Abstract

Weighted group algebras have been studied extensively in Abstract Harmonic Analysis where complete characterizations have been found for some important properties of weighted group algebras, namely amenability and Arens regularity. One of the generalizations of weighted group algebras which may be considered is weighted hypergroup algebras. Defining weighted hypergroups, analogous to weighted groups, we study Arens regularity and isomorphism with operator algebras for them. We also examine our results on three classes of discrete weighted hypergroups constructed by conjugacy classes of FC groups, the dual space of compact groups, and hypergroup structure defined by orthogonal polynomials.

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1 Introduction

Roughly speaking, a discrete hypergroup is a set equipped with an extra structure, which leads to the construction of a Banach algebra on the Banach space of all absolutely summable functions on the hypergroup. This binary operation takes the Dirac measures of each two elements of the hypergroup to a finitely-supported probability measure and therefore has a probabilistic taste, since one may roughly express that the outcome of the action of two elements of a hypergroup is chosen ‘randomly’.

Not only were discrete hypergroups defined as a generalization of (discrete) groups, but also one may show that some objects related to locally compact groups may be studied as discrete hypergroups. For instance, if \( G \) is an FC group (i.e. every conjugacy class is finite), then the set of all conjugacy classes of \( G \), denoted by \( \text{Conj}(G) \), forms a commutative discrete hypergroup. Also, for a compact group \( G \), the set of equivalence classes of irreducible unitary representations of \( G \), denoted by \( \widehat{G} \) and called the dual of the group \( G \), is a commutative discrete hypergroup. On one hand, these examples as well as hypergroups defined by orthogonal polynomials, connect the studies done on hypergroups to different topics in abstract harmonic analysis. On the other
hand, the similarities of hypergroups with groups suggest that one may be able to generalize
the studies on groups to hypergroups.

One of the topics related to hypergroups which has been initiated based on a similar study
on groups is “weighted hypergroups” and “weighted hypergroup algebras”. The weighted hy-
pergroup algebra, as a Banach algebra can be subject of study for different properties of Banach
algebras. In this manuscript, we study Arens regularity and isomorphism with operator alge-ras of weighted hypergroup algebras. To recall, the second dual of a Banach algebra can be
equipped by two algebraic actions to form Banach algebras, we call a Banach algebra ‘Arens
regular’ if these two actions coincide. Also a Banach algebra $A$ is called an operator algebra
if there is a Hilbert space $H$ such that $A$ is a closed subalgebra of $\mathcal{B}(H)$. We particularly focus on
and examine various classes of weighted hypergroup algebras with respect to these properties.
One may note that, for the specific weight $\omega \equiv 1$, the weighted case is reduced back to regular
hypergroups and their algebras. The first studies over weighted hypergroup algebras may be
tracked back to [2, 9, 10].

The paper is organized as follows. We start this paper by Section 2 wherein we give the defi-
nition of discrete hypergroups consistently and briefly go through three examples of hypergroup
structures mentioned before. Section 3 is devoted to weights on discrete hypergroups, their cor-
corresponding algebras, and their examples. In Subsection 3.1, we study weighted hypergroup
algebras, $\ell^1(H, \omega)$, for discrete hypergroups $H$ and hypergroups weights $\omega$. Subsequently, we
introduce some weights which are related to the growing rate of finitely generated hypergroups.
We continue this section by studying some examples. First in in Subsection 3.2, we study
weights and their properties on Conj($G$), as a hypergroup, for FC groups $G$. As examples of
these weights, if $(G, \sigma)$ is a weighted discrete FC group for some group weight $\sigma$, then $Z\ell^1(G, \sigma)$,
the center of $\sigma$-weighted group algebra, is isometrically algebraic isomorphic to $\ell^1($Conj($G), \omega_{\sigma})$
for some hypergroup weight $\omega_{\sigma}$ which is generated using $\sigma$. We will introduce and study more
examples of hypergroup weights on Conj($G$) here. In Subsection 3.3, we introduce and study
some hypergroup weights on the dual of compact groups. Finally, we close the section by Sub-
section 3.4, in which we construct some weights on some simple polynomial hypergroups. The
weights defined in this sections form a source of interesting examples with the respect to the
main results of the paper.

Arens regularity of weighted group algebras has been studied by Craw and Young in [6].
They showed that a locally compact group $G$ has a weight $\omega$ such that $L^1(G, \omega)$ is Arens regular
if and only if $G$ is discrete and countable. They also characterized the Arens regularity of
weighted group algebras with respect to one feature of the (group) weight, called 0-clusterness
as described in [7]. In Section 4, the Arens regularity of weighted hypergroup algebras for
discrete hypergroups is studied and it is shown that the strong 0-clusterness of the corresponding
hypergroup weight results in the Arens regularity of the weighted hypergroup algebra (strong
0-clusterness implies 0-clusterness, [7]).

Injectivity and Isomorphism of weighted group algebras to operator algebras has been stud-
ied before, see [20, 25]. In Section 5, studying the hypergroup case, we demonstrate that for
hypergroup weights which are weakly additive and whose inverse is 2-summable over the hyper-
group, the weighted hypergroup algebra is injective and hence an isomorphism to an operator algebra exists. To do so, we apply some results regarding Littlewood multipliers of hypergroups. This machinery lets us to examine classes of hypergroup weights which are not weakly additive as well, namely exponential weights, as it is studied in Subsection 5.1.

2 Discrete hypergroups and examples

2.1 Definition

Let $H$ be a discrete set. Let $\ell^1(H)$ denote the Banach space of all functions $f : H \to \mathbb{C}$ such that

$$\|f\|_1 := \sum_{x \in H} |f(x)| < \infty.$$  \hfill (2.1)

Let $c_c(H)$ and $c_0(H)$ respectively denote the space of all finitely supported and vanishing at infinity elements of $\ell^\infty(H)$. We call $H$ a hypergroup if the following conditions hold.

(H1) There exists an associative binary operation $\ast$ called convolution on $\ell^1(H)$ under which $\ell^1(H)$ is a Banach algebra. Moreover, for every $x, y \in H$, $\delta_x \ast \delta_y$ is a positive measure with a finite support and $\|\delta_x \ast \delta_y\|_{\ell^1(H)} = 1$.

(H2) There exists an element (necessarily unique) $e$ in $H$ such that $\delta_e \ast \delta_x = \delta_x \ast \delta_e = \delta_x$ for all $x$ in $H$.

(H3) There exists a (necessarily unique) bijection $x \to \check{x}$ of $H$ called involution satisfying $(\delta_x \ast \delta_y)(\check{\check{x}}) = \delta_y \ast \delta_x$ for all $x, y \in H$.

(H4) $e$ belongs to $\text{supp}(\delta_x \ast \delta_y)$ if and only if $y = \check{x}$.

We call a hypergroup $H$ commutative if $\ell^1(H)$ forms a commutative algebra. To facilitate the notation, for each pair $x, y \in H$ and $f \in c_c(H)$, the value of the measure of $\delta_x \ast \delta_y$ on $f$ is denoted by $f(\delta_x \ast \delta_y)$. As mentioned in [3, 1.1.2], for each pair $f, g \in \ell^1(H)$,

$$f \ast g(z) = \sum_{x \in H} \sum_{y \in H} \delta_x \ast \delta_y(z) f(x) g(y) \quad (z \in H).$$

We can define a left translation on $\ell^\infty(H)$ by $L_x f : H \to \mathbb{C}$ where $L_x f(y) := f(\delta_x \ast \delta_y)$ for each $f$ in $\ell^\infty(H)$ and $x, y \in H$. Note that $L_x f \in c_c(H)$ for $f \in c_c(H)$. Similar to the group case a non-zero, positive, left invariant linear functional $h$ (possibly unbounded) on $c_c(H)$ is called a Haar measure i.e. $h(L_x f) = h(f)$ for all $f \in c_c(H)$ and $x \in H$. For a discrete hypergroup the existence of a Haar measure is proved and it is unique up to multiplication by a positive constant. Furthermore, for a discrete hypergroup with a Haar measure $h : H \to (0, \infty)$ such that $h(e) = 1$, $h(x) = (\delta_x \ast \delta_x(e))^{-1}$ for all $x \in H$. 

3
We define \( L^1(H, h) \) (sometimes denoted by \( L^1(H) \) if there is no risk of confusion) to be the Banach space of integrable functions on \( H \) with respect to the Haar measure \( h \). Further, for each \( f, g \in c_c(H) \) and \( y \in H \), let us define
\[
f \ast_h g(y) := \sum_{x \in H} f(x)g(\delta_x \ast \delta_y)h(x) \quad \tilde{f}(x) := \overline{f(x)}.
\]

One may extend \( \ast_h \) and \( \sim \) to \( L^1(H, h) \). \( L^1(H, h) \) equipped with the convolution \( \ast_h \) forms a Banach algebra. Also the map \( f \to fh \), \( L^1(H, h) \to \ell^1(H) \) is an isometric algebra isomorphism from the Banach algebra \( L^1(H, h) \) onto the Banach algebra \( \ell^1(H) \), by [17, Theorem 1.8]. Due to this isomorphism, we focus our study on \( \ell^1(H) \) without loss of generality.

2.2 The conjugacy classes of FC groups

Let \( G \) be a (discrete) group with the group algebra \( \ell^1(G) \) and \( \text{Conj}(G) \) is the set of all conjugacy classes of \( G \). We denote the centre of the group algebra by \( Z\ell^1(G) \). The group \( G \) is called an FC or finite conjugacy group if for each \( C \in \text{Conj}(G) \), \( |C| < \infty \). Let \( \Psi \) denotes a linear mapping from \( c_c(G) \cap Z\ell^1(G) \) to \( c_c(\text{Conj}(G)) \) such that for each \( f \in Z\ell^1(G) \cap c_c(G), \Psi(f)(C) = |C|f(C) \) for \( C \in \text{Conj}(G) \) where \( \Psi(f)(C) := f(x) \) for (every) \( x \in C \). Through \( \Psi \) one can equip \( \text{Conj}(G) \) with an convolution \( \ast \) which is
\[
\delta_C \ast \delta_D = \frac{1}{|C||D|} \sum_{E \in C, E \subseteq CD} \alpha_{C \ast D} E \delta_E.
\]

And therefore, \( \text{Conj}(G) \) for an FC group \( G \) forms a hypergroup when \( \tilde{G} := \{x^{-1} : x \in C \} \). Then the mapping \( \Psi \) can be extended to an isometric Banach algebra isomorphism between \( \ell^1(\text{Conj}(G)) \) and \( Z\ell^1(G) \). For the Haar measure, \( h \), on \( \text{Conj}(G) \), \( h(C) = (\delta_C \ast \delta_C(e))^{-1} = |C| \) for \( C \in \text{Conj}(G) \).

As an extension of finite product of hypergroups (or in particular groups), let \( \{H_i\}_{i \in I} \) be a family of discrete hypergroups, then \( H := \bigoplus_{i \in I} H_i \) where for each \( x \in H \), \( x = (x_i)_{i \in I} \) where \( x_i \) is the identity of the hypergroup \( H_i, e_{H_i}, \) for all \( i \in I \) except finitely many. \( H \) is called restricted direct product of \( \{H_i\}_{i \in I} \) which is a hypergroup (or a group if for every \( i \), \( H_i \) is a group).

Example 2.1 For a family of finite groups \( (G_i)_{i \in I} \), let \( G := \bigoplus_{i \in I} G_i \) be the restricted direct product of \( (G_i)_{i \in I} \). Then \( G \) is a discrete FC group and \( \text{Conj}(G) \) equals the hypergroup generated by the restricted direct product of \( \text{Conj}(G_i)_{i \in I} \), \( \text{Conj}(G) = \bigoplus_{i \in I} \text{Conj}(G_i) \).

2.3 The dual of compact groups

Let \( G \) be a compact group and \( \hat{G} \) denotes the set of all irreducible unitary (necessary finite-dimensional!) representations of a compact group \( G \), up to unitary equivalence relation. For
each two unitary irreducible representations $\pi, \sigma \in \hat{G}$, with corresponding dimensions $d_{\pi}$ and $d_{\sigma}$, respectively, we know that $\pi \otimes \sigma$ forms a new unitary representation of $G$ whose dimension is $d_{\pi}d_{\sigma}$. This new representation can be decomposed as a direct product of finitely many irreducible unitary representations $\pi_1, \ldots, \pi_n$ with respective positive constants $m_1^{\pi,\sigma}, \ldots, m_n^{\pi,\sigma} \in \mathbb{N}$, i.e. $\pi \otimes \sigma \cong \bigoplus_{i=1}^n m_i^{\pi,\sigma} \pi_i$. So one may define a convolution on $c_c(\hat{G})$ by

$$\delta_{\pi} \ast \delta_{\sigma} := \sum_{i=1}^n \frac{m_i^{\pi,\sigma}d_{\pi}}{d_{\pi}d_{\sigma}} \delta_{\pi_i}. \quad (2.3)$$

Then $\hat{G}$ equipped with the discrete topology, the convolution (2.3), and the involution resulting from complex conjugate forms a discrete commutative hypergroup. Also for each $\pi \in \hat{G}$, the Haar measure is defined by $h(\pi) = d_{\pi}^2$, where $d_{\pi}$ is the dimension of the representation $\pi$.

**Example 2.2** Let $SU(2)$ be the compact Lie group of $2 \times 2$ special unitary matrices on $\mathbb{C}$, and let $\hat{SU}(2)$ be the hypergroup of all irreducible representations on $SU(2)$. It is known that $\hat{SU}(2) = (\pi_\ell)_{\ell \in \mathbb{N}_0}$ where $\mathbb{N}_0 := \{0, 1, 2, \cdots \}$ and the dimension of $\pi_\ell$ is $\ell + 1$. Moreover, for all $\ell, \ell'$, $\pi_\ell = \pi_{\ell'}$ and $\pi_{\ell} \otimes \pi_{\ell'} \cong \pi_{|\ell-\ell'|} \oplus \pi_{|\ell-\ell'|+2} + \cdots + \pi_{\ell+\ell'}$. This tensor decomposition is called "Clebsch-Gordan" decomposition formula. So using Definition 2.3 and Clebsch-Gordan formula, we have that

$$\delta_{\pi_\ell} \ast \delta_{\pi_{\ell'}} = \sum_{r=|\ell-\ell'|}^{\ell+\ell'} \frac{(r+1)}{(\ell+1)(\ell'+1)} \delta_{\pi_r}$$

where $\sum_2$ denotes the sum by twos. Also $\pi_\ell = \pi_{\ell'}$ and $h(\pi_\ell) = (\ell + 1)^2$ for all $\ell$.

**Example 2.3** Suppose that $\{G_i\}_{i \in I}$ is a non-empty family of compact groups for arbitrary indexing set $I$. Let $G := \prod_{i \in I} G_i$ be the product of $\{G_i\}_{i \in I}$ i.e. $G := \{(x_i)_{i \in I} : x_i \in G_i\}$ equipped with the product topology. Then $G$ is a compact group and [12, Theorem 27.43] implies that $\hat{G}$ is nothing but the (discrete) space of all $\pi = \bigotimes_{i \in I} \pi_i$ such that each $\pi_i \in \hat{G}_i$ and $\pi_i$ are trivial representations except for finitely many $i \in I$. Moreover, for each $\pi = \bigotimes_{i \in I} \pi_i \in \hat{G}$, $d_{\pi} = \prod_{i \in I} d_{\pi_i}$. Hence for $\pi_k = \bigotimes_{i \in I} \pi_i^{(k)} \in \hat{G}$ for $k = 1, 2$, one can show that

$$\delta_{\pi_1} \ast \delta_{\pi_2}(\pi) = \prod_{i \in I} \delta_{\pi_i^{(1)}}(\pi) *_{\hat{G}_i} \delta_{\pi_i^{(2)}}(\pi) \quad \text{for all } \pi = \bigotimes_{i \in I} \pi_i \in \hat{G},$$

where $*_{\hat{G}_i}$ is the hypergroup convolution of $\hat{G}_i$ for each $i \in I$.

