FIRST PASSAGE TIMES FOR SUBORDINATE BROWNIAN MOTIONS

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Abstract. Let $X_t$ be a subordinate Brownian motion, and suppose that the Lévy measure of the underlying subordinator has a completely monotone density. Under very mild conditions, we find integral formulae for the tail distribution $P(\tau_x > t)$ of first passage times $\tau_x$ through a barrier at $x > 0$, and its derivatives in $t$. As a corollary, we examine the asymptotic behaviour of $P(\tau_x > t)$ and its $t$-derivatives, either as $t \to \infty$ or $x \to 0^+$. 

1. Introduction

The present article complements and extends the results of the recent paper [28], where spectral theory for a class of Lévy processes killed upon leaving a half-line was developed. In a closely related paper [29], first passage times were studied for a rather general class of one-dimensional Lévy processes. In the present article, more detailed properties of first passage times are established for processes considered in [28]: symmetric Lévy processes, whose Lévy measure has a completely monotone density function on $(0, \infty)$. More precisely, we prove asymptotic formulae, regularity, and estimates of the tail distribution $P(\tau_x > t)$ of the first passage time through a barrier at the level $x$ for a Lévy process $X_t$: 

$$\tau_x = \inf \{ t \geq 0 : X_t \geq x \}, \quad x \geq 0,$$

as well as its derivatives in $t$. Alternatively, the results can be stated in terms of the supremum functional $M_t = \sup_{s \in [0,t]} X_s$, since we have $P(\tau_x > t) = P(M_t < x)$ for all $t, x \geq 0$.

In [28], a formula was given for generalised eigenfunctions $F_\lambda(x)$ of the transition semigroup of the killed process. As an application, the distribution of first passage times was expressed in terms of the eigenfunctions $F_\lambda(x)$. The full statement of this result was only announced, and a formal proof was given under more restrictive conditions. In the present paper, we provide the proof in the general case (Theorem 1.6). The expression for the distribution of $\tau_x$ is then used to find estimates and asymptotic expansion of $(d/dt)^n P(\tau_x > t)$. This requires detailed analysis of the eigenfunctions $F_\lambda(x)$.

The double Laplace transform (in $t$ and $x$) of $P(\tau_x > t)$ is known for general Lévy processes since 1957 due to the result of Baxter and Donsker (Theorem 1 in [4]). For symmetric Lévy processes $X_t$ with Lévy-Khintchin exponent $\Psi(\xi)$,

$$\int_0^\infty \int_0^\infty e^{-\xi x - z t} P(\tau_x > t) dx dt = \frac{1}{\xi \sqrt{z}} \exp \left( - \frac{1}{\pi} \int_0^\infty \frac{\xi \log(z + \Psi(\zeta))}{\zeta^2 + \zeta^2} d\zeta \right).$$

However, the double Laplace transform in (1.1) has been inverted only for few special cases. It is a classical result that for the Brownian motion, $\tau_x$ is the $(1/2)$-stable subordinator. An explicit formula for the distribution of $\tau_x$ was found for the Cauchy process (the symmetric $1$-stable process) by Darling [10], for a compound Poisson process with

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The main result of this article is an explicit, applicable for numerical computations expression for $P(\tau_x > t)$ for a class of symmetric Lévy processes, which includes symmetric $\alpha$-stable processes, relativistic $\alpha$-stable processes and geometric $\alpha$-stable processes (in particular, the variance gamma process) and many others. More precisely, the following assumption is in force throughout the article.

**Assumption 1.1.** Any of the following equivalent conditions is satisfied (see Proposition 2.13 in [28]):

(a) $X_t$ is a subordinate Brownian motion, $X_t = B_{Z_t}$, and the Lévy measure of the subordinator $Z_t$ has a completely monotone density. Here $B_s$ is the one-dimensional Brownian motion (Var $B_s = 2s$), $Z_t$ is a subordinator (nonnegative Lévy process), and $B_s$ and $Z_t$ are independent processes;

(b) $X_t$ is a symmetric Lévy process, whose Lévy measure has a completely monotone density on $(0, \infty)$;

(c) $X_t$ is a Lévy process with Lévy-Khintchine exponent $\Psi(\xi) = \psi(\xi^2)$ for some complete Bernstein function $\psi(\xi)$.

We assume that $X_t$ is non-trivial, that is, $X_t$ is not constantly 0.

**Remark 1.2.** All explicit formulae and estimates proved in this article are given in terms of the complete Bernstein function $\psi(\xi)$. Translation to the Lévy-Khintchine exponent $\Psi(\xi) = \psi(\xi^2)$ is immediate, but usually results in less elegant expressions.

In this article, the term *explicit formula* is used for an expression involving a finite number of (absolutely convergent) integrals, elementary functions and the function $\psi$. The complete Bernstein function $\psi$ extends to a holomorphic function on $C\setminus(-\infty, 0]$ (see Preliminaries). Sometimes (namely, in the formula for $F_{\lambda}(x)$) we also use this holomorphic extension of $\psi$.

Our proofs are based on the following two theorems.

**Theorem 1.3** (Corollary 4.2 in [29]). We have

$$
\int_0^\infty e^{-\xi^2}P(\tau_x > t)dx = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 + \xi^2} \frac{\psi'(\lambda^2)}{\sqrt{\psi(\lambda^2)}} \times
$$

$$
\times \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\lambda^2 - \xi^2}{\xi^2 + \zeta^2} d\zeta \right) e^{-t\psi(\lambda^2)}d\lambda
$$

for all $t, \xi > 0$. \qed

We remark that the above result is proved in [29] for all symmetric Lévy processes with Lévy-Khintchine exponent $\Psi(\xi)$ having strictly positive derivative on $(0, \infty)$.

The transition semigroup $P_t^{(0,\infty)}$ (acting on $L^p((0, \infty))$ for any $p \in [1, \infty]$) of the process $X_t$ killed upon leaving the half-line $(0, \infty)$, and its $L^2((0, \infty))$ generator $A_{(0,\infty)}$, are defined
formally, for example, in [28]. These notions are only required in the statements of Theorems 1.4 and 1.10 and therefore they are not discussed in detail below.

**Theorem 1.4 (Theorem 1.1 in [28])**. For every \( \lambda > 0 \), there is a bounded continuous function \( F_\lambda \) on \((0, \infty)\) which is the eigenfunction of \( P_t^{(0,\infty)} \), that is,

\[
P_t^{(0,\infty)} F_\lambda (x) = e^{-t \psi(\lambda^2)} F_\lambda (x)
\]

for all \( t, x > 0 \). The function \( F_\lambda \) is characterised by its Laplace transform:

\[
\mathcal{L} F_\lambda (\xi) = \frac{\lambda}{\lambda^2 - \xi^2} \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + \zeta^2} \log \frac{\psi'(\lambda^2) (\lambda^2 - \zeta^2)}{\psi(\lambda^2) - \psi(\zeta^2)} \, d\zeta \right) \quad (1.3)
\]

for \( \xi \in \mathbb{C} \) such that \( \text{Re} \xi > 0 \). Furthermore, for \( x > 0 \) we have

\[
F_\lambda (x) = \sin(\lambda x + \vartheta_\lambda) - G_\lambda (x),
\]

where the phase shift \( \vartheta_\lambda \) belongs to \([0, \pi/2]\), and the correction term \( G_\lambda (x) \) is a bounded, completely monotone function on \((0, \infty)\). More precisely, we have

\[
\vartheta_\lambda = -\frac{1}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 - \zeta^2} \log \frac{\psi'(\lambda^2) (\lambda^2 - \zeta^2)}{\psi(\lambda^2) - \psi(\zeta^2)} \, d\zeta,
\]

and \( G_\lambda \) is the Laplace transform of a finite measure \( \gamma_\lambda \) on \((0, \infty)\). When \( \psi(\xi) \) extends to a function \( \psi^+(\xi) \) holomorphic in the upper complex half-plane \( \{ \xi \in \mathbb{C} : \text{Im} \xi > 0 \} \) and continuous in \( \{ \xi \in \mathbb{C} : \text{Im} \xi \geq 0 \} \), and furthermore \( \psi^+(-\xi) \neq \psi(\lambda) \) for all \( \xi > 0 \), then the measure \( \gamma_\lambda \) is absolutely continuous, and

\[
\gamma_\lambda (d\xi) = \frac{1}{\pi} \left( \text{Im} \frac{\lambda \psi'(\lambda^2)}{\psi(\lambda^2) - \psi^+(-\xi^2)} \right)
\times \exp \left( -\frac{1}{\pi} \int_0^\infty \frac{\xi}{\xi^2 + \zeta^2} \log \frac{\psi'(\lambda^2) (\lambda^2 - \zeta^2)}{\psi(\lambda^2) - \psi(\zeta^2)} \, d\zeta \right) \, d\xi
\]

for \( \xi > 0 \).

We introduce the following two conditions:

\[
\sup_{\xi > 0} \frac{\xi |\psi''(\xi)|}{\psi'(\xi)} < 2,
\]

and, given \( t_0 > 0 \),

\[
\int_1^\infty \sqrt{\frac{\psi'(\xi^2)}{\psi(\xi^2)}} e^{-t_0 \psi(\xi^2)} \, d\xi < \infty.
\]

**Remark 1.5.** (a) By Proposition 2.3(b), for every complete Bernstein function \( \psi \) the supremum in (1.7) is not greater than 2. Condition (1.7), is needed only to assert that \( \sup_{\lambda > 0} \vartheta_\lambda < \pi/2 \), and can be replaced by the latter.

(b) When \( \psi(\xi) \) is unbounded (that is, \( X_t \) is not a compound Poisson process) and regularly varying of order \( \varrho_0 \) at 0 and \( \varrho_\infty \) at \( \infty \), then \( \varrho_0, \varrho_\infty \in [0, 1] \) and

\[
\lim_{\xi \to 0+} (\xi |\psi''(\xi)|/\psi'(\xi)) = 1 - \varrho_0, \quad \lim_{\xi \to \infty} (\xi |\psi''(\xi)|/\psi'(\xi)) = 1 - \varrho_\infty
\]

(see [8]). Hence (1.7) is automatically satisfied.

(c) Assumption (1.8) is a rather mild growth condition on \( \psi(\xi) \) for large \( \xi \). By Proposition 2.3(a), \( \sqrt{\psi'(\xi^2)/\psi(\xi^2)} \leq 1/\xi \). Hence, (1.8) is satisfied for all \( n \geq 0 \) and \( t_0 > 0 \) whenever \( \psi(\xi) \geq c |\log \xi| (\xi > 0) \) for some \( c > 0 \). When \( \psi(\xi) = f(\log(1 + \xi)) \), then (1.8) is equivalent to \( \int_0^\infty \sqrt{f'(s)/f(s)} e^{-t_0 f(s)} \, ds \leq \infty \). For example, if \( \psi(\xi) = \log(1 + \log(1 + \xi)) \), then (1.8) holds if and only if \( t_0 > 1/2 \).
In particular, many processes frequently found in literature, including symmetric $\alpha$-stable processes, relativistic $\alpha$-stable processes and geometric $\alpha$-stable processes, satisfy (1.7) and (1.8). These and some other examples are discussed in Section 7.

The following are the main results of the article. The first of them provides a formula for $P(\tau_x > t)$ and its derivatives in $t$, and it was proved in [28] under more restrictive assumptions. The full statement of the theorem (for $n = 0$ and $n = 1$) was announced in [28] as Theorem 1.8.

**Theorem 1.6.** If (1.7) and (1.8) hold for some $t_0 > 0$, then for all $n \geq 0$, $t > t_0$ ($t \geq t_0$ if $n = 0$) and $x > 0$,

$$(-1)^n \frac{d^n}{dt^n} P(\tau_x > t) = \frac{2}{\pi} \int_0^\infty \sqrt{\psi(\lambda^2)} \left(\psi(\lambda^2)^n e^{-t \psi(\lambda^2)} F_\lambda(x)\right) d\lambda.$$  \hspace{1cm} (1.9)

In Theorem 4.6 in [29] it is proved that

$$\frac{1}{20000} \min\left(1, \frac{1}{\sqrt{t \psi(1/x^2)}}\right) \leq P(\tau_x > t) \leq 10 \min\left(1, \frac{1}{\sqrt{t \psi(1/x^2)}}\right)$$  \hspace{1cm} (1.10)

for all $t, x > 0$ for a relatively wide class of symmetric Lévy processes $X_t$ (similar results for many asymmetric processes are also available in [29]). Our second result generalises these estimates to derivatives of $P(\tau_x > t)$ in $t$, for a restricted class of processes and $t$ large enough. For similar bounds with slightly different assumptions, see Lemma 6.2, Corollary 6.4, Remark 6.5 and Proposition 7.1.

**Theorem 1.7.**  
(a) If (1.7) and (1.8) hold for some $t_0 > 0$, then the distribution of $\tau_x$ is ultimately completely monotone, that is, for each fixed $n \geq 0$, $(-1)^n(d^n/dt^n)P(\tau_x > t) \geq 0$ for $t$ large enough.

(b) If $\varphi = \sup_{x > 0}(|\psi'(\xi)|/\psi'(\xi)) < 1$ (cf. (1.7)), then there are positive constants $c_1(n)$, $c_2(n)$, $c_3(n, \varphi)$ such that

$$\frac{c_1(n)}{t^{n+1/2} \sqrt{\psi(1/x^2)}} \leq (-1)^n \frac{d^n}{dt^n} P(\tau_x > t) \leq \frac{c_2(n)}{t^{n+1/2} \sqrt{\psi(1/x^2)}}$$  \hspace{1cm} (1.11)

for $n \geq 0$, $x > 0$ and $t \geq c_3(n, \varphi)/\psi(1/x^2)$. When $c_2(n)$ is replaced by a constant $\tilde{c}_2(n, \varphi)$, then the upper bound in (1.11) holds for all $t, x > 0$.

Next, we study the asymptotic behaviour of $P(\tau_x > t)$ as $t \to \infty$ or $x \to 0^+$. The function $V(x)$ is given by an explicit formula; it is the renewal function of the ascending ladder-height process (see Preliminaries). The case $n = 0$ has been previously studied in [17].

**Theorem 1.8.**  
(a) If $\psi$ is unbounded, and (1.7) and (1.8) hold for some $t_0 > 0$, then for all $n \geq 0$ and $x \geq 0$,

$$\lim_{t \to \infty} \left(t^{n+1/2} \frac{d^n}{dt^n} P(\tau_x > t)\right) = \frac{(-1)^n \Gamma(n + 1/2)}{\pi} V(x).$$  \hspace{1cm} (1.12)

The convergence is locally uniform in $x \in [0, \infty)$.

(b) If $\psi$ is unbounded, regularly varying of order $\varrho \in [0, 1]$ at infinity, and (1.7) and (1.8) hold for some $t_0 > 0$, then for all $n \geq 0$ and $t > t_0$ ($t \geq t_0$ if $n = 0$),

$$\lim_{x \to 0^+} \left(\sqrt{\psi(1/x^2)} \frac{d^n}{dt^n} P(\tau_x > t)\right) = \frac{(-1)^n \Gamma(n + 1/2)}{\pi \Gamma(1 + \varrho)} \frac{1}{\varrho^{n+1/2}}.$$  \hspace{1cm} (1.13)
The convergence is uniform in $t \in (t_0, \infty)$.

**Remark 1.9.** The asymptotic behaviour of $(d/dt)\mathbb{P}(\tau_x > t)$ as $t \to \infty$ was completely described for (possibly asymmetric) stable Lévy processes in [14], Theorem 1. In our Theorem 1.8(a), we consider a class of symmetric, but not necessarily stable Lévy processes, and derivatives of higher order are included.

