Extended Thermodynamics for dense gases and macromolecular fluids, obtained through a non relativistic limit

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Abstract

In this paper we consider the 14 moments model of Extended Thermodynamics for dense gases and macromolecular fluids. Solutions of the restrictions imposed by the entropy principle and that of Galilean relativity for such a model have been until now obtained in literature only in an approximate manner up to a certain order with respect to thermodynamic equilibrium; for more restrictive models they have been obtained up to whatever order, but by using Taylor expansions around equilibrium and without proving convergence. Here we have found an exact solution without using expansions. The idea has been to write firstly a relativistic model, for which it is easy to impose the Einsteinian relativity principle, and then taking its non relativistic limit.

1 Introduction

The 14 moments model of Extended Thermodynamics for dense gases and macromolecular fluids was firstly studied by Kremer in [1], up to second order with respect to equilibrium. The balance equations to describe this model are

\[
\begin{align*}
\partial_t F + \partial_k F_k &= 0, \\
\partial_t F_i + \partial_k G_{ki} &= 0, \\
\partial_t F_{ij} + \partial_k G_{kij} &= P_{<ij>}, \\
\partial_t F_{ill} + \partial_k G_{kill} &= P_{ill},
\end{align*}
\]

(1)

where the independent variables are \( F, F_i, F_{ij}, F_{ill}, F_{iill} \) and are symmetric tensors. \( P_{<ij>}, P_{ill}, P_{iill} \) are productions and they too are symmetric tensors. The fluxes \( G_{ki}, G_{kij}, G_{kill}, G_{kiill} \) are constitutive functions and are symmetric over all indexes, except for \( k \).

In ideal gases, we have also the conditions \( G_{ki} = F_{ki}, G_{ill} = F_{ill}, G_{iill} = F_{iill} \) and, moreover, \( G_{kiij} \) and \( G_{kiill} \) are symmetric over all couples of indexes; then the present case is less restrictive.

We want that our system (1) is a symmetric hyperbolic one, with all consequent nice mathematical
properties. To this end, we impose the entropy principle; in the next section it will be proved that it is equivalent to the following equations

\[ F = \frac{\partial h'}{\partial \lambda}, \quad F_i = \frac{\partial h'}{\partial \lambda_i}, \quad F_{ij} = \frac{\partial h'}{\partial \lambda_{ij}}, \]

\[ F_{i\ell} = \frac{\partial h'}{\partial \lambda_{i\ell}}, \quad F_{i\ell j} = \frac{\partial h'}{\partial \lambda_{i\ell j}}, \quad F_{i\ell jk} = \frac{\partial h'}{\partial \lambda_{i\ell jk}}, \]

\[ G = \frac{\partial \phi'}{\partial \lambda}, \quad G_{\ell} = \frac{\partial \phi'}{\partial \lambda_{\ell}}, \quad G_{\ell i} = \frac{\partial \phi'}{\partial \lambda_{\ell i}}, \]

\[ G_{\ell i j} = \frac{\partial \phi'}{\partial \lambda_{\ell i j}}, \quad G_{\ell i j k} = \frac{\partial \phi'}{\partial \lambda_{\ell i j k}}, \]

where we have taken into account the definition \( G_{\ell} = F_{\ell} \), from which the following compatibility conditions holds

\[ \frac{\partial \phi'}{\partial \lambda} = \frac{\partial h'}{\partial \lambda}. \]

Moreover, also in the next section, it will be proved that the Galilean relativity principle is equivalent to the following two other conditions

\[ 0 = \frac{\partial h'}{\partial \lambda} \lambda_i + 2 \lambda_{ij} \frac{\partial h'}{\partial \lambda_j} + \lambda_{jpp} \left( \frac{\partial h'}{\partial \lambda_{rs}} \delta_{ij} + 2 \frac{\partial h'}{\partial \lambda_{ij}} \right) + 4 \lambda_{ppqq} \frac{\partial h'}{\partial \lambda_{i\ell jk}} + h' \delta_{ik}. \]

In other words, we have to find \( h' \) and \( \phi'_{\ell} \) satisfying the equations 3 and 4; after that, 1–5 are useful to obtain \( \lambda, \lambda_i, \lambda_{ij}, \lambda_{i\ell}, \lambda_{i\ell jk} \) as functions of our independent variables \( F, F_i, F_{ij}, F_{i\ell}, F_{i\ell jk} \). Lastly, 7–10 will give the constitutive functions \( G_{k\ell}, G_{kij}, G_{k\ell i j}, G_{k\ell i j k} \).

Now, the restrictions 3 and 4 have been imposed by Kremer in 1 only up to secondo order with respect to equilibrium. In 2 Carrisi and Pennisi have imposed them up to whatever order (for the more restrictive model of ideal gases), but by using Taylor’s expansions around equilibrium, without worrying about convergence problems. Here we have found the exact solution without using expansions. The idea to obtain this result has been the following

• Firstly we have assumed a relativistic model for which it is easy to impose the Einsteinian relativity principle; the results for this model can be found in section 3;

• After that, we have shown, in sect. 4, how to take the non relativistic limit of the model in sect.3; to this end we have used a methodology which is easy to find for ideal gases because in this case we have suggestions from the kinetic theory of gases; we have adopted this methodology also for our more general case. Obious, the validity of this assumption has to be tested at the end by verifying that our results satisfy truly the eqs. 4 and 5. We will be obtain that eqs. 4 are identically satisfied. Instead of this, the condition 3 will have still to be imposed; it is the only condition for which there is no correspondence between the relativistic case and the classical one.

• Then, in sect. 5 we have taken effectively this limit and found that \( h' \) and \( \phi'_{\ell} \) are determined by the following eqs. 5 and 6, except for 4 scalar arbitrary functions \( H_0, H_1, H_2, H_3 \) which depend on the scalars 7.
There remain the further condition (3), the requirement of convexity for the function $h'$ and the problem of subsystems. We have exploited them, but don’t report here the results, for the sake of brevity. We assure only that we have found the exact solution of (3) without using expansions.

We close this section by reporting the results of sect. 5; in this way it will be not necessary to search them throughout the paper and they will be available for the applications. They are

$$\phi^k = H_0 V_0^k + H_1 V_1^k + H_2 V_2^k + H_3 V_3^k,$$

$$h' = 8H_0 X_1 - H_1 X_2 - \frac{2}{3}H_2 X_3 - \frac{1}{2}H_3 X_4,$$

with

$$V_0^k = -2\lambda_{kl}$$

$$V_1^k = -2\lambda_{kh} \lambda_{hl} + 4\lambda_{ppkl} \lambda_k + \frac{4}{5}\lambda_{ll} \lambda_{kl}$$

$$V_2^k = -2\lambda_{kh}^2 \lambda_{hl} + \frac{6}{5}\lambda_{ll} \lambda_{ka} \lambda_{all} + 4\lambda_{ka} \lambda_a \lambda_{ppkl} +$$

$$\quad + \frac{11}{25}\lambda_{ll}^2 \lambda_{kl} - \lambda_{kl} \lambda_a \lambda_{all} + \lambda_k \lambda_{all} \lambda_{all} +$$

$$\quad + (tr \lambda_{ab}^2) \lambda_{kl} - \frac{12}{5}\lambda_{ppkl} \lambda_k \lambda_{kl}$$

$$V_3^k = 2\lambda_{ppkl} \left(2\lambda_{kh} \lambda_{h} - tr \lambda_{ab}^2 \lambda_k - \frac{8}{5}\lambda_{ll} \lambda_{ka} \lambda_a + \frac{17}{25}\lambda_{ll}^2 \lambda_k \right) +$$

$$\quad + (\lambda_{kh} \lambda_k)(\lambda_{all} \lambda_{all}) - \frac{4}{5}\lambda_{ll} (\lambda_{all} \lambda_{all}) \lambda_k + \frac{17}{25}\lambda_{ll}^2 \lambda_{ka} \lambda_{all} +$$

$$\quad - (\lambda_a \lambda_{all}) \lambda_k \lambda_{kl} + (tr \lambda_{ab}^2) \lambda_k \lambda_{all} + \frac{4}{5}\lambda_{ll} (\lambda_a \lambda_{all}) \lambda_{kl} +$$

$$\quad + \frac{8}{5}\lambda_{ll} \lambda_{kh} \lambda_{hl} + \frac{74}{375}\lambda_{ll} \lambda_{kl} - \frac{4}{5}\lambda_{ll} (tr \lambda_{ab}^2) \lambda_{kl} + (\lambda_{ab} \lambda_{all} \lambda_{all}) \lambda_k +$$

$$\quad - (\lambda_{ab} \lambda_a \lambda_{bl}) \lambda_{kl} + \frac{2}{3}(tr \lambda_{ab}^3) \lambda_{kl} - 2\lambda_{kh}^3 \lambda_{hl},$$

$$X_1 = \lambda_{ppkl},$$

$$X_2 = 2\lambda_{all} \lambda_{all} - \frac{16}{5}\lambda_{ppkl} \lambda_{ll},$$

$$X_3 = 8\lambda_{ppkl} \left(\frac{11}{50}\lambda_{ll}^2 - \frac{1}{2} tr \lambda_{ab}^2 \right) + 2\lambda_{ab} \lambda_{all} \lambda_{bl} - \frac{6}{5}\lambda_{ll} \lambda_{all} \lambda_{all},$$

$$X_4 = 2\lambda_{ab}^2 \lambda_{all} \lambda_{bl} - tr \lambda_{ab}^2 \lambda_{cll} \lambda_{cll} - \frac{8}{5}\lambda_{ll} \lambda_{ab} \lambda_{all} \lambda_{bl} +$$

