The information-theoretical viewpoint on the physical complexity of classical and quantum objects and their dynamical evolution

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Charles Bennett’s measure of physical complexity for classical objects, namely logical-depth, is used in order to prove that a chaotic classical dynamical system is not physically complex.

The natural measure of physical complexity for quantum objects, quantum logical-depth, is then introduced in order to prove that a chaotic quantum dynamical system is not physically complex too.

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References
I. INTRODUCTION: THE SHALLOWNESS OF RANDOM OBJECTS

Despite denoting it with the term "complexity", Andrei Nikolaevich Kolmogorov [1], [2], [3], [4], [5] introduced the notion denoted nowadays by the school of Paul Vitanyi [6] as "plain-Kolmogorov-complexity" (that I will denote with the letter K from here and beyond) in order of obtaining an intrinsic measure of the amount of information of that object and not as a measure of the amount of physical-complexity of that object.

That the amount of information and the amount of physical complexity of an object are two completely different concepts became further clear after the introduction by Gregory Joseph Chaitin of the notion denoted nowadays by the school of Paul Vitanyi [6] as "prefix-Kolmogorov-complexity" and denoted by the school of Chaitin and Cristian S. Calude simply as algorithmic information [7] (and that I will denote with the letter I from here and beyond) and the induced notion of algorithmic-randomness:

an algorithmically random object has a very high algorithmic information but is certainly not physically-complex.

Such a simple consideration is indeed sufficient to infer that algorithmic information can be seen in no way as a measure of physical complexity.

A natural measure of physical complexity within the framework of Algorithmic Information Theory, the logical depth, was later introduced by Gregory Chaitin and Charles Bennett [8], constituting what is nowadays generally considered as the algorithmic information theoretic viewpoint as to physical complexity, though some author can still be found who not only ignores that, as it was clearly realized by Brudno himself [9], [10], [11], the chaoticity of a dynamical system (defined as the strict positivity of its Kolmogorov-Sinai entropy) is equivalent to its weak algorithmic chaoticity (defined as the condition that almost all the trajectories, symbolically codified, are Brudno-algorithmically random) but is weaker than its strong algorithmic chaoticity (defined as the condition that almost all the trajectories, symbolically codified, are Martin-Löf-Solovay-Chaitin-algorithmically random), but uses the notions of chaoticity and complexity as if they were synonymous, a thing obviously false since, as we have seen, the (weak) algorithmic randomness of almost all the trajectories of a chaotic dynamical system implies exactly the opposite, namely that its trajectories are not complex at all.

Indeed it is natural to define complex a dynamical system such that almost all its trajectories, symbolically codified, are logical deep.

So, despite the the still common fashion to adopt the terms "chaoticity" and "complexity" as synonymous, one has that that every chaotic dynamical system is shallow, as I will show in section II and section III.

The key point of such an issue is so important to deserve a further repetition with the own words of Charles Bennett [8] illustrating the physical meaning of the notion of logical depth and the reason why it is a good measure of physical complexity:

"The notion of logical-depth developed in the present paper was first described in [12], and at greater length in [13] and [14]; similar notions have been independently introduced by [15] ("potential"), [16] ("incomplete sequence"), [17] ("hitting time") and Koppel, this volume ("sophistication"). See also Wolfram’s work on "computational irreducibility" [18] and Hartmanis’ work on time- and space-bounded algorithmic information [19]."

"We propose depth as a formal measure of value. From the earliest days of information theory it has been appreciated that information per se is not a good measure of message value. For example a typical sequence of coin tosses has high information content but little value; an ephemeris, giving the positions of the moon and planets every day for a hundred years, has no more information than the equations of motions and initial conditions from which it was calculated, but saves its owner the effort of recalculating these positions. The value of a message thus appears to reside not in its information (its absolutely unpredictable parts), nor in its obvious redundancy (verbatim repetitions, unequal digit frequencies), but rather in what might be called its buried redundancy - parts predictable only with difficulty, things the receiver could in principle have figured out without being told, but only at considerable cost in money, time or computation. In other words the value of a message is the amount of mathematical or other work plausibly done by its originator, which its receiver is saved from having to repeat".

The quantum analogue of such a notion, i.e. quantum logical depth, is introduced in section IV.

The definition of the physical complexity of a quantum dynamical system is then introduced in section V where it is shown that in the quantum case, as in the classical case, a physically complex dynamical system is not chaotic.
II. THE DEFINITION OF THE PHYSICAL COMPLEXITY OF STRINGS AND SEQUENCES OF CBITS

I will follow from here and beyond the notation of my Phd-thesis [11]; consequentially, given the binary alphabet \( \Sigma := \{0, 1\} \), I will denote by \( \Sigma^* \) the set of all the strings on \( \Sigma \) (i.e. the set of all the strings of cbits) by \( \Sigma^\infty \) the set of all the sequences on \( \Sigma \) (i.e. the set of all the sequences of cbits) and by \( CHAITIN - RANDOM(\Sigma^\infty) \) its subset consisting of all the Martin-Löf-Solovay-Chaitin random sequences of cbits.

I will furthermore denote strings by an upper arrow and sequences by an upper bar, so that I will talk about the string \( \vec{x} \in \Sigma^* \) and the sequence \( \bar{x} \in \Sigma^\infty \); \( |\vec{x}| \) will denote the length of the string \( \vec{x} \), \( x_n \) will denote the \( n^{th} \)-digit of the string \( \vec{x} \) or of the sequence \( \bar{x} \) while \( \vec{x}_n \) will denote its prefix of length \( n \).

I will, finally, denote by \( <_l \) the lexicographical-ordering relation over \( \Sigma^* \) and by \( \text{string}(n) \) the \( n \)th string in such an ordering.

Fixed once and for all a universal Chaitin computer \( U \), let us recall the following basic notions:

Given a string \( \vec{x} \in \Sigma^* \) and a natural number \( n \in \mathbb{N} \):

**Definition II.1**
*canonical program of \( \vec{x} \):*

\[
\vec{x}^* := \min_{<_l} \{ \vec{y} \in \Sigma^* : U(\vec{y}) = \vec{x} \} \tag{2.1}
\]

**Definition II.2**
\( \vec{x} \) is \( n \)-compressible:

\[
|\vec{x}^*| \leq |\vec{x}| - n \tag{2.2}
\]

**Definition II.3**
\( \vec{x} \) is \( n \)-incompressible:

\[
|\vec{x}^*| > |\vec{x}| - n \tag{2.3}
\]

**Definition II.4**
*halting time of the computation with input \( \vec{x} \):*

\[
T(\vec{x}) := \begin{cases} 
\text{number of computational steps after which } U \text{ halts on input } \vec{x}, & \text{if } U(\vec{x}) = \downarrow \\
+\infty, & \text{otherwise.}
\end{cases} \tag{2.4}
\]

We have at last all the ingredients required to introduce the notion of *logical depth* as to strings.

