Interval Semantics
for Standard Floating-Point Arithmetic

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Abstract

If the non-zero finite floating-point numbers are interpreted as point intervals, then
the effect of rounding can be interpreted as computing one of the bounds of the result
according to interval arithmetic. We give an interval interpretation for the signed zeros
and infinities, so that the undefined operations $\pm 0 \times \pm \infty$, $\pm \infty - \pm \infty$, $\pm \infty / \pm \infty$, and
$\pm 0 / \pm 0$ become defined.

In this way no operation remains that gives rise to an error condition. Mathemat-
ically questionable features of the floating-point standard become well-defined sets of
reals. Interval semantics provides a basis for the verification of numerical algorithms.
We derive the results of the newly defined operations and consider the implications for
hardware implementation.

Keywords: IEEE floating-point standard, interval arithmetic, NaNs, exceptions

1 Introduction

IEEE Standard 754-1985 for Binary Floating-Point Arithmetic [4] achieved great success
by finding a synthesis of the best features of the existing processors and causing these to be
widely adopted in a short time. It was not to be expected that IEEE Standard 754 had a
remedy for the fact that none of the existing processors was based on a coherent approach
to the fundamental problem of how to approximate on digital processors the operations of
arithmetic on reals. An example of the ad-hoc approach taken in IEEE Standard 754 is the
introduction of the infinities. Mathematically, there is one advantage: division of a non-
zero number by zero becomes defined. But it introduces more exceptions than it removes:
$0 \times \pm \infty$, $\pm \infty - \pm \infty$, $\pm \infty / \pm \infty$, while $0/0$ remains undefined.

Any improvement to the standard needs to be based on a mathematically convincing
approach to the following problem:

\footnote{Its successor, IEEE Standard 754-2008 for Floating-Point Arithmetic [5], does not make changes that are
relevant to this paper.}
How to map the arithmetical structure of the reals to a closed, exception-free algebra on a set of computer-representable quantities.

In this paper we review results from [3] (see there for earlier references) that allow such an algebra to be based on intervals. Surprisingly, most of the standard carries over unchanged to our algebra. What does not carry over are the error conditions: there are none.

Our proposal is based on the idea of interpreting floating-point numbers as sets of reals. The sets of reals include the reals themselves by identifying the singleton sets \( \{ x \} \) with \( x \) itself, for all reals \( x \). As we will show, this defines an arithmetic that is compatible with arithmetic on the reals. Where it deviates is that division by zero becomes defined.

But there is a more important advantage. Any arithmetic that intends to approximate real arithmetic with a finite set of values necessarily introduces uncertainty. For example, in binary floating-point arithmetic the result of dividing 1 by 10 leaves uncertainty concerning the digits from a certain point onwards. Also, as there is necessarily a greatest floating-point number \( M \), there has to be uncertainty about any result that exceeds this maximum. Hence there need to be floating-point numbers that are interpreted as \( \{ x \in \mathbb{R} \mid M \leq x \} \) and as \( \{ x \in \mathbb{R} \mid -M \geq x \} \).

In Section 2 we show how the theoretical advantages of arithmetic on sets of reals becomes practical by restricting these sets to be intervals. We also establish notation and terminology for intervals and floating-point numbers. In Section 3 we define the central feature of this paper: an interval interpretation of each floating-point number. Here we take the view that zero has finite precision. In Section 4 we state theorems that establish that to a large extent the existing floating-point standard already implements the arithmetic operations on floating-point numbers interpreted as intervals. Section 5 treats the case where interval semantics changes the existing definition; Section 6 derives the undefined cases according to interval semantics. Section 7 is a short discussion of infinite-precision zeroes. Finally, in Section 8 we survey the consequences of the new semantics for hardware implementation.

2 Preliminaries

2.1 Floating-point numbers

If \( x \) is a floating-point number greater than \(-\infty\) (less than \(+\infty\)), then \( x^- \) (\( x^+ \)) is the next smaller (greater) floating-point number. The constant \( m \) is defined as the least positive floating-point number and \( M \) as the greatest finite floating-point number.

