ON THE DYNAMICS OF A DURABLE COMMODITY MARKET

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Abstract. Disequilibria phenomenon appears in the economic model of durable stocks proposed by A. Panchuck and T. Puu in [7]. In this paper, assuming that agents have the same utility functions, we give not only bounds of the disequilibrium but also prove the existence of a compact set of no-trade points such that it does not depend on the initial stock distribution. We also give a description of the nature of ω-limit sets in the general case proving that disequilibrium points can be attained as limit points of orbits.

1. Introduction and model statement. A stock market model was introduced by Puu in [8] and studied by Panchuk and Puu in [7]. In a broad sense, in this framework, stock refers to any commodity that remains on the market and it only changes owner. Housing market is an example. A detailed description of the model can be found in [7, 8], here we make a brief summary introducing the basic facts which are necessary to understand our results.

Two agents and two commodities are considered. Normalizing the quantities of each commodity and denoting them for each stock as $x$ and $y$, we have that $0 \leq x, y \leq 1$. If $(x, y)$ is the distribution for the first agent, then $(1 - x, 1 - y)$ is the asset for the second one. It is assumed that prices of stocks are $p_x = 1$ whereas $p_y \in (0, \infty)$, thus only $p = p_y$ is taken into account. Agents will be interested in trading according to the satisfaction level defined by Cobb-Douglas utility functions,

$$U(x, y) = x^\alpha y^{1-\alpha}, \quad V(x, y) = (1 - x)^\beta (1 - y)^{1-\beta},$$

where $0 < \alpha, \beta < 1$ are the elasticity coefficients. For a current distribution $(x, y)$ and a price $p$ the agents trade looking for enhancing their own satisfaction level, with budget limits given by $l := x + py$, and thus, the options for the next state are on the line

$$L_l = \{(x, y) \in [0, 1] \times [0, 1] : x + py = l\}.$$

The Edgeworth box, [5] is used in [7, 8] to visualize the distribution of the stocks for both agents (see Figure 1).

The budget line is the same for both agents and the highest possible satisfaction level is obtained at the points in which the budget line is tangent to indifference curves defined as $U(x, y) = c_1$ and $V(x, y) = c_2$, where $c_1$ and $c_2$ are constants.
We denote by \((x_1, y_1)\) and \((x_2, y_2)\) the optimal preferences of the agents, which are determined by maximizing their levels of utility restricted to a fixed budget \(l\), namely
\[
\begin{align*}
x_1 &:= x_1(x, y, p) := \alpha (x + py) = \alpha l, \\
y_1 &:= y_1(x, y, p) := (1 - \alpha) \left( \frac{x}{p} + y \right) = \frac{1 - \alpha}{p} x_1 = \frac{1 - \alpha}{p} l, \\
x_2 &:= x_2(x, y, p) := 1 - \beta (1 - x + p(1 - y)) = 1 - \beta (1 + p - l), \\
y_2 &:= y_2(x, y, p) := 1 - (1 - \beta) \left( \frac{1 - x}{p} + 1 - y \right) = 1 - \frac{1 - \beta}{p \beta} (1 - x_2) \\
   &= 1 - \frac{1 - \beta}{p} (1 + p - l),
\end{align*}
\]
see [7] for more details. The game used to select the new state follows the rules described in [7, 8] in order to try to improve the level of utility of both agents, such that if the trade does not improve levels of both agents the state remains. Thus, the final choice will be selected according to the piecewise linear map
\[
\Phi(p, x, y) = \Phi_p(x, y) = \begin{cases} 
(x_1, y_1) & \text{if } (x_1 - x_2)(x_1 - x) \leq 0, \\
(x_2, y_2) & \text{if } (x_2 - x_1)(x_2 - x) \leq 0, \\
(x, y) & \text{if } (x - x_1)(x - x_2) < 0.
\end{cases}
\]