$$\quad + \frac{17}{25}\lambda_{ll}^2 \lambda_{all} \lambda_{all} + 8\lambda_{ppkl} \left(- \frac{37}{375}\lambda_{ll}^3 + \frac{2}{5}\lambda_{ll} (tr \lambda_{ab}^2) - \frac{1}{3} tr \lambda_{ab}^3 \right),$$

$$X_5 = -\frac{2}{5}\lambda_{ll}^2 + 16\lambda_{ppkl} \lambda - 4\lambda_a \lambda_{all} + 2 tr \lambda_{ab}^2,$$

$$X_6 = 4\lambda_{ab} \lambda_{all} + 8\lambda_{ppkl} \left(- \frac{4}{5}\lambda_{ll} \lambda + \frac{1}{2}\lambda_a \lambda \right) +$$

$$\quad + \frac{8}{5}\lambda_{ll} \lambda_{all} \lambda_{a} - \frac{4}{5}\lambda_{ll} tr \lambda_{ab}^2 + \frac{8}{75}\lambda_{ll}^3 - 4\lambda_{ab} \lambda_a \lambda_{bl} + \frac{4}{3} tr \lambda_{ab}^3,$$
\[ X_7 = \frac{8}{15} (tr \lambda_{ab}^3) \lambda_{ll} - \frac{14}{25} \lambda_{ll}^2 tr \lambda_{ab}^2 + \frac{46}{375} \lambda_{ll}^4 + 4k a_b \lambda_{a} \lambda_{ll} \lambda_{bl} + + 2 (tr \lambda_{ab}^2) \lambda_{d} \lambda_{dll} - (\lambda_{a} \lambda_{all})^2 - \frac{12}{5} \Lambda \lambda_{ll} \lambda_{a} \lambda_{all} + + (\lambda_{a} \lambda_{a})(\lambda_{ll} \lambda_{bl}) - 4 \lambda_{ab}^2 \lambda_{a} \lambda_{bl} + - 8 \lambda_{ppll} \left( \lambda_{ll} \lambda_{a}^2 - \frac{1}{2} \lambda_{ab} \lambda_{a} \lambda_{b} - \frac{11}{25} \lambda_{ll}^2 + \frac{3}{10} \lambda_{ll} \lambda_{a} \lambda_{a} \right) + + \frac{12}{5} \lambda_{ll} \lambda_{a} \lambda_{bl} - \frac{22}{25} \lambda_{ll} \lambda_{a} \lambda_{all} . \]

\[ X_8 = - \frac{34}{25} \lambda_{ll}^2 \lambda_{a} \lambda_{bl} + 2 (tr \lambda_{ab}^2) \lambda_{cd} \lambda_{dll} + \frac{16}{5} \lambda_{ll} \lambda_{a} \lambda_{bl} + + \frac{148}{375} \lambda_{ll} \lambda_{a} \lambda_{all} - \frac{8}{5} \lambda_{ll} (tr \lambda_{ab}^2) \lambda_{cd} \lambda_{dll} + \frac{4}{3} (tr \lambda_{ab}^2) \lambda_{cd} \lambda_{dll} - 4 \lambda_{ab}^2 \lambda_{a} \lambda_{bl} + + 2 \lambda_{ppll} \left( 2 \lambda_{ab} \lambda_{a} \lambda_{b} - (tr \lambda_{cd}^2) \lambda_{a} \lambda_{a} - \frac{8}{5} \lambda_{ll} \lambda_{ab} \lambda_{a} \lambda_{b} + \frac{17}{25} \lambda_{ll}^2 \lambda_{a} \lambda_{a} \right) + + (\lambda_{ab} \lambda_{a} \lambda_{b})(\lambda_{dll} \lambda_{dll}) - \frac{4}{5} \lambda_{ll} \lambda_{a} \lambda_{all} \lambda_{ll} \lambda_{bl} - - 2 (\lambda_{a} \lambda_{all})(\lambda_{ll} \lambda_{bl}) + + \frac{4}{5} \lambda_{ll} \lambda_{a} \lambda_{all} + \frac{16}{5} \lambda_{ll} \lambda_{a} \lambda_{bl} \lambda_{ll} \lambda_{bl} + + \frac{34}{25} \Lambda \lambda_{ll} \lambda_{a} \lambda_{ll} + 16 \lambda_{ppll} \left( - \frac{37}{375} \lambda_{ll}^3 + \frac{2}{5} \lambda_{ll} (tr \lambda_{ab}^2) - \frac{1}{3} tr \lambda_{ab}^2 \right) + + \frac{4}{75} \lambda_{ll}^2 (tr \lambda_{ab}^2) - \frac{8}{125} \lambda_{ll}^2 (tr \lambda_{ab}^2) + \frac{4}{15} \frac{37}{625} \lambda_{ll}^2 . \]

It is interesting that \( \phi' \) has been determined except for 4 scalar functions \( H_0, H_1, H_2, H_3 \). Instead of this, if we have used only the representation theorems without imposing the entropy principle and the Galilean invariance, we would have obtained that \( \phi' \) was depending on 6 arbitrary scalar functions, being a linear combination of \( \lambda_{ll}, \lambda_{ab} \lambda_{bl}, \lambda_{ll}^2 \lambda_{all}, \lambda_{k}, \lambda_{a} \lambda_{a}, \lambda_{ll}^2 \lambda_{a} \lambda_{a} \). We note that also \( h' \) is determined in terms of \( H_0, H_1, H_2, H_3 \). These are arbitrary functions of the 8 scalars \( X_1 - X_8 \). Instead of this, if we have used only the representation theorems without imposing the entropy principle and the Galilean invariance, we would have obtained that all the scalar function are arbitrary functions of the following 14 scalars \( \lambda_{ll}, tr \lambda_{ab}^2, tr \lambda_{ab}^3, \lambda_{all} \lambda_{ll}, \lambda_{all} \lambda_{a}, \lambda_{all} \lambda_{bl}, \lambda_{ab} \lambda_{ab} \lambda_{bl}, \lambda_{a} \lambda_{a} \lambda_{b}, \lambda_{ab} \lambda_{al} \lambda_{bl}, \lambda_{ab} \lambda_{a} \lambda_{b}, \lambda_{ab} \lambda_{al} \lambda_{bl}, \lambda_{ab} \lambda_{al} \lambda_{bl}, \lambda_{ab} \lambda_{a} \lambda_{b}, \lambda_{ab} \lambda_{al} \lambda_{bl}, \lambda_{ab} \lambda_{a} \lambda_{b}, \lambda_{ppll}, \lambda \). It is useful now to verify these results: To this end we can substitute eqs. (5), (6) and (7) into (4) and obtain that they are identically satisfied. The corresponding calculations are long, so it will be useful to subdivide them with the following steps.

- Firstly we can verify that (4) is satisfied with \( X_i \) instead of \( h' \), for \( i = 1, \cdots , 8 \); consequently, for the theorem on derivation of composite functions, it will be satisfied by whatever function of \( X_i \), as \( h' \) is. But we have to note that, if we simplify \( X_8 \) through the Hamilton-Kayley theorem

\[ \lambda_{ab}^3 = \lambda_{ll} \lambda_{ab}^2 + \frac{1}{2} (tr \lambda_{ab}^2 - \lambda_{ll}^2) \lambda_{ab} + \left( \frac{1}{3} tr \lambda_{cd}^2 - \frac{1}{2} \lambda_{ll} tr \lambda_{cd}^2 + \frac{1}{6} \lambda_{ll}^2 \right) \delta_{ab} \]
it will become more comlicate to verify that \( X_8 \) is a solution, because it will be necessary also to use the identity

\[
0 = \delta_{ij} \left( -\lambda_{ab}^2 \lambda_{alt} \lambda_{bdl} + \lambda_{ilt} \lambda_{abl} \lambda_{bdl} + \frac{1}{2} \lambda_{alt} \lambda_{alt} tr \lambda_{cd} - \frac{1}{2} \lambda_{alt} \lambda_{alt} \lambda_{d} \right) + \\
\lambda_{ij} \left( -\lambda_{ab} \lambda_{alt} \lambda_{bdl} + \lambda_{ilt} \lambda_{alt} \lambda_{alt} \right) - \lambda_{ij}^2 \lambda_{alt} \lambda_{alt} + \lambda_{ilt} \lambda_{jll} \frac{1}{2} (\lambda_{lt}^2 - tr \lambda_{cd}) + \\
-2 \lambda_{ilt} \lambda_{l(i} \lambda_{j)b} \lambda_{bdl} + \lambda_{ia} \lambda_{alt} \lambda_{jb} \lambda_{bdl} + 2 \lambda_{ilt} (\lambda_{j}^2) \lambda_{bdl}
\]

which can be easily proved in the reference frame where \( \lambda_{ab} \) has the diagonal form.

- The second step is to verify that (11) is satisfied in the case \( H_0 = 1, H_1 = 0, H_2 = 0, H_3 = 0 \). Similarly for the case \( H_0 = 0, H_1 = 1, H_2 = 0, H_3 = 0 \); then for the case \( H_0 = 0, H_1 = 0, H_2 = 1, H_3 = 0 \) and, lastly, for the case \( H_0 = 0, H_1 = 0, H_2 = 0, H_3 = 1 \). Consequently, it will be satisfied for all constant values of \( H_0, H_1, H_2, H_3 \). After that it is satisfied also in the general case: In fact, for the property on derivation of a product, the terms where \( H_0, H_1, H_2, H_3 \) are not differentiated will simplify, for what above said, and it remains

\[
\sum_{j=0}^{3} V_j^k \cdot \sum_{i=1}^{8} \frac{\partial H_j}{\partial X_i}
\]

for the right hand side of (11) written with \( X_i \) instead of \( h' \); the result is zero for what already verified in the first step.

