Supplementary Materials for

Information encoding in the spatial correlations of entangled twin beams

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Sci. Adv. 9, eadf9161 (2023)
DOI: 10.1126/sciadv.adf9161

This PDF file includes:

Sections S1 to S4
Figs. S1 to S6
References
In order to show the dependence of the spatial cross-correlation on the angular spectrum of the pump given by Eq. (2) in the main text, we start with the interaction Hamiltonian for the four-wave mixing (FWM) process, which can be written as \[ \hat{H} = i\hbar \int d^3r \chi^{(3)}(r) \hat{E}_p^+(r, t) \hat{E}_p^-(r, t) \hat{E}_c(r, t) + \text{H.c.} \] (S1)

Here \( \chi^{(3)} \) is the third order nonlinear response of the atomic medium and the subscripts \( p, pr, \) and \( c \) denote pump, probe, and conjugate fields, respectively. We have also assumed that the two required pump photons come from a single pump beam, as is the case for our experiment. These two pump photons are simultaneously converted into a pair of photons, one for the probe and conjugate fields, respectively. We have also assumed that the two required pump photons come from a single pump beam, as is the case for our experiment. These two pump photons are simultaneously converted into a pair of photons, one for the probe and conjugate fields, respectively.

\[ E_{\text{p}}^+(r, t) = E_\text{p}(\rho, z) e^{i(k_p^z z - \omega_p t)}, \] (S2)

where \( k_p^z \) denotes the \( z \)-component of the pump’s \( k \)-vector, \( \rho \) is the two-dimensional position in the transverse plane at the center of the nonlinear medium, and \( E_\text{p}(\rho, z) \) is a complex function representing the slowly varying envelop of the pump’s electric field. In our experiments, the waist of the pump is placed at the center of the \( \chi^{(3)} \) nonlinear medium (atomic vapor cell) and its transverse spatial distribution is assumed to remain unchanged throughout the length of the nonlinear medium. As a result, the slowly varying field amplitude is assumed to be independent of \( z \), that is \( E_\text{p}(\rho, z) = E_\text{p}(\rho) \).

**FIG. S1.** Four-wave mixing. (A) Schematic diagram of the FWM process used in the experiment in which two photons are absorbed from a single pump field to generate quantum correlated probe and conjugate fields. (B) Phase matching condition for the configuration in which two pump photons are absorbed from a single pump field. Phase matching needs to be satisfied for an efficient FWM process and leads to momentum correlations between the generated probe and conjugate photons. The non-collinear configuration results from the response of the atomic medium used for the FWM process \([48]\).

In momentum space (\( k \)-space) the field operators for the probe and conjugate can be written as

\[ \hat{E}_p^-(r, t) = \left( \frac{1}{2\pi} \right)^{3/2} \int dk_{pr} e^{-i(k_{pr} \cdot r - \omega_{pr} t)} \hat{a}^\dagger_{k_{pr}}, \] (S3)

\[ \hat{E}_c(r, t) = \left( \frac{1}{2\pi} \right)^{3/2} \int dk_{c} e^{-i(k_{c} \cdot r - \omega_{c} t)} \hat{b}^\dagger_{k_{c}}, \] (S4)

where \( \hat{a}^\dagger_{k_{pr}} \) (\( \hat{b}^\dagger_{k_{c}} \)) is the creation operator for a probe (conjugate) photon with a spatial profile given by a plane wave and \( \hbar k \) momentum. Using Eqs. (S2), (S3), and (S4), the FWM interaction Hamiltonian can be rewritten as

\[ \hat{H} = i\hbar \left( \frac{1}{2\pi} \right)^3 \int d^3r dk_{pr} dk_{c} \chi^{(3)}(r) E_\text{p}^2(\rho) e^{2i(k_p^z z - \omega_p t)} e^{-i(k_{pr} \cdot r - \omega_{pr} t)} e^{-i(k_{c} \cdot r - \omega_{c} t)} \hat{a}^\dagger_{k_{pr}} \hat{b}^\dagger_{k_{c}} + \text{H.c.} \] (S5)

\[ = i\hbar \Gamma \int d^3r dk_{pr} dk_{c} E_\text{p}^2(\rho) e^{-i(k_{pr}^z z - \omega_{pr} t)} e^{i\Delta k z} \hat{a}^\dagger_{k_{pr}} \hat{b}^\dagger_{k_{c}} + \text{H.c.}, \] (S6)

where we have assumed that \( \chi^{(3)} \) is spatially independent so that it can be taken outside the integral and absorbed into \( \Gamma \equiv \chi^{(3)} / (2\pi)^3 \). The term \( \Delta k = 2k_p^z - k_{pr}^z - k_c^z \) denotes the longitudinal phase-mismatch, as shown in Fig. S1B, and \( k_{pr}^z (k_c^z) \) is the probe (conjugate) momentum vector in the transverse plane. In obtaining Eq. (S6) we have assumed the angular frequency mismatch \( (\Delta \omega = 2\omega_p - \omega_{pr} - \omega_c) \) to be zero.

