Heterogeneity of in situ stress: A Review

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Abstract. Campaigns to determine the state of in situ stress form an integral part of almost all underground rock engineering projects, but stress measurements obtained at different locations on a project site generally display spatial variability. However, there is no international consensus on techniques to objectively quantify such spatial variability, and the nebulous description of “stress heterogeneity” is often used. Our review shows that there are no consistent and universally agreed definitions for stress heterogeneity: the existing definitions are overly simplistic and lack a robust statistical treatment of variability in stress tensors. We demonstrate that stress data can be partitioned into homogeneous stress domains using the k-means algorithm in multivariate stress space, and the resulting clusters characterised using multivariate statistics. This allows us to propose a clear and unambiguous definition of stress heterogeneity. Our analyses also suggest that completely meaningful partitioning of stress data requires development of new algorithms and cluster validity indices.

1. Introduction
Campaigns to determine the state of in situ stress form an integral part of almost all underground rock engineering projects, and often comprise the application of relatively expensive measurement techniques such as hydraulic fracturing and borehole overcoring. The importance of such campaigns is heightened when designing critical infrastructure such as underground nuclear waste repositories, as these designs require high confidence in the stress state. When stress measurements are obtained at different locations on a project site, the results are generally found to display spatial variability. However, there is no international consensus on techniques to objectively quantify such spatial variability, and the nebulous description of “stress heterogeneity” is often used.

Although the term stress heterogeneity is commonly found in the civil, petroleum and mining related rock engineering literature on in situ stress, a consistent definition of it seems to be missing. Similarly, a formal means of characterising a stress field as either homogeneous or heterogeneous is also absent. For emphasis, we borrow the quote from the classical text The Logic of Chance, by John Venn: “What is meant by a homogeneous class? is a pertinent and significant enquiry, but applying this condition to any simple cases its meaning is readily stated” [1].

Stress in its tensorial form is difficult to quantify statistically, but a recently-introduced multivariate model eases this difficulty [2,3,4]. Nevertheless, how should we perform these statistical calculations? For example, should all measurements across a project be combined in one group and a mean stress state calculated? Or should the region of the project be partitioned into domains and the stress state determined for each domain? In the former case we are assuming the stress state to be statistically homogeneous, but in the latter we are implicitly considering it to be heterogeneous. But what is
heterogeneity of \textit{in situ} stress? How is it defined? How is it quantified? How do we determine whether the \textit{in situ} stress state is homogeneous or heterogeneous?

In this paper, we aim to open a conversation on what stress heterogeneity is. We begin our review of the concept as it has been considered in the literature. From this, and using illustrated examples of stress fields together with analysis using concepts from multivariate statistics and cluster analysis, we propose a new definition of homogeneous and heterogeneous stress fields. We conclude with an example application of these to real \textit{in situ} stress data.

2. Review of homogeneous and heterogeneous stress states

The ideas of homogeneous, anisotropic and heterogeneous stress fields as proposed by Stephansson and Zang [5] are shown in Figure 1. From this, we see that a homogeneous stress field is one where the stresses do not vary throughout the domain of interest and are aligned with the far-field boundary stresses. An anisotropic stress field differs only in that the stresses within the domain are aligned with some internal plane of anisotropy rather than the boundary stresses. Heterogeneous stress fields result when the stress state is perturbed by defects in a rock mass such as geological inclusions, cavities and the like. Overall, these definitions seem to primarily rely on the geological conditions prevailing at a particular site, and although useful do not provide a quantitative context for defining stress heterogeneity. Also, from the viewpoint of a definition of stress heterogeneity, it is difficult to objectively distinguish between homogeneous and anisotropic stress fields, since in both cases there is uniformity of stress within the zone of interest.

![Figure 1. Homogeneous, anisotropic and heterogeneous stress fields (after [5]).](image)

An explicit definition of heterogeneous stress fields using stress components was presented by Ramsay & Lisle [6]. These authors proposed that if any of the stress components vary within a zone of interest, the resulting stress field is heterogeneous; homogeneous conditions represent uniformity of all stress components. In our view, this definition is overly simplistic as we now explain.

It is widely accepted that the magnitude of the vertical stress component in the ground is a function of depth. While at any particular site there may be local deviations from this due to perturbations caused by, for example, discontinuities in the rock, overall we expect the vertical stress to increase linearly with depth. Indeed, such a variation would usually be considered representative of homogeneous, not heterogeneous conditions, and suggests that a straightforward functional spatial change in stress conditions should not be considered as indicative of stress heterogeneity. This may be explored further by reference to Figure 2, which Ramsay and Lisle presented as an example of a heterogeneous stress field. Certainly, the orientations and magnitudes of the principal stresses shown in Figure 2(a) indicate a regular change across this region, which represents heterogeneity according to Ramsay and Lisle. However, Figure 2(b) shows both the Cartesian components of stress together with their simple functional relations, and thus shows that the relations are directly comparable to that for vertical stress. We are therefore prompted to ask, if a simple linear variation of vertical stress is indicative of homogeneous conditions, then is the same true for other stress components?
Figure 2. An example stress field from synthetic data (after [6]).

