Periodic orbits for a discontinuous vector field arising from a conceptual model of glacial cycles

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Abstract

Conceptual climate models provide an approach to understanding climate processes through a mathematical analysis of an approximation to reality. Recently, these models have also provided interesting examples of nonsmooth dynamical systems. Here we develop a new conceptual model of glacial cycles consisting of a system of three ordinary differential equations defining a discontinuous vector field. Our model provides a dynamical systems framework for a mechanism previously shown to play a crucial role in glacial cycle patterns, namely, an increased ice sheet ablation rate during deglaciations. We use ad hoc singular perturbation techniques to prove the existence of a large periodic orbit crossing the discontinuity boundary, provided the ice sheet edge moves sufficiently slowly relative to changes in the snow line and temperature. Numerical explorations reveal the periodic orbit exists when the time constant for the ice sheet edge has more moderate values.

Keywords: nonsmooth vector field, limit cycle, glacial cycle

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1. Introduction

In recent years an extensive theory of discontinuous differential equations has been built, based largely on the work of Filippov [7]. Although the general theory is far from complete
Conceptual models of the Earth’s ice sheets began with the work of Budyko [5] and Sellers [30]. Using a simple equation incorporating latitudinal variation of the incoming solar radiation and the ice-albedo feedback effect, Budyko was able to show the existence of a stable small ice cap. Widiasih [35] introduced an equation for the motion of the edge of the ice sheet (called the ice line), enabling the use of dynamical systems theory to examine the ice line dynamics. Subsequently, she and McGehee [19] introduced an approximation to the Budyko–Widiasih model, reducing the model to a system of two ordinary differential equations.

Many aspects of the Earth’s glacial cycles remain a mystery. For roughly the last million years, the Earth’s climate has experienced extensive cold periods with large ice sheets covering much of North America and Eurasia. These glacial periods have been followed by relatively rapid transitions to warmer temperatures as the ice sheets retreat. These so-called interglacial periods, which are shorter in duration, are typically followed by a slow descent into the subsequent glacial period, leading to the asymmetric sawtooth pattern evident in the paleoclimate data [27]. These glacial/interglacial cycles have occurred with a periodicity of roughly 100 000 years.

The explanation for the 100 000 year cycles usually appeals in some way to the work of Milankovitch [20], who examined the impact of variations in the Earth’s orbital parameters on the incoming solar radiation (called Milankovitch cycles). It turns out that the eccentricity of the Earth’s orbit changes with a period of about 100 000 years, corresponding with the glacial cycles of the last million years. However, the paleoclimate record shows that, for the period between five million and one million years ago, the glacial cycles occurred with a period of about 41 000 years, corresponding instead to changes in the Earth’s obliquity (axial tilt). Since the Milankovitch cycles have not changed significantly over the last five million years, it is clear that the Earth’s climate cycles cannot be explained as a simple response to the Milankovitch forcing.

Many models have been proposed to explain the response of the Earth’s climate to the Milankovitch cycles. One of the earliest was in a paper of Maasch and Saltzman [17], where they proposed that, for the last million years, the Earth’s climate system has had an intrinsic oscillation of 100 000 years, independent of the Milankovitch cycles. Dynamically speaking, the autonomous system has a stable periodic orbit with a period of about 100 000 years. When the Milankovitch cycles are introduced, the system becomes nonautonomous and falls into synchronization with the eccentricity cycles. The change that occurred one million years ago can be explained as slow changes in the parameters of the Earth’s climate system creating a Hopf bifurcation in the autonomous system, changing the response to the forcing.

Many other conceptual models have been introduced to study nonlinear climate feedbacks as they relate to the glacial cycles. These models often include a threshold that, once exceeded, triggers a switch from, say, a glacial to an interglacial period. In previous studies the threshold variable has been ice volume [12, 13]; a combination of Milankovitch forcing and ice volume [23, 25]; a function of ice volume, Milankovitch forcing and area of the Antarctic ice sheet [24]; and a switch connected to the rapid formation of sea ice [31]. In related work, a relaxation model is subjected to astronomical forcing to produce oscillations in global ice volume in [2]. In many of these works, model parameters are adjusted in attempts to generate output that qualitatively mirrors the paleoclimate data.

Starting with the McGehee–Widiasih version of Budyko’s equation [19] and inspired by the work of Källén et al [14], we consider Widiasih’s ice line variable to be the albedo
where the snow cover remains throughout the year, and introduce a new variable, corresponding to the extent of the ice sheet. The thinking here is that, during the glacial advance, the snow is accumulating (forming the glacier) at a lower latitude than the ice sheet edge, and hence that the albedo line is at a lower latitude than the ice sheet edge. On the other hand, the work of Wright and Stefanova [36] indicates that, during the glacial retreat, superglacial forests growing on the thinner, stagnant ice nearer the ice sheet margins create an albedo line at a latitude higher than that of the ice sheet edge. These phenomena give rise to what we think of as two distinct climate regimes, one occurring during the glacial advance and the other occurring during the glacial retreat. The boundary between these two regimes corresponds to a discontinuity in the vector field, allowing us to view the model as a Filippov system.

Our model is a system of three ordinary differential equations with a discontinuity boundary consisting of a plane in the three-dimensional state space. Using techniques reminiscent of singular perturbation theory, we show the existence of a large periodic orbit crossing the discontinuity boundary.

In terms of the theory of glacial cycles, this work represents only the beginning, analogous to the establishment of a periodic orbit in the Maasch and Satzman model. Missing is an analysis of the response to the Milankovitch forcing. The forcing will be easy to introduce into the model, since it is based on the Budyko model, where the Milankovitch cycles can be easily incorporated [18].

In terms of the mathematical theory, there may be some work to be done in studying the response of a Filippov system to time variation of the parameters. In addition, our analysis points to the need to develop a theory of singular perturbations for Filippov systems.

The paper is organized as follows. In the next section we introduce Budyko’s model and its coupling with a dynamic ice line. In section 3 we review the quadratic approximation introduced in [19], which we then augment by including a snow line independent of the edge of the ice sheet, as presented in section 4. This in turn leads to a discontinuous three-dimensional vector field, the dynamics of which we analyze in section 5. We point to several problems of mathematical interest generated by the study of the model and we summarize this work in the concluding section.

### 2. A coupled temperature-ice line model

Budyko’s energy balance model concerns the average annual temperature in latitudinal zones in a world assumed to be symmetric about the equator. While Budyko focused solely on equilibrium temperature distributions, we consider the time-dependent equation [11]

$$ R \frac{\partial T(y, t)}{\partial t} = Q_0(y)(1 - \alpha(y)) - (A + BT) - C(T - T). \tag{1} $$

In equation (1), $y$ is the sine of the latitude, chosen for convenience (the area of an infinitesimal latitudinal strip at $y$ is proportional to $dy$). Note that due to symmetry considerations, $y \in [0, 1]$, with $y = 0$ the equator and $y = 1$ the North Pole.

The function $T = T(y, t)$ (°C) is the annual mean surface temperature on the circle of latitude at $y$, and $R$ is the heat capacity of the Earth’s surface (with units $J \text{ (m}^2 \text{°C}^{-1}$). The left-hand side of (1) represents the change in energy stored in the Earth’s surface at $y$; units on each side of (1) are Watts per meter squared ($W \text{ m}^{-2}$).
\( Q \) denotes the mean annual incoming solar radiation (or insolation), a parameter depending on the eccentricity of Earth’s orbit [18]. The function \( s(y) \), which depends upon the obliquity of Earth’s orbit [18], accounts for the distribution of insolation across latitude, and satisfies

\[
\int_0^1 s(y) \, dy = 1.
\]

The function \( \alpha(y) \) represents the planetary albedo, a measure of the extent to which insolation is reflected back into space. Thus the \( Qs(y)(1 - \alpha(y)) \)-term represents the energy from the sun absorbed at the surface at ‘latitude’ \( y \).

