Comparison of Model Order Reduction Methods for Optimal Sensor Placement for Thermo-Elastic Models *

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In this paper we consider an optimal sensor placement problem for a thermo-elastic solid body model. Temperature sensors are placed in a near-optimal way so that their measurements allow an accurate prediction of the thermally induced displacement of a point of interest (POI). Low-dimensional approximations of the transient thermal field are used which allow for efficient calculations. Four model order reduction (MOR) methods are applied and also compared with respect to the accuracy of the estimated POI displacement and the location of the sensors obtained.

**Keywords:** thermo-elastic model; optimal sensor placement; model order reduction

1. Introduction

This paper is concerned with an optimal sensor placement problem for thermo-elastic solid body models. The goal is to place temperature sensors in a near-optimal way so that their measurements allow an accurate prediction of the thermally induced deformation at a particular point of interest (POI) in real time. Our main motivation and example used throughout the paper is the prediction of the deflection of the tool center point (TCP) of a machine tool under thermal loading.

Sensor placement problems for partial differential equations (PDEs) are difficult optimization problems due to their large scale and the intricate structure of the objective, which normally involves the estimator covariance matrix of the quantity estimated, or its asymptotic inverse, the Fisher information matrix. In our approach we exploit the particular structure of thermo-elastic models. While the evolution of the temperature $T$ is relatively slow, the deformation $u$ is instantaneous. Moreover, we can neglect mechanically induced heat sources, so that the coupling is $T \mapsto u$ is uni-directional.

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In the optimal sensor placement problem, our objective is to choose sensor locations in such a way that the measurements contain the maximum amount of information with regard not to the reconstruction of the temperature field itself, but rather with regard to the subsequent prediction of the thermally induced displacement at a point of interest. In order to reduce the complexity of the placement problem we apply model order reduction (MOR) for the temperature field. In this paper we compare four model order reduction techniques, namely Proper Orthogonal Decomposition (POD), Balanced Truncation (BT) and two different Moment Matching (MM) approaches. The focus of this paper is the application of modern model order reduction methods to the sensor placement procedure. Furthermore the results of the sensor placement for these different MOR methods for the underlying temperature dynamics are compared.

Let us put our work into perspective. Model order reduction techniques are frequently used in sensor placement problems involving PDEs. We mention reaction-diffusion or convection-diffusion problems (Alonso, Frouzakis, and Kevrikidis (2004); Green (2006); Armaou and Demetriou (2006); Alonso et al. (2004); García et al. (2007)) as well as fluid dynamics applications (Mokhasi and Rempfer (2004); Cohen, Siegel, and McLaughlin (2006); Wilcox (2006); Yildirim, Chryssostomidis, and Karniadakis (2009)), where POD is often the method of choice. In another class of applications involving mechanical deformation of large structures (Kammer (1991); Yi, Li, and Gu (2011); Meo and Zumpano (2005); Sun and Büyüköztürk (2015)), modal analysis is employed as a reduction technique.

The references above fall into three categories concerning the actual placement strategy, viz. forward, backward and simultaneous placement procedures. In the forward sequential setting, which we also adopt, sensors are placed one by one such that one obtains a maximal gain of information for each sensor. In the backward approach, one begins with a set of all possible sensor locations and sequentially removes the ones which contribute the least amount of information. Both are greedy algorithms. Simultaneous placement procedures can only be used, due to the combinatorial complexity, when the number of potential locations is relatively small.

The goal in all of the above references is to reconstruct the respective field of the underlying PDE from a limited number of measurements. In our setting, this would correspond to recovering the entire temperature field from a number of temperature measurements. By contrast, the objective in this paper is to predict the mechanical displacement induced by that temperature field. This approach changes the metric w.r.t. which we measure the gain of information; see Körkel et al. (2008), Koevoets et al. (2007) and Herzog and Riedel (2015). In Koevoets et al. (2007), thermally induced displacements are estimated using temperature measurements as well. However, in contrast to our work, only a small number of potential sensor locations is considered, and modal analysis is used. In Herzog and Riedel (2015), we presented a sensor placement approach based on POD. One drawback of this simulation based MOR technique is its dependence on the thermal load cycle used to obtain the snapshots. The present paper is an extension of this work and we compare four different MOR techniques, three of which do not depend on simulations and are thus more widely applicable.

The paper is organized as follows. In the remainder of this section we state the thermoelastic model. The sequential placement strategy is described in detail in section 3. The model order reduction techniques under consideration are presented in section 4. Numerous numerical experiments are provided in section 5 which allow a comparison of the MOR techniques in the context of sensor placement problems.

We introduced a separate section describing the model.
2. Thermo-Elastic Forward Model

Before describing the sensor placement and model order reduction procedures, we first introduce the model problem to be considered. We denote the solid body whose thermo-mechanical displacement we consider by $\Omega$, and its surface by $\Gamma$. In our application $\Omega$ is the column of a machine tool shown in Figure 1(a). We consider the linear heat equation,

$$
\begin{aligned}
\rho_c \dot{T} - \text{div}(\lambda \nabla T) &= 0 \quad \text{in } \Omega \times (0, t_{\text{end}}), \\
\lambda \frac{\partial}{\partial n} T + \alpha(x)(T - T_{\text{ref}}) &= r(x, t) \quad \text{on } \Gamma \times (0, t_{\text{end}}), \\
T(\cdot, 0) &= T_0 \quad \text{in } \Omega.
\end{aligned}
$$

(1)

The boundary conditions represent a simple model for the free convection occurring at the surface $\Gamma$, and the heat transfer coefficient may vary across the surface. The right hand side $r(x, t)$ represents thermal loads, e.g., due to electric drives. All symbols occurring in (1) are summarized in Table 1.

