A NOTE ON THE ARRANGEMENT OF SUBGROUPS IN THE AUTOMORPHISM GROUPS OF SUBMODULE LATTICES OF FREE MODULES

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Abstract. A complete description of subgroups in the general linear group over a semilocal ring containing the group of diagonal matrices was obtained by Z.I.Borewicz and N.A.Vavilov. It is shown in the present paper that a similar description holds for the intermediate subgroups of the group of all automorphisms of the lattice of right submodules of a free finite rank $R$–module over a simple Artinian ring containing the group consisting of those automorphisms which leave invariant an appropriate sublattice. Bibliography: 12 titles.

§ 1. Introduction

The following result is due to Z.I.Borewicz and N.A.Vavilov (see [2] for a preliminary version).

Theorem [9]. Let $R$ be a semilocal ring such that the decomposition of the factor-ring $R/J(R)$ in the direct sum of simple Artinian rings does not include either fields containing less than seven elements, or the full matrix ring $M(2, \mathbb{F}_2)$. Then for every intermediate subgroup $F$, $D(n, R) \leq F \leq GL(n, R)$, there exists a unique $D$–net $\sigma$

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of two-sided ideals in $R$ of order $n$ such that $G(\sigma) \leq F \leq N(\sigma)$, where $N(\sigma)$ is the normaliser of the net subgroup $G(\sigma)$ in $\text{GL}(n, R)$.

One may find a wealth of background information and many further related references in the surveys [10], [11].

This theorem was generalised by A.Z. Simonian [8], A.A. Panin and A.V. Yakovlev [6], A.A. Panin [4], [5]. It was shown that these results can be stated in terms of a Galois correspondence between sublattices of the lattice of (right) submodules of a free module $R^n$ and subgroups of $\text{GL}(n, R)$ considered as automorphism groups of this lattice.

One can try to describe subgroups of the group of all automorphisms of this lattice containing the group of “diagonal” automorphisms (i.e., the group consisting of those automorphisms which leave invariant its appropriate sublattice).

We solve this problem for a simple Artinian ring in the present paper.

§ 2. Preliminaries

Let $L$ be a lattice and let $G$ be a subgroup of the group $\text{Aut}(L)$ of all automorphisms of the lattice $L$. Consider a subgroup $F$ of the group $G$ and a sublattice $M$ of the lattice $L$. By definition, put

$$L(F) = \{ l \in L \text{ such that } f(l) = l \text{ for every } f \in F \},$$

$$G(M) = \{ g \in G \text{ such that } g(m) = m \text{ for every } m \in M \}.$$  

Let $L$ be a complete modular lattice and $L_0$ be a sublattice of $L$ which is a finite Boolean algebra such that $1_L = 1_{L_0}$ and $0_L = 0_{L_0}$. Consider a subgroup $G$ of the group $\text{Aut}(L)$. Let $H = G(L_0)$, $L'_0 = L(H)$. By $e_1, e_2, \ldots, e_n$ we denote the atoms of $L_0$.

For every $x \in L$ the collection $([x]_1, \ldots, [x]_n)$, where $[x]_i = (x + \sum_{j \neq i} e_j) \cdot e_i$, is called the support of $x$ and is denoted by $[x]$. The support can be defined equivalently as the minimal (with respect to the natural ordering) collection $(x_1, \ldots, x_n)$, where $x_i \leq e_i$ and $x \leq \sum_{i=1}^n x_i$ (see [5], [6]).

Certain subgroups of type $G(M)$, where $M$ is a sublattice of $L'_0$, are of special interest. The following definition was introduced in [6].

**Definition.** A net collection in $L'_0$ is a collection of elements $\tau = (\tau_{ij})_{i,j=1}^n$ such that for every indices $i, j, k$ and $g \in G$:

1. $\tau_{ij} \leq e_j$
2. $\tau_{ii} = e_i$
3. $\tau_{ij} \in L'_0$
4. $[g(e_i)]_j \leq \tau_{ij}$ if and only if $[g(\tau_{ki})]_j \leq \tau_{kj}$.

For every net collection $\tau = (\tau_{ij})$ in $L'_0$ the lattice generated by elements $\sum_{j=1}^n \tau_{ij}$, $i = 1, \ldots, n$ is called the lattice associated with $\tau$ and is denoted by $K_\tau$. 

