MARCINKIEWICZ-TYPE DISCRETIZATION OF $L^p$-NORMS UNDER THE NIKOLSKII-TYPE INEQUALITY ASSUMPTION

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Abstract. The paper studies the Marcinkiewicz-type discretization problem for integral norms on subspaces of $L^p$. Certain close to optimal results are obtained on subspaces for which the Nikolskii-type inequality for $L^\infty$ norm is valid. The proofs use the recent developments of the chaining technique due to R. van Handel.

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1. Introduction

Let $L$ be an $N$-dimensional subspace of $L^p = L^p(\mu)$ with respect to some probability measure $\mu$. In this paper we consider the following problem of sampling discretization (or the Marcinkiewicz-type discretization problem, see [17] and [18], where this notion was introduced): for which $m$ there are points $x_1, \ldots, x_m$ and numbers $c, C > 0$ such that

$$c \|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^{m} |f(x_j)|^p \leq C \|f\|_p^p$$

for every $f \in L$, where

$$\|f\|_p := \left( \int |f|^p \, d\mu \right)^{1/p}.$$ 

The obvious bound is $m \geq N$, so we want to obtain some conditions on the subspace $L$ such that the sampling discretization problem could be solved with the number of points $m$ close to the dimension of the subspace (ideally, with $m = O(N)$). This and similar problems have been extensively studied in recent years (see [4], [5], [3], [11], [17], [18], and [20]). Moreover, some close questions were studied from the convex geometry point of view (see [1], [6], [7], [12], [13]). The first classical result of such type was obtained by Marcinkiewicz for the trigonometric polynomials (see for example [22, Theorem 1.3.6]).

The case $p = 2$ is now well studied, but still is far from full understanding. The following general result is proved in Rudelson’s works [12] and [13] (see also the discussion in [3], [17], and [18], where the Rudelson’s assumptions were reformulated in terms of the Nikolskii-type inequality). Assume that a subspace $L \subset L^2$ is such that for some constant $M$ one has $\|f\|_\infty \leq MN^{1/2} \|f\|_2$ for any $f \in L$. Then in the problem of the sampling discretization one can take $m = O(N \log N)$ points. We note that the above condition is equivalent to the following one: there is an orthonormal basis $\{u_1, \ldots, u_N\}$ in $L$ such that for any point $x$ one has $|u_1(x)|^2 + \ldots + |u_N(x)|^2 \leq M^2 N$. Moreover, the methods of the recent breakthrough solution of the Kadison-Singer problem (see [10]) allow one to obtain the best possible (in terms of order) result concerning the Marcinkiewicz-type discretization for the multivariate trigonometric polynomials with frequencies from any set $Q$ (see [18, Theorem 1.1]).

The case $p = 1$ was considered in [17] and the general case $p \neq 2$ was recently studied in [4] and [5]. In particular, Theorem 2.2. from [4] asserts that for $p \leq 2$ one can take $m = O(N \log^3 N)$ points in the sampling discretization problem for the $N$-dimensional subspace $L$ of $L^p(\mu)$ provided that for some number $M$ one has $\|f\|_\infty \leq MN^{1/2} \|f\|_2$ for every $f \in L$. 

This result is similar to the stated above Rudelson’s result. We note that for \( p > 2 \) the cited papers do not provide such general results.

In this paper we also study the Marcinkiewicz-type discretization problem for \( p \neq 2 \). Our goal is to provide a sharper result for \( p \in (1, 2) \) and to prove some general results for \( p > 2 \). The condition that we impose on the \( N \)-dimensional subspace \( L \subset L^p \) is of Nikolskii type:

\[
\|f\|_\infty \leq MN^{1/\max(p,2)}\|f\|_{\max(p,2)}, \quad \forall f \in L.
\]

We note that for \( p < 2 \) it coincides with that introduced above. The main results of the present paper are stated in Theorems 4.3, 4.5, and 4.7 and in their simplest form can be formulated as follows.

**Theorem.** Let \( p \in [1, \infty) \), and let \( L \) be an \( N \)-dimensional subspace of \( L^p \). If for some number \( M \) one has

\[
\|f\|_\infty \leq MN^{1/\max(p,2)}\|f\|_{\max(p,2)}
\]

for all \( f \in L \), then there are

\[
m = \begin{cases} 
O(N[\log N]^p), & p > 2 \\
O(N[\log N]^2), & p \in (1, 2) \\
O(N[\log N]^{7/2}), & p = 1
\end{cases}
\]

points \( x_1, \ldots, x_m \) such that

\[
\frac{1}{2}\|f\|_p^p \leq \frac{1}{m}\sum_{j=1}^m |f(x_j)|^p \leq \frac{3}{2}\|f\|_p^p
\]

for any \( f \in L \).

The approach that we use here is based on the Talagrand’s generic chaining technique (see [14]) and combines ideas of [7] on the symmetrization argument, some new developments in chaining technique from [9], and some known Talagrand’s bounds for the entropy numbers from [15] and [16], which can also be found in his book [14]. It should be mentioned that the chaining technique has already been used in various works on sampling discretization (see [17], [18], [19], [12], [13], and [7]) and proved to be a powerful tool in this area.

Further the paper is organized as follows. In the second section we recall the basic notions of the chaining technique, formulate some extensions of the results from [9], and give some technical lemmas, that are used further. In the third section we obtain bounds for the expectation of the random variable

\[
\sup_{f \in B_p} \left| \frac{1}{m}\sum_{j=1}^m |f(X_j)|^p - \|f\|_p^p \right|
\]

under the assumptions on the rate of decay of the entropy numbers of the unit ball

\[
B_p := \{ f \in L : \|f\|_p \leq 1 \}
\]

with respect to the discrete uniform norm \( \|f\|_\infty, X := \max_{1 \leq j \leq 1} |f(X_j)| \) for a fixed set of points \( X := \{X_1, \ldots, X_m\} \). Finally, in the fourth section we prove the main results of the paper concerning the Marcinkiewicz-type discretization in subspaces of \( L^p \). Appendices A and B contain the proofs of the extensions of the results from [9], which we are using in the paper. However, we note that they repeat the proofs from [9] almost word for word and are presented here only for the readers’ convenience.

Throughout the paper the symbols \( C, C_1, C_2, \ldots \) denote absolute constants, whose values may vary from line to line. Similarly, the symbols \( C(a, b, c, \ldots), C_1(a, b, c, \ldots), C_2(a, b, c, \ldots), \ldots \) denote numbers, whose values depend only on parameters \( a, b, c, \ldots \), and also may vary from line to line. If the random variable \( X \) has the distribution \( \mu \), we write \( \mathbb{E}_X f(X) \) (or simply \( \mathbb{E} f(X) \)) in place of the integral \( \int f \, d\mu \).
2. Generic chaining, van Handel’s approach and auxiliary lemmas

We recall the basic facts from the generic chaining theory (see [14]).

Let $X_f$ be a random process with $f \in (F, \varrho)$, where $\varrho$ is a quasi-metric on $F$, i.e. it has all the properties of a metric but in place of usual triangle inequality we have the following relaxed triangle inequality

$$\varrho(f, g) \leq R(\varrho(f, h) + \varrho(h, g))$$

for some constant $R > 0$ for all $f, g, h \in F$. Assume that there are numbers $K > 0$ and $\alpha > 0$ such that

$$P(|X_f - X_g| \geq K t^{1/\alpha} \varrho(f, g)) \leq 2e^{-t}$$

for all $t > 0$.

**Definition 2.1.** An admissible sequence of $F$ is an increasing sequence $(F_k)$ of partitions of $F$ such that $|F_k| \leq 2^{2k}$ for all $n \geq 1$ and $|F_0| = 1$. For $f \in F$ let $F_k(f)$ denote the unique element of $F_k$ that contains $f$.

**Definition 2.2.** Let $\alpha > 0$ and $\theta \geq 1$. Let

$$\gamma_{\alpha, \theta}(F, \varrho) := \left(\inf \sup_{f \in F} \sum_{k=0}^{\infty} \left[2^{k/\alpha} \text{diam}(F_k(f))\right]^{\theta}\right)^{1/\theta},$$

where the infimum is taken over all admissible sequences of $F$ and where $\text{diam}(G) := \sup_{f, g \in G} \varrho(f, g)$.

The quantity $\gamma_{\alpha, \theta}(F, \varrho)$ is called the chaining functional. If the metric $\varrho$ is induced by a norm $\| \cdot \|$, we will also use the notation $\gamma_{\alpha, \theta}(F, \| \cdot \|)$ in place of $\gamma_{\alpha, \theta}(F, \varrho)$.

We need the following fundamental result (see [14, Theorem 2.2.22]).

