A $\mathbb{Z}$-BASIS FOR THE CLUSTER ALGEBRA ASSOCIATED TO AN AFFINE QUIVER

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Abstract. The canonical bases of cluster algebras of finite types and rank 2 are given explicitly in [4] and [14] respectively. In this paper, we will deduce $\mathbb{Z}$-bases for cluster algebras for affine types $\tilde{A}_{n,n}$, $\tilde{D}$ and $\tilde{E}$. Moreover, we give an inductive formula for computing the multiplication between two generalized cluster variables associated to objects in a tube.

1. Introduction

Cluster algebras were introduced by S. Fomin and A. Zelevinsky [11] in order to develop a combinatorial approach to study problems of total positivity. By the work of [13], the link between acyclic cluster algebras and representation theory of quivers found its general framework in [1] where the authors introduced the cluster category. Let $Q$ be an acyclic quiver with vertex set $Q_0 = \{1, 2, \cdots, n\}$. Let $A = \mathbb{C}Q$ be the path algebra of $Q$ and we denote by $P_i$ the indecomposable projective $\mathbb{C}Q$-module with the simple top $S_i$ corresponding to $i \in Q_0$ and $I_i$ the indecomposable injective $\mathbb{C}Q$-module with the simple socle $S_i$. Let $\mathcal{D}b(Q)$ be the bounded derived category of $\text{mod}\mathbb{C}Q$ with the shift functor $T$ and the AR-translation $\tau$. The cluster category associated to $Q$ is the orbit category $\mathcal{C}(Q) := \mathcal{D}b(Q)/F$ with $F = T \circ \tau^{-1}$. Let $\mathbb{Q}(x_1, \cdots, x_n)$ be a transcendental extension of $\mathbb{Q}$. The Caldero-Chapoton map of an acyclic quiver $Q$ is the map $X^Q_Q : \text{obj}(\mathcal{C}(Q)) \to \mathbb{Q}(x_1, \cdots, x_n)$ defined in [3] by the following rules:

1. if $M$ is an indecomposable $\mathbb{C}Q$-module, then
   $$X^Q_M = \sum_e \chi(\text{Gr}_e(M)) \prod_{i \in Q_0} x^{-\langle e, s_i \rangle}_{i}^{\dim M - e};$$

2. if $M = TP_i$ is the shift of the projective module associated to $i \in Q_0$, then
   $$X^Q_M = x_i;$$

3. for any two objects $M, N$ of $\mathcal{C}_Q$, we have
   $$X^Q_{M \oplus N} = X^Q_M X^Q_N.$$  

Here, we denote by $\langle -, - \rangle$ the Euler form on $\mathbb{C}Q$-mod and $\text{Gr}_e(M)$ is the $e$-Grassmannian of $M$, i.e. the variety of submodules of $M$ with dimension vector $e$. For any object $M \in \mathcal{C}(Q)$, $X^Q_M$ will be called the generalized cluster variable for $M$.

We note that the indecomposable $\mathbb{C}Q$-modules and $TP_i$ for $i \in Q_0$ exhaust the indecomposable objects of the cluster category $\mathcal{C}(Q)$:

$$\text{ind} - \mathcal{C}(Q) = \text{ind} - \mathbb{C}Q \cup \{TP_i : i \in Q_0\}.$$

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Each object \( M \) in \( \mathcal{C}(Q) \) can be uniquely decomposed in the following way:

\[
M = M_0 \oplus TP_M
\]

where \( M_0 \) is a module and \( P_M \) is a projective module. The module \( M_0 \) can be recovered using the following homological functor:

\[
H^0 = \text{Hom}_{\mathcal{C}(Q)}(\mathbb{C}Q, -) : \mathcal{C}(Q) \rightarrow \mathbb{C}Q - \text{mod}
\]

Hence we can rewrite \( M = M_0 \oplus TP_M \) as:

\[
M = H^0(M) \oplus TP_M.
\]

Let \( R = (r_{ij}) \) be a matrix of size \(|Q_0| \times |Q_0|\) satisfying

\[
r_{ij} = \dim_{\mathbb{C}} \text{Ext}^1(S_i, S_j)
\]

for any \( i, j \in Q_0 \). The Caldero-Chapoton map can be reformulated by the following rules [10][12]:

1. \[
X_{\tau P} = X_{TP} = x^{\dim P/\text{rad} P}, X_{\tau I} = X_{-I} = x^{\dim \text{soc} I}
\]

for any projective \( \mathbb{C}Q \)-module \( P \) and any injective \( \mathbb{C}Q \)-module \( I \);

2. \[
X_M = \sum_{\mathcal{C}} \chi(\text{Gr}_{\mathcal{C}}(M)) x^{R^T + (\dim M - 2)R^T - \dim M}
\]

where \( M \) is a \( \mathbb{C}Q \)-module, \( R^T \) is the transpose of the matrix \( R \) and \( x^v = x_1^{v_1} \cdots x_n^{v_n} \) for \( v = (v_1, \ldots, v_n) \in \mathbb{Z}^n \).

Let \( \mathcal{A}H(Q) \) be the subalgebra of \( \mathbb{Q}(x_1, \ldots, x_n) \) generated by

\[
\{ X_M, X_{\tau P} \mid M, P \in \text{mod} - \mathbb{C}Q, P \text{ is projective module} \}.
\]

Let \( \mathcal{E}H(Q) \) be the subalgebra of \( \mathcal{A}H(Q) \) generated by

\[
\{ X_M \mid M \in \text{ind} - \mathbb{C}Q, \text{Ext}^1_{\mathbb{C}Q}(M, M) = 0 \}.
\]

In [3], the authors showed that the Caldero-Chapoton map induces a one to one correspondence between indecomposable objects in \( \mathcal{C}(Q) \) without self-extension and the cluster variables of the cluster algebra \( \mathcal{A}(Q) \). Hence, one can view \( \mathcal{E}H(Q) \) as the cluster algebra for a quiver \( Q \). When \( Q \) is a simply laced Dynkin quiver, \( \mathcal{A}H(Q) \) is coincides with \( \mathcal{E}H(Q) \). In [1], the authors showed a \( \mathbb{Z} \)-basis of \( \mathcal{A}H(Q) \). When \( Q \) is the Kronecker quiver, \( \mathcal{A}H(Q) \) is also coincides with \( \mathcal{E}H(Q) \). In [6], the authors gave a \( \mathbb{Z} \)-basis of \( \mathcal{A}H(Q) \) called the semicanonical basis. When \( Q \) is the quiver of type \( \tilde{A}_n \), \( \mathcal{A}H(Q) \) is still equal to \( \mathcal{E}H(Q) \) and a \( \mathbb{Z} \)-basis is given in [8](also see Section 6.3).

Now let \( Q \) be an affine quiver of type \( \tilde{A}_n, \tilde{D} \) or \( \tilde{E} \). There are many references about the theory of representations of affine quivers, for example, see [2] and [7]. The main goal of this paper is to give a \( \mathbb{Z} \)-basis of \( \mathcal{A}H(Q) \) for an affine quiver \( Q \). We will prove the following theorem.

**Theorem 1.1.** Let \( Q \) be an affine quiver. A \( \mathbb{Z} \)-basis of \( \mathcal{A}H(Q) \) is the following set:

\[
\{ X_L, X_{T \oplus R} | \dim(T_1 \oplus R_1) \neq \dim(T_2 \oplus R_2), \text{Ext}^1_{\mathbb{C}Q}(T, R) = 0, \text{Ext}^1_{\mathbb{C}Q}(L, L) = 0 \}
\]

where \( L \) is any non-regular exceptional object, \( R \) is 0 or any regular exceptional module and \( T \) is 0 or any indecomposable regular module with self-extension and in addition if \( Q \neq \tilde{A}_{1,1} \) and \( \dim(T \oplus R) = m\delta \) for some \( m \in \mathbb{N} \), then \( R = 0 \) and \( T \) is an indecomposable module of dimension vector \( m\delta \) in a non-homogeneous tube.

Moreover, we give an inductive formula for computing the multiplication between two generalized cluster variables associated to objects in a tube in Section 6. Our tools are the following cluster multiplication theorems.
Theorem 1.2. \[15\] (1) For any $A$-modules $V_{\xi}', V_{\eta'}$ we have
\[
d^i(\xi', \eta')X_{V_{\xi'}, V_{\eta'}} = \int_{\lambda \neq \xi' \oplus \eta'} \chi([\hat{\text{Ext}}^1_A(V_{\xi'}, V_{\eta'})]_{\Lambda})X_{\lambda}
\]
\[+ \int_{\gamma, \beta} \chi([\hat{\text{Hom}}_A(V_{\eta'}, \tau V_{\xi'})]_{\Lambda \oplus \gamma})X_{V_{\gamma}}X_{\tau^{-1}V_{\lambda}}\]
where $V_{\xi'}$ has no projective direct summand and $d^i(\xi', \eta') = \dim \text{Ext}_A(V_{\xi'}, V_{\eta'})$.

(2) For any $A$-module $V_{\xi'}$ and $P \in \rho$ is projective \(\text{Then}\)
\[
d(\rho, \xi')X_{V_{\xi'}, X}^{\dim P'/\text{rad} P} = \int_{\delta, \eta'} \chi([\hat{\text{Hom}}_A(V_{\eta'}, I)_{\Lambda \oplus \delta}])X_{V_{\eta'}}X_{\delta}^{\dim P'/\text{rad} P'}
\]
where $I = D\text{Hom}_A(P, A)$, $I' \in \rho'$ injective, $P' \in \rho'$ projective, and $d(\rho, \xi') = \dim \text{Hom}_A(V_{\rho}, V_{\xi'})$.

Theorem 1.3. \[5\] Let $Q$ be an acyclic quiver and $M$ any indecomposable non-projective $A$-module, then
\[
X_MX_{\tau M} = 1 + X_E
\]
where $E$ is the middle term of the Auslander-Reiten sequence ending in $M$.

