Article

Spectra of Self-Similar Measures

Yong-Shen Cao \(^1,2\), Qi-Rong Deng \(^1\) and Ming-Tian Li \(^1\)*

\(^1\) Center for Applied Mathematics of Fujian Province, Fujian Key Laboratory of Mathematical Analysis and Applications (FJKLMAA), School of Mathematics and Statistics, Fujian Normal University, Fuzhou 350117, China
\(^2\) School of Computing and Information Science, Fuzhou Institute of Technology, Fuzhou 350506, China

* Correspondence: limtwd@fjnu.edu.cn

Abstract: This paper is devoted to the characterization of spectrum candidates with a new tree structure to be the spectra of a spectral self-similar measure \(\mu_{N,D}\) generated by the finite integer digit set \(D\) and the compression ratio \(N^{-1}\). The tree structure is introduced with the language of symbolic space and widens the field of spectrum candidates. The spectrum candidate considered by Łaba and Wang is a set with a special tree structure. After showing a new criterion for the spectrum candidate with a tree structure to be a spectrum of \(\mu_{N,D}\), three sufficient and necessary conditions for the spectrum candidate with a tree structure to be a spectrum of \(\mu_{N,D}\) were obtained. This result extends the conclusion of Łaba and Wang. As an application, an example of spectrum candidate \(\Lambda(N,B)\) with the tree structure associated with a self-similar measure is given. By our results, we obtain that \(\Lambda(N,B)\) is a spectrum of the self-similar measure. However, neither the method of Łaba and Wang nor that of Strichartz is applicable to the set \(\Lambda(N,B)\).

Keywords: spectrality; tree structure; self-similar measure; orthogonal basis

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1. Introduction

Let \(\mu\) be a probability measure on \(\mathbb{R}^d\) with compact support \(K\). We say that \(\mu\) is a spectral measure if there exists a countable set \(\Lambda \subset \mathbb{R}^d\) such that the set of exponential functions \(E_\Lambda := \{\exp 2\pi i \langle \lambda, x \rangle : \lambda \in \Lambda\}\) is an orthogonal basis of \(L^2(\mu)\). In this case, \(\Lambda\) is called a spectrum of \(\mu\) and \((\mu, \Lambda)\) is called a spectral pair. In particular, if \(\mu\) is the normalized Lebesgue measure restricted on \(K\), we say \(K\) is a spectral set.

In [1], Fuglede introduced the notion of a spectral set in the study of the extendability of the commuting partial differential operators and raised the famous conjecture: \(K\) is a spectral set if and only if \(K\) is a translational tile. Although the conjecture was finally disproven for the case that \(K \subset \mathbb{R}^d\) with \(d \geq 3\) and is still open for \(\mathbb{R}^d\) with \(d \leq 2\), it has led to the development of harmonic analysis, operator theory, tiling theory, convex geometry, etc.

In 1998, Jorgensen and Pedersen [2] discovered the first singular, non-atomic spectral measure—the middle-forth Cantor measure—and proved the middle-third Cantor measure is not a spectral measure. Following this discovery, there has been much research on the spectrality of self-similar (or self-affine) measures and Moran-type self-similar (or self-affine) measures (see for example [3–21] and the references therein).

Consider the iterated function system (IFS) \(\{\phi_j\}_{j=1}^q\) given by

\[\phi_j(x) = \frac{1}{N}(x + d_j),\]
where $N$ is an integer with $|N| > 1$ and $D = \{d_j\}_{j=1}^q$ is a finite subset of $\mathbb{R}$. It is well known (see [22] or [23]) that there exists a unique probability measure $\mu_{N,D}$ satisfying

$$\mu_{N,D}(E) = \frac{1}{q} \sum_{j=1}^{q} \mu_{N,D}(\phi_j^{-1}(E)),$$

for Borel set $E$ of $\mathbb{R}$.

The measure $\mu_{N,D}$ is called the self-similar measure of the IFS $\{\phi_j\}_{j=1}^q$ and is supported on the set

$$T(N,D) = \left\{ \sum_{k=1}^{\infty} d_k N^{-k} : d_k \in D, k \geq 1 \right\},$$

which is the attractor of $\{\phi_j\}_{j=1}^q$. Given a finite set $S \subset \mathbb{Z}$ with $\not\exists S \in \not\exists D$, we say $(\frac{1}{N}D, S)$ is a compatible pair if the matrix $[\frac{1}{N} \exp(2\pi i \frac{k}{N} s)]_{d \in D, s \in S}$ is a unitary matrix. In other words, $(\frac{1}{N}D, S)$ is a spectral pair. For a finite set $A$ in $\mathbb{R}$,

$$\delta_A := \frac{1}{|A|} \sum_{a \in A} \delta_a,$$

where $\delta_a$ is the Dirac measure at $a$. Write

$$\Lambda(N,S) = \left\{ \sum_{j=0}^{k} s_j N^j : k \geq 0, s_j \in S \right\}.$$

Using the dominated convergence theorem, Strichartz [24] proved that $\mu_{N,D}$ is a spectral measure with a spectrum $\Lambda(N,S)$ under the conditions that $(\frac{1}{N}D, S)$ is a spectral pair with $0 \in S$ and the Fourier transform of $\delta_{\frac{1}{N}D}$ does not vanish on $T(N,S)$. By using the Ruelle transfer operator, Laba and Wang in [3] removed the condition that the Fourier transform of $\delta_{\frac{1}{N}D}$ does not vanish on $T(N,S)$. Furthermore, they obtained the following conclusion:

**Theorem 1.** (Laba and Wang). Let $N \in \mathbb{N}$ with $|N| > 1$, $D \subset \mathbb{Z}$ with $0 \in D$, and $\gcd(D) = 1, 0 \in S \subset \mathbb{Z}$. If $(\frac{1}{N}D, S)$ is a compatible pair, then $(\mu_{N,D}, \Lambda(N,S))$ is not a spectral pair if and only if there exist integers $m \geq 1$, $\{s_j\}_{j=0}^{m-1} \subset S$ and $\{\eta_j\}_{j=0}^{m-1} \subset \mathbb{Z}\setminus\{0\}$ such that $\eta_{j+1} = N^{-1}(\eta_j + s_j)$ for $0 \leq j \leq m - 1$, where $\eta_m := \eta_0, \eta_m := \eta_0$.

It is well known that to prove the spectrality of the invariant measure $\mu_{N,D}$, the first key step is to construct a suitable spectrum candidate. In this process, the set $\Lambda(N,S) = S + NS + N^2S + \cdots$ (finite sum) is the natural spectrum candidate to be considered. Form Theorem 1, we conclude that $\Lambda(N,S)$ is not a spectrum of $\mu_{N,D}$ if and only if there is a periodic orbit $\{\eta_j\}_{j=0}^{m-1} \subset \mathbb{Z}\setminus\{0\}$ under the dual IFS $\{\psi_i(x) = \frac{1}{N}(x + s_i) : s_i \in S\}$. The following example implies that the natural spectrum candidate has a weak point. When $D = \{0, 1\}$, the invariant measure $\mu_{2,D}$ is just the Lebesgue measure on the unit interval with the unique spectrum $Z$. However, $\Lambda(2, \{0, 1\}) = \mathbb{N} \neq Z$ in this case. In other words, the natural candidate $\Lambda(2, \{0, 1\})$ is not a spectrum of $\mu_{2,D}$. Actually, any set with form $S + 2S + 2^2S + \cdots$ (finite sum) is not a spectrum of $\mu_{2,D}$. In this case, one needs to consider the spectrum candidate with a more general form $S_1 + NS_2 + N^2S_3 + \cdots$ (finite sum), where $(\frac{1}{N}D, S_i)$ are compatible pairs. Moreover, it is well known that a spectral self-similar (or self-affine) measure has more than one spectrum in general. The results in [7,9–11] show that one may consider spectrum candidates with a tree structure. It is worth mentioning that Li [16] obtained a simplified form of Theorem 1. It is the best of our understanding, partial results have been obtained in the case of a higher-dimensional space. Developing the method in [3], Dutkay and Jorgensen [14] obtained a sufficient condition for the spectral pair of self-affine measures, and Li [19] obtained a necessary condition for the natural spectrum candidate to be a spectrum of a self-affine measure.
Motivated by the above results, we considered a class of spectrum candidates with a tree structure (defined in Section 2) and obtained three necessary and sufficient conditions for such spectrum candidates not to be the spectra of \( \mu_{N,D} \) (Theorem 2), which generalizes Łaba and Wang’s result.

The most difficult part of the proof of Theorem 2 is that the first statement implies the second. For this purpose, we show a new criterion for \( \Lambda \) to be a spectrum of \( \mu_{N,D} \). As an application, we give an example involving a self-similar measure \( \mu \) and a spectrum candidate \( \Lambda(N,B) \) with a tree structure in Section 4. By Theorem 2, we obtain \((\mu, \Lambda(N,B))\) is a spectral pair. However, neither the criterion of Łaba and Wang (Theorem 1) nor that of Strichartz [24] is applicable to this set \( \Lambda(N,B) \).

2. Preliminaries

In this section, we shall recall some basic properties of spectral measures and introduce the tree structure using symbolic space.

Let \( \mu \) be a probability measure on \( \mathbb{R} \). The Fourier transform of \( \mu \) is defined by

\[
\hat{\mu}(\xi) = \int e^{-2\pi i \xi x} \, d\mu(x), \quad x \in \mathbb{R}.
\]

We write \( Z(\hat{\mu}) = \left\{ \xi : \hat{\mu}(\xi) = 0 \right\} \). For a discrete set \( \Lambda \subset \mathbb{R} \), write \( E_\Lambda = \left\{ \exp(2\pi i x \lambda) : \lambda \in \Lambda \right\} \) for a family of exponential functions in \( L^2(\mu) \). Then, \( E_\Lambda \) is an orthogonal family of \( L^2(\mu) \) if and only if

\[
\Lambda - \Lambda \subset Z(\hat{\mu}) \cup \{0\}.
\]

Define

\[
Q_\Lambda(\xi) = \sum_{\lambda \in \Lambda} |\hat{\mu}(\lambda + \xi)|^2, \quad x \in \mathbb{R}.
\]

By using the Parseval identity, Jorgenson and Pederson ([2]) obtained the following basic criterion for the orthogonality of \( E_\Lambda \) in \( L^2(\mu) \).