**Theorem 2.3.** Under the above assumption (2.1) there are positive constants $c, C$ such that for any $f_0 \in F$ one has

$$P\left(\sup_{f \in F} |X_f - X_{f_0}| \geq R K \gamma_{\alpha, 1}(F, \varrho)(ct)^{1/\alpha}\right) \leq Ce^{-t}.$$ 

In particular,

$$\mathbb{E} \sup_{f \in F} |X_f - X_{f_0}| \leq C(\alpha) R K \gamma_{\alpha, 1}(F, \varrho).$$

**Definition 2.4.** Recall the definition of the entropy numbers:

$$e_k(F, \varrho) := \inf \{\varepsilon : \exists f_1, \ldots, f_{2^k} \in F : F \subset \bigcup_{j} B_\varepsilon(f_j)\},$$

where $B_\varepsilon(f) := \{g : \varrho(f, g) < \varepsilon\}$.

If the metric $\varrho$ is induced by a norm $\| \cdot \|$, we will also use the notation $e_k(F, \| \cdot \|)$ in place of $e_k(F, \varrho)$. We note here that sometimes the other definition of the entropy numbers is used with $2^k$ points in place of $2^{2k}$.

We also need the following property of the entropy numbers in an $N$-dimensional space (see estimate (7.1.6) in [22] and Corollary 7.2.2 there). Assume that $\varrho$ is induced by a norm $\| \cdot \|$. Then for $k > k_0$ one has

$$e_k(F, \| \cdot \|) \leq 3 \cdot 2^{2k_0/2} e_{k_0}(F, \| \cdot \|) 2^{-2k/2}.$$ 

We will apply the following result from [9].
Theorem 2.5. Let \( q \geq 2, p > 1, \alpha > 0 \). Let \( L \) be a linear space endowed with a \( q \)-convex norm \(| \cdot |\) with a constant \( \eta \), i.e.

\[
\left\| \frac{f + g}{2} \right\| \leq \max(\|f\|, \|g\|) - \eta\|f - g\|^q
\]

for any \( f, g \) with \( \|f\| \leq 1, \|g\| \leq 1 \). Let \( \varrho \) be a quasi-metric on \( L \) such that

\[
\varrho(f, g) \leq R(\varrho(f, h) + \varrho(h, g)); \quad \varrho\left(f, \frac{f + g}{2}\right) \leq \varkappa \varrho(f, g)
\]

for all \( f, g, h \in L \) for some constants \( R, \varkappa > 0 \). Assume that there is a metric \( d \) on \( L \) and for each \( h \in L \) there is a norm \(| \cdot |_h \) on \( L \) such that

\[
c_1d(f, g)^p \leq \varrho(f, g) \leq c_2\left(\|f - g\|_h + d(f, g)(d(f, h)^{p-1} + d(h, g)^{p-1})\right)
\]

for some numbers \( c_1, c_2 \). Then there is a number \( C := C(q, p, \alpha, R, \varkappa, c_1, c_2) \) such that for the unit ball \( B := \{\|f\| \leq 1\} \) one has

\[
\gamma_{\alpha, 1}(B, \varrho) \leq C\left(\eta^{-1/q}\left[\sup_{\|f\| \leq 1} \sum_{k=0}^{\infty} (2k/\alpha e_k(B, d_f))^{q/(q-1)}\right]^{(q-1)/q} + [\gamma_{\alpha, p}(B, d)]^p\right),
\]

where \( d_h(f, g) := \|f - g\|_h \).

The quasi-metric \( \varrho \) in the above theorem can appear from the expressions of the following type

\[
\tilde{\varrho}(f, g) := \left(\int \|f|^p - |g|^p\|^p\, d\nu\right)^{1/r} = \|f|^p - |g|^p\|_r
\]

for \( p > 1, r \in (1, \infty] \). Indeed, set

\[
\varrho(f, g) := \|f - g\|((|f|^{p-1} + |g|^{p-1})\|_r; \quad \|f\|_h := \|f||h|^{-1}\|_r; \quad d(f, g) := \|f - g\|^{1/p} = \|f - g\|_{pr}.
\]

It can be readily verified that \( \tilde{\varrho}(f, g) \leq p\varrho(f, g) \).

Lemma 2.6. For quasi metric \( \varrho \), metric \( d \) and norms \(| \cdot |_h \) defined above we have

\[
\varrho(f, g) \leq C_1(p)(\varrho(f, h) + \varrho(h, g)); \quad \varrho\left(f, \frac{f + g}{2}\right) \leq \varrho(f, g);
\]

\[
C_2(p)d(f, g)^p \leq \varrho(f, g) \leq C_2(p)\left(\|f - g\|_h + d(f, g)(d(f, h)^{p-1} + d(h, g)^{p-1})\right)
\]

for some numbers \( C_1(p), C_2(p), C_3(p) \), dependent only on \( p > 1 \).

Proof. We note that \( (|f| + |g|)^{p-1} \leq 2^{p-1}\max\{|f|^{p-1}, |g|^{p-1}\} \leq 2^{p-1}(|f|^{p-1} + |g|^{p-1}) \) for \( p > 1 \). Thus,

\[
2^{1-p}|f - g|^p \leq |f - g|(|f|^{p-1} + |g|^{p-1}) = |f - g|(|f - h + h|^{p-1} + |g - h + h|^{p-1})
\]

\[
\leq 2^{p-1}|f - g|(|f - h|^{p-1} + |h|^{p-1} + |g - h|^{p-1} + |h|^{p-1})
\]

\[
\leq 2^p(|f - g||h|^{p-1} + |f - g||f - h|^{p-1} + |f - g||g - h|^{p-1})
\]

implying

\[
2^{1-p}d(f, g)^p \leq D(f, g) \leq 2^p\left(\|f - g\|_h + d(f, g)(d(f, h)^{p-1} + d(g, h)^{p-1})\right).
\]

Next,

\[
D\left(f, \frac{f + g}{2}\right) = 2^{-1}\|f - g|\ left(\frac{|f + g}{2}|^{p-1}\right)\right\|_r
\]

\[
\leq 2^{-1}\|f - g|\ left(\frac{|f + g}{2}|^{p-1} + |f + g|^{p-1}\right)\|_r \leq D(f, g).
\]
Finally,
\[ |f - g|(|f|^p + |g|^p) \leq 2^{p-1}(|f - h|(|f|^p + |h|^p) + |h - g|(|h|^p + |g|^p)) \]
\[ = 2^{p-1}(|f - h|(|f|^p + |h|^p) + |h - g|(|h|^p + |g|^p)) \]
\[ = 2^{p-1}(|f - h|(|f|^p + |h|^p) + |h - g|(|h|^p + |g|^p) + |f - h||h - g|). \]

We now note that \( ab^{p-1} \leq a^p + b^p \). Thus,
\[ |f - h||h - g|^p + |h - g||f - h|^p \leq 2(|f - h|^p + |h - g|^p) \]
\[ \leq 2^p (|f - h||f|^p + |h - g||h|^p) \]
and
\[ D(f, g) \leq 4^p (D(f, h) + D(h, g)). \]

The lemma is proved.

\( \square \)

**Remark 2.7.** We note that in [9] only a special case of Theorem 2.5 was considered (see Theorem 7.3 there), but the proof of Theorem 2.5 repeats the argument there almost verbatim. We will provide the details in Appendix A for the readers’ convenience.

We need the following bound (see [14, Theorem 4.1.4] and [9, Theorem 5.8]).

**Theorem 2.8.** Let \( B := \{ x : \| x \| \leq 1 \} \) where the norm \( \| \cdot \| \) is \( q \)-convex with constant \( \eta \). Let \( d \) be a metric which is generated by another norm. Then for any \( \alpha > 0 \) there is a number \( C(\alpha, q) \) such that
\[ \gamma_{\alpha, q}(B, g) \leq C(\alpha, q) \eta^{-1} \sup_{k \geq 0} 2^{k/\alpha} e_k(B, g). \]

We also need the following extension of the above result.

**Theorem 2.9.** Let \( B := \{ x : \| x \| \leq 1 \} \) where the norm \( \| \cdot \| \) is \( q \)-convex with constant \( \eta \). Let \( d \) be a metric which is generated by another norm. Then for any \( \alpha > 0 \) and for any \( p \in [1, q) \) there is a number \( C(\alpha, p, q) \) such that
\[ \gamma_{\alpha, p}(B, d) \leq C(\alpha, p, q) \eta^{-p/q} \left( \sum_{k \geq 0} (2^{k/\alpha} e_k(B, d))^{pq/(q-p)} \right)^{1/p-1/q}. \]

The proof again repeats the argument form [9, Theorem 5.8] almost verbatim. We present the proof in Appendix B for the readers’ convenience.

Finally, we will use the following technical bound.

