ON THE DIFFERENTIAL TRANSCENDENCE OF THE ELLIPTIC HYPERGEOMETRIC FUNCTIONS

CARLOS E. ARRECHE, THOMAS DREYFUS, AND JULIEN ROQUES

Abstract. We apply the differential Galois theory for difference equations in order to prove a criterion ensuring that any nonzero solution of a given order two difference equation is differentially transcendental. We then apply our result to the elliptic analogue of the hypergeometric equation.

Contents

Introduction 2
1. Difference Galois theory 3
2. Parametrized Difference Galois theory 5
2.1. General facts 5
2.2. Differential transcendence criteria 6
3. Difference equations over elliptic curves 9
3.1. The base field 9
3.2. Theta functions 10
3.3. Irreducibility of the \(\sigma\)-Galois groups 11
4. Application to the elliptic hypergeometric functions 11
4.1. The elliptic hypergeometric functions 11
4.2. The elliptic hypergeometric equation 12
4.3. Irreducibility of the \(\sigma\)-Galois group of the elliptic hypergeometric function 12
4.4. Differential transcendence of the elliptic hypergeometric functions 14
References 15

Date: September 17, 2018.
2010 Mathematics Subject Classification. 39A06,12H05.
Key words and phrases. Linear difference equations, difference Galois theory, elliptic curves, differential algebra.
Introduction

The elliptic hypergeometric functions form a common analogue of classical hypergeometric functions and $q$-hypergeometric functions, which have been a focus of intense study in the last 200 years within the theory of special functions and are ubiquitous in physics and mathematics. The general theory of these elliptic hypergeometric functions was initiated by Spiridonov in [Spi16] and has been a dynamic field of research, see for instance [vdB+07, FR09, M+09, Rai10, Ros02]. In the intervening years a number of remarkable analogues of known properties and applications of classical and $q$-hypergeometric functions have been discovered for the elliptic hypergeometric functions; see [Spi16] for more details.

In this work we develop a criterion to decide differential transcendence for elliptic hypergeometric functions. More precisely, our main result is that for “generic” values of the parameters, in a sense made precise in Section 4, the elliptic hypergeometric functions are differentially transcendental, i.e. they do not satisfy any polynomial differential equations with elliptic function coefficients, see Definition 2.3. Our algebraic proof of this result is based on differential Galois theory for difference equations [HS08], which associates a geometric object to such a difference equation, the Galois group, that encodes the polynomial differential equations that may be satisfied by the solutions. There is a Galois correspondence that implies in particular that the larger the group, the fewer the polynomial differential relations that exist among the solutions. As a preliminary result, we prove in Theorem 2.4 a criterion that ensures that the Galois group is large enough to force every nonzero solution to be differentially transcendental. Then we apply Theorem 2.4 to the elliptic hypergeometric function solution of equation (4.2) discovered in [Spi16] by interpreting the latter as a second-order linear difference equation over an elliptic curve.

Our strategy here is in the tradition of other applications of differential Galois theory for difference equations of [HS08] to questions about shift difference equations [Arr17], $q$-difference equations, [DHR16], deterministic finite automata and Mahler functions [DHR18], lattice walks in the quarter plane [DHRS18, DR17, DHRS17], and shift, $q$-dilation, and Mahler difference equations in general [AS17]. In order to apply our criterion in practice, one needs to check that there are no telescoper relations of a certain kind and that a certain Riccati equation has no solutions. In recent years, the algorithmic solution of these two problems has attracted the attention of many researchers independently of the question of differential transcendence, see for example [Pet92, Hen97, Hen98, Roq18, DR15] for the Riccati equations, see also [Tie05, Nis18], and [Abr95, CS12] for the telescopers. We hope that our results will motivate the development of new algorithms to handle the remaining cases.

The paper is organized as follows. In Section 1, we recall some facts about the difference Galois theory developed in [vdPS97]. To a difference equation is associated an algebraic group. The larger the group, the fewer the algebraic relations that exist among the solutions of the difference equation. In Section 2, we recall some facts about the differential Galois theory for difference equations of [HS08]. Here the Galois group is a linear differential
algebraic group, that is, a group of matrices defined by a system of algebraic
differential equations in the matrix entries. This group encodes the poly-
ominal differential relations among the solutions of the difference equation.
In this section we prove a criterion to ensure that every nonzero solution
of a given second-order difference equation is differentially transcendental;
see Theorem 2.4. In Section 3 we restrict ourselves to the situation where
the coefficients of the difference equation are elliptic functions. We recall
some results from [DR15], where the authors explain how to compute the
difference Galois group of [vdPS97] for order two equations with elliptic
coefficients. This computation was inspired by Hendricks’ algorithm, see
[Hen97]. In Section 4, we follow [Spi16] in defining the elliptic analogue of
the hypergeometric equation (4.2) and, under a certain genericity assump-
tion, we prove that its nonzero solutions are differentially transcendental,
see Theorem 4.3.

1. Difference Galois theory

For details on what follows, we refer to [vdPS97, Chapter 1]. Unless
otherwise stated, all rings are commutative with identity and contain the
field of rational numbers. In particular, all fields are of characteristic zero.
A \( \sigma \)-ring (or difference ring) \((R, \sigma)\) is a ring \(R\) together with a ring auto-
morphism \(\sigma : R \to R\). If \(R\) is a field then \((R, \sigma)\) is called a \(\sigma\)-field. When
there is no possibility of confusion the \(\sigma\)-ring \((R, \sigma)\) will be simply denoted
by \(R\). There are natural notions of \(\sigma\)-ideals, \(\sigma\)-ring extensions, \(\sigma\)-algebras,
\(\sigma\)-morphisms, etc. We refer to [vdPS97, Chapter 1] for the definitions.

The ring of \(\sigma\)-constants \(R^{\sigma}\) of the \(\sigma\)-ring \((R, \sigma)\) is defined by
\[
R^{\sigma} := \{ f \in R \mid \sigma(f) = f \}.
\]

We now let \((K, \sigma)\) be a \(\sigma\)-field. We assume that the field of constants
\(C := K^{\sigma}\) is algebraically closed and that the characteristic of \(K\) is 0.

We consider a difference equation of order two with coefficients in \(K\):
\[
\sigma^2(y) + a\sigma(y) + by = 0 \quad \text{with} \quad a \in K \quad \text{and} \quad b \in K^*.
\]
and the associated difference system:
\[
\sigma Y = AY \quad \text{with} \quad A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \in \text{GL}_2(K).
\]

By [vdPS97, §1.1], there exists a \(\sigma\)-ring extension \((R, \sigma)\) of \((K, \sigma)\) such that
1) there exists \(U \in \text{GL}_2(R)\) such that \(\sigma(U) = AU\) (such a \(U\) is called a
fundamental matrix of solutions of (1.2));
2) \(R\) is generated, as a \(K\)-algebra, by the entries of \(U\) and \(\det(U)^{-1}\);
3) the only \(\sigma\)-ideals of \((R, \sigma)\) are \(\{0\}\) and \(R\).

