Quasi-bound state in the $\bar{K}NNN$ system

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Abstract

The paper is devoted to the $\bar{K}NNN$ system, which is an exotic system consisting of an antikaon and three nucleons. Dynamically exact four-body Faddeev-type equations were solved and characteristics of the quasi-bound state in the system were evaluated. Three antikaon-nucleon and three nucleon-nucleon potentials were used, so the dependence of the four-body pole positions on the two-body interaction models was studied. The resulting binding energies $B_{\bar{K}^{-}ppn}^{\text{Chiral}} \sim 30.5 - 34.5$ MeV obtained with chirally motivated and $B_{\bar{K}^{-}ppn}^{\text{SIDD}} \sim 46.4 - 52.0$ MeV obtained with phenomenological antikaon-nucleon potentials are close to those obtained for the $K^{-}pp$ system with the same $\bar{K}N$ and $NN$ potentials, while the four-body widths $\Gamma_{\bar{K}^{-}ppn} \sim 38.2 - 50.9$ MeV are smaller.

I. INTRODUCTION

Attractive nature of $\bar{K}N$ interaction lead to suggestions, that quasi-bound states can exist in few-body systems consisting of antikaons and nucleons [1]. In particular, a deep and relatively narrow quasi-bound state was predicted in the lightest three-body $\bar{K}NN$ system [2]. Many theoretical calculations of the system were performed after that using different methods and inputs. All of them agree, that the quasi-bound state really exists in spin-zero state of $\bar{K}NN$, usually denoted as $K^{-}pp$, but predict quite different binding energies and widths of the state.

The experimental situation is unsettled as well: several candidates for the $K^{-}pp$ state were reported by different experiments [3] – [5], while other experiments left the matter unsettled [6], [7]. However, the measured binding energies and especially decay widths of the state differ from each other and are far from all theoretical predictions. The most recent results by J-PARK E15 experiment [8], [9] for the binding energy are comparable to some theoretical predictions, but the width of the $K^{-}pp$ quasi-bound state is still too large.

In recent years we performed a series of calculations of different states of the three-body $\bar{K}NN$ and $\bar{K}\bar{K}N$ systems [10], using dynamically exact Faddeev-type equations in AGS form with coupled $\bar{K}NN$ and $\pi\Sigma N$ channels. In particular, we evaluated $K^{-}pp$ quasi-bound state binding energy and width using three different models of $\bar{K}N$ interaction. The same was done for the $\bar{K}\bar{K}N$ system. We also demonstrated, that there is no quasi-bound

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states, caused by pure strong interactions, in spin one state of $KNN$ system, which is $K^{-}np$. In addition, we calculated near-threshold amplitudes of $K^{-}$ elastic scattering on deuteron. Finally, we evaluated $1s$ level shift in kaonic deuterium, which is an atomic state, caused by presence of the strong $KN$ interaction in comparison to the pure Coulomb state.

Four-body $KNNN$ system is another system with strangeness, which could lead more light on the question of antikaon-nucleon interaction. Some theoretical calculations of the quasi-bound state were already performed \[1\,11–13\], but more accurate calculations are needed. We calculated binding energy and width of the state using four-body Faddeev-type equations by Grassberger and Sandhas \[14\]. Only these dynamically exact equations in momentum representation can treat energy-dependent $KN$ potentials, necessary for the this system, exactly. We used our two-body antikaon-nucleon potentials, constructed for the three-body AGS calculations \[10\], as the input.

II. FOUR-BODY FADDEEV-TYPE EQUATIONS

The three-body Faddeev-type equations in Alt-Grassberger-Sandhas form

$$ U_{\alpha\beta}(z) = (1 - \delta_{\alpha\beta})G_{0}^{-1}(z) + \sum_{\gamma=1}^{3} (1 - \delta_{\alpha\gamma})T_{\gamma}(z)G_{0}(z)U_{\gamma\beta}(z) $$

(1)

define the three-body transition operators $U_{\alpha\beta}(z)$, which describe process $\beta + (\alpha\gamma) \rightarrow \alpha + (\beta\gamma)$. The $G_{0}(z)$ in Eq. (1) is three-body Green function, Faddeev partition indices $\alpha, \beta = 1, 2, 3$ simultaneously define a particle ($\alpha$) and the remained pair ($\beta\gamma$), $\alpha \neq \beta \neq \gamma$. The operator $T_{\alpha}(z)$ is a two-body $T$-matrix, describing interaction in the ($\beta\gamma$) pair.

A separable potential $V_{\alpha}$ leading to a separable $T$-matrix

$$ V_{\alpha} = \lambda_{\alpha}|g_{\alpha}\rangle \langle g_{\alpha}| \rightarrow T_{\alpha}(z) = |g_{\alpha}\rangle \tau_{\alpha}(z) \langle g_{\alpha}| $$

(2)

allows to write the three-body AGS equations in the form

$$ X_{\alpha\beta}(z) = Z_{\alpha\beta}(z) + \sum_{\gamma=1}^{3} Z_{\alpha\gamma}(z)\tau_{\gamma}(z)X_{\gamma\beta}(z) $$

(3)

with new transition $X_{\alpha\beta}$ and kernel $Z_{\alpha\beta}$ operators, defined by

$$ X_{\alpha\beta}(z) = \langle g_{\alpha}|G_{0}(z)U_{\alpha\beta}(z)G_{0}(z)|g_{\beta}\rangle, $$

(4)

$$ Z_{\alpha\beta}(z) = (1 - \delta_{\alpha\beta})\langle g_{\alpha}|G_{0}(z)|g_{\beta}\rangle. $$

(5)
Here for simplicity the one-term separable potentials Eq. (2) used, while in general $V_\alpha$ can consist of $N$ terms.

The four-body Faddeev-type Grassberger-Sandhas equations were derived in Ref. [14]

$$U_{\alpha\beta}^{\sigma\rho}(z) = (1 - \delta_{\sigma\rho})\delta_{\alpha\beta}G_0^{-1}(z)T_{\alpha\beta}(z)G_0^{-1}(z) + \sum_{\tau,\gamma} (1 - \delta_{\sigma\gamma})U_{\alpha\gamma}T_{\tau\gamma}(z)G_0(z)U_{\beta\gamma}^{\tau\rho}$$

(6)

In addition to the free Green function $G_0(z)$, which now acts in four-body space, and the two-body $T$-matrix $T_{\alpha}(z)$, three-body $U_{\alpha\beta}(z)$ and four-body $U_{\alpha\beta}^{\sigma\rho}(z)$ operators enter the system Eq. (6). The high indices $\sigma, \rho, \tau$ define a partition, which could be $3 + 1$ or $2 + 2$ type, while the low indices $\alpha, \beta$ define two-body subsystems of the particular three-body subsystem, denoted by the high index.

