ON THE SUPPORT OF
THE FREE ADDITIVE CONVOLUTION

By
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Abstract. We consider the free additive convolution of two probability mea-
sures $\mu$ and $\nu$ on the real line and show that $\mu \boxplus \nu$ is supported on a single interval
if $\mu$ and $\nu$ each has single interval support. Moreover, the density of $\mu \boxplus \nu$ is
proven to vanish as a square root near the edges of its support if both $\mu$ and $\nu$
have power law behavior with exponents between $-1$ and 1 near their edges. In
particular, these results show the ubiquity of the conditions in our recent work on
optimal local law at the spectral edges for addition of random matrices [5].

1 Introduction

The classical convolution of two probability measures is of key interest in proba-
bility theory as it gives the law of the sum of two independent random variables.
In analogy, Voiculescu [33] introduced in free probability theory the free additive
convolution. Let $\mu$ and $\nu$ be two Borel probability measures on the real line. Then
the free additive convolution of $\mu$ and $\nu$, denoted $\mu \boxplus \nu$, is the law of $X+Y$ where $X$
and $Y$ are freely independent, self-adjoint, non-commutative random variables with
laws $\mu$ and $\nu$. Though conceptually related, the classical convolution and the free
convolution behave strikingly different. For example, the classical convolution of
two pure point measures is always pure point, while the free convolution always
has a non-vanishing absolutely continuous part. In particular, choosing $\mu = \nu$ as
centered Bernoulli distribution, the free convolution $\mu \boxplus \mu$ is an absolutely con-
tinuous measure, while the classical convolution is regularizing only in the sense
that the $n$-fold convolution of $\mu$ becomes, upon rescaling, Gaussian in the limit
of large $n$. Note that the analogous central limit theorem for the free additive
convolution yields Wigner’s semicircle law in the limit.

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This basic example fittingly illustrates that, in contrast to the classical convolution, it is hard to infer, based upon intuition and heuristics, qualitative properties of the free additive convolution measure. Part of the reason is that there is no simple formula for the free additive convolution measure; it can only be obtained as implicit solution to certain systems of equations. Thence the following seemingly simple question turns out quite difficult to answer: If $\mu$ and $\nu$ are both supported on a single interval, is $\mu \boxplus \nu$ then also supported on a single interval? Interestingly, while regularity properties of the free convolution measure have been extensively studied [15, 8, 9], this natural problem apparently has not been studied in the literature. The aim of this note is to answer this question in the affirmative for a large class of initial measures motivated by random matrix theory.

The emergence of the semicircle law indicates a strong link between free probability and random matrix theory. Voiculescu discovered in [34] that random matrices not only provide examples of asymptotically free random variables, but can also be used to generate freeness. Conjugating symmetric matrices by independent Haar unitary matrices furnishes asymptotically free random variables. The prime example is the setup of the addition of two deterministic matrices in randomly chosen relative basis. In other words, if $A = A_N$ and $B = B_N$ are two sequences of deterministic Hermitian matrices of size $N$ and $U$ is Haar distributed on the unitary group $U(N)$, then $A$ and $UBU^*$ are asymptotically free in the limit of large $N$ and the asymptotic eigenvalue distribution of $A + UBU^*$ is given by the free additive convolution of the limiting eigenvalue distributions of $A$ and $B$. Given this convergence result, it is natural to ask about the speed of convergence and whether the convergence also holds on local scales. We have recently answered both questions by deriving a so-called local law for the Green functions; cf. [5, 6, 7]. However, the study of local laws for the model mentioned above crucially relies, in contrast to the global scale, on detailed regularity properties and the qualitative behavior of the deterministic free convolution measures. An objective of the present paper is to derive these decisive properties of the free convolution in the most relevant cases arising in random matrix theory.

We will focus on the free additive convolution of a class of Jacobi type measures. These are measures supported on a single interval with density behaving as a power law with exponent between -1 and 1 near the edges; see Assumption 2.1 below. Wigner’s semicircle law as well as the Marchenko–Pastur law are included in this class. Our main result, Theorem 2.2, asserts that the free additive convolution of two Jacobi type measures is supported on a single interval and that its density vanishes as a square root at the two endpoints. These are the main conditions
on the free convolution measure we required in our recent paper on the optimal local law at the spectral edges [5]. Theorem 2.2 shows that these assumptions are natural.

The square root behavior at the edge is ubiquitous for densities arising in random matrix theory. The same phenomenon has been extensively studied for Wigner type random matrices [1], and more recently for the underlying Dyson equation in a general non-commutative setup [3]. Under some regularity condition on the matrix of variances, single interval support for the density has also been shown [1, Theorem 2.11]; see also [3, Corollary 9.4] for a generalization. Despite these similarities in the statements, the approach used for the Dyson equation is very different from the methods in the current work; simply the structures of the underlying defining equations are not comparable.

Our proofs rely on methods from function theory. Albeit being introduced as an algebraic operation, the free additive convolution can be studied with complex analysis. The Stieltjes transform of the free additive convolution is related to the Stieltjes transforms of the original measures through analytic subordination. The existence of analytic subordination functions off the real line was observed in [35, 19] and may directly be used to define the free additive convolution [12, 20]; see Subsection 2.1 below. Function theory then provides powerful tools to study the free additive convolution and its regularization properties in great generality, that is, for very general Borel probability measures; see [15, 8, 9, 11] and references therein. Specializing to Jacobi measures, we can analyze the boundary behavior of the subordination function on the real line and extract the qualitative behavior of the free convolution measure claimed in Theorem 2.2.

**Organization of the paper.** In Section 2, we state our main results in detail, give the full definition of the free additive convolution and embed our paper in the literature. In Section 3, we derive estimates on the Stieltjes transform of the free convolution and localize the subordination function on the real line. This information is then used in Section 4 to characterize regular edges and to prove the Theorem 2.2.

**Notation.** We use $c$ and $C$ to denote strictly positive constants. Their values may change from line to line. We denote by $\mathbb{C}^+$ the upper half-plane in $\mathbb{C}$, i.e., $\mathbb{C}^+ := \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$.

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2 Main results

Let \( \mu_\alpha \) and \( \mu_\beta \) be two Borel probability measures on \( \mathbb{R} \). In this paper we study support and regularity properties of the free additive convolution measure, \( \mu_\alpha \boxplus \mu_\beta \); see Subsection 2.1 for the precise definition of the free additive convolution. We focus on the case when \( \mu_\alpha \) and \( \mu_\beta \) are both absolutely continuous and have single interval support. Moreover, we assume that they are of Jacobi type by which we mean that they vanish as a power-law at the edges of the support. More precisely, we will assume that \( \mu_\alpha \) and \( \mu_\beta \) satisfy the following.

Assumption 2.1. The measures \( \mu_\alpha \) and \( \mu_\beta \) are compactly supported and centered probability measures that are absolutely continuous with respect to Lebesgue measure with density functions \( \rho_\alpha \) and \( \rho_\beta \) such that:

(i) Both density functions \( \rho_\alpha \) and \( \rho_\beta \) have single non-empty interval supports, \( [E_\alpha^{-}, E_\alpha^{+}] \) and \( [E_\beta^{-}, E_\beta^{+}] \), respectively.

(ii) The density functions have a power law behavior: there are exponents \( -1 < t_\alpha^{-}, t_\alpha^{+} < 1 \) and \( -1 < t_\beta^{-}, t_\beta^{+} < 1 \) such that

\[
C^{-1} \leq \frac{\rho_\alpha(x)}{(x - E_\alpha^{-})^{t_\alpha^{-}}(E_\alpha^{+} - x)^{t_\alpha^{+}}} \leq C, \quad \text{for a.e. } x \in [E_\alpha^{-}, E_\alpha^{+}],
\]

\[
C^{-1} \leq \frac{\rho_\beta(x)}{(x - E_\beta^{-})^{t_\beta^{-}}(E_\beta^{+} - x)^{t_\beta^{+}}} \leq C, \quad \text{for a.e. } x \in [E_\beta^{-}, E_\beta^{+}],
\]

hold for some positive constant \( C > 1 \).

We will explain in Section 2.1 that under these conditions the free convolution measure \( \mu_\alpha \boxplus \mu_\beta \) is known to have a continuous and bounded density \( \rho \). Our following main result shows that \( \rho \) is supported on a single interval with square root singularities at the edges.

Theorem 2.2. Assume that \( \mu_\alpha \) and \( \mu_\beta \) satisfy Assumption 2.1, in particular, these measures have single interval support. Then their free additive convolution \( \mu_\alpha \boxplus \mu_\beta \) is also supported on a single compact interval that we denote by \( [E_{-}, E_{+}] \). Moreover, \( E_{-} < 0, E_{+} > 0 \) and there exists \( C > 1 \) such that

\[
C^{-1} \leq \frac{\rho(x)}{\sqrt{x - E_{-}} \sqrt{E_{+} - x}} \leq C, \quad \forall x \in [E_{-}, E_{+}],
\]

where \( \rho \) denotes the continuous density function of \( \mu_\alpha \boxplus \mu_\beta \).
Remark 2.3. It can be checked from our proofs that the constant $C$ in (2.2) depends only on certain control parameters, namely on the constant in (2.1), on the exponents in (2.1), the second moments of $\mu_\alpha$ and $\mu_\beta$, and on the constant $c > 0$ serving as a lower bound in

\begin{equation}
|m_{\mu_\alpha}(E)| \geq c, \quad |m_{\mu_\beta}(E')| \geq c
\end{equation}

for all $E \in [E_\alpha^\alpha - 1, E_\alpha^\alpha] \cup [E_\alpha^\alpha, E_\alpha^\alpha + 1]$ and for all $E' \in [E_\beta^\beta - 1, E_\beta^\beta] \cup [E_\beta^\beta, E_\beta^\beta + 1]$.

Notice that the Stieltjes transforms of $\mu_\alpha$ and $\mu_\beta$ can be extended as non-tangential limits to the real axis Lebesgue-almost everywhere. Outside the support of the measure $\mu_\alpha$, respectively $\mu_\beta$, these extensions are real valued and analytic. Since $\mu_\alpha$ and $\mu_\beta$ have single interval supports, the Stieltjes transforms cannot have any zeros outside their supports. In particular, there is indeed a positive lower bound $c$ in (2.3) on the indicated intervals.

