On the existence and approximation of a dissipating feedback

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Abstract

Given a matrix $A \in \mathbb{R}^{n \times n}$ and a tall rectangular matrix $B \in \mathbb{R}^{n \times q}$, $q < n$, we consider the problem of making the pair $(A, B)$ dissipative, that is the determination of a feedback matrix $K \in \mathbb{R}^{q \times n}$ such that the field of values of $A - BK$ lies in the left half open complex plane. We review and expand classical results available in the literature on the existence and parameterization of the class of dissipating matrices, and we explore new matrix properties associated with the problem. In addition, we discuss various computational strategies for approximating the dissipating matrix $K$ of minimal Frobenius or Euclidean norms.

Keywords: Passivation of a matrix pair, matrix stabilization, stabilizing feedback, matrix nearness problems, constrained gradient flow.

2000 MSC: 15A18, 65K05

1. Introduction

A linear dynamical system

$$\dot{x} = Ax$$

with $A \in \mathbb{R}^{n \times n}$ is said to be dissipative if the matrix $A$ has a field of values $W(A) = \{ z \in \mathbb{C} : z = x^*Ax, x \in \mathbb{C}^n, \|x\| = 1 \}$ contained in the left half open complex plane $\mathbb{C}^-$. Here $x^*$ stands for the conjugate transpose of the complex vector $x$, and $\|x\|$ is the Euclidean norm. Under certain hypotheses, dissipativity implies passivity - passivity is the property that a system requires no external energy to operate. A non-passive system is transformed into a passive one by means of controls in the form $u = Kx$. We shall call such a $K$ a “dissipating feedback matrix”.

The considered problem is of great relevance in many applications; see, e.g., \cite{HP10, WMcK07}. Several interesting examples are described in \cite{L79}, while

\footnotesize
\(^1\)Analogously, a system is said to be weakly dissipative if $W(A)$ is contained in the closed left half complex plane $\mathbb{C}^-$, which includes the imaginary axis, and $W(A) \cap i\mathbb{R} \neq \emptyset$. 

Preprint submitted to Elsevier
I linear models for real life mechanical, electrical and electromechanical control systems are considered in [FPEN86].

We are interested in the problem of finding a (possibly weakly) dissipating feedback matrix $K$ to a non-dissipative linear control system of the form

$$
\begin{align*}
\dot{x} &= Ax - Bu \\
u &= Kx.
\end{align*}
$$

Hence, the existence of a dissipating feedback matrix $K$ ensures that the closed-loop linear system $\dot{x} = (A - BK)x$ is dissipative. The feedback matrix $K$ is called weakly dissipating if $W(A - BK) \subset \mathbb{C}^{-}$ and $W(A - BK) \cap i\mathbb{R} \neq \emptyset$. For real data, weak dissipativity clearly implies that the field of values boundary passes through the origin.

In matrix terms the problem can be stated as follows:

Given $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times q}$, with $q < n$, find a matrix $K \in \mathbb{R}^{q \times n}$ such that the field of values of $A - BK$ is contained in the left half open (closed) complex plane.

Throughout we assume that $A$ is stable, that is its eigenvalues are all in $\mathbb{C}^{-}$, however $W(A)$ has nonzero intersection with the right half open complex plane.

The problems of existence and representation of a feedback matrix have been extensively analyzed in the control community; a widely used result stated as [SIG98, Theorem 2.3.12] ensures the existence of $K$ under hypotheses on the data, while providing a parameterization of all dissipating matrices. We revisit this parameterization, and observe that it may not include all possible feedback matrices. By using an alternative proof of their existence, we thus propose alternative parametrizations of dissipating matrices, and highlight the actual degrees of freedom associated with the problem.

The concept of dissipativity is tightly related to other definitions of stability, which are largely investigated in the Control community. For real data, dissipativity of $A$ corresponds to ensuring that $\frac{1}{2}\lambda_{\text{max}}(A + A^T) < 0$, where $\lambda_{\text{max}}(\cdot)$ is the rightmost eigenvalue of the argument matrix. Weak dissipativity requires that $\frac{1}{2}\lambda_{\text{max}}(A + A^T) \leq 0$. The quantity $\mu_2(A) = \frac{1}{2}\lambda_{\text{max}}(A + A^T)$ is called the numerical abscissa (see, e.g., [D59, S06]), and it monitors the exponential stability (alas contractivity) property of the system solution $x(t)$, since it holds

$$
\|x(t_2)\| \leq e^{\mu_2(A)(t_2 - t_1)}\|x(t_1)\|.
$$

Clearly, if $\mu_2(A) \leq 0$, then $\|x(t_2)\| \leq \|x(t_1)\|$ and the system is said to be exponentially stable. In particular, concepts like $(M, \beta)$-stability are introduced and characterized (see e.g. [HPW02, PP92]), meaning - for a given matrix $G$ - that $\|e^{Gt}\| \leq Me^{\beta t}$. For the system (2) the matrix is $G = A - BK$. In our setting, if $K$ is such that the field of values $W(A - BK)$ is all in $\mathbb{C}^{-}$, then the system is $(1, \mu_2(A - BK))$-stable.

Dissipativity may also be associated with the numerical solution of the algebraic Riccati equation, when using a projection method. A projection method...
determines an approximate solution by first projecting the data onto a computed approximation space of much smaller dimension than that of the data \( A, B \), and then solving a reduced Riccati equation with data \( \tilde{A}, \tilde{B} \) of significantly smaller size. In this context, it was shown in [S16] that if there exists \( K \) such that \( A - BK \) is dissipating, then the projected pair \((\tilde{A}, \tilde{B})\) is stabilizable, thus ensuring that the sought after solution of the reduced Riccati equation exists.

The feedback matrix \( K \) may be required to have additional properties, such as a small norm. Following standard approaches, we formulate this problem as an optimization procedure with inequality matrix constraints, thus falling into a linear matrix inequalities (LMI) framework [BEFB94, SIG98]. As an alternative we explore the use of a functional approach, which is a variant of the method recently proposed in [GL17]. Numerical experiments on selected data illustrate the performance of the tested methods.

In addition to the notation already introduced, the following definitions will be used throughout. \( I_n \) denotes the identity matrix of dimension \( n \), and the subscript is removed when clear from the context. For a square matrix \( B \), \( \text{Sym}(B) = (B + B^T)/2 \) denotes its symmetric part. We denote by \( \| \cdot \|_F \) the Frobenius norm on \( \mathbb{R}^{n \times n} \) and by \( \langle X, Y \rangle = \text{trace}(X^T Y) = \sum_{i,j=1}^n x_{ij} y_{ij} \) the corresponding inner product. Moreover, \( \| \cdot \|_2 \) denotes the matrix norm induced by the Euclidean vector norm.

2. Known existence results and parameterization

Conditions on the existence of a dissipating matrix have been known for quite some time in the Control community. A thorough and insightful discussion is available in the monograph [SIG98]. The following fundamental theorem provides existence conditions for the matrix \( K \) such that \( BKC + (BKC)^* + Q < 0 \), for given \( B, C \) and a symmetric \( Q \) [SIG98, Theorem 2.3.12].

Theorem 2.1. Let the matrices \( B \in \mathbb{C}^{n \times q} \), \( C \in \mathbb{C}^{k \times n} \) and \( Q = Q^* \in \mathbb{C}^{n \times n} \) be given. Then the following statements are equivalent:

(i) There exists a matrix \( K \) satisfying \( BKC + (BKC)^* + Q < 0 \).