Given a string \( \vec{x} \in \Sigma^* \) and two natural number \( s,t \in \mathbb{N} \):

**Definition II.5**
*logical depth of \( \vec{x} \) at significance level \( s \):*

\[
D_s(\vec{x}) := \min\{T(\vec{y}) : U(\vec{y}) = \vec{x}, \vec{y} \text{ s-incompressible} \} \tag{2.5}
\]

**Definition II.6**
\( \vec{x} \) is \( t \)-deep at significance level \( s \):

\[
D_s(\vec{x}) > t \tag{2.6}
\]
Definition II.7

\(\bar{x}\) is \(t\)-shallow at significance level \(s\):

\[
D_s(\bar{x}) \leq t
\]  

(2.7)

I will denote the set of all the \(t\)-deep strings as \(t - DEEP(\Sigma^*)\) and the set of all the \(t\)-shallow strings as \(t - SHALLOW(\Sigma^*)\).

Exactly as it is impossible to give a sharp distinction among \(Chaitin-random\) and \(regular\) strings while it is possible to give a sharp distinction among \(Martin-L"of-Solovay-Chaitin-random\) and \(regular\) sequences, it is impossible to give a sharp distinction among \(deep\) and \(shallow\) strings while it is possible to give a sharp distinction among \(deep\) and \(shallow\) sequences.

Given a sequence \(\bar{x} \in \Sigma^\infty\):

Definition II.8

\(\bar{x}\) is strongly deep:

\[
\text{card}\{ n \in \mathbb{N} : D_s(\bar{x}(n)) > f(n) \} < \aleph_0 \quad \forall s \in \mathbb{N}, \forall f \in REC - MAP(\mathbb{N}, \mathbb{N})
\]  

(2.8)

where, following once more the notation adopted in [11], \(REC - MAP(\mathbb{N}, \mathbb{N})\) denotes the set of all the (total) recursive functions over \(\mathbb{N}\).

To introduce a weaker notion of depth it is necessary to fix the notation as to reducibilities and degrees:

denoted the \(Turing\ reducibility\) by \(\leq_T\) and the polynomial time \(Turing\ reducibility\) by \(\leq_P^T\) [20] let us recall that there is an intermediate constrained-reducibility among them: the one, called \(recursive\ time\ bound\ reducibility\), in which the halting-time is constrained to be not necessarily a polynomial but a generic recursive function; since \(recursive\ time\ bound\ reducibility\) may be proved to be equivalent to \(truth-table\ reducibility\) (I demand to [21], [7] for its definition and for the proof of the equivalence) I will denote it by \(\leq_{tt}\).

A celebrated theorem proved by Peter Gacs in 1986 [22] states that every sequence is computable by a Martin L"of-Solovay-Chaitin-random sequence:

Theorem II.1

\(Gacs'\ Theorem:\)

\[
\bar{x} \leq_T \bar{y} \quad \forall \bar{x} \in \Sigma^\infty, \forall \bar{y} \in CHAITIN - RANDOM(\Sigma^\infty)
\]  

(2.9)

This is no more true, anyway, if one adds the constraint of recursive time bound, leading to the following:

Definition II.9

\(\bar{x}\) is weakly deep:

\[
\exists \bar{y} \in CHAITIN - RANDOM(\Sigma^\infty) : \neg(\bar{x} \leq_{tt} \bar{y})
\]  

(2.10)

I will denote the set of all the strongly-deep binary sequences by \(STRONGLY - DEEP(\Sigma^\infty)\) and the set of all the weakly-deep binary sequences as \(WEAKLY - DEEP(\Sigma^\infty)\).

Shallowness is then once more defined as the opposite of depth:

Definition II.10

\(strongly\-shallow\ sequences\ of\ cbits:\)

\[STRONGLY - SHALLOW(\Sigma^\infty) := \Sigma^\infty - (STRONGLY - DEEP(\Sigma^\infty))\]

(2.11)

Definition II.11

\(weakly\-shallow\ sequences\ of\ cbits:\)

\[WEAKLY - SHALLOW(\Sigma^\infty) := \Sigma^\infty - (WEAKLY - DEEP(\Sigma^\infty))\]

(2.12)

Weakly-shallow sequences of cbits may also be characterized in the following useful way [8]:

[8]:
Theorem II.2

*Alternative characterization of weakly-shallow sequences of cbits:*

\[ \bar{x} \in W E A K L Y - S H A L L O W ( \Sigma^\infty) \iff \exists \mu_{\text{recursive}} : \bar{x} \in \mu - R A N D O M ( \Sigma^\infty) \]  \hspace{1cm} (2.13)

where, following once more the notation of \[\text{II}, \quad \mu - R A N D O M ( \Sigma^\infty) \] denotes the set of all the Martin-Löf random sequences w.r.t. the measure \( \mu \).

As to sequences of cbits, the considerations made in section \[\text{I}\] may be thoroughly formalized through the following:

**Theorem II.3**

*Weak-shallowness of Martin Löf - Solovay - Chaitin random sequences:*

\[ C H A I T I N - R A N D O M ( \Sigma^\infty) \cap W E A K L Y - D E E P ( \Sigma^\infty) = \emptyset \]  \hspace{1cm} (2.14)

PROOF:

Since the Lebesgue measure \( \mu_{\text{Lebesgue}} \) is recursive and by definition:

\[ C H A I T I N - R A N D O M ( \Sigma^\infty) = \mu_{\text{Lebesgue}} - R A N D O M ( \Sigma^\infty) \]  \hspace{1cm} (2.15)

the thesis immediately follows by theorem \[\text{II.2} \] ■
III. THE DEFINITION OF THE PHYSICAL COMPLEXITY OF CLASSICAL DYNAMICAL SYSTEMS

Since much of the fashion about complexity is based on a spread confusion among different notions, starting from the basic difference among plain Kolmogorov complexity $K$ and algorithmic information $I$, much care has to be taken.

Let us start from the following notions by Brudno:

**Definition III.1**

*Brudno algorithmic entropy* of $\bar{x} \in \Sigma^\infty$:

$$B(\bar{x}) := \lim_{n \to \infty} \frac{K(\bar{x}(n))}{n} \quad (3.1)$$

At this point one could think that considering the asymptotic rate of algorithmic information instead of plain Kolmogorov complexity would result in a different definition of the algorithmic entropy of a sequence.

That this is not the case is the content of the following:

**Theorem III.1**

$$B(\bar{x}) = \lim_{n \to \infty} \frac{I(\bar{x}(n))}{n} \quad (3.2)$$

**PROOF:**

It immediately follows by the fact that:

$$|I(\bar{x}(n)) - K(\bar{x}(n))| \leq o(n) \quad (3.3)$$

**Definition III.2**

$\bar{x} \in \Sigma^\infty$ is Brudno-random:

$$B(\bar{x}) > 0 \quad (3.4)$$

I will denote the set of all the Brudno random binary sequences by $BRUDNO(\Sigma^\infty)$.