In [13], Theorem 3, an analogous problem is studied when one-dimensional distributions of the Lévy process $X_t$ belong to the domain of attraction of a (possibly asymmetric) stable law. Again, this partially overlaps with Theorem 1.8(a), but neither result generalizes the other one.

Finally, Theorem 1.4 is complemented by the following completeness result. It was proved in [28] under an extra assumption that the operator $\Pi$, defined in the statement of the theorem, is injective. Using methods developed partially in [29], we show that $\Pi$ is always injective, and therefore the theorem holds in full generality. The present statement was announced in [28] as Theorem 1.3.

**Theorem 1.10.** For $f \in C_c((0, \infty))$ and $\lambda > 0$, let
\[
\Pi f(\lambda) = \int_0^\infty f(x) F_\lambda(x) dx.
\]
Then $\sqrt{2/\pi} \Pi$ extends to a unitary operator on $L^2((0, \infty))$, which diagonalises the action of $P_t^{(0,\infty)}$:
\[
\Pi P_t^{(0,\infty)} f(\lambda) = e^{-t\psi(\lambda^2)} \Pi f(\lambda) \quad \text{for } f \in L^2((0, \infty)).
\]
Furthermore, $f \in D(A_{(0,\infty)}; L^2)$ if and only if $\psi(\lambda^2) \Pi f(\lambda)$ is in $L^2((0, \infty))$, and
\[
\Pi A_{(0,\infty)} f(\lambda) = -\psi(\lambda^2) \Pi f(\lambda) \quad \text{for } f \in D(A_{(0,\infty)}; L^2).
\]

**Remark 1.11.** In [28], the generalised eigenfunction expansion of Theorem 1.10 was the key step in the proof of (a restricted version of) Theorem 1.6. Here, the proofs of Theorems 1.6 and 1.10 are independent.

We conclude the introduction with a brief description of the structure of the article. In Preliminaries, we recall the notion of complete Bernstein and Stieltjes functions, their properties and a Wiener-Hopf type transformation $\psi \mapsto \psi^\dagger$. Some new ideas are developed here, e.g. a type of continuity of the mapping $\psi \mapsto \psi^\dagger$ is proved. We also give some simple estimates related to Laplace transforms of monotone functions, and introduce the renewal function $V(x)$ of the ascending ladder-height process. In Section 3 we prove Theorem 1.10. Next two sections contain estimates and properties of $\vartheta_\lambda$ and $F_\lambda(x)$, respectively (see Theorem 1.4), which are essential to the derivation of the main results. In Section 6 we prove Theorems 1.6–1.8. Finally, some examples are studied in Section 7.

2. **Preliminaries**

2.1. **Complete Bernstein functions.** A function $f(z)$ is said to be a complete Bernstein function (CBF in short) if
\[
f(z) = c_1 + c_2 z + \frac{1}{\pi} \int_{0+}^\infty \frac{z}{z + s} m(ds),
\]
where $c_1, c_2 \geq 0$, and $m$ is a Radon measure on $(0, \infty)$ such that $\int \min(s^{-1}, s^{-2}) m(ds) < \infty$. A function $g(z)$ is said to be a Stieltjes function if
\[
g(z) = \frac{c_1}{z} + c_2 + \frac{1}{\pi} \int_{0+}^\infty \frac{1}{z + s} \tilde{m}(ds),
\]
where $c_1, c_2 \geq 0$, and $\mu$ is a Radon measure on $(0, \infty)$ such that $\int \min(1, s^{-1})\mu(ds) < \infty$.

Complete Bernstein and Stieltjes functions are often defined on $(0, \infty)$. However, (2.1) and (2.2) define holomorphic functions on $\mathbb{C} \setminus (-\infty, 0]$. We always identify functions on $(0, \infty)$ with their holomorphic extensions to $\mathbb{C} \setminus (-\infty, 0]$.

We list some basic properties of complete Bernstein and Stieltjes functions.

**Proposition 2.1** (Proposition 2.18 in [28], and Corollary 6.3 in [31]).

(1) Let $f$ be a complete Bernstein function with representation (2.1). Then

$$c_1 = \lim_{z \to 0^+} f(z), \quad c_2 = \lim_{z \to \infty} \frac{f(z)}{z},$$

(2.3)

and

$$m(ds) = \lim_{\varepsilon \to 0^+} (\text{Im} f(-s + i\varepsilon) ds),$$

(2.4)

with the limit understood in the sense of weak convergence of measures.

(2) Let $g$ be a Stieltjes function with representation (2.2). Then

$$c_1 = \lim_{z \to 0^+} (zg(z)), \quad c_2 = \lim_{z \to \infty} g(z),$$

(2.5)

and

$$\tilde{\mu}(ds) + \pi c_1 \delta_0(ds) = \lim_{\varepsilon \to 0^+} (-\text{Im} g(-s + i\varepsilon) ds),$$

(2.6)

with the limit understood in the sense of weak convergence of measures.

**Proposition 2.2** (see [31]). Suppose that $f$ and $g$ are not constantly equal to 0.

(a) the following conditions are equivalent: $f(z)$ is CBF, $z/f(z)$ is CBF, $f(z)/z$ is Stieltjes, $1/f(z)$ is Stieltjes;

(b) $f(z)$ is CBF if and only if $f(z) \geq 0$ for $z \geq 0$, and $f$ extends to a holomorphic function in $\mathbb{C} \setminus (-\infty, 0]$ such that $\text{Im} f(z) > 0$ when $\text{Im} z > 0$;

(c) if $f, g$ are CBF and $a, c > 0$, then also $f(z) + g(z), cf(z), f(cz), (z-a)/(f(z)-f(a))$ (extended continuously at $z = a$) and $f(g(z))$ are CBF.

**Proposition 2.3** (Proposition 2.21 in [28]). If $f$ is a complete Bernstein function, then

(a) $0 \leq zf'(z) \leq f(z)$ for $z > 0$;

(b) $0 \leq -zf''(z) \leq 2f'(z)$ for $z > 0$;

(c) $|f(z)| \leq (\sin(\varepsilon/2))^{-1} f(|z|)$ for $z \in \mathbb{C}$, $|\text{Arg } z| \leq \pi - \varepsilon$, $\varepsilon \in (0, \pi)$;

(d) $|zf'(z)| \leq (\sin(\varepsilon/2))^{-1} f(|z|)$ for $z$ as in (c);

(e) $|f(z)| \leq c(f, \varepsilon)(1 + |z|)$ for $z$ as in (c).

Throughout the article, $\psi$ usually denotes the complete Bernstein function in the Lévy-Khintchine exponent of $X_t$. Some preliminary results and definitions, however, are valid for more general functions $\psi$. If this is the case, we explicitly state all assumptions on $\psi$ each time it is mentioned.

Following [28, 29], we define

$$\psi^\dagger(\xi) = \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\xi \log \psi(\zeta^2)}{\xi^2 + \zeta^2} d\zeta \right),$$

(2.7)

for any positive function $\psi$ for which the integral converges. In this case, $\psi^\dagger(\xi)$ is defined at least when $\text{Re } \xi > 0$. By a simple substitution, for $\xi > 0$,

$$\psi^\dagger(\xi) = \exp \left( \frac{1}{\pi} \int_0^\infty \frac{\log \psi(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta \right).$$

(2.8)
This definition is extended continuously by \( \xi \) and both \( \psi(\xi) \) and \( \xi/\psi(\xi) \) are increasing on \((0, \infty)\), then
\[
e^{-2C/\pi} \sqrt{\psi(\xi^2)} \leq \psi^\dagger(\xi) \leq e^{2C/\pi} \sqrt{\psi(\xi^2)},
\]
(2.9)
where \( C \approx 0.916 \) is the Catalan constant. Note that \( e^{2C/\pi} \leq 2 \).

If, in addition, \( \psi(\xi) \) is regularly varying at \( \infty \), then
\[
\lim_{\xi \to \infty} \frac{\psi^\dagger(\xi)}{\sqrt{\psi(\xi^2)}} = 1.
\]
(2.10)
An analogous statement for \( \xi \to 0 \) holds for \( \psi(\xi) \) regularly varying at \( 0 \).

In particular, (2.9) holds for any CBF. Likewise, (2.10) holds for any regularly varying CBF.

The estimate (2.9) for CBFs was obtained independently in [24], Proposition 3.7, while (2.10) for CBFs was derived in [21], Proposition 2.2.

**Proposition 2.5** (Proposition 3.4 in [28]). Whenever both sides of the following identities make sense, we have:

(a) \( (1/\psi)^\dagger = 1/\psi^\dagger \), \( (\psi o)^\dagger = (\psi^\dagger)^o \) \( (o \in \mathbb{R}) \) and \( (\psi_1 \psi_2)^\dagger = \psi_1^\dagger \psi_2^\dagger \);
(b) \( (e^c)^\dagger = e^{c \psi^\dagger} \) for \( c > 0 \);
(c) when \( \psi(\xi) = \xi^k \), then \( \psi(\xi) = \xi^k \);
(d) if appropriate limits of \( \psi \) exist, then the corresponding limits of \( \psi^\dagger \) exist, and
\[
\lim_{\xi \to 0^+} \psi^\dagger(\xi) = (\lim_{\xi \to 0^+} \psi(\xi))^{1/2}, \quad \lim_{\xi \to \infty} \psi^\dagger(\xi) = (\lim_{\xi \to \infty} \psi(\xi))^{1/2}.
\]
The first part of the following result was independently proved in [23], Proposition 2.4.

**Lemma 2.6** (Lemma 3.8 in [28]). If \( \psi(\xi) \) is a CBF, then also \( \psi^\dagger(\xi) \) is a CBF, and
\[
\psi^\dagger(\xi) \psi^\dagger(-\xi) = \psi(-\xi^2)
\]
(2.11)
for \( \xi \in \mathbb{C} \setminus \mathbb{R} \).

**Proposition 2.7.** Let \( \psi_n \) be a sequence of CBFs. If \( \psi_n(\xi) \to \psi(\xi) \) as \( n \to \infty \) for all \( \xi > 0 \), then \( \psi_n^\dagger(\xi) \to \psi^\dagger(\xi) \) locally uniformly in \( \xi \in \mathbb{C} \setminus (-\infty, 0] \).

**Proof.** For any CBFs \( \psi^\dagger(\xi), \psi(\xi) \), we have
\[
\frac{\psi^\dagger(\xi)}{\psi(\xi)} = \exp \left( \frac{1}{\pi} \int_{0}^{\infty} \frac{\xi \log(\psi(\xi^2)/\psi(\xi^2))}{\xi^2 + \xi^2} d\xi \right)
\]
(2.12)
when \( \text{Re} \xi > 0 \). Furthermore, by monotonicity and concavity of \( \psi_n, \psi_n(1, \xi) \leq \psi_n(1, 0) \leq \psi_n(1, \xi^2) \leq \psi_n(1, \xi^2) \leq \psi_n(1, \xi^2) \leq \psi_n(1, \xi^2) \leq \psi_n(1, \xi^2) \leq \psi_n(1, \xi^2) \). Hence, by dominated convergence, (2.12) gives convergence of \( \psi_n^\dagger(\xi) \to \psi^\dagger(\xi) \) when \( \text{Re} \xi > 0 \). By Corollary 7.6(b) in [34], \( \psi_n^\dagger(\xi) \to \psi^\dagger(\xi) \) for all \( \xi \in \mathbb{C} \setminus (-\infty, 0] \). Pointwise convergence of CBFs is automatically locally uniform on \( \mathbb{C} \setminus (-\infty, 0] \): by Proposition 2.3(d,e), when \( \text{Arg} \xi \in (-\pi + \epsilon, \pi - \epsilon) \), we have
\[
|\psi_n^\dagger(\xi)| \leq c_3(\epsilon) |\psi_n(\xi)| \leq c_4(\epsilon, f) \max(|\xi|^{-1}, 1),
\]
which proves that \( \psi_n^\dagger(\xi) \) are locally equicontinuous in \( \mathbb{C} \setminus (-\infty, 0] \).

As in [28], for a nonvanishing, increasing and differentiable function \( \psi \) with strictly positive derivative, we denote
\[
\psi_\lambda(\xi) = \frac{1 - \xi/\lambda^2}{1 - \psi(\xi)/\psi(\lambda^2)}, \quad \lambda, \xi > 0, \lambda^2 \neq \xi.
\]
(2.13)
This definition is extended continuously by \( \psi_\lambda(\lambda^2) = \psi(\lambda^2)/(\lambda^2 \psi(\lambda^2)) \), and when \( \psi \) extends holomorphically to \( \mathbb{C} \setminus (-\infty, 0] \), also \( \psi_\lambda(\xi) \) is defined for \( \xi \in \mathbb{C} \setminus (-\infty, 0] \).
simplicity, we denote $\psi^1_\lambda(\xi) = (\psi^1_\lambda)\xi$. By Proposition 2.2(c), if $\psi(\xi)$ is a CBF, then $\psi^1_\lambda(\xi)$ (and hence also $\psi^1_\lambda$) is a CBF for any $\lambda > 0$.

Although (2.13) is generally used for complete Bernstein functions $\psi$, we will occasionally need this definition for a negative function $\psi(\xi) = -\xi^\theta$ with $\theta \in (-1, 0)$.

**Corollary 2.8.** If $\psi(\xi)$ is a CBF, then
$$e^{-2c/\pi \sqrt{\psi}(\xi^2)} \leq \psi^1_\lambda(\xi) \leq e^{2c/\pi \sqrt{\psi}(\xi^2)}.$$ \hfill □

**Corollary 2.9.** If $\psi$ is a CBF, then the function $\psi^1_\lambda(\xi)$ is jointly continuous in $\lambda > 0$, $\xi \in \mathbb{C} \setminus (-\infty, 0)$.

The following monotonicity property of $\psi^1_\lambda(\xi)$ plays an important role.

**Proposition 2.10.** Suppose that for twice differentiable, nonvanishing and increasing functions $\psi(\xi)$ and $\tilde{\psi}(\xi)$ we have
$$-\frac{\psi''(\zeta)}{\psi'(\zeta)} \leq -\frac{\tilde{\psi}''(\zeta)}{\tilde{\psi}'(\zeta)}, \quad \zeta > 0. \quad (2.14)$$

Then
$$\frac{\psi_\lambda(\xi^2)}{\psi_\lambda(\lambda^2)} \leq \frac{\tilde{\psi}_\lambda(\xi^2)}{\tilde{\psi}_\lambda(\lambda^2)}, \quad 0 < \lambda < \xi,$$
$$\frac{\psi_\lambda(\xi^2)}{\psi_\lambda(\lambda^2)} \geq \frac{\tilde{\psi}_\lambda(\xi^2)}{\tilde{\psi}_\lambda(\lambda^2)}, \quad 0 < \xi < \lambda.$$

**Proof.** Integration in $\zeta$ of both sides of (2.14) yields that
$$\frac{\psi'(\zeta)}{\psi'(\xi_1)} \leq \frac{\tilde{\psi}'(\zeta)}{\tilde{\psi}'(\xi_1)}, \quad 0 < \xi_1 < \zeta,$$
and by another integration in $\zeta$,
$$\frac{\psi(\xi_2) - \psi(\xi_1)}{\psi'(\xi_1)} \leq \frac{\tilde{\psi}(\xi_2) - \tilde{\psi}(\xi_1)}{\tilde{\psi}'(\xi_1)}, \quad 0 < \xi_1 < \xi_2.$$

Taking $\xi_1 = \lambda^2$ and $\xi_2 = \xi^2$, we obtain
$$\frac{\psi_\lambda(\xi^2)}{\psi_\lambda(\lambda^2)} = \frac{\psi'(\lambda^2)(\lambda^2 - \xi^2)}{\psi'(\lambda^2) - \psi'(\xi^2)} \leq \frac{\tilde{\psi}'(\lambda^2)(\lambda^2 - \xi^2)}{\tilde{\psi}'(\lambda^2) - \tilde{\psi}'(\xi^2)} = \frac{\tilde{\psi}_\lambda(\xi^2)}{\tilde{\psi}_\lambda(\lambda^2)}, \quad 0 < \xi < \lambda.$$

The other part is proved in a similar manner. \hfill □

Finally, recall that $\psi_\lambda(\lambda^2) = \psi(\lambda^2)/(\lambda^2 \psi'(\lambda^2))$. This gives the following simple estimate:
$$\frac{\psi_\lambda(\xi^2)}{\psi_\lambda(\lambda^2)} = \frac{\psi'(\lambda^2)(\lambda^2 - \xi^2)}{\psi'(\lambda^2) - \psi'(\xi^2)} \leq \frac{\psi'(\lambda^2) \max(\lambda^2, \xi^2)}{[\psi'(\lambda^2) - \psi'(\xi^2)]}, \quad \lambda, \xi > 0 \quad (2.15)$$
when $\psi$ is a CBF.

