HOMOGENIZATION OF PERIODIC MULTI-DIMENSIONAL STRUCTURES: THE LINEARLY ELASTIC/PERFECTLY PLASTIC CASE

NADIA ANSINI AND FRANÇOIS BILLE EBOSISSE

Abstract. In this paper we study the asymptotic behaviour via Γ-convergence of some integral functionals $F_{\varepsilon}$ which model some multi-dimensional structures and depend explicitly on the linearized strain tensor. The functionals $F_{\varepsilon}$ are defined in particular classes of functions with bounded deformation while the limit problem is set in the usual framework of Sobolev spaces or $BD(\Omega)$. We also construct an example of such functionals showing that under some special assumptions we can have non local effects.

1 Introduction

In recent years there has been an increasing interest in the description of media whose microscopic behaviour takes into account lower dimensional or multi-dimensional structures and can be modeled by suitable integral functionals with respect to periodic measures (see [9], [22], [1], [6], [4]). Zhikov studied in [22] the homogenization of functionals $F_{\varepsilon}$ defined as

$$F_{\varepsilon}(u, \Omega) = \int_{\Omega} f\left(\frac{x}{\varepsilon}, \nabla u\right) d\mu_{\varepsilon}$$

on $C^{\infty}(\Omega; \mathbb{R}^m)$, where $\mu_{\varepsilon}$ is defined by $\mu_{\varepsilon}(B) := \varepsilon^n \mu(\frac{1}{\varepsilon}B)$ with $\mu$ a fixed 1-periodic Radon measure and $f$ is a Borel function 1-periodic in the first variable (see also Braides and Chiadò Piat [1] for the case $\mu = \chi_E$ with $E$ periodic, and Bouchitté, Buttazzo and Seppecher [4] for relaxation results in the case of general $\mu$). On
the other hand, following the approach of Ambrosio, Buttazzo and Fonseca [1],
which is somehow complementary to the “smooth approach” described above,
Ansini, Braides and Chiod` o Piat studied in [4] the asymptotic behaviour of energy
functionals concentrated on periodic multi-dimensional structures, of the form
\[
F_\varepsilon(u, \Omega) = \int_{\Omega} f \left( \frac{x}{\varepsilon}, \frac{Du}{d\mu_\varepsilon} \right) d\mu_\varepsilon.
\]

In this case the problem is set in the framework of Sobolev spaces \( W^{1,p}_{\mu_\varepsilon}(\Omega; \mathbb{R}^m) \) with respect to the measure \( \mu_\varepsilon \) of [1]. We recall that \( W^{1,p}_{\mu_\varepsilon}(\Omega; \mathbb{R}^m) \) is the space of functions \( u \in L^p(\Omega; \mathbb{R}^m) \) whose distributional derivative is a measure absolutely continuous with respect to \( \mu_\varepsilon \) with \( p \)-summable density \( dDu/d\mu_\varepsilon \). A homogeniza-
tion theorem for \( F_\varepsilon \) has been proved under a standard growth condition of order \( p \) on \( f \) and a notion of \( p \)-homogenizability introduced for the measure \( \mu \) (see [4] Theorem 3.5).

In the context of linear elasticity or perfect plasticity, in place of con-
sidering energies depending on the deformation gradient \( Du \), it is more appropriate to
consider energy functionals depending explicitly on the linearized strain tensor \( E_u \). Our goal in this paper is to study the asymptotic behaviour of functionals of the type
\[
F_\varepsilon(u, \Omega) = \int_{\Omega} f \left( \frac{x}{\varepsilon}, \frac{dE_u}{d\mu_\varepsilon} \right) d\mu_\varepsilon
\]
defined in a particular class of functions with bounded deformation denoted by
\( \text{LD}^p_{\mu_\varepsilon}(\Omega) \) (introduced in Section 3). More precisely, \( \text{LD}^p_{\mu_\varepsilon}(\Omega) \) is the space of func-
tions \( u \in L^p(\Omega; \mathbb{R}^n) \), whose deformation tensor \( E_u \) is a measure absolutely continuous with respect to \( \mu_\varepsilon \) with \( p \)-summable density \( dE_u/d\mu_\varepsilon \). Using both classical and fine properties of functions with bounded deformation and the same assump-
tions as in [4] with a modified definition of \('p\)-homogenizable measure’, we prove in the first part of the paper, a homogenization theorem (Theorem 5.1). Precisely,
we show the existence of the \( \Gamma \)-limit of the functionals \( F_\varepsilon \) with respect to \( L^p \-
convergence in the Sobolev space \( W^{1,p}(\Omega; \mathbb{R}^m) \), and with respect to \( L^1 \)-convergence in \( BD(\Omega) \) (the space of functions with bounded deformation in \( \Omega \), that is the space of functions \( u \in L^1(\Omega; \mathbb{R}^n) \) whose deformation tensor \( E_u \) is a Radon measure with finite total variation in \( \Omega \), see [2]). We show that the \( \Gamma \)-limit admits an integral representa-
tion
\[
F_{\text{hom}}(u, \Omega) = \int_{\Omega} f_{\text{hom}}(E_u) \, dx
\]
in \( W^{1,p}(\Omega; \mathbb{R}^n) \); moreover, if \( f \) is convex then
\[
F_{\text{hom}}(u, \Omega) = \int_{\Omega} f_{\text{hom}}(E_u) \, dx + \int_{\Omega} f_\infty \left( \frac{dE^s_u}{d|E^s_u|} \right) \, d|E^s_u|
\]
in \( BD(\Omega) \), where \( E_u \) is the density of the absolutely continuous part and \( E^s_u \) is the singular part of \( E_u \) with respect to the Lebesgue measure; \( f_{\text{hom}} \) is described.
by an asymptotic formula and $f_{\text{hom}}^\infty$ denotes the recession function of $f_{\text{hom}}$ (see (5)).

In the second part of this paper we show that when the scaling argument leading to the functionals $F_\varepsilon$ does not apply, non local effects can arise. More precisely, we consider functionals of the type

$$F_\varepsilon^\gamma(u, \Omega) = \varepsilon^\gamma \int_\Omega f\left(\frac{x}{\varepsilon}, \frac{dE u}{d\mu_\varepsilon}\right) d\mu_\varepsilon,$$

which in the previous approach tend to the null functional when $\gamma > 0$, and we construct an explicit example showing that, with a suitable choice of $\gamma$, $\mu_\varepsilon$ and of the convergence with respect to which the $\Gamma$-limit is computed, we have a limit functional of a non local nature.

2 Notation and preliminaries

In the sequel $\mathbb{M}^{n \times n}$ stands for the space of $n \times n$ matrices and $\mathbb{M}^{n \times n}_{\text{sym}}$ for the space of $n \times n$ symmetric matrices. The letter $c$ will stand for an arbitrary fixed strictly-positive constant independent of the parameters under consideration, whose value may vary from line to line. The symbols $(\cdot, \cdot)$ and $|\cdot|$ stand for the Euclidean scalar product and the Euclidean norm. The Hausdorff $k$-dimensional measure and the Lebesgue measure in $\mathbb{R}^n$ are denoted by $H^k$ and $L^n$ respectively. We write $|E|$ for the Lebesgue measure $L^n$ of $E$.

Given a matrix-valued measure $\mu$ on $\Omega$, we adopt the notation $|\mu|$ for its total variation (see Federer [14]). The measure $\mu L F$ is defined by $(\mu L F)(B) = \mu(B \cap F)$. We write $\mu \ll \lambda$ to mean that the measure $\mu$ is absolutely continuous with respect to the positive measure $\lambda$. We denoted by $\frac{d\mu}{d\lambda}$ the Radon-Nikodym derivative of $\mu$ with respect to $\lambda$.

$L^p_\lambda(\Omega; \mathbb{R}^N)$ stands for the usual Lebesgue space of $p$-summable $\mathbb{R}^N$-valued functions with respect to $\lambda$. If $u \in L^1(\Omega; \mathbb{R}^n)$ then $Du$ denotes its distributional gradient. We say that $u \in L^1(\Omega; \mathbb{R}^n)$ is a function of bounded variation, and we write $u \in BV(\Omega; \mathbb{R}^n)$, if all its distributional first derivatives $D_i u_j$ are Radon measures with finite total variation in $\Omega$; we denote by $Du$ the $\mathbb{M}^{n \times n}$-valued measure whose entries are $D_i u_j$.

We will use the following notion of Sobolev space with respect to a measure $\lambda$, which is a finite Borel positive measure on $\Omega$, introduced by Ambrosio, Buttazzo and Fonseca [1]

$$W^{1,p}_\lambda(\Omega; \mathbb{R}^n) = \left\{ u \in L^p(\Omega; \mathbb{R}^n) : u \in BV(\Omega; \mathbb{R}^n), Du \ll \lambda, \frac{dDu}{d\lambda} \in L^p_\lambda(\Omega; \mathbb{M}^{n \times n}) \right\}$$

for all $1 \leq p \leq +\infty$. 
Let \( u \in L^1(\Omega; \mathbb{R}^n) \), and let \( Eu \) be the symmetric part of the distributional gradient of \( u \); i.e.,

\[
Eu := E_{ij}u, \quad E_{ij}u := \frac{1}{2}(D_iu_j + D_ju_i).
\]

The space \( LD(\Omega) \) is defined as the set of all functions \( u \in L^1(\Omega; \mathbb{R}^n) \) such that \( E_{ij}u \in L^1(\Omega) \) for any \( i, j = 1, \ldots, n \).

We say that \( u \in L^1(\Omega; \mathbb{R}^n) \) is a function with bounded deformation, and we write \( u \in BD(\Omega) \), if \( E_{ij}u \) is a Radon measure with finite total variation in \( \Omega \) for any \( i, j = 1, \ldots, n \).

For every \( u \in BD(\Omega) \) we consider the Radon-Nikodym decomposition of \( Eu \), with respect to the Lebesgue measure \( \mathcal{L}^n \), into a singular part \( Esu \) and an absolutely continuous part \( Ea = Eu \mathcal{L}^n \), with density \( \frac{dEu}{d\mathcal{L}^n} \).