**Lemma 2.10.** Let \( a, b > 0 \). Then there is a number \( C(a, b) \) such that for any \( N \geq 2 \) one has
\[ \sum_{k \geq \log N} (2^{ak} 2^{-2k/N})^b \leq C(a, b) N^{ab}. \]

**Proof.** Note that
\[ N^{-ab} \sum_{k \geq \log N} (2^{ak} 2^{-2k/N})^b = \sum_{k \geq \log N} (2^{a(k-\log N) - 2k/\log N})^b. \]
There is a number \( c(a) > 0 \) such that \( ax - 2^x \leq -x + c(a) \) for any \( x > 0 \). Thus, the last expression is estimated by
\[ 2^{bc(a)} \sum_{k \geq \log N} (2^{-(k-\log N)})^b \leq C(a, b). \]

The lemma is proved. \( \square \)
3. Discretization under the entropy numbers decay assumptions

Here we follow the ideas of Guédon and Rudelson from [7]. Let $B$ be a set of functions. Consider the following random variables:

$$V_p(B) := \sup_{f \in B} \left| \frac{1}{m} \sum_{j=1}^{m} |f(X_j)|^p - \|f\|_p^p \right|, \quad R_p(f) = \sum_{j=1}^{m} |f(X_j)|^p$$

We start with the following symmetrization argument.

Lemma 3.1. Assume that for every fixed $X := (X_1, \ldots, X_m)$

$$E_{\varepsilon, \sup_{f \in B}} \left| \sum_{j=1}^{m} \varepsilon_j |f(X_j)|^p \right| \leq \Theta(X) \sup_{f \in B} (R_p(f))^{1-r},$$

with $r \in (0, 1)$, where $\varepsilon_1, \ldots, \varepsilon_m$ are independent symmetric Bernoulli random variables with values $\pm 1$. Then

$$E \varepsilon_p(B) \leq 2^{1/r} m^{-1} E \Theta^{1/r} + 2r^{-1} (m^{-1} E \Theta^{1/r})^r \left( \sup_{f \in B} E |f(X_1)|^p \right)^{1-r}.$$

Proof. Let $X'_1, \ldots, X'_m$ be independent copies of $X_1, \ldots, X_m$. We note that

$$m E \varepsilon_p(B) = E \sup_{f \in B} \left| \sum_{j=1}^{m} (|f(X_j)|^p - E |f(X'_j)|^p) \right| \leq E_X \sup_{f \in B} \left| \sum_{j=1}^{m} (|f(X_j)|^p - |f(X'_j)|^p) \right|$$

$$= E_X E_{\varepsilon} \sup_{f \in B} \left| \sum_{j=1}^{m} \varepsilon_j (|f(X_j)|^p - |f(X'_j)|^p) \right| \leq 2E_X \sup_{f \in B} \left| \sum_{j=1}^{m} \varepsilon_j |f(X_j)|^p \right|$$

$$\leq 2\Theta \sup_{f \in B} (R_p(f))^{1-r} \leq 2(\Theta^{1/r})^r m^{1-r} \left( E \sup_{f \in B} \frac{1}{m} \sum_{j=1}^{m} |f(X_j)|^p \right)^{1-r}$$

$$\leq 2(\Theta^{1/r})^r m^{1-r} \left( E \varepsilon_p(B) + \sup_{f \in B} E |f(X_1)|^p \right)^{1-r}.$$

Thus, $E \varepsilon_p(B) \leq 2(\Theta^{1/r})^r m^{-r} \left( E \varepsilon_p(B) + \sup_{f \in B} E |f(X_1)|^p \right)^{1-r}$ and

$$E \varepsilon_p(B) \leq 2^{1/r} m^{-1} E \Theta^{1/r} + 2r^{-1} (m^{-1} E \Theta^{1/r})^r \left( \sup_{f \in B} E |f(X_1)|^p \right)^{1-r}.$$

Indeed, if for some positive numbers $v, a, b$ and some $r \in (0, 1)$ one has $v \leq a(v + b)^{1-r}$, then by convexity and Young’s inequality one has $a(v + b)^{1-r} \leq av^{1-r} + ab^{1-r} \leq ra^{1/r} + (1 - r)v + ab^{1-r}$ and $v \leq a^{1/r} + r^{-1} ab^{1-r}$.

We will use the following lemma (see [8, Lemma 4.3]).

Lemma 3.2. Let $\varepsilon_1, \ldots, \varepsilon_m$ be independent symmetric Bernoulli random variables with values $\pm 1$. Then for every $q \geq 2$ there is a number $C_q$, depending only on $q$, such that

$$P \left( \left| \sum_{j=1}^{m} \varepsilon_j \alpha_j \right| \geq C_q \left( \sum_{j=1}^{m} |\alpha_j| q' \right)^{1/q'} t \right) \leq 2e^{-t},$$

where $q' = q/(q - 1)$.

Theorem 3.3. Let $p \in [1, \infty)$ and let $L$ be some subspace of $L^p$. Let $B_p$ be the unit ball in $L$, i.e. $B_p = \{ f \in L : \|f\|_p \leq 1 \}$ and let the set of points $X = \{X_1, \ldots, X_m\}$ be fixed. Consider the
norm $\|f\|_{\infty, X} := \max_{1 \leq j \leq m} |f(X_j)|$. Assume, that there is a number $N$ such that for the entropy numbers one has the following bound

$$e_k(B_{p}, \gamma \cdot \|\cdot\|_{\infty, X}) \leq M \begin{cases} 2^{-k/p}, & k \leq \log N, \\ N^{-1/p}2^{-k/N}, & k \geq \log N. \end{cases}$$

Then there is a number $C(p)$, which depends only on $p$, such that

1) for $p \geq 2$, one has

$$\mathbb{E}_e \sup_{f \in B_p} \left|\sum_{j=1}^m \varepsilon_j |f(X_j)|^p\right| \leq C(p)M[\log N]^{1-1/p} \sup_{f \in B_p} \left(\sum_{j=1}^m |f(X_j)|^p\right)^{1-1/p},$$

2) for $p \in (1, 2)$, one has

$$\mathbb{E}_e \sup_{f \in B_p} \left|\sum_{j=1}^m \varepsilon_j |f(X_j)|^p\right| \leq C(p)M^{p/2}[\log N]^{1/2} \sup_{f \in B_p} \left(\sum_{j=1}^m |f(X_j)|^p\right)^{1/2},$$

3) and for $p = 1$ one has

$$\mathbb{E}_e \sup_{f \in B_p} \left|\sum_{j=1}^m \varepsilon_j |f(X_j)|\right| \leq (1)M^{1/2} \log N \sup_{f \in B_p} \left(\sum_{j=1}^m |f(X_j)|\right)^{1/2}.$$

Proof. For any $q \geq 2$, by Lemma 3.2 we have the estimate (2.1) with $\alpha = q$ and with the quasi-metric

$$\varrho_q(f, g) := \left(\sum_{j=1}^m |f(X_j) - g(X_j)|\left(|f(X_j)|^{p-1} + |g(X_j)|^{p-1}\right)^q\right)^{1/q'}.$$ 

Thus, by Theorem 2.3 the bound for the expectation will follow from the bound for the chaining functional $\gamma_{q, 1}(B_p, \varrho_q)$.

1) Firstly, consider the case $p \geq 2$. In that case we take $q = p$. Let

$$\|f\|_h = \left(\sum_{j=1}^m |f(X_j)|^{pp'}|h(X_j)|^p\right)^{1/pp'} \leq \|f\|_{\infty, X} \left(\sum_{j=1}^m |h(X_j)|^p\right)^{1-1/p}$$

and

$$d(f, g) = \left(\sum_{j=1}^m |f(X_j) - g(X_j)|^{pp'}\right)^{1/\left(pp'\right)} \leq \|f - g\|_{\infty, X} \left(\sum_{j=1}^m |f(X_j) - g(X_j)|^p\right)^{(1-1/p)1/p}.$$ 

Since the $L^p$-norm is $p$-convex for $p \geq 2$, we can apply van Handel’s bound from Theorem 2.5. Thus,

$$\gamma_{p, 1}(B_p, \varrho) \leq \mathcal{C}_1(p) \left(\sup_{h \in B_p} \left(\sum_{k=0}^\infty (2^{k/p}e_k(B_p, \|\cdot\|_h))^{pp'}\right)^{1/pp'} + [\gamma_{p^2, p}(B_p, d)]^{pp'}\right).$$

Note that

$$\sup_{h \in B_p} \left(\sum_{k=0}^\infty (2^{k/p}e_k(B_p, \|\cdot\|_h))^{pp'}\right)^{1/pp'} \leq \sup_{h \in B_p} \left(\sum_{j=1}^m |h(X_j)|^p\right)^{1-1/p} \left[\sum_{k=0}^\infty (2^{k/p}e_k(B_p, \|\cdot\|_{\infty, X}))^{pp'}\right]^{1/pp'}$$

and

$$\left[\sum_{k=0}^\infty (2^{k/p}e_k(B_p, \|\cdot\|_{\infty, X}))^{pp'}\right]^{1/pp'} \leq M \left[\sum_{k \leq \log N} 1 + N^{-p'/p} \sum_{k \geq \log N} (2^{k/p}2^{-2k/N})^{pp'}\right]^{1/pp'}$$

$$= M \left[\log N + C_2(p)\right]^{1/pp'} \leq C_3(p)M[\log N]^{1-1/p}.$$
We also note that, since the $L^p$-norm is $p$-convex for $p \geq 2$, by Theorem 2.8 one has
\[
\left[\gamma_{p^2, p}(B_p, d)\right]^p \leq C_4(p) \sup_{k \geq 0} \left[2^{k/p} e_k(K, d)\right]^p \\
\leq C_5(p) \sup_{h \in B_p} \left(\sum_{j=1}^{m} |h(X_j)|^p\right)^{1-1/p} \sup_{k \geq 0} 2^{k/p} e_k(B_p, \|\cdot\|_{\infty, X}) \\
\leq C_6(p) M \sup_{h \in B_p} \left(\sum_{j=1}^{m} |h(X_j)|^p\right)^{1-1/p}.
\]

The theorem is proved for $p \geq 2$.