Let $\psi(\xi)$ be the complete Bernstein function related with the Lévy-Khintchine exponent $\Psi(\xi)$ of the Lévy process $X_t$, $\Psi(\xi) = \psi(\xi^2)$. With the notation of this section, for $\lambda, \xi > 0$ we have (see Remark 4.12 in [28], and the proof of Theorem 4.1 in [29])
$$\int_0^\infty e^{-tx} \mathbf{P}(\tau_x > t) dx = 2 \pi \int_0^\infty \frac{\lambda}{\lambda^2 + \xi^2} \frac{\lambda \psi'(\lambda^2)}{\psi(\lambda^2)} \psi^1_\lambda(\xi)e^{-t\psi(\lambda^2)} d\lambda; \quad (2.16)$$
$$\vartheta_\lambda = \frac{1}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 - \xi^2} \log \frac{\psi_\lambda(\xi^2)}{\psi_\lambda(\lambda^2)} d\xi = \text{Arg} \psi^1_\lambda(i\lambda); \quad (2.17)$$
\[
\mathcal{L}F_\lambda(\xi) = \frac{\lambda}{\lambda^2 + \xi^2} \frac{\psi_\lambda^* (\xi)}{\sqrt{\psi_\lambda (\lambda^2)}};
\]

(2.18)

\[
\gamma_\lambda (d\xi) = \frac{1}{\pi} \frac{\lambda \psi' (\lambda^2)}{\psi (\lambda^2) - \psi^+ (-\xi^2)} d\xi;
\]

(2.19)

for the last equality the assumption of the final part of Theorem 1.4 is required.

2.2. Estimates for the Laplace transform. This short section contains some rather standard estimates for the inverse Laplace transform, similar to those used in [29].

Proposition 2.11. Let \( a > 0, c \geq 1 \). If \( f \) is nonnegative and \( f(x) \leq cf(a) \max(1, x/a) \) \((x > 0)\), then for any \( \xi > 0 \),

\[
f(a) \geq \frac{\xi \mathcal{L} f(\xi)}{c(1 + (a\xi)^{-1}e^{-a\xi})}.
\]

Proof. We have

\[
\xi \mathcal{L} f(\xi) = \int_0^a \xi e^{-\xi x} f(x) dx + \int_a^\infty \xi e^{-\xi x} f(x) dx
\]

\[
\leq cf(a) \int_0^a \xi e^{-\xi x} dx + \frac{cf(a)}{a} \int_a^\infty \xi x e^{-\xi x} dx
\]

\[
= cf(a)(1 - e^{-a\xi}) + \frac{cf(a)}{a\xi} (1 + a\xi)e^{-a\xi}
\]

\[
= cf(a)(1 + (a\xi)^{-1}e^{-a\xi}). \quad \square
\]

Proposition 2.12. If \( f \) is nonnegative and increasing, then for \( a, \xi > 0 \),

\[
f(a) \leq e^{a\xi} \xi \mathcal{L} f(\xi).
\]

Proof. As before,

\[
\xi \mathcal{L} f(\xi) = \int_0^a \xi e^{-\xi x} f(x) dx + \int_a^\infty \xi e^{-\xi x} f(x) dx
\]

\[
\geq f(a) \int_a^\infty \xi e^{-\xi x} dx = f(a)e^{-a\xi}. \quad \square
\]

Proposition 2.13. Let \( b \geq a > 0 \), and suppose that \( f \) is nonnegative and increasing on \((0, b)\), and \( f(x) \geq m \) for \( x \geq b \). Then for any \( \xi > 0 \),

\[
f(a) \leq \frac{e^{a\xi} \xi \mathcal{L} f(\xi) - me^{-(b-a)\xi}}{1 - e^{-(b-a)\xi}}.
\]

Proof. By Proposition 2.12 applied to \( f(x)1_{(0,a)}(x) + f(a)1_{[a,\infty)}(x) \),

\[
f(a) \leq e^{a\xi} \left( \int_0^\infty e^{-\xi x} f(x) dx + \int_a^\infty e^{-\xi x} (f(a) - f(x)) dx \right)
\]

\[
\leq e^{a\xi} \left( \int_0^\infty e^{-\xi x} f(x) dx + \int_0^a e^{-\xi x} (f(a) - m) dx \right)
\]

\[
= e^{a\xi} \xi \mathcal{L} f(\xi) + e^{-(b-a)\xi} (f(a) - m). \quad \square
\]
2.3. Elements of fluctuation theory. In the sequel we will need the function $V(x)$, which is described by its Laplace transform, $LV(\xi) = \xi / \psi(\xi)$. By Proposition 2.2(a) and Lemma 2.6, $\xi / \psi(\xi)$ is a complete Bernstein function. Hence, $V$ is a Bernstein function, i.e. $V$ is nonnegative and $V'$ is completely monotone (see [34]). By Theorem 4.4 in [29],

$$\frac{1}{5} \frac{1}{\psi(1/x^2)} \leq V(x) \leq 5 \frac{1}{\sqrt{\psi(1/x^2)}}, \quad x > 0.$$ 

Suppose that $\psi$ extends to a function $\psi^+$ holomorphic in the upper complex half-plane and continuous in the region $\text{Im} \xi \geq 0$. Then by Proposition 4.5 in [29],

$$V(x) = bx + \frac{1}{\pi} \int_0^\infty \text{Im} \left( -\frac{1}{\psi^+(-\xi^2)} \right) \frac{\psi^+ (\xi)}{\xi} (1 - e^{-x\xi}) d\xi, \quad x > 0,$$

where $b = (\lim_{\xi \to 0^+} (\xi / \psi(\xi)))^{1/2}$.

Probabilistically, $V(x)$ is the renewal function for the ascending ladder-height process $H_s$ corresponding to $X_t$. When $\psi$ is unbounded, then $X_t$ satisfies the absolute continuity condition and $V(x)$ is (unique up to a multiplicative constant) increasing harmonic function for $X_t$ on $(0, \infty)$, cf. [35].

3. Eigenfunction expansion

In this section we prove a peculiar integral identity, which (together with the results of [28]) yields a short proof of Theorem 1.10.

Lemma 3.1. Let $\psi(\xi)$ be an arbitrary positive, continuously differentiable function on $(0, \infty)$ with strictly positive derivative, and such that $(\log (1 + \psi(\xi^2)))/\xi^2$ is integrable on $(0, \infty)$. Then for all $\xi_1, \xi_2 > 0$,

$$\int_0^\infty \frac{\lambda^4 \psi'(\lambda^2)(\xi_1 + \xi_2) \psi^3(\lambda_1) \psi^3(\lambda_2)}{\psi(\lambda^2)(\lambda^2 + \xi_1^2)(\lambda^2 + \xi_2^2)} d\lambda = \frac{\pi}{2}. \quad (3.1)$$

Proof. As in the proof of Theorem 4.1 in [29], for $\xi > 0$ and $z \in \mathbb{C} \setminus (-\infty, 0]$ we define

$$\varphi(\xi, z) = \exp \left( -\frac{1}{\pi} \int_0^\infty \frac{\xi \log (1 + \psi(\xi^2)/z)}{\xi^2 + \xi^2} d\xi \right),$$

where log is the principal branch of the logarithm. Clearly, for each $\xi > 0$, $\varphi(\xi, z)$ is a holomorphic function of $z$, and $\varphi(\xi, z) > 0$ when $z > 0$. As it was observed in [29], when $\text{Im} z > 0$, we have $\text{Arg}(1 + \psi(\xi^2)/z) \in (-\pi, 0)$, so that

$$\text{Arg} \varphi(\xi, z) = -\frac{1}{\pi} \int_0^\infty \frac{\xi \text{Arg}(1 + \psi(\xi^2)/z)}{\xi^2 + \xi^2} d\xi \in (0, \pi/2).$$

Hence, for any $\xi_1, \xi_2 > 0$, $\text{Im}(\varphi(\xi_1, z) \varphi(\xi_2, z)) > 0$ when $\text{Im} z > 0$. By Proposition 2.2(b), $f(z) = \varphi(\xi_1, z) \varphi(\xi_2, z)$ is a CBF of $z$.

By monotone convergence, $f(z)$ converges to 0 when $z \to 0^+$, and by dominated convergence, $\lim_{z \to \infty} f(z) = 1$. It follows that the constants $c_1, c_2$ in the representation (2.1) for the CBF $f(z)$ vanish. Furthermore, for any $\xi > 0$, $\varphi(\xi, z)$ extends to a continuous function $\varphi^+(\xi, z)$ in the region $\text{Im} z \geq 0$, given by the formula

$$\varphi^+(\xi, z) = \exp \left( -\frac{1}{\pi} \int_0^\infty \frac{\xi \log^-(1 + \psi(\xi^2)/z)}{\xi^2 + \xi^2} d\xi \right),$$

where $\log^-(z)$ is the continuous extension of $\log z$ to the region $\text{Im} z \leq 0$. The measure $m$ in the representation (2.1) for the function $f(z) = \varphi(\xi_1, z) \varphi(\xi_2, z)$ is therefore given by
\( m(ds) = \pi^{-1} \operatorname{Im}(\varphi^+(\xi_1, -s)\varphi^+(\xi_2, -s))ds \) (Proposition 2.1(a)). By (2.1) and monotone convergence,

\[
\frac{1}{\pi} \int_0^\infty \frac{\operatorname{Im}(\varphi^+(\xi_1, -s)\varphi^+(\xi_2, -s))}{s} ds = \frac{1}{\pi} \lim_{z \to \infty} \int_0^\infty \frac{z \operatorname{Im}(\varphi^+(\xi_1, -s)\varphi^+(\xi_2, -s))}{z + s} \frac{ds}{s}, \tag{3.2}
\]

Note that if \( \psi(\xi) \) is bounded on \((0, \infty)\) and \( s \geq \sup_{\xi > 0} \psi(\xi) \), then \( \varphi^+(\xi, -s) \) is real, so that the integrand in (3.2) vanishes for such \( s \). Hence, we may substitute \( \psi(\lambda^2) \) for \( s \) in (3.2) to obtain that

\[
\frac{2}{\pi} \int_0^\infty \lambda \psi'(\lambda^2) \frac{\varphi^+(\xi_1, -\psi(\lambda^2))\varphi^+(\xi_2, -\psi(\lambda^2))}{\psi(\lambda^2)} d\lambda = 1.
\]

Finally, in the proof of Theorem 4.1 in [29] (formula (4.3) therein) it is proved that

\[
\varphi^+(\xi, -\psi(\lambda^2)) = \frac{\lambda(\lambda + \xi i)\psi(\xi)}{\lambda^2 + \xi^2}.
\]

Since \( \operatorname{Im}((\lambda + \xi i)(\lambda + \xi^2 i)) = \lambda(\xi_1 + \xi_2) \), the lemma is proved.

\[\square\]

**Remark 3.2.** Lemma 3.1 was conjectured in a preliminary version of [28].

**Proof of Theorem 1.10** In [28], Theorem 1.10 was proved under an extra assumption that the operator \( \Pi \) defined in (1.14) is injective. Below we show that this condition is always satisfied.

Let \( \varepsilon(\xi) = e^{-\xi x}1_{(0, \infty)}(x) \ (\xi > 0) \). By (2.18), for \( \lambda > 0 \),

\[
\Pi \varepsilon(\lambda) = \int_0^\infty e^{-\xi x} F_\lambda(x) dx = LF_\lambda(\xi) = \sqrt{\frac{\lambda^2 \psi'(\lambda^2)}{\psi(\lambda^2)}} \frac{\lambda}{\lambda^2 + \xi^2} \psi(\xi).
\]

By Lemma 3.1,

\[
\langle \Pi \varepsilon(\xi), \Pi \varepsilon(\xi) \rangle_{L^2(0, \infty)} = \int_0^\infty \frac{\lambda^4 \psi'(\lambda^2)\psi(\xi_1)\psi(\xi_2)}{\psi(\xi^2)(\lambda^2 + \xi_1^2)(\lambda^2 + \xi_2^2)} d\lambda = \frac{\pi}{2(\xi_1 + \xi_2)} = \frac{\pi}{2} \langle \varepsilon(\xi), \varepsilon(\xi) \rangle_{L^2(0, \infty)}.
\]

The functions \( \varepsilon(\xi) \) form a linearly dense set in \( L^2((0, \infty)) \). By approximation, \( \sqrt{2/\pi} \Pi \) is a unitary operator on \( L^2((0, \infty)) \). In particular, \( \Pi \) is injective.

\[\square\]

4. **Estimates of \( \vartheta_\lambda \)**

As it will become clear in the next section, upper bounds for the phase shift \( \vartheta_\lambda \) (which is defined by (1.5)) are crucial for applications of Theorems 1.4 and 1.10 when \( \vartheta_\lambda \) is close to \( \pi/2 \), estimates of the eigenfunctions \( F_\lambda(x) \) are problematic. Recall that by (1.5) and (2.17),

\[
\vartheta_\lambda = \operatorname{Arg} \psi(\lambda) = -\frac{1}{\pi} \int_0^\infty \frac{1}{1 - \xi^2} \log \frac{\psi(\lambda^2 \xi^2)}{\psi(\lambda^2)} d\xi, \quad \lambda > 0. \tag{4.1}
\]

Here \( \psi(\xi) \) is a complete Bernstein function in the Lévy-Khintchine exponent of \( X_t \). By a substitution \( \zeta = 1/s \) in the integral over \( (1, \infty) \), one obtains that (see Proposition 4.16
Proposition 4.2

Proposition 4.3.

When $\psi(\xi) = \xi^{\alpha/2}$ ($0 < \alpha \leq 2$), that is, when $X_t$ is the symmetric $\alpha$-stable process, we have $\vartheta_\lambda = (2-\alpha)/2$ (Example 6.1 in [28]). In the general case, $0 \leq \vartheta_\lambda < \pi/2$, and $\vartheta_\lambda \leq (\pi/2)\sup_{\xi>0}(\psi'(\xi)/\psi(\xi))$ (Proposition 4.1 of [28]). This estimate is sufficient for our needs, and it is significantly improved below. By (4.1), we have the following monotonicity result. Note that $\psi$ need not be a complete Bernstein function in Propositions 4.1 and 4.2.

