I. INTRODUCTION AND OVERVIEW OF RESULTS

The primordial power of gravity waves, or tensor metric perturbations, is a direct probe of the inflation energy scale [1–4], unknown at present. Observation of their B-mode signature [5, 6] in polarization of the cosmic microwave background (CMB) is a target of several developed experiments. Gravity wave modes are generically frozen while the Hubble scale is smaller than the mode wavelengths. Yet, a mode starts to evolve as soon as it “enters” the Hubble horizon. This evolution determines the pattern of the tensor signatures in the CMB spectra. The modes processed by the evolution are to be constrained by ground and space-based interferometers, resonant mass detectors, or timing of millisecond pulsars.

While numerical calculations are helpful in theoretical studies and may be necessary for data analysis, exact analytical results provide a framework for our understanding of physical phenomena. Quantitative analytic description of gravity wave evolution in the realistic cosmological setting, roughly established today, is the subject of this paper. A consistent, involving no fitting and ad hoc approximations, study in elementary formulas with clear interpretation can be performed to several percent accuracy.

We invoke both the Fourier and real space views of cosmological dynamics. The advantage of Fourier decomposition is decoupling of modes with different wavenumbers in linear order. There are also merits in the complementary real space approach. First, the real space formulation manifests the locality of interactions and causality of solutions. Causality imposes severe restrictions on the admissible evolution, and real space is a preferred starting ground to explore the implications. Second, only in real space we have an appealing formulation of the full, non-linear, cosmological dynamics. Understanding of the linear regime in real space is thus essential for connecting linear and non-linear theories.

Evolution of a tensor (transverse traceless, helicity ±2) mode $h_{ij}$ of the metric (12) with a wavevector $k$ obeys a wave equation, e.g. [7–9],

$$\ddot{h}_{ij} + 2\dot{h}_{ij} + k^2 h_{ij} = 16\pi G a^2 \Sigma_{ij}. \tag{1}$$

The mode dynamics is affected by the Hubble expansion rate $\mathcal{H} \equiv \dot{a}/a$ and the tensor component of matter anisotropic stress $\Sigma_{ij}$ (20). After the horizon entry the mode undergoes continuous oscillations, which are damped by the Hubble expansion. On subhorizon scales ($k \gg \mathcal{H}$), the damping becomes adiabatic, $h \propto 1/a$, and the oscillation period settles at $2\pi/k$. The asymptotic amplitude and phase of the subhorizon oscillations depend on the mode evolution during the horizon entry.

The modes with $k \gtrsim k_{\text{eq}} \equiv \tau_{\text{eq}}^{-1} \sim 10^{-2}\text{Mpc}^{-1}$ enter the horizon when photons and neutrinos make up a substantial fraction of the overall energy density. This applies to all the modes accessible to direct detection and to the modes exciting the CMB multipoles with $l \gtrsim d_{\text{dec}}/\tau_{\text{eq}} \approx 130$. These modes also contribute appreciably to the lower multipoles of CMB polarization up to the rise of the reionization bump at $l \sim 10$.

If anisotropic stress were negligible then during radiation domination $h^{(\text{rad},0)} = j_0(\varphi) = \sin \varphi/\varphi$. Here and for the entire paper

$$\varphi \equiv k\tau \tag{2}$$
and mode normalization is $h \to 1$ as $a \to 0$. In real space, the neutrinoless radiation era evolution of gravity waves is described by a top-hat Green’s function $h^{(\text{rad})} = \frac{\tau}{2\pi} \theta(\tau - |x|)$.

However, recently Weinberg [10] showed that the contribution of free-streaming neutrinos to anisotropic stress and the consequent gravity wave damping [8, 11] is by no means tiny. The subhorizon amplitude of tensors and the induced by them CMB polarization were found suppressed by a factor $A_0 \approx 0.8$ [10] on all scales entering the horizon from neutrino decoupling to matter domination, $\mathcal{H}_{\nu,\text{dec}}^{-1} \sim 0.1 \text{kpc} \to k_{\text{eq}}^{-1}$. All tensor power spectra are suppressed on these scales by $A_0^2$.

Weinberg’s result was derived by calculating the anisotropic stress $\Sigma$ of free streaming neutrinos as a line of sight integral over the metric perturbation $h$. Hence, eq. (1) was converted into a closed integro-differential equation. The source on its right hand side, however, was a complicated integral, which depended on $h_{ij}(\tau)$ nonlocally and should be computed numerically at every evolution step. The amplitude and phase of the subhorizon oscillations could then be found as the ultimate output of numerical evolution with this source from superhorizon to subhorizon scales.

A real space approach provides an instant proof that regardless of the abundance of neutrinos, or presence of additional exotic species, the phase of subhorizon tensor oscillations in the radiation era is unshifted with respect to the zero-stress solution $h^{(\text{rad})}$:

$$h^{(\text{rad})} \to A_0 \sin \varphi / \varphi \quad \text{(3)}$$

as $\varphi \to \infty$. By causality of tensor metric perturbations$^3$, their Green’s function $h(\tau, x)$ which is localized at the origin at $\tau \to 0$ should remain zero beyond the particle horizon:

$$h(\tau, x) = 0 \text{ for } |x| > \tau. \quad \text{(4)}$$

(We imply that tensor perturbations develop entirely from primordial metric inhomogeneities, i.e., there are no primordial “isocurvature” tensor excitations.) Subhorizon oscillations of the Fourier modes are mapped to small-scale discontinuous features of the Green’s function at the horizon $|x| = \tau$. Only when the phase shift is absent [phase of eq. (35) is zero] the corresponding, step-like, discontinuity [eq. (36)] is consistent with condition (4).

The amplitude suppression factor $A_0$ is determined by the Green’s function jump at the particle horizon. In the order linear in $R_\nu \equiv \rho_\nu/\rho$, when neutrino stress is calculated ignoring its feedback to the metric, we find

$$A_0 = 1 - \frac{5}{9} R_\nu. \quad \text{(5)}$$

For three neutrino flavors ($R_\nu \approx 0.405$ for $N_{\nu, \text{eff}} \approx 3.04$ [14–16]) the agreement of eq. (5) with a numerical value $A_0^{\text{CMBFAST}} \approx 0.76$ is 2%. Fourier transformation of the Green’s function, which we derive in the $O(R_\nu)$ order, gives an analytic expression for the evolution of tensor modes in the radiation era. Good match of corresponding equations (34,45) with the full numerical CMBFAST [17] calculations is demonstrated on the right panel of Fig. 1.

The amplitude and phase of gravity wave oscillations are affected by changes in the Hubble expansion during the subsequent cosmological epochs. On the smallest scales ($k/\mathcal{H} \to \infty$) the amplitude and phase shift evolve adiabatically (infinitesimally over an oscillation period). Then the amplitude decays as $1/a$ while the phase shift remains frozen. Hence,

$$h(\tau, k \to \infty) = A(\tau) \sin \varphi / \varphi = A(\tau) h^{(\text{rad})}(\varphi), \quad \text{(6)}$$

where $A(\tau)$ is given by eq. (52). In real space, this small-scale behavior corresponds to a step discontinuity of Green’s function at the particle horizon,

$$h^{(\text{disc})}(\tau, x) = \frac{A(\tau)}{2\pi} \theta(\tau - |x|) = A(\tau) h^{(\text{rad})}(\tau, x). \quad \text{(7)}$$

The modes entering the horizon in the matter era ($k \ll k_{\text{eq}}$) encounter negligible anisotropic stress. For them during matter domination

$$h^{(\text{mat})} = 3 j_1(\varphi)/\varphi = 3 \left(\sin \varphi / \varphi^3 - \cos \varphi / \varphi^2\right). \quad \text{(8)}$$

The respective Green’s function is

$$h^{(\text{mat})} = \frac{3}{4\pi} \left(1 - \frac{x^2}{\tau^2}\right) \theta(\tau - |x|). \quad \text{(9)}$$

Real space results (7) and (9) suggest a compact analytic expression that correctly describes the tensor evolution in both the small and large scale limits and is qualitatively acceptable in the intermediate range:

$$h \sim A(\tau) h^{(\text{rad})} + [1 - A(\tau)] h^{(\text{mat})}. \quad \text{(10)}$$

At any $\tau$, the first term guarantees the proper $k \to \infty$ behavior (6), while on superhorizon scales $h(k \to 0) = 1$.
Here, a “transfer function” $T$ is considered by Turner, White and Lidsey [18] to parameterize the tensor evolution by Turner, White and Lidsey [18] considers an ansatz $h^{(\text{mat,0})}$ for a mode that enters the horizon in the matter era, when $A(\tau)$ decays. This simple parameterization matches well the numerical tensor evolution to the CMB last scattering and beyond, Figs. 2 and 3. Yet, despite the close match, formula (10) is only qualitative for a finite $k$. A systematic treatment, presented further, reveals important effects eluding this minimal description.