The empirically-derived \((A + BT)\)-term models the outgoing longwave radiation (OLR) emitted by the Earth [9], while the transport of heat energy across latitudes is modeled by the \( C(T - T) \)-term, in which

\[
T = \int_0^1 T(y, t) \, dy
\]

is the global annual mean surface temperature. The constants \( A \), \( B \) and \( C \) are positive and empirical (for a more detailed introduction to Budyko’s EBM see [19, 32]).

In [19] the equilibrium temperature profiles of (1) are shown to be given by

\[
T^*(y) = \frac{1}{B + C} \left( Qs(y)(1 - \alpha(y)) - A + \frac{C}{B} (Q(1 - \sigma) - A) \right),
\]

where

\[
\sigma = \int_0^1 \alpha(y)s(y) \, dy.
\]

Letting \( p_0(y) = 1 \) and \( p_2(y) = \frac{1}{2}(3y^2 - 1) \) denote the first two even Legendre polynomials, we make use of the expression

\[
s(y) = s_0p_0(y) + s_2p_2(y), \quad s_0 = 1, \quad s_2 = -0.482,
\]

which is uniformly within 2% of the actual values of \( s(y) \) [21], in all that follows.

In Budyko’s model one assumes the planet has an ice cap, with ice at all latitudes above a certain latitude \( y = \eta \), and no ice south of \( y = \eta \). The edge of the ice sheet \( \eta \) is called the ice line. The albedo function is then given by

\[
\alpha(y) = \begin{cases} 
\alpha_1, & \text{if } y < \eta \\
\alpha_2, & \text{if } y > \eta 
\end{cases}, \quad \alpha_1 < \alpha_2,
\]

where \( \alpha_1 \) and \( \alpha_2 \) represent the albedos of surface with no ice cover and that having an ice cover, respectively. With this choice of albedo function, and using expression (4), equilibrium temperature profiles (2) are even, piecewise quadratic functions having a discontinuity at \( \eta \). We also note \( \eta \) serves to parametrize (3), and hence equilibrium functions (2), as well. In particular, there are infinitely many equilibrium temperature functions, one for each value of \( \eta \). We write \( T^*_\eta(y) \) when we wish to emphasize this parametrization.

Defining \( T^*_\eta(y) = \frac{1}{2}(\lim_{y \to -\eta} T^*(y) + \lim_{y \to \eta} T^*(y)) \), one finds the temperature at equilibrium at the ice line is given by

\[
T^*_\eta(\eta) = \frac{1}{B + C} \left( Qs(\eta)(1 - \alpha_0) - A + \frac{C}{B} (Q(1 - \sigma(\eta)) - A) \right).
\]
Budyko was interested in the existence of $\eta$-values for which
$T^{\eta}(\eta) = \eta^*$, where $T_c$ is a critical temperature above which ice melts and below which ice forms. Specifically, he investigated the relationship between the number of such $\eta$-values and the parameter $Q$.

As mentioned in the introductory section, Budyko was motivated by the study of the positive ice-albedo feedback: If temperatures were to decrease, an existing ice sheet would grow, increasing the albedo and further lowering temperatures, leading to an ever larger ice sheet. A warming climate would lead to a smaller ice sheet and reduced albedo, thereby raising temperatures and further reducing the size of the ice sheet.

Budyko found that for a range of $Q$-values there exist $12\eta < \eta^*$ with $T_{c} = -10^{\circ}C$, $i = 1, 2$, with $\eta_1$ in lower latitudes and $\eta_2$ nearer the North Pole. If $Q$ decreased sufficiently, however, there were no such $\eta$-values, with the implication being that the ice line would be positioned at the equator for these $Q$-values (a so-called Snowball Earth event). This was viewed as a consequence of ice-albedo feedback. Notably, previous analyses of Budyko’s model lacked any treatment of the stability of the ‘preferred’ equilibrium temperature distributions having $T^{\eta}(\eta) = T_c$ (or, indeed, of $T^{\eta}(y)$ for any $\eta$) from a dynamical systems perspective, that is, in the infinite-dimensional setting intrinsic to (1).

Additionally, Budyko’s model lacks any mechanism by which the ice line $\eta$ is allowed to respond to changes in temperature. This limitation was remedied by E. Widiasih in [35] through the addition of an ODE for the evolution of $\eta$, leading to the integro differential system

$$R \frac{\partial T}{\partial t} = Q(y)(1 - \alpha(y, \eta)) - (A + BT) - C(T - T_c) \quad (7a)$$

$$\frac{d\eta}{dt} = \rho(T(\eta, t) - T_c). \quad (7b)$$

Here, $\rho > 0$ is a parameter governing the relaxation time of the ice sheet. The temperature distribution $T(y, t)$ evolves according to Budyko’s equation (7a), while the dynamics of $\eta$ are determined by the temperature at the ice line, relative to the critical temperature. The ice sheet retreats toward the pole if $T(\eta, t) > T_c$, and moves equatorward if $T(\eta, t) < T_c$.

Working in the infinite-dimensional setting and with parameters as in Budyko’s work (the left column in table 1), Widiasih proved the existence of equilibrium temperature–ice line pairs $(T_\eta(y), \eta_i, i = 1, 2$, for $\rho$ sufficiently small. The ice line $y = \eta_2$ corresponds to a stable, small ice cap, while ice line $y = \eta_1$ corresponds to a large, unstable ice sheet (in particular,
there are no oscillations of the ice sheet in the model). This work provided Budyko’s pioneering and influential model with a modern dynamical systems perspective.

3. The approximation of McGehee and Widiasih

Recall equilibrium solutions of Budyko’s equation (1) are even and piecewise quadratic, with a discontinuity at $\eta$, when using albedo function (5) and expression (4). This motivated the introduction of a quadratic approximation to system (7) in [19]. We outline this approach here, referring the reader to [19] for all computational details.

Let $X$ denote the space of even, piecewise quadratic functions having a discontinuity at $\eta$. The four-dimensional linear space $X$ can be parametrized by the new variables $w_0, z_0, w_2$ and $z_2$ by letting

$$T(y) = \begin{cases} 
  w_0 + \frac{1}{2} z_0 + \left(w_2 + \frac{1}{2} z_2\right) p_2(y), & y < \eta \\
  w_0 - \frac{1}{2} z_0 + \left(w_2 - \frac{1}{2} z_2\right) p_2(y), & y > \eta \\
  w_0 + w_2 p_2(\eta), & y = \eta
\end{cases} \quad (8)$$

The choice of $T(\eta)$ in (8) coincides with $T(\eta) = \frac{1}{2}(\lim_{y\to\eta^-} T(y) + \lim_{y\to\eta^+} T(y))$, to maintain consistency with (6). One can show that, in these variables,

$$T = \int_0^1 T(y)dy = w_0 + z_0 \left(\eta - \frac{1}{2}\right) + z_2 p_2(\eta), \text{ where } p_2(\eta) = \int_0^\eta p_2(y)dy. \quad (9)$$

