Fully Implicit Multidimensional Hybrid Upwind Scheme for Coupled Flow and Transport

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Abstract

Robust and accurate fully implicit finite-volume schemes applied to Darcy-scale multiphase flow and transport in porous media are highly desirable. Recently, a smooth approximation of the saturation-dependent flux coefficients based on Implicit Hybrid Upwinding (IHU) has been proposed to improve the nonlinear convergence in fully implicit simulations with buoyancy. Here, we design a truly multidimensional extension of this approach that retains the simplicity and robustness of IHU while reducing the sensitivity of the results to the orientation of the computational (Cartesian) grid. This is achieved with the introduction of an adaptive, local coupling between the fluxes that takes the flow pattern into account. We analyze the mathematical properties of the proposed methodology to show that the scheme is monotone in the presence of competing viscous and buoyancy forces and yields saturations remaining between physical bounds. Finally, we demonstrate the efficiency and accuracy of the scheme on challenging two-dimensional two-phase examples with buoyancy, with an emphasis on the reduction of the grid orientation effect.

Keywords: Porous media, Two-phase flow and transport, Implicit finite-volume schemes, Grid-orientation effect, Truly multidimensional schemes, Upwinding

1. Introduction

The numerical simulation of subsurface flow requires the design of accurate and robust discretization schemes for the highly nonlinear partial differential equations (PDEs) governing coupled flow and transport in porous media. The high heterogeneity characterizing geological porous media constitutes a challenge for the computational models used in practice. In realistic models, the rock properties (permeability and porosity) can vary by several orders of magnitude, which results in a vast range of flow velocities and time scales in the hyperbolic transport of species. High localized flow velocities impose severe restrictions on the time step size for explicit time integration methods. Therefore, a fully implicit, backward-Euler method guaranteeing unconditional stability is often preferred in the case of strongly nonlinear problems with high heterogeneity.

However, constructing accurate and efficient fully implicit schemes for subsurface flow simulation is a challenging task. In particular, the nonlinearity of the saturation-dependent coefficients present in the flux terms makes the design of high-resolution schemes difficult. Therefore, these coefficients are often approximated with robust first-order upwind schemes such as the commonly used Phase-Potential Upwinding (PPU, Peaceman, 1977; Aziz and Settari, 1979; Sammon, 1988; Brenier and Jaffré, 1991). But, even limited to the family of first-order schemes, the choice of an upwinding strategy has a significant impact on the accuracy of the flow predictions and on the computational cost of the numerical scheme.

Previous authors have shown that nonlinear convergence issues can result from the approximation of the interfacial phase flux (Wang and Tchelepi, 2013; Li and Tchelepi, 2015) and can consequently undermine the performance of the scheme since the fully implicit discretization requires solving large nonlinear systems at each time step. Specifically, when the approximation is based on PPU, convergence

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difficulties may arise when a small change in the primary variables – pressure and saturation – causes an abrupt change from cocurrent flow to countercurrent flow, and vice versa. This issue, referred to as the “flip-flopping” of the phase fluxes, was recently addressed with Implicit Hybrid Upwinding (IHU) in Lee et al. (2015); Lee and Efendiev (2016, 2018); Hamon and Tchelepi (2016); Hamon et al. (2016, 2018); Moncorgé et al. (2018). The IHU strategy is based on a separate evaluation of the viscous, buoyancy, and capillary parts to achieve a differentiable flux in the highly nonlinear transport problem. This reduces the flip-flopping of the phase fluxes and significantly improves the nonlinear convergence of Newton’s method while producing a similar accuracy as that of the standard PPU scheme.

The computational challenges are even greater in multiphase flows where the displacing phase has greater mobility than the displaced phase. Both the standard PPU and IHU upwinding procedures have limited accuracy for adverse mobility ratios because the saturation-dependent coefficients are approximated dimension-by-dimension. That is, the construction of the flux at an interface is entirely based on the saturation information contained in the two adjacent control volumes. A well-known issue resulting from this approach is the high sensitivity of the flow pattern to the orientation of the computational grid. In practice, these dimension-by-dimension schemes are therefore inaccurate when the flow is not aligned with the orientation of the computational grid. This unphysical grid orientation effect has been studied extensively for porous media problems (Yanosik and McCracken 1979; Pruess and Bodvarsson 1983; Shubin and Bell 1984; Brand et al. 1991). So-called multidimensional schemes taking into account the saturation information in an extended stencil around each interface have been designed to overcome this limitation of two-point upwinding. Following the work of Colella (1990); Koren (1991); Roe and Sidlikover (1992); LeVeque (1997); Berger et al. (2003) in computational fluid dynamics, the multidimensional schemes tailored for subsurface flow simulation include Chen et al. (1993); Arbogast and Huang (2006); Hurtado et al. (2007); Lamine and Edwards (2010, 2013, 2015); Eymard et al. (2012).

But, few of these schemes can be efficiently combined with a fully implicit discretization targeting highly nonlinear problems with competing viscous and buoyancy forces. In Kozdon et al. (2011a,b); Keilegavlen et al. (2012), the authors design a multidimensional scheme based on PPU that remains robust in fully implicit simulations of coupled multiphase flow and transport, and that is provably monotone in the presence of buoyancy. In this paper, we build on their approach to construct an efficient and accurate multidimensional IHU scheme. As in Kozdon et al. (2011a), we neglect capillary forces and focus on the mixed elliptic-hyperbolic incompressible two-phase flow and transport problem with viscous and buoyancy forces discretized on two-dimensional Cartesian grids. We propose a scheme that reduces the sensitivity of the results to the orientation of the computational grid while preserving a robust behavior of the Newton solver in the presence of competing viscous and buoyancy forces.

This is achieved by adapting the methodology previously used to construct the multidimensional PPU scheme. Specifically, to compute the viscous numerical flux, we introduce a local coupling between the phase fluxes belonging to the same interaction region of the dual grid. The buoyancy flux is computed separately and is upwinded based on fixed density differences. Importantly, this approach yields a numerical flux that is a monotone function of the saturations in the presence of viscous and buoyancy forces. We also prove that the saturation solution to the proposed fully implicit scheme remains between physical bounds, zero and one. Using numerical examples, we demonstrate that the proposed scheme is significantly less sensitive to the orientation of the grid than the two-point PPU and IHU approaches. We also illustrate the robustness of the multidimensional IHU by applying the scheme to challenging nonlinear test cases with large time steps.

We first describe the mathematical model describing coupled multiphase flow and transport with viscous and buoyancy forces in Section 2. Then, in Section 3, we proceed with the description of the fully implicit finite-volume scheme and we define the dual grid as the union of interaction regions. These interaction regions are used to introduce the truly multidimensional IHU in Section 4 and to study its mathematical properties in Section 5. Finally, using the nonlinear solver described in Section 6, we demonstrate the accuracy and robustness of the multidimensional scheme using numerical examples with buoyancy in Section 7.
2. Governing equations

In this work, we consider two immiscible and incompressible fluid phases flowing in an incompressible porous medium. Mass conservation for phase $\ell$ is expressed as

$$
\phi \frac{\partial S_\ell}{\partial t} + \nabla \cdot \mathbf{u}_\ell = q_\ell \quad \forall \mathbf{x} \in \Omega \subset \mathbb{R}^2, \quad \forall \ell \in \{1, 2\},
$$

where $\phi(\mathbf{x})$ is the porosity of the medium and $t$ is time. The source/sink term is denoted by $q_\ell$, with the convention that $q_\ell > 0$ for injection, and $q_\ell < 0$ for production. The saturation $S_\ell(\mathbf{x}, t)$ represents the fraction of the pore volume occupied by phase $\ell$, with the constraint that the sum of saturations is equal to one,

$$
\sum_\ell S_\ell = 1.
$$

Using the multiphase extension of Darcy’s law, the phase velocity $\mathbf{u}_\ell$ is written as a function of the phase potential gradient $\nabla \Phi_\ell$ as

$$
\mathbf{u}_\ell = -k\lambda_\ell \nabla \Phi_\ell = -k\lambda_\ell (\nabla p - \mathbf{g}_\ell) \quad \text{with} \quad \mathbf{g}_\ell = \rho_\ell \mathbf{g} \nabla z, \quad \forall \ell \in \{1, 2\},
$$

where capillary forces are neglected. In (3), $p(\mathbf{x}, t)$ is the pressure, $k(\mathbf{x})$ is the scalar rock permeability, $\lambda_\ell(S_1, S_2) = k_\ell(S_1, S_2)/\mu_\ell$ is the phase mobility – defined as the phase relative permeability divided by the phase viscosity –, $\rho_\ell$ is the phase density, $g$ is the gravitational acceleration, and $z$ is the depth.

We require that the mobilities be monotone functions of the saturations, which is realistic and holds for two-phase relative permeability models such as the Corey model [Corey, 1954].

