LARGE SCALE ABSOLUTE EXTENSORS

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Abstract. Asymptotic dimension was defined by M. Gromov for metric spaces in a manner dual to the covering dimension for topological spaces. Since the covering dimension can be characterized via extension properties of maps to spheres, it makes sense to seek analogs of that phenomenon in the large scale category. There have been two approaches in that direction so far (A. Dranishnikov’s using extension of proper asymptotically Lipschitz functions to euclidean spaces, and by Repovš-Zarichnyi using maps to open cones). Our approach is specifically designed to seek relation to the covering dimension of the Higson corona $\nu(X)$ in case of proper metric spaces $X$. As an application we recover results of Dranishnikov-Keesling-Uspenskiy (dim($\nu(X)$) $\leq$ asdim($X$)) and Dranishnikov (dim($\nu(X)$) = asdim($X$) if asdim($X$) $< \infty$).

1. Introduction

Asymptotic dimension of metric spaces was introduced by M. Gromov [9] as a means of exploring the large scale properties of the space and has been studied extensively during the last two decades.

A related concept is Property A of G. Yu (see [15], [11], and [18] for definitions and basic results) and its similarity to the asymptotic dimension was explained in [3]. In [4] the concept of large scale paracompactness was introduced as an analog of classical paracompactness. It turns out, for spaces of bounded geometry, Property A and large scale paracompactness coincide. Large scale paracompactness is defined by existence of $\delta$-partitions of unity for arbitrary $\delta > 0$. Those are $(\delta, \delta)$-Lipschitz maps $f: X \to K$ from $X$ to a simplicial complex $K$ (with the standard $l_1$-metric) such that point-inverses of stars of vertices are uniformly bounded and their Lebesgue number is at least $\frac{1}{\delta}$.

It is a natural idea to study large scale extensors $K$ of a metric space $X$ defined as follows: for any $\epsilon > 0$ there is $\delta > 0$ such that any $(\delta, \delta)$-Lipschitz map $f: A \subset X \to K$ extends to a $(\epsilon, \epsilon)$-Lipschitz map $g: X \to K$. It turns out, for bounded metric spaces $K$, it is a coarse invariant. For arbitrary metric spaces $K$ another approach is needed.

One of the ways of studying asymptotic dimension is through slowly oscillating functions. Let $X, Y$ be metric spaces and $x_0 \in X$. We say that a function $f: X \to Y$ is slowly oscillating if for any $R, \epsilon > 0$ there is a $N > 0$ such that for any $x \in X$ with $d(x_0, x) > N$ the diameter of the set $f(B(x_0, R))$ is less than...
The set $C_h(X)$ of all slowly oscillating bounded complex-valued functions on $X$ is a closed $C^*$-subalgebra of the $C^*$-algebra $C(X)$ of all bounded functions on $X$ and thus corresponds to a compactification $\overline{X}$ of $X$ which is called the Higson compactification of $X$ [8]. We know [8] that for a proper metric space $X$, the covering dimension of the Higson corona $\nu(X) := \overline{X} \setminus X$ does not exceed the asymptotic dimension of $X$. Also, see [6], $\dim(\nu(X)) = \asdim(X)$ if $X$ is a proper metric space of finite asymptotic dimension $\asdim(X)$. Our approach gives alternative proofs of those results.

In this paper we introduce the concept of large scale absolute extensor of a metric space. $K$ is a large scale absolute extensor of $X$ ($K \in \ls-AE(X)$) if for any subset $A$ of $X$ and any slowly oscillating function $f: A \to K$ there is a slowly oscillating extension $g: X \to K$. We characterize large scale absolute extensors of a space in terms of extensions of $(\epsilon, R)$-continuous functions. It turns out that being large scale absolute extensor of a space is a coarse invariant of the space. In the later part of the paper we find necessary and sufficient conditions for a sphere $S^m$ to be a large scale extensor of $X$. This is done by comparing existence of Lipschitz extensions in a finite range of Lipschitz constants to existence of Lebesgue refinements in a finite range of Lebesgue constants. We characterize asymptotic dimension of the space in terms of spheres being large scale absolute extensors of the space.

2. Basic concepts

2.1. The coarse category. Let us modify slightly the concept of a function being bornologous from [17].

**Definition 2.1.** Given $\alpha: (0, \infty) \to (0, \infty)$ and a function $f: (X, d_X) \to (Y, d_Y)$ of metric spaces we say $f$ is $\alpha$-Lipschitz if $d_Y(f(x), f(y)) \leq \alpha(d_X(x, y))$ for any $x, y \in X$.

$f: (X, d_X) \to (Y, d_Y)$ is bornologous if it is $\alpha$-Lipschitz for some $\alpha: (0, \infty) \to (0, \infty)$.

Notice $\lambda$-Lipschitz functions correspond to $\alpha$ being the dilation $\alpha(x) = \lambda \cdot x$ and $(\lambda, C)$-Lipschitz functions (or asymptotically Lipschitz functions) correspond to $\alpha(x) = \lambda \cdot x + C$.

We identify two bornologous functions $f: X \to Y$ that are within finite distance from each other and that leads to the coarse category (or the large scale category).

A function $f: X \to Y$ is a coarse embedding (or a large-scale embedding) if there are functions $\alpha: (0, \infty) \to (0, \infty)$ and $\beta: (0, \infty) \to (0, \infty)$ such that

$$\alpha(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq \beta(d_X(x, x'))$$

for all $x, x' \in X$.

A function $f: X \to Y$ is a coarse equivalence (or a coarse isomorphism) if it is a coarse embedding and there is a constant $D$ such that any point of $f(X)$ is $D$-close to a point of $Y$.

2.2. Continuity. The aim of this section is to generalize the concept of continuity so that they can be equally applied to the large scale and to the small scale. We are also interested in duality between the large scale category and the small scale category (the uniform category).
Definition 2.2. [5] A function $f : (X, d_X) \to (Y, d_Y)$ of metric spaces is $(\epsilon, \delta)$-continuous if $d_X(x, y) < \delta$ implies $d_Y(f(x), f(y)) < \epsilon$ for all $x, y \in X$.

Remark 2.3. The concept of $(\epsilon, \delta)$-continuity coincides with the concept of $f$ having $(\delta, \epsilon)$-variation (see [13]).

The following is a dualization of the standard definition of uniformly continuous functions:

Proposition 2.4. A function $f : (X, d_X) \to (Y, d_Y)$ of metric spaces is large scale continuous if and only if for every $\delta > 0$ there is $\epsilon > 0$ such that $f$ is $(\epsilon, \delta)$-continuous.

