Spin(7)-manifolds with three-torus symmetry

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Abstract

Metrics of exceptional holonomy are vacuum solutions to the Einstein equation. In this paper we describe manifolds with holonomy contained in Spin(7) preserved by a three-torus symmetry in terms of tri-symplectic geometry of four-manifolds. These complement examples that have appeared in the context of domain wall problems in supergravity.

1 Introduction

Metrics of exceptional holonomy have received much attention from both mathematicians and physicists over the years. The mathematical motivation for studying exceptional holonomy metrics began with Berger’s classification of Riemannian holonomy groups [3], though their existence was first shown much later in Bryant’s paper [5]. Significant results then followed, in particular it is worth mentioning the complete exceptional holonomy metrics discovered by Bryant and Salamon [7] and Joyce’s construction [18, 17] of compact Riemannian manifolds with holonomy G2 and Spin(7). In this paper we focus on holonomy Spin(7)-metrics. From the physical perspective one motivation for studying these metrics comes from superstring theories [1, 8, 9, 14, 24]. Recently, we [21] used the notion of multi-moment map to study torsion-free G2-metrics admitting an isometric action of T2. In this paper we use similar ideas to study holonomy Spin(7)-metrics with T3 symmetry; symmetry groups of rank three fit well with Reidegeld’s study [23] of cohomogeneity one Spin(7)-metrics.

The paper is organised as follows. In section 2 we briefly explain the notion of multi-moment maps for geometries with a closed four-form and T3 symmetry. We then study how to reduce a torsion-free Spin(7)-manifold to a tri-symplectic four-manifold and thereafter, in section 3 explain how to obtain all torsion-free Spin(7)-manifolds with free T3 symmetry starting

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from tri-symplectic four-manifolds. In the final section of the paper we present two examples illustrating our reduction and reconstruction procedures. One of the examples complements previous ones that have appeared in the context of domain wall problems in supergravity theories [12, 13].

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2 Reduction via multi-moment maps

In [21] we developed a notion of multi-moment map for geometries with a closed three-form. We explain in [20] how this idea generalises to higher degree forms. For a manifold $Y$ endowed with a closed four-form $\Phi$ and an action of a three-torus preserving $\Phi$ the definition is particularly simple. A multi-moment map, for $T^3$ acting on $(Y, \Phi)$, is an invariant function $\nu: Y \to \mathbb{R}$ such that

$$d\nu = \Phi(U_1, U_2, U_3, \cdot),$$ (2.1)

where the vector fields $U_1, U_2$ and $U_3$ generate the $T^3$ action. Following [21, Theorem 3.1(i)] one finds that such a multi-moment map is guaranteed to exist provided that $\check{b}_1(Y) = 0$.

Remark 2.1. Note that closedness of the one-form $U_1 \cdot U_2 \cdot U_3 \cdot \Phi$ follows by applying Cartan’s formula:

$$0 = L_{U_1}\Phi = d(U_1 \cdot \Phi), \quad 0 = L_{U_2}(U_1 \cdot \Phi) = d(U_2 \cdot U_1 \cdot \Phi), \quad 0 = L_{U_3}(U_2 \cdot U_1 \cdot \Phi) = d(U_3 \cdot U_2 \cdot U_1 \cdot \Phi).$$ △

Let us now recall the fundamental aspects of $Spin(7)$-geometry following [5]. On $\mathbb{R}^8$ we consider the four-form $\Phi_0$ given by

$$\Phi_0 = e_{1234} + (e_{12} + e_{34})(e_{56} + e_{78}) + (e_{13} - e_{24})(e_{57} - e_{68}) - (e_{14} + e_{23})(e_{38} + e_{67}) + e_{5678},$$ (2.2)

where $e_1, \ldots, e_8$ is the standard dual basis and wedge signs have been omitted. The stabiliser of $\Phi_0$ is the compact 21-dimensional Lie group

$$Spin(7) = \{ g \in GL(8, \mathbb{R}) : g^*\Phi_0 = \Phi_0 \}.$$

This group preserves the standard metric $g_0 = \sum_{i=1}^8 e_i^2$ on $\mathbb{R}^8$ and the volume form $vol_0 = e_{12345678}$. These tensors are uniquely determined by $\Phi_0$ via the relations $14 vol_0 = \Phi_0^2$ and $(Y \wedge X, \Phi_0) \wedge (Y \wedge X, \Phi_0) = \Phi_0 = 6 ||X \wedge Y||^2 vol_0$, cf. [19]. The form $\Phi_0$ is self-dual, meaning $\star \Phi_0 = \Phi_0$.

A $Spin(7)$-structure on an eight-manifold $Y$ is given by a four-form $\Phi \in \Omega^4(Y)$ which is linearly equivalent at each point to $\Phi_0$. It determines a
To relate these to the $\text{Spin}(7)$-structure is called \textit{torsion-free} if the form $\Phi$ is parallel with respect to the Levi-Civita connection, meaning $\nabla^\text{LC} \Phi = 0$. This happens precisely when $\Phi$ is closed. One then calls $(Y, \Phi)$ a torsion-free $\text{Spin}(7)$-manifold. In this situation the metric $g$ has holonomy contained in $\text{Spin}(7)$ and is Ricci-flat. In particular, $g$ is real-analytic in harmonic coordinates.

Since a torsion-free $\text{Spin}(7)$-manifold comes equipped with a closed four-form, we may study multi-moment maps for such manifolds. Assume that $(Y, \Phi)$ has a three-torus symmetry, generated by vector fields $U_i$, necessarily real-analytic \cite[Theorem 2.3]{16}, and that there is a non-constant $U$ on some open set $\Theta$ forms $\sum \Phi$. Then $d\nu = \Phi(U_1, U_2, U_3, \cdot)$ is non-zero if and only if $U_1$, $U_2$ and $U_3$ are linearly independent, cf. \cite{11}. So $T^3$ acts locally freely on some open set $Y_0 \subset Y$.

