Learning Algorithm for Relation-Substitutable Context-Free Languages

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Abstract. We generalized the class of $k,l$-substitutable languages (Yoshinaka, 2008). Each language in the generalized class is closed under a good substitutability. The substitutability is defined by a recognizable equivalence relation. We show the convergence of our generalized learning algorithm. The size of the characteristic sample is smaller than Yoshinaka’s. Keywords. Context-free languages; Grammatical inference; Identification in the limit; Learning algorithms from text; Formal languages.

1 Introduction

In grammatical inference, efficient learning algorithms for context-free languages are one of the most topical issues. One goal is to find a reasonable language class for expressing natural languages. As Gold showed, even the class of regular languages is not identifiable in the limit from positive data. Early in 1980s, the first non-trivial learning strategy for a class of regular languages has been found. Clark and Eyraud have shown that substitutable context-free languages are polynomial-time identifiable in the limit from positive data. Yoshinaka gave a definition of a languages class “$k,l$-substitutable languages”, which has hierarchy structure. We noticed that a $k,l$-substitutable language is represented by a language with respect to substitution of some recognizable equivalence relation.

2 Preliminaries

$\#S$ is the cardinal of a set $S$. The power set is $\mathcal{P}(S) = \{X \mid X \subseteq S\}$. The set of natural numbers is $\mathbb{N} = \{0, 1, 2, \ldots\}$, and $\omega$ is its cardinality. And $\mathcal{P}_F(S) := \{X \mid X \subseteq S \land \#X < \omega\}$. The Cartesian product of set $S$ is $S^k = \{(x_1, \ldots, x_k) \mid 1 \leq i \leq k, x_i \in S\}$. Alphabet is a finite set. $\Sigma^*$ is the free monoid over alphabet $\Sigma$. A language is a subset of $\Sigma^*$. For a language $L \subseteq \Sigma^*$, the distribution over $x \in \Sigma^*$ is defined by $D_L(x) = \{(l, r) \in (\Sigma^*)^2 \mid lxxr \in L\}$. For $a \in \Sigma$ and $w \in \Sigma^*$, $\#(a, w)$ is the number of occurrence of ‘$a$’ in the string $w$. We denote the empty string by $\lambda$. A language $L \subseteq \Sigma^*$ is regular iff there exists a finite monoid $M$, a subset $F \subseteq M$, and a homomorphism $h : \Sigma^* \rightarrow M$ such that $L = h^{-1}(F)$. A
context-free grammar (CFG) is defined by a quadruple $G = (V, \Sigma, P, S)$ where $V$ is a finite set of nonterminals, $P$ is a finite set of members in form of $A \rightarrow \beta$ for some $A \in V$ and $\beta \in (V \cup \Sigma)^*$. We write $\alpha A \gamma \Rightarrow_G^* \alpha \beta \gamma$ if $A \rightarrow \beta \in P$ for $\alpha, \gamma \in (V \cup \Sigma)^*$. And $\Rightarrow^*$ is the reflexive transitive closure of $\Rightarrow$. We denote $L(\alpha) := \{w \in \Sigma^*| \alpha \Rightarrow^*_G w\}$, specially $L(S)$ is denoted by $L(G)$. And $L \subseteq \Sigma^*$ is a context-free language if for some CFG $G$, $L = L(G)$. If $L(G) = L(G')$, then two CFGs are equivalent, denoted by $G \approx G'$. The Chomsky normal form (CNF) of a CFG is the CFG $(\Sigma, V, P, S)$ such that each member of $P$ is in the form of $S \rightarrow \lambda$, $A \rightarrow BC$, or $A \rightarrow a$ for some $A, B, C \in V$ and $a \in \Sigma$. And there is a Fact: $\forall G :$ CFG $\exists G' :$ CNF-CFG, $G \approx G'$.

