On distributive laws in derived bracket construction

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Abstract

We will introduce new relations which naturally arises between Lie-operad and Leibniz-operad. We prove that the relations are “distributive laws” of the Lie operad over the Leibniz operad in the sense of M. Markl (1996). By using the distributive law, we define a new operad, which is called a Lie-Leibniz operad. The Lie-Leibniz operad is Koszul, because the Lie operad and the Leibniz operad are both Koszul. The Koszul dual operad of Lie-Leibniz operad is also studied. In addition, we will prove that the Lie-Leibniz operad naturally becomes a resolution over the Lie operad.

1 Introduction

Leibniz algebras of Loday [8, 9] and their homotopy versions are the key concepts in higher dimensional topological field theory. Hence it is interesting work to study the operad of Leibniz algebras (See Ginzburg-Kapranov [3], Loday [9] and so on, for the details of operads).

It is known that the operad of Leibniz algebras $\mathcal{L}eib$ is isomorphic to $\mathcal{L}ie \otimes \mathcal{P}erm$ (cf. Vallette [15]), where $\mathcal{L}ie$ is the operad of Lie algebras and $\mathcal{P}erm$ is the one of permutation algebras of Chapoton [2]. This implies that between Lie operad and Leibniz operad there are two natural binary-quadratic (bq) relations,

$$[1, (2, 3)] = ([1, 2], 3) + (2, [1, 3]), \tag{1}$$

$$[(1, 2), 3] = ([1, 2], 3) - ([2, 1], 3), \tag{2}$$

where $(1, 2)$ is the generator of Lie operad, $[1, 2]$ and $[2, 1]$ are the generators of Leibniz operad. We call a Leibniz algebra a Lie-Leibniz algebra (or Lie-Loday algebra), if it has a Lie bracket satisfying (1) and (2). These relations naturally arise in derived bracket construction (Kosmann-Schwarzbach [6, 7], see also
Hence we call (1) and (2) the natural relations\(^1\).

We will prove that the natural relations (1) and (2) are distributive laws of \(\text{Lie}\) over \(\text{Leib}\) in the sense of Markl [10] and Beck [1]. In [10] it was proved that if two Koszul operads are unified by distributive laws, then the deduced operad is again Koszul. Because \(\text{Lie}\) and \(\text{Leib}\) are both Koszul, the operad of Lie-Leibniz algebras, which is denoted by \(LL\), is Koszul. We also prove that by the derived bracket construction the operad \(LL\) can be decomposed into the form of tensor product

\[
LL = \text{Lie} \otimes D
\]

where \(D\) is a new operad defined in Section 2.2.

We shortly describe a connection between Lie-Leibniz algebras and Courant algebroids, or 3-dimensional topological field theory of AKSZ type (cf. Ikeda [5], see also Roytenberg [12]). The Courant algebroids are defined as vector bundles over smooth manifolds, \(B \to M\), equipped with Leibniz brackets \([\cdot, \cdot]\) of degree \(-1\), symplectic structures \((\cdot, \cdot)\) of degree \(-2\), and derivation representations \(\rho : B \to TM\) of degree \(-1\), satisfying,

\[
\begin{align*}
[x_1, [x_2, x_3]] &= [[x_1, x_2], x_3] + [x_2, [x_1, x_3]], \\
\rho(x_1)(x_2, x_3) &= ([x_1, x_2], x_3) + (x_2, [x_1, x_3]), \\
\rho(x_3)(x_1, x_2) &= ([x_1, x_2], x_3) + ([x_2, x_1], x_3),
\end{align*}
\]

where \(x_1, x_2, x_3\) are smooth sections of the bundle and where the degrees of variables are all +1. By (3), the space of sections becomes a (graded-)Leibniz algebra. Because the product \((\cdot, \cdot)\) is a symplectic structure, i.e., Poisson bracket, it is a Lie bracket. In (5), because the degrees of variables are +1,

\[
([1, 2] - [2, 1])(x_1, x_2) = [x_1, x_2] + [x_2, x_1].
\]

