(q; l, λ)—deformed Heisenberg algebra: representations, special functions and quantization

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Abstract. This paper addresses a new characterization of Sudarshan’s diagonal representation of the density matrix elements ρ(z′, z), derived from the constructed (q; l, λ)—deformed boson coherent states. The induced ρ(z′, z) self-reproducing property with the associated self-reproducing kernel K(z′, z) is computed and analyzed. An explicit construction, including the recursion relation of novel generalized continuous (q; l, λ)—Hermite polynomials is provided. New classes of these polynomials are deduced with the exact resolution of the moment problems giving their orthogonality weight functions. Besides, the Berezin-Klauder-Toeplitz quantization of classical phase space observables and relevant normal and anti normal forms are determined and discussed.

Keywords: Deformed Heisenberg algebra, coherent states, q—Hermite polynomials, density matrix, reproducing kernel, hypergeometric function, diagonal representation, Berezin-Klauder-Toeplitz quantization

30 January 2013

1. Introduction

The Heisenberg algebra, generated by the identity operator 1 and two mutually adjoint operators, b and its Hermitian conjugate $b^\dagger$ (also called annihilation and creation operators in Physics literature), satisfying the commutation relations

$$[b, b^\dagger] = 1, \quad [b, 1] = 0 = [b^\dagger, 1],$$

(1)

where $[A, B] := AB - BA$, plays a central role in the investigation of physical systems and in mathematics.

Defining the operator $N := b^\dagger b$, known as the number operator, the commutation relations (1) induce the two following properties:

$$[N, b] = -b \quad \text{and} \quad [N, b^\dagger] = b^\dagger.$$  

(2)
Let $\mathcal{F}$ be a Fock space and $\{|n\rangle \mid n \in \mathbb{N} \cup \{0\}\}$ be its orthonormal basis. The actions of $b$, $b^\dagger$ and $N$ on $\mathcal{F}$ are given by

$$b|n\rangle = \sqrt{n}|n-1\rangle, \quad b^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \text{and} \quad N|n\rangle = n|n\rangle$$ (3)

where $|0\rangle$ is a normalized vacuum:

$$b|0\rangle = 0, \quad \langle 0|0 \rangle = 1.\quad (4)$$

From (3) the states $|n\rangle$ for $n \geq 1$ are built as follows:

$$|n\rangle = \frac{1}{\sqrt{n!}}(b^\dagger)^n|0\rangle, \quad n = 1, 2, \ldots (5)$$

satisfying the orthogonality and completeness conditions:

$$\langle m|n \rangle = \delta_{m,n}, \quad \sum_{n=0}^{\infty} |n\rangle\langle n| = 1.\quad (6)$$

The generalization of the canonical commutation relations (1) was suggested long before the discovery of quantum groups, by Heisenberg to achieve the regularization for nonlinear spinor field theory. The issue was considered as small additions to the canonical commutations relations [11, 29]. Snyder, investigating the infrared catastrophe of soft photons in the Compton scattering, raised this issue and built a non-commutative Lorentz invariant space-time where the non-commutativity of space operators is proportional to non-linear combinations of phase space operators [26]. Further the deformation of the harmonic oscillator algebra whose applications in physics are presently rather technical but nonetheless very promising [32], possesses an important and useful representation theory in connection to that of their classical limit algebra. The investigation under the form $aa^\dagger - qa^\dagger a = 1$, $q > 1$ of the one-parameter deformed Heisenberg algebras in theoretical physics originated from the study of the dual resonance models of strong interactions [2].

From the other side, there are some hopes that, in physical studies of nonlinear phenomena, the deformed oscillator can play the same role as the usual boson oscillator in non-relativistic quantum mechanics. This could explain why various quantum deformations of boson oscillator commutation relations have attracted a great attention during the last few years (see [6] and references therein). This might be also due to the fact that there exist correspondences between quantum groups, quantum algebras, statistical mechanics, quantum field theory, conformal field theory, quantum and nonlinear optics and non commutative geometry, etc. Furthermore, such a connection is extended to coherent states deducible from the study of quantum groups and, therefore, from the deformation of Heisenberg algebra. Recently [9], a deformation of the Heisenberg algebra by a set of parameters was introduced with a new family of generalized coherent states respecting the Gazeau-Klauder’s criteria.

Besides, Parthasarathy and Sridhar studied a $q-$analogue of the diagonal representation of the density matrix using the $q-$boson coherent states and gave the generalization of the self-reproducing property of density matrix elements $\rho(z', z)$ and the associated kernel $K(z', z)$ (see [21, 22] for more details).
The present work addresses a new characterization of Sudarshan’s diagonal representation of the density matrix [?], derived from the constructed \((q;l,\lambda)\)-deformed boson coherent states. The \((q;l,\lambda)\)-generalization of the self-reproducing property of density matrix elements \(\rho(z',z)\) and associated self-reproducing kernel \(K(z',z)\) are computed and analyzed. New families of generalized Hermite polynomials associated with the position and momentum operators as well as the main relevant operator properties are investigated and discussed.

The paper is organized as follows. In section 2, we first give main useful definitions and results on the \((q;l,\lambda)\)-deformed oscillator algebra and related coherent states. Then we deduce resulting new features used in the sequel. In section 3, we provide an explicit construction, including the recursion relation of generalized continuous \((q;l,\lambda)\)-Hermite polynomials generated by polynomial expansion of the deformed position and momentum operators in associated Fock space basis. Particular classes of deformed Hermite polynomials are deduced with explicit resolution of the moment problems giving their orthogonality weight functions. In the section 4, diagonal representation of density matrix using the \((q;l,\lambda)\)-CS is computed. Reproducing kernel \(K(z,\zeta)\) and its properties are investigated. In the section 5, matrix elements of normal and anti normal forms and mean operator values are determined. The Berezin-Klauder-Toeplitz quantization (also called CS quantization) of classical phase space observables is performed in section 6. Furthermore, the angle and time evolution operators and semi-classical phase space trajectories are discussed. Finally, we end by a conclusion in the section 7, followed by appendices on some computational details.

2. On the \((q;l,\lambda)\)-deformed oscillator algebra

In this section, for the clarity of our development, we first recall main definitions and results on the \((q;l,\lambda)\)-deformed oscillator algebra [9], and then deduce resulting new features used in the sequel.

Definition 2.1 The \((q;l,\lambda)\)-deformed oscillator algebra is defined as the associative algebra generated by the operators \(\{1, a, a^\dagger, N\}\) satisfying the commutation relations [9]

\[
\begin{align*}
    aa^\dagger - a^\dagger a & = l^2 q^{\lambda-N-1}, \\
    [N, a^\dagger] & = a^\dagger, \\
    [N, a] & = -a,
\end{align*}
\]

where \(\varphi\) is the structure function, \(l\) and \(\lambda\) are complex numbers with \(l \neq 0\) and \(q > 0\).

This algebra carries out a Hopf algebra structure (see Appendix A). The operator products \(aa^\dagger\) and \(a^\dagger a\) are obtained from [7] and are given by

\[
\begin{align*}
aa^\dagger & = l^2 q^{\lambda} \frac{1 - q^{-N-1}}{q - 1}, \\
a^\dagger a & = l^2 q^{\lambda} \frac{1 - q^{-N}}{q - 1}.
\end{align*}
\]

Proposition 2.2 [9] The orthonormalized basis of the Fock space \(\mathcal{F}\) is given by

\[
|n\rangle := \frac{q^{\frac{l}{2}(\frac{N}{2})}}{\sqrt{\gamma_n(q;q)_n}} a^\dagger^n |0\rangle, \quad n = 0, 1, 2, \ldots
\]
where \( \gamma = l^2q^{\lambda-1}/(1-q); \) the \( q \)-shifted factorial \( (q;q)_n \) is defined as: \( (z;q)_n :=\prod_{k=0}^{n-1}(1-zq^k) \), \( n = 1,2,\ldots \) with \( (z;q)_0 := 1 \) by convention. The states \( |n\rangle \) satisfy the orthogonality and completeness conditions
\[
\langle m|n \rangle = \delta_{m,n}, \quad \sum_{n=0}^{\infty} |n\rangle \langle n| = 1.
\]
Moreover, the actions of the operators \( a,a^\dagger,aa^\dagger, a^\dagger a \) and \( N \) on \( |n\rangle \) are given by
\[
|n\rangle = \sqrt{\varphi(n)|n-1\rangle}, \quad a^\dagger |n\rangle = \sqrt{\varphi(n+1)|n+1\rangle}, \quad (11)
\]
\[
aa^\dagger |n\rangle = \varphi(n+1)|n\rangle, \quad a^\dagger a|n\rangle = \varphi(n)|n\rangle, \quad N|n\rangle = n|n\rangle. \quad (12)
\]

