SYMPLECTIC FOLIATED FILLINGS OF SPHERE COTANGENT BUNDLES

FRANCISCO PRESAS AND SUSHMITA VENUGOPALAN

Abstract. We classify symplectically foliated fillings of certain contact foliated manifolds. We show that up to symplectic deformation, the unique minimal symplectically foliated filling of the foliated sphere cotangent bundle of the Reeb foliation in $S^3$ is the associated disk cotangent bundle. En route to the proof, we study another foliated manifold, namely the product of a circle and an annulus with almost horizontal foliation. In this case, the foliated unit cotangent bundle does not have a unique minimal symplectic filling. We classify the foliated fillings of this manifold up to symplectic deformation equivalence using combinatorial invariants of the filling.

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1. Introduction

A classical question in differential topology was to understand the cobordism classes of closed differentiable manifolds [26]. Cobordisms preserving an additional geometric structure have been the subject of more recent research. One such example in symplectic topology is the question of determining the classes of oriented symplectic cobordisms. An oriented symplectic cobordism between two contact manifolds is a smooth cobordism with a symplectic structure, such that the boundaries have neighbourhoods symplectomorphic to their symplectizations.

The most studied situation is the cobordism class of the empty set. Since the cobordism is oriented, we have to distinguish between two cases: symplectic cobordisms from the empty set to a contact manifold that are called \textit{symplectic fillings}, and from the contact manifold to the empty set called \textit{symplectic caps}. Any contact 3-manifold has a symplectic cap [12], but the same is not true for symplectic fillings. In fact, the flexible subclass of contact structures, the overtwisted ones [3, 10] do not admit symplectic fillings (see [23] and the references therein for a whole historical account). Combining these two results, it is clear that the oriented cobordism relation is not an equivalence relation.

Another question is to classify all the possible cobordisms between two contact manifolds. Many contact 3-folds have been shown to possess unique minimal symplectic fillings. Examples include $S^3$, $\mathbb{R}P^3$ and some Lens spaces $L(p, 1)$, see [11], [19] and [18]. The Lens space $L(4, 1)$ has two minimal symplectic fillings. The 3-torus, viewed as the unit cotangent bundle of the 2-torus, has a standard contact structure. The fillings of this contact manifold are well-understood [28]. Recently, there has been some progress in studying the fillings of unit cotangent bundles of higher genus surfaces [24]. In higher dimensions the obstructions to fillability are less understood but it is clear that not any stably complex cobordism is symplectically representable [14].

In another direction one can study a different kind of geometric structures over a differentiable manifold: foliations. A natural question is to understand the class of foliated cobordisms. We restrict ourselves to foliations of codimension one. In this case, on an (unfoliated) cobordism, any distribution transverse to the boundary can be deformed to a foliation transverse to the boundary [25].

The uniqueness question in this context is how many different foliated manifolds have a given foliated manifold as their boundary. Firstly, for any given manifold, there is an infinite number of manifold fillings. Secondly, for a given foliated manifold and a fixed manifold filling, there is an infinite number of foliations in the interior that restrict to the fixed foliation on the boundary. Indeed, if we start with a filling that does not have Reeb components, we choose a loop transverse to the leaves, and replace it with a Reeb component. Thus we have produced a filling that is not conjugate to the initial one. This procedure can be applied repeatedly to construct a sequence of pairwise non-conjugated foliated fillings.

In this article, we study a cobordism category that is the combination of the two previous ones. The objects of study are \textit{foliated contact manifolds} – these are manifolds equipped with a codimension 1 foliation and a codimension 2 distribution
tangent to the foliation, such that on any leaf the distribution is a contact structure. The cobordisms are equipped with a codimension one foliation and a closed two form $\omega$, which restricts to a symplectic form on the leaves. As in the unfoliated case, the cobordism is directed. On each leaf, a neighbourhood of its intersection with the convex resp. concave boundary has the structure of a positive resp. negative symplectization. A (unwritten) Corollary of the techniques developed in [7] is that a contact foliation with an overtwisted leaf does not admit a foliated filling. The result is on expected lines, since it is a foliated analogue of the result in [23].

The goal of this article is to address the uniqueness problem. As in the contact case, the more rigid examples are the unit cotangent bundles. Unit cotangent bundles have a canonical contact structure, and the corresponding disk cotangent bundles are natural symplectic fillings. Analogously, for a foliated manifold, the unit cotangent bundle of the leaves, called the foliated unit cotangent bundle is a foliated manifold with a contact structure on the leaves. The foliated disk cotangent bundle is a filling for the foliated unit cotangent bundle. The main example we discuss is the unit cotangent bundle of $S^3$ with the Reeb foliation $F_{\text{Reeb}}$ that is foliated diffeomorphic to $S^3 \times S^1$ with foliation $F_{\text{Reeb}} \times S^1$. In this case, we show that the symplectic filling is unique up to symplectic deformation equivalence, echoing results from unfoliated examples. Another example we consider is the product of a two-dimensional annulus with almost horizontal foliation and the torus $T^2$, i.e the foliated unit cotangent bundle of the product of an annulus with almost horizontal foliation and $S^1$. The boundary leaves are copies of $T^3$, and Wendl [28] has proved that the fillings of $T^3$ are diffeomorphic to $D^2 \times T^2$. The foliated case is a surprising departure, and we observe new phenomenon not seen in unfoliated examples. However, even in this case, up to symplectic deformation class, the fillings are classified by certain combinatorial invariants.

As previously mentioned, we study foliated fillings that are strong symplectic. A codimension-one foliation $F$ in an odd-dimensional manifold $M$ is called strong symplectic if there is a closed two-form on $M$ that restricts to a symplectic form on the leaves of the foliation. The terminology is to distinguish it from a weak symplectic foliation, on which there is a two-form that restricts to a symplectic form on leaves, but is not required to be closed. In the case of 3-folds with a codimension one foliation, tautness of the foliation is equivalent to the existence of a strong symplectic form. It is reasonable to expect that the strong symplectic condition is the analogue of tautness in higher dimensions. From an analytic point of view, the condition is necessary for Gromov compactness to hold, since the cohomological condition gives a common bound in the energy of homologous holomorphic curves tangent to different leaves.

A filling is minimal if it does not have any embedded symplectic sphere in a leaf whose self-intersection in the leaf is $-1$. We will show that if there exists such a sphere, it is always part of an $S^1$-family and it can be blown down, see Proposition 2.3.
Theorem 1. Let \((S^3, F_{\text{Reeb}})\) be the Reeb foliation. Any minimal strong symplectic filling of the sphere cotangent bundle \(S(T^*F_{\text{Reeb}})\) is symplectic deformation equivalent to the disk cotangent bundle \(D(T^*F_{\text{Reeb}})\).

Theorem 1 is proved in Section 5.2. Our next result (proved in Section 5.1) says that in a number of cases, a compact leaf in the contact foliation bounds a compact leaf in the symplectic filling.

Theorem 2. Suppose \((M^n, F^3)\) is a foliated manifold with a leafwise contact structure \(\xi^2 \subset TF\). Suppose \(W^{n+1}\) is a strong symplectic manifold with boundary \(M\), and which is leafwise a strong symplectic filling. Further, let \(L_M\) be a leaf of \(M\) that is contactomorphic to either \(S^3\), \(\mathbb{R}P^3\), a Lens space \(L(p,1)\), or \(T^3\) with the standard contact structure. Then, \(L_M\) bounds a compact leaf \(L\) of \(W\). In all cases except when \(L_M = L(4,1)\), the leaf \(L\) in the filling is the canonical filling of \(L_M\). If \(L_M = L(4,1)\), the leaf \(L\) can be one of the two possible minimal fillings of \(L(4,1)\) (see [18]).

Again this result illustrates a contrast between smooth foliations, and those with a strong symplectic form. If we have a smooth foliated filling, a compact leaf intersecting the boundary can typically be destroyed by perturbation. This is true even for codimension one foliations. For instance, if we assume that the filling is a taut foliation, we consider a loop transverse to the foliation and replace it by a Reeb component. As a result, all the leaves become open and the new compact leaf does not touch the boundary. The following is a consequence of Theorem 2, and is proved in Section 5.1.

Corollary 1.1. Suppose \((M^n, F^3)\) is a foliated manifold with a leafwise contact structure \(\xi^2 \subset TF\). Suppose \(W^{n+1}\) is a strong symplectic manifold with boundary \(M\), and which is leafwise a strong symplectic filling. Further, assume that any leaf of \(M\) is contactomorphic to either \(S^3\), \(\mathbb{R}P^3\), a Lens space \(L(p,1)\), or \(T^3\) with the standard contact structure. Then all the leaves of \(W\) are compact and intersect the boundary.

Theorem 2 is useful in analyzing fillings of the foliated unit cotangent bundle of \((S^3, F_{\text{Reeb}})\), which is the product \((S^3, F_{\text{Reeb}}) \times S^1\). The foliation contains a compact leaf \(T^3\), which disconnects the manifold into two copies of \((S^1 \times D^2) \times S^1\), where in both pieces the solid torus \((S^1 \times D^2)\) has the Reeb foliation.
A standard technique for proving the uniqueness of fillings is by constructing a holomorphic foliation of the filling, see [16], [19] etc. We do not know of a way to find a holomorphic subfoliation of the filling of the Reeb component \( (S^1 \times D^2, F_{\text{Reeb}}) \times S^1 \). Therefore, we take an indirect approach. Given any filling \( W_{\text{Reeb}} \), we perform a surgery operation to transform it into a filling of \( M_{\text{ah}} := (A^2, F_{\text{ah}}) \times T^2 \), where \( F_{\text{ah}} \) is an almost horizontal foliation, see Figure 1. The contact structure is the one canonically associated to the unit foliated cotangent bundle associated to \( A^2 \times S^1 \).

Suppose \( W_{\text{ah}} \) is a filling of the almost horizontally foliated manifold \( M_{\text{ah}} \). In order to analyze \( W_{\text{ah}} \) using holomorphic curves, it is useful to attach the symplectization \( R_{\geq 0} \times M_{\text{ah}} \) to the boundary of \( W_{\text{ah}} \). The result is a symplectically foliated manifold \( W_{\infty}^{\text{ah}} \) with a cylindrical end. The manifold \( W_{\infty}^{\text{ah}} \) is a Lefschetz fibration with a foliated base space \( R \times (A^2, F_{\text{ah}}) \), and whose regular fibers are cylinders. Singular points of the fibration are circles transverse to the foliation, so that on any leaf the set of singular points is discrete. There are no singular values on the boundary \( R \times \partial A \).

The combinatorial type of a filling consists of the following data.

(a) (Dehn twists on boundary leaves, \( k_{\pm} \in \mathbb{Z} \)) The boundary leaves \( \pi^{-1}(R \times \partial A) \) are Lefschetz fibrations with no singular points. Denote the outer and inner boundaries of \( A \) by \( \partial_+ A \) and \( \partial_- A \) respectively. There are canonical trivializations of the fibration near the ends \( \{ \pm \infty \} \times \mathcal{M} \) obtained via the identification \( R_{\geq 0} \times M_{\text{ah}} \to W_{\infty}^{\text{ah}} \). Therefore the trivialized bundles \( \pi^{-1}((-\infty, \epsilon) \times \partial_{\pm} A) \) and \( \pi^{-1}((\epsilon, \infty) \times \partial_{\pm} A) \) are glued by an element of the mapping class group of the cylinder, namely \( k_{\pm} \in \pi_1(R \times S^1) \simeq \mathbb{Z} \).

(b) (Combinatorial data of singular loci) Each connected component of the singular point set of the Lefschetz fibration projects to a closed embedded loop \( \gamma \) in the solid torus \( R \times A \). Let \( \Gamma \) be the set of connected components of singular values. The combinatorial data of the singular loci is the multi-set of braid types \( \{ [\gamma] : \gamma \in \Gamma \} \) of the loops of singular values. By braid type, we mean the homotopy equivalence class of embedded loops in the solid torus \( R \times A \).

Further, if the leaves in \( (A, F_{\text{ah}}) \) are oriented by the arrows in Figure 1, then, a monodromy calculation yields the relation

\[
k_{+} - k_{-} = \sum_{\gamma \in \Gamma} \text{wind}(\gamma),
\]

where \( \text{wind}(\gamma) \) is the winding number of the loop in the solid torus, or in other words, the number of strands in the braid \( \gamma \).

**Theorem 3.** Minimal symplectic fillings of \( S(T^*(S^1 \times (A, F_{\text{ah}}))) \) are classified up to symplectic deformation equivalence by the combinatorial type.

The proof of this theorem is given in Section 4.4. We remark that if \( k_{+} \neq k_{-} \), the boundary leaves of the filling \( W_{\text{ah}} \) correspond to two fillings of \( T^3 \) that are not symplectically deformation equivalent.

We remark that the braid type of the set of all the singular values of the Lefschetz fibration, taken together, is not an invariant. It is only the braid type of a connected component that is a combinatorial invariant. We will show that the linking between
different connected components is not preserved by symplectic deformation equivalence.

We believe that the techniques developed in this article apply, with some modifications, to the case of weak symplectic fillings (in the sense of weak foliated symplectic structure). We believe that in this case, the combinatorial invariant classifying different non-equivalent fillings is provided by the integers $\text{wind}(\gamma)$, $\gamma \in \Gamma$. In other words, the weak symplectic fillings are classified by the number of connected components of the braid and the number of strands of each connected component. This just measures the immersion class of the braid. Weak deformations allow us to unknot the connected components of the braid, unlike in the strong case.

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2. Background

2.1. Symplectic and contact foliations. A $k$-dimensional foliation on a manifold $M^n$ is an integrable distribution $\mathcal{F} \subset TM$ of rank $k$. We assume that manifolds are oriented and foliations are co-oriented. A strong symplectic form on a foliated manifold $(M,\mathcal{F})$ is a closed two-form $\omega \in \Omega^2(M)$ whose restriction to leaves $\omega|_{\mathcal{F}}$ is symplectic. The pair $(\mathcal{F}, \omega)$ is called a strong symplectic foliation on $M$. Two such forms $\omega_0$ and $\omega_1$ are equivalent if $\omega_0|_{\mathcal{F}} = \omega_1|_{\mathcal{F}}$. The equivalence class of a strong symplectic form $\omega_0$ is an affine space. Indeed, if $\omega_0$ and $\omega_1$ are equivalent strong symplectic forms then $(1-t)\omega_0 + t\omega_1$ lies in the same equivalence class for all $t \in \mathbb{R}$. A trivial strong symplectic foliation is a product of a $(n-k)$-dimensional manifold and a $k$-dimensional symplectic manifold, and the strong symplectic form is defined by pullback. As opposed to a strong symplectic form, a weak symplectic form is a two-form $\omega \in \Omega^2(M)$ whose restriction to the leaves $\omega|_{\mathcal{F}}$ is symplectic. The form $\omega$ is not required to be closed in $M$.

On a foliated manifold $(M,\mathcal{F})$, a foliated contact structure is a sub-distribution $\xi \subset \mathcal{F}$, whose rank is one less than the rank of $\mathcal{F}$, and that is a contact structure on the leaves. A foliated contact form is a one-form $\alpha \in \Omega^1(M)$ that satisfies $\ker \alpha \cap \mathcal{F} = \xi$.

A foliated symplectomorphism is a foliated diffeomorphism $\phi : (W_0, \mathcal{F}_0, \omega_0) \to (W_1, \mathcal{F}_1, \omega_1)$ between (strong or weak) symplectic foliations that satisfies $\phi^*\omega_1|_{\mathcal{F}_0} = \omega_0|_{\mathcal{F}_0}$. A foliated contactomorphism is a foliated diffeomorphism $\phi : (M_0, \mathcal{F}_0, \xi_0) \to (M_1, \mathcal{F}_1, \xi_1)$ satisfying $\phi^*\xi_1 = \xi_0$. The word ‘foliated’ is sometimes dropped when it is clear from the context.

