Scotogenic S3 symmetric generation of realistic neutrino mixing

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Abstract

Realistic neutrino mixing is achieved at one-loop level radiatively using S3 × Z2 symmetry. The model comprises of two right-handed neutrinos, maximally mixed to produce the structure of the left-handed Majorana neutrino mass matrix characterized by θ13 = 0, θ23 = π/4 and any value of θ12 particular to the Tribimaximal (TBM), Bimaximal (BM) and Golden Ratio (GR) or other mixings. A small deviation from this maximal mixing between the two right-handed neutrinos could generate non-zero θ13, shifts of the atmospheric mixing angle θ23 from π/4 and also could correct the solar mixing angle θ12 by a small amount altogether in a single step. In this scotogenic mechanism of generating non-zero θ13 by shifting from maximal mixing in the right-handed neutrino sector, two Z2 odd inert scalar SU(2)L doublets were used, the lightest of which can serve as a dark matter candidate.

I Introduction

Neutrinos oscillate owing to their massive nature as established by the oscillation experiments. The mass eigenstates and flavour eigenstates are different and are related by the Pontecorvo, Maki, Nakagawa, Sakata – PMNS – matrix:

\[
U = \begin{pmatrix}
c_{12}c_{13} & s_{12}c_{13} & -s_{13}e^{-i\delta} \\
-c_{23}s_{12} + s_{23}s_{13}c_{12}e^{i\delta} & c_{23}c_{12} + s_{23}s_{13}s_{12}e^{i\delta} & -s_{23}c_{13} \\
-s_{23}s_{12} - c_{23}s_{13}c_{12}e^{i\delta} & -s_{23}c_{12} + c_{23}s_{13}s_{12}e^{i\delta} & c_{23}c_{13}
\end{pmatrix}. \tag{1}
\]

Here \(c_{ij} = \cos \theta_{ij}\) and \(s_{ij} = \sin \theta_{ij}\). Needless to mention that the mass eigenstates are non-degenerate.

Non-zero \(\theta_{13}\), though small in comparison to the other mixing angles was discovered in 2012 by the short-baseline reactor anti-neutrino experiments \[1\]. Before these non-zero \(\theta_{13}\) results, models were studied in literature that correspond to Tribimaximal (TBM), Bimaximal (BM) and Golden Ratio (GR) mixings (that we now onwards collectively refer as popular lepton mixings). All these mixings have \(\theta_{13} = 0, \theta_{23} = \pi/4\) and tuning \(\theta_{12}^0\) to the specific values as shown in Table 1 produced the different mixing patterns viz. TBM, BM and GR.

Setting \(\theta_{13} = 0\) and \(\theta_{23} = \pi/4\) in Eq. (1) will yield a general structure for all popular mixing as:

\[
U^0 = \begin{pmatrix}
\cos \theta_{12}^0 & \sin \theta_{12}^0 & 0 \\
-\frac{\sin \theta_{12}^0}{\sqrt{2}} & \frac{\cos \theta_{12}^0}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
-\frac{\sin \theta_{12}^0}{\sqrt{2}} & \frac{\cos \theta_{12}^0}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}. \tag{2}
\]
Table 1: The values $\theta_{12}^0$ corresponding to various popular lepton mixings namely, TBM, BM, and GR patterns.

| Model | TBM | BM | GR |
|-------|-----|----|----|
| $\theta_{12}^0$ | $35.3^\circ$ | $45.0^\circ$ | $31.7^\circ$ |

The current 3$\sigma$ global fit \cite{2,3} for $\theta_{13}$, $\theta_{23}$ and $\theta_{12}$ as from NuFIT3.2 of 2018 \cite{2} are:

$$\begin{align*}
\theta_{12} &= (31.42 - 36.05)^\circ, \\
\theta_{23} &= (40.3 - 51.5)^\circ, \\
\theta_{13} &= (8.09 - 8.98)^\circ.
\end{align*}$$

So popular mixing and non-zero $\theta_{13}$ observations are not in harmony. Several model-building exercises have been taking place since the observation of non-zero $\theta_{13}$ to include it in the popular mixing framework. In \cite{4}, the possibility of smallness of $\theta_{13}$ and $\Delta m_{solar}^2$ to have a common origin was explored. In some efforts \cite{5} a dominant component was characterized by larger oscillation parameters such as $\Delta m_{atmos}^2$ and $\theta_{23} = \pi/4$, whereas the smaller mixing parameters viz. non-zero $\theta_{13}, \theta_{12}$, solar splitting and deviation of atmospheric mixing from maximality were produced by a smaller see-saw \cite{6} component as perturbation to the dominant one\footnote{For some earlier models with similar goals, see \cite{7}.}. In \cite{8,9} the mixing angle $\theta_{13} = 0$ was produced using various symmetries and non-vanishing $\theta_{13}$ was produced by perturbation to these symmetric forms.

