Fractional Isoperimetric Noether’s Theorem
in the Riemann–Liouville Sense*

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Abstract

We prove Noether-type theorems for fractional isoperimetric variational problems with
Riemann–Liouville derivatives. Both Lagrangian and Hamiltonian formulations are obtained.
Illustrative examples, in the fractional context of the calculus of variations, are discussed.

Keywords: calculus of variations; isoperimetric constraints; fractional calculus; variational
principles of physics; invariance; Noether’s theorem.

1 Introduction

During the last fifteen years, the fractional calculus of variations and fractional mechanics have
increasingly attracted the attention of many researchers — see, e.g., [8,12,15,21,24,30,31] and
references therein. For the state of the art, we refer to the recent book [25].

One of the oldest and interesting class of variational problems are the isoperimetric problems
[38]. Isoperimetry in mathematical physics has roots in the Queen Dido problem of the calculus
of variations, and has recently been subject to several investigations in the context of fractional
calculus [1,23,27,28]. Here we prove Noether-like theorems for fractional isoperimetric problems
of the calculus of variations, both in Lagrangian (Theorem 5) and Hamiltonian (Theorem 7) forms.

Noether’s universal principle establishes a relation between the existence of symmetries and the
existence of conservation laws, and is one of the most beautiful results of the calculus of variations
and mechanics [26, 35] and optimal control [7, 34, 37]. Noether’s principle has been proved as
a theorem in various contexts [6, 36]. What is important to remark here is that Noetherian
conservation laws appear naturally in closed systems, and that in practical terms such systems
do not exist: forces that do not store energy, so-called non-conservative or dissipative forces, are
always present in real systems. In presence of external non-conservative forces, Noether’s theorem
and respective conservation laws cease to be valid. However, it is still possible to obtain a Noether-
type theorem which covers both conservative (closed system) and non-conservative cases. Roughly
speaking, one can prove that Noether’s conservation laws are still valid if a new term, involving the
non-conservative forces, is added to the standard constants of motion [10]. The seminal work [11]
makes use of the notion of fractional Euler–Lagrange extremal introduced by [32,33] to prove a
Noether-type theorem that combines conservative and non-conservative cases. Another fractional
Noether-type theorem is found in [3]. Fractional versions of Noether’s theorem for isoperimetric
problems are the subject of the present work.

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The text is organized in four sections. Section 2 recalls the definitions from fractional calculus needed in the sequel and fix the notations. Our results are formulated and proved in Section 3; we use a fractional operator to generalize the classical concept of conservation law in mechanics and we obtain a general fractional version of Noether’s theorem valid along the fractional isoperimetric Euler–Lagrange extremals (Theorem 5); then we consider a more general fractional isoperimetric optimal control problem, obtaining the corresponding fractional Noether’s theorem in Hamiltonian form (Theorem 7). Section 4 illustrates and discusses the new results with examples.

2 Preliminaries on Fractional Calculus

In this section we fix notations by collecting the necessary definitions of fractional derivatives in the sense of Riemann–Liouville [5, 16, 18, 29].

Definition 1. (Riemann–Liouville fractional integrals) Let \( f \) be defined on the interval \([a, b]\). For \( t \in [a, b] \), the left Riemann–Liouville fractional integral \( \mathcal{I}_a^\alpha f \) and the right Riemann–Liouville fractional integral \( \mathcal{I}_b^\alpha f \) of order \( \alpha \), \( \alpha > 0 \), are defined by

\[
\mathcal{I}_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \theta)^{\alpha-1} f(\theta) d\theta,
\]

\[
\mathcal{I}_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\theta - t)^{\alpha-1} f(\theta) d\theta,
\]

where \( \Gamma \) is the Euler gamma function.

Definition 2. (Riemann–Liouville derivatives) Let \( f \) be defined on the interval \([a, b]\). For \( t \in [a, b] \), the left Riemann–Liouville fractional derivative \( \mathcal{D}_a^\alpha f \) and the right Riemann–Liouville fractional derivative \( \mathcal{D}_b^\alpha f \) of order \( \alpha \) are defined by

\[
\mathcal{D}_a^\alpha f(t) = D^n \mathcal{I}_a^{\alpha-n} f(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{d}{dt} \right)^n \int_a^t (t - \theta)^{\alpha-n-1} f(\theta) d\theta,
\]

\[
\mathcal{D}_b^\alpha f(t) = (-D)^n \mathcal{I}_b^{\alpha-n} f(t) = \frac{1}{\Gamma(n-\alpha)} \left( -\frac{d}{dt} \right)^n \int_t^b (\theta - t)^{\alpha-n-1} f(\theta) d\theta,
\]

where \( n \in \mathbb{N} \) is such that \( n - 1 \leq \alpha < n \), and \( D \) is the usual derivative.

