Abelian Anomalies in Nonlocal Regularization

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Abstract

Nonlocal regularization of QED is shown to possess an axial anomaly of the same form as other regularization schemes. The Noether current is explicitly constructed and the symmetries are shown to be violated, whereas the identities constructed when one properly considers the contribution from the path integral measure are respected. We also discuss the barrier to quantizing the fully gauged chiral invariant theory, and consequences.

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INTRODUCTION

A scheme for the nonlocal regularization of gauge theories has recently been introduced which, aside from preserving the physical aspects of gauge invariance, is also finite, Poincaré invariant, and perturbatively unitary, without changing the dimension of spacetime or altering the pole structure (particle content). It also avoids the ambiguities associated with defining $\gamma^5$ in fractional spacetimes, and therefore provides an ideal stage for examining chiral anomalies. An important aspect of nonlocal regularization is that the Lagrangian is presented in regulated form at the beginning. In local field theory, the regularization method is invoked at the level of the calculation of diagrams, which leads to many of the inherent problems and ambiguities. In this paper, we shall examine the ABJ anomaly in the nonlocal regularization of QED to determine how the anomaly will manifest itself.

Because the consequences of the chiral anomaly are well known (fermion doubling to retain renormalizability, the correct rate for $\pi^0 \to \gamma\gamma$), it would be truly surprising for the nonlocal regularization to bypass it consistently, while retaining any sort of reasonable local limit. Here we will show that nonlocal QED produces an anomaly in the perturbative expansion, and that when the Ward identities are correctly derived by considering the Jacobian of the measure under a local axial transformation, they are satisfied and a consistent local limit is obtained.

In section 2, we develop conventions and briefly review results from local QED. Section 3 develops an equivalent classical theory as a precursor to nonlocal quantization, following which the quantization is performed and Ward identities and relevant loop corrections are obtained in section 4. In section 5, we discuss the failed attempt to gauge the full chiral invariance.
I. LOCAL QED

We begin by briefly reviewing local QED in order to establish conventions and the method we will follow in developing the anomaly in the nonlocal theory. The standard Lagrangian for local QED is written as

\[
L = \bar{\psi} (i\gamma \cdot \partial - m) \psi - \frac{1}{4e^2} F^2 - \bar{\psi} A\psi \\
\equiv \bar{\psi} S^{-1} \psi + \frac{1}{2e^2} A^\mu D^-_{\mu} A^\nu - \bar{\psi} A'\psi,
\]

which possesses the infinitesimal gauge invariance:

\[
\delta A^\mu = \theta^\mu, \quad \delta \psi = -i \theta \psi,
\]

giving rise to the conserved vector current \( \bar{\psi} \gamma^\mu \psi \). We have introduced the inverse propagators into the Lagrangian in order to clarify notation later and in doing so we are assuming that the trivial gauge fixing procedure has been performed on the photon. In the chiral limit, this local Lagrangian also has a global axial invariance \( \delta \psi = -i \omega \gamma^5 \psi \) (\( \omega = const. \)), and the associated Noether current:

\[
J^\mu_5 = \bar{\psi} \gamma^\mu \gamma^5 \psi,
\]

is classically conserved. When fermion masses are present the equations of motion give:

\[
\partial_\mu J^\mu_5 = 2i m J_5.
\]

It is well known that this current is no longer conserved when one quantizes the theory, and QED is then said to have an anomaly. (This persists when \( m \neq 0 \) as the current does not obey the classical equations of motion \( (1.4) \).) This result is easily seen by computing the second order correction to the axial current coupling to two photons \( \left(1,4\right) \). It was realized by Fujikawa \( \left(1,5\right) \), that one could understand the anomaly in the path integral by considering carefully the transformation properties of the properly regulated measure, and although the anomaly cannot be removed from the theory, it is possible to generate consistent Ward
identities by considering the Jacobian of the following local, infinitesimal change of fermion
variables [7]:

$$\delta \psi(x) = -i \omega(x) \gamma^5 \psi(x).$$  \hspace{1cm} (1.5)$$

Writing the generating functional:

$$Z[S_\mu, \bar{\eta}, \eta] = \int d\mu[A^\mu, \psi, \bar{\psi}]$$

$$\times \exp[i \int d^4 x (L + S_\mu A^\mu + \bar{\eta} \psi + \bar{\psi} \eta)],$$  \hspace{1cm} (1.6)$$
onespace

one transforms the fermionic degrees of freedom as in Eq. (1.5), and generates Ward identities for the axial current by setting the infinitesimal variation of the generating functional to zero:

$$\left. \frac{\delta}{\delta \omega} Z[S_\mu, \bar{\eta}, \eta] \right|_{\omega=0} = 0.$$  \hspace{1cm} (1.7)$$

