Eluder Dimension and Generalized Rank

Gene Li\textsuperscript{1} Pritish Kamath\textsuperscript{1} Dylan J. Foster\textsuperscript{2} Nathan Srebro\textsuperscript{1}

gene@ttic.edu pritish@ttic.edu dylanf@mit.edu nati@ttic.edu

\textsuperscript{1}Toyota Technological Institute at Chicago, \textsuperscript{2}MIT

Abstract

We study the relationship between the eluder dimension for a function class and a generalized notion of rank, defined for any monotone “activation” $\sigma: \mathbb{R} \rightarrow \mathbb{R}$, which corresponds to the minimal dimension required to represent the class as a generalized linear model. When $\sigma$ has derivatives bounded away from 0, it is known that $\sigma$-rank gives rise to an upper bound on eluder dimension for any function class; we show however that eluder dimension can be exponentially smaller than $\sigma$-rank. We also show that the condition on the derivative is necessary; namely, when $\sigma$ is the relu activation, we show that eluder dimension can be exponentially larger than $\sigma$-rank.

1 Introduction

Russo and Van Roy (2013) introduced the notion of eluder dimension for a function class and used it to analyze algorithms (based on the Upper Confidence Bound (UCB) and Thompson Sampling paradigms) for the multi-armed bandit problem with function approximation. Since then, eluder dimension has been extensively used to construct and analyze the regret of algorithms for contextual bandits and reinforcement learning with function approximation (see, e.g. Wen and Van Roy, 2013; Osband and Van Roy, 2014; Wang et al., 2020; Ayoub et al., 2020; Du et al., 2020b; Foster et al., 2020; Jin et al., 2021; Dong et al., 2021). While eluder dimension has mainly been used to analyze upper bounds on regret, recently Foster et al. (2020) provided lower bounds for contextual bandits in terms of eluder dimension, if one is hoping for instance-dependent regret bounds.

The main question motivating this paper is

\textit{Which function classes have “small” eluder dimension?}

Russo and Van Roy (2013) established upper bounds on eluder dimension for (i) function classes for which inputs have finite cardinality (the “tabular” setting), (ii) linear functions over $\mathbb{R}^d$ of bounded norm, and (iii) generalized linear functions over $\mathbb{R}^d$ of bounded norm, with any activation that has derivatives bounded away from 0. Apart from these function classes (and those that can be embedded into these), understanding of eluder dimension has been limited. Indeed, one might wonder whether a function class has “small” eluder dimension only if it can be realized as a class of (generalized) linear models!

Our Results. In this work, we shed light on the relationship between eluder dimension and notions of generalized-rank (analogous to the notion of dimension complexity) that capture the realizability of any function class as a (generalized) linear model. Informally, for an activation $\sigma: \mathbb{R} \rightarrow \mathbb{R}$ and a function class $\mathcal{F} \subseteq (X \rightarrow \mathbb{R})$, the $\sigma$-rank is the smallest dimension $d$ needed to express any function in $\mathcal{F}$ as a generalized linear function in $\mathbb{R}^d$ with activation $\sigma$ (see Definition 3). Our main results are as follows:

1. We show that eluder dimension can be exponentially smaller than $\sigma$-rank for any monotone activation $\sigma$, not just those with derivatives bounded away from 0. (Theorem 7).

2. We show that the condition on the derivative being bounded away from 0 is necessary for $\sigma$-rank to be an upper bound on eluder dimension. Namely, when $\sigma$ is the relu activation, we show that eluder dimension can be exponentially larger than $\sigma$-rank (Theorem 8).\footnote{This result was independently established by Dong et al. (2021).}
2 Eluder Dimension and Star Number

Eluder dimension is a “sequential” notion of complexity for function classes, originally defined by Russo and Van Roy (2013). Informally speaking, it characterizes the longest sequence of adversarially chosen points one must observe in order to accurately estimate the function value at any other point. We consider a variant of the original definition, proposed by Foster et al. (2020), that is never larger and is sufficient to analyze all the applications of eluder dimension in literature.

**Definition 1.** For any function class $\mathcal{F} \subseteq (\mathcal{X} \to \mathbb{R})$, $f^* \in \mathcal{F}$, and scale $\varepsilon \geq 0$, the exact eluder dimension $\text{Edim}_{f^*}(\mathcal{F}, \varepsilon)$ is the largest $m$ such that there exists $(x_1, f_1), \ldots, (x_m, f_m) \in \mathcal{X} \times \mathcal{F}$ satisfying:

$$\forall i \in [m] : |f_i(x_i) - f^*(x_i)| > \varepsilon, \quad \text{and} \quad \sum_{j<i} (f_i(x_j) - f^*(x_j))^2 \leq \varepsilon^2.$$  

Then for all $\varepsilon > 0$:

**Lemma**

1. The eluder dimension is $\text{Edim}_{f^*}(\mathcal{F}, \varepsilon) = \sup_{\varepsilon' \geq \varepsilon} \text{Edim}_{f^*}(\mathcal{F}, \varepsilon')$.

2. $\text{Edim}(\mathcal{F}, \varepsilon) := \sup_{f^* \in \mathcal{F}} \text{Edim}_{f^*}(\mathcal{F}, \varepsilon)$ and $\text{Edim}(\mathcal{F}, \varepsilon) := \sup_{f^* \in \mathcal{F}} \text{Edim}_{f^*}(\mathcal{F}, \varepsilon)$.

This definition is never larger than the original definition of Russo and Van Roy (2013), which asks for a witnessing pair of functions $f_i, f'_i \in \mathcal{F}$ (the above restricts $f'_i = f^*$). Hence, all lower bounds on our variant of eluder dimension immediately apply to the original definition. Moreover, all upper bounds on eluder dimension in this paper can also be shown to hold for the other definition (unless stated otherwise).

We also consider the closely related notion of star number defined by Foster et al. (2020), which generalizes a combinatorial parameter introduced in the active learning literature by Hanneke and Yang (2015) (we will denote it as $\text{Sdim}$ for consistency). The only difference between the definitions of eluder dimension and star number is that $\sum_{j<i}$ is replaced by $\sum_{j \neq i}$, which makes the star number a “non-sequential” notion of complexity better suited for i.i.d. learning problems.

