Lensing and imaging by a stealth defect of spacetime

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Abstract

We obtain the geodesics for the simplest possible stealth defect which has a flat spacetime. We, then, discuss the lensing properties of such a defect, and the corresponding image formation. Similar lensing properties can be expected to hold for curved-spacetime stealth defects.

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I. INTRODUCTION

A particular Skyrmion spacetime defect has been studied recently in a series of papers: the self-consistent Anz"{a}tze for the fields were established in Ref. [1], the origin of a possible negative asymptotic gravitational mass was discussed in Ref. [2], and the details of a special defect solution with zero asymptotic gravitational mass were given in Ref. [3]. This last defect solution, with positive energy density of the matter fields but a vanishing asymptotic gravitational mass, has been called a “stealth defect.”

It was stated in Sec. 4 of Ref. [3] that “assuming the existence of this particular type of spacetime-defect solution without long-range fields, an observer has no advance warning if he/she approaches such a stealth-type defect solution (displacement effects of background stars are negligible, at least initially).” The goal of the present article is to expand on the parenthetical remark of the previous quote. We study, in particular, the geodesics of the special defect solution of Fig. 5 in Ref. [3], which has a flat spacetime. For completeness, we have also performed a simplified calculation of the geodesics for a curved-spacetime stealth defect and present the results in App. A.

II. GEODESIC EQUATIONS

The topology and coordinatization of the spacetime manifold considered has been reviewed in Sec. 2 and App. A of Ref. [4] (see also Ref. [5] for a detailed discussion of the differential structure). Very briefly, the spatial part of the manifold is obtained by removing the interior of a ball in three-dimensional Euclidean space and by identifying antipodal points on the boundary of this ball (the defect surface has topology $\mathbb{R}P^2 \sim S^2/\mathbb{Z}_2$). As to the coordinatization, there are three coordinate charts. Here, we focus on the chart-2 coordinates, the other charts being similar. Moreover, we use dimensionless coordinates, all lengths being measured in units of $1/(e f)$ for the theory as defined in Ref. [3].

The metric of the vacuum solution reads as follows (cf. Sec. 3 of Ref. [4]):

$$ds^2 \mid^{(\text{vac. sol.)}} = -\left(1 - l/\sqrt{w}\right) (dt)^2 + \frac{1 - y_0^2/w}{1 - l/\sqrt{w}} (dy)^2 + w \left[(dz)^2 + \sin^2 z (dx)^2\right], \quad (2.1a)$$

$$w \equiv y_0^2 + y^2, \quad (2.1b)$$

$$y_0 \equiv e f b > 0, \quad (2.1c)$$

where $y_0$ corresponds to the dimensionless version of the defect length scale $b > 0$. Note that we only show the dimensionless coordinates of $n = 2$ chart, one of which is the dimensionless quasi-radial coordinate $y \in (-\infty, \infty)$ with $y = 0$ corresponding to the defect surface.
For a globally regular solution, the constant $l$ in (2.1a) takes the following values:

$$l \in (-\infty, y_0).$$

(2.2)

With $l = 0$ for the stealth-defect solution from Sec. 2.4 and Fig. 5 in Ref. [3], we have the metric

$$ds^2 \bigg|^{(\text{vac. sol. } l=0)} = -(dt)^2 + A(y) (dy)^2 + w \left[ (dz)^2 + \sin^2 z (dx)^2 \right],$$

(2.3)

with

$$A(y) = \frac{y^2}{y_0^2 + y^2}.$$

(2.4)

Then, the nonvanishing Christoffel symbols are

$$\Gamma^y_{yy} = \frac{A'}{2A},$$

(2.5a)

$$\Gamma^y_{zz} = -\frac{w'}{2A},$$

(2.5b)

$$\Gamma^y_{xx} = -\frac{w' \sin^2 z}{2A},$$

(2.5c)

$$\Gamma^z_{yz} = \Gamma^z_{zy} = \frac{w'}{2w},$$

(2.5d)

$$\Gamma^z_{xx} = -\sin z \cos z,$$

(2.5e)

$$\Gamma^x_{yx} = \Gamma^x_{xy} = \frac{w'}{2w},$$

(2.5f)

$$\Gamma^x_{zz} = \Gamma^x_{xz} = \cot z,$$

(2.5g)

where the prime stands for differentiation with respect to $y$. The first three Christoffel symbols are divergent at the defect surface, but our results will show that the motion of a particle can still be regular.

