Differential Complex of Poisson Manifold and Distributions

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Abstract

We study various homological structures associated with Poisson algebra, the canonical differential complex for singular Poisson structure and the analogue of the star operator for such manifolds.

Give the interpretation of the classical Koszul differential of exterior forms, as the supercommutator with some second order element.

Describe the space of invariant distributions on manifold with singular Poisson structure.

1 Lie superalgebra structure on the space of multiderivations of a commutative algebra. Poisson cohomologies

Let $A$ be a real or complex vector space. For each integer $k$ let $L^k(A)$ be the space of multilinear antisymmetric maps from $A^k$ into $A$. Let $L^0(A)$ be $A$ and $L(A) = \oplus_{k=0}^{\infty} L^k(A)$.

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There is a natural Lie algebra structure on the space \( L^1(A) \) defined by the commutator of the composition. The composition on this space can be extended to the operator on \( L(A) \) called the compositional product (see \([1]\)):

for \( \alpha \in L^m(A) \) and \( \beta \in L^n(A) \) define \( \alpha \circ \beta \in L^{m+n-1}(A) \) as

\[
(\alpha \circ \beta)(u_1, \ldots, u_{m+n-1}) = \\
= \sum_{s(1) < \cdots < s(n)} \text{sgn}(s) \alpha(\beta(u_{s(1)}, \ldots, u_{s(n)}), u_{s(n+1)}, \ldots, u_{s(m+n-1)})
\]

As a result, the commutator on \( L^1(A) \) can be extended to the supercommutator on the space \( L(A) \) as

\[
[\alpha, \beta] = (-1)^{(m+1)n} \alpha \circ \beta + (-1)^m \beta \circ \alpha
\]

This bracket satisfies the conditions needed for the space \( L(A) \) be a superalgebra:

for \( \alpha \in L^m(A) \), \( \beta \in L^n(A) \) and \( \gamma \in L^k(A) \)

(a) \( [\alpha, \beta] = (-1)^{mn} [\beta, \alpha] \);
(b) \( (-1)^{mk} [[\alpha, \beta], \gamma] + (-1)^{mn} [[\beta, \gamma], \alpha] + (-1)^{nk} [[\gamma, \alpha], \beta] = 0 \).

An element \( \mu \in L^2(A) \) satisfying the condition \( [\mu, \mu] = 0 \), defines a Lie algebra structure on \( A \):

for \( a, b \in A \) let \( [a, b]_{\mu} = \mu(a, b) \)

We call such element involutive.

An involutive element \( \mu \in L^2(A) \) defines a linear operator \( \partial_\mu : L(A) \longrightarrow L(A) \), of degree +1: \( \partial_\mu(\alpha) = [\mu, \alpha] \). From the condition (b) and the involutiveness of the element \( \mu \) follows that the operator \( \partial_\mu \) is a coboundary operator, i.e. \( \partial_\mu \circ \partial_\mu = 0 \).

Let \( A \) be a commutative algebra over the field of the real or complex numbers. In this case, the space \( L(A) \) has a structure of an exterior algebra under the multiplication operator defined by the classical formula

for \( \alpha \in L^m(A) \), and \( \beta \in L^n(A) \), let

\[
(\alpha \wedge \beta)(u_1, \ldots, u_{m+n}) = \\
= \frac{1}{m!n!} \sum_{s} \text{sgn}(s) \alpha(u_{s(1)}, \ldots, u_{s(m)})\beta(u_{s(m+1)}, \ldots, u_{s(m+n)})
\]

2
We call an element $\alpha \in L(A)$ \textit{multiderivation} if for any set of elements $a, a_1, \ldots, a_m \in A$:

$$\alpha(aa_1, a_2, \ldots, a_k) = a\alpha(a_1, \ldots, a_k) + a_1\alpha(a, a_2, \ldots, a_k)$$

The subspace of all multiderivations in $L^m(A)$ we denote by $\text{Der}^m(A)$. Also, we put: $\text{Der}^0(A) = A$ and $\text{Der}(A) = \bigoplus_{k=0}^{\infty} \text{Der}^k(A)$.

The subspace $\text{Der}(A)$ in the space $L(A)$ is closed as under the operator of exterior multiplication defined by the formula (2), so under the bracket defined by the formula (3). Moreover, these two structures are interconnected by the following property:

\begin{equation}
\text{for } \alpha \in \text{Der}^m(A), \beta \in \text{Der}^n(A) \text{ and } \gamma \in \text{Der}(A) \text{ we have}
\end{equation}

\[ [\alpha, \beta \gamma] = [\alpha, \beta] \land \gamma + (-1)^{(m+1)n} \beta \land [\alpha, \gamma] \]

For any integer $k$, we have the subspace $\wedge^k \text{Der}^1(A)$ in $\text{Der}^k(A)$, which is the set of the elements of the type $v_1 \land \ldots \land v_k$, where each $v_i, \ i = 1, \ldots, k$ is an element of the space $\text{Der}^1(A)$. The subalgebra $\wedge^k \text{Der}^1(A)$ in the algebra $\text{Der}(A)$ is also closed under the bracket $[\ , \ ]$ and the exterior multiplication. For the restriction of the bracket on the algebra $\wedge^k \text{Der}^1(A)$, the following explicit formula can be used:

\begin{equation}
[u_1 \land \ldots \land u_m, \ v_1 \land \ldots \land v_n] = \\
\sum_{i,j} (-1)^{m+i+j-1} [u_i, \ v_j] \land u_1 \land \ldots \land \widehat{u_i} \land \ldots \land u_m \land v_1 \land \ldots \land \widehat{v_j} \land \ldots \land v_n
\end{equation}

where $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ are the elements of the space $\text{Der}^1(A)$.

An involutive element $P \in \text{Der}^2(A)$ defines a bracket on $A$ which, at the same time, is a biderivation. Such a structure on a commutative algebra is called the \textit{Poisson structure}. As the subspace $\text{Der}(A)$ is closed under the supercommutator, it will be invariant under the action of the operator $\partial_P$, and therefore we have the subcomplex $(\text{Der}(A), \partial_P)$ of the complex $(L(A), \partial_P)$. From the formula (3) for the bracket $[\ , \ ]$, on $\text{Der}(A)$, follows that the operator $\partial_P : \text{Der}(A) \rightarrow \text{Der}(A)$ is an antidifferential, that is

\text{for each } u \in \text{Der}^m(A) \text{ and } v \in \text{Der}^n(A) \text{ we have}

$$\partial_P(u \land v) = \partial_P(u) \land v + (-1)^m u \land \partial_P(v)$$
Therefore, on the cohomologies of the complex \((\text{Der}(A), \partial_P)\) can be induced the structure of exterior algebra from \(\text{Der}(A)\). The cohomology algebra of the complex \((\text{Der}(A), \partial_P)\) is called the cohomology algebra of the Poisson algebra \((A, P)\).

