Infrared Evolution and Phase Structure of a Gauge Theory Containing Different Fermion Representations

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We study the evolution of an asymptotically free vectorial SU(N) gauge theory from the ultraviolet to the infrared and the resultant phase structure in the general case in which the theory contains fermions transforming according to several different representations of the gauge group. We discuss the sequential fermion condensation and dynamical mass generation that occur, and comment on the effect of bare fermion mass terms.

I. INTRODUCTION

The phase structure of a non-Abelian gauge theory depends on its fermion content. Here we consider an asymptotically free vectorial gauge theory (in (3 + 1) dimensions, at zero temperature and chemical potential) with an SU(N) gauge group and fermions corresponding to several different representations of the gauge group. We denote the running gauge coupling of the theory as \( g(\mu) \), with \( \alpha(\mu) = g(\mu)^2/(4\pi) \), where \( \mu \) is the Euclidean energy/momentum scale (which will often be suppressed in the notation). Since the gauge interaction is asymptotically free, at a sufficiently high energy scale \( \mu, \alpha(\mu) \) is small and the theory is perturbatively calculable. We will study a theory which contains several Dirac fermions transforming according to different representations of SU(N). We denote a representation as \( R \), the set of fermion representations in the theory as \( \{ R \} \equiv \{ R_1, ..., R_k \} \), the number of Dirac fermions in each representation \( R_i \) as \( N_{R_i} \), and the set of these numbers as \( \{ N_R \} \equiv \{ N_{R_1}, ..., N_{R_k} \} \). [1]. We will first consider the case in which all of these fermions are massless or have bare masses in the high-scale Lagrangian that are small compared with the scale where \( \alpha \) grows to a size of order unity and the theory becomes strongly coupled. One interesting property of this type of theory is that it can exhibit fermion condensations at different energy scales, with fermions with larger quadratic Casimir invariants condensing and gaining dynamical masses at higher scales. This theory could arise from a larger one which is a chiral gauge theory, in which fermion masses would generically be forbidden. However, if we consider the theory by itself, then, since it is vectorial, and hence fermion mass terms do not violate the SU(N) gauge symmetry, it is natural to consider a more complicated situation in which some fermions have masses that are comparable to or greater than the scale where the coupling \( \alpha \) grows to O(1). We shall also briefly comment on this latter possibility.

Although our work is an abstract field-theoretic study, not an effort to construct a phenomenological model, we note that there has been considerable interest recently in the analysis of vectorial non-Abelian gauge theories with fermions in higher-dimensional representations, partly motivated by technicolor model-building [2-8]. We note in passing that in the early development of the Standard Model, the possibility was considered that the color SU(3)c sector might contain not just quarks but also other fermions transforming as higher-dimensional representations of the color group [4]. Fermions in higher-dimensional representations have also been used in constructions of chiral gauge theories, but here we restrict our consideration to vectorial gauge theories.

II. GENERAL THEORETICAL FRAMEWORK

A. Beta Function

In this section we review the general theoretical framework that we will use in our calculations. The beta function of the theory is denoted \( \beta = dg/d\ln \mu \). In terms of \( \alpha \), this can be written as

\[
\frac{d\alpha}{dt} = -\frac{\alpha^2}{2\pi} \left[ b_1 + \frac{b_2 \alpha}{4\pi} + O(\alpha^2) \right]
\]

(2.1)

where the coefficient \( b_1 \) arises at \( \ell \)-loop order in perturbation theory, and the first two coefficients, \( b_1 \) and \( b_2 \), are scheme-independent. These are [5]

\[
b_1 = \frac{1}{3} \left[ 11C_2(G) - 4 \sum R N_R T(R) \right]
\]

(2.2)

and [6]

\[
b_2 = \frac{1}{3} \left[ 34C_2(G)^2 - 4 \sum R (5C_2(G) + 3C_2(R)) N_R T(R) \right].
\]

(2.3)

Here \( C_2(R) \) is the quadratic Casimir invariant and \( T(R) \) is the trace invariant for the representation \( R \), with \( C_2(G) = C_2(\text{adj.}) \) and \( C_2(SU(N)) = N \) (see appendix). The condition that the theory be asymptotically free, i.e., that \( b_1 > 0 \), yields the upper bound

\[
\sum R N_R T(R) < \frac{11N}{4}.
\]

(2.4)
Since all of the terms on the left-hand side contribute positively, this implies the upper bound on the number of fermions in each representation \( N_R < N_{R,\text{max}} \), where
\[
N_{R,\text{max}} = \frac{11N}{4T(R)}. \tag{2.5}
\]

Here and below, we implicitly carry out an analytic continuation of \( N_R \) from non-negative integers to non-negative real numbers; however, it is understood that physically they are, of course, non-negative integers. If there are few fermions, then also \( b_2 > 0 \), so that the two-loop beta function has a zero only at the origin, \( \alpha = 0 \).

A sufficient increase in the numbers of fermions in various representations leads to a reversal in the sign of \( b_2 \) from positive to negative, while still satisfying the condition of asymptotic freedom, \( 2.4 \). For a set of fermion representations \( \{N_R\} \) with this property, the two-loop beta function has a zero away from the origin at
\[
\alpha_{IR} = -\frac{4\pi b_1}{b_2} = \frac{4\pi b_1}{|b_2|}. \tag{2.6}
\]

For the theory with a single type of fermion representation, we denote the value of \( N_R \) where \( b_2 = 0 \) as \( N_{R,\text{IR}} \). This is
\[
N_{R,\text{IR}} = \frac{17C_2(G)^2}{2[5C_2(G) + 3C_2(R)]T(R)}. \tag{2.7}
\]

The fact that \( N_{R,\text{IR}} < N_{R,\text{max}} \) is evident because for \( N = N_{R,\text{IR}}, b_1 \) has the positive (i.e., asymptotically free) value
\[
b_1 = \frac{C_2(G) [6C_2(G) + 11C_2(R)]}{5C_2(G) + 3C_2(R)} > 0
\]
for \( N_R = N_{R,\text{IR}} \).

If \( b_2 < 0 \), so that there is an infrared zero of the beta function, then as the scale \( \mu \) decreases from large values, \( \alpha(\mu) \) increases toward this value. The infrared behavior then depends on whether or not the value of the coupling \( \alpha_{IR} \) is sufficiently large as to cause spontaneous chiral symmetry breaking \[8\]. If the properties of the theory are such that no fermion condensates form, then this is an exact infrared fixed point (IRFP) of the (perturbatively calculated) renormalization group equation for \( \alpha \). If, on the other hand, some fermions do condense, so that they get dynamically generated masses and are integrated out of the low-energy effective theory applicable below the scale(s) of condensation, then, since the beta function changes, the original value of \( \alpha_{IR} \) is only an approximate IRFP. Since the coefficients \( b_1 \) and \( b_2 \) are the maximal set of coefficients in the beta function that are scheme-independent, it follows that conclusions obtained from the two-loop beta function should be at least qualitatively reliable physically. However, since we will deal with values of \( \alpha_{IR} \) of order unity, i.e., strongly coupled gauge interactions, it is understood that there are inevitably significant theoretical uncertainties in the results. In this context, we recall that the two-loop perturbative beta function is an asymptotic expansion in \( \alpha \) and does not include a number of important effects, including confinement and instantons. Indeed, instanton effects involve factors like \( \exp(-c\pi/\alpha) \) (where \( c \) is a constant), which cannot be seen to any order of perturbation theory. Moreover, it should be noted that even if there is no zero of the two-loop beta function away from the origin, i.e., a perturbative IRFP, the beta function may exhibit a nonperturbative slowing of the running associated with the fact that at energy scales below the confinement scale, the physics is not accurately described in terms of the Lagrangian degrees of freedom (fermions and gluons) \[9, 11\]. We observe that one can calculate \( \alpha_{IR} \) more accurately using the higher-order coefficients of the beta function. Finally, although an asymptotically free vectorial SU(\(N\)) gauge theory of the type that we consider here does not require an ultraviolet completion, it could, as remarked above, arise as the low-energy effective field theory resulting from the breaking of a larger, chiral, gauge symmetry. In this case, one would also want to assess the effects of residual higher-dimensional operators from this larger gauge theory (e.g., \[12\]).

