On exact discretization of the cubic-quintic Duffing oscillator

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Abstract
Application of intersection theory to construction of \( n \)-point finite-difference equations associated with classical integrable systems is discussed. As an example, we present a few exact discretizations of one-dimensional cubic and quintic Duffing oscillators sharing form of Hamiltonian and canonical Poisson bracket up to the integer scaling factor.

1 Introduction
A completely integrable system on symplectic manifold \( M \) with form \( \omega \) of dimension \( 2n \) is defined by \( n \) smooth functions \( f_1, \ldots, f_n \) in involution
\[
\{f_i, f_j\} = 0, \quad i, j = 1, \ldots, n
\]
with the independent differentials \( df_i \) at each cotangent space \( T^*_x(M), x \in M \).

If \( c \in \mathbb{R}^n \) is a regular value of \( f = (f_1, \ldots, f_n) \), then the corresponding level \( X = f^{-1}(c) \) is a smooth \( n \)-dimensional Lagrangian submanifold of \( M \). Geometrically this means that locally around the regular value \( c \) the map \( f : M \to \mathbb{R}^n \) collecting the integrals of motion is a Lagrangian fibration, i.e. it is locally trivial and the fibers are Lagrangian submanifolds.

Let us consider a finite-difference equation
\[
\mathcal{Y}(P_{k-\ell}, \ldots, P_k, \ldots P_{k+m}; k) = 0, \quad k, \ell, m \in \mathbb{Z}.
\] (1.1)
relating \( \ell + m + 1 \) points \( P_i \) of submanifold \( X \). Ordinary finite difference equations of this type can be viewed as a dynamical \( \ell + m + 1 \)-point map, see [15].

Because all points \( P_i \) in (1.1) belong to the given Lagrangian submanifold we may suppose that the corresponding map preserves functions \( f_i \) and symplectic form \( \omega \) up to a scaling factor. Finite-difference equation sharing integrals of motion with the continuous time system and the symplectic structure is the so-called exact discretization of integrable systems. Nowadays, refactorization in the Poisson-Lie groups is viewed as one of the most universal constructions of finite difference equations (1.1), see discussion in [3, 4, 15, 18, 19, 27] and references within.

The idea is to identify \( \ell + m + 1 \) points \( P_{k-\ell}, \ldots, P_{k+m} \) in (1.1) with intersection points of \( X \) with auxiliary curve \( Y \). If \( X \) and \( Y \) are algebraic, then we can consider the standard equation for their intersection divisor
\[
\text{div}(X \cdot Y) = 0
\]
as the finite-difference equation (1.1) for the corresponding completely integrable system. Here \( \text{div}(X \cdot Y) \) is the intersection divisor of two algebraic varieties and \( = \) is a suitable equivalence relation [6, 11, 13, 16]. Our main objective is to study properties of such \( \ell + m + 1 \)-point finite-difference equations for different integrable systems [32, 33, 34, 35, 36]. In this paper we restrict ourselves by consideration of cubic and quintic nonlinear Duffing oscillators in order to clarify our viewpoint on relations between the exact discretizations and the intersection divisors.

Thus, we consider integrable systems on two-dimensional plane \( M \) with a pair coordinates \( q, p \) and symplectic form \( \omega = dp \wedge dq \). Because any smooth curve on the plane is a Lagrangian submanifold, we can directly apply classical intersection theory [2, 12] to exact discretization of one-dimensional...
Hamiltonian systems with the algebraic Hamilton function. Below we consider Hamiltonians $H(q, p)$ associated with hyperelliptic curves $X$ on the projective plane defined by equation

$$X : \quad y^2 = a_{2g+1}x^{2g+1} + \cdots + a_1x + a_0, \quad a_j \in \mathbb{C},$$

at $g = 1, 2$ and various intersections of $X$ with the line, quadric and cubic on the plane defined by

$$Y : \quad y = \mathcal{P}(x), \quad \mathcal{P}(x) = b_mx^m + \cdots + b_0, \quad m = 1, 2, 3.$$ (1.2)

From now on $x$ and $y$ mean coordinates on the projective plane, whereas $q$ and $p$ are coordinates on the phase space $M$.

## 2 Cubic oscillator

The Hamilton function

$$H(q, p) = p^2 - a_4q^4 - a_3q^3 - a_2q^2 - a_1q$$ (2.1)

and canonical Poisson bracket $\{q, p\} = 1$ determine Hamiltonian equations

$$\dot{q} = \frac{\partial H}{\partial p} = 2p, \quad \dot{p} = -\frac{\partial H}{\partial q} = 4a_4q^3 + 3a_3q^2 + 2a_2q + a_1$$ (2.2)

and equation of motion

$$\ddot{q} = 8a_4q^3 + 6a_3q^2 + 4a_2q + 2a_1,$$ (2.3)

for the generalized oscillator with the cubic nonlinearity [22].

At $a_3 = a_1 = 0$ this integrable system is called a cubic Duffing oscillator without forcing. Duffing oscillators have received remarkable attention in recent decades due to the variety of their engineering applications. For instance magneto-elastic mechanical systems, large amplitude oscillations of centrifugal governor systems, nonlinear vibration of beams, plates and fluid flow induced vibration, seismic waves before earthquake, ecology or cancer dynamics, financial fluctuations and so on are modeled by the nonlinear Duffing equations.

In the numerical integration of nonlinear differential equations, discretization of the nonlinear terms poses extra ambiguity in reducing the differential equation to a discrete difference equation. For instance, in the framework of the standard-like discretization differential equation

$$\ddot{q} + Aq + Bq^3 = 0$$ (2.4)

can be transformed to the finite-difference equation

$$\frac{q_{n+1} - 2q_n + q_{n-1}}{h^2} + Aq_n + B(q_{n+1} + q_{n-1})q_n^2 = 0,$$

where $h$ is a discrete time interval [23, 24]. This equation may be reduced to the expression with the mapping function $F(q_n)$

$$q_{n+1} - 2q_n + q_{n-1} = F(q_n), \quad F(q_n) = \frac{-Aq_n - Bq_n^3}{h^{-2} + 1/2Bq_n^2}$$

or to the area preserving map on the plane

$$p_{n+1} = p_n + \phi(q_n), \quad q_{n+1} = q_n + p_{n+1},$$

where $\phi(q_n)$ is a rational control function [20, 21]. This integrable map admits the invariant integral

$$\tilde{H} = p_{n+1}^2 + Aq_nq_{n+1} + Bq_n^2q_{n+1}^2,$$

see details in [20, 21, 23, 24, 25, 26, 27], but it is not exact discretization of the Duffing oscillator, i.e. trajectories of the discrete flow do not coincide with the trajectories of the continuous flow.
2.1 Exact discretization and intersection divisors

In order to get exact discretization sharing integrals of motion with the continuous time system and the Poisson bracket we can start with the well-known analytical solutions $q(t)$ of the Duffing equation (2.4), which are expressed via Jacobi elliptic functions.

