Subdominant pseudoultrametric on graphs

A. A. Dovgoshey and E. A. Petrov

Abstract. Let \((G, w)\) be a weighted graph. We find necessary and sufficient conditions under which the weight \(w: E(G) \to \mathbb{R}^+\) can be extended to a pseudoultrametric on \(V(G)\), and establish a criterion for the uniqueness of such an extension. We demonstrate that \((G, w)\) is a complete \(k\)-partite graph, for \(k > 2\), if and only if for any weight that can be extended to a pseudoultrametric, among all such extensions one can find the least pseudoultrametric consistent with \(w\). We give a structural characterization of graphs for which the subdominant pseudoultrametric is an ultrametric for any strictly positive weight that can be extended to a pseudoultrametric.

Bibliography: 14 titles.

Keywords: weighted graph, infinite graph, ultrametric space, shortest path metric, complete \(k\)-partite graph.

§ 1. Introduction

In this work by a graph we mean a pair \((V, E)\) consisting of a nonempty set \(V\) and a (possibly empty) set \(E\) whose elements are unordered pairs of distinct points of \(V\). Given a graph \(G = (V, E)\), the set \(V = V(G)\) is referred to as the set of vertices and \(E = E(G)\) is the set of edges.

We shall mostly use the notation employed in the monograph [1]. Let us introduce some definitions. A graph \(H\) is a subgraph of a graph \(G, H \subseteq G\), if

\[ V(H) \subseteq V(G) \quad \text{and} \quad E(H) \subseteq E(G). \]

Recall that a graph \(G\) is called complete if any two distinct vertices \(u, v\) are adjacent, \(\{u, v\} \in E(G)\). A graph \(G\) is finite if \(|V(G)| < \infty\); if \(E(G) = \emptyset\), then \(G\) is an empty graph. A finite nonempty graph \(P \subseteq G\) is called a path (in \(G\)) if the vertices of \(P\) can be arranged in a sequence \((v_1, v_2, \ldots, v_n)\) in such a way that

\[ \{\{v_i, v_j\} \in E(P)\} \iff |i - j| = 1. \]

We shall identify the path \(P\) with the sequence \((v_1, v_2, \ldots, v_n)\) and say that \(P\) joins \(v_1\) and \(v_n\). A finite subgraph \(C \subseteq G\) is called a cycle if \(|V(C)| \geq 3\) and its vertices can be arranged in a cyclic sequence \((v_1, v_2, \ldots, v_n)\), \(n = |V(C)|\), in such a way that

\[ \{\{v_i, v_j\} \in E(C)\} \iff (|i - j| = 1 \quad \text{or} \quad |i - j| = n - 1). \]
Two vertices in a graph are \textit{connected} if there is a path that joins them. A \textit{connected graph} is a graph in which any pair of vertices is connected.

A graph \( G = (V, E) \) endowed with a function \( w : E \to \mathbb{R}^+ = [0, +\infty) \) is called a \textit{weighted graph}; the function \( w \) is the \textit{weight} or the \textit{weight function}. Weighted graphs will be denoted by \((G, w)\).

Now let us recall the necessary definitions of the theory of metric spaces. An \textit{ultrametric} on a set \( X \) is a function \( d : X \times X \to \mathbb{R}^+ \) such that for any \( x, y, z \in X \):

(i) \( d(x, y) = d(y, x) \),
(ii) \( d(x, y) = 0 \iff (x = y) \),
(iii) \( d(x, y) \leq \max\{d(x, z), d(z, y)\} \).

If condition (ii) is replaced with the weaker condition (ii\textsuperscript{′}) \( d(x, x) = 0 \), then \( d \) is called a \textit{pseudoultrametric}. Inequality (iii) is often referred to as the \textit{strong triangle inequality}.

A function \( d : X \times X \to \mathbb{R}^+ \) satisfying the standard triangle inequality and conditions (i)–(ii\textsuperscript{′}) is called a \textit{pseudometric}.

If \((G, w)\) is a weighted graph and

\[
2 \max_{e \in E(C)} w(e) \leq \sum_{e \in E(C)} w(e),
\]

for all cycles \( C \subseteq G \), then there exists a pseudometric \( d : V(G) \times V(G) \to \mathbb{R}^+ \) such that

\[ w(\{x, y\}) = d(x, y), \]

for all \( \{x, y\} \in E(G) \). For weighted graphs \((G, w)\) in which \( G \) is connected, an example of such a pseudometric is the well-known ‘shortest path pseudometric’

\[
d_w(x, y) = \begin{cases} 0 & \text{if } x = y, \\
\inf_{P \in \mathcal{P}_{x,y}} \sum_{e \in P} w(e) & \text{if } x \neq y,
\end{cases}
\]

where \( \mathcal{P}_{x,y} \) is the set of all paths joining \( x \) and \( y \) in \( G \). This result was established in [2]. In the same work, the following problem was formulated: what are the conditions under which a weight \( w \) is pseudoultrametrizable, that is, there exists a pseudoultrametric \( d \) which extends \( w \) in the sense that (1.2) holds for all edges \( \{x, y\} \) of the graph \( G \)?

In this work we obtain the following results.

- **Theorem 2.** A criterion for pseudoultrametrizability of \( w \) is the presence of at least two edges of maximum weight in each cycle. This answers the question formulated above.

- **Theorem 1, Theorem 3, Corollary 1.** The function

\[
\rho_w(x, y) := \begin{cases} 0 & \text{if } x = y, \\
\inf_{P \in \mathcal{P}_{x,y}} \max_{e \in P} w(e) & \text{if } x \neq y,
\end{cases}
\]

determines the subdominant pseudoultrametric for an arbitrary weight \( w \); it is the greatest pseudoultrametric for a pseudoultrametrizable \( w \).
• Theorem 5, Corollary 7. For a connected $G$ the existence of a strictly positive pseudoultrametrizable weight $w$ which is extendable by no ultrametric is equivalent to the existence of two vertices $u^*, v^* \in V(G)$ and a sequence of paths $F_i \in \mathcal{P}_{u^*, v^*}$ for which $\limsup_{i \to \infty} V(F_i)$ is an independent subset of $V(G)$.

• Theorem 6. $G$ is a complete $k$-partite graph if and only if for any pseudoultrametrizable weight $w$ there exists a least pseudoultrametric extending it.

• Theorem 7. A criterion for the uniqueness of the extension in terms of the $\varepsilon$-connectivity of the graph vertices.

The starting point for the proof of the main results in this paper is formula (1.4), which can be treated as the ‘natural ultrametric analogue’ of the shortest path pseudometric. Indeed, it can be shown that the metric space $(X, d)$ is ultrametric if and only if $d^\alpha$ is a metric for any $\alpha > 0$ (see [3]). Now replacing $\sum_{e \in P} w(e)$ in (1.3) with $\left(\sum_{e \in P} w^\alpha(e)\right)^{1/\alpha}$ and passing to the limit as $\alpha \to \infty$, we obtain the expression $\max_{e \in P} w(e)$, which is involved in (1.4).

The above reasoning ‘explains’ why many of the results of this work are similar to the corresponding results of [2]. More formally, the analogy between the shortest path pseudometric and the subdominant pseudoultrametric is drawn in Corollaries 4–6 and Proposition 1.

Note that the problem of extending a weight that is specified on the graph edges to an ultrametric or, in the more general case, to a metric of some special form, naturally arises in the study of the problem of determining whether a data set obtained from an experiment over pairs of objects exhibits properties which are interesting to the researcher.

§ 2. Subdominant pseudoultrametric

In the lemma below and later on we identify the pseudoultrametric space $(X, d)$ with the complete weighted graph $(G, w_d)$ such that $V(G) = X$ and

$$w_d\{x, y\} = d(x, y),$$

for any pair of distinct points $x, y \in X$.