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§ 3. The main theorem

Let $L$ be the lattice of right submodules of a free $R$–module $M$ of rank $n \geq 2$, where $R$ is a semilocal ring such that the field $\mathbb{F}_2$ of two elements does not occur in the decomposition of the factor-ring $R/J(R)$ in the direct sum of simple Artinian rings. Consider a basis $\bar{e}_1, \ldots, \bar{e}_n$ of $M$ and let $e_i$ be spanned by $\bar{e}_i$. Let $L_0$ be the sublattice of $L$ generated by $e_1, \ldots, e_n$. One can consider $\text{PGL}(n, R)$ as a subgroup of $\text{Aut}(L)$. Since we are only interested in description of the intermediate subgroups, it does not make a difference to speak about $\text{PGL}(n, R)$ or $G = \text{GL}(n, R)$ here, with $H = G(L_0)$ playing the role of $\text{D}(n, R)$ in our case.

It is clear that $\sigma$ is a net collection in $L_0'$ if and only if the transposed collection of ideals $\sigma^T$ is a $D$–net over $R$, and in this case the net subgroup $G(\sigma^T)$ equals $G(K_{\sigma})$.

The main result of this article is the following

**Theorem 1.** Let $T$ be a skew field, $R = M(m, T)$ a full matrix ring over $T$ not equal to $\mathbb{F}_2, \mathbb{F}_3, \mathbb{F}_4, \mathbb{F}_5, M(2, \mathbb{F}_2)$, and $G = \text{Aut}(L)$ the group of all automorphisms of $L$. Let $H = G(L_0)$, $L_0' = L(H)$. Then for every subgroup $F$, $H \subseteq F \subseteq G$, there exists a unique $D$–net $\sigma$ of two-sided ideals in $R$ of order $n$ such that $G(K_{\sigma}) \subseteq F \subseteq \mathcal{N}_G G(K_{\sigma})$, where $K_{\sigma}$ is the lattice associated with the net collection $\sigma^T$ in $L_0'$.

**Proof.** First, note that $\sigma^T$ is still a net collection in $L_0'$ for every $D$–net $\sigma$ over $R$.

It is clear that the lattice of right $M(m, T)$–submodules of $M(m, T)^n$ is isomorphic to the lattice of right $T$–subspaces of $T^{nm}$.

Let $n \geq 3$. Using the Fundamental Theorem of Projective Geometry [1], we can assume that $G = \text{GL}(n, R) \rtimes \text{Aut}(T)$ and $H = \text{D}(n, R) \rtimes \text{Aut}(T)$. Thus each intermediate subgroup of $G$ containing $H$ is of the form $F \rtimes \text{Aut}(T)$, where $\text{D}(n, R) \subseteq F \subseteq \text{GL}(n, R)$. If the ring $R$ satisfies the conditions of Theorem 1, then for every such $F$ there exists a unique $D$–net $\sigma$ of two-sided ideals in $R$ of order $n$ such that $G(\sigma) \subseteq F \subseteq \mathcal{N}(\sigma)$. It is clear that $G(K_{\sigma}) = G(\sigma) \rtimes \text{Aut}(T)$. We have to prove that $G(K_{\sigma})$ is normal in $F \rtimes \text{Aut}(T)$ (it is not evident since $\text{Aut}(T)$ is not a normal subgroup of $G$). If $f \in F$, then $f^{-1}hf \in G(K_{\sigma})$ for every $h \in D(n, R)$, therefore $f(\sum_{i=1}^n \sigma_{ij}) \in L_0'$. Since $R$ is a simple ring, we obtain $L_0' = L_0$. Hence $f^{-1}\varphi f \in G(K_{\sigma})$ for every $\varphi \in \text{Aut}(T)$. The rest is trivial.

