Aspects of the competition between atom-field and field-environment couplings under the influence of an external source in dispersive Jaynes-Cummings model

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We give a fully analytical description of the dynamics of an atom dispersively coupled to a field mode in a dissipative environment fed by an external source. The competition between the unitary atom-field (which leads to entanglement) and the dissipative field-environment couplings are investigated in detail. We find the time evolution of the global atom-field system for any initial state and we show that atom-field steady state is at most classically correlated. For an initial state chosen, we evaluate the purity loss of the global system and of atomic and field subsystems as a function of time. We find that the source will tend to compensate for the dissipation of the field intensity and to accelerate decoherence of the global and atomic states. Moreover, we show that the degree of entanglement of the atom-field system, for the particular initial state chosen, can be completely quantified by concurrence. Analytical expression for time evolution of the concurrence is given.

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I. INTRODUCTION

Quantum mechanics of open systems has lived a revival of interest especially after the early days of quantum computation when it has been realized that maintaining quantum coherence is an essential ingredient to fully exploit the new possibilities opened by the applications of quantum mechanics in computational physics. Devices using unique quantum mechanical features can perform information processing in a much more efficient way than those at work nowadays. The key ingredients of quantum computing devices with computational capabilities that superseded these classical counterpart are basically:

1. the linear structure of their state space;
2. the unitary character of their dynamical evolution and
3. the tensorized form of multiparticle states.

The last one, in particular, represents a striking departure from classicality due to entanglement, since combining different systems results in an exponential growth of the available coding space; moreover, the tensor product structure is the very basis of many efficient quantum manipulations.

Of course, all this holds just for closed quantum systems. Real world quantum systems interact with their environment to a greater or lesser extent. No matter how weak the coupling to such an environment, the evolution of quantum subsystems is eventually affected by non-unitary features such as decoherence, dissipation and heating. From a mathematical point of view, the relevant state space, given by density matrices, has now a convex structure and the allowed quantum dynamics is described by completely positive (CP) maps. Initial pure states preparation are typically corrupted on extremely short time scales due to quantum coherence loss that turns them into mixed states: the initial information irreversibly leaks out from the system into the very large number of uncontrollable degrees of freedom of the environment.

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The above mentioned limitations are of course common to the many devices proposed for quantum computation \cite{1,2}. In the present contribution we will restrict ourselves to a detailed study of cavity quantum electrodynamics (cavity QED) \cite{3}. In this case one typically has a superconducting cavity with a coherent state fed into it. The interesting dynamics for quantum computation or for studying basic features of quantum mechanics is related to the (unitary) atom-field coupling. It will produce several typical quantum features such as entanglement and superpositions of states \cite{4}. However, it is impossible to avoid that from the moment a coherent field is fed into the cavity, it will couple to the environment and the effects of the latter will tend to destroy all the typically quantum mechanical features due to the unitary coupling. One of the aims of the present paper is to investigate the relation between the time scales of these two competing processes. Another question which, to our knowledge, has not been satisfactorily answered so far is the following: the atom-field coupling depends crucially on the presence of the field in the cavity. If one is working at zero temperature it is clear that asymptotically the field will go to its vacuum state, rendering the unitary atom-field entanglement impossible quite independently of the environment. The basic question is: can one circumvent this problem by adding an external source which will be present during the whole process? How does it affect all other time scales of the process?

Besides the fundamental issues addressed by the present work, the results obtained here can also have purely practical purposes. In fact, recently de Oliveira, Moussa and Mizrahi \cite{5} proposed a control mechanism of a mesoscopic superposition of states (“Schrödinger cat” state) in a dissipative cavity by the coupling with an external source. It is a variant of the scheme for creation and monitoring of coherent superpositions of field states described in \cite{3}. In Ref. \cite{6}, Gerry studied the interaction of an atom with both a quantized cavity field and an external classical field. The interaction between atom and cavity field has a dispersive character. As a result, various forms of superpositions of cavity field states can be produced. Further, robust coherent states may be generated in the steady state of the cavity if dissipation is included. The present work adds to those contributions in the sense that the source is treated quantum mechanically and dissipation is taken into account while atom and field interact. We show that the presence of an external source will attenuate dissipative effects, since it will maintain a constant field intensity in the cavity. However, since decoherence strongly depends on the field intensity, the presence of an external source will tend to increase typical decoherence time scales.

Moreover, a rather new aspect treated here is the quantification of quantum correlations in this dissipative dynamics. It is not difficult to find measures of the degree of entanglement between subsystems of a global system whose state is pure. The situation dramatically changes, however, if the global state is characterized by a statistical mixture. For two qubits systems it is possible to provide for a quantitative measure of entanglement using the notion of concurrence \cite{8}. For a particular initial state, we show that the atom-field system can be mapped onto a system composed by two two-dimensional subsystems. We therefore give analytical expressions for the time evolution of the entanglement between atom and field measured by the concurrence. To our knowledge, it is the first time that the time evolution of the concurrence is obtained for a dissipative system.

This article was organized as follows: in Section II we find the time development of the atom-field state for any initial condition. In Section III, we calculate the time evolution of an initial state chosen and we obtain the idempotency defect or linear entropy of global and reduced density operators in order to study the purity loss of the compound system and of the atomic and field subsystems. We obtain correlation measures of the global state. In particular, the degree of entanglement between atom and field is evaluated by finding an expression for concurrence. Finally, the appendix describes the method used to obtain expressions for the solutions of different equations of motion that appear in Section II.

II. THE DYNAMICS OF THE SYSTEM ATOM-FIELD IN THE PRESENCE OF A RESONANT EXTERNAL SOURCE

Before handling the complete problem of the dispersive atom-field interaction in a dissipative environment with an external source, let us consider the time evolution of a single driven electromagnetic field mode coupled to a thermal reservoir at null temperature. The evolution of the field state, described by the density operator $\hat{\rho}_f$ in the interaction picture, is governed by the master equation

$$\frac{d}{dt} \hat{\rho}_f (t) = -i \left[ F e^{-i(\omega_S - \omega_0)t} a^\dagger + F^* e^{i(\omega_S - \omega_0)t} a, \hat{\rho}_f (t) \right]$$

$$+ k \left[ 2 \hat{a} \hat{\rho}_f (t) \hat{a}^\dagger - \hat{a}^\dagger \hat{a} \hat{\rho}_f (t) - \hat{\rho}_f (t) \hat{a}^\dagger \hat{a} \right],$$

where $k$ is the damping constant and $\omega_0$ and $\omega_S$ are the mode and source frequencies, respectively. The contribution of the external source is identified by the commutator in the right hand side of Eq. (1): the coupling between source
and field is given by the constant $F$. Without loss of generality, let us assume that the source is resonant with the mode, i.e., $\omega_S = \omega_0$. Hence, the master equation can be written as

$$\frac{d}{dt}\hat{\rho}_f(t) = (\mathcal{L}_S + \mathcal{D})\hat{\rho}_f(t),$$

where we have defined

$$\mathcal{L}_S \equiv -i \left[ F\hat{a}^\dagger + F^*\hat{a}, \cdot \right] = -i F \left[ (\hat{a}^\dagger \cdot) - (\cdot \hat{a}^\dagger) \right] - i F^* \left[ (\hat{a}) - (\cdot \hat{a}) \right]$$

and

$$\mathcal{D} \equiv k (2\mathcal{J} - \mathcal{M} - \mathcal{P}).$$

The superoperators in Eq. (3) are $\mathcal{M} \cdot \equiv \hat{a}\hat{a}^\dagger \cdot$, $\mathcal{P} \cdot \equiv \hat{a}\hat{a}^\dagger \cdot$, $\mathcal{J} \cdot \equiv \hat{a} \cdot \hat{a}^\dagger$. The term $\mathcal{D}$ is called “dissipator” and contains the non-unitary contributions to the dynamics of $\hat{\rho}_f$. The formal solution of the equation (3) is given by the expression

$$\hat{\rho}_f(t) = e^{(\mathcal{L}_S+\mathcal{D})t}\hat{\rho}_f(0),$$

where $\hat{\rho}_f(0)$ represents the field state at $t = 0$. The solution (5) can be rewritten as

$$\hat{\rho}_f(t) = \hat{D} \left[ -i F \frac{F}{k} (1 - e^{-kt}) \right] \exp \left( \mathcal{D}t \right) \hat{D}^\dagger \left[ -i F \frac{F}{k} (1 - e^{-kt}) \right].$$

Here, $\hat{D}$ is the displacement operator of the Heisenberg-Weyl group, defined as

$$\hat{D} (\alpha) = \exp \left( \alpha\hat{a}^\dagger - \alpha^*\hat{a} \right) = \exp \left( -\frac{|\alpha|^2}{2} \right) e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}} = \exp \left( -\frac{|\alpha|^2}{2} \right) e^{-\alpha^*\hat{a}} e^{\alpha\hat{a}^\dagger}.$$

Note that the coherent state

$$\hat{\rho}^{\text{stat}}_f = | -i F/k \rangle \langle -i F/k |$$

is the stationary state of the dynamics described by the master equation (3). In fact, it is easy to show that $(\mathcal{L}_S + \mathcal{D}) \hat{\rho}^{\text{stat}}_f = 0$. Besides, since $\lim_{t \to \infty} \exp (\mathcal{D}t) \hat{\rho}_f (0) = | 0 \rangle \langle 0 |$, whichever it may be the initial state $\hat{\rho}_f (0)$, then $\hat{\rho}_f (t \to \infty) = \hat{\rho}^{\text{stat}}_f$. In the other words, in the limit $t \to \infty$, the field state converges to the state $\hat{\rho}^{\text{stat}}_f$. Therefore, as a result of the coupling between the field mode and an external source, a stationary coherent state is produced.