As an alternative to eq. (10), a very popular parameterization of tensor evolution by Turner, White and Lidsey [18] considers an ansatz $h^{\text{TWL}} = T(k/k_{eq}) h^{(\text{mat,0})}$. Here, a “transfer function” is $T(y) = [1 + 1.34y + 2.5y^2]^{1/2}$, with its last two coefficients fixed by a numerical fit at $z = 0$. This parameterization is rather misleading. Most importantly, the ansatz $T(k/k_{eq}) h^{(\text{mat,0})}$ is motivated by an erroneous statement that “for $\tau \gg \tau_{eq}$, the temporal behavior of all modes is given by $3j_1(k\tau)/k\tau^2$. On the contrary, both subhorizon solutions of the wave equation in the matter era, $h^{(\text{mat,0})} \rightarrow -3\cos\varphi/\varphi^2$ and the complimentary $n \rightarrow 3\sin\varphi/\varphi^2$, eq. (80), decay as $1/\alpha$. A mode with $k \gtrsim k_{eq}$ for $\tau \gg \tau_{eq}$ is described by their linear combination, which is determined by the evolution before matter domination. In fact, only the complimentary solution contributes when $k \gg k_{eq}$, eq. (6). This ironic feature and the importance of the phase shift for the CMB signatures of gravity waves was emphasized by Ng and Speliotopoulos [19].

Other shortcomings of the TWL transfer function are disregard of the neutrino-induced suppression of the modes with $k \gtrsim k_{eq}$ and of the dynamical effects caused by the residual radiation density around the CMB last scattering ($\rho_{\text{rad}}/\rho|_{\text{dec}} \approx 0.25$ for the WMAP parameters [20]). The second problem was alleviated by Wang’s [21] time-dependent transfer functions which contain two additional fitting parameters. Nevertheless, his parameterizations continued to evolve to the incorrect $\cos\varphi/\varphi^2$ form even for the shortest wavelengths.

We obtain the functional form of the tensor evolution at a finite $k$ by analytically solving the dynamical equations under controlled approximations. In addition to leading naturally to adequate simple analytical description, derivation, as opposed to fitting, also offers better means of identifying the physics responsible for evolution features. For earlier works that applied different methods but were guided by the same intentions see Refs. [19, 22, 23].

The modes that enter the horizon before equality ($k\tau_{eq} < 1$) are likewise affected by the subdominant radiation density. The effects in this case are less dramatic. In the linear in $k\tau_{eq}$ order the impact of the residual radiation is fully accounted for by a shift $\tau \rightarrow \tau + \tau_e$ in the radiation-free transfer function: $h(\tau) \approx h^{(\text{mat,0})}(\tau + \tau_e)$. This result is obtained by observing that for $\tau \lesssim \tau_{eq}$ the large-scale mode amplitudes are essentially frozen and that for $\tau \gg \tau_{eq}$ the background expands as $a(\tau) \approx a^{(\text{mat,0})}(\tau + \tau_e)$. The advance of the tensor evolution on the large scales by the constant conformal time increment $\tau_e$ is consistent with numerical computations (Fig. 5).

We analyze the subhorizon evolution with the WKB expansion in the orders of $k/H$. To start with an order above the adiabatic approximation, we factor out the $1/a$ decay by considering a variable $\nu \propto aH$, eq. (49) [24]. Under the horizon, the amplitude of $\nu$ oscillations freezes. However, their phase shift grows during radiation domination as $\frac{1}{2\kappa_{\nu}} \ln \varphi + \text{const}$, where $\tau_e = \tau_{eq}/(\sqrt{2} - 1) \sim 260 \text{ Mpc}$ is given by eq. (47). The oscillation amplitude and the additive constant in the phase are set by the mode evolution during the horizon entry, described next.

Matching the next-to-adiabatic WKB order to the superhorizon primordial fluctuations is rather subtle. For initial oscillations the WKB approximation breaks down and $v^{\text{WKB}}$ diverges. Ng and Speliotopoulos [19] approached the problem by changing the evolution variable to $\ln \tau$, now varying from $-\infty$. They matched the growing mode in the “forbidden” superhorizon region to the “allowed” subhorizon one by applying the standard WKB matching procedure at a turning point. Their result reproduced accurately the phase of subhorizon oscillations but significantly underestimated the amplitude. Pritchard and Kamionkowski [23] stretched the WKB approach by retaining the decaying mode in the forbidden region. The obtained in terms of special functions amplitude remained in poor agreement with numerical calculations.

We consider the WKB solution for $v(\tau)$ only on subhorizon scales. To normalize its amplitude and fix the phase, during the horizon entry we, instead, treat the matter correction to the Hubble expansion perturbatively in $\rho_{\text{mat}}/\rho$. Both the WKB and perturbative approximations are valid and both solutions can be matched when a mode has entered the horizon but the matter density is still negligible. The result, in terms of elementary functions,

$$h = A(\tau) \sin[\varphi - \Delta \varphi(\tau)]/\varphi$$

with the amplitude suppression (72) and phase shift (70) is in excellent agreement with cmbfast computations (Fig. 4). As a comparison, the matching procedure of [23] reproduces the amplitude of subhorizon oscillations for $k = 0.1677 \text{ Mpc}^{-1}$ to 13% [23]. For the same $k$ and same cosmological parameters, our formula achieves 1%.

The evolution of the modes that enter the horizon after equality ($k\tau_{eq} > 1$) is likewise affected by the subdominant radiation density. The effects in this case are less dramatic. In the linear in $k\tau_{eq}$ order the impact of the residual radiation is fully accounted for by a shift $\tau \rightarrow \tau + \tau_e$ in the radiation-free transfer function: $h(\tau) \approx h^{(\text{mat,0})}(\tau + \tau_e)$. This result is obtained by observing that for $\tau \lesssim \tau_{eq}$ the large-scale mode amplitudes are essentially frozen and that for $\tau \gg \tau_{eq}$ the background expands as $a(\tau) \approx a^{(\text{mat,0})}(\tau + \tau_e)$. The advance of the tensor evolution on the large scales by the constant conformal time increment $\tau_e$ is consistent with numerical computations (Fig. 5).
Tensor metric perturbations were connected to the CMB temperature and polarization spectra in the original works [25–28], a full systematic approach was developed in [29], and a review of the physical picture and typical analytic approximations given recently in [23]. We do not elaborate this connection in the present article. Yet, we remember that generation of CMB polarization and its curl-type $B$ component, clean from linear scalar contribution [5, 6], requires photon scattering on free electrons, to polarize photons, but moderate optical depth, for the polarized photons to reach the Earth. These conditions are met inside the surface of CMB last scattering ($z \sim 1090 \pm 100$) and after reionization ($z \lesssim 20 - 6$). The tensors also contribute to CMB temperature anisotropy, primarily, through the integrated Sachs-Wolfe mechanism [25], induced by $h_{ij}$ after the last scattering ($z \lesssim 1090$).

Among other results of this paper, we note simple equation (15) that describes fully general-relativistic transport of radiation of decoupled ultrarelativistic particles. The radiation intensity is quantified by a variational transport equation (A11) that describes fully general-relativistic radiation transport is affected only by spatial gradients of the metric and not by metric temporal change. As a result, the intensity and its statistical distribution are time-independent on superhorizon scales in any gauge. This fully nonlinear conservation does not require adiabaticity of cosmological perturbations. If particles have ever been in thermal equilibrium and decoupled on superhorizon scales, a superhorizon value of their intensity $I$ is simply related to metric perturbation on a hypersurface of uniform particle density, eq. (A20). In the locally Minkowski metric $I$ reduces to the regular particle intensity $dE/ (dVd\Omega_b)$.

The rest of the paper is organized as follows. Sec. II reviews the dynamical equations. Sec. III solves the coupled evolution of gravity waves and neutrinos during the radiation era. Sec. IV considers the evolution of tensor perturbations through the radiation-matter transition to the matter era. In Appendix A we describe propagation of decoupled particles in an arbitrary metric and in linear theory. In Appendix B we confirm by explicit integration of the real space dynamics that no amount of free streaming neutrinos can change the gravity wave phase in the radiation era.

In this paper, space-time coordinates $x^\mu = (\tau, x)$ correspond to the perturbed Friedmann-Robertson-Walker metric

$$g_{\mu\nu} = a^2(\tau) \left[ \eta_{\mu\nu} + h_{\mu\nu}(x^\nu) \right], \quad (12)$$

with $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$. Zero of conformal time $\tau$ is set by the condition $\tau(a \rightarrow 0) \equiv 0$ in the radiation era. The scale factor $a \equiv 1$ at present. Background curvature, if any, is ignored for the considered redshifts. Figures compare the analytic formulas with numerical computations assuming a flat model with $\Omega_m = 0.3, \Omega_b h^2 = 0.022$, and $h = 0.7$.

## II. DYNAMICS

The linear evolution of gravity waves is described by wave equation (1). The gravity waves are driven by neutrino anisotropic stress, which develops after neutrinos decouple ($z < z_{\nu \text{dec}} \sim 10^{10}$). The driving is dynamically relevant while neutrinos contribute noticeably to the total energy density ($z \gtrsim z_\nu \approx 3200$). At those redshifts, all the three standard neutrino generations may be assumed ultrarelativistic, given the current cosmological bounds on the neutrino masses [20, 30–36]. Then neutrino dynamics is approachable in the full general relativity for an arbitrary metric, as discussed in Appendix A and summarized by Sec. II A next.

### A. General relativistic free streaming

We describe the transport of ultrarelativistic neutrinos by intensity $I$, defined in Appendix A as their energy-integrated phase space distribution. $I(x^\mu, n_i)$ is a function of spacetime coordinates and direction of neutrino propagation. The direction is specified by the covariant spatial components $n_i$ of a null vector $n^\mu \propto dx^\mu/d\tau$ which is normalized as $\sum_{i=1}^3 n_i^2 = 1$.