Proposition 4.1. Suppose that $\psi$ and $\tilde{\psi}$ are arbitrary twice-differentiable, increasing and nonvanishing functions (not necessarily complete Bernstein functions). Define $\vartheta_\lambda$ by (4.2), and define $\tilde{\vartheta}_\lambda$ in a similar way, using $\tilde{\psi}$. We assume that the integrals in the definitions of $\vartheta_\lambda$ and $\tilde{\vartheta}_\lambda$ are convergent and finite. If $-\psi''(\xi)/\psi'(\xi) \leq -\tilde{\psi}''(\xi)/\tilde{\psi}'(\xi)$ for all $\xi > 0$, then $\vartheta_\lambda \leq \tilde{\vartheta}_\lambda$ for all $\lambda > 0$.

Proposition 4.2 (see Example 6.1 in [28]). Let $\varrho \neq 0$ and $\psi(\xi) = \xi^\varrho$ if $\varrho > 0$, $\psi(\xi) = -\xi^\varrho$ if $\varrho < 0$. Define $\vartheta_\lambda$ by (4.2). Then $\vartheta_\lambda = (1-\varrho)\pi/4$.

From now on, $\psi$ is again the complete Bernstein function such that $\Psi(\xi) = \psi(\xi^2)$ is the Lévy-Khintchine exponent of the process $X_t$.

Proposition 4.3. We have

$$
\left(\inf_{\xi>0} \frac{\xi|\psi''(\xi)|}{\psi'(\xi)}\right)^2 \frac{\pi}{4} \leq \vartheta_\lambda \leq \left(\sup_{\xi>0} \frac{\xi|\psi''(\xi)|}{\psi'(\xi)}\right)^2 \frac{\pi}{4}.
$$

Proof. Denote the supremum in (4.3) by $1 - \varrho$. By Proposition 2.3(b), $\varrho \in [-1, 1]$. Suppose that $\varrho$ is non-zero, and let $\psi(\xi) = \xi^\varrho$ if $\varrho > 0$, $\tilde{\psi}(\xi) = -\xi^\varrho$ if $\varrho < 0$. Observe that $\tilde{\psi}$ is increasing and $-\xi\tilde{\psi}''(\xi)/\tilde{\psi}'(\xi) = (1-\varrho)$, so that

$$
-\psi''(\zeta)/\psi'(\zeta) \leq -\tilde{\psi}''(\zeta)/\tilde{\psi}'(\zeta), \quad \zeta > 0.
$$

The upper bound in (4.3) follows by Propositions 4.1 and 4.2. When $\varrho = 0$, we simply consider $\tilde{\psi}(\xi) = -\xi^{-\varepsilon}$ and let $\varepsilon \to 0^+$. Finally, the lower bound is proved in a similar manner.

We conclude this section with a local estimate of $\vartheta_\lambda$, which depends solely on $\psi(\lambda^2)$, $\psi'(\lambda^2)$ and $\psi''(\lambda^2)$. This result is used only in the construction of an irregular example in Subsection 7.6. We need the following technical result.

Proposition 4.4. For $a > 0$,

$$
-\frac{1}{\pi} \int_0^\infty \frac{1}{1-x^2} \log \frac{1+a^2x^2}{1+a^2} \; dx = \arctan a.
$$

For $b \in (0, 1)$,

$$
-\frac{1}{\pi} \int_0^1 \frac{1}{1-x^2} \log(1-b^2(1-x^2)) \; dx = \frac{(\arcsin b)^2}{\pi}.
$$
Proof. The function $\log((1 + a^2 x^2)/(1 + a^2))$ is holomorphic in the upper half-plane with a branch cut along $(ia^{-1}, i\infty)$. By an appropriate contour integration and limit procedure (we omit the details),

$$2 \int_0^\infty \frac{1}{1 - x^2} \log \frac{1 + a^2 x^2}{1 + a^2} \, dx = \int_0^\infty \frac{1}{1 - x^2} \log \frac{1 + a^2 x^2}{1 + a^2} \, dx$$

$$= \int_{ia^{-1}}^{ia} \frac{2\pi i}{1 - x^2} \, dx = -2\pi \int_{1/a}^{\infty} \frac{1}{1 + y^2} \, dy = -2\pi \arctan a.$$

For the second equality, we have by Taylor expansion and beta integral,

$$- \int_0^\infty \frac{1}{1 - x^2} \log(1 - b^2(1 - x^2)) \, dx = \sum_{n=1}^\infty \frac{b^{2n}}{2n} \int_0^1 (1 - s)^{n-1}s^{-1/2} \, ds = \sum_{n=1}^\infty \frac{\Gamma(n)\Gamma(1/2)b^{2n}}{2n\Gamma(n + 1/2)} = (\arcsin b)^2,$$

see e.g. p. 108 in [31] for the last identity. \qed

Proposition 4.5. We have

$$\vartheta_\lambda \leq \frac{\pi}{2} - \arcsin \sqrt{\frac{\lambda^2 \psi'(\lambda^2)}{\psi(\lambda^2)}}$$

and

$$\vartheta_\lambda \geq \left( \arcsin \sqrt{\frac{\lambda^2 |\psi''(\lambda^2)|}{2\psi'(\lambda^2)}} \right)^2 + \left( \arcsin \sqrt{\frac{\lambda^2 \psi(\lambda^2) |\psi''(\lambda^2)|}{2\psi'(\lambda^2)(\psi(\lambda^2) - \lambda^2 \psi'(\lambda^2))}} \right)^2 - \left( \arcsin \sqrt{\frac{\lambda^4 |\psi''(\lambda^2)|}{2(\psi(\lambda^2) - \lambda^2 \psi'(\lambda^2))}} \right)^2.$$

It is easy to see that these bounds are sharp for $\psi(\xi) = c_1 \xi/(c_2 + \xi)$ and $\psi(\xi) = c \xi$ (and, in fact, these are all CBFs $\psi$ for which equalities hold), see examples in Section 7.

Proof. Note that $\psi_\lambda(0) = 1$ and $\psi_\lambda(\lambda^2) = \psi(\lambda^2)/(\lambda^2 \psi'(\lambda^2))$. Let $a^2 = \psi_\lambda(\lambda^2) - 1$. Since $\psi_\lambda$ is concave, we have $\psi_\lambda(\lambda^2 \zeta^2) \geq 1 + a^2 \zeta^2$ for $\zeta < 1$, and $\psi_\lambda(\lambda^2 \zeta^2) \leq 1 + a^2 \zeta^2$ for $\zeta > 1$. Hence, by Proposition 4.4,

$$\vartheta_\lambda = -\frac{1}{\pi} \int_0^\infty \frac{1}{1 - \zeta^2} \log \frac{\psi_\lambda(\lambda^2 \zeta^2)}{\psi_\lambda(\lambda^2)} \, d\zeta$$

$$\leq -\frac{1}{\pi} \int_0^\infty \frac{1}{1 - \zeta^2} \log \frac{1 + a^2 \zeta^2}{1 + a^2} \, d\zeta = \arctan a.$$

Since $\arctan a = \pi/2 - \arcsin(1/(1 + a^2)^{1/2})$, this proves the upper bound.

For the lower bound, let $a_1^2 = \psi_\lambda(\lambda^2) = 1 + a^2$ and $a_2^2 = \lambda^2 \psi_\lambda'(\lambda^2)/\psi_\lambda(\lambda^2)$. By a short calculation, $a_2^2 = \lambda^2 |\psi''(\lambda^2)|/(2\psi'(\lambda^2))$. Since $\psi_\lambda(\xi)$ is concave, its graph lies below the tangent line at $\xi = \lambda^2$, so that

$$\psi_\lambda(\lambda^2 \zeta^2) \leq \psi_\lambda(\lambda^2) - \lambda^2 \psi_\lambda'(\lambda^2)(1 - \zeta^2)$$

$$= a_1^2(1 - a_2^2(1 - \zeta^2)).$$
A similar argument for the CBF \(\xi/(\psi_\lambda(\xi) - 1)\) and a short calculation give
\[
\frac{\xi}{\psi_\lambda(\xi) - 1} \leq \frac{\lambda^2}{a^2} + \frac{a^2 - a_1^2a_2^2}{a^4} (\xi - \lambda^2)
\]
\[= \frac{a_2^2\xi - a_1^2a_2^2(\xi - \lambda^2)}{a^4},\]
which for \(\xi = \lambda^2/\zeta^2\) \((\zeta > 0)\) reduces to
\[
\psi_\lambda(\lambda^2/\zeta^2) \geq 1 + \frac{a_2^2\xi - a_1^2a_2^2(\xi - \lambda^2)}{a^2 + a_1^2a_2^2(1-\xi^2)}.
\]
Therefore, by (4.2),
\[
\vartheta_\lambda = \frac{1}{\pi} \int_0^1 \frac{1}{1 - \xi^2} \log \frac{\psi_\lambda(\lambda^2/\zeta^2)}{\psi_\lambda(\lambda^2)} \, d\zeta
\]
\[\geq \frac{1}{\pi} \int_0^1 \frac{1}{1 - \xi^2} \log \left( \frac{1 - a_1^2a_2^2(1-\xi^2)}{(1 - a_1^2(1-\xi^2))(1 - a_1^2a_2^2(1-\xi^2))} \right) \, d\zeta.
\]
Triple application of Proposition 4.4 yields
\[
\pi\vartheta_\lambda \geq (\arcsin(a_2))^2 + (\arcsin(a_1a_2/a))^2 - (\arcsin(a_2/a))^2. \quad \square
\]

5. Properties of \(F_\lambda(x)\)

Some basic estimates of the eigenfunctions \(F_\lambda(x)\) have already been established in [28]. However, for the proof of Theorems 1.6–1.8, more detailed properties of \(F_\lambda(x)\) are required. Recall that by (1.3) and (2.18),
\[
\mathcal{L}F_\lambda(\xi) = \frac{\lambda}{\lambda^2 + \xi^2} \frac{\psi_\lambda'(\xi)}{\sqrt{\psi_\lambda(\lambda^2)}}, \quad \lambda > 0, \, \text{Re} \, \xi > 0, \quad (5.1)
\]
and by (1.4),
\[
F_\lambda(x) = \sin(\lambda x + \vartheta_\lambda) - G_\lambda(x), \quad \lambda > 0, \, x > 0.
\]
The phase shift \(\vartheta_\lambda\) was studied in detail in the previous section. Recall that the correction term \(G_\lambda(x)\) is a completely monotone function, \(G_\lambda(x) = \mathcal{L}G_\lambda(x)\), and \(G_\lambda\) is a finite measure on \((0, \infty)\). In many important cases, \(G_\lambda\) is given explicitly by (1.6) and (2.18); however, in this section these identities are not used. We extend the definition of \(F_\lambda(x)\) by letting \(F_\lambda(x) = 0\) for \(x \leq 0\).

It was proved in Proposition 4.22 in [28] that
\[
\mathcal{L}F_\lambda(\xi) \leq \frac{\lambda + \xi}{\lambda^2 + \xi^2} \leq \frac{2}{\lambda + \xi}, \quad \lambda, \xi > 0. \quad (5.2)
\]
By combining (5.1) and Proposition 2.4 we obtain the following more detailed estimate.

Corollary 5.1. We have
\[
e^{-2c/\pi} \frac{\lambda}{\lambda^2 + \xi^2} \sqrt{\frac{\psi_\lambda(\xi^2)}{\psi_\lambda(\lambda^2)}} \leq \mathcal{L}F_\lambda(\xi) \leq e^{2c/\pi} \frac{\lambda}{\lambda^2 + \xi^2} \sqrt{\frac{\psi_\lambda(\xi^2)}{\psi_\lambda(\lambda^2)}} \quad (5.3)
\]
for \(\lambda, \xi > 0. \quad \square
Hölder continuity of $F_\lambda$ was already studied in Lemma 4.24 in [28]. However, when \( \psi \) does not have a power-type growth at infinity, $F_\lambda$ fails to be Hölder continuous. On the other hand, uniform estimate for $F_\lambda(x)$ for small $x > 0$ is crucial in the proof of Theorem 1.6. We generalize the estimates of [28] in the two lemmas below.

Since $G_\lambda(x)$ is completely monotone, we have $G'_\lambda(x) \leq 0$ and $G''_\lambda(x) \geq 0$ for $\lambda, x > 0$. It follows that the function $F_\lambda(x) = \sin(\lambda x + \theta_\lambda) - G_\lambda(x)$ is increasing on $[0, (\pi/2 - \theta_\lambda)/\lambda]$ and concave on $[0, (\pi - \theta_\lambda)/\lambda]$ (this explains the role of upper bounds for $\theta_\lambda$). Furthermore, $0 \leq G_\lambda(x) \leq \sin \theta_\lambda$ (Lemma 4.21 in [28]), and so $-1 - \sin \theta_\lambda \leq F_\lambda(x) \leq 1$.

**Lemma 5.2.** When $\lambda > 0$ and $0 < \lambda x \leq (\pi/2 - \theta_\lambda)/2$, we have

\[
\frac{e^{1-2c/\pi}}{2(e+1)} \lambda x \sqrt{\frac{\psi_\lambda(1/x^2)}{\psi_\lambda(\lambda^2)}} \leq F_\lambda(x) \leq \left( 2e^{1+2c/\pi} + \frac{4e}{\pi/2 - \theta_\lambda} \right) \lambda x \sqrt{\frac{\psi_\lambda(1/x^2)}{\psi_\lambda(\lambda^2)}}.
\]

**Proof.** Let $\lambda > 0$, $b = (\pi/2 - \theta_\lambda)/\lambda$ and $0 < a \leq b$. By concavity of the sine function and convexity of $G_\lambda$, for any $x > a$ we have

\[
\frac{\sin(\lambda x + \theta_\lambda) - \sin \theta_\lambda}{x} \leq \frac{\sin(\lambda a + \theta_\lambda) - \sin \theta_\lambda}{a},
\]

\[
\frac{\sin \theta_\lambda - G_\lambda(x)}{x} \leq \frac{\sin \theta_\lambda - G_\lambda(a)}{a}.
\]

Hence $F_\lambda(x) \leq (x/a) F_\lambda(a)$ when $x > a$. This and monotonicity of $F_\lambda$ on $[0, b]$ yield

\[
F_\lambda(x) \leq F_\lambda(a) \max(1, x/a)
\]

for all $x > 0$. By Proposition 2.11 for $\xi > 0$ we have

\[
F_\lambda(a) \geq \frac{\xi LF_\lambda(\xi)}{1 + (a \xi)^{-1} e^{-a \xi}}.
\]

For $\xi = 1/a$, this gives

\[
F_\lambda(a) \geq \frac{e}{e+1} \frac{LF_\lambda(1/a)}{a}.
\]

On the other hand, $F_\lambda(x)$ is increasing in $x \in (0, b)$, and clearly $F_\lambda(x) \geq -2$ for all $x \geq b$. Proposition 2.13 gives

\[
F_\lambda(a) \leq \frac{e^a \xi LF_\lambda(\xi) + 2e^{-(\pi/2 - \theta_\lambda - \lambda a)\xi/\lambda}}{1 - e^{-(\pi/2 - \theta_\lambda - \lambda a)\xi/\lambda}}
\]

for all $\xi > 0$. Let $\xi = 1/a$. Using the inequality $1 - e^{-s} \geq s/(s+1)$ ($s > 0$) in the denominator and $e^{-1/s} \leq s$ ($s > 0$) in the numerator, we obtain that

\[
F_\lambda(a) \leq \frac{1}{1 - e^{-(\pi/2 - \theta_\lambda - \lambda a)/(\lambda a)}} \left( \frac{e LF_\lambda(1/a)}{a} + 2e^{-(\pi/2 - \theta_\lambda - \lambda a)/(\lambda a)} \right)
\]

\[= \frac{e}{1 - e^{-(\pi/2 - \theta_\lambda - \lambda a)/(\lambda a)}} \left( \frac{LF_\lambda(1/a)}{a} + 2e^{-(\pi/2 - \theta_\lambda)/(\lambda a)} \right)
\]

\[\leq \frac{e(\pi/2 - \theta_\lambda)}{\pi/2 - \theta_\lambda - \lambda a} \left( \frac{LF_\lambda(1/a)}{a} + \frac{2\lambda a}{\pi/2 - \theta_\lambda} \right).
\]
The above bounds for $F_{\lambda}(a)$ combined with (5.3) give
\[
e^{1-2c/\pi} \frac{\lambda/\tau}{e + 1} \frac{\sqrt{\psi_{\lambda}(1/\tau)}}{\psi_{\lambda}(\tau^2)} \leq F_{\lambda}(a)
\]
\[
\leq \frac{e(\pi/2 - \vartheta_{\lambda})}{\pi/2 - \vartheta_{\lambda} - \lambda} \left( e^{\sqrt{\psi_{\lambda}(1/\tau)}} \psi_{\lambda}(\tau^2) + \frac{2\lambda a}{\pi/2 - \vartheta_{\lambda}} \right).
\]

