Optimal time-decay estimates for a diffusive Oldroyd-B model

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Abstract. In this paper, we study the optimal time decay rates for the higher order spatial derivatives of solutions to a diffusive Oldroyd-B model. As pointed out in the Sect. 1.2 of Huang et al. (J Differ Equ 306:456–491, 2022), how to establish the optimal decay estimate for the highest-order spatial derivatives of the solution to this model is still an open problem. Motivated by Wang and Wen (Sci China Math 65:1199–1228, 2022), we give a positive answer to this problem via some delicate analyses on the low- and high-frequency parts of the solution.

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1. Introduction

In this paper, we study the following diffusive Oldroyd-B system

\[
\begin{align*}
\partial_t u + u \cdot \nabla u + \nabla p - \epsilon \Delta u &= \kappa \text{div} \tau, \\
\partial_t \tau + u \cdot \nabla \tau - \mu \Delta \tau + \beta \tau &= Q(\nabla u, \tau) + \alpha \mathbb{D} u, \\
\text{div} u &= 0, \\
(u, \tau)(x, 0) &= (u_0, \tau_0),
\end{align*}
\]

(1.1)
on \mathbb{R}^3 \times (0, \infty), which is used to describe the motion of viscoelastic fluids. Here, \( u = (u_1, u_2, u_3)^T : \mathbb{R}^3 \to \mathbb{R}^3 \) is the velocity field of fluid, symmetric matrix \( \tau \in S_3(\mathbb{R}) \) is the tangential part and non-Newtonian part of the stress tensor, \( p \in \mathbb{R} \) is the pressure function of the fluid, \( \mathbb{D} u = \frac{1}{2} \left( \nabla u + (\nabla u)^T \right) \) is the deformation tensor, and

\[ Q(\nabla u, \tau) = \Omega \tau - \tau \Omega + b(\mathbb{D} u \tau + \tau \mathbb{D} u) \]

admits the usual bilinear form with the skew-symmetric part of velocity gradient \( \Omega = \frac{1}{2} \left( \nabla u - (\nabla u)^T \right) \) and constant \( b \in [-1, 1] \). The parameters \( \kappa, \beta \) and \( \alpha \) satisfy that \( \kappa, \beta, \alpha > 0 \). Moreover, \( \epsilon \geq 0 \) is the viscosity coefficient of the fluid, and \( \mu \geq 0 \) is the center-of-mass diffusion coefficient. The system (1.1) was first proposed by Oldroyd in 1958 [20].

As pointed out by Bhave et al. [2], the diffusion coefficient \( \mu \) is significantly smaller than other effects, the diffusive term \( \mu \Delta \tau \) is ignored in the classical Oldroyd-B model (the non-diffusive model). However, in the recent work by Málek et al. [18], the authors showed that the stress diffusion term can be interpreted either as a consequence of a nonlocal energy storage mechanism or as a consequence of a nonlocal entropy production mechanism.

The non-diffusive Oldroyd-B model [i.e., (1.1) with \( \mu = 0 \)] has been well studied by many authors in the field of mathematics and physics. For the existence results of strong solution, one can refer to the early works by Guilloté and Saut [9], Fernández-Cara et al. [8], and by Molinet and Talhouk [19, and
the recent works by Hieber et al. [10], Fang et al. [7], and by Zi et al. [25]. Lions and Masmoudi [15] obtained a global existence result for weak solutions of the corotational model \((b = 0)\). For the results about blow-up criteria, one can refer to the works by Chemin and Masmoudi [3], Lei et al. [13], and by Sun and Zhang [22]. The large time behaviors of the solutions are investigated by Hieber et al. [11] and by Huang et al. [12]. For a detailed review of those works, one can refer for instance to the Introduction in [12], the review papers by Lin [14] and by Renardy and Thomases [21], and the references therein.

However, there are only a few works on the mathematical theory of the diffusive Oldroyd-B model \([\text{i.e., (1.1)}\) with \(\mu > 0\)]. In the case \(\epsilon, \mu > 0\), Constantin and Klögl [4] obtained the global existence and uniqueness of strong solutions in \(\mathbb{R}^2\). In the case \(\epsilon = 0, \mu > 0\), Elgindi and Rousset [6] obtained a global wellposedness result provided that the initial data are small in \(H^s(\mathbb{R}^2)(s > 2)\). This result was extended to 3D case with small initial data in \(\mathbb{R}^3\) by Elgindi and Liu [5]. Recently, Huang et al. [12] proved the global wellposedness to the Cauchy problem in 3D and deduced some time decay estimates. Liu et al. [16] proved a similar results for the compressible counterpart of this model. Here, we restate the result of [12] in the following Proposition.

**Proposition 1.1.** [12] Assume that \((u_0, \tau_0) \in H^3(\mathbb{R}^3)\). For any given \(\epsilon \geq 0\) and \(\mu > 0\), there exists a sufficiently small constant \(0 < \varepsilon_0 \leq 1\) satisfying \((3.4)\) such that the Cauchy problem \((1.1)\) admits a unique global solution \((u, \tau) \in C([0, +\infty) ; H^3(\mathbb{R}^3))\) satisfying

\[
\|u(t)\|_{H^3}^2 + \|\tau(t)\|_{H^3}^2 + \int_0^t \left( \epsilon \|\nabla u(s)\|_{H^3}^2 + \|\nabla u(s)\|_{H^2}^2 + \mu \|\nabla \tau(s)\|_{H^3}^2 + \|\tau(s)\|_{H^3}^2 \right) ds \leq C_1 \varepsilon_0^2, \tag{1.2}
\]

for \(t \geq 0\), provided that \(\|u_0\|_{H^3} + \|\tau_0\|_{H^3} \leq \varepsilon_0\), where \(\varepsilon_0\) is a constant independent of \(\epsilon\) and \(t\), and the positive constant \(C_1\) may depend on \(\mu\) but independent of \(\epsilon\) and \(t\).

Assume further that \((u_0, \tau_0) \in L^1(\mathbb{R}^3)\). Then, the following upper time-decay estimates hold:

\[
\|\nabla^k u(t)\|_{L^2} \leq C_2 (1 + t)^{-\frac{k}{2} - \frac{3}{2}}, \quad \|\nabla^{k_1} \tau(t)\|_{L^2} \leq C_2 (1 + t)^{-\frac{k_1}{2} - \frac{3}{2}}, \tag{1.3}
\]

and

\[
\|\nabla^3 u(t)\|_{L^2} + \|\nabla^2 \tau(t)\|_{H^1} \leq C_2 (1 + t)^{-\frac{3}{2}}, \tag{1.4}
\]

for any \(t \geq 0\), where \(k = 0, 1, 2\), \(k_1 = 0, 1\), and the constant \(C_2\) depends only on \(\|(u_0, \tau_0)\|_{H^3 \cap L^1}\) and \(C_1\).

