Non-highest weight representations of the current algebra $\mathfrak{so}(1, n)$, and Laplace Operators.

M. Zyskin *

March, 1999

Abstract

We constructed canonical non-highest weight unitary irreducible representation of $\mathfrak{so}(1, n)$ current algebra as well as canonical non-highest weight non-unitary representations. We constructed certain Laplacian operators as elements of the universal enveloping algebra, acting in representation space. We speculated about a possible relation of those Laplacians with the loop operator for the Yang-Mills.

*IHES, Le Bois-Marie 35, route de Chartres F-91440, Bures-sur-Yvette, France; zyskin@ihes.fr
1 Introduction

Recently, there was a lot of interest in string theories on AdS and other manifolds with boundaries, and to their relationship to Yang Mills quantum field theories on the boundary. String theory on AdS has a symmetry group, corresponding to isometries of target space, whose bosonic part is $SO(1,n)$. There should be conserved Noether currents corresponding to such symmetries. Also, such string theories should be described by conformal field theories. If the conformal field theory is unitary, the currents are holomorphic, and give rise to a Kac-Moody algebra. From analyticity on world sheet, the space of states in such conformal theory, associated to points on worldsheet, should be in highest weight modules of the Kac-Moody algebra, and then there is a Virasoro algebra coming via the Sugawara construction [8]. Namely, the currents are (closed) one forms on the world sheet $j^a(\cdot)_\mu dx^\mu$, the integrals of which over paths gives conserved charges, with values in Lie algebra; and ”equal time” commutation relations of currents should be such as to reproduce the Lie algebra commutators of charges; if we integrate the current over a path $\Gamma$ on world sheet with weights $\psi(\gamma)$, $\gamma \in \Gamma$, and use commutation relations for currents, we will get

$$[\int_\Gamma \psi j^a d\gamma, \int_\Gamma \xi j^b d\gamma] = f^{abc} \int_\Gamma \psi \xi j^c d\gamma + \tilde{k} \int_\Gamma \psi' \xi d\gamma;$$

and in particular, taking a closed path, parametrised by $\theta$, and $\psi, \xi$ of the form $e^{in\theta}$, we get that Fourier coefficients of currents satisfy commutation relations of a Kac-moody algebra.

In our case, the algebra is $so(1,n)$, which is a real form of a complex algebra; still it can be promoted to a Kac-Moody-like loop algebra in the usual way; it is generated by $e_i \otimes_R P(s)$, with $e_i \in so(1,n)$, $P(s)$ Laurent polynomials, and the bracket $[e_i \otimes P(s), e_j \otimes G(s)] = [e_i, e_j] \otimes P(s)G(s) + \tilde{k} (e_i, e_j) \text{ Res}(\frac{dP}{ds}G)$, where $(..)$ is an invariant symmetric form on $so(1,n)$; this form is not positive-definite. Highest weight module for such loop algebra is not unitary.

If we think about boundary states, those associated with the boundary of world sheet disk, say states associated to a circle, we cannot deduce from analyticity that the state must be highest weight, as functions $\{e^{in\theta}\}$ on a circle, unlike functions $\{z^n\}$ at the origin, are equally good for positive or negative n, and in fact there are two possible candidates for representations: one non-unitary highest weight, and another unitary, not highest weight. Non highest weight representations are little studied in the framework of the conformal field theory; in particular, there is no meaningful way to have a Sugawara construction of the Virasoro algebra; instead, it is quite natural to get certain infinite dimensional laplacians in the universal enveloping algebra.
2 Unitary irreducible representation of $\hat{so}(1, n)$ loop algebra and unitary action of operators in universal enveloping algebra