Assume now that $a \in (0, b/2]$. Then $a \leq b/2 = (\pi/2 - \vartheta_{\lambda})/(2\lambda) \leq 1/\lambda$, and therefore $1/\lambda^2 \leq \lambda^2 + 1/\lambda^2 \leq 2/\lambda^2$. It follows that
\[
e^{1-2c/\pi} \frac{\lambda/\tau}{e + 1} \frac{\sqrt{\psi_{\lambda}(1/\tau)}}{\psi_{\lambda}(\tau^2)} \leq F_{\lambda}(a)
\]
\[
\leq \frac{e(\pi/2 - \vartheta_{\lambda})}{\pi/2 - \vartheta_{\lambda} - \lambda} \left( e^{\sqrt{\psi_{\lambda}(1/\tau)}} \psi_{\lambda}(\tau^2) + \frac{2\lambda a}{\pi/2 - \vartheta_{\lambda}} \right).
\]

Since $\psi_{\lambda}(\xi)$ is increasing on $(0, \infty)$ and $\lambda < 1/\tau$, we have $1 \leq \sqrt{\psi_{\lambda}(1/\tau^2)}/\psi_{\lambda}(\tau^2)$. Finally, $\lambda a \leq \lambda b/2 = (\pi/2 - \vartheta_{\lambda})/2$, so that $(\pi/2 - \vartheta_{\lambda})/(\pi/2 - \vartheta_{\lambda} - \lambda a) \leq 2$. Formula (5.4) follows.

Since $F_{\lambda}(x) = \sin(\lambda x + \vartheta_{\lambda}) - G_{\lambda}(x)$ for a completely monotone $G_{\lambda}$, the modulus of continuity of $F_{\lambda}$ is described by the behavior of $F_{\lambda}(x)$ for small $x$. More precisely, we have the following result.

**Lemma 5.3.** We have
\[
|F_{\lambda}(x) - F_{\lambda}(y)| \leq \min \left( \frac{30\lambda|x - y|}{\pi/2 - \vartheta_{\lambda}} \sqrt{\frac{\psi_{\lambda}(1/|x - y|^2)}{\psi_{\lambda}(\lambda^2)}}, 4 \right)
\]
for all $\lambda > 0$ and $x, y \geq 0$.

**Proof.** The inequality $|F_{\lambda}(x) - F_{\lambda}(y)| \leq 4$ is clear. For $x, y > 0$ and $\lambda > 0$,
\[
|F_{\lambda}(x) - F_{\lambda}(y)| \leq |G_{\lambda}(x) - G_{\lambda}(y)| + \lambda|x - y|
\]
\[
\leq |G_{\lambda}(|x - y|) - G_{\lambda}(0^+)| + \lambda|x - y|
\]
\[
\leq |F_{\lambda}(|x - y|)| + 2\lambda|x - y|.
\]

The same inequality is obviously true also when $x = 0$ or $y = 0$, so from now on we assume that $x, y \geq 0$.

Suppose that $\lambda|x - y| \leq (\pi/2 - \vartheta_{\lambda})/2$. Then, by (5.6) and (5.4),
\[
|F_{\lambda}(x) - F_{\lambda}(y)| \leq \left( 2e^{1+2c/\pi} + \frac{4e}{\pi/2 - \vartheta_{\lambda}} \right) \lambda|x - y| \sqrt{\frac{\psi_{\lambda}(1/|x - y|^2)}{\psi_{\lambda}(\lambda^2)}} + 2\lambda|x - y|.
\]

As in the proof of Lemma 5.2, $1 \leq \sqrt{\psi_{\lambda}(1/|x - y|^2)}/\psi_{\lambda}(\lambda^2)$ (indeed $\lambda < 1/|x - y|$, and $\psi_{\lambda}$ is increasing), so that
\[
|F_{\lambda}(x) - F_{\lambda}(y)| \leq \left( 2e^{1+2c/\pi} + \frac{4e}{\pi/2 - \vartheta_{\lambda}} + 2 \right) \lambda|x - y| \sqrt{\frac{\psi_{\lambda}(1/|x - y|^2)}{\psi_{\lambda}(\lambda^2)}}.
\]

Finally, the parenthesised expression does not exceed $(\pi e^{1+2c/\pi} + 4e + \pi)/(\pi/2 - \vartheta_{\lambda})$, and (5.5) follows.
It remains to consider $\lambda|x-y| > (\pi/2 - \vartheta_\lambda)/2$. We claim that in this case, the minimum in (5.5) is equal to 4. By Proposition 2.2(a), $f(\xi) = \xi/\psi(\xi)$ is a CBF. Hence, $f(x)/f(a) \geq \min(1, x/a)$. We obtain

$$\frac{\lambda|x-y|}{\pi/2 - \vartheta_\lambda} \sqrt{\frac{\psi(1/|x-y|^2)}{\psi(\lambda^2)}} = \frac{1}{\pi/2 - \vartheta_\lambda} \sqrt{\frac{\lambda^2/\psi(\lambda^2)}{(1/|x-y|^2)/\psi(1/|x-y|^2)}} \geq \frac{\min(\lambda|x-y|, 1)}{\pi/2 - \vartheta_\lambda} > \frac{1}{2}.$$  

This proves our claim, and the proof is complete. □

The behavior of $F_\lambda(x)$ when $x \to 0^+$ (with fixed $\lambda > 0$) was studied in Lemma 4.27 in [23]: if $\psi$ is an unbounded CBF regularly varying of order $\alpha \in [0, 1]$ at infinity, then $F_\lambda(x)$ is regularly varying of order $\alpha$ at 0, and

$$\lim_{x \to 0^+} \left( \sqrt{\psi(1/x^2)} F_\lambda(x) \right) = \frac{\sqrt{\lambda^2 \psi'((\lambda^2)}{\Gamma(1+\alpha)} , \quad \lambda > 0.\tag{5.7}$$  

The next result shows that as $\lambda \to 0^+$ with fixed $x > 0$, $F_\lambda(x)$ behaves as $\lambda \sqrt{\psi'((\lambda^2)} V(x)$, where $V(x)$ is the renewal function of the ascending ladder-height process, $LV(\xi) = \xi/\psi(\xi)$ (see Preliminaries).

**Proposition 5.4.** Suppose that $\psi$ is an unbounded CBF. The function $F_\lambda(x)$ is jointly continuous in $\lambda > 0, x \geq 0$. Furthermore, if $\limsup_{\lambda \to 0^+} \vartheta_\lambda < \pi/2$, then

$$\lim_{\lambda \to 0^+} \frac{F_\lambda(x)}{\lambda \sqrt{\psi'((\lambda^2)} = V(x), \quad x \geq 0,$$

and the convergence is locally uniform in $x \geq 0$. In other words, $F_\lambda(x)/((\lambda \sqrt{\psi'((\lambda^2)}$ extends to a continuous function in $\lambda, x \geq 0$.

**Proof.** Roughly speaking, we use estimates of $LF_\lambda$ to show $L^2$ continuity of $\psi(x)$ in $\lambda > 0$, and then estimates of $F_\lambda$ to replace $L^2$ convergence by locally uniform convergence.

Let $\lambda_n \to \lambda > 0$. By Corollary 2.9, $LF_{\lambda_n}(1 + i\xi)$ converges to $LF_\lambda(1 + i\xi)$ pointwise for $\xi \in \mathbb{R}$. By (5.2) and dominated convergence, $LF_{\lambda_n}(1 + i\xi)$ converges to $LF_\lambda(1 + i\xi)$ in $L^2(\mathbb{R})$. Note that $LF_\lambda(1 + i\xi)$ is the Fourier transform of $e^{-\lambda V(x)}$. By Plancherel’s theorem, $e^{-\lambda F_{\lambda_n}(x)}$ converges to $e^{-\lambda F_\lambda(x)}$ in $L^2(\mathbb{R})$. By (5.5), the sequence $F_{\lambda_n}(x)$ is equicontinuous in $x$ (here we use the assumption that $\psi$ is unbounded), and hence it converges to $F_\lambda(x)$ locally uniformly in $x \geq 0$. The first part of the proposition is proved.

As $\lambda \to 0^+$, the functions $(\lambda^2/\psi((\lambda^2)) \psi(\xi))$ converge pointwise to the CBF $\xi/\psi(\xi)$.

Therefore, by Proposition 2.7, $\lambda^2/\psi((\lambda^2)) \psi(\xi)$ converges to $\xi/\psi(\xi)$ ($\xi \in \mathbb{R}$). We conclude that $LF_\lambda(1 + i\xi)/(\lambda \sqrt{\psi'((\lambda^2)}$ converges to $LV(1 + i\xi)$ ($\xi \in \mathbb{R}$). Again, we obtain $L^2(\mathbb{R})$ convergence of $e^{-\lambda F_\lambda(x)/(\lambda \sqrt{\psi'((\lambda^2)}$ to $e^{-\lambda V(x)}$. The proof will be complete if we show that the family of functions $F_\lambda(x)/(\lambda \sqrt{\psi'((\lambda^2)}$ is equicontinuous as $\lambda \to 0^+$.

By (5.5), we have

$$|F_\lambda(x) - F_\lambda(y)| \leq \frac{30|x-y|}{\pi/2 - \vartheta_\lambda} \sqrt{\frac{\psi(1/|x-y|^2)}{\psi((\lambda^2)) \psi(\lambda^2)}} \leq \frac{30}{\pi/2 - \vartheta_\lambda} \sqrt{\frac{1 - \lambda^2|x-y|^2}{\psi(1/|x-y|^2) - \psi((\lambda^2))}.\tag{5.8}$$


Fix $\lambda_0 > 0$ small enough. By the assumption, $\pi/2 - \vartheta_\lambda$ is bounded below by a positive constant for $\lambda \in (0, \lambda_0)$. It follows that for $\lambda \in (0, \lambda_0)$ and $x, y \geq 0$ satisfying $|x - y| < 1/\lambda_0$, the expression on the right hand side of (6.8) is bounded above by $c/\sqrt{\psi(1/|x - y|)}^2 - \psi(\lambda_0^2)$ (with $c$ depending only on $\psi$ and $\lambda_0$). This upper bound does not depend on $\lambda \in (0, \lambda_0)$, and since $\psi$ is unbounded, it converges to 0 as $|x - y| \to 0$. □

6. SUPREMA AND FIRST PASSAGE TIMES FOR COMPLETE BERNSTEIN FUNCTIONS

Below we prove Theorems 1.6–1.8. We remark that the condition (1.7) is used only to assert that $\sup_{x > 0} \vartheta_\lambda < \pi/2$, and can be replaced by the latter condition.

Proof of Theorem 1.6 First we consider $n = 0$, that is, we will show that

$$\mathbb{P}(\tau_x > t) = \frac{2}{\pi} \int_0^\infty \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} e^{-t\psi(\lambda^2)} F_\lambda(x) d\lambda. \tag{6.1}$$

We claim that the integral in (6.1) converges, and that $e^{-t\xi} e^{-t\psi(\lambda^2)} F_\lambda(x) \sqrt{\psi'(\lambda^2)/\psi(\lambda^2)}$ is jointly integrable in $\lambda, x > 0$ for any $\xi > 0$. Assuming this is true, the proof is quite straightforward. Indeed, by Fubini, the Laplace transform (in $x$) of the right hand side of (6.1) is then

$$\frac{2}{\pi} \int_0^\infty \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} e^{-t\psi(\lambda^2)} \mathcal{L} F_\lambda(x) d\lambda.$$ 

By Theorem 1.3 (see (2.16)) and Theorem 1.4 (see (2.18)), this is equal to the Laplace transform (in $x$) of $\mathbb{P}(\tau_x > t)$, and the result follows by the uniqueness of the Laplace transform. Hence it is enough to prove our claim.

Let $t \geq t_0$, $\Theta = \sup_{x > 0} \vartheta_\lambda$ and $\lambda_0 = (\pi/2 - \Theta)/(2x)$. By the assumption (1.7) and Proposition 1.3, $\Theta < \pi/2$, and hence $\lambda_0 > 0$. Note that $\lambda_0 x \leq \pi/4 < 1$. Since $|F_\lambda(x)| \leq 2$ and $\sqrt{\psi'(\lambda^2)/\psi(\lambda^2)} \leq 1/\lambda$, we have

$$\int_{\lambda_0}^\infty \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} e^{-t\psi(\lambda^2)} |F_\lambda(x)| d\lambda$$

$$\leq 2 \int_{\min(\lambda_0, 1)}^1 \frac{e^{-t\psi(\lambda^2)}}{\lambda} d\lambda + 2 \int_{1}^\infty \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} e^{-t\psi(\lambda^2)} d\lambda$$

$$\leq 2 \max(0, -\log \lambda_0) + 2 \int_{1}^\infty \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} e^{-t\psi(\lambda^2)} d\lambda,$$

which is finite by assumption (1.8). We now consider $\lambda \in (0, \lambda_0)$. By Lemma 5.2 we have

$$0 \leq \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} e^{-t\psi(\lambda^2)} F_\lambda(x) \leq c_1(\Theta) \lambda x \sqrt{\frac{\psi'(\lambda^2) \psi(1/x^2)}{\psi(\lambda^2)\psi(\lambda^2)}}.$$ 

Furthermore, by (2.15) (recall that $\lambda < \lambda_0 < 1/x$),

$$\lambda x \sqrt{\frac{\psi'(\lambda^2) \psi(1/x^2)}{\psi(\lambda^2)\psi(\lambda^2)}} \leq \frac{\lambda \psi'(\lambda^2)}{\sqrt{\psi(\lambda^2)(\psi(1/x^2) - \psi(\lambda^2))}}.$$
Integration and substitution $z = \psi(\lambda^2)$ give
\[
\int_0^\lambda \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} e^{-\psi(\lambda^2)} |F_\lambda(x)| d\lambda \leq c_1(\Theta) \int_0^\lambda \frac{\lambda \psi'(\lambda^2)}{\sqrt{\psi(\lambda^2)(\psi(1/x^2) - \psi(\lambda^2))}} d\lambda
\]
\[
= c_1(\Theta) \int_0^\lambda \frac{1}{\sqrt{z(\psi(1/x^2) - z)}} dz \leq \frac{c_1(\Theta)}{2} \pi.
\]
We conclude that
\[
\int_0^\infty \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} e^{-\psi(\lambda^2)} |F_\lambda(x)| d\lambda \leq c_2(\psi, t) + 2 \max(0, \log x),
\]
which shows that the integral in (6.1) is absolutely convergent. This also shows that
\[
\int_0^\infty \int_0^\infty e^{-\xi x} \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} e^{-\psi(\lambda^2)} |F_\lambda(x)| d\lambda d\xi < \infty,
\]
as desired. The proof in the case $n = 0$ is complete.