2 The principles of entropy and of Galilean relativity

We want that our system (11) is symmetric and hyperbolic, with all the consequent nice mathematical properties. To this end we impose that all the solution of eqs. (11) satisfy also the entropy inequality

\[
\frac{\partial h}{\partial t} + \frac{\partial \phi_k}{\partial x_k} = \sigma \geq 0.
\]

This is equivalent to assume the existence of Lagrange Multipliers \( \lambda, \lambda_i, \lambda_{ij}, \lambda_{ilt}, \lambda_{ill} \) such that

\[
dh = \lambda dF + \lambda_i dF_i + \lambda_{ij} dF^{ij} + \lambda_{ilt} dF^{ilt} + \lambda_{ill} dF^{ill}, \\
d\phi_k = \lambda dF_k + \lambda_i dG_{ik} + \lambda_{ij} dG_{ijk} + \lambda_{ilt} dG_{iltk} + \lambda_{ill} dG_{illk}
\]

(8)

besides a residual inequality which we leave for the sake of brevity. The Lagrange Multipliers are also called "mean field". Let us now impose the Galilean relativity principle by considering the following change of independent variables

\[
F_{i_1i_2...i_n} = \sum_{k=0}^{n} \binom{n}{k} m_{(i_1i_2...i_k)} v_{i_{k+1}...i_n}
\]

(9)

which can be found in [3] and that, applied to our case, becomes

\[
F = m, \\
F_i = mv_i + m_i, \\
F_{ij} = mv_i v_j + m_{ij} + 2m_i v_j, \\
F_{ilt} = m_{ilt} v_i + m_{ilt} v_i + mv^2 v_i + m_{il} v_i^2 + 2m_{ilt} v_i, \\
F_{ill} = m_{ill} + mv^4 + 4m_i v_i^2 + 2m_{ilt} v_i^2 + 4m_{ill} v_i + 4m_{ill} v_i.
\]

(10)
Also in [3] we can find how change the constitutive functions $G_{k_1\ldots i_n}$ when the reference frame changes, i.e.,

$$H_{k_1i_2\ldots i_n} = \sum_{j=0}^{n-1} \binom{n}{j} M_{k(i_1\ldots i_j i_{j+1}\ldots i_n)}$$  \hspace{1cm} (11)

where the functions $H_{k_1\ldots i_n}$ are defined by

$$G_{k_1\ldots i_n} = v_k F_{i_1\ldots i_n} + H_{k_1\ldots i_n}.$$  \hspace{1cm} (12)

It is interesting that (11) looks like (9), except that they don’t act on the index $k$. In our particular case, eqs. (13) become

$$H_k = M_k,$$

$$H_{ki} = M_k v_i + M_{ki},$$

$$H_{kij} = M_k v_i v_j + 2 M_{k(i v_j)} + M_{kij},$$

$$H_{kijl} = M_k v_i v_j v_k + 2 M_{kl ji v_l} + M_{kijl},$$

$$H_{kijl} = M_k v^4 + 4 M_{k[i v_j v_k} + 2 M_{kij v_l] + M_{kijl} + M_{kijl}.$$  \hspace{1cm} (13)

Consequently, the functions $G_{k_1\ldots i_n}$ transform as follows

$$G_{k_i} = m_i v_i + m_k v_i + M_k v_i + M_{ki},$$

$$G_{kij} = m_i v_i v_j + m_{ij} v_k + 2 m_{i(i v_j) v_k} + M_k v_i v_j + M_{k(i v_j)} + M_{ij},$$

$$G_{kijl} = m_{iijl} v_k + m_{ij ij v_k} + 4 m_{iij v_k} + m_{iij v_k} + m_{ijl} v_k +$$

$$+ 2 m_{ijl v_k} + M_{kij v_l} + M_{kl ij v_l} + M_{kijl} + M_{kijl}.$$  \hspace{1cm} (14)

We note that from (12) and (11), for $n = 0$, and from $G_k = F_k$ it follows $M_k = F_k - F v_k$. This and (10) yields $M_k = m_k$. The new variables $m, m_i, m_{ij}, m_{iij}, m_{ijl}$ and $M_i, M_{ij}, M_{ijl}, M_{ijll}$ have the same symmetries of $F_{i_1\ldots i_n}$ and $G_{k_1\ldots i_n}$.

Let us now substitute into eqs. (9) the expressions which we have above found for the variables and the constitutive functions. In this way eqs. (9) become

$$d h = \lambda^l d m + \lambda^l_i d m_i + \lambda^l_{ij} d m_{ij} + \lambda^l_{iij} d m_{iij} + \lambda^l_{iijl} d m_{iijl} +$$

$$+ \lambda^l_{iijl} d m_{iijl} + (\lambda^l_i m + 2 \lambda^l_{ij} m_{ij} + \lambda^l_{ijl} m_{ijl}) d v_i +$$

$$+ 2 \lambda^l_{ijl} m_{ijl} + 4 \lambda^l_{ppq} m_{ppq} d v_i,$$  \hspace{1cm} (15)

$$d(\phi_k) = (\lambda^l d m + \lambda^l_i d m_i + \lambda^l_{ij} d m_{ij} + \lambda^l_{iij} d m_{iij} +$$

$$+ \lambda^l_{iijl} d m_{iijl}) v_k + \lambda^l d M_k + \lambda^l_i d M_{ki} + \lambda^l_{ij} d M_{ki} +$$

$$+ \lambda^l_{ijl} d M_{kijl} + \lambda^l_{ijijl} d M_{kijl} + (\lambda^l_i m + \lambda^l_{ij} m_{ij} +$$

$$+ \lambda^l_{ijl} m_{ijl} + \lambda^l_{ppq} m_{ppq}) d v_k + (\lambda^l_i M_k + 2 \lambda^l_{ij} M_{kj} +$$

$$+ \lambda^l_{ijl} M_{kijl} + 2 \lambda^l_{ppq} M_{ppq} + 4 \lambda^l_{ppq} M_{ppq} d v_i,$$  \hspace{1cm} (16)
with

\[ \lambda^I = \lambda + \lambda_iv_i + \lambda_{ij}v_iv_j + \lambda_{itt}v_i^2 + \lambda_{ppqq}v^4, \]
\[ \lambda_i = \lambda_i + 2\lambda_{ij}v_j + \lambda_{itt}v^2 + 2\lambda_{ppp}v_iv_j + 4\lambda_{ppqq}v_i^2, \]
\[ \lambda_{ij} = \lambda_{ij} + \lambda_{hpp}v_k\delta_{ij} + 2\lambda_{ppp}(v_j) + 4\lambda_{ppqq}v_jv_i + 2\lambda_{hhpp}v^2\delta_{ij}, \]
\[ \lambda_{ill} = \lambda_{ill} + 4\lambda_{hhpp}v_i, \]
\[ \lambda_{iill} = \lambda_{ppqq}. \]

The Galilean relativity principle imposes that \( h, \phi_k - hv_k, M_{ki}, M_{kij}, M_{kilt}, M_i \) don’t depend on \( v_i \). For this condition on \( h \) and \( \phi_k - hv_k \) we obtain

\[ \frac{\partial h}{\partial v_i} = 0 = m\lambda^I_i + 2\lambda^I_{ij}m_j + \lambda^I_{ppp}(m_{il}\delta_{ij} + 2m_{ij}) + 4\lambda^I_{ppqq}m_{iill}, \]
\[ \frac{\partial (\phi_k - hv_k)}{\partial v_i} = 0 = M_k\lambda^I_i + 2M_{kj}\lambda^I_{ij} + M_{kilt}\lambda^I_{pp} + 2M_{kilt}\lambda^I_{pp} + 4\lambda^I_{ppqq}M_{kilt} + h'd_{ik}, \]

where \( h' \) is defined by

\[ h' = \lambda^I_i m_i + \lambda^I_{il}m_{iil} + \lambda^I_{iill}m_{iill} + \lambda^I_{iill}m_{iill} - h; \]

In this way eqs. (15) and (16) become respectively

\[ dh^I = \lambda^I_dm_i + \lambda^I_{il}m_{iil} + \lambda^I_{iill}m_{iill} + \lambda^I_{iill}m_{iill} \]
\[ d\phi_k^I = \lambda^I_i dM_k + \lambda^I_{il}dM_{ki} + \lambda^I_{iill}dM_{kilt} + \lambda^I_{ppqq}dM_{kilt} \]

with \( \phi_k^I = \phi_k - hv_k \). Let us also define

\[ \phi_k' = \lambda^I_i M_k + \lambda^I_{il}M_{ki} + \lambda^I_{iill}M_{kilt} + \lambda^I_{iill}M_{kilt} - \phi_k^I; \]

so that eqs. (18) become

\[ dh' = m\lambda^I_i + m_j\lambda^I_{ij} + m_{iil}\lambda^I_{iill} + m_{iill}d\lambda^I_{iill} \]
\[ d\phi_k' = M_kd\lambda^I_i + M_{ki}d\lambda^I_{ij} + M_{kilt}d\lambda^I_{iill} + M_{kilt}d\lambda^I_{iill} \]

from which, by taking \( \lambda^I_i, \lambda^I_{ij}, \lambda^I_{iill}, \lambda^I_{iill} \) as independent variables and, by taking the derivatives with respect to the various components of the mean field, it follows

\[ m = \frac{\partial h'}{\partial \lambda^I_i}, \quad m_i = \frac{\partial h'}{\partial \lambda^I_{ij}}, \quad m_{iil} = \frac{\partial h'}{\partial \lambda^I_{iill}}, \]
\[ m_{iill} = \frac{\partial h'}{\partial \lambda^I_{iill}}, \quad m_{iill} = \frac{\partial h'}{\partial \lambda^I_{iill}}, \]
\[ M_k = \frac{\partial \phi_k'}{\partial \lambda^I_i}, \quad M_{ki} = \frac{\partial \phi_k'}{\partial \lambda^I_{ij}}, \quad M_{kilt} = \frac{\partial \phi_k'}{\partial \lambda^I_{iill}}, \]
\[ M_{kilt} = \frac{\partial \phi_k'}{\partial \lambda^I_{iill}}. \]
These last equations are noting more than (2), but in the new reference frame. By substituting in (17) from (20) we obtain equations whose expression, in the previous reference frame, are (4). Consequently, (2) and (4) are equivalent to the principles of entropy and of Galilean relativity. Other interesting aspects can be found in [4], which are adapted for the present case in [5] and [6]. In order to find the general solution of (4), let us write a relativistic counterpart of our equations.