Remark 1. If \( \alpha \) is an integer, then from (1) and (2) one obtains the standard derivatives, that is,

\[
\mathcal{D}_a^\alpha f(t) = \left( \frac{d}{dt} \right)^\alpha f(t), \quad \mathcal{D}_b^\alpha f(t) = \left( -\frac{d}{dt} \right)^\alpha f(t).
\]

Theorem 1. Let \( f \) and \( g \) be two continuous functions on \([a, b]\) and \( p > 0 \). The following property holds for all \( t \in [a, b] \): \( \mathcal{D}_a^p (f(t) + g(t)) = \mathcal{D}_a^p f(t) + \mathcal{D}_a^p g(t) \).

Remark 2. In general, the Riemann–Liouville fractional derivative of a constant \( c \) is not equal to zero. More precisely, one has

\[
\mathcal{D}_a^\alpha c = \frac{c}{\Gamma(1-\alpha)} (t-a)^{-\alpha}.
\]

Remark 3. The left Riemann–Liouville fractional derivative of order \( p > 0 \) of function \((t-a)^v\), \( v > -1 \), is given by

\[
\mathcal{D}_a^p (t-a)^v = \frac{\Gamma(v+1)}{\Gamma(-p+v+1)} (t-a)^{v-p}.
\]

The reader interested in additional background on fractional calculus and more general fractional operators is referred to [4, 11, 20, 25]. For applications in physics see [10].
3 Main Results

In [1] a formulation of the Euler–Lagrange equations was given for isoperimetric problems of the calculus of variations with fractional derivatives in the sense of Riemann–Liouville. Here we prove a fractional version of Noether’s theorem valid along the fractional isoperimetric Euler–Lagrange extremals. For that we introduce an appropriate fractional operator that allow us to generalize the classical concept of conservation law. Under the extended fractional notion of conservation law, we begin by proving in §3.1 a fractional Noether theorem without changing the time variable \( t \), i.e., without transformation of the independent variable (Theorem 4). In §3.2 we proceed with a time-reparameterization technique to obtain the fractional Noether’s theorem in its general form (Theorem 5). Finally, in §3.3 we consider more general fractional isoperimetric optimal control problems, obtaining the corresponding fractional Noether’s theorem in Hamiltonian form (Theorem 7).

3.1 On the fractional isoperimetric Riemann–Liouville conservation of momentum

We begin by defining the fractional isoperimetric problem under consideration.

**Problem 1. (The fractional isoperimetric problem)** The fractional isoperimetric problem of the calculus of variations in the sense of Riemann–Liouville consists to find the stationary functions of the functional

\[
I[q(\cdot)] = \int_a^b L(t, q(t), aD_\alpha^\alpha q(t)) \, dt
\]

subject to \( k \in \mathbb{N} \) isoperimetric equality constraints

\[
\int_a^b g_j(t, q(t), aD_\alpha^\alpha q(t)) \, dt = l_j, \quad j = 1, \ldots, k,
\]

and 2n boundary conditions

\[
q(a) = \phi, \quad q(b) = \psi,
\]

where \( [a, b] \subset \mathbb{R}, \ a < b, \ 0 < \alpha < 1, \ l_j, \ j = 1, \ldots, k, \) are \( k \) specified real constants, and the admissible functions \( q : t \mapsto q(t) \) and the Lagrangian \( L : (t, q, v) \mapsto L(t, q, v) \) are assumed to be functions of class \( C^2 \):

\[
q(\cdot) \in C^2([a, b]; \mathbb{R}^n), \\
L(\cdot, \cdot, \cdot) \in C^2([a, b] \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}).
\]

**Remark 4.** When \( \alpha \to 1 \), Problem 1 is reduced to the classical isoperimetric problem of the calculus of variations:

\[
I[q(\cdot)] = \int_a^b L(t, q(t), \dot{q}(t)) \, dt \longrightarrow \min, \\
\int_a^b g_j(t, q(t), \dot{q}(t)) \, dt = l_j, \quad j = 1, \ldots, k,
\]

subject to the boundary conditions [\( \phi, \psi \)]. For a modern account to isoperimetric variational problems see [2, 9, 24].