2.1

We introduce a space $\mathfrak{s}$ of real valued functions on a circle, which is a vector space over real numbers with basis $\{ e[n] \equiv s^n \}_{n \in \mathbb{Z}}$, and with some positive definite inner product

$$
(e[n], e[m]) = F[n, m]
$$

$$
F[n, m] = F[m, n]; F[n, m] \in \mathbb{R}^+,
$$

(1)

extended by linearity to the whole space. Usual multiplication of functions give rise to a product, $e[n] \cdot e[m] = e[m+n]$. If we want multiplications by $s \equiv e[1]$ and $s^{-1} \equiv e[-1]$ to be self-adjoint operators

$$
(e[n], s \cdot e[m]) = (s \cdot e[n], e[m])
$$

$$
(e[n], s^{-1} \cdot e[m]) = (s^{-1} \cdot e[n], e[m]),
$$

then from (1) we have a condition

$$
F[n, m + 1] = F[n + 1, m].
$$

Therefore in this case we should have

$$
F[n, m] = \tilde{F}[n + m, n - m \text{ mod } 2]
$$

(2)

where $n \text{ mod } 2$ is 0 for even $n$ and 1 for odd $n$. Such scalar products exist:

**Example:** consider the inner product

$$
(e[n], e[m]) = \delta_{n,m} \alpha(n)
$$

(3)

where $\alpha(n) > 0$ for all $n \in \mathbb{Z}$. Such scalar product is obviously positive-definite. From the definition of the adjoint operator $(e[n], s \cdot e[m]) = (s^* e[n], e[m])$ is easily follows that in this case

$$
 s^* e[n] = \frac{\alpha(n)}{\alpha(n-1)} e[n-1]
$$
2.2

The loop algebra $\hat{\mathfrak{s}\mathfrak{o}}(1,n)$ has an interesting unitary, non-highest weight irreducible representation, which we believe is important and which we describe below. Consider the Hilbert space $\mathfrak{h}$ of real valued functions $\phi(X)$ on a sphere $S^n$: $X_1^2 + X_2^2 + \ldots + X_n^2 = 1$ with integral zero, $\int \phi(X) d\omega_X = 0$, and with the scalar product

$$\langle \phi, \psi \rangle = \int \int \phi(X) \psi(X') \ln (1 - (X, X')) d\omega_X d\omega_{X'}$$  \hspace{1cm} (4)

It is possible to show that for functions on a sphere with zero integral such scalar product is positive-definite.

Let us introduce a Hilbert space $\mathfrak{H}$ of maps $\phi(X,s)$ from $S^1$ into $\mathfrak{h}$. For Laurent polynomials,

$$\phi(X,s) = \sum_{n=-N}^{M} \phi_n(X) e^{-n}(s)$$

where $\phi_n(X) \in \mathfrak{h}$, and $\{e[n]\}_{n \in \mathbb{Z}}$ is the basis in $\mathfrak{h}$, we define the scalar product in $\mathfrak{H}$ as follows:

$$\langle \phi(), \psi() \rangle = \sum_{m,n} (e[-n], e[-m]) \int \int \phi_n(X) \psi_m(X') \ln (1 - (X, X')) d\omega_X d\omega_{X'},$$

and then take the completion.

We introduce also bosonic Fock space $\mathfrak{F}$ of symmetric tensor products $\mathfrak{F} = \bigoplus S \mathfrak{H}^\otimes m$, which is also a Hilbert space with the scalar product

$$\langle \phi()^\otimes m, \psi()^\otimes m \rangle = m! \langle \phi() , \psi() \rangle^m$$ \hspace{1cm} (5)

or more generally,

$$\langle \eta_1 \otimes \eta_2 \ldots \otimes \eta_m, \xi_1 \otimes \xi_2 \ldots \otimes \eta_m \rangle = \begin{cases} 0, & m \neq n \\ \sum_{\{\sigma\}} \prod_{i=1}^{n} <\eta_i, \xi_{\sigma_i}> , & m = n \end{cases}$$ \hspace{1cm} (6)

We also introduce $\text{Exp}(\phi())$ in a completion of $\mathfrak{F}$,

$$\text{Exp}(\phi(X,s)) = 1 + \sum_{m=1}^{\infty} \frac{1}{m!} \phi(X,s)^\otimes m$$ \hspace{1cm} (7)

It’s easy to see that

$$\langle \text{Exp}(\phi()), \text{Exp}(\phi()) \rangle = \exp(\langle \phi(), \phi() \rangle)$$ \hspace{1cm} (8)
2.3 Unitary irreducible representation of loop algebra $\tilde{so}(1, n)$ via first order differential operators acting in $\mathfrak{H}$

We consider maps $S^1 \to \mathfrak{g}$ from a circle into the Lie algebra $so(1, n)$, with the point-wise Lie bracket,

$$\{I_1, I_2\}(s) = \{I_1(s), I_2(s)\}$$  \hspace{1cm} (9)

