $F_i f$ preserves the distinguished elements. For example, in the bi-pointed case we have $F_2 f(l \otimes \bot_X) = l \otimes f(\bot_X) = l \otimes \bot_Y$ and $F_2 f(r \otimes \top_X) = r \otimes f(\top_X) = r \otimes \top_Y$.

Moreover, the above definitions give rise to endofunctors on $\textbf{MS}_i^S$, $\textbf{MS}_i^L$ and $\textbf{MS}_i^C$, where $i = 2, 3$; which we also identify by $F_i$, as follows. First, for a given $i$-pointed metric space $(X_i, d)$, $M_i \times X_i$ is given the metric

$$d((m, x), (n, y)) = \begin{cases} \frac{1}{2}d(x, y), & m = n; \\ 1, & m \neq n. \end{cases}$$

We now consider the quotient metric on $M_i \otimes X_i$. Though the quotient metric in general is only a pseudo-metric, in our case it is indeed a metric and the distance between two elements can be computed explicitly as follows (see [4] for proofs).
Lemma 1. The distance $d(m \otimes x, n \otimes y)$ is calculated by the following formulas.
For the bi-pointed case:

$$d(m \otimes x, n \otimes y) = \begin{cases} 
\frac{1}{2}d(x, y), & m = n; \\
\frac{1}{2}d(x, T) + \frac{1}{2}d(\perp, y), & m \neq n.
\end{cases}$$

For the tri-pointed case:

$$d((a \otimes x), (a \otimes y)) = \frac{1}{2}d(x, y)$$

$$d((a \otimes x), (b \otimes y)) = \frac{1}{2}\min\{d(x, L) + d(T, y), d(x, R) + 1 + d(R, y)\}$$

A formula similar to the last one holds for other cases of $a, b$ with $a \neq b$.

Given a $(i$-pointed) metric space $(X_i, d)$, we use the same letter $d$ to identify the metrics on $M \times X_i$ and $M \otimes X_i$, since it is understood from the context to which $d$ we refer.

Consider the bi-pointed case. Clearly, by definition, the metric on $M_2 \otimes X_2$ is one bounded and $d(l \otimes \perp, r \otimes T) = 1$. Therefore $M_2 \otimes X_2$ is a bi-pointed metric space and hence $(M_2 \otimes X_2, d)$ is an object in the category of bi-pointed metric spaces with any choice of morphisms (Lipschitz, short or continuous). One can similarly verify that $M_3 \otimes X$ is a tri-pointed metric space with the metric given in Lemma 1.

To define $F_i$ on the categories $\text{MS}_i^C$, $\text{MS}_i^L$ and $\text{MS}_i^S$, one needs to show that it acts on morphisms in the expected way. What we need follows from i, ii and iii of the following lemma.

Lemma 2. Let $X$ and $Y$ be two $i$-pointed metric spaces. If $f : X \to Y$ has any of the following properties then so does $F_i f$.

i). Continuous

ii). Lipschitz

iii). Short map

iv). Isometric embedding
Proof. (i). Let $\epsilon > 0$ be arbitrary. Since $f$ is continuous, $\exists \delta > 0$ such that $d(f(x), f(y)) < \epsilon$, whenever $d(x,y) < \delta$. Choose $\delta_0 = \min\{\delta, \frac{1}{4}\}$.

Suppose that $d_\mathcal{M}X(a \otimes x, b \otimes y) < \delta_0$. Then we have $d_X(x, L_X) < \delta$ and $d_X(y, T_X) < \delta$. By the continuity of $f$ we have $d_Y(f(x), f(L_X)) < \epsilon$ and $d_Y(f(y), f(T_X)) < \epsilon$. Thus, $d_\mathcal{M}Y(F_i f(a \otimes x), F_i f(b \otimes y)) < \epsilon$, which is the required condition for $F_i f$ to be continuous. This completes the proof.

(ii). Let $f$ be a Lipschitz continuous function with the Lipschitz constant $k$. Note that

$$
\min\{d(f(x), L) + d(T, f(y)) \, , \, d(f(x), R) + 1 + d(R, f(y))\}
= \min\{d(f(x), f(L)) + d(f(T), f(y)) \, , \, d(f(x), f(R)) + 1 + d(f(R), f(y))\}
\leq \min\{kd(x, L) + kd(T, y) \, , \, kd(x, R) + kd(T, L) + kd(R, y)\}
= k \min\{d(x, L) + d(T, y) \, , \, d(x, R) + d(T, L) + d(R, y)\}
$$

Therefore,

$$
d(F_3 f(a \otimes x), F_3 f(b \otimes y)) = d(a \otimes f(x), b \otimes f(y)) \leq kd(a \otimes x, b \otimes y)
$$

This completes the proof.

(iii) is proved in [4] and the proof of (iv) is similar to that of (ii). □

One can now define endofunctors $F_i$ on $\mathcal{MS}_i^S$, $\mathcal{MS}_i^C$ and $\mathcal{MS}_i^L$, as supported by Lemma 2.

Authors of [4] have computed the initial algebra $(G_i, h_i)$ and the final co-algebra $(S_i, \psi_i)$ of $F_i$ on $\mathcal{MS}_i^S$ and they have shown that the final co-algebra is obtained by the Cauchy completion of the initial algebra. It turns out that the initial algebra and the final co-algebra of $F_i$ on $\mathcal{Set}_i$ are the same as that of $F_i$ on $\mathcal{MS}_i^S$ after forgetting the metric structure. Moreover, [4] exhibits a bi-Lipschitz isomorphism between two co-algebras of $F_3$ on $\mathcal{MS}_3^L$, one being the Sierpinski gasket, and raises the question whether either of these co-algebras of $F_3$ is the final co-algebra for the endofunctor $F_3$ on $\mathcal{MS}_3^L$. This study was initiated based on this question.