**Remark 1.2.** For readers’ convenience, we remark that estimates in \((1.3)\) are partial conclusions of Theorem 1.2 in [12] \{see, [12], Theorem 1.2, part (i)\}. And \((1.4)\) is a byproduct of the proof of \((1.3)\) which was stated in Lemma 4.5 of [12] \{see (4.15) in p. 478 of [12]\}.

**Remark 1.3.** For a proof of Proposition 1.1, see Theorems 1.1 and 1.2 in [12]. The assumption \((3.4)\) for \(\varepsilon_0\) is also assumed in [12] via a qualitative statement “\(\varepsilon_0\) is sufficiently small constant”. Here, we give the quantitative assumption \((3.4)\) is in order to clarify the proof, see the proof of Lemma 3.1 for details.

**Remark 1.4.** Besides the results stated in Proposition 1.1, the authors also proved a similar result for the non-diffusive model in [12]. From the proofs in [12], one can find that the non-diffusive model \((\epsilon > 0, \mu \geq 0)\) is much easier to handle than the diffusive model \((\epsilon \geq 0, \mu > 0)\). Therefore, in the present paper, we only deal with the diffusive model.

**Remark 1.5.** In the previous works (e.g., [12,16] and the references therein), the optimal time decay estimate for the highest-order spatial derivatives of \(\tau\) is not obtained for both \((1.1)\) and the compressible system.
1.1. Main result

The main result of the present paper is to investigate the optimal decay estimate for the highest-order spatial derivatives of the solution obtained in Proposition 1.1.

**Theorem 1.6.** Under the assumptions of Proposition 1.1, for any given $\epsilon \geq 0$ and $\mu > 0$, the solution $(u, \tau)$ of problem (1.1) enjoys the following further optimal decay estimates

$$
\|\nabla^3 u(t)\|_{L^2} + \|\nabla^2 \tau(t)\|_{L^2} \leq C_3 (1 + t)^{-\frac{3}{4}}, \quad \|\nabla^3 \tau(t)\|_{L^2} \leq C_5 (1 + t)^{-\frac{11}{14}}.
$$

**Remark 1.7.** The conclusion of Theorem 1.6 is optimal in the sense that the decay estimates for $(u, \tau)$ are the same as the decay estimates for solution of the linearized system (2.3). This result gives an positive answer to the open problem asked in the Sect. 1.2 of [12], refer to page 461 in [12] for details.

The main ideas of this work are as follows. Motivated by Wang and Wen [24], we make some delicate estimates to remove the low-frequency part of the energy functional to obtain the optimal decay estimate for $\|\nabla^3 u\|_{L^2}$, see Lemma 3.1 for details. The main difficulties arise in the decay estimate of $\|\nabla^3 \tau\|_{L^2}$ due to the fact that the decay rate of $\|\nabla^3 \tau\|_{L^2}$ should be fast than that of $\|\nabla^3 u\|_{L^2}$ as indicated by decay properties of the linearized problem, see Sects. 1.2 and 1.3 of [12] for a detailed illustration. Our observation here is that the high-frequency part $\|\nabla^3 (u^h, \tau^h)\|_{L^2}$ decays faster than $\|\nabla^3 (u, \tau)\|_{L^2}$. Then, we make full use of the low–high-frequency decomposition technique to prove our main result, see Lemma 3.2 for details.

The rest of the paper is organized as follows. In Sect. 2, we recall some known results which will be used in the proof of the main Theorem. In Sect. 3, we improve the decay rates of the solution successively and prove Theorem 1.6.

2. Preliminaries

**Notations**

In this paper, some standard notations are used. We use $C$ to denote the generic positive constant which may depend on the initial value and some other known constants but independent of time $t$ and the variable parameter $\epsilon$. We also use $B_i, C_i (i = 0, 1, 2, \ldots)$ to denote the specific constants which are necessary to clarify the proofs. Some other notations are stated as below:

“$G \lesssim F$” means that there exists a positive constant $C$ such that “$G \leq CF$”. We simply use the notations $\| \cdot \|_{L^p}$ and $\| \cdot \|_{H^k}$ ($1 \leq p \leq \infty, k = 1, 2, \ldots$) for the norm of spaces $L^p(\mathbb{R}^3)$ and $H^k(\mathbb{R}^3)$. As usual, $\langle \cdot, \cdot \rangle$ is the inner-product in $L^2(\mathbb{R}^3)$. For a Banach space $X$, we write $\|f\|_X^2 + \|g\|_X^2$ as $\|(f, g)\|_X^2$.

For $f \in L^2(\mathbb{R}^3)$, we denote the Fourier transform of $f$ by

$$
\hat{f}(\xi) := \mathcal{F} [f](\xi) := (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix \cdot \xi} f(x) dx.
$$

Let $\Lambda := \sqrt{-\Delta}$ and $\mathcal{P}$ be the Leray projector which can be represented via the Fourier transform as follows

$$
\Lambda f = \mathcal{F}^{-1}[|\xi|^{2} |\hat{f}|], \quad \text{and} \quad (\mathcal{P} u)^i = \mathcal{F}^{-1} \left[ \left( \delta_{j,k} - \frac{\xi_j \xi_k}{|\xi|^2} \right) (\hat{u}^k) \right].
$$