(ii) The following two conditions hold

\[
B^\dagger Q(B^\dagger)^* < 0 \quad \text{or} \quad BB^* > 0 \\
(C^\ast)^\dagger Q((C^\ast)^\dagger)^* < 0 \quad \text{or} \quad C^*C > 0.
\]

Suppose the above statements hold. Let \( B = B_t B_r \), \( C = C_t C_r \) be the full rank factorizations of \( B \) and \( C \), respectively. Then all matrices \( K \) in statement (i) are given by

\[
K = B_t^+ HC_t^+ + Z - B_r^+ B_r Z C_t^+ C_r^+.
\]
where $Z$ is an arbitrary matrix and

$$
H := -R^{-1}B_{\ell}^*\Phi C_r^*(C_r^*\Phi C_r^*)^{-1} + S^{1/2}L(C_r^*\Phi C_r^*)^{-1/2}
$$

$$
S := R^{-1} - R^{-1}B_{\ell}^*(\Phi - \Phi C_r^*(C_r^*\Phi C_r^*)^{-1}C_r^*\Phi)B_{\ell}R^{-1},
$$

where $L$ is an arbitrary matrix such that $\|L\| < 1$ and $R$ is an arbitrary positive definite matrix such that $\Phi := (B_{\ell}R^{-1}B_{\ell}^* - Q)^{-1} > 0$.

Here $Q$ plays the role of $A + A^*$, so that item (i) precisely corresponds to our setting. In the theorem statement, $(B^\perp)^*$ is the matrix spanning the null space of $B^*$. The theorem thus says that $K$ exists if and only if $Q$ is negative definite on the Kernel of $B^*$ and on the Kernel of $C$, or otherwise, if $B$ $(C)$ has full row (column) rank. The theorem also provides a parameterization of dissipating matrices.

The following corollary specializes the result to our case, where $C = I$; see also [SIG98, Corollary 2.3.9].

**Corollary 2.1.** Assume $C = I$ and $B$ full column rank, so that $B_{\ell} = B$ and $B_r = I$. With the notation of Theorem 2.1, we have

$$
K = H = -R^{-1}B^* + R^{-1}L\Phi^{-1}
$$

where $L \in \mathbb{R}^{q \times n}$ is an arbitrary matrix such that $\|L\| < 1$ and $R \in \mathbb{R}^{q \times q}$ is an arbitrary positive definite matrix such that $\Phi := (BR^{-1}B^* - Q)^{-1} > 0$.

The role of the matrix $R$ is to push into the positive half complex plane the indefinite matrix $-Q = -(A + A^*)$. In [SIG98], the choice $R = \rho I_q$ for some large enough $\rho$ is considered sufficient to be able to obtain a positive definite $\Phi$. However, by doing so, some degrees of freedom may be lost. In particular, if one is interested in a norm minimizing $K$, a full symmetric positive definite $R$ should be considered.

By going through the proof of the previous theorem, it is possible to show that there exist dissipating matrices that are not represented by the parameterization $K$ given above. To this end, we first deepen our understanding of the quantities involved in the classical parametrization in terms of invariant subspaces. This will allow us to capture the role of the free matrices $R$ and $L$. For
\[ K = -R^{-1}B^* + R^{-\frac{1}{2}}L\Phi^{-\frac{1}{2}} \] we have

\[
BK + K^*B^* + Q = [I, K^*] \begin{bmatrix} Q & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix}
\]

\[
= \Phi^{-\frac{1}{2}}[\Phi^\frac{1}{2}, -\Phi^\frac{1}{2}BR^{-\frac{1}{2}} + L^*] \begin{bmatrix} I & 0 \\ 0 & R^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} Q & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & R^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} \Phi^\frac{1}{2} \\ -\Phi^\frac{1}{2}BR^{-\frac{1}{2}} + L \\ \Phi^{-\frac{1}{2}} \end{bmatrix}
\]

\[
= \Phi^{-\frac{1}{2}}[I, -\Phi^\frac{1}{2}BR^{-\frac{1}{2}} + L^*] \begin{bmatrix} \Phi^\frac{1}{2} & 0 \\ 0 & R^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} Q & B \\ B^* & 0 \end{bmatrix} \begin{bmatrix} \Phi^\frac{1}{2} & 0 \\ 0 & R^{-\frac{1}{2}} \end{bmatrix} \begin{bmatrix} I \\ -\Phi^\frac{1}{2}BR^{-\frac{1}{2}} + L \\ \Phi^{-\frac{1}{2}} \end{bmatrix}
\]

where \( \tilde{Q} = \Phi^\frac{1}{2}Q\Phi^\frac{1}{2} \) and \( \tilde{B} = \Phi^\frac{1}{2}BR^{-\frac{1}{2}} \). Therefore, \( BK + K^*B^* + Q < 0 \) if and only if

\[
[I, -\tilde{B} + L^*] \begin{bmatrix} \tilde{Q} & \tilde{B} \\ \tilde{B}^* & 0 \end{bmatrix} \begin{bmatrix} I \\ -\tilde{B}^* + L \end{bmatrix} < 0,
\]

so that the block diagonal matrix with \( \Phi^\frac{1}{2} \) and \( R^{-\frac{1}{2}} \) provides an arbitrary scaling of the original saddle point matrix; see also Remark 3.3 later on. Since \( \Phi = (BR^{-1}B^* - Q)^{-1} \), it follows that

\[
\tilde{BB}^* - \tilde{Q} = \Phi^\frac{1}{2}(BR^{-1}B^* - Q)\Phi^\frac{1}{2} = I.
\]

After simple algebra we can thus write

\[
[I, -\tilde{B} + L^*] \begin{bmatrix} \tilde{Q} & \tilde{B} \\ \tilde{B}^* & 0 \end{bmatrix} \begin{bmatrix} I \\ -\tilde{B}^* + L \end{bmatrix} = (\tilde{Q} - \tilde{BB}^*) + (L\tilde{B}^* + \tilde{B}L^* - \tilde{BB}^*)
\]

\[
= -I + (L\tilde{B}^* + \tilde{B}L^* - \tilde{BB}^*).
\]

The first matrix product is negative definite if and only if \(-I + (L\tilde{B}^* + \tilde{B}L^* - \tilde{BB}^*) < 0 \). Assume all data are real, and let \( x \) be such that \( ||x|| = 1 \). If \( \tilde{B}^*x = 0 \) then \(-x^*x + 2x^*\tilde{L}\tilde{B}^*x - ||\tilde{B}^*x||^2 = -1 < 0 \) and the inequality is obtained. If \( ||B^*x|| \neq 0 \), and under the hypothesis that \( ||L|| < 1 \) we obtain

\[
-x^*x + 2x^*\tilde{L}\tilde{B}^*x - ||\tilde{B}^*x||^2 \leq -1 + 2||x^*L||||\tilde{B}^*x|| - ||\tilde{B}^*x||^2
\]

\[
< -1 + 2||\tilde{B}^*x|| - ||\tilde{B}^*x||^2 = -(||\tilde{B}^*x|| - 1)^2 < 0.
\]

In summary, we see that the role of the \( q \times q \) matrix \( R \) is to define the positive definite matrix \( \Phi \) so that (3) holds. Moreover, the matrix \( L \) yields the “if” statement. However, it seems that \( L \) does not necessarily need to have norm less than one for the desired inequality to be satisfied. The following example

\[\text{\footnotesize \cite{SIG93}}\]
illustrates one such case. In other words, the part of the statement in Theorem 2.1 stating that all matrices $K$ have the given parametrization only considers a subset of all possible dissipating matrices.

