The Laplace transform of the lognormal distribution

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Abstract

We study the analytical properties of the Laplace transform of the lognormal distribution. Two integral expressions for the analytic continuation of the Laplace transform of the lognormal distribution are provided, one of which takes the form of a Mellin-Barnes integral. As a corollary, we obtain an integral expression for the characteristic function; we show that the integral expression derived by Leipnik in [11] is incorrect. We present two approximations for the Laplace transform of the lognormal distribution, both valid in $\mathbb{C} \setminus (-\infty, 0]$. In the last section, we discuss how one may use our results to compute the density of a sum of independent lognormal random variables.

Keywords: lognormal distribution, Laplace transform, characteristic function, analytic continuation, Mellin transform, series approximation

2010 Mathematics Subject Classification: Primary 60E10, Secondary 62E17.

1 Introduction

A positive random variable $X$ is said to have a lognormal distribution with parameters $\mu \in \mathbb{R}$ and $\sigma > 0$, written $X \sim LN(\mu, \sigma^2)$, if it has probability density function given by

$$f(x; \mu, \sigma) = \frac{1}{\sqrt{2\pi} \sigma x} \exp \left[ -\frac{(\ln x - \mu)^2}{2\sigma^2} \right], \quad x > 0.$$  

The lognormal distribution has a wide range of applications in the natural sciences and fields like finance, actuarial science, economics and engineering. Integral transforms, such as the Laplace and Fourier transforms, of the lognormal distribution have received considerable attention in the literature for several decades. The Laplace transform of $X$, henceforth denoted by $\phi$, is defined by

$$\phi(z; \mu, \sigma) := \mathbb{E} [e^{-zX}] = \int_0^\infty e^{-zx} f(x; \mu, \sigma) dx, \quad \text{Re}(z) \geq 0. \quad (1)$$

The characteristic function of $X$, henceforth denoted by $\varphi$, is the restriction of $\phi$ to the imaginary axis:

$$\varphi(t; \mu, \sigma) := \mathbb{E} [e^{itX}] = \phi(-it; \mu, \sigma), \quad t \in \mathbb{R}. \quad (2)$$

Since these integral transforms have no known closed form, there has been substantial effort to put forth viable approximation methods (see [1] for a thorough overview and numerical comparison of several

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methods). Some authors, such as Barouch and Kaufman [3], Barakat [2], Holgate [9], and Leipnik [11], have proposed series representations for (2). Others, including Gubner [8] and Tellambura and Senaratne [12], have proposed numerical integration methods for computing (1). Gubner’s numerical integration procedure reduces oscillations of the integrand by deforming the contour of integration. Tellambura and Senaratne improved upon Gubner’s method by deriving the steepest-descent contour and by providing two, related, closed-form contours.

More recently, Asmussen et al. [1] used a modified version of Laplace’s method to derive an asymptotically equivalent, closed-form approximation for (1). Moreover, Asmussen et al. [1] constructed a Monte Carlo estimator and, based on this framework, Laub et al. [10] generalized the approach to approximate the Laplace transform of a finite sum of dependent lognormals.

There are several disadvantages with existing methods in the literature. Examples include:

• The majority of methods are only valid, at most, for arguments in the right half plane, \( \{ z \in \mathbb{C} : \text{Re}(z) \geq 0 \} \). As a result, one must exclude some efficient paths of integration when performing an inversion of the Laplace transform.

• It appears that there are no convergent series representations in the literature that are valid on the entire domain of analyticity. Since \( \phi \) is not analytic at the origin, the taylor series representation centered at any point will have finite radius of convergence. For example, the formal Taylor series of \( \phi \), centered at the origin, is given by

\[
\sum_{n=0}^{\infty} \frac{(-z)^n}{n!} e^{\mu n + \frac{\sigma^2}{2} n^2}.
\]  
(3)

It is easy to see that the series (3) diverges for all \( z \neq 0 \).

• In 1991, Leipnik [11] presented the following expression for the characteristic function: Let \( X \sim \text{LN}(0, \sigma^2) \), then, for \( t > 0 \) and \( 0 < k < 1 \), the characteristic function is given by

\[
\varphi(t; 0, \sigma) = \frac{1}{2\pi} \int_{-\infty}^{k+i\infty} \sin(\pi s) \Gamma(s) e^{-(\ln t + i\pi^2 s + \frac{\sigma^2}{2})s} ds.
\]  
(4)

It has been reported that the right-hand side of (4), and the subsequent series for \( \varphi \) derived in [11], are unreliable in numerical computations (see [6], and [1]). We claim that the result is incorrect. To see that (4) is incorrect, observe that the integrand is entire and that one may take \( k \in \mathbb{R} \). After shifting the contour of integration to the left of the origin (taking \( k < 0 \)), it is easy to see that the expression in (4) is \( O(t^{\left|k\right|}) \), as \( t \to 0 \). Hence, the expression converges to 0 as \( t \to 0 \), violating the fact that the characteristic function must converge to 1 as \( t \to 0 \).

Leipnik obtains (4) by first deriving a functional differential equation, and then solving it using a method due to de Bruijn. In this method, the differential equation is transformed into a forward difference-differential equation and an ansatz solution is posed. It appears that Leipnik imposed an inconvenient condition on the ansatz; specifically, in equation (25) of [11], he imposed the condition \( S(z-1) = -S(z) \) when he could have taken \( S(z-1) = S(z) \). As a result, Leipnik searched for an anti-periodic solution for \( S \) and ultimately obtained \( S(z) = \sin \pi z \) rather than \( S(z) = 1 \).

In this paper, we explore the analytic continuation of the Laplace transform of the lognormal distribution and present new, efficient, series approximations of \( \phi \) that are valid on \( \mathbb{C} \setminus (-\infty, 0] \). In Sections
2 and 3, we provide two (integral) expressions of the analytic continuation to \( \mathbb{C} \setminus (−∞, 0] \), one of which takes the form of a Mellin-Barnes integral. As a corollary, we obtain an integral expression for the characteristic function \( \phi \) (this expression is stated by Dufresne in [6] without proof). In the third section of this paper, we exploit the Mellin-Barnes integral expression and use knowledge of the gamma function to derive series approximations for \( \phi \) that are valid for arguments in \( \mathbb{C} \setminus (−∞, 0] \).

