ON THE COMPARISON OF THE DIRICHLET AND NEUMANN COUNTING FUNCTIONS

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To Mikhail Shlëmovich Birman on his 80-th birthday

INTRODUCTION

Let \( N_N(\lambda) \) and \( N_D(\lambda) \) be the counting functions of the Dirichlet and Neumann Laplacian on a domain \( \Omega \subset \mathbb{R}^n \). If \( \lambda \) is not a Dirichlet or Neumann eigenvalue then

\[
N_N(\lambda) = N_D(\lambda) + g^-(\lambda),
\]

where \( g^-(\lambda) \) denotes the number of negative eigenvalues of the Dirichlet-to-Neumann map at \( \lambda \in \mathbb{R} \). The equality (*) was proved in [Fr1] for domains with sufficiently smooth boundaries. L. Friedlander also noticed that (*) immediately implies Payne’s conjecture for the Laplacian on a bounded domain, according to which the \( (k + 1) \)th Neumann eigenvalue does not exceed the \( k \)th Dirichlet eigenvalue. Later R. Mazzeo remarked that (*) remains valid for domains with smooth boundaries in any Riemannian symmetric space of noncompact type and gave a geometric explanation of Friedlander’s result [M].

For irregular boundaries, the Dirichlet-to-Neumann map may not be well-defined and then (*) does not make sense. In 2004 N. Filonov suggested another proof of Payne’s conjecture for the Laplacian [F]. This proof does not use (*) and works for nonsmooth boundaries. The author assumed that the resolvent of the Neumann Laplacian on \( \Omega \) is compact but this condition can be removed (see Remark 1.9).

The aim of this note is to show that (*) holds for abstract operators in a Hilbert space \( H \), provided that the Dirichlet-to-Neumann map is understood in a proper sense. Traditionally, one assumes that the Dirichlet-to-Neumann map is a family of operators acting in the same space and depending on the spectral parameter \( \lambda \) (see Subsection 1.3). In our understanding, it is a family of operators \( B_\lambda \) generated by the restrictions of the same sesquilinear form to different subspaces \( G_\lambda \subset H^1 \). The identity (*) is proved with the use of special isomorphisms between the subspaces \( G_\lambda \) with different values of \( \lambda \).

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This approach is close in spirit to Birman’s paper [B1] on self-adjoint extensions of symmetric operators. In particular, it removes technical problems related to nonsmooth boundaries and allows one to extend Payne’s conjecture to all operators generated by differential quadratic forms with constant coefficients on an arbitrary domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$ (see Corollary 1.13). Another advantage of our scheme is that, unlike the classical Dirichlet-to-Neumann map, the operators $B_\lambda$ do not blow up as $\lambda$ passes through isolated eigenvalues. This enables one to perform more detailed analysis of the relation between their properties and spectral characteristics of the Dirichlet and Neumann problems.

The paper is constructed as follows. In Section 1 we introduce some necessary notation and state the main results. Note that the notation is deliberately chosen as if $A$ is a second order elliptic differential operator acting in the Sobolev spaces on a domain, subject to the Dirichlet or Neumann boundary condition (even though $H$ does not have to be a function space and the ellipticity is irrelevant). In Section 2 we prove some simple auxiliary lemmas on abstract self-adjoint operators. Section 3 is devoted to the proof of main statements. Finally, Section 4 contains some remarks and by-product results, which are not needed in our proofs but may be of interest in themselves.

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1. Basic notation and main results

1.1. Notation. We shall always be assuming that $\lambda, \mu \in \mathbb{R}$ and $z \in \mathbb{C}$.

Let $H$ be an infinite dimensional separable complex Hilbert space. As usual, $(\cdot, \cdot)$ and $\| \cdot \|$ are the inner product and norm in $H$, and $+ \mathbb{N}$ denotes a direct sum in $H$. Let

- $H^1$ be a dense subspace of $H$;
- $a[\cdot]$ be a closed positive quadratic form on $H^1$ and $a[\cdot, \cdot]$ be the corresponding sesquilinear form;
- $A_N$ be the self-adjoint operator in $H$ generated by the form $a[\cdot]$.

We shall consider $H^1$ as a Hilbert space provided with the inner product $a[\cdot, \cdot]$. Let

- $H^1_0$ be a closed subspace of $H^1$ which is dense in $H$;
- $A_D$ be the self-adjoint operator in $H$ generated by the restriction of $a[\cdot]$ to $H^1_0$.

Further on we shall write $B$ instead of $N$ or $D$ in the case where the corresponding statement holds or definition refers to the both operators $A_N$ and $A_D$. In particular, we shall be using the following notation.

- $\sigma(A_B)$ and $\sigma_{ess}(A_B)$ denote the spectrum and the essential spectrum of $A_B$.
- $\lambda_{B, \infty} := \inf \sigma_{ess}(A_B)$.
- $\lambda_{B, 1} \leq \lambda_{B, 2} \leq \lambda_{B, 3} \ldots$ are the eigenvalues of the operator $A_B$ lying in the interval $(-\infty, \lambda_{B, \infty})$ and counted with their multiplicities.
• $\chi_{A}$ denotes the characteristic function of the Borel set $A \subset \mathbb{R}$, so that
• $\chi_{A}(A_{B})$ is the spectral projection of $A_{B}$ corresponding to $A$.
• $E_{B}(z)$ be the orthogonal projection onto $\ker (A_{B} - zI)$ and $E_{B}(z) := I - E_{B}(z)$.
• $N_{B}(\lambda) := \dim \chi_{(-\infty, \lambda)}(A_{B}) H$ is the left continuous counting function of the operator $A_{B}$.

The Rayleigh–Ritz variational formula implies that $N_{D}(\lambda) \leq N_{N}(\lambda)$ or, in other words, $0 < \lambda_{N,j} \leq \lambda_{D,j}$ for all $j = 1, 2, \ldots, \infty$. We have $N_{B}(\lambda) = \# \{ j : \lambda_{B,j} < \lambda \}$ whenever $\lambda \leq \lambda_{B,\infty}$ and $N_{B}(\lambda) = \infty$ otherwise.

Let
• $H_{A}^{1}$ be the set of vectors $u \in H^{1}$ such that the functionals $v \rightarrow a[u, v]$ on $H_{0}^{1}$ are $H$-continuous;
• $A$ be the operator acting from $H_{A}^{1}$ to $H$ such that $(Au, v) = a[u, v]$ for all $v \in H_{0}^{1}$;
• $G_{z} := \{ u \in H_{A}^{1} : Au = zu \}$ where $z \in \mathbb{C}$;
• $b[u, v] := a[u, v] - (Au, v)$ and $b[u] := b[u, u]$ where $u \in H_{A}^{1}$ and $v \in H^{1}$.

Since the operator $A$ is $H^{1}$-closed, $G_{z}$ are closed subspaces of $H^{1}$. Denote
• $B_{\lambda} := (I - \lambda \Pi_{A}^{-1})|_{G_{\lambda}}$ where
• $\Pi_{A}$ is the $H^{1}$-orthogonal projection onto $G_{\lambda}$ (an explicit formula for $\Pi_{A}$ is given in Subsection 2.3).

We shall consider $B_{\lambda}$ as an operator in $G_{\lambda}$. Obviously,

$$a[B_{\lambda}u, v] = a[u, v] - \lambda (u, v) = b[u, v], \quad \forall u, v \in G_{\lambda}.$$ (1.1)

Therefore $B_{\lambda}$ is a bounded self-adjoint operator in the Hilbert space $G_{\lambda}$ provided with the inner product $a[\cdot, \cdot]$.