2.2. Symplectic foliated filling. The concept of symplectic fillings of contact manifolds extends to the foliated setting. Given a contact foliation $(M, \mathcal{F}_M, \xi)$, a strong symplectic foliated filling consists of a strong symplectic foliation $(W, \mathcal{F}, \omega)$ whose boundary is transverse to the foliation, and a foliated diffeomorphism

$$i : (M, \mathcal{F}_M) \to (\partial W, \mathcal{F}|_{\partial W})$$
such that the leaves of $W$ are strong symplectic fillings of the leaves of $M$. In particular, there is a vector field, called the Liouville vector field,

$$Y \in \Gamma(\mathcal{O}_p(\partial W), T\mathcal{F}), \quad Y \pitchfork \partial W,$$

which satisfies

$$L_Y \omega|_F = \omega|_F, \quad (\ker(\iota_Y \omega) \cap F)|_M = \xi.$$

The word ‘strong’ in the terminology ‘strong symplectic foliated filling’ is to be interpreted as strong symplectic foliation. The fact that it is also a leafwise strong filling is suppressed.

Fillings can also be defined for contact foliated manifolds $(M, \mathcal{F}_M, \xi)$ with boundary if the boundary is tangent to the foliation. The filling $(W, \mathcal{F}, \omega)$ is a manifold with corners. Its codimension one boundary splits into two components

$$\partial_1 W = \partial_F W \cup \partial_0 W,$$

where the first is tangent to the foliation $\mathcal{F}$ and the second is transverse to the foliation. The contact foliation $M$ can be identified to the transverse boundary $\partial_F W$.

In addition to the properties of fillings in the previous paragraph, we additionally require that the tangential boundary $\partial_F W$ is a filling of $\partial M$. Thus the two components of the codimension one boundary of $W$ intersect in a codimension two corner, which is $\partial M$.

Cylindrical ends can be attached to a symplectic filling as follows. Given a foliated filling $(W, \mathcal{F}, \omega)$ of a contact foliation $(M, \mathcal{F}_M, \xi)$ and a Liouville vector field $Y$, $\alpha := \iota_Y \omega$ is a foliated contact form on $M$. For a small $\epsilon > 0$, there is a symplectic embedding

$$((-\epsilon, 0] \times M, d(e^t \alpha)) \to W, \quad (t, m) \mapsto \phi^t_Y(m),$$

where $\phi^t_Y$ is the time $t$ flow of $Y$. Attaching cylindrical ends to the filling $W$ produces the foliated symplectic manifold

$$(3) \quad W^\infty := W \cup_{\partial W \approx [0, \infty) \times M} [0, \infty) \times M,$$

which we call the extended symplectic filling. The identification $i : \partial W \to M$ extends naturally to a projection $i : W^\infty \setminus K \to M$, where $K$ is a compact set.

The next lemma shows that any two extended fillings of $M$ are related by a foliated symplectomorphism outside a compact set.

**Lemma 2.1.** Suppose $(W_0, \omega_0)$ and $(W_1, \omega_1)$ are fillings of the contact foliated manifold $(M, \mathcal{F}_M, \xi)$, whose boundaries are equipped with identifications $i_l : \partial W_l \to M$, $l = 0, 1$. Then, there are compact sets $K_l \subset W^\infty_l$, for $l = 0, 1$, and a foliated symplectomorphism $\phi : W^\infty_0 \setminus K_0 \to W^\infty_1 \setminus K_1$ satisfying $i_1 \circ \phi = i_0$.

**Proof.** Suppose, for $l = 0, 1$, $\alpha_l \in \Omega^1(M)$ is the contact form induced by the Liouville vector field in $W_l$. Since they represent the same contact structure on $M$, there is a positive function $F : M \to \mathbb{R}_{>0}$ such that $\alpha_1|_{\mathcal{F}_M} = F \alpha_0|_{\mathcal{F}_M}$. Then, $\phi$ can be defined on the cylindrical ends of $W_1^\infty$ as

$$W_1^\infty \setminus W_1 \supset \mathbb{R} \times M \to \mathbb{R} \times M \subset W_0^\infty \setminus W_0, \quad (a, m) \mapsto (a + F(m), m).$$

$\square$
The foliated fillings \((W_0, \mathcal{F}_0, \omega_0)\) and \((W_1, \mathcal{F}_1, \omega_1)\) are symplectic deformation equivalent relative to ends if the map \(\phi\) in Lemma 2.1 extends to a foliated diffeomorphism \(\phi : W^\infty_0 \to W^\infty_1\) and there is a path of strong symplectic forms \(\varpi_t\) on \(W^\infty_0\) and a compact set \(K \subset W^\infty_0\) such that

\[
\varpi_0 = \omega_0, \quad \varpi_1 = \phi^* \omega_1, \quad \varpi_t|_{\mathcal{F}_0} = \omega_0|_{\mathcal{F}_0} \quad \forall t \text{ on } W^\infty_0 \setminus K.
\]

The fillings \((W_0, \omega_0)\) and \((W_1, \omega_1)\) are symplectomorphic relative to ends here if \(\phi\) extends to a foliated symplectomorphism \(\phi : W^\infty \to W^\infty_1\). The terms 'symplectomorphism' and 'symplectic deformation equivalence', when used in the context of non-compact manifolds, mean that they are relative to non-compact ends.

2.3. Foliated cotangent bundle. We now give a class of examples for symplectic and contact foliations. Let \(X\) be a manifold of dimension \(n\), and \(\mathcal{F}\) is a foliation of dimension \(k\). The leafwise cotangent bundle \(T^* \mathcal{F}\) is a vector bundle \(\pi : T^* \mathcal{F} \to X\) with pullback foliation \(\pi^* \mathcal{F}\). We claim that the canonical symplectic form on the leaves extends to a closed two-form on \(T^* \mathcal{F}\). By choosing a splitting of the tangent bundle \(TX = T\mathcal{F} \oplus TF^\perp\), a cotangent vector on the foliation \(\xi \in T^* \mathcal{F}\) extends to a cotangent vector \(\hat{\xi}\) in \(T^* X\). Define a one-form \(\alpha \in \Omega^1(M)\) as follows

\[
\xi \in T^* \mathcal{F}, \quad \alpha_\xi(v) := \hat{\xi}(d\pi(v)).
\]

The two-form \(\omega := d\alpha\) is closed and restricts to the canonical symplectic form on the leaves of \(T^* \mathcal{F}\). When the foliation on \(X\) is unambiguous, the foliated cotangent bundle is denoted by \(FT^* X\).

In a similar way, the sphere cotangent bundle \(S(T^* \mathcal{F})\) is a foliated contact manifold. Then,

\[
S(T^* \mathcal{F}) := \{v \in T^*_x \mathcal{F} : x \in X, \|v\|_g = 1\},
\]

with contact form \(\alpha|_{S(T^* \mathcal{F})}\), and the norm \(\|\|_g\) on the fibers of \(T^* \mathcal{F}\) is induced by a leafwise Riemannian metric on \((X, \mathcal{F})\). The contact structure on \(S(T^* \mathcal{F})\) is independent of the choice of the leafwise metric \(g\). This is because \(S(T^* \mathcal{F})\) can be defined as the double cover of the projectivization \(\mathbb{P}(T^* \mathcal{F})\) with contact structure \(\xi\) given by

\[
\xi_0 := \{v \in T_p \mathcal{F}_{\mathbb{P}(T^* \mathcal{F})} : p(\pi_\mathcal{F} v) = 0\}, \quad p \in \mathbb{P}(T^* \mathcal{F}).
\]

The disk bundle \(D(T^* \mathcal{F})\) associated to the cotangent space \(T^* \mathcal{F}\) is a leafwise strong filling for the unit sphere cotangent bundle \(S(T^* \mathcal{F})\).

2.4. Foliated blow-ups and blow-downs. Let \((M, \mathcal{F}, \omega)\) be a five-manifold with a strong symplectic foliation of codimension one. A foliated family of exceptional spheres is an embedded family \(i : S^1 \times \mathbb{P}^1 \hookrightarrow M\) such that \(\{t\} \times \mathbb{P}^1\) is an exceptional sphere in a leaf.

Symplectic blow-up can be carried out at a point in the symplectic manifold with dimension at least four. Analogously on a foliated symplectic manifold, blow-up is carried out along a closed transversal, i.e. a loop \(\gamma : S^1 \to M\) that is transverse to leaves. By Proposition 2.2 below, the foliated symplectic form \(\omega\) can be altered without changing \(\omega|_{\mathcal{F}}\) so that the transversal is tangent to the line field \(\ker \omega\). A neighbourhood of the loop is now just a product \(S^1 \times \mathbb{D}^4\) with the trivial symplectic foliation, and the blow-up can be carried out fiberwise. This process is called foliated
blow-up, and it can be carried out on any strong symplectic foliation. Therefore, uniqueness results for foliated symplectic fillings can only be obtained by restricting attention to minimal fillings, i.e. ones which do not contain any symplectic exceptional sphere.

In the non-foliated case (see McDuff [21]), exceptional spheres can be blown down. If a manifold with a strong symplectic foliation contains an exceptional sphere $S$ in a leaf, then it is part of a foliated $S^1$-family of exceptional spheres. This is seen as follows. Choose a leafwise tame almost complex $J$ for which the given sphere is holomorphic. By automatic transversality in four dimensions, the moduli space of embedded holomorphic $(1,1)$-spheres is 1-dimensional. By Gromov compactness in the foliated setting, the moduli space is a finite number of copies of $S^1$. Given a foliated $S^1$-family of exceptional spheres, there is a foliated symplectic form $\omega_1$ such that $\omega_1|_F = \omega|_F$, and for which the restriction of $(\mathcal{F},\omega_1)$ to a neighbourhood of the spheres is a trivial symplectic foliation, see Proposition 2.3 below. Blow-down can then be performed fiberwise.

### 2.5. Hamiltonian perturbation of a foliated symplectic form.

In this section, we prove that if a subset $S$ of a strong symplectic foliated manifold $(M^{2n+1},\mathcal{F}^{2n},\omega)$ satisfies certain hypotheses, then there is a foliated symplectic form $\omega_1$ that is same as $\omega$ on $\mathcal{F}$, and for which $S$ is a trivial symplectic foliation. The discussion is restricted to codimension one foliations. To prove this result, it is helpful to take the viewpoint that certain well-behaved symplectic foliations are symplectic fibrations over a one-manifold. A symplectic fibration, is a fiber bundle $\pi : M \to B$ with a smoothly varying fiberwise symplectic form $\omega_b \in \Omega^2(\pi^{-1}(b))$, such that the fibers are symplectomorphic. The product $B \times (N,\omega)$ is a trivial symplectic fibration. Given an open set $Z$ in a strong symplectic manifold $(M,\mathcal{F},\omega)$ such that $(Z \to Z/\mathcal{F},\omega|_{Z/\mathcal{F}})$ is a trivial symplectic fibration, there is a closed extension $\omega_1 \in \Omega^2(Z)$ of $\omega|_{Z/\mathcal{F}}$. The following proposition says that in case the fibers of $Z \to Z/\mathcal{F}$ are simply connected, $\omega_1 \in \Omega^2(Z)$ extends to a closed form on all of $M$ in a way that $\omega|_{\mathcal{F}} = \omega_1|_{\mathcal{F}}$.

**Proposition 2.2.** Suppose $(M,\mathcal{F},\omega)$ is a codimension one strong symplectic foliation, $I$ is the interval $[0,1]$ or $S^1$, and $i : I \times U \to M$ is a foliated embedding such that $i((t) \times U)$ is an open simply-connected set in a leaf, and the leafwise symplectic form $i^*\omega$ is independent of $t$ for all $t \in I$. For any compact subset $V \subset U$, there is an extension $\omega_1 \in \Omega^2(M)$ of the leafwise symplectic form $\omega|_{\mathcal{F}}$, whose pull-back $i^*(\mathcal{F},\omega_1)$ is a trivial strong symplectic foliation on the product $I \times V$.

**Proof.** Let $\tilde{\omega} \in \Omega^2(I \times U)$ be the pullback of $i^*\omega \in \Omega^2(U)$. The forms $\tilde{\omega}$ and $i^*\omega$ agree on the foliation. So, we can write

$$\tilde{\omega} - i^*\omega = \alpha \wedge dt,$$

where $\alpha := i_\# (\tilde{\omega} - i^*\omega)$.

By the closedness of the forms $\tilde{\omega}, i^*\omega$, we obtain $d\alpha \wedge dt = 0$. This condition implies $d\alpha|_{\mathcal{F}} = 0$. Since $U$ is simply connected, there is a function $f : I \times U \to \mathbb{R}$ so that $df|_{\mathcal{F}} = \alpha|_{\mathcal{F}}$, and consequently,

$$\tilde{\omega} - i^*\omega = df \wedge dt \quad \text{on } I \times U.$$
Let $\eta : U \to [0,1]$ be a cut-off function that is 1 on $V$ and 0 outside a neighbourhood of $V$ in $U$. The required form $\omega_1$ is defined as $\omega + (i^{-1}r(d(\eta f) \wedge dt)$ on the image of $i$, and as $\omega$ outside $\text{Im}(i)$.

**Proposition 2.3.** (Family of exceptional spheres has a product neighbourhood) Suppose $(M^5,\mathcal{F},\omega)$ is a strong symplectic foliation. Let $Z \subset M$ be such that the restriction $(\mathcal{F},\omega)|_Z$ is a fibration by symplectic spheres, and each of the spheres is an embedded (-1)-sphere in the leaf containing it. There is a closed form $\omega_1 \in \Omega^2(M)$ and a constant $\epsilon > 0$ such that

$$\omega_1|_\mathcal{F} = \omega|_\mathcal{F}$$

and additionally the following holds. Let $\omega_\epsilon \in \Omega^2(\mathcal{O}_P(-1))$ be a symplectic form on the blow-up of the origin in $\mathbb{C}^2$, for which the exceptional divisor $E$ has volume $\epsilon$. There is a foliated diffeomorphism $f : S^1 \times \mathcal{O}_P(E) \to \mathcal{O}_P(Z)$ that maps $S^1 \times E$ to $Z$, and such that $f^*\omega_1 = \omega_\epsilon$.

**Proof.** The result is an application of Proposition 2.2, for which we need to show that a neighbourhood $\mathcal{O}_P(Z)$, with leafwise form $\omega|_\mathcal{F}$ is a trivial symplectic fiber bundle over $S^1$. Firstly, since $M$ is a strong symplectic foliation, every sphere in $Z$ has the same volume. We take $\epsilon$ to be the volume of the spheres. Let $\sigma \in \Omega^2(S^2)$ be a symplectic form such that $\int_{S^2}\sigma = \epsilon$. The fibration $(\mathcal{F}|_Z,\omega|_\mathcal{F}\cap Z)$ is a trivial fibration with fibers $(S^2,\sigma)$. This is because $\text{Symp}(S^2,\sigma)$ is homotopic to $\text{Diff}(S^2)$, which is connected.

Next, we claim that there is a neighbourhood $\mathcal{O}_P(Z)$ such that $(\mathcal{O}_P(Z),\omega|_\mathcal{F})$ is a trivial fibration with fibers $(\mathcal{O}_P(E),\omega_\epsilon)$. For a single leaf of $\mathcal{O}_P(Z)$, we obtain a symplectomorphism to $\mathcal{O}_P(E)$ via the symplectic neighbourhood theorem. The isomorphism for the $S^1$-bundle follows from the fact that an $S^1$-family of rank two symplectic vector bundles on $S^2$ is just the product. This is indeed the case, because the space of bundle isomorphisms of a rank two symplectic vector bundle is contractible, and so, an $S^1$-family is trivial. \hfill $\Box$

### 3. Holomorphic foliations

We define holomorphic curves on foliated symplectic manifolds with cylindrical ends and analyze symplectic fillings in the almost horizontal case.

#### 3.1. Holomorphic curves.

**Definition 3.1.** (Leafwise cylindrical almost complex structures)

(a) On a foliated symplectic manifold $(W,\mathcal{F},\omega)$, a leafwise almost complex structure is a bundle automorphism $J : T\mathcal{F} \to T\mathcal{F}$ compatible with $\omega|_\mathcal{F}$ and satisfying $J^2 = -\text{Id}$.

(b) (Almost complex structures on symplectizations) Suppose $(M,\mathcal{F},\xi)$ is a foliated contact manifold, and let $\alpha \in \Omega^1(M)$ be a leafwise contact form whose Reeb vector field is $R_\alpha$. In the symplectization $\mathbb{R} \times M$, the tangent space of the leaves admits a splitting $T(\mathbb{R} \times \mathcal{F}) = \mathbb{R} \oplus R_\alpha \oplus \xi$, where $\xi = \ker \alpha \subset T\mathcal{F}$. Let $\partial_\alpha$ denote the unit vector field in the $\mathbb{R}$-direction. The space of cylindrical almost complex structures, denoted by $\mathcal{J}_\alpha(\mathbb{R} \times M)$, consists of leafwise compatible almost complex structures $J$ on $\mathbb{R} \times M$ that satisfy
(i) $J$ is invariant under $\mathbb{R}$-translations in $\mathbb{R} \times M$,
(ii) $J\partial_a = R_\alpha$,
(iii) and $J\xi = \xi$.