B. Results from Approximate Solution of Dyson-Schwinger Equation for Fermion Propagator

A solution of the Dyson-Schwinger (DS) equation for the propagator of a fermion \( \psi \) in the representation \( R \) of the gauge group, with zero bare mass, in the approximation of one-gluon (also called ladder) exchange, yields a nonzero, dynamically generated mass if the coupling \( \alpha(\mu) \) exceeds a critical value \( \alpha_{R,\text{cr}} \) given by \[13, 16\]
\[
\frac{3C_2(R)\alpha_{R,\text{cr}}}{\pi} = 1. \tag{2.9}
\]
In the same ladder approximation, the anomalous dimension for the fermion (bilinear) mass operator is \( \gamma = 1 \) at \( \alpha = \alpha_{\text{cr},R} \). Some lattice studies have reported initial results on measurements of \( \gamma \) \[23, 24\]. Corrections to the one-gluon exchange approximation have been analyzed and found not to be too large \[17\]. To assess the implications of these corrections for the boundary of the chirally symmetric phase, one also calculates \( \alpha_{IR} \) to the corresponding higher order. Since the dynamically generated mass for this fermion is the coefficient of the bilinear fermion operator in an effective Lagrangian, this indicates the formation of a condensate of the fermions in the representation \( R \), and associated spontaneous chiral symmetry breaking (S\(\chi\)SB) by the gauge interaction, as \( \alpha \) increases through the critical value \( \alpha_{R,\text{cr}} \). Some early studies with lattice simulations of chiral symmetry breaking were carried out for SU(2) and SU(3) for various fermion representations in \[19, 21\]. There has been considerable recent lattice work, mainly on the group SU(3) with fermions in the fundamental representation or rank-2 symmetric (sixtet) representation and on SU(2) with
fermions in the adjoint (equivalent to rank-2 symmetric) representation. Some of the rapidly increasing number of papers reporting results from numerical lattice simulations include Refs. 23, 34. To our knowledge, there have not been lattice studies of chiral symmetry breaking in a theory containing dynamical fermions in two or more different representations (simultaneously present).

The analysis of the gauge coupling evolution and chiral symmetry realization in vectorial asymptotically free gauge theories has been of particular interest in the context of technicolor (TC) theories 25, especially in the context of the most promising such theories, which exhibit a slowly running (“walking”) gauge coupling associated with an approximate infrared fixed point of the renormalization group 15 (see also 18). In the actual application to theories of dynamical electroweak symmetry breaking, one must embed the technicolor sector in a larger theory, extended technicolor (ETC) in order to give masses to quarks and leptons and to account for their generational structure 20. A necessary property of TC/ETC theories is that the ETC symmetry must break in a series of stages to the TC symmetry, which is an asymptotically free, vectorial theory that becomes strongly coupled on the TeV scale, producing bilinear technifermion condensates that break the electroweak gauge symmetry. 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C. βDS Method for Determining Chiral Phase Boundary

Here we recall a method to estimate the critical value, \( N_{R, cr} \), of the number of fermions in a single representation \( R \) beyond which the theory goes from a phase with spontaneous chiral symmetry breaking to a phase without such breaking 10. The method combines an analysis of the beta function and coupling constant evolution into the infrared with an expression for the critical coupling from an approximation solution of the DS equation, and hence we call it the βDS method.

Let us first consider the theory with \( N_R \) fermions transforming according to a single representation \( R \). If \( N_R \) is sufficiently small that \( b_2 > 0 \), then as the reference scale \( \mu \) decreases from large values, \( \alpha(\mu) \) increases until it exceeds the critical value \( \alpha_{R, cr} \) at which there is the formation of a bilinear condensate of the fermions

\[
\langle \bar{\psi} \psi \rangle \equiv \sum_{j=1}^{\dim(R)} \langle \bar{\psi}_j \psi_j \rangle = \sum_{j=1}^{\dim(R)} \langle \bar{\psi}_{j, L} \psi_{j, R} \rangle + h.c. \quad (2.10)
\]

(For the gauge group SU(2), the condensate can be written in terms of a product of same-chirality fermions, as discussed below.) If \( N_R \) is sufficiently large that \( b_2 < 0 \), then the two-loop beta function has an infrared zero at \( \alpha_{IR} \). The value of \( \alpha_{IR} \) is a monotonically decreasing function of \( N_R \), with partial derivative

\[
\frac{\partial \alpha_{IR}}{\partial N_R} = \frac{12\pi T(R) C_2(G) [7C_2(G) + 11C_2(R)]}{[17C_2(G)^2 - 2N_R(5C_2(G) + 3C_2(R))]^2} \quad (2.11)
\]

If the theory only has one type of fermion representation \( R \), then as \( N_R \) increases through a critical value \( N_{R, cr} \), and the value of \( \alpha_{IR} \) decreases through the critical value \( \alpha_{R, cr} \), the condensate vanishes and the theory goes over to one without any spontaneous breaking of chiral symmetry. Setting

\[
\alpha_{IR} = \alpha_{R, cr} \quad (2.12)
\]

yields a solution for the critical number \( N_{R, cr} \) for this case where the theory has fermions in only one representation, \( R \). Stated in other terms, if \( N_R < N_{R, cr} \), then as the theory evolves into the infrared, \( \alpha(\mu) \) eventually increases above the critical value \( \alpha_{R, cr} \), the fermions condense and gain dynamical masses of order the condensation scale, and the evolution further into the infrared of the low-energy effective theory applicable below this scale is governed by a different beta function. Thus, as noted above, in this case, \( \alpha_{IR} \) is only an approximate infrared fixed
point. Here, with fermions in a single representation, below the condensation scale, the beta function would be that of the pure gauge theory with no fermions, and hence would not have a perturbative infrared fixed point. The only light degrees of freedom in this theory would be the Nambu–Goldstone bosons (NGB’s) resulting from the breaking of the global chiral symmetry by the fermion condensates, and these, being derivatively coupled, become non-interacting as the energy scale goes to zero. If, on the other hand, \( N_R > N_{R,cr} \), then \( \alpha_{IR} < \alpha_{IR,cr} \), so that no condensates form, there is thus no spontaneous chiral symmetry breaking, and \( \alpha_{IR} \) is an exact infrared fixed point. As \( N_R \) increases to \( N_{R,max} \) so that \( b_1 \) decreases to zero, the value of \( b_2 \) approaches a nonzero value, so that \( \alpha_{IR} \to 0 \). The value of \( b_2 \) at \( N_R = N_{R,max} \) is \( N(7N + 11C_2(R)) \).

We next consider the general case of massless fermions transforming according to several different types of representations, denoted, as above, by the set of numbers \( \{N_R\} \). As the reference scale \( \mu \) decreases from large values where the coupling \( \alpha(\mu) \) is small, this coupling increases. There are then two possibilities: (i) \( b_2 > 0 \), so that the two-loop beta function does not have an infrared zero, and the coupling \( \alpha(\mu) \) increases until it exceeds the critical value for fermion condensation; (ii) \( b_2 < 0 \), so that the two-loop beta function does have an infrared zero, and \( \alpha(\mu) \) increases toward this value. Under category (ii) there are two subcategories, just as there were for the case of a single type of fermion representation, (iia) the numbers \( \{N_R\} \) are sufficiently small that \( \alpha_{IR} \) is greater than the critical value for some condensate to form, and (iib) the numbers \( \{N_R\} \) are sufficiently large so that \( \alpha_{IR} \) is less than the critical value for any condensate to form. In cases (iia) and (iib), \( \alpha_{IR} \) is an approximate and exact infrared fixed point, respectively.

Let us assume that the set \( \{N_R\} \) is such that either case (i) or case (iia) holds. Then as the scale \( \mu \) decreases from large values, the coupling \( \alpha(\mu) \) increases sufficiently so that there is condensation in the most attractive channel (MAC). For a channel in which fermions of representations \( R_1 \) and \( R_2 \) form a condensate transforming as \( R_{\text{cond.}} \),

\[
R_1 \times R_2 \to R_{\text{cond.}} \quad (2.13)
\]
a measure of the attractiveness is

\[
\Delta C_2 = C_2(R_1) + C_2(R_2) - C_2(R_{\text{cond.}}) . \quad (2.14)
\]

The maximimization of \( \Delta C_2 \) implies that in a vectorial gauge theory, the most attractive channels are always of the form

\[
R \times \tilde{R} \to 1 \quad (2.15)
\]

for various \( R \), which preserve the gauge invariance. Furthermore, for channels of the form (2.15), \( \Delta C_2 = 2C_2(R) \), so that the criterion for the critical coupling is, in the one-gluon exchange approximation to the DS equation,

\[
\frac{3\alpha \Delta C_2}{2\pi} = \frac{3\alpha C_2}{\pi} = 1 , \quad (2.16)
\]
as in Eq. (2.9). It follows that as the theory evolves from high scales \( \mu \) to lower scales, as \( \alpha(\mu) \) increases, if it exceeds a critical value for condensation, the one-gluon exchange approximation predicts that this will occur first in the channel (2.15) with the largest value of \( C_2(R) \). Let us denote the scale where this occurs as \( \Lambda_1 \).