Let
• $\sigma(B_{\lambda})$ and $\sigma_{\text{ess}}(B_{\lambda})$ be the spectrum and essential spectrum of $B_{\lambda}$;
• $G_{\lambda}^{0} := \ker B_{\lambda}$, $G_{\lambda}^{-} := \chi_{(-\infty, 0)}(B_{\lambda}) G_{\lambda}$ and $G_{\lambda}^{+} := \chi_{(0, +\infty)}(B_{\lambda}) G_{\lambda}$,

where $\chi_{(-\infty, 0)}(B_{\lambda})$ and $\chi_{(0, +\infty)}(B_{\lambda})$ are the corresponding spectral projections of the operator $B_{\lambda}$.

Finally, let
• $\mathcal{H}_{0}$ be the subspace of $H$ spanned by all common eigenvectors of $A_{N}$ and $A_{D}$;
• $\mathcal{H}$ be the $H$-orthogonal complement of $\mathcal{H}_{0}$;
• $n_{N,D}(\lambda) := \dim E_{B}^{0}(\lambda) \mathcal{H}_{0}$ and $n_{D}(\lambda) := \dim E_{B}^{0}(\lambda) \mathcal{H} = \dim E_{B}^{0}(\lambda) H - n_{N,D}(\lambda)$.

Clearly, $\mathcal{H}$ and $\mathcal{H}_{0}$ are invariant subspaces of the operators $A_{N}$ and $A_{D}$, whose intersections with $H^{1}$ are $H^{1}$-orthogonal. Similarly, $G_{\lambda} \cap \mathcal{H}$ and $G_{\lambda} \cap \mathcal{H}_{0}$ are invariant subspaces of $B_{\lambda}$. We have $G_{\lambda} \cap \mathcal{H}_{0} = G_{\lambda}^{0} \cap \mathcal{H}_{0} = E_{B}(\lambda) \mathcal{H}_{0}$ and $B_{\lambda}|_{G_{\lambda} \cap \mathcal{H}_{0}} = 0$.

In particular, $G_{\lambda} \cap \mathcal{H}_{0} = \{ 0 \}$ whenever $\lambda$ is not an eigenvalue corresponding to a common eigenvector of the operators $A_{N}$ and $A_{D}$.

1.2. Main results. The following lemma implies that the restriction $B_{\lambda}|_{\mathcal{H}}$ analytically depends on $\lambda$ outside the intersection of the essential spectra $\sigma_{\text{ess}}(A_{N})$ and $\sigma_{\text{ess}}(A_{D})$. 

Lemma 1.1. The $H^1$-orthogonal projection onto $G_\lambda \cap \mathcal{H}$ is an analytic operator-valued function of $\lambda$ on the set $\mathbb{R} \setminus (\sigma_{\text{ess}}(A_N) \cap \sigma_{\text{ess}}(A_D))$.

One can easily show that
\[
E_N(\lambda)H + E_D(\lambda)H \subset G^0_\lambda, \quad \forall \lambda \in \mathbb{R},
\]
(see Subsection 3.3). The next lemma is less obvious.

Lemma 1.2. If $\lambda \not\in \sigma_{\text{ess}}(A_N) \cup \sigma_{\text{ess}}(A_D)$ then we have $G^0_\lambda = E_D(\lambda)H + E_N(\lambda)H$. If $\lambda \not\in \sigma_{\text{ess}}(A_N) \cup \sigma_{\text{ess}}(A_D)$ then the point 0 does not belong to the essential spectrum of the operator $B_\lambda$.

Lemmas 1.1 and 1.2 imply

Theorem 1.3. Let $\lambda \not\in \sigma_{\text{ess}}(A_N) \cup \sigma_{\text{ess}}(A_D)$. Then for each sufficiently small $\varepsilon > 0$ there exists $\delta > 0$ such that the intersection $(-\varepsilon, \varepsilon) \cap \sigma(B_\mu)$ consists of

1. $n_N(\mu) + n_D(\mu) + n_{N,D}(\mu)$ zero eigenvalues if $\mu = \lambda$,
2. $n_D(\lambda)$ negative and $n_N(\lambda)$ positive eigenvalues if $\mu \in (\lambda - \delta, \lambda)$,
3. $n_N(\lambda)$ negative and $n_D(\lambda)$ positive eigenvalues if $\mu \in (\lambda, \lambda + \delta)$

(as usual, the eigenvalues are counted according to their multiplicities).

Remark 1.4. By Lemma 1.2, if $\lambda \not\in \sigma_{\text{ess}}(A_N) \cup \sigma_{\text{ess}}(A_D)$ then $[-\varepsilon, \varepsilon] \cap \sigma(B_\mu)$ is empty for all sufficiently small $\varepsilon, \delta > 0$ and all $\mu \in (\lambda - \delta, \lambda + \delta)$. By Lemma 1.1, the eigenvalues $\nu_j(\mu)$ of the restrictions $B_\mu|_{G_\mu \cap \mathcal{H}}$ lying in $(-\varepsilon, \varepsilon)$ are continuous functions of $\mu \in (\lambda - \delta, \lambda + \delta)$. Therefore, if $\varepsilon$ and $\delta$ are small enough then $\nu_j(\mu) \in (-\varepsilon, \varepsilon)$ for some $\mu \in (\lambda - \delta, \lambda + \delta)$ if and only if $\nu_j(\lambda) = 0$. Theorem 1.3 states that $n_D(\mu)$ eigenvalues $\nu_j(\mu)$ change their sign from minus to plus and $n_N(\mu)$ eigenvalues $\nu_j(\mu)$ change their sign from plus to minus as $\mu$ passes through the eigenvalue $\lambda$. At the point $\lambda$ all these eigenvalues are equal to zero and, in addition, there are $n_{N,D}(\lambda)$ zero eigenvalues of the restriction $B_\lambda|_{G_\lambda \cap \mathcal{H}_0}$.

Remark 1.5. A similar result was obtained in Fr1 and M for differential operators on domains with smooth boundaries under the additional assumption that their spectra are discrete. Theorem 1.3 holds in the abstract setting and remains valid for $\lambda$ lying in the gaps of the essential spectra.

Corollary 1.6. Let $a < b$. If $[a, b] \cap \sigma_{\text{ess}}(A_N) = \emptyset$ and $[a, b] \cap \sigma_{\text{ess}}(A_D) = \emptyset$ then
\[
\dim G^-_a = \dim G^-_b + \dim \chi_{(a,b)}(A_N)\mathcal{H} - \dim \chi_{(a,b)}(A_D)\mathcal{H}.
\]

If $a < \inf \sigma(A_N)$ then $G^-_a = \{0\}$ and, according to the next theorem, the equality (1.3) remains valid for $b \in [\lambda_{N,\infty}, \lambda_{D,\infty})$.

Theorem 1.7. $N_N(\lambda) = N_D(\lambda) + n_D(\lambda) + \dim G^-_\lambda$ for all $\lambda < \lambda_{D,\infty}$.

Remark 1.8. By Theorem 1.7, $N_N(\lambda) = \dim G^-_\lambda$ for all $\lambda$ lying below $\sigma(A_D)$. In the case where $A_N$ and $A_D$ are self-adjoint extensions of the same symmetric operator defined on $\mathcal{D}(A_N) \cap \mathcal{D}(A_D)$, the above identity was obtained by M.S. Birman [B1] (see also [B2]). Theorem 1.7 extends Birman’s result to all $\lambda < \lambda_{D,\infty}$ in a slightly more general setting (see Subsection 4.1).
Remark 1.9. N. Filonov noticed in [F1] that, for the Laplacian on an arbitrary domain \( \Omega \subset \mathbb{R}^n \),

\begin{equation}
(1.4) \quad a[u] \leq \lambda \|u\|^2, \quad \forall u \in \chi_{[0,\lambda]}(A_D)H + E_N(\lambda)H + G_N^0 + G_N^-.
\end{equation}

Similar arguments show that (1.4) holds for any pair of abstract operators \( A_D \) and \( A_N \) (see Subsection 3.2). The estimate (1.4) immediately implies that

\begin{equation}
(1.5) \quad N_N(\lambda) \geq N_D(\lambda) + n_D(\lambda) + \dim G_N^-, \quad \forall \lambda \in \mathbb{R}.
\end{equation}

The inequality (1.5) is sufficient to prove Payne’s conjecture for the Laplacian on a bounded domain (see the proof of Corollary 1.13).

