WEIGHTED PERIODIC AND DISCRETE PSEUDO-DIFFERENTIAL OPERATORS

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Abstract. In this paper, we study elements of symbolic calculus for pseudo-differential operators associated with the weighted symbol class \( M^m_{\rho, \Lambda}(T \times \mathbb{Z}) \) (associated to a suitable weight function \( \Lambda \) on \( \mathbb{Z} \)) by deriving formulae for the asymptotic sums, composition, adjoint, transpose. We also construct the parametrix of \( M \)-elliptic pseudo-differential operators on \( T \). Further, we prove a version of Gohberg’s lemma for pseudo-differential operators with weighted symbol class \( M^m_{\rho, \Lambda}(T \times \mathbb{Z}) \) and as an application, we provide a sufficient and necessary condition to ensure that the corresponding pseudo-differential operator is compact on \( L^2(T) \). Finally, we provide Gårding’s and Sharp Gårding’s inequality for \( M \)-elliptic operators on \( \mathbb{Z} \) and \( T \), respectively, and present an application in the context of strong solution of the pseudo-differential equation \( T \sigma u = f \) in \( L^2(T) \).

1. Introduction

The theory of pseudo-differential operators plays an important role in modern mathematics due to the fact that it has drawn a significant motivation from partial differential equations, signal processing, and time-frequency analysis, see [20, 28, 13]. Pseudo-differential operators acting on functions defined on smooth manifolds are an essential generalization of differential operators. The study of pseudo-differential operators originated in 1960s with the works of Kohn and Nirenberg [22] and Hörmander [20] in the study of singular integral differential operators, mainly for inverting differential operators to solve elliptic differential equations. Ever since the theory is a key tool, mostly for its connections with mathematical physics and in many areas of harmonic analysis, quantum field theory, and the index theory.

In general, using the Mikjlin-Hörmander theorem for Fourier multipliers, pseudo-differential operators associated to the class \( S^0_{1,0} \) are \( L^p \)-bounded. It is also well known that the \( L^2 \)-boundedness property holds for pseudo-differential operators associated with the symbols, \( S^0_{\rho,\delta} \) with \( 0 < \delta < \rho \leq 1 \) (see [2, 20]). However, the situation becomes completely different...
for the case $p \neq 2$. Fefferman [12] proved that the pseudo-differential operators whose symbols belong to the class $S^{0}_{\rho,0}$ with $0 < \rho < 1$ are not in general $L^p$-bounded for $p \neq 2$. To avoid the difficulty described above, Taylor in [34] introduced a suitable symbol sub-class $M^{m}_{\rho,0}$ of $S^{0}_{\rho,0}$ and developed symbolic calculus for the associated pseudo-differential operators. Further, Garello and Morando [15, 16] introduced subclass $M^{m}_{\rho,\Lambda}$ of $S^{0}_{\rho,0}$, which are just a weighted version of the symbol class introduced by Taylor and developed the symbolic calculus for the associated pseudo-differential operators with many applications to study the regularity of multi-quasi-elliptic operators.

Considerable attention has been devoted in the past sixteen years to study various properties of pseudo-differential operators associated with the symbol class $M^{m}_{\rho,\Lambda}$ on $\mathbb{R}^n$ in various directions by several researchers. For instance, the symbolic calculus, parametrix, and $L^p$, $1 < p < \infty$, boundedness of pseudo-differential operators with symbol in $M^{m}_{\rho,\Lambda}$, $m > 0$ has been studied by Wong [36]. Further, Kalleji [21] constructed a weighted symbol class $M^{m}_{\rho,\Lambda}(\mathbb{T}^n \times \mathbb{Z}^n)$, $m \in \mathbb{R}$ associated to a suitable weight function $\Lambda$ on $\mathbb{Z}^n$ and study minimal and maximal extensions, among some other results for pseudo-differential operators associated with symbol in $M^{m}_{\rho,\Lambda}(\mathbb{Z}^n \times \mathbb{T}^n)$. We also note that recently, the authors in [23] constructed and studied $M$-elliptic pseudo-differential operators on $\mathbb{Z}^n$ with symbol in $M^{m}_{\rho,\Lambda}(\mathbb{Z}^n \times \mathbb{T}^n)$ which is just a weighted version of the Hörmander symbol class $S^{0}_{\rho,\Lambda}(\mathbb{Z}^n \times \mathbb{T}^n)$ on $\mathbb{Z}^n$, introduced by Botchway, Kabiliti and Ruzhansky [3]. More details about discrete pseudo-differential operators can be found in [4, 5, 8, 9]. This paper investigates $M$-elliptic pseudo-differential operators on $\mathbb{T}$. In particular, we investigate elements of symbolic calculus for pseudo-differential operators associated with the class $M^{m}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$ by deriving formulae for the asymptotic sums of symbols, composition, adjoint, transpose. We also construct the parametrix of $M$-elliptic pseudo-differential operators on $\mathbb{T}$. e can extend this result for $\mathbb{T}^n$.

Another important result in analysis is the Gohberg lemma due to its application in spectral theory and the singular integral equations. Gohberg lemma was first obtained by Gohberg [17] to investigate integral operators. In 1970, the Gohberg lemma for pseudo-differential operators with bounded symbols was obtained by Grušhin [18]. Later, for symbols in the Hörmander class $S^{0}_{1,0}(S^1 \times \mathbb{Z})$, an analogue of Gohberg’s lemma has been proved in [24] to prove the spectral invariance. Further, this theory has been extended to compact Lie group by the first author and Ruzhansky in [11]. Recently, for symbols in the Hörmander class $S^{0}_{1,0}(\Omega \times \mathbb{Z})$, Ruzhansky and Publo [27] investigated a “non-harmonic version” of Gohberg’s lemma, and provided a sufficient and necessary condition to ensure that the corresponding pseudo-differential operator is a compact operator in $L^2(\Omega)$. In this manuscript, we also establish a version of the Gohberg lemma but for the weighted symbol class $M^{0}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$. As an application of Gohberg’s lemma, we also provide a sufficient and necessary condition to ensure that the corresponding pseudo-differential operator is compact on $L^2(\mathbb{T})$. Particularly, we evaluate the norm of $T_\sigma - K$, where $K$ is a compact operator and $\sigma \in M^{0}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$, and give estimates for the essential spectrum of such operator $T_\sigma$. Using the relation between the lattice quantization and the toroidal quantization developed in [3], we prove Gohberg’s lemma for the weighted symbol class $M^{0}_{\rho,\Lambda}(\mathbb{Z} \times \mathbb{T})$ on $\mathbb{Z}$.

Garding’s type inequality plays a crucial role in the study of several problems related to initial value problem of parabolic type. Investigation of Garding’s inequality for strongly elliptic operators was first proved by Gårding [14] to derive the existence of solutions of the Dirichlet problem for elliptic operators as well as to study the distribution of the eigenvalues. After that, considerable attention has been devoted by several researchers to studying Gårding’s inequality for pseudo-differential operators associated with the Hörmander
symbols with applications to PDE in different contexts. For example, Gårding’s inequality for pseudo-differential operators associated with the Hörmander symbols on \( \mathbb{R}^n \) with \( 0 \leq \delta \leq \rho \leq 1 \), on compact Lie group with matrix-valued symbols, and in context of non-harmonic analysis on general smooth manifolds can be found in \([34, 31, 10, 6]\). Here we would like to note that, recently, the first and second authors proved the Gårding’s inequality for \( SG \) \( M \)-elliptic operators to obtain results about the existence and uniqueness of solutions of the parabolic type IVP. On the other hand, in order to deal with non-elliptic problems, Hörmander proved in \([19]\) sharp Gårding’s inequality for operators with symbols having nonnegative real part. The sharp Gårding’s inequality and its generalizations become an essential tool to investigate the existence of solutions to a wide class of boundary value problems and to analyze the global solvability and the local well-posedness of the Cauchy problem for evolution equations. The sharp Gårding’s inequality on \( \mathbb{R}^n \) is one of the most important tools of the microlocal analysis with numerous applications in the theory of PDE \([19]\). Notably, the sharp Gårding’s inequality requires the condition imposed on the full symbol. Further, the sharp Gårding’s inequality for the Kohn-Nirenberg classes \( S^m_{\rho,\Lambda}(G) \) proved in \([29]\). Recently, the authors in \([7]\) extended these inequality for the Hörmander classes \( S^m_{\rho,\delta}(G) \) for all \( 0 \leq \delta < \rho \leq 1 \). In this paper, we prove the Gårding’s and sharp Gårding’s inequality for \( M \)-elliptic pseudo-differential operator with symbol in \( M^0_{\rho,\Lambda}(T \times Z) \) on \( T \) and \( M^0_{\rho,\Lambda}(Z \times T) \) on \( Z \), respectively. We also present an application of Gårding’s inequality in the context of strong solution of the pseudo-differential equation \( T_\sigma u = f \) in \( L^2(T) \).

The presentation of this manuscript is divided into six sections, including the introduction as follows:

- In Section 2 we first recall some of the basics of Fourier analysis and important properties of periodic pseudo-differential operators on \( T \). We also recall the weighted symbol class \( M^m_{\rho,\Lambda}(T \times Z) \), \( m \in \mathbb{R} \) associated to a suitable weight function \( \Lambda \) on \( Z \) from \([21]\).
- In Section 3 we study elements of symbolic calculus for pseudo-differential operators associated with symbol in the weighted class \( M^m_{\rho,\Lambda}(Z \times T) \) by deriving formulae for the asymptotic sums, composition, adjoint, transpose. By recalling the definition of \( M \)-ellipticity for symbols we construct the parametrix of \( M \)-elliptic pseudo-differential operators.
- In Section 4 we study compact \( M \)-elliptic pseudo-differential operators on \( T \). We prove a version of Gohberg’s lemma for pseudo-differential operators on \( T \) and \( Z \) with symbol in the weighted symbol class \( M^0_{\rho,\Lambda}(T \times Z) \) and \( M^0_{\rho,\Lambda}(Z \times T) \), respectively. Further, we provide a sufficient and necessary condition to ensure the compactness of a pseudo-differential operator on \( L^2(T) \) (also on \( \ell^2(Z) \)) with symbol in \( M^0_{\rho,\Lambda}(T \times Z) \) (respectively in \( M^0_{\rho,\Lambda}(Z \times T) \)).
- In Section 5 we prove Gårding’s and Sharp Gårding’s inequality for \( M \)-elliptic operators on \( Z \) and \( T \), respectively.
- In Section 6 we discuss an application of Gårding’s inequality in the context of strong solution of the pseudo-differential equation \( T_\sigma u = f \) in \( L^2(T) \).

