Abstract

We study acceleration phenomena in monostable integro-differential equations with weak Allee effect. Previous works have shown its occurrence and obtained correct upper bounds on the rate of expansion, but precise lower bounds were left open. In this paper, we provide a sharp lower bound of acceleration for a large class of dispersion operators. Indeed, our results cover fractional Laplace operators and standard convolutions in a unified way, which is also a contribution of this paper. To achieve this, we construct a refined sub-solution that captures the expected dynamics of the accelerating solution, and this is here the main difficulty.

Keywords: nonlocal operators, fractional laplacian, acceleration, interface dynamics.

1 Introduction.

In this paper, we are interested in describing quantitatively propagation phenomena in the following (non-local) integro-differential equation:

\[ u_t(t, x) = D[u](t, x) + u^\beta(t, x)(1 - u(t, x)) \quad \text{for} \quad t > 0, x \in \mathbb{R}, \]

\[ u(0, \cdot) = 1 \chi_{(-\infty, 0]}, \]

where

\[ D[u](t, x) := P.V. \left( \int_{\mathbb{R}} [u(t, y) - u(t, x)]J(x - y) \, dy \right) \]

with \( J \) is a nonnegative function satisfying the following properties.

**Hypothesis 1.1.** Let \( s \in (0, 1) \). The kernel \( J \geq 0 \) is symmetric and is such that there exists positive constants \( J_0, J_1 \) and \( R_0 \geq 1 \) such that

\[ \int_{|z| \leq 1} J(z)|z|^2 \, dz \leq 2 J_1 \quad \text{and} \quad \frac{J_0}{|z|^{1+2s}} \mathbb{1}_{\{|z| \geq 1\}} \geq J(z) \geq \frac{J_0^{-1}}{|z|^{1+2s}} \mathbb{1}_{\{|z| \geq R_0\}}. \]

The operator \( D[\cdot] \) describes the dispersion process of the individuals. Roughly, the kernel \( J \) gives the probability of a jump from a position \( x \) to a position \( y \), so that the tails of \( J \) are of crucial importance to quantify the dynamics of the population. As a matter of fact, the parameter \( s \) will thus appear in the rates we...
obtain later. One may readily notice that our hypothesis on \( J \) allows to cover the two broad types of integro-differential operators \( D[u] \) usually considered in the literature which are the fractional laplacian \((-\Delta)^\beta u\) and the standard convolution operators with integrable kernels often written \( J \ast u - u \). This universality is one main contribution of this paper.

The parameter \( \beta \) above describes a weak Allee effect that the population overcomes. A biological description and discussion about the origin and relevance of such an effect may be found in a book by Courchamp \textit{et al.} \cite{16} but also in \cite{6, 22, 8}. In crude terms, the Allee effect means that a too small population will not have enough strength to survive and expand. This effect is said to be weak whenever the growth rate of a very small population is eventually extremely small but still positive as opposed to a strong Allee effect leading to negative growth rates for small populations. In the sequel, and without further notice, we take \( \beta > 1 \) (again, yielding small growth rates for small densities). Moreover, we could generalise our results to nonlinearities with the same behaviour around zero and one but we choose to stick to this exact

\[ \alpha | \cdot | \]

\[ J \]

\[ u \]

expression to avoid unnecessary computations, we will not duplicate here this construction. However, they were unable to provide a matching lower bound: their lower bound was growing faster than linearly, but not fast enough.

Let us review existing works that are relevant to understand the importance of our work. Propagation phenomena in reaction diffusion and integro-differential equations has been the object of intense studies in the last decades. Starting from the Fisher-KPP equation \cite{23} and related works by \textit{e.g.} Aronson and Weinberger \cite{7}, the quantitative description of spreading gave birth to various mathematical tools and techniques such as travelling waves, accelerating profiles, transition fronts, among many others. When \( \beta = 1 \), that is a Fisher-KPP nonlinearity, it is known that \eqref{1.1} exhibits some propagation phenomena: starting with some nonnegative nontrivial compactly supported initial data, the corresponding solution \( u(t, x) \) converges to 1, its stable steady state, at large time and locally uniformly in space. This is referred as the \textit{hair trigger effect} \cite{7}.