The impact of neutrinos on the metric is determined by their energy-momentum tensor $T^\mu_\nu$. For a given intensity $I$,

$$T^\mu_\nu = \int \frac{d^2n_i}{\sqrt{-g}} \frac{n^\mu n^\nu}{n^0} I(x^\mu, n_i), \quad (13)$$

eq \text{eq. (A9)}, \text{where } n_0(n_i) \text{ can be found from the null condition } n_\mu n^\mu = 0. \text{Provided the considered perturbations are superhorizon during neutrino decoupling, the initial conditions are set on a hypersurface of uniform neutrino density by (A20)}

$$I^{(u)}(x, n_i) = \frac{\text{const}}{\left[ n_i n_{j(3)} g_{ij}^{(u)}(x) \right]^2}. \quad (14)$$

Afterward, the neutrino intensity evolves according to transport equation (A11)

$$n^\mu \frac{\partial I}{\partial x^\mu} = \frac{1}{2} n^\mu n^\nu g_{\mu\nu, i} \left( 4n_i I - \frac{\partial I}{\partial n_i} \right). \quad (15)$$

The considered intensity $I$ is gauge dependent.\(^4\) However, by separating the particle and metric dynamics, the

\(^4\) Similarly to the majority of the traditional cosmological variables, e.g. [7], $I(x^\mu, n_i)$ defines a “gauge-independent” variable in any fixed gauge.
presented description achieves, first, compact transport equation in the full theory. Second, time independence of the intensity for superhorizon inhomogeneities, with negligible $g_{\mu\nu,i}$. (The metric, which is partly determined by a gauge condition, may nevertheless evolve even on superhorizon scales.) In a locally Minkowski metric, $I$ reduces to the conventional intensity of neutrino radiation.

In linearized theory, the intensity perturbation $I \equiv \delta I/I$ obeys transport equation (A13)

$$\dot{I} + n_i \mathcal{I}_i = 2n_i \hat{n}^\mu \hat{n}^\nu h_{\mu\nu,i},$$

with $\hat{n}^\mu \equiv (1, n_i)$ and initial conditions (A21),

$$\mathcal{I}^{(u)} = 2n_i n_j h_{ij}^{(u)},$$

The perturbed components of the neutrino energy-momentum tensor are determined by $I$ from eqs. (A14)-(A17).

**B. Tensor modes**

Tensor linear perturbations over the flat Friedmann-Robertson-Walker metric satisfy the conditions $h_{00} = h_{0i} = 0$ and

$$\partial_t h_{ij} = 0, \quad \sum_i h_{ii} = 0.$$ (18)

The tensor perturbations evolve according to a wave equation, e.g. [7–9],

$$\dot{h}_{ij} + 2H h_{ij} - \nabla^2 h_{ij} = 16\pi G a^2 \Sigma_{ij},$$ (19)

where they are sourced by the tensor (transverse) component of anisotropic stress

$$\Sigma_{ij} \equiv T_j^i - \frac{1}{3} \delta_j^i T^k_k.$$ (20)

Neutrino contribution to anisotropic stress follows from eq. (A17) by substituting the explicit linear solution for neutrino intensity (A23). When $\tau_{in}$ in this solution is set to zero and the above tensor conditions are applied, eq. (A23) gives [10]

$$\frac{1}{2} \mathcal{T}(\tau, x) = n_i n_j h_{ij}(\tau, x) - \int_0^\tau d\tau' n_i n_j \dot{h}_{ij}(\tau', x'),$$ (21)

with $x' = x + n(\tau' - \tau)$.

The impact of photon anisotropic stress on tensor metric perturbations is not entirely negligible. While local photon anisotropy is washed out by Thomson scattering before the recombination, the anisotropy develops after photons decouple at $z_{\gamma, dec} \sim 1090$. At this redshift, radiation constitutes as much as a quarter of the total energy density. The generated photon stress affects the modes that enter the horizon shortly after the decoupling. Numerical calculations show up to 7% consequent suppression of the metric modes with $k \sim 0.005 \text{Mpc}^{-1}$. However, since the photon stress was insignificant during decoupling, CMB polarization is almost unsuppressed. CMBFAST predicts less than 0.5% suppression of the polarization correlation $C_l^{BB}$. The photon stress also affects little CMB temperature — the corresponding suppression of the tensor component of $C_l^{TT}$ is less than 1%. We therefore neglect photon anisotropic stress in the following analysis. Even more so, we can disregard the stress of nonrelativistic species (baryons and CDM) because of their tiny velocity dispersion.

Now we consider a single plane wave, not necessarily harmonic, and choose the coordinates $y$ and $z$ in a plane of constant $h_{ij}$: $h_{ij} = h_{ij}(\tau, x)$. Then for tensor perturbations, by the first of conditions (18), $h_{xi} = h_{ix} = 0$ and $\Sigma_{xi} = \Sigma_{ix} = 0$. For the remaining components of $\Sigma_{ij}$ with $i, j \in \{y, z\}$ we can take the average in eq. (A17) over the orientation of $(n_y, n_z)$ under a fixed $n_x \equiv \mu$.

The result is

$$\Sigma_j(\tau, x) = -\frac{\mu}{4} \int_0^\tau d\tau' \int_1^{-1} d\mu (1 - \mu^2)^2 h_{ij}(\tau', x'),$$ (22)

where $x' \equiv x + \mu(\tau' - \tau)$. Fourier transformation of this expression reproduces eq. (16) of Ref. [10].

From now on, we imply a decomposition of a gravity wave $h_{ij}$ over some polarization basis, e.g., into + and $\times$ components,

$$h_{ij} = \sum_{\lambda = 1, 2} e^{(\lambda)}_{ij} h_\lambda.$$ (23)

Since the evolution imposed by eqs. (19) and (22) is identical for all the polarization components, we consider an arbitrary component $h_\lambda$ and drop the subscript $\lambda$ later.

**C. Plane tensor Green’s functions**

The evolution of any linear perturbation can be described by superposing plane Green’s functions [37, 38] $h(\tau, x)$, which satisfy the initial condition

$$h(\tau \rightarrow 0, x) = \delta(x).$$ (24)

For example, a harmonic plane wave mode $h(\tau, k) e^{ikx}$ with the initial condition $h(\tau \rightarrow 0, k) = 1$ is described by the Fourier integral

$$h(\tau, k) = \int_{-\infty}^\infty dx e^{-ikx} h(\tau, x).$$ (25)

Since $h \wedge \dot{h}$ decays, or “squeezes” [39], on superhorizon scales, the initial $\dot{h}$ need not be specified precisely. It is sufficient to require a finite $\tau \rightarrow 0$ limit of $\int_{-\infty}^\infty dx \dot{h}(x)$, to ensure that we work with a non-decaying mode.

The initial condition (24) has several immediate consequences. First, by the causality of gravity wave propagation

$$h(\tau, x) = 0 \text{ for } |x| > \tau.$$ (26)
Second, by parity conservation during the considered evolution, the initially even perturbation (24) remains even: \( h(\tau, -x) = h(\tau, x) \). Third, the \( k \to 0 \) limit of eq. (25), time-independence of \( h(\tau, k \to 0) \), and initial condition (24) add up to a sum rule

\[
\int_{-\infty}^{\infty} dx \ h(\tau, x) = h(\tau, k \to 0) = 1 \quad (27)
\]

for all \( \tau \).

### III. RADIATION ERA

The magnitude and phase of the tensor modes with \( k \gg k_{\text{eq}} \) is set by their dynamics in the radiation epoch. At that time neutrino anisotropic stress plays an active role. The coupled dynamics of gravity waves and neutrinos during radiation domination can be solved analytically in real space. Analytic expressions for the radiation era evolution of Fourier tensor modes then follow by Fourier transformation.

#### A. Self-similarity

The coupled evolution of gravity waves and neutrinos with the specified Green’s function initial conditions involves no external dynamical scales during radiation domination, when \( H = 1/\tau \). Then the plane tensor Green’s function, whose dimension is \( \tau^{-1} \), takes a self-similar form

\[
h(\tau, x) = \frac{\tilde{h}(\chi)}{\tau}, \quad \chi \equiv \frac{x}{\tau} \quad (28)
\]

The initial condition (24) requires the normalization

\[
\int_{-1}^{1} d\chi \ \tilde{h}(\chi) = 1, \quad (29)
\]

as evident from the sum rule (27). Note that integration over \( \chi \) is restricted to \( \chi \in [-1, 1] \) because by causality (26) \( \tilde{h}(\chi) \) vanishes identically beyond this interval.

