Towards unification of quark and lepton flavors in $A_4$ modular invariance

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Abstract

We study quark and lepton mass matrices in the $A_4$ modular symmetry towards the unification of the quark and lepton flavors. We adopt modular forms of weights 2 and 6 for quarks and charged leptons, while we use modular forms of weight 4 for the neutrino mass matrix which is generated by the Weinberg operator. We obtain the successful quark mass matrices, in which the down-type quark mass matrix is constructed by modular forms of weight 2, but the up-type quark mass matrix is constructed by modular forms of weight 6. The viable region of $\tau$ is close to $\tau = i$. Lepton mass matrices also work well at nearby $\tau = i$, which overlaps with the one of the quark sector, for the normal hierarchy of neutrino masses. In the common $\tau$ region for quarks and leptons, the predicted sum of neutrino masses is 87–120 meV taking account of its cosmological bound. Since both the Dirac CP phase $\delta_{CP}^\ell$ and $\sin^2 \theta_{23}$ are correlated with the sum of neutrino masses, improving its cosmological bound provides crucial tests for our scheme as well as the precise measurement of $\sin^2 \theta_{23}$ and $\delta_{CP}^\ell$. The effective neutrino mass of the $0\nu\beta\beta$ decay is $\langle m_{ee} \rangle = 15–31$ meV. It is remarked that the modulus $\tau$ is fixed at nearby $\tau = i$ in the fundamental domain of SL(2,Z), which suggests the residual symmetry $Z_2$ in the quark and lepton mass matrices. The inverted hierarchy of neutrino masses is excluded by the cosmological bound of the sum of neutrino masses.

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1 Introduction

The standard model (SM) was well established by the discovery of the Higgs boson. However, the flavor theory of quarks and leptons is still unknown. In order to understand the origin of the flavor structure, many works have appeared by using the discrete groups for flavors. In the early models of quark masses and mixing angles, the $S_3$ symmetry was used [1,2]. It was also discussed to understand the large mixing angle [3] in the oscillation of atmospheric neutrinos [4]. For the last twenty years, the discrete symmetries of flavors have been developed, that is motivated by the precise observation of flavor mixing angles of leptons [5–13].

Many models have been proposed by using the non-Abelian discrete groups $S_3$, $A_4$, $S_4$, $A_5$ and other groups with larger orders to explain the large neutrino mixing angles. Among them, the $A_4$ flavor model is an attractive one because the $A_4$ group is the minimal one including a triplet irreducible representation, which allows for a natural explanation of the existence of three families of leptons [14–20]. However, the variety of models is so wide that it is difficult to obtain a clear evidence of the $A_4$ flavor symmetry.

Recently, a new approach to the lepton flavor problem appeared based on the invariance under the modular group [21], where the model of the finite modular group $\Gamma_3 \simeq A_4$ has been presented. This work inspired further studies of the modular invariance approach to the lepton flavor problem. It should be emphasized that there is a significant difference between the models based on the $A_4$ modular symmetry and those based on the usual non-Abelian discrete $A_4$ flavor symmetry. Yukawa couplings transform non-trivially under the modular group and are written in terms of modular forms which are holomorphic functions of the modulus $\tau$.

The modular group includes the finite groups $S_3$, $A_4$, $S_4$, and $A_5$ [22]. Therefore, an interesting framework for the construction of flavor models has been put forward based on the $\Gamma_3 \simeq A_4$ modular group [21], and further, based on $\Gamma_2 \simeq S_3$ [23]. The proposed flavor models with modular symmetries $\Gamma_4 \simeq S_4$ [24] and $\Gamma_5 \simeq A_5$ [25] have also stimulated studies of flavor structures of quarks and leptons. Phenomenological discussions of the neutrino flavor mixing have been done based on $A_4$ [26,27], $S_4$ [28,29], $A_5$ [30], and $T'$ [31] modular groups, respectively. In particular, the comprehensive analysis of the $A_4$ modular group has provided a distinct prediction of the neutrino mixing angles and the CP violating phase [27]. The $A_4$ modular symmetry has been also applied to the $SU(5)$ grand unified theory (GUT) of quarks and leptons [32], while the residual symmetry of the $A_4$ modular symmetry has been investigated phenomenologically [33]. Furthermore, modular forms for $\Delta(96)$ and $\Delta(384)$ were constructed [34], and the extension of the traditional flavor group is discussed with modular symmetries [35]. Moreover, multiple modular symmetries are proposed as the origin of flavor [36]. The modular invariance has been also studied combining with the generalized CP symmetries for theories of flavors [37]. The quark mass matrix has been discussed in the $S_3$ and $A_4$ modular symmetries as well [38,40]. Besides mass matrices of quarks and leptons, related topics have been discussed in the baryon number violation [38], the dark matter [41,42] and the modular symmetry anomaly [43].

In this work, we study both quark and lepton mass matrices in the $A_4$ modular symmetry. If flavors of quarks and leptons are originated from a same two-dimensional compact space, quarks and leptons have the same flavor symmetry and the same value of the modulus $\tau$. Therefore, it is challenging to reproduce observed three Cabibbo-Kobayashi-Maskawa (CKM) mixing angles and the CP violating phase while observed large mixing angles in the lepton sector within the framework of the $A_4$ modular invariance with the common $\tau$. This work provides a new aspect in order to
discover the unification theory of the quark and lepton flavors. We have already discussed the quark mass matrices in terms of $A_4$ modular forms of weight 2 \cite{10}. It has been found that quark mass matrices of $A_4$ do not work unless Higgs sector is extended by $A_4$ triplet representations. In this paper, we propose to adopt modular forms of weight 6 in addition to modular ones of weight 2 for quarks with the Higgs sector of SM. We use modular forms of weight 4 for the neutrino mass matrix, which is generated by the Weinberg operator. The common value of the modulus $\tau$ is successfully obtained by using observed four CKM matrix elements and three lepton mixing angles. We also predict the CP violating Dirac phase of leptons, which is expected to be observed at T2K and NOνA experiments \cite{44,45}, and the sum of neutrino masses$^1$.

The paper is organized as follows. In section 2, we give a brief review on the modular symmetry and modular forms of weights 2, 4 and 6. In section 3, we present the model for quark mass matrices in the $A_4$ modular symmetry. In section 4, we show numerical results for the CKM matrix. In section 5, we discuss the lepton mass matrices and present numerical results. Section 6 is devoted to a summary. In Appendix A, the tensor product of the $A_4$ group is presented. In Appendix B, we present how to obtain Dirac CP phase, Majorana phases and effective mass of the $0\nu\beta\beta$ decay.