The first approximation we present in Section 4 is a convergent series for which the error term is uniformly bounded on \( \mathbb{C} \setminus (−∞, 0] \) by a constant that can be made arbitrarily small (by choice of a parameter). Furthermore, the approximation is asymptotic to \( \phi \) as the magnitude of the argument decreases to zero. The second approximation we present is a sum which improves as the parameter \( \sigma \rightarrow \infty \). The terms of the series/sum are composed of expressions involving error functions and/or Hermite polynomials. The approximations are used to compute \( \phi \) for several real arguments and the results compared to the values obtained by way of numerical integration.

In the last section, we discuss how one may use the analytic continuation of \( \phi \) to compute the density of a sum of independent lognormals via Laplace inversion. By deforming the contour of the Bromwich integral to a Hankel contour, we obtain a real integral with an integrand which decays exponentially. The result is an integral which is easily evaluated numerically.

## 2 The analytic continuation of the Laplace transform of the lognormal distribution

The integral definition of the function \( \phi \), given by (1), is finite when Re \( (z) \geq 0 \) and it is well known that it is analytic in the right half plane \( \mathbb{C}^+ = \{ z \in \mathbb{C} : \text{Re} (z) > 0 \} \). It will be convenient for us to express this function as

\[
\phi(z; \mu, \sigma) = C(\mu, \sigma) \int_0^\infty \frac{1}{x} \exp \left[ -zx - \frac{1}{2\sigma^2} (\ln x)^2 + \frac{\mu}{\sigma^2} \ln x \right] dx,
\]

where \( C(\mu, \sigma) := (2\pi\sigma^2)^{-1/2} \exp (-\mu^2/2\sigma^2) \). Noting that the integral in (5) is finite for all \( \mu \in \mathbb{C} \), we define

\[
\Phi(z, w; \sigma) := C(w, \sigma) \int_0^\infty \frac{1}{x} \exp \left[ -zx - \frac{1}{2\sigma^2} (\ln x)^2 + \frac{w}{\sigma^2} \ln x \right] dx, \quad (z, w) \in \overline{\mathbb{C}^+} \times \mathbb{C}.
\]

Since \( \Phi(z, w; \sigma) = \phi(z; \mu, \sigma) \), the function \( \Phi \) is an extension of \( \phi \). Here \( \overline{\mathbb{C}^+} \) denotes the closure of \( \mathbb{C}^+ \), and throughout this paper we take the logarithm to be complex with the principal branch. The main result of this section is given in the following theorem. It provides us with an expression for \( \phi(z; \mu, \sigma) \) which is analytic on \( \mathbb{C} \setminus (−∞, 0] \).

**Theorem 1.** Let \( \sigma > 0 \) and let \( \Phi \) be defined by (6). Then the Laplace transform of \( X \sim LN(\mu, \sigma^2) \) is analytically continued to \( \mathbb{C} \setminus (−∞, 0] \) by the equation

\[
\phi(z; \mu, \sigma) = \Phi(1, \mu + \ln z; \sigma).
\]

**Proof.** Fix \( \sigma > 0 \) and let \( F(z, w, x) = x^{-1} \exp \left[ -zx - (\ln x)^2/2\sigma^2 + w \ln x/\sigma^2 \right] \) so that

\[
\Phi(z, w; \sigma) = C(w, \sigma) \int_0^\infty F(z, w, x) dx.
\]
The function $C(\cdot, \sigma)$ is entire, and, for each $z \in \mathbb{C}^+$, $F(z, \cdot, \cdot)$ is continuous on $\mathbb{C} \times (0, \infty)$, and, for each pair $(z, x) \in \mathbb{C}^+ \times (0, \infty)$, $F(z, \cdot, x)$ is entire. Thus, for each $n \in \mathbb{N}$, and for each $z \in \mathbb{C}^+$, the function $\Phi_n(z, \cdot ; \sigma)$ defined by

$$\Phi_n(z, w; \sigma) := C(w, \sigma) \int_{\frac{1}{n}} F(z, w, x) dx, \quad w \in \mathbb{C}$$

is entire. Since $\Phi_n(z, \cdot ; \sigma) \to \Phi(z, \cdot ; \sigma)$ uniformly on compact subsets of $\mathbb{C}$, the function $\Phi(z, \cdot ; \sigma)$ is entire. To prove the theorem, we make the formal substitution $ct = x$ in (6) which yields

$$\Phi(z, w; \sigma) = C(w, \sigma) \exp \left( -\frac{1}{2\sigma^2} (\ln c)^2 + \frac{w}{\sigma} \ln c \right) \int_0^\infty \frac{1}{t} \exp \left[ -zt - \frac{1}{2\sigma^2} (\ln t)^2 + \frac{(w - \ln c)}{\sigma^2} \ln t \right] dt$$

where (8) holds provided $cz \in \mathbb{C}^+$. Setting $c = 1/z$ we obtain

$$\Phi(z, w; \sigma) = \Phi(1, w + \ln z; \sigma).$$

Therefore, since $\Phi(z, \cdot ; \sigma)$ is an entire function for each $z \in \mathbb{C}^+$, setting $w = \mu$ yields an analytic continuation for the Laplace transform of $X$ defined by (1).

3 The analytic continuation of $\phi$ as a Mellin-Barnes integral

In this section we derive an alternate expression for $\phi$, the Laplace transform of $X \sim \text{LN}(\mu, \sigma^2)$, in the form of a Mellin-Barnes integral. As a consequence, we obtain the corresponding expression for the characteristic function of the lognormal random variable $X$. As far as we are aware, there is no other explicit proof of the result in the literature. For convenience, we will often write $f(x)$ and $\phi(z)$ instead of $f(x; \mu, \sigma)$ and $\phi(z; \mu, \sigma)$ with the understanding that $\mu$ and $\sigma$ are the parameters of $X$.

