The outflow of granular material from orifices of thin vertical sidewalls

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Abstract. In this work we report experiments carried out to describe the features of the mass flow rate of dry cohesionless granular material that flows, only due to gravity, out from circular orifices at the sidewall of silos with thin wall thickness, w. This study shows that the classical Hagen's correlation for the mass flow rate is essentially correct for this case. Finally, we show graphically, in a three-dimensional plot, the evolution of the flow rate for continuous variations of the parameters \(D\) and \(w\).

1. Introduction
The discharge of non-cohesive granular material from the bottom exit of containers, due only to gravity action is a very interesting phenomenon because it is independent of filling depth and non-linearly dependent on the orifice size. The first fundamental proposal of a valid formula for the flow rate of grains emerging from the bottom, \(\dot{m}_0\), was developed by Gotthilf Hagen \cite{1} a German hydraulic engineer who in 1852 discovered experimentally a relationship of the form: \(\dot{m}_0 = \rho g^{1/2} D^{5/2}\), where \(\rho\) is the bulk density, \(g\) is the acceleration due to gravity and \(D\) is the diameter of the circular orifice. Notoriously, this formula constitutes the foundation of the hourglass theory \cite{2–4}. Industrially, it is the basis to estimate controlled granular flows as the dosage of powders and granules \cite{5, 6} among others.

Despite the enormous utility of Hagen's correlation, only a few studies have been conducted to test its validity in the discharge of grains through orifices on the vertical sidewalls of bins \cite{7–14}. To the best of our knowledge, Franklin and Johanson \cite{7} were the first researchers who studied experimentally the
problem of the gravity-driven lateral outflow of granular material from boxes. In their studies, they used galvanized sheet metal 0.238 cm thickness and they found that the flow rate, for orifices at the vertical sidewall, scales as \( \dot{m} = c \rho g^{1/2} D^{5/2} \) where \( c \) is a factor dependent on the angle of repose of the granular material, \( \theta_r \). In such work, no consideration of the wall thickness was invoked. Later, Bagrintsev and Koshkovskii \[8\] used oval and circular exit holes, made on the transparent plastic walls of vertical cylinders whose thicknesses were 0.1 and 0.45 cm. They found that the mass flow rate scales approximately as \( \dot{m}_0 = c \rho g^{1/2} D^{7/2} \) where now \( c \) is a parameter that depends on the geometrical features of the exit hole; they also qualitatively observed that "to obtain the greatest possible outflow capacity through a circular hole in a cylindrical tube it is necessary to use a tube of minimum allowable wall thickness". Later, Chang et al \[9\] and Davies and Foye \[10\] carried out experiments in rectangular vessels with rectangular exit slots. Chang et al did not report the dimensions of the wall thickness of their containers and Davies and Foye \[10\] have reported the use of mild steel sheets 0.12 cm thick. In spite of it, all of these authors essentially found that the mass discharge obeys the relation \( \dot{m} = \rho g^{1/2} D^{5/2} \), as Sheldon and Durian \[11\], who used circular exit holes on steel cans 0.025 cm wall thickness and square Aluminium tubing 0.03 cm wall thicknesses. Nevertheless, none of the previously referred works analyzed the effect of the wall thickness on \( m \) systematically.

Recently, we have found \[12–14\], by means of experiments, a general correlation for the flow rate of granular material through orifices in the sidewalls that involves the important effect of the wall thickness, \( w \). Such a general formula predicts that when the wall thickness is very small, Hagen's correlation must necessarily be valid. We get this result by assuming, as a case, the consideration of negligible thickness in our formula of the mass discharge from sidewalls of any thickness. Systematic experiments reported here will validate such prediction.

Experiments were performed by using polydisperse granular material because the use of a monodisperse granular material could induce instabilities in the flow \[15\].

To reach our goal, the plan of this work is as follows. First, in the next Section \( \S 2 \) we will analyse through our formula \[12-14\] the main results to estimate the mass flow rate from orifices with very thin lateral walls. Then, in Section \( \S 3 \), we are going to report experiments with discharge rates in this limit case. We found that the experiments fit a Hagen-like correlation valid when \( w \rightarrow 0 \), well. All those results will allow us, in Section \( \S 4 \), to give a geometrical and physical interpretation of the mass flow rate formula for any thickness \( w \), including the case when \( w = 0 \). Finally, in the last Section we will give the main conclusions of the study here tackled.