## 2 Modular group and modular forms of weights 2, 4, 6

The modular group $\bar{\Gamma}$ is the group of linear fractional transformations $\gamma$ acting on the modulus $\tau$, belonging to the upper-half complex plane as:

$$\tau \longrightarrow \gamma \tau = \frac{a \tau + b}{c \tau + d}, \text{ where } a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1, \text{ Im}[\tau] > 0,$$

which is isomorphic to $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{I, -I\}$ transformation. This modular transformation is generated by $S$ and $T$,

$$S: \tau \longrightarrow -\frac{1}{\tau}, \quad T: \tau \longrightarrow \tau + 1,$$

which satisfy the following algebraic relations,

$$S^2 = I, \quad (ST)^3 = I. \quad (3)$$

We introduce the series of groups $\Gamma(N)$ ($N = 1, 2, 3, \ldots$), called principal congruence subgroups, defined by

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}. \quad (4)$$

For $N = 2$, we define $\bar{\Gamma}(2) \equiv \Gamma(2)/\{I, -I\}$. Since the element $-I$ does not belong to $\Gamma(N)$ for $N > 2$, we have $\bar{\Gamma}(N) = \Gamma(N)$. The quotient groups defined as $\Gamma_N \equiv \bar{\Gamma}/\bar{\Gamma}(N)$ are finite modular groups. In this finite groups $\Gamma_N$, $T^N = I$ is imposed. The groups $\Gamma_N$ with $N = 2, 3, 4, 5$ are isomorphic to $S_3$, $A_4$, $S_4$ and $A_5$, respectively \cite{22}.

\footnote{In order to reduce the number of parameters, we discussed another $A_4$ quark/lepton mass matrices after this work \cite{46}. However, the sum of neutrino masses is predicted to be larger than the cosmological upper-bound, 120 meV. Indeed, a model with two parameters fewer than the present model gives 140 meV for the sum of neutrino masses.}

\section*{References}

- [10]
- [44]
- [45]
- [22]
Modular forms of level $N$ are holomorphic functions $f(\tau)$ transforming under $\Gamma(N)$ as:

$$f(\gamma \tau) = (c\tau + d)^k f(\tau), \quad \gamma \in \Gamma(N),$$

where $k$ is the so-called as the modular weight.

Superstring theory on the torus $T^2$ or orbifold $T^2/Z_N$ has the modular symmetry $[47][52]$. Its low energy effective field theory is described in terms of supergravity theory, and string-derived supergravity theory has also the modular symmetry. Under the modular transformation of Eq.(1), the supergravity theory is given simply by

$$\phi(\tau) \rightarrow (c\tau + d)^{-k_I} \rho^{(I)}(\gamma) \phi(\tau),$$

where $-k_I$ is the modular weight and $\rho^{(I)}(\gamma)$ denotes a unitary representation matrix of $\gamma \in \Gamma$.

In the present article we study global supersymmetric models, e.g., minimal supersymmetric extensions of the Standard Model (MSSM). The superpotential which is built from matter fields and modular forms is assumed to be modular invariant, i.e., to have a vanishing modular weight. For given modular forms this can be achieved by assigning appropriate weights to the matter superfields.

The kinetic terms are derived from a Kähler potential. The Kähler potential of chiral matter fields $\phi^{(I)}$ with the modular weight $-k_I$ is given simply by

$$K_{\text{matter}} = \frac{1}{|i(\bar{\tau} - \tau)|^{k_I}} |\phi^{(I)}|^2,$$

where the superfield and its scalar component are denoted by the same letter, and $\bar{\tau} = \tau^*$ after taking the vacuum expectation value (VEV) $[3]$. Therefore, the canonical form of the kinetic terms is obtained by changing the normalization of parameters $[27]$.

For $\Gamma_3 \simeq A_4$, the dimension of the linear space $\mathcal{M}_k(\Gamma(3))$ of modular forms of weight $k$ is $k + 1 [53][57]$, i.e., there are three linearly independent modular forms of the lowest non-trivial weight $2$. These forms have been explicitly obtained $[21]$ in terms of the Dedekind eta-function $\eta(\tau)$:

$$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \quad q = \exp(i2\pi\tau),$$

where $\eta(\tau)$ is a so called modular form of weight $1/2$. In what follows we will use the following basis of the $A_4$ generators $S$ and $T$ in the triplet representation:

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

where $\omega = \exp(i\pi/3)$. The modular forms of weight 2 transforming as a triplet of $A_4$ can be written in terms of $\eta(\tau)$ and its derivative $[21]$:

$$Y_1(\tau) = \frac{i}{2\pi} \left( \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \frac{\eta'((\tau + 2)/3)}{\eta((\tau + 2)/3)} - \frac{27\eta'(3\tau)}{\eta(3\tau)} \right),$$

$$Y_2(\tau) = \frac{-i}{\pi} \left( \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega^2 \eta'((\tau + 2)/3) \right),$$

$$Y_3(\tau) = \frac{-i}{\pi} \left( \frac{\eta'((\tau + 1)/3)}{\eta((\tau + 1)/3)} + \omega \eta'((\tau + 2)/3) \right).$$

$\text{The most general Kähler potential consistent with the modular symmetry possibly contains additional terms, as recently pointed out in Ref. [54]. However, we consider only the simplest form of the Kähler potential.}$
They satisfy also the constraint \([21]\):

\[
(Y_2(\tau))^2 + 2Y_1(\tau)Y_3(\tau) = 0 .
\]  

The overall coefficient in Eq. (10) is one possible choice. It cannot be uniquely determined. The triplet modular forms of weight 2 have the following \(q\)-expansions:

\[
Y_3^{(2)} = \begin{pmatrix} Y_1(\tau) \\ Y_2(\tau) \\ Y_3(\tau) \end{pmatrix} = \begin{pmatrix} 1 + 12q + 36q^2 + 12q^3 + \ldots \\ -6q^{1/3}(1 + 7q + 8q^2 + \ldots) \\ -18q^{2/3}(1 + 2q + 5q^2 + \ldots) \end{pmatrix} .
\]  

They satisfy also the constraint \([21]\):

\[
(Y_2(\tau))^2 + 2Y_1(\tau)Y_3(\tau) = 0 .
\]  