If the separable potentials Eq. (2), leading to the corresponding separable $T$-matrices, is used, the system Eq. (6) can be rewritten in the same way, as the three-body one. The new system of equations

$$\bar{U}_{\alpha\beta}^{\sigma\rho}(z) = (1 - \delta_{\sigma\rho})(\bar{G}_0^{-1})_{\alpha\beta}(z) + \sum_{\tau,\gamma,\delta} (1 - \delta_{\sigma\tau})\bar{T}_{\alpha\gamma}(z)(\bar{G}_0)_{\gamma\delta}(z)\bar{U}_{\beta\delta}^{\tau\rho}(z)$$

(7)

contains new operators

$$\bar{U}_{\alpha\beta}^{\sigma\rho}(z) = \langle g_\alpha | G_0(z)U_{\alpha\beta}^{\sigma\rho}(z)G_0(z)|g_\beta \rangle,$$

(8)

$$\bar{T}_{\alpha\beta}(z) = \langle g_\alpha | G_0(z)U_{\alpha\beta}(z)G_0(z)|g_\beta \rangle,$$

(9)

$$(\bar{G}_0)_{\alpha\beta}(z) = \delta_{\alpha\beta}T_{\alpha}(z).$$

(10)

It is seen that the four-body system with separable potentials Eq. (7) looks similarly to Eq. (1), which describes a three-body system with arbitrary potentials. This analogy can be used for further modification of the equations. Namely, if the three-body $T$-matrices $\bar{T}_{\alpha\beta}(z)$ in Eq. (7) are presented in a separable form

$$\bar{T}_{\alpha\beta}(z) = |\bar{g}_\alpha^\tau\bar{\tau}_{\alpha\beta}(z)|\bar{g}_\beta^\tau|,$$

(11)

the four-body equations Eq. (7) can be rewritten as

$$\bar{X}_{\alpha\beta}^{\sigma\rho}(z) = \bar{Z}_{\alpha\beta}^{\sigma\rho}(z) + \sum_{\tau,\gamma,\delta} \bar{Z}_{\alpha\gamma}^{\tau\rho}(z)\bar{\tau}_{\gamma\delta}(z)\bar{X}_{\beta\delta}^{\tau\rho}(z)$$

(12)
with new transition $\bar{X}_{\alpha\beta}^\sigma$ and kernel $\bar{Z}_{\alpha\beta}^\sigma$ operators, defined by

$$
\bar{X}_{\alpha\beta}^\sigma(z) = \langle \bar{g}_\alpha^\sigma | (\bar{G}_0)_{\alpha\alpha}(z) \bar{U}_{\alpha\beta}^\sigma(z)(\bar{G}_0)_{\beta\beta}(z) | \bar{g}_\beta^\rho \rangle, 
$$
(13)

$$
\bar{Z}_{\alpha\beta}^\sigma(z) = (1 - \delta_{\sigma\rho}) \langle \bar{g}_\alpha^\sigma | (\bar{G}_0)_{\alpha\beta}(z) | \bar{g}_\beta^\rho \rangle.
$$
(14)

We solved the four-body equations Eq. (12) with separable two-body $T$-matrices being an input and three-body $T$-matrices $\bar{T}_{\alpha\beta}^\tau$ represented in separable form.

### III. SEPARABLE THREE-BODY AMPLITUDES

We used $KN$ and $NN$ potentials, which are separable ones by construction. Therefore, separable versions of three-body and 2+2 amplitudes, entering the equations (12), should be constructed. These amplitudes are described by three-body AGS equations Eq.(3), which being written in momentum basis for $s$-wave interactions have a form

$$
X_{\alpha\beta}(p, p'; z) = Z_{\alpha\beta}(p, p'; z) + \sum_{\gamma=1}^{3} 4\pi \int_{0}^{\infty} Z_{\alpha\gamma}(p, p''; z) \tau_{\gamma}(p''; z) X_{\gamma\beta}(p'', p'; z) p''^2 dp''.
$$
(15)

Here $p, p'$ and $z$ are relative momenta and three-body energy. It is possible to evaluate eigenvalues $\lambda_n$ and eigenfunctions $g_{n\alpha}(p; z)$ of the system Eq.(15) from

$$
g_{n\alpha}(p; z) = \frac{1}{\lambda_n} \sum_{\gamma=1}^{3} 4\pi \int_{0}^{\infty} Z_{\alpha\gamma}(p, p'; z) \tau_{\gamma}(p'; z) g_{n\gamma}(p'; z) p'^2 dp'.
$$
(16)

with normalization condition

$$
\sum_{\gamma=1}^{3} 4\pi \int_{0}^{\infty} g_{n\gamma}(p'; z) \tau_{\gamma}(p'; z) g_{n\gamma}(p'; z) p'^2 dp' = -\delta_{nn'}.
$$
(17)

Knowledge of the eigenvalues and eigenfunction allows us to write down Hilbert-Schmidt expansion (HSE) of the kernel functions $Z_{\alpha\beta}$:

$$
Z_{\alpha\beta}^{\text{HSE}}(p, p'; z) = -\sum_{n=1}^{\infty} \lambda_n g_{n\alpha}(p; z) g_{n\beta}(p'; z),
$$
(18)

which leads to the separable three-body amplitude

$$
X_{\alpha\beta}^{\text{HSE}}(p, p'; z) = -\sum_{n=1}^{\infty} \frac{\lambda_n}{1 - \lambda_n} g_{n\alpha}(p; z) g_{n\beta}(p'; z).
$$
(19)
Since the kernel function $Z_{\alpha \beta}$ and three-body amplitude $X_{\alpha \beta}$ are energy-dependent functions, being off energy shell in the four-body equations Eq. (12), it is necessary to solve eigen-equations Eq. (16) with normalization condition Eq. (17) for every value of the three-body energy during the four-body calculations. It is timely consuming work. Instead of this we used Energy Dependent Pole Expansion/Approximation (EDPE/EDPA) method, suggested in [17] specially for the four-body GS equations. It needs solution of the eigen-equations Eq. (16) only once, for a fixed energy $z_{\text{fix}}$. Usually it is chosen to be the binding energy $z_{\text{fix}} = E_B$ if a bound state in the system exists or $z_{\text{fix}} = 0$ if not. After that energy dependent form-factors

$$g_{m\alpha}(p; z) = \sum_{\gamma=1}^{3} \frac{4\pi}{3} \int_{0}^{\infty} Z_{\alpha \gamma}(p, p'; z) \tau_{\gamma}(p'; z_{\text{fix}}) g_{n\gamma}(p'; z_{\text{fix}}) p'^2 dp'$$  \hspace{1cm} (20)

and propagators

$$\left(\Theta(z)\right)^{-1}_{mn} = \sum_{\gamma=1}^{3} \frac{4\pi}{3} \int_{0}^{\infty} g_{m\gamma}(p'; z) \tau_{\gamma}(p'; z_{\text{fix}}) g_{n\gamma}(p'; z_{\text{fix}}) p'^2 dp'$$  \hspace{1cm} (21)

$$- \sum_{\gamma=1}^{3} \frac{4\pi}{3} \int_{0}^{\infty} g_{m\gamma}(p'; z) \tau_{\gamma}(p'; z) g_{n\gamma}(p'; z) p'^2 dp'$$

are calculated. Finally, the separable three-body amplitude can be written in a form

$$X_{\alpha \beta}^{\text{EDPE}}(p, p'; z) = \sum_{m,n=1}^{\infty} g_{m\alpha}(p; z) \Theta_{mn}(z) g_{n\beta}(p'; z).$$  \hspace{1cm} (22)

If only one term is taken in the sums in Eq. (22), the Energy Dependent Pole Expansion turns into Energy Dependent Pole Approximation. It is seen, that EDPE method needs only one solution of the eigenvalue equations Eq. (16) and calculations of the integrals Eqs. (20, 21) after that. According to the authors, the method converges faster than Hilbert-Schmidt expansion, it is accurate already with one term.