2.3. Slowly oscillating functions.

Proposition 2.5. Two metrics $d_X$ and $\rho_X$ on $X$ are large scale equivalent if and only if they have the same bounded sets and any function $f : X \to K$ that is slowly oscillating with respect to one metric is also slowly oscillating with respect to the other metric.

Proof. Assume $d_X$ and $\rho_X$ on $X$ have the same bounded sets. Notice $d_X$ and $\rho_X$ on $X$ are large scale equivalent if and only if any family of sets that is uniformly bounded with respect to one metric is also uniformly bounded with respect to the other metric. Use that to show that any function $f : X \to K$ that is slowly oscillating with respect to one metric is also slowly oscillating with respect to the other metric.

Assume any function $f : X \to K$ that is slowly oscillating with respect to one metric is also slowly oscillating with respect to the other metric. Suppose $d_X$ and $\rho_X$ are not large scale equivalent. Without loss of generality assume there is a sequence $\{(x_n, y_n)\}$ in $X \times X$ such that $\{d_X(x_n, y_n)\}$ is bounded, $\rho_X(x_n, y_n) \to \infty$ and $x_n \to \infty$ with respect to metric $d_X$. The function $f : X \to [0, 1]$ sending all $x_n$'s to 0 and sending all $y_n$'s to 1 is slowly oscillating with respect to $\rho_X$, so it can be extended to a slowly oscillating function $F : (X, \rho_X) \to [0, 1]$. Notice $F$ is not slowly oscillating with respect to $d_X$, a contradiction. ■

Proposition 2.6. Two metrics $d_X$ and $\rho_X$ on $X$ are uniformly equivalent if and only if any function $f : K \to X$ that is slowly oscillating with respect to one metric is also slowly oscillating with respect to the other metric.

Proof. If metrics $d_X$ and $\rho_X$ on $X$ are not uniformly equivalent, then there is $\epsilon > 0$ such that distances from $x_n$ to $y_n$ with respect to one metric are converging to 0 yet distances from $x_n$ to $y_n$ with respect to the other metric are all greater than $\epsilon$. Put $K = \{0, 1\} \times \{n^2\}_{n=1}^\infty$ and define $f : K \to X$ by $f(0, n^2) = x_n$, $f(1, n^2) = y_n$ for all $n \geq 1$. Notice $f$ is slowly oscillating with respect to one metric only. ■

Definition 2.7. Given a metric space $(K, d_K)$ its $M$-micro-version is $(K, d^M_K)$, where $d^M_K(x, y) = d_K(x, y)$ if $d_K(x, y) \leq M$ and $d^M_K(x, y) = M$ if $d_K(x, y) \geq M$.

Definition 2.8. Given a metric space $(X, d_X)$ its $M$-macro-version is $(X, d^M_X)$, where $d^M_X(x, y) = d_X(x, y)$ if $d_X(x, y) \geq M$ and $d^M_X(x, y) = 0$ if $0 < d_X(x, y) < M$.

We will use the following characterization of the Higson compactification from [13].
Proposition 2.9. Suppose that $X$ is a noncompact proper metric space. The Higson compactification $h(X)$ is the unique compactification of $X$ such that if $Y$ is any compact metric space and $f : X \to Y$ is continuous, then $f$ has a continuous extension to $h(X) = X \cup \nu(X)$ if and only if $f$ is slowly oscillating.

2.4. Concept related to covers. One can introduce the concept of dimension of a cover of a set.

Definition 2.10. If $\mathcal{U}$ is a family of subsets of a set $X$, then $\dim(\mathcal{U}) \leq n$ means that each $x \in X$ is contained in at most $(n + 1)$ elements of $\mathcal{U}$. Equivalently, the multiplicity $m(\mathcal{U})$ of $\mathcal{U}$ is at most $n + 1$.

Definition 2.11. The Lebesgue number $\text{Leb}(\mathcal{U})$ of a cover $\mathcal{U}$ of $X$ is the supremum of all $r \geq 0$ such that every $r$-ball $B(x,r)$ is contained in some element of $\mathcal{U}$.

If the Lebesgue number of $\mathcal{U}$ is at least $R$, we express it by saying $\mathcal{U}$ is $R$-Lebesgue.

Definition 2.12. The diameter $\text{diam}(\mathcal{U})$ of a family of sets in a metric space $X$ is the supremum of distances $d(x,y)$, where $x$ and $y$ belong to the same element of $\mathcal{U}$.

$\mathcal{U}$ is uniformly bounded (or $M$-bounded) if $\text{diam}(\mathcal{U}) < \infty$ (if $\text{diam}(\mathcal{U}) < M$).

3. Dualizing covering dimension

There are three major ways to define covering dimension for topological spaces $X$:

1. In terms of open covers of $X$;
2. In terms of pushing maps $f : X \to K$ (from $X$ to a CW complex $K$) into the $n$-skeleton $K^{(n)}$ of $K$;
3. In terms of extending maps $f : A \to S^n$ (from a closed subset $A$ of $X$ to the $n$-sphere) over the whole $X$.

One way of generalizing 1) to the coarse category was given by [9] (see below). 2) was generalized in [4] and [6] (see also [16]) has a generalization of 3) involving $R^{n+1}$ instead of $S^n$.

This paper is devoted to a different way of generalizing 3), one that uses extensions of slowly oscillating functions instead of continuous maps. It turns out that generalization is related to an alternative way of dualizing 1).

There are two ways of defining covering dimension of a space $X$ using covers:

a. any open cover of $X$ admits an open refinement of multiplicity at most $n + 1$,
b. any finite open cover of $X$ admits an open refinement of multiplicity at most $n + 1$.

Definition b) works well in case of normal spaces $X$ and is equivalent to $S^n$ being an absolute extensor of $X$. Definition a) works well in case of paracompact spaces $X$ and is equivalent to b) in that case.

We consider the following possible dualizations of the above two cases to the large scale category.

A. For every $r > 0$ there is $s > 0$ such that any cover $\mathcal{U}$ of $X$ of Lebesgue number at least $s$ admits a refinement $\mathcal{V}$ of Lebesgue number at least $r$ and multiplicity at most $n + 1$. 
B. For every $r > 0$ there is $s > 0$ such that any finite cover $\mathcal{U}$ of $X$ of Lebesgue number at least $s$ admits a refinement $\mathcal{V}$ of Lebesgue number at least $r$ and multiplicity at most $n + 1$.

To compare A) to Gromov’s original definition of asymptotic dimension, let us recall one of the many equivalent definitions of asymptotic dimension of a metric space:

**Definition 3.1.** [2] A metric space $X$ is of asymptotic dimension at most $n$ if and only if for every $\lambda > 0$ there exists a uniformly bounded cover of $X$ with Lebesgue number at least $\lambda$ and multiplicity at most $n + 1$.

