Thermodynamic q-Distributions That Aren’t

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Bosonic q-oscillators commute with themselves and so their free distribution is Planckian. In a cavity, their emission and absorption rates may grow or shrink—and even diverge—but they nevertheless balance to yield the Planck distribution via Einstein’s equilibrium method, (a careless application of which might produce spurious q-dependent distribution functions). This drives home the point that the black-body energy distribution is not a handle for distinguishing q-excitations from plain oscillators. A maximum cavity size is suggested by the inverse critical frequency of such emission/absorption rates at a given temperature, or a maximum temperature at a given frequency. To remedy fragmentation of opinion on the subject, we provide some discussion, context, and references.

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I. INTRODUCTION

The excitations of various systems, such as nonlinear masers, interacting-magnon-cavities, etc. [Floratos & Tomaras, Celeghini et al. II, Bullough & Bogoliubov, Chaichian, Ellinas, & Kulish] are often modelled through q-oscillators, most usefully for cyclotomic values of the deformation parameter. In some contrast, for generic values of the deformation parameter, as always in quantum algebras, q-oscillators amount to a mere change of coordinates [Curtright & Zachos, Curtright, Ghandour, & Zachos], advantageous or not, for describing a given physical situation. This is a summary of our unpublished April 1992 notes. Subsequently, we appreciated that the paradox discussed below had also already been resolved correctly by [Bullough & Bogoliubov, Agarwal & Chaturvedi] and, incorrectly, by [Lee & Yu] and a number of other authors. Because the bibliography is so fragmented, however, and because the same points are repeatedly being discussed, often clarified, but often confused despite the extant bibliography, we decided to summarize some conventional thinking below, and to provide some references. For another broad survey of the subject, the reader may consult [Chaichian et al.].

Bosonic q-oscillators are bosons (they commute with each other), and thus free collections of them are described by the conventional Bose-Einstein distribution. On the other hand, it is possible to see (e.g. Polychronakos, Yan) that for special values of the deformation parameter $q$ the deformed bosons discontinuously start to exclude each other and collapse to fermions. This has suggested the notion that the interactions which are naturally systematized by the q-oscillator formalism, and which thus suggest q-dependent partition functions and distributions, should somehow interpolate between the Fermi and the Bose-Einstein distribution.

Unfortunately, an easy, but specious, way to readily produce q-dependent density distribution functions is via Einstein’s celebrated method of balancing the equilibrium absorption and emission rates from a black-body wall of a cavity without any direct reference to the hamiltonian of the gas of q-modes in the cavity. A naive application of this technique would appear to yield q-dependent distributions for free modes, which, however, as mentioned,

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are mere bosons. Nevertheless, this paradox is resolved by correct taking of thermal averages, and the conventional Planck-Bose-Einstein distribution ensues, without much bearing on the coordinates used, as expected.

The minimum value of the frequency for which the absorption/emission rate of quasi-modes involved converges suggests an inverse maximal cavity size $1/\omega_c$, below which q-oscillators are not a good description of the system in question, at a given temperature; alternately, a maximum temperature for a given frequency; but, depending on the physics problem addressed, it may also indicate a singularity of the coordinate description employed.

II. BRIEF REVIEW OF Q-HEISENBERG ALGEBRAS

The $q$-Heisenberg algebra, which is ultimately traceable to unpublished work of Heisenberg through [Ram-plerch et al.], is derivable though a map from SU(2)$_q$.

Consider the following formal contraction of SU(2)$_q$:

$$[J_0, J_+] = J_+$$
$$[J_+, J_-] = \frac{1}{2} \left( q^{2J_0} - q^{-2J_0} \right) / (q - q^{-1})$$
$$[J_-, J_0] = J_- ,$$

(2.1)

namely [Chaichian & Ellinas] and [Ng] (contrast to [Celeghini et al.]):

$$b \equiv q^{J_0} J_- \sqrt{2(q - 1/q)} ,$$
$$b^\dagger \equiv J_+ q^{J_0} \sqrt{2(q - 1/q)} ,$$

(2.2)

so that

$$[J_0, b^\dagger] = b^\dagger ,$$
$$[b, J_0] = b ,$$

(2.3)

and hence

$$bb^\dagger - q^2 b^\dagger b = 1 - q^{4J_0} .$$

(2.4)

The last term on the r.h.s. vanishes e.g. for $|q| > 1$, $J_0 << 0$ in an infinite-dimensional representation (Schwinger’s contraction), to yield the $q$-oscillator algebra [Cigler, Kuryshkin, Jannussis et al., Macfarlane, Biedenharn, Kulish & Damashinsky] of

• TYPE b:

$$bb^\dagger - q^2 b^\dagger b = 1 .$$

(2.5)

The conventional (Bargmann holomorphic) realization [Cigler, Alvarez-Gaume et al., Ruegg, Floratos & Tomaras, Bracken et al.] for this algebra is $b^\dagger = x$ and $b = D_q x$, where $D_q$ is the quantum derivative, i.e. the slope of the chord to the graph of a function between $x$ and $qx$:

$$D_q f(x) \equiv \frac{f(qx) - f(x)}{x(q - 1)} .$$

(2.6)

(For the spacetime, $q$-Hermite polynomial, realization of the q-oscillator system see [Atakishiev & Suslov, Floreanini & Vinet, van der Jeugt, Minahan].)