In general, we apply a similar convention throughout paper: when we state that some constant depends on the two input measures $\mu_\alpha$ and $\mu_\beta$, we mean that it depends only on the above control parameters.

Remark 2.4. The assumption that the exponents $t_\alpha^\alpha, t_\alpha^\beta, t_\beta^\beta$ and $t_\beta^\alpha$ in (2.1) are bigger than $-1$ is necessary to have finite measures. In general, the assumption that they are smaller than 1 is necessary to have a square root behavior at the edges of the free convolution. Indeed, if one of the exponents exceeds 1, it can happen that an edge behavior other than the square root emerges; see [24, 25] for a detailed analysis of a special case when one of the measures is the semicircle law and the other has a convex behavior at the endpoints of the supports. However, we still expect that the free additive convolution of two Jacobi measures with general exponents is supported on a single interval. We point out that in most applications the endpoint exponents are strictly below 1, so our theorem applies.

2.1 Free additive convolution. In this subsection we review the definition of the free additive convolution in detail. We start with the Stieltjes transform: For any probability measure $\mu$ on $\mathbb{R}$, its Stieltjes transform is defined as

$$m_\mu(z) := \int_{\mathbb{R}} \frac{1}{x - z} \, d\mu(x), \quad z \in \mathbb{C}^+.$$  

We further denote by $F_\mu$ the negative reciprocal Stieltjes transform of $\mu$, i.e.,

$$F_\mu(z) := -\frac{1}{m_\mu(z)}.$$  

Note that $F_\mu : \mathbb{C}^+ \to \mathbb{C}^+$ is analytic and satisfies

$$\lim_{\eta \to \infty} \frac{F_\mu(i\eta)}{i\eta} = 1.$$
Conversely, if
\[ F : \mathbb{C}^+ \to \mathbb{C}^+ \]
is an analytic function with \( \lim_{\eta \to \infty} F(i\eta)/i\eta = 1 \), then \( F \) is the negative reciprocal Stieltjes transform of a probability measure \( \mu \), i.e., \( F(z) = F_\mu(z) \), for all \( z \in \mathbb{C}^+ \); see, e.g., [2].

Voiculescu introduced the free additive convolution of Borel probability measures on \( \mathbb{R} \) in the groundbreaking paper [33] in an algebraic setup as the distribution of the sum of two freely independent non-commutative random variables. Our starting point is the following result which can be used to define the free additive convolution in an analytic setup.

**Proposition 2.5** (Theorem 4.1 in [12], Theorem 2.1 in [20]). Given two Borel probability measures, \( \mu_\alpha \) and \( \mu_\beta \), on \( \mathbb{R} \), there exist unique analytic functions,
\[ \omega_\alpha, \omega_\beta : \mathbb{C}^+ \to \mathbb{C}^+, \]
such that:

(i) for all \( z \in \mathbb{C}^+ \), \( \text{Im} \omega_\alpha(z), \text{Im} \omega_\beta(z) \geq \text{Im} z \), and
\[ \lim_{\eta \to \infty} \frac{\omega_\alpha(i\eta)}{i\eta} = \lim_{\eta \to \infty} \frac{\omega_\beta(i\eta)}{i\eta} = 1; \]

(ii) for all \( z \in \mathbb{C}^+ \),
\[ F_{\mu_\alpha}(\omega_\beta(z)) = F_{\mu_\beta}(\omega_\alpha(z)), \quad \omega_\alpha(z) + \omega_\beta(z) - z = F_{\mu_\alpha}(\omega_\beta(z)). \]

The analytic function \( F : \mathbb{C}^+ \to \mathbb{C}^+ \) defined by
\[ F(z) := F_{\mu_\alpha}(\omega_\beta(z)) = F_{\mu_\beta}(\omega_\alpha(z)) \]
is by part (i) of Proposition 2.5 the negative reciprocal Stieltjes transform of a probability measure \( \mu \), called the free additive convolution of \( \mu_\alpha \) and \( \mu_\beta \) and denoted by
\[ \mu \equiv \mu_\alpha \boxplus \mu_\beta. \]
The functions \( \omega_\alpha \) and \( \omega_\beta \) are referred to as the subordination functions. The subordination phenomenon was first noted by Voiculescu [35] in a generic situation and in full generality by Biane [19].

Choosing \( \mu_\alpha \) arbitrary and \( \mu_\beta \) as delta mass at \( x \in \mathbb{R} \), it is easy to check that \( \mu_\alpha \boxplus \mu_\beta \) simply is \( \mu_\alpha \) shifted by \( x \). We therefore exclude this trivial case from our considerations. Moreover, by a simple shift we may without lost of generality assume that \( \mu_\alpha \) and \( \mu_\beta \) are centered measures; see Assumption 2.1.
The atoms of $\mu_a \boxplus \mu_\beta$ are determined as follows. A point $w \in \mathbb{R}$ is an atom of $\mu_a \boxplus \mu_\beta$ if and only if there exist $x, y \in \mathbb{R}$ such that $w = x + y$ and $\mu_a((x)) + \mu_\beta((y)) > 1$; see [15, Theorem 7.4]. Thus in particular under Assumption 2.1, the free additive convolution does not have any atoms. The boundary behavior of the subordination functions $\omega_a$ and $\omega_\beta$ was studied by Belinschi in a series of papers [8, 9, 10] where he proved the following two results. For sake of simplicity, we limit the discussion to compactly supported measures.

**Proposition 2.6** (Theorem 3.3 in [9], Theorem 6 in [10]). Let $\mu_a$ and $\mu_\beta$ be compactly supported Borel probability measures on $\mathbb{R}$, none of them being a single point mass. Then the subordination functions $\omega_a, \omega_\beta : \mathbb{C}^+ \to \mathbb{C}^+$ extend continuously to $\mathbb{C}^+ \cup \mathbb{R}$ as functions with values in $\mathbb{C}^+ \cup \mathbb{R} \cup \{\infty\}$.

In Theorem 4.1 of [9], Belinschi proved that the singular continuous part of $\mu_a \boxplus \mu_\beta$ is always zero, and that the absolutely continuous part of $\mu_a \boxplus \mu_\beta$ is always nonzero and admits a continuous density function. We denote this density function by $\rho$. Summing up, we have under Assumption 2.1 the following regularity result:

**Lemma 2.7.** Let $\mu_a$ and $\mu_\beta$ satisfy Assumption 2.1. Then the free additive convolution measure $\mu_a \boxplus \mu_\beta$ is absolutely continuous with respect to Lebesgue measure and admits a continuous and bounded density function $\rho$ that is real analytic wherever strictly positive.

**2.2 Previous results.** We start by results concerning the supports of free convolution measures. Biane studied in [18] the free convolution of the semicircle law with an arbitrary probability measure: Denote by

$$
\sigma_t(dx) = \frac{1}{2\pi t} \sqrt{4t - x^2} 1_{[-2\sqrt{t}, 2\sqrt{t}]}(x) dx
$$

the density of the semicircle law of variance $t$. Given a probability measure $\lambda$ on $\mathbb{R}$, we obtain a one-parameter family of probability measures, the so-called semi-circular flow, by setting

$$
\mu_t := \lambda \boxplus \sigma_t, \quad t > 0.
$$

Biane proved that the number of connected components of $\mu_t$ is a non-increasing function of $t$ and that the continuous density of $\mu_t$ satisfies

$$
|\mu_t(x)| \leq \left( \frac{3}{4\pi^3 t^2 |x - x_0|} \right)^{1/3}, \quad x \in \mathbb{R},
$$

where $x_0$ is the closed point to $x$ in the complement of the interior of the support of $\mu_t$. For further results of the semi-circular flow we refer to [31].
In the appendix to [13], Biane obtained support properties of freely stable laws.

Voiculescu proved in [32] the free central limit theorem for the addition of freely independent non-commutative random variables in terms of convergence of moments. The convergence to the limiting semicircle distribution turned out to be much stronger: already after a finite number of free convolutions, the distribution of the finite free sum becomes absolutely continuous with respect to Lebesgue measure. This so-called superconvergence was established first by Bercovici and Voiculescu in [14] and subsequently refined by Kargin [22] and Wang [36].

The $n$-fold free convolution power $\lambda \boxplus n$ of a probability measure $\lambda$ on $\mathbb{R}$ can be embedded in a one-parameter family $\{\lambda \boxplus t, t \geq 1\}$ with the semigroup structure $\lambda \boxplus t_1 \boxplus \lambda \boxplus t_2 = \lambda \boxplus (t_1 + t_2)$; $t_1, t_2 \geq 1$; see [14, 29]. Huang proved in [21] that the support of $\lambda \boxplus t$, $t > 1$, consists of at most finitely many atoms and countably many intervals and that the number of the components of the support of $\lambda \boxplus t$ is a decreasing function of $t$. We mention that the system of subordination equations in both cases, the semicircular flow and the free convolution semigroup, reduce to a single equation rendering the support analysis much simpler.

The results in [14, 18, 21, 23, 36] seemingly suggest that convolving freely reduces the number of connected components in the support. Yet superconvergence results to freely stable laws in [17] and results in [16] might betoken that the situation is in general not quite as clear.

For the free addition of two Jacobi measures, Olver and Rao proved in [30] that if $\mu_\alpha$ is a Jacobi measure with $\gamma_\alpha = 1/2$ and $\mu_\beta$ is a Jacobi measure whose Stieltjes transform is single-valued, then $\mu_\alpha \boxplus \mu_\beta$ is a Jacobi measure which exhibits a square root behavior at its edges.

In [5], Section 3, we studied the behavior of free additive convolution at the smallest and largest endpoints of its support. We showed that under similar conditions to the current Assumption 2.1 and the additional assumption that at least one of the following two bounds,

$$
\sup_{z \in \mathbb{C}^+} |m_{\mu_\alpha}(z)| \leq C, \quad \sup_{z \in \mathbb{C}^+} |m_{\mu_\beta}(z)| \leq C,
$$

holds, for some positive constant $C$, that $\mu_\alpha \boxplus \mu_\beta$ vanishes as a square root at the smallest and largest endpoint of its support. Theorem 2.2 overpasses these results by removing the unnatural assumption in (2.8), but more importantly, it asserts that the free additive convolution, under Assumption 2.1, has only two edges, i.e., $\mu_\alpha \boxplus \mu_\beta$ is supported on a single interval, and its density is strictly positive inside the support.
Finally, we mention that the linear stability of the system (2.5) was first effectively studied by Kargin in [23] under some genericity conditions. In [4], we showed that these conditions are fulfilled in the regular bulk and in [5] we extended the stability results to square root edges where the system is only quadratically stable.

