SATAKE DIAGRAMS
AND
REAL STRUCTURES ON SPHERICAL VARIETIES

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Abstract. With each antiholomorphic involution $\sigma$ of a connected complex semisimple Lie group $G$ we associate an automorphism $\epsilon_\sigma$ of its Dynkin diagram. The definition of $\epsilon_\sigma$ is given in terms of the Satake diagram of $\sigma$. Let $H \subset G$ be a self-normalizing spherical subgroup. If $\epsilon_\sigma = \text{id}$ then we prove the uniqueness and existence of a $\sigma$-equivariant real structure on $G/H$ and on the wonderful completion of $G/H$.

1. Introduction and statement of results

In this paper, we consider real structures on complex manifolds acted on by complex Lie groups. A real structure on a complex manifold $X$ is an antiholomorphic involutive diffeomorphism $\mu : X \to X$. Suppose a complex Lie group $G$ acts holomorphically on $X$ and let $\sigma : G \to G$ be an involutive antiholomorphic automorphism of $G$ as a real Lie group. A real structure $\mu : X \to X$ is said to be $\sigma$-equivariant if $\mu$ satisfies $\mu(g \cdot x) = \sigma(g) \cdot \mu(x)$ for all $g \in G, x \in X$. We start with homogeneous manifolds of arbitrary complex Lie groups. In Section 2 we prove that a $\sigma$-equivariant real structure on $X = G/H$ exists and is unique if $H$ is self-normalizing and $\sigma(H)$ and $H$ are conjugate by an inner automorphism of $G$. The conjugacy of $H$ and $\sigma(H)$ is also necessary for the existence of a $\sigma$-equivariant real structure.

Assume $G$ is connected and semisimple and denote by $\mathfrak{g}$ the Lie algebra of $G$. In Section 3 with any antiholomorphic involution $\sigma : G \to G$ we associate an automorphism class $\epsilon = \epsilon_\sigma \in \text{Aut}(\mathfrak{g})/\text{Int}(\mathfrak{g})$ acting on the Dynkin diagram in the following way. We choose a Cartan subalgebra of the real form $\mathfrak{g}_0 \subset \mathfrak{g}$ and the root ordering as in the classical paper of I. Satake [13]. Let $\Pi_\circ$ (resp. $\Pi_\bullet$) be the set of compact (resp. non-compact) simple roots, $\kappa : \Pi_\circ \to \Pi_\circ$ the involutory self-map associated with $\sigma$. Denote by $W_\bullet$ the subgroup of the Weyl group $W$ generated by simple reflections $s_\alpha$, where $\alpha \in \Pi_\bullet$, and let $w_\bullet$ be the
element of maximal length in $W$. Then $\epsilon(\alpha) = -w_\alpha(\alpha)$ for $\alpha \in \Pi$, and $\epsilon(\alpha) = \kappa(\alpha)$ for $\alpha \in \Pi_v$. On the Satake diagram, $\epsilon$ interchanges the white circles connected by two-pointed arrows and permutes the black ones as the outer automorphism of order 2 for compact algebras $A_n(n \geq 2)$, $D_n(n \text{ odd})$, $E_6$, and identically otherwise.

Let $B \subset G$ be a Borel subgroup. Then $\sigma$ acts on the character group $\mathcal{X}(B)$ in a natural way. Namely, $\sigma(B) = cBc^{-1}$ for some $c \in G$ and, given $\lambda \in \mathcal{X}(B)$, the character

$$B \ni b \mapsto \lambda^c \sigma(b) := \lambda(c^{-1}\sigma(b)c)$$

is in fact independent of $c$. In Section 4, we show that the arising action coincides with the one given by $\epsilon_\sigma$.

In Section 5, we consider equivariant real structures on homogeneous spherical spaces. It turns out that, under some natural conditions on a spherical subgroup $H \subset G$, the homogeneous space $G/H$ possesses a $\sigma$-equivariant real structure. More precisely, we have the following result.

**Theorem 1.1.** Assume $\epsilon_\sigma = \text{id}$. Then any spherical subgroup $H \subset G$ is conjugate to $\sigma(H)$ by an inner automorphism of $G$, i.e., $\sigma(H) = aHa^{-1}$ for some $a \in G$. The map

$$\mu_0 : G/H \to G/H, \quad \mu_0(g \cdot H) := \sigma(g) \cdot a \cdot H,$$

is correctly defined, antiholomorphic and $\sigma$-equivariant. Moreover, if the subgroup $H$ is self-normalizing then: (i) $\mu_0$ is involutive, hence a $\sigma$-equivariant real structure on $G/H$; (ii) such a structure is unique.

In Section 6, we prove a similar theorem for wonderful varieties. Wonderful varieties were introduced by D.Luna [9], and we recall their definition in Section 6. Wonderful varieties can be viewed as equivariant completions of spherical varieties with certain properties. If such a completion exists, it is unique. Furthermore, if $H$ is a self-normalizing spherical subgroup of a semisimple group $G$ then, by a result of F.Knop [7], $G/H$ has a wonderful completion.

**Theorem 1.2.** Let $H$ be a self-normalizing spherical subgroup of $G$ and let $X$ be the wonderful completion of $G/H$. If $\epsilon_\sigma = \text{id}$ then there exists one and only one $\sigma$-equivariant real structure $\mu : X \to X$.

**Remark.** Assume that $\sigma$ defines a split form of $G$. Then it is easily seen that $\epsilon_\sigma = \text{id}$. In the split case Theorems 1.1 and 1.2 are joint results with S.Cupit-Foutou [3]. In this case, the $\sigma$-equivariant real structure on a wonderful variety $X$ is called **canonical**. Assume in addition that $X$ is strict, i.e. all stabilizers (and not just the principal
one) are self-normalizing, and equip $X$ with its canonical real structure. Then \[3\] contains an estimate of the number of orbits of the connected component $G_0^\sigma$ on the real part of $X$.