The number operator

$$N \equiv \ln \left( 1 + (q^2 - 1)b^\dagger b \right) / \ln q^2$$

so that

$$[N]_b \equiv \frac{q^{2N} - 1}{q^2 - 1} = b^\dagger b$$

(2.7)

may be introduced [Macfarlane], hermitean for real $q$, s.t.

$$[N, b^\dagger] = b^\dagger ,$$
$$[N, b] = -b ,$$
$$[b, b^\dagger] = [N + 1]_b - [N]_b = q^{2N} .$$

(2.8)

1 For $q$-coherent and q-squeezed states, see [Arik & Coon, Jannussis et al., Biedenharn, Bracken et al., Nelson et al., Celeghini et al.].
This $q$-oscillator algebra can then be mapped to the alternate form of
• TYPE $\alpha$:

$$\alpha = q^{-N}b, \quad \alpha^\dagger = b^\dagger q^{-N} \quad \Rightarrow \quad [\alpha, \alpha^\dagger] = q^{-2N}, \quad (2.9)$$

with

$$\alpha^\dagger \alpha = (1 - q^{-2N})/(1 - q^{-2}) \equiv [N]_\alpha, \quad \alpha \alpha^\dagger = [N + 1]_\alpha, \quad \alpha^\dagger - q^{-2} \alpha \alpha^\dagger = 1. \quad (2.10)$$

Note the reflection to the previous q-oscillator type via $q \mapsto 1/q$. Alternatively, map to the popular form of
• TYPE $a$:

$$a = q^{-N/2}b, \quad a^\dagger = b^\dagger q^{-N/2} \quad \Rightarrow \quad aa^\dagger - qa^\dagger a = q^{-N}, \quad (2.11)$$

with

$$a^\dagger a = [N]_a \equiv q^N - q^{-N} \quad \Rightarrow \quad [a, a^\dagger] = [N + 1]_a - [N]_a. \quad (2.12)$$

An alternate, less symmetric (non-hermitean) map

$$a = q^{-N}b, \quad a^\dagger = b^\dagger, \quad (2.13)$$

which leads to the same algebra, produces the holomorphic realization $a^\dagger = x$ and $a = D_q$, where now

$$D_q f(x) \equiv f(qx) - f(x/q) x(q - 1/q). \quad (2.14)$$

These three types are then largely equivalent, and likewise equivalent to classical oscillators for generic $q$. Deforming functionals for the $q$-Heisenberg algebra of type $\alpha$ are [Cigler,Kuryshkin,Jannussis et al.,Polychronakos,Song]:

$$a^\dagger = \sqrt{[N]_\alpha} A^\dagger, \quad a = A \sqrt{[N]_\alpha}, \quad (2.15)$$

where the classical oscillator algebra is

$$[A, A^\dagger] = 1, \quad N = A^\dagger A, \quad (2.16)$$

consistent with the above.

Mutatis mutandis,

$$\alpha^\dagger = \sqrt{[N]_\alpha} A^\dagger, \quad \alpha = A \sqrt{[N]_\alpha}, \quad (2.17)$$

$$b^\dagger = \sqrt{[N]_b} A^\dagger, \quad b = A \sqrt{[N]_b}. \quad (2.18)$$

It then follows directly that these q-algebras dictate, for $N|n\rangle = n|n\rangle$:

$$\langle n + 1|a^\dagger|n\rangle = \sqrt{[n + 1]_a}, \quad (2.19)$$

and its hermitean conjugate:

$$\langle n - 1|a|n\rangle = \sqrt{[n]_a}, \quad (2.20)$$

and similarly

$$\langle n - 1|b|n\rangle = \sqrt{[n]_b}, \quad \langle n - 1|\alpha|n\rangle = \sqrt{[n]_\alpha}. \quad (2.21)$$

It follows by inspection of these deforming maps that all these excitations, $a^\dagger, A^\dagger, b^\dagger, \alpha^\dagger$, commute among themselves, and so do their annihilation operators, like good bosons. The deformation maps detailed above are invertible for generic $q$, i.e. not a root of unity, and thus they may be effectively regarded as partial resummation of a perturbation series.