Remark 1.10. The equality \( N_N(\lambda) = N_D(\lambda) + n_D(\lambda) + \dim G_N^- \) remains valid for all \( \lambda > \lambda_{D,\infty} \) because \( N_N(\lambda) = N_D(\lambda) = \infty \). However, as was pointed out by N. Filonov, it may not be true for \( \lambda = \lambda_{D,\infty} \).

Remark 1.11. Let \( \lambda = \lambda_{D,k} < \lambda_{N,\infty} \). Theorem 1.7 implies that the number of eigenvalues \( \lambda_{N,j} \) lying below \( \lambda_{D,k} \) is equal to \( k - 1 + n_D(\lambda_{D,k}) + \dim G_{D,k}^- \). Therefore

1. \( \lambda_{N,k+q_k+p_k-1} < \lambda_{D,k} \), where \( p_k := \dim G_{D,k}^- \) and \( q_k := n_D(\lambda_{D,k}) \).

If \( n_D(\lambda_{D,k}) = 0 \) then \( \lambda_{N,k+q_k+p_k} = \lambda_{D,k} \); if \( n_D(\lambda_{D,k}) \neq 0 \) then \( q_k \geq 1 \). Thus we always have

2. \( \lambda_{N,k+p_k} \leq \lambda_{D,k} \).

Note that the estimates (1) and (2) are actually consequences of (1.3). These estimates and Lemma 1.2 imply that

3. \( \lambda_{N,k+1} \leq \lambda_{D,k} \) whenever there exists a vector \( u \in G_{D,k}^- \), such that \( b[u] \leq 0 \) and \( u \not\in D(A_D) \);

4. \( \lambda_{N,k+1} < \lambda_{D,k} \) whenever \( n_D(\lambda_{D,k}) \geq 1 \) and there exist two vectors \( u_1, u_2 \in G_{D,k}^- \), such that \( b[u_1] \leq 0, b[u_2] \leq 0 \) and the linear subspace spanned by \( u_1 \) and \( u_2 \) does not contain Neumann eigenvectors.

Indeed, if \( G_{D,k}^- \geq 1 \) then (3) and (4) follow from (2) and (1) respectively. If \( G_{D,k}^- = \{0\} \) then \( u \in G_{D,k}^0 \) and \( u_1, u_2 \in G_{D,k}^0 \). The inclusion \( u \in G_{D,k}^0 \) implies that \( \lambda_{D,k} \) is also a Neumann eigenvalue and, consequently, \( \lambda_{N,k+1} = \lambda_{D,k} \). The inclusions \( u_1, u_2 \in G_{D,k}^0 \) imply that \( n_D(\lambda_{D,k}) \geq 2 \) (otherwise a linear combination of \( u_1 \) and \( u_2 \) would belong to \( E_N(\lambda_{D,k})H \).

Lemma 1.2 and Theorem 1.7 also imply

**Corollary 1.12.** If \( \lambda < \lambda_{D,\infty} \) and \( \lambda \not\in \sigma(A_N) \cup \sigma(A_D) \) then the number of negative eigenvalues of the self-adjoint operator \( R'(\lambda) := (A_N - \lambda I)^{-1} - (A_D - \lambda I)^{-1} \) in \( H \) coincides with \( N_N(\lambda) - N_D(\lambda) \).

Obviously, the number of negative eigenvalues of the operator \( (A_B - \lambda I)^{-1} \) jumps by \( n_B(\lambda_0) + n_{N,D}(\lambda_0) \) as \( \lambda \) passes through an eigenvalue \( \lambda_0 \). Corollary 1.12 shows that the corresponding jump for \( R'(\lambda) \) is equal to \( n_N(\lambda_0) - n_D(\lambda_0) \), as if \( R'(\lambda) \) were the orthogonal sum of the operators \( (A_N - \lambda I)^{-1} \) and \( -(A_D - \lambda I)^{-1} \).
1.3. The Dirichlet-to-Neumann map. In the theory of boundary value problems, it is often possible to construct a linear isomorphism $W : G_\lambda \to \mathcal{H}$, where $\mathcal{H}$ is a Hilbert space of functions defined on the boundary. Then one can consider the operator $WB_\lambda W^{-1} : \mathcal{H} \to \mathcal{H}$ instead of $B_\lambda$. Clearly, these two operators have the same eigenvalues. If $H^1, H^1_0$ are the Sobolev spaces and $Wv$ is the restriction of $v$ to the boundary then $WB_\lambda W^{-1}$ is usually called the Dirichlet-to-Neumann map. This scheme works under certain smoothness conditions on the boundary and the coefficients, whereas our approach does not rely on the existence of an auxiliary operator $W$ and does not require any additional assumptions.

1.4. Applications to boundary value problems. Let $\Omega$ be an arbitrary open subset of $\mathbb{R}^n$ with $n \geq 2$. Consider a differential operator $L$ acting from the space of $m$-vector functions $C^\infty(\Omega, \mathbb{C}^m)$ into the space of $l$-vector functions $C^\infty(\Omega, \mathbb{C}^l)$ and denote by $L^*$ its formal adjoint. Let us assume that the form $\int_\Omega |Lu(x)|^2 \, dx$ with domain $C^\infty(\Omega, \mathbb{C}^m) \cap L_2(\Omega, \mathbb{C}^m)$ is strictly positive and closable in $H = L_2(\Omega, \mathbb{C}^m)$, and denote its closure by $a[u]$. If $H^1 := \mathcal{D}(a)$ and $H^1_0$ is the $H^1$-closure of $C^\infty_0(\Omega)$ then $A = L^*L$ and $A_B$ is the differential operator $A$ with the corresponding boundary condition.

Corollary 1.13. Let $L$ be an operator with constant coefficients. Then $\lambda_{N,k+1} \leq \lambda_{D,k}$ for all eigenvalues $\lambda_{D,k} \in (0, \lambda_{N,\infty})$. If at least one Dirichlet eigenfunction corresponding to $\lambda_{D,k}$ does not satisfy the Neumann boundary condition then $\lambda_{N,k+1} < \lambda_{D,k}$.

Remark 1.14. Our proof of Corollary 1.13 uses the exponential functions $u_\xi(x) = e^{ix \cdot \xi}$ and is very similar to the proof of the Payne conjecture given in [Fr1]. The main difference is that L. Friedland considered the Dirichlet-to-Neumann map and therefore had to assume that the boundary is smooth enough.

Remark 1.15. If $A$ is the Laplacian on a convex $n$-dimensional domain with sufficiently smooth boundary then $\lambda_{N,k+n} < \lambda_{D,k}$. This estimate was obtained in [LV]. Later L. Friedlander found another proof, based on the fact that $G_\lambda \cup G_\lambda^-$ contains all first order derivatives $D_ju$ of the Dirichlet eigenfunctions $u \in E_D(\lambda)H$ (the derivatives obviously belong to $G_\lambda$, and the estimate $b[D_ju] \leq 0$ is a consequence of the convexity). The inclusion $D_ju \in G_\lambda \cup G_\lambda^-$ also implies that $N_N(\lambda) \geq N_D(\lambda) + 2n_D(\lambda)$ (see [Fr2] for details).

2. Further notation and auxiliary results

2.1. The inverse $A_N^{-1}$ is a bounded self-adjoint operator in $H^1$ because $a[A_N^{-1}u, v] = (u, v)$ for all $u, v \in H^1$. Since $a[u, v] = \lambda(u, v)$ for all $v \in E_N(\lambda)H$ and $v \in H^1$, its spectral projections $E_N(\lambda)$ are $H^1$-orthogonal.

Let

- $\Pi_0$ be the orthogonal projection in $H^1$ onto $H^1_0$.

