SEPARATE CONTINUITY OF THE LEMPERT FUNCTION
OF THE SPECTRAL BALL

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Abstract. We find all matrices \( A \) from the spectral unit ball \( \Omega_n \) such that the Lempert function \( l_{\Omega_n}(A, \cdot) \) is continuous.

The characteristic polynomial of a \( n \times n \) complex matrix \( A \) is

\[
P_A(t) := \det(tI_n - A) = t^n + \sum_{j=1}^{n} (-1)^j \sigma_j(A)t^{n-j},
\]

where \( I_n \) is the unit matrix. Let \( r(A) := \max\{|\lambda| : P_A(\lambda) = 0\} \) be the spectral radius of \( A \). The spectral unit ball is the pseudoconvex domain \( \Omega_n := \{A : r(A) < 1\} \).

Let \( \sigma(A) := (\sigma_1(A), \ldots, \sigma_n(A)) \). The symmetrized polydisk is the bounded domain \( G_n := \sigma(\Omega_n) \subset \mathbb{C}^n \), which is hyperconvex (see [3]) and hence taut.

We are interested in two-point Nevanlinna–Pick problems with values in the spectral unit ball, so let us consider the Lempert function of a domain \( D \subset \mathbb{C}^m \): for \( z, w \in D \),

\[
l_D(z, w) := \inf\{ |\alpha| : \exists \varphi \in \mathcal{O}(\mathbb{D}, D) : \varphi(0) = z, \varphi(\alpha) = w \},
\]

where \( \mathbb{D} \subset \mathbb{C} \) is the unit disk. For general facts about this function, see for instance [4]. The Lempert function is symmetric in its arguments, upper semicontinuous and decreases under holomorphic maps, so for \( A, B \in \Omega_n \),

\[
(1) \quad l_{\Omega_n}(A, B) \geq l_{G_n}(\sigma(A), \sigma(B)).
\]

The domain \( G_n \) is taut, so its Lempert function is continuous.

The systematic study of the relationship between Nevanlinna–Pick problems valued in the symmetrized polydisk or spectral ball began with [1]. In particular, it showed that when both \( A \) and \( B \) are cyclic (or non-derogatory) matrices, i.e. they admit a cyclic vector (see other equivalent properties in [5]), then equality holds in (1). It follows that \( l_{\Omega_n} \) is continuous on \( \mathcal{C}_n \times \mathcal{C}_n \), where \( \mathcal{C}_n \) denotes the (open) set of cyclic matrices. On the other hand, in general, if equality holds in (1) at \((A, B)\), then \( l_{\Omega_n} \) is continuous at \((A, B)\).

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(see [6] Proposition 1.2). The converse is also true, since \( l_{\Omega_n} \) is an upper semicontinuous function, \( l_{\Omega_n}^\pm \) is a continuous function and (1) holds.

The goal of this note is to study the continuity of \( l_{\Omega_n} \) separately with respect to each argument. In [6], the authors looked for matrices \( B \) such that \( l_{\Omega_n}(A,\cdot) \) is continuous at \( B \) for any \( A \). They conjecture that this holds for any \( B \in \mathcal{C}_n \), and prove it for \( n \leq 3 \) [6 Proposition 1.4], and the converse statement for all dimensions (see [6 Theorem 1.3]).

In the present paper, we ask for which \( A \) the function \( l_{\Omega_n}(A,\cdot) \) is continuous at \( B \) for any \( B \) (or simply, continuous on the whole \( \Omega_n \)). By [2, Proposition 4], for any matrix \( A \in \mathcal{C}_n \) with at least two different eigenvalues, the function \( l_{\Omega_n}(A,\cdot) \) is not continuous at any scalar matrix. On the other hand, \( l_{\Omega_n}(0, B) = r(B) \) and hence \( l_{\Omega_n}(A,\cdot) \) is a continuous function for any scalar matrix \( A \) (since the automorphism \( \Phi_\lambda(X) = (X - \lambda I)(I - \lambda X)^{-1} \) of \( \Omega_n \) maps \( \lambda l_n \) to 0, where \( \lambda \in \mathbb{D} \)).

We have already mentioned that if \( A \in \Omega_n \) \((n \geq 2)\), then the following conditions are equivalent:

(i) the function \( l_{\Omega_n} \) is continuous at \((A,B)\) for any \( B \in \Omega_n \);
(ii) \( l_{\Omega_n}(A,\cdot) = l_{\Omega_n}(\sigma(A),\sigma(\cdot)) \).

Consider also the condition

(iii) \( A \in \mathcal{C}_2 \) has two equal eigenvalues.

