Computation of Minimal Homogeneous Generating Sets and Minimal Standard Bases for Ideals of Free Algebras

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Abstract. Let $K\langle X \rangle = K\langle X_1, \ldots, X_n \rangle$ be the free algebra generated by $X = \{X_1, \ldots, X_n\}$ over a field $K$. It is shown that with respect to any weighted $\mathbb{N}$-gradation attached to $K\langle X \rangle$, minimal homogeneous generating sets for finitely generated graded (two-sided) ideals of $K\langle X \rangle$ can be algorithmically computed, and that if an ungraded (two-sided) ideal $I$ of $K\langle X \rangle$ has a finite Gröbner basis $\mathcal{G}$ with respect to a graded monomial ordering on $K\langle X \rangle$, then a minimal standard basis for $I$ can be computed via computing a minimal homogeneous generating set of the associated graded ideal $\langle LH(I) \rangle$.

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1. Introduction and Preliminary

Throughout this paper, $K$ denotes a commutative field, $K^* = K - \{0\}$, algebras are meant associative $K$-algebras. Unless otherwise stated, ideals of algebras are meant two-sided ideals, and an ideal generated by a subset $S$ is denoted by $\langle S \rangle$. Moreover, we use $\mathbb{N}$ to denote the additive monoid of nonnegative integers.

Let $A = \oplus_{q \in \mathbb{N}} A_q$ be an $\mathbb{N}$-graded algebra over a field $K = A_0$, not necessarily commutative. Then it is a well-known fact that minimal homogeneous generating sets of graded one-sided ideals in $A$ give rise to very important invariants such as Betti numbers which are determined by the unique number of generators and the unique number of generators with the same degree. It is equally well known that if $A = K[x_1, \ldots, x_n]$ is the commutative polynomial $K$-algebra in $n$ variables, then minimal homogeneous generating sets for graded ideals of $A$ can be effectively computed by using a computer algebra system such as CoCoA [Coc], and consequently minimal
standard bases (i.e. Macaulay bases) for ungraded ideals of \( A \) can be computed effectively as well (see [KR], Definition 4.2.13, Theorem 4.6.3).

Let \( K\langle X \rangle = K \langle X_1, \ldots, X_n \rangle \) be the noncommutative free \( K \)-algebra generated by \( X = \{ X_1, \ldots, X_n \} \), and \( \mathcal{B} = \{ 1, \ X_i \cdots X_is \ | \ X_is \in X, \ s \geq 1 \} \) the standard \( K \)-basis of \( K\langle X \rangle \). For convenience, elements of \( \mathcal{B} \) are referred to as *monomials* and denoted by lower case letters \( w, u, v, s, \ldots \). Equip \( K\langle X \rangle \) with a weighted \( \mathbb{N} \)-gradation \( K\langle X \rangle = \bigoplus_{q \in \mathbb{N}} K\langle X \rangle_q \) by assigning each \( X_i \) a *positive degree* \( d_{gr}(X_i) = m_i, \ 1 \leq i \leq n \), that is, for each \( w = X_{i_1} \cdots X_{i_a} \in \mathcal{B}, d_{gr}(w) = d_{gr}(X_{i_1}) + \cdots + d_{gr}(X_{i_a}) = m_{i_1} + \cdots + m_{i_a} \), and for each \( q \in \mathbb{N}, K\langle X \rangle \) has the degree-\( q \) homogeneous part \( K\langle X \rangle_q = K \text{-span}\{ w \in \mathcal{B} \ | \ d_{gr}(w) = q \} \). If \( f \in K\langle X \rangle_q \) is a nonzero homogeneous element of degree \( q \), then we write \( d_{gr}(f) = q \).

Let \( I \) be a finitely generated graded two-sided ideal of \( K\langle X \rangle \). Then it follows from ([Li1], Proposition 3.5) that as with a one-sided graded ideal, any two minimal homogeneous generating sets of \( I \) have the same number of generators, and any two minimal homogeneous generating sets of \( I \) contain the same number of homogeneous elements of degree \( n \) for all \( n \in \mathbb{N} \). Based on the Buchberger-Bergman-Mora algorithm for computing Gröbner bases in free algebras (see Algorithm 1 presented below), in this paper we first show that the methods and algorithms, developed in ([CDNR], [KR]) for computing minimal homogeneous generating sets of graded submodules in free modules over commutative polynomial algebras, can be adapted for computing minimal homogeneous generating sets of \( I \) (Section 2). Secondly, in consideration of the relation with standard bases in \( K\langle X \rangle \), we show that if an ungraded ideal \( I \) of \( K\langle X \rangle \) has a finite Gröbner basis \( \mathcal{G} \) with respect to a graded monomial ordering \( \prec_{gr} \), then a minimal standard basis of \( I \), which has similar properties as mentioned above for a minimal homogeneous generating set, can be computed via computing a minimal homogeneous generating set of the associated graded ideal \( \langle \mathcal{LH}(I) \rangle \) of \( I \) (Section 3).

Concerning the Gröbner basis theory for the free \( K \)-algebra \( K\langle X \rangle = K\langle X_1, \ldots, X_n \rangle \), we now recall from ([Mor], [Gr]) some basic facts as follows. Let \( \prec \) be a monomial ordering on \( \mathcal{B} \), which is by definition a well-ordering \( \prec \) on \( \mathcal{B} \) satisfying: \( u \prec v \) implies \( wus \prec wvs \) for all \( w, u, v, s \in \mathcal{B} \); \( v \neq u \) and \( v = wus \) implies \( u \prec v \) for all \( u, v, w, s \in \mathcal{B} \). If \( f \in K\langle X \rangle \) is such that \( f = \sum_{i=1}^{m} \lambda_i w_i \) with \( \lambda_i \in K^* \), \( w_i \in \mathcal{B} \), and \( w_1 \prec w_2 \prec \cdots \prec w_m \), then we write \( \text{LM}(f) = w_m \) for the leading monomial of \( f \), and we write \( \text{LC}(f) = \lambda_m \) for the leading coefficient of \( f \).

