Non-revelation Mechanism Design

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Abstract

We consider mechanism design and redesign for markets like Internet advertising where many frequent, small transactions are organized by a principal. Mechanisms for these markets rarely have truth-telling equilibria. We identify a family of winner-pays-bid mechanisms for such markets that exhibit three properties. First, equilibria in these mechanisms are simple. Second, the mechanisms’ parameters are easily reoptimized from the bid data that the mechanism generates. Third, the performance of mechanisms in the family is near the optimal performance possible by any mechanism (not necessarily within the family). Our mechanisms are based on batching across multiple iterations of an auction environment, and our approximation bound is asymptotically optimal, with loss inversely proportional to the cube root of the number of iterations batched. Our analysis methods are of broader interest in mechanism design and, for example, we also use them to give new sample complexity bounds for mechanism design in general single-dimensional agent environments.

1 Introduction

This paper builds a theory for mechanism design and redesign for markets like Internet advertising where many frequent, small transactions are organized by a principal. In these environments bidders place bids in advance and, as goods become available, the bids of the relevant bidders are entered into an auction to determine the allocation of goods. For practical reasons, these market mechanisms do not have truth-telling equilibria and, thus, it is challenging to reason about their design and, when bid data is available, their redesign. Our theory focuses on designing practical, e.g., winner-pays-bid, mechanisms for the canonical model of auction theory, i.e., the independent private value model.

Much of the theory of mechanism design is developed via the revelation principle (Myerson, 1981) which observes that existence of a mechanism with good equilibrium implies the existence of one with a good truth-telling equilibrium. There are many practical constraints, however, that prevent mechanisms with truth-telling equilibrium from being adopted, see Ausubel and Milgrom (2006). Instead, practitioners often employ non-revelation mechanisms such as those with winner-pays-bid (i.e., first-price) semantics. While it is possible to undo the revelation principle and identify a corresponding non-revelation mechanism, the resulting mechanisms are generally complex and require fine-grained distributional knowledge, see Appendix A.

The design of good non-revelation mechanisms is well understood most generally in i.i.d. position auctions where the efficient equilibrium is unique (Chawla and Hartline, 2013). Relaxing

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the assumption of identical bidders, however, the studied non-revelation mechanisms for these environments generally leave a constant factor of the optimal welfare and revenue on the table, see [Syrgkanis and Tardos, 2013], [Hartline et al., 2014]. Moreover, in more complex environments, such as that of single-minded combinatorial auctions, the studied non-revelation mechanisms can very far from optimal (super-constant approximation, see [Dütting and Kesselheim, 2015]).

In this paper we identify a family of practical, non-revelation mechanisms for general environments, including single-minded combinatorial auctions, that exhibit three properties. First, these mechanisms use limited distributional information to obtain performance near that of the optimal mechanism, a $1 + \epsilon$ approximation for any desired $\epsilon$. Second, these mechanisms can be easily parameterized and reoptimized from the equilibrium bid data they generate. Mechanisms from such a family can be tuned as fundamentals of the market evolve. Third, the equilibria in these mechanisms have a simple focal equilibrium, and are therefore easy to analyze.

The *iterated population model* is a standard interpretation of the independent private value model. In this model there are a collection of populations, and each population consists of a continuum of bidders which induces a distribution of values. The strategies of these bidders induce a distribution of bids. In each iteration, one bidder from each population is drawn independently and uniformly at random to participate in a mechanism. For example, the mechanism might be the first price auction where the highest bidder wins and pays her bid. Notice that a single stage of this iterated population model is equivalent to the standard independent private value model that is pervasive in auction theory. Our model simply is an explicit extension of this standard model to an iterated environment.

Our approach yields the following conclusions. First, while the classical theory tends to design and analyze mechanisms for the stage environment, ignoring our assumption that the auction is iterated over many stages, there is an advantage to linking the decisions that are made across the stages. Second, the family of mechanisms that we identify are *rank-based*, i.e., based only on the order of bids from bidders in the same population and not the cardinal values of the bids. Thus, mechanisms in this family look, to each population of bidders, like the i.i.d. position auctions that [Chawla and Hartline, 2013] prove have simple, unique, and efficient equilibria. Third, in relation to data-driven methods in mechanism design, our approach uses samples from the bid distribution in two distinct ways: (a) the family of mechanisms is parameterized and these parameters can be optimized from bid data using the simple and statistically efficient methods for counterfactual estimation developed by [Chawla et al., 2014], cf. the literature on sample complexity in mechanism design; and (b) samples from the bid distribution are used within the execution of the mechanism to calculate the ranks of bidders (from which the mechanism determines its outcome), cf. the literature on prior-independent mechanism design.

The second observation above, that the equilibria for each population looks like an i.i.d. position auction which only admits the efficient equilibrium, gives us a “revelation principle” for non-revelation mechanisms. The efficient equilibrium assigns bidders to positions in the same order as their values. Thus, the outcome (and, e.g., welfare) of the mechanism is the same as it would be if the bidders reported their values (even though doing so is not an equilibrium). Consequently, our approach gives the benefits of revelation mechanism design, namely that analysis is straightforward from bidder values, but for simple and practical mechanisms that do

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1 A similar observation was made by [Jackson and Sonnenschein, 2007] in the case where each agent is returning repeatedly to the mechanism. Our conclusion is conceptually different from theirs as the decisions we are linking are between distinct bidders from the same population.
not have truthtelling as an equilibrium.

Our analysis is based on several natural new approximation results that compare pricing and ranking mechanisms. This analysis is reminiscent of the correlation gap work of [Yan (2011)] and may be of independent interest.

1.1 Motivating Example: Bayesian Single-Minded Combinatorial Auction

To illustrate the difficulties that arise when designing non-revelation mechanisms for single-dimensional agents, consider the stage environment of the single-minded combinatorial auction. In each stage $t$, there are $m$ items for sale, and the bidders from population $i$ desire some publicly-known bundle $S_i$. A bidder in population $i$ obtains value $v_i^t$ if they receive all the items in $S_i$, and 0 otherwise. A winner-pays-bid mechanism chooses an allocation (e.g. “allocate the feasible set maximizing the sum of bids”) and charges winners their bids.

An important special case of the single-minded combinatorial auction is when there is a single item, desired by all the bidders. The natural winner-pays-bid mechanism for a single item is the first-price auction. When agents are i.i.d., the first-price auction has a unique, efficient equilibrium (e.g., [Chawla and Hartline, 2013]). With asymmetric distributions, however, the equilibria of the first-price auction have long been known to be inefficient ([Vickrey, 1961]). The best-known welfare bounds guarantee only a $\frac{1}{e}$-fraction of the optimal welfare ([Syrgkanis and Tardos, 2013]), while there are examples with welfare as much as a 1.15-factor away from optimal ([Hartline et al., 2014]).

For the general single-minded combinatorial setting, i.e., with multiple items and bidders desire distinct bundles of items, Bayes-Nash equilibrium welfare can be even worse. The highest-bids-win winner-pays-bid mechanism allocates to the feasible subset of bids with the highest total. Computational issues aside, the equilibria of the winner-pays-bid mechanism with this rule can have welfare as far as a factor of $m$ away from optimal. The literature has considered also a greedy winner-pays-bid mechanism, where bidders are allocated greedily based on their bid weighted by $\sqrt{1/|S_i|}$. The equilibrium welfare of this greedy mechanism can be as low as a $\sqrt{m}$-factor of the optimal welfare ([Borodin and Lucier, 2010]). Both of these mechanisms are unparameterized by distributional knowledge; [Dütting and Kesselheim, 2013] prove that no unparameterized mechanism can be shown to obtain better than a $\sqrt{m}$ fraction of the optimal welfare via the standard proof method.

When the value distributions of the bidders are known, there are winner-pays-bid mechanisms that achieve optimal welfare or revenue in equilibrium. Identifying such a mechanism requires a delicate undoing of the revelation principle, which we outline in Appendix A. The resulting mechanism is complex and parameterized by details of the value distributions. As such, these “un-revelation” mechanisms are more of a theoretical novelty than a practical suggestion.

This paper designs non-revelation mechanisms, including those with winner-pays-bid semantics, for arbitrary single-dimensional settings, including the single-minded combinatorial auction, in the iterated population environment. Our mechanisms have almost optimal welfare or revenue in their unique BNE. Furthermore, our mechanisms require much less information on the part of the designer than the un-revelation mechanism of Appendix A. We simply require that the designer know expected order statistics of the value distribution (or virtual value distribution, for revenue), rather than the full distributional knowledge required by the un-revelation mechanism. These statistics can be easily estimated from previous iterations of the mechanism.
1.2 Approach and Results

Our approach is based on linking decisions for bidders from each population across the iterations of the stage environment. This linking of decisions replaces competition between bidders in distinct populations, which is asymmetric, with competition between bidders in the same population, which is symmetric. This linking is achieved by considering the family of mechanisms that determine their outcome only from the rank of a bidder among others from the same population. Such a mechanism cannot know a bidder’s value exactly but has a posterior distribution over values, obtained by conditioning on the bidder’s rank. The optimal mechanism in this family optimizes as if the bidders’ values were equal to the expectation of their respective posterior distributions given their ranks. Our approximation result then shows that little welfare (or revenue) is lost by optimizing with respect to these conditional expected values rather than the exact values. When there are \( n \) populations and decisions are linked across \( T \) iterations, the loss is bounded by a \( O\left(\sqrt{n/T}\right) \) fraction of the optimal welfare (or revenue).

To prove our linking result, we build up a series of natural approximation lemmas which we believe to be of independent interest. As evidence, we use these results to prove a sample complexity result for general feasibility environments. For regular value distributions, using \( \Theta(n^5 \epsilon^{-8}) \) sampled profiles from the true value distributions, we give a computationally efficient procedure to obtain a mechanism with expected revenue at least a \( (1 - \epsilon) \)-fraction of optimal. For unbounded regular distributions, this is the only known polynomial sample complexity result for arbitrary feasibility environments. Our procedure coarsens value space based on the samples and estimates conditional virtual values for bidders in each coarse region.

1.3 Related Work

Many previous papers have studied the welfare and revenue properties of non-truthful mechanisms, or their \textit{price of anarchy}. Syrgkanis and Tardos (2013) give worst-case welfare approximation bounds for a large family of non-truthful auctions, including first-price and all-pay auctions. Borodin and Lucier (2010) derive similar worst-case results for winner-pays-bid mechanisms based on greedy allocation rules, and Devanur et al. (2015) derive non-truthful mechanisms whose equilibria are simple to learn via no-regret algorithms. Moreover, Hartline et al. (2014) extend many of these welfare analyses to the revenue objective when there is sufficient competition or reserve prices. Many of these proofs are via the smoothness framework of Roughgarden (2009). Unfortunately, Dütting and Kesselheim (2015) prove limits on smoothness-based welfare bounds for environments such as the single-minded combinatorial auction. All of these mechanisms are unparameterized by distributional knowledge (except for the reserve price result). We get asymptotically welfare optimal mechanisms – bypassing this and other lower bounds – by allowing the designer to better adapt the stage mechanism based on bids in other stages.

The mechanisms we consider for the iterative population model link decisions for bidders from the same population across distinct iterations. This linking of decisions results in the perceived competition in the mechanism to be among bidders within the same population. Competitions for rank among independent and identically distributed bidders are strategically simple (e.g., Chawla and Hartline, 2013). Linking of decisions has been previously considered in the context of social choice (Jackson and Sonnenschein, 2007) and principal-agent delegation (Frankel, 2014). In both of these papers the decisions being linked are the multi-dimensional reports of a single agent and this linking of decisions enables the principal to obtain the first-best outcome in the limit. Our analysis improves upon these results by bounding the convergence...
rate to the first-best outcome.

Our mechanism changes the nature of the competition in the iterative population model so that bidders compete across iterations but within each population instead of within an iteration but across populations. This approach has been employed in revelation mechanism design to reduce Bayesian mechanism design to Bayesian algorithm design. Hartline and Lucier (2010) give such a reduction for single-dimensional agents and both Hartline et al. (2011) and Bei and Huang (2011) give reductions for single- and multi-dimensional agents. Notably, the reduction of Hartline et al. (2011) for single-dimensional agents gives a mechanism that falls within the family of mechanisms that we develop, albeit with a truth telling payment scheme. The approximation of welfare and revenue of our mechanism is independent of the payment scheme; therefore, their approximation result implies the same approximation result for our mechanisms. The approximation bound that we derive improves on theirs in that it is multiplicative and does not assume bidders values fall within a bounded range.

In so far as our main problem is harder than the main problem of sample complexity in mechanism design (as we have samples from the bid distribution not the value distribution), our results have implications on that literature. The question in this literature is to identify the number of samples necessary to select a mechanism that guarantees a multiplicative \((1 - \epsilon)\) approximation to the optimal revenue. This question was studied for single-item auctions by Cole and Roughgarden (2014) and extended to downward-closed environments by Devanur et al. (2016). The latter paper shows that the sufficient number of samples is \(\tilde{O}(n^2\epsilon^{-4})\). Our methods extend this result by relaxing downward closure, but the sample complexity bound worsens to \(\tilde{O}(n^5\epsilon^{-8})\). All of these mechanisms are computationally tractable and apply to unbounded but regular distributions (see Section 2 for a definition of regularity).

