Statistical criteria for search of heavy neutrino in tritium spectrum

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Abstract. The method of quasi-optimal weights is applied to constructing (quasi-)optimal criteria for search of heavy (4th generation) neutrino contribution in experimental $\beta$-decay spectra. Various approaches to searching for anomalous contributions in spectra are discussed. In particular the tritium $\beta$-decay spectrum (for instance, in Troitsk-$\nu$-mass, Mainz Neutrino Mass and KATRIN experiments) is analyzed using the derived special criteria. The power functions constructed for each criteria show the efficiency of the derived quasi-optimal criteria as statistical instruments for detecting the anomalous contributions in the spectra. The overall sensitivity of the criteria is estimated.

1. Introduction
Since the suggestion by B. Pontecorovo [1] of using the tritium $\beta$-decay for searching for the neutrino mass the $\beta$-spectrum became the most precise instrument in direct neutrino mass measurements. The extraordinary precision in neutrino mass upper limit was achieved in Troitsk-$\nu$-mass [2] and Mainz neutrino mass experiments [3] and soon-to-be-started electron antineutrino mass measurements by KATRIN experiment [4, 6] are aimed at one order of magnitude improvement of the limit. However after more than half a century the tritium $\beta$-decay spectrum still reveals further opportunities for investigation and hunting for physics beyond the Standard Model.

Recently it was proposed to study the whole tritium spectrum for the traces of a heavy neutrino admixture. Neutrinos with the mass of several keV are candidates for Dark Matter and these masses are not excluded by other experiments or astrophysical observations. Therefore an investigation of tritium spectrum within the energies of some keVs below the end-point becomes a new task for the experiments in Troitsk and Karlsruhe.

Note that the measurements of the tritium spectrum end-point can also provide information on so-called light sterile neutrinos (possible additional generation of neutrino with mass of about 1 eV). Corresponding studies have been performed by both Troitsk and Mainz collaborations. The data analysis methods used therein are in general similar to those used in the measurements of the electron antineutrino effective mass.

Anomalous contributions in experimental spectra come either from an incorrect estimation of systematical errors or from a real underestimated phenomenon that can be related to new physics. An illustration comes directly from the search for neutrino mass in Troitsk-$\nu$-mass experiment where a step-like anomaly has been observed and remained unexplained before the new analysis with re-estimated systematics and new theoretical description of the setup.
was performed. Thus anomalies in experimental spectra give us profound understanding of experimental setups, or indicate new physics beyond the Standard Model. For instance, in tritium $\beta$-decay spectrum possible additional interactions can lead to a step-like anomaly near the end-point while the existence of the forth neutrino (with the mass of a few keV) induces a kink structure in the region of several keV from the end-point.

This particular work aims at constructing a new statistical instrument for the search of the kink structure. To search for anomaly one should exploit a statistically reliable inference about presence or absence of the anomaly. Such inference can be provided by special statistical criteria, constructed for each particular situation and accounting for any available information about the theoretical model or experimental setup. The most common approaches here are as follows:

(i) Direct fit with additional parameters (the mass of neutrino and the mixing parameter, the amplitude and the position of the step) [2], [5], [7].

The approach requires good knowledge of the experimental setup and all possible systematic uncertainties. The sensitivity of direct fit to the parameters of anomaly depends critically on the systematics. The presence or absence of the kink can not be reliably proved without proving that all uncertainties are taken into account.

(ii) Searching for the kink with various filters [7].

Using the knowledge of the form of the anomalous contribution (the kink) one can look for it in the spectrum by searching for a rapid change in the rate or the break of the derivative of the experimental spectrum. Scanning the spectrum with a pair of "windows" or filters can give a hint on the position of anomaly and is less sensitive to the systematics

(iii) Wavelet analysis [8].

The new approach exploiting the wavelet analysis of the spectrum appears to be independent from the systematic uncertainties in a wide range of parameters. It requires a thorough selection of the type of wavelets, the modelling of the anomalous contribution and corresponding spectra. However it has been shown to be rather sensitive to the kink structure and at the same time tolerate rather large systematic errors.

(iv) Constructing special statistical criteria for the kink from the heavy keV neutrino accounting for the uncertainties of other parameters (based on the method of quasi-optimal weights [9]).