We now find upper and lower bounds for the derivatives in $t$ of $P(\tau_x > t)$. Note that the estimates of $P(\tau_x > t)$ are covered by [29] for a wider class of Lévy processes. We begin with a simple technical result.

**Proposition 6.1.** (a) Let $\gamma(a; x) = \int_0^x e^{-s/a} ds$ ($a, x > 0$) be the lower incomplete gamma function. Then
\[
\frac{\min(1, x^a)}{ae} \leq \gamma(a; x) \leq \frac{\min(\Gamma(a + 1), x^a)}{a}, \quad a, x > 0. \tag{6.2}
\]
(b) We have
\[
\int_0^x \frac{e^{-s/a}}{\sqrt{x-s}} ds \leq \frac{\min(2\Gamma(a) + \pi a e^{-a}, 2(1/a + 1)x^a)}{\sqrt{2x}}, \quad a, x > 0. \tag{6.3}
\]

**Proof.** For the part (a), we simply have
\[
\gamma(a; x) = \int_0^x e^{-s/a} ds \geq \frac{1}{a} \int_0^{\min(1, x)} s^{a-1} ds = \frac{\min(1, x^a)}{ae},
\]
and
\[
\gamma(a; x) = \int_0^x e^{-s/a} ds \leq \min \left( \Gamma(a), \int_0^x s^{a-1} ds \right) = \min \left( \Gamma(a), \frac{x^a}{a} \right).
\]
To prove (b), we split the integral into two parts. First,
\[
\int_0^{x/2} \frac{e^{-s/a}}{\sqrt{x-s}} ds \leq \sqrt{\frac{2}{x}} \int_0^{x/2} e^{-s/a} ds
\]
\[
= \sqrt{\frac{2}{x}} \gamma(a; x/2) \leq \sqrt{\frac{2}{x}} \min \left( \Gamma(a), \frac{x^a}{a} \right).
\]
Next, \( s^a e^{-s} \) attains its maximum at \( s = a \). Hence,

\[
\int_{x/2}^{x} \frac{e^{-s} s^{a-1}}{\sqrt{x - s}} \, ds \leq a^a e^{-a} \int_{x/2}^{x} \frac{1}{\sqrt{s^2(x - s)}} \, ds
\]

\[
\leq a^a e^{-a} \sqrt{\frac{2}{x}} \int_{x/2}^{x} \frac{1}{\sqrt{s(x - s)}} \, ds = \frac{\pi a^a e^{-a}}{2} \sqrt{\frac{2}{x}}.
\]

Finally,

\[
\int_{x/2}^{x} \frac{e^{-s} s^{a-1}}{\sqrt{x - s}} \, ds \leq a^{a-1} \int_{x/2}^{x} \frac{1}{\sqrt{x - s}} \, ds = \sqrt{2} x^{a-1/2}.
\]

It follows that

\[
\int_{0}^{x} \frac{e^{-s} s^{a-1}}{\sqrt{x - s}} \, ds \leq \sqrt{\frac{2}{x}} \min\left(\frac{\Gamma(a + 1)}{a}, x^a\right) \leq \sqrt{\frac{2}{x}} \left( \min\left(\frac{\Gamma(a + 1)}{a}, x^a\right) + \min\left(\frac{\pi a^a e^{-a}}{2}, x^a\right) \right),
\]

which gives (6.3).

\[\square\]

**Lemma 6.2.** Suppose that (1.7) and (1.8) hold for some \( t_0 > 0 \). For \( n \geq 0 \), \( t > t_0 \), \( t \geq t_0 \) if \( n = 0 \) and \( \lambda_0 > 0 \), denote (cf (1.8))

\[
I_n(t_0, \lambda_0) = \frac{4}{\pi} \int_{\lambda_0}^{\infty} e^{-t_0 \psi(\lambda^2)} \sqrt{\psi'(\lambda^2) \psi(\lambda^2)} \, \lambda \, d\lambda.
\]

Let \( \Theta = \sup_{\xi > 0} \vartheta_\xi \) and \( \lambda_0(x) = (\pi/2 - \Theta)/(2x) \). If \( t > t_0 \) \( (t \geq t_0 \) if \( n = 0 \) \) and \( x > 0 \), then

\[
C_1(n, \Theta) \min\left(\left(\frac{\psi(1/x^2)}{t_n+1/2 \sqrt{\psi(1/x^2)}}\right)^n, \frac{1}{t_n+1/2 \sqrt{\psi(1/x^2)}}\right) + I_n(t, \lambda_0(x))
\]

\[
\leq (-1)^n \frac{d^n}{d\sigma^n} P(\tau > t) \quad \text{(6.4)}
\]

\[
\leq C_2(n, \Theta) \min\left(\left(\frac{\psi(1/x^2)}{t_n+1/2 \sqrt{\psi(1/x^2)}}\right)^n, \frac{1}{t_n+1/2 \sqrt{\psi(1/x^2)}}\right) - I_n(t, \lambda_0(x)).
\]

**Remark 6.3.**

1. When (1.8) holds for some \( t_0 > 0 \), then clearly \( I_0(t_0, \lambda_0) < \infty \) for any \( \lambda_0 > 0 \). Furthermore, for any \( n \geq 0, t > t_0 \) and \( \lambda_0 > 0 \), we have \( e^{-(t-t_0)\psi(\lambda^2)} \leq c(n, \lambda_0)(\psi(\lambda^2))^n \) for \( \lambda \geq \lambda_0 \). Hence, (1.8) implies also that \( I_n(t, \lambda_0) < \infty \) for any \( n \geq 0, t > t_0 \) and \( \lambda_0 > 0 \).

2. When \( t > t_0 \), then \( I_n(t, \lambda_0) \leq e^{-(t-t_0)\psi(\lambda_0^2)} I_n(t_0, \lambda_0) \). Hence, \( I_n(t, \lambda_0) \), if finite, decays exponentially fast as \( t \to \infty \). Therefore, \( I_n(t, \lambda_0) \) can be considered as the ‘error term’; see Theorem 1.8.

\[\square\]

**Proof of Lemma 6.2.** By the assumption (1.7) and Proposition 4.3, \( \Theta < \pi/2 \). Let \( k = (\pi/2 - \Theta)/2 \in (0, \pi/4) \), so that \( \lambda_0 = \lambda_0(x) = k/x \). Denote the integrand in (1.9) by \( f_{n,t,x}(\lambda) \),

\[
f_{n,t,x}(\lambda) = \sqrt{\frac{\psi'(\lambda^2)}{\psi(\lambda^2)}} (\psi(\lambda^2))^n e^{-t\psi(\lambda^2)} F_\lambda(x).
\]
By Lemma 5.2 for \( \lambda \in (0, \lambda_0) \) we have

\[
f_{n,t,x}(\lambda) \geq (\psi(\lambda^2))^{n-1/2} \sqrt{\psi'(\lambda^2)} e^{-t \psi(\lambda^2)} \frac{e^{1-2c/\pi} \lambda \psi(\lambda^2)}{2(e + 1)} \sqrt{\frac{\psi(1/x^2)}{\psi(1/x^2)}}
\]

\[
= \frac{e^{1-2c/\pi} \lambda \psi(\lambda^2)}{2(e + 1)} e^{-t \psi(\lambda^2)} (\psi(\lambda^2))^{n-1/2} \sqrt{1 - \lambda^2 x^2} \sqrt{\psi(1/x^2)} - \psi(\lambda^2)
\]

where \( \gamma \) is the lower incomplete gamma function. By (6.2),

\[
\int_0^{\lambda_0} f_{n,t,x}(\lambda) d\lambda \geq \frac{e^{1-2c/\pi} \sqrt{1 - k^2}}{4(e + 1) \sqrt{\psi(1/x^2)}} \int_0^{\psi(\lambda_0^2)} \frac{z n^{-1/2} e^{-tz} dz}{\psi(1/x^2)}
\]

where \( z = \psi(\lambda^2) \).

Since \( \lambda x < k \) and \( \psi(\lambda_0^2) = \psi(k^2/x^2) \geq k^2 \psi(1/x^2) \) (\( \psi \) is nonnegative and concave, and \( k < 1 \)), we obtain (with \( z = \psi(\lambda^2) \))

\[
\int_0^{\lambda_0} f_{n,t,x}(\lambda) d\lambda \geq \frac{e^{1-2c/\pi} \sqrt{1 - k^2}}{4(e + 1) \sqrt{\psi(1/x^2)}} \frac{\min(1, (k^2/t \psi(1/x^2))^{n+1/2})}{(n + 1/2) e}.
\]

(6.5)

In a similar way,

\[
f_{n,t,x}(\lambda) \leq \left( 2e^{1+2c/\pi} + \frac{2e}{k} \right) \lambda \psi(\lambda^2) e^{-t \psi(\lambda^2)} (\psi(\lambda^2))^{n-1/2} \sqrt{\psi(1/x^2)} - \psi(\lambda^2),
\]

and so

\[
\int_0^{\lambda_0} f_{n,t,x}(\lambda) d\lambda \leq \left( e^{1+2c/\pi} + \frac{e}{k} \right) \int_0^{\psi(1/x^2)} \frac{z n^{-1/2} e^{-tz}}{\sqrt{\psi(1/x^2)} - z} dz
\]

\[
= \left( e^{1+2c/\pi} + \frac{e}{k} \right) \frac{1}{tn} \int_0^{\psi(1/x^2)} \frac{s n^{-1/2} e^{-s}}{\sqrt{t \psi(1/x^2)} - s} ds.
\]

By (6.3), with \( c_1(n) = 2\Gamma(n + 1/2) + \pi((n + 1/2)/e)^{n+1/2} \) and \( c_2(n) = 2(1/(n + 1/2) + 1) \),

\[
\int_0^{\lambda_0} f_{n,t,x}(\lambda) d\lambda \leq \left( e^{1+2c/\pi} + \frac{e}{k} \right) \frac{\min(c_1(n), c_2(n)(t \psi(1/x^2))^{n+1/2})}{tn \sqrt{2t \psi(1/x^2)}}.
\]

(6.6)

Hence, we found a two-sided estimate for the integral of \( f_{n,t,x}(\lambda) \) over \((0, \lambda_0)\). The integral over \((\lambda_0, \infty)\) is highly oscillatory, and therefore difficult to estimate. For this reason, we are satisfied with a simple bound obtained using the inequality \( |F_{\lambda}(x)| \leq 2 \).

\[
\frac{2}{\pi} \int_0^{\lambda_0} |f_{n,t,x}(\lambda)| d\lambda \leq \frac{4}{\pi} \int_0^{\lambda_0} (\psi(\lambda^2))^{n-1/2} \sqrt{\psi'(\lambda^2)} e^{-t \psi(\lambda^2)} d\lambda = I_n(t, \lambda_0).
\]

(6.7)

The lower bound in (6.4) is a consequence of (6.5) and (6.7), and the upper bound in (6.4) follows from (6.6) and (6.7). \( \square \)
We remark that in the statement of the lemma, we can take
\[ C_1(n, \Theta) = \frac{e^{1-2C/\pi} \sqrt{1-\pi^2/16 (\pi/4 - \Theta)^2}^{n+1}}{2\pi^e(e + 1)(n + 1/2)} \geq (\pi/2 - \Theta)^{2n+1}/17(n + 1/2)^2, \]
\[ C_2(n, \Theta) = \frac{e^{\sqrt{2}}}{\pi} \left( \frac{e^{2C/\pi} + \frac{2}{\pi/2 - \Theta}}{(2\Gamma(n + 1/2) + \pi((n + 1/2)/e)^{n+1/2})} \right) \leq \frac{15n!}{\pi/2 - \Theta}. \]
Note that \( C_1(n, \Theta) \) decreases with \( \Theta \), while \( C_2(n, \Theta) \) increases with \( \Theta \). The notation of Lemma 6.2, namely \( I_n(t_0, \lambda_0) \), \( C(n, \Theta) \) and \( C_2(n, \Theta) \), is kept in the remaining part of the section.

**Corollary 6.4.** Let \( \varepsilon > 0, n \geq 1, t_0 > 0, x_0 > 0 \). If the conditions (1.7) and (1.8) hold true, then there are positive constants \( C_3(n, \Theta), C_4(n, \Theta), C_5(\varepsilon, n, \Theta, I_n(t_0, \lambda_0(x_0))) \) (here \( \Theta = \sup_{x > 0} \vartheta_\lambda(x) \)) such that
\[ \frac{C_3(n, \Theta)}{t^{n+1/2} / \psi(1/x^2)} \leq (-1)^n \frac{d^n}{dt^n} P(\tau_x > t) \leq \frac{C_4(n, \Theta)}{t^{n+1/2} / \psi(1/x^2)} \] (6.8)
when \( x \in (0, x_0] \) and
\[ t \geq \max (1 + \varepsilon)t_0, \frac{1}{\psi(1/x^2)}, \frac{C_5(\varepsilon, n, \Theta, I_n(t_0, \lambda_0(x_0)))}{(\psi(1/x^2))^{1+\varepsilon}}. \]

**Proof.** This is a combination of Lemma 6.2 and Remark 6.3. As in the proof of Lemma 6.2, we let \( k = (\pi/2 - \Theta)/2 \), so that \( \lambda_0 = \lambda_0(x) = k/x \). Recall that since \( k < 1 \), we have \( \psi(k^2/x^2) \geq k^2 \psi(1/x^2) \). It follows that for \( n \geq 1, \ varepsilon > 0, x > 0 \) and \( t > (1 + \varepsilon)t_0 \), we have
\[ \frac{I_n(t, \lambda_0)}{I_n(t_0, \lambda_0)} \leq e^{-(t-t_0)\psi(\lambda_0)} \leq e^{-\varepsilon(1+\varepsilon)t\psi(1/x^2)} \]
\[ \leq e^{-k^2\varepsilon/(1+\varepsilon)t\psi(1/x^2)} \leq \frac{c_2(\varepsilon, n, \Theta)}{(t\psi(1/x^2))^{n+1/2+n/\varepsilon}}. \]
Fix \( A > 0 \). When \( t(\psi(1/x^2))^{1+\varepsilon} \geq A^2 \), we obtain
\[ I_n(t, \lambda_0) \leq \frac{c_2(\varepsilon, n, \Theta)}{A^n t^{n+1/2} / \psi(1/x^2)} I_n(t_0, \lambda_0). \]
Hence, by (6.4), if \( t(\psi(1/x^2)) \geq 1 \), we have (with the constants \( C_1(n, \Theta) \) and \( C_2(n, \Theta) \) of Lemma 6.2)
\[ \left( C_1(n, \Theta) - \frac{c_2(\varepsilon, n, \Theta)}{A^n} \right) \frac{1}{t^{n+1/2} / \psi(1/x^2)} \leq (-1)^n \frac{d^n}{dt^n} P(\tau_x > t) \]
\[ \leq \left( C_2(n, \Theta) + \frac{c_2(\varepsilon, n, \Theta)}{A^n} \right) \frac{1}{t^{n+1/2} / \psi(1/x^2)}. \]
When \( x \in (0, x_0] \), then \( I_n(t_0, \lambda_0(x)) \leq I_n(t_0, \lambda_0(x_0)) \). Hence, (6.8) holds if \( A > (C_5(\varepsilon, n, \Theta, I_n(t_0, \lambda_0(x_0))))^{1/\varepsilon} \) for some \( C_5 \).