Note that, these results can be extended easily from \( T \) to the \( n \)-dimensional torus \( T^n \) given by \( T^n = T \times \cdots \times T \), and similarly from \( Z \) to \( Z^n \).
2. Preliminaries

In this section, we first recall some notation and basic properties of periodic Fourier analysis and pseudo-differential operators on \( \mathbb{T} \). We also recall the toroidal symbol class \( \mathcal{S}^m_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z}) \) in the view of Ruzhansky-Turunen theory \cite{ruzhansky2022fourier} as well as the weighted symbol class \( \mathcal{M}^m_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z}) \) from \cite{ruzhansky2020wavelet}. We refer \cite{ruzhansky2022fourier,ruzhansky2020wavelet,mishin2017pseudo,ruzhansky2020wsh,ruzhansky2018pseudo} for more details and the study of various operator theoretical properties of pseudo-differential operators on \( \mathbb{T} \).

The Fourier transform \( \hat{f} \) of a function \( f \in L^1(\mathbb{T}) \) is defined by
\[
\hat{f}(k) = \int_{\mathbb{T}} e^{2\pi i k x} f(x) \, dx, \quad k \in \mathbb{Z},
\]
where \( dx \) is the normalized Haar measure on \( \mathbb{T} \). The above periodic Fourier transform can be extended to \( L^2(\mathbb{T}) \) using the standard density arguments. We normalize the Haar measures on \( \mathbb{T} \) in such a manner so that the following Plancherel formula holds:
\[
\int_{\mathbb{T}} |\hat{f}(x)|^2 \, dx = \sum_{k \in \mathbb{Z}} |f(k)|^2.
\]

The inverse of the periodic Fourier transform is given by
\[
f(x) = \sum_{k \in \mathbb{Z}} e^{-2\pi i k x} \hat{f}(k), \quad x \in \mathbb{T},
\]
where \( f \) belongs to a suitable function space, namely, the Schwartz space of \( \mathbb{Z} \), \( S(\mathbb{Z}) \), the space of rapidly decaying functions from \( \mathbb{Z} \to \mathbb{C} \).

**Definition 2.1. (Forward and backward differences \( \Delta_k \) and \( \tilde{\Delta}_k \))**
Let \( \sigma : \mathbb{Z} \to \mathbb{C} \). We define the forward and backward partial difference operators \( \Delta_k \) and \( \tilde{\Delta}_k \), respectively, by
\[
\Delta_k \sigma(k) := \sigma(k+1) - \sigma(k),
\]
\[
\tilde{\Delta}_k \sigma(k) := \sigma(k) - \sigma(k-1).
\]

Let us now recall the Hörmander symbol class, \( \mathcal{S}^m_{\rho}(\mathbb{T} \times \mathbb{Z}) \), on \( \mathbb{T} \), which is same as defined in \cite{ruzhansky2022fourier}.

**Definition 2.2.** Let \( m \in \mathbb{R} \) and \( \rho > 0 \). We say that a function \( \sigma : \mathbb{T} \times \mathbb{Z} \to \mathbb{C} \) belongs to \( \mathcal{S}^m_{\rho}(\mathbb{T} \times \mathbb{Z}) \) if \( \sigma(x,k) \) is smooth in \( x \) for all \( k \in \mathbb{Z} \), and for all \( \alpha, \beta \in \mathbb{N}_0 \), there exists a positive constant \( C_{\alpha,\beta} \) such that

\[
\left| \Delta^\alpha_k \partial^\beta_x \sigma(x,k) \right| \leq C_{\alpha,\beta}(1+|k|)^{m-\rho|\alpha|}, \quad (x,k) \in \mathbb{T} \times \mathbb{Z}.
\]

For the symbol \( \sigma \in \mathcal{S}^m_{\rho}(\mathbb{T} \times \mathbb{Z}) \), the corresponding pseudo-differential operator associated with \( \sigma \) is given by
\[
(T_\sigma f)(x) := \sum_{k \in \mathbb{Z}} e^{-2\pi i k x} \sigma(x,k) \hat{f}(k), \quad x \in \mathbb{T}.
\]
We denote \( \text{OPS}^m_{\rho}(\mathbb{T} \times \mathbb{Z}) \) be the set of all operators corresponding to the symbol class \( \mathcal{S}^m_{\rho}(\mathbb{T} \times \mathbb{Z}) \).

**Definition 2.3.** Let \( \Lambda \) be a positive function. We say that \( \Lambda \) is a weight function if there exist suitable positive constants, \( \mu_0 \leq \mu_1 \) and \( C_0, C_1 \) such that
\[
C_0(1+|k|)^{\mu_0} \leq \Lambda(k) \leq C_1(1+|k|)^{\mu_1}, \quad k \in \mathbb{Z}.
\]

Furthermore, we assume that there exists a real constant \( \mu \) such that \( \mu \geq \mu_1 \) and for all \( \alpha, \gamma \in \mathbb{N}_0 \) with \( \gamma \in \{0,1\} \), we can find a positive constant \( C_{\alpha,\gamma} \) such that
\[
\left| k^\gamma \Delta^\alpha_k \Lambda(k) \right| \leq C_{\alpha,\gamma} \Lambda(k)^{1-\frac{1-\alpha}{\mu}}, \quad k \in \mathbb{Z}. \quad (2.1)
\]
**Definition 2.4.** Let \( m \in \mathbb{R} \) and \( \rho \in \left( 0, \frac{1}{m} \right] \). Then the toroidal symbol class \( S_{\rho, A}^m (T \times Z) \) is the set of all functions \( \sigma : T \times Z \to \mathbb{C} \) which are smooth in \( x \) for all \( k \in \mathbb{Z} \), and for all \( \alpha, \beta \in \mathbb{N}_0 \), there exists a positive constant \( C_{\alpha, \beta} \) such that

\[
|\Delta_k \partial_x^\beta \sigma(x, k)| \leq C_{\alpha, \beta} \Lambda(k)^{m - \rho \alpha}, \quad (x, k) \in T \times Z.
\]  

(2.2)

As usual we set

\[
S_{\rho, A}^\infty (T \times Z) := \bigcup_{m \in \mathbb{R}} S_{\rho, A}^m (T \times Z)
\]

and

\[
S_{\rho, A}^{-\infty} (T \times Z) := \bigcap_{m \in \mathbb{R}} S_{\rho, A}^m (T \times Z).
\]

Let \( \sigma \in S_{\rho, A}^m (T \times Z) \). Define a pseudo-differential operator \( T_\sigma \) associated with symbol \( \sigma \) by

\[
T_\sigma f(x) = (2\pi)^{-1} \sum_{k \in \mathbb{Z}} \int_T e^{i(x-y) \cdot k} \sigma(x, k) f(y) dy, \quad x \in T
\]

(2.3)

for every \( f \in C^\infty (T) \). We write \( \text{Op} (S_{\rho, A}^m (T \times Z)) \) for the class of pseudo-differential operators associated with the symbol class \( S_{\rho, A}^m (T \times Z) \).

Now, we will describe the main ingredient of this paper, namely the symbol class \( M_{\rho, A}^m (T \times Z) \).

**Definition 2.5.** For \( m \in \mathbb{R} \) and \( \rho \in \left( 0, \frac{1}{m} \right] \), the symbol class \( M_{\rho, A}^m (T \times Z) \) consists of all functions \( \sigma : T \times Z \to \mathbb{C} \) which are smooth in \( x \) for all \( k \in \mathbb{Z} \), and for \( \gamma \in \{0, 1\} \),

\[
k^\gamma \Delta_k \sigma(x, k) \in S_{\rho, A}^m (T \times Z).
\]

In the same manner, we write \( \text{Op} (M_{\rho, A}^m (T \times Z)) \) for the class of pseudo-differential operators associated with the symbol class \( M_{\rho, A}^m (T \times Z) \).

**Remark 2.6.** For every \( m \in \mathbb{R} \) and \( \rho \in \left( 0, \frac{1}{m} \right] \), we have

\[
M_{\rho, A}^m (T \times Z) \subset S_{\rho, A}^m (T \times Z).
\]

**Lemma 2.7.** For every \( m \in \mathbb{R} \) and \( 0 < \rho \leq \frac{1}{m} \), there exists a positive integer \( N_0 \) such that

\[
S_{\rho, A}^{m - N_0} (T \times Z) \subset M_{\rho, A}^m (T \times Z) \subset S_{\rho, A}^{-\infty} (T \times Z).
\]

(2.4)

More precisely, \( N_0 := \left( \frac{1}{\mu_0} - \rho \right) \). Moreover,

\[
\bigcap_{m \in \mathbb{R}} M_{\rho, A}^m (T \times Z) = \bigcap_{m \in \mathbb{R}} S_{\rho, A}^m (T \times Z) = S_{\rho, A}^{-\infty} (T \times Z).
\]

(2.5)

**Proof** The proof of the above lemma is similar to the proof of Lemma 3.10 in [23]. □

3. **Symbolic calculus and parametrix for** \( M_{\rho, A}^m (T \times Z) \)

In this section we study the symbolic calculus for pseudo-differential operators associated with symbol in \( M_{\rho, A}^m (\mathbb{Z}^n \times \mathbb{T}^n) \) by deriving formulae for composition, adjoint, transpose of the operators. We also construct the parametrix of \( M \)-elliptic pseudo-differential operators. We start this section with the following result related to the asymptotic sums of symbols.
Theorem 3.1. Let \( \{m_j\}_{j \in \mathbb{N}_0} \) be a strictly decreasing sequence of real numbers such that \( m_j \to -\infty \) as \( j \to \infty \). Suppose \( \sigma_j \in M_{p, \Lambda}^{m_j}(\mathbb{T} \times \mathbb{Z}), j \in \mathbb{N}_0 \). Then there exists a symbol \( \sigma \in M_{p, \Lambda}^{m_0}(\mathbb{T} \times \mathbb{Z}) \) such that

\[
\sigma(x, k) \sim \sum_{j=0}^{\infty} \sigma_j(x, k),
\]

i.e.,

\[
\sigma(x, k) - \sum_{j=0}^{N-1} \sigma_j(x, k) \in M_{p, \Lambda}^{m_N}(\mathbb{T} \times \mathbb{Z}),
\]

for every positive integer \( N \).

