Heavy-Quark Mass and Heavy-Meson Decay Constants from QCD Sum Rules

Wolfgang LUCHA*, Dmitri MELIKHOV* † and Silvano SIMULA**

*Institute for High Energy Physics, Austrian Academy of Sciences, Nikolsdorfgasse 18, A-1050 Vienna, Austria
†Faculty of Physics, University of Vienna, Boltzmanngasse 5, A-1090 Vienna, Austria
**INFN, Sezione di Roma III, Via della Vasca Navale 84, I-00146 Rome, Italy

Abstract. We present a sum-rule extraction of decay constants of heavy mesons from the two-point correlator of heavy-light pseudoscalar currents. Our primary concern is to control the uncertainties of the decay constants, induced by both input QCD parameters and limited accuracy of the sum-rule method. Gaining this control is possible by applying our novel procedure for the extraction of hadron observables utilizing Borel-parameter-depending dual thresholds. For the charmed mesons, we obtain $f_D = (206.2 \pm 7.3_{(\text{OPE})} \pm 5.1_{(\text{syst})})$ MeV and $f_{D_s} = (245.3 \pm 15.7_{(\text{OPE})} \pm 4.5_{(\text{syst})})$ MeV. In the case of the beauty mesons, the decay constants prove to be extremely sensitive to the exact value of the $b$-quark MS mass $m_b(m_b)$. By matching our sum-rule prediction for $f_B$ to the lattice outcomes, the very accurate $b$-mass value $m_b(m_b) = (4.245 \pm 0.025)$ GeV is found, which yields $f_B = (193.4 \pm 12.3_{(\text{OPE})} \pm 4.3_{(\text{syst})})$ MeV and $f_{B_s} = (232.5 \pm 18.0_{(\text{OPE})} \pm 2.4_{(\text{syst})})$ MeV.

Keywords: nonperturbative QCD, QCD sum rules, quark–hadron duality, continuum threshold, charmed or beauty meson, heavy-quark mass

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QUARK–HADRON DUALITY

The calculation of the decay constants $f_p$ of ground-state heavy pseudoscalar mesons $P$ by QCD sum rules [1, 2] is a complicated problem: First, a reliable operator product expansion (OPE) for the “Borelized” correlation function of two pseudoscalar heavy-light currents has to be found. Second, even if all the parameters of this OPE are known precisely, the knowledge of only a truncated OPE for the correlator allows to extract bound-state observables with only finite accuracy, reflecting an inherent uncertainty of the QCD sum-rule approach. Controlling this uncertainty constitutes a delicate problem for actual applications [3].

Recall one essential feature of the sum-rule extractions of decay constants: the quark–hadron duality assumption entails a (merely approximate) relation between hadronic ground-state contribution and OPE with the “QCD-level” correlator cut at some effective continuum threshold $s_{eff}$:

$$f_Q^2 M_Q^4 \exp(-M_Q^2 \tau) = \Pi_{\text{dual}}(\tau, s_{\text{eff}})$$

$$\equiv \int_{(m_Q+m)^2}^{s_{\text{eff}}} ds \exp(-s \tau) \rho_{\text{pert}}(s) + \Pi_{\text{power}}(\tau).$$

Here, the perturbative spectral density $\rho_{\text{pert}}(s)$ is obtained as a series expansion in powers of the strong coupling $\alpha_s$:

$$\rho_{\text{pert}}(s) = \rho^{(0)}(s) + \frac{\alpha_s}{\pi} \rho^{(1)}(s) + \frac{\alpha_s^2}{\pi^2} \rho^{(2)}(s) + \cdots.$$ 

Obviously, in order to extract a decay constant $f_Q$ one has to find a way to fix the effective continuum threshold $s_{\text{eff}}$.

