OPTIMAL CONTROL OF EVOLUTION DIFFERENTIAL INCLUSIONS WITH POLYNOMIAL LINEAR DIFFERENTIAL OPERATORS

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Abstract. In this paper we have introduced a new class of problems of optimal control theory with differential inclusions described by polynomial linear differential operators. Consequently, there arises a rather complicated problem with simultaneous determination of the polynomial linear differential operators with variable coefficients and a Mayer functional depending on high order derivatives. The sufficient conditions, containing both the Euler-Lagrange and Hamiltonian type inclusions and transversality conditions are derived. Formulation of the transversality conditions at the endpoints of the considered time interval plays a substantial role in the next investigations without which it is hardly ever possible to get any optimality conditions. The main idea of the proof of optimality conditions of Mayer problem for differential inclusions with polynomial linear differential operators is the use of locally-adjoint mappings. The method is demonstrated in detail as an example for the semilinear optimal control problem and the Weierstrass-Pontryagin maximum principle is obtained. Then the optimality conditions are derived for second order convex differential inclusions with convex endpoint constraints.

1. Introduction. This paper concerns with the special kind of optimal control problem with differential inclusions, where the left hand side of the evolution inclusion is polynomial linear differential operators (PLDO’s) with variable coefficients. In general, in the last decade, discrete and continuous time processes with higher order ordinary and partial differential inclusions found wide application in the field of mathematical economics and in problems of control dynamic system optimization and differential games (see, [21] and references therein). For example, as is pointed out in [21], boundary value problems for second and fourth-orders differential equations play a very important role in both theory and application. In particular, fourth-order linear differential equations, subjected to some boundary conditions, arise in the mathematical description of some physical systems, for example, the mathematical models of deflection of beams. Moreover, optimization of differential inclusions with the third order non self-adjoint operators arise in the discussion of processes that proceed without conservation of energy; in problems with friction, in the theory of open resonators, in problems of inelastic scattering, and others [22].

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We notice that, the problems accompanied with the higher-order discrete and differential inclusions are more complicated due to the higher order derivatives and their discrete analogs; the difficulty is rather to construct adjoint inclusions and transversality conditions. A convenient procedure for eliminating this complication in optimal control theory involving higher order derivatives is a formal transformation of these problems to the system of first order differential inclusions/equations. It appears that in practice returning to the original higher order problem and expressing the obtained optimality conditions by the original problem dataset, in general, is very difficult. In accordance with these difficulties on the whole in literature only the qualitative properties of second order differential inclusions are investigated (see [1], [2], [8] and references therein).

The paper [1] gives necessary and sufficient conditions ensuring the existence of a solution to the second order differential inclusion with Cauchy initial-value problem. Furthermore, second order interior tangent sets are introduced and studied to obtain such conditions. The paper [2] studies, in the context of Banach spaces, the problem of three boundary conditions for both second order differential inclusions and second order ordinary differential equations. The results are obtained in several new settings of Sobolev-type spaces involving Bochner and Pettis integrals. In the paper [8] are proved the existence of viable solutions to the Cauchy problem for a second order differential inclusions.

Some qualitative properties and optimization of first order discrete and continuous time processes with lumped and distributed parameters has been expanding in all directions at an astonishing rate during the last few decades (see [3],[4],[6],[7],[9]-[12],[25],[27],[28] and their references).

The optimization of higher order differential inclusions was first developed by Mahmudov in [16]-[23]. Since then this problem has attracted many author’s attentions (see [5] and their references). In the papers [16], [20] are studied a new class of problems of optimal control theory with Sturm-Liouville type differential inclusions involving second and fourth orders linear self-adjoint differential operators.

The main idea of the proof of optimality conditions of Mayer problem for differential inclusions with PLDO’s is the use of locally-adjoint mappings and discrete-approximate method, where the problem \( (P_V) \) is replaced by the sth-order discrete approximate problem. But in this paper to avoid some long calculations, derivation of these conditions for sth-order discrete-approximate problem are omitted. In particular case, our adjoint inclusion is an immediate generalization of the Euler-Lagrange inclusion for a first order differential inclusions.

The present paper is ordered in the following manner.

In Section 2 are given the necessary facts and supplementary results from the book of Mahmudov [13]; Hamiltonian function and locally adjoint mapping (LAM) are introduced and the problem with initial point constraints for PLDO’s governed by time-dependent set-valued mapping are formulated. In Section 3, we present the main results; on the basis of transversality conditions at the endpoints of the considered time interval are proved the sufficient conditions of optimality for differential inclusions with PLDO’s and with initial point constraints. In particular, it is shown that our problem involve optimization of so-called Sturm-Liouville type differential inclusions. To the best of our knowledge, there is no paper which considers optimality conditions for these problems in the literature and we aim to fill this gap. Therefore, the novelty of our formulation of the problem is justified. For the
establishment of the Euler-Lagrange type and Hamiltonian inclusions and transversality conditions are used construction of suitable rewriting the primal PLDO and rearrangement of its integration. The case of variable coefficients of PLDO’s turns out to be more complicated, unless transversality assumptions at the endpoints of the considered time interval are imposed. Notice that the proof relies on consideration of a convex case, even though the result remains true for nonconvex problem, too. In Section 4 the problem governed by PLDO’s with constant coefficients is considered. Furthermore, practical applications of these results are demonstrated by optimization of some semilinear with respect to the state variable optimal control problems for which the Weierstrass-Pontryagin maximum condition [26] is obtained. Our results allow us to simplify enough the proof of the maximum principle, to ob-
some Mahmudov’s adjoint inclusion being generalization of Euler-Lagrange
inclusions and non-functional endpoint constraints. By using second order suitable
problems for which the Weierstrass-Pontryagin maximum condition [26] is obtained.
by optimization of some semilinear with respect to the state variable optimal control
A set-valued mapping $F$ is called closed if its $gphF$ is a closed subset in
$R^{2n}$. The domain of a set-valued mapping $F$ is denoted by $domF$ and is defined
as $domF = \{ x : F(x) \neq \emptyset \}$. A set-valued mapping $F$ is convex-valued if $F(x)$ is a convex set for each $x \in domF$.

The Hamiltonian function and argmaximum set corresponding to a set-valued
mapping $F$ are defined by the following relations

$$H_F(x, v^*) = \sup_v \{ \langle v, v^* \rangle : v \in F(x) \}, \quad v^* \in R^n,$$

$$F_A(x, v^*) = \{ v \in F(x) : \langle v, v^* \rangle = H_F(x, v^*) \},$$

respectively. For a convex $F$ we put $H_F(x, v^*) = -\infty$ if $F(x) = \emptyset$. In other terms, $H_F(x, v^*)$ is the support function to the set $F(x)$, evaluated at $v^*$. As usual, $intM$ denotes the interior of the set $M \subseteq R^{2n}$ and $riM$ denotes the relative interior of a set $M$, i.e. the set of interior points of $M$ with respect to its affine hull $AffM$. A convex cone $K_M(z_0), z_0 = (x^0, v^0)$ is called a cone of tangent directions at a point $z_0 \in M$ to the set $M$, if from $\tilde{z} = (\tilde{x}, \tilde{v}) \in K_M(z_0)$ it follows that $\tilde{z}$ is a tangent vector to the set $M$ at a point $z_0 \in M$, i.e., there exists such function $q : R^n \rightarrow R^{2n}$ that $z_0 + \alpha \tilde{z} + q(\alpha) \in M$ for sufficiently small $\alpha > 0$ and $\alpha^{-1}q(\alpha) \rightarrow 0$, as $\alpha \downarrow 0$.

