On $m$-Kropina Metrics of Scalar Flag Curvature

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Abstract

Singular Finsler metrics, such as Kropina metrics and $m$-Kropina metrics, have a lot of applications in the real world. In this paper, we consider a special class of singular Finsler metrics: $m$-Kropina metrics which are defined by a Riemannian metric and a 1-form on a manifold. We show that an $m$-Kropina metric ($m \neq -1$) of scalar flag curvature must be locally Minkowskian in dimension $n \geq 3$. For $m = -1$, we respectively characterize Kropina metrics which are of scalar flag curvature and locally projectively flat in dimension $n \geq 3$ by some equations, and obtain some principles and approaches of constructing non-trivial examples of Kropina metrics of scalar flag curvature.

Keywords: $m$-Kropina Metric, Scalar Flag Curvature, (non-)Projective Flatness

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1 Introduction

It is well-known that the local structure of Riemann metrics of constant sectional curvature has been solved. The Beltrami Theorem in Riemann geometry states that a Riemann metric is locally projectively flat if and only if it is of constant sectional curvature. In Finsler geometry, the flag curvature is a natural extension of the sectional curvature in Riemann geometry, and a Riemann metric of scalar flag curvature is nothing but of constant sectional curvature in dimension $n \geq 3$. It is known that every locally projectively flat Finsler metric is of scalar flag curvature. However, the converse is not true. There are regular or singular Finsler metrics of constant flag curvature which are not locally projectively flat ([1] [19]). Therefore, it is a natural problem to study and classify Finsler metrics of scalar flag curvature. This problem is far from being solved for general Finsler metrics. Thus we shall investigate some special classes of Finsler metrics. Recent studies on this problem are concentrated on Randers metrics, square metrics and some other special $(\alpha, \beta)$-metrics.

Randers metrics are among the simplest Finsler metrics in the following form

$$F = \alpha + \beta,$$

where $\alpha$ is a Riemannian metric and $\beta$ is a 1-form satisfying $\|\beta\|_\alpha < 1$. After many mathematician’s efforts, Bao-Robles-Shen finally classify Randers metrics of constant flag curvature by using the navigation method ([1]). Further, Shen-Yildirim classify Randers metrics of weakly isotropic flag curvature ([12]). There are Randers metrics of scalar flag curvature which are neither of weakly isotropic flag curvature nor locally projectively flat ([2] [17]). So far, the problem of classifying Randers metrics of scalar flag curvature still remains open.

Recently, square metrics have been shown to have many special geometric properties. A square metric is defined in the following form

$$F = \frac{(\alpha + \beta)^2}{\alpha},$$

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where \( \alpha \) is a Riemannian metric and \( \beta \) is a 1-form with \( \|\beta\|_\alpha < 1 \). In [11], Shen-Yildirim determine the local structure of all locally projectively flat square metrics of constant flag curvature. L. Zhou shows that a square metric of constant flag curvature must be locally projectively flat (20). Later on, the present author and Z. Shen further prove that a square metric in dimension \( n \geq 3 \) is of scalar flag curvature if and only if it is locally projectively flat, and they also classify closed manifolds with a square metric of scalar flag curvature in dimension \( n \geq 3 \) (19).

In [15], the present author further studies a larger class of \((\alpha, \beta)\)-metrics \( F = \alpha \phi(\beta/\alpha) \) including square metrics, where \( \phi(s) \) is determined by the following known ODE

\[
\{1 + (k_1 + k_3)s^2 + k_2s^4\} \phi''(s) = (k_1 + k_2s^2)\{\phi(s) - s\phi'(s)\},
\]

where \( k_1, k_2, k_3 \) are constant with \( k_2 \neq k_1 k_3 \). For this class, he proves that if \( \beta \) is closed and the dimension \( n \geq 3 \), then \( F \) is of scalar flag curvature if and only if \( F \) is locally projectively flat, and in particular, he shows that \( \beta \) must be closed for a subclass of \( \phi(s) = 1 + a_1s + \epsilon s^2 \) with \( a_1 \) and \( \epsilon \neq 0 \) being constant. Moreover, he obtains the local and in part the global classifications to those metrics of scalar flag curvature.

The Finsler metrics mentioned above are regular. It seems hard to classify a general regular Finsler metric of scalar flag curvature, even for a general regular \((\alpha, \beta)\)-metric. On the other hand, singular Finsler metrics, such as Kropina metrics and \( m \)-Kropina metrics, have a lot of applications in the real world. In this paper, we will study \( m \)-Kropina metrics of scalar flag curvature in dimension \( n \geq 3 \). An \( m \)-Kropina metric has the following form

\[
F = \alpha^{1-m} \beta^m, \quad m \neq 0, 1.
\]

When \( m = -1 \), \( F \) is called a Kropina metric ([9]). There have been some research papers on Kropina metrics ([11] [10] [14], [16]–[19]). \( m \)-Kropina metrics naturally appear in characterizing a class of singular \((\alpha, \beta)\)-metrics which are locally projectively flat ([16] [17]) and locally projectively flat with constant flag curvature ([18]).

**Theorem 1.1** Let \( F = \alpha^{1-m} \beta^m \) be an \( m \)-Kropina metric \((m \neq -1)\) on an \( n \geq 3 \)-dimensional manifold \( M \), where \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is Riemannian and \( \beta = b_i(x)y^i \) is a 1-form. Then \( F \) is of scalar flag curvature if and only if \( F \) can be written in the form

\[
F = \tilde{\alpha}^{1-m} \tilde{\beta}^m,
\]

where \( \tilde{\alpha} \) is a flat Riemann metric and \( \tilde{\beta} \) is a 1-form which is parallel with respect to \( \tilde{\alpha} \) with constant length \( \|\tilde{\beta}\|_{\tilde{\alpha}} = 1 \). In this case, \( F \) is locally Minkowskian, and \( \tilde{\alpha} \) and \( \tilde{\beta} \) can be locally written as

\[
\tilde{\alpha} = |y|, \quad \tilde{\beta} = y^1,
\]

and further \( \alpha, \beta \) are related with \( \tilde{\alpha}, \tilde{\beta} \) by

\[
\alpha = c^{-m} \tilde{\alpha}, \quad \beta = c^{1-m} \tilde{\beta},
\]

where \( c = c(x) > 0 \) is a scalar function.

In [17], the present author proves that for an \( n \geq 3 \)-dimensional locally projectively flat \( m \)-Kropina metric \((m \neq -1)\), it has the same conclusion as in Theorem 1.1. Meanwhile, in [10], the present author and Z. Shen gives the same fact for an \( n \geq 2 \)-dimensional \( m \)-Kropina metric \((m \neq -1)\) of constant flag curvature. Therefore, Theorem 1.1 generalizes the corresponding results in [17] [10]. Besides, in two-dimensional case, the present author also obtains the same conclusion if an \( m \)-Kropina metric \((m \neq -1)\) is just Douglasian ([10]).
When \( m = -1 \), \( F = \alpha^2 / \beta \) is a Kropina metric. In Section 4 below, we respectively give a general characterization for Kropina metrics which are of scalar flag curvature and locally projectively flat in dimension \( n \geq 3 \) (see Theorem 4.1 and Theorem 4.2 below), and then in Section 5 we use Theorem 4.1 to prove again the known result for Kropina metrics of constant flag curvature (see Corollary 5.1). However, it seems very difficult to classify Kropina metrics of scalar flag curvature, even when they are of locally projectively flat (cf. [10] [17]). In the following, we will show some methods, including some applications of Corollary 4.3 below, of constructing non-trivial Kropina metrics of scalar flag curvature. Kropina metrics are related with Randers metrics in some extent. Let \( F = \alpha^2 / \beta \) be a Kropina metric with \( ||\beta||_\alpha = 1 \) and then define

\[
\bar{\alpha}^2 := \frac{(1 - \bar{b}^2)\alpha^2 + \bar{b}^2 \beta^2}{(1 - \bar{b}^2)^2}, \quad \bar{\beta} := -\frac{\bar{b}\beta}{1 - \bar{b}^2},
\]

where \( 0 < \bar{b} = \bar{b}(x) < 1 \) is a scalar function. It is clear that \( \bar{b} = ||\bar{\beta}||_{\bar{\alpha}}. \)

**Theorem 1.2** Let \( F = \alpha^2 / \beta \) be an \( n(\geq 2) \)-dimensional Kropina metric with \( ||\beta||_\alpha = 1 \) and define a Riemann metric \( \bar{\alpha} \) and a 1-form \( \bar{\beta} \) by (4). Then \( \bar{F} = \bar{\alpha} + \bar{\beta} \) is a Randers metric. If \( F \) is of scalar flag curvature for any scalar \( b \), then \( F \) is also of scalar flag curvature. Further, if \( F \) is of weakly isotropic flag curvature, then \( F \) is of constant flag curvature.