Lemma 1. Let $(X, d)$ be a pseudoultrametric space. Then for any cycle $C$ with $V(C) \subseteq X$ there exist at least two different edges $e_1, e_2$ such that

$$w_d(e_1) = w_d(e_2) = \max_{e \in E(C)} w_d(e).$$

Proof. Let $q(C)$ be the number of edges in cycle $C$. If $q(C) = 3$, then (2.2) follows from the strong triangle inequality. Assume that (2.2) is true for $q(C) \leq n$, but that there exists a cycle with $q(C) = n + 1$ which contains exactly one edge $e_1 = \{x, y\}$ such that

$$w_d(e_1) = \max_{e \in C} w_d(e).$$

Let $z$ be the vertex of cycle $C$ which is adjacent to $y$ but distinct from $x$. In view of the uniqueness of the edge of maximum weight, we have

$$d(y, z) < d(x, y).$$
Hence, because of the strong triangle inequality, this implies \( d(x, z) = d(x, y) \). Let \( C_1 \) be the cycle for which

\[
V(C_1) = V(C) \setminus \{y\}, \quad E(C_1) = (E(C) \setminus \{\{x, y\}, \{y, z\}\}) \cup \{\{x, z\}\}.
\]

Then \( q(C_1) = n \) and \( \{x, z\} \) is the unique edge of maximum weight, which contradicts the inductive hypothesis.

**Remark 1.** It is quite possible that the lemma proved above is well known. In any case, the situation where the graph contains two edges of maximum length occurs very often in the study of ultrametrics and their generalizations. For instance, so-called 2-ultrametric spaces are characterized by the fact that in any four-point subspace the length of at least two edges is equal to the diameter of the subspace; see [4].

We now pass to the definition of the subdominant pseudoultrametric.

On the set \( \mathfrak{F} \) of pseudometrics specified on \( X \) we introduce a partial order \( \preceq \) as follows:

\[
d_1 \preceq d_2 \iff \forall x, y \in X : d_1(x, y) \leq d_2(x, y).
\]  

(2.3)

In [5], for a given metric space \( (X, d) \), the subdominant ultrametric was defined as the greatest element of the partially ordered set \( (\mathfrak{F}_d, \preceq) \), where \( \mathfrak{F}_d \subseteq \mathfrak{F} \) is the set of ultrametrics \( \delta \) such that

\[
\delta(x, y) \leq d(x, y),
\]

for all \( x, y \in X \). We generalize this definition to weighted graphs.

**Definition 1.** Let \( (G, w) \) be a nonempty weighted graph and \( \mathfrak{F}_{w,u} \) be the family of all pseudoultrametrics \( \rho \) such that

\[
\rho(u, v) \leq w(\{u, v\}),
\]

for any edge \( \{u, v\} \in E(G) \). If the partially ordered set \( (\mathfrak{F}_{w,u}, \preceq) \) contains the greatest element, then this element is called the subdominant pseudoultrametric for the weight \( w \).

Note that \( \mathfrak{F}_{w,u} \neq \emptyset \) since the zero pseudoultrametric

\[
\rho(u, v) = 0 \quad \forall u, v \in V(G)
\]

belongs to \( \mathfrak{F}_{w,u} \).

We now pass to the construction of the subdominant pseudoultrametric.

Let \( u, v \) be two distinct vertices of a connected weighted graph \( (G, w) \). As in (1.3), we denote by \( \mathcal{P}_{u,v} \) the set of paths joining \( u \) and \( v \). On the Cartesian square \( V(G) \times V(G) \) we define a function \( \rho_w \) by means of formula (1.4).

**Theorem 1.** For any nonempty connected weighted graph \( (G, w) \) the function \( \rho_w \) is the subdominant pseudoultrametric.

**Proof.** We verify the validity of the strong triangle inequality

\[
\rho_w(u, v) \leq \max\{\rho_w(u, p), \rho_w(p, v)\}
\]  

(2.4)

for any three distinct vertices \( u, v, p \in V(G) \).
Let \( \varepsilon \) be an arbitrary positive number. There exist paths \( P_1 \in \mathfrak{P}_{u,p} \) and \( P_2 \in \mathfrak{P}_{p,v} \) such that
\[
\rho_w(u, p) + \varepsilon \geq \max_{e \in P_1} w(e), \quad \rho_w(p, v) + \varepsilon \geq \max_{e \in P_2} w(e).
\] (2.5) The subgraph \( G_1 \subseteq G \) with \( V(G_1) = V(P_1) \cup V(P_2) \) and \( E(G_1) = E(P_1) \cup E(P_2) \) is connected. Let \( P_3 \) be a path in \( G_1 \) that joins \( u \) and \( v \). Then, taking into account (2.5), we derive
\[
\rho_w(u, v) \leq \max_{e \in P_3} w(e) \leq \max \{ \max_{e \in P_1} w(e), \max_{e \in P_2} w(e) \} \leq \max \{ \rho_w(u, p) + \varepsilon, \rho_w(p, v) + \varepsilon \}.
\]

Now, letting \( \varepsilon \) tend to zero, we establish (2.4).

It remains to verify that the pseudoultrametric \( \rho_w \) is subdominant. Assume that there exist \( \rho \in \mathfrak{S}_{w,u} \) and \( v_1, v_2 \in V(G) \) such that
\[
\rho(v_1, v_2) > \rho_w(v_1, v_2).
\] (2.6) Together with (1.4) this inequality implies the existence of a path \( P \in \mathfrak{P}_{v_1,v_2} \) such that
\[
\rho(v_1, v_2) > \max_{e \in P} w(e).
\]

Note that \( \rho(u, v) \leq w(\{u, v\}) \) for all \( \{u, v\} \in E(G) \). Consequently, the path \( P \) does not contain \( \{v_1, v_2\} \). In the pseudoultrametric space \( (V(P), \rho) \) we now consider the cycle \( C \) with
\[
V(C) = V(P), \quad E(C) = E(P) \cup \{v_1, v_2\}.
\]
Then \( \{v_1, v_2\} \) is the only edge of cycle \( C \) on which \( \max \{ \rho(x, y) : \{x, y\} \in E(C) \} \) is attained. This contradicts Lemma 1. Therefore, for any \( v_1, v_2 \in V(G) \) and any \( \rho \in \mathfrak{S}_{w,u} \) we have the inequality \( \rho(v_1, v_2) \leq \rho_w(v_1, v_2) \), which means that \( \rho_w \) is the greatest element of the partially ordered set \( (\mathfrak{S}_{w,u}, \preceq) \).

**Remark 2.** If \( G \) is a finite graph and a weight \( w \) is defined by some metric as in (1.2), then the subdominant pseudoultrametric \( \rho_w \) is in fact an ultrametric. For complete graphs \( G \) this classical case was considered as early as [6]. An efficient procedure for the calculation of the subdominant ultrametric for finite metric spaces can be found in [7] and [8].

We now investigate the relationship between the shortest path pseudometric and the subdominant pseudoultrametric.

By analogy with Definition 1 we introduce the following

**Definition 2.** Let \((G, w)\) be a nonempty weighted graph and \( \mathfrak{S}_{w,m} \) be the set of all pseudometrics \( d \) such that
\[
d(u, v) \leq w(\{u, v\}),
\] (2.7) for all edges \( \{u, v\} \in E(G) \). The subdominant pseudometric (for the weight \( w \)) is the greatest element of the partially ordered set \( (\mathfrak{S}_{w,m}, \preceq) \), provided that such an element exists.

**Proposition 1.** Let \((G, w)\) be a nonempty connected weighted graph. Then the pseudometric \( d_w \) defined by formula (1.3) is the subdominant pseudometric.
Proof. Inequality (2.7) holds for any \( \{u, v\} \in E(G) \). This follows from (1.3) and the fact that the two-element sequence \( u, v \) is a path belonging to \( \mathcal{P}_{u,v} \). Consequently, \( d_w \in \mathfrak{F}_{w,m} \). Assume that there exist \( d \in \mathfrak{F}_{w,m} \) and \( u, p \in V(G) \) for which

\[
d(u, p) > d_w(u, p).\]

Then one can find a path \( (u = x_1, \ldots, x_n = p) \in \mathcal{P}_{u,p} \) such that

\[
d(u, p) > \sum_{i=1}^{n-1} w(x_i, x_{i+1}). \tag{2.8}\]

Since \( d \in \mathfrak{F}_{w,m} \), we have \( w(x_i, x_{i+1}) \geq d(x_i, x_{i+1}) \) for \( i = 1, \ldots, n - 1 \). In view of (2.8), this gives

\[
d(x_1, x_n) \geq \sum_{i=1}^{n-1} d(x_i, x_{i+1}),
\]

which contradicts the triangle inequality.

Corollary 1. Let \((G, w)\) be a nonempty connected weighted graph. Then the pseudoultrametric \( \rho_w \) constructed by the rule (1.4) is the subdominant pseudoultrametric for \( d_w \), which is to say that \( \rho_w \leq d_w \) and \( \rho \leq \rho_w \) for any pseudoultrametric \( \rho \) satisfying the condition \( \rho \leq d_w \).