For the commutative algebra \(A\), consider the space of the first order differential operators from \(A\) to itself, denoted by \(\text{Diff}^1(A)\). By definition, \(\text{Diff}^1(A)\) is the subspace of the space \(\text{Hom}(A, A)\) consisting of the mappings \(\varphi : A \rightarrow A\), such that, for any \(a \in A = \text{Hom}_A(A, A) \subset \text{Hom}(A, A)\), we have that \([\varphi, a] \in \text{Hom}_A(A, A) = A\).

As it is well-known \(\text{Diff}^1(A) = \text{Der}^1(A) \oplus A\):
\[
\varphi(uv) - u\varphi(v) = (\varphi - \varphi(1))(uv) - u(\varphi - \varphi(1))(v)
\]
So, we have that \((\varphi - \varphi(1))(uv) - u(\varphi - \varphi(1))(v) = c(u)v\). Consequently, putting \(v = 1\) we get \(c = \varphi - \varphi(1)\), and therefore, the mapping
\[
X = \varphi - \varphi(1) : A \rightarrow A
\]
is an element of the space \(\text{Der}^1(A)\), and any element \(\varphi \in \text{Diff}^1(A)\), can be decomposed as \(\varphi = (\varphi - \varphi(1)) + \varphi(1)\), where \(\varphi - \varphi(1) \in \text{Der}^1(A)\) and \(\varphi(1) \in A\).

The anticommutative algebra structure described above, can be considered as well in the case of \(L(B)\), where \(B = \text{Diff}^1(A)\). The space \(B\) is equipped with the natural structure of Lie algebra, defined by the commutator \([u, v] = u \circ v - v \circ u\) which means that there exists an involutive element \(\mu \in L^2(B)\). So, we can consider the complex \((L(\text{Diff}^1(A)), \partial_\mu)\).

For each positive integer \(n\) let \(\Omega^n(A)\) be the subspace of the space \(L^n(\text{Diff}^1(A))\), consisting of such mappings
\[
\omega : \text{Diff}^1(A) \times \ldots \times \text{Diff}^1(A) \rightarrow \text{Diff}^1(A)
\]
that
\begin{itemize}
  \item \(\omega\) takes values in \(A = \text{Hom}_A(A, A) \subset \text{Diff}^1(A)\);
  \item \(\omega(u_1, \ldots, u_n) = 0\) if at least one of the elements \(u_1, \ldots, u_n \in \text{Diff}^1(A) = \text{Der}^1(A) \oplus A\) is in \(A\);
  \item \(\omega\) is an \(A - \text{multilinear}\), i.e. \(\omega(au_1, \ldots, u_n) = a\omega(u_1, \ldots, u_n)\), for any \(a \in A\) and \(u_1, \ldots, u_n \in \text{Der}(A)\).
\end{itemize}
Using the classical terminology, it can be said that the elements of the subspace \( \Omega^n(A) \) are differential forms of the order \( n \) on the commutative algebra \( A \).

**Theorem 1** The subspace \( \Omega(A) = \bigoplus_{n=0}^{\infty} \Omega^n(A) \) in the space \( L(\text{Diff}^1(A)) \) is invariant under the action of the operator \( \partial_\mu = [\mu, \cdot] \), and the restriction of the operator \( -\partial_\mu \) on \( \Omega(A) \) coincides with the classical differential on space of differential forms.

**Proof.** By the definition of the supercommutator, for \( \omega \in \Omega^n(A) \) we have the following

\[
\begin{align*}
[\mu, \omega](u_1, \ldots, u_{n+1}) &= (-1)^n \sum (-1)^{n+1-i} \mu(\omega(u_1, \ldots, \hat{u_i}, \ldots, u_{n+1}), u_i) + \\
&\quad + \sum (-1)^{i+j-1} \omega(\mu(u_i, u_j), u_1, \ldots, \hat{u_i}, \ldots, \hat{u_j}, \ldots, u_{n+1}) = \\
&\quad = \sum (-1)^{i-1}[\omega(u_1, \ldots, \hat{u_i}, \ldots, u_{n+1}), u_i] + \\
&\quad + \sum (-1)^{i+j-1} \omega([u_i, u_j], u_1, \ldots, \hat{u_i}, \ldots, \hat{u_j}, \ldots, u_{n+1}) = \\
&\quad = \sum (-1)^i u_i \omega(u_1, \ldots, \hat{u_i}, \ldots, u_{n+1}) + \\
&\quad + \sum (-1)^{i+j-1} \omega([u_i, u_j], u_1, \ldots, \hat{u_i}, \ldots, \hat{u_j}, \ldots, u_{n+1}) = \\
&\quad = - (\sum (-1)^{i-1} u_i \omega(u_1, \ldots, \hat{u_i}, \ldots, u_{n+1})) + \\
&\quad + \sum (-1)^{i+j} \omega([u_i, u_j], u_1, \ldots, \hat{u_i}, \ldots, \hat{u_j}, \ldots, u_{n+1}) = \\
&\quad = -(d\omega)((u_1, \ldots, u_{n+1}))
\end{align*}
\]

To summarize, we can state that the subspace \( \Omega(A) \) in the space \( L(\text{Diff}^1(A)) \) is not closed under the operation of supercommutator, but it is invariant under the action of the operator \( [\mu, \cdot] \), where \( \mu \in L^2(\text{Diff}^1(A)) \) is the element defined by the commutator in the Lie algebra \( \text{Diff}^1(A) \). It can be defined the operation of the exterior multiplication in the space \( \Omega(A) \) by the formula \( \tilde{p} \), after which the operator \( d \) becomes the antiderivation of degree +1 of the algebra \( \Omega(A) \).