That is, with this one-gluon approximation to the Dyson–Schwinger equation, the fermion with the largest value of \( C_2(R) \) has the smallest value of \( \alpha_{IR,cr} \) and hence forms a condensate at this highest condensation scale. Associated with this condensation, the fermions transforming according to this representation gain a dynamical mass of order \( \Lambda_1 \). In the low-energy effective field theory that is applicable at scales below \( \Lambda_1 \), these fermions are then integrated out, and the theory evolves in a manner determined by a new beta function, calculated without these fermions.

For sets of numbers \( \{N_R\} \) for which case (iia) holds, we again find that the partial derivative of \( \alpha_{IR} \) with respect to one of the numbers \( N_R \), denoted \( N_{R_i} \), with the others, \( N_{R_j} \) with \( j \neq i \), held fixed, is negative:

\[
\frac{\partial \alpha_{IR}}{\partial N_{R_i}} < 0 . \quad (2.17)
\]

Hence, the same logic applies as before. We can start with a set of fermions \( \{N_R\} \) which is such that \( b_2 < 0 \), so that there is an infrared zero of the two-loop beta function, and the numbers \( N_R \) are sufficiently small that \( \alpha_{IR} \) is large, and the theory forms chiral-symmetry breaking condensates. We can then increase one of the numbers, \( N_{R_i} \), with the others held fixed. As we do this, \( \alpha_{IR} \) decreases, and eventually decreases through the critical value given in Eq. (2.12) for condensation in the channel \( R_i \times \bar{R}_i \to 1 \), at which point this condensate vanishes. The condition in Eq. (2.12) then defines a critical value \( N_{R_i,crit} \). However, in contrast to the simpler case of the theory with fermions in only a single representation, now the critical value \( N_{R_i,cr} \) for a given \( R_i \) depends on the values of the numbers, \( N_{R_j} \), \( j \neq i \), of fermions transforming according to other representations of the gauge group. Another tool that has been applied to analyze chiral symmetry breaking is a conjectured inequality concerning thermal degrees of freedom [31, 32].

### D. Global Chiral Symmetry

For a vectorial SU(\(N\)) theory with \( N \neq 2 \) with massless fermions in a set of representations \( \{R\} \equiv \{R_1, R_2, ..., R_k\} \), such that the numbers of (Dirac) fermions are \( \{N_R\} \equiv N_{R_1}, N_{R_2}, ..., N_{R_k} \}, \) the formal (classical) global chiral symmetry is \( \prod_{i=1}^{k} U(N_{R_i})_L \times U(N_{R_i})_R \). For each \( R_i \), the group \( U(N_{R_i})_L \times U(N_{R_i})_R \) can be rewritten as

\[
SU(N_{R_i})_L \times SU(N_{R_i})_R \times U(1)_{R_i,v} \times U(1)_{R_i,A} . \quad (2.18)
\]

The vectorial global symmetry \( U(1)_{R_i,v} \) represents the conservation of fermion number for the fermions in the
representation $R_i$. Each of the $k$ axial global symmetries $U(1)_{R_i,A}$ is broken by SU($N$) instantons \cite{36}, with divergences of the corresponding axial-vector currents $\partial_{\lambda} J_{R_i}^{A,\lambda} \propto q/(4\pi)T(R_i)F_{\mu\nu}^{a} \tilde{F}^{a,\mu\nu}$. From these $k$ broken symmetries $U(1)_{R_i,A}$, $i = 1, ..., k$, one can construct $k - 1$ linear combinations that are conserved in the presence of instantons, which we denote $U(1)_{s,A}$, $s = 1, ..., k - 1$ with currents $J_{s}^{A,\lambda}$. Let us define

$$J_{R_i}^{A,\lambda} = \frac{J_{R_i}^{A,\lambda}}{T(R_i)} . \quad (2.19)$$

One of the conserved currents is (up to a normalization factor)

$$J_{A}^{A,\lambda} \propto J_{R_1}^{A,\lambda} - J_{R_2}^{A,\lambda} . \quad (2.20)$$

The others are constructed by Gram-Schmidt orthonormalization. For example, for $k = 3$, the other one is

$$J_{2}^{A,\lambda} \propto 1/2 \left[ J_{R_1}^{A,\lambda} + J_{R_2}^{A,\lambda} - 2 J_{R_3}^{A,\lambda} \right] . \quad (2.21)$$

Thus the actual (continuous) nonanomalous global symmetry of the theory, before any fermion condensates form, is

$$G_{global} = \prod_{i=1}^{k} SU(N_{R_i})_{L} \times SU(N_{R_i})_{R} \times U(1)_{R_i,V}$$

$$\times \left[ \prod_{s=1}^{k-1} U(1)_{A,s} \right] . \quad (2.22)$$

The resultant realization of this global symmetry depends on the gauge coupling evolution and whether the coupling $\alpha$ increases above the critical value for condensation of the fermions in the $R_i$ representation. As an example, let us assume that all of the fermions condense, at the respective different scales $\Lambda_{N_{R_i}}$, $i = 1, ..., k$. For our discussion here we shall label the representation with the largest value of the $C_{2}(R_i)$ as $R_i$; from the one-gluon exchange approximation, it then follows that the $R_i$ fermions condense at the highest scale, $\Lambda_i$. In accordance with the most-attractive channel arguments recalled above, the fermion condensate of the form $\langle \bar{\psi}_{R_i} \psi_{R_i} \rangle$ preserves the global $U(1)_{R_i,V}$ and breaks the non-Abelian global symmetry from $SU(N_{R_i})_{L} \times SU(N_{R_i})_{R}$ to its diagonal, vectorial subgroup, $SU(N_{R_i})_{V}$. This condensate also breaks each of the $k - 1$ $U(1)_{s,A}$ axial symmetries. In the low-energy effective field theory applicable at scales $\mu < \Lambda_1$, with the fermions in the $R_i$ representation having gained dynamical masses of order $\Lambda_i$ and having been integrated out, one can construct $k - 2$ appropriate linear combinations of the former $k - 1$ $U(1)_{s,A}$ axial symmetries that exclude the $R_i$ fermions and are preserved in the presence of instantons. We denote these as $U(1)_{s,A}'$. The continuous global symmetry group of this low-energy effective theory below $\Lambda_{R_i}$ is then

$$G'_{global} = \prod_{i=2}^{k} SU(N_{R_i})_{L} \times SU(N_{R_i})_{R} \times \prod_{i=1}^{k} U(1)_{R_i,V} \times \prod_{s=1}^{k-2} U(1)_{A,s}' . \quad (2.23)$$

The number of broken generators of continuous global Lie algebras at the first scale is $N^{2}_{R_i} - 1$ from the breaking of the non-Abelian group, plus one for the breaking of one linear combination of the $k - 1$ nonanomalous axial $U(1)$ symmetries, for a total of $N_{NGB,A_{R_i}} = N^{2}_{R_i}$ Nambu-Goldstone bosons resulting from this first level of fermion condensation. One repeats this process at each of the various condensation scales. The NGB's produced at each level couple derivatively, and hence become progressively more weakly interacting as powers of $\mu/\Lambda_{R_i}$, where $f_{R_i}$ is the generalization of the pion decay constant applicable to the condensation of the $R_i$ fermions.

In the case $N = 2$, because SU(2) has only (pseudo)real representations, the analysis of the global symmetry is different than in the case of SU($N$) with $N \neq 2$. If, for example, one has an SU(2) theory with $N_f$ (Dirac) fermions in the fundamental representation, then one can reexpress these fermions as a set of $2N_f$ chiral (say, left-handed) fermions, and the covariant derivative term has the form $\bar{\psi}_{\alpha} \gamma_{\mu} D_\mu \psi_{\alpha}$, where $\psi$ is a $2N_f$-dimensional vector of left-handed fermions. It follows that the formal (classical) global symmetry in this case is $U(2N_f)_L$, or equivalently, $SU(2N_f)_L \times U(1)_L$. The $U(1)_L$ is broken by the SU(2) instantons \cite{36}, so that the nonanomalous global symmetry is $SU(2N_f)_L$. The condensates are of the form $\langle \bar{\psi}_{\alpha} \gamma_{\mu} \psi_{\alpha} T C \psi_{\beta}^{\dagger} \rangle$, where $\epsilon_{\alpha\beta}$ is the antisymmetric tensor density for SU(2) and $1 \leq p \leq 2N_f$. If the fermions are in the rank-2 symmetric (equivalently, the adjoint) representation, of the form $\psi_{p,L}^{ab}$ with $1 \leq p \leq 2N_f$, then the condensate are of the form $\langle \epsilon_{ab} \bar{\psi}_{p,L}^{\dagger} T C \psi_{p',L}^{\dagger} \rangle$, and so forth for higher-dimensional representations. These condensates break the SU(2$N_f$)$_L$ down to its symplectic subgroup, Sp(2$N_f$)$_L$. In this case there are thus $N_{NGB} = 2N^2_f - N_f - 1$ Nambu-Goldstone bosons.
III. SU(N) GAUGE THEORY WITH FERMIONS IN A SINGLE REPRESENTATION

In this section we review some results on an SU(N) gauge theory with fermions in a single representation, which will serve as a useful background for our analysis of the theory with fermions in multiple different representations.

