The equilibrium points and stability of grid-connected synchronverters

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Abstract—Virtual synchronous machines are inverters with a control algorithm that causes them to behave towards the power grid like synchronous generators. A popular way to realize such inverters are synchronverters. Their control algorithm has evolved over time, but all the different formulations in the literature share the same “basic control algorithm”. We investigate the equilibrium points and the stability of a synchronverter described by this basic algorithm, when connected to an infinite bus. We formulate a fifth order model for a grid-connected synchronverter and derive a necessary and sufficient condition for the existence of equilibrium points. We show that the set of equilibrium points with positive field current is a two-dimensional manifold that can be parametrized by the corresponding pair \((P, Q)\), where \(P\) is the active power and \(Q\) is the reactive power. This parametrization has several surprising geometric properties, for instance, the prime mover torque, the power angle and the field current can be seen directly as distances or angles in the \((P, Q)\) plane. In addition, the stable equilibrium points correspond to a subset of a certain angular sector in the \((P, Q)\) plane. Thus, we can predict the stable operating range of a synchronverter from its parameters and from the grid voltage and frequency. Our stability result is based on the intrinsic two time scales property of the system, using tools from singular perturbation theory. We illustrate our theoretical results with two numerical examples.

Index Terms—Virtual synchronous machine, frequency droop, inverter, synchronverter, Park transformation, saturating integrator, singular perturbation method.

I. INTRODUCTION

Most distributed generators are connected to the utility grid via inverters that rely on various control algorithms to maintain synchronism. They usually offer no inertia, and behave as controlled current sources that produce fluctuating power. Numerous researchers are investigating how the future power grids should be controlled when inverters become dominant, offering competing control algorithms for grid-forming converters, see for instance the recent study [28]. One of the proposed approaches is to emulate the behavior of synchronous generators (SG), so that an inverter-based grid behaves like one based on SG, see for instance [4], [7], [9], [16], [19], [23], [28], [31], [36]. This has many advantages, such as backward compatibility with the current grid, well known black start and fault ride-through procedures, and well tested primary and secondary frequency and voltage support algorithms. Following [4], inverters that behave towards the utility grid like synchronous machines are called virtual synchronous machines (VSM).

One particular type of VSM are the synchronverters, introduced in [36], [37]. This type of inverter has attracted considerable attention, see for instance [1], [2], [3], [8], [23], [26], [33], [35], and the recent survey [29]. The hardware of a synchronverter is similar to that of a conventional three phase inverter (with any number of DC levels, most commonly 3), the novelty is in the control algorithm. The only hardware difference is that some fast acting energy storage (typically, capacitors) is required on the DC bus, to provide the energy pulses (both positive and negative) needed for the emulation of rotor inertia. We base our modelling on the simplified circuit diagram of a grid-connected inverter in Fig. 1. Even though the synchronverter control scheme has evolved over time, all the formulations present in the literature share the same “basic control algorithm”. We base our modelling on this basic algorithm (see Fig. 2 in Sect. [1] for more details).

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AC side in an ingenious way, leading to aGAS of the VSM connected to an infinite bus. In the context of microgrids, stability results are derived in [6], [25], [32], and the importance of accurate modeling has been discussed, among others, in [30] and in the recent review [22]. The paper [18] presents an interesting equilibrium point analysis for microgrids interfaced via solid-state transformers. This analysis is then employed to develop a power sharing algorithm between the inverters.

This paper investigates the local asymptotic stability of a VSM functioning according to the basic synchronverter algorithm, when connected to a powerful grid modelled as an infinite bus. For this purpose, we formulate a fifth order grid-connected synchronverter model. This model is an extension of the fourth order model developed and analyzed in [3], [20], where the rotor (or field) current was assumed to be constant (thus ignoring the reactive power control loop). Using advanced mathematical methods, different sufficient conditions for almost global asymptotic stability of the fourth order model were derived in [3], [20]. Here we include the field current as a state variable and we investigate the stability of the equilibrium points of the resulting fifth order system.

We derive a novel geometric representation of the fourth and fifth order models’ equilibrium points. We use extensively the mapping of equilibrium points into the power plane, where the coordinates are \( P \) (the active power) and \( Q \) (the reactive power). (In the language of differential geometry, the manifold of equilibrium points with positive field current is diffeomorphic to the power plane.) We show that, for a fixed prime mover torque and for field current values in a “reasonable” range, the image of the fourth order model equilibrium points in the power plane moves on a circle. The radius of this circle depends on the prime mover torque at the equilibrium. In the same geometric representation, we identify a stability sector for the fifth order model equilibrium points. This sector allows to determine a priori if certain reference values of active and reactive power will generate stable (or unstable) fifth order model equilibrium points.

The paper is organized as follows. In Sect. II we recover the fourth order grid-connected synchronverter model from [20], [21], and we extend it to a fifth order one, adding the field current to the state vector. In Sect. III the equilibrium points of the fourth order model are studied and the novel geometric representation is introduced. In Sect. IV we study the equilibrium points of the fifth order model and their representation in the power plane. In Sect. V we use results from Sect. III and IV to find a sufficient condition ensuring the stability of the fifth order model equilibrium points, employing singular perturbation methods developed in [15]. Based on this result, we characterize the power plane region corresponding to stable fifth order model equilibrium points. Finally, in Sect. VI we use two numerical examples to illustrate our novel geometric representation and our theoretical derivations.

II. MODELLING THE GRID-CONNECTED SYNCHRONVERTER

In this section we construct the basic fifth order model of the synchronverter, following the terminology and notation of [20], [21], [37]. Note that the paper [21] has proposed five modifications to the synchronverter algorithm from [37], to improve its stability and performance. Of these, we adopt here only the two most important ones: a substantial increase of the effective size of the filter inductors, by using virtual inductors, and the improved anti-windup field current controller.

Our analysis is based on a simplified model of a synchronverter, given in Fig. 2. This model is simplified because it does not take into account the various low-pass filters that are included to reduce high frequency noise, and it also ignores most of the saturation blocks included in the algorithm (see [5], [21]) (however, the saturating integrator contained in the field current controller is considered). We also ignore start-up procedures and various protections. Including these features would result in a high-order model that is practically impossible to analyze rigorously. Moreover, ignoring the aforementioned features does not significantly alter the steady-state behaviour of the system, making our stability analysis relevant also for higher order models.

We proceed as follows: first we recover the fourth order model from [21] (where the field current \( i_f \) was assumed to be a parameter). Then, we extend this model by including \( i_f \) as a state variable, obtaining a fifth order model.

\[ \begin{align*}
  P &= \text{formula for } T_m \\
  Q &= \text{formula for } T_m \\
  g &= \text{equations for } T_m, e \text{ and } Q \\
  \theta &= \text{field current controller} \\
  v &= \sqrt{\frac{2}{3}} V \sin \theta_g,
\end{align*} \]

where \( V \) is a positive constant or a slowly changing signal (this is the rms value of the line voltage).

Denote by \( M_f > 0 \), the peak mutual inductance between the virtual rotor winding and any one stator winding, by \( i_f \) the variable field current (or rotor current) and by \( e \) the vector...
of electromotive forces, also called the internal synchronous voltage. We rewrite [37, eq. (4)]:

\[ e = M_i f_i \omega \sin \theta - M_i \frac{d i_f}{d \tau} \cos \theta \]  

(2)

and we note that the variable current \( i_f \) governs the amplitude of \( e \). We apply the Park transformation

\[ U(\theta) = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos \theta & \cos(\theta - \frac{2\pi}{3}) & \cos(\theta + \frac{2\pi}{3}) \\ -\sin \theta & -\sin(\theta - \frac{2\pi}{3}) & -\sin(\theta + \frac{2\pi}{3}) \end{bmatrix} \]

(1)

to [2]. For any \( \mathbb{R}^3 \)-valued signal \( v \), the first two components of \( U(\theta)v \) are called the \( dq \) coordinates of \( v \), denoted by \( v_d, v_q \).

By using the notation \( \begin{bmatrix} \theta \\ d \theta/d\tau \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \) and the internal synchronous voltage \( e \), we represent the internal \( v_d \) and \( v_q \) as virtual. Only phase \( a \) is shown. This is taken from Fig. 3 in [21].