Our contribution is in two directions. In Section 2, we show that the final co-algebra of $F_i$ on $\mathcal{MS}_i^S$ is same as that on $\mathcal{MS}_i^C$. Along the way we show
how the mediating morphism from a co-algebra to the final co-algebra can be obtained by the limit of a sequence obtained by iterating the co-algebra. In Section 3 we show that the initial algebra of \( F_i \) on \( \text{MS}^S_i \) is not the initial algebra of \( F_i \) on \( \text{MS}^C_i \). We still do not know whether the initial algebra of \( F_i \) on \( \text{MS}^C_i \) exists. Moreover, in Section 4, we show that the final co-algebra and initial algebra of \( F_i \) on \( \text{MS}^S_i \) are not the final co-algebra and initial algebra of \( F_i \) on \( \text{MS}^L_i \). In the case of \( F_i \) on \( \text{MS}^L_i \), we do not know whether the final co-algebra and initial algebra exist. However, the results suggests a negative answer to the question that was raised in [4], and mentioned in the above paragraph.

2. Final co-algebra for \( F_i \) on \( \text{MS}^C_i \)

In this section we consider \( F_i \) defined on \( \text{MS}^C_i \). Authors of [4] have computed the initial algebra and final co-algebra of the endofunctor \( F_i \) on \( \text{MS}^S_i \) and shown that the final co-algebra is given by the Cauchy completion of the initial algebra. Let us briefly recall how this is done. Consider the following initial chain starting from the initial object \( I \), where all the maps are isometric embeddings.

\[
\begin{align*}
I & \rightarrow M_i \otimes I \\
M_i \otimes I & \rightarrow M_i^2 \otimes I \\
M_i^2 \otimes I & \rightarrow M_i^3 \otimes I \\
& \vdots \\
M_i^n \otimes I & \rightarrow M_i^{n+1} \otimes I \\
& \vdots
\end{align*}
\]

Take \( C = \bigcup M_i^n \otimes I \). Define a relation on \( C \) as follows. Let \( x, y \in C \). Then \( x \in M_i^r \otimes I \) and \( y \in M_i^s \otimes I \) for some \( r, s \). Without lost of generality take \( s > r \). The relation is defined by \( x \approx y \) iff \( f(x) = y \); where \( f = M_i^{s-1} \otimes ! \circ \cdots \circ M_i^{r} \otimes ! \). Let \( \sim \) be equivalence closure of \( \approx \). Take \( G = C/\sim \). This \( G \) is the colimit of the above chain and also the carrier set of the initial algebra. The morphisms arising in the colimit are given by \( C_n : M_i^n \otimes I \rightarrow G \), where \( C_n(x) = [x] \), which are isometric embeddings. By Adamek’s Theorem (see [2]) the initial algebra is given by \( (G, g : M_i \otimes G \rightarrow G) \), where \( g : M_i \otimes G \rightarrow G \) is given by \( g(m \otimes [x]) = [m \otimes x] \). The carrier set of the final co-algebra is the Cauchy completion of \( G \) which we denote by \( S \). Throughout this paper we will consider \( G \) as a dense subset of the complete metric space \( S \).

Authors of [4] have also shown that the initial algebra and final co-algebra of \( F_i \) on \( \text{Set}_i \) are the same as that of \( F_i \) on \( \text{MS}^S_i \), leaving out the metric structure. One can make use of this fact to compute the mediating morphism.
at the set level from a given co-algebra to the final coalgebra, as the limit of a sequence. We demonstrate it for the tri-pointed case, as the construction for the bi-pointed case is similar. Let \((X, e)\) be any co-algebra for \(F_3\) on \(\text{Set}_3\), where \(e\) is a set function. By iterating this coalgebra, we obtain the following chain.

\[
X \xrightarrow{e} M_3^0 \otimes X \xrightarrow{M_3^0 \otimes e} M_3^2 \otimes X \xrightarrow{M_3^2 \otimes e} M_3^3 \otimes X \cdots M_3^{n-1} \otimes X \xrightarrow{M_3^{n-1} \otimes e} M_3^n \otimes X \cdots
\]

Set \(M_3^0 \otimes X = X\) and \(M_3^0 \otimes e = e\). For an \(x \in X\), we can iterate this \(x\) and obtain the sequence \((\chi_n)_{n=0,1,\ldots}\) such that \(\chi_n = m_1 \otimes m_2 \otimes \ldots \otimes m_n \otimes x_n \in M_3^n \otimes X\), given by \(\chi_0 = x\), \(\chi_1 = M_3^0 \otimes e(\chi_0) = e(x)\) and \(\chi_n\) is given inductively by \(\chi_n = M_3^{n-1} \otimes e(\chi_{n-1}) = M_3^{n-1} \otimes e(m_1 \otimes m_2 \otimes \ldots \otimes m_{n-1} \otimes x_{n-1})\). Then we have a corresponding sequence \((\theta_n)\) in \(G\) as follows. \(\theta_1 = [m_1 \otimes \overline{x_1}]\), \(\theta_2 = [m_1 \otimes m_2 \otimes \overline{x_2}], \theta_3 = [m_1 \otimes m_2 \otimes m_3 \otimes \overline{x_3}], \ldots\); where \(\overline{x_i}\) is chosen to be \(T\) or \(L\) or \(R\) provided that the value of \(m_i\) is \(a\) or \(b\) or \(c\) respectively. Since \(G\) is also the carrier set arising in the initial algebra of \(F_i\) on \(\text{MS}_i\), it has a metric structure. Now we shall prove that the sequence \((\theta_n)\) is a Cauchy sequence in \(G\).