From the geodesic equation with affine parameter $\lambda$, we find

$$0 = \frac{d^2 t}{d\lambda^2},$$

(2.6a)

$$0 = \frac{d^2 y}{d\lambda^2} + \Gamma^y_{yy} \left( \frac{dy}{d\lambda} \right)^2 + \Gamma^y_{zz} \left( \frac{dz}{d\lambda} \right)^2 + \Gamma^y_{xx} \left( \frac{dx}{d\lambda} \right)^2,$$

(2.6b)

$$0 = \frac{d^2 z}{d\lambda^2} + 2\Gamma^z_{yz} \frac{dy}{d\lambda} \frac{dz}{d\lambda} + \Gamma^z_{xx} \left( \frac{dx}{d\lambda} \right)^2,$$

(2.6c)

$$0 = \frac{d^2 x}{d\lambda^2} + 2\Gamma^x_{yx} \frac{dy}{d\lambda} \frac{dx}{d\lambda} + 2\Gamma^x_{xz} \frac{dx}{d\lambda} \frac{dz}{d\lambda}.$$
We can choose the normalization of $\lambda$ so that the solution of (2.6a) has
\[
\frac{dt}{d\lambda} = 1.
\tag{2.7}
\]
Then, $\lambda$ can be replaced by $t$ in (2.6b)–(2.6d). Since the metric is spherically symmetric, we need only consider the case $z = \pi/2$. Our calculation follows Sec. 8.4 of Ref. [6].

For the case $dx/dt \neq 0$, divide (2.6d) by $dx/dt$ and use the Christoffel symbols from (2.5). We, then, have
\[
\frac{d}{dt} \left( \ln \frac{dx}{dt} + \ln w \right) = 0,
\tag{2.8}
\]
which gives a real constant (up to a sign),
\[
J \equiv w \frac{dx}{dt}.
\tag{2.9}
\]

With (2.5), (2.9) and multiplying (2.6b) by $2A dy/dt$, we find
\[
\frac{d}{dt} \left[ A \left( \frac{dy}{dt} \right)^2 + \frac{J^2}{w} \right] = 0.
\tag{2.10}
\]
Hence, there is the following constant of motion:
\[
E \equiv A \left( \frac{dy}{dt} \right)^2 + \frac{J^2}{w}.
\tag{2.11}
\]
By elimination of $t$ from (2.9) and (2.11), we get $y$ as a function of $x$,
\[
A \frac{(dy)^2}{w^2} + \frac{1}{w} = \frac{E}{J^2}.
\tag{2.12}
\]
From (2.9), (2.11) and $z = \pi/2$, the metric (2.3) along the geodesic can now be written as
\[
ds^2 = (-1 + E)(dt)^2.
\tag{2.13}
\]
In other words, we have
\[
E = 1, \quad \text{for a massless particle,}
\tag{2.14a}
E \in [0, 1), \quad \text{for a massive particle,}
\tag{2.14b}
\]
where the case $E = 0$ corresponds to $dy/dt = 0$, as will be discussed in Sec. III.
III. RADIAL GEODESICS

Consider the geodesic equation for a particle moving solely in the negative $y$ direction (going from right to left in Fig. 1), i.e., $dx/dt = 0$. From the definition of $J$ in (2.9), it then follows that $J = 0$, even though $J$ was initially defined as a nonzero quantity. The corresponding energy-type constant of motion is

$$E = \frac{y^2}{y_0^2 + y^2} \left(\frac{dy}{dt}\right)^2.$$  

(3.1)

The solutions of (3.1) are

$$y = \pm \sqrt{-y_0^2 + \left(+\sqrt{E} t + C_1\right)^2},$$  

(3.2a)

$$y = \pm \sqrt{-y_0^2 + \left(-\sqrt{E} t + C_2\right)^2},$$  

(3.2b)

where $C_1$ and $C_2$ are two real constants.

