UNITARY LIE ALGEBRAS AND LIE TORI OF TYPE BCₕ, r ≥ 3

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ABSTRACT. A Lie Λ-torus of type Xₕ is a Lie algebra with two gradings – one by an abelian group Λ and the other by the root lattice of a finite irreducible root system of type Xₕ. In this paper we construct a centreless Lie Λ-torus of type BCₕ, which we call a unitary Lie Λ-torus, as it is a special unitary Lie algebra of a nondegenerate Λ-graded hermitian form of Witt index r over an associative torus with involution. We prove a structure theorem for centreless Lie Λ-tori of type BCₕ, r ≥ 3, that states that any such Lie torus is bi-isomorphic to a unitary Lie Λ-torus, and we determine necessary and sufficient conditions for two unitary Lie Λ-tori to be bi-isomorphic. The motivation to investigate Lie Λ-tori came from the theory of extended affine Lie algebras, which are natural generalizations of the affine and toroidal Lie algebras. Every extended affine Lie algebra possesses an ideal which is a Lie n-torus of type Xₙ for some irreducible root system Xₙ, where by an n-torus we mean that the group Λ is a free abelian group of rank n for some n ≥ 0. The structure theorem above enables us to classify centreless Lie n-tori of type BCₙ, r ≥ 3. We show that they are determined by pairs consisting of a quadratic form κ on an n-dimensional Z₂-vector space and of an orbit of the orthogonal group of κ. We use that result to construct extended affine Lie algebras of type BCₙ, r ≥ 3. Our article completes a large project involving many earlier papers and many authors to determine the centreless Lie n-tori of all types.

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1. INTRODUCTION

In Lie theory it is known that a finite-dimensional simple Lie algebra over a (not necessarily algebraically closed) field of characteristic 0 having a root system of type $\text{BC}_r$ or $\text{Br}$, $r \geq 3$, relative to a maximal split toral subalgebra is isomorphic to the special unitary Lie algebra of a nondegenerate hermitian form of Witt index $r$ ([S, Chap. V], [T]). In this paper, we prove an infinite-dimensional graded analogue of that result. More specifically, we show that any centreless Lie torus of type $\text{BC}_r$ for $r \geq 3$ is bi-isomorphic to the special unitary Lie algebra of a nondegenerate graded hermitian form of Witt index $r$.$^1$

The notion of a Lie torus was first introduced by Y. Yoshii ([Y3], [Y4]). An equivalent definition was later formulated by E. Neher in [N1]. By definition, a Lie $\Lambda$-torus $L$ of type $\Delta$ has two compatible gradings, one a root grading by the root lattice of a finite irreducible (not necessarily reduced) root system $\Delta$ and the other an external grading by an arbitrary abelian group $\Lambda$. Because of the double grading, there is a natural notion of equivalence for Lie tori called bi-isomorphism.

The motivation for the study of Lie tori came from extended affine Lie algebras (EALAs), which are natural generalizations of the affine and toroidal Lie algebras. There is a construction due to Neher of a family of EALAs starting from a centreless Lie $\Lambda$-torus with $\Lambda$ isomorphic to $\mathbb{Z}^n$ for some $n \geq 0$ (or what is referred to as a centreless Lie $n$-torus). Moreover, any EALA occurs in one such family [N2]. So an understanding of the structure of centreless Lie $n$-tori yields a corresponding understanding of the structure of EALAs. More generally, for an arbitrary torsion-free abelian group $\Lambda$, centreless Lie $\Lambda$-tori that possess an invariant form can be

$^1$For Lie tori, and more generally for root graded Lie algebras, type $\text{BC}_r$ contains type $\text{Br}$ as a special case (see Remark 3.3.3). In contrast, for finite-dimensional simple Lie algebras and for extended affine Lie algebras the convention is that type $\text{Br}$ and type $\text{BC}_r$ are disjoint. Consequently, extended affine Lie algebras of type $\text{Br}$ and $\text{BC}_r$ are constructed from Lie tori of type $\text{BC}_r$ (see Section 7.3).
used to construct what are called invariant affine reflection algebras. (See [N3, §6.7] and Remark 8.1.3 below.)

This paper is devoted to describing the structure of Lie Λ-tori of type BC

r

, r \geq 3. In Chapters 2–5 we present the necessary background on root graded Lie algebras, Lie tori, and associative tori with involution, as well as hermitian forms and unitary Lie algebras over associative tori with involution.

Chapter 6 contains the main results of the paper. We first construct a centreless Lie Λ-torus of type BC

r

, which we call a unitary Lie Λ-torus, as it is a special unitary Lie algebra S of a nondegenerate Λ-graded hermitian form of Witt index r over an associative torus with involution. The root grading of S is the root space decomposition relative to an ad-diagonalizable subalgebra of S, and the external grading is induced from the Λ-grading of the hermitian form. We then prove a structure theorem (Theorem 6.3.1) for centreless Lie Λ-tori of type BC

r

, r \geq 3, that states that any such Lie torus is bi-isomorphic to a unitary Lie Λ-torus. Our next main result (Theorem 6.6.1) provides necessary and sufficient conditions for two unitary Lie Λ-tori to be bi-isomorphic.

In Chapter 7 we specialize to the case when Λ is isomorphic to \(\mathbb{Z}^n\). We obtain a classification of centreless Lie n-tori of type BC

r

, r \geq 3, in Theorem 7.2.4. These Lie tori are determined by pairs consisting of a quadratic form \(\kappa\) on a \(n\)-dimensional vector space over \(\mathbb{Z}_2\) and an orbit under a certain action of the orthogonal group of \(\kappa\). We apply Neher’s method and our results to construct maximal EALAs of type B

r

 and BC

r

, r \geq 3. Using coordinates in the grading group \(\Lambda\) (\(\cong \mathbb{Z}^n\)), we give explicit expressions for the product and invariant bilinear form on the EALA, in the spirit of [BGK]. Chapter 8 contains some concluding remarks and some possible directions for future investigations.

By treating the case of type BC

r

, r \geq 3, our structure theorem completes a program to describe the centreless Lie n-tori of all types. This effort, which has involved many authors, began in 1993 with the seminal paper on EALAs of type A

r

, r \geq 3, by Berman, Gao and Krylyuk [BGK]. (For an overview of the program and relevant references see [AF, §7–11].)

Our work on Lie algebras graded by the root system BC

r

 was started together with Yun Gao. Due to other commitments, he felt he could not devote time to this present project and urged us to proceed without him. We value his contributions to our monograph [ABG] and to the initial investigations that ultimately led to our present paper; and we thank him for his enthusiastic support of our efforts.

2. PRELIMINARIES

2.1. Notational conventions.

We begin with some conventions and definitions that will be used throughout the paper.

All algebras and vector spaces are over \(F\), a field of characteristic different from 2. In Sections 4.2, 7.2, 7.3 and Chapters 3, 6, 8 we assume that \(F\) has characteristic 0, and we prove the main results of the paper under that hypothesis. This additional assumption on the field will always be stated explicitly.

Unless indicated to the contrary, all associative algebras are unital, and by a module for an associative algebra \(A\), we mean a right module for \(A\). If \(X\) is an \(A\)-module, then \(gl_A(X)\) is the Lie algebra with underlying space \(End_A(X)\) under the commutator product. The centre of an associative or Lie algebra \(A\) is denoted
by \( Z(\mathcal{A}) \). A Lie algebra \( \mathcal{L} \) is said to be centreless if \( Z(\mathcal{L}) = 0 \). The centroid of any algebra \( \mathcal{A} \) is the associative algebra \( \text{Cent}(\mathcal{A}) \) consisting of all endomorphisms of \( \mathcal{A} \) that commute with all left and right multiplications. If \( \mathcal{A} \) is a unital associative algebra, then \( Z(\mathcal{A}) \) and \( \text{Cent}(\mathcal{A}) \) are isomorphic under the map which sends \( \alpha \) to left multiplication by \( \alpha \).

If \( S \) is any subset of a group \( \Lambda \), then \( \langle S \rangle \) stands for the subgroup generated by \( S \).

2.2. **Associative algebras with involution and hermitian forms.**

An **associative algebra with involution** is a pair \( (\mathcal{A}, -) \) consisting of an associative algebra \( \mathcal{A} \) and a period 2 anti-automorphism \( "-" \) of \( \mathcal{A} \). We adopt the notation

\[
\mathcal{A}_+ = \{ \alpha \in \mathcal{A} \mid \overline{\alpha} = \alpha \} \quad \text{and} \quad \mathcal{A}_- = \{ \alpha \in \mathcal{A} \mid \overline{\alpha} = -\alpha \},
\]

for the symmetric and skew-symmetric elements relative to the involution, so that \( \mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_- \). The **centre** of \( (\mathcal{A}, -) \) is defined as

\[
Z(\mathcal{A}, -) = Z(\mathcal{A}) \cap \mathcal{A}_+.
\]

If \( (\mathcal{A}, -) \) is an associative algebra with involution, a map \( \xi : X \times X \rightarrow \mathcal{A} \) is called a **hermitian form** over \( (\mathcal{A}, -) \) if \( X \) is a (right) \( \mathcal{A} \)-module and \( \xi : X \times X \rightarrow \mathcal{A} \) is a bi-additive map such that

\[
\xi(x, \alpha, y) = \overline{\alpha} \xi(x, y), \quad \xi(x, y, \alpha) = \xi(x, y)\alpha \quad \text{and} \quad \xi(y, x) = \overline{\xi(x, y)}
\]

for \( \alpha \in \mathcal{A} \) and \( x, y \in X \). If \( Y \) is an \( \mathcal{A} \)-submodule of \( X \), then

\[
Y^\perp := \{ x \in X \mid \xi(x, y) = 0 \text{ for all } y \in Y \}
\]

is an \( \mathcal{A} \)-submodule of \( X \). The form \( \xi \) is nondegenerate if \( X^\perp = 0 \). If \( \alpha \in \mathcal{A} \), we say that \( \xi \) represents \( \alpha \) if \( \xi(x, x) = \alpha \) for some \( x \in X \).

2.3. **Graded structures.**

Let \( \Lambda \) be an additive abelian group.

2.3.1. We have the following basic terminology:

(a) A vector space \( X \) over \( \mathbb{F} \) is **\( \Lambda \)-graded** if \( X \) has a decomposition \( X = \bigoplus_{\sigma \in \Lambda} X^\sigma \) into subspaces indexed by \( \Lambda \).\(^2\) If \( x \in X \), by \( \deg_A(x) = \sigma \) we mean that \( x \in X^\sigma \). The **\( \Lambda \)-support** of \( X \) is

\[
\text{supp}_\Lambda(X) = \{ \sigma \in \Lambda \mid X^\sigma \neq 0 \}.
\]

If the subgroup \( \langle \text{supp}_\Lambda(X) \rangle \) of \( \Lambda \) generated by \( \text{supp}_\Lambda(X) \) equals \( \Lambda \), then \( X \) is said to have full support in \( \Lambda \).\(^3\) When \( \dim_{\mathbb{F}}(X^\sigma) \) is finite for all \( \sigma \in \Lambda \), then \( X \) is said to have finite graded \( \mathbb{F} \)-dimension, and when \( \dim_{\mathbb{F}}(X^\sigma) \leq 1 \) for all \( \sigma \in \Lambda \), then \( X \) is called finely \( \Lambda \)-graded. If \( L \) is a subgroup of an abelian group \( \Lambda \) and \( X \) is an \( L \)-graded vector space, we regard \( X \) as an \( \Lambda \)-graded vector space by setting \( X^\sigma = 0 \) for \( \sigma \in \Lambda \setminus L \).

(b) An algebra \( \mathcal{A} \) is **\( \Lambda \)-graded** if \( \mathcal{A} = \bigoplus_{\sigma \in \Lambda} \mathcal{A}^\sigma \) is graded as a vector space and \( \mathcal{A}^\sigma \mathcal{A}^\tau \subseteq \mathcal{A}^{\sigma+\tau} \) for \( \sigma, \tau \in \Lambda \).

\(^2\)We write the degrees of the graded spaces as superscripts except in the case of root gradings (see Section 3.2), where it is more customary to use subscripts.

\(^3\)Often we assume a graded space has full support, since if this condition is not satisfied, we can always replace \( \Lambda \) by the group \( \langle \text{supp}_\Lambda(X) \rangle \).
(c) An **associative algebra with involution** \( (A, -) \) is \( \Lambda \)-**graded** if \( A \) is \( \Lambda \)-graded as an algebra, and the involution preserves the grading. Then \( A_- \) and \( A_+ \) are graded subspaces of \( A \), and we set

\[
\Lambda_- = \Lambda_-(A, -) = \text{supp}_\Lambda(A_-) \quad \text{and} \quad \Lambda_+ = \Lambda_+(A, -) = \text{supp}_\Lambda(A_+)
\]

so that

\[
\text{supp}_\Lambda(A) = \Lambda_- \cup \Lambda_+.
\]

Note that in general \( \Lambda_- \) and \( \Lambda_+ \) may not be subgroups of \( \Lambda \).

(d) If \( A \) is a \( \Lambda \)-graded algebra and \( A' \) is a \( \Lambda' \)-graded algebra, an **isograded-isomorphism** of \( A \) onto \( A' \) is a pair \((\varphi, \varphi_{\mathfrak{gr}})\), where \( \varphi : A \to A' \) is an algebra isomorphism, \( \varphi_{\mathfrak{gr}} : \Lambda \to \Lambda' \) is a group isomorphism and \( \varphi(A^\sigma) = A'^{\varphi_{\mathfrak{gr}}(\sigma)} \) for \( \sigma \in \Lambda \).

If such a pair exists, we say that \( A \) and \( A' \) are **isograded-isomorphic**. If \( A \) has full support in \( \Lambda \), then \( \varphi_{\mathfrak{gr}} \) is determined by \( \varphi \) and we can abbreviate the pair \((\varphi, \varphi_{\mathfrak{gr}})\) as \( \varphi \). When \( \Lambda = \Lambda' \) and \( \varphi_{\mathfrak{gr}} = \text{id} \), then \( A \) and \( A' \) are said to be **graded-isomorphic**.

The notions of isograded-isomorphic and graded-isomorphic for graded associative algebras with involution are defined similarly (by insisting that the map \( \varphi \) respects the involutions).

(e) If \( A \) is a \( \Lambda \)-graded associative algebra, an \( A \)-**module** \( X \) is \( \Lambda \)-graded if \( X = \bigoplus_{\sigma \in \Lambda} X^\sigma \) is graded as a vector space and \( X^\sigma A^\tau \subseteq X^{\sigma+\tau} \) for \( \sigma, \tau \in \Lambda \).

(f) If \( (A, -) \) is a \( \Lambda \)-graded associative algebra with involution, a **hermitian form** \( \xi : X \times X \to A \) over \( (A, -) \) is said to be \( \Lambda \)-**graded** if the \( A \)-module \( X \) is \( \Lambda \)-graded and \( \xi(X^\sigma, X^\tau) \subseteq A^{\sigma+\tau} \) for \( \sigma, \tau \in \Lambda \). We then say that \( \xi \) is of **finite graded \( F \)-dimension** (resp. **finely \( \Lambda \)-graded** if \( X \) is of finite graded \( F \)-dimension (resp. **finely \( \Lambda \)-graded**).

(g) If \( X \) is a \( \Lambda \)-graded \( A \)-module, we set

\[
\text{End}_A(X)^\sigma = \{ T \in \text{End}_A(X) \mid T(X^\tau) \subseteq X^{\sigma+\tau} \text{ for } \tau \in \Lambda \},
\]

for \( \sigma \in \Lambda \), and we let \( \text{End}_A^\Lambda(X) = \bigoplus_{\sigma \in \Lambda} \text{End}_A(X)^\sigma \). Then \( \text{End}_A^\Lambda(X) \) is a \( \Lambda \)-graded associative algebra under composition. We further let \( \mathfrak{gl}_A^\Lambda(X) \) be the \( \Lambda \)-graded Lie algebra with underlying graded space \( \text{End}_A^\Lambda(X) \) under the commutator product. We say that the gradings on \( \text{End}_A^\Lambda(X) \) and \( \mathfrak{gl}_A^\Lambda(X) \) are **induced by** the grading on \( X \). If \( \text{supp}_\Lambda(X) \) is finite or if \( X \) is a finitely generated \( A \)-module, then \( \text{End}_A(X) = \text{End}_A^\Lambda(X) \) is \( \Lambda \)-graded [NvO, Cor. 2.4.4 and 2.4.5], and hence \( \mathfrak{gl}_A(X) = \mathfrak{gl}_A^\Lambda(X) \) is \( \Lambda \)-graded.

3. **ROOT GRADED LIE ALGEBRAS AND LIE TORI**

Lie tori are root graded Lie algebras with additional structure. In this chapter, we recall the notions of root graded Lie algebras and Lie tori.

We suppose throughout the chapter that \( F \) is a field of characteristic \( 0 \) and that \( \Delta \) is a finite irreducible (not necessarily reduced) root system in a finite-dimensional vector space \( F \Delta \) (defined for example as in [Bo, Chap. VI, §1, Def. 1]).

\[\text{(N1)}\] and in other papers on Lie tori and extended affine Lie algebras, it has been convenient to adopt the convention that \( 0 \) is a root. However, for compatibility with [ABG], we do not do that here.
3.1. Root systems.

Our notation for root systems is standard. Let

\[ Q = Q(\Delta) := \text{span}_\mathbb{Z}(\Delta) \]

be the root lattice of \( \Delta \), and

\[ \Delta_{\text{ind}} := \left\{ \mu \in \Delta \mid \frac{1}{2} \mu \notin \Delta \right\} \]

be the set of indivisible roots in \( \Delta \). For \( \mu \in \Delta \), \( \mu^\vee \) will denote the coroot of \( \mu \). That is, \( \mu^\vee \) is the element of the dual space of the vector space \( \mathbb{F}\Delta \) so that \( \nu \mapsto \nu - \langle \nu | \mu^\vee \rangle \mu \) is the reflection corresponding to \( \mu \) in the Weyl group of \( \Delta \), where \( \langle \cdot | \cdot \rangle \) is the natural pairing of \( \mathbb{F}\Delta \) with its dual space.

The root system \( \Delta \) has type \( X_r \), where \( X_r = A_r, B_r, C_r, D_r, E_6, E_7, E_8, F_4, G_2 \) or \( BC_r \). If \( X_r \neq BC_r \), then \( \Delta \) is reduced (that is, \( 2\mu / \mu \notin \Delta \) for \( \mu \in \Delta \) and \( \Delta = \Delta_{\text{ind}} \)). On the other hand, if \( X_r = BC_r \), then \( \Delta_{\text{ind}} \) is an irreducible root system of type \( B_r \) (see 3.1.1 below).

3.1.1. If \( \Delta \) has type \( BC_r \), we may choose a \( \mathbb{Z} \)-basis \( \varepsilon_1, \ldots, \varepsilon_r \) for \( Q \) so that

\[ \Delta = \{ \pm \varepsilon_i \mid 1 \leq i \leq r \} \cup \{ \pm (\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq r \} \cup \{ \pm 2\varepsilon_i \mid 1 \leq i \leq r \} \]

and

\[ \Delta_{\text{ind}} = \{ \pm \varepsilon_i \mid 1 \leq i \leq r \} \cup \{ \pm (\varepsilon_i \pm \varepsilon_j) \mid 1 \leq i < j \leq r \} \]

For this basis, we define a permutation \( i \mapsto \bar{i} \) of \( \{1, \ldots, 2r\} \) by \( \bar{i} = 2r + 1 - i \) and set \( \varepsilon_{\bar{i}} = -\varepsilon_i \) for \( 1 \leq i \leq r \), so that

\[ \varepsilon_i = -\varepsilon_{\bar{i}} \quad \text{for all } 1 \leq i \leq 2r, \]

and

\[ \Delta = \{ \varepsilon_i \mid 1 \leq i \leq 2r \} \cup \{ \varepsilon_i + \varepsilon_j \mid 1 \leq i, j \leq 2r, j \neq \bar{i} \}, \]

\[ = \{ \varepsilon_i \mid 1 \leq i \leq 2r \} \cup \{ \varepsilon_i + \varepsilon_j \mid 1 \leq i \leq 2r, j \neq \bar{i} \}, \]

where the expressions on the last line are unique.

3.1.2. When \( \Delta \) is of type \( BC_r \) in various examples (such as in Sections 4.2 and 6.1 below), we do not assume a priori that a choice of basis for \( Q \) as in 3.1.1 has been made, but rather instead use a basis that arises naturally.

3.2. Root graded Lie algebras.

**Definition 3.2.1.** ([ABG, Chap. 1]) A \( \Delta \)-graded (or root graded) Lie algebra (with grading subalgebra of type \( \Delta_{\text{ind}} \)) is a \( Q \)-graded Lie algebra \( \mathcal{L} = \bigoplus_{\mu \in Q} \mathcal{L}_\mu \) over \( \mathbb{F} \) satisfying

(RG1): \( \mathcal{L} \) has a split simple subalgebra \( \mathfrak{g} \) with splitting Cartan subalgebra \( \mathfrak{h} \), and there exists an \( \mathbb{F} \)-linear isomorphism \( \mu \mapsto \tilde{\mu} \) of \( \mathbb{F}\Delta \) onto \( \mathfrak{h}^* \) such that the root system of \( \mathfrak{g} \) with respect to \( \mathfrak{h} \) is \( \Delta_{\text{ind}} \), and such that \( \mathcal{L}_\mu = \{ x \in \mathcal{L} \mid [h, x] = \tilde{\mu}(h)x \text{ for } h \in \mathfrak{h} \} \) for \( \mu \in Q \);

(RG2): \( \text{supp}_Q(\mathcal{L}) \subseteq \Delta \cup \{0\} \);

(RG3): \( \mathcal{L} \) is generated as a Lie algebra by the spaces \( \mathcal{L}_\mu, \mu \in \Delta \).

In that case, we say that \( (\mathfrak{g}, \mathfrak{h}) \) is the grading pair for \( \mathcal{L} \), \( \mathfrak{g} \) is the grading (simple) subalgebra of \( \mathcal{L} \), and \( \mathfrak{h} \) is the grading ad-diagonalizable subalgebra of \( \mathcal{L} \). Also, when \( \Delta \) has type \( X_r \), we often refer to a \( \Delta \)-graded Lie algebra as an \( X_r \)-graded Lie algebra.

3.2.2. If \( \mathcal{L} \) is a \( \Delta \)-graded Lie algebra with grading pair \( (\mathfrak{g}, \mathfrak{h}) \), then \( Z(\mathcal{L}) \subseteq \mathcal{L}_0 \) and \( \mathcal{L}/Z(\mathcal{L}) \) is \( \Delta \)-graded Lie algebra with the induced \( Q \)-grading and with grading pair \( (\pi(\mathfrak{g}), \pi(\mathfrak{h})) \), where \( \pi : \mathcal{L} \to \mathcal{L}/Z(\mathcal{L}) \) is the canonical map.
In this paper we are primarily interested in the case that $\Delta$ is of type $BC_r$ for $r \geq 3$. Such $\Delta$-graded Lie algebras are described in [ABG, Chap. 2 and 3], and we will recall that description in Section 4.2.

3.3. Lie tori.

Let $\Lambda$ be an arbitrary additive abelian group and $Q$ be the root lattice of a root system $\Delta$. If $L$ is a $(Q \times \Lambda)$-graded Lie algebra, we write $L_\sigma^\mu$ for the $(\mu, \sigma)$-component of $L$ (rather than $L^{(\mu, \sigma)}$ or $L_{(\mu, \sigma)}$) and adopt the notation

$$L_\sigma = \bigoplus_{\mu \in Q} L_\sigma^\mu$$

for $\sigma \in \Lambda$ and

$$L^\mu = \bigoplus_{\sigma \in \Lambda} L_\sigma^\mu$$

for $\mu \in Q$.

In this way, $L = \bigoplus_{\mu \in Q} L^\mu = \bigoplus_{\sigma \in \Lambda} L_\sigma$ is both a $Q$-graded algebra and a $\Lambda$-graded algebra, and these gradings are compatible in the sense that each $L_\mu$ is $\Lambda$-graded (or equivalently, each $L_\sigma$ is $Q$-graded). Conversely, compatible gradings by $Q$ and $\Lambda$ on an algebra $L$ determine a $(Q \times \Lambda)$-grading on $L$.

Next we present the definition of a Lie torus following [N1].

Definition 3.3.1. A Lie $\Lambda$-torus of type $\Delta$ is a $(Q \times \Lambda)$-graded Lie algebra $L$ over $\mathbb{F}$ satisfying:

(LT1): $\text{supp}_Q(L) \subseteq \Delta \cup \{0\}$.

(LT2): (i) $L_0^\mu \neq 0$ for $\mu \in \Delta_{\text{ind}}$.

(ii) If $\mu \in \Delta$, $\sigma \in \Lambda$, and $L_\sigma^\mu \neq 0$, then $L_\sigma^\mu = \mathbb{F} e_\mu^\sigma$ and $L_{-\mu}^\sigma = \mathbb{F} f_\mu^\sigma$, where

$$[[e_\mu^\sigma, f_\mu^\sigma], x_\tau^\nu] = \langle \nu | \mu^\vee \rangle x_\tau^\nu$$

for all $x_\tau^\nu \in L_\tau^\nu$, $\nu \in Q$, $\tau \in \Lambda$.

(LT3): $L$ is generated as a Lie algebra by the spaces $L_\mu$, $\mu \in \Delta$.

(LT4): $L$ has full support in $\Lambda$.

The $Q$-grading (resp. the $\Lambda$-grading) on $L$ is called the root grading (resp. the external grading) of $L$. If $\Delta$ has type $X_r$, we often refer to a Lie $\Lambda$-torus of type $\Delta$ as a Lie $\Lambda$-torus of type $X_r$. We use the term Lie torus when it is not necessary to specify either $\Lambda$ or $\Delta$.

Remark 3.3.2. Suppose that $L$ is a Lie $\Lambda$-torus of type $\Delta$.

(a) It is known (see [N1]) that $L$ is a $\Delta$-graded Lie algebra, as defined in Section 3.2. (See Proposition 3.4.1 (g) below for the case when $L$ is centreless.)