### 2.4 Polynomial hypergroups

Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $(a_n)_{n \in \mathbb{N}_0}$ and $(c_n)_{n \in \mathbb{N}_0}$ be sequences of non-zero real numbers and $(b_n)_{n \in \mathbb{N}_0}$ be a sequence of real numbers with the property $a_0 + b_0 = 1$ and $a_n + b_n + c_n = 1$ for $n \geq 1$. If
$(R_n)_{n \in \mathbb{N}_0}$ is a sequence of polynomials defined by

\begin{align*}
R_0(x) &= 1, \\
R_1(x) &= \frac{1}{a_0}(x - b_0), \\
R_1(x)R_n(x) &= a_n R_{n+1}(x) + b_n R_n(x) + c_n R_{n-1}(x), \quad n \geq 1,
\end{align*}

then it is proved in [4] that there exists a probability measure $\pi$ on $\mathbb{R}$ with respect to that the sequence $(R_n)_{n \in \mathbb{N}_0}$ satisfy the orthogonality relation. Moreover,

\begin{align*}
R_n(x)R_m(x) = \sum_{|k| = |n - m|} g(n, m; k) R_k(x)
\end{align*}

where $g(n, m; k) \in \mathbb{R}^+ \cup \{0\}$ for all $|n - m| \leq k \leq n + m$. Let $*$ to be defined on $\mathbb{N}_0$ to $\ell^1(\mathbb{N}_0)$ such that

\begin{align*}
\delta_n * \delta_m := \sum_{|k| = |n - m|} g(n, m; k) \delta_k
\end{align*}

and $\bar{n} = n$. Then $(\mathbb{N}_0, *, \bar{\cdot})$ is a discrete commutative hypergroup which is called polynomial hypergroup structure defined on $\mathbb{N}_0$. On the other hand, every hypergroup structure defined on $\mathbb{N}_0$, so that $\bar{n} = n$ for every $n \in \mathbb{N}_0$, is constructed by a sequence of orthogonal polynomials. In particular, the hypergroup structure of $\tilde{SU}(2)$ (presented in Example 2.2) can be rendered by a family of Chebyshev polynomials as a polynomial hypergroup structure on $\mathbb{N}_0$. This class of hypergroups has been of interest for many studies on discrete hypergroups, for example see [8, 13, 16, 15, 18, 19].

Example 2.4 Let $\mathbb{N}_0$ be equipped with the hypergroup convolution $\delta_n * \delta_m := \frac{1}{2}\delta_{|n-m|} + \frac{1}{2}\delta_{n+m}$. This hypergroup structure is called Chebyshev polynomial of the first type. One can show that the hypergroup algebra of $\mathbb{N}_0$ is isomorphic to the subalgebra of symmetric functions on the Fourier algebra of the torus, i.e. $Z_{\pm 1}A(\mathbb{T}) := \{f + \tilde{f} : f \in A(\mathbb{T})\}$. Here $\tilde{f}(x) = f(-x)$.

3 Weighted discrete hypergroups and examples

In this section we study weights on discrete hypergroups, their corresponding algebras, and their examples. Specially we are interested to see concrete examples of weights defined on the classes of commutative discrete hypergroups which were mentioned in Section 2.

3.1 General Theory

The following definition is a specific case of the weighted hypergroups defined in [9]; here, we focus mainly on discrete hypergroups.
**Definition 3.1** Let $H$ be a discrete hypergroup. We call a function $\omega : H \to (0, \infty)$ a weight if, for every $x, y \in H$, $\omega(\delta_x * \delta_y) \leq \omega(x)\omega(y)$ where

$$\omega(\delta_x * \delta_y) := \sum_{t \in H} \omega(t)\delta_x * \delta_y(t).$$

Then we call $(H, \omega)$ a weighted hypergroup. Let $\ell^1(H, \omega)$ be the set of all complex functions on $H$ such that

$$\|f\|_{\ell^1(H, \omega)} := \sum_{t \in H} |f(t)|\omega(t) < \infty.$$ 

Then one can easily observe that $(\ell^1(H, \omega), \| \cdot \|_{\ell^1(H, \omega)})$ forms a Banach algebra which is called a weighted hypergroup algebra.

- It is easy to see that if $\omega$ is a positive function on $H$ such that $\omega(t) \leq \omega(x)\omega(y)$ for all $t, x, y \in H$ where $t \in \text{supp}(\delta_x * \delta_y)$, then $\omega$ is a weight on $H$. We call this a central weight. We will show later that not all hypergroup weights are central. (See Examples 3.11 and 3.12)

- A hypergroup weight $\omega$ on $H$ is called weakly additive, if for some $C > 0$, $\omega(\delta_x * \delta_y) \leq C(\omega(x) + \omega(y))$ for all $x, y \in H$.

- Two weights $\omega_1$ and $\omega_2$ are called equivalent if there are constants $C_1, C_2$ such that $C_1\omega_1 \leq \omega_2 \leq C_2\omega_2$.

**Example 3.2** Let $\{H_i\}_{i \in I}$ be a family of discrete hypergroups with corresponding weights $\{\omega_i\}_{i \in I}$ such that $\omega_i(e_{H_i}) = 1$ for all $i \in I$ except finitely many. Then $\omega(x)_{i \in I} := \prod_{i \in I} \omega_i(x_i)$ forms a hypergroup weight on the restricted direct product of hypergroups $H_i$.

Let $H$ be a discrete hypergroup. For each finite subset $F$ of $H$, we define

$$F^n := \bigcup \{\text{supp}(\delta_{x_1} * \cdots * \delta_{x_n}) : \text{ for all } x_1, \ldots, x_n \in F\}.$$ 

A hypergroup $H$ is called a finitely generated hypergroup if there exists a finite subset $F \subseteq H$, called a generator, such that $H = \bigcup_{n \in \mathbb{N}} F^n$. Let $F$ be a finite symmetric generator of $H$ i.e. $x \in F$ implies that $\bar{x} \in F$. Then we define

$$\tau_F : H \to \mathbb{N} \cup \{0\}$$

by $\tau_F(x) = \inf\{n \in \mathbb{N} : x \in F^n\}$ for all $x \neq e$ and $\tau_F(e) = 0$. Moreover, since $F$ is symmetric, $\tau_F(\bar{x}) = \tau_F(x)$. It is straightforward to verify that if $F'$ is another finite symmetric generator of $H$, then for some constants $C_1, C_2$, $C_1\tau_{F'} \leq \tau_F \leq C_2\tau_{F'}$. If there is no risk of confusion, we may just use $\tau$ instead of $\tau_F$. For each pair $x, y \in H$ and each $t \in \text{supp}(\delta_x * \delta_y)$, $t$ belongs to $F^{\tau(x) + \tau(y)}$ so that $\tau(t) \leq \tau(x) + \tau(y)$.
**Definition 3.3** For a given $\beta \geq 0$, $\omega_\beta(x) := (1 + \tau(x))^\beta$ forms a central weight on $H$ which is called a Polynomial weight. Similarly, for given $C > 0$ and $0 \leq \alpha \leq 1$, $\sigma_{\alpha,C}(x) := e^{C\tau(x)^\alpha}$ is a central weight on $H$ which is called an Exponential weight.

### 3.2 Weights on $\text{Conj}(G)$

For a (discrete) group $G$, as a hypergroup, a weight is a mapping $\sigma : G \to (0, \infty)$ such that $\sigma(xy) \leq \sigma(x)\sigma(y)$ for all $x, y \in G$. Then $(G, \sigma)$ is called a weighted group. Therefore, $\ell^1(G, \sigma)$ equipped by the convolution and the weighted norm is a Banach algebra called a weighted group algebra.

We need the following lemma to prove more about weights on $\text{Conj}(G)$.

**Lemma 3.4** Let $\omega$ be a function defined on $\text{Conj}(G)$. Then

$$
\omega(\delta_C * \delta_D) = \frac{1}{|C||D|} \sum_{t \in C} \sum_{s \in D} \omega(C_{ts}) \quad (C, D \in \text{Conj}(G)),
$$

where $\omega(\delta_x * \delta_y)$ is defined in (3.1).

**Proof.** The proof is a straightforward calculation based on (2.2) as follows:

$$
\omega(\delta_x * \delta_y) = \sum_{E \in \text{Conj}(G)} \omega(E) \delta_C * \delta_D(E) = \sum_{E \in \text{Conj}(G)} \omega(E) \frac{|E|}{|C||D|} \alpha_{C,D} E
$$

$$
= \sum_{C_t \in \text{Conj}(G), C_t \subseteq CD} \frac{1}{|C||D|} \alpha_{C_t} C_t \omega(C_t)
$$

$$
= \sum_{t \in G} \frac{\omega(C_t)}{|C||D|} \sum_{s \in G} 1_C(s)1_D(s^{-1}t)
$$

$$
= \sum_{t \in G, s \in G} \frac{\omega(C_{st})}{|C||D|} 1_C(s)1_D(t)
$$

$$
= \frac{1}{|C||D|} \sum_{t \in D} \sum_{s \in C} \omega(C_{st}).
$$

The following proposition lets us apply group weights to generate hypergroup weights on $\text{Conj}(G)$.

**Proposition 3.5** Let $G$ be a FC group possessing a weight $\sigma$. Then the mean function $\omega_\sigma$ defined as $\omega_\sigma(C) = |C|^{-1} \sum_{t \in C} \sigma(t)$ for every $C \in \text{Conj}(G)$ is a weight on the hypergroup $\text{Conj}(G)$. 
Proof. By Lemma 3.4, it suffices to show that

$$\frac{1}{|C||D|} \sum_{t \in C} \sum_{s \in D} \omega_\sigma(C_{ts}) \leq \omega_\sigma(C) \omega_\sigma(D)$$

for all $C, D \in \text{Conj}(G)$. For some $C \in \text{Conj}(G)$, let $1_C$ denote the characteristic function on $C$ and $*$ denotes the convolution of $\ell^1(G)$. Using weighted group algebra $\ell^1(G, \sigma)$, one gets

$$\sum_{t \in D} \sum_{s \in C} \omega_\sigma(C_{st}) = \sum_{t \in G} \sum_{s \in G} 1_C(s)1_D(t) \omega_\sigma(C_{st})$$

$$= \sum_{t \in G} \sum_{s \in G} 1_C(s)1_D(s^{-1}t) \omega_\sigma(C_t)$$

$$= \sum_{t \in G} 1_C * 1_D(C_t) \omega_\sigma(C_t)$$

$$= \sum_{E \in \text{Conj}(G)} 1_C * 1_D(E) |E| \omega_\sigma(E)$$

$$= \sum_{E \in \text{Conj}(G)} 1_C * 1_D(E) \sum_{s \in E} \sigma(s)$$

$$= \sum_{t \in G} 1_C * 1_D(t) \sigma(t) = \|1_C * 1_D\|_{\ell^1(G, \sigma)}$$

$$\leq \|1_E\|_{\ell^1(G, \sigma)} \|1_D\|_{\ell^1(G, \sigma)} = |C| \omega_\sigma(C) |D| \omega_\sigma(D),$$

because $\|1_E\|_{\ell^1(G, \sigma)} = \sum_{t \in E} \sigma(t) = \omega_\sigma(E) |E|$ for every $E \in \text{Conj}(G)$. □

We call $\omega_\sigma$ the weight derived from $\sigma$. When $(G, \sigma)$ is a weighted FC group, we define

$$Z\ell^1(G, \sigma) = \{ f \in \ell^1(G, \sigma), \ f(yxy^{-1}) = f(x) \ \forall x, y \in G \}$$

which is the center of the Banach algebra $\ell^1(G, \sigma)$; hence, it is a commutative Banach algebra.

Corollary 3.6 Let $(G, \sigma)$ be a weighted FC group, and $\omega_\sigma$ on $\text{Conj}(G)$ be the weight derived from $\sigma$. Then the weighted hypergroup algebra $\ell^1(\text{Conj}(G), \omega_\sigma)$ is isometrically isomorphic to $Z\ell^1(G, \sigma)$.

Proof. By Proposition 3.5, $\ell^1(\text{Conj}(G), \omega_\sigma)$ is a weighted hypergroup algebra. Let us define $\Psi : Z\ell^1(G, \sigma) \to \ell^1(\text{Conj}(G), \omega_\sigma)$ such that for each $f \in Z\ell^1(G, \sigma)$, $\Psi(f)(C) = |C| f(C)$ for all $C \in \text{Conj}(G)$. Note that $\Psi$ is an algebra homomorphism, since the hypergroup algebra convolution corresponds to the image of the group convolution restricted to $Zc_c(G) = c_c(G) \cap \ell^1(G)$ through the mapping $\Psi$. Due to the definition of $\omega_\sigma$, clearly $\Psi$ is an isometry as well. □
Remark 3.7 Let $G$ be an FC group and $\omega$ be a central weight on $\text{Conj}(G)$. Then the mapping $\sigma_\omega$, defined on $G$ by $\sigma_\omega(x) := \omega(C_x)$, forms a group weight on $G$. And $\ell^1(\text{Conj}(G), \omega)$ as a Banach algebra is isometrically isomorphic to $Z\ell^1(G, \sigma_\omega)$.

Example 3.8 Let $G$ be a discrete FC group. The mapping $\omega(C) = |C|$, for $C \in \text{Conj}(G)$, is a central weight on $\text{Conj}(G)$. Clearly, if $E \subseteq CD$, we have $|E| \leq |C||D|$.

Example 3.9 Let $G = \bigoplus_{i \in I} G_i$ for a family of finite groups $(G_i)_{i \in I}$ as in Example 2.1. Given $C = (C_i)_{i \in I} \in \text{Conj}(G)$, define $I_C := \{i \in I : C_i \neq e_{G_i}\}$. For each $\alpha > 0$, we define a mapping $\omega_\alpha(C) := (1 + |C_{i_1}| + \cdots + |C_{i_n}|)^\alpha$ where $i_j \in I_C$. We show that $\omega_\alpha$ is a central weight on $\text{Conj}(G)$. Let $E \subseteq CD$ for some $E, C, D \in \text{Conj}(G)$. One can easily show that for each $i \in I$, $E_i \subseteq C_i D_i$; $I_E \subseteq I_C \cup I_D$. Therefore,

$$\omega_\alpha(C) = (1 + \sum_{i \in I_E} |E_i|)^\alpha \leq (1 + \sum_{i \in I_G} |C_i||D_i|)^\alpha \quad \text{(by Example 3.8)}$$

$$\leq \left(1 + \sum_{i \in I_C} |C_i|\right)^\alpha \left(1 + \sum_{i \in I_D} |D_i|\right)^\alpha = \omega_\alpha(C)\omega_\alpha(D).$$

A group $G$ is called a group with finite commutator group or FD if its derived subgroup is finite. It is immediate that for a group $G$, for every $C \in \text{Conj}(G)$, $|C| \leq |G'|$ when $G'$ is the derived subgroup of $G$. Therefore, the order of conjugacy classes of an FD group are uniformly bounded by $|G'|$. The converse is also true, that is for an FC group $G$, if the order of conjugacy classes are uniformly bounded, then $G$ is an FD group, see [24, Theorem 14.5.11]. The following proposition implies that every hypergroup weight on the conjugacy classes of an FD group which is constructed by a group weight (as given in Proposition 3.5) is equivalent to a central weight.

Proposition 3.10 Let $(G, \sigma)$ be a weighted FD group. Then the hypergroup weight $\omega_\sigma(C) := |G|^2 \omega(C)$, for $C \in \text{Conj}(G)$, forms a central weight. Here $\omega_\sigma$ is defined in Proposition 3.5.