Morgenstern and Roughgarden (2015) have previously considered sample complexity for general environments (not necessarily downward closed) and their work identified a family of mechanisms for which the mechanism with the highest revenue on the sample is an additive \(\epsilon\) off from optimal, with the number of samples bounded by \(\tilde{O}(n^5\epsilon^{-3})\). Unfortunately, identifying an optimal mechanism for the sample can be computationally hard, even for simple environments like single-dimensional matching markets (e.g. Briest (2008)). Moreover, in general environments the additive \(\epsilon\) loss can be much worse than a multiplicative \((1 - \epsilon)\) approximation. The mechanism for which we derive our sample complexity bound is in the family of mechanisms identified by Morgenstern and Roughgarden (2015).

Finally, the optimal mechanism in our family of mechanisms is identified via a characterization of optimal rank-based mechanisms from Chawla et al. (2014).

2 Preliminaries

This work considers a special case of the single-dimensional independent private value model of mechanism design, in which a stage environment is iterated many times. The stage environment has \(n\) agents, with agent \(i\) having value \(v_i \in [0, \infty)\) drawn from distribution \(F_i\). Equivalently, one can think of \(i\) being drawn uniformly at random from a continuum population with value distribution \(F_i\). The joint distribution on values is is the product \(F = F_1 \times \cdots \times F_n\). Service to agent \(i\) is denoted by \(x_i = 1\) and no service by \(x_i = 0\). A stage allocation is denoted by \(x = (x_1, \ldots, x_n)\). The set of allowable allocations \(x\) is given by set system \(X \subseteq \{0, 1\}^n\) (notably, we do not require that \(X\) be downward closed). Agent \(i\) can be assigned a non-negative payment denoted \(p_i\) and her utility is linear in allocation and payment as \(u_i = v_i x_i - p_i\).
We focus our study on a *batched environment* which has $T$ independent copies of the stage environment. The batched environment is an independent private value model with $nT$ agents. Importantly, the $T$ agents from population $i$ are independent and identically distributed (i.i.d.) according to $F_i$. The set of feasible allocations for the batched environment is $X^n$. Values, allocations, payments, and utilities in the batched environment are denoted as $v^T_i$, $x^T_i$, $p^T_i$, and $u^T_i$ for agent $i$ in stage $t$. In this paper we contrast mechanism design for the stage environment with that for the batched environment and demonstrate that samples from the bid distribution allow an iterated stage environment to be converted into a batched environment. Below we give definitions and basic results from auction theory focusing on the stage environment; however, the batched environment satisfies analogous definitions.

A mechanism takes as input a profile of bids $b = (b_1, \ldots, b_n)$ and outputs a feasible allocation $x \in X$ and agent payments $p$. A mechanism consists of an allocation algorithm $\tilde{x}(b)$, which maps bid profiles to a feasible allocation, and a payment rule $\tilde{p}(b)$, which maps bid profiles to a non-negative payment for each agent. A standard allocation algorithm is highest-bids-win which is defined by $\tilde{x}(b) \in \arg\max_{x \in X} \sum_i b_i x_i$. We consider payment rules defined directly from the allocation algorithm via standard payment semantics. Winner-pays-bid semantics has payment rule $\tilde{p}_i(b) = b_i \tilde{x}_i(b)$, and all-pay semantics has payment rule $\tilde{p}_i(b) = b_i$. Mechanisms with these semantics do not have truth-telling as an equilibrium.

We analyze non-revelation mechanisms in Bayes-Nash equilibrium (BNE): each agent’s report to the mechanism is a best response to the distribution of bids induced by other agents’ strategies. The strategy of agent $i$ is denoted $s_i$ and maps the agent’s value to a bid. The mechanism $(\tilde{x}, \tilde{p})$, the agents’ strategies $s$, and distribution of values $F$ induce interim allocation and payment rules. Agent $i$’s interim allocation rule is $x_i(v_i) = \mathbb{E}_{v_i \leftarrow [\tilde{x}_i(s(v))]}$ and interim payment rule $p_i(v_i) = \mathbb{E}_{v_i \leftarrow [\tilde{p}_i(s(v))]}$. Myerson (1981) characterized the interim allocation and payment rules that arise in BNE. These results are summarized in the following theorem.

**Theorem 1** [Myerson, 1981]. *Interim allocation and payment rules are induced by a Bayes-Nash equilibrium of a mechanism with onto strategies and values drawn from a product distribution if and only if for each agent $i$,

1. (monotonicity) allocation rule $x_i(v_i)$ is monotone non-decreasing in $v_i$, and
2. (payment identity) payment rule $p_i(v_i)$ satisfies $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(z) dz + p_i(0)$.

This paper studies the objectives of welfare and revenue. The welfare of a mechanism is $\mathbb{E}\sum_i v_i x_i(v_i))$. The optimal mechanism for welfare allocates the value-maximizing feasible set, which is monotone and therefore implementable with payments via Theorem 1. The revenue of a mechanism is given by $\mathbb{E}\sum_i p_i(v_i))$. Our revenue analysis is based on the standard virtual value characterization of Myerson (1981):

**Lemma 2.** In BNE, the ex ante expected payment of an agent satisfies $\mathbb{E}_{v_i}[p_i(v_i)] = \mathbb{E}_{v_i}[\phi_i(v_i) x_i(v_i)]$, where $\phi_i(v_i) = v_i - \frac{1-F(v_i)}{f(v_i)}$ is the Myerson virtual value for value $v_i$.

It follows from Lemma 2 that the equilibrium revenue of a mechanism is given by $\sum_i \mathbb{E}_v[p_i(v_i)] = \sum_i \mathbb{E}_v[\phi_i(v_i) x_i(v_i)]$. Regular distributions are those for which the virtual value functions are monotone non-decreasing. For regular distributions, the optimal mechanism allocates the virtual value-maximizing feasible set.

This paper develops mechanisms for the batched environment and equilibria in these mechanisms that look to each agent like an i.i.d. rank-by-bid position auction. A position auction
is defined by \( n \) position weights \( w_1 \geq \ldots \geq w_n \) and an outcome is an assignment of agents to positions. If agent \( i \) is assigned to position \( j \) her allocation is \( x_i = 1 \) with probability \( w_j \) and zero otherwise. The rank-by-bid allocation algorithm assigns agents to positions assortatively by bid. The following theorem shows that Bayes-Nash equilibria in rank-by-bid position auctions are straightforward.

**Theorem 3** [Chawla and Hartline (2013)]. In i.i.d. position environments, the rank-by-bid winner-pays-bid and all-pay auctions have a unique and welfare-maximizing BNE (in which agents are assigned to positions in order of their true values).

### 3 Surrogate-Ranking Mechanisms

In this section we define a family of mechanisms for the batched environment and discuss their equilibria and optimization. Recall that the batched environment has \( T \) stages and each stage environment has \( n \) agents that are drawn uniformly from each of \( n \) populations. Agents from the same population are i.i.d. Our mechanisms will have allocation rules from the following family:

**Definition 4.** The surrogate ranking algorithm (SRA) for the batched environment is parameterized by \( nT \) surrogate values, denoted \( \Psi_i = \{\psi^1_i \geq \ldots \geq \psi^T_i\} \) for each population \( i \), and a stage allocation algorithm \( \hat{x} \) that maps a profile of \( n \) surrogate values to a feasible allocation \( x \in X \).

The algorithm on bids from each of the \( nT \) agents works as follows:

1. For each population \( i \) and stage \( t \), compute the rank \( r(i,t) \) of the bid of the agent in population \( i \) and stage \( t \) with respect to the \( T - 1 \) other bids from population \( i \).

2. For each stage \( t \), allocate to agents in stage \( t \) according to \( \hat{x}(\psi^{r(1,t)}_1, \ldots, \psi^{r(n,t)}_n) \).

As described in Section \ref{sec:algorithms}, we consider mechanisms that are defined by an allocation algorithm and a payment semantic. Our analysis will include the surrogate ranking mechanisms (SRMs) with both winner-pays-bid and all-pay semantics.

In Section \ref{sec:notation}, we present notation and concepts which will aid in reasoning about surrogate ranking algorithms. Surrogate-ranking mechanisms can be implemented with either winner-pays-bid or all-pay semantics, and in Section \ref{sec:characteristics}, we show that both implementations have a focal and well-behaved BNE. We derive the welfare- and revenue-optimal surrogate ranking algorithms in Section \ref{sec:optimal}.

#### 3.1 Surrogate-Based Stage Algorithms

In this section we discuss stage algorithms based on surrogate values (as in Definition \ref{def:sra}). In such an algorithm, agent \( i \)'s surrogate values are denoted by \( \Psi_i = \{\psi^1_i \geq \ldots \geq \psi^T_i\} \), and these are coupled with a stage allocation algorithm \( \hat{x} \) which maps a profile of \( n \) surrogate values to a feasible allocation \( x \in X \). The following definition characterizes the outcome of such a stage allocation algorithm when the profile of surrogate values is uniformly distributed.

**Definition 5.** For \( nT \) surrogate values, denoted \( \Psi_i = \{\psi^1_i \geq \ldots \geq \psi^T_i\} \) for each population \( i \), and a stage allocation algorithm \( \hat{x} \) that maps a profile of \( n \) surrogate values to a feasible allocation in \( X \), the characteristic weights \( (w^1_i, \ldots, w^T_i) \) for population \( i \) are defined by calculating the allocation probability of each surrogate when the surrogates of other populations are drawn uniformly at random, i.e., \( w^j_i = \mathbb{E}[\hat{x}_i(\psi^j_i, \psi_{-i})] \) for each surrogate \( j \) and uniform random \( \psi_{-i} \).
In what follows, we discuss several natural surrogate-based stage algorithms that, as we will see, induce uniform surrogate distributions. The agents in a mechanism based on such an algorithm effectively compete for allocation probabilities equal to the characteristic weights. The input distribution to the stage algorithm $\hat{x}$ is jointly determined by two factors: the rule used to select surrogate values, and the distribution of bids input to this rule. Formally:

**Definition 6.** Given surrogate values $\Psi_i = \{\psi_1^i, \ldots, \psi_T^i\}$ for agent $i$, a surrogate selection rule for $\Psi_i$ is a function $\sigma_i$ mapping bids for $i$ to surrogate values in $\Psi_i$.

Our analysis will focus on two particular surrogate selection rules (and their associated algorithms). The sample ranking rule samples bids according to some distribution and chooses a surrogate value based on the rank of agent $i$’s bid among the samples. The binning rule divides bid space into $T$ intervals of equal probability according to some distribution and maps a bid in the $j$th highest interval to the $j$th highest surrogate value. Formally (with the subscript for population $i$ omitted):

**Definition 7.** Given distribution $G$ and set of surrogate values $\Psi$, with $\psi_1 \geq \ldots \geq \psi_T$, the sample-ranking selection rule for $G$ and $\Psi$ draws $T-1$ samples from $G$, computes the rank of $r$ of input bid $b$ among the $T-1$ samples, and outputs surrogate value $\psi_r$.

**Definition 8.** Given distribution $G$, set of surrogate values $\Psi$ with $\psi_1 \geq \ldots \geq \psi_T$, and partitioning $\mathcal{I} = \{I_1, \ldots, I_T\}$ of $G$’s support into intervals of equal probability, the binning selection rule for $G$ and $\Psi$ maps input bid $b$ to $\psi^j$ for the $j$ for which $I_j \ni b$.

The surrogate ranking algorithm (Definition 4) implements the sample ranking selection rule with samples being drawn from the symmetric equilibrium bid distribution for each population. If $i$’s input bid is distributed according to the same distribution, then their rank among the samples will be uniformly distributed. This property will in turn mean that agents’ allocations are determined by their characteristic weights. Formally:

**Definition 9.** Given a distribution over bids $G$ and a surrogate selection rule $\sigma$ with surrogates $\Psi = \{\psi_1, \ldots, \psi_T\}$, $\sigma$ induces uniformity for $G$ if $P_{b \sim G}[\sigma(b) = \psi^j] = 1/T$ for all $j \in \{1, \ldots, T\}$.

The following lemmas are immediate from the definitions.

**Lemma 10.** The sample ranking and binning surrogate selection rules for any distribution induce uniformity for inputs drawn from that same distribution.

**Lemma 11.** The composition of a stage allocation algorithm with a surrogate selection rule that induces uniformity for its BNE bid distribution allocates to bidders according to its characteristic weights. Specifically, a bidder from population $i$ who is assigned surrogate value $\psi^j_i$ is allocated with probability $w^j_i$.

### 3.2 Equilibrium Analysis

In this section, we show that from each agent’s perspective, surrogate-ranking mechanisms hide asymmetry that might be present in the stage settings. In particular, they induce a rank-based position auction among agents from each population, forcing agents to compete for the characteristic weights of their population’s surrogate values. With pay-your-bid or all-pay payment semantics, they therefore inherit the equilibrium of rank-based position auctions, which is shown to be unique in Chawla and Hartline (2013). Formally:
Theorem 12. For a monotone\(^2\) stage allocation algorithm \(\mathbf{x}\), a set of \(T\) surrogate values \(\Psi_i = \{\psi^1_i \geq \ldots \geq \psi^T_i\}\) for each population \(i\), and their characteristic weights \(w^1_i\) for each \(i\) and \(j\), there is a BNE in the winner-pays-bid (resp. all-pay) SRM where for each population \(i\), the agents in population \(i\) bid according to the unique BNE of the i.i.d. rank-based winner-pays-bid (resp. all-pay) position auction with position weights \(w^1_i,\ldots,w^T_i\) and value distribution \(F_i\).

