Study of symmetry in F(R) theory of gravity

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Abstract

An action in which the Ricci scalar is nonminimally coupled with a scalar field and contains higher order curvature invariant terms carries a conserved current under certain conditions that decouples geometric part from the scalar field. The conserved current relates the pair of arbitrary coupling parameters \( f(\phi) \) and \( \omega(\phi) \) with the gravitational field variable, where \( \omega(\phi) \) is the Brans-Dicke coupling parameter. The existence of such conserved current may be helpful to sketch the cosmological evolution from its early age till date in a single frame.

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1 Introduction

In the recent years, modified theory of gravity is being treated all around with increasing importance, with the anticipation that such a theory of gravity which includes higher order curvature invariant terms may help in the understanding of the dark energy origin (for a review, see [1]). In this connection, various modified \( F(R) \) theories of gravity have been constructed and applied to the description of the late-time cosmic acceleration together with the check for local tests (see eg., [2] and the references therein). \( F(R) \) theory originated in the early sixties with the work of Utiyama and B.De-Witt [3]. They demonstrated for the first time that for renormalizability at one-loop level, the classical action should be supplemented by higher order curvature invariant terms. Later, the search for a viable semiclassical or quantum model for the dark energy component and its dynamical properties has turn out to be one of the most important topics of string cosmology, which attempts to reveal the cosmological implications of string effective actions. In particular, quantum corrections to the Einstein-Hilbert action naturally arise for the closed string, and modify the gravitational interaction in a way which might be testable in the near future. More recently, there has been many calculations showing that, when quantum corrections or String/M-theory are considered, the effective classical gravitational action admits higher order corrections in curvature invariants [4] - [9].

Higher-order gravity terms can generate non-trivial effects in a four-dimensional spacetime. If the dilaton varies in spacetime, gradient terms appear and higher order curvature invariant terms interact with a non-constant \( \alpha' \)-coupling. Also, compactification of the 26 or 10 - dimensional target space is encoded in residual modulus fields which, in general, will evolve in time. Both the cases finally yield a dynamical field non-minimally coupled to gravity. Dilaton and modulus cosmologies in higher order gravity were considered, for example, in [10] - [27]. In connection with the recent observations, dark energy models with higher order curvature invariant terms were inspected, in the case of fixed moduli [28], while a dynamical dilaton as a dark energy candidate was studied in [29, 30].

Although \( F(R) \) theories offer a chance to explain the acceleration of the universe, they are not free of problems. For instance, the application of their metric formulation to the homogeneous and isotropic Robertson-Walker geometry yields a fourth order nonlinear coupled (with the scalar field) differential equation for the scale factor \( a(t) \), which in general cannot be analytically solved even for the simplest form of \( F(R) \). Symmetry plays an important role to find analytical solutions, eg., it has been possible to find exact analytical solutions for \( R^2 \) gravity, by invoking Nöther symmetry [31]. Further, it has been shown [32], [33], that there exists a conserved current, other than Nöther current, for a general scalar tensor theory of gravity, nonminimally coupled to a scalar field under certain conditions. The use of such conserved current in tackling the field equations has also been demonstrated in [33].
Additionally, it has also been shown [33] that the same conserved current exists even for higher order theory of gravity, under a different condition. The importance of such symmetry had not been expatiated there. In the present work, we therefore would like to demonstrate the use of such symmetry in the context of exploring exact analytical solutions of higher order theory of gravity. It has been shown that under the assumption on the existence of such symmetry, the fourth order differential equation decouples scalar field from the gravitational field variables. Thus, it becomes less difficult to tackle the field equations. The study with a maximally symmetric formalism in $F(R)$ theory of gravity in the background of homogeneous and isotropic space time has been taken up recently [34]. However, the symmetry under consideration in the present work is independent of background space-time and the action has got a more general form.

In the following section we review our earlier work [33], i.e., find the conserved current for a general action corresponding to $F(R)$ theory of gravity. The action under consideration does not incorporate Gauss-Bonnet term in 4-dimensional space-time. Nevertheless, adding terms of the type $R^n$ is no less important, since they can produce early times inflation [35], and late time cosmic acceleration [36], [37]. In section 3, we take up $R^2$ theory of gravity as an example to demonstrate how it works suitably to explain cosmological evolution from early Universe till date.