Three-body form-factors $g_{\alpha}$ in Eqs. (20, 21, 22) are denoted as $\bar{g}_{\alpha}^\rho$ in the four-body equations, while three-body energy-dependent functions $\Theta(z)$ are denoted as $\bar{\tau}_{\alpha \beta}^\rho$ (four-body equations are written down for EDPA, i.e. with only one term is taken in Eq. (22), $m = n = 1$).

We tried different versions of numerical treatment of Eqs. (16, 17) and Eqs. (20, 21, 22). The best result was obtained when the eigen-equations Eq. (16) with normalization condition Eq. (17) were solved using such number of integration points, which is enough for exact calculations of the corresponding binding energy of the three-body subsystem. For this sake 20 point is enough. The whole set of the eigenvalues and the corresponding eigenfunctions
(in this case 20 eigenfunctions for each of 20 eigenvalues) was evaluated. We used ZGEEV subroutine from Intel oneAPI Math Kernel Library - Fortran, which computes the eigenvalues and, optionally, the left and/or right eigenvectors for an n-by-n complex nonsymmetric matrix. After that the first term in Eq.\((22)\) with \(m = n = 1\) gives the correct binding energy (more exactly, \(\Theta_{11}(z)_{z \to E_B} \infty\)) since we strictly set the first eigenvalue \(\lambda_1 = 1\) (of course, when binding energy in the system exists). We checked how the amplitudes converge, and found that, really, the first term already gives the result close to the original value of \(X_{\alpha\beta}(p, p'; z)\) at the discrete set of momenta, used in Eqs.\((16-17)\).

**IV. FOUR-BODY EQUATIONS FOR THE \(\bar{K}NNN\) SYSTEM**

Two types of partitions for a four-body system: \(3 + 1\) and \(2 + 2\), - for the \(\bar{K}NNN\) system are: \(|\bar{K} + (NNN)\rangle\), \(|N + (\bar{K}NN)\rangle\) and \(|(\bar{K}N) + (NN)\rangle\). We started by writing down the system Eq.\((12)\) for 18 channels considering three nucleons as nonidentical particles. Four-body asymptotic states are denoted by \(\sigma_\alpha\) indices with \(\sigma = 1, 2, 3\) stands for \(|\bar{K} + (NNN)\rangle\), \(|N + (\bar{K}NN)\rangle\) and \(|(\bar{K}N) + (NN)\rangle\) partitions, correspondingly, and \(\alpha = N_iN_j\) or \(\bar{K}N_i\) \((i, j = 1, 2, 3, \ i \neq j)\) denotes the pair in the two- or three-body subsystem:

\[
\begin{align*}
\tilde{g}^1_{N_iN_j} : & |\bar{K} + (N_1 + N_2N_3)\rangle, |\bar{K} + (N_2 + N_3N_1)\rangle, |\bar{K} + (N_3 + N_1N_2)\rangle, \\
\tilde{g}^2_{N_iN_j} : & |N_1 + (\bar{K} + N_2N_3)\rangle, |N_2 + (\bar{K} + N_3N_1)\rangle, |N_3 + (\bar{K} + N_1N_2)\rangle, \\
\tilde{g}^3_{\bar{K}N_i} : & |N_1 + (N_2 + \bar{K}N_3)\rangle, |N_2 + (N_3 + \bar{K}N_1)\rangle, |N_3 + (N_1 + \bar{K}N_2)\rangle, \\
& |N_1 + (N_3 + \bar{K}N_2)\rangle, |N_2 + (N_1 + \bar{K}N_3)\rangle, |N_3 + (N_2 + \bar{K}N_1)\rangle, \\
\tilde{g}^3_{N_iN_j} : & |(N_2N_3) + (\bar{K} + N_1)\rangle, |(N_3N_1) + (\bar{K} + N_2)\rangle, |(N_1N_2) + (\bar{K} + N_3)\rangle, \\
\tilde{g}^3_{\bar{K}N_i} : & |(\bar{K}N_1) + (N_2 + N_3)\rangle, |(\bar{K}N_2) + (N_3 + N_1)\rangle, |(\bar{K}N_3) + (N_1 + N_2)\rangle
\end{align*}
\]

After antisymmetrization, necessary for a system with identical fermions, only five states, plotted in Fig.\([1]\) remains. The kernel functions \(\tilde{Z}_{\alpha\beta}^{\sigma\rho}\) of the system of four-body equations Eq.\((12)\) can be seen in Fig.\([2]\) (\(\tilde{Z}_{\alpha\beta}^{\sigma\rho}\) carrys only one bottom index due to \(\delta_{\alpha\beta}\) function in it’s definition).

Looking for a quasi-bound state needs solution of the homogeneous system of equations, which can be written in a matrix form

\[
\tilde{X} = \tilde{Z} \tilde{\tau} \tilde{X}
\]  \(\text{(24)}\)
FIG. 1. States of the four-body system equations.

FIG. 2. Kernel functions $\tilde{Z}^{\alpha \rho}_{\omega}$ of the four-body system equations.

with

$$\tilde{X}^{\rho}_{\alpha} = \begin{pmatrix} \tilde{X}^{1}_{NN} \\ \tilde{X}^{2}_{NN} \\ \tilde{X}^{2}_{KN} \\ \tilde{X}^{3}_{NN} \\ \tilde{X}^{3}_{KN} \end{pmatrix}, \quad \tilde{Z}^{\alpha \rho}_{\omega} = \begin{pmatrix} 0 & \tilde{Z}^{12}_{NN} & 0 & \tilde{Z}^{13}_{NN} & 0 \\ \tilde{Z}^{21}_{NN} & 0 & 0 & \tilde{Z}^{23}_{NN} & 0 \\ 0 & 0 & \tilde{Z}^{22}_{KN} & 0 & \tilde{Z}^{23}_{KN} \\ \tilde{Z}^{31}_{NN} & \tilde{Z}^{32}_{NN} & 0 & 0 & 0 \\ 0 & 0 & \tilde{Z}^{32}_{KN} & 0 & 0 \end{pmatrix}, \quad (25)$$

$$\tilde{\tau}^{\rho}_{\alpha \beta} = \begin{pmatrix} \tau^{1}_{NN,NN} & 0 & 0 & 0 & 0 \\ 0 & \tau^{2}_{NN,NN} & \tau^{2}_{NN,KN} & 0 & 0 \\ 0 & \tau^{2}_{KN,NN} & \tau^{2}_{KN,KN} & 0 & 0 \\ 0 & 0 & 0 & \tau^{3}_{NN,NN} & \tau^{3}_{NN,KN} \\ 0 & 0 & 0 & \tau^{3}_{KN,NN} & \tau^{3}_{KN,KN} \end{pmatrix}. \quad (26)$$