The arguments of the calculus of variations assert that by using the Lagrange multiplier rule, Problem 1 is equivalent to the following augmented problem [14, §12.1]: to minimize

\[
I[q(\cdot), \lambda] = \int_a^b F(t, q(t), aD_\alpha^\alpha q(t), \lambda) \, dt
\]

\[
:= \int_a^b \left[ L(t, q(t), aD_\alpha^\alpha q(t)) - \lambda \cdot g(t, q(t), aD_\alpha^\alpha q(t)) \right] \, dt
\]
subject to (5). The augmented Lagrangian

$$F := L - \lambda \cdot g,$$

(9)

$$\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k,$$

has an important role in our study.

The notion of extremizer (a local minimizer or a local maximizer) to Problem 1 is found in [1]. Extremizers can be classified as normal or abnormal.

Definition 3. An extremizer of Problem 1 that does not satisfy

$$\partial_2 g (t, q(t), a D_{\alpha}^\alpha q(t)) + \partial_3 g (t, q(t), a D_{\alpha}^\alpha q(t), \lambda) = 0,$$

(10)

where $\partial_i g$ denotes the partial derivative of $g(\cdot, \cdot, \cdot)$ with respect to its $i$th argument, is said to be a normal extremizer; otherwise (i.e., if it satisfies (10) for all $t \in [a, b]$), is said to be abnormal.

Next theorem summarizes the main result of [1].

Theorem 2. (see [1]) If $q(\cdot)$ is a normal extremizer to Problem 1, then it satisfies the following fractional isoperimetric Euler–Lagrange equation in the sense of Riemann–Liouville:

$$\partial_2 F (t, q(t), a D_{\alpha}^\alpha q(t), \lambda) + \partial_3 F (t, q(t), a D_{\alpha}^\alpha q(t), \lambda) = 0,$$

(11)

t \in [a, b], where $F$ is the augmented Lagrangian (9) associated with Problem 1.

Remark 5. When $\alpha \to 1$, the fractional isoperimetric Euler–Lagrange equation (11) is reduced to the classical isoperimetric Euler–Lagrange equation

$$\partial_2 F (t, q(t), \dot{q}(t), \lambda) - \frac{d}{dt} \partial_3 F (t, q(t), \dot{q}(t), \lambda) = 0$$

(see, e.g., [25, §4.2]).

Theorem 2 leads to the concept of isoperimetric fractional extremal in the sense of Riemann–Liouville.

Definition 4. (Fractional isoperimetric extremal) A function $q(\cdot)$ that is a solution of (11) is said to be a fractional isoperimetric Riemann–Liouville extremal for Problem 1.

In order to prove a fractional isoperimetric Noether’s theorem, we adopt a technique used in [10,12,34]. For that, we use [8] to introduce the notion of variational invariance and formulate a necessary condition of invariance without transformation of the independent variable $t$.

Definition 5. (Invariance of [8] without transforming $t$) Functional [8] is invariant under an $\varepsilon$-parameter group of infinitesimal transformations $\tilde{q}(t) = q(t) + \varepsilon \xi(t, q) + o(\varepsilon)$ if

$$\int_{t_a}^{t_b} F (t, q(t), a D_{\alpha}^\alpha q(t), \lambda) \, dt = \int_{t_a}^{t_b} F (t, \tilde{q}(t), a D_{\alpha}^\alpha \tilde{q}(t), \lambda) \, dt$$

(12)

for any subinterval $[t_a, t_b] \subseteq [a, b]$.

The next theorem establishes a necessary condition of invariance.

Theorem 3. (Necessary condition of invariance) If functional [8] is invariant in the sense of Definition 5 then

$$\partial_2 F (t, q(t), a D_{\alpha}^\alpha q(t), \lambda) \cdot \xi(t, q(t)) + \partial_3 F (t, q(t), a D_{\alpha}^\alpha q(t), \lambda) \cdot a D_{\alpha}^\alpha \xi(t, q(t)) = 0.$$
Proof. Having in mind that condition (12) is valid for any subinterval \([t_a, t_b] \subseteq [a, b]\), we can get rid off the integral signs in (12). Differentiating this condition with respect to \(\varepsilon\), then substituting \(\varepsilon = 0\), and using the definitions and properties of the fractional derivatives given in Section 2, we arrive to the intended conclusion:

\[
0 = \partial_2 (L - \lambda \cdot g) (t, q, a D^\alpha q) \cdot \xi (t, q) + \partial_3 (L - \lambda \cdot g) (t, q, a D^\alpha q) \cdot \eta (t, q) + \partial_6 (L - \lambda \cdot g) (t, q, a D^\alpha q) \cdot \gamma (t, q)
\]

Expression (14) is equivalent to (13).

