Probing electron Zitterbewegung in Topological Insulators

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Zitterbewegung is an inherent hallmark of the Dirac equation which arises from the superposition between the positive and negative energy part of the spinor states. This exotic quantum phenomenon is not observed very recently in traped ions and cold atom systems, but still a challenging task for electron systems. Here we present a general theory about the electron Zitterbewegung in a topological insulator in the presence of an in-plane electric field. We demonstrate theoretically that the topological edge states in topological insulators come from electron Zitterbewegung and the quantum spin Hall effect and quantum anomalous Hall effect can be viewed as a manifestation of anomalous electron Zitterbewegung.

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The Dirac equation, a combined theory of the quantum mechanics and the special relativity, predicts the existence of the electron spin and the antimatter, which were demonstrated experimentally in the last century. However, the Dirac equation also predicts some astonishing effects, for instance, ‘Zitterbewegung’ and Klein’s paradox. Zitterbewegung(ZB), an unexpected trembling motion of electron, was first predicted by E. Schrödinger from the Dirac equation in 1930 [1]. The ZB will be completely absent if there is no superposition between the positive- and negative-energy part of the spinor states. This rapid quivering motion around the mean position of electron at the speed of light $c$ is proposed to be a possible origin of electron spin and magnetic moment [2].

The ZB is a quantum phenomenon which can even appear in non-relativistic cases, such as a particle moving in a crystal [3] and a Cooper pair in superconductors [4, 5], but only observed experimentally in the trapped ion [6, 7] and cold atom [8] systems. However, this prediction is still never observed for electrons in free space, because this rapid trembling motion displays an extremely high frequency $\omega_Z = 2m_0 \omega_c / \hbar \approx 1.5 \times 10^{12}$ Hz and a tiny amplitude $\lambda_Z = \hbar / m_0 c \approx 3.9 \times 10^{-4}$ nm. The detection of the oscillation with such high frequency and negligibly small amplitude is beyond the reach of the state-of-the-art experimental technique. It is natural to imagine that the frequency of the ZB should be decreased when the energy gap $2m_0 \omega_c^2$ decreases.

Semiconductors are good candidates for observing the electron ZB phenomenon since its bandgaps range from 0 to 10 eV, which is much smaller than the energy gap ($\approx 1$ MeV) for electrons in the free space. Very recent theoretical works demonstrated that in semiconductor quantum structures the electron ZB can be generated by the spin-orbit interactions (SOIs) which couples spin-up and spin-down subbands and leads to an oscillatory motion of the wave packet [3]. This ZB behavior is a natural consequence of the SOIs which is a relativistic effect and generates an entanglement between the spin-up and spin-down subbands. The similar jitter behaviors are also found in single-wall semiconducting carbon nanotubes and single layer and bilayer graphene [3]. Although there are many theoretical works predicting the ZB behavior in different semiconductor quantum systems, the experimental measurement is a very challenging task and extremely difficult. This is because that there are many electrons ($10^{10}$ $\sim 10^{12}$/cm$^2$) in these quantum structures, the ZB of each electron shows different phases which smear out this oscillation finally.

The time-reversal invariant topological insulator (TI) is a new state of quantum matter possessing an insulating bulk and helical metallic edge or surface states, which shows a single massless Dirac dispersion at the $\Gamma$ point of the Brillouin zone. TIs are distinguished from a normal band insulator by a nontrivial topological invariant characterizing its band structure. The quantum spin Hall effect was proposed independently in graphene [12] and HgTe quantum wells [13]. The existence of edge states and surface states was confirmed by the recent experiment in HgTe quantum wells [14] and the angle-resolved photoemission spectroscopy experiments [15, 16].

Now we show that the electron quantum dynamics in topological insulators gives naturally the origin of the edge states based on a general theory of electron ZB. The quantum spin Hall effect and quantum anomalous Hall effect can be viewed as the manifestation of the electron ZB in TIs.