And we now define our learning criterion. This is identification in the limit from positive data defined by Gold(3). Assume $\mathbb{R}$ be some finite descriptions (grammars), say subclass context-free grammars. Assume $\mathcal{L}$ be a function from $\mathbb{R}$ to $P(\Sigma^*)$ (languages). A learning algorithm is a computable function $A$ from $\mathcal{P}_F(\Sigma^*)$ to $\mathbb{R}$. For a language $L \subseteq \Sigma^*$, a presentation is an infinite sequence $w_1, w_2, \ldots$ such that $\forall w \in L \exists n \in \mathbb{N}$, $w = w_n$. The $n$-th hypothesis grammar is $R_n = A(w_1, \ldots, w_n)$. $A$ is said to identify the class $\mathcal{L}$ of languages in the limit from positive data iff $\forall L \in \mathcal{L}$ presentation for $L$, $\exists N \in \mathbb{N}$, $L = L(R_N) \wedge \forall n \geq N, L(R_n) = L(R_N)$.

We also require that the algorithm needs only polynomial amounts of data and computation. Collin de la Higuera4 proposed a measure of it, the existence of a “core” of the target language. Our learning algorithm in this paper uses the characteristic set proposed in Clark and Eyraud’s learning algorithm.

**Characteristic Set(Clark and Eyraud, 2007).** For a CFG $G = (V, \Sigma, P, S)$, $\alpha \in (V \cup \Sigma)^*$, $A \in V$, $\omega(\alpha) := \min\{w \in \Sigma^*| \alpha \Rightarrow^*_G w\}$, $\chi(A) := \{(x, z) \in (\Sigma^*)^2| S \Rightarrow^*_G xAz\}$, where “min” is defined by the lexicographic order on $\Sigma^*$. For $\beta \in (V \cup \Sigma)^*$, if $A \rightarrow \beta \in P$ then $\chi(\beta) := \min\{\chi(A)| A \rightarrow \beta \in P\}.$ The characteristic set is defined by $CS(G) := \{\omega(\beta) \circ \chi(A)| A \rightarrow \beta \in P\}$, where $y \circ \langle x, z \rangle := xyz$.

**Remark 1.** $\forall A \in V$, $\omega(A) \circ \chi(A) \in CS(G)$. Because $\omega(A) \circ \chi(A) = \omega(\gamma') \circ \chi(A)$ for some $\gamma'$ s.t. $A \rightarrow \gamma' \in P$.

Then we recall two fundamental classes of languages with some restrictive substitutability:

**Substitutable languages(Clark and Eyraud, 2007).** $L \subseteq \Sigma^*$ is substitutable iff $\forall x, y \in \Sigma^*, D_L(x) \cap D_L(y) \neq \emptyset \Rightarrow D_L(x) = D_L(y)$.

**k,l-Substitutable languages(Yoshinaka, 2008).** $L \subseteq \Sigma^*$ is $k,l$-substitutable iff $\forall x, y \in \Sigma^*, u \in \Sigma^k, v \in \Sigma^l, D_L(uxv) \cap D_L(uyv) \neq \emptyset \Rightarrow D_L(uxv) = D_L(uyv)$. 
3 Class Definition and Comparison with Yoshinaka’s k, l-substitutable Class

In this section we show that our class is a generalization of the class of Yoshinaka(2008)’s. Assume ∼ be a recognizable equivalence relation on Σ∗. We generalized k, l-substitutable class as follows;

**Definition 2.** For a fixed recognizable equivalence relation ∼, L ⊆ Σ∗ is ∼-substitutable iff ∀x, y ∈ Σ∗, DL(x) ∩ DL(y) ≠ ∅ ∧ x ∼ y ⇒ DL(x) = DL(y).

And we use the following class notations;

**Definition 3.** Ck,l := {L ⊆ Σ∗ | L is k, l-substitutable}, C∼ := {L ⊆ Σ∗ | L is ∼-substitutable}.

First, we show the following fact;

**Claim.** ∀k, l ∈ N, ∃ ∼, Ck,l = C∼.

**Proof.** Assume x ∼k,l y := x = y ∨ ∃u ∈ Σk, v ∈ Σl, {x, y} ∈ uΣ∗v. Clearly ∼k,l is an equivalence relation of finite index. And each equivalence class is in form of {w} or uΣ∗v, so it is regular. Thus ∼k,l is a recognizable relation.

