An Odd Variant of Euler Sums

Ce Xu\textsuperscript{a,b,*} Weiping Wang\textsuperscript{c,†}

\textsuperscript{a} Multiple Zeta Research Center, Kyushu University
Motooka, Nishi-ku, Fukuoka 819-0389, Japan
\textsuperscript{b} School of Mathematical Sciences, Xiamen University
Xiamen 361005, P.R. China
\textsuperscript{c} School of Science, Zhejiang Sci-Tech University,
Hangzhou 310018, P.R. China

Abstract For positive integers $p_1, p_2, \ldots, p_k, q$ with $q > 1$, we define the Euler $T$-sum $T_{p_1 p_2 \cdots p_k, q}$ as the sum of those terms of the usual infinite series for the classical Euler sum $S_{p_1 p_2 \cdots p_k, q}$ with odd denominators. Like the Euler sums, the Euler $T$-sums can be evaluated according to the Contour integral and residue theorem. Using this fact, we obtain explicit formulas for Euler $T$-sums with repeated arguments analogous to those known for Euler sums. Euler $T$-sums can be written as rational linear combinations of the Hoffman $t$-values. Using known results for Hoffman $t$-values, we obtain some examples of Euler $T$-sums in terms of (alternating) multiple zeta values. Moreover, we prove an explicit formula of triple $t$-values in terms of zeta values, double zeta values and double $t$-values. We also define alternating Euler $T$-sums and prove some results about them by the Contour integral and residue theorem. Furthermore, we define another Euler type $T$-sums and find many interesting results. In particular, we give an explicit formulas of triple Kaneko-Tsumura $T$-values of even weight in terms of single and the double $T$-values. Finally, we prove a duality formula of Kaneko-Tsumura’s conjecture.

Keywords: Multiple zeta value; Hoffman $t$-value; Euler $T$-sum; Contour integral; Residue theorem; Kaneko-Tsumura $T$-zeta values.

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\textsuperscript{*Email: 19020170155420@stu.xmu.edu.cn; 9ma18001g@math.kyushu-u.ac.jp
\textsuperscript{†Email: wpingwang@yahoo.com, wpingwang@zstu.edu.cn

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1 Introduction and Notations

For positive integers \( s_1, \ldots, s_k \) with \( s_1 > 1 \), the multiple zeta value (MZV for short) is defined by

\[
\zeta(s_1, s_2, \ldots, s_k) := \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}}. \tag{1.1}
\]

The study of multiple zeta values began in the early 1990s with the works of Hoffman [14] and Zagier [25]. The study of multiple zeta values have attracted a lot of research in the area in the last two decades. For detailed history and applications, please see the book of Zhao [26].

In a recent paper [16], Hoffman introduced and studied a new kind of multiple zeta values

\[
t(s_1, s_2, \ldots, s_k) := \sum_{n_1 > \cdots > n_k \geq 1} \frac{1}{n_1^{s_1} n_2^{s_2} \cdots n_k^{s_k}} \tag{1.2}
\]

which is called multiple \( t \)-values. As its normalized version,

\[
\tilde{t}(s_1, s_2, \ldots, s_k) := 2^{s_1 + \cdots + s_k} t(s_1, s_2, \ldots, s_k) \tag{1.3}
\]

we call it multiple \( \tilde{t} \)-values. In both these definitions, we call \( k \) the “depth” and \( s_1 + \cdots + s_k \) the “weight”.

In this paper we consider the odd variant of Euler sums

\[
T_{p_1 p_2 \cdots p_k, q} := \sum_{n=1}^{\infty} h_{n-1}^{(p_1)} h_{n-2}^{(p_2)} \cdots h_{n-1}^{(p_k)} \frac{1}{(n-1/2)^q}, \tag{1.4}
\]

which we call Euler \( T \)-sums, where \( p_j \in \mathbb{N} \) (\( j = 1, 2, \ldots, k \)) and \( 2 \leq q \in \mathbb{N} \) with \( p_1 \leq p_2 \leq \cdots \leq p_k \). Here \( h_n^{(p)} \) is defined for \( n \in \mathbb{N}_0, p \in \mathbb{N} \) by

\[
h_n^{(p)} := \sum_{k=1}^{n} \frac{1}{(k-1/2)^p}, \quad h_0^{(p)} := 0, \quad h_n := h_n^{(1)}. \tag{1.5}
\]

The classical Euler sum was introduced by Flajolet and Salvy [10], which is defined by

\[
S_{p_1 p_2 \cdots p_k, q} := \sum_{n=1}^{\infty} H_n^{(p_1)} H_n^{(p_2)} \cdots H_n^{(p_k)} \frac{1}{n^q}, \tag{1.6}
\]
where \( H^{(p)}_n \) is harmonic number of order \( p \) defined by

\[
H^{(p)}_n := \sum_{k=1}^{n} \frac{1}{n^p}, \quad H^{(p)}_0 := 0, \quad H_n := H^{(1)}_n. \tag{1.7}
\]

In the definitions of (1.4) and (1.6), the quantity \( s_1 + \cdots + s_k + q \) is called the “weight” of the sum, and the quantity \( k \) is called the “degree”. The linear sums \( S_{p,q} \) was the first considered by Euler in 1742 (see [3] for a discussion). Classical Euler sums may be studied through a profusion of methods: combinatorial, analytic and algebraic. There are many other researches on Euler sums and Euler type sums. Some related results for Euler sums may be seen in the works of [2, 5, 9, 11, 17, 22] and references therein.

Since repeated summands in partitions are indicated by powers, we denote, for instance, the sum

\[
T_{1225}^{15,q} := \sum_{n=1}^{\infty} \frac{n^2 h^2_{n-1} \left(h^2_{n-1}\right)^3}{(n-1/2)^q} h_{n-1}. \tag{5}
\]

As remarked by Flajolet and Salvy [10], every Euler sum of weight \( w \) and degree \( k \) is a \( \mathbb{Q} \)-linear combination of MZVs of weight \( w \) and depth at most \( k + 1 \) (explicit formula see our previous paper [24]). According to the definitions of Hoffman \( t \)-value and Euler \( T \)-sum, it is clear that every Euler \( T \)-sum of weight \( w \) and degree \( k \) is a \( \mathbb{Q} \)-linear combination of Hoffman \( t \)-value of weight \( w \) and depth at most \( k + 1 \). Because, by the methods of [15, 24], we may easily deduce the following relation

\[
T_{1i_2\cdots i_m,q} = \sum_{\xi \in C_m} \sum_{\sigma \in S_m} \ell(q, J_1(I^{(m)}_{\sigma}), J_2(I^{(m)}_{\sigma}), \ldots, J_p(I^{(m)}_{\sigma})),
\]

where \( \xi := (\xi_1, \xi_2, \ldots, \xi_p) \in C_m \) (\( C_m \) is a set of all compositions of \( m \)) and a permutation \( \sigma \in S_m \) (\( S_m \) is a symmetric group of all permutations on \( m \) symbols), \( I^{(m)}_{\sigma} = (i_{\sigma(1)}, \ldots, i_{\sigma(m)}) \), and

\[
J_c(I^{(m)}_{\sigma}) = i_{\sigma(1)} + \cdots + i_{\sigma(c)} \quad \text{for} \quad c = 1, 2, \ldots, p.
\]

The motivation for this paper arises from the results of Flajolet and Salvy. In [10], Flajolet and Salvy used the method of contour integral to evaluated the classical Euler sums \( S_{p_1p_2\cdots p_k,q} \). Contour integration is a classical technique for evaluating infinite sums by reducing them to a finite number of residue computations. They used the method to found many interesting results. In particular, they proved the famous result that a nonlinear Euler sum \( S_{p_1p_2\cdots p_k,q} \) reduces to a combination of sums of lower orders whenever the weight \( p_1 + p_2 + \cdots + p_k + q \) and the order \( k \) are of the same parity. In this paper, we will extend their method to Euler \( T \)-sums and find many similar results.

The main purpose of this paper is study the reducible formulas of Euler \( T \)-sums and type \( T \)-sums by the method of Contour integral. We will prove that a nonlinear Euler \( T \)-sum \( T_{p_1p_2\cdots p_k,q} \) reduces to a combination of \( \log(2) \), Euler \( T \)-sums with depth \( \leq k - 1 \), multiple zeta values with depth \( \leq k \) whenever the weight \( p_1 + p_2 + \cdots + p_k + q \) and the order \( k \) are of the same parity.

The remainder of this paper is organized as follows. In the second section we provide some asymptotic formulas of \( \Psi(1/2-s) \). Then we apply it and contour integral to evaluate the linear and nonlinear Euler \( T \)-sums. Specially, we establish the explicit formulas of linear \( T \)-sum \( T_{p,q} \).
with \( p+q \) odd, quadratic \( T \)-sums \( T_{p_1p_2,q} \) with \( p_1+p_2+q \) even and cubic \( T \)-sum \( T_{1,q} \) with \( q \) even. Further, we prove that all Euler \( T \)-sum \( T_{p_1p_2,\cdots,p_k,q} \) can be expressed in terms of a combination of \( \log(2) \), Euler \( T \)-sums with degree \( \leq k-1 \), multiple zeta values with depth \( \leq k \) whenever the weight \( p_1+p_2+\cdots+p_k+q \) if the weight \( p_1+p_2+\cdots+p_k+q \) and order \( k \) are of the same parity. In the third section, we define an alternating Euler \( T \)-sums and evaluate the linear and a quadratic alternating Euler \( T \)-sums. In the fourth section, we define an Euler type \( T \)-sums \( \tilde{S}_{p_1p_2,\cdots,p_k,q} \), which is defined by

\[
\tilde{S}_{p_1p_2,\cdots,p_k,q} := \sum_{n=1}^{\infty} \frac{h_{n_1}^{(p_1)} h_{n_2}^{(p_2)} \cdots h_{n_k}^{(p_k)}}{n^q}. \tag{1.8}
\]

Then, we establish many relations of the sum by using the Contour integral. In particular, we prove a general formula of quadratic sums \( \tilde{S}_{p_1p_2,q} \) with \( p_1+p_2+q \) even. According to the relation of \( \tilde{S}_{p_1p_2,q} \) and triple Kaneko-Tsumura \( T \)-value, we can obtain a formula of triple \( T \)-value with weight even. In the last section, we prove a duality identity of Kaneko-Tsumura’s conjecture, and establish a relation between the double \( T \)-values and the double \( t \)-values.