3 Relativistic extended thermodynamics for dense gases and macromolecular fluids

Let us consider the balance equations

\[ \partial_\alpha T^{\alpha\beta} = 0, \quad \partial_\alpha A^{\alpha\beta\gamma} = I^{\beta\gamma}. \]  

(21)

where \( T^{\alpha\beta} \) isn’t symmetric, while \( A^{\alpha\beta\gamma} \) and \( I^{\beta\gamma} \) are symmetric only with respect to the indexes \( \beta\gamma \). The first of these is the conservation law of momentum-energy, while the trace of the second one is the conservation law of mass, so that

\[ I^{\beta\gamma} g_{\beta\gamma} = 0. \]  

(22)

The entropy principle is expressed in terms of \( h^\alpha \), called (entropy density - entropy flux density) tensor, such that

\[ \partial_\alpha h^\alpha = \sigma \geq 0 \]  

(23)

for every solution of eqs. (21).

For Liu Theorem [7] eq. (21) is equivalent to assuming the existence of the Lagrange multipliers \( \lambda_\beta, \lambda_{\beta\gamma} \) such that

\[ \partial_\alpha h^\alpha - \sigma - \lambda_\beta \partial_\alpha T^{\alpha\beta} - \lambda_{\beta\gamma} (\partial_\alpha A^{\alpha\beta\gamma} - I^{\beta\gamma}) = 0 \]  

(24)

for every value of the independent variables. By differentiating it becomes

\[ dh^\alpha = \lambda_\beta dT^{\alpha\beta} + \lambda_{\beta\gamma} dA^{\alpha\beta\gamma}; \quad -\sigma + \lambda_{\beta\gamma} I^{\beta\gamma} = 0. \]  

(25)

If we define \( h'^\alpha \) by

\[ h^\alpha = -h'^\alpha + \lambda_\beta T^{\alpha\beta} + \lambda_{\beta\gamma} A^{\alpha\beta\gamma}; \]  

(26)

then eq. (25) can be rewritten as

\[ dh'^\alpha = T^{\alpha\beta} d\lambda_\beta + A^{\alpha\beta\gamma} d\lambda_{\beta\gamma}. \]  

(27)

This last equation, by taking \( \lambda_\beta, \lambda_{\beta\gamma} \) as independent variables, becomes

\[ T^{\alpha\beta} = \frac{\partial h'^\alpha}{\partial \lambda_\beta}, \quad A^{\alpha\beta\gamma} = \frac{\partial h'^\alpha}{\partial \lambda_{\beta\gamma}} \]  

(28)

It follows that from the knowledge of \( h'^\alpha \) we obtain \( T^{\alpha\beta} \) and \( A^{\alpha\beta\gamma} \); we have only a condition on \( h'^\alpha \); it has to satisfy the Einsteinian relativity principle. For well known representation theorems as [8] and [9], we have that

\[ h^\alpha = h_0 \lambda^\alpha + h_1 \lambda^{\alpha\gamma} \lambda_\gamma + h_2 \lambda^{\alpha\gamma} \lambda_\gamma + h_3 \lambda^{\alpha\gamma} \lambda_\gamma \]  

(29)
where the following definitions have been used

\[ \lambda^{\alpha\gamma} = \lambda^{\alpha\beta} \lambda^{\beta\delta} g^{\delta\gamma}, \quad 3^{\alpha\gamma} = \lambda^{\alpha\beta} \lambda^{\beta\gamma} \]

with \( h \) scalar functions depending on

\[ Q_1 = \lambda_\beta, \quad Q_2 = 2^{\alpha\gamma} g_{\alpha\gamma}, \quad Q_3 = 3^{\alpha\gamma} g_{\alpha\gamma}, \quad Q_4 = 4^{\alpha\gamma} g_{\alpha\gamma}, \quad P_0 = \lambda_\beta \lambda^\beta, \quad P_1 = \lambda_\beta \lambda^\gamma \lambda^\beta, \quad P_2 = \lambda_\beta \lambda^\omega \lambda^\gamma, \quad P_3 = \lambda_\beta \lambda^\omega \lambda^\gamma. \tag{30} \]

4 The non relativistic limit of the previous model

In order to take this limit, let us consider a modified procedure of that used in [10], [11] for ideal gases. We assume that this procedure holds also for macromolecular gases; this assumption doesn’t lead to wrong results because we have already verified, at the end of sect. 1, that these results are correct. Let us subdivide the procedure in two parts.

4.1 A first transformation in 3-dimensional form

Eq. (21) for \( \beta = 0 \), \( i \) becomes

\[
\frac{1}{c} \partial_t T^{00} + \partial_k T^{0k} = 0, \tag{31}
\]

\[
\frac{1}{c} \partial_t T^{0i} + \partial_k T^{ki} = 0. \tag{32}
\]

Similarly, eq. (21) for \( \beta\gamma = 00, \beta\gamma = 0i, \beta\gamma = ij \) becomes

\[
\frac{1}{c} \partial_t A^{000} + \partial_k A^{k00} = I^{00}, \quad \frac{1}{c} \partial_t A^{00i} + \partial_k A^{k0i} = I^{0i}, \quad \frac{1}{c} \partial_t A^{0ij} + \partial_k A^{kij} = I^{ij}. \tag{33}
\]

Now we adopt the following change of variables

\[
T^{00} = m_0^4 c F_2, \quad T^{k0} = m_0^4 G_2^k, \quad T^{0i} = m_0^4 F^i_2, \quad T^{ki} = \frac{m_0^4 c}{G_2^k}
\]

\[
A^{000} = m_0^5 c^2 F_3, \quad A^{00i} = m_0^5 c F_3^i, \quad A^{0ij} = m_0^5 F_3^{ij}
\]

\[
A^{k00} = m_0^5 c G_3^k, \quad A^{k0i} = m_0^5 G_3^k, \quad A^{kij} = \frac{1}{c} m_0^5 G_3^{kij}
\]

\[
I^{00} = c R m_0^5, \quad I^{0i} = Q^i m_0^5, \quad I^{ij} = \frac{1}{c} p^{ij} m_0^5. \tag{37}
\]

In their terms eqs. (31), (32) and (33) become

\[
\partial_t F_2 + \partial_k G_2^k = 0, \quad \partial_t F_2^i + \partial_k G_2^{ki} = 0, \quad \partial_t F_3 + \partial_k G_3^k = R \tag{38}
\]

\[
\partial_t F_3^i + \partial_k G_3^{ki} = Q^i, \quad \partial_t F_3^{ij} + \partial_k G_3^{kij} = p^{ij}.
\]
The last of these can be also subdivided in
\[ \partial_t F_3^{ll} + \partial_k G_3^{kl} = p^{ll}, \quad \partial_t F_3^{<ij>} + \partial_k G_3^{k<ij>} = p^{<ij>}, \]
while eq. (23) can be subdivided in
\[ \frac{1}{c} \partial_t h^0 + \partial_k h^k = \sigma. \]  

With the following changes of names
\[ h^0 = m_0^3 h, \quad h^k = \frac{m_0^3}{c} \phi^c, \quad \sigma = \frac{m_0^3}{c} \sigma^*. \]
eq (41) becomes
\[ \partial_t h + \partial_k \phi^k = \sigma^*. \]  

### 4.2 A first transformation of the Lagrange Multipliers

EQ. (25) for \( \alpha = 0 \) becomes
\[ dh^0 = \lambda_0 dT^{00} + \lambda_i dT^{0i} + \lambda_{00} dA^{000} + 2\lambda_{0i} dA^{00i} + \lambda_{ij} dA^{0ij}. \]  

By using eq. (41) and (44)-(46), it transforms into
\[ d(m_0^3 h) = \lambda_0 m_0 c d(F_2) + \lambda_i m_0 d(F_2^i) + \lambda_{00} m_0^2 c^2 d(F_3) + \\
+ 2\lambda_{0i} m_0^2 d(F_3^i) + \lambda_{ij} m_0^2 d(F_3^{ij}) \]
from which
\[ dh = \lambda_0 m_0 c d(F_2) + \lambda_i m_0 d(F_2^i) + \lambda_{00} m_0^2 c^2 d(F_3) + \\
+ 2\lambda_{0i} m_0^2 d(F_3^i) + \lambda_{ij} m_0^2 d(F_3^{ij}) = \\
\ell dF_2 + \ell_i dF_2^i + \eta dF_3 + \mu_i dF_3^i + \mu_{ij} dF_3^{ij}, \]
with
\[ \ell = \lambda_0 m_0 c, \quad \ell_i = \lambda_i m_0, \quad \eta = \lambda_{00} m_0^2 c^2, \quad \mu_i = 2\lambda_{0i} m_0^2 c, \quad \mu_{ij} = \lambda_{ij} m_0^2. \]  