We say that \( x \in \Omega \) belongs to \( J_u \), the jump set of \( u \), if and only if there exist a unit normal \( \nu \in S^{n-1} \) and two vectors \( a \) and \( b \) in \( \mathbb{R}^n \) such that

\[
\lim_{\rho \to 0^+} \frac{1}{\rho^n} \int_{B^+_\rho(x,\nu)} |u(y) - a| \, dy = 0
\]

\[
\lim_{\rho \to 0^+} \frac{1}{\rho^n} \int_{B^-_\rho(x,\nu)} |u(y) - b| \, dy = 0
\]

where \( B^\pm_\rho(x,\nu) = \{ y \in B_\rho(x) : (y - x, \pm \nu) > 0 \} \) and \( B_\rho(x) \) is the open ball of center \( x \) and radius \( \rho \). The triplet \( (a, b, \nu) \) is uniquely determined up to a change of sign of \( \nu \) and a permutation of \( (a, b) \). For every \( x \in J_u \) we define \( u^+(x) = a \), \( u^-(x) = b \) and \( \nu_u(x) = \nu \). The singular part \( Esu \) can be written as the sum of \( Esu \big|_{J_u} \) and of \( Esu \big|_{\Omega \setminus J_u} \); the first part, called the jump part, can be represented by

\[
E^su \big|_{J_u} = (u^+ - u^-) \otimes \nu_u \mathcal{H}^{n-1} \mathcal{L} J_u \tag{1}
\]

while the second part, called the Cantor part, vanishes on any Borel set which is \( \sigma \)-finite with respect to \( \mathcal{H}^{n-1} \) (see [2] Remark 4.2 and Proposition 4.4). We call intermediate topology on \( BD(\Omega) \) that defined by the distance

\[
\|u - v\|_{L^1(\Omega; \mathbb{R}^n)} + |Eu(\Omega) - Ev(\Omega)|. \tag{2}
\]

If \( u \in L^1(\Omega; \mathbb{R}^n) \) is such that \( E_{ij}u \in L^p(\Omega) \) for any \( i, j = 1, \ldots, n \) and \( \Omega \) has a locally Lipschitz boundary, then we have Korn’s inequality for all \( 1 < p < +\infty \)

\[
\sum_{i,j=1}^n \int_\Omega |D_iu_j(x)|^p \, dx \leq c \int_\Omega \left( |u(x)|^p + |Eu(x)|^p \right) \, dx; \tag{3}
\]

hence, this space is none other than \( W^{1,p}(\Omega; \mathbb{R}^n) \) (see Chapter 1, Section 1 in [20]). For a general exposition of the theory of functions of bounded deformation we refer to [18], [19], [16], [17], [3], [2], [20], [2].
If \( u \in L^1(\Omega; \mathbb{R}^n) \), we denote by \( \tilde{u} \) the \textit{precise representative} of \( u \), whose components are defined by
\[
\tilde{u}_i(x) = \limsup_{\rho \to 0^+} \int_{B_{\rho}(x)} u_i(y) \, dy.
\] (4)

Let \( f : \mathbb{R}^k \to [0, +\infty] \) be a convex function. We define the recession function \( f^\infty \) of \( f \) as
\[
f^\infty(\xi) = \lim_{t \to \infty} \frac{f(t\xi)}{t}
\]
for every \( \xi \in \mathbb{R}^k \). (5)

It is well-known (see, for instance, [11]) that this limit exists, and defines a convex, subadditive and positively homogeneous of degree one function.

We recall the definition of De Giorgi’s \( \Gamma \)-convergence in \( L^p \) spaces. Given a family of functionals \( F_j : L^p(\Omega; \mathbb{R}^n) \to [0, +\infty] \), \( j \in \mathbb{N} \), for \( u \in L^p(\Omega; \mathbb{R}^n) \), we define
\[
\Gamma(L^p) \liminf_{j \to +\infty} F_j(u) = \inf \left\{ \liminf_{j \to +\infty} F_j(u_j) : u_j \rightharpoonup^L u \right\},
\]
and
\[
\Gamma(L^p) \limsup_{j \to +\infty} F_j(u) = \inf \left\{ \limsup_{j \to +\infty} F_j(u_j) : u_j \rightharpoonup^L u \right\};
\]
if these two quantities coincide then their common value is called the \( \Gamma \)-limit of the sequence \( (F_j) \) at \( u \), and is denoted by \( \Gamma(L^p) \lim_{j \to +\infty} F_j(u) \). It is easy to check that \( l = \Gamma(L^p) \lim_{j \to +\infty} F_j(u) \) if and only if
(a) for every sequence \( (u_j) \) converging to \( u \) we have
\[
l \leq \liminf_{j \to +\infty} F_j(u_j);
\]
(b) there exists a sequence \( (u_j) \) converging to \( u \) such that
\[
l \geq \limsup_{j \to +\infty} F_j(u_j).
\]

We say that \( (F_\varepsilon) \) \( \Gamma(L^p) \)-converges to \( l \) at \( \varepsilon \) as \( \varepsilon \to 0 \) if for every sequence of positive numbers \( (\varepsilon_j) \) converging to \( 0^+ \) there exists a subsequence \( (\varepsilon_{j_k}) \) for which we have
\[
l = \Gamma(L^p) \lim_{k \to +\infty} F_{\varepsilon_{j_k}}(u).
\]

We recall that the \( \Gamma \)-upper and lower limits defined above are \( L^p \)-lower semi-continuous functions. For a comprehensive study of \( \Gamma \)-convergence we refer to [12] and [8], while a detailed analysis of some of its applications to homogenization theory can be found in [11].
3 The space $LD^p_\lambda(\Omega)$

In this section we define the analog of $W^{1,p}_\lambda(\Omega; \mathbb{R}^n)$ when the gradient is replaced by the linearized strain tensor.

**Definition 3.1** Let $\lambda$ be a finite Borel positive measure on the open set $\Omega \subset \mathbb{R}^n$, and let $1 \leq p \leq +\infty$. We define the space

$$LD^p_\lambda(\Omega) = \left\{ u \in L^p(\Omega; \mathbb{R}^n) : u \in BD(\Omega), Eu << \lambda, \frac{dEu}{d\lambda} \in L^p_\lambda(\Omega; \mathbb{M}_{sym}^{n \times n}) \right\}.$$  \hfill (6)

We will use the notation $LD_\lambda(\Omega)$ instead of $LD^1_\lambda(\Omega)$.

**Proposition 3.2** (i) The spaces $LD^p_\lambda(\Omega)$ and $LD^p_{\lambda'}(\Omega)$ coincide whenever $|\lambda - \lambda'|(\Omega \setminus B) = 0$ for some $\mathcal{H}^{n-1}$-negligible Borel subset $B$ of $\Omega$.

(ii) The measure $\lambda$ in Definition 3.1 can always be assumed concentrated on a Borel set where its $(n-1)$-dimensional upper density is finite.

**Proof.** Point (i) easily follows from the fact that BD functions do not charge $\mathcal{H}^{n-1}$-negligible sets (see Remark 3.3 in [2]). Point (ii) follows from Remark 2.3 in [1]. \hfill $\square$

In the following proposition we prove a Leibniz-type formula for the densities with respect to a measure $\lambda$. This formula will be used in the proof of the fundamental estimate, Proposition 5.3.

**Proposition 3.3** If $u \in LD^p_\lambda(\Omega)$, $v \in W^{1,\infty}_\lambda(\Omega)$ and $\tilde{u} \circ \lfloor Dv \rfloor d\lambda_\lambda \in L^1_\lambda(\Omega; \mathbb{M}_{sym}^{n \times n})$ then $uv \in LD^p_\lambda(\Omega)$, and

$$\frac{dE(uv)}{d\lambda} = \frac{dEu}{d\lambda} + \tilde{u} \circ \lfloor Dv \rfloor d\lambda.$$  \hfill (7)

**Proof.** By definition, functions in $LD^p_\lambda(\Omega)$ have bounded deformation. Using the characterization of the spaces $BV(\Omega)$ and $BD(\Omega)$ by means of one-dimensional sections (see Proposition 3.2 in [2]) we have

$$u^\xi_y \in BV(\Omega^\xi_y), \quad v_{y,\xi} \in BV(\Omega^\xi_y) \quad \mathcal{H}^{n-1}$$. a.e. \quad y \in \Omega^\xi$$

where

$$u^\xi_y(t) = u^\xi(y + t\xi) = (u(y + t\xi), \xi), \quad v_{y,\xi}(t) = v(y + t\xi) \quad \forall t \in \Omega^\xi_y.$$ 

Hence by the chain rule formula for $BV$ functions (see [7] Section 1.8, [8] Theorem 3.93 and Example 3.94) we have

$$(uv)^\xi_y = u^\xi_y v_{y,\xi} \in BV(\Omega^\xi_y)$$

and

\[ D(u^\xi y, v^\xi y) = \tilde{v}^\xi y D_u^\xi y + \tilde{u}^\xi y D_v^\xi y \quad H^{n-1} - \text{a.e.} \quad y \in \Omega^\xi. \]

By Proposition 3.2 in [2] and by the structure theorem for BV functions (see [3] Section 1.8), we can prove that \( uv \in BD(\Omega) \) and

\[ (Euv\xi, \xi) = (\tilde{v} Eu\xi, \xi) + (\tilde{u} \odot Dv\xi, \xi) \quad \forall \xi \in \mathbb{R}^n. \]

By choosing \( \xi = \xi_i + \xi_j \), where \( \xi_1, \ldots, \xi_n \) is a basis of \( \mathbb{R}^n \), we get

\[ E(uv) = \tilde{v} Eu + \tilde{u} \odot Dv. \quad (8) \]

Since the measures in the left hand-side of (8) are absolutely continuous with respect to \( \lambda \) with densities in \( L^p(\Omega; M_{sym}^{n \times n}) \), we finally get \( uv \in LD^p(\Omega) \) and (7) is proved.

**Remark 3.4** Note that in (7) it is necessary to consider the precise representatives of \( u \) and \( v \), since the measure \( \lambda \) may take into account also sets of zero Lebesgue measure.

### 4 Choice of the measure and some examples

Let \( \mu \) be a non-zero positive Radon measure on \( \mathbb{R}^n \) which is 1-periodic; i.e.,

\[ \mu(B + e_i) = \mu(B) \]

for all Borel subsets \( B \) of \( \mathbb{R}^n \) and for all \( i = 1, \ldots, n \). We will assume the normalization

\[ \mu([0, 1)^n) = 1. \quad (9) \]

For all \( \varepsilon > 0 \) we define the \( \varepsilon \)-periodic positive Radon measure \( \mu_\varepsilon \) by

\[ \mu_\varepsilon(B) = \varepsilon^n \mu\left( \frac{1}{\varepsilon} B \right) \quad (10) \]

for all Borel sets \( B \). Note that by (9) the family \( (\mu_\varepsilon) \) converges locally weakly* in the sense of measures to the Lebesgue measure as \( \varepsilon \to 0 \).

