2-KAC-MOODY ALGEBRAS

RAPHAËL ROUQUIER

Abstract. We construct a 2-category associated with a Kac-Moody algebra and we study its 2-representations. This generalizes earlier work [ChRou] for $\mathfrak{sl}_2$. We relate categorifications relying on $K_0$ properties as in the approach of [ChRou] and 2-representations.

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1. **Introduction**

Over the past ten years, we have advocated the idea that there should exist monoidal categories (or 2-categories) with an interesting “representation theory”: we propose to call “2-representation theory” this higher version of representation theory and to call “2-algebras” those “interesting” monoidal additive categories. The difficulty in pinning down what is a 2-algebra (or a Hopf version) should be compared with the difficulty in defining precisely the meaning of quantum groups (or quantum algebras). The analogy is actually expected to be meaningful: while quantization turns certain algebras into quantum algebras, “categorification” should turn those algebras into 2-algebras. Dequantization is specialization $q \to 1$, while “decategorification” is the Grothendieck group construction — in the presence of gradings, it
leads to a quantum object. A large part of geometric representation theory should, and can, be viewed as a construction of “irreducible” 2-representations as categories of sheaves.

The starting point of the study of 2-representation theory of Lie algebras was the definition in 2003 of \( \mathfrak{sl}_2 \)-categorifications with Joseph Chuang and its study for \( \mathfrak{sl}_2 \) in [ChRou].

A crucial feature of 2-representation theory is the construction of a machinery that produces new categories out of some given categories (with extra structure). We believe this should be viewed as an algebraic counterpart of the construction of moduli spaces as families of sheaves or other objects on a variety. The following oversimplified diagram explains how our algebraic constructions would reproduce the various counting invariants based on moduli spaces, bypassing the moduli spaces and the difficulties of their construction and the construction of their invariants.

\[
\text{Variety } X \xrightarrow{\sim} \text{Moduli space } \mathcal{M} \text{ of objects on } X
\]

\[
\text{Category of sheaves on } X \xrightarrow{\sim} \text{Category of sheaves on } \mathcal{M}
\]

While our focus here is on classical algebraic objects (related in some way to 2-dimensional geometry), it is our belief that there should be 2-algebras associated with 3-dimensional geometry, possibly non-commutative, and that their higher representation theory would provide the proper algebraic framework for the various counting invariants (Gromov-Witten, Donaldson-Thomas,...).

In this paper, we define a 2-category \( \mathcal{A}(\mathfrak{g}) \) associated with a Kac-Moody algebra \( \mathfrak{g} \). Modulo some Hecke algebra isomorphisms, the generalization from type \( A \) (finite or affine) defined in joint work with Joseph Chuang is quite natural.

In [Rou2], we define and study tensor structures on the 2-category of 2-representations of \( \mathcal{A}(\mathfrak{g}) \) on dg-categories, with aim the construction of 4-dimensional topological quantum field theories. Our 2-categories associated with Kac-Moody algebras provide a solution to the question raised by Crane and Frenkel [CrFr] of the construction of “Hopf categories”.

The 2-category \( \mathcal{A}(\mathfrak{g}) \) “categorifies” (a completion of) the \( \mathbb{Z} \)-form \( U_\mathbb{Z}(\mathfrak{g}) \) of the enveloping algebra of \( \mathfrak{g} \). Consequently, a 2-representation of \( \mathcal{A}(\mathfrak{g}) \) on an exact or a triangulated category \( \mathcal{V} \) gives rise to an action of \( U_\mathbb{Z}(\mathfrak{g}) \) on \( K_0(\mathcal{V}) \). This gives a hint at the very non-semi-simplicity of the theory of 2-representations of \( \mathcal{A}(\mathfrak{g}) \). The presence of gradings actually gives rise to a “categorification” of the associated quantum group.

The Hecke algebras used in [ChRou] are replaced by nil Hecke algebras associated with Cartan matrices. Some of their specializations occur naturally as endomorphisms of correspondences for quiver varieties [Rou3]. In type \( A \), they occur when decomposing representations of (degenerate) affine Hecke algebras according to the spectrum of the polynomial subalgebra, and not just the center. These nil Hecke algebras can be defined by generators and relations and they also have a simple construction as a subalgebra of a wreath product algebra.

We construct more generally a flat family of “Hecke” algebras over the space of matrices over \( k[u, v] \) which are hermitian with respect to \( u \leftrightarrow v \). They are filtered with associated graded algebra a wreath product of a polynomial algebra by a nil Hecke algebra. They satisfy the PBW property.

Consider a monoidal category or a 2-category defined by generators and relations. A difficulty in 2-representation theory is to check the defining relations in examples. The philosophy of
was, instead of defining first the monoidal category, to describe directly what a 2-representation should be, using the action on the Grothendieck group. A key result of this paper is to provide a similar approach for Kac-Moody algebras. We show, under certain finiteness assumptions, that it is enough to check the relations $[e_i, f_j] = \delta_{ij} h_i$ on $K_0$. This is needed to show that the earlier definition of Chuang and the author of type $A$-categorifications coincides with the more general notion defined here. It is also a crucial ingredient for the construction of algebraic and geometric 2-representations in \cite{Rou3}.

Let us describe in more detail the constructions and results of the paper.

We set up some of the formalism to deal with 2-categories, presentations by generators and relations and 2-representations in \cite{2.2}. An important role is played by biadjoint pairs and \cite{2.3} develops the theory of symmetric algebras over non-commutative rings. In \cite{3.1} we gather classical results on Hecke algebras of type $A$: affine, degenerate affine and nil affine. We introduce Hecke algebras associated with hermitian matrices in \cite{3.2} and show they satisfy a PBW Theorem. We provide specializations associated with Cartan matrices and further specializations associated with quivers (with automorphisms).

Given a Cartan datum, we construct in \cite{4.1.3} a 2-category $\mathcal{A}$ with set of objects the weight lattice $X$ and with 1-arrows generated by $E_s : \lambda \to \lambda + \alpha_s$ and $F_s : \lambda \to \lambda - \alpha_s$. The 2-arrows are generated by units and counits of dual pairs $(E_s, F_s)$ and by $x_s \in \text{End}(E_s)$ and $\tau_{st} \in \text{Hom}(E_sE_t, E_tE_s)$. We impose relations so that there is an action of the nil Hecke algebras associated with the Cartan matrix on products $E_{s_1} \cdots E_{s_n}$ induced by $x_s$ and $\tau_{st}$. Finally, we invert certain maps relating $E_sF_t, F_tE_s$ and a multiple of $1$ — this accounts for the decomposition of $[e_s, f_t]$ in the corresponding Kac-Moody algebra $\mathfrak{g}$. There is a morphism of algebras from a completion of the $\mathbb{Z}$-form of the enveloping algebra of $\mathfrak{g}$ to the Grothendieck group of $\mathcal{A}$. The category $\mathcal{A}$ is defined over a base ring with indeterminates and a specialization of these leads to a graded category. There is a morphism from the completed quantized enveloping algebra of $\mathfrak{g}$ to the graded Grothendieck group.

We introduce integrable 2-representations of $\mathcal{A}$ in \cite{5.1.1}. We show that for integrable 2-representations of $\mathcal{A}$, there is a canonical adjunction $(F_s, E_s)$, giving rise to an action of a 2-category $\mathcal{A}'$ (\cite{4.1.3} and Theorem \ref{5.2}). We provide a construction of a 2-representation $\mathcal{V}(\lambda)$ with lowest weight $\lambda \in -X^+$ (\cite{5.1.2}) and show that lowest weight integrable 2-representations admit Jordan-Hölder filtrations (Theorem \ref{5.8}). The case of $\mathfrak{sl}_2$ is crucial for several proofs and \cite{5.2} is a study of its 2-representations $\mathcal{V}(\lambda)$. We also introduce three involutions $I, D$ and $i$ that allow to swap $E_s$ and $F_s$ in particular (\cite{4.2.1} and \cite{5.3.4}).

In \cite{5.3.3} we show that in the case of abelian categories over a field with finite composition series, the notion of $\mathfrak{sl}_2$-categorifications of \cite{ChRon} coincides with that of a 2-representation of $\mathcal{A}(\mathfrak{sl}_2)$. We generalize this to type $A$ (finite or affine) in \cite{5.3.8}. This builds on the isomorphisms between (degenerate) affine Hecke algebras and Hecke algebras associated with Cartan matrices of type $A$ constructed in \cite{3.2.6}. This provides a powerful way to construct 2-representations. We extend to general Kac-Moody algebras two key facts: the relations of type “[e_s, f_t] = 0 when $s \neq t$” are a consequence of the other axioms (\cite{5.3.5}) and for abelian categories as above, the relations of type “[e_s, f_s] = h_s” follow from their $K_0$ version (\cite{5.3.6}).

The main results of this paper have been announced at seminars in Orsay, Paris and Kyoto in the Spring 2007. Certain specializations of the nil Hecke algebras associated with quivers and the resulting monoidal categories associated with “half” Kac-Moody algebras have been introduced independently by Khovanov and Lauda \cite{KhoLau1, KhoLau2}. The relations between
Hecke algebras associated with affine type $A$ Cartan matrices and representations of finite Hecke algebras of type $A$ have been studied independently by Brundan and Kleshchev \cite{BrK}.

2. Preliminaries

2.1. Notations and conventions. Given $n \in \mathbb{Z}$, we put $[n] = \frac{v^n - v^{-n}}{v - v^{-1}}$, $[n]! = \prod_{i=1}^{n} [i]$ for $n \in \mathbb{Z}_{\geq 0}$. We put also

$$\binom{i}{n} = \frac{\prod_{a=0}^{n-1} (v^i - a - v^{-n-i})}{\prod_{a=1}^{n} (v^n - v^{-n})} \text{ for } i \in \mathbb{Z}.$$

Given $\Omega$ a finite interval of $\mathbb{Z}$, we denote by $\mathfrak{S}(\Omega)$ the symmetric group on $\Omega$, viewed as a Coxeter group with generating set $\{s_i = (i, i+1)\}$ where $i$ runs over the non-maximal elements of $\Omega$. We denote by $w(\Omega)$ the longest element of $\mathfrak{S}(\Omega)$. Given $E$ a family of disjoint intervals of $\Omega$, we put $\mathfrak{S}(E) = \prod_{E' \subseteq E} \mathfrak{S}(\Omega')$ and we denote by $\mathfrak{S}(\Omega)^E$ (resp. $E\mathfrak{S}(\Omega)$) the set of minimal length representations of $\mathfrak{S}(\Omega)/\mathfrak{S}(E)$ (resp. $\mathfrak{S}(E) \setminus \mathfrak{S}(\Omega)$). We put $\mathfrak{S}_n = \mathfrak{S}[1, n]$. Given $w \in \mathfrak{S}_n$, we put $\delta_w = \delta_{1,w}$.

Let $k$ be a commutative ring. We write $\otimes$ for $\otimes_k$. Given $M$ a graded $k$-module and $i$ an integer, we denote by $M(i)$ the graded $k$-module given by $M(i)_n = M_{n+i}$.

Given $P = \sum_{i \in \mathbb{Z}} p_i v^i \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ a Laurent polynomial with non-negative coefficients, we put $P_k = \bigoplus_{i \in \mathbb{Z}} k^{|i|}(-i)$. Given $k'$ a $k$-algebra and $M$ a $k$-module, we put $k'M = k' \otimes_k M$. We also put $PM = Pk \otimes M$.

An $A$-algebra is an algebra $B$ endowed with a morphism of algebras $A \to B$. Given $B$ an $A$-algebra, we say that a $B$-module is relatively $A$-projective if it is a direct summand of $B \otimes_A M$ for some $A$-module $M$.

Categories are denoted by calligraphic letters $\mathcal{A}, \mathcal{B}, \mathcal{C}$, etc. and 2-categories are denoted by gothic letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$, etc.

We denote by $\text{Ob}(\mathcal{A})$ or by $\mathcal{A}$ the set of objects of a category (or of a 2-category) $\mathcal{A}$. Given $a$ an object, we will denote by $a$ or $1_a$ or $\text{id}_a$ the identity of $a$.

Given $F, G : \mathcal{A} \to \mathcal{B}$ two functors, a morphism $F \to G$ is the data of a compatible collection of arrows $F(a) \to G(a)$ for $a \in \mathcal{A}$ and we call these natural morphisms.

We say that an endofunctor $F$ of an additive category $\mathcal{C}$ is locally nilpotent if for every $M \in \mathcal{C}$, there is $n > 0$ such that $F^n(M) = 0$.

We denote by $\text{Sets}$ (resp. $\mathfrak{A}b$) the category of sets (resp. of abelian groups). We denote by $\mathcal{A}\text{-Mod}$ the category of $A$-modules, by $\mathcal{A}\text{-mod}$ the category of finitely generated $A$-modules and by $\mathcal{A}$-free is full subcategory of free $A$-modules of finite rank. Here, module means left module. Given $\mathcal{A}$ an additive category, we denote by $\text{Comp}^b(\mathcal{A})$ the category of bounded complexes of objects of $\mathcal{A}$ and by $\text{Ho}^b(\mathcal{A})$ the associated homotopy category.

We denote by $\mathfrak{C}\text{at}$ (resp. $\mathfrak{A}\text{dd}$, $\mathfrak{L}\text{in}_k$, $\mathfrak{A}b$, $\mathfrak{T}\text{ri}$) the strict 2-category of categories (resp. of additive categories, of $k$-linear categories, of abelian categories with exact functors, of triangulated categories). When $k$ is a field, we denote by $\mathfrak{A}b_k^f$ the 2-category of $k$-linear abelian categories all of whose objects have finite composition series and such that $k = \text{End}(V)$ for any simple object $V$ (1-arrows are $k$-linear exact functors).
2.2. 2-Categories. We set up in this section the appropriate formalism for 2-representation theory. At first, we recall the more classical setting of representation theory as a study of functors.

2.2.1. Categories. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two categories. We denote by \( \mathcal{H} \text{om}(\mathcal{A}, \mathcal{B}) \) the category of functors \( \mathcal{A} \to \mathcal{B} \): we think of these as representations of \( \mathcal{A} \) in \( \mathcal{B} \). For example, if \( \mathcal{A} \) has a unique object * and \( \mathcal{B} = \text{Sets} \), the category \( \mathcal{H} \text{om}(\mathcal{A}, \mathcal{B}) \) is equivalent to the category of sets acted on by the monoid \( \text{End}(*) \).

Given \( a \in \mathcal{A} \), we have a functor \( \text{Hom}(a, -) : \mathcal{A} \to \text{Sets} \) (the regular representation when \( \mathcal{A} \) has a unique object).

We put \( \mathcal{A}^v = \mathcal{H} \text{om}(\mathcal{A}^{\text{opp}}, \text{Sets}^{\text{opp}}) \). The functor

\[
\mathcal{A} \to \mathcal{A}^v, \quad M \mapsto \text{Hom}(M, -)
\]

is fully faithful (Yoneda’s Lemma) and we identify \( \mathcal{A} \) with a full subcategory of \( \mathcal{A}^v \) through this embedding.

Assume \( \mathcal{A} \) is enriched in abelian groups. The additive closure of \( \mathcal{A} \) is the full additive subcategory \( \mathcal{A}^a \) of the category of functors \( \mathcal{A}^{\text{opp}} \to \mathcal{A}^{\text{opp}} \) generated by objects of \( \mathcal{A} \). Given \( \mathcal{A}' \) an additive category, the restriction functor gives an equivalence from the category of additive functors \( \mathcal{A}^a \to \mathcal{A}' \) to the category of functors enriched in abelian groups \( \mathcal{A} \to \mathcal{A}' \).

Assume \( \mathcal{A} \) is an additive category. We denote by \( \mathcal{A}^i \) the idempotent completion of \( \mathcal{A} \). Given \( \mathcal{A}' \) an idempotent-complete additive category, restriction gives an equivalence from the category of additive functors \( \mathcal{A}^i \to \mathcal{A}' \) to the category of functors enriched in abelian groups \( \mathcal{A} \to \mathcal{A}' \).

Let \( M \in \mathcal{A} \) and let \( L \) be a right \( \text{End}(M) \)-module. We denote by \( L \otimes_{\text{End}(M)} M \) the object of \( \mathcal{A}^{\text{opp}} \) defined by \( \text{Hom}_{\text{End}(M)^{\text{opp}}}(L, \text{Hom}(M, -)) \).

Given \( A \) a ring, the category of \( A \)-modules in \( \mathcal{A} \) is the category of additive functors \( A \to \mathcal{A} \), where \( A \) is the category with one object * and with \( \text{End}(*) = A \). An object of that category is an object \( M \) of \( \mathcal{A} \) endowed with a morphism of rings \( A \to \text{End}(M) \).

Given an \( A \)-module \( M \) in \( \mathcal{A} \) and \( L \) a right \( A \)-module, we put \( L \otimes_A M = (L \otimes_A \text{End}(M)) \otimes_{\text{End}(M)} M \). For example, there is a canonical isomorphism \( \mathbb{Z}^n \otimes_{\mathbb{Z}} M \cong M^n \).

Let \( B \) be a commutative ring endowed with a morphism \( B \to \mathbb{Z}(A) \) and let \( A \) be a \( B \)-algebra. We denote by \( \mathcal{A} \otimes_B \mathcal{A} \) the additive category with same objects as \( \mathcal{A} \) and \( \text{Hom}_{\mathcal{A} \otimes_B \mathcal{A}}(M, N) = \text{Hom}_A(M, N) \otimes_B A \), where \( B \) acts via \( \mathbb{Z}(A) \). Let \( \mathcal{A}' \) be \( B \)-linear category. We denote by \( \mathcal{A} \otimes_B \mathcal{A}' \) the additive closure of the category with set of objects \( \text{Ob}(\mathcal{A}) \times \text{Ob}(\mathcal{A}') \) and with \( \text{Hom}((M, M'), (N, N')) = \text{Hom}_A(M, N) \otimes_B \text{Hom}_{\mathcal{A}'}(M', N') \). Given \( \mathcal{A}'' \) a \( B \)-linear category, there is an equivalence between \( \mathcal{H} \text{om}_{\mathcal{A} \otimes_B \mathcal{A}'}(\mathcal{A} \otimes_B \mathcal{A}'', \mathcal{A}'') \) and the category of \( B \)-bilinear functors \( \mathcal{A} \times \mathcal{A}' \to \mathcal{A}'' \).

An equivalence relation \( \sim \) on a category is a relation on arrows such that \( f \sim f' \) implies \( fg \sim f'g \) and \( gf \sim gf' \) (whenever this makes sense). Given \( \mathcal{A} \) a category and \( \sim \) a relation on arrows of \( \mathcal{A} \), we have a quotient category \( \mathcal{A}/\sim \) with same objects as \( \mathcal{A} \). The quotient functor \( \mathcal{A} \to \mathcal{A}/\sim \) induces a fully faithful functor \( \mathcal{H} \text{om}(\mathcal{A}/\sim, \mathcal{B}) \to \mathcal{H} \text{om}(\mathcal{A}, \mathcal{B}) \) for any category \( \mathcal{B} \).

A functor is in the image if and only if two equivalent arrows have the same image under the functor. The construction depends only on the equivalence relation on \( \mathcal{A} \) generated by \( \sim \).

Let \( k \) be a commutative ring and \( \mathcal{A} \) a \( k \)-linear category. Given \( S \) a set of arrows of \( \mathcal{A} \), let \( \sim = \sim_S \) be the coarsest equivalence relation on \( \mathcal{A} \) such that \( f \sim 0 \) for every \( f \in S \) and \( \{(f, g) \mid f \sim g\} \) is a \( k \)-submodule of \( \text{Hom}(a, a') \oplus \text{Hom}(a, a') \). We denote by \( \mathcal{A}/S = \mathcal{A}/\sim \).
the quotient \(k\)-linear category: a \(k\)-linear functor \(A \to B\) factors through \(A/S\), and then the factorization is unique up to unique isomorphism, if and only if it sends arrows in \(S\) to 0.

Given \(A\) a category, we denote by \(kA\) the \(k\)-linear category associated with \(A\): there is a canonical functor \(A \to kA\) and given a \(k\)-linear category \(B\) and a functor \(F : A \to B\), there is a \(k\)-linear functor \(G : kA \to B\) unique up to unique isomorphism such that \(F = G \cdot \text{can}\).

Let \(I = (I_0, I_1, s, t)\) be a quiver: this is the data of

- a set \(I_0\) (vertices) and a set \(I_1\) (arrows)
- maps \(s, t : I_1 \to I_0\) (source and target).

We denote by \(P = P(I)\) the set of paths in \(I\), i.e., sequences \((b_1, \ldots, b_n)\) of elements of \(I_1\) such that \(t(b_i) = s(b_{i-1})\) for \(1 < i \leq n\). It comes with maps \(s : P \to I_0\), \(b_{i-1} \mapsto s(b_i)\) (source) and \(t : P \to I_0\), \(b_i \mapsto t(b_1)\) (target). We write \(b_1 \cdots b_n\) for the element \((b_1, \ldots, b_n)\) of \(P\).

We denote by \(C(I)\) the category generated by \(I\). Its set of objects is \(I_0\) and \(\text{Hom}(i, j) = (s, t)^{-1}(i, j)\). Composition is concatenation of paths.

Let \(A\) be a category. The category of diagrams of type \(I\) in \(A\) is canonically isomorphic to the category of functors \(C(I) \to A\) (the isomorphism is given by restricting the functor).

A graded category is a category endowed with a self-equivalence \(T\). Given \(M\) an object with isomorphism class \([M]\), we put \(v[M] = [T^{-1}(M)]\).

The 2-category of graded \(k\)-linear categories is equivalent to the 2-category of \(k\)-linear categories enriched in graded \(k\)-modules:

- Let \(\mathcal{C}\) be a graded \(k\)-linear category. We define \(\mathcal{D}\) as the category with objects those of \(\mathcal{C}\) and with \(\text{Hom}_\mathcal{D}(V, W) = \bigoplus_i \text{Hom}_\mathcal{C}(V, T^iW)\). The composition of the maps of \(\mathcal{D}\) coming from maps \(f : V \to T^iW, g : W \to T^jX\) of \(\mathcal{C}\) is the map coming from \(T^i(g) \circ f : V \to T^{i+j}X\).
- Let \(\mathcal{D}\) be a \(k\)-linear category enriched in graded \(k\)-modules. Define \(\mathcal{C}\) as the category with objects families \(\{V_i\}_{i \in \mathbb{Z}}\) with \(V_i\) an object of \(\mathcal{D}\) and \(V_i = 0\) for almost all \(i\). We put \(\text{Hom}_\mathcal{C}(\{V_i\}, \{W_i\}) = \bigoplus_{i,j} \text{Hom}_\mathcal{D}(V_i, W_j)_{j-i}\). We define \(T(\{V_i\})_n = V_{n+1}\).

2.2.2. Definitions. Our main reference for basic definitions and results on 2-categories is [Gra](#). (cf also [Le](#) for the basic definitions).

**Definition 2.1.** A 2-category \(\mathfrak{A}\) is the data of

- a set \(\mathfrak{A}_0\) of objects
- categories \(\text{Hom}(a, a')\) for \(a, a' \in \mathfrak{A}_0\)
- functors \(\text{Hom}(a_1, a_2) \times \text{Hom}(a_2, a_3) \to \text{Hom}(a_1, a_3), (b_1, b_2) \mapsto b_2b_1\) for \(a_1, a_2, a_3 \in \mathfrak{A}\)
- functors \(\text{End}(a) \in \mathfrak{A}\)
- natural isomorphisms \((b_3b_2)b_1 \sim b_3(b_2b_1)\) for \(b_i \in \text{Hom}(a_i, a_{i+1})\) and \(a_1, \ldots, a_4 \in \mathfrak{A}\).
- natural isomorphisms \(bI_n \sim b\) for \(b \in \text{Hom}(a', a)\) and \(a, a' \in \mathfrak{A}\)
- natural isomorphisms \(I_n b \sim b\) for \(b \in \text{Hom}(a', a)\) and \(a, a' \in \mathfrak{A}\)


such that the following diagrams commute

\[
\begin{array}{ccc}
(b_4 b_3) b_1 & \xrightarrow{\text{can}(b_4, b_3, b_2) \cdot b_1} & b_4 (b_3 b_2) b_1 \\
\downarrow \text{can}(b_4, b_3, b_2, b_1) & & \downarrow \text{can}(b_4, b_3, b_2, b_1) \\
(b_4 b_3) (b_2 b_1) & \xrightarrow{\text{can}(b_4, b_3, b_2 b_1) \cdot b_1} & b_4 (b_3 (b_2 b_1)) \\
\downarrow \text{can}(b_4, b_3, b_2 b_1) & & \downarrow \text{b}_4 \cdot \text{can}(b_4, b_3, b_2 b_1) \\
\end{array}
\]

\[
\begin{array}{ccc}
(b_2 I_a) b_1 & \xrightarrow{\text{can}(b_2, I_a, b_1) \cdot b_2} & b_2 (I_a b_1) \\
\downarrow \text{can}(b_2) \cdot b_1 & & \downarrow \text{b}_2 \cdot \text{can}(b_1) \\
b_2 b_1 & & \end{array}
\]

Note that 2-categories are called bicategories in [Gra]. A strict 2-category is a 2-category where the associativity and unit isomorphisms are identity maps: \((b_3 b_2) b_1 = b_3 (b_2 b_1)\) and \(b I_a = b, I_a b = b\) (called 2-category in [Gra]).

Let \(\mathcal{A}\) be a 2-category. Its 1-arrows (resp. 2-arrows) are the objects (resp. arrows) of the categories \(\text{Hom}(a, a')\)

Given \(b : a \to a'\) and \(b' : a' \to a''\) two 1-arrows, we denote by \(b' b : a \to a''\) their composition. The composition of 2-arrows \(c\) and \(c'\) (viewed as arrows in a category \(\text{Hom}(a, a')\)) is denoted by \(c' \circ c\). Given \(a, a'\) and \(a''\) three objects of \(\mathcal{A}\), \(b_1, b_2 : a \to a'\), \(c : b_1 \to b_2\) and \(b_1', b_2' : a' \to a''\), \(c' : b_1' \to b_2'\), we denote by \(c' c : b_1 b_1' \to b_2 b_2'\) the “juxtaposition”.

We say that a 1-arrow \(b : a_1 \to a_2\) is

- an equivalence if there is a 1-arrow \(b' : a_2 \to a_1\) and isomorphisms \(I_{a_1} \simeq b' b\) and \(b b' \simeq I_{a_2}\)
- fully faithful if given any object \(a''\), the functor \(\text{Hom}(a'', b) : \text{Hom}(a'', a_1) \to \text{Hom}(a'', a_2)\) is fully faithful.

Note that these notions coincide with the usual notions for \(\mathcal{A} = \mathcal{C}at, \mathcal{A} = \mathcal{A}dd, \mathcal{A} = \mathcal{A}b\) or \(\mathcal{A} = \mathcal{T}ri\).

Given a 2-category \(\mathcal{A}\), we denote by \(\mathcal{A}_{\leq 1}\) the category with objects those of \(\mathcal{A}\) and with arrows the isomorphism classes of 1-arrows of \(\mathcal{A}\).

The opposite 2-category \(\mathcal{A}^{\text{opp}}\) of \(\mathcal{A}\) has same set of objects as \(\mathcal{A}\) and \(\text{Hom}_{\mathcal{A}}(a, a') = \text{Hom}_{{\mathcal{A}^{\text{opp}}}}(a', a)\), while the rest of the structure is inherited from that of \(\mathcal{A}\).

The reverse 2-category \(\mathcal{A}^{\text{rev}}\) of \(\mathcal{A}\) has same set of objects as \(\mathcal{A}\) and \(\text{Hom}_{\mathcal{A}}(a, a') = \text{Hom}_{{\mathcal{A}^{\text{rev}}}}(a', a)\).

The composition

\(\text{Hom}_{\mathcal{A}}(a_1, a_2) \times \text{Hom}_{\mathcal{A}}(a_2, a_3) \to \text{Hom}_{\mathcal{A}}(a_1, a_3)\)

is given by \((b_1, b_2) \mapsto b_1 b_2\) (composition in \(\mathcal{A}\)). The rest of the structure is inherited from that of \(\mathcal{A}\).

**Definition 2.2.** A 2-functor \(R : \mathcal{A} \to \mathcal{B}\) between 2-categories is the data of

- a map \(R : \text{Ob}(\mathcal{A}) \to \text{Ob}(\mathcal{B})\)
- functors \(R : \text{Hom}(a, a') \to \text{Hom}(R(a), R(a'))\) for \(a, a' \in \mathcal{A}\)
- natural isomorphisms \(R(b_2) R(b_1) \sim R(b_2 b_1)\) for \(b_1, b_2 1\)-arrows of \(\mathcal{A}\)
Definition 2.4. A morphism such that the following diagrams commute

\[ \begin{array}{c}
    (R(b_3)R(b_2))R(b_1) \xrightarrow{\text{can}(b_3,b_2).R(b_1)} R(b_3b_2)R(b_1) \xrightarrow{\text{can}(b_3b_2,b_1)} R((b_3b_2)b_1) \\
    \quad \xrightarrow{R(\text{can}(b_3,b_2))} R(b_3(b_2b_1)) \\
  \end{array} \]

\[ \begin{array}{c}
    R(b_3)(R(b_2)R(b_1)) \xrightarrow{R(b_3)-\text{can}(b_2,b_1)} R(b_3b_2b_1) \xrightarrow{\text{can}(b_3b_2b_1)} R(b_3(b_2b_1)) \\
    \quad \xrightarrow{R(\text{can}(b_2,b_1))} R(b_2b_1b_3) \\
  \end{array} \]

\[ \begin{array}{c}
    R(b)I_{R(a)} \xrightarrow{R(b)-\text{can}(a)} R(b)R(I_a) \xrightarrow{\text{can}(h,I_a)} R(I_a')R(b) \\
    \quad \xrightarrow{R(\text{can}(b))} R(I_{a'}b) \\
  \end{array} \]

When the 2-arrows are identity maps \( I_{R(a)} = R(I_a) \), we say that the 2-functor is strict (called strict pseudo-functor in [Gra]).

Definition 2.3. A morphism of 2-functors \( \sigma : R \rightarrow R' \) is the data of

- 1-arrows \( \sigma(a) : R(a) \rightarrow R'(a) \)
- natural isomorphisms \( R'(b)\sigma(a_1) \xrightarrow{\sim} \sigma(a_2)R(b) \) for all 1-arrows \( b : a_1 \rightarrow a_2 \)

such that the following diagrams commute

\[ \begin{array}{c}
    (R'(b_2)R'(b_1))\sigma(a_1) \xrightarrow{\text{can}(R'(b_2),R'(b_1),\sigma(a_1))} R'(b_2)(R'(b_1)\sigma(a_1)) \xrightarrow{R'(b_2)-\text{can}(b_1)} R'(b_2)(\sigma(a_2)R(b_1)) \\
    \quad \xrightarrow{\text{can}(R'(b_2),\sigma(a_2),R(b_1))^{-1}} R'(\sigma(a_2))R(b_1) \\
  \end{array} \]

\[ \begin{array}{c}
    R'(b_2b_1)\sigma(a_1) \xrightarrow{\text{can}(b_2,b_1).\sigma(a_1)} \sigma(a_3)(R(b_2)R(b_1)) \xrightarrow{\text{can}(\sigma(a_3),R(b_2),R(b_1))} (\sigma(a_3)R(b_2))R(b_1) \\
    \quad \xrightarrow{\text{can}(\sigma(a_3),R(b_2),R(b_1))^{-1}} (\sigma(a_3)R(b_2))R(b_1) \\
  \end{array} \]

\[ \begin{array}{c}
    I_{R'(a)}\sigma(a) \xrightarrow{\text{can}} \sigma(a) \xrightarrow{\text{can}^{-1}} \sigma(a)I_{R(a)} \\
    \quad \xrightarrow{\text{can}(\sigma(a))} \sigma(a)R(I_a) \\
  \end{array} \]

These are quasi-natural transformations with invertible 2-arrows in [Gra].

Definition 2.4. A morphism \( \gamma : \sigma \rightarrow \tilde{\sigma} \), where \( \sigma,\tilde{\sigma} : R \rightarrow R' \) are morphisms of 2-functors, is the data of 2-arrows \( \gamma(a) : \sigma(a) \rightarrow \tilde{\sigma}(a) \) for \( a \in \mathbb{A} \) such that the following diagrams commute

\[ \begin{array}{c}
    R'(b)\sigma(a_1) \xrightarrow{R'(b)\gamma(a_1)} R'(b)\tilde{\sigma}(a_1) \\
    \quad \xrightarrow{\text{can}} \tilde{\sigma}(a_2)R(b) \\
  \end{array} \]

\[ \begin{array}{c}
    \sigma(a_2)R(b) \xrightarrow{\gamma(a_2)R'(b)} \tilde{\sigma}(a_2)R(b) \\
    \quad \xrightarrow{\text{can}} \tilde{\sigma}(a_2)R(b) \\
  \end{array} \]
These are called modifications in \([\text{Gra}]\).

We denote by \(\text{Hom}(\mathfrak{A}, \mathfrak{B})\) the 2-category of 2-functors \(\mathfrak{A} \to \mathfrak{B}\). When \(\mathfrak{B}\) is a strict 2-category, then \(\text{Hom}(\mathfrak{A}, \mathfrak{B})\) is strict as well.

Given a property of functors, we say that a 2-functor \(F: \mathfrak{A} \to \mathfrak{B}\) has locally that property if the functors \(\text{Hom}(a, a') \to \text{Hom}(F(a), F(a'))\) have the property for all \(a, a'\) objects of \(\mathfrak{A}\).

A 2-functor \(F: \mathfrak{A} \to \mathfrak{B}\) is a 2-equivalence if there is a 2-functor \(G: \mathfrak{B} \to \mathfrak{A}\) and equivalences \(\text{id}_a \sim GF\) and \(FG \sim \text{id}_b\). This is equivalent to the requirement that \(F\) is locally an equivalence and every object of \(\mathfrak{B}\) is equivalent to an object in the image of \(F\).

Every 2-category is 2-equivalent to a strict 2-category, but there are 2-functors between strict 2-categories that are not equivalent to strict ones.

Given \(a\) an object of \(\mathfrak{A}\), then \(\text{End}(a)\) is a monoidal category. Conversely, a monoidal category gives rise to a 2-category with a single object \(*\), and the notion of monoidal functor coincides with that of 2-functor (i.e., there is a 1, 2, 3-fully faithful strict 3-functor from the 3-category of monoidal categories to that of 2-categories).

Let \(k\) be a commutative ring. A \(k\)-linear 2-category is a 2-category \(\mathfrak{A}\) that is locally \(k\)-linear and such that juxtaposition is \(k\)-linear. Given \(\mathfrak{A}\) and \(\mathfrak{B}\) two \(k\)-linear 2-categories, we denote by \(\text{Hom}(\mathfrak{A}, \mathfrak{B})\) the 2-category of \(k\)-linear 2-functors \(\mathfrak{A} \to \mathfrak{B}\). This is the locally full sub-2-category of 2-functors obtained by requiring the functors in the definition of 2-functors to be \(k\)-linear.