Price model proposed by Puu in [8] and studied by Panchuk and Puu in [7] leans on the excess supply/demand of the \(x\)-stock and follows the rule
\[
f(p, x, y) = pe^{\delta(x_2 - x_1)},
\]
where \(x_1\) and \(x_2\) were defined in (1), and \(\delta > 0\) is the adjustment step length. Notice that price cannot be negative.
The curve $L_B$ given by the equalities
\[
y = \frac{(1 - \alpha)\beta x}{\alpha(1 - \beta) + (\beta - \alpha)x},
\]
\[
p = \frac{\alpha(1 - \beta) + (\beta - \alpha)x}{\beta x}
\]
defines the fixed points of the map $f \times \Phi$, where $\times$ denotes the product of two maps, that is, $(f \times \Phi) : [0, +\infty) \times [0, 1]^2 \to [0, +\infty) \times [0, 1]^2$ given by
\[
(f \times \Phi)(p, x, y) = (f(p, x, y), \Phi(p, x, y)) = (f(p, x, y), \Phi p(x, y)).
\]
Observe that the equalities (3) and (4) follow from $x_1 = x_2 = x$, namely
\[
\begin{cases}
\alpha l = x, \\
1 - \beta(1 + p - l) = x.
\end{cases}
\]
Since $l = \frac{x_1}{\alpha}$, then $1 - \beta \left(1 + p - \frac{x_1}{\alpha}\right) = x$ and hence $1 - \beta - \beta p + \frac{\beta x}{\alpha} = x$ and
\[
p = \frac{\alpha - \alpha\beta - \alpha x + \beta x}{\beta x} = \frac{\alpha(1 - \beta) + x(\beta - \alpha)}{\beta x}.
\]
On the other hand, now $\alpha l = \alpha(x + py) = \alpha x + \frac{\alpha(1 - \beta) + x(\beta - \alpha)}{\beta}y = x$, and finally
\[
y = \frac{(1 - \alpha)\beta x}{\alpha(1 - \beta) + x(\beta - \alpha)}.
\]
In [7] it is highlighted that there exists the phenomenon known as indeterminacy of equilibria. Numerical simulations included in [7] show that some orbits sticks or freezes at points in $[0, 1]^2$, outside of the curve of fixed points given by equations (3)-(4), while the price changes. These points are called no-trade points or disequilibrium points [7]. Here we make an analytic characterization of this phenomenon in the homogeneous case in which both agents have the same utility function, that is, $\alpha = \beta$. As a consequence, we prove that disequilibrium points can be obtained also as limit points of orbits, which however is natural is quite difficult, perhaps impossible, to be observed in a numerical simulation and not only by the freezing phenomenon stated in [7].

The notation is standard from dynamical systems. For a continuous map $\varphi : X \to X$ on a metric space and $x \in X$, we denote its orbit, given by the sequence $\varphi^n(x)$, by $\text{Orb}(x, \varphi)$. The set of limit points of $\varphi^n(x)$ is called the $\omega$–limit set with respect to $\varphi$ and $x$ and it is denoted by $\omega(x, \varphi)$. An $\omega$–limit set $\omega(x, \varphi)$ is said to be orbit enclosing if $\text{Orb}(x, \varphi) \subset \omega(x, \varphi)$.

The rest of the paper will consist on two sections, the first one devoted to the homogenous case and the second one devoted to the non–homogeneous one. In the homogeneous case we will characterize analytically the $\omega$–limit sets of all the orbits, and as a consequence we explain when the projection of an orbit of the map $f \times \Phi$, equation (5), remains unaltered when time evolves. The last section will be devoted to the non–homogenous case, where a characterization of $\omega$–limit sets is given.

2. The homogenous case $\alpha = \beta$. In the homogeneous case $\alpha = \beta$, that is, both agents have the same utility functions. Then, equations (1) are rewritten simply in terms of $\alpha$ whereas the map (2) has the same form. The curve $L_B$ turns out to be
the line \( x = y \). The main difference that makes this particular case so special is that now price’s law follows the function

\[
f(p) = pe^{\delta(1-\alpha)}e^{-\delta p},
\]

which has been obtained by substituting the values of \( x_1 \) and \( x_2 \), from equation (1) for \( \alpha = \beta \). Observe that price evolution for \( \alpha = \beta \) follows a one-dimensional Ricker map (see [9]) in which stocks values are not involved, see Figure 2. Only parameters \( \alpha \) and \( \delta \) are involved in the iteration process of the price but not the current budget. This fact allows us to study the dynamics of the price independently of stocks composition. Let us recall some properties on the Ricker map.