However, our $\bar{K}N$ and $NN$ potentials, which we use for the $\bar{K}NNNN$ system calculations, are isospin- and spin-dependent interaction models, in addition $V_{NN}$ is a two-term potential.
Due to this, elements of the matrices $\mathbf{Z}_{\alpha}^{\sigma\rho}$ in Eq. (25) and $\mathbf{\tau}_{\alpha\beta}^{\rho}$ in Eq. (26), entering the antisymmetrized equations Eq. (24), are matrices themselves containing elements with additional indices $\mathbf{Z}_{\alpha(m_3,n_3)}^{\sigma\rho(m_3,n_3)}$ and $\mathbf{\tau}_{\alpha\beta(m_2,m'_2)}^{\rho \alpha \beta(m_2,m'_2)}$. Particular forms of nine elements-matrices $\mathbf{Z}_{\alpha}^{\sigma\rho}$ of Eq. (25) and nine elements-matrices $\mathbf{\tau}_{\alpha\beta}^{\rho}$ of Eq. (26) are presented in the Appendix. The additional indices $m_3, n_3$ denote number of a separable term of the three-body or $2+2$ amplitudes (at the first step only one separable term was used for the three-body $KNN$, $NNN$ and $2+2$ $\bar{K}N + NN$ amplitudes in Eq. (22), so that $m_3 = n_3 = 1$). Separable indices $m_2, n_2$ of the two-body subsystems (i.e. potentials) are $m_2 = 1$ for $V_{\bar{K}N}$ and $m_2 = 1, 2$ for $V_{NN}$. The remained indices $i, i'$ and $s, s'$ are two-body isospins and spins, correspondingly.

The unknown four-body amplitudes in Eq. (25) have a general form $X_{\alpha\beta(m_2,n_2)}^{\sigma\rho(m_3,n_3)}(i,ss';i',ss')$. Finally, the system to be solved consists of 18 coupled equations.

V. TWO-BODY INTERACTIONS AND THREE-BODY SUBSYSTEMS

A. Two-body input: $\bar{K}N$ and $NN$ potentials

Both $\bar{K}N$ and $NN$ potentials, which we used, are separable isospin- and spin-dependent ones in $s$-wave. Three our separable antikaon-nucleon potentials were constructed for our three-body calculations of the $KNN$ and $KKN$ systems [10]. They are: two phenomenological potentials with coupled $\bar{K}N - \pi\Sigma$ channels, having one- $V_{\bar{K}N}^{1,SIDD}$ or two-pole $V_{\bar{K}N}^{2,SIDD}$ structure of the $\Lambda(1405)$ resonance [18] and a chirally motivated model $V_{\bar{K}N}^{Chiral}$ with coupled $\bar{K}N - \pi\Sigma - \pi\Lambda$ channels and the two-pole structure [19]. All three potentials describe low-energy $K^-p$ scattering, namely: elastic $K^-p \rightarrow K^-p$ and inelastic $K^-p \rightarrow MB$ cross-sections and threshold branching ratios $\gamma, R_c, R_n$. They also reproduce $1s$ level shift of kaonic hydrogen caused by the strong $\bar{K}N$ interaction in comparison to the pure Coulomb level, measured by SIDDHARTA experiment [20]: $\Delta_{1s}^{SIDD} = -283 \pm 36 \pm 6$ eV, and its width $\Gamma_{1s}^{SIDD} = 541 \pm 89 \pm 22$ eV. All the experimental data are described by three our potentials with equally high accuracy. In addition, elastic $\pi\Sigma$ cross-sections with isospin $I_{\pi\Sigma} = 0$, provided by all three potentials, have a bump in a region of the $\Lambda(1405)$ resonance (according to PDG [21]: $M_{\Lambda(1405)}^{PDG} = 1405.1^{+1.3}_{-1.0}$ MeV, $\Gamma_{\Lambda(1405)}^{PDG} = 50.5 \pm 2.0$ MeV). The poles
corresponding to the Λ(1405) resonance are situated at

\[ z^{1,\text{SIDD}}_{\Lambda(1405)\rightarrow 1} = 1426 - i\, 48 \text{ MeV} \quad (27) \]
\[ z^{2,\text{SIDD}}_{\Lambda(1405)\rightarrow 1} = 1414 - i\, 58 \text{ MeV}, \quad z^{2,\text{SIDD}}_{\Lambda(1405)\rightarrow 2} = 1386 - i\, 104 \text{ MeV} \quad (28) \]

for the phenomenological potentials with one- and two-pole structure, correspondingly \[19\], and at

\[ z^{\text{Chiral}}_{\Lambda(1405)\rightarrow 1} = 1417 - i\, 33 \text{ MeV}, \quad z^{\text{Chiral}}_{\Lambda(1405)\rightarrow 2} = 1406 - i\, 89 \text{ MeV} \quad (29) \]

for the chirally motivated potential \[22\].

These three antikaon-nucleon potentials with coupled \( \bar{K}N - \pi\Sigma \) channels were used in their original form in three-body AGS equations \[10\] with coupled \( \bar{K}NN - \pi\Sigma N \) three-body channels. By this the channel coupling was taken into account in a direct way. The four-body GS equations Eq.\[12\] are too complicated for introducing additional particle channels and performing coupled-channel calculations. Due to this we used the exact optical versions of our \( \bar{K}N \) potentials \[23\], which are one-channel \( V_{\bar{K}N(-\pi\Sigma-\pi\Lambda)} \). They have exactly the same elastic part as the potential with coupled channels, while all in-elasticity is taken into account in an energy-dependent imaginary part of the potential. It was demonstrated in our three-body calculations \[23\] \[24\], that such exact optical potentials give quite accurate results for \( K^-d \) scattering length or quasi-bound state position and width of the \( K^-pp \) system in comparison with the results obtained with the coupled-channel form of interaction models. Due to this we assume that it is a good approximation for the four-body calculations as well.