Because \(\rho(x)\) is a derivation, the left-hand sides of (4) and (5) are regarded as Leibniz brackets, \(\rho(x_1)(x_2, x_3) = [x_1, (x_2, x_3)]\) and \(\rho(x_3)(x_1, x_2) = [(x_1, x_2), x_3]\). In fact, (4) and (5) are geometric examples of the distributive laws (1) and (2). Thus we get a slogan:

\textit{Courant brackets are distributive laws}

\(^1\)The relation (1) arises in other context, i.e., the study of tri-algebras. See Gubarev-Kolesnikov’s work [4] for the detailed study of this direction.
This provides a new picture for the study of Courant algebroids.

In [11] (see also [13]), Roytenberg and Weinstein studied the skewsymmetrized bracket of Courant bracket,

\[ \{x_1, x_2\} := \frac{1}{2}([x_1, x_2] - [x_2, x_1]). \]

This bracket satisfies a Jacobi identity up to homotopy,

\[ \{\{x_1, x_2\}, x_3\} + \{\{x_3, x_1\}, x_2\} + \{\{x_2, x_3\}, x_1\} = dT(x_1, x_2, x_3), \tag{6} \]

where \( d \) is a differential, which is defined as the dual of \( \rho \) above, and the Jacobi anomaly \( T \) has the following form,

\[ T(x_1, x_2, x_3) := \frac{1}{3} \left( ([x_1, x_2], x_3) + ([x_3, x_1], x_2) + ([x_2, x_3], x_1) \right). \]

It has been proved that Courant algebroids have special strong-homotopy (sh) Lie algebra structures. We will prove that the operad of Lie-Leibniz algebras is a resolution over the Lie-operad. This proposition provides an operad theoretical description for Roytenberg-Weinstein’s sh-Lie algebra theory.

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2 Main results

ASSUMPTIONS.
The ground field is \( \mathbb{K} \) of \( \text{char}(\mathbb{K}) := 0 \) and \( \mathbb{Q} \subset \mathbb{K} \).
The Leibniz algebras are left.
The operads \( \mathcal{P} \) are algebraic and \( \mathcal{P}(1) := \mathbb{K} \).

2.1 Distributive laws

In this subsection we recall distributive laws of Markl [10].

Let \( \mathcal{P} = (E_\mathcal{P}, R_\mathcal{P}) \) and \( \mathcal{Q} = (E_\mathcal{Q}, R_\mathcal{Q}) \) be the binary-quadratic (bq) operads, where \( E_\mathcal{P} \) and \( R_\mathcal{P} \) are respectively the space of generators and the one of bq-relations.

We consider a linear mapping,

\[ \delta : \mathcal{Q}(2) \odot \mathcal{P}(2) \Rightarrow \mathcal{P}(2) \odot \mathcal{Q}(2), \tag{7} \]

where \( \odot \) is the grafting of trees\(^2\). Let \( \Delta := < x - \delta(x) > \) be the graph of the map.

One defines a new bq-operad by

\[ \mathcal{P} \mathcal{Q} := (E_\mathcal{P} \oplus E_\mathcal{Q}, R_\mathcal{P} \oplus \Delta \oplus R_\mathcal{Q}). \]

\(^2\)The arity of \( \mathcal{Q}(2) \odot \mathcal{P}(2) \) is 3.
We assume that the degree of $E_Q$ is +1. Then $PQ$ becomes a graded operad, in particular,

$$PQ(4) = PQ^0(4) \oplus PQ^1(4) \oplus PQ^2(4) \oplus PQ^3(4).$$

It is easy to see that $PQ^0(4) \cong P(4)$ and $PQ^3(4) \cong Q(4)$. The map $\delta$ or the relation $\Delta$ is called a distributive law, if naturally

$$(P \odot Q)(4) \cong PQ^1(4) \oplus PQ^2(4),$$

where $\overline{P} := (P(2), P(3))$. There exists a natural homogeneous epimorphism,

$$epi : (P \odot Q)(4) \to PQ^1(4) \oplus PQ^2(4),$$

which is defined by the universality of the grafting product. Hence the above definition says that this map is mono.