We determine \( \tilde{h}(\chi) \) by solving eqs. (19) and (22). We use that \( \tilde{h} = -(\chi \tilde{h})/\tau^2 \), \( \tilde{h} = (\chi^2 \tilde{h})''/\tau^3 \), \( \nabla^2 \tilde{h} = \tilde{h}''/\tau^3 \), where primes denote derivatives, and in the radiation era \( \pi Ga^2 = 3/\rho^2 \) by the Friedmann equation. With these substitutions, eqs. (19) and (22) give:

\[
\left[ (\chi^2 - 1) \tilde{h} \right]' = \frac{3R_{\nu}}{2} \int_0^{\tau} \frac{d\tau_1}{\tau_1^2} \int_{-1}^{1} d\mu (1 - \mu^2)^2 \frac{d[\chi_1 \tilde{h}(\chi_1)]}{d\chi_1}, \quad (30)
\]

where \( \chi_1 \equiv (\chi - \mu) \tau_1 + \mu \) and \( R_{\nu} \equiv \rho_{\nu}/\rho \). Carefully integrating over \( d\tau_1 \) by parts and remembering the causality (26) we obtain:

\[
\left[ (\chi^2 - 1) \tilde{h} \right]' = \frac{3R_{\nu}}{2} \theta(1 - |\chi|) \int_{-1}^{1} d\mu [K(\mu, \chi) - K(\chi, \mu)], \quad (31)
\]

where \( \theta \) is the Heaviside step function and

\[
K(\mu, \chi) = \frac{(1 - \mu^2)^2 \tilde{h}(\chi)}{\mu - \chi} \quad (32)
\]

Equation (31)-(32) is singular at \( \chi = \pm 1 \) and \( \chi = \mu \). The real-space singularities are treated consistently with the Fourier mode approach by the standard formalism of generalized functions (see discussion in Sec. IV.C of Ref. [12]). In the following analysis of the tensor sector it is sufficient to remember that, first, the \( d\mu \) integral in eq. (31) should be taken in the Cauchy principal value. Second, an equation \((\chi - a)A = B\) for any generalized functions \(A(\chi)\) and \(B(\chi)\) should be resolved as \(A = B/(\chi - a) + \text{const}\ \delta(\chi - a)\), where the last term is the Dirac delta function. The delta function prefactor can be determined from the properties imposed on \(A\) by the initial conditions; for example, from the sum rule (27).

#### B. Phase shift

Without neutrino anisotropic stress \((R_{\nu} = 0)\), the even normalized solution of eq. (31) is

\[
\tilde{h}^{(0)} = \frac{1}{2} \theta(1 - |\chi|). \quad (33)
\]

Fourier transformation of the corresponding Green’s function (28) gives the standard neutrinoless radiation era result

\[
h^{(0)}(k\tau) = \frac{\sin(k\tau)}{k\tau}. \quad (34)
\]

Although neutrinos in the radiation era are, in fact, among the dominant species, the gravitational impact of their anisotropic stress becomes negligible on subhorizon scales. The corresponding asymptotic \((k\tau \gg 1)\) homogeneous solution of mode evolution eq. (1) is

\[
h(k\tau) = \frac{A_0 \sin(k\tau + \varphi_0)}{k\tau} + O \left( (k\tau)^{-2} \right). \quad (35)
\]

We prove that, regardless of an \( R_{\nu} \) value, \( \varphi_0 = 0 \).

The amplitude and phase of the transfer function oscillations in \( k\tau \) are fixed fully by the Green’s function discontinuity at \( |\chi| = 1 \). The \( k\tau \to \infty \) behavior of the Fourier modes (35) maps to the following discontinuous terms in real space [40, 41]:

\[
\tilde{h}^{(\text{disc})}(\chi) = \frac{A_0}{2} \cos \varphi_0 \theta(1 - |\chi|) - \frac{\sin \varphi_0}{\pi} \ln |1 - \chi^2|. \quad (36)
\]

Regardless of the continuous part of \( \tilde{h}(\chi) \), \( \tilde{h} \equiv 0 \) for \(|\chi| > 1\), as required by causality, only if \( \sin \varphi_0 = 0 \), i.e. if the logarithmic term in eq. (36) vanishes identically. Appendix B confirms that the explicit solution of eq. (31) has this property for any \( R_{\nu} \).
FIG. 1: Evolution of tensor metric perturbations in the radiation era as described by a real-space Green’s function, left, and Fourier modes, right. For both presentations, the displayed solutions are the neutrinoless, unsourced by any anisotropic stress (dashed) and the linear in $\rho_\nu/\rho$ for three neutrino flavors (solid). The Fourier modes on the right panel are additionally compared with the full Boltzmann cmbfast calculation (bold dots).

Since $\sin \varphi_0 = 0$, either $\varphi_0 = 0$ or $\varphi_0 = \pi$. If both $A_0$ and $\varphi_0$ in eq. (35) change continuously in $R_\nu$ and $\varphi_0 \equiv 0$ at $R_\nu = 0$ then $\varphi_0 = 0$ for any $R_\nu$.

C. First-order solution

We look for an explicit Green’s function $\bar{h}(\chi)$ as a series in the powers of $R_\nu$. In the zeroth order $\bar{h}$ is described by a top-hat function (33). The $O(R_\nu)$ order is addressed next; it will match the full numerical Boltzmann evolution to a few percent. Since $\bar{h} \equiv 0$ for $|\chi| > 1$, only the interval $\chi \in [-1, 1]$ is considered.

Substitution of the zeroth-order $\bar{h}$ into the right hand side of eq. (31) gives for the $O(R_\nu)$ correction

$$\left[ (\chi^2 - 1) \bar{h}(\chi) \right]' =$$

$$= \frac{3R_\nu}{2} \left[ 2\chi^4 - \frac{11}{3} \chi^2 + 1 + (\chi^2 - 1)^2 \chi \ln \frac{1 - \chi}{1 + \chi} \right].$$

After one integration and division of both sides of the obtained equation by $\chi^2 - 1$,

$$\bar{h}^{(1)'} = R_\nu \left[ \chi \left( \frac{\chi^2}{2} - 1 \right) + \frac{1}{4} (\chi^2 - 1)^2 \ln \frac{1 - \chi}{1 + \chi} \right]$$

(38)

where $\chi \in (-1, 1)$.

As discussed at the end of Sec. III A, the division by $\chi^2 - 1$ creates delta-function spikes in $\bar{h}^{(1)'}$ at $\chi = \pm 1$.

The delta-function prefactors are determined by observing that the spikes control the constant term in $\bar{h}(1) = \int \bar{h}^{(1)'} d\chi$ for $\chi \in [-1, 1]$. This term is to be fixed by $\bar{h}$ normalization (29), by which

$$\int_{-1}^{1} \bar{h}^{(1)}(\chi) = 0.$$  (39)

Integrating eq. (38) and fixing the constant term as described we find that

$$\bar{h}^{(1)} = \frac{R_\nu}{60} \left\{ 6\chi^4 - 23\chi^2 + \frac{1}{3} + 16 \ln 2 +$$

$$+ (3\chi^4 - 10\chi^2 + 15) \chi \ln \frac{1 - \chi}{1 + \chi} - 8 \ln(1 - \chi^2) \right\}$$

(40)

for $\chi \in [-1, 1]$ and $\bar{h}^{(1)} = 0$ otherwise.

As anticipated, $\bar{h}^{(1)}$ discontinuity at $|\chi| = 1$ is of the form (36), where $\varphi_0 = 0$ and

$$A_0^{(1)} = \lim_{|\chi| \to 1 - 0} 2\bar{h}^{(1)} = -\frac{5}{9} R_\nu.$$  (41)

The obtained $A_0$ determines the subhorizon radiation era solution for tensor modes, eq. (35), as

$$h(k\tau) = \left( 1 - \frac{5}{9} R_\nu + O(R_\nu^2) \right) \frac{\sin k\tau}{k\tau} + O((k\tau)^{-2}).$$  (42)

We also note that the leading behavior of the $O((k\tau)^{-2})$ correction follows from the general result (B7),

$$h(k\tau) = A_0 \left\{ \frac{\sin k\tau}{k\tau} - R_\nu \frac{\cos k\tau}{k^2\tau^2} [1 + o(1)] \right\},$$  (43)

valid in the radiation era at all orders of the $R_\nu$ expansion.

5 If we demand that the amplitude and phase change continuously in $R_\nu$, we should, in principle, allow the amplitude to become negative. The following calculations show that in a realistic range of $R_\nu$ the amplitude varies moderately and remains positive.
It is possible to find the entire radiation era evolution of a tensor Fourier mode \( h(k\tau) \) in \( O(R_\nu) \) order. Introducing \( \varphi \equiv k\tau \), we calculate

\[
h^{(1)}(\varphi) = \int_{-1}^{1} d\chi \tilde{h}^{(1)}(\chi) e^{-i\varphi\chi},
\]

with \( \tilde{h}^{(1)}(\chi) \) given by eq. (40). The result is

\[
h^{(1)}(\varphi) = \frac{R_\nu}{\varphi} \left( A^{(1)} \sin \varphi - B^{(1)} \cos \varphi \right),
\]

where

\[
A^{(1)} = -\frac{5}{9} - \frac{1}{\varphi^2} + \frac{34}{\varphi^2} \text{Ci}(2\varphi) - 4 \left( \frac{1}{\varphi^3} - \frac{3}{\varphi^5} \right) \text{Si}(2\varphi),
\]

\[
B^{(1)} = \frac{1}{\varphi} + \frac{12}{\varphi^2} \text{Si}(2\varphi) + 4 \left( \frac{1}{\varphi^3} - \frac{3}{\varphi^5} \right) \text{Ci}(2\varphi).
\]

Here, \( \text{Ci} x \equiv \int_0^x \cos \frac{\pi t}{2} \, dt = -\ln x - \gamma + ci \, x \) and \( \text{Si} x \equiv \int_0^x \sin \frac{\pi t}{2} \, dt = \frac{x}{2} + si \, x \). Explicit verification with Mathematica confirms that in the \( O(R_\nu) \) order the obtained \( h^{(0)} + h^{(1)} \) satisfies Weinberg’s integro-differential equation for a tensor Fourier mode, eq. (21) in Ref. [10].