Proof. Let $E \subseteq CD$ for some $C, D, E \in \text{Conj}(G)$. Note that for each $t \in E$, there are some $x \in C$ and $y \in D$ such that $t = xy$, so one gets that

$$\omega_\sigma(E) = \frac{1}{|E|} \sum_{t \in E} \sigma(t) \leq \frac{1}{|E|} \sum_{t \in E} \sum_{x \in C} \sigma(x) \sum_{y \in D} \sigma(y) \leq \sum_{x \in C} \sigma(x) \sum_{y \in D} \sigma(y).$$

Hence, $\omega_\sigma(E) \leq |C|\omega(C) |D|\omega(D) \leq |G|^2 \omega_\sigma(C)\omega_\sigma(D)$, and so, $\omega_\sigma(E) \leq \omega_\sigma(C)\omega_\sigma(D)$. □

In contrast to Proposition 3.10, we will see in the following examples that there exist weights on FC groups with infinite derived subgroups which are not equivalent to any central weight.

Example 3.11 Let $S_3$ be the symmetric group of order 6. Table 1 summarizes the support of $\delta_C * \delta_D$ for all $C, D \in \text{Conj}(S_3)$. Let $\omega$ be defined on $\text{Conj}(S_3)$ by $\omega(C_e) = 1$, $\omega(C_{(12)}) = 2,$
and \( \omega(C_{(123)}) = 5 \). One may verify that \( \omega \) forms a weight on \( \text{Conj}(S_3) \). In fact by applying Lemma 3.4, it is sufficient to check that the following inequalities hold for \( \omega \) to be a weight on \( \text{Conj}(S_3) \).

(i) \( \frac{1}{3} \omega(C_e) + \frac{2}{3} \omega(C_{(123)}) \leq \omega(C_{(12)})^2 \).

(ii) \( 1 \leq \omega(C_e), \omega(C_{(123)}) \).

(iii) \( \frac{1}{2} \omega(C_{(123)}) + \frac{1}{2} \omega(C_e) \leq \omega(C_{(123)})^2 \).

On the other hand, since \( 5 = \omega(C_{(123)}) \leq \omega(C_{(12)})^2 = 4 \), \( \omega \) is not a central weight.

**Example 3.12** We generate the restrict direct product \( G = \bigoplus_{n \in \mathbb{N}} S_3 \). Let us define the weight \( \omega' := \prod_{n \in \mathbb{N}} \omega \) on \( \text{Conj}(G) \) where \( \omega \) is the hypergroup weight on \( \text{Conj}(S_3) \) defined in Example 3.11.

For each \( N \in \mathbb{N} \), define \( D_N := \prod_{n \in \mathbb{N}} D_{n}^{(N)} \in \text{Conj}(G) \) where \( D_{n}^{(N)} = C_{(123)} \) for all \( n \in 1, \ldots, N \) and \( D_{n}^{(N)} = C_e \) otherwise. One can verify that \( D_N \in \text{supp}(\delta_{E_N} * \delta_{E_N}) \) for \( E_N = \prod_{n \in \mathbb{N}} E_{n}^{(N)} \in \text{Conj}(G) \) with \( E_{n}^{(N)} = C_{(12)} \) for all \( n \in 1, \ldots, N \) and \( E_{n}^{(N)} = C_e \) otherwise. Therefore

\[
\frac{\omega'(D_N)}{\omega'(E_N)^2} = \prod_{n=1}^{N} \frac{\omega'(C_{(123)})}{\omega'(C_{(12)})^2} = (5/4)^N \rightarrow \infty
\]

where \( N \rightarrow \infty \). Hence, \( \omega' \) is not equivalent to any central weight.

We close this subsection with the following proposition which lets us construct weights on the conjugacy classes of quotient groups applying weights of the conjugacy classes of the main group. Some arguments in the proof of the following proposition are similar to the ones in [23, Proposition 3.6.11].

**Proposition 3.13** Let \( G \) be a group, \( N \) a normal subgroup of \( G \), and \( \omega \) a weight on \( \text{Conj}(G) \) such that there is some \( \delta > 0 \) such that \( \omega(C) > \delta \), for any \( C \in \text{Conj}(G) \). Then the mapping \( \tilde{\omega} : \text{Conj}(G/N) \rightarrow \mathbb{R}^+ \) defined by \( \tilde{\omega}(C_{xN}) := \inf\{\omega(C_{yN}) : y \in N\} \), for \( C_{xN} \in \text{Conj}(G/N) \), forms a weight on \( \text{Conj}(G/N) \).
Proof. Let \( T : \ell^1(G) \to \ell^1(G/N) \) be the Reiter’s map defined in [23, (3.4.10)] by \( T f(xN) := \sum_{t \in N} f(\lambda x t) \) for \( f \in \ell^1(G) \) which is an onto algebra homomorphism. For each \( f \in Z\ell^1(G) \) and \( g \in \ell^1(G) \), note that \( T f * T g = T(f * g) = T(g * f) = T g * T f \). Since \( T \) is onto, this implies that \( T(Z(\ell^1(G)) \subseteq Z\ell^1(G/H) \). Let us also denote by \( T \) the restriction of \( T \) to \( Z\ell^1(G) \); hence, \( T : Z\ell^1(G) \to Z\ell^1(G/N) \). Therefore, \( T \) can be seen as a mapping \( T : \ell^1(\text{Conj}(G)) \to \ell^1(\text{Conj}(G/N)) \).

Claim. We claim that for each \( x \in G \), \( T(\delta_{C_x}) = \alpha_{C_{xN}} \delta_{C_{xN}} \) for some \( 0 < \alpha_{C_{xN}} \leq 1 \).

Since \( \omega \) is away from zero by some \( \delta > 0 \), one can easily show that \( \ell^1(\text{Conj}(G), \omega) \) is a subalgebra of \( \ell^1(\text{Conj}(G)) \). So, let us define \( T_\omega := T|_{\ell^1(\text{Conj}(G), \omega)} \) and \( A := \text{Im}(T_\omega) \) equipped with the quotient norm i.e. for each \( f \in \ell^1(\text{Conj}(G), \omega) \), \( \|T_\omega(f)\|_q := \inf\{\|f - k\|_{\ell^1(\text{Conj}(G), \omega)}, k \in \text{Ker}T_\omega\} \).

Note that, \( T_\omega(\delta_{C_x} - \delta_{C_{xy}}) = \alpha_{C_{xN}}(\delta_{C_{xN}} - \delta_{C_{xyN}}) = 0 \) for all \( x \in G \) and \( y \in N \). For each \( x \in G \) and \( y \in N \) \( \|T_\omega(\delta_{C_x})\|_q \geq \alpha_{C_{xN}} \|\delta_{C_{xN}}\|_q \geq \alpha_{C_{xN}} \|\omega(\delta_{C_{xN}})\|_q \) for some \( 0 = \|f\|_{\ell^1(\text{Conj}(G/N), \hat{\omega}^{-1})} = \|f\|_{\ell^1(\text{Conj}(G/N), \hat{\omega}^{-1})}. \)

Also, since \( A \) is equipped with the quotient topology, \( T_\omega^* \) is an isometry. Hence,
\[
\|\varphi\|_{A^*} = \|T_\omega^*(\varphi)\|_{\ell^\infty(\text{Conj}(G), \omega^{-1})} = \sup_{C_x \in \text{Conj}(G)} \frac{|\varphi(T_\omega(\delta_{C_x}))|}{\|T_\omega(\delta_{C_x})\|_q} \geq \alpha_{C_{xN}} \|\varphi(\delta_{C_{xN}})\|_{\ell^\infty(\text{Conj}(G/N), \hat{\omega}^{-1})}.
\]

So \( \|\cdot\|_{A^*} = \|\cdot\|_{\ell^\infty(\text{Conj}(G/N), \hat{\omega}^{-1})}. \) Consequently, \( \ell^1(\text{Conj}(G/N), \hat{\omega}) \) and \( \ell^1(\text{Conj}(G), \omega)/\text{Ker}T_\omega \) are isomorphic, as two Banach algebras. Thus,
\[
\|\delta_{C_{xN}} \ast \delta_{C_{yN}}\|_{\ell^1(\text{Conj}(G/N), \hat{\omega})} \leq \|\delta_{C_{xN}}\|_{\ell^1(\text{Conj}(G/N), \omega)} \|\delta_{C_{yN}}\|_{\ell^1(\text{Conj}(G/N), \hat{\omega})}.
\]
which equals to this fact that \( \tilde{\omega}(\delta_{C_xN} \ast \delta_{C_yN}) \leq \tilde{\omega}(C_xN)\tilde{\omega}(C_yN) \). This shows that \( \tilde{\omega} \) is a weight on \( \text{Conj}(G/N) \).

**Proof of Claim.** For each \( C_x \in \text{Conj}(G) \) and \( C_z \in \text{Conj}(G/N) \), applying \( \Psi \) the isomorphism from \( Z\ell^1(G) \) onto \( \ell^1(\text{conj}(G)) \), one gets

\[
T(\delta_{C_x})(C_zN) = T \circ \Psi \left( \frac{1}{|C_x|} 1_{C_x} \right)(zN) = \frac{1}{|C_x|} \sum_{t \in N} 1_{C_x}(zt). \tag{3.3}
\]

First, assume that \( C_{xN} \neq C_{zN} \). Toward a contradiction, one may assume that for some \( t \in N, \, zt \in C_x \). Without loss of generality, let \( zt = x \). But, this implies that \( zN = xN \) which is a contradiction. Therefore, if \( C_{xN} \neq C_{zN} \), \( T(\delta_{C_x})(C_{zN}) = 0 \). On the other hand, one may complete the equation (3.3) as follows.

\[
T(\delta_{C_x})(C_{zN}) = \frac{1}{|C_x|} \sum_{t \in N} 1_{C_x}(zt) = \frac{1}{|C_x|} \sum_{y \in C_x} \sum_{t \in N} \delta_y(zt).
\]

So, if for some \( y \in C_x, \, zt = y, \sum_{t \in N} \delta_y(zt) = 1; \)

\[
0 < \frac{1}{|C_x|} \leq \frac{1}{|C_x|} \sum_{y \in C_x} \sum_{t \in N} \delta_y(zt) \leq 1 = \delta_{C_{xN}}(C_{xN}).
\]

For some \( 0 < \alpha_{C_{xN}} \leq 1 \), we have proved the claim. \( \square \)

**Remark 3.14** In Proposition 3.13, if the weight \( \omega \) is central, \( \sigma_\omega(x) := \omega(C_x) \) (for \( x \in G \)) defines a weight on the group \( G \), and therefore, applying Reiter’s map, one can show that \( \tilde{\omega} \) is a central weight on \( G/N \) which respects the conjugacy classes of \( G/N \); equivalently, it is a weight on \( \text{Conj}(G/N) \).

### 3.3 Weights on duals of compact groups

In this subsection, \( G \) is a compact group. We recall that for each \( \pi \in \hat{G} \) and \( f \in L^1(G) \),

\[
\hat{f}(\pi) := \int_G f(x) \overline{\pi(x)} dx
\]

is the Fourier transform of \( f \) at \( \pi \). Let \( VN(G) \) denote the group von Neumann algebra of \( G \), i.e. the von Neumann algebra generated by the left regular representation of \( G \) on \( L^2(G) \). It is well-known that the predual of \( VN(G) \), denoted by \( A(G) \), is a Banach algebra of continuous functions on \( G \); it is called the Fourier algebra of \( G \). Moreover, for every \( f \in A(G) \),

\[
\|f\| := \sum_{\pi \in \hat{G}} d_\pi \|\hat{f}(\pi)\|_1 < \infty,
\]
where \( \| \cdot \|_1 \) denotes the trace-class operator norm.

In an attempt to find the noncommutative analogue of weights on groups, Lee and Samei in [21] defined a weight on \( A(G) \) to be a densely defined (not necessarily bounded) operator \( W \) affiliated with \( VN(G) \) and satisfying certain properties mentioned in [21, Definition 2.4] (see also [22]). Specially they assume that \( W \) has a bounded inverse \( W^{-1} \) which belongs to \( VN(G) \).

Applying a weight \( W \) on \( A(G) \), they introduced Beurling-Fourier algebra, as a function algebra on \( G \), denoted by \( A(G,W) \). Indeed, \( A(G,W) \) is the subalgebra of \( A(G) \) which consists of those functions \( f \in A(G) \) such that

\[
\| f \|_{A(G,W)} := \sum_{\pi \in \hat{G}} d_\pi \| \hat{f}(\pi) \circ W \|_1 < \infty.
\]

The algebra \( A(G,W) \) equipped with the defined norm forms a Banach algebra with respect to the pointwise multiplication. In [21], the authors also studied Arens regularity and isomorphism to operator algebras for Beurling-Fourier algebras.

In the case that \( G \) is an abelian group (with the dual group \( \hat{G} \)), one may show that the weight \( W \) on \( A(G) \) forms a mapping \( \omega_W : \hat{G} \to (0, \infty) \) where \( \omega_W \) is a group weight on \( \hat{G} \) i.e. \( \omega_W(z_1 z_2) \leq \omega_W(z_1) \omega_W(z_2) \) for all \( z_1, z_2 \in \hat{G} \). Moreover, \( A(G,W) \) is isometrically isomorphic to the weighted group algebra corresponded to \( \omega_W, L^1(\hat{G}, \omega_W) \). In the following definition, we generalize this idea to a general compact group.

**Definition 3.15** Let \( G \) be a compact group and \( W \) a weight on \( A(G) \). We define a function \( \omega_W : \hat{G} \to (0, \infty) \) by

\[
\omega_W(\pi) := \frac{\| I_\pi \circ W \|_1}{d_\pi} \quad (\pi \in \hat{G}),
\]

where \( \| \cdot \|_1 \) denotes the trace norm and \( I_\pi \) is the identity matrix corresponding to the Hilbert space of \( \pi \).

As a specific class of weights on the Fourier algebra of a compact group \( G \), in [21] (and independently, in [22]), central weights on \( A(G) \) are defined. Indeed, [21, Theorem 2.12] implies that each central weight \( W \) can be represent by a unique function \( \omega_W : \hat{G} \to (0, \infty) \) such that \( \omega_W(\sigma) \leq \omega_W(\pi_1) \omega_W(\pi_2) \) for all \( \pi_1, \pi_2, \sigma \in \hat{G} \) where \( \sigma \in \text{supp}(\delta_{\pi_1} * \delta_{\pi_2}) \). In this specific case of operator weights, \( \omega_W \) matches with our definition in Definition 3.15 and is actually a central weight on the hypergroup \( \hat{G} \) as we defined before. So in this particular case, \( \omega_W \) is a hypergroup weight is immediate. In the following we show that the same is true for a general operator weight as well.

Let us recall that \( ZA(G) := \{ f \in A(G) : f(xy^{-1}) = f(x) \text{ for all } x \in G \} \) which is a Banach algebra with pointwise product and \( \| \cdot \|_{A(G)} \). Note that for the operator weights \( W \) where \( \omega_W(\pi) = 1 \) for every \( \pi \in \hat{G} \), \( ZA(G,W) = ZA(G) \).
**Theorem 3.16** Let $G$ be a compact group and $W$ a weight on $A(G)$. Then $\omega_W$ is a weight on the hypergroup $\hat{G}$ and the weighted hypergroup algebra $\ell^1(\hat{G}, \omega_W)$ is isometrically isomorphic to $ZA(G, W)$.