**Example 2.1.** Let $P = \text{Com}$ be the commutative associative operad and let $Q = \text{Lie}$ the Lie operad. Then the derivation condition

$$\text{Lie}(2) \odot \text{Com}(2) \Rightarrow \text{Com}(2) \odot \text{Lie}(2),$$

$$[1, 23] \Rightarrow [1, 2]3 + 2[1, 3]$$

defines a distributive law. The induced operad $\text{ComLie} (=: \text{Poiss})$ is the Poisson operad.

### 2.2 Lie-Leibniz algebras

We consider a 3-dimensional vector space generated by the binary trees,

$$< [1, 2], [1, 2]_t, (1, 2) >.$$

An $S_2$-module structure is defined on the 3-space by

$$[1, 2]_t = [2, 1],$$

$$(1, 2) = -(2, 1).$$

The operad of Lie-Leibniz algebras, which is denoted by $\mathcal{L}\mathcal{L}$, is generated over the 3-space, whose bq-relations are

$$[1, [2, 3]] - [[1, 2], 3] - [2, [1, 3]],$$

$$[1, (2, 3)] - ([1, 2], 3) - (2, [1, 3]),$$

$$[(1, 2), 3] - ([1, 2], 3) + ([2, 1], 3),$$

$$((1, 2), 3) - ([1, 2], 3) - (2, [1, 3]),$$

$$[1, (2, 3)] - ([1, 2], 3) - (2, [1, 3]).$$
or equivalently,

\[
\begin{align*}
[[2, 3], 1]_t - [[1, 2], 3] - [[3, 1]_t, 2]_t, & \quad (13) \\
[[2, 3], 1]_t - ([1, 2], 3) + ([3, 1]_t, 2), & \quad (14) \\
((1, 2), 3) - ([1, 2], 3) + ([1, 2]_t, 3), & \quad (15) \\
((1, 2), 3) + ((3, 1), 2) + ((2, 3), 1), & \quad (16)
\end{align*}
\]

The formulas (13)-(16) will be used in the next section.

**Definition 2.2.** The algebras over the operad \(LL\) are called the Lie-Leibniz algebras or Lie-Loday algebras. The bidegree of Lie-Leibniz algebra is \((i, j)\), if the degree of Lie bracket is \(i\) and the one of Leibniz bracket is \(j\).

**Example 2.3** ([6, 7]). Let \((g, (\cdot, \cdot), d)\) be a dg-Lie algebra with a Lie bracket \((\cdot, \cdot)\) of degree 0 and a differential of degree +1. Define a Leibniz bracket by

\[
[x, y] := -(-1)^{|x|}(dx, y).
\]

Then \((g, (\cdot, \cdot), [\cdot, \cdot])\) becomes a graded Lie-Leibniz algebra of bidegree \((0, 1)\).

The Leibniz bracket in above example is called a **derived bracket**. In [14] it has been proved that on the level of operad the universal Leibniz bracket is given as the derived bracket. That is, the generators of Leibniz operad \([1, 2]\) and \([1, 2]_t(= [2, 1])\) are the derived brackets of the one of Lie operad,

\[
\begin{align*}
[1, 2] &= (d1, 2), \\
[1, 2]_t &= -(1, d2).
\end{align*}
\]

Here \(d\) is a formal differential. The derived bracket construction derives the natural relations (10) and (11), for example (11) is

\[
((1, 2), 3) = (d1, 2), 3) = ((d1, 2), 3) + (1, d2), 3) = ([1, 2], 3) - ([2, 1], 3).
\]