**Example 2.1.** Consider $Q = \text{diag}(\alpha, -\alpha)$, with $\alpha > 0$, and $B = e_1 = [1; 0]$. Let us take $R^{-1} = \tilde{\alpha}$ with $\tilde{\alpha} > \alpha$. Then

$$
\Phi = (BR^{-1}B^* - Q)^{-1} = \text{diag}(\frac{1}{\tilde{\alpha} - \alpha}, \frac{1}{\alpha}) > 0,
\tilde{B} = \Phi^{\frac{1}{2}}BR^{-\frac{1}{2}} = \frac{\sqrt{\alpha}}{\sqrt{\tilde{\alpha} - \alpha}}e_1
$$

with $\|\tilde{B}\| = \frac{\sqrt{\alpha}}{\sqrt{\tilde{\alpha} - \alpha}} > 1$ for all choices of $\alpha > 0$ and $\tilde{\alpha} > \alpha$. By taking $L = \frac{1}{2}\tilde{B}$ we can select $\alpha$ and $\tilde{\alpha}$ so that $\|L\| \geq 1$, while in [4] for this choice of $L$ we have that $-I + (LB^* + BL^* - BB^*) = -I$, which is clearly negative definite.

3. An invariant subspace perspective for the parametrization of the dissipating matrix

In this section we provide a different perspective, that allows us to determine a richer parametrization of dissipating matrices. We first restate the existence condition in terms of an eigenvalue problem. To this end we need to recall a standard result on structured (saddle point) matrices.

**Proposition 3.1.** ([CC84]) If the matrix $-(A + A^T)$ is positive definite on the kernel of $B^T$, then the matrix

$$
\mathcal{M} = \begin{bmatrix} -(A + A^T) & B \\ B^T & 0 \end{bmatrix}
$$

(5)

has exactly $n$ positive and $q$ negative eigenvalues.

We can state the existence result of a dissipating feedback matrix $K$ by using a quite different proof, which sheds light into different properties of the matrix $K$. In particular, similarities with the solution matrix of the Riccati equations can be readily observed; see, e.g., [S16] and references therein.

**Theorem 3.1.** The matrix $A + A^T$ is negative definite on the kernel of $B^T$ if and only if there exists a matrix $K \in \mathbb{R}^{q \times n}$ such that $W(A - BK) \subset \mathbb{C}^-$.  

**Proof.** We first prove that if the condition on $A + A^T$ holds, then there exists a matrix $K$ such that $W(A - BK) \subset \mathbb{C}^-$.  

Proving that $W(A - BK) \subset \mathbb{C}^-$ for some $K$ corresponds to stating that the symmetric matrix $(A - BK) + (A - BK)^T$ is negative definite. We can write

$$(A - BK) + (A - BK)^T = (A + A^T) - BK - K^TB = -[I, K^T] \begin{bmatrix} -(A + A^T) & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} I \\ K \end{bmatrix}.$$  

Therefore, if $K$ is chosen so that the matrix $\mathcal{M}$ in (5) is positive definite onto the space spanned by the columns of $[I; K]$, then $(A - BK) + (A - BK)^T$ is
negative definite. Using Proposition 3.1 it is possible to determine an invariant subspace of $\mathcal{M}$ corresponding to the $n$ positive eigenvalues of $\mathcal{M}$. We next show that this gives the sought after matrix $K$. Let the orthonormal columns $[X; Y]$ span this invariant subspace, with $X \in \mathbb{R}^{n \times n}$ and $Y \in \mathbb{R}^{q \times n}$. Then we have

$$\mathcal{M} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} X \\ Y \end{bmatrix} \Lambda$$

with $\Lambda$ diagonal and positive definite. Moreover, multiplying from the left by $[X^T, Y^T]$ we can write

$$\Lambda = [X^T, Y^T] \mathcal{M} \begin{bmatrix} X \\ Y \end{bmatrix} = -X^T(A + A^T)X + X^TBY + Y^TB^TX =: S + S^T,$$

where $S = X^T(-AX + BY)$. Since $\Lambda$ is positive definite, we have that $S + S^T$ is also positive definite, that is the field of values of $S$ is all in the positive right half open complex plane. In particular, this implies that $S$ is nonsingular, and thus $X$ is nonsingular. Therefore we can define $K := YX^{-1}$. Then collecting $X$ and $X^T$ on both sides of the right-most expression in (7),

$$0 < \Lambda = X^T(-A + A^T) + BYX^{-1} + X^{-T}Y^TB^T)X = X^T(-A + A^T + BK + K^TB^T)X = X^T[I, K^T]\mathcal{M} \begin{bmatrix} I \\ K \end{bmatrix} X.$$

Since the eigenvalues of the two congruent matrices

$$X^T[I, K^T]\mathcal{M} \begin{bmatrix} I \\ K \end{bmatrix} X$$

and

$$[I, K^T]\mathcal{M} \begin{bmatrix} I \\ K \end{bmatrix}$$

have the same sign, this implies that $[I, K^T]\mathcal{M} \begin{bmatrix} I \\ K \end{bmatrix}$ is also positive definite.

We finally prove the converse by negating that $-(A + A^T)$ is positive definite on $\ker(B^T)$. Suppose then that there exists $x \in \ker(B^T)$ such that $x^T(A + A^T)x \geq 0$, then we have

$$x^T((A + A^T) - BK - K^TB^T)x = x^T(A + A^T)x \geq 0,$$

which means that $W(A - BK) \not\subset \mathbb{C}^-$ independently of $K$, completing the proof.

The proof in constructive, since it determines one such $K$ explicitly. Indeed, for small matrices a dissipating feedback matrix $K$ can be computed by first determining the eigenvector matrix $[X; Y]$ corresponding to all positive eigenvalues of $\mathcal{M}$, and then setting $K = YX^{-1} \in \mathbb{R}^{q \times n}$.

Remark 3.1. From its construction, it follows that $K = YX^{-1}$ is full (row) rank, equal to $q$. Indeed, we first notice that $\text{rank}(K) = \text{rank}(Y)$. Moreover, the second block row of (6) yields $B^TX = YA$. Since both $\Lambda$ and $X$ are square and full rank, we obtain $\text{rank}(Y) = \text{rank}(B)$.  

7
Remark 3.2. From the previous remark it also follows that since \( B^T X \Lambda^{-1} = Y \) and \( X \) is nonsingular, we have \( K = Y X^{-1} = B^T X \Lambda^{-1} X^{-1} \), that is, \( K \) can be written as \( K = B^T W \) for some nonsingular matrix \( W \). Other strategies discussed in the following will also determine a similar form, but with possibly singular \( W \).

3.1. New parametrizations of dissipating matrices

The parametrization in Corollary 2.1 depends on two matrices, \( R \) and \( L \), giving at most \( q(q+1)/2 + nq \) degrees of freedom. However, by generalizing the setting of our Theorem 3.1, we can see that dissipating matrices can be parametrized by a larger number of degrees of freedom, therefore many more such matrices can be defined than those introduced in Corollary 2.1.

By generalizing the representation of Theorem 3.1, we next present two different parametrizations of the possible families of dissipating feedback matrices.

Proposition 3.2. Assume that the condition of Theorem 3.1 holds. Let \( M = \text{Qblkdiag}(\Lambda_+, \Lambda_-)Q^T \) be the eigendecomposition of \( M \), where \( \Lambda_+ \) (\( \Lambda_- \)) is diagonal with all the \( n \) positive (\( q \) negative) eigenvalues of \( M \). Partition further \( Q = [Q_{11}, Q_{12}; Q_{21}, Q_{22}] \) with \( Q_{11} \in \mathbb{R}^{n \times n} \) nonsingular. Then for any \( H_2 \in \mathbb{R}^{q \times n} \) such that \( \alpha = \|H_2(I - Q_{12}H_2)^{-1}Q_{11}\|^2 \) satisfies \( \min \lambda(\Lambda_+) > \alpha \max \lambda(|\Lambda_-|) \), the feedback matrix

\[
K = Q_{21}Q_{11}^{-1} + (Q_{22} - Q_{21}Q_{11}^{-1}Q_{12})H_2,
\]

is dissipating.

The proof is postponed to Appendix A.

Proposition 3.2 shows that as long as it is possible to separate the negative and positive eigenvalues of \( M \), a different matrix \( K \) can be obtained. Different values of \( \alpha \) yield different values of \( \|K\| \).

