Channel Estimation Theory of Low-Noise Multiple Parameters: Attainability Problem of the Cramér-Rao Bounds

M. Hotta* 1 and T. Karasawa† 2
* Department of Physics, Faculty of Science, Tohoku University,
Sendai, 980-8578, Japan
† National Institute of Informatics,
2-1-2 Hitotsubashi, Chiyoda-ku, Tokyo, 101-8430, Japan

Abstract

For decoherence processes induced by weak interactions with the environment, a general quantum channel with one noise parameter has been formulated. This channel is called low-noise channel and very useful for investigating the parameter estimation in the leading order. In this paper, we formulate the low-noise channel with multiple unknown parameters in order to address the simultaneous attainability of the Cramér-Rao bound for the parameters estimation. In general, the simultaneous achievement of the Cramér-Rao bound for multi-parameter estimations suffers from non-commutativity of optimal measurements for respective parameters. However, with certain exceptions, we show that the Cramér-Rao bound for output states of dissipative low-noise channels can be always attained in the first order of the parameters as long as $D \leq N - 1$, where $D$ and $N$ denote the
number of the parameters and the dimension of the system, respectively. This condition is replaced by $D \leq N^2 - 1$ if it is allowed to set the entanglement with ancilla systems in its input state and to perform the non-local measurement on the composite system.
1 Introduction

Decoherence processes driven by weak interactions with the environment can be described in a unified way by the low-noise channel formulation proposed in Ref. [1], which has a wide range of applications for the estimation of small noise parameters on quantum channels. An important example of its application is the noise estimation in quantum computing [2] in which small noises induced by the environment can lead to a serious obstacle to its implementation. One of the best way to eradicate the noise effect is to apply an appropriate error correcting code to the system. In this scheme, one first needs to know properties of the noises to design the code. However, because the noises are generated by very complicated many-body interactions, it is difficult to calculate the properties theoretically from first principles. Therefore, it is effective for finding out the properties to measure the noise experimentally and to estimate them from the experimental data. With this method, the estimation of such very weak noises may suffer from large ambiguity of experimental errors. Thus the estimation theory of the low-noise channel will play a significant role in order to improve the performance of the estimation. Another example to which the estimation theory of the low-noise channel can be applied is found in rare processes of the elementary particle physics [3]. Many theories of new particles has predicted the existence of very weak interactions which generate rare decays and rare reactions of particles. However, in real experiments, the number of signals indicating new physics is generally small, compared with that of standard-model signals. Thus, siding with the standard model, the evidential signals can be regarded as low-noise background datum. The estimation theory of the low-noise channel is expected to improve accuracy of estimation of such tiny reaction rates and to assist in the finding of new physics.

An estimation theory for the low-noise channel has been formulated for one noise parameter, termed a low-noise parameter, which is assumed to be unknown and very small [1]. The increase of the Fisher information due to the prior entanglement with non-local output measurement, which is called the ancilla-assisted enhancement, has been analyzed for the ancilla-extended version of the low-noise channel [1]. The low-noise channel has been also extended to the channel on the many-body system [4], and this work shows that the maximum of the Fisher information can be attained by a factorized input state.
In this paper, we address the estimation problem of multiple low-noise parameters \( \epsilon = (\epsilon^1, \cdots, \epsilon^D) \), where \( D \) is the number of the parameters. We assume that all of \( \epsilon^\mu \) are non-negative and take very small values of the same order. In later analysis, \( O(\epsilon) \) denotes the order of \( \epsilon^\mu \), and the orders of products of the parameters, \( \prod_{\mu=1}^{D} (\epsilon^\mu)^n\mu \), are denoted by \( O(\epsilon^{n_1+\cdots+n_D}) \).

Due to the multi-parameter degrees of freedom, a large variety of physical phenomena can be analyzed by use of this low-noise channel.

The low-noise channel are dependent on the unknown parameters \( \epsilon \) which are estimated by measurements. Consider an input state \( \rho_{\text{in}} \), which is independent of \( \epsilon \), into the low-noise channel \( \Gamma_\epsilon \). The output state \( \rho = \rho_{\text{out}}(\epsilon) \) is obtained by
\[
\rho_{\text{out}}(\epsilon) = \Gamma_\epsilon[\rho_{\text{in}}].
\] (1)

If the purity of the output states is decreased, then the low-noise channel is called dissipative. In what follows, we focus on the dissipative low-noise channel.

To estimate the low-noise parameters, a POVM measurement with \( D \) measurement values are performed on the output state \( \rho_{\text{out}}(\epsilon) \). The measurement values are regarded as estimation values for \( \epsilon^\mu \) and this POVM is called an estimator. When the expectation value of the estimation value is equal to the true value of the parameter \( \epsilon^\mu \) for any \( \mu \), the estimator is called locally unbiased. In the present paper the locally unbiased estimator is defined in the vicinity of \( \epsilon = 0 \). Note that by virtue of the Naimark theorem [5, 6], the POVM measurement is equivalent to a measurement of commutative observables (projection-valued measures) in a composite system comprising the original system and an ancilla system.

One of the fundamental quantities in the low-noise estimation theory is the Fisher information matrix of an output state of the low-noise channel \( \Gamma_\epsilon \). It is well known as the Cramér-Rao inequality that the inverse of the Fisher information matrix is a lower bound of the mean-square-error matrix of the locally unbiased estimators [5, 6]. In this paper, we focus our attention on the achievement problem of the lower bound for the dissipative low-noise channel. In the case with one low-noise parameter, the bound can be attained by using certain estimators. In contrast, the bound cannot be always achieved if \( D > 1 \). Even if a certain unbiased estimator is optimal for the parameter estimation of \( \epsilon^1 \), the estimator is not always optimal for other parameters \( \epsilon^2, \cdots, \epsilon^D \), because of the non-commutativity of optimal estimators. It is an important open problem to reveal the necessary and sufficient condition
for the attainability of the bound for the multi-parameter estimation. The solution for general channels has not been found to date. Recent progress has been reported on the multi-parameter estimation for special families of quantum channels, in particular, SU(N) channels [7, 8, 9] and a generalized Pauli channel [10].