Let \(\epsilon > 0\) and choose \(N \in \mathbb{N}\) such that \(N > \frac{\ln \frac{1}{2}}{\ln 2}\). Now for \(q > p > N\), consider \(d_G(\theta_p, \theta_q) = d_G([m_1 \otimes \cdots \otimes m_p \otimes \overline{x_p}], [m_1 \otimes \cdots \otimes m_q \otimes \overline{x_q}]\). The right side of this equality is equal to \(d_G([m_1 \otimes \cdots \otimes m_p \otimes l_{p+1} \otimes \cdots \otimes l_q \otimes \overline{y_q}], [m_1 \otimes \cdots \otimes m_q \otimes \overline{x_q}]\); where \(m_1 \otimes \cdots \otimes m_p \otimes l_{p+1} \otimes \cdots \otimes l_q \otimes \overline{y_q} \sim m_1 \otimes \cdots \otimes m_q \otimes \overline{x_q}\). As \(C_q\) defined above is an isometric embedding, \(d_G([m_1 \otimes \cdots \otimes m_p \otimes l_{p+1} \otimes \cdots \otimes l_q \otimes \overline{y_q}], [m_1 \otimes \cdots \otimes m_q \otimes \overline{x_q}]\) is equal to \(d_{M \otimes l}([m_1 \otimes \cdots \otimes m_p \otimes l_{p+1} \otimes \cdots \otimes l_q \otimes \overline{y_q}, m_1 \otimes \cdots \otimes m_q \otimes \overline{x_q}]\) which is less than or equal to \(\frac{1}{2^q}\) (see Lemma 15 of [4]). But \(\frac{1}{2^q} < \frac{\epsilon}{2^p} < \epsilon\). Thus we have \(d_G(\theta_p, \theta_q) < \epsilon\) which is the required condition for \((\theta_n)\) to be a Cauchy sequence. Thus one can consider the limit \(\lim_{n \to \infty} \theta_n\) in \(S\).

**Lemma 3.** \(\lim_{n \to \infty} \theta_n\) is independent of the choice of the sequence \((\chi_n)\).

**Proof.** Suppose we consider two choices for \(\chi_n; \chi_n = m_1 \otimes m_2 \otimes \cdots \otimes m_n \otimes x_n = m'_1 \otimes m'_2 \otimes \cdots \otimes m'_n \otimes y_n\). Thus initially we have \(m_1 \otimes x_1 = m'_1 \otimes y_1\). Without loss of generality take \(m_1 = a\). Then there are two possibilities for \(m'_1\) to be, namely \(b\) or \(c\), and we have \(a \otimes L = b \otimes T\) or \(a \otimes R = c \otimes T\) respectively. Consider the case \(a \otimes L = b \otimes T\). Then the corresponding sequences \(\theta_n, \theta'_n \in G\) given by \(\theta_1 = [a \otimes L], \theta_2 = [a \otimes b \otimes L], \theta_3 = [a \otimes b \otimes \cdots \otimes b \otimes L], \ldots\). 7
and \( \theta'_1 = [b \otimes T], \theta'_2 = [b \otimes a \otimes T], \theta'_n = [b \otimes a \otimes \cdots a \otimes T], \cdots \) are equal. Similarly we can show \( \theta_n = \theta'_n \) for the other cases. Thus their limits are the same.

Thus one can define a function \( f : X \rightarrow S \) by \( f(x) = \lim_{n \rightarrow \infty} \theta_n \), which is well defined according to the above lemma.

**Proposition 1.** The mediating morphism \( f \) for a given co-algebra is given by \( f(x) = \lim_{k \rightarrow \infty} \theta_k \).

**Proof.** We only need to prove that \((M_i \otimes f) \circ e = \psi \circ f\) to show that the following diagram commutes, where \( \psi : S \rightarrow M \otimes S \) is given by \( \psi([m_1 \otimes \cdots \otimes m_k \otimes \overline{x}_k]) = m_1 \otimes [m_2 \otimes \cdots \otimes m_k \otimes \overline{x}_k] \). Notice that \( S \) is the Cauchy completion of the \( G \) and we consider \( G \) as the dense subset of \( S \). Thus \( \psi \) found in [4] is written above form.

\[
\begin{array}{ccc}
X & \xrightarrow{e} & M_i \otimes X \\
\downarrow f & & \downarrow M_i \otimes f \\
S & \xrightarrow{\psi} & M_i \otimes S
\end{array}
\]

For \( x \in X \) and \( e(x) = m_1 \otimes x_1 \), \((M_i \otimes e)(m_1 \otimes x_1) = m_1 \otimes m_2 \otimes x_2, \cdots \). Then we have \( f(x) = \lim_{k \rightarrow \infty} [m_1 \otimes \cdots \otimes m_k \otimes \overline{x}_k] \) and \( f(x_1) = \lim_{k \rightarrow \infty} [m_2 \otimes \cdots \otimes m_k \otimes \overline{x}_k] \). Now \( \psi(f(x)) = \psi\left(\lim_{k \rightarrow \infty} [m_1 \otimes \cdots \otimes m_k \otimes \overline{x}_k]\right) = \lim_{k \rightarrow \infty} \psi([m_1 \otimes \cdots \otimes \overline{x}_k]) = \lim_{k \rightarrow \infty} (m_1 \otimes [m_2 \otimes \cdots \otimes m_k \otimes \overline{x}_k]) \). Now \( M_i \otimes fe(x) = M_i \otimes f(m_1 \otimes x_1) = m_1 \otimes f(x_1) \) which is equal to \( m_1 \otimes \lim_{k \rightarrow \infty} [m_2 \otimes \cdots \otimes m_k \otimes \overline{x}_k] \). But \( m_1 \otimes \lim_{k \rightarrow \infty} \theta_k = \lim_{k \rightarrow \infty} m_1 \otimes \theta_k \). Thus the above diagram commutes.