Applying the Park transformation to (1), we get the \( dq \) representation of the grid voltage as

\[ v_d = -V \sin \delta, \quad v_q = -V \cos \delta. \]  

(3)

The term \( e_d \) can be neglected, because the rate of change of the field current is small, so that \( e_d < e_q \). Thus, in the synchronverter algorithm from [37], the approximation \( e_d = 0 \) is adopted, and our analysis will follow this. (We remark that we did simulation experiments with \( e_d \) as in [3], and the results were practically the same as for \( e_d = 0 \).)

The following equation comes from the definition of \( \delta \): \n
\[ \frac{d \delta}{d \tau} = \omega - \omega_e. \]  

(11)

The fourth order grid-connected synchronverter model, which considers \( i_f \) as a given parameter, can be constructed by combining the equations (7)-(11). Its state vector is

\[ x = [i_d \quad i_q \quad \omega \quad \delta]^T \in \mathbb{R}^4. \]  

(12)

We write it as a nonlinear dynamical system:

\[ \dot{H}x = A(x, i_f)x + f(x), \]  

(13)

where

\[ H = \begin{bmatrix} L & 0 & 0 & 0 \\ 0 & L & 0 & 0 \\ 0 & 0 & J & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad f(x) = \begin{bmatrix} V \sin \delta \\ V \cos \delta \\ Tm + Dp \omega_e \\ -\omega_e \end{bmatrix}, \]

and

\[ A(x, i_f) = \begin{bmatrix} -R & \omega L & 0 & 0 \\ -\omega L & -R & -mi_f & 0 \\ 0 & mi_f & -Dp & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \]

We now derive the fifth order basic model of a synchronverter, by including \( i_f \) into the vector of the state variables. The instantaneous inverter output reactive power is

\[ Q = v_q i_d - v_d i_q = V[i_q \sin \delta - i_d \cos \delta], \]  

(14)
For this, we replace the integrator from (16) with a droop coefficient, \( Q_{\text{set}} \) is the desired reactive power, and \( V \) is as in [1]. Then, the field current \( i_f \) evolves according to

\[
M_f \frac{di_f}{dt} = \frac{\bar{Q} - Q}{K},
\]

(16)

see [21 eq.(15)], where \( K > 0 \) is a large constant. We want to make sure that \( i_f \) stays in a reasonable operating range \([u_{\text{min}}, u_{\text{max}}]\). (We will say more about this range in Sect. V.) For this, we replace the integrator from (16) with a saturating integrator (see [15], obtaining [21] eq.(21)):

\[
\frac{di_f}{dt} = \mathcal{J} \left( i_f, \frac{\bar{Q} - Q}{K} \right),
\]

(17)

where \( \mathcal{J} = KM_f \). Denoting \( w = \frac{\bar{Q} - Q}{K} \), the function \( \mathcal{J} \) is defined by

\[
\mathcal{J}(i_f, w) = \begin{cases} 
  w^+ & \text{if } i_f \leq u_{\text{min}}, \\
  w & \text{if } i_f \in (u_{\text{min}}, u_{\text{max}}), \\
  w^- & \text{if } i_f \geq u_{\text{max}},
\end{cases}
\]

where \( w^+ = \max\{w, 0\} \), \( w^- = \min\{w, 0\} \), so that \( w = w^+ + w^- \). This means that as long as \( i_f \) is in the range \([u_{\text{min}}, u_{\text{max}}]\), we have \( \frac{di_f}{dt} = \frac{\bar{Q} - Q}{K} \). However, if \( i_f \) reaches one of the end points of \([u_{\text{min}}, u_{\text{max}}]\), it is not allowed to continue out of this interval. (Note that in [17] we use \( \bar{K} \) in place of \( K \) because, differently from [21], here \( i_f \) is the state, not \( M_f/i_f \).) Using a saturating integrator in place of a usual one is needed in practice, and also in our stability proof in Sect. V.

The fifth order grid-connected synchronverter model can be constructed by combining (13), (14), and (17) as:

\[
\mathbf{H} \mathbf{z} = \mathbf{A}(\mathbf{z}) \mathbf{z} + \mathbf{f}(\mathbf{z}), \quad \frac{di_f}{dt} = \mathcal{J} \left( i_f, \frac{\bar{Q} - Q}{K} \right),
\]

(18)

with \( \mathbf{z} \) from (12), and with the state \( \mathbf{z} = [i_f]^T \in \mathbb{R}^3 \). Clearly, we mean that \( \mathbf{A}(\mathbf{z}) = \mathbf{A}(\mathbf{z}, i_f) \). If we ignore the saturating feature of \( \mathcal{J} \) in (17), and we use (16) (with \( Q \) from (14)) instead of (17) for the evolution of \( i_f \) (this is true for \( i_f \in (u_{\text{min}}, u_{\text{max}}) \)), i.e.,

\[
m \frac{di_f}{dt} = k_i d \cos \delta - k_q \sin \delta + \frac{1}{\bar{K}} \bar{Q},
\]

(19)

then we get the fifth order non-saturated model

\[
\mathbf{H} \mathbf{z} = \mathbf{A}(\mathbf{z}) \mathbf{z} + \mathbf{f}(\mathbf{z}),
\]

(20)

where

\[
\mathbf{H} = \begin{bmatrix} \mathbf{H} & \mathbf{0} \\ \mathbf{0} & m \end{bmatrix}, \quad \mathbf{f}(\mathbf{z}) = \begin{bmatrix} f(\mathbf{z}) \\ \frac{1}{\bar{K}} \bar{Q} \end{bmatrix}, \quad k = \sqrt{\frac{3}{2} V},
\]

\[
\mathbf{A}(\mathbf{z}) = \begin{bmatrix} \mathbf{A}(\mathbf{z}) & \mathbf{0} \\ k \cos \delta & -k \sin \delta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\]

with \( \mathbf{H}, \mathbf{A}, \) and \( f \) as defined after (13).

An extension of the model (20) to include the effect of measurement errors, has been derived in [11]. This is needed to analyze the sensitivity of the currents \( i_d, i_q \) with respect to the measurement errors. The paper [11] also presents the linearization of the model (20), a typical application example with relevant Bode plots, and experimental results.

The instantaneous active power \( P \) to the grid is

\[
P = v_d i_d + v_q i_q = -V [i_d \sin \delta + i_q \cos \delta]
\]

(21)

(see also [21 eq.(17)]), but this is not computed in the control algorithm, except possibly for monitoring. It is easy to derive from (14) and (21) that

\[
P^2 + Q^2 = V^2 (\bar{i}_d^2 + \bar{i}_q^2).
\]

(22)

We derive a nice formula linking the \( dq \) currents and the powers \( P \) and \( Q \). We know from (14) and (21) that

\[
\begin{bmatrix} P \\ Q \end{bmatrix} = -V \begin{bmatrix} \cos \delta & \sin \delta \\ -\sin \delta & \cos \delta \end{bmatrix} \begin{bmatrix} i_q \\ i_d \end{bmatrix}.
\]

(23)

By inverting the matrix, we obtain

\[
\begin{bmatrix} i_q \\ i_d \end{bmatrix} = -\frac{1}{V} \begin{bmatrix} \cos \delta & -\sin \delta \\ \sin \delta & \cos \delta \end{bmatrix} \begin{bmatrix} P \\ Q \end{bmatrix}.
\]

(24)

In Sect. III we will study the equilibrium points of the fourth order model (13), and in Sect. V we will extend the study to the fifth order non-saturated model (20). Finally, the model (18) will be used in Sect. VI to derive local exponential stability results for the grid-connected synchronverter.

### III. Equilibrium Points of the 4th Order Grid-Connected Synchronverter

In this section we study the equilibrium points of the fourth order model (13) for the grid-connected synchronverter. Thus, \( i_f \) is treated as a parameter here (i.e., there is no field current controller for the reactive power \( Q \)). Our main results is a geometric representation of the equilibrium points of (13) in the power plane: We find that, for “reasonable” values of \( i_f \), the images of the corresponding equilibrium points of (13) through the mappings \( P \) and \( Q \) from (23) move on a circle in the power plane. The radius and centre of this circle depend on the synchronverter parameters, on the grid voltage, and on the prime mover torque at the equilibrium. In addition, the point (\( P, Q \)) determines the power angle \( \delta \) at the equilibrium.

Finally, we establish a crucial results for the stability analysis of Sect. V: We find the interval of those field currents \( i_f > 0 \) for which the reactive power \( Q \) corresponding to the relevant equilibrium point is increasing (as a function of \( i_f \)).