Let \( u, v \in \mathcal{B} \). We say that \( u \) *divides* \( v \), denoted \( u \mid v \), if \( v = wus \) for some \( w, s \in \mathcal{B} \). As in the commutative case, if a monomial ordering \( \prec \) on \( \mathcal{B} \) is given, then the division of monomials extends to a division algorithm of dividing an element \( f \) by a finite subset of nonzero elements \( G = \{ g_1, \ldots, g_t \} \) in \( K\langle X \rangle \), which gives rise to a representation \( f = \sum_{i,j} \lambda_{ij} w_{ij} g_i u_{ij} + r \), where \( \lambda_{ij} \in K \), \( w_{ij}, u_{ij} \in \mathcal{B} \), \( g_i \in G \), satisfying \( \text{LM}(w_{ij} g_i u_{ij}) \preceq \text{LM}(f) \) for all \( \lambda_{ij} \neq 0 \), and if \( r \neq 0 \) such that \( r = \sum_k \mu_k v_k \) with \( \mu_k \in K^* \), \( v_k \in \mathcal{B} \), then \( \text{LM}(r) \preceq \text{LM}(f) \) and \( \text{LM}(g_j) \nmid v_k \) for all \( k \). We write \( \overline{f}^G = r \) and call it a *remainder* of \( f \) on division by \( G \). If \( \overline{f}^G = 0 \), then we say that \( f \) is *reduced to zero* on division by \( G \). A nonzero element \( f \in K\langle X \rangle \) is said to be *normal* (mod
if \( \overline{f}^G \) if \( f = \overline{f}^G \). Moreover, a subset \( G \) of nonzero elements in \( K\langle X \rangle \) is said to be LM-reduced if \( \text{LM}(g_i) \nmid \text{LM}(g_j) \) for all \( g_i \neq g_j \) in \( G \).

Given a monomial ordering \( \prec \) on \( B \) and a subset \( G \) of nonzero elements in \( K\langle X \rangle \), let \( I = \langle G \rangle \) be the ideal of \( K\langle X \rangle \) generated by \( G \). If for any nonzero element \( f \in I \), there is a \( g_i \in G \) such that \( \text{LM}(g_i)|\text{LM}(f) \), then \( G \) is called a Gröbner basis of \( I \). For a graded ideal \( I \) of \( K\langle X \rangle \), a Gröbner basis \( G \) of \( I \) consisting of homogeneous elements is called a homogeneous Gröbner basis of \( I \). A Gröbner basis \( G \) is said to be minimal if \( \text{LM}(g_i) \nmid \text{LM}(g_j) \) for all \( g_i \neq g_j \) in \( G \).

Let \( f, g \in K\langle X \rangle \) be two nonzero elements. If there are monomials \( u, v \in B \) such that

1. \( \text{LM}(f)u = v\text{LM}(g) \), and
2. \( \text{LM}(f) \nmid v \) and \( \text{LM}(g) \nmid u \),

then the element

\[
o(f, u; v, g) = \frac{1}{\text{LC}(f)}(f \cdot u) - \frac{1}{\text{LC}(g)}(v \cdot g)
\]

is referred to as an overlap element of \( f \) and \( g \).

**Theorem** (Termination theorem in the sense of [Mor] and [Gr]) Let \( G = \{g_1, \ldots, g_m\} \) be an LM-reduced subset of \( K\langle X \rangle \). then \( G \) is a Gröbner basis for the ideal \( I = \langle G \rangle \) if and only if for each pair \( g_i, g_j \in G \), including \( g_i = g_j \), every overlap element \( o(g_i, u; v, g_j) \) of \( g_i \) and \( g_j \) has the property \( o(g_i, u; v, g_j) = 0 \), that is, \( o(g_i, u; v, g_j) \) is reduced to 0 by the division by \( G \).

If a given LM-reduced subset \( G = \{g_1, \ldots, g_t\} \) of \( K\langle X \rangle \) is not a Gröbner basis for the ideal \( I = \langle G \rangle \), then the well-known **Buchberger-Bergman-Mora Algorithm** computes a (possibly infinite) Gröbner basis for \( I \). For the use of next section we recall this algorithm as follows.

**Algorithm 1**

**INPUT:** \( G_0 = \{g_1, \ldots, g_t\} \)

**OUTPUT:** \( G = \{g_1, \ldots, g_m, \ldots\} \), a Gröbner basis for \( I \)

**INITIALIZATION:** \( G := G_0 \), \( O := \{o(g_i, g_j) \mid g_i, g_j \in G_0\} \)

**BEGIN**

\[ \text{WHILE } O \neq \emptyset \text{ DO} \]

Choose any \( o(g_i, g_j) \in O \)
\[ O := O - \{o(g_i, g_j)\} \]
\[ o(g_i, g_j) \in O \]

IF \( r \neq 0 \) THEN
\[ O := \{o(g, r), o(r, g), o(r, r) \mid g \in G\} \]
\[ G := G \cup \{r\} \]

END

;
2. Computation of Minimal Homogeneous Generating Sets

Let $K\langle X \rangle = K\langle X_1, \ldots, X_n \rangle$ be the free $K$-algebra generated by $X = \{X_1, \ldots, X_n \}$ and $\mathcal{B}$ the standard $K$-basis of $K\langle X \rangle$. Fix a weighted $\mathbb{N}$-gradation $K\langle X \rangle = \oplus_{q \in \mathbb{N}} K\langle X \rangle_q$ for $K\langle X \rangle$ by assigning each $X_i$ a positive degree $d_{gr}(X_i) = m_i$, $1 \leq i \leq n$. Let $\prec$ be a monomial ordering on $\mathcal{B}$. Based on Algorithm 1 presented in Section 1, in this section we show that the methods and algorithms, developed in ([CDNR], [KR]) for computing minimal homogeneous generating sets of graded submodules in free modules over commutative polynomial algebras, can be adapted for computing minimal homogeneous generating sets of a finitely generated graded two-sided ideal $I$ of $K\langle X \rangle$. All notions, notations, and conventions given in Section 1 are maintained.

2.1. Definition Let $G = \{g_1, \ldots, g_t \}$ be a subset of homogeneous elements of $K\langle X \rangle$, $I = \langle G \rangle$ the graded ideal generated by $G$, and let $n \in \mathbb{N}$, $G_{\leq n} = \{g_j \in G \mid d_{gr}(g_j) \leq n \}$. If, for each nonzero homogeneous element $f \in I$ with $d_{gr}(f) \leq n$, there is some $g_i \in G_{\leq n}$ such that $\text{LM}(g_i) | \text{LM}(f)$ with respect to $\prec$, then we call $G_{\leq n}$ an $n$-truncated Gröbner basis of $I$.