The following proposition shows that A) makes a good definition for asymptotic dimension for any metric space.

**Proposition 3.2.** For a metric space $X$, asdim $X \leq n$ if and only for every $R > 0$ there is $S > 0$ such that any $S$-Lebesgue cover of $X$ admits a $R$-Lebesgue refinement of multiplicity at most $n + 1$.

**Proof.** If asdim $X \leq n$ then for any $R > 0$ using the definition 3.1 we get an uniformly bounded $R$-Lebesgue cover $\mathcal{U}$ of multiplicity at most $n + 1$. Any $S$-Lebesgue cover $\mathcal{V}$ with $S > \text{mesh}(\mathcal{U})$ will have $\mathcal{U}$ as a refinement. Conversely, for any $R > 0$ get a $S > 0$ using hypothesis. Then the cover of $X$ by $S$-balls around each point has a $R$-Lebesgue refinement of multiplicity $n + 1$.

We will see in a later section the closest we can get to definition B).

4. Large scale absolute extensors

**Definition 4.1.** A metric space $K$ is a large scale absolute extensor of a metric space $X$ (notation: $K \in \text{l-s-AE}(X)$) if for any subset $A$ of $X$ and any slowly oscillating function $f: A \to K$ there is an extension $g: X \to K$ of $f$ that is slowly oscillating.

**Proposition 4.2.** The real line $\mathbb{R}$ is not a large scale absolute extensor of itself.

**Proof.** Let $A$ be the subset of $\mathbb{R}$ consisting of squares of all integers. The inclusion $i: A \to \mathbb{R}$ is slowly oscillating as any sequence $(x_n, y_n) \in A \times A$ diverging to infinity such that $\{|x_n - y_n|\}$ is bounded must be on the diagonal of $A \times A$ starting from some $n$. Suppose $i$ extends to a slowly oscillating function $f: \mathbb{R} \to \mathbb{R}$. There is $M > 0$ such that $|f(n + 1) - f(n)| < \frac{1}{2}$ for $n > M$. Therefore,

$$(n + 1)^2 - n^2 = |f((n + 1)^2) - f(n^2)| \leq \sum_{i=n^2}^{(n+1)^2-1} |f(i + 1) - f(i)| < \frac{(n + 1)^2 - n^2}{2}$$

for $n > M$, a contradiction.

The following result shows that one may restrict attention to complete metric spaces $K$ when discussing large scale absolute extensors.

**Proposition 4.3.** If $L$ is dense in $K$, then the following conditions are equivalent for any metric space $X$:

a. $K$ is a large scale extensor of $X$,

b. $L$ is a large scale extensor of $X$. 

Proof. Suppose \( f: (X,A) \to (K,L) \). By an approximation \( g: X \to L \) of \( f \) at infinity we mean a function \( g \) such that \( g|A = f|A \) and \( d(f(x), g(x)) \to 0 \) as \( x \to \infty \). To construct \( g \) pick \( x_0 \in X \) and for each \( x \in X \setminus A \) pick \( g(x) \in L \) such that \( d_K(g(x), f(x)) < \frac{1}{1+d(x,x_0)} \).

a) \( \Rightarrow \) b). Suppose \( f: A \subset X \to L \) is slowly oscillating and choose an extension \( g: X \to K \) of \( f \) that is slowly oscillating. Choose an approximation \( h: X \to L \) of \( g \) at infinity such that \( h|A = g|A \). Notice \( h \) is a slowly oscillating extension of \( f \).

b) \( \Rightarrow \) a). Suppose \( f: A \subset X \to K \) is slowly oscillating and choose an approximation \( g: A \to L \) of \( f \) at infinity. Notice \( g \) is a slowly oscillating, so it has a slowly oscillating extension of \( h: X \to L \). Paste \( f \) and \( h|(X \setminus A) \) to obtain a slowly oscillating extension \( F: X \to K \) of \( f \). \( \blacksquare \)

**Corollary 4.4.** Being a large scale absolute extensor of a metric space \( X \) is an invariant in the uniform category. Being a compact large scale absolute extensor of a metric space \( X \) is an invariant in the topological category.

Our next result shows one can reduce investigation of large scale extensors to bounded metric spaces \( K \) and discrete metric spaces \( X \).

**Corollary 4.5.** Given metric spaces \( X \) and \( K \) the following conditions are equivalent:

a. \( K \) is a large scale extensor of \( X \).

b. Any micro-version of \( K \) is a large scale extensor of \( X \).

c. \( K \) is a large scale extensor of any macro-version of \( X \).

The following result gives a characterization of large scale absolute extensors in terms of the extensions of \((\cdot,\cdot)-continuous\) functions. Its importance lies in the fact that it relates \( K \) being a large scale extensor of \( X \) to the behavior of all bounded subsets of \( X \).

**Theorem 4.6.** The following conditions are equivalent:

a. \( K \in ls-AE(X) \),

b. For all \( M, \epsilon > 0 \) there is \( n, R, \delta > 0 \) such that for any bounded subset \( B \) of \( X \setminus B(x_0,n) \) any \((\delta, R)-continuous\) function \( f: A \subset B \to K \) extends to an \((\epsilon, M)-continuous\) function \( g: B \to K \).

c. For all \( M, \epsilon > 0 \) there is \( R, \delta > 0 \) such that for any bounded subset \( B \) of \( X \) any \((\delta, R)-continuous\) function \( f: A \subset B \to K \) extends to an \((\epsilon, M)-continuous\) function \( g: B \to K \).

d. For all \( M, \epsilon > 0 \) there is \( R, \delta > 0 \) such that any \((\delta, R)-continuous\) function \( f: A \subset X \to K \) extends to an \((\epsilon, M)-continuous\) function \( g: X \to K \).

**Proof.** a) \( \Rightarrow \) b). Suppose there is \( M, \epsilon > 0 \) with the property that for any choice of \( n, R, \delta > 0 \) there is a bounded subset \( B \) of \( X \setminus B(x_0,n) \) and an \((\delta, R)-continuous\) function \( f: A \subset B \to K \) but no extension \( g: B \to K \) of \( f \) is \((\epsilon, M)-continuous\).

By induction, as described below, choose a sequence of functions \( f_n: A_n \subset B_n \to K \) that are \((\frac{\delta}{n}, n)-continuous\), do not extend over \( B_n \) to an \((\epsilon, M)-continuous\) function, and the distance between any two points \( x \in B_i \), \( y \in B_j \) is at least \( i \) if \( i < j \).