In order to highlight the dependence of the FWM on the transverse spatial profile of the pump, we further write the interaction Hamiltonian as

\[ \hat{H} = i\hbar 2\pi \Gamma \int d^3r dk_{pr} dk_{c} \left[ \frac{1}{2\pi} \int d\rho E_\text{p}^2(\rho) e^{-i(k_{pr}^z z - \omega_{pr} t)} \right] e^{i\Delta k z} \hat{a}^\dagger_{k_{pr}} \hat{b}^\dagger_{k_{c}} + \text{H.c.}, \] (S7)

\[ = i\hbar 2\pi \Gamma \int d^3r dk_{pr} dk_{c} \chi(k_{pr}^z + k_c^z) e^{i\Delta k z} \hat{a}^\dagger_{k_{pr}} \hat{b}^\dagger_{k_{c}} + \text{H.c.}, \] (S8)
where we have introduced the function \( \mathbb{K}(k_{pr}^+ + k_c^-) = \frac{1}{(2\pi)^3} \int d\rho E_0^2(\rho)e^{-i(k_{pr}^+ + k_c^-)\cdot \rho} \), which contains all the information about the pump transverse spatial profile and represents the transverse Fourier transform of \( E_0^2(\rho) \). We can further simplify this expression by writing the pump’s transverse spatial distribution in momentum space, such that

\[
E_0(\rho) = \frac{1}{2\pi} \int dk_{pr}^+ \mathcal{E}_o(k_{pr}^+) e^{i k_{pr}^+ \cdot \rho}, \quad (S9)
\]

where \( \mathcal{E}_o(k_{pr}^+) \) represents the angular spectrum of the pump. We can then substitute Eq. (S9) in the expression for \( \mathbb{K} \) to rewrite it as

\[
\mathbb{K}(k_{pr}^+ + k_c^-) = \frac{1}{(2\pi)^3} \int d\rho \left( \int dk_{pr}^+ \mathcal{E}_o(k_{pr}^+) e^{i k_{pr}^+ \cdot \rho} \int dk_{pr}^+ \mathcal{E}_o(k_{pr}^+) e^{-i(k_{pr}^+ + k_c^-)\cdot \rho} \right) e^{-i(k_{pr}^+ + k_c^-)\cdot \rho} = \frac{1}{(2\pi)^3} \int d\rho \left( \int dk_{pr}^+ \mathcal{E}_o(k_{pr}^+) \mathcal{E}_o(k_{pr}^+) \int d\rho e^{i(k_{pr}^+ + k_c^-)\cdot \rho} \right) e^{-i(k_{pr}^+ + k_c^-)\cdot \rho} = \frac{1}{(2\pi)} \int dk_{pr}^+ \mathcal{E}_o(k_{pr}^+) \mathcal{E}_o(k_{pr}^+) \delta(k_{pr}^+ \cdot (k_{pr}^+ + k_c^- - k_p^-)) = \frac{1}{(2\pi)} \Phi(k_{pr}^+ + k_c^-), \quad (S10)
\]

where \( \Phi(k_{pr}^+ + k_c^-) = \int dk_{pr}^+ \mathcal{E}_o(k_{pr}^+) \mathcal{E}_o(k_{pr}^+ + k_c^- - k_p^-) \) is the convolution of the angular spectra of the two pump photons involved in the FWM process. Finally, substituting Eq. (S10) into Eq. (S8) and integrating over the length \( L \) of the medium (from \(-L/2 \) to \( L/2 \)) allows us to write the interaction Hamiltonian as

\[
\hat{H} = i\hbar \Gamma L \int dk_{pr}dk_c \Phi(k_{pr}^+ + k_c^-) \text{sinc}(\Delta k_z L/2) \hat{a}_p^{\dagger} \hat{b}_c^{\dagger} + \text{H.c.}. \quad (S11)
\]

\[
i\hbar \Gamma L \int dk_{pr}dk_c \mathbb{F}(k_{pr},k_c) \hat{a}_p^{\dagger} \hat{b}_c^{\dagger} + \text{H.c.}, \quad (S12)
\]

where we have defined the two-photon amplitude function \( \mathbb{F}(k_{pr},k_c) \equiv \Phi(k_{pr}^+ + k_c^-) \text{sinc}(\Delta k_z L/2) \), which takes into account the angular spectrum of the pump through function \( \Phi \) and the phase matching condition through the sinc function. Note that the interaction Hamiltonian is very similar to the one for parametric down-conversion (PDC) \([49–51]\), except that in PDC the function \( \Phi \) is directly given by the angular spectrum of the single pump photon involved in the process instead of the convolution of the angular spectra of the two pump photons involved in FWM.

With this interaction Hamiltonian we can approximate the state of the generated twin beams by using first order perturbation theory to write the twin beam state (TBS) wavefunction as

\[
|\Psi_{TBS}\rangle = e^{-\frac{\hat{H}t}{\hbar}} |\Psi_0\rangle \approx \left( 1 - \frac{i\hat{H}t}{\hbar} \right) |\Psi_0\rangle = |\Psi_0\rangle + C_1 \int dk_{pr}dk_c \mathbb{F}(k_{pr},k_c) \hat{a}_p^{\dagger} \hat{b}_c^{\dagger} |\Psi_0\rangle + \cdots \quad (S13)
\]

\[
|\Psi_0\rangle = \int dk_{pr}dk_c \mathbb{F}(k_{pr},k_c) \hat{a}_p^{\dagger} \hat{b}_c^{\dagger} |\Psi_0\rangle, \quad (S14)
\]

where we have assumed the input state to be the multimode vacuum state \( |\Psi_0\rangle = |\{0\}_{k_{pr}}, \{0\}_{k_c}\rangle \) with notation \( |\{0\}_{k}\rangle = \prod_k |0_k\rangle \) and \( C_1 = \Gamma L t \) with interaction time \( t \). It is important to note that even though we used the process for the present experiment, the seed does not have an impact on the eigenmodes of the FWM and only serves to generate bright probe and conjugate beams with Gaussian profiles that serve as local oscillators for measuring the correlations in the FWM eigenmodes, as described in the main text. The form of the TBS wavefunction given in Eq. (S15) is the same as the one from type-I SPDC \([49, 53]\) with the angular spectrum of pump field replaced with the convolution of the angular spectra of the two pump photons involved in the FWM process. The second term in Eq. (S15) is responsible for the momentum correlations between the probe and conjugate beams with a distribution dictated by \( \mathbb{F}(k_{pr},k_c) \), such that if a single \( k \)-vector for the probe is measured the distribution of the corresponding correlated \( k \)-vectors for the conjugate can be engineered by changing the angular spectrum of the pump.