Plugging (8) and (4) into (1) and equating the coefficients of $p_0(y)$ (that is, the constant terms) and $p_2(y)$, respectively, yields the four equations

$$R \left(\dot{w}_0 + \frac{1}{2} \dot{z}_0\right) = Q(1 - \alpha_1) - A - (B + C) \left(w_0 + \frac{1}{2} z_0\right) + CT \quad (10a)$$

$$R \left(\dot{w}_0 - \frac{1}{2} \dot{z}_0\right) = Q(1 - \alpha_2) - A - (B + C) \left(w_0 - \frac{1}{2} z_0\right) + CT \quad (10b)$$

$$R \left(\dot{w}_2 + \frac{1}{2} \dot{z}_2\right) = Q s_2(1 - \alpha_1) - (B + C) \left(w_2 + \frac{1}{2} z_2\right) \quad (10c)$$

$$R \left(\dot{w}_2 - \frac{1}{2} \dot{z}_2\right) = Q s_2(1 - \alpha_2) - (B + C) \left(w_2 - \frac{1}{2} z_2\right) \quad (10d)$$

(two equations each for $y < \eta$ and $y > \eta$). Adding and subtracting equations (10a)–(10b) and (10c)–(10d) and substituting in expression (9) for $T$, one arrives at

$$R \dot{w}_0 = Q(1 - \alpha_0) - A - B w_0 + C \left(\eta - \frac{1}{2}\right) z_0 + z_2 P_2(\eta)$$

$$R \dot{z}_0 = Q(\alpha_2 - \alpha_1) - (B + C) z_0$$

$$R \dot{w}_2 = Q s_2(1 - \alpha_0) - (B + C) w_2$$

$$R \dot{z}_2 = Q s_2(\alpha_2 - \alpha_1) - (B + C) z_2. \quad (11)$$
Let \( L = Q/(B + C) \). Note system (11) admits a globally attracting invariant line \( \mathcal{L} \) on which 
\( z_0 = L(\alpha_2 - \alpha_1) \), \( w_2 = Ls_2(1 - \alpha_0) \), and \( z_2 = Ls_2(\alpha_2 - \alpha_1) \) are each at equilibrium. One can show that on \( \mathcal{L} \) system (11) reduces to the single equation
\[
R\dot{w} = -B(w - F(\eta)),
\]
where we have let \( w = w_0 \) for ease of notation, and where
\[
F(\eta) = \frac{1}{B} \left( Q(1 - \alpha_0) - A + CL(\alpha_2 - \alpha_1) \left( \eta - \frac{1}{2} + s_2P_2(\eta) \right) \right).
\]
We note \( F(\eta) \) is a cubic polynomial. Thus, equation (12) provides the approximation to equation (7a).

For equation (7b), one computes the expression for \( T(\eta) \) on the invariant line \( \mathcal{L} \). As detailed in [19], equation (7b) can be written in the form

\[
\dot{\eta} = \rho(w - G(\eta)),
\]

where

\[
G(\eta) = -Ls_2(1 - \alpha_0)p_2(\eta) + T_c,
\]

(14)

when restricting to \( \mathcal{L} \). Hence the infinite-dimensional system (7) is approximated by the system of ODEs

\[
\begin{align*}
\dot{w} &= -\tau(w - F(\eta)) \\
\dot{\eta} &= \rho(w - G(\eta)),
\end{align*}
\]

(15)

with \( F(\eta) \) given by (13), \( G(\eta) \) the quadratic polynomial (14), and \( \tau = B/R \). One can show that for fixed \( \eta \) the variable \( w \) is a translate of the global average temperature [19].

Phase planes for system (15) are shown in figure 1. Note the existence of equilibrium solutions is independent of \( \rho \) and \( \tau \). Moreover, in [19] it is shown that there exist a stable equilibrium point with ice line near the pole and a saddle equilibrium point with a large ice cap, for all \( \rho > 0 \) (as in figure 1(a)) and for standard parameter values (left column in table 1). This simple two-dimensional model captures the essence of the dynamics of the infinite-dimensional Budyko–Widiasih model (7).

As mentioned above, system (15) does not allow for glacial cycles, given that \( \eta \) either approaches an equilibrium position or the equator over time. Motivated by the role played by the relative sizes of the accumulation and ablation zones in glacial advance and retreat, we introduce a new snow line variable, independent of the edge of the ice sheet, thereby supplementing equations (15). We present this model in the following section.

4. Addition of a snow line

The accumulation and ablation of ice play a fundamental role in the theory of glacial cycles, serving to control the terminus advance and retreat, the ice volume, and the geometry of the surface of the ice sheet [3]. Significant motivation for our model stems from the study [1], in which numerical simulations of the glacial cycles over the past 400 kyr were carried out using an ice sheet model coupled to a general circulation model. These simulations typically produced two stable equilibrium ice line positions for a given set of parameters, one small and the other much larger. In addition, the larger the ice sheet at equilibrium, the larger was the ablation zone, leading to increased ice sheet instability with regard to Milankovitch forcing.

Most importantly, however, Abe-Ouchi et al found the fast retreat of the ice sheet was due to significantly enhanced ablation (due in turn to delayed isostatic rebound [1]). That is, the ablation rate for a large, advancing ice sheet was necessarily much smaller than the ablation rate for a retreating ice sheet, in order to faithfully reproduce the last four glacial cycles. While we do not model the response of the lithosphere to the size of the ice sheet, we do incorporate the findings in [1] into our conceptual model, namely, that retreating ice sheets exhibit an increased ablation rate. For further insight into the important role played by ablation rates (and the ratio of ablation rates to accumulation rates) in modeling glacial cycles see, for example, [8, 14, 22, 29, 31] and [33].
Motivation for our model was also drawn from a model of the thermohaline circulation presented in [34]. This simple ‘flip-flop’ conceptual model exhibits oscillations in the temperature and salinity of a well-mixed ocean layer. The model vector field has a line of discontinuity that produces a switch to the alternate regime when intersected by a trajectory. Similarities between our model and that presented in [34] will become evident below.

Consider once again system (15). Recall that for fixed $\eta$-values, $w$ is a translate of the global average temperature. We introduce independent snow and ice lines so as to incorporate accumulation and ablation zones. The model continues to depend upon latitude as discussed in sections 2 and 3.

We begin by recasting the role played by $\eta$, interpreting $\eta$ henceforth as the snow line. We let $\xi$ denote the ice line, that is, the edge of the ice sheet (see figure 2). The ablation zone has extent $\eta - \xi$ (when $\eta > \xi$), while the accumulation zone has size $1 - \eta$.

We continue to use albedo function (5). We note the albedo of the portion of the ice sheet in the ablation zone is much smaller than that of the ice sheet lying poleward of $\eta$. As mentioned previously, superglacial forests nearer the ice sheet margins reduce albedo [36]. Additionally, as the old ice emerges and melts in the ablation zone, a surface layer of dust that was originally laid down with snowfall high on the ice sheet is created [4]. Locally increased winds further cover the surface in the ablation zone with dust and debris [26]. Albedo value $\alpha_1$ in (5) might then be interpreted as an average of the albedos of the planet’s surface south of $\xi$ and the darkened (relative to snow) ice surface in the ablation zone.

The use of independent ice and snow lines was also partially due to the desire to include ice volume in this latitude-dependent model (A strong argument for the inclusion of ice volume in any EBM is given in [22]). While our model does not explicitly include volume, the latter can be recovered from $\xi$ were we to assume a specific ice sheet shape as, for example, in [33]. Finally, the model allows for different time scales for the snow line and the (more slowly moving) ice line.