Let us define three two-forms on $Y_0$ by

$$\omega_1 = U_2 \wedge U_3 \wedge \Phi,$$  
$$\omega_2 = U_3 \wedge U_1 \wedge \Phi,$$  
$$\omega_3 = U_1 \wedge U_2 \wedge \Phi.$$  

To relate these to the $\text{Spin}(7)$-structure we introduce two $\mathbb{R}^3$-valued one-forms $\theta = (\theta_1, \theta_2, \theta_3)$ and $\Theta = (\Theta_1, \Theta_2, \Theta_3)$. The one-form $\theta$ is defined by the formula $\theta = U^i G^{-1}$, where $U^i$ has entries $U^i_j = g(U_i, \cdot)$, and $G^{-1} = (g^{ij})$ denotes the inverse of the matrix $G = (g_{ij})$ has entries $g_{ij} = g(U_i, U_j)$. Note that $\theta_i(U_j) = \delta_{ij}$. The second $\mathbb{R}^3$-valued one-form is given by the formula $\Theta = h^2 U^i$, where $h$ is the positive real-analytic function $h = \sqrt{\det(G^{-1})}$; componentwise we have $\Theta_i = h^2 \sum_{j=1}^{3} g_{ij} \theta_j$.

**Proposition 2.2.** On $Y_0$, the four-form $\Phi$ is

$$\Phi = d\nu \wedge (2 \theta_2 \wedge \theta_3 \wedge \theta_1 + \Theta_1 \wedge \omega_1 + \Theta_2 \wedge \omega_2 + \Theta_3 \wedge \omega_3)$$  
$$+ \theta_3 \wedge \theta_2 \wedge \omega_1 + \theta_1 \wedge \theta_3 \wedge \omega_2 + \theta_2 \wedge \theta_1 \wedge \omega_3 + *(d\nu \wedge \theta_3 \wedge \theta_2 \wedge \theta_1),$$  

(2.3)

**Proof.** Working locally at a point and using the $T^3$-action we may write the first three standard basis elements of $\mathbb{R}^3$ as $E_1 = k_1 U_1$, $E_2 = k_2 U_1 + \ell_2 U_2$, $E_3 = k_3 U_1 + \ell_3 U_2 + m_3 U_3$ for appropriate functions $k_1, \ldots, m_3$. Now, using (2.2), we get $k_1 \ell_2 \omega_3 = -e_{34} - e_{56} - e_{78}$, $k_1 m_3 \omega_2 - k_1 \ell_3 \omega_3 = -e_{24} + e_{57} - e_{68}$ and $-\ell_2 m_3 \omega_1 + k_2 m_3 \omega_2 + (\ell_2 k_3 - k_2 \ell_3) \omega_3 = e_{14} - e_{58} - e_{67}$. We therefore have

$$\ell_2 m_3 \omega_1 = -e_{14} + e_{58} + e_{67} - \frac{k}{h_1}(e_{24} - e_{57} + e_{68}) - \frac{k}{h_1}(e_{34} + e_{56} + e_{78})$$  
$$k_1 m_3 \omega_2 = -e_{24} + e_{57} - e_{68} - \frac{k}{h_1}(e_{34} + e_{56} + e_{78})$$  
$$k_1 \ell_2 \omega_3 = -e_{34} - e_{56} - e_{78}.$$  

Next, we write $\theta_1 = k_1 e_1 + k_2 e_2 + k_3 e_3$, $\theta_2 = \ell_2 e_2 + \ell_3 e_3$ and $\theta_3 = m_3 e_3$. Also
note that \( h\,dv = e_4 \). We then find

\[
e_{1234} = dv \wedge \theta_3 \wedge \theta_2 \wedge \theta_1, \quad e_{5678} = \ast(dv \wedge \theta_3 \wedge \theta_2 \wedge \theta_1),
\]

\[
\theta_3 \wedge \theta_2 \wedge \omega_1 = e_{1234} - e_{23}(e_{58} + e_{67}) - \frac{k_2}{k_1}e_{23}(e_{57} - e_{68}) + \frac{k_3}{k_1}e_{23}(e_{56} + e_{78}),
\]

\[
\theta_1 \wedge \theta_3 \wedge \omega_2 = e_{1234} + e_{13}(e_{57} - e_{68}) - \frac{k_2}{k_1}e_{13}(e_{56} + e_{78})
\]

\[
+ \frac{k_3}{k_1}e_{23}(e_{57} - e_{68}) - \frac{k_3}{k_1}e_{23}(e_{56} + e_{78}),
\]

\[
\theta_2 \wedge \theta_1 \wedge \omega_3 = e_{1234} + e_{12}(e_{56} + e_{78}) - \frac{k_3}{k_1}e_{23}(e_{56} + e_{78})
\]

\[
+ \frac{k_3}{k_1}e_{23}(e_{56} + e_{78}),
\]

\[
dv \wedge (\Theta_1 \wedge \omega_1 + \Theta_2 \wedge \omega_2 + \Theta_3 \wedge \omega_3) = -e_{14}(e_{58} + e_{67}) - e_{24}(e_{57} - e_{68})
\]

\[
+ e_{34}(e_{56} + e_{78}),
\]

and the given expression for \( \Phi \) follows. \( \square \)