Noticing that every $w \in \mathcal{B}$ is a homogeneous element of $K\langle X \rangle$, verification of the lemma below is straightforward.

2.2. Lemma Let $\mathcal{G} = \{g_1, \ldots, g_t \}$ be a homogeneous Gröbner basis for the graded ideal $I = \langle \mathcal{G} \rangle$ of $K\langle X \rangle$ with respect to the given monomial ordering $\prec$ on $\mathcal{B}$. For each $n \in \mathbb{N}$, put $\mathcal{G}_{\leq n} = \{g_j \in \mathcal{G} \mid d_{gr}(g_j) \leq n \}$, $I_{\leq n} = \bigcup_{q=0}^{n} I_q$ where each $I_q$ is the degree-$q$ homogeneous part of $I$, and let $I(n) = \langle I_{\leq n} \rangle$ be the graded ideal generated by $I_{\leq n}$. The following statements hold. (i) $\mathcal{G}_{\leq n}$ is an $n$-truncated Gröbner basis of $I$. Thus, if $f \in K\langle X \rangle$ is a homogeneous element with $d_{gr}(f) \leq n$, then $f \in I$ if and only if $f^{\mathcal{G}_{\leq n}} = 0$, i.e., $f$ is reduced to zero on division by $\mathcal{G}_{\leq n}$.

(ii) $I(n) = \langle \mathcal{G}_{\leq n} \rangle$, and $\mathcal{G}_{\leq n}$ is an $n$-truncated Gröbner basis of $I(n)$.

Convention In what follows, we let $o(f, g)$ represent any overlap element of two nonzero elements $f, g \in K\langle X \rangle$.

In light of Algorithm 1, an $n$-truncated Gröbner basis is characterized as follows.

2.3. Proposition Let $I = \langle G \rangle$ be the graded ideal of $K\langle X \rangle$ generated by a finite set of nonzero homogeneous elements $G = \{g_1, \ldots, g_m \}$. Without loss of generality, we assume that $G$ is LM-reduced (see Section 1). For each $n \in \mathbb{N}$, put $G_{\leq n} = \{g_j \in G \mid d_{gr}(g_j) \leq n \}$. The following statements are equivalent with respect to the given monomial ordering $\prec$ on $\mathcal{B}$.

(i) $G_{\leq n}$ is an $n$-truncated Gröbner basis of $I$. 

(ii) For each \((g_i, g_j) \in G \times G\), every overlap element \(o(g_i, g_j)\) of \(d_{\text{gr}}(o(g_i, g_j)) \leq n\) is reduced to zero on division by \(G_{\leq n}\), i.e., \(o(g_i, g_j)G^{\leq n} = 0\).

**Proof** Recall that if \(\text{LM}(g_i) = vw\) and \(\text{LM}(g_j) = wu\) for some \(u, v, w \in B\) with \(w \neq 1\), then the corresponding overlap element of \(g_i\) and \(g_j\) is

\[
o(g_i, u; v, g_j) = \frac{1}{\text{LC}(g_i)} g_i u - \frac{1}{\text{LC}(g_j)} v g_j
\]

which is obviously a homogeneous element in \(I\). If \(d_{\text{gr}}(o(g_i, g_j)) \leq n\), then it follows from (i) that (ii) holds.

Conversely, suppose that (ii) holds. To see that \(G_{\leq n}\) is an \(n\)-truncated Gröbner basis of \(I\), let us run (Algorithm 1) with the initial input data \(G\). Without optimizing Algorithm 1 we may certainly assume that \(G \subseteq \mathcal{G}\), thereby \(G_{\leq n} \subseteq \mathcal{G}_{\leq n}\), where \(\mathcal{G}\) is the new input set returned after a certain pass through the WHILE loop. On the other hand, by the construction of \(o(g_i, g_j)\) we know that if \(d_{\text{gr}}(o(g_i, g_j)) \leq n\), then \(d_{\text{gr}}(g_i) \leq n\), \(d_{\text{gr}}(g_j) \leq n\). Hence, the assumption (ii) implies that Algorithm 1 does not give rise to any new element of degree \(\leq n\) for \(\mathcal{G}\). Therefore, \(G_{\leq n} = \mathcal{G}_{\leq n}\). By Lemma 2.2 we conclude that \(G_{\leq n}\) is an \(n\)-truncated Gröbner basis of \(I\).

\[\square\]

2.4. Corollary Let \(I = \langle G \rangle\) be the graded ideal of \(K\langle X \rangle\) generated by a finite set of nonzero homogeneous elements \(G = \{g_1, \ldots, g_m\}\). Suppose that \(G_{\leq n} = \{g_j \in G \mid d_{\text{gr}}(g_j) \leq n\}\) is an \(n\)-truncated Gröbner basis of \(I\) with respect to the given monomial ordering \(\prec\) on \(B\).

(i) If \(g \in K\langle X \rangle\) is a nonzero homogeneous element of \(d_{\text{gr}}(g) = n\) such that \(\text{LM}(g_i) \nmid \text{LM}(g)\) for all \(g_i \in G_{\leq n}\), then \(G' = G_{\leq n} \cup \{g\}\) is an \(n\)-truncated Gröbner basis for both the graded ideals \(I' = I + \langle g \rangle\) and \(I'' = \langle G' \rangle\) of \(K\langle X \rangle\).

(ii) If \(n \leq n_1\) and \(g \in K\langle X \rangle\) is a nonzero homogeneous element of \(d_{\text{gr}}(g) = n_1\) such that \(\text{LM}(g_i) \nmid \text{LM}(g)\) for all \(g_i \in G_{\leq n}\), then \(G' = G_{\leq n} \cup \{g\}\) is an \(n_1\)-truncated left Gröbner basis for the graded submodule \(I' = \langle G' \rangle\) of \(K\langle X \rangle\).