Similarly, eq. (25) for \( \alpha = k \) becomes
\[ dh^k = \lambda_0 dT^{k0} + \lambda_i dT^{ki} + \lambda_{00} dA^{k00} + 2\lambda_{0i} dA^{k0i} + \lambda_{ij} dA^{kij}. \]  

By using eq. (41) and (41), (36), it transforms into
\[ d(m_0^3 \phi^k) = \lambda_0 m_0 d(G_2^k) + \lambda_i m_0 d(G_2^{ki}) + \lambda_{00} m_0^2 c^2 d(G_3^k) + \\
+ 2\lambda_{0i} m_0^2 d(G_3^{ki}) + \lambda_{ij} m_0^2 d(G_3^{kij}) \]
which can be rewritten also as
\[ d\phi^k = \lambda_0 c m_0 d(G_2^k) + \lambda_i m_0 d(G_2^{ki}) + \lambda_{00} c^2 m_0^2 d(G_3^k) + \\
+ 2\lambda_{0i} m_0^2 d(G_3^{ki}) + \lambda_{ij} m_0^2 d(G_3^{kij}) = \\
\ell dG_2^k + \ell_i dG_2^{ki} + \eta dG_3^k + \mu_i dG_3^{ki} + \mu_{ij} dG_3^{kij}, \]
where we have used (46). The equations (45) and (48) represent the entropy principle for the system (38); moreover, eq. (46) gives the transformation of the Lagrange multipliers.

Let us now find the transformation of $h^\alpha$; eq. (26) for $\alpha = 0$ is

$$h^0 = -h^0 + \lambda_0 T^{00} + \lambda_i T^{0i} + \lambda_{00} A^{000} + 2\lambda_{00} A^{0i0} + \lambda_{ij} A^{0ij}.$$  (49)

By using eqs. (41) and (45)-(46), eq. (49) becomes

$$m_0^3 h = -h^0 + \ell \tilde{m}_0^3 F_2 + \ell_i \tilde{m}_0^3 F_2^i + \eta \tilde{m}_0^3 F_3 + \mu_i \tilde{m}_0^3 F_3^i + \mu_{ij} \tilde{m}_0^3 F_3^{ij}.$$  (50)

This can be rewritten as

$$h = -h^0 + \ell \tilde{m}_0^3 F_2 + \ell_i \tilde{m}_0^3 F_2^i + \eta \tilde{m}_0^3 F_3 + \mu_i \tilde{m}_0^3 F_3^i + \mu_{ij} \tilde{m}_0^3 F_3^{ij}$$  (51)

or

$$h = -h^0 + \ell \tilde{m}_0^3 F_2 + \ell_i \tilde{m}_0^3 F_2^i + \eta \tilde{m}_0^3 F_3 + \mu_i \tilde{m}_0^3 F_3^i + \mu_{ij} \tilde{m}_0^3 F_3^{ij}$$  (52)

where we have defined

$$h^0 = m_0^3 h^0.$$  (53)

In this way we have found the counterpart of (41) for $h^0$. Let us now find the transformation of $h^k$ for $\alpha = k$; from eq. (26) we find

$$h^k = -h^0 + \lambda_0 T^{k0} + \lambda_i T^{ki} + \lambda_{00} A^{k00} + 2\lambda_{00} A^{k0i} + \lambda_{ij} A^{kij}.$$  (54)

This, by using again eqs. (41), (46) and (34)-(36), can be written as

$$\frac{m_0^3}{c} G^k = -h^0 + \frac{\ell}{m_0^4} m_0^4 G_2^k + \frac{\ell_i m_0^4}{m_0^4} G_2^k + \frac{\eta}{m_0^6 c^2} m_0^6 G_3^k + \frac{\mu_i}{m_0^5 c} m_0^5 G_3^k + \frac{\mu_{ij}}{m_0^4 c} m_0^4 G_3^{ij}$$

or

$$\phi^k = -\phi^0 + \ell G_2^k + \ell_i G_2^{ki} + \eta G_3^k + \mu_i G_3^{ki} + \mu_{ij} G_3^{ij}$$  (55)

with

$$h^k = \frac{m_0^3}{c} \phi^k.$$  (56)

Eqs. (53) and (56) are the counterparts of (41) for $h^0$ and $h^k$. Eqs. (52) and (55) are the counterpart of eq. (26) for the system (38). Let us finish by considering the counterpart of mass conservation (22), that is $-I^{00} + I^{ij} \delta_{ij} = 0$ or, by use of (37), $-c R m_0^5 + \frac{1}{2} \alpha^{ij} \tilde{m}_0^5 \delta_{ij} = 0$. In other words,

$$R = \frac{1}{c^2} \alpha^{ij} \delta_{ij}.$$  (57)

4.3 The second transformation in 3-dimensional form

4.3.1 Suggestions from the kinetic theory for ideal gases

For ideal gases the variables $F_2, F_2^i, F_3, F_3^i, F_3^{ij}$ have counterparts in statistical mechanics where they are defined as moments of the distribution function $\tilde{f}$, i.e., by means of the following integrals

$$F_2 = \int \tilde{f} \gamma^6 d\mathbf{u}, \quad F_2^i = \int \tilde{f} \gamma^6 u^i d\mathbf{u},$$  (58)

$$F_3 = \int \tilde{f} \gamma^7 d\mathbf{u}, \quad F_3^i = \int \tilde{f} \gamma^7 u^i d\mathbf{u}, \quad F_3^{ij} = \int \tilde{f} \gamma^7 u^i u^j d\mathbf{u}.$$
where $\gamma$ is the Lorentz factor $(1 - \frac{u^2}{c^2})^{-\frac{1}{2}}$.

If we take the limits of these expressions for $c \to \infty$ we obtain $F_2 = F_3$, $F_2^i = F_3^i$ so that they will be no more independent variables; to avoid this problem, we take suitable invertible linear combinations of the equations before taking the limits.

In particular, as first equation we take the linear combination of (38) through the coefficients 1 and $\frac{1}{c^2}$, respectively; so we obtain the following eq. (69) which is the **conservation law of mass** with

$$F = F_3 - \frac{1}{c^2} F_3^{ll}, \quad G^k = G_3^k - \frac{1}{c^2} G_3^{kll};$$

moreover, we have taken into account eq.(57).

As second equation we take (38) which can be written as the following eq. (69) which is the **conservation law of momentum** with

$$F^i = F_2^i, \quad G^{ki} = G_2^{ki};$$

As third equation we take the linear combination of (38), (38), (39) through the coefficients $2c^2$, $-2c^2$, 2 respectively; so it becomes

$$\partial_t F^{ll} + \partial_k G^{kll} = 0$$

which is the **conservation law of energy**, with

$$F^{ll} = 2c^2 (F_2 - F_3 + \frac{1}{c^2} F_3^{ll}), \quad G^{kll} = 2c^2 (G_2^k - G_3^k + \frac{1}{c^2} G_3^{kll}).$$

As fourth equation we take (39) which we write as

$$\partial_t F^{<ij>} + \partial_k G^{k<ij>} = p^{<ij>}$$

where

$$F^{<ij>} = F_3^{<ij>}, \quad G^{k<ij>} = G_3^{k<ij>}.$$ 

We transform furtherly eq. (63) adding to it eq. (61) multiplied by $\frac{\delta^{ij}}{3}$ and obtaining the following eq. (69) with

$$F^{ij} = F_3^{ij} + \frac{c^2 \delta^{ij}}{3} (2F_2 - 2F_3 + \frac{F_3^{ll}}{c^2}),$$

$$G^{kij} = G_3^{kij} + \frac{c^2 \delta^{ij}}{3} (2G_2^k - 2G_3^k + \frac{G_3^{kll}}{c^2}).$$

Equation (69) encloses both (61) and (63). As other equation we take the linear combination of (38), (38), (39) through the coefficients $2c^2$ and $-2c^2$, respectively; so we obtain the following eq. (69) with

$$F^{<ij>} = 2c^2 (F_3^i - F_3^j), \quad G^{k<ij>} = 2c^2 (G_3^{k<ij>}), \quad p^{<ij>} = 2c^2 Q^i.$$ 

Finally, as last equation we take the linear combination of (38), (38), (39) through the coefficients $-8c^4$, $+8c^4$, $-4c^2$ respectively; so we obtain the following eq. (69) with

$$F^{<ij>} = -8c^4 F_2 + 8c^4 F_3 - 4c^2 F_3^{ll},$$

$$G^{k<ij>} = -8c^4 G_2^k + 8c^4 G_3^k - 4c^2 G_3^{kll},$$

$$p^{<ij>} = 8c^4 R - 4c^2 p^{ll} + 4c^2 p^{ll} = 4c^2 p^{ll}.$$
where eq. (57) has been used. The complete system is
\[ \begin{align*}
\partial_t F + \partial_k G^k &= 0, \quad \partial_t F^i + \partial_k G^{ki} = 0, \quad \partial_t F^{ij} + \partial_k G^{kij} = p^{<ij>}, \\
\partial_t F^{ill} + \partial_k G^{k^{ill}} &= p^{ill}, \quad \partial_t F^{iill} + \partial_k G^{k^{iill}} = p^{iill}.
\end{align*} \tag{69} \]

Note that the only difference between this system and (1) is that in (69) intervenes \( G^k \); while in (1) there is \( F^k \); for this reason we will have to impose the further condition \( G^k = F^k \).