**Assumption 1.** (Phase mobilities). For $\ell \in \{1, 2\}$, the mobility of phase $\ell$ is positive, and a differentiable function of the saturations. Furthermore, the mobility of phase $\ell$ is increasing with respect to its saturation and decreasing with respect to the saturation of the other phases:

$$
\frac{\partial \lambda_\ell}{\partial S_\ell} \geq 0 \quad \text{and} \quad \frac{\partial \lambda_\ell}{\partial S_m} \leq 0 \quad \forall m \neq \ell.
$$

The system of mixed elliptic-hyperbolic governing PDEs can be written in two equivalent forms, where the linearly independent primary variables are the pressure, $p$, and the wetting-phase saturation, denoted by $S = S_2$ for simplicity. In the remainder of this paper, we use the saturation constraint (2) to write the saturation-dependent properties as a function of $S$ only. The first form of the governing PDEs is obtained by inserting the expression of the phase velocities given by Darcy’s law (3) into the mass conservation equations (1),

$$
\phi \frac{\partial S_\ell}{\partial t} - \nabla \cdot (k\lambda_\ell (\nabla p - \mathbf{g}_\ell)) = q_\ell \quad \forall \ell \in \{1, 2\}.
$$

The second form of the governing PDEs is split into a flow problem and a transport problem. The flow problem is obtained by summing (1) over all phases and using the saturation constraint (2) to obtain:

$$
\nabla \cdot \mathbf{u}_T(\mathbf{x}, p, S) = \sum_\ell q_\ell = q_T,
$$

where the total velocity, $\mathbf{u}_T$, is defined as the sum of the phase velocities:

$$
\mathbf{u}_T(\mathbf{x}, p, S) = \sum_\ell \mathbf{u}_\ell(\mathbf{x}, p, S) = -k\lambda_T \nabla p + k \sum_\ell \lambda_\ell \mathbf{g}_\ell.
$$

We assume that the following assumption on the total mobility holds:

**Assumption 2.** (Total mobility). The total mobility $\lambda_T = \sum_\ell \lambda_\ell$ is bounded away from zero:

$$
0 < \epsilon \leq \lambda_T(S) \leq \chi \quad \forall S \in [0, 1].
$$
Equation (6), referred to as the pressure equation, is parabolic when compressibility is taken into account [Trangenstein and Bell (1989a,b)], and elliptic in the incompressible case that we will exclusively consider. The flow problem governs the evolution of the pressure variables as a function of space and time. It is coupled to the highly nonlinear transport of species, derived next. Equation (7) is used to express the pressure gradient as a function of $u^T$ and the weights $g^\ell (\ell \in \{1,2\})$, in order to eliminate the pressure variable from (3). We obtain

$$u^\ell(x, p, S) = \frac{\lambda^\ell}{\lambda^T} u^T(x, p, S) + k \sum_m \frac{\lambda_m \lambda^\ell}{\lambda^T} (g^\ell - g_m).$$

This allows us to rewrite the system of PDEs in the following fractional flow formulation which governs the hyperbolic transport of species as

$$\phi \frac{\partial S^\ell}{\partial t} + \nabla \cdot \left( \frac{\lambda^\ell}{\lambda^T} u^T(x, p, S) + k \sum_m \frac{\lambda_m \lambda^\ell}{\lambda^T} (g^\ell - g_m) \right) = q^\ell \quad \forall \ell \in \{1,2\}. \tag{10}$$

The system composed of (9) and (10) contains a redundant equation. Although it is possible to solve both equations in (10) by relaxing the saturation constraint, we enforce the saturation constraint and solve (9) along with (10) only for $\ell = 2$. We highlight that in the two equivalent systems of governing PDEs of (5), or (6) and (10), the flow is tightly coupled to the highly nonlinear transport of species through the mobility terms. Next, we employ a fully-implicit finite-volume scheme to discretize the system of governing PDEs, with an emphasis on the truly multidimensional computation of the numerical flux aiming at reducing grid orientation effects. As explained below, the proposed methodology is based on the fractional flow formulation and builds on the Implicit Hybrid Upwinding approach [Lee et al. 2015] to construct the flux approximation.

3. Numerical scheme

3.1. Implicit finite-volume scheme

We consider a grid consisting of $M$ control volumes discretizing the domain $\Omega$. We write $S_i = S_{2,i}$ to refer to the wetting-phase saturation in control volume $i$. We assume fixed positive injection terms balanced by variable (i.e., pressure-controlled) negative production terms. The governing PDEs are discretized using a fully implicit finite-volume scheme for $n \in \mathbb{N}^+$:

$$
\begin{cases}
\sum_{j \in \text{adj}(i)} u_{T,ij}^{n+1} = q_{T,i}(p_i, S_i) \\
V_i \phi_i \frac{S_{1,i}^{n+1} - S_{1,i}^n}{\Delta t^n} + \sum_{j \in \text{adj}(i)} F_{2,ij}^{n+1} = q_{2,i}(p_i, S_i),
\end{cases}
\tag{11}
$$

where $\Delta t^n = t^{n+1} - t^n$ is the time step, $V_i$ is the volume of control volume $i$, and $\text{adj}(i)$ is the set of neighbors of control volume $i$. $F_{\ell,ij}^{n+1}$ is the numerical flux of phase $\ell$, and approximates the phase velocity integrated over interface $(ij)$ between two control volumes $i$ and $j$ as follows

$$F_{\ell,ij}^{n+1} \approx \int_{\Gamma_{ij}} u_{\ell,ij}^{n+1}(x, p, S) \cdot n_{ij} \, d\Gamma_{ij} \tag{12}
$$

$$= \int_{\Gamma_{ij}} \frac{\lambda^\ell}{\lambda^T} u^T(x, p, S) \cdot n_{ij} \, d\Gamma_{ij} + \sum_m \int_{\Gamma_{ij}} k \frac{\lambda_m \lambda^\ell}{\lambda^T} (g^\ell - g_m) \cdot n_{ij} \, d\Gamma_{ij}, \tag{13}$$

where $n_{ij}$ is the outward normal to the interface $\Gamma_{ij}$. With the convention of (11), $F_{\ell,ij}^{n+1} \geq 0$ means that the positive mass flux is from control volume $i$ to control volume $j$.

The flux defined in (13) can be decomposed into two terms. The first term is a static transmissibility coefficient depending on the rock properties and the grid geometry in a stencil involving multiple control volumes in the neighborhood of the interface. This term is independent of the saturations and...
can be computed in a preprocessing step. In this work, we limit the scope of the numerical study to Cartesian grids and therefore exclusively consider a Two-Point Flux Approximation (TPFA) to compute this transmissibility term. An extension to Multi-Point Flux Approximations (MPFA [Aavatsmark et al., 1998; Edwards and Roberts 1998; Aavatsmark, 2002; Zhou et al., 2011]) based on a larger stencil as in Keilegavlen et al. (2012); Lamine and Edwards (2013); Souza et al. (2018) will be considered in future work. The second term is a highly nonlinear, non-convex function of the saturations whose computation has a significant impact on the robustness and the accuracy of the numerical scheme. The severe nonlinearity inherent to multiphase flows has limited the applicability of high-resolution schemes to subsurface flow simulation. Instead, monotone first-order upwinding schemes are often the discretization strategy of choice for this dynamic term.

But, commonly used two-point upwinding schemes such as Phase-Potential Upwinding (PPU) and Implicit Hybrid Upwinding (IHU) suffer from a strong sensitivity to the orientation of the computational grid for flows involving adverse mobility ratios. This leads to inaccurate predictions when the flow is not aligned with the grid, which is often the case in practical simulations. In this work, we address this issue with a multidimensional evaluation of the saturation-dependent coefficients based on a larger stencil that adapts to the flow direction. Our approach extends the original IHU approach to improve the accuracy of the fully implicit numerical scheme while retaining the robustness of the IHU scheme in the presence of strong buoyancy. The approximation of the saturation-dependent coefficients in the IHU scheme relies on the introduction of a dual grid and the definition of interaction regions, described next.

### 3.2. Integration region framework

In the proposed multidimensional IHU scheme, the upwinding is performed with an adaptive stencil containing at least two control volumes in the neighborhood of the interface under consideration. To define this stencil, we adapt the approach previously used in Kozdon et al. (2011a); Keilegavlen et al. (2012) and use a dual grid made of the union of interaction regions as illustrated in Fig. 1. Each control volume in the primal grid is part of $n_{ir}$ interaction regions in the dual grid. For simplicity, this work focuses on two-dimensional Cartesian grids and scalar permeability tensors, and therefore we have $n_{ir} = 4$ in the interior control volumes. As in Kozdon et al. (2011a), an interaction region is locally labeled counterclockwise with the superscript $(m)$, with $m \in \{1, \ldots, n_{ir}\}$, starting from the bottom left corner of the control volume. In addition, the $n_{ir}$ control volume segments that make up interaction region $(m)$ are labeled with subscript $k \in \{1, \ldots, n_{ir}\}$ as are the half interfaces. The pressure (respectively, the saturations) in subcell $k$ of interaction region $(m)$ is denoted by $p_k^{(m)}$ (respectively, $S_k^{(m)}$). Because we are developing first-order schemes, we can assume that these quantities are equal to the control-volume-centered variables, that is,

$$S_{\ell,i} = S_{\ell,k}^{(k+2)} \quad \text{and} \quad p_i = p_k^{(k+2)} \quad k \in \{1, \ldots, n_{ir}\}. \tag{14}$$

Note that the superscript $k$ is defined cyclically on $\{1, \ldots, n_{ir}\}$, and that, with a slight abuse of notation, we write $k + 2$ instead of $\text{mod}(k + 2 - 1, n_{ir}) - 1$ in the previous equations.

Considering the control volume interface $(ij)$ made up of half interface $k$ in interaction region $(k + 2)$, and of half interface $k - 2$ in interaction region $(k + 1)$, we define the rock- and geometric transmissibilities as

$$T_k^{(k+2)} = T_k^{(k+1)} = \frac{1}{2} T_{ij}, \tag{15}$$

where

$$T_{ij} = T_{ji} = \frac{1}{k_i \partial V_{ij} + k_j \partial V_{ji}} \cdot \frac{\hat{d}_{ij}}{k_i \partial V_{ij} + \hat{d}_{ji}}. \tag{16}$$

In (16), $\partial V_{ij} = \partial V_{ji}$ is the area of interface $(ij)$ and $k_i$ (respectively, $k_j$) denotes the absolute permeability in control volume $i$ (respectively, $j$). The quantity $\hat{d}_{ij}$ (respectively, $\hat{d}_{ji}$) denotes the distance from the center of control volume $i$ (respectively, $j$) to interface $(ij)$. We write the numerical flux, $F_{\ell,ij}$, as the difference of two numerical fluxes at the corresponding half interfaces:

$$F_{\ell,ij} = F_{\ell,k}^{(k+2)} - F_{\ell,k}^{(k+1)}, \tag{17}$$
where we dropped the superscripts denoting the time levels. Similarly, the total velocity at interface \((ij)\), \(u_{T,ij}\), can be written as the difference of two total velocities at the corresponding half interfaces:

\[
u_{T,ij} = \bar{\omega}_{T,k}^{(k+2)} - \bar{\omega}_{T,k}^{(k+1)} - 2.
\]