**Proof.** See \([9]\). \( \square \)

**Definition 2.3** The \( (q;l, \lambda) \)-Jackson’s differential operator \( q_{\partial^l\lambda} \) acting on the space of analytic functions is defined as \([9]\)
\[
q_{\partial^l\lambda} f(y) := l^2q^\lambda \frac{\bar{f}(y) − f(q^{-1}y)}{(q−1)y}.
\]

The deformed \( (q;l, \lambda) \)-coherent states (CS) associated with the algebra \([7]\) are constructed in \([9]\):
\[
|z\rangle_{l,\lambda} = N_{l,\lambda}^{-1/2}(|z|^2) \sum_{n=0}^{\infty} q^{n(n-1)/2}z^n \frac{q^n(\varphi(q)/q)_n}{\sqrt{\varphi(q)/q)_n}}|n\rangle, \quad z \in \mathbb{D}_{l,\lambda}, \quad (14)
\]
where
\[
N_{l,\lambda}(t) = \sum_{n=0}^{\infty} q^n(t)^n \frac{q^n(\varphi(q)/q)_n}{\sqrt{\varphi(q)/q)_n}} = \frac{1}{(-t/q^\lambda;q)_\infty}, \quad (15)
\]
and
\[
\mathbb{D}_{l,\lambda} = \{ z \in \mathbb{C} : |z| < R_{l,\lambda} \}, \quad R_{l,\lambda} = \begin{cases} \infty & \text{if } 0 < q < 1 \\ \frac{l^2q^{\lambda}}{q-1} & \text{if } q > 1, \end{cases} \quad (16)
\]
with \( (z;q)_\infty := \prod_{k=0}^{\infty}(1-zq^k) \). \( R_{l,\lambda} \) is the convergence radius of the series \( N_{l,\lambda}(t) \) which is a holomorphic function with simple zeros at \( x_k = l^2q^{\lambda-1−k}/(q − 1), \ k = 0,1,\ldots \)

The CS \([13]\) are not orthogonal as we can see from the product of two CS \(|z\rangle_{l,\lambda} \) and \(|z'\rangle_{l,\lambda} \)
\[
\langle l,\lambda|z'|z\rangle_{l,\lambda} = \frac{(-z'/z; \gamma)_{\infty}}{\sqrt{(-z'/z; \gamma)_{\infty}(-z'/z; \gamma)_{\infty}}}. \quad (17)
\]

Besides, it is proved in \([9]\) that they solve the identity, i.e,
\[
\int_{\mathbb{D}_{l,\lambda}} d\mu_{l,\lambda}(\bar{z}, z) |z\rangle_{l,\lambda} |z\rangle_{l,\lambda} |z\rangle_{l,\lambda} = 1, \quad (18)
\]
where
\[
d\mu_{l,\lambda}(\bar{z}, z) = \begin{cases} \frac{1}{\eta ln q^\lambda} \frac{N_{l,\lambda}(\bar{z}z)}{N_{l,\lambda}(\bar{z}zq^{-1})} \frac{dz^2}{\pi}, & 0 < q < 1 \\ \frac{1}{2\pi} \frac{d^2x}{1+q}, & q > 1 \end{cases}, \quad 0 < x < |z| < \frac{l^2q^{\lambda}}{q-1}, \ \theta = arg(z), \quad (19)
\]
with \( \eta = l^2q^{\lambda}/(1-q) \).
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3. Deformed Hermite polynomials associated with the position and momentum operators

In [9], it is proved that the deficiency indices of the position and momentum operators, $Q = (a^\dagger + a)/\sqrt{2}$ and $P = i(a^\dagger - a)/\sqrt{2}$, is (1, 1). Therefore, they are no longer essentially self-adjoint but have each a one-parameter family of self-adjoint extensions instead. In this case, the deficiency subspaces $N_x$, $Im(x) \neq 0$, are one-dimensional. Associate now these operators to the generalized vectors

$$|x\rangle := \sum_0^\infty q_n(x)|n\rangle,$$

$$|p\rangle := \sum_0^\infty p_n(p)|n\rangle,$$

respectively, such that their actions are realized as follows:

$$Q|x\rangle = x|x\rangle, \quad P|p\rangle = p|p\rangle$$

and analyze their various relevant representations.

3.1. In the position representation: (q; l, λ)–deformed Hermite polynomials

Here two cases deserve examination depending on the q–value range.

- **Case 1:** $0 < q < 1$

By using the equations (11), (20) and (22), we readily prove the following recurrence relation obeyed by the Fock space basis coefficients $q_n$:

$$\sqrt{2(1-q)}xq_n(x) = (l^2 q^{\lambda-n}(1-q^{-1})^{1/2}q_{n+1}(x) + (l^2 q^{\lambda-n}(1-q^n))^{1/2}q_{n-1}(x), \quad (23)$$

imposing the initial conditions $q_{-1}(x) := 0$, $q_0(x) := 1$. By setting $2^{1/2}y = (1-q)^{1/2}x$ and $\psi_n(y|q) = q_n(\sqrt{2(1-q)^{-1}}y)$, the equation (43) can be re-expressed as

$$2y\psi_n(y|q) = (l^2 q^{\lambda-n}(1-q^{-n+1}))^{1/2}\psi_{n+1}(y|q) + (l^2 q^{\lambda-n}(1-q^n))^{1/2}\psi_{n-1}(y|q).$$

Putting now $\psi_n(x|q) = (l^2 q^{\lambda})^{-\frac{n}{2}}q^{\frac{n(n+1)}{4}}(q; q)_n^{-1/2}h_n(x; l, \lambda|q)$ transforms the formula (24) into the new recursive relation

$$2xh_n(x; l, \lambda|q) = h_{n+1}(x; l, \lambda|q) + l^2 q^{\lambda}(q^{-n+1})h_{n-1}(x; l, \lambda|q) \quad (25)$$

defining a novel family of (q; l, λ)–deformed Hermite polynomials, i.e. $\{h_n(x; l, \lambda|q), n = 0, 1, 2, ..., \}$. The Fock space basis vector coefficients $q_n(x)$ giving the eigenvectors of the operator $Q = (a^\dagger + a)/\sqrt{2}$ by the expansion (20) are then explicitly given by

$$q_n(x) = (l^2 q^{\lambda})^{-\frac{n}{2}}q^{\frac{n(n+1)}{4}}(q; q)_n^{-1/2}h_n(\sqrt{2^{-1}(1-q)}x; l, \lambda|q). \quad (26)$$

In the particular case, when $\lambda = 0$ and $l = 1$, (26) provides the recursive relation (1.7) obtained in [4] (replacing $q^{-1}$ by $q$, $q > 1$) for the continuous $q$–Hermite polynomials $h_n(x|q) = i^{-n}H_n(ix|q)$ for $q > 1$. In this case, the expansion coefficients $q_n$ reduce to

$$q_n(x) = q^{n(n+1)/4}(q; q)_n^{-1/2}h_n(\sqrt{2^{-1}(q-1)}x|q) \quad (27)$$
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satisfying the orthogonality relation \[4\]

\[
\int_{-\infty}^{\infty} q_m(b^{-1} \sinh u)q_n(b^{-1} \sinh u)d\mu(u) = \delta_{n,m},
\]

(28)

where \(b = \sqrt{2/(q - 1)}\),

\[
h_n(\sinh u|q) = \sum_{k=0}^{n} (-1)^k q^{k(k-n)} \left[ \frac{n}{k} \right]_q e^{(n-2k)u},
\]

(29)

and the measure \(d\mu(u)\) is given by

\[
d\mu(u) = \frac{du}{(q; q)^\infty \ln q^{-1} \prod_{k=1}^{\infty} (1 + 2 \cosh 2u q^k + q^{2k})}.
\]

(30)

- **Case 2:** \(q > 1\)

From the same equations (11), (20) and (22), we arrive at the following recursive relation

\[
\sqrt{2(q - 1)} x q_n(x) = (l^2 q^\lambda (1 - q^{-n-1}))^{1/2} q_{n+1}(x) + (l^2 q^\lambda (1 - q^{-n}))^{1/2} q_{n-1}(x),
\]

(31)

with the initial conditions \(q_{-1}(x) := 0, q_0(x) := 1\). Performing the same development as above, this recursive relation is re-arranged in the form

\[
2x \tilde{h}_n(x; l, \lambda|q) = \tilde{h}_{n+1}(x; l, \lambda|q) + l^2 q^\lambda (1 - q^{-n}) \tilde{h}_{n-1}(x; l, \lambda|q),
\]

(32)

where \(\{\tilde{h}_n(x; l, \lambda|q)\; n = 0, 1, 2, \ldots\}\) constitutes a new family of \((q; l, \lambda)\)–deformed Hermite polynomials determining the coefficients \(q_n(x)\) (20) as follows:

\[
q_n(x) = (l^2 q^\lambda)^{-\frac{1}{2}} (q^{-1}; q^{-1})^{-1/2} \tilde{h}_n(\sqrt{2-1}(q - 1)x; l, \lambda|q).
\]

(33)

reverting a simpler form for specific values \(\lambda = 0\) and \(l = 1\), i.e.

\[
q_n(x) = (q^{-1}; q^{-1})^{-1/2} \tilde{h}_n(\sqrt{2-1}(q - 1)x|q),
\]

(34)

where \(\tilde{h}_n(\sqrt{2-1}(q - 1)x|q)\) are the continuous \(q\)–Hermite polynomials taking with the base \(q^{-1}\) [31]. The polynomials [31] satisfy the following orthogonality relation

\[
\int_{-c}^c q_m(x)q_n(x)\tilde{\mu}(x)dx = \delta_{n,m},
\]

(35)

where \(c = \sqrt{2/(q - 1)}\) and the measure \(\tilde{\mu}(x)\) is given by

\[
\tilde{\mu}(x) = \frac{(q^{-1}; q^{-1})^\infty \prod_{k=0}^{\infty} (1 - 2((q - 1)x^2 - 1)q^{-k} + q^{-2k})}{\pi \sqrt{2 - (q - 1)x^2}}.
\]

(36)

3.2. In the momentum representation: \((q; l, \lambda)\)–deformed Hermite polynomials

Two \(q\)–value situations merit also to be examined in this section.