**Definition 2.** For any function class $\mathcal{F} \subseteq (\mathcal{X} \to \mathbb{R})$, $f^* \in \mathcal{F}$, and scale $\varepsilon \geq 0$, the exact star number $\text{Sdim}_{f^*}(\mathcal{F}, \varepsilon)$ is the largest $m$ such that there exists $(x_1, f_1), \ldots, (x_m, f_m) \in \mathcal{X} \times \mathcal{F}$ satisfying:

$$\forall i \in [m] : |f_i(x_i) - f^*(x_i)| > \varepsilon, \quad \text{and} \quad \sum_{j \neq i} (f_i(x_j) - f^*(x_j))^2 \leq \varepsilon^2.$$  

Then for all $\varepsilon > 0$:

1. The star number is $\text{Sdim}_{f^*}(\mathcal{F}, \varepsilon) = \sup_{\varepsilon' \geq \varepsilon} \text{Sdim}_{f^*}(\mathcal{F}, \varepsilon')$.

2. $\text{Sdim}(\mathcal{F}, \varepsilon) := \sup_{f^* \in \mathcal{F}} \text{Sdim}_{f^*}(\mathcal{F}, \varepsilon)$ and $\text{Sdim}(\mathcal{F}, \varepsilon) := \sup_{f^* \in \mathcal{F}} \text{Sdim}_{f^*}(\mathcal{F}, \varepsilon)$.

It immediately follows from these definitions that the star number is never larger than eluder dimension. On the other hand, the star number can be arbitrarily smaller than eluder dimension.

**Proposition 1.** For all $\mathcal{F}$, $f^* \in \mathcal{F}$ and scale $\varepsilon \geq 0$, it holds that\(^2\)

$$\text{Sdim}_{f^*}(\mathcal{F}, \varepsilon) \leq \text{Edim}_{f^*}(\mathcal{F}, \varepsilon) \leq \min\{|\mathcal{X}|, |\mathcal{F}|-1\}.$$  

**Proposition 2** (simplified from Foster et al. (2020, Prop 2.3)). For the class of threshold functions given as $\mathcal{F}_n^\text{th} := \{f_i : [n] \to \{0, 1\} \mid i \in [n+1]\}$, where $f_i(x) := \mathbb{1}\{x \geq i\}$, and for $f^* = f_{n+1}$, it holds for all $\varepsilon < 1$ that $\text{Sdim}_{f^*}(\mathcal{F}, \varepsilon) = 2$ and $\text{Edim}_{f^*}(\mathcal{F}, \varepsilon) = n$.

\(^2\)For the definition of eluder dimension considered by Russo and Van Roy (2013), an upper bound of $\min\{|\mathcal{X}|, (|\mathcal{F}|)^{1/2}\}$ holds, which can be tight. This upper bound holds because the witnessing pair of functions $(f_i, f'_i)$ has to be distinct for each $i$. 


### 3 Generalized Rank

*Dimension complexity* has been studied extensively in combinatorics, learning theory, and communication complexity (see e.g. Alon et al., 1985; Forster, 2002; Arriaga and Vempala, 2006; Alon et al., 2016). The classical notion of dimension complexity, also known as *sign rank*, corresponds to the smallest dimension required to embed the input space such that all hypotheses in the function class under consideration are realizable as halfspaces. We consider a generalized notion of rank that is specified for any particular activation $\sigma : \mathbb{R} \to \mathbb{R}$, and captures to the smallest dimension required to represent the function class as a generalized linear model when $\sigma$ is the activation. In the following, let $B_d(R) := \{ x \in \mathbb{R}^d \mid \|x\|_2 \leq R \}$.

**Definition 3.** For any $\sigma : \mathbb{R} \to \mathbb{R}$, the $\sigma$-*rank* $\sigma\text{-}rk(F, R)$ of a function class $F \subseteq (\mathcal{X} \to \mathbb{R})$ at scale $R > 0$ is the smallest dimension $d$ for which there exists mappings $\phi : \mathcal{X} \to B_d(1)$ and $w : F \to B_d(R)$ such that

$$
\text{for all } (x, f) \in \mathcal{X} \times F : f(x) = \sigma(\langle w(f), \phi(x) \rangle),
$$

or $\infty$ if no such $d$ exists. For a collection of activation functions $\Sigma \subseteq (\mathbb{R} \to \mathbb{R})$, the $\Sigma$-*rank* is

$$
\Sigma\text{-}rk(F, R) := \min_{\sigma \in \Sigma} \sigma\text{-}rk(F, R).
$$

**Examples.** We present some examples of $\Sigma\text{-}rk$ that motivate our definition above.

1. **Threshold activation.** $\text{sign}(z)$ yields the classic notion of *sign-rank* (equivalent to *dimension complexity*, as already mentioned). In this case, the scale $R$ is irrelevant, so we denote $\text{sign-rk}(F) := \text{sign-rk}(F, R)$ for any $R$. Note that this quantity is meaningful only for $F \subseteq (\mathcal{X} \to \{-1, 1\})$.

2. **Identity activation.** For $\text{id}(z) := z$, $\text{id-rk}(F, R)$ is the smallest dimension needed to represent each $f \in F$ as a (norm-bounded) linear function. We abbreviate $\text{rk} := \text{id-rk}$, as this corresponds to the standard notion of rank of the matrix $(f(x))_{f,x}$ (albeit with the additional norm constraint).

3. **Monotone activations.** For $L \geq \mu \geq 0$, $\mathcal{M}_\mu^L$ consists of all activations $\sigma$ such that for all $z < z'$, it holds that $\mu \leq \frac{\sigma(z') - \sigma(z)}{z' - z} \leq L$ (for differentiable $\sigma$, this is equivalent to $\mu \leq \sigma'(z) \leq L$ for all $z \in \mathbb{R}$). An important special case is when $\mu = 0$, and a particular $\sigma \in \mathcal{M}_0^L$ of interest is the rectified linear unit (ReLU) defined as $\text{relu}(z) := \max\{z, 0\}$. For ease of notation, we denote $\mathcal{M}_\mu := \mathcal{M}_\mu^1$.

While we will always be explicit about the Lipschitz constant, note that the scale of the Lipschitz constant $L$ (and $\mu$) is interchangeable with the scale of $R$. In particular,

$$
\mathcal{M}_\mu^{L} \cdot \text{rk}(F, R) = \mathcal{M}_{\mu/L} \cdot \text{rk}(F, RL).
$$

4. **All activations.** $\Sigma^{\text{all}}$ consists of all activations $\sigma$. We mention this notion of rank only in passing, and we will not focus on it for the rest of the paper.

