Cotorsion pairs and a $K$-theory localization theorem

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The problem: background

Algebraic $K$-theory is very hard to compute! For that reason, it’s useful to have results relating the $K$-theory groups of different categories.
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A major tool in this direction: finding homotopy fiber sequences

$$K(A) \rightarrow K(B) \rightarrow K(C)$$

relating the $K$-theory spaces of categories $A, B$ and $C$. 

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relating the $K$-theory spaces of categories $A$, $B$ and $C$.

These induce long exact sequences

$$\cdots \to K_{n+1}(C) \to K_n(A) \to K_n(B) \to K_n(C) \to K_{n-1}(A) \to \cdots$$

ending in $K_0(A) \to K_0(B) \to K_0(C) \to 0$. 

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The problem: background

Among these results:

**Localization (Quillen)**

Let \( \mathcal{A} \) be a Serre subcategory of an abelian category \( \mathcal{B} \). Then there exists a quotient abelian category \( \mathcal{B}/\mathcal{A} \) such that

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Very useful! **But:** only applies to *abelian* cat’s, while many of the cat’s of interest to $K$-theory are exact (e.g. $K(R) = K(\text{proj})$).
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**Exact localization (Cárdenas, Schlichting)**

Same as Quillen’s, now $\mathcal{A} \subseteq \mathcal{B}$ are exact, and $\mathcal{A}$ is Serre + more conditions. Then $\mathcal{B}/\mathcal{A}$ is exact, and there is a homotopy fiber sequence

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**Exact localization (Cárdenas, Schlichting)**

Same as Quillen’s, now $A \subseteq B$ are exact, and $A$ is Serre + more conditions. Then $B/A$ is exact, and there is a homotopy fiber sequence

$$K(A) \to K(B) \to K(B/A)$$

Allows for more applications, but may still be quite restrictive!
The problem: background

For example, given a well-behaved ring $R$, one would like to apply the Localization theorem to

$$\text{proj} \subseteq R\text{-mod}$$

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and obtain a long exact sequence relating $K(R) = K(\text{proj})$ and $G(R) = K(R\text{-mod})$.

However, $\text{proj}$ is never a Serre subcategory unless $R$ is such that every finitely generated $R$-module is projective, in which case the comparison is trivial!
The problem: my goal

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Immediate obstruction: this property is vital when proving that the quotient category $\mathcal{B}/\mathcal{A}$ is exact.

We take a different approach: instead of looking for an exact quotient category $\mathcal{B}/\mathcal{A}$ whose morphisms encode the vanishing of $\mathcal{A}$ on $K$-theory, we construct a *Waldhausen* category structure on $\mathcal{B}$ whose weak equivalences encode the vanishing of $\mathcal{A}$ on $K$-theory.
Abelian, exact, and Waldhausen categories

**Abelian categories:** they all behave like categories of modules over a ring. Every map has a kernel and a cokernel, and we have short exact sequences.

Prototype example: the category of abelian groups.

**Exact categories:** they are nicely behaved full subcategories $E \subseteq A$ of an abelian category. Since we don't have all objects, some maps in $E$ won't have kernel/cokernel in $E$, and so we have a restricted class of short exact sequences.

Prototype example: the category of projective modules over a ring.

The monos (epis) that do have a cokernel (kernel) in the exact category are called *admissible*. 
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**Waldhausen categories:** they are pointed categories, together with distinguished classes of morphisms (cofibrations and weak equivalences) + axioms.

**Example:** exact (and abelian) categories can be seen as Waldhausen categories, with cofibrations = admissible monos, and weak equivalences = isos.

Prototype example: chain complexes, with cofibrations = degreewise monos, and weak equivalences = quasi-isos.
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Cotorsion pairs

As in abelian categories, one can construct the Yoneda bifunctor $\operatorname{Ext}^1_{\mathcal{E}}$ in any exact category $\mathcal{E}$:

$\operatorname{Ext}^1_{\mathcal{E}}(A, B)$ is the abelian group of equivalence classes of extensions

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**Definition: cotorsion pair**

A **cotorsion pair** in an exact category $\mathcal{E}$ is a pair $(\mathcal{P}, \mathcal{I})$ of two classes of objects that are the orthogonal complement of each other with respect to the $\text{Ext}^1_{\mathcal{E}}$ functor.
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(i) $P \in \mathcal{P}$ if and only if $\text{Ext}^1_{\mathcal{E}}(P, I) = 0$ for every $I \in \mathcal{I}$,
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Interpretation: $\text{Ext}^1(P, \mathcal{I}) = 0$ iff for every epi $g$ with $\ker g \in \mathcal{I}$,

\[
\begin{array}{c}
X \\
\rightarrow \\
\downarrow g \\
\quad g
\end{array}
\]

\[
\begin{array}{c}
P \\
\rightarrow
\end{array}
\]

\[
\begin{array}{c}
Y
\end{array}
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Theorem (Hovey)