2. Equivariant real structures

A real structure on a complex manifold $X$ is an antiholomorphic involutive diffeomorphism $\mu : X \to X$. The set of fixed points $X^\mu$ of $\mu$ is called the real part of $X$ with respect to $\mu$. If $X^\mu \neq \emptyset$ then $X^\mu$ is a closed real submanifold in $X$ and

$$\dim_\mathbb{R}(X^\mu) = \dim_\mathbb{C}(X).$$

Suppose a complex Lie group $G$ acts holomorphically on $X$ and let $\sigma : G \to G$ be an involutive antiholomorphic automorphism of $G$ as a real Lie group. The fixed point subgroup $G^\sigma$ is a real form of $G$. A real structure $\mu : X \to X$ is said to be $\sigma$-equivariant if

$$\mu(gx) = \sigma(g) \cdot \mu(x) \quad \text{for all } g \in G, x \in X.$$ 

For such a structure the set $X^\mu$ is stable under $G^\sigma$. We are interested in equivariant real structures on homogeneous manifolds and on their equivariant embeddings.

**Theorem 2.1.** Let $G$ be a complex Lie group, let $\sigma : G \to G$ be an antiholomorphic involution, and let $H \subset G$ be a closed complex Lie subgroup. If there exists a $\sigma$-equivariant real structure on $X = G/H$ then $\sigma(H)$ and $H$ are conjugate by an inner automorphism of $G$. Conversely, if $\sigma(H)$ and $H$ are conjugate and $H$ is self-normalizing then a $\sigma$-equivariant real structure on $X$ exists and is unique.

**Proof.** Suppose first that $\mu : X \to X$ is a $\sigma$-equivariant real structure. Let $x_0 = e \cdot H$ be the base point and let $\mu(x_0) = g_0 \cdot H$. For $h \in H$ one has

$$\mu(x_0) = \mu(hx_0) = \sigma(h) \cdot \mu(x_0),$$

showing that $\sigma(H) \subset g_0 H g_0^{-1}$. To prove the opposite inclusion, observe that $g_0 \cdot \mu(x_0) = \mu(x_0)$ is equivalent to $\mu(\sigma(g_0) \cdot x_0) = \mu(x_0)$. This implies $\sigma(g_0) \cdot x_0 = x_0$, so that $\sigma(g_0) \in H$ and $g_0 \in \sigma(H)$, hence $g_0 H g_0^{-1} \subset \sigma(H)$.

To prove the converse, assume that $H$ is self-normalizing and

$$g_0 H g_0^{-1} = \sigma(H)$$

for some $g_0 \in G$. Let $r_{g_0}$ be the right shift $g \mapsto g g_0$. We have a map $\mu : X \to X$, correctly defined by $\mu(g \cdot H) = \sigma(g) g_0 \cdot H$. The
commutative diagram

\[
\begin{array}{ccc}
G & \xrightarrow{\sigma} & G \\
\downarrow & & \downarrow \\
X = G/H & \xrightarrow{\mu} & X = G/H
\end{array}
\]

where the vertical arrows denote the canonical projection \( g \mapsto g \cdot H \), shows that the map \( \mu \) is antiholomorphic. It is also clear that \( \mu \) is a \( \sigma \)-equivariant map, i.e., \( \mu(gx) = \sigma(g) \cdot \mu(x) \) for all \( g \in G \). Therefore \( \mu^2 \) is an automorphism of the homogeneous space \( X \), i.e., \( \mu^2 \) is a biholomorphic self-map of \( X \) commuting with the \( G \)-action. Since \( H \) is self-normalizing, we see that \( \mu^2 = \text{id} \). Thus \( \mu \) is a \( \sigma \)-equivariant real structure on \( X \). If \( \mu' \) is another such structure then \( \mu \cdot \mu' \) is again an automorphism of \( X = G/H \), so \( \mu \cdot \mu' = \text{id} \) and \( \mu' = \mu \). □

Example. Let \( B \) be a Borel subgroup of a semisimple complex Lie group \( G \) and let \( X = G/B \) be the flag manifold of \( G \). It follows from Theorem 2.1 that a \( \sigma \)-equivariant real structure \( \mu : X \to X \) exists for any \( \sigma : G \to G \). One has \( X^\mu \neq \emptyset \) if and only if the minimal parabolic subgroup of \( G \) is solvable or, equivalently, if the real form has no compact roots.

3. Automorphism \( \epsilon_\sigma \)

Let \( \mathfrak{g} \) be a complex semisimple Lie algebra, \( \mathfrak{g}_0 \) a real form of \( \mathfrak{g} \), and \( \sigma : \mathfrak{g} \to \mathfrak{g} \) the corresponding antilinear involution. In this section we define the automorphism \( \epsilon_\sigma \) of the Dynkin diagram of \( \mathfrak{g} \), cf. [1, 2] and [11], §9. We start by recalling the notions of compact and non-compact roots, see e.g. [12], Ch. 5.