**Proposition 3.** $\Sigma$-rank satisfies the following for all $F \subseteq (\mathcal{X} \to \mathbb{R})$.

(i) For all $R \leq R'$, $\Sigma\text{-}rk(F, R) \geq \Sigma\text{-}rk(F, R')$.

(ii) For all $\Sigma_1 \subseteq \Sigma_2$ and $R > 0$, $\Sigma_1\text{-}rk(F, R) \geq \Sigma_2\text{-}rk(F, R)$.

---

4Note that only the product of the scales of $\phi$ and $w$ is relevant. The definition remains equivalent if we let $\phi : \mathcal{X} \to B_d(R_\phi)$ and $w : F \to B_d(R_w)$ for any $R_\phi$ and $R_w$ such that $R = R_\phi \cdot R_w$.

4To prove upper bounds on eluder dimension, it suffices for this condition to hold only when $|z| \leq R$, (see e.g. Russo and Van Roy, 2013). Since we fix $\sigma$ in our definition first and then consider $\sigma$-rank at different scales $R$, this weaker condition complicates our definitions. Note that at any specific scale $R$, we can always modify $\sigma$ to satisfy the required constraint everywhere by extending it linearly whenever $|z| > R$. 

---
Proposition 4. For all $F \subseteq (X \to \mathbb{R})$, $R > 0$ and $\mu \in (0, 1]$, we have:

$$rk(F, R) \geq M_{\mu}rk(F, R) \geq M_{0}rk(F, R) \geq \text{sign}rk(F) - 1,$$

where the last inequality is meaningful only for $F \subseteq (X \to \{-1, 1\})$. Moreover, for each inequality above, there exists a function class $F$ which exhibits an infinite gap between the two quantities.

Proof. The first two inequalities follow from immediately from Proposition 3. For the last inequality, let $\phi : \mathcal{X} \to \mathbb{R}^d$ and $w : F \to \mathbb{R}^d$ be the mappings that witness $\sigma\text{-}rk(F, R) = d$ for some $\sigma \in \mathcal{M}$. Thus, we have that for all $(x, f) \in \mathcal{X} \times F$, it holds that, $f(x) = \sigma((w(f), \phi(x)))$. Let $t \in \mathbb{R}$ be any value such that $\sigma(t) = 0$. From monotonicity of $\sigma$ and the fact that $F$ is $\{-1, 1\}$-valued, we have that for all $(x, f) \in \mathcal{X} \times F$, $f(x) = \text{sign}((w(f), \phi(x)) - t)$. Thus, $\text{sign}rk(F) \leq \text{sign}rk(F, R) + 1$. We next move to the examples witnessing the separations.

$\blacktriangleright$ $rk \gg M_{\mu}\text{-}rk$: Consider $\mathcal{X} = [0, 1]$ and $F^{\exp} := \{f_\theta : x \mapsto \sigma(\theta \cdot x) \mid |\theta| \leq 1\}$, where $\sigma(\cdot)$ is defined piecewise as follows: $\sigma(z) = e^z$ for $z \in [0, 1]$; outside of $[0, 1]$, we extend the function linearly with slope 1 when $z < 0$ and slope $e$ when $z > 1$. Since $\sigma \in \mathcal{M}_\mu$ we have $M_{\mu}\text{-}rk(F^{\exp}, 1) = 1$ and hence $M_{1/\epsilon}\text{-}rk(F, e) = 1$ (from Equation 3).

Consider the points $\{x_j := j/d \mid j \in \{1, \ldots, d\}\}$ and the functions $\{f_{\theta_j} := f_{i/d} \mid j \in \{1, \ldots, d\}\}$. The matrix $A$ given by $A_{ij} := f_{\theta_j}(x_i) = (e^{j/d})^i$ is a Vandermonde matrix, and hence rank$(A) = d$. Since $d$ can be chosen to be arbitrarily large, it follows that $rk(F^{\exp}, R) = \infty$ for all $R > 0$.

$\blacktriangleright$ $M_{\mu}\text{-}rk \gg M_{0}\text{-}rk$: Consider $\mathcal{X} = [0, 1]$ and $F^{\text{relu}} := \{f_{a,b} : x \mapsto \text{relu}(ax + b) \mid a^2 + b^2 \leq 1\}$, the class of ReLUs with biases in 1 dimension. Clearly, $M_{0}\text{-}rk(F^{\text{relu}}, \sqrt{2}) = 2$.

Suppose for contradiction that for some $\sigma \in \mathcal{M}_\mu$, it holds that $\sigma\text{-}rk(F^{\text{relu}}, R) = d$ with mappings $\phi : \mathcal{X} \to B_0(1)$ and $w : F^{\text{relu}} \to B_0(R)$ for some $R > 0$. Consider $n = d + 2$ points $0 < x_1 < x_2 < \cdots < x_n < 1$. For each $i$, $f_i(x) := \text{relu}(x - x_{i-1}) \in F^{\text{relu}}$ (let $x_0 := 0$) satisfies $f_i(x_j) = \sigma((w(f_i), \phi(x_j))) = 0$ for all $j < i$, and $f_i(x_j) = \sigma((w(f_i), \phi(x_j))) > 0$ for all $j \geq i$. That is, $\langle w(f_i), \phi(x_j) \rangle = \sigma^{-1}(0)$ for all $j < i$ (since $\sigma : \mathbb{R} \to \mathbb{R}$ is strictly monotone, $\sigma^{-1}(0)$ is uniquely defined) and $\langle w(f_i), \phi(x_j) \rangle > \sigma^{-1}(0)$ for all $j \geq i$. Consider the matrix $A \in \mathbb{R}^{n \times n}$ given by $A_{ij} := \langle w(f_i), \phi(x_j) \rangle$. By definition, rank$(A) \leq d$.

On the other hand, we have that $A - \sigma^{-1}(0) \cdot J$ is an upper-triangular matrix with non-zero diagonals, where $J \in \mathbb{R}^{n \times n}$ is the all-1s matrix. It follows that rank$(A) \geq n - 1$ and hence $n \leq d + 1$. This is a contradiction and hence $M_{\mu}(F^{\text{relu}}, R) = \infty$ for all $\mu, R > 0$.

$\blacktriangleright$ $M_{0}\text{-}rk \gg \text{sign}\text{-}rk$: Consider $\mathcal{X} = [0, 1]$ and $F^{\text{th}} := \{f_t : x \mapsto \text{sign}(x - t) \mid t \in [0, 1]\}$, the class of Thresholds. Clearly, $\text{sign}rk(F^{\text{th}}) = 2$.