For a convex set-valued mapping $F$ the set-valued mapping $F^* : R^n \rightrightarrows R^n$ defined by

$$F^*(v^*; (x, v)) := \{ x^* : (x^*, -v^*) \in K^*_{gphF}(x, v) \},$$

$$K_{gphF}(x, v) = cone[gphF - (x, v)], \quad (x, v) \in gphF,$$
is called the LAM to $F$ at a point $(x, v) \in \operatorname{gph} F$, where $K^* = \{ z^* : \langle \bar{z}, z^* \rangle \geq 0, \forall \bar{z} \in K \}$ denotes the dual cone to the cone $K$, as usual. Below by using the Hamiltonian function, associated to a set-valued mapping $F$ we will define another LAM. Thus, the LAM to "nonconvex" mapping $F$ is defined as follows

$$F^*(v^*; (x, v)) := \{ x^* : H_F(x^1, v^*) - H_F(x, v^*) \leq \langle x^*, x^1 - x \rangle, \forall x^1 \in \mathbb{R}^n \},$$

$$(x, v) \in \operatorname{gph} F, v \in F_A(x, v^*).$$

Clearly, for the convex mapping $F$ the Hamiltonian function $H_F(\cdot, \cdot, v^*)$ is concave and the latter definition of LAM coincide with the previous definition of LAM (Theorem 2.1 [13]). Note that prior to the LAM the notion of coderivative has been introduced for set-valued mappings in terms of the basic normal cone to their graphs by Mordukhovich [24] and for the smooth and convex maps the two notions are equivalent.

In Section 3 our goal is to give Euler-Lagrange and Hamiltonian optimality conditions for the following general Mayer problem governed by ordinary evolution differential inclusions with PLDOs and with initial point constraints

$$\text{minimize } \varphi(x(1), x'(1), \ldots, x^{(s-1)}(1)),$$  

$$(P_V) \quad \begin{array}{l}
Lx(t) \in F(x(t), t), \text{ a.e. } t \in [0, 1], \\
x(0) \in Q_0, x'(0) \in Q_1, \ldots, x^{(s-1)}(0) \in Q_{s-1},
\end{array}$$

where $Lx = \sum_{k=1}^s p_k(t)D^k x$ is a PLDO of degree $s$ with variable coefficients $p_k : [0, 1] \to \mathbb{R}^1$ and $D^k, k = 1, \ldots, s$ is the operator of $k$-th order derivatives. In what follows for each $k$ a scalar function $p_k$ is $k$th-order continuously differentiable function, $p_k(t) \neq 0$ on $[0, 1]$ identically, $F(\cdot, t) : \mathbb{R}^n \Rightarrow \mathbb{R}^n$ is time dependent set-valued mapping, $\varphi : (\mathbb{R}^n)^* \to \mathbb{R}^1$ is continuous function and $Q_j \subseteq \mathbb{R}^n, j = 0, 1, \ldots, s-1$ are nonempty subsets of $\mathbb{R}^n, s(s \geq 2)$, is an arbitrary fixed natural number. It is required to find an arc $\tilde{x}(t)$ of the problem (1)-(3) for the $s$ th-order differential inclusions satisfying (2) almost everywhere (a.e.) on a time interval $[0, 1]$ and the initial point constraints (3) on $[0, 1]$ that minimizes the Mayer cost functional $\varphi(x(1), \ldots, x^{(s-1)}(1))$. We label this problem as $(P_V)$. A feasible trajectory $x(\cdot)$ is absolutely continuous function on $[0, 1]$ together with the higher order derivatives until $s-1$, for which $x^{(s)}(\cdot) \equiv \frac{d^s x(t)}{dt^s} \in L^1([0, 1])$. Obviously, such class of functions is a Banach space, endowed with the different equivalent norms.

**Remark 2.1.** Notice that to get sufficient condition of optimality for the Mayer problem $(P_V)$ described by ordinary evolution differential inclusions with PLDOs and with initial point constraints we use discrete-approximate method, where the problem $(P_V)$ is replaced by the following $sth$-order discrete approximate problem:

$$\text{minimize } \varphi(1 - (s - 1)h, \Delta x(1 - (s - 1)h), \ldots, \Delta^{s-1} x(1 - (s - 1)h)),$$

$$\sum_{k=1}^s p_k(t) \Delta^k x(t) \in F(x(t), t), \text{ } t = 0, h, \ldots, 1 - sh,$$

$$\Delta^k x(0) \in Q_k, k = 0, \ldots, s - 1.$$

Here $k$ th-order difference operator is defined as follows

$$\Delta^k x(t) = \frac{1}{h^k} \sum_{s=0}^k (-1)^s C_k^s x(t + (k - s)h), \quad C_k^s = \frac{k!}{s!(k - s)!}, \quad t = 0, \ldots, 1 - h.$$
Thus by using the method of approximation [13],[14],[24] we can establish necessary and sufficient conditions for rather complicated. Then by passing to the limit in necessary and sufficient conditions of this problem as \( h \to 0 \), we can construct the optimality condition for Mayer problem \((P_V)\) described by higher order differential inclusions with PLDOs and with initial point constraints. But in this paper to avoid some long calculations, derivation of these conditions for \( s \) th-order discrete-approximate problem are omitted.

3. Sufficient conditions of optimality for evolution differential inclusions with PLDOs. In this section we start our discussion with a presentation and study of sufficient optimality conditions for problem \((P_V)\). First of all we associate with the problem \((P_V)\) the following so-called the \( s \)th-order Euler-Lagrange type differential inclusion with PLDO and the transversality conditions at the endpoints \( t = 0 \) and \( t = 1 \):

\[
(i) \quad L^* x^*(t) \in F^*(x^*(t); (\ddot{x}(t), L\dot{x}(t)), t), \text{ a.e. } t \in [0, 1], \text{ where }
\]

\[
L^* x^*(t) = \sum_{k=1}^{s}(−1)^kD^k[p_k(t)x^*(t)]
\]

is the adjoint PLDO of the primal operator \( L \).