Let $\phi(\xi)$ be a smooth cut-off function satisfying that

$$
0 \leq \phi(x) \leq 1, \quad \text{and} \quad \left\{ \begin{array}{ll}
\phi(\xi) = 1, & \text{for } |\xi| \leq \frac{R}{2}, \\
\phi(\xi) = 0, & \text{for } |\xi| \geq R,
\end{array} \right.
$$
where $R > 0$ is a constant defined in Lemma 2.2. Set
\[ \phi_0(\xi) := \phi(\xi), \phi_1(\xi) := 1 - \phi(\xi), \phi_0(\xi) := 1 - (1 - \phi(\xi))^2, \tilde{\phi}_1(\xi) := \phi_1^2(\xi) = (1 - \phi(\xi))^2. \]
Using the above notations, for $f \in L^2(\mathbb{R}^3)$, we can define the low-frequency and high-frequency decomposition as follows
\[
\frac{2}{f(x) = f^f(x) + f^h(x) = f^\ell(x) + f^\hat{\ell}(x),}
\]
where
\[
f^f(x) := \phi_0(D)f(x), f^h(x) := \phi_1(D)f(x), f^\ell(x) := \tilde{\phi}_0(D)f(x), f^\hat{\ell}(x) := \tilde{\phi}_1(D)f(x),
\]
with the convention that $\phi(D)f(x) := \mathcal{F}^{-1}[-\phi\hat{f}](x)$ for smooth $\phi$. Then, we use the followings notations
\[
\Lambda^k\phi(D)f(x) := \mathcal{F}^{-1}[[|\xi|^k\phi\hat{f}]](x).
\]
As a consequence of Plancherel’s theorem, for $f \in H^3(\mathbb{R}^3)$, we have that
\[
\|\nabla^k f^\ell\|_{L^2} + \|\nabla^k f^\hat{\ell}\|_{L^2} + \|\nabla^k f^\hat{\ell}\|_{L^2} \lesssim \|\nabla^k f\|_{L^2}, \quad \text{for } k = 0, 1, 2, 3,
\]
and
\[
\|\nabla^k f^\ell\|_{L^2} \lesssim \|\nabla^k f^\hat{\ell}\|_{L^2} \lesssim \|\nabla^{k+1} f^\hat{\ell}\|_{L^2} \lesssim \|\nabla^{k+1} f\|_{L^2}, \quad \text{for } k = 0, 1, 2.
\]
In order to supplement the dissipation of $u$, similar to the treatment in [12], applying the Leray projector $\mathbb{P}$ and the operator $\Lambda^{-1}\mathbb{P}\text{div}$ to (1.1) and (1.1)$_2$, respectively, and denoting by
\[
\sigma := \Lambda^{-1}\mathbb{P}\text{div}\tau \text{ with } (\hat{\sigma})^j = i \left( \delta_{j,k} \frac{\xi_j\xi_k}{|\xi|^2} \right) \frac{\xi_i}{|\xi|} (\hat{\tau})^{j,k},
\]
we obtain
\[
\begin{cases}
\partial_t u - \epsilon \Delta u - \kappa \Lambda \sigma = \mathcal{F}_1, \\
\partial_t \sigma - \mu \Delta \sigma + \beta \sigma + \frac{\alpha}{2} \Lambda u = \mathcal{F}_2,
\end{cases}
\]
where the nonlinear terms are stated as below:
\[
\mathcal{F}_1 = -\mathbb{P}(u \cdot \nabla u), \quad \mathcal{F}_2 = -\Lambda^{-1}\mathbb{P}\text{div}(u \cdot \nabla \tau) + \Lambda^{-1}\mathbb{P}\text{divQ}(\nabla u, \tau).
\]
Next, we consider the linearized system of (2.2):
\[
\begin{cases}
\partial_t u - \epsilon \Delta u - \kappa \Lambda \sigma = 0, \\
\partial_t \sigma - \mu \Delta \sigma + \beta \sigma + \frac{\alpha}{2} \Lambda u = 0, \\
(u, \sigma)(x, 0) = (u_0, \sigma_0)(x).
\end{cases}
\]
For system (2.3), we have the following Lemmas 2.1, 2.2 and 2.3 which are proved in [12].

**Lemma 2.1.** (Lemma 2.1 in [12]) Fourier transform of the solution to system (2.3) can be solved by
\[
\hat{u}^j = (G_3(\xi, t) - \epsilon |\xi|^2 G_1(\xi, t)) \hat{u}_0^j + \kappa |\xi| G_1(\xi, t) \hat{\sigma}_0^j,
\]
\[
\hat{\sigma}^j = -\frac{\alpha}{2} |\xi| G_1(\xi, t) \hat{u}_0^j + (G_2(t, \xi) + \epsilon |\xi|^2 G_1(\xi, t)) \hat{\sigma}_0^j,
\]
for $j = 1, 2, 3$, and
\[
G_1(\xi, t) = \frac{e^{\lambda_+ t} - e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \quad G_2(\xi, t) = \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}, \quad G_3(\xi, t) = \frac{\lambda_+ e^{\lambda_+ t} - \lambda_- e^{\lambda_- t}}{\lambda_+ - \lambda_-}.
\]

**Lemma 2.2.** (Proposition 2.3 and Lemma 4.5 in [12]) There exist positive constants $R = R(\alpha, \kappa, \beta)$, $\theta = \theta(\alpha, \kappa, \beta)$ and $K = K(\alpha, \kappa, \beta)$ such that
\[
|G_1(\xi, t)| + |G_3(\xi, t)| \leq Ke^{-\theta|\xi|^2 t}, \quad |G_2(\xi, t)| \leq K \left( |\xi|^2 e^{-\theta|\xi|^2 t} + e^{-\frac{\alpha t}{2}} \right), \quad \text{for any } |\xi| \leq R \text{ and } t > 0.
\]
Lemma 3.1. (Lemma 4.1 in [12]) Letting \((u, \sigma)\) be the solution of problem (2.2), then we have the following time-decay estimates, for the low-frequency part of \((u, \sigma)\):

\[
\left( \int_{|\xi| \leq R} |\xi|^{2k} |\hat{u}|^2 \, d\xi \right)^{\frac{1}{2}} \lesssim (1 + t)^{-\frac{2}{3} - \frac{k}{6}} \max \left\{ \|u_0\|_{L^1} + \int_0^t (1 + t - s)^{-\frac{3}{2} - \frac{k}{2}} \left\| (F_1, F_2) \right\|_{L^1} \, ds \right\} + \int_0^t (1 + t - s)^{-\frac{1}{2}} \left\| (F_1, F_2) \right\|_{L^2} \, ds.
\]

Next, we recall some useful inequalities in the following Lemmas.

Lemma 2.4. [1, 23] Let \(f \in H^2(\mathbb{R}^3)\). Then, we have

\[
\|f\|_{L^q} \lesssim \|f\|_{H^1}, \quad 2 \leq q \leq 6,
\]

\[
\|f\|_{L^6} \lesssim \|\nabla f\|_{L^2},
\]

\[
\|f\|_{L^\infty} \lesssim \|\nabla f\|_{L^2}^\frac{1}{2} \|\nabla^2 f\|_{L^2}^\frac{1}{2} \lesssim \|\nabla f\|_{H^1}.
\]

Lemma 2.5. [17] Let \(k \geq 1\) be an integer and \(f, g \in L^\infty(\mathbb{R}^3) \cap H^k(\mathbb{R}^3)\), it holds that

\[
\|\nabla^k(fg)\|_{L^2} \lesssim \|f\|_{L^\infty} \|\nabla^k g\|_{L^2} + \|\nabla^k f\|_{L^2} \|g\|_{L^\infty},
\]

and

\[
\|\nabla^k(fg) - f\nabla^k g\|_{L^2} \lesssim \|\nabla f\|_{L^\infty} \|\nabla^{k-1} g\|_{L^2} + \|\nabla^k f\|_{L^2} \|g\|_{L^\infty}.
\]