## 2 Evaluations of Euler \( T \)-sums

In [23], the second author defined a parametric digamma (or Psi) function \( \Psi(-s; a) \) by

\[
\Psi(-s; a) + \gamma := \frac{1}{s-a} + \sum_{k=1}^{\infty} \left( \frac{1}{k+a} - \frac{1}{k+a-s} \right), \quad (s \in \mathbb{C}, \ a \in \mathbb{C} \setminus \mathbb{N}^-). \tag{2.1}
\]

The function \( \Psi(-s; a) \) is meromorphic in the entire complex plane with a simple pole at \( s = n+a \) for each negative integer \( n \). In here, we let

\[
\Psi(-s) := \Psi(-s; -1/2) + \gamma = \frac{1}{s+1/2} + \sum_{k=1}^{\infty} \left( \frac{1}{k-1/2} - \frac{1}{k-1/2-s} \right).
\]

From Theorems 1.1-1.3 and Corollary 2.4 in [23], by direct calculations we can obtain the following identities (\( 2 \leq p \in \mathbb{N} \))

\[
\Psi \left( \frac{1}{2} - s \right) \xrightarrow{s \rightarrow n} \frac{1}{s-n} + H_n + 2 \log(2) + \sum_{j=1}^{\infty} \left( (-1)^j H_n^{(j+1)} - \zeta(j+1) \right) (s-n)^j \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{2.2}
\]

\[
\Psi \left( \frac{1}{2} - s \right) \xrightarrow{s \rightarrow n-1/2} h_n + \sum_{j=1}^{\infty} \left( (-1)^j h_n^{(j+1)} - \overline{\zeta}(j+1) \right) (s+1/2-n)^j \quad (n \in \mathbb{N}_0), \tag{2.3}
\]

\[
\Psi \left( \frac{1}{2} - s \right) \xrightarrow{s \rightarrow -(n-1/2)} h_{n-1} + \sum_{j=1}^{\infty} \left( h_{n-1}^{(j+1)} - \overline{\zeta}(j+1) \right) (s-1/2+n)^j \quad (n \in \mathbb{N}), \tag{2.4}
\]

\[
\Psi^{(p-1)} \left( \frac{1}{2} - s \right) \xrightarrow{(p-1)! \rightarrow n} \frac{1}{(s-n)^{p-1}} \left( 1 + (-1)^p \sum_{j=p}^{\infty} \binom{j-1}{p-1} \left( \zeta(j) + (-1)^j H_n^{(j)} \right) (s-n)^j \right) \quad (n \in \mathbb{N}_0), \tag{2.5}
\]
\[
\frac{\Psi^{(p-1)} \left( \frac{1-s}{2} \right)}{(p-1)!} s \to -n \begin{pmatrix} -n-1/2 \end{pmatrix} (-1)^p \sum_{j=p}^{\infty} \left( \frac{j-1}{p-1} \right) \left( \bar{t}(j) + (-1)^j h_n^{(j)} \right) (s-n+1/2)^{j-p} \quad (n \in \mathbb{N}),
\]
(2.6)

\[
\frac{\Psi^{(p-1)} \left( \frac{1-s}{2} \right)}{(p-1)!} s \to -n \begin{pmatrix} -n-1/2 \end{pmatrix} (-1)^p \sum_{j=p}^{\infty} \left( \frac{j-1}{p-1} \right) \left( \bar{t}(j) - h_n^{(j)} \right) (s+n-1/2)^{j-p} \quad (n \in \mathbb{N}).
\]
(2.7)

We also deduce that from [23]

\[
\pi \tan(\pi s) s \to -n \begin{pmatrix} -n-1/2 \end{pmatrix} = \frac{1}{s - \frac{2n-1}{2}} + 2 \sum_{k=1}^{\infty} \zeta(2k) \left( s - \frac{2n-1}{2} \right)^{2k-1} \quad (n \in \mathbb{Z}).
\]
(2.8)

**Lemma 2.1** ([10]) Let \( \xi(s) \) be a kernel function and let \( r(s) \) be a rational function which is \( O(s^{-2}) \) at infinity. Then

\[
\sum_{\alpha \in O} \text{Res}(r(s) \xi(s))_{s=\alpha} + \sum_{\beta \in S} \text{Res}(r(s) \xi(s))_{s=\beta} = 0,
\]
(2.9)

where \( S \) is the set of poles of \( r(s) \) and \( O \) is the set of poles of \( \xi(s) \) that are not poles of \( r(s) \). Here \( \text{Res}(r(s))_{s=\alpha} \) denotes the residue of \( r(s) \) at \( s = \alpha \). The kernel function \( \xi(s) \) is meromorphic in the whole complex plane and satisfies \( \xi(s) = o(s) \) over an infinite collection of circles \( |s| = \rho_k \) with \( \rho_k \to \infty \).

In below, we use the identities (2.2)-(2.8) and residue theorem to evaluate some Euler T-sums. We need the formula ([1, 13])

\[
\pi \tan(\pi s) = 2 \sum_{k=1}^{\infty} \bar{t}(2k)s^{2k-1} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}2^{2k}(2^{2k} - 1)B_{2k} \pi^{2k}}{(2k)!} s^{2k-1},
\]
(2.10)

where \( B_{2k} \) is Bernoulli numbers. By (2.10), we have

\[
\lim_{s \to n} \frac{d^p}{ds^p} (\pi \tan(\pi s)) = \lim_{t \to 0} \frac{d^p}{dt^p} (\pi \tan(\pi t)) = \lim_{t \to 0} \frac{d^p}{dt^p} \left( 2 \sum_{k=1}^{\infty} \bar{t}(2k)s^{2k-1} \right) = (1 - (-1)^p)p!\bar{t}(p+1).
\]
(2.11)

Next, let \( \text{Res}[f(s), s = \alpha] \) to denote the residue of \( f(s) \) at \( s = \alpha \).

**Lemma 2.2** If a meromorphic function \( F(s) \) has pole of order \( m \) at \( s = \alpha \), then

\[
\text{Res}[F(s), s = \alpha] = \lim_{s \to \alpha} \frac{1}{(m-1)!} \frac{d^{m-1}}{ds^{m-1}} [(s - \alpha)^m F(s)] = \lim_{s \to \alpha} \frac{1}{m!} \frac{d^m}{ds^m} [(s - \alpha)^{m+1} F(s)].
\]

**Proof.** This proof is very simple, so we omitted. \( \square \)
2.1 Linear Euler T-sums

In this subsection, we will prove the linear T-sum $T_{p,q}$ can be expressed in terms of $\log(2)$, zeta values and $\tilde{t}$-values with $p + q$ odd ($q \geq 2$).

**Theorem 2.3** For positive integer $q > 1$,
\[
(1 + (-1)^q)T_{1,q} = (-1)^{q+1}\tilde{t}(q + 1) + (1 - (-1)^q)2\log(2)\tilde{t}(q) - 2\sum_{2k_1 + k_2 = q, \ k_1, k_2 \geq 1} \tilde{t}(2k_1)\zeta(k_2 + 1). \tag{2.12}
\]

**Theorem 2.4** For positive integers $p, q > 1$,
\[
(1 - (-1)^{p+q})T_{p,q} = (-1)^{p+q}\tilde{t}(p + q) - (-1)^p(1 + (-1)^q)\tilde{t}(p)\tilde{t}(q) - (-1)^p \sum_{k=0}^{p-1}((-1)^k - 1)\left(\frac{p + q - k - 2}{q - 1}\right)\tilde{t}(k + 1)\zeta(p + q - k - 1) + 2(-1)^p \sum_{2k_1 + k_2 = q + 1, \ k_1, k_2 \geq 1} \left(\frac{k_2 + p - 2}{p - 1}\right)\tilde{t}(2k_1)\zeta(k_2 + p - 1). \tag{2.13}
\]