**Proof** If \(g \in K\langle X \rangle\) is a nonzero homogeneous element of \(d_{\text{gr}}(g) = n_1 \geq n\) and \(\text{LM}(g_i) \nmid \text{LM}(g)\) for all \(g_i \in G_{\leq n}\), then it is straightforward to see that \(d_{\text{gr}}(H) > n\) for every nonzero \(H \in \{o(g_i, g), o(g, g_i), o(g, g) \mid g_i \in G\}\). Hence both (i) and (ii) hold by Proposition 2.3. \[\square\]

2.5. Proposition (Noncommutative analogue of ([KR], Proposition 4.5.10)) Given a finite set of nonzero homogeneous elements \(F = \{f_1, \ldots, f_m\} \subseteq K\langle X \rangle\), where \(d_{\text{gr}}(f_1) \leq d_{\text{gr}}(f_2) \leq \cdots \leq d_{\text{gr}}(f_m)\), for each fixed \(n_0 \in \mathbb{N}\), the following algorithm computes an \(n_0\)-truncated Gröbner basis \(G = \{g_1, \ldots, g_t\}\) for the graded ideal \(I = \langle F \rangle\) of \(K\langle X \rangle\), such that \(d_{\text{gr}}(g_1) \leq d_{\text{gr}}(g_2) \leq \cdots \leq d_{\text{gr}}(g_t)\).
Algorithm 2

**INPUT**: $F = \{f_1, ..., f_m\}$

**OUTPUT**: $G = \{g_1, ..., g_t\}$

**INITIALIZATION**: $O := \emptyset$, $W := F$, $G := \emptyset$, $t' := 0$

**BEGIN**

$n := \text{min}\{d_{gr}(f_i), d_{gr}(o(g_{\ell}, g_q)) \mid f_i \in W, o(g_{\ell}, g_q) \in O\}$

**IF** $n \leq n_0$ **THEN**

$O_n := \{o(g_{\ell}, g_q) \in O \mid d_{gr}(o(g_{\ell}, g_q)) = n\}$, $W_n := \{f_j \in W \mid d_{gr}(f_j) = n\}$

$O := O - O_n$, $W := W - W_n$

**WHILE** $O_n \neq \emptyset$ **DO**

Choose any $o(g_{\ell}, g_q) \in O_n$

$O_n := O_n - \{o(g_{\ell}, g_q)\}$

$\overline{o(g_{\ell}, g_q)} = r$

**IF** $r \neq 0$ **THEN**

$t' := t' + 1$, $g_{\ell'} := r$

$O := O \cup \left\{ o(g_{\ell}, g_q) \mid o(g_{\ell}, g_q) \in \left\{ o(g_{i}, g_{\ell'}), o(g_{\ell'}, g_{i}), o(g_{\ell'}, g_{\ell'}), \text{ where } g_i \in G, 1 \leq i < t' \right\}, \quad d_{gr}(o(g_{\ell}, g_q)) \leq n_0 \right\}$

$G := G \cup \{g_{\ell'}\}$

**END**

**ELSE**

**WHILE** $W_n \neq \emptyset$ **DO**

Choose any $f_j \in W_n$

$W_n := W_n - \{f_j\}$

$\overline{f_j} = r$

**IF** $r \neq 0$ **THEN**

$t' := t' + 1$, $g_{\ell'} := r$

$O := O \cup \left\{ o(g_{\ell}, g_q) \mid o(g_{\ell}, g_q) \in \left\{ o(g_{i}, g_{\ell'}), o(g_{\ell'}, g_{i}), o(g_{\ell'}, g_{\ell'}), \text{ where } g_i \in G, 1 \leq i < t' \right\}, \quad d_{gr}(o(g_{\ell}, g_q)) \leq n_0 \right\}$

$G := G \cup \{g_{\ell'}\}$

**END**

**END**

**Proof** For each fixed $n \leq n_0$, by the definition of an overlap element it is clear that $O_n$ is finite. Hence the algorithm terminates after $O_{n_0}$ and $W_{n_0}$ are exhausted. Note that both the WHILE loops append new elements to $G$ by taking the nonzero normal remainders on division by $G$. With a fixed $n$, by the definition of an overlap element and the normality
of $g_{i'} \pmod{\mathcal{G}}$, it is straightforward to check that in both the WHILE loops every nonzero $H \in \{o(g_i, g_{i'}), o(g_{i'}, g_i), o(g_{i'}, g_{i''})\}$ has $d_{\text{gr}}(H) > n$. For convenience, let us write $I(n)$ for the ideal generated by $\mathcal{G}$ which is obtained after $W_n$ is exhausted in the second WHILE loop. If $n_1$ is the first number after $n$ such that $\mathcal{O}_{n_1} \neq \emptyset$, and for some $o(g_t, g_q) \in \mathcal{O}_{n_1}$, $r = \overline{o(g_t, g_q)}^\mathcal{G} \neq 0$ in a certain pass through the first WHILE loop, then we note that this $r$ is still contained in $I(n)$. Hence, after $\mathcal{O}_{n_1}$ is exhausted in the first WHILE loop, the obtained $\mathcal{G}$ generates $I(n)$ and $\mathcal{G}$ is an $n_1$-truncated Gröbner basis of $I(n)$. Noticing that the algorithm starts with $\mathcal{O} = \emptyset$ and $\mathcal{G} = \emptyset$, inductively it follows from Proposition 2.3 and Corollary 2.4 that after $W_{n_1}$ is exhausted in the second WHILE loop, the obtained $\mathcal{G}$ is an $n_1$-truncated Gröbner basis of $I(n_1)$. Since $n_0$ is finite and all the generators of $I$ with $d_{\text{gr}}(f_j) \leq n_0$ are processed through the second WHILE loop, the eventually obtained $\mathcal{G}$ is an $n_0$-truncated Gröbner basis of $I$. Finally, the fact that the degrees of elements in $\mathcal{G}$ are non-decreasingly ordered follows from the choice of the next $n$ in the algorithm.

**Remark** Note that in Proposition 2.5 we did not assume that the subset $F$ is LM-reduced. The reason is that the algorithm starts with $\mathcal{O} = \emptyset$ and $\mathcal{G} = \emptyset$, while $\mathcal{G}$ starts to get its members from the second WHILE loop, and then, the new $\mathcal{G}$ obtained after each pass through the WHILE loops is clearly LM-reduced.