**Proof.** Let $\mathcal{X}(G)$ denotes the linear span of all the characters of $G$. First define a linear mapping $T : \mathcal{X}(G) \to c_c(\hat{G})$ by $T(\chi_\pi) = d_\pi \delta_\pi$ for each $\pi \in \hat{G}$. Note that for each $f \in \mathcal{X}(G)$, $f = \sum_{i=1}^n \alpha_i \chi_{\pi_i}$ for finitely many $\pi_i$ in $\hat{G}$ and $\alpha_i \in \mathbb{C}$. In this case,

$$
\|T(f)\|_{\ell^1(\hat{G}, \omega)} = \sum_{i=1}^n |\alpha_i| d_{\pi_i, \omega}(\pi_i)
= \sum_{i=1}^n |\alpha_i| d_{\pi_i} \|I_{\pi_i} \circ W\|_1 / d_{\pi_i}
= \sum_{i=1}^n d_{\pi_i} \|\alpha_i I_{\pi_i} \circ W\|_1
= \sum_{i=1}^n d_{\pi_i} \|\alpha_i \hat{\chi}_{\pi_i}(\pi_i) \circ W\|_1 = \|f\|_{A(G, W)}.
$$

Therefore, $T$ forms a norm preserving linear mapping. To show that $T$ is an algebra homomorphism, note that

$$
T(\chi_\pi \chi_\sigma) = T(\sum_{i=1}^n m_i \chi_{\sigma_i}) = \sum_{i=1}^n m_i d_{\sigma_i, \delta_\sigma_i} = d_{\pi_1, \delta_\pi_1} * d_{\pi_2, \delta_\pi_2} = T(\chi_\pi_1) * T(\chi_\pi_2).
$$

It is known that $\mathcal{X}(G)$ is dense in $ZA(G, W)$ and clearly $c_c(\hat{G})$ is dense in $\ell^1(\hat{G}, \omega_W)$. So $T$ can be extended as an algebra isomorphism from $ZA(G, W)$ onto $\ell^1(\hat{G}, \omega_W)$ which preserves the norm. In particular, $\ell^1(\hat{G}, \omega_W)$ forms an algebra with respect to its weighted norm and the convolution, and so $\omega_W$ is actually a hypergroup weight on $\hat{G}$. □

**Lemma 3.17** Let $G$ be a compact group and $\hat{G}$ be the set of all irreducible representations of $G$ as a discrete commutative hypergroup. Then $\omega_\beta(\pi) = d_\pi^\beta = h(d_\pi)^{\beta/2}$ is a central weight for each $\beta \geq 0$.

**Proof.** Since for every pair $\pi, \sigma \in \hat{G}$, the dimension of $\pi \otimes \sigma$ which is $d_\pi d_\sigma$ is equivalent to $\sum_{i=1}^n m_{i}^{\pi, \sigma} d_{\pi_i}$ for some $m_{i}^{\pi, \sigma} > 0$ and $(\pi_i)_{i=1}^n \subseteq \hat{G}$, for each $\pi_{i_0} \in \text{supp}(\delta_\pi * \delta_\sigma)$ one gets that $d_\pi d_\sigma = m_{i_0}^{\pi, \sigma} d_{\pi_{i_0}} \geq d_{\pi_{i_0}}$. □
Example 3.18 (Central weights on SU(2)) As mentioned in Lemma 3.17, for each \( \beta > 0 \), \( \omega_\beta(\pi_\ell) := d_\pi^{\beta/2} = (\ell + 1)^\beta \) (for \( \pi_\ell \in \hat{SU}(2) \)), is a central weight on \( \hat{SU}(2) \).

Example 3.19 (Lifting weights from \( \mathbb{Z} = \hat{T} \) to SU(2)) Let \( \sigma \) be a weight on the group \( \mathbb{Z} \) i.e. \( \sigma(m+n) \leq \sigma(m)\sigma(n) \). We define

\[
\omega_\sigma(\pi_\ell) := \frac{1}{\ell + 1} \sum_{t=-\ell}^{\ell} \sigma(r) \quad (\ell \in \mathbb{N}_0).
\]

Recall that elements of \( \hat{SU}(2) \) can be regarded as \( \pi_\ell \) when \( \ell \in \mathbb{N}_0 \). Suppose that \( m, n \in \mathbb{N}_0 \) and without loss of generality \( n \geq m \). Then,

\[
\omega_\sigma(\pi_m)\omega_\sigma(\pi_n) = \frac{1}{m+1} \sum_{t=-m}^{m} \sigma(t) \frac{1}{n+1} \sum_{s=-n}^{n} \sigma(s)
\]

\[
\geq \frac{1}{(m+1)(n+1)} \sum_{t=-m}^{m} \sigma(t + s) \quad (\dagger)
\]

\[
= \frac{1}{(m+1)(n+1)} \sum_{t=-n-m}^{n+m} \sigma(s) \quad (\dagger)
\]

\[
= \sum_{t=-n-m}^{n+m} \frac{(t+1)}{(m+1)(n+1)} \left( \frac{1}{t+1} \sum_{s=-t}^{t} \sigma(s) \right)
\]

\[
= \sum_{t=-n-m}^{n+m} \frac{(t+1)}{(m+1)(n+1)} \omega_\sigma(\pi_t)
\]

\[
= \omega_\sigma(\delta_{\pi_m} \ast \delta_{\pi_n}).
\]

To show that the summations (\( \dagger \)) and (\( \ddagger \)) are equal, let us arrange (\( \ddagger \)) as follows.

\[
\begin{array}{cccccccc}
\sigma(-m-n) & +\sigma(-m-n+2) & \cdots & +\sigma(-m+n) & +\sigma(-m+n) \\
+\sigma(-m-n+2) & +\sigma(-m-n+4) & \cdots & +\sigma(-m+n) & +\sigma(-m+n) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
+\sigma(m-2) & +\sigma(m) & \cdots & +\sigma(m+n-4) & +\sigma(m+n-2) \\
+\sigma(m) & +\sigma(m+2) & \cdots & +\sigma(m+n-2) & +\sigma(m+n) \\
\end{array}
\]

but the sum of all the entries in the first column and the last row is equal to

\[
\sum_{t=-n-m}^{n+m} \sigma(s).
\]
The next column and row give
\[ \sum_{s=m-n+2}^{m+n-2} \sigma(s), \]
and so on. So by doing this finitely many times, we get (†).

**Example 3.20 (Non-central weights on SU(2))** Let us define \( \omega_a : \hat{\text{SU}}(2) \to \mathbb{R}^+ \) such that \( \omega_a(\pi_\ell) := a^{\ell+1}/(\ell + 1) \) for a fixed constant \( a \geq (\sqrt{5} + 1)/2 \). We show that \( \omega_a \) is a weight on \( \hat{\text{SU}}(2) \). For a pair of \( \ell, \ell' \in \mathbb{N}_0 := \{0, 1, 2, 3\ldots\} \), without loss of generality suppose that \( \ell \geq \ell' \). So for \( \sum \), adding by twos, we have
\[
\frac{\ell + \ell'}{2} \sum_{r=\ell-\ell'} (r + 1) \omega_a(\pi_r) = \sum_{r=\ell-\ell'} a^{r+1}
\]
\[ = a^{\ell-\ell'+1} \sum_{r=0}^{2\ell'} a^r
\]
\[ = a^{\ell-\ell'+1} \frac{a^{2\ell'+2} - 1}{a^2 - 1}
\]
\[ = a^{\ell+\ell'+3} \frac{a^{\ell-\ell'+1} - 1}{a^2 - 1}
\]
\[ \leq a^{\ell+\ell'+2} \left( \frac{a}{a^2 - 1} \right)
\]
\[ \leq \frac{a}{a^2 - 1} \omega_a(\ell)(\ell + 1) \omega_a(\ell')(\ell' + 1).
\]
But since \( a \geq (\sqrt{5} + 1)/2 \), \( a/(a^2 - 1) \leq 1 \); therefore,
\[
\omega_a(\delta_{\pi_\ell} * \delta_{\pi_{\ell'}}) \leq \omega_a(\pi_\ell) \omega_a(\pi_{\ell'}).
\]

Note that
\[
\frac{\omega_a(\pi_{2\ell})}{\omega_a(\pi_\ell)^2} = \frac{a^{2\ell+1}}{2\ell + 1} \left( \frac{a^{\ell+1}}{\ell + 1} \right)^2 \to \infty
\]
when \( \ell \to \infty \); while \( \pi_{2\ell} \in \text{supp}(\delta_{\pi_\ell} * \delta_{\pi_{\ell'}}) \). Hence, not only is \( \omega_a \) a non-central weight but also it is not equivalent to any central weight.

**Remark 3.21** Let \( \sigma(n) = a^n \) for some \( a \geq 1 \) on \( \mathbb{Z} \). Clearly, \( \sigma \) is a weight on \( \mathbb{Z} \) and therefore, one may consider the weight \( \omega_\sigma \) as defined in Example 3.19. For each \( \ell \in \mathbb{N}_0 \) and \( a \geq (1 + \sqrt{5})/2 \),
\[
\omega_\sigma(\pi_\ell) = \frac{1}{\ell + 1} \sum_{r=-\ell}^{\ell} a^r = \frac{a^{-\ell}}{\ell + 1} \frac{a^{2\ell+2} - 1}{a^2 - 1} = \omega_a(\pi_\ell) \frac{a^{2\ell+2} - 1}{a^{2\ell+1}(a^2 - 1)}.
\]
where \( \omega_a \) is the weight defined in Examples 3.20. But for every \( \ell \in \mathbb{N}_0 \) and \( a > 1 \),

\[
\frac{1}{a+1} \leq \frac{a^{2\ell+2} - 1}{a^{2\ell+1}(a^2 - 1)} \leq \frac{a}{a^2 - 1}.
\]

This implies that the weights \( \omega_a \) and \( \omega_a \) are equivalent.

**Example 3.22** Let \( \sigma \) be a weight defined on \( \mathbb{Z} \) by

\[
\sigma(\ell) := \begin{cases} 
1 & 0 \leq \ell \\
(1 - \ell)^\beta & \ell < 0 
\end{cases} \quad (\ell \in \mathbb{Z})
\]

for some \( \beta \geq 0 \). One can simply check that \( \sigma \) is actually a weight on \( \mathbb{Z} \). Therefore, one may apply the construction in Example 3.19 to construct a hypergroup weight \( \omega_{\sigma} \) on \( \widehat{SU}(2) \). But one may also observe that the weight \( \omega_{\sigma} \) is equivalent to the weight \( \omega_\beta(\ell) := (\ell + 1)^\beta \). We will see in Section 4, that this particular weight will give interesting classes of examples.

**Example 3.23** [22, Proposition 4.11]

Let \( G \) be a compact group and \( \omega \) be a central weight on the hypergroup \( \widehat{G} \). Let \( N \) be a closed subgroup of \( G \). It is known that for each \( \pi \in \widehat{G} \), the restriction \( \pi \) to \( N \), that we denote by \( \pi|_N \) here, is a finite dimensional unitary representation of \( N \) and therefore it can be decomposed by finitely many elements of \( \widehat{N} \). For representations \( \sigma \in \widehat{N} \) and \( \pi \in \widehat{G} \), \( \sigma \leq \pi|_N \) means that \( \sigma \) is equivalent to a subrepresentation of the irreducible decomposition of \( \pi|_N \). Note that for each representation \( \sigma \in \widehat{N} \), there is at least one \( \pi \in \widehat{G} \) such that \( \sigma \leq \pi|_N \) (see [12, Theorem 27.46]).

Then for each closed subgroup \( N \) of \( G \), \( \omega_N(\sigma) := \inf_{\pi \in \widehat{G}, \sigma \leq \pi|_N} \omega(\pi) \) for \( \sigma \in \widehat{N} \) is a well defined positive function on \( \widehat{N} \). We show that \( \omega_N \) is actually a hypergroup weight on \( \widehat{N} \). To do so, for a given \( \epsilon > 0 \), note that if \( \sigma_1, \sigma_2 \in \widehat{N} \) there are \( \pi_1, \pi_2 \in \widehat{G} \) such that \( \sigma_i \leq \pi_i|_N \) and \( \omega(\pi_i) < \omega_N(\sigma_i) + \epsilon \) for \( i = 1, 2 \). For each \( \sigma \in \text{supp}(\delta_{\sigma_1} * \delta_{\sigma_2}) \), \( \sigma \leq \sigma_1 \otimes \sigma_2 \leq \pi_1|_N \otimes \pi_2|_N \); hence,

\[
\omega_N(\sigma) = \inf_{\pi \in \widehat{G}, \sigma \leq \pi|_N} \omega(\pi) \leq \inf_{\pi \leq \pi_1 \otimes \pi_2, \sigma \leq \pi|_N} \omega(\pi) \leq \omega(\pi_1)\omega(\pi_2) \leq (\omega_N(\sigma_1) + \epsilon)(\omega_N(\sigma_2) + \epsilon).
\]

Since \( \epsilon > 0 \) is arbitrary, it implies that \( \omega_N \) is a central weight on \( \widehat{N} \).

### 3.4 Weights on polynomial hypergroups

**Example 3.24** The set \( F = \{0, 1\} \) is a symmetric generator of any polynomial hypergroup structure on \( \mathbb{N}_0 \) and \( \tau_F \) is the map defined in (3.2). Since \( 1 \in F^1 \), \( \tau_F(1) = 1 \). Now suppose that \( \tau_F(k) = k \) for some \( k \in \mathbb{N} \). Then by [17, Proposition 5.2], \( g(k, 1; k + 1) \neq 0 \); therefore, \( k + 1 \in \text{supp}(\delta_k \ast \delta_1) \subseteq F^{k+1} \). But \( k + 1 \notin F^k \). Thus \( \tau_F(k + 1) = k + 1 \), and so by induction, \( \tau_F(n) = n \) for all \( n \in \mathbb{N}_0 \). In particular, \( \mathbb{N}_0 \) is a finitely generated hypergroup. Consequently,
for each $\beta \geq 0$, we can define a polynomial weight $\omega_\beta$ on $\mathbb{N}_0$ where $\omega_\beta(n) = (1 + n)^\beta$ for $n \in \mathbb{N}_0$. Also, for each $0 \leq \alpha \leq 1$ and $C > 0$, we can define an exponential weight $\sigma_{\alpha,C}$ on $\mathbb{N}_0$ where $\sigma_{\alpha,C}(n) = e^{Cn^\alpha}$ for $n \in \mathbb{N}_0$. Using these two classes of weights we can generate a variety of weighted hypergroup algebras.

Recall that $\widehat{SU}(2)$ is a particular example of polynomial hypergroup so-called Chebyshev polynomials. So the argument presented in Example 3.19 can be applied to construct hypergroup weights on polynomial hypergroups admitted by weights on $\mathbb{Z}$. To do so, we should replace the summation by twos with a new pattern of summation respecting the hypergroup structure and modify the constants.

In the following, we present a construction for the Chebyshev polynomial hypergroup of the first type (presented in Example 2.4). Suppose that $\sigma$ be a weight on $\mathbb{Z}$. Define

$$\omega_\sigma(n) = (\sigma(n) + \sigma(-n)) \quad (n \in \mathbb{N}_0).$$

One can easily verify that $\omega_\sigma$ is a hypergroup weight on $\mathbb{N}_0$ when it is equipped with the Chebyshev polynomial hypergroup structure of the first type.

**Example 3.25** Let $\mathbb{N}_0$ be equipped with the Chebyshev polynomial hypergroup structure of the first type. Given any $a \geq 1$, let $\sigma(\ell) := a^{\ell}$, for $\ell \in \mathbb{Z}$. Then the weight $\omega_\sigma$, constructed by (3.6), forms a central weight on $\mathbb{N}_0$ such that $\omega_\sigma(n) = 2a^n$ for $n \in \mathbb{N}_0$.

**Example 3.26** Let $f : \mathbb{N}_0 \to \mathbb{R}^+$ be an increasing function such that $f(0) = 1$. Then

$$\sigma_f(\ell) := \begin{cases} 1 & \ell \geq 0 \\ f(-\ell) & \ell < 0 \end{cases} \quad (\ell \in \mathbb{Z})$$

forms a group weight on $\mathbb{Z}$. Then $\omega_f(n) = f(n) + 2$, the hypergroup weight (3.6) constructed by $\sigma_f$, is a central weight on $\mathbb{N}_0$ when it is equipped with the Chebyshev polynomial hypergroup structure of the first type.

