Space-Constrained Arrays for Massive MIMO

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Abstract—We analyse the behavior of a massive multi-user MIMO (MU-MIMO) system comprising a base station (BS) equipped with one of five antenna topologies whose spatial apertures are either unconstrained, or space-constrained. We derive the normalized mean interference (NMI) with a ray-based channel model, as a metric for topology comparison. Using an existing derivation for a horizontal uniform rectangular array (HURA), we provide closed-form NMI equations for a uniform linear array (ULA) and uniform circular array (UCirA). We then derive the same for a vertical URA (VURA) and uniform cylindrical array (UCylA). Results for the often-considered unconstrained case confirm the prior understanding that topologies with wider azimuth footprints aid performance. However, in the constrained case, performance is dictated by the angular resolution of the topology, particularly in elevation. We confirm the behavioral patterns predicted by the NMI by observing the same patterns in the signal-to-interference-and-noise-ratio (SINR) with minimum mean-squared error (MMSE) processing.

Index Terms—Millimeter wave communication, mobile communication, MIMO.

I. INTRODUCTION

As a result of standardization activities [2], massive multi-user MIMO (MU-MIMO) will become a key technology in next-generation cellular systems. Antenna topology and spacing are important design elements and may be subject to constraints on the spatial aperture of the base station. Several publications examine the effects of antenna spacing for a specific topology, for example a uniform linear array (ULA) [3], [4], or horizontal uniform rectangular array (HURA) [5]. Those which compare topologies, such as [6], [7], consider equal antenna spacing rather than equal spatial apertures.

Such comparisons can be misleading for two reasons. Firstly, in the unconstrained case, topologies with larger spatial apertures in the azimuth domain are advantageous. Angular variation of incident rays is greatest in the azimuth plane, hence larger azimuthal apertures are known to increase spatial diversity within the channel, improving performance [6]. Secondly, topologies with larger azimuth footprints in the unconstrained case will have smaller antenna spacing in the constrained case, increasing antenna correlation and harming performance. Therefore, antenna configurations should be considered within constrained apertures in the interest of both practicality and fairness of comparison.

With this aim, we focus on a metric which we refer to as the normalized mean interference (NMI) between two arbitrary users. This has been shown in [1] to additionally serve as an indication of the ergodic cell-wide channel correlation, and performance with zero-forcing (ZF) and minimum-mean-squared error (MMSE) processing. Closed-form expressions for similar metrics are derived and analyzed under space constraints in [3], [7], [8] for a ULA, HURA, and/or uniform circular array (UCirA). The authors of [8] additionally simulate the NMI for a vertical URA (VURA). The limitation of the closed-form results in [3], [7], [8] is the assumption that the angles of arrival (AoA) of the incoming rays are uniformly distributed within a given angular spread. Measurements at 2.53 GHz presented in [9] demonstrate that angles are more accurately modeled using a clustered ray-based model with Gaussian or Laplacian distributed cluster central angles and Laplacian subray offsets. The assumption of uniform AoAs severely underestimates the correlation and inter-user interference in the channel. This is illustrated in [1] and [10], which derive equations for the NMI for a HURA and a ULA with arbitrary angular distribution, examining both uniform and Gaussian/Laplacian angles. A VURA is also examined in [1] through simulation, but closed-form results are not provided for this topology.

The generic results in [1] and [10] have yet to be examined under space constraints.

We extend the work in [1] and [10] by deriving the NMI with a generic ray-based channel model for additional topology models, and use the results to compare behavior with and without space constraints. More specifically:

- We derive the NMI for a VURA, a UCirA, a uniform cylindrical array (UCylA), and an alternative ULA model to that in [1], [10];
- Using channel parameters derived from measurement in [9], we examine and compare the NMI of five topologies with and without space constraints;
- Based on the NMI trends, we confirm that larger azimuth apertures of horizontal topologies aid performance in the unconstrained case. For constrained azimuth apertures, topologies with a vertical component become advantageous by improving the elevation angular resolution.