Our main results are related to the attainability problem for the estimation in the dissipative low-noise channel. To explain the results, let us diagonalize the output state $\rho(\epsilon)$ as

$$\rho(\epsilon) = p_0(\epsilon)|0(\epsilon)\rangle\langle 0(\epsilon)| + \sum_{n=1}^{N-1} p_n(\epsilon)|n(\epsilon)\rangle\langle n(\epsilon)|,$$

where the first eigenvalue $p_0(\epsilon)$ is $O(\epsilon^0)$, the other eigenvalues $p_{n\neq 0}(\epsilon)$ are $O(\epsilon)$, and $N$ denotes the dimension of the Hilbert space of the quantum system (the reason why this expression is available is shown in Sec. 4). The Fisher information of the output state for the channel has a leading term of order of $O(\epsilon^{-1})$. We first prove in the case with $D \leq N-1$ that all output states of the dissipative low-noise channel have locally unbiased estimators by which the lower bound is attained in $O(\epsilon)$ assuming that

$$\det_{\mu\nu} \left[ \sum_{n=0}^{N-1} \partial_\mu \sqrt{p_n(\epsilon)} \partial_\nu \sqrt{p_n(\epsilon)} \right] \neq 0 \quad (2)$$

in the vicinity of $\epsilon = 0$, where $\partial_\mu$ stand for $\partial/\partial^\mu$. In order to explain the meaning of Eq. (2), let us consider a map $(\sqrt{p_0(\epsilon)}, \cdots, \sqrt{p_{N-1}(\epsilon)})$ which generates a manifold by running the $D$ parameters of $\epsilon$. Clearly, if $D > N-1$, Eq. (2) does not hold because the target space defined by

$$\left\{ (\sqrt{p_0}, \cdots, \sqrt{p_{N-1}}) \left| p_i \geq 0, \sum_{i=0}^{N-1} p_i = 1 \right. \right\}$$

has $N-1$ dimensions and the parameterization of the manifold by $\epsilon$ becomes degenerate. On the other hand, when $D \leq N-1$, it is noticed that imposition of Eq. (2) is not so difficult. If the dimension of the manifold is equal to $D$, the relation in Eq. (2) automatically holds. Therefore, the condition in Eq. (2) is not anomalous and can be satisfied for many standard low-noise channels.

Next, we point out that, although the lower bound is not always achieved even in $O(\epsilon^0)$ when $D > N-1$, the entanglement with ancilla systems can
make the lower bound attainable again in \( O(\epsilon) \) when \( N^2 - 1 \geq D \). As is expected from the fact that the dissipative low-noise channel comprise a rather general class of quantum channels, these results are very surprising because they reveals that an enormous number of channels can overcome possible non-commutativity among the optimal estimators to attain the Cramér-Rao lower bound.

## 2 Brief Review of State-Parameter Estimation

We start with a brief review of a parameter estimation theory of quantum states, since the channel estimation theory is based on the state estimation theory. Let us consider a state \( \rho \) dependent on \( D \) unknown parameters \( \theta^\mu \),

\[
\rho = \rho(\theta) = \rho(\theta^1, \cdots, \theta^D).
\]

(3)

The parameters \( \theta^\mu \) are estimated by measuring the state \( \rho \) with a positive operator-valued measure (POVM) \( \{ \Pi_x | x = (x^1, \cdots, x^D) \} \) which satisfies

\[
\Pi_x \geq 0, \quad \int \Pi_x d^D x = 1,
\]

(4)

by definition. The POVM \( \Pi_x \) is called the estimator for the parameter estimation. When the condition

\[
\int x^\mu \text{Tr}[\Pi_x \rho(\theta)] d^D x = \theta^\mu
\]

is satisfied, the estimator is called unbiased. For the unbiased estimator, the mean-square-error matrix \( V^{\mu\nu} \) is defined by

\[
V^{\mu\nu}[\Pi_x] := \int (x^\mu - \theta^\mu)(x^\nu - \theta^\nu) \text{Tr}[\Pi_x \rho(\theta)] d^D x.
\]

(6)

As is well known, there exists a universal lower bound of the error matrix \( V^{\mu\nu} \), called the Cramér-Rao bound [5, 6], as

\[
V \geq J^{-1},
\]

(7)
where $J$ stands for the Fisher information matrix for $\rho$ defined by

$$J_{\mu \nu} = \frac{1}{2} \text{Tr}[\rho(L_\mu L_\nu + L_\nu L_\mu)],$$

and $L_\mu$'s are the symmetric logarithmic derivative (SLD) defined by

$$L_\mu^\dagger = L_\mu, \quad \frac{\partial}{\partial \theta_\mu} \rho = \frac{1}{2} (L_\mu \rho + \rho L_\mu).$$

Eq. (7) is called the Cramér-Rao inequality and implies that for an arbitrary $D$-dimensional vector $\vec{u}$, the following relation is always satisfied,

$$\vec{u} V \vec{u} \geq \vec{u} J^{-1} \vec{u}.$$  

Note that the equality is not always attainable. This aspect is related to the non-commutativity of quantum measurements. When $D = 1$, measurement of the SLD operator $L_1$ is optimal for $\theta^1$ estimation, that is, the equality $V = J^{-1}$ is satisfied. When $D = 2$, each measurement of the SLD operator $L_\mu$ is optimal for $\theta^\mu$ estimation independently. However, the observables $L_1$ and $L_2$ do not generally commute to each other,

$$[L_1, L_2] \neq 0.$$ 

This non-commutativity of SLD’s makes achievement conditions of the bound nontrivial. For the pure-state case ($\rho(\theta) = |\phi(\theta)\rangle \langle \phi(\theta)|$), it has been proven in Ref. [11] that the bound can be always attained. However, generally, the necessary and sufficient condition for the bound achievement has not been known.

The state estimation theory explained thus far can be applied to the channel estimation done as follows. Let us consider a $\theta^\mu$-parameterized channel $\Gamma_\theta$. The channel generates an output state $\rho(\theta) = \Gamma_\theta[\rho_{in}]$ from a input state $\rho_{in}$ independent of $\theta^\mu$. The parameter $\theta^\mu$ on the channel is estimated by use of the state estimation theory for the output state $\rho(\theta)$.

The Fisher information matrix $J[\rho_{in}]$ for the output state changes as the input state $\rho_{in}$ changes. Hence, it is important to find input states which optimize $J$. For this problem, there is a useful theorem [12] related to choice of the input state. There exists a pure input state $|\phi\rangle \langle \phi|$ for which the Fisher information matrix $J$ is the maximum over all input states $\rho_{in}$, that is, for an arbitrary vector $\vec{u}$, the following relation holds.

$$\vec{u} J[|\phi\rangle \langle \phi|] \vec{u} \geq \vec{u} J[\rho_{in}] \vec{u}.$$  

5
As was seen in Eq. (7), the mean-square-error matrix \( V \) is lower bounded by the inverse of \( J \), therefore, without loss of generality, we may concentrate on pure input states in later discussions.

As mentioned above, the Cramér-Rao bound for the output states cannot be always attained and the necessary and sufficient condition for its achievement in channel estimation is a crucial open problem. For unitary channels, the achievability problems of the bound can be completely solved by use of the result in Ref. [11], since output states of the channels for pure input states remain pure, and thus the estimation problem of the channels is reduced to that of the output pure states. To date, the solution of the lower-bound achievement has not been found for general channels.

3 Multi-Parameter Low-Noise Channel

In this section, we define the low-noise channel \( \Gamma_\epsilon \) with unknown multi-parameter \( \epsilon \). This channel is a natural extension of the low-noise channel with one parameter first introduced in Ref. [1]. The definition is given by a Kraus representation as

\[
\rho(\epsilon) = \Gamma_\epsilon[\rho] = \sum_\alpha B_\alpha(\epsilon)\rho B_\alpha^\dagger(\epsilon) + \sum_{\mu=1}^{D} e^\mu \sum_{a=1}^{K_\mu} C_{\mu a}(\epsilon)\rho C_{\mu a}^\dagger(\epsilon),
\]

where the Kraus operators satisfy the following four conditions.