This Lie algebra $\tilde{so}(1, n)$ is given by generators and relations

$$\{I_{ij}(f(s)), I_{kl}(g(s))\} = \delta_{jk}I_{il}(fg(s)) - \delta_{jl}I_{ik}(fg(s)) + \delta_{il}I_{jk}(fg(s)) - \delta_{ik}I_{jl}(fg(s))$$

$$\{I_{ij}(f(s)), I_{k,0}(g(s))\} = \delta_{jk}I_{i,0}(fg(s)) - \delta_{ik}I_{j,0}(fg(s))$$

$$\{I_{i,0}(f(s)), I_{j,0}(g(s))\} = I_{i,j}(fg(s))$$  \hspace{1cm} (10)

This algebra has the following representation in $\mathfrak{H}$: for any $v = \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n \in \mathfrak{H}$, with $\{\eta_i\} \in \mathfrak{H}$,

$$T[I(f(s)), v] = (D[I(f(s))] + \beta[I(f(s))] - \beta[I(f(s))^* 1])\cdot v \equiv$$

$$\equiv \sum_i \eta_1 \otimes \eta_2 \otimes \ldots \otimes (D[I(f(s)]\eta_i) \otimes \ldots \otimes \eta_n + \beta[I(f(s))] \otimes \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n -$$

$$- \sum_i \beta[I(f(s)^* 1)]\cdot \eta_i \otimes \eta_2 \otimes \ldots \otimes \eta_i \otimes \ldots \otimes \eta_n$$  \hspace{1cm} (11)

Here $D$ are derivations, which satisfy the Leibnitz rule in $\mathfrak{H}$,

$$D[I(f(s))] \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n = \sum_k \eta_1 \otimes \eta_2 \otimes \ldots \otimes (D[I(f(s))]\eta_k) \otimes \ldots \otimes \eta_n$$

and are given by the following differential operators in $\mathfrak{H}$:

$$D[I_{ij}(f(s))] \phi(X, s) = f(s) (X_i \partial_j - X_j \partial_i) \phi(X, s),$$

$$D[I_{i,0}(f(s))] \phi(X, s) = f(s) (\partial_i - X_i(X \partial) - (n - 1)X_i) \phi(X, s),$$

$$D[I_{i,0}(f(s))] 1 = 0;$$  \hspace{1cm} (12)

here $\partial_i \equiv \frac{\partial}{\partial_i}$ in $\mathbb{R}^n$, so that $[\partial_i, X_j] = \delta_{i,j}$;

and $\beta$ acts via tensor multiplication in $\mathfrak{H}$, with

$$\beta[I_{ij}(f(s))] = 0$$

$$\beta[I_{i,0}(f(s))] = -(n - 1)f(s)\xi_i(X),$$  \hspace{1cm} (13)
where $\xi_i(X)$ is a function on a sphere equal to the $i$-th coordinate, $\xi_i(X) = X_i$.

Such $\beta$ is a one-cocycle:

$$D[I_1]\beta[I_2] - D[I_2]\beta[I_1] = \beta[I_1, I_2]$$

(14)

**Proposition 2.1.**

1) Differential operators $D[I(f(s))]$ are correctly defined in $\mathfrak{S}$

2) The operators $D[I(f(s))]^*$, which are adjoint in $\mathfrak{S}$ to operators $D[I(f(s))]$, are equal to $(-D[I(f(s)^*)])$, where $f^*(s)$ is the adjoint operator to the multiplication by $f(s)$ in $\mathfrak{S}$

$$D[I(f(s))]^* = (-D[I(f(s)^*)])$$

in particular, if multiplication by $s$ is self-adjoint in $\mathfrak{S}$, see (2), then $D[I(f(s))]$ are skew self-adjoint in $\mathfrak{S}$.

Operators $D[I(f(s)^*)]$ act naturally in $\mathfrak{S}$, if $\eta = \sum \eta_n g_n$, with $\{\eta_n\} \subset \mathfrak{h}$, $\{g_n\} \subset \mathfrak{s}$, then

$$D[I(f(s)^*)] \eta = \sum D[I] \eta_n f(s)^* g_n,$$

where $D[I]$ are the differential operators (12).