Moreover, in many cases, the convergence to 1 can be precisely characterised. Indeed, when the dispersion kernel \( J \) is exponentially bounded, travelling waves are known to exist and solutions of the Cauchy problem typically propagate at a constant speed \cite{30, 32, 14, 20, 19, 33}. On the other hand, when the dispersion kernel \( J \) has heavy tails, travelling waves do not exist and then the Cauchy problem exhibits an acceleration phenomenon \cite{29, 33, 24}. More precisely, Garnier \cite{24} gave the first acceleration estimates and then Bouin \textit{et al.} \cite{11} provided sharp level sets for convolution operators; a group around Cabré and Roquejoffre studied in details the fractional Fisher-KPP equation concluding to an exponential propagation behaviour \cite{13, 12}.

Related but different since playing with the tails of the initial data, acceleration phenomena for positive solutions of a local Cauchy problem also appear in reaction diffusion equations \cite{26}. We emphasise that in this paper, the acceleration is only due to the structure of the dispersal operator.

When an Allee effect is introduced, the study of propagation is more subtle. Alfaro started the program with a paper about the interplay between heavy tailed initial data and Allee effect in local reaction-diffusion equations \cite{2}. Coville \textit{et. al.} \cite{20, 19, 18} have proved existence of travelling fronts when the dispersal kernel \( J \) is exponentially bounded and the Cauchy problem typically does not lead to acceleration \cite{34}. When not, the competition between heavy tails and the Allee effect leads to intense discussions. For algebraic decaying kernels, Alfaro and Coville \cite{3} provide the exact separation between existence and non existence of travelling waves. This in turn provides the exact separation between non acceleration and acceleration in the Cauchy problem. In the same spirit, Gui and collaborators discuss the existence of travelling waves and the possibility of acceleration for a fractional equation with Allee effect in \cite{25}. However, in this latter paper, no precise rate of acceleration were given. Before reviewing the last-to-date results on \eqref{1.1}, let us also mention that acceleration phenomenon also appears in some porous medium equations \cite{27, 31, 4, 5}.

As far as \eqref{1.1} is concerned, when \( J \propto | \cdot |^{-(1+2s)} \) or integrable with a finite first moment, bounds on the expansion of the level sets of \( u \) have been already obtained in \cite{21, 3} showing a delicate interplay between the tails of \( J \) and the power \( \beta \). Namely, Coville \textit{et al.} obtained a sharp upper bound of acceleration when \( J \propto | \cdot |^{-(1+2s)} \) or integrable with a finite first moment and assuming that \( \beta < 1 + \frac{1}{2s-1} \): they got an expansion at most \( t^{\frac{\beta}{2s(\beta-s)}} \). It is worth mentioning that the construction made in \cite{3, 21} to obtain this bound is very robust and can be straightforwardly adapted to the problem considered here for kernel satisfying \eqref{1.1}. So to avoid unnecessary computations, we will not duplicate here this construction. However, they were unable to provide a matching lower bound: their lower bound was growing faster than linearly, but not fast enough.

We do not state the exact exponents they get to avoid misunderstandings while reading this paper, but refer
to [21] where they are explicit.

Let us observe that by straightforward application of the strong maximum principle, the solution to (1.1) takes values in [0, 1] only. Moreover, since the initial data is decreasing, at all times \( t \in \mathbb{R}^+ \), the function \( x \mapsto u(t, x) \) is decreasing over \( \mathbb{R} \), from one to zero. To follow the propagation, we may thus follow level sets of height \( \lambda \in (0, 1) \),

\[
x_\lambda(t) := \sup \{ x \in \mathbb{R}, u(t, x) \geq \lambda \}.
\]

Our first main result is the following.