By Theorem 1.2 it is possible to construct non-trivial Kropina metrics of scalar flag curvature in dimension \( n \geq 3 \), using the known examples of Randers metrics of scalar flag curvature (see [2] [7]). However, for a given Randers metric \( \bar{F} = \bar{\alpha} + \bar{\beta} \) of scalar flag curvature, we should first make sure that the following two limits make sense

\[
\lim_{b \to 1} (1 - \bar{b}^2)(\bar{\alpha}^2 - \bar{\beta}^2) \quad \text{(Riemannian)}, \quad \lim_{b \to 1} (1 - \bar{b}^2)\bar{\beta} \neq 0,
\]

and then we can get a Kropina metric of scalar flag curvature. Logically, in Theorem 1.2 \( F \) might be also of scalar flag curvature even if \( \bar{F} \) is not of scalar flag curvature.

Next we show another principle of constructing Kropina metrics of scalar flag curvature.

**Theorem 1.3** Let \( F = \alpha^2 / \beta \) be an \( n(\geq 3) \)-dimensional Kropina metric of scalar flag curvature and \( \eta \) be a closed 1-form with \( ||\eta||_\alpha \) sufficiently small. Then \( \bar{F} = \alpha^2 / \beta + \eta \) is also a Kropina metric of scalar flag curvature. In particular, if \( F \) is of constant flag curvature, then we obtain a family of Kropina metrics \( \bar{F}'s \) of scalar flag curvature which are generally neither locally projectively flat nor of constant flag curvature, and \( \bar{F} \) is locally projectively flat if and only if \( \bar{F} \) can be locally written in the form

\[
\bar{F} = \frac{|y|}{y^1} + \eta.
\]

It is easy to conclude [10] by [18]. We should know in Theorem 1.3 that

\[
\bar{F} = \alpha^2 / \beta + \eta = \frac{\alpha^2 + \eta\beta}{\beta}
\]

is a Kropina metric for suitable \( \eta \) since \( \alpha^2 + \eta\beta \) can be still Riemannian. If \( F \) in Theorem 1.3 is of constant flag curvature, then the local structure of \( \bar{F} \) can be determined since \( F \) can be determined locally. Take \( \eta = \langle x, y \rangle \) with \( x \) close to origin, and then \( \bar{F} \) in (5) is a projectively flat Kropina metric with its scalar flag curvature given by

\[
K = \frac{3}{4} \frac{|y|^4(y^1)^4}{(\eta y^1 + |y|^2)^4}.
\]

Additionally, using Corollary 4.3 below and a warped product method, we obtain a family of Kropina metrics which are locally projectively flat (see Proposition 6.2 below).
2 Preliminaries

In local coordinates, the geodesics of a Finsler metric \( F = F(x, y) \) are characterized by

\[
\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx^i}{dt}) = 0,
\]

where

\[
G^i := \frac{1}{4} g^{ij} \{ [F^2]_{x^k y^j} y^k - [F^2]_{x^i} \}.
\]

(6)

For a Finsler metric \( F \), the Riemann curvature \( R_y = R^i_k(y) \frac{\partial}{\partial y^i} \otimes dx^k \) is defined by

\[
R^i_k := 2 \frac{\partial G^i}{\partial x^k} - y^j \frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^i \frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^j}{\partial y^j} \frac{\partial G^i}{\partial y^k}.
\]

(7)

The Ricci curvature is the trace of the Riemann curvature, \( \text{Ric} := R^m_m \). A Finsler metric is called of scalar flag curvature if there is a function \( K = K(x, y) \) such that

\[
R^i_k = K F^2 (\delta^i_k - y^i y_k), \quad y_k := (1/2 F^2) y^i y^i.
\]

A Finsler metric \( F \) is said to be projectively flat in \( U \), if there is a local coordinate system \((U, x^i)\) such that \( G^i = P y^i \), where \( P = P(x, y) \) is called the projective factor.

In projective geometry, the Weyl curvature and the Douglas curvature play a very important role. We first give their definitions. Put

\[
A^i_k := R^i_k - R \delta^i_k, \quad R := \frac{R^m_m}{n - 1}.
\]

Then the Weyl curvature \( W^i_k \) are defined by

\[
W^i_k := A^i_k - \frac{1}{n + 1} \frac{\partial A^m_m}{\partial y^m} y^i.
\]

The Douglas curvature \( D^i_{jk} \) are defined by

\[
D^i_{jk} := \frac{\partial^3}{\partial y^k \partial y^j \partial y^l} \left( G^i - \frac{1}{n + 1} G^m_m y^i \right), \quad G^m_m := \frac{\partial G^m}{\partial y^m}.
\]

The Weyl curvature and the Douglas curvature both are projectively invariant. A Finsler metric is called a Douglas metric if \( D^i_{jk} = 0 \). A Finsler metric is of scalar flag curvature if and only if \( W^i_k = 0 \). It is known that a Finsler metric in dimension \( n \geq 3 \) is locally projectively flat if and only if \( W^i_k = 0 \) and \( D^i_{jk} = 0 \) ([5]).

In literature, an \((\alpha, \beta)\)-metric \( F = F(x, y) \) is defined as follows

\[
F = \alpha \phi(s), \quad s = \frac{\beta}{\alpha},
\]

where \( \phi(s) \) is some suitable function, \( \alpha = \sqrt{a_{ij}(x)y^iy^j} \) is a Riemann metric and \( \beta = b_i(x)y^i \) is a 1-form. If we take \( \phi(s) = 1 + s \), then we get the well-known Randers metric \( F = \alpha + \beta \).

In applications, there are a lot of singular Finsler metrics. In this paper, we will discuss a class of singular \((\alpha, \beta)\) Finsler metrics—\(m\)-Kropina metrics.

An \(m\)-Kropina metric \( F = \alpha^{1-m} \beta^m \) is defined by taking \( \phi(s) = s^m \), where \( m \neq 0, 1 \) is real. In particular, it is called a Kropina metric when \( m = -1 \). For an \(m\)-Kropina metric
In this section, we will prove Theorem 1.1 by using the deformation (9) on $m$-Kropina metric $F = \alpha^{1-m} \beta^m$, it has an important property of deformation on $\alpha$ and $\beta$. We introduce it as follows. Define a new pair $(\bar{\alpha}, \bar{\beta})$ by

$$\bar{\alpha} := b^m \alpha, \quad \bar{\beta} := b^{m-1} \beta,$$

which appears first in [10]. It is interesting that under the deformation (9), the $m$-Kropina metric $F = \alpha^{1-m} \beta^m$ can also be rewritten as

$$F = \bar{\alpha}^{1-m} \bar{\beta}^m,$$  \hspace{2cm} (10)

and moreover, $\bar{\beta}$ satisfies

$$||\bar{\beta}||_{\bar{\alpha}} = 1.$$  \hspace{2cm} (11)

It has been shown that the deformation (9) plays an important role on the study of $m$-Kropina metrics ([10] [16]–[18]). In this paper, we will also use it.

For a Riemannian $\alpha = \sqrt{a_{ij}y^iy^j}$ and a 1-form $\beta = b_iy^i$, define

$$r_{ij} := \frac{1}{2}(b_{ij} + b_{ji}), \quad s_{ij} := \frac{1}{2}(b_{ij} - b_{ji}), \quad r^i_{\ j} := a^{ik}r_{kj}, \quad s^i_{\ j} := a^{ik}s_{kj},$$

$$q_{ij} := r_{ik}s^k_j, \quad t_{ij} := s_{ik}s^k_j, \quad r^i_{\ j} := b^i\!_j, \quad s^i_{\ j} := b^i\!_j s_{ij},$$

where we define $b^i := a^{ij}b_{ij}$, $(a^{ij})$ is the inverse of $(a_{ij})$, and $\nabla \beta = b_{ij}y^jdx^i$ denotes the covariant derivatives of $\beta$ with respect to $\alpha$. Here are some of our conventions in the whole paper. For a general tensor $T_{ij}$ as an example, we define $T_{i0} := T_{ij}y^j$ and $T_{00} := T_{ij}y^iy^j$, etc. We use $a_{ij}$ to raise or lower the indices of a tensor.

