The Kapustin–Witten equations on ALE and ALF gravitational instantons

Ákos Nagy1 · Gonçalo Oliveira2

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Abstract
We study solutions to the Kapustin–Witten equations on ALE and ALF gravitational instantons. On any such space and for any compact structure group, we prove asymptotic estimates for the Higgs field. We then use it to prove a vanishing theorem in the case when the underlying manifold is \( \mathbb{R}^4 \) or \( \mathbb{R}^3 \times S^1 \) and the structure group is SU(2).

Keywords Kapustin–Witten equation · Gravitational instantons · Complex monopoles

Mathematics Subject Classification 53C07 · 58D27 · 58E15 · 70S15

1 Introduction

Background

Let us fix a smooth, oriented, Riemannian 4-manifold, \((M, g)\). Let \( \Lambda^*_\mathbb{C} M = \Lambda^* M \otimes \mathbb{C} \) be the complexified exterior algebra bundle. Let \( G \) be a compact Lie group and \( P \to M \) a smooth principal \( G \)-bundle over \( M \). Let \( G_\mathbb{C} \) be the complex form of \( G \). We then have a principal \( G_\mathbb{C} \)-bundle \( P_\mathbb{C} = P \times_G G_\mathbb{C} \), defined via the adjoint action of \( G \) on \( G_\mathbb{C} \). The Hodge star operator \( * \) can be extended in two inequivalent ways to \( \Lambda^*_\mathbb{C} M \otimes g_P \cong \Lambda^* M \otimes g_{P_\mathbb{C}} \), either as a complex linear operator or as a conjugate linear operator. In
this paper, we consider complexified instantons using the conjugate linear extension. We investigated the complex linear extension in [13].

Let us denote the conjugate linear extension of the Hodge star operator by $\hat{\star}$. The Kapustin–Witten equations can be viewed as complexified self-duality equations as follows: Let $(\nabla, \Phi)$ be a pair consisting of a connection on $P_C$ and a section of $\Lambda^1 \otimes g_{P_C}$ (the Higgs field). Then $\nabla^C := \nabla + i \Phi$ is a connection on $P_C$. In [7], Kapustin and Witten introduced a family of complexified self-duality equations, parametrized by $\theta \in \mathbb{R}$, as

$$\hat{\star} \left( e^{i\theta} F_{\nabla^C} \right) = e^{i\theta} F_{\nabla^C}. \quad (1.1)$$

We recommend [2] for an introduction to the Kapustin–Witten equations.

Note that when $\Phi = 0$ everywhere and $2\theta \equiv 0 \pmod{\pi}$, Eq. (1.1) reduces to the classical self-duality equation on $P$. As is standard in complex gauge theory, we break the structure group down from $G_C$ to $G$ by adding the Coulomb type equation $d^*_\nabla \Phi = 0$. Since $g_{P_C}$ has a canonical real structure, one can separate the real and imaginary parts of Eq. (1.1) and get the following system of equations:

$$\cos(\theta) \left( F_{\nabla} - \frac{1}{2} [\Phi \wedge \Phi] \right)^- - \sin(\theta) d^-_{\nabla} \Phi = 0, \quad (1.2a)$$
$$\sin(\theta) \left( F_{\nabla} - \frac{1}{2} [\Phi \wedge \Phi] \right)^+ + \cos(\theta) d^+_{\nabla} \Phi = 0, \quad (1.2b)$$
$$d^*_\nabla \Phi = 0. \quad (1.2c)$$

Equations (1.2a) to (1.2c) are called the $\theta$-Kapustin–Witten equations. When $\theta = \frac{\pi}{4}$, Eqs. (1.2a) and (1.2b) can be rewritten as the following single equation:

$$F_{\nabla} = \ast d_{\nabla} \Phi + \frac{1}{2} [\Phi \wedge \Phi]. \quad (1.3)$$

There is a Yang–Mills–Higgs type energy functional corresponding to the Kapustin–Witten Eq. (1.3). This functional, which we call the Kapustin–Witten energy is

$$\mathcal{E}_{KW}(\nabla, \Phi) = \int_M \left( |F_{\nabla}|^2 + |\nabla \Phi|^2 + \frac{1}{4} |[\Phi \wedge \Phi]|^2 \right) \text{vol.} \quad (1.4)$$

Similarly to instantons, solutions to the Kapustin–Witten Eq. (1.3) with finite Kapustin–Witten energy are, at least formally, absolute minimizers of (1.4). When $M$ is closed and $2\theta \not\equiv 0 \pmod{\pi}$, then all solutions to the $\theta$-Kapustin–Witten Eqs. (1.2a) to (1.2c) satisfy that $\nabla$ is flat, $\Phi$ is $\nabla$-parallel, and $[\Phi \wedge \Phi]$ vanishes identically; cf. [2, Corollary 3.3].