**Proposition 2.2.**

1) Operators $T[I(f(s))]$ in $\mathfrak{S}$ have the property $T[I(f(s))]^* = -T[I(f(s)^* \cdot 1)]$; in particular, if multiplication by $s$ is self-adjoint in $\mathfrak{S}$, see (2), then $T[I(f(s))]$ are skew self-adjoint in $\mathfrak{S}$.

2) Operators $T[I(f(s))]$ satisfy the commutation relation of the algebra, that is for any $v \in \mathfrak{S}$

$$(I_1[f_1]I_2[f_2] - I_2[f_2]I_1[f_1]).v = [I_1, I_2][f_1 \cdot f_2].v$$

**Proof:** to prove 1), use (11) and (8) to show that for any $v = \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n$ and $w = \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n$ in $S^n\mathfrak{S}$

$$<w, D[I(f(s))].v> = <w, \sum_i \eta_1 \otimes \eta_2 \otimes \ldots \otimes (D[I(f(s))]\eta_i) \ldots \otimes \eta_n> =$$

$$= \sum_{\{\sigma\}} \sum_i <\xi_{\sigma(i)}, D[I(f(s))]\eta_i> \prod_{j \neq i} <\xi_{\sigma(j)}, \eta_j> =$$

$$= \sum_{\{\sigma\}} \sum_i <(-D[I(f^*(s))])\xi_{\sigma(i)}, \eta_i> \prod_{j \neq i} <\xi_{\sigma(j)}, \eta_j> =$$

$$= <(-D[I(f^*(s))]), w, v>$$

Since in $\mathfrak{S}$ $<S^n\mathfrak{S}, S^m\mathfrak{S}>$ is zero for $m \neq n$, the above equality is true for any $v, w \in \mathfrak{S}$

Similarly, since for any $v = \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_{n-1}$ in $S^{n-1}\mathfrak{S}$ and $w = \xi_1 \otimes \xi_2 \otimes \ldots \otimes \xi_n$
in $S^n\mathfrak{f}$,

$$<w, \beta[I(f(s))] \otimes v> = \sum_{\{\sigma\}} \sum_i <\xi_{\sigma(i)}, \beta[I(f(s))]> \prod_{j \neq i} <\xi_{\sigma(j)}, \eta_j> =$$

$$<\beta[I(f(s)]>, w, v>,$$

and also for any $v \in S^n\mathfrak{f}$ and $w \in S^{n-1}\mathfrak{f}$,

$$<w, (-<\beta[I(f^*(s)] 1]), v> = < -\beta[I(f^*(s)] 1] \otimes w, v>$$

Collecting all terms, we obtain 1)

2) can be verified by a straitforward computation.

2.4 Representation of the Universal Enveloping Algebra in $\mathfrak{g}$

Elements in the universal enveloping algebra, which are polynomials in generators, are represented by operators acting in $\mathfrak{g}$ given by corresponding polynomials of operators $T[I(f(s))]$. The commutation relations of the algebra are satisfied due to Proposition.

2.5 Operators $L_m$

**Definition: operators $L_m$**

Define $L_m$ to be elements of the universal enveloping algebra given by

$$L_m = \left( \sum_{k=\pm \infty}^{\pm \infty} \sum_{i<j} T[I_{ij}(e[-k])] T[I_{ij}(e[k+m])] - \right.$$

$$\left. - \sum_i T[I_{i,0}(e[-k])] T[I_{i,0}(e[k+m])] \right) - C(m)1,$$

$$C(m) = \sum_{k=\pm \infty}^{\pm \infty} \sum_i <\beta[I_{i,0}(e[-k]^* 1)], \beta[I_{i,0}(e[k+m])]>$$

(15)

where operators $T[I(f(s))]$ are defined in (11), and $\{e[k]\}$ is the basis in $\mathfrak{g}$, see (1).