For an $m$-Kropina metric $F = \alpha^{1-m} \beta^m$, by (9) we get

$$G^i = G^i_\alpha - \frac{m}{(m-1)s}a^{is} + \frac{m}{2(m-1)s}\left[(m-1)s_{00} + 2m\alpha s_0\right](b^i - 2\alpha^{-1}sy^i).$$  \hspace{2cm} (12)

Then by (8) and (12), we can get the expressions of the Weyl curvature tensor $W^i_k$ for an $n$-dimensional $m$-Kropina metric $F = \alpha^{1-m} \beta^m$. Assume $F$ is of scalar flag curvature, and then multiplying $W^i_k = 0$ by

$$(n^2 - 1)(m - 1)^2\alpha^4s^3[(m + 1)s^2 - m\beta^2]^5,$$

we have

$$A_0 + A_1s + A_2s^2 + \cdots + A_{13}s^{13} = 0,$$

where $s := \beta/\alpha$, and $A_i$'s include some powers of $\alpha$ and derivatives of $\beta$ with respect to $\alpha$. We will rewrite [13] in other forms as required in the following proof.

3 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1 by using the deformation (9) on $m$-Kropina metrics $(m \neq -1)$.

Lemma 3.1 $\beta$ is closed $\iff t_{ij} = 0 \iff t^k_k = 0$.

Lemma 3.2 (9) For a scalar function $c = c(x)$, the following holds for some $k$,

$$\alpha b_k - sy_k \neq 0 \mod (s + c).$$

5
Lemma 3.3 Let $F = \alpha^{1-m} \beta^m$ be an $m$-Kropina metric ($m \neq -1$) of scalar flag curvature on an $n(\geq 3)$-dimensional manifold $M$. Then $r_{00}$ satisfies

$$r_{00} = 2\tau \left[ mb^2 \alpha^2 - (m + 1) \beta^2 \right] - \frac{2(m + 1)}{(m - 1)b^2} \beta s_0,$$

(14)

where $\tau = \tau(x)$ is a scalar function.

Proof : Since $F = \alpha^{1-m} \beta^m$ is of scalar flag curvature, we have (13), and further (14) can be written as

$$C_i^i \left[ mb^2 - (m + 1)s^2 \right] + 24(n-2)(m+1)^3 y_i (ab_k - sy_k) s^8 \left[ (m-1) sr_{00} + 2m \alpha s_0 \right]^2 = 0,$$

(15)

where $C_i^i$ have the following form

$$f_0(s) + f_1(s) \alpha + f_2(s) \alpha^2 + \cdots + f_{n_j}(s) \alpha^{n_j} = 0,$$

(16)

where $n_j$ are some integers and $f_i(s)$'s are polynomials of $s$ with coefficients being homogeneous polynomials in $(y_i)$. It follows from (15) that

$$(n-2)(m+1)^3 y_i (ab_k - sy_k) s^8 \left[ (m-1) sr_{00} + 2m \alpha s_0 \right]^2 \equiv 0 \pmod{mb^2 - (m + 1)s^2}.$$  (17)

Since $n > 2$ and $m \neq -1$, by (17) and using Lemma 3.2 we have

$$(m-1) sr_{00} + 2m \alpha s_0 \equiv 0 \pmod{mb^2 - (m + 1)s^2}.$$  (18)

Then (18) implies

$$\alpha [ 2m \alpha s_0 + (m-1) sr_{00} ] = -(f + \sigma \alpha) \alpha^2 \left[ mb^2 - (m + 1)s^2 \right],$$

which is rewritten as

$$mb^2 \alpha^3 + m(fb^2 + 2s_0) \alpha^2 - (m + 1) \sigma \beta^2 \alpha - \beta \left[ (m + 1) f \beta - (m - 1) r_{00} \right] = 0,$$

(19)

where $f$ is a 1-form and $\sigma = \sigma(x)$ is a scalar function. Then by (19) we get

$$m(fb^2 + 2s_0) \alpha^2 - \beta \left[ (m + 1) f \beta - (m - 1) r_{00} \right] = 0.$$  (20)

Thus (20) shows that there is a $\tau = \tau(x)$ such that

$$fb^2 + 2s_0 = -2(m - 1)b^2 \beta \tau \beta.$$  

Now plug the above into (20) and then we obtain $r_{00}$ given by (14). Q.E.D.

Lemma 3.4 Let $F = \alpha^{1-m} \beta^m$ be an $m$-Kropina metric ($m \neq -1$) of scalar flag curvature on an $n(\geq 3)$-dimensional manifold $M$. Then we have

$$t_k^k = -\frac{2s_0 \beta}{b^2},$$

(21)

$$D_i^k \beta + m^6 (n + 1)\beta^8 \alpha^{12} b_k T^i = 0.$$  (22)

Proof : Since $F = \alpha^{1-m} \beta^m$ is of scalar flag curvature, we have (13), and further we can rewrite (13) as

$$D_i^k \beta + m^6 (n + 1)\beta^8 \alpha^{12} b_k T^i = 0,$$  

(22)
where $D_k^i$ are polynomial in $(y^i)$ and $T^i$ are defined by

$$T^i := m[(n - 1)(b^i t_0 - s^i s_0 - b^2 t_0) + y^i(b^2 t^j_j + 2s_j s^j)]\alpha^2 + 2(m + 1)(b^2 t_{00} + s_0^2)y^i.$$

Now it follows from (22) that there are polynomials $f^i$ in $(y^i)$ of degree two such that

$$T_i - f_i \beta = 0.$$  \hfill (23)

Contracting (23) by $y^i$ we get

$$m(2s_k s^k + b^2 t_k)\alpha^4 + [(2 + 3m - nm)(b^2 t_{00} + s_0^2) + m(n - 1)\beta t_0]\alpha^2 - f_0 \beta = 0. \hfill (24)$$

Then by (21), we have $f_0 = \theta \alpha^2$ for some 1-form $\theta = \theta_i(x)y^i$. Plugging it into (24) gives

$$0 = 2m(2s_k s^k + b^2 t_k) a_{ij} + 2(2 + 3m - nm)(b^2 t_{ij} + s_i s_j) + m(n - 1)(b_i t_j + b_j t_i) - (b_i \theta_j + b_j \theta_i). \hfill (25)$$

Further contracting (25) by $a^{ij}$ yields

$$(2 + 3m) b^2 t_k + 2(1 + 2m)s_k s^k - b^k \theta_k = 0. \hfill (26)$$

Now it is easy to follow from (26) and (27) that (21) holds. Q.E.D.

**Proof of Theorem 1.1**.

Let $F = \alpha^{1-m} \beta^m$ be an $m$-Kropina metric ($m \neq -1$) of scalar flag curvature. Then under the deformation (9), $F = \alpha^{1-m} \beta^m$ is also an $m$-Kropina metric of scalar flag curvature. So we obtain Lemma 3.3 and Lemma 3.4 under $\alpha$ and $\beta$.

Note that $\bar{b}^2 = 1$, and then by (14) we have

$$\tilde{r}_{ij} = 2\tilde{r}[m\tilde{a}_{ij} - (m + 1)\tilde{b}_i \tilde{b}_j] - \frac{m + 1}{m - 1}(\tilde{b}_i \tilde{s}_j + \tilde{b}_j \tilde{s}_i), \hfill (28)$$

We will prove $\tilde{r}_{ij} = 0$ by (28). This fact is essentially proved in [10] [17]. For convenience, we give the proof here. Contracting (28) by $\tilde{b}^j$ and using $||\tilde{\beta}||_\alpha = constant = 1$ we have

$$\tilde{r}_j + \tilde{s}_j = -2\tilde{r}_b_j - \frac{2}{m - 1}\tilde{s}_j = 0. \hfill (29)$$

Contracting (29) by $\tilde{b}^j$ we get $\tilde{r} = 0$ and then by (29) again we have $\tilde{s}_j = 0$. Thus by (29) again we have

$$\tilde{r}_{ij} = 0.$$ \hfill (30)

Next by (21) we have

$$\tilde{r}_k = -2\tilde{s}_k \tilde{s}^k. \hfill (30)$$

Since we have proved $\tilde{s}_k = 0$, we have $\tilde{r}_k = 0$ by (30). Thus Lemma 3.1 implies that $\tilde{\beta}$ is closed. Thus by this fact and $\tilde{r}_{ij} = 0$, we obtain that $\tilde{\beta}$ is parallel with respect to $\alpha$. Thus (2) naturally holds, and (3) follows from (2), where $c := ||\tilde{\beta}||_\alpha$. Q.E.D.
4 Kropina metrics of scalar flag curvature

4.1 Main results

In this section, we will characterize Kropina metrics of scalar flag curvature in terms of the covariant derivatives of $\beta$ with respect to $\alpha$ and the Riemann (or Weyl) curvature of $\alpha$. Since the deformation $\bar{\beta}$ with $m = -1$ keeps $F = \alpha^2/\beta$ unchanged formally, we can assume $||\beta||_\alpha = 1$ without loss of generality.