First choose bounded subsets \( A_1 \subset B_1 \subset X \setminus B(x_0,1) \) and a \((1,1)-continuous\) function \( f_1: A_1 \subset B_1 \to K \) that does not extend over \( B_1 \) to an \((\epsilon, M)-continuous\) function.
function. Now suppose functions \( \{ f_i \}_1^\infty \) have been chosen satisfying the above conditions, and we want to create \( f_{n+1} \). Choose \( N \) such that \( B_n \subset B(x_0, N) \) and choose bounded subsets \( A_{n+1} \subset B_n+1 \subset X \setminus B(x_0, N+n) \) and a function \( f_{n+1}: A_{n+1} \subset B_n+1 \to K \) that is \( (\frac{1}{n+1}, n+1) \)-continuous and does not extend over \( B_n+1 \) to an \( (\epsilon, M) \)-continuous function.

Paste all \( f_n \) to \( f: A = \bigcup_{n=1}^\infty A_n \to K \) and notice \( f \) is slowly oscillating. Therefore it extends to a slowly oscillating \( g: X \to K \). Since each \( g|B_n \) is not \( (\epsilon, M) \)-continuous, there are points \( x_n, y_n \in B_n \) for each \( n \geq 1 \) such that \( d_X(x_n, y_n) \leq M \) but \( d_K(f(x_n), f(y_n)) > \epsilon \). That contradicts \( g \) being slowly oscillating.

b) \( \implies c) \). Suppose \( M, \epsilon > 0 \). Choose \( n, S, \mu > 0 \) such that for any bounded subset \( B \) of \( X \setminus B(x_0, n) \) any function \( f: A \subset B \to K \) that is \( (\mu, S) \)-continuous extends to an \( (\epsilon, M) \)-continuous function \( g: B \to K \).

Choose \( m, T, \lambda > 0 \) such that for any bounded subset \( B \) of \( X \setminus B(x_0, m) \) any function \( f: A \subset B \to K \) that is \( (\lambda, T) \)-continuous extends to a function \( g: B \to K \) that is \( (\mu, S) \)-continuous. We may increase \( T \) and \( S \), so assume \( T > S > M \).

Put \( \delta = \min(\mu/2, \lambda) \) and put \( R = m + 3T \). Assume \( f: A \subset B \to K \) is \( (\delta, 4R) \)-continuous and \( B \) is bounded.

Case 1: \( A \cap B(x_0, R) = \emptyset \). Extend \( f \) over \( A \cup B(x_0, m+2T) \setminus B(x_0, m) \to K \) by sending \( B(x_0, m + 2S) \setminus B(x_0, m) \) to a set of diameter 0. Notice any such extension \( f_1 \) is \( (\lambda, T) \)-continuous. Extend \( f_1 \) to \( h: B \cup B(x_0, m + 2T) \setminus B(x_0, m) \to K \) that is \( (\mu, S) \)-continuous. Extend \( h \) over \( B(x_0, m + 2T) \) by requiring that set is sent to a subset of diameter 0. Notice any such extension is \( (\mu, S) \)-continuous.

Case 2: There is a point \( x_1 \in A \cap B(x_0, R) \). Notice \( \text{diam}(f(A \cap B(x_0, R))) < \delta \) and let \( f_1: A_1 = A \cup B(x_0, m + 2S) \setminus A \subset \{ f(x_1) \} \). Observe \( f_1 \) is \( (\mu, S) \)-continuous. Indeed, the most relevant case is that of points \( y \in A \setminus B(x_0, m + 2S) \) and \( x \in B(x_0, m + 2S) \) such that \( d_X(x, y) \leq S \). In that case \( d_X(y, x_1) < R + m + 2S + S < 4R \), so \( d_K(f_1(y), f_1(x)) < 2\delta \leq \mu \). Extend \( f_1|A_1 \setminus B(x_0, m) \) to \( g_1: B \cup B(x_0, m + 2S) \setminus B(x_0, m) \to K \) so that \( g_1 \) has \( (M, \epsilon) \)-continuous. Pasting \( g_1 \) with \( f_1 \) gives an extension of \( f \) over \( B \) that is \( (\lambda, T) \)-continuous.

c) \( \implies d) \). Suppose \( M, \epsilon > 0 \). Choose \( S, \mu > 0 \) such that for any bounded subset \( B \) of \( X \) any function \( f: A \subset B \to K \) that is \( (\mu, S) \)-continuous extends to a function \( g: B \to K \) that is \( (\epsilon, M) \)-continuous.

Choose \( T, \lambda > 0 \) such that for any bounded subset \( B \) of \( X \) any function \( f: A \subset B \to K \) that is \( (\lambda, T) \)-continuous extends to a function \( g: B \to K \) that is \( (\mu, S) \)-continuous. We may increase \( T \), so assume \( T > S \).

Put \( \delta = \min(\mu/2, \lambda) \) and put \( R = 3T \). Assume \( f: A \to K \) is \( (\delta, 4R) \)-continuous. Put \( C_k = B(x_0, (2k + 2)R) \setminus B(x_0, (2k - 1)R) \) for \( k \geq 0 \). There is an extension \( g_k: C_k \to K \) of \( f|C_k \cap A \) that is \( (\mu, S) \)-continuous.

Paste \( g_k|B(x_0, (2k + 1)R) \setminus B(x_0, 2kR) \) with \( g_{k+1}|(B(x_0, (2k + 3)R) \setminus B(x_0, (2k + 2)R)) \) and with \( f|(B(x_0, (2k + 3)R) \setminus B(x_0, 2kR)) \cap A \) to obtain a function that is \( (\mu, S) \)-continuous, so it extends over \( B(x_0, (2k + 3)R) \setminus B(x_0, 2kR) \) to a function \( h_k \) that is \( (\epsilon, M) \)-continuous. Pasting all \( h_k \) together produces an extension of \( f \) that is \( (\epsilon, M) \)-continuous.