3. Proof of Theorem 1.6

Noting that

\[
\langle \Lambda^3 u^h, \Lambda^2 \sigma^h \rangle \leq \|\Lambda^3 u^h\|_{L^2} \|\Lambda^2 \sigma^h\|_{L^2} \leq B_0 \left( \|\Lambda^3 u^h\|_{L^2}^2 + \|\Lambda^2 \sigma^h\|_{L^2}^2 \right),
\]

define the temporal energy functional that

\[
\mathcal{H}_3(t) := \alpha \|\nabla^3 u\|_{L^2}^2 + \kappa \|\nabla^3 \sigma\|_{L^2}^2, \quad \mathcal{H}_3^h(t) := \alpha \|\nabla^3 u^h\|_{L^2}^2 + \kappa \|\nabla^3 \sigma^h\|_{L^2}^2 + \eta_1 \langle \Lambda^3 u^h, \Lambda^2 \sigma^h \rangle,
\]

\[
\mathcal{H}_3^h(t) := \alpha \|\nabla^3 u^h\|_{L^2}^2 + \kappa \|\nabla^3 \sigma^h\|_{L^2}^2, \quad \mathcal{H}_3^h(t) := \alpha \|\nabla^3 u^h\|_{L^2}^2 + \kappa \|\nabla^3 \sigma^h\|_{L^2}^2 + \eta_1 \langle \Lambda^3 u^h, \Lambda^2 \sigma^h \rangle,
\]

where \( 0 < \eta_1 \leq \frac{1}{27\sigma_0} \min \{\alpha, \kappa\} \) is a constant defined in the proof of Lemma 3.1, to ensure that

\[
\frac{1}{2} \mathcal{H}_3(t) \leq \mathcal{H}_3(t) \leq 2 \mathcal{H}_3(t), \quad \text{and} \quad \frac{1}{2} \mathcal{H}_3^h(t) \leq \mathcal{H}_3^h(t) \leq 2 \mathcal{H}_3^h(t).
\]

The proof of Theorem 1.6 consists of Lemmas 3.1 and 3.2. To begin with, we have following optimal decay estimate for \(\|\nabla^3 u\|_{L^2}\) and \(\|\nabla^2 \tau\|_{L^2}\) which is not obtained in [12]. A similar result of Lemma 3.1 for the compressible model was proved in [16].

Lemma 3.1. Under the assumptions of Theorem 1.6, it holds that

\[
\|\nabla^3 u(t)\|_{L^2} + \|\nabla^2 \tau(t)\|_{L^2} + \|\nabla^3 \tau(t)\|_{L^2} \lesssim (1 + t)^{-\frac{9}{4}}.
\]
Proof. To begin with, applying $\nabla^3$ to (1.1), and then multiplying (1.1) by $\alpha \nabla^3 u$ and (1.1) by $\kappa \nabla^3 \tau$, we have from integration by parts and the cancellation relation that

$$\frac{1}{2} \frac{d}{dt} \left( \alpha \| \nabla^3 u \|_{L^2}^2 + \kappa \| \nabla^3 \tau \|_{L^2}^2 \right) + \alpha \| \nabla^4 u \|_{L^2}^2 + \kappa \mu \| \nabla^4 \tau \|_{L^2}^2 + \kappa \beta \| \nabla^3 \tau \|_{L^2}^2$$

$$= -\alpha \langle \nabla^3 (u \cdot \nabla u), \nabla^3 u \rangle - \kappa \langle \nabla^3 (u \cdot \nabla \tau), \nabla^3 \tau \rangle - \kappa \langle \nabla^2 Q(\nabla u, \tau), \nabla^4 \tau \rangle =: \sum_{p=1}^{3} I_p. \quad (3.1)$$

Using the incompressible condition, (1.2), Hölder inequality, Cauchy inequality, Lemmas 2.4 and 2.5, we can estimate $I_1, I_2$ and $I_3$ as follows. For $I_1$, we have

$$I_1 = -\alpha \langle \nabla^3 (u \cdot \nabla u), \nabla^3 u \rangle$$

$$= -\alpha \langle \nabla^3 (u \cdot \nabla u) - (u \cdot \nabla) \nabla^3 u, \nabla^3 u \rangle$$

$$\lesssim \| \nabla^3 (u \cdot \nabla u) - (u \cdot \nabla) \nabla^3 u \|_{L^2} \| \nabla^3 u \|_{L^2}$$

$$\lesssim \left( \| \nabla u \|_{L^\infty} \| \nabla^3 u \|_{L^2} + \| \nabla^3 u \|_{L^2} \| \nabla u \|_{L^\infty} \right) \| \nabla^3 u \|_{L^2}$$

$$\lesssim \varepsilon_0 \| \nabla^3 u \|_{L^2}^2.$$ 

Similarly, for $I_2$, we have

$$I_2 = -\kappa \langle \nabla^3 (u \cdot \nabla \tau), \nabla^3 \tau \rangle$$

$$= -\kappa \langle \nabla^3 (u \cdot \nabla \tau) - (u \cdot \nabla) \nabla^3 \tau, \nabla^3 \tau \rangle$$

$$\lesssim \| \nabla^3 (u \cdot \nabla \tau) - (u \cdot \nabla) \nabla^3 \tau \|_{L^2} \| \nabla^3 \tau \|_{L^2}$$

$$\lesssim \left( \| \nabla u \|_{L^\infty} \| \nabla^3 \tau \|_{L^2} + \| \nabla \tau \|_{L^\infty} \| \nabla^3 u \|_{L^2} \right) \| \nabla^3 \tau \|_{L^2}$$

$$\lesssim \varepsilon_0 \left( \| \nabla^3 u \|_{L^2}^2 + \| \nabla^3 \tau \|_{L^2}^2 \right),$$

And, for $I_3$, we have

$$I_3 = -\kappa \langle \nabla^2 Q(\nabla u, \tau), \nabla^4 \tau \rangle$$

$$\lesssim \langle \nabla u \|_{L^2}, | \nabla^4 \tau | \rangle + \langle \nabla^2 u \|_{L^6}, | \nabla^4 \tau | \rangle + \langle \nabla^3 u \|_{L^4}, | \nabla^4 \tau | \rangle$$

$$\lesssim \left( \| \nabla u \|_{L^2} \| \nabla^2 \tau \|_{L^6} + \| \nabla^2 u \|_{L^6} \| \nabla \tau \|_{L^4} + \| \nabla \tau \|_{L^3} \| \nabla^2 u \|_{L^2} \| \nabla \tau \|_{L^\infty} \right) \| \nabla^4 \tau \|_{L^2}$$

$$\leq \frac{K\mu}{2} \| \nabla^4 \tau \|_{L^2}^2 + C\varepsilon_0^2 \left( \| \nabla^3 u \|_{L^2}^2 + \| \nabla^3 \tau \|_{L^2}^2 \right).$$

