Final-state Interaction Effects on Inclusive Two-particle Production in Electron-positron Annihilation

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The final-state interaction effects on the inclusive two-particle production in electron-positron annihilation are investigated within the context of the one-photon annihilation approximation. Such effects are characterized by one structure function in the decomposition of the hadronic tensor. On the basis of the positivity, we derived an inequality to bound this structure function. The price to access it experimentally is to polarize longitudinally one of the initial-state beam, to say, the electron beam, and measure the corresponding single spin asymmetry. By combining the Callan-Gross relation with our positivity analysis, we obtain an upper bound for the single spin asymmetry considered.

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The electron-positron annihilation has been and will continue to be a Holy Land in the study of the hadron production dynamics. Usually, people assumes a physical picture for the electron-positron annihilation into hadrons like this: The electron and positron annihilates into a quasi-free quark-antiquark pair via a virtual vector boson, and the quark-antiquark pair hadronizes into the particles that reach the detector. At high energies, the final-state particles form two jets, roughly collinear with the parent quark-antiquark pair. [Three or more jets can also be formed, but such events are rare in comparison with two-jet events.] Depending upon how many and what kind of particles are measured, the inclusive particle production in electron-positron annihilation supplies us with an ideal place in which to measure various quark fragmentation functions. Among others, one can measure a pair of particles, one in each jet. For this category of reactions, we have the following channels in mind: $e^+e^- \rightarrow \pi^+\pi^-X$, $e^+e^- \rightarrow \pi^0\pi^0X$, $e^+e^- \rightarrow p\bar{p}X$, $e^+e^- \rightarrow \Lambda\bar{\Lambda}X$, $e^+e^- \rightarrow B\bar{B}X$, $e^+e^- \rightarrow D\bar{D}X$, and so on.

The study of inclusive two-particle production in electron-positron annihilation has a history of more than 20 years. In the early 1980s, Collins and Soper [1] discussed the “almost back-to-back” two-pion production in electron-positron annihilation, indicating that it can be employed to measure the transverse momentum dependence of the quark fragmentation function. Later, Fong and Webber [2] systematically predicted the two- as well as one-particle distributions in electron-positron annihilation, and pointed out the importance of measuring two-particle correlations. Artru [3], and Chen, Goldstein, Jaffe and Ji [4] observed that $e^+e^- \rightarrow \Lambda\bar{\Lambda}X$ can be used to extract the quark transversity distribution function, with the $\Lambda$ and $\bar{\Lambda}$ polarization measured. Recently, Boer, Jacob and Mulders [5] developed a systematic quantum chromodynamics (QCD) factorization of the process of type $e^+e^- \rightarrow hhX$, up to order $1/Q$, in terms of a full set of quark fragmentation functions. Hopefully, the B-factory under construction at KEK can be used to study the above processes as well as $e^+e^- \rightarrow B\bar{B}X$. At the planned Tau-Charm factory [6], one will have much chance to access $e^+e^- \rightarrow D\bar{D}X$. The advantage in choosing two detected particles to be a particle-antiparticle pair lies in that the fragmentation function for the chosen particle can be related to
that for its anti-particle by the charge conjugation.

In the naive quark-parton model, the original quark and antiquark hadronizes independently after they are generated, so there would be no correlation between two hadrons that belong to two different jets. In this work, we look into this matter in the quantum field setting. As a result, we find that due to the final-state interactions, there is a structure function describing the correlation between two detected particle, though they fall into separate jets. Such a fact can be easily understood within the context of QCD: Two jets originated from the quark and antiquark need to neutralize their colors in the end of hadronization, so they cannot fragments independently as assumed in the naive quark-parton model.

To be specific, we consider

\[ e^-(k_1) + e^+(k_2) \rightarrow h(p_1) + \bar{h}(p_2) + X, \]

where two inclusively detected hadrons, \( h \) and \( \bar{h} \), are in two opposite jets. We work in the c.m. frame, with \( \hat{z} \)-axis in the traveling direction of \( h \) and \( \hat{y} \)-axis along the normal of its production plane determined by \( k_1 \times p_1 \). We work with the one-photon annihilation approximation, which is applicable up to the intermediate energies. As usual, we define

\[ Q^2 \equiv -q^2, \quad z_1 \equiv \frac{2p_1 \cdot q}{q^2}, \quad z_2 \equiv \frac{2p_2 \cdot q}{q^2}, \]

where \( q \) is the momentum of the virtual time-like photon. It is straightforward to show that the cross section can be expressed as the following differential form