2.2 Intervals

“Real intervals” are to be interpreted according to the following definition: a real interval is a closed, connected set of reals. According to a well-known result in topology, such sets take the following forms: \( \{ x \in \mathbb{R} \mid x \leq b \} \), \( \{ x \in \mathbb{R} \mid a \leq x \leq b \} \), \( \{ x \in \mathbb{R} \mid a \leq x \} \), \( \mathbb{R} \) and
∅. Here $a$ and $b$ are reals such that $a \leq b$. We denote these sets as $(-\infty, b]$, $[a, b]$, $[a, +\infty)$, $(-\infty, +\infty) = \mathbb{R}$, and $\emptyset$.

If $x$ is an interval, $x_l$, the left bound of $x$, is the greatest lower bound of $x$ as a set of reals, if it has one, otherwise $x_l$ is $-\infty$; $x_r$, the right bound of $x$, is the least upper bound of $x$ as a set of reals, if it has one, otherwise $x_r$ is $+\infty$. An interval may consist of a single real number $s$, hence written $[s, s]$. This is called a point interval, sometimes also called degenerate interval.

### Arithmetic on real intervals

The naïve approach to interval arithmetic is embodied in the following tentative definition: for all intervals $X$ and $Y$,

$$
X + Y = \{x + y \in \mathbb{R} \mid x \in X \land y \in Y\}
$$

$$
X - Y = \{x - y \in \mathbb{R} \mid x \in X \land y \in Y\}
$$

$$
X \ast Y = \{x \ast y \in \mathbb{R} \mid x \in X \land y \in Y\}
$$

$$
X/Y = \{x/y \in \mathbb{R} \mid x \in X \land y \in Y\}
$$

The naïve approach gets everything right except for division when $0 \in Y$.

This problem can be remedied by invoking in the interval arithmetic operations, the definitions of the inverse operation $/$ on the reals as a multiplication. Even though this is not necessary for the other operations, in the following definition (still tentative) these operations have been modified in the same way.

$$
X + Y = \{z \in \mathbb{R} \mid \exists x \in X, y \in Y. x + y = z\}
$$

$$
X - Y = \{z \in \mathbb{R} \mid \exists x \in X, y \in Y. y + z = x\}
$$

$$
X \ast Y = \{z \in \mathbb{R} \mid \exists x \in X, y \in Y. x \ast y = z\}
$$

$$
X/Y = \{z \in \mathbb{R} \mid \exists x \in X, y \in Y. y \ast z = x\}
$$

The operations defined in this way have the advantage of being defined for all intervals. But the division operation again gives a problem: although now division is everywhere defined, it may fail to yield an interval, as in $[1, 1]/[-1, +1] = (-\infty, -1] \cup [+1, +\infty)$. This is not an acceptable result, as we need a closed, exception-free algebra.

As we insist on the simplicity of intervals for the inclusion of the sets of reals that arise in computation, we need to enclose some of these sets in the smallest interval containing them. Hence the introduction of the interval hull operator $\square$ where $\square S$ is defined as the least interval containing $S$, for any set $S$ of reals.

For the sake of uniformity we include the interval hull operator also in those cases where its argument is already an interval. Hence the following, definitive, definition:

$$
X + Y = \square\{z \in \mathbb{R} \mid \exists x \in X, y \in Y. x + y = z\}
$$

$$
X - Y = \square\{z \in \mathbb{R} \mid \exists x \in X, y \in Y. y + z = x\}
$$

(1)
\[ X \times Y = \{ z \in \mathbb{R} \mid \exists x \in X, y \in Y . x \times y = z \} \]
\[ X/Y = \{ z \in \mathbb{R} \mid \exists x \in X, y \in Y . y \div z = x \} \]

In the general case it is quite complicated, for the multiplication and division formulas, to express the bounds of the result intervals in terms of operations on the operand bounds. In this paper we only need to consider the case where at least one operand is a point interval. Another simplifying circumstance is that, when the bounds of an operand are not equal, their signs are equal.

With definition (1), real interval arithmetic is a closed, exception-free algebra. It is defined for all interval operands and always yields an interval.