We briefly sketch the argument showing $M_{0}\text{-}rk(F, R) = \infty$ for any $R > 0$. Suppose for some $\sigma \in \mathcal{M}_0$ it holds that $\sigma\text{-}rk(F^{\text{th}}, R) = d$. Then it is possible to realize $F^{\text{th}}$ as halfspaces with margin, since $\sigma$ is 1-Lipschitz. This implies that $F^{\text{th}}$ is online learnable with a finite mistake bound (via the Perceptron algorithm). This is a contradiction since $F^{\text{th}}$ is not online learnable with a finite mistake bound (see e.g. Shalev-Shwartz and Ben-David, 2014, Lemma 21.6 and Ex. 21.4). $\square$

4 Eluder Dimension versus Generalized Rank

We compare eluder dimension and star number with each notion of generalized rank: $rk$, $M_{\mu}\text{-}rk$ (for $\mu > 0$) and $M_{0}\text{-}rk$. Our results are summarized in Figure 1.
Eluder vs. rk and $\mathcal{M}_\mu$-rk. Russo and Van Roy (2013) (see also Osband and Van Roy, 2014) provided upper bounds on eluder dimension for linear and generalized linear function classes. We restate this result, with a slight improvement and include the proof with precise dependence on problem parameters in Appendix A for completeness.

**Proposition 5** (cf. Russo and Van Roy (2013), Prop. 6, 7; Osband and Van Roy (2014), Prop. 2, 4). For any function class $\mathcal{F} \subseteq (\mathcal{X} \rightarrow \mathbb{R})$ and $\varepsilon > 0$:

(i) For all $R > 0$, $\text{Edim}(\mathcal{F}, \varepsilon) \leq \text{rk}(\mathcal{F}, R) \cdot O\left(\log \frac{R}{\varepsilon}\right)$.

(ii) For all $L, \mu, R > 0$, $\text{Edim}(\mathcal{F}, \varepsilon) \leq \mathcal{M}_\mu$-rk($\mathcal{F}, R$) $\cdot O\left(\frac{L^2}{\mu^2} \log \left(\frac{RL}{\varepsilon}\right)\right)$.

This result has been used to prove upper bounds on eluder dimension of various function classes beyond generalized linear models; for example, the class of bounded degree polynomials, by taking the feature map $\phi(x)$ to be the vector of monomials. The upper bound in Part (i) is in fact tight (up to constants) for the class of linear functions, as shown below. This trivially implies the optimality of the bound in Part (ii) up to the factor of $(L/\mu)^2$ which to the best of our knowledge is open.

**Proposition 6** (Mahajan and Lovett (2021)). For any $R > 0$, $\mathcal{X} := B_d(1)$ and $\mathcal{F} := \{f_\theta: x \mapsto \langle \theta, x \rangle \mid \theta \in B_d(R)\}$, it holds that:

$$\text{Edim}(\mathcal{F}, \varepsilon) \geq \Omega\left(d \log \left(\frac{R}{\varepsilon}\right)\right).$$

**Proof.** We exhibit a sequence $(x_1, \theta_1), \ldots, (x_m, \theta_m)$ that witnesses the claimed lower bound on eluder dimension with $\theta^* = 0$. It suffices to consider the case of $R = 1$, as this is just a matter of scaling relative to $\varepsilon$. First, consider the case of $d = 1$. For any $\alpha \in (\varepsilon, \sqrt{2}\varepsilon)$ and $k := \lceil \log_2(1/\alpha) \rceil$, let $x_i := 1/2^{(k-i)}$ and $\theta_i = \alpha \cdot 2^{k-i}$ for $i \in \{0, \ldots, k\}$. For each $i$, it holds that $\theta_i x_i = \alpha > \varepsilon$ and $\sum_{j=1}^{\ell} (\theta_i x_j)^2 \leq \alpha^2/2 < \varepsilon^2$. Since $|x_i| \leq 1$ and $|\theta_i| \leq 1$ we get $\text{Edim}(\mathcal{F}, \varepsilon) \geq k + 1 \geq \Omega(\log(1/\varepsilon))$.

For $d > 1$, consider $d$ copies of the above 1 dimensional sequence repeated in each dimension. Namely, consider the sequence $(x_{ij}, \theta_{ij})_{i \in [d], j \in [k]}$ with $x_{ij} := e_i/2^{k-j}$ and $\theta_{ij} := \alpha 2^{k-j} \cdot e_i$ (where $e_i$ is the $i$-th standard basis vector). Since $x_{ij}, \theta_{ij} \in B_d(1)$, we have $\text{Edim}(\mathcal{F}, \varepsilon) \geq d(k + 1) = \Omega\left(d \log(1/\varepsilon)\right)$. \qed

Eluder vs. $\mathcal{M}_0$-rk. It turns out that eluder dimension and $\mathcal{M}_0$-rk are incomparable. That is, there exists a function class for which eluder dimension is exponentially smaller than $\mathcal{M}_0$-rk (and hence $\mathcal{M}_\mu$-rk and rk by Proposition 4). Moreover, there exists a different function class for which eluder dimension (even the star number) is exponentially larger than relu-rk (and hence $\mathcal{M}_0$-rk).
Theorem 7. For $\mathcal{X} = \{-1, 1\}^d$ and $\mathcal{F}^\oplus := \{f_S: x \mapsto \prod_{i \in S} x_i \mid S \subseteq [d]\}$, it holds that

(i) $\mathcal{M}_0\text{-}\text{rk}(\mathcal{F}^\oplus, R) \geq 2d^2/2 - 1$ for all $R > 0$.

(ii) $\text{Sdim}(\mathcal{F}^\oplus, \varepsilon) \leq \text{Edim}(\mathcal{F}^\oplus, \varepsilon) \leq d$ for all $\varepsilon \geq 0$.

Proof. Part (i). From Proposition 4, we have that $\mathcal{M}_0\text{-}\text{rk}(\mathcal{F}^\oplus, R) \geq \text{sign-rk}(\mathcal{F}^\oplus) - 1$ for any $\sigma \in \mathcal{M}_0$. The proof is now complete by noting a well known result that $\text{sign-rk}(\mathcal{F}^\oplus) \geq 2d^2/2$ (Forster, 2002).