Let \( \mathfrak{g}_0 = \mathfrak{t} + \mathfrak{p} \) be a Cartan decomposition. The corresponding Cartan involution extends to \( \mathfrak{g} = \mathfrak{g}_0 + i \cdot \mathfrak{g}_0 \) as an automorphism \( \theta \) of the complex Lie algebra \( \mathfrak{g} \). Clearly, \( \theta^2 = \text{id} \) and \( \sigma \cdot \theta = \theta \cdot \sigma \). Pick a maximal abelian subspace \( \mathfrak{a} \subset \mathfrak{p} \) and denote by \( \mathfrak{m} \) its centralizer in \( \mathfrak{t} \). Let \( \mathfrak{h}^+ \) be a maximal abelian subalgebra in \( \mathfrak{m} \). Then \( \mathfrak{h} = \mathfrak{h}^+ + \mathfrak{a} \) is a maximal abelian subalgebra in \( \mathfrak{g}_0 \) and any such subalgebra containing \( \mathfrak{a} \) is of that form. The Cartan subalgebra \( \mathfrak{t} = \mathfrak{h} + i \cdot \mathfrak{h} \subset \mathfrak{g} \) is stable under \( \theta \) and \( \sigma \). On the dual space \( \mathfrak{t}^* \), we have the dual linear transformation \( \theta^\mathfrak{t} \) and the dual antilinear transformation \( \sigma^\mathfrak{t} \):

\[
\theta^\mathfrak{t}(\gamma)(A) = \gamma(\theta A), \quad \sigma^\mathfrak{t}(\gamma)(A) = \overline{\gamma(\sigma A)} \quad (\gamma \in \mathfrak{t}^*, A \in \mathfrak{t}).
\]

Let \( \Delta \) be the set of roots of \( (\mathfrak{g}, \mathfrak{t}) \) and let \( \Sigma \) be the sets of roots of \( \mathfrak{g} \) with respect to \( \mathfrak{a} \otimes \mathbb{C} = \mathfrak{a} + i \cdot \mathfrak{a} \). Put \( \mathfrak{t}_\mathbb{R} = i \cdot \mathfrak{h}^+ + \mathfrak{a} \). This is a maximal real subspace of \( \mathfrak{t} \) on which all roots take real values. Choose a basis \( v_1, \ldots, v_r, v_{r+1}, \ldots, v_l \) in \( \mathfrak{t}_\mathbb{R} \) such that \( v_1, \ldots, v_r \) form a basis of \( \mathfrak{a} \) and introduce the lexicographic ordering in the dual real vector
spaces \( t_{\mathbb{R}}^* \) and \( a^* \). Then \( \Delta \subset t_{\mathbb{R}}^* \), \( \Sigma \subset a^* \), and \( \varrho(\Delta \cup \{0\}) = \Sigma \cup \{0\} \) under the restriction map \( \varrho : t_{\mathbb{R}}^* \to a^* \). Let \( \Delta^\pm, \Sigma^\pm \) be the sets of positive and negative roots with respect to the chosen orderings, \( \Pi \subset \Delta^+, \Theta \subset \Sigma^+ \) the bases, \( \Delta_\bullet = \{ \alpha \in \Delta \mid \varrho(\alpha) = 0 \} \), \( \Delta_o = \Delta \setminus \Delta_\bullet \).

The roots from \( \Delta_\bullet \) and \( \Delta_o \) are called compact and non-compact roots, respectively. Let \( \Delta^\pm_\bullet, \Sigma^\pm_\bullet \) be the sets of positive and negative roots with respect to the chosen orderings, \( \Pi_\bullet \subset \Delta^+_\bullet \), \( \Theta_\bullet \subset \Sigma^+_\bullet \) the bases, \( \Delta^\bullet = \{ \alpha \in \Delta \mid \varrho(\alpha) = 0 \} \), \( \Delta_o = \Pi \cap \Delta_\bullet \) and \( \Pi_o = \Pi \cap \Delta_o \). One shows that \( \Delta_\bullet \) is a root system with basis \( \Pi_\bullet \). Also, \( \varrho(\Delta^\pm_o) = \Sigma^\pm_\bullet, \theta^T(\Delta^\pm_o) = \Delta^\mp_o \) and \( \varrho(\Pi_o) = \Theta_\bullet \). Furthermore, one has an involutory self-map \( \omega : \Pi_o \to \Pi_o \), defined by

\[
\theta^T(\alpha) = -\omega(\alpha) - \sum_{\gamma \in \Pi_\bullet} c_{\alpha \gamma} \gamma,
\]

where \( c_{\alpha \gamma} \) are non-negative integers. The Satake diagram is the Dynkin diagram on which the simple roots from \( \Pi_\bullet \) are denoted by black circles, the simple roots from \( \Pi_o \) by white circles, and two white circles are connected by a two-pointed arrow if and only if they correspond to the roots \( \alpha \) and \( \omega(\alpha) \neq \alpha \).

Let \( W \) be the Weyl group of \( g \) with respect to \( t \) considered as a linear group on \( t^* \). The subgroup of \( W \) generated by the reflections \( s_\alpha \) with \( \alpha \in \Pi_\bullet \) is denoted by \( W_\bullet \). The element of maximal length in \( W_\bullet \) with respect to these generators is denoted by \( w_\bullet \). Note that \( -w_\bullet(\alpha) \in \Pi_\bullet \) if \( \alpha \in \Pi_\bullet \). Let \( \iota : g \to g \) be an inner automorphism such that \( \iota(t) = t \), acting as \( w_\bullet \) on \( t^* \). Since \( w_\bullet^2 = \text{id} \), we have

\[
(i^{\pm}t)^T = w_\bullet.
\]

**Proposition 3.1.** The self-map of \( \Pi \), defined by

\[
\epsilon_{\sigma}(\alpha) = \begin{cases} 
-w_\bullet(\alpha) & \text{if } \alpha \in \Pi_\bullet, \\
\omega(\alpha) & \text{if } \alpha \in \Pi_o, 
\end{cases}
\]

is an automorphism of the Dynkin diagram.