(c) (Manifold with cylindrical ends) A non-compact foliated symplectic manifold $(W^\infty, \mathcal{F}, \omega)$ has cylindrical ends if its non-compact ends are symplectomorphic to $([0, \infty) \times M, d(e^\alpha))$, where $(M, \mathcal{F}_M, \alpha)$ is a foliated contact manifold. The space of cylindrical almost complex structures, denoted by $J_\alpha(W)$, consists of leafwise compatible almost complex structures $J$ on $W$ such that for some $a > 0$, $J|_{[a, \infty) \times M}$ is a cylindrical almost complex structure.

Suppose $\Sigma$ is a compact Riemann surface, $\Gamma \subset \Sigma$ is a finite set of points, and $\Sigma^\circ := \Sigma \setminus \Gamma$ is a punctured Riemann surface. We consider $J$-holomorphic maps $u : \Sigma^\circ \to W$ that are tangent to the foliation, and whose punctures are asymptotic to the periodic orbits $\gamma_1, \ldots, \gamma_k$ in $\mathcal{F}$. Then,

$$E(u) = \int_{u^{-1}(W)} u^* \omega + \int_{u^{-1}(W^\infty \setminus W)} u^* d\alpha + \sum_i \int_{\gamma_i} \alpha$$

In particular, energy is the pairing of the relative homology class $u_* [\Sigma, \partial \Sigma] \in H_2(W, \cup_i \gamma_i)$ and the relative de Rham cohomology class $(\omega, \alpha) \in H^2(W, \cup_i \gamma_i)$.

The last statement in the proposition can be seen by deforming $u$ to a map

$$\pi : \Sigma \to W := \begin{cases} u, & \text{on } u^{-1}(W), \\ (0, \pi_M \circ u) & \text{on } u^{-1}(W^\infty \setminus W). \end{cases}$$

Here, we identify the end $W^\infty \setminus W$ to $[0, \infty) \times M$, and $\pi_M : W^\infty \setminus W \to M$ is the projection to $M$. Then, by (5), $E(u) = \int_{\Sigma} \pi^* \omega + \int_{\partial \Sigma} \alpha$.

**Definition 3.3.** (Holonomy transport) Suppose $\gamma : S^1 \to L$ is a loop in the leaf $L$ of a foliated manifold $(W, \mathcal{F})$, where the foliation is of codimension one. Suppose $\tau : (-\epsilon, \epsilon) \to \mathbb{R}$ is a transversal through $\gamma(0)$. The transversal can be transported along $\gamma$ via foliated charts of $W$, and the return map $h : \tau \to \tau$ is a homotopy invariant, called the holonomy transport $\pi_1(\mathcal{F}) \to \text{Homeo}(\tau)$. See p.141 in Calegari [5].

We will work with holomorphic curves in leaves of a symplectic foliation, for whom the holonomy transport is trivial. In that case, results about the manifold structure of the moduli space of holomorphic curves in a foliated symplectic manifold are similar to the results in the non-foliated case. Suppose $u : \Sigma \to (W^\infty, \mathcal{F})$ is a foliated $J$-map, whose image lies in a leaf $L \subset W^\infty$. The leafwise linearized Cauchy-Riemann operator is defined between the spaces

$$D_u^\mathcal{F} : \Gamma(\Sigma, u^* T\mathcal{F}) \to \Omega^{0,1}(\Sigma, u^* T\mathcal{F}).$$
A foliated $J$-map $u$ is unobstructed if $D^F_u$ is surjective.

**Proposition 3.4.** Suppose $(W^{2n+1},\mathcal{F}^{2n})$ is a foliated symplectic manifold with cylindrical ends, and with a leafwise almost complex structure $J: T\mathcal{F} \to T\mathcal{F}$. Suppose $u: \Sigma \to W$ is a foliated $J$-holomorphic map from a punctured Riemann surface $\Sigma$. If the holonomy transport is trivial on $u_*\pi_1(\Sigma)$, and if $u$ is unobstructed, there is a manifold chart on the space of $J$-holomorphic curves in a neighbourhood of $u$.

Further the manifold has a codimension one foliation.

**Proof.** Let $B := \text{Map}_F(\Sigma, W)$ denote the space of finite energy smooth maps $u: \Sigma \to W$ whose images lie in leaves and have trivial holonomy transport. Consider a map $u \in B$. Let $t$ be a transverse coordinate defined in the neighbourhood of the image of $u$, whose level sets are tangent to the foliation. There is a splitting of the tangent space

$$T_uB = \Gamma(\Sigma, u^*T\mathcal{F}) + \mathbb{R}\partial_t. \quad (6)$$

In a neighbourhood of a map $u \in B$, we can define a transverse coordinate $t_B$ as $t \circ u$. After extending $B$ to Sobolev completions of maps, it is a foliated Banach manifold.

Let $E \to B$ be an infinite dimensional vector bundle whose fiber at $u \in B$ is $E_u := \Omega^{0,1}(\Sigma, u^*T\mathcal{F})$. The moduli space of foliated holomorphic maps is the zero set of the section $\overline{\mathcal{D}}: B \to E, \ u \mapsto \overline{\mathcal{D}}u$.

The linearization of the operator $\overline{\mathcal{D}}$ at $u$ is

$$D_u := (D\overline{\mathcal{D}})_u : T_uB \to \Omega^{0,1}(\Sigma, u^*T\mathcal{F}), \ \xi \mapsto \frac{d}{dt} \mathcal{F}^{-1}_\xi \overline{\mathcal{D}}(\exp_u t\xi),$$

where $\mathcal{F}_\xi : T_uB \to T_{\exp_u t\xi}B$ is parallel transport using the metric $\omega(\cdot, J\cdot) + dt^2$.

Using the splitting $\xi = \xi_F + c\partial_t$ given by $(6)$, we can rewrite

$$(D\overline{\mathcal{D}})_u \xi = D^F_u \xi_F + c(\nabla_{\partial_t} J) \circ du \circ j.$$

Thus, we see that $(D\overline{\mathcal{D}})_u$ is onto if $D^F_u$ is onto. The surjectivity of $D^F_u$ also implies that the foliation $t_B$ restricts to a foliation $t_{Map}$ in a neighbourhood of $u$ in $\overline{\mathcal{D}}^{-1}(0)$.

After extending the operator $D^F_u$ to Sobolev completions as in [27], we can say that $(D\overline{\mathcal{D}})_u$ is Fredholm and has index $\text{ind}(D\overline{\mathcal{D}})_u = \text{ind} D^F_u + 1$. $\square$

### 3.1.1. Monotonicity for $J$-curves.

We recall a monotonicity result for $J$-holomorphic curves, which holds if the metric on the manifold is reasonably behaved.

**Definition 3.5.** (Tame symplectic manifold) A symplectic manifold $(W, \omega, J)$ is tame if the induced metric is complete, has an upper bound on the sectional curvature and has positive injectivity radius.

If $W$ is a manifold with boundary, we require the boundary to be compact, and the injectivity radius to be positive outside a collar neighbourhood of $\partial W$.

**Proposition 3.6.** (Bound on the diameter in terms of area [20, Proposition 4.4.1]) Suppose $(W, \omega, J)$ is a tame symplectic manifold. There is a constant $C > 0$ for which the following is satisfied. For any compact set $K \subset W$, and any compact
connected (possibly nodal) $J$-curve $C = f(\Sigma)$ such that $C_1$ intersects $K$ and $\partial C = f(\partial \Sigma) \subset K$, $C$ is contained in a neighbourhood of $K$ of size $C_1 \text{area}_{\omega}(C)$.

3.2. Holomorphic curves in symplectization. A finite energy holomorphic foliation of a symplectization $\mathbb{R} \times M$ is a foliation of $\mathbb{R} \times M$ by $J$-holomorphic curves, where each $J$-curve is tangent to $\mathcal{F}$ (hence it is a subfoliation of $\mathcal{F}$), and $J$ is a cylindrical leafwise almost complex structure. The holomorphic foliation is $\mathbb{R}$-invariant, and every leaf that is not an an orbit cylinder is asymptotic to Reeb orbits in the positive end $\{\infty\} \times M$. Note that by definition, for any Reeb orbit $\gamma$ in $M$, $\mathbb{R} \times \gamma$ is a holomorphic curve in the symplectization $\mathbb{R} \times M$.

In this section, we study a holomorphic subfoliation of the foliated unit cotangent bundle of a three-manifold with an almost horizontal foliation. The foliated three-manifold is $X := S^1 \times (A^2, \mathcal{F}_{ah})$. In the almost horizontal foliation on the two-dimensional annulus $A$, the two boundary components of $A$ are compact leaves, all the other leaves are non-compact, and the leaves are orientable. The foliation can be concretly described as follows. The annulus $A$ is the quotient of the strip $[-1,1] \times \mathbb{R}$ by the translation action of $\mathbb{Z}$ on the second coordinate. The foliation $\mathcal{F}_{ah}$ is the descent of a foliation on the strip whose leaves are $\{y + c = \tan \frac{\pi x}{2}\}_{c \in \mathbb{R}}$, and $\{x = \pm 1\}$. This foliation has the property that there is a coordinate

$$q_2 : A^2 \to \mathbb{R}/\mathbb{Z}$$

whose level sets are transverse to leaves, which we call the leaf coordinate. The foliation on the 3-manifold $X$ is the pull-back of $\mathcal{F}_{ah}$, which we also sometimes denote by $\mathcal{F}_{ah}$.

**Proposition 3.7.** (Holomorphic foliation on symplectization) Let $(X, \mathcal{F}) := S^1 \times (A^2, \mathcal{F}_{ah})$. There is a contact form $\alpha$ on the foliated unit cotangent bundle $M := S(FT^*X)$ and a cylindrical almost complex structure $J \in J_\alpha(\mathbb{R} \times M)$ for which $\mathbb{R} \times M$ has a finite energy holomorphic foliation by cylinders.

**Proof.** As discussed in Section 2.3, a leafwise contact form on the sphere cotangent bundle $S(FT^*X)$ is determined by a choice of leafwise Riemannian metric on $X = S^1 \times A^2$. Let $q_1$ be a coordinate function on $S^1$ and let $q_2 : A^2 \to S^1$ be the leaf coordinate on $A^2$. Then, we choose the leafwise Riemannian metric to be $dq_1^2 + dq_2^2$. In addition, the manifold $X$ also has a transverse coordinate $t : X \to \mathbb{R}$ to the foliation. (The level sets of $t$ are not tangent to the foliation.)

We next choose suitable coordinates on the sphere cotangent bundle. Let $(p_1, p_2)$ be the co-ordinates on the fibers of $T^*X$ corresponding to the coordinates $(q_1, q_2)$ on the leaves of $X$, so that the symplectic form on $T^*X$ is $\sum_i dp_i \wedge dq_i$. The unit cotangent bundle $M$ is the hypersurface $\{p_1^2 + p_2^2 = 1\}$ of $T^*X$. Thus $M$ has coordinates $(q_1, q_2, \theta, t)$, where $t$ is the coordinate transverse to the foliation, and $\theta$ is given by $(\cos \theta, \sin \theta) = (p_1, p_2)$. Using the Liouville vector field $v = \sum_i p_i \partial_{p_i}$, we obtain a contact form and Reeb vector field on $M$:

$$\alpha = \cos \theta dq_1 + \sin \theta dq_2, \quad R_\alpha = \cos \theta \partial_{q_1} + \sin \theta \partial_{q_2}.$$
Here the vector field $\partial_{q_2}$ on the annulus is defined so that $dq_2(\partial_{q_2}) = 1$ and $\partial_{q_2} \in F_{ah}$. The Reeb orbits are confined to level sets of $\theta$. Two of the level sets $\theta = 0, \pi$ are foliated by closed Reeb orbits – these are Morse-Bott submanifolds.

We now describe the foliation by holomorphic curves. Choose an almost complex structure $J_0 \in J_\alpha(M)$ as

\begin{equation}
J_0 \partial_a = R_\alpha, \quad J_0 \partial_\theta = -\sin \theta \partial_{q_1} + \cos \theta \partial_{q_2}.
\end{equation}

The leaves of the holomorphic foliation are the connected components of the fibers of the map

\begin{equation}
\mathbb{R} \times M \to \mathbb{R} \times (A, F_{ah}), \quad (a, (q_1, q_2, \theta, t)) \mapsto (e^a \sin \theta, (q_2, t)).
\end{equation}

All the leaves are finite energy cylinders. The fibers corresponding to $\theta = 0, \pi$ are Reeb cylinders. For every other cylinder, the ends are asymptotic to a Reeb orbit in $\{\theta = 0\}$ and one in $\{\theta = \pi\}$.

**Remark 3.8.** (A geometric interpretation of the Reeb vector field) A geodesic on a Riemannian manifold lifts to a flow line of the Reeb vector field on the unit cotangent bundle, see Geiges [13, Theorem 1.5.2]. The leaves of the 3-manifold $X$ have a Riemannian metric $dq_1^2 + dq_2^2$. Geodesics are curves $\gamma : I \to M$ lying on a leaf for whom the ratio $\frac{dq_2(\gamma')}{dq_1(\gamma')}$ is constant. A geodesic $\gamma$ for which the ratio is $\lambda$ lifts to a Reeb flow line $(\gamma, \theta = \tan^{-1}(\lambda))$ in the foliated unit cotangent bundle $M$. On a non-compact leaf of $M$, the only flow lines that close up are those for which $q_2 = \text{const}$, and so, $\theta = 0, \pi$.

**Remark 3.9.** (a) (On the moduli space of cylinders in the symplectization) Let $\mathcal{M}_{\mathbb{R} \times M}$ denote the leaf space of the above holomorphic foliation minus the Reeb cylinders. The holomorphic foliation is invariant under $\mathbb{R}$-translation on $\mathbb{R} \times M$. The moduli space $\mathcal{M}_{\mathbb{R} \times M}/\mathbb{R}$ has two components, namely $\mathcal{M}_{\mathbb{R} \times M}/\mathbb{R}$, consisting of maps with $\theta$ coordinate in $(0, \pi)$ and $(-\pi, 0)$ respectively. Further, by (8) each cylinder projects to a point in $(A, F_{ah})$, and in fact the maps

\begin{equation}
\mathcal{M}_{\mathbb{R} \times M}/\mathbb{R} \to (A, F_{ah})
\end{equation}

are foliated diffeomorphisms. Finally, we observe that cylinders in $\mathcal{M}_{\mathbb{R} \times M}$ project to embedded cylinders in $M$, and this projection is invariant under the $\mathbb{R}$-action on $\mathcal{M}_{\mathbb{R} \times M}$. The cylinders foliate $M - \mathcal{P}$ where $\mathcal{P}$ is the union of Morse-Bott tori $\{\theta = 0\}$, $\{\theta = \pi\}$. Thus, there is a map

\begin{equation}
\pi : M - \mathcal{P} \to \mathcal{M}_{\mathbb{R} \times M}/\mathbb{R},
\end{equation}

whose fibers are cylinders.

