What is interesting in $\eta$ and $\eta'$ Meson Decays?

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Abstract. An introduction to the physics of $\eta$ and $\eta'$ meson decays is given. A historical account of the discovery of the mesons is presented. It is followed by an overview and classification of the common decay modes and the relevance of the mesons for modern hadron and particle physics. In more detail the hadronic decay modes are discussed and in particular some interesting features of the $\eta \to 3\pi^0$ decay are presented. The last section briefly reviews and compares reactions used to produce the $\eta$ and $\eta'$ mesons for the studies of their decays.

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HISTORICAL NOTE

The $\eta$ meson was discovered in 1961 by Pevsner et al. [1] (paper submitted 10th November) in the year when The Eightfold Way was formulated and the $\omega$ meson was discovered. The $\eta'$ meson was discovered independently by two groups in 1964: Kalbfleisch et al. [2] (paper submitted 9th April) and Goldberg et al. [3] (paper submitted 15th April). In that year the quark model was proposed and the $\Omega^-$ particle was discovered.

The earliest predictions of $\eta$ and $\eta'$ can be traced back to works within the Sakata model [4] where the nucleons and the $\Lambda$ particle were used as the building blocks (so called sakatons). In this scheme the isospin 0 neutral mesons were needed to accompany pions and kaons, the only pseudoscalars known at that time. The new mesons were decomposed as follows (Okun 1957 [5, 6], Ikeda et al. [7] and Yamaguchi [8, 9]):

$$\pi^{0'} = \Lambda \bar{\Lambda}, \quad \pi^{0'} = (p\bar{p} - n\bar{n})/\sqrt{2}.$$  

The relation between $\pi^{0'}$ and $\pi^{\pm,0}$ would be entirely analogous to the relation between $\Lambda$ and $\Sigma^{\pm,0}$. Since the mesons were an unavoidable consequence of the approach their absence was used to criticize the model (Sakurai [10] 1960).

In The Eightfold Way Gell-Mann refers to $\eta$ (named $\chi^0$) in the following phrase ([11], March 15, 1961):

"... The most clear-cut new prediction for the pseudoscalar mesons is the existence of $\chi^0$, which should decay into $2\gamma$ like $\pi^0$, unless it is heavy enough to yield $\pi^+ + \pi^- + \gamma$ with appreciable probability. (In the latter case, we must have $(\pi^+ + \pi^-)$ in an odd state.) $\chi^0 \to 3\pi$ is forbidden by conservation of I and C. For a sufficiently heavy $\chi^0$ the decay $\chi^0 \to 4\pi$ is possible, but hampered"

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1 In modern terminology: $\pi^{0'} - \eta'; \pi^{0''} - \eta'$. 
by centrifugal barriers.

The events which lead to the $\eta$ discovery and interpretation are nicely summarized by Rosenfeld in an account of the early years of the Particle Data Group (PDG) [12]:

A meson (apparently unrelated because it decayed into three pions) [1] was discovered late in 1961, and properly identified [13] as the predicted pseudoscalar meson, the $\eta$, early in 1962. This completed the first meson octet, but by latter standards it attracted little attention (no press conference, no flurry of theoretical papers, and no 1962 edition of UCRL-803[2]).

In a retrospective on the origins of the quark model Zweig [15] comments on the fate of the predictions within the Sakata model: “...Their striking prediction of the existence of the $\eta$ was not mentioned by the experimental groups that discovered and studied this key particle”. Finally Okun summarizes the events in the year 1964 in the following way [16]:

In 1964 the $\eta'$-meson and the $\Omega$-hyperon were discovered [17, 18]. Earlier this year G. Zweig [19] and M. Gell-Mann [20] replaced the integer charged sakatons by fractionally charged particles (aces – Zweig; quarks – Gell-Mann). This allowed them to construct not only the octet and singlet of mesons, but also the octet and decuplet of baryons.

**MAIN DECAY MODES**

The basic facts about the $\eta$ and $\eta'$ mesons from the most recent issue of the PDG report [21] are summarized in table 1. The main decays fall into two distinctive classes: hadronic decays into three pseudoscalar mesons and radiative decays. This fact was pointed out already in 1962 by Gell-Mann [22] in case of the $\eta$ meson decays:

The forbidden decay rates into $3\pi^0$ and $\pi^+ + \pi^- + \pi^0$ are difficult to estimate, except that $3\pi^0/(\pi^+ + \pi^- + \pi^0) \leq \frac{3}{2}$. The remaining neutral decays are expected, however, to represent $\chi \rightarrow 2\gamma$. The decay $\chi \rightarrow 2\gamma$ may be described roughly on the assumption that the important intermediate steps are $\chi \rightarrow 2\rho^0$ (followed by $\rho^0 \rightarrow \gamma, \rho^0 \rightarrow \gamma$) and $\chi \rightarrow 2\omega$ (followed by $\omega \rightarrow \gamma, \omega \rightarrow \gamma$). We now wish to estimate the ratio of this rate to that of hitherto unobserved charged decay mode $\chi \rightarrow \pi^+ + \pi^- + \gamma$, which should be dominated by $\chi \rightarrow 2\rho^0$ followed by $\rho^0 \rightarrow \gamma, \rho^0 \rightarrow \pi^+ + \pi^-$. The description proposed here for the radiative decays is based on Vector Meson Dominance (VMD) model formulated by Sakurai in 1960 [10] and represented by the diagrams in Fig. 1. Using the intermediate state with two vector mesons one can describe at least qualitatively the main features of the radiative decays. This approach fails however when trying to describe decays into three pseudoscalars. The $\eta, \eta' \rightarrow \pi\pi\pi$ decays do not conserve isospin since Bose symmetry forbids the three pions with $J^P = 0^-$ to

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2 Precursor of the PDG reports first issued in 1957 [14] and than often updated.
FIGURE 1. Vector Meson dominance model for eta anomalous decays.