- **Case 1:** \(0 < q < 1\)

The above equations (11), (21) and (22) result in the following recursion relation:

\[- \sqrt{2(1-q)} pp_n(p) = i(l^2 q^{\lambda-n-1}(1 - q^{n+1}))^{1/2} p_{n+1}(p) - i(l^2 q^{\lambda-n}(1 - q^n))^{1/2} p_{n-1}(p),\]

(37)

with the initial conditions \(p_{-1}(p) := 0, p_0(p) := 1\).
Similarly to the case in the position representation, setting $2^{1/2}y = (1 - q)^{1/2}p$, $\hat{\psi}_n(y|q) = p_n(\sqrt{2/(1 - q)}y)$ and $\hat{\psi}_n(y|q) = i^n(l^2q^\lambda - \frac{n}{2})q^{n(n+1)/4}(q; q)_{n}^{-1/2}\chi_n(y; l, \lambda|q)$ with $i = \sqrt{-1}$ leads to the three term recursion relation satisfied by $\chi_n(y; l, \lambda|q)$

$$2y\chi_n(y; l, \lambda|q) = \chi_{n+1}(y; l, \lambda|q) + l^2q^\lambda(q^{-n} - 1)\chi_{n-1}(y; l, \lambda|q), \quad (38)$$

defining a new family of $(q; l, \lambda)$-deformed polynomials, i.e. $\{\chi_n(y; l, \lambda|q), \quad n = 0, 1, 2, ...\}$. The Fock space basis vector coefficients $p_n(p)$ giving the eigen-vectors of the operator $P = i(a^\dagger - a)/\sqrt{2}$ by the expansion (21) are then explicitly given by

$$p_n(p) = (i^{-1}lq^{\lambda/2})^{-n}q^{n(n+1)/4}(q; q)_{n}^{-1/2}\chi_n(\sqrt{2^{-1}(1 - q)}p; l, \lambda|q). \quad (39)$$

In the specific case of $\lambda = 0$ and $l = i$, (39) coincides with the recursion relation (1.7) in [4] (if $q^{-1}$ is replaced by $q$, $q > 1$) for the continuous $q$--Hermite polynomials $\chi_n(p|q) = i^{-n}H_n(ip|q)$ when $q > 1$. In this case,

$$p_n(p) = q^{n(n+1)/4}(q; q)_{n}^{-1/2}\chi_n(\sqrt{2^{-1}(1 - q)}p|q) \quad (40)$$

satisfying the orthogonality relation [4]

$$\int_{-\infty}^{\infty} p_m(b^{-1}\sinh v)p_n(b^{-1}\sinh v)dv = \delta_{n,m} , \quad (41)$$

where $b = \sqrt{2/(1 - q)}$ and the measure $dv(v)$ is given by

$$dv(v) = \frac{dv}{(q; q)_\infty \ln q^{-1} \prod_{k=1}^{\infty}(1 + 2 \cosh 2vq^k + q^{2k})} \quad (42)$$

**Case 2: $q > 1$**

In this case, the equations (21), (22) also allow to produce the following recursion relation

$$-\sqrt{2(q - 1)}pp_n(p) = i(l^2q^\lambda(1 - q^{-n-1}))^{1/2}p_{n+1}(p) - i(l^2q^\lambda(1 - q^{-n}))^{1/2}p_{n-1}(p), \quad (43)$$

with the initial conditions $p_{-1}(p) := 0$, $p_0(p) := 1$. Furthermore, following step by step the above development, we arrive at the relation

$$2y\hat{\chi}_n(y; l, \lambda|q) = \hat{\chi}_{n+1}(y; l, \lambda|q) + l^2q^\lambda(1 - q^{-n})\hat{\chi}_{n-1}(y; l, \lambda|q), \quad (44)$$

determining a new family of $(q; l, \lambda)$-deformed Hermite polynomials, i.e. $\{\hat{\chi}_n(y; l, \lambda|q), \quad n = 0, 1, 2, ...\}$ giving the coefficients $p_n(p)$ : 

$$p_n(p) = (i^{-1}lq^{\lambda/2})^{-n}(q^{-1}; q^{-1})_{n}^{-1/2}\hat{\chi}_n(\sqrt{2^{-1}(q - 1)}p; l, \lambda|q). \quad (45)$$

In the situation, when $\lambda = 0$ and $l = i$, (45) simplifies to

$$p_n(p) = (q^{-1}; q^{-1})_{n}^{-1/2}\hat{\chi}_n(\sqrt{2^{-1}(q - 1)}p|q), \quad (46)$$

where $\hat{\chi}_n(\sqrt{2^{-1}(q - 1)}p|q)$ are the continuous $q$--Hermite polynomials taking with the base $q^{-1}$ [31], with the orthogonality relation

$$\int_{c}^{c} p_m(p)p_n(p)\hat{\nu}(p)dp = \delta_{n,m}, \quad (47)$$

where $c = \sqrt{2/(q - 1)}$ and the measure $\hat{\nu}(p)$ is given by

$$\hat{\nu}(p) = \frac{(q^{-1}; q^{-1})_{\infty}\prod_{k=0}^{\infty}(1 - 2((q - 1)p^2 - 1)q^{-k} + q^{-2k})}{\pi\sqrt{q - 1} \sqrt{2 - (q - 1)p^2}} \quad (48)$$
4. Diagonal representation

4.1. Diagonal representation of the density matrix

The definition of the density matrix uses the overcompleteness of Fock space states as follows:

$$\rho := \sum_{n,m=0}^{\infty} \rho(n,m) |n\rangle \langle m|.$$  \hfill (49)

Letting $z = re^{i\theta}$, $0 \leq \theta \leq 2\pi$ and $0 < r < R_{l,\lambda}$ and making use of (14) amount to

$$\mathcal{N}_{l,\lambda}(r^2) |re^{i\theta}\rangle_{l,\lambda,l,\lambda} \langle re^{i\theta}| = \sum_{n,m=0}^{\infty} \frac{r^{n+m} e^{i(n-m)\theta} q^{\frac{1}{2}(\gamma_n^2)+\frac{1}{2}(l^2)}}{\sqrt{\gamma^{n+m}(q;q)_n(q;q)_m}} |n\rangle \langle m|.$$  \hfill (50)

Now by multiplying (50) by $e^{is\theta}$ and performing the integral with respect to the angle $\theta$ in the Lebesgue sense, the $r -$ integration being the " $q; l, \lambda -$ integration" we obtain

$$\int_{0}^{2\pi} \frac{d\theta}{2\pi} \mathcal{N}_{l,\lambda}(r^2) e^{is\theta} |re^{i\theta}\rangle_{l,\lambda,l,\lambda} \langle re^{i\theta}| = \sum_{n,m=0}^{\infty} \frac{r^{n+m} q^{\frac{1}{2}(\gamma_n^2)+\frac{1}{2}(l^2)}}{\sqrt{\gamma^{n+m}(q;q)_n(q;q)_m}} |n\rangle \langle m| \delta_{s,m-n}.$$  \hfill (51)