Indeed, let us consider the equation (2.4) with initial condition

$$q(0) = \alpha, \quad \dot{q}(0) = 0.$$ 

For $B > 0$ and $A > -\alpha^2 B$ periodic solution is

$$q(t) = \alpha \, \text{cn} \left( 2(A + 2\alpha^2 B)^{1/2} t ; m \right), \quad m = \frac{\alpha^2 B}{A + 2\alpha^2 B}.$$ 

For $B > 0$ and $-B\alpha^2 < A < -2\alpha^2$ periodic solution reads as

$$q(t) = \alpha \, \text{dn} \left( 2B^{1/2} t ; m \right), \quad m = 2 \left( 1 + \frac{A}{2\alpha^2} \right).$$ 

For $B < 0$ and $A > -2\alpha^2$ periodic solution has the form

$$q(t) = \alpha \, \text{sn} \left( 2(A + \alpha^2 B)^{1/2} t ; m \right), \quad m = -\frac{\alpha^2 B}{A + \alpha^2 B}.$$ 

Here $\text{cn}(z; m)$ and $\text{sn}(z; m)$ are the Jacobi elliptic functions. Discussion of the non-periodic solution can be also found in [23, 24].

Following [3] we can construct exact discretizations of the Duffing equation using these explicit solutions and well-known addition theorems for Jacobi elliptic functions, for instance

$$\text{sn}(X + Y) = \frac{\text{sn}X \, \text{cn}Y \, \text{dn}Y + \text{sn}Y \, \text{cn}X \, \text{dn}X}{1 - m^2 \text{sn}^2 X \, \text{sn}^2 Y}.$$ 

However, it is more easy and convenient to apply standard algorithms of the intersection theory for this purpose.

In order to apply the intersection theory to the exact discretization of the cubic oscillator we put $H(q, p) = E$ and consider the corresponding level curve $X$ on the projective plane defined by equation

$$X: \quad y^2 = f(x), \quad f(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0. \tag{2.5}$$

where $E = a_0$. Any partial solution $q(t)$ and $p(t)$ of the Hamiltonian equations at $t = t_i$ is a point $P_i = (x_i, y_i)$ of $X$ with absissa $x_i = q(t_i)$ and ordinate $y_i = p(t_i)$. It allows us to study relations between points of $X$ instead of relations between solutions of the differential equations.

Let $X$ be a smooth nonsingular algebraic curve on a projective plane. Prime divisors are rational points on $X$ denoted $P_i = (x_i, y_i)$ and $P_{\infty}$ is a point at infinity. Divisor

$$D = \sum m_i P_i, \quad m_i \in \mathbb{Z}$$

is a formal sum of prime divisors, and the degree of divisor $D$ is a sum $\deg D = \sum m_i$ of multiplicities of points in support of the divisor. Group of divisors is an additive Abelian group under the formal addition rule

$$\sum m_i P_i + \sum n_i P_i = \sum (m_i + n_i) P_i.$$ 

Two divisors $D, D' \in \text{Div}X$ are linearly equivalent

$$D \approx D'$$

if their difference $D - D'$ is principal divisor

$$D - D' = \text{div}(\psi) \equiv 0 \mod \text{Prin}X,$$
i.e. divisor of rational function \( \psi \) on \( X \).

Intersection divisor of \( X \) with some auxiliary smooth nonsingular plane curve \( Y \)

\[
div(X \cdot Y) = 0 \mod \text{PrinX}
\]
is equal to zero with respect to the linear equivalence of divisors. It allows us to identify intersection divisor with some finite-difference equation (1.1)

\[
\mathcal{Y}(P_{k-\ell}, \ldots, P_k, \ldots, P_{k+m}; k) = \sum_{i=k-\ell}^{k+m} m_i P_i + \sum_{i} n_i P_i^{(k)} = 0 \mod \text{PrinX}.
\]

Here we divide intersection divisor in two parts

\[
div(X \cdot Y) = \sum_{i=k-\ell}^{k+m} m_i P_i + \sum_{i} n_i P_i^{(k)}
\]

where prime divisors \( P_i^{(k)} \) are parameters of discretization implicitly depending on \( k \).

2.2 Examples of the intersections

Let us consider the intersection of plane curve \( X \) with a parabola

\[
Y : \quad y = P(x), \quad P(x) = b_2 x^2 + b_1 x + b_0
\]

and the corresponding intersection divisor \( div(X \cdot Y) \) of degree four, see [2], p.113 or [12], p.166. Following Abel we substitute \( y = P(x) \) into (2.5) and obtain the so-called Abel polynomial

\[
\psi(x) = P(x)^2 - f(x).
\]

Divisor of this polynomial on \( X \) coincides with \( div(X \cdot Y) \), i.e. roots of this polynomial are abscissas of intersection points \( P_1, P_2, P_3 \) and \( P_4 \) forming support of the intersection divisor \( div(X \cdot Y) \).

At \( b_2 = \sqrt{a_4} \) one of the intersection points is \( P_\infty \), see examples in Figure 1.

