Twisting structures and strongly homotopy morphisms\footnote{With an appendix by P.-E. Parent (University of Ottawa)}

Kathryn Hess\footnote{K.H. thanks the Institut Mittag-Leffler (Djursholm, Sweden) for its hospitality during a crucial phase of this research and the Midwest Topology Network for its financial support.}, Jonathan Scott

\textit{Institut de géométrie, algèbre et topologie (IGAT)}
\textit{École Polytechnique Fédérale de Lausanne}
\textit{CH-1015 Lausanne}
\textit{Switzerland}

\textit{Department of Mathematics}
\textit{Cleveland State University}
\textit{2121 Euclid Ave., RT 1515}
\textit{Cleveland OH 44115-2214 USA}

\textit{Department of Mathematics and Statistics}
\textit{University of Ottawa}
\textit{585 King Edward Ave.}
\textit{Ottawa ON K1N 6N5 Canada}

Abstract

In an application of the notion of twisting structures introduced by Hess and Lack \cite{HL}, we define twisted composition products of symmetric sequences of chain complexes that are degreewise projective and finitely generated. Let $\mathcal{Q}$ be a cooperad, and let $\mathbf{B}_P$ denote the bar construction on an operad $P$. To each morphism of cooperads $g : \mathcal{Q} \to \mathbf{B}_P$ is associated a $P$-co-ring, $K(g)$, which generalizes the two-sided Koszul and bar constructions. When the counit $K(g) \to P$ is a quasi-isomorphism, we show that the Kleisli category for $K(g)$ is isomorphic to the category of $P$-algebras and of their morphisms up to strong homotopy, and we give the classifying morphisms for both strict and homotopy $P$-algebras. Parametrized morphisms of (co)associative chain (co)algebras up to strong homotopy are also introduced and studied, and a general existence theorem is proved. In the appendix, we study the co-ring associated to the canonical morphism of cooperads $\mathcal{A}^\perp \to \mathbf{B}_\mathcal{A}$, which is exactly the two-sided Koszul resolution of the associative operad $\mathcal{A}$, also known as the Alexander-Whitney co-ring.

Keywords: operad, strong homotopy, twisting cochain, Kleisli category, bar construction, co-ring
1. Introduction

When working with a given type of algebra, it is often possible to form a “standard construction”, a co-free coalgebra of some sort in which the algebraic structure is encoded in a quadratic differential. For example, for an associative or Lie algebra we have respectively the bar construction and the Chevalley-Eilenberg complex \[4\]. We can furthermore reconstruct the original algebra from its standard construction, at least if 2 is invertible in the ground ring. Thus we often identify the algebra and its standard construction.

The family of higher homotopies that constitute a strongly homotopy-multiplicative map \( f : A \Rightarrow A' \) can be “rolled up” into a single morphism of coassociative coalgebras, \( F : BA \rightarrow BA' \). Similarly, if \( C_\ast(-) \) denotes the Chevalley-Eilenberg complex of a DG Lie algebra, which is a commutative associative coalgebra, then a morphism of coalgebras, \( F : C_\ast(L) \rightarrow C_\ast(L') \) ’is’ a strong homotopy Lie morphism.

More generally, let \( \mathcal{P} \) be a quadratic operad with quadratic dual cooperad \( \mathcal{P}^\perp \). A \( \mathcal{P} \)-algebra \( A \) has a standard ‘bar’ construction \( B^\mathcal{P}A \) that is a co-free connected \( \mathcal{P}^\perp \)-coalgebra, provided with a quadratic differential that encodes the algebraic structure of \( A \). A strong homotopy (henceforth abbreviated SH) \( \mathcal{P} \)-morphism from \( A \) to another \( \mathcal{P} \)-algebra \( A' \) is then a morphism of \( \mathcal{P}^\perp \)-coalgebras, \( B^\mathcal{P}A \rightarrow B^\mathcal{P}A' \). We remark that since we are using \( \mathcal{P}^\perp \) instead of the quadratic dual operad of Ginzburg and Kapranov \[9\], our bar constructions are shifted by one degree from the classical constructions.

Dually, there is a standard ‘cobar’ construction for \( \mathcal{P} \)-coalgebras, that takes values in \( \mathcal{P}^! \)-algebras, and we obtain the notion of SH morphisms of \( \mathcal{P} \)-coalgebras.

The goal of this article is to describe the various categories with SH morphisms “operadically”, when working in the category \( \text{dgProj} \) of degreewise finitely generated and projective chain complexes over a commutative ring \( R \).

Our first result reads as follows (Proposition \[4.8\] and Theorem \[4.10\]).

**Theorem 1.1.** Let \( \mathcal{P} \) be a Koszul operad in \( \text{dgProj} \). The two-sided Koszul resolution \( K(\mathcal{P}) \) of \( \mathcal{P} \) is naturally a \( \mathcal{P} \)-co-ring, and therefore induces a comonad on the category of \( \mathcal{P} \)-algebras in \( \text{dgProj} \). The associated Kleisli category, with \( \mathcal{P} \)-algebras as objects and morphisms parametrized by \( K(\mathcal{P}) \), is isomorphic to the category of \( \mathcal{P} \)-algebras in \( \text{dgProj} \) and their SH morphisms.

A similar result for \( \mathcal{P}^\perp \)-coalgebras holds as well.

**Theorem 1.2.** With the notation and hypotheses above, the resolution \( K'(\mathcal{P}) \) is naturally a \( \Omega \mathcal{P}^\perp \)-co-ring and therefore induces a comonad on the category of \( \Omega \mathcal{P}^\perp \)-algebras in \( \text{dgProj} \). The associated Kleisli category, with \( \Omega \mathcal{P}^\perp \)-algebras...
as objects, and morphisms parametrized by $K'(\mathcal{P})$, is isomorphic to the category of $\mathcal{P}_\infty$-algebras in $\text{dgProj}$ and their SH morphisms.

We consider moreover parametrization of SH morphisms of associative algebras (respectively, coassociative coalgebras) over cooperads of chain coalgebras (respectively, over operads of chain algebras) (Definition 5.3). We prove that such parametrized SH categories also admit an operadic, (co)Kleisli description (Theorem 5.18), which then enables us to establish useful existence results for natural, parametrized SH structures (Theorems 5.19 and 5.20).

In Section 2, we present our main tool in the study of morphism sets: the notion of twisting structures. While twisting cochains in the category of differential graded symmetric sequences go back as early as [8], twisting structures on a category allow for the definition of twisted products via classifying morphisms rather than twisting cochains, and can therefore be applied more generally. We define twisting structures and twisted products, prove the expected adjunction relations, and define the standard (dual) constructions associated to a classifying morphism. One result of the adjunction relations turns out to be critical to our development: a standard construction associated to any classifying map admits a natural comonoidal structure.

In Section 3, we review differential graded symmetric sequences and (co)operads. We use the bar construction with coefficients to define a twisting structure on the category of symmetric sequences of chain complexes.

In Section 4, we recall the dual cooperad $\mathcal{P}^\perp$ of a weight-graded operad $\mathcal{P}$; the inclusion $\kappa_\mathcal{P}: \mathcal{P}^\perp \to \mathcal{B}\mathcal{P}$ is a classifying morphism. We show that if $\mathcal{P}$ is Koszul (i.e., the inclusion $\kappa_\mathcal{P}$ is a quasi-isomorphism), then the Kleisli category for the standard construction $K(\kappa_\mathcal{P})$ is isomorphic to the category of $\mathcal{P}$-algebras and SH morphisms. Considering instead the unit of the adjunction, $\eta: \mathcal{P}^\perp \to \mathcal{B}\Omega\mathcal{P}^\perp$, we obtain the $\mathcal{P}_\infty$ category. We then briefly discuss the general, non-Koszul, case, where we use the identity on $\mathcal{B}\mathcal{P}$ and the unit $\mathcal{B}\mathcal{P} \to \mathcal{B}\Omega\mathcal{B}\mathcal{P}$, respectively, for our classifying morphisms.

In Section 5, we introduce and discuss the concept of parametrized SH morphisms of (co)associative coalgebras.

In the first appendix, P.-E. Parent discusses the special case of the Koszul resolution of the associative operad, $\mathcal{A}$, and shows that an algebra with an SH-comultiplicative diagonal is not the same thing as an algebra with cup-$i$ products, as one might have expected.

The second appendix is devoted to the somewhat technical proof of the existence of parametrized SH morphisms of (co)associative (co)algebras.

1.1. Comparison to other approaches

The original work in this direction, done in the category of spaces, was by Iwase and Mimura [18], who considered $A_\infty$-maps of $A_\infty$-spaces.

Markl [23] considered the coloured operad, or multicategory, $\mathcal{P}(\cdot \to \cdot)$, whose algebras are morphisms of $\mathcal{P}$-algebras. An SH $\mathcal{P}$-algebra morphism is then an algebra for a cofibrant replacement of $\mathcal{P}(\cdot \to \cdot)$; Markl shows, among other things, that SH morphisms are homotopy invariant. We can relate our
co-rings to the multicategory approach, and answer Markl’s Problem 26 of [22]: as described below, we can construct a map $\mathcal{P}(a \to c) \to \mathcal{P}(b \to c) * \mathcal{P}(a \to b)$ (where * is the free product) such that the resulting composition of SH morphisms is associative.

Let $\mathcal{P}$ be an operad, and let $K$ be the $\Omega B\mathcal{P}$-co-ring associated to the classifying morphism $B\mathcal{P} \to B\Omega B\mathcal{P}$. A category enriched in symmetric sequences is a special case of a multicategory. Consider two such categories: the first, $M$, has two objects $x$ and $y$, with hom objects $M(x, x) = M(y, y) = \mathcal{P}'$, $M(x, y) = K(\mathcal{P}')$. The second, $N$, has three objects, $a$, $b$, and $c$, with hom objects $N(a, a) = N(b, b) = N(c, c) = \mathcal{P}'$, and $N(a, b) = N(b, c) = K(\mathcal{P}')$, $N(a, c) = K(\mathcal{P}') \circ_{\mathcal{P}'} K(\mathcal{P}')$. An enriched functor $F : M \to N$ is defined by $F(x) = a$, $F(y) = c$, and $F : M(x, y) \to N(a, c)$ is the diagonal in $K(\mathcal{P}')$. Then pulling back along $F$ provides the desired associative composition in the SH category, from the multicategory point of view. Note that in our formulation of homotopy morphisms, it is obvious how to define their composition and indeed the whole category structure.

Coloured operads have the advantage that they can be applied to many more situations than can homotopy morphisms. Berger and Moerdijk [2] consider coloured operads in general, and study conditions under which, in particular, SH morphisms are rectifiable.

Leinster [21] defines the category of homotopy algebras over an operad $\mathcal{P}$ to be the category of colax representations, $X : \hat{\mathcal{P}} \to \mathcal{M}$, where $\hat{\mathcal{P}}$ is the strict monoidal category with the nonnegative integers as objects, $+$ and $0$ for the monoidal structure, and $\hat{\mathcal{P}}(m, n) = \mathcal{P} \circ_{\mathcal{P}'} m(n)$ (see Section 3.1 for the graded tensor product $\circ$ of symmetric sequences). The morphisms are monoidal transformations. In this formulation, morphisms do indeed commute with algebraic structure weakly, but homotopy invariance is not immediately obvious to us.

1.2. Acknowledgments

The authors would like to express their heartfelt appreciation to Paul-Eugène Parent for his helpful participation in the early stages of this project and in particular for his contribution to our understanding of the Alexander-Whitney co-ring, as detailed in the first appendix to this article. This has proved to be a very long-term project, which has evolved significantly over the past six years. Earlier versions of our approach to describing strongly homotopy morphisms via co-rings can be found on the arXiv [13]. Results from these earlier manuscripts, in particular concerning the Alexander-Whitney co-ring, which have already been applied in various articles and theses (e.g., [16], [15], [12], [17], [25] and [3]), are also stated and proved here.

1.3. Notation

If $C$ is a category, and $A$ and $B$ are objects in $C$, then $C(A, B)$ denotes the set of morphisms from $A$ to $B$.

If $T = (T, \mu, \eta)$ is a monad on a category $C$, then $C_T$ denotes the Kleisli category determined by $T$, with $\text{Ob } C_T = \text{Ob } C$ and $C_T(A, B) = C(A, TB)$. If
$f \in C_T(A, B)$ and $g \in C_T(B, C)$, then their composite in $C_T$ is defined to be
the composite of

$$A \xrightarrow{f} TB \xrightarrow{Tg} T^2C \xrightarrow{\mu_C} TC$$

in $C$.

Dually, if $K = (K, \Delta, \varepsilon)$ is a comonad on $C$, then $K_C$ denotes the coKleisli category
determined by $K$, with $\text{Obj} K_C = \text{Obj} C$ and $\text{C}(A, B) = C(KA, B)$. If
$f \in K_C(A, B)$ and $g \in K_C(B, C)$, then their composite in $K_C$ is defined to be
the composite of

$$KA \xrightarrow{\Delta_K} K^2A \xrightarrow{Kf} KB \xrightarrow{g} C$$

in $C$.

### 2. Twisting structures

The goal of this section is to introduce a categorical structure that conveniently
generalizes both twisting cochains from differential graded coalgebras to
differential graded algebras and twisting functions from simplicial sets to simplicial groups. Hess and Lack first formulated such a definition in [11], though
in an even more highly categorical manner.

Throughout this section $(M, \otimes, I)$ denotes a monoidal category that admits
equalizers and coequalizers. Let $\text{Mon}$ and $\text{Comon}$ denote the categories of
monoids and of comonoids in $M$. If $A$ is a monoid in $M$, then $A\text{Mod}$ and
$\text{Mod} A$ are the categories of left $A$-modules and of right $A$-modules. Similarly,
$C\text{Comod}$ and $\text{Comod} C$ denote the categories of left and right comodules over
a comonoid $C$.

The following definitions are classical.

**Definition 2.1.** Let $A$ be a monoid in $M$. Let $(M, \rho)$ be a right $A$-module,
and let $(N, \lambda)$ be a left $A$-module. The tensor product of $M$ and $N$ over $A$ is
the coequalizer

$$M \otimes_A N := \text{coequal}(M \otimes A \otimes N \xrightarrow{\rho \otimes N} M \otimes N).$$

Let $C$ be a comonoid in $M$. Let $(M, \rho)$ be a right $C$-comodule, and let $(N, \lambda)$
be a left $C$-comodule. The cotensor product of $M$ and $N$ over $C$ is the equalizer

$$M \square_C N := \text{equal}(M \otimes N \xrightarrow{\rho \otimes N} M \otimes C \otimes N).$$

We often consider objects of $M$ that are endowed with either two actions or
two coactions or an action and a coaction, for which we introduce the following
notation.

**Notation 2.2.** Let $A$ and $A'$ be monoids in $M$, and let $C$ and $C'$ be comonoids
in $M$. We consider the following classes of objects in $M$ that are endowed with
two structures.
\[ \text{Mix}_{A'} =_{A} \text{Mod}_{A'} = \{(M, \rho, \lambda) \mid (M, \rho) \in \text{Mod}_{A'}, (M, \lambda) \in _{A} \text{Mod}, \lambda(A \otimes \rho) = \rho(\lambda \otimes A') \} \]

\[ \text{Mix}_{C} = \{(M, \rho, \lambda) \mid (M, \rho) \in \text{Comod}_{C}, (M, \lambda) \in _{A} \text{Mod}, (\lambda \otimes C)(A \otimes \rho) = \rho \lambda \} \]

\[ \text{Mix}_{A} = C = \{(M, \rho, \lambda) \mid (M, \rho) \in \text{Mod}_{A}, (M, \lambda) \in C \text{Comod}, (C \otimes \rho)(\lambda \otimes A) = \lambda \rho \} \]

\[ C_{\text{Mix}} = C_{\text{Comod}} = \{(M, \rho, \lambda) \mid (M, \rho) \in \text{Comod}_{C'}, (M, \lambda) \in C_{\text{Comod}}, (\lambda \otimes C')(C \otimes \rho) = (C \otimes \rho) \lambda \} \]

Tensor and cotensor products must commute, in the sense of the following definition, if we wish to define twisting structures.

**Definition 2.3.** The monoidal category \( M \) is *twistable* if the tensor and cotensor products defined above restrict and corestrict to bifunctors

\[ - \otimes _{A} : \text{Mix}_{A} \times \text{Mix}_{A'} \rightarrow \text{Mix}_{Y} \]

and

\[ _{C} \square - : \text{Mix}_{C} \times \text{Mix}_{C'} \rightarrow \text{Mix}_{Y}, \]

for all (co)monoids \( X \) and \( Y \). Furthermore, the tensoring and cotensoring must be associative up to isomorphism in the obvious sense.

**Example 2.4.** The categories of sets and of simplicial sets are clearly twistable, as is the category \( \text{dgProj} \) of degree-wise finitely generated, projective chain complexes over a commutative ring \( R \). We prove in section 3.4.1 that the category of symmetric sequences in \( \text{dgProj} \) is also twistable (Theorem 3.10).

**Remark 2.5.** If \( M \) is twistable, then we can define a category \( \text{Mix} \) with

\[ \text{Ob Mix} = \text{Ob Mon} \cup \text{Ob Comon} \]

and

\[ \text{Mix}(X, Y) = \text{Mix}_{Y} / \cong, \]

where \( \cong \) denotes the isomorphism relation. Composition of morphisms in \( \text{Mix} \) is given by tensoring over monoids and cotensoring over comonoids.

There are functors

\[ J : \text{Mon} \rightarrow \text{Mix} \quad \text{and} \quad \tilde{J} : \text{Mon}^{op} \rightarrow \text{Mix} \]

specified on objects by \( J(A) = A = \tilde{J}(A) \). On morphisms we set

\[ J(f : A \rightarrow A') = fA' \in \text{Mix}(A, A') \]
where \( f A' \) denotes the isomorphism class of \( A' \) seen as a right \( A' \)-module via its own multiplication and as left \( A \)-module via \( f \), while
\[
\tilde{J}(f : A \to A') = A'_f \in \text{Mix}(A', A),
\]
where \( A'_f \) denotes the isomorphism class of \( A' \) seen as a left \( A' \)-module via its own multiplication and as a right \( A \)-module via \( f \).

Similarly, there are functors
\[
\text{co}J : \text{Comon} \to \text{Mix} \quad \text{and} \quad \text{co}\tilde{J} : \text{Comon}^{\text{op}} \to \text{Mix}
\]
specified on objects by \( \text{co}J(C) = C = \text{co}\tilde{J}(C) \). On morphisms we set
\[
\text{co}J(g : C \to C') = C_g \in \text{Mix}(C, C'),
\]
where \( C_g \) denotes the isomorphism class of \( C \) seen as a left \( C \)-comodule via its own comultiplication and as right \( C \)-comodule via \( g \), while
\[
\text{co}\tilde{J}(g : C \to C') = gC \in \text{Mix}(C', C),
\]
where \( gC \) denotes the isomorphism class of \( C \) seen as a right \( C \)-comodule via its own multiplication and as a left \( C' \)-comodule via \( f \).

The following definition is a slight variant of that formulated in [11].

**Definition 2.6.** A right twisting structure on a twistable monoidal category \( M \) consists of a functor
\[
B : \text{Mon} \to \text{Comon}
\]
together with natural transformations
\[
E : \text{co}J \circ B \Rightarrow J
\]
and
\[
\tilde{E} : \tilde{J} \Rightarrow \text{co}\tilde{J} \circ B
\]
of functors from \( \text{Mon} \) (respectively, \( \text{Mon}^{\text{op}} \)) to \( \text{Mix} \) and natural morphisms for all monoids \( A \)
\[
\delta_A : BA \to EA \otimes A \tilde{E}A
\]
of \( BA \)-bicomodules and
\[
\mu_A : \tilde{E}A \square BA \to A
\]
of \( A \)-bimodules such that the following diagrams commute.

\[
\begin{array}{ccc}
\tilde{E}A & \cong & \tilde{E}A \square BA \\
\text{EA} \otimes A \tilde{E}A & \xrightarrow{\mu_A \otimes A \tilde{E}A} & \text{EA} \square BA
\end{array}
\]
The choice of terminology above is motivated by the existence of right twisting structures when $\mathbf{M}$ is the category of either connected chain complexes or reduced simplicial sets, where the right twisting structure can be defined by either twisting cochains or twisting functions, both of which are classical notions. The details of both of these cases can be found in [11]. In this paper we generalize the chain complex case in section 3.4, showing that the category of symmetric sequences of chain complexes admits a right twisting structure.

**Remark 2.7.** It may be helpful to unfold the definition above. If $(B, E, \tilde{E}, \delta, \mu)$ is a right twisting structure on $\mathbf{M}$, then for all monoids $A$,

$$EA \cong BA \Box EA \quad \delta_A \Box EA \quad EA \otimes \tilde{E}A \Box EA \quad \Delta_A \Box EA \Box \mu_A \quad EA$$

Moreover, for all monoid morphisms $f : A \to A'$, the naturality of $E$ and of $\tilde{E}$ implies that

$$EA \otimes f A' \cong BABf \Box EA'$$

and

$$A'f \otimes \tilde{E}A \cong \tilde{E}A' \Box Bf BA.$$  

For any monoid $A$, we think of $EA$ and $\tilde{E}A$ as the total spaces of the “universal right $A$-bundle” and the “universal left $A$-bundle” in $\mathbf{M}$, respectively. This vision of their role motivates the following definition.

**Definition 2.8.** Let $C$ be a comonoid, and let $A$ be a monoid in a twistable monoidal category $\mathbf{M}$ endowed with a right twisting structure. A **classifying morphism** between $C$ and $A$ is a morphism of comonoids $g : C \to BA$.

Given a right $C$-comodule $V$ and a left $A$-module $W$, we define the **twisted product** of $V$ and $W$ with respect to the classifying morphism $g$ as

$$V \otimes_g W = V_g \Box BA \otimes_{A} W,$$

where $V_g$ is $V$ considered as a right $BA$-comodule via $g$. If $X$ is a right $A$-module and $Y$ is a left $C$-comodule, then

$$X \otimes_g Y = X \otimes \tilde{E}A \Box_B (gY),$$

where $gY$ is $Y$ considered as a left $BA$-comodule via $g$.

The **right $A$-bundle induced by $g$** is $C_g \otimes_g A \in c\text{Mix}_A$, while the **left $A$-bundle induced by $g$** is $A \otimes_g C \in A\text{Mix}_C$. 

8
The next theorem and its corollary are essential to establishing our operadic characterization of strongly homotopy maps.