**Proof.** We must find an automorphism \( \phi : g \to g \) preserving \( t \) and \( \Pi \), which acts on \( \Pi \) as \( \epsilon_{\sigma} \). Let \( \eta \) be the Weyl involution of \( g \) acting as \( -\text{id} \) on \( t \) and let \( \phi = \eta \cdot \theta \cdot \iota \). Then \( \phi \) acts on \( \Delta \) by

\[
\alpha \mapsto -w_\bullet(\theta^T(\alpha)).
\]

If \( \alpha \in \Pi_\bullet \) then \( \theta^T(\alpha) = \alpha \), and so \( \phi \) sends \( \alpha \) to \( -w_\bullet(\alpha) = \epsilon_{\sigma}(\alpha) \). Now, if \( \alpha \in \Pi_o \) then

\[
-w_\bullet(\theta^T(\alpha)) = w_\bullet(\omega(\alpha)) + \sum_{\gamma \in \Pi_\bullet} c_{\alpha \gamma} w_\bullet(\gamma)
\]

by the definition of \( \omega \). The simple reflections in the decomposition of \( w_\bullet \) correspond to the elements of \( \Pi_\bullet \). Applying these reflections to
Proposition 3.2. \( \omega(\alpha) \in \Pi_o \) one by one, we see that the right hand side is the sum of \( \omega(\alpha) \) and a linear combination of elements of \( \Pi_* \), whose coefficients must be nonnegative. Therefore \(-w_*(\theta^T(\Pi)) \subset \Delta^+ \). Since \(-w_* \cdot \theta^T \) arises from \( \phi \), this is an automorphism of \( \Delta \). Thus \(-w_*(\theta^T(\Pi)) \) is a base of \( \Delta \), hence \(-w_*(\theta^T(\Pi)) = \Pi \). In particular, \(-w_*(\theta^T(\alpha)) \in \Pi \), and so we obtain \(-w_*(\theta^T(\alpha)) = \omega(\alpha) = \epsilon_\sigma(\alpha) \). □

Proposition 3.3. Extend \( \epsilon_\sigma \) to a linear map of \( t^* \) and denote the extension again by \( \epsilon_\sigma \). Then \( w_* \) and \( \theta^T \) commute and

\[
\epsilon_\sigma = -w_* \theta^T = -\theta^T w_* .
\]

Proof. We already proved that \( \epsilon_\sigma \) equals \(-w_* \theta^T \) on \( \Pi \), so it suffices to show that \( \epsilon_\sigma \) also equals \(-\theta^T w_* \) on \( \Pi \). For \( \alpha \in \Pi_* \) we have \(-w_*(\alpha) \in \Pi_* \) and \( \theta^T \alpha = -\alpha \). Thus \( w_* \theta^T \alpha = -w_* \alpha = \theta^T w_* \alpha \). For \( \alpha \in \Pi_o \) we have

\[
w_*(\alpha) = \alpha + \sum_{\gamma \in \Pi_*} d_{\alpha \gamma} \gamma, \quad d_{\alpha \gamma} \geq 0 ,
\]

by the definition of \( w_* \). Applying \( \theta^T \) we get

\[
\theta^T w_*(\alpha) = \theta^T(\alpha) + \sum_{\gamma \in \Pi_*} d_{\alpha \gamma} \gamma = -\omega(\alpha) - \sum_{\gamma \in \Pi_*} (c_{\alpha \gamma} - d_{\alpha \gamma}) \gamma ,
\]

hence \(-\theta^T w_*(\alpha) \in \Delta^+ \). But \(-\theta^T w_* \) is an automorphism of \( \Delta \). Namely, define an automorphism \( \phi' : g \to g \) by \( \phi' = \eta \cdot \iota \cdot \theta \). Then \( \phi'(t) = t \) and the dual to \( \phi'|_t \) is \(-\theta^T w_* \). Therefore \(-\theta^T w_*(\Pi) = \Pi \), so that \( c_{\alpha \gamma} = d_{\alpha \gamma} \) and \(-\theta^T w_*(\alpha) = \omega(\alpha) = \epsilon_\sigma(\alpha) \). □

Remark. If \( g \) is a complex simple Lie algebra considered as a real one, then the Dynkin diagram of its complexification is disconnected and has two isomorphic connected components. Furthermore, \( \Pi_* = \emptyset \) and \( \omega : \Pi_o \to \Pi_o \) maps each component of the Satake diagram onto the other one. In particular, \( \epsilon_\sigma \neq \text{id} \). If \( g \) is simple and has no complex structure, then it is easy to find the maps \( \epsilon_\sigma \) for all Satake diagrams, see [11], Table 5. Let \( l \) be the rank of \( g \). It turns out that \( \epsilon_\sigma = \text{id} \) for \( \sigma \) defining \( \mathfrak{sl}_{l+1}(\mathbb{R}) \), \( \mathfrak{sl}_m(\mathbb{H}) \), \( l = 2m - 1 \), \( \mathfrak{so}_{p,q} \), \( p + q = 2l \), \( l \equiv p(\text{mod} 2) \), \( \mathfrak{u}_l^*(\mathbb{H}) \), \( l = 2m \), \( \text{E}7, \text{E}8 \) or any real form of \( \text{B}_l, \text{C}_l \), \( \text{F}_4 \) and \( \text{G}_2 \). For the remaining real forms \( \epsilon_\sigma \neq \text{id} \).
4. Action of $\sigma$ on $\mathcal{X}(B)$

Let $G$ be a complex semisimple Lie group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus. The Lie algebras are denoted by the corresponding German letters. We want to apply the results of the previous section to the automorphisms of $\mathfrak{g}$ which lift to $G$. Suppose $\sigma$ is an antiholomorphic involutive automorphism of $G$ and denote again by $\sigma$ the corresponding antilinear involution of $\mathfrak{g}$. The automorphisms $\eta, \theta$ and $\iota$ lift to $G$ and the liftings are denoted by the same letters. Recall that $\epsilon_\sigma$ is originally defined by its action on $\Pi$ as an automorphism class in $\text{Aut}(\mathfrak{g})/\text{Int}(\mathfrak{g})$. The linear map induced by $\epsilon_\sigma$ on $t^*$ is denoted again by $\epsilon_\sigma$. The automorphism $\phi : \mathfrak{g} \to \mathfrak{g}$, $\phi = \eta \cdot \theta \cdot \iota$, leaves $t$ stable and acts on $t^*$ as $\epsilon_\sigma$, see Propositions 3.1 and 3.2. Since $\sigma$ and $\phi$ are globally defined, we may consider their actions on the character groups of $T$ or $B$.