**Remark 3.27** Recall that for the abelian compact group $\mathbb{T}$, $A(\mathbb{T}, \sigma)$ is isometrically isomorphic to weighted group algebra $\ell^1(\mathbb{Z}, \sigma)$ through the Fourier transform. Therefore, applying the argument briefly mentioned in Example 2.4, we can see that $\ell^1(\mathbb{N}_0, \sigma)$ is isomorphic to the symmetric subalgebra of $A(\mathbb{T}, \sigma)$, that is $Z_{\pm 1}A(\mathbb{T}, \sigma) := \{f + \hat{f} : f \in A(\mathbb{T}, \sigma)\}$.

## 4 Arens regularity

In [14, Chapter 4], Kamyabi-Gol applied the topological center of hypergroup algebras to prove some results about the hypergroup algebras and their second duals. For example, in [14, Corollary 4.27], he showed that for a (not necessarily discrete and commutative) hypergroup $H$ (which possesses a Haar measure), $L^1(H)$ is Arens regular if and only if $H$ is finite.
Arens regularity of weighted group algebras has been studied by Craw and Young in [6]. They showed that a locally compact group \( G \) has a weight \( \omega \) such that \( L^1(G, \omega) \) is Arens regular if and only if \( G \) is discrete and countable. The monograph [7] presents a thorough report of the Arens regularity of weighted group algebras. In the following we adapt the machinery developed in [7, Section 8] for weighted hypergroups. In [7, Section 3], the authors study repeated limit conditions and give a rich variety of results for them. Here, we will use some of these results.

First let us recall the following definitions. Let \( A \) be a Banach algebra. For \( f, g \in A, \phi \in A^* \), and \( F, G \in A^{**} \), we define the following module actions.

\[
\langle f \cdot \phi, g \rangle := \langle \phi, gf \rangle, \quad \langle \phi \cdot f, g \rangle := \langle \phi, fg \rangle
\]

\[
\langle \phi \cdot F, f \rangle := \langle F, f \cdot \phi \rangle, \quad \langle F \cdot \phi, f \rangle := \langle F, \phi \cdot f \rangle
\]

\[
\langle F \triangleleft G, \phi \rangle := \langle G, \phi \cdot F \rangle, \quad \langle G \square F, \phi \rangle := \langle G, F \cdot \phi \rangle.
\]

Let \( F, G \in A^{**} \), and let \( (f_\alpha) \) and \( (g_\beta) \) be nets in \( A \) such that \( f_\alpha \to F \) and \( g_\beta \to G \) in weak* topology. One may show that for products \( \square \) and \( \triangleleft \) of \( A^{**} \),

\[
F \square G = w^* - \lim_{\alpha} w^* - \lim_{\beta} f_\alpha g_\beta \quad \text{and} \quad F \triangleleft G = w^* - \lim_{\beta} w^* - \lim_{\alpha} f_\alpha g_\beta.
\]

The Banach space \( A^{**} \) equipped with either the multiplication \( \square \) or the multiplication \( \triangleleft \) forms a Banach algebra. The Banach algebra \( A \) is called Arens regular if two actions \( \square \) and \( \triangleleft \) coincide.

Let \( c_0(H, \omega^{-1}) := \{ f : H \to \mathbb{C} : f \omega^{-1} \in c_0(H) \} \). Note that \( \ell^1(H, \omega) \) is the dual of \( c_0(H, \omega^{-1}) \). Hence, \( \ell^1(H, \omega)^{**} \) can be decomposed as \( \ell^1(H, \omega) \bigoplus c_0(H, \omega^{-1})^\perp \) when \( c_0(H, \omega^{-1})^\perp := \{ F \in \ell^1(H, \omega)^{**} : \langle F, \phi \rangle = 0 \ \text{for all} \ \phi \in c_0(H, \omega^{-1}) \} \). To see this decomposition, let \( F \in \ell^1(H, \omega)^{**} \), it is clear that \( f := F|_{c_0(H, \omega^{-1})} \in \ell^1(H, \omega) \) and consequently \( \Phi := F - f \in c_0(H, \omega^{-1})^\perp \). Therefore, \( F = (f, \Phi) \in \ell^1(H, \omega) \bigoplus c_0(H, \omega^{-1})^\perp \).

**Proposition 4.1** Let \((H, \omega)\) be a weighted hypergroup. Then \( \ell^1(H, \omega) \) is Arens regular if the multiplications \( \square \) and \( \triangleleft \) restricted to \( c_0(H, \omega^{-1})^\perp \) are void.

**Proof.** Now let \( F = (f, \Phi) \) and \( G = (g, \Psi) \) belong to \( \ell^1(H, \omega)^{**} \). First, note that based on the definition of these module actions, presented before, \( F \square \Psi = f \diamond \Psi \) and \( \Phi \square g = \Phi \diamond g \). According to the assumption of the proposition, \( \Phi \square \Psi = \Phi \diamond \Psi = 0 \). Thus

\[
F \square G = (f, \Phi) \square (g, \Psi) = (fg, f \diamond \Psi + \Phi \square g) = (fg, f \diamond \Psi + \Phi \diamond g) = F \diamond G.
\]

Let us define the bounded function \( \Omega_\omega : H \times H \to (0, 1] \) by

\[
\Omega_\omega(x, y) := \frac{\omega(\delta_x * \delta_y)}{\omega(x) \omega(y)} \quad (x, y \in H).
\]

(4.1)

If there is no risk of confusion, we may use \( \Omega \) instead of \( \Omega_\omega \).
For a weighted group \((G, \sigma)\), the Arens regularity of weighted group algebras has been characterized completely; [7, Theorem 8.11] proves that it is equivalent to the 0-clusterness of the function \(\Omega_\sigma\) on \(G \times G\), that is

\[
\lim \lim n \lim m \Omega_\sigma(x_m, y_n) = \lim \lim m \lim n \Omega_\sigma(x_m, y_n) = 0
\]

whenever \((x_m)\) and \((y_n)\) are sequences in \(G\), each consisting of distinct points, and both repeated limits exist. A stronger version of 0-clusterness is called strong 0-clusterness (see [7, Section 3]).

We define strongly 0-cluster functions as presented in [7, Definition 3.6] for the discrete topology spaces.

**Definition 4.2** Let \(X\) and \(Y\) be two sets and \(f\) is a bounded function on \(X \times Y\) into \(C\). Then \(f\) **0-clusters strongly** on \(X \times Y\) if

\[
\lim_{x \to \infty} \limsup_{y \to \infty} f(x, y) = \lim_{y \to \infty} \limsup_{x \to \infty} f(x, y) = 0.
\]

Let us recall the Banach space isomorphism \(\kappa : \ell^1(H, \omega) \to \ell^1(H)\) where \(\kappa(f) = f \omega\) for each \(f \in \ell^1(H, \omega)\). Note that for \(\kappa^{**} : \ell^1(H, \omega)^{**} \to \ell^1(H)^{**}\) and \(\Phi \in c_0(H, \omega)^\perp\), one gets \(\langle \kappa^{**}(\Phi), \phi \rangle = \langle \Phi, \kappa^*(\phi) \rangle\) which is equal to 0 for all \(\phi \in c_0(H)\). Therefore \(\kappa^{**}(\Phi) \in c_0(H)^\perp\). The converse is also true and straightforward to show (which we do not use here so we do not mention).

The following theorem is a generalization of [7, Theorem 8.8]. In the proof we use some techniques of the proof for [21, Theorem 3.16].

**Theorem 4.3** Let \((H, \omega)\) be a weighted hypergroup and let \(\Omega\) **0-cluster strongly** on \(H \times H\). Then \(\Phi \Box \Psi = 0\) and \(\Phi \oint \Psi = 0\) whenever \(\Phi, \Psi \in c_0(H, 1/\omega)^\perp\).

**Proof.** Let us show the theorem for \(\Phi \Box \Psi\); the proof for the other action is similar. Let \(\Phi, \Psi \in c_0(H, 1/\omega)^\perp\). By Goldstine’s theorem, there are nets \((f_\alpha)_\alpha, (g_\beta)_\beta \subseteq \ell^1(H)\) such that \(f_\alpha \to \kappa^{**}(\Phi)\) and \(g_\beta \to \kappa^{**}(\Psi)\) in the weak* topology of \(\ell^1(H)^{**}\) while \(\sup_\alpha \|f_\alpha\|_1 \leq 1\) and \(\sup_\beta \|g_\beta\|_1 \leq 1\). So for each \(\psi \in \ell^\infty(H), \kappa^*(\psi) = \psi \omega \in \ell^\infty(H, 1/\omega)\) and \(\Phi \Box \Psi \in \ell^1(H, \omega)^{**}\); hence,

\[
\langle \psi \omega, \kappa^{**}(\Phi \Box \Psi) \rangle = \langle \kappa^*(\psi), \Phi \Box \Psi \rangle = \lim_\alpha \lim_\beta \langle \psi \omega, \kappa^{-1}(f_\alpha) * \kappa^{-1}(g_\beta) \rangle = \lim_\alpha \lim_\beta \langle \psi \omega, f_\alpha / \omega * g_\beta / \omega \rangle.
\]
Thus
\[ |\langle \psi, \kappa_\beta^*(\Phi \boxtimes \Psi) \rangle| = \lim_{\alpha} \lim_{\beta} |\langle \psi, f_\alpha/\omega * g_\beta/\omega \rangle| \]
\[ = \lim_{\alpha} \lim_{\beta} \left| \sum_{y \in H} \psi(y) \omega(y) \sum_{x,z \in H} \frac{f_\alpha(x) g_\beta(z)}{\omega(x) / \omega(z)} \delta_x \delta_z \right| \]
\[ \leq \limsup_{\alpha} \limsup_{\beta} \sum_{x,z \in H} \left| \frac{f_\alpha(x) g_\beta(z)}{\omega(x) / \omega(z)} \right| \sum_{y \in H} \psi(y) \omega(y) \delta_x \delta_z \]
\[ \leq \limsup_{\alpha} \limsup_{\beta} \|\psi\|_{\ell^\infty(H)} \sum_{x,z \in H} \|f_\alpha(x)\| g_\beta(z) \sum_{y \in H} \frac{\omega(y)}{\omega(x) / \omega(z)} \delta_x \delta_z \]
\[ = \limsup_{\alpha} \limsup_{\beta} \|\psi\|_{\ell^\infty(H)} \sum_{x,z \in H} |f_\alpha(x)||g_\beta(z)| \Omega(x,z). \]

For a given \( \epsilon > 0 \), since by the hypothesis \( \lim_x \limsup_z \Omega(x,z) = 0 \), there is a finite set \( A \subseteq H \) such that for each \( x \in A^c := H \setminus A \) there exists a finite set \( B_x \subseteq H \) such that for each \( z \in B_x^c := H \setminus B, |\Omega(x,z)| \leq \epsilon. \) First note that
\[ \limsup_{\alpha} \limsup_{\beta} \sum_{x \in A^c} \sum_{z \in B_x^c} |f_\alpha(x)||g_\beta(z)| \Omega(x,z) \leq \limsup_{\alpha} \limsup_{\beta} \epsilon \|f_\alpha\|_1 \|g_\beta\|_1 \leq \epsilon. \]

Also according to our assumption about \( \Phi \) and \( \Psi \) and since for each \( x \in H, \delta_x \in c_0(H,1/\omega) \), \( \lim_{\alpha} f_\alpha(x) = 0 \) and \( \lim_{\beta} g_\beta(x) = 0. \) So for the given \( \epsilon > 0 \), there is \( \alpha_0 \) such that for all \( \alpha \prec \alpha_0, |f_\alpha(x)| < \epsilon/|A| \) for all \( x \in A. \) Moreover, for each \( x \in A^c \) there is some \( \beta^*_0 \) such that for all \( \beta \) where \( \beta^*_0 \prec \beta, |g_\beta(z)| < \epsilon/|B_x| \) for all \( z \in B_x \) (this is possible since \( A \) and \( B_x \) are finite).

Therefore, since \( |\Omega(x,z)| \leq 1, \)
\[ \limsup_{\alpha} \limsup_{\beta} \sum_{x \in A^c} \sum_{z \in H} |f_\alpha(x)||g_\beta(z)| \Omega(x,z) \leq \limsup_{\beta} \epsilon \|g_\beta\|_1 = \epsilon \]
and
\[ \limsup_{\alpha} \limsup_{\beta} \sum_{x \in A^c} \sum_{z \in B_x} |f_\alpha(x)||g_\beta(z)| \Omega(x,z) \leq \limsup_{\alpha} \sum_{x \in A^c} |f_\alpha(x)| \limsup_{\beta} \sum_{z \in B_x} |g_\beta(z)| \]
\[ \leq \limsup_{\alpha} \epsilon \|f_\alpha\|_1 = \epsilon. \]

But
\[ \sum_{x,z \in H} |f_\alpha(x)||g_\beta(z)| \Omega(x,z) = \sum_{x \in A^c} \sum_{z \in B_x} |f_\alpha(x)||g_\beta(z)| \Omega(x,z) \]
\[ + \sum_{x \in A,z \in H} |f_\alpha(x)||g_\beta(z)| \Omega(x,z) \]
\[ + \sum_{x \in A^c,z \in B_x} |f_\alpha(x)||g_\beta(z)| \Omega(x,z), \]
and so, one gets that $|\langle \psi \Omega, \kappa^{**}(\Phi \Box \Psi) \rangle| \leq 3\epsilon \|\psi\|_\infty$. But since $\epsilon > 0$ was arbitrary, this implies that $\Phi \Box \Psi = 0$.

\section*{Theorem 4.4} Let $(H, \omega)$ be a discrete weighted hypergroup and consider the following conditions:

(1) $\Omega$ 0-clusters strongly on $H \times H$.

(2) $\Phi \Box \Psi = \Phi \cdot \Psi = 0$ for all $\Phi, \Psi \in c_0(H, 1/\omega)^\perp$.

(3) $\ell^1(H, \omega)$ is Arens regular.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3).

\textbf{Proof.} (1) $\Rightarrow$ (2) by Theorem 4.3. (2) $\Rightarrow$ (3) is implied from Proposition 4.1.

\section*{Remark 4.5} Since in hypergroups, the cancellation does not necessarily exist, the argument of [6, Theorem 1] cannot be applied to show (3) implies (1).

\section*{Example 4.6} Let $N_0$ be equipped with Chebyshev polynomial hypergroup structure of the first type and $\sigma_f$ be the group weight defined in Example 3.26 for an increasing function $f$. One can easily check that if

$$\lim_{x \to \infty} \limsup_{y \to \infty} \frac{\omega(x) \omega(y)}{\omega(x) \omega(y)} = 0,$$

then $\Omega_{\sigma_f}$ 0-clusters strongly on $N_0 \times N_0$; hence, $\ell^1(N_0, \sigma_f)$ is Arens regular. Indeed, by Remark 3.27, $Z_\pm A(T, \sigma_f)$ is Arens regular. But note that $A(T, \sigma_f)$ (which is isomorphic to $\ell^1(Z, \sigma_f)$ through the Fourier transform) is not Arens regular, as $\Omega_{\sigma_f}$ does not 0-cluster strongly on $Z \times Z$ (see [7, Theorem 8.11]).

\section*{Corollary 4.7} Let $(H, \omega)$ be a weighted discrete hypergroup such that $\omega$ is a weakly additive weight. If $1/\omega \in c_0(H)$, then $\ell^1(H, \omega)$ is Arens regular.

\textbf{Proof.} We have

$$\lim_{x \to \infty} \limsup_{y \to \infty} \frac{\omega(x) \omega(y)}{\omega(x) \omega(y)} \leq \limsup_{y \to \infty} \sup_{x \to \infty} C \frac{\omega(x) + \omega(y)}{\omega(x) \omega(y)} = C \limsup_{x \to \infty} \sup_{y \to \infty} \frac{1}{\omega(x)} + \frac{1}{\omega(y)} = 0.$$

Therefore $\Omega$ 0-clusters strongly on $H \times H$ and hence $\ell^1(H, \omega)$ is Arens regular by Theorem 4.4.
Corollary 4.8 Let $H$ be an infinite finitely generated hypergroup. Then for each polynomial weight $\omega_{\beta}$ ($\beta > 0$) on $H$ defined in Definition 3.3, $\ell^1(H, \omega_{\beta})$ is Arens regular.