Here, we reiterate that the scheme relies on a fully implicit discretization and that all the variables in the total velocity are evaluated at time \(n + 1\). Next, we consider an interaction region \((m)\) and we detail the discretization of the half-interface-based quantities on the right-hand side of (17) and (18). The scheme is constructed to reduce the sensitivity of the results to the orientation of the computational grid by introducing a local coupling between the fluxes in each interaction region. A numerical flux at a given half interface can therefore be a function of all the saturations in the interaction region, leading to a stencil with a maximum of nine points in the discretization of the mobilities. To guarantee that the resulting scheme is well behaved, we require that for a fixed total velocity field, the fluxes satisfy the following monotonicity constraint in interaction region \((m)\):

\[
\frac{\partial (\bar{\omega}_{\ell,k}^{(m)} - \bar{\omega}_{\ell,k}^{(m-1)})}{\partial S^{(m)}_{\ell,j} \neq k} \leq 0 \quad \forall \ k \in \{1, \ldots, n_{ir}\}, \ \forall \ \ell \in \{1, 2\}.
\]

It follows from a mass conservation statement written at half interfaces \(k\) and \(k - 1\) that whenever the monotonicity constraint holds we also have

\[
\frac{\partial (\bar{\omega}_{\ell,k}^{(m)} - \bar{\omega}_{\ell,k}^{(m-1)})}{\partial S^{(m)}_{\ell,k}} \geq 0 \quad \forall \ \ell \in \{1, 2\}.
\]

A similar monotonicity condition is used in Kozdon et al. (2011a); Keilegavlen et al. (2012). In Section 4 we use this property to show that the saturation solution of the fully implicit scheme remains between physical bounds, 0 and 1.

4. Multidimensional Implicit Hybrid Upwinding

To construct the approximation, we follow the Implicit Hybrid Upwinding (IHU) approach (Eymard et al., 1989; Lee et al., 2015; Lee and Efendiev, 2016, 2018; Hamon and Tchelepi, 2016; Hamon et al., 2016, 2018; Moncorgé et al., 2018) and split the numerical flux at half interface \(k\) into a viscous part, \(\bar{V}_{\ell,k}\), and a buoyancy part, \(\bar{G}_{\ell,k}\), as follows

\[
\bar{F}_{\ell,k} = \bar{V}_{\ell,k} + \bar{G}_{\ell,k},
\]
where we omitted the superscript denoting the interaction region in the right-hand side. The viscous and buoyancy parts are then evaluated separately, based on physical considerations, to achieve a differentiable flux when the total velocity field is fixed. The proposed discretization of $V_{\ell,k}$ and $G_{\ell,k}$, explained next, attenuates grid orientation effects when the flow is not aligned with the computational grid, but reduces to the original IHU when the flow is aligned with the computational grid. We show in the next sections that it satisfies the monotonicity property outlined above, and that it preserves a robust nonlinear convergence behavior even in the presence of strong buoyancy forces.

Multiple strategies have been used to reflect the local flow pattern in the construction of multidimensional upwind schemes for porous media flow. In the IMPES fractional-flow based schemes of Lamine and Edwards (2010, 2013, 2015); Edwards (2011), the flow orientation is exploited in the computation of weighted averages of saturations, which are then used to evaluate mobility ratios. Instead, in the IMPES and fully implicit multidimensional PPU schemes of Kozdon et al. (2011a,b); Keilegavlen et al. (2012), the authors design a flux approximation based on (5) by directly computing weighted averages of mobilities to account for the flow orientation in each interaction region. Our methodology differs from these two approaches. Given the structure of the Implicit Hybrid Upwinding flux, we employ weighted averages of mobility ratios to obtain a fully implicit multidimensional IHU scheme satisfying the monotonicity property. The specifics of the discretization are given below.

4.1. Viscous term

In the viscous term $V_{\ell,k}$, the mobility ratio at half interface $k$ is approximated based on the sign of the total velocities in the interaction region, evaluated fully implicitly with the discretization detailed in Section 4.3. Specifically, we write the flux at half interface $k$ as

$$V_{\ell,k} = \bar{\chi}_{\ell,k} \bar{u}_{T,k}.$$  \hspace{1cm} (22)

The interfacial quantity $\bar{\chi}_{\ell,k}$ approximates the mobility ratio of the viscous term at half interface $k$ using a weighted average of the mobility ratio evaluated in the control volume and at the previous interface to reflect the local flow pattern in the interaction region, that is,

$$\bar{\chi}_{\ell,k} = \begin{cases} (1 - \omega_k^V) \chi_{\ell,k} + \omega_k^V \bar{\chi}_{\ell,k-1} & \text{if } \bar{u}_{T,k} > 0 \\ (1 - \omega_k^V) \chi_{\ell,k+1} + \omega_k^V \bar{\chi}_{\ell,k+1} & \text{otherwise}, \end{cases}$$  \hspace{1cm} (23)

where $\chi_{\ell,k}$ denotes the mobility ratio evaluated in subcell $k$:

$$\chi_{\ell,k} = \frac{\lambda_{\ell,k}}{\lambda_{T,k}} = \frac{\lambda_{\ell}(S_{\ell,k})}{\lambda_{T}(S_{\ell,k})}.$$  \hspace{1cm} (24)

The weighting coefficient $\omega_k^V$ is a function of the primary variables – pressure and saturation – in the interaction region and is constructed as the ratio of the total velocities at two consecutive interfaces:

$$\omega_k^V = \begin{cases} \varphi \left( \max \left(0, \frac{\bar{u}_{T,k-1}}{\bar{u}_{T,k}} \right) \right) & \text{if } \bar{u}_{T,k} > 0 \\ \varphi \left( \max \left(0, \frac{\bar{u}_{T,k+1}}{\bar{u}_{T,k}} \right) \right) & \text{if } \bar{u}_{T,k} < 0 \\ 0 & \text{otherwise}, \end{cases}$$  \hspace{1cm} (25)

where $\varphi : \mathbb{R}^+ \mapsto [0, 1]$ is a limiter function used to guarantee that $\omega_k^V \in [0, 1]$ and enforce monotonicity with respect to saturation. With this definition, it follows that the multidimensional viscous term (22) reduces to the viscous term in the original one-dimensional IHU scheme whenever $\varphi \equiv 0$. Different limiters have been proposed in previous work, such as the Tight Multi-D Upstream (TMU) limiter (Schneider and Raw, 1986; Hurtado et al., 2007) defined by $\varphi_{TMU}(r) = \min(1, r)$ and the Smooth Multi-D Upstream (SMU) limiter (Hurtado et al., 2007) computed with $\varphi_{SMU}(r) = r/(1 + r)$. For a constant (uniform) flow field, the SMU limiter corresponds to the upwind scheme of Koren (1991) and the TMU limiter corresponds to the narrow scheme of Roe and Sidikover (1992). In the absence of buoyancy, SMU aligns
the numerical diffusion tensor with the flow direction and TMU minimizes the transverse numerical diffusion (Kozdon et al., 2009). In addition, the TMU limiter introduces a discontinuity in the derivative of the weight $\omega_k^r$ that could undermine the nonlinear convergence behavior of the scheme. To overcome this issue, we propose a variant of the SMU limiter defined as

$$\varphi^{SMU4}(r) = \frac{r^4 + r^3 + r^2 + r}{r^4 + r^3 + r^2 + r + 1}.$$  

This limiter, referred to as fourth-order SMU (SMU4), is shown in Fig. 2. In the MultiD-IHU scheme, we use the SMU4 limiter, that is, we set $\varphi \equiv \varphi^{SMU4}$. We note that the SMU4 limiter satisfies the symmetry property

$$\varphi\left(\frac{1}{r}\right) = \frac{\varphi(r)}{r}. \tag{27}$$  

This property, also satisfied by the SMU and TMU limiters, ensures that backward-facing and forward-facing fluxes are treated the same way. It is also used to prove the monotonicity of the buoyancy term introduced in Section 4.2.

![Figure 2: Multidimensional limiters discussed in this work, including the Tight Multi-D Upstream (TMU) limiter of Schneider and Raw (1986), the Smooth Multi-D Upstream (SMU) limiter of Hurtado et al. (2007), and the fourth-order SMU limiter of (26).](image)

This methodology leads to a viscous flux that is a differentiable function of the saturations in the interaction region for a fixed total velocity field. We will show with numerical examples in Section 7 that the scheme remains well-behaved when the total velocity is a space- and time-dependent function of the primary variables. Equations (23) and (25) introduce a local coupling of the viscous terms in the interaction region that can be written in a compact matrix form as

$$A\chi_\ell = B\chi_\ell, \tag{28}$$

where $A = \{a_{jk}\} \in \mathbb{R}^{n_ir}$ has unit diagonal elements ($a_{kk} = 1$) and one non-zero off-diagonal term per row defined as

$$a_{k(k-1)} = \begin{cases} -\omega_k^r & \text{if } \bar{u}_{T,k} \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad a_{k(k+1)} = \begin{cases} 0 & \text{if } \bar{u}_{T,k} \geq 0 \\ -\omega_k^r & \text{otherwise} \end{cases} \tag{29}$$

and $B = \{b_{jk}\} \in \mathbb{R}^{n_ir}$ is non-negative, such that

$$b_{kk} = \begin{cases} 1 - \omega_k^r & \text{if } \bar{u}_{T,k} \geq 0 \\ 0 & \text{otherwise} \end{cases}, \quad b_{k(k+1)} = \begin{cases} 0 & \text{if } \bar{u}_{T,k} \geq 0 \\ 1 - \omega_k^r & \text{otherwise} \end{cases} \tag{30}$$

The matrix $A$ is non-singular whenever it is strictly diagonally dominant for at least one row, i.e., whenever there exists $k \in \{1, \ldots, n_ir\}$ such that $\omega_k^r < 1$, which was never observed in the numerical
examples. Therefore the interfacial mobility ratios are determined by solving the following local \((n_{ir} \times n_{ir})\) linear system

\[
\bar{\chi}_\ell = A^{-1} B \chi_\ell.
\]  

(31)

With this approach, each mobility ratio can depend on more than two saturations in the interaction region. This leads to an improved accuracy when the flow is not perpendicular to the interface while preserving the monotonicity of the viscous flux in the sense of [19] (Appendix A). Finally, we mention here that the matrices \(A\) and \(B\) are also used to obtain the derivatives of the mobility ratios with respect to pressure and saturation as shown below,

\[
\frac{\partial \bar{\chi}_\ell}{\partial \tau_j} = A^{-1} \left( \frac{\partial A}{\partial \tau_j} \chi_\ell + \frac{\partial B}{\partial \tau_j} \chi_\ell + B \frac{\partial \chi_\ell}{\partial \tau_j} \right),
\]

(32)

where \(\tau_j\) represents the pressure in control volume \(j\), \(p_j\), or the wetting-phase saturation in control volume \(j\), \(S_j\). The matrices \(\partial A/\partial \tau_j\) and \(\partial B/\partial \tau_j\) contain the derivatives of the coefficients of \(A\) and \(B\), respectively.