A. Fundamental Representation

For the SU(N) theory with \( N_F \) Dirac fermions in the fundamental representation \( F \) (= \( \Box \) in Young tableau notation), the condition for asymptotic freedom yields the upper bound \( N_F < N_{F,max} \), where

\[
N_{F,max} = \frac{11N}{2} .
\] (3.1)

The coefficient \( b_2 \) changes sign from positive to negative as \( N_F \) increases through the value

\[
N_{F,IR} = \frac{34N^3}{13N^2 - 3} ,
\] (3.2)

which is always less than \( N_{F,max} \). For \( N_{F,IR} < N_F < N_{F,max} \), the beta function has a zero away from the origin at

\[
\alpha_{IR} = \frac{4\pi(11N - 2N_F)}{-34N^2 + N_F(13N - 3N^{-1})} .
\] (3.3)

The estimate for the critical value for condensation (from Eq. (2.8)) is

\[
\alpha_{F,cr} = \frac{2\pi N}{3(N^2 - 1)} .
\] (3.4)

Setting \( \alpha_{IR} = \alpha_{F,cr} \) and solving for \( N_F \), one obtains the critical value of \( N_F \)

\[
N_{F,cr} = \frac{2N(50N^2 - 33)}{5(5N^2 - 3)} .
\] (3.5)

As \( N \to \infty \), this has the series expansion

\[
N_{F,cr} = N \left[ 1 - \frac{3}{50N^2} - \frac{9}{250N^4} - O \left( \frac{1}{N^6} \right) \right] .
\] (3.6)

For \( N = 2 \), \( N_{F,cr} \approx 8 \) and for \( N = 3 \), \( N_{F,cr} \approx 12 \). Recent lattice measurements for the \( N = 3 \) case are in broad agreement, to within the uncertainties, with this prediction [23].

The DS equation analysis is semi-perturbative in the sense that it contains polynomial dependence on \( \alpha \), and it neglects nonperturbative effects associated with confinement and instantons. The DS equation is an integral equation, and the standard analysis of this equation involves an integration over Euclidean loop momentum \( k \) from \( k = 0 \) to \( k = \infty \). If the theory confines, then the lower bound for the Euclidean loop momentum should actually not be \( k = 0 \), but instead \( k = k_{min} \approx r_c^{-1} \) where \( r_c \) is the spatial confinement scale [9]. The use of \( k = 0 \) thus overestimates the tendency toward SYSB. Instantons enhance SYSB, and the neglect of instanton effects amounts to an underestimate of the tendency toward SYSB; since these two neglected aspects of the physics - confinement and instantons - produce errors that are of opposite sign as regards the tendency for SYSB, it is plausible that these errors tend to cancel out, so this may help to explain why the usual DS analysis may be reasonably accurate [9], at least in the case \( N = 3 \) where recent lattice results are broadly consistent with it.

B. Rank-2 Symmetric and Antisymmetric Representations

In this section we consider the two separate cases of the SU(N) theory with (i) \( N_S \) fermions in the symmetric rank-2 representation, \( S \equiv \Box \Box \) and (ii) \( N_A \) fermions in the antisymmetric rank-2 representation, \( A \equiv \Box \). Since a number of formulas are similar for these two cases, we include them together in this section. In the case of \( R = \Box \Box \) our analysis applies for any \( N \geq 2 \), while for \( R = \Box \) we take \( N \geq 4 \), since for \( N = 2, \Box \) is the singlet and for \( N = 3, \Box \) (is not a distinct representation, but is instead equivalent to \( \Box \)). For the SU(N) theory with \( N_s \) Dirac fermions in the symmetric rank-2 representation \( \Box \Box \), the condition for asymptotic freedom yields the upper bounds \( N_S < N_{S,max} \), where

\[
N_{S,max} = \frac{11N}{2(N + 2)}
\] (3.7)

and \( N_A < N_{A,max} \), where

\[
N_{A,max} = \frac{11N}{2(N - 2)}.
\] (3.8)

The coefficient \( b_2 \) changes sign from positive to negative as \( N_S \) and \( N_A \) increase through the respective values

\[
N_{S,IR} = \frac{17N^3}{(N + 2)(8N^2 + 3N - 6)}
\] (3.9)

and

\[
N_{A,IR} = \frac{17N^3}{(N - 2)(8N^2 - 3N - 6)},
\] (3.10)

which are always less than the respective values \( N_{S,max} \) and \( N_{A,max} \).

For the theory with just \( N_S \) fermions in the \( S \) representation, and \( N_{S,IR} < N_S < N_{S,max} \), the beta function

\[
\alpha_{IR,S} = \frac{2\pi(11N - 2N_S(N + 2))}{-17N^2 + N_S(8N^2 + 19N - 12N^{-1})} .
\] (3.11)
The estimate for the critical value for condensation (from Eq. \(\text{(2.9)}\)) is

\[
\alpha_{S,cr} = \frac{\pi N}{3(N + 2)(N - 1)}. \tag{3.12}
\]

Setting \(\alpha_{IR,S} = \alpha_{S,cr}\) and solving for \(N_S\), we obtain the critical value of \(N_S\),

\[
N_{S,cr} = \frac{N(83N^2 + 66N - 132)}{5(N + 2)(4N^2 + 3N - 6)}. \tag{3.13}
\]

For \(N \rightarrow \infty\), this has the series expansion

\[
N_{S,cr} = \frac{83}{20} - \frac{649}{80N} + \frac{5027}{320N^2} + O\left(\frac{1}{N^3}\right). \tag{3.14}
\]

For \(N = 2\), \(N_{S,cr} \simeq 2.1\), while for \(N = 3\), \(N_{S,cr} \simeq 2.5\). Some lattice measurements for the \(N = 3\) case are reported in [24]. As \(N\) increases from 2 to \(\infty\), \(N_{S,cr}\) increases monotonically from \(83/40 \simeq 2.08\) to \(83/20 = 4.15\). (As before, although we quote the exact fractions and give the floating-point numbers to three significant figures, we emphasize that because of the strong-coupling nature of the physics and the approximations involved, these numbers have estimated theoretical uncertainties of O(1). This applies to all such estimates of \(N_{R,cr}\) values in this paper.)

For the theory with just \(N_A\) fermions in the \(A\) representation, and \(N_{A,IR} < N_A < N_{A,max}\), the beta function \(\text{(2.7)}\) has a zero away from the origin at

\[
\alpha_{IR,A} = \frac{2\pi(11N - 2N_A(N - 2))}{-17N^2 + N_A(8N^2 - 19N + 12N^{-1})}. \tag{3.15}
\]

The estimate for the critical value for condensation (from Eq. \(\text{(2.9)}\)) is

\[
\alpha_{A,cr} = \frac{\pi N}{3(N - 2)(N + 1)}. \tag{3.16}
\]

Setting \(\alpha_{IR,A} = \alpha_{A,cr}\) and solving for \(N_A\), we obtain the critical value

\[
N_{A,cr} = \frac{N(83N^2 - 66N - 132)}{5(N - 2)(4N^2 - 3N - 6)}. \tag{3.17}
\]

For \(N \rightarrow \infty\), this has the series expansion

\[
N_{A,cr} = \frac{83}{20} + \frac{649}{80N} + \frac{5027}{320N^2} + O\left(\frac{1}{N^3}\right). \tag{3.18}
\]

As \(N\) increases from 3 to \(\infty\), \(N_{A,cr}\) decreases monotonically from \(417/35 \simeq 11.9\) to \(83/20 \simeq 4.15\).