In the sequel \( f : \mathbb{R}^n \times M^{n \times n} \to [0, +\infty) \) will be a fixed Borel function 1-periodic in the first variable and satisfying the growth condition of order \( p \geq 1 \): there exist \( 0 < \alpha \leq \beta \) such that

\[ \alpha |A|^p \leq f(x, A) \leq \beta (1 + |A|^p) \quad (11) \]

for all \( x \in \mathbb{R}^n \) and \( A \in M^{n \times n} \).

For every bounded open set \( \Omega \), we define the functionals at scale \( \varepsilon > 0 \) as

\[ F_\varepsilon(u, \Omega) = \begin{cases} \int_{\Omega} f\left( \frac{x}{\varepsilon}, \frac{dEu}{d\mu_\varepsilon} \right) d\mu_\varepsilon & \text{if } u \in LD^p_{\mu_\varepsilon}(\Omega) \\ +\infty & \text{otherwise.} \end{cases} \quad (12) \]
Now we consider some additional assumptions on the measure \( \mu \), in order to prove the existence and the integral representation of the \( \Gamma \)-limit of the functionals \( F_\varepsilon \) as \( \varepsilon \to 0 \). In the sequel we will point out that these conditions are necessary and sufficient.

We assume:

(i) \textit{(existence of cut-off functions)} there exist \( K > 0 \) and \( \delta > 0 \) such that for all \( \varepsilon > 0 \), for all pairs \( U, V \) of open subsets of \( \mathbb{R}^n \) with \( U \subset \subset V \), and \( \text{dist}(U, \partial V) \geq \delta \varepsilon \), and for all \( u \in \text{LD}^p_{\mu_\varepsilon}(V) \) there exists \( \phi \in W^{1,\infty}(V) \) with \( 0 \leq \phi \leq 1 \), \( \phi = 1 \) on \( U \), \( \phi = 0 \) in a neighbourhood of \( \partial V \), such that

\[
\int_V \left| \frac{dD\phi}{d\mu_\varepsilon} \otimes \tilde{u} \right|^p \, d\mu_\varepsilon \leq \frac{K}{(\text{dist}(U, \partial V))^p} \int_{V \setminus U} |u|^p \, dx .
\] (13)

Such a \( \phi \) will be called a \textit{cut-off function between} \( U \) and \( V \);

(ii) \textit{(existence of periodic test-functions)} for all \( i, j = 1, \ldots, n \), there exists \( z_{ij} \in \text{LD}^p_{\mu,\text{loc}}(\mathbb{R}^n) \) such that \( x \mapsto z_{ij}(x) - x_j e_i \) is 1-periodic.

\textbf{Remark 4.1} Note that if \( \mu \) is \( p \)-homogenizable in the sense of Definition 3.2 in [4]; i.e., if there exists \( z_i \in W^{1,p}_{\mu,\text{loc}}(\mathbb{R}^n) \) such that \( x \mapsto z_i(x) - x_i \) is 1-periodic, then the functions \( z_{ij} = z_j e_i \) trivially satisfy the condition (ii) above but the converse is not true.

\textbf{Remark 4.2} Note that the Lebesgue measure trivially satisfies properties (i), (ii). Note that property (i) depends on \( \mu \) and \( p \).

We consider in our context the measure \( \mu \) of Examples 3.1(a) and (b) in [4].

\textbf{Example 4.3} (\textit{Perfectly-rigid bodies connected by springs.})

We consider

\[ E = \{ y \in \mathbb{R}^n : \exists i \in \{1, \ldots, n\} \text{ such that } y_i \in \mathbb{Z} \}, \]

that is, the union of all the boundaries of cubes \( Q_i = i + (0,1)^n \) with \( i \in \mathbb{Z}^n \). \( E \) is an \((n-1)\)-dimensional set in \( \mathbb{R}^n \). We set

\[ \mu(B) = \frac{1}{n} \mathcal{H}^{n-1}(B \cap E) \]

for all Borel sets \( B \). For every \( \varepsilon > 0 \) we have

\[ \mu_\varepsilon(B) = \frac{1}{n} \varepsilon \mathcal{H}^{n-1}(B \cap \varepsilon E) . \]

If \( u \in \text{LD}^p_{\mu_\varepsilon}(\Omega) \) then \( EU = 0 \) on every connected component of each \( \varepsilon Q_i \cap \Omega \), so in this case \( \text{LD}^p_{\mu_\varepsilon}(\Omega) \) consists of functions which are rigid displacements on these
sets; i.e., \( u_i = R_ix + c_i \) on each \( \epsilon Q_i \cap \Omega \) with \( R_i \) a \( n \times n \) skew symmetric matrix, and \( c_i \in \mathbb{R}^n \). Hence by (3), we have

\[
\frac{dEu}{d\mu} = \frac{n}{\epsilon} \frac{dEu}{d\mathcal{H}^{n-1}} = \frac{n}{\epsilon} (u_i - u_j) \odot (i - j) \text{ on } \partial(\epsilon Q_i) \cap \partial(\epsilon Q_j) \cap \Omega.
\]

In this case the functionals \( F_\epsilon \) take the form

\[
\epsilon \int_{\Omega \cap E} g \left( \frac{1}{\epsilon} \frac{dEu}{d\mathcal{H}^{n-1}} \right) d\mathcal{H}^{n-1}.
\]

Note that if \( \Omega \) is bounded then \( \text{LD}^p_{\mu_\epsilon}(\Omega) = \text{LD}^\infty_{\mu_\epsilon}(\Omega) \) for all \( p \) if the number of connected components of each \( \Omega \cap \epsilon Q_i \) is finite.

Comparing with Example 3.1(a) in [4], we get that \( W^{1,p}_{\mu_\epsilon}(\Omega; \mathbb{R}^n) \subset \text{LD}^p_{\mu_\epsilon}(\Omega) \).

The measure \( \mu \) satisfies the conditions (i) and (ii) for all \( p \geq 1 \). In fact, to prove (i) we consider the same cut-off function in Example 3.4(a) of [4]

\[
\phi(x) = 1 - \left( \frac{1}{C} \left[ \inf \{|x - y|_\infty : y \in U_\epsilon \} \right] \right) \wedge 1,
\]

where fixed \( \epsilon > 0 \), \( U_\epsilon = \bigcup \{ \epsilon Q_i : \epsilon Q_i \cap U \neq \emptyset \} \), \( |x - y|_\infty = \max_{1 \leq i \leq n} |x_i - y_i| \), and

\[
C = \left[ \inf \left\{|x - y|_\infty : x \in U_\epsilon, y \in \partial V \right\} \right] - 2
\]

(we denote \([t]\) the integer part of \( t \)). Note that \( |dD\phi/d\mu_\epsilon| \leq n/(C\epsilon) \leq c/\text{dist}(U, \partial V) \) for some constant \( c \) independent of \( U \) and \( V \).

Interpreting \( u_{\pm} \) as traces of Sobolev functions defined on each cube \( Q_i \), we have

\[
\left( \iint_{\partial Q_i} |u_{\pm}|^p d\mathcal{H}^{n-1} \right)^{1/p} \leq c \|u\|_{W^{1,p}(Q_i)},
\]

hence by a scaling argument and by Korn’s inequality (3)

\[
\left( \epsilon \iint_{\partial \epsilon Q_i} |u_{\pm}|^p d\mathcal{H}^{n-1} \right)^{1/p} \leq c \left( \iint_{\epsilon Q_i} |u|^p dx \right)^{1/p} + \frac{1}{\epsilon} \left( \iint_{\epsilon Q_i} |Eu|^p dx \right)^{1/p}
\]

\[
= c \left( \iint_{\epsilon Q_i} |u|^p dx \right)^{1/p}
\]

where \( c \) depends only on the cube. If \( p = 1 \) we can apply the trace inequality in \( \text{LD}(Q_i) \)

\[
\int_{\partial Q_i} |u_{\pm}| d\mathcal{H}^{n-1} \leq c \int_{Q_i} |u| dx + |Eu|(Q_i),
\]

so we get

\[
\epsilon \iint_{\partial \epsilon Q_i} |u_{\pm}|^\text{dH}^{n-1} \leq c \int_{\epsilon Q_i} |u| dx.
\]
Hence for all $p \geq 1$

$$\varepsilon \int_{\partial \varepsilon Q_i} |u|^p d\mathcal{H}^{n-1} \leq c \int_{\varepsilon Q_i} |u|^p dx.$$  

For two cubes

$$\varepsilon \int_{\partial \varepsilon Q_i \cap \partial \varepsilon Q_j} |\tilde{u}|^p d\mathcal{H}^{n-1} \leq \varepsilon \int_{\partial \varepsilon Q_i \cap \partial \varepsilon Q_j} (|u_i|^p + |u_j|^p) d\mathcal{H}^{n-1} \leq c \int_{\varepsilon Q_i \cup \varepsilon Q_j} |u|^p dx$$

so that

$$\int_V |D\phi \otimes \tilde{u}|^p d\mu \leq \frac{c^p \varepsilon}{\text{dist}(U, \partial V)^p} \int_{\varepsilon E \cap E^\circ \cap \text{spt} D\phi} |\tilde{u}|^p d\mathcal{H}^{n-1} \leq 2n \frac{c^p \varepsilon}{\text{dist}(U, \partial V)^p} \int_{V \setminus U} |u|^p dx.$$  

The proof of (i) is then complete. To verify (ii) we apply Remark 4.1 to Example 3.4(a) in [4] and take simply $z_{ij}(x) = [x_j]e_i$.