Namely, since $\sigma(B)$ is also a Borel subgroup, we have $\sigma(B) = cBc^{-1}$ for some $c \in G$. The action of $\sigma$ on the character group $\mathcal{X}(B)$, given by

$$\lambda \mapsto \lambda^\sigma, \quad \lambda^\sigma(b) = \overline{\lambda(c^{-1}\sigma(b)c)} \quad (b \in B),$$

is correctly defined. For, if $d \in G$ is another element such that $\sigma(B) = dBd^{-1}$ then $d^{-1}c \in B$, hence $\lambda(d^{-1}\sigma(b)d) = \lambda(d^{-1}c)\lambda(c^{-1}\sigma(b)c)c(d^{-1}d) = \lambda(c^{-1}\sigma(b)c)$.

Also, we have the right action of the automorphism group $\text{Aut}(G)$ on $\mathcal{X}(B)$, defined in the same way. Namely, for an automorphism $\varphi : G \to G$ we put

$$\lambda^\varphi(b) = \lambda(c^{-1}\varphi(b)c) \quad (b \in B),$$

where $c$ is chosen so that $\varphi(B) = cBc^{-1}$.

For two Borel subgroups $B_1, B_2$ the character groups are canonically isomorphic. Moreover, if $\lambda_1 \in \mathcal{X}(B_1)$ corresponds to $\lambda_2 \in \mathcal{X}(B_2)$ under the canonical isomorphism then $\lambda_1^\sigma$ corresponds to $\lambda_2^\sigma$ and $\lambda_1^\varphi$ corresponds to $\lambda_2^\varphi$.

Clearly, $\lambda^\varphi = \lambda$ for $\varphi \in \text{Int}(G)$, so we obtain the action of $\text{Aut}(G)/\text{Int}(G)$ on $\mathcal{X}(B)$. In particular, we write $\epsilon_\sigma(\lambda)$ instead of $\lambda^\sigma$.

**Lemma 4.1.** For any $\lambda \in \mathcal{X}(B)$ one has

$$\lambda^\sigma = \epsilon_\sigma(\lambda).$$

**Proof.** Choose $t$ and $b = b^+$ as in Section 3. Then $\sigma(B) = \iota(B)$ by Proposition 3.3. Let $d\lambda$ be the differential of a character $\lambda$ at the neutral point of $T$. Since $\lambda^\sigma(t) = \overline{\lambda(\iota^{-1}\sigma(t))}$ for $t \in T$, we have $d\lambda^\sigma = \sigma^Tw_\bullet d\lambda$. 
On the other hand, $\epsilon_\sigma(\lambda) = \lambda^\phi$, where $\phi = \eta \cdot \theta \cdot \iota$. In the course of the proof of Proposition 3.1 we have shown that $\phi$ preserves $b^+$. Thus

$$\epsilon_\sigma(\lambda)(t) = \lambda(\eta \theta(t)) = \lambda(\theta(t))^{-1} \quad (t \in T),$$

hence $d\epsilon_\sigma(\lambda) = -w \cdot \theta^T d\lambda = -\theta^T w \cdot d\lambda$ by Proposition 3.2. Since $\theta^T = -\sigma^T$ on $t^*_R$, it follows that $d\epsilon_\sigma(\lambda) = d\lambda^\sigma$.

**Remark.** The automorphism class $\epsilon_\sigma$ has the following meaning for the representation theory, see [2]. Let $V$ be an irreducible $G$-module with highest weight $\lambda$. Denote by $V^*$ the complex dual to the space of antilinear functionals on $V$. Then $G$ acts on $V^*$ in a natural way, the action being antiholomorphic. This action combined with $\sigma$ is then holomorphic, the corresponding $G$-module is irreducible and has highest weight $\epsilon_\sigma(\lambda)$.

5. **Spherical homogeneous spaces**

Let $X = G/H$ be a spherical homogeneous space. We fix a Borel subgroup $B \subset G$ and recall the definitions of Luna-Vust invariants of $X$, see [10].

For $\chi \in \mathcal{X}(B)$ let $(B)C(X)_\chi \subset C(X)$ be the subspace of rational $B$-eigenfunctions of weight $\chi$, i.e.,

$$(B)C(X)_\chi = \{ f \in C(X) \mid f(b^{-1}x) = \chi(b)f(x) \quad (b \in B, x \in X) \}.$$

Since $X$ has an open $B$-orbit, this subspace is either trivial or one-dimensional. In the latter case we choose a non-zero function $f_\chi \in (B)C(X)_\chi$. The weight lattice $\Lambda(X)$ is the set of $B$-weights in $C(X)$, i.e.,

$$\Lambda(X) = \{ \chi \in \mathcal{X}(B) \mid (B)C(X)_\chi \neq \{0\} \}.$$

Let $\mathcal{V}(X)$ denote the set of $G$-invariant discrete $\mathbb{Q}$-valued valuations of $C(X)$. The mapping

$$\mathcal{V}(X) \to \text{Hom}(\Lambda(X), \mathbb{Q}), \quad v \mapsto \{ \chi \mapsto v(f_\chi) \}$$

is injective, see [10, 7], and so we regard $\mathcal{V}(X)$ as a subset of $\text{Hom}(\Lambda(X), \mathbb{Q})$. It is known that $\mathcal{V}(X)$ is a simplicial cone, see [5, 4].