**Theorem 2.** Let $X \sim \text{LN}(\mu, \sigma^2)$ and $k > 0$. Then the Laplace transform of $X$ has integral expression

$$\phi(z) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s)e^{-\mu+ln z(s) + \frac{z^2}{2\sigma^2}} ds, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

**Proof.** The Mellin transform of $\phi$, denoted by $M[\phi; \cdot ]$, is defined by

$$M[\phi; s] = \int_0^\infty z^{s-1} \phi(z) dz, \quad s = k + it.$$ 

Using the definition of the Laplace transform, Fubini’s theorem, and the fact that

$$\int_0^\infty z^{s-1} e^{-z x} dz = x^{-s} \Gamma(s), \quad \text{Re}(s) > 0,$$
we have
\[ M[\phi; s] = \int_0^\infty z^{s-1} \left( \int_0^\infty e^{-zx} f(x) \, dx \right) \, dz \]
\[ = \int_0^\infty f(x) \left( \int_0^\infty z^{s-1} e^{-zx} \, dz \right) \, dx \]
\[ = \Gamma(s) \int_0^\infty x^{-s} f(x) \, dx \]
\[ = \Gamma(s)e^{-\mu s + \frac{\sigma^2}{2} s^2}, \quad \text{Re}(s) > 0. \]

This also shows that \( z^{k-1} \phi(z) \in L^1(0, \infty) \) for \( k > 0 \). Furthermore, \( \phi \) is continuous on \((0, \infty)\) and so, by Mellin’s inversion formula ([14] Pg.46, Theorem 28),
\[ \phi(z) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} z^{-s} M[\phi; s] \, ds = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s)e^{-(\mu + \ln z + s \alpha(t))} e^{\frac{\sigma^2}{2} s^2} \, ds, \quad z \in (0, \infty). \]

We can extend this function to take arguments in \( \mathbb{C} \setminus (-\infty, 0] \), and, in fact, it is analytic on this set. Therefore, our new expression for \( \phi \) must agree with the analytic continuation given in Section 2 by the uniqueness of analytic continuation.

Since \( \varphi(t) = \phi(-it) \), we have the following corollary to Theorem 2

**Corollary 1.** Let \( X \sim \text{LN}(\mu, \sigma^2) \) and \( k > 0 \). Then the characteristic function of \( X \) has integral expression
\[ \varphi(t) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s)e^{-(\mu + \ln|t| - s\alpha(t)) + \frac{\sigma^2}{2} s^2} \, ds, \quad t \in \mathbb{R} \setminus \{0\}. \] (11)

4 Series approximations and numerical computation of \( \phi \)

In the first subsection we present series approximations which may be used to compute \( \phi \) on \( \mathbb{C} \setminus (-\infty, 0] \). In the second subsection, we present numerical results using the series approximations and compare the error using numerical integration as a benchmark.

4.1 Series approximations

The following theorem introduces a convergent series that approximates \( \phi \) with an error that can be made arbitrarily small. Note that the approximation bears some resemblance to the results of Barouch and Kaufman [3] who investigated series approximations of the characteristic function.

**Theorem 3.** Let \( X \sim \text{LN}(\mu, \sigma^2) \) and \( \alpha \geq 1 \). Then the Laplace transform of \( X \) has expression
\[ \phi(z) = \sum_{n=0}^\infty \frac{(-z)^n}{n!} e^{\mu n + \frac{\sigma^2}{2} n^2} \cdot \frac{1}{2} \erfc \left( \frac{\mu + \ln(z/\alpha) + \sigma^2 n}{\sqrt{2\sigma}} \right) + O(e^{-\alpha}), \quad z \in \mathbb{C} \setminus (-\infty, 0]. \] (12)

Furthermore,
\[ \phi(z) \sim \sum_{n=0}^\infty \frac{(-z)^n}{n!} e^{\mu n + \frac{\sigma^2}{2} n^2} \cdot \frac{1}{2} \erfc \left( \frac{\mu + \ln(z/\alpha) + \sigma^2 n}{\sqrt{2\sigma}} \right), \quad \text{as } z \to 0. \] (13)

The function \( \erfc \) is the complimentary error function.
Proof. Let \( \alpha \geq 1 \). For \( \Re(s) > 0 \), we may write \( \Gamma(s) = \gamma(s, \alpha) + \Gamma(s, \alpha) \) where \( \gamma(\cdot, \alpha) \) and \( \Gamma(\cdot, \alpha) \) are the upper and lower incomplete gamma functions defined by

\[
\gamma(s, \alpha) := \int_0^\infty t^{s-1}e^{-t}dt, \quad \text{and} \quad \Gamma(s, \alpha) := \int_\alpha^\infty t^{s-1}e^{-t}dt.
\]

Substituting this sum into (10), and replacing \( \gamma(\cdot, \alpha) \) with the power series expansion

\[
\gamma(s, \alpha) = \sum_{n=0}^\infty \frac{(-1)^n}{n!} \alpha^{s+n},
\]

we obtain

\[
\phi(z) = \sum_{n=0}^\infty \frac{(-\alpha)^n}{n!} \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{1}{(s+n)} e^{[\ln \alpha - (\mu + \ln z)]s + \frac{\sigma^2}{2} s^2} ds + \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s, \alpha)e^{-(\mu + \ln z)s + \frac{\sigma^2}{2} s^2} ds,
\]

where \( k > 0 \) (when necessary). The interchange of summation and integration in the first term is justified by Fubini’s theorem and the fact that the integral can be bounded by a Gaussian integral, independent of \( n \). To complete the proof, we need to:

i) compute \( \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{1}{(s+n)} e^{[\ln \alpha - (\mu + \ln z)]s + \frac{\sigma^2}{2} s^2} ds, \quad n \in \mathbb{N} \cup \{0\} \), and

ii) bound \( \left| \frac{1}{2\pi} \int_{k-i\infty}^{k+i\infty} \Gamma(s, \alpha)e^{-(\mu + \ln z)s + \frac{\sigma^2}{2} s^2} ds \right| \).