To complete this process, one has to carefully consider the definition of the path integral with regards to the measure and some form of regulation procedure [8]. One then finds that the naively trivial Jacobian actually produces

$$d\mu[\psi, \bar{\psi}] \rightarrow d\mu[\psi, \bar{\psi}] \exp[-i \frac{1}{8\pi^2} \int dx \omega \tilde{F}^{\mu\nu} F_{\mu\nu} dx],$$  \hspace{1cm} (1.8)$$

where we have introduced the dual field strength as $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$. Combined with the transformation properties of the Lagrangian:

$$\delta L = 2i m \omega J_5 + J_5^\mu \partial_\mu \omega,$$  \hspace{1cm} (1.9)$$

this non-invariance of the measure leads to the anomalous weak operator conservation law

$$J_{5,\mu}^\mu = 2i m J_5 - \frac{1}{8\pi^2} \tilde{F}^{\mu\nu} F_{\mu\nu}.$$  \hspace{1cm} (1.10)$$

One can then perform the simplest quantum correction to the axial current in the presence of two external photons via fig. [9], and after imposing vector current conservation and taking into account the form of the pseudoscalar ($\gamma^5$) and pseudovector ($-i \gamma^\mu \gamma^5$) couplings, one finds
\[ p_\mu \Gamma_5^{\mu\alpha\beta} = 2m\Gamma_5^{\alpha\beta} - \frac{i}{2\pi^2} \epsilon^{\mu\nu\alpha\beta} q_{1\mu} q_{2\nu}, \]  

(1.11) consistent with (1.10).

At this point, it is worth stressing that in local theories it does not make any difference whether one includes an explicit axial coupling into the Lagrangian, since the structure of the theory does not change. In the case of nonlocal regularization, we will find that the axial coupling must be present in the beginning in order to generate a classical theory that respects (vector) current conservation.

II. NONLOCAL QED

A. Shadow Field Formalism

We will begin by introducing two types of propagators, smeared by an entire function that possesses strong convergence properties in the Euclidean regime:

\[ \hat{S}(p) = E^2(p)S(p) \]
\[ = -\int_1^\infty \frac{dx}{\Lambda^2} \exp\left(x\frac{p^2 - m^2}{\Lambda^2}\right)(\not{p} + m) \]
\[ \tilde{S}(p) = (1 - E^2(p))S(p) \]
\[ = -\int_0^1 \frac{dx}{\Lambda^2} \exp\left(x\frac{p^2 - m^2}{\Lambda^2}\right)(\not{p} + m), \]  

(2.1)

where

\[ E(p^2) = \exp\left(\frac{p^2 - m^2}{2\Lambda^2}\right). \]  

(2.2)

Here, we have given their Schwinger parameterized form. This is especially useful when calculating diagrams, since one merely writes the local graph in Schwinger parameter form, and restricts the range of parameter integrals appropriate for the process in question. For example, when one calculates single loop graphs, the unit hypercube is removed from the volume of integration. We now construct the auxiliary Lagrangian:
\[ L_{Sh} = \bar{\psi} \hat{S}^{-1} \psi + \bar{\phi} \hat{S}^{-1} \phi + \frac{1}{2e^2} A^\mu D^{-1}_{\mu\nu} A^\nu - (\bar{\psi} + \bar{\phi}) \bar{\Lambda}(\psi + \phi). \quad (2.3) \]

This particular choice of Lagrangian corresponds to a nonlocal regularization of QED, in which the classical theory retains the smearing on the internal photon lines, and internal fermion lines are ‘localized’. It is a simple matter to convince oneself of this, since we have by construction a Lagrangian that generates tree diagrams in which every \( \psi \) coupling has a \( \phi \) coupling corresponding to it, and so every tree process will consist of two separate graphs for each fermion line, one with a ‘hatted’ propagator and one with a ‘barred’ one, the two adding to give the local propagator. This guarantees decoupling of longitudinal photons at the classical level \([1]\). (We could have ‘localized’ the entire theory by introducing ‘shadow’ fields for the photon as well as the fermions, in which case the classical theory would be identical to the local theory, and we would have a viable physical theory of QED with a fundamental scale.)

It is simple to see that (2.3) is invariant under:

\[ \delta A^\mu = \theta^\mu, \]
\[ \delta \psi = -i E^2 \theta(\psi + \phi), \]
\[ \delta \phi = -i (1 - E^2) \theta(\psi + \phi). \]

The conserved Noether current (generalizing the local vector current) is given by

\[ J^\mu = (\bar{\psi} + \bar{\phi}) \gamma^\mu (\psi + \phi). \quad (2.4) \]

The shadow fields are introduced merely as a device to generate the nonlocal action and symmetries in a compact form. They do not have a pole in their propagator and hence are not propagating degrees of freedom, and they should not be included in asymptotic states.

To generate the action in terms of physical fields alone, we must integrate them out of the action by forcing them to obey their equations of motion. We have

\[ \phi = \bar{S}_A(\psi + \phi) = (1 - \bar{S}_A)^{-1} \bar{S}_A \psi, \quad (2.5) \]
and the Lagrangian, gauge transformations and Noether current are given by:

\[ L = \bar{\psi} S^{-1} \psi + \frac{1}{2e^2} A^\mu \hat{D}_{\mu\nu} A^\nu \\
- \bar{\psi} (1 - A S)^{-1} A (1 - \bar{S} A)^{-1} \psi \\
+ \bar{\psi} A \bar{S} (1 - A \bar{S})^{-1} S^{-1} (1 - \bar{S} A)^{-1} \bar{S} A \psi \\
= \bar{\psi} S^{-1} \psi + \frac{1}{2e^2} A^\mu \hat{D}_{\mu\nu} A^\nu - \bar{\psi} A (1 - \bar{S} A)^{-1} \psi, \quad (2.6) \]

\[ \delta \psi = -i E^2 \theta (1 - \bar{S} A)^{-1} \psi, \quad (2.7) \]

\[ J^\mu = \bar{\psi} (1 - A \bar{S})^{-1} \gamma^\mu (1 - \bar{S} A)^{-1} \psi. \quad (2.8) \]

These results reproduce the classical theory described in [1] (up to a rescaling of the electromagnetic field strength \( A \rightarrow eA \), and a sign convention on the coupling).