Applying a standard linear Lagrangian finite element discretization in space, see e.g. Braess (2007); Grossmann, Roos, and Stynes (2007), the thermal boundary value problem (1) can be written in terms of a linear time-invariant (LTI) control system

$$
E_{\text{th}} \dot{T} = A_{\text{th}} T + B_{\text{th}} z,
$$

$$
T(\cdot, 0) = T_0.
$$

(2)

Here $T \in \mathbb{R}^{n_{\text{th}}}$ represents the temperature field in the nodal Lagrangian basis. The so-called system inputs $z \in \mathbb{R}^{m}$ represent all external influences to the system such as the temperatures $T_{\text{ref}}$ of the surrounding media and the thermal loads $r(x, t)$. The matrices $E_{\text{th}} \in \mathbb{R}^{n_{\text{th}} \times n_{\text{th}}}$, $A_{\text{th}} \in \mathbb{R}^{n_{\text{th}} \times n_{\text{th}}}$, and $B_{\text{th}} \in \mathbb{R}^{n_{\text{th}} \times m}$ denote the mass matrix, the stiffness matrix, and the input matrix, respectively.
The linear elasticity model is based on the balance of forces,

\[- \text{div} \sigma(\varepsilon(u), T) = \mathbf{f} \quad \text{in } \Omega. \quad (3)\]

We employ an additive split of the stress tensor $\sigma$ into its mechanically and thermally induced parts. Together with the usual homogeneous and isotropic stress-strain relation, we obtain the following constitutive law, see (Boley and Weiner 1960, Sec. 1.12), (Eslami et al. 2013, Sec. 2.8):

\[
\begin{align*}
\sigma(u, T) &= \sigma^{\text{el}}(\varepsilon(u)) + \sigma^{\text{th}}(T), \\
\sigma^{\text{el}}(\varepsilon(u)) &= \frac{E}{1+\nu} \varepsilon(u) + \frac{E\nu}{(1+\nu)(1-2\nu)} \text{trace}(\varepsilon(u)) I, \\
\sigma^{\text{th}}(T) &= -\frac{E}{1-2\nu} \beta (T-T_{\text{ref}}) I.
\end{align*}
\] (4)

Herein, $\varepsilon$ denotes the linearized strain tensor,

\[\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^\top),\]

and $I$ the $3 \times 3$ identity tensor. Moreover, $E$ and $\nu$ denote Young’s modulus and Poisson’s ratio; see Table 1.

| Symbol | Meaning                  | Value  | Units          |
|--------|--------------------------|--------|----------------|
| $T$    | temperature              |        | K              |
| $T_0$  | initial temperature      |        | K              |
| $r$    | thermal surface load     |        | W m$^{-2}$     |
| $\rho$ | density                  | 7 250  | kg m$^{-3}$    |
| $c_p$  | specific heat at constant pressure | 500  | J kg$^{-1}$ K$^{-1}$ |
| $\lambda$ | thermal conductivity   | 46.8   | W K$^{-1}$ m$^{-1}$ |
| $T_{\text{ref}}$ | ambient temperature | 20     | °C             |
| $\alpha$ | heat transfer coefficient | 0 to 12 | W K$^{-1}$ m$^{-2}$ |
| $\mathbf{u}$ | displacement         |        | m              |
| $\sigma$ | stress                  |        | N m$^{-2}$     |
| $\varepsilon$ | strain                  |        | 1              |
| $\nu$  | Poisson’s ratio          | 0.3    | 1              |
| $E$    | modulus of elasticity    | 114 \times 10^9 | N m$^{-2}$ |
| $\beta$ | thermal volumetric expansion coefficient | 1.1 \times 10^{-5} | K$^{-1}$ |

Table 1. Table of symbols occurring in the thermal (1) and the elasticity model (4).

In order to simplify the notation, we consider from now on the temperature relative to the constant $T_{\text{ref}}$, i.e., we write $T$ instead of $T - T_{\text{ref}}$. Using the same finite element discretization as for the thermal model (1) and noting that there are no further external forces in the system, i.e., $\mathbf{f} = 0$, system (3) becomes

\[A_{\text{el}} \mathbf{u} - A_{\text{th}} T = 0, \quad (5)\]

where $u \in \mathbb{R}^{n_{\text{el}}}$ is the coefficient vector representing the spatially discretized displacement field and the matrices $A_{\text{el}} \in \mathbb{R}^{n_{\text{el}} \times n_{\text{el}}}$ and $A_{\text{th}} \in \mathbb{R}^{n_{\text{el}} \times \text{rich}}$ denote the elasticity stiffness.
matrix and the thermo-elastic coupling matrix, respectively. Note that since we use the same piecewise linear finite element (FE) discretization for both the thermal and elasticity equations, we have $n_{el} = 3n_{th}$. Since we are interested in the thermally induced deformations of the machine column, we consider the coupling of the thermal model (2) with the elasticity equation (5) and obtain the coupled thermo-elastic system

$$
\begin{bmatrix}
E_{th} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\dot{T} \\
\dot{u}
\end{bmatrix} =
\begin{bmatrix}
A_{th} & 0 \\
A_{thel} - A_{el}
\end{bmatrix}
\begin{bmatrix}
T \\
u
\end{bmatrix} +
\begin{bmatrix}
B_{th} \\
0
\end{bmatrix} z,
$$

(6)

$$
\Leftrightarrow \quad E \dot{x} = Ax + Bz
$$

for the combined state $x = (T, u)$, which is of dimension $n = n_{th} + n_{el} = 4n_{th}$. Note that the coupled thermo-elastic model (6) and the thermal model (2) are basically of the same form. Moreover, the coupled system is a so-called differential algebraic equation (DAE) of index 1. However, since we are interested in determining the displacement $u$ at certain points of interest, specifically the tool center point in our application, we additionally define the output equation

$$
y = Cx,
$$

(7)

where $C \in \mathbb{R}^{q \times n}$ maps the state vector $x$ to the points of interest, the so-called system outputs $y$. Finally, the thermo-elastic model is described by the LTI control system (6)–(7). In the following section, we present the sensor placement procedure based on this model.

3. Sequentially Optimal Sensor Placement

With the sensor placement problem we seek to choose optimal positions of temperature sensors on the surface $\Gamma$. The goal is to reconstruct from these measurements the temperature field $T$ as an intermediate quantity, and subsequently to predict the displacement $u(x_{TCP})$ at the point of interest, i.e., the tool center point (TCP). As was mentioned in the introduction such placement problems are challenging for a number of reasons. In the following subsections, we reduce the complexity of the problem step by step so that it becomes tractable. The material in this section closely follows (Herzog and Riedel 2015, Sec. 2), where a sequential placement approach was proposed in the context of POD-reduced temperature dynamics. Major parts of this technique can be used in combination with other MOR methods as well. In the interest of keeping this paper self-contained, however, we briefly recall the main steps.