**Remark 2.3.** The functions \( k_1, \ldots, m_3 \) from the proof of Proposition 2.2 are related to \( G \) in the following way

\[
G = \begin{pmatrix}
\frac{1}{k_1} & -\frac{k_2}{k_1^2} & \frac{k_2 k_3 - k_3 k_2}{k_1^2 m_3} \\
-\frac{k_2}{k_1^2} & \frac{k_2^2}{k_1^2} + \frac{1}{k_2^2} & \frac{k_2 k_3 (k_2 - k_3)}{k_1^2 m_3} - \frac{\ell_2}{\ell_2^2 m_3} \\
\frac{k_2 k_3 - k_3 k_2}{k_1^2 m_3} & \frac{k_2 k_3 (k_2 - k_3)}{k_1^2 m_3} + \frac{\ell_2}{\ell_2^2 m_3} & \frac{1}{m_3}
\end{pmatrix},
\]

and for \( G^{-1} = (g^{ij}) \) we have

\[
G^{-1} = \begin{pmatrix}
\frac{k_1}{k_2^2} + \frac{k_3}{k_2^2} & \frac{k_2 k_3}{k_2^2} & \frac{k_3}{k_2^2} \\
\frac{k_2 + k_3}{k_2} & \frac{k_2}{k_2} & \frac{k_3}{k_2} \\
\frac{k_3}{k_2} & \frac{k_3}{k_2} & \frac{1}{k_2}
\end{pmatrix}.
\tag{2.4}
\]

Now suppose that \( t \in v(Y_0) \) is a regular value for \( v: Y_0 \to \mathbb{R} \). Then \( \mathcal{X}_t = v^{-1}(t) \) is a real-analytic hypersurface and has unit normal \( N = h(dv)^2 \). We shall denote by \( i \) the inclusion \( \mathcal{X}_t \hookrightarrow Y_0 \).

**Definition 2.4.** The \( T^3 \) reduction of \( Y_0 \) at level \( t \) is the four-manifold

\[
M = v^{-1}(t)/T^3 = \mathcal{X}_t/T^3.
\]

This quotient space is a tri-symplectic manifold.

**Proposition 2.5.** The \( T^3 \) reduction \( M \) carries three pointwise linearly independent symplectic forms defining the same orientation.

**Proof.** Consider the real-analytic two-forms \( \omega_1, \omega_2 \) and \( \omega_3 \) on \( Y_0 \). These forms are \( T^3 \)-invariant and closed since for instance \( \mathcal{L}_{U_1} \omega_1 = \mathcal{L}_{U_1} (U_{2,3} \Phi) = 0 \) and \( d\omega_1 = d(U_{2,3} \Phi) = \mathcal{L}_{U_2} (U_{3,3} \Phi) = 0 \), respectively. Furthermore,
as \( U_1 \omega_1 = -dv \), etc., their pull-backs to \( \lambda_i = r^{-1}(t) \) are basic. Thus they descend to three closed forms \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) on \( M \).

The proof of Proposition 2.2 shows that at a point \( k_1 \ell_3 m_3 \sigma_1 = k_1(e_{86} + e_{67}) + k_2(e_{57} - e_{68}) - k_3(e_{56} + e_{78}), k_1 \ell_3 m_3 \sigma_2 = \ell_2(e_{57} - e_{68}) - \ell_3(e_{56} + e_{78}) \) and \( k_1 \ell_3 m_3 \sigma_3 = -m_3(e_{56} + e_{78}) \). Consequently, \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are non-degenerate symplectic forms defining the same orientation. \( \square \)

The symplectic triple \( (\sigma_1, \sigma_2, \sigma_3) \) on \( M \) defines a matrix \( Q = (q_{ij}) \) given by \( \sigma_i \wedge \sigma_j = 2q_{ij} \text{vol}_M \), where \( \text{vol}_M \) is the induced volume form on \( M \).

**Proposition 2.6.** The matrices \( G \) and \( Q \) are related via \( G^{-1} = h^2 Q \). In particular, \( \text{vol}_M = \frac{k^3}{6} \sum_{i,j=1}^3 g_{ij} \sigma_i \wedge \sigma_j \). Moreover, for any positive smooth function \( \lambda \) on \( M \), the redefinitions \( \tilde{Q} = \lambda^2 Q, \tilde{G} = \lambda G, \tilde{h}^2 = \det(\tilde{G}^{-1}) \) retain the relation \( G^{-1} = \tilde{h}^2 \tilde{Q} \).

**Proof.** Working locally at a point and using the \( T^3 \)-action, as in the proof of Proposition 2.2, we have

\[
\sigma_1 \wedge \sigma_2 = \frac{2k_1 \ell_3 + k_3 \ell_3}{h^2} \text{vol}_M, \quad \sigma_1 \wedge \sigma_3 = \frac{2k_1 m_3}{h^2} \text{vol}_M, \quad \sigma_2 \wedge \sigma_3 = \frac{2 \ell_3 m_3}{h^2} \text{vol}_M, \quad \sigma_1 \wedge \sigma_2 \wedge \sigma_3 = 2 \text{vol}.
\]

where \( \text{vol}_M = e_{5678} \) is induced volume form on \( M \). The relation between \( Q \) and \( G^{-1} \) now follows directly from the expression (2.4), and it immediately implies the last two assertions of the proposition. \( \square \)

As we shall see below, the above behaviour of \( G \) and \( Q \) with respect to rescaling plays a subtle role in the description of induced geometry on the hypersurface \( \lambda_i \).