\[
(ii) \sum_{k=0}^{s-1}(-1)^{s-k}D^{s-k-1}[p_{s-k}(0)x^*(0)] \in K^*_0(\ddot{x}(0));
\]

\[
\sum_{k=0}^{s-2}(-1)^{s-k-1}D^{s-k-2}[p_{s-k}(0)x^*(0)] \in K^*_1(\ddot{x}'(0));
\]

\[
\text{.................................}
\]

\[
D[p_s(0)x^*(0)] - p_{s-1}(0)x^*(0) \in K^*_0(\ddot{x}^{s-2}(0));
\]

\[
- p_s(0)x^*(0) \in K^*_0(\ddot{x}^{s-1}(0)).
\]

\[
(iii) \left( \sum_{k=0}^{s-1}(-1)^{s-k}D^{s-k-1}[p_{s-k}(1)x^*(1)], \sum_{k=0}^{s-2}(-1)^{s-k-1}D^{s-k-2}[p_{s-k}(1)x^*(1)], \ldots, D[p_s(1)x^*(1)] - p_{s-1}(1)x^*(1), - p_s(1)x^*(1) \right) \in \partial \varphi(\ddot{x}(1), \ldots, \ddot{x}^{(s-1)}(1)).
\]

Later on we assume that \( x^*(t), t \in [0, 1] \) is absolutely continuous function with the higher order derivatives until \( s-1 \) and \( x^{(s)}(\cdot) \in L^1([0, 1]) \).

At last we formulate the condition ensuring that the LAM \( F^* \) is nonempty at a given point:

\[
(iv) \quad L\ddot{x}(t) \in F_A(\ddot{x}(t), x^*(t)), \text{ a.e. } t \in [0, 1] \text{ or, equivalently, }
\]

\[
(L\ddot{x}(t), x^*(t)) = H_F(\ddot{x}(t), x^*(t)), \text{ } L\ddot{x}(t) \in F(\ddot{x}(t), t).
\]

We are now ready for the main result, which gives sufficient conditions of optimality for evolution differential inclusions with PLDOs.

**Theorem 3.1.** Let \( \varphi : (\mathbb{R}^n)^s \to \mathbb{R}^1 \) be continuous and convex function and \( F(\cdot, t) : \mathbb{R}^n \to (\mathbb{R}^n)^s \) be a convex set-valued mapping. Moreover let \( Q_j, j = 0, \ldots, s-1 \) be convex sets. Then for optimality of the trajectory \( \ddot{x}(t) \) in the problem \((P_V)\) with evolution differential inclusions and PLDOs it is sufficient that there exists an absolutely
continuous function \( x^*(t), t \in [0,1] \) with the higher order derivatives until \( s - 1 \), satisfying a.e. the Euler-Lagrange type differential inclusion with PLDOs (i), (iv) and transversality conditions (ii), (iii) at the endpoints \( t = 0 \) and \( t = 1 \).

**Proof.** It is not hard to see that by using the Theorem 2.1 [13] in term of Hamiltonian function from the condition (i) we derive the important inequality

\[
H_F(x(t),x^*(t),t) - H_F(\tilde{x}(t),x^*(t),t) \leq \langle L^*x^*(t),x(t) - \tilde{x}(t) \rangle
\]

or its useful reformulation

\[
H_F(x(t),x^*(t),t) - H_F(\tilde{x}(t),x^*(t),t) \leq \left\langle \sum_{k=1}^{s} (-1)^{k} \frac{d^k(p_k(t)x^*(t))}{dt^k}, x(t) - \tilde{x}(t) \right\rangle
\]

(4)

Further using the definition of the Hamiltonian function, (4) can be converted to the inequality

\[
0 \geq \langle Lx(t) - L\tilde{x}(t), x^*(t) \rangle - \langle L^*x^*(t), x(t) - \tilde{x}(t) \rangle
\]

or

\[
0 \geq \left\langle \sum_{k=1}^{s} p_k(t) \frac{d^k(x(t) - \tilde{x}(t))}{dt^k}, x^*(t) \right\rangle - \left\langle \sum_{k=1}^{s} (-1)^{k} \frac{d^k(p_k(t)x^*(t))}{dt^k}, x(t) - \tilde{x}(t) \right\rangle.
\]

(5)

Integrating (5) over the interval \([0,1]\) we have

\[
0 \geq \int_{0}^{1} \left[ \left\langle \sum_{k=1}^{s} p_k(t) \frac{d^k(x(t) - \tilde{x}(t))}{dt^k}, x^*(t) \right\rangle - \left\langle \sum_{k=1}^{s} (-1)^{k} \frac{d^k(p_k(t)x^*(t))}{dt^k}, x(t) - \tilde{x}(t) \right\rangle \right] dt.
\]

(6)

Let us denote

\[
G = \sum_{k=1}^{s} \left\langle \frac{d^k(x(t) - \tilde{x}(t))}{dt^k}, p_k(t)x^*(t) \right\rangle - \sum_{k=1}^{s} \left\langle (-1)^{k} \frac{d^k(p_k(t)x^*(t))}{dt^k}, x(t) - \tilde{x}(t) \right\rangle.
\]

In what follows our approach lies in reducing \( G \) in a relationship consisting of \( s \) sums from \( k(k = 1, \ldots, s) \) to \( s \) of suitable derivatives of scalar products; thus, after some transformations we can deduce an important representation for a first term of \( G \) as follows

\[
\sum_{k=1}^{s} \left\langle \frac{d^k(x(t) - \tilde{x}(t))}{dt^k}, p_k(t)x^*(t) \right\rangle = \sum_{k=1}^{s} \left[ \frac{d}{dt} \left\langle x^{k-1}(t) - \tilde{x}^{k-1}(t), p_k(t)x^*(t) \right\rangle \right]
\]

\[
- \sum_{k=2}^{s} \left[ \frac{d}{dt} \left\langle x^{k-2}(t) - \tilde{x}^{k-2}(t), \frac{d(p_k(t)x^*(t))}{dt} \right\rangle \right]
\]

\[
+ \sum_{k=3}^{s} \left[ \frac{d}{dt} \left\langle x^{k-3}(t) - \tilde{x}^{k-3}(t), \frac{d^2(p_k(t)x^*(t))}{dt^2} \right\rangle \right]
\]
\[- \cdots + \sum_{k=s-2}^{s} \left[ \frac{d}{dt} \left\langle x^{(s-2)}(t) - \tilde{x}^{(s-2)}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \right] \]
\[+ \sum_{k=s-1}^{s} \left[ \frac{d}{dt} \left\langle x^{(s-1)}(t) - \tilde{x}^{(s-1)}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \right] \]
\[+ \frac{d}{dt} \left\langle x(t) - \tilde{x}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \]
\[+ \sum_{k=1}^{s} \left[ \frac{d^k}{dt^k} \left\langle \tilde{x}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \right]. \]  

(7)