TABLE 1. Main properties of the $\eta$ and $\eta'$ mesons \cite{21}.

| Decay                  | $\eta$         | $\eta'$        |
|------------------------|----------------|----------------|
| $M_\eta$               | $547.51 \pm 0.18$ MeV | $M_{\eta'} = 957.78 \pm 0.14$ MeV |
| $\Gamma_\eta$         | $1.30 \pm 0.07$ keV | $\Gamma_{\eta'} = 0.203 \pm 0.016$ MeV |
| $\eta \to \gamma \gamma$ | 39%            | $\eta' \to \pi^+ \pi^- \eta$ | 44% |
| $\eta \to \pi^0 \pi^0 \pi^0$ | 32%            | $\eta' \to \rho^0 \gamma$ | 29% |
| $\eta \to \pi^+ \pi^- \pi^0$ | 23%            | $\eta' \to \pi^0 \pi^0 \eta$ | 21% |
| $\eta \to \pi^+ \pi^- \gamma$ | 5%             | $\eta' \to \omega \gamma$ | 3%  |

occur in the isoscalar state. If one acknowledge this fact and consider that the final state has isospin 1 then a possible intermediate state could be $\pi^\pm \rho^{\mp}$ ($\rho^0$ is forbidden by $C$ conservation). The $\pi^\pm$ and $\rho^{\mp}$ have to be in a relative $p$-wave and cannot contribute to the neutral decay mode into three $\pi^0$. This is obviously in contrast with the experiment since the neutral mode is 1.41 times more frequent than $\pi^+ \pi^- \pi^0$ and close to the upper boundary quoted by Gell-Mann. The solution would be to assume that the two body intermediate states include also scalar mesons. The scalar mesons were discovered later and even today their properties are not well established. Therefore the hadronic decays of $\eta$ and $\eta'$ into three pseudoscalar mesons provide a source of information on the low energy scalar interactions.

MODERN PERSPECTIVE

The $\eta$ and $\eta'$ mesons play a special role in understanding low energy Quantum Chromodynamics (QCD). They are isoscalar members of the nonet of the lightest pseudoscalar mesons. The systematical way of dealing with that regime of QCD is provided by Chiral Perturbation Theory (CHPT) \cite{23}. The global symmetry of QCD where the three light quarks are massless is the chiral symmetry – $SU(3)_L \times SU(3)_R$ which is spontaneously broken to $SU_V(3)$. The octet of Goldstone pseudoscalar mesons, arising from the broken symmetry, is identified with pions, kaons and eta. The particles are used as degrees of freedom in CHPT and a systematical expansion is performed in powers of momenta of the mesons. At each level of the calculations one has to introduce new counterterms (free parameters of the theory) in order to restore renormalizability. Direct breaking of the chiral symmetry is accounted by expanding in the quark masses. Standard CHPT provides an accurate description of the strong and electroweak interactions.
of the pseudoscalar mesons at low energies [24, 25]. Due to weak $K_L - \eta$ mixing a good understanding of $\eta$ decays is a prerequisite for calculation of Standard Model contributions to rare kaon decays such as $K_L \to \pi^0 e^+ e^- \ [26, 27]$. CP violation in the flavor conserving processes can be tested in the $\eta$ decays which are analogous to those of $K_L$: $\eta \to \pi \pi$, $\eta \to \pi^0 e^+ e^-$ and in the angular asymmetry between the $\pi^+ \pi^-$ and the $e^+ e^-$ decay planes in the $\eta \to \pi^+ \pi^- e^+ e^-$ decay. Conversion decays of $\eta$ provide information about the pseudoscalar formfactor needed for example for light-by-light contribution to the anomalous magnetic moment of the muon. It was also pointed out that $\eta, \eta' \to 3 \pi$ decays provide valuable information on light quark masses.

Most of the complications of the $\eta$ treatment in the standard CHPT are due to mixing of the $\eta$ and $\eta'$ fields. The $\eta'$ is most esoteric meson of the pseudoscalar nonet, since it is closely related to the axial $U(1)$ anomaly and remains massive even in the chiral limit. It was observed by Witten [28] that in QCD with infinite number of colors [29] the $\eta_0$ state is massless and the global $SU(3)_L \times SU(3)_R$ symmetry is replaced by $U(3)_L \times U(3)_R$. Corresponding effective Lagrangians were studied by e.g. [30, 31, 32, 33]. A systematical treatment of the $1/N_c$ expansion was introduced into CHPT by Kaiser and Leutwyler [34]. An alternative approach is $U(3)$ CHPT of Beisert and Borasoy [35]. With that tool mixing of $\eta$ and $\eta'$ has received a solid theoretical foundation. In particular it has been recognized both from phenomenology and theory side that two mixing angles are required to describe the mixing in the singlet and octet basis [36, 37, 38, 39, 40].

Due to the large mass of $\eta'$ light vector and scalar mesons could be produced in the decays. Importance of vector mesons is seen in radiative decay modes; $\eta' \to \rho \gamma$ and $\eta' \to \omega \gamma$. Contributions of light scalar mesons $\sigma, f_0, a_0$ should play a significant role in the decays into $\eta \pi \pi$ and $\pi \pi \pi$ [41, 42] but it is not as apparent due to the width of the mesons and the offshell behavior. Description of vector and scalar resonances is beyond the scope of the standard CHPT but there is continuous progress in the theoretical treatment. For example the VMD model is being incorporated into the theory [43, 44] and unitarity constrains are implemented by dispersion relations [45, 46].

In conclusion in the early stages of CHPT development, $\eta$ and specially $\eta'$ were treated as an additional complication or by hiding their effects were hidden in the low energy constants [47]. Refinements and extensions of the CHPT methods stimulated by the need for precise calculations for kaons and eta decays provided tools which enable to include the $\eta'$ in an elegant and consistent way and to perform reliable calculations for the processes. One should mention a recent theoretical programme for simultaneous treatment of $\eta$ and $\eta'$ decays within the chiral unitary approach [35, 48]. It is based on chiral perturbation theory and the unitarization using the Bethe-Salpeter equation. It is consistent with the constraints from chiral symmetry and the chiral anomaly, while at the same time it matches to the one loop expansion of CHPT. It is discussed by Borasoy and Nißler in the contributions to this Symposium.