Combining the above three estimate with (3.1), we obtain

$$\frac{d}{dt} \left( \alpha \| \nabla^3 u \|_{L^2}^2 + \kappa \| \nabla^3 \tau \|_{L^2}^2 \right) + \kappa \mu \| \nabla^4 \tau \|_{L^2}^2 + \kappa \beta \| \nabla^3 \tau \|_{L^2}^2 \leq B_1 \varepsilon_0 \left( \| \nabla^3 u \|_{L^2}^2 + \| \nabla^3 \tau \|_{L^2}^2 \right). \quad (3.2)$$

Next, to supplement the dissipation of $u$, multiplying $\Lambda^3 \phi_1(D)(2.2)_1$ and $\Lambda^2 \phi_1(D)(2.2)_2$ by $\Lambda^2 \sigma_h$ and $\Lambda^3 u_h$, respectively, then summing the result up, we have, from integration by parts, that

$$\frac{d}{dt} \left( \Lambda^3 u_h, \Lambda^2 \sigma_h \right) + \frac{\alpha}{2} \| \Lambda^3 u_h \|_{L^2}^2$$

$$= \kappa \| \Lambda^3 \sigma_h \|_{L^2}^2 - (\epsilon + \mu) \langle \Lambda^3 u_h, \Lambda^4 \sigma_h \rangle - \beta \langle \Lambda^3 u_h, \Lambda^2 \sigma_h \rangle + \langle \Lambda^2 \mathcal{F}_1, \Lambda^3 \sigma_h \rangle + \langle \Lambda^3 u_h, \Lambda^2 \mathcal{F}_2 \rangle$$

$$\lesssim \left( \| \Lambda^4 \sigma_h \|_{L^2} + \| \Lambda^3 u_h \|_{L^2} \right) \| \Lambda^4 \sigma_h \|_{L^2} + \| \Lambda^3 u_h, \Lambda^2 \mathcal{F}_2 \|.$$
where the last two terms can be estimated as follows,
\[
|\langle \Lambda^2 F_1^h, \Lambda^3 \sigma^h \rangle| \lesssim \|\Lambda^4 \sigma^h\|_{L^2} \left( \|u\|_{L^\infty} \|\nabla^3 u\|_{L^2} + \|\nabla u\|_{L^3} \|\nabla^2 u\|_{L^6} + \|\nabla^2 u\|_{L^5} \|\nabla u\|_{L^3} \right)
\lesssim \|\Lambda^4 \sigma^h\|_{L^2} \|\nabla u\|_{H^1} \|\nabla^3 u\|_{L^2},
\]
and
\[
|\langle \Lambda^3 u^h, \Lambda^2 F_2^h \rangle| \lesssim \|\Lambda^3 u^h\|_{L^2} \|\Lambda^2 F_2^h\|_{L^2} \\
\lesssim \|\nabla^3 u\|_{L^2} \left( \|u\|_{L^\infty} \|\nabla^3 \tau\|_{L^2} + \|\nabla u\|_{L^3} \|\nabla^2 \tau\|_{L^6} + \|\nabla^2 u\|_{L^5} \|\nabla \tau\|_{L^3} \right)
+ \|\nabla^3 u\|_{L^2} \left( \|\nabla u\|_{L^3} \|\nabla^2 \tau\|_{L^6} + \|\nabla^2 u\|_{L^5} \|\nabla \tau\|_{L^3} + \|\nabla^3 u\|_{L^2} \|\nabla \tau\|_{L^\infty} \right)
\lesssim \|\nabla^3 u\|_{L^2} \left( \|\nabla u\|_{H^1} \|\nabla^3 \tau\|_{L^2} + \|\nabla \tau\|_{H^1} \|\nabla^3 u\|_{L^2} \right)
\lesssim \|\nabla(u, \tau)\|_{H^1}^2 \left( \|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 \tau\|_{L^2}^2 \right).
\]
Combining the above three inequalities, we obtain that
\[
\frac{d}{dt} \langle \Lambda^3 u^h, \Lambda^2 \sigma^h \rangle + \frac{\alpha}{2} \|\Lambda^3 u^h\|_{L^2}^2 \leq \frac{\alpha}{4} \|\Lambda^3 u^h\|_{L^2}^2 + C \|\Lambda^4 \tau^h\|_{L^2}^2 + C \|\nabla(u, \tau)\|_{H^1}^2 \left( \|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 \tau\|_{L^2}^2 \right),
\]
which implies that
\[
\frac{d}{dt} \langle \Lambda^3 u^h, \Lambda^2 \sigma^h \rangle + \frac{\alpha}{4} \|\Lambda^3 u^h\|_{L^2}^2 \leq B_2 \|\Lambda^3 \tau^h\|_{L^2}^2 + B_2 \varepsilon_0 \left( \|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 \tau\|_{L^2}^2 \right). \tag{3.3}
\]
Set \( \eta_1 = \min \left\{ \frac{1}{2B_0} \min\{\alpha, \kappa\}, \frac{\kappa}{2B_2} \right\} \). Then, multiplying (3.3) by \( \eta_1 \), summing the result with (3.2), and assuming
\[
\varepsilon_0 \leq \min \left\{ \frac{\alpha}{16B_2}, \frac{\alpha \eta_1}{16B_1}, \frac{\kappa \beta}{4B_2 \eta_1}, \frac{\kappa \beta}{4B_1} \right\}, \tag{3.4}
\]
we obtain that
\[
\frac{d}{dt} \tilde{H}_3(t) + \frac{\kappa \beta}{2} \|\nabla^4 \tau\|_{L^2}^2 + \frac{\kappa \beta}{2} \|\nabla^3 \tau\|_{L^2}^2 + \frac{\eta_1 \alpha}{8} \|\Lambda^3 u^h\|_{L^2}^2 \leq (B_1 + B_2) \varepsilon_0 \|\nabla^3 u^\ell\|_{L^2}^2,
\]
which, together with the fact \("\|\nabla^3 u\|_{L^2}^2 = \|\Lambda^3 u\|_{L^2}^2 \leq \|\Lambda^3 u^\ell\|_{L^2}^2 + \|\Lambda^3 u^h\|_{L^2}^2"\), implies
\[
\frac{d}{dt} \tilde{H}_3(t) + \min \left\{ \frac{\beta}{2}, \frac{\eta_1}{8} \right\} \left( \kappa \|\nabla^3 \tau\|_{L^2}^2 + \alpha \|\nabla^3 u\|_{L^2}^2 \right) \leq \left( (B_1 + B_2) \varepsilon_0 + \frac{\eta_1 \alpha}{8} \right) \|\nabla^3 u^\ell\|_{L^2}^2. \tag{3.5}
\]
Next, letting \( \eta_2 := \frac{1}{2} \min \left\{ \frac{\beta}{2}, \frac{\eta_1}{8} \right\} \), \( B_3 := (B_1 + B_2) \varepsilon_0 + \frac{\eta_1 \alpha}{8} \), we have, from (3.5), that
\[
\frac{d}{dt} \tilde{H}_3(t) + \eta_2 \tilde{H}_3(t) \leq B_3 \|\nabla^3 u^\ell\|_{L^2}^2. \tag{3.6}
\]
Using Hölder’s inequality, Sobolev’s inequality, (1.3), (1.4) and Lemma 2.3, we have that