3.1 Displacement Estimation

Our first step to make the problem tractable is to apply model order reduction to the temperature state space. That is, we replace the temperature $T$ by an approximation

$$
T \approx V \hat{T},
$$

where $V \in \mathbb{R}^{n \times r}$ with $r \ll n$. The particular choice of the matrix $V$ depends on the model order reduction technique; see section 4. Exploiting the linearity of the elasticity
equation (5), we can express the elasticity field in terms of the reduced temperature \(\hat{T}\),
\[
\mathbf{u} = S \mathbf{T} \approx SV \hat{T}.
\]
Herein, \(S = A_{el}^{-1}A_{thel}\) is the solution operator for the elasticity equation. Then the displacement at the point of interest is
\[
\mathbf{u}(x_{TCP}) = C_{el} \mathbf{u} \approx C_{el} SV \hat{T} \in \mathbb{R}^3
\]
with an observation matrix \(C_{el} \in \mathbb{R}^{3 \times n_{el}}\) which is related to the output equation \(y = Cx\) in (7) by
\[
C = \begin{bmatrix}
C_{th} & 0 \\
0 & C_{el}
\end{bmatrix}
\]
with \(C_{th} \in \mathbb{R}^{q_{th} \times n_{th}}\). Since in the present application only the displacement of the TCP is observed, we have \(q_{th} = 0\) and \(C = [0, C_{el}]\) holds.

To estimate the TCP displacement (8) from temperature measurements, we first estimate the reduced temperature \(\hat{T}\). This estimation will be instantaneous, i.e., at any given moment in time we estimate the temperature field using only measurements from that time instance. Therefore we suppress the dependence on time in our notation. The estimation of the reduced temperature field is achieved by solving the least squares problem
\[
\text{Minimize} \quad \frac{1}{2} \sum_{i=1}^{n_{sensors}} \left( \sum_{j=1}^{r} \hat{\mathbf{T}}_j v_j(x_i) - \tilde{T}_i \right)^2.
\]

Here \(\tilde{T}_i\) denotes the \(i\)-th temperature measurement, acquired at location \(x_i\) on the surface \(\Gamma\), where \(i = 1, \ldots, n_{sensors}\). We restrict these locations to the vertices of the finite element mesh on the surface. Notice moreover that each row of the reduced basis matrix \(V\) corresponds to one nodal degree of freedom in the temperature finite element space. Hence we can identify the \(j\)-th column of \(V\) with a function \(v_j\). This justifies the notation \(v_j(x_i) = V_{ij}\).

Next let \(X = [x_1, \ldots, x_{n_{sensors}}]\) be the vector of all sensor positions selected. Then the solution of the linear least-squares problem (9) is given by the normal equation
\[
J(X)^\top J(X) \hat{T} = J(X)^\top \tilde{T}.
\]
Here \(J(X)\) denotes the Jacobian of the residuals w.r.t. the unknowns (the components of the reduced temperature \(\hat{T}\)), i.e.,
\[
J(X) = \begin{bmatrix}
v_1(x_1) & \cdots & v_r(x_1) \\
\vdots & \ddots & \vdots \\
v_1(x_{n_{sensors}}) & \cdots & v_r(x_{n_{sensors}})
\end{bmatrix} \in \mathbb{R}^{n_{sensors} \times r}.
\]
The vector of measurements is
\[
\tilde{T} = [\tilde{T}_1, \ldots, \tilde{T}_{n_{sensors}}]^\top.
\]
The solution of (10) is unique if \(J(X)\) has full rank, which we assume. To ensure this, it is necessary to have \(n_{sensors} \geq r\). The solution of (10) provides an estimator for the
reduced temperature,

\[ \theta \hat{\phi}(X) = (J(X)^\top J(X))^{-1} J(X)^\top \tilde{T}. \]

Using the relation (8), an estimator for the TCP displacement \( u(x_{TCP}) \) is given by

\[ \theta_{u(x_{TCP})}(X) = C_{el} SV \theta \hat{\phi}(X) = C_{el} SV (J(X)^\top J(X))^{-1} J(X)^\top \tilde{T}. \] (11)

3.2 Covariance of the Estimator

In order to assess the quality of the estimator we consider the covariance matrix of \( \theta_{u(x_{TCP})}(X) \). We assume the measurement errors to be i.i.d. random variables with normal distribution \( \mathcal{N}(0, \mu^2) \), i.e., with zero mean and the variance \( \mu^2 \). This means that the measurements \( \tilde{T} \) are normally distributed with associated covariance matrix

\[ \text{Cov}_{\tilde{T}} = \mu^2 I. \]

The estimator \( \theta \hat{\phi}(X) \) for the reduced temperature field is a linear transformation of the measurements \( \tilde{T} \), thus its covariance can be written as

\[ \text{Cov}_{\tilde{T}}(X) = (J(X)^\top J(X))^{-1} J(X)^\top \text{Cov}_{\tilde{T}} J(X)(J(X)^\top J(X))^{-1} \]
\[ = \mu^2 (J(X)^\top J(X))^{-1} \in \mathbb{R}^{r \times r}. \]

Analogously we obtain the covariance of the estimator \( \theta_{u(x_{TCP})}(X) \) for the TCP displacement from

\[ \text{Cov}_{u(x_{TCP})}(X) = \mu^2 C_{el} SV (J(X)^\top J(X))^{-1} (C_{el} SV)^\top \in \mathbb{R}^{3 \times 3}. \] (12)

Notice that the covariance \( \text{Cov}_{u(x_{TCP})}(X) \) does not depend on the current thermal state of the machine. However it does depend on the sensor positions selected (encoded in \( X \)) as well as on the choice of the reduced-order basis matrix \( V \).