**Theorem 2.9.** Let $C$ and $C'$ be comonoids, and let $A$ and $A'$ be monoids in a twistable monoidal category $\mathbf{M}$ endowed with a right twisting structure. Let $g: C \to BA$ and $g': C' \to BA'$ be classifying morphisms. If

$$(g, g')_*: C\text{-Comod}_{C'} \to A\text{-Mod}_{A'}$$

denotes the functor specified by

$$(g, g')_*(N) = A \otimes_N g \otimes g' A'$$

for all $(C, C')$-bicomodules $N$, and

$$(g, g')^*: A\text{-Mod}_{A'} \to C\text{-Comod}_{C'}$$

denotes the functor specified by

$$(g, g')^*(M) = C \otimes_M g \otimes g' C',$$

then the functors

$$(g, g')_*: C\text{-Comod}_{C'} \rightleftarrows A\text{-Mod}_{A'}: (g, g')^*$$

form an adjunction.

**Proof.** It clearly suffices to consider the universal case, i.e., $C = BA, C' = BA'$ and $g = Id_{BA}, g' = Id_{BA'}$.

We define a unit natural transformation $\eta: Id \Rightarrow (g, g')^* (g, g')_*$ by

$$\eta_N = \delta_{BA} \square N \square \delta_{BA'}: N \to EA \otimes \tilde{EA} \square N \square \tilde{EA} \otimes E A'$$

and a counit natural transformation $\varepsilon: (g, g')_*(g, g')^* \Rightarrow Id$ by

$$\varepsilon_M = \mu_A \otimes_M \mu_{A'}: \tilde{EA} \square EA \otimes M \square \tilde{EA}' \square E A' \to M.$$  

Since $(\mu_A \otimes \tilde{EA})(\tilde{EA} \square \delta_{BA}) = Id_{\tilde{E}A}$ and $(EA \otimes \mu_A)(\delta_{BA} \square E A) = Id_{E A}$, we can show easily that

$$\varepsilon_{(g, g)^*} \circ (g, g')_* \eta$$

is the identity natural transformation on $(g, g')_*$ and that

$$(g, g')^* \varepsilon \circ \eta_{(g, g')^*}$$

is the identity natural transformation on $(g, g')^*$. In other words, $\eta$ and $\varepsilon$ are the unit and counit of an adjunction.

Specializing to $g = Id_I$ or $g' = Id_I$, we obtain the next result.
Corollary 2.10. Let $C$ and $A$ be comonoid and a monoid in a twistable monoidal category $M$ endowed with a right twisting structure. If $g : C \to BA$ is a classifying morphism, then there are adjunctions

$$(g_\ell)_* : C \mathsf{Comod} \rightleftarrows A \mathsf{Mod} : (g_\ell)^*$$

and

$$(g_r)_* : \mathsf{Comod}_C \rightleftarrows \mathsf{Mod}_A : (g_r)^*,$$

where

$$(g_\ell)_*(N) = A \otimes_g N \quad \text{and} \quad (g_\ell)_*(N') = N' \otimes_g A$$

for all left $C$-comodules $N$ and right $C$-comodules $N'$, and

$$(g_r)^*(M) = C \otimes_g M \quad \text{and} \quad (g_r)^*(M') = M' \otimes_g C,$$

for all left $A$-modules $M$ and right $A$-comodules $M'$.

The following proposition plays a critical role in the rest of this paper. Recall that if $A$ is a monoid in a monoidal category $M$, then an $A$-co-ring is a comonoid in the bimodule category $(\mathsf{Mod}_A, \otimes)$. Dually, if $C$ is a comonoid in $M$, then a $C$-ring is monoid in the bicomodule category $(\mathsf{Comod}_C, \Box)$.

Proposition 2.11. Let $(B, E, \tilde{E}, \delta, \mu)$ be a right twisting structure on a twistable monoidal category $M$. Let $g : C \to BA$ be a classifying morphism.

1. If $N$ is a comonoid in $(C \mathsf{Comod}_C, \Box, C)$, then $A \otimes_g N \otimes_g A$ admits a natural $A$-co-ring structure.

2. If $M$ is a monoid in $(A \mathsf{Mod}_A, \otimes, A)$, then $C \otimes_g M \otimes_g C$ admits a natural $C$-ring structure.

Proof. (1) Since $N$ is a comonoid, $N \Box (-)$ is the underlying functor of a comonad on $C \mathsf{Comod}$. Since $(A \otimes_g (-), C \otimes_g (-))$ form an adjoint pair, $A \otimes_g N \Box C \otimes_g (-) = A \otimes_g N \otimes_g (-)$ is also the underlying functor of a comonad. It follows that $A \otimes_g N \otimes_g A$ is an $A$-co-ring, with structure maps coming from the comultiplication and counit of the comonad applied to $A$.

The proof of (2) is dual to that of (1).

Considering $C$ as a bicomodule over itself and $A$ as a bimodule over itself, in the obvious way, we obtain important special cases of the constructions considered in the proposition above.

Definition 2.12. Let $g : C \to BA$ be a classifying morphism. Treating $A$ as a bimodule over itself, and $C$ as a bicomodule over itself, we obtain the standard construction on $g$

$$K(g) = A \otimes_g C \otimes_g A.$$ 

and the dual standard construction on $g$

$$T(g) = C \otimes_g A \otimes_g C.$$
Observe that $C$ is a comonoid in $\langle C\text{-Comod}_C, □_C, C \rangle$, with the unique isomorphism $C \overset{\cong}{\to} C □_C$ as comultiplication. Similarly, $A$ is a monoid in $\langle A\text{-Mod}_A, ⊗_A, A \rangle$, where the multiplication is $A ⊗_A A \overset{\cong}{\to} A$. The next proposition is therefore an immediate consequence of Proposition 2.11.

**Proposition 2.13.** For any classifying morphism $g : C \to BA$, the standard construction $K(g)$ is an $A$-co-ring, and the dual standard construction $T(g)$ is a $C$-ring; these structures are natural in $A$ and in $C$.

For future reference, we note that any twisted product with respect to a classifying morphism $g$ can be computed in terms of the standard construction $K(g)$ and the dual standard construction $T(g)$. The proof consists of straightforward calculation.

**Proposition 2.14.** Let $g : C \to BA$ be a classifying morphism.

1. If $M$ is a right $A$-module and $N$ is a left $A$-module, then
   \[ M \otimes_A K(g) \otimes_N N \cong M \otimes_A C \otimes g N. \]

2. If $U$ is a right $C$-comodule and $V$ is a left $C$-comodule, then
   \[ U □_C T(g) □_C V \cong U \otimes g T(g) \otimes g V. \]

The isomorphisms above are natural in all variables.

**Remark 2.15.** There is a strictly dual notion of left twisting structures $(Ω, P, \tilde{P}, µ, δ)$ on twistable model categories. If $\textbf{M}$ admits both left and right twisting structures, and $(Ω, B)$ is an adjoint pair of functors, then the natural transformations $P$ and $\tilde{P}$ can be deduced from $E$ and $\tilde{E}$, as the right and left $ΩC$-bundles induced by the unit $η : C \to BΩC$ of the $(Ω, B)$-adjunction, i.e.,

\[ PC := Cη □_{BΩC} EΩC ∈ \text{cMix}_{ΩC} \]

and

\[ \tilde{PC} := EΩC □_{BΩC} η C ∈ \text{ΩMix}_{C}. \]

The natural transformations $E$ and $\tilde{E}$ can be similarly deduced from $P$ and $\tilde{P}$, using the counit of the $(Ω, B)$-adjunction. Moreover, given classifying morphisms $g : C \to BA$ and $g' : C' \to BA'$, an adjunction

\[ c\text{Comod}_C \rightleftarrows A\text{Mod}_A \]

can be constructed using the transpose $g^h : ΩC \to A$ of $g$, as well as $PC$ and $P\tilde{C}$. 

11
3. Symmetric sequences of chain complexes

In this section, we recall the definition of the monoidal category of symmetric sequences with the composition product. Operads are monoids in this category; with some hypotheses, we can consider cooperads to be comonoids. The functors $E$ and $\tilde{E}$ are the two acyclic bar constructions as defined in [9, 7]; we recall the definitions, define the morphisms necessary for our twisting structures, and verify that the various identities hold.

3.1. Symmetric sequences

We work in the closed symmetric monoidal category $\text{dgM}$ of differential graded (DG) modules over an arbitrary commutative ring, $R$, furnished with the (graded) tensor product $\otimes = \otimes_R$. The graded hom functor $\text{Hom}(B, -)$ is right adjoint to $- \otimes B$ for all DG modules $B$. The linear dual of $B$ is the DG module $B^\# = \text{Hom}(B, R)$. We denote by $\Sigma_n$ the symmetric group of permutations of $[n] = \{1, \ldots, n\}$.

The suspension $sX$ of a DG module $X$ is defined by $(sX)_n \cong X_{n-1}$, $\partial(sx) = -s\partial(x)$. For the sake of the Koszul convention, we treat $s$ as a symbol of degree 1 and as a natural isomorphism of degree 1. Thus $s^n = s \circ \cdots \circ s$ ($n$ times) has degree $n$. The inverse $s^{-1}$ is called the desuspension. If $f : X \to Y$ is a linear map of degree $k$, then $sf : sX \to sY$ is the linear map of degree $k$ defined by $(sf)(x) = (-1)^k f(x)$. If $X$ and $Y$ are DG modules, then an isomorphism $s^2(X \otimes Y) \cong s X \otimes s Y$ is defined by $s^2(x \otimes y) \mapsto (-1)^{\deg x} sx \otimes sy$.

A symmetric sequence of DG modules is a sequence $X = (X_n)$, where $X_n$ is a right module over the symmetric group $\Sigma_n$, for all $n \geq 0$. The parameter $n$ is referred to as the arity. A morphism of symmetric sequences $\varphi : \mathcal{K} \to \mathcal{Y}$ is a sequence of morphisms, $(\varphi_n : X_n \to Y_n)$, where each $\varphi_n$ is $\Sigma_n$-equivariant. The category of symmetric sequences and their morphisms is denoted $\text{dgM}^{\Sigma}$. We say that $\mathcal{K}$ is connected if $\mathcal{K}(0) = 0$, and projective if each $\mathcal{K}(n)$ is $R[\Sigma_n]$-projective.

When convenient, we may index our symmetric sequences by finite sets in the usual way [6]. Let $I$ be a finite set of cardinality $n$. If $\mathcal{K}$ is a symmetric sequence, we set $\mathcal{K}(I) = \bigoplus_{n} \mathcal{K}(n) / \sim$, where the direct sum is over all bijections $\alpha : [n] \to I$, and $(x \cdot \sigma)_{\alpha} \sim x_{\alpha \sigma^{-1}}$. Note that this is the same as the colimit of the diagram with one map $X_{\beta} \to X_{\beta \sigma}$, $x_{\beta} \mapsto (x\sigma)_{\beta \sigma}$, for each $\beta : [n] \to I$ and for each $\sigma \in \Sigma_n$.

The graded tensor product of symmetric sequences $\mathcal{K}$ and $\mathcal{Y}$ is the symmetric sequence $\mathcal{K} \otimes \mathcal{Y}$ defined by

$$\mathcal{K} \otimes \mathcal{Y}(I) = \bigoplus_{I_1, I_2 = I} \mathcal{K}(I_1) \otimes \mathcal{Y}(I_2).$$

The composition product of symmetric sequences is then given by

$$\mathcal{K} \circ \mathcal{Y} = \bigoplus_{m \geq 0} \mathcal{K}(m) \otimes \mathcal{Y}^{\otimes m}.$$
The composition product is associative \[20, 28\]. It has a unit: the symmetric sequence \(J\) defined by
\[
J(n) = \begin{cases} 
R & \text{if } n = 1, \\
0 & \text{otherwise.}
\end{cases}
\]
By \[27\] for example, \((\text{dgM}^\Sigma, \circ, J)\) is a right-closed monoidal category.

### 3.2. Operads

An operad is a monoid in \((\text{dgM}^\Sigma, \circ, J)\). Thus, an operad is a symmetric sequence \(P\) equipped with an associative multiplication \(\gamma : P \circ P \to P\) that is unital with respect to the unit \(\eta : J \to P\). A morphism of operads is then a morphism of monoids. We denote by \(\text{Op}\) the category of operads and operad morphisms.

An augmented operad is an operad \(P\) along with a morphism of operads, \(\epsilon : P \to J\). The augmentation ideal of \(P\) is \(\tilde{P} = \ker \epsilon\).

**Example 3.1.** The associative operad \(A\) is defined as the symmetric sequence \(A(n) = R[\Sigma_n]\) with \(\Sigma_n\) acting on the right by multiplication. The composition product is defined by feeding permutations within blocks into a block permutation.

**Example 3.2.** The suspension operad \(S\) plays an important role in the theory of quadratic operads. Let \(S(n)\) be the free graded \(R\)-module generated by an element \(s_{n-1}\) of degree \(n - 1\), equipped with the sign representation of \(\Sigma_n\). Suppose \(\vec{m} = (m_1, \ldots, m_n) \in I_{n,m}\). There exists \(\sigma \in \Sigma_n\) such that \(\sigma\vec{m} = (m_{\sigma^{-1}(1)}, \ldots, m_{\sigma^{-1}(n)})\) is non-increasing. Denote by \(\sigma\vec{m} \in \Sigma_m\) the block permutation determined by \(\sigma\) and \(\vec{m}\). Let \(\kappa(\sigma, \vec{m})\) be the sign introduced by the Koszul rule in the map,
\[
s_{m_1-1} \otimes \cdots \otimes s_{m_n-1} \mapsto \kappa(\sigma, \vec{m})s_{m_{\sigma^{-1}(1)}-1} \otimes \cdots \otimes s_{m_{\sigma^{-1}(n)}-1}.
\]
Clearly, \(\kappa(\rho\sigma, \vec{m}) = \kappa(\rho, \sigma\vec{m})\kappa(\sigma, \vec{m})\). In particular, \(\kappa(\sigma, \vec{m}) = \kappa(\sigma^{-1}, \sigma\vec{m})\).
Define
\[
\gamma(\vec{m})(s_{n-1} \otimes s_{m_1-1} \otimes \cdots \otimes s_{m_n-1}) = \alpha(\sigma, \vec{m})s_{m-1},
\]
where
\[
\alpha(\sigma, \vec{m}) = (\text{sgn}\sigma)(\text{sgn}\sigma\vec{m})\kappa(\sigma, \vec{m}).
\]
We need to verify that \(\alpha(\sigma, \vec{m})\) is independent of \(\sigma\). If \(\rho \in \Sigma_n\) is another permutation such that \(\rho\vec{m}\) is non-increasing, then \(\rho\vec{m}\) and \(\sigma\vec{m}\) are equal as lists. Thus \(\rho = \pi\sigma\), where \(\pi\) fixes \(\sigma\vec{m}\). Since \(\text{sgn}\) is a group homomorphism, and since \((\pi\sigma)\vec{m} = \pi\sigma\vec{m}\), it follows that
\[
\alpha(\rho, \vec{m}) = \alpha(\pi, \sigma\vec{m})\alpha(\sigma, \vec{m}).
\]
Thus it suffices to show that $\alpha(\pi, \sigma \bar{m}) = 1$. Let $k_i = m_{\sigma^{-1}(i)}$, for $i = 1, \ldots, n$. Since $\sigma \bar{m}$ is non-increasing, $\pi$ is the composition of transpositions $\tau_j = (j, j+1)$, where $k_j = k_{j+1}$. Thus $\text{sgn}\tau_j = -1$ and $\text{sgn}(\tau_j)\bar{k} = (-1)^{k_j}$. Now, $\kappa(\tau_j, \bar{k})$ is the sign introduced by interchanging two copies of $s_{k_j-1}$, so $\kappa(\tau_j, \bar{k}) = (-1)^{k_j-1}$. Thus
\[\alpha(\tau_j, \bar{k}) = (-1)(-1)^{k_j}(-1)^{k_j-1} = 1.\]
It follows that $\alpha(\pi, \sigma \bar{m}) = 1$, and so (1) is independent of the choice of $\sigma$.

Furthermore, the $\gamma \bar{m}$ have the equivariance properties to define a product $\gamma : \mathcal{J} \circ \mathcal{J} \to \mathcal{J}$, making $\mathcal{J}$ into an operad. The unit is defined by the isomorphism $\eta : \mathcal{J}(1) \cong \mathcal{J}(1)$.

### 3.2.1. Algebras

The Schur functor $T : \text{dgM}^S \to \text{End}(\text{dgM})$ is defined on objects by
\[T \mathcal{X}(V) = \bigoplus_{n \geq 0} \mathcal{X}(n) \otimes V^\otimes n\]
for $\mathcal{X} \in \text{dgM}^S$. A morphism of symmetric sequences $\varphi : \mathcal{X} \to \mathcal{Y}$ induces a natural transformation of Schur functors, $T \varphi : T \mathcal{X} \Rightarrow T \mathcal{Y}$.

The functor $T$ is monoidal: $T \mathcal{X} \circ \mathcal{Y} \cong T \mathcal{X} \circ T \mathcal{Y}$. If $\mathcal{P}$ is an operad, then $T \mathcal{P}$ is the underlying functor of a monad $T \mathcal{P}$, with unit defined by the obvious inclusion on the $n = 1$ summand, and multiplication defined by the product $\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$. A $\mathcal{P}$-algebra is an algebra for the monad $T \mathcal{P}$. Thus a $\mathcal{P}$-algebra is a differential graded module $A$, equipped with structure morphisms for all $n \geq 0$,
\[\lambda_n : \mathcal{P}(n) \otimes A^\otimes n \to A\]
that are equivariant, associative and unital. We will use the notation
\[p(a_1, \ldots, a_n) := \lambda_n(p \otimes a_1 \otimes \cdots \otimes a_n)\]
for all $p \in \mathcal{P}(n)$, $a_1, \ldots, a_n \in A$.

A morphism of $\mathcal{P}$-algebras is a chain map that preserves the structure morphisms. The category of $\mathcal{P}$-algebras and morphisms is denoted $\mathcal{P}\text{-Alg}$.

Let $V \in \text{dgM}$. The free $\mathcal{P}$-algebra on $V$ is the DG module $T \mathcal{P}(V)$, with structure morphism and unit coming from those of the monad $T \mathcal{P}$.

### 3.2.2. Derivations

Let $A$ be a $\mathcal{P}$-algebra. Let $g : A \to A$ be a linear map of degree $k$. We say that $g$ is a $\mathcal{P}$-derivation if
\[g(p(a_1, \ldots, a_n)) = \sum_{i=1}^n (-1)^{\ell_i} p(a_1, \ldots, g(a_i), \ldots, a_n),\]
(where $\ell_i = \text{deg} a_1 + \cdots + \text{deg} a_{i-1}$) for all $n \geq 0$, $p \in \mathcal{P}(n)$, and $a_1, \ldots, a_n \in A$. 

14
3.2.3. Normalization

(See [1].) We may suppose that our operad $\mathcal{P}$ satisfies $\mathcal{P}(0) = 0$, by taking the augmentation ideals of our algebras. More fully, we note that $P = \mathcal{P}(0)$ is the initial $\mathcal{P}$-algebra. Indeed, in arity zero, the composition product is the sum of morphisms

$$\gamma : \mathcal{P}(n) \otimes \mathcal{P}(0) \otimes \cdots \otimes \mathcal{P}(0) \to \mathcal{P}(0)$$

that are precisely the structure maps of a $\mathcal{P}$-algebra. Let $A$ be a $\mathcal{P}$-algebra. Then the structure morphism $\mathcal{P}(0) \otimes A \to A$ provides a map, $\eta_A : P \to A$. Since the structure morphism in $A$ is associative, $\eta_A$ is a morphism of $\mathcal{P}$-algebras. Since this is the only possible $\mathcal{P}$-algebra morphism from $P$ to $A$, it follows that $P$ is initial.

Consider the category of augmented $\mathcal{P}$-algebras: the objects are $\mathcal{P}$-algebra morphisms $\varepsilon_A : A \to \mathcal{P}$; morphisms are commutative triangles

$$\begin{array}{ccc}
A & \xrightarrow{\varphi} & B \\
\downarrow{\varepsilon_A} & & \downarrow{\varepsilon_B} \\
P & & \\
\end{array}$$

We set $IA = \ker \varepsilon_A$; this is the augmentation ideal of $A$. As in the classical case, $\varepsilon_A \circ \eta_A = 1_A$, so $A \cong P \oplus IA$. Let $\hat{\mathcal{P}}(n) = \mathcal{P}(n)$ if $n > 0$, while $\hat{\mathcal{P}}(0) = 0$. Then $\hat{\mathcal{P}}$ is a sub-operad of $\mathcal{P}$, and $IA$ is a $\hat{\mathcal{P}}$-algebra.

3.2.4. Algebras as left modules

Let $z : \text{dgM} \to \text{dgM}^\Sigma$ be the functor defined by

$$z(V)(n) = \begin{cases} 
V & \text{if } n = 0 \\
0 & \text{otherwise.}
\end{cases}$$

Then $z$ restricts to define a functor $\mathcal{P}\text{-Alg} \to \mathcal{P}\text{Mod}$. Indeed, it is an easy exercise to show that $z(T_{\mathcal{P}}(A)) \cong A \circ z(A)$, so applying $z$ to the structure morphism $T_{\mathcal{P}}(A) \to A$, we obtain the structure morphism

$$\mathcal{A} \circ z(A) \cong z(T_{\mathcal{P}}(A)) \to z(A),$$

so $z(A)$ is a left $\mathcal{P}$-module. It is immediate from the definition that $z$ is full and faithful.

Since $\mathcal{P}\text{-Alg}$ embeds in $\mathcal{P}\text{Mod}$, it is often easier to state and prove results about left modules rather than algebras.

3.3. Cooperads

A cooperad is a symmetric sequence $\mathcal{D}$ along with a counit $\varepsilon : \mathcal{D}(1) \to R$ and comultiplications

$$\psi_n : \mathcal{D}(n) \to \mathcal{D}(m) \otimes \mathcal{D}(n_1) \otimes \cdots \otimes \mathcal{D}(n_m)$$

(2)
for all $\bar{n} = (n_1, \ldots, n_m) \in I_{m,n}$, that are coassociative, counital, and equivariant.

If $\mathcal{D}$ is connected, then the above morphisms yield a sequence of $\Sigma_n$-equivariant morphisms,

$$\tilde{\psi}_n : \mathcal{D}(n) \to \bigoplus_{m=1}^{n} \left( \mathcal{D}(m) \otimes \mathcal{D}[m,n] \right)^{\Sigma_m}$$

that we may compose with the natural map from fixed points to orbits, to obtain a morphism of symmetric sequences, $\psi : \mathcal{D} \to \mathcal{D} \circ \mathcal{D}$. In fact, by [4, Proposition 1.1.15], the $\psi_n$ may be recovered from $\psi$ and the two notions are equivalent.

Henceforth, we assume that our cooperads are connected, and so a cooperad is simply a comonoid with respect to the composition product.

A morphism of cooperads is a morphism of symmetric sequences, $\varphi : \mathcal{D} \to \mathcal{K}$, that commutes with the structure morphisms (2). We denote by $\textbf{CoOp}$ the category of cooperads and cooperad morphisms.