![Figure 1: Intersection of curve \( X \) with parabola \( Y \) : \( y = \sqrt{a_4} x^2 + b_1 x + b_0 \)]

In this case polynomial \( \psi(x) \) is equal to

\[
\psi(x) = (2b_1 b_2 - a_3)x^3 + (2b_0 b_2 + b_1^2 - a_2)x^2 + (2b_0 b_1 - a_1)x + b_0^2 - a_0
\]

Equating coefficients of \( \psi \) one gets relation between abscissas of the remaining rational points \( P_1, P_2 \) and \( P_3 \) in support of the intersection divisor

\[
x_1 + x_2 + x_3 = -\frac{2b_0 b_2 + b_1^2 - a_2}{2b_1 b_2 - a_3}.
\]
If \( P_1 \neq P_2 \) as in Figure 1a, we can define parabola \( Y \) using the Lagrange interpolation by any pair of points \((P_1, P_2), (P_1, P_3)\) or \((P_2, P_3)\). For instance, taking the following pair of points \((P_1, P_2)\) one gets

\[
P(x) = b_2x^2 + b_1x + b_0 = \sqrt{a_4}(x - x_1)(x - x_2) + \frac{(x - x_2)y_1}{x_1 - x_2} + \frac{(x - x_1)y_2}{x_2 - x_1},
\]

which allows us to determine \( b_2, b_1, b_0 \) as functions on \( x_{1,2} \) and \( y_{1,2} \). Substituting coefficients of \( P(x) \) into the equation we obtain an explicit expression for abscissa \( x_3 \) as a function of coordinates \( x_{1,2} \) and \( y_{1,2} \)

\[
x_3 = -x_1 - x_2 + \phi(x_1, y_1, x_2, y_2), \quad \phi = \frac{2b_0b_2 + b_1^2 - a_2}{2b_1b_2 - a_3}
\]  (2.8)

If we have a double intersection point, for instance \( P_1 = P_3 \) as in Figure 1b, then

\[
x_2 = -2x_1 + \phi(x_1, y_1), \quad \phi = -\frac{2b_0b_2 + b_1^2 - a_2}{2b_1b_2 - a_3},
\]  (2.9)

where function \( \phi(x_1, y_1) \) is defined by \( P(x) \) due to the Hermite interpolation

\[
P(x) = b_2x^2 + b_1x + b_0 = \sqrt{a_4}(x - x_1)^2 + \frac{(x - x_1)(4a_4x_1^3 + 3a_3x_1^2 + 2a_2x_1 + a_1)}{2y_1} + y_1.
\]

In modern terms, we consider two partitions of the intersection divisor

\[
div(X \cdot Y) = (P_1 + P_2) + P_3 + P_\infty \quad \text{and} \quad div(X \cdot Y) = (2P_1) + P_2 + P_\infty.
\]

Using brackets (,) we separate a part of the intersection divisor which is necessary for polynomial interpolation of auxiliary curve \( Y \). Because

\[
div(X \cdot Y) = 0,
\]

these partitions can be rewritten as addition and doubling of prime divisors

\[-P_3 = P_1 + P_2, \quad -P_2 = 2P_1,
\]

where we use standard hyperelliptic inversion \((x, y) \rightarrow (x, -y)\), see Figure 1.

At \( b_2 \neq \sqrt{a_4} \) support of the intersection divisor consists of four rational points \( P_1 \neq P_\infty \) up to multiplicity. Let us consider the following partitions of this divisor

\[
div(X \cdot Y) = (P_1 + P_2 + P_3) + P_4, \quad div(X \cdot Y) = (2P_1 + P_2) + P_3, \quad div(X \cdot Y) = (3P_1) + P_2,
\]

see Figure 2. In the first case parabola \( Y \) is defined by the Lagrange interpolation using three ordinary points \( P_1, P_2 \) and \( P_3 \). In the second and third cases parabola \( Y \) is defined by the Hermite interpolation using either double and ordinary points \( 2P_1, P_2 \) or one triple point \( 3P_1 \), respectively.

In the first case abscissa of the fourth intersection point is

\[
x_4 = -x_1 - x_2 - x_3 + \varphi(x_1, x_2, x_3, y_1, y_2, y_3), \quad \varphi = -\frac{a_3 - 2b_1b_2}{a_4 - b_2^2},
\]  (2.10)

where function \( \varphi \) is defined using coefficients of quadratic polynomial \( P(x) = b_2x^2 + b_1x + b_0 \)

\[
P(x) = \frac{(x - x_2)(x - x_3)y_1}{(x_1 - x_2)(x_1 - x_3)} + \frac{(x - x_1)(x - x_3)y_2}{(x_2 - x_1)(x_2 - x_3)} + \frac{(x - x_1)(x - x_2)y_3}{(x_3 - x_1)(x_3 - x_2)}.
\]  (2.11)

In the second case expression for the abscissa looks like

\[
x_3 = -2x_1 - x_2 + \varphi(x_1, x_2, y_1, y_2), \quad \varphi = -\frac{a_3 - 2b_1b_2}{a_4 - b_2^2}.
\]  (2.12)
Here function $\varphi$ is defined via coefficients of the same polynomial $P(x) = b_2x^2 + b_1x + b_0$ and Hermite interpolation formulæ

$$P(x) = \frac{(x - x_1)^2y_2 - (x - 2x_1 + x_2)(x - x_2)y_1}{(x_1 - x_2)^2} + \frac{(x - x_1)(x - x_2)(4a_4x_1^3 + 3a_3x_1^2 + 2a_2x_1 + a_1)}{2y_1(x_1 - x_2)}$$

In the third case, when we consider tripling the prime divisor on $X$

$$(x_2, y_2) = 3(x_1, y_1),$$

second abscissa is equal to

$$x_2 = -3x_1 + \varphi(x_1, y_1), \quad \varphi = -\frac{a_3 - 2b_1b_2}{4 - b_2^2}, \quad (2.13)$$

where function $\varphi$ is defined via coefficients of the polynomial

$$P(x) = b_2x^2 + b_1x + b_0 = -\frac{(x - x_1)^2(4a_4x_1^3 + 3a_3x_1^2 + 2a_2x_1 + a_1)^2}{8y_1^2}$$

$$+ \frac{(x - x_1)(6a_4x_1^2 + 3a_3x_1 + a_2) - 2a_4x_1^3 + a_2x_1 + a_1}{2y_1} + y_1. \quad (2.14)$$

At $b_2 = 0$ we have the intersection divisor of $X$ with line $Y$, which can be represented in the following form

$$\text{div}(X \cdot Y) = (P_1 + P_2) + P_3 + P_4.$$ 

It means that line $Y$ is interpolated by two points $P_1$ and $P_2$

$$P(x) = b_1x + b_0 = \frac{x - x_2}{x_1 - x_2}y_1 + \frac{x - x_1}{x_2 - x_1}y_2,$$

whereas abscissas of remaining two points $P_3$ and $P_4$ are the roots of polynomial

$$\psi(x) = \frac{(x - x_1)(x - x_2)}{(x - x_1)(x - x_2)} = a_4x^2 + (a_4(x_1 + x_2) + a_3)x + a_4(x_1^2 + x_1x_2 + x_2^2) + a_3(x_1 + x_2) + a_2 - b_2^2.$$