We shall now discuss the effects of an axial coupling on QED, and explore the consequences of loop effects on the classical equations of motion. But we must first consider the effects of an axial coupling in this regularization scheme. Consider the nonlocal regularized version of the global axial invariance:

\[ \delta \psi = -i E^2 \omega \gamma^5 (\psi + \phi), \]
\[ \delta \phi = -i (1 - E^2) \omega \gamma^5 (\psi + \phi). \quad (2.9) \]

If we consider graphs containing corresponding current insertions—even at the classical level—we no longer have current conservation (axial nor vector), and the longitudinal degree of freedom of the photon therefore does not decouple. This can be seen by computing any tree process with two or more axial current insertions, and coupling in an external longitudinal photon (or axial boson), and observing that these tree processes are not ‘localized’. No axial couplings occur in the shadow field equations of motion and, therefore, any diagram with two adjacent axial insertions will not receive the ‘barred’ fermion propagator contribution in the full nonlocal Lagrangian. This is trivially due to the fact that we did not include the axial coupling in the construction of the nonlocal Lagrangian, and consequently have not guaranteed decoupling and gauge invariance in its presence.
B. Nonlocal QED with Axial Couplings

We now generate a nonlocal Lagrangian that respects current conservation and decoupling at tree level in the presence of axial couplings, and therefore contains the physics of QED. In order to do this we begin again at the local level with

\[
L = \bar{\psi} S^{-1} \psi + \frac{1}{2e^2} A^\mu D^{-1}_{\mu\nu} A^\nu \\
- \bar{\psi} A \psi - \bar{\psi} B \gamma^5 \psi - iC \bar{\psi} \gamma^5 \psi,
\]

where we have included an axial vector coupling to some field(s) \( B^\mu \) and (for convenience) a pseudoscalar coupling to a pseudoscalar field \( C \) (neither field having U(1) vector transformation properties nor additional photon interactions). We then repeat the shadow field construction to generate the Lagrangian:

\[
L_{Sh} = \bar{\psi} \hat{S}^{-1} \psi + \frac{1}{2e^2} A^\mu \hat{D}^{-1}_{\mu\nu} A^\nu \\
- (\bar{\psi} + \bar{\phi}) \hat{A}(\psi + \phi) - (\bar{\psi} + \bar{\phi}) \hat{B} \gamma^5 (\psi + \phi) \\
- iC (\bar{\psi} + \bar{\phi}) \gamma^5 (\psi + \phi),
\]

and integrate out the shadow fields (defining \( \Gamma = \hat{A} + \hat{B} \gamma^5 + iC \gamma^5 \)):

\[
\phi = (1 - \bar{\phi} S) \Gamma^{-1} \bar{S} \Gamma \psi,
\]

\[
L_{NL} = \bar{\psi} \hat{S}^{-1} \psi + \frac{1}{2e^2} \hat{A}^\mu \hat{D}^{-1}_{\mu\nu} \hat{A}^\nu - \bar{\psi} \Gamma (1 - \bar{S} \Gamma)^{-1} \psi.
\]

Then, invariance under the transformation:

\[
\delta \psi = -ieE^2 \theta (1 - \bar{\phi} S) \Gamma^{-1} \psi
\]

is guaranteed. The shadow field equations give the following classically conserved vector and axial vector Noether currents:

\[
J^\mu = (\bar{\psi} + \bar{\phi}) \gamma^\mu (\psi + \phi) \\
= \bar{\psi} (1 - \Gamma \bar{S})^{-1} \gamma^\mu (1 - \bar{S} \Gamma)^{-1} \psi,
\]

\[
J_5^\mu = (\bar{\psi} + \bar{\phi}) \gamma^5 \gamma^\mu (\psi + \phi) \\
= \bar{\psi} (1 - \Gamma \bar{S})^{-1} \gamma^5 \gamma^\mu (1 - \bar{S} \Gamma)^{-1} \psi.
\]
Let us also define the pseudoscalar density:

\[ J_5 = (\bar{\psi} + \bar{\phi})\gamma^5(\psi + \phi) \]
\[ = \bar{\psi}(1 - \Gamma S)^{-1}\gamma^5(1 - \bar{S}\Gamma)^{-1}\psi. \]  
(2.16)

The variation of the generated Lagrangian under (2.9) gives the same result as in the local case:

\[ \delta L = 2i\omega m(\bar{\psi} + \bar{\phi})\gamma^5(\psi + \phi) \equiv 2i\omega m J_5, \]  
(2.17)

and the equations of motion also give (1.4). We have thus explicitly seen one major difference between local theories and their nonlocal ‘extensions’, namely, that it is necessary to consider the effect of all of the interaction terms when constructing the nonlocal Lagrangian, otherwise we cannot guarantee that the classical action will display the symmetries of the unregulated theory.