C. Adjoint Representation

For the case of \(N_{Adj}\) Dirac fermions, or equivalently, \(2N_{Adj,Maj}\) Majorana fermions, in the adjoint representation \(Adj\), the condition for asymptotic freedom is \(N_{Adj} < N_{Adj,max}\), where

\[
N_{Adj,max} = \frac{11}{4}. \tag{3.19}
\]

i.e., \(N_{Adj} \leq 2\). Majorana fermions in the adjoint representation of the gauge group appear naturally in supersymmetric theories. In the present non-supersymmetric context, we shall restrict ourselves to adjoint fermions of Dirac type. The coefficient \(b_2\) changes sign from positive to negative as \(N_{Adj}\) increases through the value

\[
N_{Adj,IR} = \frac{17}{16}. \tag{3.20}
\]

For \(N_{Adj,IR} < N_{Adj} < N_{Adj,max}\), the beta function has a zero away from the origin at

\[
\alpha_{IR} = \frac{2\pi(11 - 4N_{Adj})}{N(-17N + 16N_{Adj})}. \tag{3.21}
\]

Setting

\[
\alpha_{Adj,cr} = \frac{\pi}{3N} \tag{3.22}
\]

equal to \(\alpha_{Adj,cr}\), one solves for

\[
N_{Adj,cr} = \frac{83}{40} = 2.075. \tag{3.23}
\]

IV. SU(2) GAUGE GROUP

For the simplest non-Abelian Yang-Mills gauge group, SU(2), we can give a rather compact general treatment that includes all possible representations. We recall that this group has only (pseudo)-real representations \(R\), which are labeled by a single Dynkin index, the non-negative integer \(p_1 = 2I\), where \(I\) will be labelled as the “isospin” (not to be confused with the actual gauged weak isospin). \(I = 1/2\) is the fundamental representation, \(\square\): \(I = 1\) is the adjoint or equivalently, rank-2 symmetric representation, \(\square\square\): \(I = 3/2\) is the rank-3 symmetric representation, and so forth. The following SU(2) relations will be useful:

\[
C_2(I) = I(I + 1) \tag{4.1}
\]

and

\[
T(I) = \frac{(2I + 1)(I + 1)}{3}. \tag{4.2}
\]

The asymptotic freedom condition \(\text{(2.21)}\) reads

\[
\sum_I N_I T(I) < \frac{11}{2}, \tag{4.3}
\]

where the sum over \(I\) is formally over all positive integral and half-integral values, but actually truncates, because of fact that \(C_2(I) > 11/2\) for \(I \geq 2\). Hence, \(\text{(4.3)}\) reduces to the Diophantine inequality

\[
\frac{1}{2}N_{1/2} + 2N_1 + 5N_{3/2} < \frac{11}{2}. \tag{4.4}
\]
The nontrivial solutions to this include cases with only one type of fermion representation present. In these cases, the allowed numbers of fermions of each type are

\[ N_{1/2} \leq 10 \]  
\[ N_1 \leq \left[ \frac{11}{4} \right] = 2 \]  
and

\[ N_{3/2} \leq \left[ \frac{11}{10} \right] = 1 \]  

where here \([\nu]\) denotes the greatest integer less than or equal to \(\nu\) and it is understood in each case that the \(N_I\)'s for other \(I\)'s are zero. We also find the following solutions of the asymptotic freedom condition with two different fermion representations present (and \(N_{3/2} = 0\)):

\[ 1 \leq N_{1/2} \leq 6, \quad N_1 = 1 \]  
and

\[ 1 \leq N_{1/2} \leq 2, \quad N_1 = 2 \]  

Substituting the general result for \(C_2(I)\) in Eq. (4.1) in Eq. (2.9), we have, in this approximation,

\[ \alpha_{IR, I=3/2} = \frac{8\pi(11 - 10N_{3/2})}{17(25N_{3/2} - 8)} \]  

Finally, among the cases with a single fermion representation present, there is the case of fermions with \(I = 3/2\). For this case, \(b_2\) reverses sign from positive to negative as \(N_{3/2}\) increases through the value \(N_{3/2} = 8/25 = 0.32\). The two-loop beta function has a zero away from the origin at

\[ \alpha_{IR, I=3/2} = \frac{8\pi(11 - 10N_{3/2})}{17(25N_{3/2} - 8)} . \]  

Setting this equal to \(\alpha_{cr, I=3/2} = 4\pi/45 \simeq 0.28\), we get the critical value

\[ N_{cr, I=3/2} = \frac{1126}{1325} \simeq 0.85 \]  

Since \(\alpha_{IR}\) decreases with increasing \(N_{3/2}\) and since the minimal nonzero value is \(N_{f, cr, I=3/2} = 1\), this predicts that with one such Dirac fermion with \(I = 3/2\), the infrared fixed point is below the value for condensation and hence is an exact IR fixed point. That is, the gauge coupling will evolve to this point without any condensate involving the \(I = 3/2\) forming, so that it does not gain any dynamical mass and remains massless. Thus, in the infrared limit of this theory the fermion is massless. We summarize our results for SU(2) in Table I

### A. SU(2) Theory with Fermions in Several Representations

We next consider the SU(2) theory with fermions in several different representations. As discussed above, the requirement of asymptotic freedom limits the possible numbers of fermions, delineated by the numbers \(N_{1/2}, N_1, N_{3/2}\). For the case of \(N_{3/2} = 0\), we have

\[ b_2 = \frac{1}{3} \left[ 136 - \frac{49N_{1/2}}{2} - 128N_1 \right] . \]  

For \(N_1 = 1\) and \(N_{1/2} = 0, 1\), it follows that \(b_2 > 0\), so that the beta function has no infrared zero away from the origin. This means that as the scale \(\mu\) decreases, \(\alpha\) increases until it exceeds the value \(\alpha_{cr, I=1}\), and the \(I = 1\) fermions condense. They are then integrated out, and the theory evolves further into the infrared as governed by the beta function with only the \(I = 1/2\) fermions present. The coupling thus increases further until it exceeds the value \(\alpha_{cr, I=1/2}\), at which point these \(I = 1/2\) fermions condense.

For \(N_1 = 1\) and \(1 \leq N_{1/2} \leq 6, b_2 < 0\) and so the beta function has an infrared zero away from the origin, at

\[ \alpha_{IR, I=1/2} = \frac{16\pi(11 - N_{1/2} - 4N_1)}{49N_{1/2} + 256N_1 - 272} . \]  

If this is less than the critical value \(2.9\), then no fermion condensates form. Setting this \(\alpha_{IR}\) equal to the smaller
of the two critical values, \( \alpha_{cr,I=1} \), one derives the condition for condensation of the \( I = 1 \) fermions. This is
\[
N_{1/2} + \frac{128}{29} N_1 < \frac{1328}{145} \approx 9.16 .
\] (4.17)

In addition to the cases with \( N_1 = 0 \) dealt with above, this condition is satisfied for \( N_1 = 1 \) and \( 1 \leq N_{1/2} \leq 4 \).

For \( N_1 = 1 \) and \( N_{1/2} = 5 \), \( \alpha_{IR} = 0.44 \), which is close enough to \( \alpha_{cr} = \pi/6 = 0.52 \) so that, given the uncertainties in the calculation, there might or might not be condensation of the \( I = 1 \) fermions. For \( N = 1 \) and \( N_{1/2} = 6 \), \( \alpha_{IR} = 0.18 \), which is below the value for condensation of both the \( I = 1 \) and \( I = 1/2 \) fermions. Hence, in this case, this is an exact infrared fixed point, and the theory evolves into the infrared without any spontaneous chiral symmetry breaking.

For \( N_1 = 2 \), the condition of asymptotic freedom, \( 2N_{1/2} + 8N_1 < 22 \), is \( N_{1/2} < 3 \). Aside from the case \( N_{1/2} = 0 \) dealt with above, for the cases \( N_{1/2} = 1 \) and \( N_{1/2} = 2 \), the two-loop beta function has an infrared zero at the respective values \( \alpha = 32\pi/289 \approx 0.35 \) and \( \alpha = 8\pi/169 \approx 0.15 \), both of which are smaller than the estimate \( \alpha_{cr,I=1} = \pi/6 \), so that the \( \beta DS \) analysis predicts that no condensate occurs and the theory evolves into the infrared in a phase without any spontaneous chiral symmetry breaking.

