A COARSE CARTAN-HADAMARD THEOREM WITH APPLICATION TO THE COARSE BAUM-CONNES CONJECTURE

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Abstract. We establish a coarse version of the Cartan-Hadamard theorem, which states that proper coarsely convex spaces are coarsely homotopy equivalent to the open cones of their ideal boundaries. As an application, we show that such spaces satisfy the coarse Baum-Connes conjecture. Combined with the result of Osajda-Przytycki, it implies that systolic groups and locally finite systolic complexes satisfy the coarse Baum-Connes conjecture.

1. Introduction

The metric on a geodesic space $X$ is said to be convex if all geodesic segments $\gamma_1: [0, a_1] \to X$ and $\gamma_2: [0, a_2] \to X$ satisfy the inequality

$$\gamma_1(ta_1), \gamma_2(ta_2) \leq (1 - t) \gamma_1(0), \gamma_2(0) + t \gamma_1(a_1), \gamma_2(a_2),$$

for all $t \in [0, 1]$, where we denote by $x_1, x_2$ the distance between $x_1$ and $x_2$. This condition generalizes metric properties of simply connected complete Riemannian manifolds with non-positive sectional curvature. A geodesic space with a convex metric is also called a Busemann non-positively curved space. Unlike Gromov’s definition of hyperbolicity of metric spaces, convexity does not behave well under coarse equivalences of geodesic spaces even if we allow bounded errors in the inequality. Indeed, the 2-dimensional vector space $\mathbb{R}^2$ with the $l_1$-metric contains fat 2-gons, and so the $l_1$-metric is not convex, although it is coarsely equivalent to the $l_2$-metric, which is convex. An idea to overcome this problem is to consider a particular subfamily of geodesics.

Let $X$ be a metric space. Let $C \geq 0$ be a constant. Let $\mathcal{L}$ be a family of geodesic segments. The space $X$ is geodesic $(C, \mathcal{L})$-coarsely convex, if $C$ and $\mathcal{L}$ satisfy the following.

(i) For $v, w \in X$, there exists a geodesic segment $\gamma \in \mathcal{L}$ with $\gamma(0) = v$ and $\gamma(v, w) = w$.

(ii) Let $\gamma, \eta \in \mathcal{L}$ be geodesic segments such that $\gamma: [0, a] \to X$, $\eta: [0, b] \to X$. For $t \in [0, a]$, $s \in [0, b]$ and for $0 \leq c \leq 1$, we have that

$$\gamma(ct), \eta(cs) \leq c \gamma(t), \eta(s) + (1 - c) \gamma(0), \eta(0) + C.$$
We say that a metric space $X$ is a \textit{geodesic coarsely convex} space if there exist a constant $C$ and a family of geodesics $\mathcal{L}$ such that $X$ is geodesic $(C, \mathcal{L})$-coarsely convex.

Being geodesic coarsely convex is not invariant under coarse equivalence yet. In Section 3, we introduce an alternative definition, we say \textit{coarsely convex}, using quasi-geodesics, and show that it is invariant under coarse equivalence. We remark that geodesic coarsely convex spaces are coarsely convex spaces. For a coarsely convex space $X$, the \textit{ideal boundary}, denoted by $\partial X$, is a set of equivalence classes of quasi-geodesic rays which can be approximated by elements of $\mathcal{L}$, equipped with a metric given by the “Gromov product”.

Suppose that $N$ is a connected, simply connected, complete, Riemannian $n$-manifold with all sectional curvatures being less than or equal to zero. It follows from the Cartan-Hadamard theorem that $N$ is diffeomorphic to the Euclidean space $\mathbb{R}^n$. We remark that the ideal boundary of $N$ is homeomorphic to the $(n-1)$-sphere $S^{n-1}$, and $\mathbb{R}^n$ is regarded as the \textit{open cone} over $S^{n-1}$. The main result of this paper is a coarse geometric analogue of this theorem.

\textbf{Theorem 1.1.} Let $X$ be a proper coarsely convex space. Then $X$ is coarsely homotopy equivalent to $O\partial X$, the open cone over the ideal boundary of $X$.

The class of geodesic coarsely convex spaces includes geodesic Gromov hyperbolic spaces [14, §2, Proposition 25] and CAT(0)-spaces, more generally, Busemann non-positively curved spaces [5][24]. We remark that this class is closed under direct product, therefore, it includes products of these spaces. An important subclass of geodesic coarsely convex spaces is a class of \textit{systolic complexes}.

Systolic complexes are connected, simply connected simplicial complexes with combinatorial conditions on links. They satisfy one of the basic feature of CAT(0)-spaces, that is, the balls around convex sets are convex. This class of simplicial complexes was introduced by Chepoi [6] (under the name of \textit{bridged complexes}), and independently, by Januszkiewich-Świątkowski [19] and Haglund [16]. Osajda-Przytycki [23] introduced \textit{Euclidean geodesics}, which behave like CAT(0) geodesics, to construct boundaries of systolic complexes. Their result implies the following.

\textbf{Theorem 1.2 ([23, Corollary 3.3, 3.4])}. The 1-skeleton of systolic complexes are geodesic coarsely convex spaces.

A group is \textit{systolic} if it acts geometrically by simplicial automorphisms on a systolic complex. Osajda-Przytycki used their result to show that systolic groups admit $EZ$-structures. This implies the Novikov conjecture for torsion-free systolic groups. Now it is natural to ask whether systolic groups satisfy the \textit{the coarse Baum-Connes conjecture}.

Let $X$ be a proper metric space. The \textit{coarse assembly map} is a homomorphism from the \textit{coarse K-homology} of $X$ to the K-theory of the \textit{Roe-algebra} of $X$. The coarse Baum-Connes conjecture [17] states that for “nice” proper metric spaces, the coarse assembly maps are isomorphisms.
As a corollary of Theorem 1.1, we have the following.

**Theorem 1.3.** Let $X$ be a proper coarsely convex space. Then $X$ satisfies the coarse Baum-Connes conjecture.

The coarse Baum-Connes conjecture is known to be true for several classes of proper metric spaces. Examples of such classes are following.

1. Geodesic Gromov hyperbolic spaces [17][31].
2. Busemann non-positively curved spaces [17][31][12].
3. Direct products of geodesic Gromov hyperbolic spaces and Busemann non-positively curved spaces [11].
4. Metric spaces which admit coarse embeddings into the Hilbert space [33].

Theorem 1.3 covers examples (1), (2) and (3) in the above list. Combining it with Theorem 1.2, we obtain the following.

**Corollary 1.4.** Let $X$ be a locally finite systolic complex. Then $X$ satisfies the coarse Baum-Connes conjecture. Especially, systolic groups satisfy the coarse Baum-Connes conjecture.

Recently, Osajda-Huang [21] showed that Artin groups of almost large-type are systolic groups, and Osajda-Prytuła [22] showed that graphical small cancellation groups are systolic groups. We remark that large-type Artin groups are of almost large-type, and it is unknown whether these groups act geometrically on CAT(0)-spaces.

**Corollary 1.5.** Artin groups of almost large type and graphical small cancellation groups satisfy the coarse Baum-Connes conjecture.

Corollary 1.4 with a descent principle implies the Novikov conjecture for systolic groups. As already mentioned, it is known to be true [23]. In fact we can show the Novikov conjecture for wider classes of groups since the coarse Baum-Connes conjecture is stable under taking product with any polycyclic group and is studied well for relatively hyperbolic groups. Let $C$ be a class of groups consisting of direct products of hyperbolic groups, CAT(0)-groups, systolic groups, and polycyclic groups. Note that a polycyclic group with a word metric is not necessarily coarsely convex. We give details in Remark 6.11.

**Theorem 1.6.** Let a finitely generated group $G$ be one of the following:

1. a member of $C$,
2. a group which is hyperbolic relative to a finite family of subgroups belonging to $C$,
3. a group which is the direct product of a group as in (2) and a polycyclic group.

Then the group $G$ satisfies the coarse Baum-Connes conjecture. Moreover, if $G$ is torsion free, then $G$ satisfies the Novikov conjecture.
To the best knowledge of the authors, it is unknown whether each group \( G \) in Theorem 1.6 admits an \( EZ \)-structure or not.

Finally, we mention some algebraic properties of groups acting geometrically on coarsely convex spaces. These are direct consequences of semihyperbolicity of coarsely convex spaces and results of Alonso and Bridson \([1]\).

**Corollary 1.7.** Let \( G \) be a group acting on a coarsely convex spaces \( X \) properly and cocompactly by isometries. Then the following hold.

1. \( G \) is finitely presented and of type \( FP_\infty \).
2. \( G \) satisfies a quadratic isoperimetric inequality.

Moreover, suppose that a system of good quasi-geodesic segments \( \mathcal{L} \) of \( X \) is \( G \)-invariant, then

3. \( G \) has a solvable conjugacy problem.
4. Every polycyclic subgroup of \( G \) contains a finitely generated abelian subgroup of finite index.

**Remark 1.8.** It is already known that systolic groups satisfy all properties mentioned in Corollary 1.7 since Januszkiewich-Świątkowski \([10]\) proved that systolic groups are biautomatic.

The organization of the paper is as follows. In Section 2 we briefly review coarse geometry, and give the definition of coarse homotopy. In Section 3 we introduce coarsely convex spaces, and we show that it is invariant under coarse equivalence. In Section 4 we construct the ideal boundary, then we introduce the *Gromov product* to define a topology on the boundary. In Section 5 we give a proof of Theorem 1.1. In Section 6 we discuss on the relation with the coarse Baum-Connes conjecture. We give a proof of Theorem 1.3.

We also show that the coarse \( K \)-homology of a coarsely convex space is isomorphic to the reduced \( K \)-homology of its ideal boundary. Then we discuss on the direct product with polycyclic groups, and on relatively hyperbolic groups. In Section 7 we show that a coarsely convex space is semihyperbolic in the sense of Alonso-Bridson, and we mention that Corollary 1.7 follows from this fact. In Section 8 we give a functional analytic characterization of the ideal boundary. As a corollary, we obtain that the ideal boundary coincides with the combing corona in the sense of Engel and Wulff \([8]\).

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2. Coarse geometry

In this section we briefly review coarse geometry. For points \( v, w \in X \), we denote by \( \overline{v, w} \) the distance between \( v \) and \( w \). For \( r \geq 0 \) and for a subset \( K \subset X \), we denote by \( B_r(K) \) the closed \( r \)-neighbourhood of \( K \) in \( X \).

2.1. **Coarse map.** Let \( X, Y \) be metric spaces. Let \( f: X \to Y \) be a map.

1. The map \( f \) is **bornologous** if there exists a non-decreasing function \( \theta: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) such that for all \( x, x' \in X \), we have
   \[
   f(x), f(x') \leq \theta(x, x').
   \]

2. The map \( f \) is **proper** if for each bounded subset \( B \subset Y \), the inverse image \( f^{-1}(B) \) is bounded.

3. The map \( f \) is **coarse** if it is bornologous and proper.

For maps \( f, g: X \to Y \), we say that \( f \) and \( g \) are **close** if there exists a constant \( C \geq 0 \) such that \( f(x), g(x) \leq C \) for all \( x \in X \). A coarse map \( f: X \to Y \) is a **coarse equivalence map** if there exists a coarse map \( g: Y \to X \) such that the composites \( g \circ f \) and \( f \circ g \) are close to the identity \( \text{id}_X \) and \( \text{id}_Y \), respectively. We say that \( X \) and \( Y \) are **coarsely equivalent** if there exists a coarse equivalence map \( f: X \to Y \).

There exists a weaker equivalence relation between coarse maps, which plays an important role for an algebraic topological approach to the coarse Baum-Connes conjecture.

**Definition 2.1.** Let \( f, g: X \to Y \) be coarse maps between metric spaces. The maps \( f \) and \( g \) are **coarsely homotopic** if there exists a metric subspace \( Z = \{(x, t) : 0 \leq t \leq T_x\} \) of \( X \times \mathbb{R}_{\geq 0} \) and a coarse map \( h: Z \to Y \), such that

1. the map \( X \ni x \mapsto T_x \in \mathbb{R}_{\geq 0} \) is bornologous,
2. \( h(x, 0) = f(x) \), and
3. \( h(x, T_x) = g(x) \).

Here we equip \( X \times \mathbb{R}_{\geq 0} \) with the \( l_1 \)-metric, that is, \( d_{X \times \mathbb{R}_{\geq 0}}((x, t), (y, s)) := \overline{x, y} + |t - s| \) for \( (x, t), (y, s) \in X \times \mathbb{R}_{\geq 0} \).

Coarse homotopy is then an equivalence relation on coarse maps. A coarse map \( f: X \to Y \) is a **coarse homotopy equivalence map** if there exists a coarse map \( g: Y \to X \) such that the composites \( g \circ f \) and \( f \circ g \) are coarsely homotopic to the identity \( \text{id}_X \) and \( \text{id}_Y \), respectively. We say that \( X \) and \( Y \) are **coarsely homotopy equivalent** if there exists a coarse homotopy equivalence map \( f: X \to Y \).

2.2. **Quasi-isometry.** Let \( \lambda \geq 1 \) and \( k \geq 0 \) be constants. Let \( X \) and \( Y \) be metric spaces. We say that a map \( f: X \to Y \) is a \((\lambda, k)\)-**quasi-isometric embedding** if for all \( x, x' \in X \), we have

\[
\frac{1}{\lambda} \overline{x, x'} - k \leq f(x), f(x') \leq \lambda \overline{x, x'} + k.
\]
Let $X' \subset X$ be a subset. For $M \geq 0$, we say that $X'$ is $M$-dense in $X$ if $X = B_M(X')$. We say that a map $f : X \to Y$ is a quasi-isometry if there exist constants $\lambda, k, M$ such that $f$ is a $(\lambda, k)$-quasi-isometric embedding and the image $f(X)$ is $M$-dense in $Y$. We say that $X$ and $Y$ are quasi-isometric if there exists a quasi-isometry $f : X \to Y$.

A $(\lambda, k)$-quasi-geodesic in $X$ is a $(\lambda, k)$-quasi-isometric embedding $\gamma : I \to X$, where $I$ is a closed connected subset of $\mathbb{R}$. If $I = \mathbb{R}_{\geq 0}$, then we say that $\gamma$ is a $(\lambda, k)$-quasi-geodesic ray, and if $I = [0, a]$, then we say that $\gamma$ is a $(\lambda, k)$-quasi-geodesic segment.

A metric space $X$ is $(\lambda, k)$-quasi-geodesic if for all $x, y \in X$, there exists a $(\lambda, k)$-quasi-geodesic segment $\gamma : [0, a] \to X$ with $\gamma(0) = x$ and $\gamma(a) = y$. We say that a metric space $X$ is quasi-geodesic if there exist constants $\lambda$ and $k$ such that $X$ is $(\lambda, k)$-quasi-geodesic. The following criterion is well-known.

**Lemma 2.2.** Let $X$ and $Y$ be quasi-geodesic spaces. Then $X$ and $Y$ are coarsely equivalent if and only if $X$ and $Y$ are quasi-isometric.

**2.3. Open cone.** Let $M$ be a compact metrizable space. The open cone over $M$, denoted by $\mathcal{O}M$, is the quotient $\mathbb{R}_{\geq 0} \times M/(\{0\} \times M)$. For $(t, x) \in \mathbb{R}_{\geq 0} \times M$, we denote by $tx$ the point in $\mathcal{O}M$ represented by $(t, x)$.

Let $d_M$ be a metric on $M$. We assume that the diameter of $M$ is at most 2. We define a metric $d_{\mathcal{O}M}$ on $\mathcal{O}M$ by

$$d_{\mathcal{O}M}(tx, sy) := |t - s| + \min\{t, s\}d_M(x, y).$$

We call $d_{\mathcal{O}M}$ the induced metric by $d_M$.

**Remark 2.3.** When we take another metric $d'_M$ on $M$ such that the diameter of $M$ is at most 2, we have the induced metric $d'_{\mathcal{O}M}$ on $\mathcal{O}M$ by $d'_M$. Then the identity map $id_{\mathcal{O}M}$ between $(\mathcal{O}M, d_{\mathcal{O}M})$ and $(\mathcal{O}M, d'_{\mathcal{O}M})$ is not necessarily a coarse equivalence map, in fact, it is not necessarily a coarse homotopy equivalence map. Nevertheless a radial contraction gives a coarse homotopy equivalence map. We refer to [17] and [31].