\[
S(f(p)) = \frac{f'''(p)}{f'(p)} - \frac{3}{2} \left( \frac{f''(p)}{f'(p)} \right)^2 < 0
\]

at any value of \( p > 0 \) in which it is properly defined. As a consequence the unimodal Ricker map \( f \) has only one metric attractor, that is, an \( \omega \)-limit set attracting the orbits in a set of full Lebesgue measure. Moreover, the metric attractor can be a periodic orbit, a solenoidal attractor (essentially a Cantor set with almost periodic dynamics) or a union of periodic intervals containing a dense orbit, see [6, Corollary 1]. In addition, using notation \( p_0 = \frac{1}{\alpha \delta} \) for the maximum of \( f \), called turning point, the dynamics of \( f \) is contained in the interval \([f^2(p_0), f(p_0)]\). The latter follows easily since for \( p > f(p_0) \) we have that \( f(p) < f^2(p_0) \), thus dynamics is on \([f^2(p_0), f(p_0)]\), see [4, Theorem 1.3] for more details. Figure 3 shows the typical bifurcation diagram of the Ricker map for \( \alpha = 0.3 \), as well as the computation of topological entropy following the algorithms from [1, 3] and the estimation of Lyapunov exponents at \( f(p_0) \). Note that positive topological entropy gives us the parameter region where topological chaos is possible, while positive Lyapunov exponents show the parameter region where this topological chaos is physically observable.

Let \( \alpha \in (0,1) \) and \( p > 0 \) be fixed. We define the set

\[
A_\alpha(p) := \left\{ (x, y) \in [0,1]^2 : \min \left\{ \frac{1-\alpha}{p\alpha} x, 1 - \frac{1-\alpha}{p\alpha} (1-x) \right\} \leq y \leq \max \left\{ \frac{1-\alpha}{p\alpha} x, 1 - \frac{1-\alpha}{p\alpha} (1-x) \right\} \right\}.
\]

See Figure 4 as an example of the shape of sets \( A_\alpha(p) \).
We have computed orbits of length equal to 50000 and we have drawn the last 250 points. On the center and right, topological entropy and estimation of Lyapunov exponents, respectively.

Figure 4. The shaded area represents $A_\alpha(p)$ for $\alpha = 0.6$ and $p = 2$ whereas the line represents the points $(x, y)$ such that for $p = 2$ the budget is $l = 1.5$.

**Lemma 1.** For each fixed $\alpha \in [0, 1]$ and $p > 0$, the set $A_\alpha(p)$ is compact, convex and not empty. In particular,

$$\{(x, x) : x \in [0, 1]\} \subseteq A_\alpha(p).$$

*Proof.* $A_\alpha(p)$ is a set limited by two parallel lines $g_1(x) = \frac{1-\alpha}{\alpha p} x$ and $g_2(x) = 1 - \frac{1-\alpha}{\alpha p} (1 - x)$, $x \in \mathbb{R}$, and the square $[0, 1]^2$. Thus $A_\alpha(p)$ is a parallelogram whenever $g_1(x) \neq g_2(x)$ and the set $\{(x, x) : x \in [0, 1]\} \subset A_\alpha(p)$ is a diagonal of the parallelogram, since $g_1(0) = 0$ and $g_2(1) = 1$. If $g_1(x) = g_2(x)$, that is, if $p = \frac{1-\alpha}{\alpha}$ then $A_\alpha(p) = \{(x, x) : x \in [0, 1]\}$.

We distinguish two types of sets $A_\alpha(p)$ as follows. Given $p > 0$, we say that the set $A_\alpha(p)$ is of type 1 whenever $\frac{1-\alpha}{\alpha} < p$ and of type 2 if $\frac{1-\alpha}{\alpha} > p$. Note that the remaining case $\frac{1-\alpha}{\alpha p} = 1$ gives us the line of fixed points $y = x$. The following lemma shows the evolution of $A_\alpha(p)$ when the price $p$ changes.

**Lemma 2.** Let $\alpha \in [0, 1]$ and $p' > p > 0$. Then

1. If the sets $A_\alpha(p)$ and $A_\alpha(p')$ are of type 1, then $A_\alpha(p) \subset A_\alpha(p')$. 

Theorem 1. \[ \text{Observations are useful to prove the next result.} \]

\[ \text{Proof.} \] Note that if \( p' > p \), then \( \frac{1 - \alpha}{p' - \alpha} < \frac{1 - \alpha}{p - \alpha} \). This immediately gives us \( \frac{1 - \alpha}{p' - \alpha} x < \frac{1 - \alpha}{p - \alpha} x \) for any \( x \in [0,1] \) and \( 1 - \frac{1 - \alpha}{p' - \alpha} (1 - x) > 1 - \frac{1 - \alpha}{p - \alpha} (1 - x) \) for all \( x \in [0,1] \). The lemma follows then easily.