The equilibrium points of (13) have been explicitly computed in [20, Sect. 3], under the assumption of a constant field current \( i_f \) in a reasonable range \( I_f \subset (0, \infty) \). For the reader’s convenience, we report those results here. In this paper, angles are regarded modulo \( 2\pi \), i.e., \( \delta \) and \( \delta + 2\pi \) are considered to be the same angle, except for certain arguments in Sect. V.

**Assumption 1:** Let \( R, L, J, m, D_p, V, \omega_h, \omega_n > 0 \) and \( T_m \in \mathbb{R} \) be given. Denote

\[
T_m = T_m + D_p (\omega_n - \omega_k).
\]

(25)
Assume that
\[ 4R\omega_k\tilde{T}_m \geq -V^2. \tag{26} \]

**Proposition 3.1:** Consider the model \[13\], with \( x \) from \[12\], and with parameters satisfying Assumption \[1\]. Denote
\[ \phi \in \left(0, \frac{\pi}{2}\right) \quad \text{such that} \quad \tan \phi = \frac{\omega_2 L}{R}, \tag{27} \]
\[ \Lambda(i_f) = -\frac{\tilde{T}_m}{m_i f} L \sqrt{p^2 + \omega_x^2} + \frac{m_i f \omega_p p}{V \sqrt{p^2 + \omega_x^2}}, \tag{28} \]
where \( p = R / L \). We define the interval \( I_f \subset (0, \infty) \) as follows:
\[ I_f = \{ i_f > 0 \mid |\Lambda(i_f)| \leq 1 \}. \]

For any \( i_f \in I_f \cup (-I_f) \), the model \[13\] has two equilibrium points, \( x_1^f \) and \( x_2^f \), with the power angles \( \delta_1^f \) and \( \delta_2^f \) satisfying:
\[ \delta_1^f = \arccos \Lambda - \phi, \quad \delta_2^f = -\arccos \Lambda - \phi, \tag{29} \]
where \( \arccos \Lambda : [-1, 1] \rightarrow [0, \pi] \). The other equilibrium of the model states \( x^f \) are given for \( j \in \{1, 2\} \) by
\[ \hat{\epsilon}_{d_j} = -\frac{\tilde{T}_m \omega_x}{m_i f} + \frac{V \sin \delta^f}{R} \quad \hat{\epsilon}^c = -\frac{\tilde{T}_m}{m_i f}, \quad \omega^c = \omega_p. \tag{30} \]

Note that if \( |\Lambda| = 1 \), then \( \delta_1^f = \delta_2^f \) and thus \( x_1^f = x_2^f \).

Note that Assumption \[1\] guarantees that \( I_f \) is nonempty.

It is clear from \[9\] that \( \tilde{T}_m \) represents the prime mover torque at equilibrium. The proof follows from [20, Sect. 3], where the notation is slightly different: what is denoted in \[20\] by \( T_m \), \( R_e \), and \( L_c \), is denoted here by \( T_m \), \( D_p \), \( \omega_p \), and \( R \), respectively. Moreover, in \[20\] it is assumed that \( i_f > 0 \), however the derivations in \[20\, Sect. 3\] remains valid also for \( i_f \in (-I_f) \). We now prove that if \( \tilde{T}_m \neq 0 \), then \( I_f \) is a closed interval. If \( \tilde{T}_m > 0 \), then \( \Lambda \) is an increasing function of \( i_f \), and our claim follows. If, instead, \( \tilde{T}_m < 0 \), then \( \Lambda \) is first decreasing for a certain interval of \( i_f \), after which it becomes increasing, and we have \( \Lambda > 0 \) for all \( i_f > 0 \). Thus, we can conclude again that \( I_f \) is a closed interval. Finally, if \( \tilde{T}_m = 0 \), then \( \Lambda \) depends linearly on \( i_f \) and it is clear that \( I_f \) is an interval (not closed). The above scenarios are depicted in Fig. 4.

**Remark 3.2:** As mentioned above, if \( \tilde{T}_m > 0 \) then \( \Lambda \) is an increasing function of \( i_f > 0 \) and
\[ \{ \Lambda(i_f) \mid i_f \in I_f \} = [-1, 1]. \]
Thus, for every \( \Lambda \in [-1, 1] \), \[28\] has two solutions:
\[ i_{f1} = \frac{\sqrt{p^2 + \omega_x^2} (AV + \sqrt{A^2V^2 + 4\omega_p R\tilde{T}_m})}{2m_i \omega_x p}, \tag{31} \]
and \( i_{f2} \) is as above, with \( - \) instead of \( + \) in front of the square root in the brackets. Clearly \( i_{f2} < 0 < i_{f1} \). Thus, there is only one positive solution of \[28\] for each fixed \( \Lambda \in [-1, 1] \).

On the contrary, if \( \tilde{T}_m < 0 \) then \( \Lambda \) is first a decreasing function of \( i_f > 0 \), and then an increasing one. Moreover, \( \Lambda(i_f) > 0 \) for all \( i_f > 0 \) (see Fig. 4). This implies that
\[ \{ \Lambda(i_f) \mid i_f \in I_f \} \subset (0, 1], \]
and that if \( \Lambda \) belongs to the above set, then \( i_{f1}, i_{f2} \) from \[31\] are both positive. Finally, for the case \( \tilde{T}_m = 0 \), \( i_{f2} = 0 \) and \( \Lambda \) is linear in \( i_f \), so that \( \{ \Lambda(i_f) \mid i_f \in I_f \} = (0, 1] \).

**Proposition 3.3:** We use the notation of Proposition \[3.1\]. Under Assumption \[1\] if for some \( i_f \in I_f \), the model \[13\] has a stable equilibrium point \( x_1^f \), then \( x_1^f = x_2^f \) and
\[ \delta_1^f = -\phi - \phi. \]

Note that if \( R \) tends to zero then \( \phi \) tends to \( \pi / 2 \), see \[27\], and the above condition becomes the famous necessary stability condition \( \delta \in (-\pi, 0] \) appearing often in the literature.

**Proof.** Denote by \( h(x) \) the right-hand side of \[13\]. Let \( A_{ij} \), \( j \in \{1, 2\} \), be the Jacobian \( A_{ij} = \partial h / \partial x \) computed at \( x_1^f \). A necessary condition for the equilibrium point \( x_1^f \) to be stable is that \( H^{-1}A_{ij} \) is a stable matrix, which implies that \( \det(H^{-1}A_{ij}) > 0 \). It can be verified easily that
\[ \det(H^{-1}A_{ij}) > 0 \iff \sin(\delta_1^f + \phi) > 0, \tag{32} \]
see \[20\, eq.(3.5)\] for the detailed derivation.

Recall the expressions of \( \delta_1^f \) and \( \delta_2^f \) from \[29\]. We have
\[ \delta_1^f + \phi \in [0, \pi], \quad \delta_2^f + \phi \in [-\pi, 0]. \]
Clearly, \[32\] holds only for \( j = 1 \). Thus, \( x_1^f = x_2^f \) and, moreover, we must have \( \delta_1^f + \phi \in (0, \pi) \).

**Proposition 3.4:** We use the notation of Proposition \[3.1\]. Consider the model \[13\], with parameters satisfying Assumption \[1\] and let \( i_f \in I_f \cup (-I_f) \), so that \[13\] has two equilibrium points. Then, at every equilibrium point of this system we have
\[ \tilde{T}_m \omega_x = P + R \frac{p^2 + Q^2}{V^2}. \tag{33} \]
Moreover, the power angle value \( \delta^c \) at the equilibrium satisfies
\[ \tan \delta^c = \frac{\omega_p L P - R Q}{R P + \omega_p L Q + V^2}. \tag{34} \]

**Proof.** In this proof, for convenience, we omit the superscript \( e \) to indicate the equilibrium point values. At an equilibrium point of \[13\] the left-hand sides of \[7\] and \[8\] are zero. We multiply these equations with \( i_d \) and \( i_q \) respectively, and we add them using \[4\], obtaining
\[ -R(i_d^2 + i_q^2) - m_i f i_d \omega - v_d i_d - v_q i_q = 0. \]
Using the formulas (10) and (21), we get
\[ R(\bar{i}_d^2 + \bar{i}_q^2) - T_m\omega + P = 0. \]
It follows from (9) that \( T_m = \bar{T}_m \), and we know from (30) that \( \omega = \omega_k \). Substituting these values above, we get
\[ \bar{T}_m\omega_k = P + R(\bar{i}_d^2 + \bar{i}_q^2). \]
Using (22) this becomes (34).