**3. Coarsely convex space**

**Definition 3.1.** Let $X$ be a metric space. Let $\lambda \geq 1$, $k \geq 0$, $E \geq 1$, and $C \geq 0$ be constants. Let $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non-decreasing function. Let $\mathcal{L}$ be a family of $(\lambda, k)$-quasi-geodesic segments. The metric space $X$ is $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex, if $\mathcal{L}$ satisfies the following.

(i) For $v, w \in X$, there exists a quasi-geodesic segment $\gamma \in \mathcal{L}$ with $\gamma : [0, a] \to X$, $\gamma(0) = v$ and $\gamma(a) = w$.

(ii) Let $\gamma, \eta \in \mathcal{L}$ be quasi-geodesic segments with $\gamma : [0, a] \to X$ and $\eta : [0, b] \to X$.

Then for $t \in [0, a]$, $s \in [0, b]$, and $0 \leq c \leq 1$, we have that

$$\gamma(ct), \eta(cs) \leq cE \gamma(t), \eta(s) + (1 - c)E \gamma(0), \eta(0) + C.$$
Let $\gamma, \eta \in \mathcal{L}$ be quasi-geodesic segments with $\gamma: [0, a] \to X$ and $\eta: [0, b] \to X$.

Then for $t \in [0, a]$ and $s \in [0, b]$, we have

$$|t - s| \leq \theta(\gamma(0), \eta(0)) + \gamma(t), \eta(s)).$$

The family $\mathcal{L}$ satisfying $[\text{i}]^q$, $[\text{ii}]^q$, and $[\text{iii}]^q$ is called a system of good quasi-geodesic segments, and elements $\gamma \in \mathcal{L}$ are called good quasi-geodesic segments.

We say that a metric space $X$ is a coarsely convex space if there exist constants $\lambda, k, E, C$, a non-decreasing function $\theta: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and a family of $(\lambda, k)$-quasi-geodesic segments $\mathcal{L}$ such that $X$ is $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex.

We remark that if $\mathcal{L}$ consists of only geodesic segments, then $\mathcal{L}$ satisfies $[\text{iii}]^q$ by the triangle inequality. Therefore geodesic $(C, \mathcal{L})$-coarsely convex spaces are $(1, 0, 1, C, \text{id}_{\mathbb{R}_{\geq 0}}, \mathcal{L})$-coarsely convex. We also remark that Gromov [15, 6.B] mentioned the inequality in $[\text{ii}]^q$.

**Proposition 3.2.** Let $X$ and $Y$ be quasi-geodesic spaces such that $X$ and $Y$ are coarsely equivalent. If $X$ is coarsely convex, then so is $Y$.

**Proof.** Let $X$ and $Y$ be quasi-geodesic spaces such that $X$ and $Y$ are coarsely equivalent. There exist a map $f: X \to Y$ and $A \geq 1$ such that $f(X)$ is $A$-dense in $Y$, and for all $x, x' \in X$,

$$\frac{1}{A} \frac{x, x'}{x} - A \leq f(x), f(x') \leq A \frac{x, x'}{x} + A.$$

Suppose that $X$ is $(\lambda, k, E, C, \theta, \mathcal{L}_X)$-coarsely convex. For points $p, q \in Y$ and a path $\gamma: [0, a] \to X$, we define a path $\gamma_{p,q}: [0, a] \to Y$ by

$$\gamma_{p,q}(0) := p, \quad \gamma_{p,q}(a) := q, \quad \gamma_{p,q}(t) := f \circ \gamma(t) \text{ for } t \in (0, a).$$

If $p, f \circ \gamma(0) \leq A$, $q, f \circ \gamma(a) \leq A$ and $\gamma$ is a $(\lambda, k)$-quasi-geodesic segment, then $\gamma_{p,q}$ is a $(A\lambda, A(k+3))$-quasi-geodesic segment in $Y$. Thus we define a family of $(A\lambda, A(k+3))$-quasi-geodesic segments in $Y$, denoted by $\mathcal{L}_Y$, as a family consisting of all quasi-geodesic segments $\gamma_{p,q}$ where $p, q$ are points in $Y$, and $\gamma$ is a quasi-geodesic segment in $\mathcal{L}_X$ such that $p, f \circ \gamma(0) \leq A$ and $q, f \circ \gamma(a) \leq A$.

We will show that $\mathcal{L}_Y$ satisfies the conditions in Definition 3.1. It is clear that $[\text{i}]^q$ holds. We consider $[\text{ii}]^q$. Let $\gamma, \eta \in \mathcal{L}_Y$ be quasi-geodesic segments such that $\gamma: [0, a] \to Y$, and $\eta: [0, b] \to Y$. Then there exist $\gamma', \eta' \in \mathcal{L}_X$ such that

$$\gamma(0), f \circ \gamma'(0) \leq A, \quad \gamma(a), f \circ \gamma'(a) \leq A, \quad \gamma(t) = f \circ \gamma'(t) \text{ for } t \in (0, a),$$

$$\eta(0), f \circ \eta'(0) \leq A, \quad \eta(b), f \circ \eta'(b) \leq A, \quad \eta(s) = f \circ \eta'(s) \text{ for } s \in (0, b).$$
For $t \in [0, a]$, $s \in [0, b]$ and $0 \leq c \leq 1$, we have that
\[
\gamma(ct), \eta(cs) = \gamma(ct), f \circ \gamma'(ct) + f \circ \gamma'(ct), f \circ \eta'(cs) + f \circ \eta'(cs), \eta(cs)
\leq A \gamma'(ct), \eta'(cs) + 3A
\leq A \left\{ cE \gamma'(t), \eta'(s) + (1 - c)E \gamma'(0), \eta'(0) + C \right\} + 3A
\leq A \left\{ cE(A \gamma(t), \eta(s)) + 3A \\
+ (1 - c)E(A \gamma(0), \eta(0) + 3A^2) + C \right\} + 3A
\leq cA^2E \gamma(t), \eta(s) + (1 - c)A^2E \gamma(0), \eta(0) + 3A^2E + AC + 3A.
\]

Finally we consider (iii). Let $\gamma, \eta \in \mathcal{L}_Y$ and $\gamma', \eta' \in \mathcal{L}_X$ be as above. Then for all $t \in [0, a]$ and $s \in [0, b]$, we have
\[
|t - s| \leq \theta(\gamma'(0), \eta'(0) + \gamma'(a), \eta'(b))
\leq \theta(A(\gamma(0), \eta(0) + \gamma(a), \eta(b)) + 6A^2).
\]

\[\square\]

The class of coarsely convex spaces is closed under direct product.

**Proposition 3.3.** Let $(X, d_X)$ and $(Y, d_Y)$ be coarsely convex metric spaces. Then the the product with the $\ell_1$-metric $(X \times Y, d_{X \times Y})$ is coarsely convex. Indeed let $\mathcal{L}^X$ and $\mathcal{L}^Y$ be systems of good quasi-geodesic segments of $X$ and $Y$, respectively. Then for any quasi-geodesic segments $\gamma \in \mathcal{L}^X$ defined on $[0, a]$ and $\eta \in \mathcal{L}^Y$ defined on $[0, b]$, the map
\[
\frac{a}{a+b} \gamma \oplus \frac{b}{a+b} \eta : [0, a+b] \ni t \mapsto \left( \gamma\left(\frac{a}{a+b}t\right), \eta\left(\frac{b}{a+b}t\right) \right) \in X \times Y
\]
is a quasi-geodesic segment of $X \times Y$, and the family of such quasi-geodesic segments $\mathcal{L}^{X \times Y}$ is a system of good quasi-geodesic segments of $X \times Y$.

**Proof.** Let $X$ and $Y$ be metric spaces. Suppose that $X$ and $Y$ are $(\lambda, k, E, C, \theta, \mathcal{L}^X)$-coarsely convex and $(\lambda', k', E', C', \theta', \mathcal{L}^Y)$-coarsely convex, respectively. It is straightforward to check that the product $X \times Y$ is $(\max\{\lambda, \lambda'\}, k + k', \max\{E, E'\}, C + C', \theta + \theta', \mathcal{L}^{X \times Y})$-coarsely convex. \[\square\]

CAT(0) spaces, more generally, Busemann non-positively curved spaces, and geodesic Gromov hyperbolic spaces are examples of geodesic coarsely convex spaces. In these examples, the set of all geodesic segments is the system of good geodesic segments. In general, this does not hold. Let $\Gamma_{\mathbb{Z}^2}$ be the Cayley graph of rank 2 free abelian group $\mathbb{Z}^2$ with the standard generating set $\{(1, 0), (0, 1)\}$. Let $\gamma_n$ be a geodesic segment defined by $\gamma_n(t) := (t, 0)$ for $0 \leq t \leq n$ and $\gamma_n(t) := (n, t-n)$ for $t > n$. We fix any constant $E \geq 1$. 
Then for $n \in \mathbb{N}$, we have
\[
\frac{\gamma_0(n), \gamma_n(n)}{2E} = \frac{1}{2E}E \gamma_0(2En), \gamma_n(2En) = 2n - n = n \to \infty \quad (n \to \infty).
\]
Thus the set of all geodesic segments in $\Gamma_{Z^2}$ does not satisfy the condition (ii) in Definition 3.1. However, since $\mathbb{Z}^2$ is coarsely equivalent to $\mathbb{R}^2$, which is geodesic coarsely convex, by Proposition 3.2, $\mathbb{Z}^2$ is coarsely convex.

**Example 3.4.** Let $V$ be a normed vector space. Then $V$ is coarsely convex. Indeed, for $p, v \in V$ with $\|v\| = 1$, and for $r > 0$, we define a geodesic segment $\gamma(p, v; r) : [0, r] \to V$ by $\gamma(p, v; r)(t) := p + tv$. Let $L_{\text{Aff}}$ be the set of all geodesic segments $\gamma(p, v; r)$ with $p, v \in V$, $\|v\| = 1$ and $r > 0$.

Clearly $L_{\text{Aff}}$ satisfies (ii) in Definition 3.1. Since $L_{\text{Aff}}$ consists only of geodesics, it also satisfies (iii). For $p, v, w \in V$, $r, l > 0$ with $\|v\| = \|w\| = 1$, and for $t \in [0, r]$, $s \in [0, l]$, $c \in [0, 1]$, we have
\[
\|(p + ctv) - (p + csw)\| = c\|(p + tv) - (p + sw)\|.
\]
Now it is easy to show that $L_{\text{Aff}}$ satisfies (iii).

**Remark 3.5.** For a map $\gamma : [a, b] \to X$, we denote by $\gamma^{-1}$, the map $\gamma^{-1} : [a, b] \to X$ defined by $\gamma^{-1}(t) := \gamma(b - (t - a))$ for $t \in [a, b]$. For $c \in [a, b]$, we denote by $\gamma|_{[a,c]}$ the restriction of $\gamma$ to $[a, c]$. Let $L$ be a family of quasi-geodesic segments in $X$. The family $L$ is symmetric if $\gamma^{-1} \in L$ for all $\gamma \in L$, and $L$ is prefix closed if $\gamma|_{[a,c]} \in L$ for all $\gamma \in L$ with $\gamma : [a, b] \to X$ and for all $c \in [a, b]$.

Let $X$ be a $(\lambda, k, E, C, \theta, L)$-coarsely convex space. Suppose that $L$ is symmetric and prefix closed. Then the following holds. Let $\gamma, \eta \in L$ be $(\lambda, k)$-quasi-geodesic segments such that $\gamma : [0, a] \to X$ and $\eta : [0, b] \to X$. For $t_1, t_2 \in [0, a]$, $s_1, s_2 \in [0, b]$ and $0 \leq c \leq 1$, we have that
\[
\gamma((1-c)t_2), \eta(cs_2 + (1-c)s_1) \leq cE \gamma(t_2), \eta(s_2) + (1-c)E \gamma(t_1), \eta(s_1) + C.
\]
It seems natural to require that $L$ is symmetric and prefix closed in the definition of the coarsely convex space. However, in the proof of Theorem 1.1, we require neither condition.

Finally, we mention other generalizations of the spaces of non-positive curvature. Alonso and Bridson formulated a notion of *semihyperbolicity* for metric spaces, and studied groups acting on semihyperbolic spaces. In Section 7, we show that a coarsely convex space is semihyperbolic.

Kar [20] introduced and studied the class of metric spaces called *asymptotically CAT(0)-spaces*. This class and the class of coarsely convex spaces share many examples. Therefore it is desirable to clarify the relation between these two classes of metric spaces.
4. Ideal Boundary

Throughout this section, let $X$ be a $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex space. We will construct the ideal boundary $\partial X$ of $X$, as the set of equivalence classes of quasi-geodesic rays which can be approximated by quasi-geodesic segments in $\mathcal{L}$.

In this section, we introduce several constants and a function. Here we summarize them.

$$
\begin{align*}
    k_1 &= \lambda + k, \\
    D &= 2(1 + E)k_1 + C, \\
    \tilde{\theta}(t) &= \theta(t + 1) + 1, \\
    D_1 &= 2D + 2,
\end{align*}
$$

$$
D_2 = E(D_1 + 2k_1), \\
D_2' = \max\{1, E(\lambda(\theta(0)) + k)\}, \\
D_3 &= 2D_2'(D_2)^2, \\
D_4 &= 2E(1 + \lambda\tilde{\theta}(1) + 2k_1 + D_1).
$$

We remark that all constants in the above list are greater than or equal to 1. We also summarize several families of quasi-geodesic segments and rays related to $\mathcal{L}$.

We define $\mathcal{L}^\infty$ as the set of all $\mathcal{L}$-approximatable maps $\gamma : \mathbb{R}_{\geq 0} \to X$ with $\gamma(t) = \gamma([t])$ for all $t \in \mathbb{R}_{\geq 0}$, where $\mathcal{L}$-approximatable maps are defined in Section 4.1. Now let $O \in X$ be a base point. The following is the list of the families related to $\mathcal{L}$ and $\mathcal{L}^\infty$.

$$
\tilde{\mathcal{L}} := \mathcal{L} \cup \mathcal{L}^\infty, \\
\mathcal{L}_O^\infty := \{\gamma \in \mathcal{L}^\infty : \gamma(0) = O\}, \\
\mathcal{L}_O := \{\gamma \in \mathcal{L} : \gamma : [0, a] \to X, a \gamma \geq 2\theta(0), \gamma(0) = O\}, \\
\tilde{\mathcal{L}}_O := \mathcal{L}_O \cup \mathcal{L}_O^\infty.
$$

4.1. Approximatable Ray. Let $\gamma : \mathbb{R}_{\geq 0} \to X$ be a map. Let $\gamma_n : [0, a_n] \to X$ be quasi-geodesic segments in $X$. A sequence $\{(\gamma_n, a_n)\}_n$ is an $\mathcal{L}$-approximate sequence for $\gamma$ if for all $n$, we have $\gamma_n \in \mathcal{L}$, $\gamma_n(0) = \gamma(0)$, and for all $l \in \mathbb{N}$ the sequence $\{\gamma_n\}_n$ converges to $\gamma$ uniformly on $\{0, 1, \ldots, l\} \subset \mathbb{R}_{\geq 0}$. A map $\gamma : \mathbb{R}_{\geq 0} \to X$ is $\mathcal{L}$-approximatable if there exists an $\mathcal{L}$-approximate sequence for $\gamma$.

**Lemma 4.1.** Let $\gamma : \mathbb{R}_{\geq 0} \to X$ be an $\mathcal{L}$-approximatable map such that $\gamma(t) = \gamma([t])$ for all $t \in \mathbb{R}_{\geq 0}$. Then $\gamma$ is a $(\lambda, k_1)$-quasi-geodesic ray, where $k_1 := \lambda + k$.