II. SYSTEM MODEL

A base station (BS) equipped with $M$ omnidirectional antennas lies at the center of a circular cell of radius $r$ and receives uplink communication from $L$ single-antenna users (UEs) positioned uniformly randomly within the cell, outside of the exclusion radius $r_0$ around the BS. We consider a single-cell

1Furthermore, the parameter settings used for the VURA placed it perpendicular to the examined ULA. This orientation puts the VURA at a disadvantage by setting it parallel to the angle around which the majority of the azimuth radiation is concentrated.
system with perfect CSI at the BS to simplify analysis, providing an upper bound on performance which is suitable for the purpose of topology comparison. We assume a ray-based model, motivated by measurements in [9], where the $M \times 1$ channel from a user $l$ to the BS, $\mathbf{h}_l$, is a summation of $S$ scattered subrays in each of $C$ scattering clusters:

$$\mathbf{h}_l = \sum_{c \in C(l)} \sum_{s=1}^S \gamma_{c,s}^{(l)} a(\phi_{c,s}^{(l)}, \theta_{c,s}^{(l)}).$$

The ray coefficient $\gamma_{c,s}^{(l)} = \sqrt{\beta_{c,s}^{(l)}} \exp(j\Theta_{c,s}^{(l)})$ contains the power, $\beta_{c,s}^{(l)}$, and uniformly distributed phase, $\Theta_{c,s}^{(l)} \sim U[0, 2\pi]$, of subray $s$ within cluster $c$ of the channel for user $l$. We model the ray powers $\beta_{c,s}^{(l)}$ as a fraction of the total link gain for user $l$ such that $\sum_{c=1}^C \sum_{s=1}^S \beta_{c,s}^{(l)} = \beta^{(l)}$. We utilize the classical path-loss and shadowing equation such that $\beta^{(l)} = AX_l(d_l/d_0)^{-\Gamma}$, for a user $d_l$ meters from the BS, where $A$ is a unitless attenuation constant representing the average attenuation at reference distance $d_0$ without shadow fading, $10^\log_{10}(X_l) \sim \mathcal{CN}(0, \sigma^2_X)$ models the effects of shadow fading, and $\Gamma$ is the path-loss exponent. The steering vectors, $a(\phi_{c,s}^{(l)}, \theta_{c,s}^{(l)})$, are functions of $\phi_{c,s}^{(l)}$ and $\theta_{c,s}^{(l)}$, rays’ azimuth angle of arrival (AAoA) and elevation angle of arrival (EAAoA), respectively, and are defined in Section IV. We measure the AAoA as the angle between the incoming ray and the $x$-axis in the azimuth $x$-$y$ plane. The EAAoA is measured as the angle between the ray and the $z$-axis. We define broadside as $\phi = 0$ in azimuth and $\theta = \pi/2$ in elevation. We implement a cluster-ray-based model wherein each AAoA, $\phi_{c,s}^{(l)} = \phi_{c,s}^{(l)} + \Delta_{c,s}$, arises from a cluster central angle $\phi_{c}^{(l)}$ and a subray offset $\Delta_{c,s}$. Similarly, $\theta_{c,s}^{(l)} = \theta_{c,s}^{(l)} + \delta_{c,s}$. Fig. 1 provides a diagram depicting all five antenna topologies, and the azimuth and elevation angles of an incident ray.

### III. Normalized Mean Interference

The interference power between two distinct users with channels $\mathbf{h}_l$ and $\mathbf{h}_{l'}$ is given by $\mathbf{h}_l^H \mathbf{h}_{l'}$. Using the channel model in (1), in [10] we simplify the NMI (denoted $\kappa$) to

$$\kappa = \frac{E_{\phi, \theta, \phi', \theta'} |\mathbf{h}_l^H \mathbf{h}_{l'}|^2 / [M^2 \beta^{(l')} \beta^{(l)}]}{\sum_{c=1}^C \sum_{s=1}^S |a(\phi_{c,s}^{(l)}, \theta_{c,s}^{(l)}) a(\phi_{c,s}^{(l')}, \theta_{c,s}^{(l')})|^2 / M^2}. \quad (2)$$

where $E_{\phi, \theta, \phi', \theta'}$ is the mean over phases, AAoAs, and EAAoAs. As the ray angles and phases are independent and identically distributed, $\kappa$ in (2) can be written as

$$\kappa = \frac{1}{M^2} \sum_{m=0}^{M-1} \sum_{n=0}^{M-1} \left| E_{\phi, \theta, \phi', \theta'} [a(\phi, \theta) \ast_m (a(\phi, \theta))_{m'}] \right|^2. \quad (3)$$