### A. Solution of the equations of motion

From now on, we will consider the field interacting dispersively with a two level atom and coupled to a zero temperature reservoir and to an external source. In the interaction picture, the evolution of the compound atom-field system, described by the density operator $\hat{\rho}$, is governed by the master equation

$$\frac{d}{dt}\hat{\rho}(t) = -i \omega \left[ (\hat{a}^\dagger \hat{a} + 1) | e \rangle \langle e | - \hat{a}^\dagger \hat{a} | g \rangle \langle g | , \hat{\rho}(t) \right]$$

$$-i \left[ F\hat{a}^\dagger + F^*\hat{a}, \hat{\rho}(t) \right]$$

$$+ k \left[ 2\hat{a}\hat{\rho}(t) \hat{a}^\dagger - \hat{a}^\dagger \hat{a}\hat{\rho}(t) - \hat{\rho}(t) \hat{a}^\dagger \hat{a} \right].$$

$e$ and $g$ are the atomic levels of interest and $\omega \equiv G^2/\delta$, where $G$ measures the coupling between atom and field and $\delta$ is the difference between the frequency of the atomic transition and the frequency of the mode (detuning). In order that the dispersive approximation remains valid in the presence of the source, the condition $|\delta|/G \gg |F|/k$ must be satisfied.

Equations of motion for the operators $\hat{\rho}_{ee}(t) = | e \rangle \langle e | \hat{\rho}(t) | e \rangle$, $\hat{\rho}_{gg}(t) = | g \rangle \langle g | \hat{\rho}(t) | g \rangle$ and $\hat{\rho}_{ge}(t) = \hat{\rho}_{eg}(t) = | g \rangle \langle e | \hat{\rho}(t) | e \rangle$ are obtained from the master equation (3). These equations have the general form

$$\frac{d}{dt}\hat{\rho}_{ij}(t) = \mathcal{L}_{ij}' \hat{\rho}_{ij}(t),$$
The arguments of the above displacement operators are

\[ L_{ij} = \mathcal{L}_{ij} + \mathcal{L}_S, \]

where \( \mathcal{L}_{ij} \) represents one of the superoperators

\[
\begin{align*}
L_{gg} &\equiv L_+ = i\omega (\mathcal{M} - \mathcal{P}) + k (2\mathcal{J} - \mathcal{M} - \mathcal{P}), \\
L_{ee} &\equiv L_- = -i\omega (\mathcal{M} - \mathcal{P}) + k (2\mathcal{J} - \mathcal{M} - \mathcal{P}), \\
L_{eg} &\equiv -i\omega (\mathcal{M} + \mathcal{P} + 1) + k (2\mathcal{J} - \mathcal{M} - \mathcal{P}),
\end{align*}
\]

and \( \mathcal{L}_S \) was defined in (3).

The formal solution of (10) is

\[ \hat{\rho}_{ij}(t) = e^{L_{ij} t} \hat{\rho}_{ij}(0), \]

where \( \hat{\rho}_{ij}(0) \) represents the operator \( \hat{\rho}_{ij} \) at \( t = 0 \). In the appendix we describe in details the method employed to obtain the expressions for the “matrix elements” of \( \hat{\rho} \) that we will show next. The “diagonal elements”, \( \hat{\rho}_{ee}(t) \) and \( \hat{\rho}_{gg}(t) \), can be expressed as

\[
\begin{align*}
\hat{\rho}_{ee}(t) &= \hat{D} \{ \beta_e(t) \} \exp (L_+ t) \hat{D}^\dagger \{ \beta_e(0) \}, \\
\hat{\rho}_{gg}(t) &= \hat{D} \{ \beta_g(t) \} \exp (L_- t) \hat{D}^\dagger \{ \beta_g(0) \}.
\end{align*}
\]

The arguments of the above displacement operators are

\[
\begin{align*}
\beta_e(t) &= \frac{F}{\omega - i k} \left[ e^{-(k+i\omega)t} - 1 \right], \\
\beta_g(t) &= -\frac{F}{\omega + i k} \left[ e^{-(k-i\omega)t} - 1 \right].
\end{align*}
\]

The expressions (14a) and (14b) are completely general. For a given initial state, the technique presented in the appendix can be used to determine the operators \( \exp (L_+ t) \hat{\rho}_{ee}(0) \) and \( \exp (L_- t) \hat{\rho}_{gg}(0) \).

The general solution of the equation of motion for the “non-diagonal element” \( \hat{\rho}_{eg} = \hat{\rho}_{ge}^\dagger \) can be written as

\[ \hat{\rho}_{eg}(t) = \exp \left\{ z(t) + |F|^2 \left( p^2 - q^2 + 2pq + |p + q|^2 \right) t \right\} \hat{D} \{ \beta_e(t) \} \]

\[
\times \exp \left\{ -2F \left[ \text{Re} p(t) - i \text{Im} q(t) \right] a \right\} \exp (L_+ t) \hat{D}^\dagger \{ \beta_g(0) \} \]

\[
\times \exp \left\{ -2F \left[ \text{Re} p(t) - i \text{Im} q(t) \right] a^\dagger \right\} \hat{D}^\dagger \{ \beta_g(t) \}.
\]

The functions \( z(t) \), \( p(t) \) and \( q(t) \) in the above expression are given by

\[
\begin{align*}
z(t) &= -\frac{2i\omega |F|^2}{(k + i\omega)^2} \left\{ t + \frac{4 \left[ e^{-(k+i\omega)t} - 1 \right] - e^{-2(k+i\omega)t} + 1}{2(k + i\omega)} \right. \\
&\left. + i \frac{\omega}{(k + i\omega)^2} \left[ \cosh [(k + i\omega)t] - 1 \right]^2 \right\}, \\
p(t) &= i \frac{k}{(k + i\omega)^2} \left\{ \cosh [(k + i\omega)t] - 1 \right\} - i \frac{\sinh [(k + i\omega)t]}{k + i\omega}, \\
q(t) &= -\frac{\omega}{(k + i\omega)^3} \left\{ \cosh [(k + i\omega)t] - 1 \right\}.
\end{align*}
\]

The analysis of the argument of the first exponential in RHS of Eq. (16) provides some understanding of the asymptotic behavior of the term \( \hat{\rho}_{eg} \). The functions defined in (17a)-(17c) produce

\[
\begin{align*}
z(t) + |F|^2 (\cdots) (t) &= -\frac{2i\omega |F|^2}{(k + i\omega)^2} \left\{ t + \frac{4 \left[ e^{-(k+i\omega)t} - 1 \right] - e^{-2(k+i\omega)t} + 1}{2(k + i\omega)} \right. \\
&\left. - \frac{|F|^2}{(k + i\omega)^2} \left[ e^{-(k+i\omega)t} - 1 \right]^2 \right. \\
&\left. + \frac{|F|^2}{k^2 + \omega^2} \left( e^{-2bt} - 2e^{-kt} \cos \omega t + 1 \right) \right\},
\end{align*}
\]
where \( \cdots = p^2 - q^2 + 2pq + |p + q|^2 \). For long times \((t \gg 1/k)\), the RHS of (18) depends linearly on \( t \). The real part of the dominant term in this regime is \(-4\omega^2k|F|^2t/(k^2 + \omega^2)^2\). It is responsible by the complete disappearance of the “non-diagonal elements” \( \hat{\rho}_{eg} \) and \( \hat{\rho}_{ge} \). Therefore, whichever may be the initial state of the compound system atom-field, the stationary regime is characterized by the state

\[
\rho_{\text{stat}} = \text{tr}_f [\hat{\rho}_{ee}(0)] |F/(ik - \omega)\rangle \langle F/(ik - \omega)| \otimes |e\rangle \langle e| + \text{tr}_f [\hat{\rho}_{gg}(0)] |F/(ik + \omega)\rangle \langle F/(ik + \omega)| \otimes |g\rangle \langle g|. \tag{19}
\]

The characteristic time of decay of “non-diagonal elements” for long times depends on the damping constant \( k \), the effective coupling between atom and field \( \omega \) and the intensity of the external source \( F \). It is easy to verify that the larger the value of \( |F| \) the faster the decay of \( \hat{\rho}_{eg} \) and \( \hat{\rho}_{ge} \). On the other hand, keeping the value of intensity constant, the decay of the “non-diagonal elements” is more rapid in the critical damping \((k/\omega = 1)\) regime for long times.

Finally, in absence of the external source, i.e., making \( F = 0 \) in Eqs. (14a), (14b) and (17), we recover the results obtained in Ref. [1].