Given \(\mathfrak{A}\) a 2-category, we denote by \(k\mathfrak{A}\) the \(k\)-linear closure of \(\mathfrak{A}\): its objects are those of \(\mathfrak{A}\) and \(\text{Hom}_{k\mathfrak{A}}(a, a') = k\text{Hom}_\mathfrak{A}(a, a')\).

Let \(b: a \to a'\) be a 1-arrow. A right adjoint (or right dual) of \(b\) is a triple \((b^\vee, \varepsilon_b, \eta_b)\) where \(b^\vee: a' \to a\) is a 1-arrow and \(\varepsilon_b: b b^\vee \to I_{a'}\) and \(\eta_b: I_a \to b^\vee b\) are 2-arrows such that the compositions

\[
\begin{align*}
    b & \longrightarrow b I_a \quad \quad b \eta_b \quad \longrightarrow \quad b(b^\vee b) \quad \longrightarrow \quad (bb^\vee)b \quad \longrightarrow \quad I_{a'} b \quad \longrightarrow \quad b
\end{align*}
\]

and

\[
\begin{align*}
    b^\vee & \longrightarrow I_a b^\vee \quad \quad \eta b^\vee \quad \longrightarrow \quad (b^\vee b) b^\vee \quad \longrightarrow \quad b^\vee (bb^\vee) \quad \longrightarrow \quad b^\vee I_{a'} \quad \longrightarrow \quad b^\vee
\end{align*}
\]

are identities. We also say that \((b, \varepsilon_b, \eta_b)\) is a left adjoint (or dual) of \(b' = b^\vee\) (and we write \(b = b^\vee\)) and we say that \((b, b^\vee, \varepsilon_b, \eta_b)\) (or simply \((b, b^\vee)\)) is an adjoint quadruple (resp. an adjoint pair).

To simplify the exposition, let us assume for the reminder of \([\text{2.2.2}]\) that \(\mathfrak{A}\) is strict. Let \((b^\vee, \varepsilon_b, \eta_b)\) be a triple such that \((\varepsilon_b b) \circ (b\eta_b)\) and \((b^\vee \varepsilon_b) \circ (\eta_b b^\vee)\) are invertible. Then, \((b^\vee, \varepsilon_b, (b^\vee((\varepsilon_b b) \circ (b\eta_b))) \circ \eta_b)\) is a right adjoint of \(b\).

Given \(b_1: a \to a'\) a 1-arrow and \((b_1, b_1')\) an adjoint pair, we have a canonical isomorphism

\[
\text{Hom}(b, b_1) \sim \text{Hom}(b_1', b^\vee), \quad f \mapsto f^\vee = (b^\vee \varepsilon_{b_1}) \circ (b^\vee f b_1') \circ (\eta_b b_1').
\]

Assume now there are dual pairs \((b, b^\vee)\) and \((b^\vee, b)\). We have an automorphism

\[
(1) \quad \text{End}(b) \sim \text{End}(b), \quad f \mapsto (f^\vee)^.\]
2.2.3. Generators and relations. An equivalence relation \( \sim \) on \( \mathfrak{A} \) is the data for every \( a, a' \) objects, for every \( b, b' : a \to a' \) of an equivalence relation on \( \text{Hom}(b, b') \) compatible with composition and juxtaposition, i.e., if \( c_1 \sim c_2 \), then given a 2-arrow \( c \), we have \( c_1 \circ c \sim c_2 \circ c \), \( c_1 \circ c \sim c \circ c_2 \), \( c_1 \circ c_2 \sim c_2 \circ c_1 \) and \( c_1 \circ c_2 \sim c_2 \circ c_1 \), whenever this makes sense. Given a relation \( \sim \) on 2-arrows of \( \mathcal{C} \), the equivalence relation generated by \( \sim \) is the coarsest refinement of \( \sim \) that is an equivalence relation.

Let \( \mathfrak{A} \) be a 2-category and \( \sim \) an equivalence relation. We denote by \( \mathfrak{A}/\sim \) the 2-category with same objects as \( \mathfrak{A} \) and with \( \text{Hom}_{\mathfrak{A}/\sim}(a, a') = \text{Hom}_{\mathfrak{A}}(a, a')/\sim \) (so, \( \mathfrak{A}/\sim \) has the same 1-arrows as \( \mathfrak{A} \)). The local quotient functors induce a strict quotient 2-functor \( \mathfrak{A}/\sim \). Given a 2-category \( \mathfrak{B} \), the quotient strict 2-functor \( \mathfrak{A} \to \mathfrak{B}/\sim \) induces a strict 2-functor \( \text{Hom}(\mathfrak{A}/\sim, \mathfrak{B}) \to \text{Hom}(\mathfrak{A}, \mathfrak{B}) \) that is locally an isomorphism. A 2-functor \( R \) is in the image if and only if two equivalent 2-arrows have the same image under \( R \).

Given \( S \) a set of 2-arrows of \( \mathfrak{A} \), we denote by \( \tilde{S} \) the smallest set of 2-arrows of \( \mathfrak{A} \) closed under juxtaposition and composition and containing \( S \) and the invertible 2-arrows.

We denote by \( \mathfrak{A}[S^{-1}] \) the 2-category with same objects as \( \mathfrak{A} \) and with \( \text{Hom}_{\mathfrak{A}[S^{-1}]}(a, a') = \text{Hom}_{\mathfrak{A}}(a, a')/[S(a, a')^{-1}] \), where \( S(a, a') \) are the 2-arrows of \( \tilde{S} \) that are in \( \text{Hom}_{\mathfrak{A}}(a, a') \) (so, \( \mathfrak{A}[S^{-1}] \) has the same 1-arrows as \( \mathfrak{A} \)).

The canonical strict 2-functor \( \mathfrak{A} \to \mathfrak{A}[S^{-1}] \) induces a strict 2-functor \( \text{Hom}(\mathfrak{A}[S^{-1}], \mathfrak{B}) \to \text{Hom}(\mathfrak{A}, \mathfrak{B}) \) that is locally an isomorphism. A 2-functor \( R \) is in the image if and only if the image under \( R \) of any 2-arrow in \( S \) is invertible.

Assume \( \mathfrak{A} \) is a \( k \)-linear 2-category. Let \( S \) be a set of 2-arrows of \( \mathfrak{A} \). Given \( a, a' \) objects of \( \mathfrak{A} \), we consider the equivalence relation \( \sim_{S(a,a')} \) on \( \text{Hom}(a, a') \). Let \( \sim \) be the coarsest equivalence relation on \( \mathfrak{A} \) that refines the relations \( \sim_{S(a,a')} \). We put \( \mathfrak{A}/S = \mathfrak{A}/\sim \).

A 2-quiver \( I = (I_0, I_1, I_2, s, t, s_2, t_2) \) is the data of
- three sets \( I_0 \) (vertices), \( I_1 \) (1-arrows) and \( I_2 \) (2-arrows)
- maps \( s, t : I_1 \to I_0 \) (source and target)
- maps \( s_2, t_2 : I_2 \to P = P(I_0, I_1, s, t) \) (source and target of 2-arrows) such that \( s(s_2(c)) = s(t_2(c)) \) and \( t(s_2(c)) = t(t_2(c)) \) for all \( c \in I_2 \).

Let \( I \) be a 2-quiver. Let \( a, a' \in I_0 \). We define a quiver \( I(a, a') = (I_0, I_1, \tilde{s}, \tilde{t}) \). We put \( \tilde{I}_0 = (s, t)^{-1}(a, a') \), the set of paths from \( a \) to \( a' \). The set \( \tilde{I}_1 \) is given by triples \( (b, c, b') \) where \( b, b' \in P \), \( c \in I_2 \) satisfy \( t(b') = s(s_2(c)), t(s_2(c)) = s(b), s(b') = a, t(b) = a' \). We put \( \tilde{s}(b, c, b') = bs_2(c)b' \) and \( \tilde{t}(b, c, b') = bt_2(c)b' \). We introduce a relation \( \sim \) on \( P(I(a, a')) \) by
\[
(b_1t_2(c_1)b_2, c_2, b_3)(b_1, c_1, b_2s_2(c_2)b_3) \sim (b_1, c_1, b_2t_2(c_2)b_3)(b_1s_2(c_1)b_2, c_2, b_3)
\]
(whenever this makes sense).

The strict 2-category \( \mathfrak{C}(I) \) generated by \( I \) is defined as follows. Its set of objects is \( I_0 \). We put \( \text{Hom}(a, a') = \mathcal{C}(I(a, a'))/\sim \). Composition of 1-arrows is concatenation of paths. Juxtaposition is given by
\[
(b_1, c_1, b'_1)(b_2, c_2, b'_2) = (b_1, c_1, b'_1b_2t_2(c_2)b'_2) \circ (b_1s_2(c_1)b'_1b_2, c_2, b'_2).
\]
Note that the category \( \mathfrak{C}(I) \leq 1 \) is \( \mathcal{C}(I_0, I_1, s, t) \).

Let \( \mathfrak{B} \) be a strict 2-category. An I-diagram \( D \) in \( \mathfrak{B} \) is the data of
- an object \( a_i \) of \( \mathfrak{B} \) for any \( i \in I_0 \)
- a 1-arrow \( b_j : a_{s(j)} \to a_{t(j)} \) for any \( j \in I_1 \)
- a 2-arrow \( c_k : b_{s_2(k)} \to b_{t_2(k)} \) for any \( k \in I_2 \)
where given \( p = (p_1, \ldots, p_n) \in P \), we put \( b_p = b_{p_1} \cdots b_{p_n} \).

![Diagram](image)

The data of \( b_j \)'s and \( c_k \)'s is the same as the data, for \( i, i' \in I_0 \), of an \( I(i, i') \)-diagram in \( \text{Hom}(a_i, a_{i'}) \).

A morphism \( \sigma : D \to D' \) is the data of

- 1-arrows \( \sigma_i : a_i \to a'_i \) for \( i \in I_0 \)
- invertible 2-arrows \( \sigma_j : b'_j \sigma_{s(j)} \rightarrowtail \sigma_{t(j)} b_j \) for every \( j \in I_1 \)

![Diagram](image)

such that for every \( k \in I_2 \) with \( s_2(k) = (j_1, \ldots, j_n) \) and \( t_2(k) = (\overline{j_1}, \ldots, \overline{j_n}) \), the following 2-arrows \( b'_{s_2(k)} \sigma_{s(j)} \to \sigma_{t(j)} b_{t_2(k)} \) are equal:
A morphism \( \gamma : \sigma \to \tilde{\sigma} \) is the data of 2-arrows \( \gamma_i : \sigma_i \to \tilde{\sigma}_i \) for \( i \in I_0 \) such that for every \( j \in I_1 \), we have \( (\gamma_{t(j)} b_j) \circ \sigma_j = \tilde{\sigma}_j \circ (b_j \gamma_{s(j)}) \), i.e., the following diagram of 2-arrows is commutative:

This gives rise to a strict 2-category \( \mathcal{H}om(I, \mathcal{B}) \) of \( I \)-diagrams in \( \mathcal{B} \).

Restriction gives a strict 2-functor \( R : \mathcal{H}om(\mathcal{C}(I), \mathcal{B}) \to \mathcal{H}om(I, \mathcal{B}) \). It is locally an isomorphism and it is surjective on objects, so it is a 2-equivalence.

2.2.4. 2-Representations. Let \( \mathcal{A} \) and \( \mathcal{B} \) be two 2-categories. We will consider 2-representations of \( \mathcal{A} \) in \( \mathcal{B} \), i.e., 2-functors \( R : \mathcal{A} \to \mathcal{B} \). We put \( \mathcal{A}-\text{Mod}(\mathcal{B}) = \mathcal{H}om(\mathcal{A}, \mathcal{B}) \), a 2-category. Given \( R : \mathcal{A} \to \mathcal{B} \), a sub-2-representation is a 2-functor \( R' : \mathcal{A} \to \mathcal{B} \) equipped with a fully faithful morphism \( R' \to R \). There is a canonical 2-equivalence \( \mathcal{H}om(\mathcal{A}^{\text{opp}}, \mathcal{B}^{\text{opp}}) \cong \mathcal{H}om(\mathcal{A}, \mathcal{B})^{\text{opp}} \).

Let \( S \) be a collection of objects of \( \mathcal{B} \). An action of \( \mathcal{A} \) on \( S \) is a 2-representation of \( \mathcal{A} \) in \( \mathcal{B} \) with image contained in \( S \). Note that if \( \mathcal{A} \) has only one object and is viewed as a monoidal category \( \mathcal{A} \) and \( S = \{ C \} \), we recover the usual notion of an action of \( \mathcal{A} \) on \( C \).

Let \( a \in \mathcal{A} \). We define a 2-functor \( \mathcal{H}om(a, -) : \mathcal{A} \to \mathcal{C}at \) by \( a' \mapsto \mathcal{H}om(a, a') \). The functor \( \mathcal{H}om(a', a'') \to \mathcal{H}om(\mathcal{H}om(a, a'), \mathcal{H}om(a, a'')) \) is given by juxtaposition. The associativity and unit maps of \( \mathcal{A} \) provide the required 2-arrows.

Let \( R : \mathcal{A} \to \mathcal{C}at \) be a 2-functor. Given \( a \) an object of \( \mathcal{A} \), there is an equivalence of categories from \( R(a) \) to the category of morphisms \( \mathcal{H}om(a, -) \to R \):

- Given \( M \) an object of the category \( R(a) \), we define a morphism \( \sigma : \mathcal{H}om(a, -) \to R \).
  The functor \( \mathcal{H}om(a, a') \to R(a') \) is \( b \mapsto R(b)(M) \). The required natural isomorphisms come from the natural isomorphisms \( R(b)R(f) \cong R(bf) \).
- Conversely, given \( \sigma : \mathcal{H}om(a, -) \to R \), we put \( M = \sigma(I_a) \).

Assume from now on that our 2-categories are \( k \)-linear.

Let \( b : a \to a' \) be a 1-arrow. A cokernel of \( b \) is the data of an object \( \text{Coker}(b) \) and of a 1-arrow \( b' : a' \to \text{Coker}(b) \) such that for any object \( a'' \), the functor \( \mathcal{H}om(b', a'') : \mathcal{H}om(\text{Coker}(b), a'') \to \mathcal{H}om(a', a'') \) is fully faithful with image equivalent to the full subcategory of 1-arrows \( b'' : a' \to a'' \) such that \( b''b = 0 \). When a cokernel of \( b \) exists, it is unique up to an equivalence unique up to a unique isomorphism.

We say that \( \mathcal{A} \) admits cokernels if all 1-arrows admit cokernels. This is the case for the 2-category of \( k \)-linear categories, of abelian categories or of triangulated categories.

We define kernels as cokernels taken in \( \mathcal{A}^{\text{rev}} \).

Assume \( \mathcal{B} \) admits kernel and cokernels.
Let \( b : a \to a' \) be a fully faithful 1-arrow. We say that it is thick if \( b \) is a kernel of \( a \to \text{Coker}(b) \).

When \( \mathcal{B} \subset \text{Lin}_k \), the notion of thickness corresponds to

- \( \text{Lin}_k \) or \( \text{Tri} \): \( a \) is closed under direct summands
- \( \mathcal{A} \): \( a \) is closed under extensions, subobjects and quotients

Let \( R, R' : \mathcal{A} \to \mathcal{B} \) be two 2-functors and \( \sigma : R' \to R \). Assume \( \sigma \) is locally fully faithful. We define \( R'' = \text{Coker} \sigma : \mathcal{A} \to \mathcal{B} \) (denoted also by \( R/R' \) when there is no ambiguity) by \( R'' : a \mapsto \text{Coker} \sigma(a) \). The composition \( \text{Hom}(a, a') \xrightarrow{R(a,a')} \text{Hom}(R(a), R(a')) \xrightarrow{\text{can}} \text{Hom}(R(a), \text{Coker} \sigma(a')) \) factors uniquely through \( \text{Hom}(\text{Coker} \sigma(a), \text{Coker} \sigma(a')) \) and this defines a functor \( \text{Hom}(a, a') \to \text{Hom}(\text{Coker} \sigma(a), \text{Coker} \sigma(a')) \). The constraints are obtained by taking quotients.

We have a Grothendieck group functor \( K_0 : \text{Tri}_{\leq 1} \to \mathcal{A} \). When \( \mathcal{B} \) is endowed with a canonical 2-functor to the 2-category of triangulated categories, we will still denote by \( K_0 \) the composite functor \( \mathcal{B}_{\leq 1} \to \mathcal{A} \). For example, \( \mathcal{B} \) is the category of exact categories or of dg-categories and we consider the derived category 2-functor. Viewing additive categories as exact categories for the split structure provides another example (this is the homotopy category functor). This gives a “decategorification” functor \( \mathcal{A} \)-Mod(\( \mathcal{B} \))$_{\leq 1} \to \text{Hom}(\mathcal{A}_{\leq 1}, \mathcal{A})$.

Let \( \mathcal{A} \) and \( \mathcal{B} \) be \( k \)-linear 2-categories. Assume \( \mathcal{B} \) is locally idempotent-complete. Let \( \mathcal{A}^i \) be the idempotent completion of \( \mathcal{A} \). The canonical strict 2-functor \( \mathcal{A} \to \mathcal{A}^i \) induces a 2-equivalence \( \mathcal{A}^i \)-Mod(\( \mathcal{B} \)) \( \sim \) \( \mathcal{A} \)-Mod(\( \mathcal{B} \))

### 2.3. Symmetric algebras

The theory of symmetric or Frobenius algebras is classical (cf eg [Bro]). We need here a version over a non-commutative base algebra and we study transitivity properties.

#### 2.3.1. Serre functors

Let \( k \) be a field and \( \mathcal{T}_1, \mathcal{T}_2 \) be two \( k \)-linear categories. Let \( E : \mathcal{T}_1 \to \mathcal{T}_2 \) be a functor and \((E, F)\) an adjoint pair, provided with bifunctorial isomorphisms

\[
\alpha(M, N) : \text{Hom}(EM, N) \sim \text{Hom}(M, FN) \text{ for } M \in \mathcal{T}_1 \text{ and } N \in \mathcal{T}_2
\]

Let \( S_i \) be a Serre functor for \( \mathcal{T}_i \), for \( i = 1, 2 \): we have bifunctorial isomorphisms

\[
\gamma_i(M, N) : \text{Hom}(M, N)^* \sim \text{Hom}(N, S_i M) \text{ for } M, N \in \mathcal{T}_i
\]
Then, \((S_2^{-1}FS_1, E)\) is an adjoint pair with defining isomorphisms given by the following commutative diagram

\[
\begin{align*}
\text{Hom}(S_2^{-1}FS_1N, M) & \xrightarrow{\sim} \text{Hom}(N, EM) \\
\gamma_2(S_2^{-1}FS_1N,M)^* & \xrightarrow{\sim} \gamma_1(N,EM)^* \\
\text{Hom}(M, FS_1N)^* & \xrightarrow{\alpha(M,S_1N)^*} \text{Hom}(EM, S_1N)^*
\end{align*}
\]

**Lemma 2.5.** Let \((E', F')\) be an adjoint pair, with \(E' : T_1 \to T_2\). Given \(f \in \text{Hom}(E, E')\), we have \(\gamma f = S_2^{-1}f^{-}\).

**Proof.** Given \(M \in T_1\) and \(N \in T_2\), we have a commutative diagram

\[
\begin{align*}
\text{Hom}(N, EM) & \xrightarrow{\gamma_1} \text{Hom}(EM, S_1N)^* \\
\text{Hom}(M, FS_1N)^* & \xrightarrow{\alpha^*} \text{Hom}(S_2^{-1}FS_1N, M) \\
\text{Hom}(N, fM) & \xleftarrow{\gamma_1} \text{Hom}(fM, S_1N)^* \\
\text{Hom}(M, f^{-}S_1N)^* & \xrightarrow{\alpha^*} \text{Hom}(S_2^{-1}f^{-}S_1N,M)
\end{align*}
\]

and the result follows. \(\square\)

2.3.2. Frobenius forms. Let \(B\) be a \(k\)-algebra and \(A\) a \(B\)-algebra. We denote by \(m : A \otimes_B A \to A\) the multiplication map.

The canonical isomorphism of \((A, B)\)-bimodules \(\text{Hom}_B(A, B) \sim \text{Hom}_A(A, \text{Hom}_B(A, B))\) restricts to an isomorphism

\[
t \mapsto \hat{t} : \text{Hom}_B(A, B) \sim \text{Hom}_A(A, \text{Hom}_B(A, B)).
\]

Let us describe this explicitly. Given \(t : A \to B\) a morphism of \((B, B)\)-bimodules, we have the morphism of \((A, B)\)-bimodules

\[
\hat{t} : A \to \text{Hom}_B(A, B)
\]

\[
a \mapsto (a' \mapsto t(a'a')).
\]

Conversely, given \(f : A \to \text{Hom}_B(A, B)\) a morphism of \((A, B)\)-bimodules, then \(f(1) : A \to B\) is a morphism of \((B, B)\)-bimodules and we have \(f = \hat{f}(1)\).

**Definition 2.6.** Let \(t : A \to B\) be a morphism of \((B, B)\)-bimodules. We say that \(t\) is a Frobenius form if \(A\) is a projective \(B\)-module of finite type and \(\hat{t} : A \to \text{Hom}_B(A, B)\) is an isomorphism.

Let \(t : A \to B\) be a Frobenius form. It defines an automorphism of \(Z(B)\)-algebras, the Nakayama automorphism:

\[
\gamma_t : A^B \sim A^B, \ a \mapsto \hat{t}^{-1}(a' \mapsto t(aa')).
\]

We have

\[t(aa') = t(a'\gamma_t(a))\]

for all \(a \in A^B\) and \(a' \in A\).

This makes \(\hat{t}\) into an isomorphism of \((A, B \otimes_{Z(B)} A^B)\)-modules

\[
\hat{t} : A_1 \sim \text{Hom}_B(A, B).
\]

We say that \(t\) is symmetric if \(\gamma_t = \text{id}_{A^B}\).
Remark 2.7. Note that if \( t(aa') = t(a'a) \) for all \( a, a' \in A \), then \( A^B = A \).

Given \( t \) and \( t' \) two Frobenius forms, there is a unique element \( z \in (A^B)^\times \) such that \( t'(a) = t(az) \) for all \( a \in A \). If in addition \( t \) and \( t' \) are symmetric, then \( z \in Z(A^B)^\times \).

2.3.3. Adjunction (Res,Ind). Let \( B \) be a \( k \)-algebra and \( A \) a \( B \)-algebra.

The data of an adjunction \((\text{Res}^A_B, \text{Ind}^A_B)\) is the same as the data of an isomorphism \( A \otimes_B - \sim \text{Hom}_B(A,-) \) of functors \( B\text{-Mod} \to A\text{-Mod} \).

Assume there is such an adjunction. The functor \( \text{Hom}_B(A,-) \) is right exact, hence \( A \) is projective as a \( B \)-module. The functor \( \text{Hom}_B(A,-) \) commutes with direct sums, hence \( A \) is a finitely generated projective \( B \)-module.

Assume now \( A \) is a finitely generated projective \( B \)-module. We have a canonical isomorphism

\[
\text{Hom}_B(A,B) \otimes_B - \sim \text{Hom}_B(A,-).
\]

So, the data of an adjunction \((\text{Res}^A_B, \text{Ind}^A_B)\) is the same as the data of an isomorphism \( f : A \sim \text{Hom}_B(A,B) \) of \((A,B)\)-bimodules. Given \( f \), let \( t = f(1) : A \to B \). This is the morphism of \((B,B)\)-bimodules corresponding to the counit \( \varepsilon : \text{Res}^A_B \text{Ind}^A_B \to \text{id}_B \).

On the other hand, we have \( f = \hat{t} \). Summarizing, we have the following Proposition.

**Proposition 2.8.** Let \( B \) be an algebra and \( A \) a \( B \)-algebra. We have inverse bijections between the set of Frobenius forms and the set of adjunctions \((\text{Res}^A_B, \text{Ind}^A_B)\):

\[
t \mapsto \text{adjunction defined by} \hat{t}
\]

\[
\text{counit} \leftrightarrow \text{adjunction}
\]

Assume we have a Frobenius form \( t : A \to B \). The unit of adjunction of the pair \((\text{Res}^A_B, \text{Ind}^A_B)\) corresponds to a morphism of \((A,A)\)-bimodules \( A \to A \otimes_B A \). The image of 1 under this morphism is the Casimir element \( \pi = \pi^A_B \in (A \otimes_B A)^A \). It satisfies

\[
(t \otimes 1)(\pi) = (1 \otimes t)(\pi) = 1 \in A.
\]

Conversely, given an element \( \pi \in (A \otimes_B A)^A \), there exists at most one \( t \in \text{Hom}_{B,B}(A,B) \) satisfying (2), and such a morphism is a Frobenius form.

Note that right multiplication induces an isomorphism \( A^B \sim \text{End}(\text{Ind}^A_B) \) and the automorphism \( \iota \) is the Nakayama automorphism \( \gamma_t \).

**Remark 2.9.** We developed the theory for left modules, but this is the same as the theory for right modules. Namely, let \( t : A \to B \) be a Frobenius form. Since \( A \) is finitely generated and projective as a \( B \)-module, it follows that \( \text{Hom}_B(A,B) \) is a finitely generated projective right \( B \)-module, hence \( A \) is a finitely generated projective right \( B \)-module. Consider the composition

\[
\hat{t} : A \xrightarrow{\sim} \text{Hom}_{B^{opp}}(\text{Hom}_B(A,B),B) \xrightarrow{\text{Hom}_{B^{opp}}(\hat{t},B)} \text{Hom}_{B^{opp}}(A,B), \quad a \mapsto (a' \mapsto t(aa')).
\]

The first map is an isomorphism since \( A \) is finitely generated and projective as a \( B \)-module. It follows that \( \hat{t} \) is an isomorphism.
2.3.4. Transitivity. Let $C$ be an algebra, $B$ a $C$-algebra and $A$ a $B$-algebra. We assume that $A$ (resp. $B$) is a finitely generated projective $B$-module (resp. $C$-module).

Given $t \in \text{Hom}_{B,B}(A,B)$, $t' \in \text{Hom}_{C,C}(B,C)$ and $t'' = t' \circ t \in \text{Hom}_{C,C}(A,C)$, we have a commutative diagram

$$
\begin{array}{cccc}
A & \text{Hom}_{C}(A,C) \\
\downarrow_{t} & \sim \\
\text{Hom}_{B}(A,B) & \text{Hom}_{B}(A,\text{Hom}_{C}(B,C))
\end{array}
$$

The units of adjunction are given by composition:

$$
A \xrightarrow{1 \otimes \pi_B^1} A \otimes_B A \xrightarrow{1 \otimes 1 \otimes \pi_C^1} A \otimes_C A, \ 1 \mapsto \pi_C^1.
$$

**Lemma 2.10.** If $t \in \text{Hom}_{B,B}(A,B)$ and $t' \in \text{Hom}_{C,C}(B,C)$ are Frobenius forms, then $t' \circ t : A \to C$ is a Frobenius form.

**Lemma 2.11.** Let $t' \in \text{Hom}_{C,C}(B,C)$ and $t'' \in \text{Hom}_{C,C}(A,C)$ be Frobenius forms. There is a unique $t \in \text{Hom}_{B}(A,B)$ such that $t'' = t' \circ t$. It is a Frobenius form and it is given by $t = \text{Hom}_{B}(A,t')^{-1}(t''(1)) \in \text{Hom}_{B,B}(A,B)$.

Let $t'' \in \text{Hom}_{C,C}(A,C)$ and $\zeta \in A^C$. Define $t' \in \text{Hom}_{C,C}(B,C)$ by $t'(b) = t''(b\zeta)$. If $t''$ is a Frobenius morphism and the pairing

$$
B \times B \to C, \ (b,b') \mapsto t''(bb')
$$

is perfect, then $t'$ is a Frobenius form.

Assume now $t$, $t'$ and $t''$ are given and let $\zeta \in A^C$. Then,

$$
t(\zeta) = 1 \iff \forall b \in B, \ t'(bt(\zeta)) = t'(b) \iff \forall b \in B, \ t''(b\zeta) = t'(b).
$$

Note that $\zeta$ is determined by $t'$ up to adding an element $\xi \in A^C$ such that $t''(B\xi) = 0$. The next lemma shows that under certain conditions on $A$, the form $t'$ is always obtained from such a $\zeta$.

**Lemma 2.12.** Assume $B$ is a quotient of $A$ as a $(B,C)$-bimodule (this is the case if $A$ is a progenerator for $B$ and $C \subset Z(A)$). Let $t \in \text{Hom}_{B,B}(A,B)$ and $t'' \in \text{Hom}_{C,C}(A,C)$ be Frobenius forms. There is a unique $t' \in \text{Hom}_{C,C}(B,C)$ such that $t'' = t' \circ t$. It is a Frobenius form.

**Proof.** Since $A$ is a progenerator for $B$, the morphism $\hat{t}'$ is determined by $\text{Hom}_{B}(A,\hat{t}')$. The unicity of $t'$ follows.

Assume $A$ is a progenerator for $B$ and $C$ is central in $A$. Since $A$ is a progenerator for $B$, there exists an integer $n$ and a surjection of $B$-modules $f : A^n \to B$. Let $m \in f^{-1}(1)$ and consider the morphism $A \to A^n, \ a \mapsto am$. The composition $g : A \to A^n \to B$ is a morphism of $B$-modules with $g(1) = 1$. Since $C$ is central, $g$ is a morphism of $(B,C)$-bimodules.

Assume now there is a surjective morphism of $(B,C)$-bimodules $h : A \to B$. Then, $h(1) \in Z(C)^\times$, let $g : A \to B, \ a \mapsto ah(1)^{-1}$. This is a morphism of $(B,C)$-bimodules with $g(1) = 1$.

Let $\zeta = \hat{t}^{-1}(g)$. We have $t(\zeta) = 1$ and we define $t'$ by $t'(b) = t''(b\zeta)$. We have $t'' = t' \circ t$, the morphism $\text{Hom}_{B}(A,\hat{t}')$ is invertible and since $A$ is a progenerator for $B$, it follows that $\hat{t}'$ is an isomorphism.

\[\square\]
2.3.5. **Bases.** Let $B$ be an algebra, $A$ a $B$-algebra and assume $A$ is free of finite rank as a $B$-module. Let $\mathcal{B}$ be a basis of $A$ as a left $B$-module: $A = \bigoplus_{v \in \mathcal{B}} Bv$.

Let $t \in \text{Hom}_{B,B}(A, B)$. Then, $t$ is a Frobenius form if and only if there exists a dual basis $\mathcal{B}^\vee = \{v^\vee\}_{v \in \mathcal{B}}$: i.e., $\mathcal{B}^\vee$ satisfies $t(v^\vee v') = \delta_{v,v'}$ for $v, v' \in \mathcal{B}$.

Assume $t$ is a Frobenius form. Then, $\mathcal{B}^\vee$ exists and is unique. It is a basis of $A$ as a right $B$-module. We have

$$t(v^\vee) = (B \ni v' \mapsto \delta_{v,v'})$$

for $v \in \mathcal{B}$.

Given $a \in A$, we have

$$a = \sum_{v \in \mathcal{B}} t(av^\vee) v = \sum_{v \in \mathcal{B}} v^\vee t(va).$$

Given $a \in A^B$, we have

$$\gamma_t(a) = \sum_{v \in \mathcal{B}} v^\vee t(av).$$

The unit of the adjoint pair $(\text{Res}^A_B, \text{Ind}^A_B)$ is given by the morphism of $(A,A)$-bimodules

$$A \to A \otimes_B A, \ 1 \mapsto \pi_B^A = \sum_{v \in \mathcal{B}} v^\vee \otimes v.$$  

Consider now $C$ an algebra and a $C$-algebra structure on $B$ such that $B$ is free of finite rank as a $C$-module. Let $\mathcal{B}'$ be a basis of $B$ as a $C$-module. Then, $\mathcal{B}'' = \mathcal{B}' \mathcal{B} = \{v'v\}_{v \in \mathcal{B}, v' \in \mathcal{B}'}$ is a basis of $A$ as a $C$-module.

Let $t' : B \to C$ be a Frobenius form. The dual basis to $\mathcal{B}''$ for the Frobenius form $t'' = t' \circ t : A \to C$ is $\mathcal{B}''' = \{v''v'^\vee\}_{v \in \mathcal{B}, v' \in \mathcal{B}'}$. Given $a \in A$, we have

$$t(a) = \sum_{v' \in \mathcal{B}'} t''(av'^\vee)v' = \sum_{v' \in \mathcal{B}'} v'^\vee t''(v'a).$$

Given $v \in \mathcal{B}$, we have

$$v^\vee = \sum_{v' \in \mathcal{B}'} (v'v)^\vee t'(v').$$

2.3.6. **Ramification.** Let $A$ be a $B$-algebra endowed with a Frobenius form $t$ and assume $A^B = A$.

The following statements are equivalent:

(a) $A$ is a projective $(A \otimes_B A^\text{opp})$-module

(b) there exists $a \in A$ such that $m((1 \otimes a \otimes 1 \otimes 1)\pi) = 1$

(c) there exists $a \in A$ such that $m((1 \otimes 1 \otimes a \otimes 1)\pi) = 1$

where $A \otimes_B A$ is viewed as a module over $((A \otimes A^\text{opp}) \otimes_B (A \otimes A^\text{opp}))$.

When $A$ is commutative, the statements (a)-(c) above are equivalent to the following two statements

(d) $A$ is étale over $B$

(e) $m(\pi) \in A^\times$. 

3. HECKE ALGEBRAS

3.1. Classical Hecke algebras. We recall in this section the various versions of affine Hecke algebras and the isomorphisms between them after suitable localizations. We consider only the case of $GL_n$: in this case, the inclusion $G_m^n \hookrightarrow G^n_s$ gives an algebraic $\mathfrak{S}_n$-equivariant map that makes it possible to avoid completions. In general, one needs to use the exponential map from the Lie algebra of a torus to the torus. All constructions and results in this section extend to arbitrary Weyl groups.

3.1.1. BGG-Demazure operators. Given $1 \leq i \leq n$, we put $s_i = (i, i + 1) \in \mathfrak{S}_n$. We define an endomorphism of abelian groups $\partial_i \in \text{End}_\mathbb{Z}(\mathbb{Z}[X_1, \ldots, X_n])$ by

$$\partial_i(P) = \frac{P - s_i(P)}{X_{i+1} - X_i}.$$ 

The formula defines endomorphisms of various localizations, for example $\mathbb{Z}[X_{1}^{\pm 1}, \ldots, X_{n}^{\pm 1}]$.