In addition to the contributions to the present Symposium a recent introduction to the topic of $\eta$ decay physics is also given by Bijnens [49] and the extensive reviews of the field could be found in the proceedings of two workshops [50, 51].
HADRONIC DECAYS

The three body decays into pseudoscalar mesons constitute the majority of all \( \eta \) decays – 55% \((\eta \rightarrow \pi\pi\pi)\) and \( \eta' \) decays – 65% \((\eta' \rightarrow \pi\pi\eta)\) as seen in table [1]. The decays \( \eta, \eta' \rightarrow \pi\pi\pi \) do not conserve isospin and the partial widths are similar to the second order electromagnetic processes (decays into \( \gamma \) pairs). It appears however that the contributions from electromagnetic processes are suppressed [52, 53, 54] and the decays are driven by an isospin violating term of the QCD Lagrangian proportional to \( m_d - m_u \).

The lowest order contribution to the decay mechanism is given by the Current Algebra graphs shown in Fig. [2] consisting of a combination of \( \eta - \pi^0 \) mixing and elementary low energy QCD process – scattering of two pseudoscalar mesons. The partial width of the \( \eta \rightarrow \pi^0\pi^+\pi^- \) decay calculated using current algebra is 66 eV [56], much below the experimental value 294 ± 16 eV [21]. The second order in the low energy expansion of the effective Lagrangian of QCD was calculated by Gasser and Leutwyler in 1984 increasing the result to 160 eV [57]. The big change implies the importance of the \( \pi\pi \) interaction in the final state. One may expect that higher loop calculations enhance the prediction for the decay further since they should give a better description of the final state \( \pi\pi \) interactions. However, such calculations would be very involved and the result might not be predictive since it requires many low energy constants which are not well known. An elegant method of including the \( \pi\pi \) interactions up to higher orders is provided by dispersion relations which connect the imaginary part of the decay amplitude with the amplitude itself. The amplitude as an analytic function is uniquely characterized by its singularities and can be represented by an integral over its discontinuities along a branch cut. The unitarity and analyticity will determine the amplitude only up to some subtraction polynomial which could be constrained using CHPT. There are two calculations using this technique [46, 45] but employing different formalism. They lead consistently to an enhancement of the decay rate by about 14%. The dependence of the decay width (\( \Gamma \)) on \( m_d - m_u \) can be expressed in the following way:

\[
\Gamma \propto \left( \frac{1}{Q} \right)^4 \tag{1}
\]

where

\[
\frac{1}{Q^2} = \frac{m_d^2 - m_u^2}{m_s^2 - \frac{1}{3}(m_d + m_u)^2}. \tag{2}
\]

The standard way to determine \( Q \) is to use the leading order expressions for the masses of the pseudoscalar mesons, together with Dashen theorem [58], which leads to the
following formula:

\[ Q^2 \approx Q_D^2 = \frac{m_K^2}{m_\pi^2} \frac{m_\pi^2 - m_K^2}{m_K^0 - m_K^2 + m_\pi^2 - m_\pi^0} \]  

(3)

– numerically \( Q_D = 24.1 \). However, \( \Gamma \) is sensitive to the exact value of \( Q \) (an uncertainty of 10\% in the decay rate translates to a 2.5\% error in \( Q \)) and the decay might provide a precise constraint for the light quark mass ratios \( [59] \). Namely \( Q \) determines the major axis of the ellipse in the \( m_u/m_s, m_d/m_s \) plane. One rewrites Eqn. \( [2] \) as:

\[ \Gamma = \left( \frac{Q_D}{Q} \right)^4 \bar{\Gamma} \]  

(4)

where \( \bar{\Gamma} \) is the decay width calculated in the CHPT assuming \( Q_D = Q \). Deviation of \( Q \) from \( Q_D \) can be studied by comparing \( \bar{\Gamma} \) to the value of the \( \Gamma \) extracted from experiment. One important prerequisite is the reliability of the \( \bar{\Gamma} \) determination. The calculations can be tested by comparing the ratio \( \Gamma(\eta \to 3\pi^0)/\Gamma(\eta \to \pi^+\pi^-\pi^0) \) and the kinematical distributions of the pions with experiment. Recently \( Q \) was derived by determining the subtraction polynomial within the dispersion relation approach of \( [45] \) from preliminary KLOE data on the \( \eta \to \pi^+\pi^-\pi^0 \) decay \( [60, 61] \), yielding the \( Q \) value 22.1 \( [62] \). In an alternative approach by Borasoy and Nißler \( [63] \), rescattering of any pair of pseudoscalar mesons is taken care of by the Bethe-Salpeter equations. The parameters beyond the one loop are constrained by fitting both to pseudoscalar scattering data and to the \( \eta, \eta' \) reactions. The conclusion is that the current experimental situation does not allow to constrain the parameters, in particular to conclude that \( Q \neq Q_D \), and therefore to provide precise information about light quark masses within this approach.

For a three body decay there are only two independent kinematical variables to describe the phase space. A convenient common choice is to use Mandelstam variables which for the decay \( 0 \to 1 + 2 + 3 \) (\( p_i \) is the fourmomentum) can be written:

\[ s_i = (p_0 - p_i)^2. \]  

(5)

This means that for example \( s_3 \) is the invariant mass squared of particles 1 and 2. The \( s_i \) can be expressed by kinetic energies of the decay products (\( T_i \)) in the center of mass system of the mother particle:

\[ s_i = (m_0 - m_i)^2 - 2T_im_0. \]  

(6)

The variables are therefore not independent:

\[ \sum_{i=1}^{3} T_i = m_0 - \sum_{i=1}^{3} m_i \equiv 3\langle T \rangle, \]  

(7)

where the introduced variable \( \langle T \rangle \) is the average kinetic energy of the outgoing particles. When any two \( s_i \) are chosen as the independent variables the phase space can be represented by a two dimensional plot (the Dalitz plot \( [64] \)). The probability density in...
such variables is proportional to the amplitude squared for the process. The boundaries
of the Dalitz plots for the $\eta$ and $\eta'$ hadronic decays are shown in Fig. 3.

