Geometric Control of Multiple Quadrotor UAVs Transporting a Cable-Suspended Rigid Body

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Abstract—This paper is focused on tracking control for a rigid body payload, that is connected to an arbitrary number of quadrotor unmanned aerial vehicles via rigid links. An intrinsic form of the equations of motion is derived on the nonlinear configuration manifold, and a geometric controller is constructed such that the payload asymptotically follows a given desired trajectory for its position and attitude. The unique feature is that the coupled dynamics between the rigid body payload, links, and quadrotors are explicitly incorporated into control system design and stability analysis. These are developed in a coordinate-free fashion to avoid singularities and complexities that are associated with local parameterizations. The desirable features of the proposed control system are illustrated by a numerical example.

I. INTRODUCTION

Aerial transport of payloads by towed cables is common in various situations, such as emergency response, industrial, and military applications. Examples of aerial towing range from emergency rescue missions where individuals are lifted from dangerous situations to the delivery of heavy equipment to the top of a tall building.

Transportation of a cable-suspended load has been studied traditionally for helicopters [1], [2]. Small unmanned aerial vehicles or quadrotors are also considered for load transportation and deployments [3], [4], [5]. However, these are based on simplified dynamics models. For example, the effects of the payload are considered as additional force and torque exerted to quadrotors, instead of considering the dynamic coupling between the payload and the quadrotor, and a pre-computed trajectory that minimizes swing motion of the payload is followed, instead of actively controlling the motion of payload and cable [4]. As such, these may not be suitable for agile load transportation where the motion of cable and payload should be actively suppressed online.

Recently, geometric nonlinear control systems are developed for the complete dynamic model of a single quadrotor transporting a cable-suspended load [6], and for multiple quadrotors transporting a common payload cooperatively [7]. It is also generalized for a quadrotor with a payload connected by flexible cable that is modeled as a serially-connected links, to incorporate the deformation of cable [8]. However, in these results, it is assumed that the payload is modeled by a point mass. Such assumption is quite restrictive for practical cases where the size of the payload is comparable to the quadrotors and the length of cables.

This paper is organized as follows. A dynamic model is presented and the problem is formulated at Section II. Control systems are constructed at Sections III and IV which are followed by a numerical example. Due to the page limit, parts of proofs are relegated to [9].
II. Problem Formulation

Consider $n$ quadrotor UAVs that are connected to a payload, that is modeled as a rigid body, via massless links (see Figure [1]). Throughout this paper, the variables related to the payload are denoted by the subscript 0, and the variables for the $i$-th quadrotor are denoted by the subscript $i$, which is assumed to be an element of $\mathcal{I} = \{1, \cdots n\}$ if not specified. We choose an inertial reference frame $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and body-fixed frames $\{\mathbf{b}_{i1}, \mathbf{b}_{i2}, \mathbf{b}_{i3}\}$ for $0 \leq j \leq n$ as follows. For the inertial frame, the third axis $\mathbf{e}_3$ points downward along the gravity and the other axes are chosen to form an orthonormal frame. The origin of the $j$-th body-fixed frame is located at the center of mass of the payload for $j = 0$ and at the mass center the quadrotor for $1 \leq j \leq n$. The third body-fixed axis $\mathbf{b}_{i3}$ is normal to the plane defined by the centers of rotors, and it points downward.

The location of the mass center of the payload is denoted by $\mathbf{x}_0 \in \mathbb{R}^3$, and its attitude is given by $R_0 \in \text{SO}(3)$, where the special orthogonal group is defined by $\text{SO}(3) = \{ R \in \mathbb{R}^{3 \times 3} | R^T R = I, \det[R] = 1 \}$. Let $\rho_i \in \mathbb{R}^3$ be the point on the payload where the $i$-th link is attached, and it is represented with respect to the zeroth body-fixed frame. The other end of the link is attached to the mass center of the $i$-th quadrotor. The direction of the link from the mass center of the $i$-th quadrotor toward the payload is defined by the unit-vector $q_i \in S^2$, where $S^2 = \{ q \in \mathbb{R}^3 | ||q|| = 1 \}$, and the length of the $i$-th link is denoted by $l_i \in \mathbb{R}$.

Let $\mathbf{x}_i \in \mathbb{R}^3$ be the location of the mass center of the $i$-th quadrotor with respect to the inertial frame. As the link is assumed to be rigid, we have $\mathbf{x}_i = \mathbf{x}_0 + R_0 \rho_i - l_i q_i$. The attitude of the $i$-th quadrotor is defined by $R_i \in \text{SO}(3)$, which represents the linear transformation of the representation of a vector from the $i$-th body-fixed frame to the inertial frame.

The corresponding configuration manifold of this system is $\mathbb{R}^3 \times \text{SO}(3) \times (S^2 \times \text{SO}(3))^n$.

The mass and the inertia matrix of the payload are denoted by $m_0 \in \mathbb{R}$ and $J_0 \in \mathbb{R}^{3 \times 3}$, respectively. The dynamic model of each quadrotor is identical to [10]. The mass and the inertia matrix of the $i$-th quadrotor are denoted by $m_i \in \mathbb{R}$ and $J_i \in \mathbb{R}^{3 \times 3}$, respectively. The $i$-th quadrotor can generates a thrust $-f_i R_i \mathbf{e}_3 \in \mathbb{R}^3$ with respect to the inertial frame, where $f_i \in \mathbb{R}$ is the total thrust magnitude and $\mathbf{e}_3 = [0,0,1]^T \in \mathbb{R}^3$. It also generates a moment $M_i \in \mathbb{R}$ with respect to its body-fixed frame. The control input of this system corresponds to $\{ f_i, M_i \}_{1 \leq i \leq n}$.

Throughout this paper, the 2-norm of a matrix $A$ is denoted by $||A||$, and its maximum eigenvalue and minimum eigenvalues are denoted by $\lambda_M[A]$ and $\lambda_m[A]$, respectively. The standard dot product is denoted by $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}^T \mathbf{y}$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$.

A. Equations of Motion

The kinematic equations for the payload, quadrotors, and links are given by

\[
\begin{align*}
\dot{\mathbf{q}}_i &= \omega_i \times \mathbf{q}_i = \dot{\omega}_i q_i, \\
\dot{R}_0 &= R_0 \dot{\Omega}_0, \quad \dot{R}_i = R_i \dot{\Omega}_i,
\end{align*}
\]

where $\dot{\mathbf{q}}_i \in \mathbb{R}^3$ is the angular velocity of the $i$-th link, satisfying $q_i \cdot \omega_i = 0$, and $\Omega_0$ and $\Omega_i \in \mathbb{R}^3$ are the angular velocities of the payload and the $i$-th quadrotor expressed with respect to its body-fixed frame, respectively. The hat map $\hat{\cdot} : \mathbb{R}^3 \rightarrow \text{so}(3)$ is defined by the condition that $\hat{x} y = x \times y$ for all $x, y \in \mathbb{R}^3$, and the inverse of the hat map is denoted by the vee map $\check{\cdot} : \text{so}(3) \rightarrow \mathbb{R}^3$.