Given $w = s_{i_1} \cdots s_{i_r}$ a reduced decomposition of an element of $\mathfrak{S}_n$, we put

$$\partial_w = \partial_{i_1} \cdots \partial_{i_r}.$$ 

This is independent of the choice of the reduced decomposition.

The $\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$-linear morphism $\partial_{w[1,n]}$ takes values in $\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$. It is a symmetrizing form for the $\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$-algebra $\mathbb{Z}[X_1, \ldots, X_n]$. We view $\mathbb{Z}[X_1, \ldots, X_n]$ as a graded algebra with deg($X_i$) = 2. Then, $\partial_{w[1,n]}$ is homogeneous of degree $-n(n - 1)$.

**Lemma 3.1.** Denote by $\pi$ the Casimir element for $\partial_{w[1,n]}$. Then $m(\pi) = \prod_{1 \leq j < i \leq n}(X_i - X_j)$.

**Proof.** The algebra $\mathbb{Z}[X_1, \ldots, X_n]$ is étale over $\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ outside $m(\pi) = 0$. So, $\prod_{1 \leq j < i \leq n}(X_i - X_j)$ is homogeneous of degree $n(n - 1)$, it follows that there is $a \in \mathbb{Z}$ such that $m(\pi) = a \prod_{1 \leq j < i \leq n}(X_i - X_j)$. On the other hand, $\partial_{w[1,n]}(m(\pi)) = n! = \partial_{w[1,n]}\left(\prod_{1 \leq j < i \leq n}(X_i - X_j)\right)$ and the lemma follows. \(\square\)

Let $A = \mathbb{Z}[X_1, \ldots, X_n] \rtimes \mathfrak{S}_n$. This algebra has a Frobenius form over $\mathbb{Z}[X_1, \ldots, X_n]$ given by

$$Pw \mapsto P\delta_{w[1,n]}$$ for $P \in \mathbb{Z}[X_1, \ldots, X_n]$ and $w \in \mathfrak{S}_n$.

By composition, we obtain a Frobenius form $t$ for $A$ over $\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ given by

$$t(Pw) = \partial_{w[1,n]}(P)\delta_{w[1,n]}$$ for $P \in \mathbb{Z}[X_1, \ldots, X_n]$ and $w \in \mathfrak{S}_n$.

The corresponding Nakayama automorphism of $A$ is the involution

$$X_i \mapsto X_{n-i+1}, \quad s_i \mapsto -s_{n-i}.$$ 

3.1.2. Degenerate affine Hecke algebras. Let $\bar{H}_n$ be the degenerate affine Hecke algebra of $GL_n$:

$$\bar{H}_n = \mathbb{Z}[X_1, \ldots, X_n] \otimes \mathbb{Z}\mathfrak{S}_n$$ as an abelian group, $\mathbb{Z}[X_1, \ldots, X_n]$ and $\mathbb{Z}\mathfrak{S}_n$ are subalgebras, and $T_i X_j = X_j T_i$ if $j - i \neq 0, 1$ and $T_i X_{i+1} - X_i T_i = 1$.

We denote here by $T_1, \ldots, T_{n-1}$ the Coxeter generators for $\mathfrak{S}_n$ and we write $T_w$ for the element $w$ of $\mathfrak{S}_n$.

Given $P \in \mathbb{Z}[X_1, \ldots, X_n]$, we have $T_i P - s_i(P)T_i = \partial_i(P)$.

We have a faithful representation on $\mathbb{Z}[X_1, \ldots, X_n] = \bar{H}_n \otimes_{\mathbb{Z}\mathfrak{S}_n} \mathbb{Z}$ where

$$T_i(P) = s_i(P) + \partial_i(P).$$
Here, $Z$ is the trivial representation of $\mathfrak{S}_n$.

The algebra $H_n$ has a Frobenius form over $\mathbb{Z}[X_1, \ldots, X_n]$ given by
\begin{equation}
PT_w \mapsto P\partial_{w-w[1,n]} \quad \text{for } P \in \mathbb{Z}[X_1, \ldots, X_n] \text{ and } w \in \mathfrak{S}_n.
\end{equation}

By composition, we obtain a Frobenius form $t$ for $H_n$ over $\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ given by
\begin{equation}
t(PT_w) = \partial_{w[1,n]}(P)\delta_{w-w[1,n]} \quad \text{for } P \in \mathbb{Z}[X_1, \ldots, X_n] \text{ and } w \in \mathfrak{S}_n.
\end{equation}

The corresponding Nakayama automorphism of $H_n$ is the involution
\begin{equation}
X_i \mapsto X_{n-i+1}, \quad T_i \mapsto -T_{n-i}.
\end{equation}

### 3.1.3. Finite Hecke algebras

Let $R = \mathbb{Z}[q^{\pm 1}]$. Let $H_n^f$ be the Hecke algebra of $\mathrm{GL}_n$: this is the $R$-algebra generated by $T_1, \ldots, T_{n-1}$, with relations
\begin{equation}
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i \quad \text{if } |i-j| > 1 \text{ and } (T_i - q)(T_i + 1) = 0.
\end{equation}

The algebra $H_n^f$ is actually symmetric, via the classical form given by $T_w \mapsto \delta_{1,w}$. In other terms, the Nakayama automorphism is inner: it is conjugation by $T_{w[1,n]}$. On the other hand, the Hecke algebra is not symmetric over $\mathbb{Z}[q]$ and the classical form induces a degenerate pairing, while the form $t_f$ above is still a Frobenius form over $\mathbb{Z}[q]$ (cf §3.1.5).

### 3.1.4. Affine Hecke algebras

Let $H_n$ be the affine Hecke algebra of $\mathrm{GL}_n$: $H_n = R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \otimes R$ $H^f$ as an $R$-module, $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$ and $H^f$ are subalgebras and
\begin{equation}
T_iX_j = X_jT_i \text{ if } j - i \neq 0, 1 \text{ and } T_iX_{i+1} - X_iT_i = (q - 1)X_{i+1}.
\end{equation}

Given $P \in \mathbb{Z}[X_1^{\pm 1}, \ldots, X_n^{\pm 1}]$, we have $T_iP - s_i(P)T_i = (q - 1)X_{i+1}\partial_i(P)$.

We have a faithful representation on $R[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] = H_n \otimes_{H_n^f} R$, where
\begin{equation}
T_i(P) = qs_i(P) + (q - 1)X_{i+1}\partial_i(P).
\end{equation}

Here $R$ denotes the one-dimensional representation of $H_n^f$ on which $T_i$ acts by $q$.

The algebra $H_n$ has a Frobenius form over $\mathbb{Z}[X_1, \ldots, X_n]$ given by (3) and a Frobenius form $t$ over $\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}$ given by (4). The corresponding Nakayama automorphism of $H_n$ is the involution
\begin{equation}
X_i \mapsto X_{n-i+1}, \quad T_i \mapsto -qT_{n-i}^{-1}.
\end{equation}

### 3.1.5. Nil Hecke algebras

Let $0H_n^f$ be the nil Hecke algebra of $\mathrm{GL}_n$: this is the $\mathbb{Z}$-algebra generated by $T_1, \ldots, T_{n-1}$, with relations
\begin{equation}
T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}, \quad T_iT_j = T_jT_i \quad \text{if } |i-j| > 1 \text{ and } T_i^2 = 0.
\end{equation}

Given $w = s_{i_k} \cdots s_{i_1}$ a reduced decomposition of an element $w \in \mathfrak{S}_n$, we put $T_w = T_{i_1} \cdots T_{i_k}$. Let $t_0$ be the linear form on $0H_n^f$ defined by $t_0(T_w) = \delta_{w-w[1,n]}$. This is a Frobenius form, with Nakayama automorphism given by $T_i \mapsto T_{n-i}$.

The nil Hecke algebra $0H_n$ is a graded algebra with $\deg T_i = -2$ and $t_0$ is homogeneous of degree $n(n - 1)$. 


Lemma 3.3. Let \( f : M \to N \) be a morphism of relatively \( \mathbb{Z} \)-projective \( 0H_n^f \)-modules. If \( T_w[1,n]f : T_w[1,n]M \to T[1,n]N \) is an isomorphism, then \( f \) is an isomorphism.

Proof. The annihilator of \( T_w[1,n] \) on a relatively \( \mathbb{Z} \)-projective module \( L \) is \((0H_n^f \leq -2)L\). Nakayama’s Lemma shows that under the assumption of the lemma, the morphism \( f \) is surjective. On the other hand, \( \ker f \) is a direct summand of \( M \), hence \( \ker f \) is relatively \( \mathbb{Z} \)-projective. Since \( T_w[1,n] \ker f = 0 \), it follows that \( \ker f = 0 \). \( \square \)

Let \( A \) be an algebra. We denote by \( \Lambda^0H_n^f \) the algebra whose underlying abelian group is \( A^{\otimes n} \otimes 0H_n^f \), where \( A^{\otimes n} \) and \( 0H_n^f \) are subalgebras and where \((a_1 \otimes \cdots \otimes a_n)T_i = T_i(a_1 \otimes \cdots \otimes a_{i-1} \otimes a_{i+1} \otimes a_i \otimes a_{i+2} \otimes \cdots \otimes a_n)\).

3.1.6. Nil affine Hecke algebras. Let \( 0H_n \) be the nil affine Hecke algebra of \( \text{GL}_n \): \( 0H_n = \mathbb{Z}[X_1, \ldots, X_n] \otimes 0H_n^f \) as an abelian group, \( \mathbb{Z}[X_1, \ldots, X_n] \) and \( 0H_n^f \) are subalgebras and
\[
T_i X_j = X_j T_i \quad \text{if} \quad j \neq i \quad \text{and} \quad T_i X_i = X_i T_i = 1 \quad \text{and} \quad T_i X_i - X_i T_i = -1.
\]
Given \( P \in \mathbb{Z}[X_1, \ldots, X_n] \), we have \( T_i P - s_i(P)T_i = PT_i - T_i s_i(P) = \partial_i(P) \).

We have a faithful representation on \( \mathbb{Z}[X_1, \ldots, X_n] = 0H_n \otimes_{0H_n^f} \mathbb{Z} \) where
\[
T_i(P) = \partial_i(P).
\]
Let \( b_n = T_w[1,n]X_1^{n-1}X_2^{n-2} \cdots X_{n-1} \). By induction on \( n \), one sees that \( \partial_w(X_1^{n-1}X_2^{n-2} \cdots X_{n-1}) = 1 \), hence \( b_n^2 = b_n \). We have an isomorphism of \( 0H_n \)-modules
\[
\mathbb{Z}[X_1, \ldots, X_n] \sim 0H_n b_n, \quad P \mapsto Pb_n.
\]
Since \( \{\partial_w(X_1^{n-1} \cdots X_{n-1})\}_{w \in \mathfrak{S}_n} \) is a basis of \( \mathbb{Z}[X_1, \ldots, X_n] \) over \( \mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n} \), it follows that the multiplication map gives an isomorphism of \( (0H_n^f, \mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}) \)-bimodules
\[
0H_n^f \otimes (\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n} X_1^{n-1} \cdots X_{n-1} b_n) \sim 0H_n b_n.
\]

Proposition 3.4. The action of \( 0H_n \) on \( \mathbb{Z}[X_1, \ldots, X_n] \) induces an isomorphism
\[
0H_n \sim \text{End}_{\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}}(\mathbb{Z}[X_1, \ldots, X_n]).
\]
Since \( \mathbb{Z}[X_1, \ldots, X_n] \) is a free \( \mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n} \)-module of rank \( n! \), the algebra \( 0H_n \) is isomorphic to a \( (n! \times n!) \)-matrix algebra over \( \mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n} \).

The restriction to \( 0H_n^f \) of any \( 0H_n \)-module is relatively \( \mathbb{Z} \)-projective.

Proof. Since \( \mathbb{Z}[X_1, \ldots, X_n] \) is a finitely generated projective \( 0H_n \)-module, the canonical map
\[
0H_n \sim \text{End}_{\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}}(\mathbb{Z}[X_1, \ldots, X_n])
\]

The \( (0H_n^f, \mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}) \)-bimodule \( \mathbb{Z}[X_1, \ldots, X_n] \) is a direct summand of \( 0H_n \). So, given \( M \) an \( \mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n} \)-module, then \( \mathbb{Z}[X_1, \ldots, X_n] \otimes_{\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}} M \) is a direct summand of \( 0H_n^f \otimes \mathbb{Z}[X_1, \ldots, X_n] \otimes_{\mathbb{Z}[X_1, \ldots, X_n]^{\mathfrak{S}_n}} M \) as an \( 0H_n^f \)-module. So, given \( N \) an \( 0H_n \)-module, then \( N \) is a direct summand of \( 0H_n^f \otimes \mathbb{Z} N \) as an \( 0H_n^f \)-module and the proposition is proven. \( \square \)
Lemma 3.3 joined with Proposition 3.4 gives a useful criterion to check that a morphism of $^0H_n$-modules is invertible. Note also that the proposition shows that $^0H_n$ is projective as a $(^0H_n, \; ^0H_n)$-bimodule.

The algebra $^0H_n$ has a Frobenius form over $\mathbb{Z}[X_1, \ldots, X_n]$ given by (3) and a Frobenius form $t$ over $\mathbb{Z}[X_1, \ldots, X_n]^{S_n}$ given by (4). The corresponding Nakayama automorphism of $^0H_n$ is the involution

$$X_i \mapsto X_{n-i+1}, \; T_i \mapsto -T_{n-i}.$$  

A special feature of the nil affine Hecke algebra, compared to the affine Hecke algebra and the degenerate affine Hecke algebra, is that the Nakayama automorphism $\gamma$ is inner, hence the nil affine Hecke algebra is actually symmetric over $\mathbb{Z}[X_1, \ldots, X_n]^{S_n}$. Indeed, when viewed as a subalgebra of $\text{End}_\mathbb{Z}(\mathbb{Z}[X_1, \ldots, X_n])$, then $^0H_n$ contains $\mathcal{G}_n$. The injection of $\mathcal{G}_n$ in $^0H_n$ is given by $s_i \mapsto (X_i - X_{i+1})T_i + 1$ (cf also (3.1.7)). We have

$$w[1, n] \cdot a \cdot w[1, n] = \gamma(a)$$ for all $a \in ^0H_n$.

It follows that the linear form $t'$ given by $t'(a) = t(aw[1, n])$ is a symmetrizing form for $^0H_n$ over $\mathbb{Z}[X_1, \ldots, X_n]^{S_n}$.

The nil affine Hecke algebra $^0H_n$ is a graded algebra with deg $X_i = 2$ and deg $T_i = -2$ and $t$ is homogeneous of degree 0. The nil affine Hecke algebra has also a bifiltration given by

$$F^{<i,j}}(^0H_n) = \mathbb{Z}[X_1, \ldots, X_n]_{\leq i} \otimes \left(^0H_n\right)_{\geq j}.$$

Note that $t(F^{<(n-1),n-1})) = 0$.

3.1.7. Isomorphisms. The polynomial representations above induce isomorphisms with the semi-direct product of the algebra of polynomials with $\mathcal{G}_n$, after a suitable localization.

Let $R' = \mathbb{Z}[X_1, \ldots, X_n, (X_i - X_j)^{-1}, (X_i - X_j - 1)^{-1}]_{i \neq j}$. We have an isomorphism of $R'$-algebras

$$R' \times \mathcal{G}_n \simeq R' \otimes_{\mathbb{Z}[X_1, \ldots, X_n]} \tilde{H}_n, \quad s_i \mapsto \frac{X_i - X_{i+1}}{X_i - X_{i+1} + 1} (T_i - 1) + 1 = (T_i + 1) \frac{X_i - X_{i+1}}{X_i - X_{i+1} - 1} - 1.$$

Let $R'_q = R[X_1^\pm, \ldots, X_n^\pm, (X_i - X_j)^{-1}, (qX_i - X_j)^{-1}]_{i \neq j}$. We have an isomorphism of $R'_q$-algebras

$$R'_q \times \mathcal{G}_n \simeq R'_q \otimes_{R[X_1^\pm, \ldots, X_n^\pm]} H_n, \quad s_i \mapsto \frac{X_i - X_{i+1}}{qX_i - X_{i+1}} (T_i - q) + 1 = (T_i + 1) \frac{X_i - X_{i+1}}{qX_i - X_{i+1}} - 1.$$

Let $^0R' = \mathbb{Z}[X_1, \ldots, X_n, (X_i - X_j)^{-1}]_{i \neq j}$. We have an isomorphism of $^0R'$-algebras

$$^0R' \times \mathcal{G}_n \simeq ^0R' \otimes_{\mathbb{Z}[X_1, \ldots, X_n]} ^0H_n, \quad s_i \mapsto (X_i - X_{i+1})T_i + 1 = T_i(X_{i+1} - X_i) - 1.$$

Let us finally note that the functor

$$M \mapsto M^{\mathcal{G}_n} : (^0R' \times \mathcal{G}_n)\text{-mod} \to (^0R' \mathcal{G}_n)\text{-mod}$$

is an equivalence of categories.

3.2. Nil Hecke algebras associated with hermitian matrices. In this section, we introduce a flat family of algebras presented by quiver and relations. To a symmetrizable Cartan datum afforded by a quiver with automorphism, we associate a member of that family.
3.2.1. Definition. Let $I$ be a set, $k$ a commutative ring and $Q = (Q_{i,j})_{i,j \in I}$ a matrix in $k[u, v]$ with $Q_{ii} = 0$ for all $i \in I$.

Let $n$ be a positive integer and $L = I^n$. We define a (possibly non-unitary) $k$-algebra $H_n(Q)$ by generators and relations. It is generated by elements $1_{\nu}, x_{i,\nu}$ for $i \in \{1, \ldots, n\}$ and $\tau_{i,\nu}$ for $i \in \{1, \ldots, n-1\}$ and $\nu \in L$ and the relations are

- $1_{\nu}1_{\nu'} = \delta_{\nu,\nu'}1_{\nu}$
- $\tau_{i,\nu} = 1_{s_i(\nu)}\tau_{i,\nu}1_{\nu}$
- $x_{a,\nu} = 1_{\nu}x_{a,\nu}1_{\nu}$
- $x_{a,\nu}x_{b,\nu} = x_{b,\nu}x_{a,\nu}$
- $\tau_{i,\nu}s_i(\nu)\tau_{i,\nu} = Q_{i,\nu+1}(x_{i,\nu}, x_{i+1,\nu+1})$
- $\tau_{i,j}(\nu)\tau_{j,\nu} = \tau_{j,s_i(\nu)}\tau_{i,\nu}$ if $|i - j| > 1$
- $\tau_{i,s_i+1}\tau_{i,s_i+1}(\nu)\tau_{i,\nu+1} - \tau_{i,\nu+1}\tau_{i,\nu+1}(\nu)\tau_{i+1,\nu} = (x_{i+2,\nu} - x_{i,\nu})^{-1}(Q_{\nu,\nu+1}(x_{i+2,\nu}, x_{i+1,\nu}) - Q_{\nu,\nu+1}(x_{i,\nu}, x_{i+1,\nu}))$ if $\nu_i = \nu_{i+2}$
- otherwise

for $\nu, \nu' \in I$, $1 \leq i, j \leq n-1$ and $1 \leq a, b \leq n$.

Remark 3.5. Note that when $I$ is finite, then $H_n(Q)$ has a unit $1 = \sum_{\nu \in L} 1_{\nu}$.

Remark 3.6. It is actually more natural to view $H_n(Q)$ as a category $\mathcal{H}_n(Q)$ with set of objects $L$ and with Hom-spaces generated by

$$x_{a,\nu} \in \text{End}(\nu) \text{ for } 1 \leq a \leq n$$

$$\tau_{i,\nu} \in \text{Hom}(\nu, s_i(\nu)) \text{ for } 1 \leq i \leq n-1$$

with the relations above.

Given $a \in 1_{\nu}H_n(Q)1_{\nu'}$, we will sometimes write $x_ia$ for $x_{i,\nu}a$ and $ax_i$ for $ax_{i,\nu}'$ and proceed similarly for $\tau_i$.

Consider the (possibly non-unitary) algebra $R_n = (k[I][x])^{\otimes n} = k[x_1, \ldots, x_n] \otimes (k[I])^{\otimes n}$. We denote by $1_s$ the idempotent corresponding to the $s$-th factor of $k[I]$ and we put $1_{\nu} = 1_{\nu_1} \otimes \cdots \otimes 1_{\nu_n}$ for $\nu \in L$.

There is a morphism of algebras $R_n \rightarrow H_n(Q) \cong \mathcal{H}_n(Q)$, $x_11_{\nu} \mapsto x_{1,\nu}$. It restricts to a morphism $R_n^{S_n} \rightarrow Z(H_n(Q))$. Note that $R_1 = H_1(Q)$ and we put $H_0(Q) = k$.

Let $J$ be a set of finite sequences of elements of $\{1, \ldots, n-1\}$ such that $\{s_i, \ldots, s_i\}_{i \in J}$ is a set of minimal length representatives of elements of $\mathcal{S}_n$. Then,

$$S = \{\tau_{i_1, s_{i_2} \cdots s_{i_r}(\nu)} \cdots \tau_{i_r, \nu}x_{1,\nu}^{a_{1,\nu}} \cdots x_{n,\nu}^{a_{n,\nu}} \}_{(i_1, \ldots, i_r) \in J, (a_1, \ldots, a_n) \in \mathbb{Z}_{\geq 0}^n, \nu \in L}$$

generates $H_n(Q)$ as a $k$-module.

The algebra $H_n(Q)$ is filtered with $1_{\nu}$ and $x_{i,\nu}$ in degree 0 and $\tau_{i,\nu}$ in degree 1. The morphism $R_n \rightarrow H_n(Q)$ extends to a surjective algebra morphism

$$k[I][x] \rightarrow H_n^f \rightarrow \text{gr} H_n(Q), \quad T_i1_{\nu} \mapsto \tau_{i,\nu}.$$
The algebra is said to satisfy the PBW (Poincaré-Birkhoff-Witt) property if that morphism is an isomorphism.

**Theorem 3.7.** Assume \( n \geq 2 \). The following assertions are equivalent

- \( H_n(Q) \) satisfies PBW
- \( H_n(Q) \) is a free \( k \)-module with basis \( S \)
- \( Q_{ij}(u, v) = Q_{ji}(v, u) \) for all \( i, j \in I \).

**Proof.** The first two assertions are equivalent, thanks to the generating family \( S \) described above.

Let \( \nu \in L \) with \( \nu_i \neq \nu_{i+1} \). We have

\[
Q_{\nu_{i+1}, \nu_i}(x_{i,s_i(\nu)}, x_{i+1,s_i(\nu)}) \tau_{i,\nu} = \tau_{i,\nu} Q_{\nu_i, \nu_{i+1}}(x_{i,\nu}, x_{i+1,\nu}) = Q_{\nu_i, \nu_{i+1}}(x_{i+1,\nu}, x_{i,s_i(\nu)}) \tau_{i,\nu}.
\]

It follows that

\[
(Q_{\nu_{i+1}, \nu_i}(x_{i,s_i(\nu)}, x_{i+1,s_i(\nu)}) - Q_{\nu_i, \nu_{i+1}}(x_{i+1,s_i(\nu)}, x_{i,s_i(\nu)})) \tau_{i,\nu} = 0.
\]

Assume \( S \) is a basis of \( H_n(Q) \). We have \( Q_{\nu_{i+1}, \nu_i}(x_{i,s_i(\nu)}, x_{i+1,s_i(\nu)}) - Q_{\nu_i, \nu_{i+1}}(x_{i+1,s_i(\nu)}, x_{i,s_i(\nu)}) = 0 \).

Consequently, \( Q_{ij}(u, v) = Q_{ji}(v, u) \) for all \( i, j \in I \).

Assume \( Q_{ij}(u, v) = Q_{ji}(v, u) \) for all \( i, j \in I \). Choose an ordering of pairs of distinct elements of \( I \). Given \( i < j \), put \( P_{ij} = Q_{ij} \) and \( P_{ji} = 1 \). The theorem follows now from Proposition 3.12 below. \( \square \)

Denote by \( Q \mapsto \bar{Q} \) the automorphism given by \( \bar{Q}_{ij}(u, v) = Q_{ji}(v, u) \). The algebras \( H_n(Q) \) form a flat family of algebras over the space of matrices \( Q \) with vanishing diagonal and hermitian with respect to the automorphism of \( k[u, v] \) swapping \( u \) and \( v \) (i.e., such that \( Q = \bar{Q} \)).

**Corollary 3.8.** Assume \( Q \) is hermitian. Let \( I' \) be a subset of \( I \) and \( Q' = (Q_{ij})_{i,j \in I'} \). Then, the canonical map \( H_n(Q') \to H_n(Q) \) is injective and induces isomorphisms \( 1_\nu H_n(Q')1_{\nu'} \cong 1_\nu H_n(Q)1_{\nu'} \) for \( \nu, \nu' \in (I')^n \).

From Proposition 3.12 below, we obtain a description of the center of \( H_n(Q) \).

**Proposition 3.9.** Assume \( Q \) is hermitian. Then, we have \( Z(H_n(Q)) = R_n^{S_n} \).

When \( |I| = 1 \), then \( H_n(Q) \) is the nil affine Hecke algebra \( 0^0 H_n \) associated with \( GL_n \).

Given \( 0 \leq l \leq n \), we have an injective morphism of \( R_n \)-algebras

\[
H_l(Q) \otimes H_{n-l}(Q) \to H_n(Q)
\]

given by \( 1_\nu \otimes 1_{\nu'} \mapsto 1_{\nu \cup \nu'}, x_{j,\nu} \otimes 1_{\nu'} \mapsto x_{j,\nu \cup \nu'}, 1_\nu \otimes x_{j,\nu'} \mapsto x_{i+j,\nu \cup \nu'}, \) etc.

Assume \( Q \) is hermitian. Let \( i_1, \ldots, i_m \) be distinct elements of \( I \) and let \( d_1, \ldots, d_m \in \mathbb{Z}_{\geq 0} \) with \( n = \sum_{r} d_r \). Let \( \nu = (i_1, \ldots, i_1, \ldots, i_m, \ldots, i_m) \). The construction above induces an isomorphism of algebras

\[
0^0 H_{d_1} \otimes \cdots \otimes 0^0 H_{d_m} \cong 1_\nu H_n(Q)1_{\nu'}.
\]

**Remark 3.10.** The algebra Khovanov and Lauda \([KhoLau] 2\) associate to a symmetrizable Cartan matrix \((a_{ij})\) corresponds to \( Q_{ij}(u, v) = u^{-a_{ij}} + v^{-a_{ij}} \) for \( i \neq j \).
Let us describe some isomorphisms between $H_n(Q)$’s.

Let $\{a_i\}_{i \in I}$ in $k$ and $\{\beta_{ij}\}_{i,j \in I}$ in $k^\times$. Let $Q'_{ij}(u,v) = \beta_{ij} \beta_{ji} Q_{ij}(\beta_{ji} u + a_j, \beta_{ii} v + a_i)$. We have an isomorphism

$$H_n(Q') \sim H_n(Q), \quad 1_\nu \mapsto 1_\nu, \quad x_{i,\nu} \mapsto \beta_{i,\nu}^{-1}(x_{i,\nu} - a_{\nu}), \quad \tau_{i,\nu} \mapsto \beta_{\nu_i,\nu_{i+1}} \tau_{i,\nu}.$$ 

The construction above provides an action of the subgroup $\{ (\beta_{ij})_{i,j} | \beta_{ij} \beta_{ji} = 1 \}$ and $\beta_{ii} = 1 \}$ of $(G_m)^{I \times I}$ on $H_n(Q)$.

Assume $Q$ is hermitian. Given $\nu \in I^n$, we define $\tilde{\nu} \in I^n$ by $\tilde{\nu}_i = \nu_{n-i+1}$. There is an involution of $H_n(Q)$

$$H_n(Q) \sim H_n(Q), \quad 1_\nu \mapsto 1_{\tilde{\nu}}, \quad x_{i,\nu} \mapsto x_{n-i+1,\nu}, \quad \tau_{i,\nu} \mapsto -\tau_{i,\nu}.$$ 

Let us finally construct a duality. There is an isomorphism

$$H_n(Q) \sim H_n(Q)^{opp}, \quad 1_\nu \mapsto 1_\nu, \quad x_{i,\nu} \mapsto x_{i,\nu}, \quad \tau_{i,\nu} \mapsto \tau_{i,\nu}.$$ 

**Remark 3.11.** One can also work with a matrix $Q$ with values in $k(u,v)$ and define $H_n(Q)$ by adding inverses of the relevant polynomials in $x_{i,\nu}$’s.

3.2.2. **Polynomial realization.** Let $P = (P_{ij})_{i,j \in I}$ be a matrix in $k[u,v]$ with $P_{ii} = 0$ for all $i \in I$ and let $Q_{ij}(u,v) = P_{ij}(u,v)P_{ji}(v,u)$.

Consider the (possibly non-unitary) $k$-algebra $A_n(I) = k[I][x] \otimes \mathfrak{S}_n$.

The following Proposition provides a faithful representation of $H_n(Q)$ on the space $R_n$. It also shows that, after localization, the algebra $H_n(Q)$ depends only on the cardinality of $I$ (assuming non-vanishing of $Q_{ij}$ for $i \neq j$).

**Proposition 3.12.** Let $\mathcal{O}' = \bigoplus_{\nu \in L} k[x_1, \ldots, x_n][\{(x_i - x_j)^{-1}\}_{i \neq j, \nu_i = \nu_j}]_\nu$. We have an injective morphism of $k$-algebras

$$H_n(Q) \rightarrow \mathcal{O}' \otimes \mathbb{Z}^{|I|}[x] \otimes A_n(I)$$

$$1_\nu \mapsto 1_\nu, \quad x_{a,\nu} \mapsto x_{a,1_\nu},$$

$$\tau_{i,\nu} \mapsto \begin{cases} (x_i - x_{i+1})^{-1}(s_1 1_\nu - 1_\nu) & \text{if } \nu_i = \nu_{i+1} \\ P_{\nu_i,\nu_{i+1}}(x_{i+1}, x_i)s_i 1_\nu & \text{otherwise} \end{cases}$$

for $1 \leq a \leq n, 1 \leq i \leq n - 1$ and $\nu \in L$. It defines a faithful representation of $H_n(Q)$ on $R_n = \bigoplus_{\nu \in L} k[x_1, \ldots, x_n]_\nu$.

Assume $P_{ij} \neq 0$ for all $i \neq j$. Let

$$\mathcal{O} = \bigoplus_{\nu \in L} k[x_1, \ldots, x_n][\{P_{\nu_i,\nu_j}(x_i, x_j)^{-1}\}_{i \neq j, \nu_i = \nu_j}, \{(x_i - x_j)^{-1}\}_{i \neq j, \nu_i = \nu_j}]_\nu.$$ 

The morphism above induces an isomorphism $\mathcal{O} \otimes_{k[I][x]} H_n(Q) \sim \mathcal{O} \otimes_{k[I][x]} A_n(I)$.

**Proof.** Let $\tau'_{i,\nu} = \begin{cases} (x_i - x_{i+1})^{-1}(s_1 1_\nu - 1_\nu) & \text{if } \nu_i = \nu_{i+1} \\ P_{\nu_i,\nu_{i+1}}(x_{i+1}, x_i)s_i 1_\nu & \text{otherwise}. \end{cases}$

Let us check that the defining relations of $H_n(Q)$ hold with $\tau_{i,\nu}$ replaced by $\tau'_{i,\nu}$. We will not write the idempotents $1_\nu$ to make the calculations more easily readable.
We have
\[
\tau_{i,s+1}(\nu)\tau_{i+1,\nu} = \\
\begin{cases}
(x_i - x_{i+1})^{-1}((x_i - x_{i+2})^{-1}(s_is_{i+1} - s_i) - (x_{i+1} - x_{i+2})^{-1}(s_{i+1} - 1)) & \text{if } \nu_i = \nu_{i+1} = \nu_{i+2} \\
P_{\nu_{i+1}, \nu_{i+2}}(x_{i+1}, x_i)(x_i - x_{i+2})^{-1}(s_{i+1} - s_i) & \text{if } \nu_{i+1} = \nu_{i+2} \neq \nu_i \\
(x_i - x_{i+1})^{-1}\left(P_{\nu_{i+1}, \nu_{i+2}}(x_{i+2}, x_{i+1})s_is_{i+1} - P_{\nu_{i+1}, \nu_{i+2}}(x_{i+2}, x_{i+1})s_{i+1}\right) & \text{if } \nu_i = \nu_{i+2} \neq \nu_{i+1} \\
P_{\nu_{i+1}, \nu_{i+2}}(x_{i+1}, x_i)P_{\nu_{i+1}, \nu_{i+2}}(x_{i+2}, x_i)s_is_{i+1} & \text{if } \nu_{i+2} \notin \{\nu_i, \nu_{i+1}\}.
\end{cases}
\]
Assume \(\nu_i = \nu_{i+1} = \nu_{i+2}\). We have
\[
\tau_{i,s+1}(\nu)\tau_{i+1,\nu} = \\
= (x_{i+1} - x_{i+2})^{-1}((x_i - x_{i+2})^{-1}(x_i - x_{i+1})^{-1}(s_{i+1}s_is_{i+1} - s_{i+1}s_i - s_is_{i+1} + s_i + s_{i+1} - 1) \\
= \tau_{i+1,s+1}(\nu)\tau_{i,s+1}(\nu)\tau_{i+1,\nu}
\]
Assume \(\nu_i = \nu_{i+1} \neq \nu_{i+2}\). We have
\[
\tau_{i,s+1}(\nu)\tau_{i+1,\nu} = \\
= (x_{i+1} - x_{i+2})^{-1}P_{\nu_{i+1}, \nu_{i+2}}(x_{i+1}, x_i)P_{\nu_{i+1}, \nu_{i+2}}(x_{i+2}, x_i)(s_{i+1}s_is_{i+1} - s_is_{i+1}) \\
= \tau_{i+1,s+1}(\nu)\tau_{i,s+1}(\nu)\tau_{i+1,\nu}
\]
Assume \(\nu_{i+1} = \nu_{i+2} \neq \nu_i\). We have
\[
\tau_{i,s+1}(\nu)\tau_{i+1,\nu} = \\
= (x_{i+1} - x_{i+2})^{-1}P_{\nu_{i+1}, \nu_{i+2}}(x_{i+1}, x_i)P_{\nu_{i+1}, \nu_{i+2}}(x_{i+2}, x_i)(s_{i+1}s_is_{i+1} - s_{i+1}s_i) \\
= \tau_{i+1,s+1}(\nu)\tau_{i,s+1}(\nu)\tau_{i+1,\nu}
\]
Assume \(\nu_i, \nu_{i+1}\) and \(\nu_{i+2}\) are distinct. We have
\[
\tau_{i,s+1}(\nu)\tau_{i+1,\nu} = \\
= P_{\nu_{i+1}, \nu_{i+2}}(x_{i+1}, x_i)P_{\nu_{i+1}, \nu_{i+2}}(x_{i+2}, x_i)s_{i+1}s_is_{i+1} \\
= \tau_{i+1,s+1}(\nu)\tau_{i,s+1}(\nu)\tau_{i+1,\nu}
\]
Assume finally \(\nu_i = \nu_{i+2} \neq \nu_{i+1}\). We have
\[
\tau_{i,s+1}(\nu)\tau_{i+1,\nu} = \\
= (x_{i+1} - x_{i+2})^{-1}P_{\nu_{i+1}, \nu_{i+2}}(x_{i+1}, x_i)\left(P_{\nu_{i+1}, \nu_{i+1}}(x_{i+2}, x_{i+1})s_{i+1}s_is_{i+1} - P_{\nu_{i+1}, \nu_{i+1}}(x_{i+1}, x_{i+1})\right)
\]
and
\[
\tau_{i+1,s+1}(\nu)\tau_{i,s+1}(\nu)\tau_{i+1,\nu} = \\
= (x_{i+1} - x_{i+2})^{-1}P_{\nu_{i+1}, \nu_{i+1}}(x_{i+1}, x_{i+1})\left(P_{\nu_{i+1}, \nu_{i+1}}(x_{i+2}, x_{i+1})s_{i+1}s_is_{i+1} - P_{\nu_{i+1}, \nu_{i+1}}(x_{i+1}, x_{i+1})\right)
\]
hence
\[
\tau_{i+1,s+1}(\nu)\tau_{i,s+1}(\nu)\tau_{i+1,\nu} - \tau_{i+1,s+1}(\nu)\tau_{i,s+1}(\nu)\tau_{i+1,\nu} = \\
= (x_{i+1} - x_{i+2})^{-1}\left(P_{\nu_{i+1}, \nu_{i+1}}(x_{i+1}, x_{i+1})P_{\nu_{i+1}, \nu_{i+1}}(x_{i+2}, x_{i+1}) - P_{\nu_{i+1}, \nu_{i+1}}(x_{i+1}, x_{i+2})P_{\nu_{i+1}, \nu_{i+1}}(x_{i+2}, x_{i+1})\right).
\]
The other relations are immediate to check.
Let $B$ be the $k$-subalgebra of $\mathcal{O} \otimes _{k(t)[z]}^{} A_n(I)$ image of the morphism. We have $\mathcal{O} \otimes _{k(t)[z]}^{} B = \mathcal{O} \otimes _{k(t)[z]}^{} A_n(I)$. The image of $S$ in $\mathcal{O} \otimes _{k(t)[z]}^{} A_n(I)$ is linearly independent over $k$. It follows that the canonical map $H_n(Q) \to B$ is an isomorphism and that $S$ is a basis of $H_n(Q)$ over $k$. 