It is well-known, cf. [22], that any orientable hypersurface in a \( \text{Spin}(7) \)-manifold carries an induced \( G_2 \)-structure. To express the \( G_2 \)-structure \( \phi = N \Phi \) on \( \lambda_i \) it is useful to rewrite \( \Phi \) in a way that abuses notation slightly, namely using the forms defined on \( M \).

\[
\Phi = dv \wedge (\theta_3 \wedge \theta_2 \wedge \theta_1 + \Theta_1 \wedge \sigma_1 + \Theta_2 \wedge \sigma_2 + \Theta_3 \wedge \sigma_3) + \theta_3 \wedge \theta_2 \wedge \sigma_1 + \theta_1 \wedge \theta_3 \wedge \sigma_2 + \theta_2 \wedge \theta_1 \wedge \sigma_3 + \text{vol}_M. \tag{2.5}
\]

From (2.5) we see that

\[
h\Phi = \theta_3 \wedge \theta_2 \wedge \theta_1 + \Theta_1 \wedge \sigma_1 + \Theta_2 \wedge \sigma_2 + \Theta_3 \wedge \sigma_3. \tag{2.6}
\]

Alternatively we may, up to orientation, specify the \( G_2 \)-structure by the four-form \( \psi = i^* \Phi = \ast \Phi \):

\[
\psi = \theta_3 \wedge \theta_2 \wedge \sigma_1 + \theta_1 \wedge \theta_3 \wedge \sigma_2 + \theta_2 \wedge \theta_1 \wedge \sigma_3 + \text{vol}_M.
\]
As the Spin(7)-structure is torsion-free, the induced real-analytic $G_2$-structure on $X_1$ is cosymplectic, meaning $d\psi = 0$.

It turns out that there is a family of smooth cosymplectic $G_2$-structures on $X_1$ obtained by scaling of the volume form on $M$:

**Proposition 2.7.** Let $(\phi, \psi)$ be the $G_2$-structure on $X_1$ described above. For any positive smooth function $\lambda$ on $M$, the changes $\lambda^2 \mathcal{Q} =: \tilde{\mathcal{Q}}$ and $\lambda G =: \tilde{G}$ of $\mathcal{Q}$ and $G$, respectively, give a new cosymplectic $G_2$-structure $(\tilde{\phi}, \tilde{\psi})$ on $X_1$:

\[
\tilde{h}\phi = \theta_3 \wedge \theta_2 \wedge \theta_1 + \tilde{\Theta}_1 \wedge \sigma_1 + \tilde{\Theta}_2 \wedge \sigma_2 + \tilde{\Theta}_3 \wedge \sigma_3, \quad (2.7)
\]

\[
\tilde{\psi} = \theta_3 \wedge \theta_2 \wedge \sigma_1 + \theta_1 \wedge \theta_3 \wedge \sigma_2 + \theta_2 \wedge \theta_1 \wedge \sigma_3 + \tilde{\text{vol}}_M, \quad (2.8)
\]

where $\tilde{h} = \det(\tilde{\mathcal{Q}})^{-\frac{1}{4}} = \lambda^{-\frac{3}{2}} h$, $\tilde{\Theta}_i = \sum_{j=1}^{3} \tilde{q}_{ij} \Theta_j = \lambda^{-2} \Theta_i$, $\tilde{\text{vol}}_M = \frac{1}{8} \sum_{i,j=1}^{3} \tilde{q}_{ij} \sigma_i \wedge \sigma_j = \lambda^{-2} \text{vol}_M$.

**Proof.** Working locally at a point, as in the proof of Proposition 2.2 we have the basis $(\hat{e}_1, \ldots, \hat{e}_4, \ldots, \hat{e}_8)$ for $T^*X_1$. We now define a new basis $(\hat{f}_1, \ldots, \hat{f}_3, \ldots, \hat{f}_8)$ for $T^*X_1$ by letting $f_i := \sqrt{\lambda} \hat{e}_i$, for $i = 1, 2, 3$, and $\hat{f}_i := \frac{1}{\sqrt{\lambda}} \hat{e}_i$, for $i = 5, \ldots, 8$. Writing $\phi$ and $\psi$ in terms of $f_i$ we have that

\[
\tilde{\phi} = -f_{123} - f_3(f_{56} + f_{78}) + f_2(f_{57} - f_{68}) + f_1(f_{58} + f_{67}),
\]

\[
\tilde{\psi} = f_{12}(f_{56} + f_{78}) + f_{13}(f_{57} - f_{68}) - f_{23}(f_{58} + f_{67}) + f_{5678},
\]

which shows that $\tilde{\phi}$ and $\tilde{\psi}$ define a $G_2$-structure with volume form $\tilde{\text{vol}}_X = \frac{1}{\sqrt{\lambda}} \text{vol}_X$. Clearly, $\tilde{\psi}$ is closed. Hence the new $G_2$-structure is also cosymplectic. \qed

### 3 Inversion via a flow

We now consider how the reduction procedure from the previous section may be inverted, constructing a Spin(7)-metric starting from a triple of symplectic forms on a four-manifold $M$. First we need a weakening of the notion of coherent symplectic triple [21] Definition 6.4].

**Definition 3.1.** A weakly coherent symplectic triple $\mathcal{C}$ on a four-manifold $M$ consists of three symplectic forms $\sigma_1, \sigma_2, \sigma_3$ that pointwise span a maximal positive subspace of $\Lambda^2 T^*M$.