**Proof.** In the context of this paper, these theorems results form applying the kernels
\[
\pi \tan(\pi s)\Psi(1/2 - s) \quad \text{and} \quad \frac{\pi \tan(\pi s)\Psi^{(p-1)}(1/2 - s)}{(p-1)!}
\]
to the base function $r(s) = s^{-q}$, respectively. Now, we only prove the formula (2.13). The identity (2.12) can be shown in a similar way. Let
\[
F_{p-1,q}(s) := \frac{\pi \tan(\pi s)\Psi^{(p-1)}(1/2 - s)}{s^q(p-1)!}.
\]
The function $F_{p-1,q}(s)$ only have poles at the $s = 0, \pm(n - 1/2), n (n \in \mathbb{N})$. At a positive integer $n$, the pole $\pm(n - 1/2)$ are simple and the residue is
\[
\text{Res}[F_{p-1,q}(s), s = n - 1/2] = -\lim_{s \rightarrow n - 1/2} \frac{\Psi^{(p-1)}(1/2 - s)}{s^q(p-1)!} = -(-1)^p\tilde{t}(p) + h_n^{(p)} \frac{(n-1/2)^q}{(n-1/2)^q},
\]
\[
\text{Res}[F_{p-1,q}(s), s = 1/2 - n] = -\lim_{s \rightarrow 1/2 - n} \frac{\Psi^{(p-1)}(1/2 - s)}{s^q(p-1)!} = -(-1)^{p+q}\tilde{t}(p) - h_{n-1}^{(p+q)} \frac{(n-1/2)^q}{(n-1/2)^q}.
\]
where we used the identities (2.6) and (2.7). For a positive integer $n$, the pole has order $p - 1$ (since $s = n$ is a zero of order one of $\tan(\pi s)$), by (2.5), (2.11) and Lemma 2.2, the residue is
\[
\text{Res}[F_{p-1,q}(s), s = n] = \frac{1}{(p-1)!} \lim_{s \rightarrow n} \frac{d^{p-1}}{ds^{p-1}} \pi \tan(\pi s) \frac{s^q}{s^q}
\]
\[
= (-1)^{p-1} \sum_{k=0}^{p-1}((-1)^k - 1)\left(\frac{p + q - k - 2}{q - 1}\right)\tilde{t}(k + 1) \frac{(p - 1)!}{(p - 1)!}.
\]
By (2.5) and (2.10) with $n = 0$, we know that if $s \to 0$, then
\[
F_{p-1,q}(s) = \frac{2}{s^{p+q-1}} \left\{ \sum_{k=1}^{\infty} \bar{t}(2k)s^{2k-2} + (-1)^p \sum_{k_1,k_2=1}^{\infty} \binom{k_2 + p - 2}{p - 1} \bar{t}(2k_1) \zeta(k_2 + p - 1) s^{2k_1 + k_2 + p - 3} \right\}.
\]
Hence, the residue of the pole of order $p + q - 1$ at 0 is found to be
\[
\text{Res}[F_{p-1,q}(s), s = 0] = (1 + (-1)^{p+q}) \bar{t}(p + q) + 2(-1)^p \sum_{2k_1 + k_2 = q + 1, k_1,k_2 \geq 1}^{\infty} \binom{k_2 + p - 2}{p - 1} \bar{t}(2k_1) \zeta(k_2 + p - 1).
\]
Summing these four contributions yields the statement of the theorem 2.4. □

Therefore, from Theorems 2.3 and 2.4, we arrive at the conclusions
\[
\bar{t}(q, 1) = T_{1,q} \in \mathbb{Q}[\log(2), \text{Zeta values}] \quad (q \text{ even}),
\]
\[
\bar{t}(q, p) = T_{p,q} \in \mathbb{Q}[\text{Zeta values}] \quad (p + q \text{ odd}),
\]
where we used the relation $\bar{t}(p) = (2^p - 1) \zeta(p)$. For even weights, a modified form of the identity holds, but without any linear Euler $T$-sum occurring. This gives back well-known nonlinear relations between $\bar{t}$-values at even arguments.

Example 2.1
\[
T_{1,2} = -\frac{7}{2} \zeta(3) + \pi^2 \log(2),
\]
\[
T_{2,3} = -\frac{31}{2} \zeta(5) + \frac{3}{2} \pi^2 \zeta(3),
\]
\[
T_{3,2} = -\frac{31}{2} \zeta(5) + 2\pi^2 \zeta(3),
\]
\[
T_{1,4} = -\frac{31}{2} \zeta(5) + \frac{1}{3} \pi^4 \log(2) - \frac{1}{2} \pi^2 \zeta(3).
\]

2.2 Quadratic Euler $T$-sums

Theorem 2.5 For positive integer $q > 1$,
\[
(1 + (-1)^q)T_{1^2,q} = \pi^2 \zeta(q) + (-1)^q \bar{t}(q + 2) + (1 - (-1)^q)4 \log(2) \bar{t}(q + 1) - 2\bar{t}(q + 1, 1)
\]
\[
= (1 + (-1)^q)4 \log(2) \bar{t}(q) - 4 \sum_{2k_1 + k_2 = q + 1, k_1,k_2 \geq 1}^{\infty} \bar{t}(2k_1) \zeta(k_2 + 1)
\]
\[
- 8 \log(2) \sum_{2k_1 + k_2 = q, k_1,k_2 \geq 1}^{\infty} \bar{t}(2k_1) \zeta(k_2 + 1)
\]
\[
+ 2 \sum_{2k_1 + k_2 + k_3 = q, k_1,k_2 \geq 1}^{\infty} \bar{t}(2k_1) \zeta(k_2 + 1) \zeta(k_3 + 1). \tag{2.14}
\]

Proof. The proof is based on the function
\[
F_{0^2,q}(s) := \frac{\pi \tan(\pi s) (\Psi(1/2 - s))^2}{s^q}
\]
Consider the function $F_{02,q}(s)$ only have poles at the $s = 0, \pm(n - 1/2), n (n \in \mathbb{N})$. By a similar argument as in the proof of (2.13), we deduce

$$
\text{Res}[F_{02,q}(s), s = n] = \frac{\pi^2}{n^q} \quad (n \in \mathbb{N}),
$$

$$
\text{Res}[F_{02,q}(s), s = n - 1/2] = -\frac{h_n^2}{(n - 1/2)^q} \quad (n \in \mathbb{N}),
$$

$$
\text{Res}[F_{02,q}(s), s = 1/2 - n] = -(1)^q\frac{h_{n-1}^2}{(n - 1/2)^q} \quad (n \in \mathbb{N})
$$

and

$$
\text{Res}[F_{02,q}(s), s = 0] = (1 + (-1)^q)\bar{\ell}(q + 2) + (1 - (-1)^q)4 \log(2)\bar{\ell}(q + 1) + (1 + (-1)^q)4 \log^2(2)\bar{\ell}(q) - 4 \sum_{2k_1 + k_2 = q + 1, k_1, k_2 \geq 1} \bar{\ell}(2k_1)\zeta(k_2 + 1) - 8 \log(2) \sum_{2k_1 + k_2 = q, k_1, k_2 \geq 1} \bar{\ell}(2k_1)\zeta(k_2 + 1)
$$

$$
- 2 \sum_{2k_1 + 1k_2 + k_3 = q, k_1, k_2, k_3 \geq 1} \bar{\ell}(2k_1)\zeta(k_2 + 1)\zeta(k_3 + 1).
$$

Thus, summing these four contributions yields the desired result. □

Hence, from (2.14),

$$
T_{12,q} + \bar{\ell}(q + 1, 1) \in \mathbb{Q}[\log(2), \text{Zeta values}] \quad (q \text{ even}).
$$

If $q = 2$, then

$$
T_{12,2} + \bar{\ell}(3, 1) = 2 \log^2(2)\pi^2.
$$

Note that $T_{12,q} = 2\bar{\ell}(q, 1, 1) + \bar{\ell}(q, 2)$.

**Theorem 2.6** For positive integer $q > 1$,

\begin{align*}
(1 - (-1)^q)T_{12,q} &= \pi^2(\zeta(q, 1) + 2 \log(2)\zeta(q)) - (q - 1)\pi^2\zeta(q + 1) + ((-1)^q + 1)\bar{\ell}(2)\bar{\ell}(q, 1) \\
&- \bar{\ell}(2)\bar{\ell}(q + 1) - \bar{\ell}(q + 2, 1) - \bar{\ell}(q + 1, 2) - (-1)^q\bar{\ell}(q + 3) \\
&+ 2(1 + (-1)^q) \log(2)\bar{\ell}(q + 2) + 2 \sum_{2k_1 + k_2 = q + 2, k_1, k_2 \geq 1} (k_2 - 1)\bar{\ell}(2k_1)\zeta(k_2 + 1) \\
&+ 4 \log(2) \sum_{2k_1 + k_2 = q + 1, k_1, k_2 \geq 1} k_2\bar{\ell}(2k_1)\zeta(k_2 + 1) \\
&- 2 \sum_{2k_1 + k_2 = q + 1, k_1, k_2 \geq 1} k_3\bar{\ell}(2k_1)\zeta(k_2 + 1)\zeta(k_3 + 1) \\
&= (2.15)
\end{align*}

**Proof.** Consider the function

$$
F_{01,q}(s) := \frac{\pi \tan(\pi s)\Psi(1/2 - s)\Psi^{(1)}(1/2 - s)}{s^q}.
$$

Then, by a similar argument as in the proof of (2.14), we can prove the theorem. □

If $q = 3$, then

$$
2T_{12,3} + \bar{\ell}(5, 1) + \bar{\ell}(4, 2) = \frac{-\pi^6}{24} + 6\pi^2 \log(2)\zeta(3).
$$
Theorem 2.7 For positive integer $p, q > 1$,

\[(1 - (-1)^{p+q})T_{p,q} = -(-1)^q (1 + (-1)^q) \tilde{t}(p) T_{1,q} - (-1)^p \tilde{t}(p) \tilde{t}(q + 1) - T_{1,p+q} - T_{p,q+1} - (-1)^{p+q} \tilde{t}(p + q + 1)\]

\[+ (1 + (-1)^{p+q})2 \log(2) \tilde{t}(p + q) - 2 \sum_{2k_1 + 2k_2 = p+q, \ k_1, k_2 \geq 1} \tilde{t}(2k_1) \zeta(k_2 + 1)\]

\[+ (-1)^{p+2} \sum_{2k_1 + 2k_2 = q+1, \ k_1, k_2 \geq 1} \binom{k_2 + p - 2}{p-1} \tilde{t}(2k_1) \zeta(k_2 + p - 1)\]

\[+ (-1)^p 4 \log(2) \sum_{2k_1 + k_2 = q+1, \ k_1, k_2 \geq 1} \binom{k_2 + p - 2}{p-1} \tilde{t}(2k_1) \zeta(k_2 + p - 1)\]

\[- (-1)^p 4 \sum_{2k_1 + k_2 = q+1, \ k_1, k_2 \geq 1} \binom{k_3 + p - 2}{p-1} \tilde{t}(2k_1) \zeta(k_2 + 1) \zeta(k_3 + p - 1)\]

\[+ (-1)^p \sum_{l=0}^{p-1} (1 - (-1)^l) \binom{p + q - l - 1}{q - 1} \tilde{t}(l + 1) \zeta(p + q - l)\]

\[- (-1)^p \sum_{l=0}^{p-1} (1 - (-1)^l) \binom{p + q - l - 2}{q - 1} \tilde{t}(l + 1)\]

\[\times \left( (S_{1,p+q-l-1} + 2 \log(2) \zeta(p + q - l - 1)) + (-1)^p \sum_{k=1}^{p-1} (-1)^{k+1} \sum_{l=0}^{p-k-1} (1 - (-1)^l) \binom{p + q - k - l - 2}{q - 1} \tilde{t}(l + 1)\]

\[\times \left( (1 - (-1)^{k+1} S_{k+1,p+q-k-l-1} - \zeta(k + 1) \zeta(p + q - k - l - 1)) \right). \] (2.16)

Proof. By computing the residues of the function

\[F_{0(p-1),q}(s) := \frac{\pi \tan(\pi s) \Psi(1/2 - s) \Psi(p-1)(1/2 - s)}{s^q},\]

we may deduce the desired formula.