d) \( \implies a) \). Suppose there is a slowly oscillating function \( f: A \subset X \to K \). For every \( n \), there is \( S_n, \mu_n > 0 \) such that any function \( g: A \subset X \to K \) that is \( (\mu_n, S_n) \)-continuous extends to an \( (\frac{1}{n}, n) \)-continuous function \( \tilde{g}: X \to K \), and
there is $T_n, \lambda_n > 0$ such that any function $g : A \subset X \to K$ that is $(\lambda_n,T_n)$-continuous extends to a $(\mu_n,S_n)$-continuous function $\bar{g} : X \to K$. We can take $T_n > S_n > n$, and also $\{T_n\}, \{S_n\}$ to be increasing sequences. We create the extension of function $f$ in two steps. First, find $R_n > 0$ such that for $x, y \in A$ with $d(x_0, x) > R_n$ and $d(x,y) < T_n$ we have $d(f(x), f(y)) < \lambda_n$. There is an extension of $f_n : A \cap (B(x_0, R_{n+1}) \setminus B(x_0, R_n)) \to K$ to a $(\mu_n,S_n)$-continuous function $g_n : A \cap (B(x_0, R_{n+1}) \setminus B(x_0, R_n)) \cup (B(x_0, R_{n+1}+n) \setminus B(x_0, R_{n+1}-(n+1))) \to K$. This defines the first stage function $g$. In the second stage there is an extension of $g_n : A \cup (B(x_0, R_{n+1}+n) \setminus B(x_0, R_{n+1}-(n+1))) \cup (B(x_0, R_{n-1}+n-1) \setminus B(x_0, R_{n-1}-(n-1))) \to K$ to a function $h_n : A \cap B(x_0, R_{n+1}+n) \setminus B(x_0, R_{n-1}-n) \to K$ that is $(\frac{1}{n}, n)$-continuous. Finally pasting all the $h_n$ we get the desired slowly oscillating extension $h : X \to K$.

**Lemma 4.7.** Suppose $\text{diam}(K) \leq M$ and $g : Y \to X$ is $\alpha$-Lipschitz, $\alpha : (0, \infty) \to (0, \infty)$. If $f : X \to K$ is $(\delta, \delta)$-Lipschitz, then $f \circ g$ is $(\epsilon, \epsilon)$-Lipschitz provided $\epsilon < M$ and $\delta < \frac{\epsilon}{\alpha(\frac{M-\epsilon}{\epsilon})+1}$.

**Proof.** We need to show $d_K(f(g(x)), f(g(y))) \leq \epsilon \cdot d_Y(x, y) + \epsilon$ for all $x, y \in Y$. It is so if $\epsilon \cdot d_Y(x, y) + \epsilon \geq M$, so assume $\epsilon \cdot d_Y(x, y) + \epsilon < M$ or $d_Y(x, y) < \frac{M-\epsilon}{\epsilon}$. In this case $d_X(g(x), g(y)) \leq \alpha(\frac{M-\epsilon}{\epsilon})$ and $d_K(f(g(x)), f(g(y))) \leq \delta \cdot \alpha(\frac{M-\epsilon}{\epsilon}) + \delta \leq \delta \cdot (\alpha(\frac{M-\epsilon}{\epsilon}) + 1) \leq \epsilon \leq \epsilon \cdot d_Y(x, y) + \epsilon$.

**Corollary 4.8.** The following conditions are equivalent for a bounded metric space $K$:

a. $K \in \text{ls-AE}(X),$

b. For all $\epsilon > 0$ there is $\delta > 0$ such that for any subset $A$ of $X$ any $(\delta, \delta)$-Lipschitz function $f : A \to K$ extends to an $(\epsilon, \epsilon)$-Lipschitz function $g : X \to K$.

**Proof.** Assume $\text{diam}(K) < M$ and $M > 1$.

a) $\implies$ b). Given $1 > \epsilon > 0$ find $S, \mu > 0$ such that any $(\mu, S)$-continuous function $f : A \to K$ extends to $F : X \to K$ that is $(\epsilon, \frac{M-\epsilon}{\epsilon})$-continuous. Put $\delta = \frac{\mu}{S+1}$. If $f : A \to K$ is $(\delta, \delta)$-Lipschitz, then it is $(\mu, S)$-continuous as $d_X(x, y) \leq S$ implies $d_K(f(x), f(y)) \leq S \cdot \delta + \delta = \delta \cdot (S + 1) = \mu$. Pick an extension $g : X \to K$ of $f$ that is $(\epsilon, \frac{M-\epsilon}{\epsilon})$-continuous. If $d_X(x, y) > \frac{M-\epsilon}{\epsilon}$, then $d_K(g(x), g(y)) \leq M \leq \epsilon \cdot d_X(x, y) + \epsilon$.

b) $\implies$ a). Suppose $S, \epsilon > 0$ and $\mu = \frac{\epsilon}{S+1}$. Pick $1 > \delta > 0$ such that for any subset $A$ of $X$ any $(\delta, \delta)$-Lipschitz function $f : A \to K$ extends to an $(\mu, S)$-Lipschitz function $g : X \to K$. Every $(\delta, \frac{M-\delta}{\epsilon})$-continuous function $f : A \subset X \to K$ is $(\delta, \delta)$-Lipschitz, so it extends to an $(\mu, S)$-Lipschitz function $g : X \to K$. Notice $g$ is $(\epsilon, S)$-continuous. By [14], $K \in \text{ls-AE}(X)$.

**Corollary 4.9.** The unit interval $I = [0, 1]$ is a large scale absolute extensor of any metric space $X$.

**Proof.** Assume $X$ is $M$-discrete for some $M > 0$ and $f : A \subset X \to I$ is $(\delta, \delta)$-Lipschitz. Notice $f$ is $(\delta + \frac{\delta}{M})$-Lipschitz. By McShane Theorem (see [13]) if $f$ extends to $g : X \to I$ that is $(\delta + \frac{\delta}{M})$-Lipschitz. By choosing $\delta$ sufficiently small, we can accomplish $g$ to be $(\epsilon, \epsilon)$-Lipschitz.

**Corollary 4.10.** The half-open interval $[0, 1)$ and the open interval $(0, 1)$ are large scale absolute extensors of any metric space $X$. 

5. **Spheres as large scale extensors**

The purpose of this section is to find necessary and sufficient conditions for a sphere $S^m$ to be a large scale extensor of $X$. This is done by comparing existence of Lipschitz extensions to existence of Lebesgue refinements (see [5.1]).

Given a cover $\mathcal{U} = \{U_s\}_{s \in S}$ of a metric space $(X, d)$ there is a natural family of functions $\{f_s\}_{s \in S}$ associated to $\mathcal{U}$: $f_s(x) := \text{dist}(x, X \setminus U_s)$. If the multiplicity $m(\mathcal{U})$ is finite, then $\mathcal{U}$ has a natural partition of unity $\{\phi_s\}_{s \in S}$ associated to it:

$$\phi_s(x) = \frac{f_s(x)}{\sum_{t \in S} f_t(x)}.$$

That partition can be considered as a barycentric map $\phi : X \to \mathcal{N}(\mathcal{U})$ from $X$ to the nerve of $\mathcal{U}$. We consider that nerve with $l_1$-metric. Recall $\mathcal{N}(\mathcal{U})$ is a simplicial complex with vertices belonging to $\mathcal{U}$ and $\{U_1, \ldots, U_k\}$ is a simplex in $\mathcal{N}(\mathcal{U})$ if and only if $\bigcap_{i=1}^k U_i \neq \emptyset$.