The popular mixings were amended at tree-level using a two-component Lagrangian with discrete symmetries $A_4$, $S_3$ in \cite{10,11}. In these models, type II see-saw yielded the dominant component that gave the popular mixing, corrections to which were offered by type I see-saw sub-dominant component. Similar enterprise just for the no solar mixing (NSM) case i.e., $\theta_{12}^0 = 0$ using $A_4$ was pursued\footnote{The dominant type II see-saw had vanishing solar splitting, thus one can make use of degenerate perturbation theory to get large solar mixing.} in \cite{12}. In \cite{13} TBM was obtained radiatively using $A_4$. Recent works with realistic neutrino mixings can be found in \cite{14,15}.

Here we discuss a radiative $S_3 \times Z_2$ model\footnote{A brief account on discrete group $S_3$ is presented in Appendix of the paper.}. Some earlier works on $S_3$ in context of neutrino mass are \cite{16,17}. Neutrino mass with $S_3 \times Z_2$ within left-right symmetry was studied in \cite{18}. A common practice \cite{19} was to find a symmetry among the three neutrinos that can produce a mass matrix that can be expressed as a linear combination of a democratic matrix $M_{dem}$ and an identity matrix $I$, like $c_1 I + c_2 M_{dem}$ with $c_1$ and $c_2$ being two complex numbers. This could serve as a reasonable scenario to start with from which some models obtained realistic mixing through perturbation to such initial structures \cite{19} whereas in some models \cite{20} various GUT symmetries or extra-dimensional theories were considered to generate these initial structures and renormalization group effects at high energies were explored to obtain realistic mixing. Another way \cite{21} of constructing $S_3$ models is to have a 3-3-1 local gauge symmetry, and later on associate it to a $(B - L)$ extension or use soft breaking of $S_3$. Since $S_3$ has irreducible representations of one-dimension and two-dimension, the latter can be used to obtain maximal mixing in the $\nu_\mu - \nu_\tau$ block \cite{22}. Collider signatures of $S_3$ flavour symmetry was vividly studied in \cite{23}. $S_3$ models are also studied in quark sector \cite{24}. 

\footnotetext[1]{For some earlier models with similar goals, see \cite{7}.} 
\footnotetext[2]{The dominant type II see-saw had vanishing solar splitting, thus one can make use of degenerate perturbation theory to get large solar mixing.} 
\footnotetext[3]{A brief account on discrete group $S_3$ is presented in Appendix of the paper.}
In this work our objective is to use $S3$ to radiatively obtain:

1. The structure of the mixing matrix of popular mixing kind as shown in Eq. (2) that is characterized by $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and $\theta_{12}^0$ of any of the alternatives displayed in Table 1.

2. Realistic neutrino mixings i.e., precisely non-zero $\theta_{13}$, shifts of atmospheric mixing angle $\theta_{23}$ from maximality and tiny corrections to the solar mixing angle $\theta_{12}$.

In this radiative $S3 \times Z_2$ model, neutrino masses and mixings are generated at one-loop. The model has two right-handed neutrinos comprising an $S3$ doublet, that are maximally mixed to obtain the structure as required by popular mixings as in Eq. (2). A small deviation from this maximal mixing in the right-handed neutrino sector could produce in a single step non-zero $\theta_{13}$, shifts of $\theta_{23}$ from $\pi/4$ and small corrections to $\theta_{12}$ as is required by the mixing to be realistic. To achieve this, two $Z_2$ odd scalars $\eta_i$, ($i = 1, 2$), were required, the lightest among them can be a good dark matter candidate. A similar analysis based on $A4$ was performed where instead of using deviations from maximal mixing between the two right-handed neutrino states to generate non-zero $\theta_{13}$, small mass splittings between two right-handed neutrinos were used in [26].

II The $S3 \times Z_2$ Model

In mass basis the left-handed neutrino Majorana mass matrix is $M_{\nu}^{\text{mass}} = \text{diag} \left( m_1, m_2, m_3 \right)$. One can transport this in its flavour basis with help of the common form of the popular lepton mixing matrix $U^0$ in Eq. (2) as:

$$M_{\nu L}^{\text{flavour}} = U^0 M_{\nu L}^{\text{mass}} U^{0T} = \begin{pmatrix} a & c & c \\ c & b & d \\ c & d & b \end{pmatrix}. \tag{4}$$

The $a, b, c$ and $d$ used here are given by:

$$a = m_1 \cos^2 \theta_{12}^0 + m_2 \sin^2 \theta_{12}^0$$

$$b = \frac{1}{2} \left( m_1 \sin^2 \theta_{12}^0 + m_2 \cos^2 \theta_{12}^0 + m_3 \right)$$

$$c = \frac{1}{2\sqrt{2}} \sin 2\theta_{12}^0 \left( m_2 - m_1 \right)$$

$$d = \frac{1}{2} \left( m_1 \sin^2 \theta_{12}^0 + m_2 \cos^2 \theta_{12}^0 - m_3 \right). \tag{5}$$

Thus,

$$\tan 2\theta_{12}^0 = \frac{2\sqrt{2}c}{b + d - a}. \tag{6}$$

It is essential for $a, b, c$ and $d$ to be non-zero for the neutrino masses to be realistic and non-degenerate.