\[
\left( \int_{|\xi| \leq R} |\xi|^6 |\hat{u}|^2 d\xi \right)^{\frac{1}{2}} \lesssim (1 + t)^{-\frac{3}{4}} \|u_0, \tau_0\|_{L^1} + \int_0^t (1 + t - s)^{-\frac{3}{4}} \|u, \tau\|_{L^2} \|\nabla (u, \tau)\|_{L^2} ds + \int_0^t (1 + t - s)^{-\frac{3}{4}} \|u, \tau\|_{L^2} \|\nabla^2 (u, \tau)\|_{H^1} ds \leq (1 + t)^{-\frac{3}{4}}.
\]

Then, using Gronwall’s inequality and (3.6), we get

\[
\mathcal{H}_3(t) \leq 2\mathcal{H}_3(0) + B_3 \int_0^t e^{-\eta_2(t-s)} \left( \int_{|\xi| \leq R} |\xi|^6 |\hat{u}(s)|^2 d\xi \right) ds \lesssim (1 + t)^{-\frac{3}{4}}.
\]

(3.7)

Now, we are in the position to improve the decay estimate of \(\|\nabla^2 \tau\|_{L^2}\). Applying \(\nabla^2\) to (1.1)2 multiplying the result by \(\nabla^2 \tau\), and integrating with respect to \(x\), we have, from (3.7), that

\[
\frac{1}{2} \frac{d}{dt} \|\nabla^2 \tau\|_{L^2}^2 + \mu \|\nabla^3 \tau\|_{L^2}^2 + \frac{\beta}{2} \|\nabla^2 \tau\|_{L^2}^2 \lesssim \alpha \|\nabla^3 u\|_{L^2}^2 + \|\nabla^2 Q(\nabla u, \tau)\|_{L^2}^2 + \|\nabla^2 (u \cdot \nabla u)\|_{L^2}^2 \lesssim \alpha \|\nabla^3 u\|_{L^2}^2 + \|\nabla^3 u\|_{L^2}^2 \|\tau\|_{L^\infty}^2 + \|\nabla u\|_{L^\infty}^2 \|\nabla^2 \tau\|_{L^2}^2 + \|\nabla^3 \tau\|_{L^2}^2 \|u\|_{L^\infty}^2 + \|\nabla \tau\|_{L^\infty}^2 \|\nabla^2 u\|_{L^2}^2 \lesssim (1 + t)^{-\frac{2}{4}},
\]

which, together with Gronwall’s inequality, implies that

\[
\|\nabla^2 \tau(t)\|_{L^2}^2 \lesssim e^{-\beta t} \|\nabla^2 \tau_0\|_{L^2}^2 + \int_0^t e^{-\beta(t-s)} (1 + s)^{-\frac{2}{4}} ds \leq C(1 + t)^{-\frac{2}{4}}.
\]

(3.8)

The proof is complete by (3.7) and (3.8).

Next, we prove the optimal decay estimate for \(\|\nabla^3 \tau\|_{L^2}\).

**Lemma 3.2.** Under the assumptions of Theorem 1.6, it holds that

\[
\|\nabla^3 \tau(t)\|_{L^2} \lesssim (1 + t)^{-\frac{11}{4}}.
\]

**Proof.** To begin with, multiplying \(\alpha \nabla^3 \phi_1(D1)\) (1.1)1 and \(\kappa \nabla^3 \phi_1(D1)\) by \(\nabla^3 u\) and \(\nabla^3 \tau\), respectively, then summing the result up, we have, from integration by parts, that

\[
\frac{1}{2} \frac{d}{dt} \left( \alpha \|\nabla^3 u\|_{L^2}^2 + \kappa \|\nabla^3 \tau\|_{L^2}^2 \right) + \alpha \|\nabla^4 u\|_{L^2}^2 + \kappa \|\nabla^4 \tau\|_{L^2}^2 + \kappa \|\nabla^3 \tau\|_{L^2}^2 + \kappa \beta \|\nabla^3 \tau\|_{L^2}^2 = \sum_{p=1}^3 \mathcal{J}_p.
\]

(3.9)
Using Parseval’s relation, the fact that $\tilde{\phi}_1(\xi) = \phi_1^2(\xi)$, we can rewrite $J_1$ in the following form
\[
J_1 = -\alpha \langle \nabla^3 (u \cdot \nabla u), \nabla^3 u^h \rangle \\
= -\alpha \langle \mathcal{F} [\nabla^3 (u \cdot \nabla u)]^h, \mathcal{F} [\nabla^3 u^h] \rangle \\
= -\alpha \langle (i\xi_j)(i\xi_m)\phi_1(\xi)\mathcal{F}[(u \cdot \nabla u)], (i\xi_j)(i\xi_m)\phi_1(\xi)\hat{u} \rangle \\
= -\alpha \langle (i\xi_j)(i\xi_m)\phi_1(\xi)\mathcal{F}[(u \cdot \nabla u)], (i\xi_j)(i\xi_m)\phi_1^2(\xi)\hat{u} \rangle \\
= -\alpha \langle \nabla^3 (u \cdot \nabla u), \nabla^3 u^h \rangle.
\]

Then, using the incompressible condition and decomposition (2.1), for $J_1$ we have
\[
J_1 = -\alpha \langle \nabla^3 (u \cdot \nabla u^\ell), \nabla^3 u^h \rangle - \alpha \langle \nabla^3 (u \cdot \nabla u^h), \nabla^3 u^h \rangle \\
=: J_{11} + J_{12}.
\]

Using Hölder inequality, Cauchy inequality, (1.3), (1.4), Lemmas 2.4, 2.5 and 3.1, for $J_{11}$, we have
\[
J_{11} = -\alpha \langle \nabla^3 (u \cdot \nabla u^\ell), \nabla^3 u^h \rangle \\
\lesssim \left( \|\nabla^3 u\|_{L^2} \|\nabla u^\ell\|_{L^\infty} + \|u\|_{L^\infty} \|\nabla^4 u^\ell\|_{L^2} \right) \|\nabla^3 u^h\|_{L^2} \\
\lesssim \left( \|\nabla^3 u\|_{L^2} \|\nabla^2 u\|_{H^1} + \|u\|_{H^1} \|\nabla^4 u^\ell\|_{L^2} \right) \|\nabla^3 u^h\|_{L^2} \\
\lesssim \left( (1 + t)^{-\frac{\alpha}{2} - \frac{\gamma}{2}} + (1 + t)^{-\frac{\gamma}{2}} \|\nabla^4 u^\ell\|_{L^2} \right) (1 + t)^{-\frac{\gamma}{2}} \\
\lesssim (1 + t)^{-\frac{\alpha}{2} + \frac{\gamma}{2}} + (1 + t)^{-\frac{\gamma}{2}} \|\nabla^4 u^\ell\|_{L^2}.
\]