Proof. Denote by \( \rho_w^* \) the subdominant pseudoultrametric for \( d_w \). It follows from (2.1) and (1.3) that \( \rho_w \leq d_w \), and hence \( \rho_w \leq \rho_w^* \). The inverse relation \( \rho_w^* \leq \rho_w \) follows from the fact that any pseudoultrametric \( \rho \) satisfying the condition \( \rho \leq d_w \) belongs to the set \( \mathfrak{F}_{w,u} \) (see Definition 1). Thus, \( \rho_w^* = \rho_w \).

Remark 3. It is interesting to find conditions under which \( d_w \) and \( \rho_w \) are a metric and an ultrametric, respectively. We shall study this in §4. Note that the problem of finding a criterion for the existence of a subdominant ultrametric was posed in [9].

§3. Pseudoultrametrization of weighted graphs

Definition 3. Let \((G, w)\) be a weighted graph and \( d : V(G) \times V(G) \to \mathbb{R}^+ \) be an ultrametric (pseudoultrametric). We say that \( d \) extends \( w \) if (1.2) holds for all \( \{x, y\} \in E(G) \).

Definition 4. A weight \( w \) is said to be ultrametrizable (pseudoultrametrizable) if there exists an ultrametric (pseudoultrametric) that extends it.

The following theorem provides a criterion for pseudoultrametrizability of a weight \( w \).

Theorem 2. Let \((G, w)\) be a nonempty weighted graph. The weight \( w \) is pseudoultrametrizable if and only if for any cycle \( C \subseteq G \) there exist at least two different edges \( e_1, e_2 \in E(C) \) such that

\[
w(e_1) = w(e_2) = \max_{e \in E(C)} w(e). \tag{3.1}\]

If \( G \) is a connected graph and \( w \) is a pseudoultrametrizable weight, then the subdominant pseudoultrametric extends \( w \).
Proof. It follows from Lemma 1 that (3.1) holds for any cycle $C \subseteq G$ if $w$ is a pseudoultrametrizable weight.

Conversely, suppose that for any cycle $C \subseteq G$ there exist $e_1, e_2 \in E(C)$ such that (3.1) holds. We show that this entails the pseudoultrametrizability of $w$. First, consider the case of a connected graph $G$. By virtue of Theorem 1 it suffices to prove that for all edges $\{u, v\} \in E(G)$ we have

$$\rho_w(u, v) = w(\{u, v\}),$$

(3.2)

where $\rho_w$ is the subdominant pseudoultrametric for $w$. By definition, we have

$$\rho_w(u, v) \leq w(\{u, v\}).$$

If the strict inequality holds, then there exists a path $P \in \mathfrak{P}_{u,v}$ such that

$$\max_{e \in P} w(e) < w(\{u, v\}).$$

The last inequality implies $\{u, v\} \notin E(P)$ since if $\{u, v\} \in E(G)$, then there exists a cycle $C$ for which

$$V(C) = V(P), \quad E(C) = E(P) \cup \{\{u, v\}\}.$$
Figure 1. Passing from a disconnected graph $G$ to a connected graph $\tilde{G}$.

graph $\tilde{G}$ that this edge is of the form $\{v_{i_0}, v_{i_2}\}$. Removing the vertex $v_{i_0}$ from the cycle $C$, we obtain a path $P$,

$$V(P) = V(C) \setminus \{v_{i_0}\}, \quad E(P) = E(C) \setminus \{\{v_{i_0}, v_{i_1}\}, \{v_{i_0}, v_{i_2}\}\},$$

which joins $v_{i_1}$ and $v_{i_2}$. Since $E(P) \subseteq E(G)$, the vertices $v_{i_1}$ and $v_{i_2}$ belong to the same connected component, which contradicts the way they have been chosen.

**Remark 4.** Condition (3.1) is obviously equivalent to the strong triangle inequality if the cycle $C$ contains exactly three vertices. Note also that

$$\sum_{e \in E(C)} w(e) \geq w(e_1) + w(e_2) = 2 \max_{e \in E(C)} w(e),$$

provided that (3.1) holds true. Thus, pseudoultrametrizability of the weight function entails its pseudometrizability, as one would expect.

**Remark 5.** It immediately follows from Theorem 2 that pseudoultrametrizability of weight is a local property, which is to say that if for any finite subgraph $H$ of a weighted graph $(G, w)$ the weight $w$ restricted to $E(H)$ is pseudoultrametrizable, then the weight $w$ itself is pseudoultrametrizable as well.

Recall that a *forest* is a graph which has no cycles and a *tree* is a connected forest.

**Corollary 2.** Let $G$ be a nonempty graph. Then $G$ is a forest if and only if any weight $w : E(G) \to \mathbb{R}^+$ is pseudoultrametrizable.

To prove the corollary it is sufficient to note that the existence of a cycle $C \subseteq G$ entails the existence of a weight $w : E(G) \to \mathbb{R}^+$ for which condition (3.1) is violated.

Let $(G, w)$ be a nonempty weighted graph with a pseudoultrametrizable weight $w$. Denote by $\mathcal{U}_w$ the family of all pseudoultrametrics on $V(G)$ which extend $w$.

**Theorem 3.** If $G$ is a connected graph, then the subdominant pseudoultrametric $\rho_w$ is the greatest element of the partially ordered set $(\mathcal{U}_w, \leq)$. Conversely, if the partially ordered set $(\mathcal{U}_w, \leq)$ has a greatest element, then the graph $G$ is connected.
Proof. Let \( \mathcal{F}_{w,u} \) be the family mentioned in Definition 1 and suppose that \( G \) is a connected graph. Then \( \mathcal{F}_{w,u} \supseteq \mathcal{U}_w \) and the subdominant pseudoultrametric \( \rho_w \) belongs to \( \mathcal{F}_{w,u} \). To prove that \( \rho_w \) is the greatest element in \( (\mathcal{U}_w, \leq) \), it suffices to verify the inclusion \( \rho_w \in \mathcal{U}_w \), which has already been established in Theorem 2.

Now assume that \( G \) is disconnected. Take points \( v_{i_0} \) and \( v_{i_1} \) belonging to distinct connected components and consider the weighted graph \( (\hat{G}, \hat{w}) \) as in the proof of Theorem 2. It is clear that \( \mathcal{U}_w \supseteq \mathcal{U}_{\hat{w}} \). Since the constants \( c_i \) in formula (3.3) can be taken arbitrarily, the last inclusion implies the relation

\[
\sup_{\rho \in \mathcal{U}_w} \rho(v_{i_0}, v_{i_1}) = +\infty.
\]

Thus, in the case of a disconnected graph, the partially ordered set \( (\mathcal{U}_w, \leq) \) contains no greatest element.

With the use of the above theorem one can easily derive a statement which is converse to Corollary 1.

**Corollary 3.** Let \((G, w)\) be a nonempty weighted graph. If there exists a subdominant pseudoultrametric for \( w \), then the graph \( G \) is connected.

**Remark 6.** In Corollaries 1 and 3 we do not require pseudoultrametrizability of the weight \( w \).

To make the analogy between \( \rho_w \) and \( d_w \) more apparent, let us denote by \( \mathcal{M}_w \) the family of all pseudometrics which extend \( w \).

As is shown in [2], in the case of a connected graph \( G \), the shortest path pseudometric \( d_w \) belongs to \( \mathcal{M}_w \) for any pseudometrizable weight \( w \). Moreover, \( \mathcal{M}_w \) being endowed with the partial order \( \preceq \) as a subset of the partially ordered set \( (\mathcal{F}, \preceq) \) (see (2.5)) the following analogue of Theorem 3 can be formulated.

**Theorem 4** (see [2]). Let \((G, w)\) be a nonempty weighted graph with a pseudometrizable weight \( w \). If \( G \) is a connected graph, then \( d_w \) is the greatest element of the partially ordered set \( (\mathcal{M}_w, \preceq) \). Conversely, if \( (\mathcal{M}_w, \preceq) \) has a greatest element, then \( G \) is a connected graph.

Theorems 3 and 4 yield

**Corollary 4.** Let \( G \) be a nonempty graph. The following conditions are equivalent:

(i) \( G \) is a connected graph;

(ii) the partially ordered set \( (\mathcal{M}_w, \preceq) \) contains the greatest element for any pseudometrizable weight \( w : E(G) \to \mathbb{R}^+ \);

(iii) the partially ordered set \( (\mathcal{U}_w, \leq) \) contains the greatest element for any pseudoultrametrizable weight \( w : E(G) \to \mathbb{R}^+ \).