The result of Theorem 3.1 corresponds to using the limiting case \( \alpha = 0 \) in Proposition 3.2, that is \( H_2 = 0 \) in the definition of \( Z \) in the proposition proof. This way, \( K \) is well defined as long as \( \lambda_{\text{min}}^+ > 0 \), that is as long as \( M \) has \( n \) strictly positive eigenvalues, as indeed shown by Theorem 3.1. Indeed, the expression \( K = Q_{21}Q_{11}^{-1} + (Q_{22} - Q_{21}Q_{11}^{-1}Q_{12})H_2 \) parametrizes \( K \) in terms of some matrix \( H_2 \) with the required conditions. This parametrization may be used for determining the feedback matrix \( K \) having certain properties, such as minimum Frobenius norm, see section 4. Due to the low number of degrees of freedom, however, this parametrization is unlikely to cover all possible feedback matrices \( K \). This concern was confirmed by some of our numerical experiments, which showed that this procedure usually determines a local minimum, which does not seem to be the global one.

The next proposition provides another, more general parametrization for the set of dissipating feedback matrices, by means of a pencil \((M, D)\), where \( D \) is a symmetric positive definite matrix playing the role of the parameter. In particular, this means that at least \( (n+q)(n+q-1)/2 \) degrees of freedom are available for the family of dissipating matrices.
Theorem 3.2. There exists a matrix $K$ such that $W(A - BK) \subset \mathbb{C}^-$ if and only if the pencil $(M, D)$ admits $n$ positive eigenvalues for some symmetric and positive definite matrix $D \in \mathbb{R}^{(n+q) \times (n+q)}$.

Proof. We first recall that the signature of the eigenvalues of $(M, D)$ is the same as that of $M$ [W73, Theorem 5].

Assume there exists $D$ symmetric and positive definite such that $M[X; Y] = D[X; Y] \Lambda$ with $\Lambda > 0$, with $[X; Y]$ $D$-orthogonal. Since $0 < \Lambda = [X^T, Y^T]M[X; Y]$, proceeding as in the discussion after (7) the nonsingularity of $X$ is ensured. Finally, setting $K := YX^{-1}$,

$$[I, K^T]M \begin{bmatrix} I \\ K \end{bmatrix} = X^{-T}[X^T, Y^T]M \begin{bmatrix} X \\ Y \end{bmatrix} X^{-1} = X^{-T}[X^T, Y^T]D \begin{bmatrix} X \\ Y \end{bmatrix} \Lambda X^{-1} = X^{-T} \Lambda X^{-1} > 0.$$

We next prove that if $K$ exists such that $W(A - BK) \subset \mathbb{C}^-$, then we can define a symmetric and positive definite matrix $D$. Let $U = [I; K]$ and define

$$D = UDUT + U_DU^T_D,$$

with $D = (UTU)^{-2}$, $[U, U_D]$ square and full rank with $U_D^TU = 0$, and for any symmetric and positive definite matrix $D_D \in \mathbb{R}^{q \times q}$. By construction we have $U^TDU = I_n$. We have thus found a subspace of dimension $n$, range($U$), such that, for any $0 \neq x \in \mathbb{R}^n$,

$$\frac{x^T U^T M U x}{x^T U^T D U x} > 0$$

which implies that the pencil $(M, D)$ has at least $n$ positive eigenvalues.

As opposed to the case $D = I$, it does not seem to be possible to ensure that $K$ has full rank, because $Y$ and $X$ depend on the matrix $D$ to be determined.

Note also that $D$ may also be viewed as the matrix defining a different inner product associated with the invariant subspace basis.

Remark 3.3. Since the matrix $D$ is somewhat arbitrary, except for being symmetric and positive definite, a block diagonal matrix could be considered. On the other hand, this simplifying strategy would significantly decrease the number of degrees of freedom, which play a role when looking for the minimal norm feedback matrix, as discussed in the next section. A similar drawback can be observed for the classical derivation highlighted in the second part of section 2: indeed, in there, a scaling with the free parameter matrix $\text{diag}(\Phi^+, R^\Phi)$ is performed, but this may prevent the parametrized family from containing the matrix of minimal norm.
4. Computing a (weakly) dissipating feedback of minimal norm

In this section we explore the possible computation of a feedback matrix of minimal norm that makes the system either dissipative or weakly dissipative. Let $\mathbb{W}^{q \times n}(A, B)$ be the set of weakly dissipating matrices for the pair $(A, B)$. The problem can thus be stated as:

Find $K_\ast \in \mathbb{W}^{q \times n}(A, B)$ such that

$$\inf_{K \in \mathbb{W}^{q \times n}(A, B)} \|K\|_\ast.$$  \hfill (8)

Here $\|\cdot\|_\ast$ stands for the Frobenius norm ($\|\cdot\|_F$) or the 2-norm ($\|\cdot\|_2$).

The following result implies that the feedback matrix of minimal norm is to be found among the weakly dissipating matrices.

**Proposition 4.1.** Assume that $\mathbb{W}(A) \cap \mathbb{C}^+ \neq \emptyset$ and let $K_1$ be a dissipating feedback matrix. Then there exists a weakly dissipating feedback matrix $K_2$ with $\|K_2\|_\ast < \|K_1\|_\ast$.

**Proof.** Let $K(\rho) = (1 - \rho)K_1$, $0 \leq \rho \leq 1$. Naturally $W(A - BK(0)) = W(A - BK_1) \subset \mathbb{C}^-$ and $W(A - BK(1)) = W(A) \not\subset \mathbb{C}^-$.

By continuity of eigenvalues of $S(\rho) = \text{Sym}(A - BK(\rho))$ we have that for sufficiently small $\rho > 0$, $\eta(S(\rho)) = \text{Sym}(A - BK(\rho)) < 0$ and there exists $\rho = \rho_0 > 0$ such that $\eta(S(\rho_0)) = 0$. Setting $K_2 = (1 - \rho_0)K_1$ determines a weakly-dissipating feedback $K_2$ with $\|K_2\|_\ast = (1 - \rho_0)\|K_1\|_\ast$.

The following result is concerned with the existence of a weakly dissipating minimizer for (8).

**Proposition 4.2.** Assume that $A + A^T$ is negative definite on the kernel of $B^T$. Then (8) is equivalent to

$$\min_{K \in \mathbb{W}^{q \times n}(A, B)} \|K\|_\ast.$$  \hfill (9)

**Proof.** Under the considered assumption, Theorem 5.1 implies the existence of a dissipating matrix $K_1$, then the set $\mathbb{W}^{q \times n}(A, B)$ is not empty. Moreover, Proposition 10 implies the existence of a weakly dissipating matrix $K_2 \in \mathbb{W}^{q \times n}(A, B)$ with $\alpha := \|K_2\|_\ast \leq \|K_1\|_\ast$. Thus we can look for the solution to (8) in the bounded and closed (and thus compact) set

$$\{K \in \mathbb{W}^{q \times n}(A, B) \text{ s.t. } \|K\|_\ast \leq \alpha\}.$$

Since $\|\cdot\|_\ast$ is a continuous function, the result follows from Weierstraß Theorem.

Note that in the case where one wishes to compute some strictly dissipating feedback it would be sufficient to replace the matrix $A$ by $A_\delta := A + \delta I$, where $\delta$ represents the maximal real part of $W(A - BK)$. Then applying the same procedure to the pair $\{A_\delta, B\}$ provides a strictly dissipating feedback.

Before we proceed with the actual computational strategies, we linger over some spectral properties of the involved matrices.
**Proposition 4.3.** Assume that $\text{Sym}(A)$ has $t$ positive eigenvalues with corresponding eigenvectors $Q_− = [q_1, \ldots, q_t]$, and that $K ∈ \mathbb{R}^{q \times n}$ is a dissipating feedback. Then it must be $\text{rank}(Q^T(B + K^T)) ≥ t$.