Now we turn our attention to (34). If we multiply both sides of (7) (at equilibrium) with \( \cos \delta \), both sides of (8) (at equilibrium) with \( \cos \delta \) and then add, we get
\[ m\omega_k i_f \cos \delta = -R[i_d \sin \delta + i_q \cos \delta] + \omega_k L \frac{Q}{V} + V. \]
In the same way, if we multiply (7) with \( \cos \delta \), (8) with \( \sin \delta \) and we subtract them, we get
\[ -m\omega_k i_f \sin \delta = \omega_k L [i_d \sin \delta + i_q \cos \delta] + R \frac{Q}{V}. \]
According to (21) the last two equations can be rewritten as
\[ m_i \omega_k \sin \delta = \omega_k L \frac{P}{V} - R \frac{Q}{V}, \]  
(35)
\[ m_i \omega_k \cos \delta = R \frac{P}{V} + \omega_k L \frac{Q}{V} + V, \]  
(36)
Since \( i_f \neq 0 \), the left sides of (35) and (36) cannot be both zero. We divide the sides of (36) by the sides of (35), which shows that (34) holds.

Remark 3.5: The equation (33) has a clear intuitive interpretation: the left-hand side is the mechanical power coming from the virtual prime mover (the frequency droop mechanism is part of the prime mover). The second term on the right-hand side is the power consumed in the output resistors \( R \) in series with each of the three phases, if we think of the model as representing a synchronous machine. (This follows from (22) and the fact that the Park transformation is unitary.)

In the following, we present the novel geometric representation of the (manifold of) equilibrium points of (13) in the \( P Q \) plane. For this, we first introduce some useful notation.

Notation. We use the notation of Proposition 3.1. Consider the model (13), with parameters satisfying Assumption 1. We define \( i_{f-} = \inf I_f \), and \( i_{f+} = \sup I_f \). (Depending on the sign of \( T_m \), \( \Lambda(i_{f-}) \) and \( \Lambda(i_{f+}) \) take different values, as discussed in Remark 3.2.) Let \( X_i(i_f), X_0(i_f) \) be the two equilibrium points of (13) corresponding to \( i_f \in I_f \), as described in (29)-(30). \( X_i(i_f), X_0(i_f) \) coincide at \( i_f = i_{f-} \) and at \( i_f = i_{f+} \). We denote by \( \delta_i(i_f) \) the power angle component of \( X_i(i_f) \), \( j \in \{1,2\} \), and by \( P_j(i_f) \) \( (Q_j(i_f)) \) the active (reactive) power at the equilibrium point \( X_j(i_f) \), for \( j \in \{1,2\} \). If \( X, Y, Z \in \mathbb{R}^2 \), then \( \delta_1; \delta_2; \delta_3 \) denotes the angle from the vector \( X - Y \) to the vector \( Z - Y \) (counterclockwise). We do not distinguish between a vector and the pair of real numbers that are its coordinates.

Theorem 3.6: Consider the model (13), with parameters satisfying Assumption 1. Then the points in \( \mathbb{R}^2 \) defined by
\[ S_j(i_f) = (P_j(i_f), Q_j(i_f)), \quad i_f \in I_f, \quad j \in \{1,2\}, \]  
are on the circle with centre \( C \) and radius \( r \) given by
\[ C = \left( -\frac{V^2}{2R}, 0 \right), \quad r^2 = \frac{V^2 + 4V^2 R T_m \omega_k}{4R^2}. \]  
(37)
Define the points \( Z, M, O \in \mathbb{R}^2 \) as
\[ Z = (R, \omega_k L), \quad M = -\frac{V^2}{||Z||^2} Z, \quad O = (0,0). \]
Then the distances \( CO, CM \) are equal and \( \tilde{O}; M; C = \tilde{C}; O; M = \delta \).

Moreover, the following holds:
\[ S_j(i_f) - M = V \frac{\cos(\phi - \delta_j(i_f))}{||Z||} \sin(\phi - \delta_j(i_f)) \]  
(39)
Proof. According to Proposition 3.4, the powers \( P_j(i_f) \) and \( Q_j(i_f) \) satisfy the quadratic equation
\[ P_j^2 + Q_j^2 + \frac{V^2}{R} P_j = \frac{\tilde{T}_m \omega_k V^2}{R}. \]
The solutions of this equation are on a circle symmetric with respect to the \( P \) axis. The formulas for the centre \( C \) and the radius \( r \) follow from standard computations.
From a routine computation we get that
\[ ||M - C|| = \frac{V^2}{2R} = ||C||. \]
One conclusion from the above is that the triangle \( COM \) is isosceles, and since the angle of \( Z \) (with respect to the \( P \) axis) is \( \delta \), we get that (38) holds (see Fig. 5(a), 5(b)).

We now prove (34). For convenience we denote \( (P, Q) = (P_j(i_f), Q_j(i_f)), \delta = \delta_j(i_f) \). From (7), (8) and (22), we have
\[ V \begin{bmatrix} R & -\omega L \cos \delta & -\omega L \sin \delta \end{bmatrix} \begin{bmatrix} P \cos \phi \sin \phi \cos \phi \sin \phi \end{bmatrix} + V \begin{bmatrix} \sin \delta \cos \delta \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} m_i \omega_k \end{bmatrix}. \]
Using the definition of \( \phi \) from (27), we have
\[ V \begin{bmatrix} R & -\omega L \cos \delta & -\omega L \sin \delta \end{bmatrix} \begin{bmatrix} \cos \phi \sin \phi \cos \phi \sin \phi \end{bmatrix} = \begin{bmatrix} m_i \omega_k \end{bmatrix}. \]
Substituting this above, commuting the first two matrices, and multiplying with the inverse of the matrix from (23), we obtain
\[ \frac{||Z||}{V} \begin{bmatrix} \cos \phi & -\sin \phi \end{bmatrix} \begin{bmatrix} Q \cos \phi \sin \phi \cos \phi \sin \phi \end{bmatrix} + V \begin{bmatrix} ||Z|| \sin \phi \cos \phi \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \begin{bmatrix} m_i \omega_k \end{bmatrix}. \]
Multiplying with the inverse of the first matrix above, and also with \( V/||Z|| \), and swapping the rows, we get
\[ \begin{bmatrix} P \cos \phi \sin \phi \cos \phi \sin \phi \end{bmatrix} + \frac{V^2}{||Z||} \begin{bmatrix} \cos(\phi - \delta) m_i \omega_k \end{bmatrix}. \]
From here, we get (37) by substituting
\[ M = -\frac{V^2}{||Z||} \begin{bmatrix} \cos \phi \sin \phi \cos \phi \sin \phi \end{bmatrix}. \]

Remark 3.7: From (38), several useful facts follow. First, taking the norms, we have that (for \( i_f \in I_f \))
\[ ||S_j(i_f) - M|| = \frac{V}{||Z||} m_i \omega_k, \]  
(40)
\[ \text{i.e., the distance from } S_j(i_f) \text{ to } M \text{ is proportional to } i_f. \]  
This implies that the level curves in the power plane for constant \( i_f \) are circles, with centre \( M \) and radius given by (40). Second,
tells us that the vector $S_1(i) - M$ forms an angle of $\phi - \delta_1$ with the $P$ axis (see Fig. 5(a)\footnote{5(b)}. Thus,

$$S_1(i) ; M : O = \phi - (\phi - \delta_1) = \delta_1.$$ 

Third, clearly $i_{f-} < i_f < i_{f+}$ for any $i_f \in I_f$. From (40), $S_1(i_{f-})$ is the point on the circle from Theorem 3.6 that is the closest to $M$, while $S_1(i_{f+})$ is the point on the same circle that is the farthest from $M$. This implies that $S_1(i_{f-})$, $M$, $C$ and $S_1(i_{f+})$ are on a straight line $\mathcal{L}$, as in Fig. 5(a)\footnote{5(b)}.

From the above facts it follows that, for increasing $i_f \in I_f$, the point $S_1(i_f)$ moves counterclockwise on the circle described in Theorem 3.6 from $S_1(i_{f-})$ to $S_1(i_{f+})$.