Since each $f_s$ is 1-Lipschitz, $\sum_{t \in S} f_t(x)$ is $2m(\mathcal{U})$-Lipschitz and each $\phi_s$ is $\frac{2m(\mathcal{U})}{\text{Leb}(\mathcal{U})}$-Lipschitz (use the fact that $\text{barycentric map}$ of functions $\{S\}$ sphere the $\text{Lipschitz}$ extension to existence of Lebesgue refinements (see [5.1]).

The following will be important in relating large scale extensors to asymptotic dimension.

**Proposition 5.1.** Suppose $X$ is a metric space, $m \geq 0$. Then the following are equivalent.

a) For any $\epsilon > 0$ there is $\epsilon > \delta > 0$ such that any $(\delta, \delta)$-Lipschitz function $f : A \to S^m$, $A$ a subset of $X$, extends to an $(\epsilon, \epsilon)$-Lipschitz function $\tilde{f} : X \to S^m$.

b) For any $s > 0$ there is $t > s > 0$ such that for any finite $m + 2$-element cover $\mathcal{U} = \{U_0, \ldots, U_{m+1}\}$ of $X$ with Lebesgue number greater than $t$, there is a refinement $\mathcal{V}$ so that $\mathcal{V}$ has Lebesgue number greater than $s$ and the multiplicity of $\mathcal{V}$ is at most $m+1$.

**Proof.** By switching to a macro-version of $X$ we may assume $X$ is 1-discrete. a) $\implies$ b). Let $s > 0$, define $\epsilon = \frac{1}{2m(m+1)}$ and get corresponding $\delta$ from hypothesis. Define $t = \frac{4(m+2)^2}{s}$. Considering any $m + 2$-element $t$-Lebesgue cover $\mathcal{U}$, get a barycentric map $\phi : X \to \mathcal{N}(\mathcal{U}) = \Delta^{m+1}$ with $\text{Lip}(\phi) \leq \delta$. There is $g : X \to \partial \Delta^{m+1}$ such that $\text{Lip}(g) \leq 2\epsilon$ and $g(x) = \phi(x)$ for all $x \in X$ so that $\phi(x) \in \partial \Delta^{m+1}$. Consider $V_i = \{x \in X \mid g_i(x) > 0\}$. Notice $V = \{V_i\}_{i=0}^{m+1}$ is of multiplicity at most $m + 1$. Also $x \in V_i$ implies $x \in U_i$, so $\mathcal{V}$ refines $\mathcal{U}$. Given $x \in X$ there is $i$ such that $g_i(x) \geq \frac{1}{m+1}$. If $d(x, y) < s$, then $|g_i(x) - g_i(y)| < \frac{1}{m+1}$ and $g_i(y) > 0$. Thus, the ball at $x$ of radius $s$ is contained in one element of $\mathcal{V}$.

b) $\implies$ a). For the proof of this direction we think of maps from $X$ to an $(m+1)$-simplex $\Delta^{m+1}$ as a partition of unity. Since we want to create a map to its boundary
$S^m = \partial \Delta^{m+1}$, a geometrical tool is the radial projection $r$ which we splice in the form of $(1 - \beta) \cdot r + \beta \cdot \phi$ with a partition of unity $\phi$ coming from a covering of $X$ of multiplicity at most $m + 1$.

It follows from [14] that there exists $C > 0$ such that given a $\lambda$-Lipschitz $f : A \to \Delta^{m+1}$ one can extend it to a $C \cdot \lambda$-Lipschitz $g : X \to \Delta^{m+1}$.

Let $\epsilon > 0$. Define $\delta_1 = \frac{\epsilon}{(m+2)^2 \cdot 82C(m+2)}$ and choose $\delta_2 < \delta_1$ such that for any finite $m + 2$-element cover with Lebesgue number greater than $t = \frac{1}{\delta_1}$ there is a refinement with Lebesgue number greater than $s = \frac{1}{\delta_2}$ and multiplicity at most $m + 1$.

We define $\delta = \min\{\delta_1, \delta_2\}$ and show below that any $(\delta, \delta)$ Lipschitz map $f : A \to S^m$ extends to $(\epsilon, \epsilon) \tilde{f} : X \to S^m$.

We first extend $f$ to a $2\delta C$ Lipschitz $g : X \to \Delta^{m+1}$.

Let $\alpha : X \to [0, 1]$ be defined as $\alpha(x) = (m + 2) \cdot \min\{g_i(x) \mid 0 \leq i \leq m + 1\}$. Notice $\text{Lip}(\alpha) \leq (m + 2)2C \cdot \delta$. Let $\beta : [0, 1] \to [0, 1]$ be defined by $\beta(z) = 3z - 1$ on $[1/3, 2/3]$, $\beta(z) = 0$ for $z \leq 1/3$ and $\beta(z) = 1$ for $z \geq 2/3$ and note that $\text{Lip}(\beta) \leq 3$.

Put $U_i = \{x \in X \mid g_i(x) > \frac{\alpha(x)}{m+2} \text{ or } \alpha(x) > 2/3\}$ and notice $\text{Leb}(U) \geq r = \frac{1}{2 \delta C(m+2)}$ as follows:

Case 1: $x \in X$ and $\alpha(x) > 3/4$. Now, for any $y \in X$ with $d(x, y) < \frac{1}{4 \delta C(m+2)}$ one has $\alpha(x) - \alpha(y) \leq 1/12$, which implies that $\alpha(y) > 2/3$. Thus, in that case the ball $B(x, \frac{1}{4 \delta C(m+2)})$ is contained in all $U_i$.

Case 2: $\alpha(x) \leq 3/4$. There is $i$ so that $g_i(x) \geq \frac{1}{m+2}$. Since $\psi_i = g_i - \frac{\alpha(x)}{m+2}$ is $4\delta C$-Lipschitz, for any $y \in X$ satisfying $d(x, y) < \frac{1}{16 \delta C(m+2)}$ one has $\psi_i(x) - \psi_i(y) < \frac{1}{4(m+2)}$ and $\psi_i(y) > 0$ as $\psi_i(x) \geq \frac{1}{4(m+2)}$.