Similarly, for $J_{12}$, we have
\[
J_{12} = -\alpha \langle \nabla^3 (u \cdot \nabla u^h), \nabla^3 u^h \rangle - (u \cdot \nabla)\nabla^3 u^h \]
\[
\lesssim \left( \|\nabla^3 u\|_{L^2} \|\nabla u^h\|_{L^\infty} + \|u\|_{L^\infty} \|\nabla^3 u^h\|_{L^2} \right) \|\nabla^3 u^h\|_{L^2} \\
\lesssim \|\nabla^2 u\|_{H^1} \|\nabla^3 u^h\|_{L^2}^2 \\
\lesssim (1 + t)^{-\frac{\alpha}{2} - \frac{\gamma}{2} \times 2} = (1 + t)^{-\frac{\alpha}{2}}.
\]

Combining the above two estimates, we have
\[
J_1 = J_{11} + J_{12} \lesssim (1 + t)^{-\frac{\alpha}{2}} + (1 + t)^{-\frac{\gamma}{2}} \|\nabla^4 u^\ell\|_{L^2}.
\]

Next, using the incompressible condition, Hölder inequality, Cauchy inequality, the decomposition (2.1), (1.3), (1.4), Lemmas 2.4, 2.5 and 3.1, we can estimate $J_2$ and $J_3$ as follows.
\[
J_2 = \kappa \langle \nabla^2 (u \cdot \nabla \tau)^h, \Delta \nabla^2 \tau^h \rangle \lesssim \left( \|\nabla^2 (u \cdot \nabla \tau)\|, \|\nabla^4 \tau^h\| \right) \\
\lesssim \left( \|\nabla^2 u\|_{L^2} \|\nabla \tau\|_{L^\infty} + \|u\|_{L^\infty} \|\nabla^3 \tau\|_{L^2} \right) \|\nabla^4 \tau^h\|_{L^2} \\
\leq \frac{\kappa H}{4} \|\nabla^4 \tau^h\|_{L^2}^2 + C \left( \|\nabla^2 u\|_{L^2} \|\nabla^2 \tau\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 \|\nabla^3 \tau\|_{L^2}^2 \right) \\
\leq \frac{\kappa H}{4} \|\nabla^4 \tau^h\|_{L^2}^2 + C(1 + t)^{-\frac{\gamma}{2} - \frac{\alpha}{2}} + C(1 + t)^{-\frac{\alpha}{2} - \frac{\gamma}{2}} \\
\leq \frac{\kappa H}{4} \|\nabla^4 \tau^h\|_{L^2}^2 + C(1 + t)^{-7}.
\]
and

\[ J_3 = -\kappa(\nabla^2(Q(\nabla u, \tau))^h, \nabla^4 \tau^h) \]

\[ \lesssim \left( \|\nabla^3 u\|_{L^2} \|\tau\|_{L^\infty} + \|\nabla u\|_{L^\infty} \|\nabla^2 \tau\|_{L^2} \right) \|\nabla^4 \tau^h\|_{L^2} \]

\[ \leq \frac{\kappa \mu}{4} \|\nabla^4 \tau^h\|_{L^2}^2 + C \left( \|\nabla^3 u\|_{L^2}^2 \|\nabla \tau\|_{H^1}^2 + \|\nabla^2 u\|_{H^1}^2 \|\nabla^2 \tau\|_{L^2}^2 \right) \]

(3.12)

\[ \leq \frac{\kappa \mu}{4} \|\nabla^4 \tau^h\|_{L^2}^2 + (1+t)^{-\frac{2}{3}} + C(1+t)^{-\frac{5}{2}} \]

\[ \leq \frac{\kappa \mu}{4} \|\nabla^4 \tau^h\|_{L^2}^2 + C(1+t)^{-8}. \]

Substituting (3.10), (3.11) and (3.12) into (3.9), we obtain that

\[
\frac{d}{dt} \left( \alpha \|\nabla^3 u^h\|_{L^2}^2 + \kappa \|\nabla^3 \tau^h\|_{L^2}^2 \right) + \kappa \mu \|\nabla^4 \tau^h\|_{L^2}^2 + \kappa \beta \|\nabla^5 \tau^h\|_{L^2}^2 \]

\[ \lesssim (1+t)^{-\frac{2}{25}} + (1+t)^{-\frac{2}{7}} \|\nabla^4 \tau^h\|_{L^2}. \]

Using Hölder’s inequality, Sobolev’s inequality, (1.3), (1.4) and Lemmas 2.3, 2.4 and 3.1, we have that

\[ \left( \int \frac{|\xi|^8 |\ddot{u}|^2 d\xi}{|\xi| \leq R} \right)^{\frac{1}{2}} \lesssim \left( (1+t)^{-\frac{2}{11}} \|(u_0, \tau_0)\|_{L^1} + \int_0^t (1+t-s)^{-\frac{12}{7}} \|(u, \tau)\|_{L^2} \|\nabla(u, \tau)\|_{L^2} ds \right) \]

\[ + \int_0^t (1+t-s)^{-2} \|(u, \tau)\|_{L^2} \|\nabla^2 (u, \tau)\|_{L^2} \|\nabla^3 (u, \tau)\|_{L^2} \] \]

\[ \lesssim (1+t)^{-\frac{2}{11}} + \int_0^t (1+t-s)^{-\frac{2}{7}} (1+s)^{-\frac{2}{7}} (1+s)^{-\frac{2}{7}} ds \]

\[ \leq (1+t)^{-\frac{1}{8}}. \]

Combining (3.13) and (3.14), we get that

\[ \frac{d}{dt} \left( \alpha \|\nabla^3 u^h\|_{L^2}^2 + \kappa \|\nabla^3 \tau^h\|_{L^2}^2 \right) + \kappa \mu \|\nabla^4 \tau^h\|_{L^2}^2 + \kappa \beta \|\nabla^5 \tau^h\|_{L^2}^2 \lesssim (1+t)^{-\frac{2}{25}}. \]

(3.15)