Any element \( p \in \text{Der}^2(A) \) defines the mapping \( \tilde{p} : A \longrightarrow \text{Der}^1(A) \) as follows

\[
\tilde{p}(a)(b) = p(a, b)
\]

which can be extended to the mapping \( \tilde{p} : \Omega(A) \longrightarrow \text{Der}(A) \) by the following formula

\[
\tilde{p}(\alpha)(a_1, \ldots, a_n) = (-1)^n \alpha(\tilde{p}(a_1), \ldots, \tilde{p}(a_n))
\]

where \( \alpha \in \Omega(A) \) and \( a_1, \ldots, a_n \in A \).
As it was mentioned, the involutiveness of the element $p$ (i.e. $[p, p] = 0$) is equivalent to the bracket $\{a, b\} = p(a, b)$ be a Lie algebra structure on $A$: 

$$[p, p](a, b, c) = 2(p(p(a, b), c) + p(p(b, c), a) + p(p(c, a), b)) = 2(\{a, b\}, c) + \{b, c\}, a + \{c, a\}, b)$$

**Lemma 2** If $p$ is involutive, the mapping $\tilde{p}: A \rightarrow Der^1(A)$ is a Lie algebra homomorphism.

**Proof.** $\tilde{p}(\{a, b\})(c) = \{\{a, b\}, c\}$; as it follows from the Jacoby identity for the bracket $\{\}$, we have 

$$\{\{a, b\}, c\} = \{a, b, c\} - \{b, a, c\} = (\tilde{p}(a)\tilde{p}(b) - \tilde{p}(b)\tilde{p}(a))(c)$$

**Theorem 3** The mapping $\tilde{p}: \Omega(A) \rightarrow Der(A)$ is a homomorphism of the complexes $(\Omega(A), d)$ and $(Der(A), \partial_p = [p, \cdot])$.

**Proof.** $\tilde{p}(d\omega)(a_1, \ldots, a_{n+1}) = (-1)^{n+1}(d\omega)(\tilde{p}(a_1), \ldots, \tilde{p}(a_{n+1})) = \sum_i(-1)^i \tilde{p}(a_i)\omega(\tilde{p}(a_1), \ldots, \tilde{p}(a_{n+1})) + \sum_{i<j}(-1)^{i+j}\omega([\tilde{p}(a_i), \tilde{p}(a_j)], \ldots, \tilde{p}(a_{i+1}), \ldots, \tilde{p}(a_{j+1}), \ldots, \tilde{p}(a_{n+1})) = (-1)^{n+1}(\sum_i(-1)^{i-1}p(a_i, \omega(\tilde{p}(a_1), \ldots, \tilde{p}(a_{i+1}), \ldots, \tilde{p}(a_{n+1})) + \sum_{i<j}(-1)^{i+j}\omega([\tilde{p}(a_i), \tilde{p}(a_j)], \ldots, \tilde{p}(a_{i+1}), \ldots, \tilde{p}(a_{j+1}), \ldots, \tilde{p}(a_{n+1})).$

On the other hand we have:

$$[p, \tilde{p}(\omega)](a_1, \ldots, a_{n+1}) = \sum_i(-1)^{i-1}p((\tilde{p}(\omega))(a_1, \ldots, a_i, \ldots, a_{n+1}, a_i) + \sum_{i<j}(-1)^{i+j-1}\tilde{p}(\omega)(p(a_i, a_j), \ldots, \tilde{p}(a_{i+1}), \ldots, \tilde{p}(a_{j+1}))) = (-1)^{n+1}(\sum_i(-1)^{i-1}p(a_i, \omega(\tilde{p}(a_1), \ldots, \tilde{p}(a_{i+1}), \ldots, \tilde{p}(a_{n+1}))) + \sum_{i<j}(-1)^{i+j-1}\omega([\tilde{p}(a_i), \tilde{p}(a_j)], \ldots, \tilde{p}(a_{i+1}), \ldots, \tilde{p}(a_{j+1}), \ldots, \tilde{p}(a_{n+1})))$$
2 Schouten bracket as the deviation of the coboundary operator from the Leibniz rule

The main result of the previous section is the fact that a supercommutator on an exterior algebra gives rise of some coboundary operator on this algebra, and even the classical differential on the exterior algebra of differential forms can be represented as a supercommutator with some second order element of some superalgebra containing the algebra of differential forms. In this section, we consider some reverse situation: a coboundary operator on some exterior algebra induces a superalgebra structure on this algebra.

Let $E$ be a real or complex $\mathbb{Z}$-graded exterior algebra with a multiplication operation denoted by $\wedge$. Let $\partial : E \longrightarrow E$ be a boundary operator ($\partial \circ \partial = 0$ and $\partial(E_i) \subset E_{i-1}$ for $i = 0, \ldots, \infty$).

The operator $\partial$ is said to be an antidifferential if for any $u \in E_m$ and $v \in E$, it satisfies the following condition

$$\partial(u \wedge v) = \partial(u) \wedge v + (-1)^m u \wedge \partial(v)$$

For any boundary operator on the exterior algebra $E$ we can define the bilinear mapping $\mathcal{[}, \mathcal{]} : E \times E \longrightarrow E$ as follows

$$\mathcal{[}, \mathcal{]}(u, v) = \partial(u) \wedge v + (-1)^m u \wedge \partial(v) - \partial(u \wedge v)$$

(5)

If the operator $\partial$ is antiderivation, the mapping defined by this formula is trivial.

In any case, we can ask the question is the bracket $\mathcal{[}, \mathcal{]}$ a Lie superalgebra structure on $E$ or not? To be so, the following conditions must be hold:

for any $u \in E_m$, $v \in E_n$ and $w \in E_k$

(\textbf{s1}) $\mathcal{[}, u, v \mathcal{]} = (-1)^{mn}[u, v]$

(\textbf{s2}) $\mathcal{[}, u, v \wedge w \mathcal{]} = [u, v] \wedge w + (-1)^{(m+1)n} v \wedge [u, w]$

(\textbf{s3}) $(-1)^{mk}[[u, v], w] + (-1)^{mn}[[v, w], u] + (-1)^{nk}[[w, u], v] = 0$
The first of these three conditions is obviously always true. The third one is also always true for the first order elements and is equivalent to \((\partial \circ \partial)(x \wedge y \wedge z) = 0\), for \(x, y, z \in E_1\); and implies that the bracket \([x, y] = -\partial(x, y)\) defines a Lie algebra structure on \(E_1\).

The condition (s2) is equivalent to the following equality for the operator \(\partial\):

\[
\begin{align*}
\partial(\alpha \wedge \beta \wedge \gamma) &= \\
&= \partial(\alpha \wedge \beta) \wedge \gamma + (-1)^m \alpha \wedge \partial(\beta \wedge \gamma) + (-1)^{(m+1)n} \beta \wedge \partial(\alpha \wedge \gamma) - \\
&\quad - (\partial \alpha \wedge \beta \wedge \gamma) + (-1)^m \alpha \wedge \beta \wedge \gamma + (-1)^{m+n} \alpha \wedge \beta \wedge \partial \gamma
\end{align*}
\]

(6)

It is easy to check by induction that the condition (s2) implies that the operator \(\partial\), on the elements of the type \(u_1 \wedge \ldots \wedge u_n \in E_n\) where \(u_1, \ldots, u_n \in E_1\) has the form

\[
\partial(u_1 \wedge \ldots \wedge u_n) = \sum_{i<j} (-1)^{i+j}[u_i, u_j] \wedge u_1 \wedge \ldots \wedge \hat{u}_i \wedge \ldots \wedge \hat{u}_j \wedge \ldots \wedge u_n
\]

and in this case, all of the above three conditions are true on the subalgebra \(\wedge E_1 = \bigoplus_{k=0}^{\infty} (\wedge^k E_1)\).