Our prime intent is to generate the form of $M_{\nu L}^{\text{flavour}}$ in Eq. (4) radiatively with one-loop. Thus one has to designate each of the fields in our model with particular $S3 \times Z_2$ quantum numbers. There are two right-handed neutrinos present in the model. Maximal mixing between these two right-handed neutrinos can be found in [25].
Leptons \[ SU(2)_L \times S_3 \times Z_2 \]

- \[ L_{eL} \equiv (\nu_e, e^-)_L \]
- \[ L_{\zeta L} \equiv (\nu_\mu, \mu^-)_L \]
- \[ N_{\alpha R} \equiv \left( \begin{array}{c} N_{1R} \\ N_{2R} \end{array} \right) \]

Scalars \[ SU(2)_L \times S_3 \times Z_2 \]

- \[ \Phi \equiv \left( \begin{array}{c} \phi_1^+ \\ \phi_2^+ \\ \phi_1^0 \\ \phi_2^0 \end{array} \right) \]
- \[ \eta \equiv \left( \begin{array}{c} \eta_1^+ \\ \eta_2^+ \\ \eta_1^0 \\ \eta_2^0 \end{array} \right) \]

| Leptons | \[ SU(2)_L \times S_3 \times Z_2 \] |
|---------|-----------------|
| \[ L_{eL} \equiv (\nu_e, e^-)_L \] | 2 | 1 | 1 |
| \[ L_{\zeta L} \equiv (\nu_\mu, \mu^-)_L \] | 2 | 2 | 1 |
| \[ N_{\alpha R} \equiv \left( \begin{array}{c} N_{1R} \\ N_{2R} \end{array} \right) \] | 1 | 2 | -1 |

| Scalars | \[ SU(2)_L \times S_3 \times Z_2 \] |
|---------|-----------------|
| \[ \Phi \equiv \left( \begin{array}{c} \phi_1^+ \\ \phi_2^+ \\ \phi_1^0 \\ \phi_2^0 \end{array} \right) \] | 2 | 2 | 1 |
| \[ \eta \equiv \left( \begin{array}{c} \eta_1^+ \\ \eta_2^+ \\ \eta_1^0 \\ \eta_2^0 \end{array} \right) \] | 2 | 2 | -1 |

Table 2: All fields along with their respective charges. We confine this model to neutrino sector only.

Neutrino fields can produce the desired form of left-handed Majorana neutrino mass matrix in Eq. (4) that corresponds to \( \theta_{13} = 0, \theta_{23} = \pi/4 \) and \( \theta_{12}^0 \) of the popular lepton mixing scenarios. After obtaining the form in Eq. (4), we will see in due course, a slight shift from this maximal mixing between the right-handed neutrino states is capable of yielding realistic neutrino mixings, viz. non-zero \( \theta_{13} \), deviation of atmospheric mixing \( \theta_{23} \) from \( \pi/4 \) as well as small corrections to solar mixing \( \theta_{12} \).

The model has the three left-handed lepton \[ SU(2)_L \] doublets \[ L_{\zeta L} \equiv (\nu_\zeta, \zeta^-)_L \] where \( \zeta = e, \mu, \tau \), out of which \( L_{\mu L} \) and \( L_{\tau L} \) comprise a doublet of \( S_3 \) whereas \( L_{eL} \) remains a singlet under \( S_3 \). Apart from these there are two Standard Model (SM) gauge singlet right-handed neutrinos \( N_{\alpha R} \), \( (\alpha = 1, 2) \) that transform as a doublet under \( S_3 \). The scalar spectrum of the model has a couple of inert \[ SU(2)_L \] doublet scalars, \( \eta_i \equiv (\eta_i^+, \eta_i^0)^T \), \( (i = 1, 2) \), forming an \( S_3 \) doublet (\( \eta \)). We also have two other \[ SU(2)_L \] doublet scalars, namely \( \Phi_j \equiv (\phi_j^+, \phi_j^0)^T \), \( (j = 1, 2) \), that are combined to form an \( S_3 \) doublet (\( \Phi \)). Besides the \( S_3 \), the model also has an unbroken \( Z_2 \) symmetry under which all other fields except the right-handed neutrinos and the scalar \( \eta \) are even. After spontaneous symmetry breaking (SSB), \( \phi_j \) get vacuum expectation value (vev), but \( \eta_i \) do not. Let \( v_j \) be the vevs of \( \phi_j^0 \) i.e., \( \langle \Phi_j \rangle \equiv v_j \), \( (j = 1, 2) \). Fields and their specific charges are shown in Table 2. We deal with the neutrino sector only in this model. The charged lepton mass matrix is diagonal in the basis in which we perform the analysis and the entire mixing comes from the neutrino sector.