4.2. Buoyancy term

In the buoyancy term \(G_{\ell,k}\), the discretization of the mobility ratio at half interface \(k\) is fully based on (fixed) density differences as in the one-dimensional IHU scheme. Our approach is based on the fact that the heaviest (respectively, lightest) phase propagates downwards (respectively, upwards). We write the buoyancy term as

\[
G_{\ell,k} = \bar{T}_k \sum_m \bar{\psi}_{\ell,m,k}(\rho_\ell - \rho_m)\Delta z_k,
\]

(33)

where \(\Delta z_k = z_{k+1} - z_k\), and where \(\bar{T}_k\) is defined in [15]. We omitted again the superscript denoting the interaction region in the right-hand side. As in the viscous term, the discrete mobility ratio at half interface \(k\) is obtained with a weighted averaging procedure that accounts for the orientation of the buoyancy force with respect to the grid. This procedure is described below for the case \((\rho_\ell - \rho_m)\Delta z_k > 0\):

\[
\bar{\psi}_{\ell,m,k} = \begin{cases} 
(1 - \omega_k^G) \frac{\lambda_{\ell,k} \lambda_{m,k+1}}{\lambda_{\ell,k} + \lambda_{m,k+1}} + \omega_k^G \frac{\lambda_{\ell,k} \lambda_{m,k+2}}{\lambda_{\ell,k} + \lambda_{m,k+2}} & \text{if } \Delta z_k \Delta z_{k+1} > 0 \\
(1 - \omega_k^G) \frac{\lambda_{\ell,k} \lambda_{m,k+1}}{\lambda_{\ell,k} + \lambda_{m,k+1}} + \omega_k^G \frac{\lambda_{\ell,k-1} \lambda_{m,k+1}}{\lambda_{\ell,k-1} + \lambda_{m,k+1}} & \text{if } \Delta z_k \Delta z_{k-1} > 0 \\
\frac{\lambda_{\ell,k} \lambda_{m,k+1}}{\lambda_{\ell,k} + \lambda_{m,k+1}} & \text{otherwise.}
\end{cases}
\]

(34)

For instance, consider that interface \(k\) is downdip \((\Delta z_k > 0)\) and assume that phase \(\ell\) is heavier than phase \(m\). We can distinguish two cases based on the depth of neighboring subcells. First, if interface \(k - 1\) is updip and interface \(k + 1\) is downdip, the heavier phase mobility \(\lambda_\ell\) is evaluated in the top control subcell \(k\), whereas the lighter phase mobility \(\lambda_m\) is evaluated in the bottom subcells \(k + 1\) and \(k + 2\). Second, if interface \(k - 1\) is downdip and interface \(k + 1\) is updip, \(\lambda_k\) is evaluated in the top control subcells \(k - 1\) and \(k\), whereas \(\lambda_m\) is evaluated in the bottom subcell \(k + 1\).

The case \((\rho_\ell - \rho_m)\Delta z_k \leq 0\) is treated analogously and is obtained by switching the phase indices \(\ell\) and \(m\) in [34]. Finally, the weighting coefficient \(\omega_k^G\) is written as:

\[
\omega_k^G = \begin{cases} 
\varphi \frac{\bar{T}_{k+1} \Delta z_{k+1}}{\bar{T}_k \Delta z_k} & \text{if } \Delta z_k \Delta z_{k+1} > 0 \\
\varphi \frac{\bar{T}_{k-1} \Delta z_{k-1}}{\bar{T}_k \Delta z_k} & \text{if } \Delta z_k \Delta z_{k-1} > 0 \\
0 & \text{otherwise,}
\end{cases}
\]

(35)

where \(\varphi\) is defined in [26] and guarantees that \(\omega_k^G \in [0, 1]\). The multidimensional buoyancy term reduces to the one-dimensional IHU buoyancy term whenever \(\varphi \equiv 0\). In the compressible case, the ratio of [35] would also contain phase densities. These densities cancel in the incompressible case considered in this work and are omitted for clarity.
Unlike the viscous term, the buoyancy term does not require a local linear solve to determine the mobility ratio since $\bar{\psi}_{\ell,m,k}$ is decoupled from $\bar{\psi}_{\ell,m,k-1}$ and $\bar{\psi}_{\ell,m,k+1}$. By construction, $\bar{G}_{\ell,k}$ is a differentiable function of the saturations, and its derivatives can be computed in a straightforward manner. We show in Appendix A that since the SMU4 limiter satisfies the symmetry property (27), the buoyancy term constitutes a monotone approximation of the buoyancy flux. Since the same property holds for the viscous term, (21) implies that $\bar{F}_{\ell,k}$ satisfies the monotonicity condition of (19). Finally, we note that by construction, we have $\bar{\psi}_{\ell,m,k} = \bar{\psi}_{m,\ell,k}$. It follows that $\bar{G}_{\ell,k} = -\bar{G}_{m,k}$, which guarantees that $\sum_{m} \bar{F}_{m,k} = \bar{u}_{T,k}$.

4.3. Total velocity discretization

In the previous paragraphs, the presentation has focused on the discretization of the mobilities present in the transport equation. To complete the formulation of the multidimensional IHU scheme, we now describe the discretization of the phase mobilities in the total velocity. Considering a half interface $k$, the total velocity, $\bar{u}_{T,k}$, is written as

$$\bar{u}_{T,k}(\Delta p_{k}, S_{k}, S_{k+1}) = \bar{T}_{k} \lambda_{WA}^{T,k} \Delta p_{k} + \bar{T}_{k} \sum_{\ell} \lambda_{WA}^{T,k} g_{\ell,k}, \quad (36)$$

where $\lambda_{WA}$ denotes the discrete mobility of phase $\ell$ at interface $k$. In this work, we employ a two-point upwinding for the approximation of the mobilities present in (36) using the control-volume-centered pressures and saturations. Specifically, we apply the discretization derived in Hamon et al. (2016) to improve the nonlinear convergence of the scheme by attenuating the flip-flopping issue in the total velocity. This methodology is based on an averaging procedure in which the discrete mobility at half interface $k$ in (36) is defined as

$$\lambda_{WA}^{T,k} = \beta_{k}(\Delta p_{k}) \lambda_{T}(S_{k}) + (1 - \beta_{k}(\Delta p_{k})) \lambda_{T}(S_{k+1}). \quad (37)$$

The weighting coefficient $\beta_{k} \in [0, 1]$ is designed such that the resulting phase mobilities are differentiable and increasing functions of the pressure difference that adapt to the balance between viscous and buoyancy forces at each interface in the computational domain. It reads

$$\beta_{k}(\Delta p_{k}) = \frac{1}{2} + \frac{1}{\pi} \arctan(\gamma_{k} \Delta \Phi_{k}). \quad (38)$$

The coefficient $\gamma_{k} \in \mathbb{R}^{+}$ relates the magnitude of viscous forces to the magnitude of buoyancy forces as follows

$$\gamma_{k} = \frac{\pi \inf_{S} \lambda_{T}(S)}{(\sum_{m} \sup_{S} \lambda_{m}(S)) \max_{m} |g_{m,k} - \frac{1}{n_{p}} \sum_{s} g_{s,k}|}, \quad (39)$$

where $\Delta \Phi_{k}$ denotes the average potential gradient at the interface, defined by:

$$\Delta \Phi_{k} = \frac{1}{n_{p}} \sum_{\ell} \Delta \Phi_{\ell,k} = \Delta p_{k} + \frac{1}{n_{p}} \sum_{\ell} g_{\ell,k}. \quad (40)$$

This discretization of $\lambda_{WA}^{T,k}$ is shown in our previous work on two-point upwinding schemes [Hamon et al., 2016] to improve the convergence of Newton’s method compared to the standard two-point PPU while achieving similar accuracy. We emphasize here that even though we only consider a simple two-point upwinding method in the total velocity, our numerical tests show that the resulting scheme exhibits very little sensitivity to the orientation of the grid due to the multidimensional treatment of the discrete mobilities in the transport equation described in the previous subsections. Next, we analyze some mathematical properties of the scheme.

5. Mathematical properties of the scheme

Here, we analyze the fully implicit scheme for coupled flow and transport constructed in Sections 3 and 4. Our analysis accounts for a space- and time-dependent total velocity discretized with (36) and
assume that for \( i \in \{1, \ldots, M\} \), the discrete system reads:

\[
\begin{align*}
& \sum_k \left( u_{T,k}^{(k-2)} - u_{T,k-1}^{(k-2)} \right) n+1 = 0 \\
& V_i \phi_i \frac{S^{n+1}_i - S^n_i}{\Delta t_n} + \sum_k \left( \bar{F}_{2,k}^{(k-2)} - \bar{F}_{2,k-1}^{(k-2)} \right) \left( \{u_{T,i}^{n+1}\}, \{S^{n+1}_i, S^{n+1}_{i,k}\}_{\ell \neq k} \right) = 0.
\end{align*}
\]

where, in the second line of (41), we have grouped the fluxes by interaction region using (17):

\[
\sum_{j \in \text{adj}(i)} F_{2,i,j}^{n+1} = \sum_k \left( \bar{F}_{2,k}^{(k+2)} - \bar{F}_{2,k-2}^{(k+1)} \right) = \sum_k \left( \bar{F}_{2,k}^{(k-2)} - \bar{F}_{2,k-1}^{(k-2)} \right).
\]

In this new formulation, it is easier to exploit the monotonicity condition satisfied by the flux in (19). This is highlighted in the following proposition, in which we show that the monotonicity of the multidimensional Implicit Hybrid Upwinding (IHU) guarantees that the saturation solution remains between 0 and 1.