(i) The channel is a TPCP map:

\[
\sum_\alpha B_\alpha^\dagger(\epsilon)B_\alpha(\epsilon) + \sum_{\mu=1}^{D} e^\mu \sum_{a=1}^{K_\mu} C_{\mu a}^\dagger(\epsilon)C_{\mu a}(\epsilon) = 1_S.
\]

(ii) \( B_\alpha(\epsilon) \) is analytic at \( \epsilon = 0 \), giving the power series expansion

\[
B_\alpha(\epsilon) = \kappa_\alpha 1_S - \sum_{(n_1, \ldots, n_D) \neq (0, \ldots, 0)} N_\alpha^{(n_1 \ldots n_D)}(\epsilon)^{n_1} \cdots (\epsilon^D)^{n_D},
\]

in the neighborhood of \( \epsilon = 0 \), where \( \kappa_\alpha \) and \( N_\alpha^{(n_1 \ldots n_D)} \) are coefficients and operators, respectively, independent of \( \epsilon \).
(iii) $\kappa_\alpha$ satisfies
\[ \sum_\alpha |\kappa_\alpha|^2 = 1. \]

(iv) $C_{\mu\alpha}(\epsilon)$ is analytic at $\epsilon = 0$, giving the power series expansion
\[ C_{\mu\alpha}(\epsilon) = M_{\mu\alpha} + \sum_{(n_1, \ldots, n_D) \neq (0, \ldots, 0)} M_{\mu\alpha}^{(n_1 \cdots n_D)} (\epsilon^1)^{n_1} \cdots (\epsilon^D)^{n_D}, \]
where $M_{\mu\alpha}$ and $M_{\mu\alpha}^{(n_1 \cdots n_D)}$ are operators independent of $\epsilon$.

The condition (i) is a natural characteristic of physical channels, because all physical channels are TPCP maps and any TPCP map has Kraus representations. The conditions (ii) and (iv) simply imply that the channel shows nonsingular behavior near $\epsilon = 0$. Therefore, taking proper limits of weak coupling with the environment, physical processes induced by the environment can always be described by this low-noise channel. From condition (iii), the channel automatically reduces to the identical channel in the limit of vanishing the parameters:
\[ \lim_{\epsilon \to +0} \Gamma_\epsilon = \text{id}_S, \]
where $\text{id}_S$ stands for the identity channel on the system $S$. This shows that the parameters $e^\mu$ can represent noise parameters caused by the environment.

The second term of the right-hand-side of Eq. (12) describes a dissipative effect by the environment in the lowest order of $\epsilon$ and plays a crucial role in the later discussion. Without loss of generality, we may assume that each of $C_{\mu\alpha}$ is not proportional to others. If not, the second term can be rewritten in the form of Eq. (12) with a smaller number of independent $C_{\mu\alpha}(\epsilon)$ by using coordinate transformation of $e^\mu$. For the sake of convenience, let us denote $K$ the total number of $C_{\mu\alpha}(\epsilon)$ as
\[ K = \sum_{\mu=1}^D K_\mu. \]

In addition, we redefine the Kraus operators as
\[ N_{1\alpha} = N_\alpha^{(100 \cdots 0)}, N_{2\alpha} = N_\alpha^{(010 \cdots 0)}, \ldots, N_{D\alpha} = N_\alpha^{(00 \cdots 01)}. \]
Then $B_\alpha(\epsilon)$ is rewritten as
\[ B_\alpha(\epsilon) = \kappa_\alpha 1 - \sum_{\mu=1}^D e^\mu N_{\mu\alpha} + O(\epsilon^2). \]
The TPCP condition (Eq. (13)) in the lowest order of the parameter \( \epsilon \) is given by
\[
\sum_a M_{\mu a}^\dagger M_{\mu a} = \sum_a \left( \kappa_\alpha N_{\mu a}^\dagger + \kappa_\alpha^* N_{\mu a} \right).
\] (19)
Due to this relation, the operator \( \sum_a \kappa_\alpha^* N_{\mu a} \) can be broken down into a sum of two terms such that
\[
\sum_a \kappa_\alpha^* N_{\mu a} = \frac{1}{2} \sum_a M_{\mu a}^\dagger M_{\mu a} + i H_\mu,
\] (20)
where \( H_\mu \) are Hermitian operators. Using Eqs. (12) and (20), the derivative of \( \rho(\epsilon) \) at \( \epsilon = 0 \) is evaluated as
\[
\partial_\mu \rho(0) = \sum_{a=1}^{K_\mu} \left[ M_{\mu a} \rho(0) M_{\mu a}^\dagger - \frac{1}{2} M_{\mu a}^\dagger M_{\mu a} \rho(0) - \frac{1}{2} \rho(0) M_{\mu a}^\dagger M_{\mu a} \right] - i[H_\mu, \rho(0)].
\] (21)
The first term of the right hand side indicates a noise effect with decoherence induced by the environment. On the other hand, the second term expresses a unitary time evolution of the system \( S \). Therefore, if the first term vanishes, the purity of the output states does not change in \( O(\epsilon) \). In the later discussions, we assume that the operators \( M_{\mu a} \) are non-vanishing and thus, in general, the second term does not disappear. This assumption implies that decoherence by the environment inevitably takes place in the system \( S \). In this case, the low-noise channel is called dissipative.

4 Channel Estimation

We now discuss the attainability of the Cramér-Rao inequality for the multi-parameter estimation in the low-noise channel. Let us consider a pure input state \( |\phi\rangle \langle \phi| \) for the low-noise channel \( \Gamma_\epsilon \). The output state \( \rho(\epsilon) = \Gamma_\epsilon[|\phi\rangle \langle \phi|] \) can be diagonalized as
\[
\rho(\epsilon) = \sum_{n=0}^{N-1} p_n(\epsilon) |n(\epsilon)\rangle \langle n(\epsilon)|
\] (22)
with an orthonormal basis \( \{ |n(\epsilon)\rangle \} \). Because of a property of the low-noise channel (See Eq. (16)), we are able to impose the following boundary conditions, \( \rho(0) = |\phi\rangle \langle \phi |, p_0(0) = 1 \), and \( p_{n\neq0}(0) = 0 \), which imply \( |0(0)\rangle = |\phi\rangle \).