**Proof.** Let $\gamma : \mathbb{R}_{\geq 0} \to X$ be an $\mathcal{L}$-approximatable map. Then there exists an $\mathcal{L}$-approximate sequence $\{(\gamma_n, a_n)\}_n$ for $\gamma$. We fix $t, s \in \mathbb{R}_{\geq 0}$. Set $i := \lfloor t \rfloor$ and $j := \lfloor s \rfloor$. Then for any $\epsilon > 0$, there exists an integer $n$ such that

$$
\frac{1}{\lambda} |i - j| - k \leq \frac{\gamma(i), \gamma_n(i)}{\gamma(j), \gamma_n(j)} < \epsilon.
$$

Since $\gamma_n$ is a $(\lambda, k)$-quasi-geodesic segment, we have

$$
\frac{1}{\lambda} |i - j| - k \leq \gamma_n(i), \gamma_n(j) \leq \lambda |i - j| + k.
$$
Then we have
\[
\frac{1}{\lambda} |t - s| - \frac{1}{\lambda} - k - 2\epsilon \leq \frac{\gamma(t) - \gamma(s)}{\lambda(t - s)} \leq \frac{\lambda |t - s| + \lambda + k + 2\epsilon}{\lambda}.
\]
Since \(\epsilon\) can be arbitrarily small, \(\gamma\) is a \((\lambda, \lambda + k)\)-quasi-geodesic ray. \(\square\)

We define a family of quasi-geodesic rays, denoted by \(L^\infty\), as a family consisting of all \(L\)-approximatable maps \(\gamma: \mathbb{R}_{\geq 0} \to X\) such that \(\gamma(t) = \gamma([t])\) for all \(t \in \mathbb{R}_{\geq 0}\). We set \(\bar{L} := L \cup L^\infty\). Let \(O \in X\) be a base point. Then we define \(L_O^\infty\) as the subset of \(L^\infty\) consisting of all quasi-geodesic rays in \(L^\infty\) stating at \(O\).

By an argument similar to that in the proof of Lemma 4.1, we have the following.

**Proposition 4.2.** Set \(I = [0, a]\) or \(I = \mathbb{R}_{\geq 0}\) and \(J = [0, b]\) or \(J = \mathbb{R}_{\geq 0}\). The family \(\bar{L}\) satisfies the following.

1. Let \(\gamma, \eta \in \bar{L}\) be quasi-geodesics with \(\gamma: I \to X\) and \(\eta: J \to X\). Then for \(t \in I\), \(s \in J\) and \(0 \leq c \leq 1\), we have
   \[
   \frac{\gamma(ct), \eta(cs)}{\lambda(t - s)} \leq cE\frac{\gamma(t), \eta(s)}{\lambda(t - s)} + (1 - c)E\frac{\gamma(0), \eta(0)}{\lambda(0)} + D,
   \]
   where \(D := 2(1 + E)k_1 + C\).

2. We define a non-decreasing function \(\tilde{\theta}: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}\) by \(\tilde{\theta}(t) := \theta(t + 1) + 1\). For \(\gamma, \eta \in \bar{L}\) with \(\gamma: I \to X\) and \(\eta: J \to X\), and for \(t \in I\), \(s \in J\), we have
   \[
   |t - s| \leq \tilde{\theta}(\gamma(0), \eta(0)) + \gamma(t), \eta(s)).
   \]

**Lemma 4.3.** Set \(I = [0, a]\) or \(I = \mathbb{R}_{\geq 0}\) and \(J = [0, b]\) or \(J = \mathbb{R}_{\geq 0}\). Let \(\gamma, \eta \in \bar{L}\) be quasi-geodesics such that \(\gamma: I \to X\) and \(\eta: J \to X\) with \(\gamma(0) = \eta(0)\). For all \(a \in I\), \(b \in J\) and \(0 \leq t \leq \min\{a, b\}\), we have
   \[
   \frac{\gamma(t), \eta(t)}{\lambda(t)} \leq E\frac{\gamma(a), \eta(b)}{\lambda} + \lambda\tilde{\theta}(\gamma(a), \eta(b)) + k_1 + D.
   \]

**Proof.** We suppose \(a \leq b\). Then
   \[
   \frac{\gamma(t), \eta(t)}{\lambda(t)} \leq E\gamma(a), \eta(a) + D
   \leq E\gamma(a), \eta(b) + \lambda|b - a| + k_1 + D
   \leq E\gamma(a), \eta(b) + \lambda\tilde{\theta}(\gamma(a), \eta(b)) + k_1 + D.
   \]

**Definition 4.4.** For quasi-geodesic rays \(\gamma\) and \(\eta\) in \(L^\infty\), we say that \(\gamma\) and \(\eta\) are equivalent if
   \[
   \sup\{\frac{\gamma(t), \eta(t)}{\lambda(t)} : t \in \mathbb{R}_{\geq 0}\} < \infty,
   \]
and we denote by \(\gamma \sim \eta\). For \(\gamma \in L^\infty\), we denote by \([\gamma]\) its equivalence class. The ideal boundary of \(X\) is the set \(\partial X := L^\infty/\sim\) of equivalence classes of quasi-geodesic rays in \(L^\infty\). The ideal boundary of \(X\) with respect to \(O\) is the set \(\partial_O X := L_O^\infty/\sim\) of equivalence classes of quasi-geodesic rays in \(L_O^\infty\).
Lemma 4.5. For \( \gamma, \eta \in \mathcal{L}_O^\infty \), if \([\gamma] = [\eta]\) then \( \overline{\gamma(t), \eta(t)} \leq D \) for all \( t \in \mathbb{R}_{\geq 0} \).

Proof. Let \( \gamma, \eta \in \mathcal{L}_O^\infty \) be quasi-geodesic rays. We suppose that there exists \( s > 0 \) such that \( \gamma(s), \eta(s) > D \). Then by Proposition 4.2 for \( 0 < c \leq 1 \), we have

\[
\overline{\gamma(s/c), \eta(s/c)} \geq \frac{1}{cE} (\overline{\gamma(s), \eta(s)} - D) \to \infty \quad (c \to 0).
\]

Thus we have \( \sup \{ \overline{\gamma(t), \eta(t)} : t \in \mathbb{R}_{\geq 0} \} = \infty \). \( \square \)

4.2. Gromov product. We define \( \mathcal{L}_O \) as the subset of \( \mathcal{L} \) consisting all quasi-geodesic segments \( \gamma \in \mathcal{L} \) with \( \gamma: [0, a_\gamma) \to X, a_\gamma \geq 2\theta(0) \) and \( \gamma(0) = O \). Set \( \tilde{\mathcal{L}}_O := \mathcal{L}_O \cup \mathcal{L}_O^\infty \).

Definition 4.6. We define a product \( \cdot | \cdot \): \( \tilde{\mathcal{L}}_O \times \tilde{\mathcal{L}}_O \to \mathbb{R}_{\geq 0} \cup \{ \infty \} \) as follows. Set \( I = [0, a] \) or \( I = \mathbb{R}_{\geq 0} \) and \( J = [0, b] \) or \( J = \mathbb{R}_{\geq 0} \). Then for \( \gamma, \eta \in \tilde{\mathcal{L}}_O \) with \( \gamma: I \to X \) and \( \eta: J \to X \), we define

\[
(\gamma \mid \eta)_O := \sup \{ t : t \in I \cap J, \overline{\gamma(t), \eta(t)} \leq D_1 \},
\]

where \( D_1 := 2D + 2 \). When it is clear which point is the base point \( O \), we write \( (\gamma \mid \eta) \) instead of \( (\gamma \mid \eta)_O \).

Lemma 4.7. Let \( \gamma, \eta \in \tilde{\mathcal{L}}_O \) be quasi-geodesics. Set \( a := (\gamma \mid \eta) \). If \( a < \infty \), then

\[
\overline{\gamma(a), \eta(a)} \leq D_1 + 2k_1.
\]

Proof. Let \( \gamma, \eta \in \tilde{\mathcal{L}}_O \) be quasi-geodesics. Set \( a := (\gamma \mid \eta) \). For any positive number \( \delta \) with \( 0 < \delta \leq a \), there exists a \( \delta' \) with \( 0 \leq \delta' \leq \delta \) and \( \overline{\gamma(a - \delta'), \eta(a - \delta')} \leq D_1 \). Now,

\[
\overline{\gamma(a - \delta'), \eta(a)} \leq \lambda \delta + k_1 \quad \text{and} \quad \overline{\eta(a - \delta'), \eta(a)} \leq \lambda \delta + k_1.
\]

Thus \( \overline{\gamma(a), \eta(a)} \leq D_1 + 2(\lambda \delta + k_1) \). Since \( \delta \) can be arbitrarily small, we have \( \overline{\gamma(a), \eta(a)} \leq D_1 + 2k_1 \). \( \square \)

Lemma 4.8. Set \( D_2 := E(D_1 + 2k_1) \). For \( \gamma, \eta, \xi \in \tilde{\mathcal{L}}_O \), we have

\[
(\gamma \mid \xi) \geq D_2^{-1} \min \{(\gamma \mid \eta), (\eta \mid \xi)\}.
\]

Proof. Set \( a := (\gamma \mid \eta) \) and \( b := (\eta \mid \xi) \). Set \( a' := D_2^{-1} \min \{a, b\} \). Then

\[
\overline{\gamma(a'), \eta(a')} \leq \frac{a'}{a} E \overline{\gamma(a), \eta(a)} + D \leq D_2^{-1} E(D_1 + 2k_1) + D = D + 1,
\]

\[
\overline{\eta(a'), \xi(a')} \leq \frac{a'}{b} E \overline{\eta(b), \xi(b)} + D \leq D_2^{-1} E(D_1 + 2k_1) + D = D + 1,
\]

\[
\overline{\gamma(a'), \xi(a')} \leq 2D + 2 = D_1.
\]

It follows that \( (\gamma \mid \xi) \geq a' \). \( \square \)

Lemma 4.9. Set \( D_2' := \max \{1, E(\lambda \theta(0) + k)\} \). We have the following.
(1) For $\gamma, \eta \in \mathcal{L}_O$ with $\gamma: [0, a] \to X$ and $\eta: [0, b] \to X$, if $\gamma(a) = \eta(b)$, then
\[
(\gamma \mid \eta) \geq D_2^{-1} \min\{a, b\}.
\]

(2) For $\gamma, \eta \in \mathcal{L}_O^\infty$, if $[\gamma] = [\eta]$, then
\[
(\gamma \mid \eta) = \infty.
\]

**Proof.** The statement (2) follows from Lemma 4.5. Thus we show (1). For $\gamma, \eta \in \mathcal{L}_O$ with $\gamma: [0, a] \to X$ and $\eta: [0, b] \to X$, we suppose $\gamma(a) = \eta(b)$. Set $d := \min\{a, b\}$. Then we have
\[
\gamma(d), \eta(d) \leq \lambda|a - b| + k \leq \lambda\theta(0) + k
\]
Then $\frac{\gamma(D_2^{-1}d), \eta(D_2^{-1}d)}{d} \leq 1 + D$. Thus $(\gamma \mid \eta) \geq D_2^{-1}d$. \(\square\)

**Lemma 4.10.** Set $D_3 := 2D_2'(D_2)^2$. We have the following.

1. Let $\gamma_1, \eta_1, \gamma_2, \eta_2 \in \mathcal{L}_O$ be quasi-geodesic segments with $\gamma_i: [0, a_i] \to X$ and $\eta_i: [0, b_i] \to X$ for $i = 1, 2$. If $\gamma_i(a_i) = \eta_i(b_i)$ for $i = 1, 2$, then
\[
D_3^{-1}(\gamma_1 \mid \gamma_2) \leq (\eta_1 \mid \eta_2) \leq D_3(\gamma_1 \mid \gamma_2).
\]

2. For quasi-geodesic rays $\gamma_1, \eta_1, \gamma_2, \eta_2 \in \mathcal{L}_O^\infty$, if $[\gamma_i] = [\eta_i]$ for $i = 1, 2$, then
\[
D_3^{-1}(\gamma_1 \mid \gamma_2) \leq (\eta_1 \mid \eta_2) \leq D_3(\gamma_1 \mid \gamma_2).
\]

3. Let $\gamma_1, \eta_1 \in \mathcal{L}_O$ be quasi-geodesic segments with $\gamma_1: [0, a_1] \to X$ and $\eta_1: [0, b_1] \to X$. Let $\gamma_2, \eta_2 \in \mathcal{L}_O^\infty$ be quasi-geodesic rays. If $\gamma_1(a_1) = \eta_1(b_1)$ and $[\gamma_2] = [\eta_2]$ then
\[
D_3^{-1}(\gamma_1 \mid \gamma_2) \leq (\eta_1 \mid \eta_2) \leq D_3(\gamma_1 \mid \gamma_2).
\]

**Proof.** We give a proof for the first statement.

Since $b_i \geq 2\theta(0)$ and $|a_i - b_i| \leq \theta(0)$ for $i = 1, 2$, we have $a_i \geq b_i/2$ for $i = 1, 2$. By Lemma 4.8 and Lemma 4.9
\[
(\gamma_1 \mid \gamma_2) \geq D_2^{-2} \min\{(\gamma_1 \mid \eta_1), (\eta_1 \mid \eta_2), (\eta_2 \mid \gamma_2)\}
\geq (D_2')^{-1} \min\{a_1, b_1, (\eta_1 \mid \eta_2), a_2, b_2\}
\geq (2D_2')^{-1}(\eta_1 \mid \eta_2).
\]
We can prove the rest of the statement in the same way. \(\square\)

**Definition 4.11.** We define a product $(\cdot \mid \cdot): (X \cup \partial O) \times (X \cup \partial O) \to \mathbb{R}_{\geq 0}$ as follows.

1. For $v, w \in X \cup \partial O$ with $v \in B_{\lambda_2\theta}(O)$ or $w \in B_{\lambda_2\theta}(O)$, we define
\[
(v \mid w) := 0.
\]
(1) For \( v, w \in X \setminus B_{\lambda \theta(0)+k}(O) \), we define
\[
(v \mid w) := \sup(\gamma \mid \eta),
\]
where the supremum is taken over all \( \gamma, \eta \in \mathcal{L}_O \) with \( \gamma: [0, a] \to X, \eta: [0, b] \to X, \gamma(a) = v \) and \( \eta(b) = w \).

(2) For \( x, y \in \partial_O X \), we define
\[
(x \mid y) := \sup(\gamma \mid \eta),
\]
where the supremum is taken over all \( \gamma, \eta \in \mathcal{L}_\infty^O \) such that \( x = [\gamma] \) and \( y = [\eta] \).

(3) For \( x \in \partial_O X \) and \( v \in X \setminus B_{\lambda \theta(0)+k}(O) \), we define
\[
(v \mid x) := \sup(\gamma \mid \eta),
\]
where the supremum is taken over all quasi-geodesic rays \( \eta \in \mathcal{L}_\infty^O \) with \( x = [\eta] \) and quasi-geodesic segments \( \gamma \in \mathcal{L}_O \) with \( \gamma: [0, a] \to X \) and \( v = \gamma(a) \). We define \( (x \mid v) := (v \mid x) \).

Lemma 4.10 implies the following.

**Lemma 4.12.** We have the following.

1. For \( v, w \in X \setminus B_{\lambda \theta(0)+k}(O) \) and for \( \gamma, \eta \in \mathcal{L}_O \) with \( \gamma: [0, a] \to X \) and \( \eta: [0, b] \to X \), if \( \gamma(a) = v \) and \( \eta(b) = w \), then
\[
(\gamma \mid \eta) \leq (v \mid w) \leq D_3(\gamma \mid \eta).
\]

2. For \( x, y \in \partial_O X \) and for \( \gamma, \eta \in \mathcal{L}_\infty^O \), if \( x = [\gamma] \) and \( y = [\eta] \), then
\[
(\gamma \mid \eta) \leq (x \mid y) \leq D_3(\gamma \mid \eta).
\]

3. For \( x \in \partial_O X \), \( v \in X \setminus B_{\lambda \theta(0)+k}(O) \), and for \( \eta \in \mathcal{L}_\infty^O \), \( \gamma \in \mathcal{L}_O \) with \( \gamma: [0, a] \to X \), if \( x = [\eta] \) and \( v = \gamma(a) \), then
\[
(\gamma \mid \eta) \leq (v \mid x) \leq D_3(\gamma \mid \eta).
\]

**Corollary 4.13.** For \( x, y, z \in (X \setminus B_{\lambda \theta(0)+k}(O)) \cup \partial_O X \), we have
\[
(x \mid z) \geq (D_2 D_3)^{-1} \min\{(x \mid y), (y \mid z)\}.
\]

**Lemma 4.14.** Let \( \gamma \in \mathcal{L}_\infty^O \) be a quasi-geodesic ray and let \( \{\gamma_n, a_n\}_n \) be an \( \mathcal{L} \)-approximate sequence for \( \gamma \). Then we have \( \lim\inf_{n \to \infty} (\gamma \mid \gamma_n) = \infty \).