(b) (Sections of the holomorphic fibration) For the fibration (9), we can choose sections

\begin{equation}
s_{\pm} : \mathcal{M}_{\mathbb{R} \times M}/\mathbb{R} \to M,
\end{equation}

such that $\theta(s_{\pm}) \in (0, \epsilon)$ and $\theta(s_{-}) \in (-\epsilon, 0)$. Further, they can be chosen so that the maps $s_{\pm} : (A, F_{ah}) \to M$ are homotopic as foliated maps.
3.3. Holomorphic curves in the filling. In this section, we describe a family of holomorphic curves on the foliated filling of the sphere cotangent bundle of the foliated 3-manifold $X := S^1 \times (A^2, F_{ah})$. Here the annulus $A^2$ has the almost horizontal foliation $F_{ah}$. The holomorphic curves are tangent to the foliation, and further they form a subfoliation of the filling. Let $M := S(FT^*X)$ be the sphere cotangent bundle with contact form $\alpha$ as in Proposition 3.7. Let $W$ be a strong symplectic filling of $M$, and $W^\infty$ be the extended filling, i.e. $W^\infty$ is obtained by attaching cylindrical ends to $W$. Let $J$ be a generic compatible cylindrical leafwise almost complex structure on $W^\infty$, which agrees with $J_+$ (from Proposition 3.7) on the ends, i.e. $J|_{[a,\infty) \times M} = J_+$. The $J_+$-holomorphic foliation on $\mathbb{R} \times M$ is translation invariant, and any curve in the holomorphic foliation of Proposition 3.7 has a lower bound on the $\mathbb{R}$-coordinate. Therefore, there is a cylinder $u_0 : \mathbb{R} \times S^1 \to \mathbb{R} \times M$ that is contained in $[a, \infty) \times M$, and thus it is also a $J$-holomorphic map to $W^\infty$. Let $\mathcal{M}$ be the connected component of the moduli space of finite energy holomorphic maps $u : \mathbb{R} \times S^1 \to W$ tangent to the foliation that contains $u_0$. We remark that the space $\mathcal{M}$ consists of maps modulo reparametrization of the domain.

The main result of this section is:

**Proposition 3.10.** (Holomorphic foliation on fillings)

(a) The space $\mathcal{M}$ is a manifold of dimension 3 and possesses codimension one foliation. Every curve in $\mathcal{M}$ is embedded and no two curves in $\mathcal{M}$ intersect.

(b) The moduli space $\mathcal{M}$ has a compactification $\overline{\mathcal{M}}$, and the foliation $F_{\mathcal{M}}$ extends to the boundary. The boundary $\overline{\mathcal{M}} \setminus \mathcal{M}$ consists of

(i) a compact foliated 2-dimensional manifold, where each point represents a leaf in the foliation of $\mathbb{R} \times M$.

(ii) A compact 1-dimensional manifold, denoted by $\mathcal{M}_{nodal}$, that is transverse to the foliation $F_{\mathcal{M}}$, where each point represents a nodal curve consisting of two embedded index zero curves in $W^\infty$. Each of the nodal curves is disjoint from the curves in $\mathcal{M}$.

(c) The collection of curves in $\mathcal{M}$ and the nodal curves in $\mathcal{M}_{nodal}$ form a foliation of $W^\infty$ outside of the one-dimensional set of nodal points. At any of the nodal points double points, where two leaves intersect transversely; these are the nodes of the nodal curves in $\overline{\mathcal{M}} \setminus \mathcal{M}$.

**Proof.** All curves in the moduli space $\mathcal{M}$ have trivial holonomy transport, because the curves are asymptotic to a Reeb orbit at the cylindrical ends, and the Reeb orbit has trivial holonomy. Therefore curves in the foliated setting behave in the same way as the non-foliated case. Genericity of the almost complex structure $J$ implies that for any holomorphic curve $u$ not contained in $[a, \infty) \times M$, $\text{ind}_F(u) \geq -1$. Consider an irreducible curve component $u$ in the compactified moduli space $\overline{\mathcal{M}}$. Under the map $u$, punctures are asymptotic to a Reeb orbit in one of the Morse-Bott tori $\{\theta = 0\}$ or $\{\theta = \pi\}$ in $M$. Each of these orbits has odd (Morse-Bott version of) Conley-Zehnder index. By the formula (A.2) in Wendl [28], $\text{ind}_F(u)$ is even, and so,
ind_F(u) \geq 0. Therefore, the proof of Theorem 7 in Wendl [28] entirely carries over, and the proposition is proved.

We make a few remarks about Gromov compactness. Convergence of holomorphic curves in sft has two phenomena: one of nodal degeneration from standard GW theory in symplectic geometry. The other is forming of multi-level buildings. In the first case, behaviour in the foliated setting is on expected lines. Sequence of foliated J-maps converge to foliated nodal J-curves. In our case, the curves are of index 2, and when there is nodal degeneration, we end up with curves with index 0. We remark that index 0 regular maps produce an $S^1$-family of curves, which is zero-dimensional in each leaf. The second case is more complicated: a sequence of J-maps on a leaf $L$ can converge to a multilevel building of J-curves, but not all levels may be in the same leaf, see [7]. We do not encounter this splitting phenomenon as we do not get curves that have non-trivial components in more than one level. □

Remark 3.11. The moduli space $\overline{M}$ is a manifold with corners. It has four (codimension one) boundary components:

- two of them are transverse to the foliation, consisting of height two curves as in (bi). We call them the top and bottom boundary, and they are canonically identified to $\mathbb{R}$-equivalence classes in the symplectization, i.e. there are standard foliated diffeomorphisms

\[(A^2, F_{ah}) \to \partial_{\pm} \overline{M} \to \mathcal{M}_{\pm}^{\mathbb{R} \times M}/\mathbb{R}.\]

- The other two boundary components are tangent to the foliation $\mathcal{F}$ and consist of curves foliating the fillings of $T^3$, the boundary components of $\mathbb{S}(FT^3(\mathbb{S}^1 \times (A, F_{ah})))$. This is a consequence of the definition of fillings of manifolds with boundary, see Section 2.2. We call these the side boundaries of $\overline{M}$, and denote them by $\partial_{\text{side}, \pm} \overline{M}$.

Proposition 3.12. There is a foliated diffeomorphism $(A^2, F_{ah}) \times [0, 1] \to \overline{M}$ that extends the canonical diffeomorphism (11) on the top and bottom boundaries.

Proof. We first show that there are maps from the annulus to the top and bottom boundaries, that are homotopic in $\overline{M}$. There is a foliated map

\[\pi : W^\infty \to \overline{M}\]

that sends a point to the curve in $\overline{M}$ on which it lies. Let $\overline{W}^\infty$ denote the compactification of $W^\infty$ by adding $\{\infty\} \times M$ to the cylindrical end. We recall that the top and bottom boundary components $\partial_{\pm} \overline{M}$ can be canonically identified to $\mathbb{R}$-equivalence classes of maps $\mathcal{M}_{\pm}^{\mathbb{R} \times M}/\mathbb{R}$ in the symplectization, see (11). Therefore, the map (12) extends to

\[\pi : \overline{W}^\infty \setminus \mathcal{P} \to \overline{M},\]

where $\mathcal{P} \subset \{\infty\} \times M$ is the union of the Morse-Bott tori. Here $\pi|_{\{\infty\} \times M}$ is defined by composing the fibration $M - \mathcal{P} \to \mathcal{M}_{\pm}^{\mathbb{R} \times M}/\mathbb{R}$ in (9) with the identification $\mathcal{M}_{\pm}^{\mathbb{R} \times M}/\mathbb{R} \to \partial_{\pm} \overline{M}$ in (11). The maps $s_{\pm} : \partial_{\pm} \overline{M} \to \overline{W}^\infty \setminus \mathcal{P}$ in (10) are sections of the fibration (13). Since these maps are homotopic in $W^\infty$, the homotopy
Lemma 3.13. Suppose $X$ is a two-manifold with boundary, and $s : [0, 1] \times S^1 \to X$ is a smooth map such that $s([0, 1] \times \{0\}) \subset \partial X$, and $s|_{\partial p([0,1] \times S^1)}$ is an embedding. Then, $X$ is a cylinder and $s$ can be homotoped relative to boundaries to a diffeomorphism.

Proof. The manifold $X$ is a cylinder because $s$ gives a homotopy between two of its boundary components $s([0] \times S^1)$, $s([1] \times S^1)$. We fix a diffeomorphism $X \to [0,1] \times S^1$, so that $s$ is an identity map near the ends. By projecting to the $S^1$, the map $s$ represents a loop in the space of degree 1 maps from $S^1$ to $S^1$, called $\text{Map}_1(S^1, S^1)$. The space $\text{Map}_1(S^1, S^1)$ is homotopy equivalent to $S^1$, and the loop represented by $s$ is classified by an integer $k \in \mathbb{Z}$. The homotopy class corresponding to any $k$ has a representative that is a diffeomorphism.

Lemma 3.14. Suppose $s : [0, 1] \times (A, F_{ah}) \to [0, 1] \times (A, F_{ah})$ is a foliated map that is a diffeomorphism in a neighbourhood of $\{0, 1\} \times A$. Then, $s$ can be homotoped to a foliated diffeomorphism in all of $[0, 1] \times A$.
The map $s$ can be viewed as a loop of foliated maps $(A,F) \rightarrow (A,F)$, and we write $s = (s_t)_{t \in S^1}$. Each element $s_t : A \rightarrow A$ in the loop splits into radial and angular parts: $s_t = (r, \theta) : A \rightarrow I \times S^1$, where $I$ is an interval. For non-compact leaves, the value of $\theta$ is determined by $r$ because of the foliatedness condition. So, the homotopy class of $s$ is determined by a loop in the space $Map_0(I,I)$ of maps from the interval to the interval that are the identity close to the boundary. This space retracts by deformation to the identity and so we can deform $s$ into a foliated diffeomorphism. □

Proposition 3.15. (No nodal curves on lateral boundary) Let $\mathcal{M}_{\text{nodal}} \subset \overline{\mathcal{M}}$ denote the one-dimensional moduli space of nodal curves as in Proposition 3.10 (bii). If $W$ is a minimal filling, $\mathcal{M}_{\text{nodal}}$ is a finite union of circles, which do not intersect the lateral boundaries $\overline{\mathcal{M}}(\partial_{\text{side}}, \pm \mathcal{M})$. Each point in this 1-dimensional manifold represents a nodal curve which is a union of two disks.

Proof. Index zero spheres are ruled out by minimality, the only other possible nodal curve is a union of disks. Neither of these disks are contained in the end $[R, \infty) \times M$, because the only curves of $\overline{\mathcal{M}}$ that are contained in the end $[R, \infty) \times M$ are those that occur in the symplectization. Further, the curves are somewhere injective because of their asymptotic behaviour. Since $J$ is generic away from the ends, neither of the components of the nodal curve have non-negative index, and are therefore index zero. So, the moduli space of nodal curves is a one-dimensional manifold transverse to the foliation on $\overline{\mathcal{M}}$.

Next, we claim that $\mathcal{M}_{\text{nodal}}$ does not intersect the lateral boundaries. The argument is as in Wendl [28]. Indeed, $W^\infty \rightarrow \overline{\mathcal{M}}$ is a Lefschetz fibration which is singular along $\mathcal{M}_{\text{nodal}}$. The monodromy map on a loop in $\partial_{\text{side}, \pm} \overline{\mathcal{M}}$ enclosing all the points of $\mathcal{M}_{\text{nodal}} \cap \partial_{\text{side}, \pm} \overline{\mathcal{M}}$ is trivial, because the bundle is trivial on the ends of the moduli space $\partial_{\text{side}, \pm} \overline{\mathcal{M}} \simeq \mathbb{R} \times S^1$. The mapping class group of the cylinder is $\mathbb{Z}$, and is generated by a single element. So the product of positive Dehn-twists cannot be identity. □

Finally, we show that by adjusting the almost complex structure on $W^\infty$, we obtain a holomorphic subfoliation that is standard in the non-compact ends. The standard filling of the sphere cotangent bundle $M_{\text{ah}} := S (FT^* X_{\text{ah}})$ is the disk cotangent bundle $W_{\text{std}} := D (FT^* X_{\text{ah}})$. By attaching cylindrical ends, we obtain the cotangent bundle $W_{\text{std}}^\infty = FT^* X_{\text{ah}}$. There is a foliated cylindrical almost complex structure on $J_0$ on $FT^* X_{\text{ah}}$ for which the fibers of the projection

$$FT^* X_{\text{ah}} \rightarrow \mathbb{R} \times (A, \mathcal{F}_{\text{ah}}), \quad (q_1, q_2, p_1, p_2, t) \mapsto (p_2, (q_2, t))$$

are holomorphic. Indeed, on the cylindrical ends $[0, \infty) \times M_{\text{ah}}$, $J_0$ can be defined as in (7). We refer to the holomorphic foliation in (16) as the standard holomorphic foliation on the cylindrical ends $[0, \infty) \times M_{\text{ah}}$.

Proposition 3.16. (Holomorphic foliation is standard in ends) Suppose $W$ is a filling of $M_{\text{ah}} := S (FT^* X_{\text{ah}})$. There is a cylindrical almost complex structure $\hat{J}$ on $W^\infty$ that is
(a) equal to $J_0$ on $[R, \infty) \times M_{ah}$ for a large $R$.
(b) Suppose $\overline{M}_J$ is the moduli space of $J$-holomorphic curves given by Proposition 3.10. The holomorphic foliation induced by the curves in $\overline{M}_J$ on $[R, \infty) \times M_{ah}$ is the same as the standard holomorphic foliation on $[R, \infty) \times M_{ah}$.

Proof. We start with an arbitrary generic cylindrical almost complex structure $J$ that is equal to $J_0$ on $[0, \infty) \times M_{ah}$. Let $\hat{M}_J$ be the moduli space of $\hat{J}$-holomorphic curves produced by Proposition 3.10. The holomorphic foliation induced by the curves in $\hat{M}_J$ on $[R, \infty) \times M_{ah}$ is the same as the standard holomorphic foliation on $[R, \infty) \times M_{ah}$.

4. Model fillings

Suppose $W_{ah}$ is a filling of the foliated unit cotangent bundle of $S^1 \times (A, F_{ah})$. The results of Section 3 can by summarized by saying that the extended filling $W_{ah}^{\infty}$ is a foliated Lefschetz fibration over $\mathbb{R} \times (A, F_{ah})$ that has a standard structure on non-compact ends. The regular fibers are cylinders $\mathbb{R} \times S^1$. In this section, we prove that such Lefschetz fibrations can be classified up to symplectic deformation equivalence using combinatorial data.

A part of the combinatorial data comes from the boundary of $W_{ah}^{\infty}$. The boundary of a Lefschetz fibration on $\mathbb{R} \times (A, F_{ah})$ consists of Lefschetz fibrations over two cylinders. The boundary components are extended fillings of $T^3$, and were studied by Wendl [28].

4.1. Model filling of $T^3$. The manifold $T^3$ has a standard contact structure, by viewing it as the unit cotangent bundle of $T^2$. The standard filling of $T^3$ is the disk cotangent bundle $\partial(T^*T^2)$. By a Luttinger surgery of the zero section $T^2 \subset T^*T^2$, one obtains a $\mathbb{Z}^2$-family of fillings. Wendl [28] proved that any filling of $T^3$ is symplectic deformation equivalent to one of the elements in this family. The $\mathbb{Z}^2$-parameter associated to a filling is called the Luttinger constant of the filling. In this section, we first describe the Luttinger surgery for a choice of parameter $(k, k_b) \in \mathbb{Z}^2$. We then prove a result about which Luttinger constants occur on the boundary of the almost horizontal filling.

The description of the Luttinger surgery is on the lines of Wendl [28], who in turn, followed Auroux-Donaldson-Katzarkov [2]. To describe the surgery, we choose an additional parameter $c \in \mathbb{R}_{>0}$, but later it will be shown that various choices lead to symplectomorphic manifolds. We denote $\sigma = (c, k, k_b) \in \mathbb{R}_{>0} \times \mathbb{Z}^2$. The surgery...
is performed on the cotangent bundle

\[ T^* \mathbb{T}^2 = \{(q_1, q_2, p_1, p_2) : \mathbb{R}^2 \times (S^1)^2 \}, \quad \omega_0 = \sum_{i=1,2} dq_i \wedge dp_i \]

along the zero section \( \{p_1 = p_2 = 0\} \). For any \( r > 0 \), let \( K_r := \{(q, p) \in T^* \mathbb{T}^2 : |p_1|, |p_2| < r\} \). There is a symplectomorphism \( \psi_\sigma : K_{2c} \setminus K_c \to K_{2c} \setminus K_c \) defined as

\[ \psi_\sigma(q_1, q_2, p_1, p_2) := (q_1 + k \chi(p_2) \beta(\frac{p_1}{c}), q_2 + k_b \chi(p_1) \beta(\frac{p_2}{c}), p_1, p_2) \]

where \( \chi : \mathbb{R} \to \{0, 1\} \) is 0 on \( \mathbb{R}_{<0} \) and 1 on \( \mathbb{R}_{\geq 0} \), and \( \beta : \mathbb{R} \to [0, 1] \) is a smooth cut-off function that is 0 on \( (-\infty, -1), 1 \) on \( [1, \infty) \) and satisfies \( \int_{-1}^1 t \beta'(t) dt = 0 \).