3.2.3. Cartan matrices. Let $C = (a_{ij})$ be a Cartan matrix, i.e.,

- $a_{ii} = 2$,
- $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$ and
- $a_{ij} = 0$ if and only if $a_{ji} = 0$.

We put $m_{ij} = -a_{ij}$. Let $\{t_{ij,r,s}\}$ be a family of indeterminates with $i \neq j \in I$, $0 \leq r < m_{ij}$ and $0 \leq s < m_{ji}$ and such that $t_{ji,s,r} = t_{ij,r,s}$. Let $\{t_{ij}\}_{i \neq j}$ be a family of indeterminates with $t_{ij} = t_{ji}$ if $a_{ij} = 0$.

Let $k = k_C = \mathbb{Z}[\{t_{ij,r,s}\} \cup \{\pm 1\}]$. Let $Q_{ii} = 0$, $Q_{ij} = t_{ij}$ if $a_{ij} = 0$ and

$$Q_{ij} = t_{ij} u^{m_{ij}} + \sum_{0 \leq r < m_{ij}} t_{ij,r,s} u^r v^s + t_{ji} v^{m_{ji}}$$

for $i \neq j$ and $a_{ij} \neq 0$.

We put $H_n(C) = H_n(Q)$. This is a $k$-algebra, free as a $k$-module.

Consider $s \neq t \in I$ and assume $n = m_{st} + 2$. Let $\nu = (t, s, \ldots, s) \in I^n$. Given $0 \leq i \leq n - 1$, let $c_i = c_i(\nu) = (s, \ldots, s, t, s, \ldots, s)$, where $t$ is in the $(i+1)$-th position. The canonical isomorphisms $0 H_i \cong 1_{(s,\ldots,s)} H_i(1_{(s,\ldots,s)})$ and $0 H_{n-i-1} \cong 1_{(s,\ldots,s)} H_{n-i-1}(1_{(s,\ldots,s)})$ give rise to a morphism of unitary algebras

$$0 H_i \otimes 0 H_{n-i-1} \to 1_{c_i(\nu)} H_n(Q) 1_{c_i(\nu)}.$$

We denote by $e_{i+1}$ the image of $b_i \otimes b_{n-1-i}$ (cf. [3.1.6]).

The following Lemma generalizes a result of Khovanov and Lauda [KhoLau2, Corollary 7].

**Lemma 3.13.** Let $P^i = H_n(Q)e_{i+1}$. Define $\alpha_{i+1,i} = e_{i+1} T_{n-1} \cdots T_{i+2} e_{i+1}$ and $\alpha_{i+1,i} = e_{i+2} T_{i+2} \cdots T_{i+1} e_{i+1}$. We have a complex $P^i$ of projective $H_n(Q)$-modules

$$0 \mapsto P^0 \xrightarrow{\alpha_{0,1}} P^1 \mapsto \cdots \mapsto P^{i-1} \xrightarrow{\alpha_{i-1,i}} P^i \mapsto \cdots \mapsto P^{n-1} \xrightarrow{\alpha_{n-2,n}} 0$$

which is homotopy equivalent to 0, with splittings given by the maps $\alpha_{i+1,i} = (-1)^{i+n} t_{st}^{-1} \alpha_{i+1,i}$.

**Proof.** Note that $b_r b_r = b_r e_{i+1} e_{i+2}$ and $b_r T_i b_r = T_i b_r$, hence $\alpha_{i+1} = \tau_{n-1} \cdots \tau_{i+2} e_{i+2}$ and $\alpha_{i+1,i} = e_{i+2} T_i \cdots T_i e_{i+1}$. We have

$$\alpha_{i-1,i} \alpha_{i,i+1} = e_i \tau_{n-1} \cdots \tau_i e_{i+2} = e_i \tau_{n-2} \cdots \tau_i e_{i+2} = 0.$$

It follows that the maps $\alpha_{i-1,i}$ provide a differential.

We have

$$\alpha_{i+1,i} \alpha_{i,i+1} = \tau_{n-1} \cdots \tau_i \tau_{n-1} e_{i+2} = \tau_{n-2} \cdots \tau_i \tau_{n-1} \cdots \tau_i e_{i+2} = 0.$$
It follows that
\[\alpha_{i,i+1}\alpha_{i+1,i} - \alpha_{i,i-1}\alpha_{i-1,i} = \partial s_1\cdots s_{i-1}s_{i+1} \left( (x_{i+2} - x_i)^{-1} (Q_{st}(x_{i+2}, x_{i+1}) - Q_{st}(x_i, x_{i+1})) \right).\]

Write \(Q_{st}(u, v) = \sum_{a,b} q_{ab} u^a v^b\) with \(q_{ab} \in \mathbb{Z}\). We have
\[\sum_{a \geq 1, b \geq 0} \partial s_1\cdots s_{i-1} (x_i^a) \partial s_{i+1}\cdots s_n (x_i^a) = (1)^{n+i} q_{n-2,0}\]
and finally \(\alpha_{i,i+1}^\prime + \alpha_{i,i-1}^\prime = 1\). 

Assume \(C\) is a symmetrizable Cartan matrix, \(i.e.,\) there is a family \((d_i)_{i \in I}\) of positive integers with \(\text{lcm}\{d_i\} = 1\) and such that \((b_{ij})\) is symmetric, for \(b_{ij} = d_i a_{ij}\).

Let \(k^*\) be the quotient of \(k\) by the ideal generated by those \(t_{i,j,r,s}\) such that \(d_i r + d_j s \neq -b_{ij}\). Let \(H_n^\bullet(C) = k^* \otimes_k H_n(C)\). The algebra \(H_n^\bullet(C)\) is graded with \(\text{deg} L_{i,\nu} = 0\), \(\text{deg} x_{i,\nu} = 2d_{\nu}\), and \(\text{deg} \tau_{i,\nu} = -b_{i\nu, i\nu+1}\).

Remark 3.14. The description of the basis \(S\) for \(H_n^\bullet(C)\) (cf Theorem 3.7) shows that the rank of the sum of the homogeneous components of \(L_{i,\nu} H_n^\bullet(C) L_{i,\nu}\) with degree less than a given integer is finite.

3.2.4. Quivers with automorphism. Let \(\Gamma\) be a quiver with a compatible automorphism [Lun, §12.1.1]: this is the data of

- a set \(\tilde{I}\) (vertices)
- a set \(H\) (edges) and a map with finite fibers \(h \mapsto [h]\) from \(H\) to the set of two-element subsets of \(\tilde{I}\)
- maps \(s : H \rightarrow \tilde{I}\) (source) and \(t : H \rightarrow \tilde{I}\) (target) such that \([s(h), t(h)] = [h]\) for any \(h \in H\)
- automorphisms \(a : \tilde{I} \rightarrow \tilde{I}\) and \(a : H \rightarrow H\) such that \(s(a(h)) = a(s(h))\) and \(t(a(h)) = a(t(h))\) and such that \(s(h)\) and \(t(h)\) are not in the same \(a\)-orbit for \(h \in H\).

We put \(I = \tilde{I}/a\). We define \(i \cdot i = 2\#(i)\) and \(i \cdot j = -\#\{h \in H | [h] \in i \cup j\}\) for \(i \neq j\) in \(I\) (note that this uses only the graph structure, not the orientation). This defines a Cartan datum and \(\left(\frac{2i^2}{i^2} \cdot \frac{ij}{ij} \right)\) is a symmetrizable Cartan matrix.

Given \(i, j \in I\), let \(d_{ij}\) be the number of orbits of \(a\) in \([h \in H | s(h) \in i \text{ and } t(h) \in j]\). We have \(d_{ij} + d_{ji} = -2(i \cdot j) / \text{lcm}(i \cdot i, j \cdot j)\) for \(i \neq j\).

Define \(P_{ij}(u, v) = (v^{l/(j \cdot j)} - u^{l/(i \cdot i)})^{d_{ij}}\) where \(l = \text{lcm}(i \cdot i, j \cdot j)\), for \(i \neq j\) and \(P_{ii} = 0\).

We have \(Q_{ij} = (-1)^{d_{ij}} \left( u^{l/(i \cdot i)} - v^{l/(j \cdot j)} \right)^{-2(i \cdot j)/l} \) for \(i \neq j\).

We put \(k = \mathbb{Z}\) and \(H_n(\Gamma) = H_n(Q)\). This is a specialization of the algebra \(H_n(C)\) introduced in \(\text{[3.2.3]}\).

The algebra \(H_n(\Gamma)\) is graded with \(\text{deg} L_{i,\nu} = 0\), \(\text{deg} x_{i,\nu} = \nu_i \cdot \nu_i\) and \(\text{deg} \tau_{i,\nu} = -\nu_i \cdot \nu_{i+1}\). As a graded algebra, it is a specialization of \(H_n^\bullet(C)\) (here, \(d_i = (i \cdot i)/2\)).
Consider another choice of orientation $s', t'$ of the graph $(\bar{I}, H, h \mapsto [h])$, compatible with the automorphism $a$. Given $i \neq j$, define

$$\beta_{ij} = \begin{cases} (-1)^{d_{ij} + d'_{ij}} & \text{if } d_{ij} \geq d'_{ij} \\ 1 & \text{otherwise.} \end{cases}$$

We have an isomorphism

$$H_n(\Gamma) \sim H_n(\Gamma'), \ 1_\nu \mapsto 1_\nu, \ x_{i,\nu} \mapsto x_{i,\nu}, \ \tau_{i,\nu} \mapsto \beta_{\nu,\nu+1} \tau_{i,\nu}.$$  

It follows that, up to isomorphism, the graded algebra $H_n(\Gamma)$ depends only on the Cartan datum. Note nevertheless that the system of isomorphisms constructed above between the algebras corresponding to different orientations is not a transitive system. Consequently, we do not define “the” algebra associated to a Cartan datum (or a graph with automorphism). Note finally that, up to isomorphism, $H_n(\Gamma)$ depends only on the Cartan matrix and a change of quiver with automorphism corresponds to a rescaling of the grading.

Note that if $\Gamma$ is the disjoint union of full subquivers $\Gamma_1$ and $\Gamma_2$, then $H_n(\Gamma) = H_n(\Gamma_1) \otimes H_n(\Gamma_2)$.

3.2.5. Type A graphs. Let $k$ be a field and $q \in k^\times$.

Assume first $q = 1$. Given $I$ a subset of $k$, we denote by $I_1$ the quiver with set of vertices $I$ and with an arrow $i \to i + 1$, whenever $i, i + 1 \in I$.

Assume now $q \neq 1$. Given $I$ a subset of $k^\times$, we denote by $I_q$ the quiver with set of vertices $I$ and with an arrow $q \to qi$, whenever $i, qi \in I$.

Note that $I_q$ has type A and we put $sl_q = gl_q$. Let us assume $I_q$ is connected. Let us describe the possible types for the underlying graph.

Assume $q = 1$. Type:

- $A_n$ if $|I| = n$ and $k$ has characteristic 0 or $p > n$.
- $\tilde{A}_{p-1}$ if $|I| = p$ is the characteristic of $k$.
- $A_\infty$ if $I$ is bounded in one direction but not finite.
- $A_{\infty,\infty}$ if $I$ is unbounded in both directions.

Assume $q \neq 1$. Denote by $e$ the multiplicative order of $q$. Type:

- $A_n$ if $|I| = n < e$.
- $\tilde{A}_{e-1}$ if $|I| = e$.
- $A_\infty$ if $I$ is bounded in one direction but not finite.
- $A_{\infty,\infty}$ if $I$ is unbounded in both directions.

3.2.6. Idempotents and representations. Let $k$ be a field and let $\Gamma$ be a quiver. We denote by $kH_n(\Gamma)\text{-}\text{Mod}_{\text{g}}$ the category of $H_n(\Gamma)$-modules $M$ such that $M = \bigoplus_\nu 1_\nu M$ and for every $\nu$, the elements $x_{i,\nu}$ act locally nilpotently on $1_\nu M$ for $1 \leq i \leq n$.

Let $I$ be a subset of $k$ and let $\Gamma = I_1$.

Let

$$\mathcal{O}' = \bigoplus_{\nu \in I^n} k[X_1, \ldots, X_n][((X_i - X_j)^{-1})_{i \neq j, \nu_i \neq \nu_j}, \{(X_i - X_j + 1)^{-1})_{i \neq j, \nu_i + 1 \neq \nu_j}],$$

a non-unitary ring. Note that this is a subring of

$$\bigoplus_{\nu \in I^n} k[X_1, \ldots, X_n][(X_i - X_j - a)^{-1}]_{i \neq j, a \neq \nu_i - \nu_j}.$$
We denote by $1_{\nu}$ the unit of the summand of $O'$ corresponding to $\nu$. We put a structure of non-unitary algebra on $O'[H_n] = O' \otimes \mathbb{Z}[X_1, \ldots, X_n] [H_n]$ by setting

$$T_i 1_{\nu} - 1_{s_i(\nu)} T_i = (X_{i+1} - X_i)^{-1} (1_{\nu} - 1_{s_i(\nu)}) .$$

Let

$$O' = \bigoplus_{\nu \in I^n} k[x_1, \ldots, x_n] \{(\nu_i - \nu_j + x_i - x_j)^{-1}\}_{i \neq j, \nu_i \neq \nu_j}, \{(\nu_i - \nu_j + 1 + x_i - x_j)^{-1}\}_{i \neq j, \nu_i \neq \nu_j} ,$$

a subring of

$$\bigoplus_{\nu \in I^n} k[x_1, \ldots, x_n] [(x_i - x_j - a)^{-1}]_{i \neq j, a \neq 0} .$$

From Proposition 3.12 and 3.1.7 we obtain the following proposition.

**Proposition 3.15.** We have an isomorphism of non-unitary algebras

$$\tilde{O}' H_n(\Gamma) \simeq \tilde{O}' H_n, \quad \tau_i 1_{\nu} \mapsto \left\{ \begin{array}{ll} (X_i - X_{i+1} + 1)^{-1} (T_i - 1) 1_{\nu} & \text{if } \nu_i = \nu_{i+1} \\ \{(X_i - X_{i+1}) T_i + 1\} 1_{\nu} & \text{if } \nu_{i+1} = \nu_i + 1 \\ \frac{X_i - X_{i+1} + 1}{X_i - X_{i+1} + 1} (T_i - 1) 1_{\nu} + 1 & \text{otherwise.} \end{array} \right.$$  

Let $M$ be a $kH_n$-module. Given $a \in k^n$, we denote by $M_a$ the $k[X_1, \ldots, X_n]$-submodule of $M$ of elements with support contained in the closed point of $\mathbb{A}_k^n$ given by $a$.

We denote by $\tilde{C}_\Gamma$ the category of $kH_n$-modules $M$ such that

$$M = \bigoplus_{a \in \Gamma^n} M_a .$$

**Theorem 3.16.** We have an equivalence of categories

$$k H_n(\Gamma)-\text{Mod}_0 \simeq \tilde{C}_\Gamma, \quad M \mapsto M$$

where $X_i$ acts on $1_{\nu} M$ by $(x_i + \nu_i)$ and $T_i$ acts on $1_{\nu} M$ by

- $(x_i - x_{i+1} + 1) T_i + 1$ if $\nu_i = \nu_{i+1}$
- $(x_i - x_{i+1} - 1)^{-1} (T_i - 1)$ if $\nu_{i+1} = \nu_i + 1$
- $(x_i - x_{i+1} + \nu_{i+1} - \nu_i + 1)(x_i - x_{i+1} + \nu_{i+1} - \nu_i)^{-1} (T_i - 1) + 1$ otherwise.

Assume $I$ is finite. Let $d : I \to \mathbb{Z}_{>0}$ be a function and $H_n(I, d)$ be the quotient of $kH_n$ by the two-sided ideal generated by $\prod_{i \in I} (X_i - i)^{d(i)}$, a degenerate cyclotomic Hecke algebra. Let $H_n(\Gamma, d)$ be the quotient of $H_n(\Gamma)$ by the ideal generated by $x_i^{d(i)}$ for $i \in I$.

**Corollary 3.17.** The construction of Theorem 3.16 induces an isomorphism of $k$-algebras $H_n(\Gamma, d) \simeq H_n(I, d)$.

Let $k$ be a field and $q \in k - \{0, 1\}$. Let $I$ be a subset of $k^\times$ and let $\Gamma = I_q$.

Let

$$O' = \bigoplus_{\nu \in I^n} k[X_1^{\pm 1}, \ldots, X_n^{\pm 1}] \{(X_i - X_j)^{-1}\}_{i \neq j, \nu_i \neq \nu_j}, \{(q X_i - X_j)^{-1}\}_{i \neq j, q \nu_i \neq \nu_j} ,$$

"
a non-unitary $k[X_1^\pm, \ldots, X_n^\pm]$-algebra. Note that this is a subring of
\[
\bigoplus_{\nu \in \mathbb{N}} k[X_1^\pm, \ldots, X_n^\pm][(X_i - aX_j)^{-1}]_{i \neq j, a \in k - \{0, \nu, \nu^{-1}\}}.
\]

We denote by $1_\nu$ the unit of the summand of $\mathcal{O}'$ corresponding to $\nu$. We put a structure of non-unitary algebra on $\mathcal{O}'H_n = \mathcal{O}' \otimes \mathbb{Z}[q_{\nu_1}, q_{\nu_2}, \ldots, q_{\nu_n}]H_n$ by setting
\[
T_i1_\nu - 1_{s_i(\nu)}T_i = (1 - q)X_iX_i^{-1}(1 - 1_{s_i(\nu)}).
\]

Let
\[
\mathcal{O}' = \bigoplus_{\nu \in \mathbb{N}} k[x_1^\pm, \ldots, x_n^\pm][((\nu_i\nu_j^{-1}x_i - x_j)^{-1})_{i \neq j, i, j \neq \nu, \nu^{-1}}, ((q\nu_i\nu_j^{-1}x_i - x_j)^{-1})_{i \neq j, \nu, \nu^{-1}}],
\]
a subring of
\[
\bigoplus_{\nu \in \mathbb{Z}} k[q^\pm, x_1^\pm, \ldots, x_n^\pm][(x_i - ax_j)^{-1}]_{i \neq j, a \in k - \{0, 1\}}.
\]

From Proposition 3.12 and Proposition 3.17, we obtain the following proposition.

**Proposition 3.18.** We have an isomorphism of non-unitary algebras
\[
\mathcal{O}'H_n(\Gamma) \cong \mathcal{O}'H_n, \quad x_i1_\nu \mapsto \nu_i^{-1}x_i1_\nu,
\]
\[
\tau_i1_\nu \mapsto \begin{cases} 
\nu_i(qX_i - X_i^{-1})(T_i - q)1_\nu & \text{if } \nu_i = \nu_i+1 \\
q^{-1}\nu_i^{-1}((X_i - X_i^{-1})T_i + (q - 1)X_i^{-1})1_\nu & \text{if } \nu_i+1 = q\nu_i \\
(X_iX_i^{-1}(T_i - q) + 1)1_\nu & \text{otherwise.}
\end{cases}
\]

Let $M$ be a $kH_n$-module. Given $a \in (k^\times)^n$, we denote by $M_a$ the $k[X_1^\pm, \ldots, X_n^\pm]$-submodule of $M$ of elements with support contained in the closed point of $\mathbb{A}^n_k$ given by $a$.

We denote by $\mathcal{C}_\Gamma$ the category of $kH_n$-modules $M$ such that
\[
M = \bigoplus_{a \in \Gamma^n} M_a.
\]

**Theorem 3.19.** We have an equivalence of categories
\[
kH_n(\Gamma)/(\text{Mod}_0) \cong \mathcal{C}_\Gamma, \quad M \mapsto M
\]
where $X_i$ acts on $1_\nu M$ by $\nu_i(x_i + 1)$ and $T_i$ acts on $1_\nu M$ by
\[
\bullet (qX_i - X_i^{-1})\tau_i + q \text{ if } \nu_i = \nu_i+1
\]
\[
\bullet (q^{-1}x_i - x_i^{-1})^{-1}(\tau_i + (1 - q)x_i) \text{ if } \nu_i+1 = q\nu_i
\]
\[
\bullet (\nu_i^2x_i - \nu_i\nu_i^2x_i)^{-1}((\nu_i\nu_i^2x_i - \nu_i\nu_i^2x_i)(\nu_i) + (1 - q)\nu_i\nu_i^2x_i) \text{ otherwise.}
\]

Assume $I$ is finite. Let $d : I \to \mathbb{Z}_{>0}$ be a function and $H_n(I, d)$ be the quotient of $kH_n$ by the two-sided ideal generated by $\prod_{i \in I} (X_i - i)^{d(i)}$, a cyclotomic Hecke algebra.

**Corollary 3.20.** The construction of Theorem 3.19 induces an isomorphism of $k$-algebras $H_n(\Gamma, d) \cong H_n(I, d)$.

**Remark 3.21.** The isomorphisms of Corollaries 3.17 and 3.20 have been constructed and studied independently by Brundan and Kleshchev [BrK]. They provide gradings on (degenerate) cyclotomic Hecke algebras.
4.1. Construction.

4.1.1. Half Kac-Moody algebras. Let $I$ be a set and $C = (a_{ij})_{i,j \in I}$ a Cartan matrix. We consider the ring $\mathbf{k}$ and the matrix $Q$ of $\mathbb{Z}$. Define $\mathcal{B} = \mathcal{B}(C)$ as the free strict monoidal $\mathbf{k}$-linear category generated by objects $E_s$ for $s \in I$ and by arrows

$$x_s : E_s \to E_s \text{ and } \tau_{st} : E_s E_t \to E_tE_s \text{ for } s, t \in I$$

with relations

1. $\tau_{st} \circ \tau_{ts} = Q_{st}(E_t x_s, x_t E_s)$
2. $\tau_{tu} E_s \circ E_t \tau_{su} \circ \tau_{st} E_u - E_u \tau_{st} \circ \tau_{su} E_t \circ E_s \tau_{tu} = \begin{cases} Q_{st}(x_se_tE_t x_t E_s) E_s - E_u Q_{st}(E_x x_t E_s) & \text{if } s = u \\ 0 & \text{otherwise.} \end{cases}$
3. $\tau_{st} \circ x_s E_t - E_s x_t \circ \tau_{st} = \delta_{st}$
4. $\tau_{st} \circ E_s E_t - x_s E_t \circ \tau_{st} = -\delta_{st}$

These relations state that the maps $x_s$ and $\tau_{st}$ give an action of the nil affine Hecke algebra associated with $C$ on powers of $E$. More precisely, we have an isomorphism of (non-unitary) algebras

$$H_n(C) \sim \bigoplus_{\nu, \nu' \in I^n} \text{Hom}_{\mathcal{B}}(E_{\nu_1} \cdots E_{\nu_k}, E_{\nu'_1} \cdots E_{\nu'_k})$$

$$1_\nu \mapsto \text{id}_{E_{\nu_1} \cdots E_{\nu_k}}$$

$$x_{i,\nu} \mapsto E_{\nu_1} \cdots \nu_{i+1} x_{\nu_{i+1}} E_{\nu_{i+2}} \cdots E_{\nu_k}$$

$$\tau_{i,\nu} \mapsto E_{\nu_1} \cdots \nu_{i+1} E_{\nu_{i+2}} \cdots E_{\nu_k}$$

Let $s \in I$ and $n \geq 0$. We have an isomorphism of algebras $\mathbf{k}^{0H_n} \sim \text{End}_{\mathcal{B}}(E_s^n)$ and we denote by $E_s^{(n)} = b_n E_s^n \in \mathcal{B}^I$ the image of the idempotent $b_n = T_{w[1,n]}X_1^{n-1}X_2^{n-2} \cdots X_{n-1} \in 0H_n^{\text{opp}}$ (cf \ref{3.1.6}). We denote also by $F_s^{(n)}$ the image of $T_{w[1,n]}X_1^{n-1}X_2^{n-2} \cdots X_{n-1} \in 0H_n^{\text{opp}}$. Note that this idempotent corresponds to the idempotent $b'_n = X_1^{n-1}X_2^{n-2} \cdots X_{n-1}T_{w[1,n]}$ of $0H_n$. Thanks to Lemma \ref{3.3} we have the following result (as in [ChRou, Lemma 5.15]).

**Lemma 4.1.** The action map is an isomorphism $0H_n b_n \otimes_{P_n^\mathbf{k}} E_s^{(n)} \sim E_s^n$. In particular, we have $E_s^n \simeq n! \cdot E_s^{(n)}$. Similarly, we have isomorphisms $b'_n \cdot 0H_n \otimes_{P_n^\mathbf{k}} F_s^{(n)} \sim F_s^n$. In particular, we have $F_s^n \simeq n! \cdot F_s^{(n)}$.

The following Proposition is a consequence of Lemma \ref{3.3} (apply $\text{Hom}_{H_n(C)}(P, -)$). It gives a categorical version of the Serre relations.

**Proposition 4.2.** Consider $s \neq t \in I$ and let $m = m_{st}$. Let $\alpha_{i,i+1} = \tau_{m+1} \cdots \tau_{i+2} \tau_{i+1}$ and $\alpha'_{i+1,i} = (-1)^{i+1}m_{st}^{-1} \tau_1 \tau_2 \cdots \tau_{i+1}$. We have a complex

$$\cdots \longrightarrow E_s^{(m-i)} E_t E_s^{(i+1)} \xrightarrow{\alpha_{i,i+1}} E_s^{(m-i+1)} E_t E_s^{(i)} \xrightarrow{\alpha_{i-1,i}} E_s^{(m-i+2)} E_t E_s^{(i-1)} \cdots$$
Remark 4.3. The first part of Proposition 4.2 generalizes [KhoLau2]. We will give a different proof of the existence of an isomorphism (second part of the Proposition) in [Rou3] in the case of integrable 2-representations.

Assume now \( C \) is symmetrizable and consider \( (d_i), (b_{ij}) \) and \( k^* \) as in 3.2.3. We put \( \mathcal{B}_0^* = \mathcal{B} \otimes_k k^* \).

The category \( \mathcal{B}_0^* \) can be enriched in graded abelian groups by setting \( \deg x_s = 2d_s \) and \( \deg \tau_{st} = -b_{st} \). We denote by \( \mathcal{B}^* \) the corresponding graded category. It follows from Theorem 3.7 and Remark 3.14 that Hom-spaces in \( \mathcal{B}^* \) are free \( k^* \)-modules of finite rank.

We put \( E_s^{(n)} = b_n E_s^{(\frac{n(n-1)}{2})d_s} \). Note that \( P_n^{(\frac{n(n-1)}{2})d_s} \) is self-dual as a graded \( P_n^{E_s} \)-module and we have

\[
E_s^n \simeq v^n \tau_{|n|!} E_s^{(n)}
\]

where \( [n]_s! = [n]! (v^{d_s}) \).

The maps \( \alpha_{ij} \) and \( \alpha'_{ij} \) of Proposition 4.2 are graded and the proposition remains true in \( \mathcal{B}^* \).

Consider finally \( \Gamma \) a quiver with a compatible automorphism and consider the specialization \( k^* \to \mathbb{Z} \) of 3.2.4. We put \( \mathcal{B}_*^* (\Gamma) = \mathcal{B}^*(C) \otimes_{k^*} \mathbb{Z} \).

4.1.2. Kac-Moody algebras. Let \( C = (a_{ij})_{i,j \in I} \) be a Cartan matrix. Let \( (X,Y,\langle -,-\rangle, \{\alpha_i\}_{i \in I}, \{\alpha_i^\vee\}_{i \in I}) \) be a root datum of type \( C \), i.e.,

- \( X \) and \( Y \) are finitely generated free abelian groups and \( \langle -,-\rangle : Y \times X \to \mathbb{Z} \) is a perfect pairing
- \( I \to X, \ i \mapsto \alpha_i \) and \( I \to Y, \ i \mapsto \alpha_i^\vee \) are injective and \( \langle \alpha_i^\vee, \alpha_j \rangle = a_{ij} \).

Associated with this data, there is a Kac-Moody algebra \( \mathfrak{g} \), a quantum group \( U_v (\mathfrak{g}) \) when \( C \) is symmetrizable, as well as completed versions [Lu]. Let us recall those we will need.

Assume \( C \) is symmetrizable. Consider the \( \mathbb{Q}(v) \)-algebra \( \mathcal{U}_v^+ (\mathfrak{g}) \) generated by elements \( e_i \) for \( i \in I \) with relations

\[
\sum_{a+b=1-a_{ij}} (-1)^a e_i^{(a)} e_j e_i^{(b)} = 0
\]

for any \( i \neq j \in I \), where \( e_i^{(a)} = e_i^{\frac{a}{|a|!}} \). We denote by \( \mathcal{U}_v^+ (\mathfrak{g}) \) the \( \mathbb{Z}[v^{\pm 1}] \)-subalgebra generated by the \( e_i^{(a)} \) for \( i \in I \) and \( a \geq 0 \). We define an algebra \( \mathcal{U}_v^- (\mathfrak{g}) \) isomorphic to \( \mathcal{U}_v^+ (\mathfrak{g}) \) with \( e_i \) replaced by \( f_i \).

Let \( \mathcal{U}_v (\mathfrak{g}) \) be the category enriched in \( \mathbb{Q}(v) \)-vector spaces with set of objects \( X \) and morphisms generated by \( e_i : \lambda \to \lambda + \alpha_i \) and \( f_i : \lambda \to \lambda - \alpha_i \) subject to the following relations:

- the relation (5) and its version with \( e_r \) replaced by \( f_r \)
- \( [e_i, f_j] 1_\lambda = \delta_{ij} \langle \alpha_i^\vee, \lambda \rangle 1_\lambda \).

Let \( \mathcal{U}_v (\mathfrak{g}) \) be the subcategory enriched in \( \mathbb{Z}[v^{\pm 1}] \)-modules of \( \mathcal{U}_v (\mathfrak{g}) \) with same objects as \( \mathcal{U}_v (\mathfrak{g}) \) and with morphisms generated by \( e_i^{(r)} \) and \( f_i^{(r)} \) for \( i \in I \) and \( r \geq 0 \).

We put \( \mathcal{U}_1 (\mathfrak{g}) = \mathcal{U}_v (\mathfrak{g}) \otimes_{\mathbb{Z}[v^{\pm 1}]} \mathbb{Z}[v^{\pm 1}]/(v - 1) \), etc.

Note that \( \bigoplus_{\lambda, \mu \in X} \text{Hom}_{\mathcal{U}_v (\mathfrak{g})} (\lambda, \mu) \) is the non-unitary ring \( \hat{\mathcal{A}} \) of [Lu] §23.2.
The category of functors (compatible with the $\mathbb{Z}[v^{\pm 1}]$-structure) $U_v(\mathfrak{g}) \to \mathbb{Z}[v^{\pm 1}]$-Mod is equivalent to the category of unital $\mathcal{A}$-$\hat{\mathcal{U}}$-modules via $V \mapsto \bigoplus_{\lambda} V(\lambda)$ and we will identify the two categories. A representation $V$ of $U_v(\mathfrak{g})$ is integrable if for every $v \in V$ and $i \in I$, there is $n_0$ such that $e_i^{(n)} v = f_i^{(n)} v = 0$ for all $n \geq n_0$.

Assume $C$ is a general Cartan matrix. The constructions above still make sense in the non-quantum case and lead to a category $U_1(\mathfrak{g})$.

We denote by $W = \{\sigma_i\}_{i \in I}$ the Weyl group of $\mathfrak{g}$.

4.1.3. 2-Kac Moody algebras. Let $\mathcal{B}_1$ be the strict monoidal $k$-linear category obtained from $\mathcal{B}$ by adding $F_s$ right dual to $E_s$ for every $s \in I$. Define

$$\varepsilon_s = \varepsilon_{E_s} : E_s F_s \to 1 \text{ and } \eta_s = \eta_{E_s} : 1 \to F_s E_s.$$  

The dual pairs $(E_s, F_s)$ provides dual pairs $(E^n_s, F^n_s)$ and the action of $0 H_n$ on $E^n_s$ induces an action of $(0 H_n)^{\text{opp}}$ on $F^n_s$. We denote by $x_s$ the endomorphism of $F_s$ induced by $x_s \in \text{End}(E_s)$ and denote also by $\tau_{st} : F_s F_t \to F_t F_s$ the morphism induced by $\tau_{st} \in \text{Hom}(E_s E_t, E_t E_s)$.