Putting $p = 3, q = 2$ in equation above gives

\[2T_{13,2} + T_{1,5} + T_{3,3} = 8 \log(2) \pi^2 \zeta(3) - \frac{7}{360} \pi^6.\]

A more general reduction results from the kernel

\[F_{(p_1-1)(p_2-1),q}(s) := \frac{\pi \tan(\pi s) \Psi(p_1-1)(1/2 - s) \Psi(p_2-1)(1/2 - s)}{(p_1-1)!(p_2-1)!},\]

but it involves a parity restriction on the weight because of its trigonometric factor.

Theorem 2.8 For positive integer $p_1, p_2, q > 1$,

\[(1 + (-1)^{p_1+p_2+q})T_{p_1,p_2,q} \in \mathbb{Q}[\text{zeta values, double zeta values, double } \tilde{t}-\text{values}].\]
We have

\[
(1 + (-1)^{p_1+p_2+q})T_{p_1,p_2,q} \\
= -T_{p_1+1,p_1+q} - T_{p_2+1,p_2+q} - (-1)^{p_1+p_2} (1 + (-1)^{p_1+q}) \bar{\ell}(p_1) \bar{\ell}(p_2) + (-1)^{p_1} ((-1)^{p_2+q} - 1) \bar{\ell}(p_1) T_{p_2,q} \\
+ (-1)^{p_2} ((-1)^{p_1+q} - 1) \bar{\ell}(p_2) T_{p_1,q} - (-1)^{p_1} \bar{\ell}(p_1) \bar{\ell}(p_2 + q) - (-1)^{p_2} \bar{\ell}(p_2) \bar{\ell}(p_1 + q) \\
- (-1)^{p_1+p_2} \sum_{l=0}^{p_1+p_2-1} ((-1)^{l - 1} \left( \frac{p_1 + p_2 + q - l - 2}{q - 1} \right) \bar{\ell}(l + 1) \zeta(p_1 + p_2 + q - l - 1) \\
+ (-1)^{p_1+p_2} \sum_{k=1}^{p_2}((-1)^{k(p_1+2)} \left( \frac{p_2 - k}{p_1 - 1} \right) \sum_{l=0}^{p_2-k}((-1)^{l - 1} \left( \frac{p_2 + q - k - l - 1}{q - 1} \right) \bar{\ell}(l + 1) \\
\times (\zeta(k + p_1 - 1) \zeta(p_2 + q - k - l) + (-1)^{k+p_1-1} S_{k+p_1-1,p_2+q-k-l}) \\
+ (-1)^{p_1+p_2} \sum_{k=1}^{p_1}((-1)^{k(p_2+2)} \left( \frac{p_2 - k}{p_1 - 1} \right) \sum_{l=0}^{p_1-k}((-1)^{l - 1} \left( \frac{p_1 + q - k - l - 1}{q - 1} \right) \bar{\ell}(l + 1) \\
\times (\zeta(k + p_2 - 1) \zeta(p_1 + q - k - l) + (-1)^{k+p_2-1} S_{k+p_2-1,p_1+q-k-l}) \\
+ (-1)^{p_1+p_2+q} \bar{\ell}(p_1 + p_2 + q) + (-1)^{p_1+2} \sum_{k_1+k_2=p_2+q+1, k_1,k_2 \geq 1} \left( \frac{p_1 + k_1 - 2}{p_1 - 1} \right) \zeta(p_1 + k_1 - 1) \bar{\ell}(2k_2) \\
+ (-1)^{p_1+2} \sum_{k_1+k_3=q+1, k_1,k_2,k_3 \geq 1} \left( \frac{p_2 + k_1 - 2}{p_2 - 1} \right) \zeta(p_2 + k_2 - 1) \bar{\ell}(2k_1) \\
+ (-1)^{p_1+p_2} \sum_{k_1+k_2+k_3=p_2+q+1, k_1,k_2,k_3 \geq 1} \left( \frac{p_1 + k_1 - 2}{p_1 - 1} \right) \left( \frac{p_2 + k_2 - 2}{p_2 - 1} \right) \zeta(p_1 + k_1 - 1) \zeta(p_2 + k_2 - 1) \bar{\ell}(2k_3).}
\]

Proof. Let

\[
F_{(p_1-1)(p_2-1),q}(s) := \frac{\pi \tan(\pi s) \Psi^{(p_1-1)}(1/2 - s) \Psi^{(p_2-1)}(1/2 - s)}{s^q(p_1 - 1)!(p_2 - 1)!}.
\]

By (2.5)-(2.8) and (2.10), we arrive at

\[
\text{Res}[F_{(p_1-1)(p_2-1),q}(s), s = n - 1/2] = -\frac{(-1)^{p_1+p_2} \bar{\ell}(p_1) \bar{\ell}(p_2) + (-1)^{p_1} \bar{\ell}(p_1) h_n^{(p_2)} + (-1)^{p_2} \bar{\ell}(p_2) h_n^{(p_1)}}{(n - 1/2)^q} \\
- \frac{h_n^{(p_1)} h_n^{(p_2)}}{(n - 1/2)^q}.
\]

\[
\text{Res}[F_{(p_1-1)(p_2-1),q}(s), s = 1/2 - n] = -\frac{(-1)^{p_1+p_2+q} \bar{\ell}(p_1) \bar{\ell}(p_2) - \bar{\ell}(p_1) h_n^{(p_2)} - \bar{\ell}(p_2) h_n^{(p_1)}}{(n - 1/2)^q} \\
- (-1)^{p_1+p_2+q} \frac{h_n^{(p_1)} h_n^{(p_2)}}{(n - 1/2)^q}.
\]

and

\[
\sum_{n=1}^{\infty} \text{Res}[F_{(p_1-1)(p_2-1),q}(s), s = n] \in \mathbb{Q}[\zeta \text{ values, double } \zeta \text{ values}],
\]

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Res\left[F_{(p_1-1)(p_2-1),q}(s), s = 0\right] \in \mathbb{Q}[\text{zeta values}].

Hence, using Lemma 2.1 and combining the four identities gives the desired result. □

If \( p_1 = p_2 = q = 2 \), then
\[ T_{2x,2} + T_{2,4} = \frac{7}{360} \pi^6. \]

By the definitions of \( \tilde{t} \)-values and \( T \)-sums, we have (\( p_1 \geq 1, p_2, q > 1 \))
\[
T_{p_1,p_2,q} = \tilde{t}(p_1,p_2) + \tilde{t}(q,p_2, p_1) + \tilde{t}(q, p_1 + p_2)
= \tilde{t}(p_2)\tilde{t}(q, p) + \sum_{n=1}^\infty \frac{h_n^{(p_1)}(\tilde{t}(p_2) - h_n^{(p_2)})}{(n-1/2)^q} - \tilde{t}(p_2 + q, p_1)
= \tilde{t}(p_2, q, p_1) + \tilde{t}(p_2)\tilde{t}(q, p_1) - \tilde{t}(p_2 + q, p_1).
\]

Hence, from Theorems 2.5-2.8, we obtain the conclusion that the Hoffman \( t \)-values of depth three can be expressed in terms of zeta values, double zeta values and double \( t \)-values.

### 2.3 Cubic and Higher Order Euler \( T \)-sums

For higher degree sums, like the cubic
\[
T_{1^3,q} := \sum_{n=1}^\infty \frac{h_n^3}{(n-1/2)^q}
\]
it is natural to consider the kernel \( \pi \tan(\pi s) (\Psi(1/2 - s))^3 \).