3 Properties of the Stieltjes transform and the subordination functions

We will repeatedly use the following integral representation for Pick functions; see, e.g., Chapter III of [2] for a reference.

**Lemma 3.1.** Let $f : \mathbb{C}^+ \rightarrow \mathbb{C}^+ \cup \mathbb{R}$ be an analytic function. Then there exists a positive Borel measure $\mu$ on $\mathbb{R}$ and $a \in \mathbb{R}, b \geq 0$ such that

$$f(\omega) = a + b \omega + \int_{\mathbb{R}} \left( \frac{1}{x - \omega} - \frac{x}{1 + x^2} \right) d\mu(x), \quad \omega \in \mathbb{C}^+, \quad (3.1)$$

and

$$\int_{\mathbb{R}} \frac{1}{1 + x^2} d\mu(x) < \infty. \quad (3.2)$$

The negative reciprocal Stieltjes transforms of $\mu_\alpha$ and $\mu_\beta$ enjoy the following properties.

**Lemma 3.2.** Let $\mu_\alpha$ and $\mu_\beta$ satisfy Assumption 2.1. Then there exist Borel measures $\hat{\mu}_\alpha$ and $\hat{\mu}_\beta$ such that

$$F_{\mu_\alpha}(\omega) - \omega = \int_{\mathbb{R}} \frac{1}{x - \omega} d\hat{\mu}_\alpha(x), \quad \omega \in \mathbb{C}^+, \quad (3.3)$$

with

$$0 < \hat{\mu}_\alpha(\mathbb{R}) = \int_{\mathbb{R}} x^2 d\mu_\alpha(x) < \infty \quad (3.4)$$

and

$$\text{supp}\, \hat{\mu}_\alpha = \text{supp}\, \mu_\alpha. \quad (3.5)$$

In particular, $\hat{\mu}_\alpha$ is a finite compactly supported Borel measure. The same statements hold true when the index $\alpha$ is replaced by $\beta$.

**Remark 3.3.** Equations (3.3) and (3.4) are well-known results; see, e.g., Proposition 2.2 in [27]. For convenience we include their proofs below.
Proof of Lemma 3.2. We start by noticing that the Stieltjes transform of \( \mu_a \) admits the following asymptotic expansion,

\[
m_{\mu_a}(i\eta) = \frac{1}{-i\eta} \int_{\mathbb{R}} \text{d}\mu_a(x) - \frac{1}{(i\eta)^2} \int_{\mathbb{R}} x \text{d}\mu_a(x) - \frac{1}{(i\eta)^3} \int_{\mathbb{R}} x^2 \text{d}\mu_a(x) + O(\eta^{-4}),
\]

\[
= \frac{1}{-i\eta} + \frac{1}{i\eta^3} \int_{\mathbb{R}} x^2 \text{d}\mu_a(x) + O(\eta^{-4}),
\]

as \( \eta \nearrow \infty \), where we used that \( \mu_a \) is a centered probability measure. Hence, taking the negative reciprocal, we find in the limit \( \eta \nearrow \infty \) that

\[
(F_{\mu_a}(i\eta) - i\eta = -\frac{1}{i\eta} \int_{\mathbb{R}} x^2 \text{d}\mu_a(x) + O(\eta^{-2}).
\]  

Next, note that \( f(\omega) := F_{\mu_a}(\omega) - \omega \) satisfies

\[
\text{Im}(f(\omega)) = \frac{\text{Im} m_{\mu_a}(\omega) - \text{Im} \omega |m_{\mu_a}(\omega)|^2}{|m_{\mu_a}(\omega)|^2} = \frac{\text{Im} \omega \left( \int_{\mathbb{R}} \frac{\text{d}\mu_a(x)}{|x-\omega|^2} / \left| \int_{\mathbb{R}} \frac{\text{d}\mu_a(x)}{|x-\omega|} \right|^2 - 1 \right)}{ \text{Im} \omega} > 0,
\]

for \( \omega \in \mathbb{C}^+ \), where we used that \( \mu_a \) is supported at more than one point by Assumption 2.1. Thus Lemma 3.1 applies to \( f \) with some measure \( \mu =: \hat{\mu}_a \). Comparing (3.6) with (3.1), we conclude that \( b = 0 \) and thus

\[
(F_{\mu_a}(\omega) - \omega = f(\omega) = a + \int_{\mathbb{R}} \left( \frac{1}{x-\omega} - \frac{x}{1+x^2} \right) \text{d}\hat{\mu}_a(x).
\]

Choosing \( \omega = i\eta \) in (3.7) and taking \( \eta \nearrow \infty \), comparison with (3.6) immediately yields that

\[
a = \int_{\mathbb{R}} \frac{x}{1+x^2} \text{d}\hat{\mu}_a
\]

and thus (3.3) holds. Furthermore, (3.4) also holds by comparing the coefficient of \( \eta^{-1} \) on the right sides of (3.3) and (3.6) in the large \( \eta \) limit.

Taking the imaginary parts in (3.7) we further obtain

\[
\text{Im}(f(\omega)) = \frac{\text{Im} m_{\mu_a}(\omega)}{|m_{\mu_a}(\omega)|^2} - \text{Im} \omega = \int_{\mathbb{R}} \frac{\text{Im} \omega}{|x-\omega|^2} \text{d}\hat{\mu}_a(x), \quad \omega \in \mathbb{C}^+.
\]

From Assumption 2.1, we know that the extension of \( m_{\mu_a} \) to \( \mathbb{R} \) is continuous and real valued outside the support of \( \mu_a \). Since in addition \( \mu_a \) is supported on a single interval, we conclude that \( m_{\mu_a} \) does not have any zeros on \( \mathbb{R} \setminus \text{supp} \mu_a \). Hence taking the limit \( \text{Im} \omega \searrow 0 \) in (3.8), we conclude by the Stieltjes inversion formula that \( \hat{\mu}_a \) is absolutely continuous with respect to Lebesgue measure on the
complement of supp $\mu_a$ with vanishing density function. Hence we must have supp $\hat{\mu}_a \subseteq$ supp $\mu_a$. In particular, $\hat{\mu}_a$ is a finite compactly supported measure.

Finally, to conclude (3.5), we need to prove the opposite containment, i.e., that supp $\hat{\mu}_a \supseteq$ supp $\mu_a$. Suppose, on the contrary, that supp $\hat{\mu}_a$ is a proper subset of supp $\mu_a = [E_\alpha^-, E_\alpha^+]$. Then we can find a non-empty open interval $I \subset \mu_a \setminus \hat{\mu}_a$ such that $f(\omega) : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ extends continuously to $I$ with $\text{Im} f(\omega) = 0$, for all $\omega \in I$. Then by the Schwarz reflection principle, $f$ extends analytically through $I$ and, hence, $m$ is meromorphic on $I$. However, since $I \subset \text{supp} \mu_a$, we have $\lim_{\eta \searrow 0} \text{Im} m_{\mu_a}(\omega + i\eta) > 0$ by Assumption 2.1, for almost all $\omega \in I$. Since $m(\omega)$ is meromorphic on $I$ and

$$\text{Im} f(\omega) = \text{Im} m(\omega)/|m(\omega)|^2, \quad \omega \in I,$$

we hence also have $\lim_{\eta \searrow 0} \text{Im} f(\omega + i\eta) > 0$ for almost all $\omega \in I$, a contradiction to $\text{Im} f(\omega) = 0$, for all $\omega \in I$. Thus $I$ must be empty and we have supp $\hat{\mu}_a = \text{supp} \mu_a$. This proves (3.5) and concludes the proof of Lemma 3.2. \hfill \Box

**Remark 3.4.** As the measures $\hat{\mu}_a$ and $\hat{\mu}_\beta$ are finite and compactly supported, we have by dominated convergence that

\begin{align}
F'_{\mu_a}(\omega) - 1 &= \int_{\mathbb{R}} \frac{1}{(x - \omega)^2} \, d\hat{\mu}_a(x), \quad F''_{\mu_a}(\omega) = \int_{\mathbb{R}} \frac{1}{(x - \omega)^3} \, d\hat{\mu}_a(x),
\end{align}

for all $\omega \in \mathbb{C}^+ \cup \mathbb{R} \setminus \text{supp} \hat{\mu}_a$; the same relations hold with the $a$ changed to $\beta$.

**3.1 Bounds on the subordination functions.** The goal of this subsection is to control the imaginary parts of the subordination functions within a sufficiently large neighborhood of the support of the free convolution measure.

We first introduce the domain of the spectral parameter $z$ we will be working on. Let $J \subset \mathbb{R}$ be the interval

\begin{align}
J := \{ E \in \mathbb{R} : E_\alpha^a + E_\beta^\beta - 1 \leq E \leq E_\alpha^\alpha + E_\beta^\beta + 1 \}.
\end{align}

Then we introduce the domain

\begin{align}
\mathcal{E} := \{ z = E + i\eta \in \mathbb{C}^+ \cup \mathbb{R} : E \in J, 0 \leq \eta \leq 1 \}.
\end{align}

Lemma 3.1 of [33] shows that supp $\mu_a \boxplus \mu_\beta \subset J$ and we can therefore restrict the discussion to that interval, respectively to $\mathcal{E}$. Finally, we use the shorthand

$$m(z) = m_{\mu_a \boxplus \mu_\beta}(z), \quad z \in \mathbb{C}^+,$$

to denote the Stieltjes transform of $\mu_a \boxplus \mu_\beta$, and also its continuous extension to $\mathcal{E}$.