The deviation operator \( \delta\rho(\epsilon) = \rho(\epsilon) - |\phi\rangle \langle \phi | \) between the input state and the output state is evaluated by use of Eq. (21) as

\[
\delta\rho(\epsilon) = \sum_{\mu} \epsilon^\mu \partial_\mu \rho(0) + O(\epsilon^2) \\
= \sum_{\mu} \epsilon^\mu \sum_a \left[ M_{\mu a} |\phi\rangle \langle \phi | M_{\mu a}^\dagger - \frac{1}{2} M_{\mu a}^\dagger M_{\mu a} |\phi\rangle \langle \phi | - \frac{1}{2} |\phi\rangle \langle \phi | M_{\mu a}^\dagger M_{\mu a} \right] \\
- i \sum_{\mu} \epsilon^\mu [H_{\mu}, |\phi\rangle \langle \phi |] + O(\epsilon^2). \tag{23}
\]

Here, let us define an \( N - 1 \) dimensional matrix by

\[
\Delta(\epsilon) = [\langle n(0) \delta\rho(\epsilon) | n'(0) \rangle]. \tag{24}
\]

Note that all of the orthonormal vectors \( |n(0)\rangle \) \((n = 1, \ldots, N - 1)\) are orthogonal to \( |\phi\rangle \) by definition. Then we see that the matrix \( \Delta(\epsilon) \) is rewritten using Eq. (23) such that

\[
\Delta(\epsilon) = \left[ \sum_{\mu a} \epsilon^\mu \langle n(0) | M_{\mu a} |\phi\rangle \langle \phi | M_{\mu a}^\dagger | n'(0) \rangle \right] + O(\epsilon^2). \tag{25}
\]

The leading term of \( \Delta(\epsilon) \) is \( O(\epsilon) \), because, as mentioned before, the Kraus operators \( M_{\mu a} \) do not vanish due to the assumption of the dissipative low-noise channel. Thus the matrix \( \Delta(\epsilon) \) possesses \( O(\epsilon) \) eigenvalues. Some eigenvalues of \( \Delta(\epsilon) \) may be higher terms as \( O(\epsilon^2) \) or exactly zero. However, the presence of such eigenvalues does not affect the following analysis at all.

The matrix \( \Delta(\epsilon) \) becomes a diagonal matrix with the basis \( \{ |n(0)\rangle \} \) in \( O(\epsilon) \) and the \( N - 1 \) eigenvalues are \( \delta p_n \) defined by \( \delta p_n(\epsilon) = p_n(\epsilon) - p_n(0) \), that is,

\[
\delta\rho(\epsilon)|n(0)\rangle = \delta p_n(\epsilon)|n(0)\rangle + O(\epsilon^2), \quad (n = 1, \ldots, N - 1). \tag{26}
\]

It is worthwhile noting that the eigenvalues \( \delta p_n(\epsilon) \) are unchanged even if we use a different orthonormal basis \( |\tilde{m}(0)\rangle := \sum_n U_{mn} |n(0)\rangle \) in the definition of \( \Delta(\epsilon) \), where \( U_{mn} \) is a unitary matrix. Thus, when one calculate \( \delta p_n(\epsilon) \), the
orthonormal basis \{\ket{n(0)}\} can be arbitrarily fixed in Eq. (26). In addition, for \( K \leq N - 1 \), non-vanishing \( \delta p_{n}(\epsilon) \)s can be obtained by solving another eigenvalue equation (See Appendix). Using the eigenvalues \( \delta p_{n}(\epsilon) \), we obtain another expression of \( \rho(\epsilon) \) such that

\[
\rho(\epsilon) = |\phi\rangle\langle\phi| + \sum_{\mu} \epsilon^{\mu} \sum_{n=0}^{N-1} (\partial_{\mu} \delta p_{n}(\epsilon) |n(\epsilon)\rangle \langle n(\epsilon)| + |\partial_{\mu} 0(\epsilon)\rangle + |\phi\rangle \langle \partial_{\mu} 0(\epsilon)|) + O(\epsilon^{2}),
\]

(27)

where \( |\partial_{\mu} 0(\epsilon)\rangle \) means \( \partial_{\mu} |0(\epsilon)\rangle \). From Eq. (27), the following relation is straightforwardly obtained,

\[
\langle n(\epsilon)| \partial_{\mu} \rho(\epsilon) |m(\epsilon)\rangle = \partial_{\mu} \delta p_{n}(\epsilon) \delta_{nm} + O(\epsilon),
\]

(28)

for \( n, m \neq 0 \).

Now we describe the Fisher information matrix for the low noise channel. From Eq. (22), the SLD’s are calculated [5] as

\[
\langle n(\epsilon)| L_{\mu}(\epsilon)|m(\epsilon)\rangle = \frac{2}{p_{n}(\epsilon) + p_{m}(\epsilon)} \langle n(\epsilon)| \partial_{\mu} \rho(\epsilon) |m(\epsilon)\rangle.
\]

(29)

Using this, we have the Fisher information matrix \( J(\epsilon) = [J_{\mu\nu}(\epsilon)] \) as

\[
J_{\mu\nu}(\epsilon) = \sum_{nm} \langle n(\epsilon)| \partial_{\mu} \rho(\epsilon) |m(\epsilon)\rangle \frac{2}{p_{n}(\epsilon) + p_{m}(\epsilon)} \langle m(\epsilon)| \partial_{\nu} \rho(\epsilon) |n(\epsilon)\rangle.
\]

(30)

If all of the eigenvalues \( \delta p_{n}(\epsilon) \) are \( O(\epsilon) \), the Fisher information matrix \( J(\epsilon) \) is evaluated from Eq. (28) as

\[
J_{\mu\nu}(\epsilon) = \sum_{n=1}^{N-1} \frac{\partial_{\mu} \delta p_{n}(\epsilon) \partial_{\nu} \delta p_{n}(\epsilon)}{\delta p_{n}(\epsilon)} + O(\epsilon^{0}),
\]

(31)

If some of the eigenvalues \( \delta p_{n}(\epsilon) \) are \( O(\epsilon^{2}) \) or exactly zeros, the terms associated with the \( O(\epsilon^{2}) \) eigenvalues in the sum of Eq. (31) can be neglected. Taking account of a fact that the leading order of \( J(\epsilon) \) is \( O(\epsilon^{-1}) \), let us define a divergent part of \( J(\epsilon) \) by

\[
J_{\mu\nu}^{\text{div}}(\epsilon) := \sum_{n=1}^{N-1} \frac{\partial_{\mu} \delta p_{n}(\epsilon) \partial_{\nu} \delta p_{n}(\epsilon)}{\delta p_{n}(\epsilon)}.
\]

(32)
It is worthwhile noting that the classical Fisher information $J^c_{\mu\nu}(\epsilon)$ defined by

$$J^c_{\mu\nu}(\epsilon) := \sum_{n=0}^{N-1} \frac{\partial_\mu p_n(\epsilon) \partial_\nu p_n(\epsilon)}{p_n(\epsilon)} = 4 \sum_{n=0}^{N-1} \frac{\partial_\mu \sqrt{p_n(\epsilon)} \partial_\nu \sqrt{p_n(\epsilon)}}{p_n(\epsilon)}$$  \hspace{1cm} (33)$$

coincides with $J^{\text{div}}_{\mu\nu}(\epsilon)$ in $O(\epsilon^{-1})$:

$$J^{\text{div}}_{\mu\nu}(\epsilon) = J^c_{\mu\nu}(\epsilon) + O(\epsilon^0).$$  \hspace{1cm} (34)$$