The Fundamental Theorem of Projective Geometry fails for $n = 2$, so one has to consider this case separately. We can assume that $R = T$ is a skew field. Then every proper nonzero subspace of $M$ is an atom of the lattice $L$, so the automorphism group of $L$ is organised very simply: it’s just the symmetric group acting on the set of atoms of $L$. Let $T$ be a skew field not equal to $\mathbb{F}_2$ and $\mathbb{F}_3$. Then $L$ contains at least 5 atoms. It’s easy to verify that there are only 5 intermediate subgroups: $H, \mathcal{N}_G H, G(M_1), G(M_2), G$, where $M_1 = \{e_1\}, M_2 = \{e_2\}$, and all subgroups except $H$ are self-normalisable. This completes the proof of Theorem 1.

If $n \geq 3$, then the assertion of Theorem 1 holds true for a somewhat broader
class of rings.

Let $R$ be a ring with the following property:

*each automorphism of $R$ leaves invariant all its two-sided ideals*

Such objects will be called *good* rings. It is clear that full matrix rings over skew fields, rings without automorphisms different from identical, and uniserial rings are good ones (recall that a ring is called uniserial, if its ideals are linearly ordered by inclusion).

**Theorem 2.** Let $R$ be a good semilocal ring such that the decomposition of the factor-ring $R/J(R)$ in the direct sum of simple Artinian rings does not include either fields containing less than seven elements, or the full matrix ring $M(2,\mathbb{F}_2)$. Let $n \geq 3$, $G = \text{Aut}(L)$ be the group of all automorphisms of $L$, $H = G(L_0)$, $L_0' = L(H)$. Then for every subgroup $F$, $H \leq F \leq G$, there exists a unique $D$–net $\sigma$ of two-sided ideals in $R$ of order $n$ such that $G(K_\sigma) \leq F \leq N_G(K_\sigma)$, where $K_\sigma$ is the lattice associated with the net collection $\sigma^T$ in $L_0'$.

**Proof.** Follow the lines of the proof of Theorem 1, using the Fundamental Theorem of Projective Geometry for rings of stable rank 2 [12].

§ 4. Final remarks

1°. The Fundamental Theorem of Projective Geometry holds in more general settings [3], [7], [12]. Therefore it is natural to try to generalise Theorems 1 and 2 in order to cover semilocal rings (i.e. to formulate the result in terms of lattices and their automorphism groups).

2°. The assertion of Theorem 1 for $n = 2$ and skew fields can be deduced from a more general theorem on the automorphisms of certain complete modular lattices, see [5].

**Definition.** For every $i \neq j$ and $x \leq e_j$ the set of $f \in G$ such that

(i) $f(e_s) = e_s$ for $s \neq i$,

(ii) $[f(e_i)]_k = \begin{cases} 0, & k \neq i, j; \\ e_i, & k = i; \\ x, & k = j \end{cases}$

is denoted by $H_{ij}(x)$ and its elements are called “transvections”.

The following result is a corollary of Theorem 3.1 [5] (see also [6] for a slightly weaker result).

**Theorem 3.** Let $L$ be a modular lattice of finite length, $L_0$ a sublattice of the same length which is a Boolean algebra with atoms $e_1, \ldots, e_n$, let $G$ be a subgroup of the group of all automorphisms of the lattice $L$, $H = G(L_0)$.

Assuming that the conditions (a) – (d) stated below are satisfied, for every subgroup $F \geq H$ of the group $G$ the group $G(K_\sigma)$ is normal in $F$. 
(a) For every $i$ there exists at least one automorphism from $H$ which changes all atoms $x \in L \setminus \{e_i\}$ such that $[x]_i = e_i$ and leaves invariant all atoms $x \in L$ with $[x]_i = 0$.

(b) If $x, y \in L$ are atoms with $[x] = [y]$, then there exists $h \in H$ such that $h(x) = y$.

(c) For every $i \neq j$ the set $H_{ij}(e_j)$ is not empty.

(d) For every $a \in G$ and $i \neq j$ the subgroup generated by $h \in H$ such that $H_{ij}(aha^{-1}(e_i)) \cap \langle a, H \rangle \neq \emptyset$, and $H_{ij}(ah_{-1}a^{-1}(e_i)) \cap \langle a, H \rangle \neq \emptyset$ is equal to $H$.

It is worth mentioning that the description of the intermediate subgroups of the general linear group over a skew field, containing the group of diagonal matrices, can also be deduced from this theorem. Moreover, it is likely that methods and results of [4], [5], [6] can bring further insight in our understanding of linear groups.

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