Nucleon-nucleon potentials \( V_{NN}^{\text{TSA-A}} \) and \( V_{NN}^{\text{TSA-B}} \) used in our three-body calculations \[10\] were used here together with the new version of the two-term separable \( NN \) potential \( V_{NN}^{\text{TSN}} \), described in \[24\]. All three potentials reproduce Argonne v18 \( NN \) phase shifts at low energies up to 500 MeV with change of sign, which means they are repulsive at short distances. They give proper singlet and triplet \( NN \) scattering lengths and deuteron binding energy. The new nucleon-nucleon potential \( V_{NN}^{\text{TSN}} \) reproduces Argonne v18 \( NN \) phase shifts of \( pp \) scattering slightly better than the previously used ones, its parameters are more natural then those of the older \( V_{NN}^{\text{TSA}} \).

In principle, Coulomb interaction also should be included in the equations, but we are interested in the quasi-bound state caused mainly by strong potentials. We assume that in such a state Coulomb interaction plays a minor role and can be neglected.
B. $3 + 1$ and $2 + 2$ partitions

We investigated $\bar{K}NNN$ system with the lowest value of the four-body isospin $I^{(4)} = 0$ and spin $S^{(4)}$ one half, which can be denoted as $K^-ppn$ or $\bar{K}^0nnp$. The total angular momentum is zero, all two-body interactions are $s$-wave ones. For the $\bar{K}NNN$ system with these quantum numbers the following three-body subsystems contribute:

- $\bar{K}NN$ with isospin $I^{(3)} = 1/2$ and spin $S^{(3)} = 0$ ($K^-pp$) or spin $S^{(3)} = 1$ ($K^-np$)
- $NNN$ with isospin $I^{(3)} = 1/2$ and spin $S^{(3)} = 1/2$ ($^3H$ or $^3He$)

Together with the $2 + 2$ partition

- $\bar{K}N + NN$ with isospin $I^{(4)} = 0$ and spin $S^{(4)} = 1/2$.

The corresponding three-body amplitudes is not an input, they are calculated during the four-body calculations.

TABLE I. Dependence of the binding energy $B_{K^-pp}^{\text{Opt}}$ (MeV) and width $\Gamma_{K^-pp}^{\text{Opt}}$ (MeV) of the quasi-bound state in the $K^-pp - \bar{K}^0np$ subsystem ($\bar{K}NN$, $S^{(3)} = 0$) on three $\bar{K}N$ and three $NN$ interaction models. Calculations were performed using the exact optical $\bar{K}N$ potentials. The binding energy is counted from the threshold energy of the $K^-pp$ system $z_{th,K^-pp} = m_K + 2m_N = 2373.485$ MeV.

| $V_{K^N}^{\text{TSA-A}}$ | $V_{K^N}^{\text{TSA-B}}$ | $V_{K^N}^{\text{TSN}}$ |
|-------------------------|-------------------------|-------------------------|
| $B_{K^-pp}^{\text{Opt}}$ | $\Gamma_{K^-pp}^{\text{Opt}}$ | $B_{K^-pp}^{\text{Opt}}$ | $\Gamma_{K^-pp}^{\text{Opt}}$ |
| $V_{1.\text{SIDD}}^{\text{KNN}}$ | 55.4 | 60.9 | 54.3 | 60.8 | 53.3 | 64.7 |
| $V_{2.\text{SIDD}}^{\text{KNN}}$ | 48.2 | 46.2 | 47.5 | 45.9 | 46.7 | 48.4 |
| $V_{\text{Chiral}}^{\text{KNN}}$ | 31.9 | 42.2 | 33.2 | 48.7 | 29.9 | 48.2 |

The three-body $\bar{K}NN$ system with different quantum numbers was studied in our previous works. In particular, quasi-bound state pole positions and widths in the $K^-pp$ system ($\bar{K}NN$ with isospin $I^{(3)} = 1/2$ and spin $S^{(3)} = 0$) were calculated in [22] with older $V_{NN}^{\text{TSA-B}}$ nucleon-nucleon potential. Recently the calculations were repeated with the new $V_{NN}^{\text{TSN}}$ [24], the results can be found in Table 4 of the paper. Binding energies $B_{K^-pp,\text{Opt}}$
and widths $\Gamma_{K^-pp,Opt}$ calculated using the exact optical potentials are shown in Table I for three antikaon-nucleon $V_{KN}^{1,\text{SIDD}}$, $V_{KN}^{2,\text{SIDD}}$, $V_{KN}^{\text{Chiral}}$ and three nucleon-nucleon $V_{NN}^{\text{TSA-A}}$, $V_{NN}^{\text{TSA-B}}$, $V_{NN}^{\text{TSN}}$ potentials. It is seen that the new $NN$ potential changed quasi-bound state positions in $K^-pp$ system by few MeV.

TABLE II. Dependence of the binding energy $B_{K^-np,Opt}^\text{Opt}$ (MeV) and width $\Gamma_{K^-np,Opt}$ (MeV) of the quasi-bound state in the $K^-np - K^0nn$ subsystem ($KNN, S^{(3)} = 1$) on three $KN$ and three $NN$ interaction models. Calculations were performed using the exact optical $KN$ potentials. The energy is counted from the $K^-d$ threshold $z_{th,K^-d} = m_K + 2m_N + E_{deu} = 2371.26$ MeV.

| Potential       | $B_{K^-np}^\text{Opt}$ | $\Gamma_{K^-np}^\text{Opt}$ | $B_{K^-np}^\text{Opt}$ | $\Gamma_{K^-np}^\text{Opt}$ |
|-----------------|------------------------|-------------------------------|------------------------|-------------------------------|
| $V_{KN}^{1,\text{SIDD}}$ | 1.6                    | 70.1                         | 0.8                    | 67.6                         |
| $V_{KN}^{2,\text{SIDD}}$ | 5.2                    | 63.7                         | 5.0                    | 61.4                         |
| $V_{KN}^{\text{Chiral}}$ | 2.6                    | 46.4                         | 2.4                    | 53.1                         |

No quasi-bound states similar to that one in $K^-pp$ were found in the $K^-np$ system, which is $KNN$ with isospin $I^{(3)} = 1/2$ and spin $S^{(3)} = 1$ in our previous calculations [19]. However, new nucleon-nucleon potential $V_{NN}^{\text{TSN}}$ changed the picture: the quasi-bound state caused purely by strong interactions can exist in the $K^-np$ system [23] in addition to the atomic kaonic deuterium. The binding energies $B_{K^-np,Opt}$ and width $\Gamma_{K^-np,Opt}$ evaluated with the exact optical versions of the $V_{KN}^{1,\text{SIDD}}$, $V_{KN}^{2,\text{SIDD}}$, $V_{KN}^{\text{Chiral}}$ potentials and three nucleon-nucleon $V_{NN}^{\text{TSA-A}}$, $V_{NN}^{\text{TSA-B}}$, $V_{NN}^{\text{TSN}}$ potentials can be seen in Table II. The binding energy of the state is so close to the $K^-d$ threshold, that relatively weak $NN$ interaction, playing in the case of strong quasi-bound state secondary role, can resolve the question of existence of the state.

