Twisted modules for vertex operator algebras and Bernoulli polynomials

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1 Introduction

This work is a continuation of a series of papers of two of the present authors, stimulating by work of Bloch. In those papers we used the general theory of vertex operator algebras to study central extensions of classical Lie algebras and super-algebras of differential operators on the circle in connection with values of ζ-functions at the negative integers. In the present paper, using general principles of the theory of vertex operator algebras and their twisted modules, we obtain a bosonic, twisted construction of a certain central extension of a Lie algebra of differential operators on the circle, for an arbitrary twisting automorphism. The construction involves the Bernoulli polynomials in a fundamental way. This is explained through results in the general theory of vertex operator algebras, including a new identity, which we call "modified weak associativity." This paper is an announcement. The detailed proofs will appear elsewhere.

More specifically, the present goal is to obtain a new general Jacobi identity for twisted operators, and for related iterates of such operators, extending the previous analogous results in the untwisted setting in our papers mentioned above. As a consequence we obtain twisted constructions of certain central extensions of Lie algebras of differential operators on the circle, combining and extending methods from [L3, L4, M1–M3, FLM1, FLM2, and DL]. In those earlier papers we used vertex operator techniques to analyze untwisted actions of the Lie algebra $\hat{D}^+$, studied in [Bl], on a module for a Heisenberg Lie algebra of a certain standard type, based on a finite-dimensional vector space equipped with a nondegenerate symmetric bilinear form. Now consider an arbitrary isometry $\nu$ of period say $p$, that is, with $\nu^p = 1$. Here we announce that the corresponding $\nu$-twisted modules carry an action of the Lie algebra $\hat{D}^+$ in terms of twisted vertex operators, parametrized by certain quadratic vectors in the untwisted module.

In particular, we extend a result from [FLM1, FLM2, DL] where actions of the Virasoro algebra were constructed using twisted vertex operators. In addition we explicitly compute certain "correction" terms for the generators of the "Cartan subalgebra" of $\hat{D}^+$ that naturally appear in any twisted construction. These correction terms are expressed in terms of special values of certain Bernoulli polynomials. They can in principle be generated, in the theory of vertex operator algebras, by the formal operator $e^{\Delta_x}$.

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involved in the construction of a twisted action for a certain type of vertex operator algebra. We generate those correction terms in an easier way, using a new “modified weak associativity” relation that is a consequence of the twisted Jacobi identity.

In [KR] Kac and Radul established a relationship between the Lie algebra of differential operators on the circle and the Lie algebra \( \hat{gl}(\infty) \); for further work in this direction, see [AFMO], [KWY]. Our methods and motivation for studying Lie algebras of differential operators, based on vertex operator algebras, are new and very different, so we do not pursue their direction.

2 The Lie algebra \( \hat{D}^+ \) and its untwisted construction

Let \( D \) be the Lie algebra of formal differential operators on \( \mathbb{C}^\times \) spanned by \( t^n D^r \), where \( D = \frac{d}{dt} \) and \( n \in \mathbb{Z}, r \in \mathbb{N} \) (the nonnegative integers). Let \( \hat{D} = \mathbb{C}c \oplus D \) be the nontrivial one–dimensional central extension (cf. [KR]) with the following commutation relations:

\[
[t^m f(D), t^n g(D)] = t^{m+n}(f(D + n)g(D) - g(D + m)f(D)) + \Psi(t^m f(D), t^n g(D))c,
\]

where \( f \) and \( g \) are polynomials and \( \Psi \) is the 2–cocycle (cf. [KR]) determined by

\[
\Psi(t^m f(D), t^n g(D)) = -\Psi(t^n g(D), t^m f(D)) = \delta_{m+n,0} \sum_{i=1}^{m} f(-i)g(m-i), \ m > 0.
\]

We consider the Lie subalgebra \( D^+ \) of \( D \) generated by the formal differential operators

\[
L_n^r = (-1)^{r+1}D^r(t^n D)^r,
\]

where \( n \in \mathbb{Z}, r \in \mathbb{N} \). The subalgebra \( D^+ \) has an essentially unique central extension (cf. [N]) and this extension may be obtained by restriction of the 2–cocycle \( \Psi \) to \( D^+ \). Let \( \hat{D}^+ = \mathbb{C}c \oplus \hat{D}^+ \) be the nontrivial central extension defined via the slightly normalized 2–cocycle \(-\frac{1}{2}\Psi\), and view the elements \( L_n^r \) as elements of \( \hat{D}^+ \). This normalization gives, in particular, the usual Virasoro algebra bracket relations

\[
[L_m^0, L_n^0] = (m-n)L_{m+n}^0 + \frac{m^3 - m}{12}\delta_{m+n,0}c.
\]

In [Bl] Bloch discovered that the Lie algebra \( \hat{D}^+ \) can be defined in terms of generators that lead to a simplification of the central term in the Lie bracket relations. Oddly enough, if we let

\[
\tilde{L}_n^r = L_n^r + \frac{(-1)^r}{2}\zeta(-1-2r)\delta_{n,0}c,
\]

then the central term in the commutator

\[
[\tilde{L}_m^r, L_n^s] = \sum_{i = \min(r,s)}^{r+s} a_i^{(r,s)}(m,n)\tilde{L}_m^i + \frac{(r+s+1)^2}{2(2(r+s)+3)!}m^{2(r+s)+3}\delta_{m+n,0}c
\]

(2.3)
is a pure monomial (here $a_i^{(r,s)}(m,n)$ are structure constants). In order to conceptualize this simplification (especially the appearance of $\zeta$–values) one constructs certain infinite-dimensional projective representations of $\mathfrak{D}^+$ using vertex operators.