The modular forms of the higher weight, \(k\), can be obtained by the \(A_4\) tensor products of the modular forms with weight 2, \(Y_3^{(2)},\) as given in Appendix A. For weight 4, that is \(k = 4\), there are five modular forms by the tensor product of \(3 \otimes 3\) as:

\[
Y_1^{(4)} = Y_1^2 + 2Y_2Y_3 , \quad Y_1^{(4)} = Y_3^2 + 2Y_1Y_2 , \quad Y_1^{(4)} = Y_2^2 + 2Y_1Y_3 = 0 ,
\]

\[
Y_3^{(4)} = \begin{pmatrix} Y_1^{(4)} \\ Y_2^{(4)} \\ Y_3^{(4)} \end{pmatrix} = \begin{pmatrix} Y_1^2 - Y_2Y_3 \\ Y_3^2 - Y_1Y_2 \\ Y_2^2 - Y_1Y_3 \end{pmatrix} ,
\]

where \(Y_1^{(4)}\) vanishes due to the constraint of Eq. (12). For weight 6, there are seven modular forms by the tensor product of \(A_4\) as:

\[
Y_1^{(6)} = Y_1^3 + Y_2^3 + Y_3^3 - 3Y_1Y_2Y_3 ,
\]

\[
Y_3^{(6)} = \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = \begin{pmatrix} Y_1^3 + 2Y_1Y_2Y_3 \\ Y_2^3 + 2Y_2Y_3^2 \\ Y_3^3 + 2Y_3Y_2^2 \end{pmatrix} , \quad Y_3^{(6)} = \begin{pmatrix} Y_1^{(6)} \\ Y_2^{(6)} \\ Y_3^{(6)} \end{pmatrix} = \begin{pmatrix} Y_3^3 + 2Y_1Y_2Y_3 \\ Y_3^3 + 2Y_2Y_3^2 \\ Y_3^3 + 2Y_3Y_2^2 \end{pmatrix} .
\]

By using these modular forms of weights 2, 4 and 6, we discuss quark and lepton mass matrices.

### 3 Quark mass matrices in the \(A_4\) modular invariance

Let us consider a \(A_4\) modular invariant flavor model for quarks. There are freedoms for the assignments of irreducible representations and modular weights to quarks and Higgs doublets.

The simplest one is to assign the triplet of the \(A_4\) group to three left-handed quarks, but three different singlets \((1, 1', 1')\) of \(A_4\) to the three right-handed quarks, \((u^c, c^c, t^c)\) and \((d^c, s^c, b^c)\), respectively, where the sum of weights of the left-handed and the right-handed quarks is \(-2\). The Higgs fields are supposed to be \(A_4\) singlets with weight 0. Then, three independent couplings appear in the superpotential of the up-type and down-type quark sectors, respectively, as follows:

\[
w_u = \alpha_u u^c H_u Y_3^{(2)} Q + \beta_u c^c H_u Y_3^{(2)} Q + \gamma_u t^c H_u Y_3^{(2)} Q ,
\]

\[
w_d = \alpha_d d^c H_d Y_3^{(2)} Q + \beta_d s^c H_d Y_3^{(2)} Q + \gamma_d b^c H_d Y_3^{(2)} Q ,
\]

\[
w = \alpha_w H_u Y_3^{(2)} Q + \beta_w c^c H_u Y_3^{(2)} Q + \gamma_w t^c H_u Y_3^{(2)} Q ,
\]
where $Q$ is the left-handed $A_4$ triplet quarks, and $H_q$ is the Higgs doublet. The parameters $\alpha_q$, $\beta_q$, $\gamma_q$ are constant coefficients. Assign the $A_4$ triplet $Q$ as $((d_L, u_L), (s_L, c_L), (b_L, t_L))$. By using the decomposition of the $A_4$ tensor product in Appendix A, the superpotentials in Eqs. (15) and (16) give the mass matrix of quarks, which is written in terms of modular forms of weight 2 as:

$$M_q = v_q \begin{pmatrix} \alpha_q & 0 & 0 \\ 0 & \beta_q & 0 \\ 0 & 0 & \gamma_q \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}^{\tau}_{RL}, \quad (q = u, d),$$

(17)

where $\tau$ of the modular forms $Y_i(\tau)$ is omitted. The constant $v_q (q = u, d)$ is the VEV of the neutral component of the Higgs field $H_q$. Parameters $\alpha_q, \beta_q, \gamma_q$ are taken to be real without loss of generality, and they can be adjusted to the observed quark masses. The remained parameter is only the modulus $\tau$. The numerical study of the quark mass matrix in Eq. (17) is rather easy. However, it is impossible to reproduce observed hierarchical three CKM mixing angles by fixing one complex parameter $\tau$.

| $SU(2)$ | $A_4$ | $-k_I$ | $Q$ | $(d^c, s^c, b^c)$ | $(u^c, c^c, t^c)$ | $H_q$ | $Y_3^{(2)}$, $Y_3^{(6)}$, $Y_3^{(6)}$ |
|---------|-------|-------|-----|--------------------|------------------|-------|-------------------|
| 2       | 3     | -2    | 1   | $(1, 1^\prime, 1^\prime)$ | $(1, 1^\prime, 1^\prime)$ | 2     | 1, 3, 3, 3'       |
|         |       |       |     |       | $(-4, -4, -4)$ |       | $k = 2, k = 6, k = 6$ |

Table 1: Assignments of representations and weights $-k_I$ for MSSM fields and modular forms.

In order to obtain realistic quark mass matrices, we use modular forms of weight 6 in Eq. (14). As a simple model, we take modular forms of weight 6 only for the up-type quark mass matrix while the down-type quark one is still given in terms of modular forms of weight 2 such as Eq. (17)\textsuperscript{3}. Then, we have six independent couplings in the superpotential of the up-quark sector as:

$$w_u = \alpha_u u^c H_u Y_3^{(6)} Q + \alpha'_u u^c H_u Y_3^{(6)} Q + \beta_u c^c H_u Y_3^{(6)} Q + \beta'_u c^c H_u Y_3^{(6)} Q + \gamma_u t^c H_u Y_3^{(6)} Q + \gamma'_u t^c H_u Y_3^{(6)} Q,$$

(18)

where assignments of representations and weights for MSSM fields are given in Table 1. The up-type quark mass matrix is written as:

$$M_u = v_u \begin{pmatrix} \alpha_u & 0 & 0 \\ 0 & \beta_u & 0 \\ 0 & 0 & \gamma_u \end{pmatrix} \begin{pmatrix} Y_1^{(6)} & Y_3^{(6)} & Y_2^{(6)} \\ Y_2^{(6)} & Y_1^{(6)} & Y_3^{(6)} \\ Y_3^{(6)} & Y_2^{(6)} & Y_1^{(6)} \end{pmatrix}^{\tau}_{RL} + \begin{pmatrix} g_{u1} & 0 & 0 \\ 0 & g_{u2} & 0 \\ 0 & 0 & g_{u3} \end{pmatrix} \begin{pmatrix} Y_1^{(6)} & Y_3^{(6)} & Y_2^{(6)} \\ Y_2^{(6)} & Y_1^{(6)} & Y_3^{(6)} \\ Y_3^{(6)} & Y_2^{(6)} & Y_1^{(6)} \end{pmatrix}^{\tau}_{RL},$$

(19)

where $g_{u1} = \alpha'_u/\alpha_u$, $g_{u2} = \beta'_u/\beta_u$ and $g_{u3} = \gamma'_u/\gamma_u$ are complex parameters while $\alpha_u$, $\beta_u$ and $\gamma_u$ are real. On the other hand, the down-type quark mass matrix is given as:

$$M_d = v_d \begin{pmatrix} \alpha_d & 0 & 0 \\ 0 & \beta_d & 0 \\ 0 & 0 & \gamma_d \end{pmatrix} \begin{pmatrix} Y_1 & Y_3 & Y_2 \\ Y_2 & Y_1 & Y_3 \\ Y_3 & Y_2 & Y_1 \end{pmatrix}^{\tau}_{RL}.$$

(20)

We will fix the modulus $\tau$ phenomenologically by using quark mass matrices in Eqs. (19) and (20).