Part (ii). For any $x \in \{-1, 1\}^d$ consider its 0-1 representation $\bar{x} \in \mathbb{F}_2^d$ (representing +1 by 0 and −1 by 1). All functions in $\mathcal{F}^\oplus$ can be simply viewed as linear functions over $\mathbb{F}_2$. Namely, any parity function is indexed by a vector $a \in \mathbb{F}_2^d$, with $f_a(x) := (-1)^{(a, \bar{x})}$. Note that $\text{Edim}(\mathcal{F}^\oplus, \varepsilon) = 0$ for all $\varepsilon \geq 2$. For any $\varepsilon < 2$, suppose $\text{Edim}_{\text{F}}(\mathcal{F}^\oplus, \varepsilon) = m$, witnessed by $(x_1, f_{a_1}), \ldots, (x_m, f_{a_m}) \in \{-1, 1\}^d$ and $f^* = f_{a^*}$. We have

- $f_a(x_i) \neq f_{a^*}(x_i)$, and
- $f_a(x_j) = f_{a^*}(x_j)$ for all $j < i$ since $\sum_{j < i} (f_a(x_j) - f_{a^*}(x_j))^2 < \varepsilon^2 < 4$ iff all the terms are 0.

Thus, we have $\langle a_i - a^*, \bar{x} \rangle = 0$ for all $\bar{x} \in \mathbb{F}_2$-span($\{\bar{x}_1, \ldots, \bar{x}_{i-1}\}$). But $\langle a_i - a^*, \bar{x}_i \rangle = 1$ and hence $\bar{x}_i$ is linearly independent of $\{\bar{x}_1, \ldots, \bar{x}_{i-1}\}$ over $\mathbb{F}_2^d$. Thus, $\{\bar{x}_1, \ldots, \bar{x}_m\}$ are all linearly independent over $\mathbb{F}_2^d$, and hence $m \leq d$.

Theorem 8. For all $R > 0$, $\mathcal{X} = \mathcal{B}_d(1)$, and $\mathcal{F}^{\text{relu}} := \{f_{\theta, b}: x \mapsto \text{relu}(\langle \theta, x \rangle + b) \mid \|\theta\|_2 + b^2 \leq R\}$, it holds that

(i) $\mathcal{M}_0\text{-}\text{rk}(\mathcal{F}^{\text{relu}}, R) \leq \text{relu-rk}(\mathcal{F}^{\text{relu}}, R) \leq d + 1$,

(ii) $\text{Edim}(\mathcal{F}^{\text{relu}}, \varepsilon) \geq \text{Sdim}(\mathcal{F}^{\text{relu}}, \varepsilon) \geq \left(\frac{R}{\delta}\right)^{d/2}$ for all $\varepsilon \in (0, \frac{2R}{\delta})$.

Proof. Part (i) is immediate from the definition. We show Part (ii) in the special case of $R = 2$; the general case follows by relatively scaling $\varepsilon$, since relu is homogeneous, namely, $\text{relu}(ax) = a \cdot \text{relu}(x)$.

Consider any $U \subseteq \mathcal{X}$ such that $\|u\|_2 = 1$ and $\langle u, v \rangle \leq 1 - \varepsilon$ for all $u, v \in U$. It holds that $\text{Sdim}_{\text{F}}(\mathcal{F}^{\text{relu}}, \varepsilon) \geq |U|$ when $f^*$ is the identically zero function, since the function $f_u(x) = \text{relu}(\langle u, x \rangle - (1 - \varepsilon))$ is such that $f_u(u) = 0$ for all $u \in U \setminus \{v\}$, whereas $f_u(u) = \varepsilon$. A standard sphere packing argument shows that such a set $U$ exists with $|U| \geq (1/2\varepsilon)^{d/2}$ for all $\varepsilon < 1/2$. In particular, the $\delta$-packing number of the unit sphere is at least $(1/\delta)^d$ (Vershynin, 2018, Cor. 4.2.13). Thus, we can find $(1/\delta)^d$ points such that each pair $u, v$ satisfies $\|u - v\|_2 \geq \delta$, or equivalently $\langle u, v \rangle \leq 1 - \delta^2/2$. Setting $\delta = \sqrt{2\varepsilon}$ proves the claimed lower bound.

Theorem 8 was independently proven in Dong et al. (2021, Thm. 5.1).

Remark. While we considered the variant of eluder dimension as defined by Foster et al. (2020), the lower bound on eluder dimension in Theorem 8 immediately holds for the notion defined by Russo and Van Roy (2013). On the other hand, the upper bound on eluder dimension in Theorem 7 can be shown to hold even with the definition of Russo and Van Roy (2013) (by replacing every instance of $a^*$ by $a^*_i$).

5 Discussion

Our results clarify the relationship between eluder dimension (and star number) and various notions of generalized-rank. Namely, apart from the upper bounds in Proposition 3, there is little relationship between eluder dimension and generalized rank. Eluder dimension can be exponentially smaller than $\sigma$-rank for any monotone $\sigma$. On the other hand, eluder dimension can be exponentially larger than $\sigma$-rank for $\sigma \in \mathcal{M}_0$, thereby showing that condition that $\sigma$ has derivatives bounded away from 0 is necessary.
Approximate Generalized Rank. In many reinforcement learning problems, the assumption of realizability may be too stringent; hence one might consider a misspecified setting, where it is assumed that the reward function (or value function) is approximated by some function class up to \( \delta \) error (see e.g. Du et al., 2020a; Van Roy and Dong, 2019; Lattimore et al., 2020; Neu and Olkhovskaya, 2020; Foster and Rakhlin, 2020; Zanette et al., 2020). A natural question to ask then is how eluder dimension is related to approximate notions of rank, with the goal of obtaining a general technique for providing regret bounds in the misspecified setting. To this end, we can define \( \delta \)-approximate \( \sigma \)-rank, \( \sigma\text{-rk}_\delta \), by replacing condition (2) with:

\[
\text{for all } (x, f) \in \mathcal{X} \times \mathcal{F} : |f(x) - \sigma((w(f), \phi(x)))| \leq \delta .
\]  

It turns out that \( Edim \) (and \( Sdim \)) are incomparable to \( \sigma\text{-rk}_\delta \). Namely, for the class of parity functions \( \mathcal{F}^\oplus \) defined in Theorem 7, \( M_0\text{-rk}_\delta(\mathcal{F}^\oplus) \geq \text{sign-rk}(\mathcal{F}^\oplus) - 1 \geq 2^{d/2} - 1 \) for any \( \delta < 1 \). On the other hand, for the class of singletons defined over \( \mathcal{X} = [d] \) and \( \mathcal{F}^{\text{sing}} := \{ f_i : x \mapsto 1(x = i) \mid i \in [d] \cup \{0\} \} \), it holds that \( Sdim(\mathcal{F}^{\text{sing}}, \varepsilon) \geq d \) for all \( \varepsilon < 1 \), whereas \( \text{rk}_\delta(\mathcal{F}^{\text{sing}}) \leq O((\log d)/\delta^2) \).