Now suppose that \((X,e)\) is an \( F_i \) co-algebra on \( MS^C_i \). Then leaving out the metric structure, we can calculate the mediating morphism at the set level as given above. The following lemma states that this mediating morphism \( f \) is continuous in the metric setting for both the bi-pointed case and the tri-pointed case. We prove only the tri-pointed case as the bi-pointed case is similar.

**Proposition 2.** Let \( I \) be the initial object and \( X \) be any object in the category \( MS^C_i \). Let \( \mu \) be the unique morphism \( \mu : I \rightarrow X \) (by initiality condition). Then \( \mu \) and all \( M^n_i \otimes \mu \)'s are isometric embeddings.
Proving $\mu$ is an isometric embedding is straightforward. The other part follows from Lemma 2.

**Lemma 4.** The mediating morphism $f$, defined above, is continuous.

**Proof.** Let $x \in X$ and $\epsilon > 0$ be arbitrary. From the definition of $f$ we have 
$$d_S(f(x), f(y)) = \lim_{p \to \infty} d_G([m_1 \otimes \cdots \otimes m_p \otimes x_p], [n_1 \otimes \cdots \otimes n_p \otimes y_p]).$$
Therefore, there is some $N \in \mathbb{N}$ such that 
$$d_S(f(x), f(y)) - \frac{\epsilon}{4} < d_G([m_1 \otimes \cdots \otimes m_p \otimes x_p], [n_1 \otimes \cdots \otimes n_p \otimes y_p])$$
for all $p > N$ and for all $y \in X$. Choose $p = \max\left\{N + 1, \frac{\ln 4}{\ln 2}\right\}$. Denote $g_p = M_p^{-1} \otimes e \circ \cdots \circ M_1 \otimes e \circ e$. Because $g_p$ is continuous at $x$, $\exists \delta_p > 0$ such that $d_{M_p \otimes I}(g_p(x), g_p(y)) < \frac{\epsilon}{4}$ whenever $d(x, y) < \delta_p$. Suppose that $d(x, y) < \delta_p$. Then we have

$$d_S(f(x), f(y)) < \frac{\epsilon}{4} + d_{M_p \otimes I}([m_1 \otimes \cdots \otimes m_p \otimes x_p], [n_1 \otimes \cdots \otimes n_p \otimes y_p])$$

$$=[ \because M_p \otimes I \xrightarrow{C_p} G \text{ isometric embedding} ]$$

$$= \frac{\epsilon}{4} + d_{M_p \otimes X}([m_1 \otimes \cdots \otimes m_p \otimes x_p, n_1 \otimes \cdots \otimes n_p \otimes y_p])$$

$$=[ \because M_p \otimes I \xrightarrow{M_p \otimes \mu} M_p \otimes X \text{ isometric embedding} ]$$

$$\leq \frac{\epsilon}{4} + d_{M_p \otimes X}([m_1 \otimes \cdots \otimes m_p \otimes x_p, m_1 \otimes \cdots \otimes m_p \otimes x_p] +$$

$$d_{M_p \otimes X}([m_1 \otimes \cdots \otimes m_p \otimes x_p, n_1 \otimes \cdots \otimes n_p \otimes y_p] +$$

$$d_{M_p \otimes X}([n_1 \otimes \cdots \otimes n_p \otimes y_p, n_1 \otimes \cdots \otimes n_p \otimes y_p])$$

$$\leq \frac{\epsilon}{4} + \frac{1}{2p} + d_{M_p \otimes X}(g_p(x), g_p(y)) + \frac{1}{2p}$$

$$< \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon$$

completing the proof. \qed

The mediating morphism is uniquely determined for a given co-algebra $(X, e)$ and hence it is unique. Therefore, $(S, \psi)$ is the final co-algebra of $F_i$ on $\text{MS}_i^C$. 

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Proposition 3. The final co-algebra of $F_i$ on $MS^C_i$ is the same as that of $F_i$ on $MS^S_i$.

3. The initial algebra of $F_i$ on $MS^C_i$ is not that of $F_i$ on $MS^S_i$

As the final co-algebra of $F_i$ on $MS^C_i$ is the same as the final co-algebra of $F_i$ on $MS^S_i$, one may wonder whether a similar result holds for the initial algebra of $F_i$ on $MS^S_i$ and $MS^C_i$. We give a negative answer to this question by giving a counter example (Example 1). First we need some preliminary results. The initial algebra of $F_i$ on $MS^S_i$ is the same as the initial algebra of $F_i$ on $Set^i$ (see [4]). Lemma 5 given below states a way to compute the mediating morphism at the set level for both the tri-pointed and bi-pointed cases. We will use this later to decide whether the mediating morphism is continuous or Lipschitz. Again, we demonstrate it only for the tri-pointed case, as the bi-pointed case is similar.