We derive equations of motion according to Lagrangian mechanics. The velocity of the $i$-th quadrotor is given by $\dot{\mathbf{x}}_i = \dot{x}_i + \dot{R}_0 \rho_i - l_i \dot{q}_i$. The kinetic energy of the system is composed of the translational kinetic energy and the rotational kinetic energy of the payload and quadrotors:

\[
\begin{align*}
T &= \frac{1}{2} m_0 ||\dot{x}_0||^2 + \frac{1}{2} J_0 \dot{\Omega}_0
+ \sum_{i=1}^{n} \frac{1}{2} m_i ||\dot{x}_i + \dot{R}_0 \rho_i - l_i \dot{q}_i||^2 + \frac{1}{2} J_i \dot{\Omega}_i, \quad (3)
\end{align*}
\]

The gravitational potential energy is given by

\[
U = -m_0 g e_3 \cdot x_0 - \sum_{i=1}^{n} m_i g e_3 \cdot (x_0 + R_0 \rho_i - l_i q_i), \quad (4)
\]

where it is assumed that the unit-vector $e_3$ points downward along the gravitational acceleration as shown at Fig.[1].

The corresponding Lagrangian of the system is $\mathcal{L} = T - U$.

Coordinate-free form of Lagrangian mechanics on the two-sphere $S^2$ and the special orthogonal group $\text{SO}(3)$ for various multibody systems has been studied in [11], [12]. The key idea is representing the infinitesimal variation of $q_i \in S^2$ in terms of the exponential map:

\[
\delta q_i = \frac{d}{de} \bigg|_{e=0} \exp(\epsilon \mathbf{e}_3) q_i = \epsilon \times q_i, \quad (5)
\]

for a vector $\xi_i \in \mathbb{R}^3$ with $\xi_i \cdot q_i = 0$. Similarly, the variation of $R_i$ is given by $\delta R_i = R_i \dot{\mathbf{n}}_i$ for $q_i \in \mathbb{R}^3$.

By using these expressions, the equations of motion can be obtained from Hamilton’s principle as follows (see Appendix [A] for more detailed derivations).

\[
\begin{align*}
M_4 &\left(\ddot{x}_0 - g e_3\right) - \sum_{i=1}^{n} m_i q_i q_i^T R_0 \dot{\rho}_i \dot{\Omega}_0 \\
&= \sum_{i=1}^{n} \left( u_i - m_i l_i \omega_i - m_i q_i q_i^T R_0 \dot{\rho}_i \right), \quad (6)
\end{align*}
\]

\[
\begin{align*}
(J_0 &- \sum_{i=1}^{n} m_i \dot{R}_0 q_i q_i^T R_0 \dot{\rho}_i) \dot{\Omega}_0 + \sum_{i=1}^{n} m_i \dot{\rho}_i \dot{R}_0^T q_i q_i^T (\ddot{x}_0 - g e_3) \\
&+ \dot{\Omega}_0 J_0 \dot{\Omega}_0 = \sum_{i=1}^{n} \dot{\rho}_i \dot{R}_0^T (u_i - m_i l_i \omega_i - m_i q_i q_i^T R_0 \dot{\rho}_i), \quad (7)
\end{align*}
\]

\[
\begin{align*}
\dot{\omega}_i &= \frac{1}{l_i} \dot{q}_i (\ddot{x}_0 - g e_3 - R_0 \dot{\rho}_i \dot{\Omega}_0 + R_0 \dot{\Omega}_0 \dot{\rho}_i) - \frac{1}{m_i l_i} \dot{q}_i \dot{u}_i, \quad (8)
\end{align*}
\]

\[
\begin{align*}
J_i \dot{\Omega}_i + \dot{\Omega}_i \times J_i \Omega_i &= M_i, \quad (9)
\end{align*}
\]

where $M_4 = m_4 I + \sum_{i=1}^{n} m_i q_i q_i^T \in \mathbb{R}^{3 \times 3}$, which is symmetric, positive-definite for any $q_i$. 

The vector \( u_i \in \mathbb{R}^3 \) represents the control force at the \( i \)-th quadrotor, i.e., \( u_i = -f_iR_i\mathbf{e}_3 \). The vectors \( u^\parallel_i \) and \( u^\perp_i \in \mathbb{R}^3 \) denote the orthogonal projection of \( u_i \) along \( q_i \), and the orthogonal projection of \( u_i \) to the plane normal to \( q_i \), respectively, i.e.,

\[
    u^\parallel_i = (I + q_i^2)u_i = (q_i \cdot u_i)q_i = q_i q_i^T u_i, \tag{10}
\]

\[
    u^\perp_i = -\hat{q}_i^T u_i = -q_i \times (q_i \times u_i) = (I - q_i q_i^T)u_i. \tag{11}
\]

Therefore, \( u_i = u^\parallel_i + u^\perp_i \).

**B. Tracking Problem**

Define a matrix \( \mathcal{P} \in \mathbb{R}^{6 \times 3n} \) as

\[
    \mathcal{P} = \begin{bmatrix} I_{3 \times 3} & \cdots & I_{3 \times 3} \\ \hat{\rho}_1 & \cdots & \hat{\rho}_n \end{bmatrix}. \tag{12}
\]

Assume the links are attached to the payload such that

\[
    \text{rank}[\mathcal{P}] \geq 6. \tag{13}
\]

This is to guarantee that there exist enough degrees of freedom in control inputs for both the translational motion and the rotational maneuver of the payload. The assumption (13) requires that the number of quadrotor is at least three, i.e., \( n \geq 3 \), since when \( n = 2 \) the above matrix \( \mathcal{P} \) has a non-empty null space spanned by \( [(\rho_1 - \rho_2)^T,(\rho_2 - \rho_1)^T]^T \).

This follows from the fact that it is impossible to generate any moment to the payload along the direction of \( \rho_1 - \rho_2 \) when \( n = 2 \).

Suppose that the desired trajectories for the position and the attitude of the payload are given as smooth curves, namely \( x_{0d}(t) \in \mathbb{R}^3 \) and \( R_{0d}(t) \in SO(3) \) during a time period. From the attitude kinematics equation, we have

\[
    \dot{\Omega}_{0d} = R_{0d}\hat{\Omega}_{0d}, \tag{14}
\]

where \( \Omega_{0d} \in \mathbb{R}^3 \) corresponds to the desired angular velocity. It is assumed that the velocity and the acceleration of the desired trajectories are bounded by known constants.

We wish to design a control input of each quadrotor \( \{f_i, M_i\}_{1 \leq i \leq n} \) such that the state of zero tracking errors becomes an asymptotically stable equilibrium of the controlled system.