As in [10], the positive three-dimensional subbundle $\Lambda^+ = \langle \sigma_1, \sigma_2, \sigma_3 \rangle \subset \Lambda^2 T^*M$ corresponds to a unique oriented conformal structure on $M$. Fix a volume form $\text{vol}_M$ on $M$ compatible with the orientation and define a $3 \times 3$-matrix $Q = (q_{ij})$ by $\sigma_i \wedge \sigma_j = 2q_{ij} \text{vol}_M$, for $i, j = 1, 2, 3$. Subsequently, denote by $h$ the positive smooth function satisfying $h^{-4} = \det(Q)$. We now
consider a $T^3$-bundle $\pi_M: \mathcal{X} \to M$ endowed with connection one-form $\theta = (\theta_1, \theta_2, \theta_3) \in \Omega^1(\mathcal{X}, \mathbb{R}^3)$. We define three one-forms $\Theta_i$ for $i = 1, 2, 3$, by the formula $\Theta_i = \sum_{j=1}^{3} q_{ij} \theta_j$. Finally, denote the curvature by $F = \pi_M^*(d\theta) \in \Omega^2(M, \mathbb{R}^3)$. With these definitions in mind we have:

**Proposition 3.2.** Let $(M', \rho')$ be a weakly coherent tri-symplectic four-manifold. Suppose that $\mathcal{X}$ is a principal $T^3$-bundle over $M$ with connection one-form $\theta = (\theta_1, \theta_2, \theta_3)$ and curvature $F$. Define a three-form $\phi$ and a four-form $\psi$ by

$$h\phi = \theta_3 \wedge \theta_2 \wedge \theta_1 + \Theta_1 \wedge \sigma_1 + \Theta_2 \wedge \sigma_2 + \Theta_3 \wedge \sigma_3,$$

$$\psi = \theta_3 \wedge \theta_2 \wedge \sigma_1 + \theta_1 \wedge \theta_3 \wedge \sigma_2 + \theta_2 \wedge \theta_1 \wedge \sigma_3 + \text{vol}_M.$$

(3.1)

Then $\phi$ determines a $G_2$-structure on $\mathcal{X}$ satisfying $\ast \phi = \psi$.

Let $A = (a_{ij})$ be the $3 \times 3$-matrix defined pointwise by the projection $F^+ = (\sigma_1, \sigma_2, \sigma_3)A$. Then the $G_2$-structure $\phi$ is cosymplectic if and only if the matrix $QA$ is symmetric:

$$QA = A^tQ$$

(3.2)

**Proof.** Write the entries of $G^{-1} := h^2 Q$ as in (2.4) and then express the functions $k_1, \ldots, k_3$ in terms of the entries $g_{ij}$ of $G^{-1} = h^2 Q$. Next, choose a conformal basis $e_5, e_6, e_7, e_8$ of $T^*M$ so that $h\omega_i$ are as in the proof of Proposition 2.2 and then write $\theta_i = k_1 e_1 + k_2 e_2 + k_3 e_3$, $\theta_2 = \ell_2 e_2 + \ell_3 e_3$, $\theta_3 = m_3 e_3$. It now follows, using Proposition 2.7 that the basis $(e_1, \ldots, e_4, \ldots, e_8)$ is a $G_2$-basis for $T^*\mathcal{X}$ with defining form $\phi$ given via (3.1).

For the final assertion we need to study the condition $d\psi = 0$. The equation $d\psi = 0$ holds if and only if one has

$$d\theta_1 \wedge \sigma_2 - d\theta_2 \wedge \sigma_1 - d\theta_1 \wedge \sigma_3 = d\theta_2 \wedge \sigma_3 - d\theta_3 \wedge \sigma_2 = 0.$$

A calculation shows that these relations correspond to the three equations

$$-a_{13}q_{12} + a_{12}q_{13} - a_{23}q_{22} + (a_{22} - a_{33})q_{23} + a_{32}q_{33} = 0,$$

$$a_{13}q_{11} + a_{23}q_{12} + (a_{33} - a_{11})q_{13} - a_{21}q_{23} - a_{31}q_{33} = 0,$$

$$-a_{12}q_{11} + (a_{11} - a_{22})q_{12} - a_{32}q_{13} + a_{21}q_{22} + a_{31}q_{32} = 0,$$

(3.3)

and these are equivalent to the condition (3.2). □

**Remark 3.3.** Condition (3.2) on $F$ is independent of the choice of orientation compatible volume form on $M$. Though the bilinear form on $\Lambda^3 T^* M$, given by wedging, is only well-defined after choosing a representative volume form, self-adjointness of the projection $F^+ \in \Lambda^+ \subset \Lambda^2 T^* M$ does not depend on the specific choice.

Provided the assumptions of Proposition 3.2 hold, we therefore obtain a family of cosymplectic $G_2$-manifolds. This is a consequence of Proposition 2.2, and contrasts with the corresponding analysis of $SU(3)$-structures on
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$T^2$-bundles over coherently tri-symplectic four-manifolds [21, Proposition 6.5]. In that situation we made a particular choice of volume form to obtain a half-flat structure.

Remark 3.4. Existence of three-torus bundles over a weakly coherent tri-symplectic four-manifold $(M, \omega)$ is related to Chern-Weil theory. One finds that for any closed two-form $F$ with integral periods, $F \in \Omega^2_Z(M, \mathbb{R})$, there exists a $T^3$-bundle $\pi: X \to M$ with connection one-form $\theta$ that satisfies $\pi_M^*(d\theta) = F$.

Studying a certain Hamiltonian flow, Hitchin [15] developed a relationship between torsion-free Spin(7)-metrics and cosymplectic $G_2$-manifolds. In particular, he derived evolution equations that describe the one-dimensional flow of a cosymplectic $G_2$-manifold along its unit normal in a torsion-free Spin(7)-manifold. In inverting our construction, one could use Hitchin’s flow on the cosymplectic structure of Proposition 3.2. However, Hitchin’s flow does not preserve the level sets of the multi-moment map: the unit normal is $h(d\nu)^\sharp$, but $\partial/\partial \nu = h^2(d\nu)^\sharp$. It is therefore more natural for us to determine the flow equations associated to the latter vector field.