(b) By [ABFP, Lem. 1.1.10], either

$$\text{supp}_Q(L) = \Delta \cup \{0\} \quad \text{or} \quad \text{supp}_Q(L) = \Delta_{\text{ind}} \cup \{0\}.$$ 

(c) If $L$ is centreless, then the centroid $\text{Cent}(L)$ of $L$ is a $\Lambda$-graded subalgebra of $\text{End}_{\mathbb{F}}^\Lambda(L)$, and $\text{supp}_\Lambda(\text{Cent}(L))$ is a subgroup of $\Lambda$, called the centroidal grading group of $L$ [BN, Prop. 3.13]

Remark 3.3.3. Suppose that $\Delta$ is a root system of type $BC_r$. The Lie $\Lambda$-tori of type $\Delta_{\text{ind}}$ are precisely the Lie $\Lambda$-tori of type $\Delta$ whose $Q$-support equals $\Delta_{\text{ind}} \cup \{0\}$. So the class of Lie $\Lambda$-tori of type $B_r$ is contained in the class of Lie $\Lambda$-tori of type $BC_r$.

We will use the following natural notion of equivalence for $(Q \times \Lambda)$-graded algebras and hence in particular for Lie tori [ABFP, Sec. 2.1].
Definition 3.3.4. Let $\mathcal{L}$ be a $(Q \times \Lambda)$-graded Lie algebra and let $\mathcal{L}'$ be a $(Q' \times \Lambda')$-graded Lie algebra (where here $\Lambda$, $Q$, $\Lambda'$, and $Q'$ can be arbitrary abelian groups). A bi-isograded-isomorphism, or a bi-isomorphism for short, of $\mathcal{L}$ onto $\mathcal{L}'$ is a triple $(\psi, \psi_{\text{rt}}, \psi_{\text{ex}})$, where $\psi : \mathcal{L} \rightarrow \mathcal{L}'$ is an algebra isomorphism, $\psi_{\text{rt}} : Q \rightarrow Q'$ and $\psi_{\text{ex}} : \Lambda \rightarrow \Lambda'$ are group isomorphisms, and $\psi(\mathcal{L}_\mu^\sigma) = \mathcal{L}_{\psi_{\text{rt}}(\mu)}^\psi(\sigma)$ for $\mu \in Q$ and $\sigma \in \Lambda$. If such a triple exists, we say that $\Delta$ of type $\psi$.

Remark 3.3.5. Suppose that $\psi$ is a bi-isomorphism of a Lie $\Lambda$-torus $\mathcal{L}$ of type $\Delta$ onto a Lie $\Lambda'$-torus of type $\Delta'$. Then $\psi_{\text{rt}}(\text{supp}_Q(\mathcal{L})) = \text{supp}_{Q'}(\mathcal{L}')$. Hence, by Remark 3.3.2 (b), if $\Delta$ and $\Delta'$ are either both reduced or both non-reduced, we have $\psi_{\text{rt}}(\Delta) = \Delta'$, so $\psi_{\text{rt}}$ is an isomorphism of the root system $\Delta$ onto the root system $\Delta'$.

3.4. Basics on centreless Lie tori.

Throughout this section, we assume that $\mathcal{L}$ is a centreless Lie $\Lambda$-torus of type $\Delta$. Following [N1], let $\mathfrak{g}$ denote the subalgebra of $\mathcal{L}$ generated by $\{\mathcal{L}_0\}_{\mu \in \Delta}$ and set $\mathfrak{h} = \sum_{\mu \in \Delta} [\mathcal{L}_0^\mu, \mathcal{L}_0^\mu]$.

Proposition 3.4.1. Assume that $\mathcal{L} = \bigoplus_{(\mu, \sigma) \in Q \times \Lambda} \mathcal{L}_\mu^\sigma$ is a centreless Lie $\Lambda$-torus of type $\Delta$.

(a) If $\mu \in \Delta_{\text{ind}}$, then $\mathcal{L}_2^\mu = 0$.

(b) $\mathfrak{g}$ is a finite-dimensional split simple Lie algebra with splitting Cartan subalgebra $\mathfrak{h}$.

(c) There is a unique linear isomorphism $\mu \mapsto \overline{\mu}$ of $\mathbb{F}\Delta$ onto $\mathfrak{h}^*$ such that $\Delta_{\text{ind}}$ is the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$ and $[e_\mu^\sigma, f_\mu^\sigma] = \overline{\mu}^\vee$ for $\mu \in \Delta_{\text{ind}}$. (Here $\overline{\mu}^\vee \in (\mathfrak{h}^*)^* = \mathfrak{h}$.)

(d) If $\mu \in Q$, then $\mathcal{L}_\mu = \{x \in \mathcal{L} \mid [h, x] = \mu(h)x \text{ for } h \in \mathfrak{h}\}$.

(e) If $\mu \in \Delta$, $\sigma \in \Lambda$, and $\mathcal{L}_\mu^\sigma \neq 0$, then $[e_\mu^\sigma, f_\mu^\sigma] = \mu^\vee$.

(f) $\mathfrak{g} = \mathcal{L}_0$ and $\mathfrak{h} = \mathcal{L}_0^\mu$.

(g) As a $Q$-graded Lie algebra, $\mathcal{L}$ is a $\Delta$-graded Lie algebra with grading pair $(\mathfrak{g}, \mathfrak{h})$.

Proof. Parts (a)–(e) and (g) were announced in [N1, Sec. 3] under the hypothesis that $\Lambda$ is a finitely generated free abelian group. A proof of (a)–(f) for arbitrary $\Lambda$ can be found in [ABFP, Prop. 6.3]. Part (g) follows from (b), (c), (d) and (LT3).

Henceforth, we will use the map $\mu \mapsto \overline{\mu}$ in Proposition 3.4.1 (c) to identify $\mathbb{F}\Delta$ and $\mathfrak{h}^*$ and will omit the tildes. Thus, $\Delta$ is a root system in $\mathfrak{h}^*$ and $\Delta_{\text{ind}}$ is the set of roots of $\mathfrak{g}$ relative to $\mathfrak{h}$. Moreover, by Proposition 3.4.1 (d), we have

$$\mathcal{L}_\mu = \{x \in \mathcal{L} \mid [h, x] = \mu(h)x \text{ for } h \in \mathfrak{h}\}$$

for $\mu \in Q$. Also, if $\mathcal{L}_\mu^\sigma \neq 0$ for $\mu \in \Delta$, $\sigma \in \Lambda$, then by Proposition 3.4.1 (e),

$$[e_\mu^\sigma, f_\mu^\sigma] = \mu^\vee,$$

where $\mu^\vee \in (\mathfrak{h}^*)^* = \mathfrak{h}$. Thus, $\{e_\mu^\sigma, \mu^\vee, f_\mu^\sigma\}$ is an $\mathfrak{sl}_2$-triple.

Following [AABGP, Sec. II.2] and [Y3], we set

$$\Lambda_\mu := \text{supp}_\Lambda(\mathcal{L}_\mu) = \{\sigma \in \Lambda \mid \mathcal{L}_\mu^\sigma \neq 0\}$$
for \(\mu \in \Delta\). The following facts are proved in [Y3, Sec. 3] (see also [ABFP, Lem. 1.1.12]):

**Lemma 3.4.2.** Suppose that \(\mathcal{L}\) is a centreless Lie \(\Delta\)-torus of type \(\Delta\) and \(\mu \in \Delta\). Then

(a) \(\Lambda_\mu\) depends only on the length of \(\mu\).

(b) If \(\mu \in \Delta_{\text{ind}}\), then \(0 \in \Lambda_\mu\) and \(-\Lambda_\mu = \Lambda_\mu\).

(c) If \(\mu, \nu \in \Delta\) with \(\Lambda_\nu\) and \(\Lambda_\mu\) nonempty, then \(\Lambda_\nu - \langle \nu | \mu \rangle \Lambda_\mu \subseteq \Lambda_{\nu - \langle \nu | \mu \rangle \mu} \).

(d) If \(\mu\) has minimum length in \(\Delta_{\text{ind}}\), then \(\langle \Lambda_\mu \rangle = \Lambda\).

The next lemma is a consequence of \(\mathfrak{sl}_2\)-theory.

**Lemma 3.4.3.** If \(\mu, \nu, \mu + \nu \in \Delta\), \(\sigma \in \Lambda_\mu\), \(\tau \in \Lambda_\nu\), and \(\sigma + \tau \in \Lambda_{\mu + \nu}\), then

\[
[\mathcal{L}_\sigma^\mu, \mathcal{L}_\tau^\nu] = \mathcal{L}_{\sigma + \tau}^{\mu + \nu},
\]

(2)

**Proof.** By assumption we have

\[
\mathcal{L}_\sigma^\mu \neq 0, \quad \mathcal{L}_\tau^\nu \neq 0 \quad \text{and} \quad \mathcal{L}_{\sigma + \tau}^{\mu + \nu} \neq 0.
\]

Suppose first that \(\nu \notin \mathbb{Z}\mu\). Let \(M = \sum_{k \in \mathbb{Z}} \mathcal{L}_{\nu + k\mu}^\tau\). Then, by (LT1) and (LT2)(ii), \(M\) is a finite-dimensional \(\mathfrak{sl}_2(\mu, \sigma)\)-module, where \(\mathfrak{sl}_2(\mu, \sigma)\) is the Lie algebra spanned by \(\{e_\sigma^\mu, e_\tau^\nu, f_\sigma^\mu\}\). Moreover, by (LT2)(ii), this module has one-dimensional weight spaces, and the eigenvalues of \(\text{ad}(\mu^\sigma)|_M\) are integers of the same parity. Hence by \(\mathfrak{sl}_2\)-theory, \(M\) is irreducible and \(\text{ad}(e_\sigma^\mu)\mathcal{L}_\tau^\nu = \mathcal{L}_{\sigma + \tau}^{\mu + \nu}\), proving the desired fact.

So we can assume that \(\nu \in \mathbb{Z}\mu\) and similarly that \(\mu \in \mathbb{Z}\nu\). Thus, \(\nu = \pm \mu\), and, since \(\mu + \nu \in \Delta\), we have \(\nu = \mu\). Now \(\text{ad}(e_\sigma^\mu) e_{2\mu}^{\sigma + \tau} = 0\) by (LT1), so \(\text{ad}(e_\sigma^\mu)\text{ad}(f_\sigma^\mu) e_{2\mu}^{\sigma + \tau} = \text{ad}(\mu^\nu) e_{2\mu}^{\sigma + \tau} = 4 e_{2\mu}^{\sigma + \tau}\). Therefore \(\text{ad}(e_\sigma^\mu)\mathcal{L}_\tau^\nu \neq 0\), again proving the conclusion. \(\square\)

**Remark 3.4.4.** If \(\omega : Q \to Q\) is in the Weyl group of \(\Delta\), then there exists an isograded isomorphism \(\psi\) from \(\mathcal{L}\) to \(\mathcal{L}\) such that \(\psi_\tau = \omega\). (To see this, one extends an inner automorphism of \(\mathfrak{g}\). See the argument in the proof of Lemma 3.8 of [AF].)

## 4. UNITARY LIE ALGEBRAS

Throughout this chapter we assume that \((\mathcal{A}, -)\) is an associative algebra with involution.

4.1. **The Lie algebras** \(\mathfrak{u}(X, \xi), \mathfrak{fu}(X, \xi), \) and \(\mathfrak{su}(X, \xi)\).

**Definition 4.1.1.** Suppose that \(\xi : X \times X \to \mathcal{A}\) is a hermitian form over \((\mathcal{A}, -)\). To construct unitary Lie algebras from \(\xi\) we will use an associative algebra with involution \((\mathcal{E}, \ast)\) that is determined by \(\xi\). We recall the definition of \((\mathcal{E}, \ast)\), following [A, Ex. 2.3], in (a) and (d) below.

(a) For \(x, y \in X\), define \(E(x, y) \in \text{End}_\mathcal{A}(X)\) by

\[
E(x, y)z = x.\xi(y, z).
\]

Then

\[
\xi(E(x, y)z, w) = \xi(z, E(y, x)w),
\]

(3)

\[
E(x, \alpha, y) = E(x, y, \overline{\alpha}), \quad \text{and}
\]

(4)

\[
E(x, y)E(z, w) = E(x, \xi(y, z), w)
\]

(5)
hold for all $\alpha \in A$ and $x, y, z, w \in X$. We set
\[ E = e(X, \xi) := \text{span}_F \{ E(x, y) \mid x, y \in X \}, \]
and note that by (5), $E$ is an associative subalgebra of $\text{End}_A(X)$; however $E$ may not be unital.

(b) Let
\[ U = u(X, \xi) := \{ T \in \text{End}_A(X) \mid \xi(Tx, y) + \xi(x, Ty) = 0 \ \forall \ x, y \in X \}. \tag{6} \]
Then $U$ is a Lie subalgebra of $\mathfrak{gl}_A(X)$, and we say that $U$ is the unitary Lie algebra of $\xi$.

(c) For $x, y \in X$, set
\[ U(x, y) := E(x, y) - E(y, x), \]
and let
\[ \mathcal{F} = \mathfrak{f}(X, \xi) := U(X, X), \]
where $U(X, X) = \text{span}_F \{ U(x, y) \mid x, y \in X \}$. It follows that
\[ U(x, \alpha) = U(x, y\overline{\alpha}) \quad \text{and} \quad U(x, y) = -U(y, x) \tag{7} \]
for $\alpha \in A$ and $x, y \in X$, and, by (3), that $U(X, X) \subseteq U$. In particular, if $x \in X$ and $a \in A_+$, then
\[ U(x.a, x) = U(x, x.a) = 0. \tag{8} \]
Moreover,
\[ [T, U(x, y)] = U(Tx, y) + U(x, Ty) \tag{9} \]
for $x, y \in X$ and $T \in U$, so that $\mathcal{F}$ is an ideal of the Lie algebra $U$, referred to as the finite unitary Lie algebra of $\xi$.

(d) Suppose $\xi$ is nondegenerate. It follows from (3) that there is a well-defined linear map $* : E \to E$ of period 2 such that $E(x, y)^* = E(y, x)$ for $x, y \in X$. Using (4) and (5), it is easy to check that $*$ is an involution of $E$. Further, by (3), we have
\[ \xi(Tx, y) = \xi(x, T^*y) \tag{10} \]
for $T \in E$ and $x, y \in X$. It is then clear that
\[ \mathcal{F} = \{ T \in E \mid T^* = -T \}, \tag{11} \]
and hence, using (10) and nondegeneracy, that
\[ \mathcal{F} = \mathfrak{u} \cap E. \tag{12} \]

(e) Finally, we let
\[ \mathcal{S} = \mathfrak{su}(X, \xi) = \mathcal{F}^{(1)}, \]
where $\mathcal{F}^{(1)} = [\mathcal{F}, \mathcal{F}]$ denotes the derived algebra of $\mathcal{F}$. The ideal $\mathcal{S}$ of $\mathfrak{u}$ is called the special unitary Lie algebra of $\xi$.

**Example 4.1.2.** Suppose that $\xi : X \times X \to A$ is a nondegenerate hermitian form over an associative division algebra with involution $(A, -)$. Then $E$ is the algebra of all finite rank endomorphisms in $\text{End}_A(X)$ that have an adjoint relative to $\xi$ [J, Prop. 1, §IV.8]. Thus, by (12), $\mathcal{F}$ is the Lie algebra of all finite rank endomorphisms in $\mathfrak{u}$.
4.1.3. If \( \xi : X \times X \to A \) is a hermitian form over \((A, \cdot)\), then \( \text{End}_A(X) \) is a left module for the center \( Z(A, \cdot) = Z(A) \cap A_+ \) with the action given by \((\mathfrak{z}T)(x) := T(x, \mathfrak{z}) = T(x, \mathfrak{z}) \) for \( \mathfrak{z} \in Z(A, \cdot), \) \( T \in \text{End}_A(X) \) and \( x \in X \). Then,
\[
\mathfrak{z}E(x, y) = E(x, \mathfrak{z}y) = E(x, y, \mathfrak{z}) \quad \text{and} \quad \mathfrak{z}U(x, y) = U(x, \mathfrak{z}y) = U(x, y, \mathfrak{z})
\]
for \( x, y \in A, \mathfrak{z} \in Z(A, \cdot) \).

4.1.4. (Gradings on \( \mathcal{E}, \mathcal{F}, \mathcal{S} \) and \( \mathcal{U} \)) Suppose that \( \xi : X \times X \to A \) is a \( \Lambda \)-graded hermitian form over \((A, \cdot)\). Since \( \mathcal{E} \) is spanned by homogeneous elements of \( \text{End}_{\Lambda}^R(X) \), \( \mathcal{E} \) is a \( \Lambda \)-graded subalgebra of \( \text{End}_{\Lambda}^R(X) \) with
\[
\mathcal{E}^\tau = \sum_{\rho + \sigma = \tau} E(X^\rho, X^\sigma)
\]
for \( \tau \in \Lambda \). Similarly, \( \mathcal{F} \) is a \( \Lambda \)-graded subalgebra of \( \mathfrak{gl}_A^R(X) \) with
\[
\mathcal{F}^\tau = \sum_{\rho + \sigma = \tau} U(X^\rho, X^\sigma)
\]
for \( \tau \in \Lambda \); and \( \mathcal{S} \) is a \( \Lambda \)-graded subalgebra of \( \mathcal{F} \). Moreover, if \( \text{supp}_A(X) \) is finite or if \( X \) is a finitely generated \( A \)-module, then \( \mathcal{U} \) is a \( \Lambda \)-graded subalgebra of \( \mathfrak{gl}_A(X) = \mathfrak{gl}_A^R(X) \).

4.2. The BC\( _r \)-graded unitary Lie algebra \( \mathfrak{bu}(X, \xi) \).

We have introduced three Lie algebras \( \mathfrak{u}(X, \xi), \mathfrak{fu}(X, \xi) \) and \( \mathfrak{su}(X, \xi) \) associated with a hermitian form \( \xi \). A fourth Lie algebra \( \mathfrak{bu}(X, \xi) \) will play a key role in the proof (but not the statement) of our structure theorem about centreless Lie tori of type BC\( _r \).

In defining \( \mathfrak{bu}(X, \xi) \) we make the following assumptions:

(i) \( \mathcal{F} \) has characteristic 0;
(ii) \( r \geq 1 \);
(iii) \( (A, \cdot) \) is an associative algebra with involution over \( \mathbb{F} \);
(iv) \( \xi : X \times X \to A \) is a hermitian form over \((A, \cdot)\) such that \( X = X_{\text{hyp}} \perp X_{\text{an}} \), where \( X_{\text{hyp}} \) and \( X_{\text{an}} \) are \( A \)-submodules of \( X \);\(^5\)
(v) \( X_{\text{hyp}} \) has an \( A \)-basis \( \{ x_i \} \) such that
\[
\xi(x_i, x_j) = \delta_{ij}
\]
for \( 1 \leq i, j \leq 2r \), where
\[
i = 2r + 1 - i; \quad \text{and}
\]
(vi) \( X_{\text{an}} \) contains an element \( v_0 \) such that \( \xi(v_0, v_0) = 1 \).

Let
\[
\mathcal{U} = \mathfrak{u}(X, \xi), \quad \mathcal{F} = \mathfrak{fu}(X, \xi), \quad \text{and} \quad \mathcal{S} = \mathfrak{su}(X, \xi)
\]
as in Definition 4.1.1, and set
\[
h_i = U(x_i, x_i) = E(x_i, x_i) - E(x_i, x_i)
\]
in \( \mathcal{F} \) for \( 1 \leq i \leq r \). It is straightforward to verify that
\[
[U(x_i, v_0), U(v_0, x_i)] = h_i,
\]
(14)
\(^5\)Later when we discuss unitary Lie tori in Chapter 6, the decomposition \( X = X_{\text{hyp}} \perp X_{\text{an}} \) will be a Witt decomposition of \( \xi \).
Definition 4.2.1. Let $B$ be the $\mu$-weight space of $\mathfrak{h}$ in $X$. Let $\{\varepsilon_1, \ldots, \varepsilon_r\}$ be the dual basis in $\mathfrak{h}^*$ of $\{h_1, \ldots, h_r\}$, and, as in 3.1.1, let $\varepsilon_i = -\varepsilon_i$ for $1 \leq i \leq r$. Then

$$X = (\bigoplus_{i=1}^{2r} X_{\varepsilon_i}) \oplus X_0,$$

$$X_{\varepsilon_i} = x_i A \text{ for } 1 \leq i \leq 2r, \text{ and } X_0 = X_{an}. \quad (16)$$

Let $Q = \mathbb{Z}\varepsilon_1 + \cdots + \mathbb{Z}\varepsilon_r$ in $\mathfrak{h}^*$. By (16), $X$ is $Q$-graded as an $A$-module (if we assign the trivial $Q$-grading $A = A_0$ to $A$), and we have

$$\text{supp}_Q(X) = \{0\} \cup \{\varepsilon_i \mid 1 \leq i \leq 2r\}.$$

Since $\text{supp}_Q(X)$ is finite, it follows that the Lie algebra $\mathfrak{gl}_A(X) = \bigoplus_{\mu \in Q} \mathfrak{gl}_A(X)_\mu$ is $Q$-graded with

$$\mathfrak{gl}_A(X)_\mu = \{T \in \mathfrak{gl}_A(X) \mid TX_\mu \subseteq X_{\mu + \nu} \text{ for } \nu \in Q\}$$

(see 2.3.1 (g)). Moreover, $\xi$ is $Q$-graded (again assigning the trivial $Q$-grading to $A$), so $\mathfrak{u}, \mathfrak{f},$ and $\mathfrak{s}$ are $Q$-graded subalgebras of $\mathfrak{gl}_A(X)$ (see 4.1.4). Notice that $\text{supp}_Q(\mathfrak{gl}_A(X)) \subseteq \{\mu \mid \mu, \nu \in \text{supp}_Q(X)\}$, so we have $\text{supp}_Q(\mathfrak{gl}_A(X)) \subseteq \Delta \cup \{0\}$, where

$$\Delta = \{\varepsilon_i \mid 1 \leq i \leq 2r\} \cup \{\varepsilon_i + \varepsilon_j \mid 1 \leq i, j \leq 2r, \ j \neq i\}.$$

Thus the supports of $\mathfrak{s}, \mathfrak{f},$ and $\mathfrak{u}$ are also contained in $\Delta \cup \{0\}$. Observe also that $\Delta$ is a root system of type $BC_r$ in $\mathfrak{h}^*$, and $Q = Q(\Delta)$ is the root lattice of $\Delta$.

We are now ready to introduce the Lie algebra $\mathfrak{bu}(X, \xi)$.

**Definition 4.2.1.** Let

$$\mathcal{B} = \mathfrak{bu}(X, \xi) := (\mathfrak{u}_\mu : \mu \in \Delta)_{\text{alg}} = \sum_{\mu \in \Delta} \mathfrak{u}_\mu + \sum_{\mu \in \Delta} [\mathfrak{u}_\mu, \mathfrak{u}_{-\mu}].$$

Then $\mathcal{B}$ is a $Q$-graded ideal of $\mathfrak{s}$ which we call the $\text{BC}_r$-graded unitary Lie algebra determined by $\xi$.

**4.2.2.** The Lie algebras $\mathcal{B}, \mathfrak{s}$ and $\mathfrak{f}$ are $Q$-graded ideals of $\mathfrak{u}$ and

$$\mathfrak{h} \subseteq \mathcal{B} \subseteq \mathfrak{s} \subseteq \mathfrak{f} \subseteq \mathfrak{u}$$

by (14). So the $Q$-gradings on the Lie algebras $\mathcal{B}, \mathfrak{s}, \mathfrak{f}$, and $\mathfrak{u}$ are the root gradings relative to the adjoint action of $\mathfrak{h}$.
4.2.3. It should be noted that, unlike $\mathcal{F}$ and $\mathfrak{S}$, the algebra $\mathcal{B}$ depends not only on $(A, -)$, $X$ and $\xi$, but also on a decomposition $X = X_{\text{hyp}} \perp X_{\text{an}}$, an $A$-basis $\{x_i\}_{i=1}^{2r}$ for $X_{\text{hyp}}$, and a distinguished element $v_0 \in X_{\text{an}}$. However, for simplicity we have suppressed this in the notation $\mathfrak{b}u(X, \xi)$.

The terminology in Definition 4.2.1 is justified by the following theorem that comes from [ABG].

**Theorem 4.2.4.** Let $F$ be a field of characteristic 0.

(a) Suppose $r \geq 1$; $\xi : X \times X \to A$ is a hermitian form over an associative algebra with involution $(A, -)$ such that $X = X_{\text{hyp}} \perp X_{\text{an}}$ where $X_{\text{hyp}}$ and $X_{\text{an}}$ are $A$-submodules of $X$; $X_{\text{hyp}}$ has an $A$-basis $\{x_i\}_{i=1}^{2r}$ such that $\xi(x_i, x_j) = \delta_{ij}$ for all $i, j = 1, \ldots, 2r$, where $i = 2r + 1 - i$, and there is an element $v_0 \in X_{\text{an}}$ with $\xi(v_0, v_0) = 1$. Then $\mathcal{B} = \mathfrak{b}u(X, \xi)$ is a $B_r$-graded Lie algebra with grading pair $(\mathfrak{g}, \mathfrak{h})$, where

$$\mathfrak{g} = \bigoplus_{1 \leq i < j \leq 2r} F U(x_i, x_j) + \bigoplus_{i=1}^{2r} F U(x_i, v_0),$$

$$\mathfrak{h} = \bigoplus_{i=1}^{r} F U(x_i, x_i).$$

(b) Conversely, if $r \geq 3$ and $\mathcal{L}$ is a $B_r$-graded Lie algebra, then $\mathcal{L}/Z(\mathcal{L})$ is isomorphic to $\mathcal{B}/Z(\mathcal{B})$ for some $B_r$-graded unitary Lie algebra $\mathcal{B}$ as in (a). More precisely, if $\mathcal{L}$ is a $B_r$-graded Lie algebra for $r \geq 3$ with grading pair $(\mathfrak{g}_\mathcal{L}, \mathfrak{h}_\mathcal{L})$ then there exists a $B_r$-graded unitary Lie algebra $\mathcal{B}$ (as in (a)) and a Lie algebra isomorphism $\psi : \mathcal{L}/Z(\mathcal{L}) \to \mathcal{B}/Z(\mathcal{B})$ so that $\psi$ maps the canonical image of $(\mathfrak{g}_\mathcal{L}, \mathfrak{h}_\mathcal{L})$ in $\mathcal{L}/Z(\mathcal{L})$ onto the canonical image of $(\mathfrak{g}, \mathfrak{h})$ in $\mathcal{B}/Z(\mathcal{B})$, where $\mathfrak{g}$ and $\mathfrak{h}$ are defined by (18) and (19).