### 4.3.2 Reasons for the above choice of coefficients.

The coefficients in the above linear combinations have been chosen for the following reasons: from [58], [59], [60], [62], [63], [67], [68] it follows

\[
F = \int \tilde{f} \gamma^7 (1 - \frac{u^2}{c^2}) d, \quad F^i = \int \tilde{f} \gamma^6 u^i d\mathbf{u} = \int \tilde{f} \gamma^5 d\mathbf{u}, \quad F^{ill} = 2c^2 \int \tilde{f} \gamma^7 (1 - \frac{u^2}{c^2}) d\mathbf{u}, \quad F^{<ij>} = \int \tilde{f} \gamma^7 (u^i u^j - \frac{1}{3} u^2 \delta^{ij}) d\mathbf{u}, \quad F^{iill} = 2c^2 \int \tilde{f} \gamma^7 u^i (1 - \frac{1}{\gamma}) d\mathbf{u},
\]

which have limits \( \int \tilde{f} d\mathbf{u}, \int \tilde{f} u^i d\mathbf{u}, \int \tilde{f} u^2 d\mathbf{u}, \int \tilde{f} u^i u^j d\mathbf{u}, \int \tilde{f} u^2 d\mathbf{u}, \int \tilde{f} u^4 d\mathbf{u} \), where we have taken into account also that
\[
\frac{1}{\gamma} = \sqrt{1 - \frac{u^2}{c^2}} = 1 - \frac{u^2}{2 c^2} - \frac{1}{8} \frac{u^4}{c^4} + \frac{u^6}{8 c^6} (\ldots) . \tag{70}
\]

From the previous expressions it follows that \( F^{ij} \) has limit \( \int \tilde{f} u^i u^j d\mathbf{u} \). Obviously, the previous properties hold in the case of ideal gases; we assume that the corresponding change of equations is appropriate also for dense gases and for macromolecular fluids. In this way generality is not lost because the Galilean relativity principle has the same form for both cases.

### 4.4 Second Transformation of the Lagrange multipliers

The change of equations in the previous section induces another one on the Lagrange multipliers and we want now to determine it.

We observe that the equations (59), (60), (65), (67), [68] give \( F, F^i, F^{ij}, F^{ill}, F^{iill} \) in terms of \( F_2, F_3, F_3^{ij}, F_3^{ill} \). Let us now take the inverse of these relations, which are
\[
F_2 = \frac{F^{ill}}{2c^2} + F, \quad F_3^i = F^{ill}, \quad F_3^{ij} = F^{ij} + \frac{\delta^{ij} F^{ppll}}{3 \frac{4c^2}{e}}.
\] \tag{71}

and these we now substitute in eq. (45); so it becomes
\[
\begin{align*}
dh &= l d \left( \frac{F^{ill}}{2c^2} + F \right) + \eta d (F + \frac{F^{ppll}}{4c^2} + \frac{F^{ill}}{c^2}) + \\
&+ \mu_i d (\frac{F^{ill}}{2c^2} + F^i) + \mu_{ij} d (F^{ij} + \frac{1}{3} \delta^{ij} \frac{F^{ppll}}{4c^2}) = \\
&= \lambda d F + \lambda_i d F^i + \lambda_{ij} d F^{ij} + \lambda_{iill} d F^{iill} \lambda_{ppll} d F^{ppll}
\end{align*} \tag{72}
With
\[
\lambda = l + \eta \quad , \quad \lambda_l = l_i + \mu_i \quad , \quad \lambda_{ij} = \mu_{ij} + \left( \frac{l}{2c^2} + \frac{\eta}{c^2} \right) \delta_{ij}
\]
(73)
\[
\lambda_{illl} = \frac{\mu_i}{2c^2} \quad , \quad \lambda_{ppll} = \frac{\eta}{4c^4} + \frac{\mu_{ll}}{12c^2}
\]
In this way we have found the first part of the entropy principle for the new system.
For the sequel it will be useful to take the inverse of eqs. (73). They are
\[
\eta = 8c^4 \lambda_{ppll} - \frac{2}{3} \lambda_{ll} + \lambda \quad , \quad \ell = -8c^4 \lambda_{ppll} + \frac{2}{3} \lambda_{ll} + \mu_i = 2c^2 \lambda_{illl}
\]
(74)
Similarly, eqs. (59) give \( G^k, G^{ki}, G^{kij}, G^{kil}, G^{kili}, G^{kl} \), in terms of \( G^k, G^{ki}, G^{kl}, G^{ki} \), \( k = 3, G^{kii} \). The inverses of these relations are
\[
G^2 = \frac{G^{kll}}{2c^2} + G^k \quad , \quad G^{kii} = G^{kii}
\]
(75)
which now we substitute into eq. (48) which now becomes
\[
d\phi^k = \ell d\left( \frac{G^{kll}}{2c^2} + G^k \right) + \ell_i dG^{ki} + \eta d\left( G^k + \frac{G^{kppll}}{4c^4} + \frac{G^{kll}}{c^2} \right) + \mu_i d\left( G^{kii} + \frac{1}{3} \delta_{ij} G^{kppll} \right) = \lambda dG^k + \lambda_i dG^{ki} + \lambda_{ij} dG^{kij} + \lambda_{ill} dG^{kil} + \lambda_{ppll} dG^{ppll}
\]
thanks to eq. (73).
In this way we have obtained the second part of the entropy principle. We deduce now the transformation of \( h' \) and \( \phi'^k \); to this end, let us take eq. (52) and substitute eqs. (71); so we obtain
\[
h = -h' + l \left( \frac{F^{ll}}{2c^2} + F \right) + l_i F^i + \eta \left( F + \frac{F^{ppll}}{4c^4} + \frac{F^{ll}}{c^2} \right) + \mu_i \left( \frac{F^{ll}}{2c^2} + F^i \right) + \mu_{ij} \left( F^{ij} + \frac{\delta_{ij} F^{ppll}}{3} \right)
\]
(77)
which, for eq. (73), can be written in the following way
\[
h = -h' + \lambda F + \lambda_i F^i + \lambda_{ij} F^{ij} + \lambda_{ll} F^{ll} + \lambda_{ppll} F^{ppll}
\]
(78)
From eqs. (55), by substituting from eqs. (75) we find
\[
\phi^k = -\phi'^k + \ell \left( \frac{G^{kll}}{2c^2} + G^k \right) + \ell_i G^{ki} + \frac{\eta}{4c^4} G^{kppll} + \frac{G^{kll}}{2c^2} + G^{ki}) + \mu_i \left( G^{kii} + \frac{1}{3} \delta_{ij} G^{kppll} \right) = -\phi'^k + \lambda G^k + \lambda_i G^{ki} + \lambda_{ij} G^{kij} + \lambda_{ill} G^{kil} + \lambda_{ppll} G^{ppll}
\]
(79)
for eqs. (73).
The equations (72) and (76) gives the entropy principle for the balance equations (69). The equations (78) and (79) give the counterparts of (52) and (55) for the new system (69). We have now to obtain \( h' \) and \( \phi'^k \) from (29) expressing them in terms of the new Lagrange multipliers and then taking the limits for \( c \rightarrow \infty \). The functions \( h' \) and \( \phi'^k \) are called "potentials".
5 Determination of the potentials

With eqs. (78) and (79) we have found that the functions $h'$ and $\phi'^k$ for the new system (69) are exactly the same obtained, through the above mentioned passages, from the 4-potential $h'^\alpha$ for the system (21). Consequently, we may obtain them from the result (29) of the relativistic system. To this end it is necessary to write the relativistic Lagrange multipliers in terms of those for the system (69). By deducing $\lambda_0$, $\lambda_i$, $\lambda_{00}$, $\lambda_{0i}$ and $\lambda_{ij}$ from (46), and by substituting in their expressions eqs. (74) we obtain

$$
\lambda_{\beta \gamma} = \frac{c^2}{m_0^2} \left[ \begin{pmatrix} -8\lambda_{pigg} & 0_j \\ 0_i & -4\lambda_{pigg}\delta_{ij} \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 & -\lambda_{ill} \\ \lambda_{ill} & 0_{ij} \end{pmatrix} \right] + \frac{1}{c^2} \begin{pmatrix} 2\lambda_{ll} & 0_j \\ 0_i & \lambda_{ij} + \frac{1}{3}\lambda_{ll}\delta_{ij} \end{pmatrix} + \frac{1}{c^4} \begin{pmatrix} -\lambda & 0_j \\ 0_i & -\lambda \delta_{ij} \end{pmatrix},
$$

$$
\lambda^\beta = \frac{c^4}{m_0} \left[ \begin{pmatrix} 8\lambda_{pigg} \\ 0_i \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 \\ -2\lambda_{ill} \end{pmatrix} + \frac{1}{c^2} \begin{pmatrix} -2\lambda_{ll} \\ 0_i \end{pmatrix} + \frac{1}{c^3} \begin{pmatrix} 0 \end{pmatrix} \right].
$$

We can now begin to evaluate the 4-vectors intervening in (29) and the scalars (30).

5.1 The scalars $Q_1 - Q_4$

Let us begin with $Q_1$. From eq. (80) we find

$$Q_1 = \frac{c^2}{m_0^2} \left( -20\lambda_{pigg} + \frac{1}{c^2} \frac{8}{3}\lambda_{ll} - \frac{4}{c^4}\lambda \right). \tag{81}
$$

Consequently, if we assume that the scalar functions depend on $Q_1$ as composite functions through $\frac{m_0^4}{2\hbar c}Q_1$, then their limits will be functions of $\lambda_{pigg} = X_1$.