**Proof.** See Appendix B.

We next show that in correspondence to a weakly dissipating matrix $K$ there is a nontrivial null space of $\text{Sym}(A − BK)$ of dimension at most $q$.

**Proposition 4.4.** Assume that $A + A^T$ is negative definite on the kernel of $B^T$. If $K$ is a weakly dissipating feedback then $\text{Sym}(A − BK)$ has a zero eigenvalue with multiplicity $m$, with $0 < m ≤ q$.

**Proof.** Using Proposition 4.1, the hypothesis ensures that there exists a weakly dissipating matrix $K$. We only need to show that $\text{Sym}(A − BK)$ has at most $q$ zero eigenvalues. Let $[B_0, N]$ be unitary, with $\text{Range}(B_0) = \text{Range}(B)$, so that $\text{Range}(N)$ is the null space of $B^T$. Let $K$ be weakly dissipating, so that $((A + A^T) − BK − K^T B^T)$ has $m > 0$ zero eigenvalues. We can write

$[B_0, N]^T ((A + A^T) + BK + K^T B^T)[B_0, N] = S_{11} S_{12} S_{T12} S_{22} =: S$,

with $S_{22} ∈ \mathbb{R}^{(n−q)×(n−q)}$ and $S_{12} ∈ \mathbb{R}^{q×(n−q)}$. By hypothesis it follows that $S_{22} = −N^T(A + A^T)N > 0$. Let $[u; v]$ be a nonzero vector such that $S[u; v] = [0; 0]$. Then it must hold that $v = −S_{22}^{-1} S_{T12} u$ with $u ∈ \mathbb{R}^q$. The eigenspace of $S$ associated with the zero eigenvalue is thus spanned by the vectors $[I_q; S_{22}^{-1} S_{T12} u]$, and there are thus at most $q$ of them, that are linearly independent, that is there are $m ≤ q$ zero eigenvalues.

### 4.1. The LMI framework

The problem [5] can be stated as an LMI optimization problem, whose actual form depends on the norm used in the minimization problem. Following standard strategies (see, e.g., [BEFB94]), if the $2$-norm is to be minimized, then the problem can be stated as

\[
\min_{K \in \mathbb{R}^{q \times n}} \|K\|_2 \quad \text{subject to} \quad A + A^T − BK − K^T B^T ≤ 0, \quad \begin{bmatrix} \gamma I_q & K \\ K^T & \gamma I_n \end{bmatrix} ≥ 0
\]

where $γ > 0$ is such that $\|K\| ≤ γ$. The problem is thus expressed in terms of the two variables $K$ and $γ$, the first of which is a rectangular matrix.

If the Frobenius norm is to be minimized, the problem becomes

\[
\min_{K \in \mathbb{R}^{q \times n}} \|K\|_F \quad \text{subject to} \quad A + A^T − BK − K^T B^T ≤ 0, \quad \begin{bmatrix} I & \text{vec}(K) \\ \text{vec}(K)^T & \gamma \end{bmatrix} ≥ 0
\]
where $\text{vec}(K)$ stacks all columns of $K$ one after the other, so that $\|K\|_F^2 \leq \gamma$; see, e.g., [DL17].

Both problems can be numerically solved by using standard LMI packages. In our computational experiments we used the Matlab version of Yalmip with the call to either SeDuMi (see [Sedumi]) or Mosek (see [Mosek]). Some of these results are reported in section 5.

4.2. A direct approach

Using Theorem 3.2 we can compute the feedback matrix $K = YX^{-1}$ of minimal norm by solving the following optimization problem:

$$\inf_{D > 0} \|YX^{-1}\|_F$$
subject to
$$M[X; Y] = D[X; Y]\Lambda \quad \text{with} \quad \Lambda > 0.$$  

This method has limitations when applied to problems of large dimensions, that is when $n \gg 1$, moreover it seems to strongly depend on the starting guess, as many local minima seem to exist.

4.3. A gradient system approach

In this section we propose a gradient-flow differential equation approach that adapts to our setting a strategy first proposed in [GL17]. Given the matrix $\text{Sym}(A)$ and identifying its $m$ rightmost eigenvalues (e.g. its positive eigenvalues), we construct a smoothly varying matrix $K$ that moves these eigenvalues to the origin, so as to make the system weakly dissipative. We look for one such feedback matrix $K$ having minimum Frobenius norm. We write $K = \varepsilon E$ with $E$ of unit Frobenius norm, and with perturbation size $\varepsilon > 0$. For a fixed $\varepsilon > 0$, we minimize the function

$$F_\varepsilon(E) = \frac{1}{2}\sum_{i=1}^{m} \left(\lambda_i(\text{Sym}(A - \varepsilon BE))\right)^2$$

constrained by $\|E\|_F = 1$,  

by solving numerically the corresponding gradient-flow differential equation. Here $\lambda_i$s are the $m$ rightmost eigenvalues of the argument symmetric matrix. We denote the obtained minimum by $E_\varepsilon$ and then look for the smallest $\varepsilon > 0$ such that $F_\varepsilon(E_\varepsilon) = 0$, which we denote by $\varepsilon_m^*$. In general, the existence of $\varepsilon_m^*$ is not guaranteed. Formally, this can be expressed as:

Solve

$$\min_{\varepsilon > 0} \min_{\|E\|_F = 1} F_\varepsilon(E).$$

Clearly, the minimum of $F_\varepsilon(E)$ is zero, that is with the optimal $K = \varepsilon_m^*E$ the matrix $\text{Sym}(A - BK)$ has $m$ coalescent eigenvalues at zero.

Due to classical results on eigenvalue interlacing of low-rank modifications of symmetric matrices [HH13], the number of positive eigenvalues of $\text{Sym}(A)$
provides a rigorous lower bound for rank($K$) in order to find an optimal weakly
dissipating feedback.

The two-phase method works as follows.

**Inner procedure.** Assume $\varepsilon > 0$ is fixed. Suppose that $E(t)$ is a smooth matrix-valued function of $t$ such that the $m$ largest eigenvalues of $\text{Sym}(A - B\varepsilon E(t))$, denoted by $\lambda_i(t)$ for $i = 1, \ldots, m$, are simple with corresponding eigenvectors $x_i(t)$ normalized to have unit 2-norm. Define $G(E) = -\sum_{i=1}^{m} \lambda_i z_i x_i^T = -ZX^T$, with $z_i = B^T x_i$. The steepest descent direction $\dot{E}$ for the functional $F_\varepsilon(E)$ is obtained by solving the gradient system (see [GL17])

$$\dot{E} = -G(E) + \beta E,$$

with $\beta = \langle G(E), E \rangle$. \hfill (17)

Note that $G(E)$ is the free gradient matrix of $F_\varepsilon(E)$. Then the following result generalizes the corresponding theorem in [GL17].

**Theorem 4.1.** The following statements are equivalent along solutions of (17), provided that the $m$ largest eigenvalues $\lambda_i$ of $\text{Sym}(A - \varepsilon BE(t))$ are simple and that there exists at least an index $i \leq m$ such that $\lambda_i \neq 0$.

1. $\frac{d}{dt} F_\varepsilon(E(t)) = 0$.
2. $\dot{E} = 0$.
3. $E$ is a real multiple of $G(E)$.

The proof follows the same lines as that of [GL17, Theorem 3.2].

Since the equilibrium of the ODE (17) has rank-$m$, we proceed similarly to [GL17, equation (19)] and replace the matrix differential equation (17) on $\mathbb{R}^{q \times n}$ by a projected differential equation onto the manifold of rank-$m$ matrices, so as to maintain the solution equilibria. To preserve the projection property in the numerical treatment, we have considered a projected Euler method on the manifold of rank-$m$ matrices (see, e.g., [HLW06, section IV.4]).