The classical information matrix $J^c_{\mu\nu}(\epsilon)$ is an induced metric of a $D$ dimensional manifold $M$ embedded in an $N-1$ dimensional space $P$ defined by

$$P := \{ (\sqrt{p_0}, \cdots, \sqrt{p_{N-1}}) | p_i \geq 0, \sum_{i=0}^{N-1} p_i = 1 \}.$$  \hspace{1cm} (35)$$

When $(\epsilon^1, \cdots, \epsilon^D)$ locally has a one-to-one correspondence with a point on the manifold $M$, the matrix $J^c_{\mu\nu}(\epsilon)$ has its inverse matrix $J^{-1}_{\mu\nu}(\epsilon)$ which is $O(\epsilon)$. If $D > N-1$, some parameters of $\epsilon$ are redundant and the parameterization of $p_n(\epsilon)$ by $\epsilon^\mu$ becomes degenerate, that is,

$$\det_{\mu\nu} \left[ \sum_{n=0}^{N-1} \partial_\mu \sqrt{p_n(\epsilon)} \partial_\nu \sqrt{p_n(\epsilon)} \right] = 0.$$  \hspace{1cm} (36)$$

In later analysis, we assume non-degeneracy of the parameterization;

$$\det_{\mu\nu} \left[ \sum_{n=0}^{N-1} \partial_\mu \sqrt{p_n(\epsilon)} \partial_\nu \sqrt{p_n(\epsilon)} \right] \neq 0$$  \hspace{1cm} (37)$$

for $D \leq N-1$. In this case, the quantum Fisher information $J(\epsilon)$ also has its inverse matrix $J^{-1}(\epsilon) = [J^{\mu\nu}(\epsilon)]$, whose order is $O(\epsilon)$, and satisfies that

$$J^{\mu\nu}(\epsilon) = J^{\text{div},\mu\nu}(\epsilon) + O(\epsilon^2).$$  \hspace{1cm} (38)$$

Now we shall prove that a locally unbiased estimator, which attains the Cramér-Rao bound in $O(\epsilon)$, can be explicitly constructed. Firstly, let us introduce commuting Hermitian operators $A_\mu(\epsilon)$ given by

$$A_\mu(\epsilon) := \sum_{n=1}^{N-1} \frac{\partial_\mu \delta p_n(\epsilon)}{\delta p_n(\epsilon)} |n(\epsilon)\rangle \langle n(\epsilon)|.$$  \hspace{1cm} (39)$$
If some of $\delta p_n(\epsilon)$ are $O(\epsilon^2)$, the corresponding terms are neglected in the sum of Eq. (37). Their contravariant operators $A^\mu(\epsilon)$ are also defined by

$$A^\mu(\epsilon) := \sum_\nu J^{\text{div},\mu\nu}(\epsilon) A^\nu(\epsilon). \quad (38)$$

Note that the order of $A^\mu(\epsilon)$ is $O(\epsilon^0)$. This operator can be written by the spectral decomposition as

$$A^\mu(\epsilon) = \sum_n x^\mu_n(\epsilon) P_n(\epsilon), \quad (39)$$

where $P_n(\epsilon)$ are projection operators satisfying $\sum_n P_n(\epsilon) = 1$. Here we define a POVM operator by

$$\Pi_x = \sum_{n=1}^{N-1} \prod_{\mu=1}^D \delta (x^\mu - x^\mu_n(\epsilon)) P_n(\epsilon). \quad (40)$$

We adopt $\Pi_x$ as an estimator for a measurement to estimate $\epsilon$. Then it is proven from Eq. (28) that the estimator satisfies

$$\int x^\mu \text{Tr}[\Pi_x \rho(\epsilon)] d^D x = \text{Tr}[A^\mu(0) \rho(\epsilon)] = \epsilon^\mu + O(\epsilon^2). \quad (41)$$

Thus the estimator $\Pi_x$ is locally unbiased for the $\epsilon$ estimation in the vicinity of $\epsilon = 0$. It is also shown that the estimator locally attains the Cramér-Rao bound as follows. Using Eq. (37) and Eq. (38), we have

$$\frac{1}{2} \text{Tr}[\rho(\epsilon) \{A^\mu(\epsilon), A^\nu(\epsilon)\}] = J^{\text{div},\mu\nu}(\epsilon) + O(\epsilon^2). \quad (42)$$

On the other hand, the mean-square-error matrix $V^{\mu\nu}$ for $\Pi_x$ is evaluated as

$$V^{\mu\nu} = \int (x^\mu - \epsilon^\mu) (x^\nu - \epsilon^\nu) \text{Tr}[\Pi_x \rho(\epsilon)] d^D x = \frac{1}{2} \text{Tr}[\rho(\epsilon) \{A^\mu(\epsilon), A^\nu(\epsilon)\}] + O(\epsilon^2).$$

Consequently, we see that the Cramér-Rao bound is actually satisfied in $O(\epsilon)$:

$$V^{\mu\nu} = J^{\mu\nu}(\epsilon) + O(\epsilon^2). \quad (43)$$
This is a main result of this paper. This implies that the lower bound can always be achieved in the estimation of the dissipative low-noise channel for $D \leq N - 1$ if it satisfies Eq. (35). Because, generally, multi-parameter estimation cannot attain the bound, this result is very significant.

In the case with $D > N - 1$, we are able to give another important discussion. The inverse matrix $J^{-1}(\epsilon)$ of the quantum Fisher information matrix may exist in this case, even though the classical Fisher information matrix $J^c_{\mu\nu}(\epsilon)$ does not have its inverse. However, even if the inverse matrix exists, it is not sufficiently suppressed in general. Its components are generally $O(\epsilon^0)$, not $O(\epsilon)$. Moreover, the Cramér-Rao bound cannot be always attained even in $O(\epsilon^0)$. The disadvantage can be remedied by using the ancilla-extension of the low-noise channel [4]. In fact, the bound becomes attainable again when $D \leq N^2 - 1$. This is the second result of this paper.

To see this, let us consider an ancilla system $A$ which has a Hilbert space with the same dimension $N$ as that of the original system $S$. Suppose that the dissipative low-noise channel is extended to $\Gamma_{\epsilon} \otimes id_A$, where $id_A$ stands for the identity channel on the ancilla system. Entangled states of the composite system $S + A$ are available as input states for the channels and a collective measurements is performed for the output states in order to estimate the $D$ parameters $\epsilon^\mu$. In this case, the dimension $N_{S+A}$ of the Hilbert space of $S + A$ is larger than that of $S$ because $N_{S+A} = N^2$. Therefore, when the number of the parameters $\epsilon^\mu$ is less than $N_{S+A}$ ($D \leq N_{S+A} - 1$), the above analysis shows that the Cramér-Rao bound can be attained in $O(\epsilon)$ assuming that the ancilla-extended channels satisfy Eq. (35) for their output states. Consequently, when $N \leq D \leq N^2 - 1$, the ancilla-assisted enhancement effect is able to make the Cramér-Rao bound achievable in $O(\epsilon)$.