Proof. The results of [ABG] provide the proof of this theorem, but a translation of notation is required to apply them. Let $C = (v_0, A)^\perp$ in $X_{\text{an}}$, and identify $X_{\text{hyp}} \oplus v_0 A$ with $A^{2r+1}$ using the ordered $A$-basis $\{x_1, \ldots, x_r, v_0, x_{r+1}, \ldots, x_{2r}\}$ for $X_{\text{hyp}} \oplus v_0 A$. Then we have $X = A^{2r+1} \oplus C$ and $\xi = \omega_{2r+1} \perp (-\chi)$, where $\omega_{2r+1}$ (resp. $-\chi$) is the restriction of $\xi$ to $A^{2r+1}$ (resp. $C$). With this translation, (a) and the first statement in (b) comprise [ABG, Thm. 3.10]. The last (more explicit) statement in (b) follows from the proof of the same theorem. (See Paragraph 2.5, Theorem 2.48, Paragraph 2.50, and the proof of Proposition 3.9 in [ABG].) \qed

**Remark 4.2.5.** Suppose that $\mathcal{B} = \mathfrak{b}u(X, \xi)$ and $\mathfrak{h}$ are as in Theorem 4.2.4 (a), and let $\pi : \mathcal{B} \to \mathcal{B}/Z(\mathcal{B})$ denote the canonical projection. Now $\mathfrak{h} \cap Z(\mathcal{B}) = 0$ (since $0 \neq U(v_0, x_i) \subseteq U(X_0, x_i) \subseteq \mathfrak{U}_i \subseteq \mathcal{B}$, and hence any element of $\mathfrak{h} \cap Z(\mathcal{B})$ lies in the kernel of $\varepsilon_i$ for all $i = 1, \ldots, r$). Thus, $\pi|_\mathfrak{h}$ is a linear isomorphism of $\mathfrak{h}$ onto $\pi(\mathfrak{h})$. Consequently, the inverse dual of this map is a linear isomorphism of $\mathfrak{h}^*$ onto $\pi(\mathfrak{h})^*$. Later we will use this map to identify $\mathfrak{h}^*$ and $\pi(\mathfrak{h})^*$. In this way, the induced $Q$-grading on $\pi(\mathcal{B})$ can be regarded as the root space decomposition relative to the adjoint action of $\pi(\mathfrak{h})$.

The next two results provide us with detailed information about the root spaces of $\mathcal{B}$, $\mathfrak{S}$, $\mathcal{F}$, and $\mathfrak{U}$ relative to the adjoint action of $\mathfrak{h}$.

**Proposition 4.2.6.**
Remark 4.2.9. Suppose that $M_\alpha$ is a monomorphism of $Z$.

Proof. Part (a) is clear since $\mu_\mu \subseteq B \subseteq U$ for $0 \neq \mu \in Q$. Part (b) follows from (13).

Proposition 4.2.7. We have

$$\mathcal{F}_0 = \sum_{i=1}^{r} U(x_i, A, x_i) + U(X_{an}, X_{an}).$$

Moreover, for each $\mu \in \Delta$, a general element of $\mathcal{F}_\mu$ can be expressed uniquely in the form indicated below.

(a) $\mu = \varepsilon_i + \varepsilon_j$, $1 \leq i \neq j \leq 2r$, $i \neq j$: $U(x_i, \alpha, x_j)$, $\alpha \in A$

(b) $\mu = 2\varepsilon_i$, $1 \leq i \leq 2r$: $U(x_i, b, x_i)$, $b \in A_{−}$

(c) $\mu = \varepsilon_i$, $1 \leq i \leq 2r$: $U(v, x_i)$, $v \in X_{an}$.

Proof. Applying Proposition 4.2.6 (b), (17), (7) and (8), we obtain (20) and the existence of the expressions in (a)–(c). Also, the uniqueness is easily checked. For example, in (c), suppose that $U(v, x_i) = 0$, where $v \in X_{an}$. Then we have $0 = U(v, x_i|x_i = v$.

The following identities, together with Proposition 4.2.7, allow us to recover $(A, −)$, $X$ and $\xi$ from the Lie algebra $B$ and its root space decomposition. Each of these identities can be checked directly using (7) and (9).

Proposition 4.2.8. Suppose that $\alpha, \beta \in A$, $v, w \in X_{an}$ and $1 \leq i, j, k \leq 2r$. Then,

$$U(x_i, \alpha, x_j) = -U(x_j, \alpha, x_i)$$

$$[U(x_i, \alpha, x_j), U(x_j, \beta, x_k)] = U(x_i, \alpha \beta, x_k)$$

$$[U(x_i, \alpha, x_j), U(v, x_i)] = -U(v, \alpha, x_j)$$

$$[U(v, x_i), U(w, x_j)] = -U(x_i, \xi(v, w), x_j)$$

Remark 4.2.9. Suppose that $r \geq 2$. It follows from Proposition 4.2.7, identity (23) (with $\alpha = 1$) and identity (24) (with $v = v_0$, $w = v_0 \alpha$ and $j = \beta$) that the ideal of $B$ generated by $\{B_{\varepsilon_i + \varepsilon_j} | 1 \leq i, j \leq 2r, 1 \leq i, j \leq 2r\}$ contains $B_\mu$ for all $\mu \in \Delta$ and hence equals $B$.

For use in the construction of extended affine Lie algebras (see Section 7.3), we now determine the centroid of the BC$_r$-graded unitary Lie algebra $B$. This has been done previously in [BN, Thm. 5.8], where $B$ is presented in a different way (using the module decomposition of $B$ relative to the adjoint action of the grading subalgebra), and we could transport that result to our setting. However, for the convenience of the reader, instead we include a direct proof in the present setup.

Proposition 4.2.10. For $\mathfrak{g} \in Z(A, −)$, define $M_{\mathfrak{g}} : \text{End}_A(X) \rightarrow \text{End}_A(X)$ by $M_{\mathfrak{g}}(T) = \mathfrak{g}T$ for $T \in \text{End}_A(X)$ (see 4.1.3). Then the map $\mathfrak{g} \rightarrow M_{\mathfrak{g}|B}$ is an algebra monomorphism of $Z(A, −)$ into $\text{Cent}$. Moreover, when $r \geq 3$, this map is an isomorphism.

Proof. One can verify directly that the map $\mathfrak{g} \rightarrow M_{\mathfrak{g}}$ is an algebra homomorphism of $Z(A, −)$ into the centroid of the associative algebra $\text{End}_A(X)$. Further, by 4.1.3, $M_{\mathfrak{g}}$ stabilizes $\mathcal{F}$ for $\mathfrak{g} \in Z(A, −)$. Therefore, $M_{\mathfrak{g}|\mathcal{F}}$ lies in the centroid of $\mathcal{F}$, so it stabilizes the root spaces of $\mathcal{F}$ relative to $\mathfrak{h}$. Thus, $M_{\mathfrak{g}}$ stabilizes $\mathcal{B}$, and so $M_{\mathfrak{g}|\mathcal{B}}$.
lies in the centroid of $\mathcal{B}$. To see that the map $\mathfrak{z} \to M_{\mathfrak{z}}|_{\mathcal{B}}$ is injective, suppose that $M_{\mathfrak{z}}|_{\mathcal{B}} = 0$ where $\mathfrak{z} \in Z(\mathcal{A})$. Then $U(v_{0}\mathfrak{z}, x_{i}) = M_{\mathfrak{z}}U(v_{0}, x_{i}) = 0$ for $1 \leq i \leq 2r$, so $v_{0}\mathfrak{z} = 0$ by the uniqueness in Proposition 4.2.7 (c). Hence, $\mathfrak{z} = \varepsilon(v_{0}, v_{0} \mathfrak{z}) = 0$.

Now let $r \geq 3$ and $T \in \text{Cent}(\mathcal{B})$. We show that $M_{\mathfrak{z}} = M_{\mathfrak{z}}$ for some $\mathfrak{z} \in Z(\mathcal{A})$. Set $J = \{1, \ldots, 2r\}$. By Proposition 4.2.7, if $i, \bar{i}, j$ are distinct in $J$, then any element of the root space of $\mathcal{B}$ corresponding to $\varepsilon_{i} + \varepsilon_{j}$ can be written uniquely in the form $U(x_{i}, \alpha, x_{j})$, where $\alpha \in \mathcal{A}$. Since $T$ must stabilize root spaces, when $i, \bar{i}, j$ are distinct in $J$, there exists a unique $\tau_{ij} \in \text{End}_{\mathcal{A}}(\mathcal{A})$ such that

$$T(U(x_{i}, \alpha, x_{j})) = U(x_{i}, \tau_{ij}(\alpha), x_{j})$$

for $\alpha \in \mathcal{A}$.

Applying $T$ to equation (22), we see that

$$\tau_{ik}(\alpha \beta) = \alpha \tau_{jk}(\beta) = \tau_{ij}(\alpha \beta)$$

(25)

for $i, \bar{i}, j, \bar{j}, k, \bar{k}$ distinct in $J$. The special cases of $\alpha = 1$ and $\beta = 1$ then tell us that

$$\tau_{ik} = \tau_{jk} = \tau_{ij}$$

(26)

when $i, \bar{i}, j, \bar{j}, k, \bar{k}$ are distinct in $J$. Now if $i, \bar{i}, j$ are distinct in $J$, then since $r \geq 3$, we can choose $k \in J$ with $i, \bar{i}, j, \bar{j}, k, \bar{k}$ distinct, in which case $\tau_{ij} = \tau_{ik} = \tau_{ij}$, and similarly $\tau_{ij} = \tau_{ij}$. So any $\tau_{ij}$ with $i, \bar{i}, j$ distinct in $J$ equals one with $i, j$ distinct in $\{1, \ldots, r\}$, and in that event, it follows easily from (26) that $\tau_{ij}$ is independent of the choice of $i, j$. We let $\tau$ denote this common map. From (25) we know that $\tau \in \text{Cent}(\mathcal{A})$, and therefore

$$\tau(\alpha) = 3\alpha = 3\mathfrak{z}$$

for $\alpha \in \mathcal{A}$, where $\mathfrak{z} = \tau(1) \in Z(\mathcal{A})$. Now applying $T$ to equation (21), we see that $U(x_{i}, \tau(\alpha), x_{j}) = -U(x_{j}, \tau(\alpha), x_{i}) = U(x_{i}, \tau(\alpha), x_{j})$ for $\alpha \in \mathcal{A}$, $i, \bar{i}, j$ distinct in $J$. So $\tau$ commutes with $\tau$. Hence $\mathfrak{z} \in Z(\mathcal{A}, -)$.

Finally, $T' := T - M_{\mathfrak{z}}$ sends the root spaces $\mathcal{B}_{\varepsilon_{i} + \varepsilon_{j}}$, with $i, \bar{j}, j$ distinct in $J$, to $0$ so $T' = 0$ by Remark 4.2.9.

5. HERMITIAN FORMS AND UNITARY LIE ALGEBRAS OVER ASSOCIATIVE TORI

The results we develop in this chapter on graded hermitian forms and unitary Lie algebras over associative tori with involution will be used in the next chapter to construct unitary Lie tori of type BC$_{r}$.

5.1. Associative tori with involution.

Suppose that $L$ is an additive abelian group.

**Definition 5.1.1.** An associative $L$-torus is an $L$-graded associative algebra $\mathcal{A}$ such that $\mathcal{A}^{\sigma}$ is spanned by an invertible element of $\mathcal{A}$ for each $\sigma \in L$. (In other language, an associative $L$-torus is a twisted group algebra of $L$ [P, §1.1.2].) If “$-$” is a graded involution on such an $\mathcal{A}$, then $\mathcal{A}$ is an associative $L$-torus with involution.

**Remark 5.1.2.** If $L$ is free of finite rank, the associative $L$-tori with involution have been classified by Yoshii in [Y2]. We will return to this setting in Section 7.1.

The following fact is stated in [BGK, Prop. 2.44 (iii)] when $L$ is free of finite rank, and for arbitrary $L$ in [N3, 7.7.1]. For the convenience of the reader, we supply a proof.
Lemma 5.1.3. If \( A \) is an associative \( L \)-torus, then \( A = Z(A) \oplus [A, A] \).

Proof. To show that \( Z(A) \cap [A, A] = 0 \), suppose the contrary. Then since \( A \) is an associative \( L \)-torus, we know that \( [A, A]^\sigma = Z(A)^\sigma \) is one-dimensional for some \( \sigma \in L \). So \( 0 \neq [\alpha, \beta] \in Z(A)^\sigma \) for some nonzero homogeneous \( \alpha, \beta \in A \). Since \( A^\sigma \) is one-dimensional, \( [\alpha, \beta] = \vartheta \alpha \beta \) for some \( \vartheta \in F^\times \). So \( 0 = [\alpha, [\alpha, \beta]] = \vartheta [\alpha, \alpha \beta] = \vartheta \alpha \beta \). But since \( \alpha, \beta \) are invertible \( \vartheta = 0 \), a contradiction.

To show that \( A = Z(A) + [A, A] \), it is enough to prove that any homogeneous element \( \alpha \) of \( A \setminus Z(A) \) is in \( [A, A] \). Now \( \alpha \beta \neq \beta \alpha \) for some nonzero homogeneous \( \beta \). So \( \beta \alpha = \delta \alpha \beta \), where \( \delta \in F^\times \) and \( \delta \neq 1 \). Then \( [\alpha \beta^{-1}, \beta] = (1 - \delta) \alpha \), so \( \alpha \in [A, A] \). \( \square \)

5.1.4. If \( A \) is an associative \( L \)-torus with involution, then since \( A \) is finely graded with support \( L \), we have \( L = L_+ \uplus L_- \), where \( \uplus \) denotes disjoint union. (See 2.3.1 (c).)

5.1.5. Now suppose that \( L \) is a subgroup of an abelian group \( \Lambda \) and \( A \) is an associative \( L \)-torus. Then \( A \) is a graded division algebra (in the sense that each nonzero homogeneous element in invertible), so \( \Lambda \)-graded \( A \)-modules have properties analogous to modules over a division algebra. In particular, if \( X \) is such a \( \Lambda \)-graded \( A \)-module, every homogeneous \( A \)-spanning set of \( X \) contains a homogeneous \( A \)-basis of \( X \); every homogeneous \( A \)-independent set in \( X \) is contained in a homogeneous \( A \)-basis of \( X \); \( X \) is a free \( A \)-module with a homogeneous \( A \)-basis; and any two homogeneous \( A \)-bases for \( A \) have the same cardinality [RTW, p. 100]. We define the rank of \( X \) over \( A \), denoted by \( \text{rank}_A(X) \), to be the cardinality of a homogeneous \( A \)-basis for \( X \).

If \( \sigma \in \Lambda \) and \( B^\sigma \) is an \( F \)-basis for \( X^\sigma \), then \( B^\sigma \) is also an \( A \)-basis for \( X^\sigma \cdot A \), so

\[
\text{rank}_A(X^\sigma \cdot A) = \text{dim}_F(X^\sigma).
\]

Let \( S = \text{supp}_\Lambda(X) \). Then \( L + S \subseteq S \), so \( S \) is the union of cosets of \( L \) in \( \Lambda \). Let \( S/L = \{ \sigma + L \mid \sigma \in S \} \) in \( \Lambda/L \). Then \( X = \bigoplus_{\sigma \in \Theta} X^\sigma \cdot A \), where \( \Theta \) is a set of representatives of the cosets in \( S/L \), and

\[
\text{rank}_A(X) = \sum_{\sigma \in \Theta} \text{dim}_F(X^\sigma),
\]

where the right-hand side uses the arithmetic of cardinal numbers.

5.2. Hermitian forms over associative tori with involution.

Throughout Section 5.2, we assume \( L \) is a subgroup of an abelian group \( \Lambda \) and \((A, -)\) is an associative \( L \)-torus with involution. Recall that we are regarding \((A, -)\) as a \( \Lambda \)-graded associative algebra with involution (as in 2.3.1 (a)).

Definition 5.2.1. Suppose that \( \xi : X \times X \to A \) is a \( \Lambda \)-graded hermitian form over \((A, -)\). We say that \( \xi \) (or \( X \)) is anisotropic\(^6\) if \( \xi(x, x) \neq 0 \) for all nonzero homogeneous \( x \in X \). An \( A \)-submodule \( Y \) of \( X \) is totally isotropic if \( \xi(y, y') = 0 \) for all \( y, y' \in Y \). When the form \( \xi \) on \( X \) is nondegenerate and \( X \) is the orthogonal direct sum of two totally isotropic graded \( A \)-submodules, then we say that \( \xi \) (or \( X \)) is hyperbolic. When \( X \) has a homogeneous \( A \)-basis \( \{x, y\} \) such that \( \xi(x, y) = 1 \) and \( \xi(x, x) = 0 = \xi(y, y) \), then \( \xi \) (or \( X \)) is called a hyperbolic plane.

\(^6\)If \( \Lambda \) is torsion-free, then \( \Lambda \) can be ordered, so it is easy to check that our graded definition of anisotropic is equivalent to the usual ungraded definition of anisotropic, namely that \( \xi(x, x) \neq 0 \) for all \( 0 \neq x \in X \) [RTW, p. 101].
5.2.2. Suppose that $\xi : X \times X \to A$ is a nondegenerate $\Lambda$-graded hermitian form over $(A, -)$, and $X$ has finite rank over $A$. If $Y$ is a graded $A$-submodule of $X$, then $Y^\perp$ is a graded $A$-submodule of $X$ and $\text{rank}_A(Y^\perp) + \text{rank}_A(Y) = \text{rank}_A(X)$. Hence if $\xi|_{Y \times Y}$ is nondegenerate, then $X = Y \oplus Y^\perp$.

Using the results of [RTW, Sec. 1], we can prove the following analogues of classical facts about finite-dimensional hermitian forms over division rings:

**Theorem 5.2.3.** Let $\xi : X \times X \to A$ be a nondegenerate $\Lambda$-graded hermitian form of finite graded $\mathbb{F}$-dimension over an associative $L$-torus with involution $(A, -)$.

(a) If $\xi$ is hyperbolic, and hence $X = W_1 \oplus W_2$, where $W_1$ and $W_2$ are totally isotropic graded $A$-submodules of $X$, then $\text{rank}_A(W_1) = \text{rank}_A(W_2)$, and $X$ is the graded orthogonal sum of hyperbolic planes, where the sum runs over an index set of cardinality $\text{rank}_A(W_1)$.

(b) If $\xi$ is anisotropic, then $X$ has an orthogonal homogeneous $A$-basis. In fact, any orthogonal set of nonzero homogeneous elements of $X$ is contained in an orthogonal homogeneous basis of $X$ over $A$.

(c) There exist $\Lambda$-graded $A$-submodules $X_{\text{hyp}}$ and $X_{\text{an}}$ of $X$ such that

$$X = X_{\text{hyp}} \perp X_{\text{an}},$$

$\xi_{\text{hyp}} = \xi|_{X_{\text{hyp}} \times X_{\text{hyp}}}$ is hyperbolic, and $\xi_{\text{an}} = \xi|_{X_{\text{an}} \times X_{\text{an}}}$ is anisotropic. Moreover, if $X = X_{\text{hyp}}' \perp X_{\text{an}}'$ is another such orthogonal decomposition, then there exists an $A$-linear graded isometry from $X$ to $X$ that maps $X_{\text{hyp}}$ to $X_{\text{hyp}}'$ and $X_{\text{an}}$ to $X_{\text{an}}'$.

**Proof.** We may assume that $X \neq 0$. We define an equivalence relation $\sim$ on $\Lambda$ by saying $\tau \sim \sigma$ if and only if $\tau \equiv \pm \sigma \pmod{L}$. If $\sigma \in \Lambda$, we set $X(\sigma) = \sum_{\tau \sim \sigma} X^\tau$.

Let $R$ denote a set of representatives of the equivalence classes for $\sim$. Then, any homogeneous element of $X$ is in $X(\sigma)$ for some $\sigma \in R$; $X = \bigoplus_{\sigma \in R} X(\sigma)$; and, if $Y$ is any graded $A$-submodule of $X$, $Y = \bigoplus_{\sigma \in R} Y \cap X(\sigma)$. Thus to prove the theorem, we can assume that $X = X(\sigma)$ for some $\sigma \in \Lambda$. Hence, we have $X = X^{\sigma}A + X^{-\sigma}A$.

Thus $\text{rank}_A(X) = \dim_{\mathbb{F}}(X^{\sigma})$ if $2\sigma \in L$, whereas $\text{rank}_A(X) = 2 \dim_{\mathbb{F}}(X^{-\sigma})$ if $2\sigma \notin L$. So $X$ has finite rank over $A$. Then (a) and (b) are easily checked (using 5.2.2). Moreover, (c) follows from [RTW, Prop. 1.4 (iv)]. However, a few words are needed to justify the use of this proposition. Indeed, it is assumed in [RTW] that the grading group $\Lambda$ is torsion-free and divisible. But the proof of [RTW, Prop. 1.4 (iv)] does not use that assumption, provided we interpret $\frac{1}{2}L$ as $\{\sigma \in \Lambda \mid 2\sigma \in L\}$.

**Definition 5.2.4.** Suppose the assumptions of Theorem 5.2.3 hold.

(a) A decomposition $X = X_{\text{hyp}} \perp X_{\text{an}}$ as in Theorem 5.2.3 (c) is said to be a *Witt decomposition* of $\xi$ (or $X$). In that case we write

$$\Lambda_{\text{hyp}} = \text{supp}_\Lambda(X_{\text{hyp}}) \quad \text{and} \quad \Lambda_{\text{an}} = \text{supp}_\Lambda(X_{\text{an}}).$$

Note that by Theorem 5.2.3 (c), these subsets of $\Lambda$ are independent of the choice of Witt decomposition.

(b) By assumption $X_{\text{hyp}} = W_1 \oplus W_2$, where $W_1$ and $W_2$ are totally isotropic $A$-submodules of $X$. The *Witt index* of $\xi$ (or $X$) is defined as $\text{rank}_A(W_1) = \text{rank}_A(W_2)$. It follows from Theorem 5.2.3 (c) that this index is independent of the choice of Witt decomposition, and it is clear that it does not depend on the decomposition $X_{\text{hyp}} = W_1 \oplus W_2$. 
In our setting, anisotropic forms have the following characterization:

Proposition 5.2.5. Suppose the assumptions of Theorem 5.2.3 hold. If $\xi$ is finely graded and $2\text{supp}_A(X) \subseteq L$, then $\xi$ is anisotropic. Moreover, the converse holds if $F$ is algebraically closed.

Proof. “$\Rightarrow$” Suppose the contrary that $\xi(x,x) = 0$ for some $0 \neq x \in X^\sigma$, $\sigma \in \Lambda$. As $2\sigma \in L$, it follows that $x.A^{-2\sigma} \neq 0$. But since $X$ is finely-graded, $X^{-\sigma} = x.A^{-2\sigma}$ and hence $\xi(x,-\sigma) = 0$, a contradiction.

“$\Leftarrow$” By the argument in the proof of Theorem 5.2.3, we can assume that $X = X^\sigma.A + x^{-\sigma}.A$ for some $\sigma \in \text{supp}_A(X)$. But $0 \neq \xi(X^\sigma,X^\sigma) \subseteq A^{2\sigma}_L$, so $2\sigma \in L$.

Hence,

$$X = X^\sigma.A,$$

so $2\text{supp}_A(X) = 2(\sigma + L) \subseteq L$. It remains to show that $\dim_F(X^\sigma) = 1$. For this fix $0 \neq a \in A^2_\Lambda$. We can define an $F$-bilinear form $f : X^\sigma \times X^\sigma \to F$ such that $\xi(x,y) = f(x,y)a$ for $x,y \in X^\sigma$. Then, $f$ is symmetric and anisotropic (in the usual ungraded sense). Hence, since $F$ is algebraically closed, $\dim(X^\sigma) = 1$. \hfill $\Box$

5.3. Unitary Lie algebras over associative tori with involution.

In Section 5.3 (except in the final Remark 5.3.9), we assume that $L$ is a subgroup of an abelian group $\Lambda$, that $(A,-)$ is an associative $L$-torus with involution, and that $\xi : X \times X \to A$ is a nondegenerate $\Lambda$-graded hermitian form of finite graded $F$-dimension over $(A,-)$; and we fix a Witt decomposition $X = X_{\text{hyp}} \perp X_{\text{an}}$ of $\xi$.

By Theorem 5.2.3 we can choose a homogeneous $A$-basis $\{x_i\}_{i \in I}$ for $X$ and a permutation $i \mapsto \bar{i}$ of period 2 of $I$, such that

$$\xi(x_i,x_j) = \delta_{ij}\gamma_i$$

for $i,j \in I$, where each $\gamma_i$ is a homogeneous invertible element of $A$, and such that

$$X_{\text{hyp}} = \bigoplus_{i \in J} x_i.A \quad \text{and} \quad X_{\text{an}} = \bigoplus_{i \in K} x_i.A.$$  

(29)

where

$$J = \{i \in I \mid \bar{i} \neq i\} \quad \text{and} \quad K = \{i \in I \mid \bar{i} = i\}.$$

In this section, we fix a choice of such a basis $\{x_i\}_{i \in I}$ and we fix a total ordering “$\leq$” of the (possibly infinite) index set $I$. Our goal is to use the basis $\{x_i\}_{i \in I}$ to deduce more information about $\mathfrak{fu}(X,\xi)$ and $\mathfrak{su}(X,\xi)$.

The elements $\gamma_i$ and the bijection $i \mapsto \bar{i}$ are uniquely determined by the basis $\{x_i\}_{i \in I}$. Also (28) implies that

$$\gamma_i = \overline{\gamma_{\bar{i}}}$$

(30)

for $i \in I$.