From the definition of $G_\lambda$ it clear that $\Pi'_0 = I - \Pi_0$ (this well known result can be found, for example, in [K] or [BS] Chapter 10, Section 3). Since $a[A_D^{-1}u, v] = (u, v)$ =
a[A_N^{-1}u, v] for all u ∈ H^1 and v ∈ H^1_0, we have A_D^{-1} = \Pi_0 A_N^{-1} and D(A_D) = \Pi_0 D(A_N).

The following simple lemma is also well known in the theory of self-adjoint extensions.

**Lemma 2.1.** We have H^1_A = G_0 + D(A_B). If w_0 ∈ G_0 and w_B ∈ D(A_B) then

\[ A(w_0 + w_B) = A_B w_B. \]

**Proof.** Obviously, G_0 + D(A_B) ⊂ H^1_A. On the other hand, if v ∈ H^1_A then there exists \( \tilde{v} \in H \) such that \((u, \tilde{v}) = a[u, v]\) for all u ∈ H^1_0. Since \((u, v) = a[u, A_N^{-1}v]\), this implies that \( \Pi_0 v = \Pi_0 A_N^{-1} \tilde{v} = A_D^{-1} \tilde{v}. \) Therefore \( v = \Pi_0' v + A_D^{-1} \tilde{v} \) and \( v = \Pi_0' (v - A_N^{-1} \tilde{v}) + A_D^{-1} \tilde{v}. \) These equalities imply the first statement.

If \( w_0 ∈ G_0 \) and \( w_B ∈ D(A_B) \) then \( a[w_0 + w_B, v] = a[w_B, v] = (A_B w_B, v), \) for all \( v ∈ H^1_1 \). This proves the second statement. \( \Box \)

2.2. By Lemma 2.1 \( H^1_A \) is dense in \( H^1 \). Since \( a[A_D^{-1} Au, v] = a[A_D^{-1} Au, \Pi_0 v] = (AA_D^{-1} Au, \Pi_0 v) = (Au, \Pi_0 v) = a[\Pi_0 u, v] \) for all \( u, v ∈ H^1_A \), this implies that \( A_D^{-1} A = \Pi_0|_{H^1_A}. \) Consequently,

\[ G_z := \ker (A - zI) = \ker A_D^{-1}(A - zI) = \ker (\Pi_0 - zA_D^{-1}) = \ker \Pi_0(I - zA_B^{-1}), \quad ∀z ∈ \mathbb{C}. \]

By (2.1), we have \((I - zA_B^{-1}) G_z ⊂ G_0 \bigcap (I - zA_B^{-1}) H^1\). On the other hand, if \((I - zA_B^{-1}) u ∈ G_0\) then \( u ∈ G_z\) because \((A - zI) u = A(I - zA_B^{-1}) u = 0. \) Therefore

\[ (I - zA_B^{-1}) G_z = G_0 \bigcap (I - zA_B^{-1}) H^1, \quad ∀z ∈ \mathbb{C}. \]

Let

- \( R_B(z) := (A_B - zI)^{-1} \) be the resolvent of \( A_B. \)

For each \( z ⊈ σ_{\text{ess}}(A_B), \) the operator \( R_B(z) E_B'(z) \) is bounded from \( H \) to \( H^1, \)

\[ \ker (R_B(z) E_B'(z)) = E_B(z) H, \quad R_B(z) E_B'(z) H ⊂ E_B'(z) D(A_B) ⊂ H^1_A \]

and \((A - zI) R_B(z) E_B'(z) = E_B'(z). \) We also have

\[ (I - zA_B^{-1})^{-1}|_{E_B'(z) H} = (I + zR_B(z)) E_B'(z), \quad ∀z ⊈ σ_{\text{ess}}(A_B), \]

where the operators in the right and left hand sides map \( E_B'(\lambda) H^1 \) onto \( E_B'(\lambda) H^1 \) and are \( H^1\)-bounded. This implies that \((I - zA_B^{-1}) H^1 = E_B'(z) H^1\) and, in view of (2.2),

\[ (I - zA_B^{-1}) G_z = G_0 \bigcap E_B'(z) H^1, \quad ∀z ⊈ σ_{\text{ess}}(A_B). \]

2.3. Denote \( T_z := (I - zA_N^{-1})|_{H^1_0 ∩ H} \) and \( T^*_z := \Pi_0(I - zA_N^{-1})|_{H}. \) Let \( Σ \) be the set of points \( z ∈ \mathbb{C} \) such that the spectrum of the operator \( T^*_z T_z : H^1_0 ∩ H → H^1_0 ∩ H \) contains the point \( 0, \) and let

- \( \Pi(z) := T_z (T^*_z)^{-1} T^*_z \) and \( \Pi'(z) := I - \Pi(z), \) where \( z ∈ \mathbb{C} \setminus Σ. \)
By (2.1), we have $G_z \cap \mathcal{H} = \ker T_z^*$. Since $T_z^* \Pi'(z) = 0$ and $\Pi'(z)u = u$ for all $u \in \ker T_z^*$, this implies that $\Pi'(z)$ is a projection onto $G_z \cap \mathcal{H}$ in $H^1 \cap \mathcal{H}$. Its $H^1$-adjoint coincides with $\Pi'(\bar{z})$; in particular, $\Pi'(\lambda)$ is the $H^1$-orthogonal projection onto $G_z \cap \mathcal{H}$ in $\mathcal{H}$. Thus we obtain
\begin{equation}
\Pi'_\lambda = \Pi'(\lambda) \oplus E_B(\lambda)|_{H^1 \cap \mathcal{H}_u}, \quad \forall \lambda \in \mathbb{R} \setminus \Sigma.
\end{equation}

2.4. If $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(A_B)$, let
- $P_B(z) := R_B(z)(D_B(z)(A - zI))$ and $P_B'(z) := I - P_B(z)$;
- $G_{z,B} := \{v \in H^1_\lambda: (A - zI)v \in E_B(z)H\}$.

The operators $P_B(z)$ and $P_B'(z)$ are projections in $H^1_\lambda$ because
\begin{equation}
P_B^2(z) = R_B(z)E_B'(z)(A_B - zI)R_B(z)E_B'(z)(A - zI) = E_B'(z)R_B(z)E_B'(z)(A - zI) = P_B(z).
\end{equation}

One can easily show that $P_B(z)H^1_\lambda = E_B'(z)D(A_B)$ and $P_B'(z)H^1_\lambda = G_{z,B}$. The subspace $G_{z,B}$ is the inverse image of $E_B(z)H$ by the map $A - zI$, whereas $G_z$ is the kernel of $A - zI$. Therefore $G_z \subset G_{z,B}$ and the dimension of the quotient space $G_{z,B}/G_z$ does not exceed $n_{B}(z) + n_{N,D}(z)$. This implies that the subspaces $G_{z,B}$ are $H^1$-closed for all $z \in \mathbb{C} \setminus \sigma_{\text{ess}}(A_B)$.

If $z \notin \sigma(A_B)$ then $P_B(z)H^1_\lambda = D(A_B)$ and $P_B'(z)H^1_\lambda = G_z = G_{z,B}$. In particular, $P_D(0)|_{H^1_\lambda} = P_0$ and $P_B'(0)|_{H^1_\lambda} = P_0$ and $H^1_\lambda = P_B'(0)H^1_\lambda + P_B(0)H^1_\lambda$ is the decomposition discussed in Lemma 2.1. By direct calculation, if $u, v \in H^1_\lambda$ and $\lambda, \mu \in \mathbb{R}$ then
\begin{equation}
b[P'_N(\lambda)u, P'_N(\mu)v] = b[u, v] - (u, E_N'(\mu)(A - \mu I)v) + ((A - \mu I)u, R_N(\mu)E_B'(z)(A - \mu I)v)
\end{equation}

and
\begin{equation}
b[P'_D(\lambda)u, P'_D(\mu)v] = b[u, v] + (E_D'(\lambda)(A - \lambda I)u, v) - (R_D(\lambda)E_B'(z)(A - \lambda I)u, (A - \lambda I)v).
\end{equation}

3. Proofs of main results

3.1. Proof of Lemma 1.1. The operator-valued function $T_z^* T_z$ is analytic. Therefore, for each $z_0 \notin \Sigma$, the inverse operator $(T_z^* T_z)^{-1}$ exists and analytically depends on $z$ in a sufficiently small neighbourhood of $z_0$ (see, for instance, [Ya, Section 1.8]). Thus it is sufficient to show that $\Sigma \subset (\sigma_{\text{ess}}(A_N) \cap \sigma_{\text{ess}}(A_D))$.