### III. The Evolution of an Uncorrelated Initial State

In order to understand the influence on the entanglement process between atom and field in dispersive JCM by the introduction of both dissipative mechanism and external source, let us calculate the time evolution of an uncorrelated initial state

\[
|\psi\rangle = \frac{1}{\sqrt{2}} (|e\rangle + |g\rangle) \otimes |-iF/k\rangle. \tag{20}
\]

This state is prepared turning on the source at \( t \to -\infty \). The source is maintained continuously pumping the field and, at \( t = 0 \), the field reaches the stationary state (18). The atom, prepared in a coherent superposition of the states \( |e\rangle \) and \( |g\rangle \), begins to interact with the field. Such state was chosen by simplicity but the results obtained can be easily extended to more general states such as \((\sin \phi |e\rangle + e^{i\chi} \cos \phi |g\rangle) \otimes |\alpha\rangle\).

#### A. The Global Density Operator

Given the initial state (20), the “matrix elements” of \( \hat{\rho}(0) \) are \( \hat{\rho}_{ee}(0) = \hat{\rho}_{gg}(0) = \hat{\rho}_{eg}(0) = \hat{\rho}_{ge}(0) = \frac{1}{2} |-iF/k\rangle \langle -iF/k| \). The state \( \hat{\rho}(t) \) is determined after finding the time evolution of each “matrix elements”. The evolved global state is given by the expression

\[
\hat{\rho}(t) = \hat{\rho}_{gg}(t) \otimes |g\rangle \langle g| + \hat{\rho}_{ge}(t) \otimes |g\rangle \langle e| + \hat{\rho}_{eg}(t) \otimes |e\rangle \langle g| + \hat{\rho}_{ee}(t) \otimes |e\rangle \langle e|. \tag{21}
\]

Equations (14a) and (14b) produce, for the “diagonal elements”,

\[
\hat{\rho}_{ee}(t) = \frac{1}{2} |\beta_{ee}'(t)|^2 \langle \beta_{ee}'(t) \rangle, \tag{22a}
\]

\[
\hat{\rho}_{gg}(t) = \frac{1}{2} |\beta_{gg}'(t)|^2 \langle \beta_{gg}'(t) \rangle. \tag{22b}
\]

where we have defined

\[
\beta_{ee}'(t) = \beta_{ee}(t) - \frac{iF}{k} e^{-(k+i\omega)t}, \tag{23a}
\]

\[
\beta_{gg}'(t) = \beta_{gg}(t) - \frac{iF}{k} e^{-(k-i\omega)t}. \tag{23b}
\]

Here, \( \beta_{ee}(t) \) and \( \beta_{gg}(t) \) are the amplitudes defined in Eqs. (15a) and (15b).

The evolution of the “non-diagonal elements” is calculated by the using of Eq. (16):

\[
\hat{\rho}_{eg}(t) = \frac{1}{2} \exp[\Phi(\omega, k, F; t)] |\beta_{ee}'(t)| \langle \beta_{gg}'(t) \rangle. \tag{24}
\]
The complex phase $\Phi (\omega, k, F; t)$ is given by the expression

$$\Phi (\omega, k, F; t) = -i \omega t + z (t) + |F|^2 \left( p^2 - q^2 + 2pq + |p + q|^2 \right) (t)$$

$$+ i \Theta (F/k, t) + \Gamma (F/k, t)$$

$$+ \frac{|F|^2}{k} \left\{ 2i \text{Re } [(p + q) (t) e^{-kt} \cos \omega t] - 2i \text{Im } [(p + q) (t) e^{-kt} \sin \omega t] \right\}$$

$$- 4e^{-(k+i\omega)t} \text{Im } q (t) + i \text{Re } p (t) \right\},$$

where the functions $\Theta (F/k, t)$ and $\Gamma (F/k, t)$ are

$$\Theta (F/k, t) = \frac{|F|^2}{k (k^2 + \omega^2)} \left[ e^{-2kt} (k \sin 2\omega t + \omega \cos 2\omega t) - \omega \right],$$

$$\Gamma (F/k, t) = -\frac{|F|^2}{k^2} (1 - e^{-2kt}) - \frac{|F|^2}{k (k^2 + \omega^2)} \times \left[ e^{-2kt} (k \cos 2\omega t - \omega \sin 2\omega t) - k \right].$$

Hence, after $t$, the state of the compound system atom-field is

$$\hat{\rho} (t) = \frac{1}{2} \left\{ |e, \beta'_e (t) \rangle \langle e, \beta'_e (t) | + |g, \beta'_g (t) \rangle \langle g, \beta'_g (t) | \right.$$  

$$+ \left\{ \exp [\Phi (\omega, k, F; t)] |e, \beta'_e (t) \rangle \langle g, \beta'_g (t) | + \text{H. c.} \right\},$$

where H. c. stands for Hermitean conjugate.

The density operator $\hat{\rho} (t)$ can be diagonalized yielding the following eigenvectors and eigenvalues

$$|\psi_+ \rangle = \frac{1}{\sqrt{2}} \left\{ |e, \beta'_e (t) \rangle + \exp [-i \text{Im } \Phi (\omega, k, F; t)] |g, \beta'_g (t) \rangle \right\},$$

$$|\psi_- \rangle = \frac{1}{\sqrt{2}} \left\{ |e, \beta'_e (t) \rangle - \exp [-i \text{Im } \Phi (\omega, k, F; t)] |g, \beta'_g (t) \rangle \right\},$$

$$\lambda_+ (t) = \frac{1}{2} \left\{ 1 + \exp [\text{Re } \Phi (\omega, k, F; t)] \right\},$$

$$\lambda_- (t) = \frac{1}{2} \left\{ 1 - \exp [\text{Re } \Phi (\omega, k, F; t)] \right\},$$

in terms of which $\hat{\rho} (t)$ can be written as

$$\hat{\rho} (t) = \lambda_+ (t) |\psi_+ \rangle \langle \psi_+ | + \lambda_- (t) |\psi_- \rangle \langle \psi_- |.$$

The purity of the state represented by a density operator $\hat{\rho}$ is conveniently measured by the idempotency defect or linear entropy [11]

$$\varsigma (t) = 1 - \text{tr } \hat{\rho}^2 (t)$$

In general, if $\hat{\rho}$ describes a pure state, $\varsigma = 0$, otherwise $\varsigma > 0$. The idempotency defect of the state of the system atom-field as a function of time is given by

$$\varsigma (t) = 1 - \lambda_+^2 (t) - \lambda_-^2 (t) = \frac{1}{2} \left\{ 1 - \exp [2 \text{Re } \Phi (\omega, k, F; t)] \right\}.$$

The idempotency defect has an upper limit which is characteristic of a complete mixture. In the case studied, this value is $1/2$. As we already discussed, the “non-diagonal elements” $\hat{\rho}_{eg} (t) = \hat{\rho}_{ge}^\dagger (t)$ vanish at $t \to \infty$. As consequence, the coherence loss of the system atom-field is complete in the stationary regime, i.e., $\varsigma (t \to \infty) = 1/2$. 
1. Short and long times behavior of coherence loss of global state

The real part of the function $\Phi(\omega, k, F; t)$ controls the behavior of the linear entropy $\zeta$ of the state of the global system $\hat{\rho}$. For long times $(kt \gg 1)$, $\Phi$ depends linearly on $t$, and $\zeta$ grows accordingly $\zeta(t) \sim 1 - \exp(-2t/\tau_{\text{dec}})$. The characteristic decoherence time in this regime is

$$\tau_{\text{dec}} = \left(\frac{k^2 + \omega^2}{4\omega k |F|^2}\right)^2 = \left\{ kD^2 \left[ \beta'_e(\infty), \beta'_g(\infty) \right] \right\}^{-1}, \quad (30)$$

where $D \left[ \beta'_e(\infty), \beta'_g(\infty) \right]$ stands for the distance in the phase space between the coherent states $|\beta'_e(\infty)\rangle$ and $|\beta'_g(\infty)\rangle$. This distance, defined by the expression $D(z, z') = |z - z'|$, measures the distinguishability between the coherent states $|z\rangle$ and $|z'\rangle$ ($z$ and $z'$ are two complex numbers). The amplitudes $\beta'_e(\infty)$ and $\beta'_g(\infty)$ are given by

$$\beta'_e(\infty) = \frac{F}{\omega - i k}, \quad \beta'_g(\infty) = \frac{F}{\omega + i k}. \quad (31)$$

Hence, for long times, the more distinguishable the stationary states $|\beta'_e(\infty)\rangle$ and $|\beta'_g(\infty)\rangle$, the more rapid the decoherence process of the global state. This result is a direct consequence of the long times behavior of the decay of the “non-diagonal elements” discussed above. In fact, the characteristic decoherence time of the global state inherits the properties of the characteristic time of decay of the “non-diagonal elements” for long times.

For short times $(kt \ll 1$ and $\omega t \ll 1)$, the linear entropy $\zeta$ grows accordingly $\zeta(t) \sim 1 - \exp\left[-2 \left(t/\tau_{\text{dec}}\right)^3\right]$, where

$$\left(\frac{\tau_{\text{dec}}}{t}\right)^3 = |F|^2 k \left[ 1 + \frac{4}{3} \left(\frac{\omega}{k}\right)^2 \right]. \quad (32)$$

In this regime, the larger the intensity of the external source $|F|^2$, the faster is the coherence loss. Besides, for a given intensity value, the global state begins to lose coherence more rapidly in the subcritical damping regime $(k/\omega < 1)$.