Combining Corollary 2 and Theorem 3 of this work with Proposition 3.3 and Corollary 3.6 of [2], we obtain

**Corollary 5.** Let \( G \) be a nonempty graph. Then the following conditions are equivalent:

(i) \( G \) is a tree;

(ii) any weight \( w : E(G) \to \mathbb{R}^+ \) is pseudometrizable and the partially ordered set \( (\mathcal{M}_w, \preceq) \) contains the greatest element;
(iii) any weight $w: E(G) \to \mathbb{R}^+$ is pseudoultrametrizable and the partially ordered set $(\mathcal{U}_w, \preceq)$ contains the greatest element.

Some other examples which illustrate the analogy between $\rho_w$ and $d_w$ are given in the next section.

**Remark 7.** If the weight $w$ is pseudometrizable, but not pseudoultrametrizable, then we can pose the problem of constructing an extension of $w$ which is as ‘ultrametric’ as possible. An appropriate measure of ‘ultrametricity’ can be introduced with the use of the so-called ‘betweenness exponent’, which is defined for a given metric $d$ as the supremum of $\alpha \geq 1$ for which $d^\alpha$ is a metric as well (see [3], [10]). Note that the notion of the betweenness exponent is easily generalized to weighted graphs; for $\alpha = 1$ it reduces to the condition (1.1) of pseudometrizability of a weight $w$, and for $\alpha = \infty$, to the condition (3.1) of pseudoultrametrizability of $w$.

§ 4. Ultrametrization of weighted graphs

In the previous section we showed that a weight $w: E(G) \to \mathbb{R}^+$ is pseudoultrametrizable if and only if for any cycle $C \subseteq G$ condition (3.1) holds true. If a pseudoultrametrizable weight $w$ is strictly positive, that is, for any $e \in E(G)$ we have

$$w(e) > 0,$$

and the graph $G$ is finite and connected, then it is easily seen that the subdominant pseudoultrametric $\rho_w$ is an ultrametric. As is shown by the following example, in the case of an infinite graph $G$ the fact that $w$ is strictly positive does not guarantee that $\rho_w$ is an ultrametric.

![Figure 2. A weighted graph $G$ with a pseudoultrametrizable positive weight $w$ for which $\mathcal{U}_w$ contains no ultrametrics.](image)

**Example 1.** Let $(G, w)$ be the infinite weighted graph depicted in Fig. 2, where

$$\varepsilon_n = w(\{u, s_n\}) = w(\{s_n, t_n\}) = w(\{t_n, v\})$$
are positive real numbers such that \( \lim_{n \to \infty} \varepsilon_n = 0 \) and \( \varepsilon_n > \varepsilon_{n+1} \) for any \( n \). The length of any cycle \( C \subseteq G \) is equal to 6 and the vertices of the cycle are \( u, s_n, t_n, v, t_m, s_m \), where \( m > n \). The cycle \( C \) has three different edges of maximum weight; hence, \( w \) is pseudoultrametrizable. It follows from the definition of \( \rho_w \) that

\[
\rho_w(u, v) = \max\{w(\{u, s_n\}), w(\{s_n, t_n\}), w(\{t_n, v\})\} = \varepsilon_n.
\]

Letting \( n \to \infty \), we obtain \( \rho_w(u, v) = 0 \).

It follows from Theorem 3 that for a connected graph \( G \) the set \( \mathcal{U}_w \) contains ultrametrics if and only if \( \rho_w \) is an ultrametric.

In this section we describe the structure of graphs \( G \) such that \( \rho_w \) is an ultrametric for any strictly positive pseudoultrametrizable weight \( w \).

Note that [11] offers a complete characterization of metric spaces \((X, d)\) such that the subdominant for \( d \) pseudoultrametric is an ultrametric.

Below we shall use the following Lemma established in [2].

**Lemma 2.** Let \( G \) be a connected graph and \( u^*, v^* \) be two nonadjacent vertices of graph \( G \). Further, let \( \tilde{F} = \{F_j\}_{j \in \mathbb{N}} \) be a sequence of paths connecting \( u^* \) and \( v^* \) which satisfy the following condition:

1. (i) for each \( e^0 \in E(G) \) there exist \( u^0 \in e^0 \) and \( i = i(e^0) \) such that
   \[
   u^0 \notin \bigcup_{k=1}^\infty V(F_{i+k}).
   \]

Then there exists a subsequence \( \{F_{jk}\}_{k \in \mathbb{N}} \) of the sequence \( \tilde{F} \) such that

2. (i2) for \( l \neq k \),
   \[
   E(F_{jl}) \cap E(F_{jk}) = \emptyset;
   \]

3. (i3) if \( C \) is a cycle in the graph \( \bigcup_{k \in \mathbb{N}} F_{jk} \) and
   \[
   k_0 = k_0(C) := \min\{k \in \mathbb{N} : E(C) \cap E(F_{jk}) \neq \emptyset\},
   \]
   then \( C \) and \( F_{jk_0} \) have at least two common edges.

**Remark 8.** Here and below, by the union \( \bigcup_{i \in \mathcal{I}} G_i \) of subgraphs \( G_i \) of a graph \( G \) we mean the subgraph \( \tilde{G} \subseteq G \) such that

\[
V(\tilde{G}) = \bigcup_{i \in \mathcal{I}} V(G_i) \quad \text{and} \quad E(\tilde{G}) = \bigcup_{i \in \mathcal{I}} E(G_i).
\]

In the following theorem we formulate the main result of this section.

**Theorem 5.** Let \( G = (V, E) \) be a nonempty connected graph. Then the following two conditions are equivalent:

1. (i) for any strictly positive pseudoultrametrizable weight \( w \) the subdominant pseudoultrametric \( \rho_w \) is an ultrametric;
2. (ii) for any pair of distinct vertices \( u^*, v^* \in V(G) \) and any sequence \( \tilde{F} \) of paths \( F_j \in \mathcal{P}_{u^*, v^*} \), \( j \in \mathbb{N} \), one can find an edge \( e^0 = \{u^0, v^0\} \in E(G) \) such that for all \( i > 0 \)
   \[
   u^0, v^0 \in \bigcup_{k=1}^\infty V(F_{i+k}).
   \]
Proof. (i) $\implies$ (ii). Assume that condition (ii) is violated. Then there exists a pair of distinct vertices $u^*, v^*$ and a sequence $\{F_j\} \subseteq \mathcal{P}_{u^*, v^*}$ such that for any edge $e^0 \in E(G)$, $e^0 = \{u^0, v^0\}$, one can find $i \in \mathbb{N}$ and at least one vertex incident with $e^0$ (for instance, $u^0$) such that

$$u^0 \notin \bigcup_{k=1}^{\infty} V(F_{i+k}).$$

We show that in this case condition (i) is violated as well. By virtue of Lemma 2 it can be assumed without loss of generality that

$$E(F_i) \cap E(F_j) = \emptyset \text{ for } i \neq j, \quad (4.2)$$

and any cycle $C$ in $\bigcup_{j \in \mathbb{N}} F_j$ has at least two common edges with $F_{k_0}$, where $k_0$ is defined as in (4.1).

Consider the graph

$$\tilde{G} = \bigcup_{i \in \mathbb{N}} F_i$$

and let $\{\varepsilon_i\}_{i \in \mathbb{N}}$ be a strictly decreasing sequence of positive real numbers such that $\lim_{i \to \infty} \varepsilon_i = 0$. We define the weight $w_1 : E(\tilde{G}) \to \mathbb{R}^+$ as follows:

$$w_1(e) := \varepsilon_i$$

if $e \in E(F_i)$. This is well-defined by virtue of (4.2). The weights of all edges of the path $F_i$, $i \in \mathbb{N}$, are equal and the weights of the edges of the path $F_{i+1}$ are strictly less than the weights of the edges of the path $F_i$, $i \in \mathbb{N}$. Any cycle $C$ that is contained in the graph $\tilde{G}$ has at least two common edges $e_1$ and $e_2$ with the path $F_{k_0}$, where $k_0$ is the least number among the labels of the paths which go through edges of the cycle $C$. Hence, the weights of these edges are maximum in this cycle and are equal to each other: $w_1(e_1) = w_1(e_2) = \varepsilon_{k_0} = \max_{e \in E(C)} w_1(e)$. By virtue of Theorem 2 the weight $w_1$ is pseudoultrametrizable. Having chosen the pseudoultrametrization $\rho_{w_1}(u, v)$ as in (1.4) we obtain

$$\rho_{w_1}(u^*, v^*) = \inf_{i \in \mathbb{N}} \max_{e \in F_i} w(e) = \inf_{i \in \mathbb{N}} \varepsilon_i = 0. \quad (4.3)$$

Thus, $\rho_{w_1}$ is a pseudoultrametric on the set $V(\tilde{G})$ which is not a metric.