**Example 4.4 (Elastic media connected by springs).**

Let $E$ be as in the previous example and let

$$\mu(B) = \frac{1}{n + 1} \left( |B| + \varepsilon \mathcal{H}^{n-1}(E \cap B) \right)$$

$$\mu_{\varepsilon}(B) = \frac{1}{n + 1} \left( |B| + \varepsilon \mathcal{H}^{n-1}((\varepsilon E \cap \varepsilon B) \right).$$

In this case the functions in $\text{LD}_{\mu_{\varepsilon}}(\Omega)$ are functions whose restriction to each $\varepsilon Q_i \cap \Omega$ belongs to $W^{1,p}(\varepsilon Q_i \cap \Omega; \mathbb{R}^n)$ when $p > 1$ by the Korn’s inequality (3) (we suppose that $\varepsilon Q_i \cap \Omega$ has a locally Lipschitz boundary) and to $\text{LD}(\varepsilon Q_i \cap \Omega)$ when $p = 1$, while the difference of the traces on both sides of $\partial(\varepsilon Q_i) \cap \partial(\varepsilon Q_j) \cap \Omega$ is $p$-summable for every $i, j \in \mathbb{Z}^n$. Hence if we compare our case with Example 3.1(b) in [4], we can conclude that $W^{1,p}_{\mu_{\varepsilon}}(\Omega; \mathbb{R}^n) = \text{LD}_{\mu_{\varepsilon}}(\Omega)$ if $p > 1$ and $W^{1,1}_{\mu_{\varepsilon}}(\Omega; \mathbb{R}^n) \subset \text{LD}_{\mu_{\varepsilon}}(\Omega)$ if $p = 1$. The functionals $F_\varepsilon$ take the form

$$\frac{1}{n + 1} \int_{\Omega} f \left( \frac{x}{\varepsilon} \right) \frac{dE_u}{dx} \, dx + \varepsilon \int_{\Omega \cap \varepsilon E} g \left( \frac{x}{\varepsilon} \right) \frac{dE_u}{\varepsilon d\mathcal{H}^{n-1}} \, d\mathcal{H}^{n-1}.$$  

The measure $\mu$ satisfies conditions (i) and (ii) for all $p \geq 1$ by Example 3.4(b) in [4].

10
5 The homogenization theorem

The homogenization theorem for the functionals in (12) takes the following form.

**Theorem 5.1** Let $\mu$ be a measure which satisfies conditions (i) and (ii) in Section 4, and for every bounded open subset $\Omega$ of $\mathbb{R}^n$ let $F_\varepsilon(\cdot, \Omega)$ be defined on $L^p(\Omega; \mathbb{R}^n)$ by (12). Then the $\Gamma$-limit

$$F_{\text{hom}}(u, \Omega) = \Gamma(L^p) \lim_{\varepsilon \to 0} F_\varepsilon(u, \Omega)$$

exists for all bounded open subsets $\Omega$ with Lipschitz boundary and for all $u \in L^p(\Omega; \mathbb{R}^n)$; it can be represented on $W^{1,p}(\Omega; \mathbb{R}^n)$ for $p \geq 1$ as

$$F_{\text{hom}}(u, \Omega) = \int_\Omega f_{\text{hom}}(Eu) \, dx,$$

where the homogenized integrand satisfies the asymptotic formula

$$f_{\text{hom}}(A) = \lim_{k \to +\infty} \inf \left\{ \frac{1}{k^n} \int_{(0,k)^n} f(x, \frac{dEu}{d\mu}) \, d\mu : \right\}
\left. \quad u \in LD^p_{\mu, \text{loc}}(\mathbb{R}^n), \ u - Ax \ \text{ \ k-periodic} \right\}$$

for all $A \in \mathbb{M}^{n \times n}_{\text{sym}}$.

Moreover, $F_{\text{hom}}(u, \Omega) = +\infty$ if $p > 1$ and $u \in L^p(\Omega; \mathbb{R}^n) \setminus W^{1,p}(\Omega; \mathbb{R}^n)$, or if $u \in L^1(\Omega; \mathbb{R}^n) \setminus BD(\Omega)$ when $p = 1$.

Furthermore, if $f$ is convex then the $\Gamma$-limit can be represented as

$$F_{\text{hom}}(u, \Omega) = \int_\Omega f_{\text{hom}}(Eu) \, dx + \int_\Omega f_{\text{hom}}^\infty \left( \frac{dEu^*}{d|Eu^*|} \right) \, d|Eu^*|$$

for all $u \in BD(\Omega)$ when $p = 1$.

**Remark 5.2** Note that we cannot replace the sets $[0,k)^n$ by the sets $(0,k)^n$ if $\mu([0,k)^n \setminus (0,k)^n) \neq 0$, see Remark 3.6 in [4].

Same examples and considerations of Remarks 3.7 and 3.8 in [4], applied to our case, show that condition (ii) for the measure $\mu$ and the assumption that $\Omega$ has a Lipschitz boundary are necessary to get a homogenization theorem. In fact, if condition (ii) fails then $f_{\text{hom}}(A) = +\infty$ if $A \neq 0$; while if $\Omega$ does not have Lipschitz boundary then the equality (15) may not hold.

The following proposition is a usual tool to prove the existence of the $\Gamma$-limit and its integral representation (see [12] Chapter 18, [11] Chapter 11).
Proposition 5.3 (Fundamental Estimate) For every $\sigma > 0$ there exists $\varepsilon_\sigma$ and $M > 0$ such that for all $U, U', V$ open subsets of $\Omega$ with $U' \subset U$ and $\text{dist}(U', V \setminus U) > 0$, for all $\varepsilon < \varepsilon_\sigma \text{dist}(U', V \setminus U)$ and for all $u \in \text{LD}^p_{\mu_\varepsilon}(\Omega)$, $v \in \text{LD}^p_{\mu_\varepsilon}(\Omega)$ there exists a cut-off function between $U'$ and $U$, $\phi \in \text{W}^{1,\infty}(U \cup V)$, such that

$$F_\varepsilon(\phi u + (1 - \phi)v, U' \cup V) \leq (1 + \sigma)(F_\varepsilon(u, U) + F_\varepsilon(v, V)) + M \left(\frac{\varepsilon}{\text{dist}(U', V \setminus U)}\right)^p \int_{(U \cap V) \setminus U'} |u - v|^p \, dx + \sigma \mu_\varepsilon((U \cap V) \setminus U').$$

Proof. By taking (17) and condition (i) into account, the proof follows exactly that of Proposition 4.1 [4].

Proposition 5.4 For every $A \in \text{M}^{n \times n}_{\text{sym}}$ there exists $z_A \in \text{LD}^p_{\mu,\text{loc}}(\mathbb{R}^n)$ such that $z_A - Ax$ is 1-periodic and satisfies

$$\int_{[0,1)^n} |z_A|^p \, d\mu \leq c |A|^p.$$

Proof. Define $z_A = \sum_{i,j=1}^n A_{ij} z_{ij}$, where $z_{ij}$ are as in condition (ii). Inequality (18) is then trivial. \(\square\)

We fix $(\varepsilon_j)$ which goes to zero. We define

$$F'(u, U) = \Gamma(L^p)-\liminf_{j \to +\infty} F_{\varepsilon_j}(u, U)$$

$$F''(u, U) = \Gamma(L^p)-\limsup_{j \to +\infty} F_{\varepsilon_j}(u, U)$$

for all $u \in L^p(\Omega; \mathbb{R}^n)$ and for all open subsets $U$ of $\Omega$.

Proposition 5.5 (Growth Condition) We have for all open subsets $U$ of $\Omega$ with $|\partial U| = 0$

$$F''(u, U) \leq c \int_U (1 + |Eu|^p) \, dx$$

for all $u \in \text{W}^{1,p}(\Omega; \mathbb{R}^n)$ if $p > 1$ and

$$F''(u, U) \leq c(|U| + |Eu|(U))$$

for all $u \in BD(\Omega)$ if $p = 1$.

Proof. This Growth Conditions can be obtained modifying the proof of Proposition 4.3 in [4]. In particular in Step 2 therein now we have to consider the affine functions $u_i(x) = A_i x + c_i$ for some $A_i \in \text{M}^{n \times n}_{\text{sym}}$ and $c_i \in \mathbb{R}^n$, in Step 3 we just have to note that piecewise affine functions are dense in $BD$ endowed with the intermediate topology (2) (see [20] Theorem 3.2 Chapter 2 Section 3). \(\square\)
Proposition 5.6 There exists a subsequence of \((\varepsilon_j)\) (not relabeled) such that for all open subsets \(U\) of \(\Omega\) with \(|\partial U| = 0\) there exists the \(\Gamma\)-limit

\[
\Gamma- \lim_{j \to +\infty} F_{\varepsilon_j}(u, U) = F(u, U),
\]

for all \(u \in W^{1,p}(\Omega; \mathbb{R}^n)\) if \(p > 1\) and for all \(u \in BD(\Omega)\) if \(p = 1\). There exists a function \(\varphi : \mathbb{M}^{n \times n} \to \mathbb{R}\) such that

\[
F(u, U) = \int_U \varphi(Eu)dx
\]

for all \(u \in W^{1,p}(\Omega; \mathbb{R}^n)\) if \(p \geq 1\); moreover if \(f\) is convex

\[
F(u, U) = \int_U \varphi(Eu)dx + \int_U \varphi(\infty) \left( \frac{dE^s u}{|d|E^s u|} \right) d|E^s u|
\]

for all \(u \in BD(\Omega)\) if \(p = 1\).

Proof. To prove the existence of the \(\Gamma\)-limit on \(W^{1,p}(\Omega; \mathbb{R}^n)\) for \(p > 1\) and \(BD(\Omega)\) for \(p = 1\), and the integral representation of the \(\Gamma\)-limit

\[
F(u, U) = \int_U \varphi(Du)dx
\]

on \(W^{1,p}(\Omega; \mathbb{R}^n)\) when \(p \geq 1\), we repeat the proof of Proposition 4.4 using Propositions 5.3 and 5.5. Moreover, we can prove that \(\varphi(Du) = \varphi(Eu)\). In fact, let \(w_j \to Ax\) be such that

\[
F(Ax, \Omega) = \lim_{j \to +\infty} F_{\varepsilon_j}(w_j, \Omega)
\]

and let \(Rx + c\) be a rigid displacement, then

\[
F(Ax + Rx + c, \Omega) \leq \liminf_{j \to +\infty} F_{\varepsilon_j}(w_j + Rx + c, \Omega)
\]

\[
= \lim_{j \to +\infty} F_{\varepsilon_j}(w_j, \Omega) = F(Ax, \Omega)
\]

so that \(\varphi(A + R) \leq \varphi(A)\). The reverse inequality follows similarly, therefore for all \(R\) \((n \times n)\) skew-symmetric matrix

\[
\varphi(A + R) = \varphi(A)
\]

which implies \(\varphi(B) = \varphi(\frac{B + B^T}{2})\) for any \(B \in \mathbb{M}^{n \times n}\).