Neutrino mass can be generated radiatively at one-loop level from Fig. (1). The neutrino mass matrix will receive contributions from the following terms of the \( S_3 \times Z_2 \) invariant scalar potential from the
scalar four-point vertex\footnote{Two \( \eta \) are created and two \( \phi \) are destroyed at the scalar four point vertex causing terms of \((\eta^\dagger \phi)(\eta^\dagger \phi)\) nature to be pertinent among other terms in the scalar potential.}:

\[
V_{\text{relevant}} \supset \lambda_1 \left\{ \left( \eta_2^\dagger \phi_2 + \eta_1^\dagger \phi_1 \right)^2 \right\} + h.c. + \lambda_2 \left\{ \left( \eta_2^\dagger \phi_2 - \eta_1^\dagger \phi_1 \right)^2 \right\} + h.c. \\
+ \lambda_3 \left\{ \left( \eta_1^\dagger \phi_2 \right) \left( \eta_2^\dagger \phi_1 \right) + \left( \eta_2^\dagger \phi_2 \right) \left( \eta_1^\dagger \phi_1 \right) \right\} + h.c. .
\]

(7)

Here all the quartic couplings \( \lambda_j \ (j = 1, 2, 3) \) are taken real.

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Figure 1: One-loop scotogenic neutrino mass generation using \( S_3 \times Z_2 \) symmetry.

At all the three vertices of Fig. (1), all symmetries are conserved. The Dirac vertices conserving \( S_3 \times Z_2 \) can be written as:

\[
\mathcal{L}_{\text{Yukawa}} = y_1 \left[ (\bar{N}^0_2 R \eta_2^0 + \bar{N}^0_1 R \eta_1^0) \nu_e \right] + y_2 \left[ (\bar{N}^0_1 R \eta_2^0) \nu_\tau + (\bar{N}^0_2 R \eta_1^0) \nu_\mu \right] + h.c.
\]

(8)

Since the left-handed neutrinos \( \nu_\zeta_L \) transform as doublet of \( S_3 \) for \( \zeta = \mu, \tau \) and invariant under \( S_3 \) if \( \zeta = e \), the Yukawa couplings involved are different for \( \zeta = \mu, \tau \) and \( \zeta = e \), namely, \( y_1 \) for \( \zeta = e \) and \( y_2 \) for \( \zeta = \mu, \tau \) respectively.

Let us now have a look at the right-handed neutrino sector. Recall we have two SM gauge singlet right-handed neutrinos, \( N_{1R} \) and \( N_{2R} \), that transform as a doublet of \( S_3 \). Thus the \( S_3 \times Z_2 \) invariant direct mass term for the right-handed neutrinos will look like:

\[
\mathcal{L}_{\text{right-handed neutrinos}} = \frac{1}{2} m_{R_{12}} \left[ N^T_{1R} C^{-1} N_{2R} + N^T_{2R} C^{-1} N_{1R} \right] .
\]

(9)

Thus \( S_3 \) symmetry allows a symmetric mass matrix with only non-zero off-diagonal terms for the right-handed neutrinos. If one allows soft breaking of \( S_3 \) at the scale where right-handed neutrinos get mass by introducing terms like:

\[
\mathcal{L}_{\text{soft}} = \frac{1}{2} \left[ m_{R_{11}} N^T_{1R} C^{-1} N_{1R} + m_{R_{22}} N^T_{2R} C^{-1} N_{2R} \right]
\]

(10)
to get non-zero diagonal entries, then one can write the right-handed neutrino mass matrix as:

\[
M_{\nu_R} = \frac{1}{2} \begin{pmatrix} m_{R_{11}} & m_{R_{12}} m_{R_{22}} \\ m_{R_{12}} & m_{R_{22}} \end{pmatrix} .
\]

(11)
The symmetric structure of the matrix in Eq. (11) also reflects its Majorana nature.

With the model ingredients ready, at this stage, we are in a position to present a basic description of the left-handed Majorana neutrino mass matrix arising from Fig. 1, the detailed expressions for which will be provided at a later stage of our analysis. To set the stage of the discussion, let us first sketchily indicate how the elements of the left-handed neutrino mass matrix will receive contributions from this one-loop diagram [27] in Fig. 1. Let us make a few simplifying assumptions to make the expressions look less complicated at the moment. For this purpose, let $\lambda$ commonly represent some combinations of the three quartic couplings given in Eq. (7) i.e., $\lambda_1$, $\lambda_2$ and $\lambda_3$. Also the splitting between the masses of $\eta_1$ and $\eta_2$ comprising the S3 doublet is neglected and $m_0$ is assumed to be the common mass of them. Further, if the real part of $\eta_j^0$ is denoted by $\eta_{Rj}$ and $\eta_{Ij}$ be the imaginary part of $\eta_j^0$, then difference between the masses of $\eta_{Rj}$ and $\eta_{Ij}$ can be taken proportional to $\lambda v_j$ and can be small in general.