Let $(X,e)$ be an algebra for $F_3$ on $Set_3$, where $e$ is a set function. Let $(G,g)$ be the initial algebra of $F_3$ in this setting (See [4]). For $x = [((m_1 \otimes \cdots m_k \otimes d)] \in G$; where $d \in \{T, L, R\}$, $\bar{x} = m_1 \otimes \cdots m_k \otimes d_X$. $d_X$ is chosen corresponding to $d$. For instance if $d = T$, then $d_X = T_X$.

Now consider the chain $M^k_i \otimes X \xrightarrow{M^k_i \otimes e} M^{k-1}_i \otimes X \cdots M_3 \otimes X \xrightarrow{e} X$. Take $g_k = e \circ M_3 \otimes e \cdots M^{k-1}_3 \otimes e$, and define $f : G \longrightarrow X$ by $f(x) = g_k(\bar{x})$.

Lemma 5. The mediating morphism for a given algebra $(X,e)$ for $F_i$ on $Set_i$, $i = 2$ or $3$, is given by $f(x) = g_k(\bar{x})$, where $g_k = e \circ M_i \otimes e \cdots M^{k-1}_i \otimes e$.

Proof. Let us first prove that $f$ is well defined. Let $[(p_1 \otimes \cdots p_r \otimes d_r)] = [(q_1 \otimes \cdots q_s \otimes d_s)]$. Thus we have $p_1 \otimes \cdots p_r \otimes d_r \sim q_1 \otimes \cdots q_s \otimes d_s$. Without loss of generality take $s > r$ and consider the following initial chain, where 0 is the initial object.

Thus $(M^{s-1}_i \otimes \circ M^{s-2}_i \otimes \cdots \otimes M^{s}_i \circ d_r) = q_1 \cdots \otimes d_s$

Since 0 is the initial object, the following diagram commutes.

$$
\begin{array}{c}
M_i \otimes X \xrightarrow{e} X \\
\uparrow M_i \otimes \eta \quad \quad \quad \uparrow \eta \\
M_i \otimes 0 \xleftarrow{!} 0
\end{array}
$$
and hence by applying $M_i \otimes -$ repeatedly we have the following commuting square.

\[
\begin{array}{c}
M_i^s \otimes X \\
\uparrow M_i^s \otimes \eta \\
M_i^s \otimes 0
\end{array}
\quad
\begin{array}{c}
M_i^r \otimes e \circ M_i^{r+1} \otimes e \circ \cdots \circ M_i^{s-2} \otimes e \circ M_i^{s-1} \otimes e \\
\uparrow M_i^r \otimes \eta \\
M_i^r \otimes 0
\end{array}
\quad
\begin{array}{c}
M_i^r \otimes X
\end{array}
\]

Thus

\[
M_i^r \circ e \circ M_i^{r+1} \circ e \circ \cdots \circ M_i^{s-2} \circ e \circ M_i^{s-1} \circ e(q_1 \otimes \cdots q_s \otimes d_{sX}) = p_1 \otimes \cdots p_r \otimes d_{rX}
\]

Now consider $f[(q_1 \otimes \cdots q_s \otimes d_s)] = e \circ M_i \circ e \circ \cdots \circ M_i^{s-1} \circ e(q_1 \otimes \cdots q_s \otimes d_{sX})$. The right side of this equation is equal to

\[
e \circ M_i \circ e \circ \cdots \circ M_i^{s-1} \circ e(M_i^r \circ e \circ \cdots \circ M_i^{s-1} \circ e(q_1 \otimes \cdots q_s \otimes d_{sX}))
\]

which is equal to $e \circ M_i \circ e \circ \cdots \circ M_i^{r-1} \circ e(p_1 \otimes \cdots p_r \otimes d_{rX}) = f[(p_1 \otimes \cdots p_r \otimes d_r)]$. Thus $f[(q_1 \otimes \cdots q_s \otimes d_s)] = f[(p_1 \otimes \cdots p_r \otimes d_r)]$ and therefore $f$ is well defined.

We are left to show that the following diagram commutes.

\[
\begin{array}{ccc}
M_i \otimes G & \overset{g}{\longrightarrow} & G \\
\downarrow M_i \otimes f & & \downarrow f \\
M_i \otimes X & \overset{e}{\longrightarrow} & X
\end{array}
\]

For any $m_0 \otimes [(m_1 \otimes m_2 \otimes \cdots \otimes m_k \otimes x_k)] \in M_i \otimes G$ and $e(m_k \otimes x_k) = x_{k-1}$; where $k = 1, 2, \ldots$, Consider the following equality.

\[
f[(m_0 \otimes m_1 \otimes \cdots \otimes m_k \otimes x_k)] = e \circ M_i \circ e \circ \cdots \circ M_i^k \circ e(m_0 \otimes m_1 \otimes \cdots \otimes m_k \otimes x_{kX})
\]

Applying $M_i^k \otimes e$ to the element $m_0 \otimes m_1 \otimes \cdots \otimes m_k \otimes x_{kX}$, the right side becomes $e \circ \cdots \circ M_i^{k-1} \circ e(m_0 \otimes m_1 \otimes \cdots \otimes m_{k-1} \otimes x_{k-1})$. Continuing this process, eventually we get $f[(m_0 \otimes m_1 \otimes m_k \otimes x_k)] = e(m_0 \otimes x_0)$. Similarly $f[(m_1 \otimes \cdots m_k \otimes x_k)] = x_0$. Thus we have $e \circ M_i \circ f \{m_0 \otimes [(m_1 \otimes \cdots m_k \otimes x_k)]\} = f \circ h \{m_0 \otimes [(m_1 \otimes \cdots m_k \otimes x_k)]\} = e(m_0 \otimes x_0)$ and hence $f \circ h = e \circ M_i \circ f$. □
Proposition 4 (See [4]). Let \( X_0 = \{\bot, \top\} \) be a bi-pointed set. Then there are isometries \( c_0, c_1, c_2, \cdots, c_n, \cdots \) such that \( M^p_2 \otimes X_0 \simeq D_n ; \forall n \in \mathbb{Z}^+ \cup \{0\} \).