**Proposition 1.** The exponential function set \( E_\Lambda \) is an orthogonal set of \( L^2(\mu) \) if and only if \( Q_\Lambda(\xi) \leq 1 \) for all \( \xi \in \mathbb{R} \), and \( E_\Lambda \) is an orthogonal basis of \( L^2(\mu) \) if and only if \( Q_\Lambda(\xi) = 1 \) for all \( \xi \in \mathbb{R} \).

Given a finite set \( D \subset \mathbb{R} \), we call

\[
m_D(\xi) = \frac{1}{\sharp D} \sum_{d \in D} \exp(2\pi i \xi d), \quad \xi \in \mathbb{R}
\]

the mask of \( D \). It is clear that it is just the Fourier transformation of the uniform probability measure on \( D \).

**Definition 1.** For two finite subsets \( D \) and \( S \) of \( \mathbb{R} \) with the same cardinality \( m \), we say \((D, S)\) is a compatible pair if

\[
\left[ \frac{1}{\sqrt{m}} \exp(2\pi i ds) \right]_{d \in D, s \in S}
\]

is a unitary matrix.

The following conclusion is well known.

**Lemma 1.** For two finite subsets \( D \) and \( S \) of \( \mathbb{R} \) with the same cardinality \( m \), the following statements are equivalent:

(i). \((D, S)\) is a compatible pair;

(ii). \( m_D(s_1 - s_2) = 0 \) for any \( s_1 \neq s_2 \in S \);

(iii). \( \sum_{s \in S} m_D(\xi + s)^2 = 1 \) for any \( \xi \in \mathbb{R} \).
In other words, \((D, S)\) is a compatible pair if and only if \(S\) is a spectrum of the uniform probability measure on \(D\).

Let \(N\) be an integer with \(|N| > 1\) and \(D = \{d_j\}_{j=1}^q\) a finite subset of \(\mathbb{Z}\) with \(0 \in D\). We denote by \(\mu_{N, D}\) the unique invariant measure with respect to the IFS \(\{\phi_j(x) = \frac{1}{q}(x + d_j) : 1 \leq j \leq q\}\) with equal probability weights, i.e.,

\[
\mu_{N, D} = \frac{1}{q} \sum_{j=1}^q \mu_{N, D} \circ \phi_j^{-1}.
\]

In the sequel, we write \(\mu = \mu_{N, D}\) for simplicity. Thus, we have

\[
\hat{\mu}(\xi) = \prod_{j=1}^\infty m_D(N^{-j}\xi), \quad \xi \in \mathbb{R}.
\]

For \(k \geq 1\), we write

\[
\hat{\mu}_k(\xi) = \prod_{j=1}^k m_D(N^{-j}\xi), \quad \xi \in \mathbb{R}.
\]

Write \(Y(m_D) = \{\xi \in \mathbb{R} : m_D(\xi) = 1\}\). When \(\gcd(D) = 1\), we have

\[
Y(m_D) = \{\xi \in \mathbb{R} : |m_D(\xi)| = 1\} = \mathbb{Z}.
\]

Now, we introduce the tree structure. First, we recall some basic notation of symbolic space. Given a positive integer \(q > 1\), write \(\Sigma_q = \{0, 1, \ldots, q-1\}\). Let \(\Sigma^* = \bigcup_{n=0}^\infty \Sigma^n_q\) stand for the set of all finite words, where \(\Sigma^0_q = \{\theta\}\) denotes the set of empty words. The length of a finite word \(\sigma\) is the number of symbols it contains and is denoted by \(|\sigma|\). The concatenation of two finite words \(\sigma\) and \(\sigma'\) is written as \(\sigma\sigma'\). We say \(\sigma\) is a prefix of \(\sigma\sigma'\). Given \(\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in \Sigma^*\) and \(1 \leq k \leq n\), write \(\sigma_k = \sigma_1 \cdots \sigma_k\). The following definition will bring convenience to us.

**Definition 2.** A sequence of finite words \(\{I_n\}_{n \geq 1} \subset \Sigma^*\) is called increasing if for any \(n \geq 1\), \(I_n\) is a prefix of \(I_{n+1}\) and \(|I_{n+1}| = |I_n| + 1\).

Let \(\mathcal{C}\) be a mapping from \(\Sigma^*\) to \(\mathbb{Z}\) satisfying \(\mathcal{C}(\emptyset) = 0\) and \(\mathcal{C}(I) = 0\) if \(I\) ends with the symbol 0. It induces a family of mapping \(\mathcal{F} = \{F_I\}_{I \in \Sigma^*}\) defined by

\[
F_I : \Sigma^* \longrightarrow \mathbb{Z},
I \longmapsto \mathcal{C}(IJ_1) + NC(IJ_2) + \cdots + N^{|I|-1}\mathcal{C}(IJ),
\]

where \(IJ_i\) is the concatenation of \(I\) and \(J_i\) for \(1 \leq i \leq |J|\). We write \(F(I) = F_\emptyset(I)\) for convenience. By a simple deduction, we have the following consistency: for any \(I, J, K \in \Sigma^*\),

\[
F_I(J) + N^{|I|}F_K = \mathcal{C}(IJ_1) + \cdots + N^{|I|-1}\mathcal{C}(IJ) + N^{|I|}\mathcal{C}(IK_1) + \cdots + N^{|K|-1}\mathcal{C}(IK) = F_K(J).
\]

**Definition 3.** We say a countable set \(\Lambda \subset \mathbb{R}\) has a \((\mathcal{C}, \mathcal{F})\) tree structure if there exists a mapping \(\mathcal{C}\) and an associated family of mappings \(\mathcal{F}\) defined in the above paragraph such that

\[
\Lambda = \bigcup_{I \in \Sigma^*} \{F(I)\}.
\]
For $I \in \Sigma^*$, let $S_I = \{ C(i) : i \in \Sigma_q \}$. According to the definition of the mapping $C$, we have $C(0) = 0 \in S_I$.

**Remark 1.** Given a sequence of finite sets $S = \{ S_n \}_{n \geq 1}$, if $S_I = S_{I+1}$ for any $I \in \Sigma^*(n \geq 0)$, we obtain

$$\Lambda = S_1 + NS_2 + N^2S_3 \cdots .$$

In particular, if $S_n = S$ for $n \geq 1$, we obtain

$$\Lambda = S + NS + N^2S \cdots ,$$

which is just the case considered by Laba and Wang in [3].

In this paper, we consider a countable set $\Lambda$ as a spectrum candidate satisfying the following three conditions:

(C1). $\Lambda$ has a $(C, F)$ tree structure.

(C2). For any $I \in \Sigma^*$, $(\frac{1}{N} D, S_I)$ is a compatible pair.

(C3). The set $S = \bigcup_{I \in \Sigma^*} S_I$ is bounded.

**Remark 2.** Since we only assume that $(\frac{1}{N} D, S_I)$ is a compatible pair with $S_I = \{ C(i) : i \in \Sigma_q \}$, the map $C$ may not be a maximal mapping defined in [8] (Definition 2.5) even if $D = \{0, 1, \cdots , q - 1\}$.

Now, we exploit some basic properties of $\Lambda$ satisfying the conditions (C1), (C2), and (C3). The first one is the uniqueness of the tree representation.

**Proposition 2.** Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $\gcd(D) = 1$. Assume that a countable set $\Lambda$ satisfies the conditions (C1), (C2), and (C3). Then, for any $I \in \Sigma^q$ and $J, K \in \Sigma^q$ with $n \geq 0$, we have $F_I(J) = F_I(K)$ if and only if $J = K$.

**Proof.** We just prove the necessity. Suppose there exist $I \in \Sigma^q$ and $J \neq K \in \Sigma^q$ with $n \geq 0$ such that $F_I(J) = F_I(K)$. Let $l$ be the smallest integer with $|J| \neq |K|$. From $F_I(J) = F_I(K)$, it follows that

$$N^{l-1}C(I|J|) + \cdots + N^nC(I|J|) = N^{l-1}C(I|K|) + \cdots + N^nC(I|K|),$$

which implies $C(I|J|) \equiv C(I|K|) \pmod{N}$. Noting $C(I|J|), C(I|K|) \in S_{|I|\|J|}$, we obtain $(\frac{1}{N} D, S_{|I|\|J|}$ is not a compatible pair, which is a contradiction to the condition (C2).

**Proposition 3.** Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $\gcd(D) = 1$. Assume that a countable set $\Lambda$ satisfies the conditions (C1), (C2), and (C3). Then, $E(\Lambda)$ is an orthogonal set of $L^2(\mu)$.