In a similar way, (10) is followed from the Jacobi identity of the Lie bracket. From this we notice that the monomials in \(LL\) are represented by the Lie operad and the formal differentials, for example,

\[
\begin{align*}
[[1, 2], 3] &\cong (\pm)((1, 2), 3) \odot (d \odot d \odot 1), \\
([1, 2], 3) &\cong ((1, 2), 3) \odot (d \odot 1 \odot 1 + 1 \odot d \odot 1), \\
[1, (2, 3)] &\cong (1, (2, 3)) \odot (d \odot 1 \odot 1),
\end{align*}
\]

where \(1\) is the identity and \((\pm)\) is an appropriate sign.
To study the operad $\mathcal{L}\mathcal{L}$ we introduce a new operad $\mathcal{D}$ which is called a **deriving operad**. The deriving operad is defined as follows. We consider an $S$-module $(d, 1 \otimes 1, 0, \ldots)$ and the free operad over the module, $\mathcal{T}(d, 1 \otimes 1)$, where $d$ is a 1-ary operator and $1 \otimes 1$ is a binary commutative product. Define an operad $\mathcal{O}$ as a quotient operad of $\mathcal{T}(d, 1 \otimes 1)$,

$$\mathcal{O} := \mathcal{F}(d, \text{Com}(2))/ (R_1, R_2)$$

where $(R_1, R_2)$ is the space of two quadratic relations such that $R_1$ is the differential properties

$$dd = 0, \quad d(1 \otimes 1) - d \otimes 1 - 1 \otimes d = 0$$

and $R_2$ is the associative law,

$$(1 \otimes 1) \otimes 1 = 1 \otimes 1 \otimes 1 = 1 \otimes (1 \otimes 1).$$

The operad $\mathcal{O}$ becomes a graded operad, $\mathcal{O} = (\mathcal{O}^i)$, whose degree is defined as the number of $d$.

**Definition 2.4** (deriving operad). *The $n$th component of $\mathcal{D}$ is by definition

$$\mathcal{D}(n) := \mathcal{O}^0(n) \oplus \cdots \oplus \mathcal{O}^{n-1}(n).$$

It is obvious that $\mathcal{D}^0 \cong \text{Com}$ (the commutative associative operad) and it is easy to prove that $\mathcal{D}^{\text{top}} \cong \mathcal{P}e\mathcal{r}m$, the operad of permutation algebras of Chapton ([2]), for the isomorphism see [14]. In general $\mathcal{D}$ has the following form.

$$\begin{align*}
\mathcal{D}^0 &= \text{Com}, \\
\mathcal{D}^1(2) &= < d \otimes 1, 1 \otimes d >, \\
\mathcal{D}^1(3) &= < d \otimes 1 \otimes 1, 1 \otimes d \otimes 1, 1 \otimes 1 \otimes d >, \\
\mathcal{D}^2(3) &= < d \otimes d \otimes 1, d \otimes 1 \otimes d, 1 \otimes d \otimes d >, \\
\cdots &= \cdots.
\end{align*}$$

From this observation, we obtain

**Proposition 2.5.** $\mathcal{L}\mathcal{L} \cong \text{Lie} \otimes \mathcal{D}.$

From $\dim \text{Lie}(n) = (n - 1)!$ and $\dim \text{Com}(n) = 1$, we obtain

$$\dim \mathcal{L}\mathcal{L}(n) = \sum_{m=1}^{n} (n - 1)! \binom{n}{m}.$$
Corollary 2.6. Let $R_{L\mathcal{L}}$ be the space of quadratic relations of $L\mathcal{L}$. Then $\dim R_{L\mathcal{L}} = 13$.

Proof. It is known that for any bq-operad $\mathcal{P} = (E, R)$,

$$3(\dim E)^2 = \dim \mathcal{P}(3) + \dim R.$$ 

In the case of $L\mathcal{L}$, $\dim E = 3$ and $\dim L\mathcal{L}(3) = 14$, which gives the identity of the lemma. \hfill \Box

The natural relations (10) and (11) define a mapping

$$\mathcal{L}ieb(2) \odot \mathcal{L}ie(2) \Rightarrow \mathcal{L}ie(2) \odot \mathcal{L}ieb(2).$$ (17)

We should prove that this is a distributive law.

