An Analytical Solution to the Coalescence Time of Compact Binary Systems

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Compact binary systems coalesce over time due to radiation of gravitational waves, following the field equations of general relativity. Conservation of energy and angular momentum gives a mathematical description for the evolution of separation between the orbiting objects and eccentricity of the orbit. We develop an analytical solution to the coalescence time for any binary system with arbitrary separation between the compact objects and eccentricity of the orbit. This result is compared with the accurate numerical calculation and is applied to a number of known compact binary systems in the Galaxy.

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Compact objects are stellar remnants, such as black holes (BHs), neutron stars (NSs) and white dwarfs (WDs). A compact binary system consists of two compact objects (BH-BH, BH-NS, BH-WD, NS-NS, NS-WD) orbiting about a common center of gravity. According to the general theory of relativity (GR) these systems radiate energy in the form of gravitational waves (GWs) [1, 2]. The radiation of GWs results in the orbital properties of the systems changing with time and leading to coalescence of the two objects.

The first indirect evidence of GW radiation was found in a binary pulsar system discovered by Hulse and Taylor, well known as Hulse-Taylor binary [3, 4]. On 14 September 2015, the first direct detection of GWs was achieved by the advanced Laser Interferometer Gravitational Wave Observatory (LIGO) [5]. This has been followed by detections of GWs by LIGO and Virgo from GW151012, GW151226, GW170104, GW170729, GW170814, GW170817, GW170818, GW170823, etc. [6]. These detections have started a new era in astronomy and astrophysics. Among the 10 GW signals detected by LIGO and Virgo during the first and second observation runs, 9 were binary BH-BH mergers [3] and GW170817 was a binary NS-NS merger [6, 7]. The NS-NS merger event was also followed by electromagnetic detection [8, 9] of a short gamma-ray burst (GRB), which confirmed the theory [13, 14] that at least some short GRBs are produced by the coalescence of two NSs [7].

With these observations, it is now more important to study the evolution and dynamics of compact binary systems. The orbital evolution of compact binaries and the radiation of GWs have been studied in detail since their first prediction by Einstein. Change in the orbital parameters, the semi-major axis and eccentricity, due to GW radiation has been studied in detail in [10, 17]. The radiation of GW causes shrinking of the orbital separation and the coalescence time can be calculated numerically when the semi-major axis is zero.

In this paper, we present an analytical estimate of the coalescence time of a binary system with an arbitrary eccentricity $0 < e_0 < 1$ and a semi-major axis $a_0$ at the time of observation. We begin by reviewing the numerical solution done by Peters [10] and then present our analytical solution of the coalescence time of a compact binary system radiating GWs.

The energy and angular momentum losses due to GW radiation by a compact binary system of masses $m_1$ and $m_2$ in a close orbit around a common centre of gravity with eccentricity $e$ and semi-major axis $a$ are given by [16, 17]

\[
\langle \frac{dE}{dt} \rangle = -\frac{32}{5} G^4 m_1 m_2 (m_1 + m_2) \frac{c^5}{a^5 (1 - e^2)^{7/2}} \times \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \tag{1}
\]

\[
\langle \frac{dL}{dt} \rangle = -\frac{32}{5} G^2 m_1^2 m_2^2 (m_1 + m_2)^{1/2} \frac{c^3 a^2 (1 - e^2)^2}{(1 + e^2)} \times \left(1 + \frac{7}{8} e^2\right) \tag{2}
\]

where $G$ is the universal gravitational constant and $c$ is the speed of light. Equations (1) and (2) are averaged over the orbital period. The change in $e$ and $a$ are found by equating the gravitational energy of the two body problem, $E = -G m_1 m_2 / 2a$ into equation (1) and the corresponding angular momentum, $L = G m_1^2 m_2 a (1 - e^2) / (m_1 + m_2)$ into equation (2) as

\[
\frac{da}{dt} = -\beta \frac{1}{a^3 (1 - e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \tag{3}
\]

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\[ \frac{de}{dt} = -\frac{19}{12} a^4 (1 - e^2)^{7/2} \left( e + \frac{121}{304} e^3 \right) \]  

(4)

where

\[ \beta = \frac{64 G^3 m_1 m_2 (m_1 + m_2)}{5 c^5} \]

An analytical solution to the coalescence time in case of a circular orbit \((e = 0)\), is given by [16],

\[ T_c(a_0) = \frac{a_0^3}{4 \beta} \]  