**Proof.** Given $\alpha \neq \beta \in \Lambda$, there exist two finite words $I, J \in \Sigma^*$ such that

$$\alpha = F(I), \quad \beta = F(J).$$

If $|I| = 1$, we add symbol 0 in the end of $I$ or $J$ to obtain $|I| = |J|$. Without loss of generality, we assume that $I, J \in \Sigma_q^m$ for some integer $n$. Let $l$ be the smallest positive integer satisfying $|I| \neq |J|$. Recall that $F(I|J|) = C(I|I|) + NC(I|2|) \cdots + N^{l-1}C(I|l|)$. Then, there exists an integer $z_0$ such that

$$N^{-1}(F(I|J|) - F(I|J|)) = \frac{1}{N}(C(I|I|) - C(I|I|)) + z_0.$$
By virtue of the condition (C2), we know that \((\frac{1}{N}D, S_{|I| - 1})\) is a compatible pair. Noting that both \(\mathcal{C}(|I|)\) and \(\mathcal{C}(|J|)\) belong to \(S_{|I| - 1}\), we obtain

\[
m_D(N^{-1}(F(I) - F(J))) = m_D(\frac{1}{N}(\mathcal{C}(I) - \mathcal{C}(J) + z_0)) = m_D(\frac{1}{N}(\mathcal{C}(I) - \mathcal{C}(J))) = 0.
\]

This leads to

\[
\hat{\mu}(\alpha - \beta) = \hat{\mu}(F(I) - F(J)) = \prod_{j=1}^{l-1} m_D(N^{-j}(F(I) - F(J))) m_D(N^{-j}(F(I) - F(J))) \prod_{j=l+1}^{\infty} (N^{-j}(F(I) - F(J))) = 0.
\]

\(\square\)

For any \(I \in \Sigma^*_N\) and \(k \geq 1\), define

\[
\Lambda_l = \{F_I(J) : J \in \Sigma^*\} \quad \text{and} \quad \Lambda^k_l : = \{F_I(J) : J \in \Sigma^k_N\}.
\]

We write \(\Lambda^k : = \Lambda^k_0\) for simplicity. It is clear that

\[\Lambda^k_l \subseteq \Lambda^{k+1}_l.\]

From the condition (C2) and Lemma 1(ii), it follows that \(E(\Lambda^k_l)\) is an orthogonal set of \(L^2(\mu_k)\). By (2), we obtain \(\#\Lambda^k_l = q^k\). Noting the fact that \(\dim(L^2(\mu_k)) = q^k\), we conclude that \(E(\Lambda^k_l)\) is an orthogonal basis of \(L^2(\mu_k)\). In other words, \(\Lambda^k_l\) is a spectrum of \(\mu_k\). By Lemma 1, we have

\[
\sum_{\lambda \in \Lambda^k_l} \prod_{j=1}^{k} m_D(N^{-j}(\xi + \lambda))^2 = \sum_{\lambda \in \Lambda^k_l} |\hat{\mu}_k(\xi + \lambda)|^2 = 1, \quad \forall \xi \in \mathbb{R}.
\]

In fact, we have the following conclusion.

**Proposition 4.** Let \(N \in \mathbb{Z}\) with \(|N| > 1\) and \(D \subseteq \mathbb{Z}\) with \(0 \in D\) and \(\gcd(D) = 1\). Assume that a countable set \(\Lambda\) satisfies the conditions (C1), (C2), and (C3). Then, \(Q_{\Lambda}(\xi) \equiv 1\) if and only if \(Q_{\Lambda}(\xi) \equiv 1\) for any \(I \in \Sigma^*,\)

**Proof.** By virtue of \(\Lambda_d = \Lambda\), the sufficiency is obvious.

Next, we prove the necessity. Given \(n \geq 1\) and \(I \in \Sigma^*_N\), write \(B_I = \{x + F(I) : x \in [0, 1]\}\) and \(B_I = \{N^{-n}(\xi + F(I)) : \xi \in [0, 1]\}\). It is easy to see that both \(B_I\) and \(\hat{B}_I\) are compact sets. Noting the fact that \(\hat{\mu}_n\) can be extended to be an entire function on the complex plane, \(\hat{\mu}_n\) has at most finitely many zero points in \(B_I\). On the other hand, recall that

\[
\Lambda = \bigcup_{I \in \Sigma^*_N} \bigcup_{J \in \Sigma^*} (F(I) + N^n F_J), \quad n \geq 1.
\]

Noting the fact that every integer is a period of \(m_D\), we have \(\hat{\mu}_n(\xi + F(I)) = \hat{\mu}_n(\xi + F(I))\) for any \(I \in \Sigma^*_N\) and \(J \in \Sigma^*_N\). Hence,
There exist a finite word $I \in \Sigma^*$.

Theorem 2. Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $gcd(D) = 1$. Assume that a countable set $\Lambda$ satisfies the conditions (C1), (C2), and (C3). Then, the following statements are equivalent:

(i). $(\lambda, \Lambda)$ is not a spectral pair.
(ii). There exists a finite word $I \in \Sigma^*$ such that $\inf_{\xi \in T} Q_{\Lambda_I}(\xi) = 0$.
(iii). There exist a finite word $J \in \Sigma^*$, a sequence of nonzero integers $\{b_i\}_{i \geq 1} \subset \mathbb{Z} \setminus \{0\}$ and a sequence of increasing finite words $\{J_{i_1} \cdots i_l\}_{l \geq 1} \subset \Sigma^*$, which has a prefix $J$ such that, for any $l \geq 1$, we have $b_{l+1} = \frac{1}{N} (b_l + \mathcal{C}(J_{i_1} \cdots i_l))$.

We shall divide the proof into three parts (iii) ⇒ (i), (i) ⇒ (ii), and (ii) ⇒ (iii).

First, we prove (iii) ⇒ (i), which plays a key role in the proof of (i) ⇒ (ii).

Proof of Theorem 2 (iii) ⇒ (i). We shall prove $Q_{\Lambda_I}(\beta_1) = 0$. Thus, from Proposition 4, the conclusion follows.

Given $\lambda \in \Lambda_I$, there exists a positive integer $m \geq 1$ and $L \in \Sigma^m$ such that

$$\lambda = F_I(L) \in \Lambda^m_I.$$  

Since the sequence $\{b_i\}_{i \geq 1}$ is nonzero, the sequence of integers $\{\mathcal{C}(J_{i_1} \cdots i_l)\}_{l \geq 1}$ has infinitely many nonzero terms. Thus, there exist infinitely many terms $I$ with $i_j \neq 0$. Take an integer $r > m$ with $i_r \neq 0$. Write $\lambda^* := F_I(K) \in \Lambda^r_I$. According to Proposition 2 and $i_r \neq 0$, we have $\lambda \neq \lambda^*$ and $\lambda \in \Lambda^m_I \subset \Lambda^r_I$. From $b_{k+1} = N^{-1}(\beta_k + \mathcal{C}(J_{i_1} \cdots i_k))(k \geq 1)$, it follows that

\[
Q_{\Lambda}(\xi) = \sum_{\lambda \in \Lambda} |b_n(\xi + \lambda)|^2 |\beta(N^{-n}(\xi + \lambda))|^2 \\
= \sum_{\lambda \in \Lambda} \sum_{J \in \Sigma^*} |b_n(\xi + F(I))|^2 |\beta(N^{-n}(\xi + F(I)) + N^n F_I(J))|^2 \\
= \sum_{\lambda \in \Lambda} |b_n(\xi + F(I))|^2 \sum_{J \in \Sigma^*} |\beta(N^{-n}(\xi + F(I)) + F_I(J))|^2 \\
= \sum_{\lambda \in \Lambda} |b_n(\xi + F(I))|^2 Q_{\Lambda_I}(N^{-n}(\xi + F(I))).
\]
\[ \beta_1 + \lambda^* = N(\beta_2 + C(J|_2) + NC(J|_3) + \cdots + N^{r-1}C(J)) \\
\quad = \cdots \\
= N^r \beta_{r+1} \in N^r \mathbb{Z}, \]
which implies \(|\hat{\mu}_r(\beta_1 + \lambda^*)|^2 = 1\). Noting (3) and \(\lambda \neq \lambda^*\), we have
\[ 1 \leq |\hat{\mu}_r(\beta_1 + \lambda_1)|^2 + |\hat{\mu}_r(\beta_1 + \lambda)|^2 \leq \sum_{\gamma \in \Lambda_t} |\hat{\mu}_r(\beta_1 + \gamma)|^2 = 1, \]
Thus, we obtain \(|\hat{\mu}_r(\beta_1 + \lambda)| = 0\). Hence,
\[ |\hat{\mu}(\beta_1 + \lambda)| = 0, \quad \forall \lambda \in \Lambda_f. \]
It follows that \(Q_{\Lambda_f}(\beta_1) = \sum_{\lambda \in \Lambda_f} |\hat{\mu}(\beta_1 + \lambda)|^2 = 0. \)

The following three lemmas play key roles in the proof of Theorem 2 (i) \(\Rightarrow\) (ii). First, we show a new criterion for \(\Lambda\) to be a spectrum of \(\mu\).

**Lemma 2.** Let \(N \in \mathbb{Z}\) with \(|N| > 1\) and \(D \subset \mathbb{Z}\) with \(0 \in D\) and \(\gcd(D) = 1\). Assume that a countable set \(\Lambda\) satisfies the conditions (C1), (C2), and (C3). If there exists a positive number \(c > 0\) such that, for any \(\xi\) and \(I \in \Sigma^\ast\), there is \(\lambda_{\xi,I} \in \Lambda_I\) satisfying
\[ |\hat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi,I})| \geq c, \]
then \((\mu, \Lambda)\) is a spectral pair.

**Proof.** Suppose \((\mu, \Lambda)\) is not a spectral pair. Then, there exists \(\xi_0 \in T\) such that \(Q_{\Lambda}(\xi_0) < 1\).