One great source of confusion in a part of the literature arises from the ignorance of the following basic result proved by Brudno himself [9]:

**Theorem III.2**

Brudno randomness is weaker than Chaitin randomness:

$$BRUDNO - RANDOM(\Sigma^\infty) \supset CHAITIN - RANDOM(\Sigma^\infty) \quad (3.5)$$

as we will see in the sequel of this section.

Following the analysis performed in [11] (to which I demand for further details) I will recall here some basic notion of Classical Ergodic Theory:

given a classical probability space $(X, \mu)$:

**Definition III.3**

*endomorphism* of $(X, \mu)$:

$$T : HALTING(\mu) \to HALTING(\mu)$$

surjective:

$$\mu(A) = \mu(T^{-1}A) \quad \forall A \in HALTING(\mu) \quad (3.6)$$

where $HALTING(\mu)$ is the halting-set of the measure $\mu$, namely the $\sigma$-algebra of subsets of $X$ on which $\mu$ is defined.

**Definition III.4**
classical dynamical system: 
a triple \((X, \mu, T)\) such that:

- \((X, \mu)\) is a classical probability space
- \(T : HALTING(\mu) \to HALTING(\mu)\) is an endomorphism of \((X, \mu)\)

Given a classical dynamical system \((X, \mu, T)\):

**Definition III.5**

\((X, \mu, T)\) is ergodic:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(A \cap T^k(B)) = \mu(A)\mu(B) \forall A, B \in HALTING(\mu)
\]

**Definition III.6**

\(n\)-letters alphabet:

\[
\Sigma_n := \{0, \cdots, n-1\}
\]

Clearly:

\[
\Sigma_2 = \Sigma
\]

**Definition III.7**

finite measurable partition of \((X, \mu)\):

\[
A = \{A_0, \cdots A_{n-1}\} n \in \mathbb{N}:
A_i \in HALTING(\mu) \quad i = 0, \cdots, n-1
A_i \cap A_j = \emptyset \quad \forall i \neq j
\mu(X - \bigcup_{i=0}^{n-1} A_i) = 0
\]

I will denote the set of all the finite measurable partitions of \((X, \mu)\) by \(\mathcal{P}(X, \mu)\).

Given two partitions \(A = \{A_i\}_{i=0}^{n-1}\) and \(B = \{B_j\}_{j=0}^{m-1}\) \(\in\) \(\mathcal{P}(X, \mu)\):  

**Definition III.8**

\(A\) is a coarse-graining of \(B\) \((A \preceq B)\):

- every atom of \(A\) is the union of atoms by \(B\)

**Definition III.9**

coarsest refinement of \(A = \{A_i\}_{i=0}^{n-1}\) and \(B = \{B_j\}_{j=0}^{m-1}\) \(\in\) \(\mathcal{P}(X, \mu)\):

\[
A \vee B \in \mathcal{P}(X, \mu)
A \vee B := \{A_i \cap B_j \mid i = 0, \cdots, n-1 \quad j = 0, \cdots, m-1\}
\]

Clearly \(\mathcal{P}(X, \mu)\) is closed both under coarsest refinements and under endomorphisms of \((X, \mu)\).

Let us observe that, beside its abstract, mathematical formalization, the definition III.7 has a precise operational meaning.

Given the classical probability space \((X, \mu)\) let us suppose to make an experiment on the probabilistic universe it describes using an instrument whose distinguishing power is limited in that it is not able to distinguish events belonging to the same atom of a partition \(A = \{A_i\}_{i=0}^{n-1}\) \(\in\) \(\mathcal{P}(X, \mu)\).

Consequentially the outcome of such an experiment will be a number:

\[
r \in \Sigma_n
\]
specifying the observed atom $A_i$ in our coarse-grained observation of $(X, \mu)$.

I will call such an experiment an operational observation of $(X, \mu)$ through the partition $A$.

Considered another partition $B = \{B_j\}_{j=0}^{n-1} \in \mathcal{P}(X, \mu)$ we have obviously that the operational observation of $(X, \mu)$ through the partition $A \lor B$ is the conjunction of the two experiments consisting in the operational observations of $(X, \mu)$ through the partitions, respectively, $A$ and $B$.

Consequentially we may consistently call an operational observation of $(X, \mu)$ through the partition $A$ an $A$-experiment.

The experimental outcome of an operational observation of $(X, \mu)$ through the partition $A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)$ is a classical random variable having as distribution the stochastic vector

\[
\begin{pmatrix}
\mu(A_0) \\
\vdots \\
\mu(A_{n-1})
\end{pmatrix}
\]

whose classical probabilistic information, i.e. its Shannon entropy, I will call the entropy of the partition $A$, according to the following:

**Definition III.10**

entropy of $A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)$:

\[
H(A) := H\left(\begin{pmatrix}
\mu(A_0) \\
\vdots \\
\mu(A_{n-1})
\end{pmatrix}\right)
\]

(3.13)

It is fundamental, at this point, to observe that, given an experiment, one has to distinguish between two conceptually different concepts:

1. the uncertainty of the experiment, i.e. the amount of uncertainty on the outcome of the experiment before to realize it

2. the information of the experiment, i.e. the amount of information gained by the outcome of the experiment

As lucidly observed by Patrick Billingsley [24], the fact that in Classical Probabilistic Information Theory both these concepts are quantified by the Shannon entropy of the experiment is a consequence of the following:

**Theorem III.3**

The soul of Classical Information Theory:

\[
\text{information gained} = \text{uncertainty removed}
\]

(3.14)

Theorem III.3 applies, in particular, as to the partition-experiments we are discussing.

Let us now consider a classical dynamical system $CDS := (X, \mu, T)$.

The $T$-invariance of $\mu$ implies that the partitions $A = \{A_i\}_{i=0}^{n-1}$ and $T^{-1}A := \{T^{-1}A_i\}_{i=0}^{n-1}$ have equal probabilistic structure. Consequentially the $A$-experiment and the $T^{-1}A$-experiment are replicas, not necessarily independent, of the same experiment, made at successive times.

In the same way the $\lor_{k=0}^{n-1} T^{-k}A$-experiment is the compound experiment consisting of $n$ repetitions $A, T^{-1}A, \cdots, T^{-(n-1)}A$ of the experiment corresponding to $A \in \mathcal{P}(X, \mu)$.

The amount of classical information per replication we obtain in this compound experiment is clearly:

\[
\frac{1}{n} H(\lor_{k=0}^{n-1} T^{-k}A)
\]

It may be proved [25] that when $n$ grows this amount of classical information acquired per replication converges, so that the following quantity:

\[
h(A, T) := \lim_{n \to \infty} \frac{1}{n} H(\lor_{k=0}^{n-1} T^{-k}A)
\]

(3.15)

exists.