The output of the Luttinger surgery is the symplectic manifold

\[ (\text{Lutt}_\sigma(T^* \mathbb{T}^2), \omega_\sigma) := (T^* \mathbb{T}^2 \setminus K_{2c}, \omega_0) \cup_{\psi_\sigma} (K_{2c}, \omega_0). \]

**Remark 4.1.** For a non-trivial Luttinger parameter \( \sigma \), the spaces \( T^* \mathbb{T}^2 \) and \( \text{Lutt}_\sigma(T^* \mathbb{T}^2) \) are symplectomorphic. The symplectomorphism is given by global coordinates \((Q_1, Q_2, P_1, P_2)\) on \( \text{Lutt}_\sigma(T^* \mathbb{T}^2) \) defined as follows: \((Q, P)\) is the standard coordinate on \( K_{2c} \), and thus on the overlap, we have \( (q, p) = \psi_\sigma(Q, P) \). The coordinates \((Q, P)\) extend to all of \( T^* \mathbb{T}^2 \setminus K_{2c} \). However, the restriction of the symplectomorphism to the ends \( \mathbb{R}_{\geq 0} \times \mathbb{T}^3 \) is not isotopic to identity. So \( T^* \mathbb{T}^2 \) and \( \text{Lutt}_\sigma(T^* \mathbb{T}^2) \) are not symplectic deformation equivalent.

**Proposition 4.2.** (Classification of \( \mathbb{T}^3 \)-fillings [28, Proposition 5.6]) Suppose \((W^\infty, \omega)\) is a filling of \( M = (\mathbb{T}^3, \alpha_0) \) with infinite ends attached, so there is a symplectomorphism to the non-compact ends \( F : [R, \infty) \times M \to W^\infty \) for some \( R > 0 \). There is a Luttinger surgery parameter \( \sigma \) such that the map \( F \) extends to a symplectic deformation equivalence \( F : W_\sigma \to W^\infty \), i.e. \( F \) is a diffeomorphism such that \((1-t)\omega_\sigma + tF^* \omega \) is symplectic on \( W_\sigma \) for all \( t \in [0, 1] \).

**Remark 4.3.** (Interpretations of Luttinger surgery parameters) There is a cylindrical almost complex structure \( J_0 \) on \( T^* \mathbb{T}^2 \) such that the fibers of the map

\[ \pi : T^* \mathbb{T}^2 \to S^1 \times \mathbb{R}, \quad (q_1, q_2, p_1, p_2) \mapsto (q_2, p_2) \]

are holomorphic cylinders. For any Luttinger parameter \( \sigma \), there is an \( \omega_\sigma \)-compatible almost complex structure on \( W_\sigma \) that is equal to \( J_0 \) on the ends \( [R, \infty) \times \mathbb{T}^3 \), such that the fibers of the map

\[ \pi : (Q_1, Q_2, P_1, P_2) \mapsto (Q_2, P_2) \]

are \( J_\sigma \)-holomorphic. The Luttinger parameters \((k, k_b)\) have the following interpretations.

- (a) (As offsets of ends of cylinders) The cylinder \( u^{(\theta, \eta)} := \{(p_2, Q_2) = (\rho, \eta)\} \) has one end asymptotic to \( \{\theta = \pi, q_2 = \eta\} \) and the other end asymptotic to \( \{\theta = 0, q_2 = \eta + k_b \beta(\rho/c)\} \).
- (b) (As Dehn twists) Let us restrict ourselves to the case when \( k_b = 0 \). We will see in Proposition 4.4 that \( k_b \) indeed vanishes in the almost horizontal case. If \( k_b = 0 \), then \((Q_2, P_2) = (q_2, p_2)\). Use the coordinates in \( K_{2c} \), (which in this case is same as the coordinates in \( \{p_2 < 2c\} \)) to trivialize the bundle
\[ \pi^{-1}\{p_2 < 2c\}, \] and the coordinates on \( T^*\mathbb{T}^2 \setminus K_e \) to trivialize \( \pi^{-1}\{p_2 > c\}\). The Luttinger surgery has the effect of gluing these trivial bundles with a Dehn twist in the overlap, i.e. cylinder in second trivialization is the cylinder in first trivialization with a positive \( k \) Dehn twist applied to it.

**Proposition 4.4.** (Luttinger constants for almost horizontal filling) Suppose \( W^\infty_{ah} \) is a foliated filling of \( M_{ah} := \mathbb{S}(FT^*X) \), where \( X := \mathbb{S}^1 \times (A^2, F_{ah}) \), and \((A^2, F_{ah})\) is the almost horizontal filling of the annulus. Denote by \( \partial_+ A \) and \( \partial_- A \) the inner and outer boundaries of the annulus. Suppose \((k^\pm, k^\pm)\) are the Luttinger parameters of the fillings of \( \mathbb{S}(FT^*(\mathbb{S}^1 \times \partial_{\pm} A)) \) (obtained by applying Proposition 4.2). Then,

\begin{itemize}
  \item[(a)] \( k^\pm_0 = 0 \),
  \item[(b)] and \( k^+ - k^- \geq 0 \) is the number of nodal curves (Proposition 3.10 (bii)) in each non-compact leaf of the filling of \( M_{ah} \).
\end{itemize}

**Proof.** There is an identification of the cylindrical end \( F : [R, \infty) \times M_{ah} \rightarrow W^\infty_{ah} \) for a large \( R > 0 \). By Proposition 4.2, the restrictions \( F_{\pm} := F|_{\mathbb{S}(FT^*(\mathbb{S}^1 \times \partial_{\pm} A))} \) extend to diffeomorphisms

\[ F_{\pm} : \text{Lutt}_{(k^\pm_0, k^\pm, c^\pm)}(T^*\mathbb{T}^2) \rightarrow \partial_{\pm} W_{ah}, \tag{18} \]

where \( \partial_{\pm} W_{ah} \) are boundary components of \( W_{ah} \). Suppose \((q_1, q_2, \theta)\) are coordinates on \( T^3 \simeq \mathbb{S}(T^*\mathbb{T}^2) \) defined as in the proof of Proposition 3.7 – i.e. the unit cotangent fiber has coordinate \( \theta \), the \( \mathbb{S}^1 \) factor in \( X \) has coordinate \( q_1 \), and the boundary of the annulus has coordinate \( q_2 \). We denote loops in these directions as \( \gamma_{q_1}, \gamma_{q_2} \) and \( \gamma_\theta \). The diffeomorphisms (18) implies that the loop \( \gamma_\theta + k^- \gamma_{q_1} + k^+ \gamma_{q_2} \) is contractible in \( W_{ah} \). The loops \( \gamma_{q_1} \) and \( \gamma_\theta \) have trivial holonomy transport and \( \gamma_{q_2} \) has non-trivial holonomy transport. This implies \( k^\pm_0 = 0 \).

The moduli space of nodal curves \( \mathcal{M}_{nodal} \) intersects each of the non-compact leaves in \( \mathcal{M} \) at a finite number of points. The non-compact leaf is of the form \( \mathbb{R} \times [0, 1] \). The moduli space of nodal curves intersects each of the non-compact leaves in \( \mathcal{M}_{nodal} \) at a finite number of points. The moduli space of nodal curves intersects each of the non-compact leaves in \( \mathcal{M}_{nodal} \) at a finite number of points.

\[ \pi^{-1}\{p_2 < 2c\}, \] and the coordinates on \( T^*\mathbb{T}^2 \setminus K_e \) to trivialize \( \pi^{-1}\{p_2 > c\}\). The Luttinger surgery has the effect of gluing these trivial bundles with a Dehn twist in the overlap, i.e. cylinder in second trivialization is the cylinder in first trivialization with a positive \( k \) Dehn twist applied to it.

4.2. **Topological Lefschetz fibrations.** In this section, we recall some standard topological definitions and results about Lefschetz fibrations.

**Definition 4.5.** A topological Lefschetz fibration consists of an oriented four-manifold \( X \), and a proper map \( \pi : X \rightarrow \mathbb{C} \) with the following property: For any \( x_0 \in X \), either \( d\pi_{x_0} \) is surjective, or there are coordinates \((z_1, z_2)\) in a neighbourhood of \( x_0 \) such that, \((z_1, z_2)(x_0) = (0, 0)\) and \( f = f(x_0) + z_1^2 + z_2^2 \). A topological Lefschetz fibration is called simple if for any critical value \( x \in \mathbb{C} \), there is exactly one critical point in \( \pi^{-1}(x) \).

We observe that in a topological Lefschetz fibration \( \pi : X \rightarrow \mathbb{C} \), if a fiber \( \pi^{-1}(x) \) has multiple critical point, then we can choose non-intersecting vanishing cycles corresponding to each of the critical points.
Definition 4.6. (Equivalence of Lefschetz fibrations) Suppose \( \pi_1 : X_1 \to \Sigma_1 \) and \( \pi_2 : X_2 \to \Sigma_2 \) are Lefschetz fibrations, where \( \Sigma_1 = \Sigma_2 = \mathbb{C} \).

(a) (Path-shrinking map) We say \( \beta : \Sigma_1 \to \Sigma_2 \) is a path-shrinking map if there is a path \( \gamma : [0, 1] \to \mathbb{C} \) that is mapped to a point \( p \) by \( \beta \), and \( \beta|_{(\Sigma_1 \setminus \gamma)} \) maps diffeomorphically onto \( \Sigma_2 \setminus \{p\} \). See Figure 2.

(b) (Lifts) If \( \beta : \Sigma_1 \to \Sigma_2 \) is a diffeomorphism, a lift is a diffeomorphism \( \alpha : X_1 \to X_2 \) satisfying

\[
\pi_2 \circ \alpha = \pi_1 \circ \beta.
\]

For a path-shrinking map \( \beta : \Sigma_1 \to \Sigma_2 \), a surjective map \( \alpha : X_1 \to X_2 \) is a lift if it satisfies (19), the restriction \( \alpha|_{\pi_1^{-1}(\Sigma_1 \setminus \gamma)} \) is a diffeomorphism onto \( \pi_2^{-1}(\Sigma_2 \setminus \{p\}) \), and the singular points in \( \pi_1^{-1}(\gamma) \) are mapped bijectively to the singular points in \( \pi_2^{-1}(p) \).

Lefschetz fibrations \( \pi_1 : X_1 \to \Sigma_1 \) and \( \pi_2 : X_2 \to \Sigma_2 \) are one-step equivalent if there is a map \( \beta : \Sigma_1 \to \Sigma_2 \) that is either a diffeomorphism or a path-shrinking map, and a lift \( \alpha : X_1 \to X_2 \) of \( \beta \). The equivalence relation on Lefschetz fibrations is the transitive closure of the one-step equivalence relation.

We need to include lifts of path-shrinking maps in our notion of equivalence, only because we allow fibers to have multiple singularities. In this aspect, our treatment differs from the standard literature [8]. The Lefschetz fibrations of the standard literature are called simple Lefschetz fibrations in this paper.

Remark 4.7. (Equivalent Lefschetz fibrations are diffeomorphic) Suppose \((\beta, \alpha)\) is an equivalence between Lefschetz fibrations \( \pi_1 : X_1 \to \mathbb{C} \) and \( \pi_2 : X_2 \to \mathbb{C} \), and \( \beta : \mathbb{C} \to \mathbb{C} \) is a path-shrinking map. Then there is a diffeomorphism \( X_1 \to X_2 \) that agrees with \( \alpha \) on \( \pi_1^{-1}(\mathbb{C} \setminus U) \), where \( U \) is a neighbourhood of the shrunk path \( \gamma \).

Definition 4.8. (Lefschetz fibration data) For a Lefschetz fibration with fiber \( F \), a simple Lefschetz fibration datum \( \Phi = (Q, b, \gamma, \mu) \) consists of

(a) a singular value set \( Q \subset \mathbb{C} \),
(b) a base point \( b \in \mathbb{C} \setminus Q \),
(c) (Arc system) a collection of disjoint neighbourhoods \( \{U_q\}_{q \in Q \cup \{\infty\}} \), and a path \( \gamma_q \) in \( \mathbb{C} \setminus S \) connecting \( b \) to a point on \( \partial U_q \).
(d) (Vanishing cycles) embedded loops $\mu_q \subset F_b$ corresponding to every singular point $q \in Q$.

A Lefschetz fibration datum differs from a simple Lefschetz datum in the following ways:

(a) the singular values form a multi-set $Q$. If a singular value $q$ occurs $k > 1$ times in $Q$, then the corresponding vanishing cycles in $\mu_q \subset F_b$ are disjoint.

(b) The neighbourhood and path $(U_q \cup \gamma_q) \setminus \{b\}$ can intersect $(U_q' \cup \gamma_q') \setminus \{b\}$ if the vanishing cycles $\mu_q$ and $\mu_q'$ are disjoint.

The braid group $B_m$ acts on the space of Lefschetz data by Hurwitz moves which we now describe. In a simple Lefschetz datum $\Phi = (Q, b, \gamma, \mu)$, the arc system induces an ordering on $Q$ by going counter-clockwise starting from $\infty$. A generator $\sigma_i \in B_m$ acts on $\Phi$ by swapping two adjacent elements $q_i, q_{i+1}$ in the arc system ordering, and changing the vanishing cycle $\mu_{q_i}$ to the conjugate $t_{\mu_{q_{i+1}}}^{-1}(\mu_{q_i})$. Here $t_{\mu_{q_{i+1}}} : F_b \to F_b$ is the Dehn twist along the vanishing cycle $\mu_{q_{i+1}}$. The other loops $\mu_q$ are left unaltered. The action extends to the space of Lefschetz data, because of the non-intersecting property of loops in case of coincident critical values.

**Definition 4.9.** (Equivalence of Lefschetz data) Lefschetz data are equivalent under

(a) homotopically moving the base point, singular values (they should remain distinct), arc system and vanishing cycles,

(b) action of a non-isotopic diffeomorphism $g : F_b \to F_b$ on all the vanishing cycles,

(c) and the action of the Braid group.

**Proposition 4.10.** There is a one-one correspondence between equivalence classes of Lefschetz fibrations on $\mathbb{C}$ and equivalence classes of Lefschetz fibration data.

The proof in case of simple Lefschetz fibrations can be found in Donaldson [8], Amoros [1]. The extension to the general case follows in a straightforward way and we omit it.

One can define a family of Lefschetz fibrations (resp. simple Lefschetz fibrations) parametrized by a compact topological space $K$. The Lefschetz data for a family is given by a map $b : K \to \text{Lef}((\mathbb{R}^2) \text{ (resp. } b : K \to \text{SLef}(\mathbb{R}^2))$.

4.3. **Combinatorial type.** In Section 3.3, we showed that the filling of the foliated unit cotangent bundle of $S^1 \times (A, F_{ah})$ is a Lefschetz fibration over $\mathbb{R} \times (A, F_{ah})$ with a standard structure on the non-compact ends. The fibers are cylinders $F = \mathbb{R} \times S^1$. In this section, we give combinatorial invariants for these fibrations that provide a classification up to symplectic deformation equivalence. We show that any connected component of singular points has a constant $\mathbb{R}$-coordinate in the fiber, which we call the level of the singular component. This property of the symplectic Lefschetz fibration plays an important role in the description of the combinatorial type.

A Lefschetz fibration $\pi : X \to \mathbb{R}^2$ is compatible with a symplectic form $\omega$ on $X$, if at the singular points of the fibration, there are coordinates $(z_1, z_2)$ on the leaf so that $\omega$ is a Kähler form, and on the complement of singular points, the fibers of $\pi$ are symplectic.
We consider Lefschetz fibrations over the foliated base manifold
\[ B := \mathbb{R} \times (A^2, \mathcal{F}_{ah}), \quad p_2 : B \to \mathbb{R}, \quad q_2 : A^2 \to \mathbb{S}^1. \]
Here \( \mathcal{F}_{ah} \) is the almost horizontal foliation on the annulus \( A^2 \), \( p_2 \) is the projection to the first coordinate, and \( q_2 \) is a \( \mathbb{S}^1 \)-valued leaf coordinate. The two-form
\[ \omega_B := dq_2 \wedge dp_2 \]
is a strong symplectic form on \( B \). The regular fibers of the Lefschetz fibration are cylinders
\[ F := \{(p_1, q_1) \in \mathbb{R} \times \mathbb{R}/\mathbb{Z}\}, \quad \omega_F := dq_1 \wedge dp_1. \]
We study foliated symplectic Lefschetz fibrations \( \pi : X \to \mathbb{R} \times A \) whose structure is standard on ends. That is, there is a foliated symplectomorphism
\[ i : X \setminus K \to ((\mathbb{R} \times A) \times F, \omega_B \oplus \omega_F) \setminus K_0 \]
that lifts the identity map on the base \( \mathbb{R} \times A \), and \( K \subset X \), \( K_0 \subset \mathbb{R} \times A \times F \) are compact sets. As a consequence of standardness on ends, there are no singular fibers on the boundary \( \mathbb{R} \times \partial A \) – the reason is as in the second half of the proof of Proposition 3.15, and is a monodromy calculation.