### 2. Negligible thickness limit

As aforementioned, the wall thickness affects the granular flow when it occurs across sidewalls, i.e., \( w \) also modulates \( \dot{m} \), the lateral outflow, provided \( D/d_g \gg 1 \) \[3\]. In Fig. 1, we depict a scheme that represents the transversal region close to a hole of size \( D \) in a vertical sidewall of the container. In Fig. 1(c) becomes evident that there is always a natural angle of wall, \( \alpha \), which can be defined as \( \alpha = \arctan(D/w) \). Meanwhile, in this same figure it can be observed that if there is no flow due to the wall thickness being wide enough, the granular material contained there will attain its angle of repose. Thus, an outflow is kept if \( \alpha > \theta_r \) (see Fig. 1(a) and 1(b)) and, conversely, the outflow should be arrested if \( \theta_r \geq \alpha \). Consequently, the mass flow rate itself must be proportional to \( (\alpha - \theta_r) \).
Figure 1. Schematic of the zone close to the vertical exit hole (transversal view): (a) very thin sidewall. There, the granular outflow is strong, (b) no negligible wall thickness, the outflow is weak and (c) no flow or arrest flow condition. The exact condition to arrest the flow is \( \alpha = \arctan(D/w) \geq \theta \) [12-14].

Another important feature to get a general relation for \( \dot{m} \) is that the mass flow rate through vertical holes is a fraction of \( \dot{m}_0 \) (the mass flow rate through bottom holes) [7-11]. Therefore, the mass flow rate dependent on \( D \) and \( w \) is a relation of the form [12-14]

\[
\dot{m} = c \dot{m}_0 \left[ \alpha - \theta_r \right] = c \dot{m}_0 \left[ \arctan \left( \frac{D}{w} \right) - \theta_r \right].
\]

(1)

where \( c \) is the dimensionless discharge parameter corresponding to the lateral outflow and \( \dot{m}_0 \) is given by the Hagen's correlation

\[
\dot{m}_0 = \rho g^{1/2} D^{5/2}.
\]

(2)

A set of experimental studies have shown that the formula (1) pretty well describes the mass flow rate of granular solids crossing a vertical wall when \( w < D \) [12-14]. A simpler formula for the lateral flow rate across thin-walled silos can be obtained from (1) expanding in powers series the function \( \arctan(D/w) \), when \( w \to 0 \), and \( D \) is a valid diameter, to first order it yields

\[
\dot{m} \approx c \dot{m}_0 \left[ \frac{\pi}{2} - \theta_r - \frac{w}{D} \right] \quad \text{for} \quad D >> w,
\]

(3)

therefore, as a direct consequence of (3), if the wall thickness is negligible (formally \( w = 0 \)) we get that

\[
\dot{m} \approx c \rho g^{1/2} D^{5/2} \left[ \frac{\pi}{2} - \theta_r \right],
\]

(4)

where we have used the relationship (2). The correlation (4) expresses that the mass flow rate for the lateral outflow essentially obeys the Hagen formula because the factor \( (\pi / 2 - \theta_r) \) is a numerical quantity, independent of \( D \) and \( w \). It confirms the experimental results reported by several authors [7, 9-11] who worked with thin face walls. Thus, the Hagen correlation results valid when the orifices are made on a very thin sidewall.

3. Experiments
To compare the behavior of the mass flow rate for both, bottom exits and lateral exits, we performed experiments with 13 cm diameter, 22 cm height and \( w = 0.0080 \pm 0.0005 \) cm thickness a cylindrical steel can. In this vessel, we drilled four circular orifices spatially distributed at the bottom and four orifices equally distributed around the silo perimeter and 8 cm above the bottom. The diameters of the holes were \( D = 0.50 \pm 0.05, 0.70 \pm 0.05, 1.10 \pm 0.05 \) and \( 1.70 \pm 0.05 \) cm. We used beach sand (mean diameter \( d_s = 0.03 \) cm, bulk density \( \rho = 1.5 \text{ gr/cm}^3 \) and angle of repose \( \theta_s = 33^\circ \pm 0.5^\circ = 0.57 \pm 0.008 \) rad) and granulated sugar (mean diameter \( d_g = 0.073 \) cm, bulk density \( \rho = 0.84 \text{ gr/cm}^3 \) and angle of repose \( \theta_s = 33.5^\circ \pm 0.5^\circ = 0.58 \pm 0.008 \) rad) as granular solids. The section of the laboratory in which the experiments were executed was climate controlled (25 \( \pm \) 1°C and 45 \( \pm \) 10% R.H.). The moisture contents of the sand and granulated sugar samples were of 0.50 \( \pm \) 0.06% and 0.015 \( \pm \) 0.005% w.b., respectively. Details of the measurement procedure of the discharge rates by using a force sensor are given elsewhere [12-14].

In the first stage of the experiments, we measured the mass flow rate of both bulk solids. In Fig. 2 we show the plot of the experimental flow rate, \( \dot{m}_{\text{Expt}} \), as a function of \( D \), for bulk materials outflowing from the bottom and lateral exits. Figure 2(a) corresponds to sand and Fig. 2(b) is for sugar. For both materials the mass flow rates behave essentially as

\[
\dot{m}_{\text{Expt}} \propto D^{5/2}.
\]

Moreover, in Fig. 3 we give plots of the respective sand and sugar experimental mass flow rates, \( \dot{m}_{\text{Expt}} \), versus \( \rho g^{1/2} D^{5/2} \) (for bottom exits) and \( \rho g^{1/2} D^{5/2} [\pi / 2 - \theta_f] \) (for lateral holes). It can be easily appreciated that the best fit for data obeys a linear relation

\[
\dot{m}_{\text{Expt}} = \begin{cases} 
  a \rho g^{1/2} D^{5/2}, & \text{for bottom exits} \\
  c \rho g^{1/2} D^{5/2} [\pi / 2 - \theta_f], & \text{for lateral exits}
\end{cases},
\]

where the dimensionless discharge coefficients have the value \( a = 0.48 \pm 0.01 \) for sand and sugar emerging from bottom exits. Similarly, the discharge coefficients for lateral outflow have approximately the same value \( c = 0.16 \pm 0.01 \) for both materials.