Here \( D_n = \left\{ \frac{p}{2^q} / 0 \leq p \leq 2^q & p, q \in \mathbb{Z}^+ \cup \{0\} \right\} \) and \( c_0 : X_0 \rightarrow D_0 = \{0, 1\} \) is given by \( c_0(\bot) = 0, c_0(\top) = 1 \). Moreover, \( c_k : M^k_2 \otimes X_0 \rightarrow D_k \) is given inductively by

\[
c_k(m_1 \otimes \cdots m_k \otimes d) = \begin{cases} 
\frac{1}{2} c_k(m_2 \otimes \cdots m_k \otimes d), & m_1 = l; \\
\frac{1}{2} (c_k(m_2 \otimes \cdots m_k \otimes d) + 1), & m_1 = r.
\end{cases}
\]

The following example states that \( (D, \phi) \) is not an initial algebra of \( F_2 \) on \( \text{MS}_2^C \) and \( \text{MS}_2^L \). We will show it for the continuous case. The same example will work for the Lipschitz case too.

Example 1. Let \( X_0 = \{\bot, \top\} \) be a bi-pointed set. Consider the function \( e : M_2 \otimes X_0 \rightarrow X_0 \) given by \( e(l \otimes 0) = 0, e(l \otimes 1) = 0 \) and \( e(r \otimes 1) = 1 \). Clearly this \( e \) is Lipschitz as it is a function from a finite metric space and it is also continuous. The initial algebra of \( F_2 \) on \( \text{MS}_2^S \) is the pair \( (D, \phi) \) where \( D \) is the dyadic rationals in the unit interval and \( \phi : M_2 \otimes D \rightarrow D \) is as follows.

\[
\phi(m \otimes x) = \begin{cases} 
x, & m = l; \\
x + \frac{1}{2}, & m = r.
\end{cases}
\]

Suppose that \( (\phi, D) \) is the initial algebra of \( F_2 \) on \( \text{MS}_2^C \). Then there is a unique continuous function \( f \) such that the expected diagram commutes. Suppose that \( f \) is continuous. Then \( \exists \delta > 0 \) such that \( d(f(x), f(y)) < \frac{1}{2} \) whenever \( \forall d(x, y) < \delta \). Choose \( n \) large enough so that \( \frac{1}{2^n} < \delta \).

Thus \( d(1, \frac{2^n - 1}{2^n}) < \delta \) and \( d\left(f(1), f\left(\frac{2^n - 1}{2^n}\right)\right) < \frac{1}{2} \). Now \( f(1) = 1 \) as \( f \) preserves the distinguished elements. Now \( \frac{2^n - 1}{2^n} \in D_n \) and from the straightforward computation we have \( c_n(r \otimes r \otimes \cdots \otimes r \otimes 0) = \frac{2^n - 1}{2^n} \). Thus the element \( \frac{2^n - 1}{2^n} \) is identified with \( r \otimes r \otimes \cdots \otimes r \otimes 0 \). Thus \( f\left(\frac{2^n - 1}{2^n}\right) \) can
be evaluated as follows.

\[
\begin{align*}
  f\left(\frac{2^n - 1}{2^n}\right) &= e_2 \circ M_2 \otimes e_2 \circ \cdots \circ M_{2^n - 2} \otimes e_2 \circ M_{2^n - 1} \otimes e_2 ((r \otimes r \cdots \otimes r \otimes 0) \\
  &= e_2 \circ M_2 \otimes e_2 \circ \cdots \circ M_{2^n - 2} \otimes e_2 ((r \otimes \cdots \otimes r \otimes 0) \\
  &= : \ : \ : \\
  &= e_2(r \otimes 0) = 0
\end{align*}
\]

Therefore we have \(d\left(f(1), f\left(\frac{2^n - 1}{2^n}\right)\right) = d(1, 0) = 1 < \frac{1}{2}\) which is obviously a contradiction. Thus the mediating morphism \(f\) is not continuous. Hence \((\phi, D)\) is not the initial algebra for \(F_2\) on \(\text{MS}_2^C\).

**Example 2.** Consider the tri-pointed set \(Y_0 = \{T, L, R\}\) and the function \(e_3 : M_3 \otimes Y_0 \rightarrow Y_0\) given by \(e_3(a \otimes T) = T, e_3(c \otimes R) = R\) and \(e_3(a \otimes L) = e_3(a \otimes R) = e_3(b \otimes L) = e_3(b \otimes R) = L\). Because \(e_3\) is a function from a finite metric space, it is Lipschitz and hence continuous too. Consider the initial algebra \((G, g)\) of \(F_3\) on \(\text{MS}_3^S\), where \(g : M_3 \otimes G \rightarrow G\) is given by \(g(m \otimes [x]) = [m \otimes x]\). See the beginning of Section 3.