Making appropriate time shifts (or setting $C_1 = C_2 = y_0$) and defining $B \equiv \sqrt{E}/y_0$, the solutions (3.2) reproduce the results of Sec. 3 in Ref. [4]. Finally, as mentioned before, we find a constant $y$ solution if $E = 0$.

IV. NONRADIAL GEODESICS

Nonradial geodesics exist in two types, those which reach the defect surface ($\mathbb{R}P^2 \sim S^2/\mathbb{Z}_2$) and those which do not.
Figure 2. Nonradial geodesic which does not cross the defect surface.

A. Geodesics without crossings of the defect surface

Outside the defect surface, the spacetime (2.3) is Minkowskian with vanishing curvature invariants [4]. So, geodesics which do not cross the defect surface should be straight lines. The following calculation will show this explicitly.

From (2.12), we find
\[ dx = \pm \int \frac{y \, dy}{(y_0^2 + y^2) \sqrt{(E/J^2)(y_0^2 + y^2) - 1}}. \]  
(4.1)

Define the quasi-radial coordinate \( y_1 \) corresponding to the point on the line closest to the defect surface (cf. Fig. 2), so that \( |y_1| \) corresponds to an “impact parameter.” Since \( d\sqrt{w}/dx \) and \( dy/dx \) vanish at \( y_1 \), (2.12) gives
\[ \frac{1}{y_0^2 + y_1^2} = \frac{E}{J^2}. \]  
(4.2)

Then, (4.1) can be written as
\[ x(y) = x(\infty) \pm \int_y^\infty \frac{y \, dy}{(y_0^2 + y^2) \sqrt{(y_0^2 + y^2)/(y_0^2 + y_1^2) - 1}}. \]  
(4.3)

At \( y = y_1 \), (4.3) gives
\[ |x(y_1) - x(\infty)| = \pi/2. \]  
(4.4)

The result (4.4) shows that these particular geodesics (nonradial and nonintersecting with the defect surface) are indeed straight lines.
B. Geodesics crossing the defect surface

Now, consider nonradial geodesics which cross the defect surface. If we use in (2.12) the replacement
\[
\frac{dy}{dx} = \frac{1}{2y} \frac{dy^2}{dx},
\]
we find the following two solutions for \(y^2\):
\[
y^2 = \frac{\tan^2(x_1 + x) + 1}{E/J^2} - y_0^2, \quad (4.6a)
\]
\[
y^2 = \frac{\tan^2(x_2 - x) + 1}{E/J^2} - y_0^2, \quad (4.6b)
\]
where \(x_1\) and \(x_2\) are two real constants.

Note that the metric (2.3) has a spherically symmetric form and that the “radial” coordinate is \(\sqrt{w} \in [y_0, \infty)\). After a shift of the constants, the solutions (4.6) can be written as
\[
\sqrt{w} \sin(x_1 - x) = \pm \frac{J}{\sqrt{E}}, \quad (4.7a)
\]
\[
\sqrt{w} \sin(x_2 + x) = \pm \frac{J}{\sqrt{E}}, \quad (4.7b)
\]
with \(\sqrt{w} \geq y_0\). Several comments on the solutions (4.7) are in order:

1. mathematically, the solutions are straight lines or straight-line segments in polar-type coordinates (\(\sqrt{w}, x\));
2. the solutions are regular at the defect surface, \(\sqrt{w} = y_0\);
3. to find the complete geodesic of a given particle among these solutions, we must remember the antipodal identifications at the defect surface \(\sqrt{w} = y_0\).