**Proposition 1.** (Saturation estimate) Consider the solution \((p, S_2)\) to the fully implicit scheme defined written in (41). Provided that the initial saturation is between physical bounds, we have the following saturation estimate, valid for \( i \in \{1, \ldots, M\}, \ell \in \{1, 2\}, \) and \( n \in \mathbb{N}^+ \):

\[
0 \leq S^{n+1}_{i,\ell} \leq 1.
\]

**Proof.** The proof is done by induction and relies on the monotonicity of the flux in the sense of (19). We assume that for \( i \in \{1, \ldots, M\}, \ell \in \{1, 2\}, n \leq n_0, \) we have

\[
0 \leq S^n_{i,\ell} \leq 1.
\]

We consider first the transport equation in control volume \( i \in \{1, \ldots, M\}. \) We take the maximum of \( S^{n_0+1}_{2,i} \) and one in the neighboring control volumes (third argument of the fluxes). Using the monotonicity of \( \bar{F}_{2,k}^{(k-2)} - \bar{F}_{2,k}^{(k-2)} \), we obtain the following inequality in control volume \( i \):

\[
V_i \phi_i \frac{S^{n_0+1}_{2,i} - 1}{\Delta t_n} + \sum_k \left( \bar{F}_{2,k}^{(k-2)} - \bar{F}_{2,k-1}^{(k-2)} \right) \left( \{u_{T,i}^{n_0+1}\}, \{S^{n_0+1}_i, \max(S^{n_0+1}_{2,j}, 1)\}_{\ell \neq k} \right) \leq 0,
\]

where we used \( S^{n_0+1}_{2,i} \) in the accumulation term. Consider now the flux terms at half interfaces \( k \) and \( k - 1 \) in interaction region \( (k - 2) \). When all the saturations in the control volumes involved in this interaction region are equal to one, we have

\[
\left( \bar{F}_{2,k}^{(k-2)} - \bar{F}_{2,k-1}^{(k-2)} \right) \left( \{u_{T,i}^{n_0+1}\}, 1, \{1\}_{\ell \neq k} \right) = \left( u_{T,k}^{(k-2)} - u_{T,k-1}^{(k-2)} \right) n+1,
\]

since \( \bar{u}_{2,k} = 1 \) in (22), and \( \bar{u}_{1,k} = 0 \) in (33), whenever all the saturations are equal to one in the interaction region. Using this in conjunction with the discrete pressure equation, we obtain

\[
V_i \phi_i \frac{1 - \bar{u}_{2,k}^{(k-2)}}{\Delta t_n} + \sum_k \left( \bar{F}_{2,k}^{(k-2)} - \bar{F}_{2,k-1}^{(k-2)} \right) \left( \{u_{T,i}^{n_0+1}\}, 1, \{1\}_{\ell \neq k} \right) = \sum_k \left( u_{T,k}^{(k-2)} - u_{T,k-1}^{(k-2)} \right)^{n+1} = 0.
\]

In (47), we take again the maximum between \( S^{n_0+1}_{2,i} \) and one in the neighboring control volumes. Using the monotonicity properties of \( \bar{F}_{2,k}^{(k-2)} - \bar{F}_{2,k-1}^{(k-2)} \), we write

\[
V_i \phi_i \frac{1 - \bar{u}_{2,k}^{(k-2)}}{\Delta t_n} + \sum_k \left( \bar{F}_{2,k}^{(k-2)} - \bar{F}_{2,k-1}^{(k-2)} \right) \left( \{u_{T,i}^{n_0+1}\}, \max(S^{n_0+1}_{2,i}, 1)\}_{\ell \neq k} \leq 0.
\]
Clearly, (45) and (48) only differ by the accumulation term and by their second argument in the flux term. One can therefore combine these equations by taking the maximum of $S_{n+1}^{n_0+1,1}$ and one to obtain

\[ V_i \phi_i \frac{\max(S_{n+1}^{n_0+1,1}, 1) - 1}{\Delta t^n} + \sum_k \left( \bar{F}_{2,k}^{(k-2)} - \bar{F}_{2,k-1}^{(k-2)} \right) \left( \{ u^{n_0+1}_{T,l} \}, \max(S_{n+1}^{n_0+1,1}, 1), \{ \max(S_{n+1}^{n_0+1,1}) \} \right) \leq 0. \]  

(49)

Finally, summing (49) over all control volumes in the computational domain, the flux terms cancel as the scheme is mass conservative. Therefore, we obtain:

\[ \sum_i V_i \phi_i \frac{\max(S_{n+1}^{n_0+1,1}, 1) - 1}{\Delta t^n} \leq 0. \]  

(50)

which gives $S_{n+1}^{n_0+1} \leq 1$ for all control volumes $i \in \{1, \ldots, M\}$. The proof for $S_{n+1}^{n_0+1} \geq 0$ is analogous.

6. Newton’s method with damping

The systems of nonlinear algebraic equations of (11) are solved with Newton’s method with damping (Deuflhard, 2011). Successive linearizations and updates are performed until convergence is reached:

\[ \text{solve } J \delta U^{\nu+1} = -R(U^{n+1,\nu}) \text{ for } \delta U^{\nu+1}, \]  

(51)

\[ \text{then } U^{n+1,\nu+1} \leftarrow U^{n+1,\nu} + \tau^{\nu+1} \delta U^{\nu+1}, \]  

(52)

where $n$ denotes the time level, $\nu$ denotes the nonlinear iteration number, $\tau^{\nu+1} \in \mathbb{R}^{N_n \times N_n}$ is a diagonal matrix of damping parameters, and $J \in \mathbb{R}^{N_n \times N_n}$ denotes the Jacobian matrix of $R$ with respect to the primary variables. In (52), $\tau^{\nu+1}$ is chosen to ensure that the maximum absolute change in saturation remains smaller than $(\Delta S)_{\text{max}} = 0.2$. Convergence is reached when the maximum value of the normalized residual drops below the tolerance:

\[ \max_{i,\ell} |R_{i,\ell}| \leq 10^{-8}, \]  

(53)

with

\[ R_{i,\ell} = \frac{V_i \phi_i (S_{n+1}^{n_0+1,1} - S_{n}^{n_0+1,1}) + \Delta t^n \left( \sum_{j \in \text{adj}(i)} F_{\ell,ij}^{n+1} - V_i q_{\ell,i} \right)}{V_i \phi_i}. \]  

(54)

If the solver fails to converge after 50 iterations, Newton’s method is restarted from the previous time step, with a time step size reduced by half. If necessary, this time step chopping is repeated. After a converged solution is obtained, the time step size is reset to its original size.

7. Numerical examples

In this section, we compare the accuracy and performance of four fully implicit schemes. The first scheme, referred to as 1D-PPU, is based on the standard two-point phase-per-phase upwinding commonly used in industrial simulators. 1D-IHU is the scheme proposed in Hamon et al. (2016). It uses a two-point hybrid upwinding combined with the procedure based on a weighted averaging of the mobilities in the total velocity. MultiD-PPU is the multidimensional scheme described in Kozdon et al. (2011a) constructed with the SMU limiter. Finally, MultiD-IHU is the multidimensional scheme based on hybrid upwinding described in Section 3. It relies on the SMU4 limiter for the viscous and buoyancy fluxes.

7.1. Three-well problem with buoyancy

We first assess the reduction of the numerical bias caused by the orientation of the grid with a three-well problem with buoyancy derived from Kozdon et al. (2011a); Keilegavlen et al. (2012). We consider a
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$x - y$ domain of dimensions $[-0.5 \text{ ft}, 0.5 \text{ ft}] \times [-0.5 \text{ ft}, 0.5 \text{ ft}]$, discretized with a $51 \times 51$ uniform Cartesian grid so that $\Delta x = \Delta y = 1/51 \text{ ft}$. The scalar absolute permeability is equal to

$$k(x, y) = \begin{cases} 
50 \text{ mD} & \text{if } r = \sqrt{x^2 + y^2} < r_0 \\
5 \times 10^{-5} \text{ mD} & \text{otherwise,}
\end{cases}$$

where $r_0 = (1 - \Delta x)/2$. The well locations are computed using a secondary coordinate system defined as $(x', y') = (x \cos \theta + y \sin \theta, -x \sin \theta + y \cos \theta)$. The injector is always in the center of the domain at $(x', y') = (0, 0)$ and is operated at a fixed rate. The two producers are placed at $(x', y') = (\pm 0.3 \sin(\pi/6), -0.3 \cos(\pi/6))$ and are both operated using a bottom-hole-pressure control. The symmetry of the flow pattern is preserved by orienting the buoyancy force in the direction that is perpendicular to the straight line connecting the two producers, going updip from the injector to the producers. The phase properties are such that the non-wetting phase is twice as light as the wetting phase, with $\rho_1 = 32 \text{ lb}_{\text{m}} \text{ ft}^{-3}$ and $\rho_2 = 64 \text{ lb}_{\text{m}} \text{ ft}^{-3}$. The constant phase viscosities are chosen to be $\mu_1 = 100 \text{ cP}$ and $\mu_2 = 1 \text{ cP}$. Finally, we use Corey-type relative permeabilities such that $k_{r1}(S) = (1 - S)^4$ and $k_{r2}(S) = S^2$. The gravity number is such that

$$N_G = \frac{|k| q_2 - q_1}{\mu_2 |u_T|} = 1.3.$$  

The saturation maps for different angles $\theta$ at $T = 0.092 \text{ PVI}$ are in Fig. 3. The constant time step size is $\Delta t \approx 0.0013 \text{ PVI}$ and corresponds to a CFL number of approximately 3.4. The saturation maps show that the 1D-PPU and 1D-IHU schemes are very sensitive to the orientation of the grid as they prefer flow along the grid directions. The proposed MultiD-IHU scheme does not exhibit this severe numerical bias and significantly reduces the grid orientation effect. We note that the MultiD-IHU saturation maps are in agreement with those generated with MultiD-PPU. These findings are confirmed by the water cuts at the two producers in Fig. 4. They show that the truly multidimensional schemes yield a consistent value of the breakthrough time at the producers when the grid is rotated, and better preserve the symmetry of the problem. This is not the case for 1D-PPU and 1D-IHU, which both predict very different water cuts for the two producers.