The following definition is useful in order to introduce an appropriate concept of fractional isoperimetric conservation law in the sense of Riemann–Liouville.

Definition 6. (cf. Definition 19 of [11]) Given two functions \(f\) and \(h\) of class \(C^1\) in the interval \([a, b]\), we introduce the following operator:

\[
D^\alpha_t (f, h) = -h \cdot t D^\alpha_t f + f \cdot a D^\alpha_t h,
\]

where \(t \in [a, b]\) and \(\alpha \in \mathbb{R}^+\).

Remark 6. In the classical context one has \(\gamma = 1\) and

\[
D^1_t (f, h) = f' \cdot h + f \cdot h' = \frac{d}{dt} (f \cdot h) = D^1_t (h, f).
\]

Roughly speaking, \(D^\alpha_t (f, h)\) is a fractional version of the derivative of the product of \(f\) with \(h\). Differently from the classical context, in the fractional case one has, in general, \(D^\alpha_t (f, h) \neq D^\gamma_t (h, f)\).

We recall that the Leibniz formula, as we know it from standard calculus, is not valid for fractional derivatives.

We now prove the fractional isoperimetric Noether’s theorem in the sense of Riemann–Liouville without transformation of the independent variable \(t\).

Theorem 4. (The Noether law of fractional momentum) If (8) is invariant in the sense of Definition 5 then

\[
D^\alpha_t [\partial_3 F (t, q(t), a D^\alpha q(t), \lambda) \cdot \xi (t, q(t))] = 0
\]

along any fractional isoperimetric Riemann–Liouville extremal \(q(t), t \in [a, b]\) (Definition 4).

Proof. We use the fractional Euler–Lagrange equations

\[
\partial_2 (L - \lambda \cdot g) (t, q, a D^\alpha q) = -t D^\alpha_t \partial_3 (L - \lambda \cdot g) (t, q, a D^\alpha q)
\]

in (13), obtaining

\[
0 = -t D^\alpha_t \partial_3 (L - \lambda \cdot g) (t, q, a D^\alpha q) \cdot \xi (t, q) + \partial_3 (L - \lambda \cdot g) (t, q, a D^\alpha q) \cdot D^\alpha_t \xi (t, q)
\]

\[
= D^\alpha_t \left( \partial_3 (L - \lambda \cdot g) (t, q, a D^\alpha q) \cdot \xi (t, q) \right).
\]

The proof is complete.

Remark 7. When \(\alpha \to 1\), we obtain from (15) the following conservation law applied to the isoperimetric problem (9–7):

\[
\frac{d}{dt} [\partial_3 F (t, q(t), \dot{q}(t), \lambda) \cdot \xi (t, q(t))] = 0
\]

along any isoperimetric Euler–Lagrange extremal \(q(\cdot)\). For this reason, we call to the fractional isoperimetric law (15) the fractional isoperimetric Riemann–Liouville conservation of momentum.
3.2 The fractional isoperimetric Noether theorem

The next definition gives a more general notion of invariance for the integral functional \( I \). The main result of this section, Theorem 5, is formulated with the help of this definition.

**Definition 7.** (Invariance of \( I \)) The integral functional \( I \) is said to be invariant under the one-parameter group of infinitesimal transformations

\[
\begin{align*}
\bar{f} = t + \varepsilon \tau(t, q) + o(\varepsilon), \\
\bar{q}(t) = q(t) + \varepsilon \xi(t, q) + o(\varepsilon),
\end{align*}
\]

if

\[
\int_{t_a}^{t_b} F(t, q(t), a D^\alpha t q(t), \lambda) \, dt = \int_{\bar{t}(t_a)}^{\bar{t}(t_b)} F(\bar{f}(t), \bar{q}(t), a D^\alpha \bar{t} q(\bar{t}), \lambda) \, d\bar{t}
\]

for any subinterval \([t_a, t_b] \subseteq [a, b]\).

Our next theorem gives a formulation of Noether’s principle to fractional isoperimetric problems in the calculus of variations in the sense of Riemann–Liouville.