For a nonradial ingoing line, it is convenient to choose coordinates, so that the end of the ingoing line has \(x = \pi/2\) (Fig. 3). In these coordinates, the ingoing line is given by
\[
\sqrt{w} \sin(x_0 - x) = -y_0 \cos x_0, \quad (4.8a)
\]
with
\[
0 < x_0 < x \leq \pi/2. \quad (4.8b)
\]
Observe that we have included the end point of the ingoing line in the above formula. We can check that the formula (4.8) indeed corresponds to one of the solutions (4.7).

In this case, there will exist, among the solutions (4.7), a unique outgoing line (Fig. 4) if the following two conditions are met:
Figure 3. Ingoing line (4.8) lying in the domain of the chart-2 coordinates. The dashed line shows the $x_3$ Cartesian axis, which, however, does not belong to the domain of the chart-2 coordinates.

Figure 4. Nonradial geodesic crossing the defect surface.

Figure 5. A family of geodesics crossing the defect surface.
1. the beginning of the outgoing line and the end of the ingoing line must be antipodal
points at the defect surface (these points are identified);

2. the complete geodesic must be a straight line if \( y_0 = 0 \).

Note that, with a nonradial ingoing line as in Fig. 4, the quantity \( J \) will change sign after
crossing the defect surface (see Ref. 5 for further discussion of the anomalous angular-
momentum behavior of scattering solutions). Based on the above two points, Fig. 5 shows
three geodesics from a continuous family of geodesics crossing the defect surface [the family
ranges continuously from a radial geodesic (dot-dashed line) to a tangent geodesic (dotted
line)].

From the particular family of geodesics as shown in Fig. 5, we obtain what may be
called a “shifting tangent geodesic” (dotted line in Fig. 5). But, from the limiting case of
the geodesic in Fig. 2 with \( y_1 \to 0^+ \), we obtain what may be called an “ongoing tangent
geodesic” (solid line in Fig. 2 pushed towards the defect surface). Hence, we conclude that
“certain geodesics at the defect surface \( Y = 0 \) cannot be continued uniquely,” as mentioned
in the second remark of Sec. VI in Ref. 2.

V. IMAGE FORMATION BY A STEALTH DEFECT

The geodesics of the stealth-defect spacetime (2.3) have been discussed in Secs. III and
IV. For a nonradial geodesic reaching the defect surface, Fig. 4 shows that the defect causes
a parallel shift in the ambient space. In this section, we will show that this shift can, in
principle, create an image of a given object.

First, consider geodesics which start from a point \( P \) at one side of the defect (Fig. 6). For
geodesics which cross the defect, there will be an intersection point \( P' \) at the other side of the
defect. In fact, \( P \) and \( P' \) are reflection points about the “center” of the defect (considered
to be obtained by surgery on the three-dimensional Euclidean space). The different paths
connecting \( P \) and \( P' \) have, in general, different values for the time-of-flight.

Next, observe that, based on the above discussion for the geodesics of a stealth-defect
spacetime, a permanent luminous object will give a real image of the object (Fig. 7). The
qualification “permanent” for the light source refers to the different time-of-flight values
mentioned in the previous paragraph.

Several additional remarks are in order. First, the image in Fig. 7 is located at the
reflection point on the other side of the defect.

Second, the image is inverted and the image size is equal to the object size. Note that
this is also the case if an object in Minkowski spacetime is located at a \( 2f \) distance from
a standard thin double-convex lens, where \( f \) is the focal length of the lens (cf. Sec. 27.3
of Ref. [7]). Recall that the time-of-flight of different paths connecting the $2f$ points of a standard lens in Minkowski spacetime is equal, due to the reduced speed of light in the lens material and the appropriate shape of the lens.