Let us explain Bloch’s construction [Bl]. As in [FLM2], consider the (infinite-dimensional) Lie algebra $\hat{\mathfrak{h}}$, the affinization of an abelian Lie algebra $\mathfrak{h}$ of dimension $d$ (over $\mathbb{C}$) with nondegenerate symmetric bilinear form $(\cdot,\cdot)$. The algebra $\hat{\mathfrak{h}}$ is spanned by $\alpha(m)$ ($\alpha \in \mathfrak{h}$, $m \in \mathbb{Z}$) and $C$ (which is central), satisfying the commutation relations

$$[\alpha(m), \beta(n)] = (\alpha, \beta) m \delta_{m+n,0} C.$$ 

The subalgebra spanned by $\alpha(m)$ ($\alpha \in \mathfrak{h}$, $m \in \mathbb{Z}$, $m \neq 0$) and $C$ is a Heisenberg Lie algebra, and $S(\hat{\mathfrak{h}}^-)$, where $S(\cdot)$ denotes the symmetric algebra and $\hat{\mathfrak{h}}^-$ is the span of the $a(m)$ with $a \in \mathfrak{h}$ and $m < 0$, carries the structure of an induced module for $\hat{\mathfrak{h}}$ with $\alpha(0)$ acting trivially and $C \mapsto 1$ (cf. [FLM2]). Then the correspondence

$$L_n^{(r)} \mapsto \frac{1}{2} \sum_{q=1}^{d} \sum_{j \in \mathbb{Z}} j^r (n-j)^r :\alpha_q(j)\alpha_q(n-j) : \quad (n \in \mathbb{Z}), \quad c \mapsto d,$$

(2.4)

where $\{\alpha_q\}$ is an orthonormal basis of $\mathfrak{h}$ and $:\cdot:\cdot$ is the usual normal ordering, which brings $\alpha(n)$ with $n > 0$ to the right, gives a representation of $\mathfrak{D}^+$. Let us denote the operator on the right-hand side of (2.4) by $L_n^{(r)}(n)$. In particular, the operators $L_n^{(0)}(m)$ ($m \in \mathbb{Z}$) give a well-known representation of the Virasoro algebra with central charge $c \mapsto d$,

$$[L_n^{(0)}(m), L_n^{(0)}(n)] = (m-n)L_n^{(0)}(m+n) + d \frac{m^3 - m}{12} \delta_{m+n,0},$$

and the construction (2.4) for those operators is the standard realization of the Virasoro algebra on a module for a Heisenberg Lie algebra (cf. [FLM2]). The appearance of zeta-values in (2.2) can be conceptualized by the following heuristic argument [Bl]: Suppose that we remove the normal ordering in (2.4) and use the relation $[\alpha_q(m), \alpha_q(-m)] = m$ to rewrite $\alpha_q(m)\alpha_q(-m)$, with $m \geq 0$, as $\alpha_q(-m)\alpha_q(m) + m$. It is easy to see that the resulting expression contains an infinite formal divergent series of the form

$$1^{2r+1} + 2^{2r+1} + 3^{2r+1} + \ldots .$$

A heuristic argument of Euler’s suggests replacing this formal expression by $\zeta(-1 - 2r)$, where $\zeta$ is the (analytically continued) Riemann $\zeta$–function. The resulting (zeta-regularized) operator is well defined and gives the action of $L_n^{(r)}$; such operators satisfy the bracket relations (2.3).

3 Twisted modules for vertex operator algebras and the “modified weak associativity” relation

The notion of twisted module for a vertex operator algebra was formalized in [FFR] and [D] (see also the geometric formulation in [FrS]), summarizing the basic properties of
the actions of twisted vertex operators discovered in [FLM1] and [FLM2] (cf. [L1]); the
main nontrivial axiom in this notion is the twisted Jacobi identity of [FLM2] and [L2] (cf.
[FLM1]). Fix a vertex operator algebra \((V, Y, \mathbf{1}, \omega)\) of central charge \(c_V \in \mathbb{C}\) (also called “rank”), or \(V\) for short (see [FLM2] and [FHL] for the definition and explicit constructions
of vertex operator algebras and modules, and for necessary “formal calculus”). Also fix
an automorphism \(\nu\) of period \(p > 0\) of the vertex operator algebra \(V\), that is, a linear
automorphism of the vector space \(V\) preserving \(\omega\) and \(\mathbf{1}\) such that
\[
\nu Y(v, x) \nu^{-1} = Y(\nu v, x) \quad \text{for} \quad v \in V,
\]
\[
\nu^p = 1_V
\]
(\(1_V\) being the identity operator on \(V\)). In addition, fix a primitive \(p\)-th root of unity \(\omega_p \in \mathbb{C}\).