Recall that \(\Lambda^n = \{C(J|_1) + NC(J|_2) + \cdots + N^{n-1}C(J) : J \in \Sigma^n\}\) for \(n \geq 1\). We write \(Q_n(\xi_0) := \sum_{\lambda \in \Lambda^n} |\hat{\mu}(\xi_0 + \lambda)|^2\). By virtue of \(\lim_{n \to \infty} \Lambda^n = \Lambda\) and \(\Lambda_n \subset \Lambda_{n+1}\) for \(n \geq 1\), we obtain
\[ \lim_{n \to \infty} Q_n(\xi_0) = Q_\Lambda(\xi_0) \text{ and } Q_n(\xi_0) \leq Q_{n+1}(\xi_0). \]

Given a positive number \(\varepsilon\) with \(\varepsilon < \frac{1}{2}(1 - Q_\Lambda(\xi_0))\), there exists an integer \(M \geq 1\) such that
\[ Q_\Lambda(\xi_0) - \varepsilon \leq Q_M(\xi_0) \leq Q_n(\xi_0) \leq Q_\Lambda(\xi_0) < 1, \quad \forall n \geq M. \]

(5)

By (1), we have
\[ \lim_{m \to \infty} \beta_m(\xi_0 + \lambda) = \hat{\mu}(\xi_0 + \lambda), \quad \forall \lambda \in \Lambda. \]

In combination with (5), we have a positive integer \(K \geq M + 1\) such that
\[ \sum_{\lambda \in \Lambda^M} |\hat{\mu}_K(\xi_0 + \lambda)|^2 \leq \sum_{\lambda \in \Lambda^M} |\hat{\mu}(\xi_0 + \lambda)|^2 + \varepsilon \leq Q_\Lambda(\xi_0) + \varepsilon. \]

According to (3), we have \(\sum_{\lambda \in \Lambda^K} |\hat{\mu}_K(\xi_0 + \lambda)|^2 = 1\). Thus,
\[ \sum_{I \in \Sigma^K \setminus \Sigma^M} |\hat{\mu}_K(\xi_0 + F(I))|^2 = \sum_{\lambda \in \Lambda^K} |\hat{\mu}_K(\xi_0 + \lambda)|^2 - \sum_{\lambda \in \Lambda^M} |\hat{\mu}_K(\xi_0 + \lambda)|^2 \]
\[ \geq 1 - Q_\Lambda(\xi_0) - \varepsilon > 0. \]

(6)

For any \(I \in \Sigma^K \setminus \Sigma^M\), there exists \(\lambda_{\xi_0,I} \in \Lambda_I\) such that
\[ |\hat{\mu}(N^{-K}(\xi_0 + F(I)) + \lambda_{\xi_0,I})| > c. \]

(7)
Write $\tilde{\Lambda} = \{ F(I) + N^K \lambda_{\tilde{e},I} : I \in \Sigma_\tilde{e} \setminus \Sigma_p = \Lambda_1 \}$. It is clear that $\tilde{\Lambda} \subset \Lambda$. Since $(\frac{1}{N}D, S_t)$ is a compatible pair for any $I \in \Sigma^*$, $C(I) = 0$ if and only if the finite word $I$ ends with the symbol $0$. Then, we have

$$\Lambda^M \cap \tilde{\Lambda} = \varnothing.$$ 

In combination with (5)–(7), we obtain

$$Q_\Lambda(\tilde{\xi}_0) = \sum_{\lambda \in \Lambda} |\tilde{\mu}(\tilde{\xi}_0 + \lambda)|^2$$

$$\geq \sum_{\lambda \in \Lambda^M} |\tilde{\mu}(\tilde{\xi}_0 + \lambda)|^2 + \sum_{\lambda \in \Lambda} |\tilde{\mu}(\tilde{\xi}_0 + \lambda)|^2$$

$$= \sum_{\lambda \in \Lambda^M} |\tilde{\mu}(\tilde{\xi}_0 + \lambda)|^2 + \sum_{I \in \Sigma^+ \setminus \Sigma_p} |\tilde{\mu}F(I + N^K \lambda_{\tilde{e},I})|^2$$

$$= \sum_{\lambda \in \Lambda^M} |\tilde{\mu}(\tilde{\xi}_0 + \lambda)|^2 + \sum_{I \in \Sigma^+ \setminus \Sigma_p} |\tilde{\mu}_K(\tilde{\xi}_0 + F(I))|^2 |\tilde{\mu}(N^{-K}(\tilde{\xi}_0 + F(I)) + \lambda_{\tilde{e},I})|^2$$

$$\geq Q_\Lambda(\tilde{\xi}_0) - \varepsilon + c^2 \sum_{I \in \Sigma^+ \setminus \Sigma_p} |\tilde{\mu}_K(\tilde{\xi}_0 + F(I))|^2$$

$$\geq Q_\Lambda(\tilde{\xi}_0) - \varepsilon + c^2 (1 - Q_\Lambda(\tilde{\xi}_0) - \varepsilon).$$

Letting $\varepsilon \to 0$, we obtain

$$0 \geq c^2 (1 - Q_\Lambda(\tilde{\xi}_0)),$$

which is a contradiction to $Q_\Lambda(\tilde{\xi}_0) < 1$. \qed

To use Lemma 2, we need the following lemma, which implies that, under some conditions for any point in $T$, there exists a path that escapes from $Z(\tilde{\mu}, T)$.

Lemma 3. Let $N \in \mathbb{Z}$ with $|N| > 1$ and $D \subset \mathbb{Z}$ with $0 \in D$ and $\gcd(D) = 1$. Assume that a countable set $\tilde{\Lambda}$ satisfies the conditions (C1), (C2), and (C3) and $\inf_{\xi \in T} Q_\Lambda(\xi) > 0$ for any $I \in \Sigma^*_p$. If $Z(\tilde{\mu}, T) \neq \varnothing$ and for any $\alpha \in Z(\tilde{\mu}, T)$ and $I \in \Sigma^*$, there exists no $K \in \Sigma^*$ with $\alpha + F_I(K) = 0$, then for any $\xi \in T$, there exist two nonnegative integers $w$ and $v$ with $1 \leq v \leq p + 1$ and a finite word $f = j_1 \cdots j_w \in \Sigma_q$ satisfying the following property:

If $w = 0$, we have

$$0 < |m_D(N^{-1}(\xi + F_I(j_1)))| < 1, \quad 1 \leq l \leq v,$$

and $|\tilde{\mu}(N^{-v}(\xi + F_I(j_1)))| > 0$.

If $w > 0$, we have

$$m_D(N^{-1}(\xi + F_I(j_1))) = 1, \quad 1 \leq l \leq w,$$

$$0 < |m_D(N^{-1}(\xi + F_I(j_1)))| < 1, \quad w + 1 \leq l \leq w + v,$$

and $|\tilde{\mu}(N^{-w-v}(\xi + F_I(j_1)))| > 0$.

Proof. First, we shall prove the existence of $w$. If $T \cap \varnothing = \varnothing$, we take $w = 0$. If $T \cap \varnothing \neq \varnothing$, since $(\frac{1}{N}D, S_t)$ is a compatible pair, by Lemma 1(iii), there exists $j_1 \in \Sigma_q$ such that

$$|m_D(N^{-1}(\xi + F_I(j_1)))| > 0.$$ 

(8)

If $|m_D(N^{-1}(\xi + F_I(j_1)))| < 1$, we take $w = 0$. If $|m_D(N^{-1}(\xi + F_I(j_1)))| = 1$, also by Lemma 1(iii), there exists $j_2 \in \Sigma_q$ such that

$$|m_D(N^{-2}(\xi + F_I(j_1)))| > 0.$$
If $|m_D(N^{-2}(\xi + F_I(j_1j_2)))| < 1$, we take $w = 1$. When $|m_D(N^{-2}(\xi + F_I(j_1j_2)))| = 1$, the process goes on. Under the process, we claim that there exists a finite sequence of symbols $\{j_n\}_{n=1}^w \subset \Sigma_d$ such that

$$m_D(N^{-1}(\xi + F_I(j_1 \cdots j_i))) = 1, \quad \forall 1 \leq i \leq w,$$

and

$$0 < |m_D(N^{-w-1}(\xi + F_I(j_1 \cdots j_{w+1})))| < 1, \quad \forall j_{w+1} \in \Sigma_d. \quad (9)$$

Otherwise, there exists an infinite sequence $\{j_i\}_{i \geq 1} \subset \Sigma_d$ such that $m_D(N^{-1}(\xi + F_I(j_1 \cdots j_i))) = 1$ for $i \geq 1$. By (2) and the hypothesis of the lemma, we have $N^{-1}(\xi + F_I(j_1 \cdots j_i)) \in \mathbb{Z}\setminus\{0\}$. According to the proof of Theorem 2(iii) $\Rightarrow$ (i), we obtain $Q_{\lambda}(\xi) = 0$, which is a contradiction to the condition $\inf_{\xi \in T} Q_{\lambda}(\xi) > 0$ for any $L \in \Lambda^\ast$.

Next, we shall prove the existence of $v$. We write $\tilde{j} := j_1 \cdots j_w$ and $\eta := N^{-w}(\xi + F_I(\tilde{j}))$, where $\tilde{j} = \emptyset$, $F_I(\tilde{j}) = 0$ and $\eta = \xi$ when $w = 0$. In what follows, we define a sequence of sets $\{Y_n\}_{n \geq 0}$ by induction on $n$. Define $Y_0 = \{\emptyset\}$, and

$$Y_n := \{L \in \Sigma_d^n : L|_{n-1} \in Y_{n-1}, 0 < |m_D(N^{-n}(\eta + F_I(L)))| < 1\}, \quad n \geq 1.$$

We have the following claim. \(\square\)

**Claim:** For $n \geq 1$, we have $\#Y_n \geq 2^n$.

**Proof.** When $n = 1$, since $(\frac{1}{N}D, S_{ij})$ is a compatible pair, there exist two symbols $l_1 \neq l_2 \in \Sigma_d$ such that

$$0 < |m_D(N^{-1}(\eta + F_I(l_k)))| < 1, \quad 1 \leq k \leq 2.$$

Thus, we obtain $\#Y_1 \geq 2$. Suppose the inequality $\#Y_n \geq 2^n$ holds as $n = k$. Let $n = k + 1$. For any $L \in Y_n$, it is clear $L|_1 \in Y_1$. By (9), we obtain $N^{-1}(\eta + F_I(L|_1)) \not\in \mathbb{Z}$. Thus, $N^{-k}(\eta + F_I(L)) \not\in \mathbb{Z}$. Since $(\frac{1}{N}D, S_{ijL})$ is a compatible pair, there exist at least two symbols $l_1 \neq l_2 \in \Sigma_d$ such that

$$0 < |m_D(N^{-n-1}(\eta + F_I(lk)))| < 1, \quad 1 \leq k \leq 2.$$

By the arbitrariness of $L \in Y_n$, we obtain $\#Y_{n+1} \geq 2^{n+1}$. Hence, the claim follows by induction. Together with Proposition 2, the above claim implies

$$\#\{N^{-p-1}(\alpha + F_I(L)) : L \in Y_{p+1}\} = \#Y_{p+1} \geq 2^{p+1} > p.$$