3.3 Sensor Placement Strategy

The precision of a linear least-squares estimator can be inferred from the eigenvalues of its covariance matrix; see for instance (Fedorov and Hackl 1997, Ch. 1) or (Seber and Wild 2005, Ch. 3). Large eigenvalues of \( \text{Cov}_{u(x_{TCP})}(X) \) indicate a high sensitivity of \( u(x_{TCP}) \) w.r.t. perturbations in the temperature measurements. For this reason it is our goal to choose sensor positions \( X \) such that the covariance becomes small in a sense to be defined. This results in the optimal sensor placement problem

\[ \text{Minimize } \Psi(\text{Cov}_{u(x_{TCP})}(X)) \]
\[ \text{subject to } X = [x_1, \cdots, x_{n_{sensors}}] \subset \Gamma. \] (13)
Common optimality criteria used in experimental design include the following,

\[
\begin{align*}
\Psi_A(Cov) &= \text{trace}(Cov), \\
\Psi_D(Cov) &= \ln(\det(Cov)), \\
\Psi_E(Cov) &= \lambda_{\text{max}}(Cov),
\end{align*}
\] (14)

where \( \lambda_{\text{max}} \) denotes the maximal eigenvalue; see for instance Uciński (2005). Different optimality criteria may produce different solutions of (13). We can interpret the different criteria in terms of the uncertainty ellipsoid for the estimates \( u(x_{TCP}) \). While \( D \)-optimal designs minimize the ellipsoid’s volume, \( A \)-optimal designs minimize the mean squared lengths of its axes and \( E \)-optimal designs minimize the length of its largest axis. We will focus in the rest of the paper on the \( D \)-optimality criterion.

Due to the fact that problem (13) is in general hard to solve, we will reduce the complexity of the problem in the following. On the one hand, taking the constraints \( x_1, \ldots, x_{n_{\text{sensors}}} \in \Gamma \) literally would require a parametrization of a complicated surface such as the one depicted in Figure 1(b). As mentioned previously, we therefore restrict the potential sensor positions to the vertices to the FE nodes to overcome this difficulty. This amounts to the selection of a finite but possibly large set \( \Gamma_{\text{finite}} \subset \Gamma \) of feasible sensor locations.

Finding globally optimal solutions of (13) with the constraints \( x_1, \ldots, x_{n_{\text{sensors}}} \in \Gamma_{\text{finite}} \) would require the solution of a large combinatorial problem. While in principle this can be achieved for sizable problems using sophisticated branch-and-bound algorithms, see for instance Uciński and Patan (2007), we proceed differently here and replace the simultaneous placement by a sequential (greedy) placement of one sensor at a time. As observed in Herzog and Riedel (2015) this will lead to slightly suboptimal solutions but it makes the overall problem tractable. In addition, the sequential placement strategy allows us to determine the total number of sensors required for a desired target precision on the fly.

In view of the sequential placement approach we need to solve a series of subproblems of the form

\[
\min_{x_i \in \Gamma_{\text{finite}}} \Psi(Cov_{u(x_{TCP})}(x_1, \ldots, x_{i-1}; x_i)).
\] (15)

Here \( x_1, \ldots, x_{i-1} \) are the locations of the sensors already placed and \( x_i \) is the currently sought position of the \( i \)-th temperature sensor. Care needs to be taken here when \( i < r \), i.e., when the number of sensors is not yet sufficient to estimate the entire temperature field in the reduced basis, which has dimension \( r \). We follow here the approach proposed in Herzog and Riedel (2015) and restrict the estimation problem to the leading \( \min\{i, r\} \) components of the reduced basis. To this end we sort the columns of the reduced basis matrix \( V \) in decreasing order of significance, which is a natural side product of the model order reduction techniques under consideration.

Altogether, the approach just described amounts to using the restricted basis matrix

\[
V_i = [v_1, \ldots, v_{\min\{i, r\}}]
\]

and the restricted Jacobian

\[
J_i(x_i) =
\begin{pmatrix}
v_1(x_1) & \cdots & v_{\min\{i, r\}}(x_1) \\
\vdots & \ddots & \vdots \\
v_1(x_{i-1}) & \cdots & v_{\min\{i, r\}}(x_{i-1}) \\
v_1(x_i) & \cdots & v_{\min\{i, r\}}(x_i)
\end{pmatrix}
\]
where only the sensor location \( x_i \) associated with the last row is subject to optimization in (15). Finally, the covariance matrix appearing in the objective in (15) is given by
\[
\text{Cov}_{u(x_{TCP})}(x_1, \ldots, x_{i-1}; x_i) = \mu^2 C_{el} S V_i (J_i(x_i)^	op J_i(x_i))^{-1} (C_{el} S V_i)^	op,
\]
(16)
compare (12). We refer the reader to Herzog and Riedel (2015) for further details.

4. Model Order Reduction Techniques

Within the sensor placement procedure, the temperature field \( T \in \mathbb{R}^{n_{th}} \) of the machine column is replaced by a low-dimensional approximation \( \hat{T} \in \mathbb{R}^r \), \( r \ll n_{th} \). The transformation is described by the basis matrix \( V \in \mathbb{R}^{n_{th} \times r} \) such that \( T \approx VT \) holds in a sense to be made precise. In order to find an appropriate low-dimensional subspace, spanned by the columns of \( V \), for the approximation of the temperature field \( T \) we apply various MOR techniques to the state-space models (2) and (6)–(7), respectively.

A sophisticated overview of the following MOR methods can, e.g., be found in Antoulas (2005). In general, modern projection based MOR is concerned with the dimension reduction of dynamical systems of the form
\[
\dot{E} \hat{x} = A \hat{x} + B z, \\
y = C \hat{x}.
\]
(17)
That is, projection based MOR aims at finding projection matrices \( V, W \in \mathbb{R}^{n \times r} \) with \( r \ll n \) defining the reduced-order model (ROM)
\[
\dot{\hat{E}} \hat{x} = \hat{A} \hat{x} + \hat{B} z, \\
\hat{y} = \hat{C} \hat{x},
\]
(18)
where
\[
\hat{E} = W^\top E V \in \mathbb{R}^{r \times r}, \\
\hat{B} = W^\top B \in \mathbb{R}^{r \times m}, \\
\hat{A} = W^\top A V \in \mathbb{R}^{r \times r}, \\
\hat{C} = C V \in \mathbb{R}^{q \times r}.
\]
The requirement \( \hat{y} \approx y \) is interpreted differently for each MOR technique. In the following we give brief introductions to POD, BT, and two moment matching procedures, namely Padé approximation (Padé) and the iterative rational Krylov algorithm (IRKA). Note that the POD and Padé method simply use one-sided projections with \( V = W \). The MOR techniques considered also differ w.r.t. whether they are applied to the temperature equation (2) only, or to the thermo-mechanically coupled control system (6)–(7). The latter also contains information about the low-dimensional output \( y = C \hat{x} \), the TCP displacement that we are ultimately interested in.