Theorem 1.4. \[5\] Let $Q$ be an acyclic quiver and $M, N$ be any two objects in $\mathcal{C}(Q)$ such that $\dim \text{Ext}_C(1)(M, N) = 1$, then
\[
X_MX_N = X_B + X_{B'}
\]
where $B$ and $B'$ are the unique objects such that there exists non-split triangles
\[
M \rightarrow B \rightarrow N \rightarrow M[1] \quad \text{and} \quad N \rightarrow B' \rightarrow M \rightarrow N[1]
\]

In \[10\], the author construct a $Z$-basis for a cluster algebra of type $\hat{A}$ referred as the semicanonical basis. It is interesting to compare it to the $Z$-bases in this paper.

2. Numerators of Laurent expansions in generalized cluster variables

In the following, we will suppose that $Q$ is one of $\hat{A}_{n, m}, \hat{D}$ and $\hat{E}$ with an orientation where every vertex is a sink or a source. For any object $M = M_0 \oplus (\oplus s_i T P_i) \in \mathcal{C}(Q)$ where $M_0$ is a module, we extend the dimension vector to the objects in the cluster category by setting
\[
\dim(M) = \dim(M_0) - (s_1, \cdots, s_n).
\]

Let $E_i[n]$ be the indecomposable regular module with quasi-socle $E_i$ and quasi-length $n$ and $X_0 = 1$. For any $\mathbb{C}Q$-module $M$, we denoted by $\dim_{\mathbb{C}}M(i)$ the $i$-th component of $\dim M$.

Definition 2.1. For $M, N \in \mathcal{C}(Q)$ with $\dim(M) = (m_1, \cdots, m_n)$ and $\dim(N) = (r_1, \cdots, r_n)$, we write $\dim(M) \preceq \dim(N)$ if $m_i \leq r_i$ for $1 \leq i \leq n$. Moreover, if there exists some $i$ such that $m_i < r_i$, then we write $\dim(M) < \dim(N)$.

Remark 2.2. Note that for the cluster multiplication formula in the Theorem 1.2(1), we have the following exact sequences:
\[
0 \rightarrow V_{\eta'} \rightarrow V_{\lambda} \rightarrow V_{\xi'} \rightarrow 0
\]
and
\[
0 \rightarrow V_{\beta} \rightarrow V_{\eta'} \rightarrow \tau V_{\xi'} \rightarrow V_{\gamma} \rightarrow 0.
\]
If $\dim \text{Ext}_A(V_{\xi'}, V_{\eta'}) \neq 0$ \(\Rightarrow \dim(V_{\beta}) + \dim(\tau^{-1}V_{\gamma}) < \dim(V_{\eta'}) + \dim(V_{\xi'}) = \dim(V_{\lambda})\)
because $\tau V_{\xi'}$ has no injective summand.
According to the definition of the Caldero-Chapoton map, we consider the Laurent expansions in generalized cluster variables 
\[ X_M = \prod_{1 \leq i \leq \text{dim}(M)} \frac{P(x)}{x_i} \] for \( M \in \mathcal{C}(Q) \) such that the integral polynomial \( P(x) \) in the variables \( x_i \) is not divisible by any \( x_i \). We define the denominator vector of \( X_M \) as \((m_1, \ldots, m_n)\) \([10]\). The following theorem is called as the denominator theorem.

**Theorem 2.3.** [4] Let \( Q \) be an acyclic quiver. Then for any object \( M \) in \( \mathcal{C}(Q) \), the denominator vector of \( X_M \) is \( \text{dim}(M) \).

According to the denominator Theorem [2.3] we can prove the following propositions.

**Proposition 2.4.** If \( M \) is \( P_i \) or \( I_i \) for \( 1 \leq i \leq n \), then \( X_M = P(x) \cdot \text{dim}(M) \) where the constant term of \( P(x) \) is 1.

**Proof.** 1) If \( i \) is a sink point, we have the following short exact sequence:
\[ 0 \rightarrow P_i \rightarrow I_i \rightarrow I' \rightarrow 0 \]

Then by the cluster multiplication theorem in Theorem 1.2, we have:
\[ X_{\tau P_i} X_{P_i} = x_1 \cdot \text{dim}(I') + 1 \]

Thus the constant term of numerator in \( X_{\tau P_i} \) as an irreducible fraction of integral polynomials in the variables \( x_i \) is 1 because of \( X_{\tau P_i} = x_i \).

If \( i \) is a source point, we have the following short exact sequence:
\[ 0 \rightarrow P' \rightarrow P_i \rightarrow I_i \rightarrow 0 \]

Similarly we have:
\[ X_{\tau P_i} X_{P_i} = X_{P'} + 1 \]

Thus we can finish it by induction on \( P' \).

2) For \( X_{I_i} \), it is totally similar. \( \square \)

Note that \( X_{\tau P_i} = x_i = \frac{1}{x_i^{-1}} \) and \( \text{dim}(\tau P_i) = (0, \ldots, 0, -1, 0, \cdots, 0) \) with \( i \)-th component 1 and others 0. Hence we denote the denominator of \( X_{\tau P_i} \) by \( x_i^{-1} \), and assert the constant term of numerator in \( X_{\tau P_i} \) is 1. With these notations, we have the following Proposition 2.5.

**Proposition 2.5.** For any object \( M \in \mathcal{C}(Q) \), then \( X_M = \frac{P(x)}{\text{dim}(M)} \) where the constant term of \( P(x) \) is 1.

**Proof.** It is enough to consider the case that \( M \) is an indecomposable module.

1) When \( M \) is an indecomposable preprojective module, then by exchange relation in Thereom 1.4 we have
\[ X_M X_{\tau M} = \prod_i X_{B_i} + 1 \]

Thus by Proposition 2.3, we can prove that \( X_M = \frac{P(x)}{\text{dim}(M)} \) where the constant term of \( P(x) \) is 1 by induction with the help of the directness of AR-quiver in the preprojective component of \( \text{mod}\mathcal{C}Q \). The discussion is similar for any indecomposable preinjective module.

2) When \( M \) is an indecomposable regular module, we only need to prove that the proposition holds for any regular simple module according to the exchange relations.

First, suppose \( M \) is in some homogeneous tube with dimension vector \( \delta \). Note that there exists a point \( e \) such that \( \dim(\delta(e)) = 1 \). Thus we have
\[ \dim \text{Ext}^1_{\mathcal{C}Q}(M, P(e)) = \dim \text{Hom}_{\mathcal{C}Q}(P(e), M) = 1. \]
Then we obtain the following two non-split exact sequences:

\[
0 \rightarrow P(e) \rightarrow L \rightarrow M \rightarrow 0
\]

and

\[
0 \rightarrow L' \rightarrow P(e) \rightarrow M \rightarrow L'' \rightarrow 0
\]

where \(L\) and \(L'\) are preprojective modules and \(L''\) is a preinjective module. Using Theorem 1.2 or Theorem 1.4, we have

\[
X_MX_{P(e)} = X_L + X_LX_{\tau^{-1}L''}
\]

where \(\dim(L' \oplus \tau^{-1}L'') < \dim(P(e) \oplus M)\).

We have already known that the constant term of the numerator in \(X_{P(e)}, X_L\) as an irreducible fraction of integral polynomials in the variables \(x_i\) is 1 by 1), then the constant term of the numerator in \(X_M\) must be 1.

Now we consider these non-homogeneous tubes. Note that by the AR formula in Theorem 1.3, we only need to prove the constant term of the numerator in \(X_{E_i}\) is 1 for \(1 \leq i \leq r\). Suppose \(M\) is a regular simple module such that \(\dim(M(e) = \dim\delta(e) = 1\). We denote \(M\) by \(E_1\), thus \(\dim E_1(e) = 1\), and \(\dim E_3(e) = 0\) for \(2 \leq i \leq r\). Therefore \(\dim Ext^1(E_2, P(e)) = 1\), then we have the following non-split exact sequences combining the relation \(\tau E_2 = E_1\)

\[
0 \rightarrow P(e) \rightarrow L \rightarrow E_2 \rightarrow 0
\]

and

\[
0 \rightarrow L' \rightarrow P(e) \rightarrow E_1 \rightarrow L'' \rightarrow 0
\]

where \(L\) and \(L'\) are preprojective modules and \(L''\) is a preinjective module. Then we have

\[
X_{E_2}X_{P(e)} = X_L + X_LX_{\tau^{-1}L''}
\]

where \(\dim(L' \oplus \tau^{-1}L'') < \dim(P(e) \oplus E_2)\).

We have already known that the constant term of the numerator in \(X_{P(e)}, X_L\) as an irreducible fraction of integral polynomials in the variables \(x_i\) is 1 by 1), then the constant term of the numerator in \(X_{E_2}\) must be 1.

Note that \(\dim E_1[2](e) = \dim E_1(e) + \dim E_2(e) = 1\), by similar discussions, we can obtain the constant term of the numerator in \(X_{E_i[2]}\) must be 1. Thus by \(X_{E_1}X_{E_2} = X_{E_1[2]} + 1\), we obtain that the constant term of the numerator in \(X_{E_i}\) must be 1. Using the same method, we can prove the constant term of the numerator in \(X_{E_i}\) must be 1 for \(3 \leq i \leq r\).

\[\square\]

3. Generalized cluster variables on tubes

Let \(Q\) be an affine quiver with the minimal imaginary root \(\delta = (\delta_i)_{i \in Q_0}\). Then tubes of indecomposable regular \(\mathbb{C}Q\)-modules are indexed by the projective line \(\mathbb{P}^1\). Let \(\lambda\) be the index of a homogeneous tube and \(M(\lambda)\) be the regular simple \(\mathbb{C}Q\)-module with dimension vector \(\delta\) in this homogeneous tube. Let \(M|i|\) be the regular module with quasi-socle \(M\) and quasi-regular length \(i\) for any \(i \in \mathbb{N}\). Let \(X_M\) be the generalized cluster variable associated to \(M\) by the reformulation of the Caldero-Chapoton map. Then we have

**Proposition 3.1.** Let \(\lambda\) and \(\mu\) be in \(\mathbb{P}^1\) such that \(M(\lambda)\) and \(M(\mu)\) are two regular simple modules of dimension vector \(\delta\). Then \(X_{M(\lambda)} = X_{M(\mu)}\).