By [2, Theorem 8], (iii) implies (ii). Theorem 1 below says that the scalar matrices and the matrices satisfying (iii) are the only cases when \( l_{\Omega_n}(A,\cdot) \) is a continuous function. Then the mentioned above result [5, Proposition 4] shows that (i) implies (iii) and hence the conditions (i), (ii) and (iii) are equivalent.

**Theorem 1.** If \( A \in \Omega_n \), then \( l_{\Omega_n}(A,\cdot) \) is a continuous function if and only if either \( A \) is scalar or \( A \in \mathcal{C}_2 \) has two equal eigenvalues.

**Proof.** Using \( \Phi_\lambda \) and an automorphisms of \( \Omega_n \) of the form \( X \rightarrow P^{-1}XP \), where \( P \) is an invertible matrix, we may assume that 0 is an eigenvalue of \( A \) and the matrix is in a Jordan form.

It is enough to prove that \( l_{\Omega_n}(A,\cdot) \) is not a continuous function if \( A \) has at least one non-zero eigenvalue or \( A \in \Omega_n \) is a non-zero nilpotent matrix and \( n \geq 3 \).

In the first case, let \( d_1 \geq \cdots \geq d_k \) be the numbers of the Jordan blocks corresponding to the pairwise different eigenvalues \( \lambda_1 = 0, \lambda_2, \ldots, \lambda_k \). We shall prove that \( l_{\Omega_n}(A,\cdot) \) is not continuous at 0. It is easy to see that \( A \) can be represented as blocks \( A_1, \ldots, A_l \) (with sizes \( n_1, \ldots, n_l \)) such that the eigenvalues of \( A_1 \) are equal to zero and the other blocks are cyclic with at least two different eigenvalues values (\( A_1 \) is missed if \( d_1 = d_2 \)). By [3 Proposition 4], we know that there are \( \{A_{i,j}\}_{j} \rightarrow 0, 1 \leq i \leq l \), such that \( \sup_{i,j} l_{\Omega_n}(A_i, A_{i,j}) := m < r(A) \). Taking \( A_j \) to be with blocks \( A_{1,j}, \ldots, A_{l,j} \), it is easy to see \( l_{\Omega_n}(A, A_j) \leq \max_i l_{\Omega_n}(A_i, A_i,j) \leq m < l_{\Omega_n}(A,0) \) which implies that \( l_{\Omega_n}(A,\cdot) \) is not continuous at 0.
Let now $A \neq 0$ be a nilpotent matrix. Then $A = (a_{ij})_{1 \leq i,j \leq n}$ with $a_{ij} = 0$ unless $j = i + 1$. Let $r = \text{rank}(A) \geq 1$. Following the proof of Proposition 4.1 in [6], let

$$F_0 := \{1\} \cup \{j \in \{2, \ldots, n\} : a_{j-1,j} = 0\} := \{1 = b_1 < b_2 < \cdots < b_{n-r}\},$$

and $b_{n-r+1} := n + 1$. We set $d_i := 1 + \#(F_0 \cap \{(n-i+2), \ldots, n\})$. The hypotheses on $A$ imply that we can choose its Jordan form so that $a_{n-1,n} = 1$, so $1 = d_1 = d_2 \leq d_3 \leq \cdots \leq d_n = \#F_0 = n - r$, $d_{j+1} \leq d_j + 1$.

Corollary 4.3 and Proposition 4.1 in [6] show that for any $C \in \mathcal{C}_n$,

$$L_{\Omega_n}(A,C) = h_{\mathcal{G}_n}(0,\sigma(C)) := \inf\{|\alpha| : \exists \psi \in \mathcal{H}(\mathbb{D},\mathcal{G}_n) : \psi(\alpha) = \sigma(C)\},$$

where

$$\mathcal{H}(\mathbb{D},\mathcal{G}_n) = \{\psi \in \mathcal{O}(\mathbb{D},\mathcal{G}_n) : \text{ord}_0 \psi_j \geq d_j, 1 \leq j \leq n\}.$$

Note that $d_j \leq j - 1$ for $j \geq 2$. Let $m := \min_{j \geq 2} \frac{d_j}{j-1}$ and choose a $k$ such that $\frac{d_k}{k-1} = m$. If $m = 1$, then $d_j = j - 1$ for all $j \geq 2$, and if furthermore $n \geq 3$, we can take $k = 3$.

With $k$ chosen as above, let $\lambda$ be a small positive number, $b = k\lambda^{k-1}$ and $c = (k - 1)\lambda^k$. Then $\lambda$ is a double zero of the polynomial $\Lambda(z) = z^{n-k}(z^k - bz + c)$ with zeros in $\mathbb{D}$. Let $B$ be a diagonal matrix such that its characteristic polynomial is $P_B(z) = \Lambda(z)$.