We define a morphism of monoids

$$h : \text{Ob}(\mathcal{B}_1) \to X, \ E_s \mapsto \alpha_s, \ F_s \mapsto -\alpha_s.$$  

Consider the strict 2-category $\mathcal{A}_1$ with set of objects $X$ and $\text{Hom}(\lambda, \lambda') = h^{-1}(\lambda' - \lambda)$, a full subcategory of $\mathcal{B}_1$. We write $E_{s, \lambda}$ for $E_s 1_\lambda$, $\varepsilon_{s, \lambda}$ for $\varepsilon_s 1_\lambda$, etc.

Let $\mathfrak{A} = \mathfrak{A}(\mathfrak{g})$ be the $k$-linear strict 2-category deduced from $\mathfrak{A}_1$ by inverting the following 2-arrows:

- when $\langle \alpha^\vee_s, \lambda \rangle \geq 0$,
  $$\rho_{s, \lambda} = \sigma_{s s} + \sum_{i=0}^{\langle \alpha^\vee_s, \lambda \rangle - 1} \varepsilon_s \circ (x^i_s F_s) : E_s F_s 1_\lambda \to F_s E_s 1_\lambda \oplus 1^{\langle \alpha^\vee_s, \lambda \rangle}_\lambda$$

- when $\langle \alpha^\vee_s, \lambda \rangle \leq 0$,
  $$\rho_{s, \lambda} = \sigma_{s s} + \sum_{i=0}^{-1 - \langle \alpha^\vee_s, \lambda \rangle} (F_s x^i_s) \circ \eta_s : E_s F_s 1_\lambda \oplus 1_{\lambda}^{-\langle \alpha^\vee_s, \lambda \rangle} \to F_s E_s 1_\lambda$$

- $\sigma_{st} : E_s F_t 1_\lambda \to F_t E_s 1_\lambda$ for all $s \neq t$ and all $\lambda$

where we define

$$\sigma_{st} = (F_t E_s \varepsilon_t) \circ (F_t \tau_{ts} F_s) \circ (\eta_t E_s F_t) : E_s F_t \to F_t E_s.$$  

Remark 4.4. The inversion of maps in the definition of $\mathfrak{A}$ accounts for the Lie algebra relations $[e_s, f_s] = h_s$ and $[e_s, f_t] = 0$ for $s \neq t$. The elements $h_\zeta$ for $\zeta \in Y$ appear only through their action as multiplication by $\langle \zeta, \lambda \rangle$ on the $\lambda$-weight space.

Assume $C$ is symmetrizable. We proceed as in §4.1.1 to define graded versions. Let $\mathfrak{A}_0^\bullet = \mathfrak{A} \otimes_k k^\bullet$. The category $\mathfrak{A}_0^\bullet$ can be enriched in graded abelian groups by setting

$$\text{deg} \varepsilon_{s, \lambda} = d_s(1 - \langle \alpha^\vee_s, \lambda \rangle) \text{ and } \text{deg} \eta_{s, \lambda} = d_s(1 + \langle \alpha^\vee_s, \lambda \rangle).$$  

We denote by $\mathfrak{A}^\bullet$ the corresponding graded 2-category.
Note that $\sigma_{st}$ is a graded map (for all $s, t \in I$), while $\rho_{s, \lambda}$ carries shifts:

$$\rho_{s, \lambda} : E_s F_s 1_\lambda \xrightarrow{\langle \alpha^\vee_s, \lambda \rangle - 1} F_s E_s 1_\lambda \oplus \bigoplus_{i=0}^{-1-\langle \alpha^\vee_s, \lambda \rangle} 1_\lambda (d_s (2i + 1 - \langle \alpha^\vee_s, \lambda \rangle)) \text{ when } \langle \alpha^\vee_s, \lambda \rangle \geq 0,$$

$$\rho_{s, \lambda} : E_s F_s 1_\lambda \oplus \bigoplus_{i=0}^{-1-\langle \alpha^\vee_s, \lambda \rangle} 1_\lambda (d_s (2i + 1 - \langle \alpha^\vee_s, \lambda \rangle)) \xrightarrow{f_s} F_s E_s 1_\lambda \text{ when } \langle \alpha^\vee_s, \lambda \rangle \leq 0.$$

We have a dual pair in $\mathcal{A}^*$

$$\left( E_s 1_\lambda, 1_\lambda F_s (d_s (1 + \langle \alpha^\vee_s, \lambda \rangle)) \right).$$

Finally, given a quiver $\Gamma$ with a compatible automorphism and associated Cartan matrix $C$, we put $\mathcal{A}^*_Z(\Gamma) = \mathcal{A}^* \otimes_k Z$ (cf 3.2.3). We put also $\mathcal{A}_Z = \mathcal{A} \otimes_k Z$.

Let us summarize: we have constructed several 2-categories with set of objects $X$ and with $\text{Hom}(\lambda, \lambda') = h^{-1}(\lambda' - \lambda)$. Given a root datum, we have a $\mathbf{k}$-linear 2-category $\mathcal{A}$ and, when $C$ is symmetrizable, we have a specialization $\mathcal{A}^*$ that is $\mathbf{k}^*$-linear and graded. Given in addition a quiver with compatible automorphism affording the Cartan matrix, we have a further specialization $\mathcal{A}^*_Z$ that is graded and $\mathbf{Z}$-linear.

**Remark 4.5.** The action of $0H_n$ on $E^n_s$ is given by

$$X_i \mapsto E^n_{s-i} x_s E^{i-1}_s$$

while the action of $0H_n^{\text{opp}}$ on $F^n_s$ is given by

$$X_i \mapsto F^n_{s-i} x_s F^{i-1}_s$$

4.1.4. **Second adjunctions.** We define “canditates” units and counits for an adjunction $(F_s, E_s)$.

Let $s \in I$ and $\lambda \in X$. Assume $\langle \alpha^\vee_s, \lambda \rangle \geq 0$. Let $\bar{\varepsilon}_{s, \lambda} : F_s E_s 1_\lambda \rightarrow 1_\lambda$ be the map whose image under

$$\text{Hom}(\sigma_{s,s}, 1_{\lambda}) : \text{Hom}(F_s E_s 1_\lambda, 1_\lambda) \xrightarrow{\langle \alpha^\vee_s, \lambda \rangle - 1} \text{Hom}(E_s F_s 1_\lambda, 1_\lambda) / \bigoplus_{i=0}^{\langle \alpha^\vee_s, \lambda \rangle} \text{End}(1_\lambda) \cdot \varepsilon_{s,i} \circ (x_s^{i} F_s)$$

coincides with $(-1)^{\langle \alpha^\vee_s, \lambda \rangle + 1} \varepsilon_{s,i} \circ (x_s^{\langle \alpha^\vee_s, \lambda \rangle} F_s)$.

Assume $\langle \alpha^\vee_s, \lambda \rangle < 0$. Let $\bar{\varepsilon}_{s, \lambda} : F_s E_s 1_\lambda \rightarrow 1_\lambda$ be the unique morphism such that

$$\bar{\varepsilon}_{s, \lambda} \circ \rho_{s, \lambda} = (0, 0, \ldots, 0, (-1)^{\langle \alpha^\vee_s, \lambda \rangle + 1}).$$

Assume $\langle \alpha^\vee_s, \lambda \rangle > 0$. Let $\hat{\varepsilon}_{s, \lambda} : 1_\lambda \rightarrow F_s E_s 1_\lambda$ be the unique morphism such that

$$\rho_{s, \lambda} \circ \hat{\varepsilon}_{s, \lambda} = (0, 0, \ldots, 0, (-1)^{\langle \alpha^\vee_s, \lambda \rangle + 1}).$$

Assume $\langle \alpha^\vee_s, \lambda \rangle \leq 0$. Let $\hat{\varepsilon}_{s, \lambda} : 1_\lambda \rightarrow E_s F_s 1_\lambda$ be the map whose image under

$$\text{Hom}(1_\lambda, \sigma_{s,s}) : \text{Hom}(1_\lambda, E_s F_s 1_\lambda) \xrightarrow{\langle \alpha^\vee_s, \lambda \rangle - 1} \text{Hom}(1_\lambda, F_s E_s 1_\lambda) / \bigoplus_{i=0}^{\langle \alpha^\vee_s, \lambda \rangle} (F_s x_s^i \circ \eta_{s,i} \cdot \text{End}(1_\lambda))$$

coincides with $(-1)^{\langle \alpha^\vee_s, \lambda \rangle} (F_s x_s^{\langle \alpha^\vee_s, \lambda \rangle}) \circ \eta_{s,i}$. 
4.1.5. Other versions. We define here 2-categories related to the ones defined in the previous section by adding generators and imposing extra symmetry conditions and relations.

We define $\mathcal{B}_1^l$ as the strict monoidal $k$-linear category obtained from $\mathcal{B}$ by adding $F_s$ left and right adjoint to $E_s$ for every $s \in I$. Define

$$\varepsilon_s^l = \varepsilon_{F_s} : F_s \cdot E_s \to 1,$$

and $
\eta_s^l = \eta_{F_s} : 1 \to E_s \cdot F_s.
$

Define also $\mathcal{A}_1^l$ and $\mathcal{A}^l$ as $\mathcal{A}_1$ and $\mathcal{A}$ were defined from $\mathcal{B}_1$. Now, we define $\mathcal{A}'$ as the $k$-linear strict 2-category obtained from $\mathcal{A}_1^l$ by adding the relation $\varepsilon_s^l = \varepsilon_s$. Note that given $X \in \mathcal{A}'$, we have $X^\vee = \vee X$.

Remark 4.6. The relation $\varepsilon_s^l = \varepsilon_s$ shows that $\varepsilon_s^l$ can be expressed in terms of the maps $x_s$, $\tau_{s,s}$, $\eta_s$ and $\varepsilon_s$. As a consequence, the adjunction $(F_s,E_s)$ is determined by the adjunction $(E_s,F_s)$ and the maps $x_s$ and $\tau_{s,s}$.

We define specializations of $\mathcal{A}'$ in the same way as those defined for $\mathcal{A}$. Note that

$$\deg \eta_{s,\lambda}^l = d_s(1 - \langle \alpha_s^\vee, \lambda \rangle) \quad \text{and} \quad \deg \varepsilon_{s,\lambda}^l = d_s(1 + \langle \alpha_s^\vee, \lambda \rangle)$$

and we have a dual pair in $\mathcal{A}^*$

$$
\left(1_{\lambda} F_s \left(-d_s(1 + \langle \alpha_s^\vee, \lambda \rangle)\right), E_s 1_{\lambda}\right).
$$

We define $\tilde{\mathcal{A}}'$ to be the quotient of $\mathcal{A}'$ given by the relations $\eta_s^l = \hat{\eta}_s$ and $(f^\vee)^\vee = f$ for every 2-arrow $f$ of $\mathcal{A}'$.

There are canonical strict 2-functors

$$\mathcal{A} \to \mathcal{A}' \to \tilde{\mathcal{A}}'.$$

4.2. Properties.

4.2.1. Symmetries. In §4.2.1, we work in $\mathcal{A}$. The map $\sigma_{st}$ can be defined using the Hecke action on $F^2$ instead of $E^2$.

Lemma 4.7. Given $s,t \in I$, we have $\sigma_{st} = (E_s F_t \xrightarrow{E_s F_t \eta_s} E_s F_t F_s E_s \xrightarrow{E_s F_t \tau_{s,s}} E_s F_t F_s F_t E_s \xrightarrow{\varepsilon_{F_t} F_t E_s} F_t E_s)$. 

Proof. The lemma follows from the commutativity of the following diagram
We define the Chevalley involution, a strict 2-equivalence of 2-categories $I : \mathcal{A}^{\text{opp}} \xrightarrow{\sim} \mathcal{A}$ satisfying $I^2 = \text{Id}$ by

$$I(1) = 1 - \lambda, \quad I(E_s) = F_s, \quad I(\varepsilon) = -\eta_s, \quad I(\tau_s) = -\tau_{ts} \text{ and } I(x_s) = x_s.$$  

Note that $I(\sigma_s) = -\sigma_{ts}$ (Lemma 3.7), $I(\rho_{s,\lambda}) = -\rho_{s,-\lambda}$ and $I(\bar{\varepsilon}) = -\bar{\eta}$.  

We define the Chevalley duality, a strict 2-equivalence of 2-categories $D : \mathcal{A}^{\text{rev}} \xrightarrow{\sim} \mathcal{A}$ satisfying $D^2 = \text{Id}$ by

$$1_\lambda \mapsto 1_\lambda, \quad E_s \mapsto F_s, \quad x_s \mapsto x_s, \quad \tau_s \mapsto \tau_{ts}, \quad \varepsilon_s \mapsto \varepsilon_s, \quad \eta_s \mapsto \eta_s.$$  

Note that $D(\sigma_s) = \sigma_{ts}$ (Lemma 4.7) and $D$ fixes $\rho_{s,\lambda}, \varepsilon$ and $\eta$.  

There is also a strict equivalence of monoidal categories

$$\mathcal{B}^{\text{rev}} \xrightarrow{\sim} \mathcal{B}, \quad E_s \mapsto E_s, \quad x_s \mapsto x_s, \quad \tau_{ts} \mapsto -\tau_{ts}.$$  

4.2.2. Relations in $\mathfrak{sl}_2$. We provide isomorphisms between sums of objects of type $E_s^m F_s^n$ and sum of objects of type $F_s^n E_s^m$.  

In this section, we work in the category $\mathcal{A}$ associated with $\mathfrak{g} = \mathfrak{sl}_2$: $I = \{s\}$ with $s \cdot s = 2$, $X = Y = \mathbb{Z}$, $\alpha_s = 1$ and $\alpha_s = 2$.  

We put $E = E_s$ and $F = F_s$. We put $\varepsilon = \varepsilon_s$ and $\eta = \eta_s$. Let $i \in \mathbb{Z}_{\geq 0}$. We define by induction $\varepsilon_m : E^m F^m \to 1$ and $\eta_m : 1 \to F^m E^m$ in $\mathcal{B}_1$. We put $\varepsilon_0 = \eta_0 = \text{id}$ and $\varepsilon_m = \varepsilon_{m-1} \circ (E^m \xrightarrow{\varepsilon} F^m)$ and $\eta_m = (F^m \circ \eta_{m-1})$.  

Given $a, b \in \mathbb{Z}_{\geq 0}$, we denote by $\mathcal{P}(a, b)$ the set of partitions with at most $a$ non-zero parts, all of which are at most $b$. Given $\mu = (\mu_1 \geq \cdots \geq \mu_a \geq 0) \in \mathcal{P}(a, b)$, we denote by $m_\mu(X_1, \ldots, X_a) = \sum \sigma X_1^{\mu_{\sigma(1)}} \cdots X_a^{\mu_{\sigma(a)}}$ the corresponding monomial symmetric function (here, $\sigma$ runs over $S_a$ modulo the stabilizer of $\mu$).  

Let $m, n, i \in \mathbb{Z}_{\geq 0}$ with $i \leq m$ and $i \leq n$ and let $\lambda \in X$. Let $r = m - n + \lambda$. Assume $r < 0$. We put

$$L(m, n, i, \lambda) = \bigoplus_{\begin{subarray}{c} w \in \mathbb{S}^m_{\leq i} \\ w' \in \mathbb{S}^n_{\leq i} \\ \mu \in \mathcal{P}(i, -r, -i) \end{subarray}} (T_w \otimes m_\mu(X_{n-i+1}, \ldots, X_n) X_{n-i+1}^{i-1} X_{n-i+2}^{i-2} \cdots X_{n-1} T_{w'}) \mathbb{Z} \subset \mathcal{H}^{0}_{m \circ \mathfrak{h}_i} H_0^{0}_{n} \mathcal{H},$$  

where the right action (resp. the left action) of $0_{H_i}$ on $0_{H_m}$ (resp. on $0_{H_n}$) is via $X_r \mapsto X_{r+m-i}$ and $T_r \mapsto T_{r+m-i}$ (resp. $X_r \mapsto X_{r+n-i}$ and $T_r \mapsto T_{r+n-i}$). The sum is direct since $T_w [1] X_{n-i+1}^{i-1} \cdots X_{n-i} T_{w[1]} = 0$ (cf. 3.1.6).  

Note that $\bigoplus_{\mu \in \mathcal{P}(i, -r, -i)} L(m, n, i, \lambda)$ is the subspace of $\mathbb{Z}[X_1, \ldots, X_i]^{S_i}$ of symmetric polynomials whose degree in any of the variables is at most $-r - i$. It has dimension $\binom{r + i}{i}$. Note that $L(m, n, 0, \lambda) = \mathbb{Z}$ and $L(m, n, i, \lambda) = 0$ if $i > 0$ and $r = 0$.  

Let $\tilde{L}(m, n, i, \lambda) = L(m, n, i, \lambda) \otimes (0_{H_{m-i}} \otimes \mathcal{H}_{n-i})^{\text{opp}}, a ((0_{H_m} \otimes \mathcal{H}_n)^{\text{opp}}, (0_{H_{m-i}} \otimes \mathcal{H}_{n-i})^{\text{opp}})$-submodule of $0_{H_m} \otimes \mathcal{H}_i 0_{H_n}$.  

When needed, we will also consider the modules $L([a, b], [a', b'], i, \lambda)$ and $\tilde{L}([a, b], [a', b'], i, \lambda)$ where $1 \leq a \leq b \leq m$ and $1 \leq a' \leq b' \leq n$, which are defined similarly.  

**Lemma 4.8.** The multiplication map induces an isomorphism

$$L(m, n, i, \lambda) \otimes (0_{H_{m-i}} \otimes (0_{H_{n-i}})^{\text{opp}}) \xrightarrow{\sim} \tilde{L}(m, n, i, \lambda).$$  

The $((0_{H_m} \otimes (0_{H_n})^{\text{opp}}, (0_{H_{m-i}} \otimes (0_{H_{n-i}}))^{\text{opp}})$-submodule $L(m, n, i, \lambda)(0_{H_{m-i}} \otimes (0_{H_{n-i}})^{\text{opp}})$ of $0_{H_m} \otimes \mathcal{H}_i 0_{H_n}$ is projective.
The canonical isomorphism

\[
0^\ast H_n^f Z[X_1, \ldots, X_{m-i}] \otimes Z[X_1, \ldots, X_{n-i}]X_{n-i+1}X_{n-i+2} \cdots X_n 0^\ast H_n^f.
\]

On the other hand, \(0^\ast H_d\) is projective as a \((0^\ast H_d^f, 0^\ast H_d)\)-bimodule (cf. [3.1.6]) and the last statement of the lemma follows. □

Let \(L'(m, n, i, \lambda) = \text{Hom}_{\mathbb{Z}}(L(n, m, i, \lambda), \mathbb{Z})\) and

\[
\bar{L}'(m, n, i, \lambda) = \text{Hom}_{0^\ast H_m^f \otimes 0^\ast H_n^f}(\bar{L}(n, m, i, \lambda), 0^\ast H_m^f \otimes 0^\ast H_n^f).
\]

The canonical isomorphism

\[
L(n, m, i, \lambda) \otimes_{\mathbb{Z}} 0^\ast H_m^f \otimes 0^\ast H_n^f \sim \bar{L}(n, m, i, \lambda)
\]

induces an isomorphism

\[
\text{Hom}_{\mathbb{Z}}(L(n, m, i, \lambda), 0^\ast H_m^f \otimes 0^\ast H_n^f) \sim \bar{L}'(m, n, i, \lambda)
\]

and composing with the canonical isomorphism

\[
L'(m, n, i, \lambda) \otimes_{\mathbb{Z}} 0^\ast H_m^f \otimes 0^\ast H_n^f \sim \text{Hom}_{\mathbb{Z}}(L(n, m, i, \lambda), 0^\ast H_m^f \otimes 0^\ast H_n^f),
\]

we obtain an isomorphism of right \((0^\ast H_m^f \otimes 0^\ast H_n^f^{\text{opp}})\)-modules

\[
L'(m, n, i, \lambda) \otimes_{\mathbb{Z}} 0^\ast H_m^f \otimes 0^\ast H_n^f \sim \bar{L}'(m, n, i, \lambda).
\]

Given \(m, n \in \mathbb{Z}_{\geq 0}\), we define by induction a map \(\sigma_{m,n} : E^m F^n \rightarrow F^n E^m\). The maps \(\sigma_{m,0}\) and \(\sigma_{0,n}\) are identities. We put \(\sigma_{m,1} = (\sigma E^{m-1}) \circ (E \sigma_{m-1,1})\) and \(\sigma_{m,n} = (F^{n-1} \sigma_{m,1}) \circ (\sigma_{m,n-1} F)\).

**Lemma 4.9.** The map \(\sigma_{m,n}\) is a morphism of \((H_m^f \otimes (H_n^f)^{\text{opp}})\)-modules. We have

\[
\sigma_{m,1} = \left( E^m F \xrightarrow{\eta E^m F} F E^m + 1 \quad F \xrightarrow{F (T_1 \cdots T_m) F} F E^m + 1 \quad F \xrightarrow{F E^m \varepsilon} F E^m \right)
\]

and

\[
\sigma_{1,n} = \left( E^m F \xrightarrow{E F^m \eta} F E^m + 1 \quad F \xrightarrow{E (T_1 \cdots T_n) E} F E^m + 1 \quad E \xrightarrow{E F^m \varepsilon} F E^m \right).
\]

Given \(a, b \in \mathbb{Z}_{\geq 0}\), we have a commutative diagram
Proof. We have a commutative diagram

and the second statement follows by induction. The third statement follows from the second one by applying the Chevalley duality (cf [4.2.1]).

Let $i \in [1, m - 1]$. Since $T_{i+1}T_1 \cdots T_m = T_1 \cdots T_m T_i$, we have a commutative diagram

It follows that $\sigma_{m,1}$ commutes with the action of $0 H^T_m$ and by induction we deduce that $\sigma_{m,n}$ commutes with $0 H^T_m$. The commutation with $(0 H^T_n)^{\text{opp}}$ follows by applying the Chevalley duality.

We have a commutative diagram

hence, we obtain a commutative diagram

The last part of the Lemma follows now by induction on $b$. $\square$
Lemma 4.11. Let \( f \) modules and \( l \) partition obtained by adding \( \phi \) modules. Let

Lemma 4.10. \( R \) where \( \mathcal{P} \)

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We prove the second part of the lemma by induction on \( n \). We write \( k \subset \mu \) if there is \( i \) such that \( \mu_i = k \) and we denote by \( \mu \setminus k \) the partition obtained by removing \( k \) to \( \mu \). We have

\[
\partial_{s_1 \cdots s_a} (X_{a+1}^l m_\mu(X_1, \ldots, X_a)) = \sum_{k \subseteq \mu} \partial_{s_1} (X_1^k \partial_{s_2 \cdots s_a} (X_{a+1}^l m_\mu \setminus k (X_2, \ldots, X_a))).
\]

By induction, we have

\[
\partial_{s_2 \cdots s_a} (X_{a+1}^l m_\mu \setminus k (X_2, \ldots, X_a)) = X_2^{l-a+1} m_\mu \setminus k (X_3, \ldots, X_{a+1}) + R,
\]

where the degree in \( X_2 \) of \( R \) is strictly less than \( l - a + 1 \). It follows that

\[
\partial_{s_1 \cdots s_a} (X_{a+1}^l m_\mu (X_1, \ldots, X_a)) = \sum_{k \subseteq \mu} X_1^{l-a} X_2^k m_\mu \setminus k (X_3, \ldots, X_{a+1}) + R' = X_1^{l-a} m_\mu (X_2, \ldots, X_{a+1}) + R',
\]

where the degree in \( X_1 \) of \( R' \) is strictly less than \( l - a \). The lemma follows.

The following Lemma is clear.

Lemma 4.11. Let \( C \) be a \( k \)-linear category, \( X, Y \) two objects of \( C \), \( L \) and \( L' \) two right \( \text{End}(X) \)-modules and \( f : L \rightarrow \text{Hom}(X, Y) \) and \( f' : L' \rightarrow \text{Hom}(X, Y) \) two morphisms of right \( \text{End}(X) \)-modules. Let \( \phi : L \otimes_{\text{End}(X)} X \rightarrow Y \) and \( \phi' : L' \otimes_{\text{End}(X)} X \rightarrow Y \) be the associated morphisms.
Consider finite filtrations on \( L \) and on \( L' \) such that \( f(L^{<i}) = f'(L'^{<i}) \) for all \( i \). Assume there are isomorphisms \( L^{<i}/L^{<i} \cong L'^{<i}/L'^{<i} \) for all \( i \) such that the following diagram commutes

\[
\begin{array}{c}
L^{<i}/L^{<i} \\
\sim \\
\Hom(X,Y)/L^{<i} \text{End}(X) \\
L'^{<i}/L'^{<i}
\end{array}
\]

Then, \( \phi \) is an isomorphism if and only if \( \phi' \) is an isomorphism.

**Lemma 4.12.** Assume \( m-n+\lambda \leq 0 \). We have an isomorphism \( \sum_{i} \text{act} \circ \left( \text{id} \otimes ((F^{-i} \eta_{i} E^{m-i}) \circ \sigma_{m-i,n-i}) \right) : \)

\[
\bigoplus_{i=0}^{\min(m,n)} L(m,n,i,\lambda) \otimes_{\mathbb{Z}} E^{m-i} F^{m-i} 1_{\lambda}(i(i-2m-\lambda)) \cong F^{m} E^{m} 1_{\lambda}.
\]

It induces an isomorphism of \((^0H_{m}^{I} \otimes (^0H_{n}^{I})^{\text{opp}})\)-modules:

\[
\bigoplus_{i=0}^{\min(m,n)} \bar{L}(m,n,i,\lambda) \otimes_{_{\mathcal{H}_{m-i}^{I} \otimes (^0H_{n-i})^{\text{opp}}}} E^{m-i} F^{m-i} 1_{\lambda}(i(i-2m-\lambda)) \cong F^{m} E^{m} 1_{\lambda}.
\]

Assume \( m-n+\lambda \geq 0 \). We have an isomorphism \( \sum_{i} (\text{id} \otimes (\sigma_{m-i,n-i} \circ (E^{m-i} \epsilon_{i} F^{m-i}))) \circ \text{act}^{*} : \)

\[
\bigoplus_{i=0}^{\min(m,n)} E^{m} F^{m} 1_{\lambda} \cong \bigoplus_{i=0}^{\min(m,n)} L'(m,n,i,\lambda) \otimes_{\mathbb{Z}} F^{m-i} E^{m-i} 1_{\lambda}(i(2n-\lambda-i)).
\]

It induces an isomorphism of \((^0H_{m}^{I} \otimes (^0H_{n}^{I})^{\text{opp}})\)-modules:

\[
E^{m} F^{m} 1_{\lambda} \cong \bigoplus_{i=0}^{\min(m,n)} \bar{L}'(m,n,i,\lambda) \otimes_{_{\mathcal{H}_{m-i}^{I} \otimes (^0H_{n-i})^{\text{opp}}}} F^{m-i} E^{m-i} 1_{\lambda}(i(2n-\lambda-i)).
\]

**Proof.** Note first that the statements for \((m,n,\lambda)\) where \( m-n+\lambda \leq 0 \) are transformed into the statements for \((n,m,-\lambda)\) by the Chevalley involution \( I \). It is immediate to check that the maps are graded and it is enough to prove the Lemma in the non-graded setting.

Assume \( m-n+\lambda \leq 0 \). Note that the first statement is equivalent to the second one (Lemma 4.8), whose map makes sense thanks to Lemma 4.9. We will drop the idempotents \( 1_{\lambda} \) to simplify notations. Note that the result holds for \( m = n = 1 \) as \( \rho_{a,\lambda} \) is invertible by definition.

Since \( \bar{L}(m,n,i,\lambda) \otimes_{\mathcal{H}_{m-i}^{I} \otimes (^0H_{n-i})^{\text{opp}}} (^0H_{m-i} \otimes (^0H_{n-i})^{\text{opp}}) \) is projective as a \((^0H_{m}^{I} \otimes (^0H_{n}^{I})^{\text{opp}}, ^0H_{m-i} \otimes (^0H_{n-i})^{\text{opp}})\)-bimodule (Lemma 4.8), it is enough to show that the second map is an isomorphism after multiplication by \( T_{w[1,m]} \otimes T_{w[1,n]} \) (Lemma 3.3).

We prove the Lemma by induction on \( n+m \). Note that the Lemma holds trivially when \( n=0 \) or \( m=0 \) as well as when \((m,n) = (1,1)\). So, we can assume \( m+n \geq 3 \).
Let us first consider the case $m - n + \lambda = 0$. Applying the Chevalley duality if necessary, we can assume that $n > 1$. By induction, we have isomorphisms

\[(7) \quad E^m F^n \oplus \tilde{L}(m, n - 1, 1, \lambda - 2) \otimes_{0} H_{m-1} \otimes (0 H_{n-2})^{opp} E^{m-1}F^{n-1} \cong F^{n-1}E^m F \]

\[\cong F^n E^m \oplus \tilde{L}'(m, 1, 1, \lambda) \otimes_{0} H_{m-1} F^{n-1}E^{m-1}.\]

By Lemma 4.9 we have a commutative diagram

\[
\begin{array}{ccc}
E^{m-1}F & \xrightarrow{\eta} & F E^m F \\
\downarrow{\sigma_{m-1,1}} & & \downarrow{\sigma_{m-1,1}} \\
F E^{m-1} & \xrightarrow{F \eta} & F^2 E^m \\
\end{array}
\]

It follows that the composition of maps in (7) has one of its components equal to

\[(8) \quad \text{act} \circ (T_{n-1}E^m) \circ (F^{n-1}\eta E^{m-1}) \circ (F^{n-2}\sigma_{m-1, n-1}) : \]

\[\tilde{L}(m, n - 1, 1, \lambda - 2) \otimes_{0} H_{n-1} \otimes (0 H_{n-2})^{opp} E^{m-1}F^{n-1} \rightarrow F^n E^m.\]

We have $(T_{w[1]} \otimes T_{w[1]}^{opp})L(m, n - 1, 1, \lambda - 2) = (T_{w[1]} \otimes T_{w[1]})\mathbb{Z}$ and it follows that the map in (8) vanishes after multiplication by $(T_{w[1]} \otimes T_{w[1]}^{opp})$. We deduce that the component $\sigma_{m, n} : E^m F^n \rightarrow F^n E^m$ of the composition of maps in (7) is an isomorphism.

- We consider now the case $n = 1$ and $m + \lambda \leq 0$. By induction, we have an isomorphism

\[E^{m-1}F E \oplus L([2, m], 1, 1, \lambda + 2) \otimes E^{m-1} \cong F E^m.\]

So, we have an isomorphism

\[(9) \quad E^m F \oplus L(1, 1, 1, \lambda) \otimes E^{m-1} \oplus L([2, m], 1, 1, \lambda + 2) \otimes E^{m-1} \cong F E^m\]

Taking the image under the Chevalley duality of the commutative diagram of Lemma 4.9 we obtain a commutative diagram

\[
\begin{array}{ccc}
E^{m-1} & \xrightarrow{\eta} & E^{m-1}F E \\
\downarrow{\eta} & & \downarrow{\eta} \\
F^{m} & \xrightarrow{F(T_{1} \cdots T_{m-1})} & F E^{m} \\
\end{array}
\]

It follows that the isomorphism (9) induces an isomorphism $(\sigma_{m, 1}, \text{act} \circ (\text{id} \otimes (\eta E^{m-1})), (\text{act} \circ (\text{id} \otimes (\eta E^{m-2})))E)$:

\[E^m F \oplus \left( \bigoplus_{0 \leq i < -\lambda} X^i T_1 \cdots T_{m-1} \mathbb{Z} \right) \otimes E^{m-1} \oplus L([2, m], 1, 1, \lambda + 2) \otimes E^{m-1} \cong F E^m.\]

Let

\[M = \left( \bigoplus_{0 \leq i < -\lambda} X^i T_1 \cdots T_{m-1} H_{m-1} \right) \oplus \bigoplus_{w \in S_{2, m-1}} T_w (X_m)^l H_{m-1}.\]
This is a $0H^f_m$-submodule of $0H_m$. We have

$$T_{w[1,m]}M = T_{w[1,m]} \left( \sum_{i<\lambda} \partial_{s_{m-1} \cdots s_1}(X^i_1)^0H_{m-1} + \sum_{i<-\lambda-m} (X^i_m)^0H_{m-1} \right)$$

$$= T_{w[1,m]} \sum_{i<-\lambda-m} (X^i_m)^0H_{m-1}$$

$$= T_{w[1,m]}\bar{L}(m, 1, 1, \lambda)^0H_{m-1}.$$  

Note that $M$ is generated by $\dim Z L(m, 1, 1, \lambda)$ elements as a right $0H_{m-1}$-module. Since $\bar{L}(m, 1, 1, \lambda)^0H_{m-1}$ is a free right $0H_{m-1}$-module of rank $\dim Z L(m, 1, 1, \lambda)$, it follows that $M$ is a free right $0H_{m-1}$-module of that rank. We have an isomorphism

$$(\sigma_{m,1}, \text{act } o (\text{id } \otimes (\eta E^{m-1}))) : E^m F \oplus M \otimes_{0H_{m-1}} E^{m-1} \cong F E^m.$$  

The isomorphism

$$(\sigma_{m,1}, \text{act } o (\text{id } \otimes (\eta E^{m-1}))) : E^m F \oplus \bar{L}(m, 1, 1, \lambda) \otimes_{0H^f_{m-1}} E^{m-1} \cong F E^m$$

becomes an isomorphism after multiplication by $T_{w[1,m]}$, since it coincides with the multiplication by $T_{w[1,m]}$ of the isomorphism (10). It follows from Lemmas 4.8 and 3.3 that the morphism (11) is an isomorphism and the lemma is proven when $n = 1$.