Third, the brightness of the image depends on the defect scale $b$ and the location of the object: the image will be brighter if $b$ is increased for unchanged object position or if the object is brought closer to the defect for unchanged defect scale.

Fourth, return to the analogy with standard lenses in Minkowski spacetime as mentioned
in the second remark and note that our defect resembles a zoom lens, with a finite range of focal lengths. If we recall the standard lens equation $1/d_{\text{object}} + 1/d_{\text{image}} = 1/f$ in Minkowski spacetime (cf. Eq. (27.12) of Ref. [7]), we see that our defect has an effective focal length $f_{\text{eff}}$ given by

$$2 f_{\text{eff}} = \sqrt{b^2 + (y_{\text{object}})^2} \in (b, \infty),$$

where $y_{\text{object}} \neq 0$ is the chart-2 quasi-radial coordinate of a small object away from the defect surface.

Fifth, if a permanent pointlike light source is placed at point $P$ of Fig. 6, then an observer at point $P'$ in the same figure will see a luminous disk (different from the Einstein ring, which the observer would see if the defect were replaced by a patch of Minkowski spacetime with a static spherical star at the center).

Sixth, it may be of interest to compare our lensing and imaging results from the spacetime defect with those from wormholes (see, e.g., Refs. [11, 12]). In both cases, there is an unusual ingredient in the physics setup: negative energy density for the wormholes and a degenerate metric for the spacetime defect.

VI. DISCUSSION

In the present article, we have studied the geodesics of the stealth-defect solution of Fig. 5 in Ref. [3], which has a flat spacetime (from the parameter choice $\tilde{\eta} \equiv 8\pi G_N f^2 = 0$ for $G_N = 0$ and $f > 0$).

Remark that the stealth-defect solution of Fig. 4 in Ref. [3] has a curved spacetime (parameter choice $\tilde{\eta} = 1/10$), which results in some additional bending of light passing near the defect surface. In fact, the bending is outwards, as the effective mass near the defect surface is negative; see the $l(w)$ panel of Fig. 4 in Ref. [3]. Still, the lensing property is essentially the same as for the flat-spacetime defect (see App. A for a simplified calculation).

In the argument of Sec. V, we considered light rays. But, with the particle–wave duality, we can also interpret the three geodesics crossing the defect surface in Fig. 6 as coherent light emitted from the source $P$ (as mentioned before, the emission is assumed to last for a long time). At the point $P'$, these coherent-light bundles have a constant phase difference, which leads to stationary interference. In this sense, our defect resembles not only a material lens in Minkowski spacetime but also some type of interferometer.

As a final comment, we contrast our lensing results with those of standard gravitational lensing. Standard gravitational lensing can be interpreted as being due to the curvature of spacetime resulting from a nonvanishing matter distribution. The lensing of Fig. 6 is, however, entirely due to the nontrivial topology from the hypothetical defect, as the spacetime manifold of this particular solution is flat.
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Appendix A: Geodesics of a curved-spacetime stealth defect

In Sec. V, we have shown that a particular defect in flat spacetime resembles a material lens in Minkowski spacetime. In this appendix, we will see that the same resemblance holds for the corresponding defect in curved spacetime.

1. General results

The general spherically symmetric Ansatz for the metric of a spacetime defect is given by the following line element [1]:

$$ds^2 \overset{(\text{gen. sol.)}}{=} -M(w) (dt)^2 + N(w) (dy)^2 + w \left[ (dz)^2 + \sin^2 z (dx)^2 \right],$$  
(A1a)

$$M(w) \equiv [\mu(w)]^2,$$  
(A1b)

$$N(w) \equiv (1 - y_0^2/w)[\sigma(w)]^2,$$  
(A1c)

with $w$ defined by (2.1b). The functions $\mu(w)$ and $\sigma(w)$ are determined by the field equations and the boundary conditions. At this moment, we do not need to know the explicit form of these functions.