Let us fix $\lambda \in \mathbb{R}$, and let $u \in \mathcal{H}$. Since $T_{\lambda}^* T_{\lambda}$ is the $H^1$-adjoint to $T_{\lambda}$, we have $T_{\lambda}^* T_{\lambda} u = 0$ if and only if $T_{\lambda} u = 0$. The latter means that $u \in \mathcal{D}(A_N) \cap \mathcal{D}(A_D)$ and $A_N u = A_D u = \lambda u$. Since $u \in \mathcal{H}$, it is only possible if $u = 0$. This implies that $\ker(T_{\lambda}^* T_{\lambda}) = \{0\}$.

Assume that the essential spectrum of the operator $T_{\lambda}^* T_{\lambda}$ contains the point 0. Then, for any given finite dimensional subspace $\mathcal{L}$, there exists a sequence of $H^1$-orthogonal vectors $u_n \in H^1_0 \cap \mathcal{H}$ such that $u_n$ are $H$-orthogonal to $\mathcal{L}$, $a[u_n] = 1$.
and \( a[T\lambda u_n, u_n] \to 0 \) as \( n \to \infty \). Clearly, \( T\lambda u_n = u_n - \lambda A_N^{-1}u_n \to 0 \) and \( \Pi_0 T\lambda u_n = u_n - \lambda A_D^{-1}u_n \to 0 \) in \( H^1 \). If \( \lambda \notin \sigma_{\text{ess}}(A_D) \) then, by (2.3), we have
\[
a[u - \lambda A_B^{-1}u] \geq C|a[u]| \text{ with some positive constant } C \text{ for all vectors } u \in H^1 \text{ which are } H\text{-orthogonal to } \mathcal{E} = E_B(\lambda)H.
\]
This implies that \( \lambda \in \sigma_{\text{ess}}(A_N) \cap \sigma_{\text{ess}}(A_D) \).

### 3.2. Proof of the estimate (1.4)
We have
\[
(3.1) \quad a[u, v] = \lambda (u, v), \quad \forall u \in E_N(\lambda)H, \quad \forall v \in H^1,
\]
\[
(3.2) \quad a[u, v] = \lambda (u, v), \quad \forall u \in D(A_D), \quad \forall v \in G_\lambda.
\]
Let \( u = u_1 + u_2 + u_3 \), where \( u_1 \in \chi_{[0, \lambda]}(A_D)H, \quad u_2 \in E_N(\lambda) \) and \( u_3 \in G_0^0 + G_\lambda^\perp \). Then (3.1) and (3.2) imply
\[
a[u] - \lambda \|u\|^2 = a[u_1] - \lambda \|u_1\|^2 + a[u_3] - \lambda \|u_3\|^2 = ((A_D - \lambda I)u_1, u_1) + a[B\lambda u_3, u_3] \leq 0.
\]

### 3.3. Proof of Lemma (1.2)
The inclusion (1.2) immediately follows from (3.1) and (3.2).

Assume that \( \lambda \notin \sigma_{\text{ess}}(A_N) \cup \sigma_{\text{ess}}(A_D) \) and \( u \in G^0_\lambda \). Then \((I - \lambda A_N^{-1})u \) is \( H^1 \)-orthogonal to \( G_\lambda \). In view of (2.5), this means that \((I - \lambda A_N^{-1})u = \Pi(\lambda)v \) for some \( v \in H^1 \cap H \). Therefore \((I - \lambda A_N^{-1})u = (I - \lambda A_N^{-1})w \) for some \( w \in H^1_0 \), which is equivalent to the inclusion \( u \in G_\lambda \cap (E_N(\lambda)H + H^1_0) = E_N(\lambda)H + E_D(\lambda)H \).

Assume now that \( \lambda \notin \sigma_{\text{ess}}(A_N) \cup \sigma_{\text{ess}}(A_D) \). If the point 0 belongs to the essential spectrum of the operator \( B_\lambda \) then there exists a sequence of vectors \( u_n \in G_\lambda \cap G_0^0 \) such that \( a[u_n] = 1 \) and \( a[B_\lambda u_n] \to 0 \) as \( n \to \infty \). Moreover, since \( \dim E_N(\lambda)H < \infty \), we can choose the sequence \( \{u_n\} \) in such a way that
\[
u_n - (T^*\lambda T\lambda)^{-1} \Pi_0(I - \lambda A_N^{-1})^2u_n \in E_N'(\lambda), \quad \forall n = 1, 2, \ldots,
\]
where \( T_\lambda \) and \( T^*\lambda \) are the operators defines in Subsection 2.3. Then, by (2.5),
\[
B_\lambda u_n = (I - \lambda A_N^{-1})u_n - \Pi_\lambda (I - \lambda A_N^{-1})u_n
\]
\[
= (I - \lambda A_N^{-1}) (u_n - (T^*\lambda T\lambda)^{-1} \Pi_0(I - \lambda A_N^{-1})^2u_n) \to 0
\]
in \( H^1 \), and (2.3) implies that
\[
a[u_n - (T^*\lambda T\lambda)^{-1} \Pi_0(I - \lambda A_N^{-1})^2u_n]
\]
\[
= a[\Pi_0' u_n] + a[\Pi_0^* u_n - (T^*\lambda T\lambda)^{-1} \Pi_0(I - \lambda A_N^{-1})^2u_n] \to 0.
\]
Therefore \( a[\Pi_0' u_n] = a[(I - A_D^{-1}A)u_n] = a[(I - \lambda A_D^{-1})u_n] \to 0 \) as \( n \to \infty \). However, this is not possible because \( \lambda \notin \sigma_{\text{ess}}(A_D) \) and \( u_n \) are orthogonal to \( E_D(\lambda)H \subset G_\lambda^0 \). The obtained contradiction proves the second statement of the lemma.

### 3.4. Proof of Theorem (1.3)
If \( \lambda \notin \sigma(A_N) \cup \sigma(A_D) \) then the theorem is obvious because, in view of Lemmas (1.2) and (1.1), we have \( n_N(\lambda) = n_D(\lambda) = n_{N,D}(\lambda) = 0 \) and \((-\varepsilon, \varepsilon) \cap \sigma(B_\mu) = \emptyset \) for all sufficiently small \( \varepsilon, \delta > 0 \) and all \( \mu \in (\lambda - \varepsilon, \lambda + \varepsilon) \).

Suppose that \( \lambda \) is an isolated eigenvalue. The first statement of the theorem is an immediate consequence of Lemma (1.2), so we only need to prove (2) and (3). Let us choose \( \varepsilon \) and \( \delta \) as explained in Remark (1.4) and assume, in addition, that \( \delta \) is so
small that $\lambda - \delta > 0$ and the interval $[\lambda - \delta, \lambda + \delta]$ does not contain any points from $\sigma(A_N) \cup \sigma(A_D)$ with the exception of $\lambda$.

