In this note we determine the obstruction to triviality of the stack of exact vertex algebroids thereby recovering the result of [GMS].

The stack $\mathcal{VA}_{\mathcal{O}_X}$ of exact vertex $\mathcal{O}_X$-algebroids is a torsor under the stack in Picard groupoids $\mathcal{EC}_{\mathcal{O}_X}$ of exact Courant $\mathcal{O}_X$-algebroids. The latter is equivalent to the stack of torsors under $\Omega^2_X \to \Omega^3_{cl}$. Therefore, $\mathcal{EC}_{\mathcal{O}_X}$-torsors are classified by $H^2(X; \Omega^2_X \to \Omega^3_{cl})$.

The goal of the present note is to determine the class of $\mathcal{VA}_{\mathcal{O}_X}$.

The first step toward this goal is to replace $\mathcal{VA}_{\mathcal{O}_X}$ by the equivalent $\mathcal{EC}_{\mathcal{O}_X}$-torsor $CE_{\mathcal{O}_X}(A_{\Omega^1_X}) \langle , \rangle$, whose (locally defined) objects are certain Courant algebroids which are extensions by $\Omega^1_X$ of the Lie $\mathcal{O}_X$-algebroid $A_{\Omega^1_X}$, the Atiyah algebra of the sheaf $\Omega^1_X$. Any such extension induces an $A_{\Omega^1_X}$-invariant symmetric pairing $\langle , \rangle$ on the Lie algebra $\text{End}_{\mathcal{O}_X}(\Omega^1_X)$.

The objects of $CE_{\mathcal{O}_X}(A_{\Omega^1_X}) \langle , \rangle$ are those for which $\langle , \rangle$ is given by the negative of the trace of the product of endomorphisms.

We show that $\mathcal{VA}_{\mathcal{O}_X}$ and $CE_{\mathcal{O}_X}(A_{\Omega^1_X}) \langle , \rangle$ are anti-equivalent as $\mathcal{EC}_{\mathcal{O}_X}$-torsors by adapting the strategy of [BD] to the present setting and making use of (the degree zero part of) the unique vertex $\Omega^\bullet_X$-algebroid constructed in [B]. It follows that the classes of $\mathcal{VA}_{\mathcal{O}_X}$ and $CE_{\mathcal{O}_X}(A_{\Omega^1_X}) \langle , \rangle$ in $H^2(X; \Omega^2_X \to \Omega^3_{cl})$ are negatives of each other. The advantage of passing to $CE_{\mathcal{O}_X}(A_{\Omega^1_X}) \langle , \rangle$ has to do with the fact that Courant algebroids are objects of “classical” nature. In particular they are $\mathcal{O}_X$-modules (as opposed to vertex algebroids).

The second step is the determination of the class of $CE_{\mathcal{O}_X}(A_{\Omega^1_X}) \langle , \rangle$ which is achieved in the more general framework. Namely, we consider a transitive Lie $\mathcal{O}_X$ algebroid $\mathcal{A}$, locally free of finite rank over $\mathcal{O}_X$, and denote by $\mathfrak{g}$ the kernel of the anchor map. Thus, $\mathfrak{g}$ is a sheaf of Lie algebras in $\mathcal{O}_X$-modules. If $\mathcal{A}$ is a Courant algebroid, such that the associated Lie algebroid is identified with $\mathcal{A}$, then $\mathfrak{g}$ is an extension of $\mathcal{A}$ by $\Omega^1_X$. The symmetric bilinear form on $\mathcal{A}$ induces an $\mathcal{A}$-invariant symmetric bilinear form on $\mathfrak{g}$.

Suppose $\mathcal{A}$, $\mathfrak{g}$ as above, and $\langle , \rangle$ an $\mathcal{A}$-invariant symmetric bilinear form on $\mathfrak{g}$. Let $CE_{\mathcal{O}_X}(\mathcal{A}) \langle , \rangle$ denote the stack of whose (locally defined) objects are pairs, consisting of a Courant algebroid $\mathcal{A}$ together with an identification of the associated Lie algebroid with $\mathcal{A}$, such that the symmetric bilinear form induced on $\mathfrak{g}$ coincides with $\langle , \rangle$. We show that, if $\mathcal{A}$ admits a flat connection locally on $X$, then $CE_{\mathcal{O}_X}(\mathcal{A}) \langle , \rangle$ has a natural structure of an $\mathcal{EC}_{\mathcal{O}_X}$-torsor and calculate its characteristic class in $H^2(X; \Omega^2_X \to \Omega^3_{cl})$. It turns out that...
it is equal to the Pontryagin class naturally associated to the pair \((A, \langle , \rangle)\). For example, the Pontryagin class of \((A_{\Omega^1_X}, \langle , \rangle)\) (where the symmetric pairing is given by the negative of the trace of the product of endomorphisms) is equal to \(-2 \text{ch}(\Omega^1_X)\). Hence, the class of \(E\mathcal{VA}_{O_X}\) in \(H^2(X; \Omega^2_X \to \Omega^{3,ct})\) is equal to \(2 \text{ch}(\Omega^1_X)\).

The paper is organized as follows. In Section 2 we recall the basic material on Courant algebroids. In Section 3 we study Courant extensions of transitive Lie algebroids and classification thereof. In Section 4 we recall the basic material on vertex algebroids and their relationship to Courant algebroids. We explain how the construction of [B] gives an example of a vertex extension (of the Atiyah algebra of the cotangent bundle) and use this example to classify exact vertex algebroids.

The fact that the Pontryagin class of a principal bundle (defined with respect to an invariant symmetric pairing on the Lie algebra of the structure group) is the obstruction to the existence of a Courant extension of the Atiyah algebra of the principal bundle was pointed out by P. Ševera in [S] together with the constructions of 3.5.

2. COURANT ALGEBROIDS

2.1. Courant algebroids. A Courant \(O_X\)-algebroid is an \(O_X\)-module \(Q\) equipped with

1. a structure of a Leibniz \(C\)-algebra

\[
[ , ] : Q \otimes_C Q \to Q ,
\]

2. an \(O_X\)-linear map of Leibnitz algebras (the anchor map)

\[
\pi : Q \to T_X ,
\]

3. a symmetric \(O_X\)-bilinear pairing

\[
\langle , \rangle : Q \otimes_{O_X} Q \to O_X ,
\]

4. a derivation

\[
\partial : O_X \to Q
\]

such that \(\pi \circ \partial = 0\)

which satisfy

\[
[q_1, f q_2] = f[q_1, q_2] + \pi(q_1)(f)q_2 \quad (2.1.1)
\]

\[
\langle [q_1, q_2] + [q_1, [q, q_2]] \rangle = \pi(q)(\langle q_1, q_2 \rangle) \quad (2.1.2)
\]

\[
[q, \partial(f)] = \partial(\pi(q)(f)) \quad (2.1.3)
\]

\[
\langle q, \partial(f) \rangle = \pi(q)(f) \quad (2.1.4)
\]

\[
[q_1, q_2] + [q_2, q_1] = \partial(\langle q_1, q_2 \rangle) \quad (2.1.5)
\]

for \(f \in O_X\) and \(q, q_1, q_2 \in Q\).

2.1.1. A morphism of Courant \(O_X\)-algebroids is an \(O_X\)-linear map of Leibnitz algebras which commutes with the respective anchor maps and derivations and preserves the respective pairings.
2.2. **The associated Lie algebroid.** Suppose that \( Q \) is a Courant \( \mathcal{O}_X \)-algebroid. Let

\[
\Omega_Q \overset{\text{def}}{=} \mathcal{O}_X \partial(\mathcal{O}_X) \subset Q,
\]

\[
\overline{Q} \overset{\text{def}}{=} Q/\Omega_Q.
\]

Note that the symmetrization of the Leibniz bracket on \( Q \) takes values in \( \Omega_Q \).

For \( q \in Q, f, g \in \mathcal{O}_X \)

\[
[q, f \partial(g)] = f[q, \partial(g)] + \pi(q)(f)\partial(g)
\]

\[
= f\partial(\pi(q)(g)) + \pi(q)(f)\partial(g)
\]

which shows that \([Q, \Omega_Q] \subseteq \Omega_Q\). Therefore, the Leibniz bracket on \( Q \) descends to the Lie bracket

\[
[ , ] : \overline{Q} \otimes_{\mathcal{O}_X} \overline{Q} \to \overline{Q}.
\]

Since \( \pi \) is \( \mathcal{O}_X \)-linear and \( \pi \circ \partial = 0 \), \( \pi \) vanishes on \( \Omega_Q \) and factors through the map

\[
\pi : \overline{Q} \to T_X.
\]

2.2.1. **Lemma.** The bracket (2.2.1) and the anchor (2.2.2) determine the structure of a Lie \( \mathcal{O}_X \)-algebroid on \( \overline{Q} \).

2.3. **Transitive Courant algebroids.** A Courant \( \mathcal{O}_X \)-algebroid is called transitive if the anchor is surjective.

2.3.1. **Remark.** A Courant \( \mathcal{O}_X \)-algebroid \( Q \) is transitive if and only if the associated Lie \( \mathcal{O}_X \)-algebroid is.

2.3.2. Suppose that \( Q \) is a transitive Courant \( \mathcal{O}_X \)-algebroid. The derivation \( \partial \) induces the \( \mathcal{O}_X \)-linear map

\[
i : \Omega^1_X \to Q.
\]

Since \( \langle q, \alpha \rangle = \iota_{\pi(q)}\alpha \), it follows that the map \( i \) is adjoint to the anchor map \( \pi \). The surjectivity of the latter implies that \( i \) is injective. Since, in addition, \( \pi \circ i = 0 \) the sequence

\[
0 \to \Omega^1_X \xrightarrow{i} Q \to \overline{Q} \to 0
\]

is exact. Moreover, \( i \) is isotropic with respect to the symmetric pairing.