The set of all $B$-stable prime divisors in $X$ is denoted by $\mathcal{D}(X)$. This is a finite set. To any $D \in \mathcal{D}(X)$ we assign $\omega_D \in \text{Hom}(\Lambda(X), \mathbb{Q})$. Namely, $\omega_D(\chi) = \text{ord}_D f_\chi$, the order of $f_\chi$ along $D$. We also write $G_D$ for the stabilizer of $D$. The Luna-Vust invariants of $X$ are given by the triple $\Lambda(X), \mathcal{V}(X), \mathcal{D}(X)$. The homogeneous space $X$ is completely determined by these combinatorial invariants. More precisely, one has the following theorem of I. Losev [8].
**Theorem 5.1.** Let $X_1 = G/H_1, X_2 = G/H_2$ be two spherical homogeneous spaces. Assume that $\Lambda(X_1) = \Lambda(X_2), \mathcal{V}(X_1) = \mathcal{V}(X_2)$. Assume further there is a bijection $j : \mathcal{D}(X_1) \to \mathcal{D}(X_2)$, such that $\omega_D = \omega_{j(D)}, G_D = G_{j(D)}$. Then $H_1$ and $H_2$ are conjugate by an inner automorphism of $G$.

We now return to equivariant real structures. Let $\sigma$ be an antiholomorphic involution of a semisimple complex algebraic group. Given a spherical subgroup $H \subset G$, observe that $\sigma(H)$ is also a spherical subgroup of $G$. Put $X_1 = G/H, X_2 = G/\sigma(H)$, and denote again by $\sigma$ the antiholomorphic map

$$X_1 \to X_2, g \cdot H \mapsto \sigma(g) \cdot \sigma(H).$$

Since the conjugate coordinate functions of $\sigma : G \to G$ are regular, we have $\sigma^* \cdot \mathbb{C}(X_2) = \overline{\mathbb{C}(X_1)}$. Choose and fix $c \in G$ in such a way that $\sigma(B) = cBc^{-1}$.

**Proposition 5.2.** $\epsilon_{\sigma}(\Lambda(X_1)) = \Lambda(X_2)$.

**Proof.** For $f \in \mathbb{C}(X_2)$ define a rational function on $X_1$ by

$$f'(x) = \overline{f(\sigma(cx))}.$$  

Note that for $b \in B$ one has $b' := \sigma(cbc^{-1}) \in B$. Furthermore, since $b_0 := \sigma(c)c \in B$, we have

$$\chi^\sigma(b) = \overline{\chi(c^{-1}\sigma(b)c)} = \overline{\chi(b_0^{-1}\sigma(c)\sigma(b)\sigma(c)^{-1}b_0)} = \overline{\chi(b')}.$$  

Now take $f = f_\chi$. Then we obtain

$$f'(b^{-1}x) = \overline{f(\sigma(cbc^{-1})x)} = \overline{f(\sigma(b^{-1})\sigma(cx))} = \overline{\chi(b')}f'(x),$$

showing that $f'$ is a $B$-eigenfunction of weight $\chi^\sigma$ on $X_1$. Since the transform $f \mapsto f'$ is invertible and $\chi^\sigma = \epsilon_{\sigma}(\chi)$ by Lemma 4.1 it follows that $\Lambda(X_2) = \epsilon_{\sigma}(\Lambda(X_1))$. \hfill $\Box$

**Proposition 5.3.** Extend $\epsilon_{\sigma}$ by duality to $\text{Hom}(\mathcal{X}(B), \mathbb{Q})$. Then $\epsilon_{\sigma}(\mathcal{V}(X_1)) = \mathcal{V}(X_2)$.

**Proof.** The map

$$\mathbb{C}(X_2) \ni f \mapsto \overline{f \circ \sigma} \in \mathbb{C}(X_1)$$

is a field isomorphism which is $\sigma$-equivariant in the obvious sense, namely,

$$(g \cdot f) \circ \sigma = \sigma(g) \cdot (f \circ \sigma) \quad (g \in G).$$

Therefore, for $v \in \mathcal{V}(X_1)$ the valuation of $\mathbb{C}(X_2)$ defined by $v'(f) = v(\overline{f \circ \sigma})$ is also $G$-invariant, i.e., $v' \in \mathcal{V}(X_2)$. Furthermore, since the function $f'$, defined in Proposition 5.2, is in the $G$-orbit of $\overline{f \circ \sigma}$, we
have $v'(f) = v(f')$. Now take $f = f_X$. Then $f'$ is a $B$-eigenfunction with weight $\epsilon_\sigma(\chi)$. Therefore $\epsilon_\sigma(v) = v'$.

For a $B$-invariant divisor $D$ on $X_1$ its image $\sigma(D)$ is a $\sigma(B)$-invariant divisor on $X_2$. Obviously, the map

$$j : \mathcal{D}(X_1) \to \mathcal{D}(X_2), \ j(D) := \sigma(c \cdot D),$$

is a bijection.

**Proposition 5.4.** For any $D \in \mathcal{D}(X_1)$ one has $\omega_{j(D)} = \epsilon_\sigma(\omega_D)$. The stabilizers of $D$ and $j(D)$ are parabolic subgroups containing $B$ and satisfying

$$\sigma(G_{j(D)}) = cG_Dc^{-1}.$$ 

Their roots systems are obtained from each other by $\epsilon_\sigma$.

**Proof.** Let $f \in \mathbb{C}(X_2)$ and let $f' \in \mathbb{C}(X_1)$ be the function defined in Proposition 5.2. Then

$$\text{ord}_{j(D)} f = \text{ord}_D f'.$$

Applying this to $f = f_X$ we obtain $\omega_{j(D)} = \epsilon_\sigma(\omega_D)$. The definition of $j$ implies readily that $\sigma(G_{j(D)}) = cG_Dc^{-1}$, and the last assertion follows from Lemma 4.1. □

Combining Propositions 5.2, 5.3, and 5.4, we get the following corollary.