Proof. We argue from the perspective of agents in an arbitrary population \(i\). Assume agents in other populations are bidding according to the position auction equilibrium for the characteristic weights of their populations. By Lemma 10, symmetric bid distributions for each population induce uniformity for each population in every stage. Lemma 11 then implies that the agent in population \(i\) who is assigned surrogate value \(\psi^j_i\) is allocated with probability \(w^j_i\). Thus, to the agents in population \(i\) bid according to the equilibria of the rank-based winner-pays-bid (resp. all-pay) auction with position weights \(w^1_i,\ldots,w^T_i\). Monotonicity of the allocation rule implies monotonicity of the position weights; it follows that the unique equilibrium of agents from population \(i\) (as guaranteed by Theorem 3) assigns the agents to positions according to their values. We conclude that the prescribed strategies are an equilibrium for agents from all populations. \(\square\)

The equilibrium of Theorem 12 is unique under the natural assumption that agents are not able to condition their strategy on the label of their stage. The uniqueness follows from the fact that such an equilibrium is necessarily symmetric within each population, and therefore induces uniformity. Hence, the equilibrium appears to agents in each population as a position auction with position weights equal to the characteristic weights. The symmetric equilibrium for such an auction is unique, by a straightforward application of revenue equivalence. This yields:

Theorem 13. The equilibria of Theorem 12 for winner-pays-bid and all-pay SRMs are unique among stage-invariant BNE.

As the equilibrium of Theorem 12 is the unique equilibrium which is symmetric among agents in a population, we will refer to it as the symmetric equilibrium of the winner-pays-bid or all-pay SRM.

Definition 14. In the symmetric equilibrium of the winner-pays-bid (resp. all-pay) SRM, agents in each population \(i\) bid according to the unique BNE of the i.i.d. winner-pays-bid (resp. all-pay) rank-by-bid position auction for the position environment with position weights equal to the characteristic weights \(w^1_i,\ldots,w^T_i\) and distribution \(F_i\).

The surrogate ranking mechanism suggests a way to design mechanisms from data. In particular, samples from the bid distribution can be used to estimate good surrogate values, which we derive in Section 3.3. Samples from the bid distribution can also be used to calculate the rank of an agent’s value, as long as that agent’s bid is from the same distribution. The surrogate ranking algorithm (Definition 4) for the batched environment obtained these samples from the agents from the same population but in different stages of the batched environment. The following algorithm for the stage environment relaxes the assumption that the stages are batched and replaces it with direct sample access to the (supposed) bid distribution. These sampled bids could be obtained, for example, by previous iterations of the stage mechanism.

\(^2\)We will see in Section 4 that assuming monotonicity is without loss.
Definition 15. The surrogate sample ranking algorithm for the stage environment is parameterized by \( nT \) surrogate values, denoted \( \Psi_i = \{ \psi^1_i \geq \ldots \geq \psi^T_i \} \) for each population \( i \), a stage allocation algorithm \( \hat{x} \) that maps a profile of \( n \) surrogate values to a feasible allocation \( x \in X \), and \( n \) bid distributions \( G_1, \ldots, G_n \). The algorithm, on bids from each of the \( n \) agents, works as follows:

1. For each agent \( i \), draw \( T - 1 \) samples from the bid distribution \( G_i \).
2. For each agent \( i \), compute the rank \( r(i) \) of the bid of the agent with respect to the \( T - 1 \) bids sampled from \( G_i \).
3. Allocate to agents according to \( \hat{x}(\psi_1^{r(1)}, \ldots, \psi_n^{r(n)}) \).

The analysis of Theorem 12 shows that if for all agents \( i \), \( G_i \) is the bid distribution for population \( i \) in the symmetric equilibrium of the surrogate ranking mechanism, then it is a best response for agent \( i \) to bid according to that same equilibrium. In fact, Theorem 13 implies that this is the unique such stationary point. To see this, assume there was another set of distributions \( \{ G_i \}_{i=1}^n \) with the property that the best response in the winner-pays-bid or all-pay surrogate sample ranking mechanisms for \( \{ G_i \}_{i=1}^n \) was for agent \( i \) to bid according to \( G_i \). Note that this would also be an equilibrium of the corresponding SRM. By Theorem 13, the symmetric equilibrium is the unique equilibrium that is stage invariant. The following theorem summarizes this argument.

Theorem 16. For a monotone stage allocation algorithm \( \hat{x} \), surrogate values \( \{ \Psi_i \}_{i=1}^n \), and the SRM symmetric equilibrium bid distributions \( G_1, \ldots, G_n \), bidding according to \( \{ G_i \}_{i=1}^n \) is a BNE of the surrogate sample ranking mechanism for \( \hat{x} \), \( \{ \Psi_i \}_{i=1}^n \), and \( \{ G_i \}_{i=1}^n \).

The design of revelation mechanisms is facilitated by the fact that the assumed equilibrium has agents bidding their true values. Thus, if the outcome of the allocation algorithm has good properties with respect to its input bids, those properties also hold with respect to the agents’ values. Because the equilibria of surrogate ranking mechanisms are monotone within each population, they assign agents in each population to surrogates in the order of their values. This is the same allocation that would be achieved if the agents bid their values. Thus, for analysis of welfare and virtual welfare, we are free to consider the surrogate ranking algorithm on the true values of the agents.

Theorem 17. The allocation of the symmetric equilibrium of a surrogate ranking mechanism is the same as the outcome of the corresponding surrogate ranking algorithm on the true values of the agents.

3.3 Optimal Surrogate-Ranking Mechanisms

In this section we derive the optimal surrogate-ranking mechanisms for welfare and revenue, assuming agents play the symmetric equilibrium. Since agents are ranked according to their true values in this equilibrium, it suffices to optimize over the underlying surrogate ranking algorithms instead. The free parameters for such algorithms are the choice of surrogate values \( \psi^j_i \) for \( i \in \{ 1, \ldots, n \} \) and \( j \in \{ 1, \ldots, T \} \) and the choice of the stage allocation algorithm \( \hat{x} \). To optimize these parameters, we consider the relaxed algorithm design problem of maximizing a generic virtual surplus quantity subject to the constraint that the algorithm must be rank-based.
We note that for monotone virtual value functions, there is an obvious solution to this problem which happens to be a SRM.

Given a Bayesian population environment with distributions $F_1, \ldots, F_n$ and $T$ stages, we define the rank-based algorithm design problem as follows: the designer must choose a stage allocation algorithm $\pi$ which takes as inputs the ranks $r_i^t = r_i^1, \ldots, r_i^T$ of agents in an arbitrary stage $t$ within each of their respective populations and outputs a (possibly randomized) feasible allocation $\pi_i(r_i^t)$ for that stage. As a constraint, $\pi_i$ must be monotone for each agent. The objective is to maximize $\mathbb{E} \left[ \sum_j \sum_i \phi_i(v_i^t) \pi_i^j(r_i^t) \right]$ for some given virtual value function $\phi_i(\cdot)$, where ranks are drawn uniformly for each population. (For example, $\phi_i(v_i^t) = v_i^t$ corresponds to welfare maximization.) Note that the restriction to a single allocation algorithm across all stages is without loss of generality, as the stages are symmetric - one may permute the labels of the stages uniformly at random before running the algorithm.

The above optimization problem can be solved by inspection. Fixing an allocation algorithm, the objective can be rewritten as $\sum_i \sum_j \mathbb{E} \left[ \phi_i(v_i^t) | r_i^t \right] \pi_i^j(r_i^t)$ by linearity of expectation. From this expression, it becomes clear that the optimal algorithm maximizes the quantity $\sum_i \mathbb{E} \left[ \phi_i(v_i^t) | r_i^t \right] \pi_i^j(r_i^t)$ in every stage. Note that if $\phi_i(\cdot)$ is monotone, then this algorithm will be monotone as well, and therefore feasible. The above proves:

**Theorem 18.** The virtual welfare-optimal rank-based algorithm for the batched population setting maximizes $\sum_i \mathbb{E} \left[ \phi_i(v_i^t) | r_i^t \right] \pi_i^j(r_i^t)$ in each stage. In particular, the welfare-optimal rank-based algorithm uses $\phi_i(v_i^t) = v_i^t$, and for regular distributions, the revenue-optimal rank-based algorithm uses $\phi_i(v_i^t) = v_i^t - \frac{1-F(v_i^t)}{f(v_i^t)}$.

Note that the rank-based algorithm design problem is a relaxation of the surrogate ranking algorithm design problem; any surrogate ranking algorithm with a monotone stage allocation algorithm is feasible. Moreover, the algorithm prescribed by Theorem 18 is a surrogate ranking algorithm. We therefore can conclude that this algorithm is optimal among surrogate ranking algorithms.

**Corollary 19.** Assuming agents play the symmetric equilibrium, the welfare-optimal surrogate-ranking mechanism uses surrogate values $v_i^t = \mathbb{E}_{v \sim F_i} \left[ v | r_i^t \right]$ and allocation algorithm $\hat{x}(v) = \arg \max_{\pi \in \Pi} \sum_i v_i x_i$. For regular distributions, the revenue-optimal surrogate-ranking mechanism uses surrogate values $v_i^t = \mathbb{E}_{v \sim F_i} \left[ v | r_i^t \right]$ and allocation algorithm $\hat{x}(v) = \arg \max_{\pi \in \Pi} \sum_i v_i x_i$.

### 3.4 Inference

We conclude this section by noting that the optimal choice of surrogate values requires mild distributional knowledge in the form of expected order statistics of the value or virtual value function. In the situation where the prior is unknown to the mechanism designer, Chawla et al. (2014, 2016) give a procedure for inferring the optimal surrogate values from samples.

In particular, Chawla et al. (2016) consider inference of the optimal surrogate values in i.i.d. position auctions with values distributed on $[0, 1]$. Because each population of a surrogate-ranking mechanisms inherits the unique BNE of the position environment with that population’s
characteristic weights, we may directly apply their procedure to samples from a batched or online SRM with possibly suboptimal parameters to estimate the optimal surrogate values. Their procedure only requires that for each population, some pair of characteristic weights differs by at least $\epsilon$, so that the equilibrium is nontrivial. Denoting by $P^k_i$ the $k$th highest optimal surrogate value for revenue for population $i$, and $V^k_i$ the $k$th highest optimal surrogate value for welfare for population $i$, we obtain

**Corollary 20 (Chawla et al. (2016)).** Consider a batched environment with $T$ stages and values drawn from $[0,1]$ for each population. Assume surrogate values for a winner-pays-bid SRM are selected such that for every population $i$, there exists some $j$ such that the characteristic weights satisfy $w_{i,j+1}^i - w_i^j > \epsilon$. Then given $N$ samples of the batched environment with $T$ stages per sample, there exist estimators $\hat{P}^k_i$ and $\hat{V}^k_i$ satisfying:

$$
\mathbb{E}[||\hat{P}^k_i - P^k_i||] \leq \Theta \left( \frac{T}{\min(k, T-k)} \sqrt{\frac{1}{N} \log \left( \frac{T}{\epsilon} \right)} \right)
$$

$$
\mathbb{E}[||\hat{V}^k_i - V^k_i||] \leq \Theta \left( \frac{T \log T}{\sqrt{N}} \log \left( \frac{T}{\epsilon} \right) + \frac{T}{N \epsilon} \right)
$$

A similar result holds for all-pay SRMs. It follows that as long as the initial choice of surrogate values is not trivial, it is easy to infer better surrogate values and reoptimize the mechanism. Note that the convergence rates of the estimators of Chawla et al. (2016) are faster than if one tried to infer the full value distributions, which would be required to implement the optimal winner-pays-bid or all-pay mechanism by undoing the revelation principle (as described in Appendix A). In Appendix E we show that errors in estimation of the optimal surrogate values also propagate to performance loss in a natural, controlled way.

### 4 Approximation Analysis

We have derived the optimal surrogate-ranking mechanisms for welfare and revenue. In this section, we show that in the symmetric equilibrium, these mechanisms' performance approaches optimality among all mechanisms for the batched environment as the number of stages, $T$, grows large. To do so, we give a reduction from rank-based mechanism design to Bayesian algorithm design. Specifically, we will show that for any monotone\footnote{The restriction to monotonicity is essentially without loss. Hartline and Lucier (2010) show how to convert a non-monotone allocation algorithm to one which is monotone and obtains higher (virtual) welfare. Their procedure requires access to the distribution over inputs to the algorithm, which we have because the designer controls the choice of surrogate values.} but possibly non-optimal stage algorithm $\hat{x}$ there is a surrogate ranking algorithm (and therefore equilibrium of the corresponding SRM) which attains welfare and revenue approaching that of $\hat{x}$ applied to the true values in each stage, rather than surrogate values. Consequently, if $\hat{x}$ is the optimal allocation algorithm, then our construction will yield a surrogate-ranking mechanism which, in its symmetric equilibrium, is nearly optimal (with large $T$). The welfare- and revenue-optimal surrogate-ranking mechanisms will only improve on the performance of the constructed mechanism.

**Theorem 21.** For any monotone stage allocation algorithm $\hat{x}$, there exists a surrogate-ranking algorithm which attains a $(1 - O(\sqrt{n/T}))-\text{fraction of the welfare and virtual welfare of } \hat{x}$.
Combining the theorem above with Theorem 17, i.e., that the allocation of the symmetric equilibrium of a surrogate ranking mechanism is the same as that of the surrogate ranking algorithm on the true values, we have the following corollaries.

**Corollary 22.** For any feasibility setting, the welfare-optimal surrogate-ranking mechanism in its symmetric equilibrium attains a \((1 - O(3\sqrt{n}/T)))\)-fraction of the surplus of the optimal mechanism.

**Corollary 23.** For any feasibility setting, the revenue-optimal surrogate-ranking mechanism in its symmetric equilibrium attains a \((1 - O(3\sqrt{n}/T)))\)-fraction of the revenue of the optimal mechanism.