As in Definition 4.1.1, we let

$$E = e(X,\xi), \quad U = u(X,\xi), \quad F = \mathfrak{fu}(X,\xi), \quad \text{and} \quad S = \mathfrak{su}(X,\xi).$$

5.3.1. (Matrix notation and the trace map) We adopt matrix notation relative to the basis $\{x_i\}_{i \in I}$, which enables us to introduce a trace map on $E$.

(a) For $i,j \in I$ and $\alpha \in A$, let $e_{ij}(\alpha)$ denote the endomorphism in $\text{End}_A(X)$ given by

$$e_{ij}(\alpha)x_i = \delta_{ij}x_i.\alpha$$

(31)

We have two maps, the involution $\alpha \mapsto \overline{\alpha}$ of $A$ and the permutation $i \mapsto \bar{i}$ of $I$. The context will make it clear which of these is intended.
for $t \in \mathbb{I}$. Then one verifies using (28) that
\[ E(x_i, \alpha, x_j, \beta) = e_{ij}(\alpha \beta \gamma_j) \] and so \[ e_{ij}(\alpha) = E(x_i, x_j, (\gamma_j^{-1} \alpha)) \] (32)
for $i, j \in \mathbb{I}$, $\alpha, \beta \in \mathcal{A}$. Thus, each $e_{ij}(\alpha)$ is in $\mathcal{E}$ and
\[ \mathcal{E} = \sum_{i,j \in \mathbb{I}} e_{ij}(\mathcal{A}). \] (33)
In fact, this sum is direct, and each element of $\mathcal{E}$ can be written uniquely in the form $\sum_{ij} e_{ij}(\alpha_{ij})$, where $\alpha_{ij} \in \mathcal{A}$ for all $i, j$. Moreover, we have
\[ e_{ij}(\alpha)e_{kl}(\beta) = \delta_{jk}e_{i\ell}(\alpha\beta) \] (34)
for $i, j, k, \ell \in \mathbb{I}$, $\alpha, \beta \in \mathcal{A}$, and so $\mathcal{E}$ can be regarded as the associative algebra of $(\mathbb{I} \times \mathbb{I})$-matrices over $\mathcal{A}$ with finitely many nonzero entries.

(b) The trace map on $\mathcal{E}$ is the additive map $\text{tr} : \mathcal{E} \to \mathcal{A}$ so that
\[ \text{tr} (e_{ij}(\alpha)) = \delta_{ij} \alpha \] (35)
for $i, j \in \mathbb{I}$, $\alpha \in \mathcal{A}$. One easily checks using (34) that
\[ \text{tr}(T_1T_2) \equiv \text{tr}(T_2T_1) \pmod{[\mathcal{A}, \mathcal{A}]} \] (36)
for $T_1, T_2 \in \mathcal{E}$. Also if $i, j \in \mathbb{I}$ and $\alpha, \beta \in \mathcal{A}$, then we have $\text{tr} (E(x_i, \alpha, x_j, \beta)) = \text{tr} (e_{ij}(\alpha \beta \gamma_j)) = \delta_{ij} \alpha \beta \gamma_j \equiv \delta_{ij} \beta \gamma_j \alpha \pmod{[\mathcal{A}, \mathcal{A}]}$ and so
\[ \text{tr} (E(x_i, \alpha, x_j, \beta)) \equiv \xi(x_j, \beta, x_i, \alpha) \pmod{[\mathcal{A}, \mathcal{A}]} \] (37)
Hence for all $x, y \in \mathcal{A}$,
\[ \text{tr} (E(x, y)) \equiv \xi(y, x) \pmod{[\mathcal{A}, \mathcal{A}]} \] (38)
Consequently, modulo $[\mathcal{A}, \mathcal{A}]$, the trace function is independent of the choice of basis $\{x_i\}_{i \in \mathbb{I}}$. (This is also easy to see directly using (36).)

(c) It follows from (37) that
\[ \text{tr} (U(x, y)) \equiv \xi(y, x) - \xi(x, y) \pmod{[\mathcal{A}, \mathcal{A}]} \] (39)
(d) Recall from 4.1.1(d) that we have an involution $*$ on $\mathcal{E}$, which satisfies $E(x, y) = E(y, x)^*$ for $x, y \in X$. One can check using (4), (30) and (32) that
\[ e_{ij}(\alpha)^* = e_{ji}(\gamma_j^{-1} \alpha \gamma_i) \] (40)
for $i, j \in \mathbb{I}$ and $\alpha \in \mathcal{A}$.

(e) For $i, j \in \mathbb{I}$ and $\alpha \in \mathcal{A}$, set
\[ u_{ij}(\alpha) = e_{ij}(\alpha) - e_{ij}(\alpha)^* = e_{ij}(\alpha) - e_{ji}(\gamma_j^{-1} \alpha \gamma_i) \] (41)
so that
\[ u_{ij}(\alpha) = -u_{ji}(\gamma_j^{-1} \alpha \gamma_i) \] (42)
Then (32) can be used to show that
\[ U(x_i, \alpha, x_j, \beta) = u_{ij}(\alpha \beta \gamma_j) \] and so \[ u_{ij}(\alpha) = U(x_i, x_j, (\gamma_j^{-1} \alpha)) \] (43)
for $i, j \in \mathbb{I}$, $\alpha, \beta \in \mathcal{A}$. Thus,
\[ \mathcal{F} = \sum_{i,j \in \mathbb{I}} u_{ij}(\mathcal{A}). \] (44)

Remark 5.3.2. If $X$ has finite rank over $\mathcal{A}$, it follows from (33) that $\mathcal{E} = \text{End}_\mathcal{A}(X)$ and hence, by virtue of (3), that $\mathcal{F} = \mathbb{U}$. 
In the next two propositions, we use $U$-operators and then matrix notation to express each element of $\mathcal{F}$ uniquely and to calculate its trace.

**Proposition 5.3.3.** We have

$$\mathcal{F} = \sum_{i,j \in I} U(x_i, A, x_j, A)$$

with

$$\text{tr} \left( U(x_i, \alpha, x_j, \beta) \right) \equiv \delta_{ij} (\alpha \overline{\gamma_j} \alpha_j - \gamma_i \beta \alpha) \pmod {[A, A]}$$

for $\alpha, \beta \in A$. Each element in $\mathcal{F}$ can be expressed uniquely in the form

$$\sum_{i,j \in I, i < j} U(x_i, \alpha_{ij}, x_j) + \sum_{i \in I} U(x_i, b_i, x_i),$$

where $\alpha_{ij} \in A$ and $b_i \in A_-$. 

*Proof.* Equation (42) follows from the definition of $\mathcal{F}$. By (38), we have

$$\text{tr} \left( U(x_i, \alpha, x_j, \beta) \right) \equiv \xi(x_j, \beta, x_i, \alpha) - \xi(x_i, \alpha, x_j, \beta) \equiv \overline{\delta_{ji}} \gamma_j \alpha - \overline{\alpha} \delta_{ji} \gamma_i \beta \equiv \delta_{ij} (\alpha \overline{\gamma_j} \alpha_j - \gamma_i \beta \alpha) \pmod {[A, A]}.$$ 

Finally, by (42), (7), and (8), each element of $\mathcal{F}$ can be expressed in the form given in (44). Uniqueness of the expression in (44) comes from (40) and (41). \square

**Proposition 5.3.4.** We have

$$\mathcal{F} = \sum_{i,j \in I} u_{ij}(A) = \left\{ \sum_{i,j \in I} e_{ij}(\alpha_{ij}) \mid \alpha_{ij} \in A, \alpha_{ji} = -\gamma_j^{-1} \alpha_{ij} \overline{\gamma_j} \gamma_i \text{ for } i, j \in I \right\},$$

with

$$\text{tr} \left( u_{ij}(\alpha) \right) \equiv \delta_{ij} (\alpha - \overline{\alpha}) \pmod {[A, A]}$$

for $\alpha, \beta \in A$ and $i, j \in I$. Each element of $\mathcal{F}$ can be expressed uniquely in the form

$$\sum_{i,j \in I, i < j} u_{ij}(\alpha_{ij}) + \sum_{i \in I} u_{ii}(b_i \gamma_i)$$

where $\alpha_{ij} \in A$ and $b_i \in A_-$. 

*Proof.* The equalities in (45) follow from (41) and (40). Also, since

$$\text{tr} \left( u_{ij}(\alpha) \right) = \delta_{ij} \alpha - \delta_{ji} \gamma_j^{-1} \overline{\alpha} \gamma_i,$$

we have (46). Finally, the last statement in the proposition follows from the last statement in Proposition 5.3.3 and (41). \square

In view of (45), we can regard $\mathcal{F}$ as a Lie algebra of skew-hermitian $(I \times I)$-matrices over $A$.\(^8\)

We now prove the main result of this chapter which gives a simple and convenient description of the Lie algebra $\mathcal{S}$ and some of its properties. This theorem was shown to hold in a special case in [AABGP, Prop. III.3.14 and Lem. III.3.21].

\(^8\)If $i = i$ and $\gamma_i = 1$ for all $i \in I$, then $\mathcal{F}$ is the usual Lie algebra of skew-hermitian matrices.
Suppose that $\xi : X \times X \to A$ is a nondegenerate $\Lambda$-graded hermitian form of finite graded $F$-dimension over an associative $L$-torus with involution $(A, -)$, $X = X_{hyp} \perp X_{an}$ is a Witt decomposition of $\xi$, and $X_{hyp} \neq 0$ and $X_{an} \neq 0$ (that is $\xi$ is not anisotropic or hyperbolic). Let $S = \mathfrak{su}(X, \xi)$ and $\mathcal{F} = \mathfrak{fu}(X, \xi)$. Then

(a) $S = \{ T \in \mathcal{F} | \text{tr}(T) \equiv 0 \pmod{[A, A]} \}$.

(b) $S$ is generated as an algebra by $U(X_{hyp}, X_{an})$.

(c) Suppose that the rank of $X$ over $A$ is infinite or finite and not divisible by char($\mathcal{F}$) (which holds in particular if char($\mathcal{F}$) = 0). Then $Z(S) = 0$.

(d) Under the assumptions of (c), let $\varpi : A \to \mathcal{F}$ be the $L$-graded projection onto $A^0 = F1 = F$ (where $F$ has the trivial grading). Then

$$(T_1 | T_2) := \varpi(\text{tr}(T_1 T_2))$$

defines a $\Lambda$-graded nondegenerate associative symmetric bilinear form on $\mathcal{F}$ whose restriction to $\mathcal{F}$ and to $S$ is nondegenerate.

Proof. Since $X_{hyp} \neq 0$ and $X_{an} \neq 0$, the sets $J$ and $K$ in (29) are nonempty. Further since $X = X_{hyp} \oplus X_{an}$ and $\xi(X_{hyp}, X_{an}) = 0$, we know that $U(X_{hyp}, X_{hyp})$ maps $X_{hyp}$ to $X_{hyp}$ and $X_{an}$ to 0; $U(X_{an}, X_{an})$ maps $X_{hyp}$ to 0 and $X_{an}$ to $X_{an}$; and $U(X_{hyp}, X_{an})$ exchanges $X_{hyp}$ and $X_{an}$. Therefore

$$\mathcal{F} = U(X_{hyp}, X_{hyp}) \oplus U(X_{an}, X_{an}) \oplus U(X_{hyp}, X_{an}).$$

(a) and (b): Let $S_1$ denote the subalgebra of $\mathcal{F}$ generated by $U(X_{hyp}, X_{an})$, and set

$$S_2 = \{ T \in \mathcal{F} | \text{tr}(T) \equiv 0 \pmod{[A, A]} \}.$$ 

To prove (a) and (b), we must verify that $S_1 = S = S_2$. We do this by showing that $S_1 \subseteq S \subseteq S_2 \subseteq S_1$.

First, if $i \in J$ and $x'' \in X_{an}$, then (9) implies that $[U(x_i, x_i), U(x_i, x'')] = U(U(x_i, x_i) x_i, x'') = U(x_i, \eta x'')$. Thus, $U(x_i, \eta x'') \in \mathcal{F}^{(1)} = S$. Since $\eta$ is invertible, $U(X_{hyp}, X_{an}) \subseteq S$ by (7) and so $S_1 \subseteq S$.

The containment $S \subseteq S_2$ is a consequence of (36).

Finally, we assume that $T \in S_2$ and show that $T \in S_1$. Since $U(X_{hyp}, X_{an}) \subseteq S_1$, we can suppose by (48) that

$$T \in (U(X_{hyp}, X_{hyp}) \oplus U(X_{an}, X_{an})) \cap S_2.$$

Now if $x', y' \in X_{hyp}$ and $x'', y'' \in X_{an}$, we have

$$[U(x', x''), U(y', y'')] = U(U(x', x'') y', y'') + U(y', U(x', x'') y'')$$

and so

$$U(x'', \xi(x', y'), y'') - U(y', x'. \xi(x'', y'')) \in S_1. \quad (49)$$

If we take $x' = x_i$ and $y' = x_i \gamma_i^{-1}$, where $i \in J$, then $\xi(x', y') = 1$ and so $U(x'', y'') = U(y', x'. \xi(x'', y'')) \in S_1$. Subtracting elements of this form from $T$, we can assume that

$$T \in U(X_{hyp}, X_{hyp}) \cap S_2. \quad (50)$$

For the rest of the proof of (b), we fix $j \in K$ and let

$$x'' = x_j, \quad \gamma = \gamma_j = \xi(x'', x'') \quad \text{and} \quad y'' = x''. \gamma^{-1}.$$
Since $\xi(x'', y'') = 1$, we have $U(x'',\xi(x', y'), x''\gamma^{-1}) - U(y', x') \in S_1$ by (49). Thus

$$U(x', y') + U(x'', x''\gamma^{-1}\xi(y', x')) \in S_1$$

(51) for $x', y' \in X_{hyp}$. Subtracting such elements from $T$, we can suppose that $T \in U(x'', x',\mathcal{A}) \cap S_2$ (of course we are no longer assuming (50)). Now by (8), we see that $T \in U(x'', x',\mathcal{A}^-) \cap S_2$, so we can assume

$$T = U(x'', x', b),$$

where $b \in \mathcal{A}^-$. Then by (38), $\text{tr}(T) = -\xi(x'', x', b) + \xi(x', x'') = -\xi(x'', x')b - b\xi(x'', x') = -\gamma b - b\gamma$ and so $\gamma b + b\gamma \in [\mathcal{A}, \mathcal{A}]$. Therefore $\gamma b \in [\mathcal{A}, \mathcal{A}]$ and so

$$T \in U(x'', x',\gamma^{-1}[\mathcal{A}, \mathcal{A}]).$$

Now for $x', y' \in X_{hyp}$ and $\alpha \in \mathcal{A}$,

$$U(x', \alpha, y') + U(x'', x',\gamma^{-1}\xi(y', x'.\alpha)) \in S_1$$

and

$$U(x', y' \xi) + U(x'', x',\gamma^{-1}\xi(y', \xi x')) \in S_1$$

by (51). Taking the difference, we get $U(x'', x',\gamma^{-1}[\xi(y', x'), \alpha]) \subseteq S_1$. Since $\xi(X_{hyp}, X_{hyp}) = \mathcal{A}$, we have $U(x'', x',\gamma^{-1}[\mathcal{A}, \mathcal{A}]) \subseteq S_1$; hence $T \in S_1$ and $S_2 \subseteq S_1$.

(c) Assume $T = \sum_{i,j \in \mathbb{Z}} e_{ij}(\alpha_{ij}) \in Z(S)$ and set $U_k := u_{kk}(1) = e_{kk}(1) = e_{kk}(1)$ for $k \in J$. By (35), $\text{tr}(U_k) = 0$, and so $U_k \in S$ by (a). The relation $U_k T = T U_k$ implies that

$$\sum_{j} e_{kj}(\alpha_{kj}) - \sum_{j} e_{kj}(\alpha_{kj}) = \sum_{i} e_{ik}(\alpha_{ik}) - \sum_{i} e_{ik}(\alpha_{ik}).$$

Hence, it must be that $\alpha_{ik} = 0 = \alpha_{ik}$ and $\alpha_{kj} = 0 = \alpha_{kj}$ for $i, j \neq k, k$ and that $\alpha_{kk} = 0 = \alpha_{kk}$ for all $k \in J$. Thus, we may assume

$$T = \sum_{k \in J, k < k} (e_{kk}(\alpha_{kk}) - e_{kk}(\gamma_k^{-1}\alpha_{kk}\gamma_k)) + \sum_{i,j \in \mathbb{K}} e_{ij}(\alpha_{ij}),$$

(52) where $\alpha_{ji} = -\gamma_j^{-1}\alpha_{ij}\gamma_i$ for $i, j \in \mathbb{K}$. Suppose $\ell \in J, \ell < \bar{\ell}$, and $m \in \mathbb{K}$. Then $u_{\ell m}(\beta) = e_{\ell m}(\beta) - e_{\ell m}(\gamma_k^{-1}\beta\gamma_k)$ belongs to $S$, as it has trace $0$ for all $\beta \in \mathcal{A}$. From $u_{\ell m}(\beta) T = T u_{\ell m}(\beta)$ we deduce

$$e_{\ell m}(\gamma_k^{-1}\beta\gamma_k) + \sum_{j \in \mathbb{K}} e_{ij}(\beta\alpha_{mj}) = e_{\ell m}(\alpha_{\ell\beta}) - \sum_{i \in \mathbb{K}} e_{i\ell}(\alpha_{im}\gamma_{\ell m}^{-1}\beta\gamma_{\ell m}).$$

(53)

Therefore, $\beta\alpha_{mj} = 0$ for all $j \in \mathbb{K}, j \neq m$, and all $\beta$, forcing $\alpha_{mj} = 0$ for all $j$ different from $m$. Moreover, it follows from (53) that $\beta\alpha_{mm} = \alpha_{m\ell}$ for all $\beta \in \mathcal{A}$. In particular when $\beta = 1$, we obtain $\alpha_{\ell m} = \alpha_{mm}$ for all $\ell \in J, m \in \mathbb{K}$. Since both $J$ and $K$ are assumed to be nonempty, there is a unique element of $\mathcal{A}$, call it $\zeta$, so that $\zeta = \alpha_{\ell m}$ for all $\ell \in J, m \in \mathbb{K}$, and $\zeta \in Z(\mathcal{A})$, as it commutes with all $\beta$. The $(m, \ell)$-term of (53) gives $\gamma_{\ell m}^{-1}\beta\zeta_{\ell} = -\zeta_{\ell m}^{-1}\beta\zeta_{\ell}$ for all $\beta \in \mathcal{A}$ Taking $\beta = 1$ and using the fact that $\zeta$ is central, we see that $\zeta = -\zeta$, so that $\zeta \in \mathcal{A}$. Substituting these results back into (52), we have

$$T = \sum_{k \in J} e_{kk}(\zeta) - e_{kk}(\gamma_k^{-1}\zeta\gamma_k) + \sum_{j \in \mathbb{K}} e_{jj}(\zeta) = \sum_{x \in \mathbb{I}} e_{xx}(\zeta).$$

(54)

Since elements of $S$ are finite sums, it must be that $\zeta = 0$ when $|\mathbb{I}| = \infty$. Thus, the centre is trivial when $\mathbb{I}$ is infinite. So we may assume that $|\mathbb{I}|$ is finite and not
divisible by the characteristic of \( F \). Then since \( \text{tr}(T) \equiv 0 \pmod{|A,A|} \) by (a), we see from (54) that \( 0 \equiv |I|\zeta \pmod{|A,A|} \). Then, \( \zeta \in [A,A] \) and \( T = R_\zeta \), where \( R_\zeta(x) = x.\zeta \) for all \( x \in X \). We have proved that \( Z(S) \subseteq \{ R_\zeta \mid \zeta \in Z(A) \cap A_- \cap [A,A] \} \).

Thus, by Lemma 5.1.3, \( Z(S) = 0 \).

(d) The same proof given for [AABGP, Lem. III.3.21] works here. That argument uses part (c) above, Lemma 5.1.3 and the equation \( \varpi(\text{tr}(T^*)) = \varpi(\text{tr}(T)) \) for \( T \in \mathcal{E} \). This last equation can be checked for example using (37).

\[ \square \]

**Corollary 5.3.6.** Under the hypotheses of Theorem 5.3.5, suppose that \( T \in \mathcal{F} \) has an expression

\[ T = \sum_{i<j} U(x_i, \alpha_{ij}, x_j) + \sum_i U(x_i, b_i, x_i) \]

as in (44) (resp. \( T = \sum_{i<j} u_{ij}(\alpha_{ij}) + \sum_i u_{ii}(b_i \gamma_i) \) as in (47)). Then \( T \in \mathcal{S} \) if and only if

\[ \sum_{i<j}(\alpha_{ii} - \alpha_{jj}) + \sum_i (b_i \gamma_i + \gamma_i b_i) \equiv 0 \pmod{|A,A|} \]

(resp., \( \sum_{i<j}(\alpha_{ii} - \alpha_{jj}) + \sum_i (b_i \gamma_i + \gamma_i b_i) \equiv 0 \pmod{|A,A|} \)).

\[ \square \]

**Proof.** This follows from Theorem 5.3.5 (a) and a calculation of \( \text{tr}(T) \) using (43) (resp. (46)).

We now use matrix notation to give an explicit description of the \( \Lambda \)-gradings on \( \mathcal{E} \) and \( \mathcal{F} \) induced by the \( \Lambda \)-grading on \( X \).

**Proposition 5.3.7.** Let \( \rho_i = \deg_\Lambda(x_i) \) for \( i \in I \). Then, \( \deg(\gamma_i) = \rho_i + \rho_i \) for \( i \in I \). Moreover, the \( \Lambda \)-gradings on \( \mathcal{E} \) and \( \mathcal{F} \) are given respectively by

\[ \deg_\Lambda(e_{ij}(\alpha)) = \rho_i - \rho_j + \deg_\Lambda(\alpha), \]

\[ \deg_\Lambda(u_{ij}(\alpha)) = \rho_i - \rho_j + \deg_\Lambda(\alpha) \]

(55)

for \( \alpha \) homogeneous in \( A \), \( i, j \in K \).

**Proof.** The first relation follows from the fact that \( \gamma_i = \xi(x_i, x_i) \). The remaining equations come from the definition of the gradings (see 4.1.4), (32) and (41).

\[ \square \]

**Remark 5.3.8.** If \( \xi \) not hyperbolic or anisotropic, Proposition 5.3.7 also tells us the \( \Lambda \)-grading on \( \mathcal{S} \), since we know the precise form of elements of \( \mathcal{S} \) by Corollary 5.3.6.

**Remark 5.3.9.** For the sake of clarity and ease of reference, in this section we have chosen to work in the context of graded hermitian forms over associative tori with involution. However, most of the results of Section 5.3 hold in a more general ungraded setting. Indeed, suppose that \( \xi : X \times X \to A \) is a hermitian form over an associative algebra with involution \((A, -)\) on a free \( A \)-module \( X \) with \( A \)-basis \( \{x_i\}_{i \in I} \) satisfying (28), where \( \gamma_i \) is an invertible element of \( A \) for \( i \in I \), and \( i \mapsto i \) is a permutation of period 2 of \( I \); and suppose that we define submodules \( X_{\text{hyp}} \) and \( X_{\text{an}} \) by (29). Then all of the results of the section hold (with the same proofs) except for Theorem 5.3.5 (c), Theorem 5.3.5 (d) and Proposition 5.3.7. The last two of these make no sense in the ungraded setting. In the case of Theorem 5.3.5 (c), the same proof shows that the following statement is true: If \( |I| \) is finite, then \( Z(\mathcal{S}) = 0 \); and if \( |I| \) is finite and not divisible by \( \text{char}(F) \), then

\[ Z(\mathcal{S}) = \{ R_\zeta \mid \zeta \in Z(A) \cap A_- \cap [A,A] \}, \]

where \( R_\zeta : X \to X \) is defined by \( R_\zeta x = x.\zeta \) for \( x \in X \). Indeed, the proof of Theorem 5.3.5 (c) establishes all of this except for the inclusion “\( \supseteq \)” which is easily checked using the fact that \( T = R_\zeta \) can be expressed as in (54).
6. UNITARY LIE TORI AND THE MAIN THEOREMS

Throughout this chapter, we assume that $F$ has characteristic 0.

We use the special unitary Lie algebra to give a construction of a centreless Lie $\Lambda$-torus of type $BC_r$ and refer to the resulting algebra as a unitary Lie $\Lambda$-torus. Then we show that if $r \geq 3$, any centreless Lie $\Lambda$-torus of type $BC_r$ is obtained in this way, and we determine when two such unitary Lie $\Lambda$-tori are bi-isomorphic.

6.1. Construction of unitary Lie $\Lambda$-tori.

**Assumptions 6.1.1.** Throughout Section 6.1 we assume that:

(H1) $r$ is an integer $\geq 1$ and $\Lambda$ is an abelian group.

(H2) $L$ is a subgroup of $\Lambda$, and $(A, -)$ is an associative $L$-torus with involution over $F$ (which we regard as $\Lambda$-graded by setting $A^\sigma = 0$ for $\sigma \in \Lambda \setminus L$).

(H3) $\xi : X \times X \to A$ is a nondegenerate $\Lambda$-graded hermitian form of finite graded $F$-dimension and Witt index $r$.

(H4) For some Witt decomposition $X = X_{\text{hyp}} \perp X_{\text{an}}$ (and hence for all Witt decompositions of $\xi$ by Theorem 5.2.3(c)),

(a) $\Lambda_{\text{hyp}} = L$,

(b) the restriction of $\xi$ to $X_0^{\text{an}}$ represents 1,

(c) $X_{\text{an}}$ is finely graded.

(H5) $X$ has full support in $\Lambda$.

**Remark 6.1.2.** It follows from Proposition 5.2.5 that (H4)(c) is redundant if $F$ is algebraically closed.

6.1.3. (The construction) Let

$$F = \text{fu}(X, \xi) \quad \text{and} \quad S = \text{su}(X, \xi),$$

so $S \subseteq F$.