Let $\mathcal{L}_\mu$ be the subspace of $G_\mu \cap \mathcal{H}$ spanned by the eigenfunction corresponding to the eigenvalues $\nu_j(\mu)$ (see Remark 1.4). By Lemma 1.2 we have

$$\mathcal{L}_\lambda = E_N(\lambda)\mathcal{H} + E_D(\lambda)\mathcal{H} \subset G_\lambda^0.$$

Therefore $E_B'(\lambda)\mathcal{L}_\lambda \subset \mathcal{L}_\lambda$, $\dim E_N'(\lambda)\mathcal{L}_\lambda = n_D(\lambda)$ and $\dim E_D'(\lambda)\mathcal{L}_\lambda = n_N(\lambda)$.

We are going to show that

$$\left| a\left[ B_\mu P_N'(\mu)u, \chi_{(-\varepsilon,\varepsilon)}(B_\mu)P_N'(\mu)u \right] - (\mu - \lambda)\|u\|^2 \right| \leq C(\lambda - \mu)^2\|u\|^2, \quad \forall u \in E_N'(\lambda)\mathcal{L}_\lambda$$

and

$$\left| a\left[ B_\mu P_D'(\mu)u, \chi_{(-\varepsilon,\varepsilon)}(B_\mu)P_D'(\mu)u \right] - (\lambda - \mu)\|u\|^2 \right| \leq C(\lambda - \mu)^2\|u\|^2, \quad \forall u \in E_D'(\lambda)\mathcal{L}_\lambda,$$

where $C$ is a constant independent of $u$ and $\mu \in (\lambda - \varepsilon, \lambda + \varepsilon)$. From (3.3) and (3.4) it follows that

$$\dim \chi_{(-\varepsilon,\varepsilon)}(B_\mu)P_B'(\mu)E_B'(\lambda)\mathcal{L}_\lambda = \dim E_B'(\lambda)\mathcal{L}_\lambda = n_N(\lambda) + n_D(\lambda) - n_B(\lambda)$$

for all $\mu$ sufficiently close to $\lambda$. Therefore the estimates (3.3) and (3.6) imply the theorem (with another positive $\delta$).

In order to prove (3.3) and (3.4), note that $b[u] = 0$ for all $u \in \mathcal{L}_\lambda$ and, in view of (2.6) and (2.7),

$$a\left[ B_\mu P_N'(\mu)u, P_N'(\mu)u \right] = b[P_N'(\mu)u] = (\mu - \lambda)\|u\|^2 + (\lambda - \mu)^2 (u, R_N(\mu)u), \quad \forall u \in \mathcal{L}_\lambda$$

and

$$a\left[ B_\mu P_D'(\mu)u, P_D'(\mu)u \right] = b[P_D'(\mu)u] = (\lambda - \mu)\|u\|^2 - (\lambda - \mu)^2 (u, R_D(\mu)u), \quad \forall u \in \mathcal{L}_\lambda.$$

Therefore, for all $\mu \in (\lambda - \delta, \lambda + \delta)$ we have

$$\left| a\left[ B_\mu P_N'(\mu)u, P_N'(\mu)u \right] - (\mu - \lambda)\|u\|^2 \right| \leq C_1^{-1}(\lambda - \mu)^2\|u\|^2, \quad \forall u \in E_N'(\lambda)\mathcal{L}_\lambda,$$
and

\[(3.10) \quad |a[B_\mu P'_D(\mu)u, P'_D(\mu)u] - (\lambda - \mu)\|u\|^2 | \leq C_1^{-1}(\lambda - \mu)^2\|u\|^2, \quad \forall u \in E'_D(\lambda)\mathcal{L}_\lambda,\]

where \(C_1\) is the distance from \([\lambda - \delta, \lambda + \delta]\) to \((\sigma(A_N \cup \sigma(A_D)) \setminus \{\lambda\}\).

Let \(S_B\) be the projections onto \(E_B(\lambda)\mathcal{H}\) in \(\mathcal{L}_\lambda\) such that \(S_N E_D(\lambda) = 0\) and \(E_N(\lambda)\mathcal{H} \cap E_D(\lambda)\mathcal{H} = \{0\}\), the projections \(S_B\) are well defined and bounded as operators from \(H\) to \(H^1\).

If \(u \in \mathcal{L}_\lambda\) then \(P'_B(\mu)u = P'_B(\mu)(u - S_Bu) = (I - (\lambda - \mu)R_B(\mu))(u - S_Bu)\) for all \(\mu \neq \lambda\) and, by (1.1),

\[a[B_\mu P'_N(\mu)u, v] = a[P'_N(\mu)u, v] - \mu(P'_N(\mu)u, v) = (\mu - \lambda)(S_Du, v), \quad \forall v \in G_\mu,\]

\[a[B_\mu P'_D(\mu)u, v] = a[P'_D(\mu)u, v] - \mu(P'_D(\mu)u, v) = (\mu - \lambda)(S_Nu, v), \quad \forall v \in G_\mu,\]

for all \(\mu \in (\lambda - \delta, \lambda + \delta)\). Since \((S_Nu, v) = \lambda^{-1}a[S_Nu, v]\) and \((S_Du, v) = \mu^{-1}a[S_Du, v]\) whenever \(v \in G_\mu\), the above identities imply that

\[(3.11) \quad a[B_\mu P'_N(\mu)u, v] = \mu^{-1}(\mu - \lambda)a[S_Du, v], \quad \forall u \in \mathcal{L}_\lambda, \forall v \in G_\mu,\]

\[(3.12) \quad a[B_\mu P'_D(\mu)u, v] = \lambda^{-1}(\mu - \lambda)a[S_Nu, v], \quad \forall u \in \mathcal{L}_\lambda, \forall v \in G_\mu.\]

In view of Lemma 1.1, \((I - \chi_{(-\varepsilon, \varepsilon)}(B_\mu)) P'_B(\mu)E'_B(\lambda)\big|_{\mathcal{L}_\lambda}\) is an analytic operator-valued function of \(\mu \in (\lambda - \delta, \lambda + \delta)\). Since this operator-valued function vanishes at \(\mu = \lambda\), we have

\[(3.13) \quad a\left[\left(I - \chi_{(-\varepsilon, \varepsilon)}(B_\mu)\right) P'_B(\mu)E'_B(\lambda)u\right] \leq C_2(\lambda - \mu)^2a[u] = C_2(\lambda - \mu)^2\lambda\|u\|^2, \quad \forall u \in \mathcal{L}_\lambda,\]

with some positive constant \(C_2\) independent of \(\mu\) and \(u\). Substituting

\[v = \left(I - \chi_{(-\varepsilon, \varepsilon)}(B_\mu)\right) P'_B(\mu)u\]

into (3.11), (3.12) and applying (3.13), we obtain

\[(3.14) \quad a[B_\mu P'_B(\mu)u, (I - \chi_{(-\varepsilon, \varepsilon)}(B_\mu)) P'_B(\mu)u] \leq C_3(\lambda - \mu)^2\|u\|^2, \quad \forall u \in E'_B(\lambda)\mathcal{L}_\lambda,\]

with some constant \(C_3\) independent of \(\mu\) and \(u\). Now (3.3) and (3.4) follow from (3.9), (3.10) and (3.14).

3.5. **Proof of Corollary 1.6** Lemma 1.1 and Theorem 1.3 imply that the function \(\dim G^-_\lambda\) is constant on every connected component of the set \(\mathbb{R} \setminus (\sigma(A_N) \cup \sigma(A_D))\). If \(\lambda \not\in \sigma_{\text{ess}}(A_N) \cup \sigma_{\text{ess}}(A_D)\) and \(\lambda \in \Lambda\) is an eigenvalue then, by Theorem 1.3,

\[(3.15) \quad \dim G^-_\mu = \begin{cases} \dim G^-_\lambda + n_D(\lambda), & \forall \mu \in (\lambda - \delta, \lambda), \\ \dim G^-_\lambda + n_N(\lambda), & \forall \mu \in (\lambda, \lambda + \delta), \end{cases}\]

provided that \(\delta > 0\) is small enough. In other words, the value of \(\dim G^-_\mu\) jumps by \(n_N(\lambda) - n_D(\lambda)\) as \(\mu\) passes through the eigenvalue \(\lambda\), and \(\dim G^-_\lambda = \dim G^-_{\lambda-0} - n_D(\lambda)\).
Summing up these jumps over all the eigenvalues lying between \( a \) and \( b \), we obtain (1.3).