These results imply that, up to a small multiplicative factor, rank-based mechanism design is nearly without loss. If agents play the symmetric equilibrium, then non-revelation mechanism design is similarly almost without loss. To prove Theorem 21, we first develop tools in a much simpler, single-stage setting with i.i.d. agents. In Section 4.2, we show that posting a price to \(T\) i.i.d. agents can be closely approximated by a multi-unit auction (with a fixed number of units that depends on the price). This result generalizes to distributions over posted pricings and rankings. In Section 4.3, we apply this result to the population setting to relate the performance of ranking algorithms to that of binning algorithms - surrogate-based algorithms which use the binning selection rule. Binning algorithms will serve as an intermediary between any target algorithm \(\hat{x}\) and the family of surrogate-ranking algorithms. Finally, we show in Section 4.4 how to convert an arbitrary stage algorithm in the population setting into a binning algorithm without much loss. Combining the above results yields the theorem.

Throughout the subsequent sections, it will often be helpful to refer to an agent’s location in their distribution directly, rather than using their values or virtual values. The quantile of an agent \(i\) with distribution \(F_i\) and value \(v_i\) is given by \(q_i(v_i) = 1 - F(v_i)\). For notational convenience, we will additionally use \(v_i(q_i)\) to write the value as a function of quantile.

### 4.1 Prior Results

Before proving the main theorem, we observe that a weaker result follows from a theorem of Hartline et al. (2011). In this paper, the authors consider an algorithm which can be interpreted as a surrogate ranking algorithm built from an arbitrary stage allocation algorithm \(\hat{x}\). They show that for agents whose values are distributed on the interval \([0, 1]\), their mechanism has an additive per-stage welfare loss of at most \(n/(4\sqrt{T})\). Because we derive the welfare-optimal surrogate-ranking mechanism, we inherit this welfare guarantee. In particular, we have:

**Corollary 24 (Hartline et al. (2011)).** In any feasibility setting, if agents have value distributions on \([0, 1]\), then in the symmetric equilibrium, the welfare-optimal surrogate-ranking mechanism loses at most an additive \(n/(4\sqrt{T})\) per stage with respect to the welfare of the optimal mechanism.

We improve on this welfare guarantee in three ways. First, we derive a guarantee for arbitrary distributions, even those with unbounded support. Second, our bounds will be multiplicative. Finally, our bounds apply to revenue in addition to welfare.

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\(^5\)With the additional assumption that virtual values are bounded below by \(-\phi\), the guarantee of Corollary 24 applies to revenue as well, with an additional factor of \(1 + \phi\) applied to the loss.
4.2 Pricing vs. Ranking

For a mechanism to maximize welfare or revenue effectively, it must be able to discriminate between agents with high and low values. To prove Theorem 21 we must show that ranking mechanisms can do this effectively. We build towards this goal by first showing that ranking mechanisms can approximate the simplest form of discrimination: posted pricing. We will describe price-posting mechanisms in terms of the location of the price $p$ via its quantile $q(p) = 1 - F(p)$. We in particular consider prices for which $q(p)$ is an integral multiple of $1/n$. Formally:

**Definition 25.** The $k/n$-price posting algorithm allocates agents if and only if their quantile is below $k/n$, for some integer $k$. This can be achieved by posting the price with quantile $k/n$.

We first show that price-posting can be approximated by ranking. Posting a price at the quantile $k/n$ will result in allocation to $k$ of the $n$ agents in expectation. The rank-based equivalent enforces this quota pointwise, allocating the $k$ highest-valued agents each time.

**Definition 26.** The top $k$-of-$n$ algorithm for $n$ agents ranks agents by value and allocates the $k$ agents with the highest values.

As the law of large numbers might suggest, these two algorithms perform comparably for large $n$ when $k$ is bounded away from the extremes (one and $n-1$). The lemma below gives a formal statement; its proof can be found in Appendix B.

**Lemma 27.** For any distribution $F$, the top $k$-of-$n$ algorithm attains a $\rho(k,n)$-fraction of the welfare of the $k/n$-price posting algorithm with $n$ agents. If $F$ is regular, then it attains a $\eta\left(\min(k,n-k),n\right)$-fraction of the revenue of the $k/n$-price posting algorithm, where

$$\rho(k,n) \approx 1 - \frac{n}{2\pi k(n-k)} \quad \text{and} \quad \eta(k,n) \approx 1 - \frac{1}{\sqrt{k}} \left( \frac{n}{n-k} \right)^{\frac{3}{2}},$$

with the error stemming from Stirling’s approximation.

We can further generalize Lemma 27 by comparing distributions over pricing algorithms with the analogous distributions over top-$k$ algorithms. As long as prices avoid the extremes of the distribution, ranking performs well with respect to pricing.

**Lemma 28.** For any value distribution $F$, consider a distribution over $k/n$-price posting algorithms for $n$ agents, where the highest price is at quantile $k/n$ and the lowest price is at quantile $k/n$. The same distribution over corresponding top-$k$-of-$n$ algorithms attains a $\rho\left(\min(\underline{k},n-k),\bar{k}\right)$-fraction of the welfare of the distribution over price-posting algorithms. If $F$ is regular, then the distribution over top-$k$-of-$n$ algorithms attains an $\eta\left(\min(\underline{k},n-k),\bar{k}\right)$-fraction of the price-posting revenue as well.

**Proof.** Lemma 27 implies that for each price in the distribution of the price-posting algorithm, there is a top-$k$-of-$n$ algorithm which approximates it and which appears with the same probability. The approximation ratio of a distribution over pairwise approximations is at least the approximation from the worst pair. Note that the approximations from Lemma 27 are symmetric about $1/2$, and are worst for very low and very high $k$. It follows that the approximation ratio is driven by the $\underline{k}/n$- and $\bar{k}/n$-price posting algorithm. \qed
4.3 Binning vs. Ranking

Lemma 28 shows that in simple settings, ranking can discriminate with almost as much accuracy as pricing. We now extend this idea to the batched population model. We compare the surrogate ranking algorithm with a surrogate binning algorithm, which composes the allocation algorithm \( \hat{x} \) with the binning surrogate selection rule (Definition 8) in each stage.

**Definition 29.** For the stage allocation algorithm \( \hat{x} \), value distributions \( \{F_i\}_{i=1}^n \), surrogate values \( \{\Psi_i\}_{i=1}^n \), and binning surrogate selection rules \( \{\sigma_i\}_{i=1}^n \), the surrogate binning algorithm is given by computing \( \tilde{x}(\mathbf{b}_t) = \hat{x}(\sigma_1(b_{1,t}), \ldots, \sigma_n(b_{n,t})) \) in each stage.

We now show that surrogate-binning algorithms can be well-approximated by ranking algorithms. The allocation rules of binning algorithms appear to agents as piecewise constant functions. Hence, each population \( i \) sees a distribution over \( k/T \)-price posting algorithms. Moreover, in the batched population environment, we may rank rather than price, just as in Lemma 28. By ranking each agent and treating the agent with rank \( j \) as if they were in the \( j \)th bin, we produce a surrogate-ranking algorithm. Lemma 28 implies that this algorithm performs almost as well as the binning algorithm, provided that the prices from the binning algorithm do not come from extreme quantiles.

**Theorem 30.** For monotone stage algorithm \( \hat{x} \) and surrogate values \( \psi_i^1 \geq \psi_i^2 \geq \ldots \geq \psi_i^T \) with \( \psi_i^1 = \psi_i^2 = \ldots = \psi_i^k \) and \( \psi_i^k = \psi_i^{k+1} = \ldots = \psi_i^T \) for each population \( i \), the surrogate ranking algorithm attains a \( \rho(\min(k, T-k), T) \)-fraction of the welfare of the binning algorithm. If distributions are regular, then the surrogate ranking algorithm attains a \( \eta(\min(k, T-k), T) \)-fraction of the binning algorithm’s virtual surplus.

To derive Theorem 30, note that by Lemmas 10 and 11, the surrogate-binning algorithm allocates agents according to its characteristic weights. Moreover, we already showed in Section 3.2 that surrogate-ranking algorithms also allocate agents according to characteristic weights. Thus if the two algorithms use the same surrogate values, these characteristic weights will be the same. Next, note that the surrogate-binning and surrogate-ranking algorithms appear to agents as distributions over pricing and top-\( k \) algorithms, respectively, with the distributions determined by marginal characteristic weights. Formally:

**Lemma 31.** Any surrogate-binning (resp. surrogate-ranking) algorithm with surrogate values \( \{\psi_i^j\}_{i=1}^n \) appears to agents in each distribution \( i \) as a distribution over price-posting (resp. top-\( k \)) algorithms. The probability of offering the price with quantile \( \hat{q} \) (resp. of allocating \( j \) units) is given by \( w_i^j = w_i^{j+1} \), where \( w_i^0 = 1 \), \( w_i^{T+1} = 0 \), and \( w_i^j \) is the characteristic weight for \( \psi_i^j \) for \( j = 1, \ldots, T \).

Finally, Theorem 30 follows from applying Lemma 28. Notice that if \( \psi_i^1 = \ldots = \psi_i^k \), the binning algorithm’s allocation rule on the first \( k \) intervals of distribution \( i \)’s quantile space will be constant. In terms of distribution \( i \)’s randomization over posted pricings, the highest nontrivial price offered has quantile \( k/n \), and the lowest has quantile \( k/n \). These extremal quantiles drive the approximation guarantees relating pricing to ranking and, thus, good approximation bounds can be obtained via Lemma 28 if there is not much loss in restricting to binning algorithms that price at moderate quantiles.
4.4 Approximately Optimal Binning

In this section we show that any monotone allocation algorithm can be converted into a binning algorithm with large first and last bin without losing much welfare or virtual welfare. Combined with Lemma 31, this construction implies Theorem 21. The approach to converting any allocation algorithm into a binning algorithm (see Hartline and Lucier, 2010) is to map the \( j \)th surrogate of population \( i \), which corresponds to an agent with value drawn from \( F_i \) conditioned on interval \([\frac{j-1}{T}, \frac{j}{T}]\) to a redraw from this conditional distribution. Such a resampling does not change the induced allocation rule for any other agents, and replaces the allocation rule on the \( j \)th quantile interval with its average.

The basic approach does not directly lead to the desired approximation bound because, for the top quantile interval, the allocation probability at the top of the interval may be much higher than its average allocation probability and higher values on the interval may be much higher than the interval’s average. For example, if the value and allocation rule are both one for an \( \epsilon \) measure and zero otherwise, then the original welfare is \( \epsilon \) and the welfare from resampling is \( \epsilon^2 \). A second issue is that we wish to be able to apply Lemma 31 to get a good approximation bound. For example, we would get a \( 1 - \Theta(\sqrt{1/k}) \) bound if the top \( k \) intervals have the same allocation probability.

To resolve both of these issues we will first modify the algorithm to treat agents with values in the top \( k \) intervals as if they had the highest value in the support of their distributions. The quantiles of the remaining agents will be rescaled. Conditioned on the value not being in the top \( k \) intervals, the value distribution after rescaling will match the original unconditioned value distribution. We refer to this transformation as top promotion. Unlike the basic resampling approach, applying this method to one agent does change the mechanism for other agents. We show that this change does not have a significant impact on the outcomes other agents receive, and approximately preserves welfare and revenue from each population.

We analyze the top promotion procedure in Section 4.4.1 and the binning algorithm that results from resampling in Section 4.4.2. The analysis focuses on the individual contribution of an arbitrary agent in an arbitrary distribution \( i \) and stage \( t \). We suppress both indices for notational convenience.

4.4.1 Top Promotion

We define and analyze a procedure for giving priority to high-valued agents to protect against distributions with infrequent, extremely high-valued bidders. For each distribution, we will treat agents with quantiles in \([0, k/T] \) as if their quantile was 0. For the remaining agents, we will rescale their quantiles such that the distribution of values on conditioned on \( q \in [k/T, 1] \) after rescaling matches the original distribution in its entirety. The resulting distribution of inputs will therefore be close enough to the original that the performance loss from this transformation will be minimal.

**Definition 32.** Given a monotone stage algorithm \( \hat{x} \) and an integer \( k \) sufficiently smaller than \( T \), the top promotion algorithm for \( \hat{x} \) runs \( \hat{x} \) on agents with quantiles transformed for each population as follows:

1. For any \( q \in [0, k/T] \), return 0.
2. For any \( q \in [k/T, 1] \), return \( (q - k/T)/(1 - k/T) \).
Before we analyze the top promotion algorithm, we prove a lemma that relates the virtual surplus of two similar allocation rules. The similarity assumption of the lemma requires that the inverses of the allocation rules be approximations of each other (formalized in the statement of the lemma). The lemma also limits virtual value functions to ones where lines from the origin cross the cumulative virtual value function \( R(q) = \int_0^q \phi(r) \, dr \) only from below, i.e., \( R(\alpha q) \geq \alpha R(q) \). This condition is always satisfied for the virtual values for revenue from Lemma 2, however, we state the lemma generally because we will not be applying it directly to virtual values for revenue.