We compute the integral in i) using differentiation with respect to a parameter. Letting

\[
F_n(w) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{1}{(s+n)} e^{w(s+n) + \frac{\sigma^2}{2} s^2} ds,
\]

we have

\[
F_n'(w) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} e^{w(s+n) + \frac{\sigma^2}{2} s^2} ds = \frac{1}{\sqrt{2\pi \sigma} e^{-w^2/2\sigma^2}},
\]

and so, for \( w \in \mathbb{R} \),

\[
F_n(w) = F_n(-\infty) + \int_{-\infty}^w F_n'(y)dy = 0 + \frac{e^{\frac{\sigma^2}{2} n^2}}{\sqrt{2\pi \sigma}} \int_{-\infty}^w e^{-\frac{(y-\frac{\sigma^2}{2} n^2)}{2\sigma^2}} dy = \frac{e^{\frac{\sigma^2}{2} n^2}}{2} \text{erfc} \left( \frac{-w + \frac{\sigma^2}{2} n}{\sqrt{2\sigma}} \right).
\]

It can be shown that the interchange of integration and differentiation is justified, and \( F_n(-\infty) = 0 \) using the dominated convergence theorem. Since the complementary error function is entire, the result extends to arguments in \( \mathbb{C} \) by analytic continuation. Therefore

\[
\frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{1}{(s+n)} e^{[\ln \alpha - (\mu + \ln z)]s + \frac{\sigma^2}{2} s^2} ds = e^{-[\ln \alpha - (\mu + \ln z)]n} F_n \left( \ln \alpha - (\mu + \ln z) \right)
\]

\[
= \alpha^{-n} z^n e^{\mu n + \frac{\sigma^2}{2} n^2} \cdot \frac{1}{2} \text{erfc} \left( \frac{\mu + \ln \frac{z}{\alpha} + \frac{\sigma^2 n}{\sqrt{2\sigma}}}{\sqrt{2\sigma}} \right)
\]

for \( \Re(s) > 0 \).
To bound the integral in ii), observe that the integrand is entire and we may choose any \( k \in \mathbb{R} \) for the vertical contour. Choosing \( k \leq 1 \), we have \(|\Gamma(s, \alpha)| \leq e^{-\alpha} \) for \( s = k + it \), \( t \in \mathbb{R} \), and so

\[
\left| \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \Gamma(s, \alpha) e^{-(\mu + \ln z)s + \frac{2^2}{2} s^2} ds \right| \leq \frac{e^{-\alpha}}{2\pi} \int_{-\infty}^{\infty} e^{-(\mu + \ln |z|)k + \text{Arg}(z)t + \frac{2^2}{2} (k^2 - t^2)} dt
\]

\[
\leq \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{2^2}{2} k^2 + \frac{2^2}{2} k^2 - \alpha - k(\mu + \ln |z|)}.
\]

We set \( k = 0 \) to obtain the error term in (12) and we choose \( k \) to be negative to show (13). \( \square \)

The following theorem presents an approximation of \( \phi \) that improves as the parameter \( \sigma \) increases.

**Theorem 4.** Let \( X \sim \text{LN}(\mu, \sigma^2) \), and \( M, N \in \mathbb{N} \). Then the Laplace transform of \( X \) has expression

\[
\phi(z) = \sum_{n=0}^{N} \frac{(-z)^n}{n!} e^{\mu n + \frac{2}{2} \sigma^2 n^2} \cdot \frac{1}{2} \text{erfc} \left( \frac{\mu + \ln z + \sigma^2 n}{\sqrt{2}\sigma} \right) + \sum_{m=0}^{M} \frac{(-1)^m a_m}{\sqrt{2\pi}\sigma^{m+1}} e^{-(\mu + \ln z)^2 / 2\sigma^2} H_m \left( -\frac{\mu + \ln z}{\sigma} \right) + O(\sigma^{-M-2}). \tag{14}
\]

The function \( \text{erfc} \) is the complimentary error function, \( H_m \) is the \( m \)th (probabilist’s) Hermite polynomial defined by

\[
H_m(x) := (-1)^m e^{x^2} \frac{d^m}{dx^m} e^{-x^2},
\]

and the coefficients \( a_m \) are defined by

\[
a_m = \frac{\Gamma(m+1)}{(m+1)!} + (-1)^{m+1} \frac{1}{m+1} \sum_{j=1}^{N} \frac{(-1)^j}{j!} \frac{1}{j^{m+1}}.
\]

**Proof.** Let \( N \in \mathbb{N} \). Recall that the function \( \Gamma \) has a simple pole at \( s = -n \), \( n = 0, 1, 2, \ldots \), with residue \( \text{Res}(\Gamma, -n) = (-1)^n / n! \). We may remove the first \( N+1 \) poles of \( \Gamma \) by writing

\[
\Gamma(s) = \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{1}{(s + n)}
\]

to obtain a function, denoted \( \gamma_N \), that is holomorphic on \( \{ s \in \mathbb{C} : |s| < N+1 \} \). Thus, we write

\[
\Gamma(s) = \gamma_N(s) + \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{1}{(s + n)},
\]

where, for \( |s| < N+1 \), we may write

\[
\gamma_N(s) = \sum_{m=0}^{\infty} a_m s^m,
\]
with \( a_m = \gamma_N^{(m)}(0)/m! \). Note that as \( \sigma \to \infty \), the mass of the integrand in the integral (10) is increasingly supported on the set \( \{ k + it : t \in (-N - 1, N + 1) \} \). Thus, we choose \( M \in \mathbb{N} \) and write

\[
\phi(z) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \left( \gamma_N(s) + \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{1}{(s+n)} \right) e^{-\mu \ln z + \frac{z^2}{n^2}} ds
\]

\[
= \sum_{n=0}^{N} \frac{(-1)^n}{n!} \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{1}{(s+n)} e^{-\mu \ln z + \frac{z^2}{n^2}} ds
\]

\[
+ \sum_{m=0}^{M} a_m \cdot \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} s^m e^{-\mu \ln z + \frac{z^2}{n^2}} ds + R_M(z), \quad (15)
\]

where,

\[ R_M(z) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} \left( \gamma_N(s) - \sum_{m=0}^{M} a_m s^m \right) e^{-\mu \ln z + \frac{z^2}{n^2}} ds. \]