**Corollary 5.5.** If $\epsilon_\sigma$ leaves stable $\Lambda(X_1), \mathcal{V}(X_1)$ and, for any $D \in \mathcal{D}(X_1)$, one has $\epsilon_\sigma(\omega_D) = \omega_D$ and $\sigma(G_D) = cG_Dc^{-1}$ then $H$ and $\sigma(H)$ are conjugate by an inner automorphism, i.e., $\sigma(H) = aHa^{-1}$, where $a \in G$. The map $g \cdot H \mapsto \sigma(g)\alpha \cdot H$ is correctly defined, antiholomorphic and $\sigma$-equivariant. Moreover, if the subgroup $H$ is self-normalizing then this map is a $\sigma$-equivariant real structure on $X_1$ and such a structure is unique.

**Proof.** The conjugacy of $H$ and $\sigma(H)$ results from Theorem 5.1. The remaining assertions follow from Theorem 2.1. □

**Proof of Theorem 1.1** It suffices to apply the above corollary in the case $\epsilon_\sigma = \text{id}$. □

**Proposition 5.6.** If $\epsilon_\sigma = \text{id}$, then any $v \in \mathcal{V}(G/H)$ is $\mu_0$-invariant, i.e., for a non-zero rational function $f \in \mathbb{C}(G/H)$ one has $v(f \circ \mu_0) = v(f)$.

**Proof.** Consider $f$ as a right $H$-invariant function on $G$ and put $f^a(g) = f(ga)$ ($g \in G$). Then $f^a$ is right $aHa^{-1}$-invariant. Since $\sigma(H) = aHa^{-1}$, we can view $f^a$ as a rational function on $X_2 = G/\sigma(H)$. Recall that we have the map $\sigma : X_1 \to X_2$. The definition of $\mu_0 : X_1 \to X_1$ implies $f \circ \mu_0 = f^a \circ \sigma$. It suffices to prove the equality $v(f \circ \mu_0) = v(f)$.
on $B$-eigenfunctions. Now, if $f$ is such a function then $f^a$ is also a $B$-eigenfunction with the same weight. In the proof of Proposition 5.3 for a given $v \in \mathcal{V}(X_1)$ we defined $v' \in \mathcal{V}(X_2)$ and proved that $\epsilon_\sigma(v) = v'$. In our setting $v = v'$, and so we obtain $v(f) = v'(f^a) = v(f^a \circ \sigma) = v(f \circ \mu_0)$.

\begin{proof}

Example. Up to an automorphism of $X = \mathbb{C}P^d$, there are two real structures $\mu_1, \mu_2 : X \to X$ for $d$ odd and one real structure $\mu_1 : X \to X$ for $d$ even. In homogeneous coordinates

$$\mu_1(z_0 : z_1 : \ldots : z_d) = (\overline{z_0} : \overline{z_1} : \ldots : \overline{z_d})$$

and

$$\mu_2(z_0 : z_1 : \ldots : z_d) = (-\overline{z_1} : \overline{z_0} : \ldots : -\overline{z_d} : \overline{z_d-1}), \ d = 2l - 1.$$ 

One has $X^{\mu_1} = \mathbb{R}P^d$ and $X^{\mu_2} = \emptyset$. Let $s_l$ be the block $(2l \times 2l)$-matrix with $l$ diagonal blocks

$$
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}.
$$

For $g \in G = \text{SL}(d + 1, \mathbb{C})$ put

$$\sigma_1(g) = \overline{g} \quad \text{and} \quad \sigma_2(g) = -s_l \overline{g} s_l \quad \text{if} \quad d + 1 = 2l.$$

Then $G^{\sigma_1} = \text{SL}(d + 1, \mathbb{R})$ (the split real form) and $G^{\sigma_2} = \text{SL}(l, \mathbb{H})$, where $d + 1 = 2l$. One checks easily that $\mu_1$ is $\sigma_1$-equivariant and $\mu_2$ is $\sigma_2$-equivariant. Note that a real structure can be $\sigma$-equivariant only for one involution $\sigma$. Therefore $X$ has no $\sigma$-equivariant real structure if $\sigma$ defines a pseudo-unitary group $\text{SU}(p, q)$, $p + q = d + 1$.

6. Wonderful embeddings

A complete non-singular algebraic $G$-variety $X$ of a semisimple group $G$ is called wonderful if $X$ admits an open $G$-orbit whose complement is a finite union of smooth prime divisors $X_1, \ldots, X_r$ with normal crossings and the closures of $G$-orbits on $X$ are precisely the partial intersections of these divisors. The notion of a wonderful variety was introduced by D.Luna [9], who also proved that wonderful varieties are spherical. The total number of $G$-orbits on $X$ is $2^r$. The number $r$ coincides with the rank of $X$ as a spherical variety. Moreover, if a spherical homogeneous space $G/H$ has a wonderful embedding then such an embedding is unique up to a $G$-isomorphism.

Theorem 6.1. Let $G$ be a complex semisimple algebraic group, $H \subset G$ a spherical subgroup, and $\sigma : G \to G$ an antiholomorphic involution. Assume that $G/H$ admits a wonderful embedding $G/H \hookrightarrow X$. If there exists a $\sigma$-equivariant real structure on $G/H$ then it extends to a $\sigma$-equivariant real structure on $X$.  

Proof. This follows from the uniqueness of wonderful embedding. Namely, let \( \varepsilon : G/H \to X \) be the given wonderful embedding. Take the complex conjugate \( \overline{X} \) of \( X \) and let \( \overline{\varepsilon} : G/H \to \overline{X} \) be the corresponding antiholomorphic map. We identify \( \overline{X} \) with \( X \) as topological spaces and endow \( \overline{X} \) with the action \((g, x) \mapsto \sigma(g) \cdot x\), which is regular. Now, take a \( \sigma \)-equivariant real structure \( \mu \) on \( G/H \) and consider the map \( \overline{\varepsilon} \circ \mu : G/H \to \overline{X} \). This is again a wonderful embedding of \( G/H \). Since two wonderful embeddings are \( G \)-isomorphic, there is a \( G \)-isomorphism \( \nu : X \to \overline{X} \) such that \( \nu \circ \varepsilon = \overline{\varepsilon} \circ \mu \). The map \( \nu \) defines a required \( \sigma \)-equivariant real structure on \( X \). □

Proof of Theorem 1.2. Let \( G/H \hookrightarrow X \) be the wonderful completion. The existence and uniqueness of a \( \sigma \)-equivariant real structure \( \mu_0 \) on \( G/H \) follows from Theorem 1.1. By Theorem 6.1 this real structure extends to \( X \), the extension being obviously unique. □

As an application of our previous results we have the following property of the \( \sigma \)-equivariant real structure \( \mu \).

