Schwarzschild black hole as particle accelerator of spinning particles

O. B. Zaslavskii

Department of Physics and Technology, Kharkov V.N. Karazin National University - 4 Svoboda Square, Kharkov 61022, Ukraine and
Institute of Mathematics and Mechanics, Kazan Federal University - 18 Kremlyovskaya St., Kazan 420008, Russia

received 2 April 2016; accepted in final form 13 May 2016
published online 27 May 2016

PACS 04.70.Bw – Classical black holes
PACS 97.60.Lf – Black holes

Abstract – It is shown that in the Schwarzschild background there exists a direct counterpart of the Bañados-Silk-West effect for spinning particles. This means that if two particles collide near the black-hole horizon, their energy in the centre-of-mass frame can grow unbounded. In doing so, the crucial role is played by the so-called near-critical trajectories when the particle parameters are almost fine-tuned. A direct scenario of the collision under discussion is possible with restriction on the energy-to-mass ratio $E/m < \frac{1}{\sqrt{3}}$ only. However, if one takes into account multiple scattering, this becomes possible for $E \geq m$ as well.

Copyright © EPLA, 2016

Introduction. – Several years ago, it was noticed that the collision of two particles falling towards the Kerr extremal black hole can lead to the unbounded growth of the energy $E_{c.m.}$ in the centre-of-mass frame [1]. This is called the Bañados-Silk-West (BSW) effect after the names of its authors. This interesting observation triggered a lot of works on this subject. It turned out that the effect exists also for nonextremal black holes [2], it is inherent to generic rotating black holes [3], etc. Quite recently, a new venue appeared for the effect in question —collision of spinning particles. It was considered in [4] for the Schwarzschild metric and in [5] for the Kerr one. As near the Kerr black hole high-energy collisions are possible even without spin, this looks like some generalization of the same BSW effect. Meanwhile, the high-energy collision in the Schwarzschild background is a qualitatively new phenomenon. For spinless particles there is no counterpart of it.

The acceleration of particles to unbounded energies in the Schwarzschild background obtained in [4] looks very much unlike the BSW effect. In particular, a relevant collision can occur far from the horizon. However, the problem is that these results are accompanied with serious physical difficulties. The main points here are the unavoidable appearance of superluminal motion and the change of the character of trajectories from time-like to space-like. (See sect. VII of [4] where all these difficulties are discussed in detail.) Therefore, being formally correct, the results of [4] leave very serious questions and doubts.

The aim of the present work is to show that the direct counterpart of the BSW effect in the Schwarzschild background (overlooked in [4]) does exist for spinning particles. In doing so, no difficulties with superluminal motion appear. Therefore, although for spinless particles such an effect is absent, for spinning ones it is safely included, with minor modifications, into the general scheme elaborated for spinless particles.

Throughout the paper, the fundamental constants are $G = c = 1$.

Basic formulas. – We consider the Schwarzschild metric

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where $f = 1 - \frac{2M}{r}$, $M$ is the black-hole mass. Let a spinning particle move in this background. We restrict ourselves by motion within the equatorial plane $\theta = \frac{\pi}{2}$ with the spin perpendicular to the plane of motion. In the framework of the Lagrangian theory of the spinning particle [6] applied to the Schwarzschild metric, one can obtain
the expression for the component of the four-momentum:

\[ P^t = \frac{X}{f}, \quad X = \frac{E - \frac{M_s J}{r_H}}{1 - \frac{M^2}{r^2}}, \quad (2) \]

\[ P^\phi = \frac{L_{\text{eff}}}{r^2}, \quad L_{\text{eff}} = \frac{J - E s}{(1 - \frac{M^2}{r^2})}, \quad (3) \]

\[ P^r = \sigma Z, \quad (4) \]

where \( \sigma = \pm 1 \) depending on the direction of the radial motion and

\[ Z = \sqrt{X^2 - f \left( m^2 + \frac{L_{\text{eff}}^2}{r^2} \right)}. \quad (5) \]