### III. Control System Design for Simplified Dynamic Model

In this section, we consider a simplified dynamic model where the attitude dynamics of each quadrotor is ignored, and we design a control input by assuming that the thrust at each quadrotor, namely \( u_i \) can be arbitrarily chosen. It corresponds to the case where each quadrotor is replaced by a fully actuated vehicle that can generates a thrust arbitrarily. The effects of the attitude dynamics of quadrotors will be incorporated in the next section.

In the simplified dynamic model given by (6)-(8), the dynamics of the payload are affected by the parallel components \( u^\parallel_i \) of the control inputs, and the dynamics of the links are directly affected by the normal components \( u^\perp_i \) of the control inputs. This motivates the following control system design procedure: first, the parallel components \( u^\parallel_i \) are chosen such that the payload follows the desired position and attitude trajectory while yielding the desired direction of each link, namely \( q_{id} \); next, the normal components \( u^\perp_i \) are designed such that the actual direction of the links \( q_i \) follows \( q_{id} \).

#### A. Design of Parallel Components

Let \( a_i \in \mathbb{R}^3 \) be the acceleration of the point on the payload where the \( i \)-th link is attached, relative to the gravitational acceleration:

\[
    a_i = \ddot{x}_0 - g\mathbf{e}_3 + R_0\hat{\Omega}_0^2\rho_i - R_0\hat{\rho}_i\hat{\Omega}_0. \tag{15}
\]

The parallel component of the control input is chosen as

\[
    u^\parallel_i = \mu_i + m_i l_i ||\omega_i||^2 q_i + m_i q_i q_i^T a_i, \tag{16}
\]

where \( \mu_i \in \mathbb{R}^3 \) is a virtual control input that is designed later, with a constraint that \( \mu_i \) is parallel to \( q_i \). Note that the expression of \( u^\parallel_i \) is guaranteed to be parallel to \( q_i \) due to the projection operator \( q_i q_i^T \) at the last term of the right-hand side of the above expression.

The motivation for the proposed parallel components becomes clear if (16) is substituted into (6)-(7) and rearranged to obtain

\[
    m_0(\ddot{x}_0 - g\mathbf{e}_3) = \sum_{i=1}^n \mu_i, \tag{17}
\]

\[
    J_0\dot{\Omega}_0 + \dot{\hat{\Omega}}_0 J_0\Omega_0 = \sum_{i=1}^n \hat{\rho}_i R_0^T \mu_i. \tag{18}
\]

Therefore, considering a free-body diagram of the payload, it is clear that the virtual control input \( \mu_i \) corresponds to the force exerted to the payload by the \( i \)-link, namely the tension of the \( i \)-th link. When there is no control force from each quadrotor, i.e., \( u^\parallel_i = 0 \), the tension of the \( i \)-th link is composed of the projected relative inertial force at the point where the \( i \)-th link is attached to the payload and the centrifugal force due to the rotation of the link. Substituting (17) and (18) back into (15), we obtain

\[
    a_i = \frac{1}{m_0} \sum_{j=1}^n \mu_j + R_0\hat{\Omega}_0^2\rho_i + R_0\hat{\rho}_j J_0^{-1}(\dot{\hat{\Omega}}_0 J_0\Omega_0 - \sum_{j=1}^n \hat{\rho}_j R_0^T \mu_j). \tag{19}
\]

Next, we determine the virtual control input \( \mu_i \). Any control scheme developed for the translational and rotational dynamics of a rigid body can be applied to (17) and (18). Here, we consider a proportional-derivative type nonlinear controller studied in [13]. Define position, attitude, and angular velocity tracking error vectors \( e_{x_0}, e_{\Omega_0}, e_{\Omega_0} \in \mathbb{R}^3 \) for the payload as

\[
    e_{x_0} = x_0 - x_{0d}, \tag{20}
\]

\[
    e_{R_0} = \frac{1}{2}(R_0^T R_0 - R_0^T R_0)\mathbf{e}_3, \tag{21}
\]

\[
    e_{\Omega_0} = \hat{\Omega}_0 - R_0^T R_{0d} \hat{\Omega}_{0d}. \tag{22}
\]
The desired resultant control force \( F_d \in \mathbb{R}^3 \) and moment \( M_d \in \mathbb{R}^3 \) acting on the payload are given in term of these error variables as

\[
F_d = m_0(-k_{x0} e_{x0} - k_{x0} e_{x0} + \dot{\bar{x}}_{0d} - \dot{e} e_{3}), \\
M_d = -k_{R0} e_{R0} - k_{\Omega 0} e_{\Omega 0} + (R_0^T R_{0d} \Omega_{0d})^\top J_0 R_{0d}^T R_{0d} \Omega_{0d} + J_0 R_{0d}^T R_{0d} \Omega_{0d},
\]

for positive constants \( k_{x0}, k_{R0}, k_{\Omega 0} \in \mathbb{R} \).

One may try to choose the virtual control input by making the expressions in the right-hand sides of (26) and (27) identical to \( F_d \) and \( M_d \), respectively. But, this is not valid as each \( \mu_i \) is constrained to be parallel to \( q_i \). Instead, we choose the desired value of \( \mu_i \), without any constraint, such that

\[
\sum_{i=1}^{n} \mu_{id} = F_d, \quad \sum_{i=1}^{n} \dot{\mu}_i R_0^T \mu_{id} = M_d, \tag{25}
\]
or equivalently, using the matrix \( P \) defined at (12),

\[
\begin{bmatrix} R_0^T \mu_{1d} \\ \vdots \\ R_0^T \mu_{nd} \end{bmatrix} = \begin{bmatrix} R_0^T F_d \\ M_d \end{bmatrix}. \tag{26}
\]

From the assumption stated at (13), there exists at least one solution to the above matrix equation for any \( F_d, M_d \). Here, we find the minimum-norm solution given by

\[
\begin{bmatrix} \mu_{1d} \\ \vdots \\ \mu_{nd} \end{bmatrix} = \text{diag}[R_0, \cdots, R_0] P^T (P P^T)^{-1} \begin{bmatrix} R_0^T F_d \\ M_d \end{bmatrix}. \tag{27}
\]

The virtual control input \( \mu_i \) is selected as the projection of its desired value \( \mu_{id} \) along \( q_i \),

\[
\mu_i = (\mu_{id} \cdot q_i) q_i = q_i q_i^T \mu_{id}, \tag{28}
\]

and the desired direction of each link, namely \( q_{id} \in \mathbb{S}^2 \) is defined as

\[
q_{id} = -\frac{\mu_{id}}{||\mu_{id}||}. \tag{29}
\]

It is straightforward to verify that when \( q_i = q_{id} \), the resultant force and moment acting on the payload become identical to their desired values.

Here, the extra degrees of freedom in control inputs are used to minimize the magnitude of the desired tension at (26), but they can be applied to other tasks, such as controlling the relative configuration of links [7]. This is referred to future investigation.