With the use of the pseudoultrametric $\rho_{w_1}$, we extend the weight function $w_1$ to the set of all edges of the graph $G$ whose ends lie in $V(\tilde{G})$. We demonstrate that the weight obtained is strictly positive as well. For this weight we shall keep the notation $w_1$.

By our assumption, for at least one end of the edge $e^0$, say $u^0$, there exists an index $i_0$ such that

$$u^0 \notin V(F_i), \quad (4.4)$$

for all $i > i_0$. Let $F$ be an arbitrary path in $\tilde{G}$ that joins $u^0$ with $v^0$, and $e \in E(F)$ be the edge of this path that is incident with $u^0$. It follows from (4.4) that

$$e \in \bigcup_{i=1}^{i_0} E(F_i).$$
Since the sequence \( \{\varepsilon_i\}_{i \in \mathbb{N}} \) decreases, we have the inequalities \( w(e) \geq \varepsilon_{i_0} \) and \( \max_{e \in E} w(e) \geq \varepsilon_{i_0} > 0 \). Hence,

\[
    w_1(\{u_0, v_0\}) = \rho_{w_1}(u^0, v^0) > 0.
\]

Let us assign a positive weight to the edges of the graph \( G \) for which \( u \notin V(\tilde{G}) \) or \( v \notin V(\tilde{G}) \). The maximum weight of an edge \( e \) in the graph \( \tilde{G} \) is \( \varepsilon_1 \). Let us take a real number \( M \geq \varepsilon_1 \) and assign the weight \( w_2(e) = M \) to each edge \( e = \{u, v\} \) of this kind. Further, define \( w_2(e) := \rho_{w_1}(u, v) \) in the case when \( e = \{u, v\} \) with \( u, v \in V(\tilde{G}) \). It is easily seen that the weight \( w_2 \) defined in this way is positive on all edges of the graph \( G \) and at the same time it is pseudoultrametrizable. Indeed, if \( C \) is a cycle in \( G \) such that all vertices of \( C \) belong to \( \tilde{G} \); then the existence of two edges in \( C \) of maximum weight follows from the fact that \( \rho_{w_1} \) is a pseudoultrametric. Now suppose that \( v \in V(C) \), but \( u \notin V(\tilde{G}) \), then the two edges of the cycle \( C \) which are incident with \( v \) have the maximum weight \( M \). Pseudoultrametrizability of \( w_2 \) follows from Theorem 2. Since \( \tilde{G} \subseteq G \) and \( w_1 = w_2|_{E(\tilde{G})} \), definition (1.4) yields

\[
    \rho_{w_2}(u, v) \leq \rho_{w_1}(u, v), \tag{4.5}
\]

for \( u, v \in V(\tilde{G}) \). Relations (4.3) and (4.5) imply \( \rho_{w_2}(u^*, v^*) = 0 \). Thus, we have obtained a strictly positive pseudoultrametrizable weight \( w_2 \) for which the subdominant pseudoultrametric \( \rho_{w_2} \) is not an ultrametric.

(ii) \( \implies \) (i). Let \( (G, w) \) be a weighted graph with a pseudoultrametrizable strictly positive weight and suppose that condition (ii) is satisfied. We need to show that the pseudoultrametric \( \rho_w \), defined on the set of vertices of the graph \( G \) is in fact an ultrametric.

Assume that there are vertices \( u^* \) and \( v^* \), \( u^* \neq v^* \), such that \( \rho_w(u^*, v^*) = 0 \). Then there exists a sequence \( \{F_k\}_{k \in \mathbb{N}} \) satisfying the condition that for any \( \varepsilon > 0 \) one can find a \( k(\varepsilon) \in \mathbb{N} \) such that

\[
    \max_{e \in F_k} w(e) < \varepsilon \tag{4.6}
\]

for all \( k \geq k(\varepsilon) \). Moreover, by virtue of condition (ii), there exists an edge \( e^0 = \{u^0, v^0\} \in E(G) \) such that \( u^0, v^0 \in \bigcup_{i=1}^{\infty} V(F_{k+1}) \) for all \( k > 0 \).

In the graph \( G_\varepsilon := \bigcup_{i=1}^{\infty} F_{k(\varepsilon)+i} \) consider a path \( P \) that joins \( u^0 \) and \( v^0 \). It follows from (4.6) and from the definition of \( G_\varepsilon \) that the inequality

\[
    w(e) < \varepsilon
\]

holds for all \( e \in P \). From the definition of the function \( \rho_w \) it follows that

\[
    \rho_w(u^0, v^0) \leq \max_{e \in P} w(e) < \varepsilon.
\]

Letting \( \varepsilon \) tend to zero, we obtain \( \rho_w(u^0, v^0) = 0 \), and since the weight \( w \) is strictly positive and pseudoultrametrizable, by Theorem 2 we have

\[
    0 < w(\{u^0, v^0\}) = \rho_w(u^0, v^0).
\]

The contradiction obtained completes the proof.
Theorem 5 established above and Theorem 3.11 from [2] imply

**Corollary 6.** Let $G$ be a nonempty connected graph. Then the following two conditions are equivalent:

(i) for any strictly positive pseudoultrametrizable weight $w: E(G) \to \mathbb{R}^+$, the subdominant pseudoultrametric $\rho_w$ is in fact an ultrametric;

(ii) for any strictly positive pseudometrizable $w: E(G) \to \mathbb{R}^+$ the shortest path pseudometric $d_w$ is in fact a metric.

With the use of Theorems 3 and 5 we can easily describe the structure properties of a graph $G$ for which there exists a strictly positive pseudoultrametrizable weight $w$ such that $\mathcal{U}_w$ contains no ultrametrics.

Let us recall some definitions.

For an arbitrary sequence of sets $\{A_n\}_{n \in \mathbb{N}}$ the upper limit of this sequence (denoted as $\limsup_{n \to \infty} A_n$) is the set of all elements $a$ such that $a \in A_n$ for infinitely many values of the index $n$, that is,

$$\limsup_{n \to \infty} A_n = \bigcap_{k=1}^{\infty} \left( \bigcup_{n=1}^{\infty} A_{n+k} \right).$$

A subset $V_0$ of vertices of a graph $G$ is said to be independent if no two vertices in $V_0$ are adjacent.

**Corollary 7.** Let $G = (V, E)$ be a nonempty connected graph. Then the following conditions are equivalent:

(i) there exists a strictly positive pseudoultrametrizable weight such that the set $\mathcal{U}_w$ contains no ultrametrics;

(ii) there exist two vertices $u^*, v^* \in V(G)$ and a sequence of paths $\{F_j\}_{j \in \mathbb{N}}$, $F_j \in \mathcal{P}_{u^*, v^*}$, such that

$$\limsup_{j \to \infty} V(F_j)$$

is an independent set.

We now give examples of graphs $G$ for which any strictly positive pseudoultrametrizable weight $w$ can be extended to an ultrametric.

**Example 2.** If each connected component of a nonempty graph $G = (V, E)$ contains at most one vertex of infinite degree, then for any strictly positive pseudoultrametrizable weight $w: E(G) \to \mathbb{R}^+$ there exists an ultrametric $\rho \in \mathcal{U}_w$.

Indeed, let $\{G_i : i \in \mathcal{I}\}$ be the collection of connected components of the graph $G$. If necessary, we embed $G$ in a connected graph $\tilde{G}$ as in the proof of Theorem 2. If some vertices $u^*$ and $v^*$ belong to the same connected component $G_i$, then one of the vertices, say $u^*$, is incident with finitely many edges $e_1, e_2, \ldots, e_n$. Then for any infinite sequence of paths $F_i \in \mathcal{P}_{u^*, v^*}$, one of the edges $e_i$ is gone through by infinitely many paths.