Shrink each $U_i$ to $V_i$ so that $m(V) \leq m + 1$ and $\text{Leb}(V) \geq s = \frac{1}{2}$. The barycentric map $\phi : X \to \partial \Delta^{m+1}$ corresponding to $V$ has $\text{Lip}(\phi) \leq 4(m+2)^2 \delta$.

Define $h(x) = \sum_{i=0}^{m+1} (g_i(x) - \frac{\alpha(x)}{m+2}) \cdot \frac{1-\beta(\alpha(x))}{1-\alpha(x)} \cdot e_i + \sum_{i=0}^{m+1} \beta(\alpha(x)) \cdot \phi_i(x) \cdot e_i$. To show $\text{Lip}(h) \leq \epsilon$ we will use the following observations.

(1) If $u, v : X \to [0, M]$, then $\text{Lip}(u \cdot v) \leq M \cdot (\text{Lip}(u) + \text{Lip}(v))$.

(2) In addition, if $v : X \to [k, M]$ and $k > 0$, then

$$\text{Lip}\left(\frac{v}{k}\right) \leq \frac{\text{Lip}(v)}{k^2}.$$  

(3) $v(x) = 1 - \alpha(x) \geq 1/3$ if $\frac{1-\beta(\alpha(x))}{1-\alpha(x)} > 0$.

Therefore $\text{Lip}\left(\sum_{i=0}^{m+1} \beta(\alpha(x)) \cdot \phi_i(x) \cdot e_i\right) \leq (m+2)(3\text{Lip}(\alpha) + \text{Lip}(\phi)) \leq (m+2)(3(m+2)2\delta C + 4(m+2)^2 \delta) \leq (m+2)(6C+4)\delta$.

Also, $\text{Lip}\left(\frac{1-\beta(\alpha(x))}{1-\alpha(x)}\right) \leq 9 \cdot 4 \cdot (m+2) \cdot 2\delta C$, so $\text{Lip}\left(\sum_{i=0}^{m+1} (g_i(x) - \frac{\alpha(x)}{m+2}) \cdot \frac{1-\beta(\alpha(x))}{1-\alpha(x)}\right) \leq (m+2) \cdot (4\delta C + 2(m+2)\delta C) \leq 76(m+2)^2 \delta C \leq 76(m+2)^2 \delta C$. So $\text{Lip}(h) \leq (m+2)^3(82C+4)\delta \leq \epsilon$.

It remains to show $h(X) \subset \partial \Delta^{m+1}$ and $h|A = f$. $h|A = f$ follows from the fact $\alpha(x) = 0$ if $x \in A$. It is clear $h(x) \in \partial \Delta^{m+1}$ if either $\beta(\alpha(x)) = 0$ or $\beta(\alpha(x)) = 1$, so assume $0 < \beta(\alpha(x)) < 1$. In that case $\phi_i(x) > 0$ implies $g_i(x) - \frac{\alpha(x)}{m+2} > 0$, so the
only possibility for \( h(x) \) to miss \( \partial \Delta^{m+1} \) is when \( g_i(x) - \frac{\alpha_i(x)}{m+2} > 0 \) for all \( i \) which is not possible.

From proposition 5.1 we get the following:

**Corollary 5.2.** If \( X \) is a metric space and \( m \geq 0 \), then the following conditions are equivalent:

a. \( S^m \) is a large scale absolute extensor of \( X \).

b. For any \( s > 0 \) there is \( t > 0 \) such that any finite \( m + 2 \)-element cover \( U = \{U_0, \ldots, U_{m+1}\} \) of \( X \) with \( \text{Leb}(U) > t \) admits a refinement \( V \) so that \( \text{Leb}(V) > s \) and the multiplicity of \( V \) is at most \( m + 1 \).

The next proposition and its corollary will be useful in the proof of 6.2.

**Proposition 5.3.** Suppose \( X \) is a metric space, \( n \geq 0 \). If for any \( s > 0 \) there is \( t > 0 \) such that every \( t \)-Lebesgue cover \( U = \{U_0, \ldots, U_{n+1}\} \) of \( X \) admits a \( s \)-Lebesgue refinement \( V \) satisfying \( m(V) \leq n + 1 \), then for any \( q > 0 \) there is \( r > 0 \) such that any \( r \)-Lebesgue cover \( W = \{W_0, \ldots, W_{n+2}\} \) of \( X \) admits a \( q \)-Lebesgue refinement \( V \) of multiplicity at most \( n + 2 \).

**Proof.** By switching to a macro-version of \( X \) we may assume \( X \) is 1-discrete. Let \( q > 0 \). By hypothesis, there is \( t > 0 \) such that any \( n + 2 \)-element \( t \)-Lebesgue cover \( U = \{U_0, \ldots, U_{n+1}\} \) of \( X \) admits a \( q \)-Lebesgue refinement \( V \) satisfying \( m(V) \leq n + 1 \).

First we show that any \( 2t \)-Lebesgue \( n + 2 \)-element cover \( U = \{U_i\}_{i=0}^{n+1} \) of \( A \subset X \) has a refinement \( V \) such that \( \text{Leb}(V) \geq q \) and \( m(V) \leq n + 1 \). Define \( U'_i = U_i \cup (X \setminus A) \) for \( i \leq n + 1 \) and notice that \( \text{Leb}(U'_i) \geq t \) as follows. If \( x \in X \), then \( B(x, t) \cap A \) is either empty or is contained in \( B(y, 2t) \) for some \( y \in A \). Since \( B(y, 2t) \cap A \subset U_i \) for some \( i \leq n + 1 \), \( B(y, 2t) \subset U'_i \) and hence \( B(x, t) \subset U'_i \). By hypothesis, there is a cover \( W \) of \( X \) such that \( W \) refines \( U' \), \( \text{Leb}(W) \geq q \), and \( m(W) \leq n + 1 \). By putting \( V = W|_A \) we get the required refinement.

Suppose \( W = \{W_0, \ldots, W_{n+2}\} \) is an \( r = 4t \)-Lebesgue cover of \( X \). Let \( A \) be the union of balls \( B(x, 2t) \) such that \( B(x, 4t) \) is not contained in \( W_{n+2} \). Define \( U_i = W_i \cap A \) for \( i \leq n + 1 \) and observe as follows that \( \text{Leb}(U_i) \geq 2t \) for \( U = \{U_i\}_{i=0}^{n+1} \) as a cover of \( A \). If \( x \in A \), then there is \( y \in X \) such that \( B(y, 4t) \) is not contained in \( W_{n+2} \) and \( x \in B(y, 2t) \). Therefore, \( B(y, 4t) \subset W_{i} \) for some \( i \leq n + 1 \) which means \( B(x, 2t) \cap A \subset B(y, 4t) \cap A \subset W_{i} \cap A = U_i \).