**Theorem 5.** (Fractional isoperimetric Noether’s theorem) If the integral functional \( I \) is invariant in the sense of Definition 7, then

\[
D^\alpha_t (F(t, q, a D^\alpha t q, \lambda) - \alpha \partial_3 F(t, q, a D^\alpha t q, \lambda) \cdot a D^\alpha t q, \tau(t, q)) + D^\alpha_t (\partial_3 F(t, q, a D^\alpha t q, \lambda), \xi(t, q)) = 0
\]

along any fractional isoperimetric Riemann–Liouville extremal \( q(\cdot) \).

**Proof.** We reparameterize the time (the independent variable \( t \)) with a Lipschitzian transformation \([\sigma_a, \sigma_b] \ni \sigma \mapsto t(\sigma) = \sigma f(\delta) \in [a, b] \) that satisfies

\[
t'_\sigma = \frac{dt(\sigma)}{d\sigma} = f(\delta) = 1 \text{ if } \delta = 0.
\]

In this way one reduces \( I \) to an autonomous integral functional:

\[
\bar{I}[t(\cdot), q(t(\cdot)), \lambda] = \int_{\sigma_a}^{\sigma_b} (L - \lambda \cdot g(t(\sigma)), q(t(\sigma)), \sigma \alpha D^\alpha_{t(\sigma)} q(t(\sigma))) t'_\sigma d\sigma,
\]

where \( t(\sigma_a) = a, t(\sigma_b) = b \),

\[
\sigma \alpha D^\alpha_{t(\sigma)} q(t(\sigma)) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{d\sigma} \right)^n \int_{\frac{\sigma}{f(\delta)}}^{\sigma f(\delta)} (\sigma f(\delta) - \theta)^{n - \alpha - 1} q(\theta f^{-1}(\delta)) d\theta
\]

\[
= \frac{(t'_\sigma)^{-\alpha}}{\Gamma(n - \alpha)} \left( \frac{d}{d\sigma} \right)^n \int_{\frac{\sigma}{f(\delta)}}^{\sigma} (\sigma - s)^{n - \alpha - 1} q(s) ds
\]

\[
= (t'_\sigma)^{-\alpha} D^\alpha_\sigma q(\sigma).
\]

Using the definitions and properties of fractional derivatives given in Section 2 we get

\[
\bar{I}[t(\cdot), q(t(\cdot)), \lambda] = \int_{\sigma_a}^{\sigma_b} (L - \lambda \cdot g(t(\sigma)), (t'_\sigma)^{-\alpha} D^\alpha_\sigma q(\sigma)) t'_\sigma d\sigma
\]

\[
= \int_{\sigma_a}^{\sigma_b} (\bar{L} f - \lambda \cdot \bar{y} f) \left( t(\sigma), q(t(\sigma)), t'_\sigma, (t'_\sigma)^{-\alpha} D^\alpha_\sigma q(t(\sigma)) \right) d\sigma
\]

\[
= \int_a^b (L - \lambda \cdot g(t, q(t), a D^\alpha t q(t))) dt
\]

\[
= I[q(\cdot), \lambda].
\]
By hypothesis, functional (19) is invariant under transformations (16), and it follows from Theorem 4 that if the integral functionals in (3) and (4) are invariant in the sense of Definition 7, then the integral functional (19) is invariant in the sense of Definition 5. It follows from Theorem 4 that

\[ \frac{\partial}{\partial t} \left( \mathcal{L}_f - \lambda \cdot \mathcal{G}_f, \xi \right) + \frac{\partial}{\partial t} \left( \mathcal{L}_f - \lambda \cdot \mathcal{G}_f, \tau \right) = 0 \]  

is an isoperimetric fractional conserved law in the sense of Riemann–Liouville. For \( \delta = 0 \) the condition (18) allow us to write that

\[ a \left( t' \sigma \right)^2 \frac{\partial}{\partial t} \sigma = a D_t^\alpha q(t), \]

and we get

\[ \partial_4 \left( \mathcal{L}_f - \lambda \cdot \mathcal{G}_f \right) = \partial_3 \left( \mathcal{L} - \lambda \cdot \mathcal{G} \right), \]  

and

\[ \frac{\partial}{\partial t} \left( \mathcal{L}_f - \lambda \cdot \mathcal{G}_f \right) = -\alpha \partial_3 \left( \mathcal{L} - \lambda \cdot \mathcal{G} \right) \cdot a D_t^\alpha q + \mathcal{L} - \lambda \cdot \mathcal{G}. \]  

Substituting the quantities (21) and (22) into (20), we obtain the isoperimetric fractional conservation law (17).