The following is the main result of this subsection.
Proposition 3.5. Assume that $\mu_\alpha$ and $\mu_\beta$ satisfy Assumption 2.1. Then there is a constant $C \geq 1$, depending on $\mu_\alpha$ and $\mu_\beta$ via their control parameters, such that

$$C^{-1}\text{Im } m(z) \leq \text{Im } \omega_\alpha(z) \leq C \text{Im } m(z),$$

$$C^{-1}\text{Im } m(z) \leq \text{Im } \omega_\beta(z) \leq C \text{Im } m(z),$$

for all $z \in \mathcal{E}$.

We split the proof of Proposition 3.5 in several steps. We start with two definitions.

Definition 3.6 ($I_\alpha, I_\beta$). Define the functions $I_\alpha$ and $I_\beta$ by setting

$$I_\alpha(\omega) := \int_{\mathbb{R}} \frac{d\mu_\alpha(x)}{|x - \omega|^2}, \quad I_\beta(\omega) := \int_{\mathbb{R}} \frac{d\mu_\beta(x)}{|x - \omega|^2}, \quad \omega \in \mathbb{C}^+.$$ 

Remark 3.7. Note that $I_\alpha$ and $I_\beta$ extend continuously to the real line outside the respective supports of $\mu_\alpha$ or $\mu_\beta$. Moreover, from (2.6) we note that

$$\text{Im } m(z) = \text{Im } \omega_\beta(z) \cdot I_\alpha(\omega_\beta(z)) = \text{Im } \omega_\alpha(z) \cdot I_\beta(\omega_\alpha(z)), \quad z \in \mathbb{C}^+.$$ 

Definition 3.8 ($\hat{I}_\alpha, \hat{I}_\beta$). Let $\hat{\mu}_\alpha$ and $\hat{\mu}_\beta$ denote the measures from Lemma 3.2. Set then

$$\hat{I}_\alpha(\omega) := \int_{\mathbb{R}} \frac{d\hat{\mu}_\alpha(x)}{|x - \omega|^2}, \quad \hat{I}_\beta(\omega) := \int_{\mathbb{R}} \frac{d\hat{\mu}_\beta(x)}{|x - \omega|^2}, \quad \omega \in \mathbb{C}^+.$$ 

Remark 3.9. Taking the imaginary part in (3.3) and using (2.5), we find from (3.15) that

$$\hat{I}_\alpha(\omega_\beta(z)) = \frac{\text{Im } \omega_\alpha(z) - \text{Im } z}{\text{Im } \omega_\beta(z)}, \quad \hat{I}_\beta(\omega_\alpha(z)) = \frac{\text{Im } \omega_\beta(z) - \text{Im } z}{\text{Im } \omega_\alpha(z)}, \quad z \in \mathbb{C}^+.$$ 

Hence, since $\text{Im } \omega_\alpha(z) \geq \text{Im } z$, $\text{Im } \omega_\beta(z) \geq \text{Im } z$ by Proposition 2.5, we further find that

$$\hat{I}_\alpha(\omega_\beta(z)) \cdot \hat{I}_\beta(\omega_\alpha(z)) \leq 1, \quad z \in \mathbb{C}^+.$$ 

The following result shows that the subordination functions are uniformly bounded, under Assumption 2.1, on $\mathcal{E}$.

Lemma 3.10 (Lemma 3.2. of [5]). Assume that $\mu_\alpha$ and $\mu_\beta$ satisfy Assumption 2.1. Then

$$|\omega_\alpha(z)| \leq C, \quad |\omega_\beta(z)| \leq C,$$

uniformly in $z \in \mathcal{E}$, with constants depending on $\mu_\alpha$ and $\mu_\beta$ via their control parameters.
From (3.18) we directly get the following estimates.

**Lemma 3.11.** Assume that \( \mu_\alpha \) and \( \mu_\beta \) satisfy Assumption 2.1. Let \( \mathcal{E} \) be the domain defined in (3.11). Then we have

\[
\inf_{z \in \mathcal{E}} I_\alpha(\omega_\beta(z)) \geq c, \quad \inf_{z \in \mathcal{E}} I_\beta(\omega_\alpha(z)) \geq c,
\]

for some constant \( c > 0 \). Similarly, we have

\[
\inf_{z \in \mathcal{E}} \hat{I}_\alpha(\omega_\beta(z)) \geq c', \quad \inf_{z \in \mathcal{E}} \hat{I}_\beta(\omega_\alpha(z)) \geq c',
\]

for some constant \( c' > 0 \). In particular, we have

\[
\sup_{z \in \mathcal{E}} \hat{I}_\alpha(\omega_\beta(z)) \leq C', \quad \sup_{z \in \mathcal{E}} \hat{I}_\beta(\omega_\alpha(z)) \leq C',
\]

for some constant \( C' \).

**Proof.** The estimates in (3.19) follow directly from the definitions of \( I_\alpha, I_\beta \) in (3.13) and the upper bounds in (3.18). The estimates in (3.20) follow from the definitions of \( \hat{I}_\alpha, \hat{I}_\beta \) in (3.15), (3.4) and the upper bounds in (3.18). Finally, (3.21) follows by combining (3.20) and (3.17). \( \square \)

The estimates in Lemma 3.11 are complemented by the following result. Define

\[
d_\alpha(\omega) := \text{dist}(\omega, \text{supp} \mu_\alpha), \quad d_\beta(\omega) := \text{dist}(\omega, \text{supp} \mu_\beta).
\]

**Lemma 3.12.** Under the assumptions of Lemma 3.11, we have the bounds

\[
\inf_{z \in \mathcal{E}} d_\alpha(\omega_\beta(z)) \geq g, \quad \inf_{z \in \mathcal{E}} d_\beta(\omega_\alpha(z)) \geq g,
\]

for a strictly positive constant \( g \) depending only on \( \mu_\alpha \) and \( \mu_\beta \). Moreover, we have that

\[
\sup_{z \in \mathcal{E}} I_\alpha(\omega_\beta(z)) \leq \frac{1}{g^2}, \quad \sup_{z \in \mathcal{E}} I_\beta(\omega_\alpha(z)) \leq \frac{1}{g^2}.
\]

**Proof.** Recall the definition of \( \hat{I}_\alpha \) in (3.15). Using (3.8) we can write

\[
\hat{I}_\alpha(\omega) = \frac{\text{Im} m_{\mu_\alpha}(\omega)}{|m_{\mu_\alpha}(\omega)|^2 \text{Im} \omega} - 1 = \frac{\int_R \frac{d\mu_\alpha(x)}{|x-\omega|^2}}{|\int_R \frac{d\mu_\alpha(x)}{x-\omega}|^2} - 1.
\]

We now claim that \( \hat{I}_\alpha(\omega) \to \infty \) as \( \omega \) approaches \( \text{supp} \mu_\alpha \) in \( \mathcal{D}_\alpha := \mathbb{C}^+ \cup \mathbb{R} \setminus \text{supp} \mu_\alpha \). To do so we distinguish two cases: we first study \( \hat{I}_\alpha(\omega) \) with \( \omega \) in a neighborhood of the edges \( E_\alpha^- \) respectively \( E_\alpha^+ \), and then study \( \hat{I}_\alpha(\omega) \) for \( \omega \) inside the bulk separately.
An elementary computation shows that, for \( \omega \in \mathbb{C}^+ \) satisfying \( |\omega - E_\alpha^-| \leq \delta \) with some (small) \( \delta > 0 \),
\[
(3.26) \quad \int_{\mathbb{R}} \frac{d\mu_a(x)}{|x-\omega|^2} \geq c \begin{cases} 
\frac{(\text{Re } \omega - E_\alpha^-)^2}{\text{Im } \omega}, & \text{if } \text{Re } \omega - E_\alpha^- > \text{Im } \omega, \\
(\text{E}_\alpha^\alpha - \text{Re } \omega)^{\alpha - 1}, & \text{if } \text{Re } \omega - E_\alpha^- < -\text{Im } \omega, \\
(\text{Im } \omega)^{\alpha - 1}, & \text{if } \text{Im } \omega \geq |\text{Re } \omega - E_\alpha^-|,
\end{cases}
\]
for some constant \( c > 0 \) depending on \( \delta \); see, e.g., Lemma 3.4 in [5].

For \( t^-_\alpha \geq 0 \), we have, for \( \omega \in \mathbb{C}^+ \) satisfying \( |\omega - E_\alpha^-| \leq \delta \), the estimate
\[
(3.27) \quad \left| \int_{\mathbb{R}} \frac{d\mu_a(x)}{x-\omega} \right| \leq C \begin{cases} 
|\log \text{Im } \omega|, & \text{if } \text{Re } \omega - E_\alpha^- > \text{Im } \omega, \\
|\log(\text{E}_\alpha^\alpha - \text{Re } \omega)|, & \text{if } \text{Re } \omega - E_\alpha^- < -\text{Im } \omega, \\
|\log \text{Im } \omega|, & \text{if } \text{Im } \omega \geq |\text{Re } \omega - E_\alpha^-|,
\end{cases}
\]
for some constant \( C \) depending on \( \delta \). For \( t^-_\alpha < 0 \), we have, for \( \omega \in \mathbb{C}^+ \) satisfying \( |\omega - E_\alpha^-| \leq \delta \),
\[
\left| \int_{\mathbb{R}} \frac{d\mu_a(x)}{x-\omega} \right| \leq C \begin{cases} 
|\log \text{Im } \omega| (\text{Re } \omega - E_\alpha^-)^{\alpha^-}, & \text{if } \text{Re } \omega - E_\alpha^- > \text{Im } \omega, \\
|\log(\text{E}_\alpha^\alpha - \text{Re } \omega)| |\text{Re } \omega - E_\alpha^-|^{\alpha^-}, & \text{if } \text{Re } \omega - E_\alpha^- < -\text{Im } \omega, \\
|\log \text{Im } \omega| (\text{Im } \omega)^{\alpha^-}, & \text{if } \text{Im } \omega \geq |\text{Re } \omega - E_\alpha^-|,
\end{cases}
\]
for some strictly positive constants \( C \) depending on \( \delta \).