- We consider finally the case $n > 1$ and $m - n + \lambda < 0$. We have an isomorphism

$$F \bigoplus_{i=0}^{n-1} L(m, n-1, i, \lambda) \otimes E^{n-i} E^{m-i-1} \cong F^n E^m.$$  

The case $n = 1$ of the lemma gives isomorphisms

$$(L(m - i, 1, 1, \lambda - 2(n - i - 1)) \otimes E^{m-i-1}) F^{n-i-1} \oplus E^{m-i} F^{n-i} \cong F^{n-i} F^{m-i-1}.$$  

Combining the previous two isomorphisms, we obtain an isomorphism

$$\bigoplus_{i=0}^{n} (L(m, [2, n], i, \lambda) \otimes L(m, [2, n], i-1, \lambda) \otimes L(m - i + 1, 1, 1, \lambda - 2(n - i))) \otimes E^{m-i} F^{n-i} \cong F^n E^m.$$  

In that isomorphism, the map $L(m, [2, n], i, \lambda) \otimes E^{m-i} F^{n-i} \to F^n E^m$ is $\text{act } o (\text{id } \otimes (F^{m-i} \eta E^{m-i}) \circ \sigma_{m-i, n-i})$. It follows from Lemma 4.9 that the map

$$L(m, [2, n], i-1, \lambda) \otimes L(m - i + 1, 1, 1, \lambda - 2(n - i)) \otimes E^{m-i} F^{n-i} \to F^n E^m$$

is

$$\text{act } o \left( \text{id } \otimes \left( \text{act } o ((T_{n-i} \cdots T_1) E^{m-i+1}) \right) \right) \circ \left( \text{id } \otimes \text{id } \otimes (F^{m-i} \eta E^{m-i}) \circ \sigma_{m-i, n-i} \right).$$

Let $i > 0$ and

$$M_i = L(m, [2, n], i, \lambda) \oplus \bigoplus_{l \leq -r+n-i} T_{n-i} \cdots T_1 X^l_i Z L(m, [2, n], i-1, \lambda) \cdot \bigoplus_{1 \leq j \leq m-i} T_j \cdots T_{m-i} Z,$$
a subgroup of $^0H_m \otimes_{^0H_i} ^0H_n$. We have shown that there is an isomorphism
\[
\text{act} \circ \left( \text{id} \otimes ((F^{m-i} \eta_{i} E^{m-i}) \circ \sigma_{m-i,n-i}) \right) : \bigoplus_{i=0}^{n} M_i \otimes E^{m-i} F^{m-i} \cong F^{n} E^{m}.
\]
Let
\[ N_i = \bigoplus_{w' \in [2,n-i+1] \in \mathcal{S}_{2,n} \atop \mu \in \mathcal{P}(i-1, r-l)} T_{n-i} \cdots T_{1} X_{1}^{i} m_{\mu}(X_{n-i+2}, \ldots, X_{n}) X_{n-i+2}^{i} \cdots X_{n-1} T_{w'} Z
\]
and
\[ N_i' = \bigoplus_{w' \in [2,n-i] \in \mathcal{S}_{2,n} \atop \mu \in \mathcal{P}(i-1, r-l)} m_{\mu}(X_{n-i+1}, \ldots, X_{n}) X_{n-i+1}^{i} \cdots X_{n-1} T_{w'} Z.
\]
We have
\[ M_i = \left( \bigoplus_{w \in \mathcal{S}_{m-i}} T_{w} Z \right) \otimes (N_i' \oplus N_i).
\]
We have
\[ T_{w[n-i+1,n]} N_i' T_{w[1,n]} = \sum_{\mu \in \mathcal{P}(i-1, r-l)} m_{\mu}(X_{n-i+1}, \ldots, X_{n}) T_{w[1,n]} Z
\]
and
\[ T_{w[n-i+1,n]} N_i T_{w[1,n]} = \sum_{\mu \in \mathcal{P}(i-1, r-l)} \partial_{s_{n-1} \cdots s_{n-i+1}} (\partial_{s_{n-1} \cdots s_{1}} (X_1^i) m_{\mu}(X_{n-i+2}, \ldots, X_{n})) T_{w[1,n]} Z
\]
where $P_{k,l}$ is a symmetric polynomial and $R_{k,\mu} = \partial_{s_{n-1} \cdots s_{n-i+1}} (X_{n-i+1}^k m_{\mu}(X_{n-i+2}, \ldots, X_{n}))$ satisfies $\deg_{s_{i}} R_{k,\mu} \leq \max(k-i+1, r-i-l-1)$ by Lemma 4.10.

Let us fix $k$ and $l$. By induction, the composite morphism
\[ E^{m-i} F^{m-i} \xrightarrow{\sigma_{m-i,n-i}} F^{m-i} E^{m-i} \xrightarrow{P_{k,l} E^{m-i}} E^{m-i} \]
is equal to
\[ \sum_{j \geq i} (m_{\mu'}(X_{n-j+1}, \ldots, X_{n-i}) X_{n-j+1}^{j-i-1} \cdots X_{n-i-1}^{i-1}) E^{m-i}) \circ (\text{id} \otimes (F^{m-j} \eta_{j-i} E^{m-j}) \circ \sigma_{m-j,n-j}) \circ f_{j,\mu'}
\]
for some $f_{j,\mu'} : E^{m-i} F^{m-i} \rightarrow E^{m-j} F^{m-j}$. We have
\[ T_{w[n-i+1,n]} m_{\mu}(X_{n-j+1}, \ldots, X_{n-i}) X_{n-j+1}^{j-i-1} \cdots X_{n-i-1}^{i-1} R_{k,\mu}(X_{n-i+1}, \ldots, X_{n}) T_{w[1,n]} = \partial_{w[n-j+1,n]} m_{\mu'}(X_{n-j+1}, \ldots, X_{n-i}) R_{k,\mu}(X_{n-i+1}, \ldots, X_{n}) T_{w[1,n]} = S_{k,\mu,\mu'}(X_{n-j+1}, \ldots, X_{n}) T_{w[1,n]},
\]
where $S_{k,\mu,\mu'}$ is a symmetric polynomial and $\deg_{s_{i}} S_{k,\mu,\mu'} \leq -r-j$ by Lemma 4.10. Note that if $j = i$ and $k \neq -r-1$, then $\deg_{s_{i}} S_{k,\mu,\mu'} \leq -r-i-1$.  

Assume \( l = -r + n - i - 1 \) and \( k = l - n + i = -r - 1 \). We have
\[
R_{k,\mu} = \partial_{s_{n-1} \cdots s_{n-i+1}} (X_{n-i+1}^{r-1} m_{\mu}(X_{n-i+2}, \ldots, X_n)) = m_{\mu \{ -r - i \}}(X_{n-i+1}, \ldots, X_n) + T(X_{n-i+1}, \ldots, X_n),
\]
where \( T \) is a symmetric polynomial with \( \deg_{s_\lambda} T \leq -r - i - 1 \) (Lemma 4.10).

We have shown that the images of \( L(m, n, i, \lambda) \) and of \( M_i \) in \( \text{Hom}(E^{m-i} F^{n-i}, F^{(n)} E^{(m)}) \) coincide modulo maps that factor through
\[
\bigoplus_{j \geq i} (T_{w[1,m]} T_{w[1,n]}^{\text{opp}}) L(m, n, i, \lambda) \otimes E^{m-j} F^{n-j} \to F^{(n)} E^{(m)}.
\]

Using Lemma 4.11 we deduce by descending induction on \( i \) that the lemma holds, using that \( \dim_{\mathbf{Z}} M_i = \dim_{\mathbf{Z}} L(m, n, i, \lambda) \) as in the case \( n = 1 \) considered earlier. \( \square \)

**Remark 4.13.** Let \( \hat{\mathcal{B}}_1 \) the \( k \)-linear category \( \mathcal{B} \times \mathcal{B}^{\text{opp}} \). Denote by \( F_{s} \) the object \( E_s \) of \( \mathcal{B}^{\text{opp}} \) and define \( \hat{h} : \text{Ob}(\hat{\mathcal{B}}_1) \to X, (M, N) \mapsto h(M) + h(N) \). Consider the 2-category \( \hat{\mathcal{A}}_1 \) with set of objects \( X \) and \( \text{Hom}(\lambda, \lambda') = \hat{h}^{-1}(\lambda' - \lambda) \). The isomorphisms of Lemma 4.12 together with \( \sigma_{st} \) for \( s \neq t \), are the first steps to provide a direct construction of a composition on the homotopy category of \( \hat{\mathcal{A}}_1 \) (after adding maps \( (M \otimes E_s, F_s \otimes N) \to (M, N) \)).

### 4.2.3. Decomposition of \( [E_s^{(m)}, F_t^{(n)}] \).

**Lemma 4.14.** Let \( s \in I \) and \( m, n \in \mathbf{Z}_{\geq 0} \). Let \( r = m - n + \langle \alpha_s^\vee, \lambda \rangle \). We have the following isomorphisms in \( \mathcal{A}^i \) and in \( \mathcal{A}^{s^i} \):
\[
E_s^{(m)} F_s^{(n)} \simeq \bigoplus_{i \in \mathbf{Z}_{\geq 0}} \begin{bmatrix} r \\ i \end{bmatrix}_s F_s^{(n-i)} E_s^{(m-i)} \quad \text{if} \quad r \geq 0
\]
\[
E_s^{(m)} F_s^{(n)} \oplus \bigoplus_{i \in 1+2\mathbf{Z}_{\geq 0}} \begin{bmatrix} i - 1 - r \\ i \end{bmatrix}_s F_s^{(n-i)} E_s^{(m-i)} \simeq \bigoplus_{i \in 2\mathbf{Z}_{\geq 0}} \begin{bmatrix} i - 1 - r \\ i \end{bmatrix}_s F_s^{(n-i)} E_s^{(m-i)} \quad \text{if} \quad r < 0
\]
\[
F_s^{(n)} E_s^{(m)} \oplus \bigoplus_{i \in 1+2\mathbf{Z}_{\geq 0}} \begin{bmatrix} i - 1 + r \\ i \end{bmatrix}_s F_s^{(n-i)} F_s^{(n-i)} \simeq \bigoplus_{i \in 2\mathbf{Z}_{\geq 0}} \begin{bmatrix} i - 1 + r \\ i \end{bmatrix}_s F_s^{(n-i)} F_s^{(n-i)} \quad \text{if} \quad r > 0
\]
\[
F_s^{(n)} E_s^{(m)} \simeq \bigoplus_{i \in \mathbf{Z}_{\geq 0}} \begin{bmatrix} -r \\ i \end{bmatrix}_s E_s^{(n-i)} F_s^{(n-i)} \quad \text{if} \quad r \leq 0
\]

Let \( t \in I - \{s\} \) and \( m, n \in \mathbf{Z}_{\geq 0} \). We have the following isomorphisms in \( \mathcal{A}^i \) and in \( \mathcal{A}^{s^i} \):
\[
F_t^{(n)} E_t^{(m)} \simeq F_t^{(n)} E_t^{(m)}.
\]

**Proof.** The first isomorphism follows from the isomorphism of \( (0H_m \otimes 0H_n^{\text{opp}}) \)-modules in Lemma 4.12. Assume \( r < 0 \). Given \( l \in \mathbf{Z}_{>0} \), we have (cf e.g. [Liu 1.3.1(e) p.9])
\[
\sum_i (-1)^i \begin{bmatrix} i - 1 - r \\ i \end{bmatrix} \cdot \begin{bmatrix} -r \\ l - i \end{bmatrix} = 0.
\]
It follows that
\[
\bigoplus_{i \in \mathbb{Z}_{\geq 0}, j \geq 0} \left[ i - 1 - r \atop i \right] F^{(m-i-j)} F^{(n-i-j)} E_{i+n-j} \simeq \bigoplus_{i \in \mathbb{Z}_{\geq 0}, j \geq 0} \left[ i - 1 - r \atop j \right] F^{(m-i-j)} F^{(n-i-j)} \oplus \bigoplus_{i \geq 1} \left[ -r \atop i \right] E^{(m-i)} F^{(n-i)},
\]
hence
\[
\bigoplus_{i \in \mathbb{Z}_{\geq 0}} \left[ i - 1 - r \atop i \right] F^{(n-i)} E^{(m-i)} \simeq \bigoplus_{i \in \mathbb{Z}_{\geq 0}} \left[ i - 1 - r \atop i \right] F^{(n-i)} E^{(m-i)} \oplus \bigoplus_{i \geq 1} \left[ -r \atop i \right] E^{(m-i)} F^{(n-i)}
\]
using the first isomorphism of the lemma for \((m - i, n - i)\). The second isomorphism of the lemma follows by applying again the first isomorphism.

The third and fourth isomorphism follow from the second and first by applying the Chevalley involution.

The isomorphisms \(\sigma_{st}\) induce an isomorphism \(E_s^m F_t^n \simeq F_t^m E_s^m\) compatible with the action of \(0 H_m \otimes 0 H_n\) (the proof in Lemma 4.9 works when \(s \neq t\)). It follows that \(E_s^{(m)} F_t^{(n)} \simeq F_t^{(m)} E_s^{(m)}\). \(\square\)

4.2.4. Decategorification. Proposition 4.2 shows that we have a morphism of algebras
\[
U_1^+(\mathfrak{g}) \to B_{\leq 1}, \ e_s^{(r)} \mapsto [E_s^{(r)}]
\]
and, when \(C\) is symmetrizable, to a morphism of \(\mathbb{Z}[v^\pm 1]\)-algebras
\[
U_1^+(\mathfrak{g}) \to B_{\leq 1}^*, \ e_s^{(r)} \mapsto [E_s^{(r)}].
\]

The defining relations for \(\mathfrak{A}\) show that we have a monoidal functor:
\[
U_1(\mathfrak{g}) \to \mathfrak{A}_{\leq 1}^i, \ \lambda \mapsto \lambda, \ e_s^{(r)} \mapsto [E_s^{(r)}], \ f_s^{(r)} \mapsto [F_s^{(r)}]
\]
and, when \(C\) is symmetrizable, a monoidal functor compatible with the \(\mathbb{Z}[v^\pm 1]\)-structure:
\[
U_v(\mathfrak{g}) \to \mathfrak{A}_{\leq 1}^*, \ \lambda \mapsto \lambda, \ e_s^{(r)} \mapsto [E_s^{(r)}], \ f_s^{(r)} \mapsto [F_s^{(r)}].
\]

5. 2-Representations

We assume in this section that the set \(I\) is finite. All results are stated over \(k\) and are related to representations of \(\mathfrak{g}\). They generalize immediately to the graded case over \(k^*\) and relate then to representations of \(U_v(\mathfrak{g})\).

5.1. Integrable representations.

5.1.1. Definition. Let \(\mathfrak{B}\) be a \(k\)-linear 2-category.

Given \(R : \mathfrak{A} \to \mathfrak{B}\) a 2-functor, we have a collection \(\{R(\lambda)\}_{\lambda \in X}\) of objects of \(\mathfrak{B}\). We say that \(R\) gives a 2-representation of \(\mathfrak{A}\) on \(\{R(\lambda)\}\). If this makes sense, we put \(\mathcal{V} = \bigoplus_{\lambda \in X} R(\lambda)\) and say that we have a 2-representation of \(\mathfrak{A}\) on \(\mathcal{V}\).

The data of a strict 2-functor \(R : \mathfrak{A} \to \mathfrak{B}\) is the same as the data of
- a family \((\mathcal{V}_\lambda)_{\lambda \in X}\) of objects of \(\mathfrak{B}\)
- 1-arrows \(E_{s,\lambda} : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda+s}\) and \(F_{s,\lambda} : \mathcal{V}_\lambda \to \mathcal{V}_{\lambda-s}\) for \(s \in I\)
- \(x_{s,\lambda} \in \text{End}(E_{s,\lambda})\) and \(\tau_{s,t,\lambda} \in \text{Hom}(E_{s,\lambda+s} E_{t,\lambda} E_{t,\lambda+s} E_{s,\lambda})\) for \(s, t \in I\)
- an adjunction \((E_{s,\lambda}, F_{s,\lambda+s}\))
such that

- relations (1)-(4) in \( \mathcal{H} \) hold
- the maps \( \rho_{s,\lambda} \) and \( \sigma_{st} \) for \( s \neq t \) are isomorphisms.

Note that there are canonical strict 2-functors that are locally equivalences

\[ \mathcal{A}' \text{-Mod}(\mathcal{B}) \to \mathcal{A}' \text{-Mod}(\mathcal{B}) \to \mathcal{A} \text{-Mod}(\mathcal{B}). \]

From now on, we assume \( \mathcal{B} \) is a locally full 2-subcategory of \( \mathit{Lin}_k \).

**Definition 5.1.** A 2-representation \( \mathfrak{A} \to \mathfrak{B} \) is integrable if \( E_s \) and \( F_s \) are locally nilpotent for all \( s \), i.e., for any \( \lambda \) and any object \( M \) of the category \( \mathcal{V}_\lambda \), there is an integer \( n \) such that \( E_{s,\lambda+n\alpha_s} \cdots E_{s,\lambda+\alpha_s} E_{s,\lambda}(M) = 0 \) and \( F_{s,\lambda-n\alpha_s} \cdots F_{s,\lambda-\alpha_s} F_{s,\lambda}(M) = 0 \).

Our main object of study is the 2-category of integrable 2-representations of \( \mathfrak{A} \) in \( k \)-linear, abelian, triangulated and dg-categories.

Given \( \mathcal{V} \) a 2-representation of \( \mathfrak{A} \), then we endow \( \mathcal{V}^{\text{opp}} \) with the structure of a 2-representation of \( \mathfrak{A} \) by using the Chevalley involution \( I \), with \( (\mathcal{V}^{\text{opp}})_\lambda = (\mathcal{V}_\lambda)^{\text{opp}} \).

Let \( \mathcal{V} \) be an integrable 2-representation of \( \mathfrak{A} \) in \( \mathit{Lin}_k \). There is an induced action of \( \text{Ho}^b(\mathfrak{A}) \) on \( \text{Ho}^b(\mathcal{V}) \).

**Lemma 5.2.** Let \( s \in I \).

- Let \( C \in \text{Ho}^b(\mathcal{V}) \). If \( \text{Hom}_{\text{Ho}^b(\mathcal{V})}(E^i_s M, C) = 0 \) for all \( M \in \text{Ho}^b(\mathcal{V}) \) such that \( F^i_s M = 0 \) and all \( i \geq 0 \), then \( C = 0 \).
- Let \( X \) be a 1-arrow of \( \text{Ho}^b(\mathfrak{A}) \) with a right dual. If \( X E^i_s(M) = 0 \) for all \( M \in \text{Ho}^b(\mathcal{V}) \) such that \( F^i_s M = 0 \) and all \( i \geq 0 \), then \( X(N) = 0 \) for all \( N \in \text{Ho}^b(\mathcal{V}) \).
- Let \( f \) a 2-arrow of \( \text{Ho}^b(\mathfrak{A}) \) between 1-arrows with right duals. If \( f(E^i_s M) \) is an isomorphism for all \( M \in \text{Ho}^b(\mathcal{V}) \) such that \( F^i_s M = 0 \) and all \( i \geq 0 \), then \( f(N) \) is an isomorphism for all \( N \in \text{Ho}^b(\mathcal{V}) \).

**Proof.** Let \( i \) be a maximal integer such that \( F^i_s C \neq 0 \). We have

\[ \text{End}(F^i_s C) \simeq \text{Hom}(E^i_s F^i_s C, C) = 0, \]

hence a contradiction and consequently \( C = 0 \).

Let \( X^\vee \) be a right dual of \( X \). Let \( M, N \in \text{Ho}^b(\mathcal{V}) \) such that \( F^i_s M = 0 \) and let \( i \geq 0 \). We have

\[ \text{Hom}(E^i_s(M), X^\vee \cdot X(N)) \simeq \text{Hom}(X E^i_s(M), X(N)) = 0, \]

and we deduce from the first statement of the Lemma that \( X^\vee X(N) = 0 \), hence \( X(N) = 0 \).

The last assertion follows from the second one by taking for \( X \) the cone of \( f \). \( \square \)

5.1.2. **Simple 2-representations.** We assume that the root datum is \( Y \)-regular, i.e., the image of the embedding \( I \to Y \) is linearly independent in \( Y \) (cf [Lan, §2.2.2]). Let \( X^+ = \{ \lambda \in X \mid (\alpha_i^+, \lambda) \in \mathbb{Z}_{\geq 0} \text{ for all } i \in I \} \). The set \( X \) is endowed with a poset structure defined by \( \lambda \geq \mu \) if \( \lambda - \mu \in \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i^+ \).

Let \( \lambda \in -X^+ \). Consider the 2-functor \( \mathcal{H} \text{Hom}(\lambda, -) : \mathfrak{A} \to \mathit{Lin}_k \) and let \( R : \mathfrak{A} \to \mathit{Lin}_k \) be the 2-subfunctor generated by the \( F_{s,\lambda} \) for \( s \in I \), i.e., \( R(\mu) \) is the \( k \)-linear full subcategory of \( \mathcal{H} \text{Hom}(\lambda, \mu) \) with objects in \( h^{-1}(\mu - \lambda + \alpha_s) F_s \). We denote by \( \mathcal{L}(\lambda) \) the quotient 2-functor, viewed as a \( k \)-linear category endowed with a decomposition \( \mathcal{L}(\lambda) = \bigoplus_{\mu \in X} \mathcal{L}(\lambda)_\mu \) and endowed with an action of \( \mathfrak{A} \).
Denote by $\mathbf{1}_\lambda$ the identity functor of $\mathcal{L}(\lambda)_\lambda$. It follows from Lemma 4.12 that $F_s E_s^{(\alpha', -\lambda)+1}\mathbf{1}_\lambda$ is isomorphic to a direct summand of $E_s^{(\alpha', -\lambda)+1} F_s \mathbf{1}_\lambda$. In particular, $F_s E_s^{(\alpha', -\lambda)+1} \mathbf{1}_\lambda = 0$. The isomorphism
\[
\text{End}(E_s^{(\alpha', -\lambda)+1}\mathbf{1}_\lambda) \simeq \text{Hom}(E_s^{(\alpha', -\lambda)}\mathbf{1}_\lambda, F_s E_s^{(\alpha', -\lambda)+1}\mathbf{1}_\lambda)
\]
shows that $E_s^{(\alpha', -\lambda)+1}\mathbf{1}_\lambda = 0$.

Since $F_s F_s \mathbf{1}_\mu$ is a direct summand of $F_s F_s \mathbf{1}_\mu$ plus a multiple of $\mathbf{1}_\mu$, it follows that every object of $\mathcal{L}(\lambda)$ is isomorphic to a direct summand of a sum of objects of the form $E_{s_1} \cdots E_{s_n} \mathbf{1}_\lambda$ for some $s_1, \ldots, s_n \in I$. In particular, every object of $\mathcal{L}(\lambda)_\lambda$ is isomorphic to a direct summand of a multiple of $\mathbf{1}_\lambda$. Since $\text{End}(\mathbf{1}_\lambda)$ is a quotient of $\text{End}(\mathbf{1}_\lambda)$, it is commutative and $\mathcal{L}(\lambda)_\lambda$ is equivalent to a full subcategory of $\text{End}(\mathbf{1}_\lambda)$-proj.

Note that when $C$ is a symmetrizable Cartan matrix, then $C \otimes K_0(\mathcal{L}(\lambda))$ is isomorphic to the simple integrable representation of $\lambda$ with lowest weight $\lambda$ [Kac, Corollary 10.4], or it is 0. We will show in [Rou3] that it is indeed non zero and determine $\text{End}(\mathbf{1}_\lambda)$.

5.1.3. Lowest weights. Let $A$ be an $\text{End}(\mathbf{1}_\lambda)$-algebra. Let $\mathcal{V} = \mathcal{L}(\lambda) \otimes_{\text{End}(\mathbf{1}_\lambda)} A$, given by $\mathcal{V}_\mu = \mathcal{L}(\lambda)_\mu \otimes_{\text{End}(\mathbf{1}_\lambda)} A$, where the map $\text{End}(\mathbf{1}_\lambda) \to Z(\mathcal{L}(\lambda)_\mu)$ is given by right multiplication. The action of $\mathfrak{A}$ on $\mathcal{L}(\lambda)$ extends to an action on $\mathcal{V}$. Similarly, if $\mathcal{A}$ is a $\text{End}(\mathbf{1}_\lambda)$-linear category, we have an action of $\mathfrak{A}$ on $\mathcal{L}(\lambda) \otimes_{\text{End}(\mathbf{1}_\lambda)} A$.

Let $\mathcal{V}$ be a 2-representation of $\mathfrak{A}$ in $\mathfrak{Lin}_K$. Given $\lambda \in X$, we denote by $\mathcal{V}_\lambda^{\mathrm{lw}}$ the full subcategory of $\mathcal{V}_\lambda$ of objects $M$ such that $F_s M = 0$ for all $s \in I$.

Lemma 5.3. If $\mathcal{V}_\lambda^{\mathrm{lw}} \neq 0$, then $\lambda \not\in -X^+$.

Proof. Assume there is $s \in I$ such that $\langle \alpha'_s, \lambda \rangle > 0$ and let $M \in \mathcal{V}_\lambda^{\mathrm{lw}}$. Then, $M$ is a direct summand of $E_s F_s M = 0$. \hfill $\square$

Assume $\lambda \not\in -X^+$. The canonical morphism of 2-representations $\mathcal{H}om(\lambda, -) \to \mathcal{V}$ associated with $M \in \mathcal{V}_\lambda^{\mathrm{lw}}$ factors through a morphism $R_M : \mathcal{L}(\lambda) \to \mathcal{V}$. Note that $R_M(\mathbf{1}_\lambda) = M$. So, we have a morphism of algebras $\text{End}(\mathbf{1}_\lambda) \to \text{End}(M)$ and this shows that the morphism above extends to a morphism of 2-representations $R_M : \mathcal{L}(\lambda) \otimes_{\text{End}(\mathbf{1}_\lambda)} \text{End}(M) \to \mathcal{V}$. We have also a canonical morphism of 2-representations $R_\lambda : \mathcal{L}(\lambda) \otimes_{\text{End}(\mathbf{1}_\lambda)} \mathcal{V}_\lambda^{\mathrm{lw}} \to \mathcal{V}$ that extends $R_M$.

Lemma 5.4. Let $\mathcal{V}$ be a 2-representation of $\mathfrak{A}$ in $\mathfrak{Lin}_K$ and $\lambda \not\in -X^+$. The morphism of 2-representations
\[
R_\lambda : \mathcal{L}(\lambda) \otimes_{\text{End}(\mathbf{1}_\lambda)} \mathcal{V}_\lambda^{\mathrm{lw}} \to \mathcal{V}
\]
is fully faithful.

Proof. Let $\lambda \not\in -X^+$ and $M \in \mathcal{V}_\lambda^{\mathrm{lw}}$. Let $\mathcal{L}_M(\lambda) = \mathcal{L}(\lambda) \otimes_{\text{End}(\mathbf{1}_\lambda)} \text{End}(M)$. Let $X$ be an object of $\mathcal{H}om(\lambda, \mu)$ that is a product of $E_s$’s. There is an object $Y$ of $\mathcal{H}om(\mu, \lambda)$ right dual to $X$. We have a commutative diagram of canonical maps
\[
\begin{array}{ccc}
\text{End}_{\mathcal{L}_M(\lambda)}(X \mathbf{1}_\lambda) & \longrightarrow & \text{Hom}_{\mathcal{L}_M(\lambda)}(\mathbf{1}_\lambda, Y X \mathbf{1}_\lambda) \\
\downarrow & & \downarrow \\
\text{End}_{\mathcal{V}}(X M) & \longrightarrow & \text{Hom}_{\mathcal{V}}(M, Y X M)
\end{array}
\]
Proposition 5.6. Let $R$ be a $2$-representation of $\Lambda$ with a right dual. If $C$ acts by $0$ on $\text{Ho}^b(\mathcal{L}(\lambda))$ for all $\lambda \in -X^+$, then $C$ acts by $0$ on $\text{Ho}^b(\mathcal{V})$ for all integrable $2$-representations $\mathcal{V}$ of $\Lambda$ in $\text{Lin}_k$.

Lemma 5.5. Let $C$ be a $1$-arrow of $\text{Ho}^b(\mathcal{A})$ with a right dual. If $C$ acts by $0$ on $\text{Ho}^b(\mathcal{L}(\lambda))$ for all $\lambda \in -X^+$, then $C$ acts by $0$ on $\text{Ho}^b(\mathcal{V})$ for all integrable $2$-representations $\mathcal{V}$ of $\mathcal{A}$ in $\text{Lin}_k$.

Proof. Let $M \in \mathcal{V}_\lambda$ such that $FM = 0$. Lemma 5.4 provides a fully faithful morphism of $2$-representations

$$R : \mathcal{V}(\lambda) \otimes_{\text{End}(1_\lambda)} \text{End}(M) \to \mathcal{V}$$

with $R(1_\lambda) \cong M$. We deduce that $C(E_iM) = 0$ for all $i$. This holds also for $\mathcal{V}$ replaced by $\text{Ho}^b(\mathcal{V})$ and Lemma 5.2 shows that $C$ acts by $0$ on $\mathcal{V}$.

The second statement follows by taking for $C$ the cone of $f$. $\square$

Proposition 5.6. Let $\mathcal{V}$ be a $2$-representation of $\mathcal{A}'$ in $\text{Lin}_k$ and $\lambda \in -X^+$. The morphism of $2$-representations

$$\sum_{\lambda \in -X^+} R_\lambda : \bigoplus_{\lambda \in -X^+} \mathcal{L}(\lambda) \otimes_{\text{End}(1_\lambda)} \mathcal{V}_{\lambda}^{\text{lw}} \to \mathcal{V}$$

is fully faithful.

Proof. Let $\lambda \in -X^+$ and $M \in \mathcal{V}_{\lambda}^{\text{lw}}$. Consider $\mu \in -X^+$ with $\mu \neq \lambda$ and let $N \in \mathcal{V}_{\mu}^{\text{lw}}$. Let $s_1, \ldots, s_m, t_1, \ldots, t_n \in I$ such that $\alpha_{s_1} + \cdots + \alpha_{s_m} + \lambda = \alpha_{t_1} + \cdots + \alpha_{t_n} + \mu$. If $m = 0$, then we have

$$\text{Hom}(M, E_{t_1} \cdots E_{t_n} N) \cong \text{Hom}(F_{t_1} M, E_{t_2} \cdots E_{t_n} N) = 0.$$ 

Assume $m > 0$. Since $F_{s_1} E_{t_1} \cdots E_{t_n} N$ is isomorphic to a direct summand of a sum of objects of the form $E_{t_1} \cdots E_{t_{i-1}} E_{t_{i+1}} \cdots E_{t_n} N$ for $t_i = s_1$, it follows by induction on $m$ that

$$\text{Hom}(E_{s_1} \cdots E_{s_m} M, E_{t_1} \cdots E_{t_n} N) = 0.$$ 

So, there are no non-zero maps between an object in the image of $R_\lambda$ and an object in the image of $R_{\mu}$. Lemma 5.4 provides the conclusion. $\square$

An immediate consequence of Proposition 5.6 is a decomposition result for additive $2$-representations generated by lowest weight objects.

Corollary 5.7. Assume $\mathcal{V}$ is an idempotent complete integrable $2$-representation and every object of $\mathcal{V}$ is a direct summand of $XM$ for some object $X$ of $\mathcal{A}'$ and $M \in \mathcal{V}$ with $F_i M = 0$ for all $i$.

Then, there is an equivalence of $2$-representations

$$\sum_{\lambda \in -X^+} R_\lambda : \bigoplus_{\lambda \in -X^+} (\mathcal{L}(\lambda) \otimes_{\text{End}(1_\lambda)} \mathcal{V}_{\lambda}^{\text{lw}})^i \to \mathcal{V}.$$
filtration by full k-algebras. We denote by \( \Phi^\text{int}(\mathfrak{B}) \) the 1, 2-full subcategory of 2-representations \( \mathcal{V} \) of \( \mathfrak{A}' \) in \( \mathfrak{B} \) which are integrable and such that \( \{ \lambda \in -X^+ | \mathcal{V}_{\lambda} \neq 0 \} \) is bounded below (i.e., there is an integer \( n \) such that given a sequence \( \lambda_1 > \lambda_2 > \cdots > \lambda_r \) of elements of \( -X^+ \) with \( \mathcal{V}_{\lambda_i} \neq 0 \) for all \( i \), then \( r \leq n \)).

Theorem 5.8. Let \( \mathcal{V} \) be an idempotent complete 2-representation in \( \Phi^\text{int}(\mathfrak{Link}) \). There is a filtration by full k-linear 2-subrepresentations

\[
0 = \mathcal{V}\{0\} \subset \mathcal{V}\{1\} \subset \cdots \subset \mathcal{V}\{n\} = \mathcal{V},
\]

there are \( \text{End}(\mathfrak{I}_{\lambda}) \)-linear categories \( \mathcal{M}_{\lambda,l} \) for \( \lambda \in -X^+ \) and isomorphisms of 2-representations

\[
\mathcal{V}\{l\}/\mathcal{V}\{l-1\} \cong \bigoplus_{\lambda \in -X^+} \left( \mathcal{L}(\lambda) \otimes_{\text{End}(\mathfrak{I}_{\lambda})} \mathcal{M}_{\lambda,l} \right)^i.
\]

Proof. We proceed by induction on the maximal length of a sequence \( \lambda_1 < \cdots < \lambda_n \) of elements of \( -X^+ \) such that \( \mathcal{V}_{\lambda_i} \neq 0 \). Let \( L \) be the set of minimal elements \( \lambda \in -X^+ \) such that \( \mathcal{V}_{\lambda} \neq 0 \). Proposition 5.6 gives a fully faithful morphism of 2-representations

\[
\bigoplus_{\lambda \in L} \mathcal{L}(\lambda) \otimes_{\text{End}(\mathfrak{I}_{\lambda})} \mathcal{V}_{\lambda} \rightarrow \mathcal{V}
\]

that is an equivalence on \( \lambda \)-weight spaces for \( \lambda \in L \). By induction, its cokernel satisfies the conclusion of the Theorem and we are done. \( \square \)

This theorem extends to abelian and (dg) triangulated settings, cf [Rou3].

5.1.5. Bilinear forms. Assume \( \mathcal{V} \) is a 2-representation of \( \mathfrak{A}' \) in \( \mathfrak{T}_{ri_k} \), where \( k \) is a field endowed with a k-algebra structure.

The action of \( \mathfrak{A}' \) on \( \mathcal{V} \) induces an action of \( U_1(\mathfrak{g}) \) on \( K_0(\mathcal{V}) \). The same holds for 2-representations in abelian or exact categories.

Assume \( \mathcal{V} \) is Ext-finite, i.e., \( \dim_k \bigoplus_{i \in \mathbb{Z}} \text{Hom}_\mathcal{V}(M,N[i]) < \infty \) for all \( M, N \in \mathcal{V} \).

We have a pairing on \( K_0(\mathcal{V}) \):

\[
K_0(\mathcal{V}) \times K_0(\mathcal{V}) \rightarrow \mathbb{Z}, \quad ([M],[N]) = \sum_i (-1)^i \dim_k \text{Hom}(M,N[i]).
\]

We have

\[
\langle e_s(v), v' \rangle = \langle v, f_s(v') \rangle \quad \text{and} \quad \langle f_s(v), v' \rangle = \langle v, e_s(v') \rangle.
\]

Note in particular that if \( L \) is a field such that the pairing is perfect on \( L \otimes K_0(\mathcal{V}) \), then \( L \otimes K_0(\mathcal{V}) \) is a semi-simple representation of \( L \otimes \mathbb{Z} U_1(\mathfrak{g}) \).