### 4.1. Model equations

The temperature-ice line-snow line model is a nonsmooth system with state space

$$B = \{ (w, \eta, \xi) : w \in \mathbb{R}, \eta \in [0, 1], \xi \in [0, 1] \}.$$
where $\xi$ denotes the ice line (edge of the ice sheet), $\eta$ denotes the snow line, and $w$ is related to global average temperature as described in section 3. To define the model equations, pick parameters $b_0 < b < b_1$ representing ablation rates, and a parameter $a$ denoting the accumulation rate. When $b(\eta - \xi) - a(1 - \eta) < 0$ we assume the ice sheet advances, and we set

$$\dot{w} = -\tau(w - F(\eta)) = f_1^I(w, \eta, \xi) \quad (16a)$$

$$\dot{\eta} = \rho(w - G_\eta(\eta)) = f_2^I(w, \eta, \xi) \quad (16b)$$

$$\dot{\xi} = \epsilon(b_0(\eta - \xi) - a(1 - \eta)) = f_3^I(w, \eta, \xi). \quad (16c)$$

The function $F(\eta)$ in (16a) is given by (13), $G_\eta(\eta)$ in (16b) is given by (14) but with $T_c = T_c^* = -5.5$ °C, and $\epsilon > 0$ is a time constant for the movement of the ice line.

When $b(\eta - \xi) - a(1 - \eta) > 0$ we assume the ice sheet retreats, and we set

$$\dot{w} = -\tau(w - F(\eta)) = f_1^R(w, \eta, \xi) \quad (17a)$$

$$\dot{\eta} = \rho(w - G_\eta(\eta)) = f_2^R(w, \eta, \xi) \quad (17b)$$

$$\dot{\xi} = \epsilon(b_1(\eta - \xi) - a(1 - \eta)) = f_3^R(w, \eta, \xi). \quad (17c)$$

The function $F(\eta)$ in (17a) is again given by (13), while $G_\eta(\eta)$ in (17b) is given by (14) with $T_c = T_c^* = -10$ °C.

The assumption that the ablation rates satisfy $b_0 < b_1$ was motivated by [1] as described above. The choice of different $T_c$-values is motivated by works such as [28] and [31]. In [28], $T_c$ is set to 0 °C in modeling extensive glacial episodes in the Earth’s past, while a linear interpolation between $T_c = -13$ °C and $T_c = -3$ °C is introduced in the glacial cycle model presented in [31].

We thus arrive at a three-dimensional system having a plane of discontinuity

$$\Sigma = \{(w, \eta, \xi) : b(\eta - \xi) - a(1 - \eta) = 0\} = \{(w, \eta, \xi) : \xi = \left(1 + \frac{a}{b}\right)\eta - \frac{a}{b} \equiv \gamma(\eta)\}. \quad (18)$$

As we will see, a trajectory in $(w, \eta, \xi)$-space passing through $\Sigma$ switches from advancing mode to glacial retreat, or vice versa, similar in spirit to the flip-flop model in [34].

Let

$$S_+ = \{(w, \eta, \xi) \in B : \xi < \gamma(\eta)\} \text{ and } S_- = \{(w, \eta, \xi) \in B : \xi > \gamma(\eta)\},$$

with $\gamma(\eta)$ as in (18). Called a discontinuity boundary (or switching boundary), $\Sigma$ is a hyperplane separating $B$ into subspaces $S_+$ and $S_-$. Consider the vector fields

$$V_{\pm} : S_{\pm} \to \mathbb{R}^3, \quad V_{\pm}(w, \eta, \xi) = (f_1^\pm(w, \eta, \xi), f_2^\pm(w, \eta, \xi), f_3^\pm(w, \eta, \xi));$$

noting $V_{\pm}$ each extend smoothly to $\Sigma$. For $x = (w, \eta, \xi) \in B$, consider the differential inclusion

$$x \in V(x), \quad \{1 - q\} V_+(x) + q V_- (x) : q \in [0, 1], \quad x \in \Sigma, \quad x \in S_+. \quad (19)$$

Solutions, while in $S_-$, have flow $\phi_-(x, t)$ given by $x = V_-(x)$, and solutions have flow $\phi_+(x, t)$ given by $x = V_+(x)$ while in $S_+$. For $x \in \Sigma$, $x$ must lie in the closed convex hull of the two vectors $V_-(x)$ and $V_+(x)$. 

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A solution to \((19)\) in the sense of Filippov is an absolutely continuous function \(x(t)\) satisfying \(\dot{x} \in V(x)\) for almost all \(t\). (Note \(x(t)\) is not defined at times for which \(x(t)\) arrives at or leaves \(\Sigma\).) Given that \(V_x\) are continuous on \(S_\pm \cup \Sigma\), the set-valued map \(V(x)\) is upper semi-continuous, and closed, convex and bounded for all \(x \in B\) and \(t \in \mathbb{R}\). This implies that for each \(x_0 \in \text{Int}(B)\) there is a solution \(x(t)\) to differential inclusion \((19)\) in the sense of Filippov, defined on an interval \([0, t_f]\), with \(x(0) = x_0\) [16].

We now turn to an analysis of the model.

5. Model analysis

We first note model equations \((16)\) and \((17)\) partially decouple: As long as a trajectory remains in \(S_- \) or \(S_+\), changes in \(\xi\) play no role in the evolution of \(w\) and \(\eta\). This simple observation will prove useful in the following analysis.

We set several parameter values as in table 1. We note most of these parameter values only serve to approximate complex aspects of the climate system. Nonetheless, these values seem appropriate for the conceptual models considered in this paper. The parameters in the left column of table 1, which are aligned with the modern climate, have appeared in several analyses of Budyko’s model ([19, 32] and references therein).

The ablation rates \(b_0\) and \(b_1\) and the accumulation rate \(a\) in table 1 are not well-constrained; their choice was motivated by the rates used in [33], for which the ratio of ablation to accumulation rates is roughly 3. The values for time constants \(\tau\) and \(\rho\) are somewhat arbitrary and play no role in the proof of our main result.

5.1. Equilibria

An equilibrium point \(Q_{+,w}^* = (w_{+,w}^*, \eta_{+,w}^*, \xi_{+,w}^*)\) for \((17)\) satisfies \(w_{+,w}^* = F(\eta_{+,w}^*) = G_s(\eta_{+,w}^*)\) and \(\xi_{+,w}^* = (1 + \frac{a}{b_1})\eta_{+,w}^* - \frac{a}{b_1}\). In [19] it was shown that, with parameters as in table 1, the (decoupled) subsystem \((17a)–(17b)\) has a ‘small ice cap’ sink \((w_{+,w}^*, \eta_{+,w}^*) \approx (5.08, 0.95)\) and a ‘large ice cap’ saddle \((w_{+,w}^*, \eta_{+,w}^*) \approx (-17.26, 0.25)\) (see figure 1(a)), and no other equilibria having \(\eta \in [0, 1]\). Hence system \((17)\) has two equilibria

\[Q_{+,w}^* = \left\{ w_{+,w}^*; \eta_{+,w}^*; \xi_{+,w}^* \left(1 + \frac{a}{b_1}\right)\eta_{+,w}^* - \frac{a}{b_1}\right\}\]

The third column of the Jacobian matrix \(J_e\) for \((17)\) is \([0 \quad 0 \quad -eb_1]^T\); combined with the results in [19], this implies \(J_e(Q_{+,w}^*_e)\) has three negative eigenvalues, while \(J_e(Q_{+,w}^*_{+,w})\) has two negative eigenvalues and one positive eigenvalue.