Next, set
\[
(3.28) \quad T(\omega) := \frac{\int_{\mathbb{R}} \frac{d\mu_a(x)}{|x-\omega|^2}}{\left| \int_{\mathbb{R}} \frac{d\mu_a(x)}{|x-\omega|^2} \right|^2},
\]
with \( \omega \in \mathcal{D}_\alpha = \mathbb{C}^+ \cup \mathbb{R} \setminus \text{supp } \mu_a \). We now distinguish the cases \( t^-_\alpha \in [0, 1) \) and \( t^-_\alpha \in (-1, 0) \).

For \( t^-_\alpha \in [0, 1) \), we conclude from (3.26) and (3.27) that there is \( c' > 0 \) such that
\[
(3.30) \quad T(\omega) \geq c' \begin{cases} 
\frac{(\text{Re } \omega - E_\alpha^-)^2}{\text{Im } \omega(\log \text{Im } \omega)^2}, & \text{if } \text{Re } \omega - E_\alpha^- > \text{Im } \omega, \\
\frac{|\text{Re } \omega - E_\alpha^-|^{\alpha^- - 1}}{|\log(\text{E}_\alpha^\alpha - \text{Re } \omega)|^2}, & \text{if } \text{Re } \omega - E_\alpha^- < -\text{Im } \omega, \\
\frac{|\text{Im } \omega|^{\alpha^- - 1}}{|\log \text{Im } \omega|^2}, & \text{if } \text{Im } \omega \geq |\text{Re } \omega - E_\alpha^-|,
\end{cases}
\]
when \( |\omega - E_\alpha^-| \leq \delta \), and hence \( T(\omega) \not\to \infty \) as \( \omega \to E' \in [E_\alpha^-, E_\alpha^- + \delta) \) in \( \mathcal{D}_\alpha \), since \( t^-_\alpha < 1 \).
For \( t_\alpha^a \in (-1, 0) \), we conclude from (3.26) and (3.28) that there is \( c' > 0 \) such that

\[
T(\omega) \geq c' \begin{cases} 
\frac{\text{Re} \omega - E_\alpha^a}{|\log \text{Im} \omega|^2 \text{Im} \omega}, & \text{if } \text{Re} \omega - E_\alpha^a > \text{Im} \omega, \\
\frac{|\log E_\alpha^a - \text{Re} \omega|^{\frac{m-1}{2}}}{\text{Im} \omega}, & \text{if } \text{Re} \omega - E_\alpha^a < -\text{Im} \omega, \\
\frac{|\text{Im} \omega|^{-\frac{m-1}{2}}}{|\log \text{Im} \omega|^2}, & \text{if } \text{Im} \omega \geq |\text{Re} \omega - E_\alpha^a|,
\end{cases}
\]  

(3.31)

when \(|\omega - E_\alpha^a| \leq \delta\), and hence \( T(\omega) \uparrow \infty \) as \( \omega \to E' \in [E_\alpha^a, E_\alpha^a + \delta) \), since \( t_\alpha^a \in (-1, 0) \).

The same argument shows, as \( t_\mu^a \in (-1, 1) \), that \( \hat{I}_a(\omega) \) diverges when \( \omega \) approaches \( (E_\alpha^a - \delta, E_\alpha^a) \) in \( D_a \).

Next, we know that the density \( \rho_a \) of \( \mu_a \) is a.e. positive on \([E_\alpha^a + \delta', E_\alpha^a + \delta] \), with \( 0 < \delta' \leq \delta \). Hence there is a constant \( c > 0 \), depending on \( \delta' \), such that

\[
\int_{\mathbb{R}} \frac{d\rho_a(x)}{|x - \omega|^2} \geq c \frac{1}{\text{Im} \omega},
\]

(3.32)

for all \( \omega \in D_a \) satisfying \( \text{dist}(\omega, [E_\alpha^a + \delta, E_\alpha^a - \delta]) \leq \delta'/2 \). On the other hand, as the density \( \rho_a \) is a.e. finite in the bulk, there are constants \( c \) and \( C \), depending on \( \delta' \), such that

\[
\left| \int_{\mathbb{R}} \frac{d\rho_a(x)}{x - \omega} \right| \leq C + c \int_{E_\alpha^a + \delta} \frac{dx}{|x - \omega|} \leq C + c |\log \text{Im} \omega|,
\]

(3.33)

for all \( \omega \in D_a \) with \( \text{dist}(\omega, [E_\alpha^a + \delta, E_\alpha^a - \delta]) \leq \delta'/2 \). Thus, from (3.29), we conclude that \( T(\omega) \) diverges when \( \omega \) tends to any \( E \in [E_\alpha^a + \delta, E_\alpha^a - \delta] \) within \( D_a \). Hence so must \( \hat{I}_a(\omega) \).

However, since \( \hat{I}_a(\omega_\beta(z)) \) is uniformly bounded for all \( z \in \mathcal{E} \) by (3.21), we conclude that \( \omega = \omega_\beta(z) \) must be separated outside \( \text{supp} \mu_a \). That is \( d_a(\omega_\beta(z)) \) must, by the continuity of \( \omega_\beta \), be uniformly bounded from below on \( \mathcal{E} \).

Once (3.23) has been established, (3.24) follows directly from the definitions of \( I_a \) and \( I_\beta \) in (3.13).

In sum, we have proved so far that there is a constant \( C \geq 1 \), depending only on \( \mu_a \) and \( \mu_\beta \) via their control parameters, such that

\[
C^{-1} \leq I_a(\omega_\beta(z)) \leq C, \quad C^{-1} \leq I_\beta(\omega_a(z)) \leq C, \quad z \in \mathcal{E}.
\]

(3.34)

Thence, recalling (3.14), we observe that the imaginary parts of \( m(z) \equiv m_{\mu_a \boxplus \mu_\beta}(z) \), \( \omega_a(z) \) and \( \omega_\beta(z) \) are all comparable on the domain \( \mathcal{E} \), which proves (3.12) for some constant \( C \geq 1 \) depending only on \( \mu_a \) and \( \mu_\beta \). This concludes the proof of Proposition 3.5.
4 Characterization of (regular) edges and Proof of Theorem 2.2

Recall that \( \rho(x) \) denotes the (continuous) density function at \( x \in \mathbb{R} \) of the free additive convolution measure \( \mu_a \boxplus \mu_\beta \) and that \( m(z) \) is its Stieltjes transform at \( z \in \mathbb{C}^+ \). From (3.14) and (3.34) we note that \( \text{Im} \, \omega_a(E+i\eta) \) and \( \text{Im} \, \omega_\beta(E+i\eta) \) vanish in the limit \( \eta \searrow 0 \) if \( \rho(E) = 0 \). The next result shows that the ratio of the imaginary parts of the subordination functions has a finite and positive limit as the spectral parameter approaches the real line.

**Lemma 4.1.** Suppose that \( \mu_a \) and \( \mu_\beta \) satisfy Assumption 2.1. Let \( E \in \mathbb{J} \); see (3.10). Then

\[
\lim_{\eta \searrow 0} \frac{\text{Im} \, \omega_a(E+i\eta)}{\text{Im} \, \omega_\beta(E+i\eta)} = \frac{I_\alpha(\omega_\beta(E))}{I_\beta(\omega_a(E))}.
\]

The limit, which is bounded from above and from below by strictly positive constants (see (3.34)), is a continuous function in \( E \). The corresponding statements hold with the roles of the indices \( \alpha \) and \( \beta \) interchanged.

**Remark 4.2.** In (4.1) we take the limit \( z = E + i\eta \to E \) in the direction of the imaginary axes, yet as \( \omega_a \) and \( \omega_\beta \) extend continuously to the real axis by Proposition 2.6, we obtain the same limit when taken along any non-tangential direction to \( \mathbb{R} \).

**Proof of Lemma 4.1.** For \( z \in \mathbb{E} \setminus \mathbb{J} \), we have \( \text{Im} \, m(z) \neq 0 \), thus from (3.14),

\[
\text{Im} \, \omega_a(z) = \frac{\text{Im} \, m(z)}{I_\beta(\omega_a(z))}, \quad \text{Im} \, \omega_\beta(z) = \frac{\text{Im} \, m(z)}{I_\alpha(\omega_\beta(z))},
\]

hence

\[
\frac{\text{Im} \, \omega_a(E+i\eta)}{\text{Im} \, \omega_\beta(E+i\eta)} = \frac{I_\alpha(\omega_\beta(E+i\eta))}{I_\beta(\omega_a(E+i\eta))} = \int_{\mathbb{R}} \frac{d\mu_a(y)}{|y-\omega_\beta(E+i\eta)|^2}, \quad \eta > 0.
\]

From Lemma 3.12 we know that \( \omega_a(z) \) and \( \omega_\beta(z) \) stay away from the support of the measures \( \mu_\beta \), respectively \( \mu_a \) for all \( z \in \mathbb{E} \), by the continuity of \( \omega_a \) and \( \omega_\beta \) and dominated convergence, we can take the limit \( \eta \searrow 0 \) in (4.3) and conclude that the limit is a finite strictly positive number by (3.19) and Lemma 3.12. Continuity of the limit is immediate.

We are now ready to characterize the (regular) edges of the measure \( \mu_a \boxplus \mu_\beta \).

For brevity, we use the following notation: Denote the set of vanishing points, \( \mathcal{V} \), of \( \mu_a \boxplus \mu_\beta \) by

\[
\mathcal{V} := \partial \{ x \in \mathbb{R} : \rho(x) > 0 \},
\]
where $\rho$ denotes the density of $\mu_\alpha \boxplus \mu_\beta$. We remark that $V$ is not necessarily the boundary of the support of $\mu_\alpha \boxplus \mu_\beta$. It may happen that $V$ contains isolated zeros: $x \in \mathbb{R}$ is called an isolated zero if $\rho(x) = 0$, and $\rho(x + \epsilon) > 0$ and $\rho(x - \epsilon) > 0$, for all $\epsilon \in (0, \epsilon_0)$, for some $\epsilon_0 > 0$. However, we will prove below in Proposition 4.7 that $\mu_\alpha \boxplus \mu_\beta$ does not have isolated zeros under Assumption 2.1.