![Figure 2](image_url)

**Figure 2.** Plot of the experimental mass flow rate \( \dot{m}_{\text{Expt}} \) as a function of the orifice diameter, \( D \), for bottom exit holes (■) and for orifices in the sidewall (∙). (a) Beach sand and (b) granulated sugar. All data fit \( \dot{m}_{\text{Expt}} \propto D^{5/2} \), error bars are of 4%. 
Figure 3. Plots of $\dot{m}_{\text{Expt}}$ versus $\rho g^{1/2}D^{5/2}$ (for bottom exit holes and lateral exit holes) because $[\pi / 2 - \theta] = 1$ and 0.99 for sand and sugar, respectively. (a) Experiments using sand and (b) with sugar. The straight lines correspond to linear fits, errors bars are of 4%.

Incidentally, the ratio of vertical to horizontal discharge rates is around one third, which is close to values reported in experiments using agricultural grains [9]. All the latest results allow us to prove that that Hagen's correlation describe fine the mass flow rate in both configurations.

4. The geometrical approach of the mass flow rate correlation

From a geometrical point of view, equation (1), which is valid for the flow rate of lateral outflow, represents a three-dimensional surface because each one of the parameters $D$ and $w$ can be varied continuously in the plane $D-w$; by using the correlation (2) in such a formula we have that its most explicit form is given by the expression

$$\dot{m}_{\text{Expt}}(D,w) = c \rho g^{1/2}D^{5/2} \left[ \arctan \frac{D}{W} - \theta \right].$$  (7)

In Fig. 4 we show for sand ($c = 0.16 \pm 0.01$) two different views of the surface $\dot{m}_{\text{Expt}}(D,w)$ which may correspond to actual values of the mass flow rate for the couple values $(D,w)$, where necessarily $D > 6d_g$ (in order to avoid possible clogging or arrest of the granular flow [3]) and $w \geq 0$.

Fig. 4 show the general nonlinear behavior of correlation (7) and in the plot on the right side we show as a part of the surface, its intersection with the plane $w=0$, which describes the curve $\dot{m}(D,0) = c \rho g^{1/2}D^{5/2} \left[ \arctan \frac{D}{W} - \theta \right]$ given in (6) and any part of which was depicted in Fig. 2(a). In this case, there is a granular flow for any value in which $D > 6d_g$.

We also observe that the three-dimensional surface of Fig. 4 shows graphically that if $w$ and $D$ are similar the flow is weak, (low values of $\dot{m}_{\text{Expt}}(D,w)$), whereas if $D >> w$, then the granular flow is strong. Thus, the maximum flow rate occurs when $w \to 0$ (actually a very thin sidewall).

A fundamental fact of the surface is that when a the mass flow rate reaches a given value, say $\dot{m}(D,w)$, only for a specific couple of values of $(D,w)$; mathematically it means that $\dot{m}(D,w)$ is an injective function or one-to-one function that preserves distinctness: it never maps distinct elements of its domain to the same element of its codomain. Clearly, the error inherent to the experiments may imply a slight “thickness” of such a surface.
Figure 4. Three-dimensional graphic representation of the actual mass flow rate for sand \( \dot{m}_{\text{Exp}} = c \rho g^{1/2} D^{5/2} \left[ \arctan(D/w) - \theta \right] \). It is important to observe that the 3D surface is valid for any set of values \((D, w)\). In this plot, \( w \) can be null \((w = 0)\), but \( D \) must be larger than \(6d_g\). e.g. the line observed in the plane \(D-w\) corresponds to very weak flows. When \( w = 0 \) the mass flow rate now is a projection on the plane \( w = 0 \) it entails that \( \dot{m}_{\text{Exp}}(D,0) = c \rho g^{1/2} D^{5/2} \) must be a curve.

5. Conclusions
This study completes the formulation of a correlation to accurately describe the mass flow rate for lateral and circular exit holes of diameter \( D \) and wall thickness \( w \). In particular, here we demonstrated that the Hagen formula (where \( \dot{m} \propto D^{5/2} \)), is suitable to quantify the flow rate across orifices on the sidewall of containers with very thin vertical walls. We also have shown that the geometrical representation of \( \dot{m}(D, w) \) allows to be defining several important characteristics of the flow. Finally, we believe that this type of studies allows us to reach a deeper understanding of the mechanics of the discharge of non-cohesive granular materials.

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