3.6. **Proof of Theorem 1.7.** Let \( \mathcal{L} \) be the subspace of \( \chi_{(-\infty,\lambda)}(A_N)H \) spanned by all the vectors \( v \in \chi_{(-\infty,\lambda)}(A_N)H \) such that

\[
\chi_{(-\infty,\lambda)}(A_D)(A-\lambda I)v = 0.
\]

The inclusion \( \mathcal{L} \subset \mathcal{D}(A_N) \) implies that \( \dim \mathcal{L} \geq N_N(\lambda) - N_D(\lambda) - n_D(\lambda) \) and \( b[v] = 0 \) for all \( v \in \mathcal{L} \). From the latter identity, (3.16) and (2.7) it follows that \( b[P_D(\lambda)v] < b[v] \) for all nonzero \( v \in \mathcal{L} \). Since \( ((A-\lambda I)v,v) < 0 \) for all nonzero \( v \in \chi_{(-\infty,\lambda)}(A_N)H \) and \( ((A-\lambda I)v,v) \geq 0 \) for all \( v \in \mathcal{D}(A_D) \) satisfying (3.16), we have \( \mathcal{L} \cap \mathcal{D}(A_D) = \{0\} \). Therefore \( \ker P_D(\lambda) \mid \mathcal{L} \subset L \cap \mathcal{D}(A_D) = \{0\} \) and, consequently,

\[
\dim P_D(\lambda) \mathcal{L} \geq N_N(\lambda) - N_D(\lambda) - n_D(\lambda).
\]

Thus we have \( \dim G_\lambda \geq \dim P_D(\lambda) \mathcal{L} \geq N_N(\lambda) - N_D(\lambda) - n_D(\lambda) \). Now the theorem follows from (1.5).

3.7. **Proof of Corollary 1.12.** We have \( R'(\lambda)H \subset G_\lambda \) and, by (1.11),

\[
a[B_\lambda R'(\lambda)u,v] = a[R'(\lambda)u,v] - \lambda(R'(\lambda)u,v) = (u,v) = a[A_N^{-1}u,v]
\]

for all \( \forall u \in H \) and \( \forall v \in G_\lambda \). The above identity implies that \( B_\lambda R'(\lambda)u = \Pi_\lambda A_N^{-1}u \) for all \( u \in H \). In view of Lemma 1.2, the operator \( B_\lambda \) is invertible and, consequently, \( R'(\lambda) = B_\lambda^{-1} \Pi_\lambda A_N^{-1} \). Since \( a[A_N^{-1}u,v] = (u,v) \), the subspace \( \Pi_\lambda A_N^{-1}H \) is \( H^1 \)-dense in \( G_\lambda \). Therefore \( R'(\lambda)H \) is an \( H^1 \)-dense subspace of \( G_\lambda \). Finally, by (3.17),

\[
a[B_\lambda R'(\lambda)u,R'(\lambda)v] = (u,R'(\lambda)v), \quad \forall u,v \in H.
\]

Thus we have \( a[B_\lambda u,u] < 0 \) on a \( k \)-dimensional subspace of \( G_\lambda \) if and only if \( (R'(\lambda)u,u) < 0 \) on a \( k \)-dimensional subspace of \( H \). Now the corollary follows from Theorem 1.7.

3.8. **Proof of Corollary 1.13.** Let \( a(\xi) \) be the full symbol of the operator \( L^*L \), and let \( \lambda_1(\xi),\ldots,\lambda_m(\xi) \) be the eigenvalues of \( a(\xi) \). Then \( \lambda_{D,k} > \lambda_* := \min_j \inf_\xi \lambda_j(\xi) \) for all \( k \) because \( a[u] \geq \lambda_* \|u\|^2 \) on \( C_0^\infty(\Omega) \). On the other hand, since \( \lambda_j(\xi) \) are continuous functions of \( \xi \), the equation \( \det(a(\xi) - \lambda I) = 0 \) has infinitely many \( \xi \)-solutions for each fixed \( \lambda > \lambda_* \). Therefore \( G_\lambda \) contains an infinite dimensional set formed by functions of the form \( u_\xi = e^{i\xi \cdot \vec{c}} \) where \( \vec{c} \in \ker(a(\xi) - \lambda I) \). For each of these functions we have \( a[u_\xi] = \lambda \|u_\xi\|^2 \). This implies that either \( \dim G_\lambda^0 \geq 1 \) or \( \dim G_\lambda^0 = \infty \). By Lemma 1.2, the latter is possible only if \( \lambda \geq \lambda_{N,\infty} \). Therefore, by Remark 1.11(2), we have \( \lambda_{N,k+1} \leq \lambda_{D,k} \) for all eigenvalues lying below \( \lambda_{N,\infty} \). If at least one Dirichlet eigenfunction corresponding to \( \lambda_{D,k} \) does not satisfy the Neumann boundary condition then \( n_D(\lambda_{D,k}) \geq 1 \) and, by Remark 1.11(1), \( \lambda_{N,k+1} \leq \lambda_{D,k} \).
4. Remarks

4.1. $A_D$ and $A_N$ as self-adjoint extensions. Denote $H^2_0 := H^1_0 \cap \mathcal{D}(A_N)$. Since $\mathcal{D}(A_D) = \Pi_0 \mathcal{D}(A_N)$ (see Subsection 2.1), we have $H^2_0 = \mathcal{D}(A_D) \cap \mathcal{D}(A_N)$.

The $H$-adjoint $A^*$ coincides with the restriction of $A$ to $H^2_0$. Indeed, if $(u, Av) = (\tilde{u}, v)$ for some $u, \tilde{u} \in H$ and all $v \in H^1_A$ then, taking $v \in \mathcal{D}(A_N)$ or $v \in \mathcal{D}(A_D)$, we obtain $u = A\tilde{u} = A_D^{-1} \tilde{u}$. Therefore $\mathcal{D}(A^*) \subset H^2_0$. On the other hand, if $u \in H^1_0$ then $(u, Av) = a[u, v]$ because $u \in H^1_0$ and $a[u, v] = (A_N u, v)$. Thus $\mathcal{D}(A^*) = H^2_0$ and $A^* = A|_{H^2_0}$.

If $H^2_0$ is not dense in $H$ then the second adjoint $A^{**}$ does not exist and the operator $A$ is not closable in $H$ (see, for example, [BS Section 3.3]).

If $H^2_0$ is dense in $H$ then $A_D$ and $A_N$ are self-adjoint extensions of $A^*$, and $A^{**}$ is the closure of $A$. Note that $\mathcal{D}(A) = H^1_0$ may be strictly smaller than $H^1 \cap \mathcal{D}(A^{**})$. Also, the $H^1$-closed subspaces $G_z$ may be strictly smaller than $\ker(A^{**} - zI)|_{H^1}$ and may not be closed in $H$ (see the next subsection).

4.2. An example. Let $\Omega$ be a bounded domain with smooth boundary, and let $H = L_2(\Omega)$ and $H^s$ be the Sobolev spaces. If $a[u] = \|\nabla u\|^2 + \|u\|^2$ and $H^0_1$ is the $H^1$-closure of $C^\infty_0(\Omega)$ then $A = -\Delta + I$, $\mathcal{D}(A) = \{u \in H^1 : Au \in H\}$ and $G_0 = \{u \in H^1 : Au = 0\}$. The self-adjoint operators $A_D$ and $A_N$ are obtained by imposing the Dirichlet and Neumann boundary conditions. The $H$-adjoint $A^*$ coincides with the restriction of $A$ to $H^2_0 := \{u \in H^2 : u|_{\partial \Omega} = \partial_n u|_{\partial \Omega} = 0\}$, where $\partial_n$ is the normal derivative. The second $H$-adjoint $A^{**}$ is the extension of $A$ to $\mathcal{D}(A^{**}) = \{u \in H : Au \in H\}$, and $\mathcal{D}(A) = H^1 \cap \mathcal{D}(A^{**})$.