Then in view of (7) in the definition of $G$ we have an efficient formula:

\[G = \sum_{k=1}^{s} \left[ \frac{d}{dt} \left\langle x^{(k-1)}(t) - \tilde{x}^{(k-1)}(t), \tilde{\mu}(t)x^*(t) \right\rangle \right] \]
\[- \sum_{k=2}^{s} \left[ \frac{d}{dt} \left\langle x^{(k-2)}(t) - \tilde{x}^{(k-2)}(t), \tilde{\mu}(t)x^*(t) \right\rangle \right] \]
\[+ \sum_{k=3}^{s} \left[ \frac{d}{dt} \left\langle x^{(k-3)}(t) - \tilde{x}^{(k-3)}(t), \tilde{\mu}(t)x^*(t) \right\rangle \right] \]
\[- \cdots + \sum_{k=s-2}^{s} \left[ \frac{d}{dt} \left\langle x^{(s-2)}(t) - \tilde{x}^{(s-2)}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \right] \]
\[+ \sum_{k=s-1}^{s} \left[ \frac{d}{dt} \left\langle x^{(s-1)}(t) - \tilde{x}^{(s-1)}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \right] \]
\[+ \frac{d}{dt} \left\langle x(t) - \tilde{x}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \]
\[+ \sum_{k=1}^{s} \left[ \frac{d^k}{dt^k} \left\langle \tilde{x}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \right]. \]  

(8)

Then taking into account the structure of $G$ in (8) we can compute the integral on the right hand side of (6) as follows:

\[\int_{0}^{1} Gdt = \sum_{k=1}^{s} \left[ \int_{0}^{1} \frac{d}{dt} \left\langle x^{(k-1)}(t) - \tilde{x}^{(k-1)}(t), \tilde{\mu}(t)x^*(t) \right\rangle \right] \]
\[- \sum_{k=2}^{s} \left[ \int_{0}^{1} \frac{d}{dt} \left\langle x^{(k-2)}(t) - \tilde{x}^{(k-2)}(t), \tilde{\mu}(t)x^*(t) \right\rangle \right] \]
\[+ \sum_{k=3}^{s} \left[ \int_{0}^{1} \frac{d}{dt} \left\langle x^{(k-3)}(t) - \tilde{x}^{(k-3)}(t), \tilde{\mu}(t)x^*(t) \right\rangle \right] \]
\[- \cdots + \sum_{k=s-2}^{s} \left[ \int_{0}^{1} \frac{d}{dt} \left\langle x^{(s-2)}(t) - \tilde{x}^{(s-2)}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \right] \]
\[+ \sum_{k=s-1}^{s} \left[ \int_{0}^{1} \frac{d}{dt} \left\langle x^{(s-1)}(t) - \tilde{x}^{(s-1)}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \right] \]
\[+ \frac{d}{dt} \left\langle x(t) - \tilde{x}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \]
\[+ \sum_{k=1}^{s} \left[ \int_{0}^{1} \frac{d^k}{dt^k} \left\langle \tilde{x}(t), (\tilde{\mu}(t)x^*(t)) \right\rangle \right]. \]
Hence as a result of integration over an interval [0, 1] we deduce that

\[
\int_0^1 G dt = \sum_{k=1}^{s} \left[ \langle x^{(k-1)}(1) - \tilde{x}^{(k-1)}(1), p_k(1)x^*(1) \rangle ight] \\
- \left[ \langle x^{(k-1)}(0) - \tilde{x}^{(k-1)}(0), p_k(0)x^*(0) \rangle \right] \\
- \sum_{k=2}^{s} \left[ \langle x^{(k-2)}(1) - \tilde{x}^{(k-2)}(1), \frac{d(p_k(1)x^*(1))}{dt} \rangle \right] \\
- \left[ \langle x^{(k-2)}(0) - \tilde{x}^{(k-2)}(0), \frac{d(p_k(0)x^*(0))}{dt} \rangle \right] \\
+ \sum_{k=3}^{s} \left[ \langle x^{(k-3)}(1) - \tilde{x}^{(k-3)}(1), \frac{d^2(p_k(1)x^*(1))}{dt^2} \rangle \right] \\
- \left[ \langle x^{(k-3)}(0) - \tilde{x}^{(k-3)}(0), \frac{d^2(p_k(0)x^*(0))}{dt^2} \rangle \right] \\
- \cdots + \sum_{k=s-2}^{s} \left[ \langle x^{(k-s+2)}(1) - \tilde{x}^{(k-s+2)}(1), (-1)^{s-3}d^{s-3}(p_k(1)x^*(1)) \rangle \right] \\
- \left[ \langle x^{(k-s+2)}(0) - \tilde{x}^{(k-s+2)}(0), (-1)^{s-3}d^{s-3}(p_k(0)x^*(0)) \rangle \right] \\
+ \sum_{k=s-1}^{s} \left[ \langle x^{(k-s+1)}(1) - \tilde{x}^{(k-s+1)}(1), (-1)^{s-2}d^{s-2}(p_k(1)x^*(1)) \rangle \right] \\
- \left[ \langle x^{(k-s+1)}(0) - \tilde{x}^{(k-s+1)}(0), (-1)^{s-2}d^{s-2}(p_k(0)x^*(0)) \rangle \right] \\
+ \langle x(1) - \tilde{x}(1), (-1)^{s-1}d^{s-1}(p_k(1)x^*(1)) \rangle \\
- \langle x(0) - \tilde{x}(0), (-1)^{s-1}d^{s-1}(p_k(0)x^*(0)) \rangle.
\]

Here by suitable rearrangement and necessary simplification we have

\[
\int_0^1 G dt = \sum_{k=1}^{s} \left[ \langle x^{(k-1)}(1) - \tilde{x}^{(k-1)}(1), p_k(1)x^*(1) \rangle \right] \\
- \sum_{k=2}^{s} \left[ \langle x^{(k-2)}(1) - \tilde{x}^{(k-2)}(1), \frac{d(p_k(1)x^*(1))}{dt} \rangle \right] \\
+ \sum_{k=3}^{s} \left[ \langle x^{(k-3)}(1) - \tilde{x}^{(k-3)}(1), \frac{d^2(p_k(1)x^*(1))}{dt^2} \rangle \right] \\
- \cdots + \sum_{k=s-2}^{s} \left[ \langle x^{(k-s+2)}(1) - \tilde{x}^{(k-s+2)}(1), (-1)^{s-3}d^{s-3}(p_k(1)x^*(1)) \rangle \right] \\
+ \sum_{k=s-1}^{s} \left[ \langle x^{(k-s+1)}(1) - \tilde{x}^{(k-s+1)}(1), (-1)^{s-2}d^{s-2}(p_k(1)x^*(1)) \rangle \right]
\]
Thus, from (6), (9) we have

\[
\begin{align*}
0 & \geq \sum_{k=1}^{s} \left\langle x^{(k-1)}(1) - \tilde{x}^{(k-1)}(1), p_{k}(1)x^{*}(1) \right\rangle \\
& - \sum_{k=2}^{s} \left\langle x^{(k-2)}(1) - \tilde{x}^{(k-2)}(1), \frac{d(p_{k}(1)x^{*}(1))}{dt} \right\rangle \\
& + \sum_{k=3}^{s} \left\langle x^{(k-3)}(1) - \tilde{x}^{(k-3)}(1), \frac{d^2(p_{k}(1)x^{*}(1))}{dt^2} \right\rangle \\
& - \cdots - \sum_{k=s-2}^{s} \left\langle x^{(k-s+2)}(1) - \tilde{x}^{(k-s+2)}(1), (-1)^{s-3} \frac{d^{s-3}(p_{k}(1)x^{*}(1))}{dt^{s-3}} \right\rangle \\
& + \sum_{k=s-1}^{s} \left\langle x^{(k-s+1)}(1) - \tilde{x}^{(k-s+1)}(1), (-1)^{s-2} \frac{d^{s-2}(p_{k}(1)x^{*}(1))}{dt^{s-2}} \right\rangle \\
& + \left\langle x(1) - \tilde{x}(1), (-1)^{s-1} \frac{d^{s-1}(p_{s}(1)x^{*}(1))}{dt^{s-1}} \right\rangle.
\end{align*}
\]