### B. Design of Normal Components

Substituting (15) into (8), the equation of motion for the \( i \)-link is given by

\[
\dot{\omega}_i = \frac{1}{I_i} \hat{\omega}_i \dot{a}_i - \frac{1}{m_i I_i} \hat{\omega}_i u^{\perp}_{id}.
\]

Here, the normal component of the control input \( u^{\perp}_{id} \) is chosen such that \( q_i \to q_{id} \) as \( t \to \infty \). Control systems for the unit-vectors on the two-sphere have been studied in [14], [15]. In this paper, we apply a control system developed in terms of the angular velocity in [15]. For the given desired direction of each link, its desired angular velocity is obtained from the kinematics equation as

\[
\omega_{id} = q_{id} \times \dot{q}_{id}. \tag{30}
\]

Define the direction and the angular velocity tracking error vectors for the \( i \)-th link, namely \( e_{i\omega}, \dot{e}_{i\omega} \in \mathbb{R}^3 \) as

\[
e_{i\omega} = q_{id} \times \dot{q}_i, \quad \dot{e}_{i\omega} = \dot{\omega}_i + \dot{q}^2_{id} \omega_{id}. \tag{31}
\]

The resulting stability properties are summarized as follows.

**Proposition 1:** Consider the simplified dynamic model defined by (9)-(18). For given tracking commands \( x_{0d}, \dot{x}_{0d}, R_{0d}, \) a control input is designed as (35). Then, there exist the values of controller gains, \( k_{x0}, k_{R0}, k_{\Omega 0}, k_{q}, k_{\omega} \) such that the zero equilibrium of tracking errors \( (\dot{e}_{x0}, \dot{e}_{x0}, \dot{e}_{R0}, \dot{e}_{\Omega 0}, e_{q}, e_{\omega}) \) is exponentially stable.

**Proof:** See Appendix B

**Remark 1:** At (28), the negative sign appeared to make the tension at each cable positive when \( q_i = q_{id} \). Assuming that the tracking errors \( e_{x0}, \dot{e}_{x0}, e_{R0}, e_{\Omega 0} \), and the variables \( \dot{x}_{0d}, \dot{\Omega}_{0d}, \Omega_{0d} \) obtained from the desired trajectories are sufficiently small, this guarantees that quadrotors remain above the payload. If desired, the negative sign at (28) can be eliminated to place quadrotors below the payload, resulting in a tracking control of an inverted rigid body multi-link pendulum, that can be considered as a generalization of a flying spherical inverted spherical pendulum [7].

### IV. Control System Design for Full Dynamic Model

The control system designed at the previous section is based on a simplifying assumption that each quadrotor can generates a thrust along any direction. However, the dynamics of quadrotor is underactuated since the direction of the total thrust is always parallel to its third body-fixed axis, while the magnitude of the total thrust can be arbitrarily changed. This can be directly observed from the expression of the total thrust, \( u_i = -f_i \hat{R}_i e_\theta \), where \( f_i \) is the total thrust
magnitude, and \( R_i e_3 \) corresponds to the direction of the third body-fixed axis. Whereas, the rotational attitude dynamics is fully actuated by the arbitrary control moment \( M_i \).

Based on these observations, the attitude of each quadrotor is controlled such that the third body-fixed axis becomes parallel to the direction of the ideal control force \( u_i \), designed in the previous section. The desired direction of the third body-fixed axis of the \( i \)-th quadrotor, namely \( b_{3_i} \in S^2 \) is given by

\[
b_{3_i} = -\frac{u_i}{\|u_i\|}.
\]

This provides two-dimensional constraint on the three-dimensional desired attitude of each quadrotor, and there remains one degree of freedom. To resolve it, the desired direction of the first body-fixed axis \( b_{1_i}(t) \in S^2 \) is introduced as a smooth function of time. Due to the fact that the first body-fixed axis is normal to the third body-fixed axis, it is impossible to follow an arbitrary command \( b_{1_i}(t) \) exactly. Instead, its projection onto the plane normal to \( b_{3_i} \) is followed, and the desired direction of the second body-fixed axis is chosen to constitute an orthonormal frame [10]. More explicitly, the desired attitude of the \( i \)-th quadrotor is given by

\[
R_{i e} = \left[ -\frac{(b_{3_i})^2 b_{1_i}}{\|b_{3_i}\|^2 \|b_{1_i}\|}, \frac{b_{3_i} b_{1_i}}{\|b_{3_i}\| \|b_{1_i}\|}, b_{3_i} \right],
\]

which is guaranteed to be an element of \( SO(3) \). The desired angular velocity is obtained from the attitude kinematics equation, \( \Omega_{i e} = (R_{i e}^T R_{i e})' \in \mathbb{R}^3 \).

Define the tracking error vectors for the attitude and the angular velocity of the \( i \)-th quadrotor as

\[
e_{R_i} = \frac{1}{2}(R_{i e}^T R_i - R_i^T R_{i e})' \quad e_{\Omega_i} = \Omega_i - R_{i e}^T R_{i e} \Omega_{i e}.
\]

The thrust magnitude is chosen as the length of \( u_i \), projected on to \(-R_i e_3\), and the control moment is chosen as a tracking controller on \( SO(3) \):

\[
f_i = -u_i \cdot R_i e_3,
\]

\[
M_i = -\frac{k_R}{\epsilon^2} e_{R_i} - \frac{k_{\Omega}}{\epsilon} e_{\Omega_i} + J_i \Omega_i - J_i (\hat{\Omega}_i R_i^T R_i \Omega_{i e} - R_{i e}^T R_{i e} \hat{\Omega}_{i e}),
\]

where \( \epsilon, k_R, k_{\Omega} \) are positive constants.

Stability of the corresponding controlled systems for the full dynamic model can be studied by showing the the error due to the discrepancy between the desired direction \( b_{3_i} \) and the actual direction \( R_i e_3 \) can be compensated via Lyapunov analysis [10], or singular perturbation theory can be applied to the attitude dynamics of quadrotors [6], [7]. For both cases, the structures of the control systems are identical, and here we use singular perturbation for simplicity.

**Proposition 2:** Consider the full dynamic model defined by (36)-(39). For given tracking commands \( x_{0_0}, R_{0_0} \) and the desired direction of the first body-fixed axis \( b_{1_0} \), control inputs for quadrotors are designed as (39) and (40). Then, there exists \( \epsilon^* > 0 \), such that for all \( \epsilon < \epsilon^* \), the zero equilibrium of the tracking errors \( e_{x_0}, e_{x_0}, e_{R_0}, e_{\Omega_0}, e_{R_i}, e_{\Omega_i} \) is exponentially stable.

**Proof:** See Appendix [C].

**V. NUMERICAL EXAMPLE**

We consider a numerical example where three quadrotors \( (n = 3) \) transport a rectangular box along a figure-eight curve, that is a special case of Lissajous figure shaped like an \( \infty \) symbol.