With the TBS wavefunction given in Eq. (S15), we are now in a position to calculate the spatial cross-correlation defined in the main text in terms of the spatial intensity fluctuations of images acquired by the EMCCD. We start by noting that a \( f \)-to-\( f \)
optical system is used to map the momentum distribution to a position distribution on the EMCCD in the far field; hence, the transverse spatial cross-correlation of the spatial intensity fluctuations in the far field can be written as

\[
\begin{align*}
  c_{\text{cross}}(x_1, x_2) & \equiv \\
  & \left. \left\langle \delta \hat{N}_{\text{pr}}(x_1; z_f) \delta \hat{N}_{\text{c}}(x_2; z_f) \right\rangle \right|_{z=z_0},
\end{align*}
\]  

(S16)

where \( z_f \) and \( z_o \) indicate the far field and near field along the propagation direction (\( z \)-axis), respectively, \( \delta \hat{N} = \hat{N} - \langle \hat{N} \rangle \) with photon number operator \( \hat{N} \), and \( x_1 \) and \( x_2 \) are two-dimensional transverse positions in the far field. The measurements are done in the far field where a photon with a transverse momentum vector \( k^z \) is mapped to spatial location \( x = f k^z / k \) by the \( f \)-to-\( f \) optical system. In the last expression, the \( z \)-component of the \( k \)-vector is given by \( k^z_j = \sqrt{k^2_j - |k^z_j|^2} \), where \( k^z_j \) is the magnitude of the corresponding \( k \)-vector.

We also note that in the bright limit the photon number fluctuations are proportional to the amplitude quadrature fluctuations of the field. This can be shown by writing the field operators as \( \hat{a} = |\alpha| + \delta \hat{a} \), with \( |\alpha| \) representing the mean value (amplitude) of the field and \( \delta \hat{a} \) the field fluctuation operator, such that

\[
\hat{N} = \hat{a}^\dagger \hat{a} = (|\alpha| + \delta \hat{a})(|\alpha| + \delta \hat{a}) \approx |\alpha|^2 + |\alpha|(\delta \hat{a}^\dagger + \delta \hat{a}),
\]  

(S17)

where we have taken advantage of the fact that in the bright limit \( |\delta \hat{a}|/|\alpha| \ll 1 \) to drop the term quadratic in the field fluctuations. Here we can identify \( \langle \hat{N} \rangle = |\alpha|^2 \) and \( \delta \hat{X} = (\delta \hat{a}^\dagger + \delta \hat{a})/\sqrt{2} \) to rewrite Eq. (S17) as

\[
\delta \hat{N} \approx \sqrt{2}|\alpha|\delta \hat{X},
\]  

(S18)

which makes it possible to write the spatial cross-correlation in terms of quadrature operators

\[
\begin{align*}
  c_{\text{cross}}(x_1, x_2) & \equiv \\
  & \left. \left\langle \delta \hat{X}_{\text{pr}}(x_1; z_f) \delta \hat{X}_{\text{c}}(x_2; z_f) \right\rangle \right|_{z=z_0}.
\end{align*}
\]  

(S19)

We can now take the first order approximation of the TBS wavefunction given in Eq. (S15) to calculate the required expectation value of the spatial cross-correlation of the quadrature fluctuations in \( k \)-space given in Eq. (S19). If we take into account that \( \left\langle X_{\text{pr},c} \right\rangle \) vanishes for \( |\Psi_{\text{TBS}}\rangle \), we have that

\[
\left\langle \delta \hat{X}_{\text{pr}}(k_1) \delta \hat{X}_{\text{c}}(k_2) \right\rangle = \left\langle \hat{X}_{\text{pr}}(k_1) \hat{X}_{\text{c}}(k_2) \right\rangle
\]  

(S20)

\[
\approx \langle \Psi_0 | \hat{X}_{\text{pr}}(k_1) \hat{X}_{\text{c}}(k_2) | \Psi_0 \rangle + 2 \int dk_{\text{pr}} dk_c \Re \left\{ C_1 F(k_{\text{pr}}, k_c) \langle \Psi_0 | \hat{X}_{\text{pr}}(k_1) \hat{X}_{\text{c}}(k_2) \hat{a}_{k_{\text{pr}}}^\dagger \hat{b}_{k_c} | \Psi_0 \rangle \right\}
\]

\[
+ |C_1|^2 \int dk_{\text{pr}} dk_c \int dk_{\text{pr}}' dk_c' F(k_{\text{pr}}', k_c) F(k_{\text{pr}}', k_c') \langle \Psi_0 | \hat{b}_{k_{\text{pr}}}^\dagger \hat{a}_{k_{\text{pr}}'} \hat{X}_{\text{pr}}(k_1) \hat{X}_{\text{c}}(k_2) \hat{a}_{k_{\text{pr}}}^\dagger \hat{b}_{k_c} | \Psi_0 \rangle
\]  