We consider the single-particle Hamiltonian of electron at low-energy regime in the presence of a uniform electric field $\mathbf{E}$

$$H = \epsilon(k) I + \sum_{i=1}^{3} d_i(k) \sigma_i - e \mathbf{E} \cdot \mathbf{r},$$

$$= \epsilon + D_0 - \mathbf{F} \cdot \mathbf{r},$$

(1)

where $\sigma_i (i = 1, 2, 3)$ is the Pauli matrix, $\epsilon(k)$ is the kinetic energy. We assume that the electric field points along the $x$ axis $\mathbf{E} = (E, 0, 0)$ without loss of any generality. The different forms of the off-diagonal term $d_i$ can be used to describe the different important systems: 1) the two-dimensional electron gas with the Rashba and

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We consider the single-particle Hamiltonian of electron at low-energy regime in the presence of a uniform electric field $\mathbf{E}$

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Dresselhaus SOIs with $d_0^0 = -\alpha k_y - \beta k_x$, $d_2^0 = \alpha k_x + \beta k_y$, $d_0^2 = 0$, where $\alpha$ and $\beta$ describe the strengths of the Rashba and Dresselhaus SOIs, respectively [3; 2] single layer graphene with $d_1^1 = v_F h k_x$, $d_2^2 = v_F h k_y$, $d_3^0 = 0$ [3; 3] bilayer graphene with $d_1^1 = -\hbar^2 (k_x^2 - k_y^2)/2m$, $d_2^2 = -\hbar^2 k_x k_y/m$, $d_3^0 = 0$ [3; 4] the two-dimensional TI HgTe quantum wells with an inverted band structure and the increasing of the order $T$. The spinor part of the low-order $T$ can see that the vectors spin operators is converted into the vector product be-
ducion ($A \otimes B$). First, we consider the simple case: for the electron position operator $y_H(t)$ evolving with the time $t$

$$y_H(t) = e^{iHt/h} ye^{-iHt/h},$$

$$= y(0) + (i/\hbar)[H, y] + \frac{(i/\hbar)^2}{2!} [H, [H, y]] + \cdots,$$

$$y(0) + (i/\hbar)e^\nu + \sum_{n=1}^{\infty} \frac{(i/\hbar)^n}{n!} T_n ,$$

where $e^\nu = [e, y] = -i\delta_{y_0} \epsilon, T_1 = [D_0^0, y] = \sum_j (-i\delta_{y_0}) \sigma_j, T_2 = [D_0^0, D_0^0] + F[D_0^0, x] = \sum_j d_j^0 d_j^0 [\sigma_j, \sigma_j] + F D_0^y = D_0^x y + F D_0^y, \cdots$, the $n$-th commutator $T_n$ can be obtained from the recursive relation, $T_n = [D_0^0, T_{n-1}] + F[T_{n-1}, x]$. Generally, the analytical expression of the above summation is very difficult to obtain because the number of the commutators increases hierarchically with increasing the order $n$.

Interestingly, the similarity between the commutation relationship $[\sigma_j, \sigma_j] = 2\epsilon_{ijk} \sigma_k$ and the vector cross production $(A \times B)_k = \epsilon_{ijk} A_j B_k$ offers us a new way to solve this problem. First, we consider the simple case: the electron ZB in the absence of an external electric field. In this simple case, the commutation between the spin operators is converted into the vector product between them. The high-order commutators $T_n$ are actually the vector product of the low-order $T_{n-1}$ and the spinor part $D_0$ of the Hamiltonian $H$, as shown in Fig. 1. We can transfer the algebra summation in Eq. (2) into the sum of commutators $T_2 n+1(T_2 n)(n = 1, 2, \cdots)$ utilizing the similarity between the commutator $[D_0^0, T_n]$ and the product vector $D_0^0 \times T_n$,

$$T_n = [D_0^0, T_{n-1}] \sim i(D_0^0 \times T_{n-1})/2$$

One can see that the vectors $T_2 n+1(T_2 n)$ point along the same direction while the lengths of the vectors $T_n$ varies with the increasing of the order $n$ as $T_2 n+1 = -|D_0^0|^2/2(T_2 n-1)$. This character allows us to get the analytical expression for the electron position operator

$$y_H(t) = y(0) + \frac{i t}{\hbar} (e^\nu + T_1) + \frac{T_2}{2|D_0^0|^2} \left[ \cos \left( 2(D_0^0)|t/\hbar| - 1 \right) \right. + \frac{i T_3}{2|D_0^0|^3} \left[ \sin \left( 2(D_0^0)|t/\hbar| - \left( 2|D_0^0||t/\hbar| \right) \right) \right] ,$$

where the matrix $S$ relates the $T_{n+1}$ and $T_n$ in the basis $\{D_0^0, D_0^0 \times \alpha \neq \beta \subset \{0, x, \ldots \}\}$, and $C_1$ is the coefficient of $T_1$ in this basis. When $F \neq 0$, $S$ can be diagonalized as $S = VEV^{-1}$, and $C_0 = S^{-1} C_1$, thus the