We note

\[\xi_{+,w}^* = \left(1 + \frac{a}{b_1}\right)\eta_{+,w}^* - \frac{a}{b_1}\left(1 + \frac{a}{b}\right)\eta_{+,w}^* - \frac{a}{b} = \gamma(\eta_{+,w}^*),\]

implying that \(Q_{+,w}^* \in S_-\). Such an equilibrium point is called a virtual equilibrium point [6]; a \(\phi_*\)-trajectory in \(S_+\) will intersect \(\Sigma\) (and switch to the flow \(\phi_\cdot\)) before having the chance to approach \(Q_{+,w}^*\).

Definition 5.1. Let \(x^* \in B\).

(i) \(x^*\) is a regular equilibrium point of \((19)\) if either \(V_e(x^*) = 0\) and \(x^* \in S_+\), or if \(V_e(x^*) = 0\) and \(x^* \in S_-\).


(ii) \( x^* \) is a virtual equilibrium point of (19) if either 
\[ V(x^*) = \eta^* \] and \( x^* \in S_{-} \), or if 
\[ V(x^*) = -\eta^* \] and \( x^* \in S_{+} \).

(iii) \( x^* \) is a boundary equilibrium point of (19) if 
\[ V(x) = \eta^* - \eta^* \] and \( x \in \Sigma \).

Similarly, one can show system (16) has two equilibria, a sink 
\( Q_{-} = (w_{-}, \eta_{-}, s_{-}, \xi_{-}) \), and 
a saddle \( Q_{+} = (w_{+}, \eta_{+}, u_{+}, \xi_{+}) \) having a two-dimensional stable manifold. Once again the sink \( Q_{-} \) is virtual as 
\[ \gamma(\xi_{-}) < 0. \]

A key observation can be gleaned from figure 1. First note subsystems (16a)–(16b) and (17a)–(17b) differ only in that \( G_0(\eta) = G_0(\eta) + (T^c - T^c) \). As indicated in figure 3, this simply moves \( \eta_{-} \) and \( \eta_{+} \) closer together, relative to \( \eta_{-} \) and \( \eta_{+} \).

Let \( \psi_{+} \) denote the flow corresponding to subsystem (17a)–(17b) (figure 1(a)), and let \( \psi_{-} \) denote the flow corresponding to subsystem (16a)–(16b) (figure 1(b)). Note the stable set of the sink \( (w_{-}, \eta_{-}, s_{-}) \) under the flow \( \psi_{-} \) is contained in the stable set of the sink \( (w_{+}, \eta_{+}, u_{+}) \) under the flow \( \psi_{+} \). Given the linear nature of equations (16c)–(17c) (along with the aforementioned decoupling), it follows that the stable set of the sink \( Q_{+} \) under the ‘advancing’ flow \( \phi_{+} \) (system (16)) is a subset of the stable set of the sink \( Q_{+} \) under the ‘retreating’ flow \( \phi_{-} \) (system (17)).

We now turn to the behavior of trajectories intersecting the discontinuity boundary.

5.2. Behavior on the discontinuity boundary \( \Sigma \)

The discontinuity boundary \( \Sigma \) given by (18) is a plane with normal vector 
\[ N = [w \ \eta \ \xi]^T = [0 \ 1 + \frac{a}{b} \ -1]^T. \]

For \( x \in \Sigma \), a computation yields \( V(x) \perp N \) if and only if
Thus $V_+$ is tangent to $\Sigma$ only on the parabola $\Lambda_+$ \( \{(w, \eta, \xi) \in \Sigma : w = g_+(\eta)\} \) (the solid curve in figure 4). Furthermore, given \( x = (w, \eta, \xi) \in \Sigma, V_+(x) \cdot \mathbf{N} < 0 \) for \( w < g_+(\eta) \), and \( V_+(x) \cdot \mathbf{N} > 0 \) for \( w > g_+(\eta) \). Referring to the orientation provided in figure 4, $\phi_+$ trajectories pass through $\Sigma$ below $\Lambda_+$ from front to back, while $\phi_-$ trajectories pass through $\Sigma$ above $\Lambda_+$ from back to front.

One can show there are points $p_1$ and $p_2$ on $\Lambda_+$, with $p_1$ near $Q_{s, +}$ and $p_2$ near $Q_{s, +}^*$, such that points on $\Lambda_+$ other than $p_1$ and $p_2$ are fold singularities \cite{15}. Points on $\Lambda_+$ between $p_1$ and $p_2$ are invisible tangencies, and points above $p_2$ and below $p_1$ are invisible tangencies.

Similar computations reveal that $V_-$ is tangent to $\Sigma$ if and only if

\[
w = G_-(\eta) + \frac{er(1 - \eta)(b_1 - b)}{\rho(a + b)} \equiv g_-(\eta).
\] (20)

that is, only on the parabola $\Lambda_- = \{(w, \eta, \xi) \in \Sigma : w = g_-(\eta)\}$ (the dashed curve in figure 4). Given \( x = (w, \eta, \xi) \in \Sigma, V_-(x) \cdot \mathbf{N} > 0 \) for \( w > g_-(\eta) \), and \( V_-(x) \cdot \mathbf{N} < 0 \) for \( w < g_-(\eta) \). Thus $\phi_-$ trajectories pass through $\Sigma$ below $\Lambda_-$ from front to back, while $\phi_-$ trajectories pass through $\Sigma$ above $\Lambda_-$ from back to front.

There are points $q_1$ and $q_2$ on $\Lambda_-$, with $q_1$ near $Q_{s, -}^*$ and $q_2$ near $Q_{s, -}^*$, such that points on $\Lambda_-$ other than $q_1$ and $q_2$ are fold singularities. Points on $\Lambda_-$ between $q_1$ and $q_2$ are invisible tangencies, and points above $q_2$ and below $q_1$ are visible tangencies.
Let
\[ \Sigma_{SL} = \{(w, \eta, \xi) \in \Sigma : g_+(\eta) < w < g_-(\eta)\}. \]
For \( x \in \Sigma_{SL}, V_s(x) \cdot N > 0 \) (so \( V_s(x) \) points into \( S_+ \)), while \( V_s(x) \cdot N < 0 \) (so \( V_s(x) \) points into \( S_- \)). The subset \( \Sigma_{SL} \) of the discontinuity boundary \( \Sigma \) is therefore a repelling sliding region [16]; Filippov’s approach does not provide for unique solutions \( x(t) \) in forward time if \( x(0) \in \Sigma_{SL} \).

Let
\[ \Sigma_+ = \{(w, \eta, \xi) \in \Sigma : w < g_+(\eta)\} \quad \text{and} \quad \Sigma_- = \{(w, \eta, \xi) \in \Sigma : w > g_-(\eta)\}. \]
Unique solutions to (19) do exist for trajectories passing through \( \Sigma_+ \) or \( \Sigma_- \), as \( V_s(x) \cdot N \) and \( V_s(x) \cdot N \) have the same sign on each of \( \Sigma_+ \) and \( \Sigma_- \). A \( \phi_t \)-trajectory in \( S_+ \) intersecting \( \Sigma_+ \) at \( x \) will cross \( \Sigma \) transversally, becoming a \( \phi_t \)-trajectory at \( x \). Similarly, a \( \phi_t \)-trajectory in \( S_- \) intersecting \( \Sigma_- \) at \( y \) will cross \( \Sigma \) transversally, becoming a \( \phi_t \)-trajectory at \( y \). In this scenario a unique (though nonsmooth) solution of (19) exists, in the sense of Filippov.