**Lemma 33.** For virtual value function \( \phi(\cdot) \) and cumulative virtual value \( R(q) = \int_0^q \phi(r) \, dr \) satisfying \( R(\alpha q) \geq \alpha R(q) \) for all quantiles \( q \) and \( \alpha \in [0, 1] \), and any two allocation rules \( \hat{x} \) and \( \hat{x} \) that satisfy \( \hat{x}^{-1}(z) \geq \hat{x}^{-1}(z) \geq \frac{1}{z} \hat{x}^{-1}(z) \), the virtual surpluses satisfy

\[
E_{q \sim U[0,1]}[\phi(q)\check{x}(q)] \geq \frac{1}{\alpha} E_{q \sim U[0,1]}[\phi(q)\hat{x}(q)].
\]

The proof can be found in Appendix C. We use Lemma 33 to compare the virtual welfare of the top promotion algorithm to the original algorithm via two observations. First, the “unallocation rules”, i.e., \( y(q) = 1 - x(1 - q) \) for allocation rule \( x(q) \), satisfy the inverse-approximation condition of the lemma. Second, the virtual surplus of the unallocation rule is given by the expected virtual value plus the negative virtual surplus of the unallocation rule. Specifically \( E_q[\phi(q)x(q)] = E_q[\phi(q)] + E_q[-\phi(1-q)y(q)] \). While virtual values for revenue always satisfy the property that rays from the origin cross the cumulative virtual value curve from below, this property does not generally hold for the negative virtual values \( -\phi(1-q) \). Regularity, i.e., monotonicity of the original virtual value function, however, implies the property for negative virtual values. These observations are formally summarized in the subsequent lemma, which we prove in Appendix C.

**Lemma 34.** The top promotion algorithm attains at least a \( (1 - \frac{k(n-1)}{T})(1 - \frac{k}{T}) \)-fraction of the welfare and virtual surplus (under regularity) of \( \hat{x} \).

### 4.4.2 Binning via Resampling

We now define a surrogate-binning algorithm which retains all but a vanishing fraction of the surplus of \( \hat{x} \). To do so, we must choose a stage allocation algorithm, as well as a surrogate value for each interval in quantile space for each distribution. Our binning algorithm will use the top promotion algorithm from Section 4.4.1 based on \( \hat{x} \) as its stage allocation algorithm. Our surrogate values will be chosen by random resampling. Formally:

**Definition 35.** The resampling algorithm for allocation algorithm \( \hat{x} \) is the surrogate binning mechanism whose stage algorithm is the top promotion algorithm for \( \hat{x} \) and with surrogate values given by:

- for \( j \in \{1, \ldots, T-k\} \), determine \( \psi_i^j \) by randomly sampling from distribution \( F_i \) conditioned on the interval \([ (j-1)/T, j/T ] \) in quantile space.
- determine \( \{ \psi_i^j \}_{j=T-k+1}^{T} \) jointly by sampling and sorting \( k \) values from \( F_i \) conditioned on the interval \([ (T-k)/T, 1 ] \) in quantile space.

**Lemma 36.** For any monotone stage allocation algorithm \( \hat{x} \) and \( k < T/2 \), the resampling algorithm attains a \( (1 - k/T)^2(1 - k(n-1)/T) \)-fraction of the welfare of \( \hat{x} \). For regular value distributions, this bound holds for virtual surplus as well.
The lemma follows from noting that the resampling algorithm’s allocation rule is a piecewise-constant approximation to the allocation rule of the top promotion algorithm. We show that the revenue of this approximation is close by integrating by parts the virtual surplus of any allocation rule $x$, $E_{q \sim U[0,1]}[-X(q)\phi'(q)] + X(1)\phi(1)$, where $X(q) = \int_0^q x(s) \, ds$ is the cumulative allocation rule for $x$. Because the allocation rule for the resampling algorithm is a piecewise constant approximation to that of the top promotion algorithm, their cumulative allocation rules are multiplicatively close everywhere, which yields the result. The welfare result holds from identical reasoning. The full proof can be found in Appendix D.

4.4.3 Proof of Theorem 21

All that remains to prove the approximation result is to compose our lemmas and select a value for the parameter $k$. First, note the binning algorithm’s surrogate values are the same for the intervals $[0, \frac{1}{T}], \ldots, [\frac{k-1}{T}, \frac{k}{T}]$ and the intervals $[\frac{T-k}{T}, \frac{T-k+1}{T}], \ldots, [\frac{T-1}{T}, 1]$. It follows that welfare and revenue loss from applying Theorem 30 are $\rho(k,n)$ and $\eta(k,n)$, respectively. Moreover, by Lemma 36, the resampling algorithm attains a $(1 - k/T)^2(1 - k(n-1)/T)$-fraction of the revenue and welfare of $\hat{x}$. Composing these lemmas and setting $k = (n/T)^2$ yields the theorem.

5 Sample Complexity Implications

Finally, we briefly outline our sample complexity result. Formally, we assume the mechanism designer has access to $m$ value profiles, $v^1, \ldots, v^m$, with each $v^j_i$ sampled independently from a regular distribution $F_i$. The mechanism designer must design an auction that when run on a fresh sample $v^{m+1}$ is guaranteed at least a $(1 - \epsilon)$ fraction of the expected revenue of the optimal mechanism designed with knowledge of the value distributions. We make no assumptions on the set of feasible allocations $X$ - it need not even be downward-closed. We first state our result:

**Theorem 37.** For any single-dimensional feasibility environment with regular bidders, $\tilde{O}(n^{5\epsilon^{-8}})$ sampled value profiles suffice to design a mechanism which obtains at least a $(1 - \epsilon)$-fraction of the optimal expected revenue.

On a high level, our mechanism resembles a surrogate binning algorithm: it attempts to divide each agent’s quantile space into $T$ uniform intervals, and treats values in the same interval identically. The mechanism then attempts to treat agents optimally, conditioned on their interval. The analysis of our mechanism reuses much of the analysis of top promotion and the resampling-based binning algorithm of Sections 4.4.1 and 4.4.2, combined with basic concentration results for regular distributions. A full proof of the result and description of the mechanism can be found in Appendix E.

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A Undoing the Revelation Principle

Good first-price and all-pay mechanisms for a given environment can be found by undoing the revelation principle (ignoring computational complexity). This construction applies to any revelation mechanism $M$. For concreteness, imagine applying this approach to a single-minded combinatorial auction problem where $M$ is the Vickrey-Clarke-Groves (VCG) mechanism. We give the all-pay version of the construction which is slightly simpler, but exhibits the same issues.

Definition 38. The all-pay unrevelation mechanism for a revelation mechanism $M$ is:

1. For each agent $i$ and value $v_i$, calculate $s_i(v_i)$ as the expected payment in $M$ when the agent’s value is $v_i$ and other agents’ values are drawn from the distribution.

2. For each agent $i$, given bid $b_i$ in the un-revelation mechanism, calculate the agent’s value as $v_i = s_i^{-1}(b_i)$.

3. Serve the agents who are served by $M$ on values $v = (v_1, \ldots, v_n)$; all agents pay their bids.

The characterization of Bayes-Nash equilibrium (Theorem 1) implies that $s_i$ is the strategy that agents will employ in equilibrium of the constructed all-pay mechanism. Thus, the all-pay mechanism has the same equilibrium outcome.

From this definition we can see why symmetric and ordinal environments (i.e., IID position environments) are special. For these environments all agents will have the same strategy function, this strategy function will order higher valued bidders higher (by monotonicity), and the ordinal environment then implies that all that is needed to select an outcome is the order of values not their cardinal values. Thus, the mechanism simplifies to simply ordering the bids and the strategy function does not need to be calculated.

Even absent computational issues in estimating the strategy functions so as to implement this mechanism, it is clear that very detailed distributional information is needed to run the unrevelation mechanism. Moreover, the resulting outcomes may be very sensitive to small errors with the inversion of the strategy function. This unrevelation mechanism is not to be considered practical.

B Proof of Lemma 27

In this appendix, we show that the revenue and welfare of the top-$k$-of-$n$ algorithm approximates the $k/n$-price posting algorithm for $n$ agents. We will do this by explicitly characterizing the worst-case distributions for each objective, and analyze the per-agent contribution to each algorithm’s surplus. For notational convenience, we suppress the subscripts on functions which would refer to our agent.
Key to the analysis will be two formulae for the expected surplus of an algorithm, in terms of its interim allocation rule \( x(\cdot) \) and the distribution’s value function \( v(\cdot) \). We have that an algorithm’s surplus is:

\[
E_{q \sim U[0,1]}[x(q)v(q)] = E_{q \sim U[0,1]}[-x'(q)V(q)],
\]

where \( V(q) = \int_0^q v(z) \, dz \), and the equality follows from integration by parts. An analogous formula holds for virtual surplus, with \( v(q) \) replaced by the Myerson virtual value at \( q \).

We will first analyze welfare, and then highlight the changes necessary for proving the result for virtual surplus. The only real difference between the two objectives is the fact that values are always positive, whereas virtual values may be negative. This changes the nature of the approximation, as allocating the wrong agent becomes actively harmful to the performance of the algorithm.

**Welfare** We begin by normalizing the per-agent surplus of the price-posting mechanism to 1. Note that for the \( k/n \)-price posting algorithm, the interim allocation rule is 1 until quantile \( k/n \), and then drops to 0. It follows from equation (1) that our normalization is equivalent to the assumption that \( V(k/n) = 1 \).

Next, we note that because \( v(\cdot) \) is positive and decreasing, \( V(\cdot) \) is increasing and concave, with \( V(0) = 0 \). Let \( x(\cdot) \) be the allocation rule of the top-\( k \)-of-\( n \) mechanism. Given our normalization, the problem of finding the worst-case distribution then becomes:

\[
\min_{V(\cdot)} E_{q \sim U[0,1]}[-x'(q)V(q)]
\]

subject to

- \( V(0) = 0 \)
- \( V(k/n) = 1 \)
- \( V(\cdot) \) concave
- \( V(\cdot) \) increasing

This program can be solved by inspection by noticing that there is pointwise minimal function satisfying the constraints of the program: namely, the optimal \( V(q) \) is linear with slope \( v(q) = n/k \) for \( q \leq k/n \), and constant at 1 for \( q \geq k/n \). This corresponds to the distribution with \( k/n \) mass on the value \( n/k \), and the rest on 0.

Having solved for the worst-case distribution, it remains to compute optimal value of the objective. Note that the allocation rule for the top-\( k \)-of-\( n \) algorithm is

\[
x(q) = \sum_{j=0}^{k-1} \left( \frac{n-1}{k} \right)^j q^j (1-q)^{n-j-1}.
\]

Combining this with our knowledge of the worst-case distribution, we can easily compute the per-agent surplus of the top-\( k \)-of-\( n \) mechanism. Omitting tedious computations, we get the following formula for the multiplicative loss per agent:

\[
\text{Loss}_W(k,n) = 1 - \sum_{i=0}^{k-1} \left( \frac{n}{i} \right) \left( \frac{k}{n} \right)^i \left( \frac{n-k}{n} \right)^{n-i} (k-i).
\]
To obtain the bound given in the statement of the lemma, note that the following sequence of inequalities holds:

\[
\text{Loss}_W(k, n) = 1 - \sum_{i=0}^{k-1} \binom{n}{i} \left(\frac{k}{n}\right)^i \left(\frac{n-k}{n}\right)^{n-i} (k-i) \\
= \sum_{i=0}^{k} \binom{n}{i} \left(\frac{k}{n}\right)^i \left(\frac{n-k}{n}\right)^{n-i} - \sum_{i=0}^{k-1} \binom{n-1}{i} \left(\frac{k}{n}\right)^i \left(\frac{n-k}{n}\right)^{n-i-1} \\
\leq \sum_{i=0}^{k} \binom{n}{i} \left(\frac{k}{n}\right)^i \left(\frac{n-k}{n}\right)^{n-i} - \sum_{i=0}^{k-1} \binom{n-1}{i} \left(\frac{k}{n}\right)^i \left(\frac{n-k}{n}\right)^{n-i} \\
= \left(\frac{n}{k}\right)^k \left(\frac{n-k}{n}\right)^{n-k}.
\]

The result for welfare follows from applying Sterling’s approximation.

**Virtual Surplus** We now adapt the above proof to virtual surplus. The main difference will be the fact that the Myerson virtual value, denoted \(\phi(q)\), can be negative. We will additionally use the fact that the Myerson virtual value is the derivative of the price-posting revenue curve. That is, \(\phi(q) = R'(q) = \frac{d}{dq}v(q)(1-q)\). It follows that cumulative virtual value has the convenient form \(R(q) = v(q)q\).

As before, we normalize the virtual surplus from price-posting to 1. This corresponds with setting \(R(q) = 1\). Subject to normalization, we use properties of revenue curves to derive the worst-case distribution for virtual surplus. We assume values are regularly distributed, which implies that \(R(q)\) is concave. Moreover, since \(R(q) = v(q)q\), we have that \(R(0) = R(1) = 0\). These properties yield the following program for the worst-case distribution:

\[
\min_{R(\cdot)} \mathbb{E}_{q \sim U[0, 1]}[-x'(q)R(q)] \\
\text{subject to } R(0) = R(1) = 0 \\
R(k/n) = 1 \\
R(\cdot) \text{ concave}
\]

Again, this may be solved by inspection. The worst-case \(R(\cdot)\) is triangular, with its apex at \((k/n, 1)\). That is, on \([0, k/n]\), \(R(q)\) has slope \(n/k\), and on \([k/n, 1]\), it has slope \(-n/(n-k)\).