FIG. 1: (color online). Schematic of electron Zitterbewegung in a HgTe quantum well with an inverted band without and with an in-plane electric field (a) and (b). Schematic of the recursive relationship between the commutators with different orders (c).
expression of the ZB can be simplified as
\[
y_x(t) = y(0) + (it/\hbar)\delta y - DC_0 + D\textbf{V}\exp(it\textbf{E}/\hbar)V^{-1}C_0.
\] (4)

In general, the theory we developed here can be applied to more general cases (see the supporting material), even for the trapped ion and cold atom systems.

Now we consider electron ZB in two-dimensional topological insulators mentioned above including a driving electric field. We consider a HgTe quantum well with an inverted band structure described by the BHZ model [13]. The predicted topological edge states are demonstrated experimentally recently [14]. The single-particle effective Hamiltonian for the electron is
\[
H = \epsilon + D^0 = Fx,
\] (5)
where \(\epsilon = C - Dk^2\), \(D^0 = (Ak_x, Ak_y, M - Bk^2) \cdot \Sigma, \Sigma = (\sigma_x, \sigma_y, \sigma_z)^T\). A, B, C, D, and M are the parameters determined by the thickness of the quantum well. In this case, \(D^0\) is expanded in the basis \(\{D^0, D^x, D^y, D^z, D^y x, D^x y, D^z x, D^z y, D^x y z, D^x y^2\}\), and the S matrix in this basis is
\[
\begin{array}{cccccccccc}
0 & 0 & 0 & 0 & -4P_{0,x} & -4P_{0,y} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 4P_{0,0} & 0 & 0 & -4P_{0,y} & -4P_{0,x} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

where \(P_{0,0} = A^2 k_x^2 + (M - Bk^2)^2\), \(P_{0,x} = -iA^2 k_x + 2iBk_x(M - Bk^2)\), \(P_{0,y} = -iA^2 k_y + 2iBk_y(M - Bk^2)\), and \(P_{0,z} = 2B(M - Bk^2)\).

Next we will show the electron ballistic orbital motion for both a topological insulator and a normal band insulator, corresponding to the positive and negative parameters M. For an electron injected in the Gaussian wave packet moving along the x axis, i.e., \(k_x = k_{0x}, k_y = 0\) at the beginning \(t = 0\), with spin pointing along the \(z\) axis, which is perpendicular to the quantum well \(\varphi(k_x) = \frac{1}{\sqrt{2\pi}} \int dk_x e^{-((k_x - k_{0x})^2)/2\Gamma^2} \phi_{k_{0x},x}(k_x)\hat{\uparrow}\), where \(\Gamma\) is the width of the wave packet in \(k\)-space and \(k_{0x}\) is the initial central wave vector in the \(x\) direction for the packet. The guiding center of the wave packet \(\langle y(t)\rangle\) can be calculated numerically utilizing the explicit expression (see Eq. (4)). The numerical results are plotted in Figs. 2(a) and 2(b). From these figures, one can see clearly the electron orbital motions are very different between the normal band insulator and topological insulator phases. For the normal insulator with \(M > 0\) (see Fig. 2(b)), the spin-up electron trajectory, i.e., the guiding center \(\langle y(t)\rangle\), shows an oscillation against the propagating time as in a conventional 2DEG with SOIs; while for the topological insulator \(M < 0\) (see Fig. 2(a)), a spin-up electron shows a surprising behavior trajectory which bends opposite to the \(y\) axis monotonously. This behavior make it possible to observe the Zitterbewegung first time in a solid. As we mentioned before, electrons in the normal band insulator oscillate with different phases, leading to smearing out of the ZB.