5.2. Simple 2-representations of \( \mathfrak{s}_2 \).

5.2.1. Symmetrizing forms. Fix a positive integer \( n \). Let \( i \) be an integer with \( 0 \leq i \leq n \). We put \( P_i = k[X_1, \ldots, X_i] \). We denote by \( H_{i,n} \) the subalgebra of \( 0 \mathfrak{H}_n \) generated by \( T_1, \ldots, T_{i-1} \) and \( P_n^{\mathfrak{S}[i+1,n]} \). This is the same as the subalgebra generated by \( 0 \mathfrak{H}_i \) and \( P_n^{\mathfrak{S}[i+1,n]} \). We have a decomposition as abelian groups

\[
H_{i,n} = 0 \mathfrak{H}_i^f \otimes_{\mathbb{Z}} \mathbb{P}_n^{\mathfrak{S}[i+1,n]}.
\]

and a decomposition as algebras

\[
H_{i,n} = 0 \mathfrak{H}_i \otimes_{\mathbb{Z}} \mathbb{Z}[X_{i+1}, \ldots, X_n]^{\mathfrak{S}[i+1,n]}.
\]
**Lemma 5.9.** The algebra $H_{i,n}$ has a symmetrizing form over $P_n^{S_n}$

$$t_i : H_{i,n} \rightarrow P_n^{S_n}(2i(n - 1))$$

$$P \cdot T_w \cdot w[1, i] \mapsto \partial_{w[1, i]w[i+1, n]}(P)\delta_{w,w[1, i]}$$

for $w \in S_i$ and $P \in P_n^{S[i+1, n]}$.

**Proof.** The decomposition (12) shows that $H_{i,n}$ has a symmetrizing form over $P_n^{S[1,i] \times S[i+1,n]}$ given by $PT_w \cdot w[1, i] \mapsto \partial_{w[1, i]}(P)\delta_{w,w[1, i]}$ for $w \in S_i$ and $P \in P_n^{S[i+1, n]}$.

The algebra $P_n$ has a symmetrizing form over $P_n^{S_n}$ given by $\partial_{w[1, n]}$ and a symmetrizing form over $P_n^{S[1,i] \times S[i+1,n]}$ given by $\partial_{w[1, i]}\partial_{w[i+1, n]}$. It follows from Lemma 2.12 that the algebra $P_n^{S[1,i] \times S[i+1,n]}$ has a symmetrizing form over $P_n^{S_n}$ given by $\partial_{w[1, n]w[1, i]w[i+1, n]}$. The lemma follows now from Lemma 2.10. □

Let $e_i(\cdots)$ (resp. $h_i(\cdots)$) denote the elementary (resp. complete) symmetric functions and put $e_i = h_i = 0$ for $i < 0$.

**Lemma 5.10.** The morphism $\partial_{s_{n+1}}$ is a symmetrizing form for the $P_n^{S[i+1,n]}$-algebra $P_n^{S[i+2,n]}$. The set \{\(X^j_{i+1}\)\}_{0 \leq j \leq n-i-1} is a basis, with dual basis \{\((-1)^j e_{n-i-j}(X_{i+2}, \ldots, X_n)\)\}.

**Proof.** The first statement follows as in the proof of Lemma 5.9 from Lemma 2.12. We have $\partial_{s_{n+1}}(h_j(X_1, \ldots, X_n)) = -h_{j-1}(X_1, \ldots, X_{n+1})$ and $\partial_{s_{n+1}}(e_j(X_{n+1}, \ldots, X_n)) = e_j(X_m, \ldots, X_n)$.

Let $k, j \in [0, n-i-1]$. We have $e_k(X_{i+2}, \ldots, X_n) = e_k(X_{i+1}, \ldots, X_n) - X_{i+1}e_{k-1}(X_{i+2}, \ldots, X_n)$, hence

$$\partial_{s_{n-1} \cdots s_{i+1}}(X^j_{i+1}e_k(X_{i+2}, \ldots, X_n)) = (-1)^{n+i+1}h_{j-n+i+1}(X_{i+1}, \ldots, X_n)e_k(X_{i+1}, \ldots, X_n) - \partial_{s_{n-1} \cdots s_{i+1}}(X^j_{i+1}e_{k-1}(X_{i+2}, \ldots, X_n))$$

By induction, we obtain

$$\partial_{s_{n-1} \cdots s_{i+1}}(X^j_{i+1}e_k(X_{i+2}, \ldots, X_n)) = (-1)^{n+i+1}(h_{j-n+i+1}e_k - h_{j-n+i+2}e_{k-1} + \cdots + (-1)^k h_{j-n+i+k}e_0),$$

where we wrote $e_j$ and $h_j$ for the functions in the variables $X_{i+1}, \ldots, X_n$. It follows from the fundamental relation between elementary and complete symmetric functions that

$$\partial_{s_{n-1} \cdots s_{i+1}}(X^j_{i+1}e_k(X_{i+2}, \ldots, X_n)) = 0 \text{ if } j + k \neq n - i - 1$$

while

$$\partial_{s_{n-1} \cdots s_{i+1}}(X^j_{i+1}e_{n-i-j}(X_{i+2}, \ldots, X_n)) = (-1)^j.$$

□

5.2.2. **Induction and restriction.** We have the usual canonical adjoint pair ($\text{Ind}_{H_{i,n}}^{H_{i+1,n}}, \text{Res}_{H_{i,n}}^{H_{i+1,n}}$). The symmetric forms on the algebras $H_{i,n}$ and $H_{i+1,n}$ described in Lemma 5.9 provide an adjoint pair ($\text{Res}_{H_{i,n}}^{H_{i+1,n}}, \text{Ind}_{H_{i,n}}^{H_{i+1,n}}$) and we will now describe the units and counits of that pair, in terms of morphisms of bimodules.

The following proposition gives a Mackey decomposition for nil affine Hecke algebras.
Proposition 5.11. Assume $i \leq n/2$. We have an isomorphism of graded $(H_{i,n}, H_{i,n})$-bimodules

$$
\rho_i : H_{i,n} \otimes_{H_{i-1,n}} H_{i,n}(2) \oplus \bigoplus_{j=0}^{n-2i-1} H_{i,n}(-2j) \xrightarrow{\sim} H_{i+1,n}
$$

$$(a \otimes a', a_0, \ldots, a_{n-2i-1}) \mapsto a T_i a' + \sum_{j=0}^{n-2i-1} a_j X_{i+1}^j.
$$

Assume $i \geq n/2$. We have an isomorphism of graded $(H_{i,n}, H_{i,n})$-bimodules

$$
\rho_i : H_{i,n} \otimes_{H_{i-1,n}} H_{i,n}(2) \xrightarrow{\sim} H_{i+1,n} \oplus \bigoplus_{j=0}^{2i-n-1} H_{i,n}(2j+2)
$$

$$
a \otimes a' \mapsto (a T_i a', a a', a X_{i+1} a', \ldots, a X_{i}^{2i-n-1} a').
$$

Proof. By [ChRou, Proposition 5.32], we know that the maps above are isomorphisms after applying $- \otimes_{P_n^{[i,n]}} k$, where $k$ is any field. So, the maps are isomorphisms after applying $- \otimes_{P_n^{[i,n]}} \mathbb{Z}$. The proposition follows now from Nakayama’s Lemma.

Let $B_i$ be a basis for $H_{i,n}$ over $P_n^{[i,n]}$ and $\{b^j\}_{b \in B_i}$ be the dual basis. The symmetrizing forms on $H_{i,n}$ and $H_{i+1,n}$ induce a canonical morphism of $(H_{i,n}, H_{i,n})$-bimodules, which is the Frobenius form of $H_{i+1,n}$ as an $H_{i,n}$-algebra:

$$
\varepsilon_i : H_{i+1,n} \rightarrow H_{i,n}(-2(n-2i-1))
$$

and a canonical morphism of $(H_{i+1,n}, H_{i+1,n})$-bimodules

$$
\eta_i : H_{i+1,n} \rightarrow H_{i+1,n} \otimes_{H_{i,n}} H_{i+1,n}(2(n-2i-1)).
$$

They give rise to the counit and unit of the adjoint pair $(\text{Res}^{H_{i,n}}_{H_{i+1,n}}, \text{Ind}^{H_{i+1,n}}_{H_{i,n}})$. Note that $t_i \circ \varepsilon_i = \varepsilon_{i+1}$.

Lemma 5.12. Let $P \in P_n^{[i+2,n]}$ and $w \in \mathcal{S}_{i+1}$. We have

$$
\varepsilon_i(PT_w s_1 \cdots s_i) = \begin{cases} 
\partial_{s_{n-1} \cdots s_{i+1}}(P)T_{w s_1 \cdots s_i} & \text{if } w \in \mathcal{S}_{i}s_1 \cdots s_1 \\
0 & \text{otherwise.}
\end{cases}
$$

We have

$$
\varepsilon_i(P) = \partial_{s_{n-1} \cdots s_{i+1}} (P(X_1 - X_{i+1}) \cdots (X_i - X_{i+1}))
$$

and

$$
\varepsilon_i(PT_i) = -\partial_{s_{n-1} \cdots s_{i+1}} (P(X_1 - X_{i+1}) \cdots (X_{i-1} - X_{i+1})).
$$

When $i < n/2$, we have

$$
\varepsilon_i(T_i) = \varepsilon_i(X_{i+1}^j) = 0 \text{ for } j < n - 2i - 1 \text{ and } \varepsilon_i(X_{i+1}^{n-2i-1}) = (-1)^{n+1}.
$$

When $i \geq n/2$, we have

$$
\varepsilon_i(T_i) = (-1)^{n+1} X_i^{2i-n} \mod \sum_{j=0}^{2i-n-1} P_n^{[i,j] \times \mathcal{S}_{i+1,n]} X_i^j).
$$
Proof. Let us consider the first equality. Let \( f : H_{i+1,n} \rightarrow H_{i,n} \) be the \( \mathbb{Z} \)-linear map sending \( PT_{w}s_{1} \cdots s_{i} \) to the second term of the equality. Note that \( f(Pa) = Pf(a) \) for all \( P \in P_{n}^{\mathbb{G}[i+1,n]} \) and \( a \in H_{i+1,n} \).

Let \( j < i \), let \( P \in P_{n}^{\mathbb{G}[i+2,n]} \) and let \( w \in \mathbb{G}_{i+1} \). If \( w \notin \mathbb{G}_{i}s_{1} \cdots s_{1} \), then

\[
f(T_{j}PT_{w}s_{1} \cdots s_{i}) = 0 = T_{j}f(PT_{w}s_{1} \cdots s_{i}).
\]

Assume now \( w \in \mathbb{G}_{i}s_{1} \cdots s_{1} \). Then,

\[
f(T_{j}PT_{w}s_{1} \cdots s_{i}) = \partial_{s_{n-1} \cdots s_{i+1}}(s_{j}(P))T_{j}T_{w}s_{1} \cdots s_{i} + \partial_{s_{n-1} \cdots s_{i+1}}(P)T_{w}s_{1} \cdots s_{i}
= T_{j}\partial_{s_{n-1} \cdots s_{i+1}}(P)T_{w}s_{1} \cdots s_{i}
= T_{j}f(PT_{w}s_{1} \cdots s_{i}).
\]

It follows that \( f \) is left \( H_{i,n} \)-linear. Since \( t_{i} \circ f = t_{i+1} \), we obtain the first equality from Lemma 2.11.

We have

\[
s_{i} \cdots s_{1} = (X_{1} - X_{i+1}) \cdots (X_{i} - X_{i+1})T_{i} \cdots T_{1} \mod F^{<(*,2i)}
\]
hence \( \varepsilon_{i}(P) = \partial_{s_{n-1} \cdots s_{i+1}}(P(X_{1} - X_{i+1}) \cdots (X_{i} - X_{i+1})) \).

We have

\[
T_{i}s_{i} \cdots s_{1} = -T_{i}s_{i-1} \cdots s_{1} = -(X_{1} - X_{i+1}) \cdots (X_{i-1} - X_{i+1})T_{i} \cdots T_{1} \mod F^{<(*,2i)}
\]
hence \( \varepsilon_{i}(PT_{i}) = -\partial_{s_{n-1} \cdots s_{i+1}}(P(X_{1} - X_{i+1}) \cdots (X_{i-1} - X_{i+1})) \).

The vanishing statements follow immediately from degree considerations.

Let \( P = X_{i+1}^{n-2i+1}(X_{1} - X_{i+1})(X_{2} - X_{i+1}) \cdots (X_{i} - X_{i+1}) \). We have

\[
P = \sum_{j=0}^{i} (-1)^{j}X_{i+1}^{n-2i+1+j}\varepsilon_{i-j}(X_{1}, \ldots, X_{i}).
\]

We have \( \partial_{s_{n-1} \cdots s_{i+1}}(X_{i+1}^{r}) = 0 \) for \( r < n - i - 1 \). It follows that

\[
\varepsilon_{i}(X_{i+1}^{n-2i+1}) = \partial_{s_{n-1} \cdots s_{i+1}}(P) = (-1)^{i}\partial_{s_{n-1} \cdots s_{i+1}}(X_{i+1}^{n-i-1}) = (-1)^{n+1}
\]
by Lemma 5.10.

Assume \( i \geq n/2 \). We have

\[
\varepsilon_{i}(T_{i}) = -\partial_{s_{n-1} \cdots s_{i+1}}((X_{1} - X_{i+1}) \cdots (X_{i-1} - X_{i+1}))
= \sum_{j=n-i-1}^{i-1} (-1)^{j+1}\partial_{s_{n-1} \cdots s_{i+1}}(X_{i+1}^{j})\varepsilon_{i-j}(X_{1}, \ldots, X_{i-1}).
\]

Since \( e_{k+1}(X_{1}, \ldots, X_{i-1}) = e_{k+1}(X_{1}, \ldots, X_{i}) - X_{i}e_{k}(X_{1}, \ldots, X_{i-1}) \), we see by induction that \( e_{k}(X_{1}, \ldots, X_{i-1}) \in (-1)^{k}X_{i}^{k} + \sum_{j < k} P_{n}^{\mathbb{G}[1,i]}X_{i}^{j} \). It follows that

\[
\varepsilon_{i}(T_{i}) = (-1)^{n+1}X_{i}^{2i-n} \mod \sum_{j=0}^{2i-n-1} P_{n}^{\mathbb{G}[1,i] \times \mathbb{G}[i+1,n]}X_{i}^{j}.
\]
Lemma 5.13. We have
\[ \eta(1) = T_i \cdots T_{s_1} \cdots s_i \pi + \cdots + T_i \cdots s_{i+1} \pi T_i \cdots T_2 + s_1 \cdots s_i \pi T_i \cdots T_1, \]
where \( \pi = \sum_{j=0}^{n-i-1} (-1)^j e_{n-i-1-j}(X_{i+2}, \ldots, X_n) \otimes X_i^j. \)
Let \( P \in P_n^R. \) We have
\[ m((1 \otimes P \otimes 1 \otimes 1)\eta(1)) = (-1)^{n+1} \partial_{s_1 \cdots s_i}(P(X_{i+1} - X_{i+3}) \cdots (X_{i+1} - X_n)) \]
and
\[ m((1 \otimes X_{i+1}^j \otimes 1 \otimes 1)\eta(1)) = m((1 \otimes T_{i+1} \otimes 1 \otimes 1)\eta(1)) = 0 \text{ for } j < 2i - n + 1 \]
and
\[ m((1 \otimes X_{i+1}^{2i-n+1} \otimes 1 \otimes 1)\eta(1)) = (-1)^{n+1}. \]
When \( i > n/2 - 1, \) we have
\[ m((1 \otimes T_{i+1} \otimes 1 \otimes 1)\eta(1)) = (-1)^n X_i^{n-2i-2} \pmod{P_n^{i+1} \otimes i+1_n X_i^{j+2}}. \]

Proof. Let \( B = \{ X_{i+1}^j \}_{0 \leq j \leq n-i-1}, \) a basis for \( \mathbb{Z}[X_{i+1}, \ldots, X_n]^{\otimes [i+2,n]} \) over \( \mathbb{Z}[X_{i+1}, \ldots, X_n]^{\otimes [i+1,n]} \)
with dual basis \( B^\vee = \{ (-1)^j e_{n-i-1-j}(X_{i+2}, \ldots, X_n) \} \) for the symmetrizing form \( \partial_{s_1 \cdots s_{i+1}} \)
(Lemma 5.10). Let \( B \) be a basis for \( \mathbb{Z}[X_{i+1}, \ldots, X_n]^{\otimes [i+2,n]} \) over \( \mathbb{Z}[X_{i+1}, \ldots, X_n]^{\otimes [i+1,n]} \)
and \( B^\vee \) the dual basis for the symmetrizing form \( \partial_{s_1 \cdots s_{i+1}}. \) Let \( \pi = \sum_{a \in B} a^\vee \otimes a \) be the Casimir element.
Let \( R = \{ 1, T_i, \ldots, T_i \cdots T_1 \}, \) a basis of \( H_i^{\otimes f} \) over \( H_i^f. \) Its dual basis for the Frobenius form
\[ T_w \mapsto \begin{cases} T_{ws_1 \cdots s_i} & \text{if } w \in S_i s_i \cdots s_1 \\ 0 & \text{otherwise} \end{cases} \]
is given by \( \{ 1^\vee = T_i \cdots T_1, \ldots, (T_i \cdots T_2) \vee = T_1, (T_i \cdots T_1) \vee = 1 \}. \)
It follows from Lemmas 5.12 and 2.11 that
\[ T_{w^1} s_1 \cdots s_i \mapsto \begin{cases} T_{ws_1 \cdots s_i} & \text{if } w \in S_i s_i \cdots s_1 \\ 0 & \text{otherwise} \end{cases} \]
extends to a Frobenius form for the \( (H_i^1 \otimes \mathbb{Z}[X_{i+1}, \ldots, X_n]^{\otimes [i+2,n]}) \)-algebra \( H_{i+1,n} \) for which the basis dual to \( R \) is \( h^\vee s_1 \cdots s_i \). Then, \( \{ ah \}_{a \in B, h \in R} \) is a basis of \( H_{i+1,n} \) as an \( H_{i,1} \)-module.
Furthermore, the dual basis for the Frobenius form \( \varepsilon_i \) is \( \{ h^\varepsilon s_1 \cdots s_i a^\vee \}_{a \in B, h \in R} \) (cf Lemma 5.12).
So, we have
\[ \eta(1) = T_i \cdots T_1 s_1 \cdots s_i \pi + \cdots + T_i \cdots s_{i+1} \pi T_i \cdots T_2 + s_1 \cdots s_i \pi T_i \cdots T_1. \]
We deduce that
\[ T_w[1,i+2] \eta(1) = T_w[1,i+2] s_1 \cdots s_i \pi T_i \cdots T_1 = (-1)^i T_{w[i+2]} \pi T_i \cdots T_1. \]
Let \( b \in H_{i+2,n}^H. \) Define
\[ f(b) = m((1 \otimes b \otimes 1 \otimes 1)\eta(1)) = \sum_{a \in B, h \in R} h^\varepsilon s_1 \cdots s_i a^\vee bah \in H_{i+2,n}. \]
We have
\[ T_w[1,i+2] f(b) = (-1)^i T_{w[i+2]} \sum_{a \in B} a^\varepsilon ba T_i \cdots T_1. \]
Since $\deg(\pi) = 2(n - i - 1)$, it follows that

$$T_{w[1,i+2]} f(b) = 0$$

for $b \in F^{<(2i-n+1),s)}$.

We have

$$\left( Q(X_1, \ldots , X_n) \otimes [i+3,n] \right) \times G_{i+2} \times G_{i+1} \times G_{i+1} = Q(X_1, \ldots , X_n) \otimes [i+1] \times \otimes [i+3,n]$$

and

$$H_{i+2}^{H_{i+1,n}} = P_n \otimes [i+1] \times \otimes [i+3,n]$$

We deduce that given $b \in H_{i+2}^{H_{i+1,n}}$, then $f(b) \in P_n$. Note that left multiplication by $T_{w[1,i+2]}$ is injective on $P_n$.

We have $m(\pi) = (X_{i+2} - X_{i+1}) \cdots (X_n - X_{i+1})$ by Lemma 3.1. Let $P \in P_n^{\otimes i}$. We have $T_{w[1,i+2]} f(P) = (-1)^i T_{w[1,i+2]} \partial s_{1 \cdots s_i} (P(X_{i+2} - X_{i+1}) \cdots (X_n - X_{i+1}))$, hence

$$f(P) = (-1)^{n+1} \partial s_{1 \cdots s_i} (P(X_{i+1} - X_{i+2}) \cdots (X_{i+1} - X_n))$$

We have

$$T_{w[1,i+2]} f(T_{i+1} P) = (-1)^i T_{w[1,i+2]} \sum_{j=0}^{n-i-2} (-1)^j e_{n-i-2-j} (X_{i+3}, \ldots , X_n) P X_j^i T_{i+1} \cdots T_1$$

hence

$$f(T_{i+1} P) = (-1)^n \partial s_{1 \cdots s_i} (P(X_{i+1} - X_{i+3}) \cdots (X_{i+1} - X_n))$$

Assume $i > n/2 - 1$. The vanishing statements are immediate consequences of the previous two equalities of the Lemma.

We have

$$f(X_{i+1}^{2i-n+1}) = (-1)^{n+1} \partial s_{1 \cdots s_i} (X_{i+1}^{2i-n+1} (X_{i+1} - X_{i+2}) \cdots (X_{i+1} - X_n))$$

$$= (-1)^{n+1} \partial s_{1 \cdots s_i} (X_{i+1}^i) = (-1)^{n+1}.$$ 

Assume $i \leq n/2 - 1$. We have

$$f(T_{i+1}) = (-1)^n \partial s_{1 \cdots s_i} ((X_{i+1} - X_{i+3}) \cdots (X_{i+1} - X_n))$$

$$= \sum_{j=i}^{n-i-2} (-1)^{i-j} \partial s_{1 \cdots s_i} (X_{i+1}^j) e_{n-i-2-j} (X_{i+3}, \ldots , X_n).$$

By induction, we see that $e_k (X_{i+3}, \ldots , X_n) \in (-1)^k X_{i+2}^k + \sum_{j<k} \mathbb{Z}[X_{i+2}, \ldots , X_n] \otimes [i+2,n] X_{i+2}^j$. Consequently,

$$f(T_{i+1}) = (-1)^n X_{i+2}^{n-2i-2} \left( \text{mod} \sum_{j=0}^{n-2i-3} P_n \otimes [i+1] \times \otimes [i+2,n] X_{i+2}^j \right).$$

□

As a consequence of Lemmas 5.12 and 5.13 we obtain a description of the units and counits $\eta_i$ and $\varepsilon_i$ through the isomorphisms of Proposition 5.11.
**Proposition 5.14.** If \( i < n/2 \) then we have a commutative diagram
\[
H_{i,n} \otimes_{H_{i-1,n}} H_{i,n}(2) \oplus H_{i,n} \oplus H_{i,n}(-2) \oplus \cdots \oplus H_{i,n}(-2(n-2i-1)) \xrightarrow{\rho_i} H_{i+1,n}
\]
\[
(0,0,\ldots,0,(−1)^{n+1}) \quad H_{i,n}(-2(n-2i-1))
\]

If \( i \geq n/2 \) then the image of \( \varepsilon_i \circ \rho_i \) in
\[
\text{Hom}_{\mathcal{H}_{i,n},\mathcal{H}_{i,n}}(H_{i,n} \otimes_{H_{i-1,n}} H_{i,n}(2), H_{i,n}(2(2i - n + 1))) / \bigoplus_{j=0}^{2i-n-1} (a \otimes a' \mapsto aX_i^j a') \cdot Z(H_{i,n})_{2(2i-n-j)}
\]
is equal to the image of the map \( a \otimes a' \mapsto (-1)^{n+1}aX_i^{2i-n}a' \).

If \( i \leq n/2 - 1 \) then the image of \( \rho_{i+1} \circ \eta_i \) in
\[
\text{Hom}_{\mathcal{H}_{i+1,n},\mathcal{H}_{i+1,n}}(H_{i+1,n}, H_{i+2,n}(2(2i - 2i - 2))) / \bigoplus_{j=0}^{n-2i-3} X_i^{j+2}Z(H_{i+1,n})_{2(n-2i-2-j)}
\]
is equal to \( (-1)^n X_i^{n-2i-2} \).

If \( i > n/2 - 1 \) then we have a commutative diagram
\[
H_{i+1,n} \xrightarrow{\eta_i} H_{i+1,n} \otimes_{H_{i,n}} H_{i+1,n}(2(n-2i-1)) \xrightarrow{\rho_{i+1}} H_{i+2,n}(2(n-2i-2)) \oplus H_{i+1,n}(2(n-2i-1)) \oplus H_{i+1,n}(2(n-2i+1)) \oplus \cdots \oplus H_{i+1,n}
\]

5.2.3. \( \mathfrak{sl}_2 \)-action. Let \( \tilde{\mathcal{L}}(-n) = H_{(n+λ)/2,n} \)-free for \( λ \in \{-n, -n + 2, \ldots, n - 2, n\} \). We define \( E = \bigoplus_{i=0}^{n-1} \text{Ind}^{H_{i+1,n}}_{H_{i,n}} \) and \( F = \bigoplus_{i=0}^{n-1} \text{Res}^{H_{i+1,n}}_{H_{i,n}} \). We have a canonical adjunction \((E,F)\). Multiplication by \( X_{i+1} \) induces an endomorphism of \( \text{Ind}^{H_{i+1,n}}_{H_{i,n}} \) and taking the sum over all \( i \), we obtain an endomorphism \( x \) of \( E \). Similarly, \( T_{i+1} \) induces an endomorphism of \( \text{Ind}^{H_{i+1,n}}_{H_{i,n}} \) and we obtain an endomorphism \( \tau \) of \( E^2 \). Propositions 5.11 and 5.14 show that this endows \( \tilde{\mathcal{L}}(-n) = \bigoplus \tilde{\mathcal{L}}(-n) \) with an action of \( \tilde{\mathfrak{A}}' \).

Let \( R = R_M : \mathcal{L}(-n) \otimes_{\text{End}(\mathcal{L}(-n))} P_n^\mathfrak{S}_n \to \tilde{\mathcal{L}}(-n) \) be the morphism of 2-representations associated with \( M = P_n^\mathfrak{S}_n \in \tilde{\mathcal{L}}(-n)_- \) (cf §5.1.3).

**Proposition 5.15.** The canonical map \( \text{End}(\mathcal{L}(-n)) \to P_n^\mathfrak{S}_n \) is an isomorphism and \( R \) induces an isomorphism of 2-representations of \( \mathfrak{A} \)
\[
\mathcal{L}(-n) \xrightarrow{\sim} \tilde{\mathcal{L}}(-n).
\]

In particular, the action of \( \mathfrak{A} \) on \( \mathcal{L}(-n) \) induces an action of \( \tilde{\mathfrak{A}}' \).

**Proof.** The canonical map \( \mathcal{L}(-n) \to F^{(n)}E^{(n)}\mathcal{L}(-n) \) is an isomorphism by Lemma 5.12. It follows that \( E^{(n)} \) induces an isomorphism \( \text{End}(\mathcal{L}(-n)) \xrightarrow{\sim} \text{End}(E^{(n)}\mathcal{L}(-n)) \). We have a commutative diagram
of canonical morphisms of $\text{End}(\mathbf{1}_{-n})$-algebras

$$\xymatrix{ \text{End}(\mathbf{1}_{-n}) \ar[r]^-{\sim} & \text{End}(E^{(n)}\mathbf{1}_{-n}) \ar[r]^-{\sim} & \text{End}(E^{(n)}M) \ar[r]^-{\sim} & P_{n}^{S_{n}} }$$

so the canonical map $\text{End}(\mathbf{1}_{-n}) \to P_{n}^{S_{n}}$ is a split surjection of $\text{End}(\mathbf{1}_{-n})$-algebras, hence it is an isomorphism. The proposition follows. \qed

5.3. Construction of representations. In this section, we show that, for integrable representations, certain axioms are consequences of others.

5.3.1. Biadjointness.

Theorem 5.16. The canonical strict 2-functor $\mathfrak{A} \to \mathfrak{A}'$ induces an equivalence from the 2-category of integrable 2-representations of $\mathfrak{A}'$ to the 2-category of integrable 2-representations of $\mathfrak{A}$.

Proof. It is enough to consider the case $\mathfrak{g} = \mathfrak{sl}_{2}$ and an integrable 2-representation $\mathcal{V}$ of $\mathfrak{A}$. The theorem will follow from the fact that the maps $\hat{\varepsilon}_{\lambda}$ are the counits of an adjoint pair $(F \text{Id}_{\lambda}, \text{Id}_{\lambda} \circ E)$. It is enough to show that

$$(E\hat{\varepsilon}) \circ (\hat{\eta}E) \text{ and } (\hat{\varepsilon}F) \circ (F\hat{\eta}) \text{ are invertible.}$$

Note that this holds for $\mathcal{V} = \mathcal{L}(\lambda)$ by Proposition 5.14. Lemma 5.5 shows that the first invertibility in (13) holds for any $\mathcal{V}$. Now, applying that result to $\mathcal{V}^{\text{opp}}$ endowed with the action of $\mathfrak{A}$ induced by $I$, we obtain that the second invertibility in (13) holds as well. \qed

A consequence of Theorem 5.16 is an extension of Lemma 5.5.

Lemma 5.17. Let $C$ be a 1-arrow of $\text{Ho}^{b}(\mathfrak{A})$. If $C$ acts by 0 on $\text{Ho}^{b}(\mathcal{L}(\lambda))$ for all $\lambda \in -X^{+}$, then $C$ acts by 0 on $\text{Ho}^{b}(\mathcal{V})$ for all integrable 2-representations $\mathcal{V}$ of $\mathfrak{A}$ in $\text{Lin}_{k}$.

Let $f$ be a 2-arrow of $\text{Ho}^{b}(\mathfrak{A})$. If $f$ is an isomorphism on $\text{Ho}^{b}(\mathcal{L}(\lambda))$ for all $\lambda \in -X^{+}$, then $f$ is an isomorphism on $\text{Ho}^{b}(\mathcal{V})$ for all integrable 2-representations $\mathcal{V}$ of $\mathfrak{A}$ in $\text{Lin}_{k}$.

5.3.2. Braid group action. We follow the construction of [ChRou §6]. Let $s \in I$ and $\lambda \in X$. Let $l = (\alpha_{s}^{\vee}, \lambda)$. We define a complex $\Theta_{s,\lambda} \in \text{Comp}(\mathcal{H}om_{\mathfrak{A}}(\lambda, \lambda - l\alpha_{s}))$ by $\Theta_{s,\lambda}^{r} = F_{s}^{(l+r)}E_{s}^{(r)}$ for $r \geq 0$ and $\Theta_{s,\lambda}^{r} = 0$ for $r < 0$. Since $b_{n-1}b_{n} = b_{n}b_{n-1} = b_{n}$, it follows that $F_{s}^{l+r}\eta_{b_{n}}E_{s}^{r} : F_{s}^{l+r}E_{s}^{r} \to F_{s}^{l+r+1}E_{s}^{r+1}$ restricts to a map

$$d^{r} : F_{s}^{(l+r)}E_{s}^{(r)} \to F_{s}^{(l+r+1)}E_{s}^{(r+1)}.$$ 

Since $b_{2}b_{2} = 0$, it follows that $d^{r+1} \circ d^{r} = 0$ and $d$ defines the differential of $\Theta_{s,\lambda}$.

Let $\mathcal{V}$ be an integrable 2-representation of $\mathfrak{A}$ in $\text{Lin}_{k}$. We define an endofunctor $\Theta_{s}$ of $\text{Comp}(\mathcal{V})$. Given $\lambda \in X$, we define $\Theta_{s} : \text{Comp}^{b}(\mathcal{V}_{\lambda}) \to \text{Comp}^{b}(\mathcal{V}_{\sigma_{s}(\lambda)})$ as the total (direct sum) complex associated with the complex of functors $\Theta_{s,\lambda} \in \text{Comp}(\mathcal{H}om_{\mathfrak{A}}(\lambda, \sigma_{s}(\lambda)))$.

Theorem 5.18. The functor $\Theta_{s}$ induces a self-equivalence of $\text{Ho}^{b}(\mathcal{V})$. 

Proof. Note that it is enough to consider the case $g = \mathfrak{sl}_2$. The functor $\Theta_s$ has left and right adjoints. The theorem holds when $V = \mathcal{L}(-n) \otimes_{p_n^*} k$ for any field $k$ by [ChRou] Theorem 6.4. So, it holds for $\mathcal{L}(-n) \otimes_{p_n^*} \mathbb{Z}$, hence for $\mathcal{L}(-n)$ by Nakayama’s Lemma. The conclusion follows now from Lemma 5.17.

Conjecture 5.19 (Chuang-R.). The functors $\Theta_s$ satisfy braid relations.

5.3.3. $\mathfrak{sl}_2$-categorifications. We recall the definition of [ChRou] §5.2.1. Let $k$ be a field.

Definition 5.20. Let $V \in \mathfrak{A}^l_k$. An $\mathfrak{sl}_2$-categorification on $V$ is the data of

- an adjoint pair $(E, F)$ of exact functors $V \to V$
- $X \in \text{End}(E)$ and $T \in \text{End}(E^2)$

such that

- the actions of $[E]$ and $[F]$ on $K_0(V)$ give a locally finite representation of $\mathfrak{sl}_2$
- classes of simple objects are weight vectors
- $F$ is isomorphic to a left adjoint of $E$
- $X$ has a single eigenvalue
- the action on $E^n$ of $X_i = E^{n-i}XE^{i-1}$ for $1 \leq i \leq n$ and of $T_i = E^{n-i-1}TE^{i-1}$ for $1 \leq i \leq n-1$ induce an action of an affine Hecke algebra with $q \neq 1$, a degenerate affine Hecke algebra or a nil affine Hecke algebra of $\text{GL}_n$.

Note that the three types of actions (affine Hecke with $q \neq 1$, degenerate affine Hecke and nil affine Hecke) are equivalent by Theorems 3.16 and 3.19. The endomorphism $T$ needs to be changed, as follows:

affine $\longrightarrow$ nil $\quad$ degenerate $\longleftrightarrow$ nil

$T \longmapsto (qEX -XE)T + q \quad T \longmapsto (EX -XE + 1)T + 1$

Note also that, in the nil case, if $a$ is the eigenvalue of $X$, then by replacing $X$ by $X - a$ one reaches the case where $0$ is the eigenvalue of $X$. As a consequence, given an $\mathfrak{sl}_2$-categorification, one can construct a new categorification by modifying $X$ and $T$ as above so that the action of $X$ and $T$ induce an action of the nil affine Hecke algebra $^0H_n$ on $\text{End}(E^n)$ and $X$ is locally nilpotent.