As mentioned in Sec. III, we only need to consider the particle moving in the equatorial plane, $z = \pi/2$. Then, the nonvanishing Christoffel symbols are

$$\Gamma_{t y}^t = \Gamma_{y t}^t = -\frac{1}{2M} \frac{dM}{dy},$$  
(A2a)

$$\Gamma_{t t}^y = -\frac{1}{2N} \frac{dM}{dy},$$  
(A2b)

$$\Gamma_{y y}^y = \frac{1}{2N} \frac{dN}{dy},$$  
(A2c)

$$\Gamma_{y x}^y = -\frac{1}{2N} \frac{dw}{dy},$$  
(A2d)

$$\Gamma_{x y}^x = \frac{1}{2w} \frac{dw}{dy}.$$  
(A2e)
With the procedure used in Sec. II, the geodetic equation gives

\[ \frac{dt}{d\lambda} = M, \]  
\[ w \frac{dx}{d\lambda} = \tilde{J}, \]  
\[ N \left( \frac{dy}{d\lambda} \right)^2 + \frac{\tilde{J}^2}{w} - \frac{M^3}{3} = \tilde{E}, \]

where \( \tilde{J} \) and \( \tilde{E} \) are real constants and \( \lambda \) is the affine parameter. By elimination of \( \lambda \) from (A3b) and (A3c), we have

\[ \frac{\tilde{J}^2 N(w)}{w^2} \left( \frac{dy}{dx} \right)^2 + \frac{\tilde{J}^2}{w} - \frac{[M(w)]^3}{3} = \tilde{E}, \]  

where the explicit \( w \)-dependence of \( N \) and \( M \) has been restored. With the replacement (4.5), condition (A4) can be written as

\[ \frac{\tilde{J}^2 N(w)}{4y^2w^2} \left( \frac{dy^2}{dx} \right)^2 + \frac{\tilde{J}^2}{w} - \frac{[M(w)]^3}{3} = \tilde{E}, \]  

with the constants \( \tilde{J} \) and \( \tilde{E} \) from (A3).

The orbit of a particle moving in the equatorial plane \( z = \pi/2 \) is described by (A5). Observe that \( N(w) \) and \( M(w) \) are functions of \( w \) and, hence, functions of \( y^2 \). If the solution of (A5) exists, \( x \) must be a function of \( y^2 \): \( x = x(y^2) \). Recall that the chart-2 coordinates have the following ranges:

\[ x \in (0, \pi), \quad y \in (-\infty, \infty), \quad z \in (0, \pi). \]  

For a particular solution \( x = x(y^2) \) in the \( z = \pi/2 \) plane of the chart-2 domain, there are then two branches: one branch with \( y \geq 0 \) and the other one with \( y \leq 0 \) (note that the point \( y = 0 \) has been included for both branches, as was done in Sec. IV). To be specific, the lines which correspond to these two branches of the solution are symmetrical about the “center” of the defect surface. If the orbit of a given particle does not cross the defect surface, then this orbit is usually described by only one of these two branches. But, if the particle crosses the defect surface, then we argue that the ingoing and outgoing lines are given by two different branches. Remark that, in flat spacetime, this argument is consistent with the two conditions for the existence of a unique outgoing line as discussed in Sec. IV B.

Based on above points, a defect in a curved spacetime resembles a material lens and has the same properties as discussed in Sec. V for the flat-spacetime case. Still, there is one exception: a black hole may occur for this defect spacetime [4]. Then, the metric (A1) is not globally regular and (A5) cannot properly describe the orbit of the particle reaching the
defect surface. In fact, the particle will be confined within the black-hole horizon once it crosses the horizon (appropriate coordinates would, for example, be the Painlevé–Gullstrand-type coordinates of App. C in Ref. [4]).