The binding energies and widths of the quasi-bound states of the $KNN$ systems with spin zero and one were calculated using the three-body AGS equations. Details of three-body calculations can be found in [23, 25]. The codes for numerical solution of the three-body AGS equations for the $KNN$ systems were than modified in order to construct separable versions of the three-body amplitudes, as described in Section III.
TABLE III. Dependence of the binding energy $B_{N_{NN}}$ (MeV) and width $\Gamma_{N_{NN}}$ (MeV) of the quasi-bound state in the $N_{NN}$ subsystem ($S^{(3)} = 1/2$) on three $NN$ models. The binding energy is counted from the threshold energy of the $N_{NN}$ system $z_{th,N_{NN}} = 3m_N = 2816.76$ MeV.

| $V_{TSA-A}^{NN}$ | $V_{TSA-B}^{NN}$ | $V_{TSN}^{NN}$ |
|------------------|------------------|----------------|
| 9.03             | 9.04             | 9.52           |

The three-body AGS equations Eq. (3) were written and numerically solved for the three-nucleon system $N_{NN}$ with three $NN$ potentials $V_{TSA-A}^{NN}$, $V_{TSA-B}^{NN}$, $V_{TSN}^{NN}$ as an input. The calculated binding energies of the system are equal for both $^3$H and $^3$He nuclei since Coulomb interaction was not taken into account. They are presented in Table III. The resulting energies are larger than known values for binding energy of triton $8.4820$ MeV and helium-3 $7.7181$ MeV nucleus. Such overestimation is typical for separable $NN$ potentials. The numerical code was afterwards changed for construction of separable version of the $N_{NN}$ amplitude.

TABLE IV. Dependence of the binding energy $B_{Opt}^{K^-p+np}$ (MeV) and width $\Gamma_{Opt}^{K^-p+np}$ (MeV) of the quasi-bound state in the $K^-p+np$ partition ($\bar{K}N+NN,S^{(4)} = 1/2$) on three $\bar{K}N$ and three $NN$ interaction models. Calculations were performed using the exact optical $\bar{K}N$ potentials. The binding energy is counted from the threshold energy of the $K^-npp$ system $z_{th,K^-npp} = m_{\bar{K}} + 3m_N = 3312.405$ MeV.

| $V_{TSA-A}^{NN}$ | $V_{TSA-B}^{NN}$ | $V_{TSN}^{NN}$ |
|------------------|------------------|----------------|
| $B_{Opt}^{K^-p+np}$ | $\Gamma_{Opt}^{K^-p+np}$ | $B_{Opt}^{K^-p+np}$ |
| 20.0             | 83.6             | 19.1           |
|                 |                  | 81.1           |
| $V_{Chiral}^{\bar{K}N}$ | $V_{Chiral}^{\bar{K}N}$ | $V_{Chiral}^{\bar{K}N}$ |
| 18.8             | 54.5             | 20.9           |
|                 |                  | 58.5           |
|                 |                  | 17.9           |
|                 |                  | 53.7           |

Finally, the partition of the $2 + 2$ type $\bar{K}N + NN$ is a system with two non-interacting pairs of particles. It is a special part for the four-body Faddeev-type equations and not a
simple sum of two-body $KN$ and $NN$ amplitudes. $KN + NN$ partition is described by a three-body system of AGS equations, therefore, it is a "three-body" amplitude. Only $KN + NN$ partition with spin one $NN$ pair has a quasi-bound state, the binding energies and widths are shown in Table IV. The separable version of the corresponding $KN + NN$ amplitude was constructed in similar way as in the case of $KNN$ and $NNN$ three-body subsystems.

VI. RESULTS AND DISCUSSION

We solved the four-body GS equations for the $K^-ppn - K^0nnp$ system with the antikaon-nucleon and nucleon-nucleon potentials described above. As the first step we used only one separable term in separable representation of the three-body $KNN$, $NNN$ and "three-body" $KN+NN$ amplitudes Eq.(22), so that we used EDPA method. The binding energies $B_{K^-ppn}$ and widths $\Gamma_{K^-ppn}$ of the $KNNN$ system evaluated using three exact optical versions of the antikaon-nucleon potentials and three nucleon-nucleon interaction models are presented in Table V.

It is seen that the binding energy $B_{K^-ppn}$ and width $\Gamma_{K^-ppn}$ of the four-body quasi-body state strongly depend on the model of antikaon-nucleon interaction. In fact, it is a property of all three-body $KNN$ and $KKN$ systems as well. Phenomenological $\bar{K}N(-\pi\Sigma)$ potentials give comparable binding energies, while the chirally motivated potential led to much more shallow state. As for the widths, the models of antikaon-nucleon interaction having two-pole structure of $\Lambda(1405)$ resonance, which are the phenomenological $V^2_{KN}^{\text{SIDD}}$ and chiral $V^{\text{Chiral}}_{KN}$, have very close widths, while one-pole $V^1_{KN}^{\text{SIDD}}$ potential lead to much larger width.

As for the dependence of the obtained results on the models of $NN$ interactions, it seem they play more visible role than in the case of $KNN$ system. Nucleon-nucleon interaction plays visible role together with the chirally motivated antikaon nucleon potential, while the results obtained with phenomenological $\bar{K}N$ interaction models depend on the $V_{NN}$ slightly.

Comparing the four-body binding energies and widths in Table V with those obtained for the three-body spin zero $KNN$ system usually denoted as $K^-pp$ Table I, it is seen that the binding energies remained almost the same or become slightly smaller after adding one neutron, while the widths become much smaller. So that the additional to $K^-pp$ neutron changes the binding energy of the system slightly, but it "tightens" the quasi-bound state.
TABLE V. Dependence of the binding energy $B_{K^-ppn}$ (MeV) and width $\Gamma_{K^-ppn}$ (MeV) of the quasi-bound state in the $K^-nnp$ system on three $\bar{K}N$ and $NN$ interaction models.

| V$_{KN}$ | $V_{TSA-A}^{NN}$ | $V_{TSA-B}^{NN}$ | $V_{TSN}^{NN}$ | Other results |
|----------|-----------------|-----------------|----------------|---------------|
| $V_{1,SIDD}^{KN}$ | 52.0 | 50.4 | 50.3 | 49.6 | 51.2 | 50.8 |
| $V_{2,SIDD}^{KN}$ | 47.0 | 39.6 | 46.4 | 38.2 | 46.4 | 39.9 |
| $V_{Chiral}^{KN}$ | 32.6 | 39.7 | 34.5 | 50.9 | 30.5 | 42.8 |

Results for the binding energies and widths of other authors are also presented in Table V. The largest binding energy and smaller width were predicted in [1] by using many-body G-matrix formalism for the few-body system and an antikaon-nucleon model, which does not reproduce actual experimental data on $K^-p$ scattering and kaonic hydrogen.

Obviously, the four-body binding energies are much larger and the corresponding widths are much smaller than those for the other, spin one state of the $\bar{K}NN$ state ($K^-np$).