Then we show that our class is strictly inclusive. For that, we introduce another example of recognizable equivalence relations;

**Symbol Counting with Bound.** For a ∈ Σ, d ∈ N, and x, y ∈ Σ∗, x ∼#a,d y := #(a, x) = #(a, y) ≤ d ∨ #(a, x) > d ∧ #(a, y) > d. ∼#a,d is a recognizable equivalence relation.

**Claim.** There exists a language L ⊆ Σ∗ which is ∼#a,d-substitutable but not k, l-substitutable for any k, l.

**Proof.** L = b∗ ∪ b∗ab∗ is a ∼#a,d-substitutable language for any d ≥ 1. For any k, l ∈ N, L ∋ bkbbl, bklabl so b ∼k,l a. And DL(a) ∩ DL(b) ⊃ (b′, b′). But DL(b) \ DL(a) ∋ (b′, abl).

And on intersection, we have the following result;

**Lemma 4.** ∀ ∼1, ∼2 ∃ ∼, C∼1 ∪ C∼2 ⊆ C∼.

**Proof.** Assume x ∼ y := x ∼1 y ∧ x ∼2 y. Then ∼ is a recognizable equivalence relation. If L ∈ L1∪L2, then L is ∼i-substitutable for some i. So DL(x)∩DL(y) ≠ ∅ ∧ x ∼ y ⇒ DL(x) = DL(y).

By lemma 4 we proved the following fact.

1 An 2-ary relation ∼ on Σ∗ is recognizable if there exist a finite monoid (M, o, 1), a monoid homomorphism h : Σ∗ → M, and a subset F of M2 such that x1 ∼ x2 if and only if (h(x1), h(x2)) ∈ F.
In this section, we give a polynomial-time identification learning algorithm for

**Claim.** \( \forall k, l \in \mathbb{N}, \exists \sim, C_{k,l} \subseteq C_{\sim} \).

For \( k, l \)-substitutable class, \( \forall k \leq k', l \leq l', C_{k,l} \subseteq C_{k',l'} \) holds. If \( k < k' \lor l < l' \), \( \subseteq \) turns to \( \subseteq \). For our substitutable class, this property and lemma [3] is generally stated by:

**Fact 5.** \( \sim \subseteq \sim' \) implies \( C_{\sim} \subseteq C_{\sim'} \).

Notice that the equivalence relation itself is *not* recognizable. The point of to widen the identifiable languages class is find a recognizable relation as near as equality:

**Remark 6.** The equivalence relation = is *not* recognizable relation on \( \Sigma^* \).

## 4 Lemma of Typing CFGs with a Finite Monoid

In this section we give a basic lemma for proving the convergence of our learning algorithm.

**Lemma 7.** Assume \( M \) be a finite monoid and \( h : \Sigma^* \rightarrow M \) is a homomorphism. For any CFG \( G \), there exists a CFG \( G' = (V', \Sigma', P', S') \) such that \( G \approx G' \) and \( \forall A \in V' \setminus \{S'\}, A \Rightarrow_{G'}^a \) implies \( p = h(w) \).

**Proof.** Assume the given CFG be \( (V, \Sigma, P, S) \) and in CNF. We define \( G' = (V', \Sigma, P', S') \) as follows: \( V' := \{ A_p \mid A \in V \land p \in M \} \cup \{ S' \}, P' := \{ A_{pq} \rightarrow B_pC_q \mid A \rightarrow BC \in P \} \cup \{ A_p \rightarrow a \mid A \rightarrow a \in P \land p = h(a) \} \). Then the following is proved by induction: (1) \( \forall A_p \in V' \setminus \{S'\}, A_p \Rightarrow_{G'}^a \) implies \( p = h(w) \). (2) \( \forall A_p \in V' \setminus \{S'\}, L(A_p) = L(A) \cap h^{-1}(p), \) thus \( L(G) = L(G') \).

Proof for (1) is as follows: for basic case of \( A_p \rightarrow a \land p = h(a) \), the statement is clear. In induction step, assume \( A_{pq} \Rightarrow B_pC_q \Rightarrow^* w_1w_2 \land B_p \Rightarrow^* w_1 \land C_q \Rightarrow^* w_2 \) and \( p = h(w_1) \land q = h(w_2) \). Thus \( pq = h(w_1)h(w_2) = h(w_1w_2) \).