**Theorem 1.2.** Assume that \( J \) satisfies Hypothesis 1.1 and that

\[\beta < 1 + \frac{1}{2s - 1}.\]

Then for any \( \lambda \in (0, 1) \), the level line \( x_\lambda(t) \) accelerates with the following rate\(^1\),

\[x_\lambda(t) \simeq \lambda t^{\frac{s}{2(s-1)}}.\]

Up to our knowledge, this is the first, sharp, unified estimate of the level sets in this context. As already explained above, previous papers were able to derive correct upper bounds but getting a precise lower bound was left open. Our contribution is thus a lower bound that matches the already known upper bounds. To give the reader a clear panorama of the rates of invasion occurring in integro-differential models, we may summarise our and previous contributions in Figure 1. Note that the condition on \( \beta \) fits and unifies all related papers [2, 25, 21].

Before explaining our strategy to get our lower bound, we would like to make some further comments. We observe that the restriction \( s < 1 \) is purely technical and from our analysis we can straightforwardly obtain a sharp estimate when \( s \geq 1 \) and \( \beta \) satisfy \( \frac{1}{2s - 1} > 1 \). The acceleration phenomena observed in this situation is a pure nonlinear effect as the behaviour of the solution of the linear problem then plays no role. The case \( s \geq 1 \) will be treated in a forthcoming companion paper.

To achieve this lower bound, our strategy consists in the construction of a new type of sub-solution that captures all the expected dynamics of the solution \( u \). In particular, it turns out to be mandatory to identify several zones of space over which the behaviour of the solution \( u \) is given by one specific part of the equation. This is something new compared to previous papers. Roughly, the dynamics close to \( t^{\frac{s}{2(s-1)}} \) are due to the nonlinearity only via the related ODE, the far-field zone is purely dissipative and has the behaviour of the linearised equation, and the transition zone between the two is a subtle interplay between the two effects. This dichotomy will be detailed and illustrated in Section 4. Lastly, and related to what just explained, we consider interesting the fact that the exponent of acceleration is a function of \( \beta \) but not the way that the solution flattens with time: it is purely related to the rate of dispersion and will be shown numerically. See Figure 2 for a schematic view of the expected behaviour of the solution.

It is worth adding that the propagation of a compactly supported initial data would lead to a certain amount of different considerations. In particular, the possibility of invasion is related to the size of the initial data due to the existence or not of the so-called hair-trigger effect. Depending on the choice of parameter, \( s \) and \( \beta \), for compactly supported initial datum, it may happen that the solution get extinct at large time, which is referred as the quenching phenomenon [1, 35], and thus no propagation occur.

The rest of the paper is organised as follows. The following Section 2 describes in broad lines the construction of the sub-solution. We proceed in deeper calculations to achieve Theorem 1.2 in Section 4.

## 2 The strategy for the construction of a sub-solution.

We are looking for a sub-solution \( \underline{u} \) to (1.1) that satisfies everywhere

\[
\underline{u} \leq D[\underline{u}] + (1 - \varepsilon)\underline{u}^\beta \quad \text{and} \quad \underline{u} \leq \varepsilon,
\]

\(^1\)We use the notation \( u \simeq \lambda v \) for the existence of a positive constant \( C_\lambda \) such that \( C_\lambda v \leq u \leq C_\lambda^{-1} v \).
Figure 1: In the green zone, the model enjoys linear propagation with existence of travelling fronts [17]: $x_\lambda(t) \sim e^{\lambda t}$. In the blue zone, we provide the sharp lower bound, upper bounds being given in [21, 3]: $x_\lambda(t) \approx t^{\frac{\beta}{2\beta - 1}}$. The orange zone is a zone of exponential propagation, after e.g Roquejoffre et al. [12], Garnier [24], Bouin et al [11]: $x_\lambda(t) \sim e^{\lambda t}$.