2) We now move on to the case $p \in (1, 2)$. In that case we take $q = 2$. Let
\[
\|f\|_h = \left(\sum_{j=1}^{m} |f(X_j)|^2 |h(X_j)|^{2p-2}\right)^{1/2} \leq \|f\|_{\infty, X}^{p/2} \left(\sum_{j=1}^{m} |f(X_j)|^{2-2p} |h(X_j)|^{2p-2}\right)^{1/2} \\
\leq \|f\|_{\infty, X}^{p/2} \left(\sum_{j=1}^{m} |f(X_j)|^p\right)^{2/p} \left(\sum_{j=1}^{m} |h(X_j)|^p\right)^{2p-2/2p}
\]
and
\[
d(f, g) = \left(\sum_{j=1}^{m} |f(X_j) - g(X_j)|^{2p}\right)^{1/(2p)} \leq \|f - g\|_{\infty, X}^{1/2} \left(\sum_{j=1}^{m} |f(X_j) - g(X_j)|^p\right)^{1/(2p)}.
\]
Since the $L^p$-norm is 2-convex for $p \in (1, 2)$, by van Handel’s theorem (Theorem 2.5), we have
\[
\gamma_{2,1}(B_p, \bar{\varrho}) \leq C_p \left[\sup_{h \in B_p} \left(\sum_{k=0}^{\infty} \left(2^{k/2} e_k(B_p, \|\cdot\|_h)\right)^2\right)^{1/2} + \left[\gamma_{2p, p}(B_p, d)\right]^p\right].
\]
Note that
\[
\sup_{h \in B_p} \left[\sum_{k=0}^{\infty} \left(2^{k/2} e_k(B_p, \|\cdot\|_h)\right)^2\right]^{1/2} \leq C_1(p) \sup_{h \in B_p} \left(\sum_{j=1}^{m} |h(X_j)|^p\right)^{1/2} \left[\sum_{k=0}^{\infty} \left(2^{k/2} e_k(B_p, \|\cdot\|_{\infty, X})\right)^2\right]^{1/2}
\]
and
\[
\left[\sum_{k=0}^{\infty} \left(2^{k/2} e_k(B_p, \|\cdot\|_{\infty, X})\right)^2\right]^{1/2} \leq M^{p/2} \left[\sum_{k \leq \log N} 1 + N^{-1} \sum_{k \geq \log N} \left(2^{k/2}-2^{k-1}/N\right)^2\right]^{1/2} \\
\leq M^{p/2} \left[\log N + C_2\right]^{1/2} \leq C_3 M^{p/2} [\log N]^{1/2}.
\]
Next, by Theorem 2.9
\[
\left[\gamma_{2p, p}(B, d)\right]^p \leq C_4(p) \left(\sum_{k=0}^{\infty} \left(2^{(k+2p)/2} e_k(B, d)\right)^{2p/(2-p)}\right)^{1-p/2} \\
\leq C_5(p) \sup_{h \in B} \left(\sum_{j=1}^{m} |h(X_j)|^p\right)^{1/2} \left(\sum_{k=0}^{\infty} \left(2^{(k+2p)/2} e_k(B, \|\cdot\|_{\infty, X})\right)^{2p/(2-p)}\right)^{1-p/2}.
\]
Note that
\[
\left(\sum_{k=0}^{\infty} \left(2^{(k+2p)/2} e_k(B, \|\cdot\|_{\infty, X})\right)^{2p/(2-p)}\right)^{1-p/2}
\]
\[
\leq M^{p/2} \left[\sum_{k \leq \log N} 1 + N^{-1/(2-p)} \sum_{k \geq \log N} \left(2^{k/2}-2^{k-1}/N\right)^{2p/(2-p)}\right]^{1-p/2} \leq C_6(p) M^{p/2} [\log N]^{1-p/2}.
\]
Since $1 - p / 2 \leq 1 / 2$, the theorem is proved for $p \in (1, 2)$.

3) Finally, we consider the case $p = 1$. In that case we again take $q = 2$ and, since

$$\left( \sum_{j=1}^{m} ||f(X_j)| - |g(X_j)||^2 \right)^{1/2} \leq \left( \sum_{j=1}^{m} |f(X_j) - g(X_j)|^2 \right)^{1/2} = \varphi(f, g)$$

we need to bound the chaining functional $\gamma_{2,1}(B, \varphi)$. Here we simply apply Dudley’s entropy bound (see [14, Proposition 2.2.10]):

$$\gamma_{2,1}(B, \varphi) \leq C \sum_{k=0}^{\infty} 2^{k/2} e_k(B, \varphi) \leq C_1 \sup_{h \in B} \left( \sum_{j=1}^{m} |h(X_j)| \right)^{1/2} \sum_{k=0}^{\infty} 2^{k/2} e_k(B, \| \cdot \|_{\infty, X})$$

$$\leq C_2 \sup_{h \in B} \left( \sum_{j=1}^{m} |h(X_j)| \right)^{1/2} M^{1/2} \log N.$$  

The theorem is proved.

Applying now Lemma 3.1 we get the following conditional result.

**Corollary 3.4.** Let $p \in [1, \infty)$, $\omega(p) := \max\{p, 2\} - 1, \omega(1) = 2$, and let $L$ be some subspace of $L^p$. Let $B_{p}$ be the unit ball in $L$, i.e. $B_{p} = \{ f \in L : ||f||_p \leq 1 \}$. Assume that there is a number $N$ such that for any set of points $X = \{X_1, \ldots, X_m\}$ for the norm $||f||_{\infty, X} := \max_{1 \leq j \leq m} |f(X_j)|$ one has

$$e_k(B, \| \cdot \|_{\infty, X}) \leq M \begin{cases} 2^{-k/p}, k \leq \log N, \\ N^{-1/p} 2^{-k/N}, k \geq \log N. \end{cases}$$

Let $\theta_m := (\mathbb{E} M^p) [\log N]^{\omega(p)} m^{-1}$. Then there is a number $\alpha(p) \geq 1$, depending only on $p$, such that

$$\mathbb{E} V_p(B) \leq \alpha(p) (\theta_m + (\theta_m)^{1/\max\{p, 2\}})$$

and for any $\varepsilon \in (0, 1)$ one has

$$P \left( \sup_{f \in B_{p}} \left\| \frac{1}{m} \sum_{j=1}^{m} |f(X_j)|^p - ||f||_p^p \right\| \leq \varepsilon \right) \geq 1/2,$$

provided that $\theta_m^{1/\max\{p, 2\}} \leq \frac{\varepsilon}{4 \alpha(p)}$.

4. Discretization under the Nikolskii-type inequality assumption

We recall the following lemma (see [14, Lemma 16.5.4] and [15]).

**Lemma 4.1.** Let $(X, \| \cdot \|)$ be a Banach space and let the norm $\| \cdot \|$ in the dual space $X^*$ be $p$-convex with some constant $\eta$ for some $p \geq 2$. For a fixed set of vectors $x := (x_1, \ldots, x_m)$, consider a norm $\|x^*\|_{\infty, x} := \max_{1 \leq j \leq m} |x^*(x_j)|$ on $X^*$. Then for some number $K(p, \eta)$, dependent only an $p$ and $\eta$, one has

$$e_k(B^*_x, \| \cdot \|_{\infty, x}) \leq K(p, \eta) \max_{1 \leq j \leq m} \|x_j\| 2^{-k/p} [\log m]^{1/p},$$

where $B^*_x := \{x^* \in X^* : \|x^*\|_x \leq 1\}$.