To verify the validity of the NMI as an indication of system performance with uplink processing, we simulate the cell-wide SINR for user $l$ with MMSE processing given by (4) in [11]

$$E_{\beta, \phi, \theta} [\text{SNR}_{\text{MMSE}}] = E_{\beta, \phi, \theta} [\frac{\mathbf{H}_l^H (\mathbf{H}_l \mathbf{H}_l^H + \frac{1}{\rho} \mathbf{I}_M)^{-1} \mathbf{h}_l}{\mathbf{H}_l^H \mathbf{h}_l}] \quad (4)$$

where $\mathbf{H}_l = [\mathbf{h}_l, \ldots, \mathbf{h}_{l-1}, \mathbf{h}_{l+1}, \ldots, \mathbf{h}_l]$, $\rho$ is the signal-to-noise ratio (SNR), $E[|s|^2]/\sigma_n^2$ symbols, $s$, and noise power, $\sigma_n^2$.

### IV. Analysis of the NMI

This section contains closed-form expressions for the NMI for all five topologies. We first provide a generic equation for the topologies which are confined to the $x$-$y$ plane (the HURA, ULA, and UCIrA), then derive a second equation for those which utilize vertical antenna placement (the UCYlA and VURA). Combining the appropriate equation and topology-specific parameters in Table I gives the NMI for each topology.

#### A. HURA, ULA, UCIrA

Consider an HURA in the azimuth $x$-$y$ plane. $M$ antennas are arranged into $M_x$ rows separated by $d_y$ wavelengths along the $x$-axis, and $M_y$ columns separated $d_y$ wavelengths apart along the $y$-axis, where $M_x M_y = M$. The steering vector for a ray approaching at angle $\phi$ in azimuth and $\theta$ in elevation is

$$a(\phi, \theta) = a_x(\phi, \theta) \otimes a_y(\phi, \theta). \quad (5)$$

The $m_x$th element of the $M_x \times 1$ vector $a_x(\cdot)$ is defined as

$$a_x(\phi, \theta) = e^{j2\pi d_y (m_y - 1) \sin \theta \cos \phi} \quad (6)$$

and the elements of the $M_y \times 1$ vector $a_y(\cdot)$ are defined as

$$a_y(\phi, \theta) = e^{j2\pi d_y (m_y - 1) \sin \theta \sin \phi}. \quad (7)$$

From [1], we see that, for steering vectors defined as in (5), (6), (7), the NMI in (3) requires expectations of the form

$$E_{\phi, \theta} [e^{j\sin \theta \sin \phi + \cos \phi}] = E_{\phi, \theta} [e^{j\sqrt{1+2\sin \theta \sin \phi \cos \phi}}] \quad (8)$$

with $z_1 = 2\pi d_y (m_y - m_y')$ for $m_x, m_x' \in [1, M_x]$, $z_2 = 2\pi d_x (m_x - m_x')$ for $m_x', m_x \in [1, M_x]$, $A = \tan^{-1}(z_2/z_1)$. The entries of the $M \times 1$ steering vector for a ULTRA situated along the $y$-axis are defined as in (7); with this, (3) requires

$$E_{\phi, \theta} [e^{j2\pi d (m - m') \sin \theta \sin \phi}] \quad (9)$$

for $m, m' \in [1, M]$. Many analysis of the ULTRA [1], [3], [7], [10] omit the $\sin \theta$ term in (9). This is equivalent to redefining the distribution of $\phi$ to account for variation in elevation. However, campaigns such as [9] measure the true elevation and azimuth angles. We use (7) and hence (9) to align with the use of measured data for $\phi$ and $\theta$. Note that (9) is (8) with $z_1 = 2\pi d (m - m')$, $z_2 = 0$, and $A = 0$. 