**Proof.** Choose a vertex \(p \in Q_0\) such that \(\delta_p = 1\). We assume that \(p\) is a sink. Let \(Q\) have the underlying graph not of type \(\tilde{A}_{n}\). Then there is unique edge \(\alpha \in Q_1\) with the head \(p\) and tail \(p'\). It is easy to check \(\delta_{p'} = 2\). Let \(P_p\) be the indecomposable projective module corresponding \(p\) and \(I(\delta - p)\) be the indecomposable preinjective module of dimension vector \(\delta - \dim S_p\). Then \(\dim_{\mathbb{C}} Ext^1_{\mathbb{C}Q}(I(\delta - p), P_p) = 2\). Given
any $\epsilon \in \text{Ext}_C^1(I(\delta - p), P_p)$, we have a short exact sequence whose equivalence class is $\epsilon$ as follows:

$$
\varepsilon : 0 \to P_p \xrightarrow{(1 \ 0)} M_\epsilon \xrightarrow{(0 \ 1)} I(\delta - p) \to 0
$$

where $(M_\epsilon)_i = (P_p)_i \oplus I(\delta - p)_i$ for any $i \in Q_0$, $(M_\epsilon)_\beta = I(\delta - p)_\beta$ for $\beta \neq \alpha$ and $(M_\epsilon)_\alpha$ is

$$(M_\epsilon)_\alpha = \begin{pmatrix} 0 & m(\epsilon, \alpha) \\ 0 & 0 \end{pmatrix}$$

where $m(\epsilon, \alpha) \in \text{Hom}_C(I(\delta - p)'_p, P_p)$. For any $\epsilon, \epsilon' \in \text{Ext}_C^1(I(\delta - p), P_p)$, $M_\epsilon \cong M_\epsilon'$ if and only if $m(\epsilon, \alpha) = tm(\epsilon', \alpha)$ for some $t \in C$. The regular simple $C$-modules (denoted by $M(\lambda)$) with dimension vector $\delta$ satisfy that $M(\lambda)_\alpha$ is as follows

$$M(\lambda)_{\lambda'} = C^2 \begin{pmatrix} 1 & \lambda \\ \lambda^* & 1 \end{pmatrix} M(\lambda)_p = C$$

where $\lambda \in C^* \setminus \{1\}$. Let $M(\lambda)$ and $M(\lambda')$ be any two regular simple $C$-modules with dimension vector $\delta$. Let $P$ be an indecomposable projective module such that $P \subseteq M(\lambda)$ and $(\dim P)_p = 0$. Then $P$ is also a submodule of $M(\lambda')$ and $(\dim P')_p = 0$. Let $P$ be an indecomposable projective module such that $P \subseteq M(\lambda)$ and $(\dim P)_p = 1$. Assume that $P_\alpha \cong C^{a+b\lambda}$, then there exists $P' \in \text{Gr}_{\lambda}(M(\lambda'))$ such that $P' \cong P$ and $P'_\alpha \cong C^{a+b\lambda}$. Since $\tau M(\lambda) = M(\lambda)$, we know $\tau^{-1}P$ and $\tau^{-1}P'$ are the submodules of $M(\lambda)$ and $M(\lambda')$ for any $i \in N$, respectively. Hence, any preprojective submodule $X$ of $M(\lambda)$ corresponds to a preprojective submodule $X'$ of $M(\lambda')$ and $X \cong X'$. Let $Q$ be of type $\tilde{A}_n$. Then there are two adjacent edge $\alpha$ and $\beta$. Any regular simple module $M(\lambda)$ satisfies that $M(\lambda)_\alpha$ is as follows:

$$C \xrightarrow{1} C \xrightarrow{\lambda} C \xrightarrow{1} C.$$ 

The discussion is similar as above. If $p$ is a source, the discussion is also similar. Therefore, there is a homeomorphism between $\text{Gr}_{\lambda}(M(\lambda))$ and $\text{Gr}_{\lambda}(M(\lambda'))$ for any dimension vector $\lambda$. By definition, $X_{M(\lambda)} = X_{M(\mu)}$. □

We note that there is an alternative proof of Proposition 3.1 in [10] Lemma 3.14.

**Proposition 3.2.** For any $m, n \in \mathbb{N}$ and $m \geq n$, we have

$$X_{M[m]}X_{M[n]} = X_{M[m+n]} + X_{M[m+n-2]} + \cdots + X_{M[m-n+2]} + X_{M[m-n]}.$$

**Proof.** When $n = 1$, we know $\dim_C \text{Ext}_C^1(M[m], M) = \dim_C \text{Hom}(M, M[m]) = 1$. The involving non-split short exact sequences are

$$0 \to M \to M[m+1] \to M[m] \to 0$$

and

$$0 \to M \to M[m] \to M[m-1] \to 0.$$ 

Thus by the cluster multiplication theorem in Theorem 1.2 or Theorem 1.3 and the fact $\tau M[k] = M[k]$ for any $k \in \mathbb{N}$, we obtain the equation

$$X_{M[m]}X_M = X_{M[m+1]} + X_{M[m-1]}.$$ 

Suppose that it is right for $n \leq k$. When $n = k + 1$, we have

$$X_{M[m]}X_{M[k+1]} = X_{M[m]}(X_{M[k]}X_{M[M[k-1]]}) = X_{M[m]}X_{M[k]}X_{M}X_{M[M[k-1]]}X_{M[k-1]} = \sum_{i=0}^{k-1} X_{M[m+k-2i]}X_{M} - \sum_{i=0}^{k-1} X_{M[m+k-1-2i]}$$
Example 3.4. Consider $r=3$, we have

$$X_{E_1} = \sum_{i=0}^{k-1} (X_M[m+k+1-2i] + X_M[m+k-1-2i]) - \sum_{i=0}^{k+1} X_M[m+k+1-2i].$$

By Proposition 3.1 and Proposition 3.2, we can define $X_n\delta := X_M[n]$ for $n \in \mathbb{N}$.

Now consider the non-homogeneous tubes $T(k)$ with rank $r_k$ for $1 \leq k \leq m$. The corresponding regular simple modules are $E_{k,1}, \cdots, E_{k,r_k}$ with $T_{E_{k,i+1}} = E_{k,i}$. In fact $m = 2$ or $3$ in our conditions. If we restrict the discussion to one tube, we will omit the index $k$ for convenience. Set $q.soc(E_{k,i}[n]) = E_{k,i}$ and $X_n\delta_{k,i} = X_{E_{k,i}[n]}$ for $n \in \mathbb{N}$.

**Proposition 3.3.** Let $E_i$ and $E_j$ be two regular simples in a non-homogeneous tube with rank $r$. Then we have

$$X_{E_i[mr]} = X_{E_j[mr]} + X_{E_{i+1}[mr-2]} - X_{E_{i+1}[mr-2]}$$

where $1 \leq i < j \leq r$ and $m \in \mathbb{N}$.

**Proof.** It is easy to prove that

$$X_{E_i[mr]}X_{E_{i+1}[mr-1]} = X_{E_i[mr]} + X_{E_{i+2}[mr-2]},$$

$$X_{E_{i+1}[mr-1]}X_{E_i} = X_{E_{i+1}[mr]} + X_{E_{i+1}[mr-2]}. $$

Hence, we have

$$X_{E_i[mr]} = X_{E_{i+1}[mr]} + X_{E_{i+1}[mr-2]} - X_{E_{i+2}[mr-2]}.$$ 

Similarly we have

$$X_{E_{i+1}[mr]} = X_{E_{i+2}[mr]} + X_{E_{i+2}[mr-2]} - X_{E_{i+3}[mr-2]},$$

$$X_{E_{i+2}[mr]} = X_{E_{i+3}[mr]} + X_{E_{i+3}[mr-2]} - X_{E_{i+4}[mr-2]},$$

$$\vdots$$

$$X_{E_{j-1}[mr]} = X_{E_j[mr]} + X_{E_j[mr-2]} - X_{E_{j+1}[mr-2]}.$$ 

Thus

$$X_{E_i[mr]} + X_{E_{i+1}[mr]} + \cdots + X_{E_{j-1}[mr]}$$

$$= (X_{E_{i+1}[mr]} + X_{E_{i+2}[mr-2]} - X_{E_{i+2}[mr-2]} + X_{E_{i+2}[mr]} + X_{E_{i+2}[mr-2]} - X_{E_{i+3}[mr-2]})$$

$$\vdots$$

$$= X_{E_{i+1}[mr]} + X_{E_{i+2}[mr]} + \cdots + X_{E_{j}[mr]} + X_{E_{j+1}[mr-2]}.$$ 

Therefore

$$X_{E_i[mr]} = X_{E_j[mr]} + X_{E_{j+1}[mr-2]} - X_{E_{j+1}[mr-2]}.$$ 

**Example 3.4.** Consider $r=3$, we have

$$X_{E_1}X_{E_2}X_{E_3} = (X_{E_1[3]} + 1)X_{E_3} = X_{E_1[3]} + X_{E_1} + X_{E_3}.$$ 

Similarly we have

$$X_{E_2}X_{E_3}X_{E_1} = X_{E_2[3]} + X_{E_2} + X_{E_1},$$

and

$$X_{E_3}X_{E_2}X_{E_1} = X_{E_2[3]} + X_{E_3} + X_{E_2}.$$ 

Therefore

$$X_{E_1[3]} = X_{E_2[3]} + X_{E_2} - X_{E_3} = X_{E_3[3]} + X_{E_2} - X_{E_1}.$$
Proposition 3.5. \( X_{nδ,1} = X_{nδ,1} + \sum_{\text{dim}L < nδ} a_L X_L \) for \( i \neq j \) in different nonhomogeneous tubes where \( a_L \in \mathbb{Q} \).