Proof. Let $F$ be a finite generator of the hypergroup $H$ containing the identity of $H$ rendering the central weight $\omega_{\beta}$. Recall that $\omega_{\beta}$ is weakly additive with constant $C = \min\{1, 2^{\beta-1}\}$. Moreover, for each $N \in \mathbb{N}$, $F^N$ is a finite subset of $H$ such that for each $x \in H \setminus F^N$, $\tau_F(x) \geq N$; hence, $\omega_{\beta}(x) = (1 + \tau_F(x))^\beta \geq (1 + N)^\beta$. Hence $1/\omega_{\beta} \in c_0(H)$ and therefore $\ell^1(H, \omega_{\beta})$ is Arens regular, by Corollary 4.7. □

Remark 4.9 Every infinite finitely generated hypergroup $H$ admits a weight for which the corresponding weighted algebra is Arens regular. On the other hand, an argument similar to [6, Corollary 1] may apply to show that for every uncountable discrete hypergroup $H$, $H$ does not have any weight $\omega$ which 0-clusters strongly.

Example 4.10 Let $\mathbb{N}_0$ be a polynomial hypergroup structure. By Example 3.24, the polynomial weight $\omega_{\beta}$ on $\mathbb{N}_0$ is of the form $\omega_{\beta}(n) = (1 + n)^\beta$ for every $n \in \mathbb{N}_0$. Therefore by Corollary 4.8, $\ell^1(\mathbb{N}_0, \omega_{\beta})$ is Arens regular. In particular $\omega_{\beta}(\pi_k) = (\ell + 1)^\beta$ is a polynomial weight on the hypergroup $\widehat{\text{SU}(2)}$. Therefore $\ell^1(\widehat{\text{SU}(2)}, \omega_{\beta})$ is Arens regular.

Example 4.11 Let $\sigma$ and $\omega_{\sigma}$ be as defined in Example 3.22 for some $\beta \geq 0$. As $\omega_{\sigma}$ is equivalent to the weight $\omega_{\beta}$, presented in Example 4.10, $\Omega_{\omega_{\sigma}}$ also 0-clusters strongly on $\widehat{\text{SU}(2)} \times \widehat{\text{SU}(2)}$. Therefore, $\ell^1(\widehat{\text{SU}(2)}, \omega_{\sigma})$ is Arens regular. On the other hand, $A(\text{SU}(2), \omega_{\sigma})$ is not Arens regular if $\beta > 0$. To observe the later fact, first note that by applying [6], we obtain that $\ell^1(\mathbb{Z}, \sigma)$ is not Arens regular, because, $\Omega_{\sigma}$ does not 0-cluster strongly. Therefore, $A(\mathbb{T}, \sigma)$ is not Arens regular. By considering $VN(\mathbb{T}, \sigma)$ and $VN(\text{SU}(2), \omega_{\sigma})$, one may verify that $VN(\mathbb{T}, \sigma)$ embeds *-weakly in $VN(\text{SU}(2), \omega_{\sigma})$ (the details of this embedding will appear in a manuscript by the second named author and et al). Hence, $A(\mathbb{T}, \sigma)$ is a quotient of $A(\text{SU}(2), \omega_{\sigma})$ and consequently $A(\text{SU}(2), \omega_{\sigma})$ is not Arens regular.

In the following, we generalize the result of Example 4.10 to all $\text{SU}(n)$, the group of all $n \times n$ special unitary matrices on $\mathbb{C}$, based on a recent study on the representation theory of $\text{SU}(n)$. As an example for Lemma 3.17, $(\widehat{\text{SU}(n)}, \omega_{\beta})$ is a discrete commutative hypergroup where $\omega_{\beta}(\pi) = d^2_{\pi}$ for some $\beta \geq 0$. It is known that there is a one-to-one correspondence between $\widehat{\text{SU}(n)}$ and $n$-tuples $(\pi_1, \ldots, \pi_n) \in N^0_n$ such that $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_{n-1} \geq \pi_n = 0$. This presentation of the representation theory of $\text{SU}(n)$ is called dominant weight. Using this presentation, we have the following formula which gives the dimension of each representation by the formula

$$d_n = \prod_{1 \leq i < j \leq n} \frac{\pi_i - \pi_j + j - i}{j - i}$$

(4.2)
where $\pi$ is the representation corresponding to $(\pi_1, \ldots, \pi_n)$. Suppose that $\pi, \nu, \mu$ are representations corresponding to $(\pi_1, \ldots, \pi_n)$, $(\nu_1, \ldots, \nu_n)$, and $(\mu_1, \ldots, \mu_n)$, respectively, such that $\pi \in \text{supp}(\delta_\nu * \delta_\mu)$. Collins, Lee, and Śniady showed in [5, Corollary 1.2], for each $n \in \mathbb{N}$, there exists some $C_n > 0$ such that

$$\frac{d_\pi}{d_\mu d_\nu} \leq 2C_n \left( \frac{1}{1 + \mu_1} + \frac{1}{1 + \nu_1} \right). \quad (4.3)$$

Applying (4.3), we prove that $\omega_\beta$ 0-clusters on $\widehat{SU}(n)$.

**Proposition 4.12** For every $\beta > 0$, $\ell^1(\widehat{SU}(n), \omega_\beta)$ is Arens regular.

**Proof.** Let $(\mu_m)_{m \in \mathbb{N}}$ and $(\nu_k)_{k \in \mathbb{N}}$ be two arbitrary sequences of distinct elements of $\widehat{SU}(n)$. Since, the elements of $(\mu_m)_{m \in \mathbb{N}}$, $(\nu_k)_{k \in \mathbb{N}}$ are distinct, $\lim_{m \to \infty} \mu_1^{(m)} = \infty \ (\lim_{k \to \infty} \nu_1^{(k)} = \infty)$ where $\mu_m = (\mu_1^{(m)}, \ldots, \mu_n^{(m)}) \ (\nu_k = (\nu_1^{(k)}, \ldots, \nu_n^{(k)}))$. For each arbitrary pair $(m, k) \in \mathbb{N} \times \mathbb{N}$, if $\pi \in \text{supp}(\delta_{\mu_m} * \delta_{\nu_k})$, we have

$$d_\pi \leq 2C_n \left( \frac{1}{1 + \mu_1^{(m)}} + \frac{1}{1 + \nu_1^{(k)}} \right) d_{\mu_m} d_{\nu_k}.$$

Hence

$$\omega_\beta(\pi) \leq (2C_n)^\beta \left( \frac{1}{1 + \mu_1^{(m)}} + \frac{1}{1 + \nu_1^{(k)}} \right)^\beta \omega_\beta(\mu_m) \omega_\beta(\nu_k).$$

Therefore

$$\omega_\beta(\delta_{\mu_m} * \delta_{\nu_k}) = \sum_{\pi \in \widehat{SU}(n)} \delta_{\mu_m} * \delta_{\nu_k}(\pi) \omega_\beta(\pi) \leq (2C_n)^\beta \left( \frac{1}{1 + \mu_1^{(m)}} + \frac{1}{1 + \nu_1^{(k)}} \right)^\beta \omega_\beta(\mu_m) \omega_\beta(\nu_k).$$

Or equivalently

$$\Omega_\beta(\mu_m, \nu_k) := \frac{\omega_\beta(\delta_{\mu_m} * \delta_{\nu_k})}{\omega_\beta(\mu_m) \omega_\beta(\nu_k)} \leq (2C_n)^\beta \left( \frac{1}{1 + \mu_1^{(m)}} + \frac{1}{1 + \nu_1^{(k)}} \right)^\beta.$$

Hence,

$$\lim_{m \to \infty} \limsup_{k \to \infty} \Omega_\beta(\mu_m, \nu_k) = \lim_{k \to \infty} \limsup_{m \to \infty} \Omega_\beta(\mu_m, \nu_k) = 0.$$

Since $\widehat{SU}(n)$ is countable, this argument implies that $\Omega_\beta$ 0-clusters strongly on $\widehat{SU}(n) \times \widehat{SU}(n)$ and, by Theorem 4.4, $\ell^1(\widehat{SU}(n), \omega_\beta)$ is Arens regular.\[\square\]

**Example 4.13** Let $\text{Aff}_p := \mathbb{Z}_p \rtimes \mathbb{Z}_p^*$ be the affine group generated with $\mathbb{Z}_p(:= \mathbb{Z}/p\mathbb{Z})$ for a prime number $p$, when for each $(a, b), (a', b') \in \text{Aff}_p$ we define $(a, b)(a', b') = (a + ba', bb')$. The following table presents the structure of conjugacy classes of $\text{Aff}_p$. 

---

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| Conjugacy classes | $C_{(0,1)}$ | $C_{(1,1)}$ | $C_{(0,y)}$ for $y \in 2, \ldots, p-1$ |
|-------------------|-------------|--------------|---------------------------------|
| Number            | 1           | 1            | $p-2$                           |
| Size              | 1           | $p-1$        | $p$                             |

Therefore, for each three conjugacy classes say $C_1, C_2, D \in \text{Conj}(\text{Aff}_p)$, $|D| \leq (|C_1| + |C_2|)$ if $D \subseteq C_1 C_2$ for each prime number $p \geq 3$. In other words, the weight $\omega_p(C) := |C|$, defined in Example 3.8, forms a central additive weight on $\text{Aff}_p$. Let $\mathcal{P}$ be the set of all odd prime numbers. Define $G = \bigoplus_{p \in \mathcal{P}} \text{Aff}_p$ and $\omega_\alpha$ to be the weight defined in Example 3.9 for some $\alpha > 0$. For $C \in \text{supp}(\delta_D * \delta_E)$ for $C, D, E \in \text{Conj}(G)$, we have

$$\omega_\alpha(C) = (1 + |C_{i_1}| + \cdots + |C_{i_n}|)^\alpha$$

$$\leq (1 + |D_{i_1}| + \cdots + |D_{i_n}| + 1 + |E_{i_1}| + \cdots + |E_{i_n}|)^\alpha$$

$$\leq M ((1 + |D_{i_1}| + \cdots + |D_{i_n}|)^\alpha + (1 + |E_{i_1}| + \cdots + |E_{i_n}|)^\alpha)$$

$$\leq C(\omega_\alpha(D) + \omega_\alpha(E))$$

for $M = \min\{1, 2^{\alpha-1}\}$, where $i_j \in I_C$ for $I_C$ defined in Example 3.9. Hence, $\omega_\alpha$ is a central weakly additive weight. Moreover, since for each $p \in \mathcal{P}$, $|C| \geq p-1$ for any non-trivial element $C \in \text{Conj}(\text{Aff}_p)$, $\lim_n \omega_\alpha(C_n) = \infty$ for each sequence of distinct elements of $\text{Conj}(G)$. Therefore by Corollary 4.7, $\ell^1(\text{Conj}(G), \omega_\alpha)$ is Arens regular.

**Example 4.14** Let $SL(2, 2^n)$ denote the finite group of special linear matrices over the field $\mathbb{F}_{2^n}$ with cardinal $2^n$, for given $n \in \mathbb{N}$. The character table of $SL(2, 2^n)$ can be found in [1]. In the following we just present the part of the character table related to the conjugacy classes of $SL(2, 2^n)$.

| Conjugacy classes | $e$ | $N$ | $c_3(x)$ | $c_4(z)$ |
|-------------------|-----|-----|----------|----------|
| Number            | 1   | 1   | $(2^n-2)/2$ | $2^{n-1}$ |
| Size              | 1   | $2^{2n} - 1$ | $2^{2n} + 2^n$ | $2^{2n} - 2^n$ |

As a direct result of the above table, for each three conjugacy classes say $C_1, C_2, D \in \text{Conj}(SL(2, 2^n))$, $|D| \leq 2(|C_1| + |C_2|)$ if $D \subseteq C_1 C_2$ for all $n$. Let us define the FC group $G$ to be the RDPF of $(SL(2, 2^n))_{n \in \mathbb{N}}$ i.e. $G := \bigoplus_{n=1}^{\infty} SL(2, 2^n)$. Therefore, similar to the previous example, for the hypergroup $\text{Conj}(G)$, the weight $\omega_\alpha$, as defined in Example 3.9, is a weakly additive weight with the constant $M = 2^{\alpha} \min\{1, 2^{\alpha-1}\}$. Moreover, since $\lim_{C \to \infty} \omega_\alpha(C) = \infty$, $\ell^1(\text{Conj}(G), \omega_\alpha)$ is Arens regular, by Corollary 4.7.

**Remark 4.15** Let $\omega$ be a central weight on $\text{Conj}(G)$ for some FC group $G$. Then there is a group weight $\sigma_\omega$, as defined in Remark 3.7, such that $\ell^1(\text{Conj}(G), \omega)$ is isometrically Banach algebra isomorphic to $Z\ell^1(G, \sigma_\omega)$. So one may also use the embedding $\ell^1(\text{Conj}(G), \omega) \hookrightarrow \ell^1(G, \sigma_\omega)$ to study Examples 4.13 and 4.14 by applying the theorems which are characterizing weighted group algebras.
Remark 4.16 Let $G$ be an FC group and $\sigma$ a group weight on $G$. We defined $\omega_\sigma$, the derived weight on $\text{Conj}(G)$ from $\sigma$ in Definition 3.5. Recall that in this case $Z\ell^1(G,\sigma)$ is isomorphic to the Banach algebra $\ell^1(\text{Conj}(G),\omega_\sigma)$, by Corollary 3.6. If $N$ is a normal subgroup of $G$, we defined a quotient mapping $T_{\omega_\sigma} : \ell^1(\text{Conj}(G),\omega_\sigma) \rightarrow \ell^1(\text{Conj}(G/N),\tilde{\omega}_\sigma)$ in Proposition 3.13 where $\tilde{\omega}_\sigma(C_{xN}) = \inf \{\omega_\sigma(C_{xy}) : y \in N \}$ ($C_{xN} \in \text{Conj}(G/N)$). Let us note that for an Arens regular Banach algebra $A$, every quotient algebra $A/I$ where $I$ is a closed ideal of $A$ is Arens regular as well (see [7, Corollary 3.15]). Therefore, if $\ell^1(\text{Conj}(G),\omega_\sigma)$ is Arens regular, for every normal subgroup $N$, $\ell^1(\text{Conj}(G/N),\tilde{\omega}_\sigma)$, which is isomorphic to $\ell^1(\text{Conj}(G),\omega_\sigma)/\text{Ker}(T_{\omega_\sigma})$, is Arens regular.

In the final result of this section, we apply some techniques of [6] to show that for restricted direct product of hypergroups product weights fail to give Arens regular algebras.

Proposition 4.17 Let $(H_i)_{i \in I}$ be an infinite family of non-trivial hypergroups and for each $i \in I$, $\omega_i$ is a weight on $H_i$ such that $\omega_i(e_{H_i}) = 1$ for all except finitely many $i \in I$. Let $H = \bigoplus_{i \in I} H_i$ and $\omega = \prod_{i \in I} \omega_i$. Then $\ell^1(H, \omega)$ is not Arens regular.

Proof. Since $I$ is infinite, suppose that $N_0 \times N_0 \subseteq I$. Define $v_n = (x_i)_{i \in I}$ where $x_i = e_{H_i}$ for all $i \in I \setminus (n,0)$ and $x_{(n,0)}$ be a non-identity element of $H_{(n,0)}$ for all $n \in \mathbb{N}$. Similarly define $u_m = (x_i)_{i \in I}$ where $x_i = e_{H_i}$ for all $i \in I \setminus (0,m)$ and $x_{(0,m)}$ be a non-identity element of $H_{(0,m)}$ for all $m \in \mathbb{N}$. Note that for each pair of elements $(n,m) \in \mathbb{N} \times \mathbb{N}$, $\text{supp}(\delta_{v_n} * \delta_{u_m})$ forms a singleton in $H$; moreover, $\omega(\delta_{v_n} * \delta_{u_m}) = \omega(v_n) \omega(u_m)$. Hence, $(\delta_{v_n} * \delta_{u_m})_{(n,m) \in \mathbb{N} \times \mathbb{N}}$ forms a sequence of distinct elements in $\ell^1(H)$.