Thus, by $p = \sharp\mathcal{Z}(\hat{\mu}, T)$, there exists a finite word $L \in Y_{p+1}$ such that

$$|\hat{\mu}(N^{-p-1}(\eta + F_I(L)))| > 0.$$

Let $v \geq 1$ be the smallest positive integer such that $|\hat{\mu}(N^{-v}(\eta + F_I(L)))| > 0$ for some $L = l_1 \cdots l_v$. By taking $J = \tilde{j}l_1 \cdots l_v$, we finish the proof. \(\square\)

**Lemma 4.** If $T \cap \mathbb{Z} \neq \emptyset$, then there exists $\alpha_1 > 0$ such that, for any integer sequence $\{\theta_i\}_{i \geq 1} \subset T \cap \mathbb{Z}$, we have

$$\prod_{i=1}^{\infty} |m_D(x_i)| \geq \alpha_1,$$

where $x_i \in B(\theta_i, N^{-1})$. 
Proof. For any \( \theta \in T \cap \mathbb{Z} \), we have \( m_D(\theta) = 1 \). On the other hand, the mask function \( m_D \) can be extended to an entire function on the complex plane. Thus, \( m_D \) is uniformly continuous on any compact set. Hence, there exists a positive constant \( c_1 \) such that

\[
|1 - m_D(x)| = |m_D(\theta) - m_D(x)| \leq c_1|x - \theta|, \quad \forall x \in \{\xi + y : \xi \in T, |y| \leq 1\}.
\]

Given a sequence \( \{\theta_i\}_{i \geq 1} \subset T \cap \mathbb{Z} \), we have

\[
|m_D(x_i)| \geq 1 - c_1|x_i - \theta_i| \geq 1 - N^{-1}c_1, \quad \forall x_i \in B(\theta_i, N^{-1}), \ i \geq 1.
\]

It is clear that there exists a positive integer \( K > 0 \) such that, for \( k \geq K \), we have \( N^{-k}c_1 < \frac{1}{2} \).

Note an elementary inequality:

\[
1 - x \geq e^{-2x}, \quad 0 \leq x \leq \frac{1}{2}.
\]

Then, we have

\[
\prod_{i=1}^{\infty}|m_D(x_i)| = \prod_{i=1}^{K}|m_D(x_i)| \prod_{i=K+1}^{\infty}|m_D(x_i)|
\geq \frac{1}{2}K \prod_{i=K+1}^{\infty}e^{-2c_1N^{-i}}
= \frac{1}{2}Ke^{-\sum_{i=K+1}^{\infty}2c_1N^{-i}}
= \frac{1}{2}Ke^{-2c_1(1/N^1)} =: \alpha_1 > 0
\]

for all \( x_i \in B(\theta_i, N^{-i}) \). The proof is complete. \( \square \)

Proof of Theorem 2(i) \( \Rightarrow \) (ii). We expect to obtain a contradiction after assuming

\[
\inf_{\xi \in T} Q_{\lambda_I}(\xi) > 0, \quad \forall I \in \Sigma^s.
\]

We shall prove that there is a positive number \( c > 0 \) such that, for any \( \xi \in T \) and \( I \in \Sigma^s \), there exists \( \lambda_{\xi,I} \in \Lambda_I \) satisfying

\[
|\bar{\mu}(N^{-|I|}(|\xi + F(I)| + \lambda_{\xi,I})| \geq c.
\]

If \( Z(\bar{\mu}, T) = \varnothing \), then \( \bar{\mu}(\xi) \) has a positive lower bound on compact set \( T \). Write \( c := \inf_{\xi \in T} |\bar{\mu}(\xi)| > 0 \). For any \( \xi \in T \) and \( I \in \Sigma^s \), take \( \lambda_{\xi,I} = 0 \in \Lambda_I \). Noting \( N^{-|I|}(|\xi + F(I)|) \in T \), we have

\[
|\bar{\mu}(N^{-|I|}(|\xi + F(I)|)| \geq c.
\]

From Lemma 2, it follows that \((\mu, \Lambda)\) is a spectral pair, which is a contradiction to the hypothesis. \( \square \)

Next, we focus on the case \( Z(\bar{\mu}, T) \neq \varnothing \). We shall deal with two cases.

Case i. For any \( \eta \in Z(\bar{\mu}, T) \) and \( I \in \Sigma^s \), there exists \( J \in \Sigma^s \) such that

\[
\eta + F_J(f) = 0.
\]

By \( |\bar{\mu}(0)| = 1 \), there exists a positive number \( \delta \) with \( 0 < \delta_1 < 1 \) such that

\[
|\bar{\mu}(x)| > \frac{1}{2}, \quad \forall x \in B(0, \delta_1).
\]
Write \( \delta := \min\{\delta_1, \frac{\delta}{4}\} \), where \( d \) denotes the smallest distance between different points in \( Z(\hat{\mu}, T) \cup (T \cap \mathbb{Z}) \), i.e., \( d := \min\{|x - y| : x \neq y \in Z(\hat{\mu}, T) \cup T \cap \mathbb{Z}\} \).

We denote the set of points that has a positive distance from the zero points of \( \hat{\mu}(\xi) \) in \( T \) by

\[
P := T \setminus \left( \bigcup_{\theta \in Z(\hat{\mu}, T)} B(\theta, \delta) \right).
\]

It is clear that \( P \) is a compact set and \( a_0 := \inf_{\xi \in P} |\hat{\mu}(\xi)| > 0 \). Write \( \alpha := \min\{\frac{1}{2}a_1, a_0\} \). Given \( \xi \in T \) and \( I \in \Sigma^* \), define \( \tilde{\xi} = N^{-1}(\xi + F(I)) \).

If \( \tilde{\xi} \in P \), we take \( \lambda_{\xi, I} = 0 \). Then,

\[
|\hat{\mu}(N^{-1}(\xi + F(I)) + \lambda_{\xi, I})| = |\hat{\mu}(\tilde{\xi})| \geq a_0 \geq \alpha. \quad (14)
\]

If \( \tilde{\xi} \notin P \), by the definition of \( P \), there exists a unique \( \theta \in Z(\hat{\mu}, T) \subset T \setminus \{0\} \) such that \( \tilde{\xi} \in B(\theta, \delta) \). According to (12), there exists \( J \in \Sigma^* \) such that

\[
\theta + F_I(J) = 0. \quad (15)
\]

Take \( \lambda_{\xi, I} = F_I(J) \). Then, we have

\[
N^{-1}(\tilde{\xi} + F_I(J)) \in B(N^{-1}(\theta + F_I(J)), N^{-1} \delta), \quad 1 \leq |I|.
\]

On the other hand, by (15), we have

\[
N^{-1}(\theta + F_I(J)) \in \mathbb{Z} \cap T, \quad 1 \leq |I|.
\]

In combination with Lemma 4 and (16), this leads to

\[
\prod_{I = 1}^{[\frac{l}{\delta}]} |m_D(N^{-1}(\theta + F_I(J)))| > a_1. \quad (17)
\]

Furthermore, by (16) we have

\[
N^{-1}(\tilde{\xi} + F_I(J)) \in B(N^{-1}(\theta + F_I(J)), N^{-1} \delta) \subset B(0, \delta_I).
\]

Then, by (13), we have \( |\hat{\mu}(N^{-1}(\tilde{\xi} + F_I(J)))| \geq \frac{1}{2} \). Together with (17), this inequality implies

\[
|\hat{\mu}(N^{-1}(\tilde{\xi} + F_I(J))) + \lambda_{\xi, I})| = |\hat{\mu}(\tilde{\xi} + F_I(J))|
\]

\[
= \prod_{I = 1}^{[\frac{l}{\delta}]} |m_D(N^{-1}(\theta + F_I(J)))| |\hat{\mu}(N^{-1}(\tilde{\xi} + F_I(J)))|
\]

\[
\geq \frac{1}{2} a_1
\]

\[
\geq \alpha. \quad (18)
\]

**Case ii:** There exist \( \eta^* \in Z(\hat{\mu}, T) \) and \( I \in \Sigma^* \) such that, for any \( J \in \Sigma^* \), we have

\[
\eta^* + F_I(J) \neq 0. \quad (19)
\]

Recall that \( \tilde{S} = \bigcup_{I \in \Sigma^*} S_I \) and \( p = \mathbb{Z}(\hat{\mu}, T) \). Let

\[
U := \bigcup_{I = 1}^{p+1} \left\{ N^{-1}(\theta + \lambda) : \lambda \in \tilde{S} + N \tilde{S} + \cdots + N^{I-1} \tilde{S}, \theta \in Z(\hat{\mu}, T) \cup (T \cap \mathbb{Z}) \right\}.
\]
Furthermore, we write
\[ V = \{ x \in U : |m_D(x)| \neq 0 \} \text{ and } W = \{ x \in V : |\hat{\mu}(x)| \neq 0 \}. \]

It is clear \( W \subset V \subset U \subset T \). Since \( (\frac{1}{2}D, S_1) \) is a compatible pair for any \( I \in \Sigma^* \), we obtain \( V \neq \emptyset \).

Next, we shall prove \( W \neq \emptyset \).

**Claim 1:** There exists \( a \in \tilde{S} \) such that
\[ 0 < |m_D(N^{-1}(\eta^* + a))| < 1. \]

**Proof.** If \( T \cap Z = \emptyset \), then we have \( \sup \{|m_D(\eta)| : \eta \in T\} < 1 \) by noting that \( T \) is compact. A trivial fact that \( N^{-1}(\eta^* + a) \in T \) for any \( a \in \tilde{S} \) implies the claim is true.

When \( T \cap Z \neq \emptyset \), suppose the claim is false. Since \( (\frac{1}{2}D, S_1) \) is a compatible pair, by Lemma 1(iii) for \( \eta^* \in Z(\hat{\mu}, T) \), there exists \( j_1 \in \Sigma_q \) such that \( m_D(N^{-1}(\eta^* + F_I(j_1))) = 1 \).