Proposition 3.5. Suppose $T^3$ acts freely on a connected eight-manifold $Y$ preserving the torsion-free Spin(7)-structure $\Phi$ and admitting a multi-moment map $\nu$. Let $M$ be the topological reduction $\nu^{-1}(t)/T^3$ for any $t$ in the image of $\nu$. Then $M$ is equipped with a $t$-dependent weakly coherent real-analytic symplectic triple $\sigma_1, \sigma_2, \sigma_3$ and the seven-manifold $X_t = \nu^{-1}(t)$ carries a cosymplectic real-analytic $G_2$-structure of the form (3.1). On $X_t$ the following evolution equation holds:

$$\psi' = d(h\phi), \quad (3.4)$$

where $'$ denotes differentiation with respect to $t$.

Conversely, given a cosymplectic real-analytic $G_2$-structure of the form (3.1) defined on a seven-manifold $X_0$. Then the flow equation (3.4) admits a unique solution on some open neighbourhood of $X_0 \times \{0\} \subset X_0 \times \mathbb{R}$, and that solution determines a torsion-free Spin(7)-structure.

Proof. We have

$$\Phi = h\nu \wedge \phi + \psi.$$ 

This has derivative

$$d\Phi = d\nu \wedge (-dh \wedge \phi - h\phi) + d\psi.$$ 

By assumption, the $G_2$-structure is cosymplectic, i.e., $d\psi = 0$ on each level set. We therefore find that $d\Phi = 0$ if and only if

$$0 = \frac{\partial}{\partial \nu} d\Phi = -d(h\phi) + \psi'.$$
Hence we have a torsion-free $\text{Spin}(7)$-structure if and only if the evolution equation (3.4) is satisfied.

Observe that equation (3.4) together with an initial cosymplectic $G_2$-structure on $\mathcal{X}_0$ already ensure that the family consists of cosymplectic structures; the time derivative of $d\psi$ vanishes according to (3.4).

We note that given real-analytic initial data, the Cauchy-Kovalevskaya Theorem applies. Therefore we obtain existence and uniqueness of a solution defined on some open neighbourhood of $\mathcal{X}_0 \times \{0\} \subset \mathcal{X}_0 \times \mathbb{R}$.

For later use, we shall rewrite the evolution equation as a set of first order differential equations for the quantities defined by data on $M$. First we note that

$$
\psi' = \sigma_1' \wedge \theta_3 \wedge \theta_2 + \sigma_2' \wedge \theta_1 \wedge \theta_3 + \sigma_3' \wedge \theta_2 \wedge \theta_1 + (\theta_2' \wedge \sigma_3 - \theta_3' \wedge \sigma_2) \wedge \theta_1 + (\theta_3' \wedge \sigma_1 - \theta_1' \wedge \sigma_3) \wedge \theta_2 + (\theta_1' \wedge \sigma_2 - \theta_2' \wedge \sigma_1) \wedge \theta_3 + \text{vol}_M',
$$

$$
d(h\phi) = d\theta_1 \wedge \theta_3 \wedge \theta_2 + d\theta_2 \wedge \theta_1 \wedge \theta_3 + d\theta_3 \wedge \theta_2 \wedge \theta_1 + \sigma_1 \wedge d\Theta_1 + \sigma_2 \wedge d\Theta_2 + \sigma_3 \wedge d\Theta_3,
$$

where

$$
\sum_{i=1}^3 \sigma_i \wedge d\Theta_i = \sum_{i,j=1}^3 \sigma_i \wedge \left(d(q^{ij}) \wedge \theta_j + q^{ij} d\theta_j\right).
$$

From these equations we get the $t$-derivatives for $\sigma_1$, $\sigma_2$, $\sigma_3$:

$$
\sigma_i' = d\theta_i, \quad \text{for} \quad i = 1, 2, 3. \quad (3.5)
$$

The $t$-derivative of the connection one-form $\theta = (\theta_1, \theta_2, \theta_3)$ is given by

$$
\theta'_i \wedge \sigma_j - \theta'_j \wedge \sigma_i = \sum_{\ell=1}^3 \sigma_\ell \wedge dq^{\ell k}, \quad \text{for} \quad \text{sgn}(ijk) = +1. \quad (3.6)
$$

The volume form $\text{vol}_M$ evolves via

$$
\text{vol}_M' = \sum_{i,j=1}^3 q^{ij} \sigma_i \wedge d\theta_j. \quad (3.7)
$$

Finally the $t$-derivatives of entries $q_{ij}$ of $Q$ may be expressed via

$$
2q_{ij}' \text{vol}_M = d\theta_i \wedge \sigma_j + \sigma_i \wedge d\theta_j - 2q_{ij} \sum_{k,\ell=1}^3 q^{k\ell} \sigma_k \wedge d\theta_\ell, \quad \text{for} \quad i, j = 1, 2, 3. \quad (3.8)
$$

Note that the equations for the entries $q_{ij}$ now determine the evolution of $h$ and $G$ via the relations $h^{-4} = \det(Q)$ and $G^{-1} = h^2 Q$, respectively. □
Remark 3.6. By solving the flow equations we obtain a holonomy Spin(7)-metric with three-torus symmetry. Indeed, if $g_M$ is the time-dependent metric in the conformal class on $M$ with volume form $\text{vol}_M$, then the Spin(7)-metric is explicitly

$$h^2 dt^2 + g_M + g_{11}\theta_1^2 + g_{22}\theta_2^2 + g_{33}\theta_3^2 + g_{12}\theta_1\theta_2 + g_{13}\theta_1\theta_3 + g_{23}\theta_2\theta_3,$$

where $G = (g_{ij}) = h^{-2}Q^{-1}$.