Ikeda [7] discusses stress heterogeneity by comparing stress measurements obtained using hydraulic fracturing at two sites in Japan, and the horizontal principal stresses measured at the sites are shown in Figure 3. With the exception of one outlier, the results from the Enzan site (EN) seem – with small variability – to follow some linear function of depth. Again, we would probably consider this to be indicative of homogeneous conditions. However, the situation at the Ashigawa site (AS) is very different: although it may be possible to separate the results into two regimes, one shallower than about 160 m and one deeper, there appears to be no obvious relation between depth and stress magnitude. So, should we consider the stress field at the AS site to be heterogeneous?

Figure 3. Comparison of stress measurements at two sites in Japan (after [7]).

These examples clearly suggest that stress heterogeneity is related to the stress state changing as a function of location, but we also know that stress is often variable even over small distances. Such local variability has been examined in a series of articles by Gao & Harrison [2][3][4], who discuss how stress tensors can be analysed using multivariate statistics [8]. These authors show how the dispersion of a group of tensors may be quantified using functions of its covariance matrix. Thus, for a stress tensor cast into multivariate form as
$s_d$ = vector of distinct stress components = $[\sigma_x \ \tau_{xy} \ \sigma_y \ \tau_{yz} \ \sigma_z]^T$, \hspace{1cm} (1)

the covariance matrix associated with $n$ such tensors is given by

$$\mathbf{\Omega} = \frac{1}{n} \sum_{i=1}^{n} (s_{d_i} - \mathbf{m}) \cdot (s_{d_i} - \mathbf{m})^T \hspace{1cm} (2)$$

where

$$\mathbf{m} = \text{mean vector} = \frac{1}{n} \sum_{i=1}^{n} s_{d_i}. \hspace{1cm} (3)$$

Two measures have been found efficacious for quantifying stress dispersion: effective variance,

$$V_e = |\mathbf{\Omega}|^{(1/p)}, \hspace{1cm} (4)$$

where $p$ is the number of rows in the covariance matrix, and Multivariate Coefficient of Variability (MCV) [9],

$$\text{MCV} = (\mathbf{m}^T \cdot \mathbf{\Omega} \cdot \mathbf{m})^{-1/2}. \hspace{1cm} (5)$$

It is now necessary to recognise the relation between variability and heterogeneity. The difference between these is presented in classical texts on probability and statistics, but is often not appreciated. Interestingly, a clear explanation may be found in the medical literature, where “statistical heterogeneity” is understood to be variability in response to medical interventions among different studies [10]. The key words in this statement are different studies. Applying this definition of statistical heterogeneity to rock stress suggests that a stress field within any particular region of interest will be homogeneous, regardless of the local variability it exhibits. Taking the medical context of “different studies” as our cue, we suggest this indicates that heterogeneity can only be defined in terms of the difference in variability between different regions of interest. For any given project, this implies that the overall region should be subdivided into smaller regions, and the variability of all such subregions calculated and compared. This points to the use of cluster analysis.

3. Cluster analysis for stress heterogeneity

The aim of cluster analysis is to discover meaningful groups in data [11]. It is increasingly used in many different fields with the goal of objectively partitioning a given dataset into domains based on some (dis)similarity metric (e.g. Euclidean distance) between parameters of interest. Many algorithms are available for cluster analysis, each with its own goals and limitations. These algorithms can be broadly classified into five groups: centroid-based (k-means, fuzzy c-means); hierarchical (agglomerative, divisive); spectral clustering; density-based; and, probability-based models [12].

Here, we investigate the application of the k-means algorithm to a synthetic dataset comprising 36 individual 2D stress tensors (Table 1), arranged in three clusters. Clustering is performed on the basis of Euclidean distance between 2D stress vectors $[\sigma_x \ \sigma_y \ \tau_{xy}]^T$. figures 4(a) and 4(b) show the data partitions for $k = 2$ and $k = 3$, and as the data are known to comprise three clusters we would expect the $k = 2$ partition to be incorrect; the figures confirm this. Figure 5(a) demonstrates how the data lie in three distinct clusters in $[\sigma_x \ \sigma_y \ \tau_{xy}]^T$ space, and that the algorithm has partitioned them correctly. Comparison of Figure 5(a) with Table 2 shows that cluster 1, which is visually the most dispersed of the three, also has the greatest values of effective variance and MCV. Thus, we conclude that a combination of a clustering algorithm with multivariate statistics allows us to identify partitions in stress data and hence characterise stress heterogeneity.
The 3D scatter plot of the assigned data is given in Figure 4(c), where the spatial coordinates of the data have been randomly assigned. The clustering continues to be correct in terms of stress space, but not in terms of spatial location. The effect of this is shown in Figure 4(c), where the spatial coordinates of the data have been randomly assigned. The clustering continues to be correct in terms of stress space, but not in terms of spatial location. Ongoing research is aiming to develop techniques that permit clustering in the dual stress-spatial location space to resolve this deficiency.