Let $I$ be a finitely generated graded ideal of $K\langle X \rangle$. We say that a homogeneous generating set $F = \{f_1, \ldots, f_m\}$ of $I$ is a minimal homogeneous generating set if any proper subset of $F$ cannot be a generating set of $I$. We now proceed to show that Algorithm 2 presented above can be further modified to compute minimal homogeneous generating sets for finitely generated graded ideals of $K\langle X \rangle$. The next proposition and its corollary are noncommutative analogues of ([KR], Proposition 4.6.1, Corollary 4.6.2).

### 2.6. Proposition

Let $I = \langle F \rangle$ be the graded ideal of $K\langle X \rangle$ generated by a finite subset of nonzero homogeneous elements $F = \{f_1, \ldots, f_m\}$, where $d_{\text{gr}}(f_1) \leq d_{\text{gr}}(f_2) \leq \cdots \leq d_{\text{gr}}(f_m)$. Put $I_1 = \{0\}, I_i = \langle F_i \rangle$, where $F_i = F - \{f_1, \ldots, f_m\}, 2 \leq i \leq m$. The following statements hold.

1. **Proof** (i) If $F$ is a minimal homogeneous generating set of $I$ if and only if $f_i \not\in I_i, 1 \leq i \leq m$. Conversely, suppose $f_i \not\in I_i, 1 \leq i \leq m$. If $F$ were not a minimal homogeneous generating set of $I$, then, there is some $i$ such that $I$ is generated by $F' = \{f_1, \ldots, f_{i-1}, f_{i+1}, \ldots, f_m\}$. Thus, there are $f_j \in F'$ and nonzero homogeneous elements $h_{jk}, h_{j\ell} \in K\langle X \rangle$ such that $f_i = \sum_{j \neq i} h_{jk} f_j h_{j\ell}$ and $d_{\text{gr}}(f_i) = d_{\text{gr}}(h_{jk}) + d_{\text{gr}}(f_j) + d_{\text{gr}}(h_{j\ell})$. Thus $d_{\text{gr}}(f_j) \leq d_{\text{gr}}(f_i)$ for all $j \neq i$ appeared in the representation of $f_i$. If $d_{\text{gr}}(f_j) < d_{\text{gr}}(f_i)$ for all $j \neq i$, then $f_i \in I_i = \langle F_i \rangle$, which contradicts the assumption. If $d_{\text{gr}}(f_i) = d_{\text{gr}}(f_j)$ for some $j \neq i$, then since $h_{jk}$ and $h_{j\ell}$ are nonzero homogeneous elements, we have $h_{jk}, h_{j\ell} \in K\langle X \rangle_{\geq d_{\text{gr}}(f_i)} - \{0\} = K^*$. Putting $i' = \ldots$
\( \max \{i, j \mid j \neq i, \ d_{gr}(f_j) = d_{gr}(f_i)\} \), we then have \( f_{i'} \in I_{i'} = \langle F_{i'} \rangle \), which again contradicts the assumption. Hence, under the assumption we conclude that \( F \) is a minimal homogeneous generating set of \( I \).

(ii) In view of (i), it is sufficient to show that \( \overline{F} \) is a homogeneous generating set of \( I \). Indeed, if \( f_i \in F - \overline{F} \), then \( f_i \in I_i \). By checking \( f_{i-1} \) and so on, it follows that \( f_i \in \langle \overline{F} \rangle \), as desired. \( \square \)

2.7. Corollary Let \( F = \{f_1, \ldots, f_m\} \) be a minimal homogeneous generating set of the graded ideal \( I \) of \( K\langle X \rangle \), where \( d_{gr}(f_1) \leq d_{gr}(f_2) \leq \cdots \leq d_{gr}(f_m) \), and let \( f \in K\langle X \rangle - I \) be a homogeneous element with \( d_{gr}(f_m) \leq d_{gr}(f) \). Then \( \widehat{F} = F \cup \{f\} \) is a minimal homogeneous generating set of the graded ideal \( \widehat{I} = I + \langle f \rangle \).

\( \square \)

Combining the foregoing results, we are ready to reach the goal of this section.

2.8. Theorem (Noncommutative analogue of ([KR], Theorem 4.6.3)) Let \( F = \{f_1, \ldots, f_m\} \) be a finite set of nonzero homogeneous elements of \( K\langle X \rangle \) with \( d_{gr}(f_1) \leq d_{gr}(f_2) \leq \cdots \leq d_{gr}(f_m) = n_0 \). Then the following algorithm returns a minimal homogeneous generating set \( F_{min} \subseteq F \) for the graded ideal \( I = \langle F \rangle \); and meanwhile it returns an \( n_0 \)-truncated Gröbner basis \( G = \{g_1, \ldots, g_t\} \) for \( I \) such that \( d_{gr}(g_1) \leq d_{gr}(g_2) \leq \cdots d_{gr}(g_t) \).

Algorithm 3

INPUT: \( F = \{f_1, \ldots, f_m\} \)

OUTPUT: \( F_{min} = \{f_{j_1}, \ldots, f_{j_r}\} \subset F \), a minimal homogeneous generating set for \( I \);
\( G = \{g_1, \ldots, g_t\} \), an \( n_0 \)-truncated Gröbner basis for \( I \);

INITIALIZATION: \( O := \emptyset \), \( W := F \), \( G := \emptyset \), \( t' := 0 \), \( F_{min} = \emptyset \)

BEGIN
\( n := \min \{d_{gr}(f_i), d_{gr}(o(g_{t}, g_{q})) \mid f_i \in W, o(g_{t}, g_{q}) \in O\} \)

IF \( n \leq n_0 \) THEN
\( O_n := \{o(g_{t}, g_{q}) \in O \mid d_{gr}(o(g_{t}, g_{q})) = n\} \), \( W_n := \{f_j \in W \mid d_{gr}(f_j) = n\} \)
\( O := O - O_n \), \( W := W - W_n \)