In different words, we can say that $h(A, T)$ gives the asymptotic rate of acquired classical information per replication of the $A$-experiment.

We can at last introduce the following fundamental notion originally proposed by Kolmogorov for K-systems and later extended by Yakov Sinai to arbitrary classical dynamical systems [26], [5], [27], [28], [29]:
Definition III.11

Kolmogorov-Sinai entropy of CDS:

\[ h_{\text{CDS}} := \sup_{A \in \mathcal{P}(X, \mu)} h(A, T) \]  
(3.16)

By definition we have clearly that:

\[ h_{\text{CDS}} \geq 0 \]  
(3.17)

Definition III.12

CDS is chaotic:

\[ h_{\text{CDS}} > 0 \]  
(3.18)

Definition III.12 shows explicitly that the concept of classical-chaos is an information-theoretic one:
a classical dynamical system is chaotic if there is at least one experiment on the system that, no matter how many
times we insist on repeating it, continues to give us classical information.

That such a meaning of classical chaoticity is equivalent to the more popular one as the sensible (i.e. exponential)
dependence of dynamics from the initial conditions is a consequence of Pesin’s Theorem stating (under mild
assumptions) the equality of the Kolmogorov-Sinai entropy and the sum of the positive Lyapunov exponents.

This inter-relation may be caught observing that:

- if the system is chaotic we know that there is an experiment whose repetition definitely continues to give
  information: such an information may be seen as the information on the initial condition that is necessary to
  furnish more and more with time if one want to keep the error on the prediction of the phase-point below a
certain bound
- if the system is not chaotic the repetition of every experiment is useful only a finite number of times, after which
every further repetition doesn’t furnish further information

Let us now consider the issue of symbolically translating the coarse-grained dynamics following the traditional way
of proceeding described in the second section of [29]; given a number \( n \in \mathbb{N} : n \geq 2 \) let us introduce the following:

Definition III.13

\( n \)-adic value:

the map \( v_n : \Sigma_n^\infty \mapsto [0, 1] \):

\[ v_n(\bar{x}) := \sum_{i=1}^{\infty} \frac{x_i}{n^i} \]  
(3.19)

the more usual notation:

\( (0.x_1 \cdots x_m \cdots)_n := v_n(\bar{x}) \bar{x} \in \Sigma_n^\infty \)  
(3.20)

and the following:

Definition III.14

\( n \)-adic nonterminating natural positional representation:

the map \( r_n : [0, 1] \mapsto \Sigma_n^\infty \):

\[ r_n((0.x_1 \cdots x_i \cdots)_n) := \bar{x} \]  
(3.21)

with the nonterminating condition requiring that the numbers of the form \((0.x_1 \cdots x_i(n-1))_n = (0.\cdots(x_i + 1)0)_n \) are mapped into the sequence \( x_1 \cdots x_i(n-1) \).

Given \( n_1, n_2 \in \mathbb{N} : \min(n_1, n_2) \geq 2 \):

Definition III.15
change of basis from $n_1$ to $n_2$:
the map $cb_{n_1,n_2} : \Sigma_{n_1}^\infty \rightarrow \Sigma_{n_2}^\infty$:

$$cb_{n_1,n_2}(\bar{x}) := r_{n_2}(v_{n_1}(\bar{x})) \quad (3.22)$$

It is important to remark that [7]:

Theorem III.4

Basis-independence of randomness:

$$RANDOM(\Sigma_{n_2}^\infty) = cb_{n_1,n_2}(RANDOM(\Sigma_{n_1}^\infty)) \quad \forall n_1, n_2 \in \mathbb{N} : \min(n_1, n_2) \geq 2 \quad (3.23)$$

Considered a partition $A = \{A_i\}_{i=0}^{n-1} \in \mathcal{P}(X, \mu)$:

**Definition III.16**  

**symbolic translator of CDS w.r.t. $A$:**

$$\psi_A : X \rightarrow \Sigma_n :$$

$$\psi_A(x) := i : x \in A_i \quad (3.24)$$

In this way one associates to each point of $X$ the letter, in the alphabet having as many letters as the number of atoms of the considered partition, labeling the atom to which the point belongs.

Concatenating the letters corresponding to the phase-point at different times one can then codify $k \in \mathbb{N}$ steps of the dynamics:

**Definition III.17**

**k-point symbolic translator of CDS w.r.t. $A$:**

$$\psi_A^{(k)} : X \rightarrow \Sigma_n^k :$$

$$\psi_A^{(k)}(x) := \sum_{j=0}^{k-1} \psi_A(T^j x) \quad (3.25)$$

and whole orbits:

**Definition III.18**

**orbit symbolic translator of CDS w.r.t. $A$:**

$$\psi_A^{(\infty)} : X \rightarrow \Sigma_n^\infty :$$

$$\psi_A^{(\infty)}(x) := \sum_{j=0}^{\infty} \psi_A(T^j x) \quad (3.26)$$

The asymptotic rate of acquisition of plain Kolmogorov complexity of the binary sequence obtained translating symbolically the orbit generated by $x \in X$ through the partition $A \in \mathcal{P}(X, \mu)$ is clearly given by:

**Definition III.19**

$$B(A, x) := B(cb_{\text{card}(A), 2}(\psi_A^{\infty}(x))) \quad (3.27)$$

We saw in definition III.1 that the Kolmogorov-Sinai entropy was defined as $K(A, x)$ computed on the more probabilistically-informative $A$-experiment; in the same way the Brudno algorithmic entropy of $x$ is defined as the value of $B(A, x)$ computed on the more algorithmically-informative $A$-experiment:

**Definition III.20**

**Brudno algorithmic entropy of (the orbit starting from) $x$:**

$$B_{\text{CDS}}(x) := \sup_{A \in \mathcal{P}(X, \mu)} B(cb_{\text{card}(A), 2}(\psi_A^{(\infty)}(x))) \quad (3.28)$$

Demanding to [11] for further details, let us recall that, as it is natural for different approaches of studying a same object, the probabilistic approach and the algorithmic approach to Classical Information Theory are deeply linked:
the partial map $D_I : \Sigma^* \to \Sigma^*$ defined by:

$$D_I(\vec{x}) := \vec{x}^*$$  \hspace{1cm} (3.29)

is by construction a prefix-code of pure algorithmic nature, so that it would be very reasonable to think that it may be optimal only for some ad hoc probability distribution, i.e. that for a generic probability distribution $P$ the average code word length of $D_I$ w.r.t. $P$:

$$L_{D_I, P} = \sum_{\vec{x} \in \text{HALTING}(D_I)} P(\vec{x}) I(\vec{x})$$  \hspace{1cm} (3.30)

won’t achieve the optimal bound, the Shannon information $H(P)$, stated by the cornerstone of Classical Probabilistic Information, i.e. the following celebrated:

**Theorem III.5**

*Classical noiseless coding theorem:*

$$H(P) \leq L_P \leq H(P) + 1$$  \hspace{1cm} (3.31)

(where $L_P$ is the minimal average code word length allowed by the distribution $P$)

Contrary, the deep link between the probabilistic-approach and the algorithmic-approach makes the miracle: under mild assumptions about the distribution $P$ the code $D_I$ is optimal as it is stated by the following:

**Theorem III.6**

*Link between Classical Probabilistic Information and Classical Algorithmic Information:*

HP:

$P$ recursive classical probability distribution over $\Sigma^*$

TH:

$$\exists c_P \in \mathbb{R}_+ : 0 \leq L_{D_I, P} - H(P) \leq c_P$$  \hspace{1cm} (3.32)

With an eye at theorem III.1 it is then natural to expect that such a link between classical probabilistic information and classical algorithmic information generates a link between the asymptotic rate of acquisition of classical probabilistic information and the asymptotic rate of acquisition of classical algorithmic information of the coarse grained dynamics of CDS observed by repetitions of the experiments for which each of them is maximal.

Demanding to [10] for further details such a reasoning, properly formalized, proves the following:

**Theorem III.7**

*Brudno’s theorem:*

HP:

CDS ergodic

TH:

$$h_{CDS} = B_{CDS}(x) \quad \forall - \mu \text{-a.e. } x \in X$$  \hspace{1cm} (3.33)

Let us now consider the algorithmic approach to Classical Chaos Theory strongly supported by Joseph Ford, whose objective is the characterization of the concept of chaoticity of a classical dynamical system as the algorithmic-randomness of its symbolically-translated trajectories.

To require such a condition for all the trajectories would be too restrictive since it is reasonable to allow a chaotic dynamical system to have a countable number of periodic orbits.

Let us then introduce the following two notions:
Definition III.21

*CDS is strongly algorithmically-chaotic:*

\[ \forall \mu - a.e. x \in X, \exists A \in \mathcal{P}(X, \mu) : c_b_{\text{card}(A), 2}(\psi^\infty_A(x)) \in \text{CHAITIN} - \text{RANDOM}(\Sigma^\infty) \] (3.34)

Definition III.22

*CDS is weak algorithmically-chaotic:*

\[ \forall \mu - a.e. x \in X, \exists A \in \mathcal{P}(X, \mu) : c_b_{\text{card}(A), 2}(\psi^\infty_A(x)) \in \text{BRUDNO} - \text{RANDOM}(\Sigma^\infty) \] (3.35)

The difference between definition III.21 and definition III.22 follows by Theorem III.2. Clearly Theorem III.7 implies the following:

**Corollary III.1**

\[ \text{chaoticity} = \text{weak algorithmic chaoticity} \]
\[ \text{chaoticity} < \text{strong algorithmic chaoticity} \]

that shows that the algorithmic approach to Classical Chaos Theory is equivalent to the usual one only in weak sense.

The plethora of wrong statements found in a part of the literature caused by the ignorance of corollary III.1 is anyway of little importance if compared with the complete misunderstanding of the difference existing among the concepts of chaoticity and complexity for classical dynamical systems; with this regards the analysis made in section II may be now thoroughly formalized introducing the following natural notions:

Definition III.23

*CDS is strongly-complex:*

\[ \forall \mu - a.e. x \in X, \exists A \in \mathcal{P}(X, \mu) : c_b_{\text{card}(A), 2}(\psi^\infty_A(x)) \in \text{STRONGLY} - \text{COMPLEX}(\Sigma^\infty) \] (3.36)

Definition III.24

*CDS is weakly-complex:*

\[ \forall \mu - a.e. x \in X, \exists A \in \mathcal{P}(X, \mu) : c_b_{\text{card}(A), 2}(\psi^\infty_A(x)) \in \text{WEAKLY} - \text{COMPLEX}(\Sigma^\infty) \] (3.37)

One has that:

**Theorem III.8**

*Weak-shallowness of chaotic dynamical systems:*

\[ \text{CDS chaotic} \Rightarrow \text{CDS weakly-shallow} \]

**PROOF:**

The thesis immediately follows combining theorem III.3 with the definitions def III.23 and def III.24.
The idea that the physical complexity of a quantum object has to be measured in terms of a quantum analogue of Bennett's notion of logical depth has been first proposed by Michael Nielsen \cite{nielsen1,nielsen2}.

Unfortunately, beside giving some general remark about the properties he thinks such a notion should have, Nielsen have not given a mathematical definition of it.

The first step in this direction consists, in my opinion, in considering that, such as the notion of classical-logical-depth belongs to the framework of Classical Algorithmic Information Theory, the notion of quantum-logical-depth belongs to the framework of Quantum Algorithmic Information Theory \cite{svozil}.

One of the most debated issues in such a discipline, first discussed by its father Karl Svozil \cite{svozil2} and rediscovered later by the following literature \cite{svozil3,svozil4,svozil5,svozil6,svozil7,svozil8,svozil9}, is whether the programs of the involved universal quantum computers have to be strings of cbits or strings of qubits.

As I have already noted in \cite{svozil9}, anyway, it must be observed that, owing to the natural bijection among the computational basis $E^\otimes$ of the Hilbert space of qubits’ strings (notions that I am going to introduce) and $\Sigma^\otimes$, one can always assume that the input is a string of qubits while the issue, more precisely restated, is whether the input has (or not) to be constrained to belong to the computational basis.

So, denoted by $H_2 := \mathbb{C}^2$ the one-qubit’s Hilbert space (endowed with its orthonormal computational basis $E_2 := \{|i> , i \in \Sigma\}$), denoted by $H_2^n := \bigotimes_{k=0}^{n} H_2$ the $n$-qubits’ Hilbert space, (endowed with its orthonormal computational basis $E_n := \{|\vec{x}> , \vec{x} \in \Sigma^n\}$), denoted by $H_2^\otimes := \bigoplus_{n=0}^{\infty} H_2^n$ the Hilbert space of qubits’ strings (endowed with its orthonormal computational basis $E_\otimes := \{|\vec{x}> , \vec{x} \in \Sigma^\otimes\}$) and denoted by $H_2^\otimes^\infty := \bigotimes_{n\in\mathbb{N}} H_2$ the Hilbert space of qubits’ sequences (endowed with its orthonormal computational rigged-basis $E_\infty := \{|\vec{x}> , \vec{x} \in \Sigma^\infty\}$), one simply assumes that, instead of being a classical Chaitin universal computer, $U$ is a quantum Chaitin universal computer, i.e. a universal quantum computer whose input, following Svozil’s original position on the mentioned issue, is constrained to belong to $E_\otimes$, and is such that, w.r.t. the natural bijection among $E_\otimes$ and $\Sigma^\otimes$, satisfies the usual Chaitin constraint of having prefix-free halting-set.