**Example 3.3.** The structure morphisms in the suspension operad $\mathcal{S}$ are isomorphisms, and so $\mathcal{S}$ is also a cooperad.

### 3.3.1. Coalgebras

Let $X \in \text{dgM}^{\Sigma}$. Let $\text{dgM}_+^{\Sigma}$ be the full subcategory of $\text{dgM}$ consisting of all chain complexes concentrated in strictly positive degrees. Let $V \in \text{dgM}_+$. We define

$$\Gamma_X(V) = \bigoplus_{n \geq 1} (X(n) \otimes V \otimes n)^{\Sigma_n}.$$

By [6, Propositions 1.1.9 and 1.1.15],

$$\Gamma : \text{dgM}^{\Sigma} \to \text{End}(\text{dgM}_+), \quad \mathcal{X} \mapsto \Gamma_{\mathcal{X}}$$

is monoidal when restricted to connected projective symmetric sequences of finite type. Thus, if $\mathcal{D}$ is a projective cooperad of finite type, then the composition diagonal in $\mathcal{D}$ makes $\Gamma_{\mathcal{D}}$ the underlying functor of a comonad, $\Gamma_{\mathcal{D}}$. A $\mathcal{D}$-coalgebra is a coalgebra over $\Gamma_{\mathcal{D}}$. Thus, a $\mathcal{D}$-coalgebra is a DG module $C$ along with structure morphisms $\rho_n : C \to \mathcal{D} \otimes C \otimes n$ for all $n$, that are appropriately counital, coassociative, and equivariant.

A $\mathcal{D}$-coalgebra $C$ is co-nilpotent if for all $c \in C$, there exists $N \geq 1$ such that $\rho_n(c) = 0$ whenever $n \geq N$.

The functor $\Gamma_{\mathcal{D}}$ is the right adjoint to the forgetful functor from co-nilpotent $\mathcal{D}$-coalgebras to $\text{dgM}_+$. That is, if $C$ is a co-nilpotent $\mathcal{D}$-coalgebra and $f : C \to V$ is a chain map, then $f$ lifts uniquely through the projection $\Gamma_{\mathcal{D}}(V) \to V$ to define a morphism of coalgebras, $\phi : A \to \Gamma_{\mathcal{D}}(V)$. To obtain $\phi$, one simply sums the composites

$$C \to \mathcal{D} \otimes C \otimes n \to \mathcal{D} \otimes V \otimes n.$$

Therefore, $\Gamma_{\mathcal{D}}(V)$ is the co-free co-nilpotent $\mathcal{D}$-coalgebra co-generated by $V$. 
3.3.2. Coderivations

Let $C$ be a $\mathcal{Q}$-coalgebra. A map of degree $k$, $g: C \to C$, is called a $\mathcal{Q}$-coderivation if the following diagram commutes for all $n \geq 0$.

\[
\begin{array}{c}
C \xrightarrow{\rho} \mathcal{Q}^\otimes n \\
g \downarrow \\
C \xrightarrow{\rho} \mathcal{Q}^\otimes n
\end{array}
\]

Let $V \in \text{dgM}$. A map $t: \Gamma(\mathcal{Q}(V)) \to V$ of degree $k$ lifts uniquely to determine a $\mathcal{Q}$-coderivation $g$ on $\Gamma(\mathcal{Q}(V))$.

3.3.3. Coalgebras as left comodules

Let $\mathcal{Q}$ be a projective cooperad of finite type. Recall the functor $z: \text{dgM} \to \text{dgM}_{\Sigma}$ of Section 3.2.4. Since $z(\Gamma(\mathcal{Q}(V))) = \mathcal{Q} \circ z(V)$, $z$ restricts to define a functor, $z: \mathcal{Q}\text{-Coalg} \to \mathcal{Q}\text{Comod}$.

3.4. A twisting structure for $\text{dgProj}^\Sigma$

We now show that a reasonable subcategory of symmetric sequences of chain complexes is twistable, and then use the operadic bar construction with coefficients to construct an explicit twisting structure.

3.4.1. Twistability of $\text{dgProj}^\Sigma$

In this section, we prove that a certain subcategory of symmetric sequences is twistable. Let $\text{Proj}$ be the full subcategory of $\mathcal{M}$ spanned by finitely generated projective $R$-modules. Then $\text{gProj}$ and $\text{dgProj}$ are the categories of graded and differential graded projective $R$-modules of finite type, respectively. Let $\text{dgProj}^\Sigma$ denote the category of symmetric sequences in $\text{dgProj}$.

Let $\mathcal{P}$ be an operad. Let $\mathcal{X}$ and $\mathcal{Y}$ be left and right $\mathcal{P}$-modules, respectively. Recall that composition over $\mathcal{P}$ is defined by a reflexive coequalizer,

\[
\begin{array}{c}
\mathcal{X} \circ \mathcal{P} \circ \mathcal{Y} \\
\mathcal{X} \circ \mathcal{Y} \\
\mathcal{X} \circ \mathcal{P} \mathcal{Y}
\end{array}
\]

where the mutual section is provided by the unit morphism $\mathcal{J} \to \mathcal{P}$. Furthermore, $- \otimes -$ commutes with reflexive coequalizers simultaneously in both factors. Since the composition product is built from colimits, we obtain the following result.

**Proposition 3.4.** [27, 2.3.12,2.3.13] Let $\mathcal{O}$, $\mathcal{P}$, and $\mathcal{Q}$ be operads.

1. [27, 2.3.12] Let $\mathcal{X}$ be an $(\mathcal{O}, \mathcal{P})$-bimodule, and let $\mathcal{Y}$ be a $(\mathcal{P}, \mathcal{Q})$-bimodule. Then $\mathcal{X} \circ \mathcal{P} \circ \mathcal{Y}$ has a natural $(\mathcal{O}, \mathcal{Q})$-bimodule structure.
Let $X$ be a right $O$-module, let $Y$ be an $(O,P)$-bimodule, and let $Z$ be a right $P$-module. Then there is a unique isomorphism

$$(X \circ_O Y) \circ_P Z \cong X \circ_O (Y \circ_P Z)$$

commuting with the natural maps $(X \circ_O Y) \circ P Z \to (X \circ_O Y) \circ P Z$ and $X \circ (Y \circ_P Z) \to (X \circ_O Y) \circ P Z$.

Dually, a coreflexive equalizer is a diagram,

$$X \xrightarrow{k} Y \xrightarrow{f} Z, \quad Z \xrightarrow{r} Y,$$

in which $rf = rg = 1_Y$ and $X \xrightarrow{k} Y$ is final among morphisms equalizing $f$ and $g$. As in the dual case, if $F: C \times C \to C$ is a bifunctor such that $F(X, -)$ and $F(-, X)$ preserve coreflexive equalizers, then $F$ preserves coreflexive equalizers simultaneously in each variable [19, Corollary 1.2.12].

**Proposition 3.5.** Let $X \in \text{dgProj}^\Sigma$ be connected (that is, $X(0) = 0$). Then $X \circ -$ and $- \circ X$ preserve coreflexive equalizers.

**Proof.** At the level of DG modules, if $M$ is projective, then $M \otimes -$ and $- \otimes M$ preserve coreflexive equalizers of DG modules, since these can be defined as kernels. Therefore $- \otimes -$ preserves coreflexive equalizers in $\text{dgProj}$. It follows that if

$$X_i \xrightarrow{k_i} Y_i \xrightarrow{f_i} Z_i, \quad Z_i \xrightarrow{r_i} Y_i,$$

is a coreflexive equalizer for $i = 1, \ldots, m$ in $\text{dgProj}$, then

$$\bigotimes_{i=1}^m X_i \xrightarrow{\bigotimes_{i=1}^m k_i} \bigotimes_{i=1}^m Y_i \xrightarrow{\bigotimes_{i=1}^m f_i} \bigotimes_{i=1}^m Z_i, \quad \bigotimes_{i=1}^m Z_i \xrightarrow{\bigotimes_{i=1}^m r_i} \bigotimes_{i=1}^m Y_i,$$

is a coreflexive equalizer.

Let $\vec{n} = (n_1, \ldots, n_m)$ be an $m$-partition of $n$ consisting of positive integers. Let $\Sigma_{\vec{n}} = \Sigma_{n_1} \times \cdots \times \Sigma_{n_m}$. In general, if $G$ is a discrete group and $H < G$ is a subgroup, then $R[G]$ is free as an $R[H]$-module, with basis $G/H$. Correspondingly, write $R[\Sigma_{\vec{n}}] = R[\Sigma_{\vec{n}}] \otimes V$, where $V$ is the free $R$-module on basis $\Sigma_{n}/\Sigma_{\vec{n}}$. Then if $\mathcal{Y} \in \text{Proj}^\Sigma$,

$$(\mathcal{Y}(n_1) \otimes \cdots \otimes \mathcal{Y}(n_m)) \otimes_{\Sigma_{\vec{n}}} R[\Sigma_{\vec{n}}] \cong \mathcal{Y}(n_1) \otimes \cdots \otimes \mathcal{Y}(n_m) \otimes V.$$

Since finite direct sums of chain complexes are naturally isomorphic to finite direct products, it follows that

$$X^\circ m(n) \xrightarrow{f^\circ m} Y^\circ m(n), \quad Y^\circ m(n) \xrightarrow{r^\circ m} Y^\circ m(n) \quad (3)$$
is a coreflexive equalizer.

Now, by [6, Lemma 1.1.16], for all \( m, n \), there exists \( \mathcal{F}_m(n) \in \text{dgProj} \) and a natural isomorphism \( \mathcal{F}^\odot_m(n) \cong R[\Sigma_m] \otimes \mathcal{F}_m(n) \), since \( \mathcal{F} \) is connected. It follows that \( h^\odot_m : \mathcal{F}^\odot_m(n) \to \mathcal{Y}^\odot_m(n) \) is of the form \( R[\Sigma_m] \otimes h_m \), where \( h_m : \mathcal{F}_m(n) \to \mathcal{Y}_m(n) \), for any morphism \( h : \mathcal{F} \to \mathcal{Y} \). Therefore, if (3) is a coreflexive equalizer, so too is

\[
\mathcal{F}_m(n) \xrightarrow{k_m} \mathcal{Y}_m(n) \xrightarrow{f_m} \mathcal{F}_m(n), \quad \mathcal{F}_m(n) \xrightarrow{r_m} \mathcal{Y}_m(n).
\]

Let \( \mathcal{W} \in \text{Proj}^\Sigma \). Then for all \( m, n \) we have a natural isomorphism \( \mathcal{W}(m) \otimes \Sigma_m \mathcal{F}^\odot_m(n) \cong \mathcal{W}(m) \otimes \mathcal{F}_m(n) \), and similarly for \( \mathcal{Y} \) and \( \mathcal{F} \). It follows that

\[
\mathcal{W} \otimes \Sigma_m \mathcal{F}^\odot_m(n) \xrightarrow{k^\odot_m} \mathcal{W} \otimes \Sigma_m \mathcal{Y}^\odot_m(n) \xrightarrow{f^\odot_m} \mathcal{W} \otimes \Sigma_m \mathcal{F}^\odot_m(n), \quad \mathcal{W} \otimes \Sigma_m \mathcal{F}^\odot_m(n) \xrightarrow{g^\odot_m} \mathcal{W} \otimes \Sigma_m \mathcal{Y}^\odot_m(n)
\]

is a coreflexive equalizer.

Finally, we need to show that

\[
\mathcal{W} \circ \mathcal{X} \xrightarrow{1o\mathcal{F}} \mathcal{W} \circ \mathcal{Y} \xrightarrow{1o\mathcal{F}} \mathcal{W} \circ \mathcal{X}, \quad \mathcal{W} \circ \mathcal{X} \xrightarrow{1o\mathcal{Y}} \mathcal{W} \circ \mathcal{Y} \quad (4)
\]

is a coreflexive equalizer. Let \( \ell : \mathcal{W} \to \bigoplus_{m \geq 1} \mathcal{W} \otimes \Sigma_m \mathcal{Y}^\odot_m \) equalize \( \sum f^\odot_m \) and \( \sum g^\odot_m \). Then \( \ell(u) = \sum \ell_m(u) \), with \( \ell_m(u) \in \mathcal{W} \otimes \Sigma_m \mathcal{Y}^\odot_m \) for all \( u \in \mathcal{W} \). It is immediate that \( \ell_m \) equalizes \( f^\odot_m \) and \( g^\odot_m \). Therefore we get morphisms \( j_m : \mathcal{W} \to \mathcal{W}(m) \otimes \Sigma_m \mathcal{F}^\odot_m \) that satisfy \( k^\odot_m j_m = \ell_m \). Since for each \( w \in \mathcal{W} \), only finitely many \( \ell_m(w) \) are nonzero, we may add the \( j_m \)'s to obtain a morphism \( j : \mathcal{W} \to \mathcal{W} \circ \mathcal{X} \) such that \( k j = \ell \). Therefore (4) is a coreflexive equalizer, and so \( \mathcal{W} \circ - \) preserves coreflexive equalizers.

The proof that \(- \circ \mathcal{Y} \) preserves coreflexive equalizers follows the same lines, but is easier.

Let \( \mathcal{D} \) be a cooperad; let \( \mathcal{X} \) be a right \( \mathcal{D} \)-comodule and let \( \mathcal{Y} \) be a left \( \mathcal{D} \)-comodule. The equalizer that defines \( \mathcal{X} \boxtimes \mathcal{Y} \) is coreflexive; the common left inverse \( \mathcal{X} \circ \mathcal{D} \circ \mathcal{Y} \to \mathcal{X} \circ \mathcal{Y} \) is provided by the counit, \( \mathcal{D} \to \mathcal{I} \). From Proposition 3.3 it follows that \( (\mathcal{X} \boxtimes \mathcal{Y}) \circ \mathcal{D} \cong \mathcal{X} \boxtimes (\mathcal{Y} \circ \mathcal{D}) \) for any \( \mathcal{X} \in \text{Proj}^\Sigma \), and likewise on the other side. We therefore obtain the following results.

**Proposition 3.6.** Let \( \mathcal{C}, \mathcal{P} \) and \( \mathcal{D} \) be cooperads in \( \text{dgProj}^\Sigma \). Let \( \mathcal{X} \) be an \((\mathcal{C}, \mathcal{P})\)-bicomodule and \( \mathcal{Y} \) a \((\mathcal{P}, \mathcal{D})\)-bicomodule, both in \( \text{dgProj}^\Sigma \). Then \( \mathcal{X} \boxtimes \mathcal{Y} \) has a natural \((\mathcal{C}, \mathcal{D})\)-bicomodule structure.
Proposition 3.7. Let $\mathcal{P}$ and $\mathcal{Q}$ be cooperads in $\text{dgProj}^\Sigma$. Let $X$ be a right $\mathcal{P}$-comodule, let $Y$ be a $(\mathcal{P}, \mathcal{Q})$-bicomodule, and let $Z$ be a left $\mathcal{Q}$-comodule, all in $\text{dgProj}^\Sigma$. Then there is a unique isomorphism

$$(X \square_{\mathcal{P}} Y) \square_{\mathcal{Q}} Z \cong X \square_{\mathcal{P}} (Y \square_{\mathcal{Q}} Z)$$

that commutes with the natural maps $(X \square_{\mathcal{P}} Y) \square_{\mathcal{Q}} Z \to (X \circ_{\mathcal{P}} Y) \circ_{\mathcal{Q}} Z$ and $X \square_{\mathcal{P}} (Y \square_{\mathcal{Q}} Z) \to X \circ_{\mathcal{P}} (Y \circ_{\mathcal{Q}} Z)$.

Our next proposition of the section shows that relative (co)composition products behave well with respect to bi(co)module structures.

Proposition 3.8. Let $\mathcal{O}$ and $\mathcal{D}$ be either operads or cooperads, or one of each, both in $\text{dgProj}^\Sigma$.

1. Let $\mathcal{P}$ be an operad. Then

$- \circ_{\mathcal{P}} -$ : $\mathcal{O} \text{Mix}_{\mathcal{P}} \times \mathcal{O} \text{Mix}_{\mathcal{D}} \to \mathcal{O} \text{Mix}_{\mathcal{D}}$.

2. Let $\mathcal{P}$ be a cooperad. Then

$- \square_{\mathcal{P}} -$ : $\mathcal{O} \text{Mix}_{\mathcal{P}} \times \mathcal{O} \text{Mix}_{\mathcal{D}} \to \mathcal{O} \text{Mix}_{\mathcal{D}}$.

Proof. The proof consists of a sequence of straightforward verifications, using natural isomorphisms

$$X \circ_{\mathcal{P}} (Y \circ_{\mathcal{P}} Z) \cong (X \circ_{\mathcal{P}} Y) \circ_{\mathcal{P}} Z$$

$$X \circ_{\mathcal{P}} (Y \circ_{\mathcal{P}} Z) \cong (X \circ_{\mathcal{P}} Y) \circ_{\mathcal{P}} Z$$

$$X \circ_{\mathcal{P}} (Y \circ_{\mathcal{P}} Z) \cong (X \circ_{\mathcal{P}} Y) \circ_{\mathcal{P}} Z$$

$$X \circ_{\mathcal{P}} (Y \circ_{\mathcal{P}} Z) \cong (X \circ_{\mathcal{P}} Y) \circ_{\mathcal{P}} Z$$

Our final proposition shows that relative cocomposition behaves well with respect to relative composition.

Proposition 3.9. Let $\mathcal{P}$ be an operad and let $\mathcal{Q}$ be a cooperad.

1. Let $X$ be a right $\mathcal{P}$-module, let $Y$ be a $(\mathcal{P}, \mathcal{Q})$-mixed module, and let $Z$ be a left $\mathcal{Q}$-comodule. Then there is a unique isomorphism

$$(X \circ_{\mathcal{P}} Y) \square_{\mathcal{Q}} Z \cong X \circ_{\mathcal{P}} (Y \square_{\mathcal{Q}} Z)$$

compatible with all relevant natural maps.
2. Let $\mathcal{X}$ be a right $\mathcal{Q}$-comodule, let $\mathcal{Y}$ be a $(\mathcal{Q}, \mathcal{P})$-mixed module, and let $\mathcal{Z}$ be a left $\mathcal{P}$-module. Then there is a unique isomorphism

$$\left(\mathcal{X} \boxdot_{\mathcal{Q}} \mathcal{Y}\right) \circ \mathcal{Z} \cong \mathcal{Y} \circ \left(\mathcal{X} \circ_{\mathcal{P}} \mathcal{Z}\right)$$

compatible with all relevant natural maps.

**Proof.** Let $\mathcal{C}$ be a category. Consider the following property.

Property (5) holds in $\mathbf{Set}$, by direct verification. The forgetful functor $\mathbf{Ab} \rightarrow \mathbf{Set}$ preserves and reflects filtered colimits and all limits. Let $\mathcal{M}$ be the category of $R$-modules. The forgetful functor $\mathcal{M} \rightarrow \mathbf{Ab}$ preserves and reflects all limits and colimits. So (5) holds in $\mathcal{M}$.

Let $\mathcal{gM}$ be the category of graded $R$-modules. Let $E_i : \mathcal{gM} \rightarrow \mathcal{M}$ be defined on objects by $E_i(X) = X_i$. Limits and colimits in $\mathcal{gM}$ are defined degree-wise, so if $F : D \rightarrow \mathcal{gM}$ is a diagram, then

$$L = (\mathrm{co})\lim F \iff E_iL = (\mathrm{co})\lim E_iF \quad \forall i.$$ 

It follows that (5) holds in $\mathcal{gM}$.

Let $W : \mathcal{dgM} \rightarrow \mathcal{gM}$ be the forgetful functor, and let $F : D \rightarrow \mathcal{dgM}$ be a diagram. Let $L = (\mathrm{co})\lim WF$. We consider the differential to be a natural transformation $\partial_i : E_iW \Rightarrow E_{i-1}W$. Thus we get a morphism $\partial_iE_iL \rightarrow E_{i-1}L$ that is a differential by naturality. It follows that $(L, \partial) = (\mathrm{co})\lim F$, and so $W$ preserves and reflects limits and colimits. Hence (5) holds in $\mathcal{dgM}$.

Let $U : \mathcal{dgM}^G \rightarrow \mathcal{dgM}$ be the forgetful functor, where $G$ is a finite group and $\mathcal{dgM}^G$ is the category of right $G$-modules. Each $g \in G$ determines a natural transformation from $U$ to $U$. Therefore, if $F : D \rightarrow \mathcal{dgM}^G$ is a diagram and $L = (\mathrm{co})\lim UF$, then we can equip $L$ with a right $G$-action compatible with $UF(D) \rightarrow L$ for all $D \in D$. Therefore $L = (\mathrm{co})\lim F$, and so (5) holds in $\mathcal{dgM}^G$.

For $i \geq 1$, let $A_i : \mathcal{dgM}^\Sigma \rightarrow \mathcal{dgM}^{\Sigma_i}$ be defined by $A_i(\mathcal{X}) = \mathcal{X}(i)$. Since limits and colimits in $\mathcal{dgM}^\Sigma$ are defined arity-wise, we have that

$$L = (\mathrm{co})\lim F \iff A_iL = (\mathrm{co})\lim A_iF \quad \forall i$$

whenever $F : D \rightarrow \mathcal{dgM}^\Sigma$ is a diagram. It follows that filtered colimits commute with finite limits in $\mathcal{dgM}^\Sigma$.

We now observe that a coequalizer (such as composition over an operad) is a filtered colimit, and an equalizer (such as cocomposition over a cooperad) is a finite limit, to complete the proof.

To summarize, we have proved the following result.

**Theorem 3.10.** The category $\mathcal{dgProj}^\Sigma$ is twistable.

21
3.4.2. Operadic bar construction

We begin by recalling the bar construction of an operad; for details, see [9]. Let \( \mathcal{P} \) be an operad. Let \( I \) and \( J \) be finite indexing sets, and let \( i \in I \). We define the \( i \)th partial composition,

\[
\circ_i : \mathcal{P}(I) \otimes \mathcal{P}(J) \to \mathcal{P}(\{ i \} \amalg J),
\]

by substituting the operation \( q \in \mathcal{P}(J) \) in the \( i \)th entry of the operation \( p \in \mathcal{P}(I) \). Likewise, we set \( I \circ_i J = (I - \{ i \}) \cup J \).

A tree is a connected directed graph \( \tau \) without loops, in which each vertex has at least one incoming edge and exactly one outgoing edge. We allow edges without a source and without a target; these are called the inputs and output of the \( \tau \), respectively. If \( \tau \) has an edge that is both an input and an output, then by connectedness, \( \tau \) is the degenerate graph \( 1 \), with no vertices and one edge. If an edge is neither an input nor an output, then it is internal. We denote the set of vertices of \( \tau \) by \( V(\tau) \) and the internal edges of \( \tau \) by \( E(\tau) \). Let \( \text{In}(\tau) \) and \( \text{Out}(\tau) \) be the set of input edges and the unique output edge of \( \tau \), respectively. For \( v \in V(\tau) \), let \( \text{In}(v) \) be the set of edges entering \( v \) and let \( \text{Out}(v) \) be the unique output edge of \( v \).