(S21)

\[
\approx \int dk_{\text{pr}} dk_c \Re \left\{ C_1 F(k_{\text{pr}}, k_c) \langle \Psi_0 | (\hat{a}_{k_{\text{pr}}} \hat{b}_{k_c} + \hat{a}_{k_{\text{pr}}}^\dagger \hat{b}_{k_c}^\dagger + \hat{a}_{k_{\text{pr}}} \hat{b}_{k_c} + \hat{a}_{k_{\text{pr}}}^\dagger \hat{b}_{k_c}^\dagger) \hat{a}_{k_{\text{pr}}}^\dagger \hat{b}_{k_c} | \Psi_0 \rangle \right\}
\]

\[
= \int dk_{\text{pr}} dk_c \Re \left\{ C_1 \Phi(k_{\text{pr}} \pm k_c^z) \sin(\Delta k_2 L/2) \right\} \delta(k_1 - k_{\text{pr}}) \delta(k_2 - k_c)
\]

\[
= \sin((2k_{\text{pr}}^z - k_1^z - k_2^z) L/2) \Re \left\{ C_1 \Phi(k_1^z + k_2^z) \right\}.
\]  

(S22)

For the case in which the correlations are measured within a small region around the optimal direction of the FWM process, the phase-mismatch (\( \Delta k_2 \)) is close to zero and the \( \sin \) function can be taken to be unity [54]. In our experiments, the regions in the far field that contribute to the measurements are selected by the spatial extent of the bright probe and conjugate beams and represent a small region of the full spatial bandwidth of the FWM process, as required to approximate the \( \sin \) function as a
constant. Finally, taking into account the mapping between position and momentum in the far field, we can express Eq. (S22) in position coordinates \((x_1, x_2)\) such that

\[
c_{\text{cross}}(x_1, x_2) \rightarrow \left\langle \delta \hat{X}_{pr}(x_1; z_f) \delta \hat{X}_c(x_2; z_f) \right\rangle \propto \mathcal{Re} \{ \Phi(x_1 + x_2) \},
\]

where we have taken constant \(C_1\) to be real without loss in generality, as the phase of the nonlinear response of the atomic medium, given by \(\chi^{(3)}\), can be set to zero and serve as a phase reference for the rest of the fields involved in the FWM process. Additionally, we note that as a result of the conservation of momentum, non-zero correlations are only possible if \(-x_1 \approx x_2\) (up to the uncertainty in transverse momentum). Thus, one can define \(\xi_+ = x_1 + x_2\) and \(\bar{x} = -x_1\) to rewrite Eq. (S23) as

\[
c_{\text{cross}}(\xi_+) \rightarrow \left\langle \delta \hat{X}_{pr}(-x) \delta \hat{X}_c(\bar{x} + \xi_+) \right\rangle \propto \mathcal{Re} \{ \Phi(\xi_+) \},
\]

which corresponds to Eq. (2) in the main text. Here we have taken into account the translationally invariant nature of the cross-correlation to drop the dependence on position \(\bar{x}\).

Section S2. IMAGE ACQUISITION AND MEASUREMENT OF SPATIAL CORRELATIONS

In order to measure the spatial cross-correlation and auto-correlations of the spatial intensity fluctuations, we acquire images of the bright probe and conjugate beams with an electron multiplying charge coupled device (EMCCD). To extract the spatial intensity fluctuations from these images, we take two frames (each with a bright probe and conjugate image) in rapid succession using the kinetics mode of the EMCCD \([30, 32]\). The timing sequence for the pump and probe pulses for the two frames is shown in Fig. S2A. The input probe and pump beams are pulsed with a time duration of 1 \(\mu\)s and 10 \(\mu\)s, respectively, with the timing synchronized with the data acquisition by the EMCCD. The probe pulse is delayed by 6 \(\mu\)s with respect to the pump pulse to avoid transient effects in the FWM. The active area of the EMCCD is divided into frames with 170 (rows) \times 512 (columns) pixels. Given the maximum charge transfer rate of 300 ns/row in the kinetics mode, the minimum time difference between two adjacent frames is 51 \(\mu\)s. The camera exposure time per frame is set to 12 \(\mu\)s, which leads to a time scale between two consecutive images of \(\sim 60 \mu\)s.

![Image Acquisition Sequence and Spatial Cross-Correlation](image)

**FIG. S2.** Image acquisition sequence and spatial cross-correlation of the spatial intensity fluctuations. (A) Images of bright probe and conjugate beams in two consecutive frames for a structured pump set to encode the OU logo. The pulse sequence, which is synchronized with the data acquisition of the EMCCD, is shown on the left. The separation between two seed probe pulses is \(\sim 60 \mu\)s. (B) The spatial intensity fluctuation images for the probe (left) and conjugate (right) beams are obtained by performing a pixel to pixel subtraction of the two consecutive frames. (C) After rotation of one of the fluctuation images, a two dimensional spatial cross-correlation is implemented to calculate the distribution of the relative spatial correlations between the probe and the conjugate.
Figure S2A shows bright probe and conjugate images acquired in two consecutive frames of the EMCCD. As can be seen, the peak region of the probe image has \( \sim 5 \times 10^4 \) photocounts per pixel. The probe and conjugate spatial intensity fluctuations, shown in Fig. S2B, are obtained by performing a pixel to pixel subtraction of the two consecutive frames. Such a differential analysis technique extracts the spatial fluctuations while reducing the common spatial classical noise present in the twin beams. Given that the two consecutive frames are taken with a time difference longer than the inverse of the bandwidth of the FWM process (\( \sim 20 \) MHz), there are no quantum correlations between them, so the subtraction has no impact on the quantum properties of the twin beams.