It is imperative to note that under S3, $\nu_e$ is invariant whereas $\nu_\zeta$ ($\zeta = \mu, \tau$) transform as doublet. This feature will manifest through the Yukawa couplings (see Eq. (8)) at the two Dirac vertices which in its turn will dictate the structure of the left-handed neutrino mass matrix. Let $z \equiv \frac{\eta_{Rj}^2}{\bar{m}_0^2}$, where $m_R$ is the average mass of the heavy right-handed neutrino states. Since $z$ always appears only in the logarithm we do not distinguish between the masses of the different right-handed neutrinos for the purpose of defining $z$ throughout. Under this assumption the second diagonal entry, for example, will have the form,

$$(M^{\text{flavour}}_{\nu L})_{22} = \lambda \frac{v_m v_n}{8\pi^2} \frac{y_2^2}{m_{R22}} \ln z - 1.$$  \hspace{1cm} (12)$$

It is noteworthy that Eq. (12) is valid in the limit $m_R^2 >> m_0^2$. For $(M^{\text{flavour}}_{\nu L})_{22}$, as noted earlier in Eq. (8), $\nu_\mu$ couples only to $N_{2R}$, thus at both the Dirac vertices $N_{2R}$ will couple with $\nu_\mu$. Hence the $(2,2)$ element of the left handed neutrino mass matrix will get contribution from $m_{R22}$ only. Also $y_2$ is the only Yukawa coupling that will appear since we are dealing with $\nu_\mu$ at both the Dirac vertices for $(M^{\text{flavour}}_{\nu L})_{22}$. From similar arguments, one can obtain expression for $(M^{\text{flavour}}_{\nu L})_{33}$ just by replacing $m_{R22}$ by $m_{R11}$ in Eq. (12).

Let us now concentrate on the off-diagonal $(2,3)$ entry. Thus one has to consider $\nu_\mu$ at one of the Dirac vertices and $\nu_\tau$ at the other. From Eq. (8), one can note that $\nu_\mu$ couples to $N_{2R}$ only whereas $\nu_\tau$ does so with $N_{1R}$. Thus at one of the Dirac vertices we will have $N_{1R}$ and $N_{2R}$ at the other. Therefore, off-diagonal entries from right-handed neutrino mass matrix will come into play and $(M^{\text{flavour}}_{\nu L})_{23}$ will get contributions from $m_{R12}$ in addition to that from $m_{R11}$ and $m_{R22}$. Needless to mention that the Yukawa coupling involved will be $y_2$ as can be seen from Eq. (8). Thus one can write,

$$(M^{\text{flavour}}_{\nu L})_{23} = \lambda \frac{v_m v_n}{8\pi^2} \frac{y_2^2 m_{R12}}{m_{R11} m_{R22}} \ln z - 1.$$  \hspace{1cm} (13)$$

While writing down Eq. (13) we are taking into account the mass insertion approximation. In similar spirit, one can write down expressions for $(1,1)$, $(1,2)$ and the $(1,3)$ entries of the left-handed Majorana neutrino mass matrix.

For notational ease, let us absorb everything else present in the RHS of expressions for the elements of the left-handed Majorana neutrino mass matrix as in Eq. (12) and Eq. (13) except the Yukawa couplings, quartic couplings and the vevs in loop contributing factors say $r_{\alpha\beta}$ given by:

$$r_{11} = \frac{1}{8\pi^2 m_{R11}} \ln z - 1.$$
\[ r_{22} \equiv \frac{1}{8\pi^2 m_{R_{22}}} \ln z - 1, \]
\[ r_{12} \equiv \frac{m_{R_{12}}}{8\pi^2 m_{R_{11}} m_{R_{22}}} \ln z - 1. \]

(14)

From Eqs. (12), (13), (14) and (7), the left-handed neutrino Majorana mass matrix radiatively generated at one-loop as shown in Fig. (1) is:

\[ M_{\nu_L}^{\text{flavour}} = \begin{pmatrix} \chi_1 & \chi_4 & \chi_5 \\ \chi_4 & \chi_2 & \chi_6 \\ \chi_5 & \chi_6 & \chi_3 \end{pmatrix} \]

(15)

where,

\[
\begin{align*}
\chi_1 & \equiv y_1^2 \left[ 4r_{12}v_1v_2(\lambda_3 + \lambda_1 - \lambda_2) + (r_{11}v_1^2 + r_{22}v_2^2)(\lambda_1 + \lambda_2) \right] \\
\chi_2 & \equiv y_2^2 \left[ r_{22}(\lambda_1 + \lambda_2)v_1^2 \right] \\
\chi_3 & \equiv y_3^2 \left[ r_{11}(\lambda_1 + \lambda_2)v_2^2 \right] \\
\chi_4 & \equiv y_4^2 \left[ r_{12}(\lambda_1 + \lambda_2)v_2^2 + 2r_{22}(\lambda_3 + \lambda_1 - \lambda_2)v_1v_2 \right] \\
\chi_5 & \equiv y_5^2 \left[ r_{12}(\lambda_1 + \lambda_2)v_1^2 + 2r_{11}(\lambda_3 + \lambda_1 - \lambda_2)v_1v_2 \right] \\
\chi_6 & \equiv y_6^2 \left[ 2r_{12}(\lambda_3 + \lambda_1 - \lambda_2)v_1v_2 \right].
\end{align*}
\]

(16)

Here \( \langle \Phi_j \rangle \equiv v_j \) with \( (j = 1, 2) \).