From the proof of (3.3), it is easy to deduce that

\[ \frac{d}{dt} \langle \Lambda^3 u^h, \Lambda^2 \sigma^h \rangle + \frac{\alpha}{4} \|\Lambda^3 u^h\|_{L^2}^2 \leq B_2 \|\Lambda^4 \tau^h\|_{L^2}^2 + \|\nabla (u, \tau)\|_{H^1}^2 \left( \|\nabla^3 u^h\|_{L^2}^2 + \|\nabla^3 \tau^h\|_{L^2}^2 \right) \]

\[ \leq B_2 \|\Lambda^4 \tau^h\|_{L^2}^2 + C(1+t)^{-7}. \]

(3.16)

Then, (3.15) along with \(\eta_1 (3.16)\) imply that

\[ \frac{d}{dt} H_3^h(t) + \eta_2 H_3^h(t) \lesssim (1+t)^{-\frac{2}{25}}. \]

(3.17)

Using Gronwall’s inequality, we get, from (3.17), that

\[ \|\nabla^3 (u^h, \tau^h)(t)\|_{L^2}^2 \lesssim H_3^h(t) \lesssim e^{-\eta_2 t} H_3^h(0) + \int_0^t e^{-\eta_2 (t-s)} (1+s)^{-\frac{2}{25}} ds \lesssim (1+t)^{-\frac{2}{25}}. \]

(3.18)
Applying $\nabla^3$ to (1.1) multiplying the result by $\nabla^3 \tau$, and integrating with respect to $x$, we get,

$$\frac{1}{2} \frac{d}{dt} \|\nabla^3 \tau\|_{L^2}^2 + \mu \|\nabla^4 \tau\|_{L^2}^2 + \beta \|\nabla^3 \tau\|_{L^2}^2$$

$$= \langle \nabla^2 (u \cdot \nabla \tau), \Delta \nabla^2 \tau \rangle - \langle \nabla^2 (Q(\nabla u, \tau)), \nabla^4 \tau \rangle$$

$$+ \langle \nabla^3 \nabla^4 u^f, \nabla^3 \tau \rangle - \langle \nabla^3 u^h, \nabla^3 \text{div} \tau \rangle =: \sum_{j=1}^{4} \mathcal{K}_j.$$  \hfill (3.19)

Using (1.3), (1.4), Lemmas 2.4 and 3.1, we have, for $\mathcal{K}_1$, that

$$\mathcal{K}_1 = \langle \nabla^2 (u \cdot \nabla \tau), \Delta \nabla^2 \tau \rangle$$

$$\lesssim \left( \|u\|_{L^\infty} \|\nabla^3 \tau\|_{L^2} + \|\nabla^2 u\|_{L^2} \|\nabla \tau\|_{L^\infty} \right) \|\nabla^4 \tau\|_{L^2}$$

$$\leq \frac{\mu}{6} \|\nabla^4 \tau\|_{L^2}^2 + C \left( \|\nabla u\|_{H^1}^2, \|\nabla^3 \tau\|_{L^2}^2 + \|\nabla^2 u\|_{L^2} \|\nabla^2 \tau\|_{H^1}^2 \right)$$

$$\leq \frac{\mu}{6} \|\nabla^4 \tau\|_{L^2}^2 + C(1 + t)^{-\frac{5}{2} - \frac{\gamma}{2} + C(1 + t)^{-\frac{5}{2} - \frac{\gamma}{2}}$$

$$\leq \frac{\mu}{6} \|\nabla^4 \tau\|_{L^2}^2 + C(1 + t)^{-7}.$$  

Similarly, for $\mathcal{K}_2$, we have

$$\mathcal{K}_2 = -\langle \nabla^2 (Q(\nabla u, \tau)), \nabla^4 \tau \rangle$$

$$\lesssim \left( \|\nabla u\|_{L^\infty} \|\nabla^2 \tau\|_{L^2} + \|\nabla^3 u\|_{L^2} \|\nabla \tau\|_{L^\infty} \right) \|\nabla^4 \tau\|_{L^2}$$

$$\leq \frac{\mu}{6} \|\nabla^4 \tau\|_{L^2}^2 + C \left( \|\nabla^2 u\|_{H^1}^2, \|\nabla^2 \tau\|_{L^2}^2 + \|\nabla^3 u\|_{L^2} \|\nabla \tau\|_{H^1}^2 \right)$$

$$\leq \frac{\mu}{6} \|\nabla^4 \tau\|_{L^2}^2 + C(1 + t)^{-\frac{5}{2} - \frac{\gamma}{2} + C(1 + t)^{-\frac{5}{2} - \frac{\gamma}{2}}$$

$$\leq \frac{\mu}{6} \|\nabla^4 \tau\|_{L^2}^2 + C(1 + t)^{-8}.$$  

For $\mathcal{K}_3$, we have

$$\mathcal{K}_3 = \langle \nabla^3 \nabla^4 u^f, \nabla^3 \tau \rangle$$

$$\lesssim \|\nabla^4 u^f\|_{L^2} \|\nabla^3 \tau\|_{L^2}$$

$$\leq \frac{\beta}{2} \|\nabla^3 \tau\|_{L^2} + C \|\nabla^4 u^f\|_{L^2}^2$$

$$\leq \frac{\beta}{2} \|\nabla^3 \tau\|_{L^2} + (1 + t)^{-\frac{11}{2}},$$

where (3.14) is used. Moreover, using (3.18), for $\mathcal{K}_4$, we have

$$\mathcal{K}_4 = -\langle \nabla^3 u^h, \nabla^3 \text{div} \tau \rangle$$

$$\lesssim \|\nabla^3 u^h\|_{L^2} \|\nabla^4 \tau\|_{L^2}$$

$$\leq \frac{\mu}{6} \|\nabla^4 \tau\|_{L^2}^2 + C \|\nabla^3 u^h\|_{L^2}$$

$$\leq \frac{\mu}{6} \|\nabla^4 \tau\|_{L^2}^2 + C(1 + t)^{-\frac{25}{2}}.$$  

Substituting the above four estimates into (3.19), we get

$$\frac{d}{dt} \|\nabla^3 \tau\|_{L^2}^2 + \mu \|\nabla^4 \tau\|_{L^2}^2 + \beta \|\nabla^3 \tau\|_{L^2}^2 \lesssim (1 + t)^{-\frac{3\gamma}{2}}.$$  \hfill (3.20)
Using Gronwall’s inequality, we get, from (3.20), that

\[ \| \nabla^3 \tau(t) \|_{L^2}^2 \lesssim e^{-\beta t} \| \nabla^3 \tau_0 \|_{L^2}^2 + \int_0^t e^{-\beta(t-s)} (1 + s)^{-\frac{11}{2}} ds \lesssim (1 + t)^{-\frac{11}{2}}. \]

The proof is complete. \(\square\)

**Remark 3.3.** The key observation in the proof of Lemma 3.2 is that the high-frequency part \((u^h, \tau^h)\) decays faster and the low-frequency part \((u^l, \tau^l)\) enjoys better regularity.

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