B. The reduced density operators

The state of the atomic subsystem (respectively, field subsystem) is described by the density operator $\hat{\rho}_a$ (respectively, $\hat{\rho}_f$). This operator is obtained by taking the partial trace of $\hat{\rho}$ with respect to the field variables (respectively, atomic variables). The atomic density operator is given by

$$\hat{\rho}_a(t) = \text{tr}_f \hat{\rho}(t) = \frac{1}{2} \left\{ |e\rangle \langle e| + |g\rangle \langle g| \right\} + \left\{ \exp\left[\Phi(\omega, k, F; t)\right] \langle \beta'_g(t) | \beta'_e(t) \rangle |e\rangle \langle g| + \text{H. c.} \right\}. \quad (33)$$

This operator can be diagonalized, yielding the following eigenvectors and eigenvalues

$$|g'\rangle = \frac{1}{\sqrt{2}} \left\{ |e\rangle + \exp\left\{ -i \text{Im} \left[ \Phi(\omega, k, F; t) + \beta'_e(t) \beta'^*_g(t) \right] \right\} |g\rangle \right\}, \quad (34a)$$

$$|e'\rangle = \frac{1}{\sqrt{2}} \left\{ |e\rangle - \exp\left\{ -i \text{Im} \left[ \Phi(\omega, k, F; t) + \beta'_e(t) \beta'^*_g(t) \right] \right\} |g\rangle \right\}, \quad (34b)$$

$$\lambda_{g'}(t) = \frac{1}{2} \left\{ 1 + \exp\left\{ \text{Re} \Phi(\omega, k, F; t) - \frac{1}{2} D^2 [\beta'_e(t), \beta'_g(t)] \right\} \right\}, \quad (35a)$$

$$\lambda_{e'}(t) = \frac{1}{2} \left\{ 1 - \exp\left\{ \text{Re} \Phi(\omega, k, F; t) - \frac{1}{2} D^2 [\beta'_e(t), \beta'_g(t)] \right\} \right\}. \quad (35b)$$

The atomic purity loss is measured by the idempotency defect:

$$\varsigma_a(t) = 1 - \text{tr} \hat{\rho}_a^2(t) = \frac{1}{2} \left\{ 1 - \exp\left\{ 2 \text{Re} \Phi(\omega, k, F; t) - D^2 [\beta'_e(t), \beta'_g(t)] \right\} \right\}. \quad (36)$$
We can recognize two distinct contributions to the coherence loss of the atomic state. These contributions are identified by the two terms in the argument of the exponential in Eq. (38). The first term, proportional to the real part of the complex phase $\Phi$, reflects the presence of dissipation and the external source. The second term is proportional to the distance in the phase space between the states $|\beta'_e(t)\rangle$ and $|\beta'_g(t)\rangle$. Hence, the coherence properties of the atomic state are affected by the presence of the thermal reservoir and the external source, even if the atom is not directly coupled to them, and by the entanglement process between atom and field.

Tracing out the global density operator in the atomic variables, we get the reduced field density operator

$$\hat{\rho}_f (t) = \frac{1}{2} \left\{ |\beta'_e(t)\rangle \langle \beta'_e(t) | + |\beta'_g(t)\rangle \langle \beta'_g(t) | \right\},$$

whose idempotency defect is

$$\varsigma_f (t) = 1 - \text{tr} \hat{\rho}_f^2 (t) = \frac{1}{2} \left\{ 1 - \exp \left\{ -D^2 [\beta'_e(t), \beta'_g(t)] \right\} \right\}.$$  \hspace{1cm} (38)

The field state is a statistical mixture of the coherent states $|\beta'_e(t)\rangle$ and $|\beta'_g(t)\rangle$. Although the amplitudes $\beta'_e(t)$ and $\beta'_g(t)$ have equal moduli, the phase between them varies in time in a complicated form. Here, the contribution to the idempotency defect is due to the entanglement process between atom and field. The asymptotic field state is not a pure state but a statistical mixture of the coherent states $|\beta'_e(\infty)\rangle$ and $|\beta'_g(\infty)\rangle$, i.e.,

$$\hat{\rho}_f (\infty) = \frac{1}{2} \left\{ |\beta'_e(\infty)\rangle \langle \beta'_e(\infty) | + |\beta'_g(\infty)\rangle \langle \beta'_g(\infty) | \right\}.$$ 

The operator $\hat{\rho}_f (t)$ can be diagonalized, yielding the following eigenvectors and eigenvalues

$$|\varphi_+\rangle = \frac{1}{N_+ (t)} \left[ |\beta'_e(t)\rangle + e^{i\chi(t)} |\beta'_g(t)\rangle \right],$$ \hspace{1cm} (39a)

$$|\varphi_-\rangle = \frac{1}{N_- (t)} \left[ |\beta'_e(t)\rangle - e^{i\chi(t)} |\beta'_g(t)\rangle \right],$$ \hspace{1cm} (39b)

$$\Lambda_+ (t) = \frac{N_+^2 (t)}{4} = \frac{1}{2} \left\{ 1 + \exp \left\{ -\frac{1}{2} D^2 [\beta'_e(t), \beta'_g(t)] \right\} \right\},$$ \hspace{1cm} (40a)

$$\Lambda_- (t) = \frac{N_-^2 (t)}{4} = \frac{1}{2} \left\{ 1 - \exp \left\{ -\frac{1}{2} D^2 [\beta'_e(t), \beta'_g(t)] \right\} \right\},$$ \hspace{1cm} (40b)

where $\chi (t) = \text{Im} [\beta'_e(t) \beta'^*_g(t)]$.

1. Short and long times behavior of the coherence loss of subsystems states

For long times ($kt \gg 1$), the coherence loss of the atomic subsystem is dominated by the linear dependence on $t$ of the function $\Phi$. In this regime, the time dependence of the atomic linear entropy $\varsigma_a$ closely follows that for the global system $\varsigma$. Hence, the contribution to the atomic coherence loss, for long times, is due to the presence of the dissipation and the external source, and the time scales of the atomic and the full system’s decoherence are the same. On the other hand, for short times ($kt \ll 1$ and $\omega t \ll 1$), the contribution for the atomic coherence loss is solely due to the entanglement process. In fact, in this regime, the dominant term in the argument of the exponential in the RHS of Eq. (38) is quadratic in $t$ and comes from the factor proportional to the distance between the states $|\beta'_e(t)\rangle$ and $|\beta'_g(t)\rangle$, namely $D^2 [\beta'_e(t), \beta'_g(t)]$. The atomic linear entropy grows accordingly $\varsigma_a \sim 1 - \exp \left\{ -\left( t/\tau^{\text{st}}_{a,\text{dec}} \right)^2 \right\}$, where

$$\tau^{\text{st}}_{a,\text{dec}} = \frac{k}{2 |F| \omega}.$$ \hspace{1cm} (41)

Since the decoherence of the atomic state for short times is due to the entanglement process, it is faster in the subcritical damping ($k/\omega < 1$). Moreover, increasing of the intensity of the source shortens the atomic decoherence characteristic time.

The decoherence of the field state, for short times, is similar to that of the atomic state. In fact, at $t \to 0$, the purity loss of both systems is mainly due to the initial entanglement process. Hence, for short times, the characteristic decoherence times of atom ($\tau^{\text{st}}_{a,\text{dec}}$) and field ($\tau^{\text{st}}_{f,\text{dec}}$) are equal and related to the unitary interaction.
As we pointed out, the linear term in \( t \) appearing in the function \( \Phi \) is responsible by the complete vanishing of the “non-diagonal elements” \( \rho_{eg} \) and \( \rho_{ge} \). Contrary to the model studied in Ref. [10] – the dispersive JCM with dissipation but without coupling to an external source – the decoherence of the full system atom-field prepared in the initial state \( |\psi\rangle \) is complete. The graphs presented in Figs. 1 and 2 clearly exhibit this behavior. In Fig. 1, the graphs of \( \zeta \), \( \zeta_f \) and \( \zeta_{sf} \) as a function of \( \omega t/\pi \) for different values of \( k/\omega \) are shown, with constant \( |F|/k \) ratio. In Fig. 2 are plotted the graphs for two different values of the ratio \( |F|/k \) in the subcritical regime.

It is interesting to note that the larger the coupling between field and external source, \( |F| \), the more rapid the coherence loss of both the full system atom-field and the atom only. Since the intensity of the injected field by the source is a measure of its “classicality”, the coherence loss of the global system becomes faster as the intensity increases. Moreover, the characteristic decoherence time \( \tau_{dec}^{lt} \) is inversely proportional to mean number of photons in the asymptotic state, \( \pi(\infty) = |F|^2/(k^2 + \omega^2) \).