\textsuperscript{3}We also take modular forms of weight 2 for the charged lepton mass matrix to give a minimal number of parameters in the lepton sector.
4 Fixing $\tau$ by observed CKM

In order to obtain the left-handed flavor mixing, we calculate $M_d^1 M_d$ and $M_u^1 M_u$. At first, we take a random point of $\tau$ and $g_{ui}$ which are scanned in the complex plane by generating random numbers. The modulus $\tau$ is scanned in the fundamental domain of the modular symmetry. In practice, the scanned range of $\text{Im} |\tau|$ is $[\sqrt{3}/2, 2]$, in which the lower-cut $\sqrt{3}/2$ is at the cusp of the fundamental domain, and the upper-cut 2 is enough large for estimating $Y_t$. On the other hand, $\text{Re} |\tau|$ is scanned in the fundamental domain $[-1/2, 1/2]$ of the modular group. Supposing $|g_{ui}|$ is of order one, we scan them in $[0, 2]$ while these phases are scanned in $[-\pi, \pi]$. Then, parameters $\alpha_q, \beta_q, \gamma_q$ ($q = u, d$) are given in terms of $\tau$ and $g_q$ after inputting six quark masses.

Finally, we calculate three CKM mixing angles and the CP violating phase in terms of the model parameters $\tau$ and $g_{ui}$. We keep the parameter sets, in which the value of each observable is reproduced within the three times of $1 \sigma$ interval of error-bars. We continue this procedure to obtain enough points for plotting allowed region.

We input quark masses in order to constrain model parameters. Since the modulus $\tau$ obtains the expectation value by the breaking of the modular invariance at the high mass scale, the quark masses are put at the GUT scale. The observed masses and CKM parameters run to the GUT scale the expectation value by the breaking of the modular invariance at the high mass scale, the quark masses are given in terms of $\tau$ model parameters $\sigma$ reproduced within the three times of $1 \sigma$ interval. We take the observed values of CKM with three times $1 \sigma$ masses. These eight real parameters are scanned to reproduce the observed three CKM mixing angles and the CP violating phase with three times $1 \sigma$ error interval in Eqs. (21), (22) and (23)\footnote{We take the observed values of CKM with three times $1 \sigma$ intervals, which are almost $3 \sigma$ in this case.}.

In our work, we adopt numerical values of Yukawa couplings of quarks at the GUT scale $2 \times 10^{16}$ GeV with $\tan\beta = 5$ in the framework of the minimal SUSY breaking scenarios $^{58,59}$:

$$
\begin{align*}
    y_d &= (4.81 \pm 1.06) \times 10^{-6}, \\
    y_s &= (9.52 \pm 1.03) \times 10^{-5}, \\
    y_b &= (6.95 \pm 0.175) \times 10^{-3}, \\
    y_u &= (2.92 \pm 1.81) \times 10^{-6}, \\
    y_c &= (1.43 \pm 0.100) \times 10^{-3}, \\
    y_t &= 0.534 \pm 0.0341,
\end{align*}
$$

\begin{equation}
(21)
\end{equation}

which give quark masses as $m_q = y_q v_H$ with $v_H = 174$ GeV.

We also use the following CKM mixing angles to focus on parameter regions consistent with the experimental data at the GUT scale $2 \times 10^{16}$ GeV, where $\tan\beta = 5$ is taken $^{58,59}$:

$$
\begin{align*}
    \theta_{12}^{\text{CKM}} &= 13.027^\circ \pm 0.0814^\circ, \\
    \theta_{23}^{\text{CKM}} &= 2.054^\circ \pm 0.384^\circ, \\
    \theta_{13}^{\text{CKM}} &= 0.1802^\circ \pm 0.0281^\circ.
\end{align*}
$$

\begin{equation}
(22)
\end{equation}

Here $\theta_{ij}^{\text{CKM}}$ is given in the PDG notation of the CKM matrix $V_{\text{CKM}}$ $^{60}$. The CP violating phase is also given as:

$$
\delta_{CP} = 69.21^\circ \pm 6.19^\circ,
$$

\begin{equation}
(23)
\end{equation}

in the PDG notation. The errors in Eqs. (21), (22) and (23) represent $1 \sigma$ interval. The CKM elements $V_{ij}$ at the GUT scale $2 \times 10^{16}$ GeV are given by using these angles and the phase.

In our model, we have four complex parameters, $\tau$, $g_{u1}$, $g_{u2}$ and $g_{u3}$ after inputting six quark masses. These eight real parameters are scanned to reproduce the observed three CKM mixing angles and the CP violating phase with three times $1 \sigma$ error interval in Eqs. (22) and (23)\footnote{We take the observed values of CKM with three times $1 \sigma$ intervals, which are almost $3 \sigma$ in this case.}.

We have succeeded to reproduce completely four observed CKM elements in the parameter ranges of Table 2. The modulus $\tau$ is close to $i$, which is the fixed point of the modular symmetry.

In order to check the consistency of our quark mass matrices and the observed CKM, we show the calculated distribution on the $|V_{ub}| - \delta_{CP}$ plane at the GUT scale in Fig. 1. The calculated $\delta_{CP}$ is uniformly distributed below the observed central value of $|V_{ub}|$ while it is almost larger than the observed central value for the upper-range of $|V_{ub}|$.
Table 2: Parameter ranges consistent with the observed CKM mixing angles and CP phase $\delta_{CP}$.

| Re[$\tau$] | Im[$\tau$] | $g_{u1}$ | Arg $g_{u1}$ | $g_{u2}$ | Arg $g_{u2}$ | $g_{u3}$ | Arg $g_{u3}$ |
|-----------|-----------|--------|-------------|--------|-------------|--------|-------------|
| [0, 0.09] | [0.99, 1.09] | [0.01, 0.86] | $[-\pi, \pi]$ | [0.14, 1.29] | $[-2.3, 1.6]$ | [0.02, 0.07] | $[-\pi, \pi]$ |

Figure 1: Distribution on $|V_{ub}|$–$\delta_{CP}$ plane, where black lines denote observed central values of $|V_{ub}|$ and $\delta_{CP}$, and red dashed-lines denote three times 1σ interval.

Figure 2: Distribution on $|V_{cb}|$–$|V_{ub}|$ plane where black lines denote observed central values of $|V_{cb}|$ and $|V_{ub}|$, and red dashed-lines denote three times 1σ interval.