Real-analyticity of the cosymplectic $G_2$-structures is a subtle matter. Bryant’s study of the Hitchin flow [6] shows that non-analytic initial half-flat $SU(3)$-structures can lead to an ill-posed Hitchin system that has no solution. △

Remark 3.7. Though the torsion-free $G_2$-manifolds studied in [21] fiber over (weakly) coherently tri-symplectic four-manifolds, they do not fit naturally into the above framework. The constructed $G_2$-flow does not preserve the Spin(7)-data. △

Summarising the results discussed so far we have:

**Theorem 3.8.** Let $(Y^8, \Phi)$ be a torsion-free Spin(7)-manifold with a free $T^3$ symmetry and admitting a multi-moment map. Then the reduction $M$ at level $t$ carries a weakly coherent real-analytic symplectic triple and the level set $X_t$ is the total space of a $T^3$-bundle over $M$ satisfying condition (3.2) on the curvature.

Conversely, let $(M, \mathcal{C})$ be a weakly coherent tri-symplectic four-manifold with a closed two-form $F \in \Omega^2_Z(M, \mathbb{R}^3)$ and a choice of orientation compatible volume form. Assume $F$ satisfies condition (3.2). When these data are real-analytic, they define a torsion-free Spin(7)-metric with $T^3$-symmetry. □

4 Examples

Let us now turn to some examples that illustrate the analysis of the previous two sections. First we show that even in the flat case $\mathbb{R}^8$, with $T^3 \subset SU(4)$, the geometry of the reduction procedure is somewhat complicated. Thereafter we study hyperKähler four-manifolds, complementing previous examples that have appeared in the physics literature [12] [13].

**Example 4.1 (The flat model $\mathbb{R}^8$).** Consider $Y = \mathbb{R}^8 = \mathbb{C}^4$ endowed with the usual four-form and the action of the standard diagonal maximal torus $T^3 \subset SU(4)$. Concretely, $\Phi$ is given by

$$\Phi = \frac{1}{4}(\frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2 + dz_3 \wedge d\bar{z}_3 + dz_4 \wedge d\bar{z}_4))^2$$

$$+ \text{Re}(dz_1 \wedge d\bar{z}_2 \wedge d\bar{z}_3 \wedge d\bar{z}_4),$$
and $T^3$ acts by $(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}) \cdot (z_1, z_2, z_3, z_4) = (e^{i\theta_1}z_1, e^{i\theta_2}z_2, e^{i\theta_3}z_3, e^{-i(\theta_1+\theta_2+\theta_3)}z_4)$.

The action is generated by the vector fields $U_{\frac{\partial}{\partial z_1}} = \Re \left\{ i(z_1 \frac{\partial}{\partial z_1} - z_4 \frac{\partial}{\partial z_4}) \right\}$, for $j = 1, 2, 3$. It follows that a multi-moment map $v$: $Y \rightarrow \mathbb{R}$ is given by

$$v(z_1, z_2, z_3, z_4) = \frac{1}{8} \Im(z_1 z_2 z_3 z_4).$$

By definition, the $T^3$ reduction of $Y$ at level $t$ is the quotient space $M_t = v^{-1}(t)/T^3$. In this case $M_0$ is singular, whereas $M_t$ is a smooth manifold for each $t \neq 0$. Indeed, considering $\Xi_t: M_t \rightarrow \mathbb{R}^4$ given by

$$\Xi_t(z_1, z_2, z_3, z_4) = \left( \frac{\|z_1^2 - z_2^2\|^2}{2}, \frac{\|z_2^2 - z_3^2\|^2}{2}, \frac{\|z_3^2 - z_4^2\|^2}{2}, \Re(z_1 z_2 z_3 z_4) \right) =: (v_1, v_2, v_3, w),$$

we have global smooth coordinates on $M_t$ for $t \neq 0$.

In this smooth case, writing $(\eta_1, \eta_2, \eta_3) = (dv_1, dv_2, dv_3)G^{-1}$, the two-forms $\sigma_1, \sigma_2, \sigma_3$ are given by

$$16\sigma_1 = \eta_1 \wedge dw + 4dv_2 \wedge dv_3, \quad 16\sigma_2 = \eta_2 \wedge dw + 4dv_3 \wedge dv_1, \quad 16\sigma_3 = \eta_3 \wedge dw + 4dv_1 \wedge dv_2.$$

These forms depend (implicitly) on $t$ via the relations $4g_{ij} = \delta_{ij}\|z_i\|^2 + \|z_4\|^2$, for $i, j = 1, 2, 3$, and $z_1 z_2 z_3 z_4 = w + 8t$. In particular $g_{ij}$ is a non-constant positive function $f$, for $i \neq j$. Thus the weakly coherent triple does not specify a coherent triple, in particular it is not a hyperKähler structure.

The (oriented) conformal class has representative metric

$$\frac{h^2}{16} dw^2 + g_{11} \eta_1^2 + g_{22} \eta_2^2 + g_{33} \eta_3^2 + f(\eta_1 \eta_2 + \eta_1 \eta_3 + \eta_2 \eta_3),$$

where $h^2 = \det(G^{-1})$.

The curvature $F = (F_1, F_2, F_3)$ of the principal $T^3$-bundle $v^{-1}(t) \rightarrow M_t$ is given by

$$F_1 = 2th^2 \eta_w \wedge ((2g_{22}g_{33} - f(g_{22} + g_{33})) \eta_1 + (g_{33} - f)(g_{22} - 2f) \eta_2 + (g_{22} - f)(g_{33} - 2f) \eta_3),$$

$$F_2 = 2th^2 \eta_w \wedge ((2g_{11}g_{33} - f(g_{11} + g_{33})) \eta_2 + (g_{11} - f)(g_{33} - 2f) \eta_3 + (g_{33} - f)(g_{11} - 2f) \eta_1),$$

$$F_3 = 2th^2 \eta_w \wedge ((2g_{11}g_{22} - f(g_{11} + g_{22})) \eta_3 + (g_{22} - f)(g_{11} - 2f) \eta_1 + (g_{11} - f)(g_{22} - 2f) \eta_2),$$

where $\eta_w = g_{w1}^{-1} dw$ satisfies $\eta_w((dw)^2) = 1$ and $\eta_w((dv_i)^2) = 0$, for $i = 1, 2, 3$. Note that $F \neq F^\ast$. 