By applying $p-$times the operator $q \partial^l_{r,\lambda}$ on the two sides of (51) and evaluating the result at $r = 0$, only the term $n + m - p = 0$ survives in the right-hand side. It results

$$\left\{ (q \partial^l_{r,\lambda})^p \int_{0}^{2\pi} \frac{d\theta}{2\pi} \mathcal{N}_{l,\lambda}(r^2) e^{is\theta} |re^{i\theta}\rangle_{l,\lambda,l,\lambda} \langle re^{i\theta}| \right\}_{r=0} = \sum_{n,m=0}^{\infty} \frac{r^{n+m} q^{\frac{1}{2}(\gamma_n^2)+\frac{1}{2}(l^2)}}{\sqrt{\gamma^{n+m}(q;q)_n(q;q)_m(q;\gamma_n^2)}} \left( \frac{l^2 q^{l-n-m} - 1}{1 - q} \right)^p |n\rangle \langle m| \delta_{s,m-n} \delta_{p,m+n},$$

furnishing

$$|n\rangle \langle m| = \left( \frac{q^{(n+m)^2+nm}}{\gamma^{n+m}(q^{1+n};q)_m(q^{1+m};q)_n} \right)^{\frac{1}{2}} \times \left\{ (q \partial^l_{r,\lambda})^{n+m} \int_{0}^{2\pi} \frac{d\theta}{2\pi} \mathcal{N}_{l,\lambda}(r^2) e^{i(n-m)\theta} |re^{i\theta}\rangle_{l,\lambda,l,\lambda} \langle re^{i\theta}| \right\}_{r=0}.$$  \hfill (53)

Therefore, we can rewrite the density matrix (49) in the form

$$\rho = \sum_{n,m=0}^{\infty} \rho(n,m) \left( \frac{q^{(n+m)^2+nm}}{\gamma^{n+m}(q^{1+n};q)_m(q^{1+m};q)_n} \right)^{\frac{1}{2}} \times \left\{ (q \partial^l_{r,\lambda})^{n+m} \int_{0}^{2\pi} \frac{d\theta}{2\pi} \mathcal{N}_{l,\lambda}(r^2) e^{i(n-m)\theta} |re^{i\theta}\rangle_{l,\lambda,l,\lambda} \langle re^{i\theta}| \right\}_{r=0},$$ \hfill (54)

generalizing the Sudarshan’s diagonal representation of the density matrix [27] recovered in this case when $q \to 1$, $l = 1$ and $\lambda = 0$.

As stated in [27], the form (54) is particularly interesting since if $O = (a^\dagger)^n a^m$ is any normal ordered operator, its expectation value in the statistical state represented by the density matrix in the diagonal form expressed in terms of $(q; l, \lambda) -$coherent states

$$\rho = \int d^2 z \phi(z) |z\rangle_{l,\lambda,l,\lambda} \langle z|,$$ \hfill (55)
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is provided by
\[
tr(\rho \mathcal{O}) = tr(\rho (a^\dagger)^n a^m) = \int_{D_{l,\lambda}} d^2z \phi(z) z^n z^m, \tag{56}
\]
where \(d^2z = dRe(z) dIm(z)\) and \(\int d^2z \phi(z) = 1\) (to ensure that \(tr \rho = 1\)). The expansion coefficients \((q; l, \lambda)\) are expressed by the formula
\[
\rho(n, m) = \frac{q^{\frac{1}{2}(q)} + \frac{1}{2}(q)}{\sqrt{\gamma_{n+m}(q; q)_n(q; q)_m}} \int_{D_{l,\lambda}} d^2z \phi(z) z^n z^m, \tag{57}
\]
which is the \((q; l, \lambda)\)-analogue of eq. (7) in [27]. In polar coordinates \(z = re^{i\theta}\), these coefficients take the form
\[
\rho(n, m) = \frac{\pi q^{\frac{1}{2}(q)} \delta_{n,m}}{\gamma_n(q; q)_n} \int_0^\infty dr \frac{\phi(r^2)}{\mathcal{N}_{l,\lambda}(r^2)} r^{2n}, \quad \text{if} \quad 0 < q < 1, \tag{58}
\]
and
\[
\rho(n, m) = \frac{\pi q^{\frac{1}{2}(q)} \delta_{n,m}}{\gamma_n(q; q)_n} \int_{R_{l,\lambda}} \frac{d^2x}{\mathcal{N}_{l,\lambda}(x)} x^n, \quad \text{if} \quad q > 1. \tag{59}
\]
The relation \((59)\) generalizes the formula given by Parthasarathy and Sridhar [21] which is recovered in our case by setting \(l = 1\) and \(\lambda = 1\). It follows from \((58)\) or \((59)\) that
\[
\sum_{n=0}^\infty \rho(n, n) = 1. \tag{60}
\]

4.2. Reproducing kernel and related properties

Consider the matrix elements of \(\rho\) in the deformed CS
\[
l_{\lambda}(z' | \rho | z)_{l,\lambda} = \sum_{n,m=0}^\infty \rho(n, m) \frac{q^{\frac{1}{2}(q)} + \frac{1}{2}(q)}{\sqrt{\gamma_{n+m}(q; q)_n(q; q)_m}} \frac{z' z^m}{\mathcal{N}_{l,\lambda}(|z|^2) \mathcal{N}_{l,\lambda}(|z'|^2)}. \tag{61}
\]

**Definition 4.1** Let \(D_{l,\lambda}\) be the open disc in \(\mathbb{C}\) considered in \((16)\). Then, define the function
\[
\rho : D_{l,\lambda} \times D_{l,\lambda} \to \mathbb{C}, \quad (z', z) \mapsto \rho(z', z), \tag{62}
\]
\[
\rho(z', z) = \sum_{n,m=0}^\infty \rho(n, m) \frac{q^{\frac{1}{2}(q)} + \frac{1}{2}(q)}{\sqrt{\gamma_{n+m}(q; q)_n(q; q)_m}} \frac{z' z^m}{\mathcal{N}_{l,\lambda}(|z|^2) \mathcal{N}_{l,\lambda}(|z'|^2)}. \tag{63}
\]

**Proposition 4.2** The function \(\rho\) \((63)\) can be re-expressed as follows
\[
\rho(z', z) = \int_{D_{l,\lambda}} d^2\zeta K(\zeta, z) \rho(z', \zeta), \tag{64}
\]
where
\[
K(\zeta, z) = \frac{l_{\lambda}(\zeta | z)_{l,\lambda}}{\pi \ln q^{-1}} \frac{1}{\eta + |\zeta|^2}, \quad \text{if} \quad 0 < q < 1, \tag{65}
\]
and
\[
K(\zeta, z) = \frac{l_{\lambda}(\zeta | z)_{l,\lambda}}{\pi(1 + \frac{2}{\eta})}, \quad \text{if} \quad q > 1, \quad 0 < x = |\zeta|^2 < \frac{l^2 q^\lambda}{q - 1}, \quad \theta = \arg \zeta. \tag{66}
\]
Proof. Using (18), the relation (63) is readily put in the form
\[ ρ(z', z) := l,λ⟨z'|ρ|z⟩l,λ = \int_{D_{l,λ}} d^2ζρ(ζ', ζ)K(ζ, z), \]  
where \( K(z, ζ) = l,λ⟨ζ|ρ|ζ⟩l,λπ ln q^{-1} |ζ|^2 \) if \( 0 < q < 1 \) and \( K(z, ζ) = l,λ⟨ζ|ρ|ζ⟩l,λπ (1 + x/η) \) if \( q > 1 \), \( 0 < x = |ζ|^2 < l^2 q^{lλ}(1 - q) \), \( θ = arg(ζ) \).

The equation (64) encodes nothing but the reproducing property of \( ρ(z', z) \) where \( K(z, ζ) \) is the reproducing kernel.

In the limit \( q \to 1 \) when \( l = 1 \), the expression (66) is reduced to the undeformed reproducing kernel given in [3]. Further, the reproducing kernel possesses required properties as stated in the following.

Proposition 4.3 The kernel given in (67) and (66)

(i) satisfies the matrix multiplication property, i.e,
\[ \int_{D_{l,λ}} d^2ζK(z, ζ)K(ζ, z') = K(z, z'), \]  
(ii) obeys the Hermiticity property, i.e, \( (K(z, ζ))^* = K(ζ, z) \),
(iii) is positive,
(iv) is an entire function, i.e, for \( t = (q - 1)l^2 q^λ \)
\[ K(z, z) = \frac{1 - q}{l^2 q^λπ ln q^{-1}} \sum_{n=0}^{∞} t^n, \quad \text{if} \quad 0 < q < 1, \]  
and
\[ K(z, z) = \frac{1}{π} \sum_{n=0}^{∞} t^n, \quad \text{if} \quad q > 1, \]

Proof.

• From the resolution of the identity (18), we have
\[ \int_{D_{l,λ}} d^2ζK(z, ζ)K(ζ, z') = \frac{1}{π ln q^{-1}} \frac{l,λ⟨ζ'|z⟩l,λ}{η + |z|^2} \int_{D_{l,λ}} d^2ζ |ζ⟩l,λl,λ⟨ζ|l,λl,λ(ζ) \]

if \( 0 < q < 1 \) and
\[ \int_{D_{l,λ}} d^2ζK(z, ζ)K(ζ, z') = \frac{1}{π} \frac{l,λ⟨ζ'|z⟩l,λ}{1 + r/η} \int_{D_{l,λ}} d^2ζ |ζ⟩l,λl,λ⟨ζ|l,λl,λ(ζ) \]

if \( q > 1 \). By setting \( |ζ|^2 = x, |z|^2 = r, η = l^2 q^λ/(1 - q) \) and using the overcompleteness of the \((q; l, λ)\)–coherent states, the proof of (i) is achieved.