Note that the last assumption implies \(R^{\sigma} = C\). Such an \(R\) is called a \(\sigma\)-
Picard-Vessiot ring, or \(\sigma\)-PV ring for short, for (1.2) over \((K, \sigma)\). It is unique
up to isomorphism of \((K, \sigma)\)-algebras. Note that a \(\sigma\)-PV ring is not always
an integral domain, but it is a direct sum of integral domains transitively
permuted by \(\sigma\).
The corresponding \(\sigma\)-Galois group \(\text{Gal}(R/K)\) of (1.2) over \((K, \sigma)\), or \(\sigma\)-Galois group for short, is the group of \((K, \sigma)\)-automorphisms of \(R\):

\[
\text{Gal}(R/K) := \{ \phi \in \text{Aut}(R/K) \mid \sigma \circ \phi = \phi \circ \sigma \}.
\]

A straightforward computation shows that, for any \(\phi \in \text{Gal}(R/K)\), there exists a unique \(C(\phi) \in \text{GL}_2(\mathbb{C})\) such that \(\phi(U) = UC(\phi)\). According to [vdPS97, Theorem 1.13], one can identify \(\text{Gal}(R/K)\) with an algebraic subgroup \(G\) of \(\text{GL}_2(\mathbb{C})\) via the faithful representation

\[
\rho : \text{Gal}(R/K) \rightarrow \text{GL}_2(\mathbb{C})
\]

\[
\phi \mapsto C(\phi).
\]

If we choose another fundamental matrix of solutions \(U\), we find a conjugate representation. In what follows, by “\(\sigma\)-Galois group of the difference equation (1.1)”, we mean “\(\sigma\)-Galois group of the difference system (1.2)”.

We shall now introduce a property relative to the base \(\sigma\)-field \((K, \sigma)\), which appears in [vdPS97, Lemma 1.19].

**Definition 1.1.** We say that the \(\sigma\)-field \((K, \sigma)\) satisfies the property \((P)\) if:

- the field \(K\) is a \(C^1\)-field;
- and the only finite field extension \(L\) of \(K\) such that \(\sigma\) extends to a field endomorphism of \(L\) is \(L = K\).

**Example 1.2.** The following are natural examples of difference fields that satisfy property \((P)\):

- **S**: Shift case with \(K = \mathbb{C}(z), \sigma : f(z) \mapsto f(z + h), h \in \mathbb{C}^*\). See [Hen97].
- **Q**: \(q\)-difference case. \(K = \mathbb{C}(z^{1/q}) = \bigcup_{\ell \in \mathbb{N}^*} \mathbb{C}(z^{1/\ell}), \sigma : f(z) \mapsto f(qz), q \in \mathbb{C}^*, |q| \neq 1\). See [Hen98].
- **M**: Mahler case. \(K = \mathbb{C}(z^{1/p}), \sigma : f(z) \mapsto f(z^p), p \in \mathbb{N}_{\geq 2}\). See [Roq18].
- **E**: Elliptic case. See Section 3, and [DR15].

The following result is due to van der Put and Singer. We recall that two difference systems \(\sigma Y = \text{AY} \) and \(\sigma Y = \text{BY}\) with \(A, B \in \text{GL}_2(K)\) are isomorphic over \(K\) if and only if there exists \(T \in \text{GL}_2(K)\) such that \(\sigma(T)A = BT\). Note that \(\sigma(Y) = \text{AY}\) if and only if \(\sigma(TY) = BTY\).

**Theorem 1.3.** Assume that \((K, \sigma)\) satisfies property \((P)\). Then the following properties relative to \(G = \rho(\text{Gal}(R/K))\) hold:

- \(G/G^0\) is cyclic, where \(G^0\) is the identity component of \(G\);
- there exists \(B \in G(K)\) such that (1.2) is isomorphic to \(\sigma Y = \text{BY}\) over \(K\).

Let \(\tilde{G}\) be an algebraic subgroup of \(\text{GL}_2(\mathbb{C})\) such that \(A \in \tilde{G}(K)\). The following properties hold:

- \(G\) is conjugate to a subgroup of \(\tilde{G}\);
- any minimal element (with respect to inclusion) in the set of algebraic subgroups \(\tilde{H}\) of \(\tilde{G}\) for which there exists \(T \in \text{GL}_2(K)\) such that \(\sigma(T)AT^{-1} \in \tilde{H}(K)\) is conjugate to \(G\);
- \(G\) is conjugate to \(\tilde{G}\) if and only if, for any \(T \in \tilde{G}(K)\) and for any proper algebraic subgroup \(\tilde{H}\) of \(\tilde{G}\), one has that \(\sigma(T)AT^{-1} \notin \tilde{H}(K)\).

---

1. Recall that \(K\) is a \(C^1\)-field if every non-constant homogeneous polynomial \(P\) over \(K\) has a non-trivial zero provided that the number of its variables is more than its degree.
Proof. The proof of [vdPS97, Propositions 1.20 and 1.21] in the special case where $K := \mathbb{C}(z)$ and $\sigma$ is the shift $\sigma(f(z)) := f(z + h)$ with $h \in \mathbb{C}^*$, extends \textit{mutatis mutandis} to the present case. \hfill $\blacksquare$

This theorem is at the heart of many algorithms to compute $\sigma$-Galois groups, see for example [Hen97, Hen98, DR15, Roq18].

2. Parametrized Difference Galois theory

2.1. General facts. A $(\sigma, \delta)$-ring $(R, \sigma, \delta)$ is a ring $R$ endowed with a ring automorphism $\sigma$ and a derivation $\delta : R \to R$ (this means that $\delta$ is additive and satisfies the Leibniz rule $\delta(ab) = a\delta(b) + \delta(a)b$) such that $\sigma \circ \delta = \delta \circ \sigma$. If $R$ is a field, then $(R, \sigma, \delta)$ is called a $(\sigma, \delta)$-field. When there is no possibility of confusion, we write $R$ instead of $(R, \sigma, \delta)$. There are natural notions of $(\sigma, \delta)$-ideals, $(\sigma, \delta)$-ring extensions, $(\sigma, \delta)$-algebras, $(\sigma, \delta)$-morphisms, etc. We refer to [HS08, Section 6.2] for the definitions.

If $K$ is a $\delta$-field, and if $y_1, \ldots, y_n$ belong to some $\delta$-field extension of $K$, then $K\{y_1, \ldots, y_n\}_\delta$ denotes the $\delta$-algebra generated over $K$ by $y_1, \ldots, y_n$ and $K(y_1, \ldots, y_n)_\delta$ denotes the $\delta$-field generated over $K$ by $y_1, \ldots, y_n$. We now let $(K, \sigma, \delta)$ be a $(\sigma, \delta)$-field. We assume that the field of $\sigma$-constants $\mathcal{C} := K^\sigma$ is algebraically closed and that $K$ is of characteristic 0.

In order to apply the $(\sigma, \delta)$-Galois theory developed in [HS08], we need to work with a base $(\sigma, \delta)$-field $L$ such that $\mathcal{C} = L^\sigma$ is $\delta$-closed.\footnote{The field $\mathcal{C}$ is called $\delta$-closed if, for every (finite) set of $\delta$-polynomials $F$ with coefficients in $\mathcal{C}$, if the system of $\delta$-equations $F = 0$ has a solution with entries in some $\delta$-field extension $L(\mathcal{C})$, then it has a solution with entries in $\mathcal{C}$. Note that when the derivation $\delta$ is trivial, \textit{i.e.} $\delta = 0$, then a field is $\delta$-closed if and only if it is algebraically closed.} To this end, the following lemma will be useful.

Lemma 2.1 ([DHR18, Lemma 2.3]). Suppose that $\mathcal{C}$ is algebraically closed and let $\bar{\mathcal{C}}$ be a $\delta$-closure of $\mathcal{C}$ (the existence of such a $\bar{\mathcal{C}}$ is proved in [Kol74]). Then the ring $\bar{\mathcal{C}} \otimes_{\mathcal{C}} K$ is an integral domain whose fraction field $L$ is a $(\sigma, \delta)$-field extension of $K$ such that $L^\sigma = \bar{\mathcal{C}}$.