Leaving out the metric structure, \((G, g)\) is also the initial algebra of \(F_3\) on \(\text{Set}_3\) and \((Y_0, e_3)\) is a \(F_3\) algebra on \(\text{Set}_3\). Thus there exists a unique \(f : G \rightarrow Y_0\) such that following diagram commutes.

\[
\begin{array}{ccc}
  M_3 \otimes G & \xrightarrow{g} & G \\
  \downarrow M_3 \otimes f & & \downarrow f \\
  M_3 \otimes Y_0 & \xrightarrow{e_3} & Y_0
\end{array}
\]

The map \(f\) was found explicitly in Lemma 5. Suppose that \(f\) is continuous at \([T] \in G\). Then \(\exists \delta > 0\) such that \(d(f([T]), f(y)) < \frac{1}{2}\), whenever \(d(T, y) < \delta\). Choose \(n\) large enough so that \(\frac{1}{2^n} < \delta\). Note that \([T] = [a \otimes T] = [a \otimes a \otimes T] = \cdots = [a \otimes a \otimes \cdots \otimes a \otimes T] = \cdots\), and \(f([T]) = f([a \otimes a \otimes \cdots \otimes a \otimes T]) = T_{Y_0}\) as the distinguished elements are preserved by \(f\). Let \(y = [a \otimes a \otimes \cdots \otimes a \otimes L]\), where \(a\) occurs \((n + 1)\) times.
Then \( d([T], y) = \frac{1}{2^{n+1}} < \frac{1}{2^n} < \delta \). However,

\[
\begin{align*}
f(y_0) &= f([a \otimes a \otimes \cdots \otimes a \otimes L]) \\
&= e_3 \circ M_3 \otimes e_3 \circ \cdots \circ e_3 \circ M_3^{n-1} \otimes e_3 \circ e_3 (a \otimes a \otimes \cdots \otimes a \otimes L) \\
&= e_3 \circ M_3 \otimes e_3 \circ \cdots \circ M_3^{n-1} \otimes e_3 (a \otimes \cdots \otimes a \otimes L) \\
&= \vdots \\
&= e_3 (a \otimes L) = L_{Y_0}.
\end{align*}
\]

Therefore \( d(f(T), f(y_0)) = d(T_{Y_0}, L_{Y_0}) = 1 < \frac{1}{2} \), which is a contradiction. Thus \( f \) is not continuous and hence \( f \) is not Lipschitz. Hence \((G, g)\) is not the initial algebra of \( F_3 \) on \( MS_3^C \) as well as on \( MS_3^L \).

4. Final co-algebra and initial algebra of \( F_i \) on \( MS_i^S \) are not that of \( F_i \) on \( MS_i^L \)

In this section we answer two questions. One is the question raised in [4], whether \((S_i, \psi_i)\), the final co-algebra of \( F_i \) on \( MS_i^S \) is the final co-algebra of \( F_i \) on \( MS_i^L \). In Section 2 we have shown that \((S_i, \psi_i)\) is the final co-algebra of \( F_i \) on \( MS_i^C \). However, we provide a negative answer to the question by showing that \((S_i, \psi_i)\) is not the final co-algebra of \( F_i \) on \( MS_i^L \). The initial algebra \((G_i, g_i)\) of \( F_i \) on \( MS_i^S \), after leaving out the metric structure, is the same as that of \( F_i \) on \( Set_1 \) (See [4]). One may ask a similar question, whether \((G_i, g_i)\) of \( F_i \) on \( MS_i^L \), after leaving out the metric structure, is the same as that of \( F_i \) on \( Set_1 \). We give a negative answer to this question too.

As a consequence of Lemma 2, we have \( F_i \) as an endofunctor on \( MS_i^L \). Recall the final co-algebra of \( F_2 \) on \( MS_2^S \) which is \((I, i)\); where \( I = [0, 1] \) and \( i : I \rightarrow M_2 \otimes I \) is given by

\[
i(x) = \begin{cases} 
I \otimes x, & x \in [0, \frac{1}{2}] \\
r \otimes x, & x \in [\frac{1}{2}, 1]. 
\end{cases}
\]

Example 3. Define \( e : I \rightarrow M_2 \otimes I \) by
One can easily show that \( e \) is Lipschitz with Lipschitz constant 2 and thus \((e, I)\) is a co-algebra in \( \text{MS}_{1}^{L} \). Since \((I, i)\) is the final co-algebra in \( \text{Set}_{2} \), after forgetting the metric structure, there exists a unique set function \( f: I \rightarrow I \) such that the expected diagram commutes.

\[
\begin{array}{ccc}
I & \leftarrow & M_{2} \otimes I \\
\uparrow f & & \uparrow M_{2} \otimes f \\
I & \rightarrow & M_{2} \otimes I \\
\end{array}
\]

By commutativity \( f \) must satisfy the following conditions.

\[
f(x) = \begin{cases} 
0 & x \in [0, \frac{1}{2}] ; \\
\frac{f(4x - 1)}{2} & x \in \left[ \frac{1}{4}, \frac{1}{2} \right] ; \\
\frac{1 + f(4x - 2)}{2} & x \in \left[ \frac{3}{4}, 1 \right] ; \\
1 & x \in \left[ \frac{3}{4}, 1 \right] .
\end{cases}
\]

Define the following families of intervals for \( n = 1, 2, 3, \cdots \).

\[
I_{n} = \left[ \frac{1}{4} + \cdots + \frac{1}{4^{n}} \right] \\
J_{n} = \left[ \frac{1}{4} + \cdots + \frac{1}{4^{n}} + \frac{3}{4^{n+1}} \right]
\]

Using the conditions the mediating morphism satisfies, we will show that \( f \) satisfies the following properties.
(a) \( f(x) = 0 \), \( \forall n \in \mathbb{N} \) and \( \forall x \in I_n \)

(b) \( f(x) = \frac{1}{2^n} \), \( \forall n \in \mathbb{N} \) and \( \forall x \in J_n \)

We shall prove these properties by induction on \( n \). First let us prove (a). For \( n = 2 \) and \( x \in I_2 \), we have \( 4x - 1 \in [0, \frac{1}{4}] \) and \( f(4x - 1) = 0 \). Thus \( f(x) = \frac{f(4x - 1)}{2} = 0 \). Suppose that \( f(x) = 0 \), \( \forall x \in I_n \). Let \( x \in I_{n+1} \). Then \( 4x - 1 \in I_n \) and \( f(4x - 1) = 0 \). Thus \( f(x) = \frac{f(4x - 1)}{2} = 0 \). Thus by induction \( f(x) = 0 \), for \( x \in I_n \).