**Proposition 2.3.** The action of $L_m$ in $\mathfrak{g}$ as a formal series is well defined.

Indeed, it is easy to check that for an $\eta \in \mathfrak{f}$ and any $k, m \in Z$

$$\left( \sum_{i<j} D[I_{ij}(e[-k])] D[I_{ij}(e[k+m])] - \sum_i D[I_{i,0}(e[-k])] D[I_{i,0}(e[k+m])] \right) \eta = 0$$
and

\[ \sum_i D[I_{i,0}(e[-k])] \beta[I_{i,0}(e[k + m])] = 0 \]

Therefore, dangerous terms disappear, and for any \( v = \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n \in \mathfrak{h} \),

\[ L_m v = \sum_k \left( \right) \]

\[ \sum_{i<j,a,b} \eta_1 \otimes \eta_2 \otimes \ldots \otimes D[I_{i,j}(e[-k])] \eta_a \otimes \ldots \otimes D[I_{i,j}(e[k + m])] \eta_b \otimes \ldots \otimes \eta_n - \]

\[ - \sum_{i,a,b} \eta_1 \otimes \eta_2 \otimes \ldots \otimes D[I_{i,0}(e[-k])] \eta_a \otimes \ldots \otimes D[I_{i,0}(e[k + m])] \eta_b \otimes \ldots \otimes \eta_n - \]

\[ - \sum_{i,a} \eta_1 \otimes \eta_2 \otimes \ldots \otimes D[I_{i,0}(e[k + m])] \eta_a \otimes \ldots \otimes \eta_n \beta[I_{i,0}(e[-k])] - \]

\[ - \sum_{i,a} \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n \beta[I_{i,0}(e[-k])] \otimes \beta[I_{i,0}(e[k + m])] + \]

\[ + \sum_{a,b} \beta[I_{i,0}(e[-k] \ast 1)] \eta_a > \eta_1 \otimes \eta_2 \otimes \ldots \otimes D[I_{i,0}(e[k + m])] \eta_b \otimes \ldots \otimes \eta_n \]

\[ + \sum_{a,b} \beta[I_{i,0}(e[k + m] \ast 1)] \eta_a > \eta_1 \otimes \eta_2 \otimes \ldots \otimes D[I_{i,0}(e[-k])] \eta_b \otimes \ldots \otimes \eta_n \]

\[ + \sum_{a} \beta[I_{i,0}(e[-k] \ast 1)] \eta_a > \eta_1 \otimes \eta_2 \otimes \ldots \otimes D[I_{i,0}(e[k + m])] \eta_a \otimes \ldots \otimes \eta_n \beta[I_{i,0}(e[-k])] - \]

\[ - \sum_{a,b} \beta[I_{i,0}(e[-k] \ast 1)] \eta_a > \beta[I_{i,0}(e[k + m] \ast 1)] \eta_b > \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_a \otimes \ldots \otimes \eta_b \otimes \ldots \otimes \eta_n \]

**Remark.** It is easy to check, that unlike in Sugawara construction of the Virasoro algebra from current algebra in highest weight modules, here a "normal ordering" of operators \( T[I] \) in \( L \) gives the same expression as the one without the normal ordering.

**Proposition 2.4.**

1) \( L_m \) commutes with all operators \( T[I(f(s))] \), with \( f(s) \in \mathfrak{g} \)

2.a) For scalar product in \( \mathfrak{g} \) such that multiplication by \( s \) is a unitary operator, (2), operators \( L_m \) are formally (as a formal series) self-adjoint; however, \( ||L_m v|| \) is infinite for a generic \( v \in \mathfrak{h} \)

2.b) For scalar products (1) in \( \mathfrak{g} \), such that multiplication by \( s \) is just a bounded operator, not necessarily a self adjoint one, operators \( L_m \) are not self adjoint; for some scalar products, \( ||L_m v|| \mathfrak{h} \) is finite for a generic \( v \in \mathfrak{h} \).
3 \quad L_0 \text{ vs. Loop Equation}

Let us modify the construction of the representation, (11) and (6), as follows: let $\mathfrak{A} = \bigoplus_n S^n$ (no symmetrization, unlike the previous construction). The inner product in $\mathfrak{A}$ is defined as

$$<\eta_1 \otimes \eta_2 \ldots \otimes \eta_n, \xi_1 \otimes \xi_2 \ldots \otimes \eta_m> = \begin{cases} 
0, & m \neq n \\
\prod_{i=1}^{n} <\eta_i, \xi_i>, & m = n
\end{cases}$$