We still consider the difference equation (1.1) and the associated difference system (1.2). By [HS08, § 6.2.1], there exists a $(\sigma, \delta)$-ring extension $(S, \sigma, \delta)$ of $(L, \sigma, \delta)$ such that
\begin{enumerate}
\item there exists $U \in \text{GL}_2(S)$ such that $\sigma(U) = AU$;
\item $S$ is generated, as an $L$-$\delta$-algebra, by the entries of $U$ and $\det(U)^{-1}$;
\item the only $(\sigma, \delta)$-ideals of $S$ are $\{0\}$ and $S$.
\end{enumerate}

Such an $S$ is called a $(\sigma, \delta)$-Picard-Vessiot ring, or $(\sigma, \delta)$-PV ring for short, for (1.2) over $(L, \sigma, \delta)$. It is unique up to isomorphism of $(L, \sigma, \delta)$-algebras. Note that a $(\sigma, \delta)$-PV ring is not always an integral domain, but it is the direct sum of integral domains that are transitively permuted by $\sigma$.

The corresponding $(\sigma, \delta)$-Galois group $\text{Gal}^\delta(S/L)$ of (1.2) over $(L, \sigma, \delta)$, or $(\sigma, \delta)$-Galois group for short, is the group of $(L, \sigma, \delta)$-automorphisms of $S$:
\[\text{Gal}^\delta(S/L) = \{ \phi \in \text{Aut}(S/L) \mid \sigma \circ \phi = \phi \circ \sigma \text{ and } \delta \circ \phi = \phi \circ \delta \}.
\]
In what follows, by “$(\sigma, \delta)$-Galois group of the difference equation (1.1)”, we mean “$(\sigma, \delta)$-Galois group of the difference system (1.2)”. A straightforward computation shows that, for any $\phi \in \text{Gal}_\delta(S/L)$, there exists a unique $C(\phi) \in \text{GL}_2(\tilde{C})$ such that $\phi(U) = UC(\phi)$. By [HS08, Proposition 6.18], the faithful representation
$$\rho^\delta : \text{Gal}^\delta(S/L) \rightarrow \text{GL}_2(\tilde{C})$$
identifies $\text{Gal}^\delta(S/L)$ with a linear differential algebraic group $G^\delta$, that is, a subgroup of $\text{GL}_2(\tilde{C})$ defined by a system of $\delta$-polynomial equations over $\tilde{C}$ in the matrix entries. If we choose another fundamental matrix of solutions $U$, we find a conjugate representation.

Let $S$ be a $(\sigma, \delta)$-PV ring for (1.2) over $L$ and let $U \in \text{GL}_2(S)$ be a fundamental matrix of solutions. Then the $L$-$\sigma$-algebra $R$ generated by the entries of $U$ and $\det(U)^{-1}$ is a $\sigma$-PV ring for (1.2) over $L$. We can (and will) identify $\text{Gal}^\delta(S/L)$ with a subgroup of $\text{Gal}(R/L)$ by restricting the elements of $\text{Gal}^\delta(S/L)$ to $R$.

**Proposition 2.2** ([HS08], Proposition 2.8). The group $\text{Gal}^\delta(S/L)$ is a Zariski-dense subgroup of $\text{Gal}(R/L)$.

### 2.2. Differential transcendence criteria.

The aim of this section is to develop a galoisian criterion for the differential transcendence of the nonzero solutions of (1.1).

**Definition 2.3.** Let $F/K$ be a $(\sigma, \delta)$-field extension. We say that $f \in F$ is differentially algebraic over $K$ if there exists $n \in \mathbb{N}$ such that $f, \ldots, \delta^n(f)$ are algebraically dependent over $K$. Otherwise, we say that $f$ is differentially transcendental over $K$.

Recall that $K$ be a $(\sigma, \delta)$-field satisfying property (P) such that $\mathcal{C} = K^\sigma$ is algebraically closed and such that $K$ has characteristic 0.

Let $\tilde{\mathcal{C}}$ be a $\delta$-closure of $\mathcal{C}$. According to Lemma 2.1, $\tilde{\mathcal{C}} \otimes_{\mathcal{C}} K$ is an integral domain and $L := \text{Frac}(\tilde{\mathcal{C}} \otimes_{\mathcal{C}} K)$ is a $(\sigma, \delta)$-field extension of $K$ such that $L^\sigma = \tilde{\mathcal{C}}$. Let $S$ be a $(\sigma, \delta)$-PV ring for (1.2) over $L$ and let $R \subset S$ be a $\sigma$-PV ring for (1.2) over $L$. We also consider a $\sigma$-PV ring $\tilde{R}$ for (1.2) over $K$.

Our differential transcendence criterion is the following.

**Theorem 2.4.** Assume that $\text{Gal}(\tilde{R}/K)$ is irreducible and that the $(\sigma, \delta)$-Galois group of $\sigma y = \text{by}$ over $L$ is $\text{GL}_1(\tilde{\mathcal{C}})$. Then any nonzero solution of (1.1) in any $(\sigma, \delta)$-field extension $F$ of $K$ is differentially transcendental over $K$.

Note that the irreducibility of $\text{Gal}(\tilde{R}/K)$ may be tested algorithmically in many contexts, see [Hen97, Hen98, DR15, Roq18]. More precisely, the group is irreducible if and only if there does not exist $u \in K$ satisfying the Riccati equation $u(\sigma(u) + a) = -b$. The following lemma gives a more tractable version of the second assumption.
Lemma 2.5 (Proposition 2.6, [DHR18]). The \((\sigma, \delta)\)-Galois group of \(\sigma y = b\) over \(L\) is a proper subgroup of \(GL_1(C)\) if and only if there exist a nonzero linear differential operator \(L\) with coefficients in \(C\) and \(g \in K\) such that
\[
L \left( \frac{\delta(b)}{b} \right) = \sigma(g) - g.
\]

The following lemma will be used in the proof of Theorem 2.4.

Lemma 2.6. Assume that \((1.1)\) has a nonzero differentially algebraic solution in a \((\sigma, \delta)\)-field extension \(F\) of \(K\). Then \((1.1)\) has a nonzero differentially algebraic solution in \(S\).

Proof of Lemma 2.6. Since any two \((\sigma, \delta)\)-PV rings for \((1.1)\) over \(L\) are isomorphic, it is sufficient to prove the lemma for some \((\sigma, \delta)\)-PV ring, not necessarily for \(S\) itself. Let \(f\) be a nonzero differentially algebraic solution of \((1.1)\) in \(F\). We consider the localization \(T\) of \(L(f, \sigma(f))\) at \(fX_{1,2} - \sigma(f)X_{1,2}\), where \(X_{1,2}, X_{2,2}\) are \(\delta\)-indeterminates over \(L(f, \sigma(f))\). This ring has a natural structure of \(L\)-\((\sigma, \delta)\)-algebra such that \(\sigma X_{1,2}/X_{2,2} = A X_{1,2}/X_{2,2}\) and \(f X_{1,2}/X_{2,2}\) is a fundamental matrix of solutions of \(\sigma Y = AY\) with coefficients in \(T\). If we let \(\mathfrak{M}\) be a maximal \(\sigma, \delta\)-ideal of \(T\), then the quotient \(T/\mathfrak{M}\) is a \((\sigma, \delta)\)-PV ring for \(\sigma Y = AY\) over \(L\) and the image of \(f\) in this quotient is differentially algebraic. Let us prove that it is nonzero. Otherwise the image of the fundamental solution in the \((\sigma, \delta)\)-PV ring \(T/\mathfrak{M}\) would have a zero first column and therefore would not be invertible, leading to a contradiction. This concludes the proof.

Proof of Theorem 2.4. Assume to the contrary that Equation \((1.1)\) has a nonzero differentially algebraic solution in a \((\sigma, \delta)\)-field extension \(F\) of \(K\). According to Lemma 2.6, there exists a nonzero differentially algebraic solution \(f\) of \((1.1)\) in \(S\).