In the case when $m_1 = m_2 \equiv m$ one uses symmetrized, dimensionless variables to
parametrize the phase space:

$$x \equiv \frac{1}{\sqrt{3}} \frac{T_1 - T_2}{\langle T \rangle}; \quad y \equiv \frac{1}{3} \left( \sum_{i=1}^3 \frac{m_i}{m} \right) \frac{T_3}{\langle T \rangle} - 1. \quad (8)$$

The kinematical boundaries in the $x,y$ plane are shown in Fig. 3 for $\eta$ and $\eta'$ hadronic
decays. Phenomenologically the mechanism of the hadronic three body $\eta$ and $\eta'$ decays
are described by an expansion of the amplitude $A(x,y)$ around the center $x = y = 0$ point
and therefore the density of the Dalitz plot can be written as:

$$|A(x,y)|^2 \propto 1 + ay + by^2 + cx + dx^2... \quad (9)$$

The term linear in $x$ vanishes in case when 1 and 2 are identical or are particle-
antiparticle pair and Charge Conjugation (C) is not violated.

One can see [57] that the pions must emerge in an $I = 1$ configuration and the following relation between amplitudes of the $A \equiv A_{\pi^+\pi^-\pi^0}$ and $\bar{A} \equiv A_{\pi^0\pi^0\pi^0}$ follows:

$$\bar{A}(s_1, s_2, s_3) = A(s_1, s_2, s_3) + A(s_2, s_3, s_1) + A(s_3, s_1, s_2). \quad (10)$$

The amplitude $A(s_1, s_2, s_3)$ derived within current algebra (lowest order of CHPT) from
graphs in Fig. 2 leads to [55]:

$$A(s_1, s_2, s_3) = A(s_3) \propto \frac{3s_3 - 4m_\pi^2}{m_\eta^2 - m_\pi^2}. \quad (11)$$
The amplitude is a linear function in $s_3$ (or in $y$). In terms of the expansion from Eqn. 9 the result corresponds to $a = -1.052$ and $b = 0.28$. For the neutral decay Eq. 10 leads to the constant amplitude $\bar{A}$ always when $A$ is a linear function of $s_3$. The result is dominated by the left graph in the Fig. 2 (pion pole diagram).

In case of the isospin conserving decays $\eta' \rightarrow \eta \pi^+ \pi^-$ and $\eta' \rightarrow \eta \pi^0 \pi^0$ the wave function must be symmetric under the exchange of the pions and the three-particle final state must be isoscalar. This leads to a simple relation:

$$A_{\pi^+ \pi^- \eta}(s_1, s_2, s_3) = A_{\pi^0 \pi^0 \eta}(s_1, s_2, s_3).$$

(12)

This leads to $r_1 \equiv \Gamma(\eta' \rightarrow \eta \pi^+ \pi^-)/\Gamma(\eta' \rightarrow \eta \pi^0 \pi^0) = 2$ due to a factor 1/2 for the case with two identical particles. The mass difference between charged and neutral pions leads to smaller phase space volume for the $\pi^+ \pi^-$ case than for the $\pi^0 \pi^0$ and decreases the ratio to $1.78 \pm 0.02$ [63]. The ratio $r_1$ has not been measured directly and the evaluation from the known branching ratios gives $2.14 \pm 0.19$ [21]. Such a large value would indicate significant isospin-violating contributions in the amplitude.

As it was stated previously for a consistent description of $\eta$ and $\eta'$ decays the final state interactions and $\eta$-$\eta'$ mixing should be taken into account. Such a complete and consistent framework [65, 63] is presented also in other contributions to this symposium.

**PECCULARITIES OF THE $\eta \rightarrow 3\pi^0$ DECAY**

For decays with three identical particles $\eta, \eta' \rightarrow 3\pi^0$ it is useful to introduce polar coordinates $(\sqrt{z}, \phi)$ in the $(x, y)$ plane to parametrize the phase space:

$$x = \sqrt{z}\sin \phi; y = \sqrt{z}\cos \phi.$$ 

(13)

The Dalitz plot has sextant symmetry and therefore one can limit the range of $\phi$ to $0 \leq \phi < 60^\circ$. The variable $z$ is then given by:

$$z = x^2 + y^2 = \frac{2}{3} \sum_{i=1}^{3} \left( \frac{T_i - \langle T \rangle}{\langle T \rangle} \right)^2.$$ 

(14)

and $0 \leq z \leq 1$. The lowest term in the expansion around the center of the Dalitz plot for the decay amplitude squared is proportional to $z$:

$$|\bar{A}(z, \phi)|^2 = c_0 (1 + 2\alpha z + ...)$$

(15)

Experimental and theoretical results on the $\alpha$ parameter are summarized in table 2. The two recent low background and high statistics experiments (about $10^6$ events in the final data sample): Crystal Ball at the AGS [66] and KLOE [67] (2005 preliminary result), were not consistent until very recently. The discrepancy seems to be removed in the recent reanalysis of the KLOE data [68].

A consequence of the fact that $m_{\pi^0} < m_{\pi^\pm}$ is that the kinematical region of the $\eta \rightarrow 3\pi^0$ decay phase space includes the threshold for the $\pi^0 \pi^0 \rightarrow \pi^+ \pi^-$ process. The
FIGURE 4. Symmetrized Dalitz plot for $\eta \to 3\pi^0$ decay ($\rho \equiv \sqrt{2}(T)$).