Acknowledgements

We thank Gaurav Mahajan for allowing us to include the proof of Proposition 6 (Mahajan and Lovett, 2021). We thank Akshay Krishnamurthy, Tengyu Ma and Ruosong Wang for helpful discussions. GL was partially supported by NSF award IIS-1764032. PK was partially supported by NSF BIGDATA award 1546500. This work was done while GL, PK, and DF were participating in the Simons program on the Theoretical Foundations of Reinforcement Learning.

References

N. Alon, P. Frankl, and V. Rödl. Geometrical realization of set systems and probabilistic communication complexity. In 26th Annual Symposium on Foundations of Computer Science, pages 277–280. IEEE Computer Society, 1985. URL https://doi.org/10.1109/SFCS.1985.30.

N. Alon, S. Moran, and A. Yehudayoff. Sign rank versus VC dimension. In 29th Annual Conference on Learning Theory, volume 49 of PMLR, pages 47–80, 2016. URL http://proceedings.mlr.press/v49/alon16.html.

R. I. Arriaga and S. S. Vempala. An algorithmic theory of learning: Robust concepts and random projection. Mach. Learn., 63(2):161–182, 2006. URL https://doi.org/10.1007/s10994-006-6265-7.

A. Ayoub, Z. Jia, C. Szepesvari, M. Wang, and L. Yang. Model-based reinforcement learning with value-targeted regression. In Proceedings of the 37th International Conference on Machine Learning, volume 119 of PMLR, pages 463–474, 2020. URL http://proceedings.mlr.press/v119/ayoub20a.html.

K. Dong, J. Yang, and T. Ma. Provable model-based nonlinear bandit and reinforcement learning: Shelve optimism, embrace virtual curvature. arXiv, 2102.04168, 2021. URL https://arxiv.org/abs/2102.04168.

S. S. Du, S. M. Kakade, R. Wang, and L. F. Yang. Is a good representation sufficient for sample efficient reinforcement learning? In 8th International Conference on Learning Representations, ICLR. OpenReview.net, 2020a. URL https://openreview.net/forum?id=r1genAVKPB.

S. S. Du, J. D. Lee, G. Mahajan, and R. Wang. Agnostic Q-learning with function approximation in deterministic systems: Near-optimal bounds on approximation error and sample complexity. In Advances in Neural Information Processing Systems, volume 33, pages 22327–22337. Curran Associates, Inc., 2020b. URL https://proceedings.neurips.cc/paper/2020/file/df5c905bcd8c3348ad1b35d7231ee2b1-Paper.pdf.

J. Forster. A linear lower bound on the unbounded error probabilistic communication complexity. J. Comput. Syst. Sci., 65(4):612–625, 2002. URL https://doi.org/10.1016/S0022-0000(02)00019-3.
D. J. Foster and A. Rakhlin. Beyond UCB: Optimal and efficient contextual bandits with regression oracles. In Proceedings of the 37th International Conference on Machine Learning, volume 119 of PMLR, pages 3199–3210, 13–18 Jul 2020. URL http://proceedings.mlr.press/v119/foster20a.html.

D. J. Foster, A. Rakhlin, D. Simchi-Levi, and Y. Xu. Instance-dependent complexity of contextual bandits and reinforcement learning: A disagreement-based perspective. arXiv, 2010.03104, 2020. URL https://arxiv.org/abs/2010.03104.

S. Hanneke and L. Yang. Minimax analysis of active learning. Journal of Machine Learning Research, 16 (109):3487–3602, 2015. URL http://jmlr.org/papers/v16/hanneke15a.html.

C. Jin, Q. Liu, and S. Miryoosefi. Bellman eluder dimension: New rich classes of RL problems, and sample-efficient algorithms. arXiv, 2102.00815, 2021. URL https://arxiv.org/abs/2102.00815.

T. Lattimore, C. Szepesvari, and G. Weisz. Learning with good feature representations in bandits and in RL with a generative model. In International Conference on Machine Learning, pages 5662–5670. PMLR, 2020. URL http://proceedings.mlr.press/v119/lattimore20a.html.

G. Mahajan and S. Lovett. Personal communication. 2021.

G. Neu and J. Olkhovskaya. Efficient and robust algorithms for adversarial linear contextual bandits. In Conference on Learning Theory, pages 3049–3068. PMLR, 2020. URL http://proceedings.mlr.press/v125/neu20b.html.

I. Osband and B. Van Roy. Model-based reinforcement learning and the eluder dimension. In Advances in Neural Information Processing Systems, volume 27. Curran Associates, Inc., 2014. URL https://proceedings.neurips.cc/paper/2014/file/1141938ba2c2b213f5505d7c424ebae5f-Paper.pdf.

D. Russo and B. Van Roy. Eluder dimension and the sample complexity of optimistic exploration. In Advances in Neural Information Processing Systems, volume 26. Curran Associates, Inc., 2013. URL https://proceedings.neurips.cc/paper/2013/file/41b6d20a38bb1b0bec75acf0845530a7-Paper.pdf.

S. Shalev-Shwartz and S. Ben-David. Understanding machine learning: From theory to algorithms. Cambridge university press, 2014. URL https://www.cs.huji.ac.il/~shais/UnderstandingMachineLearning/.

B. Van Roy and S. Dong. Comments on the Du-Kakade-Wang-Yang lower bounds. arXiv, 1911.07910, 2019. URL https://arxiv.org/abs/1911.07910.

R. Vershynin. High-dimensional probability: An introduction with applications in data science, volume 47. Cambridge university press, 2018. URL https://www.math.ucsd.edu/~rvershyn/papers/HDP-book/HDP-book.html.

R. Wang, R. R. Salakhutdinov, and L. Yang. Reinforcement learning with general value function approximation: Provably efficient approach via bounded eluder dimension. In Advances in Neural Information Processing Systems, volume 33, pages 6123–6135. Curran Associates, Inc., 2020. URL https://proceedings.neurips.cc/paper/2020/file/440924c5948e05070663f88e69e8242b-Paper.pdf.