**Theorem 2.9** For positive integer \( q > 1 \),
\[
(1 + \frac{1}{(-1)^q})T_{1^3,q} = 3\pi^2(\zeta(q, 1) + 2 \log(2)\zeta(q)) - (q - 3)\pi^2\zeta(q + 1) - 3T_{1^2,q+1} - 3\tilde{t}(q + 2, 1) - \tilde{t}(q + 3) + \text{Res}[F_{0^3,q}(s), s = 0],
\]
where \( F_{0^3,q}(s) \) defined by
\[
F_{0^3,q}(s) := \frac{\pi \tan(\pi s) (\Psi(1/2 - s))^3}{s^q},
\]
and
\[
\text{Res}[F_{0^3,q}(s), s = 0] = (1 - \frac{1}{(-1)^q})\tilde{t}(q + 3) + (1 + \frac{1}{(-1)^q})6 \log(2)\tilde{t}(q + 2)
+ (1 - \frac{1}{(-1)^q})12 \log^2(2)\tilde{t}(q + 1) + (1 + \frac{1}{(-1)^q})8 \log^3(2)\tilde{t}(q)
- 6 \sum_{2k_1 + k_2 = q + 2, k_1, k_2 \geq 1} \tilde{t}(2k_1)\zeta(k_2 + 1) - 24 \log(2) \sum_{2k_1 + k_2 = q + 1, k_1, k_2 \geq 1} \tilde{t}(2k_1)\zeta(k_2 + 1)
- 24 \log^2(2) \sum_{2k_1 + k_2 = q, k_1, k_2 \geq 1} \tilde{t}(2k_1)\zeta(k_2 + 1)
+ 6 \sum_{2k_1 + k_2 + k_3 = q + 1, k_1, k_2, k_3 \geq 1} \tilde{t}(2k_1)\zeta(k_2 + 1)\zeta(k_3 + 1)
+ 12 \log(2) \sum_{2k_1 + k_2 + k_3 = q, k_1, k_2, k_3 \geq 1} \tilde{t}(2k_1)\zeta(k_2 + 1)\zeta(k_3 + 1)
\]

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We define alternating Euler T-sums with degree $\leq k - 1$, multiple zeta values with depth $\leq k$ whenever the weight $p_1 + p_2 + \cdots + p_k + q$ and the order $k$ are of the same parity.
In (1.1)-(1.3), we put a bar on top of \( s_j \) (\( j = 1, \cdots k \)) if there is a sign \((-1)^{n_j}\) appearing in the denominator on the right, which are called the alternating MZVs, alternating multiple \( t \)-values and multiple \( \tilde{t} \)-values, respectively. For example,

\[
\zeta(s_1, s_2, s_3) = \sum_{n_1 > n_2 > n_3 \geq 1} (-1)^{n_1+n_3} n_1^{s_1} n_2^{s_2} n_3^{s_3}, \quad \tilde{t}(s_1, s_2) = \sum_{n_1 > n_2 \geq 1} (-1)^{n_1} (n_1 - 1/2)^{s_1} (n_2 - 1/2)^{s_2}.
\]

Some results of alternating MZVs, readers may consult [4,6–8,18] and references therein.

For convenience, we let

\[
\tilde{t}(s) := -\tilde{t}(\bar{s}) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n - 1/2)^s}.
\]

In this section, we will discuss the alternating Euler \( T \)-sums and find some evaluations of closed form of it. We need the formula (see [23])

\[
\frac{\pi}{\cos(\pi s)} \quad s \to n = 1/2 \quad (-1)^n \frac{1}{s - \frac{2n-1}{2}} - 2 \sum_{k=1}^{\infty} \zeta(2k) \left( s - \frac{2n-1}{2} \right)^{2k-1}.
\]

(3.2)

From [13],

\[
\frac{\pi}{\cos(\pi s)} = 2 \sum_{k=0}^{\infty} \tilde{t}(2k+1) s^{2k} = \sum_{k=0}^{\infty} \frac{(-1)^k E_{2k} \pi^{2k+1}}{(2k)!} s^{2k},
\]

(3.3)

where \( E_{2k} \) is Euler number. Then, we compute

\[
\lim_{s \to n} \frac{d^p}{ds^p} \frac{\pi}{\cos(\pi s)} = (-1)^n (1 + (-1)^p) p! \tilde{t}(p+1).
\]

(3.4)

### 3.1 Linear Alternating Euler \( T \)-sums

**Theorem 3.1** For positive integer \( q \),

\[
(1 + (-1)^{q+1}) \tilde{T}_{1,q} = \frac{(-1)^{q+1}}{q+1}(q+1) - \pi \zeta(q+1) + (-1)^q - 1)2 \log(2) \tilde{t}(q) + 2 \sum_{2k_1+k_2=q-2, k_1, k_2 \geq 1} \tilde{t}(2k_1 + 1) \zeta(k_2 + 2).
\]

(3.5)

**Theorem 3.2** For positive integer \( p \) and \( q > 1 \),

\[
(1 + (-1)^{p+q}) \tilde{T}_{p,q} = \frac{(-1)^{p+q}}{p+q}(p+q) + (-1)^p(1 - (-1)^q) \tilde{t}(p) \tilde{t}(q) + (-1)^p \sum_{k=0}^{p-1} \frac{(-1)^k + 1}{q+1} \left( \frac{p+q-k-2}{q-1} \right) \tilde{t}(k+1) \zeta(p+q-k-1)
\]

\[
- 2(-1)^p \sum_{2k_1+k_2=q-1, k_1, k_2 \geq 0} \left( \frac{p+k_2-1}{p-1} \right) \tilde{t}(2k_1 + 1) \zeta(p+k_2).
\]

(3.6)
Proof. The proofs of Theorems 3.1-3.2 are similar as the proofs of Theorems 2.3-2.4. We only prove the Theorem 3.2. Consider the function

$$G_{p-1,q}(s) := \pi \Psi^{(p-1)}(1/2 - s)$$

By (2.5)-(2.7), (3.2)-(3.4), these residues are \((n \in \mathbb{N})\)

$$\text{Res}[G_{p-1,q}(s), s = n] = (-1)^{p-1} \sum_{k=0}^{p-1} \left((-1)^k + 1\right) \frac{p - q - k - 2}{q - 1} \tilde{t}(k + 1) \frac{(-1)^n}{n^{p+q-k-1}},$$

$$\text{Res}[G_{p-1,q}(s), s = n - 1/2] = \frac{(-1)^p \tilde{t}(p) + h_n(p)}{(n - 1/2)^q} (-1)^n,$$

$$\text{Res}[G_{p-1,q}(s), s = 1/2 - n] = -\frac{\tilde{t}(p) - h_n(p)}{(n - 1/2)^q} (-1)^{n+p+q},$$

$$\text{Res}[G_{p-1,q}(s), s = 0] = (1 - (-1)^{p+q}) \tilde{t}(p + q) + 2(-1)^p \sum_{2k_1+k_2=q-1, k_1,k_2 \geq 0} \left(p + k_2 - 1\right) \tilde{t}(2k_1 + 1) \zeta(p + k_2).$$

Thus, by Lemma 2.1 and summing these four contributions yields the desired result. \(\Box\)

Letting \(q = 1\) in (3.5) and \(p = q = 2\) in (3.6) give

$$\tilde{T}_{1,1} = \tilde{t}(1, 1) = 2G - \frac{1}{2} \pi \log(2),$$

$$\tilde{T}_{2,2} = \tilde{t}(2, 2) = \frac{1}{2} \tilde{t}(4) - \frac{7}{4} \pi \zeta(3),$$

where \(\tilde{t}(2) = 4G\), \(G\) is Catalan’s constant.

### 3.2 Quadratic Alternating Euler T-sums

#### Theorem 3.3
For positive integer \(q > 1\),

$$(1 + (-1)^{q+1}) \tilde{T}_{1,2,q} \equiv (q - 2) \pi \zeta(q + 1) - 2 \pi \zeta(q, 1) - 4 \pi \log(2) \zeta(q) - 2 \tilde{t}(q + 1, 1)$$

$$+ (-1)^q \tilde{t}(q + 2) - (1 + (-1)^q) 4 \log(2) \tilde{t}(q + 1) - (1 - (-1)^q) 4 \log^2(2) \tilde{t}(q) - 2 \sum_{2k_1+k_2+k_3=q+1, k_1,k_2,k_3 \geq 1} \tilde{t}(2k_1 - 1) \zeta(k_2 + 1) \zeta(k_3 + 1)$$

$$+ 4 \sum_{2k_1+k_2=q+2, k_2 \geq 2} \tilde{t}(2k_1 - 1) \zeta(k_2 + 1)$$

$$+ 8 \log(2) \sum_{2k_1+k_2=q+1, k_2 \geq 2} \tilde{t}(2k_1 - 1) \zeta(k_2 + 1). \quad (3.7)$$

Proof. The proof is based on the function

$$G_{0^2,q}(s) := \pi \Psi(1/2 - s)^2 \frac{1}{\cos(\pi s)^q}$$
and the usual residue computation. By a similar argument as in the proof of the above theorem, we may easily deduce the \((3.7)\).

If letting \(q = 1, 3\) in \((3.7)\), then
\[
T_{12,1} = \frac{\pi^3}{12} - \frac{\pi}{2} \log^2(2) - \tilde{t}(2, 1),
\]
\[
T_{12,3} = \frac{\pi^5}{90} + \frac{7}{2} \pi \log(2) - \log^2(2) \pi^3 - \pi \zeta(3, 1) - \tilde{t}(4, 1).
\]

Similarly, by considering the function
\[
G_{(p_1-1)(p_2-1)\ldots(p_k-1),q}(s) := \frac{\pi \Psi^{(p_1-1)}(1/2-s) \Psi^{(p_2-1)}(1/2-s) \ldots \Psi^{(p_k-1)}(1/2-s)}{\cos(\pi s)s^q(p_1-1)! (p_2-1)! \cdots (p_k-1)!}
\]
and using the residue computations, many other relations can be established.

### 4 Other Euler Type \(T\)-Sums

In this section, we define an Euler type \(T\)-sums
\[
\overline{S}_{p_1p_2\ldots p_k,q} := \sum_{n=1}^{\infty} \frac{h_{n}^{(p_1)} h_{n}^{(p_2)} \cdots h_{n}^{(p_k)}}{n^q}.
\]

Then, we apply the contour integral to establish many relations of \(\overline{S}_{p_1p_2\ldots p_k,q}\). Moreover, we can use the quadratic sum \(\overline{S}_{p_1p_2,q}\) to evaluate the triple Kaneko-Tsumura \(T\)-values with weight even.