The definition of the logical depth of a string of qubits is then straightforward: given a vector $|\psi> \in H_2^\otimes^\infty$ and a string $\vec{x} \in \Sigma^\otimes$:

**Definition IV.1**

canonical program of $|\psi>$:

$$|\psi>* := \min_{\vec{y} \in \Sigma^\otimes} \{ U(\vec{y}) = |\psi> \}$$  \hspace{1cm} (4.1)

**Definition IV.2**

halting time of the computation with input $|\vec{x}>$:

$$T(\vec{x}) := \begin{cases} \text{number of computational steps after which U halts on input } \vec{x} , & \text{if } U(\vec{x}) = \downarrow \\ +\infty, & \text{otherwise.} \end{cases}$$  \hspace{1cm} (4.2)

**Definition IV.3**

logical depth of $|\psi>$ at significance level $s$:

$$D_s(|\psi>) := \min\{T(\vec{y}) : U(\vec{y}) = |\psi>, \vec{y} \text{-s-incompressible} \}$$  \hspace{1cm} (4.3)

**Definition IV.4**

\footnote{as it should be obvious, the unusual locution rigged-basis I am used to adopt is simply a shortcut to denote that such a "basis" has to be intended in the mathematical sense it assumes when $H_2^\otimes^\infty$ is considered as endowed with a suitable rigging, i.e. as part of a suitable rigged Hilbert space $S \subset H_2^\otimes^\infty \subset S'$ as described in \cite{svozil10,svozil11}.}
$|\psi\rangle$ is t-deep at significance level $s$:

$$D_s(|\psi\rangle) > t \quad (4.4)$$

**Definition IV.5**

$|\psi\rangle$ is t-shallow at significance level $s$:

$$D_s(|\psi\rangle) \leq t \quad (4.5)$$

I will denote the set of all the t-deep strings of qubits as $t = DEEP(H_2^\otimes *)$.

Let us observe that a sharp distinction among depth and shallowness of qubits’ strings is impossible; this is nothing but a further confirmation of the fact, so many times shown and analyzed in [11], that almost all the concepts of Algorithmic Information Theory, both Classical and Quantum, have a clear, conceptually sharp meaning only when sequences are taken into account.

The great complication concerning sequences of qubits consists in that their mathematically-rigorous analysis requires to give up the simple language of Hilbert spaces passing to the more sophisticated language of noncommutative spaces; indeed, as extensively analyzed in [11] adopting the notion of noncommutative cardinality therein explicitly introduced, the fact that the correct noncommutative space of qubits’ sequences is the hyperfinite $II_1$ factor:

**Definition IV.6**

noncommutative space of qubits’ sequences:

$$\Sigma_{NC}^\infty := \bigotimes_{n=0}^\infty (M_2(C), \tau_{unbiased}) = R \quad (4.6)$$

and not the noncommutative space $B(H_2^\otimes \infty)$ of all the bounded linear operators on $H_2^\otimes \infty$ (that could be still managed in the usual language of Hilbert spaces) is proved by the fact that, as it must be, $\Sigma_{NC}^\infty$ has the continuum noncommutative-cardinality:

$$\text{card}_{NC}(\Sigma_{NC}^\infty) = \aleph_1 \quad (4.7)$$

while $B(H_2^\otimes \infty)$ has only the countable noncommutative cardinality:

$$\text{card}_{NC}(\Sigma_{NC}^\star) = \aleph_0 \quad (4.8)$$

While the definition of a strongly-deep sequence of cbits has no natural quantum analogue, the definition of a weakly-deep sequence of qubits is straightforward.

Denoted by $\text{RANDOM}(\Sigma_{NC}^\infty)$ the space of all the algorithmically random sequences of qubits, for whose characterization I demand to [11], let us observe that the equality between truth-table reducibility and recursive time bound reducibility existing as to Classical Computation may be naturally imposed to Quantum Computation in the following way:

given two arbitrary mathematical quantities x and y:

**Definition IV.7**

$x$ is quantum-truth-table reducible to $y$:

$$x \leq_{Qtt} y := x \text{ is U-computable from } y \text{ in bounded U-computable time} \quad (4.9)$$

Given a sequence of qubits $\bar{a} \in \Sigma_{NC}^\infty$:

**Definition IV.8**

---

2 following Miklos Redei’s many remarks [40], [41] mentioned in [11], about how Von Neumann considered his classification of factors as a theory of noncommutative cardinalities though he never thought, as well as Redei, that the same $\aleph$’s symbolism of the commutative case could be adopted.
\( \bar{a} \) is weakly-deep:

\[
\exists \bar{b} \in \text{RANDOM}(\Sigma_{\infty}^{NC}) : \neg (\bar{a} \leq_{\mu} \bar{b}) \tag{4.10}
\]

Denoted the set of all the weakly-deep sequences of qubits as \( W\text{EAKLY} - D\text{EEP}(\Sigma_{\infty}^{NC}) \):

**Definition IV.9**

set of all the weakly-shallow sequences of qubits:

\[
W\text{EAKLY} - S\text{HALLOW}(\Sigma_{\infty}^{NC}) := \Sigma_{\infty}^{NC} - (W\text{EAKLY} - D\text{EEP}(\Sigma_{\infty}^{NC})) \tag{4.11}
\]

It is natural, at this point, to conjecture that an analogue of theorem II.2 exists in Quantum Algorithmic Information Theory too:

**Conjecture IV.1**

*Alternative characterization of weakly-shallow sequences of qubits:*

\[
\bar{a} \in W\text{EAKLY} - S\text{HALLOW}(\Sigma_{\infty}^{NC}) \iff \exists \omega \in S(\Sigma_{\infty}^{NC}) U - \text{computable} : \bar{a} \in \omega - \text{RANDOM}(\Sigma_{\infty}^{NC}) \tag{4.12}
\]

where \( \omega - \text{RANDOM}(\Sigma_{\infty}^{NC}) \) denotes the set of all the \( \omega \)-random sequences of qubits w.r.t. the state \( \omega \in S(\Sigma_{\infty}^{NC}) \) to be defined generalizing the definition of \( \text{RANDOM}(\Sigma_{\infty}^{NC}) \) to states different by \( \tau_{\text{unbiased}} \) along the lines indicated in [11] as to the definition of the laws of randomness \( L_{\text{RANDOMNESS}}(\Sigma_{\infty}^{NC}, \omega) \) of the noncommutative probability space \( (\Sigma_{\infty}^{NC}, \omega) \).