V. SU(3) GAUGE GROUP

It is also of interest to investigate properties of a (vectorial, asymptotically free) SU(3) gauge theory with multiple fermion representations. We recall that the representations of SU(3) are labelled by a set of two Dynkin indices \((p_1, p_2)\), where \( p_i \) are non-negative integers. We use the following results from group theory. The dimension of the representation is
\[
\text{dim}(p_1, p_2) \equiv d(p_1, p_2) = (1+p_1)(1+p_2)
\left(1 + \frac{p_1 + p_2}{2}\right).
\] (5.1)

The quadratic Casimir invariant is
\[
C_2(p_1, p_2) = \frac{1}{3} \left[ p_1^2 + p_2^2 + p_1 p_2 + 3(p_1 + p_2)\right] .
\] (5.2)

and the trace invariant is
\[
T(p_1, p_2) = \frac{\text{dim}(p_1, p_2) C_2(p_1, p_2)}{8} .
\] (5.3)

The asymptotic freedom condition \([\text{27}]\) reads
\[
\sum_R N_R T(R) < \frac{33}{4} ,
\] (5.4)

where, again, the sum over representations truncates because for sufficiently large values of \( p_1 \) and/or \( p_2 \), \( T(p_1, p_2) > 33/4 \). We find that it is satisfied by the following nonsinglet representations labelled by their dimension and values of \((p_1, p_2)\):

\[
R_{(p_1, p_2)} = 3(1,0), 6(2,0), 8(1,1), 10(3,0). \] (5.5)

The asymptotic freedom condition \((5.4)\) thus can be written explicitly as the Diophantine inequality
\[
\frac{1}{2} N_3 + \frac{5}{2} N_6 + 3N_8 + 152 N_{10} < \frac{33}{4} .
\] (5.6)

In the case where the theory has fermions in only one of these representations \( R = (p_1, p_2) \), the upper bounds on the corresponding number \( N_R \) are, in addition to \( N_3 \leq \frac{33}{2} \),
\[
N_6 \leq \left[ \frac{33}{10} \right] = 3
\] (5.7)
\[
N_8 \leq \left[ \frac{11}{4} \right] = 2
\] (5.8)

and
\[
N_{10} \leq \left[ \frac{11}{10} \right] = 1 .
\] (5.9)

In each of these inequalities, it is understood that the \( N_R \)'s for other representations are zero.

For the case of multiple fermion representations, we find that the asymptotic freedom condition is satisfied for the following combinations of two fermion representations (where \( N_R \)'s that do not appear are zero):

\[
1 \leq N_3 \leq 11, \quad N_6 = 1
\] (5.10)
\[
1 \leq N_3 \leq 6, \quad N_6 = 2
\] (5.11)
\[
N_3 = 1, \quad N_6 = 3
\] (5.12)
\[
1 \leq N_3 \leq 10, \quad N_8 = 1
\] (5.13)
\[
1 \leq N_3 \leq 4, \quad N_8 = 2
\] (5.14)
\[
1 \leq N_6 \leq 2, \quad N_8 = 1
\] (5.15)

and

\[
N_3 = 1, \quad N_{10} = 1 .
\] (5.16)

We also find the asymptotic freedom condition allows the following combination of three fermion representations:

\[
1 \leq N_3 \leq 5, \quad N_6 = 1, \quad N_8 = 1 .
\] (5.17)

It is straightforward to calculate the values of \( b_2 \) for each of the various sets \( \{N_R\} \) involving one or several different fermion representations. As before, if \( b_2 > 0 \), then \( \alpha \) definitely increases past \( \alpha_{R,cr} \) for at least one of the fermion representations \( R \), and one analyzes the sequential condensations accordingly. If \( b_2 < 0 \), then one determines whether the behavior is of type (ia) or (ib) in the classification discussed above, i.e. whether \( \alpha_{IR} \) is greater than \( \alpha_{R,cr} \) for some \( R \) or \( \alpha_{IR} \) is less than the minimum \( \alpha_{R,cr} \). All of these types of behavior are exhibited by various sets \( \{R\} \) among those allowed by asymptotic freedom.
VI. RELATIVE SCALES OF CONDENSATION

In an SU(N) gauge theory with $N_R$ fermions in a single representation $R$ and with a small, perturbatively calculable value of $\alpha(\mu_{UV})$ at some high scale, $\mu_{UV}$, provided that $N_R$ is sufficiently small that there exists a scale $\mu = \Lambda_R$ at which $\alpha(\mu)$ increases beyond the critical value $\alpha_{R,cr}$ for condensation, then one can estimate this scale by integrating the renormalization group equation, with the leading-order result

$$\Lambda_R \simeq \mu_{UV} \exp \left[ -\frac{2\pi}{b_1(R)} \left( \alpha(\mu_{UV})^{-1} - \alpha_{R,cr}^{-1} \right) \right], \quad (6.1)$$

where we have indicated explicitly the dependence of the beta function coefficient $b_1 = (1/3)(11N_c - 4N_R T(R))$ on $R$. One can, of course, calculate $\Lambda_R$ to greater accuracy by including higher-order terms in the beta function, as well as estimates of important physics effects not included in the perturbative beta function, such as instantons, but this leading-order result will be sufficient for our discussion here. From Eq. (6.1), it follows that if one compares an SU(N) theory with fermions in the single representation $R_i$ with a different SU(N) theory with fermions in the single representation $R_j$, the ratio of the condensation scales, $\Lambda_{R_i}/\Lambda_{R_j}$, depends on all of the parameters $\mu_{UV}$, $N_{R_i}$, and $N_{R_j}$, as well as the ladder estimates for the respective critical couplings, $\alpha_{R_i,cr}$ and $\alpha_{R_j,cr}$. For fixed values of $\mu_{UV}$, $\alpha_{UV}(\mu)$, and $N_{R_i}$, one may ask how $\Lambda_{R_i}/\mu_{UV}$ depends on $R_i$. There are two countervailing effects that are relevant here: (i) as the dimension $\dim(R_i)$ of a representation $R_i$ increases, the value of $C_2(R_i)$ also tends to increase (although the dependence is not necessarily monotonic), and hence the critical value of the coupling, $\alpha_{R_i,cr}$ decreases; if this increase were the only effect, then $\Lambda_{R_i}/\mu_{UV}$ would increase with increasing size of $R_i$. However, there is an effect that goes in the opposite direction, namely, (ii) as the dimension $\dim(R_i)$ of the representation $R_i$ increases, the value of $T(R_i)$ also increases, thereby reducing $b_1(R_i)$, and slowing down the increase of $\alpha$ as $\mu$ descends from $\mu_{UV}$. Indeed, a sufficient increase in the size of the representation $R_i$ for a fixed $N_{R_i}$ can even change the infrared behavior of the theory to preclude any spontaneous chiral symmetry breaking and condensate formation. Thus, for a general $R_i$, one cannot draw a very robust conclusion about how, for fixed values of $\mu_{UV}$, $\alpha(\mu_{UV})$, and $N_{R_i}$, the condensation scale $\Lambda_{R_i}$ depends on the size of $R_i$.

In situations in which the theory has fermions in two or more different representations and these form condensates at different mass scales, it is of interest to calculate the ratio(s) of these scales. In carrying out this analysis, one acknowledges that, owing to the fact that the theory is strongly coupled at these scales, it is only possible to obtain rough estimates of such a ratio of condensation scales. Let us consider the SU(N) theory with the specific set of Dirac fermions $\{N_R\} = \{N_{R_1}, N_{R_2}\}$, say, where the $R_i$, $i = 1, 2$ are two different (non-singlet) representations of SU(N). Without loss of generality, we label the representations such that $C_2(R_1) > C_2(R_2)$. As always, we require that this set $\{N_R\}$ have the property that the theory is asymptotically free, and here we also require that the set is such that condensates of both types of fermions occur, since otherwise there is no ratio to estimate. Again, we assume that at the high reference scale $\mu_{UV}$ the coupling $\alpha(\mu_{UV})$ is small and the theory is perturbatively calculable. As $\mu$ decreases from $\mu_{UV}$, the first condensation occurs when $\alpha(\mu) = \alpha_{R_i,cr}$, where $\alpha_{R_i,cr}$ was given in Eq. (2.9), from the solution of the Dyson-Schwinger equation in the approximation of one-gauge-boson exchange. Solving the renormalization group equation to leading order, we have, for the scale at which this condensation occurs the result

$$\Lambda_1 \simeq \mu_{UV} \exp \left[ -\frac{2\pi}{b_1} \left( \alpha(\mu_{UV})^{-1} - \alpha_{R_i,cr}^{-1} \right) \right],$$

from Eq. (6.1),

$$\Lambda_1 \simeq \mu_{UV} \exp \left[ -\frac{2\pi}{b_1} \left( \alpha(\mu_{UV})^{-1} - \frac{3C_2(R_1)}{\pi} \right) \right], \quad (6.2)$$

where $b_1$ is given by the appropriate special case of Eq. (6.1), with the full set $\{N_{R_1}, N_{R_2}\}$ of fermions. The $N_{R_i}$ fermions in the condensates gain dynamical masses of order $\Lambda_1$ and are integrated out of the low-energy effective field theory applicable for $\mu < \Lambda_1$. The coupling $\alpha(\mu)$ continues to grow, as governed by the beta function of this low-energy effective theory, which differs from that of the high-scale theory by the removal of the $N_{R_i}$ fermions in the representation $R_1$. Insofar as the coupling $\alpha$ is not too large to prevent one from using the perturbative beta function to track its evolution reliably for $\mu < \Lambda_1$, one has