For a set \( I \), an \( I \)-labelled tree is a tree \( \tau \) equipped with a bijection \( I \xrightarrow{\cong} \text{In}(\tau) \). If \( I = \{1, \ldots, n\} \), then we refer to an \( I \)-labelled tree simply as an \( n \)-tree.

An isomorphism of trees is a bijection of vertex sets that preserves the edge relations and the labelling. The category of labelled \( n \)-trees and isomorphisms is denoted \( T(n) \); we note that \( T(n) = \amalg_{r \geq 0} T_{(r)}(n) \) where \( T_{(r)}(n) \) is the full subcategory of \( n \)-trees with \( r \) vertices. The trivial tree, \( 1 \), is the degenerate 1-tree with no vertices. Note that \( \Sigma_n \) acts on \( T(n) \) by permuting the labels of the entries, and that this action preserves each \( T_{(r)}(n) \). Thus \( T = (T(n)) \) forms a symmetric sequence in the category of small categories. Grafting of trees, that is, identifying output edges of \( \sigma_1, \ldots, \sigma_n \) with the input edges of the \( n \)-tree \( \tau \) to obtain the tree \( \tau(\sigma_1, \ldots, \sigma_n) \), defines an operad structure on \( T \).

Let \( \mathcal{X} \) be a symmetric sequence. If \( \tau \) is a tree, then we set

\[
\tau(\mathcal{X}) = \bigotimes_{v \in V(\tau)} \mathcal{X}(\text{In}(v)).
\]

We think of an element of \( \tau(\mathcal{X}) \) as being the tree \( \tau \), with each vertex \( v \) labeled with an element of \( \mathcal{X}(\text{In}(v)) \). We set

\[
\text{FO}(\mathcal{X})(n) = \text{colim}_{\tau \in T(n)} \tau(\mathcal{X}).
\]

The free operad on \( \mathcal{X} \) is the symmetric sequence \( \text{FO}(\mathcal{X}) = (\text{FO}(\mathcal{X})(n)) \). The grafting of trees determines a composition operation in \( \text{FO}(\mathcal{X}) \).

We can also define a pruning operation, dual to grafting, on the symmetric sequence \( \text{FO}(\mathcal{X}) \) to obtain the free cooperad \( \text{FC}(\mathcal{X}) \). Let \( \tau \) be a tree. A full subtree \( \sigma \) of \( \tau \) is a subtree such that \( \text{In}_{\sigma}(v) = \text{In}_{\tau}(v) \) for all \( v \in V(\sigma) \). Let \( \sigma \subseteq \tau \) be a full subtree containing \( \text{Out}(\tau) \); we allow \( \sigma = 1 \) and \( \sigma = \tau \). For each
\( e_i \in \text{In}(\sigma) \cap \text{In}(\tau) \), set \( \sigma_i = 1 \). If \( e_i \in \text{In}(\sigma) \cap E(\tau) \), then let \( \sigma_i \) be the largest subtree of \( \tau \) such that \( e_i = \text{Out}(\sigma_i) \). Then

\[
\Delta_\sigma(\tau) = \sigma \otimes \sigma_1 \otimes \cdots \otimes \sigma_n.
\]

This pruning operation, summed over all full subtrees \( \sigma \subseteq \tau \) containing \( \text{Out}(\tau) \), determines the composition diagonal in \( FC(\mathcal{X}) \). For example, consider the tree \( \tau \):

![Tree diagram]

labelled with elements of \( \mathcal{X}(2) \) at each node. The tree \( \tau \) has three nontrivial subtrees containing \( \text{Out}(\tau) \), and so the “reduced” composition diagonal of \( \tau \) is as pictured below.

![Reduced composition diagonal]

Let \( \mathcal{P} \) be a connected, augmented operad with augmentation ideal \( \hat{\mathcal{P}} \). The bar construction \( B\mathcal{P} \), forgetting differentials, is the cofree cooperad \( FC(s\hat{\mathcal{P}}) \). The differential is a perturbation of the internal differential (from the differential \( \mathcal{P} \)) by a bar differential. The bar differential of \( \tau \) has one term for each internal edge \( sp \xleftarrow{e} sq \), where \( p, q \in \hat{\mathcal{P}} \). We collapse the edge \( e \) to a vertex that we label by \( -s(p \circ e q) \).

If \( \mathcal{M} \) is a right \( \mathcal{P} \)-module and \( \mathcal{N} \) is a left \( \mathcal{P} \)-module, then the bar construction with coefficients, \( B(\mathcal{M}, \mathcal{P}, \mathcal{N}) \), is the symmetric sequence \( \mathcal{M} \circ B(\mathcal{P}) \circ \mathcal{N} \), with differential perturbed by two terms, \( d_L \) and \( d_R \), that come from the actions of \( \mathcal{P} \) on \( \mathcal{M} \) and \( \mathcal{N} \), respectively, that are defined as follows. We consider an
element of $B(M, P, N)$ to be a tree $\tau$ with a partition $V(\tau) = V_L \amalg V_P \amalg V_R$, where $V_L$ consists of the root vertex, and $V_R$ consists of all the targets of $\operatorname{In}(\tau)$. The root vertex $v$ is labeled with an element of $M(\operatorname{In}(v))$. The vertices $w \in V_P$ are labelled with elements of $N(\operatorname{In}(w))$. The vertices $u \in V_L$ are labelled with elements of $sP^{\prime}(\operatorname{In}(u))$. 

The perturbation $d_L$ has one term for each edge $e \in \operatorname{In}(v)$, where $v$ is the unique element of $V_L$. The edge $e$ is collapsed, and the root vertex is relabelled with $x \circ_1 u$, where $x \in M$ is the label of $v$ and $sp$ is the label of the source $u$ of $e$.

The perturbation $d_R$ has one term for each vertex $w$ that is the target of an edge starting in $V_R$. Let $sp$ be the label of $w$. Suppose $\operatorname{In}(w) = \{e_1, \ldots, e_n\}$, where $sp \overset{e_i}{\mapsto} y_i$ for $y_i \in N$. Then the contribution to $d_R$ for $e_i$ comes from collapsing the subtree with root $w$ to a vertex labeled with $-p(y_1, \ldots, y_n)$.

For example, the following picture shows the differential of $x \circ_1 sp \circ_1 y$, $x \in M(2)$, $sp \in sP(2)$, $y \in N(2)$.

![Diagram](image)

Note that twistability implies that if $M$ is an $(P', P)$-bimodule and $N$ is an $(P, P'')$-bimodule, then $B(M, P, N)$ is an $(P', P'')$-bimodule.

Let $P$ be a connected, augmented operad. Set

$$\tilde{E}P = B(P, P, J) \in BMix_BP$$

and

$$EP = B(J, P, P) \in BMix_BP.$$ 

We define a natural morphism of $B_P$-bicomodules

$$\delta_P : B(P) \to EP \circ \tilde{E}P.$$ 

It suffices to construct a natural map $\tilde{\delta}_P : B(P) \to B(P) \circ P \circ B(P)$ such that $\operatorname{Im}(\tilde{\delta}_P) \subseteq \ker(d_R \circ 1 + 1 \circ d_L)$, where $d_R$ is the component of the differential in $EP$ that comes from the left action of $P$ on itself, and $d_L$ is the component of the differential in $\tilde{E}P$ that comes from the right action of $P$ on itself. As a morphism of symmetric sequences, forgetting differentials, $\tilde{\delta}_P$ is the composite

$$B_P \xrightarrow{\Delta} B_P \circ B_P \xrightarrow{\cong} B_P \circ J \circ B_P \xrightarrow{\log_{1+1}} B_P \circ P \circ B_P.$$ 

24
Concretely, if $\Delta(\tau) = \sum \sigma \otimes \{\sigma_1 \otimes \cdots \otimes \sigma_m\}$, where $\tau$ is a tree with vertices labeled with elements of $sP$, then

$$\delta_P(\tau) = \sum \sigma \otimes 1^\otimes m \otimes \{\sigma_1 \otimes \cdots \otimes \sigma_m\}.$$ 

**Proposition 3.11.** $\operatorname{Im}(\delta_P) \subseteq \ker(d_R \circ 1 + 1 \circ d_L)$.

**Proof.** We show that there is a one-to-one correspondence between the terms of $(1 \circ d_L)\delta_P$ and the terms of $(d_R \circ 1)\delta_P$, with opposite signs.

The image of $\delta_P$ is spanned by elements of the form

$$\sigma \otimes \{1^\otimes n\} \otimes \{\sigma_1 \otimes \cdots \otimes \sigma_n\},$$

where $\sigma, \sigma_1, \ldots, \sigma_n \in B\mathcal{P}$.

From the definitions,

$$1 \circ d_L(\sigma \otimes 1^\otimes n \otimes \{\sigma_1 \otimes \cdots \otimes \sigma_n\})$$

$$= \sum_{i=1}^{n} \sigma \otimes \{1^\otimes (i-1) \otimes p_i \otimes 1^\otimes (n-i)\} \otimes \{\sigma_1 \otimes \cdots \otimes (\rho_1 \otimes \cdots \otimes \rho_{m_i}) \otimes \cdots \otimes \sigma_n\}$$

where $\sigma_i$ has root labeled by $p_i$ of arity $m_i$, that roots the subtrees $\rho_1, \ldots, \rho_{m_i}$.

Let $\sigma'$ be the subtree generated by $\sigma$ and the additional vertex $p_i$ from the $i$th term above. Since $\sigma$ is an $n$-tree and $p_i$ has arity $m_i$, $\sigma'$ has arity $n + m_i - 1$.

We find, in $\delta_P(\tau)$, the term

$$\Phi = \sigma' \otimes 1^\otimes (n + m_i - 1) \otimes \{\sigma_1 \otimes \cdots \otimes (\rho_1 \otimes \cdots \otimes \rho_{m_i}) \otimes \cdots \otimes \sigma_n\}.$$ 

Recall that $d_R(\sigma')$ has one component for each vertex which is the target only of ingoing edges; $p_i$ is one such vertex. The contribution to the differential is $-\sigma \otimes 1^\otimes (i-1) \otimes p_i \otimes 1^\otimes (n-i)$; thus $(d_R \circ 1)(\Phi)$ contains the term

$$-\sigma \otimes \{1^\otimes (i-1) \otimes p_i \otimes 1^\otimes (n-i)\} \otimes \{\sigma_1 \otimes \cdots \otimes (\rho_1 \otimes \cdots \otimes \rho_{m_i}) \otimes \cdots \sigma_n\}$$

that cancels with the $i$th term in $1 \circ d_L(\sigma \otimes 1^\otimes n \otimes \{\sigma_1 \otimes \cdots \otimes \sigma_n\})$.

To construct the natural morphism of $\mathcal{P}$-bimodules

$$\mu_{\mathcal{P}} : \overline{\mathcal{E}}_{B\mathcal{P}} \to \mathcal{P},$$

it suffices to construct a map $\mu_{\mathcal{P}} : \mathcal{P} \circ B\mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ in such a way that $\operatorname{Im}(d_L \circ 1 + 1 \circ d_R) \subseteq \ker \mu_{\mathcal{P}}$. We define $\mu_{\mathcal{P}}$ as the composite

$$\mathcal{P} \circ B\mathcal{P} \circ \mathcal{P} \xrightarrow{1 \circ 1 \circ 1} \mathcal{P} \circ \mathcal{P} \circ \mathcal{P} \xrightarrow{\Delta \circ 1 \circ \Delta} \mathcal{P} \circ B\mathcal{P} \circ 1 \circ \mathcal{P} \xrightarrow{\delta_P} \mathcal{P}.$$ 

**Proposition 3.12.** $\operatorname{Im}(d_L \circ 1 + 1 \circ d_R) \subseteq \ker \mu_{\mathcal{P}}$. 

25
Proof. Since the differentials reduce weight by one, and the augmentation $\epsilon : B\mathcal{P} \to \mathcal{J}$ kills everything of nonzero weight, we only need to concern ourselves with elements of $B\mathcal{P} \circ B\mathcal{P} \circ \mathcal{P}$ of the form

$$\Phi = p \otimes \{1^{\otimes (i-1)} \otimes \tau_i \otimes 1^{\otimes (n-i)} \} \otimes \{q_1 \otimes \cdots \otimes q_i \otimes \cdots \otimes q_n \}$$

where $\tau_i$ is represented by an $m$-tree with one vertex labeled by $sr \in s\tilde{P}(m)$ and $q_i = q_{i1} \otimes \cdots \otimes q_{im}$. From the definitions,

$$\hat{\mu}_L \Phi = (p \circ \iota)(q_1, \ldots, q_i, \ldots, q_n)$$

while

$$\hat{\mu}_R \Phi = -p(q_1, \ldots, r(q_i), \ldots, q_n).$$

By associativity of the composition product $\gamma$, $\hat{\mu}_L \Phi + \hat{\mu}_R \Phi = 0$ as desired.

Theorem 3.13. The category $\text{dgProj}^\Sigma$ admits a right twisting structure $(B, E, \tilde{E}, \delta, \mu)$.

Proof. Having identified $\tilde{E} \mathcal{P} \square_{B\mathcal{P}} E \mathcal{P} \cong \mathcal{P} \circ B\mathcal{P} \circ \mathcal{P}$ and $E \mathcal{P} \circ \tilde{E} \mathcal{P} \cong B\mathcal{P} \circ \mathcal{P} \circ B\mathcal{P}$ as above, the verification that the two diagrams of Definition 2.6 commute is a straightforward diagram chase using that the diagonal $\Delta$ of $B\mathcal{P}$ is counital and the composition product $\gamma$ of $\mathcal{P}$ is unital.

Viewing chain complexes as symmetric sequences of chain complexes concentrated in arity 0, we obtain the following immediate consequence of the theorem above.

Corollary 3.14. The category $\text{dgProj}$ admits a right twisting structure.

The twisting structure on $\text{dgM}^\Sigma$ induces important adjunctions on the level of (co)algebras over (co)operads. Recall the definition of twisted products with respect to classifying morphisms (Definition 2.8).

Definition 3.15. Let $\mathcal{Q}$ be a cooperad, and let $\mathcal{P}$ be an operad, both in $\text{dgProj}^\Sigma$. Let $g : \mathcal{Q} \to B\mathcal{P}$ be a classifying morphism. The $g$-cobar construction

$$\Omega_g : \mathcal{Q} \text{-Comod} \to \mathcal{P} \text{-Alg}$$

and the $g$-bar construction

$$B_g : \mathcal{P} \text{-Alg} \to \mathcal{Q} \text{-Comod}$$

are given by $\Omega_g : M = \mathcal{P} \circ g \mathcal{M}$ and $B_g : N = \mathcal{Q} \circ g \mathcal{N}$.

Remark 3.16. Since $z : \text{dgProj} \to \text{dgProj}^\Sigma$ is fully faithful when restricted to both $\mathcal{P}\text{-Alg}$ and $\mathcal{Q}\text{-Coalg}$, the functors of Definition 3.15 restrict to define

$$\Omega_g : \mathcal{Q} \text{-Coalg} \to \mathcal{P} \text{-Alg} \quad \text{and} \quad B_g : \mathcal{P} \text{-Alg} \to \mathcal{Q} \text{-Coalg}.$$
The next result follows immediately from Corollary 2.10, but is important enough to be formulated as a separate statement.

**Proposition 3.17.** Let \( Q \) be a cooperad, and let \( P \) be an operad, both in \( dgProj^{\Sigma} \). For any classifying morphism \( g : Q \to B_P \), the \( g \)-cobar construction \( \Omega_g \) is left adjoint to the \( g \)-bar construction \( B_g \).

We prove in the next section that when \( g \) is the canonical classifying morphism of a quadratic operad, then \((\Omega_g, B_g)\) is the usual cobar/bar adjunction.

4. Categories with morphisms up to strong homotopy

In this section, we consider classifying morphisms \( g : Q \to B_P \) with the property that the counit of the associated standard construction, \( \epsilon : K(g) \to P \), is a quasi-isomorphism. This is the case with quadratic Koszul operads and their resolutions; we show that the corresponding Kleisli categories are isomorphic to the classic “strong homotopy” categories. In general, the following argument, translated from Markl [23] where it is presented in the language of coloured operads, shows that the morphism sets of the Kleisli category are homotopy-invariant, and therefore model categories of algebras and morphisms up to strong homotopy.

**Notation 4.1.** If \( Q \) is a coaugmented cooperad in \( dgProj^{\Sigma} \) and \( g : Q \to B_P \) is a classifying morphism, then the coaugmentation in \( Q \), along with the unit in \( P \), define a coaugmentation \( \eta_g : P \to K(g) \). Let \( F : K(g) \circ P \mathcal{M} \to \mathcal{N} \) be a morphism of left \( P \)-modules. The underlying morphism \( F_0 : \mathcal{M} \to \mathcal{N} \) associated to \( F \) is the composite,

\[
\mathcal{M} \cong P \circ_{\mathcal{M}} \eta_{g}\circ P \to K(g)\circ_{\mathcal{M}} F_0 \to \mathcal{N}.
\]

**Proposition 4.2.** [23, Proposition 35] Let \( Q \) be a coaugmented cooperad and \( P \) an augmented operad, both in \( dgProj^{\Sigma} \). Let \( g : Q \to B_P \) be a classifying morphism such that \( \epsilon : K(g) \to P \) is a surjective quasi-isomorphism. Let \( F : K(g) \circ P \mathcal{M} \to \mathcal{N} \) be a morphism of left \( P \)-modules.

If \( f : \mathcal{M} \to \mathcal{N} \) is a morphism of symmetric sequences homotopic to \( F_0 \), then \( f \) is the underlying morphism of a morphism of left \( P \)-modules, \( f : K(g) \circ P \mathcal{M} \to \mathcal{N} \).

**Proof.** Let \( \mathcal{I} \) be the symmetric sequence concentrated in arity 1, with \( \mathcal{I}(1) = R\{e_0, e_1, s\} \) and \( \partial s = e_1 - e_0 \). Then \( F_0 \) and \( f \) are chain homotopic if and only if there exists a morphism of symmetric sequences \( \Phi : \mathcal{I} \circ \mathcal{I} \to \mathcal{I} \) such that \( \Phi(e_0 \otimes -) = f \) and \( \Phi(e_1 \otimes -) = F_0 \).

The symmetric sequence \( \mathcal{I} \) is a coaugmented cooperad. Indeed, the diagonal is defined by \( \psi(e_i) = e_i \otimes e_i \) for \( i = 0, 1 \), and \( \psi(s) = s \otimes e_1 + e_0 \otimes s \). The counit is defined by \( \epsilon(e_i) = 1 \) for \( i = 0, 1 \) and \( \epsilon(s) = 0 \). The coaugmentation is defined by \( \eta(1) = e_0 \).
Since $\mathcal{F}$ is concentrated in arity 1, $\mathcal{F} \circ \mathcal{D}$ is also a coaugmented cooperad. The classifying morphism $g$ extends to a classifying morphism $g' : \mathcal{F} \circ \mathcal{D} \to B\mathcal{P}$, via $g'(e_i \otimes -) = g$ for $i = 0, 1$, and $g'(s \otimes -) = 0$. Define the coaugmentation by $\eta'(1) = e_0 \otimes \eta(1)$ If $\epsilon : K(g) \to \mathcal{P}$ is a surjective quasi-isomorphism, then so too is $\epsilon' : K(g') \to \mathcal{P}$.

Let $\text{Hom}_\Sigma(\mathcal{M}, \mathcal{N})$ be the internal morphism symmetric sequence in $\text{dgM}$; that is, $\text{Hom}_\Sigma(\mathcal{M}, \mathcal{N})(n) = \prod_m \text{Hom}_{R[\Sigma^m]}(\mathcal{M}^{\otimes n}(m), \mathcal{N}(m))$. It suffices to construct a $\mathcal{P}$-bimodule morphism $\bar{F}^\sharp : K(g) \to \text{Hom}_\Sigma(\mathcal{M}, \mathcal{N})$ such that $\bar{F}^\sharp \eta = f^\sharp$, where $f^\sharp : \mathcal{F} \to \text{Hom}_\Sigma(\mathcal{M}, \mathcal{N})$ is the right adjoint of $f$. We will then have that $\bar{F}$, the left adjoint of $\bar{F}^\sharp$, has $f$ as its underlying morphism.

Define $H : \mathcal{F} \circ \mathcal{D} \to \text{Hom}_\Sigma(\mathcal{M}, \mathcal{N})$ by $H(e_0 \otimes \eta(r)) = f^\sharp(r)$ for $r \in \mathcal{F}$, $H(e_1 \otimes q) = F^\sharp(q)$ for all $q \in \mathcal{D}$, and $H(s \otimes q) = 0$ for all $q \in \mathcal{D}$. Then $H$ extends uniquely to a $\mathcal{P}$-bimodule morphism $K(g') \to \text{Hom}_\Sigma(\mathcal{M}, \mathcal{N})$, also called $H$.

We note that $K(g)$ is cellular, in the sense that it is almost free as a $\mathcal{P}$-bimodule, and has an increasing filtration such that the differential strictly reduces filtration degree. Furthermore, since we are assuming that $\mathcal{P}$ and $\mathcal{D}$ are projective as symmetric sequences, so too is $K(g)$. Therefore the solid square

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\eta'} & K(g') \\
\downarrow{\eta} & & \downarrow{\epsilon'} \\
K(g) & \xrightarrow{\epsilon} & \mathcal{P}
\end{array}
$$

has the lifting $\sigma$ as indicated by the dotted arrow. We set $\bar{F}^\sharp = H \sigma$ to complete the proof.

4.1. Strongly homotopy morphisms of $\mathcal{P}$-algebras

We now use the right twisting structure for symmetric sequences of chain complexes to provide two operadic descriptions of strongly homotopy morphisms of algebras and of coalgebras over a quadratic operad. We begin by recalling the basic theory of weight-graded operads.

4.1.1. Weight-graded operads

Weight-graded operads, first considered by Fresse in [1], allow the construction of quadratic dual cooperads and Koszul resolutions for non-quadratic operads. A weight grading on a chain complex $X$ is a grading $X \cong \bigoplus_{s \geq 0} X(s)$, where the differential preserves each $X(s)$. If $X$ and $Y$ are weight-graded, then $X \otimes Y$ is weight-graded, with $(X \otimes Y)(n) = \bigoplus_{s+t=n} X(s) \otimes Y(t)$, so weight-graded chain complexes form a symmetric monoidal category in which we can define symmetric sequences and operads.