The dependence of \( \zeta(t) \) with the ratio \( k/\omega \) is more complicated. Since the characteristic decoherence time directly depends on the characteristic dissipation time, \( 1/k \), one would expect the decoherence of the global system to be slower in a less dissipative environment. This conjecture seems to be verified if one compares, in Fig. 1, the curves of \( \zeta(t) \) for \( k/\omega = 0.2 \) and \( k/\omega = 1.2 \). Note that \( \zeta \) reaches the saturation rapidly in the case \( k/\omega = 1 \). Hence, we expect that the larger the value of \( k/\omega \), the faster the saturation of \( \zeta \). But it does not happen: as shown in Fig. 1, for \( k/\omega = 1 \), \( \zeta \) reaches the plateau at \( t = \pi/\omega \), while for \( k/\omega = 5 \), the plateau is reached at \( t = 3\pi/2\omega \), approximately. Roughly, the coherent superposition between the states \( |e, \beta'_e(t)\rangle \) and \( |g, \beta'_g(t)\rangle \) that the unitary contribution tries to create, is transformed by non-unitary mechanism into a statistical mixture. The more distinguishable the states that form the superposition, the faster the coherence loss. A measure of distinguishability is provided by the distance in the phase space between the states \( |\beta'_e(t)\rangle \) and \( |\beta'_g(t)\rangle \). The amplitudes \( \beta'_e(t) \) and \( \beta'_g(t) \) are inversely proportional to \( \omega \pm ik \); hence, on the one hand, if the increasing of \( k \) favours decoherence, on the other hand, high dissipation taxes rapidly diminishes the separability between the states \( |\beta'_e(t)\rangle \) and \( |\beta'_g(t)\rangle \) and the decoherence becomes slower.

As the model studied in Ref. [10], the atom is more influenced by non-unitary dynamics. In fact, the contribution to the purity loss of the atomic state results both from the interaction between atom and field and from the presence of the dissipative environment. These different contributions can be identified by the terms proportional to \( \text{Re } \Phi(\omega,k,F; t) \) and to \( D^2 [\beta'_e(t), \beta'_g(t)] \) in the argument of the exponential in Eq. (40). Hence, the purity loss of the atomic state is complete, as one can verify in the graphs in Fig. 1, specially for the cases \( k/\omega = 1 \) and \( k/\omega = 5 \). On the other hand, the decoherence of the field state only results from the interaction between atom and field. However, contrary to the model studied in Ref. [10], the asymptotic state of the field is formed by a statistical mixture of the coherent states \( |\beta'_e(\infty)\rangle \) and \( |\beta'_g(\infty)\rangle \). The asymptotic value of the linear entropy \( \zeta_f \) is larger in the critical regime \( (k/\omega = 1) \) than in the subcritical and supercritical regimes.

It is worth to note that the linear entropy of the field \( \zeta_f \) exhibits local maxima and minima, specially in the critical regime. These local maxima and minima correspond to the instants \( (t_c) \) of maximum and minimum distance between the states \( |\beta'_e(t)\rangle \) and \( |\beta'_g(t)\rangle \). These critical instants can be calculated by the zeros of the time derivative of

\[
D^2 [\beta'_e(t), \beta'_g(t)] = \frac{4|F|^2}{k^2(k^2 + \omega^2)} [k( e^{-kt} \cos \omega t - 1) - \omega e^{-kt} \sin \omega t]^2. \tag{42}
\]

In this way, the critical instants shall satisfy

\[
k( e^{-kt_c} \cos \omega t_c - 1) - \omega e^{-kt_c} \sin \omega t_c = 0 \tag{43}
\]

or

\[
\cos \omega t_c = 0. \tag{44}
\]

When \( t_c \) satisfies (44), the distance in (42) vanishes, i.e., \( \beta'_e(t_c) = \beta'_g(t_c) \). In this case, \( \zeta_f \) is null, the field is found in a pure state and the global state disentangles. On the other hand, if \( \omega t_c = (2n + 1) \pi/2 \), \( n \) integer, Eq. (44) is satisfied. Now, we can have a local maximum or a local minimum depending of the signal of the second time derivative of \( D^2 [\beta'_e(t), \beta'_g(t)] \). At the critical instants, we have

\[
\text{sgn} \left\{ \frac{d^2}{dt^2} D^2 [\beta'_e(t), \beta'_g(t)] \bigg|_{t=t_c} \right\} = \text{sgn} \left[ \sin \omega t_c (k + \omega e^{-kt_c} \sin \omega t_c) \right].
\]

We consider two different cases:
1. Critical and supercritical regimes \((k/\omega \geq 1)\): If \(\omega t_c = (2n + 1)\pi/2, n\) even, we have a local minimum, otherwise, we have a local minimum, but the value of \(\varsigma_f\) is not null, i.e., the field state is characterized by a statistical mixture.

2. Subcritical regime \((k/\omega < 1)\): If \(\omega t_c = (2n + 1)\pi/2, n\) even, we have a local maximum. On the other hand, if \(n\) is odd, in order the signal of the second time derivative to be positive, \(t_c\) shall satisfy

\[
t_c > \frac{1}{k} \ln \frac{\omega}{k} = t_{\text{trans}}.
\]

Hence, if \(t_c = (2n + 1)\pi/2\omega < t_{\text{trans}}, n\) odd, we have a local maximum, otherwise, if \(t_c > t_{\text{trans}}\), we have a local minimum. Note that the local minima corresponding to \(t_c > t_{\text{trans}}, \varsigma_f\) is not null.

### D. Measuring correlations

According to Werner [12], the density operator \(\hat{\rho}\) which represents the state of a bipartite system \(A + B\) is said disentangled (or separable) iff

\[
\hat{\rho} = \sum_i p_i \hat{\rho}_A^i \otimes \hat{\rho}_B^i, \tag{45}
\]

where \(\hat{\rho}_A^i (\hat{\rho}_B^i)\) are density operators on the state space of the system \(A(B)\). \(p_i\) are non-negative real numbers, such as \(\sum_i p_i = 1\). If \(\hat{\rho}\) cannot be written in form (45), the state is said entangled or quantum correlated. Moreover, we can demand both \(\hat{\rho}_A^i\) and \(\hat{\rho}_B^i\) to be a pure state.

The state of each subsystem is described by the reduced density operators, \(\hat{\rho}_A \equiv \text{tr}_B \hat{\rho}\) and \(\hat{\rho}_B \equiv \text{tr}_A \hat{\rho}\). However, in general, the global state cannot be determined from the reduced states. In short, \(\hat{\rho} \neq \hat{\rho}_A \otimes \hat{\rho}_B\). An important information about the global state is lost in the partial tracing out procedure. This information is related to the local (classical) and non-local (quantum) correlations between the two subsystems, \(A\) and \(B\). We can ask about the “distance” between the global state \(\hat{\rho}\) and the corresponding completely uncorrelated state \(\hat{\rho}_A \otimes \hat{\rho}_B\) as a measure of the total correlation of the state \(\hat{\rho}\). A possible choice of distance is given by the Hilbert-Schmidt metric, hence, the total correlation measure of the state \(\hat{\rho}\) is defined as

\[
c (\hat{\rho}) \equiv d^2 (\hat{\rho}, \hat{\rho}_A \otimes \hat{\rho}_B) = ||\hat{\rho} - \hat{\rho}_A \otimes \hat{\rho}_B||^2 \equiv \text{tr} (\hat{\rho} - \hat{\rho}_A \otimes \hat{\rho}_B)^2.
\]

Returning to dispersive JCM, the expression for \(c(\hat{\rho})\) can be written as

\[
c (\hat{\rho}) = \frac{\varsigma_f (t)}{2} \{1 + [1 - 2\varsigma (t)][1 + 2\varsigma_f (t)]\}. \tag{46}
\]

\(c(\hat{\rho})\) is a non-negative quantity: if the field is found in a pure state, we have \(c(\hat{\rho}) = 0\) and the global state is characterized by a completely uncorrelated state.

The correlation measure above defined does not distinguish classical and quantum correlations. In order to evaluate the entanglement of the system atom-field, we chose the concurrence [8] as measure of degree of entanglement. It has been proven to be a reasonable entanglement measure for mixed states of bipartite systems composed by two-level subsystems. Since the reduced state of the field has rank no greater than two, we can effectively consider the global system atom-field formed by two two-level subsystems at each instant of time \(t\). If the density matrix \(\hat{\rho}\) represents the state of two two-level systems \(A\) and \(B\), the concurrence is defined as

\[
C (\hat{\rho}) = \max \{0, x_1 - x_2 - x_3 - x_4\},
\]

where \(x_1 \geq x_2 \geq x_3 \geq x_4\) are the eigenvalues of the matrix \(\sqrt{\hat{\rho}^{1/2} \hat{\rho} \hat{\rho}^{1/2}}\). The matrix \(\tilde{\rho}\) is given by

\[
\tilde{\rho} = (\hat{\sigma}_y \otimes \hat{\sigma}_y) \hat{\rho}^* (\hat{\sigma}_y \otimes \hat{\sigma}_y).
\]

Here, \(\hat{\rho}^*\) represents the complex conjugation of \(\hat{\rho}\) in a fixed basis. \(\hat{\sigma}_y\) is the Pauli pseudo-matrix

\[
\hat{\sigma}_y = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}
\]

in the same basis. Note that \(0 \leq C (\hat{\rho}) \leq 1\). The upper limit indicates maximum entanglement; the lower limit is characteristic of separable states.
The contribution of the unitary process for the concurrence can be recognized by the presence of the square root of unitary interaction between them and the dissipative dynamics due to the coupling between field and environment. We conclude that the degree of entanglement between atom and field results from the competition of two processes: the decay of the concurrence in the critical regime is more rapid than the corresponding decay in the supercritical regime. In the subcritical regime, at the instants when atom and field are disentangled, the asymptotic global state is completely uncorrelated [10]. These statements remain true even the source is eliminated. In this case, the field evolves to the vacuum state and is known as Baker-Hausdorff formula. In short, it can be rewritten as

\[ \rho_{t} = e^{\hat{A} t} \rho_{0} e^{-\hat{A} t} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \hat{A}, \ldots \hat{A} \right] \hat{B}. \]  

Keeping the value of \(|F|/k \) constant, the maximum value of \( c(\hat{\rho}(\infty)) \) occurs in the critical regime.