Remark 8. When \( \alpha \to 1 \), we obtain from (17) the isoperimetric Noether’s conservation law:

\[ \frac{d}{dt} \left[ \partial_4 F(t, q, \dot{q}) \cdot \xi(t, q) + (F(t, q, \dot{q}) - \partial_3 F(t, q, \dot{q}) \cdot \dot{q}) \tau(t, q) \right] = 0 \]

along any Euler–Lagrange extremal \( q \) of problem (6)–(7).

3.3 Optimal control of fractional isoperimetric systems

We now adopt the Hamiltonian formalism to generalize Theorem 5 to the fractional optimal control setting. The fractional isoperimetric optimal control problem in the sense of Riemann–Liouville is introduced, without loss of generality, in Lagrange form:

\[ I[q(\cdot), u(\cdot)] = \int_a^b \mathcal{L}(t, q(t), u(t)) \, dt \longrightarrow \min \]  

subject to the fractional differential system

\[ a D_t^\alpha q(t) = \varphi(t, q(t), u(t)), \]  

isoperimetric equality constraints

\[ \int_a^b g_j(t, q(t), u(t)) \, dt = l_j, \quad j = 1, \ldots, k, \]

and initial condition

\[ q(a) = q_a. \]

The Lagrangian \( \mathcal{L} : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \), the fractional velocity vector \( \varphi : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) and \( g : [a, b] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^k \), are assumed to be functions of class \( C^1 \) with respect to all their arguments, and \( l_j, \ j = 1, \ldots, k, \) are specified real constants. We also assume, without loss of generality, that \( 0 < \alpha \leq 1 \). In conformity with the calculus of variations, we are considering that the control functions \( u(\cdot) \) take values on \( \mathbb{R}^m \).

Definition 8. The fractional differential system (24) is called a fractional control system in the sense of Riemann–Liouville.
Remark 9. The fractional functional of the calculus of variations \( \mathcal{J} \) is obtained from (23) by choosing \( \varphi(t,q,u) = u \). In that case (25) is reduced to (1).

Definition 9. (Fractional isoperimetric process) An admissible pair \((q(\cdot), u(\cdot))\) that satisfies the fractional control system (21) and the fractional isoperimetric constraints (22) is said to be a fractional isoperimetric process in the sense of Riemann–Liouville.

Theorem 6. If \( (q(\cdot), u(\cdot)) \) is a fractional isoperimetric process in the sense of Riemann–Liouville, solution to problem (23)–(26), then there exists a co-vector function \( p(\cdot) \in \mathcal{P}_C^1([a,b];\mathbb{R}^n) \) such that for all \( t \in [a,b] \) the quadruple \((q(t), u(t), p(t), \lambda)\) satisfies the following conditions:

- the isoperimetric Hamiltonian system
  \[
  \begin{cases}
  _aD_t^\alpha q(t) = \partial_1 \mathcal{H}(t,q(t),u(t),p(t),\lambda), \\
  _aD_t^\alpha p(t) = \partial_2 \mathcal{H}(t,q(t),u(t),p(t),\lambda);
  \end{cases}
  \]

- the isoperimetric stationary condition
  \[
  \partial_3 \mathcal{H}(t,q(t),u(t),p(t),\lambda) = 0;
  \]

where the Hamiltonian \( \mathcal{H} \) is defined by
\[
\mathcal{H}(t,q,u,p,\lambda) = L(t,q,u) - \lambda \cdot g(t,q,u) + p \cdot \varphi(t,q,u).
\]

Proof. Minimizing (23) subject to (24) and (25) is equivalent, by the Lagrange multiplier rule, to minimize
\[
J[q(\cdot),u(\cdot),p(\cdot),\lambda] = \int_a^b [\mathcal{H}(t,q(t),u(t),p(t),\lambda) - p(t) \cdot _aD_t^\alpha q(t)] \, dt
\]
with \( \mathcal{H} \) given by (27). Theorem 6 follows by applying the fractional Euler–Lagrange optimality condition to the equivalent functional (28).

Remark 10. When \( \alpha \to 1 \), Theorem 6 coincides with the Pontryagin Maximum Principle for optimal control problems with isoperimetric constraints (cf. [24] §13.12 and [37], Theorem 2.1).