2.3.3. **Definition.** A connection on a transitive Courant \( \mathcal{O}_X \)-algebroid \( Q \) is a \( \mathcal{O}_X \)-linear isotropic section of the anchor map \( Q \to T_X \).

2.3.4. **Definition.** A flat connection on a transitive Courant \( \mathcal{O}_X \)-algebroid \( Q \) is a \( \mathcal{O}_X \)-linear section of the anchor map which is morphism of Leibniz algebras.

2.3.5. **Remark.** As a consequence of (2.1.5) a flat connection is a connection.

2.4. **Exact Courant algebroids.**
2.4.1. **Definition.** The Courant algebroid $\mathcal{Q}$ is called *exact* if the anchor map $\pi : \mathcal{Q} \to T_X$ is an isomorphism.

We denote the stack of exact Courant $\mathcal{O}_X$-algebroids by $\mathcal{ECA}_{\mathcal{O}_X}$.

2.4.2. A morphism of exact Courant algebroids induces a morphism of respective extensions of $T_X$ by $\Omega^1_X$, hence is an isomorphism of $\mathcal{O}_X$-modules. It is clear that the inverse isomorphism is a morphism of Courant $\mathcal{O}_X$-algebroids. Therefore, $\mathcal{ECA}_{\mathcal{O}_X}$ is a stack in groupoids.

2.4.3. The evident morphism $\mathcal{ECA}_{\mathcal{O}_X} \to \text{Ext}^1_{\mathcal{O}_X}(T_X, \Omega^1_X)$ is faithful. The natural structure of a $\mathbb{C}$-vector space in categories on $\text{Ext}^1_{\mathcal{O}_X}(T_X, \Omega^1_X)$ restricts to one on $\mathcal{ECA}_{\mathcal{O}_X}$. In particular, $\mathcal{ECA}_{\mathcal{O}_X}$ is a stack in Picard groupoids.

2.4.4. **Connections.** Suppose that $\mathcal{Q}$ is an exact Courant $\mathcal{O}_X$-algebroid. Let $\mathcal{C}(\mathcal{Q})$ denote the sheaf of (locally defined) connections on $\mathcal{Q}$.

2.4.5. **Lemma.** $\mathcal{C}(\mathcal{Q})$ is an $\Omega^2_X$-torsor.

**Proof.** The difference of two sections of the anchor map $\mathcal{Q} \to T_X$ is a map $T_X \to \Omega^1_X$ or, equivalently, a section of $\Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X$. The difference of two isotropic sections gives rise to a skew-symmetric tensor, i.e. a section of $\Omega^2_X$. $\square$

2.4.6. **Curvature.** For a (locally defined) connection $\nabla$ the formula

$$(\xi, \xi_1, \xi_2) \mapsto \iota_\xi(\nabla(\xi_1), \nabla(\xi_2)) - \nabla(\iota_\xi(\xi_1, \xi_2))$$

defines a section, denoted $c(\nabla)$, of $\Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X$ called the curvature of the connection $\nabla$.

A connection $\nabla$ is called flat if $c(\nabla) = 0$.

2.4.7. **Lemma.**

1. The tensor $c(\nabla)$ is skew-symmetric, i.e. a section of $\Omega^3_X$.
2. The differential form $c(\nabla)$ is closed.
3. For $\alpha \in \Omega^2_X$, $c(\nabla + \alpha) = c(\nabla) + d\alpha$.

2.4.8. **Exact Courant algebroids with connection.** Pairs $(\mathcal{Q}, \nabla)$, where $\mathcal{Q} \in \mathcal{ECA}_{\mathcal{O}_X}$ and $\nabla$ is a connection on $\mathcal{Q}$ give rise in the obvious way to a stack in Picard groupoids which we denote $\mathcal{ECA}_\nabla$.

The “zero” object in $\mathcal{ECA}_\nabla$ is “the Courant algebroid”, $\mathcal{Q}_0 = \Omega^1_X \oplus T_X$ with the obvious connection, the symmetric pairing given by the duality pairing between $\Omega^1_X$ and $T_X$, and the bracket characterized by the fact the connection is flat.

Note that the pair $(\mathcal{Q}, \nabla)$ has no non-trivial automorphisms. The assignment $(\mathcal{Q}, \nabla) \mapsto c(\nabla)$ gives rise to the morphism of Picard groupoids

$$c : \mathcal{ECA}_\nabla \to \Omega^{3,cl}_X,$$  \hspace{1cm} (2.4.1)

where $\Omega^{3,cl}_X$ is viewed as discrete, i.e. the only morphisms are the identity maps.
2.4.9. **Lemma.** The morphism (2.4.1) is an equivalence.

*Proof.* The inverse is given by the following construction. Let \([ , ]_0\) denote the Leibniz bracket on \(Q_0\). For \(H \in \Omega X^{3,cl} , \xi_1, \xi_2 \in T_X\) let

\[
[\xi_1, \xi_2]_H = [\xi_1, \xi_2]_0 + \iota_{\xi_1} \wedge \iota_{\xi_2} H.
\]

This operation extends uniquely to a Leibniz bracket \([ , ]_H\) on \(\Omega X^1 \oplus T_X\) which, together with the symmetric pairing induced by the duality pairing, is a structure of an exact Courant algebroid. Let \(Q_H\) denote this structure.

The obvious connection on \(Q_H\) has curvature \(H\). \(\square\)

2.4.10. **Classification of Exact Courant algebroids.** Suppose that \(Q\) is an exact Courant \(O_X\)-algebroid. The assignment \(\nabla \mapsto c(\nabla)\) gives rise to the morphism

\[
c : C(Q) \to \Omega X^{3,cl}.
\]

The formula \(c(\nabla + \alpha) = c(\nabla) + d\alpha\) means that \(c\) is a morphism of sheaves with an action of \(\Omega X^2\), where \(\alpha \in \Omega X^2\) acts on \(H \in \Omega X^{3,cl}\) by \(H \mapsto H + d\alpha\). Thus, the pair \((C(Q), c)\) is a torsor under \((\Omega X^2 \xrightarrow{d} \Omega X^{3cl})\).

2.4.11. **Lemma.** The correspondence \(Q \mapsto (C(Q), c)\) establishes an equivalence of stacks in \(\mathbb{C}\)-vector spaces in categories

\[
\mathcal{ECA}_{O_X} \to (\Omega X^2 \xrightarrow{d} \Omega X^{3cl}) - \text{tors}.
\]

In particular, the \(\mathbb{C}\)-vector space of isomorphism classes of exact Courant algebroids is canonically isomorphic to \(H^1(X; \Omega X^2 \to \Omega X^{3cl})\).

2.4.12. **Locally trivial exact Courant algebroids.** An exact Courant algebroid is said to be **locally trivial** if it admits a flat connection locally on \(X\).

Let \(\mathcal{ECA}_{O_X}^{loc.triv.}\) denote the stack of locally trivial exact Courant \(O_X\)-algebroids.

2.4.13. For an exact Courant \(O_X\)-algebroid let \(C^{fl}(Q)\) denote the sheaf of flat connections on \(Q\).

The sheaf \(C^{fl}(Q)\) is locally nonempty if and only if \(Q\) is locally trivial, in which case it is a torsor under \(\Omega X^{2cl}\). The correspondence \(Q \to C^{fl}(Q)\) establishes an equivalence of stacks \(\mathcal{ECA}_{O_X}^{loc.triv.} \to \Omega X^{2cl} - \text{tors}\).

### 3. Courant extensions of Lie algebroids

Suppose that \(A\) a Lie \(O_X\)-algebroid.

#### 3.1. Courant extensions.

3.1.1. **Definition.** A Courant extension of \(A\) is a Courant \(O_X\)-algebroid \(\hat{A}\) together with the isomorphism \(\hat{A} = A\) of Lie \(O_X\)-algebroids.
3.1.2. **Morphisms of Courant extensions.** A morphism of Courant extensions of $\mathcal{A}$ is a morphism of Courant $\mathcal{O}_X$-algebroids which is compatible with the identifications.

Let $\mathcal{CE}_{\mathcal{O}_X}(\mathcal{A})$ denote the stack of Courant extensions of $\mathcal{A}$.

3.1.3. For a Lie (respectively Courant) $\mathcal{O}_X$-algebroid $\mathcal{A}$ let $\mathfrak{g}(\mathcal{A})$ denote the kernel of the anchor map. Then, $\mathfrak{g}(\mathcal{A})$ is a Lie (respectively Courant) $\mathcal{O}_X$-algebroid with the trivial anchor natural in $\mathcal{A}$.

If $\hat{\mathcal{A}}$ is a Courant extension of $\mathcal{A}$, then $\mathfrak{g}(\hat{\mathcal{A}})$ is a Courant extension of $\mathfrak{g}(\mathcal{A})$ Hence, there is a morphism $\mathfrak{g}(\ ) : \mathcal{CE}_{\mathcal{O}_X}(\mathcal{A}) \to \mathcal{CE}_{\mathcal{O}_X}(\mathfrak{g}(\mathcal{A}))$.

3.2. **Courant extensions of transitive Lie algebroids.** From now on we assume that $\mathcal{A}$ is a transitive Lie $\mathcal{O}_X$-algebroid and $\mathfrak{g}(\mathcal{A})$ (equivalently $\mathfrak{g}$) is a locally free $\mathcal{O}_X$-module of finite rank.