In the three dynamical regimes, the concurrence vanishes in the asymptotic limit (this is noticeable in critical and supercritical regimes). In fact, the non-unitary mechanism completely destroys any trace of entanglement between atom and field, despite of the continuous pumping of the field by the external source. Since the global state evolves to a complete statistical mixture, both the eigenvalues of the density operator \( \rho(\infty) \) are equal to \( 1/2 \). Hence, the long times behavior of the concurrence \( C(\hat{\rho}) \) is mainly governed by the factor \( |\lambda_+ (t) - \lambda_-(t)| \) in Eq. (47). Note that the decay of the concurrence in the critical regime is more rapid than the corresponding decay in the supercritical regime. In the subcritical regime, at the instants when atom and field are disentangled, \( C(\hat{\rho}) \) is null, as expected. In these instants, the field is found in a pure state, \( \hat{\rho}_f \) has rank equal to unit and \( \text{det} \hat{\rho}_f = \lambda_+ (t) \lambda_-(t) \) disappears. We can conclude that the degree of entanglement between atom and field results from the competition of two processes: the unitary interaction between them and the dissipative dynamics due to the coupling between field and environment. The contribution of the unitary process for the concurrence can be recognized by the presence of the square root of \( \lambda_+ (t) \lambda_-(t) \) in Eq. (47), whereas the effects of the non-unitary mechanism are carried in factor \( |\lambda_+ (t) - \lambda_-(t)| \).

These statements remain true even the source is eliminated. In this case, the field evolves to the vacuum state and the asymptotic global state is completely uncorrelated [10].

APPENDIX A: ON THE SOLUTIONS OF THE EQUATIONS OF MOTION OF THE “MATRIX ELEMENTS” OF GLOBAL DENSITY OPERATOR

As we discussed above, the solutions of the equations of motion (10) of the “matrix elements” \( \hat{\rho}_{ij} = \langle i | \hat{\rho} (t) | j \rangle \), \( i, j = (e, g) \), are given by the general formula (13). Each dynamical generator \( \mathcal{L}_{ij}' \) is a linear combination of elements of some Lie algebra and the action of the exponential \( e^{\mathcal{L}_{ij}' t} \) (so-called “Lie exponential”) on the initial state \( \hat{\rho}_0 \) might be easily evaluated if one expresses it as an ordered product of exponentials of elements of the corresponding algebra [13, 14, 16]. We obtain the suitable similarity transformation of the Lie exponentials involved in the solutions of the equations of motion for \( \hat{\rho}_{ee} \), etc., by using the technique developed by Wilcox [13] known as parameter differentiation method.

1. The parameter differentiation method

The parameter differentiation method [13] uses the Baker-Hausdorff formula to expand a Lie exponential in an ordered product of exponentials. If \( \hat{A} \) and \( \hat{B} \) are two operators that do not commute, the expression

\[ e^{\hat{A} t} \hat{B} e^{-\hat{A} t} = \hat{B} + \frac{\hat{A}}{2!} [\hat{A}, \hat{B}] + \cdots + \frac{1}{n!} \left[ \hat{A}, \cdots [\hat{A}, \hat{B}] \cdots \right] + \cdots \]  

is known as Baker-Hausdorff formula. In short, it can be rewritten as

\[ e^{\hat{A} t} \hat{B} e^{-\hat{A} t} = e^{\left[ \hat{A}, \cdot \right] t} \hat{B} = \sum_{n=0}^{\infty} \frac{1}{n!} \left[ \hat{A}, \cdots \hat{A} \right]^n \hat{B}. \]
The superoperator $[\hat{A}, \cdot]^n$ represents the recurrent application of the commutator $[\hat{A}, \cdot]$,

$$[\hat{A}, \cdot]^n = [\hat{A}, \cdot]^{n-1} [\hat{A}, \cdot],$$

$$[\hat{A}, \cdot] = 0,$$

$$[\hat{A}, \cdot] = 1$$

and

$$[\hat{A}, \cdot] \hat{B} = [\hat{A}, \hat{B}].$$

Let us consider the $n$-dimensional Lie algebra $\mathcal{A}_n = \{\hat{A}_1, \hat{A}_2, \ldots, \hat{A}_n\}$, where the commutator between any pair $\hat{A}_i, \hat{A}_j \in \mathcal{A}_n$ is expressed as a linear combination of elements of $\mathcal{A}_n$, i.e.,

$$[\hat{A}_i, \hat{A}_j] = \sum_{k=1}^n C_{i,j}^k \hat{A}_k.$$

(A3)

The coefficients $\{C_{i,j}^k\}$ are real or complex numbers called structure constants of the algebra $\mathcal{A}_n$. We define a Lie exponential as an exponential of any linear combination $\mathcal{L} = a_1 \hat{A}_1 + a_2 \hat{A}_2 + \cdots + a_n \hat{A}_n = \sum_{i=1}^n a_i \hat{A}_i$ of elements of $\mathcal{A}_n$, i.e.,

$$\exp (\mathcal{L}t) \equiv \exp \left\{ \left( a_1 \hat{A}_1 + a_2 \hat{A}_2 + \cdots + a_n \hat{A}_n \right) t \right\},$$

(A4)

where $\{a_i\}_{i=1, \ldots, n}$ are the coefficients of the linear combination and $t$ is a real or complex parameter. The exponential $\exp (\mathcal{L}t)$ can be expressed as an ordered product of exponentials,

$$\exp (\mathcal{L}t) = \exp \left\{ f_1 (a_1, \cdots, a_n; t) \hat{A}_1 \right\} \times \cdots \times \exp \left\{ f_n (a_1, \cdots, a_n; t) \hat{A}_n \right\}.$$  

(A5)

Here, $f_i (a_1, \cdots, a_n; t), i = 1, \ldots, n$, is a function of the coefficients $\{a_i\}_{i=1, \ldots, n}$ and of the parameter $t$. The parameter differentiation method allow us to determine these functions.

Differentiating both sides of (A5) with respect to the parameter $t$, we get

$$\frac{d}{dt} \exp (\mathcal{L}t) = \mathcal{L} \exp (\mathcal{L}t)$$

$$= \left( f_1 (a_1, \cdots, a_n; t) \hat{A}_1 \exp \left\{ f_1 (a_1, \cdots, a_n; t) \hat{A}_1 \right\} \right) \times \cdots \times \left( f_n (a_1, \cdots, a_n; t) \hat{A}_n \exp \left\{ f_n (a_1, \cdots, a_n; t) \hat{A}_n \right\} \right)$$

$$+ \cdots + \left( f_n (a_1, \cdots, a_n; t) \hat{A}_n \exp \left\{ f_n (a_1, \cdots, a_n; t) \hat{A}_n \right\} \right) \times \cdots \times \hat{A}_n \exp \left\{ f_n (a_1, \cdots, a_n; t) \hat{A}_n \right\}$$

$$= \sum_{i=1}^n f_i (a_1, \cdots, a_n; t) \hat{B}_{i1} \exp \left\{ f_1 (a_1, \cdots, a_n; t) \hat{A}_1 \right\}$$

$$\times \cdots \times \hat{B}_{in} \exp \left\{ f_n (a_1, \cdots, a_n; t) \hat{A}_n \right\}.$$  

(A6)

Here, dot indicates derivatives with respect to $t$, and the operators $\hat{B}_{ij}, i, j = 1, \cdots, n$, are given by

$$\hat{B}_{ij} = \begin{cases} \hat{A}_i, & \text{if } i = j \\ 1, & \text{if } i \neq j \end{cases}.$$
The repeated application of the Baker-Hausdorff formula \((A1)\) allows us to move the operators \(\hat{A}_i, i = 2, \ldots, n,\) to the left position in each term of the sum appearing in \((A6)\). For instance, the second term of this sum is

\[
\hat{f}_2 (a_1, \ldots, a_n; t) \exp \left\{ f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\} \hat{A}_2 \exp \left\{ f_2 (a_1, \ldots, a_n; t) \hat{A}_2 \right\} \\
\times \cdots \times \exp \left\{ f_n (a_1, \ldots, a_n; t) \hat{A}_n \right\}.
\]

To move the operator \(\hat{A}_2\) to the left of the exponential in \(\hat{A}_1\), we write

\[
\exp \left\{ f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\} \hat{A}_2 = \exp \left\{ f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\} \hat{A}_2 \\
\times \exp \left\{ -f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\} \\
\times \exp \left\{ f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\}.
\]