By [Hen97, Lemma 4.1] combined with Theorem 1.3, one of the following three cases holds

- Gal(\(\hat{R}/K\)) is reducible.
- Gal(\(\hat{R}/K\)) is irreducible and imprimitive.
- Gal(\(\hat{R}/K\)) contains \(SL_2(C)\).

Since Gal(\(\hat{R}/K\)) is irreducible by assumption, only the last two cases may occur. Then we split our study in two cases depending on whether Gal(\(\hat{R}/K\)) is imprimitive or not.

Let us first assume that Gal(\(\hat{R}/K\)) is imprimitive. It follows from Theorem 1.3 and [Hen97, Section 4.3] that \((1.1)\) is equivalent over \(K\) to
\[
(2.1)\quad \sigma^2(y) + ry = 0
\]
for some \(r \in K^\ast\). More precisely, let
\[
\sigma Y = BY \text{ with } B = \begin{pmatrix} 0 & 1 \\ -r & 0 \end{pmatrix} \in GL_2(K),
\]
be the system associated to \((2.1)\). Then there exists \(T \in GL_2(K)\) such that \(\sigma(T)A = BT\). Let \(T = (t_{i,j})\). Since \(\sigma Y = AY\) if and only if \(\sigma(T)Y = BTY\),
we obtain that \( t_{1,1}f + t_{1,2}\sigma(f) \) satisfies (2.1) with \( (t_{1,1}, t_{1,2}) \neq (0, 0) \). Let us prove that \( t_{1,1}f + t_{1,2}\sigma(f) \) is non zero. If \( t_{1,1}f + t_{1,2}\sigma(f) = 0 \), then \( f \neq 0 \) implies \( t_{1,1}t_{1,2} \neq 0 \) and then \( \sigma(f)/f \) is solution of the Riccati equation \( u(\sigma(u) + a) = -b \), which contradicts the irreducibility of \( \text{Gal}(\bar{R}/\mathbb{K}) \) by [DR15, Lemma 13].

Since \( f \) is differentially algebraic over \( \mathbb{K} \), we have that \( \sigma(f) \), and hence also \( t_{1,1}f + t_{1,2}\sigma(f) \), are differentially algebraic over \( \mathbb{L} \). By [HS08, Proposition 6.26], this implies that the \( (\sigma^2, \delta) \)-Galois group of (2.1) over \( \mathbb{L} \) is a strict subgroup of \( \text{GL}_1(\mathbb{C}) \). By Lemma 2.5 there exist a nonzero \( D \in \mathbb{C}[\delta] \) and \( h \in \mathbb{K} \) such that

\[
(2.2) \quad D = \sigma^2(h) - h = \sigma(\sigma(h) + h) - (\sigma(h) + h).
\]

Taking the determinant in \( \sigma(T)A = BT \) allows us to deduce the existence of \( p \in \mathbb{K}^* \) such that \( b = \sigma(p)\sigma(r) \), and therefore the \( (\sigma, \delta) \)-Galois groups for \( \sigma(y) = ry \) and \( \sigma(y) = by \) are the same. Consequently, by Lemma 2.5 and the assumption on the \( (\sigma, \delta) \)-Galois group of \( \sigma y = by \) over \( \mathbb{L} \), for any nonzero \( D \in \mathbb{C}[\delta] \) and any \( g \in \mathbb{K} \), we have \( D(\frac{\delta Y}{\sigma}) \neq \sigma(g) - g \). This is in contradiction with (2.2).

Assume now that \( \text{Gal}(\bar{R}/\mathbb{K}) \) is not imprimitive, so it contains \( \text{SL}_2(\mathbb{C}) \). By [DHR18, Proposition 2.10], we deduce that

\[
G_{m} := \left\{ \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \mid c \in \tilde{\mathbb{C}}^* \right\} \subset \text{Gal}(S/\mathbb{L}).
\]

Let \( n \in \mathbb{N} \) be as small as possible such that there exists \( 0 \neq P \in \mathbb{L}[X_0, \ldots, X_n] \) with \( P(f, \delta(f), \ldots, \delta^n(f)) = 0 \), and suppose that this \( P \) has smallest possible total degree \( d \in \mathbb{N} \). For \( c \in \tilde{\mathbb{C}}^* \), let \( \phi_c \in G_{m} \) with corresponding matrix \( \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix} \). For all \( c \in \tilde{\mathbb{C}}^* \), we find

\[
\phi_c P(f, \delta(f), \ldots, \delta^n(f)) = P(\phi_c(f), \phi_c(\delta(f)), \ldots, \phi_c(\delta^n(f))) = P(cf, \delta(cf), \ldots, \delta^n(cf)) = 0.
\]

Since \( \tilde{\mathbb{C}} \) is differentially closed, there exists \( c \in \tilde{\mathbb{C}}^* \) such that \( \delta(c) = 0 \) and \( c^d \neq 1 \). Since \( \delta(cf) = c\delta(f) \) for such a \( c \), we have that

\[
c^d P(f, \delta(f), \ldots, \delta^n(f)) = P(cf, c\delta(f), \ldots, c\delta^n(f)) = 0,
\]

and we find that \( P \) must be homogeneous of degree \( d \), for otherwise the total degree \( d \) would not be minimal. We may further assume that the degree \( d_n \) of \( X_n \) in \( P \) is as small as possible. Again since \( \tilde{\mathbb{C}} \) is differentially closed, there exists \( c \in \tilde{\mathbb{C}} \) such that \( \delta^2(c) = 0 \) but \( \delta(c) \neq 0 \). But then

\[
0 = P(cf, \delta(cf), \ldots, \delta^n(cf)) = P(cf, c\delta(f) + \delta(c)f, \ldots, c\delta^n(f) + \delta(c)\delta^{n-1}(f))
\]

\[
= c^d P(f, \delta(f), \ldots, \delta^n(f)) + Q(f, \delta(f), \ldots, \delta^n(f)) = Q(f, \delta(f), \ldots, \delta^n(f))
\]

for some nonzero homogeneous polynomial \( Q \in \mathbb{L}[X_0, \ldots, X_n] \) of total degree \( d \) in which the degree of \( X_n \) is strictly smaller than \( d_n \). This contradiction concludes the proof. □
3. Difference equations over elliptic curves

In this section we will be mainly interested in difference equations
\begin{equation}
\sigma^2(y) + a\sigma(y) + by = 0,
\end{equation}
with \(a, b \in M_p\), where

- \(M_p\) denotes the field of meromorphic functions over the elliptic curve \(C^*/p^Z\) for some \(p \in C^*\) such that \(|p| < 1\), i.e., the field of meromorphic functions on \(C^*\) satisfying \(f(z) = f(pz)\);
- \(\sigma\) is the automorphism of \(M_p\) defined by
  \[\sigma(f)(z) := f(qz)\]
for some \(q \in C^*\) such that \(|q| \neq 1\) and \(p^Z \cap q^Z = \{1\}\).

Note that this choice ensures that \(\sigma\) is non-cyclic.

3.1. The base field. The difference Galois groups of linear difference equations over elliptic curves have been studied in [DR15]. In loc. cit. the elliptic curves are given by quotients of the form \(C/\Lambda\) for some lattice \(\Lambda\). However, in the present work, we are mainly interested in difference equations on elliptic curves given by quotients of the form \(C^*/p^Z\) for some \(p \in C^*\) such that \(|p| < 1\). The translation between elliptic curves of the form \(C/\Lambda\) and elliptic curves of the form \(C^*/p^Z\) is standard, namely by using the fact that if \(\Lambda = Z + \tau Z\) with \(\Im(\tau) > 0\) and \(p = e^{2\pi i\tau}\) then the map \(C \to C^*: w \mapsto e^{2\pi i\tau}\) induces an isomorphism \(C/\Lambda \simeq C^*/p^Z\).