for some $\varepsilon \in (0, 1)$. Indeed, this would give, if $u \leq u(t', \cdot)$, for some $t'$

$$u_t \leq D[u] + (1 - \varepsilon)u^\beta \leq D[u] + (1 - u)u^\beta$$

and thus $u$ is a subsolution to $u$. We construct an at least of class $C^2$ function $u$ piecewise,

$$u := \varepsilon, \quad \text{on} \ \{x \leq X(t)\},$$

$$u := \phi, \quad \text{else},$$

with $\phi(t, X(t)) = \varepsilon$. The point $X(t)$ is unknown at that stage. We expect $\phi$ to solve an ODE of the form $n' = n^\beta$ near $x = X(t)$ and to look like a solution of the standard fractional Laplace equation with Heaviside initial data at the far edge. A natural candidate would be given by

$$w(t, x) := \left[ \left( \frac{\kappa t}{x^{2s}} \right)^{1-\beta} - \gamma(\beta - 1)t \right]^{\frac{1}{\beta - 1}}. \quad (2.3)$$

Note that this function is well defined for $t \geq 1$ and $x > X_0 := \kappa \frac{x}{(\beta - 1)t^{1-\frac{1}{\beta - 1}}} \gamma(\beta - 1)t^{1-\frac{1}{\beta - 1}}$ and has visually the structure of a solution to the ODE $n' = n^\beta$. The expected decay in space of a solution of the standard fractional Laplace equation with Heaviside initial data being at least of order $tx^{-2s}$, such a $w$ would have the good asymptotics. Let us define $X(t)$ such that $w(t, X(t)) = \varepsilon$, that is

$$X(t) = (\kappa t)^{\frac{1}{\beta}} \left[ x^{1-\beta} + \gamma(\beta - 1)t \right]^{\frac{1}{\beta - 1}}. \quad (2.4)$$
The positive constants $\kappa$ and $\gamma$ are for the moment free parameters to be chosen later on. One may observe that $X(t)$ moves with the speed that we expect in Theorem 1.2. However, taking $\phi$ as this $w$ would not lead to a $C^2$ function at $x = X(t)$. To remedy this issue, we complete our construction by taking $\phi$ such that

$$u(t, x) := \begin{cases} 
\varepsilon & \text{for all } x \leq X(t), \\
3 \left(1 - \frac{1}{\varepsilon} w(t, x) + \frac{1}{3\varepsilon^2} w^2(t, x)\right) w(t, x) & \text{for all } x > X(t),
\end{cases}$$

for $t > 1$.

3 Flattening estimates and consequence

3.1 A decay estimate.

Our aim here is to establish some flattening estimated on the solutions to (1.1) and analyse its consequences. More precisely, we first show

**Proposition 3.1.** Assume that $J$ satisfies (1.1), then there exists a constant $c_0$ such that for all $t$ the solution $u(t, x)$ to (1.1) satisfies

$$\lim_{x \to \infty} \frac{x^{2s}}{t} u(t, x) \geq c_0.$$

To obtain such type of estimates, let us observe that since $f \geq 0$, the comparison principle ensures $u(t, x) \geq v(t, x)$ where $v(t, x)$ is the solution of the following Cauchy problem:

$$v_t(t, x) = D[v](t, x) \quad \text{for } t > 0, x \in \mathbb{R},$$

$$v(0, \cdot) = 1_{(-\infty,0]},$$

So to obtain such behaviour for $u$ it is then enough to prove such behaviour for $v$. To obtain such estimate, we go back to the way estimates on the heat kernel are found in such a case [9, 10, 28]. The Fourier symbol
of the operator $D[\cdot]$ is
\[
\forall \xi \in \mathbb{R}, \quad W(\xi) := \int_{\mathbb{R}} (\cos(\xi y) - 1) J(y) \, dy.
\]

Note that if $J(y) \propto |y|^{-1-2s}$, that is $D[\cdot]$ is a fractional Laplacian, then $W(\xi) = |\xi|^{2s}$ whereas if $J$ is an integrable function with unit mass, as in convolution models, then $W(\xi) = J(\xi) - 1$. The presence of a singularity at 0 for $J$ has an influence on large frequencies $\xi$, whereas the tail of $J$ influences small frequencies. As a consequence, for small $\xi$, write,
\[
\int_{\mathbb{R}} (\cos(\xi y) - 1) J(y) \, dy = \int_{|y| \leq R_0} (\cos(\xi y) - 1) J(y) \, dy + \int_{|y| \geq R_0} (\cos(\xi y) - 1) J(y) \, dy.
\]