Proof Let \( \sigma_j \in M_{p, \Lambda}^{m_j}(\mathbb{T} \times \mathbb{Z}) \). Then from Remark 2.6, we have \( \sigma_j \in S_{p, \Lambda}^{m_j}(\mathbb{T} \times \mathbb{Z}) \). Consider \( \psi \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \psi \leq 1 \) and

\[
\psi(k) = \begin{cases} 
1, & \text{if } |k| \geq 1 \\
0, & \text{if } |k| \leq \frac{1}{2}.
\end{cases}
\]

Let \( (\epsilon_j)_{j=0}^{\infty} \) be a sequence of positive real numbers such that \( \epsilon_j > \epsilon_{j+1} \to 0 \). Define \( \psi_j \in C^\infty(\mathbb{R}) \), by \( \psi_j(k) := \psi(\epsilon_j k) \). It is clear that if \( \alpha \geq 1 \), then the support of \( \Delta_k^\alpha \psi_j \) is bounded. Since \( \sigma_j \in S_{p, \Lambda}^{m_j}(\mathbb{T} \times \mathbb{Z}) \), so using discrete Leibniz formula, we have

\[
\left| \Delta_k^\alpha \partial_x^\beta (\psi_j(k)\sigma_j(x, k)) \right| \leq C_{j, \alpha, \Lambda}(k)^{m_j - \rho \alpha},
\]

where \( C_{j, \alpha, \Lambda} \) is a positive constant. This means that, \( \psi_j(k)\sigma_j(x, k) \in S_{p, \Lambda}^{m_j}(\mathbb{T} \times \mathbb{Z}) \). Note that, when \( j \) is large enough, \( \Delta_k^\alpha (\psi_j(k)\sigma_j(x, k)) \), (where \( \alpha \in \mathbb{N}_0 \)), vanishes for any fixed \( k \in \mathbb{Z} \). This justifies the definition

\[
\sigma(x, k) := \sum_{j=0}^{\infty} \psi_j(k)\sigma_j(x, k), \quad (x, k) \in \mathbb{T} \times \mathbb{Z}.
\]

Clearly, \( \sigma \in S_{p, \Lambda}^{m_0}(\mathbb{T} \times \mathbb{Z}) \). Further, we have

\[
\left| \Delta_k^\alpha \partial_x^\beta \left( \sigma(x, k) - \sum_{j=0}^{N-1} \sigma_j(x, k) \right) \right|
\]

\[
\leq \sum_{j=0}^{N-1} \left| \Delta_k^\alpha \partial_x^\beta \{(\psi_j(k) - 1)\sigma_j(x, k)\} \right| + \sum_{j=N}^{\infty} \left| \Delta_k^\alpha \partial_x^\beta \{(\psi_j(k)\sigma_j(x, k))\} \right|.
\]

Since \( \epsilon_j > \epsilon_{j+1} \) and \( \epsilon_j \to 0 \) as \( j \to \infty \), so \( \sum_{j=0}^{N-1} \left| \Delta_k^\alpha \partial_x^\beta \{(\psi_j(k) - 1)\sigma_j(x, k)\} \right| \) vanishes, whenever \( |k| \) is large. Thus, there exists a positive constant \( C_{r, \alpha, \Lambda} \) such that

\[
\sum_{j=0}^{N-1} \left| \Delta_k^\alpha \partial_x^\beta \{(\psi_j(k) - 1)\sigma_j(x, k)\} \right| \leq C_{r, \alpha, \Lambda}(k)^{-r}
\]

for any \( r \in \mathbb{R} \). On the other hand, one can easily show that

\[
\sum_{j=N}^{\infty} \left| \Delta_k^\alpha \partial_x^\beta \{(\psi_j(k)\sigma_j(x, k))\} \right| \leq C'_{\alpha, \Lambda}(k)^{m_N - \rho \alpha},
\]
where \( C'_{N\alpha\beta} \) is a positive constant. This shows that for every \( N \in \mathbb{N} \), we have
\[
\sigma(x, k) - \sum_{j=0}^{N-1} \sigma_j(x, k) \in S_{\rho, \Lambda}^{mN} (T \times Z).
\]

Since \( m_j \to -\infty \), as \( j \to \infty \), using left inclusions in (2.4), we have \( \sigma - \sum_{j=0}^{N-1} \sigma_j \in S_{\rho, \Lambda}^{mN} (T \times Z) \subset M_{\rho, \Lambda}^m (T \times Z) \) for a sufficiently large \( N \). Hence \( \sigma(x, k) \in M_{\rho, \Lambda}^m (T \times Z) \). Furthermore, for all \( N \geq 2 \) and \( N' > N \)
\[
\sigma - \sum_{j=0}^{N-1} \sigma_j = \sum_{j=N}^{N'-1} \sigma_j + r_{N'}
\]
with \( r_{N'} \in S_{\rho, \Lambda}^{mN'} (T \times Z) \). By choosing a sufficiently large \( N' \) so that \( m_{N'} < m_N - N_0 \), we have \( r_{N'} \in S_{\rho, \Lambda}^{mN'-N_0} (T \times Z) \subset M_{\rho, \Lambda}^{mN} (T \times Z) \) and therefore \( \sigma - \sum_{j=0}^{N-1} \sigma_j \in M_{\rho, \Lambda}^{mN} (T \times Z) \). This completes the proof of the theorem. \( \square \)

The following results on the basic symbolic calculus of pseudo-differential operators with weighted \( M \)-symbols on \( T \times Z \) are analogs of results for pseudo-differential operators with symbols in \( S^m (T^n \times Z^n) \) given in [28] and symbols in \( M_{\rho, \Lambda}^m (Z^n \times T^n) \) given in [23].

**Theorem 3.2.** Let \( \sigma \in M_{\rho, \Lambda}^m (T \times Z) \) and \( \tau \in M_{\rho, \Lambda}^\mu (T \times Z) \). Then \( T_\sigma T_\tau = T_\lambda \), where \( \lambda \in M_{\rho, \Lambda}^{m+\mu} (T \times Z) \) and
\[
\lambda \sim \sum_{|\alpha|} \frac{(-i)^{|\alpha|}}{\alpha!} (\Delta_k^\alpha \sigma)(\partial_x^\alpha \tau).
\]
Here the asymptotic expansion means that
\[
\lambda - \sum_{|\alpha| < N} \frac{(-i)^{|\alpha|}}{\alpha!} (\Delta_k^\alpha \sigma)(\partial_x^\alpha \tau) \in M_{\rho, \Lambda}^{m+\mu-pN} (T \times Z),
\]
for every positive integer \( N \).

**Theorem 3.3.** Let \( \sigma \in M_{\rho, \Lambda}^m (T \times Z) \). Then the formal adjoint \( T_\sigma^* \) of \( T_\tau \) is the pseudo-differential operator \( T_\tau \), where \( \tau \in M_{\rho, \Lambda}^\mu (T \times Z) \) and
\[
\tau \sim \sum_{|\alpha|} \frac{(-i)^{|\alpha|}}{\alpha!} \Delta_k^\alpha \partial_x^\alpha \sigma.
\]
Here the asymptotic expansion means that
\[
\tau - \sum_{|\alpha| < N} \frac{(-i)^{|\alpha|}}{\alpha!} \Delta_k^\alpha \partial_x^\alpha \sigma \in M_{\rho, \Lambda}^{m-\mu-pN} (T \times Z),
\]
for every positive integer \( N \).

Let \( \sigma \in M_{\rho, \Lambda}^m (T \times Z) \), where \( m \in \mathbb{R} \). Then \( \sigma \) is said to be \( M \)-elliptic if there exist positive constants \( C \) and \( R \) such that
\[
|\sigma(x, k)| \geq C\Lambda(k)^m, \quad \forall x \in T, k \in Z,
\]
with \( |k| \geq R \). Naturally, a pseudo-differential operator \( T_\sigma \) corresponding to such \( \sigma \), is said to be \( M \)-elliptic.

The following lemma is analogous to the Lemma 2 in [15], which can be proved in the similar way. So we skip the proof here.
Lemma 3.4. Let $\sigma(x, k) \in M_{\rho, \Lambda}^m(\mathbb{T} \times \mathbb{Z})$ and $\psi(x, k) \in C^\infty(\mathbb{T} \times \mathbb{R})$, be such that there exist two sufficiently large positive constants $R'' > R'$ so that $\psi(x, k) = 0$, for all $x \in \mathbb{T}$ and $|k| \leq R'$, and $\psi(x, k) = 1$, for all $x \in \mathbb{T}$ and $|k| \geq R''$. Then $q(x, k) = \frac{\psi(x, k)}{\rho(x, k)} \in M_{\rho, \Lambda}^{-\rho N}(\mathbb{T} \times \mathbb{Z})$.