As to sequences of qubits, the considerations made in the section I may be thoroughly formalized, at the prize of assuming the conjecture IV.1 as an hypothesis, through the following:

**Theorem IV.1**

*Weak-shallowness of random sequences of qubits:*

**HP:**

Conjecture IV.1 holds

**TH:**

\[
\text{RANDOM}(\Sigma_{\infty}^{NC}) \cap W\text{EAKLY} - D\text{EEP}(\Sigma_{\infty}^{NC}) = \emptyset \tag{4.13}
\]

**PROOF:**

Since the unbiased state \( \tau_{\text{unbiased}} \) is certainly U-computable and by definition:

\[
\text{RANDOM}(\Sigma_{\infty}^{NC}) = \tau_{\text{unbiased}} - \text{RANDOM}(\Sigma_{\infty}^{NC}) \tag{4.14}
\]

the assumption of the conjecture IV.1 as an hypothesis immediately leads to the thesis \( \blacksquare \)
V. THE DEFINITION OF THE PHYSICAL COMPLEXITY OF QUANTUM DYNAMICAL SYSTEMS

As we have seen in section III the Kolmogorov-Sinai entropy $h_{KS}(CDS)$ of a classical dynamical system $CDS := (X, \mu, T)$ has a clear physical information-theoretic meaning that we can express in the following way:

1. an experimenter is trying to obtain information about the dynamical evolution of $CDS$ performing repeatedly on the system a given experiment $exp \in EXPERIMENTS$,
2. $h(exp, CDS)$ is the asymptotic rate of acquisition of classical information about the dynamics of $CDS$ that he acquires replicating $exp$
3. $h_{KS}(CDS)$ is such an asymptotic rate, computed for the more informative possible experiment:

$$h_{KS}(CDS) = \sup_{exp \in EXPERIMENTS} h(exp, CDS) \quad (5.1)$$

Let us now pass to analyze quantum dynamical systems, for whose definition and properties I demand to [11].

Given a quantum dynamical system $QDS$ the physical information-theoretical way of proceeding would consist in analyzing the same experimental situation in which an experimenter is trying to obtain information about the dynamical evolution of $QDS$ performing repeatedly on the system a given experiment $exp \in EXPERIMENTS$:

1. to define $h(exp, QDS)$ as the asymptotic rate of acquisition of information about the dynamics of $QDS$ that he acquires replicating the experiment $exp$
2. to define the dynamical entropy of $QDS$ as such an asymptotic rate, computed for the more informative possible experiment:

resulting in the following:

Definition V.1
dynamical entropy of $QDS$:

$$h_{d.e.}(QDS) = \sup_{exp \in EXPERIMENTS} h(exp, QDS) \quad (5.2)$$

Definition V.2

$QDS$ is chaotic:

$$h_{d.e.}(QDS) > 0 \quad (5.3)$$

The irreducibility of Quantum Information Theory to Classical Information Theory, caused by the fact that theorem [13] doesn’t extend to the quantum case owing to the existence of some non-accessible information about a quantum system (as implied by the Grönwald-Lindblad-Holevo Theorem) and the consequent irreducibility of the qubit to the cbit [12], [11], would then naturally lead to the physical issue whether the information acquired by the experimenter is classical or quantum, i.e. if $h_{d.e.}(QDS)$ is a number of cbits or a number of qubits.

Such a physical approach to quantum dynamical entropy was performed first by Göran Lindblad [43] and later refined and extended by Robert Alicki and Mark Fannes resulting in the so called Alicki-Lindblad-Fannes entropy [44].

Many attempts to define a quantum analogue of the Kolmogorov-Sinai entropy pursued, instead, a different purely mathematical approach consisting in generalizing noncommutatively the mathematical machinery of partitions and coarsest-refinements underlying the definition [11], obtaining mathematical objects whose (eventual) physical meaning was investigated subsequently.

This was certainly the case as to the Connes-Narnhofer-Thirring entropy, the entropy of Sauvageot and Thouvenot and Voiculescu’s approximation entropy [15], [16].

As to the Connes-Narnhofer-Thirring entropy, in particular, the noncommutative analogue playing the role of the classical partitions are the so called Abelian models whose (eventual) physical meaning is rather obscure since, as it has been lucidly shown by Fabio Benatti in his very beautiful book [17], they don’t correspond to physical experiments performed on the system, since even a projective-measurement (i.e. a measurement corresponding to a Projection Valued Measure) cannot, in general, provide an abelian model, owing to the fact that its reduction-formula corresponds
to a decomposition of the state of the system if and only if the measured observable belongs to the centralizer of the state of the system.

It may be worth observing, by the way, that the non-existence of an agreement into the scientific community as to the correct quantum analogue of the Kolmogorov-Sinai entropy and hence on the definition of quantum chaoticity shouldn’t surprise, such an agreement lacking even for the well more basic notion of quantum ergodicity. Zelditch’s quantum ergodicity [48] (more in the spirit of the original Von Neumann’s quantum ergodicity [49] to which it is not anyway clear if it reduces exactly as to quantum dynamical systems of the form \((A, \omega, \alpha)\) with \(\text{card}_{\text{NC}}(A) \leq \aleph_0\) and \(\alpha \in \text{INN}(A)\)) differing from Thirring’s quantum ergodicity [50] adopted both in [47] and in [44].

Returning, now, to the physical approach based on the definition V.1, the mentioned issue whether the dynamical entropy \(h_{\text{d.e.}}(QDS)\) is a measure of classical information or of quantum information (i.e. if it is a number of cbits or qubits) is of particular importance as soon as one tries to extend to the quantum domain Joseph Ford’s algorithmic approach to Chaos Theory seen in section III:

1. in the former case, in fact, one should define quantum algorithmic chaoticity by the requirement that almost all the trajectories, symbolically codified in a suitable way, belong to \(\text{BRUDNO}(\Sigma^\infty)\) for quantum weak algorithmic chaoticity and to \(\text{CHAITIN} – \text{RANDOM}(\Sigma^\infty)\) for quantum strong algorithmic chaoticity

2. in the latter case, instead, one should define quantum algorithmic chaoticity by the requirement that almost all the trajectories, symbolically codified in a suitable way, belong to \(\text{RANDOM}(\Sigma^\infty_{\text{NC}})\)

In any case one would then be tempted to conjecture the existence of a Quantum Brudno’s Theorem stating the equivalence of quantum chaoticity and quantum algorithmic chaoticity, at least in weak sense, for quantum ergodic dynamical systems.

The mentioned issue whether the dynamical entropy \(h_{\text{d.e.}}(QDS)\) is a measure of classical information or of quantum information (i.e. if it is a number of cbits or qubits) is of great importance also as to the definition of a deep quantum dynamical system (i.e. a physically-complex quantum dynamical system):

1. in the former case, in fact, one should define a strongly (weakly) - deep quantum dynamical system as a quantum dynamical system such that almost all its trajectories, symbolically codified in a suitable way, belong to \(\text{STRONGLY} – \text{DEEP}(\Sigma^\infty)\) (\(\text{WEAKLY} – \text{DEEP}(\Sigma^\infty)\)).