Let us consider now eq. $Q_2$. From eq. (80) we find

$$Q_2 = \frac{c^4}{m_0} \left( 112\lambda_{pigg}^2 + 0\left(\frac{1}{c}\right) \right),
$$

where $0\left(\frac{1}{c}\right)$ has limit zero when $c$ goes to infinity. Therefore, if we assume that the scalar functions depend on $Q_2$ as composite functions through $\frac{m_0^4}{\hbar c}Q_2$, then their limits will be functions of $112\lambda_{pigg}^2$. But this is not independent from $X_1$, so that this result is too much restrictive. The idea is now to find a number $k$ such that $\frac{m_0^4}{\hbar c}(Q_2 + kQ_1^2)$ has zero limit for $c$ going to infinity. We find $k = -\frac{7}{25}$. After that, we see that

$$Q_2 - \frac{7}{25}Q_1^2 = \frac{c^2}{m_0^2} \left( -2\lambda_{alt}\lambda_{alt} + \frac{16}{5}\lambda_{pigg}\lambda_{ll} + 0\left(\frac{1}{c}\right) \right). \tag{82}
$$

Therefore, if we assume that the scalar functions depend on $Q_1$ and $Q_2$ as composite functions through $\frac{m_0^4}{2\hbar c}Q_1$ and $\frac{m_0^4}{\hbar c}(Q_2 - \frac{7}{25}Q_1^2)$, then their limits will be functions of $X_1$ ed $X_2$. 

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5.2 Limits of the 4-vectors in the expression of the relativistic potential

From the eqs. (53), (56) and (80) it follows that the term \( h_0 \lambda^\alpha \) contributes to \( \phi^{k\ell} \) the term \(-2H_0 \lambda_{k\ell\ell} \) and to \( h' \) the term \( 8H_0 \lambda_{p\ell\ell} \), where \( H_0 \) is the limit of \( h_0 \) for \( c \) going to infinity.

But, from eqs. (56) and (80) it follows that the term \( h_1 \lambda^\alpha \lambda_\gamma \) contributes to \( \phi^{k\ell} \) the term \( \frac{c^3}{m_0^3} h_1 16 \lambda_{p\ell\ell} \lambda_{k\ell\ell} \) which is parallel to the previous one. In order not to lose generality, it is better to look for a number \( a \) such that \( (\lambda^\alpha \lambda_\gamma + a Q_1 \lambda^\alpha) \frac{m_0^3}{c^3} \) has zero limit. We find \( a = -\frac{2}{5} \). After that, we obtain

\[
\lambda^\alpha \lambda_\gamma - \frac{2}{5} Q_1 \lambda^\alpha = \frac{c^3}{m_0^3} \left[ \begin{pmatrix} X_2 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 \\ V_1^i \\ 0 \end{pmatrix} \right] + \frac{1}{c} \begin{pmatrix} 0 \\ V_1^i \\ 0 \end{pmatrix} \]

Now a linear combination, with arbitrary coefficients, of \( \lambda^\alpha, \lambda^\alpha \lambda_\gamma, \lambda^2 \lambda^\alpha \lambda_\gamma, 3 \lambda^\alpha \lambda_\gamma, 4 \lambda^\alpha \lambda_\gamma \) is also a linear combination, with arbitrary coefficients, of \( \lambda^\alpha, \lambda^\alpha \lambda_\gamma - \frac{2}{5} Q_1 \lambda^\alpha, \lambda^2 \lambda^\alpha \lambda_\gamma, 3 \lambda^\alpha \lambda_\gamma \) so we can suppose that in (29) there is \( \lambda^\alpha \lambda_\gamma - \frac{2}{5} Q_1 \lambda^\alpha \) instead of \( \lambda^\alpha \lambda_\gamma \); After that, from eqs. (53) and (56) it follows that the term \( h_1 (\lambda^\alpha \lambda_\gamma - \frac{2}{5} Q_1 \lambda^\alpha) \) contributes to \( \phi^{k\ell} \) the term \( H_1 (-2\lambda_\ell h_1 + 4 \lambda_{p\ell\ell} \lambda_{k\ell\ell} \) and to \( h' \) the terms \( H_1 X_2, \) where \( H_1 \) is the limit of \( \frac{c^3}{m_0^3} h_1 \) for \( c \) going to infinity. Proceeding furtherly in this way, we search the numbers \( a_1, a_2, a_3 \), such that

\[
\frac{m_0^3}{c^3} \left[ \begin{pmatrix} \alpha \gamma \lambda_\gamma & a_1 Q_1 \lambda^\alpha \lambda_\gamma + [a_2 Q_1^2 + a_3 (Q_2 - \frac{7}{25} Q_1^2)] \lambda^\alpha \end{pmatrix}, has zero limit for \( c \) going to infinity. We find \( a_1 = -\frac{3}{5}, a_2 = \frac{2}{25}, a_3 = -\frac{1}{2} \). After that, we obtain

\[
\alpha \gamma \lambda_\gamma - \frac{3}{5} Q_1 \alpha \gamma \lambda_\gamma + \left[ \frac{2}{25} Q_1^2 \lambda^\alpha - \frac{1}{2} (Q_2 - \frac{7}{25} Q_1^2) \right] \lambda^\alpha = \frac{c^3}{m_0^3} \left[ \begin{pmatrix} X_3 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 \\ V_2^i \\ 0 \end{pmatrix} \right] + \frac{1}{c} \begin{pmatrix} 0 \\ V_2^i \\ 0 \end{pmatrix} \]
This 4-vector can replace \( \lambda_\alpha \gamma_\gamma \) in (29); so we find that it, together with the factor \( h_2 \), contributes to \( \phi^{4k} \) the term \( H_2(-2\lambda_{kk}^2\lambda_{all} + \frac{6}{5}\lambda_{ll}^3\lambda_{ka} + 4\lambda_{ka}\lambda_{lpql} - \frac{11}{25}\lambda_{ll}^2\lambda_{ll} - \lambda_{lll}\lambda_{all} + \lambda_{kll}\lambda_{all} + \lambda_{kall}\lambda_{all} + (tr\lambda_{ab}^2)\lambda_{lll} - \frac{4}{5}\lambda_{lll}^2\lambda_{lll}\lambda_{kll}) \) and to \( h' \) the term \( H_2X_3 \), with \( H_2 \) limit of \( \frac{e^4}{m_0^2}h_2 \) for \( c \) going to infinity.

Finally, we search the numbers \( a_4, a_5, a_6, a_7, a_8, a_9 \) such that

\[
\frac{m_0^4}{e^2}
\left\{ \lambda_\alpha \gamma_\gamma + a_4Q_1 \lambda_\alpha \gamma_\gamma + [a_5Q_1^2 + a_6(Q_2 - \frac{7}{25}Q_1^2)] \lambda_\alpha \gamma + [a_7Q_3 + a_8Q_1^2 + a_9Q_1(Q_2 - \frac{7}{25}Q_1^2)] \lambda_\alpha \right\},
\]
has zero limit for \( c \) going to infinity. We find \( a_4 = -\frac{1}{5}, a_5 = \frac{1}{8}, a_6 = -\frac{1}{4}, a_7 = -\frac{1}{3}, a_8 = \frac{1}{16}, a_9 = \frac{2}{3} \). After that we have the result

\[
\begin{align*}
3\lambda_\alpha \gamma_\gamma - \frac{4}{5}Q_1\lambda_\alpha \gamma_\gamma + \left[ \frac{1}{5}Q_1^2 - \frac{1}{2}(Q_2 - \frac{7}{25}Q_1^2) \right] \lambda_\alpha = \\
\left[ -\frac{1}{3}Q_3 + \frac{1}{75}Q_1^3 + \frac{2}{5}Q_1(Q_2 - \frac{7}{25}Q_1^2) \right] \lambda_\alpha = \\
= \frac{c^3}{m_0^2} \left[ \begin{pmatrix} X_4 \\ X_3 \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right].
\end{align*}
\]

It may replace \( 3\lambda_\alpha \gamma_\gamma \) in (29), so that it, together with the factor \( h_3 \), contributes to \( \phi^{4k} \) the term \( H_3\left(2\lambda_{ppql}(-2\lambda_{kk}^2\lambda_{ll} - tr\lambda_{ab}^2)\lambda_{ll} - \frac{17}{25}\lambda_{lll}^2\lambda_{ka} - (\lambda_{lll}\lambda_{all} + \lambda_{kll}\lambda_{all} + \lambda_{kall}\lambda_{all} + (tr\lambda_{ab}^2)\lambda_{lll} + \frac{17}{25}\lambda_{lll}^2\lambda_{lll}\lambda_{kll} + \lambda_{lll}\lambda_{all}\lambda_{lll} + \lambda_{lll}\lambda_{all}\lambda_{lll}) \right) \) and to \( h' \) the term \( H_3X_4 \), with \( H_3 \) limit of \( \frac{e^4}{m_0^2}h_3 \) for \( c \) going to infinity.