Figure 1 shows the neutrinoless (dashed) and the linear in \( R_\nu \) (solid) wave solutions in real and Fourier presentations. Here, \( R_\nu \) is 0.405, corresponding to the three standard neutrinos. The Fourier modes are additionally compared with a full Boltzmann calculation with cmbfast [17] (dots). As seen, the \( O(R_\nu) \) analytical solution provides a reasonable accuracy for all \( k \tau \).

For the standard \( R_\nu \) value, the \( O(R_\nu) \) tensor amplitude suppression factor is \( A_0 = 1 - \frac{5}{9} R_\nu \simeq 0.77 \). It is in excellent agreement with the cmbfast calculation, giving \( A_0^{(\text{CMBFAST})} \simeq 0.76 \) during radiation domination (redshift \( z = 10^7 \), the same result at \( z = 10^8 \)), and with numerical solutions of Weinberg’s integro-differential equation in Fourier space, \( A_0^{(\text{Weinb})} \simeq 0.80 \) [10] and \( A_0^{(\text{PK})} \simeq 0.81 \) [23].

**IV. TO THE MATTER ERA**

Most of the detectable gravity waves effects are generated after the radiation era. The modes probed by interferometers and pulsar timing enter the horizon in the early radiation era, and their subsequent evolution is described excellently by the adiabatic approximation \( h \propto \sin(k\tau)/a \). The modes probed by CMB anisotropy and polarization enter when radiation and matter densities are comparable, and the post-radiation era evolution of these modes is more subtle. Note that their CMB imprints are left primarily at redshifts at which dark energy plays little role in the background expansion.

**A. Minimal analytic solution**

For the studied redshifts, when dark energy density can be assumed small, we describe the Hubble expansion by a radiation-matter model with \( \rho = [\rho_\gamma, 0/(1 - R_\nu)]/a^4 + \rho_{\nu, 0}/a^2 \). The corresponding scale factor \( a \), governed by the Friedmann equation, evolves as

\[
a = \left( 2\bar{\tau} + \bar{\tau}^2 \right) a_{eq}.
\]

Here, \( a_{eq} = [\rho_\gamma, 0/(1 - R_\nu)]/\rho_{\nu, 0} \) is the value of \( a \) at radiation-matter equality, and \( \bar{\tau} \) is conformal time in units of characteristic conformal time of equality

\[
\bar{\tau} \equiv \tau/\tau_e, \quad \tau_e \equiv 2 \sqrt{\frac{a_{eq}}{H_0^2 \Omega_m}}.
\]

Note that by eq. (46) \( \tau(a_{eq}) \equiv \tau_e = (\sqrt{2} - 1) \tau_e \).

First, we address the modes that enter the horizon long before equality \( (k \gg \tau_e^{-1}) \) although after neutrino decoupling. After a mode has become subhorizon \( (k \gg \bar{H}) \), we ignore the anisotropic stress. Then the mode evolves according to a homogeneous wave equation

\[
\ddot{h} + 2\bar{H} \dot{h} + k^2 h = 0.
\]

The Hubble redshift \( 2\bar{H} \) of the graviton energy is accounted for by substitution \( h \equiv \frac{\text{const}}{a} \nu \), where now [24]

\[
\ddot{\nu} + \left( k^2 - \frac{\bar{\alpha}}{a} \right) \nu = 0.
\]

In the small-scale limit \( k/\bar{H} \to \infty \), the term \( \bar{\alpha}/a \sim \bar{H}^2 \) can be dropped. In this (adiabatic) approximation \( \nu \) oscillates harmonically in \( \varphi = k\tau \) with constant amplitude. Correspondingly, the amplitude of \( h \) oscillations decays as \( 1/a \).

We normalize the amplitude and fix the phase shift of the considered short-scale modes by subhorizon radiation era solution (42). As a result,

\[
h(\tau, k \to \infty) = A(\tau) h^{(\text{rad,0})}(\varphi),
\]

where

\[
h^{(\text{rad,0})} = j_0(\varphi) = \frac{\sin \varphi}{\varphi},
\]

is the neutrinoless radiation era solution, and \( A(\tau) \propto \varphi/a \) equals

\[
A(\tau) = \frac{1 - \frac{5}{9} R_\nu}{1 + \frac{5}{9} \bar{\tau}}.
\]

The last formula neglects the \( O(R_\nu^2) \) correction in eq. (42) and uses \( a(\tau) \) form (46). We note that the Fourier short-scale evolution (50) maps onto the following Green’s function discontinuity, c.f. eqs. (35)-(36):

\[
h^{(\text{disc})}(\tau, x) = \frac{A(\tau)}{2\bar{\tau}} \theta(\tau - |x|) = A(\tau) h^{(\text{rad,0})}(\tau, x).
\]

Next, we consider the opposite, large-scale limit \( (k \ll \tau_e^{-1}) \), in which mode evolution starts only in the matter era. Then anisotropic stress of relativistic species is
negligible and, similarly to the radiation era, the tensor dynamics is scale-free. Solving eq. (19), where now $\Sigma = 0$ and $H = 2/\tau$, with a normalized Green’s function of the form (28), we obtain

$$h^{(\text{mat,0})} = \frac{3}{4\tau} \left(1 - \frac{x^2}{\tau^2}\right) \theta(\tau - |x|).$$  

(54)

Fourier transforming, we arrive at a well-known tensor transfer function in a matter dominated universe

$$h^{(\text{mat,0})} = 3j_1(\varphi)/\varphi = 3\left(\frac{\sin \varphi}{\varphi^3} - \frac{\cos \varphi}{\varphi^2}\right).$$  

(55)

Finally, we combine the Green’s function discontinuity (53), describing the short modes, with the matter era evolution (54) on the larger scales by a “minimal” analytic formula. Namely, we look for an expression of the form $(a + bx^2) \theta(\tau - |x|)$ that is normalized ($\int h \, dx = 1$) and jumps to $A(\tau)/(2\tau)$ at the horizon $|x| = \tau$. Such an expression is unique and is given by

$$h \sim A(\tau) h^{(\text{rad,0})} + [1 - A(\tau)] h^{(\text{mat,0})}.$$  

(56)

If we view formula (56) as a transfer function $h(\tau, k)$ in Fourier space, with $h^{(\text{rad,0})}$ (51) and $h^{(\text{mat,0})}$ (55), we observe that $h$ approaches unity for either $\tau \to 0$ or $k \to 0$. This formula also reproduces the correct, in $O(R_c)$, asymptotic behavior for $k \to \infty$ at any fixed $\tau$.

Figure 2 compares the approximation (56) (solid curve) with numerical cmbfast formula (56) (solid), numerical cmbfast calculation (bold dots), the neutrino- and adiabatically-damped radiation era solution (dashed), and the matter era solution (dash-dotted).

![FIG. 2: Tensor modes at the redshift of CMB last scattering, $z = 1090$. The plots are generated with: “minimal” analytical formula (56) (solid), numerical cmbfast calculation (bold dots), the neutrino- and adiabatically-damped radiation era solution (dashed), and the matter era solution (dash-dotted).](image)

B. Entering before equality ($k\tau_{eq} > 1$)

First, to determine the role of the radiation-matter transition in mode evolution, we ignore neutrino anisotropic stress. The derived formula will ultimately be corrected for it. Without anisotropic stress, an exact formal expression for tensor evolution in the radiation-matter background can be given in terms of radial prolate spheroidal functions [22, 42]. Yet, a cumbersome form of this expression (an infinite series of Hankel functions with coefficients involving more infinite series), calls for more insightful and practical description.

We continue to parameterize tensor mode evolution by the dimensionless oscillation phase $\varphi \equiv k\tau$ and use the perturbation variable $v \propto ah$. We normalize $v$ by $v(\varphi) = \varphi + o(\varphi)$ as $\varphi \to 0$. Then the initial condition
\( h(\tau \to 0) = 1 \) and explicit \( a(\tau) \) solution (46) fix \( a h / v \) to
\[
h = \frac{2a_{\text{eq}}}{a c_\phi} v = \frac{v}{(1 + \frac{1}{2} \frac{v}{a})} \varphi. \tag{58}
\]
In the first of these equalities and later
\[
\varphi_c \equiv k \tau_c. \tag{59}
\]
Source-free dynamical equation (49) for \( v(\varphi) \) in the radiation-matter background reads
\[
v'' + \left( 1 - \frac{1}{\varphi_c + \frac{1}{2} \varphi^2} \right) v = 0. \tag{60}
\]
Here and below primes denote derivatives with respect to \( \varphi \). In a purely radiation background, in which \( a(\tau) \) and \( v'' + v = 0 \), the solution is \( v(\varphi_{\text{rad},0}) = \sin \varphi \). To account for massive matter deep in the radiation era \((\varphi \ll \varphi_c)\) we treat \((\dot{a}/a)v \) eq. (49) perturbatively, approximating it by \((\dot{a}/a)v_{\text{rad},0} \). Then, neglecting \( \varphi^2 \) against \( \varphi \varphi_c \), we have
\[
v'' + v \approx \frac{\sin \varphi}{\varphi_c}, \tag{61}
\]
A substitution \( v = A(\varphi) \sin \varphi \) leads to a first-order differential equation for \( A'(\varphi) \). That equation has a unique regular at \( \varphi = 0 \) solution
\[
A' = \frac{1}{2 \varphi_c} \ln \left( 2 \varphi + \gamma - \text{ci}(2 \varphi) \right), \tag{62}
\]
where \( \gamma = 0.577216... \) is the Euler constant and \( \text{ci} x \equiv -\int_x^\infty \cos y dy / y \) is the cosine integral. Integrating eq. (62), we obtain:
\[
v(\varphi_{\text{rad}}) = A(\varphi) \sin \varphi - B(\varphi) \cos \varphi, \tag{63}
\]
where
\[
A = 1 + \frac{1}{2 \varphi_c} \left[ \frac{\pi}{2} + \text{si}(2 \varphi) \right], \tag{64}
\]
\[
B = \frac{1}{2 \varphi_c} \left[ \ln(2 \varphi) + \gamma - \text{ci}(2 \varphi) \right], \tag{65}
\]
and \( x \equiv -\int_x^\infty \sin y dy / y \) is the sine integral.