**Proposition 4.3.** Suppose that $\mu_\alpha$ and $\mu_\beta$ satisfy Assumption 2.1. Then we have

$$
|F'_{\mu_\alpha}(\omega_\beta(z)) - 1)(F'_{\mu_\beta}(\omega_\alpha(z)) - 1)| \leq 1,
$$

for all $z \in \mathbb{C}^+ \cup \mathbb{R}$; see (3.11). Moreover, we have equality in (4.5) with $z = E + i\eta \in \mathcal{E}$ if and only if the spectral parameter $z$ satisfies

$$
E \in V, \quad \eta = 0.
$$

In fact, for such $E$, we have that

$$
(F'_{\mu_\alpha}(\omega_\beta(E)) - 1)(F'_{\mu_\beta}(\omega_\alpha(E)) - 1) = 1.
$$

**Proof.** Inequality (4.5) was proved in [12] using the concept of Denjoy–Wolff points. Here we give an elementary direct argument. Recalling (3.9), we indeed note that

$$
|F'_{\mu_\alpha}(\omega_\beta(z)) - 1)(F'_{\mu_\beta}(\omega_\alpha(z)) - 1)| = \left| \int_{\mathbb{R}} \frac{d\hat{\mu}_\alpha(x)}{(x - \omega_\beta(z))^2} \int_{\mathbb{R}} \frac{d\hat{\mu}_\beta(x)}{(x - \omega_\alpha(z))^2} \right|
$$

$$
\leq \left( \int_{\mathbb{R}} |(x - \omega_\beta(z))^2 \int_{\mathbb{R}} |(x - \omega_\alpha(z))^2 \right)
$$

$$
= \hat{I}_\alpha(\omega_\beta(z))\hat{I}_\beta(\omega_\alpha(z))
$$

$$
\leq 1,
$$

for all $z \in \mathcal{E}$, where we used (3.15) and (3.17). We are now interested in the case when we have equality in (4.5).

Next, assuming equality in (i) and (ii), we show that (4.6) holds. First, recall from (3.5) that the measures $\mu_\alpha$ and $\hat{\mu}_\alpha$ have the same support. Since $\mu_\alpha$ is supported on an interval, so must $\hat{\mu}_\alpha$ be. Similarly for $\hat{\mu}_\beta$. Since the subordination functions are uniformly bounded on $\mathcal{E}$, we get equality in (i) of (4.7) if and only if $z$ is such that $\text{Im} \omega_\alpha(z) = \text{Im} \omega_\beta(z) = 0$. This entails, as $\text{Im} \omega_\alpha(z) \geq \text{Im} z$, $\text{Im} \omega_\beta(z) \geq \text{Im} z$, $z \in \mathbb{C}^+$ by Proposition 2.5 (and continuous extension to the real line), that such a $z \in \mathcal{E}$ must lie on the real line, i.e., $\eta = 0$.

To get the first part of (4.6), we note that from the definition of $\hat{I}_\alpha$ and $\hat{I}_\beta$ in (3.15) and (3.16), we have

$$
\hat{I}_\alpha(\omega_\beta(z)) = \frac{\text{Im} \omega_\alpha(z)}{\text{Im} \omega_\beta(z)} - \frac{\text{Im} z}{\text{Im} \omega_\beta(z)} = \frac{I_\alpha(\omega_\beta(z))}{I_\beta(\omega_\alpha(z))} - \frac{\text{Im} z}{\text{Im} \omega_\beta(z)},
$$
where we used (4.3) to get the second equality, and similarly with the roles of \(\alpha\) and \(\beta\) interchanged. Thus with \(z = x + i\eta_0, x \in \mathbb{R}\), we can take, by Lemma 4.1, \(\eta_0\) to zero to get
\[
(4.9) \quad \tilde{I}_\alpha(\omega_\beta(x)) = \frac{I_\alpha(\omega_\beta(x))}{I_\beta(\omega_\alpha(x))} - \lim_{\eta_0 \searrow 0} \frac{\eta_0}{\Im \omega_\beta(x + i\eta_0)}.
\]
Since \(\tilde{I}_\alpha(\omega_\beta(x)) > 0\), the right side is strictly positive as well.

Let now \(x = E\) be such that we have equality in (i) and (ii) in (4.7). Then we have
\[
1 = \tilde{I}_\alpha(\omega_\beta(E)) \tilde{I}_\beta(\omega_\alpha(E)) = \left(\frac{I_\alpha(\omega_\beta(E))}{I_\beta(\omega_\alpha(E))} - \lim_{\eta_0 \searrow 0} \frac{\eta_0}{\Im \omega_\beta(E + i\eta_0)}\right) \left(\frac{I_\beta(\omega_\alpha(E))}{I_\alpha(\omega_\beta(E))} - \lim_{\eta_0 \searrow 0} \frac{\eta_0}{\Im \omega_\alpha(E + i\eta_0)}\right).
\]
As both factors in the above product are positive, we therefore conclude that
\[
(4.10) \quad \lim_{\eta_0 \searrow 0} \frac{\eta_0}{\Im \omega_\beta(E + i\eta_0)} = \lim_{\eta_0 \searrow 0} \frac{\eta_0}{\Im \omega_\alpha(E + i\eta_0)} = 0.
\]

Summarizing, we have so far proved that if there is \(z = E + i\eta \in \mathcal{E}\) such that we have equality in (i) and (ii), then \(\eta = 0, \Im \omega_\alpha(E) = \Im \omega_\beta(E) = 0\), and (4.10) hold.

It remains to show that such \(E\) belongs to \(\mathcal{V}\). From (3.14), we see that \(\Im \omega_\alpha(E) = 0\) implies \(\Im m(E) = 0\), i.e., \(E \in \{x \in \mathbb{R} : \rho(x) = 0\}\). Moreover, from (4.10), we have that
\[
(4.11) \quad \lim_{\eta_0 \searrow 0} \frac{\Im \omega_\beta(E + i\eta_0)}{\eta_0} = \infty,
\]
which in turn means by (3.14) and by (3.34) that
\[
(4.12) \quad \lim_{\eta_0 \searrow 0} \frac{\Im m(E + i\eta_0)}{\eta_0} = \infty.
\]
However, if \(E\) were in the complement of the support of \(\rho(x)dx = d\mu_\alpha \boxplus \mu_\beta(x)\), then
\[
(4.13) \quad \frac{\Im m(E + i\eta_0)}{\eta_0} = \int_\mathbb{R} \frac{d\rho(x)}{|x - E|^2 + \eta_0^2}
\]
would remain bounded as \(\eta_0 \searrow 0\). Thus \(E \in \text{supp } \mu_\alpha \boxplus \mu_\beta \cap \{x \in \mathbb{R} : \rho(x) = 0\}\), i.e., \(E \in \mathcal{V}\).

We now prove the converse: (4.6) implies equality in (i) and (ii) in (4.7). Since \(\rho(E) = 0\), we also have \(\Im m(E) = 0\) and \(\Im \omega_\alpha(E) = \Im \omega_\beta(E) = 0\) by Proposition 3.5. This gives equality in (i).
Next, as $E \in \mathcal{V}$, there is $\epsilon_0 > 0$ such that $\rho(E - \epsilon) > 0$ or $\rho(E + \epsilon) > 0$, for all $0 < \epsilon \leq \epsilon_0$. Assume first that $\rho(E - \epsilon) > 0$, $0 < \epsilon \leq \epsilon_0$. Then, for a fixed $\epsilon > 0$, we have by (3.14) and (3.34) that
\begin{equation}
\lim_{\eta_0 \searrow 0} \frac{\eta_0}{\Im \omega_a(E - \epsilon + i\eta_0)} = \lim_{\eta_0 \searrow 0} \frac{\eta_0}{\Im \omega'_\beta(E - \epsilon + i\eta_0)} = 0.
\end{equation}
Thus, by (3.16), for any $\eta_0 > 0$,
\begin{align*}
\widehat{I}_a(\omega_\beta(E - \epsilon + i\eta_0)) \widehat{I}_\beta(\omega_a(E - \epsilon + i\eta_0)) &= \left(\frac{\Im \omega_a(E - \epsilon + i\eta_0)}{\Im \omega'_\beta(E - \epsilon + i\eta_0)} - \frac{\eta_0}{\Im \omega'_\beta(E - \epsilon + i\eta_0)}\right) \\
&\quad \times \left(\frac{\Im \omega'_\beta(E - \epsilon + i\eta_0)}{\Im \omega_a(E - \epsilon + i\eta_0)} - \frac{\eta_0}{\Im \omega_a(E - \epsilon + i\eta_0)}\right)
\end{align*}
Taking the limit $\eta_0 \searrow 0$, for fixed $\epsilon > 0$, we find from (4.14) that
\begin{equation}
\widehat{I}_a(\omega_\beta(E - \epsilon)) \widehat{I}_\beta(\omega_a(E - \epsilon)) = \frac{\Im \omega_a(E - \epsilon)}{\Im \omega'_\beta(E - \epsilon)} = 1.
\end{equation}
Next, using the continuity of $\omega_a$ and $\omega_\beta$, and that they are separated from the support of $\mu_\beta$, respectively $\mu_a$, we have by continuity that
\begin{equation}
\widehat{I}_a(\omega_\beta(E)) \widehat{I}_\beta(\omega_a(E)) = \lim_{\epsilon \searrow 0} \widehat{I}_a(\omega_\beta(E - \epsilon)) \widehat{I}_\beta(\omega_a(E - \epsilon)) = 1,
\end{equation}
and we obtain equality in (ii) assuming that $\rho(E - \epsilon) > 0$, $0 < \epsilon \leq \epsilon_0$. The case $\rho(E + \epsilon) > 0$, $0 < \epsilon \leq \epsilon_0$, is handled in the same way and we find that in either case we have equality in (ii). Thus we have proved that (4.6) implies equality in (i) and (ii) in (4.7).

Finally, since $\omega_a(E)$ and $\omega_\beta(E)$ are real valued for any $E \in \mathcal{V}$ (see (3.14) and use (3.34)), and since they are separated from the support of the respective measure $\mu_\beta$ and $\mu_a$, we conclude that $(F'_{\mu_\beta}(\omega_\beta(E)) - 1)(F'_{\mu_a}(\omega_a(E)) - 1)$ is real valued, and hence by (3.9) positive. Thence we must have
\begin{equation}
(F'_{\mu_\beta}(\omega_\beta(E)) - 1)(F'_{\mu_a}(\omega_a(E)) - 1) = 1
\end{equation}
for all $E \in \mathcal{V}$. This finishes the proof of Proposition 4.3. 

\begin{proof}
\end{proof}

We need one more technical lemma before we can move on to the proof of Theorem 2.2.