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**Table I**

| Parameters for Topology-Specific Solutions for the NMIQ |
|-----------------|-----------------|-----------------|------------------|-----------------|
| $a$             | $b$             | $\alpha$        | $\beta$          | $\gamma$        |
| N/A             | N/A             | $\sin(\psi_{m_y}) - \sin(\psi_{m_y'})$ | $\cos(\psi_{m_y}) - \cos(\psi_{m_y'})$ | $\arctan(z_2/z_1)$ |
| $A$             | $\arctan(z_2/z_1)$ |                      | $\arctan(b/a)$   |                      |
| $z_1$           | $2\pi d_y (m_y - m_y')$ | $\sqrt{a^2 + b^2 \pi d_y (\sin(\pi/M_y))}$ |                      |                      |
| $z_2$           | $2\pi d_x (m_x - m_x')$ | $0$               | $2\pi d_x (m_x - m_x')$ | $0$ |

*Fig. 1. Depiction of antenna topologies under space constraints with $M = 25$.**
Finally, the steering vector entries for a UCirA with \( M_r = M \) antennas spaced \( d_r \) wavelengths apart in a circle in the \( x\)-\( y \)-plane are defined as in [7]:

\[
(a_r(\phi, \theta))_m = e^{j \frac{\pi d_r}{2} \sin \theta \cos (\phi - \Psi_m)}, \tag{10}
\]

where \( \Psi_m = 2\pi m / M_r \). Using this definition, followed by a cosine expansion and the simplification in (8), (3) requires

\[
\mathbb{E}_{\phi, \theta}[e^{j \frac{\pi d_r}{2} \sin \theta \cos (\phi - \Psi_m)}] = \mathbb{E}_{\phi, \theta} e^{j \frac{\pi d_r}{2} \sqrt{\phi^2 + b^2} \sin \theta \sin \phi + \tan^{-1}(b/a)}],
\]

This is (8) with \( a = \sin \Psi_m, b = \cos \Psi_m \), \( z_1 = \pi d_r \sqrt{a^2 + b^2} / \sin(\pi/M_r), z_2 = 0, A = \tan^{-1}(b/a). \)

Both (9) and (11) require the solution for (8) derived in [1]:

\[
\kappa_{\text{HURA}} = \frac{1}{M^2} \sum_{m_s, m'_s, m_y, m'_y} |I(A, z_1, z_2)|^2, \tag{12}
\]

with

\[
I(A, z_1, z_2) = \sum_{n=-\infty}^{\infty} \sum_{n'=\infty}^{\infty} (-1)^p(n) \psi(n) \chi(2n') e^{jn A} J_{|n|} \times J_{|n'|} \left( \sqrt{\frac{z_1^2 + z_2^2}{4}} \right) J_{|n|} \times J_{|n'|} \left( \sqrt{\frac{z_1^2 + z_2^2}{4}} \right)
\]

where \( p(n) = \min(n, 0) \), \( J_n(\cdot) \) is the \( n \)th order Bessel function of the first kind, and we define \( \sum_{m_r, m'_r} = \sum_{m_r=0}^{M_r-1} \sum_{m'_r=0}^{M_r-1} \). Here, \( \psi(n) = \mathbb{E}_{\phi, \theta} \exp(jn \phi) \) and \( \chi(n) = \mathbb{E}_{\phi, \theta} \exp(jn \theta) \) are the characteristic functions of the azimuth and elevation angular probability density functions. In [1] we show that the infinite summations in (13) (and later in (20)) can be truncated to a handful of terms while maintaining exceptional accuracy due to the rapid decay of the characteristic functions for realistic angular distributions. The NMI for a ULA and UCirA follow from (12), as explained in Result 1.

Result 1: \( \kappa_{\text{UCylA}} \) and \( \kappa_{\text{UCirA}} \) are given by the right-hand side of (12) with \( M_r = 1, M_y = M \) and \([A, z_1, z_2]\) as in Table I.

B. UCylA, VURA

Consider a UCylA with \( M_z \) layers stacked vertically with \( d_z \) wavelength spacing, \( m_r \) antennas spaced \( d_r \) apart, each comprising a UCirA with \( M_r \) antennas spaced \( d_r \) apart. In this case,

\[
a(\phi, \theta) = a_r(\phi, \theta) \oplus a_z(\phi), \tag{14}
\]

with the entries of the \( M_y \times 1 \) vector \( a_r(\phi, \theta) \) given by (10) and those of the \( M_z \times 1 \) steering vector \( a_z(\phi) \) defined as

\[
a_z(\phi) = e^{2\pi d_r (m-1) \cos \theta}, \tag{15}
\]

For a UCylA with steering vectors as in (14), (3) requires

\[
\mathbb{E}_{\phi, \theta} e^{j \frac{\pi d_r}{2} \sin \theta \cos (\phi - \Psi_{m_r})} e^{j \frac{\pi d_r}{2} \sin \theta \cos (\phi - \Psi_{m_z})} e^{j \frac{\pi d_r}{2} \sin \theta \cos (\phi + A)} e^{jz_2 \cos \theta}.
\]