Using the above lemma, we obtain the parametrix of the elliptic operators.

Theorem 3.5. A symbol $\sigma$ is elliptic in $M_{\rho, \Lambda}^m(\mathbb{T} \times \mathbb{Z})$ if and only if there exists a symbol $\tau$ in $M_{\rho, \Lambda}^{-m}(\mathbb{T} \times \mathbb{Z})$ such that

$$T_\tau T_\sigma = I + R$$

and

$$T_\sigma T_\tau = I + S,$$

where $R$ and $S$ are pseudo-differential operators with symbols in $\bigcap_{m \in \mathbb{R}} M_{\rho, \Lambda}^m(\mathbb{T} \times \mathbb{Z})$, and $I$ is the identity operator.

Proof First suppose that there exists a symbol $\tau$ in $M_{\rho, \Lambda}^{-m}(\mathbb{T} \times \mathbb{Z})$ such that (3.1) and (3.2) are true. From (3.2), we can conclude that

$$I - T_\sigma T_\tau \in \text{Op}(M_{\rho, \Lambda}^{-\infty}(\mathbb{T} \times \mathbb{Z})).$$

Hence, by Theorem 3.2, we have

$$1 - \sigma(x, k) \tau(x, k) \in M_{\rho, \Lambda}^{-\rho}(\mathbb{T} \times \mathbb{Z}),$$

So, we can find two positive constants $C$ and $C_0$ such that

$$|1 - \sigma(x, k) \tau(x, k)| \leq C_0 \Lambda(k)^{-\rho} \leq C(1 + |k|)^{-\rho m}, \quad (x, k) \in \mathbb{T} \times \mathbb{Z}.$$

Let $R \in \mathbb{N}$ such that $C(1 + R)^{-\rho m} < \frac{1}{2}$. Then it follows that

$$|\sigma(x, k) \tau(x, k)| \geq \frac{1}{2}, \quad \forall \ |k| \geq R,$$

and hence

$$|\sigma(x, k)| \geq \frac{1}{2 |\tau(x, k)|} \geq \frac{1}{2 C_0 (\Lambda(k))^m}, \quad \forall \ |k| \geq R,$$

since

$$|\tau(x, k)| \leq C'(\Lambda(k))^{-m}, \quad (x, k) \in \mathbb{T} \times \mathbb{Z}.$$

Hence $\sigma$ is an $M$-elliptic symbol of order $m$.

Conversely, let us assume that $\sigma$ is an $M$-elliptic symbol of order $m$. Then there exist positive constants $C$ and $R$ such that

$$|\sigma(x, k)| \geq C \Lambda(k)^m,$$

for all $x \in \mathbb{T}$ and for all $k \in \mathbb{Z}$ with $|k| \geq R$. The idea is to find a sequence of symbols $\tau_j \in M_{\rho, \Lambda}^{-m-\rho}(\mathbb{T} \times \mathbb{Z}), j = 0, 1, 2, \ldots$. Let us assume that this can be done. Then, by Theorem 3.1, there exists a symbol $\tau \in M_{\rho, \Lambda}^m(\mathbb{T} \times \mathbb{Z})$ such that $\tau \sim \sum_{j=0}^{\infty} \tau_j$, and, by Theorem 3.2, the symbol $\lambda$ of the product $T_\tau T_\sigma$ is in $M_{\rho, \Lambda}^0(\mathbb{T} \times \mathbb{Z})$ such that

$$\lambda - \sum_{|\gamma| < N} \frac{(-i)^{|\gamma|}}{\gamma!} (\partial_\nu^\gamma \sigma) (\Delta_\nu^\gamma \tau) \in M_{\rho, \Lambda}^{-\rho N}(\mathbb{T} \times \mathbb{Z}), \quad (3.3)$$

for every positive integer $N$. Also $\tau \sim \sum_{j=0}^{\infty} \tau_j$ implies that

$$\tau - \sum_{j=0}^{N-1} \tau_j \in M_{\rho, \Lambda}^{-m-\rho N}(\mathbb{T} \times \mathbb{Z}), \quad (3.4)$$
for every positive integer \( N \). Hence, by (3.3) and (3.4),
\[
\lambda - \sum_{|\gamma| < N} \frac{(-i)^{|\gamma|}}{\gamma!} (\partial_x^{|\gamma|}) (\Delta^N_k \tau_j) \in M_{-p}^N (T \times \mathbb{Z}),
\]
for every positive integer \( N \). But we can write
\[
\sum_{|\gamma| < N} \frac{(-i)^{|\gamma|}}{\gamma!} (\Delta^N_k \tau_j) (\partial_x^{|\gamma|}) \\
= \tau_0 \sigma + \sum_{l=1}^{N-1} \tau_l \sigma + \sum_{|\gamma| + j = l, j < N} \frac{(-i)^{|\gamma|}}{\gamma!} (\Delta^N_k \tau_j) (\partial_x^{|\gamma|}) \\
+ \sum_{|\gamma| + j \geq N, j < N} \frac{(-i)^{|\gamma|}}{\gamma!} (\Delta^N_k \tau_j) (\partial_x^{|\gamma|}).
\]
To find a sequence of symbols \( \tau_j \in M_{p,\Lambda}^{-m-\rho_j} (T \times \mathbb{Z}), j = 0, 1, 2, \ldots \), we choose \( \psi \) to be any function in \( C^\infty (\mathbb{R}^n) \) such that \( \psi(k) = 1 \), for \( |k| \geq 2R \) and \( \psi(k) = 0 \), for \( |k| \leq R \). Define
\[
\tau_0(x,k) = \begin{cases} 
\frac{\psi(k)}{\sigma(x,k)}, & |k| > R, \\
0, & |k| \leq R, 
\end{cases} \quad (x,k) \in T \times \mathbb{Z}.
\]
From Lemma 3.4, it is clear that \( \tau_0 \in M_{-p,\Lambda}^{-m} (T \times \mathbb{Z}) \). Now define, \( \tau_l \), for \( l \geq 1 \), inductively by
\[
\tau_l = - \left\{ \sum_{|\gamma| + j = l} \frac{(-i)^{|\gamma|}}{\gamma!} (\Delta^N_k \tau_j) (\partial_x^{|\gamma|}) \right\} \tau_0.
\]
Then it can be shown that \( \tau_j \in M_{p,\Lambda}^{-m-\rho_j} (T \times \mathbb{Z}), j = 0, 1, 2, \ldots \). Now, by (3.7), \( \tau_0 \sigma = 1 \), for \( |k| \geq 2R \). The second term on the right hand side of (3.6) vanishes for \( |k| \geq 2R \) by (3.7) and (3.8). Also the third term there,
\[
(\Delta^N_k \tau_j) (\partial_x^{|\gamma|}) \in M_{p,\Lambda}^{-pN} (T \times \mathbb{Z}),
\]
whenever \( |\gamma| + j \geq N \). Hence, by (3.6),
\[
\sum_{|\gamma| < N} \frac{(-i)^{|\gamma|}}{\gamma!} \sum_{j=0}^{N-1} \Delta^N_k \tau_j (\partial_x^{|\gamma|}) - 1 \in M_{p,\Lambda}^{-pN} (T \times \mathbb{Z}),
\]
for every positive integer \( N \). Thus, by (3.5) and (3.9),
\[
\lambda - 1 \in M_{p,\Lambda}^{-pN} (T \times \mathbb{Z}),
\]
for every positive integer \( N \). Hence, if we pick \( R \) to be the pseudo-differential operator with symbol \( \lambda - 1 \), then the proof of (3.1) is complete.

By a similar argument, we can find another symbol, \( \kappa \in M_{-m} (T \times \mathbb{Z}) \), such that
\[
T_{\sigma} T_{\kappa} = I + R',
\]
where $R'$ is a pseudo-differential operator with symbol in $\bigcap_{m \in \mathbb{R}} M_{\rho, \Lambda}^m (\mathbb{T} \times \mathbb{Z})$. By (3.1) and (3.10),

$$T_\kappa + RT_\kappa = T_\tau + T_\tau R'.$$

Since $RT_\kappa$ and $T_\tau R'$ are pseudo-differential operators with symbols in $\bigcap_{m \in \mathbb{R}} M_{\rho, \Lambda}^m (\mathbb{T} \times \mathbb{Z})$, it follows that

$$T_\kappa = T_\tau + R''_\kappa,$$

(3.11)

where

$$R''_\kappa = T_\tau R' - RT_\kappa,$$

is another pseudo-differential operator with symbol in $\bigcap_{m \in \mathbb{R}} M_{\rho, \Lambda}^m (\mathbb{T} \times \mathbb{Z})$. Hence, by (3.10) and (3.11),

$$T_\sigma T_\tau = I + S,$$

where

$$S = R' - T_\sigma R''_\kappa.$$

Since $S$ is a pseudo-differential operator with symbol in $\bigcap_{m \in \mathbb{R}} M_{\rho, \Lambda}^m (\mathbb{T} \times \mathbb{Z})$, it follows that (3.2) is proved. \qed

4. Compact $M$-elliptic pseudo-differential operators

This section is devoted to study the compact $M$-elliptic pseudo-differential operators on $\mathbb{T}$ and $\mathbb{Z}$. First, we prove Gohberg’s lemma for pseudo-differential operators on $\mathbb{T}$ and $\mathbb{Z}$ with symbol in the weighted symbol class $M_{\rho, \Lambda}^0 (\mathbb{T} \times \mathbb{Z})$ and $M_{\rho, \Lambda}^0 (\mathbb{Z} \times \mathbb{T})$, respectively. Using Gohberg’s lemma, we provide a necessary and sufficient condition to ensure the compactness of a pseudo-differential operator on $L^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$. We start this section by recalling the definition of Fredholm operators.