If two vertices $u^*$ and $v^*$ belong to distinct connected components, then there exists an edge of the form $\{v_{i_0}, v_i\}$ which belongs to all of the paths $F_i \in \mathcal{P}_{u^*, v^*}$. Applying Theorem 5 to the graph $\tilde{G}$ we establish the existence of an ultrametric $\rho \in \mathcal{U}_w$. 
Example 3. If a nonempty graph $G$ is a tree, then the subdominant pseudoultrametric $\rho_w$ is in fact an ultrametric for any strictly positive weight $w$: $E(G) \rightarrow \mathbb{R}^+$. Indeed, it is a well-known characteristic property of trees that for any two distinct vertices $u^*, v^* \in V(G)$ there exists exactly one path in $G$ that connects them. Consequently, any sequence $\tilde{F}$ of paths $F_j \in \mathcal{P}_{u^*, v^*}$ is a constant sequence $F_1 = \cdots = F_n = F_{n+1} = \cdots$. This automatically entails the validity of condition (ii) in Theorem 5.

§ 5. The least element in $\mathcal{U}_w$ and the uniqueness of the extension of a weight to a pseudoultrametric

In this section we demonstrate that complete $k$-partite graphs $G$ and only this kind of graphs possess the property that for any pseudoultrametrizable weight $w: E(G) \rightarrow \mathbb{R}^+$ the partially ordered set $(\mathcal{U}_w, \leq)$ contains the least element. Then, we give a criterion for the uniqueness of the extension of a weight $w$ to a pseudoultrametric $\rho: V(G) \times V(G) \rightarrow \mathbb{R}^+$ in the case of an arbitrary nonempty graph $G$. A criterion for the existence of such an extension was obtained above in Theorem 2.

Recall that a graph $G$ is said to be $k$-partite (where $k$ is an arbitrary cardinal) if its vertex set $V(G)$ can be partitioned into $k$ nonempty disjoint subsets $V_\alpha$, $V_\alpha = \bigcup_{\alpha \in I} V_\alpha$, $\alpha \in I$, $|I| = k$, $V_{\alpha_i} \cap V_{\alpha_j} = \emptyset$ for $i \neq j$, in such a way that for any edge $\{x, y\} \in E(G)$ the ends $x$ and $y$ belong to different parts $V_\alpha$. If, in addition, for any two points $x$ and $y$ belonging to different parts $V_\alpha$ we have the inclusion $\{x, y\} \in E(G)$, then $G$ is called the complete $k$-partite graph. Clearly, for $k = 1$ the complete $k$-partite graph is empty, $E(G) = \emptyset$, and for $k \geq 2$ it is connected.

The following lemma is a reformulation of Lemma 3.2 in [12].

Lemma 3. Let $G$ be a graph with $V(G) \neq \emptyset$. Then $G$ is the complete $k$-partite graph for a certain $k \geq 1$ if and only if $G$ contains no induced subgraphs which are isomorphic to the graph $H$ depicted in Fig. 3.

We shall denote by TM (twice-max) the set of all unordered pairs $p, q$ of distinct nonadjacent vertices of a graph $(G, w)$ such that any path $P \in \mathcal{P}_{p, q}$ contains at least two different edges $e_1$ and $e_2$ satisfying the condition $w(e_1) = w(e_2) = \max_{e \in P} w(e)$.

![Figure 3](image)

Figure 3. If $E(H)$ is endowed with an arbitrary weight $w$ such that $w(\{u, v\}) > 0$, then $(\mathcal{U}_w, \leq)$ contains no least element.
Theorem 6. The following conditions are equivalent for any nonempty graph $G$:

(i) for any pseudoultrametrizable weight $w: E(G) \to \mathbb{R}^+$ the partially ordered set $(\mathcal{U}_w, \leq)$ contains the least pseudoultrametric $\rho_{0,w}$, that is,

$$\rho_{0,w}(u, v) \leq \rho(u, v),$$  \hspace{1cm} (5.1)

for all $\rho \in \mathcal{U}_w$ and all $u, v \in V(G)$;

(ii) $G$ is a complete $k$-partite graph with $k \geq 2$.

If condition (ii) is satisfied and $w$ is a pseudoultrametrizable weight, then for $u \neq v$,

$$\rho_{0,w}(u, v) = \begin{cases} 0 & \text{if } \{u, v\} \in \text{TM}, \\ \max_{e \in F} w(e) & \text{if } \{u, v\} \notin \text{TM}, \end{cases}$$  \hspace{1cm} (5.2)

where $F$ is an arbitrary path from $\mathfrak{P}_{u,v}$ such that $\max_{e \in F} w(e)$ is attained at only one edge.

Proof. (i) $\implies$ (ii). Suppose that condition (ii) is violated. Then by Lemma 3 one can find vertices $u, v$, $\{u, v\} \in E(G)$, and a vertex $p \in V(G)$, $u \neq p \neq v$, such that $\{u, p\} \notin E(G)$ and $\{v, p\} \notin E(G)$. Define the weight $w(e) = 1$ for all $e \in E(G)$ and consider two pseudoultrametrics on the vertex set $V(G)$ of the graph $(G, w)$:

- $\rho_1(u, p) = \rho_1(p, u) = \rho_1(s, s) = 0$ for all $s \in V(G)$ and $\rho_1(s, t) = 1$ otherwise;
- $\rho_2(v, p) = \rho_2(p, v) = \rho_2(s, s) = 0$ for all $s \in V(G)$ and $\rho_2(s, t) = 1$ otherwise.

It is easily seen that $\rho_1, \rho_2 \in \mathcal{U}_w$. Assuming that there exists a least pseudoultrametric $\rho \in \mathcal{U}_w$ we arrive at a contradiction:

$$1 = \rho(u, v) \leq \rho(u, p) + \rho(p, v) \leq \min\{\rho_1(u, p), \rho_2(u, p)\} + \min\{\rho_1(p, v), \rho_2(p, v)\} = 0.$$

Thus, we have shown that there exists a pseudoultrametrizable weight $w$ for which the partially ordered set $\mathcal{U}_w$ contains no least pseudoultrametric.

(ii) $\implies$ (i). Suppose that condition (ii) is satisfied and let $(G, w)$ be an arbitrary weighted graph with a pseudoultrametrizable weight. As has been noted above, since $k \geq 2$, the graph $G$ is connected. We show that $\rho_{0,w}$ defined by (5.2) is the least element of the partially ordered set $\mathcal{U}_w$.

Note that the function $\rho_{0,w}$ for a pseudoultrametrizable weight $w$ is well defined. Indeed, suppose that $\{u, v\} \notin \text{TM}$ and let there exist two different paths $F_1, F_2 \in \mathfrak{P}_{u,v}$ each containing exactly one edge of maximum weight. Let $\rho \in \mathcal{U}_w$. In the pseudoultrametric space $(V(G), \rho)$ consider the cycle that consists of the path $F_1$ and edge $\{u, v\}$. By virtue of Lemma 1 we conclude that $\max_{e \in F_1} w(e)$ is the unique possible value of $\rho(u, v)$. Similarly, considering the cycle that consists of the path $F_2$ and edge $\{u, v\}$, we derive $\rho(u, v) = \max_{e \in F_2} w(e)$ for any $\rho \in \mathcal{U}_w$.

At the same time, for any edge $\{u, v\} \in E(G)$ we have $\rho_{0,w}(u, v) = w(\{u, v\})$, since in this case $\{u, v\} \notin \text{TM}$ and $(u, v)$ is one of the paths that join the vertices $u$ and $v$.

We show that the function $\rho_{0,w}$ is indeed a pseudoultrametric. To do this it suffices to establish the strong triangle inequality

$$\rho_{0,w}(x, y) \leq \max\{\rho_{0,w}(x, z), \rho_{0,w}(z, y)\}$$  \hspace{1cm} (5.3)

for any distinct vertices $x, y, z \in V(G)$. First, we shall demonstrate that this inequality holds in the case where at least two pairs of these vertices are adjacent.
Indeed, if we have three pairs of adjacent vertices, then (5.3) follows from the pseudoultrametrizability of the weight \( w \). If only two pairs of vertices are adjacent and the weights of the corresponding edges are different, then, according to (5.2), the weight of the remaining edge is equal to the maximum of these two weights. Inequality (5.3) holds as well. Now assume that only two pairs of vertices are adjacent and the weights of the corresponding edges are equal, \( w(\{x, z\}) = w(\{z, y\}) = \rho_{0, w}(x, z) = \rho_{0, w}(z, y) = a \), but \( \rho_{0, w}(x, y) = b > a \). Then there exists a path \( F \in \mathcal{P}_{x, y} \) with exactly one edge \( e_0 \) of maximum weight. Consider the cycle that consists of the path \( F \) and edges \( \{x, z\} \) and \( \{z, y\} \) (or one of these edges, depending on whether \( F \) contains one of them). This cycle contains only one edge \( e_0 \) of maximum weight, which in view of Theorem 2 contradicts the pseudoultrametrizability of the weight \( w \).