We also present the distribution of CKM elements $|V_{cb}|$ and $|V_{ub}|$ at the GUT scale in Fig. 2. The magnitude of $|V_{ub}|$ is predicted to be in the whole region of the three times 1σ interval while the calculated $|V_{cb}|$ is mostly distributed in the lower-range of the three times 1σ interval.

In Fig. 3, we show the plot of Re[$\tau$] and Im[$\tau$], which will be compared with the case of leptons. The allowed region of the modulus $\tau$ is close to $i$, but is clearly deviated from it. The modulus $\tau = i$ is the fixed point of the modular symmetry. Indeed, $\tau = i$ is invariant under the S transformation $\tau = -1/\tau$, where the subgroup $Z^S_2 = \{I,S\}$ of $A_4$ is preserved. This region of $\tau$ is discussed in connection with $\tau$ of the lepton mass matrices in the next section.

In Table 3, we present one parameter set and calculated CKM elements, which is the best-fit point, that is, its $\chi^2$ is minimum. The magnitudes of $g_{qi}$ are at most of order $O(0.5)$. Ratios of $\alpha_q/\gamma_q$ and $\beta_q/\gamma_q$ ($q = u, d$) correspond to the observed quark mass hierarchy.

We also present the mixing matrices of up-type quarks and down-type quarks for the sample of Table 3. They are given as:

Figure 3: The allowed region on Re[$\tau$]–Im[$\tau$] plane, where observed CKM mixing angles and $\delta_{CP}$ are reproduced. The solid curve is the boundary of the fundamental domain, $|\tau| = 1$. 

We also present the distribution of CKM elements $|V_{cb}|$ and $|V_{ub}|$ at the GUT scale in Fig. 2. The magnitude of $|V_{ub}|$ is predicted to be in the whole region of the three times 1σ interval while the calculated $|V_{cb}|$ is mostly distributed in the lower-range of the three times 1σ interval.
\[
\begin{array}{|c|c|}
\hline
\tau & -0.038 + 1.05i \\
g_{u1} & -0.147 + 0.118i \\
g_{u2} & -0.091 - 0.425i \\
g_{u3} & 0.027 + 0.0197i \\
\alpha_u/\gamma_u & 4.33 \times 10^{-5} \\
\beta_u/\gamma_u & 3.85 \times 10^{-3} \\
\alpha_d/\gamma_d & 1.45 \times 10^{-2} \\
\beta_d/\gamma_d & 4.26 \times 10^{-3} \\
|V_{us}| & 0.225 \\
|V_{cb}| & 0.029 \\
|V_{ub}| & 0.0030 \\
\delta_{CP} & 76.9^\circ \\
\chi^2 & 0.46 \\
\hline
\end{array}
\]

Table 3: Numerical values of parameters and output of CKM parameters at the best-fit point.

\[
V_u \approx \begin{pmatrix}
0.987 & -0.017 + 0.142i & 0.056 - 0.039i \\
0.029 + 0.148i & 0.960 & -0.236 - 0.014i \\
-0.048 - 0.004i & 0.241 - 0.021i & 0.969
\end{pmatrix},
\]

\[
V_d \approx \begin{pmatrix}
0.992 & 0.083 - 0.062i & 0.057 - 0.043i \\
-0.065 - 0.049i & 0.962 & -0.259 + 0.003i \\
-0.076 - 0.058i & 0.251 + 0.002i & 0.963
\end{pmatrix},
\]

where $V_{CKM} = V_u^\dagger V_d$. It is noted that these are presented in the diagonal base of the generator $S$, where we can see the hierarchical structure of mixing. The diagonal base of $S$ is realized by the unitary transformation of $S$ in Eq. (9), $V_S S V_S^\dagger = \text{diag}[1, -1, -1]$, where

\[
V_S \equiv \begin{pmatrix}
\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\
\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}.
\]

Then, the mixing matrix $V_q$ in the original base of $S$ is transformed to $V_S V_q$ because the quark mass matrix is transformed as $V_S M_q^\dagger M_q V_S^\dagger$.

In conclusion, our quark mass matrices with the $A_4$ modular symmetry reproduce the observed CKM mixing matrix very well at nearby $\tau = i$. This successful result encourages us to investigate the lepton flavors in the same framework. We discuss the lepton sector with the $A_4$ modular symmetry in Section 5.
5 Lepton mass matrix in the $A_4$ modular invariance

5.1 Lepton mass matrix

The modular $A_4$ invariance also gives the lepton mass matrix in terms of the modulus $\tau$ which is common to both quarks and leptons if flavors of quarks and leptons are originated from a same two-dimensional compact space. We assign the $A_4$ representation and weight for leptons in Table 4, where the left-handed lepton doublets compose a $A_4$ triplet and the right-handed charged leptons are $A_4$ singlets. The weights of the leptons are assigned like the down-type quarks in Table 1.

| SU(2) | $L$ | $(\epsilon^c, \mu^c, \tau^c)$ | $H_u$ | $H_d$ | $Y_r^{(2)}, Y_r^{(4)}$ |
|-------|-----|-----------------|------|------|-----------------|
| $A_4$ | 2   | 1               | 2    | 2    | 2               |
| $-k_I$| 3   | $(1, 1^*, 1')$  | 1    | 1    | $3, \{3, 1, 1'\}$ |

Table 4: Assignments of representations and weights $-k_I$ for MSSM fields and modular forms of weight 2 and 4.

Then, the superpotential of the charged lepton mass term is given in terms of modular forms of weight 2, $Y_3^{(2)}$ since weights of the left-handed leptons and the right-handed charged leptons are $-2$ and 0, respectively. It is given as:

$$w_E = \alpha_ee^cH_dY_3^{(2)}L + \beta_\mu\mu^cH_dY_3^{(2)}L + \gamma_\tau\tau^cH_dY_3^{(2)}L,$$

(26)

where $L$ is the left-handed $A_4$ triplet leptons. Taking $(e_L, \mu_L, \tau_L)$ in the flavor base the charged lepton mass matrix $M_E$ is simply written as:

$$M_E = v_d\begin{pmatrix} \alpha_e & 0 & 0 \\ 0 & \beta_\mu & 0 \\ 0 & 0 & \gamma_\tau \end{pmatrix}\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}_{RL},$$

(27)

where coefficients $\alpha_e$, $\beta_\mu$ and $\gamma_\tau$ are real parameters.

Suppose neutrinos to be Majorana particles. By using the Weinberg operator, the superpotential of the neutrino mass term, $w_\nu$, is given as:

$$w_\nu = -\frac{1}{\Lambda}(H_uH_uLLY_r^{(4)})_1,$$

(28)

where $\Lambda$ is a relevant cutoff scale. Since the left-handed lepton doublet has weight $-2$, the superpotential is given in terms of modular forms of weight 4, $Y_3^{(4)}$, $Y_1^{(4)}$ and $Y_1^{(4)}$.