If $\mathcal{P}$ is any connected operad, then we can give it the canonical weight grading, defined by

$$
\mathcal{P}(r)(n) = \begin{cases} 
\mathcal{P}(n) & r = n - 1 \\
0 & \text{otherwise}
\end{cases}
$$
If $M$ is a weight-graded symmetric sequence, then its grading induces weight gradings on the free operad and cooperad on $M$. Essentially the same argument that shows that $T_d(V)(d) = T_d(V_1)$ and $T_d^>(V)(d) = 0$ for the tensor algebra on a connected chain complex $V$, establishes that $F_c(r)(M) = F_c(r)(M(1))$ and $F_c^>(r)(M)(r) = 0$. (Recall that $F_c(r)(M)$ is the sub symmetric sequence generated by trees with $r$ vertices.)

The bar construction $BP$ of a weight-graded operad is therefore naturally weight-graded. Set

$$P_{(s)} = H_s(B_s(P)(s)),$$

where the homological grading is with respect to the bar wordlength, that is, the number of vertices in a representative tree. Since $F_{(r)}^>(sP)(s) = 0$, we have that

$$P_{(s)} = \ker(d : B_s(P)(s) \to B_{s-1}(P)(s)),$$

and so $P$ is a sub weight-graded symmetric sequence of $BP$. Fresse proves that in fact, $P$ is a sub weight-graded (in fact, quadratic) cooperad of $BP$. The inclusion

$$\kappa_P : P \to BP$$

is the canonical classifying morphism of $P$.

**Remark 4.3.** We take this opportunity to observe that $P(2) \cong sP(2)$. If $P$ is quadratic and $P^!$ is its quadratic dual, then $P \cong P \otimes P^!$.

If $M$ is a left $P$-module, then the $P$-bar construction of $M$ is the cofree $P$-module, $B_P(M) = P \otimes M$, with internal differential from $P$ and $M$, perturbed by the unique coderivation determined by the composite

$$P(2) \otimes M(k) \otimes M(\ell) \xrightarrow{x \otimes 1 \otimes 1} P(2) \otimes M(k) \otimes M(\ell) \to M(k + \ell).$$

Observe that this makes sense for a general weight-graded operad, since $P$ is always quadratic. For a full treatment of bar and cobar constructions, see for example Fox and Markl, [5], or Getzler and Jones [8].

If $Q$ is a weight-graded cooperad, then the operadic cobar construction on $Q$ [ref], $\Omega Q$, is a weight-graded operad. We define

$$Q^!(2) = \coker(d : \Omega^{s-1}Q(s) \to \Omega^sQ(s)).$$

Fresse [7] shows that $Q^!$ is a weight-graded (in fact, quadratic) quotient operad of $\Omega Q$.

Let $M$ be a left $Q$-comodule. The $Q$-cobar construction of $M$ is the free $Q$-module, $\Omega_Q(M) = Q^! \circ M$, whose internal differential is perturbed by the unique derivation determined by

$$C \to Q(2) \otimes C \otimes C \to Q^!(2) \otimes C \otimes C.$$

Again, $Q^!$ is quadratic because it is a quotient of a quadratic operad, so the above equation makes sense.
Recall [7, 5.2.8], [8, Definition 2.23] that a weight-graded operad is called 
Koszul if the canonical classifying morphism $\kappa_\mathcal{P} : \mathcal{P}^\perp \to B\mathcal{P}$ is a quasi-
isomorphism. Recall moreover the bar and cobar constructions of Definition 3.15.

**Proposition 4.4.** Let $\mathcal{P}$ be a weight-graded Koszul operad in $\mathrm{dgProj}^{\Sigma}$ with canonical classifying morphism $\kappa_\mathcal{P} : \mathcal{P}^\perp \to B\mathcal{P}$. Then

$$\Omega_\mathcal{P}^\perp = \Omega_{\kappa_\mathcal{P}} \quad \text{and} \quad B\mathcal{P} = B_{\kappa_\mathcal{P}}.$$ 

**Proof.** For brevity, we set $Q = \mathcal{P}^\perp$. The isomorphism $Q \cong B\mathcal{P}$ is the corestriction of the composite

$$Q \cong Q \circ Q \circ \kappa_{\mathcal{P}} \circ \kappa_{\mathcal{P}}.$$ 

Let $\mathcal{M}$ be a left $\mathcal{P}$-module. We observe that

$$E\mathcal{P} \circ \mathcal{M} = B(\mathcal{J}; \mathcal{P}) \circ \mathcal{M} \cong B(\mathcal{J}; \mathcal{P})$$

by [8, 4.1.2]. Thus it suffices to show that the map

$$Q \circ \mathcal{M} \to Q \circ B(\mathcal{J}; \mathcal{P})$$

commutes with differentials. The map obviously commutes with the internal differential in $Q$, if any, and with the differential in $B\mathcal{P}$. Since $\Im \kappa_\mathcal{P} \subset F^r(s\mathcal{P}(2))$, and $Q$ is quadratic, it suffices to show that

$$Q(2) \otimes \mathcal{M}(n_1) \otimes \mathcal{M}(n_2) \xrightarrow{\kappa \otimes 1 \otimes 1} B\mathcal{P}(2) \otimes \mathcal{M}(n_1) \otimes \mathcal{M}(n_2)$$

$$d$$

$$Q(1) \otimes \mathcal{M}(n_1 + n_2) \xrightarrow{\kappa \otimes 1} B\mathcal{P}(1) \otimes \mathcal{M}(n_1 + n_2)$$

commutes. From the definitions, $Q(2) = B\mathcal{P}(2) = s\mathcal{P}(2)$, and both differentials come from desuspending then applying the action of $\mathcal{P}$ on $\mathcal{M}$, so the diagram commutes.

The proof for cobar constructions is similar, and exploits the fact that $(\mathcal{P}^\perp)^\perp = \mathcal{P}$ for Koszul operads $\mathcal{P}$.

Propositions 4.4 and 3.17 yield another proof of the following well known result.

**Proposition 4.5.** [8, Theorem 2.25] Let $\mathcal{P}$ be a Koszul operad in $\mathrm{dgProj}^{\Sigma}$. Then the $\mathcal{P}^\perp$-cobar and $\mathcal{P}$-bar constructions are adjoint functors,

$$\Omega_{\mathcal{P}^\perp} : \mathcal{P}^\perp\text{-Coalg} \rightleftarrows \mathcal{P}\text{-Alg} : B_{\mathcal{P}}.$$
4.1.2. The Kleisli category description of strong homotopy

In this section, we observe that the standard construction, \( K(P) \), for the canonical classifying morphism, \( \kappa_P : P^\perp \to B_P \) is the two-sided Koszul resolution of \( P \), for a weight-graded operad \( P \). We then show that the Kleisli category associated to the comonad determined by \( K(P) \) is isomorphic to the category of \( P \)-algebras and strongly homotopy \( P \)-algebra morphisms.

**Definition 4.6.** Let \( P \) be a weight-graded Koszul operad. The objects of the category \( P \text{-Alg}_{sh} \) are all \( P \)-algebras, while the morphisms are strongly homotopy morphisms of \( P \)-algebras, i.e.,

\[
P \text{-Alg}_{sh}(A, E) = P^\perp \text{Coalg}(B_PA, B_PE).
\]

Dually, the objects of the category \( P^\perp \text{-Coalg}_{sh} \) are all \( P^\perp \)-coalgebras, while the morphisms are strongly homotopy morphisms of \( P^\perp \)-coalgebras, i.e.,

\[
P^\perp \text{-Coalg}_{sh}(A, E) = P \text{-Alg}(\Omega P^\perp A, \Omega P^\perp E).
\]

Recall the canonical classifying morphism \( \kappa_P : P^\perp \to B_P \) constructed in Section 4.1.1 and the standard construction \( K(g) \) for any classifying morphism \( g \), from Definition 2.12. It is easy to see from the definitions that the underlying \( P \)-bimodule of the standard construction \( K(\kappa_P) \) is the two-sided Koszul resolution of \( P \); see Fresse [7]. For this reason, we write \( K(P) \) instead of \( K(\kappa_P) \) and are motivated to make the following definition.

**Definition 4.7.** The **two-sided dual Koszul resolution** of \( P \) is the dual standard construction applied to the classifying morphism \( \kappa_P \):

\[
T(P) = P^\perp \circ_{\kappa_P} P \circ_{\kappa_P} P^\perp,
\]

which is a \( P^\perp \)-bicomodule.

**Proposition 4.8.** If \( P \) is a Koszul operad in \( \text{dgProj}^\Sigma \), then \( K(P) \) is a \( P \)-co-ring and \( T(P) \) is a \( P^\perp \)-ring.

**Proof.** The proposition is an instance of Proposition 2.13 for the classifying morphism \( \kappa_P : P^\perp \to B_P \).

**Notation 4.9.** Let

\[
K_P = K(P) : P \text{-Mod} \to P \text{-Mod}.
\]

Observe that \( K_P \) is (the underlying functor of) a comonad, since \( K(P) \) is a co-ring. Moreover there is an induced comonad on \( P \text{-Alg} \), denoted \( K_P \), with underlying endofunctor \( K(P) \circ z(-) \).

Similarly, there is a monad on \( P^\perp \text{-Comod} \) with underlying endofunctor

\[
T_P = T(P) \square_{\perp} : P^\perp \text{-Comod} \to P^\perp \text{-Comod},
\]

which in turn restricts and corestricts to a monad on \( P^\perp \text{-Coalg} \), denoted \( T_P \), with underlying endofunctor \( T(P) \square_{\perp} z(-) \).
Recall the definition of the (co)Kleisli category determined by a (co)monad (Notation 1.3).

**Theorem 4.10.** If $\mathcal{P}$ is a Koszul operad in $\text{dgProj}^\Sigma$, then there are isomorphisms of categories

$$\Theta_{\mathcal{P}} : \mathcal{P}\text{-Alg}_{sh} \xrightarrow{\cong} \mathcal{K}_\mathcal{P} \mathcal{P}\text{-Alg}$$

and

$$\Upsilon_{\mathcal{P}} : \mathcal{P}\mathcal{P}^{-}\text{-Coalg}_{sh} \xrightarrow{\cong} \mathcal{P}\mathcal{P}^{-}\text{-Coalg}_{\mathcal{T}_\mathcal{P}},$$

which are the identity on objects.

**Proof.** The definition of $\Theta_{\mathcal{P}}$ on morphisms follows from the sequence of natural isomorphisms below, where $A$ and $B$ are $\mathcal{P}$-algebras.

$$\mathcal{P}\text{Mod}(K(\mathcal{P}) \circ z(A), z(E))$$

$$\cong \mathcal{P}\text{Mod}(\mathcal{P} \circ_{\mathcal{P}} \mathcal{P}\mathcal{P}^{-} \circ_{\mathcal{P}} z(A), z(E))$$

$$\cong \mathcal{P}\mathcal{P}^{-}\text{Comod}(\mathcal{P}\mathcal{P}^{-} \circ_{\mathcal{P}} z(A), \mathcal{P}\mathcal{P}^{-} \circ_{\mathcal{P}} z(E))$$

$$\cong \mathcal{P}\mathcal{P}^{-}\text{Comod}(z(B \mathcal{P} A), z(B \mathcal{P} E))$$

$$\cong \mathcal{P}\mathcal{P}^{-}\text{Coalg}(B \mathcal{P} A, B \mathcal{P} E).$$

Similarly, the definition of the functor $\Upsilon_{\mathcal{P}}$ on morphisms follows from the sequence of isomorphisms below, where $C$ and $D$ are $\mathcal{P}\mathcal{P}^{-}$-coalgebras.

$$\mathcal{P}\mathcal{P}^{-}\text{Comod}(z(C), T(\mathcal{P}) \square_{\mathcal{P}\mathcal{P}^{-}} z(D))$$

$$\cong \mathcal{P}\mathcal{P}^{-}\text{Comod}(z(C), \mathcal{P}\mathcal{P}^{-} \circ_{\mathcal{P}} \mathcal{P} \circ_{\mathcal{P}} z(D))$$

$$\cong \mathcal{P}\text{Mod}(\mathcal{P} \circ_{\mathcal{P}} z(C), \mathcal{P} \circ_{\mathcal{P}} z(D))$$

$$\cong \mathcal{P}\text{Alg}(z(\Omega \mathcal{P}\mathcal{P}^{-} C), z(\Omega \mathcal{P}\mathcal{P}^{-} D))$$

$$\cong \mathcal{P}\text{-Alg}(\Omega \mathcal{P}\mathcal{P}^{-} C, \Omega \mathcal{P}\mathcal{P}^{-} D).$$

Note that in both of the sequences of isomorphisms above, Corollary 2.10 and Proposition 4.4 play a key role.

### 4.2. Strongly homotopy morphisms of $\mathcal{P}_\infty$-algebras, the Koszul case

Let $\mathcal{P}$ be an operad. A $\mathcal{P}_\infty$-algebra is defined to be an algebra over a cofibrant replacement $\mathcal{P}'$ of $\mathcal{P}$. If $\mathcal{P}$ is Koszul, then we may choose our cofibrant replacement to be $\Omega: \mathcal{P}_{\mathcal{P}^{-}}$, the operadic cobar construction on the quadratic dual cooperad $\mathcal{P}_{\mathcal{P}^{-}}$ [3, 4.2.14]. The unit of the operadic bar-cobar adjunction defines a classifying morphism,

$$\eta_{\mathcal{P}^{-}} : \mathcal{P}_{\mathcal{P}^{-}} \rightarrow \mathbf{B}\Omega \mathcal{P}_{\mathcal{P}^{-}}.$$  

Observe that the associated standard construction, $K(\eta_{\mathcal{P}^{-}})$, is a $\Omega: \mathcal{P}_{\mathcal{P}^{-}}$-co-ring, by Proposition 2.13.
Let $A$ be a $\Omega \mathcal{P}^\perp$-algebra. By analogy with the homotopy-associative case \cite{29}, we define the \textit{bar-tilde construction} $\tilde{B}_P(A)$ to be the cofree $\mathcal{P}^\perp$-coalgebra $\Gamma_{\mathcal{P}^\perp}(A)$, with differential perturbed by the composition
\[
\mathcal{P}^\perp(n) \otimes A^\otimes n \to s^{-1} \mathcal{P}^\perp(n) \otimes A^\otimes n \to \Omega(\mathcal{P}^\perp)(n) \otimes A^\otimes n \to A.
\]
Here we are using the fact that the operad $\Omega \mathcal{P}^\perp$ is generated by the symmetric sequence $s^{-1} \mathcal{P}^\perp$.

**Proposition 4.11.** Let $P$ be a Koszul operad in $\text{dgProj}^\Sigma$. If $A$ is a $\Omega \mathcal{P}^\perp$-algebra, then $B_{\eta_{\mathcal{P}^\perp}}(A) = \tilde{B}_P(A)$.

**Proof.** Definitions 2.8 and 3.15 imply that $B_{\eta_{\mathcal{P}^\perp}}(A) = \mathcal{P}^\perp \circ \eta_{\mathcal{P}^\perp} \circ \mathcal{E}_{\mathcal{P}^\perp} \circ \Omega_{\mathcal{P}^\perp} (\mathcal{P}^\perp)(A) \cong \tilde{B}_P(A)$.

Let $\mathcal{P}_{\text{Alg}_\infty}$ denote the category of $\mathcal{P}_\infty$-algebras and morphisms up to strong homotopy, i.e., the full subcategory of $\mathcal{P}^\perp$-$\text{Coalg}$ spanned by the almost-cofree coalgebras. (A DG coalgebra is \textit{almost cofree} if the underlying coalgebra of graded modules is cofree.)

**Theorem 4.12.** Let $\mathcal{P}$ be a Koszul operad in $\text{dgProj}^\Sigma$. The $\eta_{\mathcal{P}^\perp}$-bar construction induces an isomorphism of categories,
\[
B_{\eta_{\mathcal{P}^\perp}} : \kappa_{(\eta_{\mathcal{P}^\perp})}(\Omega \mathcal{P}^\perp\cdot \text{-Alg}) \to \mathcal{P}_{\text{-Alg}_\infty},
\]
where $\kappa_{(\eta_{\mathcal{P}^\perp})}$ denotes the comonad on $\Omega \mathcal{P}^\perp\cdot \text{-Alg}$ with underlying endofunctor $\kappa_{(\eta_{\mathcal{P}^\perp})} \circ \mathcal{P}^\perp \circ \eta_{\mathcal{P}^\perp} (\cdot)$.

**Proof.** If $A$ and $E$ are $\Omega \mathcal{P}^\perp$-algebras, then
\[
\kappa_{(\eta_{\mathcal{P}^\perp})}(\Omega \mathcal{P}^\perp\cdot \text{-Alg})(A,E) = \Omega \mathcal{P}^\perp \text{Mod}(K(\eta_{\mathcal{P}^\perp}) \circ \eta_{\mathcal{P}^\perp} \circ \mathcal{E}_{\mathcal{P}^\perp} \circ \Omega_{\mathcal{P}^\perp} (\mathcal{P}^\perp)(z(A),z(E)))
\]
\[
= \Omega \mathcal{P}^\perp \text{Mod}(\Omega \mathcal{P}^\perp \circ \eta_{\mathcal{P}^\perp} \circ \mathcal{P}^\perp \circ \eta_{\mathcal{P}^\perp} \circ \mathcal{E}_{\mathcal{P}^\perp} \circ \Omega_{\mathcal{P}^\perp} (\mathcal{P}^\perp)(z(A),z(E))
\]
\[
\cong \mathcal{P}^\perp \text{Comod}(\mathcal{P}^\perp \circ \eta_{\mathcal{P}^\perp} \circ \mathcal{P}^\perp \circ \eta_{\mathcal{P}^\perp} \circ \mathcal{E}_{\mathcal{P}^\perp} \circ \Omega_{\mathcal{P}^\perp} (\mathcal{P}^\perp)(z(A),z(E)) = \mathcal{P}^\perp_{\text{-Coalg}}(B_{\eta_{\mathcal{P}^\perp}}(A),B_{\eta_{\mathcal{P}^\perp}}(E)),
\]
where the isomorphism $(\ast)$ is an instance of Corollary 2.10. It follows that $B_{\eta_{\mathcal{P}^\perp}}$ is full and faithful.

To see that $B_{\eta_{\mathcal{P}^\perp}}$ is essentially surjective on almost-cofree coalgebras, observe first that if $(C,d)$ is almost-cofree, then there is a graded module $X$ such that $C \cong \Gamma_{\mathcal{P}^\perp}(X)$. Moreover, the differential $d$ is determined by a map of degree $-1$, $d : \Gamma_{\mathcal{P}^\perp}(X) \to X$. The composite
\[
\Gamma_{s^{-1}\mathcal{P}^\perp}(X) \cong s^{-1}\Gamma_{\mathcal{P}^\perp}(X) \xrightarrow{dx} X
\]
extends in the obvious way to an action of $\Omega \mathcal{P}^\perp$ on $X$, endowing $X$ with the structure of a $\Omega \mathcal{P}^\perp$-algebra.
4.3. The nonKoszul case

Let \( \mathcal{P} \) be any operad. The standard construction on the identity map \( \text{id} : B\mathcal{P} \to B\mathcal{P} \) is simply the two-sided bar construction,

\[
K(\text{id}) = B(\mathcal{P}, \mathcal{P}, \mathcal{P}).
\]

Moreover, the bar construction on a left \( \mathcal{P} \)-module \( \mathcal{M} \) is the bar construction with coefficients,

\[
B_{\text{id-}\mathcal{M}} = B(\mathcal{P}, \mathcal{P}, \mathcal{M}).
\]

The category of \( \mathcal{P} \)-algebras and strongly homotopy \( \mathcal{P} \)-morphisms is thus the full subcategory of \( \mathcal{B}_\mathcal{P}\text{Comod} \) spanned by objects of the form \( B(\mathcal{P}, \mathcal{P}, Z(A)) \), which is isomorphic to the Kleisli category for the comonad with underlying functor \( B(\mathcal{P}, \mathcal{P}, -) \). Note that this description reduces to the classical case, of left \( A \)-modules and strongly homotopy \( A \)-linear maps.

To work with \( \mathcal{P}_\infty \)-algebras, we consider algebras over the resolution \( \Omega B\mathcal{P} \) of \( \mathcal{P} \). We take as our classifying morphism the unit of the adjunction, \( \eta : B\mathcal{P} \to B\Omega B\mathcal{P} \). In this case, the standard construction is

\[
K(\eta) = B(\Omega B\mathcal{P}; \mathcal{P}; \Omega B\mathcal{P}).
\]

In both cases, the augmentation is a surjective quasi-isomorphism, and so the resulting Kleisli categories have homotopy-invariant morphism sets, by Proposition 4.2.

5. Strong homotopy with parameters

In this section we study a parametrized version of strong homotopy for morphisms of (co)associative (co)algebras. We begin by explaining how to parametrize strong homotopy, in terms of the usual cobar/bar adjunction. We then give a Kleisli-type operadic description of parametrized strong homotopy, which we apply to prove a useful existence theorem.

5.1. Introducing parameters

Let \( F : C \rightleftarrows D : U \) be a pair of adjoint functors. Let \( \text{Monad}_C \) and \( \text{Comonad}_D \) denote the categories of monads on \( C \) and of comonads on \( D \), respectively. It is well known and easy to show that \( UF : C \to C \) is a monad and \( FU : D \to D \) is a comonad. More generally, there are functors

\[
\text{Monad}_D \to \text{Monad}_D : T \mapsto UTF
\]

and

\[
\text{Comonad}_C \to \text{Comonad}_C : K \mapsto FKU.
\]

We view these functors as providing parametrized families of monads and comonads arising from the \((F,U)\)-adjunction.
We now apply this point of view to the usual cobar/bar adjunction for the associative operad \( \mathcal{A} \) (Example 3.1), which is a slight variation on the \( (\Omega_{\mathcal{A}^\perp}, B_{\mathcal{A}}) \)-adjunction of Propositions 4.4. Recall from Example 3.2 the symmetric sequence \( \mathcal{S} \), which gives rise to the operadic suspension functor
\[
\mathcal{S} = \mathcal{S} \otimes - : \text{dgM}^\Sigma \to \text{dgM}^\Sigma,
\]
which is clearly invertible. It is easy to check the following useful properties of operadic suspension.

**Lemma 5.1.** Operadic suspension is a strongly monoidal functor, i.e., for all symmetric sequences \( \mathcal{X} \) and \( \mathcal{Y} \),
\[
\mathcal{S}(\mathcal{X} \circ \mathcal{Y}) \cong \mathcal{S}\mathcal{X} \circ \mathcal{S}\mathcal{Y}.
\]
In particular, operadic suspension induces endofunctors
\[
\mathcal{S} : \text{Op}_{\text{dgM}} \to \text{Op}_{\text{dgM}} \quad \text{and} \quad \mathcal{S} : \text{CoOp}_{\text{dgM}} \to \text{Comon}_{\text{dgM}}.
\]
Moreover, if \( \mathcal{P} \) is an operad and \( \mathcal{Q} \) is a cooperad, then operadic suspension induces a functor
\[
\mathcal{S} : \text{Mod}_{\mathcal{P}} \to \text{Mod}_{\mathcal{S}\mathcal{P}} \quad \text{and} \quad \mathcal{S} : \text{Comod}_{\mathcal{Q}} \to \text{Comod}_{\mathcal{S}\mathcal{Q}}.
\]
Recall that \( \mathcal{A}^\perp = \mathcal{S}\mathcal{A}^\sharp \).