Remark 3.8: It follows from the formula for $r$ in (37) that, depending on the sign of $\tilde{T}_m$, three scenarios are possible for the points $M$ and $O$ from Theorem 3.6. If $\tilde{T}_m > 0$ then $M, O$ are inside the circle, while if $\tilde{T}_m < 0$ ($\tilde{T}_m = 0$) then $M, O$ are outside (on) the circle. The case $\tilde{T}_m > 0$ is the most common.

Theorem 3.9: We use the notation of Theorem 3.6. Let Assumption 1 hold. Then:

(a) If $\omega_k L > R$, then $M$ is to the right of $C$. There is a unique $i_{f0} \in I_f$ for which $S_1(i_{f0}) = (-V^2/2R, r)$ and

$$\frac{d}{di_f} Q_1(i_f) > 0 \quad \text{for} \quad i_f \in I^+_f = (i_{f-}, i_{f0}).$$

(b) If $\omega_k L \leq R$, then $M$ is to the left of (or directly below) $C$. There is a unique $i_{f0} \in I_f$ for which $S_1(i_{f0}) = (-V^2/2R, r)$ and

$$\frac{d}{di_f} Q_1(i_f) > 0 \quad \text{for} \quad i_f \in I^+_f = (i_{f0}, i_{f+}).$$

Theorems 3.6, 3.9 are illustrated in Fig. 5(a)\footnote{5(b)} As discussed in Remark 3.7 we see that $S_1(i_f)$ moves counterclockwise on the circle, for increasing $i_f$, from $S_1(i_{f-})$ to $S_1(i_{f+})$, while $S_2(i_f)$ moves clockwise between the same two endpoints. The movement of $S_2(i_f)$ is symmetric to the one of $S_1(i_f)$, with respect to the line $\mathcal{L}$.

Note that the case $\omega_k L > R$ is the most common.

Proof of Theorem 3.9 Let $\tilde{T}_m \geq 0$. In the case (a), an elementary computation shows that the $P$-coordinate of $M$ is larger than that of $C$:

$$-V^2R \geq -V^2R.$$

Hence, $M$ is to the right of $C$, as stated. Note that this implies that the slope of $\mathcal{L}$ is negative, as in Fig. 5(a)\footnote{5(b)}

As discussed in Remark 3.7 $S_1(i_f)$ moves counterclockwise on the circle for increasing $i_f \in I_f$. Since $i_f$ is proportional to the distance from $S_1$ to $M$, $Q_1(i_f)$ is strictly increasing (with positive derivative) for $i_f \in I^+_f = (i_{f-}, i_{f0})$, where $i_{f0}$ is the field current for which $Q_1(i_f)$ reaches its maximum value, namely $r$. From Fig. 5(a)\footnote{5(b)} we see that $i_{f0}$ is the unique field current for which $P_1(i_{f0}) = -V^2/2R$.

We move now to case (b). We perform the same elementary computation as before, reaching the opposite conclusion for $\omega_k L \leq R$, namely, that $M$ is to the left of (or directly below) $C$. Thus, for $\omega_k L < R$, the slope of $\mathcal{L}$ is positive (as depicted in Fig. 5(b)\footnote{5(b)}, and for $\omega_k L = R$, $\mathcal{L}$ is vertical.

In the proof of (b), the interval on which $Q_1$ is increasing is from $i_{f0}$, where $Q_1$ is at its minimum, until $i_{f+}$. We see from Fig. 5(b)\footnote{5(b)} that $S_1(i_{f0}) = (-V^2/2R, r)$.

The proof of the case $\tilde{T}_m < 0$ is similar.

Remark 3.10: For $\tilde{T}_m < 0$ both the solutions $i_{f1}, i_{f2}$ of (28), with $\Delta \in \{\Lambda(i_f) : i_f \in I_f\}$, are positive (see Remark 3.2). This has an intuitive geometrical meaning. Fixing $\Delta$ is similar to fixing $\lambda = \arccos \Lambda$, i.e., the angle $S_1(i_f) : M : C$ in Fig. 6. Since $M$ is outside of the circle (see Remark 3.9), the line passing through the points $M$ and $S_1(i_f)$ cuts the circle in another point, namely, $S_1(i_f)$. The values $i_{f1}, i_{f2}$ are the two (positive) solutions of (28) mentioned above. The case $s_1 = s_1'$ in Fig. 6 corresponds to the value of $\Lambda$ for which the square root in (31) is zero, i.e., $i_{f1} = i_{f2}$.

Remark 3.11: The point $S_1(i_f)$ moves counterclockwise on the circle described in Theorem 3.6 from $S_1(i_{f-})$ to $S_1(i_{f+})$ for increasing $i_f \in I_f$, as discussed in Remark 3.7. When $\tilde{T}_m \geq 0$, this implies that $\delta_1(i_f)$ is decreasing from $\delta_1(i_{f-}) = \pi - \phi$, to $\delta_1(i_{f+}) = -\phi$. However, when $\tilde{T}_m < 0$ then this is not true. Indeed $\delta_1(i_{f+}) = \delta_1(i_{f-}) = -\phi$.

IV. EQUILIBRIUM POINTS OF THE FIFTH ORDER GRID-CONNECTED SYNCHRONVERTER

In this section we study the equilibrium points of the fifth order grid-connected synchronverter model (20). Using the results for the fourth order model (13) from the previous section, we derive a necessary and sufficient condition for the existence of the equilibrium points of (20) (where $i_f$ is a state variable) and we compute them explicitly. As in Sect. III we consider the grid to be an infinite bus, with constant $V, \omega_k$.

The fifth order model (13) or (20) is shown as a block diagram in Fig. 7, with the fourth order model (13) as a block.

Assumption 2: Let $R, L, J, m, D_p, D_q, V, \omega_k, \omega_k, v_{set} > 0$ and $\tilde{T}_m, Q_{set} \in \mathbb{R}$ be given.

Our first result concerns mainly the equation that must be satisfied by the active power $P$ at an equilibrium point of (20).

Proposition 4.1: Consider the model (20), with parameters satisfying Assumption 2. Recall $\tilde{Q}$ from (15) and $\tilde{T}_m$ from (23).

A necessary condition for this system to have equilibrium points is

$$4R^2\tilde{Q}^2 \leq V^2 + 4RV^2\tilde{T}_m\omega_k.$$ \hspace{1cm} (41)

At every equilibrium point of this system we have

$$\omega = \omega_k, \quad T_e = \tilde{T}_m, \quad Q = \tilde{Q}, \hspace{1cm} (42)$$

and $P$ satisfies the equation

$$\tilde{T}_m\omega_k = \frac{P + R\omega_k^2 + Q^2}{V^2}. \hspace{1cm} (43)$$

Remark 4.2: A formula equivalent to (43) has appeared in \cite{21} eq.(24)], but instead of a mathematical proof it was derived from a physical balance equation. As proposed in \cite{21}, this formula can be used in the synchronverter algorithm to determine the value of the parameter $\tilde{T}_m$, if the reference values $P_{set}$ and $Q_{set}$ are given and if some estimate (for instance, zero) is adopted for the differences $\omega_k - \omega_k$ and $v_{set} - \sqrt{2/3}V$. 

The case $\omega_p L > R$, described in Theorem 3.9(a). Note that (43) is a second order equation in $T$. For this reason, it is similar to assume that

$$T \mapsto \omega_p L \leq R$$

Indeed, if the estimate zero is adopted for these differences

\[ (17) \text{ or } (19) \]

the coefficients depend on the parameters of the system. For this equation to have a real solution, by elementary algebra, the condition (41) must be satisfied. Hence, if (41) does not hold, then the system cannot have equilibrium points. \hfill \blacksquare

Remark 4.3: The equilibrium points of (20) come in symmetric pairs. Indeed, if $\mathbf{z}' = [\alpha' \beta' \gamma' \delta']^T$ is such an equilibrium point, then also

$$\mathbf{z} = [-\alpha' - \beta' \gamma' \delta' + \pi - \delta'']^T$$

is an equilibrium point. The intuition behind this is clear: if we rotate the rotor by $180^\circ$ and at the same time invert the current $i_f$ in the rotor, then by the symmetry of the rotor nothing has really changed. The replacement of the rotor angle $\theta$ with $\theta + \pi$ causes $i_d$ and $i_q$ to change sign, while the currents in the stationary frame remain unchanged. We see from (23) that the active and reactive powers $P, Q$ at $\mathbf{z}'$ and at $\mathbf{z}$ are the same.