Let us prove the integral representation of the \(\Gamma\)-limit on \(BD(\Omega)\) whenever \(f\) is convex. We consider the functional defined in \(L^1_{\text{loc}}(\Omega; \mathbb{R}^n)\)

\[
G(u) = \begin{cases} 
\int_\Omega \varphi(Eu)dx & \text{if } u \in C^1(\Omega; \mathbb{R}^n) \\
+\infty & \text{otherwise},
\end{cases}
\]
and we introduce
\[ G(u, U) = \inf \left\{ \liminf_{h \to +\infty} G(u_h, U) : u_h \in C^1(\Omega; \mathbb{R}^n) \quad u_h \to u \quad \text{in} \quad L^1_{\text{loc}}(\Omega; \mathbb{R}^n) \right\} \]
the relaxed functional of \( G \). It is well known that \( \phi \) is convex and it is easy to check that \( \phi(A) \geq c|A| \) for every \( A \in \mathbb{M}^{n \times n}_{\text{sym}} \), hence by the lower semicontinuity and relaxation theorems for functionals of measures (see for instance \([15], [11]\)), we obtain
\[ G(u, U) = \int_U \varphi(\mathcal{E}u) \, dx + \int_U \varphi^\infty \left( \frac{dE^*u}{d\|E^*u\|} \right) d\|E^*u\| \]
for every \( u \in BD(\Omega) \) (see \([20]\) Section 5). Since \( F(\cdot, U) \leq G(\cdot, U) \) in \( W^{1,1}(\Omega; \mathbb{R}^n) \), by the lower semicontinuity of the \( \Gamma \)-limit we obtain
\[ F(u, U) \leq \int_U \varphi(\mathcal{E}u) \, dx + \int_U \varphi^\infty \left( \frac{dE^*u}{d\|E^*u\|} \right) d\|E^*u\| \]
for all \( u \in BD(\Omega) \). The reverse inequality is obtained by a convolution argument. In fact we consider \( U_k = \{ x \in U : d(x, \partial U) > \frac{1}{k} \} \), \( \rho_k \) with \( \text{spt} \rho_k \subset B(0, \frac{1}{k}) \) and \( u_k = u * \rho_k \). For \( y \in B(0, \frac{1}{k}) \) and \( k \) large enough we have that \( U_k \subset y + U \).

Since \( F(\cdot, U) \) is convex for all \( U \in A(\Omega) \) and \( F(u^y, U_k) \leq F(u, U) \) with \( u^y(x) = u(x - y) \), by Jensen’s inequality
\[ F(u \ast \rho_k, U_k) \leq F(u, U) \]
on the other hand, we also have
\[ \lim_{k \to +\infty} F(u_k, U_k) = G(u, U) \]
hence we can conclude that
\[ F(u, U) = \int_U \varphi(\mathcal{E}u) \, dx + \int_U \varphi^\infty \left( \frac{dE^*u}{d\|E^*u\|} \right) d\|E^*u\| \]
as desired. \( \square \)

**Proposition 5.7 (Homogenization Formula)** For all \( A \in \mathbb{M}^{n \times n}_{\text{sym}} \) there exists the limit in \((\mathbb{R})\) and we have \( \varphi(A) = f_{\text{hom}}(A) \).

**Proof.** It can be obtain repeating the proof of the Proposition 4.5 of \([4]\) but defining
\[ g_k(A) = \inf \left\{ \frac{1}{k^n} \int_{(0,k)^n} f(x, \frac{dE_u}{d\mu}) d\mu : u \in \text{LD}^p_{\mu, \text{loc}}(\mathbb{R}^n), \ u - Ax \ k\text{-periodic} \right\} \]
for all \( A \in \mathbb{M}^{n \times n}_{\text{sym}} \) and \( k \in \mathbb{N} \). \( \square \)
Proof of Theorem 5.1. It remains to check the coercivity of the \( \Gamma \)-limit. By the growth condition on \( f \) and a comparison argument, it is enough to prove this for \( f(A) = |A|^p \). We know that the \( \Gamma \)-limit \( F_{\text{hom}} \) exists for all \( u \in L^p(\Omega; \mathbb{R}^n) \) and for all sets \( R \) in the countable family \( \mathcal{R} \) of all finite unions of open rectangles of \( \Omega \) with rational vertices, in this case \( F_{\text{hom}} \) is also convex. For all \( U', U \in A(\Omega) \) such that \( U' \subset \subset U \) there exists \( R \in \mathcal{R} \) such that \( U' \subset \subset R \subset \subset U \). Reasoning as in the previous proof, for \( y \in B(0, \frac{1}{k}) \) and \( k \) large enough we have that \( R \subset y + U \) hence

\[
F_{\text{hom}}(u_k, R) \leq F'(u, U)
\]

and

\[
\liminf_{k \to +\infty} F_{\text{hom}}(u_k, U') \leq F'(u, U) \tag{19}
\]

with \( u_k = u * \rho_k \) (see [12] Chapter 23).

It will be enough then to prove that \( f_{\text{hom}}(A) \geq c|A|^p \). In fact for any \( u \in L^p(\Omega; \mathbb{R}^n) \setminus W^{1,p}(\Omega; \mathbb{R}^n) \) when \( p > 1 \) by (19) \( F'(u, U) \geq c \liminf_{k \to +\infty} \int_{U'} |Du_k|^p \, dx \) by the arbitrariness of \( U' \), we get \( F_{\text{hom}}(u, U) = +\infty \). Similarly, if \( p = 1 \) for all \( u \in L^1(\Omega; \mathbb{R}^n) \setminus BD(\Omega) \) we have \( |Eu|(\Omega) = +\infty \), let \( \Omega' \subset \subset \Omega \) we get by (19) that \( F'(u, \Omega) \geq c \liminf_{k \to +\infty} |Eu_k|(\Omega') \)

by arbitrariness of \( \Omega' \) we obtain \( F_{\text{hom}}(u, \Omega) = +\infty \).

Since \( f_{\text{hom}} \) is positively homogeneous of degree \( p \), to prove that \( f_{\text{hom}}(A) \geq c|A|^p \), it is sufficient to check that \( f_{\text{hom}}(A) \neq 0 \) if \( A \neq 0 \). To this aim, let \( u_\epsilon \to Ax \) be such that \( F_\epsilon(u_\epsilon, (0,1)^n) \to f_{\text{hom}}(A) \). If \( f_{\text{hom}}(A) = 0 \) then by a “Poincaré-type” inequality for \( BD \) functions (Proposition 2.3 Chapter 2 of [20]), by Hölder’s inequality and a scaling argument we obtain that

\[
0 = f_{\text{hom}}(A) = \lim_{\epsilon \to 0} \int_{(0,1)^n} \left| \frac{dEu_\epsilon}{d\mu_\epsilon} \right|^p \, d\mu_\epsilon
\]

\[
\geq \lim_{\epsilon \to 0} c \left( \int_{(0,1)^n} |u_\epsilon - Ru_\epsilon| \, dx \right)^p
\]

where the constant \( c \) depends only on \( \Omega \) and \( Ru_\epsilon \) is a rigid displacement. Hence \( Ru_\epsilon \to Ax \) in \( L^1 \), and we get a contradiction because \( A \) is a symmetric matrix. \( \Box \)

6 Non local effects

Theorem 5.1 shows the \( \Gamma(L^p) \)-convergence of the functionals \( F_\epsilon \) to \( F_{\text{hom}} \) in \( W^{1,p}(\Omega; \mathbb{R}^n) \) and that the \( \Gamma \)-limit is local; in fact we have represented \( F_{\text{hom}} \) as the integration over \( \Omega \) of a local density of energy of the form \( f_{\text{hom}}(Eu) \).
Now, if we consider

\[ F_\varepsilon^\gamma(u, \Omega) = \varepsilon^\gamma \int_\Omega f \left( \frac{dEu}{d\mu_\varepsilon} \right) d\mu_\varepsilon \]

then \( \Gamma(L^p) - \lim_{\varepsilon \to 0} F_\varepsilon^\gamma(u, \Omega) = 0 \) on \( W^{1,p}(\Omega; \mathbb{R}^n) \), when \( \gamma > 0 \). In this case, however, no coerciveness result may hold for sequences \( (u_\varepsilon) \) with \( \sup_{\varepsilon > 0} F_\varepsilon^\gamma(u_\varepsilon, \Omega) < +\infty \) in any norm.

We will show with an example that a more complex notion of convergence may have to be introduced and that the \( \Gamma \)-limit functionals may be of a non-local nature.

Let \( \Omega = \omega \times (0,1) \) be a ‘cylindrical’ domain where \( \omega \) is a connected open subset of \( \mathbb{R}^2 \).

We define \( \varepsilon D_i \) to be a two dimensional disk centered at \( x_i = (\varepsilon i_1 + \frac{\varepsilon}{2}, \varepsilon i_2 + \frac{\varepsilon}{2}) \) of radius \( \varepsilon/4 \)

\[ \varepsilon E_i^2 = \varepsilon D_i \times (0,1) \quad \varepsilon E^2 = \bigcup_{i \in I_\varepsilon} \varepsilon E_i^2 \]

where \( i = (i_1, i_2) \in I_\varepsilon = \{ i \in \mathbb{Z}^2 : \varepsilon E_i^2 \subset \Omega \} \),

\[ \varepsilon E^1 = \Omega \setminus \varepsilon E^2. \]

We call \( E = D_0 \times (0,1) \).

We consider the measures

\[ \mu_\varepsilon(B) = \varepsilon \mathcal{H}^2(B \cap \partial \varepsilon E^2) \]

and the functionals

\[ F_\varepsilon^\gamma(u, \Omega) = \varepsilon^\gamma \int_\Omega \left| \frac{dEu}{d\mu_\varepsilon} \right|^2 d\mu_\varepsilon. \]

Note that, up to normalization, \( \mu_\varepsilon \) is the same measure of Example 4.3.

In this case \( LD^2_{\mu_\varepsilon}(\Omega) \) consists of functions which are rigid displacements on the sets \( \varepsilon E^1 \) and \( \varepsilon E^2 \); i.e., \( u \in LD^2_{\mu_\varepsilon}(\Omega) \) if and only if there exist \( a_i, b_i, c, d \in \mathbb{R}^3 \) such that

\[ u = c \land x + d \quad \text{on} \quad \varepsilon E^1 \]
\[ u = a_i \land x + b_i \quad \text{on} \quad \varepsilon E_i^2 \]

for each \( i \in I_\varepsilon \). We use the notation \( x = (x_\alpha, x_3) \in \mathbb{R}^3, x_\alpha = (x_1, x_2) \).