Proof. Denote \( δ = (v_1, v_2, \ldots, v_n) \) and \( \text{dim}(E_{i,j}) = (v_{j1}, v_{j2}, \ldots, v_{jn}) \), then \( δ = \sum_{1 \leq j \leq r} \text{dim}(E_{i,j}) \). Thus by the cluster multiplication theorem in Theorem 1.2 and the fact that for any dimension vector there is at most one exceptional module up to isomorphism, we have

\[
X_{S_1}^{v_1} X_{S_2}^{v_2} \cdots X_{S_n}^{v_n} = (X_{S_1}^{v_1} X_{S_2}^{v_2} \cdots X_{S_n}^{v_n}) \cdots (X_{S_1}^{v_1} X_{S_2}^{v_2} \cdots X_{S_n}^{v_n}) = (a_1 X_{E_{i,1}} + \sum_{\text{dim}L < \text{dim}E_{i,1}} a_L X_L) \cdots (a_r X_{E_{i,r}} + \sum_{\text{dim}L < \text{dim}E_{i,r}} a_L X_L)
\]

Thus we have

\[
a_1 \cdots a_r X_{δ,1} = a_1 \cdots a_r X_{δ,1} + \sum_{\text{dim}M < \text{dim}X_M} a_M X_M.
\]

Similarly

\[
X_{S_1}^{v_1} X_{S_2}^{v_2} \cdots X_{S_n}^{v_n} = (b_1 X_{E_{j,1}} + \sum_{\text{dim}T < \text{dim}E_{j,1}} b_T X_T) \cdots (b_r X_{E_{j,r}} + \sum_{\text{dim}T < \text{dim}E_{j,r}} b_T X_T)
\]

Thus we have

\[
a_1 \cdots a_n X_{δ,1} = b_1 \cdots b_n X_{δ,1} + \sum_{\text{dim}N < \text{dim}N'} b_N X_{N'} - \sum_{\text{dim}M < \text{dim}M'} a_M X_M.
\]

Therefore by Proposition 2.5 and the denominator theorem in Theorem 2.3, we have:

\[
X_{δ,1} = X_{δ,1} + \sum_{\text{dim}N < \text{dim}N'} b_N X_{N'}.
\]

Now, suppose the proposition holds for \( k \leq n \), then on the one hand

\[
X_{nδ,1} X_{δ,1} = X_{(n+1)δ,1} + \sum_{\text{dim}L' < (n+1)δ} b_L X_L'.
\]

On the other hand by induction

\[
X_{nδ,1} X_{δ,1} = (X_{nδ,1} + \sum_{\text{dim}L < nδ} a_L X_L) X_{δ,1}
\]

Therefore

\[
X_{(n+1)δ,1} = X_{(n+1)δ,1} + \sum_{\text{dim}L' < (n+1)δ} b_L X_L'.
\]

Thus the proof is finished. \( \square \)

Proposition 3.6. \( X_{nδ} = X_{nδ,1} + \sum_{\text{dim}L < nδ} a_L X_L \), where \( a_L \in \mathbb{Q} \).
A \textit{Z}-Basis for the Cluster Algebra Associated to an Affine Quiver

Proof. Suppose \( Q \) be of type \( \tilde{D} \) or \( \tilde{E} \) and \( \delta = (v_1, v_2, \cdots, v_n) \), using the same method in Proposition 3.5 we have

\[
X_{S_1}^{v_1} X_{S_2}^{v_2} \cdots X_{S_n}^{v_n} = (a_1 X_{E_{i,1}} + \sum_{\dim E' < \dim E_{i,1}} a_{L'} X_{E'} + a_{L'} X_{E'} + \cdots + a_{L''} X_{E''})
\]

\[
= a_1 \cdots a_r X_{\delta_{i,1}} + \sum_{\dim M < \delta} a_M X_M.
\]

We note that there exists a submodule \( L_1 \) satisfying the following short exact sequence:

\[
0 \to L_1 \to \delta \to L_2 \to 0
\]

where \( L_1, L_2 \) are preprojective, preinjective modules respectively. Thus by Proposition 3.3 and Proposition 3.5 we have

\[
X_{S_1}^{v_1} X_{S_2}^{v_2} \cdots X_{S_n}^{v_n} = (b_1 X_{L_1} + \sum_{\dim L' < \dim L_1} b_{L'} X_{L'}(b_{2} X_{L_2} + \sum_{\dim L' < \dim L_2} b_{L'} X_{L'}))
\]

\[
= b_1 b_2 X_{\delta} + \sum_{k=1}^{m} b_{k,1} X_{\delta_{k,1}} + \sum_{\dim N < \delta} b_{N} X_{N}
\]

\[
= b_1 b_2 X_{\delta} + b X_{\delta_{i,1}} + \sum_{\dim N' < \delta} b_{N'} X_{N'}.
\]

Thus

\[
a_1 \cdots a_r X_{\delta_{i,1}} + \sum_{\dim M < \delta} a_M X_M = b_1 b_2 X_{\delta} + b X_{\delta_{i,1}} + \sum_{\dim N' < \delta} b_{N'} X_{N'}.
\]

Therefore by Proposition 2.3 and the denominator theorem in [12], we have

\[
X_{\delta} = X_{\delta_{i,1}} + \sum_{\dim M' < \delta} a_{M'} X_{M'}.
\]

Then we can finish the proof by induction as in the proof of Proposition 3.5.

Now we assume \( Q \) is of the form \( \tilde{A} \). We give an alternative proof of the difference property in [10]. Let \( Q \) be a quiver as follows [7]:

\[
\begin{array}{c}
\text{c1} \quad \cdots \quad \text{cp} \\
\text{a} \quad \rightarrow \\
\text{d1} \quad \cdots \quad \text{dq} \\
\text{b} \quad \leftarrow \\
\end{array}
\]

Let \( \lambda \in \mathbb{C}^* \) and \( M(\lambda) \) be the regular simple module as follows:

\[
\begin{array}{c}
\text{C} \quad \cdots \quad \text{C} \\
\text{1} \quad \downarrow \quad \lambda \\
\text{C} \quad \leftarrow \\
\end{array}
\]
Its proper submodules $M_0(\lambda)$ are of the forms as follows:
\[ C \rightarrow \cdots \rightarrow C \]
\[ \lambda \downarrow \quad \downarrow \quad \quad \quad \quad \quad \quad \downarrow \quad \lambda \quad \downarrow \]
\[ C \rightarrow \cdots \rightarrow C \]

Let $M(0)$ be the regular module as follows
\[ C \rightarrow \cdots \rightarrow C \]
\[ \lambda \downarrow \quad \downarrow \quad \quad \quad \quad \quad \downarrow \quad \lambda \quad \downarrow \]
\[ C \rightarrow \cdots \rightarrow C \]

Its proper submodules $M_0(0)$ are of the following two forms:
\[ C \rightarrow \cdots \rightarrow C \]
\[ \lambda \downarrow \quad \downarrow \quad \quad \quad \quad \quad \downarrow \quad \lambda \quad \downarrow \]
\[ C \rightarrow \cdots \rightarrow C \]

and
\[ 0 \rightarrow \cdots \rightarrow 0 \]
\[ \lambda \downarrow \quad \downarrow \quad \quad \quad \quad \quad \downarrow \quad \lambda \quad \downarrow \]
\[ 0 \rightarrow \cdots \rightarrow 0 \]

The proper submodules with the second form lie in the non-homogeneous tube indexed by 0 with quasi socle $S_{d_q}$. Hence, there is a bijection between the submodules of $M(\lambda)$ and the submodules of $M(0)$ of the first form. It is easy to conclude the difference property in [10]:
\[ X_{M(0)} = X_{M(\lambda)} + X_{q.radM(0)/S_{d_q}}. \]

\[ \square \]

**Proposition 3.7.** If $\dim(T_1 \oplus R_1) = \dim(T_2 \oplus R_2)$ where $R_i$ are 0 or any regular exceptional modules, $T_i$ are 0 or any indecomposable regular modules with self-extension in non-homogeneous tubes and there are no extension between $R_i$ and $T_i$, then
\[ X_{T_1 \oplus R_1} = X_{T_2 \oplus R_2} + \sum_{\dim R < \dim(T_2 \oplus R_2)} a_R X_R \]
where $a_R \in \mathbb{Q}$.  

Proof. Suppose \( \dim(T_1 \oplus R_1) = (d_1, d_2, \cdots, d_n) \), using the same method in Proposition 3.3, we have
\[
X_{S_1}^{d_1} X_{S_2}^{d_2} \cdots X_{S_n}^{d_n} = (a_1 X_{E_{i_1}, i} + \sum_{dim L' < dim E_{i_1}} a_{L'} X_{L'}) \cdots (a_s X_{E_{i_s}, i} + \sum_{dim L'' < dim E_{i_s}} a_{L''} X_{L''})
\times (a_{R_1} X_{R_1} + \sum_{dim L'' < dim R_1} a_{L''} X_{L''})
= aX_{T_1 \oplus R_1} + \sum_{dim L < dim(T_1 \oplus R_1)} a_L X_L.
\]
\[
X_{S_1}^{d_1} X_{S_2}^{d_2} \cdots X_{S_n}^{d_n} = (b_1 X_{E_{i_1}, i} + \sum_{dim M' < dim E_{i_1}} b_{M'} X_{M'}) \cdots (b_s X_{E_{i_s}, i} + \sum_{dim M'' < dim E_{i_s}} b_{M''} X_{M''})
\times (b_{R_2} X_{R_2} + \sum_{dim M'' < dim R_2} b_{M''} X_{M''})
= bX_{T_2 \oplus R_2} + \sum_{dim L < dim(T_2 \oplus R_2)} b_M X_M.
\]
Thus
\[
aX_{T_1 \oplus R_1} + \sum_{dim L < dim(T_1 \oplus R_1)} a_L X_L = bX_{T_2 \oplus R_2} + \sum_{dim L < dim(T_2 \oplus R_2)} b_M X_M.
\]
Therefore by Proposition 2.5 and the denominator theorem in Theorem 2.3, we have
\[
X_{T_1 \oplus R_1} = X_{T_2 \oplus R_2} + \sum_{dim R < dim(T_2 \oplus R_2)} a_R X_R.
\]

4. A \( \mathbb{Z} \)-basis for the cluster algebra of the alternating quiver of \( A_{n,n}, D \) or \( E \)

Recall that for an acyclic quiver, the matrix \( B \) associated to \( Q \) is the antisymmetric matrix given by
\[
b_{ij} = |i \rightarrow j \in Q_1| - |j \rightarrow i \in Q_1|
\]
where \( 1 \leq i, j \leq n \).