**Definition 3.1** A \(\mathbb{Q}\)-graded \(\nu\)-twisted \(V\)-module \(M\) is a \(\mathbb{Q}\)-graded vector space,
\[
M = \coprod_{n \in \mathbb{Q}} M(n); \quad \text{for} \quad v \in M(n), \ \text{wt} \ v = n,
\]
such that
\[
M(n) = 0 \quad \text{for} \quad n \ \text{sufficiently negative},
\]
\[
\dim M(n) < \infty \quad \text{for} \quad n \in \mathbb{Q},
\]
equipped with a linear map
\[
Y_M(\cdot, x) : V \rightarrow (\text{End } M)[[x^{1 \over p}, x^{-1 \over p}]]
\]
\[
v \mapsto Y_M(v, x) = \sum_{n \in \mathbb{Z} / p\mathbb{Z}} v_n^\nu x^{-n-1}, \quad v_n^\nu \in \text{End } M,
\]
where \(Y_M(v, x)\) is called the **twisted vertex operator** associated with \(v\), such that the
following conditions hold:

** truncation condition:** For every \(v \in V\) and \(w \in M\)
\[
\nu_n^\nu w = 0
\]
\[
\text{for} \quad n \in \mathbb{Z} / p\mathbb{Z} \ \text{sufficiently large};
\]

** vacuum property:**
\[
Y_M(1, x) = 1_M;
\]

** Virasoro algebra conditions:** Let
\[
Y_M(\omega, x) = \sum_{n \in \mathbb{Z}} L_M(n) x^{-n-2}.
\]

Then
\[
[L_M(m), L_M(n)] = (m-n) L_M(m+n) + c_V m^3 - m \over 12 \delta_{m+n,0} 1_M,
\]

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\[ L_M(0)v = (\text{wt } v)v \]

for every homogeneous element \( v \), and

\[ Y_M(L(-1)u, x) = \frac{d}{dx} Y_M(u, x); \]

**Jacobi identity:** For \( u, v \in V \),

\[
\begin{align*}
x_0^{-1} & \delta \left( \frac{x_1 - x_2}{x_0} \right) Y_M(u, x_1)Y_M(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) Y_M(v, x_2)Y_M(u, x_1) \\
&= \frac{1}{p} x_2^{-1} \sum_{r=0}^{p-1} \delta \left( \frac{x_1 - x_0}{x_2} \right)^{1/p} Y_M(Y(v^r u, x_0)v, x_2), \quad (3.4)
\end{align*}
\]

where \( \delta(x) = \sum_{n \in \mathbb{Z}} x^n \) is the formal delta-function.

Here and below we use the “binomial expansion convention”—the notational device according to which binomial expressions are understood to be expanded in nonnegative integral powers of the second variable. An important property of a twisted module is that when restricted to the fixed-point subalgebra \( \{ v \in V \mid \nu v = v \} \), it is a true module: the twisted Jacobi identity \((3.4)\) reduces to the untwisted one, as in [FLM2].

The main commutativity and associativity properties of twisted vertex operators \([L]\), together with the fact that these properties are equivalent to the Jacobi identity, can be reformulated as follows:

**Theorem 3.2** Let \( M \) be a vector space (not assumed to be graded) equipped with a linear map \( Y_M(\cdot, x) \) \((3.1)\) such that the truncation condition \((3.2)\) and the Jacobi identity \((3.4)\) hold. Then for \( u, v \in V \) and \( w \in M \), there exist \( k(u, v) \in \mathbb{N} \) and \( l(u, w) \in \frac{1}{p} \mathbb{N} \) and a (non-unique) element \( F(u, v, w; x_0, x_1, x_2) \) of \( M((x_0, x_1^{1/p}, x_2^{1/p})) \) such that

\[
\begin{align*}
x_0^{k(u, v)} F(u, v, w; x_0, x_1, x_2) & \in M[[x_0]]((x_1^{1/p}, x_2^{1/p})), \\
x_1^{l(u, w)} F(u, v, w; x_0, x_1, x_2) & \in M[[x_1^{1/p}]]((x_0, x_2^{1/p})). \quad (3.5)
\end{align*}
\]

and

\[
\begin{align*}
Y_M(u, x_1)Y_M(v, x_2)w &= F(u, v, w; x_1 - x_2, x_1, x_2), \\
Y_M(v, x_2)Y_M(u, x_1)w &= F(u, v, w; -x_2 + x_1, x_1, x_2), \\
Y_M(Y(\nu^{-s} u, x_0)v, x_2)w &= \lim_{x_1^{1/p} \to \omega_p(x_2+x_0)^{1/p}} F(u, v, w; x_0, x_1, x_2) \quad (3.6)
\end{align*}
\]