In this section, we show that the level of a singular point can be meaningfully defined as the \( \mathbb{R} \)-coordinate of the vanishing cycle. We also prove that the level is constant on a connected component of singular points.

Parallel transport can be defined on the foliated symplectic fibration \( X \to B \) using the closed two-form \( \omega \in \Omega^2(X) \). The form \( \omega \) induces a splitting of \( T_x X \) for any \( x \in X \) as
\[ T_x X = TF_x \oplus H_x, \quad H_x := \{\xi \in T_x M : \omega(\xi, \tau) = 0 \forall \tau \in TF_x\}. \]
On a path \( \gamma : [0, 1] \to B \) not passing through singular values of the fibration, the resulting parallel transport map \( F_{\gamma(0)} \to F_{\gamma(1)} \) is a symplectomorphism.

**Lemma 4.11.** Suppose \( b \in B \) and \( F_b \) is a fiber of \( X \to B \) for which the \( p_1 \) coordinate is well-defined in \( \{p_1 > a_+\} \cup \{p_1 < a_-\} \) for some constants \( a_- \leq a_+ \). Then, the compact region of \( F_b \) between the levels \( a_+ \), \( a_- \) has volume \( a_+ - a_- \).

**Proof.** Take a point \( b_0 \in B \) for which the fiber \( F_{b_0} \subset X \setminus K \) and thus has the standard form, and the \( p_1 \) coordinate is well-defined for all of \( F_{b_0} \). If \( F_b \) is a regular fiber, the lemma is proved by parallel transporting along a path from \( b_0 \) to \( b \) not passing through a singular point. Parallel transport is also well-defined on regions of the singular fiber away from the singular point, and so the lemma can be proved in a similar way. \( \square \)

**Definition 4.12.** (Level of a singular point) Suppose \( x \) is a singular point of the Lefschetz fibration \( \pi : X \to B \), and \( b := \pi(x) \). Assume \( x \) is the only singular point in the fiber \( F_b \), and so, the fiber consists of two disks – an upper disk \( F_b^\uparrow \) and a lower disk \( F_b^\downarrow \). Suppose the \( p_1 \) coordinate is well-defined on \( F_b \) for \( p_1 > a_+ \) and \( p_1 < a_- \). Then, the level of the singular point \( x \) is
\[ \text{level}(x) := a_+ - \int_{F_b^\downarrow \cap \{p_1 \leq a_+\}} \omega. \]
As a consequence of Lemma 4.11, the level of $x$ can equivalently be defined as

$$\text{level}(x) := a - \int_{F_b \cap \{p_1 \geq a\}} \omega.$$  

In case of multiple singular points on a fiber, the definition of level can be extended in an obvious way.

**Remark 4.13.** Using Lemma 4.11, it is possible to extend the definition of $p_1$ to (22)

$$p_1 : X \to \mathbb{R}$$

such that for any fiber $F_b$, and constants $b_\leq b_+$,

$$\int_{F_b \cap \{b_\leq p_1 \leq b_+\}} \omega_F = b_+ - b_-,$$

and on a singular fiber, the level sets of $p_1$ are connected. As a result, on a singular fiber, a level set of $p_1$ is either a circle contained in one of the irreducible components or a singular point. The $p_1$-coordinate of a singular point is equal to its level. The extension of $p_1$ is uniquely defined only up till Hamiltonian diffeomorphism of the fiber.

The next lemma shows that parallel transport preserves the $p_1$-coordinate up to Hamiltonian diffeomorphism of the fiber.

**Lemma 4.14.** Suppose $F_{b_0}, F_{b_1}$ are regular fibers of $X \to B$ over $b_0, b_1 \in B$. Let $\Phi^\gamma : F_{b_0} \to F_{b_1}$ be the parallel transport map along a path $\gamma$ in $B$ from $b_0$ to $b_1$. Then, there is a Hamiltonian diffeomorphism of $F_{b_1}$ that maps the loop $\Phi^\gamma(\{p_1 = a\} \cap F_{b_0})$ to $\{p_1 = a\} \cap F_{b_1}$.

**Proof.** (Proof of Lemma 4.14) For the proof of the lemma, we define an offset function along a path – it measures the amount by which parallel transport changes the $p_1$ coordinate. Suppose $\gamma_0 : [0, 1] \to B$ is a path of regular values, with a parallel transport map $\Phi^\gamma : F_{\gamma(0)} \to F_{\gamma(1)}$. The offset function $s^\gamma : [0, 1] \to \mathbb{R}$ is defined by the following condition: for any $a \in \mathbb{R}$ the loop $\Phi^\gamma(\{p_1 = a\})$ is Hamilton isotopic to $\{p_1 = a + s^\gamma(t)\} \subset F_{\gamma(t)}$. The value of $s^\gamma(t)$ does not depend on $a$ because parallel transport preserves volume on the fiber.

If the path $\gamma$ is in a leaf, then $s^\gamma \equiv 0$. Indeed, the symplectic form on the leaf is standard on the cylindrical ends, and parallel transport by the standard symplectic form preserves the $p_1$-coordinate in the ends. This is not true if $\gamma$ does not lie on a leaf because the identification $i$ on the non-compact ends is only a foliated symplectomorphism, and not an equivalence of strong symplectic two-forms.

The offset function is preserved by foliated homotopy of paths. Two paths $\gamma_0$ and $\gamma_1$ in $B$ are related by *foliated homotopy* if there exists a smooth $\Gamma : [0, 1] \times [0, 1] \to B$ such that $\Gamma(0, \cdot) = \gamma_0, \Gamma(1, \cdot) = \gamma_1$ and for any $\tau$, the image of $\Gamma(\cdot, \tau)$ is contained in a leaf. For such a pair of paths, $s^{\gamma_0} = s^{\gamma_1}$ – this is because parallel transport map induced by a contractible loop is a Hamiltonian isotopy, see Theorem 6.21 in [22], and the offset function is constant along a leaf.

We now prove the lemma by showing that the offset function vanishes for any path. Because of the homotopy-invariance of the offset function, it is enough to describe a
path $\gamma$ in the base $(A \times [0, 1], F_{\text{ah}})$ by its projection to the annulus. Consider a path $\gamma : [0, 1] \to A$ such that $\gamma(0)$ is in the compact leaf $\partial_+ A$ and the rest of the path maps to non-compact leaves. For any small $\epsilon_0 > 0$, the path $\gamma$ is foliated homotopy equivalent to a path $\gamma' : [0, 1] \to A$, such that $\gamma'$ is a reparametrization of $\gamma|[0, \epsilon]$ for some $\epsilon < \epsilon_0$. Since the offset function is continuous, and $s \gamma = s \gamma'$, we conclude that $s \gamma \equiv 0$. Any path in $A$ that does not intersect the inner leaf $\partial_- A$ is foliated homotopy equivalent to a subset of the path $\gamma$ or $-\gamma$. Therefore, the offset function vanishes along any path, concluding the proof of the lemma.

Lemma 4.15. On any connected component of singular points, the level is constant.

Proof. Let $\gamma : [0, 1] \to B$ be a path of singular values of the Lefschetz fibration. We assume that the fibers $F_{\gamma(t)}$ has one singular point, the proof of the general case is similar. Each singular fiber consists of two disks. Let $F_{\gamma(t)}^u$ be the upper disk. Then, $F_{\gamma(t)}^u := \cup_t F_{\gamma(t)}^u$ is a disk bundle over $[0, 1]$. Suppose $\Phi_t : F_{\gamma(0)}^u \to F_{\gamma(t)}^u$ is the parallel transport map in $F_{\gamma(t)}^u$ in the direction $\ker(\omega|F_{\gamma(t)})$. Suppose the level set $\{p_1 = a\}$ is a loop in each of the disks $F_{\gamma(t)}^u$, which we denote by $\mu_{t,a}$. Then, by Lemma 4.14, $\Phi_t(\mu_{0,a})$ is Hamilton-equivalent to the loop $\mu_{t,a}$ in the fiber $F_{\gamma(t)}$. So, both loops enclose an equal area in the disk $F_{\gamma(t)}$. Since, parallel transport preserves volume in the fiber, we can conclude that the quantity $\int_{F_{\gamma(t)}^u \cap \{p_1 \leq a\}} \omega$ is $t$-independent. So, level$(\gamma(t))$ is $t$-independent.

We now describe the data associated to a symplectic Lefschetz fibration $X \to B$ with a standard structure on ends. When the fibration is restricted to the interior leaves of $\mathbb{R} \times A$, it is an $S^1$-family of Lefschetz fibrations. The mapping class group of the fiber is $\mathbb{Z}$, which is Abelian. Therefore, Hurwitz moves do not alter the vanishing cycles. As a result, the base and arc system information in the topological Lefschetz fibration datum are not relevant. The vanishing cycle is same for all singular points. However, in the symplectic case, we additionally keep track of the level data.

To each boundary leaf $\pi^{-1}(\mathbb{R} \times \partial_{\pm} A)$, we associate an integer $k_{\pm}$, which is the Dehn twist of the fiber along a line $\{c\} \times \mathbb{R}$, $c \in \partial_{\pm} A$. This quantity is indeed meaningful, because via the map $i$, the fibers at the ends of the line $\{c\} \times \mathbb{R}$ have fixed identifications to $F$.

Suppose $\Gamma$ is the set of connected components of singular points in the fibration $X \to B$. Each component $C_j$, $j \in \Gamma$ projects to a loop $\gamma_j$ in the base $\mathbb{R} \times A$. The loop $\gamma_j$ does not have self-intersections because the level is constant on the singular value component $C_j$. Let $\text{wind}(\gamma_j)$ be the winding number of $\gamma_j$ in $\mathbb{R} \times A$. Then, by a holonomy calculation,

\begin{equation}
(23) \quad k_+ - k_- = \sum_{j \in \Gamma} \text{wind}(\gamma_j).
\end{equation}

Definition 4.16. (Symplectic Lefschetz datum) The \textit{symplectic Lefschetz datum} for a Lefschetz fibration $X \to B$ that is standard on ends consists of

(a) a finite collection of embedded loops $\{\gamma_j : S^1 \to B\}_{j \in \Gamma}$;

(b) Dehn twists $k_{\pm} \in \mathbb{Z}$ on the boundary leaves satisfying (23);
(c) level $\gamma_j \in \mathbb{R}$ for each loop $j \in \Gamma$. If two loops $\gamma_{j_1}, \gamma_{j_2}$ intersect in $\mathbb{R} \times A$, then, their levels cannot coincide: $\text{level} \gamma_{j_1} \neq \text{level} \gamma_{j_2}$.

Two embedded loops $\gamma_0, \gamma_1 : S^1 \to \mathbb{R} \times A$ have the same braid type if they are connected by a homotopy of embedded loops. Two loops with the same braid type necessarily have the same winding number.

**Definition 4.17.** (Combinatorial type) The combinatorial type of a Lefschetz fibration $\pi : X \to B$ is

(a) the braid type of each component of singular points, and

(b) the Dehn twists $k_\pm$ on the boundary leaves.

**Proposition 4.18.** Suppose $\Phi_0, \Phi_1$ are symplectic Lefschetz data for fibrations on $\mathbb{R} \times (A, \mathcal{F}_{ah})$ and whose regular fibers are cylinders. Suppose $\Phi_0, \Phi_1$ have the same combinatorial type. Then, there is a $[0, 1]$-family $\Phi_t$ of symplectic Lefschetz data that connects $\Phi_0$ to $\Phi_1$.

**Proof.** The proof is trivial in the case the symplectic Lefschetz data are homotopic. The reason such a homotopy may not exist is if the linking of loops $\{\gamma^0_j\}_{j \in \Gamma}$ of singular values in $\Phi_0$ is different from the linking of $\{\gamma^1_j\}_{j \in \Gamma}$ in $\Phi_1$. In that case, we first construct a homotopy from the loops $\{\gamma^0_j\}_{j \in \Gamma}$ to $\{\gamma^1_j\}_{j \in \Gamma}$, allowing intersections between distinct loops for isolated values of $t$. Then, we produce a homotopy of the level data in a way that if $\gamma^t_{j_1}$ intersects $\gamma^t_{j_2}$, then, level$^t_{j_1} \neq \text{level}^t_{j_2}$. □

**Remark 4.19.** The linking between the singular value loops is not preserved in a homotopy of symplectic Lefschetz data. However, the braid type of a single loop is preserved. This is because the level is constant on a braid type.

### 4.4. Symplectic forms on Lefschetz fibrations

The question of existence and uniqueness of compatible symplectic forms on Lefschetz fibrations has been addressed by Gompf [15]. However, our situation differs from the standard treatment firstly because of non-compact fibers. The second point of difference is that the $S^1$-family of base manifolds $\mathbb{R}^2$ are leaves of the almost horizontal foliation, which imposes a certain condition on the non-compact ends. In this section, we construct a symplectic Lefschetz fibration corresponding to any given symplectic Lefschetz data. We also show that Lefschetz fibrations with the same combinatorial data are symplectic deformation equivalent.

**Proposition 4.20.** (Existence of symplectic form on Lefschetz fibrations) Corresponding to any symplectic Lefschetz fibration $X$ on $B := (A \times \mathbb{R}, \mathcal{F}_{ah}, \omega_B)$, with fiber $F$, and which is standard on ends (see (20)).

**Proof.** We first construct a smooth fibration $X$ corresponding to the given data, that is equipped with a diffeomorphic identification of non-compact ends as in (20). We start with the trivial $F$-bundles on $\{|p_1| > R\}$, for a large $R$. On the boundary $\mathbb{R} \times \partial_2 A$, we glue in the bundle $[-R, R] \times S^1 \times F$, where the gluing is trivial along $\{-R\} \times S^1$, and with $k_\pm$-Dehn twist at $\{R\} \times S^1$. The side boundaries can be thickened so that the bundle is now defined on $(\mathbb{R} \times A) \setminus (S^1 \times B)$. Here $B \subset \mathbb{R}^2$ is
a large ball, $S^1 \times B$ has the product foliation, it does not intersect the boundary of $\mathbb{R} \times A$, and it contains all the singular points of the Lefschetz data. By a parametric version of Proposition 4.10, the fibration $X|_{S^1 \times B}$ can be constructed as prescribed by the data. Finally this can be glued in to the fibration on the complement $(\mathbb{R} \times A) \setminus (S^1 \times B)$, because the monodromies match.

We adapt Gompf’s method [15, Theorem 10.2.18] to construct a foliated symplectic form. In that method, one needs a closed two-form which represents the right cohomology class on fibers. In our set-up this closed two-form $\zeta \in \Omega^2(X)$ is required to satisfy the following conditions:

(a) on any fiber, the form $\zeta - i^*\omega_0$ vanishes outside a compact set.
(b) For any regular fiber $F$, there is a constant $a_0$ such that for all $a \geq a_0$,

$$\int_{F \cap \{p_1 \leq a\}} \zeta = 2a.$$

(c) On a singular fiber $F$ with singularities at levels $l_1 < \cdots < l_k$ and components $F_0, \ldots, F_k$ (here $F_i$ is compact if $1 \leq i \leq k - 1$), we require

$$\int_{F_i} \zeta = l_i - l_{i-1}, \quad 1 \leq i \leq k - 1,$n
$$\int_{F_0 \cap \{p_1 \geq a\}} \zeta = l_1 + a, \quad \int_{F_k \cap \{p_1 \leq a\}} \zeta = a - l_k,$$n

for any sufficiently large $a > 0$.