To prove (b), first consider the case \( n = 1 \) and let \( x \in J_1 \). We then have \( 4x - 1 \in [\frac{3}{4}, 1] \) and \( f(4x - 1) = 1 \). Thus \( f(x) = \frac{f(4x - 1)}{2} = \frac{1}{2} \). Now suppose that for any \( x \in J_n \), \( f(x) = \frac{1}{2^{n-1}} \). Let \( x \in J_{n+1} \). We have \( 4x - 1 \in J_n \) and \( f(4x - 1) = \frac{1}{2^n} \). Thus \( f(x) = \frac{f(4x - 1)}{2} = \frac{1}{2^{n+1}} \). Thus by induction \( f(x) = \frac{1}{2^n} \), \( x \in J_n \).

With (a) and (b) being proved, to show that \( f \) is not Lipschitz, suppose to the contrary that \( f \) is Lipschitz. Then we have some \( k > 0 \) such that \( d(e(x), e(y)) \leq kd(x, y) \), \( \forall x, y \in I \). Choose \( x = \frac{1}{4} + \cdots + \frac{1}{4^n} + \frac{1}{4^{n+1}} \) and \( y = \frac{1}{4} + \cdots + \frac{1}{4^n} + \frac{3}{4^{n+1}} \). Then \( f(x) = 0 \) and \( f(y) = \frac{1}{2^n} \). From the Lipschitz condition, we have \( \frac{1}{2^n} \leq k \frac{2}{4^{n+1}} \), \( \forall n \in \mathbb{N} \); which implies \( k \geq 2.2^n \), \( \forall n \in \mathbb{N} \). Hence \( k \) is not bounded, which is a contradiction. Therefore \( f \) is not Lipschitz.

Thus we have the following proposition.

**Proposition 5.** \((I, i)\) is not the final co-algebra of \( F_2 \) on \( \text{MSL}_2 \).

**Example 4.** Consider the tri-pointed set \( \Delta = \{(x, 0) / x \in [0, 1]\} \cup \{(\frac{1}{2}, \sqrt{3})\} \), whose distinguished elements are given by \( T_\Delta = (\frac{1}{2}, \sqrt{3}) \) and \( L_\Delta = (0, 0) \), \( R_\Delta = (1, 0) \), and the metric is given by the euclidean metric on \( \mathbb{R}^2 \).
Define $e' : \triangle \rightarrow M_3 \odot \triangle$ by

$$e'(x, y) = \begin{cases} 
  a \odot \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right), & (x, y) = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right); \\
  b \odot (0, 0), & x \in \left[ 0, \frac{1}{4} \right] & y = 0; \\
  b \odot (4x - 1, 0), & x \in \left[ \frac{1}{4}, \frac{1}{2} \right] & y = 0; \\
  c \odot (4x - 2, 0), & x \in \left[ \frac{1}{2}, \frac{3}{4} \right] & y = 0; \\
  c \odot (1, 0), & x \in \left[ \frac{3}{4}, 1 \right] & y = 0.
\end{cases}$$

This $e'$ is a Lipschitz map with Lipschitz constant 2 and hence $(e', \triangle)$ is an $F_3$ co-algebra.

Suppose $(S, \psi)$ is the final co-algebra. Then $(S, \sigma)$ is also a final co-algebra as they are isomorphic (see [4]). Now, as in Example 3, there exists a unique Lipschitz map $g : \triangle \rightarrow S$ such that the following diagram commutes.

$$\begin{array}{rcl}
S & \xleftarrow{\sigma^{-1}} & M_3 \odot S \\
\uparrow g & & \uparrow M_3 \odot g \\
\triangle & \xrightarrow{e'} & M_3 \odot \triangle
\end{array}$$

By commutativity, $g$ must satisfy the following condition.

$$g(x, 0) = 0, \ x \in \left[ 0, \frac{1}{4} \right], \ g(x, 0) = 1, \ x \in \left[ \frac{3}{4}, 1 \right] \quad \text{and}$$

$$g(x, 0) = \begin{cases} 
  \frac{g(4x - 1, 0)}{2}, & x \in \left[ \frac{1}{4}, \frac{1}{2} \right]; \\
  1 + \frac{g(4x - 2, 0)}{2}, & x \in \left[ \frac{3}{4}, 1 \right].
\end{cases}$$

Using these specific properties of this mediating morphism, $g$ will satisfy the properties given below.

(a) $g(x, 0) = 0, \ x \in I_n, \ \forall \ n \in \mathbb{N}$.  

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(b) \( g(x, 0) = \frac{1}{2^n}, x \in J_n, \forall n \in \mathbb{N}. \)

From these properties it follows, as in Example 3, that \( g \) is not Lipschitz. Hence, neither \((S, \psi)\) nor \((S, \sigma)\) can be the final co-algebra.

**Proposition 6.** \((S, \psi)\) is not the final co-algebra of \( F_3 \) on \( \text{MS}_{2}^{L} \).

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