### III. QUANTIZING THE THEORY

#### A. Generating the Measure

We now wish to quantize the theory described by (2.13) in the path integral formalism, and see to what extent classical current conservation is respected. Doing this requires finding an invariant measure that respects the full nonlocal gauge invariance described by (2.14), since the trivial measure is no longer invariant. We therefore need a method to generate an invariant measure in order to consistently quantize the theory and retain the nonlocal invariance in the quantum regime, thereby guaranteeing decoupling. The simplest way to do this is to derive conditions on the invariant measure by considering how the trivial measure transforms under the nonlocal regularization gauge transformations, and requiring that an additional measure contribution compensates.

We write the full invariant measure as the product of the trivial measure and an exponentiated action term:
\[ \mu_{\text{inv}}[\phi] = D[\phi] e^{i S_{\text{meas}}[\phi]}, \quad (3.1) \]

and then perform a gauge transformation and require that the full measure be invariant. Functionally integrating to derive the measure yields:

\[ \delta \mu_{\text{inv}}[\phi] = \mu_{\text{inv}}[\phi] (i \delta S_{\text{meas}} + Tr \left[ \frac{\partial}{\partial \phi} \delta \phi \right]) = 0. \quad (3.2) \]

The trace appears as the only surviving diagonal terms of the Jacobian determinant of the infinitesimal transformation (when dealing with fermions, the Grassman derivatives will produce the necessary extra minus sign that corresponds to the inverse determinant). We then have

\[ \delta S_{\text{meas}} = i Tr \left[ \frac{\partial}{\partial \phi} \delta \phi \right]. \quad (3.3) \]

This procedure only determines the measure up to gauge invariant terms, but we feel that any such terms have no place in the measure, since they properly belong in the Lagrangian and we generate only the minimal measure necessary for invariance. We also note that there is, in general, a fair degree of arbitrariness in constructing the form of the measure. Each choice produces an equivalent theory, and corresponds to maintaining a different gauge condition after radiative corrections. The measure is also constrained to be an entire function of the 4-momentum invariants for the particular process, ensuring that no additional degrees of freedom become excited in the quantum regime. We shall see later on that it is possible to quantize the full chiral invariance, if we are prepared to give up this constraint or add additional particle content at the classical level.

Specifically, the nontrivial contributions to the measure at second order come from:

\[ \delta \psi = -i E^2 \partial S_{\text{A}} \psi, \quad (3.4) \]

and, as given in the original paper [1], produce the necessary contribution to vacuum polarization in order to satisfy the Ward identity and keep the photon transverse:

\[ S_m = -\frac{\Lambda^2}{4\pi^2} \int \frac{dp dq}{(2\pi)^4} (2\pi)^4 \delta^4(p + q) M_\nu(q) A^\mu(p) A_\mu(q), \quad (3.5) \]
\[ M_v(q) = \int_0^{1/2} dt (1 - t) \exp(t \frac{q^2}{\Lambda^2} - \frac{1}{1 - t \Lambda^2}), \]  
(3.6)

(where the massless limit will be denoted by \( M_v^0 \)).

The new piece comes at third order (\( BA^2 \)) and, as we shall see, is required for decoupling of the longitudinal photon from the induced A-V-V interaction. It comes from considering the transformation of the trivial measure under

\[ \delta \psi = -i E^2 \theta \bar{\Sigma} \bar{S} \Gamma \bar{\Sigma} \psi, \]
(3.7)

giving

\[ S_{meas} = -\frac{i}{2\pi^2} \int \frac{dp dq_1 dq_2}{(2\pi)^6} (2\pi)^4 \delta^4(p + q_1 + q_2) \]
\[ \epsilon^{\mu \nu \alpha \beta} B_{\mu}(p) q_{1 \nu} A_{\alpha}(q_1) A_{\beta}(q_2) [M_a(p; q_1, q_2) + M_a(q_1; p, q_2)] \]
(3.8)

where

\[ M_a(p; q_1, q_2) = \int_0^1 \int_0^1 \frac{dxdy}{(1 + x + y)^3} \]
\[ \times \exp\left[ \frac{xy}{1 + x + y \Lambda^2} + \frac{x}{1 + x + y \Lambda^2} + \frac{y}{1 + x + y \Lambda^2} - (1 + x + y) \frac{m^2}{\Lambda^2}\right] \]
(3.9)

(where again the case of massless fermions will be denoted by \( M_a^0 \)). This term in the action produces a Feynman rule:

\[ -i \Gamma^{\mu \nu \alpha \beta}_{5meas} = -\frac{1}{2\pi^2} \epsilon^{\mu \nu \alpha \beta}[q_{2 \nu}(M_a(p; q_1, q_2) + M_a(q_2; p, q_1)) - q_{1 \nu}(M_a(p; q_1, q_2) + M_a(q_1; p, q_2))]. \]
(3.10)

We now have the measure necessary to preserve the vector invariance in the QED sector (to third order) in the presence of axial interactions. As we will see in the next section, this measure will not help preserve axial vector current conservation and the theory has an anomaly.