Using the allocation rule of the top-\(k\)-of-\(n\) mechanism from the welfare proof, we can compute the multiplicative loss per agent for revenue as:

\[
\text{Loss}_R(n, k) = \frac{n}{k(n-k)} \sum_{i=0}^{k} \binom{n}{i} \left(\frac{k}{n}\right)^i \left(\frac{n-k}{n}\right)^{n-i} (k-i).
\]
The following sequence of equations yields the result:

\[
\text{Loss}_{R}(n, k) = \frac{n}{k(n-k)} \left[ \sum_{i=0}^{k} \binom{n}{i} \left( \frac{k}{n} \right)^i \left( \frac{n-k}{n} \right)^{n-i} - \sum_{i=0}^{k-1} \binom{n}{i} \left( \frac{k}{n} \right)^i \left( \frac{n-k}{n} \right)^{n-i-1} \right]
\]

\[
= \frac{n}{(n-k)} \left[ \sum_{i=0}^{k} \binom{n}{i} \left( \frac{k}{n} \right)^i \left( \frac{n-k}{n} \right)^{n-i} - \sum_{i=0}^{k-1} \binom{n-1}{i} \left( \frac{k}{n} \right)^i \left( \frac{n-k}{n} \right)^{n-i-1} \right]
\]

\[
\leq \frac{n}{(n-k)} \left[ \sum_{i=0}^{k} \binom{n}{i} \left( \frac{k}{n} \right)^i \left( \frac{n-k}{n} \right)^{n-i} - \sum_{i=0}^{k-1} \binom{n-1}{i} \left( \frac{k}{n} \right)^i \left( \frac{n-k}{n} \right)^{n-i} \right]
\]

\[
= \frac{n}{(n-k)} \left( \frac{n}{k} \right)^k \left( \frac{n-k}{n} \right)^{n-k}
\]

Applying Stirling’s approximation and noting symmetry about \(n/2\) yields the result.

C Proofs from Section 4.4.1

Proof of Lemma 33 The virtual surplus can be rewritten as \(\int_{0}^{1} \phi(q)x(q) dq = \int_{0}^{1} R(x^{-1}(z)) dz\). This follows from an integration by parts and then change of variables to integrate the vertical axis rather than the horizontal axis as follows:

\[
\int_{0}^{1} \phi(q)x(q) dq = R(1)x(1) - R(0)x(0) + \int_{0}^{1} R(q)(-x'(q)) dq = \int_{0}^{1} R(1) dz - \int_{0}^{1} R(x^{-1}(z)) dz.
\]

Now consider two arbitrary quantiles \(q_1\) and \(q_2\) satisfying \(\frac{1}{2}q_1 \leq q_2 \leq q_1\). By assumption, we have \(R(q_2) \geq q_2 R(q_1)/q_1 \geq \frac{1}{2}q_1\). The assumption on the approximation of the two allocation rules, namely \(\hat{x}^{-1}(z) \geq \tilde{x}^{-1}(z) \geq \frac{1}{2}x^{-1}(z)\) for all \(z \in [0, 1]\), and the expected virtual surplus written as \(\int_{0}^{1} R(x^{-1}(z)) dz\) suffice to prove the lemma.

Proof of Lemma 34 Consider the perspective of an agent \(i\) in an arbitrary stage \(t\). We argue that their individual contribution to surplus cannot decrease by more than a \(k(n-1)/T\) fraction. This hinges on two key steps: we first analyze the effect of the procedure when applied to just one population \(i\), and show that there is no loss to welfare, and at most a \(k/T\)-fraction of revenue lost. We then analyze the effect on \(i\)'s welfare and revenue when the procedure is applied to other populations as well. Multiplying these approximation factors results in the total loss.

First, note that applying the top promotion algorithm to just population \(i\) only increases the probability that the agent is allocated, as her promoted quantile is always lower than her original quantile. Since values are always positive, this will only improve the welfare contribution of distribution \(i\).

Analysis of revenue is more complicated because virtual values may be negative. Consequently, a stronger allocation rule need not improve the revenue from population \(i\). We nonetheless can show that revenue is not hurt by too much by applying Lemma 33 to the “unallocation rule” \(y(q) = 1 - x(1 - q)\). Note that \(y(q)\) is decreasing in \(q\) and that \(\int_{0}^{1} \phi(1 - q) y(q) dq\) is the total negative virtual surplus left unallocated by \(x\).
Formally, note that the expected revenue from any allocation rule \( x \) is can be written as \( \int_0^1 \phi(q)x(q)\,dq = \int_0^1 \phi(q)\,dq - \int_0^1 \phi(q)(1-x(q))\,dq \). Specifically, let \( x \) be the allocation rule of the top promotion algorithm, and \( \hat{x} \) the allocation rule of the original stage algorithm. Moreover, define \( y(q) = q - x(1-q) \) and \( \hat{y}(q) = 1 - \hat{x}(1-q) \) to be the corresponding “unallocation rules.” We will show that

\[
\int_0^1 -\phi(1-q)y(q)\,dq \geq \frac{k}{T} \int_0^1 -\phi(1-q)\hat{y}(q)\,dq.
\]

(4)

Since as a basic fact about virtual values, \( \int_0^1 \phi(q)\,dq \geq 0 \), this will prove that \( \int_0^1 \phi(q)x(q)\,dq \geq (1 - \frac{k}{T}) \int_0^1 \phi(q)\hat{x}(q)\,dq \).

To prove (4), note that for any quantile \( q \) receiving allocation \( \hat{x}(q) \) from the stage algorithm, the quantile receiving this probability under the top promotion algorithm will be \( 1 - \frac{k}{T}(1 - q) \). It follows that \( \hat{y}^{-1}(z) \geq y^{-1}(z) = \frac{k}{T}\hat{y}^{-1}(z) \) for all \( z \in [0,1] \). Moreover, note that \(-\phi(1-q)\) is decreasing in \( q \). This implies that \( R(q)/q \geq -\phi(1-q) \), where \( R(q) = \int_0^q -\phi(1-q)\,dq \). We may therefore apply Lemma 33, which yields (4).

We have shown that for both welfare and revenue, the top promotion algorithm preserves a \((1 - k/T)\)-fraction (or better, for welfare) of the objective of \( x \) from a single population \( i \), when applied to only agents in \( i \). We now analyze the effect on the welfare and revenue of the agent from population \( i \) from top promotion applied to the \( n-1 \) other populations. The key observation in our analysis is that the distribution of the quantiles of agents from the other population is nearly unchanged by top promotion. In particular, note that the probability that one or more agents from distributions other than \( i \) with quantiles set to \( 0 \) by the top promotion algorithm is at most \((n-1)k/T \) by the union bound. Conditioned on there being no such agents, the distribution of quantiles from the top promotion algorithm remains uniform. It follows that the revenue and welfare conditioned on this event is identical to the revenue from \( \hat{x} \). On the other hand, in the event that there is one or more agents from populations other than \( i \) who have top quantiles (which are promoted), the conditional welfare and virtual welfare is non-negative.

The (virtual) surplus from population \( i \) from the top promotion algorithm can be written as the approximation from promotion population in \( i \), i.e. \((1 - k/T)\), composed with the loss from applying the procedure to all other agents, i.e. \((1 - k(n-1)/T)\). The bound in the statement of the lemma results.

\[\square\]

D Proof of Lemma 36

We first characterize the allocation rule of the resampling algorithm for an arbitrary agent. Before being input to the top promotion algorithm, the resampling algorithm redraws the agent’s quantile uniformly at random from their interval in quantile space, say \([\frac{q}{T}, \frac{q+1}{T}]\). It follows that the agent’s allocation will be the average allocation of all agents in \([\frac{q}{T}, \frac{q+1}{T}]\). As such, the allocation rule of the resampling algorithm will be a piecewise constant approximation to that of the top promotion algorithm, with each piece being an average of the top promotion algorithm’s allocation probabilities for some interval.

To analyze the performance of such an approximation, we use integration by parts. We will argue for revenue, but the analysis is identical for welfare. Let \( x(q) \) be the allocation probability for our agent, given that their quantile is \( q \), under the top promotion algorithm, and let \( \mathcal{F}(q) \) be that same agent’s allocation probability under the resampling algorithm. The expected revenue from our agent under the top promotion algorithm is \( \mathbb{E}_{q \sim U[0,1]}[x(q)\phi(q)] \), and the expected revenue under the resampling algorithm is \( \mathbb{E}_{q \sim U[0,1]}[\mathcal{F}(q)\phi(q)] \). Integrating both
of these expectations by parts, we get that the expected revenues are $E_{q \sim U[0,1]}[-X(q)\phi'(q)] + X(1)\phi(1)$ and $E_{q \sim U[0,1]}[-\bar{X}(q)\phi'(q)] + \bar{X}(1)\phi(1)$, respectively, where $X(q) = \int_0^q x(s)\,ds$, and $\bar{X}(q) = \int_0^q \pi(s)\,ds$. Because $\phi(\cdot)$ is assumed to be nonincreasing, and because $X(1) = \bar{X}(1)$, it suffices to show that the cumulative allocation rule $\bar{X}(\cdot)$ approximates $X(\cdot)$.

We first note that on the quantiles $q \in [0,k/T]$, $\bar{X}(q) = X(q)$, as both allocation rules are piecewise constant; the top promotion algorithm treats all these quantiles as 0. Now consider a quantile $q \in [\frac{j}{T},1]$. Note that because the resampling algorithm’s allocation rule replaces that of the top promotion algorithm with its average on each interval, the cumulative allocations $X(q)$ and $\bar{X}(q)$ will coincide at the breakpoints of intervals. That is, $X(q) = \bar{X}(q)$ for $q \in \{\frac{j}{T}, \frac{j+1}{T}, \ldots, \frac{T-k}{T}, 1\}$.

Now consider some $q \in [\frac{j}{T}, \frac{j+1}{T}]$ for $j \in \{k/T, \ldots, (T-k-1)/T\}$. Because allocations are nonnegative, $\bar{X}$ is nondecreasing. We may therefore lowerbound $\bar{X}(q)$ by $\bar{X}(\frac{j}{T})$. Moreover, because $\hat{x}$ is monotone, $x(\cdot)$ is nonincreasing, so $X(\cdot)$ is concave. We have therefore that $X(q) \leq X_j(\frac{j}{T}) + \frac{T}{T-k}X_j(\frac{j}{T})(q - \frac{j}{T})$. The ratio between the upper and lower bounds is largest at $q = \frac{j+1}{T}$ for which it is $(j+1)/j$. It follows that on $[\frac{j}{T}, \frac{j+1}{T}]$, $\bar{X}(q) \geq \frac{j+1}{T}X(q)$.

From the above analysis, we may conclude that on the intervals $[\frac{j}{T}, \frac{j+1}{T}], \ldots, [\frac{T-k+1}{T}, \frac{T-k}{T}]$, the worst approximation occurs on the interval $[\frac{k}{T}, \frac{k+1}{T}]$, and is $\frac{k}{T-k+1}$. It remains to analyze the approximation on the interval $[\frac{T-k}{T}, 1]$. By the same reasoning as the previous paragraph, the worst-case approximation is at $q = 1$, and is $1 - k/T$.

It follows that the resampling algorithm attains a $(1 - k/T)$-fraction of the surplus of the top promotion algorithm, and therefore a $(1 - k/T)^2(1 - k(n-1)/T)$-fraction of the surplus of $\hat{x}$.

### E Sample Complexity in General Feasibility Environments

In this appendix, we describe a the samples-based mechanism of Theorem 37, and analyze its performance. We first discuss the high-level strategy in more detail. Given the earlier results of this paper, we must first mention an obvious approach. It would suffice, in light of the performance bound of Corollary 23 to estimate the surrogate values used in the optimal SRM - that is, the expected order statistics of the virtual value distribution - and then to simulate the $T-1$ copies of the stage environment using samples. Estimating the optimal surrogate values can, in fact, be done with polynomially many samples. Chawla et al. (2014) show the expected $k$th order statistic of the virtual values of a distribution $F$ to be the difference in revenue between a $k$-unit and a $k-1$ unit, highest-bidders-win auction with values drawn i.i.d. from $F$. The revenue of such auctions is straightforward functions of the revenue curve, and for regular distributions it is possible to accurately estimate the revenue curve by estimating the revenue at a net of quantiles and interpolating.

Unfortunately, the polynomial upper bounds one obtains from this approach are quite large. Informally, note that to obtain a $(1 - \epsilon)$ approximation using a SRM, one would need $T = \Omega(n\epsilon^{-3})$. This translates to $n\epsilon^{-3}$ parameters which must be estimated for each distribution, and some of these parameters correspond to extreme (and therefore difficult to estimate) order statistics. We therefore reduce the estimation demands by binning rather than ranking, akin to skipping the final ranking step in the proof of Theorem 21. We derive a mechanism which divides value space into “bins,” estimates conditional virtual values for those bins, and maximizes conditional virtual surplus. The resultant mechanism requires $T = O(n\epsilon^{-2})$ parameters per
agent, and these parameters are easier to estimate. Intuitively, while ranking is useful for Bayes-Nash incentives, it introduces unnecessary error for estimation.

The mechanism that we produce turns out to be a “t-level auction” in the sense of Morgenstern and Roughgarden (2015). With a clever use of samples, we avoid solving the computationally difficult empirical risk minimization problem that their approach requires. Consequently, our result can be thought of an algorithmic improvement on their framework.

The rest of the section is structured as follows. In Section E.1 we show how to estimate values with quantiles near \( j/T \) for \( j \in \{1, \ldots, T - 1\} \). These values will serve as breakpoints in our mechanism’s coarsening of value space. In Section E.2 we show how to translate the closely estimated breakpoints of Section E.1 into close estimates of the expected virtual value of agents conditioned on each of the intervals defined by the breakpoints. These expected virtual values will serve as surrogate values in our mechanism. In Section E.3 we extend the analysis in Section 4.4 to non-uniform bins with the noisy surrogate value estimates from the previous section. In Section E.4 we show that errors in estimating surrogate values propagate in a natural way. Finally, in Section E.5 we compose these results and set parameters carefully to produce Theorem 37.