Hence

\[ \frac{dEu}{d\mu_\varepsilon} = \frac{1}{\varepsilon} \frac{dEu}{d\mathcal{H}^2} = \frac{1}{\varepsilon} (c \land x + d - a_i \land x - b_i) \odot \nu \quad \text{on} \quad \partial (\varepsilon E_i^2). \]
Definition 6.1 Let \( u_\epsilon \in LD^{2}_{\mu_\epsilon}(\Omega) \). We say that \( u_\epsilon \) converges to \((u_1, u_2) \in L^2(\Omega; \mathbb{R}^3) \times L^2(\Omega; \mathbb{R}^3) \) if and only if

\[
\lim_{\epsilon \to 0} \int_{\varepsilon E^1} |u_\epsilon - u_1|^2 \, dx = 0 \tag{20}
\]

\[
\lim_{\epsilon \to 0} \int_{\varepsilon E^2} |u_\epsilon - u_2|^2 \, dx = 0 . \tag{21}
\]

We will study the \( \Gamma \)-limit \( F \) of \( F_\gamma^{\epsilon} \) with respect to the convergence introduced in Definition 6.1 (see Theorem 6.4). The domain of \( F \) will be the set of pairs \((u_1, u_2)\) such that \( u_1 \) is a rigid displacement and \( u_2 \) is in the space \( U \) of functions whose ‘vertical sections are rigid displacements’, introduced in the following proposition.

Proposition 6.2 Let \( u_\epsilon \in LD^{2}_{\mu_\epsilon}(\Omega) \) and \( u_2 \in L^2(\Omega; \mathbb{R}^3) \).

\[
\lim_{\epsilon \to 0} \int_{\varepsilon E^2} |u_\epsilon - u_2|^2 \, dx = 0
\]

if and only if \( u_2 \in U \) where

\[
U = \left\{ v \in L^2(\Omega; \mathbb{R}^3) : \forall \eta > 0 \ \exists J \subset \mathbb{Z}^2 \text{ and } \exists A^k \land x + B^k \text{ on } T^k_\eta \ \forall k \in J \text{ such that } \right. \\
\left. \bigcup_{k \in J} T^k_\eta \cap \Omega = \Omega \ \text{ and } \sum_{k \in J} \int_{T^k_\eta \cap \Omega} |v(x) - A^k \land x - B^k|^2 \, dx \leq o(\eta) \right\}.
\]

Proof. Let \( u_\epsilon \in LD^{2}_{\mu_\epsilon}(\Omega) \), by definition \( u_\epsilon = a_{\epsilon,i} \land x + b_{\epsilon,i} \text{ on } \varepsilon E^2_i \). Let \( h \in \mathbb{N} \) and \( \eta > 0 \) such that \( \eta = h \varepsilon \), we extend \( a_{\epsilon,i} \land x + b_{\epsilon,i} \) to \( T^k_\eta \) for each \( i \in I_k = \{ i \in \mathbb{Z}^2 : \varepsilon E^2_i \subset T^k_\eta \} \), hence we can construct a rigid displacement on \( T^k_\eta \)

\[
A^k_\epsilon \land x + B^k_\epsilon = \frac{1}{h^2} \sum_{i \in I_k} a_{\epsilon,i} \land x + b_{\epsilon,i}.
\]

Let us suppose that \( u_\epsilon \) satisfies condition (21),

\[
\int_{T^k_\eta \cap \varepsilon E^2} \left| u_2(x) - A^k_\epsilon \land x - B^k_\epsilon \right|^2 \, dx \leq \epsilon \left( \sum_{j \in I_k} \int_{\varepsilon E^2_j} \left| u_2(x) - a_{\epsilon,j} \land x - b_{\epsilon,j} \right|^2 \, dx \\
+ \sum_{j \in I_k} \int_{\varepsilon E^2_j} \left| a_{\epsilon,j} \land x + b_{\epsilon,j} - \frac{1}{h^2} \sum_{i \in I_k} a_{\epsilon,i} \land x + b_{\epsilon,i} \right|^2 \, dx \right). \tag{22}
\]
Let us estimate the last term in (22)

\[
\sum_{j \in I_k} \int_{E_j^2} \left| a_{\varepsilon,j} \wedge x + b_{\varepsilon,j} - \frac{1}{h^2} \sum_{i \in I_k} a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} \right|^2 dx 
\]

\[
\leq \epsilon \left( \sum_{j \in I_k} \int_{E_j^2} \left| a_{\varepsilon,j} \wedge x + b_{\varepsilon,j} - u_2(x) \right|^2 dx 
+ \sum_{i,j \in I_k} \frac{1}{h^2} \int_{E_i^2} \left| a_{\varepsilon,i} \wedge (x + x_j - x_i) + b_{\varepsilon,i} - u_2(x) \right|^2 dx 
+ \sum_{i,j \in I_k} \frac{1}{h^2} \int_{E_j^2} \left| u_2(x) - u_2(x + x_i - x_j) \right|^2 dx \right).
\]

For each \( x \in E_j^2 \) we have that \( x + x_i - x_j \in E_i^2 \), hence with a change of coordinates we get

\[
\sum_{j \in I_k} \int_{E_j^2} \left| a_{\varepsilon,j} \wedge x + b_{\varepsilon,j} - \frac{1}{h^2} \sum_{i \in I_k} a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} \right|^2 dx 
\leq \epsilon \left( \sum_{i \in I_k} \int_{E_i^2} \left| a_{\varepsilon,i} \wedge x + b_{\varepsilon,i} - u_2(x) \right|^2 dx 
+ \sum_{i,j \in I_k} \frac{1}{h^2} \int_{E_i^2} \left| a_{\varepsilon,i} \wedge (x_j - x_i) \right|^2 dx 
+ \sum_{i,j \in I_k} \frac{1}{h^2} \int_{E_j^2} \left| u_2(x) - u_2(x + x_i - x_j) \right|^2 dx \right).
\]

(23)

Now if we denote \( \Lambda \) the set of all translations of the type \( x_i - x_j \) with \( i, j \in I_k \) we get that

\[
\sum_{i,j \in I_k} \frac{1}{h^2} \int_{E_j^2} \left| u_2(x) - u_2(x + x_i - x_j) \right|^2 dx \]

(24)

\[
\leq \sum_{\tau \in \Lambda} \frac{1}{h^2} \sum_{r \in C(k)} \int_{E_j^2} \left| u_2(x) - u_2(x + \tau) \right|^2 dx
\]

where \( C(k) = \{(k_1, k_2), (k_1 \pm 1, k_2), (k_1, k_2 \pm 1), (k_1 \pm 1, k_2 \pm 1)\} \).
Since $|\Lambda| = c h^2$, by (24) we have
\[
\sum_{k \in J} \sum_{i,j \in I} \frac{1}{h^2} \int_{E_j^i} \left| u_2(x) - u_2(x + x_i - x_j) \right|^2 dx \\
\leq c \sum_{\tau \in \Lambda} \frac{1}{h^2} \left\| u_2(\cdot) - u_2(\cdot + \tau) \right\|^2_{L^2(\Omega; \mathbb{R}^3)} \\
\leq c \sup_{|\tau| \leq \sqrt{2} \eta} \left\| u_2(\cdot) - u_2(\cdot + \tau) \right\|^2_{L^2(\Omega; \mathbb{R}^3)}.
\]

Let us consider the cubes $Q_{\varepsilon,i}^j = (\varepsilon i + (0,1)^2) \times (\varepsilon j + (0,\varepsilon))$ for $i \in I_\varepsilon$, and $j \in J_\varepsilon = \{j \in Z : Q_{\varepsilon,i}^j \cap E_j^i \neq \emptyset\}$. Since $u_2 \in L^2(\Omega; \mathbb{R}^3)$, we can assume that there exists a sequence $(u_{\varepsilon,2})$ which is constant on each $Q_{\varepsilon,i}^j$ such that
\[
\lim_{\varepsilon \to 0} \int_{\Omega} |u_2 - u_{\varepsilon,2}|^2 dx = \lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \sum_{j \in J_\varepsilon} \int_{Q_{\varepsilon,i}^j \cap \Omega} |u_2 - u_{\varepsilon,2,i,j}|^2 dx = 0
\]
where $u_{\varepsilon,2,i,j}$ is the value of $(u_{\varepsilon,2})$ on $Q_{\varepsilon,i}^j$.

So by (21) we get
\[
\lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \sum_{j \in J_\varepsilon} \int_{Q_{\varepsilon,i}^j \cap \Omega^2} |u_{\varepsilon} - u_{\varepsilon,2,i,j}|^2 dx = 0.
\]

Note that the $L^2$-norm on the set $\mathcal{R}$ of rigid displacements is equivalent to the norm on $\mathcal{R}$
\[
\|a \land x + b\|_{\mathcal{R}} = (|a|^2 + |b|^2)^{1/2},
\]
hence by (27)
\[
\lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \sum_{j \in J_\varepsilon} \varepsilon^3 |a_{\varepsilon,i}|^2 + \varepsilon^3 |b_{\varepsilon,i} - u_{\varepsilon,2,i,j}|^2 = 0
\]
which implies that
\[
\lim_{\varepsilon \to 0} \sum_{i \in I_\varepsilon} \varepsilon^2 |a_{\varepsilon,i}|^2 = 0
\]
and
\[
\sum_{i \in I_\varepsilon} \varepsilon^2 |b_{\varepsilon,i}|^2 \leq c
\]
for each $\varepsilon > 0$ small enough.

Since $|x_j - x_i| \leq \eta$, by the equivalence of the norms we have
\[
\sum_{i,j \in I_k} \frac{1}{h^2} \int_{E_j^i} \left| a_{\varepsilon,i} \land (x_j - x_i) \right|^2 dx \\
\leq c \sum_{i,j \in I_k} \frac{\varepsilon^2}{h^2} \eta^2 |a_{\varepsilon,i}|^2 \\
= c \eta^2 \sum_{i \in I_k} \varepsilon^2 |a_{\varepsilon,i}|^2.
\]
Note that $\sum_{k \in J} \sum_{i \in I_k} = \sum_{i \in I_k}$.