5 Examples

Let us consider a two-parameter dissipative low-noise channel $\Gamma_{\epsilon}$ for a system $S$ with $N = 3$ and $D = 2$ which is a simple example satisfying $K \leq N - 1$, and focus on the case with $K_1 = K_2 = 1$. Suppose that an input state $|\phi\rangle$ go through the channel. We apply the method described in Appendix to solve the eigenvalues of the $\delta p_{\pm}(\epsilon)$ of the output state. In this
present system, the matrix \( \Lambda(\epsilon) \) defined by Eq. (55) can be written as

\[
\Lambda(\epsilon) = \left[ \sqrt{\epsilon^2 \langle \phi | (M_\mu - \langle \phi | M_\mu | \phi \rangle) (M_\nu - \langle \phi | M_\nu | \phi \rangle) | \phi \rangle} \right],
\]

where \( M_\mu \) denotes \( M_{\mu 1} \) with \( \mu = 1, 2, 3 \). The following eigenvalues is obtained by solving Eq. (54),

\[
\delta p_\pm(\epsilon) = \frac{1}{2} \left[ \epsilon^1 \delta M_{11} + \epsilon^2 \delta M_{22} \pm \sqrt{(\epsilon^1 \delta M_{11} - \epsilon^2 \delta M_{22})^2 + 4\epsilon^1 \epsilon^2 \delta M_{12} \delta M_{21}} \right] = O(\epsilon),
\]

where \( \delta M_{\mu \nu} \) are variance-matrix elements defined by

\[
\begin{bmatrix}
\delta M_{11} & \delta M_{12} \\
\delta M_{21} & \delta M_{22}
\end{bmatrix}
= \left[ \langle \phi | (M_\mu - \langle \phi | M_\mu | \phi \rangle) (M_\nu - \langle \phi | M_\nu | \phi \rangle) | \phi \rangle \right].
\]

The inverse of the Fisher information matrix can be explicitly calculated as

\[
J_{11} = \frac{(\epsilon^1)^3 \delta M_{11} \det \delta M + (\epsilon^1)^2 \epsilon^2 \delta M_{22} [3\delta M_{12} \delta M_{21} - 2\delta M_{11} \delta M_{22}] + \epsilon^1 (\epsilon^2)^2 \delta M_{22}^3}{(\delta M_{11} \delta M_{22} - \delta M_{12} \delta M_{21}) (\epsilon^1 \delta M_{11} - \epsilon^2 \delta M_{22})^2},
\]

\[
J_{22} = \frac{(\epsilon^2)^3 \delta M_{22} \det \delta M + (\epsilon^2)^2 \epsilon^1 \delta M_{11} [3\delta M_{12} \delta M_{21} - 2\delta M_{11} \delta M_{22}] + \epsilon^2 (\epsilon^1)^2 \delta M_{11}^3}{(\delta M_{11} \delta M_{22} - \delta M_{12} \delta M_{21}) (\epsilon^1 \delta M_{11} - \epsilon^2 \delta M_{22})^2},
\]

\[
J_{12} = J_{21} = -\epsilon^1 \epsilon^2 \frac{\delta M_{12} \delta M_{21}}{\delta M_{11} \delta M_{22} - \delta M_{12} \delta M_{21} (\epsilon^1 \delta M_{11} - \epsilon^2 \delta M_{22})^2}.
\]

Hence, for input states which satisfy

\[
\delta M_{11} \delta M_{22} - \delta M_{12} \delta M_{21} = O(\epsilon^0),
\]

\[
n^1 \delta M_{11} - n^2 \delta M_{22} = O(\epsilon^0),
\]

with \( n^\mu := \epsilon^\mu / \epsilon \), the inverse matrix \( J^{-1} \) behaves as \( O(\epsilon) \). Thus, the Cramér-Rao lower bound can be attained in \( O(\epsilon) \), since one can make an locally unbiased estimator to attain the bound by following the procedure in the previous section.

Next, let us give another example for \( N = 2 \) and \( D = 2 \) to show the ancilla-assisted improvement as mentioned in Sec. 4. Now suppose that an input state \( \rho_{in} \) goes through a dissipative low-noise channel \( \tilde{\Gamma}_\epsilon \) defined by

\[
\tilde{\Gamma}_\epsilon [\rho_{in}] = (1 - \epsilon^1 - \epsilon^2) \rho_{in} + \epsilon^1 \sigma_x \rho_{in} \sigma_x + \epsilon^2 \sigma_z \rho_{in} \sigma_z,
\]

\[14\]
where $\epsilon^\mu$ are unknown low-noise parameters and $\sigma_x$ and $\sigma_y$ are the Pauli matrices. Its output state $\rho(\epsilon)$ can be expressed by a Bloch representation as

$$\tilde{\rho}(\epsilon) = \tilde{\Gamma}_\epsilon \rho_{in} = \frac{1}{2} + \frac{1}{2} \vec{y} \cdot \vec{\sigma},$$

(45)

where $\vec{y}$ is a Bloch vector and $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. The SLD for the output states is exactly solved and written as

$$L_\mu = l_{o,\mu} + \vec{l}_\mu \cdot \vec{\sigma},$$

(46)

where

$$l_{o,\mu} = -\frac{1}{2} \frac{\partial_{\mu} |\vec{y}|^2}{1 - |\vec{y}|^2},$$

and

$$\vec{l}_\mu = \partial_{\mu} \vec{y} + \frac{1}{2} \frac{\partial_{\mu} |\vec{y}|^2}{1 - |\vec{y}|^2} \vec{y}.$$  

The Fisher information matrix $J_{\mu\nu}$ is directly calculated as

$$J_{\mu\nu} = \partial_\mu \vec{y} \cdot \partial_\nu \vec{y} + \frac{1}{4} \frac{\partial_\mu |\vec{y}|^2 \partial_\nu |\vec{y}|^2}{1 - |\vec{y}|^2}.$$  

(47)