**Proof.** Let \( \gamma \in \mathcal{L}_\infty^O \) be a quasi-geodesic ray and let \( \{\gamma_n, a_n\}_n \) be an \( \mathcal{L} \)-approximate sequence. Then for \( R \in \mathbb{N} \), there exists \( N > 0 \) such that for all \( n > N \), we have \( (\gamma(R), \gamma_n(R) < 1 \leq D_1 \). Thus \( (\gamma \mid \gamma_n) \geq R \). \( \square \)
Lemma 4.15. Let $v, w \in X \setminus B_{2\theta(0)+k}(O)$ be points. Let $\eta \in \mathcal{L}_O^\infty$ be a quasi-geodesic ray and $\gamma_v, \gamma_w \in \mathcal{L}_O$ be quasi-geodesic segments such that $\gamma_v: [0, a_v] \to X$, $\gamma_w: [0, a_w] \to X$, $\gamma_v(a_v) = v$ and $\gamma_w(a_w) = w$. Then we have

\[(\gamma_w | \eta) \geq \frac{(\gamma_w | \eta) - \theta(\overline{v, w})}{E(E(\overline{v, w} + \lambda\theta(\overline{v, w}) + k_1) + D_1 + D + 2k_1)}.
\]

Proof. We denote by $S$ the right hand side of (1). Set $a := (\gamma_v | \eta)$ and $b := \min\{a_v, a_w\}$. Since $b \geq a_v - |a_v - a_w| \geq (\gamma_v | \eta) - \theta(\overline{v, w})$, we have $\min\{a, b\} \geq S$. Then

\[
\frac{S}{a} E \eta(a), \gamma_v(a) + \frac{S}{b} E \gamma_v(b), \gamma_w(b) + 2D
\]

\[
\leq \frac{S}{a} E(D_1 + 2k_1) + \frac{S}{b} E(E(\overline{v, w} + \lambda\theta(\overline{v, w}) + k_1) + D) + 2D
\]

\[
\leq 2 + 2D = D_1.
\]

This completes the proof. \qed

Corollary 4.16. Set $D_4 := 2E(1 + \lambda\theta(1) + k_1) + D_1 + D + 2k_1)$. For $x \in \partial X$ and $v, w \in X \setminus B_{2\theta(0)+k}(O)$, if $(v | x) \geq 2D_3\theta(1)$ and $\overline{v, w} \leq 1$, then we have $(w | x) \geq (D_3D_4)^{-1}(v | x)$.

4.3. Topology on $X \cup \partial O X$. For all positive integers $n \geq 1$, we set

\[
V_n := \{(x, y) \in \partial O X \times \partial O X : (x | y) > n\}
\]

\[
\cup \{(x, v) \in \partial O X \times X : (v | x) > n\}
\]

\[
\cup \{(v, x) \in X \times \partial O X : (v | x) > n\}
\]

\[
\cup \{(v, w) \in X \times X : (v | w) > n\}
\]

\[
\cup \{(v, w) \in X \times X : \overline{v, w} < n^{-1}\}.
\]

For given $n \in \mathbb{N}$, we take $m \in \mathbb{N}_{>0}$ with $m > D_2D_3D_4(\theta(1) + 1)n$. Then by Corollary 4.13 and Corollary 4.16 for all $(p, q) \in V_m$ and $(q, r) \in V_m$, we have $(p, r) \in V_n$. It follows that the family $\{V_n\}_{n \in \mathbb{N}}$ forms a fundamental system of entourages of a uniform structure on $X \cup \partial O X$. (see [2, Chapter II, §1.1]), which is metrizable (see [3, Chapter IX, §2.4]).

We remark that for $x \in X \cup \partial O X$, the family $\{V_n[x]\}_{n \in \mathbb{N}}$ is a fundamental system of neighbourhoods of $x$. Here $V_n[x]$ is defined by

\[
V_n[x] := \{y \in X \cup \partial O X : (x, y) \in V_n\}.
\]

For $v \in X$, if $n > \lambda(\overline{O, v} + k_1)$, then the set $\{(v, y) \in X \times (X \cup \partial O X) : (v | y) > n\}$ is empty. Thus $V_n[x] = \{w \in X : \overline{v, w} < n^{-1}\}$. It follows that the inclusion $X \hookrightarrow X \cup \partial O X$ is a topological embedding.
4.4. Construction of quasi-geodesic rays. From now on, we always assume that the coarsely convex space $X$ is proper, that is, all closed bounded subsets are compact.

For a sequence $\{v_n\}$ in $X$ which goes to infinity, we will construct a sequence $\{N_n\}_n$ in $\mathbb{N}$ and a sequence of quasi-geodesic segments $\gamma_{N_n} \in \mathcal{L}_O$ connecting $O$ to $v_{N_n}$, which converges to a quasi-geodesic ray uniformly on every finite subsets of $\mathbb{N}$.

**Proposition 4.17.** Let $\{v_n\}_n$ be a sequence in $X$ such that $\lim_{n \to \infty} \overline{O, v_n} = \infty$. Then there exists a $(\lambda, k_1)$-quasi-geodesic ray $\gamma \in \mathcal{L}_O^\infty$ starting at $O$, and a sequence $\{N_n\}_n$ in $\mathbb{N}$ such that $\liminf_{n \to \infty} (v_{N_n} \mid [\gamma]) = \infty$.

**Proof.** Let $\{v_n\}_n$ be a sequence in $X$ such that $\lim_{n \to \infty} \overline{O, v_n} = \infty$. We choose $(\lambda, k)$-quasi-geodesic segments $\gamma_n : [0, a_n] \to X$ in $\mathcal{L}_O$ such that $\gamma_n(a_n) = v_n$.

By induction, for all $l \geq 0$, we will construct a subsequence $\{\gamma[l; n]\}_n$ of $\{\gamma_n\}_n$, and a sequence $\{v_i^\infty\}_i$ in $X$ with $v_0^\infty = O$ satisfying the following.

1. $\gamma[0; n] = \gamma_n$ for all $n \geq 0$.
2. For all $l \geq 0$, the sequence $\{\gamma[l + 1; n]\}_n$ is a subsequence of $\{\gamma[l; n]\}_n$.
3. For all $l \geq 0$, the sequence $\{\gamma[l; n]\}_n$ converges uniformly on $\{0, 1, \ldots, l\}$ to the map $t \mapsto v_i^\infty \in X$.

First, we define $\gamma[0, n] = \gamma_n$ for all $n \geq 0$. Now we suppose that we have constructed a sequence $v_0^\infty, v_1^\infty, \ldots, v_l^\infty$ and a family of subsequences $\{\gamma[0; n]\}_n, \ldots, \{\gamma[l; n]\}_n$ satisfying the above conditions. Since $\gamma[l; n]$ is a $(\lambda, k)$-quasi-geodesic segment for all $n \geq 0$, we have

$$
\overline{O, \gamma[l, n](l + 1)} \leq \lambda(l + 1) + k
$$

for all $n \geq 0$. By the properness of $X$, there exits a sequence $\{N_n^l\}_n$ of integers such that the sequence $\{\gamma[l, N_n^l](l + 1)\}_n$ converges. Thus we set $v_i^\infty := \lim_{n \to \infty} \gamma[l, N_n^l](l + 1)$ and set $\gamma[l + 1, n] := \gamma[l, N_n^l]$. Then the sequence of the maps $\{\gamma[l + 1; n]\}_n$ converges uniformly on $\{0, 1, \ldots, l + 1\}$ to the map $i \mapsto v_i^\infty$.

Now we define a map $\gamma : \mathbb{R}_{\geq 0} \to X$ by $\gamma(t) := v_i^\infty$. We claim that for all $l \in \mathbb{N}$, the sequence of maps $\{\gamma[n; n]\}_n$ converges uniformly on $\{0, 1, \ldots, l\}$ to the map $\gamma$. We fix $l \in \mathbb{N}$. Let $m$ be any integer with $m > l$. Since $\{\gamma[m; n]\}_n$ is a subsequence of $\{\gamma[l; n]\}_n$, for all $a \in \mathbb{N}$, there exists $k(l, m, a) \in \mathbb{N}$ such that $\gamma[m; a] = \gamma[l; k(l, m, a)]$. We remark that the map $a \mapsto k(l, m, a)$ is increasing. By (3), for any $\epsilon > 0$, there exists $n(l) \in \mathbb{N}$ such that for all $n > n(l)$ and $i \in \{0, 1, \ldots, l\}$, we have $v_i^\infty, \gamma[l, n](i) < \epsilon$. Then let $n$ be an integer with $n > \max\{l, n(l)\}$. Since $k(l, n, n) > n(l)$, we have

$$
\overline{v_i^\infty, \gamma[n; n](i)} = \overline{v_i^\infty, \gamma[l; k(l, n, n)]} < \epsilon
$$

for all $i \in \{0, 1, \ldots, l\}$. This completes the proof of the claim.
For \( n \in \mathbb{N} \), let \( N_n \) be the integer such that \( \gamma_{N_n} = \gamma[n;n] \). It follows that \( \{(\gamma_{N_n}, a_{N_n})\}_n \) is an \( \mathcal{L} \)-approximate sequence for \( \gamma \). Thus \( \gamma \in \mathcal{L}_{\partial}^\infty \) and by Lemma 4.1, \( \gamma \) is a \((\lambda, k_1)\)-quasi-geodesic ray. By the construction, we have \( \gamma_{N_n}(a_{N_n}) = v_{N_n} \). Since \( (v_{N_n} | [\gamma]) \geq (\gamma_{N_n} | \gamma) \), by Lemma 4.14 we have \( \liminf_{n \to \infty} (v_{N_n} | [\gamma]) = \infty \).

**Proposition 4.18.** For a proper coarsely convex space \( X \), the uniform space \( X \cup \partial_O X \) is compact.

**Proof.** Since \( X \cup \partial_O X \) is metrizable, it is enough to show that every infinite sequence of points has a convergent subsequence. Let \( \{p_n\}_n \) be a sequence in \( X \cup \partial_O X \). By choosing a subsequence, we can assume either of the following holds.

(a) \( p_n \in X \) for all \( n \).
(b) \( p_n \in \partial_O X \) for all \( n \).

First we consider the case (a). We can suppose \( \lim_n \overline{O,p_n} = \infty \). By Proposition 4.17, there exists a quasi-geodesic ray \( \gamma \in \mathcal{L}_{\partial}^\infty \), and a sequence \( \{N_n\}_n \) in \( \mathbb{N} \) such that \( \liminf(p_{N_n} | [\gamma]) = \infty \). This shows that the subsequence \( \{p_{N_n}\}_n \) converges to \( [\gamma] \).

Next we consider the case (b). We choose quasi-geodesic rays \( \eta_n \in \mathcal{L}_{\partial}^\infty \) such that \( p_n = [\eta_n] \). For each \( n \in \mathbb{N} \), we set \( v_n := \eta_n(n) \).

Let \( \eta'_n \in \mathcal{L}_O \) be a quasi-geodesic segment such that \( \eta'_n : [0, a_n] \to X \) and \( \eta'_n(a_n) = v_n \). Since \( \eta'_n(a_n), \eta_n(a_n) = \eta_n(n), \eta_n(a_n) \leq \lambda(\theta(0)) + k_1 \), we have

\[
(v_n | p_n) \geq (\eta'_n | \eta_n) \geq \frac{a_n}{E(\max\{\lambda(\theta(0)) + k_1, 1\})} \geq \frac{\overline{O,v_n} - k}{\lambda E(\lambda(\theta(0)) + k_1 + 1)} \to \infty.
\]

By Proposition 4.17, there exists a quasi-geodesic ray \( \gamma \in \mathcal{L}_{\partial}^\infty \), and a sequence \( \{N_n\}_n \) in \( \mathbb{N} \) such that \( \liminf(v_{N_n} | [\gamma]) = \infty \). By Lemma 4.13, we have

\[
(p_{N_n} | [\gamma]) \geq (D_2D_3)^{-1} \min\{(v_{N_n} | p_{N_n}), (v_{N_n} | [\gamma])\} \to \infty.
\]

This shows that the subsequence \( \{p_{N_n}\}_n \) converges to \( [\gamma] \).

4.5. **Metric on the ideal boundary.** Let \( \epsilon > 0 \) be a positive number. For \( x, y \in \partial_O X \), we define \( \rho_{\epsilon}(x, y) := (x | y)^{-\epsilon} \). It immediately follows that

1. for \( x, y \in \partial_O X \), we have \( \rho_{\epsilon}(x, y) = 0 \) if and only if \( x = y \),
2. for \( x, y \in \partial_O X \), we have \( \rho_{\epsilon}(x, y) = \rho_{\epsilon}(y, x) \),
3. for \( x, y, z \in \partial_O X \), we have

\[
\rho_{\epsilon}(x, z) \leq (D_2D_3)^{\epsilon} \max\{\rho_{\epsilon}(x, y), \rho_{\epsilon}(y, z)\}.
\]

Therefore, \( \rho_{\epsilon} \) is a quasi-metric. There exists a standard method, so called the chain construction, to obtain a metric which is equivalent to \( \rho_{\epsilon} \). For detail, see [30].

**Proposition 4.19.** Let \( \epsilon \) be a positive number such that \( (D_2D_3)^{\epsilon} \leq 2 \). Then there exists a metric \( d_{\epsilon} \) on \( \partial X \) such that \( 1/(2K)\rho_{\epsilon}(x, y) \leq d_{\epsilon}(x, y) \leq \rho_{\epsilon}(x, y) \) for all \( x, y \in \partial_O X \), where \( K := (D_2D_3)^{\epsilon} \).
4.6. Replacement of the base point. Here we will construct a map \( \Phi_O : \partial X \to \partial_O X \).

Let \( \gamma \in \mathcal{L}^\infty \) be a quasi-geodesic ray. Set \( v_n := \gamma(n) \) for \( n \in \mathbb{N} \). By Proposition 4.17, there exists a quasi-geodesic ray \( \gamma_O \in \mathcal{L}^\infty_O \) and a sequence \( \{N_n\}_n \) in \( \mathbb{N} \) such that

\[
\liminf(v_{N_n} | [\gamma_O]) = \infty.
\]

We define a map \( \Phi_O : \partial X \to \partial_O X \) by \( \Phi_O([\gamma]) := [\gamma_O] \) for \( \gamma \in \mathcal{L}^\infty \). By the following lemma, the map \( \Phi_O \) is well-defined.

**Lemma 4.20.** Let \( O' \in X \) be a point. For a quasi-geodesic ray \( \gamma = \mathcal{L}^\infty_O \) starting at \( O' \), we have

\[
\sup \{ \gamma(t), \gamma_O(t) : t \in \mathbb{R}_{\geq 0} \} \leq C_{OO'},
\]

where \( C_{OO'} := E(\lambda(\hat{\theta}(O,O')) + \overline{O,O'} + D_1 + 3k_1) + 2D \).

**Proof.** Let \( \gamma \in \mathcal{L}^\infty_O \) be a quasi-geodesic ray. Set \( v_n := \gamma(n) \) for \( n \in \mathbb{N} \). By Proposition 4.17, for any \( R > 0 \) there exists \( N \) such that \( (v_N | [\gamma_O]) \geq RD_3 \). Let \( \gamma_N \in \mathcal{L}_O \) be a quasi-geodesic segment such that \( \gamma_N : [0,a_N] \to X \) and \( \gamma_N(a_N) = v_N \). Set \( a := (\gamma_N | \gamma_O) \). Then \( a \geq R \) by Lemma 4.12 and \( \gamma_N(a), \gamma_O(a) \leq D_1 + 2k_1 \) by Lemma 4.7. Thus it follows that for all \( t \in [0,R] \), we have

\[
(2) \quad \gamma_N(t), \gamma_O(t) \leq E(\gamma_N(a), \gamma(a)) + D \leq E(D_1 + 2k_1) + D.
\]

Since \( \gamma_N(a_N) = v_N = \gamma(N) \), for all \( t \in [0,a_N] \),

\[
\begin{align*}
\gamma(t), \gamma_N(t) &\leq E(\gamma(a_N), \gamma_N(a_N) + \overline{O,O'}) + D \\
&\leq E(\gamma(N), \gamma_N(a_N) + |N - a_N| + k_1 + \overline{O,O'}) + D \\
&\leq E(\lambda(\hat{\theta}(O,O')) + k_1 + \overline{O,O'}) + D.
\end{align*}
\]

Combined with (2), we have \( \gamma(t), \gamma_O(t) \leq C_{OO'} \) for all \( t \in [0,R] \). Since \( R \) is arbitrary, we complete the proof of the Lemma.