By virtue of Lemma 3 the case where exactly two vertices are adjacent is impossible. Therefore, it may be assumed that the vertices \( x, y \) and \( z \) are pairwise nonadjacent. Let \( \rho_{0, w}(x, y) = a > 0 \) (if \( \rho_{0, w}(x, y) = 0 \), then inequality (5.3) is obvious). Since \( a > 0 \), there exists a path \( F \in \mathcal{P}_{x, y} \) with only one edge \( e_0 = \{u_0, v_0\} \) of maximum weight \( w(e_0) = a \). First let us suppose that \( F \) does not go through the point \( z \). It follows from (ii) that one of the vertices \( u_0, v_0 \), say \( u_0 \), is adjacent to \( z \). Consider a path \( F_1 \in \mathcal{P}_{x, z} \) which consists of the edge \( \{u_0, z\} \) and a part of path \( F \), and a path \( F_2 \in \mathcal{P}_{y, z} \) which consists of the edge \( \{u_0, z\} \) and the rest of path \( F \). Without loss of generality we may assume that \( \{u_0, v_0\} \in E(F_2) \). If \( w(\{u_0, z\}) > a \), then \( \{u_0, z\} \) is the only edge of maximum weight for each of the paths \( F_1, F_2 \). Then (5.2) implies

\[
\rho_{0, w}(x, z) = w(\{u_0, z\}) = \rho_{0, w}(z, y) > a,
\]

which, in turn, gives (5.3). Therefore, we assume that \( w(\{u_0, z_0\}) \leq a \). If the last inequality is strict, then \( \{u_0, v_0\} \) is the only edge of maximum weight in \( F_2 \). Hence,

\[
\rho_{0, w}(y, z) = w(\{u_0, v_0\}) = a,
\]

and (5.3) is satisfied. If \( w(\{u_0, z_0\}) = a \), then \( \{u_0, z_0\} \) is the only edge of maximum weight in \( F_1 \) and we again arrive at inequality (5.3). In the case where the path \( F \) visits the point \( z \), and thereby splits into two paths \( F_1 \in \mathcal{P}_{x, z} \) and \( F_2 \in \mathcal{P}_{z, y} \), we see that one of the values \( \rho_{0, w}(x, z), \rho_{0, w}(z, y) \) is equal to \( a \), which again entails (5.3).

It remains to show that inequality (5.1) holds for all \( \rho \in \mathcal{U}_w \) and all \( u, v \in V(G) \). Indeed, this is trivial if \( \rho_{0, w}(u, v) = 0 \). Suppose that \( \rho_{0, w}(u, v) = a > 0 \), then \( \{u, v\} \notin \mathrm{TM} \) and there exists a path \( F \in \mathcal{P}_{u, v} \) with only one edge \( e_0 \) of maximum weight \( w(e_0) = a \). Assume that \( \rho(u, v) < a \). Then \( \rho(x, y) \) is not a pseudoultrametric because there exists a cycle which contains exactly one edge of maximum weight.

As an application of Theorem 6 we shall obtain a characterization of star graphs. Recall that a star is a complete bipartite graph one of whose parts is a single-point set.

**Corollary 8.** The following conditions are equivalent:

(i) any weight \( w: E(G) \to \mathbb{R}^+ \) is pseudoultrametrizable and the partially ordered set \( (\mathcal{U}_w, \preceq) \) contains the least element;

(ii) \( G \) is a star.
The implication (ii) $\implies$ (i) follows from Theorem 6 and Corollary 5. Suppose that condition (i) is satisfied. Then, again with the use of Theorem 6 and Corollary 5, we establish at the same time that $G$ is a tree and a complete $k$-partite graph with $k \geq 2$. If $k \geq 3$, then $G$ contains a triangle formed by three vertices belonging to three different parts of the graph $G$. However, there are no cycles in trees, and hence $k = 2$. If each of the parts of $G$ contains at least two points, then it is easy to find a quadrilateral (4-cycle) $C \subseteq G$, which again contradicts the acyclicity of $G$. Thus, $G$ is a complete bipartite graph one of whose parts consists of a single point.

We now turn to conditions for the uniqueness of the extension of a weight $w$ to a pseudoultrametric. We introduce the following

**Definition 5.** Let $(G, w)$ be a nonempty weighted connected graph, and $u$ and $v$ be two distinct nonadjacent vertices of this graph. The vertices $u$ and $v$ are said to be well chained if for any $\varepsilon > 0$ there exists a path $u = u_1, u_2, \ldots, u_n = v$ such that $\{u_i, u_{i+1}\} \in E(G)$ and $w(\{u_i, u_{i+1}\}) \leq \varepsilon$ for $i = 1, \ldots, n - 1$.

The collection of all such pairs $\{u, v\}$ will be denoted by $\text{WCh}$. 

**Remark 9.** The concept of well-chained points is frequently used in the theory of metric continua [13] and plays an important role in the study of problems related to connectivity in metric spaces; see, for instance, [14].

**Remark 10.** ‘Well-chainedness’ occurs naturally in the study of subdominant ultrametrics; see [8] and [11]. In particular, for a strictly positive pseudoultrametrizable weight $w$ and a connected $G$ it can easily be shown that vertices $u$ and $v$, $u \neq v$, are well chained if and only if $\rho_w(u, v) = 0$.

**Theorem 7.** Let $(G, w)$ be a nonempty weighted connected graph with a pseudoultrametrizable weight $w$. The set $\mathcal{U}_w$ consists of a single element if and only if

$$\text{TM} \subseteq \text{WCh}. \quad (5.4)$$

**Proof.** Suppose that (5.4) is satisfied and let us show that $w$ can be uniquely extended to a pseudoultrametric. To do this it suffices to establish the equality

$$\rho(u, v) = \rho_w(u, v) \quad (5.5)$$

for any pair of distinct nonadjacent vertices $u$, $v$ and any $\rho \in \mathcal{U}_w$. Let $u$ and $v$ be distinct nonadjacent vertices such that $\{u, v\} \notin \text{TM}$. In this case, by the same arguments as in the proof of the consistency of Definition (5.2), we show that the value of $\rho(u, v)$ does not depend on the choice of $\rho \in \mathcal{U}_w$, and hence equality (5.5) is valid. Now consider the case where $\{u, v\} \in \text{TM}$. In view of the inclusion (5.4), the vertices $u$ and $v$ are well chained. Then by the definition of the subdominant pseudoultrametric $\rho_w$ we obtain

$$\rho_w(u, v) = 0. \quad (5.6)$$

Since $G$ is a connected graph, by Theorem 3 we have $\rho(u, v) \leq \rho_w(u, v)$. Moreover, $0 \leq \rho(u, v)$ for all $u, v \in V(G)$. The last two estimates and equality (5.6) imply (5.5).
Now suppose that inclusion (5.4) is not valid. We shall demonstrate that then
\( w \) can be extended to a pseudoultrametric in several ways.