The sets \( A_\alpha(p) \) are useful to describe the global dynamics of the map as follows.

**Proposition 1.** Let \( \alpha \in [0,1], p', p > 0 \) and \( n \in \mathbb{N} \). Then

1. \( (f \times \Phi)(\{p\} \times [0,1]^2) = \{f(p)\} \times A_\alpha(p). \)
2. \( (f \times \Phi)(\{p'\} \times A_\alpha(p)) = \{f(p')\} \times (A_\alpha(p) \cap A_\alpha(p')). \)
3. \( (f \times \Phi)^n(\{p\} \times [0,1]^2) = \{f^n(p)\} \times \cap_{i=0}^{n-1} A_\alpha(f^i(p)). \)

**Proof.** (1) is a consequence of the definition of \( \Phi \). So, we prove (2). Fix \((x_0, y_0) \in A_\alpha(p)\). We assume that \( y_0 > x_0 \) (the other case is analogous). If \((x_0, y_0) \in A_\alpha(p')\) then, by the definition of \( \Phi \) we have that \( \Phi(p', x_0, y_0) = (x_0, y_0) \) and therefore \((x_0, y_0) \in A_\alpha(p) \cap A_\alpha(p')\). Now, assume that \((x_0, y_0) \notin A_\alpha(p')\). Then \((f \times \Phi)(p', x_0, y_0) = (f(p'), x_1, y_1)\) with \((x_1, y_1) \in A_\alpha(p')\), because \((x_1, y_1)\) is the intersection point of the budget line \( x + p'y = x_0 + p'y_0 \) with the boundary line of \( A_\alpha(p') \) fulfilling \( y \geq x \) all \((x, y)\) within this line. Note that it is immediate that \((x_1, y_1)\) belongs to \( A_\alpha(p)\), since \((x_1, y_1)\) belongs to the segment limited by \((x_0, y_0)\) and \(\left(\frac{x_0 + y_0 p'}{1 + p'}, \frac{x_0 + y_0 p'}{1 + p'}\right)\) which is contained in \( A_\alpha(p)\), see Lemma 1, then (2) is proved. (3) is a direct consequence of (1) and (2), and the proof finishes.

Given \( p > 0 \) and \( n \in \mathbb{N} \), we consider the sets

\[ \Omega_1(p, n) = \left\{ i \in \{0, 1, \ldots, n - 1\} : \frac{1 - \alpha}{\alpha} < \frac{f^i(p)}{i} \right\}, \]

\[ \Omega_2(p, n) = \left\{ i \in \{0, 1, \ldots, n - 1\} : \frac{1 - \alpha}{\alpha} > \frac{f^i(p)}{i} \right\}, \]

and

\[ T_1(p, n) = \left\{ f^i(p) : i \in \Omega_1(p, n) \right\}, \]

\[ T_2(p, n) = \left\{ f^i(p) : i \in \Omega_2(p, n) \right\}. \]

Then it is easy to see that

\[ A_{\alpha, p}(n) := \cap_{i=0}^{n-1} A_\alpha(f^i(p)) = (\cap_{i \in \Omega_1(p, n)} A_\alpha(p')) \cap (\cap_{i \in \Omega_2(p, n)} A_\alpha(p')). \]

Observe also that \( f^i(p) = \frac{1 - \alpha}{\alpha} \) has been left out thanks to Lemma 1. These observations are useful to prove the next result.

**Theorem 1.** Let \( \alpha \in [0,1], p > 0 \) and \( n \in \mathbb{N} \). Let \( p_m := \inf_{n \geq 0} T_1(p, n) \) and \( p_M := \sup_{n \geq 0} T_2(p, n) \). Then,

\[ A_{\alpha, p} := \bigcap_{n \geq 0} P_{xy}(f \times \Phi)^n(\{p\} \times [0,1]^2)) = A_\alpha(p_m) \cap A_\alpha(p_M), \]

where \( P_{xy} \) is the projection dropping the first coordinate.