**Figure 3:** Saturation maps for different angles $\theta$ at $T = 0.092 \text{ PVI}$ in the three-well problem with buoyancy. The CFL number of these simulations is 3.4. Outside the dashed circle, the control volumes have an absolute permeability set to $5 \times 10^{-5} \text{ mD}$. The white stars show the location of the wells.
The nonlinear behavior for different angles is summarized in Table 1. We run the simulations with a small time step size leading to a CFL number of 3.4, and with a larger time step size corresponding to a CFL number of 59.7. Given that we have set the maximum number of Newton iterations to a large value — i.e., 50 —, none of the schemes require time step cuts. As in Kozdon et al. (2011a), we observe that the multidimensional schemes require fewer Newton iterations than the one-dimensional schemes, even though the cost per iteration of the multidimensional schemes is significantly larger. With MultiD-IHU, the reduction in the number of iterations summed over the five cases compared to 1D-PPU is 12.6% for a CFL number of 3.4, and reaches 12.3% for a larger CFL number of 59.7. We finally observe that MultiD-IHU consistently converges faster than MultiD-PPU in both time stepping configurations.

| Angle θ | Small CFL | Large CFL |
|---------|-----------|-----------|
|         | 1D-PPU    | 1D-IHU    | MultiD-PPU | MultiD-IHU | 1D-PPU | 1D-IHU | MultiD-PPU | MultiD-IHU |
| 0       | 267       | 253       | 241        | 231        | 44     | 44     | 40         | 40         |
| π/12    | 270       | 258       | 244        | 234        | 47     | 48     | 43         | 42         |
| π/8     | 270       | 259       | 244        | 236        | 48     | 47     | 42         | 40         |
| π/6     | 270       | 258       | 248        | 239        | 48     | 47     | 43         | 41         |
| π/4     | 267       | 245       | 243        | 234        | 48     | 46     | 43         | 43         |

Table 1: Total number of Newton iterations for different angles at $T = 0.092$ PVI in the three-well problem with buoyancy. There was no time step cut for any of the schemes considered here.
7.2. Two-phase flow in a heterogeneous medium

Here, we evaluate the reduction of the grid orientation effect achieved with MultiD-IHU on a heterogeneous porous medium. The two-dimensional \( x - y \) domain of dimensions \([-75 \text{ ft}, 75 \text{ ft}] \times [-75 \text{ ft}, 75 \text{ ft}]\) is discretized with 101 \( \times \) 101 uniform Cartesian control volumes. Outside a disc of radius \( r_0 = (150 - \Delta x)/2 \) centered in \((x_0, y_0) = (x'_0, y'_0) = (0, 0)\), we set a constant absolute permeability of \(10^{-10} \text{ mD}\). Inside the disc, the permeability field is given by

\[
k(x', y') = 200 \left(1 + \frac{\cos(3x'\pi)}{2} \cos(3y'\pi) \cos(3\bar{x}\pi) \cos(3\bar{y}\pi)\right)^3,
\]

where \(\bar{x} = x' \cos \left(\frac{\pi}{4}\right) + y' \sin \left(\frac{\pi}{4}\right)\) and \(\bar{y} = -x' \sin \left(\frac{\pi}{4}\right) + y' \cos \left(\frac{\pi}{4}\right)\). The permeability map obtained from (57) can be seen in Fig. 5(a). It exhibits eight preferential paths originating from the center of the domain. This map is rotated to test the sensitivity of the schemes to the orientation of the grid using the same technique as in Section 7.1. The height of each control volume in the domain is a function of the radius \(r = \sqrt{x'^2 + y'^2}\), computed with

\[
h(x', y') = h_{\text{ref}} + h_{\text{bump}}(x', y') \quad \text{where} \quad h_{\text{bump}}(x', y') = 20 \sin \left(\frac{1 + \min(1, \frac{r}{r_0})\pi}{2}\right),
\]

where \(h_{\text{ref}}\) denotes the height of the domain boundaries. Here, \(h(x'_a, y'_a) > h(x'_b, y'_b)\) means that \((x'_a, y'_a)\) is deeper than \((x'_b, y'_b)\). With (59), \((x'_a, y'_a) = (0, 0)\) is at the top of the domain while the domain boundaries are at the bottom as shown in Fig. 5(b). The domain is initially saturated with the lighter non-wetting phase. A well placed in control volume \((x'_0, y'_0) = (0, 0)\) injects a heavy and mobile wetting phase into a domain fully saturated with the light non-wetting phase. We fix the pressure in the control volumes surrounding the disc. The phase properties are the same as in Section 7.1. The gravity number is \(N_G = 12.9\), with counter-current flow at about 25% of the interfaces.

![Figure 5](image.png)

**Figure 5:** Logarithm of the absolute permeability field in (a). It is computed with (57) within the disc of radius \(r_0\) centered in \((x'_0, y'_0) = (0, 0)\). The smallest (respectively, largest) permeability is 30.7 mD (respectively, 675 mD). The height \(h_{\text{bump}}\) obtained from (59) is shown in (b). Outside the disc, the absolute permeability is set to \(10^{-10}\) mD (not shown in the figure) and the domain is flat.

The saturation maps after 0.06 PVI are in Fig. 6. They have been obtained with a constant time step of \(\Delta t \approx 0.0021\) PVI corresponding to a CFL number of 21.1. The saturation pattern is clearly biased by the orientation of the grid with both 1D-PPU and 1D-IHU. This is the case for all angles, but particularly for \(\theta = \pi/12\) and \(\theta = \pi/8\), for which the grid orientation effect is aggravated by the fact that the high-permeability channels are aligned with the main directions of the grid. This nonphysical behavior is significantly attenuated – though not completely eliminated – with the multidimensional schemes. We note that MultiD-PPU and MultiD-IHU yield similar saturation patterns far from the well, even though MultiD-PPU better reduces the numerical biasing caused by the orientation of the grid near the well.

The nonlinear behavior of the four schemes is detailed in Table 2 for two constant time step sizes, namely \(\Delta t \approx 0.0021\) PVI corresponding to CFL number of 21.1, and \(\Delta t \approx 0.01\) PVI for a CFL number of 106.1. The results are consistent with those of Section 7.1. Specifically, MultiD-IHU is the most efficient
Figure 6: Saturation maps for different values of the angle $\theta$ after 0.06 PVI for the heterogeneous two-phase problem. The CFL number of these simulations is 21.1. The white stars show the location of the injector.

scheme in terms of Newton iterations, with a total reduction of 10.4% compared to 1D-PPU for short time steps, and of 13.7% for larger time steps.

| Angle $\theta$ | 1D-PPU | 1D-IHU | MultiD-PPU (SMU) | MultiD-IHU (SMU4) |
|----------------|--------|--------|------------------|-------------------|
| $0$            | 167    | 161    | 153              | 146               |
| $\pi/12$       | 166    | 161    | 154              | 151               |
| $\pi/8$        | 167    | 161    | 154              | 153               |
| $\pi/6$        | 167    | 161    | 156              | 151               |
| $\pi/4$        | 166    | 161    | 154              | 145               |

Table 2: Total number of Newton iterations for different angles at $T = 0.06$ PVI in the heterogeneous injection problem with buoyancy.

7.3. Gravity segregation with low-permeability barriers

Finally, we consider a $x-y$ domain of dimensions $[-75 \text{ ft}, 75 \text{ ft}] \times [-75 \text{ ft}, 75 \text{ ft}]$, in which low-permeability layers are placed to slow down the upward migration of the lighter non-wetting phase. The domain is tilted with an angle of $\pi/3$ in the direction that is perpendicular to the low-permeability layers. We use a uniform Cartesian grid consisting of $101 \times 101$ control volumes. In the disc of radius $r_0 = (150 - \Delta x)/2\text{ ft}$ that occupies the center of domain, the absolute permeability is $5 \times 10^{-9} \text{ mD}$ in the horizontal layers and $50 \text{ mD}$ everywhere else. Outside the disc, the absolute permeability is also equal to $5 \times 10^{-9} \text{ mD}$. The absolute permeability field is shown in Fig. 7. We use the same technique as in Sections 7.1 and 7.2 to compute the location of the low-permeability layers as we rotate the grid.
The non-wetting phase is initially saturating the bottom of the domain. Specifically, the initial saturation field is given by

\[ S_{1}^{\text{init}}(x,y) = \begin{cases} 1 & \text{if } -75 < y' < -45 \text{ and } \sqrt{x^2 + y^2} \leq r_0 \\ 0 & \text{otherwise.} \end{cases} \quad (60) \]

where the \( y' \)-axis is increasing from \(-75\) ft to \(75\) ft when going from bottom to top. The phase densities are given by \( \rho_1 = 32 \text{ lb/ft}^3 \) and \( \rho_2 = 64 \text{ lb/ft}^3 \). The constant phase viscosities are chosen to be \( \mu_1 = 2 \text{ cP} \) and \( \mu_2 = 1 \text{ cP} \). Finally, we use Corey-type relative permeabilities such that \( k_{r1}(S) = (1 - S)^2 \) and \( k_{r2}(S) = S^{1.5} \).

Figure 8 compares the saturation maps produced by the different schemes after 6000 days once the plume has reached the top of the domain. As in the previous examples, the truly multidimensional schemes drastically reduce the grid orientation effect compared to the schemes based on two-point upwinding. This is particularly visible in the upper part of the domain where the shape of the plume predicted by 1D-PPU and 1D-IHU varies significantly as the grid is rotated and becomes very large for an angle \( \theta = \pi/4 \). For this example, the shape of the plume is best preserved for all grid orientations by MultiD-IHU.