\textbf{Lemma 4.4.} Let $\omega_a$ and $\omega_\beta$ be the subordination functions associated to $\mu_a$ and $\mu_\beta$ by Proposition 2.5. Then there exist finite Borel measures $\nu_a$ and $\nu_\beta$ on $\mathbb{R}$ such that
\begin{equation}
\omega_a(z) - z = \int_{\mathbb{R}} \frac{d\nu_a(x)}{x - z}, \quad \omega_\beta(z) - z = \int_{\mathbb{R}} \frac{d\nu_\beta(x)}{x - z},
\end{equation}
for any \( z \) outside the corresponding supports, where

\[
0 < \nu_\alpha(\mathbb{R}) = \int_{\mathbb{R}} x^2 d\mu_\alpha(x), \quad 0 < \nu_\beta(\mathbb{R}) = \int_{\mathbb{R}} x^2 d\mu_\beta(x).
\]

Moreover, we have

\[
\text{supp } \nu_\alpha = \text{supp } \nu_\beta = \text{supp } \mu_\alpha \boxplus \mu_\beta,
\]

thus \( (4.17) \) holds for any \( z \in \mathbb{C} \setminus \text{supp } \mu_\alpha \boxplus \mu_\beta \).

**Remark 4.5.** The measure \( \nu_\alpha \) is referred to as the subordination distribution of \( \mu_\alpha \boxplus \mu_\beta \) with respect to \( \mu_\beta \). Analogously, \( \nu_\beta \) is the subordination distribution of \( \mu_\alpha \boxplus \mu_\beta \) with respect to \( \mu_\alpha \); see \cite{26}. Statement \( (4.18) \) is a particular simple case of the results in \cite{26}; see also Theorem 1.2 in \cite{28}. For completeness we include an elementary proof of \( (4.18) \).

**Proof of Lemma 4.4.** The proof is similar to the proof of Lemma 3.2. We start from \( \omega_\alpha(i\eta) + \omega_\beta(i\eta) - i\eta = F_{\mu_\alpha}(\omega_\beta(i\eta)), \eta > 0 \). By \( (2.4) \) we have \( \lim_{\eta \to \infty} \omega_\alpha(i\eta)/i\eta = 1 \) and we can expand \( F_{\mu_\alpha}(\omega_\beta(i\eta)) \) around infinity similar to \( (3.6) \). On the other hand, as \( z \mapsto \omega_\alpha(z) - z \) is a self-map of the upper half plane by Proposition 2.5 it admits the representation \( (3.1) \) with a measure \( \nu_\alpha := \mu_\beta \). By comparison we directly find \( (4.17) \) and \( (4.18) \) for \( \omega_\alpha \). For \( \omega_\beta \) these equations are established in the same way.

Finally by \( (4.17) \), the functions \( \omega_\alpha(z) \) and \( \omega_\beta(z) \) are analytic outside \( \text{supp } \nu_\alpha \), respectively \( \text{supp } \nu_\beta \), and \( \text{Im } \omega_\alpha(E) = 0 \) for \( E \in \mathbb{R} \setminus \text{supp } \nu_\alpha \), and \( \text{Im } \omega_\beta(E) = 0 \) for \( E \in \mathbb{R} \setminus \text{supp } \nu_\beta \). We have established in Lemma 3.12 that \( \omega_\alpha(z) \) and \( \omega_\beta(z) \), \( z \in \mathcal{E} \), stay away from the supports of the measures \( \mu_\beta \), respectively \( \mu_\alpha \). Thus \( m_{\mu_\alpha}(\omega_\beta(z)) \) and \( m_{\mu_\beta}(\omega_\alpha(z)) \) are analytic outside \( \text{supp } \nu_\alpha \), respectively \( \text{supp } \nu_\beta \), with \( \text{Im } m_{\mu_\alpha}(\omega_\beta(E)) = 0 \), for \( E \in \mathbb{R} \setminus \text{supp } \nu_\beta \), and \( \text{Im } m_{\mu_\beta}(\omega_\alpha(E)) = 0 \), for \( E \in \mathbb{R} \setminus \text{supp } \nu_\alpha \). By analytic subordination we have

\[
m(z) = m_{\mu_\alpha}(\omega_\beta(z)) = m_{\mu_\beta}(\omega_\alpha(z)), \quad z \in \mathbb{C}^+.
\]

Thus, since the subordination functions continuously extend to the real line, we have

\[
\text{Im } m(E) = \text{Im } m_{\mu_\alpha}(\omega_\beta(E)) = \text{Im } m_{\mu_\beta}(\omega_\alpha(E)) = 0
\]

and we conclude that \( \text{supp } \nu_\alpha = \text{supp } \nu_\beta \) as well as \( \text{supp } \mu_\alpha \boxplus \mu_\beta \subseteq \text{supp } \nu_\alpha \). Finally, let \( E \in \mathbb{R} \setminus \text{supp } \mu_\alpha \boxplus \mu_\beta \). Then \( m(z) \) is analytic in a neighborhood of \( E \) and we have \( \text{Im } m(E) = \text{Im } m_{\mu_\alpha}(\omega_\beta(E)) = 0 \). Since, by Lemma 3.12, \( \omega_\beta(E) \) is outside \( \text{supp } \mu_\alpha \), we also have \( \text{Im } \omega_\beta(E) = 0 \). Recalling \( (4.17) \), we conclude that \( E \not\in \text{supp } \nu_\beta \). Thus we have \( \text{supp } \nu_\alpha \subseteq \text{supp } \mu_\alpha \boxplus \mu_\beta \) and we conclude that \( (4.19) \) holds. \( \square \)
**Remark 4.6.** The subordination functions extend continuously to $\mathbb{R}$ and they are real analytic outside the support of $\mu_a \boxplus \mu_\beta$. As $\nu_a$ and $\nu_\beta$ are finite measures by (4.18), dominated convergence asserts that

\[ \omega'_a(E) - 1 = \int_{\mathbb{R}} \frac{d\nu_a(x)}{(E - x)^2}, \quad \omega'_\beta(E) - 1 = \int_{\mathbb{R}} \frac{d\nu_\beta(x)}{(E - x)^2}, \]

\[ E \in \mathbb{R} \setminus \text{supp}\mu_a \boxplus \mu_\beta. \]

In particular, the subordination functions are strictly increasing in $E$ on $\mathbb{R} \setminus \text{supp}\mu_a \boxplus \mu_\beta$. Moreover, from (4.17), we have $\lim_{E \to \pm\infty} \omega_a(E) = \pm\infty$ and the same holds true for $\omega_\beta$.

Having established Proposition 4.3, we are now ready to determine the support of the free convolution measure. Recall the notation $\mathcal{V} = \partial \{ x \in \mathbb{R} : \rho(x) > 0 \}$ from (4.4).

**Proposition 4.7.** Suppose that $\mu_a$ and $\mu_\beta$ satisfy Assumption 2.1. Then there exist finite numbers $E_- < 0$ and $E_+ > 0$ such that $\mathcal{V} = \{ E_- , E_+ \}$ and

\[ \{ x \in \mathbb{R} : \rho(x) > 0 \} = (E_-, E_+). \]

In particular, we have $\text{supp } \mu_a \boxplus \mu_\beta = [E_-, E_+]$.

**Proof of Proposition 4.7.** From Proposition 4.3 we know that a point $E$ belongs to $\mathcal{V}$ if and only if $(F'_{\mu_a}(\omega_\beta(E)) - 1)(F'_{\mu_\beta}(\omega_a(E)) - 1) = 1$. Using (3.9), we rewrite this condition as

\[ \hat{I}_a(\omega_\beta(E))\hat{I}_\beta(\omega_a(E)) = \int_{\mathbb{R}} \frac{d\hat{\mu}_a(y)}{(y - \omega_\beta(E))^2} \int_{\mathbb{R}} \frac{d\hat{\mu}_\beta(y)}{(y - \omega_a(E))^2} = 1. \]

We will now look for solutions to (4.22) for $E \in J$. For ease of notation set

\[ f(E) := \int_{\mathbb{R}} \frac{d\hat{\mu}_a(y)}{(y - \omega_\beta(E))^2} \int_{\mathbb{R}} \frac{d\hat{\mu}_\beta(y)}{(y - \omega_a(E))^2}, \quad E \in \mathbb{R}. \]

By (4.20) we know that the subordination functions are strictly increasing outside the support of $\mu_a \boxplus \mu_\beta$ and we also know from (4.17) that $\omega_a(E) = E + o(1)$ and $\omega_\beta(E) = E + o(1)$, as $E \searrow -\infty$. Hence $f(E) = o(1)$ as $E \searrow -\infty$. We now increase $E$ starting from $-\infty$ and note that $f(E)$ is monotone increasing, as $\omega_a(E)$ and $\omega_\beta(E)$ are.

By (4.5), we know that $|f(E)| \leq 1$, for all $E \in \mathbb{R}$. Yet, we also know from the paragraph below (3.27) in the proof of Lemma 3.12 that $\hat{I}_a(\omega)$ and $\hat{I}_\beta(\omega)$ both diverge when $\omega$ approaches the lower endpoints of the measures $\mu_a$, respectively $\mu_\beta$. 


We therefore conclude by monotonicity of \( f \) that there is only one solution, \( E_- \), to (4.22) such that \( \omega_\alpha(E_-) < E_\beta^- \) and \( \omega_\beta(E_-) < E_\alpha^- \). The point \( E_- \) is the first point from the left reaching the support of \( \mu_\alpha \boxplus \mu_\beta \), i.e., it is the leftmost endpoint, and we therefore conclude that \( E_- \in \mathcal{J} \) as \( \text{supp} \mu_\alpha \boxplus \mu_\beta \subset \mathcal{J} \); cf. remark below (3.11).

The same reasoning, reducing \( E \) from \( \infty \), shows that there is only one solution, \( E_+ \), to (4.22) such that \( \omega_\alpha(E_+) > E_\beta^+ \) and \( \omega_\beta(E_+) > E_\alpha^- \). Moreover, \( E_+ \) must be the right most endpoint of the support of \( \mu_\alpha \boxplus \mu_\beta \) and hence \( E_+ \in \mathcal{J} \).