TABLE 2. Experimental and theoretical results for the parameter $\alpha$.

| $\alpha$            | Comment                  | Ref.  |
|---------------------|--------------------------|-------|
| -0.052 $\pm$ 0.017(stat) $\pm$ 0.010(syst) | Exp (CBarrel)            | [69]  |
| -0.031 $\pm$ 0.004  | Exp (CBall)              | [66]  |
| -0.013 $\pm$ 0.004(stat) $\pm$ 0.005(syst) | Exp (KLOE), prel.       | [60]  |
| -0.027 $\pm$ 0.004(stat) $\pm$ 0.006(syst) | Exp (KLOE), prel.       | [68]  |
| -0.026 $\pm$ 0.01(stat) $\pm$ 0.01(syst)  | Exp (WASA)               | [70]  |
| 0                   | Current Algebra          | [56]  |
| +0.015              | CHPT,1loop               | [57, 71]| 
| -0.007...-0.014     | CHPT+dispersive          | [45]  |
| -0.007              | UCHPT                    | [72]  |
| -0.031 $\pm$ 0.003  | UCHPT/fit                | [73]  |

boundaries corresponding to the $\pi^0\pi^0$ invariant mass ($M_{\pi\pi}$) equal $2m_{\pi^+}$ are illustrated by dashed lines in Fig. 4. Recently a cusp in $M_{\pi\pi}$ was observed in $K^+ \to \pi^+\pi^0\pi^0$ decay by the NA48/2 collaboration [74]. Initial interpretation of the effect was given by Cabibbo [75] and the process could provide a precise determination of a combination of the $a_0$, $a_2$ the $I = 0$ and $I = 2$ pion-pion s-wave scattering lengths. A similar effect should be present in the $K_L \to 3\pi^0$, $\eta \to 3\pi^0$ and $\eta' \to \pi^0\pi^0\eta$ decays.

The cusp effect in $\eta \to 3\pi^0$ has been studied in a one loop approximation by Belina [76]. The following expression for the amplitude is obtained:

$$A(s_1,s_2,s_3) \propto 1 + k_1 \sum_{i=1}^3 f(s_i;m_{\pi^0}) + k_2 \sqrt{\frac{2}{3}} \sum_{i=1}^3 f(s_i;m_{\pi^-})$$  \hspace{1cm} (16)

where

$$16\pi^2 f(s,m) = \begin{cases} 2 - 2\sqrt{-\Delta} \arctan \sqrt{-\frac{1}{\Delta}} ; & 0 < s < 4m^2 \\
2 + \sqrt{\Delta} \left[ \ln \frac{1-\sqrt{\Delta}}{1+\sqrt{\Delta}} + i\pi \right] ; & 4m^2 < s \end{cases}$$  \hspace{1cm} (17)
\[ \eta \quad \pi^0 \quad \pi^+ \quad \pi^- \quad \pi^0 \]

**FIGURE 5.** Lowest order \( \pi^+ \pi^- \rightarrow \pi^0 \pi^0 \) rescattering contribution in the \( \eta \rightarrow \pi^0 \pi^0 \pi^0 \) decay.

**FIGURE 6.** (left) \( M_{\pi\pi} \) distribution (divided by the phase space): line – CHPT calculations from Belina [76], points – CELSIUS/WASA experimental results [70]. (right) \( |A(z)|^2 / c_0 \): solid line – Belina; dashed line – \( \alpha = -0.031 \); dotted line – \( \alpha = 0 \); points – preliminary data from KLOE [68].

\[
\Delta = 1 - \frac{4m^2}{s}
\]

The \( k_1 \) and \( k_2 \) parameters are related to the \( \pi\pi \) scattering lengths \( a_0 \) and \( a_2 \):

\[
k_2 = \frac{32\pi}{3} (a_0 - a_2); \quad k_1 = \frac{16\pi}{3} (a_0 + 2a_2).
\]

(18)

Here the values \( a_0 - a_2 = 0.265 \) and \( a_0 + 2a_2 = 0.13 \) will be used in order to estimate the magnitude of the effect.

The predictions for the \( M_{\pi\pi} \) distribution divided by the \( \eta \rightarrow 3\pi^0 \) phase space in the region below the \( 2m_{\pi^+} \) mass (279 MeV) reveals a deviation up to few percent (left part of Fig. 6). The observation of the cusp would require a resolution of a few MeV/c\(^2\) in \( M_{\pi\pi} \) (e.g. the resolution obtained by CELSIUS/WASA of 6 MeV/c\(^2\) (RMS) would be enough) and a data sample of about \( 10^7 \) \( \eta \rightarrow 3\pi^0 \) decays [70]. Since the required data samples are only a few times larger than these collected so far by Crystal Ball, KLOE or WASA-at-COSY [77] one should expect that the effect will be observed in the near future.

Another aspect of the cusp effect is how the \( z \) distribution and in the result the extracted \( \alpha \) parameter could be affected. The predictions from [76] are in Fig. 6 (right) given by solid line. The influence of the cusp shows up in the region of \( 0.6 < z < 0.9 \) with the maximal deviation of 2\% for \( z \approx 0.75 \). One can also observe that \( \alpha > 0 \) for one loop CHPT as given in table 2. For a comparison a line represents the Crystal Ball [66] result and the points are extracted from the newest KLOE paper [68] (from the Fig. 4). It is therefore clear that the cusp effect cannot be ignored for the precise determination of \( \alpha \). One can for example exclude the cusp region by fitting \( \alpha \) only for \( 0 \leq z \leq 0.6 \) or include the theoretical predictions to describe the effect.
In addition to the cusp, at the vicinity of the same \( M_{\pi\pi} = 2m_{\pi^+} \) boundary pionium \((\pi^+\pi^- \text{ atom})\) should be created. Pionium was observed in the DIRAC experiment and its lifetime measured \([78]\) leading to an \(\alpha_0 - \alpha_2\) determination. The production rate of the pionic atoms in the decays of \( K, \eta, \eta' \) was studied by Wycech and Green \([79]\) and Silagadze \([80]\). Since pionium decays predominantly into \( \pi^0\pi^0 \) pair: \( \Gamma \approx \Gamma_{\pi^0\pi^0} \) \((81, 82)\) and therefore should be observed in \( \eta \rightarrow \pi^0\pi^0\pi^0 \), \( \eta' \rightarrow \pi^0\pi^0\eta \) and \( K^{\pm} \rightarrow \pi^0\pi^0\pi^\pm \) decays:

\[
\frac{\Gamma(P_0 \rightarrow A_{2\pi}P_3)}{\Gamma(P_0 \rightarrow \pi^+\pi^-P_3)} = \frac{\pi}{R} \frac{\alpha_{em}^3 m_0^2}{m_0^2 |a(x,y)|^2} \sqrt{\frac{1}{4} \left( 1 - \frac{m^2}{m_0^2} + \frac{m_3^2}{m_0^2} \right)^2 - \frac{m_3^2}{m_0^2}}, \tag{19}
\]

where \( x \) and \( y \) are Dalitz variables, corresponding to the atom creation:

\[ x = 0, \quad y = \frac{3}{2} \frac{m_2^2 - 4m^2 + m_3^2}{m_0(m_0 - 2m - m_3)} \tag{20} \]

\( R \) is a dimensionless remnant of the three-particle phase space integral defined in \([80]\). The result for the branching ratios is:

\[
BR(\eta \rightarrow A_{2\pi} \pi^0) \approx 2 \times 10^{-8}; BR(\eta' \rightarrow A_{2\pi} \eta) \approx 6 \times 10^{-7}; BR(K^{\pm} \rightarrow A_{2\pi} \pi^\pm) \approx 5.5 \times 10^{-7}.
\]

That corresponds to the fractions:

\[
\frac{\Gamma(\eta \rightarrow A_{2\pi} \pi^0)}{\Gamma(\eta \rightarrow 2\pi^0 \pi^0)} \approx 6 \times 10^{-8}, \quad \frac{\Gamma(\eta' \rightarrow A_{2\pi} \eta)}{\Gamma(\eta' \rightarrow 2\pi^0 \eta)} \approx 3 \times 10^{-6}, \quad \frac{\Gamma(K^{\pm} \rightarrow A_{2\pi} \pi^\pm)}{\Gamma(K^{\pm} \rightarrow 2\pi^0 \pi^\pm)} \approx 3 \times 10^{-5}.
\]

In case of \( K^{\pm} \rightarrow \pi^0\pi^0\pi^\pm \) statistics \( \sim 10^8 \) is available and there are some hints that the pionium is observed \([83]\). The signature would be a tiny peak in \( M_{\pi\pi} \) distribution at \( M_{\pi\pi} = 2m_{\pi^+} \).

**SOURCES OF THE \( \eta \) AND \( \eta' \) MESONS**

The \( \eta \) and \( \eta' \) mesons can be copiously produced in \( \pi^- p, \gamma p, pp \) or \( pd \) interactions not too far from the meson production thresholds. The mesons in this case originate from decays of nucleon isobars (\( \eta \) mainly from \( N^*(1535) \)). Crystal Ball at BNL (Brookhaven) employed \( \pi^- p \rightarrow n\eta \) reaction \([84]\) and after the move of the detector to MAMI (Mainz) \([85]\) the \( \eta \) and \( \eta' \) mesons are produced in the \( \gamma p \rightarrow p\eta, \eta' \) reaction close to threshold. In WASA at CELSIUS the \( \eta \) mesons were produced in \( pd \rightarrow ^3He\eta \) at threshold \([86]\) and in \( pp \rightarrow pp\eta \) reactions \([87]\). In WASA-at-COSY \( \eta' \) will be produced in \( pp \rightarrow pp\eta' \) reaction \([88, 89]\).

Radiative decays of \( \phi \) mesons produced at the resonance peak in \( e^+e^- \) colliders are used by CMD-2 \([90]\), SND, KLOE \([91]\) experiments. In the Crystal Barrel experiment the mesons were produced in \( p\bar{p} \) annihilations at rest. In addition there are high energy hadronic reactions such as peripheral \( \pi^- p \rightarrow n\eta, \eta' \) \([92, 93]\) production used in experiments on meson spectroscopy performed with GAMS2000 spectrometer \([94]\) and
TABLE 3. Parameters of close to threshold $\eta$ and $\eta'$ meson production reactions. $T_b$, $p_b$ are beam kinetic energy and momentum and $\beta$ velocity of the CMS at threshold. $Q$ is CMS excess energy corresponding to an optimal beam energy and $\sigma$ is reaction cross section. The last column $\sigma_T$ indicates total inclusive cross section of the processes originating from the given initial state.

| Reaction | $T_b$ | $p_b$ | $\beta$ | $Q$ | $\sigma (\sigma_{max})$ | $\sigma_T$ |
|----------|-------|-------|---------|-----|------------------------|-----------|
| $pp \rightarrow pp\eta$ | 1.253 | 1.981 | 0.63 | 40 | 10\(\mu\)b | 40mb |
| $pp \rightarrow pp\eta'$ | 2.404 | 3.208 | 0.75 | 45 | 300\(\mu\)b | 40mb |
| $pd \rightarrow ^3\text{He}\eta$ | 0.891 | 1.569 | 0.42 | 2 | 400\(\mu\)b | 80mb |
| $\pi^- p \rightarrow n\eta$ | 0.559 | 0.684 | 0.42 | 2.8 mb | 16\(\mu\)b | 300\(\mu\)b |
| $\gamma p \rightarrow p\eta$ | 0.706 | 0.706 | 0.43 | 58 | 10\(\mu\)b | 150\(\mu\)b |
| $\gamma p \rightarrow p\eta'$ | 1.447 | 1.447 | 0.61 | 27 | 150\(\mu\)b | 150\(\mu\)b |

recently by the VES collaboration [95]. A unique possibility to measure the radiative width of the mesons is provided by $\eta$, $\eta'$ formation in $\gamma(\gamma^*)$ processes as e.g. in the $e^- e^+ \rightarrow e^- e^+ \eta, \eta'$ reaction.