We assume that \( P^t > 0 \) (this is the direct counterpart of the forward-in-time condition in case of spinless particles). Derivation of eqs. (2)–(5) can be found in different works, e.g., see eqs. (29)–(31) in [7]. In [8], equations of motion in a more general case of the Kerr background are considered. They coincide with eqs. (8)–(10) of [4], where a reader can also find further relevant literature on equations of motion for spinning particles. Here, \( J \) is the total angular momentum, \( E \) being the Killing energy, \( s \) the spin per unit mass. One can check that these equations agree with those for the Kerr black hole if one puts \( a = 0 \) in the corresponding equations of motion (see, e.g., eqs. (25), (26) of [5]).

Then, one can calculate the energy in the centre-of-mass frame. According to the standard definition,

\[ E_{\text{c.m.}} = -P_{\mu}P^\mu = m_1^2 + m_2^2 + 2\alpha, \quad (6) \]

where \( P_{\mu} = P_{\mu}^1 + P_{\mu}^2 \) is the total momentum of two particles in the point of collision,

\[ \alpha = -P_{1\mu}P_{2\mu}. \quad (7) \]

For spinless particles, one can identify \( \alpha = m_1 m_2 \gamma \), where \( \gamma = -u_{1\mu}u^{2\mu} \) has the meaning of the Lorentz factor of relative motion. However, for the spinning ones, such a simple interpretation is not possible since, in general, \( P_{\mu} \neq m u_{\mu} \), where \( u_{\mu} \) is the four-velocity.

Direct calculation gives us

\[ \alpha = \frac{X_1 X_2 - Z_1 Z_2}{f} \left( \frac{(L_1_{\text{eff}}) (L_2_{\text{eff}})}{r^2} \right). \quad (8) \]

Two cases should be separated here. The potential divergences can occur if for one of the particles (say, particle 1) \( 1 - \frac{M^2}{r_0^2} = 0 \) in some point \( r_0 \). This case was analyzed in [4]. However, as is mentioned above, this leads to a number of physical difficulties and it is unclear how to resolve them. In the present letter, we consider another case in which collision occurs near the horizon outside it. Therefore, we require \( r_0 < r_H = 2M \). Then, the only potential origin of divergencies consists of collisions near the horizon where \( f \) is small.

As usual in the BSW effect, the crucial point is suitable classification of the trajectories. We call a particle usual if \( X_H \neq 0 \). (Hereafter, subscript "H" means that the corresponding quality is taken on the horizon \( r = r_H \).) It is, by definition, critical if \( X_H = 0 \). This means that for the critical particle

\[ E = \frac{M s J}{r_H^2} = 0. \quad (9) \]

Then, near the horizon,

\[ X \approx \frac{3M s J (r - r_H)}{r_H^2 (1 - \frac{M^2}{r_H^2})}. \quad (10) \]

Thus, in the point of collision \( r = r_c \) (hereafter subscript "\( c \)" corresponds to the point of collision \( X_c = O(f_c) \)), the second terms in \( Z^2 \) (5) dominates, so the expression inside the square root becomes negative. This means that such a particle cannot reach the horizon.

And, a particle is called near-critical if \( X_H = O(\sqrt{r_c - r_H}) \). Correspondingly,

\[ E = \frac{M s J}{r_H^2} = O(\sqrt{r_c - r_H}) \quad (11) \]
as well.

It is easy to see that if both particles are usual, \( \gamma \) is finite, the effect is absent. If one of the particles is critical, it cannot reach the horizon at all, so the effect is absent as well. The most interesting case arises when particle 1 is near-critical, whereas particle 2 is usual. Let

\[ X_H = a_1 \sqrt{f_c} + O(f_c), \quad (12) \]

where \( a_1 \) is some finite nonvanishing coefficient.