Now we turn to demonstrate theoretically the edge states in a spin Hall bar with a finite width. For an realistic case, i.e., the HgTe quantum well sample with finite size, one can image that the electron will be bounced back when it reaches the boundary of the spin Hall bar. From the expression of the spin-orbit force \(f_x \sim [H, [H, y]] \sim A[M + B(k_y^2 - k_x^2)]\sigma_x - 2ABk_xk_y\sigma_y - \sigma_x^2\sigma_x\), one can see that the last term \(-A^2 k_x\sigma_x\) the spin-orbit force always bends the spin-up (down) electron downward (upward). Therefore the electron bounced back from the boundary will bend to the boundary again, and forms a edge channel in the vicinity of the boundary finally. This spin-orbit force could make the edge state robust against the non-magnetic impurity scattering. The spin-orbit force will lead to the spin Hall effect in a 2DEG in the presence of SOIs. Notice that the energy of incident electron here is \(E = C + M - (B + D)(T^2 + k_{0x}^2)/2\), which locates inside the bulk gap.

To explicitly get the edge states in quasi-one-dimensional systems, we can map the infinite 2D system \((x, Y \in (-\infty, +\infty))\) into a quasi-one-dimensional system with a finite width \((x, y \in (-1, +1))\) using the transformation \(\pm 1/(1 + y) = Y \pm 1\), where \(y\) and \(Y\) are the dimensionless coordinates for the infinite and finite systems, respectively. For \(Y > 0\) (for \(Y < 0\) case, we can adopt the same transformation \(-1/(1 + y) = Y - 1\) \(k_Y = -i\delta_Y = (y - 1)^2k_{0y}\), then the original 2D Hamiltonian

\[
FIG. 2: (color online). The calculated guiding center \(\langle y(t)\rangle\) for a spin-up electron injected in a Gaussian wave packet for topological insulator \((M < 0)\) (a) and normal band insulator \((M > 0)\) (b). The red and blue curves denote the trajectories of spin-up and spin-down electrons, respectively. The relevant parameters for the graphene \(\nu\) and topological insulator \(A, B, C, D, M\) are taken from Ref. [8] and [20]. An external dc electrical field in experiments is applied along the x axis \(E = 1.0 \times 10^{-4}\text{mV/nm}\), i.e., \(F = 1.0 \times 10^{-4}\text{meV/\text{nm}}\), the width of the Gaussian wave packet is \(\Gamma = 1.0 \times 10^{-3}\text{nm}^{-1}\) and the incident momentum is \(k_{0x} = 0.005\text{nm}^{-1}\).
nian can be written as

\[
H_0^y = A_0 \sigma^z + B k_x^2 - B k_y^2 \sigma^z,
\]

where \(A' = A(y-1)^2\) and the last term of Eq. \(V(y) = -B'\{(y-1)^4 + 2(y-1)^3 - 1\}k_y^2\sigma^z = -Bk_y^2\sigma^z v(y)\) can be viewed as the confining potential without any loss of the generality. Thus a spin-up electron will feel a confining potential \(V(y)\), and the ZB solution can be obtained by converting the 2D solution into this confined system utilizing the above coordinate transformation. The spin-up and -down edge states appear at the vicinity of the opposite boundaries of the HgTe spin Hall bar in the topological phase \((M < 0)\) when \(y \to \pm 1\). For the normal band insulator phase \((M > 0)\), one can see that the electron exhibits an normal oscillation in the bar (see Fig. 3).

Finally, we would like to point out that the quantum anomalous spin Hall effect can also be understood from the electron ZB in TIs. Considering a ferromagnetic TI, a giant Zeeman splitting results in a spin-up inverted and a spin-down normal band structure, we assume these magnetic ions polarized along \(z\) axis. This situation corresponds to the BHZ Hamiltonian with the positive and negative \(M\) for spin-up and spin-down electrons, respectively. From the above discussions, the spin-up (down) electrons show normal oscillating (bending) trajectory (see Fig. 2). Therefore, the edge states only appear for spin-down electrons, i.e., the quantum anomalous spin Hall effect. Similar behavior can also be found in three-dimensional topological insulators, because the form of the Hamiltonian of 3D TI is almost the same as that of 2D TI. Therefore one can expect that the bending trajectory of electrons in 3D TIs will lead to the topological surface states.