**Floating-point intervals** A floating-point interval is a real interval where the bounds are restricted to floating-point numbers. Every finite set \( F \) of reals defines a finitary approximation to real arithmetic by mapping every real interval to the least floating-point interval containing it.

The interval hull introduced above is with respect to the set of real intervals. From now on we are only concerned with the set of floating-point intervals. Accordingly, the interval hull \( \Box \) in Equations (1) will be taken with respect to the floating-point intervals. The arithmetic operations define a closed, exception-free algebra on floating-point intervals.

### 3 A proposed semantics for floating-point numbers

We propose to interpret floating-point numbers as intervals according to the following cases.

- Each non-zero finite floating-point number \( x \) is interpreted as the point interval \([x, x]\); that is, the singleton set \( \{x\} \).

- The floating-point number \(+\infty\) is interpreted as the real interval \([M, +\infty)\).

  As \( M \) is the greatest finite floating-point number, there is uncertainty about any result greater than that. Hence, it is desirable to have a notation for the set of all those values. For technical reasons \( M \) is included in this set\(^2\). The floating-point number \(+\infty\), being different from any finite floating-point number, is an appropriate representation for \([M, +\infty)\).

- The floating-point number \(-\infty\) is interpreted as \((-\infty, -M]\) for reasons similar to the previous case.

- The floating-point number \(-0\) is interpreted as the floating-point interval \([-m, 0]\), where \( m \) is the least positive floating-point number. The floating-point number \(+0\) is interpreted as the floating-point interval \([0, m]\).

\(^2\) This is mathematically correct (the result is included in the set), but not as informative as when the lower bound of the interval were allowed to be open. This we don’t allow so as to keep interval representations simple.
It might be thought that \([0,0]\) is the most appropriate interpretation. Consider, however, the result of \(a/0\), which would be interpreted as \([a,a]/[0,0]\), and which is the empty set for finite non-zero \(a\) according to Equation (1). The interpretation proposed here follows the idea that zero is a kind of dual to infinity: infinitely small versus infinitely large. Just as we do not try to give infinite precision to the infinitely large, this interpretation of the zeroes renounces infinite precision for the infinitely small. Thus we refer to this semantics as \textit{finite-precision zeroes}. In Section 5 it will be seen that this interpretation of the zeros interacts well with the other interval interpretations. In Section 7 we explore the ramifications of infinite-precision zeroes.

4 IEEE Standard 754 already implements a fragment of interval arithmetic

The justification of our proposal is that for finite floating-point operands, IEEE Standard 754 already implements interval arithmetic, in the sense of the following theorem.

\textbf{Theorem 1} Suppose that the rounding mode is towards \(+\infty\) \((-\infty\)) and that \(x\) and \(y\) are finite floating-point numbers. The result of the floating-point operation \(x \circ y\) is the upper (lower) bound of \([x, x] \circ [y, y]\) as computed in interval arithmetic, where \(\circ\) is any of \(+\), \(-\), and \(*\).

\textit{Proof}: The effect of the rounding mode is described as follows in the IEEE Standard 754 [4]:

When rounding toward \(+\text{INFINITY}\) the result shall be the format’s value (possibly \(+\text{INFINITY}\)) closest to and no less than the infinitely precise result.

With slightly different wording IEEE Standard 754R says [5]:

With \ldots roundTowardPositive, the result shall be the format’s floating-point number (possibly \(+\infty\)) closest to and no less than the infinitely precise result.

The “infinitely precise result” is the result according to real arithmetic. The “format’s value closest to and no less than” is the right bound of the interval resulting from applying the interval hull operator when the “format’s values” constitute the floating-point numbers used in the approximation operator. A similar reasoning applies when the rounding mode is toward \(-\infty\). □

A similar result holds when the rounding mode is toward 0. No such result holds when rounding is towards nearest, as it is not known in which direction rounding takes place. This issue is addressed in Section 8.