Remark 4.4: The system (20) has an exceptional set of equilibrium points corresponding to the point $M$ defined in Theorem 3.6. Indeed, when the circle defined in Theorem 3.6 passes through the point $M$ (this happens for $\hat{T}_m = 0$), and the values of $P$ and $Q$ are the coordinates of $M$, namely

$$P = -\frac{V^2 R}{R^2 + \omega_p^2 L^2}, \quad Q = -\frac{V^2 \omega_p L}{R^2 + \omega_p^2 L^2}.$$ 

then if we choose $i_f = 0$ and any angle $\delta''$, we get an equilibrium point of (20). This can be checked through a somewhat tedious computation (using (24)), which shows that for $i_f = 0$ and any $\delta''$, (7) and (8) hold with zero on the left-hand side. The other equilibrium equations are easily seen to hold. Thus, for $\hat{T}_m = 0$ and $P, Q$ as in (45) we have infinitely many equilibrium points.

The physical interpretation of these equilibrium points is as follows: here the rotor has no current and hence no magnetic field, so that its angle is irrelevant for what happens in the stator windings. The SG now consists of only the stator windings connected to the power grid, consuming power. The practical importance of the exceptional set of equilibrium points discussed above is very small, along with all the equilibrium points that correspond to negative $i_f$. Indeed, the

Indeed, if the estimate zero is adopted for these differences (which is, a priori, our best guess), then

$$T_m \omega_n = P_{set} + R^2 \frac{P_{set}^2 + Q_{set}^2}{V^2}.$$ \hspace{1cm} (44)

For this reason, it is similar to assume that $T_m, Q_{set}$ are given (as in Proposition 4.1) or that $P_{set}, Q_{set}$ are given.

Note that (41) is equivalent to $|\hat{Q}| \leq r$, where $r$ is the radius of the circle from Proposition 3.6. Indeed, $|\hat{Q}| > r$ would be an infeasible requirement, as is clear from Fig. 5(a) or (b).

\textbf{Proof.} We omit the superscript $e$ to indicate the equilibrium point values. If the system is at an equilibrium point, then from (11) we see immediately that $\omega = \omega_g$, from (9) we see that $T_c = T_m$ and from (16) we see that $Q = \hat{Q}$. Thus, we have proved all the parts of (42).

Equation (43) follows from (33), substituting $Q = \hat{Q}$ from (42). Note that (43) is a second order equation in $P$, where

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actual field current controller employs a saturating integrator (see [17]), which constrains the $i_f$ values to an interval of positive numbers (contained in $I_f$). This is a safety feature that prevents the system from leaving its normal operating range.

**Theorem 4.5:** We work with the notation of Proposition 4.1. Then the model (20), with parameters satisfying Assumption 2, has equilibrium points if and only if (41) is satisfied. Suppose that the condition (41) is true, and let us denote by $P_l$ and $P_r$, the two real solutions of (43), so that $P_l < P_r$, and $\frac{\partial^2 P}{\partial \omega^2} = -\frac{V^2}{2R}$. At every equilibrium point we have $P = P_l$ or $P = P_r$.

Recall the exceptional point $M$ discussed in the last remark. Assume that the equilibrium point is such that $(P, \tilde{Q}) \neq M$. Then the angle $\delta^*$ satisfies

$$\tan \delta^* = \frac{\omega_l LP - R \tilde{Q}}{\omega_l LQ + V^2}.$$  

(46)

If the angle $\delta$ is measured modulo $2\pi$, and (41) holds with strict inequality, then the model (20) has precisely four equilibrium points. Two of them, denoted by $z^e_l$ and $z^e_r$, have the property that $\tilde{r}_l^e > 0$. At $z^e_l$, $P = P_r$, and at $z^e_r$, $P = P_l$. There are also the two symmetric equilibrium points $z^e_l$ and $z^e_r$ where $\tilde{r}_l^e < 0$, as described in Remark 4.3. If (41) holds with equality, then $P_l = P_r = -V^2/2R$, and the model has precisely two equilibrium points, which are a symmetric pair.

**Remark 4.6:** We see from (46) that to any $(P, \tilde{Q}) \neq M$ in the power plane correspond two possible equilibrium angles, that differ by $\pi$. This is true also if the denominator is zero, in that case $\delta^* = \pm \pi/2$. For the exceptional pair $M$, the right-hand side of (46) is 0/0, so that $\delta^*$ could take any value, in accordance with Remark 4.4.

**Proof.** We omit the superscript $e$ to indicate the equilibrium point values. Assume that (41) holds, so that (43) has two real solutions, $P_l$ and $P_r$, with $P_l < P_r$. We know from Proposition 4.1 that at every equilibrium point, $P = P_l$ or $P = P_r$. Equation (46) follows from (34), substituting $Q = \tilde{Q}$ from (42). For each choice of $P$ (either $P_l$ or $P_r$) such that $(P, \tilde{Q}) \neq M$, this equation has precisely two solutions modulo $2\pi$, that differ by an angle of $\pi$. (In the extreme case when the denominator in (46) is zero, then the solutions are $\pm \pi/2$.)

Suppose that (41) holds with strict inequality, which implies that $P_l < P_r$, and suppose that $(P, \tilde{Q}) \neq M$. Then we obtain four candidate equilibrium angles $\delta$ (two for $P = P_l$ and two for $P = P_r$). We now show that each of these four angles actually corresponds to an equilibrium point $z^e_l = (i_l, i_q, \omega, \delta, i_f)$. From (24) we see that at any equilibrium point

$$\begin{pmatrix} i_l \\ i_q \end{pmatrix} = -\frac{1}{V} \begin{pmatrix} \cos \delta & -\sin \delta \\ \sin \delta & \cos \delta \end{pmatrix} \begin{pmatrix} P \\ \tilde{Q} \end{pmatrix},$$

where $P = P_l$ or $P = P_r$. From (10) and (42) we see that at any equilibrium point,

$$T_m = -m i_l q.$$  

Thus, if $T_m \neq 0$, $i_f$ can be computed from here. If $T_m = 0$, then (at the equilibrium) should be used instead, as long as $(P, \tilde{Q}) \neq M$. The exceptional case when $(P, \tilde{Q}) = M$ leads to $i_f = 0$ and arbitrary $\delta$, as discussed in Remark 4.4.

It is easy to see that the points $z = (i_l, i_q, \omega, \delta, i_f)$ computed as described are indeed equilibrium points, and they come in two symmetric pairs, as described in Remark 4.3.

When we have equality in (41), then $P_l = P_r = -V^2/2R$. Correspondingly, there are only two solutions for (46) (modulo $2\pi$) and they differ by $\pi$. The currents $i_d, i_q, i_f$ are computed as before, and we obtain two equilibrium points (a symmetric pair), one with $i_f > 0$ and the other one with $i_f < 0$.

**Remark 4.7:** Under the conditions of the last theorem, it is easy to see that $P_l \geq 0$ if and only if

$$R \tilde{Q}^2 \leq V^2 T_m \omega_k,$$

(47)

and $P_r = 0$ if and only if we have equality in (47). Note that (47) implies (41) and we always have $P_i < 0$. If an equilibrium point corresponds to $P_i > 0$ and $\tilde{Q} = 0$, then $\tan \delta^* > 0$ (this means that $\delta^* \in (0, \pi/2) \cup (\pi, 3\pi/2)$). Indeed, this can be seen directly from (46). (These facts are clear from Fig. 5(a) and 5(b).)

**Remark 4.8:** As mentioned at the end of Remark 4.4, the real system (13) can never reach the two equilibrium points with $\tilde{r}_l^e \leq 0$, due to the saturating integrator used in the field current controller (see [17]).

V. STABILITY OF THE GRID-CONNECTED SYNCHRONVERTER

In this section we investigate the stability of the grid-connected synchronverter model (18) using [15] Theorem 4.3], which is based on singular perturbation theory. Our main result in Theorem 5.2 proves that, under reasonable assumptions, there exists a $\kappa > 0$ such that if $K > \frac{1}{\kappa}$, then the fifth order model (18) has a (locally) exponentially stable equilibrium point with a “large” domain of attraction. This stable equilibrium point “corresponds” to $x^e_l$ from Proposition 3.1. After stating our main result, we offer a visual representation of the stability region of the fifth order model (18), based on the geometric description introduced in Proposition 3.6.