Finally, consider a VURA with \( M_z \) vertically stacked rows \( d_z \) apart, each having \( M_y \) antennas with spacing \( d_y \) parallel to the \( y \)-axis and \( M_y M_z = M \). The steering vectors here are

\[
a(\phi, \theta) = a_y(\phi, \theta) \oplus a_z(\phi), \tag{17}
\]

Using the entries of the \( M_y \times 1 \) and \( M_z \times 1 \) vectors defined in (7) and (15). For this topology, (3) requires

\[
\mathbb{E}_{\phi, \theta} e^{j2\pi d_y (m_y - m'_y) \sin \phi + d_z (m_z - m'_z) \cos \theta}.
\]

Hence, \( \kappa_{\text{VURA}} \) also requires (16) with \( z_1 = 2\pi d_y (m_y - m'_y), z_2 = 2\pi d_z (m_z - m'_z), \) and \( A = 0. \)

The solutions for \( \kappa_{\text{UCylA}} \) and \( \kappa_{\text{VURA}} \) are given in Lemma 1.

Lemma 1: For a UCylA,

\[
\kappa_{\text{UCylA}} = \frac{1}{M^2} \sum_{m_r, m'_r} \sum_{m_y, m'_y} |V(A, z_1, z_2)|^2
\]

with

\[
V(A, z_1, z_2) = \sum_{n=-\infty}^{\infty} \sum_{n'=\infty}^{\infty} (-1)^p(n) e^{jn A} \sum_{n'=\infty}^{\infty} \chi(2n')
\]

using the previous definitions of \( p(n), \psi(n), \) and \( \chi(n), \)

\[
G(n, n', \hat{n}, z_2) = J_2(n' + \hat{n}) \frac{4}{\pi} \sum_{n'=1}^{\infty} (-1)^{(n'-n'-\hat{n})} J_{2n'} z^2 (n' + \hat{n})^2 (2n'^2 - 1)^2 - 4(n' + \hat{n})^2. \tag{21}
\]

For a VURA, \( \kappa_{\text{VURA}} \) is given by (19) with \( M_r = M_y \) and \([A, z_1, z_2]\) given in Table I.

Proof: The proof is given in the Appendix.

V. NUMERICAL RESULTS

This section provides a comparison of the NMI for five topologies. We consider angular distributions obtained from measurements reported in [9]. Central cluster angles are Gaussian distributed in azimuth and Laplacian distributed in elevation while subray angles are Laplacian distributed in both cases. Angular distribution parameters are given in Table II.

We consider \( L = 4 \) users in a cell with \( r = 100 \) m, pathloss as in Section II with \( \Gamma = 3.8, \sigma_{sf} = 5.5 \) dB, \( d_0 = 1 \) m, and \( A = 1 \). We assume cluster power is equally distributed among subrays \( \beta_C^{(l)} = \beta_1^{(l)}/S(2) \) where \( \beta_C^{(l)} \) is set to exponentially decay from \( \beta_1^{(l)} \) to \( \beta_\text{C}^{(l)} \) with ratio \( \beta_C^{(l)} / \beta_1^{(l)} = 0.1 [1]. \) The UL SNR \( \rho \) in (4) is chosen such that the average received SNR at the BS, \( \rho/\beta_1^{(l)} \), has a median of \(-5 \) dB.

Consider a \( D/\sqrt{2} \times D/\sqrt{2} \) square in the \( x\)-\( y \)-plane, meaning the largest dimension (the diagonal) is length \( D \). Under space-constraints, antennas are arranged with the maximum uniform space spacing possible such that the topology’s azimuth footprint fits within the space-constraint square, as shown in Fig. 1. Table III provides the resulting antenna spacing, \( d \), given \( D \). For simplicity, we assume \( d_\text{d} = d_y = d_z = d \) within a given topology, and \( M_y = M_z = M_x = \sqrt{M} \) for all.