$$\alpha(\mu)^{-1} = \alpha^{-1}(\Lambda_1) + \frac{b_1(R_2)}{2\pi} \ln \left( \frac{\Lambda_1}{\mu} \right), \quad (6.3)$$

where $b_1(R_1)$ is the value of $b_1$ from Eq. (2.2) for the low-energy effective field theory with only $N_{R_i}$ fermions in $R_2$ present. Then, given our assumptions about the set $\{N_{R_1}, N_{R_2}\}$, at a lower scale $\Lambda_2$, condensation occurs for the fermions in the representation $R_2$, when $\alpha(\mu) = \alpha_{R_2,cr}$. Solving for the ratio of these two condensations in this rough approximation, we obtain

$$\frac{\Lambda_2}{\Lambda_1} \simeq \exp \left[ -\frac{6}{b_1(R_2)} \left( C_2(R_1) - C_2(R_2) \right) \right], \quad (6.4)$$

where $b_1(R_2) = (1/3)(11N - 4N_{R_2}T(R_2))$. As an example, consider the SU(2) theory with $R_1$ and $R_2$ being the $I = 1$ and $I = 1/2$ representations, respectively, and numbers $N_{R_1} = 1$ and of $N_{R_2} = 1/2$ for which there are two condensations, as indicated in Table I. Then

$$\frac{\Lambda_{1/2}}{\Lambda_1} \simeq \exp \left[ -\frac{45}{4(11 - N_{1/2})} \right]. \quad (6.5)$$

As $N_{1/2}$ increases from 2 to 4, this ratio $\Lambda_{1/2}/\Lambda_1$ decreases from about 0.3 to 0.2. These are comparable to
the sort of ratios of condensation scales that would characterize the sequential breaking of reasonably ultraviolet-complete extended-technicolor theories (e.g., [40, 41]).

VII. EFFECTS OF NONZERO INTRINSIC MASSES FOR FERMIONS

In the discussion up to this point we have assumed that the fermions have zero intrinsic masses in the Lagrangian describing the high-scale physics, and the only masses that they acquire arise dynamically if they are involved in condensates that form as the gauge interaction becomes sufficiently strongly coupled in the infrared. This is a well-motivated assumption if the vectorial gauge theory arises as a low-energy effective field theory from an ultraviolet completion which is a chiral gauge theory. This is natural if the latter theory becomes strongly coupled, since it can then form fermion condensates that self-break it down to the vectorial subgroup symmetry. However, one may also choose to focus on the vectorial gauge theory as an ultraviolet-complete theory in itself. In a vectorial gauge theory, an intrinsic (bare) mass term for a fermion ψ, \( \mathcal{L}_m = -m \bar{\psi} \psi \), is allowed by the gauge invariance. (For an SU(2) theory, with fermions written as left-handed chiral fields, the gauge-invariant mass term can be expressed in a Majorana form, e.g., for the fundamental representation, \( m^i \psi^i_L T C \psi^i_R \).) Hence, one may consider a more general situation in which the fermions may have such intrinsic masses in the high-scale Lagrangian. Quantum chromodynamics (QCD) provides an example of this, in which the quarks have hard (also called current-quark) masses \( m_\text{u} \) that span a large range, from \( m_\text{u} \) of a few MeV to \( m_\text{t} \simeq 172 \) GeV. In particular, this range extends both far below and far above, the scale \( \Lambda_{\text{QCD}} \simeq 300 \) MeV where the QCD coupling \( \alpha_s(\mu) \) becomes \( O(1) \) and the theory confines and spontaneously breaks chiral symmetry.

The main effect of intrinsic fermion masses here is the same as in QCD; as the reference scale \( \mu \) decreases below the value of such a mass of some fermion \( m_f \), the beta function changes from one that includes this to one that excludes this in the set of light, active fermions. For a theory with a set \( \{N_R\} \) such that \( b_2 < 0 \) at a high scale, and hence evolution toward an approximate or exact infrared fixed point, the reduction of one or more numbers \( N_R \) can reverse the sign of \( b_2 \), making it positive and removing this infrared fixed point. Indeed, in principle, a theory could have sufficiently large numbers of fermions in various representations \( \{N_R\} \) that it is not asymptotically free at a high energy scale above the fermion masses, but as this scale decreases below some of these masses, the modified beta function describing the gauge coupling evolution in the result low-energy effective field theory is asymptotically free.

VIII. CONCLUSIONS

In this paper we have studied the evolution of an asymptotically free vectorial SU(\( N \)) gauge theory from high scales to the infrared and the resultant phase structure in the general case in which the theory contains fermions transforming according to several different representations of the gauge group. Using information from the beta function and results from an approximate analysis of the Dyson-Schwinger equation for the fermion(s), we have investigated examples that illustrate a wide range of possible behavior. In one type of model, the theory contains sufficiently few fermions that the coupling \( \alpha \) increases as the reference scale decreases, but the 2-loop beta function does not have an infrared zero away from the origin. In this case, as \( \alpha \) increases and exceeds a critical value for the formation of a condensate of fermions with the largest \( C_2(R) \), this forms, the fermions gain dynamical masses, and these fermions are then integrated out of the low-energy effective field theory applicable below this highest condensation scale. In the low-energy theory, the coupling \( \alpha \) continues to evolve, but according to a different beta function, and there is then condensation of the fermions with the next largest value of \( C_2(R) \), and so forth. In another type of model, the theory contains enough fermions in various representations that the beta function does have an infrared zero. In this case, there are two main categories of behavior. In one type, the value of \( \alpha_{IR} \) is larger than the critical value for condensation of the fermions with the largest \( C_2(R) \), so this condensation occurs, and is followed by sequential condensation(s) at lower scales. In a second type, the value of \( \alpha_{IR} \) is sufficiently small that there are no condensates formed, there is no spontaneous chiral symmetry breaking, and \( \alpha_{IR} \) is an exact infrared fixed point of the renormalization group. We have given explicit examples of each of these types of behavior in the case of an SU(2) gauge theory. We have also briefly discussed the effects of nonzero intrinsic fermion masses.

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[1] No confusion should result from the use of \( R \) for both representation and the right-handed chiral component of a fermion; the meaning will be clear from the context.

[2] K. Lane and E. Eichten, Phys. Lett. B 222, 274 (1989); D. Hong, S. Hsu, and F. Sannino, Phys. Lett. B 507, 89 (2004); F. Sannino and K. Tuominen, Phys. Rev. D 71, 051901(R) (2005); N. D. Christensen and R. Shrock, Phys. Lett. B 632, 92 (2006); D. D. Dietrich and F. Sannino, Phys. Rev. D 75, 085018 (2007); R. Faedi, M. T. Frandsen, T. A. Ryttov, and F. Sannino, Phys. Rev. D 76, 055005 (2007); F. Sannino, Phys. Rev. D 79, 096007 (2009); T. A. Ryttov and F. Sannino, arXiv:0906.0307.
Our normalizations for the quadratic Casimir and trace invariants of a Lie group are standard. The quadratic Casimir invariant $C_2(R)$ for the representation $R$ is given by\[ \sum_{a=1}^{g} \sum_{j=1}^{d_{i}} [D_R(T_a)]_{ij} [D_R(T_a)]_{ik} = C_2(R) \delta_{hk}, \]
where $a, b$ are group indices, $\rho(G)$ is the order of the group, $T_a$ are the generators of the associated Lie algebra, and $D_R(T_a)$ is the matrix form of the $T_a$ in the representation $R$. The trace invariant $T(R)$ is defined by \[ \sum_{i=1}^{d_{R}} [D_R(T_a)]_{ii} = T(R)(\delta_{ab}). \]

[3] F. Sannino, arXiv:0901.0931

[4] See, e.g., G. Karl, Phys. Rev. D 14, 2374 (1976); W. J. Marciano, Phys. Rev. D 21, 2425 (1980) and references therein.

[5] D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); G. ’t Hooft, unpublished.

[6] W. E. Cassey, Phys. Rev. Lett. 33, 244 (1974); D. R. T. Jones, Nucl. Phys. B 75, 531 (1974).

[7] Our normalizations for the quadratic Casimir and trace invariants of a Lie group are standard. The quadratic Casimir invariant $C_2(R)$ for the representation $R$ is given by
\[ \sum_{a=1}^{g} \sum_{j=1}^{d_{i}} [D_R(T_a)]_{ij} [D_R(T_a)]_{ik} = C_2(R) \delta_{hk}, \]
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\[ \sum_{i=1}^{d_{R}} [D_R(T_a)]_{ii} = T(R)(\delta_{ab}). \]

[8] T. Banks and A. Zaks, Nucl. Phys. B 196, 189 (1982).