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\[\text{FIG. 3: (color online). The trajectories of the electron Gaussian wave packets in a spin Hall bar mapped from 2D infinite system. The inset shows the transverse confining potential. For the topological insulator (\(M < 0\)), the edge state appears in the vicinity of the QSH bar; while for the normal band insulator (\(M > 0\)), the trajectory shows a quivering behavior.}\]
Calculation for $y_H(t)$ in spin-orbit coupled systems

We consider the single-particle Hamiltonian of electron in the presence of a uniform electric field $E$

\[
H = \epsilon(k)I + d^0(k)\sigma^i - eE \cdot r
= \epsilon + D^0 - F \cdot r,
\]

(S1)

where $\sigma_i (i = 1, 2, 3)$ is the Pauli matrix, $\epsilon(k)$ is the kinetic energy. We assume that the electric field points along the $x$ axis $E = (E, 0, 0)$ without loss of any generality.

In the following, we will derive the electron position operator $y_H(t)$ evolving with time in the Heisenberg picture.

\[
y_H(t) = e^{iHt/\hbar} y e^{-iHt/\hbar} = y(0) + \frac{(it/\hbar)^2}{2!} [H, y] + \frac{(it/\hbar)^3}{3!} [H, [H, y]] + \cdots
= y(0) + \frac{(it/\hbar)^2}{2} \epsilon y + \sum_{n=1}^{\infty} \frac{(it/\hbar)^n}{n!} T_n,
\]

(S2)

here we show the notions explicitly:

\[
\epsilon y = [\epsilon, y] = -i\partial_k \epsilon, \\
\epsilon^x y = [\epsilon, x] = \partial_k \epsilon.
\]

(S3)

\[
T_1 = [D^0, y] = (-i\partial_k d^0_i)\sigma^i,
T_2 = [D^0, D^y] + F[D^y, x]
= d^0_i \sigma^i + Fd^y_i \sigma^i
= 2i\epsilon \delta_{ij} d^0_j \sigma^i
+ Fd^y_i \sigma^i
= D^0 y + FD^y x, ...
\]

(S4)

The $n$-th commutator $T_n$ can be obtained from the $T_{n-1}$,

\[
T_n = [D^0, T_{n-1}] + F[T_{n-1}, x],
\]

(S5)

and:

\[
[D^0, D^a] = D^{0 \times a}, \\
[D^{a \times b}, x] = D^{ax \times b} + D^{ax \times b}, \\
D^{a \times (b \times c)} = (2i)^2 (P_{a,b} D^b - P_{a,b} D^b),
\]

(S6)

where $a, b, c \in \{0, x, y, x^2, xy, \cdots\}$, which can be any combination of $x, y$; $P_{a,b} = D^a \cdot D^b = d^a_i d^b_i$ is the “dot” product of $D^a$ and $D^b$.

The above relations indicate that all $T_n$ are linear combination of $\{D^a, D^{a \times b}; a \neq b \in \{0, x, y, x^2, xy, \cdots\}\}$, the
recursion relation between \( T_n \) and \( T_{n-1} \) can be represented by the matrix \( S \) in the vector space spanned by \( \{ D^a, D^{b \times c} \} \):

\[
T_n = \sum_{\alpha=a,b,c} C_{n-1,\alpha}[D^0, D^\alpha] + FC_{n-1,\alpha}[D^\alpha, x]
\]

\[
= \sum_{\alpha=a,b,c} C_{n-1,\alpha}D^0 \times \alpha + FC_{n-1,\alpha}D^\alpha \times x
\]

\[
= \sum_{\beta=a,b,c} \left( \sum_{\alpha=a,b,c} S_{\beta \alpha} C_{n-1,\alpha} \right) D^\beta,
\]  
(S7)

or in the compact matrix form:

\[
T_n = \mathbf{D} \cdot \mathbf{C}_n = \mathbf{D} \cdot \mathbf{S} \mathbf{C}_n - 1
\]

\[
= \cdots = \mathbf{D} \cdot \mathbf{S}^{n-1} \mathbf{C}_1,
\]  
(S8)

\[
\mathbf{D} = (D^0, D^x, D^y, \cdots),
\]

\[
\mathbf{C}_n = (C_{n,0}, C_{n,x}, C_{n,y}, \cdots)^T.
\]

thus

\[
y_H(t) = y(0) + (it/\hbar)\epsilon y + \frac{(it/\hbar)^2 \epsilon \epsilon_y}{2}
\]

\[
+ \mathbf{D} \sum_{n=1} \frac{(it/\hbar)^n}{n!} \mathbf{S}^{n-1} \mathbf{C}_1.
\]  
(S9)