To calculate the spatial cross-correlations of the spatial intensity fluctuations, we first rotate one of the fluctuation images by \( 180^\circ \) as required by the conservation of momentum, see Eq. (S24). This allows us to obtain the translationally invariant form of \( \Phi \), and hence the spatial cross-correlation function \( c_{\text{auto}}(\xi) \) by directly evaluating a two-dimensional spatial cross-correlation between the probe and conjugate spatial intensity fluctuation images. As illustrated in the top part of Fig. S2C, this is done by taking a portion of the spatial intensity fluctuations of the probe and using it to calculate the cross-correlation with the spatial intensity fluctuations of the conjugate as a function of the relative displacement, \( \xi \), between corresponding pixels. The process is repeated 2,000 times and the resulting spatial cross-correlations are then averaged to obtain the distribution shown in the bottom part of Fig. S2C. The same procedure is used to calculate the auto-correlation functions, except that the same fluctuation image (either for the probe or conjugate) is used as the two images needed to calculate the correlation, as illustrated in the top part of Fig. S2C, with no rotation of either image.

### Section S3. IMPLEMENTATION OF COMPUTER GENERATED HOLOGRAM

As shown in the main text, we are able to engineer the distribution of the spatial correlations between the twin beams by using a structured pump beam. This requires imparting a specific angular spectrum on the pump, as dictated by Eq. (S24). The desired pump structure, determined by \( \Phi(k^\perp_{pr} + k^\perp_c) \), is implemented via the numerical computation of a suitable phase pattern \( \phi \) that is transferred to the pump beam with a spatial light modulator (SLM). The phase structured pump beam then goes through a 4f-imaging system such that it is mapped to the center of the Rb vapor cell (pump beam waist location). A computer generated hologram (CGH) is used to impart the necessary phase distribution for a given target. The goal then is to calculate the necessary CGH with phase distribution \( \phi(\rho) \), with \( \rho = (\rho_1, \rho_2) \) the transverse position coordinate at the center of the cell, such that the pump field in the far field, \( E_{\text{out}} \), matches the amplitude and phase of the target field distribution \( T \), that is

\[
E_{\text{out}} = E_o \left( \frac{f k^\perp_c}{k} \right) + E_o \left( \frac{f k^\perp_{\text{pr}}}{k} \right) = \mathcal{F} \left\{ \left[ E_o(\rho)e^{i\phi(\rho)} \right]^2 \right\} = T,
\]

where \( E_o(\rho) \) is the pump field incident on the SLM, which for our case has a Gaussian profile and flat wavefront, \( * \) denotes the convolution operation, and \( \mathcal{F} \) represents a Fourier transform.

Figure S3 shows the procedure to calculate and optimize the CGH to obtain the required angular spectrum for the pump. The incident pump field on the SLM, \( E_{\text{in}} \), is multiplied with an initial guess phase pattern \( \phi \) to initialize the input field, \( E_{\text{in}} \). This initial field is then used as a starting point to calculate \( E_{\text{out}} \) and subsequently used to optimize the phase distribution \( \phi \) by assigning a cost to any deviations from the target pattern \( T \) and using a minimization algorithm to reduce that cost. A low cost value gives a high degree of overlap between \( E_{\text{out}} \) and \( T \) and ensures an optimal angular spectrum for the pump field. For the optimization of \( E_{\text{out}} \), we use a conjugate minimization algorithm [34] coupled with the mixed-region-amplitude-freedom (MRAF) approach. In MRAF, the transverse plane of the target field is divided into a signal and a noise region and the cost function is only evaluated over the signal region, thus allowing \( E_{\text{out}} \) to take any values outside this region. As a result, a mismatch between \( T \) and \( E_{\text{out}} \) in the noise region does not affect the cost values.

For the minimization using the conjugate gradient minimization algorithm, we define the cost function (C) as

\[
C = 10^d \left[ 1 - \sum_{\text{pixels}}^* \text{Re} \left\{ \mathbf{T} \cdot E_{\text{out}} \right\} \right]^2,
\]

where \( d = 10 \), \( \mathbf{T} \) represents the complex conjugate of the target field, and \( \cdot \) denotes the point-wise multiplication. Following the MRAF approach, the cost function is evaluated over the signal region only, as indicated by an asterisk over the summation. Depending on the spatial resolution and/or size of the grid over which phase \( \phi \) is defined, the cost function represents a surface in \( N^2 \)-dimensional space for an \( N \times N \) grid size. We choose a \( 512 \times 512 \) grid size to minimize \( C \), which results in the optimization of \( 512^2 \) independent phase values. To achieve this, the gradient of the cost function, \( \partial C/\partial \phi \), is calculated on the multi-dimensional cost surface based on which a conjugate direction is chosen, see Fig. S3. While descending along a
**FIG. S3. Calculation and optimization of the computer generated hologram.** The flowchart shows the procedure for calculating the optimal $E_{\text{out}}$ using the conjugate minimization approach coupled with the MRAF algorithm. The pump beam is modified with a phase pattern at the SLM plane such that $E_{\text{out}}$ in the far-field plane matches the desired target field $T$. The procedure is initialized with an initial guess phase $\phi$ and the corresponding $E_{\text{out}}$ is compared with target pattern $T$ to estimate an initial cost using the cost function $C$. The cost function is then iteratively minimized along various conjugate directions, $d$, until the cost function stagnates. After implementing a final phase-compression algorithm to eliminate the zero order from the SLM reflection, the final phase pattern is converted into BMP format and exported to the SLM.