**Corollary 4.21.** The map \( \Phi_O : \partial X \to \partial_O X \) is bijective.

We equip \( \partial X \) with the topology such that the map \( \Phi_O \) is a homeomorphism. This topology does not depend on the choice of \( O \). Indeed, by following lemmata, we can show that the composite \( \Phi_{OO'} := \Phi_O \circ \Phi_O^{-1} \) is continuous.

**Lemma 4.22.** Set \( D_{OO'} := E(E(D_1 + 2k_1) + D + 2C_{OO'}) \). For \( \gamma, \eta \in \mathcal{L}^\infty_O \), we have

\[
(\gamma_O | \eta_O) \geq D_{OO'}^{-1}(\gamma | \eta)_O.
\]
Proof. Let \( t > 0 \) be any positive number with \((\gamma | \eta)|_O > t\). Then we have \( \gamma(t), \eta(t) \leq E(D_1 + 2k_1) + D\). By Lemma 4.20, we have

\[
\frac{\gamma_O(t), \eta_O(t)}{\leq \frac{\gamma_O(t), \gamma(t) + \gamma(t), \eta(t) + \eta(t), \eta_O(t)}{\leq E(D_1 + 2k_1) + D + 2C_{OO'}}.
\]

Then \( \gamma_O(D_{OO'}^{-1}t), \eta_O(D_{OO'}^{-1}t) \leq D + 1 \leq D_1\). Thus we have \((\gamma_O | \eta_O)|_O \geq D_{OO'}^{-1}t\). Since \( t \)
is any positive number with \((\gamma | \eta)|_O \geq t\), we have \((\gamma_O | \eta_O)|_O \geq D_{OO'}^{-1}(\gamma | \eta)|_O\).

By the same argument in the proof of the above lemma, we have the following.

Lemma 4.23. For \( \gamma \in L_{OO'}^\infty \) and \( v \in X \), we have

\[
(\gamma_O | v)_O \geq D_{3}^{-1}D_{OO'}^{-1}(\gamma | v)|_O .
\]

Corollary 4.24. The map \( \Phi_{OO'} : X \cup \partial_O X \to X \cup \partial_O X \) defined as an extension by the identity on \( X \) of the map \( \Phi_{OO'} : \partial_O X \to \partial_O X \) is a homeomorphism.

Proof. By Corollary 4.21, \( \Phi_{OO'} \) is a bijection between the compact metrizable spaces. By Lemma 4.22 and Lemma 4.23, the map \( \Phi_{OO'} \) is continuous, therefore it is a homeomorphism.

Corollary 4.25. Let \( G \) be a group and let \( X \) be a \((\lambda, k, E, C, \theta, \mathcal{L})\)-coarsely convex space. We suppose that \( G \) acts on \( X \) properly and cocompactly by isometries, and \( \mathcal{L} \) is invariant under the action of \( G \). Then the action of \( G \) extends continuously to the ideal boundary \( \partial X \).

4.7. Examples. Let \( X \) be a proper geodesic Gromov hyperbolic space and let \( \mathcal{L} \) be a set of all geodesic segments. Then \( X \) is a coarsely convex space with the system of good geodesic segments \( \mathcal{L} \). The Gromov boundary of \( X \) is homeomorphic to the ideal boundary \( \partial X \). In fact, the Gromov boundary is identical to \( \partial X \) as a set. It is easy to show that usual topology of the Gromov boundary coincides with the one given in Section 4.3.

Next we consider the ideal boundary of the Euclidean plane \( \mathbb{R}^2 \). Let \( \mathcal{L}_{\mathbb{R}^2} \) be a set of all geodesic segments in \( \mathbb{R}^2 \). Then \( \mathbb{R}^2 \) is a coarsely convex space with the system of good geodesic segments \( \mathcal{L}_{\mathbb{R}^2} \). We consider the visual compactification of \( \mathbb{R}^2 \). Namely, we define an embedding \( \varphi : \mathbb{R}^2 \to D^2 = \{ v \in \mathbb{R}^2 : \| v \| \leq 1 \} \) by \( \varphi(v) = v/(1 + \| v \|) \) for \( v \in \mathbb{R}^2 \). We can identify the ideal boundary \( \partial \mathbb{R}^2 \) with \( S^1 \subset D^2 \) as a set. For \( x \in S^1 \subset D^2 \), we define a geodesic ray \( \eta_x : \mathbb{R}_{\geq 0} \to \mathbb{R}^2 \) by \( \eta_x(t) = tx \). Now for \( x, y \in S^1 \), let \( \theta \) be the angle between \( \eta_x \) and \( \eta_y \). Then we have

\[
\sin \frac{\theta}{2} = \frac{D}{2(x | y)}
\]

where \( D \) is a constant defined in Proposition 4.2 (1). This shows that the topology on \( \partial \mathbb{R}^2 \) coincides with that of \( S^1 \).
As mentioned in Proposition 3.3, a direct product of coarsely convex spaces is also coarsely convex. The ideal boundary of the product space is given by the join of the ideal boundaries of the factors.

Here we recall the definition of the join. Let $W_1$ and $W_2$ be topological spaces. Then we consider the following equivalence relation $\sim$ on a space $W_1 \times [0,1] \times W_2$. If $(w_1,s,w_2), (w'_1,s',w'_2)$ satisfy one of three conditions

1. $w_1 = w'_1, s = s', w_2 = w'_2$,
2. $w_1 = w'_1, s = s' = 0$,
3. $s = s' = 1, w_2 = w'_2$,

then they are equivalent, that is, $(w_1,s,w_2) \sim (w'_1,s',w'_2)$. We call the quotient space $W_1 \ast W_2 := W_1 \times [0,1] \times W_2/ \sim$ the join of $W_1$ and $W_2$, and denote by $(1-s)w_1 \oplus sw_2$ the element whose representative is $(w_1,s,w_2)$.

**Proposition 4.26.** Let metric spaces $(X,d_X)$ and $(Y,d_Y)$ be coarsely convex with systems of good quasi-geodesic segments $\mathcal{L}_X$ and $\mathcal{L}_Y$, respectively. Take base points $O_X \in X$ and $O_Y \in Y$. For the product with the $\ell_1$-metric $(X \times Y, d_{X \times Y})$, which is coarsely convex with $\mathcal{L}^{X \times Y}$ defined as in Proposition 3.3, the boundary $\partial_{(O_X,O_Y)}(X \times Y)$ is homeomorphic to the join of $\partial_{O_X}X$ and $\partial_{O_Y}Y$.

**Proof.** Put $Z = X \times Y$ and $O_Z = (O_X,O_Y)$. Let $(\mathcal{L}^X_{O_X})^\infty$, $(\mathcal{L}^Y_{O_Y})^\infty$ and $(\mathcal{L}^Z_{O_Z})^\infty$ be the families of all approximatable rays of $\mathcal{L}^X$, $\mathcal{L}^Y$ and $\mathcal{L}^Z$ from base points, respectively. Note that these families does not change if we replace $\mathcal{L}^X$, $\mathcal{L}^Y$ and $\mathcal{L}^Z$ with their prefix-closures, respectively. We assume that $\mathcal{L}^X$, $\mathcal{L}^Y$ and $\mathcal{L}^Z$ are prefix-closed without loss of generality. Take the quotient maps

$$
\pi_X: (\mathcal{L}^X_{O_X})^\infty \to \partial_{O_X}X, \quad \pi_Y: (\mathcal{L}^Y_{O_Y})^\infty \to \partial_{O_Y}Y, \quad \pi_Z: (\mathcal{L}^Z_{O_Z})^\infty \to \partial_{O_Z}Z.
$$

**Figure 1.** The ideal boundary of $\mathbb{R}^2$
We have the natural map between joins

\[ \pi_X \star \pi_Y : (\mathcal{L}^X_{O_X})^{\infty} \star (\mathcal{L}^Y_{O_Y})^{\infty} \to \partial_{O_X} X \star \partial_{O_Y} Y \]

\[ (1 - s)\gamma \oplus s\eta \mapsto (1 - s)\gamma \oplus s\eta. \]

Now we consider the map

\[ \iota : (\mathcal{L}^X_{O_X})^{\infty} \star (\mathcal{L}^Y_{O_Y})^{\infty} \to (\mathcal{L}^Z_{O_Z})^{\infty} \]

defined as \( \iota((1 - s)\gamma \oplus s\eta)(t) = (\gamma((1 - s)t), \eta(st)) \) for any \( t \in \mathbb{R}_{\geq 0} \). This is well-defined. Indeed for any \((1 - s)\gamma \oplus s\eta\), we can take \( \gamma_i \in \mathcal{L}^X \) with domain \([0, a_i]\) and \( \eta_i \in \mathcal{L}^Y \) with domain \([0, b_i]\) such that they approximate \( \gamma \) and \( \eta \), respectively, and satisfy \( s = \frac{b_i}{a_i + b_i} \) by noting prefix-closedness of \( \mathcal{L}^X \) and \( \mathcal{L}^Y \). Then \( \iota((1 - s)\gamma \oplus s\eta) \) is approximated by \((1 - s)\gamma_i \oplus s\eta_i \in \mathcal{L}^Z\).

Now we can see that \( \iota \) induces the map

\[ \bar{\iota} : \partial_{O_X} X \star \partial_{O_Y} Y \to \partial_{O_Z} Z \]

\[ (1 - s)\gamma \oplus s\eta \mapsto [(1 - s)\gamma \oplus s\eta], \]

which satisfies \( \pi_Z \circ \iota = \bar{\iota} \circ (\pi_X \star \pi_Y) \) and is a homeomorphism. \( \square \)

5. Main result

The aim of this section is to give a proof of Theorem 1.1. An outline of the proof is parallel to that for the case of Gromov hyperbolic spaces by Higson and Roe [17]. However, we need to modify the arguments in order to overcome some difficulties which do not appear in the case of Gromov hyperbolic spaces.

Here we summarize the strategy used in [17]. Let \( Y \) be a proper geodesic Gromov hyperbolic space. Higson and Roe defined an “exponential map” \( \exp: O_{\partial O} Y \to Y \). They first constructed a coarse homotopy between the open cone \( O_{\partial O} Y \) and the image of the exponential map \( \exp(O_{\partial O} Y) \). Then they constructed a coarse homotopy between \( \exp(O_{\partial O} Y) \) and \( Y \). Here they used the fact that the image \( \exp(O_{\partial O} Y) \) is quasi-convex and the nearest point projection onto \( \exp(O_{\partial O} Y) \) is bornologous. In fact, \( \exp(O_{\partial O} Y) \) is a “coarsely deformation retract” of \( Y \).

Now let \( X \) be a proper coarsely convex space. We introduce a modified exponential map \( \exp_{\epsilon} : O_{\partial O} X \to X \) by replacing the parameter \( t \) by \( t^\frac{1}{\epsilon} \). We first construct a coarse homotopy between \( O_{\partial O} X \) and \( \exp_{\epsilon}(O_{\partial O} X) \).

Then we construct a coarse homotopy between \( \exp(O_{\partial O} X) \) and \( X \). Here we need quite different arguments, since the image \( \exp_{\epsilon}(O_{\partial O} X) \) is not quasi-convex, and the nearest points projection is not bornologous, in general. In Section 5.5 we construct the coarse homotopy using a contraction toward the base point with an appropriate proportion, which is not necessarily a coarsely deformation retract.
5.1. **Setting.** Let $X$ be a proper $(\lambda,k,E,C,\theta,\mathcal{L})$-coarsely convex space. We fix a positive number $0 < \epsilon < 1$ such that $(D_2D_3)^\epsilon \leq 2$, where $D_2$ and $D_3$ are constants defined in Section 4. We set $K := (D_2D_3)^\epsilon$. Let $d_\epsilon$ be a metric given by Proposition 4.19. We remark that the diameter of $(\partial OX,d_\epsilon)$ is less than or equal to 1 since $(\gamma | \eta) \geq 1$ for all $\gamma, \eta \in \mathcal{L}_O$. Thus the induced metric $d_{\partial OX}$ on the open cone $\partial OX$ is well-defined.

5.2. **Exponential map.** We define an exponential map $\exp_\epsilon : \partial OX \to X$ as follows. For each $x \in \partial OX$, we choose a quasi-geodesic ray $\eta_x \in \mathcal{L}_O^\infty$ with $x = [\eta_x]$. Then for $t \in \mathbb{R}_{\geq 0}$, we define $\exp_\epsilon(tx) := \eta_x(t^{1/\epsilon})$. We remark that $\exp_\epsilon$ is proper, however, not necessarily bornologous. Therefore, we need to modify $\exp_\epsilon$ by combining with a radial contraction.

**Definition 5.1.** Let $r : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a Lipschitz map with Lipschitz constant less than or equal to 1 such that $r(0) = 0$ and $r(t) \to \infty$ when $t \to \infty$. The radial contraction associated to $r$ is a map $\phi : \partial OX \to \partial OX$ defined by $\phi(tx) := r(t)x$ for $tx \in \partial OX$.

We remark that any radial contraction is coarsely homotopic to the identity.

**Definition 5.2.** Let $V$ be a topological space and $M$ be a metric space. A map $f : V \to M$ is pseudocontinuous if there exists $r > 0$ such that for any $x \in V$, the inverse image $f^{-1}(B_r(f(x)))$ is a neighborhood of $x$. Here $B_r(f(x))$ is the closed ball of radius $r$ centered at $f(x)$.

**Proposition 5.3** (Higson-Roe[17]). Let $f : \partial OX \to X$ be a proper pseudocontinuous map. There exists a radial contraction $\phi : \partial OX \to \partial OX$ such that the composite $f \circ \phi$ is a coarse map.

For proof, see [17] Lemma 4.2 or [31] 4.7.5.

**Lemma 5.4.** The map $\exp_\epsilon : \partial OX \to X$ is pseudocontinuous.

**Proof.** Since the map $\exp_\epsilon$ is a composite of the continuous map $tx \mapsto t^{1/\epsilon}x$ and $\exp_1$, it is enough to show that $\exp_1$ is pseudocontinuous.

For $tx \in \partial OX$ with $0 \leq t \leq 1$, a neighbourhood $\{sy \in \partial OX : 0 \leq s < 2, y \in \partial OX\}$ of $tx$ is contained in $\exp_1^{-1}(B_{3\lambda+2k_1}(\exp_1(tx)))$.

Thus we will show that for $x, y \in X$ and $t, s \in [1,\infty)$, if $d_{\partial OX}(tx, sy) < (2KD_3)^{-1}$, then $\exp_1(tx), \exp_1(sy) \leq E(D_1 + 2k_1) + D + \lambda + k_1$.

We take quasi-geodesic rays $\eta_x, \eta_y$ as in the definition of the exponential map. We assume that $s \geq t$. Since $d_{\partial OX}(tx, sy) = |s - t| + td_\epsilon(x, y) < (2KD_3)^{-1}$, we have $|s - t| < 1$ and $td_\epsilon(x, y) < (2KD_3)^{-1}$. Set $a := (\eta_x | \eta_y)$. We have

$$a = (\eta_x | \eta_y) \geq D_3^{-1}(x | y) \geq D_3^{-1}(2Kd_\epsilon(x, y))^{-\frac{1}{\epsilon}} > t^{\frac{1}{\epsilon}} \geq t.$$
Therefore
\[
\exp_1(tx), \exp_1(sy) \leq \eta_x(t), \eta_y(t) + \eta_y(s) \\
\leq E \eta_x(a), \eta_y(a) + D + \lambda |s - t| + k_1 \\
\leq E(D + 2k_1) + D + \lambda + k_1.
\]

\[\square\]

**Corollary 5.5.** There exists a radial contraction \( \phi : \partial O \to \partial O \) such that the composite \( \exp \circ \phi \) is a coarse map.