**Proof.** Fix \( n \in \mathbb{N} \). By Lemma 2 (1) we have that \( \cap_{p' \in T_1(p, n)} A_\alpha(p') = A_\alpha(p_m, n) \) where \( p_m, n := \min\{f^i(p) : 0 \leq i \leq n - 1\} \). Similarly, by Lemma 2 (2) we have that \( \cap_{p' \in T_1(p, n)} A_\alpha(p') = A_\alpha(p_m, n) \) where \( p_M, n := \max\{f^i(p) : 0 \leq i \leq n - 1\} \). Taking into account that \( p_M = \sup_n p_{M, n} \) and \( p_m = \inf_n p_{m, n} \), the proof concludes. \( \square \)
Given \( p > 0 \), Theorem 1 characterizes the orbit of any point \((x, y) \in [0, 1]^2\). The dynamics on \([0, 1]^2\) is quite simple because the accumulation points of \( P_{xy}(f \times \Phi)^n(p, x, y) \) are from the set \( A_\alpha(p_m) \cap A_\alpha(p_M) \). Figure 5 shows graphically how the set is constructed. In addition, if the initial condition is within this set, the positions of both traders remain unchanged along time.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{The set \( A_{\alpha,p_0} \) is constructed for \( \alpha = \beta = 0, \delta = 7 \) and \( p_0 = \frac{1}{\alpha \delta} \) as the interior white set limited by the lines and containing the diagonal of the square.}
\end{figure}

**Remark 1.** Let \( \alpha > 0 \) fixed and \( p_\alpha = \frac{1-\alpha}{\alpha} \), then \( A_\alpha(p_\alpha) = \{(x, x) : x \in (0, 1)\} \). Thus, if \( p_\alpha \in \omega(p, f) \) then each point \((x, y)\) converges to a fixed point in the line \( \{(x, x) : x \in [0, 1]\} \), since there exists \( n \in \mathbb{N} \) such that \( p_\alpha = f^n(p) \) and this implies that \( (x_{n+1}, y_{n+1}) \in A_\alpha(p_\alpha) \), that is \( x_{n+1} = y_{n+1} \) and therefore \( (x_m, y_m) = (x_{n+1}, y_{n+1}) \) for \( m > n \).

Note that \( f(p) = p \) and \( p \geq 0 \) if and only if \( p = p_\alpha = \frac{1-\alpha}{\alpha} \). As a consequence of this fact and Theorem 1, we have the following results.

**Corollary 1.** Assume that \( p_\alpha \in \omega(p, f) \) for some \( p > 0 \). Then \( A_{\alpha,p} = A_\alpha(p_m) \cap A_\alpha(p_M) = \{(x, x) : x \in [0, 1]\} \).

**Corollary 2.** Assume that \( p_\alpha \) is an attracting fixed point of \( f \). Then for almost all \( p > 0 \) we have that \( A_{\alpha,p} = A_\alpha(p_m) \cap A_\alpha(p_M) = \{(x, x) : x \in [0, 1]\} \).

**Proof.** As \( f \) has negative Schwarzian derivative, for almost all \( p \in [f^2(p_0), f(p_0)] \) we have that \( \omega(p, f) = \{p_\alpha\} \). Since every orbit starting from a positive initial condition outside \([f^2(p_0), f(p_0)]\) eventually lies in this set, Corollary 1 finishes the proof.

**Remark 2.** When the fixed point \( p_\alpha \) is repulsive, we can find \( p > 0 \) such that \( A_{\alpha,p} \neq \{(x, x) : x \in [0, 1]\} \). For that we need to find parameter values such that \( \omega(p, f) \) is either a periodic orbit or a periodic sequence of intervals containing a dense orbit with period greater than 1. Then, taking an initial condition close to \( p_\alpha \) should be enough to construct such example. For instance, take \( \alpha = 0.3, \delta = 4 \) and \( p = 2.4 > p_\alpha \). Then \( A_{0.3,2.4} = A_{0.3}(2.4) \cap A_{0.3}(f(2.4)) \) is not the diagonal (see Figure 6).
Figure 6. The set \( A_{0.3,2.4} \) is shadowed light. Isolated lines defines the set, indicating that it cannot be approached by as a limit, and orbits stick on it.