2 Conserved current for an action containing higher order curvature invariant terms

The 4-dimensional string effective action under loop level approximation [38, 39] containing additional higher order curvature invariant terms can be expressed as,

$$A = \int \left[ f(\phi) R + BF(R) - \frac{\omega(\phi)}{\phi} \phi_{\mu \nu} \phi^{\mu \nu} - V(\phi) - \kappa L_m \right] \sqrt{-g} \, d^4x,$$

(1)

where, $F(R)$ is an arbitrary function of Ricci scalar, $B$ is the coupling constant and $L_m$ is the matter field Lagrangian, while $f(\phi)$ and $\omega(\phi)$ are coupling parameters, the later is of Brans-Dicke [40] origin. Field equations corresponding to the above action are,

$$f \left( R_{\mu \nu} - \frac{1}{2} g_{\mu \nu} R \right) + \frac{f}{2} g_{\mu \nu} - f_{;\alpha} g_{\mu \nu} - \frac{\omega}{\phi} \phi_{;\mu \nu} + \frac{1}{2} g_{\mu \nu} \left( \frac{\omega}{\phi} \phi_{;\alpha} \phi^{\alpha} + V(\phi) \right)$$

$$+ B \left( (F_R) R_{\mu \nu} - \frac{1}{2} F g_{\mu \nu} - (F_R)^{\alpha ; \alpha} g_{\mu \nu} - (F_R) ; \mu \nu \right) = \frac{\kappa}{2} T_{\mu \nu}$$

(2)

$$RF' + 2 \frac{\omega}{\phi} \phi_{;\mu} + \left( \frac{\omega}{\phi} - \frac{\omega}{\phi^2} \right) \phi^{\mu} \phi_{;\mu} - V'(\phi) = 0$$

(3)

In the above, $F_R$ denotes derivative of the function $F(R)$ with respect to $R$ and $T_{\mu \nu}$ is the energy momentum tensor corresponding to the matter field (barotropic fluid) Lagrangian, $L_m$. It is practically impossible to solve the above set of field equations even in the homogeneous and isotropic background geometry. In the homogeneous and isotropic background geometry, these are only a pair of independent field equations, with at least five field variables (if $F(R)$ is known a-priori), viz., $a, \phi, f(\phi), \omega(\phi)$ and $V(\phi)$, in the early Universe, and more $(\rho, p)$, in the radiation and matter dominated era. Thus, not only it is difficult to solve the above set of field equations due to the presence of fourth order coupled nonlinear differential equations, but also it requires at least three additional equations (assumptions) to obtain exact analytical solutions. One of the assumptions may be invoking the presence of dynamical symmetry, since it turns out to be a very powerful tool in this regard. It decouples the equations at one hand and gives additional equations, to make things a little bit tractable. In view of this discussion, let us now proceed to find the condition required to have some dynamical symmetry of the theory, directly from the field equations.

The trace of equation (2) is the following,

$$RF - 3f_{;\mu}^{\mu} - \frac{\omega}{\phi} \phi^{\mu} \phi_{;\mu} - 2V - B[R(F_R) + 3(F_R)^{\alpha ; \alpha} - 2F] = \frac{\kappa}{2} T^{\mu}_{\mu}.$$

(4)

Now since,

$$f_{;\mu}^{\mu} = f'' \phi^{\mu} \phi_{;\mu} + f' \phi_{;\mu}^{\mu},$$


so, multiplying equation (3) by $f$ and equation (4) by $f'$ and eliminating $Rff'$ between the two, we obtain,

\[
\left[3f'^2 + 2f\frac{\omega}{\phi}\right]_{,\mu} + \left[3f'^2 + 2f\frac{\omega}{\phi}\right]^{-\frac{1}{2}} Bf'[\nabla(F_{,\mu} + 3F_{,\mu})^\alpha_{\alpha} - 2F] + \frac{\kappa}{2} f' T_\mu^\mu - f³ \left(\frac{V}{f²}\right)' = 0.\tag{5}
\]