Let $X$ and $Y$ be two complex Banach spaces. A closed linear operator $A : X \to Y$ with dense domain $D(A)$ is said to be Fredholm if the null space $N(A)$ of $A$, and the null space $N(A^*)$ of the adjoint $A^*$ of $A$ are finite-dimensional, and the range space $R(A)$ of $A$ is a closed subspace of $Y$. For a Fredholm operator $A$, the index $i(A)$ of $A$ is defined by

$$i(A) = \dim N(A) - \dim N(A^*).$$

Let $X$ be a complex Banach space and $A : X \to X$ be a closed linear operator with dense domain $D(A)$. Then the spectrum $\Sigma(A)$ of $A$ is defined by

$$\Sigma(A) = \mathbb{C} \setminus \rho(A),$$

where $\rho(A)$ is the resolvent set of $A$ given by

$$\rho(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is bijective} \},$$

and $I$ is the identity operator on $X$. The essential spectrum $\Sigma_w(A)$ of $A$ (defined in [35]) is given by

$$\Sigma_w(A) = \mathbb{C} \setminus \Phi_w(A),$$

where

$$\Phi_w(A) = \{ \lambda \in \mathbb{C} : A - \lambda I \text{ is Fredholm} \}.$$

For a detailed study on essential spectrum of an operator, we refer to [32, 33].

For $-\infty < s < \infty$, let $J_s$ be the pseudo-differential operator with symbol $\sigma_s$ given by

$$\sigma_s(k) = (\Lambda(k))^{-s}, \quad k \in \mathbb{Z}.$$
Clearly, $\sigma_s \in M^{-s}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$. In literature, $J_s$ is called the Bessel potential of order $s$. For $s \in \mathbb{R}$ and $1 < p < \infty$, we define the Sobolev space, $H^{s,p}_\Lambda$ by

$$H^{s,p}_\Lambda = \{ u \in \mathcal{D}'(\mathbb{T}) : J_{-s} u \in L^p(\mathbb{T}) \}.$$ 

Then $H^{s,p}_\Lambda$ is a Banach space in which the norm $\| \cdot \|_{s,p}$ is given by

$$\| u \|_{s,p,\Lambda} = \| J_{-s} u \|_{L^p(\mathbb{T})}, \quad u \in H^{s,p}_\Lambda.$$ 

Note that $H^{0,p}_\Lambda = L^p(\mathbb{T})$.

The following well-known compact Sobolev embedding theorem is just the weighted version of [24, Theorem 2.5].

**Theorem 4.1.** Let $s < t$. Then the inclusion $i : H^{t,p}_\Lambda \hookrightarrow H^{s,p}_\Lambda$ is compact for $1 < p < \infty$.

The following boundedness result on weighted Sobolev space can be found in [25], Theorem 2.5 for $M^{0}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$ class.

**Proposition 4.3.** Let $\sigma \in M^{0}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$ be such that

$$\lim_{|k| \to \infty} \left\{ \sup_{x \in [-\pi, \pi]} |\sigma(x, k)| \right\} = 0.$$ 

Then

$$\Sigma_c(T_\sigma) = \{0\}.$$ 

A bounded linear operator $A$ on a complex separable and infinite-dimensional Hilbert space $X$ is essentially normal if $AA^* - A^*A$ is compact.

The next result is about the essential normality of pseudo-differential operators with weighted symbol of order 0.

**Proposition 4.4.** Let $\sigma \in M^{0}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$. Then the bounded linear operator $T_\sigma : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is essentially normal.

**Proof** Let $\tau \in M^{0}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$ be such that $T_\tau = T_\gamma$. Then using Theorem 3.2, we have

$$T_\tau T_\sigma = T_\gamma \quad \text{and} \quad T_\sigma T_\tau = T_{\tilde{\gamma}},$$

where $\gamma$ and $\tilde{\gamma}$ are symbols of order 0. Moreover, $\gamma - \sigma \tau \in M^{-\rho}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$ and $\tilde{\gamma} - \sigma \tau \in M^{-\rho}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$. Therefore, $\gamma - \tilde{\gamma} \in M^{-\rho}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$. Hence, by Theorem 4.1 and 4.2, we get

$$T_\sigma T_\tau - T_\tau T_\sigma = T_{\gamma - \tilde{\gamma}} : L^2(\mathbb{T}) \to H^{0,2}_\Lambda \hookrightarrow L^2(\mathbb{T})$$

is compact, which completes the proof.

The following theorem is known as Gohberg’s lemma in the literature.

**Theorem 4.5.** Let $\sigma \in M^{0}_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z})$. Then for all compact operators $K$ on $L^2(\mathbb{T})$,

$$\| T_\sigma - K \|_* \geq d, \quad (4.1)$$

where

$$d = \limsup_{|k| \to \infty} \left\{ \sup_{x \in [-\pi, \pi]} |\sigma(x,k)| \right\}.$$ 

Here $\| \cdot \|_*$ denotes the norm in the $C^*$-algebra of all bounded linear operators on $L^2(\mathbb{T})$. 
Proof Let \( u \) be a nonzero function in \( C^\infty(\mathbb{T}) \). Then

\[
(T_\sigma u)(x) = (2\pi)^{-1} \sum_{k \in \mathbb{Z}} \left\{ \int_{-\pi}^{\pi} e^{ik(x-y)} \sigma(x, k) u(y) dy \right\}
\]

\[
= (2\pi)^{-1} \int_{-\pi}^{\pi} (\mathcal{F}_2 \sigma)(x, x-y) u(y) dy
\]

\[
= (2\pi)^{-1} \int_{-\pi}^{\pi} (\mathcal{F}_2 \sigma)(x, y) u(x-y) dy, \quad x \in [-\pi, \pi],
\]

where \( \mathcal{F}_2 \sigma \) is the Fourier transform of \( \sigma \) with respect to the second variable in the sense of distribution. Since \( \sigma \in C^\infty(\mathbb{T} \times \mathbb{Z}) \), it follows that for all \( k \in \mathbb{Z} \), there exists \( x_k \in [-\pi, \pi] \) such that

\[
|\sigma(x_k, k)| = \sup_{x \in [-\pi, \pi]} |\sigma(x, k)|.
\]

By the definition of \( d \), there exists a sequence \( \{(x_{km}, k_m)\}_{m=1}^\infty \) such that

\[
|k_m| \to \infty,
\]

and

\[
|\sigma(x_{km}, k_m)| \to d
\]

as \( m \to \infty \). For \( m = 1, 2, \ldots \), we define the function \( u_{km} \) on \( \mathbb{T} \) by

\[
u_{km}(x) = e^{ikmx} u(x-x_{km}), \quad x \in [-\pi, \pi].
\]

Then

\[
\|u_{km}\|_{L^2(\mathbb{T})} = \|u\|_{L^2(\mathbb{T})}, \quad m = 1, 2, \ldots,
\]

and moreover, using the Riemann-Lebesgue lemma, one can easily show that \( u_{km} \to 0 \) weakly as \( k \to \infty \). Let \( K : L^2(\mathbb{T}) \to L^2(\mathbb{T}) \) be a compact operator and \( \epsilon \) be an arbitrary positive number. Then

\[
\|K u_{km}\|_{L^2(\mathbb{T})} \to 0
\]

as \( m \to \infty \), and hence, for sufficiently large \( m \),

\[
\|K u_{km}\|_{L^2(\mathbb{T})} \leq \epsilon \|u\|_{L^2(\mathbb{T})}.
\]

Lemma 4.6. \( \|\sigma(\cdot, k_m) u_{km} - T_\sigma u_{km}\|_{L^2(\mathbb{T})} \to 0 \) as \( m \to \infty \).

We assume the lemma for a moment and continue with the proof of Theorem 4.5. By Lemma 4.6, we have

\[
\|\sigma(\cdot, k_m) u_{km}\|_{L^2(\mathbb{T})} - \|T_\sigma u_{km}\|_{L^2(\mathbb{T})} \leq \epsilon \|u\|_{L^2(\mathbb{T})},
\]

for sufficiently large \( m \). Since \( \sigma \in C^\infty(\mathbb{T} \times \mathbb{Z}) \), so all derivatives of \( \sigma(\cdot, k_m) \) exist and are bounded, and hence, there exists a positive number \( \delta \) such that for all \( x \in [-\pi + x_{km}, \pi + x_{km}] \) with \( |x - x_{km}| < \delta \), we have

\[
|\sigma(x, k_m) - \sigma(x_{km}, k_m)| < \epsilon.
\]

Choose \( u \in C^\infty(\mathbb{T}) \) be such that

\[
u(x) = 0, \quad |x| \geq \delta.
\]
Then $u_{km}(x) = 0$ for all $x$ in $[-\pi + x_{km}, \pi + x_{km}]$ with $|x - x_{km}| \geq \delta$. So,
\[
\left\| \sigma (x_{km}, k_m) u_{km} \right\|_{L^2(T)} - \left\| \sigma (\cdot, k_m) u_{km} \right\|_{L^2(T)} \\
\leq \left\| \sigma (x_{km}, k_m) u_{km} - \sigma (\cdot, k_m) u_{km} \right\|_{L^2(T)} \\
= \left\{ \int_{-\pi + x_{km}}^{\pi + x_{km}} |\sigma (x, k_m) - \sigma (x_{km}, k_m)|^2 |u_{km}(x)|^2 \, dx \right\}^{1/2} \\
= \left\{ \int_{\{x \in [-\pi + x_{km}, \pi + x_{km}]: |x - x_{km}| < \delta\}} |\sigma (x, k_m) - \sigma (x_{km}, k_m)|^2 |u_{km}(x)|^2 \, dx \right\}^{1/2}.
\]
Hence, by (4.4),
\[
|\sigma (x_{km}, k_m)| \left\| u \right\|_{L^2(T)} - \left\| \sigma (\cdot, k_m) u_{km} \right\|_{L^2(T)} \leq \epsilon \left\| u \right\|_{L^2(T)}.
\]
Thus, by (4.2), (4.3), and (4.5), for sufficiently large $m$, we get
\[
\left\| u \right\|_{L^2(T)} \left\| T_{\sigma} - K \right\|_* \geq \left\| (T_{\sigma} - K) u_{km} \right\|_{L^2(T)} \\
\geq \left\| T_{\sigma} u_{km} \right\|_{L^2(T)} - \left\| Ku_{km} \right\|_{L^2(T)} \\
\geq \left\| T_{\sigma} u_{km} \right\|_{L^2(T)} - \epsilon \left\| u \right\|_{L^2(T)} \\
\geq \left\| \sigma (\cdot, k_m) u_{km} \right\|_{L^2(T)} - 2\epsilon \left\| u \right\|_{L^2(T)} \\
\geq |\sigma (x_{km}, k_m)| \left\| u \right\|_{L^2(T)} - 3\epsilon \left\| u \right\|_{L^2(T)} \\
= (|\sigma (x_{km}, k_m)| - 3\epsilon) \left\| u \right\|_{L^2(T)}.
\]
Letting $m \to \infty$, we get
\[
\left\| T_{\sigma} - K \right\|_* \geq d - 3\epsilon.
\]
Finally, using the fact that $\epsilon$ is an arbitrary positive number, we have
\[
\left\| T_{\sigma} - K \right\|_* \geq d.
\]