Z. Wen and B. Van Roy. Efficient exploration and value function generalization in deterministic systems. In Advances in Neural Information Processing Systems, volume 26. Curran Associates, Inc., 2013. URL https://proceedings.neurips.cc/paper/2013/file/bad5f33780c42f258887a9d07405083-Paper.pdf.

A. Zanette, A. Lazaric, M. Kochenderfer, and E. Brunskill. Learning near optimal policies with low inherent bellman error. In International Conference on Machine Learning, pages 10978–10989. PMLR, 2020. URL http://proceedings.mlr.press/v119/zanette20a.html.
A Proof of Proposition 5

Proposition 5 follows from Claim 10 and Claim 11 below, where for clarity, we keep track of the norms of \( \phi \) and \( w \) separately. Our improvement comes about from the following lemma, which is inspired by a similar step in Russo and Van Roy (2013).

**Lemma 9** (Inspired by Russo and Van Roy (2013)). For all \( k \geq 1 \) and \( \alpha, \beta > 0 \), if \((1 + \alpha)^k \leq 1 + \beta k\), then \( k \leq \frac{e^2}{\varepsilon} \cdot \frac{1 + \alpha}{\alpha} \ln \left( \frac{2^{3(1+\alpha)}}{\alpha} \right) \).

**Proof.** We consider two cases. In both cases, we use that \( \ln(1 + \alpha) \geq \frac{\alpha}{1+\alpha} \) holds for all \( \alpha > 0 \). Also note that \((1 + \alpha)^k > 1 + k\alpha \) and hence \( \beta > \alpha \).

**Case 1:** If \( \beta k < 1 \), we have \((1 + \alpha)^k \leq 2 \) and hence \( k \leq \frac{\ln 2}{\ln(1+\alpha)} \leq \ln 2 \cdot \frac{1+\alpha}{\alpha} \). Since \( \beta > \alpha \), we have that \( \ln 2 \leq \frac{e^2}{\varepsilon} \cdot \ln \left( \frac{2^{3(1+\alpha)}}{\alpha} \right) \), thereby completing the proof for this case.

**Case 2:** If \( \beta k \geq 1 \), we have \((1 + \alpha)^k \leq 2\beta k \). Taking logarithms, we have \( k \ln(1 + \alpha) \leq \ln k + \ln 2 \beta \). Hence, we have \( \frac{k\alpha}{1+\alpha} \leq \ln \left( \frac{k\alpha}{1+\alpha} \right) + \ln \left( \frac{2^{3(1+\alpha)}}{\alpha} \right) \). Using \( \ln x \leq \frac{x}{\varepsilon} \) for all \( x \geq 0 \), we get for \( x = \frac{k\alpha}{1+\alpha} \) that \( k \cdot \frac{\alpha}{1+\alpha} \cdot (1 - \frac{1}{k}) \leq \ln \left( \frac{2^{3(1+\alpha)}}{\alpha} \right) \), thereby completing the proof. \( \square \)

**Claim 10.** Suppose \( \text{rk}(F, R_{\phi}R_w) = d \) is witnessed by mappings \( \phi : X \to B_d(R_{\phi}) \) and \( w : F \to B_d(R_w) \).

Then \( \text{Edim}(F, \varepsilon) \leq \frac{ae}{\varepsilon^2} \cdot d \cdot \log \left( \frac{4R_d^2R_{w}^2}{\varepsilon} \right) \) for all \( \varepsilon < R_{\phi}R_w \).

**Proof.** Suppose \( \text{Edim}_{f^*}(F, \varepsilon) = m \) witnessed by the sequence \((x_1, f_1), \ldots, (x_m, f_m) \in X \times F\), for some \( f^* \in F \) and \( \varepsilon > 0 \). Denote \( w_i := w(f_i) - w(f^*) \), and \( \phi_i := \phi(x_i) \). It follows that \( w_i \in B_d(2R_w) \) and \( \phi_i \in B_d(R_{\phi}) \). From Equation 1, we have that for all \( i \in [m] \):

\[
\max_{w \in \mathbb{R}^d} \left\{ \langle w, \phi_i \rangle : \sum_{j < i} \langle w, \phi_j \rangle^2 \leq \varepsilon^2, \|w\|_2 \leq 2R_w \right\} > \varepsilon
\]

(5)

Let \( V_i := \lambda I + \sum_{j < i} \phi_j \phi_j^\top \). The above equation implies that for all \( i \in [m] \):

\[
\max_{w \in \mathbb{R}^d} \left\{ \langle w, \phi_i \rangle : \|w\|_{V_i}^2 \leq \varepsilon^2 + \lambda \cdot 4R_{w}^2 \right\} > \varepsilon,
\]

which via convex duality and setting \( \lambda := \varepsilon^2/(4R_{w}^2) \), further implies that for all \( i \in [m] \):

\[
\|\phi_i\|_{V_i^{-1}} > \frac{\varepsilon^2}{\varepsilon^2 + \lambda \cdot 4R_{w}^2} = \frac{1}{2},
\]

We will use a potential argument to track the quantity \( \det(V_i) \). First, we have the upper bound:

\[
\det(V_{m+1}) \leq \left( \frac{\text{tr}(V_{m+1})}{d} \right)^d \leq \left( \frac{\lambda d + mR_{\phi}^2}{d} \right)^d = \lambda^d \left( 1 + \frac{mR_{\phi}^2}{\lambda d} \right)^d = \lambda^d \left( 1 + \frac{m}{d} \cdot \frac{4R_{w}^2R_{\phi}^2}{\varepsilon^2} \right)^d,
\]

using, AM-GM inequality, the linearity of trace, and the definition of \( \lambda \). We also have the lower bound, using the Matrix Determinant Lemma:

\[
\det(V_{m+1}) = \det(V_m + \phi_m \phi_m^\top) = \det(V_m) \cdot (1 + \|\phi_m\|_{V_m^{-1}}^2) \geq \lambda^d (3/2)^m,
\]

where the last step follows by induction. Combining the upper and lower bounds, we get that \((3/2)^{m/d} \leq 1 + \frac{m}{d} \cdot \frac{4R_{w}^2R_{\phi}^2}{\varepsilon^2} \). The claim follows from an application of Lemma 9 with \( k = \frac{ae}{\varepsilon}, \alpha = \frac{1}{2} \) and \( \beta = \frac{4R_{w}^2R_{\phi}^2}{\varepsilon^2} \). \( \square \)
Claim 11. For \( \sigma \in \mathcal{M}_\mu^L \), suppose \( \sigma \cdot \text{rk}(\mathcal{F}, R_\phi R_w) = d \) is witnessed by mappings \( \phi : \mathcal{X} \to \mathcal{B}_d(R_\phi) \) and \( w : \mathcal{F} \to \mathcal{B}_d(R_w) \). Then \( \text{Edim}(\mathcal{F}, \varepsilon) \leq \frac{3\varepsilon}{\varepsilon^2-1} \cdot d \cdot \frac{L^2}{\mu^2} \cdot \log \left( \frac{24R_\phi^2 R_w^2 L^2}{\varepsilon^2} \right) \) for all \( \varepsilon < R_\phi R_w L \).