Witten conjectures that the moduli spaces of Eq. (1.3) have applications to low dimensional topology; cf. [19,20]. Related work has been done recently by, for example, Taubes [15,16], Mazzeo and Witten [11,12], He and Mazzeo [4], and He and Walpuski [5].
In this paper, we consider finite energy solutions to the $\theta$-Kapustin–Witten Eqs. (1.2a) to (1.2c) on certain noncompact, complete, Ricci-flat, Riemannian 4-manifolds, called ALE and ALF gravitational instantons.

Let us, briefly, introduce these classes spaces: Let $(M, g)$ be a (noncompact), smooth, and oriented Riemannian 4-manifold with Levi–Civita connection denoted by $\nabla^{LC}$. For all $R > 0$ and $x_0 \in M$, let $B_R(x_0) \subset M$ be the (closed) geodesic ball of radius $R$ around $x_0$, and let $S_R(x_0) := \partial B_R(x_0)$.

**Definition 1.1 (ALE and ALF gravitational instantons)** Let $(M, g)$ be as above and fix $x_0 \in M$. Let $Y$ be a compact 3-manifold, which is a $\mathbb{T}^k$-fibration over a closed base $B$ with projection $\pi_Y$, where $k = 0$, or 1, together with a connection on $Y$ when $k = 1$, and a metric $g_B$ on $B$. Assume that the end of $X$ is modeled on $Y \times [R_0, \infty)$, that is, there exists $R_0 > 0$, such that $B_{R_0}(x_0)$ is a smooth, compact manifold with boundary and $M - B_{R_0}(x_0)$ is diffeomorphic to $Y \times [R_0, \infty)$. Moreover, there exists a diffeomorphism $\phi : M - B_{R_0}(x_0) \rightarrow Y \times [R_0, \infty)$, such that for $j = 0, 1, 2$, we have

$$\lim_{R \to \infty} R^j \left\| \left( \nabla^{LC} \right)^j \left( g - \phi^*(dR^2 + g_T^k + R^2 \pi_Y^*(g_B)) \right) \right\|_{L^\infty(S_R)} = 0.$$

We call $(M, g)$ Asymptotically Locally Euclidean (ALE), if $k = 0$, and Asymptotically Locally Flat (ALF), if $k = 1$. An ALE or ALF 4-manifold is called a gravitational instanton, if it is Ricci-flat.

**Remark 1.2** Note that we do not require $(M, g)$ to be hyperkähler, or even complex. For example, the Euclidean–Schwarzschild manifold can be considered.

The prototypical example of an ALE gravitational instanton is $\mathbb{R}^4$ with its canonical flat metric. Other examples are given by the construction of Kronheimer [8].

The prototypical example of an ALF gravitational instanton is $\mathbb{R}^3 \times S^1$ with its canonical flat metric. Other important examples include the Euclidean–Schwarzschild, the multi-Taub–NUT, and the Atiyah–Hitchin manifolds. Many more (hyperkähler) examples are given via the Gibbons–Hawking construction [3].

**Main results**

Our first main theorem is an asymptotic bound on the Higgs field, $\Phi$, when the underlying manifold, $(M, g)$ is an ALE or ALF gravitational instanton. The proof uses ideas of [6, Theorem 10.3] adapted to the 4-dimensional setting and to curved geometries.

**Main Theorem 1** Let $(\nabla, \Phi)$ be a finite energy solution to the $\theta$-Kapustin–Witten Eqs. (1.2a) to (1.2c) on an ALE or ALF gravitational instanton $(M, g)$. Then there is a constant $c \geq 0$, such that

$$\lim_{R \to \infty} \inf_{S_R} |\Phi| = \lim_{R \to \infty} \sup_{S_R} |\Phi| = \lim_{R \to \infty} \sup_{M_R} |\Phi| = c. \quad (1.5)$$

Furthermore, if $c = 0$, then $\Phi = 0$ everywhere.
Combining Main Theorem 1 with [14, Theorem 1.1], we prove the following result.

**Corollary 1.3** Let \((\nabla, \Phi)\) be a finite energy solutions to the \(\theta\)-Kapustin–Witten Eqs. (1.2a) to (1.2c) on \(M = \mathbb{R}^4\) or \(\mathbb{R}^3 \times S^1\) with its flat metric, and let \(G = SU(2)\). Then \(\nabla \Phi = [\Phi \wedge \Phi] = 0\) and one the following statements are true:

1. If \(\nabla\) is flat, then \(\Phi\) is \(\nabla\)-parallel, and \([\Phi \wedge \Phi] = 0\).
2. If \(\nabla\) is not flat, then \(2\theta \equiv 0 \pmod{\pi}\), \(\nabla\) is an (anti-)instanton, and \(\Phi = 0\).