We now shortly discuss how one can obtain this lemma from the greedy approximation theory. Without loss of generality, we assume that $\|x_j\| = 1, \forall j \in \{1, \ldots, m\}$. Let $U$ be a convex hull of $\pm x_1, \ldots, \pm x_m$. Firstly, like in Talagrand’s work [15, Lemma 3.3], we note that, by Proposition 2 from [2] and its iterations, the desired bound follows from the bound

$$e_k(U, B) \leq K(p, \eta) 2^{-k/p} [\log m]^{1/p}.$$
where $B := \{x \in X : \|x\| \leq 1\}$. The bound for the entropy numbers can be deduced from the bound for the best $n$-term approximation: let $D = \{y_j\}$ be a set of $r$ points, then
\[
\sigma_n(U, D) := \sup_{y \in U} \inf_{\{c_j\}, |A|=n} \|y - \sum_{j \in A} c_jy_j\|.
\]

It is known (see [22, Theorem 7.4.3] and [21, Theorem 3.1]) that, if $\sigma_n(U, D) \leq An^{-\omega}$ for every $n$, then for $k \leq \log r$ one has $e_k(U, B) \leq C(\omega)[\log 2r]^{\omega/2 - \omega k}$. We note that the ball $B$ is $p' = p/(p-1)$-smooth. Now taking $D = \{\pm x_1, \ldots, \pm x_m\}$ and applying Weak Chebyshev Greedy Algorithm (see [23, Section 6.2]), we get (see [23, Theorem 6.8]) that $\sigma_n(U, D) \leq C(p, \eta)n^{-1/p}$. Thus, for $k \leq \log m$,
\[
e_k(U, B) \leq C_1(p, \eta)[\log 4m]^{1/p}2^{k/p} \leq C_2(p, \eta)[\log m]^{1/p}2^{-k/p}.
\]

From Lemma 4.1 we get the following corollary.

**Corollary 4.2.** Let $p > 2$, and let $L$ be an $N$-dimensional subspace of $L^p$. Let $B_p$ be the unit ball in $L$, i.e. $B_p = \{f \in L : \|f\|_p \leq 1\}$. Assume, that there is a constant $M$ such that for any $f \in L$ one has
\[
\|f\|_\infty \leq M\|f\|_p.
\]

Then there is a number $C(p)$ depending only on $p$, such that $\theta_m = C(p)M^p[\log N]^{p-1}[\log m]m^{-1}$ where $\theta_m$ was defined in Corollary 3.4.

**Proof.** We apply the above lemma for the space $X = L^*$, then $X^* = L$ and for any set of points $X = \{X_1, \ldots, X_m\}$ for the norm $\|f\|_{\infty, X}$ one has
\[
e_k(B_p, \|\cdot\|_{\infty, X}) \leq C(p)\max_{1 \leq j \leq m, \|f\|_p \leq 1} |(f(X_j))[2^{-k/p}[\log m]^{1/p} \leq C(p)M2^{-k/p}[\log m]^{1/p}.
\]

By estimate (2.2), we actually have
\[
e_k(B_p, \|\cdot\|_{\infty, X}) \leq C(p)M[\log m]^{1/p} \begin{cases} 2^{-k/p}, & k \leq \log N; \\ N^{-1/p}2^{-k/N}, & k \geq \log N. \end{cases}
\]

Thus, $\theta_m = C(p)M^p[\log N]^{p-1}[\log m]m^{-1}$. \qed

The following theorem provides the Marcinkiewicz-type discretization in $L^p$ for $p > 2$.

**Theorem 4.3.** Let $p > 2, a, b > 0$, and let $L$ be an $N$-dimensional subspace of $L^p$. Let $B_p$ be the unit ball in $L$, i.e. $B_p = \{f \in L : \|f\|_p \leq 1\}$. Assume that there is a constant $M \leq aN^b$ such that for any $f \in L$ one has
\[
\|f\|_\infty \leq M\|f\|_p.
\]

Then, for any $\varepsilon \in (0, 1)$, there is a number $C(a, b, p, \varepsilon)$ such that for any
\[
m \geq C(a, b, p, \varepsilon)M^p[\log N]^p,
\]

one has
\[
P\left(\sup_{f \in B_p} \left| \frac{1}{m} \sum_{j=1}^m |f(X_j)|^p - \mathbb{E}|f(X)|^p \right| \leq \varepsilon \right) \geq 1/2.
\]

Thus, there are $m = C(a, b, p, \varepsilon)M^p[\log N]^p$ points $x_1, \ldots, x_m$ such that
\[
(1 - \varepsilon)\|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(x_j)|^p \leq (1 + \varepsilon)\|f\|_p^p
\]

for any $f \in L$.

**Proof.** By the above corollary, $\theta_m = C(p)M^p[\log N]^{p-1}[\log m]m^{-1}$. Taking $C(a, b, p, \varepsilon)$ big enough and taking $m \geq C(a, b, p, \varepsilon)M^p[\log N]^p$ we get $\theta_m \leq \frac{e^2}{16a(p)^2}$ and the result follows from Corollary 3.4. \qed
We now consider the case $p \in (1, 2)$.

**Theorem 4.4.** Let $p \in (1, 2)$, and let $L$ be an $N$-dimensional subspace of $L^p$. Let $B_p$ be the unit ball in $L$, i.e. $B_p = \{ f \in L : \| f \|_p \leq 1 \}$. Assume that there is a constant $M \geq 2$ such that for any $f \in L$ one has

$$\| f \|_\infty \leq M \| f \|_2.$$ 

Then there is a number $C(p)$ depending only on $p$, such that

$$\theta_m = C(p) M^2 \log N [\log M]^{1-p/2} [\log m]^{p/2} m^{-1}$$

where $\theta_m$ was defined in Corollary 3.4.

**Proof.** Since $\| f \|^2 \leq M^2 \| f \|^{2-p} \| f \|_p$, we have $\| f \|_\infty \leq M^2/p \| f \|_p$ for any $f \in L$. Thus, we have $e_0(B_p, \| \cdot \|_\infty, X) \leq CM^{2/p}$ for any set of points $X = \{ X_1, \ldots, X_m \}$. We further use the following known property (see [14, Lemma 16.8.9]) of the entropy numbers:

$$e_{k+1}(B_p, \| \cdot \|_\infty, X) \leq 2 e_k(B_p, \| \cdot \|_2) e_k(B_2, \| \cdot \|_\infty, X),$$

where $B_2 := \{ f \in L : \| f \|_2 \leq 1 \}$. We will also use the following classical Sudakov bound for the entropy numbers of the euclidian ball with respect to some norm $\| \cdot \|_r$:

$$e_k(B_2, \| \cdot \|_r) \leq C 2^{-k/2} \mathbb{E}_g \| \sum_{k=1}^N g_k u_k \|_r.$$

Here $g = (g_1, \ldots, g_N)$ is the standard Gaussian vector and $\{ u_1, \ldots, u_N \}$ is any orthonormal basis in $L$. By this bound

$$e_k(B_2, \| \cdot \|_\infty, X) \leq C 2^{-k/2} \mathbb{E}_g \| \sum_{k=1}^N g_k u_k \|_{\infty, X},$$

where $C$ is a numerical constant. We now note that

$$\mathbb{E}_g \| \sum_{k=1}^N g_k u_k \|_{\infty, X} = \mathbb{E}_g \max_{1 \leq j \leq m} \left\| \sum_{k=1}^N g_k u_k (X_j) \right\| \leq C \max_{1 \leq j \leq m} \left\{ \left( \sum_{k=1}^N |u_k(X_j)|^2 \right)^{1/2} [\log m]^{1/2} \right\}$$

where we have used the known bound for the expectation of the maximum of Gaussian random variables (see [14, Proposition 2.4.6]). Since

$$\max_{1 \leq j \leq m} \left\{ \left( \sum_{k=1}^N |u_k(X_j)|^2 \right)^{1/2} \right\} = \sup_{|a_1|^2 + \ldots + |a_n|^2 = 1} \left\| \sum_{k=1}^N a_k u_k (X_j) \right\| \leq M \sup_{|a_1|^2 + \ldots + |a_n|^2 = 1} \left\| \sum_{k=1}^N a_k u_k \right\|_\infty = M,$$

we get

$$e_k(B_2, \| \cdot \|_{\infty, X}) \leq C M 2^{-k/2} [\log m]^{1/2}.$$

For any $r > 1$, we also have

$$e_k(B_2, \| \cdot \|_r) \leq C 2^{-k/2} \mathbb{E}_g \| \sum_{k=1}^N g_k u_k \|_r.$$