**Definition 5.2.** The classical cobar construction \( \Omega : \mathcal{A}^\sharp\text{-Coalg} \to \mathcal{A}\text{-Alg} \) is the composite functor
\[
\mathcal{A}^\sharp\text{-Coalg} \xrightarrow{T} \mathcal{A}^\perp\text{-Coalg} \xrightarrow{\Omega_{\mathcal{A}^\perp}} \mathcal{A}\text{-Alg},
\]
and the classical bar construction \( B : \mathcal{A}\text{-Alg} \to \mathcal{A}^\sharp\text{-Coalg} \) is the composite functor
\[
\mathcal{A}\text{-Alg} \xrightarrow{B_{\mathcal{A}}} \mathcal{A}^\perp\text{-Coalg} \xrightarrow{\Omega_{\mathcal{A}^\perp}^{-1}} \mathcal{A}^\sharp\text{-Coalg}.
\]
It is an easy exercise to show that the classical cobar and bar constructions are indeed the usual, well-known reduced cobar and bar constructions.

The parameters we consider are constructed from symmetric sequences in the following way. Recall the Schur functor \( T : \text{dgM}^\Sigma \to \text{End(}\text{dgM}) \) from Section 3.2.1 and the related functor \( \Gamma : \text{dgM}^\Sigma \to \text{End(}\text{dgM}_\perp) \) from Section 3.3.1. If \( \mathcal{P} \) is an operad in the category \( \mathcal{A}\text{-Alg} \) of associative chain algebras, i.e., an operad in \( \text{dgM} \) with a compatible level monoid structure, then \( T_{\mathcal{P}} \) is the endofunctor underlying a monad \( T_{\mathcal{P}} \) on \( \mathcal{A}\text{-Alg} \). Similarly, if \( \mathcal{Q} \) is a cooperad in the category \( \mathcal{A}^\sharp\text{-Coalg} \) of coassociative chain coalgebras, then \( \Gamma_{\mathcal{Q}} \) is the endofunctor underlying a comonad \( \Gamma_{\mathcal{Q}} \) on \( \mathcal{A}^\sharp\text{-Coalg} \). Consequently, there are functors
\[
\text{CoOp}_{\mathcal{A}^\sharp\text{-Coalg}} \to \text{Comon}_{\mathcal{A}\text{-Alg}} : \mathcal{Q} \mapsto \Omega\Gamma_{\mathcal{A}}B
\]
and
\[
\text{Op}_{\mathcal{A}\text{-Alg}} \to \text{Monad}_{\mathcal{A}^\sharp\text{-Coalg}} : \mathcal{P} \mapsto BT_{\mathcal{P}}\Omega,
\]
giving us families of (co)monads parametrized by (co)operads.
Definition 5.3. Let $\mathcal{Q}$ be a cooperad in $\mathcal{A}^\#\text{-Coalg}$, and let $A, B \in \mathcal{A}\text{-Alg}$. A $\mathcal{Q}$-parametrized strongly homotopy morphism from $A$ to $B$ is a morphism
$$\Omega \Gamma_\mathcal{Q} B(A) \to B$$
in $\mathcal{A}\text{-Alg}$. These are the morphisms in the coKleisli category
$$\mathcal{A}\text{-Alg}_{sh, \mathcal{Q}} := \Omega \Gamma_\mathcal{Q} B \mathcal{A}\text{-Alg}.$$

Dually, let $\mathcal{P}$ be an operad in $\mathcal{A}\text{-Alg}$, and let $C, D \in \mathcal{A}^\#\text{-Coalg}$. A $\mathcal{P}$-parametrized strongly homotopy morphism from $C$ to $D$ is a morphism
$$C \to BT \Omega(D)$$
in $\mathcal{A}^\#\text{-Coalg}$. These are the morphisms in the Kleisli category
$$\mathcal{A}^\#\text{-Coalg}_{sh, \mathcal{P}} := \mathcal{A}^\#\text{-Coalg}_{BT \Omega}.$$

Example 5.4. We obtain the usual strongly homotopy morphisms of algebras (respectively, of coalgebras) if we set $\mathcal{Q}$ (respectively, $\mathcal{P}$) equal to $J$.

The parametrized categories defined above admit natural monoidal structures. The \textit{level tensor product} of symmetric sequences $\mathcal{X}$ and $\mathcal{Y}$ is defined by $(\mathcal{X} \otimes \mathcal{Y})(n) = \mathcal{X}(n) \otimes \mathcal{Y}(n)$. The constant symmetric sequence $\mathcal{C}$, where $\mathcal{C}(n) = R$ with the trivial $\Sigma_n$-action for all $n$, is the neutral object for this product. A (co)monoid with respect to the level tensor product is called, unsurprisingly, a \textit{level (co)monoid}.

Proposition 5.5. 1. If $\mathcal{Q}$ is a cooperad in $\mathcal{A}^\#\text{-Coalg}$, then $\mathcal{A}\text{-Alg}_{sh, \mathcal{Q}}$ has a natural monoidal structure.
2. If $\mathcal{P}$ is an operad in $\mathcal{A}\text{-Alg}$, then $\mathcal{A}^\#\text{-Coalg}_{sh, \mathcal{P}}$ has a natural monoidal structure.

Proof. Recall that both $\Omega$ and $B$ are monoidal and op-monoidal, i.e., there are natural transformations of functors in $\text{dgM}$
$$\Omega(-) \otimes \Omega(-) \Rightarrow \Omega(- \otimes -) \Rightarrow \Omega(-) \otimes \Omega(-)$$
and
$$B(-) \otimes B(-) \Rightarrow B(- \otimes -) \Rightarrow B(-) \otimes B(-)$$
that are appropriately associative and unital.

To prove (1), note that if $(\mathcal{X}, \Delta)$ is any level comonoid, then $\hat{\mathcal{X}}$ is op-monoidal, i.e., there is an appropriately associative and unital natural transformation
$$\Gamma_{\mathcal{X}}(- \otimes -) \Rightarrow \Gamma_{\mathcal{X}}(-) \otimes \Gamma_{\mathcal{X}}(-),$$
given by summing up the natural maps
$$\mathcal{X}(n) \otimes (A \otimes B)^{\otimes n} \xrightarrow{\Delta \otimes \text{Id}} \mathcal{X}(n) \otimes \mathcal{X}(n) \otimes (A \otimes B)^{\otimes n} \cong (\mathcal{X}(n) \otimes A)^{\otimes n} \otimes (\mathcal{X}(n) \otimes B)^{\otimes n}.$$
It follows that $\Omega \Gamma_{\mathcal{Q}} B$ is op-monoidal, since $\mathcal{Q}$ is a level comonoid.

There is therefore a monoidal structure on $\mathcal{A}\text{-Alg}_{\mathcal{Q}}$, which is the usual monoidal product on objects. If $f : \Omega \Gamma_{\mathcal{Q}} BA \rightarrow E$ and $f' : \Omega \Gamma_{\mathcal{Q}} BA' \rightarrow E'$ represent elements of $\mathcal{A}\text{-Alg}_{\mathcal{Q}}(A, E)$ and $\mathcal{A}\text{-Alg}_{\mathcal{Q}}(A', E')$, then the monoidal product of $f$ and $f'$ is equal to the composite

$$\Omega \Gamma_{\mathcal{Q}} B(A \otimes A') \rightarrow \Omega \Gamma_{\mathcal{Q}} B(A) \otimes \Omega \Gamma_{\mathcal{Q}} B(A') \xrightarrow{f \otimes f'} E \otimes E'.$$

The proof of (2) is strictly dual and left to the reader.

5.2. Diffraction and parametrized strong homotopy

In this section we provide an operadic description of the categories $\mathcal{A}\text{-Alg}_{\mathcal{Q}}$ and $\mathcal{A}^\sharp\text{-Coalg}_{\mathcal{P}}$, analogous to Theorem 4.10. This description is given in terms of the diffraction of $\mathcal{Q}$ (respectively, $\mathcal{P}$), denoted $\Phi(\mathcal{Q})$ (respectively, $\Psi(\mathcal{P})$), which is an $\mathcal{A}$-co-ring (respectively, a $\mathcal{A}^\sharp$-ring).

Our choice of terminology is motivated by the isomorphisms established below in Theorem 5.18, which imply the existence of the following bijective correspondence. Let $A$ and $E$ be associative chain algebras. If $\mathcal{Q}$ is a cooperad in the category of chain coalgebras, then the set of morphisms of $\mathcal{A}^\sharp$-coalgebras from $\Gamma_{\mathcal{Q}}(BA)$ to $BE$ is in bijective correspondence with the set of morphisms of left $\mathcal{A}$-modules from $\Phi(\mathcal{Q}) \circ z(A)$ to $z(E)$. In other words, a “$\mathcal{Q}$-parametrized” map on the bar constructions can be “diffracted” into its component pieces on the underlying algebras. Adding up the component pieces, we obtain a $\Phi(\mathcal{Q})$-parametrization of a morphism between the underlying algebras.

One advantage to working with $\Phi(\mathcal{Q})$ is that it lends itself well to existence proofs of $\mathcal{Q}$-parametrized strongly homotopy morphisms of algebras by acyclic models methods. We formulate one such existence result and its dual in Theorems 5.19 and 5.20.

We need below a few fundamental results about $\mathcal{A}$-bimodules and $\mathcal{A}^\sharp$-comodules. We use the following notation throughout the rest of this section.

**Notation 5.6.** For all $n \geq 1$, let

$$\delta_n = 1 \cdot \text{Id}_{\{1,...,n\}} \in \mathcal{A}(n) = R[\Sigma_n],$$

and let $\delta_n^\sharp$ denote the dual element of $\mathcal{A}^\sharp(n)$. Let

$$\alpha_n = s^{1-n} \delta_n \in \mathcal{S}^{-1}\mathcal{A}(n) \quad \text{and} \quad \alpha_n^\sharp = s^{n-1} \delta_n^\sharp \in \mathcal{A}^\sharp^{-1}(n).$$

For all $\vec{n} = (n_1, ..., n_m) \in \mathbb{N}^m$, let

$$\delta_{\vec{n}} = \delta_{n_1} \otimes \cdots \otimes \delta_{n_m} \in \mathcal{A}[\vec{n}],$$

$$\delta_{\vec{n}}^\sharp = \delta_{n_1}^\sharp \otimes \cdots \otimes \delta_{n_m}^\sharp \in \mathcal{A}^\sharp[\vec{n}],$$

$$\alpha_{\vec{n}} = \alpha_{n_1} \otimes \cdots \alpha_{n_m} \in \mathcal{S}^{-1}\mathcal{A}[\vec{n}]$$

and

$$\alpha_{\vec{n}}^\sharp = \alpha_{n_1}^\sharp \otimes \cdots \alpha_{n_m}^\sharp \in \mathcal{A}^\sharp^{-1}[\vec{n}].$$
Remark 5.7. Let $\text{Mon}_\otimes$ denote the category of level monoids in $\text{dgM}^\Sigma$. The functor $- \circ \mathcal{A} : \text{dgM}^\Sigma \to \text{Mod}_\mathcal{A}$ restricts and corestricts to a functor

$$- \circ \mathcal{A} : \text{Mon}_\otimes \to \mathcal{A}\text{Mod}_\mathcal{A}.$$ 

Because $\mathcal{X} \circ \mathcal{A}$ is a free right $\mathcal{A}$-module, a left $\mathcal{A}$-action

$$\lambda : \mathcal{A} \circ \mathcal{X} \circ \mathcal{A} \to \mathcal{X} \circ \mathcal{A}$$ 

that is a morphism of right $\mathcal{A}$-modules is determined by a morphism of symmetric sequences

$$\lambda : \mathcal{A} \rightarrow \mathcal{X} \circ \mathcal{A}.$$ 

Moreover, since $\mathcal{A}$ is generated by $\delta_2$, it suffices to specify

$$\lambda \left( \delta_2 \otimes (x \otimes x') \right) \in \mathcal{X} \circ \mathcal{A}$$

for all $x \in \mathcal{X}(m)$, $x' \in \mathcal{X}(m')$, and $m, m' \geq 0$ to define a left $\mathcal{A}$-action. We choose to define $\lambda$ by setting

$$\lambda \left( \delta_2 \otimes (x \otimes x') \right) = \begin{cases} xx' \otimes \delta_2 \otimes m : m = m' \\ 0 : m \neq m', \end{cases}$$

for all $x \in \mathcal{X}(m)$, $x' \in \mathcal{X}(m')$ and $m, m' \geq 0$.

Similarly, the functor $- \circ \mathcal{A}^\sharp : \text{dgM}^\Sigma \to \text{Comod}_\mathcal{A}^\sharp$ restricts and corestricts to a functor

$$- \circ \mathcal{A}^\sharp : \text{Comon}_\otimes \to \mathcal{A}^\sharp\text{Comod}_\mathcal{A}^\sharp,$$

where the left $\mathcal{A}^\sharp$-coaction on $\mathcal{X} \circ \mathcal{A}^\sharp$ is expressed in terms of the level comultiplication on $\mathcal{X}$.

We can now define the diffracting functor in terms of operadic suspension and twisting structures and then study its properties.

Recall the right twisting structure $(B, E, \tilde{E}, \delta, \mu)$ on the category of symmetric sequences of chain complexes (section 3.4) and the definition of the adjoint functors

$$(g, g')_* : c\text{Comod}_{C^\vee} \to \mathcal{A}\text{Mod}_{\mathcal{A}'}; (g, g')^*$$

induced by classifying morphisms $g : C \to BA$ and $g' : C' \to BA'$ (Theorem 2.9), for any right twisting structure $(B, E, \tilde{E}, \delta, \mu)$. Let

$$\kappa : \mathcal{A} \to B\mathcal{A}$$

denote the canonical classifying morphism from Section 4.1.

**Definition 5.8.** The monoid diffracting functor

$$\Psi : \text{Mon}_\otimes \to \mathcal{A}^\sharp\text{Comod}_\mathcal{A}^\sharp$$
is equal to the composite

\[ \text{Mon} \xrightarrow{\xi} \text{Mod} \xrightarrow{(\kappa, \lambda, \kappa)} \text{Comod} \xrightarrow{\xi^{-1}} \text{Mon} \xrightarrow{\xi} \text{Mod} \].

The comonoid diffracting functor

\[ \Phi : \text{Comon} \xrightarrow{\xi} \text{Mod} \]

is equal to the composite

\[ \text{Comon} \xrightarrow{\xi} \text{Mod} \xrightarrow{\xi^{-1}} \text{Comod} \xrightarrow{(\kappa, \lambda, \kappa)} \text{Mod} \].

**Remark 5.9.** For every level monoid \((\mathcal{X}, \mu)\),

\[ \Psi(\mathcal{X}) = (\mathcal{A} \circ (\mathcal{S}^{-1} \circ \mathcal{S}^{-1} \circ \mathcal{A}) \circ \mathcal{A}^2) \circ \mathcal{A}^2, d_\Phi \),

where \(d_\Phi\) is specified as follows. If \(x', x'' \in \mathcal{X}(m')\) and \(x'' \in \mathcal{X}(m'')\), then for all \(n', n'' \geq 1\), the perturbation of the internal differential that gives rise to \(d_\Phi \left( \delta_2 \theta \right) \left( (s^{1-m'} x' \otimes \alpha_{\vec{n}'} \otimes (s^{1-m''} x'' \otimes \alpha_{\vec{n}''}) \right) \) is

\[
\begin{cases}
\pm s^{1-m'} (x' x'') \otimes \alpha_{\vec{n}'+\vec{n}''} & : m' = m'' \\
0 & : m' \neq m'',
\end{cases}
\]

where the sign is given by the Koszul rule.

On the other hand, for all \(x \in \mathcal{X}(m)\) and \(\vec{n} \in I_{n,m}\), the perturbation of the internal differential that determines \(d_\Phi \left( (s^{1-m} x \otimes \alpha_{\vec{n}}) \otimes (s^{1-m} x \otimes \alpha_{\vec{n}}) \right) \) is \(s^{1-m} x \otimes \alpha_{\vec{n}}\), where \(d_\Phi(n_1, ..., n_m) = (n_1, ..., n_k + 1, ..., n_m)\), and the sign is determined by the Koszul rule. It suffices to specify these values of the differential, since the underlying bicomodule is cofree and since \(\mathcal{A}^2\) is cogenerated by \(\delta_2\).

**Remark 5.10.** For every level comonoid \((\mathcal{X}, \Delta)\),

\[ \Phi(\mathcal{X}) = (\mathcal{A} \circ (\mathcal{S} \circ \mathcal{S} \circ \mathcal{A}) \circ \mathcal{A} \circ d_\Phi), \]

where \(d_\Phi\) is specified as follows. Let \(x \in \mathcal{X}(m)\), \(\vec{n} \in I_{m,n}\), and \(\alpha_{\vec{n}} \in \mathcal{A}^2[\vec{n}]\) (cf., Conventions [5.6]). Write \(\Delta(x) = x_i \otimes x^i\) (using the Einstein summation convention).

Then

\[
d_\Phi(s^{m-1} x \otimes \alpha_{\vec{n}}^i) = \pm s^{m-1} d x \otimes \alpha_{\vec{n}}^i + \delta_2 \sum_{\vec{n}'+\vec{n}''=\vec{n}} \pm (s^{m-1} x_i \otimes \alpha_{\vec{n}''}^i) \otimes (s^{m-1} x^i \otimes \alpha_{\vec{n}''}^i) + s^{m-1} x \otimes \sum_{0<k<n} \pm \alpha_{\vec{n}''}^i \otimes (\delta_1^{k-1} \otimes \delta_2 \otimes \delta_1^{k-1} - k - 1),
\]

where \(\vec{n} = (n_1, ..., n_j - 1, ..., n_m)\) if \(\sum_{i=1}^{j-1} n_i < k \leq \sum_{i=1}^{j} n_i\), and the signs are again determined by the Koszul rule.
In order to construct Kleisli categories associated to $\Psi(\mathcal{X})$ for a level monoid $X$, it must be an $\mathcal{A}$-ring, which is a consequence of the next proposition. Let $\text{Ring}_{\mathcal{A}}$ denote the category of $\mathcal{A}$-rings.

**Proposition 5.11.** The functor $- \circ \mathcal{A} : \text{Mon}_{\otimes} \to \mathcal{A}\text{Mod}_{\mathcal{A}}$ restricts and corestricts to a functor

$$- \circ \mathcal{A} : \text{Op}_{\mathcal{A}}\text{-Alg} \to \text{Ring}_{\mathcal{A}}.$$

**Proof.** Let $\mathcal{P}$ be an operad in the category of associative chain algebras, with multiplication map $\gamma : \mathcal{P} \circ \mathcal{P} \to \mathcal{P}$ and unit map $\eta : J \to \mathcal{P}$. In particular, $\gamma$ is a morphism of level monoids and therefore induces a morphism of $\mathcal{A}$-bimodules

$$(\mathcal{P} \circ \mathcal{A}) \circ (\mathcal{P} \circ \mathcal{A}) \cong \mathcal{P} \circ \mathcal{P} \circ \mathcal{A} \xrightarrow{\gamma \circ \mathcal{A}} \mathcal{P} \circ \mathcal{A},$$

which is the desired multiplication map on $\mathcal{P} \circ \mathcal{A}$. The unit of $\mathcal{P} \circ \mathcal{A}$ is just $\eta \circ \mathcal{A}$.

Proposition 5.11 and Proposition 2.11(2) together imply the following result.

**Corollary 5.12.** If $\mathcal{P}$ is an operad in the category of associative chain algebras, then $\Psi(\mathcal{P})$ is naturally an $\mathcal{A}$-ring.

As usual, a dual version of Proposition 5.11 holds as well; we leave its strictly dual proof to the reader. Let $\text{CoRing}_{\mathcal{A}^{\sharp}}$ denote the category of $\mathcal{A}^{\sharp}$-co-rings.

**Proposition 5.13.** The functor $- \circ \mathcal{A}^{\sharp} : \text{Comon}_{\otimes} \to \mathcal{A}^{\sharp}\text{Comod}_{\mathcal{A}^{\sharp}}$ restricts and corestricts to a functor

$$- \circ \mathcal{A}^{\sharp} : \text{CoOp}_{\mathcal{A}^{\sharp}}\text{-Coalg} \to \text{CoRing}_{\mathcal{A}^{\sharp}}.$$

Proposition 5.13 and Proposition 2.11(1) together imply the following result.

**Corollary 5.14.** If $\mathcal{Q}$ is a cooperad in the category of coassociative chain coalgebras, then $\Phi(\mathcal{Q})$ is naturally an $\mathcal{A}^{\sharp}$-co-ring.

**Remark 5.15.** Any simplicial or topological cooperad gives rise to a cooperad in the category of coassociative chain coalgebras, upon application of the normalized chains functor. Moreover, since $\mathcal{A}^{\sharp}(n)$ is the dual Hopf algebra to $R[\Sigma_n]$ for all $n$, $\mathcal{A}^{\sharp}$ is itself a cooperad in the category of coassociative chain coalgebras.

Dually, if we apply the normalized cochains functor to a cooperad in the category of simplicial sets or topological spaces, then we obtain an operad in the category of associative chain algebras, of which the associative operad itself is another important example.
Remark 5.16. From the definitions above, one can deduce a formula for the comultiplication $\psi_Q : \Phi(Q) \to \Phi(Q) \circ A \Phi(Q)$, where $Q$ is any cooperad in the category of coassociative chain coalgebras. Observe that since the $\mathcal{A}$-bimodule underlying $\Phi(Q)$ is free, $\psi_Q$ is determined by its image on $S_Q \otimes A^\perp$.

Let $x \in \mathcal{D}(m)$, and let $\tilde{n} \in I_{m,n}$. If $m$ does not divide $n$, then $\psi_Q(s^{m-1}x \otimes \alpha^\sharp_{n_i}) = 0$.

If there exists a natural number $\ell$ such that $n = m\ell$ and $\tilde{n} \in I_{n,m}$, then let

$$K_{\tilde{n},\ell} = I_{\ell,n_1} \times \cdots \times I_{\ell,n_m},$$

and for all $(\tilde{n}_1, ..., \tilde{n}_m) \in K_{\tilde{n},\ell}$, with $\tilde{n}_i = (n_{i_1}, ..., n_{i_\ell})$, let

$$\tilde{n}^i = (n_{i_1}, ..., n_{i_\ell}),$$

for all $1 \leq j \leq \ell$. In terms of this notation we have for $x \in \mathcal{D}(m)$ and $\tilde{n} \in I_{n,m}$ that

$$\psi_Q(s^{m-1}x \otimes \alpha^\sharp_{n_i}) = \sum_{i,K_{\tilde{n},\ell},\ell \in I_{m,\ell}} \pm s^{m-1}x_i \otimes \alpha^\sharp_{n_i} \otimes (s^{m-1}x_{i_1} \otimes \alpha^\sharp_{n_{i_1}}) \otimes \cdots \otimes (s^{m-1}x_{i_\ell} \otimes \alpha^\sharp_{n_{i_\ell}}),$$

where the signs are determined by the Koszul rule and

$$\Delta^{(\ell+1)}(x) = \sum_{i} x_{i_0} \otimes \cdots \otimes x_{i_\ell}.$$

We leave it as a (rather technical) exercise for the reader to obtain analogous formulas for the multiplication $\Psi(\mathcal{P}) \boxtimes \Psi(\mathcal{P}) \to \Psi(\mathcal{P})$.