Specific conjugate direction, the cost function is then reduced in finite size steps until a minimum is reached. The resulting phase distribution $\phi_{\text{min}}$ is then used to calculate a new gradient and a corresponding conjugate direction for further minimization. This process is iterated until the cost function stagnates. A large constant ($10^4$) in the cost function provides faster convergence while avoiding local minima.

Once the optimization algorithm is finalized, the final phase distribution $\phi_{\text{min}}$ is further phase compressed for zero-order suppression [56]. This is needed due to the limited efficiency of the SLM for higher spatial frequencies of the phase distribution. Such a reduction in efficiency with increasing spatial frequency results in a portion of the field reflecting from the SLM without acquiring the calculated phase changes. The portion of the field that does not experience a phase change presents itself as a zero order in the Fourier plane. The overall phase values can be adjusted to suppress this effect. Finally, after compression, the resulting phase distribution is converted to an 8-bit format image and exported to the SLM.
In order to be useful for applications in secure quantum communications, it is important for the encoded information to only be accessible through joint measurements of the twin beams and not through individual beam measurements. To address this point, we start by determining the explicit dependence of the auto-correlation, $c_{\text{auto}}(x_1, x_2)$, on the angular spectrum of the pump. In analogy to Eq. (S19) for the cross-correlation, we can write

$$c_{\text{auto}}(x_1, x_2) \stackrel{\text{bright limit}}{\longrightarrow} \int d\xi_c \Re \{ \Phi(x_1 + \xi_c) \Phi^*(x_2 + \xi_c) \}.$$  

(S29)

This equation determines the shape of the auto-correlation and thus provides insight into the amount of information it contains. Since it is non-invertible in general, it is impossible to extract the angular spectrum of the pump to reconstruct the information encoded in the cross-correlation. A similar derivation for the auto-correlation of the conjugate gives the same expression as the one for the probe.

We can now use Eq. (S31) to simulate the auto-correlation and find that for all the cases we consider it is localized and has the same shape, as shown in Fig. S4. We can also perform the analysis described in Section S22 for the measured probe and conjugate individually to evaluate their spatial auto-correlations. The results are shown in Fig. S4 and agree well with the simulations. As can be seen, the auto-correlations for all cases are almost identical and do not reveal the information encoded through the angular spectrum of the pump.

Furthermore, as mentioned in the main text, the fact that the encoded information is not present in the auto-correlations, whose shape is effectively independent of the angular spectrum of the pump (see Fig. S4), is a result of having highly multi-spatial mode twin beams in which a large number of the modes contribute roughly equally to the spatial correlations. In order to show that this is the case, we start with the general wavefunction for spatially multimode twin beams, which can be written in terms of the

$$\Psi(\mathbf{k}) = \sum_{\mathbf{k}} \delta(\mathbf{k}) a_{\mathbf{k}1}^\dagger a_{\mathbf{k}2}^\dagger |\Psi_0\rangle$$

(S29)

where $|\Psi_0\rangle$ is the initial state of the system. The delta function term arises from the perfect correlations between every transverse position of the probe field with itself for the vacuum and one-photon Fock states. As can be seen, both the second term contains information on the angular spectrum of the pump, and thus the encoded information. We thus omit the delta function term for the rest of the derivation.

As described in Section S1 for our experiments measurements are limited to a small region along the optimal direction for the FWM, such that the phase-mismatch $(\Delta k_z)$ is close to zero. Therefore, the auto-correlation function in the far-field can be written as

$$c_{\text{auto}}(x_1, x_2) \stackrel{\text{bright limit}}{\longrightarrow} \int d\xi_c \Re \{ \Phi(x_1 + \xi_c) \Phi^*(x_2 + \xi_c) \}.$$  

(S30)

In this expression, the integration over $x_c$ traces over the conjugate position degree of freedom.

As opposed to the cross-correlation, for the auto-correlation we expect a non-zero correlation only when $x_1 \approx x_2$. Thus, one can define $\xi_- = x_2 - x_1, \xi' \equiv x_1 + x_2$ to rewrite Eq. (S30) as

$$c_{\text{auto}}(\xi_-) \stackrel{\text{bright limit}}{\longrightarrow} \int d\xi' \Re \{ \Phi(\xi') \Phi^*(\xi_- + \xi') \}.$$  

(S31)
FIG. S4. Spatial correlations. Spatial cross-correlation and auto-correlations for the probe and conjugate for the cases when no pattern is encoded (top row), the OU logo is encoded (middle row), and h is encoded (bottom row). The required phase patterns to encode information on the twin beams for (A) a flat wavefront (no information), (D) the OU logo, and (G) symbol h are transferred to the pump beam through a CGH implemented on an SLM. The encoded information is extracted through the corresponding cross-correlations (B), (E), and (H), respectively. On the other hand, the auto-correlations for both the probe and conjugate for all cases, (C), (F), and (I), remain unchanged even when information is encoded in the distribution of the spatial correlations. The measured auto-correlations are consistent with their corresponding simulations, which further show that the auto-correlation remains unchanged for different pump angular spectra. Therefore, each beam by itself does not contain the encoded information, which can only be extracted via the spatial cross-correlation between the two beams. All figures, except for the CGH patterns, are shown in the EMCCD pixel basis. The center portion of the auto-correlations was removed as it contains an artificial maximum due to the use of the same images to calculate it.