WHILE \( O_n \neq \emptyset \) DO

Choose any \( o(g_{t}, g_{q}) \in O_n \)
\( O_n := O_n - \{o(g_{t}, g_{q})\} \)
\( o(g_{t}, g_{q})^{r} = r \)
IF \( r \neq 0 \) THEN
\[
t' := t' + 1, \quad g_{t'} := r
\]
\[
\mathcal{O} := \mathcal{O} \cup \left\{ o(g_\ell, g_q) \mid o(g_\ell, g_q) \in \left\{ o(g_i, g_{t'}), o(g_{t'}, g_i), o(g_{t'}, g_{t'}), \right\}, \ \text{where} \ g_i \in \mathcal{G}, \ 1 \leq i < t' \right\}, \]
\[
G := G \cup \{ g_{t'} \}
\]
END
ELSE
WHILE \( W_n \neq \emptyset \) DO
Choose any \( f_j \in W_n \)
\[
W_n := W_n - \{ f_j \}
\]
\[
f_j : R
\]
IF \( r \neq 0 \) THEN
\[
F_{\text{min}} := F_{\text{min}} \cup \{ f_j \}
\]
\[
t' := t' + 1, \quad g_{t'} := r
\]
\[
\mathcal{O} := \mathcal{O} \cup \left\{ o(g_\ell, g_q) \mid o(g_\ell, g_q) \in \left\{ o(g_i, g_{t'}), o(g_{t'}, g_i), o(g_{t'}, g_{t'}), \right\}, \ \text{where} \ g_i \in \mathcal{G}, \ 1 \leq i < t' \right\}, \]
\[
G := G \cup \{ g_{t'} \}
\]
END
END
END

**Proof** By Proposition 2.5 we know that this algorithm terminates and the eventually obtained \( G \) is an \( n_0 \)-truncated homogeneous Gröbner basis for the ideal \( I \), in which the degrees of elements are ordered non-decreasingly. It remains to prove that the eventually obtained \( F_{\text{min}} \) is a minimal homogeneous generating set of the ideal \( I \).

As in the proof of Proposition 2.5, let us first bear in mind that for each \( n \), in both the WHILE loops every new appended \( o(g_\ell, g_q) \) has \( d_{gr}(o(g_\ell, g_q)) > n \). Moreover, for convenience, let us write \( G(n) \) for the \( G \) obtained after \( \mathcal{O}_n \) is exhausted in the first WHILE loop, and write \( F_{\text{min}}[n], G[n] \) respectively for the \( F_{\text{min}}, G \) obtained after \( W_n \) is exhausted in the second WHILE loop. Since the algorithm starts with \( \mathcal{O} = \emptyset \) and \( G = \emptyset \), if, for a fixed \( n \), we check carefully how the elements of \( F_{\text{min}} \) are chosen during executing the second WHILE loop, and how the new elements are appended to \( G \) after each pass through the first or the second WHILE loop, then it follows from Proposition 2.3 and Corollary 2.4 that after \( W_n \) is exhausted, the obtained \( F_{\text{min}}[n] \) and \( G[n] \) generate the same ideal, denoted \( I(n) \), such that \( G[n] \) is an \( n \)-truncated Gröbner basis of \( I(n) \). We now use induction to show that the eventually obtained \( F_{\text{min}} \) is a minimal homogeneous generating set of the ideal \( I = \langle F \rangle \). If \( F_{\text{min}} = \emptyset \), then it is a minimal generating set of the zero ideal. To proceed, we assume that \( F_{\text{min}}[n] \) is a minimal homogeneous generating
set for \( I(n) \) after \( W_n \) is exhausted in the second WHILE loop. Suppose that \( n_1 \) is the first number after \( n \) such that \( O_{n_1} \neq \emptyset \). We complete the induction proof below by showing that \( F_{\min}[n_1] \) is a minimal homogeneous generating set of \( I(n_1) \).

If in a certain pass through the first WHILE loop, \( r = \overline{o(g_l, g_q)^G} \neq 0 \) for some \( o(g_l, g_q) \in O_{n_1} \), then we note that \( r \in I(n) \). It follows that after \( O_{n_1} \) is exhausted in the first WHILE loop, we have \( I(n) = \langle G(n_1) \rangle \) such that \( G(n_1) \) is an \( n_1 \)-truncated Gröbner basis of \( I(n) \). Next, assume that \( W_{n_1} = \{ f_{j_1}, \ldots, f_{j_s} \} \neq \emptyset \) and that the elements of \( W_{n_1} \) are processed in the given order during executing the second WHILE loop. Since \( G(n_1) \) is an \( n_1 \)-truncated Gröbner basis of \( I(n) \), if \( f_{j_1} \in W_{n_1} \) is such that \( r_1 = \overline{f_{j_1}^G(n_1)} \neq 0 \), then \( f_{j_1}, r_1 \in K \langle X \rangle - I(n) \). By Corollary 2.4, we conclude that \( G(n_1) \cup \{ r_1 \} \) is an \( n_1 \)-truncated Gröbner basis for \( I(n) + \langle r_1 \rangle \); and by Corollary 2.7, we conclude that \( F_{\min}[n] \cup \{ f_{j_1} \} \) is a minimal homogeneous generating set of \( I(n) + \langle r_1 \rangle \). Repeating this procedure, if \( f_{j_2} \in W_{n_1} \) is such that \( r_2 = \overline{f_{j_2}^G(n_1) \cup \{ r_1 \}} \neq 0 \), then \( f_{j_2}, r_2 \in K \langle X \rangle - (I(n) + \langle r_1 \rangle) \). By Corollary 2.4, we conclude that \( G(n_1) \cup \{ r_1, r_2 \} \) is an \( n_1 \)-truncated Gröbner basis for \( I(n) + \langle r_1, r_2 \rangle \); and by Corollary 2.7, we conclude that \( F_{\min}[n] \cup \{ f_{j_1}, f_{j_2} \} \) is a minimal homogeneous generating set of \( I(n) + \langle r_1, r_2 \rangle \). Continuing this procedure until \( W_{n_1} \) is exhausted we see that the resulted \( G[n_1] = G \) and \( F_{\min}[n_1] = F_{\min} \) generate the same module \( I(n_1) \) such that \( G[n_1] \) is an \( n_1 \)-truncated Gröbner basis of \( I(n_1) \) and \( F_{\min}[n_1] \) is a minimal homogeneous generating set of \( I(n_1) \), as desired. As all elements of \( F \) are eventually processed by the second WHILE loop, we conclude that the finally obtained \( G \) and \( F_{\min} \) have the properties that \( I = \langle G \rangle \), \( G \) is an \( n_0 \)-truncated Gröbner basis of \( I \), and \( F_{\min} \) is a minimal homogeneous generating set of \( I \).

\[ \square \]

2.9. Corollary Let \( F = \{ f_1, \ldots, f_m \} \) be a finite set of nonzero homogeneous elements of \( K \langle X \rangle \) with \( d_{gr}(f_1) = d_{gr}(f_2) = \cdots = d_{gr}(f_m) = n_0 \).