The nonlinear iteration count is given in Table 3. For this buoyancy-dominated example, the reduction
in the number of nonlinear iterations with MultiD-PPU compared to 1D-PPU is much smaller than in the previous sections. We also note that 1D-IHU exhibits a better nonlinear behavior than MultiD-PPU. MultiD-IHU still achieves a very robust nonlinear behavior that remains insensitive to the orientation of the computational grid. For a CFL number of 8.9 (respectively, 17.8), MultiD-IHU reduces the total number of nonlinear iterations summed over all cases by 19% (respectively, 25.8%) compared to 1D-PPU. It also reduces the number of Newton iterations compared to MultiD-PPU, and performs slightly better than 1D-IHU.

| Angle | Smaller CFL | Larger CFL |
|-------|-------------|------------|
|       | 1D-PPU      | 1D-IHU     | MultiD-PPU | MultiD-IHU | 1D-PPU | 1D-IHU | MultiD-PPU | MultiD-IHU |
| 0     | 253         | 238        | 257        | 241        | 199    | 187    | 202        | 182        |
| π/12  | 307         | 239        | 292        | 235        | 244    | 171    | 231        | 167        |
| π/8   | 290         | 234        | 285        | 231        | 225    | 174    | 222        | 163        |
| π/6   | 300         | 236        | 290        | 233        | 226    | 171    | 228        | 158        |
| π/4   | 281         | 229        | 277        | 229        | 212    | 167    | 214        | 151        |

Table 3: Total number of Newton iterations for different angles after 6000 days in the gravity segregation problem with low-permeability layers. There was no time step cut for any of the schemes considered here.

8. Summary of results and conclusions

We have proposed a robust and accurate fully implicit finite-volume scheme applied to the equations governing coupled multiphase flow and transport with buoyancy. Specifically, we have presented a truly multidimensional extension of Implicit Hybrid Upwinding (IHU), referred to as MultiD-IHU, that reduces the numerical biasing due to the orientation of the underlying grid while retaining the monotonicity and robustness of IHU in the presence of competing viscous and buoyancy forces. This extension builds on the methodology proposed in Kozdon et al. (2011a) for Phase-Potential Upwinding and is based on an extended stencil that adapts to the local flow pattern for the evaluation of the phase mobility ratios.

We have tested the proposed scheme on two-phase flow numerical examples including competing viscous and buoyancy forces and characterized by the presence of counter-current flow. We have shown that MultiD-IHU reduces the grid orientation effect as well as MultiD-PPU for the two first numerical examples, and leads to a more significant reduction than MultiD-PPU for the gravity segregation case with low-permeability layers. We have also demonstrated that MultiD-IHU retains the robust nonlinear behavior of 1D-IHU, even for large time steps. This is particularly the case for the third numerical example, in which MultiD-IHU reduces the number of nonlinear iterations by up 25.8% compared to 1D-PPU and by up 25.2% compared to MultiD-PPU. The computational gain due to the reduction in nonlinear iterations will help offset the added per-iteration cost associated with the larger stencils of multidimensional schemes.

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Appendix A. Numerical flux monotonicity

In this section, we show that the MultiD-IHU scheme is monotone in the sense of [19]. We consider first the monotonicity of the viscous flux by analyzing the sign of the derivatives with respect to saturation of

\[ \hat{V}_{t,k} - \hat{V}_{t,k-1} = \hat{\chi}_{t,k} \hat{u}_{T,k} - \hat{\chi}_{t,k-1} \hat{u}_{T,k-1}, \] (A.1)
Taking the derivatives with respect to saturation in (A.6) gives

\[ \frac{\partial (\bar{\chi}_{\ell,k} - \bar{\chi}_{\ell,k-1})}{\partial S_{\ell,j \neq k}} = (c_{k,j} \bar{u}_{T,k} - c_{k-1,j} \bar{u}_{T,k-1}) \chi_{\ell,j}. \]  

(A.2)

Using the assumption on the sign of the total velocities and the non-negativity of the non-negative matrix \( C \equiv A^{-1} B \), (A.2) gives

\[ \frac{\partial (\bar{\chi}_{\ell,k} - \bar{\chi}_{\ell,k-1})}{\partial S_{\ell,j \neq k}} = (c_{k,j} \bar{u}_{T,k} - c_{k-1,j} \bar{u}_{T,k-1}) \frac{\partial \chi_{\ell,j}}{\partial S_{\ell,j \neq k}} \leq 0. \]  

(A.3)

In the second case, we assume that \( \bar{u}_{T,k} \geq 0 \) and \( \bar{u}_{T,k-1} \leq 0 \). Using the definition of the upwinding of the mobility ratio, we can write

\[ \bar{\chi}_{\ell,k} - \bar{\chi}_{\ell,k-1} = \chi_{\ell,k} \bar{u}_{T,k} - \chi_{\ell,k} \bar{u}_{T,k-1}, \]  

(A.4)

which yields

\[ \frac{\partial (\bar{\chi}_{\ell,k} - \bar{\chi}_{\ell,k-1})}{\partial S_{\ell,j \neq k}} = 0. \]  

(A.5)

The third case is such that \( \bar{u}_{T,k} \geq 0 \) and \( \bar{u}_{T,k-1} \geq 0 \). Using the weighted average of (23), we write

\[ \bar{\chi}_{\ell,k} - \bar{\chi}_{\ell,k-1} = (1 - \omega_k^V) \chi_{\ell,k} \bar{u}_{T,k} + (\omega_k^V \bar{u}_{T,k} - \bar{u}_{T,k-1}) \chi_{\ell,k-1} \]

\[ = (1 - \omega_k^V) \chi_{\ell,k} \bar{u}_{T,k} + (\omega_k^V \bar{u}_{T,k} - \bar{u}_{T,k-1}) \sum_{j=1}^{n_{ir}} c_{k-1,j} \chi_{\ell,j}. \]  

(A.6)

Taking the derivatives with respect to saturation in (A.6) gives

\[ \frac{\partial (\bar{\chi}_{\ell,k} - \bar{\chi}_{\ell,k-1})}{\partial S_{\ell,j \neq k}} = (\omega_k^V \bar{u}_{T,k} - \bar{u}_{T,k-1}) c_{k-1,j} \frac{\partial \chi_{\ell,j}}{\partial S_{\ell,j \neq k}} \leq 0, \]  

(A.7)

since the definition of the limiter guarantees that \( (\omega_k^V \bar{u}_{T,k} - \bar{u}_{T,k-1}) \leq 0 \).

Next, we consider the monotonicity of the buoyancy term by studying the sign of the derivative with respect to saturation of

\[ \bar{G}_{\ell,k} - \bar{G}_{\ell,k-1} = \bar{T}_k \bar{\psi}_{\ell,m,k}(\rho_\ell - \rho_m) \Delta z_k - \bar{T}_k \bar{\psi}_{\ell,m,k-1}(\rho_\ell - \rho_m) \Delta z_{k-1}. \]  

(A.8)

We assume that \( \rho_\ell - \rho_m > 0 \) and \( \Delta z_k > 0 \). The other cases can be treated analogously and will be omitted for brevity. On a Cartesian grid, we only have to study two configurations. In the first configuration, we assume that \( \Delta z_{k+1} \Delta z_k > 0 \) and \( \Delta z_k \Delta z_{k-1} < 0 \), which yields

\[ \bar{G}_{\ell,k} = \bar{T}_k \left( (1 - \omega_k^G) \frac{\lambda_{\ell,k} \lambda_{m,k+1}}{\lambda_{\ell,k} + \lambda_{m,k+1}} + \omega_k^G \frac{\lambda_{\ell,k} \lambda_{m,k+2}}{\lambda_{\ell,k} + \lambda_{m,k+2}} \right) (\rho_\ell - \rho_m) \Delta z_k, \]  

(A.9)

\[ \bar{G}_{\ell,k-1} = \bar{T}_k \left( (1 - \omega_k^G) \frac{\lambda_{\ell,k} \lambda_{m,k-1}}{\lambda_{\ell,k} + \lambda_{m,k-1}} + \omega_k^G \frac{\lambda_{\ell,k} \lambda_{m,k-2}}{\lambda_{\ell,k} + \lambda_{m,k-2}} \right) (\rho_\ell - \rho_m) \Delta z_{k-1}. \]  

(A.10)

Noting that \( m \neq \ell \), the sign of the derivatives of the mobility ratios is given by

\[ \frac{\partial}{\partial S_{\ell,j \neq k}} \left( \frac{\lambda_{\ell,k} \lambda_{m,j}}{\lambda_{\ell,k} + \lambda_{m,j}} \right) = \frac{\lambda_{\ell,k}^2}{\lambda_{\ell,k} + \lambda_{m,j}} \frac{\partial \lambda_{m,j}}{\partial S_{\ell,j \neq k}} \leq 0. \]  

(A.11)

We now use the assumptions on the densities and the depth to obtain

\[ \frac{\partial (\bar{G}_{\ell,k} - \bar{G}_{\ell,k-1})}{\partial S_{\ell,j \neq k}} \leq 0. \]  

(A.12)
In the second configuration, we assume that $\Delta z_{k+1} \Delta z_k < 0$ and $\Delta z_k \Delta z_{k-1} > 0$. This case yields

$$G_{\ell,k} = T_k \left(1 - \omega^G_k \right) \frac{\lambda_{\ell,k} \lambda_{m,k+1}}{\lambda_{\ell,k} + \lambda_{m,k+1}} + \omega^G_k \frac{\lambda_{\ell,k-1} \lambda_{m,k+1}}{\lambda_{\ell,k-1} + \lambda_{m,k+1}} \left(\rho_\ell - \rho_m\right) \Delta z_k,$$

(A.13)

$$G_{\ell,k-1} = \bar{T}_{k-1} \left(1 - \omega_{k-1}^G \right) \frac{\lambda_{\ell,k-1} \lambda_{m,k}}{\lambda_{\ell,k-1} + \lambda_{m,k}} + \omega_{k-1}^G \frac{\lambda_{\ell,k-1} \lambda_{m,k+1}}{\lambda_{\ell,k-1} + \lambda_{m,k+1}} \left(\rho_\ell - \rho_m\right) \Delta z_{k-1}.$$

(A.14)