In [ChRou], the case of nil affine Hecke algebras wasn’t considered. The equivalence of the definitions explained above shows that the results of [ChRou] generalize to this setting. It can also be seen directly that all constructions, results and proofs in [ChRou] involving degenerate affine Hecke algebras carry over to nil affine Hecke algebras. A key point is the commutation relation between $T_i$ and a polynomial: that relation is the same for the degenerate affine Hecke algebra and the nil affine Hecke algebra. The definition of $c_n^+$ [ChRou] §3.1.4] needs to be modified: we define $c_n = T_{w[1,n]}$. Note that $T_{w[1,n]}^2 = 0$ for $n \geq 2$. Given $M$ a projective $k(^0H_n^1)$-module, we have $c_n M = \{m \in M \mid T_w m = 0 \text{ for all } w \in \mathfrak{S}_n - \{1\}\}$.

Remark 5.21. We haven’t included the parameters $a$ and $q$ in the definition, as they are not needed here.

Let $k$ be a field and $V \in \mathfrak{A}^l_k$ endowed with an $\mathfrak{sl}_2$-categorification. Let $V = \mathcal{C} \otimes K_0(V)$. The weight space decomposition $V = \bigoplus_{\lambda \in \mathbb{Z}} V_\lambda$ induces a decomposition $V = \bigoplus_{\lambda} V_\lambda$, where
\[ \mathcal{V}_\lambda = \{ M \in \mathcal{V} | [M] \in \mathcal{V}_\lambda \} \] Proposition 5.5]. Let \( x = X \) and

\[
\tau = \begin{cases} 
(qEX - XE)^{-1}(T - q) & \text{affine case} \\
(EX - XE + 1)^{-1}(T - 1) & \text{degenerate affine case} \\
T & \text{nil affine case}.
\end{cases}
\]

**Theorem 5.22.** The construction above defines an integrable 2-representation of \( \mathfrak{A}(\mathfrak{sl}_2) \) on \( \mathcal{V} \).

Conversely, a integrable 2-representation of \( \mathfrak{A}(\mathfrak{sl}_2) \) on \( \mathcal{V} \) gives rise to an \( \mathfrak{sl}_2 \)-categorification on \( \mathcal{V} \).

This provides an equivalence between the 2-category of \( \mathfrak{sl}_2 \)-categorifications and the 2-category of integrable 2-representations of \( \mathfrak{A}(\mathfrak{sl}_2) \) in \( \mathcal{A}^f_k \).

**Proof.** By [ChRou, Theorem 5.27], the maps \( \rho_{s,\lambda} \) are invertible and the result follows. \( \square \)

In the isotypic case, we have a stronger result:

**Theorem 5.23.** Let \( k \) be a field and \( \mathcal{V} \in \mathcal{A}^f_k \). Assume given an \( \mathfrak{sl}_2 \)-categorification on \( \mathcal{V} \) such that \( C \otimes K_0(\mathcal{V}) \) is a multiple of an irreducible representation of \( \mathfrak{sl}_2(C) \). Then, the construction of Theorem 5.22 gives rise to a 2-representation of \( \mathfrak{A}'(\mathfrak{sl}_2) \) on \( \mathcal{V} \).

**Proof.** Theorems 5.16 and 5.22 provide an action of \( \mathfrak{A}' \). Let \( \lambda \in X \) be minimum such that \( \mathcal{V}_\lambda \neq 0 \). Note that the theorem holds for \( \mathcal{L}(\lambda) \) by Proposition 5.15.

Let \( N \in \mathcal{V}_{\lambda+2i} \) for some \( i \geq 0 \). Let \( N' \) be the cokernel of \( \varepsilon_i(N) : E^iF^iN \to N \). We have \( F^iN' = 0 \), hence \( [N'] = 0 \) in \( K_0(\mathcal{V}) \) since the only non-zero elements of \( C \otimes K_0(\mathcal{V}) \) killed by \( [F] \) are in the \( \lambda \)-weight space. So, \( N' = 0 \) and we deduce that \( N \) is a quotient of \( E^i(F^iN) \).

Let \( M \in \mathcal{V}_\lambda \). Proposition 5.6 provides a fully faithful morphism of 2-representations

\[
R : \mathcal{L}(\lambda) \otimes_{\text{End}(\mathfrak{L})} \text{End}(M) \to \mathcal{V}
\]

with \( R(1_\lambda) \simeq M \). Since the theorem holds for \( \mathcal{L}(\lambda) \), it follows that the relations defining \( \mathfrak{A}' \) hold when applied to \( E^iM \), for every \( i \). It follows that they hold for every quotient of \( E^iM \). We deduce that the relations hold on \( \mathcal{V} \). \( \square \)

5.3.4. **Involution \( \iota \).** Let \( \mathcal{V} \) be an integrable 2-representation of \( \mathfrak{A}' \) in \( \mathcal{A}^f_k \).

Let \( (\mathcal{V})_{\lambda} = \mathcal{V}_{-\lambda} \), let \( E^s_s = F^s_s \) and \( F^s_s = F^s_s \). Let \( x_{st} \in \text{End}(E^s_s) \) corresponding to \( x_{st} \in \text{End}(E^s_s) \) and let \( \tau'_{st} \in \text{Hom}(E^s_sE^t_t, E^t_tE^s_s) \) corresponding to \( -\tau_{st} \in \text{Hom}(E^s_tE^t_t, E^s_tE^t_t) \).

The adjunction \( (F^s_s, E^s_s) \) gives an adjoint pair \( (E^s_s, F^s_s) : \eta^s_s = \eta^s_s \) and \( \varepsilon^s_s = \varepsilon^s_s \).

**Proposition 5.24.** The construction above defines a 2-representation of \( \mathfrak{A}' \) on \( \mathcal{V} \).

**Proof.** The relations (1)-(4) in [3.3.3] are clear. Let us show that the maps \( \rho_{s,\lambda} \) on \( \mathcal{V} \) are isomorphisms. Thanks to Lemma 5.17, it is enough to do so for \( \mathcal{V} = \mathcal{L}(-n) \) for some \( n > 0 \).

Given a field \( k \), consider the canonical 2-representation of \( \mathfrak{A}' \) on \( \mathcal{W} = \bigoplus (H^i_{t,n} \otimes p^e_{n} k) \)-mod. The category \( \mathcal{W} \) is endowed with a structure of \( \mathfrak{sl}_2 \)-categorification. It follows from Theorem 5.22 that the maps \( \rho_{s,\lambda} \) are isomorphisms for \( \mathcal{W} \).

We conclude now as in the proof of Proposition 3.11 that the maps \( \rho_{s,\lambda} \) are isomorphisms for \( \mathcal{L}(-n)^t \).

We are left with proving the invertibility of \( \sigma_{st} \) for \( s \neq t \). This is a consequence of Theorem 5.25 below. \( \square \)
Note that $\mathcal{V} \rightarrow \mathcal{V}'$ induces a strict endo 2-functor of the 2-category of integrable 2-representations of $\mathfrak{A}'$ that is a 2-equivalence.

5.3.5. Relation $[E_s, F_t] = 0$ for $s \neq t$. Let $\{\mathcal{V}\}_{t \in X}$ be a family of $k$-linear categories endowed with the data of

- functors $E_s : \mathcal{V}_\lambda \rightarrow \mathcal{V}_{\lambda+s}$ and $F_s : \mathcal{V}_\lambda \rightarrow \mathcal{V}_{\lambda-s}$ for $s \in I$
- $x_s \in \text{End}(E_s)$ and $\tau_{st} \in \text{Hom}(E_s E_t, E_t E_s)$ for $s, t \in I$
- an adjunction $(E_s, F_s)$ for $s \in I$

such that

- relations (1)-(4) in [5.1.1] hold
- the maps $\rho_{s, \lambda}$ are isomorphisms.

**Theorem 5.25.** The data above defines a 2-representation of $\mathfrak{A}'$ on $\mathcal{V} = \bigoplus \mathcal{V}_\lambda$.

Theorem 5.16 provides maps $\epsilon_s^l$ and $\eta_s^l$ and we only have to show the invertibility of the maps $\sigma_{st}$ for any $s \neq t \in I$. Note that the construction of [5.3.4] provide a category $\mathcal{V}'$ satisfying the same properties as the category $\mathcal{V}$.

Let $s \neq t \in I$. We write $Q_{ts}(u, v) = \sum_{a, b} q_{ab} u^a v^b$ with $q_{ab} \in k$. Let $\lambda \in X$ and $r \geq 0$. Consider the morphism

$$\psi : 0 H_{r+1} \rightarrow \text{End}(E_s E_t^r 1_\lambda)$$

$$h \mapsto (E_s E_t^r \xrightarrow{\eta} F_t E_t E_s E_t^r \xrightarrow{F_t \tau_{st} \epsilon_s^l} F_t E_t E_t^r + 1 \xrightarrow{F_t E_t h} F_t E_s E_t^r + 1 \xrightarrow{F_t \tau_{st} \epsilon_s^l} F_t E_t E_s E_t^r \xrightarrow{\epsilon_s^l} E_s E_t^r).$$

**Lemma 5.26.** Let $a \leq -(\alpha^\vee, \lambda) \rightarrow r - 1$. We have

$$\psi(X^a_{t+1} T_{w[1,r+1]}) = \begin{cases} T_{w[1,r]} q_{m_{ts}, 0} (-1)^{a + m_{ts} + 1} & \text{if } a \leq -(\alpha^\vee, \lambda) - r - 1 \\ 0 & \text{otherwise.} \end{cases}$$

**Proof.** Assume first $r = 0$. We have

$$\psi(X^a) = \sum_{a, \beta \geq 0} q_{\beta, \alpha} (\epsilon^l \circ (F_t X^{a+\beta}) \circ \eta) X^\alpha$$

If $\epsilon^l \circ (F_t X^{a+\beta}) \circ \eta \neq 0$, then $a + \beta \leq -(\alpha^\vee, \lambda) + m_{ts} - 1$, hence $\beta = m_{ts}$ and $a = -(\alpha^\vee, \lambda) - 1$, and $\epsilon^l \circ (F_t X^{a+\beta}) \circ \eta = (-1)^{(a^\vee, \lambda) + m_{ts} + 1}$. The result follows.

We assume now $r > 0$. We have

$$(\tau_{st} E_t) \circ (E_t \tau_{ts}) \circ (\tau_{ts} E_t) = (E_t \tau_{ts}) \circ (\tau_{ts} E_t) \circ (E_t \tau_{st}) + \sum_{\beta \geq 0, m_{ts} > \alpha + \alpha_2 \geq 0} q_{\alpha_1 + \alpha_2 + 1, \beta} (X^{\alpha_1} E_t E_t) (E_t X^{\beta} E_t) (E_t E_s X^\alpha).$$

Let

$$f_a = (E_s E_t^r \xrightarrow{\tau_{ts} \epsilon_s^l} E_t E_s E_t^r - 1 \xrightarrow{\eta^l} F_t E_t E_t E_t^r - 1 \xrightarrow{F_t \tau_{st} \epsilon_s^l} F_t E_t E_t E_t^r - 1 \xrightarrow{F_t \tau_{st} \epsilon_s^l} E_t E_t E_t^r \xrightarrow{\epsilon_s^l} E_t E_t E_t^r)$$

and

$$g_b = E_s E_t^r \xrightarrow{\eta^l} F_t E_t E_t E_t^r \xrightarrow{F_t \tau_{ts} \epsilon_s^l} F_t E_t E_t E_t^r \xrightarrow{\epsilon_s^l} E_t E_t E_t^r.$$
We have $X^a_{r+1}T_w[1,r+1] = T_w[1,r]X^a_{r+1}T_r \cdots T_1$, hence

$$
\psi(X^a_{r+1}T_w[1,r+1]) = T_w[1,r]f_aT_r \cdots T_1 + \sum_{\beta \geq 0} q_{\alpha_1+\alpha_2+1,\beta} T_w[1,r]g_{\alpha} \partial_{s_1 \cdots s_r-1}(X^{\alpha_2}_r)(X^\beta E_t^r).
$$

We have a commutative diagram (Lemma 4.4 and Chevalley duality)

$$
\begin{array}{c}
E_t \xrightarrow{\eta E_t} F_t E_t^2 \\
E_t \downarrow \quad F_t \downarrow \\
E_t F_t E_t \xrightarrow{\sigma E_t} F_t E_t^2 \xrightarrow{\varepsilon t E_t} E_t
\end{array}
$$

- Assume first $\langle \alpha^\vee_1, \lambda \rangle + 2r - m_t_s < 0$. The diagram (14) shows that the composition

$$
E_t E_\alpha E_t^{-1} \xrightarrow{\eta \bullet} F_t E_t^2 E_\alpha E_t^{-1} \xrightarrow{F_t T \bullet} F_t E_t^2 E_\alpha E_t^{-1} \xrightarrow{\varepsilon t \bullet} E_t E_\alpha E_t^{-1}
$$

vanishes. Since $X^a_2 T = TX^a_1 + \sum_{c=0}^{a-1} X^c_2 X^{a-1-c}_1$, it follows that

$$
E_t E_\alpha E_t^{-1} \xrightarrow{\eta \bullet} F_t E_t^2 E_\alpha E_t^{-1} \xrightarrow{F_t(X^a_1 \circ T) \bullet} F_t E_t^2 E_\alpha E_t^{-1} \xrightarrow{\varepsilon t \bullet} E_t E_\alpha E_t^{-1}
$$

equals

$$
\sum_{c=0}^{a-1} (\varepsilon^t \circ (F_t X^c) \circ \eta) X^{a-1-c} E_t E_t^{-1},
$$

hence

$$
T_w[1,r]f_a T_r \cdots T_1 = \sum_{0 \leq \alpha, \beta \leq m_t_s} T_w[1,r]g_{\alpha, \beta} X^{\beta}_E (\varepsilon^t \circ (F_t X^c) \circ \eta) \partial_{s_1 \cdots s_r-1}(X^{a-1-c+\alpha}_r)
$$

If $\varepsilon^t \circ (F_t X^c) \circ \eta \neq 0$, then $c \geq -\langle \alpha^\vee_1, \lambda \rangle - 2r + m_t_s - 1$. If $\partial_{s_1 \cdots s_r-1}(X^{a-1-c+\alpha}_r) \neq 0$, then $a-1-c+\alpha \geq r-1$. If both of those terms are non zero, then $\alpha \geq -\langle \alpha^\vee_1, \lambda \rangle - r + (m_t_s - \alpha) - 1$, hence $\alpha = -\langle \alpha^\vee_1, \lambda \rangle - r - 1$, $\alpha = m_t_s$ and $a-1-c = r-1 - m_t_s$. In particular, we have $r > m_t_s$. So, we have

$$
T_w[1,r]f_a T_r \cdots T_1 = \begin{cases}
T_w[1,r]q_{m_t_s,0}(-1)^{\langle \alpha^\vee_1, \lambda \rangle + m_t_s + 1} & \text{if } a = -\langle \alpha^\vee_1, \lambda \rangle - r - 1 \text{ and } r > m_t_s \\
0 & \text{otherwise}.
\end{cases}
$$

If $g_{\alpha} \partial_{s_1 \cdots s_r-1}(X^{\alpha_2}_r) \neq 0$, then $a + \alpha_1 \geq -\langle \alpha^\vee_1, \lambda \rangle - 2r + m_t_s - 1$ and $\alpha_2 \geq r - 1$, hence $a \geq -\langle \alpha^\vee_1, \lambda \rangle - r - 2 + (m_t_s - \alpha_1 - \alpha_2) \geq -\langle \alpha^\vee_1, \lambda \rangle - r - 1$. We obtain $a = -\langle \alpha^\vee_1, \lambda \rangle - r - 1$, $\alpha_2 = r - 1$ and $\alpha_1 + \alpha_2 = m_t_s - 1$. In particular, $r \leq m_t_s$. So, we have

$$
\begin{align*}
\sum_{\beta \geq 0} q_{\alpha_1+\alpha_2+1,\beta} T_w[1,r]g_{\alpha} \partial_{s_1 \cdots s_r-1}(X^{\alpha_2}_r)(X^\beta E_t^r) &= \\
= \begin{cases}
T_w[1,r]q_{m_t_s,0}(-1)^{\langle \alpha^\vee_1, \lambda \rangle + m_t_s + 1} & \text{if } a = -\langle \alpha^\vee_1, \lambda \rangle - r - 1 \text{ and } r \leq m_t_s \\
0 & \text{otherwise}.
\end{cases}
\end{align*}
$$
So, we have shown that

\[
\psi(X_r^a T_{w[1,r+1]}) = \begin{cases} 
T_{w[1,r]}q_{mt_0}(-1)^{(\langle \alpha^\vee, \lambda \rangle + m_{ts} + 1)} & \text{if } a = -\langle \alpha^\vee, \lambda \rangle - r - 1 \\
0 & \text{otherwise.}
\end{cases}
\]

- Assume now \( \langle \alpha^\vee, \lambda \rangle + 2r - m_{ts} \geq 0 \). We can assume that \( \langle \alpha^\vee, \lambda \rangle + r < 0 \), for otherwise the lemma is empty. So, we have \( r > m_{ts} \).

If \( \partial_{s_1 \ldots s_{r-1}}(X^\alpha) \neq 0 \), then \( \alpha_2 \geq r - 1 \), hence \( m_{ts} \geq r \), which is impossible. So,

\[
\sum_{\alpha_2 + \alpha_2 + 1 \geq 0} q_{\alpha_1 + \alpha_2 + 1, \alpha} T_{w[1,r]}q_{\alpha + \alpha_1} \partial_{s_1 \ldots s_{r-1}}(X^\alpha)(X^\beta E_t^r) = 0.
\]

Let \( \mu = \lambda + r \alpha_t + \alpha_s \). The diagram \((\text{14})\) shows that there are elements \( z_i \in Z(V_\mu) \) with \( z_i(\alpha, \mu) = (-1)^{(\alpha^\vee, \mu)} + 1 \) such that

\[
(E_t^i 1_{\mu - \alpha_t} \overset{\eta}{\longrightarrow} F_t E_t^i 1_{\mu - \alpha_t} \overset{F_t T}{\longrightarrow} F_t E_t^i 1_{\mu - \alpha_t} \overset{\varepsilon}{\longrightarrow} E_t^i 1_{\mu - \alpha_t}) = \sum_{i=0}^{\langle \alpha^\vee, \mu \rangle} z_i X_i.
\]

So,

\[
T_{w[1,r]}f_a T_{r-1} \cdots T_1 = \sum_{0 \leq \alpha < \alpha - 1 \atop 0 \leq \alpha \leq m_{ts}} T_{w[1,r]}q_{\alpha, \beta} X_{E_t}^\beta (\varepsilon^l \circ (F_t X^c) \circ \eta) \partial_{s_1 \ldots s_{r-1}}(X^\alpha - c + \alpha) + \sum_{0 \leq \alpha \leq m_{ts} \atop 0 \leq \beta \leq m_{st}} T_{w[1,r]}q_{\alpha, \beta} X_{E_t}^\beta \sum_{i=0}^{\langle \alpha^\vee, \mu \rangle} z_i \partial_{s_1 \ldots s_{r-1}}(X^\alpha + i + \alpha).
\]

We have

\[
a - 1 + m_{ts} \leq -\langle \alpha^\vee, \lambda \rangle - r - 2 + m_{ts} \leq r - 2
\]

hence \( \partial_{s_1 \ldots s_{r-1}}(X^\alpha - c + \alpha) = 0 \) for all \( a, c \geq 0 \) and \( \alpha \leq m_{ts} \).

We have \( a + \langle \alpha^\vee, \mu \rangle + m_{ts} \leq r - 1 \). If \( \partial_{s_1 \ldots s_{r-1}}(X^\alpha + i + \alpha) \neq 0 \), then \( a = -\langle \alpha^\vee, \lambda \rangle - r - 1 \), \( i = \langle \alpha^\vee, \mu \rangle \) and \( \alpha = m_{ts} \).

We have shown that

\[
T_{w[1,r]}f_a T_{r-1} \cdots T_1 = \begin{cases} 
T_{w[1,r]}q_{mt_0}(-1)^{(\langle \alpha^\vee, \lambda \rangle + m_{ts} + 1)} & \text{if } a = -\langle \alpha^\vee, \lambda \rangle - r - 1 \text{ and } r > m_{ts} \\
0 & \text{otherwise.}
\end{cases}
\]

The lemma follows.

\textbf{Proof of Theorem 5.25} Let \( N \in V_\lambda \) such that \( F_t N = 0 \). Define

\[
L = \bigoplus_{i \leq -\langle \alpha^\vee, \lambda \rangle - r} T_w X_{r+1}^i Z \text{ and } L' = \bigoplus_{i \leq -\langle \alpha^\vee, \lambda \rangle - r} X_{r+1}^i T_w Z.
\]

We have \( L \simeq L(r + 1, 1, 1, \lambda) \) (cf \((\text{12.22})\)). We have an isomorphism (Lemma \((\text{4.12})\))

\[
\text{act} \circ (\text{id} \otimes \eta E_t^r) : L \otimes_z E_t^r N \sim F_t E_t^{r+1} N.
\]

Similarly, applying Lemma \((\text{4.12})\) to \( V_\mu \), we obtain an isomorphism

\[
(id \otimes \varepsilon^l E_t^r) \circ \text{act}^* : F_t E_t^{r+1} N \sim L^* \otimes_z E_t^r N.
\]
We have a commutative diagram

\[
\begin{array}{cccccccc}
L \otimes E_s E_t^r & \xrightarrow{\text{act}(\id \otimes E_s \eta)} & E_s F_t E_t^{r+1} & \xrightarrow{\eta} & F_t E_t E_t^{r+1} & \xrightarrow{F_t \tau_t \cdot} & F_t E_s E_t E_t^{r+1} & \xrightarrow{F_t \tau_s \cdot} & F_t E_s E_t E_t^{r+1} \\
\text{id} \otimes \eta \cdot & & & & & & & & \text{id} \\
L \otimes F_t E_t E_s E_t^r & \xrightarrow{F_t \tau_s \cdot} & L \otimes F_t E_s E_t^{r+1} & & & & & & \\
\end{array}
\]

and a commutative diagram

\[
\begin{array}{cccccccc}
F_t E_s E_t^{r+1} & \xrightarrow{} & F_t E_s E_t E_t^{r+1} & \xrightarrow{\eta} & E_t E_s E_t^{r+1} & \xrightarrow{} & E_s F_t E_t^{r+1} & \xrightarrow{\epsilon_t} & E_s E_t^r \\
\text{act}^* & & & & & & & & (\id \otimes E_s \epsilon_t \cdot) \text{act}^* \\
L^t \otimes F_t E_t E_s E_t^r & \xrightarrow{} & L^t \otimes F_t E_t E_s E_t^{r+1} & \xrightarrow{F_t \tau_s \cdot} & L^t \otimes F_t E_t E_s E_t^{r+1} & \xrightarrow{\epsilon_t} & L^t \otimes E_s E_t^r \\
F_t E_s \eta \cdot & & & & & & & & F_t E_s \epsilon_t \cdot \\
L^t \otimes F_t E_t E_s E_t^r & \xrightarrow{} & L \otimes F_t E_t E_s E_t^{r+1} & \xrightarrow{\text{act}} & F_t E_t E_s E_t^{r+1} & \xrightarrow{\text{act}^*} & L^t \otimes F_t E_t E_s E_t^{r+1} & \xrightarrow{F_t \tau_s \cdot} & L^t \otimes E_s E_t^r \\
\end{array}
\]

We have a commutative diagram

\[
\begin{array}{cccccccc}
L \otimes E_s E_t^r & \xrightarrow{\text{act}(\id \otimes E_s \eta \cdot)} & E_s F_t E_t^{r+1} & \xrightarrow{\sigma_{st} \cdot} & F_t E_s E_t^{r+1} & \xrightarrow{\sigma_{ts} \cdot} & E_t E_s F_t^{r+1} & \xrightarrow{\text{act}^*} & L^t \otimes E_s E_t^r \\
\eta \cdot & & & & & & & & \epsilon_t \\
L \otimes F_t E_t E_s E_t^r & \xrightarrow{F_t \tau_s \cdot} & L \otimes F_t E_t E_s E_t^{r+1} & \xrightarrow{\text{act}} & F_t E_t E_s E_t^{r+1} & \xrightarrow{\text{act}^*} & L^t \otimes F_t E_t E_s E_t^{r+1} & \xrightarrow{F_t \tau_s \cdot} & L^t \otimes E_s E_t^r \\
\end{array}
\]

We will show that the top horizontal composition in the diagram above is an isomorphism when applied to \(N\):

\[
f : L \otimes E_s E_t^r N \xrightarrow{\sim} L^t \otimes E_s E_t^r N.
\]

It is enough to show that the map \(\gamma\) obtained from \(f\) by left multiplication by \(T_{w[1,r+1]}\) is invertible, as in the proof of Lemma \[4.12\]. Lemma \[5.26\] shows that the map

\[
E_t E_t^{(r)} \xrightarrow{X^a \otimes \id} T_{w[1,r+1]}(L \otimes E_s E_t^r) \xrightarrow{\gamma} T_{w[1,r+1]}(L^t \otimes E_s E_t^r) \xrightarrow{\langle X^a', - \rangle} E_t E_t^{(r)}
\]

is 0 for \(a + a' < -\langle \alpha_t', \lambda \rangle - r - 1\) and it is an isomorphism for \(a + a' = -\langle \alpha_t', \lambda \rangle - r - 1\). So, \(\gamma\) is an isomorphism and \(f\) as well. Consequently, the composition \(\sigma_{ts} \circ \sigma_{st}\) is an isomorphism when applied to \(E_t^r N\). We conclude from Lemma \[5.17\] that it is an isomorphism on all objects of \(\mathcal{V}\).

We apply now the result above to \(\mathcal{V}\): it shows that \(\sigma_{ts}\) has a left inverse. So, \(\sigma_{ts}\) is invertible, hence \(\sigma_{st}\) is invertible as well. \(\Box\)
5.3.6. Control from $K_0$.

**Theorem 5.27.** Consider a root datum with associated Kac-Moody algebra $\mathfrak{g}$ and associated ring $k$.

Let $k$ be a field that is a $k$-algebra and $\mathcal{V} \in \mathfrak{A}_k$. Assume given

- an adjoint pair $(E_s, F_s)$ of exact functors $\mathcal{V} \to \mathcal{V}$ for every $s \in I$
- $x_i \in \text{End}(E_s)$ and $\tau_{st} \in \text{Hom}(E_s E_t, E_tE_s)$ for every $s, t \in I$.
- a decomposition $\mathcal{V} = \bigoplus_{\lambda \in X} \mathcal{V}_\lambda$.

We assume that

- $F_s$ is isomorphic to a left adjoint of $E_s$
- $E_s(\mathcal{V}_\lambda) \subset \mathcal{V}_{\lambda + \alpha_s}$ and $F_s(\mathcal{V}_\lambda) \subset \mathcal{V}_{\lambda - \alpha_s}$
- $\{[E_s], [F_s]\}_{s \in I}$ induce an integrable representation of $\mathfrak{g}$ on $V = K_0(\mathcal{V})$
- relations (1)-(4) in §4.1.3 hold

Then, the data above defines an integrable action of $\mathfrak{A}(\mathfrak{g})$ on $\mathcal{V}$.

**Proof.** This is a consequence of Theorems 5.22 and 5.25. □

5.3.7. Type A. Let $k$ be a field. Let $q \in k^\times$ and let $I$ be a subset of $k$. Assume $0 \not\in I$ if $q \neq 1$ and consider the corresponding Lie algebra $\mathfrak{s}(t_q)$ as in §3.2.5.

Let $\mathcal{V}$ be a $k$-linear category. Consider

- an adjoint pair $(E, F)$ of endofunctors of $\mathcal{V}$
- $X \in \text{End}(E)$ and $T \in \text{End}(E^2)$.

Assume there are decompositions $E = \bigoplus_{i \in I} E_i$ and $F = \bigoplus_{i \in I} F_i$, where $X - i$ is locally nilpotent on $E_i$ and $F_i$.

When $q = 1$, we put $x_i = X - i$ (acting on $E_i$) and

$$
\tau_{ij} = \begin{cases} (E_iX -XE_j + 1)^{-1}(T - 1) & \text{if } i = j \\
(E_iX -XE_j)T + 1 & \text{if } i = j + 1 \\
\frac{E_iX -XE_j}{E_iX -XE_{j+1}}(T - 1) + 1 & \text{otherwise}
\end{cases}
$$

(restricted to $E_iE_j$).

When $q \neq 1$, we put $x_i = i^{-1}X$ (acting on $E_i$) and

$$
\tau_{ij} = \begin{cases} i(qE_iX -XE_j)^{-1}(T - q) & \text{if } i = j \\
q^{-1}i^{-1}(E_iX -XE_j)T + i^{-1}(1 - q^{-1})XE_j & \text{if } i = qj \\
\frac{E_iX -XE_j}{qE_iX -XE_j}(T - q) + 1 & \text{otherwise}
\end{cases}
$$

(restricted to $E_iE_j$).

Assume that there is a decomposition $\mathcal{V} = \bigoplus_{\lambda \in X} \mathcal{V}_\lambda$ such that

- $E_i(\mathcal{V}_\lambda) \subset \mathcal{V}_{\lambda + \alpha_i}$ and $F_i(\mathcal{V}_\lambda) \subset \mathcal{V}_{\lambda - \alpha_i}$
- $E_i$ and $F_i$ are locally nilpotent
- when $\langle \alpha^\vee, \lambda \rangle \geq 0$, the map $\sigma_{ss} + \sum_{i=0}^{\langle \alpha^\vee, \lambda \rangle} \varepsilon_s \circ (x^i_s F_s) : E_s F_s(M) \to F_s E_s(M) \oplus M^{\langle \alpha^\vee, \lambda \rangle}$ is invertible for $M \in \mathcal{V}_\lambda$
- when $\langle \alpha^\vee, \lambda \rangle \leq 0$, the map $\sigma_{ss} + \sum_{i=0}^{\langle \alpha^\vee, \lambda \rangle} (F_s x^i_s) \circ \eta_s : E_s F_s(M) \oplus M^{\langle \alpha^\vee, \lambda \rangle} \to F_s E_s(M)$ is invertible for $M \in \mathcal{V}_\lambda$. 
Theorem 5.28. The data above defines an action of $A_{\mathbb{Z}}(\mathfrak{sl}_{k}) \otimes k$ on $\mathcal{V}$.

Proof. The $x_i$'s and $\tau_{ij}$'s satisfy the relations (1)-(4) in §4.1.1 thanks to Propositions 3.15 and 3.18. The invertibility of $\sigma_s$ for $s \neq t$ follows from Theorem 5.25. □

5.3.8. $\mathfrak{sl}$-categorifications. Let $k$ be a field. Let $q \in k^\times$ and let $I$ be a subset of $k$. Assume $0 \notin I$ if $q \neq 1$ and consider the corresponding Lie algebra $\mathfrak{sl}_q$ as in §3.2.3.

Let $\mathcal{V} \in \mathfrak{sl}_q^I$.

Definition 5.29 (Chuang-Rouquier). An $\mathfrak{sl}_q^I$-categorification on $\mathcal{V}$ is the data of
- an adjoint pair $(E, F)$ of exact functors $\mathcal{V} \to \mathcal{V}$
- $X \in \text{End}(E)$ and $T \in \text{End}(E^2)$
- a decomposition $\mathcal{V} = \bigoplus_{\lambda \in X} \mathcal{V}_\lambda$.

Given $i \in k$, let $E_i$ (resp. $F_i$) be the generalized $i$-eigenspace of $X$ acting on $E$ (resp. $F$). We assume that
- $E = \bigoplus_{i \in I} E_i$
- the action of $\{[E_i], [F_i]\}_{i \in I}$ on $K_0(\mathcal{V})$ gives an integrable representation of $\mathfrak{sl}_q^I$
- $E_i(\mathcal{V}_\lambda) \subset \mathcal{V}_{\lambda + \alpha_i}$ and $F_i(\mathcal{V}_\lambda) \subset \mathcal{V}_{\lambda - \alpha_i}$
- $F$ is isomorphic to a left adjoint of $E$
- the action on $E^n$ of $X_i = E^{n-i}XE^{i-1}$ for $1 \leq i \leq n$ and of $T_i = E^{n-i}TE^{i-1}$ for
  - an affine Hecke algebra if $q \neq 1$
  - a degenerate affine Hecke if $q = 1$.

Consider an $\mathfrak{sl}_q^I$-categorification on $\mathcal{V}$.

Theorem 5.30. Assume given an $\mathfrak{sl}_q^I$-categorification on $\mathcal{V}$. The construction of §5.3.7 gives rise to an action of $A_{\mathbb{Z}}(\mathfrak{sl}_q^I) \otimes k$ on $\mathcal{V}$.

Conversely, an integrable action of $A_{\mathbb{Z}}(\mathfrak{sl}_q^I) \otimes k$ on $\mathcal{V}$ gives rise to an $\mathfrak{sl}_q^I$-categorification on $\mathcal{V}$.

Proof. The morphisms $\rho_{s,\lambda}$ are invertible by [ChRou, Theorem 5.27]. The theorem follows now from Theorem 5.28. □

Remark 5.31. A setting for categorifications of $\mathfrak{sl}_2$ [Lau] and $\mathfrak{sl}_n$ [KhoLau3] has been proposed recently. While they do not check its compatibility with the older definition above, its 2-representations should give a full 2-subcategory of those above, related to $\mathfrak{A}'$.

Note that it is straightforward to define a notion of $\mathfrak{sl}_q^I$-categorifications:

Definition 5.32. An $\mathfrak{sl}_q^I$-categorification on $\mathcal{V}$ is the data of
- an adjoint pair $(E, F)$ of exact functors $\mathcal{V} \to \mathcal{V}$
- $X \in \text{End}(E)$ and $T \in \text{End}(E^2)$.

Given $i \in k$, let $E_i$ (resp. $F_i$) be the generalized $i$-eigenspace of $X$ acting on $E$ (resp. $F$). We assume that
- $E = \bigoplus_{i \in I} E_i$
- the action of $\{[E_i], [F_i]\}_{i \in I}$ on $K_0(\mathcal{V})$ gives an integrable representation of $\mathfrak{sl}_q^I$
- classes of simple objects are weight vectors
- $F$ is isomorphic to a left adjoint of $E$
• the action on $E^n$ of $X_i = E^{n-i}X_i E^{i-1}$ for $1 \leq i \leq n$ and of $T_i = E^{n-i-1}TE^{i-1}$ for $1 \leq i \leq n - 1$ induce an action of
  - an affine Hecke algebra if $q \neq 1$
  - a degenerate affine Hecke if $q = 1$.

When $I_q$ has no component of type $\tilde{A}_n$, then the notion of $\mathfrak{sl}_{I_q}$-categorification coincides with that of $\mathfrak{sl}_{I_q}$-categorification (put $\mathcal{V}_\lambda = \{ M \in \mathcal{V} | [M] \in \mathcal{V}_\lambda \}$).

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Mathematical Institute, University of Oxford, 24-29 St Giles’, Oxford, OX1 3LB, UK
E-mail address: rouquier@maths.ox.ac.uk