• (ii), (iii) and (iv) \((z = ζ)\) are immediate from equations (65) and (66). \( □ \)

Let now \( F \) and \( ˜F \) be the normal ordering and anti-normal ordering operators, defined as \( F := a^{σ}a^{ν} \) and \( ˜F := a^{ν}a^{σ} \), respectively. Then, as treated above, the expectation value of \( F \) is given by
\[ tr(ρF) = \int_{D_{l,λ}} d^2zφ(z)z^{σ}z^{ν}. \]  

\( (q; l, λ)\)–deformed Heisenberg algebra: representations, special functions and quantization
(q;l,λ)—deformed Heisenberg algebra: representations, special functions and quantization

This can be identified as the expectation value of the complex classical function \( z^n z^\sigma \) for a probability distribution \( \phi(z) \) over the complex domain \( D_{l,\lambda} \). For \( \phi(z) = \phi_1(\theta)\phi_2(r) \), (71) takes the form

\[
tr(\rho F) = \int_0^{2\pi} \frac{d\theta}{2} e^{i(\nu-\sigma)}\phi_1(\theta) \int_0^\infty dr^2 \phi_2(r)r^{2\nu} \quad \text{if} \quad 0 < q < 1, \quad (72)
\]

and

\[
tr(\rho F) = \int_0^{2\pi} \frac{d\theta}{2} e^{i(\nu-\sigma)}\phi_1(\theta) \int_0^\infty dq^2 \phi_2(q)q^{2\nu} \quad \text{if} \quad q > 1. \quad (73)
\]

As matter of concrete illustration, consider now a \((q;l,\lambda)\)—deformed Gaussian function \( \phi_2(r) = \frac{1}{\pi}N_{l,\lambda}^{-1}(r^2) \) with \( \phi_1(\theta) = 1 \) and \( \phi_2(x) = \frac{1}{\pi}N_{l,\lambda}^{-1}(x) \) with \( \phi_1(\theta) = 1 \), in the above equations, respectively. Then we get

\[
tr(\rho F) = \ln q^{-1} \gamma^{\nu+1} q^{-(\nu+1)}(q;\nu), \quad \text{if} \quad 0 < q < 1. \quad (74)
\]

and

\[
tr(\rho F) = (-\gamma)^\nu q^{-1}(q^{-1};\nu), \quad \text{if} \quad q > 1. \quad (75)
\]

Similarly to (71), the expectation value of \( \tilde{F} \) is given by

\[
tr(\rho \tilde{F}) = \int_{D_{l,\lambda}} d^2z\phi(z)_{l,\lambda}(z|a^\dagger a^\dagger|z)_{l,\lambda}. \quad (76)
\]

Turning back to polar coordinates with \( z = \sqrt{x}e^{i\theta} \), and assuming \( \phi(z) = \phi_1(\theta)\phi_2(x) \), the latter expression is explicitly evaluated as

\[
tr(\rho \tilde{F}) = q^{-\nu(2)}\gamma^{\nu}(q;\nu)\sum_{n=0}^{\infty} \frac{q^{(2)}(q;\nu)}{(q;\nu)^n} \int_0^{2\pi} \frac{d\theta}{2} e^{i(\nu-\sigma)}\phi_1(\theta) \int_0^{R_{l,\lambda}} x^n \phi_2(x)q^2 \frac{d\lambda}{N_{l,\lambda}(x)}. \quad (77)
\]

For now \( \phi_1(\theta) = 1 \), \( \phi_2(x) = 1/\pi N_{l,\lambda}(x) \),

\[
tr(\rho \tilde{F}) = q^{-\nu(2)}\gamma^{\nu}(q;\nu)\nu(q^{-1};\nu)^2 \mathcal{O}_\infty(-q^{-\nu};q^{1+\nu}|q), \quad q > 1, \quad (78)
\]

with

\[
\mathcal{O}_\infty(x; q^{1+m}|q) := \sum_{n=0}^{\infty} \frac{q^{(2)}(q;\nu)}{(q;\nu)^n} x^n J^{(1)}_0(2i(q^{-1+\nu}/q)), \quad (79)
\]

where the function \( J^{(1)}_0(z;q) := \sum_{n=0}^{\infty} \frac{(-1)^n(\hat{z})^2n}{(q;q)_n} \) is the \( q \)—deformed Bessel function [18].

Provided the above results, it becomes easier to compute the expectation value of the deformed harmonic oscillator Hamiltonian operator, \( H = aa^\dagger + a^\dagger a \), exploiting (74), (75) and (77) as follows:

\[
tr(\rho H) = \begin{cases} 
\ell^2 q^{\lambda-1} \left( 1 + \ln q^{-2\ell^2q^{\lambda-2}/(1-q)} \right), & \text{if} \quad 0 < q < 1, \\
\ell^2 q^{\lambda-1} (1 + 2q^{-2}), & \text{if} \quad q > 1.
\end{cases} \quad (80)
\]

In the same vein, the expectation values of the deformed position and momentum operators translate into

\[
tr(\rho Q) = \sqrt{2} \int_{D_{l,\lambda}} d^2z \phi(z) Re(z), \quad tr(\rho P) = \sqrt{2} \int_{D_{l,\lambda}} d^2z \phi(z) Im(z). \quad (81)
\]
Suppose for simplicity \( \phi(z) = \phi_{1,1}(\theta)\phi_{2,1}(|z|) \), \( \varphi_i(r, \theta) \) and \( \tilde{\varphi}_i(x, \theta) \) such that

\[
\varphi_i(r, \cos \theta) := \int_0^{2\pi} \frac{d\theta}{\sqrt{2}} \cos \theta \phi_{1,i}(\theta) \int_0^\infty r dr^2 \phi_{2,i}(r), \quad 0 < q < 1, \tag{82}
\]

\[
\tilde{\varphi}_i(x, \cos \theta) := \int_0^{2\pi} \frac{d\theta}{\sqrt{2}} \cos \theta \phi_{1,i}(\theta) \int_0^{R_{1,\lambda}} x^{1/2} \phi_{2,i}(x) d^{1,\lambda}_x, \quad q > 1, \tag{83}
\]

where \( i = Q, P \) and use the polar coordinates to detail expressions in (81) as follows:

\[
tr(\rho Q) = \varphi_Q(r, \cos \theta), \quad tr(\rho P) = \varphi_P(r, \sin \theta) \tag{84}
\]

if \( 0 < q < 1, \; |z| = r, \; 0 \leq \theta \leq 2\pi \) and

\[
tr(\rho Q) = \tilde{\varphi}_Q(x, \cos \theta), \quad tr(\rho P) = \tilde{\varphi}_P(x, \sin \theta) \tag{85}
\]

if \( q > 1, \; 0 < |z|^2 = x < R_{1,\lambda}, \; 0 \leq \theta \leq 2\pi \). Therefore,

(i) For the simpler case of \( \phi_{1,Q}(\theta) = \phi_{1,P}(\theta) = 1 \), the expectation values of \( Q \) and \( P \) are reduced to zero, i.e

\[
tr(\rho Q) = tr(\rho P) = 0.
\]

(ii) For \( \phi_{1,Q}(\theta) = 1/\sqrt{2} \cos \theta, \; \phi_{1,P}(\theta) = 1/\sqrt{2} \sin \theta \), the expectation values of \( Q \) and \( P \) can be re-expressed in terms of \( \phi_{2,i}(|z|) \), i.e

\[
tr(\rho Q) = \pi \int_0^\infty r dr^2 \phi_{2,Q}(r), \quad tr(\rho P) = \pi \int_0^\infty r dr^2 \phi_{2,P}(r) \tag{86}
\]

if \( 0 < q < 1 \) and

\[
tr(\rho Q) = \pi \left( \frac{l^2 q^\lambda}{q - 1} \right)^{1/2} \sum_{n=0}^\infty q^{-3n/2} \phi_{2,Q} \left( \frac{l^2 q^\lambda n}{q - 1} \right), \tag{87}
\]

\[
tr(\rho P) = \pi \left( \frac{l^2 q^\lambda}{q - 1} \right)^{1/2} \sum_{n=0}^\infty q^{-3n/2} \phi_{2,P} \left( \frac{l^2 q^\lambda n}{q - 1} \right) \tag{88}
\]

if \( q > 1 \).