\textbf{Theorem 2} Suppose that the rounding mode is towards \(+\infty\) \((-\infty\)) and that \(x\) and \(y\) are finite floating-point numbers with \(y \neq 0\). The result of the floating-point operation \(x/y\) is the upper (lower) bound of \([x, x]/[y, y]\) as computed in interval arithmetic.
The statement of this theorem and its proof are similar to those of Theorem 1. The present theorem is only separate because of the need to exclude the possibility that \( y = 0 \). These two theorems express what we mean when we say that IEEE Standard 754 already implements a fragment of interval arithmetic.

Interesting things happen when we allow \( x \) or \( y \) to become infinite, or \( y \) to become 0 in \( x/y \). These conditions will be discussed in Section 6.

5 Modification of interval arithmetic resulting from interval semantics

Assigning, as we have done, an interval as meaning to every floating-point number has to cause deviations from IEEE Standard 754. This is because in interval arithmetic every operation has a defined result, and this is not the case in IEEE Standard 754. Under interval semantics some operations that are defined change; some undefined ones become defined. In this section we review the operations that are defined in IEEE Standard 754.

- \(+\infty \ast +\infty = [M, +\infty) \ast [M, +\infty) = \Box[M^2, +\infty)\) according to floating-point interval arithmetic. As \( M^2 \) is greater than the greatest floating-point number, the \( \Box \) operator widens its argument so that we get \(+\infty \ast +\infty = [M, +\infty)\). The resulting interval is represented by the floating-point number \(+\infty\). Thus interval semantics maintains the result according to IEEE Standard 754.

- \(+\infty \ast -\infty = (-\infty, -M] \sim -\infty\) according to similar reasoning as above.

- For a positive finite floating-point number \( a \), \( a\ast +\infty = [a, a]\ast [M, +\infty)\). This evaluates to \([M, +\infty) \sim +\infty\) if \( a \geq 1 \), otherwise to \(((a \ast M)^-, +\infty)\). Thus under interval semantics \( a\ast +\infty \) evaluates, if \( a < 1 \), to \((a \ast M)^-\) or to \(+\infty\), depending on rounding mode. The IEEE Standard 754 result is maintained by interval semantics if \( a \geq 1 \). It is not maintained if \( a \) is sufficiently less than 1. We see that IEEE Standard 754 acts as if \( \infty \) is infinitely precise. According to interval semantics only selected finite reals are infinitely precise, and \( \infty \) has limited precision, and this shows by multiplying it with a sufficiently small number.

- \(\infty \ast \infty = [M, \infty) + [M, +\infty) = \Box[M + M, \infty) = [M, \infty) \sim \infty\), which conforms to IEEE Standard 754.

- For finite floating-point number \( a \), \( a + \infty = [a, a] + [M, \infty) = \min((a+M)^-, M), +\infty\). Thus it conforms to IEEE Standard 754 for nonnegative \( a \). As \( a \) approaches \(-M\), the result approaches \([0, +\infty)\).

- \(+0 + (+0) = [0, m] + [0, m] = [0, 2m]\)

- \(+0 + (-0) = [0, m] + [-m, 0] = [-m, m]\)
• For finite positive floating-point \( a \),
  \[
  a/\infty = [a, a]/[M, \infty) = \square(0, (a/M)^+)] = [0, (a/M)^+].
  
  For moderate values of \( a \), this is close to the proposed interval interpretation of +0. For very large \( a \), it approaches to \([0, 1]\). This is because +\( \infty \) harbours a considerable degree of uncertainty, as it stands for all reals greater than \( M \).

• For finite positive floating-point \( a \),
  \[
  +\infty/a = [M, \infty)/[a, a] = \min(M, (M/a)^-), \infty).
  
  Departs moderately from IEEE Standard 754, and does so in a meaningful way. For \( a \) equal to one, it is the interval interpretation of +\( \infty \). As \( a \) increases, the lower bound of the result of \( \infty/a \) decreases until the result becomes \([1, \infty)\) for the greatest finite value of \( a \). This is in accordance with the notion that the interval interpretation of \( \infty \) is not infinitely precise.

• +\( \infty \)/0 = [M, \infty)/[0, m] = \square[M/m], +\( \infty \) = [M, +\( \infty \)] \sim +\( \infty \), which gives the IEEE Standard 754 result by mathematical reasoning rather than by fiat.