When \( F = 0 \), i.e., in the absence of an electric field, \( y_H(t) \) can be calculated without diagonalize \( S \) because

\[
\mathbf{S}^{2k+1} = P_{0,0} \mathbf{S}^{2k-1} = P_{0,0}^k \mathbf{S},
\]

\[
\mathbf{S}^{2k+2} = P_{0,0} \mathbf{S}^{2k} = P_{0,0}^k \mathbf{S}^2,
\]

so that

\[
y_H(t) = y(0) + (it/\hbar)\epsilon y + \frac{(it/\hbar)^2 \epsilon \epsilon_y}{2}
\]

\[
+ \sin \left( \frac{it P_{1/0}^{1/2}}{\hbar} \right) \mathbf{D} \cdot \mathbf{S}^2 \mathbf{C}_1
\]

\[
+ \cos \left( \frac{it P_{1/0}^{1/2}}{\hbar} \right) \mathbf{D} \cdot \mathbf{S} \mathbf{C}_1,
\]  
(S10)

this result is consistent with previous results with zero field.

When \( F \neq 0 \), \( \mathbf{S} \) can be diagonalize as

\[
\mathbf{S} = \mathbf{V} \mathbf{E} \mathbf{V}^{-1},
\]

and

\[
y_H(t) = y(0) + (it/\hbar)\epsilon y
\]

\[
- \mathbf{D} \cdot \mathbf{C}_0 + \mathbf{D} \cdot \mathbf{V} \exp(it \mathbf{E}/\hbar) \mathbf{V}^{-1} \mathbf{C}_0,
\]  
(S11)

where \( \mathbf{C}_0 = \mathbf{S}^{-1} \mathbf{C}_1 \).

Generalization of the calculation

Consider the generalized Hamiltonian under an external force field \( \mathcal{F}_r \)

\[
H = \epsilon + \mathcal{D}^0 - \mathcal{F}_r,
\]  
(S12)
where $\mathcal{D}^0$ is the generalized spin-dependent operator in $k$-space, while belongs to the closed algebra $\mathcal{A}$ satisfying:

for any $\mathcal{D}^a, \mathcal{D}^b \in \mathcal{A}$

$$[\mathcal{D}^a, \mathcal{D}^b] = C_{abc} \mathcal{D}^c, \quad (S13(a))$$

$$[\mathcal{D}^a, \mathcal{F}_r] = B_{ab} \mathcal{D}^b, \quad (S13(b))$$

where $C_{abc}$ is the structure constant ($k$-dependent) of the closed algebra $\mathcal{A}$; $B_{ab}$ is the structure constant of the extended closed algebra include both $\mathcal{A}$ and $\mathcal{F}_r$.

In the same way,

$$T_n = \mathcal{D} \cdot C_n = \mathcal{D} \cdot \mathcal{S} C_{n-1}$$

$$= \cdots = \mathcal{D} \cdot \mathcal{S}^{n-1} C_1,$$

$$\mathcal{D} = (\mathcal{D}^0, \mathcal{D}^x, \mathcal{D}^y, \mathcal{D}^z),$$

$$C_n = (C_{n,0}, C_{n,x}, C_{n,y}, \cdots)^T.$$  \hspace{1cm} (S14)

and

$$y_H(t) = y(0) + \frac{(it/\hbar)\gamma y}{2} + \frac{(it/\hbar)^2 \gamma F y}{2}$$

$$+ \mathcal{D} \cdot \sum_{n=1}^{\infty} \frac{(it/\hbar)^n}{n!} \mathcal{S}^{n-1} C_1.$$  \hspace{1cm} (S15)

In spite of the position operator, other operators like velocity and spin can be calculated in the similar way.