\textit{Well under the horizon} \((\varphi \gg 1)\), the sine and cosine integrals in expressions (64)-(65) tend to zero. Then
\[
v(\varphi_{\text{sub}}) \approx \left( 1 + \frac{\pi}{4 \varphi_c} \right) \sin \varphi - \frac{1}{2 \varphi_c} \left[ \ln(2 \varphi) + \gamma \right] \cos \varphi. \tag{66}
\]
We see that at a large but finite \( \varphi_c \equiv k \tau_c \) the oscillation phase is shifted from the \( \varphi \) oscillations of the adiabatic limit. The shift is caused by nonrelativistic matter contributing to the background expansion. As the matter role increases in time, the shift grows logarithmically even after the tensor mode has become subhorizon.

We can easily extend the mode subhorizon evolution \textit{beyond the radiation era} using the WKB approach\(^6\). To that end, we combine the prefactors \( A \) and \( B \) of eq. (63) into a varying complex amplitude \( C \equiv A - i B \):
\[
v = \text{Im} \left( C e^{i \varphi} \right). \tag{67}
\]
\( C(\varphi) \) satisfies a second-order differential equation, following from eq. (60). After the horizon entry \( C(\varphi) \) changes slowly, hence, we neglect \( C''(\varphi) \) in that equation. The remaining terms are
\[
C' = -\frac{i}{2 \dot{\varphi} \varphi_c + \varphi^2} C. \tag{68}
\]
Integrating (68) and setting the integration constant to match the mode (67) to radiation era solution (66), we find
\[
v(\varphi) = \left( 1 + \frac{\pi}{4 \varphi_c} \right) \sin(\varphi - \Delta \varphi), \tag{69}
\]
with
\[
\Delta \varphi(\varphi_{\text{sub}}) = \frac{1}{2 \varphi_c} \left[ -\ln \left( \frac{1}{4 \varphi_c} + \frac{1}{2 \varphi_c} \right) + \gamma \right]. \tag{70}
\]
Finally, we incorporate the damping by neutrinos during the horizon entry in the radiation era. We treat the correction caused by neutrino anisotropic stress as perturbation which is independent of the perturbation induced by the matter term in eq. (61). In this approximation, neglecting interference of the two corrections, neutrinos additionally suppress the oscillation amplitude by the factor in eq. (42) but they do not affect the oscillation phase. Adjusting the \( v(\varphi_{\text{sub}}) \) amplitude in eq. (69) and substituting the result into eq. (58), we thus have:
\[
h = \frac{A(\tau, k)}{\varphi} \sin[\varphi - \Delta \varphi(\tau, k)], \tag{71}
\]
where on subhorizon scales
\[
A_{\text{sub}} \approx \left( 1 - \frac{5}{2} R_\nu + \frac{\pi}{4 \varphi_c} \right) \left( 1 + \frac{\pi}{2 \varphi_c} \right), \tag{72}
\]
and \( \Delta \varphi \) is given by eq. (70).

\textit{Deep in the matter era} \((\varphi \gg \varphi_c)\), the obtained \( A_{\text{sub}} \) decays inversely with time as
\[
A_{\text{sub, mat}} \approx 2 \varphi_c (1 - \frac{5}{2} R_\nu) + \frac{\pi}{2 \varphi_c}, \tag{73}
\]
while the \( \Delta \varphi_{\text{sub}}(\varphi_{\text{mat}}) \) growth saturates at
\[
\Delta \varphi_{\text{sub, mat}} \approx \frac{\ln(4 \varphi_c) + \gamma}{2 \varphi_c}. \tag{74}
\]
Since \( \varphi_c = k \tau_c \), the phase shift of tensor oscillations at fixed time and \( k \to \infty \) approaches its asymptotic zero

\(6\) Since the adiabatic decay \( h \sim 1/a \) on subhorizon scales is already accounted for by the \( v \) prefactor, e.g. eq. (58), the presented first-order WKB calculation of \( v \) is equivalent to the second-order WKB analysis of the tensor metric perturbation \( h \).
er according to eq. (43) neutrino anisotropic stress induces a decaying shift of oscillations toward higher $\varphi$. When the neutrino source is cut off by matter domination at a finite $\varphi \sim \varphi_e$, the oscillations remain shifted by $\Delta \nu \varphi = O(R_\nu / \varphi_e)$.

C. Entering after equality ($k\tau_{eq} < 1$)

Now we address the modes that start to evolve after radiation becomes subdominant to nonrelativistic matter. We determine the tensor evolution by retaining only the leading, linear in $\varphi_e = k\tau_e$, corrections due to the residual effects of radiation. We could treat the radiation-induced corrections perturbatively, similarly to treating the subdominant matter effects in the previous subsection. However, the derivation becomes more elegant and the result is easier to interpret if, instead, we account for the radiation energy by a simple change of variables, as described next.

We start by rewriting the scale factor evolution (46) as

$$ a = a_{eq} \left[ (\tau + \tau_e)^2 - \tau_e^2 \right] / \tau_e^2. $$

(75)

In terms of the evolution variable $\varphi = k\tau$,

$$ a = \frac{\rho_{eq}}{\varphi_e^2} \left( \varphi^2 - \varphi_e^2 \right) , $$

(76)

where

$$ \varphi \equiv k(\tau + \tau_e) = \varphi + \varphi_e. $$

(77)

Next, in eq. (76) we drop the last $\varphi_e^2$ term and in the gravitational wave eq. (1) ignore the anisotropic stress of radiation, both being $O(\varphi_e^2)$ corrections. Now, with $a(\tau) \propto \varphi^2$, we observe that all the remaining contribution of radiation to eq. (1) can be accounted for by considering a matter-only scenario but replacing $\varphi$ by the shifted evolution variable $\varphi$, whose range is $(\varphi_e, +\infty)$. In other words, a solution of full eq. (1) can be written as

$$ h(\tau, k) = h_{(mat,0)}(\varphi) + O(\varphi_e^2), $$

(78)

where $h_{(mat,0)}$ is a matter-only solution with so far unspecified initial conditions.

Although the omission of the last $\varphi_e^2$ in eq. (76) has a big impact on $H$ when $\tau \lesssim \tau_e$, the considered large-wavelength perturbation is then frozen, whether we apply the true or modified Hubble expansion. Therefore, as can be easily verified, the $\varphi_e^2$ omission leads to a minor $O(\varphi_e^2)$ change in the mode evolution.

In view of the last remark, we continue to impose the initial conditions $h = 1$ and $h' = 0$ at $\varphi = 0$, i.e., at $\varphi = \varphi_e$. These conditions fix the constants $C_1$ and $C_2$ in the general solution

$$ h(\tau, k) \approx C_1 h_{(mat,0)}(\varphi) + C_2 n(\varphi), $$

(79)

where $h_{(mat,0)}$ is the regular matter era solution (55) and $n$ is a complimentary linearly independent solution. To
be specific, we take
\[ n(\varphi) = 3 \left( \frac{\cos \varphi}{\varphi^3} + \frac{\sin \varphi}{\varphi^2} \right). \]  

(80)

Then \( h^{(\text{mat,0})} \) and \( n \) are comparable on subhorizon scales. Given that \( h^{(\text{mat,0})}(\varphi_c) = 1 + O(\varphi_c^2) \) and \( n(\varphi_c) = O(\varphi_c^3) \), the initial conditions fix the weights in eq. (79) as
\[ C_1 = 1 + O(\varphi_c^2), \quad C_2 = O(\varphi_c^5). \]  

(81)

We conclude that up to \( O(\varphi_c^2) \) corrections
\[ h(\tau, k) \approx h^{(\text{mat,0})}(\varphi) = 3 \left( \frac{\sin 2\varphi}{6} - \frac{\cos 2\varphi}{4} \right). \]  

(82)

This result can also be derived by a regular perturbative in \( \varphi_c \) analysis.