We shall now recall some constructions and results from [DR15], restated in the "\(C^*/p^Z\) context" via the above identification between \(C/\Lambda\) and \(C^*/p^Z\). For \(k \in N^*\) we denote by \(C_k^*\) the Riemann surface of \(z^{1/k}\), and we let \(z_k\) be a coordinate function on each \(C_k^*\) such that \(z_{dk}^d = z_k\) for every \(d \in N^*\). We will write \(C_k^* = C^*\) and \(z_1 = z\).

We let \(M_{p,k}\) denote the field of meromorphic functions on \(C_k^*\) satisfying \(f(p^qz_k) = f(z_k)\), or equivalently the field of meromorphic functions on the elliptic curve \(C_k^*/p^Z\). The \(d\)-power map \(C_{dk}^* \to C_k^*\) : \(\xi \mapsto \xi^d\) induces an inclusion of function fields \(M_{p,k} \hookrightarrow M_{p,dk}\) for each \(k, d \in N^*\). We denote by \(K\) the field defined by
\[K := \bigcup_{k \geq 1} M_{p,k}.
\]
We endow \(K\) with the non-cyclic field automorphism \(\sigma\) defined by
\begin{equation}
\sigma(f)(z_k) := f(q_kz_k)
\end{equation}
where \(q_k = q \in C^*\) is such that \(|q| \neq 1\) and \(p^Z \cap q^Z = \{1\}\), and \(q_k \in C_k^*\) defines a compatible system of \(k\)-th roots of \(q_1 = q\) such that \(q_{dk}^d = q_k\) for every \(d \in N^*\) (cf. [Hen98, Section 2]). Then (\(K, \sigma\)) is a difference field and we have the following properties.

**Proposition 3.1** ([DR15], Proposition 5). The field of constants of (\(K, \sigma\)) is \(K^\sigma = C\).

**Proposition 3.2** ([DR15], Proposition 6). The difference field (\(K, \sigma\)) satisfies property (\(P\)) (see Definition 1.1).

**Remark 3.3.** The field \(M_p = M_{p,1}\) equipped with the automorphism \(\sigma\) does not satisfy property (\(P\)). This is why we work over \((K, \sigma)\) instead of \((M_p, \sigma)\).
Corollary 3.4. The conclusions of Theorem 1.3 are valid for \((K, \sigma)\).

3.2. Theta functions. We shall now recall some basic facts and notations about theta functions extracted from [DR15, Section 3] (but stated in the “\(\mathbb{C}^* / \mathbb{Z}^\nu\) context”, see the beginning of the previous section). For the proofs, we refer to [Mum07, Chapter I]. We still consider \(p \in \mathbb{C}^*\) such that \(|p| < 1\).

We consider the infinite product
\[
(z ; p)_\infty = \prod_{j \geq 0} (1 - zp^j).
\]

The theta function defined by
\[
(3.3) \quad \theta(z ; p) = (z ; p)_\infty (pz^{-1} ; p)_\infty
\]
satisfies
\[
(3.4) \quad \theta(pz ; p) = \theta(z^{-1} ; p) = -z^{-1} \theta(z ; p).
\]

Let \(\Theta_k\) be the set of holomorphic functions on \(\mathbb{C}_k^*\) of the form
\[
c \prod_{\xi \in \mathbb{C}_k^*} \theta(\xi z_k ; k)^{n_{\xi}}
\]
with \(c \in \mathbb{C}^*\) and \((n_{\xi})_{\xi \in \mathbb{C}_k^*} \in \mathbb{N}^{|\mathbb{C}_k^*|}\) with finite support. We denote by \(\Theta_k^{\text{quot}}\) the set of meromorphic functions on \(\mathbb{C}_k^*\) that can be written as a quotient of two elements of \(\Theta_k\). We have
\[
M_{p,k} \subset \Theta_k^{\text{quot}}.
\]

We define the divisor \(\text{div}_k(f)\) of \(f \in \Theta_k^{\text{quot}}\) as the following formal sum of points of \(\mathbb{C}_k^*/\mathbb{Z}^\nu\):
\[
\text{div}_k(f) := \sum_{\lambda \in \mathbb{C}_k^*/\mathbb{Z}^\nu} \text{ord}_\lambda(f)[\lambda],
\]
where \(\text{ord}_\lambda(f)\) is the \((z_k - \xi)\)-adic valuation of \(f\), for an arbitrary \(\xi \in \lambda\) (it follows from (3.4) that this valuation does not depend on the chosen \(\xi \in \lambda\)). For any \(\lambda \in \mathbb{C}_k^*/\mathbb{Z}^\nu\) and any \(\xi \in \lambda\), we set
\[
[\xi]_k := [\lambda].
\]
Moreover, we will write
\[
\sum_{\lambda \in \mathbb{C}_k^*/\mathbb{Z}^\nu} n_\lambda[\lambda] \leq \sum_{\lambda \in \mathbb{C}_k^*/\mathbb{Z}^\nu} m_\lambda[\lambda]
\]
if \(n_\lambda \leq m_\lambda\) for all \(\lambda \in \mathbb{C}_k^*/\mathbb{Z}^\nu\). We also introduce the weight \(\omega_k(f)\) of \(f\) defined by
\[
\omega_k(f) := \prod_{\lambda \in \mathbb{C}_k^*/\mathbb{Z}^\nu} \lambda^{\text{ord}_\lambda(f)} \in \mathbb{C}_k^*/\mathbb{Z}^\nu
\]
and its degree \(\deg_k(f)\) given by
\[
\deg_k(f) := \sum_{\lambda \in \mathbb{C}_k^*/\mathbb{Z}^\nu} \text{ord}_\lambda(f) \in \mathbb{Z}.
\]
Example 3.5. Consider $\theta = \theta(z;p)$ defined above. Then it follows from (3.3) that $\text{div}_1(\theta) = [1]$, since $\theta(z;p)$ has a zero of multiplicity one at each point of the subgroup $p^\mathbb{Z} \subset \mathbb{C}^*$. However, since $z = z^k$, we have that

$$\text{div}_k(\theta) = \sum_{i,j=0}^{k-1} \left[ \zeta_k^i \sqrt[p^j]{p} \right],$$

where $\zeta_k \in \mathbb{C}_k^*$ denotes a primitive $k$-th root of unity and $\sqrt[p^j]{p}$ is the $j$-th power of an arbitrary choice $\sqrt[p^j]{p}$ of $k$-th root of $p$.

Similarly, for any $f(z) \in M_p = M_{p,1}$ we have that $\text{div}_k(f) = \varphi_k^*(\text{div}_1(f))$, where $\varphi_k : C_1^*/p^2 \to C_1^*/p^2$ denotes the $k$-power map and $\varphi_k^*$ denotes the induced pull-back map on divisors.

3.3. Irreducibility of the $\sigma$-Galois groups. One of the assumptions of Theorem 2.4 concerns the irreducibility of the $\sigma$-Galois group. The main tool used in this paper in order to study the irreducibility of the $\sigma$-Galois group of (3.1) over $K$ is the following result.