For a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules let $\mathcal{F}^\vee$ denote the dual $\text{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$.

3.2.1. Suppose that $\hat{\mathcal{A}}$ is a Courant extension of $\mathcal{A}$. Then, the exact sequence

$$0 \to \Omega^1_X \to \hat{\mathcal{A}} \to \mathcal{A} \to 0$$

is canonically associated to the Courant extension $\hat{\mathcal{A}}$ of $\mathcal{A}$. Since a morphism of Courant extensions of $\mathcal{A}$ induces a morphism of the associated extensions of $\mathcal{A}$ by $\Omega^1_X$, it is an isomorphism of the underlying sheaves, and it is clear that the inverse isomorphism is a morphism of Courant $\Omega$-extensions of $\mathcal{A}$.

Therefore, $\mathcal{CE}_{\mathcal{O}_X}(\mathcal{A})$ is a stack in groupoids.

3.2.2. **Remark.** $\mathcal{CE}_{\mathcal{O}_X}(\mathcal{T}_X)$ is none other than $\mathcal{ECA}_{\mathcal{O}_X}$.

3.2.3. Suppose that $\hat{\mathcal{A}}$ is a Courant extension of $\mathcal{A}$. Let $\mathfrak{g} = \mathfrak{g}(\mathcal{A})$ and $\hat{\mathfrak{g}} = \mathfrak{g}(\hat{\mathcal{A}})$ for short.

Since $\langle \hat{\mathfrak{g}}, \Omega^1_X \rangle = 0$, the pairing on $\hat{\mathcal{A}}$ induces the pairing

$$\langle , \rangle : \hat{\mathfrak{g}} \otimes_{\mathcal{O}_X} \mathcal{A} \to \mathcal{O}_X$$

and the pairing

$$\langle , \rangle : \mathfrak{g} \otimes_{\mathcal{O}_X} \mathfrak{g} \to \mathcal{O}_X .$$

These yield, respectively, the maps $\hat{\mathfrak{g}} \to \mathcal{A}^\vee$ and $\mathfrak{g} \to \mathfrak{g}^\vee$. Together with the projection $\hat{\mathfrak{g}} \to \mathfrak{g}$ and the map $\mathcal{A}^\vee \to \mathfrak{g}^\vee$ adjoint to the inclusion $\mathcal{A} \to \mathfrak{A}$ they fit into the diagram

$$
\begin{array}{ccc}
\hat{\mathfrak{g}} & \longrightarrow & \mathcal{A}^\vee \\
\downarrow & & \downarrow \\
\mathfrak{g} & \longrightarrow & \mathfrak{g}^\vee
\end{array}
$$

3.2.4. **Lemma.** The diagram (3.2.2) is Cartesian.

3.2.5. **Corollary.** $\hat{\mathfrak{g}}$ is canonically isomorphic to $\mathcal{A}^\vee \times_{\mathfrak{g}^\vee} \mathfrak{g}$.
3.3. Central extensions of Lie algebras. We maintain the notations introduced above, i.e. \( A \) is a transitive Lie \( \mathcal{O}_X \)-algebroid and \( g \) denotes \( g(A) \) so that there is an exact sequence

\[
0 \to g \overset{i}{\to} A \overset{\pi}{\to} T_X \to 0.
\]

Hence, \( g \) is a Lie algebra in \( \mathcal{O}_X \)-modules. We assume that \( g \) is locally free of finite rank.

Suppose in addition that \( g \) is equipped with a symmetric \( \mathcal{O}_X \)-bilinear pairing

\[
\langle \cdot, \cdot \rangle : g \otimes \mathcal{O}_X g \to \mathcal{O}_X
\]

which is invariant under the adjoint action of \( A \), i.e., for \( a \in A \) and \( b, c \in g \)

\[
\pi(a)(\langle b, c \rangle) = \langle [a, b], c \rangle + \langle b, [a, c] \rangle
\]

holds.

3.3.1. From Lie to Leibniz. The map \( i : g \to A \) and the pairing on \( g \) give rise to the maps

\[
A^\vee \overset{i^\vee}{\to} g^\vee \overset{\langle \cdot, \cdot \rangle}{\leftarrow} g
\]

Let \( \widehat{g} = A^\vee \times_g^\vee g \) and let \( \text{pr} : \widehat{g} \to g \) denote the canonical projection. A section of \( \widehat{g} \) is a pair \((a^\vee, b)\), where \( a^\vee \in A^\vee \) and \( b \in g \), which satisfies \( i^\vee(a^\vee)(c) = \langle b, c \rangle \) for \( c \in g \).

The Lie algebra \( g \) acts on \( A \) (by the restriction of the adjoint action) by \( \mathcal{O}_X \)-linear endomorphisms and the map \( i : g \to A \) is a map of \( g \)-modules. Therefore, \( A^\vee \) and \( g^\vee \) are \( g \)-modules in a natural way and the map \( i^\vee \) is a morphism of such. Hence, \( \widehat{g} \) is a \( g \)-module in a natural way and the map \( \text{pr} \) is a morphism of \( g \)-modules.

As a consequence, \( \widehat{g} \) acquires the canonical structure of a Leibniz algebra with the Leibniz bracket \([\widehat{a}, \widehat{b}]\) of two sections \( \widehat{a}, \widehat{b} \in \widehat{g} \) given by the formula \([\widehat{a}, \widehat{b}] = \text{pr}(\widehat{a})(\widehat{b})\).

We define a symmetric \( \mathcal{O}_X \)-bilinear pairing

\[
\langle \cdot, \cdot \rangle : \widehat{g} \otimes \mathcal{O}_X \widehat{g} \to \mathcal{O}_X
\]

as the composition of \( \text{pr} \otimes \text{pr} \) with the pairing on \( g \).

The inclusion \( \Omega^1_X \overset{\pi^\vee}{\to} A^\vee \) gives rise to the derivation \( \partial : \mathcal{O}_X \to \widehat{g} \).

3.3.2. Lemma. The Leibniz bracket, the symmetric pairing and the derivation defined above endow \( \widehat{g} \) with the structure of a Courant \( \Omega^1_X \)-extension of \( g \) (in particular, a Courant \( \mathcal{O}_X \)-algebroid with the trivial anchor map).

3.3.3. Lemma. The isomorphism of Corollary 3.2.5 is an isomorphism of Courant extensions of \( g(A) \).

3.3.4. Suppose that \( \nabla \) is a connection on \( A \). \( \nabla \) determines

1. the isomorphism \( g \oplus T_X \overset{\nabla}{\to} A \) by \( a + \xi \mapsto i(a) + \nabla(\xi) \), where \( a \in g \) and \( \xi \in T_X \);
2. the isomorphism \( \phi^\vee : \widehat{g} \to \Omega^1_X \oplus g \) by \( (a^\vee, b) \mapsto \nabla^\vee(a^\vee) + b \), where \( a^\vee \in A^\vee \), \( b \in g \), \( i^\vee(a^\vee) = \langle b, \cdot \rangle \), \( \nabla^\vee : A^\vee \to \Omega^1_X \) is the transpose of \( \nabla \) and \( i^\vee : A^\vee \to \mathcal{O}_X g \) is the transpose of \( i \).
Let $[\ , \]_{\nabla}$ denote the Leibniz bracket on $\Omega^1_X \oplus \mathfrak{g}$ induced by $\phi_{\nabla}$. A simple calculation (which is left to the reader) shows that it is the extension of the Lie bracket on $\mathfrak{g}$ by the $\Omega^1_X$-valued (Leibniz) cocycle $c_{\nabla}(\ , \ : \mathfrak{g} \otimes \mathfrak{g} \to \Omega^1_X$ determined by $\iota_{\xi}(a, b) = \langle [a, \nabla(\xi)], b \rangle$ (where $\xi \in \mathcal{T}_X$, $a, b \in \mathfrak{g}$, and the bracket is computed in $\mathcal{A}$). In other words, $c_{\nabla}(\ , \ )$ is the composition

$$\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\nabla^0 \otimes \text{id}} \Omega^1_X \otimes \mathfrak{g} \times \mathfrak{g} \xrightarrow{\text{id} \otimes (\ , \ )} \Omega^1_X,$$

where $\nabla^0$ is the connection on $\mathfrak{g}$ induced by $\nabla$.

Suppose that $A \in \Omega^1_X \otimes_{\mathcal{O}_X} \mathfrak{g}$. The automorphism of $\Omega^1_X \oplus \mathfrak{g}$ given by $(\alpha, a) \mapsto (\alpha + \langle A, a \rangle, a)$ is the isomorphism of Courant algebroids $(\Omega^1_X \oplus \mathfrak{g}, \{\ , \}_{\nabla}) \to (\Omega^1_X \oplus \mathfrak{g}, \{\ , \}_{\nabla + A})$ which corresponds to the identity map on $\hat{\mathfrak{g}}$ under the identifications $\phi_{\nabla}$ and $\phi_{\nabla + A}$.

### 3.4. The action of $\mathcal{EC}_\mathcal{A}_{\mathcal{O}_X}$. As before, $\mathcal{A}$ is a transitive Lie $\mathcal{O}_X$-algebroid locally free of finite rank over $\mathcal{O}_X$, $\mathfrak{g}$ denotes $\mathfrak{g}(\mathcal{A})$, $\{\ , \}$ is an $\mathcal{O}_X$-bilinear symmetric $\mathcal{A}$-invariant pairing on $\mathfrak{g}$, $\hat{\mathfrak{g}}$ is the Courant extension of $\mathfrak{g}$ constructed in 3.3.1.