4.1 Proper Orthogonal Decomposition

Proper orthogonal decomposition (POD) is a simulation based MOR technique for linear or nonlinear time-dependent problems. As in Herzog and Riedel (2015), we apply it only to the thermal system (2). System (2) is simulated using a typical loading cycle \( z(t) \) and snapshots \( T(t_1), \ldots, T(t_{n_{\text{snapshots}}}) \) are stored. Following standard POD procedure, see e.g.,
Kunisch and Volkwein (2001), we set up the Gramian matrix

\[ G = T^\top E_{\text{th}} T, \]

where \( T = [T(t_1), ..., T(t_{n_{\text{snapshot}}})] \) is the matrix of snapshots and \( E_{\text{th}} \) denotes the mass matrix representing the \( L_2(\Omega) \) inner product. Moreover, let \( D = \text{diag}(d_j) \) be the diagonal weight matrix with entries

\[ d_1^2 = \frac{\delta t_1}{2}, \quad d_j^2 = \frac{\delta t_j + \delta t_{j+1}}{2}, \quad d_{n_{\text{snapshot}}}^2 = \frac{\delta t_{n_{\text{snapshot}}}-1}{2}. \]

depending on the time steps \( \delta t_j = t_{j+1} - t_j \). Now let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} \geq \cdots \geq \lambda_{n_{\text{snapshot}}} \geq 0 \) be the eigenvalues of the generalized eigenvalue problem

\[ DGDx = \lambda E_{\text{th}} x \]

with associated eigenvectors \( \{v_i\}, i = 1, \ldots, n_{\text{snapshot}} \), which are orthonormal w.r.t. the \( E \)-inner product, i.e., \( v_i^\top E_{\text{th}} v_j = \delta_{ij} \) holds. Then the projection matrix \( V \) defining the POD MOR technique is chosen as

\[ V = [v_1, \ldots, v_r]. \]

It is well known that the eigenvalues of \( DGD \) typically decay exponentially for snapshots drawn from the heat equation. Therefore a few eigenvectors usually suffice. Commonly, the truncation threshold \( r \) is chosen such that the ratio of the energies contained in the bases of the reduced and full models is near 1, i.e.,

\[ \frac{\sum_{i=1}^r \lambda_i}{\sum_{i=1}^{n_{\text{snapshot}}} \lambda_i} \approx 1. \]

Clearly, one drawback of POD in the context of our problem is that the snapshots, and thus the reduced basis, cover only temperature states present in the set of snapshots. Temperature states obtained from a simulation with different inputs are not necessarily well represented in \( \text{span}(V) \).

### 4.2 Balanced Truncation

In contrast to POD, balanced truncation (BT) is based on the input/output behavior of the system under consideration and does not depend on simulations of the system. The question arises which system BT should be applied to, since an output equation is always required. Normally, BT relies on the dimension of both the input as well as the output to be significantly smaller than the state dimension; see however Benner and Schneider (2013) for recent advances to overcome this limitation. Since the output of the temperature equation (2) is defined by the temperature sensors, whose locations we are seeking, we cannot directly apply BT solely to the temperature equation with all possible sensor locations as output. However we can use the thermally induced displacement at the TCP (8) as the output of (2). Thus, instead of the full thermo-mechanical model (6)–(7), we consider the equivalent control system

\[ E_{\text{th}}T = A_{\text{th}} T + B_{\text{th}} z, \]

\[ y = C_{el} A_{\text{el}}^{-1} A_{\text{th}} T =: \tilde{C} T \]

\[ \text{(19)} \]
describing the thermo-elastic behavior, which is of the same structure as (6)–(7), but of dimension $n_{th}$ instead of $n = 4n_{th}$. Note that the information of the elasticity model is entirely contained in the modified output matrix $\tilde{C}$.

The idea of balanced truncation model order reduction is to identify those states that require the least energy to be controlled, and at the same time yield the most energy through observation; see Enns (1984); Moore (1981). These states are going to be preserved within the low-dimensional approximation $\tilde{T}$. The states which are difficult to excite and/or difficult to observe are neglected.

The determination of the projection subspaces, represented by $V$ and $W$, is based on the controllability and observability of the underlying control system (19). That is, in order to identify the dominant states, the controllability and observability Gramians $P$ and $Q$ have to be computed. In practice, the Gramians can be found as the solutions of the generalized controllability and observability algebraic Lyapunov equations (ALEs)

$$A_{th}P_{th}E_{th}^\top + E_{th}P_{th}A_{th}^\top + B_{th}B_{th}^\top = 0,$$

$$A_{th}^\top Q_{th}E_{th} + E_{th}^\top Q_{th}A_{th} + C^\top C = 0,$$

respectively. As in the case of POD, $E_{th}$ represents the $L_2(\Omega)$ inner product. Then $P = Q = \text{diag}(\lambda_1, \ldots, \lambda_n)$ is referred to as a balanced realization, where the $\lambda_i$ are the so-called Hankel singular values (HSVs). Such a realization can be obtained by the application of a certain state transformation. Given such a balanced realization, the HSVs reveal the dominant states of the system and can be computed as the positive square roots of the eigenvalues of the product $P_{th}E_{th}^\top Q_{th}$. Note that in general, the solutions $P$ and $Q$ of the ALEs (20) are neither equal nor in diagonal form. Still, computing the projection matrices $V$, $W$ by using, e.g., the square-root method (SRM), see Glover (1984); Laub et al. (1987), the balancing transformation is carried out implicitly. Often Cholesky-like factorizations of the Gramians $P$, $Q$ are computed within the SRM. For large-scale problems, i.e., models with large state dimension as is the case under consideration here, low-rank approximations of the Gramians are recommended to use in order to efficiently solve the large-scale ALEs (20); see Benner, Kürschner, and Saak (2013a,b, 2014/15); Kürschner (2016). That is, the Gramians are computed in the form $P = RR^\top$ and $Q = SS^\top$ with rank($R$), rank($S$) $\ll n$. Then, a singular value decomposition (SVD)