We conjecture that Corollary 1.3 holds on an arbitrary ALE or ALF gravitational instanton.

**Organization of the paper**

In Sect. 2, we compute second-order equations that are satisfied by solutions to the \(\theta\)-Kapustin–Witten Eqs. (1.2a) to (1.2c). While these equations are known in the literature, we include their proof for clarity and completeness. These are used in the proofs of Main Theorem 1. In Sect. 3, we study the analytic properties of the Kapustin–Witten energy density. In Sect. 4, we recall a few useful properties of ALE and ALF gravitational instantons. Finally, in Sect. 5 we present the proofs of Main Theorem 1 and Corollary 1.3.

**2 The second-order Kapustin–Witten equations**

For the next lemma, let \((x_1, x_2, x_3, x_4)\) be a local, normal chart on \(M\) at an arbitrary point, and let

\[
\mathbf{j}_\Phi := \sum_{i=1}^{4} \left[ \nabla \Phi_i, \Phi_i \right] \in \Gamma \left( \Lambda^1 \otimes g_P \right).
\]

be the supercurrent generated by \(\Phi\).

**Lemma 2.1** Let \((M, g)\) be any Riemannian 4-manifold, \(P \rightarrow M\) a principal \(G\)-bundle, and regard the Ricci tensor of \((M, g), Ric_g\), as an endomorphism of \(\Lambda^1 \otimes g_P\). If \((\nabla, \Phi)\) is a solution to the \(\theta\)-Kapustin–Witten Eqs. (1.2a) to (1.2c) on \(P \rightarrow M\), it also satisfies the following system of second-order equation:

\[
\nabla^* \nabla \Phi = -\frac{1}{2} \ast \left( \ast (g_\ast (\Phi \wedge \Phi)) \wedge \Phi \right) - Ric_g (\Phi), \quad (2.1a)
\]

\[
d^* \mathbf{F}_\nabla = \mathbf{j}_\Phi. \quad (2.1b)
\]

**Remark 2.2** When \((M, g)\) is Ricci-flat, then Eqs. (2.1a) and (2.1b) are the Euler–Lagrange equations of the Kapustin–Witten energy (1.4).

**Remark 2.3** In Main Theorem 1, one can replace the condition that \((\nabla, \Phi)\) is a solution to the \(\theta\)-Kapustin–Witten Eqs. (1.2a) to (1.2c) with the assumption that it only solves
the second-order Kapustin–Witten Eqs. (2.1a) and (2.1b), and the conclusions still hold, while in Corollary 1.3, one can now conclude that $\nabla \Phi = [\Phi \wedge \Phi] = 0$ and $\nabla$ is a Yang–Mills connection.

**Proof** Using the Weitzenböck formula and $d^*_\nabla \Phi = 0$, we get

$$\nabla^* \nabla \Phi = d^*_\nabla d\nabla \Phi - *[(*F_\nabla) \wedge \Phi] - \text{Ric}_g(\Phi). \quad (2.2)$$

We now present the computation when $2\theta \neq 0 \pmod{\pi}$, thus $t = \tan(\theta)$ is defined and nonzero. (When $2\theta \equiv 0 \pmod{\pi}$, the computation is similar, and simpler.) In this case we may rewrite Eqs. (1.2a) and (1.2b) as

$$d_{\nabla}^\pm \Phi = \mp t^{\pm 1} \left( F_\nabla - \frac{1}{2}[\Phi \wedge \Phi] \right)^\pm. \quad (2.3)$$

Then, writing $d_{\nabla}^\pm \Phi = \frac{1}{2} (d\nabla \Phi \pm *d\nabla \Phi)$, and adding these two equations we find

$$d\nabla \Phi = \frac{t^{-1} - t}{t + t^{-1}} d\nabla \Phi - \frac{2}{t + t^{-1}} * \left( F_\nabla - \frac{1}{2}[\Phi \wedge \Phi] \right). \quad (2.4)$$

On the other hand, multiplying Eq. (2.3) by $t^{\mp 1}$ and adding up the resulting equations yields (after dividing by $t + t^{-1}$)

$$d\nabla \Phi = \frac{t^{-1} - t}{t + t^{-1}} * d\nabla \Phi - \frac{2}{t + t^{-1}} * \left( F_\nabla - \frac{1}{2}[\Phi \wedge \Phi] \right), \quad (2.5)$$

which by rearranging can also be read as

$$F_\nabla = \frac{1}{2}[\Phi \wedge \Phi] + \frac{t^{-1} - t}{2} d\nabla \Phi - \frac{2}{t + t^{-1}} * d\nabla \Phi. \quad (2.6)$$