**Definition 4.1.** [4][9] Let \( Q \) be an acyclic quiver with associated matrix \( B \). \( Q \) is called graded if there exists a linear form \( \epsilon \) on \( \mathbb{Z}^n \) such that \( \epsilon(B\alpha_i) < 0 \) for any \( 1 \leq i \leq n \) where \( \alpha_i \) denotes the \( i \)-th vector of the canonical basis of \( \mathbb{Z}^n \).

**Theorem 4.2.** [4] Let \( Q \) be a graded quiver and \( \{ M_1, \cdots, M_r \} \) a family objects in \( \mathcal{C}(Q) \) such that \( \dim(M_i) \neq \dim(M_j) \) for \( i \neq j \), then \( X_{M_1}, \cdots, X_{M_r} \) are linearly independent over \( Q \).

In the section, we still suppose that \( Q \) is one of \( \tilde{A}_{n,n}, \tilde{D} \) and \( \tilde{E} \) with an orientation where every vertex is a sink or a source. Note that the quiver \( Q \) we consider is graded. We will prove the following theorem.

**Theorem 4.3.** A \( \mathbb{Z} \)-basis for \( A_{\mathbb{Z}}(Q) \) is the following set denoted by \( \mathcal{S}(Q) \):
\[
\{ X_L, X_{T \oplus R} | \dim(T_1 \oplus R_1) \neq \dim(T_2 \oplus R_2), \operatorname{Ext}^1_{\mathcal{C}(Q)}(T, R) = 0, \operatorname{Ext}^1_{\mathcal{C}(Q)}(L, L) = 0 \}
\]
where \( L \) is any non-regular exceptional object, \( R \) is 0 or any regular exceptional module and \( T \) is 0 or any indecomposable regular module with self-extension.
Proof. Note that for any objects \( M, N \in \mathcal{C}(Q) \), the final results of their multiplication \( X_M X_N \) must be \( \mathbb{Q} \)-combinatorics of \[
\{ X_L, X_T \oplus R | \text{Ext}_{\mathcal{C}(Q)}^1(T, R) = 0, \text{Ext}_{\mathcal{C}(Q)}^1(L, L) = 0 \}
\]
where \( L \) is any non-regular exceptional object, \( R \) is 0 or any regular exceptional module and \( T \) is 0 or any indecomposable regular module with self-extension.

Hence from those Propositions in Section 3, we can easily find that \( X_M X_N \) is a \( \mathbb{Q} \)-combinatorics of \( S(Q) \). Then the proof is finished by the following Proposition 4.7 and Proposition 4.8.

Remark 4.4. By Proposition 4.7 in the following, we can also rewrite the \( \mathbb{Z} \)-basis as \( \{ X_M(d_1, \ldots, d_n) : (d_1, \ldots, d_n) \in \mathbb{Z}^n \} \), where \( M(d_1, \ldots, d_n) = L \) or \( T \oplus R \) associated to the element in \( S(Q) \).

Corollary 4.5. \( S(Q) \) is a \( \mathbb{Z} \)-basis of the cluster algebra \( \mathcal{E}H(Q) \).

Proof. It is obvious that \( X_{E_i} \in \mathcal{E}H(Q) \). Then by Proposition 6.2 in [9], one can show that \( X_{E_i[n]} \in \mathcal{E}H(Q) \) for any \( n \in \mathbb{N} \). Thus by Proposition 4.6, we can prove that \( X_{m_\delta \in \mathcal{E}H(Q)} \). By definition, \( X_L \in \mathcal{E}H(Q) \) for \( L \) satisfying \( \text{Ext}_{\mathcal{C}(Q)}^1(L, L) = 0 \). Thus \( S(Q) \subset \mathcal{E}H(Q) \). It follows that \( S(Q) \) is a \( \mathbb{Z} \)-basis of the cluster algebra \( \mathcal{E}H(Q) \) by Theorem 4.3.

According to Theorem 4.3 and Corollary 4.5, we have Corollary 4.5. \( \mathcal{E}H(Q) = \mathcal{A}H(Q) \).

Firstly, by Theorem 4.2 we need to prove the dimension vectors of these objects associated to the corresponding elements in \( S(Q) \) are different.

Proposition 4.7. Let \( M \) be a regular module associated to some element in \( S(Q) \) and \( L \) be a non-regular exceptional object in \( \mathcal{C}(Q) \). Then \( \dim(M) \neq \dim(L) \).

Proof. If \( L \) contains some \( \tau P_i \) as its direct summand, we know that
\[
\dim(\tau P_i) = (0, \ldots, 0, -1, 0, \cdots, 0)
\]
where the \( i \)-th component is \(-1\). Suppose \( L = \tau P_i \oplus \tau P_i \cdots \tau P_i \oplus N \) where \( N \) is an exceptional module. Because \( L \) is an exceptional object, \( X_{\tau P_i} X_N = X\tau P_i \oplus N \) i.e. \( \dim_\mathcal{C} \text{Hom}(P_i, N) = 0 \). Thus we have \( \dim_\mathcal{C} N(i) = 0 \) and \( \dim_\mathcal{C} (\tau P_i \oplus \tau P_i \cdots \tau P_i \oplus N)(i) \leq -1 \). However, \( \dim(M) \geq 0 \). Therefore, \( \dim(M) \neq \dim(L) \).

If \( L \) is a module. Suppose \( \dim(M) = \dim(L) \). Because \( L \) is an exceptional module, we know that \( M \) belongs to the orbit of \( L \) and then \( M \) is a degeneration of \( L \). Hence, there exists some \( \mathbb{C}Q \)-module \( U \) such that
\[
0 \rightarrow M \rightarrow L \oplus U \rightarrow U \rightarrow 0
\]
is an exact sequence. Choose minimal \( U \) so that we cannot separate the following exact sequence
\[
0 \rightarrow 0 \rightarrow U_1 \rightarrow U_1 \rightarrow 0
\]
from the above short exact sequence. Thus \( M \) has a non-zero map to every direct summand of \( L \). Therefore \( L \) has no preprojective modules as direct summand because \( M \) is a regular module.

Dually there exists a \( \mathbb{C}Q \)-module \( V \) such that
\[
0 \rightarrow V \rightarrow V \oplus L \rightarrow M \rightarrow 0
\]
is an exact sequence. We can choose minimal \( V \) so that one cannot separate the following exact sequence
\[
0 \rightarrow V_1 \rightarrow V_1 \rightarrow 0 \rightarrow 0
\]
from the above short exact sequence. Thus every direct summand of \( L \) has a non-zero map to \( M \). Therefore \( L \) has no preinjective modules as direct summand because \( M \) is a regular module.

Therefore \( L \) is a regular exceptional module, it is a contradiction. \( \square \)

Secondly, we need to prove that \( S(Q) \) is a \( \mathbb{Z} \)-basis.

**Proposition 4.8.** \( X_M X_N \) belongs to \( \mathbb{Z}S(Q) \) for any \( M, N \in C(Q) \).

**Proof.** According to these above discussions, for any objects \( M, N \in C(Q) \), we have

\[
X_M X_N = b_L X_L + \sum_{\dim(L) < \dim(M \oplus N)} b_L' X_{L'}
\]

where \( \dim(L) = \dim(M \oplus N) \), \( X_L, X_{L'} \in S(Q) \) and \( b_L, b_L' \in \mathbb{Q} \).

Therefore by Proposition 4.3 and the denominator theorem in Theorem 2.3, we have \( b_L = 1 \). Note that there exists a partial order on these dimension vectors by Definition 2.1. Thus in these remained \( L' \), we choose these maximal elements denoted by \( L_1', \ldots, L_s' \). Then by \( b_L = 1 \) and the coefficients of Laurent expansions in generalized cluster variables are integers, we obtain that \( a_{L_1'}, \ldots, a_{L_s'} \) are integers. Using the same method, we have \( b_{L'} \in \mathbb{Z} \).

We denote \( X^{-d}_{\tilde{S}_i} = X^d_{\tau P_i} \) for \( d \in \mathbb{N} \). Then we have the following result.

**Proposition 4.9.**

\[
X^d_{S_1} X^d_{S_2} \cdots X^d_{S_n} = X_M(d_1, \ldots, d_n) + \sum_{\dim(M) < (d_1, \ldots, d_n)} b_L X_{L}
\]

where \( X_M(d_1, \ldots, d_n) \), \( X_L \in S(Q) \), \( \dim M = (d_1, \ldots, d_n) \in \mathbb{Z}^n \) and \( b_L \in \mathbb{Z} \).

**Proof.** By Proposition 2.3, Theorem 2.3 and Theorem 4.3 \( \square \)

Note that \( \{X_M(d_1, \ldots, d_n) : (d_1, \ldots, d_n) \in \mathbb{Z}^n \} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{AH}(Q) \), then we have the following Corollary 4.10 by Proposition 4.9:

**Corollary 4.10.** \( \{X^d_{S_1} X^d_{S_2} \cdots X^d_{S_n} : (d_1, \ldots, d_n) \in \mathbb{Z}^n \} \) is a \( \mathbb{Z} \)-basis of \( \mathcal{AH}(Q) \), further, there is an isomorphism of \( \mathbb{Z} \)-algebras:

\[
\mathcal{AH}(Q) \simeq \mathbb{Z}[X_{S_1}, \cdots, X_{S_n}, X_{\tau P_1}, \cdots, X_{\tau P_n}]\]

**Remark 4.11.** In \([9]\), G. Dupont has proved the following isomorphism of \( \mathbb{Z} \)-algebras:

\[
\overline{\mathcal{A}(\tilde{A}_n)} \simeq \mathbb{Z}[X_{S_1}, \cdots, X_{S_n}, X_{\tau P_1}, \cdots, X_{\tau P_n}]\]

where \( n > 0 \).

### 5. A \( \mathbb{Z} \)-basis for the cluster algebra of type \( \tilde{A}_{n,n}, \tilde{D} \) or \( \tilde{E} \)

In this section, we will deduce a \( \mathbb{Z} \)-basis for the cluster algebra of \( \tilde{A}_{n,n}, \tilde{D} \) or \( \tilde{E} \) which is independent of their orientations.