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In the singular case $t = 0$, the three-torus collapses in three different ways: to a point, a circle or a two-torus. At the origin $(z_1, z_2, z_3, z_4) = 0$ the three-torus collapses to a point. Next, if $z_i = z_j = z_k = 0$ for exactly three different indices, then the torus collapses to a circle. In terms of the quadruple $(v_1, v_2, v_3, w)$ this collapsing happens for $w = 0$ when $v_1, v_2, v_3$ satisfy one of the following constraints: $(v_1 = v_2 = v_3 \leq 0)$, $(v_1 = v_2 = 0, v_3 \geq 0)$, $(v_1 = v_3 = 0, v_2 \geq 0)$ or $(v_2 = v_3 = 0, v_1 \geq 0)$. Finally, if $z_i = z_j = 0$ for exactly two different indices, the $T^3$ collapses to a two-torus. This happens for $w = 0$ when $v_1, v_2, v_3$ satisfy one of: $(v_1 = v_2 \leq 0)$, $(v_1 = v_3 \leq 0)$, $(v_1 = 0, v_2, v_3 \geq 0)$, $(v_2 = v_3 \leq 0)$, $(v_2 = 0, v_1, v_3 \geq 0)$ or $(v_3 = 0, v_1, v_2 \geq 0)$. \hfill $\blacklozenge$

**Example 4.2 (Examples from hyperKähler manifolds).** Let $M$ be a hyperKähler four-manifold. Then $M$ comes equipped with three symplectic forms $\sigma_1, \sigma_2, \sigma_3$ that satisfy the relations $\sigma_i \wedge \sigma_j = \delta_{ij} \tilde{\sigma}_k^2$ for $i, j, k = 1, 2, 3$. Choosing the volume form $\text{vol}_M^0 = \frac{1}{2} \sigma_1^2$, we have that $Q = \text{diag}(1, 1, 1)$. The compatible hyperKähler metric is denoted by $g^0_M$.

Let $\sigma = (\sigma_1, \sigma_2, \sigma_3)$ denote the hyperKähler triple and assume there is a constant matrix $A = (a_{ij})$ such that $\sigma A \in \Omega^2(M, \mathbb{R}^3)$. Then we may construct a $T^3$-bundle over $M$ with connection one-form $\theta$ that has curvature $F = \sigma A$. The total space $X_0$ of this bundle carries the $G_2$-structure of Proposition 3.2 which is now cosymplectic if and only if $A$ is symmetric. The associated metric on $X_0$ is complete if the hyperKähler base manifold is complete.

We shall illustrate how one may solve the flow equations (3.5–3.8) starting from the above data at $t = 0$. As an a priori simplifying assumption we consider the case when $F' = 0$, i.e., the curvature is $t$-independent. Then the differential equations for the symplectic triple simply reads $\sigma' = \Omega A$, where $\Omega = \sigma(0)$. Integrating, we find that $\sigma(t) = \Omega(1 + tA)$. We next solve the equations (3.7) and (3.8). First we observe that the volume develops according to the equation $\text{vol}_M' = v \text{vol}_M^0$, where $v = 2 \text{Tr}(Q^{-1}(1 + tA)A)$. We may therefore write $\text{vol}_M(t) = V(t) \text{vol}_M^0$, where $V' = v$ and $V(0) = 1$. The equation for $Q'$ now takes the form $VQ' = 2(1 + tA) - vQ$. It follows that we must find the unique solution of the differential equation $\ln(V)' = 2\text{Tr}((1 + tA)^{-1}A)$, $V(0) = 1$. We find that $V(t) = \det(1 + tA)^2$. Consequently, $\text{vol}_M$ and $Q$ take the form $\text{vol}_M(t) = \det(1 + tA)^2 \text{vol}_M^0$ and $Q(t) = (1 + tA)^2$. Note also that $h(t) = \det(1 + tA)$ and that $d\theta(t) = 0$. The latter observation implies, by (3.6), that the connection one-form is $t$-independent, $\theta(t) = \theta$.

The above solution is defined on $X_0 \times I$, where the interval $I \subset \mathbb{R}$ is determined by non-degeneracy of the matrix $1 + tA$ and $0 \in I$. By uniqueness of the solution on $X_0 \times I$, we deduce that the condition $F' = 0$ already follows from the initial data, i.e., it is not a simplifying assumption.

The torsion-free $\text{Spin}(7)$-structure corresponding to the above solution
has associated metric $g$ given by

$$h^2(t)dt^2 + h(t)g^0_M + h(t)^{-2}\left(\sum_{i=1}^3 q^{ii}(t)\theta^2_i + \sum_{1 \leq i < j \leq 3} q^{ij}(t)\theta_i\theta_j\right). \tag{4.1}$$

If the initial hyperKähler four-manifold is complete, we may describe completeness properties of $g$ in terms of the matrix $A$. Provided $g$ remains finite and non-degenerate, completeness corresponds to completeness of $h(t)^2dt^2$ on $I$, cf. [1]. We now find that $g$ is half-complete, cf. [2], if and only if $A$ does not have two eigenvalues of opposite sign; the metric is complete only for $A = 0$.

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