3.4.1. Let $\mathcal{CE}_{\nabla}\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\{\ , \}}$ denote the stack of Courant extensions of $\mathcal{A}$ which induce the given pairing $\langle \ , \ \rangle$ on $\mathfrak{g}$. Clearly, $\mathcal{CE}_{\nabla}\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\{\ , \}}$ is a stack in groupoids.

Note that, if $\hat{\mathcal{A}}$ is in $\mathcal{CE}_{\nabla}\mathcal{T}_{\mathcal{O}_X}(\mathcal{A})_{\{\ , \}}$, then $\mathfrak{g}(\hat{\mathcal{A}})$ is canonically isomorphic to $\hat{\mathfrak{g}}$.

3.4.2. Suppose that $Q$ is an exact Courant $\mathcal{O}_X$-algebroid and $\hat{\mathcal{A}}$ is a Courant extension of $\mathcal{A}$. Let $\hat{\mathcal{A}} + Q$ denote the push-out of $\hat{\mathcal{A}} \times_{\mathcal{T}_X} Q$ by the addition map $\Omega^1_X \times \Omega^1_X \xrightarrow{\oplus} \Omega^1_X$. Thus, a section of $\hat{\mathcal{A}} + Q$ is represented by a pair $(a, q)$ with $a \in \hat{\mathcal{A}}$ and $q \in Q$ satisfying $\pi(a) = \pi(q) \in \mathcal{T}_X$. Two pairs as above are equivalent if their (componentwise) difference is of the form $(\alpha, -\alpha)$ for some $\alpha \in \Omega^1_X$.

For $a_i \in \hat{\mathcal{A}}$, $q_i \in Q$ with $\pi(a_i) = \pi(q_i)$ let

$$[(a_1, q_1), (a_2, q_2)] = [(a_1, a_2), [q_1, q_2]], \quad \langle (a_1, q_1), (a_2, q_2) \rangle = \langle a_1, a_2 \rangle + \langle q_1, q_2 \rangle$$

(3.4.1)

These operations are easily seen to descend to $\hat{\mathcal{A}} + Q$. Note that the compositions

$$\Omega^1_X \to \hat{\mathcal{A}} \to \hat{\mathcal{A}} \times_{\mathcal{T}_X} Q \to \hat{\mathcal{A}} + Q$$

and

$$\Omega^1_X \to Q \to \hat{\mathcal{A}} \times_{\mathcal{T}_X} Q \to \hat{\mathcal{A}} + Q$$

coincide; we denote their common value by

$$i : \Omega^1_X \to \hat{\mathcal{A}} + Q.$$

(3.4.2)

3.4.3. Lemma. The formulas (3.4.1) and the map (3.4.2) determine a structure of Courant extension of $\mathcal{A}$ on $\hat{\mathcal{A}} + Q$. Moreover, the map $\mathfrak{g}(\hat{\mathcal{A}}) \to \hat{\mathcal{A}} + Q$ defined by $a \mapsto (a, 0)$ induces an isomorphism $\mathfrak{g}(\hat{\mathcal{A}} + Q) \cong \mathfrak{g}(\hat{\mathcal{A}})$ of Courant extensions of $\mathfrak{g}(\mathcal{A})$ (by $\Omega^1_X$).
3.4.4. Lemma. Suppose that $\hat{A}^{(1)}$, $\hat{A}^{(2)}$ are in $CE_{X \circ X}(\mathcal{A}) \langle , \rangle$. Then, there exists a unique $Q$ in $ECA_{O_X}$, such that $\hat{A}^{(2)} = \hat{A}^{(1)} + Q$.

Proof. Let $Q$ denote the quotient of $\hat{A}^{(2)} \times_A \hat{A}^{(1)}$ by the diagonally embedded copy of $\hat{g}$. Then, $Q$ is an extension of $\mathcal{T}$ by $\Omega^1_X$. There is a unique structure of an exact Courant algebroid on $Q$ such that $\hat{A}^{(2)} = \hat{A}^{(1)} + Q$. □

3.5. Courant extensions of flat connections. Suppose that $\mathcal{A}$ is a transitive Lie $O_X$-algebroid, locally free over $O_X$, $g = g(\mathcal{A})$, $\langle , \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $g$.

3.5.1. Suppose that $\nabla$ is a flat connection on $\mathcal{A}$ and $Q$ is an exact Courant algebroid. For $a, b \in g$, $q, q_1, q_2 \in Q$ let

$$
\langle a, b \rangle_{\nabla, Q} = \langle a, b \rangle_g, \quad \langle q_1, q_2 \rangle_{\nabla, Q} = \langle q_1, q_2 \rangle_Q, \quad \langle a, q \rangle_{\nabla, Q} = 0,
$$

(3.5.1)

and

$$
\left[ a, b \right]_{\nabla, Q} = [a, b]_{\nabla}, \quad \left[ q_1, q_2 \right]_{\nabla, Q} = [q_1, q_2]_Q, \quad \left[ q, a \right]_{\nabla, Q} = \nabla^g(\pi(q))(a)
$$

(3.5.2)

taking values in $g \oplus Q$, where $[ , ]_{\nabla}$ is the Leibniz bracket on $\Omega^1_X \oplus g$ as in 3.3.4, $\nabla^g$ is the connection on $g$ induced by $\nabla$ and $\pi$ is the anchor of $Q$.

3.5.2. Lemma. The formulas (3.5.1) and (3.5.2) define a structure of a Courant extension of $\mathcal{A}$ on $g \oplus Q$.

3.5.3. Corollary. Suppose that $\mathcal{A}$ admits a flat connection locally on $X$. Then, $CE_{X \circ X}(\mathcal{A}) \langle , \rangle$ is locally non-empty, hence a torsor under $ECA_{O_X}$.

3.5.4. Notation. We denote $g \oplus Q$ together with the Courant algebroid structure given by (3.5.1) and (3.5.2) by $\hat{\mathcal{A}}_{\nabla, Q}$.

3.5.5. Conversely, suppose that $\hat{\mathcal{A}}$ is a Courant extension of $\mathcal{A}$, and $\nabla$ is a flat connection on $\hat{\mathcal{A}}$.

Let $Q_{\nabla, \hat{\mathcal{A}}} \subset \hat{\mathcal{A}}$ denote the pre-image of $\nabla(T_X)$ under the projection $\hat{\mathcal{A}} \to \mathcal{A}$.

3.5.6. Lemma.

(1) The (restrictions of the) Leibniz bracket and the symmetric pairing on $\hat{\mathcal{A}}$ endow $Q_{\nabla, \hat{\mathcal{A}}}$ with a structure of an exact Courant algebroid.

(2) $Q_{\nabla, \hat{\mathcal{A}}} \cap Q_{\nabla} = 0$.

(3) The projection $\hat{\mathcal{A}} \to \mathcal{A}$ restricts to an isomorphism $Q_{\nabla, \hat{\mathcal{A}}} \to g$.

(4) $\hat{g} = Q_{\nabla, \hat{\mathcal{A}}} + \Omega^1_{\hat{\mathcal{A}}}$.

(5) The induced isomorphism $\hat{g} \sim g \oplus \Omega^1_X$ coincides with the one constructed in 3.3.4.
3.5.7. Lemma. The isomorphism $\hat{\mathcal{A}} \cong \mathfrak{g} \oplus \mathcal{Q}_{\nabla,\hat{\mathcal{A}}}$ induced by $\nabla$ is an isomorphism $\hat{\mathcal{A}} \cong \hat{\mathcal{A}}_{\nabla,\mathcal{Q},\hat{\mathcal{A}}}$ (of Courant extensions of $\mathcal{A}$).

3.5.8. Change of connection. Suppose that $\nabla$ is a flat connection on $\mathcal{A}$ and $A \in \Omega^1_X \otimes_{\mathcal{O}_X} \mathfrak{g}$ satisfies the Maurer–Cartan equation $\nabla A + \frac{1}{2} [A, A] = 0$, so that the connection $\nabla + A$ is also flat. Suppose that $\mathcal{Q}$ is an exact Courant algebroid. To simplify notations, let $\hat{\mathcal{A}} = \hat{\mathcal{A}}_{\nabla,\mathcal{Q}}$.

By Lemma 3.5.7, we have the isomorphism $\hat{\mathcal{A}} \cong \hat{\mathcal{A}}_{\nabla + A, \mathcal{Q}_{\nabla + A, \hat{\mathcal{A}}}}$. Recall that a closed 3-form $H \in \Omega^3_X$ defines an exact Courant algebroid $\mathcal{Q}_H$ equipped with a connection, whose curvature is equal to $H$.

Since $A \in \Omega^1_X \otimes_{\mathcal{O}_X} \mathfrak{g}$ satisfies the Maurer–Cartan equation, the 3-form $\langle A, [A, A] \rangle$ is closed.

3.5.9. Lemma. In the notations introduced above, there is an isomorphism $\mathcal{Q}_{\nabla + A, \hat{\mathcal{A}}} \cong \mathcal{Q} + \mathcal{Q}_{\langle A, [A, A] \rangle}$.

Proof. We identify $\mathcal{A}$ with $\mathfrak{g} \oplus T_X$ using the flat connection $\nabla$. In terms of this identification, the image of $T$ under the connection $\nabla + A$ consists of pairs $(A(\xi), \xi)$, where $\xi \in T_X$. Therefore, $\mathcal{Q}_{\nabla + A, \hat{\mathcal{A}}} \subset \mathfrak{g} \oplus \hat{\mathcal{A}}$ consists of pairs $(A(\xi), q)$, where $\xi \in T_X$ and $\pi(q) = \xi$.