Any other solution, \( E' \), to (4.22) must lie in \((E_-, E_+)\) and has to satisfy either

\[
\omega_\alpha(E') < E_\beta^- \quad \text{and} \quad \omega_\beta(E') > E_\alpha^- ,
\]

or

\[
\omega_\alpha(E') > E_\beta^+ \quad \text{and} \quad \omega_\beta(E') < E_\alpha^- .
\]

Yet, we now show that this cannot happen for \( \omega_\alpha \) and \( \omega_\beta \) solutions to the subordination equations (2.5).

Arguing by contradiction, assume there is a solution, \( E' \), to (4.22) with \( E_- < E' < E_+ \). Then \( E' \) satisfies either (4.24) or (4.25). Assume first that (4.24) is satisfied. Since \( \mu_\beta \) is supported on a single interval, we must have \( F_{\mu_\beta}(\omega_\alpha(E')) < 0 \). This can, for example, be seen from the representation

\[
-1 \frac{1}{F_{\mu_\beta}(\omega)} = m_{\mu_\beta}(\omega) = \int_{E_\beta^-}^{E_\beta^+} \frac{d\mu_\beta(x)}{x - \omega}, \quad \omega \in \mathbb{R},
\]

and the observation that \( m_{\mu_\beta}(\omega) \) is strictly positive on \((\infty, E_\beta^-)\). But, on the other hand, since \( E_\alpha^- < \omega_\beta(E') \), we must have \( 0 < F_{\mu_\alpha}(\omega_\beta(E')) \) since \( \mu_\alpha \) is supported on a single interval. Hence, we must have

\[
F_{\mu_\beta}(\omega_\alpha(E')) < 0 < F_{\mu_\alpha}(\omega_\beta(E')).
\]

However, by subordination we have \( F_{\mu_\beta}(\omega_\alpha(E)) = F_{\mu_\alpha}(\omega_\beta(E)) \), for all \( E \in \mathbb{R} \), contradicting (4.27). We therefore conclude that there is no solution \( E' \) to (4.22) satisfying (4.24).

The same argument shows that there cannot be a solution \( E' \) to (4.22) satisfying (4.25). Hence the only solutions to (4.22) are \( E_- \) and \( E_+ \). We thus have \( \mathcal{V} = \{E_-, E_+\} \), so that

\[
\{x \in \mathbb{R} : \rho(x) > 0\} = (E_-, E_+).
\]

In particular, \( \text{supp} \mu_\alpha \boxplus \mu_\beta = [E_-, E_+] \) and there are no isolated zeros. This concludes the proof of Proposition 4.7. \( \Box \)
Proposition 4.8. Suppose that $\mu_{\alpha}$ and $\mu_{\beta}$ satisfy Assumption 2.1, in particular the measures $\mu_{\alpha}$ and $\mu_{\beta}$ are both supported on a single interval. Let the single interval support of $\mu_{\alpha} \boxplus \mu_{\beta}$ be denoted by $[E_-, E_+]$ as in Proposition 4.7. Then there are strictly positive constants $\gamma_{\beta}^-$ and $\gamma_{\beta}^+$ such that

$$ω_{\beta}(z) = ω_{\beta}(E_-) + \gamma_{\beta}^- \sqrt{E_- - z} + O(|z - E_-|),$$

for $z$ in a neighborhood of $E_-$, where we choose the square root such that $\text{Im} \, ω_{\beta}(x) > 0$, $x > E_-$. Similarly, we have

$$ω_{\beta}(z) = ω_{\beta}(E_+) + \gamma_{\beta}^+ \sqrt{z - E_+} + O(|z - E_+|),$$

for $z$ in a neighborhood of $E_+$, where we choose the square root such that $\text{Im} \, ω_{\beta}(x) > 0$, $x < E_+$. The same conclusions apply to $ω_{\alpha}$ with strictly positive constants $γ_{\alpha}^-$ and $γ_{\alpha}^+$.

Remark 4.9. The proof of Proposition 4.8 follows a similar strategy as the proof of Lemma 3.7 in [5]. Theorem 2.2 will be a direct consequence of Proposition 4.8 and the subordination equations.

Proof of Proposition 4.8. We focus on the lower edge $E_-$ and prove (4.28). Equation (4.29) is proved in the analogous way. We start by rewriting the subordination equation (2.5) in the form of a fixed point equation. Using Lemma 3.12, and the fact that $|F'_{\mu_{\beta}}(ω)| > 0$, $ω \in \mathbb{R} \setminus \text{supp} \, \mu_{\beta}$, as follows from (3.9), we conclude by the analytic inverse function theorem that the functional inverse $F_{\mu_{\beta}}^{-1}$ of $F_{\mu_{\beta}}$ is analytic in a neighborhood of $F_{\mu_{\beta}}(ω_{\alpha}(E_-))$. Thus the function

$$\tilde{z}(ω) := -F_{\mu_{\alpha}}(ω) + ω + F_{\mu_{\beta}}^{-1}(ω_{\alpha}(E_-)) \circ F_{\mu_{\alpha}}(ω)$$

is well-defined and analytic in a neighborhood of $ω_{\beta}(E_-)$. It further follows from (2.5) that $ω_{\beta}(z)$ is a solution $ω = ω_{\beta}(z)$ to the equation $z = \tilde{z}(ω)$ (with $\text{Im} \, ω_{\beta}(z) ≥ \text{Im} \, z$). Moreover, we have $ω_{\alpha}(z) = F_{\mu_{\beta}}^{-1}(ω_{\alpha}(ω_{\beta}(z)))$.

As argued in the proof of Proposition 4.7, the lower edge $E_-$ satisfies

$$(F'_{\mu_{\alpha}}(ω_{\beta}(E_-)) - 1)(F'_{\mu_{\beta}}(ω_{\alpha}(E_-)) - 1) = 1,$$

as well as $ω_{\alpha}(E_-) < E_{\beta}$ and $ω_{\beta}(E_-) < E_{\alpha}$. The function $\tilde{z}(ω)$ can then be analytically continued to a neighborhood of $ω_{\beta}(E_-)$ with a Taylor expansion,

$$\tilde{z}(ω) = E_- + \tilde{z}'(ω_{\beta}(E_-))(ω - ω_{\beta}(E_-)) + \frac{1}{2} \tilde{z}''(ω_{\beta}(E_-))(ω - ω_{\beta}(E_-))^2 + O((ω - ω_{\beta}(E_-))^3).$$
We compute from (4.30) that
\[
\tilde{z}(\omega) = -F''_{\mu_a}(\omega) + 1 + \frac{1}{F'_{\mu_a} \circ F^{(-1)}_{\mu_a} \circ F'_{\mu_a}(\omega)} F''_{\mu_a}(\omega).
\]
(4.33)

It is straightforward to check that \(\tilde{z}'(\omega_\beta(E_-)) = 0\) as \(E_-\) is a solution to (4.31). Yet, we claim that \(\tilde{z}''(\omega_\beta(E')) < 0\). From (4.33) we compute
\[
\tilde{z}''(\omega) = -F''_{\mu_a}(\omega) + \frac{1}{F'_{\mu_a} \circ F^{(-1)}_{\mu_a} \circ F'_{\mu_a}(\omega)} F''_{\mu_a}(\omega)
\]
and thus by choosing \(\omega = \omega_\beta(z)\), we get
\[
(4.34) \quad \tilde{z}''(\omega_\beta(z)) = -\frac{F''_{\mu_a}(\omega_\beta(z))}{F'_{\mu_a}(\omega_\beta(z))} (F'_{\mu_a}(\omega_\beta(z)) - 1) - \frac{F''_{\mu_a}(\omega_\beta(z))}{(F'_{\mu_a}(\omega_\beta(z)))^3} (F''_{\mu_a}(\omega_\beta(z)))^2.
\]

Next, recall (3.9) and that \(\omega_\alpha(E_-) < E_-^\alpha\) as well as \(\omega_\beta(E_-) < E_-^\beta\). Thus
\[
(4.35) \quad F'_{\mu_a}(\omega_\alpha(E_-)) > 1, \quad F'_{\mu_a}(\omega_\beta(E_-)) > 1,
\]
\[
F''_{\mu_a}(\omega_\alpha(E_-)) > 0, \quad F''_{\mu_a}(\omega_\beta(E_-)) > 0,
\]
which implies upon choosing \(z = E_-\) in (4.34) that
\[
(4.36) \quad c \leq -\tilde{z}''(\omega_\beta(E_-)) \leq C.
\]

Hence choosing \(\omega = \omega_\beta(z)\) in the Taylor expansion of \(\tilde{z}(\omega)\) in (4.32) (thus \(\tilde{z}(\omega_\beta(z)) = z\)) and using \(\tilde{z}'(\omega_\beta(E_-)) = 0, \tilde{z}''(\omega_\beta(E_-)) \neq 0\), we get
\[
(4.37) \quad \omega_\beta(z) = \omega_\beta(E_-) + \sqrt{-\frac{2}{z''(\omega_\beta(E_-))}} \sqrt{E_- - z} + O(\sqrt{E_- - z}),
\]
for \(z\) in a neighborhood of \(E_-\), where we choose the square root such that \(\text{Im} \, \omega_\beta(x) > 0, x > E_-\). Choosing \(\gamma_\beta := (-2/z''(\omega_\beta(E_-)))^{1/2}\), we obtain (4.28). \(\square\)

Finally, we complete the proof of Theorem 2.2.

**Proof of Theorem 2.2.** It suffices to recall from (3.14) that
\[
(4.38) \quad \text{Im} \, m_{\mu_a \oplus \mu_\beta}(z) = \text{Im} \, \omega_\beta(z) \cdot I_\alpha(\omega_\beta(z)), \quad z \in \mathbb{C}^+,
\]
and that \(I_\alpha(\omega_\beta(z))\) is, by (3.19) and (3.24), uniformly bounded from below and above for all \(z \in \mathcal{E}\). Theorem 2.2 now directly follows from Proposition 4.8 and the Stieltjes inversion formula. \(\square\)
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