Fig. 2(a) examines the system without space constraints by plotting the NMI for each topology vs \( M \) for \( d = 0.5 \). In

| Parameter | Values |
|-----------|--------|
| cluster angle mean, \( \mu_z \) (azimuth), \( \mu_\text{e} \) (elevation) | 0°, 90° |
| cluster angle variance, \( \sigma_\text{z}^2 \) (azimuth), \( \sigma_\text{e}^2 \) (elevation) | (14.4°)^2, (1.9°)^2 |
| subray angle variance, \( \sigma_\text{y}^2 \) (azimuth), \( \sigma_\text{\text{z}}^2 \) (elevation) | (6.24°)^2, (1.37°)^2 |
Table III

Antenna Spacing Under Space Constraints

|        | ULA     | UCirA   | HURA    | UCylA   | VURA    |
|--------|---------|---------|---------|---------|---------|
| $D/M$  | $D/2\sin(\pi/M)$ | $D/2\sin(\pi/M)$ | $D/2\sin(\pi/M)$ | $D/2\sin(\pi/M)$ |

Fig. 3. Normalized interference power vs angular separation in: (a) azimuth ($\{\phi_2, \theta_2\} = \{\phi_1 + d\phi, \theta_1\}$); (b) elevation ($\{\phi_2, \theta_2\} = \{\phi_1, \theta_1 + d\theta\}$); (c) both ($\{\phi_2, \theta_2\} = \{\phi_1 + d\phi, \theta_1 + d\theta\}$). $M = 100, D = 7.77$.

Fig. 2(b) we observe the NMI under space constraints with $D$ equal to that of a 144-antenna HURA with $d = 0.5$. In addition to the angular distributions from Table II, we include results for the case of AAoAs and EAoAs independently uniformly distributed on $[-\pi, \pi]$, as assumed in [3], [7], [8].

Firstly, Fig. 2 illustrates the importance in using appropriate angular distributions for topology comparison. In both Fig. 2(a) and Fig. 2(b) the uniform AAoAs and EAoAs give drastically lower values of the NMI and reveal little to no topology-specific trends. This is the most spatially diverse angular distribution, hence any topology-specific characteristics which promote spatial diversity under less diverse conditions will be obscured in the uniform case. In Fig. 2(b), the ULA, HURA, and UCirA have noticeably larger NMI values than the VURA and UCylA, as they have significantly smaller inter-element spacing under space constraints (see Table III). Increased endfire radiation in the uniform AoA distribution also inflates the ULA NMI by increasing the average similarity of rays (see [1], [10]).

Secondly, the considerable difference in NMI trends between Fig. 2(a) and Fig. 2(b) speaks to the importance of observing topology behavior under space constraints. In Fig. 2(a), the topology-specific benefits are obscured by the significant advantage afforded to those with larger azimuthal apertures; only once this advantage is removed can the effects of antenna arrangement be observed.

For measurement-based angular distributions in Fig. 2(a), the NMI trends decrease in the following order: UCylA, HURA, VURA, UCirA, ULA. The azimuthal apertures increase in this order, showing how greatly this affects performance without space constraints [6]. Under space constraints in Fig. 2(b), the NMI values of different topologies become more similar and the ordering of trends is completely changed, decreasing in the following order: UCirA, HURA, ULA, VURA, UCylA. Hence, under azimuthal space constraints, the topologies with more elevation variation assist performance.

To assist in explaining this marked change, we observe the normalized interference power of two rays, $I = \mathbb{E}_\Theta[|h_1^* h_2|^2/(\beta_1 \beta_2)] = |a(\phi_1, \theta_1)^H a(\phi_2, \theta_2)|^2$, in Fig. 3. One ray approaches at broadside ($\{\phi_1, \theta_1\} = \{0, \pi/2\}$) and the other at some offset angle $d\phi$ in azimuth and/or $d\theta$ in elevation. This illustrates the angular resolution, where lower interference at smaller angular separations implies greater resolution.

Fig. 3(a) indicates that all topologies exhibit similar azimuth resolution, while Fig. 3(b) illustrates the vast differences in elevation resolution. The ULA, HURA, and UCirA, with no vertical antenna placement, require significantly large angular separation in elevation to achieve reasonable interference reduction. Most measurements suggest small elevation angular variance, meaning large elevation angular separation is unlikely [9]. In contrast, the VURA and UCylA achieve excellent interference reduction with only a few degrees of separation, explaining the dominance of vertical topologies under space constraints. Fig. 3(c) shows the global angular resolution when the rays are separated by a common angle $d\phi = d\theta$ in azimuth and elevation, hence capturing the trends in both Fig. 3(a) and Fig. 3(b). Note that the initial rate of decay out to $8^\circ$ separation follows almost the same ordering as the NMI shown in Fig. 2(b). The only difference is that the HURA/UCirA ordering is reversed due to the ripples present at wider separations. In summary, constraining the azimuth footprint reduces the differences in the azimuth resolution between topologies so performance is determined by elevation resolution, despite the reduced angular diversity in elevation.