Define the reflected quiver \( \sigma_i(Q) \) by reversing all the arrows ending at \( i \). The mutations can be viewed as generalizations of reflections i.e. if \( i \) is a sink or a source in \( Q_0 \), then \( \mu_i(Q) = \sigma_i(Q) \) where \( \mu_i \) denotes the mutation in the direction \( i \). Thus there is a natural isomorphism of cluster algebras

\[
\Phi : \mathcal{A}(Q) \longrightarrow \mathcal{A}(Q')
\]

where \( Q' \) is a quiver mutation equivalent to \( Q \), and \( \Phi \) is called the canonical cluster algebras isomorphism in \([10]\).

Now suppose \( Q \) is an acyclic quiver and \( i \) is a sink in \( Q_0 \). Let \( Q' = \sigma_i(Q) \) and \( R_{i}^+ : \mathcal{C}(Q) \longrightarrow \mathcal{C}(Q') \) be the extended BGP-reflection functor defined in \([17]\).
Denote by $X^Q_i$ (resp. by $X^\sigma_i Q$) the Caldero-Chapoton map associated to $Q$ (resp. to $\sigma_i Q$).

Then the following hold.

**Lemma 5.1.** [17] Let $Q$ be an acyclic quiver and $i$ be a sink in $Q$. Then $R^+_i$ induces a triangle equivalence

$$R^+_i : C(Q) \longrightarrow C(\sigma_i Q)$$

**Lemma 5.2.** [10] Let $Q$ be an affine quiver and $i$ be a sink in $Q$ and $M$ be an indecomposable regular $CQ$-module with quasi-composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_s = M$$

then $R^+_i M$ is a regular $C\sigma_i Q$-module with quasi-composition series

$$0 = R^+_i M_0 \subset R^+_i M_1 \subset \cdots \subset R^+_i M_s = R^+_i M$$

In particular, $R^+_i$ send quasi-socles to quasi-socles, quasi-radicals to quasi-radicals and preserves quasi-lengths.

**Lemma 5.3.** [10] Let $Q$ be an affine quiver and $i$ be a sink in $Q$. Denote by $\Phi_i : A(Q) \longrightarrow A(\sigma_i Q)$ the canonical cluster algebra isomorphism and by $R_i^+ : C(Q) \longrightarrow C(\sigma_i Q)$ the extended BGP-reflection functor. Then

$$\Phi_i(X^Q_M) = X^{\sigma_i Q}_{R^+_i M}$$

where $M$ is any rigid object in $C(Q)$ or any regular module in non-homogeneous tubes of rank $r > 1$.

In the following, we will suppose that $Q$ is one of $\tilde{A}_{n,n}, \tilde{D}$ and $\tilde{E}$ with an orientation where every vertex is a sink or a source, and $Q'$ is an acyclic quiver of $\tilde{A}_{n,n}, \tilde{D}$ or $\tilde{E}$. Then there exists an admissible sequence of sinks $(i_1, \cdots, i_s)$ such that $Q' = \sigma_{i_s} \cdots \sigma_{i_1}Q$. Denote by $\Phi : A(Q) \longrightarrow A(Q')$ the canonical cluster algebra isomorphism and by $R^+ = R^+_s \cdots R^+_1 : C(Q) \longrightarrow C(Q')$. Note that $R^+$ is an equivalence of triangulated categories by Lemma 5.1. Then we have the following main theorem.

**Theorem 5.4.** A $Z$-basis for the cluster algebra of $Q'$ is the following set denoted by $B(Q')$:

$$\{X_L, X_{T \oplus R} | \dim(T_i \oplus R_1') \neq \dim(T_2 \oplus R_2'), \text{Ext}^1_{C(Q')}(T', R') = 0, \text{Ext}^1_{C(Q')}(L', L') = 0\}$$

where $L'$ is any non-regular exceptional object, $R'$ is any regular exceptional module, $T'$ is any regular indecomposable module with self-extension and in addition if $Q' \neq \tilde{A}_{1,1}$ and $\dim(T' \oplus R') = m\delta$ for some $m \in \mathbb{N}$, then $R' = 0$ and $T'$ is an indecomposable module of dimension vector $m\delta$ in a non-homogeneous tube.

**Proof.** If $Q' = \tilde{A}_{1,1}$, it is obvious that

$$\{X_M, X_{n\delta} : M \in \text{obj}C(Q), \text{Ext}^1(M, M) = 0\}$$

is a $Z$-basis for cluster algebra of $\tilde{A}_{1,1}$, which is called the semicanonical basis in [6].

If $Q' \neq \tilde{A}_{1,1}$, then by Theorem 4.3 we have obtained a $Z$-basis for cluster algebra of $Q$:

$$\{X_L, X_{T \oplus R} | \dim(T_i \oplus R_1) \neq \dim(T_2 \oplus R_2), \text{Ext}^1_{C(Q)}(T, R) = 0, \text{Ext}^1_{C(Q)}(L, L) = 0\}$$

where $L$ is any non-regular exceptional object, $R$ is any regular exceptional module and $T$ is any indecomposable regular module with self-extension.

If $\dim(T \oplus R) = m\delta$ for some $m \in \mathbb{N}$, then we take $R = 0$ and $T$ an indecomposable module of dimension vector $m\delta$ in a non-homogeneous tube. We denote
this $\mathbb{Z}$-basis by $\mathcal{B}(Q)$. Thus $\Phi(\mathcal{B}(Q))$ is a $\mathbb{Z}$-basis for the cluster algebra of $Q'$ because $\Phi : \mathcal{A}(Q) \to \mathcal{A}(Q')$ is the canonical cluster algebras isomorphism. Then by Lemma 5.1, Lemma 5.2 and Lemma 5.3 we know that $\Phi(\mathcal{B}(Q))$ is exactly the basis $\mathcal{B}(Q')$ in Theorem 5.4. □

6. SOME SPECIAL CASES

In this section, we will give some examples for special cases to explain the $\mathbb{Z}$-basis explicitly.

6.1. $\mathbb{Z}$-basis for finite type. For finite type, we know that there are no regular parts in $\mathcal{B}(Q)$. Thus the $\mathbb{Z}$-bases are exactly

$$\{X_M : M \in \text{obj}C(Q), \text{Ext}^1(M, M) = 0\}$$

which coincides with the canonical basis for a cluster algebra of finite type in [4].

6.2. $\mathbb{Z}$-basis for the Kronecker quiver. Consider the Kronecker quiver, we note that $R = 0$ and $T = n\delta$ for $n \geq 1$. Thus a $\mathbb{Z}$-basis is

$$\{X_M, X_{n\delta} : M \in \text{obj}C(Q), \text{Ext}^1(M, M) = 0\}$$

which is called the semicanonical basis in [6]. If we modify:

$$z_1 := X_\delta, \ z_n := X_{n\delta} - X_{(n-2)\delta}.$$  

Then $\{X_M, z_n : M \in \text{obj}C(Q), \text{Ext}^1(M, M) = 0\}$ is the canonical basis for cluster algebra of Kronecker quiver in [14].

6.3. $\mathbb{Z}$-basis for $\tilde{D}_4$. Let $Q$ be the tame quiver of type $\tilde{D}_4$ as follows

```
2
3 ----> 1 ----> 5
 \ | \\
 4
```

We denote the minimal imaginary root by $\delta = (2, 1, 1, 1, 1)$. The regular simple modules of dimension vector $\delta$ are

```
C
\alpha_1
C \alpha_2
C^2 \alpha_3 \alpha_4
C
```

with linear maps

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \alpha_4 = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}$$

where $\lambda/\mu \in \mathbb{P}^1, \lambda/\mu \neq 0, 1, \infty$. Let $M$ be any regular simple $\mathbb{C}Q$-module of dimension vector $\delta$ and $M[i]$ be the regular module with top $M$ and regular length $i$ for any $i \in \mathbb{N}$. Let $X_M$ be the generalized cluster variable associated to $M$ by the reformulation of the Caldero-Chapoton map. Then we have

**Proposition 6.1.** $X_M = \frac{1}{x_1x_2x_3x_4x_5} + \frac{4}{x_1x_2x_3x_4x_5} + \frac{x_1^2 + 4x_1 + 6}{x_2x_3x_4x_5} + \frac{x_2x_3x_4x_5 + 2}{x_1} + \frac{1}{x_1}.$
We define $X_{n\delta} := X_{M[n]}$ for $n \in \mathbb{N}$. Now, we consider three non-homogeneous tubes labelled by the subset $\{0, 1, \infty\}$ of $\mathbb{P}^1$. Let $X_{\delta_1}, X_{\delta_2}, X_{\delta_3}, X_{\delta_4}, X_{\delta_5}, X_{\delta_6}$ be generalized cluster variables associated to regular modules in non-homogeneous tubes of dimension vector $\delta$. The regular simple modules in non-homogeneous tubes are denoted by $E_1, E_2, E_3, E_4, E_5, E_6$, where

$$\dim(E_1) = (1, 1, 1, 0, 0), \quad \dim(E_2) = (1, 0, 0, 1, 1), \quad \dim(E_3) = (1, 1, 0, 1, 0),$$

$$\dim(E_4) = (1, 0, 1, 0, 1), \quad \dim(E_5) = (1, 0, 1, 1, 0), \quad \dim(E_6) = (1, 1, 0, 0, 1).$$

We note that $\{E_1, E_2\}, \{E_3, E_4\}$ and $\{E_5, E_6\}$ are pairs of the regular simple modules in the bottom of non-homogeneous tubes labelled by $1, \infty$ and $0$, respectively.

Let $E_i[n]$ be the indecomposable regular module with top $E_i$ and regular length $n$ for $1 \leq i \leq 6$. We set $X_{n\delta_i} := X_{E_i[2n]}$ for $1 \leq i \leq 6$.

**Proposition 6.2.** [8] For any $n \in \mathbb{N}$, we have $X_{n\delta_1} = X_{n\delta_2} = X_{n\delta_3} = X_{n\delta_4} = X_{n\delta_5} = X_{n\delta_6}$ and $X_{n\delta_1} = X_{n\delta} + X_{(n-1)\delta}$ where $X_0 = 1$.