For $i = 1, 2$, $\xi_i \in T_X$, $q_i \in \mathcal{Q}$ satisfying $\pi(q_i) = \xi_i$, we calculate the Leibniz bracket in $\hat{\mathcal{A}}$ using (3.5.1) and (3.5.2):

$$[(A(\xi_1), q_1), (A(\xi_2), q_2)]_{\nabla, \mathcal{Q}} = (A([\xi_1, \xi_2]), [(A(\xi_1), \nabla(\bullet)_\xi), A(\xi_2)] + [q_1, q_2]_{\mathcal{Q}}) = (A([\xi_1, \xi_2]), \iota_{\xi_1, \xi_2}([A, A] + [q_1, q_2])_{\mathcal{Q}})$$

The latter formula shows that the assignment $(A(\pi(q)), q) \mapsto (q, \pi(q))$ viewed as a morphism of extensions of $T_X$ by $\Omega^1_X$ $\mathcal{Q}_{\nabla + A, \hat{\mathcal{A}}} \to \mathcal{Q} + (\Omega^1_X \oplus T_X)$

is, in fact a morphism of exact Courant algebroids $\mathcal{Q}_{\nabla + A, \hat{\mathcal{A}}} \to \mathcal{Q} + \mathcal{Q}_{\langle A, [A, A] \rangle}$.

3.5.10. Proposition. For $\mathcal{A}$ as above, $\nabla$ a flat connection on $\mathcal{A}$ the assignment $\mathcal{Q} \mapsto \hat{\mathcal{A}}_{\nabla, \mathcal{Q}}$ gives rise to a morphism $\mathcal{ECA}_{\mathcal{O}_X} \to \mathcal{ECA}_{\mathcal{O}_X}(\mathcal{A})_{\langle \cdot , \cdot \rangle}$ of $\mathcal{ECA}_{\mathcal{O}_X}$-torsors, i.e. a trivialization of $\mathcal{ECA}_{\mathcal{O}_X}(\mathcal{A})_{\langle \cdot , \cdot \rangle}$.

Proof. This amounts to showing that, for $i = 1, 2$, $\mathcal{Q}_i \in \mathcal{ECA}_{\mathcal{O}_X}$, there is a canonical isomorphism $\hat{\mathcal{A}}_{\nabla, \mathcal{Q}_1 + \mathcal{Q}_2} \cong \hat{\mathcal{A}}_{\nabla, \mathcal{Q}_1} + \mathcal{Q}_2$ which possesses associativity properties. This follows from (3.5.1) and (3.5.2) and the definition of the $\mathcal{ECA}_{\mathcal{O}_X}$-action. We leave details to the reader.

3.5.11. Corollary. There is an isomorphism $\hat{\mathcal{A}}_{\nabla + A, \mathcal{Q}} \cong \hat{\mathcal{A}}_{\nabla, \mathcal{Q}} + \mathcal{Q}_{\langle A, [A, A] \rangle}$. 

3.6. The Pontryagin class. Suppose that $\mathcal{A}$ is a transitive Lie algebroid, locally free of finite rank over $\mathcal{O}_X$, $g = g(\mathcal{A})$, and $\langle \ , \ \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $g$. We assume from now on that $\mathcal{A}$ admits a flat connection locally on $X$.

3.6.1. Suppose that $\mathcal{U} = \{U_i\}_{i \in I}$ is a covering of $X$ by open subsets such that $\mathcal{A}|_{U_i}$ admits a flat connection. Let $\nabla_i$ denote a flat connection on $\mathcal{A}|_{U_i}$. Let $A_{ij} = \nabla_j - \nabla_i$, $A_{ij} \in \Omega^1_X \otimes \mathcal{O}_X \mathcal{g}(U_i \cap U_j)$.

Let $H_{ij} = \langle A_{ij}, [A_{ij}, A_{ij}] \rangle$, $H_{ij} \in \Omega^3_{\mathcal{X}}(U_i \cap U_j)$; let $H \in \check{C}^1(\mathcal{U}; \Omega^3_{\mathcal{X}})$ denote the corresponding cochain.

Let $B_{ijk} = -\langle A_{ij} \wedge A_{jk} \rangle - \langle A_{jk} \wedge A_{ki} \rangle + \langle A_{ki} \wedge A_{ij} \rangle$, $B_{ijk} \in \Omega^2_X(U_i \cap U_j \cap U_k)$; let $B \in \check{C}^2(\mathcal{U}; \Omega^2_X)$ denote the corresponding cochain.

3.6.2. Lemma.

1. $\check{\partial}B = dB = 0$, $\check{\partial}H = dB$, i.e. $H + B$ is a 2-cocycle in $\check{C}^*(\mathcal{U}; \Omega^2_X \to \Omega^3_{\mathcal{X}})$.

2. The class of $H + B$ in $H^2(X; \Omega^2_X \to \Omega^3_{\mathcal{X}})$ does not depend on the choices of the open cover and of the locally defined flat connections.

3.6.3. Notation. We denote the class of $H + B$ in $H^2(X; \Omega^2_X \to \Omega^3_{\mathcal{X}})$ by $n(\mathcal{A}, \langle \ , \ \rangle)$.

3.7. The Pontryagin class via obstruction theory.

3.7.1. We begin by a brief outline of the cohomological classification of $\mathcal{E} \mathcal{C} \mathcal{A}_{\mathcal{O}_X}$-torsors.

Suppose that $\mathcal{S}$ is a torsor under $\mathcal{E} \mathcal{C} \mathcal{A}_{\mathcal{O}_X}$. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a covering of $X$ by open subsets such that $\mathcal{S}(U_i)$ is non-empty for all $i \in I$. In this case there are isomorphisms $\mathcal{E} \mathcal{C} \mathcal{A}_{\mathcal{O}_X}|_{U_i} \cong \mathcal{S}|_{U_i}$ of $\mathcal{E} \mathcal{C} \mathcal{A}_{\mathcal{O}_X}|_{U_i}$-torsors. A choice of such gives rise to the objects $\mathcal{Q}_{ij} \in \mathcal{E} \mathcal{C} \mathcal{A}_{\mathcal{O}_X}(U_i \cap U_j)$, isomorphisms $\mathcal{Q}_{ij} = -\mathcal{Q}_{ji}$ and flat connections $\nabla_{ijk}^0$ on $\mathcal{Q}_{ijk} = \mathcal{Q}_{ij} + \mathcal{Q}_{jk} + \mathcal{Q}_{ki}$.

We may assume (by passing to a refinement of $\mathcal{U}$), that the exact Courant algebroids $\mathcal{Q}_{ij}$ admit connections. Let $\nabla_{ij}$ denote a connection on $\mathcal{Q}_{ij}$ whose curvature we denote by $H_{ij} \in \Omega^3_{\mathcal{X}}(U_i \cap U_j)$. Let $H \in \check{C}^1(\mathcal{U}; \Omega^3_{\mathcal{X}})$ denote the corresponding cochain.

The connections $\nabla_{ij}$ give rise to connections on the exact Courant algebroids $\mathcal{Q}_{ijk}$. Hence there are 2-forms $B_{ijk} \in \Omega^2_X(U_i \cap U_j \cap U_k)$ such that $\nabla_{ijk} = \nabla_{ijk}^0 + B_{ijk}$. Let $B \in \check{C}^2(\mathcal{U}; \Omega^2_X)$ denote the corresponding cochain.

It is clear that $dH = \check{\partial}B = 0$ and $\check{\partial}H = dB$, i.e. $H + B$ is a 2-cocycle in $\check{C}^*(\mathcal{U}; \Omega^2_X \to \Omega^3_{\mathcal{X}})$. One checks easily that the class of $H + B$ in $H^2(X; \Omega^2_X \to \Omega^3_{\mathcal{X}})$ is independent of the choices of the covering $\mathcal{U}$ and the connections $\nabla_{ij}$. Moreover, the above construction establishes a bijection (in fact an isomorphism of $\mathcal{C}$-vector spaces) between the isomorphism classes of torsors under $\mathcal{E} \mathcal{C} \mathcal{A}_{\mathcal{O}_X}$ and $H^2(X; \Omega^2_X \to \Omega^3_{\mathcal{X}})$.

3.7.2. Theorem. Suppose that $\mathcal{A}$ is a transitive Lie $\mathcal{O}_X$-algebroid, locally free of finite rank over $\mathcal{O}_X$ which admits a flat connection locally on $X$, and $\langle \ , \ \rangle$ is an $\mathcal{A}$-invariant symmetric pairing on $g(\mathcal{A})$ The isomorphism class of the $\mathcal{E} \mathcal{C} \mathcal{A}_{\mathcal{O}_X}$-torsor $\mathcal{E} \mathcal{T} \mathcal{O}_X(\mathcal{A}) \langle \ , \ \rangle$ corresponds to $n(\mathcal{A}, \langle \ , \ \rangle)$. In particular, Courant extensions of $\mathcal{A}$ which induce $\langle \ , \ \rangle$ on $g(\mathcal{A})$ exist globally on $X$ if and only if $n(\mathcal{A}, \langle \ , \ \rangle) = 0$. 