5.3. **Logarithmic map.** We define a logarithmic map

\[ \log^\epsilon : \exp_\epsilon(\partial O \to \partial O \] as follows. For \( v \in \exp_\epsilon(\partial O \to \partial O \), we choose a geodesic ray \( \gamma_v \in \mathcal{L}_\infty \) and parameter \( t_v \in \mathbb{R}_{\geq 0} \) such that \( \gamma_v(t_v) = v \). Then we define \( \log^\epsilon(v) := t' \gamma_v \).

**Proposition 5.6.** The logarithmic map \( \log^\epsilon : \exp_\epsilon(\partial O \to \partial O \) is a coarse map.

**Proof.** It is easy to see that the map \( \log^\epsilon \) is proper, thus we will show that it is bornologous.

Let \( v, w \in \exp_\epsilon(\partial O \to \partial O \). We take quasi-geodesic rays \( \gamma_v, \gamma_w \in \mathcal{L}_\infty \) and parameters \( t_v, t_w \in \mathbb{R}_{\geq 0} \) as in the definition of the map \( \log^\epsilon \). Set \( T := \min\{t_v, t_w\} \). First we suppose that \( T < 1 \). Then we have \( d_{\partial O \to \partial O}(\log^\epsilon(v), \log^\epsilon(w)) \leq t'_v + t'_w < 1 + (1 + \tilde{\theta}(\gamma_v, \gamma_w))\epsilon \).

Now we suppose that \( T \geq 1 \). Then by an elementary calculus,

\[ |t'_v - t'_w| \leq \epsilon |t_v - t_w| \leq \epsilon \tilde{\theta}(\gamma_v, \gamma_w) \]

Now we will show that

\[ (\gamma_v \mid \gamma_w) \geq \frac{T}{E \tau(\gamma_v, \gamma_w)} \]

where \( \tau : \mathbb{R} \to \mathbb{R} \) is an increasing map defined by \( \tau(t) := E(t + \lambda \tilde{\theta}(t) + k_1) + D \).

If \( T \leq (\gamma_v \mid \gamma_w) \) then (4) immediately follows. Thus we suppose that \( T > (\gamma_v \mid \gamma_w) \). By Lemma 4.3 we have \( \gamma_v(T), \gamma_w(T) \leq \tau(\gamma_v, \gamma_w) \). Set \( c := (E \tau(\gamma_v, \gamma_w))^{-1} \). Then

\[ \frac{\gamma_v(cT), \gamma_w(cT)}{\gamma_v(T), \gamma_w(T)} \leq D + 1 \]

Thus \( (\gamma_v \mid \gamma_w) \geq cT = T(E \tau(\gamma_v, \gamma_w))^{-1} \). Combined with (3) and (4),

\[ d_{\partial O \to \partial O}(\log^\epsilon(v), \log^\epsilon(w)) = |t'_v - t'_w| + T'd_\epsilon([\gamma_v], [\gamma_w]) \leq \epsilon \tilde{\theta}(\gamma_v, \gamma_w) + T'(\gamma_v \mid \gamma_w)^{-\epsilon} \leq \epsilon \tilde{\theta}(\gamma_v, \gamma_w) + (E \tau(\gamma_v, \gamma_w))^\epsilon \]

\[ \square \]
5.4. **Coarse homotopy between** $\mathcal{O}\partial_O X$ **and** $\exp_\epsilon(\mathcal{O}\partial_O X)$.

**Lemma 5.7.** The composite $\log^\epsilon \circ \exp_\epsilon \circ \phi$ is coarsely homotopic to the identity $\text{id}_{\mathcal{O}\partial_O X}$.

**Proof.** Since the radial contraction $\phi$ is coarsely homotopic to the identity, it is enough to show that $\log^\epsilon \circ \exp_\epsilon$ is close to the identity.

For $x \in \partial_O X$, let $\eta_x \in \mathcal{L}_O^\infty$ be the quasi-geodesic ray representative for $x$ chosen in the definition of the exponential map. Thus $x = [\eta_x]$. For $t \in \mathbb{R}_{\geq 0}$, set $v := \exp_\epsilon(tx) = \eta_x(t^{\frac{1}{\epsilon}})$.

Let $\gamma_v \in \mathcal{L}_O^\infty$ and $t_v \in \mathbb{R}_{\geq 0}$ be the quasi-geodesic ray and the parameter, respectively, associated to $v$ chosen in the definition of the logarithmic map. Thus we have $\gamma_v(t_v) = \eta_x(t^{\frac{1}{\epsilon}})$ and

$$\log^\epsilon \circ \exp_\epsilon(tx) = \log^\epsilon(v) = t_v^{\epsilon}[\gamma_v].$$

Set $a := \min\{t^{\frac{1}{\epsilon}}, t_v\}$. By Lemma 4.3, we have $\eta_x(a), \gamma_v(a) \leq E(\lambda \tilde{\theta}(0) + k_1) + D$. Set $c := (E^2(\lambda \tilde{\theta}(0) + k_1) + DE)^{-1}$. Then we have $\eta_x(ca), \gamma_v(ca) \leq D + 1$. This implies

$$(\eta_x \mid \gamma_v) \geq ca \geq c(t^{\frac{1}{\epsilon}} - \tilde{\theta}(0)).$$

First we supposed that $a \geq \tilde{\theta}(0) + 1$. Then by an elementary calculus,

$$|t - t_v^{\epsilon}| \leq \epsilon \left|t^{\frac{1}{\epsilon}} - t_v\right| \leq \epsilon \tilde{\theta}(0).$$

We remark that $t^{\frac{1}{\epsilon}} - \tilde{\theta}(0) \geq 1$ since $t^{\frac{1}{\epsilon}} \geq a$. Then we have

$$d_{\mathcal{O}\partial_O X}(t_v^{\epsilon}[\gamma_v], t[\eta_x]) \leq |t - t_v^{\epsilon}| + \min\{t_v^{\epsilon}, t\}d_\epsilon([\gamma_v], [\eta_x])$$

$$\leq \epsilon \tilde{\theta}(0) + t\rho_\epsilon([\gamma_v], [\eta_x])$$

$$\leq \epsilon \tilde{\theta}(0) + t\left(t^{\frac{1}{\epsilon}} - \tilde{\theta}(0)\right)^{-\epsilon} c^{-\epsilon}.$$

The second term in (5) is bounded from above by a universal constant.

Next we suppose that $a < \tilde{\theta}(0) + 1$. Then we have

$$d_{\mathcal{O}\partial_O X}(t_v^{\epsilon}[\gamma_v], t[\eta_x]) \leq t_v^{\epsilon} + t < 2(\tilde{\theta}(0) + 1)^\epsilon.$$

These show that $\log^\epsilon \circ \exp_\epsilon$ is close to the identity.

**Lemma 5.8.** The composite $\exp_\epsilon \circ \phi \circ \log^\epsilon$ is coarsely homotopic to the identity on $\exp_\epsilon(\mathcal{O}\partial_O X)$.

**Proof.** Since $\phi : \mathcal{O}\partial_O X \rightarrow \mathcal{O}\partial_O X$ is coarsely homotopic to the identity, the map $\exp_\epsilon \circ \phi \circ \log^\epsilon$ is coarsely homotopic to the map $\exp_\epsilon \circ \log^\epsilon$. Thus it is enough to show that $\exp_\epsilon \circ \log^\epsilon$ is close to the identity.

For $v \in \exp_\epsilon(\mathcal{O}\partial_O X)$, let $\gamma_v \in \mathcal{L}_O^\infty$ and $t_v \in \mathbb{R}_{\geq 0}$ are the quasi-geodesic ray and the parameter, respectively, chosen in the definition of the logarithmic map. Thus $\gamma_v(t_v) = v$.

Set $x = [\gamma_v]$. Let $\eta_x \in \mathcal{L}_O^\infty$ be the quasi-geodesic ray representative of $x$ chosen in the definition of the exponential map. Now we have $\exp_\epsilon \circ \log^\epsilon(v) = \exp_\epsilon(t_v^{\epsilon}[\eta_x]) = \eta_x(t_v)$. 

\[\square\]
Since $[\gamma_v] = x = [\eta_x]$, by Lemma 4.5 we have $\gamma_v(t_v), \eta_x(t_v) \leq D$. This shows that $\exp \circ \log^v$ is close to the identity.

Summarizing the argument above, we obtain the following.

**Proposition 5.9.** The map $\exp_v \circ \phi : \mathcal{O}\partial O X \to \exp_v(\mathcal{O}\partial O X)$ is a coarse homotopy equivalence map.

### 5.5. Coarse homotopy between $\exp_v(\mathcal{O}\partial O X)$ and $X$.

Set $D_5 := 2D_1 + 2k_1$, $D_6 := ED_5 + D$ and $Y := B_{D_6}(\exp_v(\mathcal{O}\partial O X))$. There exists a subset $X(0) \subset X$ such that $X(0)$ is 2-dense in $X$ and 1-discrete, that is, for all $v, w \in X(0)$, if $v \neq w$ then $\overline{v, w} \geq 1$, and, for all $v \in X$, there exists $v' \in X(0)$ with $\overline{v, v'} \leq 2$. We can assume that $X(0) \cap Y$ is 2-dense in $Y$. We fix a map $\iota : X \to X$ such that $\iota(v) \in X(0)$ and $\overline{\iota(v), v} \leq 2$ for all $v \in X$, and $\iota(v) \in X(0) \cap Y$ for all $v \in Y$. The purpose of this subsection is to prove the following.

**Proposition 5.10.** The inclusion $Y \hookrightarrow X$ is a coarse homotopy equivalence map.

For $v \in X(0)$, we choose a quasi-geodesic segment $\gamma_v \in \mathcal{L}_O$ and a parameter $T_v \in \mathbb{R}_{\geq 0}$ such that $\gamma_v(0) = O$ and $\gamma_v(T_v) = v$. Set $s_v := \sup \{ t : \gamma_v(t), \exp_v(\mathcal{O}\partial O X) \leq D_5 \}$. We remark that $s_v \geq 0$ since $O = \gamma_v(0) \in \exp_v(\mathcal{O}\partial O X)$.

**Lemma 5.11.** For each $N \geq 0$, the cardinality of the set $\{ v \in X(0) : s_v \leq N \}$ is finite.

**Proof.** We suppose that $\{ v \in X(0) : s_v \leq N \} = \infty$. Since $X = X \cup \partial O X$ is compact and $X(0)$ is uniformly discrete, we can choose a sequence $v_i \in \{ v \in X(0) : s_v \leq N \}$ which converges to a point $x \in \partial O X$. We choose a quasi-geodesic ray $\eta \in \mathcal{L}_O^\infty$ such that $x = [\eta]$.

For sufficiently large $n$, we have $(v_n, x) \in V_{D_3N}$, where $V_{D_3N}$ is an entourage of the uniform structure defined in Section 4.3. Let $\gamma_{v_n} \in \mathcal{L}_O$ be a quasi-geodesic segment for $v_n$ chosen in the beginning of Section 5.5. Set $a := (\gamma_{v_n} | \eta)$. We have $\gamma_{v_n}(a), \eta(a) \leq D_1 + 2k_1 \leq D_5$. It follows that

$$s_{v_n} \geq a = (\gamma_{v_n} | \eta) \geq D_3^{-1}(v_n | x) > N.$$  

This contradicts that $v_n \in \{ v \in X(0) : s_v \leq N \}$.

For each positive integer $n \in \mathbb{N}$, we define a sequence $l(n)$ by

$$l(n) := \max \{ T_v : v \in X(0), n \leq s_v < n + 1 \}.$$  

By Lemma 5.11 each $l(n)$ is finite. We choose a subsequence $n_i$ satisfying the following

$$l(n_1) > 1,$$

$$l(n_{i+1}) - l(n_i) > 1, \quad (i \geq 1),$$

$$l(n_i) > l(n), \quad (i \geq 1, 1 \leq n < n_i).$$

**Lemma 5.12.** For $v \in X(0)$ and $i \geq 1$, if $l(n_i) \leq T_v$, then we have $n_i \leq s_v$. 

Proof. For \( v \in X^{(0)} \), let \( i \) be an integer such that \( l(n_i) \leq T_v \). For \( n \in \mathbb{N} \) with \( n \leq s_v < n + 1 \), we have \( T_v \leq l(n) \). We suppose that \( n < n_i \). Then we have \( l(n) < l(n_i) \leq T_v \). This is a contradiction. Thus we have \( n_i \leq s_v \).

We define a map \( \chi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \) by

\[
\chi(t) = \begin{cases} 
0 & 0 \leq t < l(n_1), \\
i & l(n_i) \leq t < l(n_{i+1}), i \geq 1.
\end{cases}
\]

Then \( \chi \) satisfies \( \chi(t) \leq t \) and \( |\chi(t) - \chi(s)| \leq |t - s| + 1 \) for \( t, s \in \mathbb{R}_{\geq 0} \). We define a map \( \varphi : X^{(0)} \to Y \) by

\[
\varphi(v) := \gamma_v(\chi(T_v)) \quad (v \in X^{(0)}).
\]

Since \( \chi(T_v) = i < n_i \), by Lemma 5.12 we have \( \chi(T) < s_v \). It follows that \( \varphi(v) \in Y \).

Lemma 5.13. The map \( \varphi \) is a coarse map.

Proof. First we show that \( \varphi \) is proper. We fix \( R > 0 \). Let \( v \in X^{(0)} \) be a point with \( \overline{\varphi(v), O} \leq R \). Since \( (1/\lambda) \chi(T_v) - k \leq \gamma_v(\chi(T_v)), \overline{O} \leq R \), we have \( \chi(T_v) \leq \lambda(R + k) \). Let \( j \in \mathbb{N} \) be an integer with \( j > \lambda(R + k) \). Then \( T_v < l(n_j) \), so \( \overline{\varphi, O} < \lambda l(n_j) + k \). This shows that \( \varphi \) is proper.

Now we show that \( \varphi \) is bornologous. For \( v, w \in X^{(0)} \), set \( i := \chi(T_v) \) and \( j := \chi(T_w) \). Then

\[
|i - j| = |\chi(T_w) - \chi(T_v)| \leq |T_w - T_v| + 1 \leq \theta(\overline{v, w}) + 1.
\]

By Lemma 4.3

\[
\gamma_v(i), \gamma_w(i) \leq E(\overline{v, w} + \lambda\overline{\varphi, w} + k_1) + D.
\]

Since \( \gamma_w \) is a \((\lambda, k)\)-quasi-geodesic segment,

\[
\gamma_w(i), \gamma_w(j) \leq \lambda|i - j| + k \leq \lambda(\theta(\overline{v, w}) + 1) + k.
\]

Then we have

\[
\overline{\varphi(v), \varphi(w)} = \gamma_v(i), \gamma_w(j) \leq E(\overline{v, w} + \lambda\overline{\varphi, w} + k_1) + D + \lambda(\theta(\overline{v, w}) + 1) + k.
\]

Therefore \( \varphi \) is bornologous.

Set \( \tilde{\varphi} := \varphi \circ i : X \to Y \). Let \( i : Y \hookrightarrow X \) be the inclusion. We will show that \( i \circ \tilde{\varphi} \) and \( \tilde{\varphi} \circ i \) are, respectively, coarsely homotopic to the identity \( \text{id}_X \) and \( \text{id}_Y \).

Indeed, since \( i \) is close to the identity, it is enough to show that \( i \circ \tilde{\varphi} \) and \( \tilde{\varphi} \circ i \) are, respectively, coarsely homotopic to the map \( i \) and the restriction \( i|_Y \) of \( i \) on \( Y \). First we construct a coarse homotopy between \( i \circ \tilde{\varphi} \) and \( i \).
Set \( Z := \{(v, t) \in X \times \mathbb{R} : 0 \leq t \leq T_{i(v)}\} \). We remark that the map \( X \ni v \mapsto T_{i(v)} \in \mathbb{R}_{\geq 0} \) is bornologous. We define a map \( H : Z \to X \) by
\[
H(v, t) := \gamma_{i(v)}(T_{i(v)} - t + \chi(t)).
\]
It is easy to see that \( H(v, 0) = i(v) \) and \( H(v, T_{i(v)}) = i \circ \tilde{\varphi}(v) \).