Assuming that $L_{\Omega_n}(A,\cdot)$ is continuous at $B$, then

$$L_{\Omega_n}(A,B) = h_{\mathcal{G}_n}(0,\sigma(B)) =: \alpha.$$

**Lemma 2.** If $L_{\Omega_n}(A,B) = \alpha$, then there is a $\psi \in \mathcal{H}(\mathbb{D},\mathcal{G}_n)$ with $\psi(\alpha) = \sigma(B)$ and

$$\sum_{j=1}^{n} \psi_j'(\alpha)(-\lambda)^{n-j} = 0.$$

**Proof.** This is analogous to the proof of Proposition 4.1 in [6]. Let $\varphi \in \mathcal{O}(\mathbb{D},\Omega_n)$ be such that $\varphi(0) = A$ and $\varphi(\bar{\alpha}) = B$. Corollary 4.3 in [6] applied to $A$ shows that $\tilde{\psi} := \sigma \circ \varphi \in \mathcal{H}(\mathbb{D},\mathcal{G}_n)$.

Now we study $\sigma_n(\varphi(\zeta)) - \sigma_n(B) = \sigma_n(\varphi(\zeta))$ near $\zeta = \alpha$. We may assume that the first two diagonal coefficients of $B$ are equal to $\lambda$. If we let $\varphi_{\lambda}(\zeta) := \varphi(\zeta) - \lambda \mathcal{I}_n$, then the first two columns of $\varphi_{\lambda}(\alpha)$ vanish, so $\sigma_n \circ \varphi_{\lambda} = \det(\varphi_{\lambda})$ vanishes to order 2 at $\alpha$. On the other hand,

$$\det(-\varphi_{\lambda}(\zeta)) = \det(\lambda \mathcal{I}_n - \varphi(\zeta)) = \lambda^n - \sum_{j=1}^{n} (-1)^j \lambda^{n-j} \tilde{\psi}_j(\zeta),$$

and since the derivative of the left hand side vanishes at $\bar{\alpha}$, the same holds for the right hand side. It remains to let $\alpha \rightarrow \bar{\alpha}$ and to use that $\mathcal{G}_n$ is a taut domain, providing the desired $\tilde{\psi}$.

**Lemma 3.** We have $\alpha^m \lesssim \lambda$; furthermore if $m = 1$ and $n \geq 3$, then $\alpha^{2/3} \lesssim \lambda$. So in all cases $\alpha \ll \lambda$. 

□
Proof. Note that there is an \( \varepsilon > 0 \) such that for \( \lambda < \varepsilon \) the map \( \zeta \rightarrow (0, \ldots, 0, k(\varepsilon \zeta)^d, (k - 1)\lambda(\varepsilon \zeta)^d, 0, \ldots, 0) \) is a competitor for \( h_{\Omega_\varepsilon}(A, B) \). So \( (\varepsilon \alpha)^d_k \leq \lambda^{k-1} \), that is, \( \alpha^m \lesssim \lambda \).

If \( m = 1 \) and \( n \geq k = 3 \), then considering the map \( \zeta \rightarrow (0, 3\lambda^{1/2}\varepsilon \zeta, 2(\varepsilon \zeta)^2, 0, \ldots, 0) \) we see that \((\varepsilon \alpha)^2 \leq \lambda^3\). □

Setting \( \psi_j(\zeta) = \zeta^d \theta_j(\zeta) \), the condition in Lemma 2 becomes

\[
a \frac{(-\lambda)^n}{\alpha} + S = 0,
\]

where \( a = (k - 1)d_k - kd_{k-1} \) and \( S = \sum_{j=1}^n \alpha^{d_j} \theta_j^j(\alpha)(-\lambda)^{n-j} \). Note that \( a \neq 0 \). Indeed, if \( m < 1 \), then \( d_k = d_{k-1} \) and hence \( a = -d_k \); if \( m = 1 \), then \( a = (k - 1)(k - 1) - k(k - 2) = 1 \). Since \( G_n \) is bounded, \( |\theta_j^j(\alpha)| \lesssim 1 \).

By Lemma 3 and the choice of \( k \), for any \( j \),

\[
a^{d_j} \lesssim \lambda^{(k-1)d_j/d_k} \leq \lambda^{j-1} \leq \lambda^{n-1}.
\]

Thus \( S \lesssim \lambda^{n-1} \). By Lemma 3 again, \( \alpha \ll \lambda \), a contradiction with (2). □

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