E.1 Estimating Breakpoints

We now consider an arbitrary value distribution \( F \) with value function \( v(\cdot) \), and show how to estimate \( v(j/T) \) for \( j \in \{1, \ldots, T - 1\} \) with a high degree of accuracy in quantile space. Specifically, our estimate for \( v(j/T) \) will have quantile in \([j/T - \epsilon, j/T + \epsilon]\) with high probability.

To obtain our estimates, consider drawing \( N = mT - 1 \) samples from \( F \). Note that the expected value of the quantile of the \((jm - 1)\)st sample is exactly \( j/T \). Call this value \( v_{jm-1} \), with quantile \( q_{jm-1} \). We will use Chernoff to bound the deviation of \( q_{jm-1} \) from its mean in terms of \( m \). Specifically, note that for any \( \epsilon \in (0, 1) \), the number of samples with quantile at most \( j/T + \epsilon \) is the sum of \( N \) iid Bernoulli random variables with mean \( j/T + \epsilon \). Note that \( q_{jm-1} > j/T + \epsilon \) only if at most \( jm - 2 \) samples overall have quantile at most \( j/T + \epsilon \). This latter event is equivalent to the number of samples with quantile at most \( j/T + \epsilon \) deviating from its mean (which is \((j/T+\epsilon)(Tm-1))\) by at least \((j/T+\epsilon)(Tm-1) - jm - 2 = \left(1 - \frac{jm - 2}{(j/T + \epsilon)(Tm - 1)}\right)(j/T+\epsilon)(Tm - 1)\).

We may bound the probability of this event using Chernoff to obtain:

\[
\Pr[q_{jm-1} \geq j/T + \epsilon] \leq e^{-\Omega\left(\left(1 - \frac{jm - 2}{(j/T + \epsilon)(Tm - 1)}\right)^2(j/T+\epsilon)(Tm - 1)\right)}
\]

This bound is worst for \( j = 1 \). Using the bound for \( j = 1 \) yields:

\[
\Pr[q_{jm-1} \geq j/T + \epsilon] \leq e^{-\Omega\left(\left(1 - \frac{m - 2}{(j/T + \epsilon)(Tm - 1)}\right)^2(1/T + \epsilon)(Tm - 1)\right)}
\]

A similar approach can bound the probability that \( q_{jm-1} < j/T - \epsilon \). This occurs only if at least \( jm - 1 \) samples have quantile at most \( j/T - \epsilon \). The latter event is equivalent to the number of samples with quantile at most \( j/T - \epsilon \) deviating from its mean of \((j/T - \epsilon)(Tm - 1))\) by at least \((jm - 1) - (j/T - \epsilon)(Tm - 1) = \left(\frac{jm - 1}{(j/T - \epsilon)(Tm - 1)} - 1\right)(j/T - \epsilon)(Tm - 1)\).

Chernoff yields:

\[
\Pr[q_{jm-1} \leq j/T - \epsilon] \leq e^{-\Omega\left(\left(\frac{jm - 1}{(j/T - \epsilon)(Tm - 1)} - 1\right)^2(j/T - \epsilon)(Tm - 1)\right)}
\]

\[
\leq e^{-\Omega\left(\left(\frac{m - 1}{(j/T - \epsilon)(Tm - 1)} - 1\right)^2(1/T - \epsilon)(Tm - 1)\right)}.
\]

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We may apply a union bound to obtain:

**Lemma 39.** Assume the following inequality holds:

\[
(Tm - 1) \min \left( \left( 1 - \frac{m-2}{(1/T + \epsilon)(m-1)} \right)^2 \left( \frac{1}{T} + \epsilon \right), \left( \frac{m-1}{(1/T - \epsilon)(m-1)^2} - 1 \right)^2 \left( \frac{1}{T} - \epsilon \right) \right) \geq \Omega \left( \frac{\ln T}{\delta} \right).
\]

Then the probability that \( q_{im-1} \in [j/T - \epsilon, j/T + \epsilon] \) for all \( j \in \{1, \ldots, T - 1\} \) is at least \( 1 - \delta \).

In subsequent sections, we will derive sensible choices for \( T \) and \( \epsilon \), which will simplify the complicated bound of Lemma 39.

### E.2 Estimating Conditional Revenues

In the previous section, we bounded the number of samples necessary to estimate a value \( \hat{q}^j \) which has quantile \( \hat{q}^j \in [j/T - \epsilon, j/T + \epsilon] \) for every \( j \in \{1, \ldots, T - 1\} \) with high probability. We now show that if we have such \( \hat{v}^j \) in hand, we may also obtain surrogate values for use in a binning mechanism in Section E.5. In particular, we will wish to estimate the expected virtual value \( \psi^j \). Note that from the definition of virtual values, this is equivalent to estimating the quantity: \( \psi^j = \frac{R(\hat{q}^j) - R(\hat{q}^{j-1})}{\hat{q}^j - \hat{q}^{j-1}} \), where \( R(q) = qv(q) \).

Note that we do not know the quantiles \( \hat{q}^j \) corresponding to the values \( \hat{v}^j \) - we have only observed the latter quantities. We do know, however, that \( \hat{q}^j \) are close to \( j/T \) for every \( j \). We will therefore estimate \( \hat{q}^j \) by \( j/T \) and \( R(\hat{q}^j) \) by \( \frac{j}{T} \hat{v}^j \). Combining these estimates and their error bounds will yield low additive error for \( \psi^j \). In particular, our additive error will be in terms of the monopoly revenue \( R^* \) for \( F \).

**Lemma 40.** Assume \( \hat{q}^j \in [j/T - \epsilon, j/T + \epsilon] \) for all \( j \), and that \( \epsilon \leq 1/(2T) \). Then for all \( j \in \{1, \ldots, T\} \), \( \hat{v}^j \in [\psi^j - \delta(\epsilon,T)R^*, \psi^j + \delta(\epsilon,T)R^*] \) for \( \delta(\epsilon,T) = O(\epsilon T^2 + \epsilon^2 T^3) \).

**Proof.** Denote by \( \hat{\psi}^j \) the estimate \( \frac{\hat{q}^j \hat{v}^j - \hat{q}^{j-1} \hat{v}^{j-1}}{\hat{q}^j - \hat{q}^{j-1}} \). We will upper- and lower-bound the difference \( \hat{\psi}^j - \psi^j \). Note that \( \hat{\psi}^j \) can be rearranged as \( j \hat{v}^j - (j-1)\hat{v}^{j-1} \).

We first upperbound the difference. We do so for \( j \in \{2, \ldots, T-1\} \) by writing:

\[
\hat{\psi}^j = \hat{\psi}^j - \hat{\psi}^{j-1} = \frac{\hat{q}^j \hat{v}^j - \hat{q}^{j-1} \hat{v}^{j-1}}{\hat{q}^j - \hat{q}^{j-1}} - \frac{\hat{q}^{j-1} \hat{v}^{j-1}}{\hat{q}^j - \hat{q}^{j-1}}
\]

\[
\geq \frac{\hat{q}^j \hat{v}^j - \hat{q}^{j-1} \hat{v}^{j-1}}{\hat{q}^j - \hat{q}^{j-1}} - \frac{\hat{q}^{j-1} \hat{v}^{j-1}}{\hat{q}^j - \hat{q}^{j-1}}
\]

\[
= \frac{\hat{q}^j \hat{v}^j - \hat{q}^{j-1} \hat{v}^{j-1}}{\hat{q}^j - \hat{q}^{j-1}} - \frac{\hat{q}^{j-1} \hat{v}^{j-1}}{\hat{q}^j - \hat{q}^{j-1}}
\]

\[
= \frac{T(\hat{q}^j \hat{v}^j - \hat{q}^{j-1} \hat{v}^{j-1})}{1 + 2\epsilon T}
\]

\[
\geq T \left( \left( \frac{j}{T} - \epsilon \right) \hat{v}^j - \left( \frac{j-1}{T} + \epsilon \right) \hat{v}^{j-1} \right) - 2\epsilon T^2 R^*
\]

\[
= \hat{\psi}^j - T \epsilon \hat{v}^j - T \epsilon \hat{v}^{j-1} - 2\epsilon T^2 R^*
\]

\[
= \hat{\psi}^j - \frac{T \epsilon}{\hat{q}^j} R(\hat{q}^j) - \frac{T \epsilon}{\hat{q}^{j-1}} R(\hat{q}^{j-1}) - 2\epsilon T^2 R^*
\]

\[
\geq \hat{\psi}^j - \frac{2T \epsilon}{\hat{q}^j} R^* - 2\epsilon T^2 R^*
\]

\[
\geq \hat{\psi}^j - (4\epsilon T^2 + 4\epsilon^2 T^3) R^*.
\]
The last inequality follows from algebra and the fact that $1/(1-x) \leq 1 + 2x$ for $x \in [0,1/2]$. The rest follow from algebra or bounds on $\hat{q}^j$ and $\hat{q}^{-1}$. For $j = 1$ and $j = T$, a similar line of reasoning holds, made simpler by the fact that $\hat{q}^0 = 0$ and $\hat{q}^T = 1$. We omit the details.

We now lowerbound $\hat{\psi}^j - \psi^j$. We write (again for $j \in \{2, \ldots, T - 1\}$):

$$\psi^j = \frac{\hat{q}^j \hat{\psi}^j - \hat{q}^{-1} \hat{\psi}^{-1}}{\hat{q}^j - \hat{q}^{-1}} \leq \frac{T (\hat{q}^j \hat{\psi}^j - \hat{q}^{-1} \hat{\psi}^{-1})}{1 - 2\epsilon T} \leq T (1 + 4\epsilon T) (\hat{q}^j \hat{\psi}^j - \hat{q}^{-1} \hat{\psi}^{-1}) \leq T \left( \left( \frac{j}{T} + \epsilon \right) \hat{\psi}^j - \left( \frac{j-1}{T} - \epsilon \right) \hat{\psi}^{-1} \right) + 4\epsilon T^2 R^*$$

$$= \hat{\psi}^j + T \epsilon \hat{\psi}^j + T \epsilon \hat{\psi}^{-1} + 4\epsilon T^2 R^*$$

$$= \hat{\psi}^j + \frac{T \epsilon}{\hat{q}^j} R(\hat{q}^j) + \frac{T \epsilon}{\hat{q}^{-1}} R(\hat{q}^{-1}) + 4\epsilon T^2 R^*$$

$$\leq \hat{\psi}^j + \frac{2T \epsilon}{T - \epsilon} R^* + 4\epsilon T^2 R^*$$

$$\leq \hat{\psi}^j + (6\epsilon T^2 + 4\epsilon^2 T^3) R^*.$$

The result follows from combining the upper and lower bounds.

E.3 Binning with Non-Uniform Bins

In Section 4.4, we analyzed surrogate binning algorithms with bins which uniformly partitioned each bidders’ quantile space. Our mechanism from samples will not be able to partition agents’ quantile spaces in exactly this way, but rather will estimate appropriate breakpoints with low error in quantile space. We must therefore generalize our performance bounds for binning algorithms to non-uniform breakpoints. To do so, we extend the proofs of Lemmas 34 and 36 in the obvious ways.

We begin by generalizing the top promotion algorithm. The algorithm will still promote the top $k$ bins from every distribution, while remapping the quantiles of the remaining bins. The breakpoints for each agent $i$ will be denoted $\hat{q}^1_i \leq \ldots \leq \hat{q}^{T-1}_i$.

Definition 41. Given a monotone allocation algorithm $\hat{x}$, the $k$-top promotion algorithm for $\hat{Q} = \{\hat{q}^1_i, \ldots, \hat{q}^{T-1}_i\}_{i=1}^n$ and $\hat{x}$ runs $\hat{x}$ on agents with quantiles transformed for each population as follows:

1. For any $q \in [0, \hat{q}^k_i]$, return 0.
2. For any $q \in [\hat{q}^k_i, 1]$, return $(q - \hat{q}^k_i)/(1 - \hat{q}_i^k)$.

A straightforward extension of the proof of Lemma 34 yields:

Lemma 42. Under regularity, the $k$-top promotion algorithm for $\hat{Q}$ and $\hat{x}$ gets a $(1 - \sum_{i=1}^n \hat{q}^k_i) (1 - \max_i \hat{q}^k_i)$-fraction of the revenue of $\hat{x}$.
The first factor in the loss comes from applying the union bound to the events that a bidder from distribution \( i \) has a promoted quantile for each \( i \). The second factor follows from the application of Lemma \ref{lem:resampling_uniform} to each bidder’s allocation rule.

As in the analysis with uniform bins, we now present and analyze a resampling-based binning mechanism based on the \( k \)-top promotion algorithm.

**Definition 43.** The resampling algorithm for allocation algorithm \( \hat{x} \) and bins defined by \( \hat{Q} \):

1. For each agent \( i \), computes their bin \( b_i \) satisfying \( v_i \in [q_i^{b_i-1}, q_i^{b_i}] \) (with \( q_i^0 = 0 \) and \( q_i^T = 1 \)).
2. For each agent \( i \) samples a quantile \( \hat{q}_i \) uniformly at random from \( [q_i^{b_i-1}, q_i^{b_i}] \).
3. Runs the \( k \)-top promotion algorithm for \( \hat{x} \) and \( \hat{Q} \) on \( \hat{q}_i \).