Proof of Lemma 4.6. Let
\[
K(x, y) = (F_{\mathbb{Z}} \sigma) (x, y), \quad x, y \in [-\pi, \pi].
\]
Then for $k = 1, 2, \ldots,$
\[
(T_{\sigma} u_{km}) (x) = (2\pi)^{-1} \int_{-\pi}^{\pi} e^{ikm(x-y)} K(x, y) u_{x_{km}} (x - y) \, dy, \quad x \in [-\pi, \pi],
\]
where $u_{x_{km}} = u(x - x_{km})$. In the view of Example 2.4 in [30], the function $q(x) = e^{-2\pi ix} - 1$ gives rise to a strongly admissible difference operator on $\mathbb{T}$ with the property that
\[
\Delta p(k) = \Delta q p(k) = p(k + 1) - p(k), \quad k \in \mathbb{Z}.
\]
Let $N$ be a positive integer. Then by the Taylor expansion formula (see [30])
\[
u_{x_{km}}(x - y) = u_{x_{km}}(x) + \sum_{\alpha=1}^{N-1} \frac{1}{\alpha!} q^{\alpha}(y) \partial^\alpha u_{x_{km}}(x) + \mathcal{O} (h(y)^N),
\]
where \( h(x) \) is the geodesic distance from \( x \) and the identity element of \( T \). Using (4.6) and (4.7), we can write

\[
T_\sigma u_{k_m}(x) = (2\pi)^{-1} \int_T e^{ik_m(x-y)} K(x, y) u_{x+k_m}(x) dy \\
+ (2\pi)^{-1} \int_T e^{ik_m(x-y)} K(x, y) \sum_{\alpha=1}^{N-1} \frac{1}{\alpha!} q^\alpha(y) \partial^\alpha u_{x+k_m}(x) dy \\
+ (2\pi)^{-1} \int_T e^{ik_m(x-y)} K(x, y) O \left( h(y)^N \right) dy.
\]

Define

\[
I_1 := \int_T e^{ik_m(x-y)} K(x, y) u_{x+k_m}(x) dy, \\
I_2 := \int_T e^{ik_m(x-y)} K(x, y) \sum_{\alpha=1}^{N-1} \frac{1}{\alpha!} q^\alpha(y) \partial^\alpha u_{x+k_m}(x) dy, \\
\]

and

\[
I_3 := \int_T e^{ik_m(x-y)} K(x, y) O \left( h(y)^N \right) dy.
\]

So, we have

\[
I_1 = \int_T e^{ik_m(x-y)} K(x, y) u_{x+k_m}(x) dy \\
= u_{k_m}(x) \sigma(x, k_m),
\]

\[
I_2 = \int_T e^{ik_m(x-y)} K(x, y) \sum_{\alpha=1}^{N-1} \frac{1}{\alpha!} q^\alpha(y) \partial^\alpha u_{x+k_m}(x) dy \\
= \sum_{\alpha=1}^{N-1} \frac{1}{\alpha!} e^{ik_mx} \partial^\alpha u_{x+k_m}(x) \int_T e^{-ik_my} K(x, y) q^\alpha(y) dy \\
= \sum_{\alpha=1}^{N-1} \frac{1}{\alpha!} e^{ik_mx} \partial^\alpha u_{x+k_m}(x) \Delta_q^\alpha \sigma(x, k_m) \\
= \sum_{\alpha=1}^{N-1} \frac{1}{\alpha!} e^{ik_mx} \partial^\alpha u_{x+k_m}(x) \Delta^\alpha \sigma(x, k_m),
\]

\[
I_3 = \int_T K(x, y) O \left( h(y)^N \right) e^{ik_m(x-y)} dy \\
= \int_T K(x, y) q^N(y) e^{ik_m(x-y)} dy \\
= e^{ik_mx} \Delta_q^N \sigma(x, k_m) \\
= e^{ik_mx} \Delta^N \sigma(x, k_m),
\]

where \( q^N(x) := O \left( h(x)^N \right) \) vanishes at the identity element of \( T \). Hence,

\[
T_\sigma u_{k_m}(x) - u_{k_m}(x) \sigma(x, k_m) = \sum_{\alpha=1}^{N-1} \frac{1}{\alpha!} e^{ik_mx} \partial^\alpha u_{x+k_m}(x) \Delta^\alpha \sigma(x, k_m) + e^{ik_mx} \Delta^N \sigma(x, k_m).
\]

Define

\[
T_N^1(x) := \sum_{\alpha=1}^{N-1} \frac{1}{\alpha!} e^{ik_mx} \partial^\alpha u_{x+k_m}(x) \Delta^\alpha \sigma(x, k_m),
\]
and
\[ T_N^2(x) := e^{ikm \Delta^N} \sigma(x, k_m). \]

Now using the fact that \( u_{x, k_m} \in C^\infty(\mathbb{T}) \), we obtain
\[
|T_N^1(x)| \leq \sum_{\alpha=1}^{N-1} C_\alpha |\Delta^\alpha \sigma(x, k_m)| \\
\leq \sum_{\alpha=1}^{N-1} C_\alpha \Lambda(k_m)^{-\rho\alpha} \\
\leq C \Lambda(k_m)^{-\rho}, \quad x \in \mathbb{T},
\]
where \( C = \sum_{\alpha=1}^{N-1} C_\alpha \). Clearly \( |T_N^1(x)| \to 0 \) uniformly on \( \mathbb{T} \) as \( m \to \infty \), and hence, \( ||T_N^1||_{L^2(\mathbb{T})} \to 0 \) as \( m \to \infty \).

Similarly, \( |T_N^2(x)| \leq C_0 \Lambda(k_m)^{-\rho N} \leq C_0 \Lambda(k_m)^{-\rho} \to 0 \) uniformly on \( \mathbb{T} \) as \( m \to \infty \), and hence, \( ||T_N^2||_{L^2(\mathbb{T})} \to 0 \) as \( m \to \infty \). This completes the proof of the theorem. \( \square \)

Now we recall the definition of the Calkin algebra, which will be used to prove the main result of this section. Let \( B(L^2(\mathbb{T})) \) and \( K(L^2(\mathbb{T})) \) denotes the \( C^* \)-algebra of bounded linear operators on \( L^2(\mathbb{T}) \) and the ideal of compact operators on \( L^2(\mathbb{T}) \), respectively. The Calkin algebra \( B(L^2(\mathbb{T}))/K(L^2(\mathbb{T})) \) is a \( * \)-algebra with respect to the product and the adjoint defined as follows:
\[
[A][B] = [AB]
\]
and
\[
[A]^* = [A^*]
\]
for all \( A \) and \( B \) in \( B(L^2(\mathbb{T})) \). Two elements \([A]\) and \([B]\) be in the Calkin algebra \( B(L^2(\mathbb{T}))/K(L^2(\mathbb{T})) \) are equal if and only if \( A - B \in K(L^2(\mathbb{T})) \).

The norm \( ||\cdot||_C \) in \( B(L^2(\mathbb{T}))/K(L^2(\mathbb{T})) \) is given by
\[
||[A]||_C = \inf_{K \in K(L^2(\mathbb{T}))} \|A - K\|_*, \quad [A] \in B(L^2(\mathbb{T}))/K(L^2(\mathbb{T})).
\]
It can be shown that \( B(L^2(\mathbb{T}))/K(L^2(\mathbb{T})) \) is a \( C^* \)-algebra. Using the Calkin algebra, (4.1) in Gohberg’s lemma is the same as
\[
||[T_\sigma]||_C \geq d.
\]

Now we are ready to prove our main theorem in this section.

**Theorem 4.7.** Let \( \sigma \in M_{p, \Lambda}^0(\mathbb{T} \times \mathbb{Z}) \). Then \( T_\sigma \) is a compact operator on \( L^2(\mathbb{T}) \) if and only if \( d = 0 \), where
\[
d = \lim_{|k| \to \infty} \sup_{x \in [-\pi, \pi]} |\sigma(x, k)|.
\]

**Proof** First, let us assume that \( d = 0 \). Then \( T_\sigma \) is compact if and only if \([T_\sigma] = 0\) in \( B(L^2(\mathbb{T}))/K(L^2(\mathbb{T})) \). By Proposition 4.4, \( T_\sigma \) is essentially normal on \( L^2(\mathbb{T}) \), which implies that \([T_\sigma]\) is normal in the Calkin algebra \( B(L^2(\mathbb{T}))/K(L^2(\mathbb{T})) \). Hence,
\[
r([T_\sigma]) = ||[T_\sigma]||_C,
\]
where \( r([T_\sigma]) \) is the spectral radius of \([T_\sigma]\), and by Proposition 4.3, we get \( \Sigma_c(T_\sigma) = \{0\} \).