By (2) and (19), we obtain \( N^{-1}(\eta^* + F_I(j_1)) \in (T \cap Z) \setminus \{0\} \). Furthermore, there exists \( j_2 \in \Sigma_q \) such that \( m_D(N^{-2}(\eta^* + F_I(j_1j_2))) = 1 \), which implies \( N^{-2}(\eta^* + F_I(j_1j_2)) \in (T \cap Z) \setminus \{0\} \).

Repeating this process, we obtain a sequence of symbols \( \{j_i\}_{i \geq 1} \subset \Sigma_q \) such that
\[ N^{-1}(\eta^* + F_I(j_1 \cdots j_i)) \in (T \cap Z) \setminus \{0\}, \quad i \geq 1. \]

By a similar argument in the proof of Theorem 2(iii) \( \Rightarrow \) (i), we obtain \( Q_{\Lambda_I}(\eta^*) = 0 \), which implies a contradiction to (11). The claim is proven.

Next, we define a sequence of sets \( \{Y_n\}_{n \geq 0} \) by induction on \( n \). Let \( Y_0 := \{\eta^*\} \), and
\[ Y_n := \{N^{-1}(\eta + a) : 0 < |m_D(N^{-1}(\eta + a))| < 1, \quad \eta \in Y_{n-1}, \quad a \in \tilde{S}\}, \quad n \geq 1. \]

By a similar argument in the proof of the claim in Lemma 3, we obtain \#\( Y_n \) \( \geq 2^p \) for \( 1 \leq n \leq p + 1 \). On the other hand, for any \( \eta \in Y_{p+1} \), there exists \( \lambda \in \tilde{S} + NS + \cdots + Np\tilde{S} \) such that \( \eta = N^{-p-1}(\eta^* + \lambda) \) and \( 0 < |m_D(\eta)| < 1 \), which implies \( Y_{p+1} \subset V \). Then, we conclude
\[ \#V \geq \#Y_{p+1} \geq 2^{p+1} > p. \]

Recall that \( p \) is the number of zero points of \( \hat{\mu}(\xi) \) on compact \( T \). Then, we obtain \( W \neq \emptyset \).

Noting that \( W \subset V \subset U \) and \( U \) is a finite set, it is obvious that both \( W \) and \( V \) are finite sets. Write
\[
\begin{align*}
\alpha_2 &:= \min \{|m_D(\eta)| : \eta \in V \} > 0, \\
\alpha_3 &:= \min \{|\hat{\mu}(\eta)| : \eta \in W \} > 0,
\end{align*}
\]

Then, there exists a positive number \( \delta_2 > 0 \) such that, for any \( \eta \in V \) and \( \omega \in W \), we have
\[
|m_D(x)| > \frac{1}{2}\alpha_2, \quad \forall x \in B(\eta, \delta_2),
\]
\[
|\hat{\mu}(x)| > \frac{1}{2}\alpha_3, \quad \forall x \in B(\omega, \delta_2).
\]

Write \( \tilde{\delta} := \min \{\delta_1, \delta_2, \frac{d}{4}\} \). We let \( \tilde{P} := T \setminus \left( \bigcup_{\theta \in Z(\hat{\mu}, T)} B(\hat{\mu}, \tilde{\delta}) \right) \) denote the set of points that has a positive distance (at least \( \tilde{\delta} \)) from the zero points of \( \hat{\mu}(\xi) \) in \( T \). It is clear that \( \tilde{P} \) is a compact set and \( \alpha_4 := \inf_{\xi \in \tilde{P}} |\hat{\mu}(\xi)| > 0 \). We write
\[ \tilde{\alpha} := \min \{\alpha_1 \frac{\alpha_3}{2} \left(\frac{\alpha_2}{2}\right)^{p+1}, \alpha_4\}, \]

where \( \alpha_1 \) comes from Lemma 4.

Given \( \xi \in T \) and \( I \in \Sigma_q^* \), write \( \tilde{\xi} := N^{-|I|}(\xi + F(I)) \).
If $\tilde{\xi} \in \tilde{P}$, we take $\lambda_{\tilde{\xi},I} = 0 \in \Lambda_I$. Then, we have

$$|\hat{\mu}(N^{-|I|}(\tilde{\xi} + F(I)) + \lambda_{\tilde{\xi},I})| = |\hat{\mu}(\tilde{\xi})| \geq \alpha_4 \geq \tilde{\alpha}. \quad (22)$$

If $\tilde{\xi} \notin \tilde{P}$, there exists $\theta \in Z(\hat{\mu}, T)$ such that $\tilde{\xi} \in B(\theta, \tilde{\delta})$. If there exists $J \in \Sigma^*$ such that $\theta + F_I(J) = 0$,

we take $\lambda_{\tilde{\xi},I} = F_I(J)$. Then, by a similar argument as (18), we have

$$|\hat{\mu}(N^{-|I|}(\tilde{\xi} + F(I)) + \lambda_{\tilde{\xi},I})| \geq \alpha. \quad (23)$$

If there is no $J \in \Sigma^*$ such that $\theta + F_I(J) = 0$,

by Lemma 3, there exist two integers $0 \leq w < \infty, 1 \leq v \leq p + 1$ and a finite word $J := j_1 \cdots j_{w+v} \in \Sigma_q^*$ such that when $w = 0$, we have

$$0 < |m_D(N^{-I}(\theta + F_I(J_i)))| < 1, \quad 1 \leq i \leq v, \quad (24)$$

and $|\hat{\mu}(N^{-v}(\theta + F_I(J)))| > 0$; when $w > 0$, we have

$$m_D(N^{-I}(\theta + F_I(J_i))) = 1, \quad 1 \leq i \leq w, \quad (25)$$

$$0 < |m_D(N^{-I}(\theta + F_I(J_i)))| < 1, \quad w + 1 \leq i \leq w + v, \quad (26)$$

and $|\hat{\mu}(N^{-v}(\theta + F_I(J)))| > 0$.

Take $\lambda_{\tilde{\xi},I} := F_I(J)$. In the case $w = 0$, since $\tilde{\xi} \in B(\theta, \tilde{\delta})$, it is obvious that

$$N^{-I}(\tilde{\xi} + F_I(J_i)) \in B(N^{-I}(\theta + F_I(J_i)), N^{-I}(\tilde{\delta})), \quad 1 \leq i \leq v. \quad (27)$$

Noting that $\theta \in Z(\hat{\mu}, T) \cup (T \cap Z)$, by (24), we obtain

$$N^{-I}(\theta + F_I(J_i)) \in V, \quad 1 \leq i \leq v.$$ 

Together with (20) and (27), the above inequality implies

$$|m_D(N^{-I}(\tilde{\xi} + F_I(J_i)))| \geq \frac{\alpha_2}{2}, \quad 1 \leq i \leq v. \quad (28)$$

Furthermore, since $N^{-v}(\theta + F_I(J)) \in V$ and $|\hat{\mu}(N^{-v}(\theta + F_I(J)))| > 0$, we have $N^{-v}(\theta + F_I(J)) \in W$ and $N^{-v}(\tilde{\xi} + F_I(J)) \in B(N^{-v}(\theta + F_I(J)), N^{-v}(\tilde{\delta}))$. From (21), it follows that

$$|\hat{\mu}(N^{-v}(\tilde{\xi} + F_I(J_i)))| \geq \frac{\alpha_3}{2}. \quad (29)$$

In combination with (28), this yields

$$|\hat{\mu}(N^{-|I|}(\tilde{\xi} + F(I)) + \lambda_{\tilde{\xi},I})| = |\hat{\mu}(\tilde{\xi} + \lambda_{\tilde{\xi},I})|$$

$$= \prod_{i=1}^\infty |m_D(N^{-I}(\tilde{\xi} + F_I(J_i)))|$$

$$= \prod_{i=1}^\infty |m_D(N^{-I}(\tilde{\xi} + F_I(J_i)))|\hat{\mu}(N^{-v}(\tilde{\xi} + F_I(J_i)))|$$

$$\geq \frac{\alpha_3}{2} \left( \frac{\alpha_2}{2} \right)^{v+1} \geq \tilde{\alpha}. \quad (30)$$
In the case \( w > 0 \), we shall divide the product into three parts

\[
|\hat{\mu}(N^{-|l|}(\tilde{\xi} + F(I)) + \lambda_{l,j})|
\]

\[
= \prod_{i=1}^{\infty} |m_D(N^{-i}(\tilde{\xi} + F_I(J)))| 
\]

\[
= \prod_{i=1}^{\infty} |m_D(N^{-i}(\tilde{\xi} + F_I(J)))(\prod_{i=1}^{w+1} |m_D(N^{-i}(\tilde{\xi} + F_I(J)))| |\hat{\mu}(N^{-w-p}(\tilde{\xi} + F_I(J))))|.
\]  

(31)

By (2) and (25), we have

\[
N^{-l}(\theta + F_I(J)) \in T \cap \mathbb{Z}, \quad 1 \leq l \leq w.
\]  

(32)

Noting \( \tilde{\xi} \in B(\theta, \tilde{\delta}) \), we have

\[
N^{-l}(\tilde{\xi} + F_I(J)) \in B(N^{-l}(\theta + F_I(J)), N^{-l} \tilde{\delta}), \quad 1 \leq l \leq w.
\]

Thus, by (10), we obtain

\[
\prod_{i=1}^{\infty} |m_D(N^{-i}(\tilde{\xi} + F_I(J)))| \geq \alpha_1.
\]  

(33)

By (32), we have \( N^{-w}(\theta + F_I(J)) \in Z(\hat{\mu}, \bar{T}) \cup (T \cap \mathbb{Z}) \). Then, by (26), we have

\[
N^{-l}(\theta + F_I(J)) \in V, \quad w + 1 \leq l \leq w + v
\]

and

\[
N^{-l}(\tilde{\xi} + F_I(J)) \in B(N^{-l}(\theta + F_I(J)), N^{-l} \tilde{\delta}), \quad w + 1 \leq l \leq w + v.
\]  

(34)

By (20) and (21), we obtain

\[
|m_D(N^{-l}(\tilde{\xi} + F_I(J)))| > \frac{\alpha_2}{2}, \quad w + 1 \leq l \leq w + v,
\]  

(35)

and

\[
|\hat{\mu}(N^{-w-p}(\tilde{\xi} + F_I(J)))| \geq \frac{\alpha_3}{2}.
\]

Together with (31), (33), and (35), the above inequality yields

\[
|\hat{\mu}(N^{-|l|}(\tilde{\xi} + F(I)) + \lambda_{l,j})| \geq \alpha_1 \left( \frac{\alpha_2}{2} \right) ^{p+1} \frac{\alpha_3}{2} \geq \tilde{\alpha}.
\]  

(36)

In combination with (14), (18), (22), (23), (30), and (36), by Lemma 2, we obtain \((\mu, \Lambda)\) is a spectral pair, which is a contradiction to our hypothesis. We finish the proof of (i) \( \Rightarrow \) (ii) in Theorem 2.