5.3. Kleisli categories and parametrized strong homotopy

In this section we give an operadic, Kleisli category description of parametrized strong homotopy of (co)associative (co)algebras, in the spirit of Theorem 4.10.

Definition 5.17. Let $Q$ be a cooperad in the category of coassociative chain coalgebras. Let $\Phi_Q$ denote the comonad on $\mathcal{A}$-Mod with underlying endofunctor

$$\Phi(Q) \circ \mathcal{A} : \mathcal{A}$-Mod \to \mathcal{A}$-Mod.$$

The induced monad on $\mathcal{A}$-Alg is also denoted $\Phi(Q)$.

Let $\mathcal{P}$ be a operad in the category of associative chain algebras. Let $\Psi_{\mathcal{P}}$ denote the monad on $\mathcal{A}$-CoMod with underlying endofunctor

$$\Psi(\mathcal{P}) \boxtimes - : \mathcal{A}$-CoMod \to \mathcal{A}$-CoMod.$$

The induced comonad on $\mathcal{A}$-CoAlg is also denoted $\Psi(\mathcal{P})$. 

41
Theorem 5.18. Let \( \mathcal{Q} \) be a cooperad in the category of coassociative chain coalgebras, and let \( \mathcal{P} \) be an operad in the category of associative chain algebras. There are isomorphisms of categories

\[
\Theta_{\mathcal{A}, \mathcal{Q}} : \mathcal{A} - \text{Alg}_{sh, \mathcal{Q}} \cong \Phi_{(\mathcal{Q})} \mathcal{A} - \text{Alg}
\]

and

\[
\Upsilon_{\mathcal{A}, \mathcal{P}} : \mathcal{A}^* - \text{Coalg}_{sh, \mathcal{P}} \cong \mathcal{A}^* - \text{Coalg}_{\mathcal{P}},
\]

which are the identity on objects.

**Proof.** The proof of this theorem strongly resembles that of Theorem 4.10. The definition of \( \Theta_{\mathcal{A}, \mathcal{Q}} \) on morphisms follows from the sequence of natural isomorphisms below, where \( A \) and \( E \) are \( \mathcal{A} \)-algebras.

\[
\Phi_{(\mathcal{Q})} \mathcal{A} - \text{Alg}(A, E) = \mathcal{A} \text{-Mod}((\Phi(\mathcal{Q}) \circ z)(A), z(E))
\]

\[
= \mathcal{A} \text{-Mod}(\mathcal{A} \circ z, \mathcal{A} \circ z(A), z(B))
\]

\[
\cong \mathcal{A}^* - \text{Comod}(\mathcal{A} \circ z, \mathcal{A}^* \circ z(A), \mathcal{A}^* \circ z(E))
\]

\[
= \mathcal{A}^* - \text{Comod}(\mathcal{A} \circ z(\mathcal{A} B), z(B E))
\]

\[
\cong \mathcal{A}^* - \text{Comod}(\mathcal{A} \circ z(\mathcal{B} A), z(B E))
\]

\[
= \mathcal{A}^* - \text{Coalg}(\Gamma \mathcal{A} B, B E)
\]

\[
= \mathcal{A}^* - \text{Coalg}(\Gamma \mathcal{B} A, B E)
\]

\[
= \mathcal{A} - \text{Alg}(\Omega \mathcal{A} B, E)
\]

\[
= \mathcal{A}^* - \text{Coalg}_{sh, \mathcal{Q}}(A, E).
\]

The proof of the dual case is similar and left to the reader.

As a consequence of Theorem 5.18, it is relatively easy to prove the following existence theorems, which is expressed in terms of acyclic models. We recall the foundations of this method before stating the theorems.

Let \( D \) be a category, and let \( M \) be a set of objects in \( D \). A functor \( X : D \rightarrow \text{dgM} \) is free with respect to \( M \) if there is a set \( \{ x_m \in X(m) \mid m \in M \} \) such that \( \{ X(f)(x_m) \mid f \in D(m, D), m \in M \} \) is a \( R \)-basis of \( X(D) \) for all objects \( D \) in \( D \). The functor \( X \) is acyclic with respect to \( M \) if \( X(m) \) is acyclic for all \( m \in M \). More generally, if \( C \) is a category with a forgetful functor \( U \) to \( \text{dgM} \) and \( X : D \rightarrow C \) is a functor, we say that \( X \) is free, respectively acyclic, with respect to \( M \) if \( UX \) is.

**Theorem 5.19.** Let \( \mathcal{Q} \) be a cooperad in \( \mathcal{A}^* - \text{Coalg} \) with \( \mathcal{Q}(0) = 0 \) and \( \mathcal{Q}(1) = R \). Let \( X, Y : D \rightarrow \mathcal{A} - \text{Alg} \) be functors, where \( D \) is a category admitting a set of models \( M \) with respect to which \( X \) is free and such that \( Y(m) \) is acyclic for every \( m \) that is a coproduct of elements of \( M \). Let \( U : \mathcal{P} - \text{Alg} \rightarrow \text{dgM} \) be the forgetful functor.
If $\tau : UX \Rightarrow UY$ is a natural transformation, then there is a natural transformation $\hat{\tau} : \Omega \hat{D}B \circ X \Rightarrow Y$ of functors from $D$ into $\mathcal{A}$-$\text{Alg}$ extending $\tau$, i.e., for all $D \in \text{Ob } D$, the following composite is equal to $\tau_D$.

$$X(D) \hookrightarrow \Omega \hat{D}B \circ X(D) \xrightarrow{\hat{\tau}_D} Y(D)$$

In other words, for each $D \in \text{Ob } D$, the natural chain map $\tau_D : X(D) \rightarrow Y(D)$ admits a natural, $\mathcal{P}$-parametrized, strongly homotopy multiplicative structure.

There is also a coalgebra version.

**Theorem 5.20.** Let $\mathcal{P}$ be an operad in $\mathcal{A}$-$\text{Alg}$ with $\mathcal{P}(0) = 0$ and $\mathcal{P}(1) = R$. Let $X, Y : D \rightarrow \mathcal{A}^\sharp$-$\text{Coalg}$ be functors, where $D$ is a category admitting a set of models $\mathcal{M}$ with respect to which $X$ is free and $Y$ is acyclic. Let $U : \mathcal{A}^\sharp$-$\text{Coalg} \rightarrow \text{dgM}$ be the forgetful functor.

If $\tau : UX \Rightarrow UY$ is a natural transformation, then there is a natural transformation $\hat{\tau} : X \Rightarrow B \hat{\mathcal{P}} \Omega Y$ of functors from $D$ into $\mathcal{A}^\sharp$-$\text{Coalg}$ lifting $\tau$, i.e., for all $D \in \text{Ob } D$, the following composite is equal to $\tau_D$.

$$X(D) \xrightarrow{\hat{\tau}_D} B \hat{\mathcal{P}} \Omega Y(D) \xrightarrow{\text{proj.}} Y(D)$$

In other words, for each $D \in \text{Ob } D$, the natural chain map $\tau_D : X(D) \rightarrow Y(D)$ admits a natural, $\mathcal{P}$-parametrized, strongly homotopy comultiplicative structure.

The proofs of these theorems are discussed in Appendix B.

**Appendix A. The Alexander-Whitney co-ring, by P.-E. Parent**

The categories $\text{DASH} = \mathcal{A}$-$\text{Alg}_{sh}$ and $\text{DCSH} = \mathcal{A}^\sharp$-$\text{Coalg}_{sh}$ are particularly important in topology. For example, Gugenheim and Munkholm showed that the Alexander-Whitney equivalence

$$C_\ast(K \times L) \rightarrow C_\ast(K) \otimes C_\ast(L)$$

of normalized chains on reduced simplicial sets is naturally the linear part of a morphism in $\text{DCSH}$, which implies, as shown in [16], that the same is true of the usual comultiplication on $C_\ast K$. We devote this section to a careful analysis of the operadic description of these categories.

Let $\mathcal{F} = K(\mathcal{A}) = (\mathcal{A} \circ \mathcal{A}^\perp \circ \mathcal{A}, \partial_\mathcal{F}) = \Phi(\mathcal{F})$. Then

$$\mathcal{F} = \{ \mathcal{F}(m) \mid m \in \mathbb{N} \}$$

with generators

$$\{ f_m = s^{m-1}u_0 \in \mathcal{F}(m)_{m-1} \mid m \in \mathbb{N} \}$$

satisfying

$$\partial f_m = \sum_{i=1}^{m-1} \delta \otimes (f_i \otimes f_{m-i}) + \sum_{i=0}^{m-2} f_{m-1} \otimes (1 \otimes i \otimes \delta \otimes 1 \otimes (m-2-i)).$$
By Proposition 4.8, \( \mathcal{F} \) is an \( \mathcal{A} \)-co-ring, which we call the Alexander-Whitney co-ring. The formula for the composition comultiplication \( \psi_\mathcal{F} \) is particularly simple. For \( n \geq 1 \), we have

\[
\psi_\mathcal{F}(f_n) = \sum_{m \leq n} \sum_{\vec{n} \in I_{m,n}} f_m \otimes (f_{n_1} \otimes \cdots \otimes f_{n_m})
\]

where \( n_i \geq 1 \) for all \( i \). In fact \( \mathcal{F} \) is a counital \( \mathcal{A} \)-co-ring, with counit \( \varepsilon : \mathcal{F} \to \mathcal{A} \) specified by \( \varepsilon(f_n) = 0 \) for all \( n > 1 \) and \( \varepsilon(f_1) = 1 \). Since \( \mathcal{F} \) is the Koszul resolution of \( \mathcal{A} \), the counit is a levelwise quasi-isomorphism.

Moreover, \( (\mathcal{F}, \partial_\mathcal{F}, \Delta_\mathcal{F}) \) is a level comonoid in the category \( \mathcal{A} \)-bimodules. Explicitly,

\[
\Delta_\mathcal{F}(f_m) = \sum_{k=1}^{m} \sum_{\vec{i} \in I_{k,m}} \left( f_k \otimes \delta^{(i_1)} \otimes \cdots \otimes \delta^{(i_k)} \right) \otimes \left( \delta^{(k)} \otimes f_{i_1} \otimes \cdots \otimes f_{i_k} \right).
\]

It is easy to check that \( \Delta_\mathcal{F} \) is coassociative.

It follows from Theorem 4.10 or Theorem 5.18 that there is a natural isomorphism

\[
\dash\langle A, A' \rangle \cong \mathcal{A} \text{-Mod}(\mathcal{F} \circ z(A), z(A'))
\]

for all associative chain algebras \( A \) and \( A' \).

On the other hand we can also characterize morphisms in \( \text{DCSH} \) in terms of \( \mathcal{F} \), as described below. This alternate characterization has already proved useful in, e.g., [16], [15], [12], [17], [25] and [3].

Let \( \mathcal{F} \) denote the comonad on \( \mathcal{A} \)-Coalg with underlying endofunctor \( - \circ \mathcal{F} \).

An \( \mathcal{A} \)-coalgebra is a chain complex \( C \), equipped with structure morphisms

\[
\psi_n : C \otimes \mathcal{A}(n) \to C \otimes^n \quad (n \geq 1)
\]

that are associative, equivariant and unital with respect to \( \mathcal{A} \to \mathcal{A} \). This definition does coincide with that of \( \mathcal{A} \)-coalgebras, since \( \mathcal{A} \) is projective. We denote by \( \mathcal{A} \text{-Coalg} \) the category of \( \mathcal{A} \)-coalgebras and morphisms that commute strictly with the structure morphisms.

The tensor functor

\[
\mathcal{F} : \text{dgM} \to \text{dgM}^\Sigma,
\]

where \( \mathcal{F}(X)(n) = X \otimes^n \), restricts and corestricts to a full and faithful functor

\[
\mathcal{F} : \mathcal{A} \text{-Coalg} \to \mathcal{A} \text{-Mod} \mathcal{A}.
\]

It follows that \( \mathcal{F} \) induces a comonad, also denoted \( \mathcal{F} \), on \( \mathcal{A} \text{-Coalg} \).

**Theorem Appendix A.1.** There is an isomorphism of categories

\[
\text{DCSH} \cong \mathcal{F} \mathcal{A} \text{-Coalg}.
\]
We will make use of the following lemma.

**Lemma Appendix A.2.** Let $f : W \otimes Y \to Z$ and $g : X \to Y$ be morphisms in $\text{dgM}$. The right adjoint to the composite

$$ W \otimes X \overset{1 \otimes g}{\to} W \otimes Y \overset{f}{\to} Z $$

is the composite

$$ W \overset{\hat{f}}{\to} \text{Hom}(Y, Z) \overset{\text{Hom}(g, Z)}{\to} \text{Hom}(X, Z), $$

where $\hat{f}$ is the right adjoint to $f$.

**Proof.** Recall that for all $w \in X$, $\hat{f}_w \in \text{Hom}(Y, Z)$ is defined by $\hat{f}_w(y) = f(w \otimes y)$. We set $g^1 = \text{Hom}(g, Z)$, so that $g^1(\phi) = \phi g$ for all $\phi \in \text{Hom}(Y, Z)$.

For all $x \in X$, we have

$$ (g^1 \hat{f}_w)(x) = g^1(\hat{f}_w)(x) = \hat{f}_w(g(x)) = f(w \otimes g(x)) = f(1 \otimes g)(w \otimes x) = (f(1 \otimes g))_n(x). $$

Therefore, the right adjoint of $f(1 \otimes g)$ is $g^1 \hat{f}$.

**Proof (Proof of Theorem Appendix A.1).** First, we show that there is a functor

$$ \text{Ind} : \text{fA-Coalg} \to \text{DCSH} $$

such that $\text{Ind}(\mathcal{F}(C)) = C$ on objects.

A morphism $\theta$ in $\text{fA-Coalg}(C, D) = \text{Mod}_{\text{fA}}(\mathcal{F}(C) \circ_{\text{fA}} \mathcal{F}, \mathcal{F}(D))$ is determined by a sequence of equivariant morphisms,

$$ C \otimes \mathcal{A}^\perp(n) \to D^\otimes n. $$

Desuspend $n$ times to obtain the equivariant morphism

$$ (s^{-1}C) \otimes \mathcal{A}^\perp(n) \to (s^{-1}D)^\otimes n. $$

Now take the right adjoint, then pass to orbits. The end result is a morphism

$$ s^{-1}C \to \mathcal{A}(n) \otimes_{\Sigma_n} (s^{-1}D)^\otimes n $$

that extends uniquely to a morphism of $\mathcal{A}$-algebras,

$$ \text{Ind}(\theta) : T_{\mathcal{A}}(s^{-1}C) \to T_{\mathcal{A}}(s^{-1}D). $$
Since the internal differentials from $C$ and $D$ automatically commute with \(\text{Ind}(\theta)\), we may suppose that they are zero. In particular, the differentials in \(\Omega(C)\) and \(\Omega(D)\) are then entirely quadratic: \(d = d_2\).

Let \(F = \text{Ind}(\theta)\); then \(Fd\) is defined by the composition

\[
s^{-1}C \xrightarrow{\delta} \mathcal{A}(2) \otimes (s^{-1}C)^{\otimes 2} \xrightarrow{1 \otimes F} \mathcal{A}(2) \otimes \mathcal{A}[2, n] \otimes (s^{-1}D)^{\otimes n} \xrightarrow{\gamma \otimes 1} \mathcal{A}(n) \otimes (s^{-1}D)^{\otimes n}. \quad (A.1)
\]

Once we identify \(\text{Hom}(X, -)\) with \(\mathcal{A}^{\otimes}(-)\), Lemma \textbf{Appendix A.2} exhibits \((A.1)\) as the right adjoint to the following composition:

\[
s^{-1}C \otimes \mathcal{A}(n) \xrightarrow{1 \otimes \gamma^f} s^{-1}C \otimes \mathcal{A}(2) \otimes \mathcal{A}[2, n] \xrightarrow{d \otimes 1} (s^{-1}C)^{\otimes 2} \otimes \mathcal{A}[2, n] \xrightarrow{\tilde{F}} (s^{-1}D)^{\otimes n} \quad (A.2)
\]

where \(W = s^{-1}C, X = \mathcal{A}(n), Y = \mathcal{A}(2) \otimes \mathcal{A}[2, n], Z = (s^{-1}D)^{\otimes n}, g = \gamma, \tilde{f} = (1 \otimes F)d, \) and \(\tilde{f}\) and \(\tilde{F}\) are the left adjoint to the cobar differential and \(F\), respectively. If we suspend \((A.2)\) \(n\) times, we obtain \((-1)^n times\) the composition,

\[
\rho \otimes 1 \quad (A.3)
\]

where \(\rho : C \otimes \mathcal{A}(2) \to C^2\) is the structure map for \(C\). The sign of \((-1)^n\) is introduced because \((1 \otimes s^{-1} \otimes 1)\) has degree \(-1\) and \(s^n\) has degree \(n\). We note that \((A.3)\) defines \(\theta d_L\).

Now, since \(\theta\) commutes with differentials, and since we are assuming that internal differentials vanish, \(\theta(d_L - d_R) = 0\). Therefore \((A.3)\) coincides with \((-1)^n \theta d_R\), namely, with \((-1)^n times\) the composition

\[
C \otimes \mathcal{A}(n) \xrightarrow{1 \otimes \gamma^f} C \otimes \mathcal{A}(n-1) \otimes \mathcal{A}[n-1, n] \xrightarrow{1 \otimes 1 \otimes s^{-1} \otimes 1} C \otimes \mathcal{A}(n-1) \otimes \mathcal{A}[n-1, n] \xrightarrow{\theta \otimes 1} D^{\otimes n-1} \otimes \mathcal{A}[n-1, n] \xrightarrow{\tilde{F}} D^{\otimes n}. \quad (A.4)
\]

This time, when we desuspend \((A.4)\) \(n\) times, we obtain \((-1)^n times\):

\[
s^{-1}C \otimes \mathcal{A}(n) \xrightarrow{1 \otimes \gamma^f} s^{-1}C \otimes \mathcal{A}(n-1) \otimes \mathcal{A}[n-1, n] \xrightarrow{\tilde{F} \otimes 1} (s^{-1}D)^{\otimes n-1} \otimes \mathcal{A}[n-1, n] \xrightarrow{d} (s^{-1}D)^{\otimes n}. \quad (A.5)
\]

The sign is introduced for essentially the same reason: \(1 \otimes 1 \otimes s^{-1}\) has degree \(-1\) while \(s^n\) has degree \(n\).

By Lemma \textbf{Appendix A.2} \((A.5)\) is left adjoint to the composite,

\[
s^{-1}C \xrightarrow{\tilde{F}} \mathcal{A}(n-1) \otimes (s^{-1}D)^{\otimes n-1} \xrightarrow{1 \otimes d} \mathcal{A}(n-1) \otimes \mathcal{A}[n-1, n] \otimes (s^{-1}D)^{\otimes n} \xrightarrow{\gamma \otimes 1} \mathcal{A}(n) \otimes (s^{-1}D)^{\otimes n}, \quad (A.6)
\]
which defines $dF$. Therefore (A.6) and (A.1) coincide, and so Ind($\theta$) is a chain map.

Let $\theta : \mathcal{T}(C) \circ_{\mathcal{A}} \mathcal{F} \to \mathcal{T}(D)$ and $\varphi : \mathcal{T}(D) \circ_{\mathcal{A}} \mathcal{F} \to \mathcal{T}(E)$ be morphisms of $(\mathcal{A}, \mathcal{A})$-bimodules. The morphism Ind($\varphi$) $\circ$ Ind($\theta$) is the sum of composites,

$$s^{-1}C \to \mathcal{A}(m) \otimes (s^{-1}D)^{\otimes m} \to \mathcal{A}(m) \otimes \mathcal{A}[m, n] \otimes (s^{-1}E)^{\otimes n} \to \mathcal{A}(n) \otimes (s^{-1}E)^{\otimes n}.$$

The left adjoint of the above composite coincides with the composite,

$$s^{-1}C \otimes \mathcal{A}^f(n) \to s^{-1}C \otimes \mathcal{A}^f(m) \otimes \mathcal{A}^f[m, n] \to (s^{-1}D)^{\otimes m} \otimes \mathcal{A}^f[m, n] \to (s^{-1}E)^{\otimes n}.$$

Suspending, we obtain the component in arity $n$ of $\varphi \circ \theta$. It follows that Ind($\varphi \circ \theta$) = Ind($\varphi$) $\circ$ Ind($\theta$).

Next, we show that Ind has an inverse functor

\[ \text{Lin} : \text{DCSH} \to \text{F}_{\mathcal{A}}\text{-Coalg} \]

that coincides with $\mathcal{T}$ on objects.

Since $\mathcal{A}$ and $\mathcal{A}^f$ are projective, the natural morphism $\pi : (\mathcal{A}^f(n) \otimes V^{\otimes n})^\Sigma_n \to (\mathcal{A}^f(n) \otimes V^{\otimes n})^\Sigma_n$ is invertible for all $n$.

Let $F : \Omega(A) \to \Omega(C)$ be a morphism in DCSV. We take the adjoint of the composite

$$s^{-1}A \overset{F}{\to} \mathcal{A}(n) \otimes_{\Sigma_n} (s^{-1}C)^{\otimes n} \overset{\pi}{\to} \text{Hom}(\mathcal{A}^f(n), (s^{-1}C)^{\otimes n})^\Sigma_n \overset{\pi^{-1}}{\to} \text{Hom}(\mathcal{A}^f(n), C^{\otimes n})^\Sigma_n$$

to obtain a $\Sigma_n$-equivariant morphism

$$s^{-1}A \otimes \mathcal{A}^f(n) \to C^{\otimes n}.$$

Suspending $n$ times, we obtain a $\Sigma_n$-equivariant morphism

$$\varphi_n : A \otimes \mathcal{A}^f(n) \to C^{\otimes n}.$$

The sequence $(\varphi_n)$ defines a morphism of symmetric sequences,

$$\mathcal{T}(A) \circ \mathcal{A}^f \to \mathcal{T}(C)$$

that extends to define a morphism of right $\mathcal{A}$-modules,

\[ \text{Lin}(F) : \mathcal{T}(A) \circ_{\mathcal{A}} \mathcal{F} \to \mathcal{T}(C). \]

The morphism Lin($F$) commutes with differentials and respects composition by symmetric arguments to those for Ind.