To define gradings on $F$ and $S$, and to study the properties of $F$ and $S$, we select an $A$-basis for $X$. Indeed, by Theorem 5.2.3(a) and (b) and by (H4)(a) and (b), we can choose a homogeneous $A$-basis $\{x_i\}_{i \in I}$ for $X$ satisfying the following properties:

$X = X_{\text{hyp}} \perp X_{\text{an}}$ is a Witt decomposition of $\xi$ with

$$X_{\text{hyp}} = \bigoplus_{i \in J} x_i A \quad \text{and} \quad X_{\text{an}} = \bigoplus_{i \in K} x_i A,$$

where $I = J \cup K$ with $J = \{1, \ldots, 2r\}; \deg_A(x_i) = 0$ for $i \in J$ and $\deg_A(x_{k_0}) = 0$ for some $k_0 \in K$; and

$$\xi(x_i, x_j) = \gamma_i \delta_{ij}$$

for $i, j \in I$, where $i \mapsto \bar{i}$ is the permutation of $I$ defined by

$$\bar{i} = 2r + i - 1 \quad \text{for} \quad i \in J \quad \text{and} \quad \bar{i} = i \quad \text{for} \quad i \in K,$$

and where $\gamma_i$ is nonzero and homogeneous in $A$ for $i \in I$ with

$$\gamma_i = 1 \quad \text{for} \quad i \in J \cup \{k_0\}.$$  \hspace{1cm} (56)

We note that $J, K, X_{\text{hyp}}, X_{\text{an}}, k_0$, the permutation $i \mapsto \bar{i}$ of $I$, and the set $\{\gamma_i\}_{i \in I}$ are completely determined by this choice of $A$-basis. We call $\{x_i\}_{i \in I}$ a compatible $A$-basis for $\xi$.

We let

$$\rho_i = \deg_A(x_i) \quad \text{for} \quad i \in I,$$  \hspace{1cm} (57)
Thus, $\Delta$ is a root system of type $BC_r$ and the ($Q \times \Lambda$)-graded Lie algebra $\mathfrak{s}$ is centreless and the ($Q \times \Lambda$)-graded unitary Lie algebra $\mathfrak{s} = \mathfrak{su}(X, \xi)$ constructed in Section 4.2, and as in that section, we let $\mathfrak{s}$ be a ($Q \times \Lambda$)-graded subalgebra of $\mathfrak{g}$ with root lattice $Q = \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_r$.

The natural action of $\mathfrak{h}$ on $X$ gives the weight space decomposition,

$$X = \bigoplus_{\mu \in Q} X_\mu$$

with $\text{supp}_Q(X) = \{0\} \cup \{e_i \mid i \in J\}$ and $X_{e_i} = x_i \mathcal{A}$ for $i \in J$ and $X_0 = X_{\text{an}}$.

It is clear that this grading is compatible with the given $\Lambda$-grading on $X$, so $X$ is ($Q \times \Lambda$)-graded with

$$X_\mu^\sigma = X_\mu \cap X^\sigma$$

for $\mu \in Q$, $\sigma \in \Lambda$. The ($Q \times \Lambda$)-grading on $X$ induces a ($Q \times \Lambda$)-grading on $\mathcal{F}$ with

$$\mathcal{F}^\nu = \bigoplus_{\rho + \sigma = \tau} \sum_{\lambda + \mu = \nu} U(X_\lambda^\rho, X_\mu^\sigma),$$

for $\nu \in Q$, $\tau \in \Lambda$ (see 6.1.4). Also, since $\mathfrak{s}$ is generated by homogeneous elements of $\mathcal{F}$, it follows that $\mathfrak{s}$ is a ($Q \times \Lambda$)-graded subalgebra of $\mathcal{F}$. The $\Lambda$-gradings on $\mathfrak{s}$ and $\mathfrak{g}$ are induced by the $\Lambda$-grading on $X$, so they do not depend on our choice of compatible basis; and the $Q$-gradings on these algebras are the root space decompositions relative to the adjoint action of $\mathfrak{h}$ (as in Section 4.2).

6.1.4. We will see in Theorem 6.6.1 that the graded Lie algebras $\mathcal{F}$ and $\mathfrak{s}$ just described are independent up to bi-isomorphism of the choice of a compatible $\mathcal{A}$-basis $\{x_i\}_{i \in I}$ for $\xi$. We will also see in Theorem 6.3.1 (a) that $\mathfrak{s}$ is a centreless Lie $\Lambda$-torus of type $BC_r$. In anticipation of these results, we adopt the following terminology:

**Definition 6.1.5.** The ($Q \times \Lambda$)-graded Lie algebra $\mathfrak{s} = \mathfrak{su}(X, \xi)$ constructed in 6.1.3 is called the *unitary Lie $\Lambda$-torus* (of type $BC_r$) constructed from $\xi$.

6.2. **Properties of unitary Lie tori.**

Our assumptions throughout this section are those of 6.1.1. We fix a compatible $\mathcal{A}$-basis $\{x_i\}_{i \in I}$ for $\xi$ and use the notation of 6.1.3. We record some properties of the ($Q \times \Lambda$)-graded Lie algebras $\mathcal{F}$ and $\mathfrak{s}$ as well as of the $\Lambda$-graded $\mathcal{A}$-module $X$.

We have the assumptions of Section 4.2, so we can construct $BC_r$-graded unitary Lie algebra $\mathfrak{d} = \mathfrak{bu}(X, \xi)$ as in that section. Also, we have the assumptions of Section 5.3, so we can apply the results of that section.

**Proposition 6.2.1.**
(a) \( S = \{ T \in \mathcal{F} \mid \text{tr}(T) \equiv 0 \pmod{[A, A]} \} \).
(b) \( S \) is generated as an algebra by \( \sum_{\xi \in J} S_{\xi} \). Consequently \( S = B \).
(c) \( Z(S) = 0 \).

Proof. Parts (a) and (c) are proved in Theorem 5.3.5 (a) and (c). For (b), we have by Proposition 4.2.7 (c) that \( U(X_{\text{hyp}}, X_{\text{an}}) = U(\sum_{\xi \in J} x_{\xi}, A, X_{\text{an}}) = \sum_{\xi \in J} S_{\xi} \). Thus, Theorem 5.3.5 (b) gives us the first statement of (b). The second assertion in (b) then follows from the definition of \( B = \mathfrak{hu}(X, \xi) \). \( \square \)

Next we describe some properties of the groups \( \Lambda \) and \( L \), the support sets for \( X \), \( X_{\text{hyp}} \) and \( X_{\text{an}} \), and the modules \( X \) and \( X_{\text{an}} \).

**Lemma 6.2.2.**

(a) \( L = \Lambda_{\text{hyp}} \subseteq \Lambda_{\text{an}} \), \( \text{supp}_X(\Lambda) = \Lambda_{\text{an}} \) and \( (\Lambda_{\text{an}}) = \Lambda \).
(b) \( 2\Lambda_{\text{an}} \subseteq L_+ \) and \( 2\Lambda \subseteq L \subseteq \Lambda \).
(c) \( \Lambda_{\text{an}} = \sum_{i \in \mathbb{N}} (\rho_i + L) = \sum_{i \in \mathbb{N}} (-\rho_i + L) \).
(d) If \( \Lambda \) is free of rank \( n \), then \( L \) is free of rank \( n \).
(e) \( \text{rank}_A(\Lambda_{\text{an}}) = |\Lambda_{\text{an}}/L| \) and \( \text{rank}_A(X) = 2r + \text{rank}_A(\Lambda_{\text{an}}) \).
(f) \( \text{rank}_A(X) \) is finite if and only if \( |\Lambda/L| \) is finite.
(g) If \( \Lambda \) is finitely generated, then \( \text{rank}_A(X) \) is finite.

Proof. (a) The first statement follows from (H4) (a) and (b); the second statement follows from the first; and then the third statement follows from (H5).
(b) If \( \sigma \in \Lambda_{\text{an}} \), then there exists a nonzero \( x \in X_{\text{an}}^* \), so \( 0 \neq \xi(x, x) \in A_{2\sigma}^2 \) and hence \( 2\sigma \in L_+ \). So \( 2\Lambda_{\text{an}} \subseteq L_+ \); hence \( 2\Lambda \subseteq L \) since \( (\Lambda_{\text{an}}) = \Lambda \).
(c) This is clear since \( \{ x_{\xi} \}_{\xi \in \mathbb{N}} \) is an \( A \)-basis for \( X_{\text{an}} \). (We know that \( \rho_i + L = -\rho_i + L \) for \( i \in \mathbb{N} \) since \( 2\Lambda \subseteq L \).)
(d) follows from (b).
(e) follows from (27) since \( X_{\text{an}} \) is finely graded.
(f) The group \( \Lambda/L \) is generated by the set \( \Lambda_{\text{an}}/L \) by (a). Hence, since \( \Lambda/L \) has exponent \( 2 \) by (b), we see that \( \Lambda/L \) is finite if and only if \( \Lambda_{\text{an}}/L \) is finite. The claim now follows from (e).
(g) Since \( \Lambda/L \) has exponent \( 2 \), (g) follows from (f). \( \square \)

The next example, which is a special case of the construction in [Y3, §7], shows that \( X \) may have infinite rank over \( A \).

**Example 6.2.3.** Suppose \( \Lambda \) is an abelian group and \( L \) is a subgroup of \( \Lambda \) containing \( 2\Lambda \). Let \( A = F[L] = \bigoplus_{\sigma \in L} F^\sigma \) (the group algebra of \( L \)) with the identity involution \( -; \) let \( X = A^{2r} \oplus F[\Lambda] \) with the natural \( A \)-grading and the natural right \( A \)-module structure; and let \( \xi : X \times X \rightarrow A \) be the unique symmetric \( A \)-bilinear form such that \( \xi(e_i, e_j) = \delta_{ij} \), where \( e_1, \ldots, e_{2r} \) is the standard basis for \( A^{2r} \), \( \xi(A^{2r}, F[\Lambda]) = 0 \), and \( \xi(t^\sigma, t^\tau) = \delta_{\sigma+L, \tau+L}^{\sigma+\tau} \) for \( \sigma, \tau \in \Lambda \). Then Assumptions 6.1.1, with \( X_{\text{hyp}} = A^{2r} \) and \( X_{\text{an}} = F[\Lambda] \), are easily verified. If \( \Lambda/L \) is infinite, \( X \) has infinite rank over \( A \).

If \( \mu \in \Delta \) and \( \sigma \in \Lambda \), then \( F^\sigma_{\mu} = S^\sigma_{\mu} \) by Proposition 4.2.6 (a). We now describe these spaces explicitly.

**Proposition 6.2.4.** Let \( \mu \in \Delta \) and \( \sigma \in \Lambda \). In each of the following cases, a general element of \( S^\sigma_{\mu} \) has a unique expression in the indicated form. Moreover, if \( \mu \) and \( \sigma \) are not covered by any of the cases below, then \( S^\sigma_{\mu} = 0 \).
(a) \( \mu = \epsilon_i + \epsilon_j, \quad 1 \leq i < j \leq 2r, \quad j \neq i, \quad \sigma \in L: \quad U(x_i, \alpha, x_j), \quad \alpha \in A^\sigma \\
(b) \quad \mu = 2\epsilon_i, \quad 1 \leq i \leq 2r, \quad \sigma \in L_-.\)
(c) \( \mu = \epsilon_i, \quad 1 \leq i \leq 2r, \quad \sigma \in \Lambda_{an}: \quad U(v, x_i), \quad v \in X^\sigma_{an}. \)

**Proof.** This follows from Proposition 4.2.7 and the definition of the \( \Lambda \)-grading on \( F \) (see 4.1.4).

6.3. **Statement of the structure theorem.**

We now state our first main theorem, a structure theorem for centreless Lie tori of type BC\(_r\), \( r \geq 3 \).

**Theorem 6.3.1 (Structure theorem).** Let \( F \) be a field of characteristic 0.

(a) Suppose that \( r \geq 1 \), \( \Lambda \) is an abelian group and \( L \) is a subgroup of \( \Lambda \). Assume that \( (A, -) \) is an associative \( L \)-torus with involution, that \( \xi: X \times X \to A \) is a nondegenerate \( L \)-graded hermitian form of finite graded \( F \)-dimension and Witt index \( r \) over \( (A, -) \), and that \( (H4) \) and \( (H5) \) in 6.1.1 hold. Let \( S = su(X, \xi) \) be the special unitary Lie algebra of \( \xi \) with a \((Q \times \Lambda)\)-grading defined using a compatible \( A \)-basis for \( \xi \) as in 6.1.3. Then \( S \) is a centreless Lie \( \Lambda \)-torus of type \( \Delta \), where \( \Delta \) is the root system of type BC\(_r\) defined by (60).

(b) Conversely, if \( r \geq 3 \), then any centreless Lie \( \Lambda \)-torus of type BC\(_r\) is bi-isomorphic to a centreless Lie \( \Lambda \)-torus \( S \) constructed as in (a).

6.4. **Proof of part (a) of the structure theorem.**

**Proof.** Under the assumptions of (a) of Theorem 6.3.1, we see that \( S \) is centreless by Proposition 6.2.1(c). Thus, what is required to be shown is that \( S \) satisfies (LT1)-(LT4) of Definition 3.3.1.

For (LT1), we have seen in Section 4.2 that \( S_\mu = 0 \) for \( \mu \in Q \setminus (\Delta \cup \{0\}) \). For (LT2)(i), we know by Proposition 6.2.4 that \( S_\mu^0 \neq 0 \) for \( \mu \in \Delta \). From Proposition 6.2.1(b) we see that (LT3) holds. For (LT4), observe that \( \text{supp}_A(X) = \Lambda_{an} \subseteq \text{supp}_A(S) \) by Lemma 6.2.2(a) and Proposition 6.2.4(c). So since \( X \) has full support in \( \Lambda \) by \( (H5) \), we have (LT4).

It remains to confirm that (LT2)(ii) holds. For this, recall that \( \mu^\vee \in (\mathfrak{h}^*)^* = \mathfrak{h} \) denotes the coroot of \( \mu \) for \( \mu \in \Delta \). Therefore,

\[
\epsilon_i^\vee = 2h_i, \quad (2\epsilon_i)^\vee = h_i, \quad \text{for } 1 \leq i \leq r, \\
(\epsilon_i - \epsilon_j)^\vee = h_i - h_j, \quad (\epsilon_i + \epsilon_j)^\vee = h_i + h_j \quad \text{for } 1 \leq i < j \leq r, \quad (62)
\]

and \( (-\mu)^\vee = -\mu^\vee \) for \( \mu \in \Delta \). Now \( [\mu^\vee, x_\nu^\tau] = \langle \nu | \mu^\vee \rangle x_\nu^\tau \) for \( x_\nu^\tau \in S_{\nu^\tau}^\sigma, \nu \in Q, \tau \in \Lambda \).

Thus, since \( Z(S) = 0 \), (LT2)(ii) is equivalent to the following statement: If \( \mu \in \Delta, \quad \sigma \in \Lambda, \) and \( S_\mu^\sigma \neq 0 \), then \( S_\mu^\sigma = F\epsilon_{\mu^\vee}^\sigma \) and \( S_{-\mu}^\sigma = Ff_{\mu^\vee}^\sigma \), where

\[
[e_{\mu^\vee}^\sigma, f_{\mu^\vee}^\sigma] = \mu^\vee.
\]

To verify that this is true, suppose that \( \mu \in \Delta, \quad \sigma \in \Lambda \) and \( S_\mu^\sigma \neq 0 \). Since \( X_{an} \) is finely graded by \( (H4)(c) \), it follows from Proposition 6.2.4 that \( S_\mu^\sigma \) and \( S_{-\mu}^\sigma \) are one-dimensional. Hence, it suffices to prove that

\[
\mu^\vee \in [S_\mu^\sigma, S_{-\mu}^\sigma]. \quad (63)
\]

For this, we may assume that \( \mu \) is one of the following: \( \epsilon_i, 2\epsilon_i, \) where \( 1 \leq i \leq r, \) or \( \epsilon_i - \epsilon_j, \epsilon_i + \epsilon_j, \) where \( 1 \leq i < j \leq r \). In each of these cases, one can check (63) directly using Proposition 6.2.4 and (62). We present only one argument, the
others being similar. Suppose that µ = εi, where 1 ≤ i ≤ r. Then, by Proposition 6.2.4(c), σ ∈ Λan. So we may choose 0 ≠ v ∈ Xan and take 0 ≠ α = ξ(v, v) ∈ A2r. Then U(v, α−1, x) ∈ Sµ, U(v, x−1, x) ∈ S−µ, and since α = we have
\[ [U(v, x), U(v, x−1, x)] = U(v, x)(v, x−1, x) + U(v, x−1, x)U(v, x)x]\n= U(−x, x) + U(v, x−1, v) = −U(x, x) + 0 = −1/2 v. □

6.5. Proof of part (b) of the structure theorem.

Proof. In order to prove part (b) of Theorem 6.3.1, we assume in this section that
\[ L = \bigoplus_{µ ∈ QL, σ ∈ Λ} L^σ_µ \]
is a centreless Lie Λ-torus of type Dn, where Dn is a root system of type BCr, r ≥ 3, and QL is the root lattice of Dn. For convenience we set J = {1, ..., 2r} and write \( \bar{i} = 2r + 1 - i \) for \( i ∈ J \).

6.5.1. (Preparation) By Proposition 3.4.1(f) and (g), L is a Dn-graded Lie algebra with grading pair \( (g, h) \), where
\[ g_L = L^0 \quad \text{and} \quad h_L = L^0. \] (64)

As in Section 3.4, we identify Dn with a root system in h^r.

By Theorem 4.2.4(b), there exists an associative algebra with involution \( (A, −) \); a hermitian form \( ξ : X × X → A \) over \( (A, −) \) with \( X = X_{hyp} ⊥ X_{an} \); an \( A \)-basis \( \{x_i\}_{i=1}^{2r} \) for \( X_{hyp} \) with \( ξ(x_i, x_j) = δ_{ij} \) for all \( i, j ∈ J \); an element \( v_0 ∈ X_{an} \) with \( ξ(v_0, v_0) = 1 \); and an isomorphism
\[ ψ : L → B/Z(B), \]
where \( B = bu(X, ξ) \). Moreover, \( ψ \) may be chosen so that
\[ ψ(g_L) = π(g) \quad \text{and} \quad ψ(h_L) = π(h), \]
where \( π : B → B/Z(B) \) is the canonical projection,
\[ g = \left( \bigoplus_{i,j ∈ J, i < j} F U(x_i, x_j) \right) + \left( \bigoplus_{i ∈ J} F U(x_i, v_0) \right), \quad h = \bigoplus_{i=1}^{r} F h_i. \] (65)

and \( h_i = U(x_i, x_i), 1 ≤ i ≤ r, \) in \( B \).

6.5.2. (The root gradings) Recall that L, as a Lie torus, comes equipped with a grading by the root lattice \( Q_L \) of the root system Dn in h^r. On the other hand, in Section 4.2 we gave B a grading by the root lattice \( Q = Q(Δ) = Zε_1 ⊕ · · · ⊕ Zε_r \) of \( Δ \), where
\[ Δ = \{ε_i \mid i ∈ J\} ∪ \{ε_i + ε_j \mid i, j ∈ J, j ≠ \bar{i}\}, \]
\( ε_1, ..., ε_r \) is the dual basis in \( h^* \) of \( h_1, ..., h_r \), and \( ε_i = −ε_i \) for \( 1 ≤ i ≤ r \). So as noted in (3.2.2), \( π(B) \) is also graded by \( Q \). We now check that \( ψ \) is an isograded-isomorphism of the \( Q_L \)-graded Lie algebra L onto the \( Q \)-graded Lie algebra \( π(B) \).

We identify \( π(h)^* \) with \( h^* \) as in Remark 4.2.5, and so \( Δ \) and \( Q \) lie in \( π(h)^* \). Next let \( ψ : h_L^* → π(h)^* \) denote the inverse dual of \( ψ|_{h_L} : h_L → π(h) \). Now, the \( Q_L \)-grading of L is the root space decomposition relative to \( h_L \) by (1), and the \( Q \)-grading of \( π(B) \) is the root space decomposition relative to \( π(h) \) by Remark 4.2.5. So it follows from the definition of \( ψ \) that
\[ ψ(L_µ) = π(B)_{\tilde{ψ}(µ)}. \]
for $\mu \in \mathfrak{h}_C^*$. Hence, we have
\[ \hat{\psi}(\text{supp}_{Q_L}(L)) = \text{supp}_{Q}(\pi(B)). \] (66)
But $\text{supp}_{Q_L}(L)$ generates $Q_L$ as a group by (LT1) and (LT2)(i), and $\text{supp}_{Q}(\pi(B))$ generates $Q$ as a group by Proposition 4.2.7. So
\[ \hat{\psi}(Q_L) = Q. \]
Thus, $\psi$ is an isograded-isomorphism of the $Q_L$-graded Lie algebra $L$ onto the $Q$-graded Lie algebra $\pi(B)$, as desired.

In addition, $\text{supp}_{Q_L}(L)$ is either $\Delta_L \cup \{0\}$ or $(\Delta_L)_{\text{ind}} \cup \{0\}$ by Remark 3.3.2 (b); whereas $\text{supp}_{Q}(\pi(B))$ is either $\Delta \cup \{0\}$ or $\Delta_{\text{ind}} \cup \{0\}$ by Proposition 4.2.7. Thus, since $\Delta_L$ and $\Delta$ are each root systems of type $BC_r$, it follows from (66) that
\[ \hat{\psi}(\Delta_L) = \Delta. \]

6.5.3. (Transfer of the $\Lambda$-grading) Now $L$, as a Lie $\Lambda$-torus, has a given $\Lambda$-grading which is compatible with the $Q_L$-grading. We use $\hat{\psi}$ to transfer the $\Lambda$-grading from $L$ to a $\Lambda$-grading on $\pi(B)$ which is compatible with the $Q$-grading on $\pi(B)$.

Henceforth, for convenience, we use $\psi$ and $\hat{\psi}$ to make the following identifications:
\[ L = \pi(B), \quad g_L = \pi(g), \quad h_L = \pi(h), \quad Q_L = Q, \quad \text{and} \quad \Delta_L = \Delta. \]
(These identifications are allowed since we are working up to bi-isomorphism.) Note in particular that $L$ and $\pi(B)$ are identified as $(Q \times \Lambda)$-graded Lie algebras. Also, since $L^0 = g_L = \pi(g)$, it follows from (64) and (65) that
\[ \pi(U(x_i, x_j)) \in L^0 \quad \text{and} \quad \pi(U(x_k, v_k)) \in L^0 \]
for $i, j \in J$, $i \neq j$, $k \in J$.

6.5.4. (Support sets) Recall that in Section 3.4 we defined $\Lambda_\mu = \text{supp}_\Lambda(L_\mu)$ for $\mu \in \Delta$. We set
\[ S = \Lambda_{\varepsilon_k} \quad \text{for} \quad k \in J \quad \text{and} \quad L = \Lambda_{\varepsilon_i + \varepsilon_j} \quad \text{for} \quad i, i, j \text{ distinct in } J. \]
By Proposition 3.4.2, these sets are well defined, and we have
\[ 0 \in S, \quad -S = S, \quad 0 \in L, \quad -L = L \]
and
\[ (S) = \Lambda. \]
Furthermore, taking $\nu = \varepsilon_1 + \varepsilon_2$ and $\mu = \varepsilon_1$ in Lemma 3.4.2 (c), we have $L - 2S \subseteq L$, and so
\[ 2S \subseteq L. \]

6.5.5. (Canonical forms for elements of $L_\mu$) Now if $i, j \in J$ with $j \neq i, \bar{i}$, then, by Proposition 4.2.7, elements of $B_{\varepsilon_i + \varepsilon_j}$ can be uniquely expressed in the form $U(x_i, \alpha, x_j)$ where $\alpha \in A$. So we define
\[ U_{ij}(\alpha) = \pi(U(x_i, \alpha, x_j)) \in L_{\varepsilon_i + \varepsilon_j} \]
for $\alpha \in A$ and $i, j \in J$ with $j \neq i, \bar{i}$ (but we do not define $U_{ii}(\alpha)$ and $U_{ii}(\alpha)$). Since $\ker(\pi) = Z(B) \subseteq B_0$, Proposition 4.2.7 tells us that a general element of $L_{\varepsilon_i + \varepsilon_j}$ has a unique expression
\[ U_{ij}(\alpha), \quad \alpha \in A. \]
Similarly if we define
\[ U_i(v) = \pi(U(v, x_i)) \in \mathcal{L}_{\epsilon_i} \]
for \( i \in J \) and \( v \in X_{\text{an}} \), then a general element of \( \mathcal{L}_{\epsilon_i} \) has a unique expression of the form
\[ U_i(v), \quad v \in X_{\text{an}}. \]
(We could also introduce unique expressions for elements of \( \mathcal{L}_{2\epsilon_i} \), but that is not needed here.)

If \( \alpha, \beta \in A, \ v, w \in X \) and \( i, \bar{i}, j, \bar{j}, k, \bar{k} \) are distinct in \( J \), we obtain the following identities by applying \( \pi \) to the identities in Proposition 4.2.8:
\[ \begin{align*}
U_{ij}(\alpha) &= -U_{ji}(\alpha) \\
[U_{ij}(\alpha), U_{jk}(\beta)] &= U_{ik}(\alpha \beta) \\
[U_{ij}(\alpha), U_i(v)] &= -U_j(v \alpha), \\
[U_i(v), U_j(w)] &= -U_{ij}(\xi(v, w)).
\end{align*} \]
(70) (71) (72) (73)

6.5.6. (The \( \Lambda \)-grading on \( A \)) Now \( \mathcal{L}_{\epsilon_1+\epsilon_2} \) is a finely \( \Lambda \)-graded vector space with support \( L \), and, by 6.5.5, the map \( \alpha \mapsto U_{ij}(\alpha) \) is a linear bijection of \( A \) onto \( \mathcal{L}_{\epsilon_1+\epsilon_2} \). Hence, there is a unique \( \Lambda \)-grading on the vector space \( A \) such that
\[ U_{12}(A^\sigma) = \mathcal{L}_{\epsilon_1+\epsilon_2}^\sigma \]
for \( \sigma \in \Lambda \); and with respect to this grading, \( A \) is finely graded with support \( L \).