More explicitly, the mass of the payload is \( m_0 = 1.5 \text{ kg} \), and its length, width, and height are 1.0 m, 0.8 m, and 0.2 m, respectively. Mass properties of three quadrotors are identical, and they are given by

\[
m_i = 0.755 \text{ kg}, \quad J_i = \text{diag}[0.0820, 0.0845, 0.1377] \text{ kgm}^2.
\]

The length of cable is \( l_i = 1 \text{ m} \), and they are attached to the following points of the payload.

\[
\rho_1 = [0.5, 0, -0.1]^T, \quad \rho_2 = [-0.5, 0.4, -0.1]^T, \quad \rho_3 = [-0.5, -0.4, -0.1]^T.
\]

In other words, the first link is attached to the center of the top, front edge, and the remaining two links are attached to the vertices of the top, rear edge (see Figure 4).

The desired trajectory of the payload is chosen as

\[
x_{0_0}(t) = [1.2 \sin(0.4\pi t), 4.2 \cos(0.2\pi t), -0.5]^T \text{ m}.
\]
The desired attitude of the payload is chosen such that its first axis is tangent to the desired path, and the third axis is parallel to the direction of gravity, it is given by

$$ R_{0a}(t) = \begin{bmatrix} \hat{x}_0 \times_0 & \hat{e}_3 & \hat{e}_3 \times_0 \end{bmatrix} . $$

Initial conditions are chosen as

$$ x_0(0) = [1, 4.8, 0]^T, \quad v_0(0) = 0_{3 \times 1}, \quad q_i(0) = e_3, \quad \omega_i(0) = 0_{3 \times 1}, \quad R_i(0) = I_{3 \times 3}, \quad \Omega_i(0) = 0_{3 \times 1} . $$

The corresponding simulation results are presented at Figures 2 and 3. Figure 2 illustrates the desired trajectory that is shaped like a figure-eight curve around two obstacles represented by cones, and the actual maneuver of the payload and quadrotors. Figure 3 shows tracking errors for the position and the attitude of the payload, tracking errors for the link directions and the attitude of quadrotors, as well as tension and control inputs. These illustrate excellent tracking performances of the proposed control system.

### APPENDIX

#### A. Lagrangian Mechanics

**a) Derivatives of Lagrangian:** Here, we develop the equations of motion for the Lagrangian given by (3) and (4). The derivatives of the Lagrangian are given by

$$ D_{i}x_{0}L = m_{T}x_{0} + \sum_{i=1}^{n} m_{i}R_{0i}\hat{\Omega}_0 \rho_i - l_{i}q_i, $$

$$ D_{\dot{q}_{i}}L = \sum_{i=1}^{n} m_{i}(l_{i}^{2}\hat{q}_{i} - l_{i}x_{0} - l_{i}R_{0i}\hat{\Omega}_0 \rho_i), $$

$$ D_{\dot{\Omega}_0}L = \dot{J}_0\Omega_0 + \sum_{i=1}^{n} m_{i}\dot{\rho}_i R_{0i}^{T}(\dot{x}_i - l_{i}q_i), $$

$$ D_{T}L = J_i\Omega_i, $$

$$ D_{x_{0}}L = m_{T}ge_{3}, $$

$$ D_{\dot{q}_{i}}L = -m_{i}l_{i}ge_{3}, $$

where \( \dot{J}_0 = J_0 - \sum_{i=1}^{n} m_{i}\hat{\rho}_i^{2} \). The variation of a rotation matrix is represented by \( \delta R_{j} = R_{ji}n_{j} \) for \( n_{j} \in \mathbb{R}^{3} \) [11]. Using this the derivative of the Lagrangian with respect to \( R_{j} \) can be written as

$$ D_{R_{0i}}L \cdot \delta R_{0i} = \sum_{i=1}^{n} m_{i}R_{0i}^{T}\hat{\Omega}_{0i} \rho_{i} \cdot (\dot{x}_{i} - l_{i}\dot{q}_{i}) + m_{i}ge_{3} \cdot R_{0i}\hat{\Omega}_{0i} \rho_{i} $$

$$ = \sum_{i=1}^{n} m_{i}\{\hat{\Omega}_{0i} R_{0i}^{T}(\dot{x}_i - l_{i}q_i) + g\dot{\rho}_i R_{0i}^{T}e_{3}\} \cdot \eta_{0} $$

$$ \equiv d_{R_{0i}}L \cdot \eta_{0}.$$  

(47)

where \( d_{R_{0i}}L(\mathbb{R}^{3})^{*} \simeq \mathbb{R}^{3} \) is referred to as left-trivialized derivatives. Substituting \( \delta R_{j} = R_{ji}n_{j} \) into the attitude kinematic equations (2) and rearranging, the variation of the angular velocity can be written as \( \delta \Omega_{j} = \eta_{j} + \Omega_{j} \times \eta_{j} \). For the variation model of \( q_{i} \) given at (3), we have \( \delta \hat{q}_{i} = \hat{\xi}_{i} \times q_{i} \) and \( \delta \hat{\xi}_{i} = \hat{\xi}_{i} \times q_{i} + \hat{\xi}_{i} \times \hat{q}_{i} \).

**b) Lagrange-d’Alember Principle:** Let \( \mathcal{L} = \int_{t_{0}}^{t_{f}} \mathcal{L} \, dt \) be the action integral. Using the above equations, the infinitesimal variation of the action integral can be written as

$$ \delta \mathcal{L} = \int_{t_{0}}^{t_{f}} D_{x_{0}}L \cdot \delta \dot{x}_{0} + D_{\dot{q}_{i}}L \cdot \delta \dot{q}_{i} + D_{\dot{\Omega}_0}L \cdot \delta \dot{\Omega}_0 $$

$$ + D_{T}L \cdot \delta T + D_{x_{0}}L \cdot \delta T + \sum_{i=1}^{n} \left[ D_{\dot{q}_{i}}L \cdot (\hat{\xi}_{i} \times q_{i} + \hat{\xi}_{i} \times \hat{q}_{i}) + D_{\dot{q}_{i}}L \cdot (\hat{\xi}_{i} \times \hat{q}_{i}) \right. $$

$$ \left. + \sum_{i=1}^{n} D_{\dot{\Omega}_0}L \cdot (\hat{\eta}_{i} + \Omega_{i} \times \eta_{i}) \right] . $$

The total thrust at the \( i \)-th quadrotor with respect to the inertial frame is denoted by \( u_{i} = -f_{i}R_{ie_{3}} \in \mathbb{R}^{3} \) and the total moment at the \( i \)-th quadrotor is defined as \( M_{i} \in \mathbb{R}^{3} \). The corresponding virtual work can be written as

$$ \delta W = \int_{t_{0}}^{t_{f}} \sum_{i=1}^{n} u_{i} \cdot \{ \delta x_{0} + R_{0i}\hat{\Omega}_0 \rho_{i} - l_{i}\hat{\xi}_{i} \times q_{i} \} + M_{i} \cdot \eta_{i} . $$