For the left-handed neutrino mass matrix in Eq. (15) to be of the form of Eq. (4), i.e., the structure needed for \( \theta_{13} = 0, \theta_{23} = \pi/4 \) and \( \theta_{12}^0 \) of the popular mixing kind, we have to set \( \chi_1 \neq \chi_2 = \chi_3 \) as well as \( \chi_4 = \chi_5 \). This is achieved when \( v_1 = v_2 = v \) and \( r_{11} = r_{22} = r \). The condition \( r_{11} = r_{22} = r \) when translated in terms of the right-handed neutrino mass matrix in Eq. (11) using Eq. (14) will lead to:

\[ M_{\nu_R} = \frac{1}{2} \begin{pmatrix} m_{\nu_{11}} & m_{\nu_{12}} \\ m_{\nu_{12}} & m_{\nu_{22}} \end{pmatrix}. \]

(17)

The matrix in Eq. (17) corresponds to maximal mixing in the right-handed neutrino sector. Thus, to get the form of left-handed neutrino mass matrix as in Eq. (4) it is necessary to have \( v_1 = v_2 = v \) as well as maximal mixing between \( N_{1R} \) and \( N_{2R} \) i.e., we have to set \( r_{11} = r_{22} = r \). Implementing these constraints to the general form of the mass matrix in Eq. (15) we get;

\[ M_{\nu_L}^{\text{flavour}} = v^2 \begin{pmatrix} y_1^2[4r_{12}\lambda_{123} + 2r_{12}] & y_1y_2[r_{12}\lambda_{12} + 2r_{12}\lambda_{123}] & y_1y_2[r_{12}\lambda_{12} + 2r_{12}\lambda_{123}] \\ y_1y_2[r_{12}\lambda_{12} + 2r_{12}\lambda_{123}] & y_2^2r_{12} & y_2^2(2r_{12}\lambda_{123}) \\ y_1y_2[r_{12}\lambda_{12} + 2r_{12}\lambda_{123}] & y_2^2(2r_{12}\lambda_{123}) & y_2^2r_{12} \end{pmatrix}. \]

(18)

Here \( \lambda_{12} \equiv \lambda_1 + \lambda_2 \) and \( \lambda_{123} \equiv \lambda_3 + \lambda_1 - \lambda_2 \). To get the form of \( M_{\nu_L}^{\text{flavour}} \) in Eq. (4), one has to identify:

\[
\begin{align*}
a & \equiv y_1^2v^2[4r_{12}\lambda_{123} + 2r_{12}] = y_1^2v^2[4r_{12}(\lambda_3 + \lambda_1 - \lambda_2) + 2r(\lambda_1 + \lambda_2)] \\
b & \equiv y_2^2v^2r_{12} = y_2^2v^2r(\lambda_1 + \lambda_2) \\
c & \equiv y_1y_2v^2[r_{12}\lambda_{12} + 2r_{12}\lambda_{123}] = y_1y_2v^2[r_{12}(\lambda_1 + \lambda_2) + 2r(\lambda_3 + \lambda_1 - \lambda_2)] \\
d & \equiv y_2^2v^2(2r_{12}\lambda_{123}) = y_2^2v^2[2r_{12}(\lambda_3 + \lambda_1 - \lambda_2)].
\end{align*}
\]

(19)

So far we are able to obtain the form of left-handed neutrino mass matrix required for \( \theta_{13} = 0, \theta_{23} = \pi/4 \) and \( \theta_{12}^0 \) of the popular mixing varieties. With this in hand, the obvious follow-up enterprise,
as mentioned earlier, will be to obtain realistic mixing viz. non-zero $\theta_{13}$, deviations of the atmospheric mixing angle $\theta_{23}$ from $\pi/4$ as well as tiny corrections to $\theta_{12}$ also. To get such realistic neutrino mixing, we have to shift from the choice of $r_{11} = r_{22} = r$, i.e., allow the two diagonal entries of the right-handed neutrino mass matrix to slightly differ from each other. In other words, let $r_{22} = r_{11} + \epsilon$, where $\epsilon$ is a small quantity. Therefore, one gets back the general form of $M_{\nu R}$ in Eq. (11) characterized by non-maximal mixing between $N_{1R}$ and $N_{2R}$. Thus setting $r_{22} = r_{11} + \epsilon$ is precisely shifting from the maximal mixing between the two right-handed neutrino states. With $v_{1} = v_{2} = v$ still valid, we can get a dominant component of $M^{flavour}_{\nu L}$ as in Eq. (18) denoted $M^{0}$ and a smaller contribution $M'$ proportional to $\epsilon$. Hence,

$$M^{flavour}_{\nu L} = M^{0} + M'$$  \hspace{1cm} (20)

with,

$$M^{0} = v^{2} \begin{pmatrix} y_{1}^{2} [4r_{12} \lambda_{123} + 2r_{11} \lambda_{12}] & y_{1}y_{2}[r_{12} \lambda_{12} + 2r_{11} \lambda_{123}] & y_{1}y_{2}[r_{12} \lambda_{12} + 2r_{11} \lambda_{123}] \\ y_{1}y_{2}[r_{12} \lambda_{12} + 2r_{11} \lambda_{123}] & y_{2}^{2}[2r_{12} \lambda_{12}] & y_{2}^{2}[2r_{12} \lambda_{123}] \\ y_{1}y_{2}[r_{12} \lambda_{12} + 2r_{11} \lambda_{123}] & y_{2}^{2}[2r_{12} \lambda_{123}] & y_{2}^{2}[2r_{12} \lambda_{12}] \end{pmatrix},$$  \hspace{1cm} (21)