2. in the latter case, instead, one should define a weakly-deep quantum dynamical system as a quantum dynamical system such that almost all its trajectories, symbolically codified in a suitable way, belong to \(\text{WEAKLY} – \text{DEEP}(\Sigma^\infty_{\text{NC}})\)

In any case, or by the theorem III.8 or by the theorem IV.1, one would be almost certainly led to a quantum analogue of theorem III.8 stating that a chaotic quantum dynamical system is weakly-shallow, i.e. is not physically complex.
VI. ACKNOWLEDGEMENTS

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VII. NOTATION

∀
∀ − µ − a.e. 
∃ 
∃! 
x = y 
x := y 
¬p 
Σ 
Σn 
Σn* 
Σn∞ 
|⃗ x|
< l 
string(n) 
⃗ x 
.
xn 
⃗ x(n) 
⃗ x∗ 
K(⃗ x) 
I(⃗ x) 
U(⃗ x) ↓
Ds(⃗ x) 
t − DEEP(Σ*)
t − SHALLOW(Σ*)
REC − MAP(N, N)
≤T 
≤P 
≤TT
CHAITIN − RANDOM(Σ∞)
HALTING(µ)

for every (universal quantificator)
for µ — almost every
exists (existential quantificator)
exists and is unique
x is equal to y
x is defined as y
negation of p
binary alphabet {0, 1}
n-letters’ alphabet
strings on the alphabet Σn
sequences on the alphabet Σn
string
length of the string ⃗ x
lexicographical ordering on Σ*
nth string in lexicographic order
sequence
concatenation operator
nth digit of the string ⃗ x or of the sequence ¯x
prefix of length n of the string ⃗ x or of the sequence ¯x
canonical string of the string ⃗ x
plain Kolmogorov complexity of the string ⃗ x
algorithmic information of the string ⃗ x
U halts on input ⃗ x
logical-depth of ⃗ x at significance level s
t-deep strings of cbits
t-shallow strings of cbits
(total) recursive functions over N
Turing reducibility
polynomial time Turing reducibility
truth-table reducibility

Martin Lòf - Solovay - Chaitin random sequences of cbits
halting set of the measure µ
| Symbol/Notation | Description |
|-----------------|-------------|
| $\mu_{\text{Lebesgue}}$ | Lebesgue measure |
| $\mu - \text{RANDOM}(\Sigma^\infty)$ | Martin Löf random sequences of cbits w.r.t. $\mu$ |
| $B(\bar{x})$ | Brudno algorithmic entropy of the sequence $\bar{x}$ |
| $\text{BRUDNO}(\Sigma^\infty)$ | Brudno random sequences of cbits |
| $\text{STRONGLY - DEEP}(\Sigma^\infty)$ | strongly-deep sequences of cbits |
| $\text{STRONGLY - SHALLOW}(\Sigma^\infty)$ | strongly-shallow sequences of cbits |
| $\text{WEAKLY - DEEP}(\Sigma^\infty)$ | weakly-deep sequences of cbits |
| $\text{WEAKLY - SHALLOW}(\Sigma^\infty)$ | weakly-shallow sequences of cbits |
| $P(X, \mu)$ | (finite, measurable) partitions of $(X, \mu)$ |
| $\preceq$ | coarse-graining relation on partitions |
| $A \bigvee B$ | coarsest refinement of $A$ and $B$ |
| $h_{\text{CDS}}$ | Kolmogorov-Sinai entropy of CDS |
| $\psi_A$ | symbolic translator w.r.t. $A$ |
| $\psi_A^{(k)}$ | k-point symbolic translator w.r.t. $A$ |
| $\psi_A^{(\infty)}$ | orbit symbolic translator w.r.t. $A$ |
| $cb_{n_1, n_2}$ | change of basis from $n_1$ to $n_2$ |
| $B_{\text{CDS}}(x)$ | Brudno algorithmic entropy of $x$'s orbit in CDS |
| $H(P)$ | Shannon entropy of the distribution $P$ |
| $L_D, p$ | average code-word length w.r.t. the code $D$ and the distribution $P$ |
| $L_P$ | minimal average code-word length w.r.t. the distribution $P$ |
| $\mathcal{H}_n$ | one-qubit’s Hilbert space |
| $\mathcal{H}_n ^\otimes$ | n-qubits’ Hilbert space |
| $\mathcal{E}_n$ | computational basis of $\mathcal{H}_n ^\otimes$ |
| $\mathcal{H}_2 ^\otimes$ | Hilbert space of qubits’ strings |
| $\mathcal{E}_2$ | computational basis of $\mathcal{H}_2 ^\otimes$ |
| $\mathcal{H}_2 ^\otimes ^\star$ | Hilbert space of qubits’ sequences |
| $\mathcal{E}_2 ^\star$ | computational rigged-basis of $\mathcal{H}_2 ^\otimes ^\star$ |
| $\mathcal{H}_2 ^\otimes ^\infty$ | bounded linear operators on $\mathcal{H}$ |
\[|\psi \rangle^*\]
\[D_s(|\psi \rangle)\]
\[t - DEEP(H_2^\otimes^*)\]
\[t - SHALLOW(H_2^\otimes^*)\]
\[S(A)\]
\[\text{card}(A)\]
\[\text{INN}(A)\]
\[\text{card}_{NC}(A)\]
\[\tau_{\text{unbiased}}\]
\[\Sigma_{NC}^\infty\]
\[R\]
\[\text{RANDOM}(\Sigma_{NC}^\infty)\]
\[\leq^Q_{\text{d.e.}}\]
\[\text{WEAKLY - DEEP}(\Sigma_{NC}^\infty)\]
\[\text{WEAKLY - SHALLOW}(\Sigma_{NC}^\infty)\]
\[L_{NC}^{\text{RANDOMNESS}}(A, \omega)\]
\[\omega - \text{RANDOM}(\Sigma_{NC}^\infty)\]
\[h_{d.e.}(QDS)\]
\[\text{id est}\]
\[\text{exempli gratia}\]

canonical program of \(|\psi \rangle\)
logical depth of \(|\psi \rangle\) at significance level \(s\)
t-deep strings of qubits
t-shallow strings of qubits
states over the noncommutative space \(A\)
cardinality of \(A\)
inner automorphisms of \(A\)
noncommutative cardinality of \(A\)
unbiased noncommutative probability distribution
noncommutative space of qubits’ sequences
hyperfinite \(II_1\) factor
random sequences of qubits
quantum truth-table reducibility
weakly-deep sequences of qubits
weakly-shallow sequences of qubits
laws of randomness of \((A, \omega)\)
random sequences of qubits w.r.t. \(\omega\)
dynamical entropy of the quantum dynamical system QDS
id est
exempli gratia
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