In this new representation, currents act on $v = \eta_1 \otimes \eta_2 \otimes \ldots \otimes \eta_n \in \mathfrak{A}$, with $\{\eta_i\} \in \mathcal{H}$, as follows:

$$T[I(f(s))].v = (D[I(f(s))] + \beta[I(f(s))] - <\beta[I(f(s)^*1]>, v \equiv \sum_{i=1}^{n} \eta_1 \otimes \eta_2 \otimes \ldots \otimes (D[I(f(s)])\eta_i) \ldots \otimes \eta_n +$$

$$+ \sum_{i=0}^{n} \eta_1 \otimes \eta_2 \otimes \ldots \eta_i \otimes \beta[I(f(s)]] \otimes \ldots \otimes \eta_n -$$

$$- \sum_{i} <\beta[I(f(s)^*1]], \eta_i> \eta_1 \otimes \eta_2 \otimes \ldots \eta_i \ldots \otimes \eta_n$$

It is easy to check that the commutation relations are satisfied, and that operators $T[I(f(s))]$ are skew self adjoint for scalar products in $s$ where multiplication by $s$ is self-adjoint. We construct operators $L_m$ in the universal enveloping algebra as before, (15).

There are similarities between two infinite-dimensional Laplacians, $L_0$ and the loop operator $\mathcal{L}$ for a Wilson loop (monodromy ) $W[x(s)]$ of the Yang-Mills connection $A(x)dx$ on $S^n$, $W[x(s)] = \int DA(x) \ P \exp \left( i \oint x(s) \sum_{\mu=1}^{4} A_{\mu} dx^{\mu} \right) \exp \left( - \int_{S^n} F \wedge *F \right)$, with the loop equation $\mathcal{L} W[x(s)] \equiv \int ds_1 ds_2 \delta(s_1 - s_2) \frac{\delta}{\delta x(s_1)} \frac{\delta}{\delta x(s_2)} W[x(s)] = 0$; for a non-self-intersecting path $\{x(s)\}$ being a consequence of the Yang Mills equation for the connection $A(x)dx$.

Let us think about $v \in \mathfrak{A}$ as about certain functions in many variables, $v(X(1), t_1, X(2), t_2, X(n), t_n) = \eta_1(X(1), t_1) \otimes \eta_2(X(2), t_2) \otimes \ldots \otimes \eta_n(X(n), t_n)$, where each
$X_{(a)}$ takes values in $S^n$. Then

$$L_0v = \left( \sum_{k=-\infty}^{\infty} \sum_{a=1}^{n} \left( \sum_{i<j} D_{X_{(a)}}{I}_{ij}(e[-k](t_a)) \right) D_{X_{(a)}}{I}_{ij}(e[k](t_a)) - \sum_i D_{X_{(a)}}{I}_{i,0}(e[-k](t_a)) \right) D_{X_{(a)}}{I}_{i,0}(e[k](t_a)) \right) v +$$

$$+ \sum_{a \neq b=1}^{n} \left( \sum_{i<j} D_{X_{(a)}}{I}_{ij}(e[-k](t_a)) \right) D_{X_{(b)}}{I}_{ij}(e[k](t_b)) - \sum_i D_{X_{(a)}}{I}_{i,0}(e[-k](t_a)) \right) D_{X_{(b)}}{I}_{i,0}(e[k](t_b)) \right) v \right)$$

$$(18)$$

$$+ \text{ (terms with not more then one derivative)}$$

where $D_{X_{(a)}}{I}(f(t_a))$ are differential operators acting as differentiation only in $X_{(a)}$ variables, with coefficients which are functions of $X_{(a)}, t_a$

The terms in $(18)$ which involve 2 differentiations in the same variable cancel out, since in fact the second order operator restricted to diagonals (we will call this restricted operator $\tilde{L}_{a}$, $a = 1, 2, \ldots n$) is zero,

$$\tilde{L}_{a} := \sum_{k=-\infty}^{\infty} \left( \sum_{i<j} D_{X_{(a)}}{I}_{ij}(e[-k](t_a)) \right) D_{X_{(a)}}{I}_{ij}(e[k](t_a)) - \sum_i D_{X_{(a)}}{I}_{i,0}(e[-k](t_a)) \right) D_{X_{(a)}}{I}_{i,0}(e[k](t_a)) \right) = 0$$