According to Lagrange-d’Alember principle, we have \( \delta \mathcal{L} = -\delta W \) for any variation of trajectories with fixed end points.
By using integration by parts and rearranging, we obtain the following Euler-Lagrange equations:

$$\frac{d}{dt}D_{x_0}L - D_{x_0}L = \sum_{i=1}^{n} u_i,$$

$$\frac{d}{dt}D_{\Omega_0}L + \Omega_0 \times D_{\Omega_0}L - dR_0L = \sum_{i=1}^{n} \dot{\rho}_i R_0^T u_i,$$

$$\dot{\Omega}_i \dot{\omega}_i - \sum_{i=1}^{n} \dot{\rho}_i R_0^T \dot{\Omega}_0 = \sum_{i=1}^{n} \dot{\rho}_i R_0^T (u_i + g_i e_3),$$

where $m_T = m_0 + \sum_{i=1}^{n} m_i \in \mathbb{R}^3$ and $\dot{\Omega}_0 = J_0 - \sum_{i=1}^{n} \dot{\rho}_i R_0^T \in \mathbb{R}^{3 \times 3}$. This can be rewritten in a matrix form as given in (52).

Next, we substitute (50) into (48) and (49) to eliminate the dependency of $\dot{\omega}_i$ in the expressions for $\ddot{x}_0$ and $\Omega_0$. Using the fact that $I + \dot{q}_i^T q_i$ for any $q_i \in S^2$ and $\dot{\Omega}_0 \dot{\Omega}_0 = -\dot{\rho}_i R_0^T \rho_i$ for any $\dot{\Omega}_0, \rho_i \in \mathbb{R}^3$, we obtain (6) and (7) after rearrangements and simplifications. It is straightforward to see that (50) is equivalent to (8).

**B. Proof of Proposition 7**

**c) Error Dynamics:** From (17) and (27), the dynamics of the position tracking error is given by

$$m_0 \ddot{e}_{x_0} = m_0 (ge_3 - \ddot{x}_{0d}) + \sum_{i=1}^{n} q_i q_i^T \mu_{id}.$$

From (25) and (23), this can be rearranged as

$$\ddot{e}_{x_0} = ge_3 - \ddot{x}_{0d} + \frac{1}{m_0} F_d + Y_x,$$

where $Y_x \in \mathbb{R}^3$ to the error caused by the difference between $q_i$ and $q_{id}$, and it is given by

$$Y_x = \frac{1}{m_0} \sum_{i=1}^{n} (q_i q_i^T - I) \mu_{id}.$$

We have $\mu_{id} = q_{id} q_{id}^T \mu_{id}$ from (28). Using this, the error term can be written in terms of $e_q$ as

$$Y_x = \frac{1}{m_0} \sum_{i=1}^{n} (q_i q_i^T \mu_{id} - (q_i q_i^T) q_i - q_{id}) = -\frac{1}{m_0} \sum_{i=1}^{n} (q_i^T \mu_{id}) q_i e_q.$$

Using (25), an upper bound of $Y_x$ can be obtained as

$$\|Y_x\| \leq \frac{1}{m_0} \sum_{i=1}^{n} \|\mu_{id}\| \|e_q\| \leq \gamma (\|F_d\| + \|M_d\|) \|e_q\|,$$

where $\gamma = \frac{1}{m_0 \sqrt{\lambda_{max}(P^TP)}}$. From (23) and (24), this can be further bounded by

$$\|Y_x\| \leq \sum_{i=1}^{n} \{ \beta (k_{x0} \|e_{x0}\| + k_{x0} \|\dot{e}_{x0}\|) \}

+ \gamma (k_{R0} \|e_{R0}\| + k_{e\Omega_0} \|e_{\Omega_0}\|) + B \} \|e_q\|,$$

for some positive constant $B$ that is determined by the given desired trajectories of the payload, and $\beta = m_0 \gamma$. Throughout the remaining parts of the proof, any bound that can be obtained from $x_{0d}, R_{0d}$ is denoted by $B$ for simplicity. In short, the position tracking error dynamics of the payload can be written as (53), where the error term is bounded by (55).

Similarly, we find the attitude tracking error dynamics for the payload as follows. Using (18), (24), and (27), the time-derivative of $J_0 e_{\Omega_0}$ can be written as

$$J_0 \dot{e}_{\Omega_0} = (J_0 e_{\Omega_0} + d) \times e_{\Omega_0} - k_{R0} e_{R0} - k_{e\Omega_0} e_{\Omega_0} + Y_R,$$

where $d = (2J_0 - \text{tr}(J_0) I) R_0^T R_0 \Omega_0$ $\Omega_0 \in \mathbb{R}^3$ [13]. Note that the term $d$ is bounded. The error term in the attitude dynamics of the payload, namely $Y_R \in \mathbb{R}^3$ is given by

$$Y_R = \sum_{i=1}^{n} \dot{\rho}_i R_0^T (q_i q_i^T - I) \mu_{id} = -\sum_{i=1}^{n} \dot{\rho}_i R_0^T (q_i^T \mu_{id}) q_i e_q,$$

Similar with (55), an upper bound of $Y_R$ can be obtained as

$$\|Y_R\| \leq \sum_{i=1}^{n} \{ \delta_i (k_{x0} \|e_{x0}\| + k_{x0} \|\dot{e}_{x0}\|) \}

+ \sigma_i (k_{R0} \|e_{R0}\| + k_{e\Omega_0} \|e_{\Omega_0}\|) + B \} \|e_q\|,$$

where $\delta_i = m_0 \frac{\|\dot{\rho}_i\|}{\sqrt{\lambda_{max}(P^TP)}}$, $\sigma_i = \frac{\delta_i}{m_0} \in \mathbb{R}$.