Let $F_K \subset F$ be a compact subset such that $\mathbb{R} \times A \times (F \setminus F_K)$ is contained in $(\mathbb{R} \times A \times F) \setminus K_0$. Choose $\zeta_F \in \Omega^2(F)$ satisfying the following conditions: (a) $\zeta_F = 0$ on $F_K$, (b) $\omega_F - \zeta_F$ is compactly supported and $\int_{F} \omega_F - \zeta_F = 0$. The two-form $\zeta_{pre} := i^*\pi_2^*\zeta_F$ satisfies the first two necessary conditions mentioned above. We add a compactly supported exact form to $\zeta_{pre}$ in order to satisfy the third condition. We demonstrate this in case $\gamma$ is a loop of singular points in $X$, and each of the singular fibers contains only one singularity at level $l$. Let $\gamma_B := \pi(\gamma)$ be the projection of the loop, and let $\mathcal{O}_B(\gamma_B) \subset \mathbb{R} \times A$ be a neighbourhood. Let $A_l := B_{2\epsilon}(l) \setminus B_{\epsilon}(l) \subset \mathbb{R}$ be a small annulus centered at $l$. The fibration $\pi^{-1}(\mathcal{O}_B(\gamma_B)) \cap \{p_1 \in A_l\}$ is trivial, and is diffeomorphic to $\mathcal{O}_B(\gamma_B) \times (A_l \times S^1)$. There exists an exact two-form $d\eta_F$ compactly supported in $A_l \times S^1$ such that $\zeta_{pre} + d\eta_F$ satisfies the condition (c) for the singular fibers in $\gamma$. Next, define a cut-off function $\eta_B : \mathcal{O}_B(\gamma_B) \to [0, 1]$ that is 1 in a neighbourhood of $\gamma_B$. The form $d(\eta_B \eta_F)$ extends by zero to all of $X$. By applying similar adjustments for all loops of singular points, we obtain the required two-form $\zeta$.

Next, we describe a covering $\cup_{\alpha} U_\alpha$ of the base $\mathbb{R} \times A$, and a symplectic form $\omega_\alpha$ on $\pi^{-1}(U_\alpha)$.

**Type 1:** Two of the sets in the covering are $\{\pm p_2 > R\} \subset \mathbb{R} \times A$. On these, the symplectic form is just $i^*\omega_0$.

**Type 2:** Amoros ([1]) constructs a symplectic form in a small neighbourhood of a singular fiber of a Lefschetz fibration. This procedure can be replicated in a
family (parametrized by an interval) to yield symplectic forms on neighbourhoods of singular fibers that agree with $i^*\omega_0$ at the ends of the fibers.

Type 3: The rest of the base space can be covered by contractible open sets. On these sets the trivialization on the ends of the fiber (20) can be extended to a trivialization of the whole fiber. Thus on each of the sets, we can produce a foliated symplectic form agreeing with $i^*\omega_0$ on the ends.

Next, we glue the forms together. For this, we show that in every chart $U_\alpha$, there is a compactly supported one-form $\eta_\alpha \in \Omega^1_c(\pi^{-1}(U_\alpha))$ such that $\omega_\alpha - \zeta = d\eta_\alpha$ on $\pi^{-1}(U_\alpha)$. Here we assume $U_\alpha$ is closed so that the 'compact support' condition only concerns the fiber direction. Firstly, observe that the forms $\eta_\alpha$ supported by construction. If $\eta_\alpha$ is of type 2, we observe that $\omega_\alpha - \zeta$ is supported function that is 1 on $[0,1]$ and $\omega_\alpha - \zeta$ is closed so that the 'compact support' condition only concerns the fiber direction. Firstly, observe that the forms $\omega_\alpha - \zeta$ are compactly supported by construction. If $U_\alpha$ is of type 1, then, $\omega_\alpha - \zeta$ is the pullback of the form $\omega_F - \zeta_F \in \Omega^2_c(F)$. Since this form integrates to zero, it is zero in $H^2_c(F)$. Therefore, there is a one-form $\eta_F \in \Omega^1_c(U_\alpha)$ such that $d\eta_F = \omega_F - \zeta_F$ which can be pulled back to $\pi^{-1}(U_\alpha)$. If $U_\alpha$ is of type 2, we observe that $\pi^{-1}(U_\alpha)$ is contractible. By Poincaré duality $H^2_c(\pi^{-1}(U_\alpha)) = 0$, and so, $\omega_\alpha - \zeta$ has a primitive. If $U_\alpha$ is of type 3, we claim that $[\omega_\alpha - \zeta] = 0$ in $H^2_c(\pi^{-1}(U_\alpha))$. This is because the forms integrate to zero on the fiber, and since $U_\alpha$ is compact and contractible, $H^2_c(F \times U_\alpha) \cong H^2_c(F) \otimes H^0(U_\alpha) \cong H^2_c(F)$.

With the primitives in hand, we define

$$\omega_{\text{pre}} := \zeta + \sum \alpha d(\rho_\alpha \eta_\alpha),$$

where $\eta_\alpha : \mathbb{R} \times A \to [0,1]$ is a partition of unity. This form agrees with $i^*\omega$ outside a compact set, and it is symplectic on fibers. Indeed, it is equal to $\omega_\alpha$ on fibers as $d\rho_\alpha$ vanishes in the fiber direction.

It remains to modify the two-form $\omega_{\text{pre}}$ to make it non-degenerate in the base direction. Unlike the compact case, we cannot add a large multiple of a symplectic form on the base. Instead, we carry out an inflation argument, wherein we add a base area form where needed and adjust it elsewhere so that the form is not disturbed outside a compact set. We assume that the compact set $K$ (from (20)) is $\{|p_1| \leq R\} \cap \{|p_2| \leq R\}$. Recall that $\omega_B = dq_1 \wedge dp_2$. Let $\eta_1 : \mathbb{R} \to [-c_1,1]$ be a compactly supported function that is 1 on $[-R,R]$, and $\int_{-R}^R \eta_1(s)ds = 0$. The constant $c_1 > 0$ is small and is to be determined. Then, there is a compactly supported function $f : \mathbb{R} \to \mathbb{R}$ such that $df = \eta_1(p_1)dp_1$. Secondly, define a compactly supported cut-off function $\eta_2 : \mathbb{R} \to [0,1]$ that is 1 on $[-R,R]$, and $|\eta_2| < c_2$ where $c_2$ is a small constant that will be determined later. Define a form

$$\omega := \omega_{\text{pre}} + Cdq_1 \wedge d(\eta_2(p_2)f(p_1)) = \omega_{\text{pre}} + Cdq_1 \wedge (f \eta_2 + \eta_1(p_1)\eta_2(p_2)dp_1).$$

The form $\omega - \omega_{\text{pre}}$ is compactly supported. We now determine the constants $C$, $c_1$ and $c_2$ so that $\omega$ is symplectic. The constant $C$ is chosen large enough so that $\omega_{\text{pre}} + C\omega_B$ is symplectic on the compact set $\{|p_1| \leq R, |p_2| \leq R\}$. This can be done because $\omega_{\text{pre}}$ is symplectic in the fiber direction. The constants $c_1$, $c_2$ are chosen so as not to disturb the non-degeneracy of the standard symplectic form on $X \setminus K$. In particular, we choose $c_1 < \frac{1}{10C}$ to control the last term in (24). This fixes $f$. Then, we choose $c_2 < 1/(10C\|f\|_{L^\infty})$ to control the second to last term in (24).

\[\square\]

Remark 4.21. (Parametric existence result) For a $[0,1]$-family of symplectic Lefschetz data, there is a family $\{X_t\}_{t \in [0,1]}$ of Lefschetz fibrations, where each element of the
family is standard in the non-compact ends. The proof is the same as the proof of Proposition 4.20.

**Proposition 4.22.** (Uniqueness of symplectic form on Lefschetz fibrations) Suppose \( \pi_k : (X_k, \omega_k) \to (\mathbb{R} \times A, F_{\text{ab}}) \) \((k = 0, 1)\) are simple symplectic Lefschetz fibrations with fiber \( F = \mathbb{R} \times S^1 \) and identification of ends \( i_k \), and of the same combinatorial type. Then, they are symplectic deformation equivalent. That is, there is a diffeomorphism \( \phi : X_0 \to X_1 \) that satisfies

(a) \( i_1 \circ \phi = i_0 \) wherever \( i_0 \) and \( i_1 \) are defined.

(b) There is a family of symplectic forms \( \{\omega_t : t \in [0, 1]\} \) on \( X_0 \) such that \( \omega_0 = \omega_0, \omega_1 = \phi^* \omega_1 \), and \( \omega_t|_\mathcal{F} \) is \( t \)-independent outside a compact subset of \( X_0 \).

**Proof.** We first prove the result assuming that \( X_0 \) and \( X_1 \) have the same symplectic Lefschetz data. In that case, there is a diffeomorphism \( \phi : X_0 \to X_1 \) satisfying the first condition in the proposition and \( \pi_1 \circ \phi = \pi_0 \). By a parametric Moser argument, we can deform \( \phi \), so that it is additionally a symplectomorphism on fibers. We remark that to apply the Moser argument on singular fibers, we use the fact that the levels of the singular values match in \( X_0 \) and \( X_1 \).

We now use an inflation argument to complete the proof. Suppose \( C > 0 \) is large enough that

\[
(1 - t)\omega_0 + t\phi^* \omega_1 + C\omega_B
\]

is symplectic for all \( t \). Then, we choose the constants \( c_1, c_2 \) and functions \( \eta_1, \eta_2, f : \mathbb{R} \to \mathbb{R} \) as in the proof of Proposition 4.20 so that

\[
(1 - t)\omega_0 + t\phi^* \omega_1 + Cd\eta_1 \wedge d(f(p_1)\eta_2(p_2))
\]

is symplectic for all \( t \in [0, 1] \). Further, we observe that the forms

\[
\omega_0 + td\eta_1 \wedge d(f(p_1)\eta_2(p_2)), \quad \phi^* \omega_1 + td\eta_1 \wedge d(f(p_1)\eta_2(p_2))
\]

are symplectic for all \( t \in [0, C] \). Thus, we have produced a path of symplectic forms connecting \( \omega_0 \) and \( \phi^* \omega_1 \).

Next, we consider the case that \( X_0 \) and \( X_1 \) have the same combinatorial type. By Proposition 4.18, there is a \([0, 1]\)-family \( \Phi_t \) of symplectic Lefschetz data whose end-points are the data of \( X_0 \) and \( X_1 \). By the parametric version of the existence result (see Remark 4.21), there is a family \( (X_t', \omega_t') \to B \) of symplectic Lefschetz fibrations that are standard on ends whose Lefschetz data is \( \{\Phi_t\}_t \). There is a sequence of diffeomorphisms \( \phi_t' : X_0' \to X_1' \) that respect the identification in the ends. The family of symplectic forms \( (\phi_t')^* \omega_t' \) on \( X_0' \) provides a symplectic deformation equivalence between \( (X_0, \omega_0) \) and \( (X_1, \omega_1) \). By the previous two paragraphs, for \( i = 0, 1 \), the Lefschetz fibrations \( (X_i, \omega_i) \) and \( (X_i', \omega_i') \) are symplectic deformation equivalent. Composing the three equivalences, the proposition is proved. \( \square \)

**Remark 4.23.** In the hypothesis of Proposition 4.22, it is assumed that the diffeomorphism \( \phi : X_0 \to X_1 \) is specified on the cylindrical ends. However, the proof also works if \( \phi \) is specified in a larger region. In particular, let \( K_0 \subset X_0, K_1 \subset X_1 \) be compact subsets, and let \( \phi : X_0 \setminus K_0 \to X_1 \setminus K_1 \) be a foliated symplectomorphism. If \( \phi \) extends to a diffeomorphism satisfying \( \pi_1 \circ \phi = \pi_0 \), then the diffeomorphism is a
symplectic deformation equivalence. The proof of Proposition 4.22 carries over to this case.

**Proof of Theorem 3.** Suppose \( W \) is a filling of \( SFT^*(S^1 \times (A, F_{ah})) \), and let \( W^\infty \) be the manifold obtained by attaching infinite ends. There is a Lefschetz fibration \( W^\infty \to \mathbb{R} \times (A, F_{ah}) \) – this is a consequence of Propositions 3.10, 3.12, 3.15 and 3.16. Lefschetz fibrations with a standard structure on ends are classified up to symplectic deformation equivalence – this is a consequence of Propositions 4.20 and 4.22. \( \square \)

### 4.5. A cohomological invariant for fillings of \( \mathbb{T}^3 \)

Having finished the proof of Theorem 3 in the previous section, we discuss a cohomological invariant of fillings of \( \mathbb{T}^3 \), and some foliated fillings. Using this invariant, Wendl’s result on the filling of \( \mathbb{T}^3 \) can be strengthened. The stronger version of the result is necessary for the proof of the result on fillings in the Reeb case.

An extended filling \( W^\infty \) of \( \mathbb{T}^3 \) is symplectic deformation equivalent to a the cotangent bundle modulo a Luttinger twist. This is proved by Wendl, see Proposition 4.2 above. Alternately, our proof (in particular Proposition 4.22) in the almost horizontally foliated case can be adapted to arrive at the same conclusion. This result can be strengthened by adding a compactly supported two-form to the standard symplectic form of \( T^* \mathbb{T}^2 \). The main observation is that if two forms \( \omega_0 \) and \( \omega_1 \) are equal outside a compact set, then, there difference \( \omega_1 - \omega_0 \) is in the class \( H^2_c(T^* \mathbb{T}^2) \), and by Poincaré lemma,

\[
H^2_c(T^* \mathbb{T}^2, \mathbb{R}) \simeq H^0(\mathbb{T}^2, \mathbb{R}) \simeq \mathbb{R}.
\]

For any \( \kappa \in \mathbb{R} \), viewed as a class in \( H^2_c(T^* \mathbb{T}^2, \mathbb{R}) \), a representative is constructed as follows. Let \((p_1, p_2) : T^* \mathbb{T}^2 \to \mathbb{R}^2\) be the coordinates on the cotangent fibers, and let \( \eta \in \Omega^2_c(\mathbb{R}^2) \) be a compactly supported form that integrates to 1. Then,

\[
(27) \quad \omega_\kappa := \omega_0 + \kappa \eta(p_1, p_2)
\]

is a symplectic form on \( T^* \mathbb{T}^2 \) if we assume \( \eta \) is supported in a region where the symplectic form \( \omega_0 \) is standard.

**Proposition 4.24.** Suppose \( W \) is a filling of \( \mathbb{T}^3 \). There exist a Luttinger constant \((k, k_0, c)\) and a constant \( \kappa \in \mathbb{R} \) so that there is a symplectomorphism\n
\[ \phi : (\text{Lutt}_\sigma(T^* \mathbb{T}^2), \omega_\sigma + \kappa \eta(p_1, p_2)) \to W^\infty, \]

that is the identity map on \( \mathbb{T}^3 \times [R, \infty) \) for a large \( R \). Here, \( \eta \in \Omega^2_c(\mathbb{R}^2) \) is a two-form that integrates to 1 and is supported in \( \{c < |p_1|, |p_2|\} \).

**Proof.** By Wendl’s result Proposition 4.4, there is a diffeomorphism\n
\[ \psi : (\text{Lutt}_\sigma(T^* \mathbb{T}^2), \omega_\sigma) \to (W^\infty, \omega) \]

that is identity on \( [R_1, \infty) \times \mathbb{T}^3 \) for a large \( R_1 \) and such that \((1 - t)\psi^* \omega + \omega_\sigma \) is a symplectic form for all \( t \in [0, 1] \). This statement continues to be true if \( \omega_\sigma \) is replaced by \( \omega_{\sigma, \kappa} := \omega_\sigma + \kappa \eta(p_1, p_2) \), where \( \eta \in \Omega^2_c(\mathbb{R}^2) \) is chosen so that the form \( \eta(p_1, p_2) \) is supported in \( \mathbb{T}^3 \times [R_1, \infty) \). We choose \( \kappa \) so that \( \omega_{\sigma, \kappa} - \psi^* \omega \) is trivial in \( H^2_c(T^* \mathbb{T}^2) \). Then, cohomology class is preserved in the family \((1 - t)\psi^* \omega + \omega_{\sigma, \kappa}, \) and the symplectomorphism is obtained via Moser’s theorem. \( \square \)
Remark 4.25. We can also add a two-form $\kappa \eta$ to strong symplectic forms on fillings of foliated unit cotangent bundles such as $S(FT^*(S^3, F_{\text{freeb}}))$ and $S(FT^*((A^2, F_{\text{ah}}) \times S^1))$. In both cases the foliated cotangent bundles are trivial. Since the filling $(W^\infty, \omega)$ can be identified symplectomorphically to the cotangent bundle in the complement of a compact set $K$, the coordinate $(p_1, p_2) : W^\infty \setminus K \to \mathbb{R}^2$ is well-defined. If the form $\eta$ is supported outside a large enough ball in $\mathbb{R}^2$, the form $\omega + \kappa \eta(p_1, p_2)$ is strong symplectic.