\[
\frac{d\sigma}{dz_1 d\cos \theta_1 d\omega d|p_{2\perp}|^2} = \frac{\alpha^2}{4\pi^2 Q^3} \frac{|p_1|}{|p_2||} L_{\mu\nu}(k_1, k_2) W^{\mu\nu}(q, p_1, p_2),
\]

where \( \theta_1 \) is the outgoing angle of \( h \) with respect to the electron beam, \( p_{2\text{parallel}} \) and \( p_{2\perp} \) are respectively the parallel and perpendicular components of \( p_2 \) with respect to \( p_1 \); \( \omega \) is the angle spanned by the \( k_1 \times p_1 \) and \( p_1 \times p_2 \) planes. \( |p_{2\parallel}| \) is related to \( |p_{2\perp}| \) by

\[ |p_{2\parallel}| = \frac{z_2 Q}{2} \left[ 1 - \left( \frac{2}{z_2 Q} \right)^2 \left( |p_{2\perp}|^2 + M^2 \right) \right]^{1/2}, \]

where \( M \) is the mass of the hadron. \( L_{\mu\nu}(k_1, k_2) \) is the well known leptonic tensor. And all the information on strong interaction is incorporated by the hadronic tensor.
\[ W^{\mu\nu}(q, p_1, p_2) = \frac{1}{4\pi} \sum_X \int d^4\xi \exp(iq \cdot \xi) \langle 0 | J^\mu(0) | h(p_1) h(p_2) X \rangle \langle h(p_1) h(p_2) X | J^\nu(\xi) | 0 \rangle. \]  

(4)

It is a common practice to parameterize various hadronic tensors in terms of Lorentz invariant structure functions. To our best knowledge, the general Lorentz decomposition of \( W^{\mu\nu}(q, p_1, p_2) \) has not yet been reported in the literature. Taking into the constraints of gauge invariance, Hermiticity, parity conservation, we can decompose our hadronic tensor as follows:

\[ W^{\mu\nu}(q, p_1, p_2) = \frac{1}{p_1 \cdot q} (-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}) W_1 + \frac{1}{q^2(p_1 \cdot q)} (p_1^\mu - \frac{p_1 \cdot q}{q^2} q^\mu) (p_1^\nu - \frac{p_1 \cdot q}{q^2} q^\nu) W_2 \\
+ \frac{1}{q^2(p_1 \cdot q)} \left[ (p_1^\mu - \frac{p_1 \cdot q}{q^2} q^\mu) p_{2\perp}^\nu + p_{2\perp}^\mu (p_1^\nu - \frac{p_1 \cdot q}{q^2} q^\nu) \right] W_3 + \frac{p_{2\perp}^\mu p_{2\perp}^\nu}{q^2(p_1 \cdot q)} W_4 \\
+ \frac{i}{q^2(p_1 \cdot q)} \left[ (p_1^\mu - \frac{p_1 \cdot q}{q^2} q^\mu) p_{2\perp}^\nu - p_{2\perp}^\mu (p_1^\nu - \frac{p_1 \cdot q}{q^2} q^\nu) \right] \hat{W}, \]  

(5)

where we introduced an auxiliary four-momentum \( p_{2\perp}^\mu = (0, p_{2\perp}, 0) \). Obviously, \( p_{2\perp}^\mu \) satisfies \( p_{2\perp}^\mu \cdot q = 0 \). \( W_1, W_2, W_3, W_4 \) and \( \hat{W} \) are five independent structure functions. The Lorentz tensor associated with \( \hat{W} \) is odd under the naive time-reversal transformation. It is this term that incorporates the final-state interaction effects on the inclusive two-particle production in electron-positron annihilation.

Now we set about deriving some constraints among our structure functions on the basis of the Hermiticity of electro-magnetic currents. The starting point is that, for an arbitrary Lorentz vector \( a^\mu \), there is always

\[ W_{\mu\nu}(q, p_1, p_2) a^\nu a^\nu \geq 0. \]  

(6)

Such an inequality can be termed the positivity of the hadronic tensor.