(5)

where \(a_0\) is the semi-major axis at the time of the observation. In the general case, with arbitrary values of \(a\) and \(e\), the coalescence time can be calculated by solving the coupled equations (4) and subsequently integrating over \(e\) (4) together. In particular, dividing equation (3) by (4) and subsequently integrating over \(e\) gives \(a\) as a function of \(e\), which is given by

\[ a(e) = \frac{k_0}{(1 - e^2)^{7/2}} \left[ 1 + \frac{121}{304} e^3 \right]^{\frac{11}{229} 121}{1181} \]  

(6)

Here \(k_0\) is a function of the initial conditions \(a = a_0\) and \(e = e_0\). By substituting equation (6) into equation (4) and integrating with respect to \(e\) and \(t\) gives the following coalescence time [16]

\[ T(a_0, e_0) = \frac{12 k_0^4}{19 \beta} \int_0^{e_0} \frac{e^{29} (1 + \frac{121}{304} e^2)^{\frac{11}{229} 121}{1181}}{(1 - e^2)^{7/2}} \, de \]  

(7)

This integral has no analytical solution, but it can be solved numerically. The numerical solution, as a ratio to the circular orbit solution \(T_c(a_0)\), is plotted in Fig. 1 with the red dotted line.

In order to obtain an analytical solution to the coalescence time \(a\) of binary system with arbitrary \(a\) and \(e\) we attempt to solve the integral in equation (7) analytically by letting

\[ u = e^{29} (1 + \frac{121}{304} e^2)^{\frac{11}{229} 121}{1181}, \quad dv = \frac{1}{(1 - e^2)^{7/2}} \, de \]

such that an integration by parts yield

\[ T(a_0, e_0) = \frac{12 k_0^4}{19 \beta} [uv - \int v du] \]

Note that the integral \(\int du\) has a complex and a real solution given by

\[ v = \begin{cases} \frac{ie}{\sqrt{e^2 + 1}} & \text{for } |e^2| > 1 \\ \frac{e}{\sqrt{1 - e^2}} & \text{otherwise} \end{cases} \]

Since we are calculating the coalescence time for systems with a closed orbit such that \(0 < e < 1\), we use the real solution to obtain

\[ T(a_0, e_0) = T_1 - \frac{12 k_0^4}{19 \beta} \int v du \]  

(8)

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig1.png}
\caption{The ratio of the coalescence time for an eccentric binary system to that of a circular one with the same semi-major axis \(a_0\) is plotted as a function of the initial eccentricity \(e_0\). The numerical solution [16] in equation (7), our first approximate analytical solution [9], our analytical solution in equation (10) and Shapiro-Teukolsky-Holes analytical solution [18] in equation (11) are plotted with red dotted, green dot-dashed, blue solid and black dashed lines, respectively.}
\end{figure}

where

\[ T_1(a_0, e_0) = \frac{12 k_0^4}{19 \beta} \frac{e^{29} (1 + \frac{121}{304} e^2)^{\frac{11}{229} 121}{1181}}{\sqrt{1 - e^2}} \]  

(9)

We take \(T_1\) to be our first approximate analytical solution of the integral in equation (7) for \(0 < e < 1\), plotted in Fig. 1 with the green dot-dashed line. The contribution from the integral \(\int v du\) in equation (8) is negligible for \(e > 0\).

When comparing our approximation \(T \approx T_1\) to the numerical solution of equation (7), we found that they have the same values for small \(e_0\) but slightly different values for \(e_0 \gtrsim 0\). We improve our analytical solution by multiplying \(T_1\) by a function of \(e_0\) which is approximately 1 for low values of \(e_0\) and becomes larger than 1 for high values of \(e_0\) (close to 1). Any function \(f(e_0)\) of the form \(1/(1 - e_0^2)^m\), where \(m\) and \(n\) are positive numbers will satisfy those conditions. In this case the best values for \(n\) and \(m\) were found to be 7/4 and 1/5 respectively, then \(f(e_0) = 1/(1 - e_0^{7/4})^{1/5}\) and the amplitude is adjusted by a factor of 39/100.
Therefore, taking only the \( T_1 \) term from equation (5) and multiplying by \( f(e_0) \) gives the following coalescence time

\[
T_c(a_0, e_0) \approx \frac{39}{100} T_1(a_0, e_0) f(e_0)
\]