Proof. Let $U = \{U_i\}_{i \in I}$ be a covering of $X$ by open subsets such that $\mathcal{A}|_{U_i}$ admits a flat connection. Let $\nabla_i$ be a flat connection on $\mathcal{A}|_{U_i}$. By Proposition 3.5.10, these give rise to trivializations of $\mathcal{C}(\mathcal{A})|_{U_i}$. The procedure outlined in 3.7.1 applied to these yields the definition of $\pi(\mathcal{A}, \langle , \rangle)$ given in 3.6. □

3.8. The Pontryagin class Atiyah style. Throughout this section $\mathcal{A}$ is a transitive Lie $\mathcal{O}_X$-algebroid locally free of finite rank over $\mathcal{O}_X$, $g$ denotes $g(\mathcal{A})$, $\langle , \rangle$ is a symmetric $\mathcal{O}_X$-bilinear $\mathcal{A}$-invariant pairing on $g$, $\hat{g}$ is the Courant extension of $g$ as in 3.3.1.

3.8.1. The Atiyah class. The (isomorphism class of the) extension

$$0 \to g \to \mathcal{A} \to T_X \to 0 \quad (3.8.1)$$

is an element of $\text{Ext}_{\mathcal{O}_X}(T_X, g)$, whose image under the canonical isomorphism $\text{Ext}_{\mathcal{O}_X}(T_X, g) \cong H^1(X; \Omega^1_X \otimes_{\mathcal{O}_X} g)$ is called the Atiyah class of $\mathcal{A}$ and will be denoted $\alpha(\mathcal{A})$.

Recall that a connection on $\mathcal{A}$ is a splitting of the extension (3.8.1). Let $\mathcal{C}(\mathcal{A})$ denote the sheaf of locally defined connections on $\mathcal{A}$. As the difference of two connections is a map $T_X \to g$, the sheaf $\mathcal{C}(\mathcal{A})$ is a torsor under $\Omega^1_X \otimes_{\mathcal{O}_X} g$. The Atiyah class $\alpha(\mathcal{A})$ is the isomorphism class of the $\Omega^1_X \otimes_{\mathcal{O}_X} g$-torsor $\mathcal{C}(\mathcal{A})$.

The cup product together with the pairing $\langle , \rangle$ give rise to the map

$$H^1(X; \Omega^1_X \otimes_{\mathcal{O}_X} g) \otimes H^1(X; \Omega^1_X \otimes_{\mathcal{O}_X} g) \to H^2(X; \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X).$$

We will denote the image of $a \otimes b$ under this map by $\langle a \sim b \rangle$.

3.8.2. Lemma. $\langle \alpha(\mathcal{A}) \sim \alpha(g) \rangle$ is the image of $\pi(\mathcal{A}, \langle , \rangle)$ under the map $H^2(X; \Omega^2_X \to \Omega^3_X) \to H^2(X; \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X)$.

3.9. Obstruction theoretic interpretation. Recall that there is an exact sequence

$$0 \to \Omega^1_X \to \hat{g} \to g \to 0.$$ 

Spliced with (3.8.1) it gives rise to the extension

$$0 \to \Omega^1_X \to \hat{g} \to \mathcal{A} \to T_X \to 0 \quad (3.9.1)$$

whose isomorphism class is an element of $\text{Ext}_{\mathcal{O}_X}^2(T_X, \Omega^1_X)$. Let $\beta = \beta(\mathcal{A}, \langle , \rangle)$ denote its image under the canonical isomorphism $\text{Ext}_{\mathcal{O}_X}^2(T_X, \Omega^1_X) \cong H^2(X; \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X)$.

The extension (3.9.1) gives rise to the stack in groupoids $\mathcal{L}$ defined as follows. For an open set $U \subseteq X$, $\mathcal{L}(U)$ is the category of pairs $(\hat{C}, \phi)$, where $\hat{C}$ is an $\Omega^1_X \otimes_{\mathcal{O}_X} \hat{g}$-torsor on $U$ and $\phi$ is a morphism of $\Omega^1_X \otimes_{\mathcal{O}_X} g$-torsors $\hat{C} \otimes_{\Omega^1_X \otimes_{\mathcal{O}_X} g} \Omega^1_X \otimes_{\mathcal{O}_X} g \to \mathcal{C}(\mathcal{A})|_U$. Equivalently, $\mathcal{L}(U)$ is the category of pairs $(\hat{A}, \psi)$, where $\hat{A}$ is an extension of $T_X$ by $\hat{g}$ and $\psi$ is a morphism of the push-out of $\hat{A}$ by the map $\hat{g} \to g$ to $\mathcal{A}$ (of extensions of $T_X$ by $g$).

As is well-known, the stack $\mathcal{L}$ is a gerbe with lien $\Omega^1_X \otimes_{\mathcal{O}_X} \Omega^2$, whose isomorphism class in $H^2(X; \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1)$ is, on one hand, $\beta$, and, on the other hand, the image of $\alpha(\mathcal{A}, \langle , \rangle)$ under the boundary map $H^1(X; \Omega^1_X \otimes_{\mathcal{O}_X} g) \to H^2(X; \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1)$ induced by the extension

$$0 \to \Omega^1_X \otimes_{\mathcal{O}_X} \Omega^1_X \to \Omega^1_X \otimes_{\mathcal{O}_X} \hat{g} \to \Omega^1_X \otimes_{\mathcal{O}_X} g \to 0.$$
3.9.1. **Lemma.** $\beta = \langle \alpha(A) \sim \alpha(A) \rangle$.

3.9.2. **Remark.** Note that there is a natural morphism of stacks $\mathcal{E}_{\mathcal{T} \mathcal{O}_X}(A \langle , \rangle) \rightarrow \mathcal{L}$ (which maps a Courant extension to the underlying $\mathcal{O}_X$-module).

4. **Vertex algebroids**

4.1. **Vertex algebroids.** A vertex $\mathcal{O}_X$-algebroid is a sheaf of $\mathbb{C}$-vector spaces $\mathcal{V}$ with a pairing

$$\mathcal{O}_X \otimes \mathbb{C} \mathcal{V} \rightarrow \mathcal{V}$$

$$f \otimes v \mapsto f \ast v$$

such that $1 \ast v = v$ (i.e. a “non-associative unital $\mathcal{O}_X$-module”) equipped with

1. a structure of a Leibniz $\mathbb{C}$-algebra $[ , ] : \mathcal{V} \otimes \mathbb{C} \mathcal{V} \rightarrow \mathcal{V}$
2. a $\mathbb{C}$-linear map of Leibniz algebras $\pi : \mathcal{V} \rightarrow \mathcal{T}_X$ (the anchor)
3. a symmetric $\mathbb{C}$-bilinear pairing $\langle , \rangle : \mathcal{V} \otimes \mathbb{C} \mathcal{V} \rightarrow \mathcal{O}_X$
4. a $\mathbb{C}$-linear map $\partial : \mathcal{O}_X \rightarrow \mathcal{V}$ such that $\pi \circ \partial = 0$

which satisfy

$$f \ast (g \ast v) - (fg) \ast v = -\pi(v)(f) \ast \partial(g) - \pi(v)(g) \ast \partial(f)$$

$$[v_1, f \ast v_2] = \pi(v_1)(f) \ast v_2 + f \ast [v_1, v_2]$$

$$[v_1, v_2] + [v_2, v_1] = \partial(\langle v_1, v_2 \rangle)$$

$$\pi(f \ast v) = f \pi(v)$$

$$\langle f \ast v_1, v_2 \rangle = f \langle v_1, v_2 \rangle - \pi(v_1)(\pi(v_2)(f))$$

$$\pi(v)(\langle v_1, v_2 \rangle) = \langle [v, v_1], v_2 \rangle + \langle v_1, [v, v_2] \rangle$$

$$\partial(fg) = f \ast \partial(g) + g \ast \partial(f)$$

$$[v, \partial(f)] = \partial(\pi(v)(f))$$

$$\langle v, \partial(f) \rangle = \pi(v)(f)$$

for $v, v_1, v_2 \in \mathcal{V}$, $f, g \in \mathcal{O}_X$.

4.1.1. A morphism of vertex $\mathcal{O}_X$-algebroids is a $\mathbb{C}$-linear map of sheaves which preserves all of the structures.

4.1.2. **Remark.** The notions “a vertex $\mathcal{O}_X$ algebroid with the trivial anchor map” and “a Courant $\mathcal{O}_X$ algebroid with the trivial anchor map” are equivalent.

4.2. **The associated Lie algebroid.** Suppose that $\mathcal{V}$ is a vertex $\mathcal{O}_X$-algebroid. Let

$$\Omega_\mathcal{V} \overset{\text{def}}{=} \mathcal{O}_X \ast \partial(\mathcal{O}_X) \subset \mathcal{V} ,$$

$$\mathcal{V} \overset{\text{def}}{=} \mathcal{V} / \Omega_\mathcal{V} .$$
It is easy to see (cf. [B]) that the action of $\mathcal{O}_X$ on $\mathcal{V}$ descends to a structure of an $\mathcal{O}_X$-module on $\mathcal{V}$ and the Leibniz bracket on $\mathcal{V}$ descends to a Lie bracket on $\mathcal{V}$. Moreover, there is an evident map $\mathcal{V} \rightarrow T_X$.

4.2.1. Lemma. The $\mathcal{O}_X$-module $\mathcal{V}$ with the bracket and the anchor as above is a Lie $\mathcal{O}_X$-algebroid.

4.3. Transitive vertex algebroids. A vertex $\mathcal{O}_X$-algebroid is called transitive if the anchor map is surjective.

4.4. Exact vertex algebroids. A vertex algebroid $\mathcal{V}$ is called exact if the map $\mathcal{V} \rightarrow T_X$ is an isomorphism. We denote the stack of exact vertex $\mathcal{O}_X$-algebroids by $\mathcal{EVA}_{\mathcal{O}_X}$.