**Proof of Theorem 1.7.** Part (a) follows directly from Corollary 6.4. For part (b), suppose that \( \|\psi''(\xi)/\psi'(\xi)\| \leq \varrho/\xi \) for some \( \varrho \in [0, 1) \) and all \( \xi > 0 \). By Proposition 4.3, with the notation of Lemma 6.2, we have \( \Theta \geq \varrho \pi/4 \).

Integrating the inequality \( -\psi''(\xi)/\psi'(\xi) \leq \varrho/\xi \), we obtain \( \psi'(\xi_1)/\psi'(\xi_2) \leq (\xi_2/\xi_1)^{\varrho} \) if \( 0 < \xi_1 < \xi_2 \). Integrating this again in \( \xi_1 \) gives \( \psi(\xi)/\psi'(\xi) \leq \xi/(1 - \varrho) \). Hence, for all
where $\Gamma(a; z)$ is the upper incomplete Gamma function. In particular, $I_n(t, \lambda_0)$ is finite, and (1.8) holds true.

Let $k = (\pi/2 - \Theta)/2 \in (\pi/8, \pi/4)$, and take $\lambda_0 = \lambda_0(x) = k/x$, as in Lemma 6.2. Recall that since $k < 1$, we have $\psi(\lambda_0^2) \geq k^2 \psi(1/x^2)$. Hence,

$$I_n(t, \lambda_0) \leq \frac{2}{\sqrt{1 - \varrho}} \frac{\Gamma(n; k^2t\psi(1/x^2))}{t^n} = \frac{2}{\sqrt{1 - \varrho}} \frac{\sqrt{t\psi(1/x^2)} \Gamma(n; (\pi/8)^2t\psi(1/x^2))}{t^{n+1/2}\sqrt{\psi(1/x^2)}}.
$$

The constant $c_3(n, \varrho)$ in the statement of the theorem is so chosen that for all $s \geq c_3(n, \varrho)$,

$$\frac{2}{\sqrt{1 - \varrho}} \sqrt{s} \frac{\Gamma(n; (\pi/8)^2s)}{\min(C_1(n, \pi/4), C_2(n, \pi/4))} \leq \frac{2}{\sqrt{1 - \varrho}} \sqrt{s} \frac{\Gamma(n; (\pi/8)^2s)}{\min(C_1(n, \pi/4), C_2(n, \pi/4))},
$$

with the constants $C_1(n, \Theta)$ and $C_2(n, \Theta)$ from Lemma 6.2. Then, by Lemma 6.2 when $t\psi(1/x^2) \geq c_3(n, \varrho)$ we have

$$\frac{C_1(n, \pi/4)}{2t^{n+1/2}\sqrt{\psi(1/x^2)}} \leq (-1)^n \frac{d^n}{dt^n} \mathbb{P}(\tau_x > t) \leq \frac{C_2(n, \pi/4)}{2t^{n+1/2}\sqrt{\psi(1/x^2)}},$$

as desired. Finally, by (6.9), we have $I_n(t, \lambda_0) \leq c(n, \varrho)/(t^{n+1/2}\sqrt{\psi(1/x^2)})$, where $c(n, \varrho) = 2\pi^2(1 - \varrho)^{-1/2} \sup_{s>0} (s^{1/2} \Gamma(n; (\pi/8)^2s))$. This and Lemma 6.2 prove that the upper bound in (1.11) holds for all $t, x > 0$, but with a constant depending on $\varrho$. □

Remark 6.5. (a) The strict inequality $\varrho < 1$ is essential to the proof. The case $\varrho = 1$ is much more complicated; see the example in Subsection 7.3.

(b) Theorem 1.7(b) can be easily generalised to the case when $|\psi''(\xi)|/|\psi'(\xi)| \leq \varrho/\xi$ only for $\xi \in (0, \varepsilon) \cup (1/\varepsilon, \infty)$, with $\varepsilon \in (0, 1)$ fixed. In this case the constant $c_3 = c_3(n, \varrho, \varepsilon)$ depends also on $\varepsilon$. We omit the details.

(c) The upper bound in (1.11) is certainly not optimal when $t\psi(1/x^2)$ is small. This is due to essential cancellations in (1.9). □

Proof of Theorem 1.8(a). We use the notation of Lemma 6.2 and take $\lambda_0 = \lambda_0(x) = (\pi/2 - \Theta)/(2x)$ for a fixed $x > 0$. Let $f_{n,t,x}(\lambda)$ be the integrand in (1.9). We have

$$t^{n+1/2} \int_{0}^{\lambda_0} f_{n,t,x}(\lambda)d\lambda = \int_{0}^{\lambda_0} \frac{F_{\lambda}(x)}{\lambda \sqrt{\psi(\lambda^2)}} t^{n+1/2}(\psi(\lambda^2))^{n-1/2}e^{-\psi(\lambda^2)} \lambda \psi'(\lambda^2)d\lambda.
$$

By Proposition 5.4, $\lim_{\lambda \to 0} F_{\lambda}(x)/(\lambda \sqrt{\psi(\lambda^2)}) = V(x)$, and therefore $F_{\lambda}(x)/(\lambda \sqrt{\psi(\lambda^2)})$ extends to a continuous function of $\lambda \in [0, \lambda_0]$. Let

$$\mu_t(d\lambda) = t^{n+1/2}(\psi(\lambda^2))^{n-1/2}e^{-\psi(\lambda^2)} \lambda \psi'(\lambda^2) 1_{[0,\lambda_0]}(\lambda)d\lambda.
$$

As $t \to \infty$, the density function of $\mu_t$ converges uniformly to 0 on $[\varepsilon, \lambda_0]$ for every $\varepsilon > 0$. Hence, $\mu_t(d\lambda)$ converges weakly to a point-mass at 0. Furthermore, by a substitution
\[ z = t\psi(\lambda^2), \]
\[ \|\mu_t\| = \frac{1}{2} \int_0^{n\psi(\lambda_0^2)} z^{n-1/2}e^{-z}dz = \frac{\gamma(n + 1/2; t\psi(\lambda_0^2))}{2}, \]
and hence \( \|\mu_t\| \) converges to \( \Gamma(n + 1/2)/2 \) as \( t \to \infty \). It follows that
\[ \lim_{t \to \infty} \left( t^{n+1/2} \int_0^\lambda f_{n,t,x}(\lambda) d\lambda \right) = \frac{\Gamma(n + 1/2)}{2} V(x). \]
Finally, by (6.7) and Remark 6.3
\[ \lim_{t \to \infty} \left| t^{n+1/2} \frac{4}{\pi} \int_0^\infty f_{n,t,x}(\lambda) d\lambda \right| \leq \lim_{t \to \infty} \left( t^{n+1/2} I_n(t, \lambda_0) \right) = 0, \]
and so (1.12) follows by (1.9). The convergence is locally uniform, since the extension of \( F_\lambda(x)/(\lambda \sqrt{\psi'(\lambda^2)}) \) is jointly continuous in \( \lambda \geq 0 \) and \( x \geq 0 \) (Proposition 5.4), and \( t^{n+1/2} I_n(t, \lambda_0(x)) \) converges to 0 locally uniformly in \( x \geq 0 \). \( \square \)

**Remark 6.6.** Alternatively, one can deduce formula (1.12) (without uniformity in \( x \)) as follows. In [17] it was proved that \( \sqrt{t} P(\tau_x > t) \) converges to \( V(x)/\sqrt{\pi} \) as \( t \to \infty \). By Theorem 1.7(a), \( P(\tau_x > t) \) is ultimately completely monotone. As it was observed in [14], Remark 4, this already implies formula (1.12) for all \( n \geq 0 \), see [8] (we omit the details). \( \square \)

**Proof of Theorem 1.8(b).** The argument is similar to the proof of part (a) of the theorem. Again we use the notation of the proof of Lemma 6.2. Let \( f_{n,t,x}(\lambda) \) be the integrand in (1.9). Fix \( t > t_0 \). We have
\[ \sqrt{\psi(1/x^2)} \int_0^\infty f_{n,t,x}(\lambda) d\lambda = \int_0^\infty \frac{\sqrt{\psi(1/x^2)} F_\lambda(x)}{\lambda \sqrt{\psi'(\lambda^2)}} (\psi(\lambda^2))^{n-1/2}e^{-t\psi'(\lambda^2)}\lambda^2 \psi'(\lambda^2) d\lambda. \]
By (5.7), \( \lim_{x \to 0^+} \sqrt{\psi(1/x^2)} F_\lambda(x)/(\lambda \sqrt{\psi'(\lambda^2)}) = 1/\Gamma(1 + \alpha) \). We will use dominated convergence for the integral over an initial interval \( (0, B) \), and a simple uniform bound on the remaining interval \( [B, \infty) \).

Let \( B > 0 \). Consider \( x \) small enough, so that \( \lambda x \leq (\pi/2 - \Theta)/2 \) and \( \psi(1/x^2) \geq 2\psi(\lambda^2) \) for \( \lambda \in (0, B) \) (recall that \( \psi \) is unbounded). By Lemma 5.2, for \( \lambda \in (0, B) \),
\[ \frac{\sqrt{\psi(1/x^2)} F_\lambda(x)}{\lambda \sqrt{\psi'(\lambda^2)}} \leq c(\Theta) \sqrt{\frac{\lambda^2 x^2 \psi(1/x^2) \psi(1/x^2)}{\psi'(\lambda^2) \psi(\lambda^2)}} = c(\Theta) \sqrt{\frac{\psi(1/x^2)(1 - \lambda^2 x^2)}{\psi(1/x^2) - \psi(\lambda^2)}} \leq c(\Theta) \frac{1}{\sqrt{1 - \psi(\lambda^2)/\psi(1/x^2)}} \leq \sqrt{2} c(\Theta). \]

Hence, by dominated convergence,
\[ \lim_{x \to 0^+} \left( \sqrt{\psi(1/x^2)} \int_0^B f_{n,t,x}(\lambda) d\lambda \right) = \frac{1}{\Gamma(1 + \alpha)} \int_0^B \left( \psi(\lambda^2) \right)^{n-1/2}e^{-t\psi'(\lambda^2)}\lambda^2 \psi'(\lambda^2) d\lambda. \]
More precisely, we have
\[
\lim_{x \to 0^+} \int_0^B \left| \sqrt{\psi(1/x^2)} f_{n,t,x}(\lambda) - \frac{(\psi(\lambda^2))^{n-1/2} e^{-t\psi(\lambda^2)} \lambda \psi'(\lambda^2)}{\Gamma(1+\alpha)} \right| d\lambda = 0,
\]
and the convergence is uniform in \( t > t_0 \), due to monotonicity of of the integrand in \( t > t_0 \). On the other hand,
\[
\frac{2}{\pi} \int_B^\infty f_{n,t,x}(\lambda) d\lambda \leq I_n(t_0, B),
\]
which converges to 0 as \( B \to \infty \), uniformly in \( x \) and \( t > t_0 \) (by Remark 6.3). Hence, by a substitution \( z = t\psi(\lambda^2) \),
\[
\lim_{x \to 0^+} \left( \sqrt{\psi(1/x^2)} \int_0^\infty f_{n,t,x}(\lambda) d\lambda \right)
= \frac{1}{\Gamma(1+\alpha)} \int_0^\infty (\psi(\lambda^2))^{n-1/2} e^{-t\psi(\lambda^2)} \lambda \psi'(\lambda^2) d\lambda
= \frac{1}{2\Gamma(1+\alpha) t^{n+1/2}} \int_0^\infty z^{n-1/2} e^{-z} dz = \frac{\Gamma(n+1/2)}{2\Gamma(1+\alpha) t^{n+1/2}}.
\]
Formula (1.13) follows now from (1.9).

7. Examples

7.1. Symmetric stable processes. These processes, corresponding to \( \Psi(\xi) = |\xi|^\alpha \) with \( \alpha \in (0, 2] \), have already been studied in Example 6.1 in [28]. In this case \( \vartheta_\lambda = (2-\alpha)\pi/8 \), and Theorem 1.6 reads
\[
(-1)^n \frac{d^n}{dt^n} P_x(\tau_x > t) = \frac{\sqrt{2\alpha} \sin(\alpha\pi/2)}{\pi} \int_0^\infty \lambda^{n-1} e^{-t\lambda^\alpha} F(\lambda x) d\lambda, \quad t, x > 0,
\]
with (see Example 6.1 in [28])
\[
F(\lambda x) = \sin(\lambda x + (2-\alpha)\pi/8) - \frac{\sqrt{2\alpha} \sin(\alpha\pi/2)}{2\pi} \int_0^\infty \frac{1}{1 + s^{2\alpha} - 2s^\alpha \cos(\alpha\pi/2)} \log \frac{1 - s^2\xi^2}{1 - s^\alpha \xi^\alpha} d\xi \exp \left( \frac{1}{\pi} \int_0^\infty \frac{1}{1 + s^2} \log \frac{1 - s^2\xi^2}{1 - s^\alpha \xi^\alpha} d\xi \right) e^{-s\lambda^\alpha} ds.
\]
Note that all above integrands are highly regular functions (for example, \( \log((1 - s^2\xi^2)/(1 - s^\alpha \xi^\alpha)) \) is a complete Bernstein function), and thus (7.2) is suitable for numerical integration. With a little effort, explicit upper bounds for numerical errors can also be computed. Together with (7.1), this gives faithful numerical bounds for \( P(\tau_x > t) \) and its derivatives in \( t \). Plots of cumulative distribution function and density function of \( \tau_x \) obtained using this method are given in Figure 1.

7.2. Processes with power-type Lévy-Khintchine exponent. In this example, we assume that the Lévy-Khintchine exponent \( \Psi(\xi) \) has the form \( \Psi(\xi) = \psi(\xi^2) \) for a complete Bernstein function \( \psi \), and \( \Psi \) is regularly varying of positive order \( \alpha_0 > 0 \) at zero, and of positive order \( \alpha_\infty > 0 \) at \( \pm \infty \) (see the first part of Table 1 for some examples). Clearly, in this case \( \psi(\xi) \) is regularly varying of orders \( \alpha_0/2 \) and \( \alpha_\infty/2 \) at 0 and \( \infty \), respectively. Hence, by Karamata’s theory of regularly varying functions (see [8]), \( -\xi \psi''(\xi)/\psi'(\xi) \) converges to \( 1 - \alpha_0/2 \) and \( 1 - \alpha_\infty/2 \) as \( \xi \to 0^+ \) and \( \xi \to \infty \), respectively.

For simplicity, we assume in addition that the supremum in (1.7) is less than one. This condition is satisfied by all examples given in the first part of Table 1. Note, however, that the condition given in Remark 6.5(b) is automatically satisfied, so that our extra
Plots prepared using gnuplot and a program with 80-bit precision floating point numbers.

Figure 1. Plots of $P(\tau_x < t)$ (black) and $\frac{d}{dt}P(\tau_x < t)$ (red) for the symmetric $\alpha$-stable Lévy process, computed with three digits of accuracy, for (a) $\alpha = 0.5$; (b) $\alpha = 1.0$; (c) $\alpha = 1.5$. Calculations are based on Theorems 1.6 and 1.4 and the following numerical integration scheme.

Suppose that $f$ is integrated over an interval $I$. First, using analytical methods, $f$ is bounded above and below on any subinterval $I'$ of $I$. Then, the integral of $f$ over $I'$ is bounded above and below by a simpler function $g$, which is integrated analytically. This yields lower and upper bounds for the integral of $f$ over $I$. A local adaptive strategy is used to divide the sub-intervals and gather bounds for the integral of over $I$. Interval arithmetic is used for nested integrals.

Plots prepared using gnuplot and a C program with 80-bit precision floating point numbers.
assumption can be easily dropped by referring to an improved version of Theorem 1.7, alluded to in Remark 6.5(b).