for \( s \in \mathbb{Z} \) (where we are using the binomial expansion convention). Conversely, let \( M \) be a vector space equipped with a linear map \( Y_M(\cdot, x) \) \((3.1)\) such that the truncation condition \((3.2)\) and the preceding statement hold, except that \( k(u, v) \in \mathbb{N} \) and \( l(u, w) \in \frac{1}{p} \mathbb{N} \) may depend on all three of \( u, v \) and \( w \). Then the Jacobi identity \((3.4)\) holds.
It is important to note that since \( k(u, v) \) can be greater than 0, the formal series
\[ F(u, v; x_1 - x_2, x_1, x_2) \]
and
\[ F(u, v; -x_2 + x_1, x_1, x_2) \]
are not in general equal. The formal limit procedure
\[ \lim_{x_1^{1/p} \to \omega_p^{(x_2 + x_0)/p}} F(u, v, w; x_0, x_1, x_2) \]
each integral power of the formal variable \( x_1^{1/p} \) in the formal series \( F(u, v, w; x_0, x_1, x_2) \) by the corresponding power of the formal series \( \omega_p^{(x_2 + x_0)/p} \) (defined using the binomial expansion convention).

Along with (3.5), the first two equations of (3.6) represent what we call “formal commutativity” for twisted vertex operators, while the first and last equations of (3.6) represent “formal associativity” for twisted vertex operators. When specialized to the untwisted case \( p = 1 (\nu = 1_V) \), these two relations lead respectively to the usual “formal commutativity” and “formal associativity” for vertex operators, as formulated in [LL] (see also [FLM2] and [FHL]). The first and last relations of the theorem also imply a significant new relation, interesting even for the untwisted case, which we call “modified weak associativity”:

**Theorem 3.3** With \( M \) and \( k(u, v) \) as in Theorem 3.2,
\[
\lim_{x_1^{1/p} \to \omega_p^{(x_2 + x_0)/p}} \left( (x_1 - x_2)^{k(u,v)} Y_M(u, x_1) Y_M(v, x_2) \right) = x_0^{k(u,v)} Y_M(Y(\nu^{-s} u, x_0)v, x_2)
\] (3.7)
for \( u, v \in V \) and \( s \in \mathbb{Z} \).

**Remark 3.4** The specialization of Theorems 3.2 and 3.3 to the case \( p = 1 (\nu = 1_V) \) and \( M = V \) gives statements describing structures and relations for the vertex operator map \( Y(\cdot, x) \) in the vertex (operator) algebra \( V \).

## 4 Homogeneous twisted vertex operators and associated relations

Homogeneous vertex operators are defined by
\[ X(v, x) = Y(x^{L(0)} v, x) \] (4.1)
for \( v \in V \) (see [FLM2]). From the Jacobi identity for vertex operators, it is not hard to obtain a Jacobi identity for homogeneous vertex operators [L3, L4, M1]:
\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) X(u, x_1)X(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) X(v, x_2)X(u, x_1)
= x_1^{-1} \delta \left( e^{y x_2/x_1} \right) X(Y[u, y]v, x_2)
\] (4.2)
where
\[
y = \log \left( 1 + \frac{x_0}{x_2} \right).
\]
The expressions \( \log(1+x) \) and \( e^y \), where \( x \) and \( y \) are formal variables, are defined by their series expansions in nonnegative powers of \( x \) and \( y \), respectively. The use of the formal variable \( y \) is natural here in particular because of the appearance of the vertex operator \( Y[u, y] \) defined and studied in \([Z1], [Z2]\):

\[
Y[u, y] = Y(e^{yL(0)}u, e^y - 1). \tag{4.3}
\]

These operators give a new vertex operator algebra isomorphic to \( V \), and the isomorphism corresponds geometrically to a change-of-coordinates transformation expressed formally as \( y \mapsto e^y - 1 \) (\([Z1], [Z2] \); see \([H1], [H2] \) for the generalization to arbitrary coordinate changes). The Jacobi identity \([4.2]\) thus suggests that there is a close relationship between homogeneous vertex operators and “cylindrical coordinates”; we will comment more on this relationship in an extended version of this announcement.

From the Jacobi identity \([4.2]\), one can obtain the commutator formula \([L3], [L4]\):

\[
[X(u, x_1), X(v, x_2)] = \text{Res}_y \delta \left( \frac{e^y x_2}{x_1} \right) X(Y[u, y]v, x_2), \tag{4.4}
\]

using a general fact concerning formal series:

\[
\text{Res}_x h(x) = \text{Res}_y \left( h(y) \frac{d}{dy} F(y) \right) \quad \text{for } h(x) \in A[[x]], \ F(x) \in xA[[x]]
\]

where \( A \) is a commutative associative algebra (or more generally, a module for it) and where the coefficient of \( x^1 \) in \( F(x) \) is invertible.