For a matrix to be semi-definite positive, it should satisfy the following sufficient and necessary conditions: All of its sub-matrices have semi-positively finite determinants. To investigate the positivity restrictions on the structure functions in our hadronic tensor, we write out explicitly the elements of \( W^{\mu\nu}(q, p_1, p_2) \):

\[ W^{00} = W^{01} = W^{02} = W^{03} = W^{10} = W^{20} = W^{30} = 0, \]  

(7)
The fact that $W^\mu\nu(q, p_1, p_2)$ is at rank three is simply due to the restriction of the current conservation condition. Therefore, we have the following nontrivial positivity inequalities:

\[ W^{11} = \frac{2}{Q^4 z_1} (Q^2 W_1 + |p_{2\perp}|^2 \cos^2 \omega W_4) \]
\[ W^{22} = \frac{2}{Q^4 z_1} (Q^2 W_1 + |p_{2\perp}|^2 \sin^2 \omega W_4) \]
\[ W^{33} = \frac{2}{Q^4 z_1} (Q^2 W_1 + |p_1|^2 W_2) \]
\[ W^{12} = W^*_{21} = \frac{2}{Q^4 z_1} |p_{2\perp}|^2 \sin \omega \cos \omega W_4 \]
\[ W^{13} = W^{31*} = \frac{2}{Q^4 z_1} |p_{2\perp}| |p_1| \cos \omega (W_3 - i\tilde{W}) \]
\[ W^{23} = W^{32*} = \frac{2}{Q^4 z_1} |p_{2\perp}| |p_1| \sin \omega (W_3 - i\tilde{W}). \]

The inequalities among the structure functions:

\[ \begin{vmatrix} W^{11} & W^{12} & W^{13} \\ W^{21} & W^{22} & W^{23} \\ W^{31} & W^{32} & W^{33} \end{vmatrix} \geq 0, \quad (14) \]

\[ \begin{vmatrix} W^{11} & W^{12} \\ W^{21} & W^{22} \end{vmatrix} \geq 0, \quad \begin{vmatrix} W^{22} & W^{23} \\ W^{32} & W^{33} \end{vmatrix} \geq 0, \quad (15) \]

\[ W^{11} \geq 0, \quad W^{22} \geq 0, \quad W^{33} \geq 0. \quad (16) \]

In our parameterization, the above six inequalities yield the following inequalities among the structure functions:

\[ W_1 \left[ (Q^2 W_1 + |p_{2\perp}|^2 W_4)(Q^2 W_1 + |p_1|^2 W_2) - |p_{2\perp}|^2 |p_1|^2 (W_3^2 + \tilde{W}^2) \right] \geq 0, \quad (17) \]

\[ W_1 (Q^2 W_1 + |p_{2\perp}|^2 W_4) \geq 0, \quad (18) \]

\[ (Q^2 W_1 + |p_{2\perp}|^2 \sin^2 \omega W_4)(Q^2 W_1 + |p_1|^2 W_2) - |p_{2\perp}|^2 |p_1|^2 \sin^2 \omega (W_3^2 + \tilde{W}^2) \geq 0, \quad (19) \]

\[ Q^2 W_1 + |p_{2\perp}|^2 \cos^2 \omega W_4 \geq 0, \quad (20) \]

\[ Q^2 W_1 + |p_{2\perp}|^2 \sin^2 \omega W_4 \geq 0, \quad (21) \]

\[ Q^2 W_1 + |p_1|^2 W_2 \geq 0, \quad (22) \]
Here some notes are in order.

1) By letting \( \sin \omega = 0 \) in (20) or \( \cos \omega = 0 \) in (21), one can fix the sign of \( W_1 \):

\[
W_1 \geq 0.
\] (23)

Therefore, (17) and (18) can be safely divided by \( W_1 \) without changing the direction of the inequality signs. On the other hand, if one assumes \( \sin \omega = 1 \) in (20) or \( \cos \omega = 1 \) in (21), these two inequalities reduce to (18).

2) Adding (20) to (21), one will gain a new inequality:

\[
Q^2 W_1 + \frac{1}{2} |p_{2\perp}|^2 W_4 \geq 0.
\] (24)

3) By setting \( \sin \omega \) to be 1 or 0 in (19), one will reproduce (17) and (18), respectively.

Some of the above inequalities have simple physical meanings. To see this, let us recall that a generic Lorentz vector can be expanded using a complete set of four independent vectors. Considering the current conservation condition, we choose one of them to be proportional to \( q \), and the rest three to be the three polarization vectors of the virtual photon:

\[
e_1^\mu = -\frac{1}{\sqrt{2}}(0, 1, +i, 0),
\] (25)

\[
e_2^\mu = +\frac{1}{\sqrt{2}}(0, 1, -i, 0),
\] (26)

\[
e_3^\mu = (0, 0, 0, 1).
\] (27)

Notice that these three polarization vectors are orthonormal, namely,

\[
e_i^* \cdot e_j = -\delta_{ij}, \text{ with } i, j = 1, 2, 3
\] (28)

In addition, they satisfy the Lorentz condition

\[
e_i \cdot q = 0, \text{ } i = 1, 2, 3.
\] (29)

Obviously, by letting \( a = q \) in (4) one has an identity \( 0 \equiv 0 \) only, which reflects the current conservation of the electro-magnetic interaction.