5.3. Section maps for the Filippov flow

One can show the curves of tangency \( \Lambda_{\pm} \) of \( V_s \) with the discontinuity boundary \( \Sigma \) do not intersect in \( B \), as in figure 4, provided
\[ \epsilon < \frac{(T_0^- - T_0^+)\rho(a + b)}{a(b_1 - b_0)}. \quad (22) \]
We assume \( \epsilon \) satisfies this bound in all that follows.

Plotted in figure 4 is the two-dimensional stable manifold \( M \) of the saddle point \( Q_{s,u,w}^* \) under the flow \( \phi_w \). Due to the \((w, \eta)\)-decoupling, the projection of \( M \) onto the \((w, \eta)\)-plane is simply the (one-dimensional) stable manifold of \((w^*, \eta^*)\) under the flow \( \psi_\epsilon \) given by subsystem (16a)–(16b) and plotted in figure 1(b). In particular, we note the plot of the surface \( M \) is independent of \( \epsilon > 0 \).

As only the stable rest points figure in any further analysis, we set \( Q_{s,u}^* = Q_{s,u,w}^* \) and \( Q_{s}^* = Q_{s,u,w}^* \) in all that follows.

\( M \) partitions \( \Sigma \) into two subsets, a ‘top’ (larger \( w \)) and a ‘bottom’ (smaller \( w \)). Denote this top subset by \( T \). For any point \( x = (w, \eta, \xi) \in T, \phi_w(x, t) = (w(t), \eta(t), \xi(t)) \rightarrow Q_w^* \). This follows from the fact \((w(t), \eta(t)) \rightarrow (w^*, \eta^*) \) as \( t \rightarrow \infty \) and hence \( \xi(t) \rightarrow \xi_w^* \) by equation (16e).

Thus \( T \subseteq W_s^t(Q_w^*) \), the stable set of the sink \( Q_w^* \) under the flow \( \phi_w \). By the observations made in section 5.1, it follows \( T \subseteq W_s^t(Q_w^*) \), the stable set of the sink \( Q_w^* \) under the flow \( \phi_w \). Again, all of this holds for any \( \epsilon > 0 \).

Let \( \Gamma_{\epsilon} = \Sigma_{\epsilon} \cap T \), the set of points in \( \Sigma \) above \( M \) and below \( \Lambda_{\pm} \) (see figure 4). Let \( \Gamma_{\epsilon} = \Sigma_{\epsilon} \cap T \), the set of points in \( \Sigma \) above \( M \) and above \( \Lambda_{\pm} \).

We have that for any \( \epsilon > 0 \), \( \Gamma_{\epsilon} \subseteq W_s^t(Q_w^*) \) and \( V_s(x) \cdot N < 0 \) for all \( x \in \Gamma_{\epsilon} \). Hence for any \( \epsilon > 0 \), the \( \phi_w \)-trajectory starting at \( x \in \Gamma_{\epsilon} \) enters \( S_- \) before necessarily intersecting \( \Gamma_{\epsilon} \) as it seeks to approach the (virtual) sink \( Q_w^* \). For any \( \epsilon > 0 \) and for each \( x \in \Gamma_{\epsilon} \), there is then a \( t = t(x, \epsilon) > 0 \) such that \( \phi_w(x, t) \in \Gamma_{\epsilon} \). We write \( t(x, \epsilon) = t(x, \epsilon) \) to emphasize the dependence of this \( t \)-value on \( \epsilon \) as well as on \( x \). We note, for future reference, that \( t(x, \epsilon) \rightarrow \infty \) as \( \epsilon \rightarrow 0^+ \).

This provides, for any \( \epsilon > 0 \), a continuous map
\[ r_{\epsilon}^- : \Gamma_{\epsilon} \rightarrow \Gamma_{\epsilon}, \quad r_{\epsilon}^-(x) = \phi_w(x, t(x, \epsilon)). \quad (23) \]
Similarly, for any \( W \neq 0 \), \( \phi \) and \( V \) are such that \( y \in \Gamma \) for all \( y \). Hence for any \( \epsilon > 0 \), the \( \phi \)-trajectory starting at \( y \in \Gamma \) enters \( S_+ \) before necessarily intersecting \( \Gamma \) as it tries to approach the (virtual) sink \( Q_+^* \). For any \( \epsilon > 0 \) and for each \( y \in \Gamma \), there exists a \( t_+=t(y, \epsilon) > 0 \) such that \( \phi(y, t) \in \Gamma_+ \). This defines a continuous map

\[
\phi_+^*(y) = \phi(y, t(y, \epsilon)),
\]

which exists for any \( \epsilon > 0 \).

We then have a continuous section map \( r_+^* \circ r_-^* : \Gamma_+ \rightarrow \Gamma_+ \), defined for any \( \epsilon > 0 \). We set \( r_+ = r_+^* \circ r_-^* \) in all that follows.

### 5.4. Existence of an attracting periodic orbit

In this section we prove Filippov system (19) has a unique attracting periodic orbit. Intuition for this result arises from consideration of the case \( \epsilon = 0 \), for which each of systems (16) and (17) has an attracting line of equilibria of the form \( (w_{-,r}, \eta_{-,r}, \gamma(\eta_{-,r})) \) respectively (see figure 5). It will be convenient to let

\[
\mathbf{Z}_+ = (w_{+,r}, \eta_{+,r}, \gamma(\eta_{+,r})) \quad \text{and} \quad \mathbf{Z}_- = (w_{-,r}, \eta_{-,r}, \gamma(\eta_{-,r}));
\]

note these points each lie in the discontinuity boundary \( \Sigma \).

When \( \epsilon = 0 \), the trajectory starting at \( \mathbf{Z}_+ \) converges to the regular equilibrium point \( (w_{+,r}, \eta_{+,r}, \gamma(\eta_{+,r})) \) under the advancing flow \( \phi_+ \), while the trajectory starting at \( \mathbf{Z}_- \) converges to the regular equilibrium point \( (w_{+,r}, \eta_{+,r}, \gamma(\eta_{+,r})) \) under the retreating flow \( \phi_- \).

For \( \epsilon > 0 \) system (19) has two virtual equilibria (as in figure 7(a)), and for \( \epsilon \) sufficiently small one might expect there to be a Filippov periodic orbit approximating the rectangular ‘orbit’ implicit in figure 5. In spirit this problem has the flavor of geometric singular perturbation problems, although in our model the existence of a discontinuity boundary precludes the use of existing theory.

We show instead that for any compact subset \( D_+ \) of \( \Gamma_+ \) containing \( \mathbf{Z}_+ \) in its interior, and for all sufficiently small \( \epsilon > 0 \), the section map satisfies (i) \( r_+(D_+) \subset D_+ \), and (ii) \( r_+ \) is a contraction map on \( D_+ \). We begin with the following proposition.
**Proposition 5.4.**

(i) Given $c_1 \in (0, 1)$ and a compact set $D_+ \subset \Gamma_+$, there exists $\epsilon_1 > 0$ such that for all $\epsilon \leq \epsilon_1$ and for all $x_0, x_1 \in D_+, \|r^+_\epsilon(x_0) - r^+_\epsilon(x_1)\| \leq c_1 \|x_0 - x_1\|$. 

(ii) Given $c_2 \in (0, 1)$ and a compact set $D_+ \subset \Gamma_+$, there exists $\epsilon_2 > 0$ such that for all $\epsilon \leq \epsilon_2$ and for all $y_0, y_1 \in D_+, \|r^+_\epsilon(y_0) - r^+_\epsilon(y_1)\| \leq c_2 \|y_0 - y_1\|$. 