Let us choose a nonzero function $v_0 \in G_0$, and define $\tilde{H}^1_0 = H^1_0 \oplus \mathcal{L}_0$, where $\mathcal{L}_0$ is the one dimensional subspace spanned by $v_0$ and $\oplus$ denotes the orthogonal sum in $H^1$. Then the corresponding operator $\tilde{A}$ is the same differential operator $-\Delta + I$ but $\mathcal{D}(\tilde{A}) = \{u \in \mathcal{D}(A) : \langle \partial_n P_N^k(0) u, v_0 \rangle_{\partial \Omega} = 0\}$, where $\langle \cdot, \cdot \rangle_{\partial \Omega}$ denotes the sesquilinear pairing between $H^{-1/2}(\partial \Omega)$ and $H^{1/2}(\partial \Omega)$. The Neumann operator remains the same, and the domain of new “Dirichlet” operator is $\mathcal{D}(\tilde{A}_D) = \mathcal{D}(\tilde{A}) \cap \tilde{H}^1_0$. Finally,

$$\mathcal{D}(\tilde{A}^*) = \mathcal{D}(A_N) \cap \tilde{H}^1_0 = \{u \in H^2 \cap \tilde{H}^1_0 : \partial_n u|_{\partial \Omega} = 0\}$$

and, consequently, $H^2_0 \subset \mathcal{D}(A^*)$.

Let $v_0 \notin H^2$. Then $v_0|_{\partial \Omega} \notin H^{3/2}(\partial \Omega)$ and $u|_{\partial \Omega} \notin H^{3/2}(\partial \Omega)$ for all $u \in \tilde{H}^1_0 \setminus H^1_0$. This implies that $H^2 \cap \tilde{H}^1_0 = H^2 \cap H^1_0$ and $\mathcal{D}(\tilde{A}^*) = \mathcal{D}(A^*) = H^2_0$. Thus we have $\tilde{A}^{**} = A^{**}$. By the above, in this case $\mathcal{D}(\tilde{A}) \neq H^1 \cap \mathcal{D}(\tilde{A}^{**})$.

The $H^1$-orthogonal complement $\tilde{G}_0 := H^1 \cap \tilde{H}^1_0 = G_0 \oplus \mathcal{L}_0$ coincides with the kernel of the functional $u \rightarrow a[v_0, u] = \langle \partial_n v_0, u \rangle_{\partial \Omega}$ defined on the space $\tilde{G}_0$. If $v_0 \notin H^2$ then this functional is not $H$-continuous and $\tilde{G}_0$ is not $H$-closed. Now (2.1) implies that $\tilde{G}_z := \ker(\tilde{A} - zI)$ are not $H$-closed for all $z \in \mathbb{C}$.

4.3. The projections $P_\xi(\lambda)$. Note that, by the spectral theorem, the right hand side of (3.7) is a nondecreasing function of $\mu$ and the right hand side of (3.8) is a nonincreasing function of $\mu$. This observation allows one to simplify the proof of
Theorem 1.3 in the case where $\dim G^-_{\lambda} < \infty$ or $\dim G^+_{\lambda} < \infty$. The monotonicity is an implicit consequence of the following result.
Lemma 4.1. Let $\Lambda$ be an arbitrary real interval, and let $v \in H^1_{\Lambda}$.

1. If $\Lambda \cap \sigma_{\text{ess}}(A_N) = \emptyset$ and $\chi_\Lambda(A_N) P_N'(0)v = 0$ then $P_N'(\lambda)v \in G_\lambda$ for all $\lambda \in \Lambda$ and $b[P_N'(\lambda)v]$ is a nondecreasing function on $\Lambda$.

2. If $\Lambda \cap \sigma_{\text{ess}}(A_D) = \emptyset$ and $\chi_\Lambda(A_D) P_D'(0)v = 0$ then $P_D'(\lambda)v \in G_\lambda$ for all $\lambda \in \Lambda$ and $b[P_D'(\lambda)v]$ is a nonincreasing function on $\Lambda$.

If, in addition, $v \notin D(A_B)$ then the function $b[P_B'(\lambda)v]$ is strictly monotone.

Proof. Since $(A - \lambda I) P_B'(\lambda)v = E_B(\lambda)(A - \lambda I)v$, the equality $\chi_\Lambda(A_B) P_B'(0)v = 0$ implies that $(A - \lambda I) P_B'(\lambda)v = -\lambda E_B(\lambda) P_B'(0)v = 0$. Thus we have $P_B'(\lambda)v \in G_\lambda$ for all $\lambda \in \Lambda$.

If $w_B \in D(A_B)$ then $P_B'(\lambda)w_B = E_B(\lambda)w_B$. Using this identity, one can easily show that

\begin{align*}
(4.1) & \quad b[P_N'(\lambda)(w + w_N)] = b[P_N'(\lambda)w] - (E_N(\lambda)(A - \lambda I)w, w_N), \\
(4.2) & \quad b[P_D'(\lambda)(w + w_D)] = b[P_D'(\lambda)w] + (w_D, E_D(\lambda)(A - \lambda I)w)
\end{align*}

for all $w \in H^1_{\Lambda}$, $w_N \in D(A_N)$ and $w_D \in D(A_D)$. Since $E_B(\lambda)P_B'(0)v = 0$ and $P_B'(\lambda)P_B'(0) = P_B'(\lambda)$ for all $\lambda \in \Lambda$, substituting $w = P_B'(0)v$, $w_B = P_B(0)v$ in (4.1), (4.2) and applying (2.6), (2.7), we obtain

\begin{align*}
& \quad b[P_N'(\lambda)v] = b[P_N'(0)v] + \lambda \|P_N'(0)v\|^2 + \lambda^2 (R_N(\lambda)P_N'(0)v, P_N'(0)v), \quad \forall \lambda \in \Lambda, \\
& \quad b[P_D'(\lambda)v] = b[P_D'(0)v] - \lambda \|P_D'(0)v\|^2 - \lambda^2 (R_D(\lambda)P_D'(0)v, P_D'(0)v), \quad \forall \lambda \in \Lambda.
\end{align*}

Now the required monotonicity results follow from the spectral theorem. \hfill \Box

Note that $P_B'(0) = P_B(0)P_B'(0) = P_B'(0) - \mu A_B^{-1} P_B'(0)$ whenever $\mu \notin \sigma(A_B)$. Therefore we have $\chi_\Lambda(A_B) P_B'(0)v = 0$ if and only if $\chi_\Lambda(A_B) P_B'(0)v = 0$ for all $\mu \notin \sigma(A_B)$.

4.4. Analytic properties of $\Pi(z)$. If the embedding $H^1_0 \hookrightarrow H$ is compact then the operator-valued functions $\Pi(z)$ and $\Pi'(z)$ introduced in Subsection 2.3 are meromorphic in the whole complex plane. Indeed, since $a[A_D^{-1}u] = (A_D^{-1}u, u)$, the compactness of the embedding $H^1_0 \hookrightarrow H$ implies that $A_D^{-1}$ is compact as an operator from $H$ to $H^1$. Consequently,

\[
T_z^* T_z - I = z^2 \Pi_0 A_N^{-2}|_{H^1_0} - 2z \Pi_0 A_N^{-1}|_{H^1_0} = z^2 A_D^{-1} A_N^{-1}|_{H^1_0} - 2z A_D^{-1}|_{H^1_0}
\]

are compact operators in $H^1_0$. Now the required result follows from the analytic Fredholm theorem (see, for example, [Ya, Section 1.8]).

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