Therefore, by Atkinson’s theorem (see [1]), the spectrum of \([T_\sigma]\) in the Calkin algebra \( B(L^2(\mathbb{T}))/K(L^2(\mathbb{T})) \) is
\[
\Sigma([T_\sigma]) = \{0\}.\]
This implies that
\[ \| [T_\sigma] \|_{C^*} = r ([T_\sigma]) = 0. \]
Hence, it follows that
\[ [T_\sigma] = 0. \]
Therefore \( T_\sigma \) is compact.

Now, to prove the converse part, assume that \( d \neq 0 \), then we need to show that \( T_\sigma \) is not compact on \( L^2(\mathbb{T}) \). Suppose that \( T_\sigma \) is compact, then putting \( K = T_\sigma \) in (4.1) will contradict our assumption that \( d \neq 0 \). This completes the proof of the theorem. \( \square \)

Now our aim is to study the Gohberg’s lemma and characterization of compact operators on \( \ell^2(\mathbb{Z}) \) with symbol in \( M^0_{\rho, \Lambda}(\mathbb{Z} \times \mathbb{T}) \) class (defined in [23]). The main ingredient is the relation between the weighted periodic and discrete symbols which can be found in [23].

**Theorem 4.8.** Let \( \sigma : \mathbb{Z} \times \mathbb{T} \to \mathbb{C} \) be a measurable function such that the pseudo-differential operator \( T_\sigma : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) is a bounded linear operator. If we define \( \tau : \mathbb{T} \times \mathbb{Z} \to \mathbb{C} \) by
\[ \tau(x, k) = \sigma(-k, x), \quad x \in \mathbb{T}, k \in \mathbb{Z}, \]
then
\[ T_\sigma = F^{-1}_\mathbb{Z} T^*_\tau F_\mathbb{Z}, \]
where \( T^*_\tau \) is the adjoint of \( T_\tau \). We also have
\[ T_\tau = F_\mathbb{Z} T^*_\sigma F^{-1}_\mathbb{Z}, \]
where \( T^*_\sigma \) is the adjoint of \( T_\sigma \).

As a corollary of Theorem 4.5 and Theorem 4.8, we obtain the following estimates for the distance between a given operator and the space of compact operators on \( \ell^2(\mathbb{Z}) \).

**Corollary 4.9.** Let \( \sigma \in M^0_{\rho, \Lambda}(\mathbb{Z} \times \mathbb{T}) \). Then for all compact operators \( K \) on \( \ell^2(\mathbb{Z}) \),
\[ \| T_\sigma - K \|_{\text{ss}} \geq d, \]
(4.10)
where
\[ d = \limsup_{|k| \to \infty} \left\{ \sup_{x \in [-\pi, \pi]} |\sigma(k, x)| \right\}. \]
Here \( \| \cdot \|_{\text{ss}} \) denotes the norm in the \( C^* \)-algebra of all bounded linear operators on \( \ell^2(\mathbb{Z}) \).

**Proof** We define \( \tau : \mathbb{T} \times \mathbb{Z} \to \mathbb{C} \) by
\[ \tau(x, k) = \sigma(-k, x), \quad x \in \mathbb{T}, k \in \mathbb{Z}. \]
Since \( \sigma \in M^0_{\rho, \Lambda}(\mathbb{Z} \times \mathbb{T}) \), we have \( \tau \in M^0_{\rho, \Lambda}(\mathbb{T} \times \mathbb{Z}) \). Then, by Theorem 4.5, we have the following estimate for all compact operators \( K' \) on \( L^2(\mathbb{T}) \):
\[ \| T_\tau - K' \|_{\text{ss}} \geq d. \]
(4.11)
We need to show that
\[ \| T_\sigma - K \|_{\text{ss}} \geq d, \]
(4.12)
for all compact operators \( K \) on \( \ell^2(\mathbb{Z}) \). Let \( K : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \) be any arbitrary compact operator. Then \( K_1 = \)
\[ \mathcal{F}_2K^*\mathcal{F}_2^{-1} : L^2(T) \to L^2(T) \text{ is a compact operator. Hence, by (4.9) and (4.11), we have} \]
\[ \|T_T - K_1\|_{L^2(T)} \geq d, \]
\[ \implies \|\mathcal{F}_2T_\sigma^*\mathcal{F}_2^{-1} - \mathcal{F}_2K^*\mathcal{F}_2^{-1}\|_{L^2(T)} \geq d, \]
\[ \implies \|\mathcal{F}_2(T_\sigma - K)^*\mathcal{F}_2^{-1}\|_{L^2(T)} \geq d, \]
\[ \implies \|T_\sigma - K\|_{L^2(T)} \geq d, \]
\[ \implies \|T_\sigma - K\|_{L^2(T)} \geq d, \]
\[ \text{and this completes the proof of the estimate (4.12).} \]

The following corollary gives us a necessary and sufficient condition for an operator to be compact on \( \ell^2(\mathbb{Z}) \) for symbol class \( M^0_{\rho,\Lambda}(\mathbb{Z} \times \mathbb{T}) \).

**Corollary 4.10.** Let \( \sigma \in M^0_{\rho,\Lambda}(\mathbb{Z} \times \mathbb{T}) \). Then \( T_\sigma \) is a compact operator on \( \ell^2(\mathbb{Z}) \) if and only if \( d = 0 \), where

\[ d = \limsup_{|k| \to \infty} \sup_{x \in [-\pi, \pi]} |\sigma(k, x)|. \]

**Proof** First, let us assume that \( d = 0 \). Define

\[ \tau(x, k) = \frac{\sigma(-k, x)}{\sigma(k, x)}, \quad x \in \mathbb{T}, k \in \mathbb{Z}. \]

Since \( \sigma \in M^0_{\rho,\Lambda}(\mathbb{Z} \times \mathbb{T}) \), we have \( \tau \in M^0_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z}) \). Hence, by Theorem 4.7, \( T_\sigma \) is compact on \( L^2(T) \). This implies that \( T_\sigma^* \) is a compact operator on \( L^2(T) \). Hence, by (4.8), \( T_\sigma \) is a compact operator on \( \ell^2(\mathbb{Z}) \).

Now, to prove the converse part, assume that \( d \neq 0 \), then we need to show that \( T_\sigma \) is not compact on \( \ell^2(\mathbb{Z}) \). Suppose that \( T_\sigma \) is compact, then putting \( K = T_\sigma \) in (4.10) will contradict our assumption that \( d \neq 0 \). This completes the proof of the theorem. \( \square \)

5. Gårding’s and sharp Gårding’s inequalities on \( \mathbb{T} \) and \( \mathbb{Z} \)

The main aim of this section is to prove Gårding’s and sharp Gårding’s inequalities for \( M \)-elliptic operators on \( \mathbb{T} \) and \( \mathbb{Z} \), respectively. First, we state the Gårding’s inequality for \( M \)-elliptic operators on \( \mathbb{T} \), which is analogous to the [3, Corollary 5.7]. The proof can be done in similar lines, so we skip the proof here.

**Theorem 5.1.** (Gårding’s inequality for \( M \)-elliptic operators on \( \mathbb{T} \))

Let \( m > 0 \). Let \( \sigma \in M^2_{\rho,\Lambda}(\mathbb{T} \times \mathbb{Z}) \) be elliptic such that \( \sigma(x, k) \geq 0 \) for all \( x \) and co-finitely many \( k \). Then there exist \( C_0, C_1 > 0 \) such that for all \( f \in H^{m,2}_\Lambda(\mathbb{T}) \), we have

\[ \text{Re}(T_\sigma f, f)_{L^2(T)} \geq C_0 \|f\|_{H^{m,2}_\Lambda(\mathbb{T})}^2 - C_1 \|f\|_{L^2(T)}^2. \]

(5.1)

Now, we will show that Theorem 5.1 implies the corresponding Gårding inequality for \( M \)-elliptic operators on \( \mathbb{Z} \). As there is no regularity concept on the lattice, the statement is given in terms of weighted \( \ell^2 \)-spaces. For this, we need the following definition:

**Definition 5.2.** For \( s \in \mathbb{R} \), let us define the weighted space \( \ell^2_{s,\Lambda}(\mathbb{Z}) \) as the space of all \( f : \mathbb{Z} \to \mathbb{C} \) such that

\[ \|f\|_{\ell^2_{s,\Lambda}(\mathbb{Z})} := \left( \sum_{k \in \mathbb{Z}} \Lambda(k)^{2s} |f(k)|^2 \right)^{1/2} < \infty. \]