Clearly Ind Lin and Lin Ind are the identity on objects. Furthermore, the algorithm that yields Lin($F$) is the reverse of the algorithm for Ind($\theta$). Therefore Ind Lin = 1 and Lin Ind = 1.
Specializing Theorems 5.19 and 5.20 to the case \( P = J = Q \), we obtain the following existence results, which have already been applied to great effect in [14], [12], [15], and [17].

**Theorem Appendix A.3.** Let \( X, Y : D \to \mathcal{T}\text{-Alg} \) be functors, where \( D \) is a category admitting a set of models \( \mathcal{M} \) with respect to which \( X \) is free and such that \( Y(m) \) is acyclic for all objects \( m \) that are coproducts of objects in \( \mathcal{M} \). Let \( U : \mathcal{T}\text{-Alg} \to \text{dgM} \) be the forgetful functor.

If \( \tau : UX \Rightarrow UY \) is a natural transformation, then there is a natural transformation \( \hat{\tau} : \Omega B \circ X \Rightarrow Y \) of functors from \( D \) into \( \mathcal{T}\text{-Alg} \) extending \( \tau \), i.e., for all \( D \in \text{Ob} D \), the following composite is equal to \( \tau_D \).

\[
X(D) \hookrightarrow \Omega BX(D) \xrightarrow{\hat{\tau}_D} Y(D)
\]

In other words, for each \( D \in \text{Ob} D \), the natural chain map \( \tau_D : X(D) \to Y(D) \) admits a natural DASH-structure.

**Theorem Appendix A.4.** Let \( X, Y : D \to \mathcal{T}\text{-Coalg} \) be functors, where \( D \) is a category admitting a set of models \( \mathcal{M} \) with respect to which \( X \) is free and \( Y \) is acyclic. Let \( U : \mathcal{T}\text{-Coalg} \to \text{dgM} \) be the forgetful functor.

If \( \tau : UX \Rightarrow UY \) is a natural transformation, then there is a natural transformation \( \hat{\tau} : X \Rightarrow B\Omega \circ Y \) of functors from \( D \) into \( \mathcal{T}\text{-Alg} \) lifting \( \tau \), i.e., for all \( D \in \text{Ob} D \), the following composite is equal to \( \tau_D \).

\[
X(D) \xrightarrow{\tau_D} B\Omega Y(D) \xrightarrow{\text{proj.}} Y(D)
\]

In other words, for each \( D \in \text{Ob} D \), the natural chain map \( \tau_D : X(D) \to Y(D) \) admits a natural DCSH-structure.

**Remark Appendix A.5.** Since the bimodule \( \mathcal{F} \) is a free \( \mathcal{T} \)-bimodule resolution of \( \mathcal{T} \), we may use it to do homological algebra. If \( \mathcal{M} \) and \( \mathcal{N} \) are right and left \( \mathcal{T} \)-modules, respectively, then

\[
\text{Tor}^\mathcal{T}(\mathcal{M}, \mathcal{N}) := H(\mathcal{M} \circ \mathcal{F} \circ \mathcal{N}).
\]

In particular, we may read off the isomorphisms

\[
\text{Tor}^\mathcal{T}(\mathcal{F}, \mathcal{F}) \cong \mathcal{T}^\perp \quad \text{and} \quad \text{Tor}^\mathcal{T}(\mathcal{F}, \mathcal{T}) \cong \mathcal{T}.
\]

**Example Appendix A.6.** We are now in a position to construct over \( R = \mathbb{F}_2 \) an example of a chain coalgebra \( M \) such that

- its cohomology algebra is realizable, i.e., there is a topological space \( X \) such that \( H^*(X; \mathbb{F}_2) \cong H^* M \) as graded algebras, but

- \( M \) is not quasi-isomorphic to the chains on any space.
This example, along with [26, Example 3.8], show that the concepts of “shc algebras” and “algebras with cup-i products” are independent of one another.

Let \( M = \mathbb{F}_2\{1, u_2, x_3, y_3, z_3, v_4, w_6\} \), where subscript indicates degree. The only non-zero differential in \( M \) is \( \partial(v) = x + y \). All elements other than \( v \) and \( w \) are primitive, while \( B\psi(v) = u \otimes u \) and \( B\psi(w) = x \otimes z + z \otimes y \). It is readily verified that \( (M, \partial, \psi) \) is a coassociative chain coalgebra.

Let \( W \) be the usual \( \mathbb{F}_2[\Sigma_2] \)-free resolution of \( \mathbb{F}_2 \). Specifically, \( W \) is generated by an element \( e_i \) with \( \partial e_i = (1 + \tau) e_{i-1} \), where \( \tau \in \Sigma_2 \) is the transposition.

Proposition Appendix A.7. There exists an equivariant morphism \( g : W \otimes M \to M \otimes M \) such that \( g(e_0 \otimes -) = \psi \).

Proof. We construct the morphism \( g \); verification that it is a chain map is routine and left to the reader.

It suffices to define \( g \) on generators. The only non-zero values that \( g \) takes on generators are \( g(e_1 \otimes w) = v \otimes z + z \otimes v \), \( g(e_3 \otimes v) = v \otimes x + y \otimes v \), and \( g(e_5 \otimes a) = a \otimes a \) for \( a \in \{u, v, w, x, y, z\} \).

By [24], \( g \) defines cup-i products in the \( \mathbb{F}_2 \)-dual \( M^\sharp \), and so \( H^*(M; \mathbb{F}_2) \) comes equipped with an action of the mod 2 Steenrod algebra. In fact, we have the following proposition.

Proposition Appendix A.8. The algebra \( H^*(M; \mathbb{F}_2) \) admits the structure of an unstable algebra over the mod 2 Steenrod algebra, where the only non-trivial operation is the \( \text{Sq}^0 \). Moreover, this algebra is isomorphic to

\[
H^*(S^2 \vee (S^3 \times S^3); \mathbb{F}_2)
\]

as unstable algebras over the mod 2 Steenrod algebra.

Proof. An easy exercise in \( \mathbb{F}_2 \)-linear algebra.

Proposition Appendix A.9. The chain coalgebra \( M \) is not realizable, i.e., \( M \) is not of the same homotopy type as \( C_*(X; \mathbb{F}_2) \) for any space \( X \).

Proof. For the duration of the proof, we suppress the coefficients from the notation. By [10], if \( X \) is a space, then the diagonal on \( C_*(X) \) is strongly homotopy-comultiplicative, that is, there is a morphism of symmetric sequences, \( \Delta : \mathcal{T}(C_*(X)) \circ \mathcal{F} \to \mathcal{T}(C_*(X) \otimes C_*(X)) \), such that \( \Delta_1 \) is the diagonal. If \( C_*(X) \) and \( M \) are connected by a sequence of chain coalgebra quasi-isomorphisms, then we may construct a morphism \( \Psi : \mathcal{T}(M) \circ \mathcal{F} \to \mathcal{T}(M \otimes M) \) such that \( \Psi_1 = \psi \). The homotopy class of such a \( \Psi \), compatible with \( \Delta \), is unique. We show that no such \( \Psi \) exists.

We attempt to define \( \Psi \) on generators \( a \otimes f_k \), for \( a \in M \) and \( k \geq 1 \). Necessarily, \( \Psi(a \otimes f_1) = \psi(a) \). We may define

\[
\Psi(w \otimes f_2) = (1 \otimes v) \otimes (z \otimes 1) + (1 \otimes z) \otimes (v \otimes 1)
\]
and \( \Psi(a \otimes f_2) = 0 \) for \( a \neq w \). Any other choice of morphism \( \Psi': M \otimes \mathcal{F}(2) \to (M \otimes M)^{\otimes 2} \) is necessarily homotopic to \( \Psi \).

Now we try to define \( \Psi \) on \( M \otimes \mathcal{F}(3) \). In order to find a value for \( \Psi(w \otimes f_3) \), we must find an element that bounds

\[
(1 \otimes z) \otimes (u \otimes 1) \otimes (u \otimes 1) + (1 \otimes u) \otimes (1 \otimes u) \otimes (z \otimes 1),
\]

but no such element exists. Therefore the diagonal on \( M \) does not extend to an \( \mathcal{F} \)-parametrized morphism.

Appendix B. Proof of Theorems 5.19 and 5.20

We prove here Theorem 5.19. The proof of Theorem 5.20 is essentially dual and therefore left to the reader. The one, slightly subtle difference in the dual case is that we no longer need the functor \( Y \) to be acyclic on coproducts of models.

According to Theorem 5.18, we need to prove the existence of a natural transformation

\[
\hat{\tau}: \Phi(\mathcal{D}) \circ z(X) \Rightarrow z(Y)
\]

of functors into the category of left \( \mathcal{A} \)-modules that extends \( \tau \). Since the symmetric sequence of graded modules underlying \( \Phi(\mathcal{D}) \circ z(X) \) is a free \( \mathcal{A} \)-module on \( \mathcal{G}_n \circ \mathcal{A} \circ z(X) \), it is sufficient to define a morphism of symmetric sequences

\[
\hat{\tau}: \mathcal{G}_n \circ \mathcal{A} \circ z(X) \to z(Y)
\]

that we then extend to a morphism of \( \mathcal{A} \)-modules

\[
\hat{\tau}: \mathcal{A} \circ \mathcal{G}_n \circ \mathcal{A} \circ z(X) \to z(Y).
\]

If \( \hat{\tau} \) commutes with the differentials, then we can conclude.

Let \( \Delta \) denote the reduced, levelwise comultiplication on \( \mathcal{D} \). It follows from the formula in Remark 5.10 for \( d \Phi \) that the differential \( \tilde{d}_\Phi \) in \( \Phi(\mathcal{D}) \circ z(X) \) is specified as follows. For all \( D \in \text{Ob} \mathcal{D} \), \( 1 \leq m \leq n \), \( q \in \mathcal{D}(m) \), \( \bar{n} \in I_{n,m} \), and \( x_i \in X(D) \) for \( 1 \leq i \leq n \),

\[
\begin{align*}
\tilde{d}_\Phi(s^{m-1} q \otimes \alpha_{\bar{n}}^r \otimes (x_1 \otimes \cdots \otimes x_n)) &= \pm s^{m-1} dq \otimes \alpha_{\bar{n}}^r \otimes (x_1 \otimes \cdots \otimes x_n) \\
+ s^{m-1} q \otimes \alpha_{\bar{n}}^r \otimes \sum_{1 \leq k \leq n} \pm(x_1 \otimes \cdots \otimes dx_k \otimes \cdots \otimes x_n) \\
+ \delta_2 \otimes \sum_{\bar{r}'+\bar{r}''=\bar{n}} \pm(s^{m-1} q_i \otimes \alpha_{\bar{r}'}^r \otimes (x_1 \otimes \cdots \otimes x_{n'})) \otimes (s^{m-1} q^i \otimes \alpha_{\bar{r}''}^r \otimes (x_{n'+1} \otimes \cdots \otimes x_n)) \\
+ s^{m-1} q \otimes \sum_{0 < k < n} \pm \alpha_{\bar{n}}^r \otimes (x_1 \otimes \cdots \otimes x_k x_{k+1} \otimes \cdots \otimes x_n),
\end{align*}
\]

where the signs are determined by the Koszul rule, \( d \) denotes the internal differential in \( \mathcal{D} \) and in \( X(D) \), and \( \Delta(q) = q_i \otimes q^i \) (using Einstein summation notation).
To prepare the proof of the existence of \( \hat{\tau} \), we define a natural, increasing bilfiltration of the symmetric sequence \( \mathfrak{S}_\mathcal{D} \circ \mathcal{A} \circ z(X(D)) \), for every object \( D \) of \( \mathbf{D} \). Let

\[
F^{s,t}(D) = \bigoplus_{k \leq s} (\mathfrak{S}_\mathcal{D} \circ \mathcal{A} \circ \mathcal{P})(k) \otimes X(D)^{\otimes k} \oplus ((\mathfrak{S}_\mathcal{D} \circ \mathcal{A} \circ \mathcal{P})(s) \otimes X(D)^{\otimes s})_{\leq s+t}
\]

and

\[
F^s(D) = \bigoplus_{k \leq s} (\mathfrak{S}_\mathcal{D} \circ \mathcal{A} \circ \mathcal{P})(k) \otimes X(D)^{\otimes k}.
\]

Observe that

\[
\hat{d}_q F^{s,t}(D) \subset \mathcal{A} \circ F^{s,t-1}(D) \forall s,t,
\]

\[
F^s(D) = \bigcup_{t \geq 0} F^{s,t}(D) = F^{s+1,t}(D) \forall t < s,
\]

and

\[
\mathfrak{S}_\mathcal{D} \circ \mathcal{A} \circ z(X(D)) = \bigcup_{s \geq 0} F^s(D).
\]

Moreover, since \( \mathfrak{S}_\mathcal{D}(1) = R \),

\[
F^1(D) = X(D)
\]

for all \( D \in \text{Ob} \, \mathbf{D} \).

We now prove the existence of \( \hat{\tau} \) by induction on \( s \) and \( t \). We start by setting \( \hat{\tau}_D \) equal to \( \tau_D \) on \( F^1(D) \), for all \( D \in \text{Ob} \, \mathbf{D} \).

Suppose now that for some \( s,t \), a morphism of symmetric sequences of graded modules \( \hat{\tau}_D : F^{s,t}(D) \to Y(D) \) has been defined naturally for all \( D \in \text{Ob} \, \mathbf{D} \), so that its extension to a morphism of left \( \mathcal{A} \)-modules is a differential map and so that its restriction to \( X(D) \) is exactly \( \tau_D \). We now show that \( \hat{\tau}_D \) can be naturally extended to \( F^{s,t+1}(D) \) for all \( D \in \text{Ob} \, \mathbf{D} \).

Note that for all \( q \in \mathfrak{S}_\mathcal{D}(k) \) and all \( \bar{s} \in I_{s,k} \),

\[
|s^{k-1} q \otimes \alpha_{\bar{s}}^t| = |q| + s - 1,
\]

since \( \mathfrak{S} \circ \mathcal{A} \circ \mathcal{P} \) is concentrated in degree 0 in every arity.

For all \( k \geq 1, r \geq 0 \) and \( D \in \text{Ob} \, \mathbf{D} \), let

\[
G_{k,r}(D) = \{ s^{k-1} q \otimes \alpha_{\bar{s}}^t \otimes (z_1 \otimes \cdots \otimes z_s) \in F^{s,t+1}(D) \mid q \in \mathfrak{S}_\mathcal{D}(k) \}_{r'} \}.
\]

Note that \( G_{k,0}(D) = \{ 0 \} \), and a natural extension of \( \hat{\tau}_D \) over \( G_{k,0}(D) \) therefore exists trivially.

For each \( k \), suppose that there is some \( r_k \geq 0 \) such that \( \hat{\tau}_D \) has been extended from \( F^{s,t}(D) \) naturally over \( G_{k,r_k}(D) \). We now prove that \( \hat{\tau}_D \) can then be extended over \( G_{k,r_k+1}(D) \), for all \( k \).

Let \( m_1, \ldots, m_s \in \mathfrak{M} \) be models with corresponding generators \( x_i \in X(m_i) \) for \( 1 \leq i \leq s \) such that

\[
\sum_{i=1}^s |x_i| = t - r_k - s + 2.
\]
Let $m = m_1 \coprod \cdots \coprod m_s$, and let $Bx_i$ denote the image of $x_i$ under the morphism $X(m_i) \to X(m)$ induced by the natural map $m_i \to m$.

Let $q \in \mathcal{O}(k)_{r_k}$, and $s_i \in I_{s,k}$. Since

$$|s^{k-1}q \otimes \alpha^k_s \otimes (Bx_1 \otimes \cdots \otimes Bx_s)| = t + 1,$$

it follows that

$$s^{k-1}q \otimes \alpha^k_s \otimes (Bx_1 \otimes \cdots \otimes Bx_s) \in F^{s,t+1}(m).$$

The formula for $\tilde{d}\Phi$ and the minimality of $r_k$ together imply then that

$$\tilde{d}\Phi(s^{k-1}q \otimes \alpha^k_s \otimes (Bx_1 \otimes \cdots \otimes Bx_s)) \in F^{s,t}(m) + G_{k,r_k},$$

and therefore, by the induction hypothesis,

$$\hat{\tau}_m(\tilde{d}\Phi(s^{k-1}q \otimes \alpha^k_s \otimes (Bx_1 \otimes \cdots \otimes Bx_s)))$$

is a well defined element of $Y(m)$, which, moreover, must be a cycle. Since $Y(m)$ is acyclic, there exists $y \in Y(m)$ such that

$$dy = \hat{\tau}_m(\tilde{d}\Phi(s^{k-1}q \otimes \alpha^k_s \otimes (Bx_1 \otimes \cdots \otimes Bx_s))).$$

We set

$$\hat{\tau}_m(s^{k-1}q \otimes \alpha^k_s \otimes (Bx_1 \otimes \cdots \otimes Bx_s)) = y.$$

Let $D \in Ob D$. We can now extend $\hat{\tau}_D$ naturally over $G_{k,r_k}(D)$ as follows.

Let $q \in \mathcal{O}(k)_{r_k}$. For any $D \in Ob D$ and any $z_1, \ldots, z_s \in X(D)$ such that

$$|s^{k-1}q \otimes \alpha^k_s \otimes (z_1 \otimes \cdots \otimes z_s)| = t + 1,$$

let $\xi_i : m_i \to D$ denote the representing morphism for $z_i$, for $1 \leq i \leq s$, i.e.,

$$X(\xi_i)(x_i) = z_i \forall 1 \leq i \leq s.$$ 

Let $m = m_1 \coprod \cdots \coprod m_s$, and $\xi = \xi_1 + \cdots + \xi_s : m \to D$. Set

$$\hat{\tau}_D(s^{k-1}q \otimes \alpha^k_s \otimes (z_1 \otimes \cdots \otimes z_s)) = Y(\xi)(\hat{\tau}_m(s^{k-1}q \otimes \alpha^k_s \otimes (Bx_1 \otimes \cdots \otimes Bx_s))).$$

It is clear that, thus defined, $\hat{\tau}_D$ is natural and commutes with the differential.

References

[1] C. Berger, I. Moerdijk, Axiomatic homotopy theory for operads, Comment. Math. Helv. 78 (2003) 805–831.

[2] C. Berger, I. Moerdijk, Resolution of coloured operads and rectification of homotopy algebras, in: Categories in algebra, geometry and mathematical physics, volume 431 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2007, pp. 31–58.
[3] M. Boyle, An Algebraic Model for the Homology of Pointed Mapping Spaces out of a Closed Surface, Ph.D. thesis, University of Aberdeen, UK, 2009.

[4] C. Chevalley, S. Eilenberg, Cohomology theory of Lie groups and Lie algebras, Trans. Amer. Math. Soc. 63 (1948) 85–124.

[5] T.F. Fox, M. Markl, Distributive laws, bialgebras, and cohomology, in: Operads: Proceedings of Renaissance Conferences (Hartford, CT/Luminy, 1995), volume 202 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1997, pp. 167–205.

[6] B. Fresse, On the homotopy of simplicial algebras over an operad, Trans. Amer. Math. Soc. 352 (2000) 4113–4141.

[7] B. Fresse, Koszul duality of operads and homology of partition posets, in: Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, volume 346 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2004, pp. 115–215.

[8] E. Getzler, J. Jones, Operads, homotopy algebra, and iterated integrals for double loop spaces, 1994. arxiv:hep-th/9403055.

[9] V. Ginzburg, M. Kapranov, Koszul duality for operads, Duke Math. J. 76 (1994) 203–272.

[10] V.K.A.M. Gugenheim, H.J. Munkholm, On the extended functoriality of Tor and Cotor, J. Pure Appl. Algebra 4 (1974) 9–29.

[11] K. Hess, S. Lack, On twisting structures, 2010. In preparation.

[12] K. Hess, R. Levi, An algebraic model for the loop space homology of a homotopy fiber, Algebr. Geom. Topol. 7 (2007) 1699–1765.

[13] K. Hess, P.E. Parent, J. Scott, Co-rings over operads characterize morphisms, 2005. arxiv:math.AT/0505559.

[14] K. Hess, P.E. Parent, J. Scott, A chain coalgebra model for the James map, Homology, Homotopy Appl. 9 (2007) 209–231.

[15] K. Hess, P.E. Parent, J. Scott, CoHochschild homology of chain coalgebras, J. Pure Appl. Algebra 213 (2009) 536–556.

[16] K. Hess, P.E. Parent, J. Scott, A. Tonks, A canonical enriched Adams-Hilton model for simplicial sets, Adv. Math. 207 (2006) 847–875.

[17] K. Hess, J. Rognes, Power maps in algebra and topology, 2010. In preparation.

[18] N. Iwase, M. Mimura, Higher homotopy associativity, in: Algebraic topology (Arcata, CA, 1986), volume 1370 of Lecture Notes in Math., Springer, Berlin, 1989, pp. 193–220.
[19] P.T. Johnstone, Sketches of an elephant: a topos theory compendium. Vol. 1, volume 43 of Oxford Logic Guides, The Clarendon Press Oxford University Press, New York, 2002.

[20] A. Joyal, Une théorie combinatoire des séries formelles, Adv. in Math. 42 (1981) 1–82.

[21] T. Leinster, Homotopy algebras for operads, 2000. arxiv:math/0002180v1 [math.QA].

[22] M. Markl, Homotopy diagrams of algebras, Rend. Circ. Mat. Palermo (2) Suppl., in: Proceedings of the 21st Winter School “Geometry and Physics” (Srní, 2001), 69, pp. 161–180.

[23] M. Markl, Homotopy algebras are homotopy algebras, Forum Math. 16 (2004) 129–160.

[24] J.P. May, A general algebraic approach to Steenrod operations, in: The Steenrod Algebra and its Applications (Proc. Conf. to Celebrate N. E. Steenrod’s Sixtieth Birthday, Battelle Memorial Inst., Columbus, Ohio, 1970), Lecture Notes in Mathematics, Vol. 168, Springer, Berlin, 1970, pp. 153–231.

[25] T. Naito, Coalgèbres d’Alexander-Whitney: un modèle algébrique pour les espaces topologiques, Ph.D. thesis, École Polytechnique Fédérale de Lausanne, Switzerland, 2009.

[26] B. Ndombol, J.C. Thomas, Steenrod operations in shc-algebras, J. Pure Appl. Algebra 192 (2004) 239–264.

[27] C.W. Rezk, Spaces of Algebra Structures and Cohomology of Operads, Ph.D. thesis, Massachusetts Institute of Technology, 1996.

[28] V.A. Smirnov, On the cochain complex of topological spaces, Mat. Sb. (N.S.) 115(157) (1981) 146–158, 160.

[29] J.D. Stasheff, Homotopy associativity of H-spaces. I, II, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid. 108 (1963) 293–312.