**Lemma 5.14.** The map \( H \) is a coarse map.

**Proof.** It is easy to show that \( H \) is proper. Thus we show that it is bornologous. We fix \((v, t), (w, s) \in Z\). We remark that \( t \geq \chi(t) \) and \( s \geq \chi(s) \). Set \( v' := i(v) \) and \( w' := i(w) \).

We suppose \( T_{w'} \geq T_{v'} \). Then
\[
\overline{H(v, t), H(w, s)} = \gamma_{v'}(T_{v'} - t + \chi(t)), \gamma_{w'}(T_{w'} - s + \chi(s))
\leq \gamma_{v'}(T_{v'} - t + \chi(t)), \gamma_{w'}(T_{w'} - t + \chi(t))
+ \gamma_{w'}(T_{w'} - t + \chi(t)), \gamma_{w'}(T_{w'} - s + \chi(s))
\leq E \gamma_{v'}(T_{v'}), \gamma_{w'}(T_{v'}) + C
+ \lambda |T_{w'} - t + \chi(t) - (T_{w'} - s + \chi(s))| + k
\leq E(\overline{v', w'} + \lambda \overline{(v', w')} + k_1 + D) + C
+ \lambda(\theta(\overline{v', w'}) + 2|t - s| + 1) + k.
\]
Since \( \overline{v', w'} \leq \overline{v, w} + 4 \), it follows that \( H \) is bornologous. \( \square \)

**Corollary 5.15.** The map \( i \circ \tilde{\varphi} \) and \( \text{id}_X \) are coarsely homotopic.

Now we construct a coarse homotopy between \( \tilde{\varphi} \circ i \) and \( \text{id}_Y \). Let \( Z' := Z \cap (Y \times \mathbb{R}_{\geq 0}) \). Let \( H' \) be the restriction of \( H \) to \( Z' \). Then the range of \( H' \) is in \( Y \). It follows that \( H' \) is a coarse map and \( H'(v, 0) = i(v), H'(v, T_{i(v)}) = \tilde{\varphi} \circ i(v) \).

**Corollary 5.16.** The map \( \tilde{\varphi} \circ i \) and \( \text{id}_Y \) are coarsely homotopic.

This completes the proof of Proposition 5.10. Combining it with Proposition 5.9, we obtain Theorem 1.1.

6. **Application to the Coarse Baum-Connes Conjecture**

6.1. **Coarse Baum-Connes conjecture.** The coarse category is a category whose objects are proper metric spaces and whose morphisms are close classes of coarse maps. Let \( X \) be a proper metric space. There are two covariant functors \( X \mapsto KX_*(X) \) and \( X \mapsto K_*^\infty(X) \) from the coarse category to the category of \( \mathbb{Z}_2 \)-graded Abelian groups. Here the \( \mathbb{Z}_2 \)-graded Abelian group \( KX_*(X) \) is called the coarse \( K \)-homology of \( X \), and the \( C^* \)-algebra \( C^*(X) \) is called the Roe algebra of \( X \). Roe [28] constructed the following coarse assembly map
\[
\mu_* : KX_*(X) \to K_*(C^*(X)),
\]
which is a natural transformation from the coarse $K$-homology to the $K$-theory of the Roe algebra. For detail, see also [17, 32] and [18].

The important feature of these functors is, both the coarse $K$-homology and the $K$-theory of the Roe-algebra are coarse homotopy invariants in the following sense.

**Proposition 6.1.** Let $X$ and $Y$ be proper metric spaces. If there exists a coarse homotopy equivalence map $f: X \to Y$, then in the following commutative diagram, two vertical homomorphisms both denoted by $f_*$ are isomorphisms

$$
\begin{array}{ccc}
K_*\pi(X) & \xrightarrow{\mu_*} & K_*\pi(C^*(X)) \\
f_* & \cong & f_* \\
K_*\pi(Y) & \xrightarrow{\mu_*} & K_*\pi(C^*(Y)).
\end{array}
$$

Coarse homotopy invariance is proved by Mayer-Vietoris principle. For details, see [18, Proposition 12.4.12] and [31, Theorem 4.3.12].

**Corollary 6.2.** Let $X$ and $Y$ are proper metric spaces. We suppose that $X$ and $Y$ are coarsely homotopy equivalent. If the coarse assembly map $\mu_*: K_*\pi(Y) \to K_*\pi(C^*(Y))$ is an isomorphism, then so is the coarse assembly map $\mu_*: K_*\pi(X) \to K_*\pi(C^*(X))$.

Let $M$ be a compact metric space. Higson-Roe [17, Section 7] showed that the coarse Baum-Connes conjecture holds for the open cone $O^M$. We remark that in [17], $M$ is assumed to be finite dimensional. However, by [10, Appendix B], we can remove this assumption.

**Theorem 6.3.** Let $M$ be a compact metric space. Then the coarse assembly map

$$
\mu_*: K_*\pi(O^M) \to K_*\pi(C^*(O^M))
$$

is an isomorphism.

**Proof of Theorem 6.3.** Let $X$ be a proper coarsely convex space. By Theorem 6.1, $X$ is coarsely homotopy equivalent to the open cone $O\partial O^X$. Then by Theorem 6.3 and Corollary 6.2 the coarse assembly map $\mu_*: K_*\pi(X) \to K_*\pi(C^*(X))$ is an isomorphism.

□

6.2. **Coarse compactification.** Let $X$ be a non-compact proper metric space. Let $\varphi: X \to \mathbb{C}$ is a function. We say that $\varphi$ is *slowly oscillating* if for any $\epsilon > 0$ and $R > 0$, there exists a bounded subset $B \subset X$ such that

$$
sup\{|\varphi(v) - \varphi(w)| : v, w \in X \setminus B, \overline{v, w} \leq R\} < \epsilon.
$$

**Definition 6.4.** Let $X$ be a proper metric space, and let $\bar{X}$ be a compactification of $X$. Then $\bar{X}$ is a *coarse compactification* if for any continuous map $\varphi: \bar{X} \to \mathbb{C}$, the restriction of $\varphi$ to $X$ is slowly oscillating.
For detail on coarse compactifications, see [29, Section 2.2], [28, Section 5.1] or [13, Section 2.2]. Let \( \bar{X} \) be a coarse compactification of \( X \). Set \( \partial X := \bar{X} \setminus X \). Then there exists a certain transgression map

\[
T_{\partial X} : KX_*(X) \to \bar{K}_{*+1}(\partial X).
\]

Here \( \bar{K}_*(\partial X) \) is the reduced K-homology of \( \partial X \). Higson-Roe constructed a homomorphism \( b : K_*(C^*(X)) \to \bar{K}_{*-1}(\partial X) \) such that \( T_{\partial X} = b \circ \mu_* \). Therefore if the transgression map (6) is injective, then so is the coarse assembly map for \( X \). See [17, 9. Appendix] and [13] for detail.

Let \( M \) be a compact metrizable space. The open cone \( OM \) has a natural compactification \( OM \cup M \) by attaching \( M \) at infinity. Indeed, we set \( CM := [0,1] \times M/\{\{0\} \times M\} \) and define \( \varphi : \mathbb{R}_{\geq 0} \to [0,1) \) by \( \varphi(t) := t/(1 + t) \). Then a map \( tx \mapsto \varphi(t)x \) gives an embedding of \( OM \) into \( CM \) with an open dense image.

**Proposition 6.5.** Let \( M \) be a compact metric space. Then the compactification \( OM \cup M \) is a coarse compactification, and the transgression map

\[
T_M : KX_*(OM) \to \bar{K}_{*+1}(M)
\]

is an isomorphism.

For the proof see [17, Proposition 4.3], [31, Lemma 4.5.3] or [13, Lemma 5.1].

**Proposition 6.6.** Let \( X \) be a proper coarsely convex space. Then \( \bar{X} = X \cup \partial X \) is a coarse compactification, where \( \partial X \) is the ideal boundary of \( X \).

**Proof.** Let \( X \) be a \((\lambda, k, E, C, \theta, \mathcal{L})\)-coarsely convex space. Let \( O \in X \) be the base point and let \( \partial X \) be the ideal boundary with respect to \( O \).

Let \( \varphi : X \to \mathbb{C} \) be a continuous map. We will show that \( \varphi \) is slowly oscillating. Since \( X \cup \partial X_O \) is compact, for any \( \epsilon > 0 \), there exists \( n > 0 \) such that if \( (p,q) \in V_n \) then \( |\varphi(p) - \varphi(q)| < \epsilon \), where \( V_n \) is an entourage of the uniform structure defined in Section 4.3.

Now we fix \( R > 1 \) and set \( d := \lambda n \{ E(R + \lambda \theta(R) + k) \} + k \). Let \( v, w \in X \setminus B_d(O) \) be points with \( \overline{v,w} \leq R \). Let \( \gamma_v, \gamma_w \in \mathcal{L}_O \) be quasi-geodesic segments such that \( \gamma_v : [0, a_v] \to X \), \( \gamma_v(a_v) = v \) and \( \gamma_w(a_w) = w \). Set \( a := \min\{a_v, a_w\} \). We remark that \( a \geq (d - k)/\lambda \geq n \). We can suppose without loss of generality that \( a = a_v = a_w \). Then

\[
\overline{\gamma_v(n), \gamma_w(n)} \leq \frac{n}{a} E(\gamma_v(a), \gamma_w(a)) + C \\
\leq \frac{\lambda n}{d - k} E(\gamma_v(a_v), \gamma_w(a_w)) + \overline{\gamma_w(a_w), \gamma_v(a_v)}) + C \\
\leq \frac{\lambda n}{d - k} E(R + \lambda \theta(R) + k) + C \\
\leq 1 + C \leq D_1.
\]

It implies that \( (v \mid w) \geq (\gamma_v \mid \gamma_w) \geq n \). Thus \( (v, w) \in V_n \), and so \( |\varphi(v) - \varphi(w)| < \epsilon \). \( \square \)
Theorem 6.7. Let $X$ be a proper coarsely convex space. Then the transgression map
$$T_{\partial X} : KX_*(X) \rightarrow \tilde{K}_{*-1}(\partial X).$$
is an isomorphism.

Proof. The statement follows immediately from the following diagram.

\[
\begin{array}{ccc}
KX_*(X) & \xrightarrow{T_{\partial X}} & \tilde{K}_{*-1}(\partial X) \\
\downarrow \cong & & \downarrow \cong \\
KX_*(\partial X) & \xrightarrow{T_{\partial X}} & \tilde{K}_{*-1}(\partial X)
\end{array}
\]

\[\square\]

6.3. Direct product with polycyclic groups. One of advantages of the coarse Baum-Connes conjecture is that the coarse Mayer-Vietoris principle holds for both sides of the coarse assembly maps. As an application of this, we have the following ([11, Proposition 7.2]).

Proposition 6.8. Let $G$ be a simply connected solvable Lie group with a lattice. We equip $G$ with a proper left invariant metric. Let $Y$ be a proper metric space. Suppose that $Y$ satisfies the coarse Baum-Connes conjecture. Then so does the direct product $Y \times G$.

Corollary 6.9. Let $G$ be a simply connected solvable Lie group with a lattice, and let $X$ be a proper coarsely convex space. Then the direct product $X \times G$ satisfies the coarse Baum-Connes conjecture.

We remark that every polycyclic group $G$ admits a normal subgroup $G'$ of finite index in $G$ which is isomorphic to a lattice in a simply connected solvable Lie group. See [26, Theorem 4.28].

6.4. Relatively hyperbolic groups. In [10], the authors studied the coarse Baum-Connes conjecture for relatively hyperbolic groups.

Theorem 6.10 ([10]). Let $G$ be a finitely generated group and $P = \{P_1, \ldots, P_k\}$ be a finite family of subgroups. Suppose that $G$ is hyperbolic relative to $P$. If each subgroup $P_i$ satisfies the coarse Baum-Connes conjecture, and admits a finite $P_i$-simplicial complex which is a universal space for proper actions, then $G$ satisfies the coarse Baum-Connes conjecture. Moreover, if $G$ is torsion-free and each subgroup $P_i$ is classified by a finite simplicial complex, then $G$ satisfies the Novikov conjecture.

Let $C$ be a class of groups consisting of all finite direct products of hyperbolic groups, CAT(0)-groups, systolic groups, and polycyclic groups. Each group $P$ in $C$ admits a finite $P$-simplicial complex which is a universal space for proper actions. We refer [7] for the case of systolic groups. If $P$ in $C$ is torsion free, then $P$ is classified by a finite simplicial complex. Now Theorem 1.6 follows from Theorem 1.3, Theorem 6.10 and Proposition 6.8.
Remark 6.11. The 3-dimensional discrete Heisenberg group never act geometrically on any coarsely convex space, since it does not satisfy any quadratic isoperimetric inequality [9, Example 8.1.1], which violate the conclusion of Corollary 1.7. Hence Theorem 1.6 does not follows directly from Theorem 1.3.

More generally, by a similar argument as the proof of [11, Theorem 1.1], we can show the following.

Theorem 6.12. Let \( m \) be a positive integer. For \( 1 \leq j \leq m \), let \( G_j \) be a group in \( C \), or, be a finitely generated group which is hyperbolic relative to a family of subgroups \( P_j = \{ P_{j,1}, \ldots, P_{j,k_j} \} \) consisting of members of \( C \). Then the direct product group

\[
G := G_1 \times \cdots \times G_m
\]

satisfies the coarse Baum-Connes conjecture. Moreover, if \( G \) is torsion-free, then \( G \) satisfies the Novikov conjecture.

7. Groups acting on a coarsely convex space

From the viewpoint of geometric group theory, it is natural to consider groups acting on coarsely convex spaces. In this section, we mention some algebraic properties of such groups, which follows immediately from semihyperbolicity of coarsely convex spaces.

Alonso and Bridson [1] introduce another formulation of “nonpositively curved space”, called semihyperbolic space. We show that a coarsely convex space is semihyperbolic in their sense.

First, we briefly review the definition and properties. Let \( X \) be a metric space. A discrete path is a map \( \gamma: [0, T_\gamma] \cap \mathbb{Z} \to X \) with \( T_\gamma \in \mathbb{N} \cup \{0\} \). For convenience, we consider \( \gamma \) as a map \( \gamma: \mathbb{N} \cup \{0\} \to X \) by setting \( \gamma(t) := \gamma(T_\gamma) \) if \( t \geq T_\gamma \). Let \( \mathcal{P}(X) \) be the set of discrete paths. We consider the endpoints map \( e: \mathcal{P}(X) \to X \times X \) given by \( e(\gamma) = (\gamma(0), \gamma(T_\gamma)) \).

A bicombing is a section \( s: X \times X \to \mathcal{P}(X) \) of the endpoints map \( e \). We denote the image of \( (x, y) \) by \( s_{(x, y)} \).

A bicombing \( s \) is said to be quasi-geodesic if there exist constants \( \lambda, k \) such that \( s_{(x, y)} \) is a \((\lambda, k)\)-quasi-geodesic segment for all \( x, y \in X \).

A bicombing \( s \) is called bounded if there exist constants \( k_1 \geq 1, k_2 \geq 0 \) such that, for all \( x, y, x', y' \in X \) and \( t \in \mathbb{N} \cup \{0\} \),

\[
\overline{s_{(x,y)}(t), s_{(x',y')}(t)} \leq k_1 \max\{ \overline{x, x'}, \overline{y, y'} \} + k_2.
\]

Definition 7.1 ([1]). A metric space \( X \) is semihyperbolic if it admits a bounded quasi-geodesic bicombing.

Alonso and Bridson [1] Theorem 1.1] showed that being semihyperbolic is invariant under quasi-isometries. Then they studied groups acting on a semihyperbolic space.
Theorem 7.2 ([1, Theorem 2.8 and Theorem 5.1]). Let $G$ be a group acting on a semihyperbolic space $X$ properly and cocompactly by isometries. Then the following holds.

1. $G$ is finitely presented and of type $FP_{\infty}$.
2. $G$ satisfies a quadratic isoperimetric inequality.

Moreover, suppose that a bicombing $s$ of $X$ is $G$-invariant, then

3. $G$ has a solvable conjugacy problem.
4. Every polycyclic subgroup of $G$ contains a finitely generated abelian subgroup of finite index.