In order to make use of the transversality condition (ii) we rewrite it in more relevant form

\[
\begin{align*}
& \left\langle x(0) - \tilde{x}(0), \sum_{k=0}^{s-1} (-1)^{s-k} D^{s-k-1}[(p_{s-k}(0)x^{*}(0))] \right\rangle \\
& + \left\langle x'(0) - \tilde{x}'(0), \sum_{k=0}^{s-2} (-1)^{s-k-1} D^{s-k-2}[(p_{s-k}(0)x^{*}(0))] \right\rangle \\
& + \left\langle x''(0) - \tilde{x}''(0), \sum_{k=0}^{s-3} (-1)^{s-k-2} D^{s-k-3}[(p_{s-k}(0)x^{*}(0))] \right\rangle \\
& + \cdots - \left\langle x^{(s-2)}(0) - \tilde{x}^{(s-2)}(0), -D[(p_{s}(0)x^{*}(0))] + p_{s-1}(0)x^{*}(0) \right\rangle \\
& - \left\langle x^{(s-1)}(0) - \tilde{x}^{(s-1)}(0), p_{s}(0)x^{*}(0) \right\rangle \geq 0; \forall x^{(k)}(0) \in K_{Q_{k}}(\tilde{x}^{(k)}(0)), k = 0, \ldots, s - 1.
\end{align*}
\]

Thus, from (6), (9) we have

\[
0 \geq \sum_{k=1}^{s} \left\langle x^{(k-1)}(1) - \tilde{x}^{(k-1)}(1), p_{k}(1)x^{*}(1) \right\rangle \\
- \sum_{k=2}^{s} \left\langle x^{(k-2)}(1) - \tilde{x}^{(k-2)}(1), \frac{d(p_{k}(1)x^{*}(1))}{dt} \right\rangle \\
+ \sum_{k=3}^{s} \left\langle x^{(k-3)}(1) - \tilde{x}^{(k-3)}(1), \frac{d^2(p_{k}(1)x^{*}(1))}{dt^2} \right\rangle \\
- \cdots - \sum_{k=s-2}^{s} \left\langle x^{(k-s+2)}(1) - \tilde{x}^{(k-s+2)}(1), (-1)^{s-3} \frac{d^{s-3}(p_{k}(1)x^{*}(1))}{dt^{s-3}} \right\rangle \\
+ \sum_{k=s-1}^{s} \left\langle x^{(k-s+1)}(1) - \tilde{x}^{(k-s+1)}(1), (-1)^{s-2} \frac{d^{s-2}(p_{k}(1)x^{*}(1))}{dt^{s-2}} \right\rangle \\
+ \left\langle x(1) - \tilde{x}(1), (-1)^{s-1} \frac{d^{s-1}(p_{s}(1)x^{*}(1))}{dt^{s-1}} \right\rangle.
\]

Using the derivative operator $D$ it is not hard to see that the relation described above can be expressed in a more compact form
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with PLDO and transversality conditions at points $t$ satisfying the conditions of Theorem 3.1, where the Euler-Lagrange type inclusion

ous function $x$ in general is nonconcave function satisfying the condition $(\ast)$. Suppose that we have the "nonconvex" problem $(\ast)$. Theorem 3.2.

$\tilde{Q}$ is the cones of tangent directions to $K$.

Furthermore, applying the definition of the transversality condition $(iii)$ for all feasible arc $x(\cdot)$ we have

$$
\varphi \left( x(1), x'(1), \ldots, x^{(s-1)}(1) \right) - \varphi \left( \tilde{x}(1), \tilde{x}'(1), \ldots, \tilde{x}^{(s-1)}(1) \right) 
\geq \sum_{k=0}^{s-1} (-1)^{s-k-1} D^{s-k-1} [p_{s-k}(1)x^*(1)], x(1) - \tilde{x}(1) 
+ \sum_{k=0}^{s-2} (-1)^{s-k-2} D^{s-k-2} [p_{s-k}(1)x^*(1)], x'(1) - \tilde{x}'(1) + \cdots 
- \overline{D[p_{s}(1)x^*(1)] - p_{s-1}(1)x^*(1), x^{(s-2)}(1) - \tilde{x}^{(s-2)}(1)} 
+ \overline{p_{s}(1)x^*(1), x^{(s-1)}(1) - \tilde{x}^{(s-1)}(1)}. 
$$

(10)

Furthermore, applying the definition of the transversality condition $(iii)$ for all feasible arc $x(\cdot)$ we have

$$
\varphi \left( x(1), x'(1), \ldots, x^{(s-1)}(1) \right) - \varphi \left( \tilde{x}(1), \tilde{x}'(1), \ldots, \tilde{x}^{(s-1)}(1) \right) 
\geq \sum_{k=0}^{s-1} (-1)^{s-k-1} D^{s-k-1} [p_{s-k}(1)x^*(1)], x(1) - \tilde{x}(1) 
+ \sum_{k=0}^{s-2} (-1)^{s-k-2} D^{s-k-2} [p_{s-k}(1)x^*(1)], x'(1) - \tilde{x}'(1) + \cdots 
- \overline{D[p_{s}(1)x^*(1)] - p_{s-1}(1)x^*(1), x^{(s-2)}(1) - \tilde{x}^{(s-2)}(1)} 
+ \overline{p_{s}(1)x^*(1), x^{(s-1)}(1) - \tilde{x}^{(s-1)}(1)}. 
$$

(11)

Then from the last two inequalities (10) and (11) for all feasible arc we have immediately $\varphi \left( x(1), x'(1), \ldots, x^{(s-1)}(1) \right) \geq \varphi \left( \tilde{x}(1), \tilde{x}'(1), \ldots, \tilde{x}^{(s-1)}(1) \right)$, that is, $\tilde{x}(\cdot)$ is optimal trajectory.

Below nonconvexity of a set-valued mapping $F(\cdot, t)$ means that its Hamilton function in general is nonconcave function satisfying the condition $(a)$.