Results for the binding energies and widths of other authors are also presented in Table V. The authors of variational calculation [11] used an energy dependent chiral $\bar{K}N$ potential, which, however, was calculated at set of fixed energies. Also the imaginary part of the quasi-bound state was calculated approximately. The result of [11] is not far from our binding energy and width evaluated with our chirally motivated antikaon-nucleon potential, taken into account exactly.
Variational calculations were also performed in [12] with energy dependent Kyoto and energy independent AY [26] KN potentials. The authors also had to fix the antikaon-nucleon energy in the Kyoto potential, and they did it in two ways, denoting the corresponding models as Kyoto-I and Kyoto-II. Huge difference between the widths obtained with these two Kyoto potentials shows that chosen procedure is not quite reliable. Besides, the binding energy and width of the four-body quasi-bound state, calculated in [12] with AY potential [26], differs drastically from the original AY results [1].

The most intriguing is the difference between our results and those of [13] since the authors solved the same four-body Faddeev-type equations, moreover, they used our phenomenological \( V^{1,\text{SIDD}}_{\bar{K}N} \) and \( V^{2,\text{SIDD}}_{\bar{K}N} \) potentials. Nucleon-nucleon PEST \( V_{NN} \) potential, which is a separable version of Paris nucleon-nucleon potential, was used there. But we do not think, that different from our’s \( NN \) potential could change the results so drastically, by \( \sim 20 \) MeV, for both: binding energy (make it much larger) and width (make it much smaller).

One of the possible reasons of differences could follow from the way of separable version of the three-body and 2 + 2 amplitudes construction. The formulas for the EDPE energy dependent form-factors and propagators in [13] differ from our Eqs. (20,21) by \( 1/\lambda \) factor on the right-hand side for the first and by \( m \leftrightarrow n \) in the second formula. We checked the formulas by comparing the approximate \( Z_{\alpha\beta}(p,p';z) \) at fixed momenta and energy with the original ones, and obtained agreement up to \( \sim 10 \) significant digits while the whole number of separable terms (which is 20 in our case) is used.

Besides, setting the binding energy for the \((\bar{K}N)_{I=0} + NN\) to the energy of the quasi-bound \( \bar{K}N \) state (\( \Lambda(1405) \) resonance), made in [13], is not quite understandable. The matter is the total four-body isospin \( I^{(4)} = 0 \) means that the \( NN \) subsystem should also have \( I_{NN} = 0 \), which is a deuteron. Due to this, the total binding energy of this partition should include both two-body energies: \( B_{\Lambda(1405)} \) and \( B_{\text{deu}} \). In fact, the energy of the \( \bar{K}N + NN \) ”three-body” system with isospins of both pairs equal to zero differs from simple sum of the two two-body energies, see Table [IV] it comes from solution of a three-body equation. As for another state of the \( \bar{K}N + NN \) partition, it does not have bound states since \( \bar{K}N \) and \( NN \) in this case have two-body isospin \( I^{(2)} = 1 \) and none of them is bound. Setting the energy in this case to deuteron binding energy, as is done in [13], is quite strange.

Our very preliminary results for the pole position of the quasi-bound state in the \( K^- p p n - \bar{K}^0 n n p \) system calculated with two phenomenological potentials were published in [27]. They
differ from those presented in Table V drastically: binding energies are much larger while the width are smaller. The reason is the different procedure of the separable representation of the three- and "three"-body amplitudes. In contrast to the present usage of the Intel MKL library subroutine, we used some hand made procedure for eigenvalues and eigenfunctions evaluation. Such difference in the procedure also can explain differences between our present results and those of [13].

Finally, the binding energy the three-body $K^-pp$ quasi-bound state evaluated in [13] with our phenomenological $V_{KN}^{1,SIDD}$ potential differ from our result by $\sim 6$ MeV and almost coinsides with the energy calculated with $V_{KN}^{2,SIDD}$. Our three-body binding energies are sufficiently different for every of the three nucleon-nucleon potentials, see Table I. Keeping this closeness of the three-body results in mind, it is hard to understand large difference (15 MeV) between the corresponding four-body results in [13].

VII. CONCLUSION

The four-body Faddeev-type GS equations for search of the quasi-bound state in the $\bar{K}NNN$ system were written down and solved. The binding energies $B_{K^-ppn}^{\text{Chiral}} \sim 30.5 - 34.5$ MeV obtained with chirally motivated and $B_{K^-ppn}^{\text{SIDD}} \sim 46.4 - 52.0$ MeV obtained with phenomenological antikaon-nucleon potentials are close to those for the $K^-pp$ system, calculated with the same $V_{KN}$ and $V_{NN}$ potentials. The widths of the four-body states $\Gamma_{K^-ppn} \sim 38.2 - 50.9$ MeV are smaller than the three-body widths of $K^-pp$. Therefore, the neutron added to the $K^-pp$ system slightly influences the binding, but tightens the system.

The four-body binding energy and width of the $\bar{K}NNN$ system strongly depend on $\bar{K}N$ potential and noticeably - on $NN$ potential, especially together with the chirally motivated antikaon-nucleon interaction model.

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Appendix: Elements of $\bar{Z}_{\alpha}^{\sigma \rho}$ and $\bar{\pi}_{\alpha \beta}^{\rho}$ matrices

After antisymmetrization we come to the system Eq. (24) of 18 coupled equations. Elements of the matrices $\bar{Z}_{\alpha}^{\sigma \rho}$ in Eq. (25) and $\bar{\pi}_{\alpha \beta}^{\rho}$ in Eq. (26) are matrices themselves with elements $\bar{Z}_{\alpha(m_2,n_2)}^{\sigma(m_3,n_3)}$ and $\bar{\pi}_{\alpha \beta(m_2,m_2')}^{\rho(s,s')}$; \(m_3, n_3\) denote number of a separable term of the three-body or 2 + 2 amplitudes \((m_3 = n_3 = 1)\), while \(m_2, n_2\) are separable indices of the potentials: \(m_2 = 1\) for \(V_{KN}\) and \(m_2 = 1, 2\) for \(V_{NN}\). Two-body isospins and spins are denoted \(i, i'\) and \(s, s'\), correspondingly.