Let a Lie algebra \(L\) be a module over some commutative real or complex \(A\), which, itself, is a module over the Lie algebra \(L\). That is: there is a Lie algebra homomorphism from \(L\) into the Lie algebra of all derivations of the algebra \(A\). Assume that these two structures: the \(A\)-module structure on \(L\) and the \(L\)-module structure on \(A\), are interconnected by the following condition:

for any \(x, y \in L\) and \(a \in A\) let \([x, ay] = x(a) \cdot y + a \cdot [x, y]\).

Let for any positive integer \(n\), \(\Omega^n_K(L, A)\) be the space of skew-symmetric, \(K\)-multilinear mappings from \(L^n\) into \(A\), where \(K\) is the field of either real or complex numbers. Using the formula \(\Box\) for the Schouten bracket on \(\wedge L = \bigoplus \wedge^k L\), we obtain that for any \(u \in \wedge^m L, v \in \wedge^n L\) and \(\omega \in \Omega^{m+n-1}_K(L, A)\):

\[
\omega([u, v]) = \omega(\partial(u) \wedge v) + (-1)^m \omega(u \wedge \partial(v)) - \omega(\partial(u \wedge v))
\]

or, in the other notations

\[
\omega([u, v]) = (-1)^{(m+1)n} i_u \omega(\partial(u)) + (-1)^m i_v \omega(\partial(v)) - \omega(\partial(u \wedge v))
\]

(7)

where, for \(\alpha \in \Omega^n_K(L, A)\) and \(x \in \wedge^q L\), under the notation \(i_x \omega\), we mean the element of the space \(\Omega^{n-q}_K(L, A)\) defined as \((i_x \alpha)(y) = \alpha(x \wedge y)\), for any \(y \in \wedge^{n-q} L\).
Using the dual notations, the expression \( \omega([u, v]) \) can be written in the following form

\[
\omega([u, v]) = (-1)^{(m+1)n}(\partial^* i_u \omega)(u) + (-1)^m(\partial^* i_v \omega)(v) - (\partial^* \omega)(u \wedge v) \quad (8)
\]

where: \( (\partial^* \alpha)(x) = \alpha(\partial x) \), for any \( \alpha \in \Omega^p_K(L, A) \) and \( x \in \wedge^p L \).

By the definition of the classical exterior differential, we have that \( d\alpha = \partial^* \alpha + \partial_1 \alpha \), where, for \( u_1, \ldots, u_{p+1} \in L \), the expression \( \partial_1 \alpha \) is defined as

\[
(\partial_1 \alpha)(u_1 \wedge \ldots \wedge u_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} u_i \alpha(u_1 \wedge \ldots \wedge \hat{u}_i \wedge \ldots \wedge u_{p+1})
\]

It is easy to verify, that

\[
(-1)^{(m+1)n}(\partial_1 i_u \omega)(u) + (-1)^m(\partial_1 i_v \omega)(v) - (\partial_1 \omega)(u \wedge v) = 0
\]

Therefore, in the expression \( 8 \) we can replace the operator \( \partial^* \) by the operator \( d \):

\[
\omega([u, v]) = (-1)^{|u|+1}|v| (di_v \omega)(u) + (-1)^{|u|} (di_u \omega)(v) - (d\omega)(u \wedge v) \quad (9)
\]

If we consider the subspace \( \Omega(L, A) = \bigoplus_{i=0}^\infty \Omega^i(L, A) \) of the space \( \Omega_K(L, A) = \bigoplus_{i=0}^\infty \Omega^i_K(L, A) \) consisting of the \( A \)-multilinear mappings, the formula \( 1 \) for the Schouten bracket is more convenient then the formulas \( 3, 4 \) and \( 5 \), as the subspace \( \Omega(L, A) \) is invariant under the action of the operator \( d \), and besides that, the formula \( 3 \) can be used as an invariant definition of the Schouten bracket in some cases. For example, for the covariant, skew-symmetric tensor fields on a smooth manifold.

As we know, an invariant element \( p \in L \wedge L \), defines an operator \( \partial_p = [p, ] : \wedge L \rightarrow \wedge L \), of degree +1, which is a coboundary operator. The dual operator

\[
\partial_p^* : \Omega_K(L, A) \rightarrow \Omega_K(L, A)
\]

\[
(\partial_p^* \omega)(x) = \omega([p, x])
\]

is an operator of degree -1 and is a boundary operator: \( \partial_p^* \circ \partial_p^* = 0 \). Using the formula \( 1 \) we obtain the following expression for \( \partial_p^* \):

\[
(\partial_p^* \omega)(u) = (d\omega)(p \wedge u) - (di_p \omega)(u) - (-1)^{|u|} (di_u \omega)(p)
\]
Or, in more brief notations

\[(\partial_p^* \omega)(u) = (i_p \circ d - d \circ i_p)(\omega)(u) - (-1)^{|u|}(di_u \omega)(p) \quad (10)\]

It is clear that the subalgebra \(\Omega(L, A)\) of the algebra \(\Omega_K(L, A)\) is not invariant under the action of the operator \(\partial_p^*\), as for \(\omega \in \Omega^n(L, A)\), \(a \in A\), and \(x \in \wedge^n L\), we have

\[(\partial_p^* \omega)(a \cdot x) = -\omega([p, a \cdot x]) = -\omega(\tilde{p}(a) \wedge x + a[p, x]) =
\]

\[= -(-1)^{|x|}(\tilde{p}(a))(p(a)) + a \cdot \omega([p, x])) = a \cdot (\partial_p^* \omega)(x) - (-1)^{|x|}(\tilde{p}(a))\]

To "correct" the operator \(\partial_p^*\), so that the algebra of differential forms \(\Omega(L, A)\) be invariant under its action, we remove the last term in the 10. The result is exactly the boundary operator of the canonical complex for Poisson manifold, which is well-known in the case when \(L\) is the Lie algebra of vector fields on some Poisson manifold \(M\), and \(A\) is the commutative algebra of smooth functions on \(M\) (see [2])

\[\partial : \Omega^m(L, A) \longrightarrow \Omega^{m-1}(L, A), \quad \partial = i_p \circ d - d \circ i_p \quad (11)\]

For \(p\), define the following bilinear mapping:

\[p : \Omega^m(L, A) \times \Omega^n(L, A) \longrightarrow \Omega^{m+n-2}(L, A) \]

\[p(\alpha, \beta) = i_p(\alpha \wedge \beta) - i_p \alpha \wedge \beta - \alpha \wedge i_p \beta \quad (12)\]

The Schouten bracket on the anticommutative graded algebra \(\Omega(L, A) = \oplus \Omega^k(L, A)\) can be defined as

\[[\alpha, \beta] = d p(\alpha, \beta) - p(d \alpha, \beta) - (-1)^{|\alpha|} p(\alpha, d\beta) \quad (13)\]

(see [3]).