From now on we fix a \( \mathbb{Q} \)-graded \( \nu \)-twisted \( V \)-module \( M \). We define homogeneous twisted vertex operators by a simple twisted generalization of \([4.1]\):

\[
X_M(u, x) = Y_M(x^{L(0)}u, x),
\]

as in \([FLM2]\). We now state the following twisted generalizations of the results above: a Jacobi identity, a commutator formula similar to \([4.4]\), and formal commutativity and associativity properties, including “modified weak associativity,” for these homogeneous twisted vertex operators.

Using the twisted Jacobi identity \([4.4]\) and the definitions \([4.1] \) and \([4.3]\), one can obtain the twisted generalization of \([4.2]\):

**Theorem 4.1** For \( u, v \in V \),

\[
x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) X_M(u, x_1) X_M(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{x_0} \right) X_M(v, x_2) X_M(u, x_1)
\]

\[
= \frac{1}{p} x_1^{-1} \sum_{r=0}^{p-1} \delta \left( \omega^{-r} \left( \frac{e^y x_2}{x_1} \right)^{1/p} \right) X_M(Y[\nu^r u, y]v, x_2) \tag{4.5}
\]

where

\[
y = \log \left( 1 + \frac{x_0}{x_2} \right). \tag{4.6}
\]
Using techniques similar to those in the untwisted case, this Jacobi identity leads to the commutator formula:

**Corollary 4.2** For \( u, v \in V \),

\[
[X_M(u, x_1), X_M(v, x_2)] = \text{Res}_y \frac{1}{p} \sum_{r=0}^{p-1} \delta \left( \omega^{-r} \left( e^y \frac{x_2}{x_1} \right)^{1/p} \right) X_M(Y[v^r u, y]v, x_2). \tag{4.7}
\]

Applying \( \lim_{x_0 \to (e^y - 1)x_2} \) to both sides of the “modified weak associativity” relation yields “modified weak associativity” for homogeneous twisted vertex operators:

**Theorem 4.3** For \( u, v \in V \), \( s \in \mathbb{Z} \), and \( k(u, v) \) as in Theorem 3.2,

\[
\lim_{x_1^{1/p} \to \omega^{-1}_y(e^y x_2)^{1/p}} \left( \frac{x_1}{x_2} - 1 \right)^{k(u, v)} X_M(u, x_1) X_M(v, x_2) = (e^y - 1)^{k(u, v)} X_M(Y[v^{-s} u, y]v, x_2). \tag{4.8}
\]

Finally, the two first equations of (3.6) can be rewritten in terms of homogeneous vertex operators, and modified weak associativity (4.8) can be used to obtain the homogeneous counterpart of the third equation of (3.6):

**Theorem 4.4** For \( u, v \in V \), \( w \in M \) and \( s \in \mathbb{Z} \), we have

\[
\begin{align*}
X_M(u, x_1) X_M(v, x_2) w &= G(u, v, w; x_1 - x_2, x_1, x_2), \\
X_M(v, x_2) X_M(u, x_1) w &= G(u, v, w; -x_2 + x_1, x_1, x_2), \\
X_M(Y[v^{-s} u, y]v, x_2) w &= \lim_{x_1^{1/p} \to \omega^{-1}_y(e^y x_2)^{1/p}} G(u, v, w; (e^y - 1)x_2, x_1, x_2), \tag{4.9}
\end{align*}
\]

where

\[
G(u, v, w; x_0, x_1, x_2) \in M((x_0, x_1^{1/p}, x_2^{1/p}))
\]

and

\[
\begin{align*}
&x_0^{k(u, v)} G(u, v, w; x_0, x_1, x_2) \in M[[x_0]][(x_1^{1/p}, x_2^{1/p})], \\
x_0^{l'(u, w)} G(u, v, w; x_0, x_1, x_2) \in M[[x_1^{1/p}]][(x_0, x_2^{1/p})], \tag{4.10}
\end{align*}
\]

for some \( k(u, v) \in \mathbb{N} \) and \( l'(u, w) \in \frac{1}{p} \mathbb{N} \). Here \( k(u, v) \) can be taken to be the same as in Theorem 3.2.

Again, along with (4.10), the first two equations of (4.4) represent what we call “formal commutativity” for homogeneous twisted vertex operators, while the first and last equations of (4.4) represent “formal associativity” for homogeneous twisted vertex operators. These two formal relations are also of interest in the untwisted case \( p = 1 \) (\( \nu = 1 \nu \)), and of course we have the same relations for the vertex operator algebra \( V \).
5 Commutator formula for iterates on twisted modules

“Modified weak associativity” for homogeneous twisted vertex operators turns out to be very useful. Formal limit operations respect products, under suitable conditions, and using this principle, one can compute, for instance, commutators of certain iterates in a natural way. For our applications, an important commutator is

\[ [X_M(Y[u_1, y_1]v_1, x_1), X_M(Y[u_2, y_2]v_2, x_2)] \]

which we would like to express in terms of similar iterates. Using “modified weak associativity” \(4.8\) and the commutator formula \(4.7\) one can prove:

**Theorem 5.1** For \(u_1, v_1, u_2, v_2 \in V\),

\[ [X_M(Y[u_1, y_1]v_1, x_1), X_M(Y[u_2, y_2]v_2, x_2)] = \]

\[ \text{Res}_y \frac{1}{p} \sum_{r=0}^{p-1} \left\{ \delta \left( \frac{\omega^r_p \left( e^{-y-x_2} \frac{x_1}{x_2} \right)^{1/p}}{x_2} \right) X_M(Y[u_1, y_1 + y]Y[\nu^{-r}v_2, -y_2]Y[v_1, y]\nu^{-r}u_2, e^{-y}x_1) \right\} \]

This generalizes the main commutator formula in \([4.3]\) and is related to a similar commutator formula in \([4.1]\–\[4.3]\).

6 Main results

We now obtain a representation of \(\hat{D}^+\) on a certain natural module for a twisted affine Lie algebra based on a finite–dimensional abelian Lie algebra (essentially a twisted Heisenberg Lie algebra), generalizing Bloch’s untwisted representation on the module \(S(\hat{h}^-)\) constructed in Section 2. This is also a generalization of the twisted Virasoro algebra construction (see \([4.1]\), \([4.2]\), \([4.3]\)).

Let \(\mathfrak{h}\) be a finite–dimensional abelian Lie algebra (over \(\mathbb{C}\)) on which there is a nondegenerate symmetric bilinear form \(\langle \cdot, \cdot \rangle\). Let \(\nu\) be an isometry of \(\mathfrak{h}\) of period \(p > 0\):

\[ \langle \nu\alpha, \nu\beta \rangle = \langle \alpha, \beta \rangle, \quad \nu^p \alpha = \alpha \]
for all $\alpha, \beta \in \mathfrak{h}$. We assume that $\nu$ preserves a rational lattice in $\mathfrak{h}$. One knows that $S(\hat{\mathfrak{h}}^-)$ carries a natural structure of vertex operator algebra, with $1 = 1$, and that $\nu$ lifts naturally to an automorphism, which we continue to call $\nu$, of period $p$ of $S(\hat{\mathfrak{h}}^-)$ (cf. [FLM2]). We proceed as in [L1], [FLM1], [FLM2] and [DL] to construct a space $S[\nu]$ that carries a natural structure of $\nu$–twisted module for $S(\hat{\mathfrak{h}}^-)$.

Recalling our primitive $p$–th root of unity $\omega_p$, for $\nu \in \mathfrak{h}$, denote by $\alpha(\nu)$, $\nu \in \mathbb{Z}$, its projection on $\mathfrak{h}(\nu)$. Define the $\nu$–twisted affine Lie algebra $\hat{\mathfrak{h}}[\nu]$ associated with the abelian Lie algebra $\mathfrak{h}$ by

$$\hat{\mathfrak{h}}[\nu] = \bigoplus_{n \in \frac{1}{p}\mathbb{Z}} \mathfrak{h}(pn) \otimes C^n \oplus \mathbb{C} \quad (6.1)$$

with

$$[\alpha \otimes C^n, \beta \otimes C^n] = \langle \alpha, \beta \rangle m \delta_{m+n,0} C \quad (\alpha \in \mathfrak{h}(pn), \beta \in \mathfrak{h}(pm), m, n \in \frac{1}{p}\mathbb{Z})$$

$$[\mathbb{C}, \hat{\mathfrak{h}}[\nu]] = 0. \quad (6.2)$$

Set

$$\hat{\mathfrak{h}}[\nu]^+ = \bigoplus_{n>0} \mathfrak{h}(pn) \otimes C^n, \quad \hat{\mathfrak{h}}[\nu]^-= \bigoplus_{n<0} \mathfrak{h}(pn) \otimes C^n. \quad (6.3)$$

The subalgebra

$$\hat{\mathfrak{h}}[\nu]_{\frac{1}{p}\mathbb{Z}} = \hat{\mathfrak{h}}[\nu]^+ \oplus \hat{\mathfrak{h}}[\nu]^- \oplus \mathbb{C} \quad (6.4)$$

is a Heisenberg Lie algebra. Form the induced (level-one) $\mathfrak{h}[\nu]$-module

$$S[\nu] = U(\hat{\mathfrak{h}}[\nu]) \otimes (\hat{\mathfrak{h}}[\nu]^+ \oplus \hat{\mathfrak{h}}[\nu]^- \oplus \mathbb{C}) \cong S(\hat{\mathfrak{h}}^-) \quad (linearly), \quad (6.5)$$

where $\hat{\mathfrak{h}}[\nu]^+ \oplus \mathfrak{h}(0)$ acts trivially on $\mathbb{C}$ and $\mathbb{C}$ acts as 1; $U(\cdot)$ denotes universal enveloping algebra. Then $S[\nu]$ is irreducible under $\mathfrak{h}[\nu]_{\frac{1}{p}\mathbb{Z}}$. We will use the notation $\alpha(n)$ ($\alpha \in \mathfrak{h}(pn)$, $n \in \frac{1}{p}\mathbb{Z}$) for the action of $\alpha \otimes C^n \in \hat{\mathfrak{h}}[\nu]$ on $S[\nu]$. As we mentioned above, the $\hat{\mathfrak{h}}[\nu]$-module $S[\nu]$ is naturally a $\nu$–twisted module for the vertex operator algebra $S(\hat{\mathfrak{h}}^-)$, and its structure and general properties are important in establishing our results described above.