Proof for (2) is as follows: \( w \in L(A_{pq}) \Rightarrow L(A) \cap h^{-1}(pq) \) is proved similarly to (1). Assume \( A \Rightarrow BC \Rightarrow^* w_1w_2 \land B \Rightarrow^* w_1 \land C \Rightarrow^* w_2 \). By I.H., \( w_1 \in L(B_{h(w_1)}) \land w_1 \in L(C_{h(w_2)}). \) Thus \( w_1w_2 \in L(A_{h(w_1)h(w_2)}) = L(A_{h(w_1w_2)}) \).

## 5 Learning Algorithm

In this section, we give a polynomial-time identification learning algorithm for \( \sim \)-substitutable languages. Assume \( L_* \) be the target \( \sim \)-substitutable language. The learning algorithm for the class is the following:

Assume we know the equivalence relation \( \sim \). Given a positive example \( K \), define

\[
\hat{V}(K) = \{ [x] \mid \exists uv \in \Sigma^*, uxv \in K \land x \neq \lambda \} \cup \{ \hat{S} \},
\]

\[
\hat{P}(K) = \{ [xy] \rightarrow [x][y] \mid [xy], [x], [y] \in \hat{V}(K) \} \cup
\{ [x] \rightarrow [x'] \mid x \sim x' \land [x], [x'] \in \hat{V}(K) \land D_K(x) \cap D_K(x') \neq \emptyset \} \cup
\{ [a] \rightarrow a \mid [a] \in \hat{V}(K) \land a \in \Sigma \} \cup
\{ \hat{S} \rightarrow [w] \mid w \in K \},
\]

where \( D_K(x) \) is as before.
and the learning algorithm is:

**Data:** A sequence of strings \( w_1, w_2, \ldots \)

**Result:** A sequence of CFGs \( G_1, G_2, \ldots \)

Let \( \hat{G} \) be a CFG generating empty language;

**For** \( n = 1, 2, \ldots \) **do**

Read the next string \( w_n \);

If \( w_n \notin \mathcal{L}(\hat{G}) \) then

\[ \text{let } \hat{G} = \langle \Sigma, V(K), \hat{P}(K), \hat{S} \rangle \text{ where } K = \{ w_1, w_2, \ldots, w_n \} \]

**end if**

output \( \hat{G} \)

**End For**

Clearly, our algorithm *never* produces illegal strings;

**Claim.** For \( K \in \mathcal{P}_F(\Sigma^*) \), \( \mathcal{L}(\hat{G}(K)) \subseteq L_* \).

**Proof.** Assume \( v([w]) := w \) be a homomorphism from \( (\Sigma \cup \hat{V}(K))^* \) to \( \Sigma^* \). We prove that \( \hat{S} \Rightarrow [w] \Rightarrow \alpha\beta\gamma \Rightarrow^* v(\alpha\beta\gamma) \) implies \( v(\alpha\beta\gamma) \in L_* \) by induction on integer \( n \). In case of \( n = 0 \), clear. Assume \( \hat{S} \Rightarrow [w] \Rightarrow^* \alpha\beta\gamma \Rightarrow^* v(\alpha\beta\gamma) \), where \( \beta \rightarrow \beta' \in P(K) \). We consider the case of the rule \( \beta \rightarrow \beta' \) is in form of \( [v(\beta)] \rightarrow [v(\beta')] \), with \( D_L(v(\beta)) \cap D_L(v(\beta')) \neq \emptyset \) \( \land \ v(\beta) \sim v(\beta') \). By induction hypothesis \( (v(\alpha), v(\gamma)) \in D_L(v(\beta')) \). Thus \( (v(\alpha), v(\gamma)) \in D_L(v(\beta')) \) by substitutability. The other case is clear.

Then we show the opposite direction: \( \mathcal{L}(\hat{G}(K)) \supseteq L_* \). This proposition holds under the condition that given positive examples contain the characteristic set of the target CFL. So we assume the following the learning premise.

**Learning Premise.** Suppose that the target language \( L_* \) is generated by a CFG \( G_* = \langle \Sigma, V, P, S \rangle \) converted to the normal form given in lemma\(^7\) And suppose that \( K \supseteq CS(G_*) \). And assume \( \lambda \notin L_* \).