**Outer procedure.** We let $E(\varepsilon)$ of unit Frobenius norm be a local minimizer of the inner optimization problem in (16) and for $i = 1, \ldots, m$ we denote by $\lambda_i(\varepsilon)$, $x_i(\varepsilon)$ and $z_i(\varepsilon)$ the corresponding largest eigenvalues, eigenvectors and $z$-vectors of $\text{Sym}(A - \varepsilon BE(\varepsilon))$. Finally we let $\varepsilon^*_m$ be the smallest value of $\varepsilon$ such that $F_\varepsilon(E(\varepsilon)) = 0$.

To determine $\varepsilon^*_m$, we are thus left with a one-dimensional root-finding problem, for which a variety of standard methods are available. Following [GL17] in our implementation we have used a Newton-like algorithm in the form

$$\varepsilon_{k+1} = \varepsilon_k - \frac{f(\varepsilon_k)}{f'(\varepsilon_k)},$$

where $f(\varepsilon) = F_\varepsilon(E(\varepsilon))$ and $' = d/d\varepsilon$. To use this iteration we need to impose the following extra assumption, which is not restrictive in practice.

**Assumption 4.1.** For $\varepsilon$ close to $\varepsilon^*_m$ and $\varepsilon < \varepsilon^*_m$, we assume that the $m$ largest eigenvalues of $\text{Sym}(A - \varepsilon BE(\varepsilon))$ are simple eigenvalues. Consequently $E(\varepsilon)$ and these eigenvalues are smooth functions of $\varepsilon$, as well as the associated vectors $x_i(\varepsilon), z_i(\varepsilon)$. 

13
Then under Assumption 4.1 the function \( f(\varepsilon) \) is differentiable and its derivative equals (see, \[GL17\], Lemma 3.5)

\[
f'(\varepsilon) = -\|G(\varepsilon)\|_F.
\]

Since the eigenvalues are assumed to be simple, the function \( f(\varepsilon) \) has a double zero at \( \varepsilon_m^* \) because it is a sum of squares, and hence it is convex for \( \varepsilon \leq \varepsilon_m^* \). This means that we may approach \( \varepsilon_m^* \) from the left by the classical Newton iteration, which satisfies \( |\varepsilon_{k+1} - \varepsilon_m^*| \approx \frac{1}{2} |\varepsilon_k - \varepsilon_m^*| \) and \( \varepsilon_{k+1} < \varepsilon_m^* \) if \( \varepsilon_k < \varepsilon_m^* \). The convexity of the function to the left of \( \varepsilon_m^* \) guarantees the monotonicity of the sequence and its boundedness.\(^3\) We refer the reader to \[GL17\] for full details.

**Remark 4.1.** Assume that for \( \varepsilon < \varepsilon_m^* \), \( \varepsilon \to \varepsilon_m^* \), \( F_\varepsilon(E(\varepsilon)) \to 0 \), and exactly \( m \) eigenvalues of \( \text{Sym}(A + \varepsilon BE(\varepsilon)) \) vanish. Let \( E_* = \lim_{\varepsilon \to \varepsilon_m^*} E(\varepsilon) \). Then, exploiting Theorem 4.1 and the rank-properties of \( E(\varepsilon) \), and passing to the limit it follows that \( E_* \) has the form

\[
E_* = ZDX^T
\]

with \( D \) a diagonal matrix and the orthonormal columns of \( X \) span the invariant space of \( \text{Sym}(A - BK_*) \) associated with the \( m \) rightmost (zero) eigenvalues, and \( Z = B^T XDX^T \) and \( K_* = \varepsilon_m^* E_* \) has rank-\( m \).

If instead \( m' > m \) eigenvalues effectively vanish, and \( m_+ \leq q \) then \( E_* \) has rank \( m_+ \).

With a Frobenius norm minimizing feedback matrix \( K_* = \varepsilon_m^* E_* \) we thus have that the matrix \( A - \varepsilon_m^* BB^T(XDX^T) \) provides a dissipative closed-loop system. This reminds us of a corresponding property of the solution \( X_* \) to the Riccati equation, and in particular, that \( A - BB^T X_* \) is associated with a stable closed-loop system.

### 4.3.1. A variant of the gradient system approach: a modified functional

The proposed functional (15) is not the only possible one. Here we shortly describe a variant that has been shown to be more effective in our experiments. Note that the associated gradient system has a very similar structure, although the gradient in this case is only continuous.

We use the notation \( a^+ = \max\{a, 0\} \). For a fixed \( \varepsilon > 0 \) we consider the minimization of the following function

\[
F_\varepsilon^+(E) = \frac{1}{2} \sum_{i=1}^m \left( \lambda_i^+ (\text{Sym}(A - \varepsilon BE)) \right)^2, \quad \text{constrained by } \|E\|_F = 1.
\]

\(^3\)A much more accurate approximation is obtained by the modified iteration \( \tilde{\varepsilon}_{k+1} = \varepsilon_k - 2f(\varepsilon_k)/f'(\varepsilon_k) \), which is such that \( |\tilde{\varepsilon}_{k+1} - \varepsilon_m^*| \approx \text{const}|\varepsilon_k - \varepsilon_m^*|^2 \); see \[GL17\].
The free gradient is continuous and has the form

\[ G^+(E) = - \sum_{i=1}^{m^+(E)} \lambda_i z_i x_i^T, \quad z_i = B^T x_i \]  \hspace{1cm} (19)

where \( m^+(E) \leq m \) is the number of positive eigenvalues among the \( m \) rightmost ones. This means that negative eigenvalues (among the \( m \) largest) do not contribute to the gradient which has rank equal to \( m^+ \). This modified strategy, which we shall call GL\((m)+\) in our numerical experiments, is able to account for more strongly varying eigenvalues, that possibly cross the origin while converging to zero as the iterations proceed.

**Remark 4.2.** An important advantage of (19) is that it no longer depends on \( m \), but only on \( m^+(E) \). In particular, if \( m \) is larger than the number of positive eigenvalues of \( \text{Sym}(A-\varepsilon BE) \) during the whole optimization process, the method is expected to converge. This also means that whenever using GL\((m)+\), by taking a sufficiently large \( m \) we expect to obtain the same results, independently of \( m \) (see Example 5.2). Only if \( m \) is chosen smaller than the final number of eigenvalues coalescing to zero we should expect an incorrect behavior. If non-convergence is observed, then one can readily increase the value of \( m \).

5. Numerical experiments

In this section we report on some of our computational experiments for determining the minimum norm feedback matrix. In particular, we analyze the behavior of the different methods we have discussed, with special emphasis on the minimization property, using both the Frobenius and the Euclidean norms.

In all examples, we checked a-priori that the system can be made dissipative, that is Theorem 3.1 holds.

The methods we are going to investigate are summarized as follows:

| Method | Description |
|--------|-------------|
| GL\((m)\) | two-step method of section 4.3 with \( m \) rightmost eigenvalues |
| LMI | Matlab basic function for the LMI problem \( (10) \) (mincx) |
| Yalmip1 | Matlab version of Yalmip with SeDuMi solver for problem \( (10) \) |
| Yalmip2 | Matlab version of Yalmip with SeDuMi solver for problem \( (12) \) |
| Pencil | minimization problem with pencil in \( (14) \) |

**Example 5.1.** We consider the following small data set

\[
A = \begin{bmatrix}
-0.2 & 1.6 & 0.2 & 2.6 & -0.4 \\
-0.2 & -0.8 & -1.2 & -0.7 & -1.8 \\
1.4 & 0.7 & -1.1 & 0.2 & 0.8 \\
0.3 & 0.8 & 0.1 & -0.1 & -0.9 \\
0.2 & -0.2 & 0.7 & -1.9 & 0.1
\end{bmatrix}, \quad B = \begin{bmatrix}
0.6 & 0.5 \\
-0.2 & 0.3 \\
0.5 & 0 \\
0.2 & 0.6 \\
0.6 & -0.6
\end{bmatrix}.
\]  \hspace{1cm} (20)
\[ \|K\|_2 \|K\|_F \]

\begin{tabular}{|l|c|c|}
\hline
Method & Minimization & \|K\|_2 & \|K\|_F \\
\hline
GL(2) & F-norm & 2.2166 & 2.3063 \\
LMI & 2-norm & 2.2166 & 2.6714 \\
Yalmip1 & 2-norm & 2.2166 & 2.5765 \\
Yalmip2 & F-norm & 2.2166 & 2.3063 \\
Pencil & F-norm & 2.2560 & 2.7585 \\
\hline
\end{tabular}

Table 1: Example 5.1.