Remark 3. It is also important to highlight that the examples of Remark 2 can be chosen in such a way that the \( \omega \)-limit set is orbit enclosing and in this case, if the \( \omega \)-limit set is a family of periodic subintervals, then the set \( A_{\alpha,p} \) is attached at the limit. For instance, take \( \alpha = 0.3, \delta = 4 \) and \( p = 1 \) we see that \( p_m = 2.54308519755389 \) and \( p_M = 1.9771882339892544 \). (see Figure 7). However, if we choose initial conditions close to \( p_\alpha \) such that \( \inf \text{Orb}(p,f) = p \) and \( \sup \text{Orb}(p,f) = f(p) \) or viceversa as in Figure 6 (see also Remark 2), then \( A_{\alpha,p} = A_\alpha(p) \cap A_\alpha(f(p)) \) and this set remains unaltered along time.

Figure 7. The set \( A_{0.3,1} \) is shadowed light. No isolated lines give us the set, which is achieved by a limit.
Remark 4. It is easy to show that both “frozen” and “non-frozen” fixed points in \([0,1]^2\) may appear for the same model parameters. For instance, Figure 8 is an example of this situation. Points at the boundary of the set of Figure 8 are attained as limit points whereas points inside the boundary remain unaltered under iteration of \(f \times \Phi\), see equation (5).

In summary, any \(\omega\)–limit set of \(\omega((p,x,y), f \times \Phi) = \omega(p,f) \times \{(x^*,y^*)\}\). When \(\omega(p,f) = \{p_\alpha\}\) we have an equilibrium point while if \(\omega(p,f) \neq \{p_\alpha\}\) we have disequilibrium points introduced in [7]. In both cases this point \((x^*, y^*)\) can be obtained as a limit point, or sticking it, that is, there exists a positive integer \(n_0\) such that \((f \times \Phi)^n(p,x,y) = (f^n(p), x^*, y^*)\) holds for all \(n > n_0\). So, the dynamics is slightly more complicated than that described in [7].

3. The non–homogenous case \(\alpha \neq \beta\). We have described the no-trade set (also disequilibrium set) when the condition \(\alpha = \beta\) is considered. Now, we note that if we take different values of \(\alpha\) and \(\beta\), \(\alpha \neq \beta\), then we can define a set \(A_{\alpha,\beta}(p)\) as the convex set in \([0,1]^2\) whose boundary is defined by the lines \(y = \frac{1-\alpha}{\alpha p}x\) and \(1-y = \frac{1-\beta}{\beta p}(1-x)\) and such that the points \((0,1)\) and \((1,0)\) do not belong to \(A_{\alpha,\beta}(p)\). Note that the lines \(y = \frac{1-\alpha}{\alpha p}x\) and \(1-y = \frac{1-\beta}{\beta p}(1-x)\) are not parallel and therefore they intersect at one point. It can be checked that this point belongs to the line given by equation (3). This intersection point can be in \([0,1]^2\) or not. It is not hard to see that the points defined by (3) are included in the sets \(A_{\alpha,\beta}(p)\).

In addition, the line (3) divides the square \([0,1]^2\) into two (forward) invariant sets \(I_1\) and \(I_2\) given by

\[
I_1 = \left\{(x,y) \in [0,1]^2 : y \geq \frac{(1-\alpha)\beta x}{\alpha(1-\beta) + (\beta-\alpha)x}\right\}
\]
and
\[
I_2 = \left\{ (x, y) \in [0, 1]^2 : y \leq \frac{(1 - \alpha)\beta x}{\alpha(1 - \beta) + (\beta - \alpha)x} \right\}.
\]

Fix \( p > 0 \) and \((x, y) \in I_1 \) (the \((x, y) \in I_2 \) is analogous) and \((p_n, x_n, y_n) = (f \times \Phi)^n(p, x, y)\) for \( n \in \mathbb{N} \). Here \( f \) depends also on \( x \) and \( y \). Since \((x_n, y_n) \in I_1 \), we have that \( x_n \) is increasing and bounded and therefore there is a limit point \( x^* \) of \( x_n \). Similarly, there is a limit point \( y^* \) for \((y_n)\). Then, we have that \( \omega((p, x, y), f \times \Phi) \subset \{(p', x^*, y^*) : p' > 0\} \). If there exists \( n_0 \in \mathbb{N} \) such that \((f \times \Phi)^{n_0}(p, x, y) = (p_{n_0}, x^*, y^*)\), then \( \omega((p, x, y), f \times \Phi) = \omega(p, f^* ) \times \{(x^*, y^*)\}, where \( f^* \) denotes the map
\[
f^*(p) := f(p, x^*, y^*).
\]

However, if such positive integer \( n_0 \) does not exists, then \((x^*, y^*)\) is reached at the limit and then the \( \omega \)-limit set of the orbit \((p_n)\), denoted by \( \Omega \), could not be an \( \omega \)-limit set of \( f^* \); at most we can state that this \( \omega \)-limit set belongs to the non–wandering set of \( f^* \) (see [2] and references therein).