Next we argue that
\[ U_{ij}(A^\sigma) = \mathcal{L}_{\epsilon_i+\epsilon_j}^\sigma \]
(74)
for \( i, \bar{i}, j, \bar{j} \) distinct and \( \sigma \in \Lambda \). Indeed, since \( \mathcal{L}_{\epsilon_i+\epsilon_j} = \bigoplus_{\sigma \in \Lambda} U_{ij}(A^\sigma) \) and \( \mathcal{L}_{\epsilon_i+\epsilon{j}} = \bigoplus_{\sigma \in \Lambda} \mathcal{L}_{\epsilon_i+\epsilon{j}}^\sigma \), it is sufficient to show that
\[ U_{ij}(A^\sigma) \subseteq \mathcal{L}_{\epsilon_i+\epsilon_j}^\sigma \]
(75)
for \( i, \bar{i}, j, \bar{j} \) distinct and \( \sigma \in \Lambda \). But by (67), we know that \( U_{ij}(1) \subseteq \mathcal{L}_{\epsilon_j+\epsilon_i}^0 \) for \( j \neq i, \bar{i} \). Thus, using (71), we have
\[ U_{ik}(A^\sigma) = [U_{ij}(A^\sigma), U_{jk}(1)] \subseteq [U_{ij}(A^\sigma), \mathcal{L}_{\epsilon_j+\epsilon_i}^0] \]
for \( i, \bar{i}, j, \bar{j}, k, \bar{k} \) distinct. Hence, if (75) holds for the pair \( (i, j) \), it also holds for \( i, k \), provided that \( i, \bar{i}, j, \bar{j}, k, \bar{k} \) are distinct. Similarly from (71) we see that if (75) holds for the pair \( (i, j) \), it also holds for the pair \( (k, j) \), provided that \( i, \bar{i}, j, \bar{j}, k, \bar{k} \) are distinct. Since (75) holds for the pair \( (1, 2) \), it now follows easily that it holds for all pairs \( (i, j) \) with \( i, \bar{i}, j, \bar{j} \) distinct.

Next, if \( \sigma, \tau \in \Lambda \), we have using (71) and (74) that
\[ U_{12}(A^\sigma A^\tau) = [U_{13}(A^\sigma), U_{32}(A^\tau)] = [\mathcal{L}_{\epsilon_1+\epsilon_3}^\sigma, \mathcal{L}_{\epsilon_3+\epsilon_2}^\tau] \subseteq \mathcal{L}_{\epsilon_1+\epsilon_2}^{\sigma+\tau} = U_{12}(A^{\sigma+\tau}), \]
and hence \( A^\sigma A^\tau \subseteq A^{\sigma+\tau} \). Thus, \( A \) is a finely \( \Lambda \)-graded associative algebra.

Using (70) and (74), we conclude that \( U_{12}(A^\overline{\sigma}) = U_{21}(A^\sigma) = \mathcal{L}_{\epsilon_2+\epsilon_1}^\sigma = U_{12}(A^\sigma) \)
for \( \sigma \in \Lambda \). Thus, the involution \( -\cdot \) on \( A \) is graded.

6.5.7. (\( A, - \) is an associative \( L \)-torus with involution) We have already established that the algebra \( (A, -) \) is a finely \( \Lambda \)-graded associative algebra with involution and that \( \text{supp}_A(A) = L \). To show that \( (A, -) \) is an associative \( L \)-torus with involution, it remains to show that \( A^\sigma \) is spanned by an invertible element for \( \sigma \in L \), since if that is true, it follows that \( L \) closed under addition and also that \( L \) is a subgroup of \( \Lambda \) using (68). Let \( \sigma \in L \). Then, \( -\sigma \in L \) and \( 0 \in L \) by (68), and thus
graded $F$ and that there exists a nonzero element $Q$ identified $L$ relative to the adjoint action of $X$. Hence, there is a unique $\Lambda$-grading on the vector space $X$ such that $U_1(X) = L_{\xi_1}$ for $\sigma \in \Lambda$; and with respect to this grading, $X$ is finely graded with support $S$. Also, arguing as in 6.5.6, we see that Assumptions 6.1.1 hold, all that remains is to show that (H4)(a), (H4)(c) and (H5) hold.

6.5.8. (The $\Lambda$-grading on $X$) Now $L_{\xi_1}$ is a finely $\Lambda$-graded vector space with support $S$, and, by 6.5.5, the map $v \mapsto U_1(v)$ is a linear bijection of $X$ onto $L_{\xi_1}$. Hence, there is a unique $\Lambda$-grading on the vector space $X$ such that

$$U_1(X) = L_{\xi_1}$$

for all $i \in J$, $\sigma \in \Lambda$, and also that $X_{\sigma} \Lambda \tau \subseteq X_{\sigma + \tau}$ for $\sigma, \tau \in \Lambda$. Hence, $X$ is a finely $\Lambda$-graded $\Lambda$-module. Note also that $U_1(v_0) = L_{\xi_1}$ by (67), so $v_0 \in X$.

To obtain a $\Lambda$-grading on the $\Lambda$-module $X$, let $\deg_\Lambda(x, \alpha) = \deg_\Lambda(\alpha) = \sigma$ for $i \in J$, $\alpha \in X_{\sigma}$ and $\sigma \in L$.

Finally, we give the $\Lambda$-module $X$ the direct sum grading with $X_{\sigma} = X_{\sigma, hyp} \oplus X_{\sigma, hyp}$ for $\sigma \in \Lambda$. It is clear that $X$ has finite graded $F$-dimension since this is true for both $X_{\sigma, hyp}$ and $X$

6.5.9. ($\xi$ is $\Lambda$-graded) Equation (73) tells us that $\xi(X_{\sigma} \Lambda \tau, X_{\sigma} \Lambda \tau) \subseteq \Lambda^{\sigma + \tau}$ for $\sigma, \tau \in \Lambda$. So the hermitian form $\xi$ on $X$ is $\Lambda$-graded. Also, if $\alpha \in \Lambda^{\sigma}$ and $\beta \in \Lambda^{\tau}$, then $\xi(x, \alpha, x, \beta) = \delta_{ij} \sigma \beta \in \Lambda^{\sigma + \tau}$, which shows that the form $\xi$ is $\Lambda$-graded on $X_{\sigma, hyp}$. Thus, $\xi$ is $\Lambda$-graded on $X$.

6.5.10. (The Witt decomposition of $\xi$) First we argue that $\xi$ is anisotropic on $X$. Let $v \in X_{\sigma}$ be nonzero for some $\sigma \in \Lambda$. Then, $\sigma \in S$ and $2\sigma \in L$ by (69), so by Lemma 3.4.3 we know that $[L_{\xi_1}, L_{\xi_2}] = L_{\xi_1 + \xi_2}$. But by (73) we have

$$[U_1(v), U_2(v)] = U_1(U_2(\xi(v, v)))$$

Thus, $\xi(v, v) \neq 0$, and the form on $X$ is anisotropic.

It follows that $\xi$ is nondegenerate. But $X_{\sigma, hyp}$ is certainly a hyperbolic space, so $X = X_{\sigma, hyp} \perp X_{\sigma}$ is a Witt decomposition of $\xi$. Therefore, the Witt index of $\xi$ is $r$.

6.5.11. (Assumptions 6.1.1 hold) We have argued that $(A, -)$ is an associative $L$-torus with involution; $\xi$ is a nondegenerate $\Lambda$-graded hermitian form of finite graded $F$-dimension with Witt decomposition $X = X_{\sigma, hyp} \perp X_{\sigma}$ and Witt index $r$; and that there exists a nonzero element $v_0 \in X_{\sigma}$ such that $\xi(v_0, v_0) = 1$. In order to see that Assumptions 6.1.1 hold, all that remains is to show that (H4)(a), (H4)(c) and (H5) hold. But the first two of these are clear. Also since $(\text{supp}_\Lambda(X_{\sigma})) = \langle S \rangle = \Lambda$, it follows that $X_{\sigma}$ has full support in $\Lambda$, and thus (H5) holds.

6.5.12. (Conclusion of the proof of Theorem 6.3.1(b)) In 6.5.2, we identified $L$ and $\pi(B)$ as $(Q \times \Lambda)$-graded Lie algebras. We have just demonstrated that the assumptions in 6.1.1 hold. Hence, by Proposition 6.2.1(b) and (c), we have $B = S$ and $Z(B) = 0$. So with the obvious identification $\pi(B) = B$, we have

$$L = \pi(B) = B = S$$

as $(Q \times \Lambda)$-graded Lie algebras.

Finally, we know that the $Q$-grading on $S$ is the root space decomposition of $S$ relative to the adjoint action of $\mathfrak{h}$ (see Section 6.1). So all that remains to be...
shown is that the $\Lambda$-grading on $S$ (which was obtained by transferring the given grading on $L$ to $S$ via the identification in 6.5.3) coincides with the $\Lambda$-grading on $S$ induced by the $\Lambda$-grading on $X$. But both of these $\Lambda$-gradings are compatible with the $Q$-grading; and by Proposition 6.2.1 (b), the Lie algebra $S$ is generated by $\sum_{i=1}^{2r} S_{\varepsilon_i}$. Hence, it suffices to verify that the two $\Lambda$-gradings agree on $S_{\varepsilon_i}$ for $i \in J$. But this is a consequence of (76) and Proposition 6.2.4 (c). □

6.6. The bi-isomorphism theorem.

Our second main theorem gives necessary and sufficient conditions for two unitary Lie tori of type BC\(_r\), \(r \geq 3\), to be bi-isomorphic. Since \(r\) is a bi-isomorphism invariant, we can for convenience fix \(r\) in the discussion.

Theorem 6.6.1 (Bi-isomorphism theorem). Assume $\mathbb{F}$ has characteristic 0. Suppose that \(r, \Lambda, L, (A, -)\) and $\xi : X \times X \to A$ (resp. \(r', r, \Lambda, L', (A', -)\) and $\xi' : X' \times X' \to A'$) satisfy (H1)–(H5) in 6.1.1 and that $\mathcal{F}$ and $S$ (resp. $\mathcal{F}'$ and $S'$) are the graded Lie algebras constructed from $\xi$ (resp. $\xi'$) as in 6.1.3 using a compatible $A$-basis for $\xi$ (respectively $\xi'$).

(a) If there exists a triple $(\theta, \theta_1, \theta_2)$ of maps, where $\theta : X \to X'$ is an $\mathbb{F}$-linear isomorphism, $\theta_1 : (A, -) \to (A', -)$ is an isomorphism of algebras with involution, and $\theta_2 : \Lambda \to \Lambda'$ is a group isomorphism, which satisfy

\[
\begin{align*}
\theta(x, \alpha) &= \theta(x), \theta(\alpha) \\
\theta_1(\xi(x, y)) &= \xi'(\theta(x), \theta(y)) \\
\theta_2(A^\sigma) &= A'^{\theta_2(\sigma)} \quad \text{and} \quad \theta(X^\sigma) = X'^{\theta_2(\sigma)}
\end{align*}
\]

for $x, y \in X$, $\alpha \in A$ and $\sigma \in \Lambda$, then $\theta_2(L) = L'$, $\mathcal{F}$ is bi-isomorphic to $\mathcal{F}'$, and $S$ is bi-isomorphic to $S'$.

(b) Conversely, if $r \geq 3$ and $S$ and $S'$ are bi-isomorphic, then there exists a triple $(\theta, \theta_1, \theta_2)$ of maps with the indicated properties.

Proof. We assume that $S$ has been constructed using a compatible $A$-basis $\{x_i\}_{i \in I}$ for $\xi$ and we adopt the notation of 6.1.3. We do the same for $\xi'$ with primed notation.

(a) Suppose we have a triple of maps $(\theta, \theta_1, \theta_2)$ as described in (a). By (79), we have $\theta_2(\text{supp}_A(A)) = \text{supp}_{A'}(A')$ so $\theta_2(L) = L'$. Next by (77)–(79),

\[X' = \theta(X_{\text{hyp}}) \oplus \theta(X_{\text{an}})\]

is a $\Lambda'$-graded orthogonal decomposition of $A'$-modules, $\theta(X_{\text{an}})$ is anisotropic, and $\{\theta(x_i)\}_{i \in J}$ is an $A'$-basis for $\theta(X_{\text{hyp}})$ relative to which the matrix of $\xi'|_{X_{\text{hyp}}}$ is the $(2r \times 2r)$-matrix $\delta_{ij}$. Then, by Theorem 5.2.3 (c), we can compose $\theta$ with a $\Lambda'$-graded $A'$-linear isometry and assume that

$\theta(X_{\text{hyp}}) = X'_{\text{hyp}}, \quad \theta(X_{\text{an}}) = X'_{\text{an}} \quad \text{and} \quad \theta(x_i) = x'_i \quad \text{for} \quad i \in J.$

Now define $\psi : \text{End}_\mathbb{F}(X) \to \text{End}_\mathbb{F}(X')$ by $\psi(T) = \theta T \theta^{-1}$. By (77), we have $\psi(E(x, y)) = E(\theta(x), \theta(y))$; hence

$\psi(U(x, y)) = U(\theta(x), \theta(y))$

for $x, y \in X$. So $\psi(\mathcal{F}) = \mathcal{F}'$. Also,

$\psi(h_i) = \psi(U(x_i, x_i)) = U(\theta(x_i), \theta(x_i)) = U(x'_i, x'_i) = h'_i$
for $1 \leq i \leq r$. So $\psi(h) = h'$, and the inverse dual $\hat{\psi} : h^* \to h'^*$ of $\psi|_h$ maps $\varepsilon_i$ to $\varepsilon'_i$ for $1 \leq i \leq r$. Thus, $\hat{\psi}(Q) = Q'$ and $\theta(X_\mu) = X'_\psi(\mu)$ for $\mu \in Q$ so that $\theta(X'_\mu) = \hat{\psi}^{\theta_2}(\sigma)$ for $\sigma \in \Lambda, \mu \in Q$. Therefore $\psi(F'_\mu) = F^{\theta_2}(\sigma)$ for $\sigma \in \Lambda, \mu \in Q$. Thus, $\psi$ is a bi-isomorphism of $F$ onto $F'$, and its restriction maps $S$ onto $S'$.

(b) For the converse, suppose that $r \geq 3$ and $\psi$ is a bi-isomorphism from $S$ to $S'$. Then, by Remark 3.3.5, $\psi_{rt} : Q \to Q'$ is an isomorphism of the root system $\Delta$ onto the root system $\Delta'$, so $\psi_{rt}(\{\varepsilon_1, \ldots, \varepsilon_{2r}\}) = \{\varepsilon'_1, \ldots, \varepsilon'_{2r}\}$. Therefore, there exists an element $\omega'$ of the Weyl group of $\Delta'$ such that $\omega'(\psi_{rt}(\varepsilon_i)) = \varepsilon'_i$ for $1 \leq i \leq 2r$. Hence, by Remark 3.4.4, we can assume that $\psi_{rt}(\varepsilon_i) = \varepsilon'_i$ for $1 \leq i \leq 2r$.

By Proposition 6.2.4(a), there exist unique linear bijections $\eta_{ij} : A \to A'$ for $i, \bar{i}, j, \bar{j}$ distinct in $J$ such that
\[
\psi(U(x_1, x_2)) = U'(x'_1, \eta_{ij}(\alpha), x'_2) \quad \text{and} \quad \eta_{ij}(A') = A'^{\theta_2(\sigma)} (80)
\]
for $\alpha \in A$ and $\sigma \in \Lambda$, where
\[
\theta_2 = \psi_{ex}.
\]
Similarly, there exist unique linear bijections $\eta_i : X_{an} \to X'_{an}$ for $i \in J$ such that
\[
\psi(U(v, x_1)) = U(\eta_i(v), x'_1) \quad \text{and} \quad \eta_i(X_{an}^\sigma) = X'_{an}^{\theta_2(\sigma)} (81)
\]
for $v \in X_{an}$ and $\sigma \in \Lambda$. Note that it follows from the second equation in (80) and in (81) that
\[
\eta_{ij}(1) = b_{ij}1 \quad \text{and} \quad \eta_i(\bar{v}_0) = c_i v'_0,
\]
where $b_{ij} \in F^*$ and $c_i \in F^*$. 

If $\alpha, \beta \in A, v, w \in X_{an}$ and $i, j, k \in J$, then applying $\psi$ to the identities in Proposition 4.2.8 yields:
\[
\eta_{ij}(\overline{\alpha}) = \eta_{ij}(\overline{\alpha}) \quad \text{if} \ i, \bar{i}, j, \bar{j} \ \text{are distinct},
\]
\[
\eta_{ij}(\alpha)\eta_{jk}(\alpha) = \eta_{ik}(\alpha\beta) \quad \text{if} \ i, \bar{i}, j, \bar{j}, k, \bar{k} \ \text{are distinct},
\]
\[
\eta_i(v)\eta_j(\alpha) = \eta_j(v, \alpha) \quad \text{if} \ i, \bar{i}, j, \bar{j} \ \text{are distinct},
\]
\[
\xi'(\eta_i(v), \eta_j(w)) = \eta_{ij}(\xi(v, w)) \quad \text{if} \ i, \bar{i}, j, \bar{j} \ \text{are distinct}.
\]

It is easy to verify now that $\theta_1 := b_{ij}^{-1}\eta_{ij}$ is independent of $i, j$; $\theta := c_i^{-1}\eta_i$ is independent of $i$; and the triple $(\theta, \theta_1, \theta_2)$ satisfies all of the required conditions, except that so far $\theta$ is only defined on $X_{an}$. (We omit the proof of these assertions since the argument is similar to the one used in the proof of Proposition 4.2.10. See also the reasoning in 6.5.6.) Finally, we extend $\theta$ to $X$, by defining $\theta : X_{hyp} \to X'_{hyp}$ by $\theta(x, \alpha) = x, \theta_i(\alpha)$ for $\alpha \in A, i \in J$. The conditions that $\theta$ must satisfy are then easily checked for this extension. \hfill \Box

6.6.2. Suppose $L$ is a centreless Lie $A$-torus of type $BC_r$, where $r \geq 3$. By Theorem 6.3.1 we know that $L$ is bi-isomorphic to a unitary Lie $A$-torus $S$ constructed from some $\xi : X \times X \to A$ over $(A, -)$. By Theorem 6.6.1, the associative $L$-torus with involution $(A, -)$ (up to isograded-isomorphism) and the $A$-rank of $X_{an}$ are bi-isomorphism invariants of $L$. We refer to $(A, -)$ as the coordinate torus and rank$_A(X_{an})$ as the anisotropic rank of $L$.

With some additional assumptions, we can give a simpler version of the bi-isomorphism theorem. For this purpose, we will use the following lemma:
Lemma 6.6.3. Suppose that $\Lambda$ is an abelian group without 2-torsion, $L$ is a subgroup of $\Lambda$ containing $2\Lambda$, and $S$ is a subset of $\Lambda$ such that $\Lambda = (L \cup S)$. Suppose further $\Lambda'$, $L'$ and $S'$ satisfy the same assumptions. Then any isomorphism of $L$ onto $L'$ mapping $2S$ onto $2S'$ extends uniquely to an isomorphism of $\Lambda$ onto $\Lambda'$ which maps $S$ onto $S'$.

Proof. The map $\sigma \mapsto 2\sigma$ is an isomorphism of $\Lambda$ onto $2\Lambda$, and we write its inverse map from $2\Lambda$ to $\Lambda$ by $\sigma \mapsto \frac{1}{2}\sigma$. We use the same notation for $\Lambda'$.

For uniqueness, suppose that $\eta_1$ and $\eta_2$ are homomorphisms from $\Lambda$ onto $\Lambda'$ that agree on $L$. Then, if $\sigma \in \Lambda$, we have $2\eta_1(\sigma) = \eta_1(2\sigma) = \eta_2(2\sigma) = 2\eta_2(\sigma)$, so $\eta_1(\sigma) = \eta_2(\sigma)$.

For existence, suppose that $\eta : L \rightarrow L'$ is an isomorphism such that $\eta(2S) = 2S'$. If suffices to show that there exists a homomorphism $\hat{\eta} : \Lambda \rightarrow \Lambda'$ such that $\hat{\eta}(S) \subseteq S'$ (since then the same argument will apply to $\eta^{-1}$). To see this, observe first that

$$\eta(2\Lambda) \subseteq 2\Lambda'.$$

Indeed, if $\sigma \in L$ we have $\eta(2\sigma) \in \eta(2L) = 2\eta(L) = 2L' \subseteq 2\Lambda'$; whereas if $\sigma \in S$ we have $\eta(2\sigma) \in \eta(2S) = 2S' \subseteq 2\Lambda'$, proving (82). Thus, we can define $\hat{\eta} : \Lambda \rightarrow \Lambda'$ by

$$\hat{\eta}(\sigma) = \frac{1}{2}\eta(2\sigma)$$

for $\sigma \in \Lambda$. Then, if $\sigma \in L$, we have $\hat{\eta}(\sigma) = \frac{1}{2}\eta(2\sigma) = \frac{1}{2}2\eta(\sigma) = \eta(\sigma)$; and if $\sigma \in S$, we have $\hat{\eta}(\sigma) = \frac{1}{2}\eta(2\sigma) \in \frac{1}{2}\eta(2S) = \frac{1}{2}2S' = S'$.

We can now prove the following corollary of Theorem 6.6.1. Note that under the assumptions of that theorem we have $2\supp_\Lambda(X) \subseteq L$ and $2\supp_\Lambda(X') \subseteq L'$ (see Lemma 6.2.2 (b)).

Corollary 6.6.4. Suppose that the hypotheses of Theorem 6.6.1 hold. In addition assume that $(\mathbb{F}^\times)^2 = \mathbb{F}^\times$ and that $\Lambda$ and $\Lambda'$ have no 2-torsion.

(a) If there exists an isograded isomorphism $\varphi$ of the $L$-graded associative algebra with involution $(A,-)$ onto the $L'$-graded associative algebra with involution $(A',-)$ such that

$$\varphi_{gr}(2\supp_\Lambda(X)) = 2\supp_{\Lambda'}(X'),$$

then $\mathcal{F}$ is bi-isomorphic to $\mathcal{F}'$ and $S$ is bi-isomorphic to $S'$. (Here we are using the notation $\varphi_{gr}$ of 2.3.1 (d).)

(b) Conversely, if $r \geq 3$ and $S$ and $S'$ are bi-isomorphic, then there exists an isograded isomorphism $\varphi$ as in (a).

Proof. We assume that $S$ has been constructed using a compatible $\mathcal{A}$-basis $\{x_i\}_{i \leq 1}$ for $\xi$ and we use the notation of 6.1.3. We make similar assumptions for $\xi'$ and $S'$ using primed notation. Note that $\mathcal{J} = \mathcal{J}' = \{1, \ldots, 2r\}$. Let $S = \supp_\Lambda(X)$ and $S' = \supp_{\Lambda'}(X')$.

(a) Our goal is to construct a triple of maps $(\theta, \theta_1, \theta_2)$, where $\theta : X \rightarrow X'$, $\theta_1 : A \rightarrow A'$ and $\theta_2 : \Lambda \rightarrow \Lambda'$ satisfy conditions (77)–(79) of Theorem 6.6.1 (a).

By Lemma 6.2.2 (a) and (b), we have the assumptions of Lemma 6.6.3 for $\varphi_{gr} : L \rightarrow L'$. Hence, there is an extension $\theta_2 : \Lambda \rightarrow \Lambda'$ of $\varphi_{gr}$ such that $\theta_2(S) = S'$. But by Lemma 6.2.2 (c), it follows that $S = \psi_{i \xi K}(\rho_i + L)$ and $S' = \psi_{i \xi' K'}(\rho'_i + L')$. So there exists a bijection $\pi : K \rightarrow K'$ such that $\theta_2(\rho_i) = \rho'_i + \sigma'_i$ where $\sigma'_i \in L'$ for
i \in K. Moreover, \( \pi(k_0) = k_0' \) and \( \sigma'_{k_0} = 0 \). We extend \( \pi \) to a bijection \( \pi : I \to I' \) by defining \( \pi(i) = i \) for \( i \in J \), and we set \( \sigma'_{i} = 0 \) for \( i \in I \). Then,
\[
\theta_2(\rho_i) = \rho'_{\pi(i)} + \sigma'_{i} \quad (83)
\]
for \( i \in I \), so
\[
\varphi_{\text{gr}}(2\rho_i) = 2\rho'_{\pi(i)} + 2\sigma'_{i}, \quad (84)
\]
for \( i \in I \). Now we choose
\[
0 \neq \beta'_i \in A^{\sigma'_{i}} \quad \text{for} \quad i \in I,
\]
with \( \beta'_i = 1 \) for \( i \in J \cup \{k_0\} \). But by (58), we have \( \gamma_i \in A^{2\rho_{i}} \) and \( \gamma'_{\pi(i)} \in A^{2\rho'_{\pi(i)}} \) for \( i \in I \). Hence, by (84) and (85), we have \( \varphi(\gamma_i) = F^{\pi}(\beta'_{i}) = F^{\pi}(\gamma'_{\pi(i)}\beta'_{i}) \) for \( i \in I \). Since by assumption \( F^{\times 2} = F^{\times} \), we can alter our choice of \( \beta'_{i} \) so that
\[
\varphi(\gamma_i) = \beta'_{i} \gamma'_{\pi(i)}\beta'_{i}. \quad (86)
\]
holds for \( i \in I \). Further, by (56), we may take \( \beta'_i = 1 \) for \( i \in J \cup \{k_0\} \), and hence
\[
\beta'_i = \beta_i, \quad (87)
\]
for \( i \in I \).

Finally, let \( \theta_1 = \varphi \), and define \( \theta : X \to X \) by
\[
\theta(\sum_{i=1}^{\ell} x_i \cdot \alpha_i) = \sum_{i=1}^{\ell} x'_{\pi(i)}(\beta'_i \varphi(\alpha_i)).
\]
It is a straightforward matter using (83)--(87) to check conditions (77)--(79) in Theorem 6.6.1 (b). We leave this to the reader.