Shrink each \( U_i \) to \( V_i \) so that the intersection of all \( V_i \) is empty and \( \text{Leb}(V) \geq q \). Define \( W'_i = V_i \) for \( i \leq n + 1 \) and \( W'_{n+2} = W_{n+2} \). The cover \( W' \) is of multiplicity at most \( n + 2 \). We show as follows that \( \text{Leb}(W') \geq q \). If \( B(x, 4t) \subset W_{n+2} \), we are done. Otherwise \( B(x, 2t) \subset A \) and there is \( i \leq n + 1 \) such that \( B(x, q) \subset V_i \) in which case \( B(x, q) \subset W'_i \).

**Corollary 5.4.** Suppose \( X \) is a metric space and \( n \geq 0 \). If \( S^n \) is a large scale absolute extensor of \( X \), then so is \( S^{n+1} \).

6. LARGE SCALE ABSOLUTE EXTENDERS AND ASYMPTOTIC DIMENSION

The following lemma is a version of the Ostrand-type definition of asymptotic dimension, but with control on Lebesque number.
Lemma 6.1. A metric space \((X,d)\) is of asymptotic dimension at most \(n\) if and only if for every \(r > 0\) there exist a uniformly bounded covering \(U\) where \(U = \bigcup_{i=1}^{n+1} U^i\) with each \(U^i\) is an \(r\)-disjoint family and the Lebesgue number of \(U\) is at least \(r\).

Proposition 6.2. Suppose \(X\) is a metric space of finite asymptotic dimension. If \(n \geq 0\), the following conditions are equivalent:

a. \(S^n\) is a large scale absolute extensor of \(X\)

b. \(\text{asdim } X \leq n\)

Proof. b) \(\Rightarrow\) a) By taking a 1-net in \(X\) we can assume \(X\) is 1-discrete. Let \(s > 0\) and by hypothesis get a uniformly bounded cover \(V\) of multiplicity \(\leq n + 1\) with Lebesgue number larger than \(s\). If \(U\) be any cover of Lebesgue number \(> t = \text{mesh}(V)\). Then \(V\) is a refinement of \(U\).

a) \(\Rightarrow\) b) In view of 5.4 we can assume that \(\text{asdim } X \leq n + 1\). Let \(s > 0\). There is \(t > s > 0\) such that any finite \(n + 2\)-element cover of \(X\) with \(L(U) > t\) admits a refinement \(V\) so that \(L(V) > s\) and the multiplicity of \(V\) is at most \(n + 1\).

As \(\text{asdim } X \leq n + 1\), there exists an uniformly bounded covering \(U\) where \(U = \bigcup_{i=1}^{n+2} U^i\) where each \(U^i\) is an \(t\)-disjoint family and the Lebesgue number of \(U\) is at least \(t\).

Define \(U_i\) to be the union of all elements of \(U^i\) and note that \(\{U_i\}_{i=1}^{n+2}\) is a \(n+2\)-element \(t\)-Lebesgue cover. By hypothesis we can get a \(S\)-Lebesgue refinement \(V\) of multiplicity \(n + 1\).

For each \(V \in V\), there is an \(1 \leq i \leq n + 2\) such that \(V \subset U_i\). Replacing \(V\) by the collection \(\{V \cap W : W \in U^i\}\) we get a \(S\)-Lebesgue uniformly bounded cover of multiplicity \(n + 1\), which implies \(\text{asdim } X \leq n\).

In section 3 we looked for a characterization of asymptotic dimension similar to using finite covers. 6.2 and 5.2 immediately gives the following:

Corollary 6.3. For a metric space \(X\) of finite asymptotic dimension, \(\text{asdim } X \leq n\) if and only if for every \(R > 0\) there is \(S > 0\) such that any finite \(n + 2\)-element \(S\)-Lebesgue cover of \(X\) admits a \(R\)-Lebesgue refinement of multiplicity at most \(n + 1\).

In the case of covering dimension, "any finite \(n+2\)-element cover" can be replaced by "any finite cover". It is not clear if that is true for asymptotic dimension.

7. Connections to the Higson Corona

Theorem 7.1. If \(X\) is a proper metric space and \(M\) is a compact ANR, then the following conditions are equivalent:

a. any continuous function \(f : A \rightarrow M\), where \(A\) is a closed subset of \(\nu(X)\), extends to a continuous function \(F : \nu(X) \rightarrow M\)

b. \(M\) is a large scale absolute extensor of \(X\).

Proof. By considering a 1-net in \(X\), we can reduce the proof to a 1-discrete \(X\). a \(\Rightarrow\) b. Suppose \(f : A \subset X \rightarrow M\) is slowly oscillating. We will describe how to extend it to a continuous function \(F : \nu(X) \rightarrow M\). By the characterizing property of \(h(X)\) the restriction \(\tilde{f} = F|_X\) is slowly oscillating and is the desired extension of \(f\).
As $M$ is an ANR, we can extend $f$ to a $g : N \to M$, where $N$ is a closed neighborhood of $A$ in $h(X)$. By hypothesis, we can extend $g|\nu(X) \cap N$ to $G : \nu(X) \to M$, then over a neighborhood $U$ of $\nu(X)$ in agreement with $g$. Now we have to define the extension on $h(X) \setminus U$, which is a compact subset of $X$, hence finite, so we can put any values for the extension there. The resulting function $F : h(X) \to M$ is continuous.

Suppose $f : A \subset \nu(X) \to M$ is continuous. Since $M$ is an ANR, $f$ can be extended to $g : N \to M$ over a neighborhood $N$ of $A$ in $h(X)$.

$g|N \cap X \to M$ is slowly oscillating, so by hypothesis it extends to a slowly oscillating $\tilde{g} : X \to M$. By $g$ $\tilde{g}$ extends to a continuous $G : h(X) \to M$. As $X$ is dense in $h(X)$, $G$ must agree on $A$ with $f$, so $G|\nu(X)$ is the required extension of $f$ over $\nu(X)$. 

Corollary 7.2. If $X$ is a proper metric space and $n \geq 0$, then the following conditions are equivalent:

a. $\dim(\nu X) \leq n$,

b. $S^n$ is a large scale absolute extensor of $X$.

In view of [6,2] we have another proof of $\dim(\nu X) = \text{asdim}(X)$ in case of $\text{asdim}(X)$ being finite.

Corollary 7.3 ([8] and [6]). If $X$ is a proper metric space and $n \geq 0$, then the following conditions are equivalent:

a. $\dim(\nu X) \leq n$,

b. $\text{asdim}(X) \leq n$.

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