Thus, $x_{3,4}$ are algebraic functions on coordinates $x_{1,2}$ and $y_{1,2}$

$$x_{3,4} = -\frac{x_1}{2} + \frac{x_2}{2} - \frac{a_4}{2a_4} + \frac{\sqrt{\alpha(x_1, x_2, y_1, y_2)}}{2a_4} \quad (2.15)$$
where
\[
\alpha(x_1, x_2, y_1, y_2) = 4a_4 \left( \frac{y_1 - y_2}{x_1 - x_2} \right)^2 - a_2 + a_3^2 - 2a_4a_3(x_1 + x_2) - a_4^2(3x_1^2 + 2x_1x_2 + 3x_2^2). \tag{2.16}
\]

In the generic cases, using intersection divisors of plane curve \(X\) with auxiliary curves
\[
Y : \quad y = b_N x^N + b_{N-1} x^{N-1} + \cdots + b_0, \quad N = 1, 2, 3, \ldots
\]
we can describe multiplication of the prime divisor on any integer \(P_1 = nP_2\), which is a key ingredient of the modern elliptic curve cryptography, and other configurations of the prime divisors entering into the intersection divisor.

All the relations between abscissas \(x_k\) \([2.7-2.15]\) are well known, here we only repeat the fairly simple calculations based on Abel’s ideas and their geometric interpretation proposed by Clebsch, see the historical comments in [16]. The modern intersection theory gives a common language for the compact description of these partial cases of intersections \([6, 11]\), whereas modern cryptography equips us with the effective algorithms for such computations \([14]\).

### 2.3 Examples of finite-difference equations

Our aim is to interpret well-studied relations between prime divisors as the finite-difference equations \([1.1]\) realizing various exact discretizations of the given Hamiltonian system and to study the properties of the corresponding discrete maps. For this purpose, we will identify partial solutions \(q(t)\) and \(p(t)\) of the Hamilton equations \(2.2\)
\[
\dot{q} = 2p, \quad \dot{p} = 4a_4 q^3 + 3a_3 q^2 + 2a_2 q + a_1
\]
at \(t = t_i\) with a prime divisor \(P_i = (x_i, y_i)\), where \(x_i = q(t_i)\) and \(y_i = p(t_i)\).

For instance, substituting
\[
x_1 = q_1, \quad y_1 = p_1, \quad x_2 = q_2, \quad y_2 = p_2, \quad x_3 = q_3, \quad y_3 = p_3
\]
in \(2.7\) and \(y_3 = -P(x_3)\) one gets finite-difference equations
\[
q_1 + q_2 + q_3 = \phi(q_1, p_1, q_2, p_2),
\]
\[
\sqrt{a_4(q_1 - q_2)(q_2 - q_3)(q_3 - q_2)} = q_3(p_1 - p_2) + p_3(q_1 - q_2) + q_1p_2 - q_2p_1,
\]
where \(\phi = \phi_1/\phi_2\) is the rational function on variables \(q_1, p_1\) and \(q_2, p_2\)
\[
\phi_1 = a_2 - a_4(q_1^2 + 4q_1q_2 + q_2^2) + 2\sqrt{a_4(q_1p_1 - 2q_1p_2 - q_2p_1)} - \frac{(p_1 - p_2)^2}{(q_1 - q_2)^2},
\]
\[
\phi_2 = 2\sqrt{a_4}\left(\frac{p_1 - p_2}{q_1 - q_2} - \sqrt{a_4(q_1 + q_2)}\right) - a_3.
\]

We can directly verify the following properties of the corresponding discrete mapping.

**Proposition 1** Relations \((2.17)\) determine 3-point mapping \(M \times M \rightarrow M\)
\[
\begin{pmatrix} q_1, q_2 \\ p_1, p_2 \end{pmatrix} \mapsto \begin{pmatrix} q_3 \\ p_3 \end{pmatrix},
\]

preserving the form of Hamiltonian \((2.7)\) and Poisson bracket, i.e. from \(\{q_1, p_1\} = 1, \{q_2, p_2\} = 1\) and \((2.17)\) will follow that \(\{q_3, p_3\} = 1\).
In order to get an iterative system of finite-difference equations we identify a part of abscissas of the intersection points with arbitrary numbers

\[ x_i = \lambda_{ik}, \quad y_i = \mu_{ik} = \pm \sqrt{f(\lambda_{ik})}, \quad \lambda_{ik} \in \mathbb{C}. \]

In this case finite-difference equations (2.13) implicitly depend on the independent variable \( k \) via parameters of discretization \( \lambda_{ik} \). For instance, addition of prime divisors

\[ P_3 = P_1 + P_2, \]

at

\[ x_1 = q_k, \quad y_1 = p_k, \quad x_3 = q_{k+1}, \quad y_3 = p_{k+1} \quad \text{and} \quad x_2 = \lambda_k, \quad y_2 = \mu_k \]
determines the following iterative system of 2-point invertible mappings

\[ q_{k+1} = -q_k - \lambda_k + \phi(q_k, \lambda_k), \quad p_{k+1} = -(b_2 q_{k+1}^2 - b_1 q_{k+1} - b_0), \quad (2.18) \]

where \( \phi \) is given by (2.18) and

\[ b_2 = \sqrt{a_4}, \quad b_1 = -\sqrt{a_4(q_k + \lambda_k)} + \frac{q_k - \mu_k}{q_k - \lambda_k}, \quad b_0 = \sqrt{a_4 q_k \lambda_k} + \frac{q_k \mu_k - \lambda_k p_k}{q_k - \lambda_k}. \]

Here \( \lambda_k \) are arbitrary numbers, whereas the corresponding ordinates

\[ \mu_k = \pm \sqrt{a_4 \lambda_k^4 + a_3 \lambda_k^3 + a_2 \lambda_k^2 + a_1 \lambda_k + H}, \quad H = \frac{1}{2} p_k^2 - a_4 q_k^4 - a_3 q_k^3 - a_2 q_k^2 - a_1 q_k \]

are the functions on the phase space \( M \). We have to use this fact to calculate Poisson bracket between variables \( q_{k+1} \) and \( p_{k+1} \) (2.13), obtained from variables \( q_k \) and \( p_k \).