A crucial albeit very trivial observation is that $s_{\text{eff}}$ must be a function of $\tau$, otherwise the l.h.s. and the r.h.s. of (1) would exhibit a different $\tau$-behaviour. The exact effective continuum threshold — corresponding to the true values of hadron mass and decay constant on the l.h.s. of (1) — is, of course, not known. Therefore, our idea of extracting hadron parameters from sum rules consists in attempting (i) to find a reliable approximation to the exact threshold $s_{\text{eff}}$ and (ii) to control the accuracy of this approximation. In a recent series of publications [4], we have constructed all the associated procedures, techniques and algorithms.

We define a dual invariant mass $M_{\text{dual}}$ and a dual decay constant $f_{\text{dual}}$ ($M_Q$ still denoting the true hadron mass) by

$$M_{\text{dual}}^2(\tau) \equiv -\frac{d}{d\tau} \log \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)),$$

$$f_{\text{dual}}^2(\tau) \equiv M_Q^{-4} \exp(M_Q^2 \tau) \Pi_{\text{dual}}(\tau, s_{\text{eff}}(\tau)).$$

In case the ground-state mass $M_Q$ is known, the deviation of the dual ground-state mass $M_{\text{dual}}$ from its actual value $M_Q$ yields an indication of the excited-state contributions picked up by our dual correlator. Assuming some specific functional shape for our effective threshold and requiring least deviation of the dual mass (2) from its actual value in the Borel window leads to a variational solution for the effective threshold. With $s_{\text{eff}}(\tau)$ at our disposal we get the decay constant from (3). The standard assumption for the effective threshold is that it is a $\tau$-independent constant. In addition to such crude approximation we also consider polynomials in $\tau$. In fact, $\tau$-dependent thresholds greatly facilitate reproducing the actual mass value. This implies that a dual correlator with $\tau$-dependent threshold isolates
the ground state much better and is less contaminated by excited states than a dual correlator with the conventional \( \tau \)-independent threshold. As consequence, the accuracies of extracted hadron observables are drastically improved. Recent experience from potential models reveals that the band of values obtained from linear, quadratic, and cubic Ansätze for the effective threshold encompasses the true value of the decay constant [4]. Moreover, we could show that the extraction procedures in quantum mechanics and in QCD are even quantitatively very similar [5]. Here, we report our results [1, 6] for heavy-meson decay constants.

### OPE AND HEAVY-QUARK MASSES

For heavy-light correlators and emerging decay constants the choice of the precise scheme adopted for defining the heavy-quark mass has a great impact. We utilize the OPE for this correlator to three-loop accuracy [7], obtained in terms of the pole mass of the heavy quark. The pole-mass scheme is standard and has been used for a long time [8]. An alternative is to reorganize the perturbative expansion in terms of the running \( \overline{\text{MS}} \) mass [9]. Since the correlator is known up to \( \mathcal{O}(\alpha_s) \), also the relation between pole and \( \overline{\text{MS}} \) mass is applied to such accuracy. Figure 1 depicts the resulting \( B \)-meson decay constant \( f_B \) for these two cases. In each case, a constant effective continuum threshold is fixed by requiring maximum stability of the found decay constant. Thus, the constant thresholds differ for pole (\( \delta_0 \)) and \( \overline{\text{MS}} \) (\( \delta_0 \)) mass schemes. Several lessons can be learnt:

(a) In the pole-mass scheme, the perturbative series for the decay constant shows no sign of convergence: each of the LO, NLO, NNLO terms contributes with similar size. Consequently, the pole-mass-scheme result for the decay constant may significantly underestimate the exact value.

(b) Reorganizing the perturbative series in terms of the \( \overline{\text{MS}} \) mass of the heavy quark yields a distinct hierarchy of the perturbative contributions [9]. Moreover, the absolute value of the decay constant extracted in this scheme turns out to be some 40\% larger than in the pole-mass case (a).

(c) Note that, in both cases, the decay constant exhibits perfect stability in a wide range of the Borel parameter \( \tau \). Thus, mere “Borel stability” is not sufficient to guarantee the reliability of some sum-rule extraction of bound-state parameters. We have pointed out this observation already several times [3]. Nevertheless, some authors still regard Borel stability as a proof of the reliability of their results.