If one takes \( a \) to be the transverse photon polarization vector, there will be
\[ e_1^{*\mu}e_{1\nu}W_{\mu\nu} = e_2^{*\mu}e_{2\nu}W_{\mu\nu} \propto Q^2W_1 + \frac{1}{2} |p_{2\perp}|^2W_4. \]  

(30)

On the other hand, \( e_1^{*\mu}e_{1\nu}W_{\mu\nu} \) and \( e_2^{*\mu}e_{2\nu}W_{\mu\nu} \) are proportional to the cross sections for the virtual photon with a transverse polarization, so (24) simply indicates that these cross sections are semi-positive.

If taking \( a \) to be the longitudinal photon polarization vector, one will have

\[ e_3^{*\mu}e_{3\nu}W_{\mu\nu} \propto Q^2W_1 + |p_1|^2W_2, \]  

(31)

so (22) implies that the cross section is semi-positive for the virtual photon with a longitudinal polarization.

Similarly, if setting the polarization of the virtual photon to be \( (e_1 + e_2)/\sqrt{2} \) or \( i(e_1 - e_2)/\sqrt{2} \), one will attain (23) and (24), which means that the cross sections for the fragmentation of circularly polarized photon are also semi-positive.

However, inequality (19), which is of most relevance to the study of the final-state interactions, has no simple interpretations. Physically, it reflects the interference effects among the fragmentation processes of the time-like photons with different polarization states.

As can be seen from Eq. (3), the contributions of the final-state interactions to the hadronic tensor are imaginary and antisymmetric. To access \( \hat{W} \), one needs to polarize one of the initial beams. We consider the case in which the electron beam is longitudinally polarized. [The case of the transversely polarized electron beam is of no practical meaning because the corresponding contribution of \( \hat{W} \) to the cross section will be proportional to \( m_e/Q \).] Correspondingly, the spin state of the incident electron can be characterized by its helicity \( \lambda \). As a result, one can deduce the following cross section formula

\[
\frac{d\sigma(\lambda)}{dz_1d\cos\theta_1dz_2d|p_{2\perp}|^2d\omega} = \frac{\alpha^2}{4\pi^2z_1Q^2} \frac{|p_1|}{|p_2|} \left[ 2Q^2W_1 + |p_1|^2\sin^2\theta_1W_2 + |p_{2\perp}|^2|p_1|\sin 2\theta_1 \cos \omega W_3 + |p_{2\perp}|^2(1 - \sin^2\theta_1 \cos^2\omega)W_4 + 4\lambda|p_{2\perp}| |p_1| \sin \theta_1 \sin \omega \hat{W} \right].
\]

(32)

In principle, one can access \( \hat{W} \) by measuring the single longitudinal spin asymmetry defined as
\[ A_L \equiv \frac{d\sigma(\lambda = +\frac{1}{2})}{dz_1 d\cos\theta_1 dz_2 d|\mathbf{p}_{2\perp}|^2 d\omega} - \frac{d\sigma(\lambda = -\frac{1}{2})}{dz_1 d\cos\theta_1 dz_2 d|\mathbf{p}_{2\perp}|^2 d\omega} + \frac{d\sigma(\lambda = +\frac{1}{2})}{dz_1 d\cos\theta_1 dz_2 d|\mathbf{p}_{2\perp}|^2 d\omega} - \frac{d\sigma(\lambda = -\frac{1}{2})}{dz_1 d\cos\theta_1 dz_2 d|\mathbf{p}_{2\perp}|^2 d\omega}. \]  

Substituting Eq. (32) into (33), one has

\[ A_L = \frac{2|\mathbf{p}_{2\perp}| |\mathbf{p}_1| \sin\theta_1 \sin\omega \hat{W}}{2Q^2W_1 + |\mathbf{p}_1|^2 \sin^2\theta_1 W_2 + |\mathbf{p}_{2\perp}| |\mathbf{p}_1| \sin 2\theta_1 \cos\omega W_3 + |\mathbf{p}_{2\perp}|^2 (1 - \sin^2\theta_1 \cos^2\omega) W_4}. \]  