Thus, using Eqs. (2.5) and (2.6), together with the Bianchi identity $d^*_\nabla * F_\nabla = 0$, we get

$$d^*_\nabla d\nabla \Phi = \frac{t^{-1} - t}{t + t^{-1}} d^*_\nabla d\nabla \Phi - \frac{2}{t - t^{-1}} d^*_\nabla \Phi * \left( F_\nabla - \frac{1}{2}[\Phi \wedge \Phi] \right)$$

$$= - \frac{t^{-1} - t}{t + t^{-1}} * \left[ F_\nabla \wedge \Phi \right] - \frac{2}{t + t^{-1}} * \left[ d\nabla \Phi \wedge \Phi \right]$$

$$= * \left[ * \left( F_\nabla - \frac{1}{2}[\Phi \wedge \Phi] \right) \wedge \Phi \right].$$

where in the last equality we replaced $F_\nabla$ using Eq. (2.6) and the Jacobi identity $[[\Phi \wedge \Phi] \wedge \Phi] = 0$. Combining the above equation with Eq. (2.2) concludes the proof of Eq. (2.1a).
Now we prove Eq. (2.1b):

\[
\begin{align*}
d^*_\nabla F_\nabla &= \frac{1}{2}d^*_\nabla [\Phi \wedge \Phi] + \frac{t^{-t^{-1}}}{2}d^*_\nabla \Phi + \frac{t^{+t^{-1}}}{2} * [F_\nabla \wedge \Phi] \\
&= j_\Phi + *([d_\nabla \Phi] \wedge \Phi] + \frac{t^{-t^{-1}}}{2} * \left([F_\nabla - \frac{1}{2}(\Phi \wedge \Phi)] \wedge \Phi \right) + \frac{t^{+t^{-1}}}{2} * [F_\nabla \wedge \Phi] \\
&= j_\Phi + *([d_\nabla \Phi] \wedge \Phi] + \frac{t^{-t^{-1}}}{2} * \left([\frac{1}{2}d_\nabla \Phi + \frac{t^{-t^{-1}}}{2} * d_\nabla \Phi] \wedge \Phi \right) \\
&+ \frac{t^{+t^{-1}}}{2} * \left([\frac{1}{2}d_\nabla \Phi - \frac{t^{+t^{-1}}}{2} * d_\nabla \Phi] \wedge \Phi \right) \\
&= j_\Phi,
\end{align*}
\]

which completes the proof. \hfill \Box

### 3 The Kapustin–Witten energy density

The Kapustin–Witten energy \((1.4)\) is the integral of

\[
e_{KW} = |F_\nabla|^2 + |\nabla \Phi|^2 + \frac{1}{q}||\Phi \wedge \Phi||^2 \geq 0. \quad (3.1)
\]

We call \(e_{KW}\) the Kapustin–Witten energy density. First we prove a decay result for \(e_{KW}\).

**Proposition 3.1** Then there is a positive number \(C = C(M, g, G)\), such that, if \((\nabla, \Phi)\) is a smooth solution to second-order Eqs. (2.1a) and (2.1b) with finite energy, then \(e_{KW}\) decays uniformly to zero at infinity.

**Proof** Let \(r_0 = \min([\text{inj}(M, g)/2, 1])\). Note that \(r_0 > 0\), since \((M, g)\) is either ALE or ALF. Let \(f = \sqrt{e_{KW}} \in L^2(B_{2r_0}(x))\). By the Weitzenböck formula and the Ricci-flatness of \((M, g)\), we have (in normal coordinates)

\[
\nabla^* \nabla (\nabla \Phi) = \nabla (\nabla^* \nabla \Phi) + \sum_{i, j=1}^4 \left(2[F_{ik}, \nabla_k \Phi_j] + [(d^*_\nabla F_\nabla)_i, \Phi_j] \right) \ dx^i \otimes dx^j.
\]

Now further using Eqs. (2.1a) and (2.1b), we get that \(\Delta |\nabla \Phi|^2 \leq C(f^2 + f) - |\nabla^2 \Phi|\). Similar computations for \(|F_\nabla|^2\) and \(||\Phi \wedge \Phi||^2\), and Kato’s inequality yields that \(f\) weakly satisfies (after maybe redefining \(C\)) the following inequality:

\[
\Delta f \leq C(f^2 + f)
\]