**Proposition 2** Relations (2.13) determine iterative system of 2-point invertible mappings

\[ \ldots \xrightarrow{\lambda_{k-2}} \left( \begin{array}{c} q_{k-1} \\ p_{k-1} \end{array} \right) \xrightarrow{\lambda_{k-1}} \left( \begin{array}{c} q_k \\ p_k \end{array} \right) \xrightarrow{\lambda_k} \left( \begin{array}{c} q_{k+1} \\ p_{k+1} \end{array} \right) \xrightarrow{\lambda_{k+1}} \ldots \]

preserving the form of Hamilton function

\[ H = \frac{p_k^2}{2} - a_4 q_k^4 - a_3 q_k^3 - a_2 q_k^2 - a_1 q_k \]

\[ = \frac{p_k^2}{2} - a_4 q_k^4 - a_3 q_k^3 - a_2 q_k^2 - a_1 q_k \quad (2.19) \]

and Poisson bracket, i.e. from \( \{ q_k, p_k \} = 1 \) and (2.18) will follow that \( \{ q_{k+1}, p_{k+1} \} = 1 \).

The proof is a straightforward calculation.

Substituting

\[ x_1 = q_k, \quad y_1 = p_k, \quad x_2 = q_{k+1}, \quad y_2 = p_{k+1} \]
in (2.9) and (2.13) one gets two other iterative systems of 2-point mappings

\[ q_{k+1} = -q_k + \phi(q_k), \quad p_{k+1} = -(b_2 q_{k+1}^2 - b_1 q_{k+1} - b_0), \]

associated with multiplication of prime divisor on integer \( (x_2, y_2) = N(x_1, y_1) \) at \( N = 2, 3 \). For the cubic Duffing oscillator at \( a_3 = a_1 = 0 \) we present these mapping explicitly

\[ N = 2, \quad q_{k+1} = \frac{p_k^2 - 2 a_4 q_k^4 - a_2 q_k^2}{2 \sqrt{a_4 q_k p_k}}, \quad p_{k+1} = \frac{q_k^4 (2 a_4 q_k^4 + a_2)^2 - (4 a_4 q_k^4 + p_k^2)^2}{4 \sqrt{a_4 q_k p_k}^2}, \]

\[ N = 3, \quad q_{k+1} = q_k + \frac{4 q_k p_k^2 (2 a_4 q_k^4 + a_2 q_k^2 - p_k^2)}{4 a_4 q_k^6 + 4 a_2 a_4 q_k^4 + a_2^2 - 8 a_4 q_k^4 - 2 a_4 q_k^2 q_k^2 + p_k^4}, \]

\[ p_{k+1} = -p_k + \frac{(q_k - q_{k+1}) (a_2 (q_{k+1} + q_k) + 2 a_4 q_k^2 (3 q_{k+1} - q_k))}{2 p_k} + \frac{q_k^2 (q_k - q_{k+1})^2 (2 a_4 q_k^2 + a_2)^2}{2 p_k} \quad (2.20) \]
Proposition 3  Relations (2.20) define two iterative systems of 2-point maps

\[ \cdots \rightarrow N\left( \begin{array}{c} q_{k-1} \\ p_{k-1} \end{array} \right) \rightarrow N\left( \begin{array}{c} q_k \\ p_k \end{array} \right) \rightarrow N\left( \begin{array}{c} q_{k+1} \\ p_{k+1} \end{array} \right) \rightarrow \cdots \]

which can be considered as the counterparts of usual geometric progression. These 2-points maps are canonical transformations of valence \( N \) preserving the form of Hamiltonian (2.17), i.e. from \( \{q_k,p_k\} = 1 \) and (2.20) will follow that \( \{q_{k+1},p_{k+1}\} = N \).

The proof is a straightforward calculation.

Let us now take intersection divisor

\[ \text{div}(X \cdot Y) = (P_1 + P_2) + P_3, \]

see Fig.2a. If we identify coordinates of all the intersection points \( P_1, \ldots, P_4 \) with the partial solutions of the Hamilton equations, relations (2.10) and \( y_4 = -p_4 \) define 4-point mapping

\[ \left( \begin{array}{c} q_1, q_2, q_3 \\ p_1, p_2, p_3 \end{array} \right) \rightarrow \left( \begin{array}{c} q_4 \\ p_4 \end{array} \right) \]

which has the standard properties.

Proposition 4  Discrete map \( M \times M \times M \rightarrow M \)

\[ q_4 = q_1 - q_2 - q_3 + \varphi(q_1, q_2, q_3, p_1, p_2, p_3), \quad p_4 = -(b_2 q_4^2 - b_1 q_4 - b_0), \]

where \( \varphi \) and \( b_k \) are given by (2.10, 2.11), preserves the form of Hamiltonian and original Poisson bracket.

The proof is a straightforward calculation.

In order to get iterative systems of finite-difference equations we identify one of the intersection points with parameter of discretization. For instance, we can substitute

\[ x_1 = q_{k-1}, \quad y_1 = p_{k-1}, \quad x_2 = q_k, \quad y_2 = p_k, \quad x_4 = q_{k+1}, \quad y_4 = -p_{k+1} \]

and

\[ x_3 = \lambda_k, \quad y_3 = \mu_k \]

in (2.10) in order to obtain a system of 3-point mappings

\[ \begin{array}{c} q_{k+1} = -q_{k-1} - q_k - \lambda_k + \varphi(q_{k-1}, q_k, \lambda_k, p_{k-1}, p_k, \mu_k) \\ p_{k+1} = -(b_2 q_{k+1}^2 - b_1 q_{k+1} - b_0) \end{array} \] (2.21)

Here \( \varphi \) is the rational function defined (2.10, 2.11).

Proposition 5  Relations (2.21) determine iterative systems of the 3-point maps

\[ \cdots \rightarrow \frac{q_{k-1}q_k}{p_{k-1}p_k}\rightarrow \frac{q_{k+1}}{p_{k+1}}\rightarrow \frac{q_{k+2}}{p_{k+2}}\rightarrow \cdots \]

preserving the form of Hamiltonian and original Poisson bracket.

The proof is a straightforward calculation in which we have to take into account that \( \mu_k = \pm \sqrt{f(\lambda_k)} \) is a function on phase space, which has nontrivial Poisson brackets with \( q_1, p_1 \) and \( q_2, p_2 \) simultaneously, see discussion in [3, 9, 18, 27].