Now we insert (31) into (29) and, summing up all the corresponding estimates obtained for different indices $k \in J$, by (27) we get

$$\sum_{k \in J} \sum_{i \in I_k} \int_{\varepsilon E_j} \left| a_{\varepsilon,j} \cap x + b_{\varepsilon,j} - \frac{1}{h^2} \sum_{i \in I_k} a_{\varepsilon,i} \cap x + b_{\varepsilon,i} \right|^2 dx$$

$$\leq c \left( \sum_{i \in I_k} \int_{E_i} \left| a_{\varepsilon,i} \cap x + b_{\varepsilon,i} - u_2(x) \right|^2 dx + \eta^2 \sum_{i \in I_k} \varepsilon^2 |a_{\varepsilon,i}|^2 \right)$$

$$+ \sup_{|\tau| \leq \sqrt{\varepsilon}} \|u_2(\cdot) - u_2(\cdot + \tau)\|_{L^2(\Omega; \mathbb{R}^3)}^2.$$  

Finally, we sum up the estimates (22) for $k \in J$ and insert (31); by (21) and (28) we get

$$\lim_{\varepsilon \to 0} \sum_{k \in J} \sum_{i \in I_k} \int_{T^{\eta}_{\varepsilon} \cap \varepsilon E^2} \left| u_2 - A^k \cap x - B^k \right|^2 dx$$

$$\leq c \sup_{|\tau| \leq \sqrt{\varepsilon}} \|u_2(\cdot) - u_2(\cdot + \tau)\|_{L^2(\Omega; \mathbb{R}^3)}^2.$$  

On the other hand it is easy to see by (28) and (29) that there exists $A^k \cap x + B^k$ such that

$$\lim_{\varepsilon \to 0} \int_{T^\eta_{\varepsilon}} \left| A^k \cap x + B^k - A^k \cap x - B^k \right|^2 dx = 0$$

for each $k \in J$, hence by (32) we can conclude that $u_2 \in \mathcal{U}$.

Conversely, if $u_2 \in \mathcal{U}$ then $\varepsilon E^2 = \varepsilon \cap \bigcup_{k \in J} T^\eta_{\varepsilon} \cap \varepsilon E^2$ and we have rigid displacements $A^k \cap x + B^k$ on each $T^\eta_{\varepsilon}$.

We define

$$a_{\varepsilon,i} \cap x + b_{\varepsilon,i} = (A^k \cap x + B^k)_{|\varepsilon E_i}$$

for each $i \in I_k$. Hence

$$\sum_{i \in I_k} \int_{\varepsilon E_i} \left| a_{\varepsilon,i} \cap x + b_{\varepsilon,i} - u_2(x) \right|^2 dx = \sum_{k \in J} \sum_{i \in I_k} \int_{T^\eta_{\varepsilon} \cap \varepsilon E^2} \left| A^k \cap x + B^k - u_2(x) \right|^2 dx$$

and by definition of $\mathcal{U}$

$$\lim_{\varepsilon \to 0} \sum_{k \in J} \sum_{i \in I_k} \int_{T^\eta_{\varepsilon} \cap \varepsilon E^2} \left| A^k \cap x + B^k - u_2(x) \right|^2 dx$$

$$= \sum_{k \in J} |E| \int_{T^\eta_{\varepsilon} \cap \Omega} \left| A^k \cap x + B^k - u_2(x) \right|^2 dx \leq o(\eta).$$

By (33), passing to the limit as $\eta \to 0$, we get

$$\lim_{\varepsilon \to 0} \sum_{i \in I_k} \int_{\varepsilon E_i} \left| a_{\varepsilon,i} \cap x + b_{\varepsilon,i} - u_2(x) \right|^2 dx = 0.$$ 

Remark 6.3 Note that, since $u_\varepsilon$ are rigid displacements, by (20) it is easy to see that $u_1$ is a rigid displacement.

For simplicity, we will denote

$$F(u_1, u_2; \Omega) = \Gamma^{-\lim}_{\varepsilon \to 0} F_\varepsilon^\gamma(u_1, u_2; \Omega)$$

for $(u_1, u_2) \in R \times U$. We will continue to write $F_\varepsilon^\gamma(u, \Omega)$ for $u \in LD_\mu^2(\Omega)$.

Theorem 6.4 For $\gamma = 2$ the functionals $F_\varepsilon^\gamma \Gamma$-converge as $\varepsilon \to 0$ to

$$F(u_1, u_2; \Omega) = c_1 \int \Omega |(u_1)_\alpha - (u_2)_\alpha|^2 \, dx + c_2 \int \Omega |(u_1)_3 - (u_2)_3|^2 \, dx$$

on $R \times U$ with respect to the convergence introduced in Definition 1.1, where $c_1 = \frac{3}{3\pi} \pi, c_2 = \frac{4}{\pi}$.

Proof. By the invariance of the functionals with respect to translations of rigid displacements and by Remark 6.3, we can always assume without loss of generality that $u_\varepsilon = u_1$ on $\varepsilon E^1$.

Let us call

$$\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} = u_1 - a_{\varepsilon,i} \wedge x - b_{\varepsilon,i}$$

hence

$$F_{\varepsilon}^\gamma(u_{\varepsilon}, \Omega) = \varepsilon^{\gamma - 1} \sum_{i \in I} \int_{\partial E_i} |(\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i}) \ast \nu|^2 \, dH^2.$$

Fix $x_3 \in (0, 1)$, we can find the following equality

$$4\varepsilon \int_{\partial E_i} |(\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i}) \ast \nu|^2 \, dH^1 - 16 \int_{E_i} |(\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i})|^2 \, dx$$

$$= \frac{\pi}{2} \varepsilon^2 \left( \left( \int_{E_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \, dx \right) \right) \left( \left( \int_{E_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \, dx \right) \right)$$

$$+ \frac{\pi}{64} \varepsilon^4 ((\alpha_{\varepsilon,i})_1^2 + (\alpha_{\varepsilon,i})_2^2 + 2(\alpha_{\varepsilon,i})_3^2).$$

Hence, if we integrate also in $x_3$, we get

$$4\varepsilon \int_{\partial E_i} |(\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i}) \ast \nu|^2 \, dH^2 - 16 \int_{E_i} |(\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i})|^2 \, dx$$

$$= \int_0^1 \frac{\pi}{2} \varepsilon^2 \left( \left( \int_{E_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \, dx \right) \right) \left( \left( \int_{E_i} \alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \, dx \right) \right)$$

$$+ \frac{\pi}{64} \varepsilon^4 ((\alpha_{\varepsilon,i})_1^2 + (\alpha_{\varepsilon,i})_2^2 + 2(\alpha_{\varepsilon,i})_3^2).$$

But

$$\lim_{\varepsilon \to 0} \sum_{i \in I} \int_{E_i} \left( (\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \, dx \right) \right) ^2 \, dx = \lim_{\varepsilon \to 0} \sum_{i \in I} \int_{E_i} \left( (\alpha_{\varepsilon,i} \wedge x + \beta_{\varepsilon,i} \, dx \right) \right) ^2 \, dx$$

21
for each \( h = 1, 2, 3 \), and

\[
\frac{\pi}{2} \varepsilon^2 \left( \int_{\varepsilon D_i} \alpha_{\varepsilon, i} \wedge x + \beta_{\varepsilon, i} \, dx_\alpha \right)_h^2 = 8 \int_{\varepsilon D_i} \left( \int_{\varepsilon E_i} \alpha_{\varepsilon, i} \wedge x + \beta_{\varepsilon, i} \, dx_\alpha \right)_h^2 \, dx_\alpha ;
\]

hence,

\[
\lim_{\varepsilon \to 0} \sum_{i \in I} \varepsilon \int_{\varepsilon E_i^2} \left( \int_{\partial_\varepsilon E_i^2} \left( \alpha_{\varepsilon, i} \wedge x + \beta_{\varepsilon, i} \right) \otimes \nu \right)^2 \, d\mathcal{H}^2 \geq 6 \int_\Omega \left| (u_1)_\alpha - (u_2)_\alpha \right|^2 \, dx + 4 \int_\Omega \left| (u_1)_3 - (u_2)_3 \right|^2 \, dx.
\]

If we pass to the limit in (34), by (35) we obtain

\[
\lim_{\varepsilon \to 0} \sum_{i \in I} \varepsilon \int_{\varepsilon E_i^2} \left( \int_{\partial_\varepsilon E_i^2} \left( \alpha_{\varepsilon, i} \wedge x + \beta_{\varepsilon, i} \right) \otimes \nu \right)^2 \, d\mathcal{H}^2 \geq 6 \int_\Omega \left| (u_1)_\alpha - (u_2)_\alpha \right|^2 \, dx + 4 \int_\Omega \left| (u_1)_3 - (u_2)_3 \right|^2 \, dx.
\]

For every sequence \( u_\varepsilon \) converging to \( (u_1, u_2) \) in the sense of Definition 6.1, by (28) we have that

\[
\lim_{\varepsilon \to 0} \sum_{i \in I} \varepsilon \int_{\varepsilon E_i^2} \left( \int_{\partial_\varepsilon E_i^2} \left( \alpha_{\varepsilon, i} \wedge x + \beta_{\varepsilon, i} \right) \otimes \nu \right)^2 \, d\mathcal{H}^2 = 6 \int_\Omega \left| (u_1)_\alpha - (u_2)_\alpha \right|^2 \, dx + 4 \int_\Omega \left| (u_1)_3 - (u_2)_3 \right|^2 \, dx.
\]

By the arbitrariness of \( u_\varepsilon \), choosing \( \gamma = 2 \)

\[
\Gamma - \liminf_{\varepsilon \to 0} F_{\varepsilon}^2(u_1, u_2; \Omega) \geq F(u_1, u_2; \Omega).
\]

Now we consider

\[
u_\varepsilon = (c \wedge x + d) \chi_{\varepsilon E_1} + (a \wedge x + b) \chi_{\varepsilon E_2}
\]
obviously it converges to \((c \wedge x + d, a \wedge x + b)\), and we call \(\alpha \wedge x + \beta = (a-c) \wedge x + (b-d)\).