**Theorem 4.1** Let $F = \alpha^2/\beta$ be an $(n \geq 3)$-dimensional Kropina metric with $||\beta||_\alpha = 1$. Denote by $\bar{R}^i_k$ the Riemann curvature tensor of $\alpha$. Then $F$ is of scalar flag curvature if and only if the following hold

\[
t_{ij} = b_it_j + b_jt_i - s_is_j + \frac{1}{n-1}\left((t^i_j + 2s^i_js_i)a_{ij} - [s^i_j - (n-3)s^i_s_i]b_ib_j\right),
\]

\[
s_{ijk} = \left\{t_j - \frac{t^i_j - (n-3)s^i_s_i}{n-1}b_j\right\}a_{ik} + r_{ik}s_j + q_{ki}b_j + s_{ijk}b_i - (i/j),
\]

\[
q_{ik} = \frac{1}{2}b^lp^l\left[(r_{lpi} - r_{lip})b_k - (i/k)\right] - \frac{1}{2}b^l(r_{lik} + r_{lik}) - r_{it}r^t_k - s_{ijk},
\]

\[
\bar{R}^i_k = \frac{(n-3)s^i_s_i - t^i_j}{n-1}(\alpha^2\delta^i_k - y^iy_k) - B_{00}\delta^i_k - B_{ik}\alpha^2 + B_{0i}y^i + B_{00}r^i_k + r^j_0r^i_k - r_{00}r^i_k,
\]

where the symbol $(i/j)$ above denotes the terms obtained from the proceeding terms by the interchange of the indices $i$ and $j$, and $\sigma_i$ and $B_{ik}$ are defined by

\[
\sigma_i := \frac{1}{2}\left\{[(n-3)s^i_s_i - (n-1)\lambda - t^i_j]b_i + 2(n-1)b^lp^l(r_{lpi} - r_{lip})\right\},
\]

\[
B_{ik} := \frac{1}{2}\left\{(r_{it}r^t_k + b^lr_{ik}) + \frac{b_k\alpha^2}{4(n-1)} + s_{ijk} + (i/k)\right\},
\]

and $\lambda = \lambda(x)$ is a scalar function. In this case, the scalar flag curvature $K$ of $F$ is given by

\[
K = \lambda s^2 + \frac{s^2}{\alpha^2}\left\{\frac{3s^2}{\alpha^2}r^2_00 + \frac{s}{\alpha}(r_{0000} + 6r_{00s0}) + 3q_{00} + 3s^2_0 - b^l(r_{l000} - r_{00l})\right\} + \frac{1}{4(n-1)}\left\{[4s^2 - 1)t^i_j - 2\left(1 + 2n\alpha^2 - 6s^2\right)s^i_s_i]\right\}.
\]

In [16] [17], the present author gives a way to characterize locally projectively flat Kropina metrics in dimension $n \geq 2$ by [35] (naturally holds for $n = 2$) and an equation on the spray $G^i$ of $\alpha$. Now using Theorem 4.1, we can obtain a different way to characterize locally projectively flat Kropina metrics by adding a Douglasian condition (see [36] below).

**Theorem 4.2** Let $F = \alpha^2/\beta$ be an $(n \geq 3)$-dimensional Kropina metric with $||\beta||_\alpha = 1$. Then $F$ is locally projectively flat if and only if (37) and the following hold

\[
s_{ij} = b_is_j - b_js_i,
\]

\[
q_{ik} = \frac{1}{2}b^l(b_ks_{i|l} - b_is_{k|i} - r_{kli} - r_{i|k}) - r_{it}r^t_k - s_{ijk},
\]

where $B_{ik}$ are defined by (38) and $\sigma_i$ are defined by

\[
\sigma_i := 2\left\{(n-3)s^i_s_i - (n-1)\lambda - t^i_j\right\}b_i + 2(n-1)(b^l s_{i|l} - 2s^i_s_i b_i).
\]

In this case, the scalar flag curvature $K$ of $F$ is given by (37).
In a special case, we have the following simpler corollary. We will construct some examples in Section 6 below by Corollary 4.3.

**Corollary 4.3** Let $F = \alpha^2/\beta$ be an $n(\geq 3)$-dimensional Kropina metric with $||\beta||_0 = 1$. Suppose

$$b_{ij} = \epsilon(a_{ij} - b_i b_j), \quad \epsilon_i = \epsilon b_i,$$

where $u = u(x), \epsilon = \epsilon(x)$ are scalar functions and $\epsilon_i := \epsilon x_i$. Then $F$ is locally projectively flat if and only if

$$\bar{R}^i_k = -\epsilon^2(\alpha^2 \delta^i_k - y^i y_k) - u(\alpha^2 b_i b_k + \beta^2 \delta^i_k - \beta y^i b_k - \beta y_k b^i).$$

In this case, the scalar flag curvature $K$ is given by

$$K = s^6[\epsilon^2(3s^2 - 4) - u].$$

### 4.2 Proof of Theorem 4.1

Assume $F$ is of scalar flag curvature. Firstly, for a Kropina metric $F = \alpha^2/\beta$, the equation (13) with $m = -1$ can be equivalently written as

$$A_3 \beta^3 + A_2 \beta^2 + A_1 \beta + (n + 1) \alpha^2 b_k [(n - 1)(t_{i0} - t_0 b_i + s_0 s_i) - (t^i_l + 2s^l s_i) y_l] = 0,$$

where $A_1, A_2, A_3$ are polynomials in $(y^i)$. Then by (44) we have

$$t_{ij} = t_j b_i - s_i s_j - \rho_i b_j - (t^i_l + 2s^l s_i) a_{ij},$$

where $\rho_i = \rho_i(x)$ are some scalar functions. By (45), using $t_{ij} = t_{ji}$ we get

$$\rho_i = \sigma b_i - (n - 1) t_i,$$

where $\sigma = \sigma(x)$ is a scalar function. Plugging (46) into (44) and then contracting (45) by $a^{ij}$, we get

$$\sigma = t^i_l - (n - 3) s^i s_i.$$

Therefore, by (45)–(47) we obtain (41). Plug $t_{ik}$ and $t_{i0}$ into (44) and then (44) can be written in the form

$$B_2 \beta^2 + B_1 \beta + 2(1 + 1) \alpha^2 b_k b_0 = 0.$$}

Similarly, using the fact that $B_0$ is divided by $\beta$ and (48) we have

$$s_{i00} = \frac{q_0 + s^l s_0 t - b^l q_{0l} - b^l s_0 t - r^l s_0}{n - 1} y_l + \left\{\frac{t^l - (n - 3) s^l s_l}{n - 1} \alpha^2 - q_{0l} + s_{00}\right\} b_l$$

$$- \alpha^2 t_i + s_0 r_{i0} - r_0 s_i - \frac{e_{0i} \beta}{n - 1},$$

where $e_{0i} = c_{ij} y^j$ are 1-forms. Plug (49) into (48) and then (48) can be written as