Part (i) above essentially follows from the fact that, as $\epsilon$ decreases to 0, $(w(t), \eta(t))$ gets ever closer to $(w^*_\epsilon, \eta^*_\epsilon)$ before the corresponding $\phi_\epsilon$-trajectory hits the discontinuity boundary (an analogous statement holds for part (ii)). We defer the proof of proposition 5.4 to the appendix.

To use proposition 5.4 to show $r_\epsilon$ is a contraction map, we need a compact set $D_+ \subset \Gamma_+$ such that $r_\epsilon(D_+) \subset D_+$. The points $Z_+$ and $Z_-$ defined in (25) prove useful here. We note that $Z_+ \in \Sigma_+$ (so $Z_+$ is below the solid curve in figure 4), which follows from the fact $w^\epsilon_{t, \epsilon} = G_\epsilon(\eta^*_\epsilon, \eta^*_\epsilon) < g_\epsilon(\eta^*_\epsilon, \eta^*_\epsilon)$, which in turn is a consequence of (20). Similarly, as $w^{\epsilon, \epsilon} = G_\epsilon(\eta^*_\epsilon, \eta^*_\epsilon) > g_\epsilon(\eta^*_\epsilon, \eta^*_\epsilon)$ by equation (21), we have $Z_- \in \Sigma_-$. ($Z_-$ is above the dashed curve in figure 4).

**Proposition 5.5.**

(i) Let $D_+$ be any compact subset of $\Gamma_+$ with $Z_+ \in \text{Int}(D_+)$. Let $D_-$ be any compact subset of $\Gamma_-$. There exists $\epsilon_3 > 0$ such that for all $\epsilon \leq \epsilon_3$, $r^+_\epsilon(D_+) \subset D_+$. 

(ii) Let $D_-$ be any compact subset of $\Gamma_-$ with $Z_- \in \text{Int}(D_-)$. Let $D_+$ be any compact subset of $\Gamma_+$. There exists $\epsilon_4 > 0$ such that for all $\epsilon \leq \epsilon_4$, $r^-_\epsilon(D_-) \subset D_-$.

**Proof.** We prove (i). Let $x = (w_0, \eta_0, \gamma(\eta_0)) \in D_+$. Pick $\delta > 0$ such that $U_\delta = B_{\delta}(Z_+) \cap \Sigma_+ \subset D_+$. Recall $\psi_\epsilon$ denotes the flow corresponding to subsystem (17a)–(17b). Note $\psi_\epsilon((w_0, \eta_0, t) = (w(t), \eta(t)) \rightarrow (w^\epsilon_{t, \epsilon}, \eta^\epsilon_{t, \epsilon})$ as $t \rightarrow \infty$. Additionally noting $\gamma(\eta)$ is continuous, there exists $T = T(x) > 0$ such that for all $t \geq T$, 

$$\|(w(t), \eta(t), \gamma(\eta(t))) - Z_+\| < \delta.$$ 

Pick $\epsilon(x) > 0$ such that $t(x, \epsilon(x)) > T$ (for $t(x, \epsilon(x))$ as in (24)). Then for all $\epsilon \leq \epsilon(x)$, $r^+_\epsilon(x) \in B_{\delta}(Z_+) \cap \Sigma_+$. 

Let $c_2 \in (0, 1)$, $c_2 < \delta / \text{diam}(D_+)$. Pick $\epsilon_2 > 0$ as in proposition 5.4(ii). Let $\epsilon_3 = \min \{\epsilon(x), \epsilon_2\}$. For $\epsilon \leq \epsilon_3$ and $y \in D_+$, 

$$\|r^+_\epsilon(y) - r^+_\epsilon(x)\| \leq c_2 \|y - x\| \leq c_2 \text{diam}(D_+) < \delta,$$

implying 

$$\|r^+_\epsilon(y) - Z_+\| \leq \|r^+_\epsilon(y) - r^+_\epsilon(x)\| + \|r^+_\epsilon(x) - Z_+\| < 2\delta.$$ 

Hence, $r^+_\epsilon(D_+) \subset U_\delta \subset D_+$. The proof of (ii) is similar.

**Theorem 5.6.** With parameters as in table 1, there exists $\hat{\epsilon} > 0$ so that for all $\epsilon \leq \hat{\epsilon}$, system (19) admits a unique attracting periodic orbit.

**Proof.** Let $D_+ \subset \Gamma_+$, with $D_+$ compact and $Z_+ \in \text{Int}(D_+)$. Let $D_+ \subset \Gamma_+$, with $D_-$ compact and $Z_- \in \text{Int}(D_-)$. Given $c_1, c_2 \in (0, 1)$, pick $\epsilon_1 > 0$ and $\epsilon_2 > 0$ as in proposition 5.4 Choose $\epsilon_3 > 0$ and $\epsilon_4 > 0$ as in proposition 5.5.

For $\epsilon \leq \hat{\epsilon} = \min\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$, $r_\epsilon = r^+_\epsilon \circ r^-_\epsilon : D_+ \rightarrow D_+$ is a contraction map with contraction factor $c_0c_2 \in (0, 1)$. Hence $r_\epsilon$ has a unique fixed point $x^*$ to which all $r_\epsilon$-orbits in $D_+$ conv-
Flowing via $\phi^{-}$ from $x^* \rightarrow r_{xy}() = -\varepsilon$, and via $\phi^{+}$ from $y^* \rightarrow r_{yx}() = + \varepsilon$, provides the desired periodic orbit. Every Filippov trajectory of system (19) passing through $D$ converges to this limit cycle.

In figure 6 we illustrate the periodic behavior of the model for three $\varepsilon$-values. When $\varepsilon$ is two orders of magnitude smaller than $\rho$, $w$ and $\eta$ move so quickly relative to $\xi$ that they essentially switch from a point near $w_{s,s}(\eta_{s,s})$ to a point near $w_{s,s}(\eta_{s,s})$, and vice versa (figure 6, top row).

The middle panel in figure 6 illustrates the periodic behavior when $\varepsilon$ is one order of magnitude smaller than $\rho$. Interestingly, the limit cycle persists when $\varepsilon$ and $\rho$ have the same order of magnitude (figure 6, bottom row).

The attracting Filippov periodic orbit is quite robust to changes in parameters introduced in our model. Note the values of time constants $\tau$ and $\rho$ played no role in the analysis (other than that played by $\rho$ in bounding $\varepsilon$ from above as in (22)). Similarly, theorem 5.6 holds for any ablation rates, provided $0 < b_0 < b < b_1$. Varying the accumulation rate $a$ from 0.5 to 1.5.
caused little change to the limit cycle in numerical experiments. Our result holds for any choice of $T_c > -10$ °C, as long as there exist two ‘advancing’ equilibria with $\eta$-coordinates $\eta_{u,-}^*$ and $\eta_{s,-}^*$ as in figure 3.

As $\epsilon \to 0^+$, the period of the limit cycle scales like $k/\epsilon$ for some $k > 0$. This is due to the fact that as $\epsilon \to 0^+$, $(w, \eta)$ essentially switches from $(w_{s,-}^*, \eta_{s,-}^*)$ to $(w_{s,+}^*, \eta_{s,+}^*)$ and back during the cycle, so that the period is governed by the linear equations (16c)–(17c) (with $\eta$ switching between $\eta_{u,-}^*$ and $\eta_{s,+}^*$).