$$(19)$$

in $\mathcal{H}$ for every $a$. We view this equation as in some sense a discrete approximation to the loop equation. The restricted operator $\tilde{L}_{a}$ consists of 2 pieces, the spin operator for the maximum compact subgroup of rotations of $S^n$, and the non-compact piece; they both come with the same infinite constant $\sum_{k=-\infty}^{\infty} 1$ (which in some regularization, is pretty small, $1 + 2\zeta(0) = 0$); but since the constant is the same for both terms, we can divide by this constant to compare. Dividing by this constant, we will get

$$\mathcal{L}_{a} := \left( \sum_{i<j} D_{X_{(a)}}{I}_{ij}(1) \right) D_{X_{(a)}}{I}_{ij}(1) - \sum_i D_{X_{(a)}}{I}_{i,0}(1) \right) D_{X_{(a)}}{I}_{i,0}(1) \right) = 0$$

The spin operator $\sum_{i<j} D_{X_{(a)}}{I}_{ij}(1) D_{X_{(a)}}{I}_{ij}(1)$ is unitary and compact; in some basis it reduces to multiplication by constants, the total spin. This spin cannot be equal to
zero, since the only eigenvalue for zero spin is the function which is identically one, and those were thrown out our Hilbert space $\mathfrak{h}$. The remaining piece is the usual laplacian operator, up to lower order terms in derivatives, in a tangent hyperplane to a sphere; thus what we observe is that restrictin of $L_0$ to the diagonals yields that the states in $\mathfrak{a}$ are eigenfunctions of the restricted to the diagonal laplacian

$$\sum_i D_{X_{(a)}}[I_{i,0}(1)] D_{X_{(a)}}[I_{i,0}(1)],$$

(20)

which is just a usual laplacian modulo terms with lower number of derivatives. Thus we obtain the following Proposition 3.5.

1) the reduction of $L_0$ in $\mathfrak{a}$ to the diagonals is identically zero
2) the above is equivalent to the fact that the restriction of the usual laplacian to the diagonals is zero on the subspace in $\mathfrak{h}$ of solutions of the equation, given by the spin operator plus lower in derivative terms

4 Conclusions

We constructed canonical non-highest weight unitary irreducible representation of $\hat{so}(1,n)$ current algebra as well as canonical non-highest weight non-unitary representations, We constructed certain Laplacian operators in the universal enveloping algebra, acting in representation space. We speculated about possible relation of those Laplacians with the loop operator for the Yang-Mills.
5 Acknowledgments

Conversations with A. Vershik, I.M. Gelfand, B. Bakalov, are appreciated. I am grate-
ful to IHES for a very stimulating atmosphere, the financial support, and hospitality.

References

[1] P.A.M. Dirak, J. Math. Phys. 4, 901,(1963)
[2] H. Sugawara, A field theory of currents, Phys. Rev 170 (1968), 1659
[3] Streater, Current commutation relations, continous tensor products, and infinitely
divisible group representations
[4] A.V. Skorohod, Integration in Hilbert Spaces
[5] Gelfand, Graev, Vershik, Russ. Math. Surv., 28 (5) 1973
[6] V. Kac, Infinite Dimensional Lie Algebras
[7] F. Beresin, Quantisation in Complex Symmetric spaces, , DAN, 1975
[8] Knizhnik, Zamolodchikov, Nucl. Phys. B247, 83, 1984
[9] Balog, Forgacs, O’Raifeartaigh, Wipf, Nucl. Phys B325 (1989), 225
M. Petropoulos, Phys. Lett. B, 236 (1990) 151
Itzhak Bars, Solution of the SL(2,R) string in curved spacetime, hep-th 9511187
[10] E.Witten, Anti-de Sitter Space, Thermal Phase Transition, And Confinement In
Gauge Theories, Adv.Theor.Math.Phys. 2 (1998) 505-532
[11] Bershadsky, Zhukov, Vaintrob, PSL(n, n) Sigma model as a Conformal Field
Theory, hep-th 9902180
[12] Berkovits, Vafa, Witten. Conformal Field Theory on AdS Background With
Ramond-Ramond Flux, hep-th 9902098
[13] A. Polyakov, talks at ENS.