We thus found that in \( O(\varphi_c^2) \) order the residual radiation density pushes the tensor modes ahead of the zeroth-order solution \( h^{(\text{mat,0})}(k\tau) \) by a constant conformal time increment \( \tau_c \simeq 260 \text{ Mpc} \). Fig. 5 shows good agreement between eq. (82) (solid) and CMBFAST integration (bold dots) for a mode with \( k = 0.2 \tau^{-1}_\text{eq} \).

In the subhorizon limit, eq. (82) reduces to
\[ h^{(\text{mat, sub})} \approx -3 \cos(\varphi + \varphi_c)/\varphi^2. \]  

(83)

The amplitude suppression factor \( A \) and the phase shift \( \Delta \varphi \) of our earlier parameterization \( h = A(\tau) \sin[\varphi - \Delta \varphi(\tau)]/\varphi \) are now equal
\[ A^{(\text{mat, sub})} \approx \frac{3}{\varphi}, \quad \Delta \varphi^{(\text{mat, sub})} \approx \frac{\pi}{2} - \varphi_c. \]  

(84)

These values for the large wavelengths can be compared with results (73) and (74), derived for the scales that enter the horizon before equality. If desired, these two sets of asymptotic formulas can be joined by various fitting ansatzes for a more accurate interpolation than provided by our minimal formula (57).

**Appendix A: Free Streaming in General Relativity**

1. **Full Theory**

Classical particles can be described by a phase space distribution \( f \) over spacetime coordinates \((\tau, x^i)\) and particle canonical momenta \( P_i \). We consider \( P_i \) as opposed to more conventional \([43-45]\) comoving proper momenta, as the primary phase space coordinates. At a minor expense of two extra terms in the linearized equations for the energy-momentum tensor, eqs. (A14) and (A17), our choice leads, first, to a simpler general-relativistic equation of \( f \) evolution. Second, to time independence of \( f \) for free particles on superhorizon scales \([46]\).

The phase space distribution of free streaming particles evolves according to the collisionless Boltzmann equation
\[ \frac{df}{d\tau} + \frac{dx^i}{d\tau} \frac{\partial f}{\partial x^i} + \frac{dP_i}{d\tau} \frac{\partial f}{\partial P_i} = 0. \]  

(A1)

The canonical momentum of a particle that has mass \( m \) and moves along a worldline \( x^\mu(\tau) = (\tau, x^i(\tau)) \) is
\[ P_i = m g_{\mu\nu} \frac{dx^\mu}{d\tau}(\tau) \frac{dx^\nu}{d\tau}(\tau), \]  

(A2)

where \( ds^2 = g_{\mu\nu} dx^\mu dx^\nu \). Natural definitions \( P_0 \equiv m g_{00} \frac{dx^0}{d\tau} = m \frac{d\tau}{d\tau} = g^{\alpha\beta} P_\alpha P_\beta \) and the calculation of \( dP_i/d\tau \) from the geodesic equation give
\[ \frac{dx^i}{d\tau} = \frac{P^i}{P^0}, \quad \frac{dP_i}{d\tau} = g_{\mu\nu} P^\mu \frac{dP^\nu}{d\tau}. \]  

(A3)

The value of \( P_0 \) at every phase space point \((x^i, P_i)\) follows from the identity
\[ -g^{\mu\nu} P_\mu P_\nu = m^2. \]  

(A4)

We specify the direction of particle propagation by the spatial components of
\[ n_\mu \equiv \frac{P_\mu}{P^0}, \quad \text{where} \quad P^2 \equiv \sum_{i=1}^3 P_i^2. \]  

(A5)

This metric-independent, therefore non-covariant, definition of \( n_\mu \) separates the infinitesimal transport of particles from the temporal change of metric. Note that by eq. (A5)
\[ \sum_{i=1}^3 n_i^2 = 1. \]  

(A6)

We also consider \( n_\mu \equiv g^{\mu\nu} n_\nu = P_\mu / P^0 \).

The energy-momentum tensor of the described particles equals
\[ T_\mu^\nu = \int \frac{n^3 P_i}{\sqrt{-g}} P^n P_\nu f. \]  

(A7)
We substitute $P_{\mu} = n_{\mu} P$ and $P^\mu = n^\mu P$, and define particle intensity $I$ in a direction $n_i$ as

$$I(x^\mu, n_i) \equiv \int P^3 dP f(x^\mu, n_i P).$$  \hfill (A8)

Then

$$T^\mu_\nu = \int \frac{d^2 n_i}{\sqrt{-g}} n^\mu n^\nu I.$$  \hfill (A9)

The dynamics of the intensity of ultra-relativistic decoupled particles is governed by a simple closed equation, derived next.

Only two of the three components of $n_i$, constrained by condition (A6), are independent. Let $\mu_\alpha = (\mu_1, \mu_2)$ be any two independent variables parameterizing $n_i$. Then partial differentiation with respect to $n_i = P_i / P$ is naturally defined as a differential operator

$$\frac{\partial}{\partial n_i} \equiv \sum_{\alpha=1,2} \left( \frac{\partial P_{\mu_\alpha}}{\partial P_i} \right) \frac{\partial}{\partial \mu_\alpha}.$$  \hfill (A10)

Integrating the Boltzmann equation (A1)-(A3) over $P^3 dP$, applying the above definition of $\partial / \partial n_i$, and remembering the condition of ultra-relativity $g^{\mu\nu} n_\mu n_\nu = 0$, we obtain

$$n^\mu \frac{\partial I}{\partial x^\mu} = \frac{1}{2} n^\mu n^\nu g_{\mu\nu,i} \left( 4n_i I - \frac{\partial I}{\partial n_i} \right).$$  \hfill (A11)

Equation (A11) is the main result of this Appendix.

As this dynamical equation shows, for superhorizon perturbations the considered intensity of decoupled particles is time independent. Indeed, when the spatial gradients $\partial I / \partial x^i$ and $g_{\mu\nu,i}$ are negligible, eq. (A11) becomes $\dot{I} = 0$. In the opposite extreme, for the locally Minkowski metric, $I$ reduces to the conventional particle intensity $dE/(dV d^2 \tilde{n})$.

2. Linear theory

For a homogeneous and isotropic distribution of particles with energy density $-T^0_0 = \rho(\tau)$ we see from eq. (A9) that $\dot{I} = \rho a^4 / (4\pi)$. We describe linear inhomogeneities over this background with relative intensity perturbation $\tilde{I}(x^\mu, n_i)$,

$$I = \rho a^4 \left( 1 + \tilde{I} \right).$$  \hfill (A12)

When the particles are ultra-relativistic and their number is conserved ($\rho a^4$ is constant), linearization of eq. (A11) gives:

$$\dot{\tilde{I}} + n_i \dot{\tilde{I}} = 2n_i \tilde{n}^\mu \tilde{n}^\nu h_{\mu\nu,i},$$  \hfill (A13)

where $\tilde{n}^\mu = (1, n_i)$. The linearly perturbed $T^\mu_\nu$ components (A9) are straightforwardly calculated from

$$n_\mu n_\nu g^{\mu\nu} = 0 \text{ and } \langle n_i \rangle = 0, \langle n_i n_j \rangle = \delta_{ij}/3, \ldots,$$

where $\langle \rangle_n$ stands for $\int d^2 n_i / 4\pi$:

$$-T^\mu_\nu = \rho \left( 1 + \tilde{I} \right) n - 2h,$$

$$T^0_0 = \langle n_i \tilde{I} \rangle_n,$$

$$T^i_0 = \frac{-T^0_0}{3} \delta^i_j + \tilde{\Sigma}^i_j.$$  \hfill (A14) \hfill (A15) \hfill (A16)

Here, $h \equiv \frac{1}{3} \sum_i h_{ii}$,

$$\tilde{\Sigma}^i_j = \rho \left[ \langle n_i n_j \rangle - \frac{1}{3} \tilde{\delta}_{ij} \tilde{I} \right] - \frac{4}{15} \tilde{h}_{ij}$$  \hfill (A17)

is anisotropic stress, and in the last formula $\tilde{h}_{ij} = h_{ij} - \tilde{\delta}_{ij} h$ is the traceless part of $h_{ij}$.

If the energy distribution of particles that stream in a direction $n_i$ is thermal and the metric is locally Minkowski then $\tilde{I}$ is proportional to the particle temperature perturbation: $\tilde{I} = 4\Delta T(n_i) / T$. Under a change of coordinates $x^\mu \rightarrow \tilde{x}^\mu = x^\mu + \alpha^\mu$ (gauge transformation), $\tilde{I}$ transforms as

$$\tilde{I}(x^\mu, n_i) - \tilde{I}(x^\mu, n_i) = -4n_i n_\mu \alpha^\mu.$$  \hfill (A18)

Similarly to the multipoles of $\Delta T(n_i) / T$, the multipoles of intensity perturbation $\tilde{I}(n_i)$ can be used to decompose the transport eq. (A13) into a computer-friendly Boltzmann hierarchy of equations for photons (with scattering terms and polarization added), neutrinos, and other cosmological species. These equations for scalar perturbations in the Newtonian gauge are given in Sec. III and Appendix A of [46] ([46] uses $D \equiv \frac{1}{4} \tilde{I}$).

The structure of these equations is considerably simpler than that of the traditional description with $\Delta T(n_i) / T$ multipoles: The dynamical matter perturbations are no longer driven by time derivatives of gravitational potentials, which are constrained non-dynamically by the same matter perturbations. Moreover, unlike the situation with the traditional approach, the change of the perturbation value during the horizon entry is controlled only by physical interactions and is not caused by gauge artifacts [46].