Theorem 1.7(b) yields the estimate

$$\frac{c_1(n)}{t^{n+1/2} \sqrt{\psi'(1/x^2)}} \leq (-1)^n \frac{d^n}{dt^n} P(\tau_x > t) \leq \frac{c_2(n)}{t^{n+1/2} \sqrt{\psi'(1/x^2)}}$$

for \( n \geq 0, x > 0 \) and \( t \geq c_3(n, \alpha)/\psi(1/x^2) \). Furthermore, by Theorem 1.8,

$$\lim_{x \to 0^+} \left( \sqrt{\psi'(1/x^2)} \frac{d^n}{dt^n} P(\tau_x > t) \right) = \frac{(-1)^n \Gamma(n + 1/2)}{\pi \Gamma(1 + \alpha_\infty/2)} \frac{1}{t^{n+1/2}}, \quad t > 0, n \geq 0,$$

$$\lim_{t \to \infty} \left( t^{n+1/2} \frac{d^n}{dt^n} P(\tau_x > t) \right) = \frac{(-1)^n \Gamma(n + 1/2)}{\pi} V(x), \quad x > 0, n \geq 0.$$

### 7.3. Slowly varying Lévy-Khintchine exponents.

When the Lévy-Khintchine exponent \( \Psi(\xi) \) has the form \( \Psi(\xi) = \psi(\xi^2) \) for a complete Bernstein function \( \psi \), and \( \Psi \) is regularly varying of order \( \alpha_0 \) at zero, and of order \( \alpha_\infty \) at \( \pm \infty \), but at least one of \( \alpha_0, \alpha_\infty \) is zero, estimates of the distribution of \( \tau_x \) become more delicate. In this case Theorem 1.7 cannot be applied, and one needs to refer to either Corollary 6.2 (which is fairly straightforward, but typically yields sub-optimal results) or the technical Lemma 6.2. Note that only \( n \geq 1 \) need to be considered, as \( n = 0 \) was studied in general in [29].

Suppose that \( |\psi'(\xi)|/\psi'(\xi) \leq 1/\xi \) for all \( \xi > 0 \) (that is, \( g = 1 \) in Theorem 1.7(b)); this condition is satisfied by all processes in the second part of Table 1. Then \( \Theta = \sup_{\lambda > 0} \vartheta_\lambda \leq \pi/4 \) by Proposition 4.3. Furthermore, by integration, \( \psi'(\xi_1)/\psi'(\xi_2) \leq \xi_2/\xi_1 \) when \( 0 < \xi_1 < \xi_2 \), and therefore (cf. the proof of Theorem 1.7(b))

$$I_n(t, \lambda_0) \leq \frac{2}{\pi \sqrt{\lambda_0^2 \psi'(\lambda_0^2)}} \int_{\lambda_0}^{\infty} e^{-t\psi(\lambda^2)} (\psi(\lambda^2))^{n-1/2} (2\lambda^2 \psi'(\lambda^2)) d\lambda$$

$$= \frac{2}{\pi} \frac{\Gamma(n + 1/2; t\psi(\lambda_0^2))}{t^{n+1/2} \sqrt{\lambda_0^2 \psi'(\lambda_0^2)}}, \quad (7.3)$$

| \( \Psi(\xi) \) | \( \alpha_0 \) | \( \alpha_\infty \) | \( X_t \) | restrictions |
|---|---|---|---|---|
| \( \xi^n \) | 2 | 2 | Brownian motion | \( \alpha \in (0, 2] \) |
| \( |\xi|^n \) | \( \alpha \) | \( \alpha \) | \( \alpha \)-stable | \( \alpha \in (0, 2] \) |
| \( c_1 |\xi|^\alpha + c_2 |\xi|^{\beta} \) | \( \alpha \) | \( \alpha \) | \( \alpha \)-stable | \( \alpha \in (0, 2] \) |
| \( (\xi^2 + m^2)^{\alpha/2} - m \) | \( \alpha \) | \( \beta \) | \( \alpha \)-stable | \( \alpha \in (0, 2], m > 0 \) |
| \( (\xi^2 + 1)^{\alpha/2} - 1)^{\beta/2} \) | \( \beta \) | \( \alpha \) | \( \alpha \)-stable | \( \alpha \in (0, 2], m > 0 \) |

Table 1. Some Lévy-Khintchine exponents \( \Psi(\xi) \), regularly varying both at 0 (of order \( \alpha_0 \)) and at \( \pm \infty \) (of order \( \alpha_\infty \)). Names of corresponding subordinate Brownian motions are given in column \( X_t \). First part of the table contains power-type functions \( \Psi \), more singular examples are given in the other parts.
where $\Gamma(a; z)$ is the upper incomplete gamma function. However, $\lambda_0^2 \psi'(\lambda_0^2)$ is no longer comparable with $\psi(\lambda_0^2)$. Nevertheless, we can combine (7.3) with Lemma 6.2 to find that
\[
\frac{C_1(n, \pi/4)}{2} \frac{1}{t^{n+1/2} \sqrt{\psi(1/x^2)}} \leq (-1)^n \frac{d^n}{dt^n} P(\tau_x > t)
\leq \left( C_1(n, \pi/4) + \frac{C_2(n, \pi/4)}{2} \right) \frac{1}{t^{n+1/2} \sqrt{\psi(1/x^2)}},
\]
provided that $t \psi(1/x^2) \geq 1$ (so that the minimum in (6.4) is $1/(t^{n+1/2} (\psi(1/x^2))^{1/2})$) and
\[
\Gamma(n + 1/2; t \psi((\lambda_0(x))^2)) \leq \frac{\pi C_1(n, \pi/4)}{4} \sqrt{\frac{(\lambda_0(x))^{2}\psi'(((\lambda_0(x))^2)}}{\psi(1/x^2)}. \tag{7.4}
\]
Here $C_1(n, \Theta)$, $C_2(n, \Theta)$ are the constants of Lemma 6.2 and $\lambda_0(x) = \pi/(8x)$.

Note that $\Gamma(n + 1/2; z) \leq c(n) e^{-z/2}$ for $z > 0$ for some $c(n)$. Furthermore, $\psi'$ is a decreasing function. After some simplification (we omit the details), this gives the following sufficient condition for (7.4):
\[
t \psi(1/x^2) \geq c'(n) \left(1 + \log \left(1 + \frac{\psi(1/x^2)}{(1/x^2) \psi'(1/x^2)}\right)\right)
\]
for some $c'(n)$. For convenience, we state this as a separate result.

**Proposition 7.1.** If $|\psi''(\xi)/\psi'(\xi)| \leq 1/|\xi|$ for all $\xi > 0$, then there are positive constants $c_1(n)$, $c_2(n)$, $c_3(n)$ such that
\[
\frac{c_1(n)}{t^{n+1/2} \sqrt{\psi(1/x^2)}} \leq (-1)^n \frac{d^n}{dt^n} P(\tau_x > t) \leq \frac{c_2(n)}{t^{n+1/2} \sqrt{\psi(1/x^2)}} \tag{7.5}
\]
for $n \geq 0$, $x > 0$ and
\[
t \geq \frac{c_3(n)}{\psi(1/x^2)} \left(1 + \log \left(1 + \frac{\psi(1/x^2)}{(1/x^2) \psi'(1/x^2)}\right)\right). \tag{7.6}
\]

For example, when $\Psi(\xi) = \log(1 + |\xi|^{\alpha})$ ($\alpha \in (0, 2)$), which corresponds to geometric stable processes, condition (7.6) reads
\[
t \geq \frac{c(\alpha, n)}{\psi(1/x^2)} (1 + \log(1 + \log(1 + 1/x))).
\]

In this case by Theorem 1.8 we also have
\[
\lim_{x \to 0^+} \left(\frac{\sqrt{\log(1/x)}}{d^n/dt^n} P(\tau_x > t) = \frac{(-1)^n \Gamma(n + 1/2)}{\pi \sqrt{\alpha/2}} \frac{1}{t^{n+1/2}}, \quad t > 0, n \geq 0,
\]
and
\[
\lim_{t \to \infty} \left(\frac{t^{n+1/2}}{d^n/dt^n} P(\tau_x > t) = \frac{(-1)^n \Gamma(n + 1/2)}{\pi} V(x), \quad x > 0, n \geq 0.
\]

### 7.4. Exponents with very slow growth. Let $\tilde{\psi}(\xi) = \log(1 + |\xi|^{\alpha})$ be the complete Bernstein function studied in the previous example, and let $\psi = \tilde{\psi} \circ \cdots \circ \tilde{\psi}$ be the $N$-fold composition of $\tilde{\psi}$ ($\alpha \in (0, 2]$, $N \geq 2$). The function $\psi$ is the Laplace exponent of the iterated geometric $\alpha$-stable subordinator, and $\Psi(\xi) = \psi(\xi^2)$ is the Lévy-Khintchine exponent of the corresponding subordinate Brownian motion. It is easy to verify that (1.7) is satisfied. However, (1.8) holds only if $N = 2$ and $t \geq 1/\alpha$ (n arbitrary). Hence, for $N = 2$ asymptotic expansion (1.13) in Theorem 1.8(b) is valid only for $t > 1/\alpha$ (part (a) of the theorem holds for all $x > 0$). When $N \geq 3$, neither part of Theorem 1.8 applies.

Note that for any $N \geq 2$ and $\alpha \in (0, 2]$, $\sup_{\xi \geq 0} (\xi |\psi''(\xi)|/\psi'(\xi)) > 1$, so these examples do not fit into the framework of Theorem 1.7(b), or even that of Subsection 7.3.
estimates of the derivatives of $P(\tau_x > t)$ for $N = 2$ and $t > 1/\alpha$ can be obtained using Lemma 6.2, see e.g. Corollary 6.4.

7.5. Compound Poisson process with Laplace distributed jumps. Let $\Psi(\xi) = \xi^2/(1 + \xi^2)$. Then the corresponding process $X_t$ is the compound Poisson process with Laplace distributed (i.e. with density function $(1/2)e^{-|x|}$) jumps occurring at unit rate. It was proved in [29] that

$$P(\tau_x > t) \approx \min \left( 1, \sqrt{1 + x^2/t} \right), \quad x > 0, t > 0.$$ 

Note that (1.7) is not satisfied, so the main results of the present article cannot be used. Indeed, we have $\vartheta_\lambda = \arctan \lambda$ (see [28], Example 6.6), and so $\sup_{\lambda > 0} \vartheta_\lambda$ is indeed equal to $\pi/2$.

7.6. Irregular example. Estimates for $\vartheta_\lambda$ are critical for Theorems 1.7 and 1.8. We have $\vartheta_\lambda \in [0, \pi/2)$, and it is easy to construct examples for which $\lim_{\lambda \to 0+} \vartheta_\lambda > \pi/4$ is possible. Below we show a rather irregular example for which $\limsup_{\lambda \to 0+} \vartheta_\lambda = \pi/2$ (but $\liminf_{\lambda \to 0+} \vartheta_\lambda = 0$).

Consider the complete Bernstein function

$$\psi(\xi) = \sum_{k=1}^\infty p_k \frac{\xi}{a_k + \xi},$$

where, for example, $p_k = 1/k!$ and $a_k = 1/(k!)^2$. Fix $q > 0$. We consider $\lambda = (qa_n)^{1/2}$ and let $n \to \infty$. We have

$$0 \leq \psi(qa_n) \left( p_n \frac{q}{1+q} \right)^{-1} - 1 = \sum_{k \neq n} p_k \frac{(1+q)a_n}{p_n a_k + qa_n} \leq \sum_{k=1}^{n-1} p_k \frac{(1+q)a_n}{p_n a_k} + \sum_{k=n+1}^\infty \frac{p_k (1+q)a_n}{q a_n} \leq (1+q) \sum_{k=1}^{n-1} \frac{k!}{n!} + \frac{1+q}{q} \sum_{k=n+1}^\infty \frac{n!}{k!} \leq \frac{1+q}{n} \sum_{k=1}^{n-1} \frac{1}{(n-k-1)!} + \frac{1+q}{q(n+1)} \sum_{k=n+1}^\infty \frac{1}{(k-n-1)!} \leq \frac{(1+q)^3e}{qn} = O(1/n).$$

Here and below the constant in the $O(\cdot)$ notation may depend on $q$. In a similar manner,

$$0 \leq \psi'(qa_n) \left( p_n \frac{1}{(1+q)^2a_n} \right)^{-1} - 1 = \sum_{k \neq n} p_k \frac{(1+q)^2a_n a_k}{p_n (a_k + qa_n)^2} \leq \sum_{k=1}^{n-1} p_k \frac{(1+q)^2a_n}{p_n a_k} + \sum_{k=n+1}^\infty \frac{p_k (1+q)^2}{q} \leq \frac{(1+q)^3e}{qn} = O(1/n),$$
and
\[
0 \leq |\psi''(qa_n)| \left( p_n \frac{2}{(1 + q)^2a_n^2} \right)^{-1} - 1 = \sum_{k \neq n} p_k \frac{(1 + q)^3a_n^2a_k}{p_n (a_k + qa_n)^3} \\
\leq \sum_{k=1}^{n-1} p_k \frac{(1 + q)^3a_n}{a_k} + \sum_{k=n+1}^{\infty} p_k \frac{(1 + q)^3}{q} \\
\leq \frac{(1 + q)^4e}{qn} = O(1/n).
\]

It follows that
\[
\psi(qa_n) = \left( p_n \frac{q}{1 + q} \right) (1 + O(1/n)),
\]
\[
\psi'(qa_n) = \left( p_n \frac{1}{(1 + q)^2a_n} \right) (1 + O(1/n)),
\]
\[
|\psi''(qa_n)| = \left( p_n \frac{2}{(1 + q)^3a_n^2} \right) (1 + O(1/n)).
\]

With the notation of the proof of Proposition 4.5 for $\lambda^2 = qa_n$ we obtain
\[
a_1^2 = \frac{\psi(\lambda^2)}{\lambda^2 \psi'(\lambda^2)} = (1 + q)(1 + O(1/n)),
\]
\[
a_2^2 = \frac{\lambda^2 |\psi''(\lambda^2)|}{2\psi'(\lambda^2)} = \frac{q}{1+q} (1 + O(1/n)),
\]
\[
a_2^2 = a_1^2 - 1 = q(1 + O(1/n)).
\]

By Proposition 4.5
\[
\limsup_{n \to \infty} \vartheta(\lambda) \leq \vartheta(\lambda) \leq \limsup_{n \to \infty} \vartheta(\lambda) = \arctan q,
\]
and, in a similar manner,
\[
\liminf_{n \to \infty} \vartheta(\lambda) \geq \frac{1}{\pi} \left( \text{arcsin} \sqrt{\frac{q}{1+q}} \right)^2 - 2 \arcsin \sqrt{\frac{1}{1+q}}
\]
\[
= \frac{\pi}{4} - \frac{1}{\pi} \left( \text{arcsin} \sqrt{\frac{1}{1+q}} + \text{arcsin} \sqrt{\frac{q}{1+q}} \right) \left( \text{arcsin} \sqrt{\frac{1}{1+q}} - \text{arcsin} \sqrt{\frac{q}{1+q}} \right)
\]
\[
= \frac{\pi}{4} - \frac{1}{\pi} \frac{\arcsin \frac{1}{1+q} - \arcsin \frac{1-q}{1+q}}{2} \arccos \frac{1-q}{1+q} \frac{1}{2} \arctan \frac{2\sqrt{q}}{1-q} = \arctan \sqrt{q}.
\]

Hence, $\lim_{n \to \infty} \vartheta(\lambda) = \arctan q$, and therefore any number in $[0, \pi/2]$ is a partial limit of $\vartheta(\lambda)$ as $\lambda \to 0^+$.

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