Note, that

$$\mathbb{E}_g \| \sum_{k=1}^N g_k u_k \|_r \leq (\mathbb{E}_X \mathbb{E}_g \| \sum_{k=1}^N g_k u_k (X) \|_r)^{1/r} \leq C_1 \sqrt{r} \left( \mathbb{E}_X \left( \sum_{k=1}^N |u_k(X)|^2 \right)^{r/2} \right)^{1/r} \leq C_1 M \sqrt{r}.$$
For a fixed $r > 2$ we now proceed like in [14, Lemma 16.8.8]. Take any $R > r$ and let $\theta \in (0, 1)$ be such that $1/r = (1 - \theta)/2 + \theta/R$. Then $\|f\|_r \leq \|f\|^{1+\theta}_r \|f\|^\theta_R$ and

$$e_k(B_2, \| \cdot \|_r) \leq 2e_k(B_2, \| \cdot \|_R)^\theta \leq C_2[2^{-k}RM^2]^{\theta/2} = C_3[2^{-k}RM^2]^{1/2-1/r+\theta/R}.$$  

Thus, since

$$[2^{-k}M^2]^{\theta/R} \leq M^{2/R}, \quad R^{\theta/R} \leq 2,$$

taking $R = 2r \log M$, we get

$$e_k(B_2, \| \cdot \|_r) \leq C_4 r^{1/2-1/r} [2^{-k}M^2 \log M]^{1/2-1/r}.$$  

By [14, Lemma 16.8.10] we get

$$e_k(B_{r'}, \| \cdot \|_2) \leq C_5 r^{1/2-1/r} [2^{-k}M^2 \log M]^{1/2-1/r}.$$  

Taking $r = p'$ we get

$$e_{k+1}(B_p, \| \cdot \|_{\infty, x}) \leq C_6 (1 - 1/p)^{1/2-1/p} [\log m]^{1/2} [\log M]^{1/p-1/2} M^{2/p} 2^{-k/p} + 1.$$  

By estimate (2.2) we have

$$e_k(B_p, \| \cdot \|_{\infty, x}) \leq C_7 (1 - 1/p)^{1/2-1/p} [\log m]^{1/2} [\log M]^{1/p-1/2} M^{2/p} \begin{cases} 2^{-k/p}, & k \leq \log N, \\ N^{-1/p} 2^{-2k/N}, & k \geq \log N. \end{cases}$$

Thus, $\theta_m := C_8 (1 - 1/p)^{p/2-1} M^2 \log N [\log M]^{-p/2} [\log m]^{p/2} m^{-1}$. The theorem is proved. ~\( \Box \)

The next theorem provides the Marcinkiewicz-type discretization in $L^p$ for $p \in (1, 2)$.

**Theorem 4.5.** Let $p \in (1, 2)$, $a, b > 0$, and let $L$ be an $N$-dimensional subspace of $L^p$. Let $B_p$ be the unit ball in $L$, i.e. $B_p = \{ f \in L: \|f\|_p \leq 1 \}$. Assume that there is a constant $M \leq aN^b$, $M \geq 2$, such that for any $f \in L$ one has

$$\|f\|_\infty \leq M\|f\|_2.$$  

Then, for any $\varepsilon \in (0, 1)$, there is a number $C(a, b, p, \varepsilon)$ such that for any

$$m \geq C(a, b, p, \varepsilon) M^2 [\log N]^2$$

one has

$$P\left( \sup_{f \in B_p} \left| \frac{1}{m} \sum_{j=1}^m |f(X_j)|^p - \|f\|_p^p \right| \leq \varepsilon \right) \geq 1/2.$$  

Thus, there are $m = C(a, b, p, \varepsilon) M^2 [\log N]^2$ points $x_1, \ldots, x_m$ such that

$$(1 - \varepsilon)\|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^m |f(x_j)|^p \leq (1 + \varepsilon)\|f\|_p^p$$

for any $f \in L$.

**Proof.** By the above theorem

$$\theta_m := C(p) M^2 \log N [\log M]^{-p/2} [\log m]^{p/2} m^{-1} \leq C(a, b, p) M^2 [\log N]^{2-p/2} [\log m]^{p/2} m^{-1}.$$  

Taking the number $C(a, b, p, \varepsilon)$ big enough and $m \geq C(a, b, p, \varepsilon) M^2 [\log N]^2$ we get $\theta_m \leq \frac{\varepsilon^2}{16C(p)^2}$ and the result follows from Corollary 3.4. The theorem is proved. ~\( \Box \)

Finally we consider the case $p = 1$. 


Theorem 4.6. Let $L$ be an $N$-dimensional subspace of $L^1$. Let $B_1$ be the unit ball in $L$, i.e. $B_1 = \{ f \in L : \|f\|_1 \leq 1 \}$. Assume there is a constant $M \geq 3$ such that for any $f \in L$ one has

$$\|f\|_\infty \leq M\|f\|_2.$$  

Then there is a number $C$, such that

$$\theta_m = CM^2[\log N]^{5/2}[\log M]^{1/2}[\log m]^{1/2}m^{-1}$$

where $\theta_m$ was defined in Corollary 3.4.

Proof. We firstly note that $\|f\|_p \leq M^{2(1-1/p)}\|f\|_1$ for any $f \in L$ for any $p \in (1, 2)$. Indeed, $\|f\|_p^p \leq \|f\|_\infty^{p-1}\|f\|_1 \leq M^{2(p-1)}\|f\|_1^p$. Thus, for any fixed set of points $X = \{X_1, \ldots, X_m\}$ one has

$$e_k(B_1, \| \cdot \|_\infty, X) \leq M^{2(1-1/p)}e_k(B_p, \| \cdot \|_\infty, X) \leq C(1 - 1/p)^{1/2-1/p}[\log m]^{1/2}[\log M]^{1/2-1/p}M^{2-1/p/2} e \in [0, 1],$$

where we have used the bound, obtained in the proof of Theorem 4.4. Let $\varepsilon = 1 - 1/p$, then

$$e_k(B_1, \| \cdot \|_\infty, X) \leq C\varepsilon^{-1/2}[\log m]^{1/2}[\log M]^{1/2}M^{2-\varepsilon}2^{-k}\varepsilon\leq \sum_{k=1}^{\log N} N^{-1/2}C\varepsilon^{-1/2}[\log m]^{1/2}[\log M]^{1/2}M^{2-\varepsilon}2^{-k}\varepsilon \leq C\varepsilon^{-1/2}[\log m]^{1/2}[\log M]^{1/2}M^{2-\varepsilon}2^{-k}\varepsilon$$

for $k \leq \log N$. Taking $\varepsilon = [\log N]^{-1}$ we get

$$e_k(B_1, \| \cdot \|_\infty, X) \leq C_1[\log m]^{1/2}[\log N]^{1/2}[\log M]^{1/2}M^{2-\varepsilon}2^{-k}\varepsilon \leq C_1[\log m]^{1/2}[\log N]^{1/2}[\log M]^{1/2}M^{2-\varepsilon}2^{-k}\varepsilon$$

for $k \leq \log N$. Thus, by estimate (2.2) we have

$$e_k(B_1, \| \cdot \|_\infty, X) \leq C_2[\log m]^{1/2}[\log N]^{1/2}[\log M]^{1/2}M^{2-\varepsilon}2^{-k}\varepsilon$$

By the definition of $\theta_m$ we have $\theta_m = C_2[\log m]^{1/2}[\log N]^{5/2}[\log M]^{1/2}M^{2-\varepsilon}2^{-k}\varepsilon$. 

Theorem 4.7. Let $a, b > 0$ and let $L$ be an $N$-dimensional subspace of $L^1$. Let $B_1$ be the unit ball in $L$, i.e. $B_1 = \{ f \in L : \|f\|_1 \leq 1 \}$. Assume there is a constant $M \geq aN^b$, $M \geq 3$ such that for any $f \in L$ one has

$$\|f\|_\infty \leq M\|f\|_2.$$  

Then, for any $\varepsilon \in (0, 1)$, there is a number $C(a, b, \varepsilon)$ such that for any

$$m \geq C(a, b, \varepsilon)M^2[\log N]^{7/2}$$

one has

$$P\left( \sup_{f \in B_p} \left| \frac{1}{m} \sum_{j=1}^{m} |f(X_j)|^p - \|f\|_p^p \right| \leq \varepsilon \right) \geq 1/2.$$

Thus, there are $m = C(a, b, \varepsilon)M^2[\log N]^{7/2}$ points $x_1, \ldots, x_m$ such that

$$(1 - \varepsilon)\|f\|_p^p \leq \frac{1}{m} \sum_{j=1}^{m} |f(x_j)|^p \leq (1 + \varepsilon)\|f\|_p^p$$

for any $f \in L$.

Proof. The proof is similar to the proofs of Theorems 4.3 and 4.5. \qed
5. Appendix A: the proof of Theorem 2.5

We again stress that the proof of Theorem 2.5 heavily follows the proof of [9, Theorem 7.3] and is presented here only for readers’ convenience. We first recall the claim of the theorem.