2. Explicit calculation

The numerical stealth-defect solution from Fig. 4 of Ref. [3] has metric functions $\sigma(w)$ and $\mu(w)$ in (A1) with approximately the following form:

\[
\sigma(w) = 1 - \frac{1}{2w}, \quad (A7a)
\]
\[
\mu(w) = 1, \quad (A7b)
\]

for $y_0 = 1$ (giving $w \equiv 1 + y^2$). We will now obtain the analytic solutions of (A3) and (A5) for the explicit choice of functions in (A7).

For the radial geodesic ($\vec{J} = 0$), the general solutions of (A3c) are

\[
\frac{2w(2w + 1)}{2w - 1} \sqrt{\frac{(-2w + 1)^2}{w^3}} = +4 t \sqrt{\tilde{E} + 1/3 + \tilde{C}_1}, \quad (A8a)
\]
\[
\frac{2w(2w + 1)}{2w - 1} \sqrt{\frac{(-2w + 1)^2}{w^3}} = -4 t \sqrt{\tilde{E} + 1/3 + \tilde{C}_2}, \quad (A8b)
\]

where $\tilde{C}_1$ and $\tilde{C}_2$ are two real constants. An example of a null radial geodesic is shown in Fig. 8.

For a nonradial geodesic, the solutions of (A5) are

\[
\pm x = \frac{1}{4} \left( (4 - D) \arctan(\sqrt{D w - 1} - \frac{\sqrt{D w - 1}}{w}) \right) + x_4, \quad (A9a)
\]

Figure 8. Null radial geodesic for the stealth defect (A1) with metric functions (A7), using dimensionless coordinates $y$ and $t$. 

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with the definition
\[ D \equiv \frac{(\tilde{E} + 1/3)}{\tilde{J}^2} \]  \hspace{1cm} (A9b)
and a real constant \( x_4 \).

For geodesics that do not cross the defect surface, we can, just as in Sec. IV A, calculate
the change in \( x \),
\[ \Delta x \equiv |x(y_1) - x(\infty)| = \frac{\pi}{2} \left( 1 - \frac{1/4}{1 + y_1^2} \right), \]  \hspace{1cm} (A10)
where \( y_1 \) corresponds to the point on the line closest to the defect surface. For small \( y_1 \) (i.e.,
the line coming close to the defect surface), (A10) shows that the line is bent away from the
defect surface. This agrees with the fact that the effective mass near the defect surface is
negative (cf. Fig. 4 in Ref. [3]).

Note that (A9) can be rewritten in the following way:
\[ \pm \frac{1}{\sqrt{D}} = \sqrt{w} \cos \left[ \frac{4(x - x_4) + \sqrt{D} w - 1/w}{4 - D} \right], \]  \hspace{1cm} (A11a)
\[ \pm \frac{1}{\sqrt{D}} = \sqrt{w} \cos \left[ \frac{4(-x + x_5) + \sqrt{D} w - 1/w}{4 - D} \right], \]  \hspace{1cm} (A11b)
with real constants \( x_4 \) and \( x_5 \). As a concrete example, we first consider the solution corre-
sponding to the upper sign on the left-hand side of (A11a), that is,
\[ \frac{1}{\sqrt{D}} = \sqrt{w} \cos \left[ \frac{4(x - x_4) + \sqrt{D} w - 1/w}{4 - D} \right]. \]  \hspace{1cm} (A12)
For given values of \( D \) and \( x_4 \), the solution (A12) has, in general, two branches (Fig. 9): one
branch lies in the upper half-plane (\( y > 0 \)) and the other in the lower half-plane (\( y < 0 \)). The
solid lines in Fig. 9 correspond to the orbits of two different particles, while the dotted lines
 correspond to the orbit of another particle. Even though the points on the solid lines which
are closest to the defect surface have \( x = \pi/2 \), these solid lines are not symmetrical about
the line \( x = \pi/2 \) for \( w \sim 2 \), as can be verified in (A12) with \( x' = \pi - x \) and \( w(x') \neq w(x) \).