**B. Radiative Corrections to the Axial Current**

When we calculate the triangle graph, we will denote the sum over graphs (with appropriate factors of \( E^2 \) and \( (1 - E^2) \)) by \( \sum_{R_n}^n \) (\( n = 3 \)), where we exclude the graph that
corresponds to three ‘barred’ fermion lines that would cause the loop to be fully localized, hence divergent. Normally regularization of loop corrections is performed at the diagram level, and in the case of the anomaly, the linear divergence leads to momentum routing ambiguities, or problems defining $\gamma^5$ in dimensional regularization. Here, the action is regulated, and it produces loop integrals that are strongly convergent, without leaving four dimensions.

Writing the amplitudes in Fig. 1 as

\[-iA^{\mu\alpha\beta} = -Tr \int \frac{d^4k}{(2\pi)^4} \sum_{R_{13}} \gamma^\mu \gamma^5 S(k) \gamma^\alpha S(k - q_1) \gamma^\beta S(p + k) \]  
\[-iB^{\mu\alpha\beta} = -Tr \int \frac{d^4k}{(2\pi)^4} \sum_{R_{13}} \gamma^\mu \gamma^5 S(-k - p) \gamma^\beta S(q_1 - k) \gamma^\alpha S(-k), \]

we first check that the longitudinal photon decouples by dotting $q_1\alpha$ into this and writing

\[\hat{q}_1 = -(k - q_1 - m) + (k - m) \] and simplifying:

\[-iq_1\alpha \Gamma^{\mu\alpha\beta}_5 = -iq_1\alpha (A^{\mu\alpha\beta} + B^{\mu\alpha\beta}) \]

\[= -8ie^{\mu\alpha\beta} \int \frac{d^4k}{(2\pi)^4} \sum_{R_{13}} \frac{k_\mu p_\nu}{(k^2 - m^2)((p + k)^2 - m^2)} - \frac{k_\alpha p_\nu - q_1\alpha p_\nu - q_1\alpha k_\nu}{((k - q_1)^2 - m^2)((p + k)^2 - m^2)} \]

\[= \frac{1}{2\pi^2} e^{\mu\alpha\beta} q_1\alpha q_2\nu [M_a(p; q_1, q_2) + M_a(q_2; p, q_1)], \]

(3.13)

(we have used $Tr[\gamma^5 \gamma^\alpha \gamma^\beta \gamma^\gamma \gamma^\nu] = -4ie^{\alpha\beta\mu\nu}$ with $e^{0123} = +1$) which is trivially seen to be cancelled by the contribution from the measure term (3.10).

In terms of the axial current conservation, dotting $p_\mu$ into the same diagram gives (after writing $\hat{p} = (p + \hat{k} - m) - (\hat{k} + m) + 2m$ which allows us to reduce the traces and immediately recognize the axial coupling term that satisfies the classical Ward identity, separating the anomalous term):

\[-ip_\mu \Gamma^{\mu\alpha\beta}_5 = -ip_\mu (A^{\mu\alpha\beta} + B^{\mu\alpha\beta}) \]

\[= -2im\Gamma^{\alpha\beta}_5 \]

\[= -8ie^{\mu\alpha\beta} \int \frac{d^4k}{(2\pi)^4} \sum_{R_{13}} \frac{k_\mu q_1\nu}{(k^2 - m^2)((p + k)^2 - m^2)} - \frac{k_\mu p_\nu - q_1\mu p_\nu - q_1\mu k_\nu}{((k - q_1)^2 - m^2)((p + k)^2 - m^2)} \]

\[= -2im\Gamma^{\alpha\beta}_5 = \frac{1}{2\pi^2} e^{\mu\alpha\beta} q_1\mu q_2\nu [M_a(q_1; p, q_2) + M_a(q_2; p, q_1)]. \]

(3.14)

After adding the measure contribution to this, we find:
\[ p_\mu \Gamma^\mu_5 = 2m \Gamma_5 \] 
\[ - \frac{i}{\pi^2} \epsilon^{\mu \nu \alpha \beta} q_{1 \mu} q_{2 \nu} (M_a(p; q_1, q_2) + M_a(q_1; p, q_2) + M_a(q_2; p, q_1)). \tag{3.15} \]

The extra piece exhibits current nonconservation and gives the local limit (1.11), after the limit \( M_a \to 1/6 \) is taken. We have seen that though we can maintain vector invariance in the presence of axial couplings, anomalies appear in the axial sector. We also note the presence of the fermion mass in the anomaly term. This means that fermion doubling will only remove the anomaly in the massless limit with the regulator on, but all mass terms are suppressed by inverse powers of the nonlocal scale, so that in the local limit even this dependence will disappear.