**Theorem 4** The bracket on \(\Omega(L, A)\) defined by the formula (13) coincides with the bracket \([[ , ]]_\delta\) which is the deviation of the operator \(\delta\) from antiderivation. That is: for any \(\alpha \in \Omega^m(L, A)\), and \(\beta \in \Omega(L, A)\), the following equality is true

\[\delta \alpha \wedge \beta + (-1)^m \alpha \wedge \beta - \delta(\alpha \wedge \beta) = dp(\alpha, \beta) - p(d \alpha, \beta) - (-1)^m p(\alpha, d\beta)\]
The proof of this theorem consists of simple verifying of the equality keeping in mind the formulas 11, 12 and 13.

For any \( a \in A \), define the element \( da \in \Omega^1(L, A) \), as \( (da)(X) = X(a) \) for any \( X \in L \). Consider the subalgebra of the \( \Omega(L, A) \) generated by \( A \) and \( dA \subset \Omega^1(L, A) \). Denote this subalgebra by \( \tilde{\Omega}(L, A) \), and the corresponding grading subspaces by \( \tilde{\Omega}^k(L, A) \) for \( k = 0, \cdots, \infty \). As it follows from the definition, each \( \tilde{\Omega}^k(L, A) \) consists of the elements of the form \( \sum_{i=1}^n a_i^i d_1^i \wedge da_k^i \). The Poisson bracket on \( A \) defined by \( p \), as \( \{ a, b \} = i_p(da \wedge db) \), for \( a, b \in A \), gives the same expression for the operator \( \delta \), as in the case when \( A \) is the algebra of smooth functions on some Poisson manifold and \( L \) is the Lie algebra of the vector fields on the same manifold (see [3]):

\[
\delta(a_0da_1 \wedge \ldots \wedge a_n) = \sum_{i=1}^n (-1)^{i+1}\{a_0, a_i\}da_1 \wedge da_i \wedge da_n + \\
+ \sum_{i<j} (-1)^{i+j}a_0d\{a_i, a_j\} \wedge da_1 \wedge \ldots \wedge da_i \wedge \ldots \wedge da_j \wedge \ldots \wedge da_n
\]  

By using of this formula, it is easy to verify that on \( \tilde{\Omega}(L, A) \) the condition 3 for \( \delta \) is true, therefore, the bracket defined by 13 or by \( [\alpha, \beta] = \delta\alpha \wedge \beta + (-1)^{\|\alpha\|}\alpha \wedge \delta\beta - \delta(\alpha \wedge \beta) \) on \( \tilde{\Omega}(L, A) \) gives a Lie superalgebra structure, which is the extension of the Lie algebra structure on \( \tilde{\Omega}^1(L, A) \). So, in the case when \( A = C^\infty(M) \) for some Poisson manifold \( M \), and \( L \) is the Lie algebra of vector fields on the same manifold, we can state that the supercomutator of differential forms on \( M \) is the deviation of the canonical boundary operator \( \delta \) from antiderifferential. An element \( x \wedge y \in L \wedge L \) defines the mapping from \( A \) into \( L, a \mapsto U_a \), as \( U_a = x(a) \cdot y - y(a) \cdot x \). It is clear that for each \( a \in A \), the expression \( U_a \) depends only on \( da \in \tilde{\Omega}^1(L, A) \). This mapping can be extended linearly for any \( p \in L \wedge L \). After that, for any \( p \in L \wedge L \) we can define the mappings \( \bar{p} : \tilde{\Omega}^k(L, A) \rightarrow \wedge^kL, k = 0, \cdots, \infty \) as follows

\[
\bar{p}(a_0da_1 \wedge \ldots \wedge da_k) = a_0U_{a_1} \wedge \ldots \wedge U_{a_k}
\]

For any fixed \( \omega \in \tilde{\Omega}^n(L, A) \), define a mappings:

\[
* : \tilde{\Omega}^k(L, A) \rightarrow \tilde{\Omega}^{n-k}(L, A)
\]

as

\[
*(a_0da_1 \wedge \ldots \wedge da_k) = a_0(i_{U_{a_k}} \circ \cdots \circ i_{U_{a_1}})\omega.
\]

In the case when \( M \) is a symplectic manifold with a symplectic form \( \alpha \), \( A = C^\infty(M) \), \( p \) is the bivector field corresponding to the form \( \alpha \), \( L \) is the
Lie algebra of vector fields on $M$, and $\omega = \alpha^{(\dim M)} / 2$, the operator $*$ is the well-known analogue of the star operator on a Riemannian manifold (see [4]).

**Theorem 5** If $\omega$ satisfies the following conditions

\[ d\omega = 0 \]
\[ da \land \omega = 0 \text{ for each } a \in A \]

then the equality $*\delta = (-1)^kd*$ is true on $\tilde{\Omega}^k(L, A)$, if and only if $(d \circ i_{U_a})\omega = 0$, for any $a \in A$.

**Proof.** on $\tilde{\Omega}^1(L, A)$ we have:

\[ (\ast\delta)(a_0 da_1) = \ast\{a_0, a_1\} = \{a_0, a_1\} \cdot \omega; \]
\[ (d\ast)(a_0 da_1) = d(a_0 i_{U_{a_1}}\omega) = da_0 \land i_{U_{a_1}}\omega + a_0 di_{U_{a_1}}\omega \]

Consequently: $(\ast\delta + d\ast)(a_0 da_1) = \{a_0, a_1\} \cdot \omega + da_0 \land i_{U_{a_1}}\omega + a_0 di_{U_{a_1}}\omega = -i_{U_{a_1}}(da_0 \land \omega) + a_0 di_{U_{a_1}}\omega = a_0 di_{U_{a_1}}\omega.$

Therefore, on the space $\tilde{\Omega}^1(L, A)$ the equality $*\delta = -d\ast$ is true if and only if $(d \circ i_{U_a})\omega = 0$ for any $a \in A$.