Theorem 2.7. The natural relations (10) and (11) define distributive laws.

Proof. The degree of $\mathcal{L}ieb(2)$ is by assumption $+1$. Then $L\mathcal{L}(4)$ becomes a graded space which is decomposed into the 4 subspaces

$$L\mathcal{L}(4) = \mathcal{L}ie(4) \oplus L\mathcal{L}^1(4) \oplus L\mathcal{L}^2(4) \oplus \mathcal{L}ieb(4).$$

From Proposition 2.5 we obtain

$$\dim L\mathcal{L}^1(4) = 24,$$ (18)

$$\dim L\mathcal{L}^2(4) = 36.$$ (19)

To prove the theorem it suffices to show that the dimension of $(\mathcal{L}ie \odot \mathcal{L}ieb)(4)$ is equal to $60(= 24 + 36)$, where $\mathcal{L}ie = (\mathcal{L}ie(2), \mathcal{L}ie(3))$.

We compute the dimension of $(\mathcal{L}ie \odot \mathcal{L}ieb)(4)$. The subspace of $(\mathcal{L}ie \odot \mathcal{L}ieb)(4)$ of degree $+2$ is

$$\mathcal{L}ie(2) \odot \left(\mathcal{L}ieb(2) \odot \mathcal{L}ieb(2)\right) \oplus \mathcal{L}ie(2) \odot \mathcal{L}ie(3).$$ (20)

The first term in (20) is generated by the 12-monomials,

$$([1, 2], [3, 4]) \quad ([1, 3], [2, 4]) \quad ([1, 4], [2, 3])$$

$$([2, 1], [3, 4]) \quad ([3, 1], [2, 4]) \quad ([4, 1], [2, 3])$$

$$([1, 2], [4, 3]) \quad ([1, 3], [4, 2]) \quad ([1, 4], [3, 2])$$

$$([2, 1], [4, 3]) \quad ([3, 1], [4, 2]) \quad ([4, 1], [3, 2]).$$

Hence we obtain

$$\dim \mathcal{L}ie(2) \odot \left(\mathcal{L}ieb(2) \odot \mathcal{L}ieb(2)\right) = 12.$$
The second term of (20) is generated by the generators of 4-types, 

\[(1, \text{Leib}(3)) \quad (2, \text{Leib}(3)) \quad (3, \text{Leib}(3)) \quad (4, \text{Leib}(3)).\]

It is known that \(\text{Leib}(n) \cong S_n\). Therefore \(\text{dim} \text{Leib}(3) = 6\), which gives

\[\text{dim} \text{Lie}(2) \odot \text{Leib}(3) = 4 \times 6 = 24.\]

Thus we obtain

\[\text{dim} \text{Lie}(2) \odot \left( \text{Leib}(2) \otimes \text{Leib}(2) \right) \odot \text{Lie}(2) \odot \text{Leib}(3) = 12 + 24 = 36.\]

This number coincides with the dimension of \(\mathcal{L}\mathcal{L}_2(4)\).

We consider the subspace of \((\mathcal{L}\mathcal{L} \odot \text{Leib})(4)\) of degree +1, which has the form

\[\text{Lie}(3) \odot \text{Leib}(2).\]

Up to the Jacobi identity on \(\text{Lie}(3)\), this subspace is generated by

\[\begin{align*}
((1, 2, 3), 4) & \quad (4, [1, 2], 3) \\
((2, 1, 3), 4) & \quad (4, [2, 1], 3) \\
((1, 3, 2), 4) & \quad (4, [1, 3], 2) \\
((3, 1, 2), 4) & \quad (4, [3, 1], 2) \\
((1, 4, 2), 3) & \quad (3, [1, 4], 2) \\
((4, 1, 2), 3) & \quad (3, [4, 1], 2)
\end{align*}\]

\[\ldots \quad \ldots,\]

totally 24 terms, this number is the same as the dimension of \(\mathcal{L}\mathcal{L}_1(4)\). \(\Box\)

By the theorems in [10], we obtain the two corollaries below.