Let us now take intersection divisor

\[ \text{div}(X \cdot Y) = (P_1 + 2P_2) + P_3, \]
see Fig. 2b. At \( a_3 = a_1 = 0 \) relation (2.12) looks like
\[
x_3 = -2x_1 - x_2 + \varphi, \quad \varphi = \frac{\varphi_1}{\varphi_2}, \quad (2.22)
\]
where
\[
\varphi_1 = 2x_1 \left( x_1 (x_1 - x_2) (2a_4 x_1^2 + a_2) - y_1^2 + y_1 y_2 \right) \left( x_1^2 - x_2^2 \right) (2a_4 x_1^2 + a_2) - 2y_1^2 + 2y_1 y_2,
\]
and
\[
\varphi_2 = 4a_4 x_1^2 (x_1 - x_2)^2 + (a_2 x_1 (x_1 - x_2) - y_1^2 + y_1 y_2)^2
\]
\[+ a_2 (x_1 - x_2) \left( 4a_2 x_1^2 (x_1 - x_2) - (5 x_1^2 y_1 - 4 x_1^2 y_2 - 3 x_1^2 y_2 y_1 + 3 x_1 x_2^2 y_1 - x_2^2 y_1) \right) \]
Substituting \( x_i = q_i \) and \( p_i = y_i, i = 1, 2, 3 \) in (2.22) and \( y_3 = -\mathcal{P}(x_3) \), one gets 3-point map which does not Poisson, i.e. bracket \( \{q_3, p_1\} \) does not function on \( q_3 \) and \( p_3 \) only.

**Proposition 6** If double point \( P_1 \) plays the role of parameter
\[
x_1 = \lambda_k, y_1 = \pm \sqrt{p_k^2 - f(q_k) + f(\lambda_k)}, \quad x_2 = q_k, y_2 = p_k, \quad x_3 = q_{k+1}, y_3 = p_{k+1},
\]
then relations (2.22) and \( y_3 = -\mathcal{P}(x_3) \) define 2-point map preserving the form of Hamiltonian and Poisson bracket.

If ordinary point \( P_2 \) plays the role of the parameter
\[
x_1 = q_k, y_1 = p_k, \quad x_2 = \lambda_k, y_2 = \pm \sqrt{p_k^2 - f(q_k) + f(\lambda_k)}, \quad x_3 = q_{k+1}, y_3 = p_{k+1},
\]
then relations (2.22) and \( y_3 = -\mathcal{P}(x_3) \) define 2-point map preserving the form of Hamiltonian and Poisson bracket up to the scaling factor, i.e. from \( \{q_k, p_k\} = 1 \) will follow that \( \{q_{k+1}, p_{k+1}\} = 2 \).

The proof is a straightforward calculation.

Let us also consider the intersection of genus one hyperelliptic curve \( X \) with line \( Y \). Substituting \( x_i = q_i \) and \( y_i = p_i \) in (2.15) and \( y_{3,4} = -\mathcal{P}(x_{3,4}) \) one gets
\[
q_{3,4} = -\frac{q_1}{2} - \frac{q_2}{2} + \frac{a_3}{2a_4} \pm \sqrt{\alpha(q_1, q_2, p_1, p_2)_2}, \quad p_{3,4} = -\frac{q_{3,4} - q_2}{q_1 - q_2} p_1 - \frac{q_{3,4} - q_1}{q_2 - q_1} p_2, \quad (2.23)
\]
where \( \alpha(q_1, q_2, p_1, p_2) \) is given by (2.16).

**Proposition 7** Relations (2.23) define invertible algebraic 4-point mapping \( M \times M \rightarrow M \times M \)
\[
\begin{pmatrix} q_1, q_2 \\ p_1, p_2 \end{pmatrix} \rightarrow \begin{pmatrix} q_3, q_4 \\ p_3, p_4 \end{pmatrix}
\]

preserving the form of Hamiltonian and canonical Poisson bracket.

The proof is a straightforward calculation.

### 3 Quintic oscillator

Let us consider Hamiltonian
\[
H(q, p) = p^2 - a_6 q^6 - a_5 q^5 - a_4 q^4 - a_3 q^3 - a_2 q^2 - a_1 q, \quad (3.1)
\]
and canonical Poisson bracket \( \{q, p\} = 1 \), which determine standard Hamilton equations
\[
\dot{q} = \{q, H\} = 2p, \quad \dot{p}_j = \{p, H\} = 6a_6 q^5 + 5a_5 q^4 + 4a_4 q^3 + 3a_3 q^2 + 2a_2 q + a_1 \quad (3.2)
\]
and Newton equation
\[ \ddot{q} = 12a_0q^5 + 10a_5q^4 + 8a_4q^3 + 6a_3q^2 + 4a_2q + 2a_1. \] (3.3)

At \( a_5 = a_3 = a_1 = 0 \) this system is the so-called cubic-quintic Duffing oscillator, which can be found in the modeling of free vibrations of a restrained uniform beam with intermediate lumped mass, the nonlinear dynamics of slender elastica, the generalized Pochhammer-Chree (PC) equation, the generalized compound KdV equation in nonlinear wave systems and so on [21 22].

We identify a common level curve \( H = E \) with the genus two hyperelliptic curve \( X \) on a projective plane
\[ X : \quad y^2 = f(x), \quad f(x) = a_6x^6 + a_5x^5 + a_4x^3 + a_3x^2 + a_2x^1 + a_0, \] (3.4)
where \( E = a_0 \). Integration of the equations (3.2) leads to the Jacobi inversion problem on the curve \( X \)
\[ \frac{dq}{p} = 2dt \quad \Rightarrow \quad \int_0^t \frac{dx}{\sqrt{f(x)}} = 2t. \] (3.5)

In [5] we can find an impressive number the explicit solutions Jacobi inversion problems when the number degrees of freedom \( n \) is equal to a genus \( g \) of hyperelliptic curve \( X \).

If \( n > g \) an analytic integration of the corresponding equations of motion is possible, but it becomes more complicated, see discussion in [8]. Thus, direct numerical integration of the equations of motion is certainly a faster way to obtain the time course of the motion. Analytical and numerical integration of the Duffing oscillator is more easy because at \( a_5 = a_3 = a_1 = 0 \) Common level curve \( X \) (3.4) is the so-called bielliptic curve. Nevertheless, even in this case in order to get suitable approximate solutions we have to apply the cumbersome numerical methods: homotopy analysis method, homotopy Pade technique, energy balance method, combination of Newton’s method and the harmonic balance method and so on, see [7] and references within.