Hence, by [17, Theorem 3.2], using the notations of the reference

\[
f = \sqrt{e_{KW}}, \quad b = f + 1, \quad q = 2 + \|e_{KW}\|_{L^{\infty}(B_{2\text{dist}(x, r_0)}(x))}^{-1}, \quad n = 4, \quad \gamma = 1, \quad a_0 = r_0, \quad \text{and} \quad a = \frac{1}{2}a_0.
\]
we get that, for some other positive number $C' = C'(M, g)$:

$$e_{KW}(x) \leq C' \|e_{KW}\|_{L^1(B_{r_0}(x), g)}. \quad (3.2)$$

By the finiteness of the energy, we get that the integral of $e_{KW}$ on $B_{r_0}(x)$ decays uniformly to zero as $\text{dist}(x, x_0) \to \infty$, and thus so does $e_{KW}(x)$, which concludes the proof. \qed

**Corollary 3.2** There is a positive number $C = C(M, g, G)$, such that

$$\forall p \in [1, \infty) \cup \{\infty\} : \|e_{KW}\|_{L^p(M, g)} \leq C \|e_{KW}\|_{L^1(M, g)}.$$

**Proof** By inequality (3.2), we get that $e_{KW} \in L^\infty(M)$ and, in fact,

$$\|e_{KW}\|_{L^\infty(M)} \leq C' \|e_{KW}\|_{L^1(M, g)}.$$

Hence, for any $p \geq 1$, using Hölder’s inequality, we get

$$\|e_{KW}\|_{L^p(M, g)} \leq \|e_{KW}\|_{L^\infty(M)}^{(p-1)/p} \|e_{KW}\|_{L^1(M, g)}^{1/p} \leq (C')^{(p-1)/p} \|e_{KW}\|_{L^1(M, g)} \leq \max\{C', 1\} \|e_{KW}\|_{L^1(M, g)},$$

thus $\|e_{KW}\|_{L^p(M, g)}$ is finite, and can be bounded by a constant independent of $p$. \qed

### 4 On the geometry of ALE and ALF gravitational instantons

In this section, we recall a few geometric properties of ALE and ALF gravitational instantons that will be used in the proof of Main Theorem 1.

For quantities $A, B$ the relation $A \lesssim B$ is equivalent to $A = O(B)$, while $A \sim B$ is equivalent to $A \lesssim B$ and $B \lesssim A$. Furthermore, let $k = 0$, if $(M, g)$ is ALE, and $k = 1$, if $(M, g)$ is ALF.

Let $x \in M, \rho$ be the radial coordinate on $T_x M$ and define $m \in C^\infty(T_x M; \mathbb{R}^\times)$ via

$$m := \exp^*_x(\text{vol}_g) \frac{1}{\rho^3 \text{vol}_g^{\times 3}}.$$  

From the Laplacian Comparison Theorem—see, for example [18, Proposition 20.7]—we have

$$\partial_\rho (\rho^{-3} m) \leq 0,$$

away from the cut locus. Let us now recall the local and global versions of the Gromov–Bishop Theorem, applied to the case of ALE and ALF gravitational instantons.

**Theorem 4.1** (The Local Bishop–Gromov Theorem for $(M, g)$) As $\rho \to 0$, the quantity $\rho^{-3} m$ converges to a constant, and as $\rho \to \infty$, we have $m \lesssim \rho^{3-k}$.

**Theorem 4.2** (The Global Bishop–Gromov Theorem for $(M, g)$) The quotient, $r^{-4} \text{Vol}(B_r(x))$, is a nonincreasing function of $r$ and converges to a constant as $r \to 0$. As $r \to \infty$, by Definition 1.1, we have $\text{Vol}(B_r(x_0)) \sim r^{4-k}.$
Theorem 4.2 and [10, Theorem 5.2] yield the following Lemma.

**Lemma 4.3** There is a smooth, positive Green’s function, $G$ on $(\mathcal{M}, g)$, and

$$\forall x \in \mathcal{M} : \frac{\text{dist}(x, \cdot)^2}{1 + \text{dist}(x, \cdot)^k} G(x, \cdot) \in L^\infty (\mathcal{M} - \{x\}).$$

Finally, we present (and prove) the appropriate Hölder–Sobolev Embedding.

**Lemma 4.4** (The Hölder–Sobolev Embedding Theorem on $(\mathcal{M}, g)$) For all $p \in (4, \infty)$, there are positive numbers, $C_{HS}$ and $R_0$, such that for any $f$ with $|\nabla f| \in L^p(\mathcal{M}, g)$ and $x, y \in \mathcal{M}$ with $\text{dist}(x, y) \geq R_0$, we have

$$|f(x) - f(y)| \leq C_{HS} \text{dist}(x, y)^{1-\frac{4-k}{p}} \|\nabla f\|_{L^p(\mathcal{M}, g)}.$$