(b) For the converse, suppose that \( \psi \) is a bi-isomorphism of \( S \) onto \( S' \). Let \( (\theta, \theta_1, \theta_2) \) be the triple promised by Theorem 6.6.1. Then, by (79), \( \varphi := \theta_1 \) is an isograded isomorphism of \( (A, -) \) onto \( (A', -) \) with \( \varphi_{\text{gr}} = \theta_2 | L \). Finally, by (79), \( \theta_2(S) = S' \), so \( \varphi_{\text{gr}}(2S) = 2S' \).

7. Lie \( n \)-tori and extended affine Lie algebras of type BC

If \( A \) is a free abelian group of finite rank \( n \), a Lie \( A \)-torus is called an \( \text{Lie} \ n \text{-torus} \). It is known that Lie \( n \)-tori are the starting point for the construction of extended affine Lie algebras (EALAs) of nullity \( n \) (see Section 7.3). With that as motivation, we apply our results from Chapter 6 to the special case of Lie \( n \)-tori to obtain a classification up to bi-isomorphism of centreless Lie \( n \)-tori of type \( \text{BC} \) for \( r \geq 3 \). We conclude by discussing the construction of EALAs from these Lie \( n \)-tori.

If \( n = 0 \), it is well known that a centreless Lie \( n \)-torus \( L \) is a finite-dimensional split simple Lie algebra (see for example [ABFP, Rem. 1.2.4]). Since that case is well understood, our focus is the case when \( n \) is a positive integer, and we make that assumption for the rest of this chapter.

7.1. Associative \( n \)-tori with involution.

If \( L \) is a free abelian group of rank \( n \), an associative \( L \)-torus with involution is called an \( \text{associative} \ n \text{-torus with involution} \). In view of Theorem 6.3.1, our first step must be to understand these graded algebras with involution. Fortunately, they have been classified by Yoshii [Y2, Thm. 2.7] using elementary quantum matrices. This classification can be formulated using quadratic forms over \( \mathbb{Z}_2 \) [AFY, AF], and we recall that point of view now.

Throughout this section, we suppose that \( L \) is a free abelian group of rank \( n \).
7.1.1. We let \( \widetilde{L} = \Lambda / 2 \Lambda \) with canonical map \( \sigma \to \tilde{\sigma} \), in which case \( \widetilde{L} \) is an \( n \)-dimensional vector space over \( \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} \). If \( \eta : \Lambda \to \Lambda' \) is an isomorphism of free abelian groups of rank \( n \), then \( \eta \) induces a vector space isomorphism \( \tilde{\eta} : \widetilde{L} \to \widetilde{L}' \) defined by \( \tilde{\eta}(\tilde{\sigma}) = \tilde{\eta}(\tilde{\sigma}) \); and any vector space isomorphism from \( \widetilde{L} \) to \( \widetilde{L}' \) is induced by some \( \eta \) in this way.

7.1.2. Suppose that \( \kappa : \widetilde{L} \to \mathbb{Z}_2 \) is a quadratic form on \( \widetilde{L} \) over \( \mathbb{Z}_2 \). Recall that this means that the map \( \kappa_p : \widetilde{L} \times \widetilde{L} \to \mathbb{Z}_2 \) defined by

\[
\kappa_p(\tilde{\sigma}, \tilde{\tau}) = \kappa(\tilde{\sigma} + \tilde{\tau}) + \kappa(\tilde{\sigma}) + \kappa(\tilde{\tau})
\]

is \( \mathbb{Z}_2 \)-bilinear. (See for example [HO, §5.2].) Then \( \kappa_p \) is an alternating form on \( \widetilde{L} \) \( (\kappa_p(\tilde{\sigma}, \tilde{\sigma}) = 0 \) for \( \tilde{\sigma} \in \widetilde{L} \) \) called the polar form of \( \kappa \). If \( \kappa_b : \widetilde{L} \times \widetilde{L} \to \mathbb{Z}_2 \) is a \( \mathbb{Z}_2 \)-bilinear form, we say that \( \kappa_b \) is compatible with \( \kappa \) if

\[
\kappa_p(\tilde{\sigma}, \tilde{\tau}) = \kappa_b(\tilde{\sigma}, \tilde{\tau}) + \kappa(\tilde{\tau}, \tilde{\sigma})
\]

for \( \tilde{\sigma}, \tilde{\tau} \in \widetilde{L} \). It is easy to check that such a \( \kappa_b \) exists (see for example [HO, 5.1.15]) and is unique up to the addition of a symmetric bilinear form on \( \widetilde{L} \).

7.1.3. If \( \kappa : \widetilde{L} \to \mathbb{Z}_2 \) is a quadratic form, we adopt the notation

\[
\text{rad}(\kappa) := \{ \tilde{\sigma} \in \widetilde{L} \mid \kappa(\tilde{\sigma}) = 0 \text{ and } \kappa_p(\tilde{\sigma}, \widetilde{L}) = 0 \}
\]

for the radical of \( \kappa \); and

\[
\text{iso}(\kappa) := \{ \tilde{\sigma} \in \widetilde{L} \mid \kappa(\tilde{\sigma}) = 0 \}
\]

for the set of isotropic vectors for \( \kappa \). The orthogonal group of \( \kappa \) is the group \( \text{O}(\kappa) := \{ \psi \in \text{GL}(\widetilde{L}) \mid \kappa(\psi(\tilde{\sigma})) = \kappa(\tilde{\sigma}) \text{ for } \tilde{\sigma} \in \widetilde{L} \} \) of isometries of \( \kappa \).

We now see that associative tori with involution are classified by quadratic forms over \( \mathbb{Z}_2 \).

Proposition 7.1.4. [AFY, Prop. 11.4] If \( (A, -) \) is an associative \( L \)-torus with involution, then there exists a unique quadratic form \( \kappa : \widetilde{L} \to \mathbb{Z}_2 \), called the mod-2 quadratic form for \( (A, -) \), such that

\[
\kappa_{\alpha_\sigma} = (-1)^{\kappa(\tilde{\sigma})} \alpha_\sigma \quad \text{and} \quad \alpha_\sigma \alpha_\tau = (-1)^{\kappa_p(\tilde{\sigma}, \tilde{\tau})} \alpha_\tau \alpha_\sigma \quad (88)
\]

for \( \alpha_\sigma \in A^\sigma \), \( \alpha_\tau \in A^\tau \), \( \sigma, \tau \in L \).\(^9\) Moreover, any quadratic form on \( \widetilde{L} \) arises from some associative \( L \)-torus with involution in this way (see the construction in 7.1.6 below). Finally, two associative \( n \)-tori are isograded isomorphic if and only if their mod-2 quadratic forms are isometric.

7.1.5. Suppose \( (A, -) \) is an associative \( L \)-torus with mod-2 quadratic form \( \kappa \). There are two subsets of \( L \) that will play a key role in what follows. First define

\[
\Gamma = \Gamma(A, -) = \text{supp}_L(Z(A, -)).
\]

Second recall that \( L_+ = L_+(A, -) = \text{supp}_L(A_+) \) (see 2.3.1). By (88) we have

\[
\Gamma = \{ \sigma \in L \mid \tilde{\sigma} \in \text{rad}(\kappa) \} \quad \text{and} \quad L_+ = \{ \sigma \in L \mid \tilde{\sigma} \in \text{iso}(\kappa) \}. \quad (89)
\]

We now describe a construction of an associative \( n \)-torus with a given mod-2 quadratic form.

\(^9\)As usual, we abuse notation and write an element \( \mu + 2\mathbb{Z} \) in \( \mathbb{Z}_2 \) as \( \mu \), in which case \((-1)^\mu \) is well defined.
7.1.6. Suppose that \( \kappa : \tilde{L} \to \mathbb{Z}_2 \) is a quadratic form and that \( \kappa_b : \tilde{L} \times \tilde{L} \to \mathbb{Z}_2 \) is a bilinear form that is compatible with \( \kappa \) (see 7.1.2). Let \( \mathcal{A} = \oplus_{\sigma \in L} \mathbb{F}t^\sigma \), the \( L \)-graded vector space with \( 0 \neq t^\sigma \in \mathcal{A}^\sigma \) for \( \sigma \in L \); and define a product and involution on \( \mathcal{A} \) by
\[
\begin{aligned}
t^\sigma t^\tau &= (-1)^{\kappa(\tilde{\sigma}, \tilde{\tau})} t^{\sigma + \tau} \quad \text{and} \quad \overline{t^\sigma} = (-1)^{\kappa(\tilde{\sigma})} t^\sigma.
\end{aligned}
\] (90)

Then, one checks that \( (\mathcal{A}, -) \) is an associative \( L \)-torus with involution whose mod-2 quadratic form is \( \kappa \). We denote \( (\mathcal{A}, -) \) by \( \text{ATL}_L(\kappa, \kappa_b) \) and call it the \textit{associative \( L \)-torus with involution determined by} \( (\kappa, \kappa_b) \). Note that by Proposition 7.1.4, \( \text{ATL}_L(\kappa, \kappa_b) \) depends up to isograded-isomorphism only on \( \kappa \).

The construction just described takes on a familiar form if we choose a \( \mathbb{Z} \)-basis \( \{\sigma_1, \ldots, \sigma_n\} \) for \( L \). Let \( t_i = t^{\sigma_i} \) for \( 1 \leq i \leq n \). Then, \( \mathcal{A} \) is generated as an algebra over \( \mathbb{F} \) by \( \{\ell_1^{\pm 1}, \ldots, \ell_n^{\pm 1}\} \); and we have \( \ell_i = (-1)^{b_i} t_i \) and \( t_i t_j = (-1)^{a_{ij}} t_j t_i \), where \( b_i = \kappa(\tilde{\sigma}_i) \in \mathbb{Z}_2 \) for \( 1 \leq i \leq n \) and \( a_{ij} = \kappa_p(\sigma_i, \sigma_j) \in \mathbb{Z}_2 \) for \( 1 \leq i \leq n \). Thus \( \text{ATL}_L(\kappa, \kappa_b) \) is the \textit{quantum torus with involution determined by the vector} \((-1)^{b_i}\) \textit{and the matrix} \((-1)^{a_{ij}}\) \textit{([AABGP, §III.3], [AG, §2])}.

Remark 7.1.7. Suppose that \( \kappa : \tilde{L} \to \mathbb{Z}_2 \) is a quadratic form. It is easy to see that it is possible to choose a bilinear form \( \kappa_b : \tilde{L} \times \tilde{L} \to \mathbb{Z}_2 \) that is compatible with \( \kappa \) and satisfies
\[
\kappa_b(\tilde{L}, \text{rad}(\kappa)) = \kappa_b(\text{rad}(\kappa), \tilde{L}) = 0.
\] (91)

In that case, if we let \( (\mathcal{A}, -) = \text{ATL}_L(\kappa, \kappa_b) \), we have the convenient property
\[
t^\sigma t^\tau = t^\tau t^\sigma = t^{\sigma + \tau} \quad \text{for} \quad \sigma \in \Gamma(\mathcal{A}, -), \tau \in L.
\] (92)

In particular, \( Z(\mathcal{A}, -) = \mathbb{F}[\Gamma] \), the group algebra of \( \Gamma \).

7.1.8. To classify associative \( L \)-tori with involution, one can proceed as follows: Fix a basis \( \{\sigma_1, \ldots, \sigma_n\} \) for \( L \). Then write down representatives of the isometry classes of quadratic forms on \( L \) using the well-known classification (see [D, Chap. I, §16] or [AFY, Rem. 5.17]). Next, for each \( \kappa \) in the list, choose \( \kappa_b \) compatible with \( \kappa \) and satisfying (91), and then construct the associative \( L \)-torus with involution \( \text{ATL}_L(\kappa, \kappa_b) \). This gives the list of associative \( L \)-tori with involution up to isograded-isomorphism, and each torus in the list satisfies (92).

Example 7.1.9. To give a concrete example, we suppose that \( n = 3 \) and fix a \( \mathbb{Z} \)-basis \( \{\sigma_1, \sigma_2, \sigma_3\} \) for \( L \). The following table lists the five possible quadratic forms \( \kappa \) on \( L \) up to isometry, and the corresponding forms \( \kappa_p \) and \( \kappa_b \), obtained as in 7.1.8.

| \( \kappa \) | \( \kappa_p \) | \( \kappa_b \) |
|----------|----------|----------|
| 0        | 0        | 0        |
| \( \ell_3 \) | 0        | 0        |
| \( \ell_2 + \ell_3 \) | \( \ell_2 \ell_3 \) | \( \ell_2 \ell_3 \) |
| \( \ell_2 \ell_3 \) | \( \ell_2 \ell_3 \) | \( \ell_2 \ell_3 \) |
| \( \ell_3 + \ell_2 \ell_3 \) | \( \ell_3 + \ell_2 \ell_3 \) | \( \ell_3 + \ell_2 \ell_3 \) |

(93)

In each row, we have given \( \kappa \) by displaying its value at \( \sum_{i=1}^{3} \ell_i \tilde{\sigma}_i \), and we have given \( \kappa_p \) and \( \kappa_b \) by displaying their values on the pairs \( \left( \sum_{i=1}^{3} \ell_i \tilde{\sigma}_i, \sum_{j=1}^{3} \ell_j \tilde{\sigma}_j \right) \). The list of associative 3-tori with involution up to bi-isomorphism is the corresponding list of tori \( \text{ATL}_L(\kappa, \kappa_b) \).
We finish this section by recording a lemma that will be needed in the next section.

**Proposition 7.1.10.** Suppose that \((A, -)\) is an associative \(L\)-torus with mod-2 quadratic form \(\kappa\). Then, the map \(\varphi \mapsto \tilde{\varphi}_{gr}\) is an epimorphism of the group of all isograded automorphisms\(^{10}\) of \((A, -)\) onto the orthogonal group \(O(\kappa)\).

**Proof.** The proof of [AY, Prop. 1.14] (iii) can be easily adapted to prove this fact. \(\Box\)

7.2. **Centreless Lie \(n\)-tori of type \(BC_r\).**

In this section, we assume that \(\text{char}(\mathbb{F}) = 0\) and \((\mathbb{F}^\times)^2 = \mathbb{F}^\times\). Note that this includes the important case when \(\mathbb{F}\) is algebraically closed of characteristic 0.

7.2.1. To construct centreless Lie \(n\)-tori as unitary Lie tori, we suppose that

(I1) \(r\) is a positive integer.

(I2) \(L\) is a free abelian group of rank \(n\) and \((A, -)\) is an associative \(L\)-torus with involution.

(I3) \(\tilde{M}\) is a subset of \(\text{iso}(\kappa)\) with \(0 \in \tilde{M}\), where \(\kappa : \tilde{L} \to \mathbb{Z}_2\) is the mod-2 quadratic form of \((A, -)\).

Beginning with these ingredients, we construct a graded hermitian form \(\xi : X \times X \to A\) over \((A, -)\) and hence a unitary Lie \(n\)-torus \(\mathfrak{su}(X, \xi)\).

First we build an abelian group \(\Lambda\) containing \(L\) that will be our grading group for \(\xi\). To do this, we regard \(L\) as a subgroup of the vector space \(QL := \mathbb{Q} \otimes \mathbb{Z}_2 L\) over \(\mathbb{Q}\) by means of the map \(\sigma \mapsto 1 \otimes \sigma\). Let \(M\) denote the inverse image under \(\tilde{\mu}\) of \(\tilde{M}\) in \(L\). Then, by (89), we have

\[
2L \subseteq M \subseteq L_+ \subseteq L,
\]

where \(L_+ = L_+(A, -)\). We let

\[
\Lambda = \left\langle \frac{1}{2} M \right\rangle
\]

in \(QL\). Then \(L\) is a subgroup of \(\Lambda\), and, since \(L \subseteq \Lambda \subseteq \frac{1}{2} L\), \(\Lambda\) is free abelian of rank \(n\).

Now let \(m\) be the order of the set \(\tilde{M}\) and \(\ell = 2r + m\). We select \(\tau_1, \ldots, \tau_\ell \in M \subseteq L_+\) such that \(\tau_i = 0\) for \(1 \leq i \leq 2r + 1\) and

\[
\tilde{\tau}_{2r+1}, \ldots, \tilde{\tau}_{\ell} \text{ are the distinct elements of } \tilde{M}.
\]

We also choose nonzero elements \(\gamma_1, \ldots, \gamma_\ell \in A_+\) with \(\gamma_i = 1\) for \(1 \leq i \leq 2r + 1\) and \(\gamma_i \in A_+^{\tau_i}\) for \(1 \leq i \leq \ell\). We call the sets \(\{\tau_i\}_{i=1}^\ell\) and \(\{\gamma_i\}_{i=1}^\ell\) compatible sets of parameters for \(\tilde{M}\).

Next let \(X = A^\ell\), a right \(A\)-module with standard basis \(e_1, \ldots, e_\ell\), and define a \(\Lambda\)-grading on \(X\) so that \(X\) is a \(\Lambda\)-graded \(A\)-module with \(\text{deg}(e_i) = \frac{1}{2} \tau_i\) for \(1 \leq i \leq \ell\). Observe that

\[
\text{supp}_\Lambda(X) = \bigcup_{i=1}^\ell \left( \frac{1}{2} \tau_i + L \right) = \frac{1}{2} M.
\] (94)

Further define a \(\Lambda\)-graded hermitian form \(\xi : X \times X \to A\) by

\[
\xi(e_i, e_j) = \delta_{ij} \gamma_i
\]

for \(1 \leq i, j \leq \ell\), where \(\tilde{i} = 2r + 1 - i\) for \(1 \leq i \leq 2r\) and \(\tilde{i} = i\) for \(2r + 1 \leq i \leq \ell\).

\(^{10}\)It is known [Y2, Lemma 1.2] that any automorphism of \((A, -)\) is isograded, but we will not use that here.
Then Assumptions 6.1.1 hold with

\[ X_{\text{hyp}} = \bigoplus_{i=1}^{2r} e_i \cdot A \quad \text{and} \quad X_{\text{an}} = \bigoplus_{i=2r+1}^{\ell} e_i \cdot A. \]

Indeed, \( X_{\text{an}} \) is finely graded since the elements \( \frac{1}{2} t_{2r+1}, \ldots, \frac{1}{2} t_{\ell} \) are distinct modulo \( L \); and it then follows that \( X_{\text{an}} \) is anisotropic by Proposition 5.2.5. The other conditions are easily checked. So we can construct the \((Q \times \Lambda)\)-graded Lie algebras \( \mathcal{F} = \mathfrak{u}(X, \xi) \) and \( \mathcal{S} = \mathfrak{s}(X, \xi) \) as in 6.1.3. In particular, \( \mathcal{S} \) is a unitary Lie \( n \)-torus of type \( BC_r \) that we say is the unitary Lie \( n \)-torus constructed from \( r \), \( L \), \( (A_i, -) \) and \( \widetilde{M} \).

**Remark 7.2.2.** It follows from Corollary 6.6.4 that \( \mathcal{F} \) and \( \mathcal{S} \) are independent, up to bi-isomorphism, of the choice of compatible sets of parameters \( \{ \tau_i \}_{i=1}^\ell \) and \( \{ \gamma_i \}_{i=1}^\ell \) for \( \widetilde{M} \).

**7.2.3.** The \((Q \times \Lambda)\)-graded Lie algebras \( \mathcal{F} \) and \( \mathcal{S} \) constructed in 7.2.1 have simple descriptions as matrix algebras using the results of Section 5.3. For this we use matrix notation relative to the homogeneous \( A \)-basis \( \{ e_i \}_{i=1}^\ell \) for \( X \). Then, by (33), we see that \( \mathcal{E} = \mathcal{E}(X, \xi) \) is equal to the algebra Mat\(_{2r}(A)\) of \( \ell \times \ell \) matrices over \( A \), and hence, since \( X \) is free of finite rank over \( A \), we have \( \mathcal{E} = \text{End}_A(A) \). Thus, by (12), we have

\[ \mathcal{F} = \sum_{i,j=1}^\ell u_{ij}(A) = \mathfrak{u}, \]

where \( u_{ij}(\alpha) = e_{ij}(\alpha) - e_{ij}(\gamma_i^{-1} \alpha \gamma_j) \) and \( \mathfrak{u} = \mathfrak{u}(X, -) \) is the unitary Lie algebra of \( \xi \). Alternatively, if we let \( J_{2r} = (\delta_{ij}) \in \text{Mat}_{2r}(F) \) and

\[ G = \begin{bmatrix} J_{2r} & 0 \\ 0 & \text{diag}(\gamma_{2r+1}, \ldots, \gamma_{\ell}) \end{bmatrix} \]

(the matrix of \( \xi \) relative to the \( A \)-basis \( \{ e_i \}_{i=1}^\ell \)), then by (11) and (39),

\[ \mathcal{F} = \{ T \in \mathfrak{gl}(A) \mid T^* = -T \}, \quad \text{where} \quad T^* = G^{-1} T^T G \quad \text{for} \ T \in \mathcal{E}, \]

\( t \) denotes the transpose, and \( \mathfrak{gl}(A) \) is Mat\(_{2r}(A)\) under the commutator product. Moreover, by Proposition 6.2.1(a),

\[ \mathcal{S} = \{ T \in \mathcal{F} \mid \text{tr}(T) \equiv 0 \mod [A, A] \}. \]

Finally, the \( Q \)-gradings on \( \mathcal{F} \) and \( \mathcal{S} \) are the root space decompositions with respect to the adjoint action of \( h = \bigoplus_{i=1}^r F u_{ii}(1) \), and the \( \Lambda \)-gradings on \( \mathcal{F} \) and \( \mathcal{S} \) are obtained by restricting the \( \Lambda \)-grading on \( \mathcal{E} \) which is given by

\[ \text{deg}_A(e_{ij}(\alpha)) = \frac{1}{2} \ell_i - \frac{1}{2} \ell_j + \text{deg}_A(\alpha) \quad (95) \]

for \( 1 \leq i, j \leq \ell \) and \( \alpha \) homogeneous in \( A \) (see Proposition 5.3.7).

We can now combine our results to prove the following:

**Theorem 7.2.4 (Classification of centreless Lie \( n \)-tori of type \( BC_r \), \( r \geq 3 \)).** Assume \( \mathcal{F} \) is a field of characteristic 0 with \( F^{\times} = F^{\times} \). Suppose that \( n \geq 1 \) and \( r \geq 3 \). Let \( L \) be a free abelian group of rank \( n \), and let \( \kappa_1, \ldots, \kappa_s \) be a list of the distinct quadratic forms over \( \mathbb{Z}_2 \) on \( L = L/2L \) up to isometry. For \( 1 \leq i \leq s \), let \( (A_i, -) \) be an associative \( L \)-torus with involution whose mod-2 quadratic form is \( \kappa_i \), and let \( \widetilde{M}_{i1}, \ldots, \widetilde{M}_{iq_i} \) be representatives of the orbits of the orthogonal group \( O(\kappa_i) \) acting on the set of subsets of iso(\( \kappa_i \)) containing 0. Then, the distinct centreless Lie \( n \)-tori of type \( BC_r \) are, up to bi-isomorphism, the unitary Lie \( n \)-tori \( \mathcal{S}_{ij} \) constructed as in 7.2.1 from \( r \), \( L \), \( (A_i, -) \) and \( \widetilde{M}_{ij}, 1 \leq i \leq s, 1 \leq j \leq q_i \).
Proof. Suppose first that two of the unitary Lie $n$-tori are bi-isomorphic, say $S$ constructed from $r$, $L$, $(A_i, -)$ and $\widetilde{M}_{ij}$; and $S'$ constructed from $r$, $L$, $(A_i', -)$ and $\widetilde{M}_{ij'}$. By Corollary 6.6.4 (b) and (94), we have an isograded isomorphism $\varphi$ from $(A_i, -)$ onto $(A_i', -)$ such that $\widetilde{\varphi}_{\gr}(\widetilde{M}_{ij}) = \widetilde{M}_{ij'}$. So by Proposition 7.1.4, $i = i'$. Also, by Proposition 7.1.10, $\widetilde{M}_{ij}$ and $\widetilde{M}_{ij'}$ are in the same orbit under $O(\kappa_i)$, and hence $j = j'$.

Next suppose that $L$ is a centreless Lie $n$-torus of type $BC_r$. By Theorem 6.3.1 (b), we can assume that $L = S$, where $S = su(X, \xi)$ is the unitary Lie $\Lambda$-torus constructed using $r, \Lambda$ (a free abelian group of rank $n$), $L'$, $(A, -)$ and $\xi : X \times X \to A$ as in 6.1.3. Also, by Lemma 6.2.2 (d), $L'$ is free of rank $n$, so we can identify it with the given group $L$. Let $S = \supp_{\Lambda}(X)$, and set

$$M := 2S \subseteq L_+ \subseteq L,$$

where $L_+ = L_+(A, -)$ (see Lemma 6.2.2 (a) and (b)). Note that $S + L_+ \subseteq S$, so $M + 2L \subseteq M$, hence

$$M$$

is the inverse image of $\widetilde{M}$ under $\sim : L \to \widetilde{L}$. (96)

By Proposition 7.1.4, there exists an isograded isomorphism $\varphi$ of the associative $L$-torus with involution $(A, -)$ onto the associative $L$-torus with involution $(A_i, -)$ for some $1 \leq i \leq s$. In addition, by (96), $\varphi_{\gr}(M) \subseteq L_+(A_i, -)$, and hence $\varphi_{\gr}(M) \subseteq L_+(A_i, -) = \text{iso}(\kappa_i)$. Since $0 \in \varphi_{\gr}(M)$, Proposition 7.1.10 tells us that there exists an isograded automorphism $\psi$ of $(A_i, -)$ such that $\psi_{\gr}(\varphi_{\gr}(M)) = \widetilde{M}_{ij}$ for some $1 \leq j \leq q_i$. Replacing $\varphi$ by $\varphi \circ \psi$, we can assume that $\varphi_{\gr}(M) = \widetilde{M}_{ij}$. Thus, letting $\widetilde{M}_{ij}$ be the inverse image of $\widetilde{M}_{ij}$ under $\sim : L \to \widetilde{L}$, we have, by (97), that $\varphi_{\gr}(M) = \widetilde{M}_{ij}$. Applying Corollary 6.6.4 (a), (94) for $S_{ij}$, and (96), we see that $L$ is bi-isomorphic to the unitary Lie $n$-torus $S_{ij}$.  

