SCHUR MULTIPLIERS OF NILPOTENT LIE ALGEBRAS

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Abstract. We consider the Schur multipliers of finite dimensional nilpotent Lie algebras. If the algebra has dimension greater than one, then the Schur multiplier is non-zero. We give a direct proof of an upper bound for the dimension of the Schur multiplier as a function of class and the minimum number of generators of the algebra. We then compare this bound with another known bound.

Keywords: Schur multiplier

1. Introduction

The Lie algebra analogue of the Schur multiplier was investigated in the dissertations of Kay Moneyhun and Peggy Batten (see [8] and [2]). Among their results is that if $\dim L = n$, then $\dim M(L) \leq \frac{1}{2} n(n-1)$ where $M(L)$ is the Schur multiplier of the Lie algebra, $L$. A number of other results bounding $\dim M(L)$ have appeared (see [3], [5], [9], [10], and [13]). In this note several other bounds are provided. In particular, we show that if $L$ is nilpotent and $\dim L > 1$, then $\dim M(L) \neq 0$. This result can be contrasted with a result of Johnson in [6] which shows that a $p$--group with trivial multiplier has restrictions placed on it. We also find an upper bound for $\dim M(L)$ as a function of class and the number of generators for $L$. This result is similar to Theorem 3.2.5 of [7]. This Lie algebra result follows as a consequence of a general theory, provided in [10] but we give a short, direct proof and contrast this bound with one found in [5].

2. Preliminaries

Suppose that $L$ is generated by $n$ elements. Let $F$ be a free Lie algebra generated by $n$ elements and $L \cong F/R$. Since $R$ is an ideal in
F, R is also free. Witt’s formula from [1] gives us
\[ \dim \frac{F^d}{F^{d+1}} = \frac{1}{d} \sum_{m \mid d} \mu(m) n^{d/m} = \ln(d) \] (2.1)
where \( \mu \) is the Möbius function. Hence, \( F/F^t \) is finite dimensional and nilpotent for all \( t \).

Let \( N \) be an ideal in \( L \) and \( S \) be an ideal in \( F \) such that \( (S + R)/R \cong N \). Recall that \( M(L) = (F^2 \cap R)/[F, R] \) ([2]). Then \( M(L/N) \cong (F^2 \cap (S + R))/[F, S + R] \). It is routine to verify that there is a natural exact sequence
\[
0 \rightarrow \frac{R \cap [F, S]}{[F, R] \cap [F, S]} \rightarrow M(L) \rightarrow M(L/N) \rightarrow \frac{N \cap L^2}{[N, L]} \rightarrow 0. \quad (2.2)
\]

Note that covers and multipliers can be computed using the GAP program [4].

3. Trivial M(L)

It is shown in [6] that if \( G \) is a finite \( p \)-group with \( M(G) = e \), then severe restrictions are placed on \( G \). For further work in the problem see [11] and also [12] for a simpler proof. We will show that if \( L \) is a nilpotent Lie algebra with \( M(L) = 0 \) then \( \dim L \leq 1 \).

Let \( L \) be a nilpotent Lie algebra generated by \( n > 1 \) elements. Hence, \( \dim L/L^2 = n \). Let \( F \) be a free Lie algebra generated by \( n \) elements with \( L \cong F/R \). Suppose that \( L \) has class \( c \). Hence, \( F^{c+2} \subset F^{c+1} \subset R \) using the result in the last section. Furthermore, \( F/F^{c+2} \) is finite dimensional and nilpotent of class \( c + 1 \). Then,
\[ n = \dim L/L^2 = \dim \frac{F/R}{(F/R)^2} = \dim \frac{F}{F^2 + R} \leq \dim F/F^2 = n. \]

**Lemma 3.1.** If \( L \) is nilpotent and \( \dim L = n > 1 \), then \( R \subset F^2 \) and \( M(L) \cong R/[F, R] \).

**Theorem 3.2.** If \( L \) is a finite dimensional nilpotent Lie algebra of dimension greater than \( 1 \) and class \( c \), then \( M(L) \neq 0 \).