**Remark 6.1** The special case where $p = 1$ ($\nu = 1_h$) corresponds to the $\mathfrak{h}$-module $S(\hat{\mathfrak{h}}^-)$ discussed in Section 2.
Choosing an orthonormal basis \( \{ \alpha_q | q = 1, \ldots, d \} \) of \( \mathfrak{h} \), we define the following two formal series acting on \( S[\nu] \):

\[
L_{\nu}^{y_1,y_2}(x) = \frac{1}{2} \sum_{q=1}^{d} \alpha_q^{\nu}(e^{y_1}x)\alpha_q^{\nu}(e^{y_2}x) + \frac{1}{2} \frac{\partial}{\partial y_1} \left( \sum_{k=0}^{p-1} e^{\frac{k(y_1+y_2)}{p}} \dim \mathfrak{h}(k) - 1 \right)
\]  

(6.6)

and

\[
\bar{L}_{\nu}^{y_1,y_2}(x) = \frac{1}{2} \sum_{q=1}^{d} \alpha_q^{\nu}(e^{y_1}x)\alpha_q^{\nu}(e^{y_2}x) + \frac{1}{2} \frac{\partial}{\partial y_1} \left( \sum_{k=0}^{p-1} e^{\frac{k(y_1+y_2)}{p}} \dim \mathfrak{h}(k) \right).
\]  

(6.7)

Remark 6.2 In the special case \( p = 1 \) and \( d = 1 \), the operators \( L_{\nu}^{y_1,y_2}(x) \) and \( \bar{L}_{\nu}^{y_1,y_2}(x) \), respectively, specialize to the operators \( L^{(y_1,y_2)}(x) \) and \( \bar{L}^{(y_1,y_2)}(x) \) of [M1], [M2].

The formal series (6.7) can be rewritten in the following form:

\[
\bar{L}_{\nu}^{y_1,y_2}(x_2) = \frac{1}{2} \lim_{x_1 \to x_2} \sum_{q=1}^{d} \left( \frac{\alpha_q^{\nu}(e^{y_1}x_1)}{e^{y_1}x_1 - 1} \right) \alpha_q^{\nu}(e^{y_1}x_1)\alpha_q^{\nu}(e^{y_2}x_2)
\]

for any fixed \( k \in \mathbb{N}, k \geq 2 \). Using “modified weak associativity” [M3], we can then identify this with a particular iterate of vertex operators:

**Proposition 6.3** With the notation as above,

\[
\bar{L}_{\nu}^{y_1,y_2}(x) = X_{S[\nu]} \left( \frac{1}{2} \sum_{q=1}^{d} Y[\alpha_q(-1)1, y_1 - y_2] \alpha_q(-1)1, e^{y_2}x \right).
\]  

(6.8)

Once this identification is made, our twisted construction of \( \hat{D}^+ \) is a simple consequence of the general theory of twisted modules for vertex operator algebras. In particular, consider the commutator formula for the untwisted operators \( L^{(y_1,y_2)}(x) \) announced in [L3] (for a proof see [M1]). Along with Proposition 6.3 and the general properties of twisted modules for vertex operator algebras, it implies the same commutator formula for the twisted operators \( \bar{L}_{\nu}^{y_1,y_2}(x) \):

**Proposition 6.4** With the notation as above,

\[
\left[ L_{\nu}^{y_1,y_2}(x_1), \bar{L}_{\nu}^{y_3,y_4}(x_2) \right] = \left[ \bar{L}_{\nu}^{y_1,y_2+y_3,y_4}(x_2), \delta \left( \frac{e^{y_1}x_1}{e^{y_3}x_2} \right) \delta \left( \frac{e^{y_2}x_1}{e^{y_4}x_2} \right) \right]
\]

= \left[ \bar{L}_{\nu}^{y_1+y_2+y_3,y_4}(x_2), \delta \left( \frac{e^{y_1}x_1}{e^{y_3}x_2} \right) \delta \left( \frac{e^{y_2}x_1}{e^{y_4}x_2} \right) \right] + \left[ \bar{L}_{\nu}^{y_1,y_2+y_3,y_4}(x_2), \delta \left( \frac{e^{y_1}x_1}{e^{y_3}x_2} \right) \delta \left( \frac{e^{y_2}x_1}{e^{y_4}x_2} \right) \right].
\]  