• For positive finite \( a \), \( a/0 = [a, a]/[0, m] = [a/m^-], \infty) \). For \( a \) not very small, this result is about the interval interpretation of +\( \infty \), which is close to what IEEE Standard 754 defines. As \( a \) approaches \( m \), this approaches \([1, \infty)\).

6 Undefined operations become defined

In Theorems 1 and 2 we showed that for ordinary cases interval semantics conform to IEEE Standard 754 outcomes. In Section 5 we reviewed cases involving infinity where IEEE Standard 754 does specify a result, but where that result may be different under interval semantics. It remains to review the cases where IEEE Standard 754 specifies that the result is NaN. As interval arithmetic is everywhere defined, interval semantics specifies a result in these cases.

• 0 \( \times +\infty = [0, m] \times [M, +\infty) = [0, +\infty)

• +\( \infty \)/0 = [M, +\( \infty \)/[M, +\( \infty \) = \square(0, +\( \infty \)) = [0, +\( \infty \)

• +\( \infty \)/-\( \infty = [M, +\( \infty \)/(-\( \infty \), -\( M \) = \square((-\( \infty \), 0)) = (-\( \infty \), 0]

• -\( \infty \)/-\( \infty = (-\( \infty \), -\( M \)/(-\( \infty \), -\( M \) = \square((0, +\( \infty \)) = [0, +\( \infty \)

• +0/0 = \square\{y \in \mathbb{R} \mid \exists x \in [0, m], y \in [0, m]. yz = x\} = [0, +\( \infty \)

• +\( \infty \)-+\( \infty = [M, +\( \infty \) - [M, +\( \infty \) = (-\( \infty \), +\( \infty \) = \mathbb{R}
7 Infinite-precision zeroes

In this section we investigate the consequences of interpreting the floating-point numbers +0 and −0 both as the interval \([0, 0]\). An important difference is that for positive finite floating-point \(a\), \(a/ + 0\) becomes \([a, a]/[0, 0]\) which equals

\[
\Box(\{z \in \mathbb{R} \mid \exists x \in [a, a], y \in [0, 0]. yz = x\}) = \emptyset.
\]

With finite-precision zeroes the empty interval never arises.

With either kind of zeroes, every operation has a defined outcome, and we have no need for NaN. But with infinite precision zeroes, we now have a likely candidate for an interval semantics for NaN: the empty set. Strong support for this idea comes from the fact that, whenever one of the operands is an empty interval, the result is empty. That is, the empty interval propagates in the same way as NaN. Also, whenever an empty interval arises, the programmer is likely to want to treat this as an exceptional condition (though it is not an error). Thus, under interval semantics, there is no INVALID OPERATION, as there is in IEEE Standard floating-point arithmetic. Yet this flag arises in the same situations, but now means the exceptional condition of an empty interval being the result.

In the remainder of this section we list the formulas involving zero of Sections 5 and 6 under infinite-precision zeroes.

From Section 5

- \(+0 + (+0) = [0, 0] + [0, 0] = [0, 0] \)
- \(+0 + (−0) = [0, 0] + [0, 0] = [0, 0] \)
- \(+∞/ + 0 = \Box(\{z \in \mathbb{R} \mid \exists x \in [M, +∞), y \in [0, 0]. yz = x\}) = \emptyset\)
- For positive finite \(a\), \(a/ + 0 = \emptyset\), as derived above.