Let us define $f_n = \delta_{v_n}$ and $g_m = \delta_{u_m}$ for all $n, m \in \mathbb{N}$. Suppose that $A := \{(v_n, u_m) : n > m\}$ and $\phi \in \ell^\infty(H)$ is the characteristic function of the subset $A$. Clearly, $\kappa^{-1}(f_n) = \omega^{-1} f_n$ and $\kappa^{-1}(g_m) = \omega^{-1} g_m$ belong to $\ell^1(H, \omega)$ for all $n, m$ and $\kappa^*(\phi) = \omega \phi \phi \in \ell^\infty(H, \omega^{-1})$, for the Banach space isomorphism $\kappa : \ell^1(H, \omega) \rightarrow \ell^1(H)$ where $\kappa(f) = f \omega$ for each $f \in \ell^1(H, \omega)$. Note that

$$
\langle \omega^{-1} f_n * \omega^{-1} g_m, \kappa^*(\phi) \rangle = \langle \omega^{-1} f_n * \omega^{-1} g_m, \omega \phi \rangle = \sum_{t \in H} (\omega^{-1} f_n * \omega^{-1} g_m)(t) \omega(t) \phi(t) = \frac{\omega(v_n * u_m)}{\omega(v_n) \omega(u_m)} \phi(\delta_{v_n} * \delta_{u_m}) = \phi(\delta_{v_n} * \delta_{u_m}) = \begin{cases} 
1 & \text{if } n > m \\
0 & \text{if } n \leq m
\end{cases}
$$

Let us recall that for each $n$ and $m$, $\|f_n\|_{\ell^1(H, \omega)} = 1$ and $\|g_m\|_{\ell^1(H, \omega)} = 1$. So $(f_n)_{n \in \mathbb{N}}$ and $(g_m)_{m \in \mathbb{N}}$, as two nets in the unit ball of $\ell^1(H, \omega)^{**}$, have two subnets $(f_\alpha)_\alpha$ and $(g_\beta)_\beta$ such that $f_\alpha$ and $g_\beta$ converge weakly* to some $F$ and $G$ in $\ell^1(H, \omega)^{**}$, respectively. Note that for the
specific element $\phi$, defined above, $\langle F \Box G, \phi \rangle = 0$ while $\langle F \diamond G, \phi \rangle = 1$. Hence $F \Box G \neq F \diamond G$ and consequently $\ell^1(H, \omega)$ is not Arens regular.

**Example 4.18** Let $G$ be the restricted direct product of an infinite family of finite groups $(G_i)_i$. By Example 2.1, $\text{Conj}(G) = \bigoplus_{i \in I} \text{Conj}(G_i)$. Also for $\omega(C_x) = \prod_{i \mid C_x} \omega_i$, $\omega$ is a weight such that $\ell^1(\text{Conj}(G), \omega)$ is not Arens regular, by Proposition 4.17.

5 Injectivity of weighted hypergroup algebras and isomorphism to operator algebras

Let $(H, \omega)$ be a weighted discrete hypergroup. In this section, we study the existence of an algebra isomorphism from $\ell^1(H, \omega)$ onto an operator algebra. A Banach algebra $\mathcal{A}$ is called an operator algebra if there is a Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is a closed subalgebra of $B(\mathcal{H})$. Let $\mathcal{A}$ be a Banach algebra and $m : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is the bilinear mapping $m(f, g) = fg$. Then $\mathcal{A}$ is called injective, if $m$ has a bounded extension from the injective tensor product $\mathcal{A} \otimes \epsilon \mathcal{A}$ into $\mathcal{A}$, where $\otimes \epsilon$ is the injective tensor product. In this case, we denote the norm of $m$ by $\|m\|_{\epsilon}$. [20, Corollary 2.2.] proves that if a Banach algebra $\mathcal{A}$ is injective then it is isomorphic to an operator algebra.

Injectiveness of weighted group algebras has been studied before. Initially Varopoulos, in [25], studied the group $\mathbb{Z}$ equipped with the weight $\sigma_\alpha(n) = (1 + |n|^\alpha)$ for all $\alpha \geq 0$. This study looked at injectiveness of $\ell^1(\mathbb{Z}, \sigma_\alpha)$. He showed that $\ell^1(\mathbb{Z}, \sigma_\alpha)$ is injective if and only if $\alpha > 1/2$. The manuscript [20], which studied the injectiveness question for a wider family of weighted group algebras, developed a machinery applying Littlewood multipliers. In particular, it partially extended Varopoulos’s result to finitely generated groups with polynomial growth. Following the structure of [20], in this section, we study the injective property of weighted hypergroup algebras.

In this section, for Banach spaces $A$ and $B$, $A \otimes \gamma B$ and respectively $A \otimes \epsilon B$ denote respectively the projective and injective tensor products of $A$ and $B$.

We know that $\ell^1(H, \omega) \otimes \gamma \ell^1(H, \omega)$ is isometrically isomorphic to $\ell^1(H \times H, \omega \times \omega)$. Moreover, $\ell^1(H \times H, \omega \times \omega)^*$ is nothing but $\ell^\infty(H \times H, \omega^{-1} \times \omega^{-1})$. Since the injective tensor norm is minimal among all cross-norm Banach space tensor norms, the identity map $\iota : \ell^1(H) \times \ell^1(H) \to \ell^1(H) \times \ell^1(H)$ may extend to a contractive mapping

$$\iota : \ell^1(H) \otimes \gamma \ell^1(H) \to \ell^1(H) \otimes \epsilon \ell^1(H).$$

Since, $\iota$ has a dense range,

$$\iota^* : (\ell^1(H) \otimes \epsilon \ell^1(H))^* \to (\ell^1(H) \otimes \gamma \ell^1(H))^* = \ell^\infty(H \times H)$$  (5.1)
is an injective mapping. Therefore, applying $\iota^*$, one may embed $(\ell^1(H) \otimes_e \ell^1(H))^* = \ell^\infty(H \times H)$, as a linear subspace of $\ell^\infty(H \times H)$.

Let $H$ be a discrete hypergroup. We define Littlewood multipliers of $H$ to be the set of all functions $f : H \times H \to \mathbb{C}$ such that there exist functions $f_1, f_2 : H \times H \to \mathbb{C}$ where $f(x, y) = f_1(x, y) + f_2(x, y)$ for $x, y \in G$ such that

$$\sup_{y \in H} \sum_{x \in H} |f_1(x, y)|^2 < \infty \quad \text{and} \quad \sup_{x \in H} \sum_{y \in H} |f_2(x, y)|^2 < \infty.$$  

We denote the set of all Littlewood multipliers by $T^2(H)$ and define the norm $\| \cdot \|_{T^2(H)}$ by

$$\|f\|_{T^2(H)} = \inf \left\{ \sup_{y \in H} \left( \sum_{x \in H} |f_1(x, y)|^2 \right)^{1/2} + \sup_{x \in H} \left( \sum_{y \in H} |f_2(x, y)|^2 \right)^{1/2} \right\},$$

where the infimum is taken over all possible decompositions $f_1, f_2$. Note that for each $f \in T^2(H)$ and a decomposition $f_1, f_2$ of that,

$$\|f\|_{\ell^\infty(H \times H)} = \sup_{x, y \in H} |f(x, y)| \leq \sup_{x, y \in H} |f_1(x, y)| + \sup_{x, y \in H} |f_2(x, y)|$$

$$\leq \sup_{y \in H} \left( \sum_{x \in H} |f_1(x, y)|^2 \right)^{1/2} + \sup_{x \in H} \left( \sum_{y \in H} |f_2(x, y)|^2 \right)^{1/2} < \infty,$$

since for discrete space $H$, $\ell^2(H) \subseteq \ell^\infty(H)$ and $\| \cdot \|_{\ell^\infty} \leq \| \cdot \|_2$. Since $f_1, f_2$, in the previous equation are arbitrary, $\|f\|_{\ell^\infty(H \times H)} \leq \|f\|_{T^2(H)}$. Hence $T^2(H) \subseteq \ell^\infty(H \times H)$. Furthermore, for each $\phi \in \ell^\infty(H \times H)$ and $f \in T^2(H)$, $f\phi \in T^2(H)$ and $\|f\phi\|_{T^2(H)} \leq \|f\|_{T^2(H)} \|\phi\|_{\ell^\infty}.$

The following theorem is the hypergroup version of [20, Theorem 2.7]. Since the proof is very similar to the group proof, we omit it here.

**Theorem 5.1** Let $I : T^2(H) \to (\ell^1(H) \otimes_e \ell^1(H))^* = \ell^\infty(H \times H)$ be the mapping which takes every element of $T^2(H)$ to itself as a bounded function on $H \times H$. Then $I(T^2(H)) \subseteq \iota^*((\ell^1(H) \otimes_e \ell^1(H))^*)$ for the mapping $\iota^*$ defined in (5.1). Moreover, $J := \iota^* \circ I : T^2(H) \to (\ell^1(H) \otimes_e \ell^1(H))^*$ is bounded and $\|J\| \leq K_\mathcal{G}$ where $K_\mathcal{G}$ is Grothendieck's constant.

From now on, we identify $(\ell^1(H) \otimes_e \ell^1(H))^*$ with its image through the mapping $\iota^*$; hence, $J$ is the identity mapping which takes $T^2(H)$ into $(\ell^1(H) \otimes_e \ell^1(H))^*$. We present our first main result of this section. This is also a generalization of [20, Theorem 3.1].

**Theorem 5.2** Let $H$ be a discrete hypergroup and $\omega$ is a weight on $H$ such that $\Omega$, defined in (4.1), belongs to $T^2(H)$. Then $\ell^1(H, \omega)$ is injective. Moreover, for $m$ as defined before, $\|m\|_e \leq K_\mathcal{G}\|\Omega\|_{T^2(H)}$.  

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Proof. Let $\Gamma_\omega : \ell^1(H \times H, \omega \times \omega) \to \ell^1(H, \omega)$ such that $\Gamma_\omega(f \otimes g) := f * g$ for $f, g \in \ell^1(H, \omega)$. The adjoint of $\Gamma_\omega$, $\Gamma_\omega^*$, can be characterized as follows.

$$\Gamma_\omega^*(\phi)(x, y) = \langle \Gamma_\omega^*(\phi), \delta_x \otimes \delta_y \rangle = \langle \phi, \Gamma_\omega(\delta_x \otimes \delta_y) \rangle = \langle \phi, \delta_x * \delta_y \rangle$$

for all $\phi \in \ell^\infty(H, \omega^{-1})$ and $x, y \in H$. Now we define $L$ from $\ell^\infty(H)$ to $\ell^\infty(H \times H)$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\ell^\infty(H, \omega^{-1}) & \xrightarrow{\Gamma_\omega^*} & \ell^\infty(H \times H, \omega^{-1} \times \omega^{-1}) \\
\downarrow P & & \downarrow R \\
\ell^\infty(H) & \xrightarrow{L} & \ell^\infty(H \times H)
\end{array}
$$

where $P(\varphi)(x) = \varphi(x)\omega(x)$ for $\varphi \in \ell^\infty(H)$ and $R(\phi)(x, y) = \phi(x, y)\omega^{-1}(x)\omega^{-1}(y)$ for $\phi \in \ell^\infty(H \times H, \omega^{-1} \times \omega^{-1})$ and $x, y \in H$. Hence, one gets

$$L(\varphi)(x, y) = R(\Gamma_\omega^* \circ P(\varphi))(x, y) = \frac{(\Gamma_\omega^* \circ P(\varphi))(x, y)}{\omega(x)\omega(y)} = \frac{\Gamma_\omega^*(\varphi \omega)(x, y)}{\omega(x)\omega(y)} = \frac{\langle \varphi \omega, \delta_x * \delta_y \rangle}{\omega(x)\omega(y)} = \sum_{t \in H} \delta_x * \delta_y(t) \frac{\omega(t)}{\omega(x)\omega(y)} \varphi(t).$$

for all $\varphi \in \ell^\infty(H)$. Hence,

$$\left| \sum_{t \in H} \delta_x * \delta_y(t) \frac{\omega(t)}{\omega(x)\omega(y)} \varphi(t) \right| \leq \sum_{t \in H} \delta_x * \delta_y(t) \frac{\omega(t)}{\omega(x)\omega(y)} |\varphi(t)| \leq \|\varphi\|_\infty \Omega(x, y)$$

So there is a function $v_\varphi : H \times H \to \mathbb{C}$ such that

$$\frac{\langle \delta_x * \delta_y, \omega \varphi \rangle}{\omega(x)\omega(y)} = v_\varphi(x, y)\|\varphi\|_\infty \Omega(x, y)$$

and $\|v_\varphi\|_\infty \leq 1$. Therefore $L(\varphi) = \Lambda(\varphi)\Omega$ where $\Lambda(\varphi)(x, y) := v_\varphi(x, y)\|\varphi\|_\infty$ for all $\varphi \in \ell^\infty(H)$. Since $\Omega$ belongs to $T^2(H) \otimes \ell^1(H)$ and $T^2(H)$ is an $\ell^\infty(H \times H)$-module, $L(\varphi) \in T^2(H)$ and $\|L(\varphi)\|_{T^2(H)} \leq \|\varphi\|_\infty \|\Omega\|_{T^2(H)}$. Therefore $L(\ell^\infty(H)) \subseteq T^2(H) \subseteq (\ell^1(H) \otimes \ell^1(H))^*$. 