**Proof.** Continuing with the notation, \( L \cong F/R, F^{c+2} \subset F^{c+1} \subset R \), and \( F/F^{c+2} \) is nilpotent. Hence, \( [F, R] \subset R \) and \( M(L) \cong R/[F, R] \neq 0 \). \( \square \)

4. An Upper Bound for \( \dim M(L) \)

Suppose \( L \) has class \( c \geq 2 \). Let \( N = L^c \) and \( S = F^c \) in Eq. 2.2. Then \( L^c \cong (F^c + R)/R \) and \([F, S] = F^{c+1} \subset R \) since \( L^{c+1} = 0 \). Hence,
Eq. 2.2 becomes
\[ 0 \to \frac{F^{c+1}}{[F, R] \cap F^{c+1}} \overset{\sigma}{\to} M(L) \to M(L/L^c) \to L^c \to 0. \] (4.1)

**Theorem 4.1.** Let \( L \) be a nilpotent Lie algebra of class \( c \) which is generated by \( n \) elements. Then
\[ \dim M(L) \leq \sum_{j=1}^{c} l_n(j + 1) \]
where \( l_n(q) = \frac{1}{q} \sum_{s|q} \mu(s)n^{q/s} \).

**Proof.** Induct on \( c \). Let \( F \) be free of rank \( n \) where \( L \cong F/R \). If \( c = 1 \), then \( F/R \) is abelian, \( F^2 \subseteq R \) and \( M(L) = F^2/[F, R] \). Since \( F^3 \subseteq [F, R] \), \( \dim M(L) \leq \dim F^2/F^3 = l_n(2) \). Now, suppose that \( c > 1 \). By induction, \( \dim M(L/L^c) \leq t = \sum_{j=1}^{c-1} l_n(j+1) \). In Eq. 4.1 let \( A = \text{Im}(\sigma) \). Then \( M(L)/A \cong B \subseteq M(L/L^c) \). Hence, \( \dim M(L)/A \leq t \). But \( F^{c+1} \subseteq R \) and \( F^{c+2} \subseteq [F, R] \cap F^{c+1} \). Thus, \( A \) is the homomorphic image of \( F^{c+1}/F^{c+2} \). Therefore \( \dim A \leq l_n(c + 1) \) by Eq. 2.1 and \( \dim M(L) \leq t + l_n(c + 1) \) as desired. \( \square \)

We compare our result to the upper bound given in [5]:

**Theorem 4.2.** If \( L \) is a Lie algebra of dimension \( n \), then
\[ \dim M(L) \leq \frac{1}{2} n(n - 1) - \dim L^2. \]

We now examine the two theorems applied to different Lie algebras.

**Example 4.3.** Let \( F \) be a free Lie algebra on 2 generators and \( L = F/F^3 \). Then \( L \) is a Lie algebra of 2 generators and class 2. So, \( L \geq L^2 \supseteq L^3 = 0 \). Then, \( \dim L/L^2 = l_2(1) = 2 \), \( \dim L^2/L^3 = l_2(2) = \frac{1}{2} [\mu(1)2^2 + \mu(2)2] = \frac{1}{2} (4 - 2) = 1 \). Thus, \( \dim L = 3 \) and by Theorem 4.1 \( \dim M(L) \leq 2 \). By Theorem 4.1
\[ \dim M(L) \leq \sum_{j=1}^{2} l_2(j + 1) = l_2(2) + l_2(3) \]
\[ = 1 + \frac{1}{3} (\mu(1)2^3 + \mu(3)2) \]
\[ = 1 + \frac{1}{3} (6) = 3. \]

Thus, the result of Theorem 4.2 proves to be a better bound than the one obtained by our new theorem.
Example 4.4. Let $F$ be a free Lie algebra of 2 generators and $L = F/F^4$. Then $L$ is a Lie algebra of 2 generators and class 3. Thus, $\dim L = 5$ and by Hardy’s Theorem, $\dim M(L) \leq 7$. By LB’s Theorem,

$$\dim M(L) \leq \sum_{j=1}^{3} l_2(j + 1) = l_2(2) + l_2(3) + l_2(4)$$

$$= 1 + 2 + \frac{1}{4}[\mu(1)2^4 + \mu(2)2^2 + \mu(4)2]$$

$$= 3 + \frac{1}{4}(16 - 4)$$

$$= 3 + 3 = 6$$

Thus, we see that in this case, our theorem creates a better upper bound for $\dim M(L)$ than the previously known result.

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