**Theorem 2.5.** Let \( q \geq 2, p > 1, \alpha > 0 \). Let \( L \) be a linear space endowed with a \( q \)-convex norm \( \| \cdot \| \) with a constant \( \eta \), i.e.

\[
\left\| \frac{f + g}{2} \right\| \leq \max(\|f\|, \|g\|) - \eta \|f - g\|^q
\]

for any \( f, g \) with \( \|f\| \leq 1, \|g\| \leq 1 \). Let \( \rho \) be a quasi-metric on \( L \) such that

\[
\rho(f, g) \leq R(\rho(f, h) + \rho(h, g)); \quad \rho\left(f, \frac{f + g}{2}\right) \leq \varpi \rho(f, g)
\]

for all \( f, g, h \in L \) for some constants \( R, \varpi > 0 \). Assume that there is a metric \( d \) on \( L \) and for each \( h \in L \) there is a norm \( \| \cdot \|_h \) on \( L \) such that

\[
c_1 d(f, g)^p \leq \rho(f, g) \leq c_2 \left( \|f - g\|_h + d(f, g)(d(f, h)^{p-1} + d(h, g)^{p-1}) \right)
\]

for some numbers \( a, b \). Then there is a number \( C := C(q, p, \alpha, R, \varpi, c_1, c_2) \) such that for the unit ball \( B := \{\|f\| \leq 1\} \) one has

\[
\gamma_{\alpha, 1}(B, \rho) \leq C \left( \eta^{-1/q} \left[ \sup_{\|f\| \leq 1} \sum_{k=0}^{\infty} (2^{k/\alpha} e_k(B, d_f))^q(1-1)^{q-1} \right]^{\eta^{-1/q}} + [\eta_{\alpha, p, r}(B, d)]^p \right),
\]

where \( e_k(B, \rho) := \|f - g\|_h \).

We recall the main tools from [9] concerning chaining through interpolation. Let

\[
K(t, f) := \inf_{g \in L}(\|g\| + t \rho(f, g))
\]

and let \( \pi_t(f) \) be any minimizer.

The following contraction principle is formulated and proved in Theorem 3.1 in [9].

**Theorem 5.1.** Assume there are functions \( s_k(f) \geq 0 \) and a number \( a > 0 \) such that

\[
e_k(B, \rho) \leq a \text{diam}(A, \rho) + \sup_{f \in B} s_k(f)
\]

for every \( k \in \mathbb{N} \) and every set \( A \subset B \). Then

\[
\gamma_{\alpha, r}(B, \rho) \leq C(\alpha) \left( a \gamma_{\alpha, r}(B, \rho) + \left[ \sup_{f \in B} \sum_{k=0}^{\infty} (2^{k/\alpha} s_k(f))^r \right]^{1/r} \right).
\]

The following theorem is Lemma 4.5 in [9].

**Theorem 5.2.** For every \( a > 0 \) one has

\[
\sup_{f \in B} \sum_{k=0}^{\infty} 2^{k/\alpha} \rho(f, \pi_{a^{2/\alpha}}(f)) \leq C(\alpha) a^{-1} \sup_{f \in B} \|f\|.
\]

Throughout this section the expression \( V \leq W \) means that there exists a number \( C := C(q, p, \alpha, R, \varpi, c_1, c_2) \) such that \( V \leq CW \).

**Lemma 5.3.** For any \( t > 0 \) and \( A \subset B \) one has

\[
\text{diam}(A_t, \| \cdot \|) \leq c(\varpi, R, q) \left( \frac{t}{\eta} \right)^{1/q} \left( \text{diam}(A, \rho) + \sup_{h \in A} \rho(h, \pi_t(h)) \right)^{1/q}
\]

where \( A_t := \{\pi_t(h) : h \in A\} \).
Proof. We note that
\[ \|\pi_t(f)\| \leq K(t, f) \leq \|u\| + tR(g(f, \pi_t(f)) + g(\pi_t(f), u)) \]
for any \( u \in L \). Thus, for fixed \( f, g \in A \) we take \( u = (\pi_t(f) + \pi_t(g))/2 \) and obtain
\[ \max(\|\pi_t(f)\|, \|\pi_t(g)\|) \leq \frac{\|\pi_t(f) + \pi_t(g)\|}{2} + tR\sup_{h \in A} g(h, \pi_t(h)) + tR\varnothing(g(\pi_t(f), \pi_t(g)). \]
By the definition of \( q\)-convexity we get
\[ \eta\|\pi_t(f) - \pi_t(g)\|^q \leq tR\sup_{h \in A} g(h, \pi_t(h)) + tR\varnothing(g(\pi_t(f), \pi_t(g)) \leq \eta t^3\varnothing(f, g) + (R + \varnothing R^2 + \varnothing R^2) \sup_{h \in A} g(h, \pi_t(h)). \]
This bound implies the statement of the lemma. \( \square \)

Remark 5.4. The lemma actually means that the set \( A_t \) is contained in some ball of radius \( c(\varnothing, R, q)(\frac{t}{\eta})^{1/q} \left( \text{diam}(A, g) + \sup_{h \in A} g(h, \pi_t(h)) \right)^{1/q} \) with respect to the norm \( \| \cdot \| \).

Lemma 5.5. Let \( (F_n) \) be an admissible sequence of \( B \) and \( a, b > 0 \). Then
\[ e_k(A, g) \lesssim b \text{diam}(A, g) + \sup_{f \in A} s_k(f) \]
for every \( k \geq 1 \) and every \( A \subset B \) where
\[ s_k(f) = (b + 1)g(f, \pi_{a2^k/\alpha}(f)) + \left( \frac{a_{2^k/\alpha}}{b\eta} \right)^{1/(q-1)} (e_{k-1}(B, d_f))^q/(q-1) + \left( \text{diam}(F_{k-1}(f), d) \right)^p. \]

Proof. For any \( F \subset B \) let
\[ A_{a2^k/\alpha}^F := \{ \pi_{a2^k/\alpha}(f) : f \in A \cap F \}, \]
let \( h_F \) be any point in \( A \cap F \) and let \( T_{k-1}^F \subset A_{a2^k/\alpha}^F \) be a net such that \( |T_{k-1}^F| \leq 2^{2^k-1} \) and
\[ \sup_{f \in A_{a2^k/\alpha}^F} d_{h_F}(f, T_{k-1}^F) \leq 4e_{k-1}(A_{a2^k/\alpha}^F, d_{h_F}). \]

Let \( T_k := \bigcup_{F \in F_{k-1}} T_{k-1}^F \). Note that \( |T_k| \leq 2^{2^k} \). We now show that
\[ \sup_{f \in A} g(f, T_k) \lesssim b \text{diam}(A, g) + \sup_{f \in A} s_k(f) \]
Let \( f \in A \) and let \( g \in T_{k-1}^{F_{k-1}(f)} \) be such that
\[ d_{h_{F_{k-1}(f)}}(\pi_{a2^k/\alpha}(f), g) \leq 4e_{k-1}(A_{a2^k/\alpha}^{F_{k-1}(f)}, d_{h_{F_{k-1}(f)}}). \]
We have
\[ g(f, T_k) \leq Rg(f, \pi_{a2^k/\alpha}(f)) + Rg(\pi_{a2^k/\alpha}(f), g) \lesssim g(f, \pi_{a2^k/\alpha}(f)) + d_{h_{F_{k-1}(f)}}(\pi_{a2^k/\alpha}(f), g) + d(\pi_{a2^k/\alpha}(f), g)(d(\pi_{a2^k/\alpha}(f), h_{F_{k-1}(f)})^p + d(h_{F_{k-1}(f)}, g)^p). \]
Let \( f' \in A \cap F_{k-1}(f) \) be such that \( g = \pi_{a2^k/\alpha}(f') \) (it exists since \( g \in T_{k-1}^{F_{k-1}(f)} \subset A_{a2^k/\alpha}^{F_{k-1}(f)} \)). Thus,
\[ d(\pi_{a2^k/\alpha}(f), g) \leq d(\pi_{a2^k/\alpha}(f), f) + d(f, f') + d(f', \pi_{a2^k/\alpha}(f')) \leq 2\sup_{h \in A} d(\pi_{a2^k/\alpha}(h), h) + \text{diam}(F_{k-1}(f), d) \lesssim \sup_{h \in A} (g(\pi_{a2^k/\alpha}(h), h))^{1/p} + \sup_{h \in A} \text{diam}(F_{k-1}(h), d). \]
and, similarly,
\[ d(π_{a2^k/α}(f), h_{F_k-1(f)})^{p-1} + d(h_{F_k-1(f)}, g)^{p-1} \leq 2(d(π_{a2^k/α}(f), h_{F_k-1(f)}) + d(h_{F_k-1(f)}, g))^{p-1} \]
\[ \lesssim (\sup_{h ∈ A}(\varrho(π_{a2^k/α}(h), h)))^{1/p} + \sup_{h ∈ A}(\text{diam}(F_k-1(h), d))^{p-1}. \]