$$C_1 \beta + (n + 1) \alpha^2 C_0 = 0.$$}

Similarly, since $C_0$ is divided by $\beta$, by (50) we have

$$(n - 1) s_{iklj} - 2(n - 1) s_{ijkl} + \cdots = f_{ik} b_j,$$
where \( f_{ik} = f_{ik}(x) \) are scalar functions. Interchange \( j, k \) in (51) we have
\[
(n - 1)s_{ij|k} - 2(n - 1)s_{ik|j} + \cdots = f_{ij}b_k,
\]
(52)
Then \( 2 \times (51) + (52) \) gives
\[
s_{ij|k} = \frac{q_j + s^l_{j|l} - b^i q_{jl} - b^i s_{jl|l} - r^l_{i|} s_{j}}{n - 1} a_{ik} + \frac{t^l_{i} - (n - 3)s^l_{i}b_j - t_j}{n - 1} a_{jk}
= \frac{2b_k c_{ij} + b_j c_{ik} - b_k f_{ij} - 2b_j f_{ik}}{3(n - 1)} - b_i q_{kj} + b_i s_{jk} + s_j r_{ik} - s_i r_{jk}.
\]
(53)
By (49) and (53) we get
\[
(f_{ij} = 2c_{ij}).
\]
(54)
By (53), plugging (54) into \( s_{ij|k} + s_{ij|k} = 0 \) we have
\[
0 = \left\{ \frac{q_j + s^l_{j|l} - b^i q_{jl} - b^i s_{jl|l} - r^l_{i|} s_{j}}{n - 1} a_{ik} + \frac{t^l_{i} - (n - 3)s^l_{i}b_j - t_j}{n - 1} b_j - t_j \right\} a_{ik} + \frac{b_i q_{kj} + b_i s_{jk} - b_j c_{ik}}{n - 1} + (i/j).
\]
(55)
Contracting (55) by \( b^i b^j \) we can first get the expression of \( b^i c_{ik} \), and then using \( b^i c_{ik} \) and contracting (55) by \( b^j \) we can get the expression of \( c_{ik} \). Now plugging \( c_{ik} \) into (55) yields
\[
0 = \left\{ \frac{b^p (b^l s_{pl|} - s^l_{pl|})}{n - 1} - s^l_{p|} f_{p|} - s_i^l + \frac{q_j + s^l_{j|l} - b^i q_{jl} - b^i s_{jl|l} - r^l_{i|} s_{j}}{n - 1} - t_j \right\} (a_{ik} - b_i b_k) + (i/j).
\]
(56)
Contracting (50) by \( a^{ik} \) we obtain
\[
s^l_{j|l} = b^l (q_{jl} + s_{jl|l}) + (n - 1)t_j + r^l_{i|} s_{j} - q_j + [(n - 1)s^l_{i} - b^p (b^l s_{pl|} - s^l_{pl|})] b_j
\]
(57)
Finally, plug (54), \( c_{ij} \) and (57) into (53) we obtain (52).

By (52), we can determine the expressions of the following quantities
\[
s_{ik|0}, s^l_{0|l}, s^l_{k|l}, s_{i0|k}, s_{0i|0}, b^p s^l_{p|l}.
\]
Plug the above quantities into (48) and then we directly obtain the Weyl curvature of \( \alpha \) given by
\[
\bar{W}^i_k = \frac{1}{n - 1} (s^l_{i|l} - q^l_{i|}) \left( \alpha^2 \delta^l_k - y^l y^k \right) + 2(q_{00} + b^l r_{00|l} + r^l r_{00}) \delta^l_k - (r^l r_{ko} + b^l r_{ko|l} + q_{ko} + q_{ok}) y^l + (s^l_{0} - g^l_{0}) y_k + (r_{ko|0} - r_{00|k}) b^l + (q_{k} - s^l_{k} \alpha^2) - r_{ko} r_{00} - r_{00} r^l_{k}.
\]
(58)
Next we use the Riemann curvature tensor \( R^i_k \) to simplify (58).

**Lemma 4.4** (58) is equivalent to the following equations
\[
q_{00} = -s_{0i|0} - r_{0i} r_{00} - b^l r_{000|l}.
\]
(59)
\[
Re^i_k = \lambda (\alpha^2 s^l_k - y^l y_k) + \left[ b^l (r_{00|l} - r_{00}) + q_{00} - s^l_{0|0} \right] \delta^l_k + r^l_{0} r_{ko} - r_{00} r^l_{k} + (q_{k} - s^l_{k} \alpha^2) + \frac{1}{2} (b^l (r_{00|k} + r_{k|0} - 2r_{ko|l}) - q_{ko} - q_{ok} + s_{k|0} + s_{0|k}) y^l
+ (s^l_{0} - g^l_{0}) y_k + (r_{ko|0} - r_{00|k}) b^l
\]
(60)
Proof : \(\implies\) : By the definition of the Weyl curvature \(\tilde{W}_{ik}\) of \(\alpha\) we have

\[
\tilde{W}_{ik} = \tilde{R}_{ik} - \frac{1}{n-1} \tilde{R}ic_{00} a_{ik} + \frac{1}{n-1} \tilde{R}ic_{k0} y_i,
\]

where \(\tilde{R}_{ik} := a_{ip} \tilde{R}^p_{ik}\) and \(\tilde{R}ic_{ik}\) denote the Ricci tensor of \(\alpha\). Using the fact \(\tilde{R}_{ik} = \tilde{R}_{ki}\) we get from (61)

\[
\tilde{W}_{ik} - W_{ki} = \frac{1}{n-1} (\tilde{R}ic_{k0} y_i - \tilde{R}ic_{00} y_k).
\]

By (58) we can get another expression of \(\tilde{W}_{ik} - W_{ki}\). Thus by (58) and (62) we have

\[
T_k y_i - T_i y_k + (n-1) [(s_{k[i} - s_{i[k} + q_{k[i} - q_{i[k})} \alpha^2 + (r_{k0|0} - r_{00|k}) b_i - (r_{i0|0} - r_{00|i}) b_k] = 0,
\]

where we define

\[
T_k := (n-2) q_{0k} - q_{k0} - (n-1) s_{k|0} - b^{l} r_{k0|l} - r_{k0} r^{l}_i - \tilde{R}ic_{k0}.
\]

Contracting (63) by \(y^k b^l\) we get

\[
(\cdots) \alpha^2 + \beta [T_0 + (n-1) b^l (r_{00|l} - r_{0l|0})] = 0.
\]

By (64) we obtain

\[
T_0 + (n-1) b^l (r_{00|l} - r_{0l|0}) = (n+1) \eta \alpha^2,
\]

where \(\eta = \eta(x)\) is a scalar function. Then it follows from the definition of \(T_i\) and (65) that

\[
\tilde{R}ic_{00} = (n-3) q_{00} - (n-1) s_{0|0} - (n+1) \eta \alpha^2 + (n-2) b^l r_{00|l} - (n-1) b^l r_{0|0} - r^{l}_i r_{00}.
\]

By (69) we can get \(\tilde{R}ic_{k0}\) and then plugging (65) and \(\tilde{R}ic_{k0}\) into (68) we obtain (60), where \(\lambda\) is defined by

\[
\lambda := -\frac{(n+1) \eta + q^l_i - s^l_i}{n-1}.
\]

Finally, summing (60) over \(i, k\) we get \(\tilde{R}ic_{00}\) and then comparing with (60) we obtain (59).

\(\iff\) : Suppose (59) and (60) hold. Summing (60) over \(i, k\) and using (59) we get \(\tilde{R}ic_{00}\) given by (60). Now as shown above, we can get (69) using (61). Q.E.D.

It is clear that no obvious way shows that \(\tilde{R}_{ik} = \tilde{R}_{ki}\) in (60). It follows from (60) that the symmetric condition \(\tilde{R}_{ik} = \tilde{R}_{ki}\) is equivalent to

\[
0 = [b^l(r_{00|k} + r_{k0|0} - 2 r_{k0|0}) - q_{0k} + q_{k0} - s_{k|0} + s_{0|k}] y_i + 2(r_{k0|0} - r_{00|k}) b_i + 2(q_{ki} + s_{ki}) \alpha^2 - (i/k).
\]

By simplifying (67) and applying for the following identities

\[
s_{ij|k} = r_{ik|j} - r_{j|k} - b_l \tilde{R}^l_{ki j},
\]

\[
b^l s_{ij|k} = b^l q_{ij} - t_k, \quad b^l b^k s_{k|l} = b^l q_l + s^l s_l = 2 s^l s_l.
\]

we can prove the following lemma. By (32) and (68) we have

\[
b^k b^l (r_{k|i} - r_{i|k}) = t_i - q_i + b^l s_{ij|l}.
\]

Lemma 4.5 (32), (67) and (68) \(\iff\) (32), (68) and (69) with \(\sigma_i\) and \(B_{ik}\) given by (32) and (69).
we can get (71). Plugging (32) and (34) into ... (74) is reduced to 
\[ \sigma_i. \] (74)

Plugging (74) into (73) yields
\[ b'(r_{l0i} + r_{l0i}) = \left[ \lambda - \frac{(n-3)s^ls_l - t^l_i}{n-1} \right] y_i - \frac{\alpha^2}{2(n-1)} \sigma_i. \] (75)