Theorem 3.6 (Proposition 17 in [DR15]). Let $G$ be the $\sigma$-Galois group of (3.1) over $K$. The following statements are equivalent:

- the group $G$ is reducible;
- the following Riccati equation has a solution in $M_{p,2}$:

$$u(\sigma(u) + a) + b = 0. \tag{3.5}$$

Moreover, if $p_1 \in \Theta_2 \cup \{0\}$ and $p_2, p_3 \in \Theta_2$ are such that

$$a = \frac{p_1}{p_3} \quad \text{and} \quad b = \frac{p_2}{p_3},$$

then any solution $u \in M_{p,2}$ of (3.5) is of the form

$$u = \frac{\sigma(r_0) r_1}{r_0 r_2},$$

for some $r_0, r_1, r_2 \in \Theta_2$ such that

(i) $\text{div}_2(r_1) \leq \text{div}_2(p_2)$,
(ii) $\text{div}_2(r_2) \leq \text{div}_2(\sigma^{-1}(p_3))$,
(iii) $\deg_2(r_1) = \deg_2(r_2)$,
(iv) $\omega_2(r_1/r_2) = q_2^\deg_2(r_0) \mod p^2$.

4. Application to the elliptic hypergeometric functions

4.1. The elliptic hypergeometric functions. We shall now introduce the elliptic hypergeometric functions following [Spi16]. Consider $p, q \in \mathbb{C}^*$ such that $|p| < 1$, $|q| < 1$, and $q^2 \cap p^2 = \{1\}$. Consider

$$(z;p,q)_\infty = \prod_{j,k \geq 0} (1 - zp^j q^k) \quad \text{and} \quad \Gamma(z;p,q) = \frac{(pq/z;p,q)_\infty}{(z;p,q)_\infty}.$$ 

We have

$$\Gamma(pz;p,q) = \theta(z;q)\Gamma(z;p,q), \quad \Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q).$$
For \( t_1, \ldots, t_8 \in \mathbb{C}^* \) satisfying the balancing condition \( \prod_{j=1}^{8} t_j = p^2 q^2 \), we set

\[
V(t_1, \ldots, t_8; p, q) = \kappa \int_{\mathbb{T}} \prod_{j=1}^{8} \frac{\Gamma(t_j z; p, q) \Gamma(t_j z^2; p, q)}{\Gamma(z^2; p, q) \Gamma(z^{-2}; p, q)} \, dz,
\]

where \( \mathbb{T} \) denotes the positively oriented unit circle and \( \kappa = \frac{(p/q)_\infty (q;p)_\infty}{4\pi i} \). For \( z \in \mathbb{C}^* \), we follow [Spi16] by setting \( t_0 = cz \), \( t_7 = c/z \), and introducing new parameters

\[
\varepsilon_j = \frac{q}{ct_j}, \quad \text{for } j = 1, \ldots, 5, \quad \varepsilon_8 = \frac{c}{t_8}, \quad \varepsilon_7 = \frac{\varepsilon_8}{q}, \quad c = \frac{\sqrt{\varepsilon_6 \varepsilon_8}}{p^2}.
\]

We denote \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_8) \). Note that we still have the balancing condition

\[
\prod_{j=1}^{8} \varepsilon_j = p^2 q^2.
\]

**Definition 4.1.** The elliptic hypergeometric function is the meromorphic function on \( \mathbb{C}^* \) defined by the following formula

\[
f_\varepsilon(z) := \frac{V(q/c\varepsilon_1, \ldots, q/c\varepsilon_5, cz, c/z, c\varepsilon_8; p, q)}{\Gamma(c^2/z\varepsilon_8; p, q) \Gamma(z/\varepsilon_8; p, q) \Gamma(c^2/\varepsilon_8; p, q) \Gamma(1/z\varepsilon_8; p, q)}.
\]

**4.2. The elliptic hypergeometric equation.** The elliptic hypergeometric function \( f_\varepsilon(z) \) satisfies the following equation

\[
A(z)(y(qz) - y(z)) + A(z^{-1})(y(q^{-1}z) - y(z)) + \nu y(z) = 0,
\]

where

\[
A(z) = \frac{1}{\theta(z^2; p) \theta(qz^2; p)} \prod_{j=1}^{8} \theta(\varepsilon_j z; p) \quad \text{and} \quad \nu = \prod_{j=1}^{6} \theta(\varepsilon_j \varepsilon_8/q; p).
\]

It is easily seen that \( A(pz) = A(z) \), so that the previous equation has coefficients in \( M_{p,1} \).

Replacing \( z \) by \( qz \) in (4.2), we obtain the following equation:

\[
\sigma^2(y) + a\sigma(y) + by = 0,
\]

with \( a = \frac{\nu - A(qz) - A(q^{-1}z^{-1})}{A(qz)}, b = \frac{A(q^{-1}z^{-1})}{A(qz)} \in M_{p,1} \).

From now on, we denote by \( G \) the \( \sigma \)-Galois group of (4.3) over \( K \) (with respect to some \( \sigma \)-PV ring).

**4.3. Irreducibility of the \( \sigma \)-Galois group of the elliptic hypergeometric function.**

**Theorem 4.2.** Assume that every multiplicative relation among the \( \varepsilon_1, \ldots, \varepsilon_8, p, q \) is induced by (4.1), in the sense that if there are integers \( \alpha_1, \ldots, \alpha_8, m, n \) such that

\[
\prod_{j=1}^{8} \varepsilon_j^{\alpha_j} = p^m q^n
\]

then \( \alpha_1 = \cdots = \alpha_8 =: \alpha \) and \( m = n = 2\alpha \) for some \( \alpha \in \mathbb{Z} \). Then \( G \) is irreducible.
Proof. To the contrary, assume that $G$ is reducible. According to Theorem 3.6, the following Riccati equation has a solution $u \in \mathcal{M}_{p,2}$:

\begin{equation}
(4.4) 
 u\sigma(u) + a + b = 0.
\end{equation}

First, note that $u \in \mathcal{M}_{p,2}$ is a solution of $(4.4)$ if and only if $v(\sigma(v) + \sigma^{-1}(a)) + \sigma^{-1}(b) = 0$ with $v = \sigma^{-1}(u) \in \mathcal{K}$. Then to simplify the expression of the divisors of $a$ and $b$, we may replace them by $\sigma^{-1}(a) = \frac{\nu - A(z) - A(z^{-1})}{A(z)}$, $\sigma^{-1}(b) = \frac{A(z^{-1})}{A(z)}$, and consider the Riccati equation satisfied by $v$. Consider $p_1 \in \Theta_2 \cup \{0\}$ and $p_2, p_3 \in \Theta_2$ such that

\[
\sigma^{-1}(a) = \frac{p_1}{p_3} \text{ and } \sigma^{-1}(b) = \frac{p_2}{p_3}.
\]

In view of the explicit expressions for $\sigma^{-1}(a)$ and $\sigma^{-1}(b)$, we see that we may take $p_2$ and $p_3$ such that

\[
\text{div}_2(p_2) = \sum_{j=1}^{8} \left[ \sqrt{\varepsilon_j} \right] + \left[ -\sqrt{\varepsilon_j} \right] + \left[ \sqrt{p\varepsilon_j} \right] + \left[ -\sqrt{p\varepsilon_j} \right]
\]

\[
+ \sum_{j=0}^{3} \left[ \sqrt[p]{p^j/q} \right] + \left[ -\sqrt[p]{p^j/q} \right] + \left[ i\sqrt[p]{p^j/q} \right] + \left[ -i\sqrt[p]{p^j/q} \right]
\]

and

\[
\text{div}_2(p_3) = \sum_{j=1}^{8} \left[ \sqrt{1/\varepsilon_j} \right] + \left[ -\sqrt{1/\varepsilon_j} \right] + \left[ \sqrt{p/\varepsilon_j} \right] + \left[ -\sqrt{p/\varepsilon_j} \right]
\]

\[
+ \sum_{j=0}^{3} \left[ \sqrt{qp^j} \right] + \left[ -\sqrt{qp^j} \right] + \left[ i\sqrt{qp^j} \right] + \left[ -i\sqrt{qp^j} \right].
\]