**Proof** We follow the proof as given in, for example, [6, Corollary 2.7]. Let $f$ be as in the statement, $x, y \in \mathcal{M}$ arbitrary, let $w$ be the midpoint of a geodesic connecting $x$ and $y$, and $R := \frac{1}{3} \text{dist}(x, y)$. Then for any $z \in \mathcal{M}$, we have $|f(x) - f(y)| \leq |f(x) - f(z)| + |f(z) - f(y)|$. If we integrate both sides of this inequality with respect to $z \in B_R(w)$, then we get

$$\text{Vol}(B_R(w)) |f(x) - f(y)| = \int_{B_R(w)} (|f(x) - f(z)| + |f(z) - f(y)|) \text{vol}(z),$$

(4.1)

and using $\gamma_{x, z}$ to denote the arc-length parametrized geodesic connecting $x$ to $z$, so $|\dot{\gamma}_{x, z}(t)| = 1$, then

$$\int_{B_R(w)} |f(x) - f(z)| \text{vol}(z) \leq \int_{B_R(w)} \int_0^{\text{dist}(x, z)} |\partial_t f(\gamma_{x, z}(t))| \text{d}t \text{vol}(z) \leq \int_{B_R(w)} \int_0^{\text{dist}(x, z)} |\nabla f(\gamma_{x, z}(t))| |\dot{\gamma}_{x, z}(t)| \text{d}t \text{vol}(z) \leq \int_0^{3R} \int_{B_R(w)} |\nabla f(\gamma_{x, z}(t))| \text{vol}(z) \text{d}t,$$

where we have used the triangle inequality to get that $\text{dist}(x, z) \leq 3R$. Now we write

$$\text{vol}(z) = m(z)d\rho \wedge \text{vol}_{\mathbb{S}^3},$$
and use inequality (4.1) to deduce that
\[ m(z) \leq \frac{\text{dist}(x, z)^3}{\text{dist}(x, \gamma_{x,z}(t))^3} m(\gamma_{x,z}(t)) \lesssim R^3 \frac{m(\gamma_{x,z}(t))}{\text{dist}(x, \gamma_{x,z}(t))^3}. \]

This, together with the fact that \( \gamma_{x,z}(t) \in B_{3R}(x) \) for \( z \in B_{R}(w) \) and all \( t \in [0, 3R] \) yields
\[
\int_{B_{R}(w)} |f(x) - f(z)| \text{vol}(z) \lesssim R^4 \int_{B_{3R}(x)} |\nabla f(\tilde{z})| \text{dist}(x, \tilde{z})^{-3} \text{vol}(\tilde{z}) \lesssim R^4 \|\text{dist}(x, \cdot)^{-3}\|_{L^{q}(B_{3R}(x), g)} \|\nabla f\|_{L^{p}(M, g)},
\]

where we have used Hölder’s inequality with conjugate exponents \( p \) and \( q \). Now, for \( \text{dist}(x, \cdot)^{-3} \) to be in \( L^{q}_{loc}(M, g) \) we must have \( p > 4 \) in which case, for \( R \gg 1 \) we find, using Theorem 4.1, that
\[
\|\text{dist}(x, \cdot)^{-3}\|_{L^{p}(B_{3R}(x), g)} = \left( \int_{B_{R}(x)} \text{dist}(x, \cdot)^{-3 p \rho^{-k}} \text{vol} + \int_{B_{3R}(x) - B_{R}(x)} \text{dist}(x, \cdot)^{-3 p \rho^{-k}} \text{vol} \right)^{\frac{p-1}{p}} \lesssim \left( 1 + \int \rho^{-3 p \rho^{-k}} \rho^{3-k} d\rho \right)^{\frac{p-1}{p}} \lesssim R^{(4-k) \frac{p-1}{p} - 3} = R^{1-k - \frac{4-k}{p}}.
\]

Using a similar trick to control the integral of \( |f(z) - f(y)| \) and inserting into Eq. (4.1) we find that for \( R \gg 1 \)
\[
|f(x) - f(y)| \lesssim \frac{R^4}{\text{vol}(B_{R}(w))} R^{1-k - \frac{4-k}{p}} \|\nabla f\|_{L^{p}(M, g)} \lesssim R^{1 - \frac{4-k}{p}} \|\nabla f\|_{L^{p}(M, g)},
\]

where we have used Theorem 4.2 to bound the volume of the balls.

\[ \square \]

5 The proofs of Main Theorem 1 and Corollary 1.3

Let \( (\nabla, \Phi) \) be a finite energy solution to the second-order Eqs. (2.1a) and (2.1b) on an ALE of ALF gravitational instanton, \((M, g)\). First of all, \( \frac{1}{2} |\Phi|^2 \) is subharmonic, because
\[
\Delta \left( \frac{1}{2} |\Phi|^2 \right) = \text{Re} \left( \langle \Phi, \nabla^{*} \nabla \Phi \rangle \right) - |\nabla \Phi|^2 = -\frac{1}{4} [\Phi \wedge \Phi]^2 - |\nabla \Phi|^2 \leq 0.
\]

Let \( G \) be the Green’s function of \((M, g)\) from Lemma 4.3. We then define a nonnegative function
\[
w(x) := \int_{M} G(x, y) \left( |\nabla \Phi(y)|^2 + \frac{1}{4} [\Phi(y) \wedge \Phi(y)]^2 \right) \text{vol}(y).
\]

(5.1)
We now can prove Main Theorem 1.