Using these expressions, we obtain

$$G_{\ell,k} - G_{\ell,k-1} = \left( T_k \left(1 - \omega^G_k \right) \frac{\lambda_{\ell,k} \lambda_{m,k+1}}{\lambda_{\ell,k} + \lambda_{m,k+1}} - \bar{T}_{k-1} \left(1 - \omega_{k-1}^G \right) \frac{\lambda_{\ell,k-1} \lambda_{m,k}}{\lambda_{\ell,k-1} + \lambda_{m,k}} \Delta z_k \right) \left(\rho_\ell - \rho_m\right)$$

$$+ \left( T_k \omega^G_k \Delta z_k - \bar{T}_{k-1} \omega_{k-1}^G \Delta z_{k-1} \right) \frac{\lambda_{\ell,k-1} \lambda_{m,k+1}}{\lambda_{\ell,k-1} + \lambda_{m,k+1}} \left(\rho_\ell - \rho_m\right).$$

(A.15)

Equation [27] guarantees that $\bar{T}_k \omega^G_k \Delta z_k - \bar{T}_{k-1} \omega_{k-1}^G \Delta z_{k-1} = 0$ whenever $\Delta z_{k+1} \Delta z_k < 0$ and $\Delta z_k \Delta z_{k-1} > 0$. Therefore the term in the second line of (A.15) cancels and we can use the sign of the derivatives of the mobility ratios to obtain the result

$$\frac{\partial (G_{\ell,k} - G_{\ell,k-1})}{\partial S_{\ell,j \neq k}} \leq 0.$$

(A.16)

References

Aavatsmark, I. (2002). An introduction to multipoint flux approximations for quadrilateral grids. *Computational Geosciences*, 6(3-4):405–432.

Aavatsmark, I., Barkve, T., and Mannseth, T. (1998). Control-volume discretization methods for 3D quadrilateral grids in inhomogeneous, anisotropic reservoirs. *SPE Journal*, 3(2):146–154.

Arbogast, T. and Huang, C.-S. (2006). A fully mass and volume conserving implementation of a characteristic method for transport problems. *SIAM Journal on Scientific Computing*, 28(6):2001–2022.

Aziz, K. and Settari, A. (1979). *Petroleum reservoir simulation*, volume 476. Applied Science Publishers London.

Berger, M. J., Helzel, C., and LeVeque, R. J. (2003). H-box methods for the approximation of hyperbolic conservation laws on irregular grids. *SIAM Journal on Numerical Analysis*, 41(3):893–918.

Brand, C. W., Heinemann, J. E., and Aziz, K. (1991). The grid orientation effect in reservoir simulation. In *SPE Symposium on Reservoir Simulation*. Society of Petroleum Engineers.

Brenier, Y. and Jaffré, J. (1991). Upstream differencing for multiphase flow in reservoir simulation. *SIAM Journal on Numerical Analysis*, 28(3):685–696.

Chen, W. H., Durlofsky, L. J., Engquist, B., and Osher, S. (1993). Minimization of grid orientation effects through use of higher order finite difference methods. *SPE Advanced Technology Series*, 1(2):43–52.

Colella, P. (1990). Multidimensional wave-oriented upwind schemes with reduced cross-wind diffusion for flow in porous media. *International Journal for Numerical Methods in Fluids*, 67(1):33–57.

Edwards, M. G. and Rogers, C. F. (1998). Finite volume discretization with imposed flux continuity for the general tensor pressure equation. *Computational Geosciences*, 2(4):259–290.

Eymard, R., Gallouët, T., and Joly, P. (1989). Hybrid finite element techniques for oil recovery simulation. *Computer Methods in Applied Mechanics and Engineering*, 74(1):83–98.

Eymard, R., Guichard, C., and Masson, R. (2012). Grid orientation effect in coupled finite volume schemes. *IMA Journal of Numerical Analysis*, 33(2):582–608.

Hamon, F. P., Mallison, B. T., and Tchelepi, H. A. (2016). Implicit hybrid upwind scheme for coupled multiphase flow and transport with buoyancy. *Computer Methods in Applied Mechanics and Engineering*, 311:599–624.
Fully Implicit Multidimensional Hybrid Upwind Scheme

Hamon, F. P., Mallison, B. T., and Tchelepi, H. A. (2018). Implicit hybrid upwinding for two-phase flow in heterogeneous porous media with buoyancy and capillarity. *Computer Methods in Applied Mechanics and Engineering*, 331:701–727.

Hamon, F. P. and Tchelepi, H. A. (2016). Analysis of hybrid upwinding for fully implicit simulation of three-phase flow with gravity. *SIAM Journal on Numerical Analysis*, 54(3):1682–1712.

Hurtado, F. S. V., Maliska, C. R., Carvalho da Silva, A. F., and Cordazzo, J. (2007). A quadrilateral element-based finite-volume formulation for the simulation of complex reservoirs. In *Latin American & Caribbean Petroleum Engineering Conference*. Society of Petroleum Engineers.

Keilegavlen, E., Kozdon, J. E., and Mallison, B. T. (2012). Multidimensional upstream weighting for multiphase transport on general grids. *Computational Geosciences*, 16(4):1021–1042.

Koren, B. (1991). Low-diffusion rotated upwind schemes, multigrid and defect correction for steady, multi-dimensional Euler flows. In *Multigrid Methods III*, pages 265–276. Springer.

Kozdon, J. E., Mallison, B. T., and Gerritsen, M. G. (2016). Analysis of hybrid upwinding for fully implicit simulation of three-phase flow with gravity. *SIAM Journal on Numerical Analysis*, 54(3):1682–1712.

Kozdon, J. E., Mallison, B. T., and Gerritsen, M. G. (2011a). Multidimensional upstream weighting for multiphase transport in porous media. *Computational Geosciences*, 15(3):399–419.

Kozdon, J. E., Mallison, B. T., Gerritsen, M. G., and Chen, W. H. (2011b). Multidimensional upwinding for multiphase transport in porous media. *SPE Journal*, 16(2):263–272.

Lamine, S. and Edwards, M. G. (2010). Higher order multidimensional upwind convection schemes for flow in porous media on structured and unstructured quadrilateral grids. *SIAM Journal on Scientific Computing*, 32(3):1119–1139.

Lamine, S. and Edwards, M. G. (2013). Higher order cell-based multidimensional upwind schemes for flow in porous media on unstructured grids. *Computer Methods in Applied Mechanics and Engineering*, 259:103–122.

Lamine, S. and Edwards, M. G. (2015). Multidimensional upwind schemes and higher resolution methods for three-component two-phase systems including gravity driven flow in porous media on unstructured grids. *Computer Methods in Applied Mechanics and Engineering*, 292:171–194.

Lee, S. H. and Efendiev, Y. (2016). C1-continuous relative permeability and hybrid upwind discretization of three phase flow in porous media. *Advances in Water Resources*, 96:209–224.

Lee, S. H. and Efendiev, Y. (2018). Hybrid discretization of multi-phase flow in porous media in the presence of viscous, gravitational, and capillary forces. *Computational Geosciences*, pages 1–19.

Lee, S. H., Efendiev, Y., and Tchelepi, H. A. (2015). Hybrid upwind discretization of nonlinear two-phase flow with gravity. *Advances in Water Resources*, 82:27–38.

Li, B. and Tchelepi, H. A. (2015). Nonlinear analysis of multiphase transport in porous media in the presence of viscous, buoyancy, and capillary forces. *Journal of Computational Physics*, 297:104–131.

Moncorgé, A., Tchelepi, H. A., and Jenny, P. (2018). Consistent upwinding for sequential fully implicit compositional simulation. In *ECMOR XVI-16th European Conference on the Mathematics of Oil Recovery*.

Peaceman, D. W. (1977). Fundamentals of numerical reservoir simulation. Elsevier.

Pruess, K. and Bodvarsson, F. S. (1983). A seven-point finite difference method for improved grid orientation performance in pattern steamfloods. In *SPE Reservoir Simulation Symposium*. Society of Petroleum Engineers.

Roe, P. L. and Sidilkover, D. (1992). Optimum positive linear schemes for advection in two and three dimensions. *SIAM Journal on Numerical Analysis*, 29(6):1542–1568.

Sammon, P. H. (1988). An analysis of upstream differencing. *SPE Reservoir Engineering*, 3(3):1053–1056.

Schneider, G. E. and Raw, M. J. (1986). A skewed, positive influence coefficient upwinding procedure for control-volume-based finite-element convection-diffusion computation. *Numerical Heat Transfer, Part A: Applications*, 9(1):1–26.

Shubin, G. R. and Bell, J. B. (1984). An analysis of the grid orientation effect in numerical simulation of miscible displacement. *Computer Methods in Applied Mechanics and Engineering*, 47(1-2):47–71.

Souza, M. R. A., Contreras, F. R. L., Lyra, P. R. M., and Carvalho, D. K. E. (2018). A higher-resolution flow-oriented scheme with an adaptive correction strategy for distorted meshes coupled with a robust
MPFA-D method for the numerical simulation of two-phase flow in heterogeneous and anisotropic petroleum reservoirs. *SPE Journal.*

Trangenstein, J. A. and Bell, J. B. (1989a). Mathematical structure of compositional reservoir simulation. *SIAM Journal on Scientific and Statistical Computing, 10*(5):817–845.

Trangenstein, J. A. and Bell, J. B. (1989b). Mathematical structure of the Black-Oil model for petroleum reservoir simulation. *SIAM Journal on Applied Mathematics, 49*(3):749–783.

Wang, X. and Tchelepi, H. A. (2013). Trust-region based solver for nonlinear transport in heterogeneous porous media. *Journal of Computational Physics, 253:*114–137.

Yanosik, J. L. and McCracken, T. A. (1979). A nine-point, finite-difference reservoir simulator for realistic prediction of adverse mobility ratio displacements. *Society of Petroleum Engineers Journal, 19*(4):253–262.

Zhou, Y., Tchelepi, H. A., and Mallison, B. T. (2011). Automatic differentiation framework for compositional simulation on unstructured grids with multi-point discretization schemes. In *SPE Reservoir Simulation Symposium.* Society of Petroleum Engineers.