Since inclusion (5.4) is not true, there exists a pair \( u_0, v_0 \) of distinct nonadjacent vertices which are not well chained, but are such that \( \{u_0, v_0\} \notin \text{WCh} \). Since \( \{u_0, v_0\} \notin \text{WCh} \), we have \( \rho_w(u_0, v_0) > 0 \), and hence there exists \( \varepsilon_0 > 0 \) such that
\[
\max_{e \in P} w(e) \geq \varepsilon_0 \tag{5.7}
\]
for any \( P \in \mathcal{P}_{u,v} \). Let \( \varepsilon_1 \) and \( \varepsilon_2 \) be two different numbers in \([0, \varepsilon_0)\). Consider the graph \( \tilde{G} \) such that
\[
V(\tilde{G}) = V(G), \quad E(\tilde{G}) = E(G) \cup \{\{u_0, v_0\}\},
\]
that is, \( \tilde{G} \) is obtained by adding an edge \( \{u_0, v_0\} \) to \( G \). On the graph \( \tilde{G} \) we define two weights \( w_i : E(\tilde{G}) \to \mathbb{R}^+ \), \( i = 1, 2 \), by the formula
\[
w_i(e) = \begin{cases} w(e) & \text{for } e \in E(G), \\ \varepsilon_i & \text{for } e = \{u_0, v_0\}. \end{cases}
\]
We verify that the weight \( w_1 \) is pseudoultrametrizable. In view of Theorem 2, it is sufficient to show that for any cycle \( C \subseteq \tilde{G} \) there exist two distinct edges \( e_1, e_2 \in E(C) \) such that
\[
\max_{e \in E(C)} w_1(e) = w_1(e_1) = w_1(e_2). \tag{5.8}
\]
If \( \{u_0, v_0\} \notin E(C) \), then condition (5.8) follows from the pseudoultrametrizability of \( w \). Suppose that \( \{u_0, v_0\} \in E(C) \). Removing the edge \( \{u_0, v_0\} \) from \( C \), we obtain a path \( P \in \mathcal{P}_{u_0,v_0} \). Since \( \{u_0, v_0\} \in \text{TM} \), there exist two different edges \( e_1 \) and \( e_2 \) in \( E(P) \) such that
\[
\max_{e \in P} w_1(e) = \max_{e \in P} w(e) = w(e_1) = w(e_2) = w_1(e_1) = w_1(e_2).
\]
By virtue of the inequalities (5.7) and \( \varepsilon_1 < \varepsilon_0 \) we see that these edges satisfy (5.8). Consequently, the weight \( w_1 : E(\tilde{G}) \to \mathbb{R}^+ \) is pseudoultrametrizable. In the same way, it can be shown that \( w_2 : E(\tilde{G}) \to \mathbb{R}^+ \) is a pseudoultrametrizable weight as well. Let \( \rho_1 \) and \( \rho_2 \) be pseudoultrametrics on \( V(\tilde{G}) = V(G) \) which extend \( w_1 \) and \( w_2 \), respectively. Then, obviously, both \( \rho_1 \) and \( \rho_2 \) extend the weight \( w \) as well. Since \( \varepsilon_1 \neq \varepsilon_2 \), we have \( w_1 \neq w_2 \), and hence \( \rho_1 \neq \rho_2 \). Therefore, if condition (5.4) is violated, then the extension of \( w \) to a pseudoultrametric on \( V(G) \) is not unique.

Remark 11. If \( G \) is a nonempty disconnected graph, then for any pseudoultrametrizable weight \( w \)
\[
\text{card}(\mathcal{U}_w) \geq \mathfrak{c}.
\]
Here, the symbol \( \mathfrak{c} \) denotes the cardinality of a continuum. This inequality follows immediately from (3.3).
Example 4. For complete $k$-partite graphs with $k \geq 2$ the uniqueness of the extension of $w$ is equivalent to the condition

$$\rho_w = \rho_{0,w},$$

(5.9)

where $\rho_w$ is the subdominant pseudoultrametric and $\rho_{0,w}$ is the pseudoultrametric defined by (5.2). Indeed, for $k \geq 2$, a complete $k$-partite graph is connected. According to Theorems 2 and 6, the pseudoultrametric $\rho_w$ is the greatest element of $(\mathcal{U}_w, \preceq)$ and $\rho_{0,w}$ is the least element of this set. Hence, for any $\rho \in \mathcal{U}_w$ we have

$$\rho_{0,w} \preceq \rho \preceq \rho_w,$$

and by virtue of (5.9) we obtain

$$\rho_{0,w} = \rho = \rho_w,$$

which means that the extension is unique. We show that (5.9) is equivalent to the inclusion (5.4). To do this we note that in the proof of Theorem 6 we established the equality

$$\rho_{0,w}(u, v) = \rho_w(u, v)$$

(5.10)

for $\rho_{0,w}(u, v) > 0$. Moreover, (5.10) should hold for adjacent vertices $u$ and $v$ since $\rho_{0,w}, \rho_w \in \mathcal{U}_w$. It follows directly from the definitions that

$$\text{WCh} = \{\{u, v\} : u \text{ and } v \text{ are not adjacent, } u \neq v \text{ and } \rho_w(u, v) = 0\};$$

(5.11)

and for complete $k$-partite graphs with $k \geq 2$,

$$\text{TM} = \{\{u, v\} : u \text{ and } v \text{ are not adjacent, } u \neq v \text{ and } \rho_{0,w}(u, v) = 0\}.$$ 

Thus, (5.9) is equivalent to the equality $\text{WCh} = \text{TM}$. By virtue of the fact that $\rho_{0,w} \preceq \rho_w$, we have the inclusion $\text{TM} \supseteq \text{WCh}$. Therefore, (5.9) is equivalent to the inverse inclusion $\text{TM} \subseteq \text{WCh}$.

Example 5. Let $G$ be a tree and $w : E(G) \to \mathbb{R}^+$ be a strictly positive weight (recall that in this case $w$ is automatically ultrametrizable).

The weight $w$ can be extended uniquely to a pseudoultrametric on $V(G)$ if and only if $\text{TM} = \emptyset$.

Indeed, any two vertices of a tree $G$ are connected by a unique path. Taking into account this uniqueness, the fact that $w$ is strictly positive, the validity of formula (5.11), and the definition of $\rho_w$, we conclude that $\text{WCh} = \emptyset$. In view of Theorem 7, it follows from this equality that the uniqueness of the extension of $w$ is equivalent to the fact that $\text{TM}$ is an empty set.

Remark 12. The equality $\text{TM} = \emptyset$ is equivalent to the uniqueness of the extension of a strictly positive pseudoultrametrizable weight $w$ to a pseudoultrametric for any $G$ which satisfies condition (ii) of Theorem 5. In the case of trees, this equality is equivalent to the following condition.

If $e_1$ and $e_2$ are two distinct edges of a tree $G$ such that $w(e_1) = w(e_2)$, then for any path $P \subseteq G$ which goes through both $e_1$ and $e_2$ there exists an edge $e_3 \in E(P)$ such that $w(e_3) > w(e_1)$. 

Bibliography

[1] J. A. Bondy and U. S. R. Murty, *Graph Theory*, Grad. Texts in Math., vol. 244, Springer-Verlag, New York 2008.

[2] O. Dovgoshey, O. Martio and M. Vuorinen, *Metrization of weighted graphs*, Reports in Math. University of Helsinki, vol. 516, 2011.

[3] O. Dovgoshey and O. Martio, “Blow up of balls and coverings in metric spaces”, *Manuscripta Math.* 127:1 (2008), 89–120.

[4] N. Jardine and R. Sibson, *Mathematical taxonomy*, Part II, Willey, London 1971.

[5] J. M. Bayod and J. Martiner-Maurica, “Subdominant ultrametrics”, *Proc. Amer. Math. Soc.* 109:3 (1990), 829–834.

[6] J. P. J. Jardine, N. Jardine and R. Sibson, “The structure and construction of taxonomic hierarchies”, *Math. Biosciences* 1:2 (1967), 173–179.

[7] R. Rammal, J. C. Angles d’Auriac and B. Doucot, “On the degree of ultrametricity”, *J. Physique* 46 (1985), 945–952.

[8] R. Rammal, G. Toulouse and M. A. Virasoro, “Ultrametricity for physicist”, *Rev. Modern Phys.* 58:3 (1986), 765–788.

[9] D. Marcu, “A study on metrics and statistical analysis”, *Studia Univ. Babes-Bolyai Math.* 49:3 (2004), 43–74.

[10] O. Dovgoshey and D. Dordovskyi, “Ultrametricity and metric betweenness in tangent spaces to metric spaces”, *p-Adic Numbers Ultrametric Anal. Appl.* 2:2 (2010), 100–113.

[11] A. J. Lemin, “On ultrametrization of general metric spaces”, *Proc. Amer. Math. Soc.* 131:3 (2003), 979–989.

[12] D. Dordovskyi, O. Dovgoshey and E. Petrov, “Diameter and diametrical pairs of points in ultrametric spaces”, *p-Adic Numbers Ultrametric Anal. Appl.* 3:4 (2011), 253–262.

[13] S. B. Nadler, jr., *Continuum theory. An introduction*, Monogr. Textbooks Pure Appl. Math., vol. 158, Marcel Dekker, New York 1992.

[14] A. G. O’Farrel, “When uniformly-continuous implies bounded”, *Irish Math. Soc. Bull.* 53 (2004), 53–56.

Aleksey A. Dovgoshey
Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Donetsk
E-mail: aleksdov@mail.ru

Evgeniy A. Petrov
Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, Donetsk
E-mail: eugeniy.petrov@gmail.com