The spatial eigenmodes of the system as $|\Psi\rangle$ can be expressed as

$$|\Psi\rangle = \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} A_{n,i} |\{n_{pr}\}_i\rangle |\{n_{c}\}_i\rangle ,$$

where $|\{n\}_i\rangle$ represents a state with $n$ photons in spatial eigenmode $i$, subscripts pr and c represent probe and conjugate, respectively, $\sum_{i=1}^{\infty} \sum_{n=0}^{\infty} |A_{n,i}|^2 = 1$, and $A_{n,i} \propto \text{sech}(s_i) \tanh^n(s_i)$ with $s_i$ the degree of squeezing of eigenmode $i$ and the proportionality constant determined by the number of eigenmodes with $s_i \neq 0$. The exact spatial profile of the eigenmodes will depend on the angular spectrum of the pump, and thus will be different for each of the cases considered in Fig. S4. To calculate the auto-correlation, we first need the reduced density matrix for one of the beams, say the probe. It is easy to show that the
density matrix for the probe beam by itself is given by

\[ \rho_{pr} = \text{Tr}_c \{ \rho_{pr,c} \} = \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} |A_{n,i}|^2 \langle \{n_{pr}\}_i \rangle \langle \{n_{pr}\}_i | , \]  

(S33)

where \( \rho_{pr,c} \) is the density matrix for the full state.

As shown in Section S1 for our experimental conditions the fluctuations of the number operator are proportional to the fluctuations of the amplitude quadrature operator. Thus, by using Eqs. (S19) and (S20), we can rewrite the auto-correlation of the spatial intensity fluctuations of the probe in the far field, Eq. (S28), as

\[ c_{\text{auto}}(x_1, x_2) \propto \left\langle \delta \hat{X}_{pr}(x_1; z_f) \delta \hat{X}_{pr}(x_2; z_f) \right\rangle = \left\langle \hat{X}_{pr}(x_1; z_f) \hat{X}_{pr}(x_2; z_f) \right\rangle , \]  

(S34)

where \( \hat{X}(x; z_f) = \hat{a}(x; z_f) + \hat{a}^\dagger(x; z_f) / \sqrt{2} \). For simplicity of notation, we drop the explicit \( z \) dependence for the quadrature and field operators as it is assumed that they are taken to be in the far field. To evaluate the auto-correlation, we first expand the field operators in terms of the eigenmodes of the probe, which we denote as \( u_i(x) \), to obtain

\[ \hat{a}(x) = \sum_{j=1}^{\infty} u_j(x) \hat{a}_j, \]  

(S35)

where we have taken the spatial dependence out of the field operator \( \hat{a}_j \), which subtracts a photon from eigenmode \( j \). With this expansion we can write

\[ \hat{X}(x) = \frac{1}{\sqrt{2}} \sum_{j=1}^{\infty} \left[ u_j(x) \hat{a}_j + u_j^*(x) \hat{a}_j^\dagger \right] , \]  

(S36)

which makes it possible to express the spatial auto-correlation after normal ordering as

\[ c_{\text{auto}}(x_1, x_2) \propto \left\langle \hat{X}_{pr}(x_1) \hat{X}_{pr}(x_2) \right\rangle = \text{Tr} \{ \rho_{pr} \hat{X}_{pr}(x_1) \hat{X}_{pr}(x_2) \} \]  

(S37)

\[ = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{i,j=1}^{\infty} |A_{n,i}|^2 \left[ u_i(x_1)u_j^*(x_2) \left( \hat{a}_{i,j} + \langle \{n_{pr}\}_i | \hat{a}_j \rangle \langle \{n_{pr}\}_i | \hat{a}_i \rangle \right) + u_i^*(x_1)u_j(x_2) \langle \{n_{pr}\}_i | \hat{a}_j^\dagger \hat{a}_j \rangle \right] \]  

\[ = \frac{1}{2} \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} |A_{n,k}|^2 \left[ u_i(x_1)u_j^*(x_2) + \frac{1}{2} \sum_{i=1}^{\infty} \sum_{n=0}^{\infty} n |A_{n,k}|^2 \left[ u_k(x_1)u_i^*(x_2) + u_k^*(x_1)u_i(x_2) \right] \right] . \]  

(S38)

Since the eigenmodes \( u_i(x) \) form a complete basis, we have from the closure relation that

\[ \sum_{i=1}^{\infty} u_i(x_1)u_i^*(x_2) = \delta(x_1 - x_2) , \]  

(S39)

which allows us to rewrite Eq. (S38) as

\[ c_{\text{auto}}(x_1, x_2) \propto \frac{1}{2} \delta(x_1 - x_2) + \sum_{n=0}^{\infty} \sum_{k=1}^{\infty} |A_{n,k}|^2 \text{Re} \left\{ u_k(x_1)u_k^*(x_2) \right\} . \]  

(S40)

In order to see the connection of this result with the one obtained from the first order expansion of the wavefunction, we can take Eq. (S30) and expand \( \Phi \) in terms of the eigenmodes for the probe \( (u_i(x)) \) and the corresponding eigenmodes for the conjugate \( (v_i(x)) \), such that \( \Phi(x_{pr} + x_c) = \sum_i \lambda_i u_i(x_{pr}) v_i(x_c) \). In this case, we can rewrite Eq. (S30) as

\[ c_{\text{auto}}(x_1, x_2) \propto \text{Re} \left\{ \sum_{i,j} \lambda_i \lambda_j^* \int d\mathbf{c} \ u_i(x_1)u_j^*(x_2) v_i(x_c) v_j^*(x_c) \right\} \]  

\[ = \sum_i \left| \lambda_i \right|^2 \text{Re} \left\{ u_i(x_1)u_i^*(x_2) \right\} , \]  