In Fig. 4(a) we compare the NMI for a fixed number of antennas across a wide range of $D$, while in Fig. 4(b) we examine the per-user cell-wide average SINR with MMSE processing using (4). Fig. 4(a) shows that the pattern observed in Fig. 2(b) is roughly consistent regardless of total size until $D$ becomes large, when the NMI values for all topologies converge near zero. When the BS antennas are spread over a large enough azimuthal footprint, the NMI becomes very small and is only negligibly affected by the arrangement of antennas. This is encouraging for concepts such as distributed MIMO where the BS aperture is very large, but antenna placement might be limited. The pattern displayed in the ergodic per-user SINR with MMSE processing agrees with the NMI ordering (as the NMI drops the SINR increases) validating the NMI as a measure of relative performance.
Expanding the exponential and using the symmetry of $\psi (\hat{J})$
\[
\hat{J} = \sum_{\pi} \sum_{\infty} z \psi \sin \int_{0}^{\pi} (z \int_{0}^{\pi} e^{\prime} J d\theta).
\]

\textbf{VI. CONCLUSION}

We provide closed-form equations for the NMI for five antenna topologies, which we use to study their behavior with and without space constraints. For fixed antenna spacing, topologies with wider azimuth footprints are advantageous. Under space constraints, the NMI is determined by the topology’s angular resolution, particularly in elevation. SINR trends with MMSE processing confirm these conclusions.

\textbf{APPENDIX}

\textbf{PROOF OF LEMMA 1}

From (16), we require an expectation of the form
\[
V(A, z_{1}, z_{2}) = E_{\theta} [E_{\phi} [\exp (j[z_{1} \sin \theta \sin (\phi + A)])] \times \exp (jz_{2} \cos \theta)].
\]

Using the analysis from [1], (22) becomes
\[
\sum_{\infty}^{\infty} \psi(n) e^{jA} E_{\theta} [J_{n}(z_{1} \sin \theta) \exp (jz_{2} \cos \theta)]
\]
\[
= \sum_{\infty}^{\infty} \sum_{\infty}^{\infty} \psi(n) e^{jA} (1)^{n'} \chi(2n')
\]
\[
\times \left\{ -1 \right\}^{\frac{1}{\pi}} \int_{0}^{\pi} J_{n}(z_{1} \sin \theta) \exp (j[z_{2} \cos \theta - 2(n') \theta)] d\theta.
\]

We substitute the Bessel function in (23) with its Fourier series and use [12, eqs. 6.681.8 and 6.681.9] to give
\[
\int_{0}^{\pi} J_{n}(z_{1} \sin \theta) \exp (j[z_{2} \cos \theta - 2(n') \theta]) d\theta
\]
\[
= \sum_{\hat{n}} (-1)^{\hat{n}} \frac{P(n)}{\hat{n}} J_{|\hat{n}|/2} \hat{n}(z_{1}/2) J_{|\hat{n}|/2} \hat{n}(z_{1}/2)
\]
\[
\times \int_{0}^{\pi} \exp (j[z_{2} \cos \theta - 2(n' + \hat{n}) \theta]) d\theta.
\]

We now require \[ \int_{0}^{\pi} \exp (j[z_{2} \cos \theta - z_{3} \theta]) d\theta \] with $z_{3} = 2(n' + \hat{n})$. Expanding the exponential and using the symmetry of $\sin(z_{2} \sin - z_{3} \theta)$ around $\pi$, this becomes
\[
(-1)^{z_{1}/2} \left[ \int_{\pi/2}^{3\pi/2} \cos(z_{2} \sin \theta) \cos(z_{3} \theta) d\theta + \int_{-\pi/2}^{\pi/2} \sin(z_{2} \sin \theta) \sin(z_{3} \theta) d\theta \right].
\]