Note that results closely related to our Theorem 5.2, with most of the proof missing, assuming that the model (13) is almost globally asymptotically stable for every constant $i_f \in [u_{\min}, u_{\max}]$, have been presented in [21] Theorem 5.1].

We introduce a function $\Xi$ that maps “reasonable” values of $i_f$ into the corresponding first equilibrium point $x^e_l$ of the fourth order model (13) (see Proposition 3.1) as

$$\Xi : I_f \to \mathbb{R}^4 \text{ such that } \Xi(i_f) = [i^e_{\delta l} i^e_{\delta r} \omega_k \delta^e_{i_f}]^T,$$

(48)

where $i^e_{\delta l}$, $i^e_{\delta r}$ and $\delta^e_{i_f}$ are given by (29), (30), so that $x^e_l \in \Xi(i_f)$. Here angles are not identified modulo $2\pi$, because we use results from singular perturbations theory that have been formulated for systems evolving on $\mathbb{R}^n$. We consider $\delta^e_{i_f} \in [-\pi, \pi].$

Recall the interval $I^+_f$ from Theorem 3.9. Then it follows from the just mentioned theorems that if (42) holds with strict inequality (so that $I^+_f$ is nonempty), then

$$\frac{d}{d\tilde{r}_f} Q_1(i_f) > 0 \quad \text{for } i_f \in I^+_f.$$  

Let $z^e_l = [i^e_{\delta l} i^e_{\delta r} \omega_k \delta^e_{i_f}]^T$ be defined as in Theorem 4.5 (i.e., $z^e_l$ is the equilibrium point of the fifth order model (20) at which $\tilde{r}_l^e > 0$ and $P = P_l$). Assume that $\tilde{r}_f \in I^+_f$. We see from Fig. 5(a) and Fig. 5(b) that this implies that the point $(P_l(\tilde{r}_f), Q_l(\tilde{r}_f))$ is to the right of the line $\mathcal{L}$ in the power plane, so that $P_l(\tilde{r}_f) = P_l$ and thus
synchronverter parameters are chosen so that the state input-output map associated to $G_i ((26) with strict inequality). Recall the function $X_i$, then $Q_1(i_f)$ is the output $Q_1(i_f)$ at $x = \Xi(i_f) = x_i^r$. Then,

$$G'(i_f) > 0, \quad \forall i_f \in U_e.$$

Proof. It follows from Theorem 3.9 that $Q_1(i_f)$ is increasing for $i_f \in I_f^+$. Thus, $G'(i_f) > 0$ for all $i_f \in U_e \subset I_f^+$. ■

Theorem 5.2: Consider the model (18), with given $R, L, J, m, D_q, D_q, V, \omega_0, \eta, \nu_{set} > 0$ and $T_m \in \mathbb{R}$, and with the state $z = [x_i^r]^T \in \mathbb{R}^5$ (x is as in (12)). We use the notation $P, T_m, i_f, I_f^+, u_{min}, u_{max}, \varepsilon, U_e, \Xi, G$ as in Proposition 5.1. Assume that (26) holds with strict inequality and that the synchronverter parameters are chosen so that $P$ has a locally exponentially stable equilibrium point for every $i_f \in U_e$.

Then, for any $\tilde{Q} \in [G(u_{min}), G(u_{max})]$, denoting $\tilde{r}_{fr} = G^{-1}(\tilde{Q})$, there exist an $\varepsilon_0 > 0$ and a $\kappa > 0$ such that: If $\tilde{K} > \frac{\varepsilon_0}{\kappa}$, then $x_i^r = (\Xi(\tilde{r}_{fr}), \tilde{r}_{fr}$) is a (locally) exponentially stable equilibrium point of the closed-loop system (18), with state space $X = \mathbb{R}^5 \times [u_{min}, u_{max}]$. Moreover, if the initial state $x(0), i_f(0) \in X$ of (18) satisfies $\|x(0) - \Xi(i_f(0))\| \leq \varepsilon_0$, then

$$x(t) \to \Xi(\tilde{r}_{fr}), \quad i_f(t) \to \tilde{i}_{fr}, \quad Q(t) \to \tilde{Q},$$

and this convergence is at an exponential rate.

Proof. The exponential stability of $P$ for each $i_f \in U_e$ (as assumed in the theorem) implies the uniform exponential stability of $P$, see [15] Remark 3.1. This, together with the result from Proposition 5.1, allows us to apply [15] Theorem 4.3, completing the proof of the theorem. ■

Remark 5.3: The local exponential stability assumption in the above theorem is true if the parameters satisfy the numerical conditions presented in [3] Theorem 1 or in [20] Theorem 6.3 (the conditions in these two references are not equivalent). Actually, [3] and [20] conclude aGAS.

We now illustrate how to derive the region of stability of the fifth order model (18) in the power plane. We assume that the inverter parameters, as well as $V$ and $\omega_0$, are known and fixed, but $P_{set}$ and $Q_{set}$ can vary. Recall the notation of Theorems 3.9, 4.5. Then the coordinates of $z_i^r = S_1(\tilde{r}_{fr})$ can be obtained from $(P_{set}, Q_{set})$ as follows:

- If $\omega_h = \omega_k$ and $\nu_{set} = \sqrt{\frac{2}{3}}V$, i.e., the grid is in nominal conditions, then $S_1(i_f) = (P_{set}, Q_{set})$.
- If the grid is not in nominal conditions, $T_m$ is computed from (44), $\tilde{T}_q$ is computed according to (15). Finally, $P$ is the larger of the two solutions of (43) and $S_1(i_f) = (P, \tilde{Q})$.

According to Proposition 4.1 $Q = \tilde{Q}$ at both the equilibrium points $\tilde{z}_i^r$ and $\tilde{z}_i^l$ of (18), and they both satisfy (43). Hence, $\tilde{z}_i^r$ and $\tilde{z}_i^l$ are located on the circle with radius $r$ and centre $C$, as shown in Fig. 9.
VI. Numerical Examples

In this section, we use two examples from the synchronous converter literature to illustrate our theoretical derivations: Example VI-A is taken from [12] and Example VI-B from [20]. The focus is the stability analysis of the fourth order model (13), and of the fifth order model (18), for varying values of \( P_{\text{set}} \) and \( Q_{\text{set}} \). We will show how the novel geometrical representation from Fig. 7(a) and (b) is indeed appearing naturally when studying the stability of the equilibrium points of (13) for \( i_f \in I_f \), and we will show how the green conic sector from Fig. 9 corresponding to the stability region of (18), depends on the value of \( \tilde{K} \).

A. Low-voltage synchronverter

We use the parameters of a synchronverter designed to supply a nominal active power of 9kW to a grid with frequency \( \omega_h = 100\pi \text{rad/sec} (50\text{Hz}) \) and line voltage \( V = 230\sqrt{3} \text{Volts} \). This is based on a real inverter that we have built, see [12]. The parameters are: \( J = 0.2 \text{Kg}\cdot\text{m}^2/\text{rad}, D_p = 3 \text{N-m/\text{rad/sec}}, L_x = 2.27\text{mH}, R_x = 0.075\Omega, K = 5 \text{kA}, n = 25, D_f = 0 \text{VAr/Volt}, m = 3.51. \) For simplicity we let \( v_{\text{set}} = \sqrt{\frac{\tilde{P}}{2}} V = 325.26 \text{Volt} \), \( Q_{\text{set}} = 0 \text{VAr} \), so that \( \tilde{Q} = 0 \). We take \( T_m = 31.69 \text{Nm} \) (according to (44), this mechanical torque corresponds to \( P_{\text{set}} = 9 \text{kW} \) and \( Q_{\text{set}} = 0 \text{VAr} \)). We have \( R = nR_x = 1.875 \Omega, L = nL_x = 56.75 \text{mH}, \) and \( \phi = 83.99^\circ \).