Our aim is to discuss exact discretization of one-dimensional oscillator [30 32 34] associated with genus two hyperelliptic curve \( X \) (3.4) which could be useful for exact numerical integration of the equations of motion.

### 3.1 Two example of intersection divisors

Let us consider intersection \( X \) (3.4) with cubic
\[ Y : \quad y = \mathcal{P}(x), \quad \mathcal{P}(x) = b_3x^3 + b_2x^2 + b_1x + b_0. \]

Substituting \( y = \mathcal{P}(x) \) into the equation (3.3) one gets Abel polynomial \( \psi(x) \). The roots of this polynomial are abscissas of the intersection points which form support of the six degree intersection divisor \( \text{div}(X \cdot Y) \)
\[ \deg \text{div}(X \cdot Y) = 6, \]
according to Bézout’s theorem. In Fig.3a we present this intersection divisor with two points at infinity
\[ \text{div}(X \cdot Y) = (P_1 + P_2) + P_3 + P_4 + 2P_\infty = 0, \]
and in the Fig.3b we present divisor with six rational ordinary points
\[ \text{div}(X \cdot Y) = (P_1 + P_2 + P_3 + P_4) + P_5 + P_6 = 0. \]

Using brackets (.) we separate a part of the intersection divisor which is necessary for Lagrange interpolation of cubic polynomial \( \mathcal{P}(x) \).

For the intersection divisor on Fig.3a we have
\[ a_6 = b_3^2, \quad \text{and} \quad a_5 = 2b_2b_3, \]
thus Abel’s polynomial \( \psi(x) = \mathcal{P}(x)^2 - f(x) \) is equal to
\[
\psi = (2b_1b_3 + b_2^2 - a_4)x^4 + (2b_0b_3 + 2b_1b_2 - a_3)x^3 + (2b_0b_2 + b_1^2 - a_2)x^2 + (2b_0b_1 - a_1)x + b_0 - a_0
\]
\[ = (2b_1b_3 + b_2^2 - a_4)(x - x_1)(x - x_2)(x - x_3)(x - x_4). \]
According to coefficients of this polynomial at $x^3$ and $x^2$ give rise to the standard equations between abscissas of the rational intersection points

$$\sum_{i=1}^{4} x_i = -\frac{a_3 - 2b_3b_0 - 2b_2b_1}{a_4 - 2b_3b_1 - b_2^2}, \quad \sum_{i\neq j} x_i x_j = -\frac{a_2 - 2b_2b_0 - b_1^2}{a_4 - 2b_3b_1 - b_2^2}.$$  

Solving these equations with respect to $x_3$ and $x_4$ one gets the following relations

$$x_{3,4} = \sigma_{\pm} (x_1, x_2, y_1, y_2)$$  

(3.6)

where

$$\sigma_{\pm} = -\frac{x_1}{2} - \frac{x_2}{2} - \frac{a_3 - 2b_3b_0 - 2b_2b_1}{2(a_4 - 2b_3b_1 - b_2^2)} \pm \frac{1}{2} \left( \frac{-3x_1^2 - 2x_1x_2 - 3x_2^2}{2(a_4 - 2b_3b_1 - b_2^2)} \right)^{1/2}.$$  

and $b_k$ are coefficients of the following cubic polynomial

$$\mathcal{P}(x) = \sqrt{a_6}(x - x_1)(x - x_2)(x + x_1 + x_2) + \frac{a_5}{2\sqrt{a_6}}(x - x_1)(x - x_2) + \frac{x - x_2}{x_1 - x_2}y_1 + \frac{x - x_1}{x_2 - x_1}y_2.$$  

(3.7)

Below we will use relations (3.6) to construct various exact discretizations of our one-dimensional integrable system.

For the intersection divisor in Fig. 3b six roots $x_1, \ldots, x_6$ of the Abel polynomial

$$\psi(x) = \mathcal{P}(x)^2 - f(x) = (b_3^2 - a_6)(x - x_1)(x - x_2)(x - x_3)(x - x_4)(x - x_5)(x - x_6).$$

satisfy to equations

$$\sum_{i=1}^{6} x_i = -\frac{a_5 - 2b_3b_2}{a_6 - b_3^2}, \quad \sum_{i\neq j} x_i x_j = -\frac{a_4 - 2b_3b_1 - b_2^2}{a_6 - b_3^2}.$$  

where $b_k$ are coefficients of the following cubic polynomial

$$\mathcal{P}(x) = \frac{(x - x_2)(x - x_3)(x - x_4)y_1}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} + \frac{(x - x_1)(x - x_3)(x - x_4)y_2}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} + \frac{(x - x_1)(x - x_2)(x - x_3)y_3}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + \frac{(x - x_1)(x - x_2)(x - x_3)y_4}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}.$$  

(3.8)
Solving these equations with respect to \( x_5 \) and \( x_6 \) one gets

\[
x_{5,6} = \tau_{\pm}(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4)
\]

where

\[
\tau_{\pm}(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = \frac{1}{2} \left( \frac{2b_2b_3 - a_5}{a_6 - b_3^2} - \sum_{i=1}^{4} x_i \right)
\]

\[
\pm \frac{1}{2} \left( -\sum_{i,j=1}^{4} x_i x_j - 2\sum_{i=1}^{4} x_i^2 - \frac{2(a_5 - 2b_2b_3)\sum_{i=1}^{4} x_i + 4a_4 - 8b_3b_1}{a_6 - b_3^2} + \frac{a_5^2 - 4a_6b_2b_3 + 4a_6b_2^2}{(a_6 - b_3^2)^2} \right)^{1/2}.
\]

These standard relations between abscissas of the intersection points may be found in [1, 2, 12]. We suppose to apply these relations to construction of the finite-difference equations (1.1) relating solutions of the equation of motion (3.3).