Then substituting (72), (74) and (75) into (67), we obtain
\[ q_{ik} - q_{ki} = s_{k[i} - s_{i[k} + \frac{b_k\sigma_i - b_i\sigma_k}{2(n-1)}. \] (76)

Using (70), by (72), (75) and (76) we get \( \sigma_i \) given by (33). Using (35), it is easy to obtain (33) by (34) and (74). By (35) again, (74) is reduced to
\[ r_{l0i} - r_{i0l} = \frac{\alpha^2\sigma_i - \sigma_0y_i}{2(n-1)}. \] (77)

and (75) becomes
\[ b'(r_{l0i} + r_{l0i}) - 2r_{i0l} = 2\left[ \lambda - \frac{(n-3)s^ls_l - t^l_i}{n-1} \right] y_i + \frac{b_k\sigma_i + \beta\sigma_k}{2(n-1)}. \] (78)

Finally, plug (33), (34), (77) and (78) into (60) and then we obtain (34) with \( B_{ik} \) defined by (56).

\[ \rightleftharpoons : \] By (33) we can easily verify (59). Next we show (60). By (35) we can get \( b'\sigma_i \). Plugging (32) and (34) into \( b'(r_{l0i} - r_{i0l}) = b'(s_{l00} + b^k\bar{R}_{kl}) \) and using (69), (70), (33) and (35) we can get (71). Plugging (32) and (34) into \( r_{i00} = s_{i00} + b^k\bar{R}_{kl} \) and using (33), (35), (30), (69) and (71) we obtain (77). By (77), we can easily get (78). Finally, by all these equations we obtain (60) from (34).

Q.E.D.

Conversely, by (31)–(36), the above proof has shown or we can directly show that the Weyl curvature of \( F \) vanishes. So \( F \) is of scalar flag curvature.

For the proof of (37), we first get \( Ric_{l00} \) by (60) and \( s^l_{0ij} \) by (32), and then plugging them into \( K = Ric/(n-1)^2 \) yields (37).

Q.E.D.
4.3 Proofs of Theorem 4.2 and Corollary 4.3

Proof of Theorem 4.2:

It is shown in \cite{ref17} that a Kropina metric $F = \alpha^2 / \beta$ is a Douglas metric if and only if \eqref{eq:38} holds. Therefore, by Theorem 4.1, we only need to use \eqref{eq:38} to simplify \eqref{eq:31}–\eqref{eq:36}. By \eqref{eq:38}, we easily get

$$t_{ij} = -s^i b_i b_j - s_i s_j, \quad t_i = -s^i b_i, \quad t^i = -2s^i s_i, \quad q_{ik} = -s_i s_k + b^i q_{ib} b_k,$$

$$q_i = s^i b_i, \quad s_{ij|k} = (r_{ik} + s_{ik}) s_j + b_i s_{j|k} - (r_{jk} + s_{jk}) s_i - b_j s_{i|k}.$$  \hfill \eqref{eq:79}

Then by \eqref{eq:38} and \eqref{eq:79}, it can be easily verified that \eqref{eq:31} and \eqref{eq:32} automatically hold.

Next we prove that if \eqref{eq:38} holds, then \eqref{eq:33} is equivalent to \eqref{eq:39}. Assume \eqref{eq:33} is true, then contracting \eqref{eq:33} by $b^k$ gives

$$b^k q_{ik} = -\frac{1}{2} b^k (b^l r_{ik|l} + s_{ij|k}),$$ \hfill \eqref{eq:80}

Then it follows from \eqref{eq:79} and \eqref{eq:80} that

$$q_{ik} = -\frac{1}{2} b^p b^l r_{ip|l} b_k - s^l s_i b_k - s_i s_k.$$ \hfill \eqref{eq:81}

Now by \eqref{eq:70}, \eqref{eq:79} and \eqref{eq:81}, we can easily conclude that \eqref{eq:33} is equivalent to

$$s_{ij|k} = -\frac{1}{2} b^l (r_{ik|l} + r_{li|k} - b^p r_{ip|l} + b_i s_{k|l} - 2b_k s_{i|l}) - r_{ir} r_k + s_i s_k,$$ \hfill \eqref{eq:82}

Thus by \eqref{eq:70} and \eqref{eq:79}, we easily get \eqref{eq:39} from \eqref{eq:81} and \eqref{eq:82}. Conversely, if \eqref{eq:39} holds, then similarly we can first obtain \eqref{eq:81} and thus \eqref{eq:82} holds. By \eqref{eq:39} and \eqref{eq:81} we get \eqref{eq:82}. Therefore, we have \eqref{eq:33}. Q.E.D.

Proof of Corollary 4.3:

By \eqref{eq:41}, we can easily verify that \eqref{eq:38} and \eqref{eq:39} automatically hold. Plug \eqref{eq:41} into \eqref{eq:40} and \eqref{eq:39} we get

$$\sigma_i = -2(n - 1) \lambda b_i, \quad B_{ik} = -\lambda b_i b_k.$$ \hfill \eqref{eq:83}

Now plugging \eqref{eq:41} and \eqref{eq:39} into \eqref{eq:44} we obtain

$$\bar{R}^i_k = -\epsilon^2 (\alpha^2 \delta^i_k - y^i y_k) + (\lambda + \epsilon^2)(\alpha^2 b^i b_k + \beta^2 \delta^i_k - \beta y^i b_k - \beta y_k b^i).$$ \hfill \eqref{eq:84}

By \eqref{eq:41} and \eqref{eq:39}, it follows from \eqref{eq:77} that

$$\lambda + u + \epsilon = 0.$$ \hfill \eqref{eq:85}

Then \eqref{eq:84} and \eqref{eq:85} imply \eqref{eq:42}, and we get \eqref{eq:43} from \eqref{eq:37}, \eqref{eq:41} and \eqref{eq:85}.

5 Kropina metrics of constant flag curvature

It has been solved for the local structure of Kropina metrics of constant flag curvature (cf. \cite{ref10} \cite{ref13} \cite{ref19}). In this section, we will use Theorem 4.1 to investigate it.
Corollary 5.1 Let $F = \alpha^2/\beta$ be an $n$-dimensional Kropina metric with $||\beta||_a = 1$. Then $F$ is of constant flag curvature if and only if $\alpha$ is of non-negative constant sectional curvature and $\beta$ satisfies $r_{00} = 0$. In this case, $F$ is flat-parallel ($\alpha$ is flat and $\beta$ is parallel), or up to a scaling on $F$, $\alpha$ and $\beta$ can be locally written as

$$\alpha = \frac{\sqrt{(1 + |x|^2) (|y|^2 - \langle x, y \rangle^2)}}{1 + |x|^2}, \quad \beta = \frac{\langle Q x + e, y \rangle}{1 + |x|^2},$$

(86)

where $Q = (q_{ij})$ is a skew-symmetric matrix, $e = (e_i)$ is a constant vector and they are related by

$$|e| = 1, \quad Qe = 0, \quad \delta_{ij} - e_i e_j = \delta_{kl} p_{ik} p_{jl}.$$  

(87)

Proof : For $n = 2$, it has been proved in [10] that $F$ is flat-parallel. So we only need to let $n \geq 3$. Assume $F$ is of constant flag curvature. Then it follows from Theorem 4.1 that its flag curvature $K$ is given by (37). Put $K = K$, which is a constant. Then by (37) we have

$$(\cdots) \alpha^2 + 12(n - 1) \beta^4 r_{00} = 0,$$

(88)

which implies $r_{00} = c \alpha^2$ for some scalar function $c = c(x)$. Since $||\beta||_a = 1$, we have $r_i + s_i = 0$. Then it is easily shown that $c = 0$ and thus $r_{00} = 0$. Now plug $r_{ij} = 0, r_{ijk} = 0, q_{ij} = 0, s_i = 0$ into (88) we have

$$(4K - 4nK - t'_l \alpha^2 + 4(n\lambda - \lambda + t'_l) \beta^2) = 0.$$  

(89)

By (89) we easily get

$$K = -\frac{t'_l}{4(n - 1)} = \frac{\lambda}{4} > 0, \quad \text{ (since } t'_l \leq 0).$$  

(90)

Using $r_{ij} = 0, s_i = 0$ and (90), it follows from (88) that

$$\tilde{R}^i_k = \lambda (\alpha^2 \delta^i_j - y^i y^j),$$

which shows that $\alpha$ is of constant sectional curvature $\lambda$. If $\lambda = 0$, then it is easy to show that $F$ is flat-parallel. If $\lambda > 0$, then using the facts that $r_{00} = 0$ and $\alpha$ is of constant sectional curvature, it follows from [3] that, up to a scaling on $F$, we can put $\lambda = 1$ and thus $\alpha, \beta$ are locally given by (89) with $Q, e$ satisfying (87).