(i) If \( F \) is LM-reduced, i.e., \( \text{LM}(f_i) \not\subset \text{LM}(f_j) \) for all \( i \neq j \), then \( F \) is a minimal homogeneous generating set of the ideal \( I = \langle F \rangle \), and meanwhile \( F \) is an \( n_0 \)-truncated Gröbner basis for \( I \).

(ii) If \( F \) is a minimal Gröbner basis of the ideal \( I = \langle F \rangle \), then \( F \) is a minimal homogeneous generating set of \( I \).

Proof By the assumption, it follows from the second WHILE loop of Algorithm 3 that \( F_{\min} = F \).

3. Computation of Minimal Standard Bases

Let \( K \langle X \rangle = K \langle X_1, \ldots, X_n \rangle \) be the free \( K \)-algebra generated by \( X = \{ X_1, \ldots, X_n \} \) and \( \mathcal{B} \) the standard \( K \)-basis of \( K \langle X \rangle \). Fix a weighted \( \mathbb{N} \)-gradation \( K \langle X \rangle = \oplus_{q \in \mathbb{N}} K \langle X \rangle_q \) for \( K \langle X \rangle \) by assigning each \( X_i \) a positive degree \( d_{gr}(X_i) = m_i \), \( 1 \leq i \leq n \). Recall that a graded monomial ordering on \( \mathcal{B} \) is a monomial ordering \( \prec \) on \( \mathcal{B} \) satisfying

\[ u, v \in \mathcal{B} \quad \text{and} \quad u \prec v \quad \text{implies} \quad d_{gr}(u) \leq d_{gr}(v). \]
A graded monomial ordering is usually denoted by $\prec_{gr}$. The most well-known graded monomial ordering on $B$ is the graded lexicographic ordering $\prec_{grlex}$.

In this section, we show that if an ungraded ideal $I$ of $K\langle X \rangle$ has a finite Gröbner basis $G$ with respect to a given graded monomial ordering $\prec_{gr}$, then a minimal standard basis for $I$ can be computed via computing a minimal homogeneous generating set of the associated graded ideal $\langle LH(I) \rangle$ of $I$ (see the definitions below). Concerning the notion of a standard basis for the ideal $I$, we have a remark given after Proposition 3.2 below. All notions, notations, and conventions used before are maintained.

Let $f = f_0 + f_1 + \cdots + f_q \in K\langle X \rangle$ with $f_i \in K\langle X \rangle_i$ and $f_q \neq 0$, and let $LH(f)$ denote the leading homogeneous element of $f$, i.e., $LH(f) = f_q$. Then every ideal $I$ of $K\langle X \rangle$ has the associated graded ideal $\langle LH(I) \rangle$ generated by the set of leading homogeneous elements $LH(I) = \{ LH(f) \mid f \in I \}$.

3.1. Definition Let $I$ be an arbitrary ideal of $K\langle X \rangle$. A subset $G$ of $I$ is said to be a standard basis for $I$, if $\langle LH(I) \rangle = \langle LH(G) \rangle$.

3.2. Proposition With respect to the fixed weighted $\mathbb{N}$-graded $K$-algebra structure $K\langle X \rangle = \oplus_{q \in \mathbb{N}} K\langle X \rangle_q$, let $K\langle X \rangle$ be equipped with the $\mathbb{N}$-grading filtration $FK\langle X \rangle = \{ F_q K\langle X \rangle \}_{q \in \mathbb{N}}$, where for each $q \in \mathbb{N}$, $F_q K\langle X \rangle = \oplus_{k \leq q} K\langle X \rangle_k$, and let $I$ be an arbitrary ideal of $K\langle X \rangle$. For a subset $G$ of $I$, the following statements are equivalent.

(i) $G$ is a standard basis of $I$;
(ii) Every nonzero element $f \in I$ has a representation

$$f = \sum_{i,j} \lambda_{ij} u_{ij} g_{ij}, \quad \lambda_{ij} \in K, \quad u_{ij}, g_{ij} \in B,$$

satisfying $d_{gr}(LH(u_{ij} g_{ij})) \leq d_{gr}(LH(f))$ for all $\lambda_{ij} \neq 0$;

(iii) Let $d_{gr}(g_j) = q_j, \quad g_j \in G$. Considering the induced filtration $FI = \{ F_q I \}_{q \in \mathbb{N}}$ of $I$ with $F_q I = I \cap F_q K\langle X \rangle$, we have

$$F_q I = \sum_{g_j \in G} \left( \sum_{k_i + q_j + k_j \leq q} F_{k_i} K\langle X \rangle g_j F_{k_j} K\langle X \rangle \right), \quad q \in \mathbb{N}.$$

Proof This is referred to the proof of ([LWZ], Lemma 2.2.3). □

By Proposition 3.2 it is clear that every standard basis $G$ of $I$ is certainly a generating set of $I$. By Definition 3.1 it is also clear that if $I$ is a graded ideal of $K\langle X \rangle$, then any homogeneous generating set $G$ of $I$ is trivially a standard basis of $I$. Nevertheless, we shall continue our discussion below for arbitrary ideals. Moreover, we specify the following