$$S_{th}^\top E_{th} R = X \Lambda Y^\top = [X_1, X_2] \text{diag}(\Lambda_1, \Lambda_2)[Y_1, Y_2]^\top$$

with decreasingly ordered singular values $\lambda_1, \ldots, \lambda_n$, reveals the $r \ll n$ dominant singular values $\lambda_1 \geq \cdots \geq \lambda_r > 0$ contained in the leading block matrix $\Lambda_1 \in \mathbb{R}^{r \times r}$. Given the SVD, the projection matrices $V$, $W$ are computed in the form

$$V = RY_1 \Lambda_1^{-\frac{1}{2}}, \quad W = SX_1 \Lambda_1^{-\frac{1}{2}}.$$

The most important advantages of BT are the guaranteed preservation of stability of the dynamical system and the easily computable error bound

$$\frac{\|y - \tilde{y}\|_2}{\|u\|_2} \leq 2 \sum_{j=r+1}^{n} \lambda_j$$

with $\lambda_j$, $j = r + 1, \ldots, n$, denoting the truncated singular values contained in $\Lambda_2$, and the $L_2$-norm $\|u\|_2^2 = \int_0^\infty u(t)^\top u(t) \, dt$. 

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It should be emphasized that in contrast to the POD approach, the elasticity equation plays a role in the MOR process through the output equation (8). Moreover, although only the matrix $V$ is needed for the sensor placement procedure, the BT method essentially needs to compute both projection matrices $V$ and $W$.

### 4.3 Moment Matching

In the context of model order reduction by moment matching (MM), the starting point is the transfer function (TF)

$$G(s) = \hat{C}(sE_{th} - A_{th})^{-1}B_{th}$$  \hspace{1cm} (21)

of the thermo-mechanical system (19). Note that we again use the system formulation (19) of dimension $n_{th}$. The TF (21) represents the input-output mapping of the underlying linear dynamical system, evaluated in frequency domain. In other words, the input-output behavior w.r.t. a certain excitation frequency $s$ is described. Under the assumption that $s_0 \in \mathbb{C}$ is not an eigenvalue of the matrix pencil $(E, A)$, the TF (21) can be written as

$$G(s) = \hat{C}(sE_{th} - A_{th})^{-1}B_{th} = \hat{C}(sE_{th} - s_0E_{th} + s_0E_{th} - A_{th})^{-1}B_{th}$$

$$= \hat{C} \left[ (s_0E_{th} - A_{th})((s - s_0)(s_0E_{th} - A_{th})^{-1}E_{th} + I) \right]^{-1}B_{th}$$

$$= \hat{C} \left[ I - (-1)(s - s_0)(s_0E_{th} - A_{th})^{-1}E \right]^{-1}(s_0E_{th} - A_{th})^{-1}B_{th}.$$  

If $s$ is sufficiently close to $s_0$, the inverse $[\cdot]^{-1}$ can in fact be treated as a Neumann series and therefore one obtains

$$G(s) = \sum_{j=0}^{\infty} \hat{C}(-sE_{th} - A_{th})^{-1}E_{th} \right) \sum_{j=0}^{\infty} m_j (s - s_0)^j,$$

where

$$m_j = \hat{C}(-s_0E_{th} - A_{th})^{-1}E_{th} \right) \sum_{j=0}^{\infty} m_j (s - s_0)^j,$$

are said to be the moments of the transfer function around $s_0$.

Moment matching MOR methods aim at finding a ROM (18) such that a certain number of moments

$$\hat{m}_j = \hat{C}(-s_0E - \hat{A})^{-1}\hat{E} \right) \sum_{j=0}^{\infty} \hat{m}_j (s - s_0)^j$$

of the TF $\hat{G}(s)$, associated with the reduced-order model (18) match the moments $m_j$ of the original TF $G(s)$. For large-scale problems the application of Krylov subspace based methods has proven to be very effective. Computing the projection matrix $\hat{V}$ in such a way that

$$\text{span}(V) = K_p \left((s_0E_{th} - A_{th})^{-1}E_{th}, \ (s_0E_{th} - A_{th})^{-1}B_{th}\right)$$  \hspace{1cm} (23)

holds, it turns out that the first $p$ moments are matched. The same holds for $W$ with

$$\text{span}(W) = K_p \left((s_0E_{th} - A_{th})^{-T}E_{th}^\top, \ (s_0E_{th} - A_{th})^{-T}\hat{C}^\top\right).$$  \hspace{1cm} (24)
Here, \( \mathcal{K}_\ell \) is a Krylov subspace defined as

\[
\mathcal{K}_\ell(Y, Z) = \text{span}\left(Z, YZ, Y^2Z, \ldots, Y^{\ell-1}Z\right).
\]

If both conditions, (23) and (24), are fulfilled, matching of \( 2p \) moments is achieved. For the choice \( s_0 = 0 \), the resulting problem is known as Padé approximation. In case \( s_0 = \infty \) the moments are called Markov parameters. In the general case \( 0 < s_0 < \infty \) the problem is widely known as a rational interpolation problem. The computation of the matrices \( V, W \) can be easily achieved by Arnoldi or Lanczos methods; see e.g., (Saad 2003, Ch. 6). A detailed description of MOR by moment matching, using Krylov subspace methods, can be found, e.g., in (Antoulas 2005, Ch. 11).

In our application, the heat equation is characterized by a rather slow evolution. Therefore, the Padé approach, approximating the system related TF at the frequency \( s_0 = 0 \), is a natural choice for a first representative of the moment matching procedures.

In addition, we will investigate the application of the iterative rational Krylov algorithm (IRKA); see Gugercin, Antoulas, and Beattie (2008). The latter describes the rational interpolation using multiple prescribed expansion points \( s_i, i = 1, 2, \ldots, r \). Note that the number of expansion points coincides with the dimension of the reduced-order model being generated by IRKA. Moreover, the expansion points are automatically determined in an \( \mathcal{H}_2 \) optimal sense. The IRKA is developed in order to find a Hermite interpolant of the TF \( G(\cdot) \) such that the first two moments are going to be matched at every expansion point. To summarize, we are going to compare moment matching approaches based on matching multiple moments (\( > 2 \)) at a single expansion point \( s_0 = 0 \) (Padé), and matching the first two moments at several expansion points (IRKA).