Given an abelian category $\mathcal{A}$, there is a one-to-one correspondence

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\{ \text{pairs of cotorsion pairs} \ (C, \mathcal{F} \cap \mathcal{Z}), (C \cap \mathcal{Z}, \mathcal{F}) \} \leftrightarrow \{ \text{abelian model structures on } \mathcal{A} \}
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Inspired by Hovey’s result, we study the relation between cotorsion pairs and Waldhausen categories.
Our main result explains how to produce a Waldhausen category from a cotorsion pair and a chosen subcategory $\mathcal{Z}$ which, in line with our original motivation, will form the class of acyclic objects.
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**Theorem (S.)**

Let $\mathcal{E}$ be an exact category, and $\mathcal{C}$, $\mathcal{Z}$ two full subcategories of $\mathcal{E}$ such that $\mathcal{Z}$ is closed under extensions and cokernels of monos, and $\mathcal{C}$ is part of a complete cotorsion pair $(\mathcal{C}, \mathcal{C}^\perp)$. Assume also that $\mathcal{C}^\perp \subseteq \mathcal{Z}$. 
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- the acyclic objects are precisely the objects of \( \mathcal{Z} \),
- weak equivalences satisfy 2-out-of-3 iff \( \mathcal{Z} \) has 2-out-of-3 for short exact sequences.
Localisation theorem

**Localization Theorem (S.)**

Let $\mathcal{B}$ be an exact category with enough injectives, and $\mathcal{A} \subseteq \mathcal{B}$ a full subcategory having 2-out-of-3 for short exact sequences and containing all injectives.

Then there exists a Waldhausen structure on $\mathcal{B}$ with admissible monos as cofibrations, denoted $(\mathcal{B}, w\mathcal{A})$, s.t. $K(\mathcal{A}) \to K(\mathcal{B}) \to K(\mathcal{B}, w\mathcal{A})$ is a homotopy fiber sequence.
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Sketch: Apply the previous theorem to construct a Waldhausen category $(\mathcal{B}, w_\mathcal{A})$ from the cotorsion pair $(\mathcal{B}, \text{inj})$. 
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Extension + 2-out-of-3 + factorizations mean we can apply Schlichting’s cylinder-free version of Waldhausen’s fibration theorem to get a homotopy fiber sequence

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Our construction is also universal in the following sense:

\[ \text{Theorem (S.)} \]

The functor \( B \rightarrow (\mathbb{B}, w_A) \) is universal among exact functors \( F : B \rightarrow C \) such that \( 0 \rightarrow FA \) is a weak equivalence for each \( A \in A \), where \( C \) is a Waldhausen category satisfying extension and 2-out-of-3.
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Examples

We can use our cotorsion pair machinery to recover familiar Waldhausen categories, like the one for chain complexes and quasi-isomorphisms.
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More interestingly, we can use the new set of constraints in our Localization theorem to compare $K(R) = K(\text{proj})$ and $G(R) = K(\text{R-mod})$ for certain classes of rings.
Examples

Let $R$ be quasi-Frobenius (a ring such that $\text{proj} = \text{inj}$). In this case, we can take $\mathcal{A} = \text{proj} = \text{inj}$, $\mathcal{B} = R\text{-mod}$ and get a Waldhausen structure on $R\text{-mod}$, $(R\text{-mod}, w_{\text{proj}})$ with monos as cofibrations, and projective-injectives as acyclic objects.

Furthermore, our Localization theorem yields a homotopy fiber sequence $K(R) \rightarrow K(R\text{-mod}) \rightarrow K(R\text{-mod}, w_{\text{proj}})$.
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Some quasi-Frobenius rings: $\mathbb{Z}/n\mathbb{Z}$, $\mathbb{k}[G]$ for $\mathbb{k}$ any field and $G$ a finite group, or any finite dimensional Hopf algebra.
Examples

Let $R$ be an Artin algebra that is also a Gorenstein ring, and let \( \text{CM} \) denote the class of maximal Cohen-Macaulay modules.

Fact: every $R$-module admits a finite resolution by objects in $\text{CM}$. Then, $K(\text{CM}) \cong G(R)$ by Quillen's Resolution theorem, and this fiber sequence again compares $K(R)$ and $G(R)$.
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Thanks for your time!