The Baker-Hausdorff formula yields

\[
\exp \left\{ f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\} \hat{A}_2 \exp \left\{ -f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\} = G_2 \left( \hat{A}_1, \ldots, \hat{A}_n; f_1 \right),
\]

where \(G_2\) is a function of \(\hat{A}_1, \ldots, \hat{A}_n\) and \(f_1\). If this procedure is performed on the other terms of the sum in \((A6)\), we find

\[
\mathcal{L} = \sum_{i=1}^{n} f_i (a_1, \ldots, a_n; t) G_i \left( \hat{A}_1, \ldots, \hat{A}_n; f_1, \ldots, f_{i-1} \right).
\]

The functions \(G_i\) are given by

\[
G_1 \left( \hat{A}_1, \ldots, \hat{A}_n \right) = \hat{A}_1,
\]

\[
G_2 \left( \hat{A}_1, \ldots, \hat{A}_n; f_1 \right) = \exp \left\{ f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\} \hat{A}_2 \\
\times \exp \left\{ -f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\},
\]

\[
G_3 \left( \hat{A}_1, \ldots, \hat{A}_n; f_1, f_2 \right) = \exp \left\{ f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\} \exp \left\{ f_2 (a_1, \ldots, a_n; t) \hat{A}_2 \right\} \\
\times \hat{A}_3 \exp \left\{ -f_2 (a_1, \ldots, a_n; t) \hat{A}_2 \right\} \\
\times \exp \left\{ -f_1 (a_1, \ldots, a_n; t) \hat{A}_1 \right\},
\]

and so on. The identity \((A8)\) yields a set of coupled differential equations for the functions \(\{f_i\}_{i=1,\ldots,n}\). The solution of this system of differential equations with the corresponding initial condition determines the functions \(\{f_i\}_{i=1,\ldots,n}\).

\[2.\] The algebra of the bosonic superoperators

The dynamical generator \(\mathcal{L}_{ij}^\prime\) which appears in the general form of equation of motion \([3]\) for the “matrix elements” \(\hat{\rho}_{ij} = \langle i | \hat{\rho} (t) | j \rangle, i, j = (e, g),\) is a linear combination of bosonic superopertors \([11, 16]\), which form a finite Lie algebra under commutation. The bosonic superoperators represent the action of creation and annihilation operators of the harmonic oscillator, \(\hat{a}^\dagger\) and \(\hat{a}\), on an operator \(\hat{O}\):

\[
a^\dagger \hat{O} = \hat{a} \hat{O}, \ a^{\dagger\dagger} \hat{O} = \hat{a}^\dagger \hat{O}, \ a^\dagger \hat{O} = \hat{O} \hat{a}, \ a^{\dagger\dagger} \hat{O} = \hat{O} \hat{a}^\dagger.
\]

The sets \(\{a^\dagger, a^{\dagger\dagger}, 1\}\) and \(\{a^\dagger, a^{\dagger\dagger}, 1\}\) constitute left and right realization of the Heisenberg-Weyl group \(hw (4)\) \([3]\), denoted \(hw_l (4)\) and \(hw_r (4)\), respectively. From the fundamental relation \(\{\hat{a}, \hat{a}^\dagger\} = 1\) and the above definitions, we derive the commutation relations between the bosonic superoperators:

\[
\{a^\dagger, a^{\dagger\dagger}\} = 1, \\
\{a^\dagger, a^{\dagger\dagger}\} = -1.
\]
An superoperator belonging to $hw_l (4)$ commutes with another belonging to $hw_r (4)$. The bilinear products of these superoperators are

\[\mathcal{M} \equiv a_1^\dagger a_1^l,\]
\[\mathcal{P} \equiv a_1^r a_1^r,\]
\[\mathcal{J} \equiv a_1^l a_1^r = a_1^r a_1^l.\]  

(A11)

By virtue of the presence of the unitary term $L_S$ [cf. Eq. (3)] due to the external source, it is convenient to define the following superoperators

\[\mathcal{X}_\pm \equiv a_1^l \pm a_1^l,\]
\[\mathcal{Y}_\pm \equiv a_1^r \pm a_1^r.\]  

(A12)

The superoperators above defined generate a finite Lie algebra. The non-null commutation relations between these superoperators are given by

\[\mathcal{J}, \mathcal{M} = \mathcal{J},\]
\[\mathcal{J}, \mathcal{P} = \mathcal{P},\]
\[\mathcal{J}, \mathcal{X}_\pm = \frac{1}{2}(\mathcal{X}_+ - \mathcal{X}_-),\]
\[\mathcal{J}, \mathcal{Y}_\pm = \frac{1}{2}(\mathcal{Y}_+ - \mathcal{Y}_-),\]
\[\mathcal{M}, \mathcal{X}_\pm = \frac{1}{2}(\mathcal{X}_+ + \mathcal{X}_-),\]
\[\mathcal{M}, \mathcal{Y}_\pm = \pm \frac{1}{2}(\mathcal{Y}_+ - \mathcal{Y}_+),\]
\[\mathcal{P}, \mathcal{X}_\pm = \pm \frac{1}{2}(\mathcal{X}_- - \mathcal{X}_+),\]
\[\mathcal{P}, \mathcal{Y}_\pm = \frac{1}{2}(\mathcal{Y}_+ + \mathcal{Y}_-),\]
\[\mathcal{X}_+, \mathcal{Y}_- = - \mathcal{X}_-, \mathcal{Y}_+ = 2.\]  

(A13)

3. On the disentanglement of the Lie exponential corresponding to the “diagonal elements”

The solution of the equation of motion for the operator $\hat{\rho}_{ee}$ is given by

\[\hat{\rho}_{ee} (t) = \exp (L'_{ee} t) \hat{\rho}_{ee} (0).\]

In terms of the superoperators above defined, the Liouvillian $L'_{ee}$ can be expressed as

\[L'_{ee} = -i\omega (\mathcal{M} - \mathcal{P}) + k \{2\mathcal{J} - \mathcal{M} - \mathcal{P} \} - i (F\mathcal{X}_- - F^*\mathcal{Y}_-) = \mathcal{L}_+ + \mathcal{L}_S,\]

where $\mathcal{L}_+$ is given in Eq. (13). From Eq. (A10), we found

\[\mathcal{L}_+, \mathcal{X}_- = -(k + i\omega) \mathcal{X}_-,\]
\[\mathcal{L}_+, \mathcal{Y}_- = -(k - i\omega) \mathcal{Y}_-.\]

Hence, the algebra generated by the set $\{\mathcal{L}_+, \mathcal{X}_-, \mathcal{Y}_-\}$ is an union of two two-dimensional Lie subalgebras.

Let us express the Lie exponential $\exp (L'_{ee} t)$ as

\[\exp (L'_{ee} t) = \exp \left[ x_-(t) \mathcal{X}_- \right] \exp \left[ y_-(t) \mathcal{Y}_- \right] \exp \left[ \lambda(t) \mathcal{L}_+ \right].\]  

(A14)

We just have to determine the functions $x_-, y_-, \lambda$, given the initial condition

\[x_-(0) = y_-(0) = \lambda(0) = 0.\]  

(A15)

Taking the derivative of both sides of Eq. (A14) with respect to $t$, and applying the well-known result

\[e^{x\hat{A}} \hat{B} e^{-x\hat{A}} = e^{\beta x} \hat{B} + (e^{\beta x} - 1) \frac{\alpha}{\beta}\]

\[\alpha \beta\]
for a two-dimensional algebra $\mathcal{A}_2 = \{\hat{A}, \hat{B}\}$, where $[\hat{A}, \hat{B}] = \alpha \hat{A} + \beta \hat{B}$, $\alpha, \beta$ are c-numbers, we find

$$\mathcal{L}'_{ee} = \left[ x_+ + \lambda (k + i\omega) x_- \right] \mathcal{X}_- + \left[ y_+ + \lambda (k - i\omega) y_- \right] \mathcal{Y}_- + \lambda \mathcal{L}_+.$$  

This identity yields the following set of differential equations

$$\dot{\lambda} = 1,$$

$$\dot{x}_- + \lambda (k + i\omega) x_- = -iF,$$  

$$\dot{y}_- + \lambda (k - i\omega) y_- = iF^*.$$  

Taking the initial conditions (A15) into account, the solution of these equations is

$$\lambda (t) = t,$$

$$x_-(t) = \frac{F}{\omega - ik} \left[ e^{-(k + i\omega)t} - 1 \right] = \beta_c (t),$$  

$$y_-(t) = \frac{F^*}{\omega + ik} \left[ e^{-(k - i\omega)t} - 1 \right] = \beta_c^*(t).$$

Note that

$$\exp (z\mathcal{X}_-) \exp (z^*\mathcal{Y}_-) = \exp \left( za^\dagger - z^* a^\dagger \right) \exp \left( -za^\dagger + z^* a^\dagger \right),$$

therefore, we can express $\hat{\rho}_{ee} (t)$ as

$$\hat{\rho}_{ee} (t) = \hat{D} [\beta_c (t)] [\exp (\mathcal{L}_+ t) \hat{\rho}_{ee} (0)] \hat{D}^\dagger [\beta_c^* (t)],$$

where $\hat{D}$ is the displacement operator of the Heisenberg-Weyl group [cf. Eq. (14)]. This procedure allows us to find an analogous expression for the operator $\hat{\rho}_{gg}$:

$$\hat{\rho}_{gg} (t) = \hat{D} [\beta_g (t)] [\exp (\mathcal{L}_- t) \hat{\rho}_{gg} (0)] \hat{D}^\dagger [\beta_g (t)].$$

$\mathcal{L}_- \mathcal{X}_- \mathcal{Y}_- \mathcal{L}_+$ are given by Eqs. (12) and (15), respectively.