Defining $\delta p = 4(1 - |\vec{y}|^2)$ which vanishes when $\epsilon = 0$, the inverse matrix $J^{\mu\nu}$ is calculated as

$$J^{11} = \frac{|\partial_2 \vec{y}|^2 + \frac{1}{\delta p} (\partial_2 |\vec{y}|^2)^2}{|\partial_1 \vec{y}|^2 |\partial_2 \vec{y}|^2 - (\partial_1 \vec{y} \cdot \partial_2 \vec{y})^2 + \frac{1}{\delta p} |\partial_2 |\vec{y}|^2 \partial_1 \vec{y} - \partial_1 |\vec{y}|^2 \partial_2 \vec{y}|^2},$$

$$J^{22} = \frac{|\partial_1 \vec{y}|^2 |\partial_2 \vec{y}|^2 - (\partial_1 \vec{y} \cdot \partial_2 \vec{y})^2 + \frac{1}{\delta p} |\partial_2 |\vec{y}|^2 \partial_1 \vec{y} - \partial_1 |\vec{y}|^2 \partial_2 \vec{y}|^2}{|\partial_1 \vec{y}|^2 |\partial_2 \vec{y}|^2 - (\partial_1 \vec{y} \cdot \partial_2 \vec{y})^2 + \frac{1}{\delta p} |\partial_2 |\vec{y}|^2 \partial_1 \vec{y} - \partial_1 |\vec{y}|^2 \partial_2 \vec{y}|^2},$$

$$J^{12} = -\frac{\partial_1 \vec{y} \cdot \partial_2 \vec{y} + \frac{1}{\delta p} \partial_1 |\vec{y}|^2 \partial_2 |\vec{y}|^2}{|\partial_1 \vec{y}|^2 |\partial_2 \vec{y}|^2 - (\partial_1 \vec{y} \cdot \partial_2 \vec{y})^2 + \frac{1}{\delta p} |\partial_2 |\vec{y}|^2 \partial_1 \vec{y} - \partial_1 |\vec{y}|^2 \partial_2 \vec{y}|^2}.  \quad (48)$$

By solving the eigenvalue equation of $J^{-1}$, we can show that one of the eigenvalues is $O(\epsilon)$, but the other is $O(\epsilon^0)$, namely,

$$\text{diag}[J^{-1}] = \begin{bmatrix} O(\epsilon^0) & 0 \\ 0 & O(\epsilon) \end{bmatrix}.  \quad (49)$$
Therefore, due to the $O(\epsilon^0)$ lower bound, it is impossible to estimate the parameters with $O(\epsilon^{1/2})$ precision. This property is briefly illustrated when $\epsilon = 0$ at which the inverse matrix can be written as

$$J^{-1}(0) = \frac{1}{\Phi} \begin{bmatrix} \partial_2|\vec{y}|^2 & \partial_1|\vec{y}|^2 \\ -\partial_1|\vec{y}|^2 & \partial_2|\vec{y}|^2 \end{bmatrix},$$

with $\Phi := |\partial_2|\vec{y}|^2\partial_1 - \partial_1|\vec{y}|^2\partial_2|\vec{y}|^2$. It is easily seen that $J^{-1}$ has a zero eigenvalue with an eigenvector $[ \partial_1|\vec{y}|^2, \partial_2|\vec{y}|^2 ]^T$, that is,

$$J^{-1}(0) \begin{bmatrix} \partial_1|\vec{y}|^2 \\ \partial_2|\vec{y}|^2 \end{bmatrix} = \vec{0}.$$

It turns out in $O(\epsilon^0)$ that the vector $[ \partial_1|\vec{y}|^2, \partial_2|\vec{y}|^2 ]^T$ corresponds to the eigenvector with the eigenvalue of $O(\epsilon)$ and the other eigenvector $[ \partial_2|\vec{y}|^2, -\partial_1|\vec{y}|^2 ]^T$ has the eigenvalue of $O(\epsilon^0)$.

In order to attain the Cramér-Rao bound of $O(\epsilon)$, let us consider an ancilla-extension of the channel, $\tilde{\Gamma}_\epsilon \otimes \text{id}_A$, and its entangled input state $|\Psi\rangle$ described by

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|++\rangle + |--\rangle),$$

(50)

where $\sigma_z|\pm\rangle = \pm|\pm\rangle$. Then, the output state is given by $(\Gamma_\epsilon \otimes \text{id}_A)[|\Psi\rangle\langle\Psi|]$. In order to evaluate the eigenvalues $\delta p_n(\epsilon)$, let us define three vectors

$$|1\rangle = \frac{1}{\sqrt{2}}(|++\rangle - |--\rangle),$$

$$|2\rangle = |+-\rangle,$n

$$|3\rangle = |-+\rangle.$$  

(51)

Note that these vectors are mutually orthogonal and all of them are orthogonal to the input state $|\Psi\rangle$. Thus it is possible to represent the matrix $\Delta(\epsilon)$ (See Eq. (24)) with a basis $\{|1\rangle, |2\rangle, |3\rangle\}$ as

$$\Delta(\epsilon) = \begin{bmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^1/2 & \epsilon^1/2 \\ 0 & \epsilon^1/2 & \epsilon^1/2 \end{bmatrix}.$$  

(52)
By estimating eigenvalues of $\Delta(\epsilon)$, it turns out that all the three eigenvalues behave as $O(\epsilon)$. In fact, the three eigenvalues are evaluated as

$$\begin{align*}
\delta p_1(\epsilon) &= \epsilon^2, \\
\delta p_2(\epsilon) &= \epsilon^1, \\
\delta p_3(\epsilon) &= 0.
\end{align*}$$

Substituting $\delta p_n(\epsilon)$ into Eq. (31) gives the Fisher information matrix such that

$$[J_{\mu\nu}] = \begin{bmatrix} \frac{1}{\epsilon} & 0 \\ 0 & \frac{1}{\epsilon^2} \end{bmatrix} + O(\epsilon^0).$$

Here the term $\partial_{\mu} \delta p_3(\epsilon) \partial_{\nu} \delta p_3(\epsilon)/\delta p_3(\epsilon)$ in Eq. (31) has been set to zero. It can be verified that a mapping onto the manifold $M$ defined by

$$(\sqrt{p_0}, \sqrt{p_1}, \sqrt{p_2}, \sqrt{p_3}) := \left(\sqrt{1 - \delta p_1(\epsilon) - \delta p_2(\epsilon) - \delta p_3(\epsilon)}, \sqrt{\delta p_1(\epsilon)}, \sqrt{\delta p_2(\epsilon)}, \sqrt{\delta p_3(\epsilon)}\right)$$

is non-degenerate;

$$\det \left[ \sum_{n=0}^{3} \frac{\partial_1 \sqrt{p_n(\epsilon)} \partial_1 \sqrt{p_n(\epsilon)}}{\sum_{n=0}^{3} \partial_2 \sqrt{p_n(\epsilon)} \partial_1 \sqrt{p_n(\epsilon)}} \sum_{n=0}^{3} \frac{\partial_1 \sqrt{p_n(\epsilon)} \partial_2 \sqrt{p_n(\epsilon)}}{\sum_{n=0}^{3} \partial_2 \sqrt{p_n(\epsilon)} \partial_2 \sqrt{p_n(\epsilon)}} \right] \neq 0.$$ 

This non-degeneracy guarantees the existence of $[J_{\mu\nu}]$. The matrix $[J_{\mu\nu}]$ is suppressed as $O(\epsilon)$. In fact, $[J_{\mu\nu}]$ is given by

$$[J_{\mu\nu}] = \begin{bmatrix} \epsilon^1 & 0 \\ 0 & \epsilon^2 \end{bmatrix} + O(\epsilon^2).$$

Calculation of the projective operators $\{|n(0)\rangle\langle n(0)| : n = 0, 1, 2, 3\}$ is simple and results in

$$\begin{align*}
|0(0)\rangle\langle 0(0)| &= |\Psi\rangle\langle \Psi|, \\
|1(0)\rangle\langle 1(0)| &= \frac{1}{2} (|+\rangle - | - \rangle) (|+\rangle - | - \rangle), \\
|2(0)\rangle\langle 2(0)| &= \frac{1}{2} (|+\rangle + | - \rangle) (|+\rangle + | - \rangle), \\
|3(0)\rangle\langle 3(0)| &= \frac{1}{2} (|+\rangle - | - \rangle) (|+\rangle - | - \rangle).
\end{align*}$$

Adopting the estimator, the ancilla-extension of the channel really makes the Cramér-Rao bound achievable in $O(\epsilon)$.
Acknowledgement

We would like to thank Masanao Ozawa for useful discussions in the first stage of this research. This research was partially supported by the SCOPE project of the MIC.