Conversely, assume $r_{00} = 0$ and $\alpha$ is of positive constant sectional curvature $1$ with $||\beta||_a = 1$. Then we have (89) and (87). Here we use Theorem 4.1 to verify that $F$ is of constant flag curvature, namely, we show that (31) - (34) hold and $K$ in (37) is a constant. It is clear that (89) naturally holds. Since $t'_l = 1 - n$ by the following proof, (34) is true by putting $\lambda = 1$. Now we verify (31) and (32), which are firstly reduced to

$$t_{ik} = \frac{t'_i (a_{ik} - b_k b_i)}{n - 1}, \quad s_{ijk} = \frac{t'_i (b_{ij} a_k - b_j a_{ik})}{n - 1}.$$  

(91)

Then we can easily verify (91) by the following computations using (87):

$$b_i = \frac{p_{ik} x^k + e_i}{1 + |x|^2}, \quad a_{ij} = \frac{(1 + |x|^2) \delta_{ij} - x^i x^j}{(1 + |x|^2)^2}, \quad t'_l = 1 - n,$$

$$t_{ij} = -\frac{\delta_{ij}}{1 + |x|^2} + \frac{x^i x^j + e_i (e_j + p_{ij} x^l) + e_j p_{ik} x^l + p_{ik} p_{jl} x^k x^l}{(1 + |x|^2)^2},$$

$$s_{ij} = \frac{(e_i + q_{ij} x^l) \delta_{lk} - (e_i + q_{il} x^j) \delta_{jk}}{(1 + |x|^2)^2} x^k (x^l q_{jk} x^l - x^j q_{ik} x^l + x^i e_j - x^j e_i).$$
Finally by (37) we have
\[ K = s^2 + \frac{4s^2 - 1}{4(n-1)} t^l. \] (92)

Since \( t^l = 1 - n \) as shown above, we have \( K = 1/4 \) by (92).

Now we have completed the proof of Corollary 5.1. Q.E.D.

6 Construction by warped product method

In this section, we show a family of examples of projectively flat Kropina metrics with \( \alpha \) in warped product form by using Corollary 4.3.

Let \( M = \mathbb{R} \times \tilde{M} \) be a product manifold, where \( \tilde{M} \) is an \((n-1)\)-dimensional manifold. Let \( \{x^A\}_{A=2}^n \) be a local coordinate system on \( \tilde{M} \). A Riemann metric \( \alpha \) of warped product type is defined as
\[ \alpha^2 = (y^1)^2 + h^2(x^1)\tilde{\alpha}^2, \] (93)
where \( \tilde{\alpha}^2 = \tilde{\alpha}_{AC}y^Ay^C \) is a Riemann metric on \( \tilde{M} \). The Riemann curvature tensors \( \bar{R} \) of \( \alpha \) and \( \tilde{\alpha} \) in (93) are related by
\[ \bar{R}^1_k = \frac{h''}{h}(y^1 y_k - \alpha^2 \delta^1_k), \] (94)
\[ \bar{R}^A_C = \bar{R}^A_C - (h')^2(\alpha^2 \delta^A_C - y^A y^C) - \frac{h''}{h}(y^1)^2 \delta^A_C, \] (95)
where \( y_k := a_{kl} y^l, \tilde{y}_C := \tilde{a}_{CA} y^A \). Define \( \eta = \eta(x^1) := \int h(x^1) dx^1 \), and then a direct computation shows that
\[ \eta_{ij} = \eta'' \alpha^2, \quad (\eta_i := \eta_{x^i}), \]
where the covariant derivative is taken with respect to \( \alpha \). The converse is proved in the following.

**Lemma 6.1** ([6]) Let \( \alpha \) be a Riemann metric on \( M \). Suppose there are two functions \( \eta \) and \( \xi \) on \( M \) with \( d\eta \neq 0 \) such that
\[ \eta_{ij} = \xi \alpha^2, \quad (\eta_i := \eta_{x^i}). \]
Then \( \alpha \) is a warped product on \( M = \mathbb{R} \times \tilde{M} \), namely, locally \( \eta \) depends only on the parameter \( x^1 \) of \( \mathbb{R} \), \( \xi = f''(x^1) \) and \( \alpha \) can be expressed as
\[ \alpha^2 = (y^1)^2 + (\eta'(x^1))^2 \tilde{\alpha}^2. \]

Now we begin the construction of examples of Kropina metrics of scalar flag curvature.

**Proposition 6.2** Let \( F = \alpha^2/\beta \) an \( n(\geq 3) \)-dimensional Kropina metric on a product manifold \( M = \mathbb{R} \times \tilde{M} \), where
\[ \alpha^2 := (y^1)^2 + h^2(x^1)\tilde{\alpha}^2, \quad \beta := y^1, \] (96)
where \( h \neq 0 \) is a smooth function on \( \mathbb{R} \) and \( \tilde{\alpha} \) is an \((n-1)\)-dimensional Riemann metric on \( \tilde{M} \). Then \( F \) is locally projectively flat if and only if \( \tilde{\alpha} \) is locally flat. In this case, the scalar flag curvature \( K \) is given by
\[ K = -\left( \frac{\beta}{\alpha} \right)^6 \left\{ \frac{h''}{h} + 3(h')^2 \left( \frac{\tilde{\alpha}}{\alpha} \right)^2 \right\}. \] (97)
Proof: For the $\alpha$ and $\beta$ defined by (96), a direct computation shows that $||\beta||_{\alpha} = 1$ and holds with
\[ \epsilon = \frac{h'}{h}, \quad u = \left(\frac{h'}{h}\right)^{\frac{1}{3}}. \] (98)
So $F$ is locally projectively flat if and only if (42) holds by Corollary 4.3.

It can be easily verified that (42) holds with $R^1_{\alpha} = \left[-(u + \epsilon^2)(y^1)^2 - \epsilon^2 h^2 \alpha^2\right] \delta_k^1 + (u + \epsilon^2)y^1 y_k = uh^2 \alpha^2 b_k$, (99)

and
\[ \tilde{R}^A_{\alpha} = \left[-(u + \epsilon^2)(y^1)^2 - \epsilon^2 h^2 \alpha^2\right] \delta_k^A + \epsilon^2 h^2 y^A \tilde{y}_C, \] (100)

where $\tilde{y}_C := \tilde{a}_CAy^A$. By (94), (99) is equivalent to
\[ u + \epsilon^2 = \frac{h''}{h}. \] (101)

which is also verified by $h_{ij} = \epsilon_i \epsilon_j$, and by (101) and (95), (100) is equivalent to
\[ \tilde{R}^A_{\alpha} = \left[-\epsilon^2 h^2 + (h')^2\right](\tilde{a}^2 \delta_k^A - y^A \tilde{y}_C) = 0 \] (102)

Now suppose $F$ is locally projectively flat. Then we have (102), namely, $\tilde{\alpha}$ is locally flat. Conversely, if $\tilde{\alpha}$ is locally flat, then by the above proof, we can easily get (42).

Finally, by (101), we obtain the scalar flag curvature $K$ given by (97). Q.E.D.

By Proposition 6.2, $F = \alpha^2/\beta$ in dimension $n \geq 3$ is locally projectively flat, where $\alpha$ and $\beta$ are defined by (96) with $h \neq 0$ being arbitrary and $\tilde{\alpha}$ being locally flat.

Proposition 6.3 Let $F = \alpha^2/\beta$ an $n(\geq 3)$-dimensional Kropina metric, where $\alpha$ and $\beta$ satisfy (77) with $||\beta||_{\alpha} = 1$, $d\epsilon \neq 0$ and $u = f(\epsilon) \neq 0$ for some function $f$. Then $F$ is locally projectively flat if and only if $\alpha$ and $\beta$ can be locally written as
\[ \alpha^2 = (y^1)^2 + h^2(\tilde{\alpha}^2), \quad \beta = y^1, \] (103)

where $\tilde{\alpha}$ is a locally flat Riemann metric and $h$ can be determined by $f$.