**Theorem 6.3.** [8] A $\mathbb{Z}$-basis for cluster algebra of $\widetilde{D}_4$ is the following set denoted by $B(Q)$:

$$\{X_i, X_{n\delta}, X_{E_i[2k+1]\otimes R}, \dim(E_i[2k+1] \oplus R_1) \neq \dim(E_j[2l+1] \oplus R_2), \quad \text{Ext}^1_{\mathcal{O}(\mathbb{C})}(L, L) = 0, \text{Ext}^1_{\mathcal{O}(\mathbb{C})}(E_i[2k+1], R) = 0, \text{and} \quad m, k, l \geq 0, 1 \leq i, j \leq 6\}$$

where $L$ is any non-regular exceptional object, $R$ is $0$ or any regular exceptional module.

7. THE INDUCTIVE MULTIPLICATION FORMULA FOR A TUBE

Now we fix a tube with rank $r$ and these regular simple modules are $E_1, \ldots, E_r$ with $r E_{i+1} = E_i$ where $E_i = E_{i+mr}$ for $1 \leq i \leq r$ and $m \in \mathbb{N}$. Let $X_{E_i}$ be the corresponding generalized cluster variable for $i = 1, \ldots, r$. With these notations, we have the following inductive cluster multiplication formula.

**Theorem 7.1.** Let $i, j, k, l, m$ and $r$ be in $\mathbb{Z}$ such that $1 \leq k \leq mr + l, 0 \leq l \leq r - 1, 1 \leq i, j \leq r, m \geq 0$.

(1) When $j \leq i$, then

1) for $k + i \geq r + j$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[(m+1)r+l+j-1]}X_{E_j[k+i-r-j]} + X_{E_i[r+j-1]}X_{E_{k+i+1}(m+1)r+l+j-k-i-1)}$.

2) for $k + i < r + j$ and $i \leq k + i - 1$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[mr+k+i+j]}X_{E_j[t+j-i]} + X_{E_i[mr+i-j+1]}X_{E_{i+j+k+k-i-j-1}}$.

3) for other conditions, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[k]}X_{E_j[mr+l]}$.

(2) When $j > i$, then

1) for $k \geq j - i$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[j-i-1]}X_{E_{k+i+1}(mr+l-j-k-i-1)} + X_{E_i[mr+i-j-1]}X_{E_j[k+i-j]}$.

2) for $k < j - i$ and $i \leq j + r - k + i - 1$, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[(m+1)r+k+i-j]}X_{E_j[t+j-r-i]} + X_{E_i[(m+1)r+i-j]}X_{E_{i+j+k+k-i-j-1}}$.

3) for other conditions, we have $X_{E_i[k]}X_{E_j[mr+l]} = X_{E_i[k]}X_{E_j[mr+l]}$.

**Proof.** We only prove (1) and (2) is totally similar to (1).

1) When $k = 1$, by $k + i \geq r + j$ and $1 \leq i, j \leq r \Rightarrow i = r$ and $j = 1$.

Then by the cluster multiplication theorem in Theorem 1.2 or Theorem 1.4, we have

$$X_{E_i}X_{E_j[mr+l]} = X_{E_i[mr+l+1]} + X_{E_2[mr+l-1]}.$$
The case for $i = r$ and $j = 1$, we have
\[ X_{E_i[2]}X_{E_1[m+r]} = (X_{E_i}X_{E_1} - 1)X_{E_1[m+r]} \]
\[ = X_{E_1}(X_{E_i[m+r+1]} + X_{E_2[m+r+1]}) - X_{E_1[m+r]} \]
\[ = X_{E_1}X_{E_1[m+r+1]} + (X_{E_1[m+r]}X_{E_2[m+r+1]} - X_{E_1[m+r]}) \]
\[ = X_{E_1}X_{E_1[m+r+1]} + X_{E_3[m+r+1]} \]
The case for $i = r$ and $j = 2$, we have
\[ X_{E_i[2]}X_{E_2[m+r]} = (X_{E_i}X_{E_1} - 1)X_{E_2[m+r]} \]
\[ = X_{E_1}(X_{E_i[m+r+1]} + X_{E_3[m+r+1]}) - X_{E_2[m+r]} \]
\[ = X_{E_1}X_{E_1[m+r+1]} + (X_{E_1[m+r]}X_{E_2[m+r+1]} - X_{E_2[m+r]}) \]
\[ = X_{E_1}X_{E_1[m+r+1]} + X_{E_3[m+r+1]} \]

For $i = r - 1 \implies j = 1$,
\[ X_{E_{r-1}[2]}X_{E_1[m+r]} = (X_{E_{r-1}}X_{E_r} - 1)X_{E_1[m+r]} \]
\[ = X_{E_{r-1}}(X_{E_r[m+r+1]} + X_{E_2[m+r+1]}) - X_{E_1[m+r]} \]
\[ = (X_{E_{r-1}[m+r+2]}X_{E_1[m+r]} + X_{E_1[m+r]}X_{E_2[m+r+1]} - X_{E_1[m+r]}) \]
\[ = X_{E_{r-1}[m+r+2]} + X_{E_{r-1}}X_{E_2[m+r+1]} - X_{E_1[m+r]} \]

Now, suppose it holds for $k \leq n$, then by induction we have
\[ X_{E_i[n+1]}X_{E_j[m+r]} \]
\[ = (X_{E_i[n]}X_{E_{n+1}} - X_{E_i[n-1]}X_{E_j[m+r]}) \]
\[ = X_{E_{i+1}}(X_{E_i[n]}X_{E_j[m+r+1]} - X_{E_i[n-1]}X_{E_j[m+r]}) \]
\[ = X_{E_{i+1}}X_{E_i[n]}X_{E_j[m+r+1]} + X_{E_i[n-1]}X_{E_j[m+r]} \]
\[ = X_{E_{i+1}}X_{E_i[m+r+2]} + X_{E_i[m+r+1]}X_{E_j[m+r+1]} - X_{E_i[m+r]} \]

2) When $k = 1$, by $i \leq l + j \leq k + i - 1 \implies i \leq l + j \leq i \implies i = l + j$.

Then by the cluster multiplication theorem in Theorem 1.2 or Theorem 1.4, we have
\[ X_{E_i}X_{E_j[m+r]} = X_{E_i}[m+r] + X_{E_j}[m+r] \]

When $k = 2$, by $i \leq l + j \leq k + i - 1 \implies i \leq l + j \leq i + 1 \implies i = l + j$ or $i + 1 = l + j$.

For $i = l + j$, we have
\[ X_{E_i[2]}X_{E_j[m+r]} = X_{E_{i+1}[2]}X_{E_j[m+r]} = (X_{E_{i+1}}X_{E_{i+1}+1} - 1)X_{E_j[m+r]} \]
\[ = (X_{E_{i+1}[m+r+1]} + X_{E_{i+1}[m+r+1]})X_{E_{i+1}+1} - X_{E_{i+1}[m+r+1]} \]
\[ = X_{E_{i+1}[m+r+2]} + X_{E_{i+1}[m+r+1]} - X_{E_{i+1}[m+r+1]} \]
\[ = X_{E_{i+1}[m+r+2]} + X_{E_{i+1}+1}X_{E_j[m+r+1]} \]

For $i + 1 = l + j$, we have
\[ X_{E_i[2]}X_{E_j[m+r]} = X_{E_{i+1}+1[2]}X_{E_j[m+r]} = (X_{E_{i+1}+1}X_{E_{i+1}+1} - 1)X_{E_j[m+r]} \]
\[ = (X_{E_{i+1}[m+r+1]} + X_{E_{i+1}[m+r+1]})X_{E_{i+1}+1} - X_{E_{i+1}[m+r+1]} \]
\[ = X_{E_{i+1}[m+r+2]}X_{E_{i+1}+1} + X_{E_{i+1}[m+r+1]}X_{E_{i+1}+1} - X_{E_{i+1}[m+r+1]} \]
\[ = X_{E_{i+1}[m+r+2]} + X_{E_{i+1}+1}X_{E_j[m+r+1]} \]
Suppose it holds for $k \leq n$, then by induction we have

$$X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}}) X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} - X_{E_{[n-1]}}) X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} - X_{E_{[n-1]}}) X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} - X_{E_{[n-1]}}) X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} - X_{E_{[n-1]}}) X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} - X_{E_{[n-1]}}) X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} - X_{E_{[n-1]}}) X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} - X_{E_{[n-1]}}) X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} - X_{E_{[n-1]}}) X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} - X_{E_{[n-1]}}) X_{E_{[mr+l]}}$$

$$= (X_{E_i[n]} X_{E_{i+1}} X_{E_{i+2}} \cdots X_{E_{[n+1]}} - X_{E_{[n-1]}}) X_{E_{[mr+l]}}$$

3) It is trivial by the definition of the Caldero-Chapton map.

Note that in the above section, we have already obtained a $\mathbb{Z}$-basis for the cluster algebra of $\tilde{A}_{n,n} \tilde{D}$ or $\tilde{E}$. Now by using Theorem 7.1 and Proposition 7.2, we can easily express $X_{E_i[m]} X_{E_j[n]}$ as a $\mathbb{Z}$-combinatorics of the basis for the cluster algebra of $\tilde{A}_{n,n} \tilde{D}$ or $\tilde{E}$ where $E_i[m]$ and $E_j[n]$ are regular modules in one fixed tube. In the following section, we will explain it by an example.

8. AN EXAMPLE

We consider a tube with rank 3. By Theorem 7.1, we can easily obtain the following proposition.