Remark 11. In the case of the fractional calculus of variations in the sense of Riemann–Liouville one has \( \varphi(t,q,u) = u \) (Remark 7) and \( \mathcal{H} = L - \lambda \cdot g + p \cdot u \). From the isoperimetric Hamiltonian system of Theorem 6 one gets \(_aD_t^\alpha q = u\) and \(_aD_t^\alpha p = \partial_2 L - \lambda \cdot \partial_2 g\), and from the stationary condition \( \partial_3 \mathcal{H} = 0 \) it follows that \( p = -\partial_1 L + \lambda \cdot \partial_3 g \). Thus, \(_aD_t^\alpha p = -_aD_t^\alpha (\partial_3 L - \lambda \cdot \partial_3 g)\). Comparing both expressions for \(_aD_t^\alpha p\), we arrive to the fractional Euler–Lagrange equations (11): \( \partial_2 L - \lambda \cdot \partial_2 g = -_aD_t^\alpha (\partial_3 L - \lambda \cdot \partial_3 g)\).

Definition 10. (Fractional isoperimetric Pontryagin extremal) A quadruple \((q(\cdot), u(\cdot), p(\cdot), \lambda)\) satisfying Theorem 6 will be called a fractional isoperimetric Pontryagin extremal in the sense of Riemann–Liouville.

The notion of variational invariance for (23)–(25) is defined with the help of the augmented functional (28).

Definition 11. (Variational invariance of (23)) We say that the integral functional (28) is invariant under the one-parameter family of infinitesimal transformations
\[
\begin{align*}
\tilde{t} &= t + \varepsilon \tau(t,q(t),u(t),p(t)) + o(\varepsilon), \\
\tilde{q}(t) &= q(t) + \varepsilon \xi(t,q(t),u(t),p(t)) + o(\varepsilon), \\
\tilde{u}(t) &= u(t) + \varepsilon \phi(t,q(t),u(t),p(t)) + o(\varepsilon), \\
\tilde{p}(t) &= p(t) + \varepsilon \chi(t,q(t),u(t),p(t)) + o(\varepsilon),
\end{align*}
\]
if
\[
[\mathcal{H}(\tilde{t},\tilde{q}(\tilde{t}),\tilde{u}(\tilde{t}),\tilde{p}(\tilde{t}),\lambda) - \tilde{p}(\tilde{t}) \cdot _aD_t^\alpha \tilde{q}(\tilde{t})] \, d\tilde{t} = [\mathcal{H}(t,q(t),u(t),p(t),\lambda) - p(t) \cdot _aD_t^\alpha q(t)] \, dt.
\]
The next result provides an extension of Noether’s theorem to the wider class of fractional isoperimetric optimal control problems.

**Theorem 7.** (Noether’s theorem in Hamiltonian form) If \( (28) \) is variationally invariant, in the sense of Definition 11, then

\[
D^{\alpha}_t \left( \mathcal{H}(t,q(t),u(t),p(t),\lambda) - (1-\alpha) p(t) \cdot \alpha D^{\alpha}_t q(t), \tau(t,q(t)) \right) - D^{\alpha}_t (p(t),\xi(t,q(t))) = 0 \quad (31)
\]

along any fractional isoperimetric Pontryagin extremal \((q(\cdot),u(\cdot),p(\cdot),\lambda)\) of problem \( (23) \)–\( (26) \).

**Proof.** The fractional isoperimetric conservation law \( (31) \) in the sense of Riemann–Liouville is obtained by applying Theorem 5 to the equivalent functional \( (28) \).

**Remark 12.** When \( \alpha \to 1 \), one gets from Theorem 7 the Noether-type theorem associated with the classical isoperimetric optimal control problem \([37, Theorem 4.1]\): invariance under a one-parameter family of infinitesimal transformations \( (29) \) implies that

\[
\mathcal{H}(t,q(t),u(t),p(t),\lambda)\tau(t,q(t)) - p(t) \cdot \xi(t,q(t)) = \text{constant}
\]

along all the Pontryagin extremals.

## 4 Examples

We illustrate our results with the help of two fractional isoperimetric problems. Example 1 considers a nonautonomous fractional isoperimetric problem of the calculus of variations; Example 2 the autonomous optimal control isoperimetric problem.