We observe that the symbol \( \sigma_s(k) = \Lambda(k)^s \) belongs to \( M^s_{\rho,\Lambda}(\mathbb{Z} \times \mathbb{T}) \), and \( f \in \ell^2_{s,\Lambda}(\mathbb{Z}) \) if and only if \( T_\sigma f \in \ell^2(\mathbb{Z}) \).
Theorem 5.3.  (Gårding’s inequality for M-elliptic operators on $\mathbb{Z}$)
Let $m > 0$. Let $\sigma \in M^{2m}_{\rho,\Lambda} (\mathbb{Z} \times \mathbb{T})$ be elliptic such that $\sigma(k,x) \geq 0$ for all $x$ and co-finitely
many $k$. Then there exist $C_1, C_2 > 0$ such that for all $f \in \ell^2_{m,\Lambda} (\mathbb{Z})$, we have
\[
\Re (T_\sigma f, f)_{\ell^2(\mathbb{Z})} \geq C_0 \|f\|^2_{\ell^2_{m,\Lambda}(\mathbb{Z})} - C_1 \|f\|^2_{\ell^2(\mathbb{Z})}.
\]
Proof  Let $\tau(k,x) = \overline{\sigma(-k,x)}$. Then by Theorem 4.8, we have
\[
T_\tau = T_\sigma^{-1} T_\sigma^* T_\sigma,
\]
and if $\sigma$ is elliptic on $\mathbb{Z} \times \mathbb{T}$, then $\tau$ is elliptic on $\mathbb{T} \times \mathbb{Z}$. Also, if $\sigma \geq 0$, then $\tau \geq 0$. Then
by Theorem 5.1, for all $g \in H^m_{\Lambda^2} (\mathbb{T})$, we have
\[
\Re (T_\tau^* g, g)_{L^2(\mathbb{T})} = \Re (T_\tau g, g)_{L^2(\mathbb{T})} \geq C_0 \|g\|^2_{H^m_{\Lambda^2}(\mathbb{T})} - C_1 \|g\|^2_{L^2(\mathbb{T})}.
\]
(5.2)
Let $f \in \ell^2_{m,\Lambda} (\mathbb{Z})$ and $g = F_\mathbb{Z} f$. Then $g \in H^m_{\Lambda^2} (\mathbb{T})$, and
\[
\|g\|_{H^m_{\Lambda^2}(\mathbb{T})} = \|f\|_{\ell^2_{m,\Lambda}(\mathbb{Z})} \quad \text{and} \quad \|g\|_{L^2(\mathbb{T})} = \|f\|_{\ell^2(\mathbb{Z})}.
\]
(5.3)
Now by Theorem 4.8,
\[
T_\sigma f = F_\mathbb{Z}^{-1} T_\sigma^* F_\mathbb{Z} f = F_\mathbb{Z}^{-1} T_\sigma^* g,
\]
so that $F_\mathbb{Z} T_\sigma f = T_\sigma^* g$. Substituting (5.3) into (5.2), we get
\[
\Re (T_\tau^* g, g)_{L^2(\mathbb{T})} \geq C_0 \|f\|^2_{\ell^2_{m,\Lambda}(\mathbb{Z})} - C_1 \|f\|^2_{\ell^2(\mathbb{Z})},
\]
\[
\Re (F_\mathbb{Z} T_\sigma f, F_\mathbb{Z} f)_{L^2(\mathbb{T})} \geq C_0 \|f\|^2_{\ell^2_{m,\Lambda}(\mathbb{Z})} - C_1 \|f\|^2_{\ell^2(\mathbb{Z})},
\]
\[
\Re (F_\mathbb{Z}^* F_\mathbb{Z} T_\sigma f, f)_{\ell^2(\mathbb{Z})} \geq C_0 \|f\|^2_{\ell^2_{m,\Lambda}(\mathbb{Z})} - C_1 \|f\|^2_{\ell^2(\mathbb{Z})}, \quad \text{since } F_\mathbb{Z}^* F_\mathbb{Z} = Id,
\]
\[
\Re (T_\sigma f, f)_{\ell^2(\mathbb{Z})} \geq C_0 \|f\|^2_{\ell^2_{m,\Lambda}(\mathbb{Z})} - C_1 \|f\|^2_{\ell^2(\mathbb{Z})},
\]
and this completes the proof. □

We now proceed to prove the sharp Gårding inequality for M-elliptic operators on $\mathbb{T}$ and $\mathbb{Z}$, respectively. For this, first we will state the sharp Gårding’s inequality for M-elliptic operators on $\mathbb{T}$ without proof as it follows similar lines to the proof of [3, Corollary 5.9].

Theorem 5.4.  (Sharp Gårding’s inequality for M-elliptic operators on $\mathbb{T}$)
Let $\sigma \in M^{m}_{\rho,\Lambda} (\mathbb{T} \times \mathbb{Z})$ be such that $\sigma(x,k) \geq 0$, for all $(x,k) \in \mathbb{T} \times \mathbb{Z}$. Then there exists a positive constant $C$ such that for all $f \in H^m_{\Lambda} (\mathbb{T})$, we have
\[
\Re (T_\sigma f, f)_{L^2(\mathbb{T})} \geq -C \|f\|^2_{H^m_{\Lambda} (\mathbb{T})}.
\]

In the following result, we prove that Theorem 5.4 implies the corresponding sharp Gårding’s inequality for M-elliptic operators on $\mathbb{Z}$.

Theorem 5.5.  (Sharp Gårding’s inequality for M-elliptic operators on $\mathbb{Z}$)
Let $\sigma \in M^{m}_{\rho,\Lambda} (\mathbb{Z} \times \mathbb{T})$ be such that $\sigma(k,x) \geq 0$ for all $(k,x) \in \mathbb{Z} \times \mathbb{T}$. Then there exists a positive constant $C$ such that for all $f \in \ell^2_{m,\Lambda} (\mathbb{Z})$, we have
\[
\Re (T_\sigma f, f)_{\ell^2(\mathbb{Z})} \geq -C \|f\|^2_{\ell^2_{m,\Lambda} (\mathbb{Z})}.
\]
Proof  Let $\tau(x,k) = \overline{\sigma(-k,x)}$. Then by Theorem 4.8, we have
\[
T_\sigma = F_\mathbb{Z}^{-1} T_\tau^* F_\mathbb{Z}.
\]
Using the same argument and notation as in the proof of Theorem 5.3, and by Theorem 5.4, we get
\[
\text{Re}(T_\sigma f, f)_{L^2(\mathbb{T})} = \text{Re} \left( F_{\sigma}^{-1} T_\sigma^* g, F_{\sigma}^{-1} g \right)_{L^2(\mathbb{Z})}
= \text{Re} \left( T_\sigma^* g, g \right)_{L^2(\mathbb{T})}
= \text{Re} \left( T_\sigma g, g \right)_{L^2(\mathbb{T})}
\geq -C \|g\|^2 \left( \frac{m}{H^m} \right)_2(\mathbb{T})
= -C \|f\|^2 \left( \frac{m}{H^m} \right)_2(\mathbb{Z}),
\]
and this completes the proof of the theorem. □

6. Applications

In this section, we present an application of Gårding’s inequality for the class $M^{\sigma}_{\rho,\Lambda} (\mathbb{T} \times \mathbb{Z})$. Let $T_{\sigma,0}$ and $T_{\sigma,1}$ are the minimal and maximal pseudo differential operator of $T_\sigma$ on $L^2(\mathbb{T})$ defined as in [21]. First, we recall the following definitions about strongly elliptic symbols and strong solutions.

**Definition 6.1.** Let $\sigma \in M^{\sigma}_{\rho,\Lambda} (\mathbb{T} \times \mathbb{Z}), m \in \mathbb{R}$. Then $\sigma$ is said to be strongly $M$-elliptic if there exist positive constants $C$ and $R$ for which
\[
\text{Re}(\sigma(x, k)) \geq C \Lambda(k)^m, \quad |k| \geq R.
\]

**Definition 6.2.** Let $\sigma \in M^{\sigma}_{\rho,\Lambda} (\mathbb{T} \times \mathbb{Z}), m > 0$, and let $f \in L^p(\mathbb{T}), 1 < p < \infty$. A function $u \in L^p(\mathbb{T})$ is a strong solution of the equation $T_\sigma u = f$ if $u \in \mathcal{D} (T_{\sigma,0})$ and $T_{\sigma,0} u = f$.

Proceeding similarly as in Theorem 18.2 of [37], we have the following result related to the strong solution of the equation $T_\sigma u = f$ on $\mathbb{T}$.

**Lemma 6.3.** Let $\sigma \in M^{\sigma}_{\rho,\Lambda} (\mathbb{T} \times \mathbb{Z}), m > 0$, be an $M$-elliptic symbol such that
\[
\text{Re} \left( T_\sigma \varphi, \varphi \right) \geq C \|\varphi\|^2_{m,2,\Lambda}, \quad \varphi \in H^m_{\Lambda} (\mathbb{T}),
\]
for some positive constant $C$. Then for every function $f$ in $L^2(\mathbb{T})$, the pseudo-differential equation $T_\sigma u = f$ has a unique strong solution $u$ in $L^2(\mathbb{T})$.

In the next result, we provide sufficient conditions for the existence and uniqueness of strong solutions in $L^2(\mathbb{T})$ for the pseudo-differential operator $T_\sigma$ with strongly elliptic symbol.

**Theorem 6.4.** Let $\sigma \in M^{\sigma}_{\rho,\Lambda} (\mathbb{T} \times \mathbb{Z}), m > 0$, be a strongly elliptic symbol. Then for all $f$ in $L^2(\mathbb{T})$ there exists a real number $\lambda_0$ such that for all $\lambda \geq \lambda_0$, the pseudo-differential equation $(T_\sigma + \lambda I) u = f$ on $\mathbb{T}$ has a unique strong solution $u$ in $L^2(\mathbb{T})$, where $I$ is the identity operator on $L^2(\mathbb{T})$.

**Proof.** From Gårding’s inequality (5.1), there exist constants $A > 0$ and $\lambda_0$ such that
\[
\text{Re} \left( T_\sigma \varphi, \varphi \right) \geq A \|\varphi\|^2_{m,2,\Lambda} - \lambda_0 \|\varphi\|^2_{0,2,\Lambda}, \quad \varphi \in H^m_{\Lambda} (\mathbb{T}).
\]
Now for any $\lambda \geq \lambda_0$, we get
\[
\text{Re} \left( (T_\sigma + \lambda I) \varphi, \varphi \right) = \text{Re} \left( T_\sigma \varphi, \varphi \right) + (\lambda - \lambda_0) \|\varphi\|^2 - \lambda_0 \|\varphi\|^2 \geq A \|\varphi\|^2_{m,2,\Lambda} + (\lambda - \lambda_0) \|\varphi\|^2 - \lambda_0 \|\varphi\|^2 \geq A \|\varphi\|^2_{m,2,\Lambda}.
\]
This shows that $T_\sigma$ satisfies the condition (6.1). Thus by Lemma 6.3, the pseudo-differential equation $(T_\sigma + \lambda I)u = f$ has a unique strong solution $u$ in $L^2(T)$ and this completes the proof. □

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