A feature of the close to threshold processes is that the particle(s) that accompanies the produced meson are emitted in a small forward cone and their momenta can be measured accurately in a dedicated detector which covers a limited range of scattering angles. A good resolution in the missing mass can therefore be achieved (typically a few MeV/c\(^2\) FWHM) and provide a clear identification of $\eta$ and $\eta'$ production. The decay particles are emitted more isotropically (since the velocity of the center of mass system is not very high) and their registration requires a detector with nearly $4\pi$ sr coverage. A typical resolution for the invariant masses of the decay products is a few ten MeV/c\(^2\). In the close to threshold processes the tagging particles and the decay particles appear in separate regions of phase space and branching ratios $\Gamma_i/\Gamma_{tot}$ can be determined directly.

Among the close to threshold reactions $\eta$, $\eta'$ production in $pp$ and $pd$ processes have the lowest ratio of the cross section to the total inclusive. The signal to background ratio can however be enhanced by increasing the resolution in the missing mass or by decreasing the beam energy to work closer to the threshold (at the price of a lower production cross section). The ideal reaction from this respect is $pd \rightarrow ^3\text{He}\eta$, where the cross raises very quickly reaching 0.4 \(\mu\)b already at excess energy of about 1 MeV above the threshold [96, 97, 98]. Together with the fact that $\eta$ is accompanied by one heavy and doubly charged particle this allows for a very clean and precise tagging by identification of the $^3\text{He}$ ion in a magnetic spectrometer. To cover the full phase space the spectrometer acceptance below one degree is sufficient. The $pp \rightarrow pp\eta, \eta'$ reactions can provide the highest useful rate of tagged $\eta$ and $\eta'$ mesons enabling studies of the rare decays which have a distinct signature like for example the decay $\eta \rightarrow e^+ e^-$ where an integrated luminosity corresponding to $10^{10}$ \(\eta\)s is needed.

For $\eta$ and $\eta'$ originating from $\phi$ radiative decays in $e^+ e^-$ colliders the general level of background is much lower than in the hadronic interactions. The physics background is mainly due to the other $\phi$ decays: $BR(\phi \rightarrow \eta \gamma) = 1.301 \pm 0.024\%$ and $BR(\phi \rightarrow \eta' \gamma) = 6.2 \pm 0.7 \times 10^{-5}$. The pseudoscalar mesons are produced antiparallel to a monoenergetic photon (with energy 360 MeV for $\eta$ and 60 MeV for $\eta'$) and have relatively low and well defined velocity. The monoenergetic photon gives in principle a very clean signature. However for neutral decays this photon can be misidentified with photons from $\eta(\eta')$
since the resolution of present electromagnetic calorimeters does not provide stringent enough constraints for the \( \eta(\eta') \) fourmomentum. When charged particles are produced, the decay system can be measured much better. At KLOE for example the invariant mass resolution for a system of charged particles is comparable to the \( pp \) missing mass resolution in the WASA detector.

The \( e^+e^- \) colliders also allow production of the mesons in \( \gamma\gamma \) interactions by the \( e^+e^- \to e^+e^-\gamma\gamma \to e^+e^-\eta, \eta' \) process. The cross section depends on the flux of the virtual photons and cross section of the \( \eta, \eta' \) production which is proportional to the radiative width of the mesons \([99]\). In all experiments so far the mesons were identified by detecting a specific decay mode that allows to measure the \( \Gamma(\gamma\gamma) \times \Gamma(i)/\Gamma_{tot} \) combination. All results on \( \eta \) and \( \eta' \) radiative widths were obtained by this indirect method \([21]\). The mesons are produced by almost real virtual photons and the outgoing electrons are emitted at very small angles. The energy of the electron and the positron will differ from the beam energy and therefore they will be deflected from the beam region by the collider dipoles. The deflection angles measured in specialized position detectors will determine their energies. If both electron and positron are measured one can tag the production of \( \eta, \eta' \) without having to rely on a decay channel. For the KLOE2 experimental programme it is planned to equip the detector with two tagging spectrometers and the first direct measurement of \( \Gamma(\eta, \eta' \to \gamma\gamma) \) would be possible. In order to produce \( \eta' \) mesons and to avoid background due to the \( \phi \) decays a higher center of mass energy is required.

In conclusion one should stress that studies of \( \eta \) and \( \eta' \) decays require high statistics precision experiments. A good understanding of systematical effects can be obtained by comparing results from experiments that use different techniques.