6. Conclusion

Building on previous work [5, 35, 19], we separated the classic Budyko ice line into two dynamic variables, the snow line and the glacial boundary, producing a system of three ordinary differential equations. Introducing two distinct climate states, one corresponding to glacial advance and the other to glacial retreat, we formulated the equations as a Filippov system with a two-dimensional plane forming the discontinuity boundary.

For the parameter values of interest here, the system has two virtual attracting equilibria, one for each of the two climate states. For example, starting with a climate where the glaciers are retreating, the system moves toward its virtual equilibrium, crossing the discontinuity boundary before reaching it. Upon crossing the boundary, the system switches climate regimes and starts heading toward the virtual equilibrium for advancing glaciers. Again, the virtual equilibrium cannot be reached and the orbit crosses the discontinuity boundary, switching back to retreating glaciers. This behavior is analogous to that found in the Welander model [34], and it produces a periodic orbit.

The proof of the existence of this periodic orbit uses ad-hoc singular perturbation techniques, with $\epsilon$ as the singular parameter. Recall that $\epsilon$ controls the speed at which the orbit approaches the discontinuity boundary $\Sigma$. By taking $\epsilon$ small enough, we were able to control the contraction parameter, giving a unique attracting fixed point for the section map, corresponding to an attracting periodic orbit for the system of differential equations. This periodic orbit represents an intrinsic cycling of the climate systems, producing the glacial cycles.

We view our model as providing a first step toward a conceptual model of the Earth’s glacial cycles. We chose parameters so that the system has two virtual equilibria, giving us an intrinsic oscillation. A next step would be to incorporate the Milankovitch cycles into the model. This would be easily accomplished, since the parameter $Q$ in equation (1) depends on the eccentricity while the parameter $s_3$ in equation (4) depends on the obliquity [18]. As these parameters vary with the Milankovitch cycles, the system might be nudged across the discontinuity boundary, creating a resonance with either eccentricity or obliquity.

Other directions suggested by this model are more mathematical in nature. For example, both of the virtual equilibria are close to the discontinuity boundary for the parameters we chose. By adjusting the parameters, one could produce a boundary equilibrium bifurcation for one or both of the equilibria (see figure 7). As one of the virtual equilibria crossed the discontinuity boundary, the periodic orbit would presumably disappear and be replaced by an attracting rest point. Moving the parameters in the other direction would find the system dramatically changing from an attracting equilibrium to a large oscillation, reminiscent of the phenomenon used by Maasch and Salzman to explain the mid-Pleistocene transition [17].

The mid-Pleistocene transition is a period roughly 1 million years ago when the length of the glacial-interglacial cycles changed from approximately 40 000 years to 100 000 years. Substantial efforts have been made to solve this ‘100 000 year problem’ (see, for example, [2, 12, 23, 25] and [31]). One might allow the parameters to vary in our model to see whether this transition can be realized. Comparisons can be made between model output and the paleoclimate data.
Figure 7. Illustration of a boundary equilibrium bifurcation as \( b_0 \) increases through \( b \). Plotted are projections of various orbits onto the \((\eta, \xi)\)-plane, and the corresponding \( \eta(t) \) and \( \xi(t) \) graphs. (a) Limit cycle. (b) \( Q_{-1}^* \in \Sigma \). (c) \( Q_{-1}^* \in S \). (d) \( Q_{-1}^*, Q_{-1}^* \in \Sigma \).
Again returning to mathematical questions, it would be interesting to explore the parameter space with the goal of discovering the origin of the periodic orbit we exhibited. Perhaps the orbit originated with a classic Hopf bifurcation, or perhaps it originated with bifurcation from a ‘fused focus’ as discussed by Filippov [7]. These explorations may or may not have implications for interpretation of the paleoclimate data, but they will almost certainly lead to interesting examples of bifurcations in Filippov systems.

Conceptual in nature, the simple model introduced here gives rise to a host of problems deserving of further study.

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Appendix

We present the proof of proposition 5.4.

Proof. We prove part (i). Let \( c_1 \in (0, 1) \) and let \( D_+ \subset \Gamma_+ \), \( D_+ \) compact. Recall \( \psi \) denotes the flow corresponding to subsystem (16a–16b). For any \( x = (w_0, \eta_0, \zeta_0) \in D_+ \), \( \psi((w_0, \eta_0), t) \rightarrow (w^*_0, \eta^*_0) \) as \( t \rightarrow \infty \). As \( E = \{ (w, \eta) : (w, \eta, \xi) \in D_+ \} \) is compact, there exists \( T_1 \) such that for all \( t \geq T_1 \) and for all \( u, v \in E \),

\[
\| \psi(u, t) - \psi(v, t) \| \leq c_1 \| u - v \|. \tag{A.1}
\]

Given \( x \in D_+ \), pick \( \epsilon(x) > 0 \) such that \( t(x, \epsilon(x)) > T_1 \) (where \( t(x, \epsilon(x)) \) is as in (23)).

By the continuity of \( \phi \) with respect to initial conditions and time, there exists \( \delta(x) > 0 \) so that for all \( y \in B_{\delta(x)}(x) \), \( t(y, \epsilon(x)) > T_1 \) where \( r_{\epsilon(x)}(y) \in \Gamma_+ \). Note for all \( \epsilon \leq \epsilon(x) \), \( t(y, \epsilon) > T_1 \).

Letting \( x \) vary, we get an open covering

\[
D_+ \subset \bigcup_{x \in D_+} B_{\delta(x)}(x)
\]

of the compact set \( D_+ \). Let \( \{ B_{\delta(x)}(x) \} : i = 1, ..., N \} \) be a finite subcover, and set \( \epsilon_i = \min \{ \epsilon(x) : i = 1, ..., N \} \). Then for any \( \epsilon \leq \epsilon_i \) and for all \( x \in D_+ \), \( t(x, \epsilon) > T_1 \).

Now let \( \epsilon \leq \epsilon_i \), let \( x_1 = (w_1, \eta_1, \gamma(\eta_1)) \), \( x_2 = (w_2, \eta_2, \gamma(\eta_2)) \in D_+ \), and let \( u = (w_1, \eta_1) \), \( v = (w_2, \eta_2) \).

Let \( r^*_{\epsilon}(x_i) = (w_i, \eta'_i, \gamma(\eta'_i)) \), \( r^*_{\epsilon}(x_2) = (w_2, \eta'_2, \gamma(\eta'_2)) \), \( u' = (w_1, \eta'_1) \) and \( v' = (w_2, \eta'_2) \). By (A.1) and our choice of \( \epsilon, \| v' - u' \|^2 \leq c_1^2 \| v - u \|^2 \). We have

\[
\| r^*_\epsilon(x_2) - r^*_\epsilon(x_1) \|^2 = \| v' - u' \|^2 + (\gamma(\eta'_2) - \gamma(\eta'_1))^2
\]

\[
= \| v' - u' \|^2 + \left( 1 + \frac{a}{b} \right)^2 (\eta'_2 - \eta'_1)^2
\]

\[
\leq c_1^2 \| v - u \|^2 + \left( 1 + \frac{a}{b} \right)^2 c_1^2 (\eta_2 - \eta_1)^2
\]

\[
= c_1^2 \| v - u \|^2 + c_1^2 (\gamma(\eta_2) - \gamma(\eta_1))^2
\]

\[
= c_1^2 \| x_2 - x_1 \|^2.
\]

Hence statement (i) holds. The proof of statement (ii) is similar. \( \square \)
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