In the light of our above findings, we adopt in the next sections the OPE formulated in terms of the \( \overline{\text{MS}} \) mass [9].

### DECAY CONSTANTS OF \( D \) AND \( D_s \)

The application of our extraction procedures leads to the following values of the charmed-meson decay constants:

\[
\begin{align*}
    f_D &= (206.2 \pm 7.3_{(\text{OPE})} \pm 5.1_{(\text{syst})} \, \text{MeV}, \\
    f_{D_s} &= (245.3 \pm 15.7_{(\text{OPE})} \pm 4.5_{(\text{syst})} \, \text{MeV}. \\
\end{align*}
\]

**FIGURE 2.** Comparison of our results for the decay constant \( f_D \) with lattice findings. For a detailed list of references, cf. [1].
The OPE-related errors in (4,5) are obtained by bootstrap studies allowing for the variation of all QCD parameters, that is, quark masses, $\alpha_s$, and condensates, in the relevant ranges. We observe perfect agreement of our predictions with the corresponding lattice results (Fig. 2). It has to be emphasized that our $\tau$-dependent effective threshold is a crucial ingredient for a successful extraction of the decay constant from the sum rule (1). Obviously, the (standard) $\tau$-independent approximation entails a much lower value for the $D$-meson decay constant $f_D$ that resides rather far from both the experimental data and the lattice outcome.

**DECAY CONSTANTS OF B AND $B_s$**

Our QCD sum-rule findings for the beauty-meson decay constants turn out to be extremely sensitive to the chosen value of the $b$-quark $\overline{\text{MS}}$ mass $m_b(m_b)$. For instance, the range $m_b(m_b) = (4.163 \pm 0.016)$ GeV [10] entails results that are barely compatible with recent lattice calculations of these decay constants (Fig. 3). Requiring our sum-rule $f_B$ result to match the average of the lattice computations provides the rather precise value of the $b$-quark $\overline{\text{MS}}$ mass

$$m_b(m_b) = (4.245 \pm 0.025) \text{ GeV}.$$ 

For this value of the $b$-quark mass, our sum-rule estimate for the $B$- and $B_s$-meson decay constants $f_B$ and $f_{B_s}$ reads

$$f_B = (193.4 \pm 12.3_{\text{(OPE)}} \pm 4.3_{\text{(syst)}}) \text{ MeV},$$

$$f_{B_s} = (232.5 \pm 18.6_{\text{OPE}} \pm 2.4_{\text{(syst)}}) \text{ MeV}.$$ 

**FIGURE 3.** Comparison of our results for the decay constant $f_B$ with lattice findings. For a detailed list of references, cf. [1].

**SUMMARY AND CONCLUSIONS**

Applying (in an attempt to improve the sum-rule method) our above modifications, we realize several serendipities:

1. The $\tau$-dependence of the effective thresholds emerges naturally when one attempts to render the duality relation exact: the dependence is evident from (1). We emphasize two facts: (a) In principle, such dependence on $\tau$ is not in conflict with any properties of quantum field theories. (b) Our analysis of $D$ mesons shows that it indeed improves decisively the quality of the related sum-rule predictions.

2. Our study of charmed mesons clearly demonstrates that using Borel-parameter-dependent thresholds leads to lots of essential improvements: (i) The accuracy of decay constants predicted by sum rules is drastically improved. (ii) It has become possible to obtain a realistic systematic error and to diminish it to the level of, say, a few percent. (iii) Our prescription brings QCD sum-rule findings into perfect agreement with both lattice QCD and experiment.

3. The beauty-meson decay constants $f_{B_{(s)}}$ are extremely sensitive to the choice of the $b$-quark mass: Matching our QCD sum-rule $f_B$ outcome to the corresponding average of lattice computations provides a truly accurate estimate of $m_b(m_b)$, in good agreement with several lattice results but, interestingly, not at all overlapping with a recent very accurate determination [10] (for details, consult Ref. [1]).

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