To complete the proof we need to:

i) compute \( \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} \frac{1}{(s+n)} e^{-\mu \ln z + \frac{z^2}{n^2}} ds \), \( n \in \{0, 1, \ldots, N\} \),

ii) compute \( a_m, \ m \in \{0, 1, \ldots, M\} \),

iii) compute \( \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} s^m e^{-\mu \ln z + \frac{z^2}{n^2}} ds \), \( m \in \{0, 1, \ldots, M\} \), and

iv) bound \( |R_M(z)| \)

The integral in i) was computed in the proof of Theorem 3. To determine ii), we need to compute \( \gamma_N^{(m)}(0) \). Note that, for \( |s| < 1 \), we may write

\[ \Gamma(s) - \frac{1}{s} = \sum_{j=0}^{\infty} b_j s^j, \]

where \( b_j = \Gamma^{(j+1)}(1)/(j+1)! \). So, for \( |s| < 1 \), we may write

\[ \gamma_N(s) = \sum_{j=0}^{\infty} b_j s^j - \sum_{n=1}^{N} \frac{(-1)^n}{n!} \frac{1}{(s+n)}. \]

Differentiating, we have

\[ \gamma_N^{(m)}(s) = \sum_{j=m}^{\infty} (j)_m b_j s^{j-m} + (-1)^{m+1} m! \sum_{n=1}^{N} \frac{(-1)^n}{n!} \frac{1}{(s+n)^{m+1}} \]

and

\[ \gamma_N^{(m)}(0) = m! b_m + (-1)^{m+1} m! \sum_{n=1}^{N} \frac{(-1)^n}{n!} \frac{1}{n^{m+1}}. \]

Therefore,

\[ a_m = \frac{\gamma_N^{(m)}(0)}{m!} = b_m + (-1)^{m+1} \sum_{n=1}^{N} \frac{(-1)^n}{n!} \frac{1}{n^{m+1}}. \]
To compute iii), we let
\[ G(w) = \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} e^{ws + \frac{x^2}{2}} ds = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{x^2}{2\sigma^2}}. \]
Then
\[ G^{(m)}(w) = \frac{d^m}{dw^m} \left( \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{x^2}{2\sigma^2}} \right) = \frac{1}{\sqrt{2\pi \sigma}} \cdot \left( \frac{-1}{\sigma} \right)^m e^{-\frac{x^2}{2\sigma^2}} H_m \left( \frac{w}{\sigma} \right), \]
and therefore,
\[ \frac{1}{2\pi i} \int_{k-i\infty}^{k+i\infty} s^m e^{-(\mu + \ln z)s + \frac{x^2}{2}} ds = G^{(m)}(-(\mu + \ln z)) = \frac{(-1)^m}{\sqrt{2\pi \sigma^{m+1}}} e^{-\frac{-(\mu + \ln z)^2}{2\sigma^2}} H_m \left( \frac{-(\mu + \ln z)}{\sigma} \right), \]
where, again, it can be shown that the interchange of integration and differentiation is justified. Finally, to show iv), we note that
\[ |\gamma_N(s) - \sum_{m=0}^M a_m s^m| \leq C |s|^M, \quad s \in i\mathbb{R}, \]
for some \( C > 0 \). To see this, observe that, for \( |s| < N \), we have
\[ |\gamma_N(s) - \sum_{m=0}^M a_m s^m| = \left| \sum_{m=M+1}^\infty a_m s^m \right| \leq \sum_{m=M+1}^\infty |a_m| |s|^m \leq C_1 |s|^M. \]
We also have
\[ |\gamma_N(s)| = \left| \frac{1}{s} - \sum_{n=1}^N \frac{(-1)^n}{n!} \frac{1}{s+n} \right| \leq \left| \frac{1}{s} \right| + \sum_{n=1}^N \frac{|(-1)^n|}{n!} \frac{1}{|s+n|} \leq C_2, \quad s \in i\mathbb{R}, \]
so that, for \( |s| \geq N \), we have
\[ \left| \gamma_N(s) - \sum_{m=0}^M a_m s^m \right| \leq |\gamma_N(s)| + \sum_{m=0}^M |a_m| |s|^m \leq C_2 + C_3 |s|^M \leq C_4 |s|^M. \]
Thus,
\[ |R_M(z)| \leq \frac{C}{2\pi} \int_{-\infty}^{\infty} |t|^M e^{\text{Arg}(z) t - \frac{x^2}{2}} dt = \frac{C}{2\pi} \int_{-\infty}^{\infty} \left| \frac{x}{\sigma} \right|^M e^{\text{Arg}(z) \frac{x}{\sigma} - \frac{x^2}{2\sigma^2}} \frac{1}{\sigma} dx \leq C' e^{-M-2}. \]

\[ 4.2 \] Numerical examples

We can compute \( \Phi(z), z \in \mathbb{C} \setminus (-\infty, 0] \), via numerical integration using either of the relations (7) or (10) given by Theorems 1 or 2, respectively. If we choose to use the former, then we need to compute \( \Phi(1, \mu + \ln z; \sigma) \) as defined by (6). For simplicity, we will discuss the computation of \( \Phi(1, a + ib; \sigma) \), for \( a, b \in \mathbb{R} \).

Making the substitution \( x \to e^x \) we have
\[ \Phi(1, a + ib; \sigma) = C(a + ib, \sigma) \int_{-\infty}^{\infty} \exp \left[ -e^x - \frac{x^2}{2\sigma^2} + \frac{(a + ib)x}{\sigma^2} \right] dx. \]
We can write this integral in the form
\[ \int_{-\infty}^{\infty} g(x)e^{itx}dx, \]  
(16)
where
\[ g(x) = \exp \left[-e^x - \frac{x^2}{2\sigma^2} + \frac{ax}{\sigma^2}\right], \text{ and } t = \frac{b}{\sigma^2}. \]

The integral (16) can be computed numerically using Filon’s quadrature method [7]. First, we determine an interval, \([x_0, x_{2N}]\), which supports most of the integrand’s mass and create a mesh consisting of \(2N+1\) points, \(x_j, \ j = 0, \ldots, 2N\). The integral is then written as a sum of \(N\) integrals over \([x_{2j}, x_{2j+2}]\), \(j = 0, \ldots, N-1:\)
\[ \int_{-\infty}^{\infty} g(x)e^{itx}dx \approx \int_{x_0}^{x_{2N}} g(x)e^{itx}dx = \sum_{j=0}^{N-1} \int_{x_{2j}}^{x_{2j+2}} g(x)e^{itx}dx. \]