A morphism of exact vertex algebroids induces a morphism of underlying extensions of $T_X$ by $\Omega^1_X$. The latter is an isomorphism and it is clear that the inverse morphism of extensions is a morphism of vertex algebroids. Hence, $\mathcal{EVA}_{\mathcal{O}_X}$ is a stack in groupoids.

4.4.1. It was shown in [B] that $\mathcal{EVA}_{\mathcal{O}_X}$ is locally non-empty and has a canonical structure of a torsor under $\mathcal{EC}_A_{\mathcal{O}_X}$. Isomorphism classes of $\mathcal{EC}_A_{\mathcal{O}_X}$-torsors form a vector space naturally isomorphic to $H^2(X; \Omega^2_X \rightarrow \Omega^3_X)$.

The purpose of this section is the determination of the isomorphism class of $\mathcal{EVA}_{\mathcal{O}_X}$. The following theorem was originally proven in [GMS] by explicit calculations with representing cocycles. Our proof, “coordinate-free” and based on Theorem 3.7.2 and the strategy proposed in [BD], appears in 4.9.6.

4.4.2. Theorem ([GMS]). The class of $\mathcal{EVA}_{\mathcal{O}_X}$ in $H^2(X; \Omega^2_X \rightarrow \Omega^3_X)$ is equal to $2 \text{ch}_2(\Omega^1_X)$.

4.4.3. Remark. Suppose that $P$ is a $GL_n$-torsor on $X$. Let $\mathcal{A}_P$ denote the Atiyah algebra of $P$. Then, $\mathfrak{g}(\mathcal{A}_P) = \mathfrak{gl}_n^P$ and the symmetric pairing on the latter given by the trace of the product of matrices is $\mathcal{A}_P$-invariant. The corresponding Pontriagin class is equal to $2 \text{ch}_2(P)$.

4.5. Vertex extensions of Lie algebroids. Suppose that $\mathcal{A}$ is a Lie $\mathcal{O}_X$-algebroid.

4.5.1. Definition. A vertex extension of $\mathcal{A}$ is a vertex algebroid $\hat{\mathcal{A}}$ together with an isomorphism $\hat{\mathcal{A}} = \mathcal{A}$ of Lie $\mathcal{O}_X$-algebroids.

4.5.2. Morphisms of vertex extensions. A morphism of vertex extensions of $\mathcal{A}$ is a morphism of vertex algebroids which is compatible with the identifications.

Let $\mathcal{VE}_{\mathcal{O}_X}(\mathcal{A})$ denote the stack of Courant extensions of $\mathcal{A}$.

4.6. Vertex extensions of transitive Lie algebroids. From now on we suppose that $\mathcal{A}$ is a transitive Lie $\mathcal{O}_X$-algebroid locally free of finite rank over $\mathcal{O}_X$. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{g}}(\mathcal{A})$.
4.6.1. Suppose that $\hat{A}$ is a vertex extension of $A$. Then, the derivation $\partial : O_X \to \hat{A}$ induces an isomorphism $\Omega^1_X \cong \Omega_{\hat{A}}$. The resulting exact sequence

$$0 \to \Omega^1_X \to \hat{A} \to A \to 0$$

is canonically associated to the vertex extension $\hat{A}$ of $A$. Since a morphism of vertex extensions of $\hat{A}$ induces a morphism of associated extensions of $A$ by $\Omega^1_X$, it is an isomorphism of the underlying sheaves. It is clear that the inverse isomorphism is a morphism of vertex extensions of $A$.

Therefore, $\mathcal{VE}_{\tau O_X}(A)$ is a stack in groupoids.

4.6.2. Remark. $\mathcal{VE}_{\tau O_X}(T_X)$ is none other than $\mathcal{EA}_{O_X}$.

4.6.3. Suppose that $\hat{A}$ is a vertex extension of $A$. Let $\hat{g} = \hat{g}(\hat{A})$ denote the kernel of the anchor map (of $\hat{A}$). Thus, $\hat{g}$ is a vertex (equivalently, Courant) extension of $g$.

Analysis similar to that of 3.2 shows that

- the symmetric pairing on $\hat{A}$ induces a symmetric $O_X$-bilinear pairing on $g$ which is $A$-invariant;
- the vertex extension $\hat{g}$ is obtained from the Lie algebroid $A$ and the symmetric $A$-invariant pairing on $g$ as in 3.3.

4.7. The action of $\mathcal{ECA}_{O_X}$. As before, $A$ is a transitive Lie $O_X$-algebroid locally free of finite rank over $O_X$, $g$ denotes $g(A)$, $\langle \ , \ \rangle$ is an $O_X$-bilinear symmetric $A$-invariant pairing on $g$, $\hat{g}$ is the Courant extension of $g$ constructed in 3.3.1.

4.7.1. Let $\mathcal{VE}_{\tau O_X}(A)\langle \ , \ \rangle$ denote the stack of Courant extensions of $A$ which induce the given pairing $\langle \ , \ \rangle$ on $g$. Clearly, $\mathcal{VE}_{\tau O_X}(A)\langle \ , \ \rangle$ is a stack in groupoids.

Note that, if $\hat{A}$ is in $\mathcal{VE}_{\tau O_X}(A)\langle \ , \ \rangle$, then $g(\hat{A})$ is canonically isomorphic to $\hat{g}$.

4.7.2. Suppose that $Q$ is an exact Courant $O_X$ algebroid and $\hat{A}$ is a vertex extension of $A$. Let $\hat{A} + Q$ denote the push-out of $\hat{A} \times_{\tau_X} Q$ by the addition map $\Omega^1_X \times \Omega^1_X \to \Omega^1_X$. Thus, a section of $\hat{A} + Q$ is represented by a pair $(a, q)$ with $a \in \hat{A}$ and $q \in Q$ satisfying $\pi(a) = \pi(q) \in T_X$. Two pairs as above are equivalent if their (componentwise) difference is of the form $(i(\alpha), -i(\alpha))$ for some $\alpha \in \Omega^1_X$.

For $a \in \hat{A}$, $q \in Q$ with $\pi(a) = \pi(q)$, $f \in O_X$ let

$$f \ast (a, q) = (f \ast a, f q), \quad \partial(f) = \partial_{\hat{A}}(f) + \partial_Q(f) \ . \quad (4.7.1)$$

For $a_i \in \hat{A}$, $q_i \in Q$ with $\pi(a_i) = \pi(q_i)$ let

$$[(a_1, q_1), (a_2, q_2)] = [(a_1, a_2), [q_1, q_2]], \quad \langle (a_1, q_1), (a_2, q_2) \rangle = \langle a_1, a_2 \rangle + \langle q_1, q_2 \rangle \quad (4.7.2)$$

These operations are easily seen to descend to $\hat{A} + Q$. 
The two maps $\Omega^1_X \to \mathcal{A} + Q$ given by $\alpha \mapsto (i(\alpha), 0)$ and $\alpha \mapsto (0, i(\alpha))$ coincide; we denote their common value by

$$i : \Omega^1_X \to \mathcal{A} + Q.$$  \hspace{1cm} (4.7.3)

4.7.3. **Lemma.** The formulas (4.7.1), (4.7.2) and the map (4.7.3) determine a structure of vertex extension of $\mathcal{A}$ on $\mathcal{A} + Q$. Moreover, the map the map $g(\mathcal{A}) \to \mathcal{A} + Q$ defined by $a \mapsto (a, 0)$ induces an isomorphism $g(\mathcal{A} + Q) \cong g(\mathcal{A})$ of vertex (equivalently, Courant) extensions of $g(\mathcal{A})$ (by $\Omega^1_X$).

4.7.4. **Lemma.** Suppose that $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}$ are in $\mathcal{ECA}_{O_X}(\mathcal{A})$. Then, there exists a unique $Q$ in $\mathcal{ECA}_{O_X}$ such that $\mathcal{A}^{(2)} = \mathcal{A}^{(1)} + Q$.

**Proof.** Let $Q$ denote the quotient of $\mathcal{A}^{(2)} \times_{\mathcal{A}} \mathcal{A}^{(1)}$ by the diagonally embedded copy of $\mathcal{g}$. Then, $Q$ is an extension of $\mathcal{T}$ by $\Omega^1_X$. There is a unique structure of an exact Courant algebroid on $Q$ such that $\mathcal{A}^{(2)} = \mathcal{A}^{(1)} + Q$. \hfill $\square$

4.8. **Comparison of $\mathcal{ECA}_{O_X}$-torsors.** Suppose that $\mathcal{A}$ is a vertex extension of the Lie algebroid $\mathcal{A}$. Let $\langle \cdot, \cdot \rangle$ denote the induced symmetric pairing on $g(\mathcal{A})$.

4.8.1. Suppose that $\mathcal{V}$ is an exact vertex algebroid. Let $\mathcal{A} - \mathcal{V}$ denote the pushout of $\mathcal{A} \times_{\mathcal{T}} \mathcal{V}$ by the difference map $\Omega^1_X \times \Omega^1_X \to \Omega^1_X$. Thus, a section of $\mathcal{A} - \mathcal{V}$ is represented by a pair $(a, v)$ with $a \in \mathcal{A}, v \in \mathcal{V}$ satisfying $\pi(a) = \pi(v) \in \mathcal{T}_X$. Two pairs as above are equivalent if their (componentwise) difference is of the form $(\alpha, \alpha)$ for some $\alpha \in \Omega^1_X$.