Using [12, eq. 8.411.2], we have
\[
\int_{\pi/2}^{3\pi/2} \cos(z_{2} \sin \theta) \cos(z_{3} \theta) d\theta = \pi J_{3}(z_{2}).
\]

For the remaining integral, [12, eq. 8.514.6] provides
\[
\sin(z_{2} \sin \theta) \sin(z_{3} \theta) = 2 \sum_{n' = 1}^{\infty} J_{3n' - 1}(z_{2}) \sin((2n' - 1) \theta) \sin(z_{3} \theta).
\]

A cumbersome but straightforward progression therefore gives
\[
\int_{\pi/2}^{3\pi/2} \sin(z_{2} \sin \theta) \sin(z_{3} \theta) d\theta = 2 \sum_{n' = 1}^{\infty} J_{3n' - 1}(z_{2}) \frac{2z_{3}}{(2n' - 1)^{2} - z_{3}^{2}}.
\]

Substituting (26) and (27) into (25), we have
\[
\int_{0}^{\pi} \exp (j[z_{2} \cos \theta - z_{3} \theta]) d\theta = (-1)^{z_{1}/2} \pi J_{3}(z_{2})
\]
\[
+ 4 \sum_{n' = 1}^{\infty} (-1)^{n'} J_{2n' - 1}(z_{2}) \frac{z_{3}}{(2n' - 1)^{2} - z_{3}^{2}}.
\]

Finally, substituting (28) into (23) gives the solution in (20).

\textbf{REFERENCES}

[1] C. L. Miller, P. J. Smith, P. A. Dmochowski, H. Tataria, and M. Matthaiou, “Analytical framework for full-dimensional massive MIMO with ray-based channels,” IEEE J. Sel. Topics Signal Process., vol. 13, no. 5, pp. 1181–1195, Sep. 2019.

[2] “Study on channel model for frequencies from 0.5 to 100 GHz, (V15.1.0),” 3rd Gener. Partnership Project (3GPP), Rep. TR 38.901, Sophia Antipolis, France, Aug. 2018.

[3] C. Masouros and M. Matthaiou, “Space-constrained massive MIMO: Hitting the wall of favorable propagation,” IEEE Commun. Lett., vol. 19, no. 5, pp. 771–774, May 2015.

[4] H. Tataria, P. J. Smith, M. Matthaiou, and P. A. Dmochowski, “Uplink analysis of large MU-MIMO systems with space-constrained arrays in Rician fading,” in Proc. IEEE ICC, May 2017, pp. 1–7.

[5] S. Biswas, C. Masouros, and T. Ratnarajah, “Performance analysis of large multiuser MIMO systems with space-constrained 2-D antenna arrays,” IEEE Trans. Wireless Commun., vol. 15, no. 5, pp. 3492–3505, May 2016.

[6] M. Z. Adam, Y. Corre, E. Björnson, and E. G. Larsson, “Performance of a dense urban massive MIMO network from a simulated ray-based channel,” EURASIP J. Wireless Commun. Netw., vol. 1, no. 1, p. 106, May 2019.

[7] X. Wu, N. C. Beaulieu, and D. Liu, “On favorable propagation in massive MIMO systems and different antenna configurations,” IEEE Access, vol. 5, pp. 5578–5593, 2017.

[8] C. D. Altamirano and C. de Almeida, “Inter-user interference reduction factor for 3-D massive MIMO systems,” in Proc. IEEE LATINCOM, Nov. 2016, pp. 1–5.

[9] S. Sagdeyov et al., “Cluster characterization of 3-D MIMO propagation channel in an urban macrocellular environment,” IEEE Trans. Wireless Commun., vol. 17, no. 8, pp. 5076–5091, Aug. 2018.

[10] C. L. Miller, P. A. Dmochowski, P. J. Smith, H. Tataria, and M. Matthaiou, “Multi-user processing for ray-based channels,” in Proc. IEEE ICC, May 2019, pp. 1–7.

[11] H. Tataria, P. J. Smith, M. Matthaiou, H. Q. Ngo, and P. A. Dmochowski, “Revisiting MMSE combining for massive MIMO over heterogeneous propagation channels,” in Proc. IEEE ICC, Jun. 2018, pp. 1–7.

[12] I. S. Gradshteyn, I. M. Ryzik, and D. Zwillinger, Table of Integrals, Series, and Products, 5th ed. Amsterdam, The Netherlands: Elsevier/Academic, 1994.