Next, from (34), the time-derivative of the angular velocity error, projected onto the plane normal to $q_i$ is given as

$$-\dot{q}_i^T e_{\omega_i} = -k_{q_i} e_{q_i} - k_{\omega_i} e_{\omega_i}.$$
tracking error variables:

\[ p_{i0} = \sum_{i=1}^{n_p} m_i l_i R_i^T 0_i + m_i l_i R_i^T 0_i, \quad m_i l_i 0_i^T 0_i = 0, \quad m_i l_i 0_i R_i 0_i. \]

For positive constants \( e_{\text{max}}, \psi, \bar{\psi}, q_i \in \mathbb{R} \), consider the following open domain containing the zero equilibrium of tracking error variables:

\[ D = \{ (e_{x0}, \dot{e}_{x0}, e_{R0}, \psi, q_i, \bar{e}_{w}) \in (\mathbb{R}^3)^4 \times (\mathbb{R}^3 \times \mathbb{R}^3)^n \mid \| e_{x0} \| < e_{\text{max}}, \Psi R_0 < \psi R_0 < 1, \psi q_i < \bar{\psi} q_i < 1 \}. \]

In this domain, we have \( \| e_{R0} \| = \sqrt{\Psi R_0 (2 - \Psi R_0)} \leq \sqrt{\Psi q_i (2 - \Psi q_i)} \leq \sqrt{\Psi R_0 (2 - \Psi R_0)} \leq \sqrt{\Psi q_i (2 - \Psi q_i)} \leq \alpha_i < 1 \), and \( \| e_{\omega} \| = \sqrt{\Psi R_0 (2 - \Psi R_0)} \leq \sqrt{\Psi q_i (2 - \Psi q_i)} \leq \sqrt{\Psi R_0 (2 - \Psi R_0)} \leq \sqrt{\Psi q_i (2 - \Psi q_i)} \leq \alpha_i < 1 \). It is assumed that \( \psi q_i \) is sufficiently small such that \( n\alpha_i \beta < 1 \).

We can show that the configuration error functions are quadratic with respect to the error vectors in the sense that

\[
\frac{1}{2} \| e_{R0} \|^2 \leq \Psi R_0 \leq \frac{1}{2 - \psi R_0} \| e_{R0} \|^2, \\
\frac{1}{2} \| e_{q_i} \|^2 \leq \psi q_i \leq \frac{1}{2 - \psi q_i} \| e_{q_i} \|^2,
\]

where the upper bounds are satisfied only in the domain \( D \).

Define a Lyapunov function as

\[
\mathcal{V} = \frac{1}{2} \| \dot{e}_{x0} \|^2 + \frac{1}{2} k_{x0} \| e_{x0} \|^2 + c_x e_{x0} \cdot \dot{e}_{x0} + \frac{1}{2} \psi_q \| q_i \|^2 + c_q q_i, \quad q_i \in \{ |e_{q_i}|, |e_{\omega_i}| \} \in \mathbb{R}^2. \]

where \( c_x, c_R, c_q \) are positive constants.

Let \( z_{x0} = [\| e_{x0} \|, \| \dot{e}_{x0} \|^T, z_{R0} = [\| e_{R0} \|, \| \psi \|, \| \psi \|] \in \mathbb{R}^2. \) The Lyapunov function satisfies

\[
\begin{align*}
\dot{z}_{x0}^T P_{x0} z_{x0} + z_{R0}^T P_{R0} z_{R0} + \sum_{i=1}^{n} z_{q_i}^T P_{q_i} z_{q_i} & \leq \mathcal{V} \\
& \leq z_{x0}^T P_{x0} z_{x0} + z_{R0}^T P_{R0} z_{R0} + \sum_{i=1}^{n} z_{q_i}^T P_{q_i} z_{q_i},
\end{align*}
\]

where the matrices \( P_{x0}, P_{R0}, P_{q1}, P_{q2}, P_{q3}, P_{R} \in \mathbb{R}^{2 \times 2} \) are given by

\[
\begin{align*}
P_{x0} &= \frac{1}{2} \begin{bmatrix} k_{x0} & -c_x \\ -c_x & 1 \end{bmatrix}, \\
P_{R0} &= \frac{1}{2} \begin{bmatrix} 2k_{R0} & -c_R \bar{\lambda} \\ -c_R \bar{\lambda} & \bar{\lambda} \end{bmatrix}, \\
P_{q1} &= \frac{1}{2} \begin{bmatrix} 2k_{q1} & -c_q \\ -c_q & 1 \end{bmatrix},
\end{align*}
\]

where \( \lambda = \lambda_m [J_0] \) and \( \bar{\lambda} = \lambda_M [J_0] \). If the constants \( c_x, c_R, c_q \) are sufficiently small, all of the above matrices are positive-definite. It follows that the Lyapunov function is positive-definite and decrescent.

The time-derivative of the Lyapunov function along (60), and (59) is given by

\[
\dot{\mathcal{V}} = -(k_{x0} - c_x) \| \dot{e}_{x0} \|^2 - c_x k_{x0} \| e_{x0} \|^2 - c_x k_{x0} \| e_{x0} \| \| \dot{e}_{x0} \| + (c_x e_{x0} + \dot{e}_{x0}) \cdot Y_x \| k_{x0} \| + c_x k_{x0} \| e_{x0} \|^2 + c_R \| e_{R0} \| \| \dot{e}_{x0} \| - k_{x0} \| e_{x0} \| \| \dot{e}_{x0} \| \| e_{x0} \| + \| e_{R0} \| + c_R \| e_{R0} \| \cdot Y_R \| e_{R0} \| + n \| e_{\omega} \|^2 - c_q k_{q0} \| e_{\omega} \| \| e_{\omega} \| - c_q k_{q0} \| e_{\omega} \| \| e_{\omega} \| - c_q k_{q0} \| e_{\omega} \| \| e_{\omega} \|.
\]

From (55), an upper bound of the fourth term of the right-hand side is given by

\[
\| (c_x e_{x0} + \dot{e}_{x0}) \cdot Y_x \| \leq \sum_{i=1}^{n} \alpha_i \beta (c_x k_{x0} \| e_{x0} \|^2 + c_x k_{x0} \| e_{\omega} \| \| e_{\omega} \| + k_{x0} \| \dot{e}_{x0} \|^2) + c_x \| e_{x0} \| + \| \dot{e}_{x0} \| (k_{R0} \| e_{R0} \| + k_{x0} \| e_{x0} \|).
\]

Similarly, using (58),

\[
\| (c_R e_{R0} + \dot{e}_{x0}) \cdot Y_R \| \leq \sum_{i=1}^{n} \alpha_i (c_R k_{R0} \| e_{R0} \|^2 + c_R k_{R0} \| e_{R0} \| \| e_{R0} \| + k_{R0} \| e_{R0} \|).
\]
+ \{c_R B\|e_{R_i}\| + (\alpha_0 \sigma_i k_{R_i} + B)\|e_{\Omega_i}\|\} \|e_{\eta_i}\|
+ \alpha_i \delta \{c_R B\|e_{R_i}\| + (\alpha_0 \sigma_i k_{R_i} + B)\|e_{\Omega_i}\|\} (k_{R_i} \|e_{x_i}\| + k_{\alpha_i} \|e_{x_i}\|)\right). \quad (63)