Theorem 3.2. Suppose that we have the "nonconvex" problem $(P_V)$, that is, $\varphi : (\mathbb{R}^n)^s \rightarrow \mathbb{R}$ and $F(\cdot, t) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ in general are nonconvex function and set-valued mapping, respectively. Moreover, suppose that $K_{Q_j}(\tilde{x}^{(j)}(0)), \tilde{x}^{(j)}(0) \in Q_j$ is the cones of tangent directions to $Q_j, j = 0, \ldots, s - 1$. Then for optimality of the trajectory $\tilde{x}(t), t \in [0, 1]$ it is sufficient that there exists an absolutely continuous function $x^*(t), t \in [0, 1]$ together with the higher order derivatives until $s - 1$, satisfying the conditions of Theorem 3.1, where the Euler-Lagrange type inclusion with PLDO and transversality conditions at points $t = 0$ and $t = 1$ consist of the following:

$\left( a \right) L^* x^*(t) \in F^* \left( x^*(t); (\tilde{x}(t), L\tilde{x}(t)), t \right), \ a.e. \ t \in [0, 1],$

$\left( b \right) \sum_{k=0}^{s-1-j} (-1)^{s-k-j} D^{s-k-1-j} [p_{s-k}(0)x^*(0)] \in K_{Q_j}^* (\tilde{x}^{(j)}(0)), j = 0, \ldots, s - 1,$

$\left( c \right) \varphi (v_0, v_1, \ldots, v_{s-1}) - \varphi (\tilde{x}(1), \tilde{x}'(1), \ldots, \tilde{x}^{(s-1)}(1))$

$\geq \sum_{j=0}^{j-1} \sum_{k=0}^{s-1-j} (-1)^{s-k-j} D^{s-k-1-j} [p_{s-k}(1)x^*(1)], v_j - \tilde{x}^{(j)}(1), \ \forall v_j \in \mathbb{R}^n,$

$\left( d \right) \langle L\tilde{x}(t), x^*(t) \rangle = H_F (\tilde{x}(t), x^*(t), t), \ a.e. \ t \in [0, 1].$
Proof. In the proof of Theorem 3.1 we have used the following inequality

\[
H_F(x(t), x^*(t), t) - H_F(\bar{x}(t), x^*(t), t) \leq \left\langle \sum_{k=1}^{s} (-1)^k \frac{d^k}{dt^k} (p_k(t)x^*(t)) \right\rangle, x(t) - \bar{x}(t) \right.
\]

(12)

Hence, from the inequality (12) immediately we have the inequality (10). Moreover, setting \( v_j = \bar{x}^{(j)}(1) (j = 0, \ldots, s - 1) \) for all feasible trajectories \( x(\cdot) \) it is not hard to see that for nonconvex \( \varphi \) the following inequality holds:

\[
\varphi(x(1), x'(1), \ldots, x^{(x-1)}(1)) - \varphi(\bar{x}(1), \bar{x}'(1), \ldots, \bar{x}^{(x-1)}(1)) \geq \sum_{j=0}^{s-1} \sum_{k=0}^{s-1-j} (-1)^{s-k-j} D^{s-k-1-j} \left[ p_{s-k}(1)x^*(1), x^{(j)}(1) - \bar{x}^{(j)}(1) \right],
\]

\( j = 0, \ldots, s - 1. \)

Then for the furthest proof we proceed by analogy with the preceding derivation of Theorem 3.1. \( \square \)

4. Some applications to optimal control problems with PLDOs. In this section we give two applications of our results. The first one is the particular Mayer problem for differential inclusions involving PLDOs with constant coefficients and the second one concerns optimization of "linear" differential inclusions with PLDOs and constant coefficients. Thus, suppose now we have the following optimization problem (for simplicity we consider a convex problem) with \( s \) th-order PLDO with constant coefficients:

\[
\begin{align*}
\text{minimize } & \varphi_0(x(1)), \\
(P_C) \quad & Lx(t) \in F(x(t), t), \text{ a.e. } t \in [0, 1], \quad Lx = D^s x + p_1 D^{s-1} x + \cdots + p_{s-1} D x \\
& x(0) = \alpha_0, \quad x'(0) = \alpha_1, \ldots, x^{(s-1)}(0) = \alpha_{s-1},
\end{align*}
\]

(13)

where \( L \) is the \( s \)th-order polynomial operator; \( p_k, k = 1, \ldots, s - 1 \) are some real constants, \( F(\cdot,t) : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a convex set-valued mapping, \( \varphi_0 : \mathbb{R}^n \rightarrow \mathbb{R}^1 \) is a continuous convex function, \( \alpha_j \in \mathbb{R}^n, j = 0, \ldots, s - 1 \) are fixed vectors. The problem \( (P_C) \) is to find a trajectory \( \tilde{x}(t) \) such that the cost functional \( \varphi_0(x(1)) \) is minimized. It should be noted that along with many others important properties the multiplication operation is commutative for PLDOs with constant coefficients.

Recall that the \( s \) th-order an adjoint PLDO with constant coefficients is defined as follows:

\[
L^* \cdot x^* = (-1)^s D^s x^* + (-1)^{s-1} p_1 D^{s-1} x^* + \cdots + p_{s-1} D x^*.
\]

Corollary 4.1. Let \( \varphi_0 : \mathbb{R}^n \rightarrow \mathbb{R}^1 \) be continuous and convex function. Moreover, let \( F(\cdot,t) \), be convex set-valued mapping. Then for the optimality of the arc \( \tilde{x}(t) \) in the problem \( (P_C) \) with PLDO it is sufficient that there exists an absolutely continuous function \( x^*(t), t \in [0, 1] \) together with the higher order derivatives until \( s - 1 \) satisfying a.e. the Euler-Lagrange type differential inclusion with PLDO

\[
L^* \cdot x^*(t) \in F^* \left( x^*(t); (\tilde{x}(t), L\tilde{x}(t)), t \right); \quad \left\langle L\tilde{x}(t), x^*(t) \right\rangle = H_F(\tilde{x}(t), x^*(t)),
\]

a.e. \( t \in [0, 1] \),

\[
L^* \cdot x^* = (-1)^s \frac{d^s}{dt^s} x^* + (-1)^{s-1} p_1 \frac{d^{s-1}}{dt^{s-1}} x^* + \cdots + p_{s-1} \frac{dx^*}{dt}.
\]
and transversality condition at the endpoint \( t = 1 \)

\[
(-1)^j \frac{d^{s-1}x^s(1)}{dt^{s-1}} \in \partial \varphi_0(\tilde{x}(1)), \quad \frac{d^j x^s(1)}{dt^j} = 0, \quad j = 0, \ldots, s - 2.
\]

Proof. We conclude this proof by returning to the condition (i), (ii), (iii) of Theorem 3.1. Clearly, a problem \((P_C)\) can be reduced to the problem of form \((P_V)\), where

\[
\varphi(x(1), x'(1), \ldots, x^{(s-1)}(1)) \equiv \varphi_0(x(1)).
\]