Finally, we shall prove Theorem 2 (ii) \( \Rightarrow \) (iii).

Since \( T \) is compact, there exists \( \xi^* \in T \) such that \( Q_{\Lambda_I}(\xi^*) = 0 \). Write

\[
X := \{ \xi \in T : \hat{\mu}(\xi) = 0 \text{ and } m_D(\xi) \neq 0 \}.
\]

It is clear that \( 0 \notin X \). Since \( \frac{1}{N} D_S(j) \) is a compatible pair, by Lemma 1, there exists an integer \( j \in \Sigma_{\eta} \) with \( m_D(\frac{1}{N}(\xi^* + F_I(j))) \neq 0 \). Noting that

\[
0 = Q_{\Lambda_I}(\xi^*) = \Sigma_{\lambda \in \Lambda_I} |\hat{\mu}(\xi^* + \lambda)|^2 \geq |m_D(\frac{1}{N}(\xi^* + F_I(j)))|^2 |\hat{\mu}(\frac{1}{N}(\xi^* + F_I(j)))|^2,
\]

we obtain \( \hat{\mu}(\frac{1}{N}(\xi^* + F_I(j))) = 0 \). By virtue of \( \xi^* \in T \), we have \( \frac{1}{N}(\xi^* + F_I(j)) \in T \). Hence, \( X \) is nonempty.
Next, we define a sequence of the subset of $X$ by induction on $n$. Define $X_0 := \{\xi^a\}$ and $X_{n+1} := \{N^{-n-1}(\xi + F_i(j)) \in X : N^{-n}(\xi + F_i(j)|_n) \in X_n, j \in \Sigma^n_{j+1}\}$, $n \geq 0$.

We have the following conclusion.

Claim 2: $\#X_{n+1} \geq \#X_n$, $n \geq 0$.

Proof. When $n = 0$, by the definition of $Q_{\Lambda_i}(\xi^a)$, we have

$$0 = Q_{\Lambda_i}(\xi^a) = \sum_{j \in \Sigma^q_i} |m_D(N^{-i}(\xi^a + F_i(j))|^2 \cdot Q_{\Lambda_i}(N^{-i}(\xi^a + F_i(j))).$$

Noting that $(\hat{\frac{1}{N}}D, S_i)$ is a compatible pair, Lemma 1(iii) implies that there exists at least one symbol $j_i \in \Sigma_q$ such that $|m_D(N^{-1}(\xi^a + F_i(j_i))| > 0$, which implies $Q_{\Lambda_i}(N^{-1}(\xi^a + F_i(j_i))) = 0$. Hence, we have $\mu(N^{-1}(\xi^a + F_i(j_i))) = 0$. This leads to $\#X_1 > \#X_0$. Suppose Claim 2 holds for $n = k - 1$. Then, $X_k$ is nonempty. For any $y \in X_k$, there exists $\bar{j} \in \Sigma^k_i$ such that $y = N^{-k}(\xi^a + F_i(\bar{j}))$ and

$$\prod_{i=1}^k |m_D(N^{-i}(\xi^a + F_i(\bar{j})))| > 0.$$

By (1) and (4), we have

$$0 = Q_{\Lambda_i}(\xi^a) = \sum_{j \in \Sigma^q_i} \prod_{i=1}^k |m_D(N^{-i}(\xi^a + F_i(\bar{j})))|^2 \cdot Q_{\Lambda_i}(N^{-k}(\xi^a + F_i(\bar{j}))).$$

Then, we obtain $Q_{\Lambda_i}(N^{-k}(\xi^a + F_i(\bar{j}))) = 0$. By a similar argument, we have

$$0 = Q_{\Lambda_i}(N^{-k}(\xi^a + F_i(\bar{j}))) = \sum_{j_{k+1} \in \Sigma_q} |m_D(N^{-k-1}(\xi^a + F_i(\bar{j}_{k+1})))|^2 \cdot Q_{\Lambda_i}(N^{-k-1}(\xi^a + F_i(\bar{j}_{k+1}))).$$

Noting that $(\hat{\frac{1}{N}}D, S_{\bar{j}})$ is a compatible pair, by Lemma 1(iii), there exists at least one symbol $j_{k+1} \in \Sigma_q$ such that

$$|m_D(N^{-k-1}(\xi^a + F_i(\bar{j}_{k+1})))| > 0.$$

Hence, $Q_{\Lambda_i}(N^{-k-1}(\xi^a + F_i(\bar{j}_{k+1}))) = 0$, which implies $\mu(N^{-k-1}(\xi^a + F_i(\bar{j}_{k+1}))) = 0$. Thus, we obtain

$$N^{-k-1}(\xi^a + F_i(\bar{j}_{k+1})) \in X_{k+1}.$$

If we consider $N^{-n-1}(\xi^a + F_i(\bar{j}_{n+1}))$ as a “next generation” of $N^{-n}(\xi^a + F_i(j))$ for $n \geq 1$, Proposition 2 implies that different points of $X_k$ have different “next generations”. Thus, we obtain $\#X_{k+1} > \#X_k$, which implies Claim 2 is true.

By noting the fact that $X$ is a subset of the finite set $Z(\mu, T)$, there exists a positive integer $h \in \mathbb{N}$ such that

$$\#X_{h+m} = \#X_h, \quad m \geq 1. \quad (37)$$

From the above argument, it follows that for any $y = N^{-n}(\xi^a + F_i(j_1 \cdots j_n)) \in X_n$, if there exists a symbols $j_{n+1} \in \Sigma_q$ such that $|m_D(N^{-n-1}(\xi^a + F_i(j_1 \cdots j_n))| > 0$, then $y$ has a “next generation” $N^{-n-1}(\xi^a + F_i(j_1 \cdots j_n)) \in X_{n+1}$. Noting that $(\hat{\frac{1}{N}}D, S_{j_1 \cdots j_{n+1}})$ is a compatible pair, by Lemma 1(iii), we have

$$\sum_{j_{n+1} \in \Sigma_q} |m_D(N^{-1}(y + C(I_{j_{n+1}})))|^2 = 1.$$
In combination with (37), we conclude that for any \( n \geq h \), there exists only one symbol \( j_{h+1} \in \Sigma_h \) such that \( m_D(N^{-1}(y + C(I)j_{h+1})) \neq 0 \). In fact, \( m_D(N^{-1}(y + C(I)j_{h+1})) = 1 \).

Then, we obtain

\[
N^{-n-1}(\xi^* + F(I)j_{h+1})) = N^{-1}(y + C(I)j_{h+1}) \in \mathbb{Z}.
\]

Continuing the process, we obtain a sequence of symbols \( \{j_{h+1}\}_{l \geq 1} \subset \Sigma_0 \), such that

\[
N^{-h-l}(\xi^* + F(I)j_{h+1} \cdots j_{h+l})) \in \mathbb{Z}, \quad l \geq 1.
\]

Define \( \beta_1 := N^{-h}(\xi^* + F(I)) \) and

\[
\beta_l := N^{-h-l+1}(\xi^* + F(I)j_{h+1} \cdots j_{h+l-1})), \quad l \geq 2.
\]

It is clear \( \beta_1 \in X_{h+1-1} \), which implies \( \beta_l \) is nonzero. Thus, the sequence of nonzero integers \( \{\beta_l\}_{l \geq 1} \) and the increasing sequence of finite words \( \{j_{h+1} \cdots j_{h+l}\}_{l \geq 1} \) with the prefix \( I \) fulfill the request. □

As a corollary of Lemma 2 and Theorem 2, we obtain another necessary and sufficient condition for \( \Lambda \) to be a spectrum of \( \mu \).

**Proposition 5.** Let \( N \in \mathbb{Z} \) with \( |N| > 1 \) and \( D \subset \mathbb{Z} \) with \( 0 \in D \) and \( \gcd(D) = 1 \). Assume that a countable set \( \Lambda \) satisfies the conditions (C1), (C2), and (C3). Then, \((\mu, \Lambda)\) is a spectral pair if and only if there exists a positive number \( c > 0 \) such that, for any \( \xi \) and \( I \in \Sigma^* \), there is \( \lambda_{\xi, I} \in \Lambda_1 \) satisfying

\[
|\hat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi, I})| \geq c.
\]

**Proof.** The sufficiency follows from Lemma 2. We just prove the necessity here. Suppose that \((\mu, \Lambda)\) is a spectral pair. By Propositions 3 and 4, we obtain, for any \( I \in \Sigma^* \),

\[
Q_{\Lambda_I}(\xi) \equiv 1, \quad \xi \in \mathbb{R}.
\]

By a similar argument in the proof of Theorem 2 (i) \( \Rightarrow \) (ii), for any \( \xi \in T \) and \( I \in T \), there exists \( \lambda_{\xi, I} \in \Lambda_1 \) such that

\[
|\hat{\mu}(N^{-|I|}(\xi + F(I)) + \lambda_{\xi, I})| \geq c.
\]

We finish the proof. □

4. An Example

In this section, we construct a self-similar measure and a set \( \Lambda(N, B) \) with a tree structure. Neither the criterion of Laba and Wang (Theorem 1) nor that of Strichartz ([24]) are applicable to this set \( \Lambda(N, B) \). However, we show that there does not exist an infinite orbit \( \{\beta_l\}_{l \geq 1} \subset \mathbb{Z} \setminus \{0\} \) associated with the dual IFS (see Theorem 3), which implies \( \Lambda(N, B) \) is a spectrum by Theorem 2.