We note for convenience that

\[
\text{div}_2(\sigma^{-1}(p_3)) = \sum_{j=1}^{8} \left[ \sqrt{q/\varepsilon_j} \right] + \left[ -\sqrt{q/\varepsilon_j} \right] + \left[ \sqrt{qp/\varepsilon_j} \right] + \left[ -\sqrt{qp/\varepsilon_j} \right]
\]

\[
+ \sum_{j=0}^{3} \left[ \sqrt[q]{qp^j} \right] + \left[ -\sqrt[q]{qp^j} \right] + \left[ i\sqrt[q]{qp^j} \right] + \left[ -i\sqrt[q]{qp^j} \right].
\]

We now consider $r_0, r_1, r_2 \in \Theta_2$ as in Theorem 3.6. For $i = 1, 2$, let

\[
\mathcal{S}_i := \{ \lambda \in \mathbb{C}_2^\ast/p\mathbb{Z} \mid \text{ord}_\lambda(r_i) \neq 0 \}
\]

denote the support of $\text{div}_2(r_i)$. For each $j \in \{1, \ldots, 8\}$ we let $\alpha_j \in \mathbb{N}$ denote the number of points in $\mathcal{S}_1$ of the form $\pm\sqrt{\varepsilon_j}$ or $\pm\sqrt{p\varepsilon_j}$. Similarly, for each $j \in \{1, \ldots, 8\}$ we let $\alpha'_j \in \mathbb{N}$ denote the number of points in $\mathcal{S}_2$ of the form $\pm\sqrt[q]{\varepsilon_j}$ or $\pm\sqrt[qp]{\varepsilon_j}$. We find that there exist $\ell_1, \ell_2 \in \{0, 1, 2, 3\}$ and $\gamma \in \mathbb{N}$ such that

\[
\omega_2(r_1/r_2) = i^{\ell_1} \sqrt[p]{2} \prod_{j=1}^{8} \sqrt[\varepsilon_j^{\alpha_j + \alpha'_j}] \sqrt[q]{-\text{deg}_2(r_2)} \sqrt[q]{\gamma} = \sqrt[q]{\text{deg}_2(r_0)} \mod p\mathbb{Z},
\]
where the second equality is obtained from property (iv) of Theorem 3.6. After taking fourth powers we see that

\[ \prod_{j=1}^{8} \xi_j^{2\alpha_j + 2\alpha_j'} = p^m q^{2\deg_2(r_2) + \gamma + 2\deg_2(r_0)} \]

for some \( m \in \mathbb{Z} \). Since every multiplicative relation among the \( \varepsilon_1, \ldots, \varepsilon_8, p, q \) is induced by (4.1), there exists \( \alpha \in \mathbb{N} \) such that \( 2\alpha_j + 2\alpha_j' = \alpha \) for every \( j \in \{1, \ldots, 8\} \) and \( m = 2\deg_2(r_2) + \gamma + 2\deg_2(r_0) = 2\alpha \). In particular, we have that \( 2\deg_2(r_2) \leq 2\alpha \). On the other hand, it follows from properties (i) and (ii) of Theorem 3.6, respectively, that \( \alpha_1 + \cdots + \alpha_8 \leq \deg_2(r_1) \) and \( \alpha_1' + \cdots + \alpha_8' \leq \deg_2(r_2) \). We note that by property (iii) of Theorem 3.6 \( 2\deg_2(r_2) = \deg_2(r_1) + \deg_2(r_2) \). Putting together these inequalities we obtain

\[ 4\alpha = \sum_{j=1}^{8} \alpha_j + \alpha_j' \leq \deg_2(r_1) + \deg_2(r_2) = 2\deg_2(r_2) \leq 2\alpha. \]

It follows from this that \( \alpha = \deg_2(r_1) = \deg_2(r_2) = 0 \). Hence, \( r_1/r_2 \) is constant and

\[ \omega_2(r_1/r_2) = 1 = \sqrt[2]{\deg_2(r_0)} \mod p \]

by property (iv) of Theorem 3.6. Since \( p \mathbb{Z} \cap q \mathbb{Z} = \{1\} \), we see that \( \deg_2(r_0) = 0 \) also.

It follows from the above that \( v \in \mathbb{C}^* \) is constant. Therefore (4.4) can be rewritten as

\[ v^2 A(z) + v(\nu - A(z) - A(z^{-1})) + A(z^{-1}) = 0, \]

i.e.

\[ (v^2 - v)A(z) + \nu v = (v - 1)A(z^{-1}). \]

But since \( \sqrt[2]{q}^{-1} \) is a pole of \( A(z) \) but not of \( A(z^{-1}) \) and, on the other hand, \( \sqrt[2]{q} \) is a pole of \( A(z^{-1}) \) but not of \( A(z) \), we obtain that \( v^2 - v = v - 1 = \nu v = 0 \), which is impossible because \( \nu \neq 0 \). This contradiction concludes the proof that \( G \) is irreducible. □

4.4. **Differential transcendence of the elliptic hypergeometric functions.** We may equip \( (K, \sigma) \) with the classical derivation \( \delta := z \frac{d}{dz} \) as in [DHRS18, Section 3.1]. Note that \( \delta \) commutes with \( \sigma \). Let \( \mathcal{C} \) be the \( \delta \)-closure of \( C \). Following Lemma 2.1, we may consider \( L := \text{Frac}(K \otimes_{\mathbb{C}} \mathcal{C}) \) and we have \( L^\sigma = \mathcal{C} \). Recall that \( f_\alpha(z) \) is meromorphic on \( \mathbb{C}^* \) and note that the latter field is a \((\sigma, \delta)\)-extension of \( K \).

**Theorem 4.3.** Assume that every multiplicative relation among the \( \varepsilon_1, \ldots, \varepsilon_8, p, q \) is induced by (4.1), in the sense that if there are integers \( \alpha_1, \ldots, \alpha_8, m, n \) such that

\[ \prod_{j=1}^{8} \xi_j^{\alpha_j} = p^m q^n \]

then \( \alpha_1 = \cdots = \alpha_8 =: \alpha \) and \( m = n = 2\alpha \) for some \( \alpha \in \mathbb{Z} \). Then \( f_\alpha(z) \) is differentially transcendental over \( K \).
Proof. According to Theorem 2.4, it is sufficient to prove that $G$ is irreducible and that the $(\sigma, \delta)$-Galois group of $\sigma y = \frac{A(q^{-1}z^{-1})}{A(qz)}$ over $L$ is $GL_1(\mathbb{C})$.

The irreducibility of $G$ was proved in Theorem 4.2. It remains to prove that the $(\sigma, \delta)$-Galois group of $\sigma y = by$ over $L$ is $GL_1(\mathbb{C})$. To the contrary, assume that it is not $GL_1(\mathbb{C})$. By Lemma 2.5, there exist a nonzero linear differential operator $L$ in $\delta$ with coefficients in $\mathbb{C}$ and $g \in K$ such that

$$L \left( \frac{\delta b}{b} \right) = \sigma(g) - g.$$  

Let $k \in \mathbb{N}^*$ such that $g \in M_{p,k}$ and consider $b$ as an element of $M_{p,k}$. Let $\omega \in \mathbb{C}_k^*/p\mathbb{Z}$ be a zero or a pole of $b$. Then it is a pole of $\frac{\delta b}{b}$. Since $L$ has constant coefficients, we get that $\omega$ is also a pole of $L \left( \frac{\delta b}{b} \right)$. Therefore, $\omega$ is a pole of $\sigma(g) - g$ and hence also a pole of $\sigma(g)$ or of $g$. Furthermore, $\sigma(g) - g$ has at least two distinct poles $\omega', \omega'' \in \mathbb{C}_k^*/p\mathbb{Z}$ such that $\omega \equiv \omega' \equiv \omega'' \pmod{q_k^2}$, where $q_k \in \mathbb{C}_k^*$ is as in (3.2). These $\omega'$ and $\omega''$ are poles of $\frac{\delta b}{b}$ and hence zeros or poles of $b$ has well. We have proved that, for every $\omega \in \mathbb{C}_k^*/p\mathbb{Z}$ that is a pole or zero of $b$, there exists $\ell \in \mathbb{Z}_{\neq 0}$ such that $\omega q^\ell_k$ is a pole or zero of $b$.