\footnote{However, for $q$ a root of unity, powers of the creation or annihilation operators will vanish [Polychronakos,Yan], and for $q = i$ fermions result, as the square of two creators or annihilators vanishes: observe that $[2] = 0$ for all three types. For a generic discussion in purely algebraic terms, see [Zachos].}
The hamiltonians formed by the above q-boson mode oscillators are as infinitely diverse as the exotic functions of the number operators $N$ (with integral eigenvalues)—or the $[N]$s—one could think of. They are interacting with the exception of the plain $N$, the provider of the truly free, linearly-spaced, spectrum. Via Bose statistics, these determine partition functions, and whence q-dependent spectral density distributions. E.g. Martin-Delgado, Bullough & Bogoliubov, Neskovic & Urosevic, Jannussis et al., Chaichian & Ellinas, Floratos & Tomaras, Manko et al., Su & Shanta et al.]

Nevertheless, for free mode-gas hamiltonians, given the absorption and emission rates from a black-body wall of a cavity, Einstein indicates how to compute density distribution functions directly from the condition of equilibrium. Specifically, let absorption and emission (spontaneous and induced) rates for q-phonons of energy $\hbar\omega$ be $C_a, C_e$. The thermalized cavity wall molecules populations are controlled by the Boltzmann distribution $\exp(-\frac{\hbar\omega}{kT})$, so that the relative population ratio of emitting to absorbing molecules is $\exp(-\frac{\hbar\omega}{kT})$. Incorporating, in the spirit of Wien’s law, $\hbar/kT$ into $\omega$, for an equilibrium distribution of excitations in the cavity,

$$C_a = C_e \ e^{-\omega} \quad (3.1)$$

holds. For ordinary photons, the QED interaction hamiltonian directly dictates $C_e \propto \langle n \rangle + 1$ and $C_a \propto \langle n \rangle$, leading to the Bose-Einstein distribution of the Planck law:

$$\langle n(\omega) \rangle = \frac{1}{e^{\omega} - 1}. \quad (3.2)$$

For the above quasi-modes, however, eqs.(2.19-21), and a linear hermitean interaction for such deformed q-boson excitations lead to

$$C_e \propto \langle n + 1 \rangle, \quad C_a \propto \langle n \rangle. \quad (3.3)$$

\textbf{An opportunity to err.} For the conventional boson case, $n$ and $\langle n \rangle$ enjoy exactly the same algebraic status, by linearity. Not so for the q-case, though: if thermal averages were overlooked, i.e. if the averages of these hyperbolic functions were the same functions of the averages $\langle n \rangle$, then one could ignore the difference between $n$ and $\langle n \rangle$, and proceed as before. The equilibrium reversibility condition would then yield the spurious distributions:

$$n_a(\omega) = \frac{1}{\ln(q^2)} \frac{1}{q - \omega}, \quad n_b(\omega) = \frac{1}{\ln(q^2)} \frac{1}{\omega - q}, \quad n_a(\omega) = \frac{1}{\ln(q^2)} \frac{q^2 - \omega}{1 - \omega}. \quad (3.4)$$

Naturally, all three distributions reduce to the classical one $n(\omega)$ as $q \to 1$. Note that as $q \to 1/q$, $n_a$ remains invariant, whence $n_a$ is real for $q = e^{i\phi}$, a pure phase; while $n_b \leftrightarrow n_a$.

For very large $\omega$,

$$n_a \sim e^{-\omega (q - 1/q) / \ln q^2}, \quad n_b \sim e^{-\omega (q^2 - 1) / \ln q^2}, \quad n_a \sim e^{-\omega (1 - 1/q^2) / \ln q^2}, \quad (3.5)$$

amounting to a mere q-dependent frequency shift of the Wien-regime distribution.

\[3\] Since any and all oscillators are deformable to each other for \textit{generic} $q$, the \textit{type} of oscillators used says little about the spectrum: via the above deforming functionals, it is evident that any hamiltonian may be represented in terms of any type of oscillators, and convenience is the dominant consideration. Nevertheless, some authors appear to suggest that the aesthetic simplicity of a hamiltonian bilinear in some type of oscillator may go beyond a trivial coordinate-system statement to somehow render it “chosen”, and its (interacting) spectrum somehow “basic”.

\[4\] For real $q$, not all infrared singularities of the above distributions are at $\omega = 0$. In particular, for $q > 1$,

$n_a$ only has a singularity at $\omega_c = 0$, but

$n_b$ diverges at $\omega_c = 2\ln q$, and goes imaginary for lower frequencies;

$n_a$ diverges at $\omega_c = \ln q$, and goes imaginary for lower frequencies.