From Section 6

- \(0 * +∞ = [0, 0] * [M, +∞) = [0, 0]. \)
- \(+0/ + 0 = +0/ − 0 = \Box(\{z \in \mathbb{R} \mid \exists x \in [0, 0], y \in [0, 0]. yz = x\}) = (−∞, +∞) \)

8 Consequences for hardware implementation

If we choose for the zeroes the finite precision semantics, there is no role for the NaNs. It might seem an advantage for hardware implementation that NaN, its associated logic, exception, and flag can be omitted. However, there seems to be no scarcity in fully IEEE Standard 754 compliant hardware implementations. Indeed, most of the logic is needed for the implementation of the algorithms for the operations, so that exploiting the absence of NaNs provides only minor relief.
An intriguing possibility is to do full interval arithmetic in hardware \[1, 6, 7\], but that is not the topic of this paper. General interval arithmetic is complex because the bounds of the same interval may differ in sign. It depends on their signs which bounds of the interval operands should be used. In this paper, we are concerned with the simple case of operating on point intervals so that the resulting interval is a point interval or it has adjacent floating-point numbers as bounds. As a result, if the upper bound is known, then the corresponding lower bound is obtained as the previous floating-point number, and vice versa. Thus, to implement interval semantics for floating-point numbers is simpler than implementing interval arithmetic in hardware.

But operands are not always point intervals: when 0 (alias \([0, m]\)) or \(+\infty\) (alias \([M, +\infty]\)) is an operand, the result may be a wide interval. Then it is not the case that the upper bound is unambiguously determined by the lower bound or vice versa. So for the hardware implementation of floating-point arithmetic according to interval semantics it is advantageous to be able to compute both bounds as much as possible in parallel.

Many present day processors consist of multiple CPU cores along with a separate floating point unit (FPU) that is IEEE 754 compliant \[2, 8\]. As with the \[8\], the CPU cores are also multithreaded. With general-purpose processors being 754 compliant, all four of the rounding modes are implemented: round to nearest, round to zero, round to \(+\infty\) and round to \(-\infty\). The FPU is implemented using a pipeline architecture with the rounding mode as part of the instruction word. This would allow the upper bound and lower bound to be computed one cycle after each other. Thereby, the computation of an interval result occurs within two cycles plus the latency of the pipeline. Before being fed to the FPU, the appropriate operands are computed by the other CPU cores that exist within the processor. The combination of these two operations yields the correct arithmetic interval result.

To achieve interval semantics when the FPU only implements the default rounding mode of round to nearest, a flag needs to be available that indicates the direction of the rounding (there is no such flag available in the processor’s FPU). In other words, after rounding of the arithmetic result, the flag will show whether a round up or truncation took place. In conjunction with the sign of the result, correct rounding can be achieved in software or in the compiler. This will require a small addition to the hardware, for which a flag can be generated by comparing the last bit of the rounded word with the first bit of the discarded portion of the pre-rounded word. If a pre-rounded binary data of the mantissa is represented by

\[
 x = b_0.b_1b_2\ldots b_r.b_{r+1}\ldots b_W
\]

where \(W\) is the pre-rounded word length and \(r\) is the word length of the data bus. Let \(R \uparrow\) represent rounding up when set active; the table below describes the result.

| \(b_r\) | \(b_{r+1}\) | \(R \uparrow\) |
|-------|-------|-------|
| 0     | 0     | 0     |
| 0     | 1     | 1     |
| 1     | 0     | 0     |
| 1     | 1     | 1     |
9 Conclusions

We considered two possible interval semantics for the floating-point zeroes: one with finite precision and one with infinite precision. According to the first, there is no role for NaNs, which is satisfying. Also, this semantics, possibly for the first time, gives a mathematical definition for the distinction between the two zeros. According to the infinite-precision semantics for the zeroes, there is a role for NaNs, namely as name for the empty interval, and this is also satisfying. Especially under this semantics it looks like the framers of IEEE Standard 754 were guided by a keen intuition in their choice of those outcomes where mathematics provided no guidance.

Numerical software should be the kind of software that is easiest to verify, as it is modelled after numerical algorithms with proven convergence properties. The IEEE standard accommodates overflow to a certain extent by allowing the infinities as actual values. However, it is difficult to verify the code in the presence of undefined values for several operations on the infinities. Interval semantics gives a mathematical interpretation to the standard floating-point values and operations that is consistent with the mathematics of the reals, yet avoids undefined results. Moreover, the distinction between +0 and −0 has always been problematic from a mathematical point of view. Interval semantics with finite-precision zeroes gives these quantities distinct meanings that are consistent with the meanings of the non-zero floating-point values.

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