[9] S. Brodsky and R. Shrock, Phys. Lett. B 666, 95 (2008).

[10] S. J. Brodsky, G. F. de Téramond, and A. Deur, arXiv:1002.3948

[11] M. Creutz, unpublished.

[12] See, e.g., M. Kurachi, R. Shrock, and K. Yamawaki, Phys. Rev. D 76, 035003 (2007); F. Sannino, arXiv:0803.0829 and references therein.

[13] K. Lane, Phys. Rev. D 10, 1353, 2605 (1974).

[14] H. D. Politzer, Nucl. Phys. B 117, 397 (1976).

[15] T. Appelquist, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. Lett. 57, 957 (1986); T. Appelquist and L. C. R. Wijewardhana, Phys. Rev. D 35, 774 (1987); Phys. Rev. D 36, 568 (1987).

[16] T. Appelquist, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. Lett. 77, 1214 (1996).

[17] T. Appelquist, K. Lane, and U. Mahanta, Phys. Rev. Lett. 61, 1553 (1988).

[18] B. Holdom, Phys. Lett. B150, 301 (1985); K Yamawaki, M. Bando, and K. Matumoto, Phys. Rev. Lett. 56, 1335 (1986).

[19] J. B. Kogut, M. Stone, H. W. Wyld, S. H. Shenker, J. Shigemitsu, and D. K. Sinclair, Nucl. Phys. B225, 326 (1983); J. B. Kogut, J. Shigemitsu, and D. K. Sinclair, Phys. Lett. B138, 283 (1984).

[20] J. B. Kogut, M. Stone, H. W. Wyld, J. Shigemitsu, S. H. Shenker, D. K. Sinclair, Phys. Rev. Lett. 48, 1140 (1982); J. B. Kogut, J. Shigemitsu, and D. K. Sinclair, Phys. Lett. B145, 239 (1984).

[21] Y. Iwasaki, K. Kanaya, S. Sakai, and T. Yoshi, Phys. Rev. Lett. 69, 21 (1992); Y. Iwasaki et al., Phys. Rev. D 69, 014507 (2004).

[22] P. H. Damgaard, U. M. Heller, A. Krasnitz, and P. Olsen, Phys. Lett. B 400, 169 (1997).

[23] T. Appelquist, G. Fleming, and E. Neil, Phys. Rev. Lett. 100, 171607 (2008); Phys. Rev. D 79, 076010 (2009); T. Appelquist et al., Phys. Rev. Lett. 104, 071601 (2010); A. Deuzeman, M. P. Lombardo, and E. Pallante, Phys. Lett. B 670, 41 (2008); arXiv:0804.4662

[24] S. Catterall, J. Giedt, F. Sannino, and J. Schneible, JHEP 0811, 009 (2008); arXiv:0910.4387

[25] S. Weinberg, Phys. Rev. D 19, 1277 (1979); L. Susskind, Phys. Rev. D 20, 2619 (1979).

[26] S. Dimopoulos and L. Susskind, Nucl. Phys. B155, 237 (1979); E. Eichten and K. Lane, Phys. Lett. B90, 125 (1980).

[27] Some examples are T. Appelquist and J. Terning, Phys. Rev. D 50, 2116 (1994); T. Appelquist and R. Shrock, Phys. Lett. B 548, 204 (2002); T. Appelquist and R. Shrock, Phys. Rev. Lett. 90, 201801 (2003); T. Appelquist, M. Piai, and R. Shrock, Phys. Rev. D 69, 015002 (2004); T. A. Ryttov and R. Shrock, arXiv:0904.2076

[28] C. Hill and E. Simmons, Phys. Rep. 381, 235 (2003). J. Martin and K. Lane, Phys. Rev. D 71, 015011 (2005).

[29] R. S. Chivukula, M. Narain, and J. Womersley, Phys. Lett. B 667, 1258 (2008) and http://pdg.lbl.gov/

[30] See talks in Workshop on Dynamical Electroweak Symmetry Breaking, Southern Denmark University, 2008, http://hep.sdu.dk/dewsb

[31] G. ’t Hooft, in Recent Developments in Gauge Theories (1979 Cargèse Summer Institute) (Plenum, New York, 1980), p. 135.

[32] A. Casher, Phys. Lett. B 83, 395 (1979).

[33] Reviews include K. Intrilligator and N. Seiberg, in D. E. Soper, ed., QCD and Beyond: Proceedings of the Theoretical Advanced Study Institute in Elementary Particle Physics, 1995 (World Scientific, Singapore, 1996), p. 223

[34] T. Appelquist, A. G. Cohen, and M. Schmaltz, Phys. Rev. D 60, 045003 (1999).

[35] T. Appelquist, A. G. Cohen, M. Schmaltz, and R. Shrock, Phys. Lett. B 459, 235 (1999).

[36] G. ’t Hooft, Phys. Rev. Lett. 37, 8 (1976); Phys. Rev. D 14, 3432 (1978).

[37] Note that, in contrast to the case of SU(2), where the value of $C_2(I)$ increases monotonically with the dimension $\dim(R_I) = 2I + 1$, this is not the case with SU(3); for example, the rank-2 symmetric representation with $(p_1, p_2) = (2, 0)$ has a smaller dimension, $\dim(R_{2,0}) = d(2,0) = 6$, than the adjoint representation $(1,1)$, with $d(1,1) = 8$, but has a larger value of $C_2$: $C_2(2,0) = 10/3$, while $C_2(1,1) = 3$.

[38] Here a “hard” quark mass is a mass that would remain in the hypothetical limit in which one turned off color interactions. If these masses are dynamically generated, then they are actually soft on the higher scale where they arise.

[39] N. D. Christensen and R. Shrock, Phys. Rev. Lett. 94, 241801 (2005).

[40] T. Appelquist and R. Shrock, Phys. Lett. B 548, 204 (2002); Phys. Rev. Lett. 90, 201801 (2003).

[41] T. Appelquist, M. Piai, and R. Shrock, Phys. Rev. D 69,
TABLE I: Some numerical results for the SU(2) theory. IRFP denotes an (exact or approximate) infrared fixed point of the renormalization group equation for $\alpha$. nIRFP means that the two-loop beta function does not have such an IRFP, i.e., a zero away from the origin. In the columns marked $c_I$ for $I = 1/2$, 1 we indicate with a y (yes) or n (no) whether the $\beta_{DS}$ method with the one-gluon (ladder) approximation to the DS equation, predicts that there is condensation of the isospin $I$ fermions. The notation $m$ means “maybe”, reflecting the substantial theoretical uncertainties in the $\beta_{DS}$ predictions due to the strong-coupling nature of the physics. If the theory with all of its massless fermions has a IRFP, this is marked as $\alpha_{IR,h}$, where $h$ stands for “highest-scale”. If the low-energy effective field theory applicable for energies below the highest condensation scale has an IRFP, this is denoted $\alpha_{IR,\ell}$, where $\ell$ stands for “lower scale”. In cases where no condensation occurs for any of the isospin $I$ fermions, $\alpha_{IR,h} = \alpha_{IR,\ell}$.

| $N_{1/2}$ | $N_1$ | $\alpha_{IR,h}$ | $\alpha_{IR,\ell}$ | $c_{1/2}$ | $c_1$ |
|----------|-------|----------------|-----------------|---------|-------|
| 1        | 0     | nIRFP         | - y             | -       | -     |
| 2        | 0     | nIRFP         | - y             | -       | -     |
| 3        | 0     | nIRFP         | - y             | -       | -     |
| 4        | 0     | nIRFP         | - y             | -       | -     |
| 5        | 0     | mIRFP         | - y             | -       | -     |
| 6        | 0     | 11.4          | - y             | -       | -     |
| 7        | 0     | 2.83          | - y             | -       | -     |
| 8        | 0     | 1.26          | - m             | -       | -     |
| 9        | 0     | 0.59          | - n             | -       | -     |
| 10       | 0     | 0.23          | - n             | -       | -     |
|          | 0     | nIRFP         | - - y           | -       | -     |
| 1        | 1     | 9.14          | IRFP y          | y y     | -     |
| 2        | 1     | 3.06          | IRFP y          | y y     | -     |
| 3        | 1     | 1.53          | nIRFP y         | y y     | -     |
| 4        | 1     | 0.84          | nIRFP y         | y y     | -     |
| 5        | 1     | 0.44          | mIRFP m         | m m     | -     |
| 6        | 1     | 0.18          | 0.18 n          | n n     | -     |
| 0        | 2     | 1.26          | - - y           | -       | -     |
| 1        | 2     | 0.59          | nIRFP y         | m       | -     |
| 2        | 2     | 0.23          | 0.23 n          | n n     | -     |