Let us now consider $b$ as an element of $M_{p,1}$. From the preceding, we deduce that for every $\omega \in \mathbb{C}_k^*/p\mathbb{Z}$, pole or zero of $b$, there exists $\ell \in \mathbb{Z}_{\neq 0}$ such that $\omega q^\ell_k$ is a pole or zero of $b$. We will use this to find a contradiction. Note that the set of zeros or poles of $b = \frac{\theta(q^{-1}z^{-1}; p)}{\theta(q^{-1}z^{-1}; p)} \times \prod_{j=1}^8 \frac{\theta(q^{-1}z^{-1}; p)}{\theta(q^{-1}z^{-1}; p)}$, seen as an element of $M_{p,1}$, is included in

$$S = \{q^{-1}z^{-1}, \ldots, q^{-1}z^{-1}, \pm q^{-1/2}, \pm q^{-1/2} \sqrt{p}, \pm q^{-3/2}, \pm q^{-3/2} \sqrt{p} \} \pmod{p\mathbb{Z}}.$$  

Furthermore, the elements of $S$ are all distinct since otherwise we would find a multiplicative relation among at most four elements among $p, q, \varepsilon_1, \ldots, \varepsilon_8$, contradicting the fact that every multiplicative relation among the $\varepsilon_1, \ldots, \varepsilon_8, p, q$ is induced by (4.1). Therefore, no simplifications occur and $S$ is exactly the set of zeros or poles of $b$. It suffices to show that for all $\ell \in \mathbb{Z}_{\neq 0}$, $S \cap \{q^\ell q^{-1} \varepsilon_1 \pmod{p\mathbb{Z}} \} = \emptyset$. Let $\ell \in \mathbb{Z}$ such that $S \cap \{q^\ell q^{-1} \varepsilon_1 \pmod{p\mathbb{Z}} \} \neq \emptyset$. If $\ell \neq 0$, then we find a multiplicative relation among at most four elements among $p, q, \varepsilon_1, \ldots, \varepsilon_8$. This contradicts the fact that every multiplicative relation among the $\varepsilon_1, \ldots, \varepsilon_8, p, q$ is induced by (4.1) and concludes the proof.$\blacksquare$

References

[Abr95] S. A. Abramov. Indefinite sums of rational functions. Proceedings of the 1995 International Symposium on Symbolic and Algebraic Computation, pages 303–308, 1995.

[Arr17] Carlos E Arreche. Computation of the difference-differential galois group and differential relations among solutions for a second-order linear difference equation. Communications in Contemporary Mathematics, 19(06):1650056, 2017.

[AS17] Carlos E Arreche and Michael F Singer. Galois groups for integrable and projectively integrable linear difference equations. Journal of Algebra, 480:423-449, 2017.
Shaoshi Chen and Michael F. Singer. Residues and telescopers for bivariate rational functions. *Adv. in Appl. Math.*, 49(2):111–133, 2012.

Thomas Dreyfus, Charlotte Hardouin, and Julien Roques. Functional relations of solutions of $q$-difference equations. *arXiv preprint arXiv:1603.06771*, 2016.

Thomas Dreyfus, Charlotte Hardouin, and Julien Roques. Hypertranscendence of solutions of mahler equations. *Journal of the European Mathematical Society (JEMS)*, 20(9):2209–2238, 2018.

Thomas Dreyfus, Charlotte Hardouin, Julien Roques, and Michael F Singer. Walks in the quarter plane, genus zero case. *arXiv preprint arXiv:1710.02848*, 2017.

Thomas Dreyfus, Charlotte Hardouin, Julien Roques, and Michael F Singer. On the nature of the generating series of walks in the quarter plane. *Inventiones mathematicae*, 213(1):139–203, 2018.

Thomas Dreyfus and Julien Roques. Galois groups of difference equations of order two on elliptic curves. *Symmetry, Integrability and Geometry: Methods and Applications*, 11(0):3–23, 2015.

Thomas Dreyfus and Kilian Raschel. Differential transcendence & algebraicity criteria for the series counting weighted quadrant walks. *arXiv preprint arXiv:1709.06831*, 2017.

J Fokko and Eric M Rains. Basic hypergeometric functions as limits of elliptic hypergeometric functions. *arXiv preprint arXiv:0902.0621*, 2009.

P. A. Hendriks. An algorithm for computing a standard form for second-order linear $q$-difference equations. *J. Pure Appl. Algebra*, 117/118:331–352, 1997. Algoritms for algebra (Eindhoven, 1996).

P. A. Hendriks. An algorithm determining the difference Galois group of second order linear difference equations. *J. Symbolic Comput.*, 26(4):445–461, 1997.

Charlotte Hardouin and Michael F. Singer. Differential Galois theory of linear difference equations. *Math. Ann.*, 342(2):333–377, 2008.

E. R. Kolchin. Constrained extensions of differential fields. *Advances in Math.*, 12:141–170, 1974.

Alphonse P Magnus et al. Elliptic hypergeometric solutions to elliptic difference equations. *SIGMA. Symmetry, Integrability and Geometry: Methods and Applications*, 5:038, 2009.

D. Mumford. *Tata lectures on theta. I*. Modern Birkhäuser Classics. Birkhäuser Boston, Inc., Boston, MA, 2007. With the collaboration of C. Musili, M. Nori, E. Previato and M. Stillman, Reprint of the 1983 edition.

Seiji Nishioka. Differential transcendence of solutions of difference riccati equations and tietze’s treatment. *Journal of Algebra*, 2018.

Marko Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. *Journal of symbolic computation*, 14(2-3):243–264, 1992.

Eric M Rains. Transformations of elliptic hypergeometric integrals. *Annals of Mathematics*, pages 169–243, 2010.

Julien Roques. On the algebraic relations between mahler functions. *Transactions of the American Mathematical Society*, 370(1):321–355, 2018.

Hjalmar Rosengren. Elliptic hypergeometric series on root systems. *arXiv preprint math/0207046*, 2002.

Vyacheslav P Spiridonov. Elliptic hypergeometric functions. *arXiv preprint arXiv:1610.01557*, 2016.

Heinrich Tietze. Über funktionalgleichungen, deren Lösungen keiner algebraischen differentialgleichung genügen können. *Monatshefte für Mathematik und Physik*, 16(1):329–364, 1905.
[vdB+07] Fokko van de Bult et al. Hyperbolic hypergeometric functions. University of Amsterdam, Amsterdam Netherlands, 2007.

[vdPS97] M. van der Put and M. F. Singer. Galois theory of difference equations, volume 1666 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1997.

The University of Texas at Dallas, Mathematical Sciences FO 35, 800 West Campbell Road, Richardson, TX 75024, USA
E-mail address: arreche@utdallas.edu

Institut de Recherche Mathématique Avancée, U.M.R. 7501 Université de Strasbourg et C.N.R.S. 7, rue René Descartes 67084 Strasbourg, FRANCE
E-mail address: dreyfus@math.unistra.fr

Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France
E-mail address: roques@math.univ-lyon1.fr