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**REFERENCES**

1. A. Pevsner, et al., *Phys. Rev. Lett.* 7, 421–423 (1961).
2. G. R. Kalbfleisch, L. W. Alvarez, A. Barbaro-Galtieri, O. I. Dahl, P. Eberhard, W. E. Humphrey, J. S. Lindsey, D. W. Merrill, J. J. Murray, A. Rittenberg, R. R. Ross, J. B. Shafer, F. T. Shively, D. M. Siegel, G. A. Smith, and R. D. Tripp, *Phys. Rev. Lett.* 12, 527–530 (1964).
3. M. Goldberg, M. Gundzik, S. Lichtman, J. Leitner, M. Primer, P. L. Connolly, E. L. Hart, K. W. Lai, G. London, N. P. Samios, and S. S. Yamamoto, *Phys. Rev. Lett.* 12, 546–550 (1964).
4. S. Sakata, *Prog. Theor. Phys.* 16, 686–688 (1956).
5. L. Okun, Some remarks concerning the compound model of fundamental particles, (1957), proc. of the Intern. Conf. on mesons and recently discovered particles, Padova - Venezia, 22 - 28 Settembre 1957, p. V-55.
6. L. Okun, *Sov. Phys. JETP* 7, 322 (1958).
7. M. Ikeda, S. Ogawa, and Y. Ohsuki, *Progress of Theoretical Physics* 22, 715–724 (1959).
8. Y. Yamaguchi, *Progress of Theoretical Physics* 19 (1958).
9. Y. Yamaguchi, *Progress of Theoretical Physics Supplement* 11, 1–36 (1959).
10. J. J. Sakurai, *Annals Phys.* 11, 1–48 (1960).
11. M. Gell-Mann, The Eightfold Way: A theory of strong interaction symmetry (1961), CTSL-20.
12. A. H. Rosenfeld, Ann. Rev. Nucl. Part. Sci. 25, 555–598 (1975).
13. P. L. Bastien, J. P. Berge, O. I. Dahl, M. Ferro-Luzzi, D. H. Miller, J. J. Murray, A. H. Rosenfeld, and M. B. Watson, Phys. Rev. Lett. 8, 114–117 (1962).
14. W. Barkas, and A. Rosenfeld, Data for elementary-particle physics (1957), UCRL 8030.
15. G. Zweig, Origins of the quark model (1980), invited talk given at 4th Int. Conf. on Baryon Resonances, Toronto, Canada, Jul 14-16.
16. L. B. Okun, The impact of the Sakata model (2006). [hep-ph/0611298]
17. P. M. Dauber, W. E. Slater, L. T. Smith, D. H. Stork, and H. K. Ticho, Phys. Rev. Lett. 13, 449–454 (1964).
18. W. Barkas, and A. Rosenfeld, Data for elementary-particle physics (1957), UCRL 8030.
19. G. Zweig, An SU(3) model for strong interaction symmetry and its breaking (1964), CERN-TH-401.
20. M. Gell-Mann, Phys. Lett. 8, 214–215 (1964).
21. W. M. Yao, et al., J. Phys. G33, 1–1232 (2006).
22. M. Gell-Mann, D. Sharp, and W. G. Wagner, Phys. Rev. Lett. 8, 261 (1962).
23. J. Bijnens, AIP Conf. Proc. 768, 153–159 (2005), hep-ph/0409068.
24. J. Bijnens, AIP Conf. Proc. 768, 153–159 (2005), hep-ph/0409068.
25. G. Zweig, An SU(3) model for strong interaction symmetry and its breaking (1964), CERN-TH-401.
26. M. Gell-Mann, Phys. Lett. 8, 214–215 (1964).
27. W. Barkas, and A. Rosenfeld, Data for elementary-particle physics (1957), UCRL 8030.
58. R. F. Dashen, *Phys. Rev.* **183**, 1245–1260 (1969).
59. H. Leutwyler, *Phys. Lett.* **B378**, 313–318 (1996), [hep-ph/9602366](http://arxiv.org/abs/hep-ph/9602366).
60. S. Giovannella, et al., KLOE results on f_0(980), a_0(980) scalars and η decays (2005), [hep-ex/0505074](http://arxiv.org/abs/hep-ex/0505074).
61. F. Ambrosino, et al., Determination of the η → π^+π^-π^0 Dalitz Plot slopes and asymmetries with the KLOE detector (2007), [arXiv:0707.2355](http://arxiv.org/abs/0707.2355).
62. B. V. Martemyanov, and V. S. Sopov, *Phys. Rev.* **D71**, 017501 (2005), [hep-ph/0502023](http://arxiv.org/abs/hep-ph/0502023).
63. B. Borasoy, and R. Nissler, *Eur. Phys. J.* **A26**, 383–398 (2005), [hep-ph/0510384](http://arxiv.org/abs/hep-ph/0510384).
64. R. H. Dalitz, *Phil. Mag.* **44**, 1068–1080 (1953).
65. N. Beisert, and B. Borasoy, *Nucl. Phys.* **A705**, 433–454 (2002), [hep-ph/0201289](http://arxiv.org/abs/hep-ph/0201289).
66. W. B. Tippens, et al., *Phys. Rev. Lett.* **87**, 192001 (2001).
67. T. Capussela, *Acta Phys. Slov.* **56**, 341–344 (2006).
68. F. Ambrosino, et al., Measurement of the slope parameter α for the η → 3π^0 decay at kloe (2007), [arXiv:0707.4137](http://arxiv.org/abs/0707.4137).
69. A. Abele, et al., *Phys. Lett.* **B417**, 193–196 (1998).
70. M. Bashkanov, et al., Measurement of the slope parameter for the η → 3π^0 decay in the pp → ppη reaction (2007), [arXiv:0708.2014](http://arxiv.org/abs/0708.2014).
71. J. Bijnens, and J. Gasser, *Phys. Scripta* **T99**, 34–44 (2002), [hep-ph/0202242](http://arxiv.org/abs/hep-ph/0202242).
72. N. Beisert, and B. Borasoy, *Nucl. Phys.* **A716**, 186–208 (2003), [hep-ph/0301058](http://arxiv.org/abs/hep-ph/0301058).
73. B. Borasoy, and R. Nissler, *Acta Phys. Slov.* **56**, 319–326 (2005), [hep-ph/0511290](http://arxiv.org/abs/hep-ph/0511290).
74. J. R. Batley, et al., *Phys. Lett.* **B633**, 173–182 (2006), [hep-ex/0511056](http://arxiv.org/abs/hep-ex/0511056).
75. N. Cabibbo, *Phys. Rev. Lett.* **93**, 121801 (2004), [hep-ph/0405001](http://arxiv.org/abs/hep-ph/0405001).
76. J. Belina, *Cusp Phenomena in η → 3π^0*, Master's thesis, Universität Bern (2006), URL [http://www.itp.unibe.ch/diploma_thesis/belina/totalcor.pdf](http://www.itp.unibe.ch/diploma_thesis/belina/totalcor.pdf).
77. P. Vlasov, et al., Preliminary results for the analysis of the pp → pp(η → 3π^0) reaction (2007), in these proceedings.