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In this case, using the following diagram with \( A = R^{-1}(\ell^1(H) \otimes \ell^1(H))^* \),

\[
\begin{array}{ccccc}
\ell^\infty(H,\omega^{-1}) & \xrightarrow{\Gamma_\omega} & A & \xrightarrow{i} & \ell^\infty(H \times H,\omega^{-1} \times \omega^{-1}) \\
P \downarrow & & R|_\nu \downarrow & & R \downarrow \\
\ell^\infty(H) & \xrightarrow{L} & (\ell^1(H) \otimes \ell^1(H))^* & \xrightarrow{i} & \ell^\infty(H \times H)
\end{array}
\]

One can easily verify that \( A = (\ell^1(H,\omega) \otimes \ell^1(H,\omega))^* \). So, we have shown that \( \Gamma^* \) is a map projecting \( \ell^\infty(H) \) into \( (\ell^1(H) \otimes \ell^1(H))^* \) as a subset of \( \ell^\infty(H \times H) \). we see that \( \Gamma_\omega^* \) is a map projecting \( \ell^\infty(H,\omega^{-1}) \) into \( (\ell^1(H,\omega) \otimes \ell^1(H,\omega))^* \). Hence, \( \Gamma_\omega^* = m^* \), where

\[
m : \ell^1(H,\omega) \otimes \ell^1(H,\omega) \to \ell^1(H,\omega)
\]

and therefore \( m \) is bounded while \( \|m\| = \|\Gamma_\omega\| = \|RT_\omega\| = \|L\| \). Moreover,

\[
\|L(\varphi)\|_{(\ell^1(H) \otimes \ell^1(H))^*} \leq \|J\| \|\Gamma^*(\varphi)\|_{T^2(H)} \leq K_{\Phi} \|\Omega\|_{T^2(H)} \|\Lambda(\varphi)\|_{\ell^\infty(H \times H)}
\]

\[
\leq K_{\Phi} \|\Omega\|_{T^2(H)} \|\varphi\|_{\ell^\infty(H)}
\]

for all \( \varphi \in \ell^\infty(H) \). Consequently, \( \|m\|_\epsilon \leq K_{\Phi} \|\Omega\|_{T^2(H)} \). \( \square \)

**Example 5.3** Let \( \omega_\beta \) be the dimension weight defined on \( \widehat{SU(n)} \) in Lemma 3.17. As we have shown in the proof of Proposition 4.12, for the polynomial weight \( \omega_\beta, \beta \geq 0, \) and \( \mu, \nu \in \widehat{SU(n)}, \)

\[
\Omega_\beta(\mu, \nu) \leq (2C_n)^\beta \left( \frac{1}{1 + \mu_1} + \frac{1}{1 + \nu_1} \right)^\beta \leq A_\beta (2C_n)^\beta \left( \frac{1}{1 + \mu_1} + \frac{1}{1 + \nu_1} \right)^\beta,
\]

where \( A_\beta = \min\{1, 2^{\beta-1}\} \). To study \( \| \cdot \|_{T^2(\widehat{SU(2)})} \) for \( \Omega_\beta \), let us note that for each \( k \in \mathbb{N} \cup \{0\} \), there are less than \( (1 + k)^{n-2} \) many \( \lambda = (\lambda_1, \ldots, \lambda_n) \in \widehat{SU(n)} \) such that \( \lambda_1 = k \). Therefore

\[
\sum_{\lambda \in \widehat{SU(n)}} \frac{1}{(1 + \lambda_1)^{2\beta}} \leq \sum_{k=0}^{\infty} \frac{(1 + k)^{n-2}}{(1 + k)^{2\beta}}
\]

where the right-hand side series converges if and only if \( 2\beta - n + 2 > 1 \). Therefore, for \( \beta > \)
\((n - 1)/2, \Omega_\beta \in T^2(\overline{SU}(n))\) and by Theorem 5.2, \(\ell^1(\overline{SU}(2), \omega_\beta)\) is injective. Moreover, note that

\[
\|\Omega_\beta\|_{T^2(\overline{SU}(n))} \leq \left\| \left(\mu, \nu \mapsto \frac{A_\beta(2C_n)^\beta}{1 + \mu_1} + \frac{A_\beta(2C_n)^\beta}{1 + \nu_1} \right) \right\|_{T^2(H)}
\]

\[
\leq \sup_{\mu \in \overline{SU}(n)} \left( \sum_{\nu \in \overline{SU}(n)} \left| \frac{A_\beta(2C_n)^\beta}{1 + \mu_1} \right| \right)^{1/2}
+ \sup_{\nu \in \overline{SU}(n)} \left( \sum_{\mu \in \overline{SU}(n)} \left| \frac{A_\beta(2C_n)^\beta}{1 + \nu_1} \right| \right)^{1/2}
\]

\[
\leq A_\beta(2C_n)^\beta \left( \sum_{k=0}^{\infty} \frac{1}{(1 + k)^{2\beta-n+2}} \right)^{1/2}.
\]

Hence, for \(A_\beta = \min\{1, 2^{\beta-1}\},\)

\[
\|m\|_\epsilon \leq K_\Phi A_\beta 2^{\beta+1} C_n^\beta \left( \sum_{k=0}^{\infty} \frac{1}{(1 + k)^{2\beta-n+2}} \right)^{1/2}.
\]

**Corollary 5.4** Let \(H\) be a discrete hypergroup and \(\omega\) is a weakly additive weight on \(H\) with a corresponding constant \(C > 0\). Then \(\ell^1(H, \omega)\) is injective if \(\sum_{x \in H} \omega(x)^{-2} < \infty\). Moreover,

\[
\|m\|_\epsilon \leq 2CK_\Phi \left( \sum_{x \in H} \frac{1}{\omega(x)^2} \right)^{1/2}.
\]

**Proof.** Suppose that \(\sum_{x \in H} \omega(x)^{-2} < \infty\). Note that for each \(t \in \text{supp}(\delta_x * \delta_y),\)

\[
\frac{\omega(t)}{\omega(x)\omega(y)} \leq C \frac{\omega(x) + \omega(y)}{\omega(x)\omega(y)} = \frac{C}{\omega(x)} + \frac{C}{\omega(y)}.
\]

Thus, for the functions \(f_1(x, y) = \omega(x)^{-1}\) and \(f_2(x, y) = \omega(y)^{-1},\)

\[
\|\Omega\|_{T^2(H)} \leq \left\| (x, y) \mapsto \frac{C}{\omega(x)} + \frac{C}{\omega(y)} \right\|_{T^2(H)}
\]

\[
\leq \left( \sup_{y \in H} \left( \sum_{x \in H} \left| \frac{C}{\omega(x)} \right| \right)^{1/2} \right) + \sup_{x \in H} \left( \sum_{y \in H} \left| \frac{C}{\omega(y)} \right| \right)^{1/2}
\]

\[
\leq 2C \left( \sum_{x \in H} \frac{1}{\omega(x)^2} \right)^{1/2}.
\]
Consequently, by Theorem 5.2, $\ell^1(H, \omega)$ is injective and
\[
\|m\| \leq 2CK \left( \sum_{x \in H} \frac{1}{\omega(x)}^2 \right)^{1/2}.
\]
\[\square\]

**Example 5.5** Let $\omega_f$ be the weight constructed by the group weight admitted by a positive increasing function $f$ (see Example 3.26). One can see that, if
\[
\sum_{n \in \mathbb{N}_0} f(n)^2 < \infty \quad \text{and} \quad \sup_{n,m \in \mathbb{N}_0} \frac{f(n+m)}{f(n) + f(m)} < \infty,
\]
then $\omega_f$ satisfies the conditions of Corollary 5.4 and therefore, $\ell^1(\mathbb{N}_0, \omega_f)$ is isomorphic to an operator algebra. On the other hand, $\ell^1(\mathbb{N}_0, \omega_f)$ can be embedded (isomorphically as a Banach algebra) into $A(T, \sigma_f)$ which is not isomorphic to any operator algebra (as it is not even Arens regular, see Example 4.6).

**Remark 5.6** Note that the assumed condition for $f$ in Example 5.5 implies the one which is required in Example 4.6 to obtain the Arens regularity. Compare it with this know fact that every Banach algebra which is isomorphic to an operator algebra is Arens regular.

**Remark 5.7** In Example 4.14, we introduced a hypergroup which was constructed by conjugacy classes of a specific group, $G = \oplus_{n=2}^{\infty} SL(2, 2^n)$. Moreover for the weight $\omega_\alpha$ defined on $\text{Conj}(G)$ in Example 3.9, we observed that $\ell^1(\text{Conj}(G), \omega_\alpha)$ is Arens regular. Also, as mentioned in Example 4.14, $\omega_\alpha$ forms a weakly additive weight on $\text{Conj}(G)$. But we can show that $\sum_{C \in \text{Conj}(G)} \omega(C)^{-2} = \infty$. Doing so, let us define $E_m$ to be the set of all $C = \oplus_{n \in \mathbb{N}} C_n \in \text{Conj}(G)$ such that $I_C = \{1, 2, \ldots, m\}$ where $I_C := \{n \in \mathbb{N} : C_n \neq e_{SL(2, 2^n)}\}$ for each $n$ in $I_C$. Moreover, for each $n \in I_C$, let $C_n = c_4(z)$ for $c_4(z)$ denoted in the conjugacy table of $SL(2, 2^n)$ in Example 4.14. Therefore,
\[
\sum_{C \in \text{Conj}(G)} \frac{1}{\omega(C)^2} \geq \sum_{m=2}^{\infty} \sum_{C \in E_m} \frac{1}{\omega(C)^2} \geq \sum_{m=2}^{\infty} \sum_{i=1}^{m-1} \frac{2^i}{(1 + 4^1 + \cdots + 4^m)^2} \geq \sum_{m=2}^{\infty} \frac{2^{m(m-1)/2}}{(4^{m+1} - 1)^2/9} = \infty.
\]

Hence, not all weakly additive weights are satisfying the other condition mentioned in Corollary 5.4.
For finitely generated hypergroups, we have introduced two classes of weights before, namely, polynomial growth weights and exponential weights. Applying this fact that polynomial weights are weakly additive, in the following, we study operator algebra isomorphism for weighted hypergroup algebras with polynomial weights. Developing a machinery which relates exponential weights to polynomial ones, we also study exponential weights in Subsection 5.1. For the case that $H$ is a group, this has been achieved in [20]

**Corollary 5.8** Let $H$ be a finitely generated hypergroup. If $F$ is a generator of $H$ such that $|F^n| \leq Dn^d$ for some $d, D > 0$ and $\omega_\beta$ is the polynomial weight on $H$ associated to $F$. Then $\ell^1(H, \omega_\beta)$ is injective if $2\beta > d + 1$. Moreover, for $C = \min\{1, 2^{\beta-1}\}$,

$$\|m\|_\epsilon \leq 2CK\mathfrak{G}\left(1 + \sum_{n=1}^{\infty} \frac{Dn^d}{(1+n)^{2\beta}}\right)^{1/2}. \tag{5.8}$$

**Proof.** To show this corollary, we mainly rely on Corollary 5.4. Recall that $\omega_\beta$ is weakly additive whose constant is $C = \min\{1, 2^{\beta-1}\}$. To show the desired bound for $\|m\|_\epsilon$, note that

$$\sum_{x \in H} \frac{1}{\omega_\beta(x)^2} = \sum_{x \in H} \frac{1}{(1 + \tau(x))^{2\beta}} = \sum_{n=0}^{\infty} \sum_{x \in F^n \setminus F^{n-1}} \frac{1}{(1+n)^{2\beta}} \leq 1 + \sum_{n=1}^{\infty} \frac{|F^n|}{(1+n)^{2\beta}} \leq 1 + \sum_{n=1}^{\infty} \frac{Dn^d}{(1+n)^{2\beta}}$$

which is convergent if $2\beta > d + 1$. Furthermore, by Corollary 5.4,

$$\|m\|_\epsilon \leq 2CK\mathfrak{G}\left(\sum_{x \in H} \frac{1}{\omega_\beta(x)^2}\right)^{1/2} \leq 2CK\mathfrak{G}\left(1 + \sum_{n=1}^{\infty} \frac{Dn^d}{(1+n)^{2\beta}}\right)^{1/2}. \tag{5.8}$$

□

**Example 5.9** For a polynomial hypergroup $\mathbb{N}_0$, as a finitely generated hypergroup with the generator $F = \{0, 1\}$, we have $|F^n| = n + 1 \leq 2n$, as we have seen before. By Corollary 5.8, for the polynomial weight $\omega_\beta$ with $\beta > 1$ associated to $F$, $\ell^1(\mathbb{N}_0, \omega_\beta)$ is injective. Corollary 5.4 implies that

$$\|m\|_\epsilon \leq 2CK\mathfrak{G}\left(\sum_{n=1}^{\infty} \frac{1}{n^{2\beta}}\right)^{1/2} \leq 2CK\mathfrak{G}\left(\sum_{n=1}^{\infty} \frac{1}{n^{2\beta}}\right)^{1/2}. \tag{5.8}$$

for $C = \min\{1, 2^{\beta-1}\}$. Recall that $\widehat{\text{SU}}(2)$ can be regarded as a specific case in this example.
Example 5.10 Let $\omega_\sigma$ be the weight constructed in Example 3.22 for some $\beta \geq 0$. As we saw before, $\omega_\sigma$ is equivalent to the polynomial weight $\omega_\beta$, and therefore by Example 5.9, $\ell^1(\hat{\mathrm{SU}}(2), \omega_\sigma)$ is isomorphic to an operator algebra if $\beta > 1$. Note that, $\ell^1(\hat{\mathrm{SU}}(2), \omega_\sigma)$ can be embedded (isomorphically as a Banach algebra) into the Banach algebra $A(\mathrm{SU}(2), \omega_\sigma)$ which is not isomorphic to any operator algebra if $\beta > 0$ as it is not Arens regular, (see Example 4.11).

5.1 Hypergroups with exponential weights

The other class of weights introduced for finitely generated hypergroups is the class of exponential weights. As we mentioned before, unlike polynomial weights, exponential weights are not necessarily weakly additive. In this subsection, following [20], we study operator algebra isomorphism of these weights by probing the cases for them $\Omega$ belongs to $T^2(H)$. The following lemma is a hypergroup adaptation for [20, Theorem 3.3] and the proof is similar to [11, Lemma B.2]; we omit the proof here.

Lemma 5.11 Suppose that $0 < \alpha < 1$, $C > 0$, and $\beta \geq \max\left\{ 1, \frac{6}{C\alpha(1-\alpha)} \right\}$. Define the functions $p : (0, \infty) \to \mathbb{R}$ and $q : (0, \infty) \to \mathbb{R}$ by

$$p(x) := Cx^\alpha - \beta \ln(1 + x), \quad q(x) := \frac{p(x)}{x}.$$

Let $H$ be a finitely generated hypergroup with a symmetric generator $F$ and $\omega : H \to (0, \infty)$ such that

$$\omega(x) = e^{p(\tau_F(x))} = e^{\tau_F(x)q(\tau_F(x))} \text{ for all } x \in H.$$

Then $\omega(t) \leq M\omega(x)\omega(y)$ for all $t, x, y \in H$ such that $t \in x \ast y$ where

$$M = \max\{ e^{p(z_1)-p(z_2)-p(z_3)} : z_1, z_2, z_3 \in [0, 2K] \cap \mathbb{N}_0 \}$$

and

$$K = \left( \frac{\beta^2}{C\alpha(1-\alpha)} \right)^{1/\alpha}.$$

Theorem 5.12 Let $H$ be a finitely generated hypergroup. If $F$ is a symmetric generator of $H$ such that $|F^n| \leq Dn^d$ for some $d, D > 0$ and $\sigma_{\alpha, C}$ is an exponential weight on $H$ for some $0 < \alpha < 1$ and $C > 0$. Then $\ell^1(H, \sigma_{\alpha, C})$ is injective.

Proof. Let $\omega_\beta$ be the weight defined in Lemma 5.11. We define a function $\omega : H \to (0, \infty)$ by

$$\omega(x) := \frac{\sigma_{\alpha, C}(x)}{\omega_\beta(x)} = e^{C\tau_F(x)^\alpha - \beta \ln(1 + \tau_F(x))} \quad (x \in H)$$
where $\omega_\beta$ is the polynomial weight defined on $H$ associated to $F$ and

$$\beta > \max\{1, \frac{6}{C\alpha(1 - \alpha)} \frac{d + 1}{2}\}.$$ 

Therefore, by Lemma 5.11, $\omega(t) \leq M\omega(x)\omega(y)$ for some $M > 0$ and all $t, x, y \in H$ such that $t \in x \ast y$. Therefore

$$\frac{\sigma_{\alpha,C}(t)}{\sigma_{\alpha,C}(x)\sigma_{\alpha,C}(y)} \leq M\frac{\omega_\beta(t)}{\omega_\beta(x)\omega_\beta(y)}.$$ 

Therefore,

$$\frac{\sigma_{\alpha,C}(t)}{\sigma_{\alpha,C}(x)\sigma_{\alpha,C}(y)} \leq M' \left(\frac{1}{(1 + \tau(x))^{\beta}} + \frac{1}{(1 + \tau(y))^{\beta}}\right)$$

for a modified constant $M' > 0$. Therefore by the proof of Corollary 5.8, $\Omega_{\sigma_{\alpha,C}} \in T^2(H)$. Hence $\ell^1(H, \sigma_{\alpha,C})$ is injective by Theorem 5.2.

\[\square\]

\textbf{Example 5.13} As a result of Theorem 5.12, and to follow Example 5.9, if $H$ is a polynomial hypergroup on $\mathbb{N}_0$, for each exponential weight $\sigma_{\alpha,C}$ for $0 < \alpha < 1$ and $C > 0$, $\ell^1(H, \sigma_{\alpha,C})$ is injective. Note that this class of hypergroups includes $\hat{SU}(2)$.

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