**Proof.** Suppose \( \text{Edim}(\mathcal{F}, \varepsilon) = m \) witnessed by the sequence \((x_1, f_1), \ldots, (x_m, f_m) \in \mathcal{X} \times \mathcal{F} \), for some \( f^* \in \mathcal{F} \) and \( \varepsilon > 0 \). Denote \( w_i := w(f_i) - w(f^*) \), and \( \phi_i := \phi(x_i) \). It follows that \( w_i \in \mathcal{B}_d(2R_w) \) and \( \phi_i \in \mathcal{B}_d(R_\phi) \). Since \( \sigma \in \mathcal{M}_\mu^L \), we have for any \( w_1, w_2, x \in \mathbb{R}^d \):

\[
\mu |\langle w_1 - w_2, x \rangle| \leq |\sigma(\langle w_1, x \rangle) - \sigma(\langle w_2, x \rangle)| \leq L |\langle w_1 - w_2, x \rangle|.
\]

Therefore, Eq. (5) can be replaced with:

\[
\max_{w \in \mathbb{R}^d} \left\{ \left| \langle w, \phi_i \rangle \right| : \sum_{j \neq i} |\langle w, \phi_i \rangle|^2 \leq \varepsilon^2/\mu^2, w \in \mathcal{B}_d(2R_w) \right\} > \varepsilon/L.
\]

(6)

Following the same steps with \( \lambda := \varepsilon^2/(4R_w^2 \mu^2) \) and \( V_i \) defined as before, we can further show that:

\[
\max_{w \in \mathbb{R}^d} \left\{ \left| \langle w, \phi_i \rangle \right| : \|w\|^2 \leq \varepsilon^2/\mu^2 + \lambda \cdot 4R_w^2 \right\} > \varepsilon/L,
\]

which implies the bound:

\[
\|\phi_i\|^2_{V_i^{-1}} > \frac{\varepsilon^2/L^2}{\varepsilon^2/\mu^2 + \lambda \cdot 4R_w^2} = \frac{1}{2} \cdot \frac{\mu^2}{L^2}.
\]

Using similar upper and lower bounds on \( \det(V_m) \) gives us that \( (1 + \mu^2/(2L^2))^{m/d} \leq 1 + \frac{\mu^2}{L^2} \cdot \frac{4R_w^2 R_\phi^2 \mu^2}{\varepsilon^2} \); the proof again concludes with an application of Lemma 9 with \( k = \frac{m^2}{d}, \alpha = \frac{\mu^2}{L^2} \) and \( \beta = \frac{4R_w^2 R_\phi^2 \mu^2}{\varepsilon^2} \).

**Discussion of prior work.** For completeness, we clarify the differences between Proposition 5 and the corresponding propositions in Russo and Van Roy (2013); Osband and Van Roy (2014).

Proposition 6 in Russo and Van Roy (2013) considers the setting exactly as in Claim 10. In our notation, the stated bound has the form \( \text{rk}(\mathcal{F}, R_\phi R_w) \cdot O(\log \left( 3 + \frac{12R_w^2}{\mu^2} \right)) \); the factor of \( R_\phi \) is missing inside the log term. As explained in Definition 3 (footnote 3), only the product \( R_\phi R_w \) is relevant as the scale of \( \phi \) and \( w \) is interchangeable.

Proposition 7 in Russo and Van Roy (2013) considers the setting exactly as in Claim 11. In our notation, the stated bound there has the form \( \mathcal{M}_\mu^L \cdot \text{rk}(\mathcal{F}, R_\phi R_w) \cdot O\left( \frac{L^2}{\mu^2} \cdot \log \left( \frac{3L^2}{\mu^2} + \frac{L^2}{\mu^2} \cdot \frac{12R_w^2}{\varepsilon^2} \right) \right) \); again the factor of \( R_\phi \) is missing. Also, the factor of \( L \) in \( R_w^2 L^2/\varepsilon^2 \) is improvable to \( \mu \).

Proposition 4 in Osband and Van Roy (2014) considers the setting analogous to Claim 11, but for vector-valued function classes, that is, \( \mathcal{F} \subseteq (\mathcal{X} \rightarrow \mathbb{R}^k) \). In the special case of \( k = 1 \), their bound in our notation, has the form \( \mathcal{M}_\mu^L \cdot \text{rk}(\mathcal{F}, R_\phi R_w) \cdot O\left( \frac{L^2}{\mu^2} \cdot \log \left( \frac{L^2}{\mu^2} + \frac{L^2}{\mu^2} \cdot \frac{R_w^2}{\varepsilon^2} \right) \right) \). The term \( R_\phi R_w \) should be \( R_\phi R_w \mu \). As shown in Equation 3, it is possible to make \( R_\phi R_w \) arbitrarily small while keeping \( L/\mu \) fixed.

Lastly, we note that Lemma 9 is slightly different than the corresponding inequality used in Russo and Van Roy (2013); Osband and Van Roy (2014) (which has \( (1 + \beta) \) in place of \( 2\beta \)). This allows us to remove the additive terms of 3 and \( 3 \frac{L^2}{\mu^2} \) inside the log factor for the \( \text{rk} \) case and the \( \mathcal{M}_\mu^L \cdot \text{rk} \) case respectively. In the \( \mathcal{M}_\mu^L \cdot \text{rk} \) case, this gives a nontrivial improvement in some regime of parameters; namely the bounds of Russo and Van Roy (2013); Osband and Van Roy (2014) would only give us a term of \( \log \left( \frac{L}{\mu} + \frac{R_w R_w L}{\varepsilon} \right) \), which can be loose when \( \mu \) is very small.