As with the uniform bins resampling algorithm of Section \ref{sec:resampling_uniform}, the performance loss of an agent from resampling is \( \min_{j \geq 2} \hat{q}_i^{j-1}/\hat{q}_i^j \). For uniform bins, this was simply \( k/(k+1) \). For non-uniform bins, the statement is slightly more complicated. Formally:

**Lemma 44.** The resampling algorithm for \( \hat{x} \) with bins defined by \( \hat{Q} \) obtains at least a 
\[
(1 - \sum_{i=1}^n \delta_i^k) \left(1 - \max_i \delta_i^k\right) \min_{j \geq k,i} (\hat{q}_i^j/\hat{q}_i^{j+1})
\]
-fraction of the revenue of \( \hat{x} \).

Finally, we note that of all the allocation algorithms which treat agents identically based on their location with respect to the bins defined by some set of breakpoints \( \hat{Q} \), there is a well-defined and easy-to-derive optimal such algorithm.

**Definition 45.** The optimal surrogate binning algorithm with bins based on \( \hat{Q} \):

1. For each agent \( i \), computes their bin \( b_i \) satisfying \( v_i \in [q_i^{b_i-1}, q_i^{b_i}] \) (with \( q_i^0 = 0 \) and \( q_i^T = 1 \)).
2. Computes surrogate values \( \psi_i^j = (R_i(q_i^j) - R_i(q_i^{j-1}))/\left(\hat{q}_i^j - \hat{q}_i^{j-1}\right) \).
3. Assigns each agent surrogate value \( \tilde{v}_i = \psi_i^{b_i} \).
4. Runs the surplus maximization algorithm on \( \tilde{v} \).

The proof of the optimality of this algorithm follows the same reasoning as that of Theorem \ref{thm:resampling_uniform} the algorithm is simply maximizing virtual surplus conditioned on the information that it is allowed to use (namely, bin labels). As a corollary to this fact and the above analysis, we have that the optimal surrogate binning algorithm based on \( \hat{Q} \) inherits the revenue guarantees of Lemma \ref{lem:resampling_uniform} for any choice of \( k \).

**Corollary 46.** The optimal surrogate binning algorithm with bins based on \( \hat{Q} \) obtains at least a 
\[
(1 - \sum_{i=1}^n \delta_i^k) \left(1 - \max_i \delta_i^k\right) \min_{j \geq k,i} (\hat{q}_i^j/\hat{q}_i^{j+1})
\]
-fraction of the optimal revenue for all \( k \in \{1, \ldots, T-1\} \).

When \( \hat{q}_i^j \) are \( \epsilon \)-close to \( j/T \) for all \( j \) and \( i \), and \( \epsilon \) is reasonably small, then Corollary \ref{cor:resampling_uniform} implies that choosing \( k = \sqrt{T/n} \) yields loss for the optimal surrogate binning algorithm which goes to 0 as \( \sqrt{n/T} \). Formally:

**Lemma 47.** Assume \( \hat{q}_i^j \in [j/T - \epsilon, j/T + \epsilon] \) for all agents \( i \) and \( j \in \{1, \ldots, T-1\} \), and further that \( \epsilon \leq \min((nT)^{-1/2}, T^{-1}) \). Then the optimal surrogate binning algorithm based on \( \hat{Q} \) obtains at least a 
\[
(1 - O(\sqrt{n/T}))
\]
fraction of the optimal revenue.
Proof. We show that choosing \( k = \sqrt{T/n} \) yields the desired performance. We do so by proving that each term loss term in the formula \( (1 - \sum_{i=1}^{n} q_i^k) (1 - \max_i q_i^k) \min_{j \geq k,i} (q_i^j/q_i^{j+1}) \) from Corollary \([16]\) is at least \( (1 - O(\sqrt{n/T})) \). We begin with \( 1 - \sum_{i=1}^{n} q_i^k \). We have:

\[
1 - \sum_{i=1}^{n} q_i^k \geq 1 - \sum_{i=1}^{n} \sqrt{\frac{T}{n}} \frac{1}{T} - n\epsilon \geq 1 - 2\sqrt{n/T}.
\]

Note that since \( \sum_{i=1}^{n} q_i^k \geq \min_i q_i^k \), the second term is also sufficiently small. Finally, we consider \( \min_{j \geq k,i} (q_i^j/q_i^{j+1}) \). We have:

\[
\min_{j \geq k,i} (q_i^j/q_i^{j+1}) \geq \min_{j \geq k} \frac{j/T - \epsilon}{(j+1)/T+\epsilon} = \frac{k/T - \epsilon}{(k+1)/T+\epsilon} = \frac{k-1}{k+2} = 1 - \frac{2}{k} = 1 - 2\sqrt{n/T}.
\]

Since each factor in the loss is \( 1 - O(\sqrt{n/T}) \), the product of these losses is of the same order. \( \square \)

### E.4 Propogation of Error

We now show that additive errors in the surrogate value estimates propogate to the performance of the algorithm in a natural way. Formally:

**Lemma 48.** Let \( \hat{\Psi} = \{\hat{\psi}_i^j\}_{i \in \{1, \ldots, n\}} \) be estimates of \( \psi_i^j = (R_i(\hat{q}_i^j) - R_i(\hat{q}_i^{j-1}))/\hat{q}_i^j - \hat{q}_i^{j-1} \) satisfying \( \hat{\psi}_i^j \in [\psi_i^j - \gamma_i, \psi_i^j + \gamma_i] \) for all \( i \) and \( j \). Let \( \text{Rev}^{\hat{\Psi}} \) be the revenue of the optimal surrogate binning algorithm using \( \hat{\Psi} \) instead of the true optimal surrogate values \( \Psi \) and bins given by \( \hat{Q} \). Let \( \text{Rev}^{\Psi} \) be revenue of the optimal surrogate binning algorithm using \( \Psi \). Then:

\[
\text{Rev}^{\hat{\Psi}} \geq \text{Rev}^{\Psi} - 2 \sum_i \gamma_i.
\]

**Proof.** Let \( \hat{x} \) be the allocation rule of the surrogate binning algorithm using the estimates \( \hat{\Psi} \), and \( x \) the allocation rule of the surrogate binning mechanism using the true optimal surrogate values \( \Psi \). Given a value profile \( v \), let \( b_i(v_i) \) be the bin assigned to agent \( i \) based on value \( v_i \). The expected revenue from \( \hat{x} \) is:

\[
E_v \left[ \sum_i E \left[ \phi_i(v) \mid v \in [\hat{q}_i^{b_i(v_i)-1}, \hat{q}_i^{b_i(v_i)}] \right] \hat{x}_i(v_i) \right] = E_v \left[ \sum_i \hat{\psi}_i^{b_i(v_i)} \hat{x}_i(v_i) \right] \geq E_v \left[ \sum_i \hat{\psi}_i^{b_i(v_i)} \hat{x}_i(v_i) \right] - \sum_i \gamma_i \geq E_v \left[ \sum_i \psi_i^{b_i(v_i)} x_i(v_i) \right] - \sum_i \gamma_i \geq E_v \left[ \sum_i \psi_i^{b_i(v_i)} x_i(v_i) \right] - 2 \sum_i \gamma_i.
\]

The first term in the final expression is simply the revenue from \( x \), completing the proof. \( \square \)
E.5 Full Mechanism and Proof of Theorem 37

We now describe our samples-based mechanism in full detail.

Definition 49. Given target loss \( \epsilon > 0 \) and sample access to regular distributions \( F_1, \ldots, F_n \), the surrogate binning mechanism from samples has expected revenue that is at least a \((1 - O(\epsilon))-\)fraction of the optimal revenue.

1. Let \( T = \Omega \left( \frac{n^2}{\epsilon^2} \right) \).
2. Let \( m = \Omega \left( \frac{n^5}{\epsilon^5} \log(\frac{n}{\epsilon}) \right) \).
3. Sample \( mT - 1 = \Omega \left( \frac{n^5}{\epsilon^5} \log(\frac{n}{\epsilon}) \right) \) value profiles (containing one value per agent).
4. For all \( i \) and \( j \), let \( \hat{v}^j_i \) be the \( jm - 1 \)th highest sample from distribution \( i \).
5. For all \( i \) and \( j \), let \( \hat{\psi}^j_i = \frac{\hat{v}^j_i - \hat{v}^{j-1}_i}{T^{-1}/(T-1)} \).
6. To determine allocation and payments on a fresh value profile \( v \):
   a. For all \( i \), let \( b_i \) be the index such that \( v_i \in [\hat{\psi}^{b_i}_i, \hat{\psi}^{b_i-1}_i] \).
   b. Output the surplus-maximizing allocation for the value profile \( (\hat{\psi}^{b_1}_1, \ldots, \hat{\psi}^{b_n}_n) \).
   c. Charge truthful payments.

Theorem 50. The surrogate binning mechanism from samples has expected revenue that is at least a \((1 - O(\epsilon))-\)fraction of the optimal revenue.

Let \( \gamma = \epsilon^5/n^3 \). To prove the theorem, we proceed in three steps. We first argue that with probability \( 1 - \epsilon \), the true quantiles of \( \hat{v}^j_i \) will be within \( \gamma \) of \( j/T \) for all \( i \). We then show that this \( \gamma \)-closeness implies an additive loss of at most \( \epsilon R^*_i / n \) in our estimates of the conditional virtual values for each bin. Since these errors propagate naturally, and since \( \sum_i R^*_i \) is an upper bound on the revenue of any mechanism for any feasibility environment, the result will follow. We begin by showing \( \gamma \)-closeness of the estimated breakpoints \( \hat{v}^j_i \) to their target quantiles \( j/T \) using Lemma 39.

Lemma 51. With probability at least \( 1 - \epsilon \), for all \( i \) and \( j \), the quantile \( \hat{\psi}^j_i \) corresponding to \( \hat{v}^j_i \) satisfies \( \hat{\psi}^j_i \in [j/T - \gamma, j/T + \gamma] \).

Proof. We will apply Lemma 39 with \( \delta = \epsilon/n \). Taking a union bound over all \( n \) distributions will yield the result. Proving the lemma amounts to showing that

\[
(Tm - 1) \min \left( \left( 1 - \frac{m-2}{(m-1)} \right)^2 \left( \frac{1}{T} + \gamma \right), \left( \frac{m-1}{m-2} \right)^2 \left( \frac{1}{T} - \gamma \right) \right) \geq \Omega \left( \frac{\ln Tn}{\epsilon} \right),
\]

with \( T \) and \( m \) as given in Definition 49. Since \( m \) and \( T \) are both large integers and because \( T \) is polynomial in \( n \) and \( \epsilon^{-1} \), we may simplify this to

\[
Tm \min \left( \left( 1 - \frac{1}{1 + T \gamma} \right)^2 \left( \frac{1}{T} + \gamma \right), \left( \frac{1}{1 + T \gamma} - 1 \right)^2 \left( \frac{1}{T} - \gamma \right) \right) \geq \Omega \left( \frac{\ln n}{\epsilon} \right).
\]

Further note that for \( x \in [0, 1] \), \( 1/(1 + x) \leq 1 - x/2 \), and that \( 1/(1 - x) \geq 1 + x \). It therefore suffices to show that

\[
Tm(T^2 \gamma^2(\frac{1}{T} - \gamma)) \geq \Omega \left( \frac{\ln n}{\epsilon} \right).
\]
Since $\gamma$ is of a strictly smaller magnitude than $T$, we need that $m \geq \Omega\left(\frac{1}{T^2 \gamma^2} \ln \frac{n}{\epsilon}\right)$, which is evident from our choice of $m$.

Let $\mathcal{E}$ denote the event that for all $i$ and $j$, the quantile $\hat{q}_i^j$ corresponding to $\hat{v}_i^j$ satisfies $\hat{q}_i^j \in [j/T - \gamma, j/T + \gamma]$. We now use Lemma 40 to show that the additive error conditioned on $\mathcal{E}$ is small.

**Lemma 52.** Conditioned on $\mathcal{E}$, the surrogate value estimates $\hat{\psi}_i^j$ satisfy $\hat{\psi}_i^j \in [\psi_i^j - \epsilon R_i^*/n, \psi_i^j + \epsilon R_i^*/n]$, where $R_i^*$ is the monopoly revenue from selling optimally to a single bidder with distribution $F_i$, $\psi_i^j = (R(\hat{q}_i^j) - R(\hat{q}_i^{j-1}))/\hat{q}_i^j - \hat{q}_i^{j-1}$, and $\hat{q}_i^j$ is the true quantile of $v_i^j$.

**Proof.** Conditioned on $\mathcal{E}$, we have that the $\hat{q}_i^j$’s are $\gamma$-close to $j/T$, where $\gamma = \epsilon^5/n^3$. By Lemma 40 we have that the additive loss is $O(\gamma T^2 + \gamma^2 T^3) R_i^* = O(\epsilon R_i^*/n)$.

**Proof of Theorem 50.** We have from combining Lemmas 47, 48, 51, and 52 that the expected revenue of our mechanism is at least $(1 - \epsilon)\text{OPT} - \frac{\epsilon}{n} \sum_i R_i^*$, where OPT is the revenue of the Bayesian optimal mechanism. Since $\sum_i R_i^* \leq n\text{OPT}$, the result follows.

**Theorem 37** follows as a corollary of Theorem 50.

**Note on Computation** Our mechanism from samples requires the designer to solve the problem of surplus maximization and to compute threshold payments. Thus, the proper way to think of our result is as reduction from Bayesian mechanism design from samples to surplus maximization (and threshold computation). This is the standard approach from the mechanism design from samples literature for complex feasibility environments, and is consistent with the approach of, e.g., Devanur et al. (2016).