**Theorem 6.2.** We keep the notations and assumptions of Theorem 1.2. Then all \( G \)-orbits on \( X \) are \( \mu \)-stable.

Proof. It suffices to show that all divisors \( X_i \) are \( \mu \)-stable. Each \( X_i \) defines a \( G \)-invariant valuation of the field \( \mathbb{C}(X) = \mathbb{C}(G/H) \). By Proposition 5.6 such a valuation is \( \mu \)-invariant. Since the divisor is uniquely determined by its valuation, it follows that \( X_i \) are \( \mu \)-stable. □

**Corollary 6.3.** Keeping the above notations and assumptions, suppose that \( \mu \) has a fixed point in the closed \( G \)-orbit \( X_1 \cap \ldots \cap X_r \subset X \). Then \( \mu \) has a fixed point in any \( G \)-orbit. In particular, the number of \( G^A \)-orbits in \( X^\mu \) is greater than or equal to \( 2^r \).

Proof. The closure of any \( G \)-orbit in \( X \) is of the form \( Y = X_{i_1} \cap \ldots \cap X_{i_k} \). We know that \( Y \) is \( \mu \)-stable and has a non-trivial intersection with \( X^\mu \). Since the real dimension of \( X^\mu \cap Y \) equals the complex dimension of \( Y \), the set \( X^\mu \) must intersect the open \( G \)-orbit in \( Y \). □

The condition \( \epsilon_\sigma = \text{id} \) is essential.

**Example.** The adjoint representation of \( \text{SL}(2, \mathbb{C}) \) gives rise to a two-orbit action on the projective plane. The closed orbit is the quadric \( C \subset \mathbb{CP}^2 \) arising from the nilpotent cone in the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \). Let \( G = \text{SL}(2, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \) and \( \sigma(g_1, g_2) = (\bar{g_2}, \bar{g_1}) \), where \( g_1, g_2 \in \text{SL}(2, \mathbb{C}) \). Note that \( G^A = \text{SL}(2, \mathbb{C}) \) considered as a real group and \( \epsilon_\sigma \neq \text{id} \). Let \( X = \mathbb{CP}^2 \times \mathbb{CP}^2 \) with each simple factor of \( G \) acting on the corresponding factor of \( X \) in the way described above. Then \( X \) is a wonderful variety of rank 2. The divisors \( X_1, X_2 \) from the definition of
a wonderful variety are $\mathbb{CP}^2 \times Q$ and $Q \times \mathbb{CP}^2$. The $\sigma$-equivariant real structure $\mu$ on $X$ is given by $\mu(z_1, z_2) = (\bar{z}_2, \bar{z}_1)$, $z_1, z_2 \in \mathbb{CP}^2$. The $G$-stable hypersurfaces $X_1, X_2$ are interchanged by $\mu$ and not $\mu$-stable.

REFERENCES

[1] D.N.Akhiezer, *On the orbits of real forms of complex reductive groups on spherical homogeneous spaces*, in: Voprosy Teorii Grupp i Homologicheskoi Algebry, Yaroslavl State Univ., Yaroslavl, 2003, pp. 4–18 (Russian).

[2] D.N.Akhiezer, *Real forms of complex reductive groups acting on quasiaffine varieties*, Amer. Math. Soc. Transl. (2), Vol. 213 (2005), pp. 1–13.

[3] D.Akhiezer, S.Cupit-Foutou, *On the canonical real structure on wonderful varieties*, arXiv: 1202.6607, to appear in Crelle journal.

[4] M.Brion, *Variétés sphériques*, Notes de la session de la S.M.F. ”Opérations hamiltoniennes et opérations de groupes algébriques”, Grenoble, 1997, pp. 1–60.

[5] M.Brion, F.Pauer, Valuations des espaces homogènes sphériques, Comment. Math. Helvetici, Vol. 62 (1987), pp. 265–285.

[6] F.Knop, *The Luna-Vust theory of spherical embeddings*. pp. 225 - 249 in: Proceedings of the Hyderabad conference on algebraic groups, Hyderabad, India Dec., 1991.

[7] F.Knop, *Automorphisms, root systems, and compactifications of homogeneous varieties*, J. Amer. Math. Soc., Vol. 9 (1996), pp.153–174.

[8] I.Losev, *Uniqueness property for spherical homogeneous spaces*, Duke Math.J., Vol.147 (2009), 2, pp. 315–343.

[9] D.Luna, *Toute variété magnifique est sphérique*, Transform. Groups, Vol.1 (1996), 3, pp. 249–258.

[10] D.Luna, T.Vust, *Plongements d’espaces homogènes*, Comment. Math. Helvetici, Vol. 58 (1983), 186–245.

[11] A.L.Onishchik, *Lectures on real semisimple Lie algebras and their representations*, ESI Lectures in Mathematics and Physics, EMS 2004.

[12] A.L.Onishchik, E.B.Vinberg, *Lie groups and algebraic groups*, Springer-Verlag, Berlin-Heidelberg-New York, 1990.

[13] I.Satake, *On representations and compactifications of symmetric Riemannian spaces*, Ann. of Math., Vol.71, No.1 (1960), 77–110

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