**Proposition 8.1.** (1) For $n \geq 3m + 1$, then

$$X_{E_2[3m+1]} X_{E_1[n]} = X_{E_2} X_{E_1[n]} + X_{E_2} X_{E_1[n+3m-3]} + X_{E_2} X_{E_1[n+3m-6]} + X_{E_1[n+3m-9]}$$

$$+ \cdots + X_{E_2} X_{E_1[n-3m+6]} + X_{E_1[n-3m+3]} + X_{E_2} X_{E_1[n-3m]}$$

(2) For $n \geq 3m + 2$, then

$$X_{E_2[3m+2]} X_{E_1[n]} = X_{E_2[n+3m+2]} X_{E_2[n+3m+1]} X_{E_2[n+3m-4]} X_{E_2[n+3m-7]}$$

$$+ \cdots + X_{E_2[n-3m+2]} X_{E_2[n-3m-1]}$$

(3) For $n \geq 3m + 3$, then

$$X_{E_2[3m+3]} X_{E_1[n]} = X_{E_2[n+3m+3]} X_{E_2[n+3m+1]} X_{E_2[n+3m-1]} X_{E_1[n+3m-3]}$$

$$+ X_{E_2[n+3m-5]} X_{E_2[n+3m-7]} + \cdots + X_{E_1[n-3m+3]}$$

$$X_{E_1[n-3m+1]} X_{E_1[n-3m-1]} + X_{E_1[n-3m-3]}$$

Proof. (1) By Theorem 7.1, then

$$X_{E_2[3m+1]} X_{E_1[n]}$$

$$= X_{E_2[n+2]} X_{E_1[n+1]} + X_{E_2} X_{E_1[n-3m]}$$

$$= X_{E_2[3m+2]} X_{E_1[n+3]} + X_{E_1[n-3m+3]} + X_{E_2} X_{E_1[n-3m]}$$

$$= X_{E_2[n+3m+3]} X_{E_1[n+3m-3]} + X_{E_2[n+3m-6]} + X_{E_1[n+3m-9]}$$

$$+ \cdots + X_{E_2} X_{E_1[n-3m+6]} + X_{E_1[n-3m+3]} + X_{E_2} X_{E_1[n-3m]}$$

(2) By Theorem 7.1, then

$$X_{E_2[3m+2]} X_{E_1[n]}$$

$$= X_{E_2[n+2]} X_{E_1[3m+3]} + X_{E_2} X_{E_2[3m+3]}$$

$$= X_{E_2[n+2]} X_{E_1[3m+3]} + X_{E_2} X_{E_2[3m+3]}$$
\(X_{E_2[3m-1]}X_{E_1[n+3]} + X_{E_2[n-3m+2]} + X_{E_2X_{E_2[n-3m-1]}}\)
\(= X_{E_2[n+5]}X_{E_1[3m-3]} + X_{E_2X_{E_2[n-3m+5]} + X_{E_2[2n-3m+2]} + X_{E_2X_{E_2[n-3m-1]}}\)
\(= X_{E_2[3m-4]}X_{E_1[n+6]} + X_{E_2X_{E_2[n-3m+8]} + X_{E_2X_{E_2[n-3m+5]} + X_{E_2[2n-3m+2]} + X_{E_2X_{E_2[n-3m-1]}}\}
\[
\vdots
\]
\(= X_{E_2[n+3m+2]} + X_{E_2X_{E_2[n+3m-1]} + X_{E_2[2n-3m+4]} + X_{E_2X_{E_2[n+3m-7]} + \cdots + X_{E_2[n+3m-2]} + X_{E_2X_{E_2[n+3m-1]}\}
\]

(3) By Theorem 7.1 then
\[X_{E_2[3m+3]}X_{E_1[n]}\]
\[= X_{E_2[n+2]}X_{E_1[3m+1]} + X_{E_2X_{E_2[n-3m-2]} + X_{E_2X_1[3m+1]} + X_{E_2X_2[n-3m-1]} + X_{E_2X_4[n-3m-3]} \]
\[= X_{E_2[3m]}X_{E_1[n+3]} + X_{E_3X_{E_3[n+3m+1]} + X_{E_2X_2[n+3m-1]} + X_{E_4[n+3m-3]} \]
\[
\vdots
\]
\[= X_{E_1[n+3m+3]} + X_{E_2X_{E_2[n+3m-1]} + X_{E_2X_4[n+3m-1]} + X_{E_4X_4[n+3m-3]} + X_{E_4X_4[n+3m-3]} \]

Corollary 8.2. When \(n = 3k + 1\), we can rewrite Proposition 8.1 as following:

(1) For \(n \geq 3m + 1\), then
\[X_{E_2[3m+1]}X_{E_1[n]} = X_{E_1[n+3m+1]} + X_{E_1[n+3m-1]} + X_{E_1[n+3m-3]} + \cdots + X_{E_1[n-3m+5]} + X_{E_1[n-3m+3]} + X_{E_1[n-3m+1]} + X_{E_1[n-3m-1]} \]

(2) For \(n \geq 3m + 2\), then
\[X_{E_2[3m+2]}X_{E_1[n]} = X_{E_2[n+3m+2]} + X_{E_2[n+3m]} + X_{E_2X_2[n+3m-2]} + X_{E_2X_2[n+3m-4]} + \cdots + X_{E_2[n+3m-2]} + X_{E_2X_2[n+3m-1]} + X_{E_2X_2[n+3m-1]} \]

(3) For \(n \geq 3m + 3\), then
\[X_{E_2[3m+3]}X_{E_1[n]} = X_{E_1[n+3m+3]} + X_{E_1[n+3m+1]} + X_{E_2X_2[n+3m-1]} + X_{E_1[n+3m-3]} + X_{E_1[n+3m-5]} + X_{E_1[n+3m-7]} + \cdots + X_{E_1[n-3m+3]} + X_{E_1[n-3m+1]} + X_{E_1[n-3m-1]} + X_{E_1[n-3m-3]} \]

Proof. When \(n = 3k + 1\), we have the following equations:
\[X_{E_2X_{E_1[n]} = X_{E_1[n]} + X_{E_2X_2[n+1]} + X_{E_2X_2[n-1]} \]
Then the proof is immediately finished from Proposition 8.1.

In the same way by using Theorem 7.1, we have the following proposition.

Propagation 8.3. (1) For \(n \geq 3m + 1\), then
\[X_{E_1[3m+1]}X_{E_1[n]} = X_{E_1[n+3m+3]} + X_{E_1[n+3m-3]} + X_{E_1X_{E_1[n+3m-6]} + X_{E_1[n+3m-9]} + \cdots + X_{E_1[n+3m-6]} + X_{E_1X_{E_1[n+3m-3]} + X_{E_1X_{E_1[n-3m-1]} \]

(2) For \(n \geq 3m + 2\), then
\[X_{E_1[3m+2]}X_{E_1[n]} = X_{E_1[2]}(X_{E_1[n+3m]} + X_{E_1[n+3m-6]} + \cdots + X_{E_1[n-3m-1]} \]

(3) For \(n \geq 3m + 3\), then
\[X_{E_1[3m+3]}X_{E_1[n]} = X_{E_1[n+3m+3]} + X_{E_2X_{E_1[n+3m]} + X_{E_1[n+3m-3]} + X_{E_1X_{E_1[n+3m-6]} + X_{E_2X_{E_1[n+3m-3]} + X_{E_2X_{E_1[n+3m-6]} + \cdots + X_{E_2X_{E_1[n+3m-3]} + X_{E_2X_{E_1[n-3m-3]} + X_{E_2X_{E_1[n+3m-3]} \]

\]
Remark 8.4. By the same method in Corollary 8.3, we can also rewrite Proposition 8.3 if we consider these different n. Here we omit it.

Remark 8.5. In fact, we can also check Proposition 8.1 Corollary 8.2 and Proposition 8.3 by induction.

If we consider the quiver is one of $\tilde{A}_{n,n}$, $\tilde{D}$ and $\tilde{E}$. Then from the above discussions and Proposition 8.3, we can easily express $X_{E_i[n]}X_{E_j[n]}$ as a $\mathbb{Z}$–combinatorics of the basis in this tube with rank 3.

9. A $\mathbb{Z}$-basis for the cluster algebra of affine type

In [10] Theorem 4.21, G. Dupont asserts that the following set is a $\mathbb{Z}$-basis for the cluster algebra of type $\tilde{A}$:

$$\{\text{cluster monomials}\} \cup \{X_{M_{\lambda}^n} \oplus E : n \geq 1, E \text{ are rigid regular modules}\}.$$ 

We will use Theorem 4.3 and the above result in [10] to give a $\mathbb{Z}$-basis for the cluster algebra of affine type.

Proof of Theorem 7.1. By Theorem 8.3 we only prove that the set $B(Q')$ is a $\mathbb{Z}$-basis for the cluster algebra of $\tilde{A}_{r,s}$ for $r \neq s$. By Theorem 3.25 in [10] and Theorem 7.1 we have

$$X_{M_{\lambda}^n} \oplus E = (X_{\delta_i,1} - X_{q \cdot \varphi M_{E_i,1}/E_i,1})^n X_E$$

$$= (X_{\delta_i,1}^n + \sum_{\dim(T' \oplus R') < \dim(T \oplus R)} a_{T' \oplus R'} X_{T' \oplus R'}) X_E$$

$$= X_{T \oplus R} + \sum_{\dim(T'' \oplus R'') < \dim(T \oplus R)} a_{T'' \oplus R''} X_{T'' \oplus R''}$$

where $a_{T' \oplus R'}, a_{T'' \oplus R''} \in \mathbb{Z}$, $\dim(T \oplus R) = \dim(M_{\lambda}^n \oplus E)$ and $X_{T \oplus R}, X_{T'' \oplus R''} \in B(Q')$. Thus from the above upper triangle equations and these dimension vectors of $M_{\lambda}^n \oplus E$ in [10] are different, we have

$$\{X_{M_{\lambda}^n} \oplus E : n \geq 1, E \text{ are rigid regular modules}\} = \{X_{T \oplus R} | \dim(T_1 \oplus R_1) \neq \dim(T_2 \oplus R_2)\},$$

where $R$ is 0 or any regular exceptional module, $T$ is any indecomposable regular module with self-extension and there are no extension between $R$ and $T$.

Also it is easy to see that

$$\{\text{cluster monomials}\} = \{X_L | Ext^1_{\mathcal{U}(Q)}(L, L) = 0\}$$

Therefore $B(Q')$ is a $\mathbb{Z}$-basis for the cluster algebra of $\tilde{A}_{r,s}$.

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