Remark As one may see from the literature on computational commutative algebra (e.g. see [KR]), if $A = K[x_1, \ldots, x_n]$ is the commutative polynomial $K$-algebra in $n$ variables, then a stan-
standard basis for an ideal \( I \) of \( A \) is nothing but the well-known Macaulay basis. While in the non-commutative case, for two-sided ideals of a \( \Gamma \)-filtered algebra \( A \), where \( \Gamma \) is an ordered semigroup with respect to a well-ordering, standard bases were introduced in [Gol] by using the induced filtration and the associated graded ideals. When a weighted \( \mathbb{N} \)-gradation \( K \langle X \rangle = \bigoplus_{q \in \mathbb{N}} K \langle X \rangle_q \) is fixed for the free algebra \( K \langle X \rangle = K \langle X_1, \ldots, X_n \rangle \), and furthermore \( K \langle X \rangle \) is equipped with the \( \mathbb{N} \)-grading filtration \( F K \langle X \rangle = \{ F_q K \langle X \rangle \}_{q \in \mathbb{N}} \), where \( F_q K \langle X \rangle = \oplus_{k \leq q} K \langle X \rangle_k \), the definition of a standard basis in the sense of [Gol] is then turned out to be Definition 3.1 above by Proposition 3.2. In this case, if \( G \) is a standard basis of an ideal \( I \) in \( K \langle X \rangle \) and if the quotient algebra \( A = K \langle X \rangle / I \) is equipped with the filtration \( FA \) induced by \( F K \langle X \rangle \), then the \( \mathbb{N} \)-filtered algebra \( A \) has the associated graded algebra \( G(A) \cong K \langle X \rangle / \langle \text{LH}(G) \rangle \). So, among other applications, the structure of standard bases for ideals of \( K \langle X \rangle \) plays an important role in the study of general PBW theory and the study of homogeneous and inhomogeneous Koszul algebras. On this aspect one may refer to ([Li2], Chapter 4) for more details.

Actually as in the commutative case with a Macaulay basis, we have the following

3.3. Proposition Let \( \prec_{gr} \) be a graded monomial ordering on \( B \) as defined in the beginning of this section, and let \( I \) be an ideal of \( K \langle X \rangle \). If \( G \) is a Gröbner basis for \( I \) with respect to \( \prec_{gr} \), then \( G \) is a standard basis for \( I \) in the sense of Definition 3.1, i.e., \( \langle \text{LH}(I) \rangle = \langle \text{LH}(G) \rangle \).

\( \square \)

Let \( I \) be an ideal of \( K \langle X \rangle \). If any proper subset of a standard basis \( G \) of \( I \) cannot be a standard basis for \( I \), then \( G \) is called a minimal standard basis. By Definition 3.1 it is clear that a subset \( G \) of \( I \) is a minimal standard basis for \( I \) if and only if \( \text{LH}(G) \) is a minimal homogeneous generating set of the graded ideal \( \langle \text{LH}(I) \rangle \). Thus, as with minimal homogeneous generating sets for graded ideals, minimal standard bases have the following properties:

1. any two minimal standard bases of \( I \) have the same number of generators; and
2. any two minimal standard bases of \( I \) contain the same number of leading homogeneous elements of degree \( n \) for all \( n \in \mathbb{N} \).

Now, it follows from Proposition 3.3 and Theorem 2.8 that we are able to give the main result of this section.

3.4. Theorem Let \( \prec_{gr} \) be a graded monomial ordering on \( B \) as defined in the beginning of this section, and let \( I \) be an ideal of \( K \langle X \rangle \). If \( G = \{ g_1, \ldots, g_m \} \) is a finite Gröbner basis for \( I \) with respect to \( \prec_{gr} \), then a minimal standard basis of \( I \) can be computed by following the steps below:

\textbf{Step 1.} With the initial input data \( F = \{ \text{LH}(g_1), \ldots, \text{LH}(g_m) \} \), run Algorithm 3 to compute a minimal homogeneous generating set \( F_{\min} \) for the graded ideal \( \langle \text{LH}(I) \rangle \), say \( F_{\min} = \{ \text{LH}(g_{j_1}), \ldots, \text{LH}(g_{j_s}) \} \).
Step 2. Write down $G = \{g_{j_1}, \ldots, g_{j_s}\}$, that is a minimal standard basis of $I$.

□

It follows from Corollary 2.9 and Theorem 3.4 that we have also the following

3.5. Corollary Let $I$ be an ideal of $K\langle X \rangle$ and let $G = \{g_1, \ldots, g_m\}$ be a finite Gröbner basis of $I$ with respect to a graded monomial ordering $\prec_{\text{gr}}$ on $\mathcal{B}$. If $G$ is a minimal Gröbner basis and $d_{\text{gr}}(\text{LH}(g_1)) = d_{\text{gr}}(\text{LH}(g_2)) = \cdots = d_{\text{gr}}(\text{LH}(g_m)) = n_0$, then $G$ is a minimal standard basis for $I$.

□

Finally, in the light of Gröbner basis theory for path algebras (i.e. quiver algebras) [Gr], we remark that the results obtained in this paper hold true for path algebras defined by finite directed graphs.

References

[CDNR] A. Capani, G. De Dominicis, G. Niesi, and L. Robbiano, Computing minimal finite free resolutions. Journal of Pure and Applied Algebra, (117& 118)(1997), 105 – 117.

[Coc] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra. Available at http://cocoa.dima.unige.it

[Gol] E. S. Golod, Standard bases and homology, in: Some Current Trends in Algebra, (Varna, 1986). Lecture Notes in Mathematics, Vol. 1352, Springer-Verlag, 1988, 88-95.

[Gr] E. L. Green, Noncommutative Grobner bases and projective resolutions, in: Proceedings of the Euroconference Computational Methods for Representations of Groups and Algebras, Essen, 1997, (Michler, Schneider, eds). Progress in Mathematics, Vol. 173, Basel, Birkhauser Verlag, 1999, 29–60.

[KR] M. Kreuzer, L. Robbiano, Computational Commutative Algebra 2. Springer, 2005.

[Li1] H. Li, On monoid graded local rings. Journal of Pure and Applied Algebra, 216(2012), 2697 – 2708.

[Li2] H. Li, Gröbner Bases in Ring Theory. World Scientific Co., 2011.

[LWZ] H. Li, Y. Wu and J. Zhang, Two applications of noncommutative Gröbner bases. Annali dell’Università di Ferrara. Sezione 7: Scienze matematiche, 45(1)(1999), 1-24.

[Mor] T. Mora, An introduction to commutative and noncommutative Gröbner bases. Theoretic Computer Science, 134(1994), 131–173.