The eigenvalues of the matrix \((A + A^T)/2\) are given by (with 4 decimal digits) \((-2.4752, -1.8301, -0.7238, 0.6506, 2.2785\)) including two positive eigenvalues. The performance of the considered methods is reported in Table 1.

The GL method was used with \(m = 2\). The dissipating matrices for GL and Yalmip2 are, respectively

\[
K_{GL} = \begin{bmatrix}
0.3690 & -0.12149 & 0.34503 & 0.1119 & 0.35065 \\
1.0340 & 0.66501 & -0.01895 & 1.3640 & -1.2432
\end{bmatrix}
\]

and

\[
K_{Yalmip2} = \begin{bmatrix}
0.3684 & -0.11954 & 0.35079 & 0.1097 & 0.3467 \\
1.0118 & 0.65736 & -0.03002 & 1.3995 & -1.2240
\end{bmatrix}
\]

showing that the two matrices are not the same, even accounting for numerical approximations, though they numerically solve the minimization problem in the same norm. Similarly, for the eigenvalues of the symmetric parts of the dissipative matrix we obtain

\[
\lambda_i(\text{Sym}(A - BK_{GL})) \in \{-2.4765, -1.8306, -0.72468, -2.4e-09, -1.3e-08\},
\]

and

\[
\lambda_i(\text{Sym}(A - BK_{Yalmip2})) \in \{-2.4743, -1.8298, -0.72353, -2.4e-10, 5.0e-10\}.
\]

Notice that because of finite precision arithmetic - the quantities actually minimized are the squares of the ones sought after - neither method is able to force the two eigenvalues to zero to machine precision. We also observe that

\[
\lambda_i(\text{Sym}(A - BK_{Yalmip1})) \in \{-2.4742, -1.8280, -0.72428, -0.69001, 2.9e-11\}
\]

that is, the minimization of the 2-norm correctly moves both positive eigenvalues of \((A + A^T)/2\), but only one is moved to zero. It is also interesting to notice that in all cases, the negative eigenvalues of \((A + A^T)/2\) are barely moved.

Finally, Figure 1 shows the field of values \(W(A - BK_{GL})\) (the plot for \(K_{Yalmip2}\) is visibly indistinguishable).

Due to the results of this example, in the following we shall focus only on the two Frobenius norm minimizing methods, which can also be more easily compared.
Example 5.2. Consider again the matrix $A$ of (20) but consider now the augmented matrix $B$

$$B = \begin{pmatrix}
0.6 & 0.5 & 1 \\
-0.2 & 0.3 & 0 \\
0.5 & 0 & 0 \\
0.2 & 0.6 & 0 \\
0.6 & -0.6 & 0
\end{pmatrix}. \tag{21}$$

The results for this new $B$ are displayed in Table 2 and they are similar to those of the previous test, in spite of the larger $B$. In this example, we also report on the behavior of GL for a different number $m$ of eigenvalues to be moved to zero. For $m = 2$ (the number of positive eigenvalues of $(A + A^T)/2$) both norms are smaller than for $m = q = 3$. The results in Table 2 show that for GL it is important to capture the actual number of positive eigenvalues of $(A + A^T)/2$ to obtain a close-to-optimal feedback matrix. □

Example 5.3. We consider the negative Grcar matrix of size $n$, defined as a Toeplitz banded matrix with unit lower bandwidth of elements equal to minus one, and upper bandwidth three, given by all ones. Its spectrum and field of values are given in Figure 2 for $n = 20$. The symmetric part of the original matrix has a large number of positive eigenvalues, so that a shifting procedure...
is adopted to have \( m = O(1) \) positive eigenvalues. To ensure that dissipation is feasible \( B \) was selected as a linear combination of all eigenvectors corresponding to positive eigenvalues of \( (A + A^T)/2 \), so that \( q = m \). The results of using \( \text{GL}(m) \) and Yasmin2 are reported in Table 3 as \( n \) and the shift vary. The reported values show that the two methods approximately return the same minimum, with Yalmip2 always being smaller. It is interesting that in some cases (incidentally corresponding to \( m = 4 \)) the discrepancy is slightly higher. A closer look reveals that for these data the positive eigenvalues occur in pairs of near eigenvalues. This seems to affect the performance of \( \text{GL}(m) \). This anomalous, though not fully unexpected behavior is explored in the next example.

| \( A - 0.6I \) | \( n \) | \( m \) | \( \text{GL}(m) \) | Yalmip2 |
|-----------------|---|---|-------------|---------|
| 50 2            | 3.499028e-02 | 3.498990e-02 |
| 100 4           | 7.339794e-01 | 7.291499e-01 |
| 150 6           | 6.275579e-01 | 6.257247e-01 |
| 200 10          | 2.448407e-01 | 2.448246e-01 |
| \( A - 0.62I \) | 100 2 | 2.181123e-02 | 2.181135e-02 |
| 150 4           | 2.904286e-01 | 2.881408e-01 |
| \( A - 0.52I \) | 20 2  | 1.627621e-02 | 1.627676e-02 |
| 40 3            | 3.019605e-01 | 3.019597e-01 |
| 45 4            | 1.931760e-01 | 1.914460e-01 |
| 50 4            | 2.275207e-01 | 2.257378e-01 |
| 100 8           | 7.909541e-01 | 7.909255e-01 |
| 150 13          | 6.278783e-01 | 6.278735e-01 |

Table 3: Example 5.3. Minimum F-norm obtained by the considered methods, as the Grcar matrix dimension varies, for different shifts. Here \( q = m \).

Figure 2: Spectrum and field of values of the matrix Grcar matrix \((n = 20)\) in Example 5.3.
Example 5.4. To deepen our understanding of the behavior of GL($m$) in case of positive clusters we consider the following class of matrices

$$A = \frac{1}{2}(A + A^T) = X\Lambda X^T, \quad \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_{n-q}, \eta_1, \ldots, \eta_q)$$

where $\lambda_i$ are uniformly distributed eigenvalues in $[-10, -10^{-2}]$ while $\eta_j \in \{1, 1 + \delta, 2, 2 + \delta, 3, 3 + \delta\}$, taken in this order as $q$ varies, so that positive clusters arise. $X$ is taken as a fixed orthonormal matrix, while $\delta \in (0, 1)$ varies, so as to increase the eigenvalue clustering. The matrix $A$ is then obtained as the lower triangular part of $A$, so that $A = (A + A^T)/2$ holds. The matrix size is $n = 20$ throughout. The matrix $B$ was taken as in the previous examples, so that $m = q$.