### 5.3 The scalars \( P_0 - P_3 \)

From eq. (30), we find \( P_0 = \frac{e^6}{m_0^2}(64\lambda_{ppql}^2 + 0(\frac{1}{c})) \) so that the limit of \( \frac{e^6}{m_0^2}P_0 \) is a function of \( X_1 \). In order to obtain a less restrictive result, we may substitute \( P_0 \) with \( P_0 + \frac{4}{25}Q_1^2m_0^2c^2 \), because this will eliminate the term \( \frac{e^6}{m_0^2}4\lambda_{ppql}^2 \) from \( P_0 \); in fact we obtain

\[
P_0 + \frac{4}{25}Q_1^2m_0^2c^2 = \frac{c^4}{m_0^2} \left(-2X_2 + 0(\frac{1}{c})\right).
\]

In this way we have a better result, even if it is still not enough; to this end let us substitute \( P_0 + \frac{4}{25}Q_1^2m_0^2c^2 \) with \( P_0 + \frac{4}{25}Q_1^2m_0^2c^2 + 2(Q_2 - \frac{7}{25}Q_1^2)m_0^2c^2 \) and the result is satisfactory because is equal to

\[
\frac{e^2}{m_0^2} \left(-\frac{2}{5}\lambda_{ll}^2 + 16\lambda_{ppql}\lambda - 4\lambda_{lll}\lambda_{all} + 2tr\lambda_{ab}^2 + 0(\frac{1}{c})\right).
\]

In this way we have found the new scalar \( X_5 \).

Let us now consider the scalar \( P_1 \) in (30). It is obvious, for eq. (85) that it is better to replace it with \( P_1 - \frac{2}{5}Q_1P_0 \); but this has limit a scalar which is a function of \( X_1, X_2, X_3 \). Briefly, let us look for the numbers \( b_1 \) and \( b_2 \) such that

\[
\frac{m_0^4}{c^4} \left\{ P_1 - \frac{2}{5}Q_1P_0 + b_1Q_1 \left(Q_2 - \frac{7}{25}Q_1^2\right) m_0^2c^2 + b_2 \left[ Q_3 - \frac{9}{10}Q_1(Q_2 - \frac{7}{25}Q_1^2) - \frac{11}{125}Q_1^3 \right] m_0^2c^2 \right\}
\]
has zero limit.
We find \( b_1 = \frac{2}{5}, b_2 = \frac{4}{3} \). After that we have

\[
P_1 = \frac{2}{5}Q_1P_0 + \frac{2}{5}Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) m_6^2 c^2 + \frac{4}{3} \left[ Q_3 - \frac{9}{10} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{11}{125} Q_1^3 \right] m_6^2 c^2 =
\]

\[
= \frac{c^2}{m_6^2} \left( X_6 + 0(\frac{1}{c}) \right),
\]

from which the new scalar \( X_6 \).
Let us consider now the scalar \( P_2 \) in Eq. (30). It is obvious, for eq. (86) that it is better replace it with

\[
P_2 = \frac{2}{5}Q_1P_1 + \left[ \frac{2}{25} Q_1^2 - \frac{1}{2} \left( Q_2 - \frac{7}{25} Q_1^2 \right) \right] P_0;
\]

but also this is not enough; then we search the numbers \( b_3 \) and \( b_4 \) such that

\[
\frac{m_6^2}{c^4} \left\{ P_2 - \frac{3}{5} Q_1 P_1 + \left[ \frac{2}{25} Q_1^2 - \frac{1}{2} \left( Q_2 - \frac{7}{25} Q_1^2 \right) \right] P_0 + \right.
\]

\[
+ b_3 Q_1 \left[ Q_3 - \frac{9}{10} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{11}{125} Q_1^3 \right] m_6^2 c^2 \right\} +
\]

\[
+ b_4 \frac{m_6^2}{c^4} \left\{ Q_4 - \frac{16}{15} Q_1 \left[ Q_3 - \frac{9}{10} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{11}{125} Q_1^3 \right] + \right.
\]

\[
- \frac{1}{2} \left( Q_2 - \frac{7}{25} Q_1^2 \right)^2 - \frac{14}{25} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{19}{625} Q_1^4 \left\} m_6^2 c^2
\]

has zero limit.
We find \( b_3 = \frac{4}{15}, b_4 = 1 \). After that it follows

\[
P_2 = \frac{3}{5} Q_1 P_1 + \left[ \frac{2}{25} Q_1^2 - \frac{1}{2} \left( Q_2 - \frac{7}{25} Q_1^2 \right) \right] P_0 + \]

\[
+ \frac{4}{15} Q_1 \left[ Q_3 - \frac{9}{10} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{11}{125} Q_1^3 \right] m_6^2 c^2 + \]

\[
+ \left\{ Q_4 - \frac{16}{15} Q_1 \left[ Q_3 - \frac{9}{10} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{11}{125} Q_1^3 \right] \right. + \right.
\]

\[
- \frac{1}{2} \left( Q_2 - \frac{7}{25} Q_1^2 \right)^2 - \frac{14}{25} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{19}{625} Q_1^4 \} m_6^2 c^2 =
\]

\[
= \frac{c^2}{m_6^2} \left( X_7 + 0(\frac{1}{c}) \right),
\]

from which the new scalar \( X_7 \).
Finally, let us consider the scalar \( P_3 \) in Eq. (33). It is obvious, for eq. (87) that it is better replace it with

\[
P_3 = \frac{4}{5} Q_1 P_2 + \left[ \frac{1}{5} Q_1^2 - \frac{1}{2} \left( Q_2 - \frac{7}{25} Q_1^2 \right) \right] P_1 + \left[ -\frac{1}{3} Q_3 + \frac{1}{15} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) \right] P_0 + \]

\[
+ b_5 Q_1 \frac{m_6^8}{c^6} \left\{ Q_4 - \frac{16}{15} Q_1 \left[ Q_3 - \frac{9}{10} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{11}{125} Q_1^3 \right] + \right.
\]

\[
- \frac{1}{2} \left( Q_2 - \frac{7}{25} Q_1^2 \right)^2 - \frac{14}{25} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{19}{625} Q_1^4 \} m_6^2 c^2
\]

\[
= \frac{c^2}{m_6^2} \left( X_7 + 0(\frac{1}{c}) \right),
\]
has zero limit.

We find \( b_5 = \frac{1}{5} \). Then we can evaluate

\[
P_3 - \frac{4}{5} Q_1 P_2 + \left[ \frac{1}{5} Q_1^2 - \frac{1}{2} (Q_2 - \frac{7}{25} Q_1^2) \right] P_1 + \left[ -\frac{1}{3} Q_3 + \frac{1}{75} Q_1^3 + \frac{2}{5} Q_1 (Q_2 - \frac{7}{25} Q_1^2) \right] P_0 +
\]

\[
+ \frac{1}{5} Q_1 \left\{ Q_4 - \frac{16}{15} Q_1 \left[ Q_3 - \frac{9}{10} Q_1 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{11}{125} Q_1^4 \right] + \right. 
\]

\[
- \frac{1}{2} \left( Q_2 - \frac{7}{25} Q_1^2 \right)^2 - \frac{14}{25} Q_1^2 \left( Q_2 - \frac{7}{25} Q_1^2 \right) - \frac{19}{625} Q_1^4 \right\} m_0^2 c^2 =
\]

\[
= \frac{c^2}{m_0^2} \left( X_8 + O(\frac{1}{c}) \right),
\]

from which the last new scalar \( X_8 \).

**NOTE:** We have obtained the decomposition (80) by considering a modified procedure of that introduced in [10], [11] for ideal gases; instead of this, if we use exactly the procedure of [10], [11], we have to substitute (80) with

\[
\lambda_{ij} = \frac{c^2}{m_0^2} \left[ \begin{pmatrix} -8 \lambda_{ppl} & 0_j \\ 0_i & -4 \lambda_{ppl} \delta_{ij} \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 & -\lambda_{ill} \\ \lambda_{ill} & 0_{ij} \end{pmatrix} + \right.
\]

\[
+ \frac{1}{c^2} \begin{pmatrix} 0 & \lambda_{ij} \\ 0_i & \lambda_{ij} \end{pmatrix} \right],
\]

\[
\lambda^2 = \frac{c^3}{m_0} \left[ \begin{pmatrix} 8 \lambda_{ppl} \\ 0_i \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 \\ -2 \lambda_{ill} \end{pmatrix} \right].
\]

Another procedure is present in literature ( [12], [13]) also for ideal gases. If we want to follow it, then we have to substitute (80) with

\[
\lambda_{ij} = \frac{c^2}{m_0^2} \left[ \begin{pmatrix} -3 \lambda_{ppl} & 0_j \\ 0_i & \lambda_{ppl} \delta_{ij} \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 & -\lambda_{ill} \\ \lambda_{ill} & 0_{ij} \end{pmatrix} + \right.
\]

\[
+ \frac{1}{c^2} \begin{pmatrix} 0 & \lambda_{ij} \\ 0_i & \lambda_{ij} \end{pmatrix} \right],
\]

\[
\lambda^2 = \frac{c^3}{m_0} \left[ \begin{pmatrix} 8 \lambda_{ppl} \\ 0_i \end{pmatrix} + \frac{1}{c} \begin{pmatrix} 0 \\ -2 \lambda_{ill} \end{pmatrix} + \frac{1}{c^2} \begin{pmatrix} -\frac{2}{3} \lambda_{ill} \\ \lambda_{ij} \end{pmatrix} \right],
\]

\[
\xi = c^4 \left[ 5 \lambda_{ppl} - \frac{2}{3} \lambda_{ill} + \frac{\lambda}{c^2} \right].
\]

We have performed calculations also with (88) and with (89) instead of (80), but we don’t report them for the sake of brevity. The interesting result is that the pertinent polynomials in \( 1/c \) are different between them and from those here obtained, but the limits for \( c \) going to infinity are the same with all 3 approaches!
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