**Example 1.** Let \( N = 6 \) and \( D = \{0, 1, 2\} \). Write \( \mu \) for the invariant measure associated with the IFS \( \{\phi_1, \phi_2, \phi_3\} \) defined by

\[
\phi_1(x) = \frac{1}{6}x, \quad \phi_2(x) = \frac{1}{6}(x + 1), \quad \phi_3(x) = \frac{1}{6}(x + 2).
\]

Let \( B_1 = \{0, 8, 22\}, B_2 = \{0, 22, 38\}, B_3 = \{0, 8, 52\}, \) and \( B_4 = \{0, 38, 52\} \). By Lemma 1, a simple induction implies that \((\frac{1}{6}D, B_i)\) is a compatible pair for \( 1 \leq i \leq 4 \). Noting

\[
\frac{1}{6}(4 + 8) = 2, \quad \frac{1}{6}(2 + 22) = 4, \quad \frac{1}{6}(4 + 8) = 2, \quad \frac{1}{6}(2 + 22) = 4, \quad \cdots,
\]
\[ \frac{1}{6} (10 + 38) = 8, \quad \frac{1}{6} (8 + 52) = 10, \quad \frac{1}{6} (10 + 38) = 8, \quad \frac{1}{6} (8 + 52) = 10, \ldots, \]

we see that both \( \Lambda(6, B_1) \) and \( \Lambda(6, B_4) \) have an infinite iterated nonzero integer sequence, where \( \Lambda(N, S) := S + NS + N^2S + \cdots \) finite sum. Thus, by Theorem 1 or by Theorem 2, we conclude that both \( \Lambda(6, B_1) \) and \( \Lambda(6, B_4) \) are not a spectrum of \( \mu \). We consider the following set defined by \( \{B_i : 1 \leq i \leq 4\} \).

\[
\begin{align*}
\Lambda(N, B) &:= B_1 + \underbrace{NB_2 + N^2B_3 + \cdots}_{B_2 \text{ and } B_3 \text{ repeat } 1 \text{ time}} \\
&+ \underbrace{N^3B_4 + N^4B_3 + N^5B_2 + N^6B_3 + N^7B_2 + \cdots}_{B_2 \text{ and } B_3 \text{ repeat } 2 \text{ times}} \\
&+ \underbrace{N^nB_1 + N^nB_2 + N^{10}B_3 + N^{11}B_2 + N^{12}B_3 + N^{13}B_2 + N^{14}B_3 + N^{15}B_2 + N^{16}B_3 + \cdots}_{B_2 \text{ and } B_3 \text{ repeat } 2^2 \text{ times}} \\
&+ \underbrace{N^{17}B_4 + N^{18}B_3 + N^{19}B_2 + \cdots + N^{32}B_3 + N^{33}B_2 + \cdots}_{B_3 \text{ and } B_2 \text{ repeat } 2^3 \text{ times}} \\
(38)
\end{align*}
\]

According to Remark 1, it is clear that Theorem 1 cannot work. We shall show \( \Lambda(N, B) \) is a spectrum of \( \mu \) by Theorem 2 in the following Theorem 3. Then, we show that Strichartz’s criterion (Theorem 2.8 in [24]) is not appropriate by proving the following Theorem 4.

Let \( A_n \) denote the set of coefficients of \( N^n (n \geq 0) \) in (38). Given two integers \( l \) and \( k \) with \( l > k \geq 0 \), we write

\[
A^l_k := A_k + NA_{k+1} + N^2A_{k+2} + \cdots + N^{l-k-1}A_{l-1}.
\]

(39)

We also write \( \Lambda^k := \Lambda_0^k \) for simplicity. For three integers \( m, n, \) and \( k \) with \( 0 \leq m < n < k \), we have

\[
\Lambda^m_n + N^{n-m} \Lambda^k_n = A_m + NA_{m+1} + \cdots + N^{n-m-1}A_{n-1} + N^{n-m}A_n + \cdots + N^{k-m-1}A_{k-1}
\]

(40)

**Theorem 3.** Given nonzero integer sequence \( \{\beta_i\}_{i \geq 1} \), then, for any integer \( M > 0 \), there exists an integer \( i \geq M \) such that

\[
\beta_{i+1} \neq N^{-1}(\beta_i + a_i),
\]

for any \( a_i \in A_i \).

**Proof.** Suppose that there exists a positive integer \( M \) such that, for any \( i > M \), we have \( \beta_{i+1} = 6^{-1}(\beta_i + a_i) \). Let \( T_0 \) be the self-similar set generated by the dual IFS \( \{ \frac{1}{5}(x + s) : s \in \bigcup_{j=1}^{4} B_j \} \).

According to the definition of the attractor \( T_0 \), there exists a positive integer \( K \) such that, for any \( i \geq K \), \( \beta_i \) belongs to a neighborhood of \( T_0 \), i.e.,

\[
\beta_i \in (-\frac{53}{5}, \frac{53}{5}).
\]

Recall a fact that \( \bigcup_{i=0}^{\infty} A_i = \{0, 8, 22, 38, 52\} \). Then, \( \beta_{K+1} = 6^{-1}(\beta_K + a_K) \) with \( a_K \in \bigcup_{j=1}^{4} B_j \) implies \( \beta_K \in \{2, 4, 6, 8, 10\} \). By noting that \( \beta_{K+2} = 6^{-1}(\beta_{K+1} + a_{K+1}) \) with \( a_{K+1} \in \bigcup_{j=1}^{4} B_j \) implies \( \beta_K \neq 6 \), hence \( \beta_K \in \{2, 4, 8, 10\} \). If \( \beta_K = 2 \), then

\[
a_K = 22, \quad a_{K+1} = 8, \quad a_{K+2} = 22, \quad a_{K+3} = 8, \ldots
\]

(40)
Hence, \(\{8, 22\} \cap A_i \neq \emptyset\) for all \(i \geq K\), which contradicts that \(\{8, 22\} \cap B_4 = \emptyset\) and \(B_4 = A_i\) for infinitely many \(i\). Hence, \(\beta_K \in \{4, 8, 10\}\).

By a similar argument for other cases, i.e., \(\beta_K \in \{4, 8, 10\}\), we always obtain a contradiction. Then, we finish the proof. \(\square\)

The following result shows that Strichartz’s method (Theorem 2.8 in [24]) is not applicable to the above set \(\Lambda(N, B)\).

**Theorem 4.** We have

\[
\liminf_{n \to \infty} \inf_{\lambda \in \Lambda^n} |m_D(N^{-n}\lambda)| = 0.
\]

**Proof.** Obviously, we need only to prove that there exists a subsequence \(\{\lambda_{n_k}\}_{k \geq 1} \subset \Lambda^{n_k}\) such that \(N^{-n_k}\lambda_{n_k}\) tends to a zero point of \(m_D\) as \(k\) tends to infinity. Let \(T_0\) be the attractor of the IFS \(\{\Phi_j(x) = \frac{1}{4}(x + j) : j \in \bigcup_{i=1}^4 B_i\}\). Thus, we have \(T_0 \subset [0, \frac{52}{3}]\).

For \(k \geq M\), we write \(n_k = 2^{2k+2} + 2k + 1\), and we take

\[
\beta_{n_k} = 38 + 52 \times 6 + 38 \times 6^2 + 52 \times 6^3 + \ldots + 38 \times 6^{2k+1} \in \Lambda_{2^{2k+1}+2k-1}^{n_k}
\]

where the coefficients 38 and 52 appear alternately. By a simple deduction, we obtain

\[
6^{-2k+1-2}(10 + \beta_{n_k}) = \frac{4}{3}.
\]

(41)

Take arbitrarily \(\alpha \in \Lambda^{2k+1+2k-1}\), and write

\[
\lambda_{n_k} = \alpha + 6^{2k+1+2k-1}\beta_{n_k}.
\]

By (40), we obtain

\[
\lambda_{n_k} \in \Lambda^{n_k}.
\]

According to the definition of \(T_0\), we have

\[
6^{-2k+1-2k+1}\alpha \in T_0,
\]

which implies

\[
|6^{-2k+1-2k+1}\alpha - 10| \leq \frac{52}{3}.
\]

In combination with (41), we have

\[
|6^{-n_k}\lambda_{n_k} - \frac{4}{3}| = |6^{-2k+1-2}(6^{-2k+1-2k+1}\alpha + \beta_{n_k}) - 6^{-2k+1-2}(10 + \beta_{n_k})| = |6^{-2k+1-2}(6^{-2k+1-2k+1}\alpha - 10)| \leq 6^{-2k+1-2} \times \frac{52}{3}.
\]

Noting the fact that \(m_D(\frac{4}{3}) = 0\), we finish the proof. \(\square\)

5. Summary and Conclusions

In this paper, we introduced a tree structure with the language of symbolic space. The natural spectrum candidate of a self-similar measure associated with an IFS is a set with a special tree structure. We obtained three equivalent conclusions for \(\Lambda\) to be a spectrum of a self-similar measure. One of them implies that there exists an infinite orbit with an element of a nonzero integer associated with the dual IFS. An example involving a self-similar measure and a spectrum candidate \(\Lambda(N, S) = S_0 + NS_1 + N^2S_2 \cdots\) showed the tree structure expands essentially the field of spectrum candidates.
It is one of the most important problems to find all spectra of a spectral measure. We are not sure that every spectrum of a self-similar measure holds a tree structure. On the other hand, the self-similar $\mu_{N,D}$ measure has another description, $\mu_{N,D} = \delta^D \mu_0 \ast \delta^D \mu_1 \ast \cdots$. It is obvious to ask if Theorem 2 holds for the Moran-type self-similar measure. As mentioned in the Introduction, the version of Theorem 1 in higher-dimensional space has not been obtained completely. It is the next research direction to prove Theorem 2 for self-affine measures.

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