Table 4 shows the results of the considered methods, minimizing the Frobenius norm. We vary both the number of positive eigenvalues of $A$ and their closeness, by tuning $\delta$. We readily see that the LMI method Yalmip2 succeeds in determining the minimum, whereas GL($m$) fails to converge in all but two cases, illustrating that the method is indeed affected by this data setting. The reason of this failure is that when (in the gradient dynamics) the $m$-th largest eigenvalue moves to the left of the uncontrollable eigenvalue $\lambda = -10^{-2}$ of Sym($A$) (the associated eigenvector $x$ is in fact such that $B^T x = 0$), we have that $\lambda = -10^{-2}$ replaces such an eigenvalue in the functional and cannot be moved to 0. Although this is a non-generic case we can figure out that almost uncontrollable eigenvalues may slow down the speed of GL($m$). This problem can be effectively solved by the variant GL($m$)+ introduced in section 4.3.1; Experiments with GL($m$)+ were thus included in Table 4. We observe that this modification provided a dramatic improvement to the method, which converged to practically the same value obtained with Yalmip2 in all cases. As this variant appears to be new, its theoretical properties still need to be analyzed; we postpone this interesting study to future research. \square

Our experience on larger data showed that GL($m$) is faster than all LMI-based methods for medium to large values of $n$. This is not unexpected, since the extremely high computational cost is one of the known drawbacks of LMI-based algorithms. Although a CPU time comparison is not the focus of this paper, which would possibly require moving to compiled languages, we believe that there is enough numerical evidence to encourage further exploration of GL($m$) and its variants towards an efficient treatment of large scale problems.

6. Conclusions

Passivating matrices are of interest for open-loop dynamical systems and have thus been analyzed in the Control literature. We have shown that their
Table 4: Example 5.4. Minimum F-norm obtained by the considered methods, as the closeness and number of positive eigenvalues vary. Here $m = q$.

| $m (= q)$ | $\delta$ | $GL(m)$ | $GL(m) +$ | Yalmip2 |
|------------|-----------|---------|-----------|---------|
| 6          | 0.00001   | -       | 5.581468e+01 | 5.581342e+01 |
| 6          | 0.001     | -       | 5.582551e+01 | 5.582426e+01 |
| 6          | 0.01      | -       | 5.592403e+01 | 5.592278e+01 |
| 6          | 0.1       | -       | 5.690962e+01 | 5.690837e+01 |
| 6          | 0.5       | 6.131648e+01 | 6.131648e+01 | 6.131648e+01 |
| 2          | 0.001     | 2.429389e+00 | 2.429389e+00 | 2.429389e+00 |
| 4          | 0.001     | -       | 5.152558e+01 | 5.152495e+01 |
| 4          | 0.01      | -       | 5.157901e+01 | 5.157837e+01 |
| 4          | 0.1       | -       | 5.211364e+01 | 5.211302e+01 |
| 4          | 0.5       | -       | 5.449942e+01 | 5.449883e+01 |

Acknowledgments

The authors thank the INdAM GNCS for support. The first author wishes also to thank the DEWS (University of L’Aquila).

Appendix A

In this Appendix we include the proof of Proposition 3.2.

**Proof.** Let us partition $Q = [Q_1, Q_2]$ conforming to the partitioning of $\Lambda_{\pm}$, and define $Z = Q_1H_1 + Q_2H_2 \in \mathbb{R}^{(n+q) \times n}$ for some $H_1 \in \mathbb{R}^{n \times n}$ nonsingular and $H_2 \in \mathbb{R}^{m \times n}$. Then

$$Z^T \mathcal{M}Z = Z^T Q_1 \Lambda_+ Q_1^T Z - Z^T Q_2 |\Lambda_-| Q_2^T Z = H_1^T \Lambda_+ H_1 - H_2^T |\Lambda_-| H_2 = H_1^T (\Lambda_+ - (H_2 H_1^{-1})^T |\Lambda_-| H_2 H_1^{-1}) H_1.$$ 

Let $\hat{H} = H_2 H_1^{-1}$, and denote with $\lambda_{\min}^+$ the smallest eigenvalue of $\Lambda_+$, and with $|\lambda_{\max}^-|$ the largest eigenvalue of $|\Lambda_-|$. Then, for any $0 \neq x \in \mathbb{R}^n$ and $y = H_1 x$
(note that $y \neq 0$ due to the nonsingularity of $H_1$), we can write

$$x^T Z^T M Z x = y^T (\Lambda_+ - \mathcal{H}^T | \mathcal{H}) y \geq (\lambda^+_{\min} - \|\mathcal{H}\|^2 | \lambda^-_{\max}|) \|y\|^2.$$ 

If $H_1, H_2$ are chosen so that $\alpha = \|\mathcal{H}\|^2$ satisfies $\lambda^+_{\min} - \|\mathcal{H}\|^2 | \lambda^-_{\max}| > 0$, then $Z^T M Z$ is positive definite.

We next show that $H_1, H_2$ can be chosen so that $Z$ has the form $Z = [I; K]$ for some $K$. Let us further partition $Q$ as

$$Q = [Q_1, Q_2] = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

so that $Z = [Q_{11}; Q_{21}] H_1 + [Q_{12}; Q_{22}] H_2$. Note that $Q_{11}$ is nonsingular, for the proof of the previous theorem. We then impose the structure of $Z$, that is

$$\begin{bmatrix} Q_{11} \\ Q_{21} \end{bmatrix} H_1 + \begin{bmatrix} Q_{12} \\ Q_{22} \end{bmatrix} H_2 = \begin{bmatrix} I \\ K \end{bmatrix}, \quad Q_{11} H_1 + Q_{12} H_2 = I, \quad K = Q_{21} H_1 + Q_{22} H_2.$$ 

It follows that $H_1 = Q_{11}^{-1} (I - Q_{12} H_2)$. Therefore, for any $H_2$ such that $I - Q_{12} H_2$ is nonsingular, the matrix $H_1$ is nonsingular, and $K$ is well defined. The statement is proved by choosing $H_2$ so that $\alpha = \|H_2 H_1^{-1}\|^2 = \|H_2 (I - Q_{12} H_2)^{-1} Q_{11}\|^2$, with $\alpha$ satisfying $\lambda^+_{\min} - \alpha |\lambda^-_{\max}| > 0$.

Finally, substituting $H_1$ is the relation $K = Q_{21} H_1 + Q_{22} H_2$ and collecting terms we obtain $K = Q_{21} Q_{11}^{-1} + (Q_{22} - Q_{21} Q_{11}^{-1} Q_{12}) H_2$.

**Appendix B**

In this Appendix we prove Proposition 13.

**Proof.** The hypotheses ensure that $-(A + A^T) + BK + K^T B^T > 0$. Let us introduce the following eigenvalue decomposition

$$J := \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} I & I \\ I & -I \end{bmatrix} =: W J W^T.$$ 

Moreover, letting $[U_+, U_-] := [B, K^T] W = \sqrt{\frac{1}{2}} [B + K^T, B - K^T]$ we have

$$0 < -(A + A^T) + BK + K^T B^T = -(A + A^T) + [B, K^T] J \begin{bmatrix} B^T \\ K \end{bmatrix} = -(A + A^T) + [B, K^T] W J W^T \begin{bmatrix} B^T \\ K \end{bmatrix} = -(A + A^T) + [U_+, U_-] J [U_+, U_-]^T.$$ 

Therefore, letting $\Lambda_- \in \mathbb{R}^{t \times t}$ denote the negative eigenvalue matrix of $-\text{Sym}(A)$ and multiplying from both sides by $Q_-,$

$$0 < Q_-^T (-(A + A^T) + BK + K^T B^T) Q_- = \Lambda_- + Q_-^T [U_+, U_-] J [U_+, U_-]^T Q_- = \Lambda_- - Q_-^T U_- U_-^T Q_- + Q_-^T U_+ U_+^T Q_-.$$
Here the term $Q^T U_+ = \sqrt{\frac{1}{2}} Q^T (B + K^T)$ has dimensions $t \times q$. Finally, we notice that the first two terms in the last expression are negative definite, so that, to satisfy the positivity constraint the matrix $Q^T U_+ U^T Q_- \Rightarrow t$ eigenvalues of $\Lambda_+ - Q^T U_- U^T Q_-$ to the non-negative half real axis. In particular, its rank must be at least $t$.

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