(S41)

where we have used the orthonormality of the eigenmodes to obtain the last expression, which is just the first order expansion of the second term on the right hand side of Eq. (S40)
If we now consider the case in which the process generates twin beams with \( M \) spatial modes that dominate and have roughly the same level of squeezing, as needed for all of them to contribute equally to the spatial correlations, then \( s_i \equiv s \) and thus \( A_{n,i} \equiv A_n \) is the same for all spatial eigenmodes \( i \). In this case, we can approximate the auto-correlation as

\[
\bar{c}_{\text{auto}}(\mathbf{x}_1, \mathbf{x}_2) \propto \frac{1}{2} \delta(\mathbf{x}_1 - \mathbf{x}_2) + \sum_{n=0}^{\infty} n|A_n|^2 \text{Re}\left\{ \sum_{k=1}^{M} u_k(\mathbf{x}_1) u_k^*(\mathbf{x}_2) \right\}.
\]  
(S42)

The summation in the curly brackets can be seen as a delta sequence such that as \( M \) tends to infinity it tends to a Dirac delta function.

We can get an estimate of the number of spatial modes our system can support by taking the ratio of the area of the pump to the area of the independently-correlated regions in the near field, i.e., at the center of the cell [49]. Taking into account the corresponding mean values. The correlation-coefficient provides a measure of similarity between correlation distributions, while the ones on the right (B) show the degree to which information is leaked into the background and are calculated by setting to zero the pixels within a circular region of 10 pixels at the center of each correlation distribution.

![Table](image)

**FIG. S5. Correlation-coefficient \( r \) between correlation distributions.** The similarity between the correlation distributions can be quantified through the correlation-coefficient. We calculate \( r \) by taking a region of 121 × 121 pixels around the center of each correlation function, which is about the size of the region where information is encoded. The values of \( r \) on the left (A) show the similarity between the different correlation distributions, while the ones on the right (B) show the degree to which information is leaked into the background and are calculated by setting to zero the pixels within a circular region of 10 pixels at the center of each correlation distribution.
which the OU logo is encoded (Fig. S4F), the h is encoded (Fig. S4I), and no information is encoded, i.e. flat pump (Fig. S4C). We find value of $r \geq 0.78$ for all cases, which indicates that the shape of the auto-correlation does not significantly change when information is encoded in the system. A big part of the degradation is due to artifacts left when removing the sharp central peak from the measured auto-correlation functions. As expected, given that the probe and conjugate auto-correlations are in theory equal to each other, the correlation-coefficient $r$ between them is large. It is also important to note that the similarity between the corresponding auto-correlations and cross-correlation is low except for the case of a flat pump, for which the correlation functions are all approximately Gaussian. This further shows that the approximately Gaussian peak at the center of the auto correlations does not contain any significant information.

Next, we check if any of the encoded information might be present in the background of the auto-correlation functions. For this we take the measured correlations and remove the central region, which contains the main Gaussian peak for the auto-correlations, by setting all pixels within a circular region with a radius of 10 pixels around the center to zero. We then take these modified correlation functions and calculate the correlation-coefficient. As shown in Fig. S5B, when we compare the modified auto-correlations with the corresponding cross-correlation we find $r < 0.1$ for all cases, mostly due to the fluctuations of the backgrounds. These results further show that there is no significant information leaked into the auto-correlation functions. It is interesting to note that even with the central region removed, the value of $r$ between the measured and simulated cross-correlations for the OU logo and h remain almost unchanged.

Another essential component for the implementation of a secure quantum communication channel using twin beams is for the temporal quantum correlations between the two modes to be preserved even when information is encoded in the distribution of their spatial correlations. To verify that this is the case, we perform temporal intensity difference squeezing measurements by bypassing the EMCCD camera and detecting the bright probe and conjugate fields with photodiodes to perform an intensity difference detection. As can be seen in Fig. S6, the level of temporal intensity difference squeezing is preserved even when the angular spectrum of the pump is modified. It is important to note that for the results shown in Fig. S6, we have subtracted the noise from the scattered pump photons, which can become significant for a structured pump. However, we have verified that if we place the additional isotropically pure $^{87}\text{Rb}$ cell to filter out the unwanted scattered pump photons in front of the balanced detection system we measure the same level of intensity-difference squeezing without the need of subtracting the background pump noise. These results show that the degree of temporal quantum correlations is not affected by the information encoded in the spatial degree of freedom.

**FIG. S6. Temporal quantum correlations.** The presence of quantum correlations in the temporal domain is verified through measurements of intensity difference squeezing. The same level of squeezing is measured, after subtraction of the scattered pump noise, independent of the information encoded in the spatial degree of freedom. The solid black trace shows the shot noise level, while the other traces show the intensity difference noise when no information is encoded (solid red trace), the OU logo is encoded (dotted green trace), and h is encoded (dashed blue trace). The same levels of temporal squeezing are measured, irrespective of the encoded information, without the need to subtract the scattered pump photons if an additional $^{87}\text{Rb}$ absorption cell is placed before the photodiodes to absorb the scattered pump, which becomes significant for the case of a structured pump beam.
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