(6.9)
In fact, this commutator formula can alternatively be directly obtained from our general commutator formula, Theorem 5.1. Formula (6.9) provides a representation of the Lie algebra $\hat{D}^+$ on the $\mathfrak{h}[\nu]$-module $S[\nu]$ via our twisted operators, generalizing the untwisted case. More precisely, let

$$L^{\nu,y_1,y_2}(x) = \sum_{n \in \mathbb{Z}, r_1, r_2 \in \mathbb{N}} L^{(r_1,r_2)}(n) x^{-n} \frac{y_1^{r_1} y_2^{r_2}}{r_1! r_2!},$$

$$L^{\nu,y_1,y_2}(x) = \frac{1}{2} \frac{d}{(y_1 - y_2)^2} + \sum_{n \in \mathbb{Z}, r_1, r_2 \in \mathbb{N}} \bar{L}^{(r_1,r_2)}(n) x^{-n} \frac{y_1^{r_1} y_2^{r_2}}{r_1! r_2!}.$$  

Then the following holds (recall the generators (2.1), (2.2) of $\hat{D}^+$):

**Theorem 6.5** Let

$$L^{(r)}(n) = L^{(r,r)}(n) \quad (n \in \mathbb{Z}, r \in \mathbb{N}),$$

$$\bar{L}^{(r)}(n) = \bar{L}^{(r,r)}(n) \quad (n \in \mathbb{Z}, r \in \mathbb{N}).$$

(a) The assignment

$$L^{(r)}_n \mapsto L^{(r)}(n), \ c \mapsto d,$$

defines a representation of the Lie algebra $\hat{D}^+$ on $S[\nu]$.

(b) The assignment

$$\bar{L}^{(r)}_n \mapsto \bar{L}^{(r)}(n), \ c \mapsto d$$

also defines a representation of the Lie algebra $\hat{D}^+$, with the central term being a pure monomial, as in (6.10).

Explicit expressions for the operators $L^{(r)}(n)$ and $\bar{L}^{(r)}(n)$, involving Bernoulli polynomials, are easy to obtain from (6.6) and (6.7):

$$L^{(r)}(n) = \frac{1}{2} \sum_{q=1}^{d} \sum_{j \in \frac{1}{p} \mathbb{Z}} j^r (n-j)^r \cdot \alpha_q(j) \alpha_q(n-j) :$$

$$-\delta_{n,0} \frac{(-1)^r}{4(r+1)} \sum_{k=0}^{p-1} \dim \mathfrak{h}(k) (B_{2(r+1)}(k/p) - B_{2(r+1)})$$

and

$$\bar{L}^{(r)}(n) = \frac{1}{2} \sum_{q=1}^{d} \sum_{j \in \frac{1}{p} \mathbb{Z}} j^r (n-j)^r \cdot \alpha_q(j) \alpha_q(n-j) :$$

$$-\delta_{n,0} \frac{(-1)^r}{4(r+1)} \sum_{k=0}^{p-1} \dim \mathfrak{h}(k) B_{2(r+1)}(k/p).$$

From our construction, the appearance of Bernoulli polynomials is seen to be directly related to general properties of homogeneous twisted vertex operators.
The next result is a simple consequence of Theorem 6.5. It describes the action of the “Cartan subalgebra” of $\hat{D}^+$ on a highest weight vector of a canonical quasi-finite $\hat{D}^+$ module; here we are using the terminology of [KR]. This corollary gives the “correction” terms referred to in the introduction.

**Corollary 6.6** Given a highest weight $\hat{D}^+$–module $W$, let $\delta$ be the linear functional on the “Cartan subalgebra” of $\hat{D}^+$ (spanned by $L_0^{(k)}$ for $k \in \mathbb{N}$) defined by

$$L_0^{(k)} \cdot w = (-1)^k \delta \left( L_0^{(k)} \right) w,$$

where $w$ is a generating highest weight vector of $W$, and let $\Delta(x)$ be the generating function

$$\Delta(x) = \sum_{k \geq 1} \frac{\delta(L(k)(0))x^{2k}}{(2k)!}$$

(cf. [KR]). Then for every automorphism $\nu$ of period $p$ as above,

$$U(\hat{D}^+) \cdot 1 \subset S[\nu]$$

is a quasi–finite highest weight $\hat{D}^+$–module satisfying

$$\Delta(x) = \frac{1}{2} \frac{d}{dx} \sum_{k=0}^{p-1} \frac{e^{\frac{ka}{p}} \dim h(k) - 1}{1 - e^x}.$$

Finally, we have an additional result (for the untwisted bosonic case an equivalent result was obtained in [Bl] and for the spinor constructions in [M2]):

**Corollary 6.7** The generating function

$$X_{S[\nu]}(\sum_{q=1}^{d} \alpha_q(-m-1)\alpha_q(-m-1)1, x), \quad (6.12)$$

$m \in \mathbb{N}$, defines the same $\hat{D}^+$–module as in Theorem 6.5. That is, every operator $L^{(r)}(n)$ (or equivalently $\bar{L}^{(r)}(n)$) can be expressed as a linear combination of the expansion coefficients of the operator (6.12), and vice versa.

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