Corollary 2.8. The operad \(\mathcal{L}\mathcal{L}\) is Koszul for any bidegree \((i, j)\).

Corollary 2.9. Given a Leibniz algebra \(\mathfrak{g}\), the free Lie algebra over \(\mathfrak{g}\), \(\mathcal{F}_{\text{Lie}}(\mathfrak{g})\), becomes the free Lie-Leibniz algebra in the category of Leibniz algebras. In particular, when \(\mathfrak{g}\) is free, it is the free Lie-Leibniz algebra.

Remark 2.10 (Poisson embedding). Since the free Poisson algebra is defined by \(\mathcal{F}_{\text{Poiss}} := \mathcal{F}_{\text{Com}} \mathcal{F}_{\text{Lie}}\), we obtain \(\mathfrak{g} \subset \mathcal{F}_{\text{Poiss}}(\mathfrak{g})\). An arbitrary Leibniz algebra can be embedded in the free Poisson algebra and the Leibniz bracket is compatible with the Poisson bracket.
3 Koszul dual algebras

We have proved $LL = \text{LieLeib}$. It is known that the dual of distributive law (i.e. the dual map $\delta^*$) is again a distributive law. Hence the Koszul dual of $\text{LieLeib}$ is $\text{ZinbCom}$, where $\text{Com} = \text{Lie}^!$ and $\text{Zinb} = \text{Leib}^!$ (the Zinbiel operad [16]). The space of generators of $\text{ZinbCom}$ is a 3-space,

$$< 1 \ast 2 , 1 \ast_t 2 , 1 \cdot 2 >,$$

where $1 \ast 2$ and $1 \cdot 2$ are respectively the duals of $[1,2]$ and $(1,2)$. The $S_2$-module structure is as follows,

$$1 \ast_t 2 = 2 \ast 1,$$
$$1 \cdot 2 = 2 \cdot 1.$$

**Proposition 3.1.** The quadratic relations of $\text{ZinbCom}$ are

\begin{align}
1 \ast (2 \ast 3) &= (1 \ast 2) \ast 3 + (2 \ast 1) \ast 3, & (21) \\
(1 \ast 2) \cdot 3 &= 1 \ast (2 \cdot 3) - (1 \cdot 2) \ast 3, & (22) \\
1 \cdot (2 \cdot 3) &= (1 \cdot 2) \cdot 3. & (23)
\end{align}

The relation (22) defines a distributive law of $\text{Zinb}$ over $\text{Com}$.

In [9] it has been proved that the Koszul dual of Leibniz identity is (21).

**Proof.** Let $R'$ be the space of relations generated by (21), (22) and (23). The relation (21) generates 6-basis, (23) generates 2-basis and (22) generates the following 6-basis,

\begin{align}
(1 \ast 2) \cdot 3 &= 1 \ast (2 \cdot 3) - (1 \cdot 2) \ast 3, \\
(2 \ast 1) \cdot 3 &= 2 \ast (1 \cdot 3) - (2 \cdot 1) \ast 3, \\
(2 \ast 3) \cdot 1 &= 2 \ast (3 \cdot 1) - (2 \cdot 3) \ast 1, \\
(3 \ast 2) \cdot 1 &= 3 \ast (2 \cdot 1) - (3 \cdot 2) \ast 1, \\
(3 \ast 1) \cdot 2 &= 3 \ast (1 \cdot 2) - (3 \cdot 1) \ast 2, \\
(1 \ast 3) \cdot 2 &= 1 \ast (3 \cdot 2) - (1 \cdot 3) \ast 2.
\end{align}

Thus we obtain

$$\dim R' = 14,$$

which satisfies the consistency condition,

$$\dim LL(3) = \dim R'.$$
The equation (22) is equivalent to
\[(1 \ast 2) \cdot 3 - (2 \cdot 3) \ast t \cdot 1 + (1 \cdot 2) \ast 3.\] (24)