Note that the optimal sensor placement strategy only requires the projection matrix \( V \). Therefore, using the Padé approximation, it suffices to compute the input Krylov subspace \( \text{span}(V) \) based on the system and input matrices \( A_{th} \) and \( B_{th} \), respectively, given by the thermal model equation (2). In contrast, similar to BT, the IRKA based approach additionally requires the specification of an output matrix in order to automatically compute optimal expansion points in the \( \mathcal{H}_2 \) sense and thus, the system (19) has to be considered. For a detailed derivation and explanation of the procedure we refer to Gugercin, Antoulas, and Beattie (2008).

5. Numerical Results and Comparison

5.1 Description of Problem Data

The following numerical experiments are based on the geometry of the prototypical Auerbach ACW 630 machine column shown in Figure 1(b). The material constants are given in Table 1. Notice that the heat transfer coefficient varies over different parts of the boundary, which are classified according to the orientation of the outer surface normal; see Figure 2(a). The machine column experiences the influence of two heat sources, see again Figure 2(a). One source originates from an electric drive mounted on the top of the machine column (surface part \( \Gamma_{r1} \)) while the other source represents the spindle driving the horizontal movement of the column (\( \Gamma_{r2} \)).

The location of the heat sources enters the matrix \( B_{th} \) in (2) and (6) and thus it has an impact on all reduced-order models. Note that the number of inputs is \( m = 3 \), because the ambient temperature \( T_{ref} \) is considered as an input in order to achieve a linear (as opposed to affine) control system (2). For the generation of the snapshots needed for the simulation based POD model order reduction technique we have to specify a typical
thermal load cycle in addition to the initial temperature. In (1) we used

\[
r(x, t) = \begin{cases} 
6700 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r1} \text{ and } 0 \leq t \leq 1800 \text{ s}, \\
2700 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r2} \text{ and } 0 \leq t \leq 1800 \text{ s}, \\
2700 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r2} \text{ and } 3600 \leq t \leq 5400 \text{ s}, \\
6700 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r1} \text{ and } 5400 < t \leq 7200 \text{ s}, \\
0 & \text{else,}
\end{cases}
\]

which represents a realistic machining scenario. This heat source gives rise to a time-dependent control input \( z(t) \) in the semi-discretized models (2) and (6). The simulation of the temperature equation (2) was done in a time interval \( t \in [0 \text{ s}, 9000 \text{ s}] \) with uniform step length \( \delta t = 60 \text{ s} \) using the implicit Euler scheme. The initial temperature for these simulations was choosen as \( T_0 \equiv T_{\text{ref}} \).

Our quantity of interest \( u(x_{TCP}) \) in the estimation problem on which the sensor placement problem is based is the displacement at the so-called tool center point. As TCP we use the tip of the main spindle (holding the tool) seen in the left of Figure 1(a). We consider the main spindle assembly as a rigid body which is thermally insulated from the machine column. Consequently, the TCP displacement is determined by the four mounting points of the sledge holding the main spindle, see Figure 2(b). The dependence of \( u(x_{TCP}) \) on the deflection in the four mounting points is actually nonlinear, but we consider its linearization; see Herzog and Riedel (2015) for details.

5.2 Comparison of Model Order Reduction Techniques for Sensor Placement

In this section we present and compare the results of the sensor placement procedure described in section 3, using the model order reduction techniques under consideration. We recall that we consider POD, BT as well as two variants of MM (Padé and IRKA), as described in section 4. In each case, we obtained the respective reduced basis \( V \) and then carried out the sequential sensor placement procedure as described in subsection 3.3, which amounts to solving a sequence of problems (15). The size of the reduced-order model was \( r = 20 \) and we placed \( n_{\text{sensors}} = 30 \) sensors in each case.

Figure 3 shows the sensor locations obtained. In the case of POD most of the sensors
Figure 3. Optimal sensor locations for the different MOR methods.

are placed near to the heat sources, see Figure 3(a). The reason for this phenomenon is that the projection subspace associated with the basis matrix \( V \) is based solely on simulations of the thermal model (2). In particular, the relative intensities and locations of the heat sources will strongly influence the computed subspace. Due to the properties of the heat equation and the heat loss through the boundary the simulation snapshots are almost zero at a certain distance from the heat sources. Therefore the POD basis also shares this property and thus the information gain of temperature measurements far away from the heat sources is negligible.

By contrast, the projection matrices \( V \) computed from the BT and the two MM procedures are set up independent of the thermal loads and initial condition. Therefore the final sensor locations are spread out more evenly across the machine column’s surface, see Figure 3(b) to Figure 3(d), to account for all possible heat source trajectories in the image space of \( B_{th} \). Despite its smaller physical extension, it can be observed that the heat source located at the lower part of the machine column attracts a significant amount of sensors. This is quite intuitive since even small deformations at the base of the machine column will naturally be transferred up along the height of the column and thus may have a sizable influence on the displacement of the TCP.

Recall that BT and IRKA are applied to the reformulation (19) of the full thermo-elastic model (6)–(7) and thus the displacement information about the quantity of interest (QOI) is contained in the ROM. In contrast, the POD and Padé procedures solely consider the thermal model (2).

Figure 4 shows the values of the objective function (15) for the different test cases as a function of the number of sensors placed. Smaller values of the objective function indicate less influence of measurement errors on the TCP displacement estimation and thus better estimation precision. Moreover, Figure 4(a)–Figure 4(d) shows the influence of the size \( r \) of the reduced-order model. For the Padé method the number \( p \) of moments to be matched was chosen as \( p \in \{5, 6, 7\} \). Based on the number of inputs \( m = 3 \) and the relation \( r = pm \) this results in reduced dimensions \( r \in \{15, 18, 21\} \) for the Padé based estimations. Regarding the evolution of the objective function, the qualitative behavior is very similar for all MOR methods. That is, the best performance in the sense of the experimental design criterion (14) is achieved for small model sizes. This result was to be expected since smaller models lead to fewer coefficients to be estimated. On the other hand, smaller models lead to larger approximation errors of the ROM, compared to the full state model. This is not accounted for in the experimental design criterion. Depending on the needs of the application or the user, a good balance between the two goals has to be found. Figure 4(e) shows a comparison of the different MOR methods for a fixed
Figure 4. Values of the optimal experimental design objective $\Psi(\text{Cov})$ for each MOR technique in dependence on the ROM size $r$ and the number of sensors placed.