In this case

\[
8 \int_{E_2^1} \left( \alpha \wedge x + \beta \right)_h^2 \, dx = \int_0^1 \frac{\pi}{2}^2 \left( \int_{E_2} \alpha \wedge x + \beta \, dx \right)_h^2 \, dx_3 + \frac{\pi}{128} \varepsilon \alpha_3^2
\]

for \(h = 1, 2\), hence by (34)

\[
\limsup_{\varepsilon \to 0} \sum_{i \in I_e} \varepsilon \int_{\partial E_2^0} \left( \alpha \wedge x + \beta \right) \circ \nu \|^2 \, d\mathcal{H}^2
\]

\[
\leq 6 \lim_{\varepsilon \to 0} \sum_{i \in I_e} \int_{E_2^0} \left( \alpha \wedge x + \beta \right)_2 \left\| \left( \alpha \wedge x + \beta \right) \right\|^2 \, dx
\]

\[
+4 \lim_{\varepsilon \to 0} \sum_{i \in I_e} \int_{E_2^0} \left( \alpha \wedge x + \beta \right)_3 \left\| \left( \alpha \wedge x + \beta \right) \right\|^2 \, dx + c \lim_{\varepsilon \to 0} \varepsilon^2 |\alpha|^2
\]

\[
= 6 |E| \left( \alpha \wedge x + \beta \right)_\alpha \left| dx + 4 |E| \int_{\Omega} (\alpha \wedge x + \beta)_3 \left\| dx . \right. \tag{40}
\]

By (38) and (40) we get

\[
\lim_{\varepsilon \to 0} \varepsilon^2 \int_{\Omega} \frac{dE u_\varepsilon}{d\mu_\varepsilon} \|^2 d\mu_\varepsilon = \frac{6 |E| \int_{\Omega} (\alpha \wedge x + \beta)_\alpha \left| dx + 4 |E| \int_{\Omega} (\alpha \wedge x + \beta)_3 \left\| dx . \tag{41}
\]

Now we fix \(\eta > 0\) and consider \(u_1 \in \mathcal{R}\) and \(v_\eta^\eta\) such that \(v_\eta^\eta \big|_{T_\eta^k} = A^k \wedge x + B^k\) with \(k \in J\). By (41) we get

\[
\limsup_{\varepsilon \to 0} F_2^2(u_1 \chi_{E_1} + v_\eta^\eta \chi_{E_2}, \Omega)
\]

\[
\leq \sum_{k \in J} \limsup_{\varepsilon \to 0} F_2^2(u_1 \chi_{E_1} + (A^k \wedge x + B^k) \chi_{E_2}, T_\eta^k \cap \Omega)
\]

\[
= \sum_{k \in J} \left[ 6 |E| \int_{T_\eta^k \cap \Omega} \left( u_1(x) - A^k \wedge x - B^k \right)_\alpha \left\| dx + 4 |E| \int_{T_\eta^k \cap \Omega} (u_1(x) - A^k \wedge x - B^k)_3 \left\| dx \right. \right. \right. \tag{42}
\]

If \(u_2 \in \mathcal{U}\) then for all \(\eta > 0\) there exists \(v_\eta^\eta\) as above such that \(\| u_2 - v_\eta^\eta \|_{L^2(\Omega; \mathbb{R}^k)} \leq o(\eta)\), since the \(\Gamma\)-upper limit is \(L^2\)-lower semicontinuous if we denote

\[
F_2''(u_1, u_2; \Omega) = \Gamma \limsup_{\varepsilon \to 0} F_2^2(u_1, u_2; \Omega)
\]
by (42) we get
\[ F''_2(u_1, u_2; \Omega) \leq \liminf_{\eta \to 0} F''_2(u_1, v_2^\eta; \Omega) \]
\[ \leq \liminf_{\eta \to 0} 6 |E| \left( \int_{\Omega} \left| (u_1(x) - v_2^\eta(x))_\alpha \right|^2 \right) dx 
+ 4 |E| \left( \int_{\Omega} \left| (u_1(x) - v_2^\eta(x))_3 \right|^2 \right) dx \]
\[ = 6 |E| \left( \int_{\Omega} \left| (u_1(x) - u_2(x))_\alpha \right|^2 \right) dx + 4 |E| \left( \int_{\Omega} \left| (u_1(x) - u_2(x))_3 \right|^2 \right) dx. \]

It follows that given \((u_1, u_2) \in \mathcal{R} \times \mathcal{U}\)
\[ \Gamma- \limsup_{\varepsilon \to 0} F^2_\varepsilon(u_1, u_2; \Omega) \leq F(u_1, u_2; \Omega) \]
so that by (39)
\[ \Gamma- \lim_{\varepsilon \to 0} F^2_\varepsilon(u_1, u_2; \Omega) = F(u_1, u_2; \Omega) \]
as desired.

If \(u_\varepsilon\) converges to \((u_1, u_2)\) in the sense of Definition 6.1 then \(u_\varepsilon\) converges weakly in \(L^2(\Omega; \mathbb{R}^3)\) to \((1-c)u_1 + cu_2\) where \(c = |E|\). If we define the energy
\[ F(u, \Omega) := \inf_{u = (1-c)u_1 + cu_2 \in \mathcal{R} \times \mathcal{U}} F(u_1, u_2; \Omega) \]
by Theorem 6.4
\[ F(u, \Omega) = \inf_{r \in \mathcal{R}} \left( \tilde{c}_1 \int_{\Omega} \left| r_\alpha - u_\alpha \right|^2 dx + \tilde{c}_2 \int_{\Omega} \left| r_3 - u_3 \right|^2 dx \right) \]
where \(\tilde{c}_1 = c_1/c^2\) and \(\tilde{c}_2 = c_2/c^2\), which explains the non local nature of our limit.

**Remark 6.5** Let us consider, up to normalization, the same measure of Example 4.4
\[ \tilde{\mu}_\varepsilon(B) = \left( |B| + \varepsilon \mathcal{H}^2(B \cap \partial \varepsilon E^2) \right) \]
and the functionals
\[ \tilde{F}^2_\varepsilon(u, \Omega) = \varepsilon^2 \int_{\Omega} \frac{dE u_\varepsilon}{d\tilde{\mu}_\varepsilon} \frac{d\tilde{\mu}_\varepsilon}{dx}. \]
In this case by Theorem 6.4 we can deduce that the \(\Gamma- \limsup_{\varepsilon \to 0} \tilde{F}^2_\varepsilon(u_1, u_2; \Omega)\) is finite for \((u_1, u_2) \in \mathcal{R} \times \mathcal{U}\).

In fact, since \(\text{LD}^2_\mu(\Omega; \mathbb{R}^3) \subset \text{LD}^2_{\mu_\varepsilon}(\Omega; \mathbb{R}^3)\), given \((u_1, u_2) \in \mathcal{R} \times \mathcal{U}\) we have
\[ \Gamma- \limsup_{\varepsilon \to 0} \tilde{F}^2_\varepsilon(u_1, u_2; \Omega) \leq \Gamma- \limsup_{\varepsilon \to 0} F^2_\varepsilon(u_1, u_2; \Omega). \]

**Acknowledgements** We wish to express our thanks to Prof. Andrea Braides for suggesting the problem and for many stimulating conversations. We also thank Prof. Luigi Ambrosio for helpful comments.
References

[1] L. Ambrosio, G. Buttazzo and I. Fonseca, Lower semicontinuity problems in Sobolev spaces with respect to a measure, *J. Math. Pures Appl.* 75 (1996), 211–224.

[2] L. Ambrosio, A. Coscia and G. Dal Maso, Fine properties of functions with bounded deformation, *Arch. Rational Mech. Anal.* 139 (1997), 201–238.

[3] L. Ambrosio, N. Fusco and D. Pallara, *Functions of Bounded Variation and Free Discontinuity Problems*, Oxford University Press, Oxford, 2000.

[4] N. Ansini, A. Braides and V. Chiadò Piat, Homogenization of periodic multidimensional structures, *Boll. Un. Mat. Ital.* (8) 2-B (1999), 735–758.

[5] G. Anzellotti and M. Giaquinta, Existence of the displacement field for an elasto-plastic body subject to Hencky’s law and von Mises yield condition, *Manuscripta Math.* 32 (1980), 101–136.

[6] G. Bouchitté, G. Buttazzo and P. Seppecher, Energies with respect to a measure and applications to low dimensional structures, *Calc. Var.* 5 (1997), 37–54.

[7] A. Braides, *Approximation of Free-Discontinuity Problems*, Springer-Verlag, Berlin, 1998.

[8] A. Braides, *Γ-convergence for Beginners*, Oxford University Press, to appear.

[9] A. Braides and V. Chiadò Piat, Remarks on the homogenization of connected media, *Nonlinear Anal.* 22 (1994), 391–407.

[10] A. Braides and A. Defranceschi, *Homogenization of Multiple Integrals*, Oxford University Press, Oxford, 1998.

[11] G. Buttazzo, *Semicontinuity, Relaxation and Integral Representation in the Calculus of Variations*, Longman, Harlow, 1989.

[12] G. Dal Maso, *An Introduction to Γ-convergence*, Birkhäuser, Boston, 1993.

[13] F. Ebobisse, Fine properties of functions with bounded deformation and applications in variational problems, Ph.D. Thesis, Univeristy of Pisa, 1999.

[14] H. Federer, *Geometric Measure Theory*, Springer-Verlag, Berlin, 1969.

[15] C. Goffman and J. Serrin, Sublinear functions of measures and variational integrals, *Duke Math. J.* 31 (1964), 159–178.

[16] R.V. Kohn, *New estimates for deformations in terms of their strains*, Ph.D. Thesis, Princeton Univ., 1979.

25
[17] H. Matthies, G. Strang and E. Christiansen, The saddle point of a differential program, *Energy Methods in Finite Element Analysis*, Wiley, New York, 1979.

[18] P.M. Suquet, Existence et régularité des solutions des équations de la plasticité parfaite, *C.R. Acad. Sci. Paris Sér. A* 286 (1978), 1201–1204.

[19] P.M. Suquet, Un espace fonctionnel pour les équations de la plasticité, *Ann. Fac. Sci. Toulouse* 1 (1979), 77–87.

[20] R. Temam, *Mathematical Problems in Plasticity*, Gauthier-Villars, Paris, 1985.

[21] R. Temam and G. Strang, Functions of bounded deformation, *Arch. Rational Mech. Anal.* 75 (1980), 7–21.

[22] V.V. Zhikov, Lavrentiev phenomenon and homogenization for some variational problems, *Composite Media and Homogenization Theory*, World Scientific, Singapore, 1995, 273–288.

[23] W.P. Ziemer, *Weakly Differentiable Functions*, Springer-Verlag, Berlin, 1989.

Nadia Ansini
SISSA/ISAS
Via Beirut 4, 34014 Trieste, Italy

François Bille Ebobisse
Dipartimento di Matematica
Università di Pisa
Via Buonarroti 2, 56127 Pisa, Italy

26