3. Initial conditions

We specify the initial conditions by following Ref. [10]. We suppose that the considered species (later, neutrinos) decouple at time $\tau_{\text{dec}}$ when the Hubble scale is much smaller than the spatial scale of the probed perturbations. Then shortly after $\tau_{\text{dec}}$ there exists a hypersurface with the following property. Every observer on this hypersurface who moves normally to it would detect the same isotropic distribution of the proper neutrino momentum $p$. This is a hypersurface of constant temperature when the particle distribution is thermal. Although the decoupled relic neutrinos are not exactly thermal,
described conditions should still be met on a hypersurface \((u)\) of uniform neutrino energy density. We choose \(u\) for an initial constant time hypersurface.

An observer who moves normally to a constant time hypersurface measures proper momentum \(p\) which is related to the canonical momentum \(P\) as

\[
p^2 = (g^{ij}P_iP_j) \quad (A19)
\]

\((g^{ij} = \text{the inverse of the 3-tensor } g_{ij})\). In terms of the variables \((A5)\), \(p^2 = (g^{ij}n_in_jP^2)\). Therefore, if the hypersurface of the uniform distribution of the proper momentum is taken for a constant time hypersurface then, according to eq. \((A8)\), the neutrino intensity varies as

\[
I^{(u)}(x, n_i) = \text{const} \left[ n_in_j(g^{ij}u(x)\right] T. \quad (A20)
\]

In linear theory and the metric \((12)\), \((g^{ij} = \alpha^{-2}(\delta_{ij} - h_{ij})\). Then comparing eq. \((A20)\) and \((A12)\) and remembering \((A6)\) we conclude that on superhorizon scales the intensity perturbation equals

\[
I^{(u)} = 2n_in_jh_{ij}^{(u)}. \quad (A21)
\]

4. Linear solution

Sourced transport equation \((A13)\) with initial condition \((A21)\) at \(\tau = \tau_{in}\) is solved by

\[
\frac{1}{2} I(\tau, x, u) = n_in_jh_{ij}^{(u)}(\tau_{in}, x_{in}) + \int_{\tau_{in}}^\tau d\tau' n_in_i\hat{n}_i\hat{n}_\mu h_{\mu\nu,i}(\tau', x'), \quad (A22)
\]

where \(x_{in} = x + n(\tau_{in} - \tau)\) and \(x' = x + n(\tau' - \tau)\). At the lower limit of the integral, \(h_{\mu\nu}\) should, strictly speaking, be evaluated on the uniform density hypersurface on which the initial conditions were set. However, for gauge-independent tensor perturbations, a hypersurface choice is irrelevant. Moreover, even for scalar perturbations the entire integral can be evaluated in any of such traditional gauges as the synchronous, Newtonian, comoving, or spatially flat gauges. Indeed, under a gauge transformation \(\tilde{x}^\mu = x^\mu + \alpha^\mu\), the integral transforms as \(-2n_in_\mu\alpha^\mu_i(\tau, x) + 2n_in_\mu\alpha^\mu_i(\tau_{in}, x_{in})\). The first of these two terms matches the transformation of the left hand side of eq. \((A22)\). And the second one is negligible for a transformation of superhorizon perturbation from the uniform density gauge to any of the mentioned gauges \([47]\).

In the integral of eq. \((A22)\), the gradient of \(h_{\mu\nu}\) can be traded for time derivatives:

\[
n_ih_{\mu\nu,i}(\tau', x') = \left( \frac{d}{d\tau'} - \frac{\partial}{\partial \tau'} \right) h_{\mu\nu}(\tau', x').
\]

The trivial integration of the full time derivative gives

\[
\frac{1}{2} I = n_in_jh_{ij}^{(u)}(\tau_{in}, x_{in}) + \hat{n}^\mu\hat{n}^\nu h_{\mu\nu}|_{\tau_{in}, x_{in}} - \int_{\tau_{in}}^\tau d\tau' \hat{n}^\mu\hat{n}^\nu h_{\mu\nu}(\tau', x'). \quad (A23)
\]

This solution is applicable to any (scalar, vector, or tensor) type of linear perturbation of ultrarelativistic species which decouple on superhorizon scales.

APPENDIX B: FORMAL PROOF OF \(\lim_{k \rightarrow \infty} \varphi_0 = 0\)

Sec. III B of the main text argues that subhorizon tensor oscillations in the radiation era \(h^{(\text{rad})}(\varphi) \rightarrow A_0 \sin(\varphi + \varphi_0)/\varphi\), \(\varphi \equiv k\tau\) are consistent with causality if only \(\varphi_0 = 0\). The unshifted oscillations match to a finite, step discontinuity of the real space tensor Green’s function at the particle horizon \(\chi = \pm 1:\)

\[
\tilde{h}(\text{disc})(\chi) = \frac{A_0}{2}\delta(1 - |\chi|). \quad (B1)
\]

Vice versa, a Green’s function discontinuity of this type implies \(\varphi_0 = 0\).

In this Appendix we verify that the solution of the real space equation \((31)\) for \(\tilde{h}(\chi)\) indeed has the form \((B1)\). We do not assume that \(R_{\nu}\) is a small parameter. To be specific, we consider the \(\tilde{h}(\chi)\) discontinuity at \(\chi = -1\).

We take \(\tilde{h}(\chi)\) of the general form \((36)\) and first show that the right hand side of eq. \((31)\) should remain finite as \(\chi \rightarrow -1 + 0\). For the second term in the integral on the right hand side of eq. \((31)^7\)

\[
\lim_{\chi \rightarrow -1 + 0} \int_{1}^{1} d\mu K(\chi, \mu) = 0. \quad (B2)
\]

The remaining integral \(\int_{-1}^{1} d\mu K(\mu, \chi)\) can be calculated explicitly. However, even without its full calculation, it is easy to see that for \(\chi \rightarrow -1\)

\[
\int_{-1}^{1} d\mu K(\mu, \chi) = -\frac{4}{3}\tilde{h}(\chi)[1 + o(1)]. \quad (B3)
\]

Here is a proof of eq. \((B2)\). This equation considers \((1 - \chi^2)^2 \int_{-1}^{1} d\mu \mu h(\mu)/(\mu - \chi)\) in the limit \(\chi \rightarrow -1 + 0\). The most divergent at \(\chi \rightarrow -1\) contribution to \(\int_{-1}^{1} d\mu \mu h(\mu)/(\mu - \chi)\) arises from the two discontinuous \(h(\mu)\) terms \((36)\). The products of either of the two corresponding integrals,

\[
\int_{-1}^{1} d\mu \mu^\alpha = \ln(\chi + 1)[1 + o(1)],
\]

\[
\int_{-1}^{1} d\mu \mu^{\ln(1 - \chi^2)/\mu - \chi} = \frac{1}{2} \ln^2(\chi + 1)[1 + o(1)],
\]

and of the prefactor \((1 - \chi^2)^2\) in the considered expression apparently vanish as \(\chi \rightarrow -1\).
Thus we can write eq. (31) as

\[
(\chi^2 - 1)\tilde{h}' = -2R_\nu \tilde{h}(\chi) [1 + o(1)].
\]  

(B4)

For \( \chi > -1 \), the last equation is formally solved by

\[
\tilde{h} = -2R_\nu \int_{-1}^{\chi} \frac{d\chi_1}{\chi_1 - 1} \int_{-1}^{\chi_1} d\chi_2 \tilde{h}(\chi_2) [1 + o(1)] + \text{const.}
\]

(B5)

Given \( \tilde{h} \) of the general form (36), the above double integral is continuous at \( \chi = -1 \). Since the prefactor of \( \tilde{h}' \) in eq. (B4) vanishes at \( \chi = -1 \), the last constant in eq. (B5) may differ for \( \chi < -1 \) and \( \chi > -1 \). The change of the constant provides a finite jump in \( \tilde{h} \) at \( \chi = -1 \). This proves eq. (B1).

1. Subleading in \( 1/k\tau \) term

The result (B5) allows to establish a simple relation between the leading and subleading in \( 1/k\tau \) terms that describe the subhorizon evolution of tensor modes during radiation domination. Evaluating the right derivatives of both sides of this equation at \( \chi = -1 \), we find a relation

\[
\tilde{h}'(-1 + 0) = R_\nu \tilde{h}(-1 + 0).
\]

(B6)

Since \( h(\chi) \) is even, we have a similar equality at \( \chi = 1 \):

\[
\tilde{h}(1 - 0) = -R_\nu \tilde{h}(1 - 0).
\]

Remembering that \( \tilde{h} \equiv 0 \) for \( |\chi| > 1 \) and considering the Fourier components of \( \tilde{h} \), we conclude that

\[
\tilde{h}^{(\text{rad})}(k\tau) = A_0 \left\{ \frac{\sin k\tau}{k\tau} - \frac{\cos k\tau}{k^2\tau^2} [1 + o(1)] \right\}.
\]

(B7)

This relation applies to all orders of the \( R_\nu \) expansion; its validity for \( O(R_\nu) \) order is evident from eqs. (42) and (45).
are summarized in Table II of [12].

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[47] This can be verified using the explicit expressions for $\alpha^\mu$, e.g., from Appendix A.3 of [12].