The construction of special criteria was proved [10, 11] to be a consistent approach to searches for anomalous contributions in experimental spectra. It exploits the method of quasi-optimal weights for deriving special statistical tests for the anomaly. The special criteria are compared by their power functions (by their efficiency). One can choose among several criteria the one that answered better to the parameters of experiment and the knowledge of systematics.

In this paper we show how the special criteria for the kink form the heavy neutrino in tritium spectrum are derived. In addition we construct the power functions for each test to prove their efficiency in search for the kink in the case when some systematic parameters are unknown.

2. Kink searches with special criteria

Similarly to the case of step-like anomaly near the end-point (for details refer to [10, 11]) one derives special criteria for the search of a heavy neutrino in $\beta$-decay spectrum. The spectrum with the contribution of heavy neutrino has the following form:

\[
\frac{d\Gamma}{dE} = \sin^2 \theta \left( \frac{d\Gamma}{dE} \right)_{m_{keV}} + \cos^2 \theta \left( \frac{d\Gamma}{dE} \right)_{m_{light}},
\]
The spectrum \( \frac{\Gamma}{dE}(E_i, m) = S_{i,m} \) is measured in a number of points with various retarding potentials (for the integral spectrum) or for various energies of emitted electrons (for the differential spectrum). The spectrum depends on the mixing parameter \( U^2 = \sin^2 \theta \) and the mass of the heavy neutrino \( m_{ck}\text{eV} \). The mass of the light neutrino is comparably small and its value is put to zero, \( m_{\text{light}} = 0 \). The data is fitted under the assumption of the null-hypothesis (i.e. the absence of the kink).

The first criterion is constructed via routines of the method of quasi-optimal moments[9]. It is by construction the Locally Most Powerful (LMP) one. "Locally" means near the null-hypothesis, that is the absence of the kink in the spectrum. For constructing this criterion we assume that the mass of the heavy neutrino and thus the position of the kink is known. The distributions for the experimental counts are \( f_i(N) = \mu_i' N^i e^{-\mu_i'} N! \), where the theoretical means are given by \( \mu_i' = U^2 S_{i,m_H} + (1 - U^2) S_{i,m=0} \).

The LMP criterion is obtained via the method of quasi-optimal weights and it is by construction the most sensitive one in case when the mass of the heavy neutrino is well-known. To derive the criterion one construct the weights as follows:

\[
\omega_i = \frac{\partial \ln f_i}{\partial U^2} = \left( \frac{N}{U^2 S_{i,m_H} + (1 - U^2) S_{i,m=0}} - 1 \right) \cdot (S_{i,m_H} + S_{i,0})
\]

Then, following the routines of the method of quasi-optimal weights [9], one compares the theoretical and the experimental means of the weights and solve the corresponding equation \( h^{\exp} = \frac{1}{M} \sum_{i=1}^{M} \omega_i = 0 \) to obtain an estimate of the mixing parameter of the heavy neutrino \( \hat{U}^2 \).

The estimate is the statistics of the Locally Most Powerful criterion. It is by construction the most efficient one, but possesses an undesired dependence on the (unknown) mass of the heavy neutrino. In the case of small values of \( \hat{U}^2 \) (that is the case with the heavy neutrino) can be also rewritten as a weighted sum of experimental counts \( \hat{U}^2 = \sum_{i=1}^{M} w_i \cdot N_i \).

The next step is to reduce the dependance of our LMP criterion to the unknown mass of heavy neutrino and the corresponding position of the kink, even loosing some sensitivity to the step itself. For this one slightly changes the weights in the sum of the LMP test (see Figs. 1, 2), to suppress the points near the position of the kink and both ends of the spectrum, saving the properties of the LMP test in the rest areas. The corresponding statistics can be obtained by tuning the weights of LMP criterion resulting in a new quasi-optimal criterion: \( S_{q-opt} = \sum_{i=1}^{k} w_i \cdot \xi_i \), where \( \xi_i = \frac{N_i - \mu_i}{\sqrt{\mu_i}} \) and \( w_i \) are chosen as shown in Fig. 2 (the absolute values of the weights are not important since the overall factor can be chosen arbitrarily). The quasi-optimal test is more robust and require no information about the exact mass of the additional neutrino.
One more criterion, \( S_{\text{pair}} = \sum_i \xi_i \cdot \xi_{i+1} \), constructed somehow speculatively, exploits the following idea: if the anomaly is a deviation of several neighbour points to one side of the fitting curve (Fig. 3) than it will increase the value of the statistics \( S \). Thus the pairwise neighbours’ correlations test can be used as a criterion for rather general class of anomalies. The universal pairwise neighbours’ correlations test can be exploited in the case of kink searches as well.