and

$$M' = \epsilon \begin{pmatrix} x & y & 0 \\ y & x' & 0 \\ 0 & 0 & 0 \end{pmatrix},$$  \hspace{1cm} (22)

where,

$$x \equiv y_{1}^{2} v^{2} \lambda_{12} = y_{1}^{2} v^{2} (\lambda_{1} + \lambda_{2})$$

$$x' \equiv y_{2}^{2} v^{2} \lambda_{12} = y_{2}^{2} v^{2} (\lambda_{1} + \lambda_{2})$$

$$y \equiv y_{1}y_{2} v^{2} \lambda_{123} = y_{1}y_{2} v^{2} (\lambda_{3} + \lambda_{1} - \lambda_{2}).$$  \hspace{1cm} (23)

$M^{0}$ in Eq. (21) will represent the form of left-handed neutrino mass matrix needed for $\theta_{13} = 0$, $\theta_{23} = \pi/4$ and $\theta_{12}$ of the popular mixing types as in Eq. (1) when we identify:

$$a' \equiv y_{1}^{2} v^{2} [4r_{12} \lambda_{123} + 2r_{11} \lambda_{12}] = y_{1}^{2} v^{2} [4r_{12} (\lambda_{3} + \lambda_{1} - \lambda_{2}) + 2r_{11} (\lambda_{1} + \lambda_{2})]$$

$$b' \equiv y_{2}^{2} v^{2} r_{11} \lambda_{12} = y_{2}^{2} v^{2} r_{11} (\lambda_{1} + \lambda_{2})$$

$$c' \equiv y_{1}y_{2} v^{2} [r_{12} \lambda_{12} + 2r_{11} \lambda_{123}] = y_{1}y_{2} v^{2} [r_{12} (\lambda_{1} + \lambda_{2}) + 2r_{11} (\lambda_{3} + \lambda_{1} - \lambda_{2})]$$

$$d' \equiv y_{2}^{2} v^{2} [2r_{12} \lambda_{123}] = y_{2}^{2} v^{2} [2r_{12} (\lambda_{3} + \lambda_{1} - \lambda_{2})]$$  \hspace{1cm} (24)

in the same spirit as was done in case of Eq. (19).

With the help of non-degenerate perturbation theory we can calculate the corrections to eigenvalues and eigenvectors of $M^{0}$ from $M'$. The unperturbed flavour basis is given by the columns of the mixing matrix $U^{0}$ as shown in Eq. (2). For ease of presentation it is useful to define,

$$\gamma \equiv (b' - 3d' - a') \quad \text{and} \quad \rho \equiv \sqrt{a'^{2} + b'^{2} + 8c'^{2} + d'^{2} - 2a'b' - 2a' d' + 2b'd'}.$$  \hspace{1cm} (25)

Thus the third ket after receiving first order corrections will take the form:

$$|\psi_{3} \rangle = \begin{pmatrix} \frac{\rho}{\sqrt{\gamma^{2} - \rho^{2}}} [\rho (\sqrt{2} y \cos 2\theta^{0}_{12} - x' \sin 2\theta^{0}_{12}) - \gamma \sqrt{2} y] \\ -\frac{1}{\sqrt{2}} [1 + \xi \epsilon] \\ \frac{1}{\sqrt{2}} [1 - \xi \epsilon] \end{pmatrix}.$$  \hspace{1cm} (26)

\(6\)We are introducing the primed notation to differentiate from the $r_{11} = r_{22} = r$ case.
Here, we have used
\[ \xi \equiv [\gamma x' + \rho(x' \cos 2\theta_{12} + \sqrt{2}y \sin 2\theta_{12})]/(\gamma^2 - \rho^2). \] (27)

If we consider CP-conserving scenario then,
\[ \sin \theta_{13} = \frac{\epsilon}{\gamma^2 - \rho^2} \left[ \rho(\sqrt{2}y \cos 2\theta_{12} - x' \sin 2\theta_{12}) - \gamma \sqrt{2}y \right]. \] (28)

Expression for non-zero \( \theta_{13} \) in terms of the parameters of our model viz. \( \epsilon \), the vacuum expectation values \( v \) and the quartic couplings \( \lambda_i \), \( (i = 1, 2, 3) \), can be obtained with help of Eqs. (24), (25) and (28).

The shift of \( \theta_{23} \) from \( \pi/4 \) can be found from Eq. (26) as
\[ \tan \varphi \equiv \tan(\theta_{23} - \pi/4) = \xi \epsilon. \] (29)

The first-order corrections to the first and second ket will contribute to changes in \( \theta_{12} \). Defining:
\[ \beta \equiv \frac{\sqrt{2} \cos 2\theta_{12} + \frac{\lambda}{\gamma^2 - \rho^2} (x - x') \sin 2\theta_{12}}{\rho}, \] (30)
will lead to corrected solar mixing angle given by,
\[ \tan \theta_{12} = \frac{\sin \theta_{12}^0 + \epsilon \beta \cos \theta_{12}^0}{\cos \theta_{12}^0 - \epsilon \beta \sin \theta_{12}^0}. \] (31)

Needless to mention, expressions for corrected \( \theta_{12} \) in Eq. (31) and deviations of \( \theta_{23} \) from maximal mixing in Eq. (29) can be translated in terms of parameters of this \( S_3 \times Z_2 \) symmetric model by applying Eqs. (24), (25), (27) and (30).