It follows that \( \partial \varphi(x(1), x'(1), \ldots, x^{(s-1)}(1)) = \partial_x \varphi_0(x(1)) \times (0, \ldots, 0) \). On the other hand since \( p_j(t) = p_{s-j}, j = 1, \ldots, s - 1 \) are constants, by sequential substitution in the transversality condition (iii) we derive that

\[
\sum_{k=0}^{s-1-j} (-1)^{s-k-j} D^{s-k-1-j} p_{s-k}(1)x^s(1) = \frac{d^{s-1-j}x^s(1)}{dt^{s-1-j}} = 0, \quad j = 1, \ldots, s - 1,
\]

and therefore for \( j = 0 \)

\[
\sum_{k=0}^{s-1} (-1)^{s-k} D^{s-k-1} p_{s-k}(1)x^s(1) = (-1)^s \frac{d^{s-1}x^s(1)}{dt^{s-1}} \in \partial \varphi_0(\tilde{x}(1)).
\]

Suppose now that we have so-called linear Mayer problem with PLDOs:

\[
\text{minimize } \varphi_0(x(1)),
\]

\[
Lx(t) \in F(x(t), t), \text{ a.e. } t \in [0, 1],
\]

\[
x^{(j)}(0) = \alpha_j, \quad j = 0, \ldots, s - 1, \quad F(x, t) = A(t)x + B(t)U,
\]

where \( \varphi_0 \) is continuously differentiable convex function, \( A(t) \) and \( B(t) \) are \( n \times n \) and \( n \times r \) continuous matrices, respectively, \( U \) is a convex closed subset of \( \mathbb{R}^r \); \( \alpha_j, j = 0, \ldots, s - 1 \) are fixed vectors, The problem is to find a controlling parameter \( \hat{u}(t) \in U \) such that the arc \( \hat{x}(t) \) corresponding to it minimizes \( \varphi_0(x(1)) \). In fact, this is optimization of Cauchy problem for "linear" differential inclusions with PLDO. The controlling parameter \( u(\cdot) \) is called admissible if it only takes values in the given control set \( U \) which is nonempty, convex, closed set.

Theorem 4.2. The arc \( \hat{x}(t) \) corresponding to the controlling parameter \( \hat{u}(t) \) is a solution to the problem \( (14), (15) \) if there exists an absolutely continuous function \( x^*(t) \) together with the higher order derivatives until \( s - 1 \), satisfying the following Euler-Lagrange type differential equation with PLDO, the transversality condition at a point \( t = 1 \) and Weierstrass-Pontryagin maximum principle:

\[
L^* x^*(t) = A^*(t)x^*(t), \text{ a.e. } t \in [0, 1],
\]

\[
(-1)^s \frac{d^{s-1}x^s(1)}{dt^{s-1}} = \varphi'_0(\tilde{x}(1)), \quad \frac{d^j x^s(1)}{dt^j} = 0, \quad j = 0, \ldots, s - 2,
\]

\[
\left\langle B(t)\hat{u}(t), x^*(t) \right\rangle = \sup_{u \in U} \left\langle B(t)u, x^*(t) \right\rangle.
\]

Proof. Obviously, the Hamiltonian is

\[
H_P(x, v^*, t) = \left\langle A(t)x, v^* \right\rangle + \sup_{u \in U} \left\langle B(t)u, v^* \right\rangle.
\]

Hence,
$$F^* (v^*; (x, \tilde{v}), t) = \partial_x H_F (x, v^*, t) = A^* (t) v^*, \tilde{v} \in F_A (x, v^*, t), \tilde{v} = A(t) x + B(t) \tilde{u},$$

where the argmaximum inclusion \( \tilde{v} \in F_A (x, v^*, t) \) means that \( \langle B(t) \tilde{u}, v^* \rangle = \sup_{u \in U} \langle B(t) u, v^* \rangle \) and in \( F^* (v^*; (x, \tilde{v}), t) \neq \emptyset \). Then by Theorem 3.1 we can write

$$L^* x^* (t) = A^* (t) x^* (t), \ L \tilde{x} (t) \in F_A (\tilde{x} (t), x^* (t), t),$$

$$\langle B(t) \tilde{u} (t), x^* (t) \rangle = \sup_{u \in U} \langle B(t) u, x^* (t) \rangle.$$

Consequently, the transversality conditions (ii) of Theorem 3.1 is unnecessary and by Corollary 4.1 \((-1)^s D^{s-1} x^* (1) = \varphi_0 (\tilde{x} (1)), \ D^j x^* (1) = 0, j = 0, \ldots, s - 2. \)

5. **Sufficient conditions of optimality for second order evolution differential inclusions with endpoint constraints.** Note that in this section the optimality conditions are given for second order convex differential inclusions \((P_M)\) with convex endpoint constraints. These conditions are more precise than any previously published ones since they involve useful forms of the Weierstrass-Pontryagin condition and second order Euler-Lagrange type adjoint inclusions. In the reviewed results this effort culminates in Theorems 5.1.

The following adjoint inclusion is the second order Euler-Lagrange type inclusion for the problem \((P_M)\)

$$\langle a_1 \rangle \left( \frac{d^2 x^* (t)}{dt^2} + \frac{dv^* (t)}{dt}, v^* (t) \right) \in F^* (x^* (t); (\tilde{x} (t), \tilde{x}' (t), \tilde{x}'' (t))) t), \ a.e. \ t \in [0, 1],$$

where

$$\langle b_1 \rangle \left( \frac{d^2 \tilde{x}^* (t)}{dt^2} \right) \in F_A (\tilde{x} (t), \tilde{x}' (t); x^* (t), t), \ a.e. \ t \in [0, 1].$$

In what follows we assume that \( x^* (t), t \in [0, 1] \) is absolutely continuous function together with the first order derivatives for which \( x^{(r)} (\cdot) \in L^p_1 ([0, 1]) \). Besides the auxiliary function \( v^* (t), t \in [0, 1] \) is absolutely continuous and \( v^* (\cdot) \in L^p_2 ([0, 1]) \).