Assigning now concrete expressions to the unknown functions, e.g. \( \phi_{2,Q}(r) = \phi_{2,P}(r) \equiv 1/\pi \mathcal{N}_{1,\lambda}(r) \), the expectation values of the deformed position and momentum operators coincide and give

\[
tr(\rho Q) = tr(\rho P) = \begin{cases} 
\left( \frac{l^2 q^\lambda}{1-q} \right)^3 \ln q^{-1}(q; q)_2, & 0 < q < 1, \\
\left( \frac{l^2 q^\lambda}{q-1} \right)^{1/2} \frac{(q^{-1} q^{-1})_\infty}{(q^{-1}/q^{-1})_\infty}, & q > 1.
\end{cases} \tag{89}
\]

To end this discussion, let us compute the matrix elements and expectation value for the Hamiltonian operator describing the propagation of light in a non-linear medium like Kerr medium. Such a Hamiltonian is usually expressed by [27]

\[
H_d := a^\dagger a + \frac{\chi}{2} a^\dagger a^2, \tag{90}
\]

by setting \( \hbar = 1 = \omega; \chi \) represents the interaction strength of the light with the non-linear medium. Its matrix elements in the Fock space states are provided by

\[
\langle r | H_d | s \rangle = \varphi(s) \left( 1 + \frac{\chi}{2} \varphi(s - 1) \right) \delta_{r,s}, \tag{91}
\]
while its expectation value is evaluated as follows:

\[ \text{tr}(\rho H) = \begin{cases} 
\gamma^2 q^{-1}(1 - q) \ln q^{-1} \left\{ 1 + \frac{1}{2} \gamma q^{-2}(1 - q^2) \right\}, & \text{if } 0 < q < 1, \\
\frac{1}{2} q^\lambda - 3 \left\{ 1 + \frac{1}{2} \right\} q^\lambda - 3 (1 + q), & \text{if } q > 1.
\end{cases} \]  

(92)

We observe that the matrix elements \[ \text{[91]} \] strongly depends on the algebra structure function, and hence on the deformed number operator.

5. Deformed \( a^n a^m \) and \( a^m a^n \) operators: matrix elements and mean values

This section deals with the computation of relevant normal and anti-normal forms in the coherent states \(|z\rangle_{L,\lambda}\) and their relation with engendered new deformed \((q;l,\lambda)\)–hypergeometric functions. As a first step in such a direction, the following preliminary result stated as a lemma reveals to be useful.

**Lemma 5.1** The matrix elements of the normal form are given by

\[ \langle r | a^m a^n | s \rangle = (-q)^n (q^{-s};q)_n q^{-\frac{1}{2} \frac{(n-m)}{2}} \sqrt{\gamma^{m+n} (q^{1+s};q)_{m-n} \delta_{r,m-n+s}}, \]  

(93)

if \( n < m \),

\[ \langle r | a^m a^n | s \rangle = (-q)^m (q^{-s-m+n};q)_m q^{-\frac{1}{2} \frac{(m-n)}{2}} \sqrt{\gamma^{m+n} (q^{1+r};q)_{n-m} \delta_{s,n-m+r}}, \]  

(94)

if \( n > m \),

while those of the anti-normal form are expressed by

\[ \langle r | a^m a^n | s \rangle = q^{-\frac{1}{2} \frac{(m)}{2}} \sqrt{\gamma^{n+m} (q^{1+r};q)_n (q^{1+s};q)_m \delta_{r+n,s+m}}. \]  

(95)

**Proof.** See appendix B.

**Proposition 5.2** The expectation values of the normal and anti-normal forms in the coherent states \(|z\rangle_{L,\lambda}\) are given, respectively, by

\[ \langle a^n a^m \rangle = (q^{-m};q)_n (-\gamma q)_m z^{m-n} \frac{\phi_1 \left( \frac{q^{1+m}}{q^{1-n}} \bigg\vert q ; \frac{|z|^2 q^{-n}}{\gamma} \right)}{\phi_1 \left( \frac{q^{1+m}}{q^{1-n}} \bigg\vert q ; \frac{|z|^2 q^{-n}}{\gamma} \right)} \]  

(96)

if \( n < m \),

\[ \langle a^n a^m \rangle = (q^{-n};q)_m (-\gamma q)_n z^{-m-n} \frac{\phi_1 \left( \frac{q^{1+n}}{q^{1-m}} \bigg\vert q ; \frac{|z|^2 q^{-m}}{\gamma} \right)}{\phi_1 \left( \frac{q^{1+n}}{q^{1-m}} \bigg\vert q ; \frac{|z|^2 q^{-m}}{\gamma} \right)} \]  

(97)

if \( n > m \), and

\[ \langle a^n a^m \rangle = q^{-\frac{1}{2}} \gamma^n (q; q)_n \left( -\frac{|z|^2}{\gamma} ; q \right)_{\infty}^{-1} \phi_1 \left( \frac{q^{1+n}}{q} \bigg\vert q ; \frac{|z|^2 q^{-n}}{\gamma} \right). \]  

(98)

Moreover, for any two integers \( n \) and \( m \),

\[ \langle a^n a^m \rangle = z^n z^m, \]  

(99)
\( (q; l, \lambda) \)-deformed Heisenberg algebra: representations, special functions and quantization

where

\[
1 \phi_1 \left( \begin{array}{c|c} A & q; t \\ B & \end{array} \right) = \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(B, q; q)_n} (A; q)_n t^n \tag{100}
\]

is the \( q \)-deformed hypergeometric function \([13]\) and \( \gamma = l^2 q^{\lambda-1}/(1-q) \).

**Proof.** By definition

\[
\langle a^n a^m \rangle := \frac{\langle l, \lambda \rangle(a^n a^m | z)_{l, \lambda}}{\langle l, \lambda \rangle(z | l, \lambda)_{l, \lambda}} = \frac{N_{l, \lambda}^{-1}(|z|^2) \sum_{r,s=0}^{\infty} q^{r+s} \gamma^{r+s} q^{r+s} (0 | a^r a^m a^s | 0)}{\sqrt{\gamma^{r+n+m}\langle q; q \rangle_{r+n+m} \delta_{r+n+m} \delta_{r+n+s+m}}}
\]

By using \([22]\), the equation \((101)\) can be rewritten as

\[
\langle a^n a^m \rangle = \frac{N_{l, \lambda}^{-1}(|z|^2) \sum_{r,s=0}^{\infty} q^{r+s} \gamma^{r+s} q^{r+s} (0 | a^r a^m a^s | 0)}{\sqrt{\gamma^{r+n+m}\langle q; q \rangle_{r+n+m} \delta_{r+n+m} \delta_{r+n+s+m}}}
\]

The proof of \((96)\) is achieved by re-expressing the sum in terms of a formula analog to \( q \)-hypergeometric function. Idem for the other expressions. \( \square \)

**6. Berezin-Klauder-Toeplitz quantization**

The Berezin-Klauder-Toeplitz quantization, (also called "anti-Wick" or coherent state quantization), of phase space observables of the complex plane, \( D_{l, \lambda} \), uses the resolution of the identity \([18]\) and is performed by mapping a function \( f \) that satisfies appropriate conditions, to the following operator in the Hilbert space \([17, 14] \) and references therein:

\[
f \mapsto A_f = \int_{D_{l, \lambda}} d \mu_{l, \lambda}(\bar{z}, z) f(z, \bar{z}) | l, \lambda \rangle (l, \lambda \langle z) = \sum_{n, n' = 0}^{\infty} (A_f)_{mm'} | n \rangle \langle n' |, \tag{103}
\]

where this integral is understood in the weak sense, i.e. it defines in fact a sesquilinear form (eventually only densely defined)

\[
B_f(\psi_1, \psi_2) = \int_{D_{l, \lambda}} d \mu_{l, \lambda}(\bar{z}, z) f(z, \bar{z}) \langle \psi_1 | l, \lambda \langle z | \psi_2 \rangle,
\]

with the matrix elements

\[
(A_f)_{mm'} = \frac{q^{n(n-1)/2} + n'(n'-1)/2}{\sqrt{\gamma^{n+n'| q; q \rangle_n q; q \rangle_{n'}}} \int_{D_{l, \lambda}} f(z, \bar{z}) z^n z'^m d \mu_{l, \lambda}(\bar{z}, z) / \langle l, \lambda \rangle(|z|^2). \tag{104}
\]

Operator \( A_f \) is symmetric if \( f(z, \bar{z}) \) is real-valued, and is bounded (resp. semi-bounded) if \( f(z, \bar{z}) \) is bounded (resp. semi-bounded). In particular, the Friedrich extension allows to define \( A_f \) as a self-adjoint operator if \( f(z, \bar{z}) \) is a semi-bounded real-valued function. Note that the original \( f(z, \bar{z}) \) is a upper or contravariant symbol, usually non-unique,
for the operator $A_f$. This problem involving the property of the function $f$ and the self-adjointness criteria of operators is thoroughly discussed in a recent work by Bergeron et al. and does not deserve further development here. So, without loss of generality, let us immediately examine different concrete expressions for the function $f$ in the line of [14] as matter of result comparison:

(i) The function $f$ only depends on $|z|^2 = t$: the matrix elements (104) take the form

$$ (A_F)_{nn'} = \frac{q^{n(n+1)/2}}{\ln q^{-1} [n]_q !} \int_0^\infty dt \frac{t^n f(t^2 q^\lambda)}{E_q ((1-q)t)}, \quad \text{if } 0 < q < 1, $$

and

$$ (A_F)_{nn'} = \frac{q^{n(n+1)/2}}{[n]_q !} \int_0^{t^2 q^\lambda/(q-1)} dt \frac{t^n f(t^2 q^\lambda)}{E_q ((1-q)t)}, \quad \text{if } q > 1. $$

(ii) The function $f$ only depends on the angle $\theta = \arg z$, i.e. $f(z, \bar{z}) = F(\theta)$: the matrix elements (104) are given by

$$ (A_F)_{nn'} = c_{n'-n}(F) \left( \frac{q^{n+1}/2}{{q q_n q_{n'} ^ n}} \right)^{1/2} (q; q)^{n_{n'}}; $$

where $c_n(F)$ are the Fourier coefficients of the function $F$:

$$ c_n(F) = \frac{1}{2\pi} \int_0^{2\pi} d\theta e^{-in\theta} F(\theta), $$

while the self-adjoint “angle” operator is defined by

$$ A_\theta = \pi I_F + i \sum_{n \neq n'} \left( \frac{q^{n+1/2} (q; q)^{n_{n'}}}{(q; q)^{n_{n'}}} \right)^{1/2} \frac{(q; q)^{n_{n'}}}{n' - n} |n\rangle \langle n'|. \quad (105) $$

(iii) The function $f(z, \bar{z}) = z$ and $f(z, \bar{z}) = \bar{z}$: the operator (103) takes the forms

$$ A_z = a, \quad a |n\rangle = \sqrt{\varphi(n)} |n-1\rangle, \quad a |0\rangle = 0, \quad (106) $$

$$ A_\bar{z} = a^\dagger, \quad a^\dagger |n\rangle = \sqrt{\varphi(n+1)} |n+1\rangle, \quad (107) $$

where the function $\varphi$ is defined in [7]. The state $|z\rangle_{l,\lambda}$ is eigen-vector of $A_z = a$ with eigenvalue $z$ like for standard CS. The operators $A_z$ and $A_\bar{z}$ satisfy the algebra (7), i.e

$$ [A_z, A_\bar{z}] = \ell^2 q^{\lambda-1-N}, $$

as required.

(iv) The function $f(z, \bar{z}) = z^\nu \bar{z}^{\nu}$, $\mu, \nu \in \mathbb{N} \cup \{0\}$: the matrix elements (104) of $A_f$ are given by

$$ (A_F)_{nn'} = \left( \frac{q^{n+1/2} (q; q)^{n_{n'}}}{(q; q)^{n_{n'}}} \right)^{1/2} (q; q)^{n_{n'}} \delta_{n-n', \mu-\nu}. \quad (108) $$

(v) The function $f$ is defined on the complex plane $\mathbb{C} = \left\{ z = \frac{a + i\nu}{\sqrt{2}} \right\}$, i.e $f(z, \bar{z}) = |z|^2$.

In this case, $f = \frac{1}{2} (A^2 + q^2)$ looks like the classical Hamiltonian of the harmonic oscillator with $\omega = 1 = 2m$ where $m$ is the particle mass. Moreover,

$$ A_{(p^2+q^2)/2} = A_{\bar{z}} = A_z A_\bar{z} = aa^\dagger = \varphi(N+1). \quad (109) $$
describes the classical phase-space trajectory. For a fixed normalized state $D$
and, in the case of quadratic expressions, $N$
where the series

while the deformed harmonic oscillator Hamiltonian quantizes as follows:

while the deformed harmonic oscillator Hamiltonian quantizes as follows:

The time evolution of the quantized version $a = A_z$ of the classical phase space point $z = (q + ip)/\sqrt{2}$ is given by

In the limit when $q \to 1, l = 1/\sqrt{2}$, one recovers the standard case $\dot{z}(t) = z\dot{\theta}$ which describes the classical phase-space trajectory. For a fixed normalized state $|z_0\rangle_{l,\lambda}$ on $D_{l,\lambda} \subset \mathbb{C}$, one can define the probability distribution on the complex plane by the map

So, the time evolution behavior $t \mapsto \rho_{|z_0\rangle_{l,\lambda}}(z, t)$ of the probability density (114) is given by

where the series $N_{l,\lambda}(x, \gamma)$ is defined as follows

Defining the function

called the lower or covariant symbol of the operator $A_f$ [19, 7], the map $f \mapsto \tilde{f}$ is an integral transform with the kernel $|A_{\tilde{f}}|^2$ which generalizes the Berezin transform.
For \( z = re^{i\theta} \) and \( f(z, \bar{z}) = F(\theta) \), one can find the lower (contravariant) symbol of a function \( F(\theta) \) as follows

\[
\hat{f}(z, \bar{z}) = i_{\lambda}(z|A_f|z)_{l,\lambda} = c_0(F) + \sum_{n \neq 0} \frac{q^{(2)}_n}{(q; q)_n} \left( \frac{(1 - q)qr^2}{l^2 q^\lambda} \right)^n S_n(r; F, q),
\]

where the function \( S_n(r; F, q) \) is given by

\[
S_n(r; F, q) = \frac{1}{N_{l,\lambda}(r^2)} \sum_{k=0}^n (-1)^{-\frac{k}{2}} (q^{-n}; q)_\frac{k}{2} \left( \frac{lq^\lambda/2e^{i\theta}}{\sqrt{1 - qr}} \right)^k c_k(F),
\]

with the conditions \( c_0(F) := c_k(e^{i\theta}) = 1 \).

In the case where the coefficients \( c_k(F) = 1/(q; q)_\frac{k}{2} \), the last function is reduced to

\[
\tilde{S}_n(r; q) = p_n(q^{-1}x'; 0|q)(e^{2i\theta}x'^{-1}; q)_\infty,
\]

where \( p_m(x; a|q) \) is the little \( q \)-Laguerre polynomial [18] and \( x' = -l^2q^\lambda e^{2i\theta}/(1 - q)qr^2 \).

Finally, for the angle operator \( A_\theta \) we have the particular series

\[
\hat{\theta}(z, \bar{z}) = \pi + i \sum_{n \neq 0} \frac{q^{(2)}_n}{(q; q)_n} \left( \frac{(1 - q)qr^2}{l^2 q^\lambda} \right)^n \tilde{S}_n(r; e^{i\theta}, q),
\]

where

\[
\tilde{S}_n(r; e^{i\theta}, q) = \frac{1}{N_{l,\lambda}(r^2)} \sum_{k=0}^n (-1)^{-\frac{k}{2}} (q^{-n}; q)_\frac{k}{2} \left( \frac{lq^\lambda/2e^{i\theta}}{\sqrt{1 - qr}} \right)^k.
\]

7. Conclusion

In this paper, we have provided an explicit construction, including the recursion relation of generalized continuous \((q; l, \lambda)-\)Hermite polynomials generated by polynomial expansion of the deformed position and momentum operators in associated Fock space basis. Particular classes of deformed Hermite polynomials have been deduced with explicit resolution of the moment problems giving their orthogonality weight functions. The diagonal representation of the density matrix using the \((q; l, \lambda)-\)CS has been computed. Reproducing kernel \( K(z, \zeta) \) and its properties have been investigated. Main matrix elements of normal and anti normal forms and mean operator values have been determined. The Berezin-Klauder-Toeplitz quantization (also called CS quantization) of classical phase space observables has been performed. Furthermore, the angle and time evolution operators and semi-classical phase space trajectories have been discussed.

Acknowledgements

This work is partially supported by the Abdus Salam International Centre for Theoretical Physics (ICTP, Trieste, Italy) through the Office of External Activities (OEA) - Prj-15. The ICMPA is also in partnership with the Daniel Iagolnitzer Foundation (DIF), France.
Appendix A. Hopf algebra structure of the \((q;l, \lambda)\)-deformed oscillator algebra

In this appendix, we show that the generalized Heisenberg-Weyl algebra, generated by the generators \(N, a, a^\dagger\) and the relations \((7)\) carries out a Hopf algebra structure.