**Proof of Main Theorem 1** The proof below is inspired by [6, Theorem 10.3].

Let \( r := \frac{1}{2} \text{dist}(x, x_0) \). Using that \(|\nabla \Phi|^2 + \frac{1}{4} |[\Phi \wedge \Phi]|^2 \leq e_{\text{KW}}\), together with Lemma 4.3 and Corollary 3.2, we have for all \( x \in M \):

\[
0 \leq w(x) \leq \int_M G(x, y)e_{\text{KW}}(y)\text{vol}(y)
= \left( \int_{M-B_r(x)} + \int_{B_r(x)} \right) G(x, y)e_{\text{KW}}(y)\text{vol}(y)
\leq r^{-1} \|e_{\text{KW}}\|_{L^1(M, g)} + \|e_{\text{KW}}\|_{L^2(B_r(x), g)}.
\]

Thus \( w \) is bounded, and furthermore, \( w(x) \) converges uniformly to zero as \( \text{dist}(x, x_0) \to \infty \). The same argument shows that the integrand in Eq. (5.1) is absolutely convergent, and thus the smoothness of \( w \) also follows. Now, by the construction of \( w \), the function \( h := w + \frac{1}{2} |\Phi|^2 \) is harmonic. Next, we show that \( h = o(\text{dist}(\cdot, x_0)) \).

For each \( R > 0 \), define

\[
m(R) := \sup_{x \in B_R(x_0)} |\Phi(x)|^2.
\]

Since \( |\Phi|^2 \) is subharmonic, the supremum is achieved at some point \( \tilde{x} \), with \( \text{dist}(\tilde{x}, x_0) = R \), that is \( m(R) = |\Phi(\tilde{x})|^2 \). Furthermore, by Kato’s inequality, \(|d(|\Phi|^2)| \leq 2|\nabla \Phi||\Phi| \leq 2e_{\text{KW}}^1 |\Phi|\), and thus, using Lemma 4.4 on \( B_R(x_0) \) with \( R \gg 1 \), we get

\[
|\Phi(\tilde{x})|^2 - |\Phi(x_0)|^2 \lesssim R^{-\frac{4+k}{p}} \|e_{\text{KW}}^{1/2}\|_{L^p(B_R(x_0), g)} \lesssim R^{-\frac{4+k}{p}} \|e_{\text{KW}}\|_{L^2(B_R(x_0), g)} \sqrt{m(R)}
\lesssim R^{-\frac{4+k}{p}} \sqrt{m(R)}.
\]

Since \( k = 0, \) or \( 1, \) we can chose, for example, \( p = \frac{3(4-k)}{2} > 4, \) and find that

\[
m(R) \leq |\Phi(x_0)|^2 + |\Phi(\tilde{x})|^2 - |\Phi(x_0)|^2 \lesssim |\Phi(x_0)|^2 + R^{1/3} \sqrt{m(R)}
\leq |\Phi(x_0)|^2 + \frac{1}{2} R^{2/3} + \frac{1}{2} m(R),
\]

and thus (for \( R \) large enough) \( m(R) = O \left( R^{2/3} \right) \), which shows that \( |\Phi|^2 \) grows strictly slower than linearly, and thus \( h = w + \frac{1}{2} |\Phi|^2 \) is harmonic and \( o(\text{dist}(\cdot, x_0)) \). Therefore, it must be constant by the gradient estimate of Cheng and Yau in [1, Section 4], which is nicely summarized in the form we need in [9, Lemma 1.5]. Let this constant be \( c \). Clearly, \( c \geq 0 \) and \( |\Phi|^2 = 2c - w \). Since \( w \) is nonnegative and converges uniformly to zero at infinity, this proves Eq. (1.5). Finally, since \( w \) is nonnegative, if \( h = c \) is zero, then so is \( \Phi \), which completes the proof of Main Theorem 1. \( \square \)
Remark 5.1 In the cases of the gravitational instantons of type ALG and ALH the proof of Main Theorem 1 given above does not work, because those manifolds do not satisfy the conditions of [10, Theorem 5.2] and hence do not have positive Green’s functions.

Finally, we prove Corollary 1.3. This is a vanishing result for finite energy Kapustin–Witten fields on $M = \mathbb{R}^4$ or $\mathbb{R}^3 \times S^1$, equipped with their flat metrics. The proof is a combination of Main Theorem 1 and [14, Theorem 1.2].