The Kaneko-Tsumura \(T\)-zeta values are defined by [19, 20]
\[
T(k_1, k_2, \ldots, k_r) := 2^r \sum_{n_1 > n_2 > \cdots > n_r > 0} \frac{1}{(2n_1 - r)^{k_1}(2n_2 - r + 1)^{k_2} \cdots (2n_r - 1)^{k_r}}, \tag{4.1}
\]
where we used the opposite convention \((n_1 > n_2 > \cdots > n_r > 0)\) of the original definition \((0 < n_1 < \cdots < n_r)\) Kaneko-Tsumura \(T\)-zeta values \((k_j \in \mathbb{N}, k_1 > 1)\). Here \(k_1 + k_2 + \cdots + k_r\) and \(r\) are called the weight and depth of the sum, respectively.

Hence, from the definitions of \(S_{p_1p_2\ldots p_k,q}\) and \(T\)-values, we have
\[
T(k_1, k_2) = \frac{1}{2^{k_1+k_2-2}} \overline{S}_{k_2,k_1}, \tag{4.2}
\]
\[
T(k_1, k_2, k_3) = \frac{1}{2^{k_1+k_2+k_3-3}} \tilde{t}(k_1) \overline{S}_{k_3,k_2} - \frac{1}{2^{k_1+k_2+k_3-3}} \overline{S}_{k_1k_3,k_2}. \tag{4.3}
\]

On the other hand, according to the relation of double \(T\)-values and double zeta values,
\[
T(k_1, k_2) = \zeta(\tilde{k_1}, k_2) + \zeta(k_1, k_2) - \zeta(k_1, \tilde{k_2}) - \zeta(\tilde{k_1}, k_2)
\]
we obtain
\[
T(q, p) = \frac{1 - (-1)^p}{2} (\zeta(p) - \zeta(\tilde{p})) + (-1)^p \sum_{k=0}^{[p/2]} \binom{m - 2k - 1}{q - 1} (\zeta(2k) - \zeta(\tilde{2k}))(\zeta(m - 2k) - \zeta(m - 2k))
\]
where \( \zeta(1) \) should be interpreted as 0 wherever it occurs, and \( \zeta(0) = \zeta(\bar{0}) = -1/2 \), \( m = p + q \) odd.

From (4.2) and (4.4), we know that the linear sum \( \tilde{S}_{p,q} \) with \( p + q \) odd can be evaluated by zeta values. In fact, considering the function

\[
F_{p-1,q}(s) := \frac{\psi(p-1)(1/2 - s)}{s^q(p-1)!} \pi \tan(\pi s)
\]

and using the residue theorem, we can also obtain the result.

By harmonic product, we give (4.5)

\[
\sum_{n=1}^{\infty} \frac{H_{n-1}^{(k_1)}}{(n-1/2)^2} = -2^{k_1+k_2-2}T(k_1,k_2) + \zeta(k_1)\tilde{t}(k_2). \tag{4.5}
\]

In fact, from [23], we also obtain (p > 1)

\[
\sum_{n=1}^{\infty} \frac{H_{n-1}}{(n-1/2)^p} = \frac{p}{2}t(p+1) - \frac{1}{2} \sum_{j=1}^{p-2} \tilde{t}(p-j)\tilde{t}(j+1) - 2\log(2)\tilde{t}(p). \tag{4.6}
\]

For nonnegative integers \( n_1, \ldots, n_p \) and \( n \), let

\[
\left( \begin{array}{c} n \\ n_1, \ldots, n_p \end{array} \right) := \frac{n!}{n_1!n_2! \cdots n_p!}, \quad (0 \leq n_1 + \cdots + n_p \leq n).
\]

**Theorem 4.1** For positive integers \( q > 1 \),

\[
(1 + (-1)^q)\tilde{S}_{1,p,q} = (-1)^{p-1} \sum_{l=0}^{p-1} ((-1)^l - 1) \left( \begin{array}{c} p + q - l - 2 \\ q - 1 \end{array} \right) \tilde{t}(l+1)\tilde{t}(p+q-l-1)
\]

\[
+ \sum_{k=1}^{p} \left( \begin{array}{c} p \\ k \end{array} \right) \sum_{|k|_{p-1}=k, \quad 0 \leq |k|_{p-1} \leq p-1.} \left( \begin{array}{c} k \\ k_1, \ldots, k_{p-1} \end{array} \right) (-1)^{p-1-|\tilde{k}|_{p-1}} \sum_{l=0}^{p-1-|\tilde{k}|_{p-1}} (-1)^{l-1}
\]

\[
\times \left( \begin{array}{c} p + q - |\tilde{k}|_{p-1} - l - 2 \\ q - 1 \end{array} \right) \tilde{t}(l+1) \sum_{n=1}^{\infty} \frac{\prod_{j=1}^{p-1} C_{n-1}^{k_j}(j)}{(n-1/2)^{p+q-|\tilde{k}|_{p-1}-l-1}} 
\]

\[
+ (-1)^{p+1} \sum_{k_1+k_2+\cdots+k_p=q, \quad k_j \in \mathbb{N}} \tilde{t}(k_1+1)\tilde{t}(k_2+1) \cdots \tilde{t}(k_p+1)
\]

\[
- 2(-1)^{p+1} \sum_{2k_1+2k_2+\cdots+k_{p+1}=q, \quad k_j \in \mathbb{N}} \zeta(2k_1)\tilde{t}(k_2+1) \cdots \tilde{t}(k_{p+1}+1), \tag{4.7}
\]

where \( |k|_{p-1} := k_1 + k_2 + \cdots + k_{p-1} (k_j \in \mathbb{N}_0), \quad |\tilde{k}|_{p-1} = k_1 + 2k_2 + \cdots + (p-1)k_{p-1} (k_j \in \mathbb{N}_0) \)

and

\[
C_n(j) = \begin{cases} 
H_n + 2\log(2), & j = 1 \\
(-1)^{j-1}H_n^{(j)} - \zeta(j), & j \geq 2.
\end{cases} \tag{4.8}
\]
For positive integers $p,q > 1$,

$$(1 + (-1)^q) \tilde{S}_{1,q} = \pi^2 \tilde{t}(q) - \sum_{k_1+k_2=q, k_1,k_2 \geq 1} \tilde{t}(k_1+1)\tilde{t}(k_2+1) + 2 \sum_{2k_1+k_2+k_3=q, k_1,k_2,k_3 \geq 1} \zeta(2k_1)\tilde{t}(k_2+1)\tilde{t}(k_3+1).$$  \hspace{1cm} (4.9)

Corollary 4.3 For positive integers $q > 1$,

$$(1 + (-1)^q) \tilde{S}_{1,q} = -2q\tilde{t}(2)\tilde{t}(q+1) + 6\tilde{t}(2)\sum_{n=1}^{\infty} \frac{H_{n-1}}{(n-1/2)^q} + 12\log(2)\tilde{t}(2)\tilde{t}(q) + \sum_{k_1+k_2+k_3=q, k_1,k_2,k_3 \geq 1} \tilde{t}(k_1+1)\tilde{t}(k_2+1)\tilde{t}(k_3+1) - 2 \sum_{2k_1+k_2+k_3+k_4=q, k_1,k_2,k_3,k_4 \geq 1} \zeta(2k_1)\tilde{t}(k_2+1)\tilde{t}(k_3+1)\tilde{t}(k_4+1).$$  \hspace{1cm} (4.10)

Setting $q$ and 4 in (4.9) and (4.10) yield

$$\tilde{S}_{1,2} = \frac{\pi^4}{8}, \quad \tilde{S}_{1,4} = \frac{\pi^6}{24} - \frac{49}{2} \zeta^2(3),$$

$$\tilde{S}_{1,2} = \frac{7}{2} \pi^2 \zeta(3), \quad \tilde{S}_{1,4} = -\frac{21}{8} \pi^4 \zeta(3) + 31\pi^2 \zeta(5).$$

Theorem 4.4 For positive integers $p,q > 1$,

$$(1 - (-1)^{p+q})\tilde{S}_{1,p,q} = (-1)^{p-1}(1 + (-1)^q)\tilde{t}(p)\tilde{S}_{1,q} + (-1)^p \sum_{l=0}^{p} ((-1)^l - 1) \left( \frac{p+q-l-1}{q-1} \right) \tilde{t}(l+1)\tilde{t}(p+q-l) + (-1)^{p-1} \sum_{l=0}^{p-1} ((-1)^l - 1) \left( \frac{p+q-l-2}{q-1} \right) \tilde{t}(l+1) \times \left( \sum_{n=1}^{\infty} \frac{H_{n-1}}{(n-1/2)^{p+q-l-1}} + 2\log(2)\tilde{t}(p+q-l-1) \right) + (-1)^{p-1} \sum_{k=1}^{p-1} \sum_{l=0}^{p-k-1} ((-1)^l - 1) \left( \frac{p+q-k-l-2}{q-1} \right) \tilde{t}(l+1).$$
Proof. By considering the function
\[ F_{0(p-1), q}(s) := \frac{\pi \tan(\pi s) \Psi(1/2 - s) \Psi^{(p-1)}(1/2 - s)}{(s + 1/2)^q(p-1)!}, \]
and using the direct residue computation. The desired formulas can be established.