Then, in the point of collision \( X_c \approx X_H + O(f_c) \) and

\[ \alpha \approx \frac{\langle X_2 \rangle_H}{\sqrt{f_c}} \left( a_1 - \sqrt{a_1^2 - m_1^2 - \frac{(L_1_{\text{eff}})_{\text{eff}}}{r_H^2}} \right), \quad (13) \]

where we neglected the last term in (8) since it remains finite. Taking into account (3), (9), we can rewrite (13) in the form

\[ \alpha \approx \frac{\langle X_2 \rangle_H}{\sqrt{f_c}} \left( a_1 - \sqrt{a_1^2 - m_1^2 - 16 \frac{M^2}{s^2} E_1^2} \right). \quad (14) \]

We see that (14) diverges when \( f_c \to 0 \). Thus, we obtained the effect of unbounded growth of energy in the centre-of-mass frame. This is the key observation of the present article.

It is worth stressing that it is the participation of a near-critical (but not exactly critical) particle which plays a crucial role. If particle 1 is critical (this was assumed in [5] for the Kerr background), it cannot reach the horizon and, again, there is no effect. It is the adjustment between the deviation of \( X_H \) from zero and the proximity to the
horizon that makes the effect possible. More precisely, the validity of (12) with small \( s \) is required. According to (11), this is equivalent to the requirement that \( E = \frac{a J}{2Mr^2} \) has the order \( \sqrt{r_c} - 2M \). One should also ensure the positive-ness of the expression inside the square root in (14).

Thus, any trajectory of this kind with \( a^2 > m^2 + 16\frac{M^2}{s^2}E^2 \) is suitable for our purpose to reach unbounded \( E_{\text{c.m.}} \).

**Avoidance of superluminal motion.** – For spinning particles, the relation between the four-velocity \( u^\mu \) and momentum \( P^\mu \) is more complicated than for spinless ones. According to eqs. (12), (13) of [4],

\[
\frac{u^r}{u^t} = \frac{P^r}{P^t},
\]

\[
\frac{u^\phi}{u^t} = \frac{(1 + 2M^2) P^\phi}{(1 - 2M^2 \frac{r^2}{s^2}) P^t}.
\]

Then, direct calculations give us eq. (22) of [4],

\[
\frac{u_\mu u^\mu}{(u^t)^2} = -(1 - \frac{2}{r})(1 - \frac{M^2}{r^2})(1 - \chi),
\]

\[
\chi = \frac{3Ms^2(j - es)^2(2 + \frac{M^2}{s^2})}{r^5(1 - \frac{M^2}{s^2})^4},
\]

\[
e = \frac{M}{r}, \quad j = \frac{J}{s}.
\]

There are potential divergences in (8) near the point \( r_0 = (Ms^2)^{1/3} \). However, this leads to difficulties connected with the inevitable change of sign of \( u_\mu u^\mu \) according to (17), superluminal motion and causality problems [4]. However, we have shown above that there is also another possibility that can lead to unbounded \( E_{\text{c.m.}} \). It is realized for collisions near the horizon. We are interested in the outside region \( r \geq 2M \) only and want to have \( u_\mu u^\mu < 0 \) everywhere in this region to avoid problems with superluminal motion. This entails the requirement \( \chi < 1 \).

Assuming the forward-in-time condition \( X \geq 0 \), we have from (10) that \( r_H^2 > Ms^2 \), so

\[
\frac{s^2}{8M^2} \equiv x < 1,
\]

and we see that \( \chi \) is a monotonically decreasing function of \( r \). Therefore, for our purpose, it is sufficient to require

\[
\chi(2M) < 1 \tag{20}
\]

since for any \( r > 2M \) we will have \( \chi(r) < \chi(2M) < 1 \) as well. As we are interested in trajectories giving unbounded \( E_{\text{c.m.}} \), one of the particles is usual while the other one is near-critical according to the explanations given above. As far as a usual particle is concerned, it is sufficient to take a spinless one, \( s = 0 \). Then, \( \chi = 0 \), so (20) is satisfied trivially. If \( s \neq 0 \) but is small enough, (20) is obeyed by continuity. For finite nonzero \( s \), \( |j - es| \) should be small enough according to (18), (20). This can be satisfied easily since \( j \) and \( e \) are independent quantities. We assume that this inequality holds true for usual particles.