By using the tensor products of $A_4$, we have

$$w_\nu = \frac{v_\mu^2}{\Lambda} \left[ \left( \frac{2\nu_e\nu_e - \nu_\mu\nu_\mu - \nu_\tau\nu_\tau}{2\mu_\mu \nu_\mu - \nu_\tau\nu_\tau - \nu_\tau\nu_\tau} \right) \otimes Y_3^{(4)} \right. + \left( \nu_e\nu_e + \nu_\mu\nu_\mu + \nu_\tau\nu_\tau \right) \otimes Y_1^{(4)} + \left( \nu_\tau\nu_\tau + \nu_\mu\nu_\mu + \nu_\tau\nu_\tau \right) \otimes Y_1^{(4)} \right],$$

(29)
where $Y_{3}^{(4)}$, $Y_{1}^{(4)}$ and $Y_{1}^{(4)}$ are given in Eq. (13), and $g_{\nu 1}$, $g_{\nu 2}$ are complex parameters. The neutrino mass matrix is written as follows:

$$M_{\nu} = \frac{v_{\nu}^{2}}{\Lambda} \begin{pmatrix}
Y_{1}^{(4)} & -Y_{3}^{(4)} & -Y_{2}^{(4)} \\
-Y_{3}^{(4)} & 2Y_{2}^{(4)} & -Y_{1}^{(4)} \\
-Y_{2}^{(4)} & -Y_{1}^{(4)} & 2Y_{3}^{(4)}
\end{pmatrix} + g_{\nu 1}Y_{1}^{(4)} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} + g_{\nu 2}Y_{1}^{(4)} \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}. \quad (30)$$

Then, the model parameters are $\alpha_e$, $\beta_e$, $\gamma_e$, $g_{\nu 1}$ and $g_{\nu 2}$. Parameters $\alpha_e$, $\beta_e$ and $\gamma_e$ are adjusted by the observed charged lepton masses. Therefore, the lepton mixing angles, the Dirac phase and Majorana phases are given by $g_{\nu 1}$ and $g_{\nu 2}$ in addition to the value of $\tau$. Since $\tau$ is restricted in the narrow range for the quark sector as seen in Fig. 3, we can give some predictions in the lepton sector. Practically, we scan $\tau$ in $|\text{Re}[\tau]| \leq 0.5$ and $|\text{Im}[\tau]| \leq 2$ like in the analysis of quark mass matrices.

### 5.2 Numerical results of leptons

We input charged lepton masses in order to constrain the model parameters. We take Yukawa couplings of charged leptons at the GUT scale $2 \times 10^{16}$ GeV, where $\tan \beta = 5$ is taken as well as quark Yukawa couplings [58, 59]:

$$y_e = (1.97 \pm 0.024) \times 10^{-6}, \quad y_\mu = (4.16 \pm 0.050) \times 10^{-4}, \quad y_\tau = (7.07 \pm 0.073) \times 10^{-3}, \quad (31)$$

where lepton masses are given by $m_\ell = y_\ell v_H$ with $v_H = 174$ GeV. We also input the lepton mixing angles and neutrino mass parameters which are given by NuFit 4.1 in Table 5 [61]. We investigate two possible cases of neutrino masses $m_i$, which are the normal hierarchy (NH), $m_3 > m_2 > m_1$, and the inverted hierarchy (IH), $m_2 > m_1 > m_3$. Neutrino masses and the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix $U_{\text{PMNS}}$ [62, 63] are obtained by diagonalizing $M_{\nu}^\dagger M_{\nu}$ and $M_{\nu}^* M_{\nu}$. We also investigate the effective mass for the $0\nu\beta\beta$ decay, $\langle m_{\text{ee}} \rangle$ (see Appendix B) and the sum of three neutrino masses $\sum m_i$ since it is constrained by the recent cosmological data, which is the upper-bound $\sum m_i \leq 120$ meV obtained at the 95% confidence level [64, 66].

| observable    | 3 $\sigma$ range for NH | 3 $\sigma$ range for IH |
|---------------|-------------------------|-------------------------|
| $\Delta m_{\text{atm}}^2$ | $(2.436-2.618) \times 10^{-3}$ eV$^2$ | $-(2.419-2.601) \times 10^{-3}$ eV$^2$ |
| $\Delta m_{\text{sol}}^2$ | $(6.79-8.01) \times 10^{-5}$ eV$^2$ | $(6.79-8.01) \times 10^{-5}$ eV$^2$ |
| $\sin^2 \theta_{23}$ | 0.433–0.609 | 0.436–0.610 |
| $\sin^2 \theta_{12}$ | 0.275–0.350 | 0.275–0.350 |
| $\sin^2 \theta_{13}$ | 0.02044–0.02435 | 0.02064–0.02457 |

Table 5: The $3\sigma$ ranges of neutrino parameters from NuFIT 4.1 for NH and IH [61].

Let us discuss numerical results for NH of neutrino masses. Since $\alpha_e/\gamma_e$ and $\beta_e/\gamma_e$ are obtained by the observed charged lepton masses when $\tau$ is fixed, we input charged lepton masses to reduce free parameters. Parameters $g_{\nu 1}$ and $g_{\nu 2}$ are constrained by four observed quantities; three mixing angles of leptons and observed mass ratio $\Delta m_{\text{sol}}^2/\Delta m_{\text{atm}}^2$. In practice, the scanned ranges of $\text{Im}[\tau]$ and $\text{Re}[\tau]$ are $[\sqrt{3}/2, 2]$ and $[-1/2, 1/2]$, respectively, like for the quark sector. Neutrino couplings $|g_{\nu 1}|$ are scanned in $[0, 10]$ while these phases are in $[-\pi, \pi]$. Indeed, we have obtained $|g_{\nu 1}| = 0.03–1.15$ and $|g_{\nu 2}| = 0.63–1.22$ in our numerical calculations.

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At first, we show the allowed region on the Re[τ]–Im[τ] plane in Fig. 4. Observed three mixing angles of leptons are reproduced at cyan, blue and red points. The sum of neutrino masses is consistent with the cosmological bound 120 meV at cyan points, but not at blue points. At red points, both CKM and PMNS are reproduced with the sum of neutrino masses below 120 meV. For comparison, we add green points for quark CKM of Fig. 3. Allowed points of leptons are almost in Im[τ] ≤ 1.12 and |Re[τ]| ≤ 0.1.

The common τ causes the feedback in the quark sector. However, the tendency of |V_{ub}|, |V_{cb}| and δ_{CP} are not so changed compared with Figs. 1 and 2 in the common region of τ.