The above bounds implies
\[ \varrho(f, T_k) \lesssim \sup_{h ∈ A}(\varrho(h, π_{a2^k/α}(h))) + e_{k-1}(A_{a2^k/α}^{F_k-1(f)}, d_{h_{F_k-1(f)}}) + \sup_{h ∈ A}(\text{diam}(F_k-1(h), d))^p. \]

We now apply Lemma 5.3 to estimate the entropy number \(e_{k-1}(A_{a2^k/α}^{F_k-1(f)}, d_{h_{F_k-1(f)}})\):
\[ e_{k-1}(A_{a2^k/α}^{F_k-1(f)}, d_{h_{F_k-1(f)}}) \lesssim \left( \frac{a^{2k/α}}{\eta} \right)^{1/q} \left( \text{diam}(A, \varrho) + \sup_{h ∈ A}(\varrho(h, π_{a2^k/α}(h))) \right)^{1/q} \sup_{h ∈ A}(e_{k-1}(B, d_{h_{F_k-1(f)}}))^q. \]

where we have used the assumption that the metric \(d_h\) is generated by a norm. Using the estimate \(x^{1/q} y ≤ b x + b^{-1/(q-1)} y^{q/(q-1)}\) we get
\[ e_{k-1}(A_{a2^k/α}^{F_k-1(f)}, d_{h_{F_k-1(f)}}) \lesssim b \text{diam}(A, \varrho) + (b + 1) \sup_{h ∈ A}(\varrho(h, π_{a2^k/α}(h))) + \sup_{h ∈ A}(\text{diam}(F_k-1(h), d))^p \]
\[ + \left( \frac{a^{2k/α}}{b^q} \right)^{1/(q-1)} \sup_{h ∈ A}(e_{k-1}(B, d_h))^q, \]

which completes the proof of the lemma. □

**Proof of Theorem 2.5**
Let \(s_k\) be as in Lemma 5.3 for \(k ≥ 1\) and let \(s_0(f) := \text{diam}(B, \varrho)\), then by Theorem 5.1 one has
\[ γ_{α,1}(B, \varrho) ≤ C(α) \left( b γ_{α,1}(B, \varrho) + \sup_{f ∈ B} \sum_{k ≥ 0} 2^{k/α} s_k(f) \right) \]
which, in our case, provides the bound
\[ γ_{α,1}(B, \varrho) \lesssim b γ_{α,1}(B, \varrho) + \text{diam}(B, \varrho) + (b + 1) \sup_{f ∈ B} \sum_{k ≥ 1} 2^{k/α} \varrho(f, π_{a2^k/α}(f)) \]
\[ + \left( \frac{a}{b^q} \right)^{1/(q-1)} \sup_{f ∈ B} \sum_{k ≥ 1} (2^{k/α} e_{k-1}(B, d_f))^q + \sup_{f ∈ B} \sum_{k ≥ 1} 2^{k/α} (\text{diam}(F_k-1(h), d))^p \]
for any admissible sequence \((F_k)\) of \(B\). Taking \(b\) sufficiently small and applying Theorem 5.2 we get
\[ γ_{α,1}(B, \varrho) \lesssim \text{diam}(B, \varrho) + a^{-1} + \left( \frac{a}{b^q} \right)^{1/(q-1)} \sup_{f ∈ B} \sum_{k ≥ 1} (2^{k/α} e_{k-1}(B, d_f))^q \]
\[ + \sup_{f ∈ B} \sum_{k ≥ 1} 2^{k/α} (\text{diam}(F_k-1(h), d))^p. \]

Taking infimum over all admissible sequences \((F_k)\) of \(B\) and taking
\[ a = \left( b^q \right)^{1/(q-1)} \sup_{f ∈ B} \sum_{k ≥ 1} (2^{k/α} e_{k-1}(B, d_f))^q \]
\[ , \]

\[ \text{we get} \]

\[ γ_{α,1}(B, \varrho) \lesssim \text{diam}(B, \varrho) + a^{-1} + \left( \frac{a}{b^q} \right)^{1/(q-1)} \sup_{f ∈ B} \sum_{k ≥ 1} (2^{k/α} e_{k-1}(B, d_f))^q \]
\[ + \sup_{f ∈ B} \sum_{k ≥ 1} 2^{k/α} (\text{diam}(F_k-1(h), d))^p. \]
we obtain
\[ \gamma_{\alpha,1}(B, \varrho) \lesssim \text{diam}(B, \varrho) + \eta^{-1/q} \left( \sup_{f \in B} \sum_{k \geq 0} \left( 2^{k/\alpha} e_k(B, d_f) \right)^{q/(q-1)} \right)^{(q-1)/q} + \gamma_{\alpha,p}(B, d)^p \]
Since \( \text{diam}(B, \varrho) \leq c_2 \text{diam}(B, d_h) + c_2 \text{diam}(B, d)^p \), we get the claim of the theorem.

6. Appendix B: the proof of Theorem 2.9

We firstly formulate the desired statement.

**Theorem 2.9** Let \( B := \{ x : \| x \| \leq 1 \} \) where the norm \( \| \cdot \| \) is \( q \)-convex with constant \( \eta \). Let \( d \) be a metric which is generated by another norm. Then for any \( \alpha > 0 \) and for any \( p \in [1, q) \) there is a number \( C(\alpha, p, q, \eta) \) such that
\[ \gamma_{\alpha,p}(B, d) \leq C(\alpha, p, q) \eta^{-p/q} \left( \sum_{k \geq 0} \left( 2^{k/\alpha} e_k(B, d) \right)^{pq/(q-p)} \right)^{1/p-1/q}. \]

Let \( K(t, f) := \inf_{g \in L} (\| d \| + t^p d(f, g)^p) \) and let \( \pi_t(f) \) be any minimizer.

We need the following lemma from [9] (see Lemma 5.9 there).

**Lemma 6.1.** For every \( a > 0 \) one has
\[ \sup_{f \in B} \sum_{k \geq 0} \left( 2^{k/\alpha} d(f, \pi_{a2^k/\alpha}(f)) \right)^p \leq c(p, \alpha)a^{-p}. \]

Similarly to the proof of Lemma 5.3 one can obtain the following lemma.

**Lemma 6.2.** For any \( t > 0 \) and \( A \subset B \) one has
\[ \text{diam}(A_t, \| \cdot \|) \leq c(p, q) \left( \frac{t}{\eta} \right)^{p/q} \left( \text{diam}(A, d) + \sup_{h \in A} d(h, \pi_t(h)) \right)^{p/q} \]
where \( A_t := \{ \pi_t(h) : h \in A \} \).

**Proof of Theorem 2.9**

From Lemma 6.2 for any \( b > 0 \) we get the bound
\[ e_k(A_t, d) \leq c(p, q) \left( \frac{t}{\eta} \right)^{p/q} \left( \text{diam}(A, d) + \sup_{h \in A} d(h, \pi_t(h)) \right)^{p/q} e_k(B, d) \]
\[ \leq c(p, q) \left( b \text{diam}(A, d) + b \sup_{h \in A} d(h, \pi_t(h)) + \left( \frac{t}{b\eta} \right)^{p/(q-p)} e_k(B, d)^{q/(q-p)} \right) \]
and
\[ e_k(A, d) \leq c(p, q) \left( b \text{diam}(A, d) + (b + 1) \sup_{h \in A} d(h, \pi_t(h)) + \left( \frac{t}{b\eta} \right)^{p/(q-p)} e_k(B, d)^{q/(q-p)} \right). \]

Taking \( t = a2^{k/\alpha} \) and applying Theorem 5.1 we get
\[ \gamma_{\alpha,p}(B, d) \leq c(\alpha, p, q) \left( b\gamma_{\alpha,p} + (b + 1) \left[ \sup_{h \in B} \sum_{k \geq 0} \left( 2^{k/\alpha} d(h, \pi_{a2^k/\alpha}(h)) \right) \right]^{1/p} \right)^{1/p} \]
\[ + \left( \frac{a}{b\eta} \right)^{p/(q-p)} \left[ \sum_{k \geq 0} \left( 2^{k/\alpha(1+p/(q-p))} e_k(B, d)^{q/(q-p)} \right) \right]^{1/p} \right). \]

Taking \( b \) sufficiently small and applying Lemma 6.1 we get
\[ \gamma_{\alpha,p}(B, d) \leq c(\alpha, p, q) \left( a^{-1} + \left( \frac{a}{\eta} \right)^{p/(q-p)} \left[ \sum_{k \geq 0} \left( 2^{k/\alpha} e_k(B, d) \right)^{pq/(q-p)} \right]^{1/p} \right). \]

Optimizing over \( a > 0 \) we get the desired bound.
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