**Example 1.** Let \( \alpha \) be a given number in the interval \((0,1)\). Consider the following fractional isoperimetric problem:

\[
\int_0^1 \left( t^4 + (0D^\alpha_t y)^2 \right) dt \to \min,
\]

\[
\int_0^1 t^2 0D^\alpha_t y dt = \frac{1}{5},
\]

\[
y(0) = 0, \quad y(1) = \frac{2}{2\alpha + 3\alpha^2 + \alpha^3}.
\]

The augmented Lagrangian is

\[
F(t,y,0D^\alpha_t y) = t^4 + (0D^\alpha_t y)^2 - \lambda t^2 0D^\alpha_t y
\]

and in \([1]\) it is proved that

\[
y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{x^2}{(t-x)^{1-\alpha}} dx = \frac{1}{\Gamma(\alpha)} \frac{2t^{\alpha+2}}{2\alpha + 3\alpha^2 + \alpha^3}
\]

is an extremal if \( \lambda = 2 \) and

\[
0D^\alpha_t y = t^2.
\]

It is easy to check the validity of our Theorem 7 for this problem: take \( \xi = 1, \tau = 1, \) and use \( (32) \)–\( (33) \)–\( (34) \) in \( (17) \) to obtain \( 0D^\alpha_t (0,1) = 0 \).

Theorem 7 gives an interesting result for autonomous fractional problems.
Example 2. Consider the autonomous fractional isoperimetric optimal control problem, i.e., the case when functions $L$, $\varphi$ and $g$ of (23)–(25) do not depend explicitly on the independent variable:

$$I[q(\cdot), u(\cdot)] = \int_a^b L(q(t), u(t)) \, dt \to \min,$$  \hfill (35)

$$\alpha D_t^\alpha q(t) = \varphi(q(t), u(t)),$$ \hfill (36)

$$\int_a^b g_j(q(t), u(t)) = l_j.$$ \hfill (37)

We will show that for the fractional problem (35)–(37) one has

$$\alpha D_t^\alpha [H(t, q(t), u(t), p(t), \lambda) + (\alpha - 1) p(t) \cdot \alpha D_t^\alpha q(t)] = 0$$ \hfill (38)

along any isoperimetric fractional Pontryagin extremal $(q(\cdot), u(\cdot), p(\cdot), \lambda)$. Indeed, as the Hamiltonian $H$ does not depend explicitly on the independent variable $t$, we can easily see that (35)–(37) is invariant under translation of the time variable: the condition of invariance (38) is satisfied with $\bar{t}(t) = t + \varepsilon$, $\bar{q}(t) = q(t)$, $\bar{u}(t) = u(t)$, and $\bar{p}(t) = p(t)$. Indeed, given that $d\bar{t} = dt$, the invariance condition (38) is verified if $\alpha D_t^\alpha \bar{q}(\bar{t}) = \alpha D_t^\alpha q(t)$. This is true because

$$\alpha D_t^\alpha \bar{q}(\bar{t}) = \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{d\bar{t}} \right)^n \int_a^{\bar{t}} (\bar{t} - \theta)^{n-\alpha-1} \bar{q}(\theta) d\theta \varepsilon
= \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{d\bar{t}} \right)^n \int_a^{\bar{t}} (t + \varepsilon - \theta)^{n-\alpha-1} \bar{q}(\theta) d\theta d\varepsilon
= \frac{1}{\Gamma(n - \alpha)} \left( \frac{d}{d\bar{t}} \right)^n \int_a^{\bar{t}} (t - s)^{n-\alpha-1} \bar{q}(s + \varepsilon) ds d\varepsilon
= \alpha D_t^\alpha \bar{q}(\bar{t} + \varepsilon) = \alpha D_t^\alpha q(t).$$ \hfill (39)

Using the notation in (29), we have $\tau = 1$, $\xi = \varrho = \varsigma = 0$. From Theorem 7 we arrive to the intended equality (38).

The Example 2 shows that in contrast with the classical autonomous isoperimetric problem of optimal control, for (35)–(37) the Hamiltonian $H$ does not define a conservation law. Instead of the classical equality $\frac{d}{dt} (H) = 0$, we have

$$\alpha D_t^\alpha [H + (\alpha - 1) p(t) \cdot \alpha D_t^\alpha q(t)] = 0,$$ \hfill (39)

i.e., fractional conservation of the Hamiltonian $H$ plus a quantity that depends on the fractional order $\alpha$ of differentiation. This seems to be explained by violation of the homogeneity of spacetime caused by the fractional derivatives, when $\alpha \neq 1$. If $\alpha = 1$, then we obtain from (39) the classical result: the Hamiltonian $H$ is preserved along all the isoperimetric Pontryagin extremals.

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