We show the allowed region on the \( \sum m_i \sin^2 \theta_{23} \) plane in Fig. 5, where colors (cyan, blue and red) of points correspond to points of τ in Fig. 4. Our prediction of the sum of neutrino masses is constrained by the cosmological bound as seen in Fig. 5. The minimal cosmological model, ΛCDM + \( \sum m_i \), provides the upper-bound \( \sum m_i < 120 \text{ meV} \) [64, 66] although it becomes weaker when the data are analysed in the context of extended cosmological models [60].

The red region, that is the common τ region for quarks and leptons, is constrained by the cos-

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**Figure 4:** Allowed regions of τ. PMNS mixing angles are reproduced at cyan, blue and red points while CKM are reproduced at green points. At cyan (blue) points, the sum of neutrino masses is below (above) 120 meV. At red points, both CKM and PMNS are reproduced with the sum of neutrino masses below 120 meV. The solid curve is the boundary of the fundamental domain, |τ| = 1.

**Figure 5:** Allowed region on \( \sum m_i \sin^2 \theta_{23} \) plane, where the horizontal solid line denotes observed best-fit one, red dashed-lines denote the bound of 3σ interval, and the vertical line is the cosmological bound, for NH. Colors of points correspond to τ in Fig. 4.

**Figure 6:** Allowed region on the \( \sum m_i - \delta_{CP} \) plane, where the vertical solid line is the cosmological bound, for NH. Colors of points correspond to points of τ in Fig. 4.
Figure 7: Predicted $\delta_{CP}^\ell$ versus $\sin^2 \theta_{23}$, where the black line denotes observed best-fit value of $\sin^2 \theta_{23}$, and red dashed-lines denote its upper(lower)-bound of $3\sigma$ interval for NH. Colors of points correspond to points of $\tau$ in Fig.4.

Figure 8: Predicted effective mass $\langle m_{ee} \rangle$ of the $0\nu\beta\beta$ decay versus $m_1$ for NH. The vertical line is the upper-bound of $m_1$, which is derived from the cosmological bound, 120 meV. Colors of points correspond to points of $\tau$ in Fig.4.

Figure 7: Predicted $\delta_{CP}^\ell$ versus $\sin^2 \theta_{23}$, where the black line denotes observed best-fit value of $\sin^2 \theta_{23}$, and red dashed-lines denote its upper(lower)-bound of $3\sigma$ interval for NH. Colors of points correspond to points of $\tau$ in Fig.4.

Figure 8: Predicted effective mass $\langle m_{ee} \rangle$ of the $0\nu\beta\beta$ decay versus $m_1$ for NH. The vertical line is the upper-bound of $m_1$, which is derived from the cosmological bound, 120 meV. Colors of points correspond to points of $\tau$ in Fig.4.

We show the allowed region on the $\sum m_i$–$\delta_{CP}^\ell$ plane in Fig.6. In the region of red points, $\delta_{CP}^\ell$ is predicted to be in the restricted ranges, $0^\circ$–$50^\circ$, $80^\circ$–$100^\circ$, $110^\circ$–$130^\circ$, $230^\circ$–$250^\circ$, $260^\circ$–$280^\circ$ and $310^\circ$–$360^\circ$. If the cosmological bound for the sum of neutrino masses will be improved, for example, it is 100 meV, $\delta_{CP}^\ell$ is predicted in the distinct range.

In Fig.7, we plot $\delta_{CP}^\ell$ versus $\sin^2 \theta_{23}$ in order to see their correlation. Since there is a significant correlation between them, the precise measurement of $\sin^2 \theta_{23}$ gives the clear prediction of $\delta_{CP}^\ell$.

We can also predict the effective mass $\langle m_{ee} \rangle$ for the $0\nu\beta\beta$ decay versus the lightest mass $m_1$ as seen in Fig.8. At the red point region, we predict $\langle m_{ee} \rangle = 15$–$31$ meV. The predicted $\langle m_{ee} \rangle$ larger than 31 meV is excluded by the sum of neutrino masses.

In Table 6, we present best-fit values of parameters and outputs, where we input the common value $\tau = -0.038 + 1.05 i$ in Table 3. In our $\chi^2$ fit, $\delta_{CP}^\ell$ is not included, but it is only an output because T2K and NO$\nu$A experiments have presented the best-fit values of $\delta_{CP}^\ell$ with opposite sign each other [44,45]. Our predicted $\delta_{CP}^\ell = 27.4^\circ$ in Table 6 is rather small. The systematic $\chi^2$ fit of both quarks and leptons will be needed by including precise data of $\delta_{CP}^\ell$ since $\delta_{CP}^\ell \simeq \pm 90^\circ$ is also predicted in Fig.6.

We also present the mixing matrices of charged leptons and neutrinos for the best-fit sample of
Table 6: Numerical values of parameters and output of PMNS parameters at the best-fit point.

| Parameter | Value               |
|-----------|---------------------|
| $\tau$   | $-0.038 + 1.05i$    |
| $g_{\nu 1}$ | $0.061 + 0.274i$  |
| $g_{\nu 2}$ | $0.671 - 0.956i$  |
| $\alpha_e/\gamma_e$ | $6.71 \times 10^{-2}$ |
| $\beta_e/\gamma_e$ | $1.24 \times 10^{-3}$ |
| $\sin^2 \theta_{12}$ | 0.305 |
| $\sin^2 \theta_{23}$ | 0.565 |
| $\sin^2 \theta_{13}$ | 0.0216 |
| $\delta_{CP}$ | 27.4° |
| $\sum m_i$ | 118 meV |
| $\langle m_{ee} \rangle$ | 30.1 meV |
| $\chi^2$ | 0.64 |

Table 6. Those are given as:

$$U_\ell \approx \begin{pmatrix} 0.989 & 0.106 - 0.073i & 0.062 - 0.043i \\ -0.087 - 0.060i & 0.960 & -0.261 + 0.003i \\ -0.087 - 0.060i & 0.250 + 0.003i & 0.962 \end{pmatrix},$$

$$U_\nu \approx \begin{pmatrix} 0.732 & 0.465 - 0.459i & 0.097 - 0.166i \\ -0.333 - 0.267i & 0.228 - 0.115i & 0.868 \\ -0.415 - 0.332i & 0.713 & -0.448 - 0.094i \end{pmatrix},$$

where $U_{PMNS} = U_\ell^\dagger U_\nu$. They are also given in the diagonal base of the generator $S$ in order to see the hierarchical structure of mixing like in the quark mixing matrix in Eq. (24), by using the unitary transformation of Eq. (25). It is noticed that the mixing matrix of charged leptons $U_\ell$ is similar to the quark mixing matrices of Eq. (24). On the other hand, two large mixing angles appear in the neutrino mixing matrix $U_\nu$.

In our numerical calculations, we have not included the RGE effects in the lepton mixing angles and neutrino mass ratio $\Delta m^2_{sol}/\Delta m^2_{atm}$. We suppose that those corrections are very small between the electroweak and GUT scales for NH of neutrino masses. This assumption is justified well in the case of $\tan \beta \leq 5$ unless neutrino masses are almost degenerate [26].