C. Nonlocal Anomaly

The analogous identities to those in Sect.\[ \text{[I]} \] may now be calculated, and consistency with the perturbative expansion verified. At this point, one of the main advantages of working with a theory that is regulated explicitly at the Lagrangian level becomes apparent, namely, the relevant transformations are already regulated, and further considerations on defining and regulating the measure are bypassed. We can consider how the naive measure transforms under the infinitesimal transformation in (3.16) to derive the Jacobian order by order in the coupling constants.

Let us start by considering the local axial transformation:

\[ \delta \psi = -iE^2 \omega \gamma^5 [1 - \bar{S} \Gamma]^{-1} \psi. \tag{3.16} \]

Under this transformation, the generating functional (1.6) transforms to:

\[ Z[J_\mu, \bar{\eta}, \eta] = \int d\mu[A^\mu, \psi, \bar{\psi}] \exp(i \int d^4x(L + \bar{\psi} \eta + \bar{\eta} \psi + \delta L + \delta \bar{\psi} \eta + \bar{\eta} \delta \psi) + i \delta S_{\text{meas}}), \tag{3.17} \]

where \( \delta L \) is given by (1.9) and to third order in couplings,

\[ \delta S_{\text{meas}} = \frac{i \Lambda^2}{2\pi^2} \int \frac{dp dq}{(2\pi)^4} (2\pi)^2 \delta^4(p + q) \omega(p)[q^\mu B_\mu(q) M_v(q) + imC(q) M_c(q)] \]
\[ + \frac{1}{2\pi^2} \int \frac{dp dq_1 dq_2}{(2\pi)^6} (2\pi)^4 \delta^4(p + q_1 + q_2) M_a(q_1; p, q_2) \omega(p) \]
\[ \epsilon^{\mu \nu \alpha \beta} q_{1 \mu} q_{2 \nu} [A_\alpha(q_1) A_\beta(q_2) + B_\alpha(q_1) B_\beta(q_2)], \tag{3.18} \]
where

\[ M_c(q) = \int_0^{\frac{1}{2}} dt \exp\left[t \frac{q^2}{\Lambda^2} - \frac{1}{1 - t \Lambda^2}\right]. \]  

(3.19)

The full invariant measure we have constructed is invariant under the transformation given in (2.14) and not (3.16). This results in the \( \delta S_{\text{meas}} \) term in (3.18), and ends up giving the ‘anomalous’ identity, since using (1.7) and (1.9), we get the weak operator identity:

\[
- i p_\mu J_5^\mu(p) = 2i m J_5(p) + \delta S_{\text{meas}} = 2i m J_5 + \frac{i \Lambda^2}{2 \pi^2} M_v(p) p_\mu B_\mu(p) - \frac{m \Lambda^2}{2 \pi^2} C(p) M_c(p)
\]

\[ + \frac{1}{2 \pi^2} \int \frac{dq_1 dq_2}{(2 \pi)^6} (2 \pi)^4 \delta^4(p + q_1 + q_2) M_a(p; q_1, q_2) \]

\[ \epsilon^{\mu \nu \alpha \beta} q_1_\mu q_2_\nu [A_\alpha(q_1) A_\beta(q_2) + B_\alpha(q_1) B_\beta(q_2)], \]  

(3.20)

resulting in the (respected) identity on the triangle graphs (3.14). Note that these identities apply only to Loop corrections generated by the Lagrangian, we do not generate identities on contributions from the measure. Operationally this is due to the fact that the presence of the measure does not affect the transformation (2.9), since we do not transform the vector fields. (The same result is true if we consider the vector analogue of (2.9). Only if we consider the transformation properties of the photon as well, do we derive the full Ward identities.)

The additional second order contributions to the current identity (3.20) remind us that the longitudinal degree of freedom of the axial vector field \( B \) does not decouple from the \( BB \) and \( BC \) two-point functions (there is no measure contribution to these processes). Strictly speaking, these are also anomalous terms, but in the local limit one would introduce counterterms into the Lagrangian and absorb them into mass redefinitions, effectively removing them from (3.20). These terms do not contribute to the triangle anomaly.

Since we began with a regulated action and imposed vector invariance, the anomaly appeared (uniquely) in the axial sector. We could also construct a conserved current from (3.20), however, as is true in the local theories, we find that the Noether current does not correspond to a conserved current in the quantum regime.
In contrast to other schemes, the anomaly relation here is perturbative: it depends on a coupling constant series. These higher order graphs are convergent and we expect any additional contributions to vanish in the local limit, reproducing the nonperturbative result of [6]. We also expect that once renormalization is performed and the local limit taken, there will be no correction to the anomalous identity (1.11) from higher order loop corrections other than those which contribute to the running of the coupling constant [8].