Proof of Corollary 1.3 Let $M$ be either $\mathbb{R}^4$ or $\mathbb{R}^3 \times S^1$, and equip it with its standard (flat) metric. Let $(\nabla, \Phi)$ be a finite energy solution to the $\theta$-Kapustin–Witten Eqs. (1.2a) to (1.2c) (or even just the second-order Eqs. (2.1a) and 2.1b) with structure group $G = SU(2)$.

If $M = \mathbb{R}^3 \times S^1$, then let us pull back $(\nabla, \Phi)$ to $\mathbb{R}^4$. In both cases, we get a smooth solution to Eqs. (2.1a) and (2.1b) on $\mathbb{R}^4$ with bounded $\Phi$. In particular, $|\Phi|$ has bounded average over spheres, and thus by [14, Theorem 1.2] we get that both $\nabla \Phi$ and $[\Phi \wedge \Phi]$ vanish identically, which yields the claims of Corollary 1.3 immediately. □

Remark 5.2 The only time $M = \mathbb{R}^4$ or $\mathbb{R}^3 \times S^1$, and $G = SU(2)$ were needed in the proof of Corollary 1.3 is when we used [14, Theorem 1.2]. Thus generalizations of this theorem would immediately provide generalizations of Corollary 1.3.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Cheng, S.-Y., Yau, S.-T.: Differential equations on Riemannian manifolds and their geometric applications. Commun. Pure Appl. Math. 28(3), 333–354 (1975)
2. Gagliardo, M., Uhlenbeck, K.: Geometric aspects of the Kapustin–Witten equations. J. Fixed Point Theory Appl. 11(2), 185–198 (2012)
3. Gibbons, G.W., Hawking, S.W.: Gravitational multi-instantons. Phys. Lett. 78B, 430 (1978)
4. He, S., Mazzeo, R.: The extended Bogomolny equations and generalized Nahm pole boundary condition. Geom. Topol. 23(5), 2475–2517 (2019)
5. He, S., Walpuski, T.: Hecke modifications of Higgs bundles and the extended Bogomolny equation. J. Geom. Phys. 146, 103487 (2019)
6. Jaffe, A., Taubes, C.H.: Vortices and Monopoles. Progress in Physics. Birkhauser, Boston (1980). MR614447 (82m:81051)
7. Kapustin, A., Witten, E.: Electric-magnetic duality and the geometric Langlands program. Commun. Number Theory Phys. 1(1), 1–236 (2007)
8. Kronheimer, P.B.: The construction of ALE spaces as hyper-Kahler quotients. J. Differ. Geom. 29(3), 665–683 (1989). MR992334 (90d:53055)
9. Li, P., Tam, L.-F.: Green’s fs functions, harmonic functions, and volume comparison. J. Differ. Geom. 41(2), 277–318 (1995)
10. Li, P., Yau, S.-T.: On the parabolic kernel of the Schrodinger operator. Acta Math. 156(3–4), 153–201 (1986)
11. Mazzeo, R., Witten, E.: The Nahm pole boundary condition. In: Katzarkov, L., Lupercio, E., Turrubiates, F.J. (eds.) The Influence of Solomon Lefschetz in Geometry and Topology, pp. 171–226. American Mathematical Society, Providence (2014)
12. Mazzeo, R., Witten, E.: The KW equations and the Nahm pole boundary condition with knots. Commun. Anal. Geom. 28(4), 871–942 (2020)
13. Nagy, A., Oliveira, G.: The Haydys monopole equation. Selecta Math. (N.S.) 26(4), Paper No. 58 (2020)
14. Taubes, C.H.: Growth of the Higgs field for solutions to the Kapustin–Witten equations on R4 (2017). arXiv:1701.03072
15. Taubes, C.H.: Sequences of Nahm pole solutions to the SU(2) Kapustin–Witten equations (2018). arXiv:1805.02773
16. Taubes, C.H.: The R-invariant solutions to the Kapustin–Witten equations on (0, ∞) × R² × R with generalized Nahm pole asymptotics (2019). arXiv:1903.03539
17. Uhlenbeck, K.K.: Removable singularities in Yang–Mills fields. Commun. Math. Phys. 83(1), 11–29 (1982). MR648355 (83e:53034)
18. Walpuski, T.: MTH931 Riemannian geometry II. https://walpu.ski/Teaching/RiemannianGeometry.pdf. Accessed 2019
19. Witten, E.: Khovanov homology and gauge theory. In: Proceedings of the Freedman Fest, pp. 291–308 (2012)
20. Witten, E.: More on gauge theory and geometric Langlands. Adv. Math. 327, 624–707 (2018)

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