Our positivity analysis has useful phenomenological implications to \( A_L \). Let us assume \( W_3 = 0 \) in (37). Then, we have

\[ |\mathbf{p}_{2\perp}| |\mathbf{p}_1| \hat{W} \leq \sqrt{(Q^2W_1 + |\mathbf{p}_{2\perp}|^2 W_4)(Q^2W_1 + |\mathbf{p}_1|^2 W_2)}. \]  

Accordingly, there is

\[ |A_L| \leq \frac{2\sqrt{(Q^2W_1 + |\mathbf{p}_{2\perp}|^2 W_4)(Q^2W_1 + |\mathbf{p}_1|^2 W_2) \sin\theta_1 \sin\omega}}{2Q^2W_1 + |\mathbf{p}_1|^2 \sin^2\theta_1 W_2 + |\mathbf{p}_{2\perp}| |\mathbf{p}_1| \sin 2\theta_1 \cos\omega W_3 + |\mathbf{p}_{2\perp}|^2 (\sin^2\theta_1 \cos^2\omega - 1) W_4}. \]  

To further bound the discovery opportunity in experimental measurement, we now input into (36) the Callan-Gross relation

\[ W_1 + \frac{z_1^2}{4} W_2 = 0 \]  

for the process considered. [Making use of the quark-parton model, one can verify Eq. (37) straightforwardly.] As a reasonable approximation, one can neglect all the \( 1/Q \)-power suppressed effects, i.e., \( W_3 \) and \( W_4 \)-terms in the denominator of the right-hand side of (36). As a result, we obtain the following upper bound to our single spin asymmetry:

\[ |A_L| \leq \frac{4M \sin\theta_1 \sin\omega}{z_1 Q(1 + \cos^2\theta_1)}. \]  

It should be stressed that (38) is an amplified bound for \( A_L \). The reason is that in deriving (35) from (17) and (23), we have assumed \( W_3 = 0 \). Since both \( W_3 \) and \( \hat{W} \) contribute at one-power suppressed level, the upper bound we derived is quite safe. Experimentally, the data
on spin asymmetries are usually subject to large statistical errors, so the above positivity constraint can be taken as a useful guide to judge the reliability of data. On the other hand, (38) also informs us that there is no large spin asymmetries in some kinematic Al regions. In addition, our positivity constraints can serve as a consistency check for the future model calculations, if any.

To be illustrative, we consider the two-pion production. We plot in Fig. 1 the upper bounds of $|A_L|$ versus the beam-referenced production angle $\theta_1$ of the first pion. In doing so, we take $Q = 4.25$ GeV at which a longitudinally polarized electron-positron collider \cite{3} is expected. Furthermore, we assume the $\mathbf{p}_1 \times \mathbf{p}_2$ plane is perpendicular to the production plane of the first pion determined by $\mathbf{k}_1 \times \mathbf{p}_1$ so as to obtain the most optimistic upper bound. To guarantee the Callan-Gross relation is reliable, we take the energy of the first pion to be ten times its mass, which corresponds to $z_1 \approx 0.66$.

In summary, the final-state interaction effects on the inclusive two-particle production is investigated in electron-positron annihilation within the context of the one-photon annihilation approximation. Such effects are characterized by one structure function in the Lorentz decomposition of the hadronic tensor. Making use of the positivity of the hadronic tensor, we derived an upper bound for this structure function. To access it experimentally, one needs to polarize the beam electron longitudinally and measure the corresponding single spin asymmetry. Further, we presented an upper bound for the considered single spin asymmetry by combining the Callan-Gross relation with the positivity constraint.

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Figure Caption

Figure 1. Plot of the upper bound of the spin asymmetry ($A_L$) for $e^-(k_1, s_1) + e^+(k_2) \rightarrow \gamma^* \rightarrow \pi^+(p_1) + \pi^-(p_2) + X$ versus the beam-referenced outgoing angle ($\theta_1$) of the first pion. To obtain the most optimistic upper bound, we assume the $p_1 \times p_2$ plane is perpendicular to the production plane of the first pion determined by $k_1 \times p_1$. The c.m. energy is set to be $Q = 4.25$ GeV, at which a longitudinally polarized electron-positron collider is expected for the future Tau-Charm Factory [6]. To guarantee the Callan-Gross relation is reliable, we take the energy of the first pion to be ten times its mass, which corresponds to $z_1 \approx 0.66$. 
Beam-Referenced Production Angle $\Theta_1$ of the First Pion

$z_1 = 0.66$
$\sin \omega = 1.$
$Q = 4.25 \text{ GeV}$