Substituting these into (61) and rearranging, $\dot{V}$ is bounded by

$$
\dot{V} \leq \sum_{i=1}^{n} -z_i^T W_{i} z_i,
$$

where $z = [\|z_{x_i}\|, \|z_{R_i}\|, \|z_{\eta_i}\|]^T \in \mathbb{R}^3$, and the matrix $W_i \in \mathbb{R}^{3 \times 3}$ is defined as

$$
W_i = \begin{bmatrix}
\lambda_m [W_{x_i}] & -\frac{1}{2} \|W_{x_i}\| & -\frac{1}{2} \|W_{z_i}\|
\frac{1}{2} \|W_{x_i}\| & \lambda_m [W_{R_i}] & -\frac{1}{2} \|W_{R_i}\|
\frac{1}{2} \|W_{z_i}\| & -\frac{1}{2} \|W_{R_i}\| & \lambda_m [W_{q_i}]
\end{bmatrix}, \quad (64)
$$

where the sub-matrices are given by

$$
W_{x_i} = \frac{1}{n} \begin{bmatrix}
-c_x k_{\alpha_i} (1 - n \alpha_i \beta_i) & -c_x k_{\alpha_i} (1 + n \alpha_i \beta_i)
-c_x k_{\alpha_i} (1 - n \alpha_i \beta_i) & -c_x k_{\alpha_i} (1 + n \alpha_i \beta_i)
0 & -c_x k_{\alpha_i} (1 + n \alpha_i \beta_i)
\end{bmatrix},
$$

$$
W_{R_i} = \frac{1}{n} \begin{bmatrix}
-\frac{1}{2} \|W_{x_i}\| & -\frac{1}{2} \|W_{x_i}\| & -\frac{1}{2} \|W_{x_i}\|
\frac{1}{2} \|W_{x_i}\| & -\frac{1}{2} \|W_{x_i}\| & -\frac{1}{2} \|W_{x_i}\|
0 & 0 & -\frac{1}{2} \|W_{x_i}\|
\end{bmatrix},
$$

$$
W_q_i = \begin{bmatrix}
\frac{c_R B}{\alpha_0 \sigma_i k_{R_i} + B} & 0 & 0
0 & \frac{c_R B}{\alpha_0 \sigma_i k_{R_i} + B} & 0
0 & 0 & \frac{c_R B}{\alpha_0 \sigma_i k_{R_i} + B}
\end{bmatrix}.
$$

If the constants $c_x, c_R, c_\eta$ that are independent of the control input are sufficiently small, the matrices $W_{x_i}, W_{R_i}, W_q_i$ are positive-definite. Also, if the error in the direction of the link is sufficiently small relative to the desired trajectory, we can choose the controller gains such that the matrix $W_i$ is positive-definite, which follows that the zero equilibrium of tracking errors is exponentially stable.

### C. Proof of Proposition 2

This proof is based on singular perturbation [16] and the attitude tracking control system developed in [10]. Let $\bar{e}_{R_i} = \frac{1}{\epsilon} e_{R_i}$. The error dynamics for $\bar{e}_{R_i}, e_{\Omega_i}$, can be written as

$$
e_{R_i} = \frac{1}{2} \text{tr}[R_i^T \dot{R_i}] - I - R_i^T \bar{R}_i, e_{\Omega_i} = J_i^{-1} (-k_{R_i} e_{R_i} - k_{\Omega_i} e_{\Omega_i}).
$$

The right-hand side of the above equations has an isolated root of $(\bar{e}_{R_i}, e_{\Omega_i}) = (0, 0),$ and they correspond to the boundary-layer system. And, the origin of the boundary-layer system is exponentially stable according to [10, Proposition 1].

More explicitly, define a configuration error function on $\text{SO(3)}$ as follows:

$$
\Psi_R = \frac{1}{2} \text{tr}[I - R_c^T R_i].
$$

From now on, we drop the subscript $i$ for simplicity, as the subsequent development is identical for all quadrotors.

Consider a domain given by $D_R = \{(R, \Omega) \in \text{SO(3)} \times \mathbb{R}^3 \mid \Psi_R < \psi_R < 2\}$. Define a Lyapunov function,

$$
W = \frac{1}{2} e_{\Omega} \cdot J e_{\Omega} + \frac{k_R}{\epsilon^2} \Psi_R + \frac{c_3}{\epsilon} e_R \cdot e_{\Omega},
$$

where $c_3$ is a positive constant satisfying

$$
c_3 < \min \left\{ \sqrt{k_R \lambda_m (J)} \cdot \frac{4 k_R \lambda_{M} (J)}{k_R^{2} \lambda_{M} (J) + 4 k_R^{2} \lambda_{M} (J)} \right\}.
$$

We can show that

$$
\zeta^T L_1 \zeta \leq \zeta^T L_2 \zeta,
$$

where $\zeta = [\|e_{\Omega}\|, \|e_{\Omega}\|] \in \mathbb{R}^2$ and the matrices $L_1, L_2 \in \mathbb{R}^{2 \times 2}$ are given by

$$
L_1 = \begin{bmatrix}
\frac{k_R}{\epsilon^2} & -\frac{c_3}{\epsilon^2} \\
-\frac{c_3}{\epsilon^2} & \lambda_m (J)
\end{bmatrix}, \quad L_2 = \begin{bmatrix}
\frac{k_R}{\epsilon^2} & c_3 \\
-\frac{c_3}{\epsilon^2} & \lambda_m (J)
\end{bmatrix}.
$$

The time-derivative of $W$ can be written as

$$
\dot{W} = (e_{\Omega} + c_3 J^{-1} e_{\eta} - k_{R_i} e_{R_i} - k_{\Omega_i} e_{\Omega_i})
+ k_{R_i} e_{R_i} + c_3 \dot{e}_{\Omega} \leq -\zeta^T \dot{U} \zeta,
$$

where the matrix $U \in \mathbb{R}^{2 \times 2}$ is

$$
U = \begin{bmatrix}
\frac{c_R k_{R_i}}{\lambda_m (J)} & -\frac{c_R k_{R_i}}{\lambda_m (J)} \\
-\frac{c_R k_{R_i}}{\lambda_m (J)} & \frac{c_R k_{R_i}}{\lambda_m (J)}
\end{bmatrix}.
$$

The condition on $c_3$ guarantees that all of matrices $L_1, L_2, U$ are positive-definite. Therefore, the zero equilibrium of the tracking errors $(\bar{e}_{R}, e_{\Omega})$ is exponentially stable, and the convergence rate is proportional to $\frac{1}{\epsilon}$. Next, we consider the reduced system, which corresponds to the translational dynamics of the point mass and the rotational dynamics of the links when $R_i = R_{c_i}$. From (39) and (36), the control force of quadrotors when $R_i = R_{c_i}$ is given by

$$
-f_i \cdot R_{c_i} e_3 = (u_i \cdot R_{c_i} e_3) R_{c_i} e_3 = (u_i - \frac{u_i}{\|u_i\|}) - \frac{u_i}{\|u_i\|} = u_i.
$$

Therefore, the reduced system is given by the controlled dynamics of the simplified model, and from Proposition 1 its origin is exponentially stable.

Then, according to Tikhonov’s theorem [16, Thm 9.3], there exists $\epsilon^* > 0$ such that for all $\epsilon < \epsilon^*$, the origin of the full dynamics model is exponentially stable.

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