In view of equation (5) we can conclude that under the following condition,

\[
B[R(F_{,\mu} + 3F_{,\mu})^\alpha_{\alpha} - 2F] = -\frac{\kappa}{2} T_\mu^\mu + \frac{f^3}{f'} \left(\frac{V}{f²}\right)',
\]

there exists a conserved current $J^\mu$,

\[
J^\mu_\mu = [(3f'^2 + 2f\frac{\omega}{\phi})\frac{\phi}{\phi}]_{,\mu} = 0,
\]

corresponding to the general form of $F(R)$ theory of gravitational action (1). In the case of homogeneous cosmology, left hand side of condition (6) contains fourth order time derivative of the scale factor $a$, as it should be, while the right hand side is an ordinary function of $\phi$, during the early Universe and the radiation dominated era, since, $T_\mu^\mu$ vanishes at those epoch. In the matter dominated era ($\rho = 0$), an additional functional dependence of the scale factor $a$ appears on the right (since $\rho = \frac{\rho_0}{a^\alpha}$, $\rho_0$ being a constant). Further the expression for the conserved current $J^\mu$ given in (7) takes the following simplified form in the case of homogeneous cosmology,

\[
\sqrt{\left(3f'^2 + 2f\frac{\omega}{\phi}\right)} \phi = \frac{c}{\sqrt{-g}}
\]

where, $c$ is a non-vanishing constant and $g$ is the determinant of the metric. Hence, the cosmological equations corresponding to $F(R)$ theory of gravity becomes somewhat easier to tackle. In addition, the last term on the right hand side of condition (6) vanishes either for dilatonic scalar, $V(\phi) = 0$, or if the potential is proportional to the square of the coupling parameter $f(\phi)$. Therefore, in general, one can conclude that the following action corresponding to $F(R)$ theory of gravity,

\[
A = \int \left[f(\phi) R + B F(R) - \frac{\omega(\phi)}{\phi} \phi_{,\mu} \phi^{\mu} - V(\phi) - \kappa L_m\right] \sqrt{-g} \, d^4x,
\]

admits the integral of motion (7), under the condition

\[
B[R F_{,\mu} + 3F_{,\mu}]^\alpha_{\alpha} - 2F] = -\frac{\kappa}{2} T_\mu^\mu,
\]

provided,

\[
V = \lambda f²,
\]

where, $\lambda$ is a constant, which vanishes for dilatonic scalar. It is interesting to note that the left hand side of condition (10), under which the conserved current (7) exists, contains only geometrical part and is overall free from the scalar field. It is also to be noted that so far we have made two assumptions, the existence of symmetry and a relation between $f(\phi)$ and $V(\phi)$. The first assumption yielded a couple of additional equations viz., (7) and (10). So, altogether we now have four equations between (2), (3), (7) and (10), out of which only three are independent, since (7) and (10) are derived in view of equations (2) and (3). So at the end, we observe that the existence of symmetry, decouples the scalar field from the geometry, increasing the possibility of obtaining analytical solutions of the field equations. Additionally, instead of the very complicated form of equation (2), one can now use three considerably simpler equations out of the four, viz., (3), (4), (7) and (10). In fact equation (4) takes further simplified form in view of equation (10), as

\[
Rf - 3f'^2_{,\mu} - \frac{\omega}{\phi} \phi_{,\mu}^\mu - 2V = 0,
\]

where, $V = \lambda f²$, as already mentioned. Thus, under different choices of the function $F(R)$, analytical solution may be found just by solving the fourth order nonlinear differential equation (10) of the scale factor $a$, in the case of Robertson-Walker metric. One more assumption is required to find the form of $f(\phi), \omega(\phi)$, and $\phi = \phi(t)$, in view of other pair of independent equations (8), and (12). In the following section we cite a nontrivial example to demonstrate the use of such symmetry.
3 Cosmic evolution with $R^2$ term

The exact solutions of the fourth order gravity equations may sometimes be found considerably easily if we consider the existence of a conserved current given in equation (7). To establish the importance of the conserved current, let us consider homogeneous and isotropic cosmological model described by the Robertson-Walker metric,

\[
ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right],
\]

for which the Ricci scalar is given by,

\[
R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right).
\]

If we now take as an example, \(F(R) = R^2\), Then, equation (10), which is the condition for the existence of conserved current reads,

\[
R^\mu_\mu = \Box R = -(\ddot{R} + 3\frac{\dot{a}}{a}R) = \frac{\kappa}{2}T^\mu_\mu.
\]

Early Universe:

In the early Universe, \(\rho = p = 0\), ie., \(T^\mu_\mu = \rho - 3p = 0\), and so, \(\Box R = 0\), in view of equation (15), ie.,

\[
\ddot{R} + 3\frac{\dot{a}}{a}R = 0.
\]

The above equation admits exponential solution for \(k = 0\), and a solution in the form, \(a = a_0t\), for \(k \neq 0\). In the second case, the Universe undergoes just the amount of inflation required to solve the problems of isotropy and homogeneity, irrespective of the form of the potential, which may be zero as in the case of dilatonic scalar. Further, there is also no need of slow roll approximation. One has to make yet another choice at this stage to find other field variables, viz, \(f(\phi), \omega(\phi)\) and \(\phi(t)\). For example, one can choose a non-zero form of the potential \(V(\phi)\), to get \(f(\phi)\). Thus \(\omega(\phi)\) and \(\phi = \phi(t)\) may be found in view of equations (8) and (12).

Radiation dominated era:

Equation (16) also holds in the Radiation dominated era as well, since, \(T^\mu_\mu = \rho - 3p = 0\). Now, if inflation is sufficient to make the Universe spatially flat, ie., \(k = 0\), then, the above equation (16) admits the solution,

\[
a = a_0t^{\frac{7}{3}}.
\]

Thus, the Radiation era of Friedmann Universe remains unaltered even in the higher order theory of gravity. One can now find the matter density \(\rho\), as \(\rho a^3 = \rho^0\), \(\rho^0\) being a constant and so the pressure, \(p = \frac{\rho}{3}\) may be evaluated. Further, Using the same form of \(f(\phi)\) as fixed in the early Universe the form of \(\omega(\phi)\), with a different evolution history of \(\phi(t)\) may be found in view of equations (8) and (12).

Matter dominated era:

In the matter dominated era, \(p = 0\) and so, equation (15) reads, \(\Box R = \frac{\kappa}{2}T^\mu_\mu = \frac{\kappa}{2}\rho\). However, since,

\[
\rho a^3 = \rho_0,
\]

where, \(\rho_0\) is a constant, so equation (15) finally takes the following form,

\[
\ddot{R} + 3\frac{\dot{a}}{a}R = -\left(\frac{\kappa\rho_0}{12B}\right) \frac{1}{a^3}.
\]

In the isotropic and homogeneous case under consideration, the above equation translates to,

\[
a^2 \dddot{a} + aaa\ddot{a} - 2a^3 = -\left(\frac{\kappa\rho_0}{72B}\right) t + l,
\]
where $l$ is a constant of integration. Equation (20) admits a solution in the form,

$$a = a_0 t^\frac{4}{3},$$

(21)

provided, $\rho_0 = \frac{320 Ba_0^3}{\kappa}$ and $l = 0$. Hence the Universe undergoes an accelerating phase in the matter dominated era, as present cosmological observations suggest. $\rho$ can now be found in view of equation (18) and as discussed above a different form of $\omega(\phi)$ and $\phi = \phi(t)$ may be found taking the same form of $f(\phi)$ fixed in the early Universe.

4 Concluding remarks

The existence of a general form of symmetry and the corresponding conserved current for a general action corresponding to $F(R)$ theories of gravity has been found in an earlier work [33]. That the existence of the integral of motion in the cosmological context makes it easier to handle the field equations for studying exact solutions has been demonstrated here in the context of $R^2$ term. In this connection the following points are noteworthy.

1. Action (1) excludes Gauss-Bonnet term, since it is topologically invariant in 4-dimensional space-time. Nevertheless, the procedure followed in section (2) may be extended in higher dimensional space-time to incorporate such term.

2. One can relax condition (11) and use equation (6) instead. In that case, one may choose a functional $\phi$ dependent form of the last term on the right hand side of equation (6), which may be found for power law or exponential type of inflation in the early Universe.

3. In principal it may be possible to find a fixed form of $f(\phi)$ and $\omega(\phi)$, which would satisfy all the field equations from early Universe through matter dominated era with different cosmological evolution of $\phi(t)$.

4. Usually, early time inflation and late time cosmic acceleration are incorporated in modified theory of gravity. That the radiation dominated era of the Friedmann Universe remains unchanged even in such theory, is an additional outcome of the present work.

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