An algebra \(C\) is a Hopf algebra if there are defined a coproduct \(\Delta\), a counit \(\epsilon\) and an anti-homomorphism of antipode \(S\) such that

\[
\Delta : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}, \quad \Delta(AB) = \Delta(A)\Delta(B) \tag{124}
\]

\[
\epsilon : \mathcal{H} \rightarrow \mathbb{C}, \quad \epsilon(AB) = \epsilon(A)\epsilon(B), \tag{125}
\]

\[
S(AB) = S(A)S(B), \tag{126}
\]

which satisfy the properties

\[
(id \otimes \Delta)\Delta(h) = (\Delta \otimes id)\Delta(h) \tag{127}
\]

\[
(id \otimes \epsilon)\Delta(h) = (\epsilon \otimes id)\Delta(h) \tag{128}
\]

\[
m(id \otimes S)\Delta(h) = m(S \otimes id)\Delta(h) = \epsilon(h)I, \tag{129}
\]

for all \(h \in \mathcal{H}\) and \(m\) is the multiplication defined as \(m : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}\). To prove this it is sufficient to show that these relations are satisfied by the generators governing the considered algebra. Using the Leibniz rule, we have

\[
q\frac{\partial}{\partial x}(fg)(x) = q\frac{\partial}{\partial x}(f(x))g(x) + q^{-1}f(x)q\frac{\partial}{\partial x}g(x), \tag{130}
\]

for \(f, g \in \mathcal{O}(D_{l, \lambda})\), the set of holomorphic functions defined on the disc \(D_{l, \lambda}\). Let the actions of coproduct \(\Delta\), counit \(\epsilon\), and antipode \(S\) on the generators of the algebra be defined as follows

\[
\Delta(a^\dagger) = c_1 l^2 q^{\lambda}a^\dagger \otimes q^{\alpha_1 N} + c_2 l^2 q^{\lambda}q^{\alpha_2 N} \otimes a^\dagger, \tag{131}
\]

\[
\Delta(a) = c_3 l^2 q^{\lambda}a \otimes q^{\alpha_3 N} + c_4 l^2 q^{\lambda}q^{\alpha_4 N} \otimes a, \tag{132}
\]

\[
\Delta(N) = c_5 N \otimes I + c_6 I \otimes N + \gamma I \otimes I, \quad \Delta I = I \otimes I, \tag{133}
\]

\[
\epsilon(a^\dagger) = c_7, \quad \epsilon(a) = c_8, \quad \epsilon(N) = c_9, \quad \epsilon(I) = I, \tag{134}
\]

\[
S(a^\dagger) = -c_{10}a^\dagger, \quad S(a) = c_{11}a, \quad S(N) = -c_{12}N + c_{13}I, \quad S(I) = I, \tag{135}
\]

where the constants \(c_k, k = 1, 2, \ldots, 13, \alpha_j = 1, 2, \ldots, 4\) and \(\gamma\) are unknown coefficients depending on the Hopf algebra properties. Then

(i) From \((127)\) and \((131)\), for \(h = a^\dagger\) we find

\[
c_1 = l^{-2}q^{\alpha_1 \gamma - \lambda}, \quad c_2 = l^{-2}q^{\alpha_2 \gamma - \lambda}, \quad c_5 = 1 = c_6. \tag{136}
\]

(ii) From \((127)\) and \((132)\), for \(h = a\) we find

\[
c_3 = l^{-2}q^{\alpha_3 \gamma - \lambda}, \quad c_4 = l^{-2}q^{\alpha_4 \gamma - \lambda}, \quad c_5 = 1 = c_6. \tag{137}
\]

(iii) A direct computation gives

\[
\Delta([a, a^\dagger]_\alpha) = (l^2 q^\lambda)^2 \left( c_1 c_3 [a, a^\dagger]_\alpha \otimes q^{(\alpha_1 + \alpha_3)N} + c_2 c_4 q^{(\alpha_2 + \alpha_4)N} \otimes [a, a^\dagger]_\alpha \right)
\]

\[
= c_2 c_3 (1 - \alpha q^{-\alpha_2 - \alpha_3}) a q^{\alpha_2 N} \otimes q^{\alpha_3 N} a^\dagger + c_1 c_4 (1 - \alpha q^{-\alpha_1 - \alpha_4})
\]
From (7) and (134), we find that

\[ \Delta([a, a^\dagger]) = \frac{q^\lambda}{q-1}(1 - \alpha - q^{-1}(1 - q\alpha)q^{-N}) \] implies that

\[ \Delta([a, a^\dagger])_a = aa^\dagger - \alpha a^\dagger a = \frac{q^\lambda}{q-1}(1 - \alpha - q^{-1}(1 - q\alpha)q^{-N}) \] (139)

Comparing (138) and (139) and setting \( 1 - \alpha q^{-\alpha_2 - \alpha_3} = 0 = 1 - \alpha q^{-\alpha_1 - \alpha_4} \), we have

\[ \alpha_2 + \alpha_3 = \alpha_1 + \alpha_4, \quad \alpha_1 + \alpha_3 = \alpha_2 + \alpha_4 = -1, \quad q^{-\gamma} = 2. \] (140)

Therefore, \( \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = -\frac{1}{2} \).

(iv) From (7) and (134), we find that

\[ c_7 = c_8 = 0. \] (141)

(v) From the identity (129) and for \( h = a^\dagger, a \) respectively, we get

\[ c_{12} = -1, \quad c_{10} = -q^{-\frac{13\alpha}{2}} = -c_{11}. \] (142)

Finally the following definition of Hopf algebra structure is in order

\[ \Delta(a^\dagger) = q^{-\frac{\gamma}{2}}(a^\dagger \otimes q^{-\frac{\gamma}{2}} + q^{-\frac{\gamma}{2}} \otimes a^\dagger) \] (143)

\[ \Delta(a) = q^{-\frac{\gamma}{2}}(a \otimes q^{-\frac{\gamma}{2}} + q^{-\frac{\gamma}{2}} \otimes a) \] (144)

\[ \Delta(N) = N \otimes I + I \otimes N + \gamma I \otimes I, \quad \Delta I = I \otimes I, \] (145)

\[ \epsilon(a^\dagger) = 0 = \epsilon(a), \quad \epsilon(N) = -\gamma, \quad \epsilon(I) = I, \] (146)

\[ S(a^\dagger) = q^{-\frac{\gamma}{2}}a^\dagger, \quad S(a) = q^{-\frac{\gamma}{2}}a, \quad S(N) = N + c_{13} I, \quad S(I) = I. \] (147)

Indeed, for \( h = N \), we have

\[ m(S \otimes \text{id})\Delta(h) = 2N + \gamma I + c_{13} = m(\text{id} \otimes S)\Delta(h), \] (148)

and

\[ (\text{id} \otimes \epsilon)(\Delta(N)) = (\epsilon \otimes \text{id})(\Delta(N)). \] (149)

Appendix B.

From (9), a direct computation gives

\[ \langle r|a^n a^\dagger m|s \rangle = \frac{q^{\frac{1}{2}(q)\frac{1}{2}(q)}}{\sqrt{\gamma^{r+s}(q; q)_r(q; q)_s}}(0|a^{r+n} a^\dagger^{m+s}|0) \]

\[ = q^{-\frac{1}{2}(q)-\frac{1}{2}(q)-\frac{r+s}{2}}\sqrt{\gamma^{m+n}(q^{1+r}; q)_r(q^{1+s}; q)_s} \delta_{r+n,s+m}, \] (150)

for any integers \( n \) and \( m \).

Let us make use of the following results to show (93) and (94)

\[ a^\dagger m a^n = a^\dagger^{m-1}a^{n-1}\varphi(N-n+1) \]

\[ = a^\dagger^{m-2}a^{n-2}\varphi(N-n+2)\varphi(N-n+1) \]

\[ \vdots \]

\[ = a^\dagger^{m-n}\prod_{k=0}^{n-1}\varphi(N-k), \quad \text{for} \quad n < m. \] (151)
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Similarly,

\[
a^{tm}a^n = \prod_{k=0}^{m-1} \varphi(N-k)a^{n-m}, \quad \text{for} \quad n > m. \tag{152}
\]

From (151) and (152), a direct computation gives

\[
\langle r | a^{tm}a^n | s \rangle = \frac{q^{\frac{1}{2}(\frac{m}{2})+\frac{1}{2}(\frac{n}{2})}}{\sqrt{\gamma^{r+s}(q; q)_r(q; q)_s}} (0 | a^r a^{tm} A^n A^s | 0) = (-q)^m(q^{-s}; q)_n q^{-\frac{1}{2}(m-n)} \frac{1}{2} \sqrt{\gamma^{m+n}(q^{1+r}; q)_m n} \delta_{r, m-n+s}, \tag{153}
\]

if \(n < m\),

\[
\langle r | a^{tm}a^n | s \rangle = \frac{q^{\frac{1}{2}(\frac{m}{2})+\frac{1}{2}(\frac{n}{2})}}{\sqrt{\gamma^{r+s}(q; q)_r(q; q)_s}} (0 | a^r a^{tm} a^n a^s | 0) = (-q)^m(q^{n-s-m}; q)_m q^{-\frac{1}{2}(n-m)} \frac{1}{2} \sqrt{\gamma^{m+n}(q^{1+r}; q)_m n} \delta_{s, n-m+r}, \tag{154}
\]

if \(n > m\).

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