In our entire analysis, we have taken \( r_{\alpha \beta} \), \( (\alpha, \beta = 1, 2) \), to be real therefore allowing no CP-violation. But one can associate Majorana phases to masses of the right-handed neutrinos, thus \( r_{\alpha \beta} \) can be complex quantities. Therefore \( \epsilon \) can also be complex that can give rise to CP-violation from Eq. (26).

In a nutshell, a radiative \( S_3 \times Z_2 \) symmetric scheme of scotogenic generation of realistic neutrino mixing is put forward. The model has two right-handed neutrinos, \( N_{1R} \) and \( N_{2R} \), which when maximally mixed can radiatively yield the form of left-handed Majorana neutrino mass matrix at one-loop characterized by \( \theta_{13} = 0, \theta_{23} = \pi/4 \) and \( \theta^0_{12} \) of any of the values specific to the tribimaximal (TBM), Bimaximal (BM) and Golden Ratio (GR) mixing collectively termed as popular lepton mixings. Small deviation from maximal mixing between the two right-handed neutrino states can produce realistic mixing angles i.e., non-zero \( \theta_{13} \), shifts of the atmospheric mixing angle \( \theta_{23} \) from \( \pi/4 \) and small corrections to \( \theta_{12} \). There are two inert \( SU(2)_L \) doublet scalar fields \( \eta_i, (i = 1, 2) \) in the model. Since the \( \eta_i \) are odd under the action of the unbroken \( Z_2 \), the lightest among these two scalars can serve as dark matter.

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A Appendix: The group \( S_3 \)

It is the permutation group of three objects \( [28] \) and therefore has \( 3! = 6 \) elements. \( S_3 \) has two generators \( A \) and \( B \) that satisfy \( A^2 = I = B^3 \) and \( (AB) (AB) = I. \) The group properties can be clearly understood from the group table shown in Table 3.
Table 3: The group table of the discrete symmetry $S_3$.

It has two one-dimensional representations $1$ and $1'$, as well as one two-dimensional representation $2$. The one dimensional representation $1$ is immune to both $A$ and $B$ whereas $1'$ flips sign when acted by $A$. In two-dimension, the group can be represented by the following matrices that obey all the properties discussed so far:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$ (A.1)

Here $\omega = e^{2\pi i/3}$ is a cube root of one. With the generators in Eq. (A.1), we can construct the rest of the members of the group as:

$$C = \begin{pmatrix} 0 & \omega^2 \\ \omega & 0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & \omega \\ \omega^2 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}.$$ (A.2)

$S_3$ is characterized by the following product rules,

$$1 \times 1' = 1', \quad 1' \times 1' = 1, \quad \text{and} \quad 2 \times 2 = 2 + 1 + 1'. \quad (A.3)$$

All the matrices $M_{ij}$ in Eqs. (A.1) and (A.2) obey,

$$\sum_{j,l=1,2} \alpha_{jl} M_{ij} M_{kl} = \alpha_{ik}.$$ (A.4)

Here $\alpha_{ij} = 0$ if $i = j$ and $\alpha_{ij} = 1$ if $i \neq j$.

Let $\Phi \equiv \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right)$ and $\Psi \equiv \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)$ be two doublets of $S_3$ which when combined according to Eq. (A.3) will yield:

$$\phi_1 \psi_2 + \phi_2 \psi_1 \equiv 1, \quad \phi_1 \psi_2 - \phi_2 \psi_1 \equiv 1', \quad \text{and} \quad \begin{pmatrix} \phi_2 \psi_2 \\ \phi_1 \psi_1 \end{pmatrix} \equiv 2.$$ (A.5)

Often, we have to work with Hermitian conjugate of the fields. Owing to the properties of the complex representations of $S_3$, [say, as for $B$ displayed in Eq. (A.1)], the hermitian conjugate of $\Phi$ is given by $\Phi^\dagger \equiv \left( \begin{array}{c} \phi_2^\dagger \\ \phi_1^\dagger \end{array} \right)$. This $\Phi^\dagger$ when combined with $\Psi$, keeping Eq. (A.3) in mind, we get,

$$\phi_2^\dagger \psi_2 + \phi_1^\dagger \psi_1 \equiv 1, \quad \phi_2^\dagger \psi_2 - \phi_1^\dagger \psi_1 \equiv 1', \quad \text{and} \quad \begin{pmatrix} \phi_1^\dagger \psi_2 \\ \phi_2^\dagger \psi_1 \end{pmatrix} \equiv 2.$$ (A.6)

Eqs. (A.5) and (A.6) play a pivotal role in determining the structure of the mass matrices in the model.
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