On each subinterval \([x_{2j}, x_{2j+2}]\), we approximate \(g(x)\) with a second order Lagrange interpolating polynomial using the data points \((x_{2j}, g_{2j}), (x_{2j+1}, g_{2j+1})\), and \((x_{2j+2}, g_{2j+2})\), where \(g_j := g(x_j)\). Thus, with \(g(x) \approx c_0^{(j)} + c_1^{(j)} x + c_2^{(j)} x^2\) on \([x_{2j}, x_{2j+2}]\), we have
\[ \int_{-\infty}^{\infty} g(x)e^{itx}dx \approx \sum_{j=0}^{N-1} \int_{x_{2j}}^{x_{2j+2}} \left(c_0^{(j)} + c_1^{(j)} x + c_2^{(j)} x^2\right)e^{itx}dx. \]  
(17)

The integrals on the right hand side of (17) can be computed explicitly. With an appropriate interval of integration, \([x_0, x_{2N}]\), and \(N\) sufficiently large, an accurate approximation of the integral (16) is obtained.

Theorem 3 was used to numerically compute \(\phi(z)\) for several real values of \(z\); Table 1 shows the results corresponding to \(\sigma = 0.0625, 0.25, 0.75,\) and 1 with \(\mu = 0\). In each case, the expression in (12) was truncated to 41 terms and evaluated using \(\alpha = 10\). Table 2 displays the absolute difference (labeled AD) between \(\phi(z)\) computed using (12) and the value of \(\phi(z)\) computed by way of numerical integration.

Theorem 4 was used to numerically compute \(\phi(z)\) for several real values of \(z\); Table 3 shows the results corresponding to \(\sigma = 1, 1.5, 2,\) and 2.5 with \(\mu = 0\). In each case, (14) was used with \(N = 5, \) and \(M = 10\). Table 4 displays the absolute difference (labeled AD) between \(\phi(z)\) computed using (14) and the value of \(\phi(z)\) computed by way of numerical integration.

5 The density of a sum of independent lognormal random variables

We have introduced two integral expressions which analytically continue the Laplace transform of \(X \sim \text{LN}(\mu, \sigma^2)\) to \(\mathbb{C} \setminus \{(-\infty, 0]\). We have also provided series approximations which may be used for numerical computations. In the last section of this paper, we consider an application which utilizes the analytic continuation of the Laplace transform of \(X\).

In this section we discuss a method to numerically compute the density of a sum of independent lognormal random variables. In this procedure, we obtain the density function by inverting the Laplace
Table 1: The function $\phi$ computed using (12), truncated to 41 terms, with $\alpha = 10$.

| $\sigma$ = 0.0625 | $\sigma$ = 0.25 | $\sigma$ = 0.75 | $\sigma$ = 1 |
|-------------------|----------------|----------------|-------------|
| $z$               | $\phi(z)$      | $\phi(z)$      | $\phi(z)$   |
| 0.5               | 0.60624        | 0.60196        | 0.57541     | 0.56171     |
| 1                 | 0.36788        | 0.36804        | 0.37469     | 0.38176     |
| 1.5               | 0.22346        | 0.22825        | 0.26086     | 0.2807      |
| 2                 | 0.13586        | 0.14342        | 0.18984     | 0.21631     |
| 3                 | 0.050369       | 0.058656       | 0.10995     | 0.14025     |
| 5                 | 0.0070017      | 0.011065       | 0.045898    | 0.072028    |
| 10                | 3.9289e-05     | 0.00028124     | 0.0096044   | 0.022991    |

Table 2: Absolute difference between $\phi(z)$ computed using (12) and $\phi(z)$ computed using numerical integration.

| $\sigma$ = 0.0625 | $\sigma$ = 0.25 | $\sigma$ = 0.75 | $\sigma$ = 1 |
|-------------------|----------------|----------------|-------------|
| $z$               | AD             | AD             | AD          |
| 0.5               | 6.572520e-14   | 6.661338e-14   | 5.155796e-10| 1.478849e-08|
| 1                 | 1.110223e-16   | 4.013456e-14   | 1.456569e-08| 9.738506e-08|
| 1.5               | 4.440892e-16   | 2.525757e-14   | 6.936278e-08| 2.349823e-07|
| 2                 | 2.775558e-17   | 1.468270e-14   | 1.760354e-07| 3.975210e-07|
| 3                 | 1.942890e-16   | 2.212219e-11   | 5.105122e-07| 7.251790e-07|
| 5                 | 1.756408e-15   | 6.980607e-08   | 1.292328e-06| 1.225385e-06|
| 10                | 1.450124e-05   | 6.055084e-06   | 2.185399e-06| 1.648996e-06|
Table 3: The function $\phi$ computed using (14) with $N = 5$, and $M = 10$.

| $z$ | $\sigma = 1$ | $\sigma = 1.5$ | $\sigma = 2$ | $\sigma = 2.5$ |
|-----|--------------|----------------|--------------|----------------|
| 0.5 | 0.56169      | 0.54186        | 0.53012      | 0.523          |
| 1   | 0.38175      | 0.39772        | 0.41216      | 0.42396        |
| 1.5 | 0.28073      | 0.31674        | 0.34538      | 0.36751        |
| 2   | 0.21634      | 0.26336        | 0.30039      | 0.32893        |
| 3   | 0.14024      | 0.19613        | 0.24163      | 0.27744        |
| 5   | 0.072008     | 0.12725        | 0.17708      | 0.21855        |
| 10  | 0.023002     | 0.062944       | 0.10844      | 0.15117        |

Table 4: absolute difference between $\phi(z)$ computed using (14) and $\phi(z)$ computed using numerical integration.

| $z$ | $\sigma = 1$ | $\sigma = 1.5$ | $\sigma = 2$ | $\sigma = 2.5$ |
|-----|--------------|----------------|--------------|----------------|
| 0.5 | 3.503349e-05| 6.444704e-07   | 2.668369e-08 | 3.269640e-09   |
| 1   | 1.716174e-05| 3.009667e-08   | 1.325727e-09 | 9.603728e-10   |
| 1.5 | 1.196279e-04| 8.780255e-07   | 2.567335e-08 | 1.195540e-09   |
| 2   | 1.371616e-04| 1.352950e-06   | 4.380613e-08 | 2.832041e-09   |
| 3   | 6.300887e-05| 1.180623e-06   | 5.771800e-08 | 4.875454e-09   |
| 5   | 2.828704e-04| 7.533040e-07   | 4.059893e-08 | 6.225437e-09   |
| 10  | 4.467548e-04| 3.262143e-06   | 4.479331e-08 | 4.615517e-09   |
transform of the sum. Using the analytic continuation of the Laplace transform, we may deform the contour of the Bromwich integral into a Hankel contour and obtain an integral for which the integrand decays exponentially.