To proof the equality $*\delta = (-1)^kd\ast$ for each $\tilde{\Omega}^k(L, A)$, the following well-known formula can be used

\[ (L_X\omega)(X_1, \cdots, X_n) = (i_Xd\omega + d i_X\omega)(X_1, \cdots, X_n) = \]
\[ = X\omega(X_1, \cdots, X_n) - \sum i_X\omega(X_1, \cdots, [X, X_i], \cdots, X_n) \]

So, we can state that the operator $*$ induces a homomorphism from the homology space $H_i(L, δ)$ of the complex $(\tilde{\Omega}(L, A), δ)$ into the cohomology space $H^{n-1}(L, A)$ of the complex $(\tilde{\Omega}(L, A), d)$.

3 Brief overview of the geometric structure of Poisson Manifolds

Further we shall consider the case when the commutative algebra $A$ is the algebra $C^\infty(M)$ for some smooth manifold $M$, $L$ is the Lie algebra of vector fields on $M$ and therefore $\Omega(M)$ is the exterior algebra of differential forms on $M$. An involutive element $p \in V^2(M)$, where $V^2(M)$ is the space of the
second-order covariant antisymmetric tensor fields on $M$, defining a Poisson algebra structure on the space $C^\infty(M)$ is called as a bivector field on the manifold $M$.

Let $\pi$ be the differential system on $M$ derived by the set of the vector fields of the type $X_f = \{ f, \cdot \}$ for $f \in C^\infty(M)$. In other words, for any point $x \in M$, the subspace $\pi(x) \subseteq T_xM$ is defined as

$$\pi(x) = \{ u \in T_xM \mid \beta(u) = (\alpha \wedge \beta)(p_x) \text{ for some } \alpha \in T_x^*M \text{ and each } \beta \in T_x^*M \}$$

The rank of the differential system $\pi$ at any point $x \in M$ (i.e. the dimension of the space $\pi(x)$) equal to the rank of the bivector field $p$ at the point $x$ ($\text{rank}(p_x) = r \iff r = 2k$, for some integer $k$ such that $\wedge^k p_x \neq 0$ and $\wedge^{k+1} p_x = 0$).

For any function $f \in C^\infty(M)$, let $\varphi_t$, $t \in \mathbb{R}$ be the one-parameter group of diffeomorphisms of the manifold $M$, corresponding to the vector field $X_f = \{ f, \cdot \}$. The bivector field $p$ is conserved by the group $\varphi_t$: the latter statement is equivalent to the following equality

$$(d(g \circ \varphi_t) \wedge d(h \circ \varphi_t))(p) \circ \varphi_t^{-1} = (dg \wedge dh)(p)$$

which itself, is equivalent to

$$\{ g \circ \varphi_t, h \circ \varphi_t \} = \{ g, h \} \circ \varphi_t$$ \hspace{1cm} (15)

the latter is a result of the Jacoby identity for the functions $f$, $g$, and $h$, which is the infinitesimal variant of [15].

It is natural to ask, is the differential system $\pi$ integrable or not. Note, that it is an involutive system

$$X, Y \in \pi \iff (X = \sum \varphi_i \{ f_i, \cdot \}, Y = \sum \psi_i \{ g_i, \cdot \}) \Rightarrow$$

$$[X, Y] = \sum (\varphi_i \{ f_i, \psi_i \} \cdot \{ g_i, \cdot \} - \psi_i \{ g_i, \varphi_i \} \{ f_i, \cdot \}) +$$

$$+ \sum \varphi_i \psi_i \{ \{ f_i, g_i \}, \cdot \} \Rightarrow [x, y] \in \pi$$

Moreover, the following theorem describes the exact condition for any bivector field $p$ the corresponding differential system $\pi$ be involutive:

**Theorem 6** The differential system $\pi$ is involutive if and only if $[p, p]|_x \in \pi|x \wedge \pi|x \wedge \pi|x$ for every point $x \in M$. 

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Proof. To prove the theorem, the following formula is useful:

\[
(\omega \wedge \alpha \wedge \beta)(X \wedge Y) = \omega(X) \cdot (\alpha \wedge \beta)(Y) + \omega(Y) \cdot (\alpha \wedge \beta)(X) - \omega(\tilde{X}(\alpha), \tilde{Y}(\beta)) + \omega(\tilde{X}(\beta), \tilde{Y}(\alpha))
\]

It is sufficient to verify this formula in the case when \(\omega = \varphi \wedge \psi\), for any \(\varphi, \psi \in \Omega^1(M)\). In this case we have the following

\[
(\varphi \wedge \psi \wedge \alpha \wedge \beta)(X \wedge Y) = (\varphi \wedge \psi)(X) \cdot (\alpha \wedge \beta)(Y) + \\
+ (\varphi \wedge \alpha)(X) \cdot (\beta \wedge \psi)(Y) + (\varphi \wedge \beta)(X) \cdot (\alpha \wedge \psi)(Y) + \\
+ (\psi \wedge \alpha)(X) \cdot (\varphi \wedge \beta)(Y) + (\psi \wedge \beta)(X) \cdot (\alpha \wedge \varphi)(Y) + \\
+ (\alpha \wedge \beta)(X) \cdot (\varphi \wedge \psi)(Y) = \\
= \omega(X) \cdot (\alpha \wedge \beta)(Y) + \omega(Y) \cdot (\alpha \wedge \beta)(X) - \\
\omega(\tilde{X}(\alpha), \tilde{Y}(\beta)) + \omega(\tilde{X}(\beta), \tilde{Y}(\alpha))
\]

The statement of the theorem, translated on the language of a local coordinate system \(\{x_1, \ldots, x_n\}\) is the following: for each \(i, j \in \{1, \ldots, n\}\) the vector field \([\tilde{p}(dx_i), \tilde{p}(dx_j)]\) takes its values in the differential system \(\pi\); which is the same thing, that \(\sigma([\tilde{p}(dx_i), \tilde{p}(dx_j)]) = 0\) for each \(\sigma \in (\pi)^\perp \subset \Omega^1(M)\).

Using the formula (9) for the Schouten bracket, we obtain:

\[
(d\sigma \wedge dx_i \wedge dx_j)(p \wedge p) = (2d\sigma)(p) \cdot (dx_i \wedge dx_j)(p) - (\sigma \wedge dx_i \wedge dx_j)([p, p]).
\]

By using of the formula (10), we obtain:

\[
(d\sigma \wedge dx_i \wedge dx_j)(p \wedge p) = (2d\sigma)(p) \cdot (dx_i \wedge dx_j)(p) - (2\sigma)(\tilde{p}(dx_i, \tilde{p}(dx_j))).
\]

Hence, we have:

\[
(\sigma \wedge dx_i \wedge dx_j)([p, p]) = -(2\sigma)(\tilde{p}(dx_i, \tilde{p}(dx_j)))).
\]

Recall that \(\sigma \in (\pi)^\perp\), the latter equality ends the proof of the theorem.