From Theorem 4.5 we know that there are four equilibrium points. We are interested in \( z_i^r, z_i^l \), i.e., those corresponding to positive \( i_f \) values at the equilibrium. These can be computed as explained in Sect. IV yielding:

\[
\begin{bmatrix}
  \delta_r \\
  \delta_l \\
  \phi_r \\
  \phi_l
\end{bmatrix}
= \begin{bmatrix}
  -15.24 \\
  -16.68 \\
  314.16 \\
  42.42^\circ \\
  0.54 \\
\end{bmatrix}, \quad
\begin{bmatrix}
  \delta_r \\
  \delta_l \\
  \phi_r \\
  \phi_l
\end{bmatrix}
= \begin{bmatrix}
  -235.04 \\
  -2.38 \\
  314.16 \\
  90.58^\circ \\
  3.81
\end{bmatrix}.
\]

We mention that if we compute the active power \( P \) at the above two equilibrium points according to (21), we get that \( P_1 = 9 \text{kW} \) at the stable equilibrium point (which is exactly \( P_{\text{set}} \)), and \( P_1 = -93.64 \text{kW} \) at the unstable equilibrium point. This corresponds to what we expected, based on Theorem 4.5.

The equilibrium points \( z_i^r \) corresponding to \( (P, \tilde{Q}) \) and \( z_i^l \) corresponding to \( (P, \tilde{Q}) \) are depicted in Fig. 10(a) on the smallest circle, which corresponds to \( T_m = 31.69 \text{Nm} \), i.e., to \( P_{\text{set}} = 9 \text{kW} \) and \( Q_{\text{set}} = 0 \text{VAr} \). For this circle, we get \( I_f = 0.37, 3.83\text{A} \). In the same figure, we also show two other circles, corresponding to the equilibrium points of (13) for \( T_m = 261.64 \text{Nm} \) (i.e., \( P_{\text{set}} = 50 \text{kW} \) and \( Q_{\text{set}} = 15 \text{kVAr} \)) and \( T_m = 614.60 \text{Nm} \) (i.e., \( P_{\text{set}} = 90 \text{kW} \) and \( Q_{\text{set}} = 25 \text{kVAr} \)), for which, respectively, we get \( I_f = 2.10, 5.56\text{A} \) and \( I_f = 3.78, 7.24\text{A} \). Note that the equilibrium point \( z_i^l \) is always unstable, which is a known fact according to Proposition 3.3 the equilibrium points \( z_i^r \) in this example are always stable (i.e., not too large) \( P_{\text{set}} \) and \( Q_{\text{set}} \) values. This can be checked by computing the eigenvalues of the linearizations.

The light green area in Fig. 10(a) indicates the stability region of the fourth order model (13), which indeed covers all the relevant \( (P, \tilde{Q}) \) values. We mention an interesting observation: it seems from our numerical results that the point \( M \) coincides with the centre of the green semidisk in Fig. 10(a) indicating the stability region of (13).

In Fig. 11(a) we show how the contour of the fifth order model (18) stability region varies for different values of \( \tilde{K} \). We use the following values: \( \tilde{K}_1 = 2.5 \text{kA-H}, \tilde{K}_2 = 14.3 \text{kA-H}, \tilde{K}_3 = 40 \text{kA-H}, \) and \( \tilde{K}_4 = 100 \text{kA-H} \). Note that \( \tilde{K}_2 \) is the value corresponding to \( K = 5 \text{kA} \), i.e., the one used above for the computation of \( z_i^r \) and \( z_i^l \). Even though the overall stability region area \( A(\tilde{K}) \) is increasing for increasing values of \( \tilde{K} \), it is not true that if \( \tilde{K}_1 > \tilde{K}_2 \), then \( A(\tilde{K}_2) \subset A(\tilde{K}_1) \), as is clear from Fig. 11(a).

We mention that, for \( \tilde{K} \rightarrow \infty \), it seems from our numerical results that the region of stability of (18) coincides with the intersection of the green sector from Fig. 9 and of the stability region of (13). This can be observed in Fig. 11(a).
where, for increasing values of $\hat{K}$, the stability region contours approach the boundary of the light green area.

**B. High-voltage synchronverter**

We consider a synchronverter from [20] that supplies a nominal active power of 500kW to a grid with frequency $\omega_e = 100\pi$ rad/sec (50Hz) and line voltage $V = 6000\sqrt{3}$ Volts. The parameters are: $J = 20.26$ Kg-m$^2$/rad, $D_p = 168.87$ N-m/(rad/sec), $L_s = 27.5$ mH, $R_e = 1.08$ $\Omega$, $K = 5000$ A, $n = 30$, $D_q = 0$ VAr/ rad, $m = 33$ H. As previously, we let $v_{set} = \sqrt{3}V = 8485.3$ Volt, $Q_{set} = 0$ VAr, so that $\hat{Q} = 0$. The mechanical torque $T_m = 1.83$ kN-m (according to (14)) corresponds to $P_{set} = 500$ kW and $Q_{set} = 0$ VAr. We have $R = nR_s = 32.4\Omega$, $L = nL_s = 825$ mH, and $\phi = 82.87^\circ$.

The two equilibrium points with positive $i_f$ values are:

$$
\begin{pmatrix}
i_{f_1} \\
i_{f_2} \\
i_{e_r} \\
i_{e_q} \\
\omega_e \\
\delta_e \\
\delta_r
\end{pmatrix} =
\begin{pmatrix}
-34.73 \\
314.16 \\
314.16 \\
-90.93^\circ \\
15.36^\circ \\
-33.29 \\
46.21^\circ
\end{pmatrix},
\begin{pmatrix}
i_{f_1} \\
i_{f_2} \\
i_{e_r} \\
i_{e_q} \\
\omega_e \\
\delta_e \\
\delta_r
\end{pmatrix} =
\begin{pmatrix}
-368.81 \\
-6.01 \\
314.16 \\
9.22
\end{pmatrix}.
$$

Again, if we compute the active power $P$ at the above two equilibrium points according to (21), we get that $P_e = 500$ kW at the stable equilibrium point (which is exactly $P_{set}$), and $P_i = -3.83$ MW at the unstable equilibrium point. In the following, we perform the same stability analysis of Example VI-A.

The equilibrium points $x_e^*$ corresponding to $(P, Q)$ and $z_e^*$ corresponding to $(P_1, Q)$ are shown in Fig. 10(b) on the smallest circle, which corresponds to $T_{m1} = 1.83$ kN-m, i.e., to $P_{set} = 500$ kW and $Q_{set} = 0$ VAr. For this circle, we get $I_f = [12.1, 9.29] A$. In the same figure, we also represent two other circles, corresponding to the equilibrium points of (13) for $T_{m2} = 18.18$ kN-m (i.e., $P_{set2} = 3000$ kW and $Q_{set2} = 200$ kVAr, and $T_{m3} = 45.19$ kN-m (i.e., $P_{set3} = 5400$ kW and $Q_{set3} = 400$ VAr), for which, respectively, we get $I_f = [7.28, 15.36] A$ and $I_f = [13.12, 21.20] A$. Also with these synchronverter values, it is clear from Fig. 10(b) that the equilibrium points $x_e^*$ are always stable for reasonable $P_{set}$ and $Q_{set}$ values. (Only for $T_{m3}$ can we see a blue arc appearing.) This is confirmed by the light green area in Fig. 10(b), indicating the stability region of the fourth order model (13). Also in this case, the point $M$ coincides with the centre of the green semidisk in Fig. 10(b), indicating the stability region of (13).

In Fig. 11(b) we show how the contours of the fifth order model (15) stability region vary for different values of $\hat{K}$. We use the following values: $\hat{K}_1 = 50$ kA-H, $\hat{K}_2 = 135$ kA-H, and $\hat{K}_3 = 300$ kA-H. Note that $\hat{K} = \hat{K}_2$ is the value corresponding to $K = 5$ kA, i.e., the one used above for the computation of $x_e^*$ and $z_e^*$. Also in this case, it is not true that if $\hat{K}_1 > \hat{K}_2$, then $A(\hat{K}_2) \subset A(\hat{K}_1)$, as is clear from Fig. 11(b). Moreover, we observe, again, that for $\hat{K} \to \infty$ the contours are approaching the green light area, indicating the intersection between the green sector from Fig. 9 and the stability region of (13).

**VII. Conclusions**

We have formulated a fifth order model for a grid-connected synchronverter, when the grid is considered to be an infinite bus. Conditions ensuring the existence of its equilibrium points have been derived, and a novel geometrical representation has been introduced. This representations links the region of stability of the fourth order model from [20, 21], with the region of stability of our fifth order model. Moreover, using singular perturbation methods, we have derived sufficient conditions guaranteeing the existence of (local) exponentially stable equilibrium points for the fifth order model. Finally, the validity of our theoretical results has been proved using two numerical examples coming from the synchronverter literature.

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