Various indices that maximise or minimise at the optimal partition have been developed to ascertain the optimal number of clusters in a dataset [12]. We use Average Silhouette Width (ASW), which is a dissimilarity-based measure of within-cluster homogeneity and between-cluster separation:

\[
\text{ASW} = \frac{1}{n} \sum_{i=1}^{n} s_i, \tag{6}
\]

where

\[
s_i = \text{silhouette coefficient} = \frac{b_i - a_i}{\max(a_i, b_i)}, \tag{7}
\]

Here, \(a_i\) is the average distance from point \(i\) to all other points in the same cluster and \(b_i\) is the minimum of the average distance between point \(i\) and all entities of another cluster. Each \(s_i\) measures how clearly a point belongs to the assigned cluster, and ranges from -1 (worst assignment) to +1 (ideal assignment). The 3D scatter plot of the assigned data is given in Figure 5(a), and Figure 5(b) shows that ASW clearly and correctly identifies \(k = 3\) as the optimal number of clusters.

### Table 1. Synthetic stress data used in the cluster analysis example.

| Cluster ID | \(x\)  | \(y\)  | Stress tensor components (MPa) | Cluster ID | \(x\)  | \(y\)  | Stress tensor components (MPa) | Cluster ID | \(x\)  | \(y\)  | Stress tensor components (MPa) |
|------------|--------|--------|-------------------------------|------------|--------|--------|-------------------------------|------------|--------|--------|-------------------------------|
| 1          | 1      | 0      | 10.14 3.29 -2.85              | 1          | 4      | 3      | 10.00 2.86 -2.62              | 2          | 3      | 4      | 4.83 1.91 -0.02              |
| 1          | 2      | 0      | 10.28 2.97 -2.22              | 1          | 5      | 3      | 9.95 2.93 -2.26              | 2          | 0      | 5      | 5.04 1.91 -0.31              |
| 1          | 3      | 0      | 9.80 2.72 -2.99              | 1          | 5      | 4      | 10.12 2.89 -2.77              | 2          | 1      | 5      | 4.89 1.94 -0.15              |
| 1          | 4      | 0      | 9.88 2.84 -1.95              | 2          | 0      | 1      | 4.84 1.98 -0.15              | 2          | 2      | 5      | 4.92 1.97 -0.15              |
| 1          | 5      | 0      | 9.76 2.66 -3.14              | 2          | 0      | 2      | 4.85 2.07 -0.21              | 2          | 3      | 5      | 5.06 1.93 -0.02              |
| 1          | 2      | 1      | 9.81 3.09 -2.38              | 2          | 1      | 2      | 4.86 1.96 -0.22              | 2          | 4      | 5      | 5.01 2.00 -0.30              |
| 1          | 3      | 1      | 9.98 3.33 -2.78              | 2          | 0      | 3      | 4.85 2.03 -0.21              | 3          | 0      | 0      | 5.08 5.03 -2.18              |
| 1          | 4      | 1      | 10.23 3.32 -2.96              | 2          | 1      | 3      | 4.90 2.04 -0.25              | 3          | 1      | 1      | 4.92 4.92 -1.89              |
| 1          | 5      | 1      | 10.18 2.90 -2.07              | 2          | 2      | 3      | 4.92 2.02 -0.16              | 3          | 2      | 2      | 4.98 4.91 -2.12              |
| 1          | 3      | 2      | 9.74 3.07 -2.00              | 2          | 0      | 4      | 5.17 2.06 -0.38              | 3          | 3      | 3      | 5.10 4.92 -2.14              |
| 1          | 4      | 2      | 10.22 3.03 -1.90              | 2          | 1      | 4      | 4.81 1.97 -0.03              | 3          | 4      | 5      | 5.03 5.01 -1.91              |
| 1          | 5      | 2      | 10.10 2.96 -2.95              | 2          | 2      | 4      | 5.20 2.02 -0.05              | 3          | 5      | 5      | 5.04 5.01 -2.14              |
As a result of the material presented above, we propose the following definition for stress heterogeneity:

If cluster analysis indicates that the stress field in a region of interest optimally comprises one cluster, then that stress field may be regarded as homogeneous. In all other situations the stress field may be regarded as heterogeneous. In all cases, multivariate statistics may be used to determine the dispersion of the stress field within the clusters.