If the rank of a differential system is constant, then its integrability follows from the Frobenius’s classical theorem; but generally, the differential system \(\pi\), is not of a constant rank. Despite this, the differential system \(\pi\) is always integrable, and it is a result of the Hermann’s theorem (see [4]), which is a generalization of the Frobenius’s theorem about the integrability of differential systems of non-constant rank. The necessary and sufficient condition for the integrability of a differential system, as it is stated in the above mentioned theorem, is the conservation of the rank of the system along the integral paths of this system. This condition is satisfied for the differential system \(\pi\), which follows from the fact that the one-parameter groups of
the Hamiltonian vector field, conserve the bivector field $p$, and therefore its rank.

An integral leaf of the differential system $\pi$ is called symplectic leaf.

The restriction of the Poisson structure on any integral leaf of the different-
ential system $\pi$ is non-singular; hence, a symplectic structure is induced by
the bivector field $p$ on such a leaf. Let us denote the symplectic form induced
by the Poisson structure on a symplectic leaf $N$ by $\omega_N$. For $x \in N$, $u \in T_xN$, and $v \in T_xN$ we have that $\omega_N(u, v) = \{f, g\}(x)$, where $u = \{f, \cdot\}|_x$, and $v = \{g, \cdot\}|_x$.

One of the indicators of the singularity of a Poisson structure is the exis-
tence of such smooth function on $M$, which commutes with all functions on
$M$ and is not constant, i.e. the center of the Lie algebra of smooth functions
$Z(M)$, does not coincide to the set of the constant functions. The elements of
the center $Z(M)$ are know as Casimir functions. From the singularity of the
Poisson structure $p$ does not follow the existence of a non-constant Casimir
function.

For instance, if one of the symplectic leaves is everywhere dense in the
manifold $M$ then a Casimir function can be only constant.

By way of illustration, consider the following

**Example:** let $M$ be a two-dimensional symplectic manifold and $p$ be
the corresponding non-singular bivector field on $M$. Let $\varphi$ be a nonconstant
smooth function on $M$. The bivector field $p_1 = \varphi \cdot p$, is involutive as well
as $p$. If the set $\varphi^{-1}(0)$ is not empty, the Poisson structure defined by $p_1$, is singular at the points of the set $\varphi^{-1}(0)$, which follows from the relation
between the bracket $\{\ , \ \}_1$ defined by $p_1$ and the bracket $\{\ , \ \}$ defined by
$p$: $\{f, g\}_1 = \varphi \cdot \{f, g\}$, for any $f, g \in C^\infty(M)$. If a function $f \in C^\infty(M)$
is a Casimir function, then we have the following

$$\varphi \cdot \{f, \cdot\} = 0 \Rightarrow \{f, \cdot\}|_{M\setminus\varphi^{-1}(0)} = 0 \Rightarrow f = \text{const}$$

If $\varphi$ is such, that the set $M\setminus\varphi^{-1}(0)$ is everywhere dense (for example, in the
case when the set $\varphi^{-1}(0)$} consists only one point $x_0$ then we have that the
function $f$ is constant everywhere on the manifold $M$. So, this is an example
of the situation when a Poisson structure is singular, but Casimir function
can be only constant.

Further, we shall extend (in some sense) the definition of the Poiss-
un bracket for distributions on a smooth manifold, and be looking for Casimir
functions in the set of distributions.
4 Distributions on Poisson manifold

Distribution on a smooth manifold $M$ is a linear function on the subspace of the space $C^\infty(M)$ consisting of the functions with a compact support. For simplicity assume that the manifold $M$ is compact, which implies that a distribution on $M$ is simply a linear function on the space $C^\infty(M)$.

Let us denote the space of all distributions on the manifold $M$ by $F(M)$. Using the classical notations, the value of a distribution $\Phi$ on a function $\phi \in C^\infty(M)$, we denote by $\langle \Phi, \phi \rangle$.

The product of a function $f$, on a distribution $\Phi$, is defined as the distribution $f \cdot \Phi$ such that, for each $\phi \in C^\infty(M)$:

$$\langle f \cdot \Phi, \phi \rangle = \langle \Phi, f \cdot \phi \rangle.$$

This operation makes the space $F(M)$ a $C^\infty(M)$-module.

For a vector field $X \in V^1(M)$ and a distribution $\Phi$, the distribution $X(\Phi)$ is defined as

$$\langle X(\Phi), \phi \rangle = -\langle \Phi, X(\phi) \rangle.$$

This action makes the $C^\infty(M)$-module, $F(M)$, a $V^1(N)$-module. That is: for any $f \in C^\infty(M)$, $X \in V^1(M)$ and $\Phi \in F(M)$, we have: $X(f \cdot \Phi) = X(f) \cdot \Phi + f \cdot X(\Phi)$, which follows from

$$\langle X(f \cdot \Phi), \phi \rangle = -\langle f \cdot \Phi, X(\phi) \rangle = -\langle \Phi, f \cdot X(\phi) \rangle = -\langle \Phi, X(\phi) \rangle \cdot f + f \cdot \langle X(\Phi), \phi \rangle.$$

Let $M$ be a Poisson manifold.

As the action of a vector field on a distribution is defined, it can be defined the Poisson bracket of a function $f$ and a distribution $\Phi$:

$$\{f, \Phi\} = X_f(\Phi),$$

where $X_f$ is the Hamiltonian vector field corresponding to the function $f$. The latter can be written as

$$\langle \{f, \Phi\}, \phi \rangle = \langle X_f(\Phi), \phi \rangle = -\langle \Phi, X_f(\varphi) \rangle = -\langle \varphi, f \rangle.$$

It makes the $C^\infty(M)$-module $F(M)$ a Lie algebra module over $C^\infty(M)$. That is:

$$\text{for } f, g \in C^\infty(M) \text{ and } \Phi \in F(M),$$

$$\{f, g \cdot \Phi\} = \{f, g\} \cdot \Phi + g \cdot \{f, \Phi\}.$$

Besides that, we have that, for any fixed $\Phi \in F(M)$, the first order differential operator $\{\Phi, \cdot\} : C^\infty(M) \rightarrow F(M)$ is such that for any $\varphi, \psi \in C^\infty(M)$:
\[ \{ \Phi, \varphi \psi \} = \varphi \{ \Phi, \psi \} + \psi \{ \Phi, \varphi \}. \]

To verify this, the following expression can be used:
\[ \{ \varphi \psi, \cdot \} = \varphi \{ \psi, \cdot \} + \psi \{ \varphi, \cdot \}. \]

After we have defined the Poisson bracket of a distribution and a smooth function on the manifold \( M \), note that, if the Poisson structure on \( M \) is singular, but has not a nonconstant center, can have such "center" in the space of the distributions. That is, there can be such a distribution \( \Phi \in F(M) \) that \( \{ \Phi, \varphi \} = 0 \) for each \( \varphi \in C^\infty(M) \). In the situation described at the end of the previous section, such distributions are the Dirac functions \( \delta_a \) for any \( a \in \varphi^{-1}(0) \), \( \delta_a(f) = f(a) \). In this case, for any \( f, g \in C^\infty(M) \), we have
\[
< \{ \delta_a, f \}, g >= < \delta_a, \{ f, g \} >= \varphi(a) \{ f, g \}(a) = 0
\]

Now, we shall describe some general construction to build a distributions "commuting" with each smooth function on the manifold \( M \).