Although we do not have an analytical proof, and in our opinion this proof would be quite difficult, we think that the following question is affirmative.

**Open problem 1.** Can disequilibrium and equilibrium points be reached as limit points following the sticking process described in [7]?

Note that we are unable to prove a similar result to that of Theorem 1, and then the description of disequilibrium points, because in the non–homogenous case the price is not homogenous for any distribution of points \((x, y)\).

At least, we can give an analogous result to that of Corollary 2. Namely, assume that \( \Omega = \{p^*\} \) where \( p^* \) is the non–zero fixed point of \( f^* \). Then \( \omega((p, x, y)) = \{(p^*, x^*, y^*)\} \) and therefore it is a fixed point of \( f \times \Phi \). Hence, this point is in curve \( \mathcal{L}_B \), that is, \((x^*, y^*)\) fulfills equation (3). It is clear that \( p^* \) must be an attracting fixed point of \( f^* \), that is, the absolute value \( |(f^*)'(p^*)| \) must be smaller than one. This condition can be easily checked: just take points from equation 3 and fix the range values of \( \delta \). By [7], the condition
\[
\delta < \frac{2}{1 - \beta + (\beta - \alpha)x}
\]
must be fulfilled and Figure 9 gives us, for fixed \( \alpha \) and \( \beta \), the shape of regions where this condition is fulfilled.

The following question remains open.

**Open problem 2.** If the fixed point \( p^* \in \Omega \), must the point \((x^*, y^*)\) be a point of the curve given by the equation (3)? Which would be an analogous result to that of Corollary 1?

4. **Conclusion.** In [7, 8] an economic model market of two durable commodities with two agents is introduced. We have considered the case in which the agents have the same utility functions for making decisions \((\alpha = \beta)\) and we have proved the existence of a compact convex set such that each point of this set is either an equilibrium or a disequilibrium point. Our results reveal that disequilibrium points can be also reached as limit points, and not only with the sticking process described in [7]. The dynamics is similar in the non–homogenous case, although a complete analytic description of non–trade points seems to be more complicated than in the homogeneous case.
Figure 9. Fix $\delta \in [0, 5]$. From left to right, in dark, stability regions are showed for parameter values $\alpha = 0.1$ and $\beta = 0.9$ (left), $\alpha = 0.3$ and $\beta = 0.7$ (middle) and $\alpha = 0.4$ and $\beta = 0.5$ (right). $\delta$ is in Y-axis and $x$ on the X-axis.

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REFERENCES

[1] L. Block, J. Keesling, S. H. Li and K. Peterson, An improved algorithm for computing topological entropy, Journal of Statistical Physics, 55 (1989), 929–939.
[2] J. S. Cánovas, On $\omega$-limit sets of non-autonomous discrete systems, Journal of Difference Equations and Applications, 12 (2006), 95–100.
[3] J. S. Cánovas and M. Muñoz-Guillermo, On the complexity of economic dynamics: An approach through topological entropy, Chaos, Solitons Fractals, 103 (2017), 163–176.
[4] W. de Melo and S. van Strien, One-dimensional Dynamics, Springer-Verlag, 1993.
[5] F. Y. Edgeworth, The pure theory of international values, Econ. J., (1894), 35–50.
[6] J. Graczyk, D. Sands and G. Świątek, Metric attractors for smooth unimodal maps, Ann. Math., 159 (2004), 725–740.
[7] A. Panchuck and T. Puu, Dynamics of a durable commodity market involving trade at disequilibrium, Commun. Nonlinear Sci. Numer. Simulat., 58 (2018), 2–14.
[8] T. Puu, Disequilibrium trade and the dynamics of stock markets, in M. Faggini, A. Parziale eds. Complexity in Economics: Cutting Edge Research, Springer, (2014), 225–245.
[9] W. E. Ricker, Stock and recruitment, Journal of The Fisheries Research Board of Canada, 11 (1954), 559–623.
[10] D. Singer, Stable orbits and bifurcations of maps of the interval, SIAM J. Appl. Math., 35 (1978), 260–267.

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