Theorem 7.2.4 reduces the classification of centreless Lie $n$-tori of type $BC_r$, $r \geq 3$, up to bi-isomorphism to two problems: the classification of $n$-dimensional quadratic forms over $\mathbb{Z}_2$ up to isometry; and, given such a quadratic form $\kappa : \widetilde{L} \to \mathbb{Z}_2$, the determination of the orbits of the orthogonal group $O(\kappa)$ acting on the set of subsets containing 0 of $\text{iso}(\kappa)$. The first of these problems has a well-known solution as we have mentioned, while the second can, at least in some cases (in particular, for small $n$), be worked out directly. We will examine the case $n = 3$ when we consider EALAs in the next section (see Example 7.3.4).

7.3. Construction of extended affine Lie algebras of type $BC_r$.

In this section, we construct some EALAs of type $BC_r$. We do not assume here that the reader is familiar with EALAs – it suffices to know that an EALA is a triple $(E, H, (\mid \mid))$ consisting of a Lie algebra $E$ with a finite-dimensional distinguished abelian Cartan subalgebra $H$ and a nondegenerate invariant symmetric bilinear form $(\mid \mid)$ satisfying a natural class of axioms [N2] which model properties of affine Kac-Moody Lie algebras. The nullity of $E$ is the rank of the finitely generated free abelian group $\Lambda$ generated by the isotropic roots of $E$, and the type of $E$ is the type of the finite irreducible root system obtained by identifying two roots of $E$ if they differ by an element of $\Lambda$. Besides [N2], the reader can consult [BGK], [AABGP], [N3] and [AF] and the references therein for more information about this topic.
Generalizing earlier work in type $A_r$ [BGK, BGKN], Neher [N2] gave a construction of a family of EALAs of nullity $n$ from a centreless Lie torus $L$ of nullity $n$. Roughly speaking one constructs an EALA $E = L \oplus C \oplus D$, where $D$ is a graded subalgebra of the Lie algebra $SCDer(L)$ of skew-centroidal derivations of $L$ and $C$ is the graded dual of $D$, and where the product on $E$ involves a 2-cocycle on $D$ with values in $C$. Varying the algebra $D$ and the 2-cocycle, one obtains a family of EALAs determined by $L$. It was announced in [N2] that any EALA occurs in the family constructed from some $L$ in this way. Moreover, it was shown in [AF, Cor. 6.3] that if two centreless Lie $n$-tori $L$ and $L'$ are bi-isomorphic, then the EALAs in the corresponding families are pairwise isomorphic.

If $L$ is a centreless Lie $n$-torus of type $\Delta$, where $\Delta$ is reduced, then the corresponding EALAs have type $\Delta$. On the other hand, if $\Delta$ has type $BC_r$ and $\text{supp}_Q(L) = \Delta \cup \{0\}$, then the corresponding EALAs have type $BC_r$. So, it follows from Remark 3.3.3 that all EALAs of nullity $n$ and type $B_r$ and $BC_r$ are obtained from a centreless Lie $n$-torus of type $BC_r$. Moreover, if $r \geq 3$, it follows from Theorem 6.3.1 that there is no loss of generality in starting with a unitary Lie $n$-torus $S$. For the sake of readers interested in working concretely with EALAs, in this section we review Neher’s construction, without proofs, in this case. Instead of constructing a family of EALAs from $S$, we focus on just one EALA, which is maximal in the sense that $D = SCDer(S)$ (the full set of skew-centroidal derivations) is used in the construction; and we use the trivial 2-cocycle on $D$ to define the multiplication. The reader familiar with [N2] will have no trouble obtaining the whole family in the same way.

Throughout this section, we will assume that $F$ is a field of characteristic 0 with $(F^\times)^2 = F^\times$, $r \geq 3$, and that $\mathcal{F} = \mathfrak{su}(X, \xi)$ and $\mathcal{S} = \mathfrak{su}(X, \xi)$ are the $(Q \times \Lambda)$-graded Lie algebras constructed from $r, L$, $(A, -)$ and $M$ as in 7.2.1 using compatible sets of parameters $\{\tau_i\}_{i=1}^\ell$ and $\{\gamma_i\}_{i=1}^\ell$, where $\ell = 2r + m$, and $m$ is the order of $M$. We let $E = \epsilon(X, \xi)$, and we view $E$, $\mathcal{F}$ and $\mathcal{S}$ as graded matrix algebras as in 7.2.3.

7.3.1. As in Theorem 5.3.5(d), we define a $\Lambda$-graded nondegenerate associative symmetric bilinear form $(\mid \mid)$ on $E$ by

$$ (T_1 \mid T_2) = \varpi(\text{tr}(T_1, T_2)) $$

for $T_1, T_2 \in \mathcal{S}$, where $\varpi : A \to F$ is the $L$-graded projection of $A$ onto $F$ (see Theorem 5.3.5(d)). Then, $(\mid \mid)$ restricts to a $\Lambda$-graded nondegenerate invariant form on $\mathcal{F}$ and $\mathcal{S}$.

Let $Z = Z(A, -); \ \kappa : L \to \mathbb{Z}_2$ be the mod-2 quadratic form of $(A, -)$; and

$$ \Gamma := \Gamma(A, -) = \text{supp}_L(Z) \quad \text{and} \quad L_+ := L_+(A, -). $$

By (89), we have

$$ \Gamma := \{\sigma \in L \mid \hat{\sigma} \in \text{rad}(\kappa)\} \quad \text{and} \quad L_+ = \{\sigma \in L \mid \hat{\sigma} \in \text{iso}(\kappa)\}. $$

Recall next that $E$ is a left $Z$-module (see 4.1.3), and we have $\chi_{ij}(\alpha) = e_{ij}(3\alpha)$ and $\hat{\chi}_{ij}(\alpha) = u_{ij}(3\alpha)$ for $3 \in Z, \ 1 \leq i, j \leq \ell$ and $\alpha \in A$. Furthermore, by Proposition 4.2.10 and Proposition 6.2.1(b), the map $3 \to M_3|_{S}$ is an isomorphism of $Z$ onto the centroid $\text{Cent}(S)$ of $S$. This map is $\Lambda$-graded so $\text{supp}_A(\text{Cent}(S)) = \Gamma$; that is, $\Gamma$ is the centroidal grading group of $S$ (see 3.3.2(c)).
Let $\text{Hom}(\Lambda, \mathbb{F})$ be the group of group homomorphisms of $\Lambda$ into $\mathbb{F}$, and let $\text{Der}_\mathbb{F}(S)$ be the algebra of derivations of $S$. For $\theta \in \text{Hom}(\Lambda, \mathbb{F})$, we define the degree derivation $\partial_\theta \in \text{Der}_\mathbb{F}(S)$ by $\partial_\theta|_{S_\sigma} = \theta(\sigma)\text{id}_{S_\sigma}$ for $\sigma \in \Lambda$. Setting
\[ \mathcal{D} = \partial_{\text{Hom}(\Lambda, \mathbb{F})}, \]
we see that $\mathcal{D}$ is an $n$-dimensional abelian subalgebra of $\text{Der}_\mathbb{F}(S)$.

Now $\text{Der}_\mathbb{F}(S)$ is a left $Z$-module under the action $zd = M_3 \circ d$ for $z \in Z$ and $d \in \text{Der}_\mathbb{F}(S)$. Moreover, the space
\[ \text{CDer}(S) = Z\mathcal{D} \]
is a subalgebra of the Lie algebra $\text{Der}(S)$, and $\text{CDer}(S)$ is a free $Z$-module of rank $n$ which is $\Gamma$-graded with $\text{CDer}(S)_{\sigma} = Z\sigma \mathcal{D}$ for $\sigma \in \Gamma$. Let
\[ \mathcal{D} := \text{SCDer}(S), \]
the $\Gamma$-graded subalgebra of $\text{CDer}(S)$ consisting of the derivations in $\text{CDer}(S)$ that are skew relative to the form $( \ | \ )$. Then $\mathcal{D}$ is called the Lie algebra of skew-centroidal derivations of $S$. Since $D^0 = \mathcal{D}$, $\dim(D^0) = n$, while $\dim(D^\sigma) = n - 1$ if $\sigma \in \Gamma \setminus 0$.

Next let
\[ C := D^{\sigma \ast} = \bigoplus_{\sigma \in \Gamma} (D^\sigma)^* \subseteq D^*, \]
be the graded-dual space of $\mathcal{D}$, where $(D^\sigma)^*$ is embedded in $D^*$ by letting its elements act trivially on $D^\tau$ for $\tau \neq \sigma$. We give the vector space $C$ a $\Gamma$-grading by setting
\[ C^\sigma = (D^{-\sigma})^*, \]
in which case, $C$ is a $\Gamma$-graded $D$-module by means of the contragradient action $"^\ast$” given by
\[ (d \cdot c)(e) = -c([d, e]), \]
for $d, e \in D$, $c \in C$. Now define $\varsigma : S \times S \rightarrow C$ by
\[ \varsigma(T_1, T_2)(d) = (dT_1|T_2). \]
Then $\varsigma$ is a $\Gamma$-graded 2-cocycle on $S$ with values in the trivial $S$-module $C$.

With these ingredients, we are ready to build an EALA. Set
\[ E = S \oplus C \oplus D, \]
where $C = D^{\sigma \ast}$. We identify $S$, $C$ and $D$ with subspaces of $E$, and define a product $[ \ | \ ]_E$ on $E$ by
\[ [T_1 + c_1 + d_1, T_2 + c_2 + d_2]_E = ([T_1, T_2] + d_1(T_2) - d_2(T_1)) + (d_1 \cdot c_2 - d_2 \cdot c_1 + \varsigma(T_1, T_2)) + [d_1, d_2] \quad (98) \]
for $T_i \in S$, $c_i \in C$, $d_i \in D$. Then, $E$ is a $\Lambda$-graded Lie algebra with the direct sum grading, and
\[ H := \frak{h} \oplus C^0 \oplus D^0 \]
is an abelian subalgebra of $E$, where $\frak{h} = \bigoplus_{i=1}^r \mathbb{F}u_{ii}(1)$. Finally we extend the bilinear form $\langle \ | \ \rangle$ on $S$ to a graded bilinear form $\langle \ | \ \rangle$ on $E$ by defining
\[ \langle T_1 + c_1 + d_1 \ | \ T_2 + c_2 + d_2 \rangle = (T_1 \ | \ T_2) + c_1(d_2) + c_2(d_1). \quad (99) \]
Then, Neher’s theorem [N2, Thm. 6] tells us that $(E, \langle \ | \ \rangle, H)$ is an EALA of nullity $n$ and type $BC_r$ or $B_r$. Moreover, by that same theorem and our Theorem 7.2.4, we have constructed a maximal EALA of nullity $n$ in each family of EALAs of nullity $n$ and type $BC_r$ or $B_r$. 
Finally, we note that the EALA $E$ is of type $B_r$ if and only if $\kappa = 0$.

7.3.2. (Computations using bases) To facilitate working with the EALA $E$ constructed in 7.3.1, we now show how to select convenient bases for $L$ and $\Lambda$ and use coordinates to obtain expressions for the product and form on $E$. This follows the original approach in [BGK] and [BGKN] which treated type $A_r$, $r \geq 2$.

We may assume that $(A, -) = \mathcal{A}(\kappa, \kappa_b)$, where $\kappa_b$ is chosen as in Remark 7.1.7, in which case we have

$$t^\sigma t^\tau = t^{\sigma + \tau} \quad \text{for } \sigma, \tau \in L.$$  \hfill (100)

Now choose a basis $\{\sigma_1, \ldots, \sigma_n\}$ for $L$ such that

$$\{\sigma_1, \ldots, \sigma_{n_1}, 2\sigma_{n_1+1}, \ldots, 2\sigma_n\} \text{ is a basis for } (M),$$  \hfill (101)

where $\sigma_1, \ldots, \sigma_{n_1} \in M$ and $0 \leq n_1 \leq n$. (To obtain such a basis, chose a basis for $\tilde{L}$ with the appropriate properties and lift to a basis for $L$.) Then, $\{\lambda_1, \ldots, \lambda_n\}$ is a basis for $\Lambda = \{\tfrac{1}{2}M\}$, where

$$\lambda_i = \frac{1}{2}\sigma_i \quad \text{for } 1 \leq i \leq n_1, \quad \text{and} \quad \lambda_i = \sigma_i \quad \text{for } n_1 + 1 \leq i \leq n.$$

Next we introduce coordinates in $D$ and $\text{CDer}(S)$. First, for $\lambda = \sum_{i=1}^n \ell_i \lambda_i \in \Lambda$, we set

$$\lambda^\# = (\ell_1, \ldots, \ell_1) \in \mathbb{F}^n.$$

Further, let $\{\theta_1, \ldots, \theta_n\}$ be the dual basis for $\{\lambda_1, \ldots, \lambda_n\}$ in $\text{Hom}(\Lambda, \mathbb{F})$, and for $s = (s_1, \ldots, s_n) \in \mathbb{F}^n$ write $\theta_s = \sum_{i=1}^n s_i \theta_i$. Then

$$\theta_s(\lambda) = \lambda^\# \cdot s$$  \hfill (102)

for $s \in \mathbb{F}^n$ and $\lambda \in \Lambda$, where $\cdot$ is the usual dot product on $\mathbb{F}^n$. Also, if $s \in \mathbb{F}^n$, let $\partial_s = \partial_{\theta_s}$. Then

$$D = \{\partial_s \mid s \in \mathbb{F}^n\};$$

$$\text{CDer}(S) = \sum_{\sigma \in \Gamma} \text{CDer}(S)^\sigma \text{ with } \text{CDer}(S)^\sigma = t^\sigma D \text{ for } \sigma \in \Gamma; \text{ and one checks using (100) that the multiplication in } \text{CDer}(S) \text{ is given by}$$

$$[t^\sigma \partial_s, t^\tau \partial_t] = (\sigma^\# \cdot s) \partial_t - (\sigma^\# \cdot r) \partial_s.$$  \hfill (103)

The homogeneous components of the algebra $D = S\text{CDer}(S) = \bigoplus_{\sigma \in \Gamma} D^\sigma$ of skew-centroidal derivations of $S$ are given by

$$D^\sigma = \bigoplus_{s \in \mathbb{F}^n, \sigma^\# \cdot s = 0} \mathbb{F} t^\sigma \partial_s.$$

for $\sigma \in \Gamma$.

To discuss $C = D^{\sigma^*}$ we define $c_\sigma(s) \in C^\sigma = (D^{-\sigma})^*$ for $s \in \mathbb{F}^n$ and $\sigma \in \Gamma$ by

$$c_\sigma(s)(t^{-\sigma} \partial_t) = s \cdot r$$  \hfill (104)

for $r \in \mathbb{F}^n$ with $\sigma^\# \cdot r = 0$. Then, $s \mapsto c_\sigma(s)$ determines a linear map of $\mathbb{F}^n$ into $C^\sigma$ with kernel $\mathbb{F}\sigma^\#$. So by counting dimensions, we see that

$$C^\sigma = \{c_\sigma(s) \mid s \in \mathbb{F}^n\}.$$

The action of $D$ on $C$ is given by

$$(t^\rho \partial_t) \cdot (c_\sigma(s)) = c_{\sigma + \rho}((\sigma^\# \cdot s + (s \cdot r)\rho^\#).$$  \hfill (105)

for $\sigma \in \Gamma$, $s \in \mathbb{F}^n$, and $\rho \in \Gamma$, $r \in \mathbb{F}^n$ with $\rho^\# \cdot r = 0$.

To calculate the action of $\text{CDer}(S)$ on $S$ using coordinates, it is natural to extend the action of $\text{CDer}(S)$ to $E = \text{Mat}_t(A)$; and then restrict to $S$. Indeed, since $E$
is $\Lambda$-graded we can define degree derivations $\{\partial^E_\theta\}_{\theta \in \text{Hom}(\Lambda, E)}$ of $E$ just as for $S$. Moreover, $\text{Dery}(E)$ is naturally a left $Z$-module and we have the $\Lambda$-graded Lie algebra $Z\partial^E_{\text{Hom}(\Lambda, E)}$ in $\text{Dery}(E)$. Letting $\partial^E_s = \sum_{i=1}^n s_i \partial^E_{\theta_i}$ for $s = (s_1, \ldots, s_n) \in \mathbb{F}^n$, we see that the restriction map $t^\sigma \partial^E_s : t^\sigma \partial^E_s$ is an algebra isomorphism sending $Z\partial^E_{\text{Hom}(\Lambda, E)}$ onto $\text{CDer}(S)$. Henceforth, we treat this map as an identification. This gives an action of $\text{CDer}(S)$ on $E$, and one checks directly using (95) and (102) that

\begin{equation}
(t^\sigma \partial^E_s)(e_{ij}(t^\tau)) = s \cdot (\tau + \frac{1}{2} t_1 - \frac{1}{2} t_2)^\# e_{ij}(t^{\sigma + \lambda})
\end{equation}

for $\sigma \in \Gamma$, $s \in \mathbb{F}^n$ with $\sigma^\# \cdot s = 0$ and $\tau \in L$. By restriction, this determines the action of $\text{CDer}(S)$ on $S$, and hence the action of $D$ on $S$.

To calculate the 2-cocycle $\zeta : S \times S \to C$, it is natural to proceed in a similar fashion and extend this map to a bilinear map $\zeta : E \times E \to C$ defined by

\[\zeta(T_1, T_2)(d) = (dT_1, T_2)\]

for $T_1, T_2 \in E$, $d \in D$; and then restrict to $S \times S$. A direct calculation using (90), (100) and (106) shows that

\begin{equation}
\zeta(e_{ij}(t^\sigma), e_{pq}(t^\tau)) = \begin{cases} 
\delta_{ip} \delta_{jq} (-1)^{\kappa (\tau, \tau)} c_{\sigma + \tau} ((\sigma + \frac{1}{2} t_1 - \frac{1}{2} t_2)^\#), & \text{if } \sigma + \tau \in \Gamma \\
0, & \text{otherwise}
\end{cases}
\end{equation}

for $\sigma, \tau \in L$ and $1 \leq i, j, p, q \leq n$. By restriction, we obtain the 2-cocycle $\zeta$ on $S$.

Equations (103), (104), (105) and (107) now allow us to calculate products in $E$ explicitly using (98) and the form on $E$ using (99).

**Example 7.3.3.** Suppose that $n = 1$. Then the EALA $E$ constructed in 7.3.1 and 7.3.2 is an affine Kac-Moody algebra. If the mod-2 quadratic form $\kappa$ of $(A, -)$ is not 0, then $\tilde{L}_+ = 0$, so $\tilde{M} = 0$. In this case $E$ has affine type $A_2^{(2)}$ in the notation of [K]. On the other hand if $\kappa = 0$, then $\tilde{L}_+ = \tilde{L}$, so $\tilde{M} = 0$ or $\tilde{L}$. The EALA $E$ is then of affine type $B_1^{(2)}$ or $D_2^{(2)}$, respectively. (In the notation of [MP], which is more natural in this context, the three affine types occurring here are in order: $BC_2^{(2)}$, $B_1^{(2)}$ and $B_2^{(2)}$.)

**Example 7.3.4.** Suppose that $r \geq 3$ and $n = 3$. Let $L$ be a free abelian group with basis $\{\sigma_1, \sigma_2, \sigma_3\}$. The five 3-dimensional quadratic forms over $\mathbb{Z}_2$ were listed in Table 93. We consider the case when $(A, -) = \text{ATGL}(\kappa, \kappa_b)$, where

\[\kappa(\sum_{i=1}^n \ell_i \tilde{\sigma}_i) = \ell_3 + \ell_1 \ell_2 \quad \text{and} \quad \kappa_b(\sum_{i=1}^n \ell_i \tilde{\sigma}_i, \sum_{i=1}^n \ell_i' \tilde{\sigma}_i) = \ell_1 \ell_2'.\]

We have $\text{rad} \kappa = 0$ and $\text{iso}(\kappa) = \{0, \tilde{\sigma}_1, \tilde{\sigma}_2, \tilde{\sigma}_1 + \tilde{\sigma}_2 + \tilde{\sigma}_3\}$. One can check (for example using facts about $O(\kappa)$ from [D, §1.16]) that, for $1 \leq m \leq 4$, $O(\kappa)$ acts transitively on the set of order $m$ subsets of $\text{iso}(\kappa)$ containing 0. So, by Theorem 7.2.4, for $1 \leq m \leq 4$ there is up to bi-isomorphism exactly one centreless Lie 3-torus $S$ of type $BC_r$ with coordinate torus $(A, -)$ and anisotropic rank $m$.

We consider the case when $m = 3$ and we select

\[\tilde{M} = \{0, \tilde{\sigma}_1, \tilde{\sigma}_2\}.\]

Then $M = \mathbb{Z}\sigma_1 \oplus \mathbb{Z}\sigma_2 \oplus 2\mathbb{Z}\sigma_3$, so the given basis $\{\sigma_1, \sigma_2, \sigma_3\}$ satisfies (101) with $n_1 = 2$. Now $\ell = 2r + 3$, and we let

\[\tau_{2r+2} = \sigma_1, \quad \tau_{2r+3} = \sigma_2, \quad \tau_i = 0 \quad \text{for } 1 \leq i \leq 2r + 1,\]
and
\[ γ_{2r+2} = t_1, \quad γ_{2r+3} = t_2, \quad γ_i = 1 \quad \text{for} \quad 1 \leq i \leq 2r + 1, \]
in which case \( \{τ_i\}_{i=1}^{ℓ} \) and \( \{γ_i\}_{i=1}^{ℓ} \) are compatible sets of parameters for \( \widetilde{M} \). We construct graded matrix algebras \( E, F \) and \( S \) from \( r, L, (A, −) \) and \( \widetilde{M} \) (using the above parameter sets) as in 7.2.3. Then
\[ S = \{ T ∈ gl_{2r+3}(A) \mid G^{-1}T^T G = -T \text{ and } \text{tr}(T) ≡ 0 \text{ (mod } [A, A])\}, \]
where
\[ G = \begin{bmatrix} J_{2r} & 0 \\ 0 & \text{diag}(1, t_1, t_2) \end{bmatrix}. \]

The \((Q × Λ)\)-gradings on \( F \) and \( S \) are as described in 7.2.3 with \( Λ = \langle \frac{1}{2} M \rangle = \langle \frac{1}{2} σ_1, \frac{1}{2} σ_2, σ_3 \rangle \). By Theorem 7.2.4, \( S \) is the unique (up to bi-isomorphism) centreless Lie 3-torus of type BC \( r \) with coordinate torus \((A, −)\) and anisotropic rank 3.

Finally, we let \( E \) be the EALA constructed as in 7.3.1. Since \( \widetilde{Γ} = \text{iso}(κ) = 0 \), we have \( Γ = 2L \) and \( Z = F[Γ] = F[2L] \). We can now use coordinates to calculate the product and form in \( E \) as in 7.3.2.

8. CONCLUSIONS

We finish with some remarks on special cases and possible generalizations of our results. Suppose that \( Λ \) is arbitrary (unless mentioned otherwise) and \( \text{char}(F) = 0 \).

8.1. Remarks.

Remark 8.1.1. (Lie tori of type \( B_r \), \( r ≥ 3 \)) If we specialize to the case when the involution \( − \) on the associative torus \( A \) is the identity, Theorems 6.3.1 and 6.6.1 yield structure and bi-isomorphism theorems for centreless \( Λ \)-Lie tori of type \( B_r \), \( r ≥ 3 \). The structure theorem for type \( B_r \) was proved (in a different form) when \( Λ = \mathbb{Z}^n \) and \( F = \mathbb{C} \) in [AG], and it is a special case of a more general structure theorem for division graded Lie algebras of type \( B_r \) due to Yoshii [Y3]. However, the bi-isomorphism theorem is new even for type \( B_r \).

Also, if we specialize to the case when the quadratic form is trivial in the classification result, Theorem 7.2.4, we obtain a classification of the centreless Lie \( n \)-tori of type \( B_r \), \( r ≥ 3 \), up to bi-isomorphism.

Remark 8.1.2. As we have noted earlier, in the construction of a family of EALAs from a centreless Lie \( n \)-torus, two centreless Lie \( n \)-tori that are bi-isomorphic give isomorphic families of EALAs [AF, Cor. 6.3]. However, the converse is not true. To obtain a one-to-one correspondence between centreless Lie \( n \)-tori and families of EALAs of nullity \( n \) one needs a coarser equivalence relation on centreless Lie \( n \)-tori called isotopy [ibid]. Therefore, it would be beneficial to have a version of the classification result, Theorem 7.2.4, with isotopy replacing bi-isomorphism. With the results for the other types of root systems in [AF] as a model, we expect that Theorem 7.2.4 will be the first main step in obtaining such a result for type BC \( r \).

Remark 8.1.3. A generalization of Lie \( Λ \)-tori of type \( Δ \) which arose out of Yoshii’s work is the class of pre-division \((Δ, Λ)\)-graded Lie algebras (see [Y3, §2], [N3, §5.1]). Such a graded Lie algebra is said to be invariant if it possesses a suitable invariant bilinear form. Correspondingly, Neher has introduced an interesting class of algebras generalizing EALAs called invariant affine reflection algebras (IARAs) [N3, §6.7]. In [N3, Thm. 6.10], Neher has announced that a family of IARAs can be
constructed from an invariant pre-division $(\Delta, \Lambda)$-graded Lie algebra whenever $\Lambda$ is torsion free. In particular, since a unitary Lie $\Lambda$-torus has an invariant form by Theorem 5.3.5, it can be used to construct a family of IARAs when $\Lambda$ is torsion free.

In light of Neher’s work discussed above, it would be useful to prove a generalization of our structure theorem and bi-isomorphism theorem for invariant pre-division $(\text{BC}_r, \Lambda)$-graded Lie algebras, $r \geq 3$. Such results would be especially interesting in the particular case of an invariant division $(\text{BC}_r, \Lambda)$-graded Lie algebra, since they would naturally include our main theorems as well as the finite-dimensional result mentioned at the beginning of this paper (when $\Lambda = 0$). We expect that the results and arguments in the division case would closely follow the ones in the present work.

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