The kernel matrices $\bar{Z}_{\alpha}^{\sigma \rho}$ consists of six elements with $\alpha = NN$:

\[
\bar{Z}_{NN}^{12} =
\begin{pmatrix}
\bar{Z}_{NN}^{12(1,1)}_{(0,11)} & 0 & \bar{Z}_{NN}^{12(1,1)}_{(1,00)} \\
0 & \bar{Z}_{NN}^{12(1,1)}_{(0,11)} & 0 \\
0 & \bar{Z}_{NN}^{12(1,1)}_{(1,00)} & \bar{Z}_{NN}^{12(1,1)}_{(1,00)}
\end{pmatrix}
\]
(A.1)

\[
\bar{Z}_{NN}^{13} =
\begin{pmatrix}
\bar{Z}_{NN}^{13(1,1)}_{(0,11)} & 0 & \bar{Z}_{NN}^{13(1,1)}_{(1,00)} \\
0 & \bar{Z}_{NN}^{13(1,1)}_{(0,11)} & 0 \\
0 & \bar{Z}_{NN}^{13(1,1)}_{(1,00)} & \bar{Z}_{NN}^{13(1,1)}_{(1,00)}
\end{pmatrix}
\]
(A.2)

\[
\bar{Z}_{NN}^{21} =
\begin{pmatrix}
\bar{Z}_{NN}^{21(1,1)}_{(0,11)} & 0 & \bar{Z}_{NN}^{21(1,1)}_{(1,00)} \\
0 & \bar{Z}_{NN}^{21(1,1)}_{(0,11)} & 0 \\
0 & \bar{Z}_{NN}^{21(1,1)}_{(1,00)} & \bar{Z}_{NN}^{21(1,1)}_{(1,00)}
\end{pmatrix}
\]
(A.3)

\[
\bar{Z}_{NN}^{23} =
\begin{pmatrix}
\bar{Z}_{NN}^{23(1,1)}_{(0,11)} & 0 & \bar{Z}_{NN}^{23(1,1)}_{(1,00)} \\
0 & \bar{Z}_{NN}^{23(1,1)}_{(0,11)} & 0 \\
0 & \bar{Z}_{NN}^{23(1,1)}_{(1,00)} & \bar{Z}_{NN}^{23(1,1)}_{(1,00)}
\end{pmatrix}
\]
(A.4)
and three elements with $\alpha = K N$:

\[
\begin{align*}
\bar{Z}^{31}_{N N} &= \left( \begin{array}{ccc}
Z^{31(1,1)}_{N N(N,1)}; & 0 & Z^{31(1,1)}_{N N(N,2)}; \\
0 & Z^{31(1,1)}_{N N(N,1)}; & 0 \\
0 & Z^{31(1,1)}_{N N(N,2)}; \\
\end{array} \right) \\
\bar{Z}^{32}_{N N} &= \left( \begin{array}{ccc}
Z^{32(1,1)}_{N N(N,1)}; & 0 & Z^{32(1,1)}_{N N(N,2)}; \\
0 & Z^{32(1,1)}_{N N(N,1)}; & 0 \\
0 & Z^{32(1,1)}_{N N(N,2)}; \\
\end{array} \right) \\
\end{align*}
\]

\[
\begin{align*}
\bar{Z}^{40}_{K N} &= \left( \begin{array}{ccc}
Z^{40(1,1)}_{K N(K,1)}; & 0 & Z^{40(1,1)}_{K N(K,2)}; \\
0 & Z^{40(1,1)}_{K N(K,1)}; & 0 \\
0 & Z^{40(1,1)}_{K N(K,2)}; \\
\end{array} \right) \\
\end{align*}
\]

Elements of $\bar{\tau}^\rho_{\alpha \beta}$ matrix Eq. (26) are parts of the NNN subsystem ($\rho = 1$):

\[
\bar{\tau}^{1}_{N N, N N} = \left( \begin{array}{ccc}
\bar{T}^{1(1)}_{N N, N N(N,1)}; & \bar{T}^{1(1)}_{N N, N N(N,1)}; & \bar{T}^{1(1)}_{N N, N N(N,1)}; \\
\bar{T}^{1(1)}_{N N, N N(N,1)}; & \bar{T}^{1(1)}_{N N, N N(N,1)}; & \bar{T}^{1(1)}_{N N, N N(N,1)}; \\
\bar{T}^{1(1)}_{N N, N N(N,1)}; & \bar{T}^{1(1)}_{N N, N N(N,1)}; & \bar{T}^{1(1)}_{N N, N N(N,1)}; \\
\end{array} \right)
\]
of the $KNN$ subsystem ($\rho = 2$):

$$
\tau^2_{NN,NN} = \begin{pmatrix}
\tau_{N,N,NN(1,1)}; & 0 & \tau_{N,N,NN(1,2)}; \\
0 & \tau_{N,N,NN(1,1)}; & 0 \\
0 & 0 & \tau_{N,N,NN(1,2)};
\end{pmatrix}
$$

(A.11)

$$
\tau^2_{NN,KN} = \begin{pmatrix}
\tau_{N,N,KN(1,1)}; & 0 & \tau_{N,N,KN(1,2)}; \\
0 & \tau_{N,N,KN(1,1)}; & 0 \\
0 & 0 & \tau_{N,N,KN(1,2)};
\end{pmatrix}
$$

(A.12)

$$
\tau^2_{KN,NN} = \begin{pmatrix}
\tau_{K,N,NN(1,1)}; & 0 & \tau_{K,N,NN(1,2)}; \\
0 & \tau_{K,N,NN(1,1)}; & 0 \\
0 & 0 & \tau_{K,N,NN(1,2)};
\end{pmatrix}
$$

(A.13)

$$
\tau^2_{KN,KN} = \begin{pmatrix}
\tau_{K,N,KN(1,1)}; & 0 & \tau_{K,N,KN(1,2)}; \\
0 & \tau_{K,N,KN(1,1)}; & 0 \\
0 & 0 & \tau_{K,N,KN(1,2)};
\end{pmatrix}
$$

(A.14)

and of the $KN + NN$ partition ($\rho = 3$):

$$
\tau^3_{NN,NN} = \begin{pmatrix}
\tau_{N,N,NN(1,1)}; & 0 & \tau_{N,N,NN(1,2)}; \\
0 & \tau_{N,N,NN(1,1)}; & 0 \\
0 & 0 & \tau_{N,N,NN(1,2)};
\end{pmatrix}
$$

(A.15)
\[
\bar{\tau}_{NN,KN}^3 = \begin{pmatrix}
\bar{\tau}_{NN,KN}^3(1)_{(00,11)} & 0 \\
0 & \bar{\tau}_{NN,KN}^3(1)_{(11,00)} \\
\bar{\tau}_{NN,KN}^3(1)_{(00,11)} & 0 \\
0 & \bar{\tau}_{NN,KN}^3(1)_{(11,00)}
\end{pmatrix}
\quad (A.16)
\]

\[
\bar{\tau}_{KN,NN}^3 = \begin{pmatrix}
\bar{\tau}_{KN,NN}^3(1)_{(00,11)} & 0 & \bar{\tau}_{KN,NN}^3(1)_{(00,11)} & 0 \\
0 & \bar{\tau}_{KN,NN}^3(1)_{(00,11)} & 0 & \bar{\tau}_{KN,NN}^3(1)_{(00,11)}
\end{pmatrix}
\quad (A.17)
\]

\[
\bar{\tau}_{KN,KN}^3 = \begin{pmatrix}
\bar{\tau}_{KN,KN}^3(1)_{(00,11)} & 0 \\
0 & \bar{\tau}_{KN,KN}^3(1)_{(11,00)}
\end{pmatrix}
\quad (A.18)
\]

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