### 3.2 Examples of finite-difference equations

Let us consider intersection divisor with four rational intersection points Fig.3a

\[
div(X \cdot Y) = (P_1 + P_2) + P_3 + P_4 + 2P_\infty,
\]

Substituting solutions of the Hamilton equations (3.2) and parameters \( \lambda_{ik} \) into the relations (3.6-3.7) we can get 4-point mapping

\[
\begin{pmatrix}
q_1, q_2 \\
p_1, p_2
\end{pmatrix} \rightarrow \begin{pmatrix}
q_3, q_4 \\
p_3, p_4
\end{pmatrix}, \quad q_{3,4} = \sigma_{\pm}(q_1, q_2, p_1, p_2), \quad p_{3,4} = -P(q_{3,4}),
\]

system of 3-point mappings depending on one parameter

\[
\begin{pmatrix}
q_k \\
p_k
\end{pmatrix} \rightarrow \begin{pmatrix}
q_{k+1} \\
p_{k+1}
\end{pmatrix}, \quad q_{k+1} = \sigma_{+}(q_k, \lambda_k, p_k, \mu_k), \quad p_{k+1} = -P(q_{k+1}),
\]

and system of invertible 2-point mappings depending on two parameter

\[
\begin{pmatrix}
q_k \\
p_k
\end{pmatrix} \rightarrow \begin{pmatrix}
q_{k+1} \\
p_{k+1}
\end{pmatrix}, \quad q_{k+1} = \sigma_{+}(\lambda_{1k}, \lambda_{2k}, \mu_{1k}, \mu_{2k}), \quad p_{k+1} = -P(q_{k+1}).
\]

In the latter case

\[
\mu_{ik} = \pm \sqrt{p_k^2 + f(\lambda_{ik}) - f(q_k)},
\]

when we calculate \( q_{k+1} \) and \( p_{k+1} \) as functions on \( q_k, p_k \) and

\[
\mu_{ik} = \pm \sqrt{p_k^2 + f(\lambda_{ik}) - f(q_{k+1})}, \quad \lambda_{ik} \in \mathbb{C},
\]

when we calculate \( q_k \) and \( p_k \) as functions on \( q_{k+1} \) and \( p_{k+1} \). Here \( f(x) \) is given by (3.4).

It is easy to check that all these mappings preserve the form of discrete Hamiltonian

\[
H = p_k^2 - a_0q_k^6 - a_3q_k^8 - a_4q_k^4 - a_3q_k^3 - a_2q_k^2 - a_1q_k.
\]

Moreover, we can directly verify the following property of the first mapping.

**Proposition 8** Mapping (3.10) preserves canonical Poisson bracket, i.e. from (3.10) and \( \{q_1, p_1\} = 1 \), \( \{q_2, p_2\} = 1 \) will follow that \( \{q_3, p_3\} = 1 \) and \( \{q_4, p_4\} = 1 \).
For the mappings depending on parameters a direct check of the conservation of Poisson bracket was not carried out.

If we take intersection divisor from Fig.3b with six rational intersection points

\[ \text{div}(X \cdot Y) = (P_1 + P_2) + P_3 + P_4 + P_5 + P_6, \]

we can use relations (3.8-3.9) to construct 6-point mapping

\[ \begin{pmatrix} q_1, q_2, q_3, q_4 \\ p_1, p_2, p_3, p_4 \end{pmatrix} \rightarrow \begin{pmatrix} q_5, q_6 \\ q_5, q_6 \end{pmatrix}, \quad q_{5,6} = \tau_{\pm}(q_1, q_2, q_3, q_4, p_1, p_2, p_3, p_4), \quad p_{5,6} = -\mathcal{P}(q_{5,6}), \quad (3.12) \]

with the following properties

**Proposition 9** Mapping (3.12) preserves the form of Hamiltonian (3.11) and canonical Poisson bracket, i.e. from \( \{q_i, p_i\} = 1, \ i = 1, \ldots, 4 \) will follow that \( \{q_5, p_5\} = 1 \) and \( \{q_6, p_6\} = 1 \).

The proof is a straightforward calculation by using modern computer algebra systems.

Replacing part of \( x_i \) on parameters \( \lambda_{ik} \) in (3.8-3.9) one also gets the systems of \( N \)-points finite difference equations preserving the form of Hamiltonian. Among them we can separate system of invertible 4-point maps depending on two parameters

\[ \begin{pmatrix} q_{k-1}, q_k \\ q_{k-1}, p_k \end{pmatrix} \rightarrow \begin{pmatrix} q_{k+1}, q_{k+2} \\ \lambda_{ik}, \lambda_{ik} \end{pmatrix}, \begin{pmatrix} q_{k+1}, q_{k+2} \\ p_{k+1}, p_{k+2} \end{pmatrix} \]

where

\[ q_{k+1} = \tau_+ \left( q_{k-1}, q_k, \lambda_{ik}, \lambda_{ik}, p_{k-1}, p_k, \mu_{ik}, \mu_{ik} \right), \quad p_{k+1} = -\mathcal{P}(q_{k+1}); \]

\[ q_{k+2} = \tau_- \left( q_{k-1}, q_k, \lambda_{ik}, \lambda_{ik}, p_{k-1}, p_k, \mu_{ik}, \mu_{ik} \right), \quad p_{k+2} = -\mathcal{P}(q_{k+2}). \]

As above, for the mappings depending on parameters a direct check of the conservation of Poisson bracket was not carried out because ordinates \( \mu_{ik} \) associated with abscissas \( \lambda_{ik} \) are nontrivial functions of the phase space, see discussion of this problem for 2-point mappings in [3, 9, 18].

### 4 Conclusion

In this paper we show how one can use the methods of the classical intersection theory to the exact discretization of the equations of motion of one-dimensional Hamiltonian systems. Similar methods are also applicable when the common level surface \( X \) of first integrals can be realised as a product of algebraic curves using either separation of variables or Lax representations for the given integrable system.

If we have the suitable Lax matrices, then refactorization in Poisson-Lie groups is viewed as one of the most universal mechanisms of integrability for integrable 2-point maps [3, 4, 15, 18, 19, 27]. In this note, we come back to the Abel and Clebsch ideas in order to study \( n \)-point finite-difference equations sharing integrals of motion and Poisson bracket up to the integer scaling factor.

Another reason to conduct these calculations is related to construction of finite-difference equations (1.1) relating points on the common level surface \( X \) of first integrals, which can not be realized as a product of the plane algebraic curves. In this generic case when we do not know the variables of separation or the Lax matrices, we can continue to study various configurations of points on algebraic surface \( X \) in the framework of the standard intersection theory [2, 3, 11, 13, 16].

We can apply exact discretizations not only to the numerical integration of the equations of motion, but also to

- construction of integrable discrete maps [15, 18, 19, 27, 37, 38];
- study of relations between different integrable systems [28, 31, 37];
- construction of new integrable systems [32, 33, 34, 36].

The main problem here is how to distinguish intersection divisors suitable for these purposes.

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