From equation (6) we obtain

\[
k_0 = \frac{a_0}{\epsilon_0^4} \frac{1 - e_0^2}{(1 + \frac{121}{304} e_0^2)^{2/5}}
\]

Substituting \( k_0(a_0, e_0), T_1 \) and multiplying the results by \( f(e_0) \) we obtained the final result,

\[
T_c(a_0, e_0) \approx \frac{a_0^4}{4\beta^2} \frac{(1 - e_0^2)^{7/2}}{(1 - e_0^{7/4})^{1/5} (1 + \frac{121}{304} e_0^2)^{2/5}}
\]

(10)

This leads to the circular orbit result exactly in equation (5) by letting \( e_0 = 0 \).

A comparison between the coalescence time using the numerical method (red dotted line) and our analytical solution (blue solid line) in equation (10) is shown in Fig. 1 for \( 0 < e_0 < 1 \). It shows that equation (10) is an accurate approximation of the numerical integral in equation (7) for \( 0 < e_0 < 1 \). Figure 1 also shows a widely used analytical approximation of the coalescence time (black dashed line) calculated by Shapiro, Teukolsky and Holes [18] as

\[
T_m(a_0, e_0) = T_c(a_0) \frac{(1 - e_0^2)^{7/2}}{(1 + \frac{121}{304} e_0^2)^{2/5}}
\]

In comparison with the numerical calculation and our analytical solution, \( T_m \) underestimates the coalescence time by up to a factor of two.

In Table I we list the coalescence time for a number of known compact binary systems in the Galaxy, calculated using our analytical solution in equation (10) denoted by \( T_c \), the numerical solution in equation (7) denoted by \( T \) and Shapiro-Teukolsky-Holes analytical solution in equation (11) denoted by \( T_m \). The binary systems include binary NS-NS systems: PSR B1913+16, PSR J0737-3039A, PSR B1534+12, PSR J1756-2251, PSR B2127+11C, PSR J1829+2456, PSR J1906+0746; and WD-NS systems: PSR J0751+1807, PSR J1141-6545, PSR J1757-5322. For all these systems the agreement between \( T_c \) and \( T \) is excellent.

\[\text{TABLE I: Coalescence time for a number of known compact binary systems in the Galaxy.} \]

| Binary System | \( m_1, m_2 \) (M⊙) | \( e_0 \) | \( a_0 \) (R⊙) | Reference | \( T_c \) (Gyr) | \( T \) (Gyr) | \( T_m \) (Gyr) |
|---------------|---------------------|----------|-------------|-----------|-------------|-------------|-------------|
| B1534+12      | 1.33, 1.35          | 0.274    | 3.280       | [19] [20] | 2.702       | 2.696       | 2.212       |
| B1913+16      | 1.44, 1.39          | 0.617    | 2.800       | [21]      | 0.293       | 0.298       | 0.136       |
| B2127+11C     | 1.35, 1.36          | 0.681    | 2.831       | [22]      | 0.211       | 0.215       | 0.087       |
| J0737-3039A&B | 1.34, 1.25          | 0.0878   | 1.2600      | [23]      | 0.0841      | 0.0840      | 0.0822      |
| J1756-2251    | 1.18, 1.4           | 0.181    | 2.700       | [24]      | 1.639       | 1.636       | 1.494       |
| J0751+1807    | 1.26, 0.12          | 0.00     | 2.28        | [25] [20] | 19.17       | 19.17       | 19.17       |
| J1757-5322    | 1.35, 0.67          | 0.00     | 3.14        | [27]      | 7.89        | 7.89        | 7.89        |
| J1141-6545    | 1.30, 0.986         | 0.172    | 1.880       | [28]      | 0.567       | 0.566       | 0.521       |
| J1829+2456    | 1.38, 1.22          | 0.139    | 6.445       | [29]      | 54.678      | 54.591      | 51.700      |
| J1906+0746    | 1.291, 1.322        | 0.085    | 1.750       | [30]      | 0.306       | 0.306       | 0.300       |

*The semi-major axis \( a_0 \) is calculated using Kepler’s law.

In summary, we have provided an analytical solution to the coalescence time for compact binary systems with arbitrary orbital separation and eccentricity. Our solution accurately reproduce the numerical results and can be useful for calculation of compact binary coalescence rates to be tested with gravitational wave detectors.

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