We believe this definition captures the essence of the problem: that across a region of interest the stress field may demonstrate clear zones in which the stresses are somehow similar, but that these zones differ from one another. We suggest using cluster analysis to identify the zones, and multivariate statistics to quantify the variability of the stresses within each zone.

Despite this definition, we note that determining whether a group of stress data comprises meaningful clusters (in terms of both cluster validity and statistical measures) appears to be a challenging task, particularly in combined 3D stress and location spaces. Furthermore, geological variability, significant measurement errors, and the usual case in rock engineering of limited stress data indicates that stress domain partitioning is not straightforward. We anticipate that ongoing research will lead to rigorous protocols for partitioning stress data for use in rock engineering.

4. Practical application

To demonstrate the application of the above techniques to actual stress data, we analyse the AS data presented previously in Figure 3. As principal stress orientation is absent from these data we have assumed that the extreme horizontal stresses \( \sigma_{H_{\text{max}}} \) and \( \sigma_{H_{\text{min}}} \) coincide with \( x \)- and \( y \)-axes respectively, so that \( \sigma_x = \sigma_{H_{\text{max}}} \) and \( \sigma_y = \sigma_{H_{\text{min}}} \). Additionally, we have assumed the vertical stress is given by \( \sigma_z = \gamma z \) with \( \gamma = 25.5 \text{kN/m}^3 \). Clustering is performed in the 3D space of \([\sigma_x \sigma_y \sigma_z]^T\), and the resulting 3D scatter plots, depth-stress cluster plots and ASW plots are shown in Figure 6.

Figure 6(a) shows how \( k = 2 \) results in two, well defined clusters. Figure 6(c) shows that these clusters form two spatial domains, separated at a depth of about 160 m. The scatter plot for \( k = 3 \) is shown in Figure 6(b), and once again indicates that obvious clusters have been formed. Comparison with Figure 6(c) shows that clusters 2 and 3 have been formed by the splitting of the earlier cluster 2, together with the moving of one datum from cluster 1 into the new cluster 2. However, Figure 6(d) shows that cluster 3 lies within cluster 2 in depth-stress space: as with Figure 4(c), disregarding spatial location during the clustering process has resulted in clusters that are not spatially separated. Table 3 presents dispersion measures for the \( k = 2 \) and \( k = 3 \) partitioning of these data, and it is useful to examine these in the context of Figure 6(a) and Figure 6(b). Firstly, considering cluster 1 in the context of moving from \( k = 2 \) to \( k = 3 \), we see that as it loses one extreme member its dispersion, as expected, reduces slightly. Secondly, and for the case of \( k = 3 \), we see that the values of \( V_e \) and MCV for visually more dispersed cluster 2 are greater than those for the visually more compact cluster 3. Taken together, these suggest that the dispersion measures applied here are meaningful in characterising stress heterogeneity.

Finally, it is important to note that Figure 6(c) shows ASW is unable to clearly identify the optimal
partition. Again, we believe this is a result of ignoring spatial location, and is indicative that a novel measure of cluster validity is needed for stress data.

Although only a preliminary analysis, the results nevertheless do suggest that k-means algorithm is efficacious for grouping stress data into meaningful clusters, that such clusters can be characterised using multivariate statistics, but that clustering must use both stress and spatial data.

Table 3. Cluster dispersion measures for AS stress data.

| $k = 2$ | $k = 3$ |
|---|---|
| Cluster | 1 | 2 |
| $V_e$, MPa$^2$ | 0.233 | 0.467 |
| MCV, % | 6.8 | 4.0 |
| Cluster | 1 | 2 | 3 |
| $V_e$, MPa$^2$ | 0.187 | 0.844 | 0.170 |
| MCV, % | 6.7 | 8.8 | 0.9 |

Figure 6. 3D scatter plots, depth-stress cluster plots and ASW plot for AS data.

5. Concluding Remarks

Our review has shown that there are no consistent and universally agreed definitions for stress heterogeneity: the existing definitions are overly simplistic and lack a robust statistical treatment of variability in stress tensors. We demonstrated that stress data can be partitioned into homogeneous stress domains using the k-means algorithm in multivariate stress space, and the resulting clusters characterised using multivariate statistics. As a result of our findings, we propose this definition:

If cluster analysis indicates that the stress field in a region of interest optimally comprises one cluster, then that stress field may be regarded as homogeneous. In all other situations the stress field may be regarded as heterogeneous. In all cases, multivariate statistics may be used to determine the dispersion of the stress field within the clusters.

Notwithstanding this, it is clear that clustering needs to take place in both stress and spatial location space in order to produce clusters that are spatially meaningful, and that new cluster validity indices are
required in order to identify optimal partitions in this dual space. These are the subject of ongoing research.

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