Finally, we discuss briefly the case of IH of neutrino masses. Indeed, there is a very small region of the common $\tau$ for quarks and leptons, which is marginal since the sum of neutrino masses is very close to the cosmological bound, 120 meV. Therefore, we omit discussions of this case.
6 Summary

We have studied both quark and lepton mass matrices in the $A_4$ modular symmetry towards the unification of quark and lepton flavors. If flavors of quarks and leptons are originated from a same two-dimensional compact space, quarks and leptons have the same flavor symmetry and the same value of the modulus $\tau$.

For the quark sector, we have adopted modular forms of weights 2 and 6. We have presented the viable model for quark mass matrices, in which the down-type quark mass matrix is constructed by modular forms of weight 2 while the up-type quark mass matrix is constructed by modular forms of weight 6. In the lepton sector, the charged lepton mass matrix is constructed by modular forms of weight 2 while modular forms of weight 4 is used for the neutrino mass matrix, which is generated by the Weinberg operator.

The viable region close to $\tau = i$ is obtained in our quark mass matrices. Lepton mass matrices also work well at nearby $\tau = i$, which overlaps with the one of the quark sector, for NH of neutrino masses. In the common $\tau$ region for quarks and leptons, the predicted sum of neutrino masses is 87–120meV taking account of its cosmological bound. Since both the Dirac CP phase $\delta^\ell_{CP}$ and $\sin^2 \theta_{23}$ are correlated significantly with the sum of neutrino masses, improving its cosmological bound provides crucial tests for our scheme as well as the precise measurement of $\sin^2 \theta_{23}$ and $\delta^\ell_{CP}$. The effective neutrino mass of the $0\nu\beta\beta$ decay is predicted to be $\langle m_{ee} \rangle = 15$–31meV. The IH of neutrino masses is almost excluded by the cosmological bound of the sum of neutrino masses.

It is remarked that the common $\tau$ is fixed at nearby $\tau = i$ in the fundamental domain of SL$(2, Z)$, which suggests the residual symmetry $Z_2$ in the quark and lepton mass matrices. Some corrections could violate the exact symmetry. It is also emphasized that the spontaneous CP violation in Type IIB string theory is possibly realized at nearby $\tau = i$, where the moduli stabilization as well as the calculation of Yukawa couplings is performed in a controlled way [07]. Thus, our phenomenological result of the modulus $\tau$ is favored in the theoretical investigation.

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Appendix

A Tensor product of $A_4$ group

We take the generators of $A_4$ group for the triplet as follows:

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

(33)

where $\omega = e^{i\frac{2\pi}{3}}$ for a triplet. In this base, the multiplication rule is

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \otimes \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = (a_1 b_1 + a_2 b_3 + a_3 b_2)_{1} \oplus (a_3 b_3 + a_1 b_2 + a_2 b_1)_{1'},$$

$$\oplus (a_2 b_2 + a_1 b_3 + a_3 b_1)_{1''},$$

$$\oplus \frac{1}{3} \begin{pmatrix} 2a_1b_1 - a_2b_3 - a_3b_2 \\ 2a_3b_3 - a_1b_2 - a_2b_1 \\ 2a_2b_2 - a_1b_3 - a_3b_1 \end{pmatrix} \oplus \frac{1}{2} \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_1b_2 - a_2b_1 \\ a_3b_1 - a_1b_3 \end{pmatrix},$$

$$1 \otimes 1 = 1, \quad 1' \otimes 1' = 1'', \quad 1'' \otimes 1'' = 1', \quad 1' \otimes 1'' = 1,$$

(34)

where

$$T(1') = \omega, \quad T(1'') = \omega^2.$$  

(35)

More details are shown in the review \[6,7\].

B Majorana and Dirac phases and $\langle m_{ee} \rangle$ in $0\nu\beta\beta$ decay

Supposing neutrinos to be Majorana particles, the PMNS matrix $U_{\text{PMNS}}$ \[62,63\] is parametrized in terms of the three mixing angles $\theta_{ij}$ ($i, j = 1, 2, 3; i < j$), one CP violating Dirac phase $\delta_{\text{CP}}$ and two Majorana phases $\alpha_{21}, \alpha_{31}$ as follows:

$$U_{\text{PMNS}} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{\text{CP}}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{\text{CP}}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{\text{CP}}} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-i\alpha_{21}} & 0 \\ 0 & 0 & e^{-i\alpha_{31}} \end{pmatrix},$$

(36)

where $c_{ij}$ and $s_{ij}$ denote $\cos \theta_{ij}$ and $\sin \theta_{ij}$, respectively.

The rephasing invariant CP violating measure of leptons \[68,69\] is defined by the PMNS matrix elements $U_{\alpha i}$. It is written in terms of the mixing angles and the CP violating phase as:

$$J_{\text{CP}} = \text{Im} \left[ U_{e1}U_{\mu 2}U_{e2}U_{\mu 1}^* \right] = s_{23}c_{23}s_{12}s_{13}c_{13}^2 \sin \delta_{\text{CP}}^\ell,$$

(37)

where $U_{\alpha i}$ denotes the each component of the PMNS matrix.
There are also other invariants $I_1$ and $I_2$ associated with Majorana phases

$$I_1 = \text{Im} \left[ U_{e1}^* U_{e2} \right] = c_{12} s_{12} c_{13} s_{21} \sin \left( \frac{\alpha_{21}}{2} \right), \quad I_2 = \text{Im} \left[ U_{e1}^* U_{e3} \right] = c_{12} s_{13} c_{23} \sin \left( \frac{\alpha_{31}}{2} - \delta_{CP} \right).$$ (38)

We can calculate $\delta_{CP}$, $\alpha_{21}$ and $\alpha_{31}$ with these relations by taking account of

$$\cos \delta_{CP} = \frac{|U_{\tau 1}|^2 - s_{12}^2 s_{23}^2 - c_{12} c_{23}^2 s_{13}^2}{2 s_{12} c_{12} c_{23} s_{23} s_{13}},$$

$$\text{Re} \left[ U_{e1}^* U_{e2} \right] = c_{12} s_{12} c_{13} s_{21} \cos \left( \frac{\alpha_{21}}{2} \right), \quad \text{Re} \left[ U_{e1}^* U_{e3} \right] = c_{12} s_{13} c_{23} \cos \left( \frac{\alpha_{31}}{2} - \delta_{CP} \right).$$ (39)

In terms of this parametrization, the effective mass for the $0\nu\beta\beta$ decay is given as follows:

$$\langle m_{ee} \rangle = \left| m_1 c_{12} s_{13}^2 + m_2 s_{12}^2 s_{13} e^{i\alpha_{21}} + m_3 s_{13}^2 e^{i(\alpha_{31} - 2\delta_{CP})} \right|. \quad (40)$$

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