Appendix

Here, we shall show that the non-vanishing $\delta p_{n}(\epsilon)$ satisfy

$$\det [\delta p_{n}(\epsilon)1_{K} - \Lambda(\epsilon)] = 0,$$

(54)

for $K \leq N - 1$, where the matrix $\Lambda$ is a $K \times K$ Hermite matrix defined by

$$\Lambda(\epsilon) = \left[ \Lambda^{\mu a, \nu b}(\epsilon) \right] = \left[ \sqrt{\epsilon^{\mu}} \left( \langle \phi | (M_{\mu a}^{\dagger} - \langle \phi | M_{\mu a}^{\dagger} | \phi \rangle) (M_{\nu b} - \langle \phi | M_{\nu b} | \phi \rangle) | \phi \rangle \right) \sqrt{\epsilon^{\nu}} \right].$$

(55)

This means that one can evaluate $\delta p_{n}(\epsilon)$ by solving Eq. (54).

The proof of Eq. (54) is as follows. We begin with a function $f(p)$ defined by

$$f(p) := \det [p1_{N-1} - \Delta(\epsilon)].$$

(56)

This function can be expressed by products of $\text{Tr} \left[ \Delta(\epsilon)^{k} \right]$ as

$$f(p) = p^{N-1} - p^{N-2} \text{Tr} \Delta(\epsilon) + \cdots$$

$$= p^{N-1} + \sum_{n=1}^{N-1} p^{N-1-n} C_{n} \left( \text{Tr} \left[ \Delta(\epsilon)^{n} \right], \text{Tr} \left[ \Delta(\epsilon)^{n-1} \right], \cdots, \text{Tr} \left[ \Delta(\epsilon) \right] \right).$$

(57)

where $\{C_{n}(\cdot)\}$ denote some coefficients expressed as a function of $\text{Tr}[\Delta(\epsilon)^{k}]$ with $k = 1, 2, \cdots, n$. In Eq. (57), the trace of $\Delta(\epsilon)^{k}$ is calculated by use of
the relation $\sum_{n=1}^{N-1} |n(0)\rangle\langle n(0)| = 1_N - |\phi\rangle\langle \phi|$ as

$$
\text{Tr} \left[ \Delta(\epsilon)^k \right] = \sum_{\mu_1 \cdots \mu_k} \sum_{a_1 \cdots a_k} \epsilon^{\mu_1} \cdots \epsilon^{\mu_k} \text{Tr}[M_{\mu_1 a_1}|\phi\rangle\langle \phi|M_{\mu_1 a_1}^\dagger (1_N - |\phi\rangle\langle \phi|)]
$$

$$
= \sum_{\mu_1 \cdots \mu_k} \sum_{a_1 \cdots a_k} \left[ \sqrt{\epsilon^{\mu_1}} |\phi\rangle M_{\mu_1 a_1}^\dagger (1_N - |\phi\rangle\langle \phi|) M_{\mu_1 a_1}|\phi\rangle \sqrt{\epsilon^{\mu_1}} \right]
$$

$$
= \text{Tr} \left[ \Lambda(\epsilon)^k \right].
$$

Thus, for $K \leq N - 1$, we obtain

$$
f(p) = p^{N-1} + \sum_{n=1}^{N-1} p^{N-1-n} C_n \left( \text{Tr} \left[ \Lambda(\epsilon)^n \right], \text{Tr} \left[ \Lambda(\epsilon)^{n-1} \right], \ldots, \text{Tr} \left[ \Lambda(\epsilon) \right] \right)
$$

$$
= p^{N-1} + \sum_{n=1}^{K} p^{N-1-n} C_n \left( \text{Tr} \left[ \Lambda(\epsilon)^n \right], \text{Tr} \left[ \Lambda(\epsilon)^{n-1} \right], \ldots, \text{Tr} \left[ \Lambda(\epsilon) \right] \right)
$$

$$
= p^{N-1-K} \left[ p^K + \sum_{n=1}^{K} p^{K-n} C_n \left( \text{Tr} \left[ \Lambda(\epsilon)^n \right], \text{Tr} \left[ \Lambda(\epsilon)^{n-1} \right], \ldots, \text{Tr} \left[ \Lambda(\epsilon) \right] \right) \right]
$$

$$
= p^{N-1-K} \det \left[ p1_K - \Lambda(\epsilon) \right].
$$

Consequently, applying this result to the eigenvalue equation for $\Delta(\epsilon)$ gives

$$
\det \left[ \delta p_n(\epsilon)1_K - \Lambda(\epsilon) \right] = 0,
$$

which completes the proof.

**References**

[1] M. Hotta, T. Karasawa and M. Ozawa, Phys. Rev. A72, 052334 (2005).

[2] For a review, M. A. Nielsen and I. L. Chuang, Quantum Computation and Quantum Information (Cambridge University Press, Cambridge, 2000).
[3] For example, such a low noise may be generated by CPT violation interactions in the flavor oscillation of Kaon systems. Related references are given in J. Ellis, J. S. Hagelin, D. V. Nanopoulous and M. Srednicki, Nucl. Phys. B241, 381 (1984).

[4] M. Hotta, T. Karasawa and M. Ozawa, J.Phys.A:Math.Gen.Vol39,14465(2006).

[5] C. W. Helstrom, Quantum Detection and Estimation Theory, Academic (New York) 1976.

[6] A. S. Holevo, it Probabilistic and Statistical Aspects of Quantum Theory, North Holland (Amsterdam) 1982.

[7] A. Fujiwara, Phys. Rev. A65, 012316 (2002).

[8] H. Imai, ” An Information Geometrical Approach to SU(3)-channel Estimation Problem”, Proceedings of The 13th Quantum Information Technology Symposium, p. 213 (2005); A. Fujiwara, ” Information Geometry of Quantum Channel Estimation”, to appear in Proceedings of 2nd International Symposium on Information Geometry and its Applications (2005).

[9] J.Kahn, Phys.Rev.A 75, 022326, (2007).

[10] A. Fujiwara and H. Imai, J. Phys. A36, 8093 (2003).

[11] K. Matsumoto, J.Phys. A35,3111 (2002) .

[12] A. Fujiwara, Phys. Rev. A63, 042304 (2001).