5. Fillings in the Reeb case

5.1. Compact leaf bounds a compact leaf.

Proof of Theorem 2. For each of the three-manifolds in this theorem, the filling is foliated by holomorphic curves. See Eliashberg [11] for $S^3$, Hind [19] for $\mathbb{RP}^3$, Hind [18] for Lens spaces $L(p, 1)$ and Wendl [28] for $T^3$. The holomorphic curves in the filling intersect the boundary $L_M$, and have an energy bound. The result is proved by showing that given an energy bound, the image of the curve is contained in a fixed diameter of the boundary. This is a consequence of applying the monotonicity result Proposition 3.6 to the leaf $L$. This result is applicable on $L$ because there is an upper bound on its curvature, and a positive lower bound on the injectivity radius. This is because the leaf $L$ is a complete submanifold with bounded second fundamental form in a compact manifold $M$. \hfill $\Box$

Next, we prove corollary 1.1 from the introduction.

Proof of Corollary 1.1. Denote by $C_\partial$ the subset of points of $W$ which belong to closed leaves which intersect the boundary. It is a closed non-empty subset of $W$ ([6], Theorem 6.1.1). Assume, by contradiction, that $C_\partial$ is not open. Then, there is a sequence of points $p_n \notin C_\partial$ in the interior of $W$ which converge to a point $p_\infty \in L \subset C_\delta$. We can assume that the points lie on a local transverse segment $l$ over $p_\infty$. The leaf $L$ is the standard symplectic filling of one of the contact manifolds in the given list. All of them share the property $\pi_1(L, \partial L) = 0$. Therefore, the holonomy transport of $l$ through any path in $L$ is trivial unless the path touches the boundary. Therefore, for $n$ large enough, $p_n$ lies in a leaf $L_n$ that has to get arbitrarily close to the boundary, so it intersects the boundary. We conclude that $p_n \in C_\delta$, thus $C_\delta$ is open. \hfill $\Box$

Our next result is a preparation for analyzing fillings in the Reeb case.

Proposition 5.1. (Compact leaf in Reeb case has the standard filling) Suppose $W$ is a filling of the foliated unit cotangent bundle $M$ of the Reeb foliation on $S^3$. Let $L_M \simeq ST^*T^2 = T^3$ denote the compact leaf in $M$. Then, $L_M$ bounds a compact leaf $L$ in $W$. There is a constant $\kappa \in \mathbb{R}$ and a symplectomorphism on the extended filling $L^\infty \subset W^\infty$:

\begin{equation}
L^\infty \to (T^*T^2, \omega_\kappa)
\end{equation}

that is identity on $L_M \times [R, \infty)$. 


Proof. The compactness of $L$ follows from Theorem 2. The Luttinger constant for the filling of $L_M$ is $(0, 0)$ for the following reason. The leaf $L_M$ partitions $M$ into two copies $M_1, M_2$, each of which is contactomorphic to $M := S(FT^*X_{reeb})$. Applying the reasoning in the proof of Proposition 4.4 (a) on the component $M_1$, we can conclude that $k_2 = 0$. By applying the same reasoning on $M_2$, we get $k_1 = 0$. Then, by Proposition 4.24, there is a symplectomorphism as in (28) for some $\kappa \in \mathbb{R}$. □

5.2. A surgery operation. In this section we provide a surgery operation to transform the Reeb component to a 3-manifold with almost horizontal foliation. This surgery is used at the level of foliated cotangent bundles to prove Theorem 1 on the fillings of the foliated unit cotangent bundle of $(S^3, \mathcal{F}_{Reeb})$.

Remark 5.2. (From Reeb to almost horizontal) Let $X_{reeb} := (S^1 \times D^2, \mathcal{F}_{reeb})$ be the solid torus with the Reeb foliation. Let $\gamma : S^1 \rightarrow X_{reeb}$ be a closed transversal that intersects every non-compact leaf exactly once. Let $U_{reeb}$ denote the thickening of the transversal, so it is $S^1 \times D^2$ with the product foliation, see Figure 3. Let $U_{out}$ be an open set that deformation retracts to $X_{reeb} \setminus U_{reeb}$, and such that $U_{reeb} \cup U_{out} = X_{reeb}$.

The almost horizontal manifold $X_{ah} := (A^2 \times S^1, \mathcal{F}_{ah})$ also has a cover by two open sets, one of which is $U_{out}$. In particular, if $\partial_+ A$ is the outer boundary of the annulus, then a neighbourhood $Op(\partial_+ A \times S^1) \subset X_{ah}$ is diffeomorphic to $U_{out}$. The other open set, which is a neighbourhood $Op(\partial_- A \times S^1)$, which is a neighbourhood of the inner boundary, is denoted by $U_{ah}$. Therefore, $X_{ah} = U_{out} \cup U_{ah}$. There is also a diffeomorphism

$$U_\cap := U_{out} \cap U_{ah} \simeq U_{out} \cap U_{reeb}.$$ 

Starting from $X_{reeb}$, the space $X_{ah}$ can be constructed by deleting $U_{reeb} \setminus U_\cap$ and gluing in $U_{ah}$, that is

$$X_{ah} = (X_{reeb} \setminus (U_{reeb} \setminus U_\cap)) \cup U_\cap U_{ah}$$

Such a surgery can be performed at the level of cotangent bundles to transform the standard filling of $S(FT^*X_{reeb})$ to a standard filling of $S(FT^*X_{ah})$.

Using the surgery construction, we prove the following proposition, which says that any filling of the foliated cotangent bundle of $X_{Reeb}$ is standard. Since $(S^3, \mathcal{F}_{Reeb})$ is made up of two copies of $X_{Reeb}$, Theorem 1 is an easy consequence of this proposition.
Proposition 5.3. (Fillings of Reeb components are standard) Suppose $W$ is a foliated filling of $\mathcal{S}(FT^*X_{\text{reeb}})$ and $W^\infty$ is the extended filling with an identification of ends given by a foliated symplectomorphism
\begin{equation}
\phi : [R, \infty) \times \mathcal{S}(FT^*X_{\text{reeb}}) \to W^\infty
\end{equation}
for some $R > 0$. Let $L \subset W$ be the boundary leaf, and let $L^\infty \subset W^\infty$ be its extension. Suppose $\phi$ extends to a symplectomorphism
\begin{equation}
\phi : (T^*\mathbb{T}^2, \omega_\kappa) \to L^\infty.
\end{equation}
for some $\kappa \in \mathbb{R}$. Then, there exists a foliated diffeomorphism
\[ \phi : (FT^*X_{\text{reeb}}, \omega_\kappa) \to W^\infty \]
that is a symplectic deformation equivalence, and which extends both (29) and (30).

Proof. The proposition is proved by applying the surgery construction of Remark 5.2 at the level of cotangent bundles. To perform the surgery, we show that the symplectomorphism (30) on $T^*\mathbb{T}^2$ extends to a foliated symplectomorphism defined on a neighbourhood $\mathcal{O}(T^*\mathbb{T}^2)$ in $(FT^*X_{\text{reeb}})$. Then, one can take the transversal $\gamma \in X_{\text{Reeb}}$ close enough to the boundary leaf $\mathbb{T}^2$ so that $FT^*X_{\text{reeb}}|\gamma$ is contained in $\mathcal{O}(T^*\mathbb{T}^2)$. This is to ensure that the region in which we perform the surgery has the standard foliated symplectic form. We give details for this step.

Firstly, we claim that $\phi$ extends to a foliated diffeomorphism on a neighbourhood of $T^*\mathbb{T}^2$ in $FT^*X_{\text{reeb}}$. The proof of this statement is analogous to that of Theorem 2.3.9 in Candel-Conlon [6], which says that for a compact leaf in a foliated manifold, the holonomy homomorphism determines a neighbourhood of the leaf up to foliated diffeomorphism. The proof uses the compactness of leaves to cover a neighbourhood of $L$ with a finite number of foliated charts. We can still do this in our case, although the leaf $T^*\mathbb{T}^2$ is not compact, and therefore the rest of the proof carries over.

By the following claim, the map $\phi$ can be deformed in a compact subset of its domain so that it is a foliated symplectomorphism.

Claim. Suppose $\mathcal{O}(T^*\mathbb{T}^2)$ is a neighbourhood of the boundary leaf $T^*\mathbb{T}^2$ in $FT^*X_{\text{reeb}}$, and $\omega_0$, $\omega_1$ are foliated symplectic forms on $\mathcal{O}(T^*\mathbb{T}^2)$ such that $\omega_0 = \omega_1$ on $T^*\mathbb{T}^2$ and on the cylindrical end $\mathcal{O}(T^*\mathbb{T}^2) \cap ([R, \infty) \times \mathcal{S}(FT^*X_{\text{reeb}}))$. Then, there are smaller neighbourhoods $\mathcal{O}_0(T^*\mathbb{T}^2)$, $\mathcal{O}_1(T^*\mathbb{T}^2)$, and a foliated isotopy $\psi : \mathcal{O}_0(T^*\mathbb{T}^2) \to \mathcal{O}_1(T^*\mathbb{T}^2)$ that is identity on $T^*\mathbb{T}^2$ and on the cylindrical end, and satisfies $\psi^*\omega_1|_\mathbb{R} = \omega_0|_\mathbb{R}.$

Proof. This is a parametric version of Moser’s theorem, and the proof is analogous. The key step is to find a compactly supported primitive for $\omega_1 - \omega_0$. After shrinking the neighbourhood $\mathcal{O}(T^*\mathbb{T}^2)$, we assume it is a closed neighbourhood diffeomorphic to $[0, \epsilon] \times T^*\mathbb{T}^2$. By Poincaré lemma, the compactly supported cohomology group is
\[ H_c^2([0, \epsilon] \times T^*\mathbb{T}^2) = H^0([0, \epsilon] \times \mathbb{T}^2) = \mathbb{R}. \]
The cohomology class $[\omega_1 - \omega_0] \in H_c^2([0, \epsilon] \times T^*\mathbb{T}^2)$ is determined by the integral of the form on any cotangent fiber. On cotangent fibers of $\mathbb{T}^2$, the form $\omega_1 - \omega_0$ integrates to zero. Therefore, $[\omega_1 - \omega_0] = 0$ and there is a compactly supported $\alpha \in$
\[ \Omega^1([0, \epsilon] \times T^*\mathbb{T}^2) \text{ such that } \omega_1 - \omega_0 = d\alpha. \] For \( t \in [0, 1] \), let \( \omega_t := (1-t)\omega_0 + t\omega_1 \). Define a time-dependent vector field \( v_t \in \text{Vect}(\mathcal{O}p(T^*\mathbb{T}^2)) \) that maps to \( \mathcal{F} \) and satisfies \( i_v \omega_t|_\mathcal{F} = \alpha|_\mathcal{F} \). We observe that \( v_t \) vanishes on the cylindrical end and on the boundary leaf \( T^*\mathbb{T}^2 \). The required diffeomorphism \( \psi \) is the time one flow of the vector field \( v_t \), and it is well-defined on a small neighbourhood of \( T^*\mathbb{T}^2 \). This finishes the proof of the Claim. \( \square \)

We now have a foliated symplectomorphism
\[ \phi : \mathcal{O}p(T^*\mathbb{T}^2) \cup ([R, \infty) \times \mathbb{S}(FT^*X_{\text{reeb}})) \to W^\infty \]
that extends (29) and (30). In fact there is a neighbourhood \( \mathcal{O}p(\mathbb{T}^2) \subset X_{\text{reeb}} \) such that \( FT^*(\mathcal{O}p(\mathbb{T}^2)) \) is contained in the domain of \( \phi \). The surgery can now be performed. Choose the transversal \( \gamma \) (see Remark 5.2) and its neighbourhood \( U_{\text{reeb}} \) within \( \mathcal{O}p(\mathbb{T}^2) \). By applying Proposition 2.2 to the image \( \phi(FT^*U_{\text{reeb}}) \), we assume that \( \phi(FT^*U_{\text{reeb}}) \cong S^1 \times T^*\mathbb{D}^2 \) is a trivial symplectic foliation. Consequently, the surgery in Remark 5.2 can be performed at the level of cotangent bundles to produce a foliated symplectic manifold
\[ W^\infty_{ah} := W^\infty \setminus \phi(FT^*(U_{\text{reeb}} \setminus U_\gamma)) \cup \partial FT^*U_\gamma \cup FT^*U_{ah}. \]
By construction, there is a foliated symplectomorphism
\[ i_{ah} : [R, \infty) \times \mathbb{S}(FT^*X_{ah}) \to W^\infty_{ah}. \]
Therefore, \( W_{ah} := W^\infty_{ah} \setminus \text{Im}(i_{ah}) \) is a filling of \( \mathbb{S}(FT^*X_{ah}) \).

We now finish the proof of the proposition. By construction, the Luttinger constant is zero for the filling of the inner torus \( \mathbb{S}(FT^*(\partial_- A \times \mathbb{S}^1)) \subset \mathbb{S}(FT^*X_{ah}) \). Therefore, \( W^\infty_{ah} \) is symplectic deformation equivalent to \( FT^*X_{ah} \) by Theorem 3. Further, by remark 4.23, the symplectic deformation equivalence \( \phi : FT^*X_{ah} \to W^\infty_{ah} \) can be constructed so that it agrees with the symplectomorphism (30) of the outer leaf \( T^*\mathbb{T}^2 \). The proof is completed by reversing the surgery (31). \( \square \)

**Proof of Theorem 1.** Suppose \( W \) is a filling of the sphere cotangent bundle \( \mathbb{S}(FT^*(\mathbb{S}^3, \mathcal{F}_{\text{reeb}})) \), and let \( W^\infty \) be the extended filling. We recall that there is an identification of cylindrical ends given by a foliated symplectomorphism
\[ \phi : [R, \infty) \times \mathbb{S}(FT^*(\mathbb{S}^3, \mathcal{F}_{\text{reeb}})) \to W^\infty \]
By Theorem 2, the compact leaf \( \mathbb{S}T^*\mathbb{T}^2 \) of the contact foliation has a compact filling \( L \subset W \). By adding cylindrical ends to \( L \), we obtain \( L^\infty \subset W^\infty \). Proposition 5.1 implies that there is a symplectomorphism
\[ \phi : (T^*\mathbb{T}^2, \omega_\kappa) \to L^\infty \]
for some \( \kappa \in \mathbb{R} \), and the maps in (33) and (32) agree on the intersection of their domains. The filling \( W^\infty \) can be split along the leaf \( L^\infty \) to yield
\[ W^\infty = W^\infty_+ \cup_{L^\infty} W^\infty_- , \]
where both \( W^\infty_+ \), \( W^\infty_- \) are fillings of \( \mathbb{S}(FT^*X_{\text{reeb}}) \). By Proposition 5.3, the filling \( W^\infty_\pm \) is standard, and there is a foliated symplectomorphism
\[ \phi_\pm : (FT^*X_{\text{reeb}}, \omega_\kappa) \to W^\infty_\pm \]
that agrees with \( \phi \) on \( T^* T^2 \) and on the cylindrical ends. The maps \( \phi_{\pm} \) patch to yield an extension of \( \phi \):

\[
\phi : (F_T S^3_{\text{Reeb}}, \omega_k) \to \mathcal{W}^{\infty}.
\]

The patching of \( \phi_{\pm} \) is continuous, but not smooth. The map \( \phi \) can be made smooth via a perturbation that is \( C^1 \)-small in the leaves and \( C^0 \)-small in the total space, so that \( \phi \) continues to be a symplectic deformation equivalence. \( \square \)

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Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, C. Nicolás Cabrera, 13-15, 28049 Madrid, Spain.

E-mail address: fpresas@icmat.es

Institute of Mathematical Sciences, CIT Campus, Taramani, Chennai 600113, India.

E-mail address: sushmita@imsc.res.in