Figure 5. Comparison of the exact (simulated by the full model) TCP displacement with the estimates (11) with simulated measurements at the respective measurement locations (left) with heat sources (26). Relative errors are shown in the right plot.

dimension ($r = 20$) of the reduced systems. Here, the POD based approach shows best performance with respect to the optimization objective.

Notice that the sequence of optimal sensor placement problems (15) operates under the assumption that only temperature states in the range of the respective basis $V$ can occur. To achieve a more meaningful comparison of the four MOR variants we created measurements from a simulation of the full model. The thermal loads in this simulation differ from those in (25), which were used to create the POD snapshots. The simulation
spans the time interval \( t \in [0 \ s, 7200 \ s] \) and we used the thermal loads

\[
r(x, t) = \begin{cases} 
6700 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r1} \text{ and } 0 \ s \leq t \leq 2400 \ s, \\
2700 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r2} \text{ and } 0 \ s \leq t \leq 2400 \ s, \\
2700 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r2} \text{ and } 2400 \ s \leq t \leq 4800 \ s, \\
6700 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r1} \text{ and } 4800 \ s < t \leq 7200 \ s, \\
0 & \text{else}
\end{cases}
\] \quad (26)

and an initial temperature of \( T_0 \equiv T_{\text{ref}} \). Figure 5(a) and Figure 5(b) show the evolution of the simulated TCP deflection, compared to its estimated position based on the \( n_{\text{sensors}} = 30 \) optimally placed sensors (in the sense of the approach in subsection 3.3) with reduced bases of dimension \( r = 20 \). For each reduced order model the simulated temperature values were evaluated at the relevant sensor locations. The TCP displacement estimate was then obtained from solving the least-squares problem (11).

Again, according to the relative errors between simulated and estimated TCP displacements, the POD approach yields the best results in this exercise. The estimation of the TCP displacement associated to the subspace spanned by the Padé procedure shows significant inaccuracies at times \( t = 120 \ s \) and \( t = 4800 \ s \). This may be due to the fact that the matching of \( p = 7 \) moments of the transfer function at a single expansion point \( s_0 = 0 \) and a resulting reduced dimension \( r = 21 \), restricted to \( r = 20 \), cannot sufficiently approximate the actual model behavior of the original full-order model at those points. Apart from these peaks, the BT, Padé and IRKA based estimates are roughly of the same order of accuracy. Note that the large relative errors at the beginning of the time interval are mainly caused by the fact that the trajectories evolve closely around zero and therefore the relative error is violated by divisions of values that are close to zero. However, it is a well known fact that the reduced-order models based on POD are in general only reliable for operating conditions near those used to generate the POD snapshots. We therefore repeat the experiment with thermal loads

\[
r_{\text{mod}}(x, t) = \begin{cases} 
3000 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r1} \text{ and } 0 \ s \leq t \leq 2400 \ s, \\
5000 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r2} \text{ and } 0 \ s \leq t \leq 2400 \ s, \\
5000 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r2} \text{ and } 2400 \ s \leq t \leq 4800 \ s, \\
3000 \text{ W m}^{-2} & \text{if } x \in \Gamma_{r1} \text{ and } 4800 \ s < t \leq 7200 \ s, \\
0 & \text{else}
\end{cases}
\] \quad (27)

which differ more significantly from the nominal loads (25) than (26). Figure 6(a) and Figure 6(b) show the estimation quality in this case. Here, we observe that the change of the intensity of the heat sources does not considerably influence the estimations compared to the previous scenario. Only the POD approach performs slightly worse and produces now similar errors than the other MOR methods.

In a final experiment the robustness of the TCP displacement estimation with respect to noisy measurements using a standard deviation of \( \sigma = 0.1 \) is analyzed in Figure 7(a) and Figure 7(b). The heat sources were chosen as in (26). The TCP evolution trajectories and the corresponding relative errors, again, reveal the superiority of the POD approach applied to the optimal experimental design framework, while the other strategies are of the same order of accuracy as before.
5.3 Discussion

In this paper we revisited an optimal sequential placement strategy for temperature sensors in order to predict thermally induced mechanical deformations from temperature measurements. To make the sensor placement procedure tractable, we applied model order reduction either to the temperature equation alone, or to the full thermo-mechanical model. The main focus of this paper was to compare the performance of different MOR methods w.r.t. the sensor placement objective and the prediction quality of the induced displacement estimation using simulated measurements at the optimized sensor positions.
Comparing the simulation results being based on the data used as the POD training scenario and a fixed model size of $r = 20$, the POD approach showed the best performance w.r.t. the experimental design objective as well as the estimation accuracy of the TCP displacement. Considering thermal loads apart from the POD training set, as well as for noisy measurement data, the POD, BT, and IRKA based MOR approaches basically show the same behavior w.r.t. the estimation accuracy. POD, as a simulation based MOR approach, depends on the snapshot ensemble to be sufficiently rich to yield a reliable ROM. Nevertheless, we found POD to work well even under significant perturbations of the heating profile used during the training phase, as compared to the other MOR schemes. Interestingly, POD shows best performance w.r.t. to the prescribed reduced order/number of sensors and the comparison criteria even though it targets only the temperature field and no information about the true QOI (the TCP displacement in our case) is included. POD especially takes into account the explicit influence of the actual acting loads and, according to that, clusters the sensors corresponding to the regions of dominant thermal interest.

The three other MOR methods which were considered do not depend on simulations. Moreover, the BT and IRKA approaches operate on an equivalent reformulation of the full thermo-mechanical model and thus include QOI information. On the other hand, these methods do not consider the actual machining process. Therefore, they are less specialized such that the corresponding ROMs need to be significantly larger in order to achieve comparable performance. Still, for drastically changed machining processes, e.g. with no action on the lower heat source, they are expected to give much better results than the POD without modification, i.e., regeneration of the POD basis by new training with the changed input situation. However, quantification of the degeneration of the POD model requires further investigation. Out of these methods, the BT and the IRKA moment matching approaches performed best, and very similarly.

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