For the near-critical particles, \( j \) and \( e \) are related according to (9). In the first approximation, neglecting the small difference between near-critical and critical trajectories, one obtains from (9), (19), (20)

\[
e^2 < \rho(x) \equiv \frac{(1 - x)^2}{6(2 + x)}. \tag{21}
\]

As \( \rho(x) \) is the monotonically decreasing function of \( x \), this entails \( \rho(x) < \rho(0) \), whence

\[
e < \sqrt{\rho(0)} = \frac{1}{2\sqrt{3}} \tag{22}
\]

According to (9), condition (21) can be rewritten in the form

\[
j < \frac{s(1 - x)}{x\sqrt{6(2 + x)}} < \frac{s}{2\sqrt{3x}} = \frac{4M^2}{\sqrt{3}s} \tag{23}
\]

Thus, there are restrictions on the relation between the total and spin momenta to avoid superluminal motion.

It follows from (9) that for the near-critical particle in (3) the quantity \( L_{\text{eff}} \approx \frac{8Ms^2}{3} > 0 \), so the relevant orbit required for the unbounded \( E_{\text{c.m.}} \) is prograde only.

As for a particle at flat infinity \( e \geq 1 \), this means that a direct scenario of high-energy collision cannot be realized for particles falling from infinity. However, it occurs for a particle that starts from the intermediate region with \( r \gtrsim 2M \), where the inequality (22) can be satisfied. Moreover, for a particle falling from infinity this is also possible in the scenario of multiple scattering instead of direct collision. This implies that a particle comes from infinity to the near-horizon region, collides there with another particle and, having obtained near-critical parameters as a result of such a collision, produces high \( E_{\text{c.m.}} \) in the next collision.

**Discussion and conclusions.** – The obtained result shows a close analogy between high-energy collisions of spinning and spinless particles near nonextremal black holes. As was found by Grib and Pavlov [2] (see also generalization in [3]), if two particles collide near the nonextremal horizon, the unbounded \( E_{\text{c.m.}} \) is possible, provided one of particles is not exactly critical but slightly deviates from the critical trajectory. In doing so, it is necessary that the deviation from the critical relation of the parameters have the same order as the small lapse function \( \sqrt{f_c} \) in the point of collision. Both for spinless and spinning particles, the effect of unbounded \( E_{\text{c.m.}} \) is absent near nonextremal black holes, if a near-critical particle falls from infinity. But it becomes possible due to the scenario of multiple collisions suggested in [2] for spinless particles.

Our result is solid since no troubles about causality and superluminal motion occur. Thus, the Schwarzschild black hole can indeed work as acceleration of spinning particles.

***

This work was funded by the subsidy allocated to Kazan Federal University for the state assignment in the sphere
of scientific activities. It was inspired by the lecture of J. W. van Holten on motion of spinning particles in the Schwarzschild background during WE-Heraeus-Seminar “Relativistic Geodesy: Foundations and Applications”.

REFERENCES

[1] Bañados M., Silk J. and West S. M., Phys. Rev. Lett., 103 (2009) 111102 (arXiv:0909.0169).
[2] Grib A. A. and Pavlov Yu. V., Pis’ma Zh. Eksp. Teor. Fiz., 92 (2010) 147 (JETP Lett., 92 (2010) 125).
[3] Zaslavskii O. B., Phys. Rev. D, 82 (2010) 083004 (arXiv:1007.3678).
[4] Armaza C., Bañados M. and Koch B., Class. Quantum Grav., 33 (2016) 105014 (arXiv:1510.01223).
[5] Guo M. and Gao S., Phys. Rev. D, 93 (2016) 084025 (arXiv:1602.08679).
[6] Hojman R. and Hojman S. A., Phys. Rev. D, 15 (1977) 2724.
[7] Hojman S. A. and Asenjo F. A., Class. Quantum Grav., 30 (2013) 025008 (arXiv:1203.5008).
[8] Saljo Motoyuki, Maeda Kei-ichi, Shibata Masaru and Mino Yasushi, Phys. Rev. D, 58 (1998) 064005.