**Proposition 1.** Let $X_j \sim \text{LN}(\mu_j, \sigma_j^2)$, $j = 1, \ldots, n$, be independent and $X = \sum_{j=1}^n X_j$. Then $X$ has density

$$f_X(x) = -\frac{1}{\pi} \int_0^\infty \text{Im} \left[ \phi(-t + i \cdot 0) \right] e^{-tx} dt, \quad x > 0.$$  \hspace{1cm} (18)

Here $\phi$ and $f_X$ denote the Laplace transform and probability density function of $X$, respectively, and $\phi(-t + i \cdot 0) = \lim_{\varepsilon \to 0^+} \phi(-t + i \varepsilon)$.

We will use the following lemma in the proof of Proposition 1

**Lemma 1.** Let $\phi_j$ denote the Laplace transform of $X_j \sim \text{LN}(\mu_j, \sigma_j^2)$. For every $k > 0$, $\phi_j(z) = O_k(|z|^{-k})$ as $|z| \to \infty$, $z \in \mathbb{C} \setminus (-\infty, 0]$. Consequently, for every $k > 0$, $\phi(z) = O_k(|z|^{-k})$ as $|z| \to \infty$, $z \in \mathbb{C} \setminus (-\infty, 0]$.

**Proof of Lemma 1.** Let $k > 0$ and $z \in \mathbb{C} \setminus (-\infty, 0]$. By Theorem 2, we have

$$\phi_j(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} z^{-(k+i\tau)} \Gamma(k+i\tau) e^{-\mu_j(k+i\tau)+\frac{s^2}{2}(k+i\tau)^2} dt,$$

and thus,

$$|\phi_j(z)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |z|^{-k} e^{\pi|\tau|} \Gamma(k) e^{-\mu_jk+\frac{s^2}{2}(k^2-\tau^2)} dt = M_{k,j}|z|^{-k}.$$  

To prove the second part of the lemma, let $k > 0$ and take $r = k/n$. Then, by the independence of the $X_j$'s and the first part of the lemma,

$$|\phi(z)| = \prod_{j=1}^n |\phi_j(z)| \leq \prod_{j=1}^n M_{r,j}|z|^{-r} = M_k|z|^{-k}$$

where, $M_k = \prod_{j=1}^n M_{r,j}$. \hfill \Box

**Proof of Proposition 5.** The density function of $X$, obtained by the inverse Laplace transform, is given by the Bromwich integral

$$f_X(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \phi(z) e^{zx} dz,$$  \hspace{1cm} (19)

for any $c > 0$. Using the analytic continuation of $\phi$, the integrand of (19) is analytic on $\mathbb{C} \setminus (-\infty, 0]$ and we can deform the contour to the contour $\Gamma_R = \gamma_1(-) + \gamma_2(-) + H_R + \gamma_2(+) + \gamma_1(+) $, shown in Figure 1, for any $R > 0$. The density function is now given by

$$f_X(x) = \frac{1}{2\pi i} \int_{\Gamma_R} \phi(z) e^{zx} dz.$$  \hspace{1cm} (20)

We will show that the contributions of the contours $\gamma_1(\pm)$ and $\gamma_2(\pm)$ go to zero as $R \to 0$. Similarly, the contributions of $\gamma_1(-)$ and $\gamma_2(-)$ go to zero and as a result $\Gamma_R \to H$, as $R \to \infty$, where $H$ is the Hankel
contour in Figure 2. We first consider the contour $\gamma_1^{(+)}$ parametrized by $z(t) = c + it$, $t \in [R, \infty)$. For every $x > 0$, we have

$$\left| \frac{1}{2\pi i} \int_{\gamma_1^{(+)}} \phi(z) e^{xz} \, dz \right| \leq \frac{1}{2\pi} \int_{R}^{\infty} \left| \phi(c + it) e^{(c+it)x} \right| \, dt$$

$$\leq \frac{1}{2\pi} e^{cx} M_2 \int_{R}^{\infty} |c + it|^{-2} \, dt = \frac{1}{2\pi} e^{cx} M_2 \int_{R}^{\infty} \frac{1}{c^2 + t^2} \, dt \to 0, \text{ as } R \to \infty,$$

where we have used lemma 1 with $k = 2$. Next we consider the contour $-\gamma_2^{(+)}$ parametrized by $z(t) = c + Rei^t$, $t \in [\pi/2, \pi - \theta_R]$, where $\theta_R \to 0$ as $R \to \infty$. Since $|z(t)| = |c + R e^{it}| \geq R - c = \left(1 - \frac{c}{R}\right) R$, there exists $R' > 0$ such that $|z(t)| \geq \frac{1}{2} R$ when $R > R'$. So for every $x > 0$, and $R > R'$, we have

$$\left| \frac{1}{2\pi i} \int_{\gamma_2^{(+)}} \phi(z) e^{xz} \, dz \right| \leq \max_{z \in \gamma_2^{(+)}} \{|\phi(z)| e^{Re(z)x}\} \cdot \frac{\pi}{2} R$$

$$\leq \max_{t \in [\pi/2, \pi - \theta_R]} \left\{M_k |z(t)|^{-k} e^{(c+R \cos t)x}\right\} \cdot \frac{\pi}{2} R$$