Let us recall the following formula for the Poisson bracket of two functions on a symplectic manifold with a symplectic form \( \omega \):
\[
\{ f, g \} \cdot \omega^n = n \cdot dg \wedge df \wedge \omega^{n-1}
\]
where \( n \) is the half-dimension of the manifold. The formula is the result of the following
\[
\{ f, g \} \cdot \omega^n = i_{U_f}(dg \wedge \omega^n) + dg \wedge i_{U_g} \omega = ndg \wedge df \wedge \omega^{n-1}
\]

Let \( N \) be a symplectic leaf in the Poisson manifold \( M \). As it was mentioned early, the restriction of the bivector field \( p \), on the leaf \( N \) is not singular and we denote the corresponding symplectic form by \( \omega_N \). Consider the following distribution on the manifold \( M \):
\[
\delta_N : C^\infty(M) \longrightarrow R, \quad < \delta_N, \varphi >= \int_N \varphi_N \cdot \omega^k
\]
where \( \varphi \in C^\infty(M) \) and \( k = \frac{1}{2} \dim N \).

**Theorem 7** For any \( \varphi \in C^\infty(M) \), we have \( \{ \delta_N, \varphi \} = 0 \).

**Proof.** By the definition of the Poisson bracket of a distribution and a smooth function we have:
for any $\varphi, \psi \in C^\infty(M)$

$$< \{\delta_N, \varphi\}, \psi> = <\delta_N, \{\varphi, \psi\}> = \int_N \{\varphi, \psi\}|_N \cdot \omega_N^k$$

Keeping in mind the fact that the Hamiltonian vector fields are tangent to the symplectic leaves, the formula [13] and the Stokes formula, we obtain

$$\int_N \{\varphi, \psi\}|_N \cdot \omega_N^k = \int_N \{\varphi|_N, \psi|_N\} \cdot \omega_N^k =$$

$$= n \int_N d\psi \wedge d\varphi \wedge \omega_N^{k-1} = n \int_{\delta_N} \psi \wedge d\varphi \wedge \omega_N^{k-1} = 0$$

Let $F_0(M)$ be the subspace of the space $F(M)$ consisting of the distributions commuting with every smooth function; $H_0(M, \delta)$ be the space of 0-dimensional homologies of the canonical complex of the Poisson manifold $M$, denoted by $(\Omega(M), \delta)$; and $H_0(M, \delta)^*$ be the space of linear functions on the space $H_0(M, \delta)$.

**Lemma 8** The spaces $F_0(M)$ and $H_0(M, \delta)^*$ are isomorphic.

**Proof.** As it follows from the definition of the Poisson bracket of a distribution and a smooth function, the space $F_0(M)$ can be defined as

$$F_0(M) = \{\Phi \in F(M) \mid <\Phi, \{f, g\}> = 0$$

for every $f, g \in C^\infty(M)\}$$

In other words, $F_0(M) = \{C^\infty(M), C^\infty(M)\}^\perp$, where $\{C^\infty(M), C^\infty(M)\}$ is the space of the sums of the type

$$\sum \{\varphi_i, \psi_i\}, \varphi_i, \psi_i \in C^\infty(M).$$

As it follows from the formula [4] for the canonical coboundary operator $\delta : \Omega(M) \rightarrow \Omega(M)$, its action on the form $\alpha = \sum \varphi_i d\psi_i \in \Omega^1(M)$ is $\delta(\alpha) = \sum \{\varphi_i, \psi_i\}$. Therefore, $\delta(\Omega^1(M)) = \{C^\infty(M), C^\infty(M)\}$. As $H_0(M, \delta) = C^\infty(M)/\delta(\Omega^1(M))$, we obtain that $\delta(\Omega^1(M))^\perp = H_0(M, \delta)^*$
Corollary 9 If $M$ is a symplectic manifold, then the space $F_0(M)$ is one-dimensional and the functional $\delta_\omega$ defined as $< \delta_\omega, \varphi > = \int_M \varphi \cdot \omega^n$, where $\varphi \in C^\infty(M)$, $\omega$ is the symplectic form and $\dim M = 2n$, forms its basis.

Proof. If $M$ is a symplectic manifold, then the mapping $*: H_0(M, \delta) \rightarrow H^{2n}(M)$, where $H^{2n}(M)$ is the $2n$-dimensional De-Rham cohomology space of $M$, is isomorphism. As $M$ is symplectic, it is an oriented manifold, therefore $H^{2n}(M) \cong R$

Let $N$ be a symplectic leaf in the Poisson manifold $M$, and $r: C^\infty(M) \rightarrow C^\infty(M)$ be the restriction mapping. It is clear that $\delta_N = r^*(\delta_{\omega_N})$, where $r^*: F(M) \rightarrow F(M)$ is the dual mapping, and $\omega_N$ is the symplectic form on $N$ induced by the Poisson structure. If the mapping $r$ is an epimorphism, then $\text{Image}(r^*) = (I_N)^\perp$, where $I_N$ is the ideal of the functions on $M$ vanishing on the submanifold $N$, and $(I_N)^\perp$ is its orthogonal subspace in the space $F(M)$.

Theorem 10 If a symplectic leaf $N$ in the Poisson manifold $M$ is such that the restriction mapping $r: C^\infty(M) \rightarrow C^\infty(M)$ is epimorphic, then the space $(I_N)^\perp \cap F_0(M)$ is one-dimensional and the element $\delta_N$ forms its basis.

Proof. As the mapping $r$ is a Poisson mapping, i.e. for each pair $\varphi$, $\psi \in C^\infty(M)$: $\pi(\{\varphi, \psi\}) = \{\pi(\varphi), \pi(\psi)\}$, $\pi^{-1}((I_N)^\perp \cap F_0(M)) = F_0(N)$, which is one-dimensional according to the Corollary 1 (see Lemma 2)

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