For $q < 1$,

$n_a$ only has a singularity at $\omega_c = 0$, but

$n_b$ diverges at $\omega_c = -2\ln q$, and goes imaginary for lower frequencies;

$n_a$ diverges at $\omega_c = -\ln q$, and goes imaginary for lower frequencies.
Multiplying by the excitation energy $\omega$ and the density of states in the cavity of volume $V$ (in three dimensions) yields the deformed spectral energy density per-frequency-interval, generalizing the law of Planck:

$$dU = \frac{V(kT)^4}{\hbar^3 c^3 \pi^2} n_{a,b,\alpha}(\omega) \omega^3 d\omega,$$

where the rescaled definition of frequency must be borne in mind.

But this would be highly paradoxical, as the free quasi-modes commute with each other, and so should obey the Bose-Einstein distribution, which is the distribution dictated directly by the partition function of a non-degenerate equally-spaced spectrum, and is independent of $q$.

### IV. NON-TRIVIAL THERMAL AVERAGES: A FREE SYSTEM OF COMMUTING QUASIMODES IS EINSTEIN-PLANCK DISTRIBUTED

Of course, the functions thermally averaged were not linear, and hence not trivial. Bullough & Bogoliubov, Agarwal & Chaturvedi compute these averages correctly. In contrast to the linear case, explicit knowledge of the (trivial) spectrum is now necessary.

One only needs to recognize that, for free excitations (whence a linear, additive, spectrum),

$$\langle [N]_b \rangle = \frac{\sum_n e^{-n\omega} q^{2n-1}}{\sum_n e^{-n\omega}} = 1/(e^\omega - q^2) = \frac{\langle N \rangle}{1 + (1 - q^2)\langle N \rangle},$$

where, for convenience, we substitute for $1/(e^\omega - 1) = \langle N \rangle$ in the final expressions.

Moreover, in some generality, merely shift the argument of the exponential by $\omega$, carry this one power outside, and compare to the above:

$$\langle [N + 1]_b \rangle = e^\omega \langle [N]_b \rangle = e^\omega/(e^\omega - q^2) = \frac{1 + \langle N \rangle}{1 + (1 - q^2)\langle N \rangle}.$$  \hspace{1cm} (4.2)

As a consequence, even though the rates themselves are not Einstein, their ratio is still the standard one. The black-body energy spectrum cannot be used as a handle for distinguishing quasi-modes from plain oscillator excitations. Emission and absorption accelerate wildly to infinity at the critical value of the frequency $\omega_c = 2 \ln q$. This does not signal an intrinsic condensation phenomenon, as the number densities are well-defined at that frequency. The interaction mechanism simply populates/depopulates high occupation numbers too rapidly. Reinstating the factors absorbed in the units of frequency, $\omega_c = 2kT \ln q / \hbar$. This minimum value of the frequency specifies an inverse maximal cavity size $1/\omega_c$, below which q-oscillators are not a satisfactory description of the system under study, at a given temperature; or else a maximal temperature for a given frequency; this may also amount to a singularity of the coordinate description employed, of use or not in simulating nonlinear systems.

For $\alpha$-modes, substitute $q \mapsto 1/q$, with the corresponding singularity for $q < 1$ and the same distribution; and for $a$-excitations,

$$\langle [N]_a \rangle = \frac{e^{\omega} - 1}{(e^\omega - q)(e^\omega - 1/q)},$$  \hspace{1cm} (4.3)

with $\omega_c = \mp \ln q$ for $q \leq 1$, and likewise the same distribution.

Misinterpretation of emission and absorption rates as number density distributions, however, Lee & Yu, eventuates in incorrect energy spectral distributions. Tuszyński et al., Song et al. follow in much the same spirit to reach spurious distributions of the type exemplified in the previous section.

Of course, for special cyclotomic values of the parameter $q$, the statistics of the particles do change, abruptly, as the Fock space is truncated Yan, Polychronakos. But since the distribution functions do not depend on $q$, they do not reflect these changes, which must be put in by hand, taking account of state multiplicities.

\hspace{1cm}  

5In fact, these references appear to miss the singularity in the absorption rate discussed above, even though their figure might hint at it.
In sharp contrast to the above discussion, for interacting hamiltonians, the partition function genuinely depends on $q$, and hence such $q$-distributions are. Nevertheless, they need not be associated to a given deformation/coordinate description, except through considerations of calculational convenience. The hyperbolic-sine-spectrum partition function has not been summed explicitly, beyond graphing/expanding to a few leading orders in $q^{-1}$, [Martin-Delgado, Bullough & Bogoliubov, Neskov & Urosevic, Manko et al.], however, this is liable to miss interesting nonanalytic structure, and thus possible condensation features that such structure may underlie.

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