Proposition 7.3. Let $X$ be a coarsely convex space. Then $X$ is semihyperbolic. Moreover, suppose that a group $G$ acts on $X$ by isometries, and $G$ preserves a system of good quasi-geodesic segments $\mathcal{L}$ of $X$, then $X$ admits $G$-invariant bounded quasi-geodesic bicombing.

Proof. Let $X$ be $(\lambda, k, E, C, \theta, \mathcal{L})$-coarsely convex. Then we can assume that $\theta$ is a large scale Lipschitz function. Indeed for every $(x, y) \in X \times X$, we take $\gamma_{x, y} \in \mathcal{L}$ whose domain is $[0, a_{x,y}]$ and $t_{x,y} \in [0, a_{x,y}]$ with $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(t_{x,y}) = y$. Then the map

$$X \times X \to \mathbb{R}_{\geq 0}; (x, y) \mapsto t_{x,y}$$

is $\theta$-bornologous. We equip $X \times X$ with the $\ell_1$-metric. Since $X$ is quasi-geodesic, so is $X \times X$. Hence we have constants $A \geq 1, B' \geq 0$ such that

$$|t_{x,y} - t_{x',y'}| \leq A(x, x') + B'$$

for any $(x, y), (x', y') \in X \times X$. When for $(x, y) \in X \times X$ we choose different $\eta_{x,y} \in \mathcal{L}$ whose domain is $[0, b_{x,y}]$ and $u_{x,y} \in [0, b_{x,y}]$ with $\eta_{x,y}(0) = x$, $\eta_{x,y}(u_{x,y}) = y$, we have $|t_{x,y} - u_{x,y}| \leq \theta(0)$. We put $B = B' + \theta(0)$. Then we have the following.

(iii)' Let $\gamma, \eta \in \mathcal{L}$ be quasi-geodesic segments with $\gamma: [0, a] \to X$ and $\eta: [0, b] \to X$.

Then for $t \in [0, a]$ and $s \in [0, b]$, we have

$$|t - s| \leq A(\gamma(0), \eta(0) + \gamma(t), \eta(s)) + B.$$

Let $\gamma, \eta \in \mathcal{L}$ be quasi-geodesic segments as in (iii)'$^q$. Put $\gamma(t) = \gamma(a)$ for any $t \geq a$ and $\eta(u) = \eta(b)$ for any $u \geq b$. We assume that $a \leq b$. Then (iii)'$^q$ implies

$$\gamma(0), \eta(0) + \gamma(a), \eta(a) \leq \gamma(0), \eta(0) + \gamma(a), \eta(b) + \eta(a), \eta(b) \leq \gamma(0), \eta(0) + \gamma(a), \eta(b) + \lambda |b - a| + k \leq \gamma(0), \eta(0) + \gamma(a), \eta(b) + \lambda(A(\gamma(a), \eta(b) + \gamma(0), \eta(0)) + B) + k \leq (\lambda A + 1)(\gamma(a), \eta(b) + \gamma(0), \eta(0)) + \lambda B + k.$$
For any \( t \leq a \), we have
\[
\gamma(t), \eta(t) \leq \frac{t}{a} E \gamma(a), \eta(a) + \frac{a-t}{a} E \gamma(0), \eta(0) + C
\]
\[
\leq E(\gamma(a), \eta(a) + \gamma(0), \eta(0)) + C
\]
\[
\leq E(\lambda A + 1)(\gamma(a), \eta(b) + \gamma(0), \eta(0)) + (E(\lambda B + k) + C).
\]

Also for any \( t \geq a \),
\[
\gamma(t), \eta(t) = \gamma(a), \eta(t) \leq \gamma(a), \eta(b) + \eta(b), \eta(t)
\]
\[
\leq \gamma(a), \eta(b) + \lambda |b - t| + k
\]
\[
\leq \gamma(a), \eta(b) + \lambda |b - a| + k
\]
\[
\leq \gamma(a), \eta(b) + \lambda(A(\gamma(a), \eta(b) + \gamma(0), \eta(0)) + B) + k
\]
\[
\leq (\lambda A + 1)(\gamma(a), \eta(b) + \gamma(0), \eta(0)) + \lambda B + k.
\]

Now we can easily construct a bounded quasi-geodesic bicombing from \( \mathcal{L} \). The second assertion follows from the construction. \(\square\)

Corollary 1.7 follows immediately from Theorem 7.2 and Proposition 7.3. Another application is given in Corollary 8.10.

An advantage of a group \( G \) acting on a coarsely convex space \( X \) is, if \( G \) preserve a system of good geodesic segments \( \mathcal{L} \) of \( X \), then \( G \) acts on the ideal boundary \( \partial X \) of \( X \), as we have already seen in Corollary 4.25. We hope more algebraic and geometric properties of the group \( G \) can be understood through the topology of \( \partial X \), such as splitting of \( G \), as in the case of hyperbolic groups by Bowditch [4] and that of CAT(0)-groups by Papasoglu-Swenson [25].

It also seems natural to ask whether the group \( G \) admits finite \( G \)-simplicial complex which is a universal space for proper actions.

8. A FUNCTIONAL ANALYTIC CHARACTERIZATION OF THE IDEAL BOUNDARY

The aim of this section is to give a functional analytic characterization of the ideal boundaries of coarsely convex spaces. As an application, we show that the ideal boundary coincides with the bicombing corona introduced by Engel and Wulff [8].

Let \( X \) be a proper metric space which is \((\lambda, k, E, C, \theta, \mathcal{L})\)-coarsely convex. Let \( O \) be a base point. We denote by \((\cdot | \cdot)\) the Gromov product with respect to the base point \( O \). We use constants defined in the beginning of Section 4.

**Definition 8.1.** We say that a function \( f : X \to \mathbb{C} \) is a Gromov function if for all \( \epsilon > 0 \), there exists \( R > 0 \) such that for \( v, w \in X \) with \((v | w) > R\), we have \(|f(v) - f(w)| < \epsilon\).
We denote by $C_g(X)$ a set of continuous Gromov functions. We will show that the set $C_g(X)$ is in fact an algebra and it is isomorphic to the algebra of all continuous functions on the ideal boundary compactification $\tilde{X} = X \cup \partial_0 X$.

Let $C(X)$ and $C(\tilde{X})$ be the algebra of continuous complex valued functions on $X$, and on $\tilde{X}$, respectively. Let $\iota: C(\tilde{X}) \to C(X)$ be a homomorphism defined by $\iota(f) = f|_X$ where $f|_X$ denotes the restriction of $f$ on $X$. We will show that in fact the image of $\iota$ lies in $C_g(X)$.

**Proposition 8.2.** For all $f \in C(\tilde{X})$, the restriction $f|_X$ is a Gromov function.

**Proof.** Let $f \in C(\tilde{X})$ be a continuous function on $\tilde{X}$. Let $\{V_n\}_{n \in \mathbb{N}}$ be the fundamental system of entourages of the uniform structure on $\tilde{X}$ defined in Section 4.3. Since $\tilde{X}$ is compact, for $\epsilon > 0$, there exists $n$ such that for $(x, y) \in V_n$, we have $|f(x) - f(y)| < \epsilon$.

Now for $v, w \in X$ with $(v | w) > n$, we have $(v, w) \in V_n$. Thus $|f(v) - f(w)| < \epsilon$. It follows that the restriction $f|_X$ is a Gromov function. \qed

Now we have shown that the restriction map $\iota: C(\tilde{X}) \to C_g(X)$ is well-defined. To show the subjectivity of $\iota$, we need the following lemma.

**Lemma 8.3.** Set $\delta_1 := \lambda(\tilde{0}) + k_1$. Let $\gamma \in \mathcal{L}_\infty^\infty$ be a quasi-geodesic ray. For $t \in \mathbb{R}_{\geq 0}$ and $\gamma_t \in \mathcal{L}_O$ with domain $[0, a_t]$ such that $\gamma_t(a_t) = \gamma(t)$, we have $(\gamma | \gamma_t) > (t - \tilde{0})/E\delta_1$.

**Proof.** Let $\gamma \in \mathcal{L}_\infty^\infty$ be a quasi-geodesic ray. For $t \in \mathbb{R}_{\geq 0}$, we choose $\gamma_t \in \mathcal{L}_O$ whose domain is $[0, a_t]$, such that $\gamma_t(a_t) = \gamma(t)$. We remark that $|a_t - t| \leq \tilde{0}$. Since $\gamma(a_t), \gamma_t(a_t) = \gamma(a_t), \gamma(t) \leq \lambda(\tilde{0}) + k_1 = \delta_1$,

we have

$$\gamma\left(\frac{a_t}{E\delta_1}\right), \gamma_t\left(\frac{a_t}{E\delta_1}\right) \leq D + 1.$$

Thus we have $(\gamma | \gamma_t) \geq (t - \tilde{0})/E\delta_1$. \qed

Now we show that the map $\iota: C(\tilde{X}) \to C_g(X)$ is surjective. Indeed, we show that every $f \in C_g(X)$ can be extended to $\tilde{X}$.

**Lemma 8.4.** Let $f: X \to \mathbb{C}$ be a continuous Gromov function. Let $\gamma \in \mathcal{L}_\infty^\infty$ be a quasi-geodesic ray. Then the limit

$$\lim_{n \to \infty} f(\gamma(n))$$

exists.

**Proof.** We will show that the sequence $\{f(\gamma(n))\}_{n \in \mathbb{N}}$ is a Cauchy sequence. For $\epsilon > 0$, there exists $N > 0$ such that for $v, w \in X$ with $(v | w) \geq (D_2D_3)^{-1}(N - \tilde{0})/E\delta_1$, we have

$$|f(v) - f(w)| < \epsilon.$$
Here $\delta_1$ is a constant defined in Lemma 8.3. Then for $m > n > N$, by Corollary 4.13 and Lemma 8.3, we have

$$\langle \gamma(n) | \gamma(m) \rangle \geq (D_2D_3)^{-1} \min \{ (\gamma(n) - [\gamma]) | (\gamma) \} \geq (D_2D_3)^{-1}(N - \tilde{\theta}(0))/E\delta_1.$$  

Thus we have $|f(\gamma(n)) - f(\gamma(m))| < \epsilon$. This complete the proof. \hfill \Box

**Lemma 8.5.** For $\gamma, \eta \in L_0^\infty$, if $\gamma \sim \eta$, then $\lim_{n \to \infty} (\gamma(n) | \eta(n)) = \infty$.

**Proof.** Let $\gamma, \eta \in L_0^\infty$ be quasi-geodesic rays with $\gamma \sim \eta$. By Corollary 4.13 and Lemma 8.3, we have

$$\langle \gamma(n) | \eta(n) \rangle \geq (D_2D_3)^{-1} \min \{ (\gamma(n) - [\gamma]) | (\gamma) \} \geq \frac{n - \tilde{\theta}(0)}{(D_2D_3)^2E\delta_1} \to \infty.$$  

\hfill \Box

**Corollary 8.6.** Let $f : X \to \mathbb{C}$ be a Gromov function. For $\gamma, \eta \in L_0^\infty$, if $\gamma \sim \eta$, then $\lim_{n \to \infty} f(\gamma(n)) = \lim_{n \to \infty} f(\eta(n))$.

Let $f : X \to \mathbb{C}$ be a continuous Gromov function. We extend $f$ to a map $\bar{f} : \bar{X} \to \mathbb{C}$ by $\bar{f}(x) := \lim_{n \to \infty} f(\gamma(n))$ where $x \in \partial OX$ and $\gamma \in L_0^\infty$ is a representative of $x$. By Lemma 8.4 and Corollary 8.6, this extension is well-defined.

**Lemma 8.7.** The above extension $\bar{f} : X \to \mathbb{C}$ is continuous.

**Proof.** We show that for each $x \in \partial OX$, the map $\bar{f}$ is continuous at $x$. We choose $\gamma \in L_0^\infty$ which is a representative of $x$. For $\epsilon > 0$, there exists $T > 0$ such that for $t \geq T$

$$|\bar{f}(x) - f(\gamma(t))| < \frac{\epsilon}{3}.$$  

Since $f$ is a Gromov function, there exists $R > 0$ such that for $u, w \in X$ with $(u | w) \geq (D_2D_3)^{-1}R$, we have

$$|f(u) - f(w)| < \frac{\epsilon}{3}.$$  

Set $T' := \max \{ T, RE\delta_1 + \tilde{\theta}(0) \}$. By Lemma 8.3, we have $(x | \gamma(T')) > R$.

First let $v \in X$ be a point with $(x | v) > R$. It follows that

$$\langle \gamma(T') | v \rangle \geq (D_2D_3)^{-1} \min \{ (\gamma(T') | x), (x | v) \} > (D_2D_3)^{-1}R.$$  

Therefore we have $|\bar{f}(x) - f(v)| \leq |\bar{f}(x) - f(\gamma(T'))| + |f(\gamma(T')) - f(v)| < \epsilon$.

Next let $y \in \partial OX$ be a point with $(x | y) > R$. We choose $\eta \in L_0^\infty$ which is a representative of $y$. There exists $S > 0$ such that for $s \geq S$

$$|\bar{f}(y) - f(\eta(s))| < \frac{\epsilon}{3}.$$
Set \( S' := \max\{S, RE\delta_1 + \tilde{\theta}(0)\} \). By Lemma 8.3 we have \((y \mid \eta(S')) > R\). This implies

\[
(\gamma(T') \mid \eta(S')) \geq (D_2 D_3)^{-2} \min\{(\gamma(T') \mid x), (x \mid y), (y \mid \eta(S'))\} > (D_2 D_3)^{-2} R.
\]

Therefore we have

\[
|\bar{f}(x) - \bar{f}(y)| \leq |\bar{f}(x) - f(\gamma(T'))| + |f(\gamma(T')) - f(\eta(S'))| + |f(\gamma(S')) - \bar{f}(y)| < \epsilon.
\]

It follows that \( \bar{f} \) is continuous at \( x \).

It follows from Lemma 8.7 that the map \( \iota : C(\bar{X}) \rightarrow C_g(X) \) is surjective. Especially, all function \( f \in C_g(X) \) is bounded. Moreover, we have the following.

**Theorem 8.8.** Let \( C_b(X) \) denote the \( C^* \)-algebra of bounded continuous complex valued functions on \( X \). The set \( C_g(X) \) is a closed \( * \)-sub-algebra of \( C_b(X) \) and the restriction map \( \iota : C(\bar{X}) \rightarrow C_g(X) \) is an isomorphism.

Engel and Wulff introduced the **combing compactifications** for proper combing spaces [8]. They also showed that a proper coarsely convex space \( X \) admits a proper combing. In fact they showed that this combing satisfies better condition, coherent and expanding [8, Lemma 3.26]. We denote by \( \bar{X}^H \) the combing compactification of \( X \).

**Corollary 8.9.** The identity map on \( X \) extends to a homeomorphism \( \bar{X} \rightarrow \bar{X}^H \) from the ideal boundary compactification to the combing compactification.

**Proof.** Engel and Wulff showed the similar statement for proper geodesic Gromov hyperbolic spaces and for Gromov boundaries [8, Lemma 3.23]. The key ingredient is a functional analytic characterization of the Gromov boundary by Roe [27, Proposition 2.1]. Now by Theorem 8.8 we can apply the argument in the proof of [8, Lemma 3.23] just replacing the Gromov product in usual sense by the one in the setting of coarsely convex spaces defined in Section 4.2.

Engel and Wulff obtained many results on groups equipped with expanding and coherent combings [8]. Here we apply one of them to groups acting on coarsely convex spaces.

Let \( G \) be a group acting geometrically on a proper coarsely convex space \( X \). Then \( G \) is finitely generated, and \( G \) is equipped with a word metric which is quasi-isometric to \( X \). By Proposition 3.2 the metric space \( G \) is coarsely convex. Let \( \partial G \) be the ideal boundary of \( G \). We denote by \( \dim(\partial G) \) the topological dimension of \( \partial G \). We also denote by \( \text{cd}(G) \) the cohomological dimension of the group \( G \). Combining [8, Corollary 7.13] with Corollary 8.9 we obtain the following.

**Corollary 8.10.** Let \( G \) be a group acting geometrically on a proper coarsely convex space. If \( G \) admits a finite model for the classifying space \( BG \), then

\[
\text{cd}(G) = \dim(\partial G) + 1.
\]
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