For $a \in \mathcal{A}, v \in \mathcal{V}$ with $\pi(a) = \pi(v)$, $f \in O_X$, let

$$f \ast (a, v) = (f \ast a, f \ast v), \quad \partial(f) = \partial_{\mathcal{A}}(f) - \partial_{\mathcal{V}}(f).$$  \hspace{1cm} (4.8.1)

For $a_i \in \mathcal{A}, v_i \in \mathcal{V}$ with $\pi(a_i) = \pi(v_i)$ let

$$[(a_1, v_1), (a_2, v_2)] = [(a_1, a_2), [v_1, v_2]], \quad \langle (a_1, v_1), (a_2, v_2) \rangle = \langle a_1, a_2 \rangle - \langle v_1, v_2 \rangle$$  \hspace{1cm} (4.8.2)

These operations are easily seen to descend to $\mathcal{A} - \mathcal{V}$.

The two maps $\Omega^1_X \to \mathcal{A} - \mathcal{V}$ given by $\alpha \mapsto (i(\alpha), 0)$ and $\alpha \mapsto (0, -i(\alpha))$ coincide; we denote their common value by

$$i : \Omega^1_X \to \mathcal{A} - \mathcal{V}.$$  \hspace{1cm} (4.8.3)

4.8.2. **Lemma.** The formulas (4.8.1), (4.8.2) together with (4.8.3) determine a structure of a Courant extension of $\mathcal{A}$ on $\mathcal{A} - \mathcal{V}$. Moreover, the map $g(\mathcal{A}) \to \mathcal{A} - \mathcal{V}$ defined by $a \mapsto (a, 0)$ induces an isomorphism $g(\mathcal{A} - \mathcal{V}) \cong g(\mathcal{A})$ of Courant extensions of $g(\mathcal{A})$ (by $\Omega^1_X$).
4.8.3. The assignment $V \mapsto \hat{A} - V$ extends to a functor

$$\hat{A} - (\bullet) : EVA_{O_X} \to C\mathcal{E}xT_{O_X}(A)_{(\cdot, \cdot)}$$

which, clearly, anti-commutes with the respective actions of $ECA_{O_X}$ on $EVA_{O_X}$ and $C\mathcal{E}xT_{O_X}(A)_{(\cdot, \cdot)}$.

4.8.4. Corollary. The functor (4.8.4) is an equivalence of stacks in groupoids. The isomorphism class of the $ECA_{O_X}$-torsors $EVA_{O_X}$ and $C\mathcal{E}xT_{O_X}(A)_{(\cdot, \cdot)}$ are opposite as elements of $H^2(X; \Omega^2_X \to \Omega^3_X)$. 

4.8.5. Remark. In fact, the above construction gives rise to the functor

$$VE \to Hom_{ECA_{O_X}}(EVA_{O_X}, C\mathcal{E}xT_{O_X}(A)_{(\cdot, \cdot)})$$

which is an equivalence.

4.9. The canonical vertex $\mathcal{O}_X$-algebroid. We will show how the construction of [B] leads to the canonical vertex extension $\hat{A}_{can}^{\Omega_1}$ of $A_{\Omega_1}$, the Atiyah algebra of the cotangent sheaf. As a consequence, we obtain the canonical equivalence

$$\hat{A}_{can}^{\Omega_1} - (\bullet) : EVA_{O_X} \to C\mathcal{E}xT_{O_X}(A_{\Omega_1})$$

which anti-commutes with the action of $ECA_{O_X}$.

4.9.1. The canonical exact vertex $\Omega^\bullet_X$-algebroid. All of the considerations regarding the Lie, Courant and vertex algebroids apply in the differential graded setting. Let $X^\sharp$ denote the dg-manifold with $\mathcal{O}_{X^\sharp}$ the de Rham complex of $X$. In [B] we showed that there exists a unique exact vertex (differential graded) $\mathcal{O}_{X^\sharp}$-algebroid which we will denote by $\mathcal{U}$. Thus, there is a short exact sequence

$$0 \to \Omega^1_{X^\sharp} \to \mathcal{U} \to \mathcal{T}_{X^\sharp} \to 0$$

(of complexes of sheaves on $X$) of which we will be interested in the short exact sequence

$$0 \to \Omega^1_{X^\sharp}(0) \to \mathcal{U}(0) \to \mathcal{T}_{X^\sharp}(0) \to 0$$

of the degree zero constituents. Note that there is a canonical isomorphism $\Omega^1_{X^\sharp}(0) \cong \Omega^1_X$.

The natural action of $\mathcal{T}_{X^\sharp}$ on $\mathcal{O}_{X^\sharp} = \Omega^\bullet_X$ restricts to the action of $\mathcal{T}_{X^\sharp}(0)$ on $\mathcal{O}_X$ and $\Omega^1_X$. The action of $\mathcal{T}_{X^\sharp}(0)$ on $\mathcal{O}_X$ gives rise to the map $\mathcal{T}_{X^\sharp}(0) \to \mathcal{T}_X$ which, together with the natural Lie bracket on $\mathcal{T}_{X^\sharp}(0)$, endows the latter with a structure of a Lie $\mathcal{O}_X$-algebroid.

The action of $\mathcal{T}_{X^\sharp}(0)$ on $\Omega^1_X$ gives rise to the map

$$\mathcal{T}_{X^\sharp}(0) \to A_{\Omega^1_X},$$

where $A_{\Omega^1_X}$ denotes the Atiyah algebra of $\Omega^1_X$.

4.9.2. Lemma. The map (4.9.1) is an isomorphism of Lie $\mathcal{O}_X$-algebroids.

4.9.3. It follows that there is a short exact sequence

$$0 \to \Omega^1_X \to \mathcal{U}(0) \to A_{\Omega^1_X} \to 0.$$
4.9.4. Lemma. The vertex $\mathcal{O}_X$-algebroid structure on $\mathcal{U}$ restricts to a structure of a vertex extension of $A_{\Omega^1_X}$ on $\mathcal{U}^{(0)}$. The induced symmetric pairing on $\text{End}_{\mathcal{O}_X}(\Omega^1_X)$ is given by the negative of the trace of the product of endomorphisms.

Proof. The first statement is left to the reader.

According to 5.3 of [B], the $\mathcal{O}_X$-vertex algebroid $\mathcal{U}^{(0)}$ is a quotient of the $(\mathcal{O}_X$-vertex algebroid) $\tilde{\mathcal{U}}^{(0)}$, where

$$\tilde{\mathcal{U}}^{(0)} = \Omega^1_X \bigoplus \left( \Omega^1_X[1] \otimes \mathcal{T}_X[-1] \bigoplus \mathcal{O}_X \otimes \mathcal{T}_X \right)$$

Moreover, under the quotient map, $\Omega^1_X \bigoplus \Omega^1_X[1] \otimes \mathcal{T}_X[-1]$ (respectively, $\Omega^1_X[-1] \otimes \mathcal{T}_X[1]$) surjects onto $\tilde{\mathcal{U}}^{(0)}$ (respectively, $\mathcal{g}(A_{\Omega^1_X}) = \text{End}_{\mathcal{O}_X}(\Omega^1_X) \cong \Omega^1_X \otimes \mathcal{O}_X \mathcal{T}_X$). The symmetric pairing on the latter is induced by the one on the former given by the formula

$$\langle \beta_1 \otimes \xi_1, \beta_2 \otimes \xi_2 \rangle = -\langle \iota_\xi \beta_2, \iota_\xi \beta_1 \rangle$$

where $\beta_i \in \Omega^1_X[-1]$ and $\xi_i \in \mathcal{T}_X[1]$.

(The formula for the symmetric pairing on $\tilde{\mathcal{U}}$ in 5.3 of [B] reads

$$\langle \beta_1 \otimes \xi_1, \beta_2 \otimes \xi_2 \rangle = -\beta_1 \tau(\xi_2)(\tau(\xi_1)(\beta_2)) - \beta_2 \tau(\xi_1)(\tau(\xi_2)(\beta_1)) - \tau(\xi_1)(\beta_2) \tau(\xi_2)(\beta_1)$$

where $\beta_i \in \mathcal{O}_X$, $\xi_i \in \mathcal{T}_X$, $\mathcal{T}_X = \mathcal{T}_X[1] \bigoplus \mathcal{T}_X$ and $\tau$ is the canonical action of $\mathcal{T}_X$ on $\mathcal{O}_X$ by derivations with $\xi \in \mathcal{T}_X[1]$ acting by contraction $\iota_\xi$. If $\beta_i \in \Omega^1_X$, then the first two summands in the formula are equal to zero for degree reasons.)

Under the canonical isomorphism $\text{End}_{\mathcal{O}_X}(\Omega^1_X) \cong \Omega^1_X \otimes \mathcal{O}_X \mathcal{T}_X$ the symmetric pairing given by (4.9.2) corresponds to the one given by the negative of the trace of the product of endomorphisms.

4.9.5. Corollary. Let $\langle , \rangle$ denote the symmetric pairing on $\mathcal{g}(A_{\Omega^1_X}) = \text{End}_{\mathcal{O}_X}(\Omega^1_X)$ given by the negative of the trace of the product of endomorphisms. Then, the isomorphism class of $\mathcal{CE}_X \tau_{\mathcal{O}_X}(A_{\Omega^1_X}, \langle , \rangle)$ is equal to $-2 \text{ch}_2(\Omega_X)$ in $H^2(X; \Omega^3_X \rightarrow \Omega^3_X)$. The claim follows from Theorem 3.7.2.

4.9.6. Proof of 4.4.2. Follows from 4.8.4 and 4.9.5.

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