The transversality conditions at the endpoint \( t = 1 \) consist of the following

$$\langle c_1 \rangle \left( v^* (1) + \frac{dx^* (1)}{dt}, -x^* (1) \right) \in \partial_{(x, u)} g (\tilde{x} (1), \tilde{x}' (1)) - K^*_M (\tilde{x} (1)) \times K^*_M (\tilde{x}' (1)).$$

Now we are ready formulate the following theorem of optimality

**Theorem 5.1.** Suppose that \( g \) is a continuous and convex function, \( F(., t) \) is a convex set-valued mapping and \( M_0, M_1 \subseteq \mathbb{R}^n \) are convex sets. Then for optimality of the feasible trajectory \( \tilde{x} (t) \) in the problem \((P_M)\) it is sufficient that there exists a pair of absolutely continuous functions \( \{ x^* (t), v^* (t) \}, t \in [0, 1] \) satisfying a.e. the second order Euler-Lagrange type inclusion \(\langle a_1 \rangle, \langle b_1 \rangle\) and the transversality condition \(\langle c_1 \rangle\) at the endpoint \( t = 1 \).
Proof. By the proof idea of Theorem 3.1 from (a₁), (b₁) we obtain the adjoint differential inclusion of second order
\[
\left( \frac{d^2 x^*(t)}{dt^2} + \frac{dv^*(t)}{dt}, v^*(t) \right) \in \partial_{(x,u)} H_F(\tilde{x}(t), \tilde{x}'(t), x^*(t), t), \ t \in [0,1].
\]

On the definition of subdifferential set of the Hamiltonian function \( H_F(\cdot, t) \) for all feasible trajectory \( x(t), t \in [0,1] \) we rewrite the last relation in the equivalent form:
\[
H_F(x(t), x'(t), x^*(t), t) - H_F(\tilde{x}(t), \tilde{x}'(t), x^*(t), t) \leq \left\langle \frac{d^2 x^*(t)}{dt^2} + \frac{dv^*(t)}{dt}, x(t) - \tilde{x}(t) \right\rangle + \left\langle v^*(t), x'(t) - \tilde{x}'(t) \right\rangle. \tag{16}
\]

Now by using definition of the Hamiltonian function, the inequality (16) can be reduced to the inequality
\[
0 \geq \left\langle \frac{d^2(x(t) - \tilde{x}(t))}{dt^2}, x^*(t) \right\rangle - \left\langle \frac{d^2 x^*(t)}{dt^2}, x(t) - \tilde{x}(t) \right\rangle - \frac{d}{dt} \left\langle v^*(t), x(t) - \tilde{x}(t) \right\rangle \tag{17}
\]

Integrating of the inequality (17) over the interval \([0,1]\) we derive that
\[
0 \geq \int_0^1 \left[ \left\langle \frac{d^2(x(t) - \tilde{x}(t))}{dt^2}, x^*(t) \right\rangle - \left\langle \frac{d^2 x^*(t)}{dt^2}, x(t) - \tilde{x}(t) \right\rangle \right] dt
+ \left\langle v^*(0), x(0) - \tilde{x}(0) \right\rangle - \left\langle v^*(1), x(1) - \tilde{x}(1) \right\rangle. \tag{18}
\]

For convenience we transform the expression in the square parentheses on the right hand side of (18) as follows
\[
\left\langle \frac{d^2(x(t) - \tilde{x}(t))}{dt^2}, x^*(t) \right\rangle - \left\langle \frac{d^2 x^*(t)}{dt^2}, x(t) - \tilde{x}(t) \right\rangle
= \frac{d}{dt} \left\langle \frac{d(x(t) - \tilde{x}(t))}{dt}, x^*(t) \right\rangle - \frac{d}{dt} \left\langle \frac{dx^*(t)}{dt}, x(t) - \tilde{x}(t) \right\rangle.
\]

Thus by elementary property of the definite integrals we can compute the integral on the right hand side of (18)
\[
\int_0^1 \left[ \left\langle \frac{d^2(x(t) - \tilde{x}(t))}{dt^2}, x^*(t) \right\rangle - \left\langle \frac{d^2 x^*(t)}{dt^2}, x(t) - \tilde{x}(t) \right\rangle \right] dt \nonumber
= \left\langle \frac{d(x(1) - \tilde{x}(1))}{dt}, x^*(1) \right\rangle - \left\langle \frac{d(x(0) - \tilde{x}(0))}{dt}, x^*(0) \right\rangle
- \left\langle \frac{dx^*(1)}{dt}, x(1) - \tilde{x}(1) \right\rangle + \left\langle \frac{dx^*(0)}{dt}, x(0) - \tilde{x}(0) \right\rangle. \tag{19}
\]

Then substituting (19) into (18), we have
0 \geq \left\langle \frac{d(x(1) - \tilde{x}(1))}{dt}, x^*(1) \right\rangle - \left\langle \frac{d(x(0) - \tilde{x}(0))}{dt}, x^*(0) \right\rangle \\
- \left\langle v^*(1) + \frac{dx^*(1)}{dt}, x(1) - \tilde{x}(1) \right\rangle + \left\langle v^*(0) + \frac{dx^*(0)}{dt}, x(0) - \tilde{x}(0) \right\rangle. \tag{20}

Now, remember that \( x(\cdot), \tilde{x}(\cdot) \) are feasible trajectories and \( x(0) = \tilde{x}(0) = x_0, x'(0) = \tilde{x}'(0) = x_1 \). Then it follows from (20) that

\[ 0 \geq \left\langle \frac{d(x(1) - \tilde{x}(1))}{dt}, x^*(1) \right\rangle - \left\langle v^*(1) + \frac{dx^*(1)}{dt}, x(1) - \tilde{x}(1) \right\rangle. \tag{21} \]

Now, thanking to the transversality conditions (c_1) at the endpoint \( t = 1 \), we can rewrite

\[ g(x(1), x'(1)) - g(\tilde{x}(1), \tilde{x}'(1)) \geq \left\langle v^*(1) + \frac{dx^*(1)}{dt}, x(1) - \tilde{x}(1) \right\rangle \]

\[ + \left\langle x^*(1) - x^*(1), x'(1) - \tilde{x}'(1) \right\rangle, x^*(1) \in K_{M_0}(\tilde{x}(1)), x^*(1) \in K_{M_1}(\tilde{x}'(1)) \]

or, equivalently,

\[ g(x(1), x'(1)) - g(\tilde{x}(1), \tilde{x}'(1)) \geq \left\langle v^*(1) + \frac{dx^*(1)}{dt}, x(1) - \tilde{x}(1) \right\rangle \]

\[ - \left\langle x^*(1), x'(1) - \tilde{x}'(1) \right\rangle. \tag{22} \]

Thus, summing the inequalities (21),(22) for all feasible trajectories \( x(\cdot) \) satisfying the initial conditions \( x(0) = x_0, x'(0) = x_1 \) and endpoint constraints \( x(1) \in M_0, x'(1) \in M_1 \) we have the needed inequality:

\[ g(x(1), x'(1)) - g(\tilde{x}(1), \tilde{x}'(1)) \geq 0 \text{ or } g(x(1), x'(1)) \geq g(\tilde{x}(1), \tilde{x}'(1)). \]

6. Conclusion. According to proposed method the problem with the differential inclusions described by polynomial linear differential operators is investigated. Obviously, this problem is an important generalization of problems with first order differential inclusions. Thus, sufficient conditions of optimality for such problems are deduced. Here existence of nonfunctional initial point or endpoint constraints generates different kind of transversality conditions. Besides, there can be no doubt that investigations of optimality conditions of problems with second and fourth order Sturm-Liouville type differential inclusions can have great contribution to the modern development of the optimal control theory with PLDOs. Consequently, there arises a rather complicated problem with simultaneous determination of the PLDOs with variable coefficients and a Mayer functional depending of high order derivatives of searched functions. Thus, it is concluded that the proposed method is reliable for solving the various optimization problems with higher order differential inclusions.

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