In Fig. 10 finally, we give a family of geodesics in order to illustrate the lensing prop-
erty of the curved-spacetime defect (cf. Fig. 6 for the lensing of the flat-spacetime defect).
Apparently, the spherical symmetry of the metric is a crucial input for the lensing behavior.
Figure 9. Geodesics in polar coordinates $(\sqrt{w}, \phi)$, where the geodesics are given by (A12). With the chart-2 coordinates $x$ and $y$, the azimuthal angle $\phi$ is defined by $\phi = x$ if $y > 0$ and $\phi = x + \pi$ if $y < 0$. The defect surface is given by the circle $w = 1$ and part of the spacetime manifold (A1), with metric functions (A7), is indicated by the shaded area. The solid lines have constants $D = 0.25$ and $x_4 = \pi/2$ and the dotted lines have constants $D = 1.25$ and $x_4 = \pi/2$. The points on the solid lines which are closest to the defect surface have polar coordinates $(2, \pi/2)$ and $(2, 3\pi/2)$, corresponding to the original coordinates $(y, x) = (\pm\sqrt{3}, \pi/2)$. 
Figure 10. A family of geodesics in polar coordinates \((\sqrt{w}, \phi)\), where the geodesics are given by (A11) with plus signs on the left-hand sides. The defect surface is given by the circle \(w = 1\). The parameters of the six curved geodesics in the upper half-plane are, from left to right, \((D = 1, \ x_4 = 2.57258)\), \((D = 1.2, \ x_4 = 2.5408766)\), \((D = 1.5, \ x_4 = 2.4784766)\), \((D = 1.5, \ x_5 = \pi - 2.4784766)\), \((D = 1.2, \ x_5 = \pi - 2.5408766)\), and \((D = 1, \ x_5 = \pi - 2.57258)\). In terms of the original \((y, x)\) coordinates, the focal points \(P\) and \(P'\) are given by \((y, x)_P = (\sqrt{8}, \pi/2)\) and \((y, x)_{P'} = (-\sqrt{8}, \pi/2)\).
[1] F.R. Klinkhamer, “Skyrmion spacetime defect,” Phys. Rev. D 90, 024007 (2014), arXiv:1402.7048.
[2] F.R. Klinkhamer and J.M. Queiruga, “Antigravity from a spacetime defect,” Phys. Rev. D 97, 124047 (2018), arXiv:1803.09736.
[3] F.R. Klinkhamer and J.M. Queiruga, “A stealth defect of spacetime,” Mod. Phys. Lett. A 33, 1850127 (2018), arXiv:1805.04091.
[4] F.R. Klinkhamer, “A new type of nonsingular black-hole solution in general relativity,” Mod. Phys. Lett. A 29, 1430018 (2014), arXiv:1309.7011.
[5] F.R. Klinkhamer and F. Sorba, “Comparison of spacetime defects which are homeomorphic but not diffeomorphic,” J. Math. Phys. 55, 112503 (2014), arXiv:1404.2901.
[6] S. Weinberg, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (Wiley & Sons, New York, 1972).
[7] R.P. Feynman, R.B. Leighton, and M. Sands, The Feynman Lectures on Physics, Volume I (Addison–Wesley, Reading MA, USA, 1963).
[8] A. Einstein, “Lens-like action of a star by the deviation of light in the gravitational field,” Science 84, 506 (1936).
[9] J. Wambsganss, “Gravitational lensing in astronomy,” Living Rev. Rel. 1, 12 (1998), arXiv:astro-ph/9812021.
[10] S. Dodelson, Gravitational Lensing (Cambridge University Press, Cambridge, England, 2017).
[11] V. Perlick, “On the exact gravitational lens equation in spherically symmetric and static space-times,” Phys. Rev. D 69, 064017 (2004), arXiv:gr-qc/0307072.
[12] A. Shatskiy, “Einstein–Rosen bridges and the characteristic properties of gravitational lensing by them,” Astron. Rep. 48, 7 (2004), arXiv:astro-ph/0407222.