Proof: We firstly show (103) by (11). Define
\[ \varphi := \int \frac{1}{f(\epsilon)} e^{\int \frac{\epsilon}{f(\epsilon)} d\epsilon} de. \] (104)

Then by (11) with $u = f(\epsilon) \neq 0$, we can easily verify that
\[ \varphi_{ij} = \epsilon e^{\frac{\epsilon}{f(\epsilon)} d\epsilon} a_{ij}, \quad (\epsilon_i := \epsilon_{x^i}). \] (105)

Obviously we have $d\varphi \neq 0$. Then by (103) and Lemma 6.1 $\alpha$ is a warped product which can be locally written as the first expression in (103) with $h(x^1) = \varphi'(x^1)$. By (104), we can define
\[ g(\varphi) := \int \frac{1}{f(\epsilon)} de. \]

Further by (11) we have
\[ \beta = \frac{e_i}{f(\epsilon)} dx^i = \frac{de}{f(\epsilon)} = d \left( \int \frac{1}{f(\epsilon)} de \right) = d(g(\varphi)) = g'(\varphi) \varphi'(x^1) dx^1. \] (106)

Then by $||\beta||_{\alpha} = 1$, $\alpha$ in (103), and (106), we must have $g'(\varphi) \varphi'(x^1) = 1$ and $\beta = y^1$.

Therefore, by Proposition 6.2 we conclude that $F$ is locally projectively flat if and only if $\tilde{\alpha}$ in (103) is locally flat. Q.E.D.
7 Proof of Theorem 1.2

Let \( F = \alpha^2 / \beta \) be an \( n(\geq 3) \)-dimensional Kropina metric with \( ||\beta||_\alpha = 1 \). Now for the given pair \( \alpha \) and \( \beta \), define a Riemann metric \( \tilde{\alpha} \) and a 1-form \( \tilde{\beta} \) by \((1)\), where \( 0 < \tilde{b} < 1 \) is a scalar function. Obviously, \( \tilde{F} := \tilde{\alpha} + \tilde{\beta} \) is a Randers metric since we can show \( ||\tilde{\beta}||_{\tilde{\alpha}} = \tilde{b} < 1 \) by \( ||\beta||_\alpha = 1 \). A simple computation shows that

\[
\lim_{\tilde{b} \to 1^-} \tilde{F} = \lim_{\tilde{b} \to 1^-} \frac{\sqrt{(1 - b^2)\alpha^2 + b^2\beta^2}}{(1 - b^2)} = \frac{\alpha^2}{2\beta} = \frac{1}{2}F. \tag{107}
\]

Let \( W^i_k \) and \( \tilde{W}^i_k \) be the Weyl curvature of \( F \) and \( \tilde{F} \) respectively. Then by \((107)\), we have

\[
\lim_{\tilde{b} \to 1^-} \tilde{W}^i_k = W^i_k.
\]

Therefore, if \( \tilde{F} \) is of scalar flag curvature for any scalar \( \tilde{b} \) (\( \tilde{W}^i_k = 0 \)), then \( F \) is also of scalar flag curvature (\( W^i_k = 0 \)).

Now assume \( \tilde{F} \) is of weakly isotropic flag curvature, namely, the scalar flag curvature \( \tilde{K} \) of \( \tilde{F} \) is in the form

\[
\tilde{K} = \frac{3\theta}{F} + \sigma,
\]

where \( \theta \) is a 1-form and \( \sigma = \sigma(x) \) is a scalar function. There is a simple way to show that \( F \) is of constant flag curvature by a known result. By taking the limit \( \tilde{b} \to 1^- \) on the flag curvature \( \tilde{K} \) of \( \tilde{F} \), \( F \) is also of weakly isotropic flag curvature. Thus by \((10)\) \((13)\), \( F \) is of constant flag curvature.

In the case of \( n \geq 3 \), we can show another direct proof. Let \((h, W)\) be the navigation data of \( \tilde{F} = \tilde{\alpha} + \tilde{\beta} \), where \( h = \sqrt{h_{ij}y^iy^j} \) is Riemannian and \( W^i \) is a vector field. Then by \((1)\) we have \( h = \alpha \) and \( W_i := h_{ij}W^j = bh_i \). By \((8)\), locally \( h \) and \( W^i \) can be written as

\[
h = \frac{\sqrt{(1 + \mu|x|^2)y^2 - \mu(x,y)^2}}{1 + \mu|x|^2}, \tag{108}
\]

\[
W^i = -2(\lambda \sqrt{1 + \mu|x|^2} + \langle d, x \rangle)x^i + \frac{2|x|^2d_i}{1 + \sqrt{1 + \mu|x|^2}} + p_{ik}x^k + e_i + \mu(x, e)x^i. \tag{109}
\]

where \( \lambda, \mu \) are constants, \( Q = (p_{ik}) \) is a skew-symmetric matrix and \( d, e \in \mathcal{R}^n \) are constant vectors. To make \( \tilde{b} \to 1^- \), we only require \( h_{ij}W^iW^j = 1 \). By \((108)\) and \((109)\), a direct computation shows that \( h_{ij}W^iW^j = 1 \) can be expressed as

\[
A\sqrt{1 + \mu|x|^2} + B = 0, \tag{110}
\]

where \( A \) and \( B \) are polynomials in \( \langle x^i \rangle \) of orders 4 and 5.

Case I: Assume \( \mu = 0 \). Then \((110)\) becomes

\[
0 = |d|^2|x|^4 + 2(2\lambda d, x + \langle d, Qx \rangle + 2\lambda^2 + \langle d, e \rangle)|x|^2 + |Qx|^2 - 4\langle d, x \rangle\langle e, x \rangle
+ 2\langle e, Qx \rangle - 4\lambda\langle e, x \rangle + |e|^2 - 1. \tag{111}
\]
Firstly by (111) we get $|e|^2 = 1$ and $Qe = 2\lambda e$. A real characteristic value of a real skew-symmetric matrix must be zero. So we have $\lambda = 0$ by $Qe = 2\lambda e$. Thus by (111) again, we easily get $d = 0$, $Q = 0$. So we have

$$\alpha = |y|, \quad \beta = \langle e, y \rangle.$$  

In this case, $F = |y|/\langle e, y \rangle$ is flat-parallel and it is locally Minkowskian.

**Case II:** Assume $\mu \neq 0$. Then by (110) we have $A = 0$, $B = 0$. For $A = 0$, its constant terms and linear terms show $|e|^2 = 1$ and $Qe = 2\lambda e$. So again we have $\lambda = 0$. Then its terms of order three gives $Qd = 0$. Now separating its terms of order 4 and 2 we obtain

$$ (4|d|^2 - \mu^2)|x|^2 + \mu(|Qx|^2 + \mu(e, x)^2 - 4\langle d, x \rangle\langle e, x \rangle) = 0, \quad (112)$$

$$ (2\langle d, e \rangle - \mu)|x|^2 + |Qx|^2 + \mu(e, x)^2 - 4\langle d, x \rangle\langle e, x \rangle = 0. \quad (113)$$

Then $\mu \times (113) - (112)$ gives $\mu\langle d, e \rangle - 2|d|^2 = 0$. Summing up the facts proved we obtain

$$ |e| = 1, \quad \lambda = 0, \quad Qd = 0, \quad Qe = 0, \quad \mu\langle d, e \rangle - 2|d|^2 = 0. \quad (114)$$

Now plugging (114) into $B = 0$, we get

$$ (2\langle d, e \rangle - \mu)|x|^2 + |Qx|^2 + \mu(e, x)^2 - 4\langle d, x \rangle\langle e, x \rangle = 0. \quad (115)$$

In (115), replace $x$ with $e$ and then we have $\langle d, e \rangle = 0$. Thus by (114) we have $d = 0$. Now by (115) we have

$$ \mu(\delta_{ij} - e_ie_j) - \delta^{kl}p_{ik}p_{jl} = 0. \quad (116)$$

Summing (116) over the indices $i,j$ we get $\mu > 0$. Now taking the limit $\hat{b} \to 1$ and using $W_i = \hat{b}b_i$, (114), (116), $d = 0$ and $\mu > 0$, up to a scaling on $F$, we get $\alpha$ and $\beta$ given by (86), where $Q, e$ satisfy (87). So it follows from (86) and (87) that $F = \alpha^2/\beta$ is of constant flag curvature $K = 1/4$ (110), which has also been shown in the proof of Corollary 5.1.

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