IV. CHIRAL GAUGE INVARIANCE

A. Classical action

We will now attempt to gauge the full chiral invariance in order to explicitly demonstrate how the nonexistence of an invariant measure in the nonlocal theory foils attempts to quantize it. We begin with the chirally invariant local theory:

\[
L = \overline{\psi} (i \partial - m) \psi - \frac{1}{4e^2} F_A^2 - \frac{1}{4g^2} F_B^2 - \overline{\psi} A \psi - \overline{\psi} B \psi \\
\equiv \overline{\psi} S^{-1} \psi + \frac{1}{2e^2} A^\mu D^{-1}_{\mu \nu} A^\nu + \frac{1}{2g^2} B^\mu D^{-1}_{\mu \nu} B^\nu \\
- \overline{\psi} A \psi - \overline{\psi} B \gamma^5 \psi,
\]

(4.1)

possessing the gauge invariance:

\[
\delta A^\mu = \theta^\mu, \quad \delta B^\mu = \omega^\mu, \\
\delta \psi = -i(\theta + \omega \gamma^5)\psi.
\]

(4.2)

Introducing the shadow fields as before, we get

\[
L_{Sh} = \overline{\psi} \hat{S}^{-1} \psi + \hat{\phi} \hat{S}^{-1} \phi + \frac{1}{2e^2} A^\mu D_{\mu \nu}^{-1} A^\nu + \frac{1}{2g^2} B^\mu D_{\mu \nu}^{-1} B^\nu \\
- (\overline{\psi} + \hat{\phi}) A (\psi + \phi) - (\overline{\psi} + \hat{\phi}) B \gamma^5 (\psi + \phi).
\]

(4.3)

Integrating them out at the classical level (defining \( \Gamma = A + B \gamma^5 \)):

\[
\phi = (1 - \tilde{S} \Gamma)^{-1} \tilde{S} \Gamma \psi,
\]

(4.4)
\[ L_{NL} = \bar{\psi} \hat{S}^{-1} \psi + \frac{1}{2e^2} A^\mu \hat{D}^{-1}_{\mu \nu} A^\nu + \frac{1}{2g^2} B^\mu \hat{D}^{-1}_{\mu \nu} B^\nu - \bar{\psi} \Gamma (1 - \hat{S} \Gamma)^{-1} \psi, \]  

\[ \delta \psi = -iE^2 (\theta + \omega \gamma^5) (\psi + \phi) \]
\[ = -iE^2 (\theta + \omega \gamma^5) (1 - \hat{S} \Gamma)^{-1} \psi, \]  

with the current definitions in (2.13) still holding.

So far, this is all classical and there is no problem with it. On attempting to quantize the theory using the path integral formalism, we will discover that the invariant measure cannot be generated by the method discussed in Sect. III A, and therefore we once more do not have a generating functional that generates graphs respecting the classical symmetries.

**B. The Measure**

Building the measure as before, we easily derive the second order pieces necessary to retain transversality of the vacuum polarization:

\[ S_{meas} = \frac{\Lambda^2}{4\pi^2} \int \frac{dp dq}{(2\pi)^4} (2\pi)^4 \delta^4(p + q) M^0(p) \]
\[ \times [A^\mu(p) A_\mu(q) + B^\mu(p) B_\mu(q)], \]  

but now we run into problems at third order. It is somewhat straightforward to show that the condition on the measure at third order is given by:

\[ \delta S_{meas} = \frac{i}{2\pi^2} \int \frac{dp dq_1 dq_2}{(2\pi)^6} (2\pi)^4 \delta^4(p + q_1 + q_2) \epsilon^{\mu \nu \alpha \beta} q_1_{\mu} M^0_{\alpha}(q_1; p, q_2) \]
\[ \times [\delta B_\alpha(p) B_\nu(q_2) B_\beta(q_1) + \delta B_\alpha(p) A_\beta(q_1) A_\nu(q_2) + \delta A_\alpha(p) A_\beta(q_1) B_\nu(q_2) + \delta A_\alpha(p) B_\beta(q_1) A_\nu(q_2)]. \]  

However, it is impossible to satisfy this condition without introducing pole structure into the measure, or extra degrees of freedom at the classical level. First, one notices that the last two terms are identical to those found in sect. III A, and give rise to (3.10). The second
term is also of the same order, and must be a variation of the same action term. It is not hard
to check that the axial variation of (3.10) gives two terms: one of the right form but
the wrong sign, and the other of the wrong form. The $B^3$ piece produces a similar problem,
in that there is no way to construct a measure that gives just this one term. In the local
limit, nothing could survive anyway due to the antisymmetry of the epsilon symbol.

As it stands, we cannot consistently quantize the theory and maintain the axial symme-
try, but there are a number of ways of proceeding from here. It is easily seen that we can
write the measure as:

$$S_{\text{meas}} = -\frac{1}{2\pi^2} \int \frac{dp dq_1 dq_2}{(2\pi)^6} \delta^4(p + q_1 + q_2)\epsilon^{\mu\nu\alpha\beta} \times M_a^0(q_1; p, q_2) B_{\sigma}(p) q_{1\mu} q_{2\alpha} \times [B_{\nu}(q_2) B_{\beta}(q_1) + A_{\nu}(q_2) A_{\beta}(q_1)].$$  (4.9)

This expression explicitly introduces additional particle content into the quantum regime,
which can be identified as the longitudinal degree of freedom of the axial boson due to the
presence of the longitudinal projector in (4.9), and so we also expect a mass to be generated
perturbatively [9].