If \( p = 2, q = 3 \), then
\[ \tilde{S}_{12,3} = -\frac{\pi^6}{16} + 49\zeta^2(3). \]

**Theorem 4.5** For positive integers \( p_1, p_2, q > 1 \),
\[
(1 + (-1)^{p_1+p_2+q})\tilde{S}_{p_1p_2,q}
= -(-1)^{p_1+p_2}(1 + (-1)^q)\tilde{t}(p_1)\tilde{t}(p_2)\zeta(q)
+ (-1)^{p_1}((-1)^{p_2+q}-1)\tilde{t}(p_1)\tilde{S}_{p_2,q} + (-1)^{p_2}((-1)^{p_1+q}-1)\tilde{t}(p_2)\tilde{S}_{p_1,q}
- (-1)^{p_1+p_2} \sum_{k_1+k_2=q+1, k_1,k_2\geq 1} \binom{k_1+p_1-2}{p_1-1} \binom{k_2+p_2-2}{p_2-1} \tilde{t}(k_1+p_1-1)\tilde{t}(k_2+p_2-1)
+ 2(-1)^{p_1+p_2} \sum_{k_1+k_2+2k_3=q+2, k_1,k_2,k_3\geq 1} \binom{k_1+p_1-2}{p_1-1} \binom{k_2+p_2-2}{p_2-1} \tilde{t}(k_1+p_1-1)\tilde{t}(k_2+p_2-1)\zeta(2k_3)
- (-1)^{p_1+p_2} \sum_{l=0}^{p_1+p_2-1} ((-1)^l-1) \binom{p_1+p_2+q-l-2}{q-1} \tilde{t}(l+1)\tilde{t}(p_1+p_2+q-l-1)
+ (-1)^{p_1+p_2} \sum_{k=1}^{p_2} (-1)^k \binom{k+p_1-2}{p_1-1} \sum_{l=0}^{p_2-k} ((-1)^l-1) \binom{p_2+q-k-l-1}{q-1} \tilde{t}(l+1)
\times \left( \zeta(k+p_1-1)\tilde{t}(p_2+q-k-l) + (-1)^{k+p_1-1} \sum_{n=1}^{\infty} \frac{H^{(k+p_1-1)}_{n-1}}{(n-1/2)^{p_2+q-k-l}} \right)
+ (-1)^{p_1+p_2} \sum_{k=1}^{p_1} (-1)^k \binom{k+p_2-2}{p_2-1} \sum_{l=0}^{p_1-k} ((-1)^l-1) \binom{p_1+q-k-l-1}{q-1} \tilde{t}(l+1)
\times \left( \zeta(k+p_2-1)\tilde{t}(p_1+q-k-l) + (-1)^{k+p_2-1} \sum_{n=1}^{\infty} \frac{H^{(k+p_2-1)}_{n-1}}{(n-1/2)^{p_1+q-k-l}} \right). \tag{4.12}
\]

Proof. By considering the function
\[ F_{(p_1-1)(p_2-1), q}(s) := \frac{\pi \tan(\pi s) \Psi^{(p_1-1)}(1/2 - s) \Psi^{(p_2-1)}(1/2 - s)}{(s + 1/2)^q(p_1-1)!(p_2-1)!}, \]
then a direct residue computation gives the desired formula.

Hence, from the relations (4.2), (4.3) and Theorems 4.1, 4.4, 4.5, we obtain the formula of
the triple $T$-value $T(p, q, r)$ of even weight in terms of single and the double $T$-values. Tsumura
also proved an explicit formula of triple $T$-value, see [21] for the detail.

Setting $p_1 = p_2 = q = 2$ yields

$$\tilde{S}_{2^2, 2} = \tilde{t}(2)\tilde{t}(4) - 2\tilde{t}^2(3) + 2\tilde{t}^2(2)\zeta(2) + 2\tilde{t}(2) \sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)}}{(n - 1/2)^2} \bigg|_{n=1} = 32\pi^2 \text{Li}_4 \left(\frac{1}{2}\right) - 98\pi^2(3) + 28\pi^2 \zeta(3) \log(2) - \frac{61\pi^6}{360} + \frac{2}{3} \pi^2 \log^4(2) - \frac{4}{3} \pi^4 \log^2(2).$$

In general, we can consider the function

$$F_{(p_1 - 1)(p_2 - 1)\cdots(p_k - 1), q}(s) := \frac{\pi \tan(\pi s) \Psi^{(p_1 - 1)}(1/2 - s) \Psi^{(p_2 - 1)}(1/2 - s) \cdots \Psi^{(p_k - 1)}(1/2 - s)}{(s + 1/2)^q(p_1 - 1)!(p_2 - 1)! \cdots (p_k - 1)!}$$

to establish more general formulas of $\tilde{S}_{p_1 p_2 \cdots p_k, q}$, but it is very difficult. Similarly, we can also consider the function

$$G_{(p_1 - 1)(p_2 - 1)\cdots(p_k - 1), q}(s) := \frac{\pi \Psi^{(p_1 - 1)}(1/2 - s) \Psi^{(p_2 - 1)}(1/2 - s) \cdots \Psi^{(p_k - 1)}(1/2 - s)}{\cos(\pi s)(s + 1/2)^q(p_1 - 1)!(p_2 - 1)! \cdots (p_k - 1)!}$$

to evaluate the alternating sum $\tilde{S}_{p_1 p_2 \cdots p_k, q}$ defined by

$$\tilde{S}_{p_1 p_2 \cdots p_k, q} := \sum_{n=1}^{\infty} \frac{h^{(p_1)}_n h^{(p_2)}_n \cdots h^{(p_k)}_n}{n^q} (-1)^n.$$

It is possible that of some other relations involving alternating Euler $T$-sums can be proved by
using the techniques of the present paper. For example, we can define an alternating $\Psi(-s)$
function

$$\Psi(-s) := \frac{1}{s + 1/2} + \sum_{k=1}^{\infty} \left( \frac{(-1)^k}{k - 1/2} - \frac{(-1)^k}{k - 1/2 - s} \right).$$

Then, consider the four functions

$$E_{(p_1 - 1)(p_2 - 1)\cdots(p_k - 1), q}(s) := \frac{\pi \tan(\pi s) \Psi^{(p_1 - 1)}(1/2 - s) \Psi^{(p_2 - 1)}(1/2 - s) \cdots \Psi^{(p_k - 1)}(1/2 - s)}{s^q(p_1 - 1)!(p_2 - 1)! \cdots (p_k - 1)!},$$

$$H_{(p_1 - 1)(p_2 - 1)\cdots(p_k - 1), q}(s) := \frac{\pi \Psi^{(p_1 - 1)}(1/2 - s) \Psi^{(p_2 - 1)}(1/2 - s) \cdots \Psi^{(p_k - 1)}(1/2 - s)}{\cos(\pi s)s^q(p_1 - 1)!(p_2 - 1)! \cdots (p_k - 1)!},$$

$$\tilde{E}_{(p_1 - 1)(p_2 - 1)\cdots(p_k - 1), q}(s) := \frac{\pi \tan(\pi s) \tilde{\Psi}^{(p_1 - 1)}(1/2 - s) \tilde{\Psi}^{(p_2 - 1)}(1/2 - s) \cdots \tilde{\Psi}^{(p_k - 1)}(1/2 - s)}{(s + 1/2)^q(p_1 - 1)!(p_2 - 1)! \cdots (p_k - 1)!},$$

$$\tilde{H}_{(p_1 - 1)(p_2 - 1)\cdots(p_k - 1), q}(s) := \frac{\pi \tilde{\Psi}^{(p_1 - 1)}(1/2 - s) \tilde{\Psi}^{(p_2 - 1)}(1/2 - s) \cdots \tilde{\Psi}^{(p_k - 1)}(1/2 - s)}{\cos(\pi s)(s + 1/2)^q(p_1 - 1)!(p_2 - 1)! \cdots (p_k - 1)!},$$

and use the contour integral to evaluate these sums

$$\sum_{n=1}^{\infty} \frac{\tilde{h}^{(p_1)}_n \tilde{h}^{(p_2)}_n \cdots \tilde{h}^{(p_k)}_n}{(n - 1/2)^q}, \quad \sum_{n=1}^{\infty} \frac{h^{(p_1)}_n h^{(p_2)}_n \cdots h^{(p_k)}_n}{(n - 1/2)^q} (-1)^n.$$
To prove the identity, we consider the function

\[
\sum_{n=1}^{\infty} \frac{\bar{h}_n^{(p_1)} \bar{h}_n^{(p_2)} \cdots \bar{h}_n^{(p_k)}}{n^q}, \quad \sum_{n=1}^{\infty} \frac{\bar{h}_n^{(p_1)} \bar{h}_n^{(p_2)} \cdots \bar{h}_n^{(p_k)}}{n^q} (-1)^n
\]

where \( \bar{h}_n^{(p)} \) is defined by

\[
\bar{h}_n^{(p)} := \sum_{k=1}^{n} \frac{(-1)^n}{(n-1/2)^p}, \quad \bar{h}_0^{(p)} := 0.
\]

5 Formulas of Kaneko-Tsumura’s Conjecture

In [20], Kaneko and Tsumura conjecture the following relation \((p \geq 2, m, q \geq 1)\)

\[
\sum_{i+j=m, \ i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} T(p+i, q+j) \in \mathcal{Z}
\]

where \( \mathcal{Z} \) is the space of usual multiple zeta values. Quite recently, T. Murakami proved the conjecture by using the motivic method employed in [12], but not gave explicit formula. In this section, we will give an explicit duality formula of the conjecture by using the residue theorem.

**Theorem 5.1** For positive integers \( m, p \) and \( q > 1 \),

\[
(-1)^{m-1} \sum_{i+j=m-1, \ i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} \sum_{n=1}^{\infty} \frac{H_n^{(m+i)}}{(n-1/2)^q+j} + (-1)^{p-1} \sum_{i+j=p-1, \ i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} \sum_{n=1}^{\infty} \frac{H_n^{(p+i)}}{(n-1/2)^q+j} = \binom{p+q+m-2}{q-1} i(p+q+m-1)
\]

\[
+ \sum_{i+j=p-1, \ i,j \geq 0} \binom{m+i-1}{i} \binom{q+j-1}{j} (-1)^i \zeta(m+i) \tilde{\tau}(q+j)
\]

\[
+ \sum_{i+j=m-1, \ i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} (-1)^i \zeta(p+i) \tilde{\tau}(q+j)
\]

\[
- \sum_{i+j=q-1, \ i,j \geq 0} \binom{m+i-1}{i} \binom{p+j-1}{j} \tilde{\tau}(m+i) \tilde{\tau}(p+j),
\]

where \( \zeta(1) \) should be interpreted as \(-2 \log(2)\) wherever it occurs, and \( \tilde{\tau}(1) := 0 \).