\[ \sum_i \xi_i \cdot \xi_{i+1} = S_{\text{pair}} \]

**Figure 4.** The power functions of the three criteria for the case when the assumed mass is 10 keV, real mass is 10 keV

**Figure 5.** The power functions of the three criteria for the case when the assumed mass is 10 keV, real mass is 7 keV

**Figure 6.** The power functions of the three criteria for the case when the assumed mass is 10 keV, real mass is 5 keV

**Figure 7.** The power functions of the three criteria for the case when the assumed mass is 10 keV, real mass is 12 keV

To answer the question, which of the derived and conventional criteria is the best for the kink-searches one refers to the standard tool of mathematical statistics, so-called power functions. These functions show simply the probability of each criterion to reject the null hypothesis while it is in fact false. The comparison of the criteria becomes than straightforward: the higher is the power function on the same plot, the better (more sensitive) is the criterion. Figs. 4–7 present the power functions of the quasi-optimal test (1), pairwise neighbours’ correlations criterion (2) and the conventional \( \chi^2 \) test for the sake of comparison (all the power functions were constructed using a simple (though preserving all the main features) model of a differential tritium beta-decay spectrum). The Fig. 4) shows that the quasi-optimal criterion is the most efficient one, the pairwise neighbours’ correlations test is less specific and therefore less sensitive. The general
\(\chi^2\) comes third since it is not tuned at all for the kink searches. In Figs. 5, 6 and 7 we show how the sensitivity of the derived tests depends on the real value of the heavy neutrino mass, that is used during the modelling (the mass assumed during the construction of the tests is 10 keV). The quasi-optimal test remain the best for the real masses of 7 keV and 5 keV and loses its efficiency with real mass at 12 keV. The pairwise neighbours’ correlations test maintains its sensitivity in a wide range of unknown parameters.

3. Conclusions
We illustrated how the recently developed approach to the search for anomalies in experimental spectra [10, 11] with account for the parameters with uncertainties can be applied to the search for heavy keV neutrino contribution to the tritium beta-decay spectrum. We showed that for each anomalous contribution the so-called Locally Most Powerful criterion can be constructed via the method of quasi-optimal weights. The LMP test is then tuned to reduce the dependence on the unknown parameters of the spectra (such as the unknown mass of the heavy neutrino). Using the power functions the constructed criteria can be compared and tested according to each specific anomalous contribution. The approach yields the criteria that are sensitive to the kink from the heavy neutrino in tritium spectra. Along with the wavelet analysis [8] and filters-approach [7] the criteria is proved to be a useful tool for future heavy neutrino searches in Troitsk [12], [13] and Karlsruhe [7].

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References
[1] Hanna G, Pontecorvo B 1949 Phys. Rev. 75 983
[2] Lobashev V 2003 Nucl. Phys. A. 719 153
[3] Kraus C et al 2005 Eur. Phys. J. C 40 447
[4] Angrik J et al (KATRIN) 2004 KATRIN Design Report 2004, Technical Report (2004) fZKA-7090, http://www.katrin.kit.edu/publikationen/- DesignReport2004-12Jan2005.pdf
[5] Aseev V et al 2011 Phys. Rev. D. 84 112003
[6] Drexlin G, Hannen V, Mertens S and Weinheimer C 2013 Advances in High Energy Physics 293986
[7] Mertens S et al. 2015 JCAP 02 020
[8] Mertens S et al 2015 Phys. Rev. D 91 042005
[9] Tkachov F 2006 Preprint arXiv:physics/0704127
[10] Lokhov A, Tkachov F and Trukhanov P 2012 Nucl. Instrum. Meth. A 686 162
[11] Lokhov A, Tkachov F and Trukhanov P 2013 Nucl. Phys. A 897 218
[12] Abdurashitov D et al 2015 JINST 10 T10005
[13] Belesev A et al 2014 J. Phys. G: Nucl. Part. Phys 41 015001