$$\leq M_k \left(\frac{1}{2} R\right)^{-k} e^{(c+R \cos (\pi/2))x} \cdot \frac{\pi}{2} R = \pi 2^{k} M_k e^{cx R^{-k+1}} \to 0, \text{ as } R \to \infty,$$

when we use lemma 1 with any $k > 1$. Thus, taking $R \to \infty$, we have

$$f_X(x) = \frac{1}{2\pi i} \int_{\gamma_1^{(-)} + \gamma_2^{(-)} + H_R + \gamma_2^{(+)} + \gamma_1^{(+)}} \phi(z) e^{xz} \, dz. \quad (21)$$
The contour $H$ is defined $\forall \delta > 0$ and $\forall \epsilon \in (0, \delta)$ and it is clear that $\theta_\epsilon \to 0$ as $\epsilon \to 0$. Rewriting (21) as

$$f_X(x) = \frac{1}{2\pi i} \left\{ \left( \int_{h_1} + \int_{h_2} + \int_{h_3} \right) \phi(z) e^{zx}dz \right\},$$

where,

$$\int_{h_1} \phi(z) e^{zx}dz = \int_\infty^{i\delta} \phi(-t - i\epsilon)e^{(-t - i\epsilon)x}dt,$n

$$\int_{h_2} \phi(z) e^{zx}dz = i\delta \int_{-\pi + \theta_\epsilon}^{\pi - \theta_\epsilon} \phi(\delta e^{it}) e^{\delta e^{it}x + i\epsilon t}dt,$n

$$\int_{h_3} \phi(z) e^{zx}dz = -\int_\infty^{i\delta} \varphi(-t + i\epsilon)e^{(t + i\epsilon)x}dt,$n

and using the fact that $\overline{\phi(z)} = \phi(\overline{z})$, we have

$$f_X(x) = \frac{1}{2\pi i} \left\{ i\delta \int_{-\pi + \theta_\epsilon}^{\pi - \theta_\epsilon} \phi(\delta e^{it}) e^{\delta e^{it}x + i\epsilon t}dt - 2i \text{Im} \left[ \int_\infty^{\infty} \phi(-t + i\epsilon)e^{(t + i\epsilon)x}dt \right] \right\} \to -\frac{1}{\pi} \text{Im} \left[ \int_0^{\infty} \phi(-t + i\cdot0)e^{-tx}dt \right], \text{ as } \epsilon, \delta \to 0 \n

$$-\frac{1}{\pi} \int_0^{\infty} \text{Im} \left[ \phi(-t + i\cdot0) \right] e^{-tx}dt.$$n

The interchange of the limit and integration can be justified by dominated convergence.

To utilize Proposition 1, one must first compute $\phi_j(-t + i\cdot0)$, $j = 1, \ldots, n$. This can be performed using the methods from Section 4, or any alternative method (for example, see [5]). Since the random variables, $X_j$, $j = 1, \ldots, n$, are independent we have

$$\phi(-t + i\cdot0) = \prod_{j=1}^{n} \phi_j(-t + i\cdot0).$$n

We can compute the integral in (18) in a similar fashion to the numerical integration method of Section 4.2.

To illustrate the method, we computed the Laplace transform of $X \sim \text{LN}(0, 1)$ using the Theorem 3 and used the inversion formula of Proposition 1 to obtain the density. Figure 3a shows a plot with both the closed form of $f_X$ and our approximation. Figure 3b shows the relative error of the approximation.
6 Conclusion

We have presented two derivations of the analytic continuation of the Laplace transform of the lognormal distribution, which we denote by \( \phi \). Since the Mellin transform of \( \phi \) has closed form, we used the Mellin inversion formula to express \( \phi \) in the form of a Mellin-Barnes integral. As a consequence, we obtained the corresponding expression for the characteristic function of the lognormal distribution. This expression is slightly different from the expression derived by Leipnik in [11]; we claim his expression is incorrect.

Using the Mellin-Barnes expression for \( \phi \), we obtained two approximations which may be used in numerical computations. The error of the first approximation (see Theorem 3) can be made arbitrarily small and the approximation is asymptotic to \( \phi \) as the magnitude of the argument goes to zero. The second approximation (see Theorem 4) improves as the parameter \( \sigma \) goes to infinity. Both approximations were shown to provide accurate results, however, we note that computation can be difficult if too many terms of the series employed.

In the last section, we showed how one may use the analytic continuation of the Laplace transform of a sum of independent lognormals to compute the density, via Laplace inversion. By deforming the vertical contour of the Bromwich integral to a Hankel contour, one may obtain a real integral for which the integrand decays exponentially. The result is an integral that can be computed numerically with ease.

The analytic continuation of the Laplace transform of the lognormal distribution has other applications. In 1977, Olof Thorin showed that the lognormal distribution is a Generalized Gamma Convolution (GGC) (see [13]). A GGC is a probability distribution \( F \) on \([0, \infty)\) with moment-generating function (mgf) of the form

\[
M(s) = \exp \left[ as + \int_0^\infty \ln \left( \frac{t}{t-s} \right) U(dt) \right], \quad s \leq 0 \text{ (or } s \in \mathbb{C} \setminus (0, \infty)),
\]

where \( a \geq 0 \) and \( U(dt) \) is a nonnegative measure, called the Thorin measure, on \((0, \infty)\) satisfying

\[
\int_{(0,1]} |\ln t| U(dt) < \infty, \text{ and } \int_{(1,\infty)} t^{-1} U(dt) < \infty,
\]
As Bondesson discusses in [4] and [5], one may compute the density of the Thorin measure using the analytic continuation of the Laplace transform of the lognormal distribution. The density, denoted here by $U$, can be computed using the formula

$$U(t) = \frac{1}{\pi} \text{Im} \left[ \frac{\phi'(-t + i \cdot 0)}{\phi(-t + i \cdot 0)} \right],$$

where $\phi(-t + i \cdot 0) = \lim_{\varepsilon \to 0^+} \phi(-t + i \varepsilon)$ (equivalently, as in Bondesson’s derivation, one may approach the negative real line from below and multiply the result by -1).

**Acknowledgment**

I would like to express my gratitude to Alexey Kuznetso for his guidance and advice.

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