**Proof.** To prove the identity, we consider the function

\[
F(s) := \frac{\Psi^{(m-1)}(1/2-s)\Psi^{(p-1)}(1/2-s)}{(s+1/2)^q(m-1)!(p-1)!}.
\]
It is obvious that the function $F(s)$ only have poles at the $s = -1/2$ and $s = n$ ($n \in \mathbb{N}_0$). At a non-negative integer $n$, by (2.2) and (2.5), we compute the residue

$$\text{Res}[F(s), s = n] = -(-1)^{m+p} \left(\frac{p + q + m - 2}{q - 1}\right) \frac{1}{(n + 1/2)^{p+q+m-1}}$$

$$- (-1)^{m+p} \sum_{i+j=p-1, i,j \geq 0} \left(\frac{m + i - 1}{i}\right) \left(\frac{q + j - 1}{j}\right) (-1)^i \zeta(m + i) + (-1)^m H_n^{(m+i)} \left(\frac{1}{n + 1/2} \right)^{q+j}$$

$$- (-1)^{m+p} \sum_{i+j=n-1, i,j \geq 0} \left(\frac{p + i - 1}{i}\right) \left(\frac{q + j - 1}{j}\right) (-1)^i \zeta(p + i) + (-1)^p H_n^{(m+i)} \left(\frac{1}{n + 1/2} \right)^{q+j}.$$ 

By (2.3) and (2.6), the residue of the pole of order $q$ is found to be

$$\text{Res}[F(s), s = -1/2] = (-1)^{m+p} \sum_{i+j=q-1, i,j \geq 0} \left(\frac{m + i - 1}{i}\right) \left(\frac{p + j - 1}{j}\right) \tilde{t}(m + i) \tilde{t}(p + j).$$

Hence, summing these two contributions yields the desired formula. □

Letting $p = m = 2, q = 3$ in (5.1), we have

$$3 \sum_{n=1}^{\infty} \frac{H_{n-1}^{(2)}}{(n - 1/2)^4} + 2 \sum_{n=1}^{\infty} \frac{H_{n-1}^{(3)}}{(n - 1/2)^3} = 112\zeta^2(3) - \frac{\pi^6}{6}.$$ 

Similarly, by considering the function

$$F(s, a) := \frac{\Psi^{(m-1)}(1/2 - s; a) \Psi^{(p-1)}(1/2 - s; a)}{(s + 1/2)^q(m - 1)!(p - 1)!}$$

with the usual residue computation, we can get the following general result.

**Theorem 5.2** For positive integers $m, p$ and $q > 1$,

$$(-1)^{m-1} \sum_{i+j=p-1, i,j \geq 0} \left(\frac{m + i - 1}{i}\right) \left(\frac{q + j - 1}{j}\right) \sum_{n=1}^{\infty} \frac{H_n^{(m+i)}}{(n + a)^{q+j}}$$

$$+ (-1)^{p-1} \sum_{i+j=m-1, i,j \geq 0} \left(\frac{p + i - 1}{i}\right) \left(\frac{q + j - 1}{j}\right) \sum_{n=1}^{\infty} \frac{H_n^{(p+i)}}{(n + a)^{q+j}}$$

$$= \left(\frac{p + q + m - 2}{q - 1}\right) \zeta(p + q + m - 1; a + 1)$$

$$+ \sum_{i+j=p-1, i,j \geq 0} \left(\frac{m + i - 1}{i}\right) \left(\frac{q + j - 1}{j}\right) (-1)^i \zeta(m + i) \zeta(q + j; a + 1)$$

$$+ \sum_{i+j=m-1, i,j \geq 0} \left(\frac{p + i - 1}{i}\right) \left(\frac{q + j - 1}{j}\right) (-1)^i \zeta(p + i) \zeta(q + j; a + 1)$$

$$- \sum_{i+j=q-1, i,j \geq 0} \left(\frac{m + i - 1}{i}\right) \left(\frac{p + j - 1}{j}\right) \zeta(m + i; a + 1) \zeta(p + j; a + 1), \quad (5.2)$$
where \( \zeta(1) \) should be interpreted as \(-2\log(2)\) wherever it occurs, and \( \zeta(1; a + 1) := 0\). Here \( \zeta(s; a + 1) \) stands for Hurwitz zeta function, which is defined by

\[
\zeta(s; a + 1) := \sum_{n=1}^{\infty} \frac{1}{(n + a)^s} \quad (\Re(s) > 1, \ a \in \mathbb{C} \setminus \mathbb{N}^-).
\]

It is clear that Theorem 5.1 is immediate corollary of Theorem 5.2 with \( a = -1/2 \). Next, we give a duality formulas of Kaneko-Tsumura conjecture.

**Theorem 5.3** For positive integers \( p, q, m \geq 2 \),

\[
(-1)^m \sum_{i+j=p-1, \ i,j \geq 0} \binom{m+i-1}{i} \binom{q+j-1}{j} T(m+i,q+j) \\
+(-1)^p \sum_{i+j=m-1, \ i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} T(p+i,q+j)
\]

\[
= \frac{1}{2^{p+q+m-3}} \left( \frac{p+q+m-2}{q-1} \right) \tilde{t}(p+q+m-1) + \frac{1}{2^{p+q+m-3}} \sum_{i+j=p-1, \ i,j \geq 0} \binom{m+i-1}{i} \binom{q+j-1}{j} \left((-1)^i + (-1)^m\right) \zeta(m+i) \tilde{t}(q+j) \\
+ \frac{1}{2^{p+q+m-3}} \sum_{i+j=m-1, \ i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} \left((-1)^i + (-1)^p\right) \zeta(p+i) \tilde{t}(q+j) \\
- \frac{1}{2^{p+q+m-3}} \sum_{i+j=q-1, \ i,j \geq 0} \binom{m+i-1}{i} \binom{p+j-1}{j} \tilde{t}(m+i) \tilde{t}(p+j). \tag{5.3}
\]

**Proof.** The result immediately follows from (4.5) and (5.1). \( \square \)

Taking \( m = p \) in (5.3), we can get the following corollary.

**Corollary 5.4** For positive integers \( p, q \geq 2 \)

\[
\sum_{i+j=p-1, \ i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} T(p+i,q+j)
\]

\[
= \frac{1}{2^{2p+q-2}} \left( \frac{2p+q-2}{q-1} \right) \tilde{t}(2p+q-1) + \frac{1}{2^{2p+q-3}} \sum_{i+j=p-1, \ i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} \left(1 - (-1)^j\right) \zeta(p+i) \tilde{t}(q+j) \\
- \frac{(-1)^p}{2^{2p+q-2}} \sum_{i+j=q-1, \ i,j \geq 0} \binom{p+i-1}{i} \binom{p+j-1}{j} \tilde{t}(p+i) \tilde{t}(p+j). \tag{5.4}
\]

Setting \( p = 2, q = 3 \) in (5.4) gives

\[
3T(2,4) + 2T(3,3) = \frac{\pi^6}{64} - \frac{49}{8} \zeta^2(3).
\]
Theorem 5.5. For positive integers $p, m$ and $q \geq 2$,

\[
(-1)^{p-1} \sum_{i+j=m-1, \ i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} \tilde{S}_{p+i,q+j} \\
+ (-1)^{m-1} \sum_{i+j=p-1, \ i,j \geq 0} \binom{m+i-1}{i} \binom{q+j-1}{j} T_{m+i,q+j} \\
= \sum_{i+j=m-1, \ i,j \geq 0} \binom{p+i-1}{i} \binom{q+j-1}{j} (-1)^i \tilde{t}(p+i) \zeta(q+j) \\
+ \sum_{i+j=p-1, \ i,j \geq 0} \binom{m+i-1}{i} \binom{q+j-1}{j} (-1)^i \tilde{t}(m+i) \tilde{t}(q+j) \\
- \sum_{i+j=q-1, \ i,j \geq 0} \binom{m+i-1}{i} \binom{p+j-1}{j} \zeta(m+i) \tilde{t}(p+j),
\] 

(5.5)

where $\zeta(1) := -2 \log(2)$ and $\tilde{t}(1) := 0$.

Proof. The proof of Theorem 5.5 is similar as the proof of Theorem 5.1. We consider the function

\[ G(s) := \frac{\Psi^{(m-1)}(1/2-s) \Psi^{(p-1)}(-s)}{(s+1)^{q}(m-1)!(p-1)!}, \]

then by a similar argument as in the proof (5.5), we deduce the desired result. \qed

Letting $(p, q, m) = (1, 2, 2)$ and $(2, 2, 2)$ yield

\[ \tilde{S}_{2,2} + 2\tilde{S}_{1,3} = T_{2,2} = \frac{\pi^4}{12}, \]
\[ \tilde{S}_{2,3} + \tilde{S}_{3,2} + T_{2,3} + T_{3,2} = 2\zeta(2)\tilde{t}(3). \]

Hence, by the relations

\[ \tilde{S}_{p,q} = 2^{p+q-2} T(q,p) \quad \text{and} \quad T_{p,q} = 2^{p+q} t(q,p) \]

we can obtain a kind of relationship of the double Kaneko-Tsumura $T$-values and the double Hoffman $t$-values.

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