The pairing $\langle, \rangle$ which defines the Koszul duality is defined as follows,
\[
\begin{align*}
\langle [[i, j], k], (a \ast b) \ast c \rangle &= \delta_{ia} \delta_{jb} \delta_{kc}, \\
\langle [[i, j], k], (a \ast t b) \ast c \rangle &= -\delta_{ia} \delta_{jb} \delta_{kc}, \\
\langle [[i, j], k], (a \ast b) \ast c \rangle &= -\delta_{ia} \delta_{jb} \delta_{kc}, \\
\langle [[i, j], k], (a \ast t b) \ast c \rangle &= \delta_{ia} \delta_{jb} \delta_{kc}, \\
\langle ([i, j], k), (a \ast t b) \cdot c \rangle &= -\delta_{ia} \delta_{jb} \delta_{kc}, \\
\langle ([i, j], k), (a \ast b) \cdot c \rangle &= \delta_{ia} \delta_{jb} \delta_{kc}, \\
\langle ([i, j], k), (a \ast b) \cdot c \rangle &= \delta_{ia} \delta_{jb} \delta_{kc}, \\
\langle ([i, j], k), (a \ast b) \cdot c \rangle &= \delta_{ia} \delta_{jb} \delta_{kc}, \\
\langle ((i, j), k), (a \cdot b) \cdot c \rangle &= \delta_{ia} \delta_{jb} \delta_{kc},
\end{align*}
\]
and all others zero, where $\delta$ is Kronecker’s delta and $(ijk), (abc) \in \{(123), (312), (231)\}$.

By a direct computation, for example
\[
\langle (13), (24) \rangle = \langle (14), (24) \rangle = \langle (15), (24) \rangle = \langle (16), (24) \rangle = 0,
\]
one can show that $R'$ is the orthogonal space of $R_{LL}$, i.e., $R' = R_{LL}^\perp$, with respect to the pairing. \(\square\)

### 4 Jacobi anomaly in $LL$

In this section we study a homotopy theory associated with $LL$. Suppose that the degree of $(1, 2)$ is 0 and the one of $[1, 2]$ is +1. The second assumption equivalently means that the formal differential in the derived bracket $[1, 2] = (d1, 2)$ has the degree +1. We denote by $s\mathcal{P}$ the shifted operad of $\mathcal{P}$. The Leibniz operad of this section is $s\mathcal{Leib}$.

We notice that there exists a natural differential operator, which is denoted by $d$, on the deriving operad $\mathcal{D}$.

**Definition 4.1.** For any $l_1 \otimes \cdots \otimes l_n \in \mathcal{D}^j(n)$, $j \leq n - 2$,
\[
d(l_1 \otimes \cdots \otimes l_n) := \sum_i (\pm) l_1 \otimes \cdots \otimes d(l_i) \otimes \cdots \otimes l_n
\]
where $d(1) = d$ and $(\pm)$ is an appropriate sign.
For example, \( d(1 \otimes d \otimes 1) = d \otimes d \otimes 1 - 1 \otimes d \otimes d \).

**Remark 4.2.** We should remark that \((D, d)\) is not a dg-operad.

It is an easy exercise to prove that \((D, d)\) is a resolution of \(s\text{Com}\), i.e.,

\[
\mathcal{D}\text{top}/\mathcal{I}\text{md} \cong s\text{Com}
\]

and \(H(D, d) = 0\). From the identities \(\mathcal{LL} \cong \text{Lie} \otimes D\) and \(\text{Lie} \otimes s\text{Com} \cong s\text{Lie}\), we obtain

**Proposition 4.3.** The complex \((\mathcal{LL}, 1 \otimes d)\) is a resolution of \(s\text{Lie}\).

Eq. (6) in Introduction is considered to be a representation of this proposition. The tensor \(T\) lives in \(\mathcal{LL}^1(3)\).

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