**Remark 8.** Any given presentation must contain \( CS(G_*) \) at some point of time.

Then we show the lemma for our main theorem;

**Lemma 9.** \( \forall A \in V \setminus \{ S \}, \forall w \in \Sigma^*, A \Rightarrow_G^* w \text{ implies } [\omega(A)] \Rightarrow_{\hat{G}(K)}^* w. \)

**Proof.** Suppose that \( A \Rightarrow_G^* BC \). By lemma\(^7\) \( h(\omega(A)) = h(\omega(BC)) \), so \( \omega(A) \sim \omega(BC) \). And notice that \( \omega(A) \cap \chi(A), \omega(BC) \cap \chi(A) \in CS(G) \). Thus \( [\omega(A)] \Rightarrow_{\hat{G}(K)} [\omega(BC)] \). And \( \omega(BC) = \omega(B)\omega(C) \).

**Theorem 10.** Under the learning premise, \( L_* \subseteq \mathcal{L}(\hat{G}(K)) \).

**Proof.** Let \( A := S \) in lemma\(^9\)
6 Polynomial Time and Data

The proof of efficiency of our algorithm is similar to Yoshinaka[5]. Notice that the judgment of “x ∼ y?” ends in linear order time. And characteristic set which our learning theory demands is strictly smaller than Yoshinaka’s.

First we show that our learning algorithm is of polynomial-time.

Lemma 11. Assume the question “w₁ ∼ w₂?” is decidable in polynomial time, then the computation of G(K) with finite K ends in polynomial time of the description of K.

Proof. We define the description Desk(K) := Σw∈K Len(w). And denote MaxLen(K) := max{|Len(w)| w ∈ K}.

1. The computation of |x| → |y| with x ∼ y costs O(#K² · {MaxLen(K)(MaxLen(K) − 1)}²) time.

2. The computation of S → [x] is just reading up the set K. This costs O(Desk(K)) time.

3. The way of dividing of the string w is Len(w) − 1, so the computation of the rule [xy] → [x][y] costs O(#K·{MaxLen(K)(MaxLen(K) − 1)}·(Len(w)−1)) time.

And notice that Desk(K) > #K, MaxLen(K).

Our characteristic set is by Clark and Eyraud’s definition, so it is polynomial data. Thus the following fact is proved;

Proposition 12. Our learning algorithm is efficient, i.e., a homomorphism-substitutable language is identifiable in the limit from polynomial time and data.

7 Example of Not Relation-Substitutable Language

The following example is given by Yoshinaka(2008), an example of not in Çₖ,ₗ for any k, l. This is also not in Çₗ for any recognizable equivalence relation ∼.

Claim. L := L(G), where G = {S → aSS, S → b}. L is not ∼-substitutable for any recognizable equivalence relation ∼.

Proof. Suppose that L is ∼-substitutable for some ∼: Σ* → M, where M is a finite monoid. By pigeon-hole principle on M², ∃N, k ∈ N, aᴺbᴺ ∼ aᴺ+kᴺ+k⁻¹. Then aᴺ+kᴺ⁻¹(aᴺ⁻¹bᴺ)(aᴺbᴺ+¹)bᴺ⁺¹ ∈ L.

aᴺ⁺¹(bᴺ⁻¹aᴺ⁻¹bᴺ)(aᴺbᴺ⁺¹)bᴺ⁺¹ = aᴺ⁺¹(bᴺ⁻¹aᴺ⁻¹bᴺ)(aᴺ+kᴺ⁺¹)bᴺ⁺¹ ∈ L, and aᴺ⁺¹(bᴺ⁻¹aᴺ⁻¹bᴺ)(aᴺbᴺ⁺¹)bᴺ⁺¹ ∈ L. But aᴺ⁺¹(bᴺ⁻¹aᴺ⁻¹bᴺ)(aᴺbᴺ⁺¹)bᴺ⁺¹ = (aᴺ⁻¹bᴺ⁺¹)bᴺ⁺¹aᴺ⁺¹bᴺ⁺¹ ∈ L.

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