4. On the disentanglement of the Lie exponential corresponding to the “non-diagonal” elements

The solution of the equation of motion for the operator $\hat{\rho}_{eg}$ is given by

$$\hat{\rho}_{eg} (t) = \exp (\mathcal{L}'_{eg} t) \hat{\rho}_{eg} (0).$$

In terms of the bosonic superoperators, $\mathcal{L}'_{eg}$ is written as

$$\mathcal{L}'_{eg} = -i\omega (\mathcal{M} + \mathcal{P} + 1) + k (2\mathcal{J} - \mathcal{M} - \mathcal{P}) - i (F\mathcal{X}_- - F^*\mathcal{Y}_-) = \mathcal{L}_{eg} + \mathcal{L}_S,$$

where $\mathcal{L}_{eg}$ is given in Eq. (12).

Let us consider the algebra generated by the following set of superoperators $\{\mathcal{L}_{eg}, F\mathcal{X}_- - F^*\mathcal{Y}_-, F\mathcal{X}_+ - F^*\mathcal{Y}_+, 1\}$. The non-null commutation relations between these elements are

$$[\mathcal{L}_{eg}, F\mathcal{X}_- - F^*\mathcal{Y}_-] = -i\omega (F\mathcal{X}_+ - F^*\mathcal{Y}_+) - k (F\mathcal{X}_- - F^*\mathcal{Y}_-),$$  

$$[\mathcal{L}_{eg}, F\mathcal{X}_+ - F^*\mathcal{Y}_+] = -(2k + i\omega) (F\mathcal{X}_- - F^*\mathcal{Y}_-) + k (F\mathcal{X}_+ - F^*\mathcal{Y}_+),$$  

$$[F\mathcal{X}_- - F^*\mathcal{Y}_-, F\mathcal{X}_+ - F^*\mathcal{Y}_+] = 4 |F|^2.$$  

The Lie exponential $\exp (\mathcal{L}'_{eg} t)$ can be expressed as

$$\exp (\mathcal{L}'_{eg} t) = e^{z(t)} \exp [p (t) (F\mathcal{X}_- - F^*\mathcal{Y}_-)] \exp [q (t) (F\mathcal{X}_+ - F^*\mathcal{Y}_+)] \exp [s (t) \mathcal{L}_{eg}],$$

where

$$z(t) = \left\{ \begin{array}{ll}
\frac{1}{2} (2k + i\omega) \mathcal{J} + i\omega (\mathcal{M} + \mathcal{P} + 1) t, & \mathcal{J} > 0,\\
0, & \mathcal{J} \leq 0,
\end{array} \right.$$  

$$p (t) = -i\omega (\mathcal{M} + \mathcal{P} + 1) - k (\mathcal{M} - \mathcal{P}) - i (2k + i\omega) \mathcal{J},$$  

$$q (t) = k (\mathcal{M} - \mathcal{P}),$$  

$$s (t) = -(2k + i\omega) \mathcal{J}.$$
where the functions to be determined \( z, p, q \) and \( s \) obey the initial condition
\[
  z(0) = p(0) = q(0) = s(0) = 0.  \tag{A21}
\]

The differentiation of (A20) with respect to \( t \) and the successive application of Baker-Hausdorff formula (A1,A2) yield
\[
  \mathcal{L}_{eg} = \dot{z} + 4q p |F|^2 - 2s q^2 (2k + i\omega) |F|^2 + 2i s q p^2 |F|^2 - 4s k p q |F|^2 \\
  + \left[ \hat{p} + s q (2k + i\omega) + s k p \right] (F \mathcal{X}_- - F^* \mathcal{Y}_-) \\
  + \left[ q + i s q p - s k q \right] (F \mathcal{X}_+ - F^* \mathcal{Y}_+) + s \mathcal{L}_{eg}.
\]

This identity yields the following set of differential equations
\[
  \dot{s} = 1, \tag{A22}
\]
\[
  \dot{q} - s (k q - i \omega p) = 0,
\]
\[
  \dot{p} + s [q (2k + i\omega) + k p] = -i,
\]
\[
  \dot{z} + 4q p |F|^2 + 2s |F|^2 \left[ \omega p^2 - q^2 (2k + i\omega) - 2k q p \right] = 0.
\]

Taking the initial condition (A21) into account, the solution of the above set of differential equations is
\[
  s(t) = t, \tag{A23}
\]
\[
  q(t) = -\frac{\omega}{(k + i\omega)^2} \left\{ \cosh \left( (k + i\omega) t \right) - 1 \right\},
\]
\[
  p(t) = i \frac{k}{(k + i\omega)^2} \left\{ \cosh \left( (k + i\omega) t \right) - 1 \right\} - \frac{i \sinh (k + i\omega) t}{k + i\omega},
\]
\[
  z(t) = -\frac{2i \omega |F|^2}{(k + i\omega)^2} \left\{ t + \frac{4 \left[ e^{-(k+i\omega)t} - 1 \right] - e^{-2(k+i\omega)t} + 1}{2(k + i\omega)} \right. \\
  \left. + i \frac{\omega}{(k + i\omega)^2} \left\{ \cosh \left( (k + i\omega) t \right) - 1 \right\}^2 \right\}.
\]

Since \( \mathcal{X}_- \) and \( \mathcal{Y}_- \) commute (as well as \( \mathcal{X}_+ \) and \( \mathcal{Y}_+ \)) we rewrite (A20) as
\[
  \exp \left( \mathcal{L}_{eg} t \right) = e^{z(t)} \exp \left[ F p(t) \mathcal{X}_- \right] \exp \left[ -F^* p(t) \mathcal{Y}_- \right] \exp \left[ F q(t) \mathcal{X}_+ \right] \\
  \times \exp \left[ -F^* q(t) \mathcal{Y}_+ \right] \exp \left[ \mathcal{L}_{eg} t \right].
\]

The commutation of the exponentials which contain the superoperators \( \mathcal{Y}_- \) and \( \mathcal{X}_+ \) in the above expression yields
\[
  \exp \left( \mathcal{L}_{eg} t \right) = \exp \left[ z(t) + 2 |F|^2 p(t) q(t) \right] \exp \left[ F p(t) \mathcal{X}_- \right] \exp \left[ F q(t) \mathcal{X}_+ \right] \\
  \times \exp \left[ -F^* p(t) \mathcal{Y}_- \right] \exp \left[ -F^* q(t) \mathcal{Y}_+ \right] \exp \left[ \mathcal{L}_{eg} t \right].
\]

Hence, the expression for \( \dot{\rho}_{eg}(t) \) can be put in the form
\[
  \dot{\rho}_{eg}(t) = \exp \left[ z(t) + 2 |F|^2 p(t) q(t) \right] \exp \left[ F (p + q) (t) \mathcal{A} \right] \exp \left[ F^* (p - q) (t) \mathcal{A}^\dagger \right] \\
  \times \exp \left[ \mathcal{L}_{eg} t \right] \dot{\rho}_{eg}(0) \exp \left[ -F^* (p + q) (t) \mathcal{A} \right] \exp \left[ -F (p - q) (t) \mathcal{A}^\dagger \right].
\]

After some algebra, we found
\[
  \dot{\rho}_{eg}(t) = \exp \left[ z(t) + |F|^2 \left( p^2 - q^2 + 2pq + |p + q|^2 \right) (t) \right] \mathcal{D} [\beta_e(t)] \\
  \times \exp \left\{ 2 F^* \left[ \text{Re} p(t) - i \text{Im} q(t) \right] \hat{a} \right\} \exp \left[ \mathcal{L}_{eg} t \right] \dot{\rho}_{eg}(0) \\
  \times \exp \left\{ -2 F \left[ \text{Re} p(t) - i \text{Im} q(t) \right] \hat{a} \right\} \hat{D}^\dagger [\beta_g(t)], \tag{A24}
\]

where \( \beta_e(t) \) and \( \beta_g(t) \) are the amplitudes defined in (15a), (15b). In Ref. [10], the action of the exponential \( \exp (\mathcal{L}_{eg} t) \) on an initial condition proportional to a coherent state has been evaluated. We employed the results obtained there to find the time development of \( \dot{\rho}_{eg} \) corresponding to the initial state (A2) studied here.
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Figures

FIG. 1: Linear entropy of the systems atom-field (solid line), atom (dotted line) and field (dashed line) as a function of $\omega t$ for different values of the ratio $k/\omega$. For all plots, we have $|F|/k = 1$.

FIG. 2: Linear entropy of the systems atom-field (solid line), atom (dotted line) and field (dashed line) as a function of $\omega t$ for different values of the ratio $|F|/k$. For all plots, we have $k/\omega = 0.2$. 
FIG. 3: Total correlation measure $c$ and concurrence $C$ of the global state $\hat{\rho}$ as a function of $\omega t$. For all plots, we have $|F|/k = 1$. 

Correlation measures
Subcritical regime

Critical and supercritical regimes