Instead we could follow [10] and generate the Wess-Zumino action through the intro-
duction of a U(1) gauge parameter $\pi$ that accounts for the fact that the fermionic measure
is not gauge invariant, and therefore all gauge configurations must be summed over. This
results in the additional integration over $\pi$ in the path integral and the Wess-Zumino term:

$$S_{W-Z} = -\frac{i}{2\pi^2} \int \frac{dp dq_1 dq_2}{(2\pi)^6} \delta^4(p + q_1 + q_2)\epsilon^{\mu\nu\alpha\beta} \times M_a^0(q_1; p, q_2) \pi(p) q_{1\mu} q_{2\alpha} \times [B_{\nu}(q_2) B_{\beta}(q_1) + A_{\nu}(q_2) A_{\beta}(q_1)].$$  (4.10)

This produces a gauge invariant generating functional, since under $\pi \rightarrow \pi + \theta$ the action is
invariant.

This, in fact, is merely a rewriting of the previous result. Formerly we excited the
longitudinal component of $B^\mu$ and thus integrated over three degrees of freedom. Here we
excite only transverse $B^\mu$, but have an additional field $\pi$, which can be identified with the longitudinal component through a Stuekelberg-type construction.

Since we expect one to be produced perturbatively, we can also introduce an axial boson mass into the classical Lagrangian:

$$L_M = \frac{1}{2} M^2 B^\mu B_\mu$$

(4.11)

explicitly breaking gauge invariance. The Stuekelberg transformation then corresponds to performing a (finite) inverse gauge transformation on all matter fields with the new auxiliary field (in this case $\pi/M$) as the gauge field:

$$B^\mu \rightarrow B^\mu - \frac{\pi^\mu}{M}, \quad \psi \rightarrow \exp(i \frac{\pi}{M} \gamma^5) \psi,$$

(4.12)

where $\delta \pi = M \omega$ supplements (4.12) to recover gauge invariance. Only the mass term is non-invariant, and after gauge fixing is performed (removing mixing terms between $B^\mu$ and $\pi^\mu$) we obtain the usual massive boson propagator in the Feynman gauge and the auxiliary field Lagrangian:

$$L_\pi = \frac{1}{2} (\pi^\mu \pi^\mu - M^2 \pi^2).$$

(4.13)

At the classical level, we now have a massive axial boson, but since the axial current is conserved, the longitudinal component remains decoupled. Nonlocalizing and quantizing, we then find that the condition (4.8) can now be satisfied with (3.10) and,

$$S_{\text{meas}} = -\frac{i}{2\pi^2} \int \frac{dp dq_1 dq_2 (2\pi)^4 \delta^4(p + q_1 + q_2) \epsilon^{\mu
u\alpha\beta}}{2\pi^6} M^0 q_1 q_2 
\times \left[ B^\mu(q_2) B^\beta(q_1) + A^\mu(q_2) A^\beta(q_1) \right].$$

(4.14)

This results in explicit coupling to the auxiliary field and hence the longitudinal component of $B^\mu$. It is a consistent measure and the Ward identities it generates are satisfied, but the local limit is nonrenormalizable and does not exist, and the massless limit also does not exist. We could apply the same method to the case where the fermion masses are nonzero.
and essentially reproduce the results obtained in the earlier sections, except that the measure would explicitly cancel off quadratic divergences such as \( \Theta,5 \) in the axial sector as well. The anomaly shows up here identically as in the 2-d Schwinger model [11].

**CONCLUSIONS**

Despite a claim to the contrary [1], the nonlocal regularization of QED does ‘suffer’ from an anomaly, and in the local limit is consistent with the results in other schemes. The advantage of the nonlocal formalism lies in the fact that the currents of interest can be constructed in regulated form. Any possible contributions from the measure needed to construct the proper Ward identities may be derived directly, without resorting to additional regularization or rerequiring a ‘proper definition’ of the measure.

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REFERENCES

[1] D. Evans, J.W. Moffat, G.Kleppe and R.P. Woodard, Phys. Rev. D43, 499 (1991).

[2] G. Kleppe and R. P. Woodard, Nucl. Phys. B388, 81 (1992).

[3] G. Kleppe and R. P. Woodard, Ann. Phys. (N. Y.) 221, 106 (1993).

[4] S. Adler, Phys. Rev., 177, 2426 (1969), J. Bell and R. Jackiw, Nuovo Cim 60A, 47 (1969).

[5] See for example, Itzykson and Zuber, Quantum Field Theory , McGraw Hill Inc. (1980).

[6] K. Fujikawa Phys. Rev. Lett., 42, 1195 (1979), K. Fujikawa, Phys. Rev., D21, 2848 (1980).

[7] For a more complete treatment, see P. Ramond, Field Theory: A Modern Primer 2nd ed., Addison-Wesley Publishing Company, Inc. (1990).

[8] S. Adler and W. Bardeen, Phys. Rev. 182, 1517 (1969).

[9] D. J. Gross and R. Jackiw, Phys. Rev. D6, 477, (1972).

[10] K. Harada and I. Tsutsui, Phys. Lett B183, 311 (1987).

[11] B. Hand, Phys. Lett. B275, 419 (1992).
FIG. 1. Graphs relevant for the calculation of the triangle Anomaly

\[ \mp - i A_{\mu\alpha\beta} \]

\[ \mp - i B_{\mu\alpha\beta} \]