The first integral in the r.h.s is of order $|\xi|^2$ with explicit constant by a direct Taylor expansion and using the hypothesis on $J$. The second one is estimated as follows. Since
\[
\int_{|y| \geq R_0} \mathcal{J}_0^{-1} \frac{\cos(\xi y) - 1}{|y|^{1+2s}} \, dy \leq \int_{|y| \geq R_0} (\cos(\xi y) - 1) J(y) \, dy \leq \int_{|y| \geq R_0} \mathcal{J}_0 \frac{\cos(\xi y) - 1}{|y|^{1+2s}} \, dy,
\]
we have that $\int_{|y| \geq R_0} (\cos(\xi y) - 1) J(y) \, dy$ is of order $|\xi|^{2s}$ with explicit estimates. As a consequence, since $s < 1$, $W$ is of order $|\xi|^{2s}$ with explicit estimates.

The all game now is to obtain good estimate on the the Green function associated to the linear equation, that is the solution defined by
\[
G_t = D[G] \quad \text{for} \quad t > 0, x \in \mathbb{R},
\]
\[
G(0, \cdot) = \delta_{x=0}, \quad \text{for } t > 0, x \in \mathbb{R},
\]

(3.7)

Indeed, note that the solution to (3.6) is then given by
\[
v(t, x) = G(t, \cdot) * 1_{(-\infty, 0)}(\cdot)(x) = \int_{x}^{+\infty} G(t, y) \, dy
\]
and getting an estimate of $G$ for very large $y$ will yield an estimate for $v$. From this computation, observe that for any $t$, we already have $\lim_{x \to -\infty} v(t, x) = 1$.

From [15, 10, 28] we can see that morally the Green function can be estimated by the Green function related to the fractional Laplacian of symbol $|\xi|^{2s}$, i.e. $G \sim G_{|\xi|^{2s}}$. From there we can then obtain the desire estimate by obtaining the estimate for the fractional case. Let us now explain how to obtain the estimates in the case of the Fractional Laplacian and recall the computation made in [28, 21, 12]. Let $p_s(t, x)$ be the heat kernel for $\Delta^s$, then the solution of (3.6) in this situation is then given by
\[
v(t, x) = \int_{\mathbb{R}} v_0(x-y) p_s(t,y) dy, \quad \forall t > 0, x \in \mathbb{R}.
\]

For the heat kernel associated to $\Delta^s$, it’s well known that there exists some constant $1 > C_1(s) > 0$ such that
\[
\frac{C_1}{t^{\frac{1}{2s}} [1 + |t^{-\frac{1}{2s}} x|^{1+2s}]} \leq p_s(t, x) \leq \frac{C_1^{-1}}{t^{\frac{1}{2s}} [1 + |t^{-\frac{1}{2s}} x|^{1+2s}]}, \quad \forall t > 0, x \in \mathbb{R}.
\]
So we get
\[
v(t, x) \geq \int_{\mathbb{R}} v_0(x-y) \cdot \frac{C_1}{t^{\frac{1}{2s}} [1 + |t^{-\frac{1}{2s}} y|^{1+2s}]} dy, \geq \int_{x}^{+\infty} \frac{C_1}{t^{\frac{1}{2s}} [1 + |t^{-\frac{1}{2s}} y|^{1+2s}]} dy,
\]
\[
\geq \int_{t^{-\frac{1}{2s}}}^{+\infty} \frac{C_1}{1 + |z|^{1+2s}} dz
\]
\[
\geq \int_{t^{-\frac{1}{2s}}}^{+\infty} \frac{C_1}{|z|^{1+2s}} dz - \int_{t^{-\frac{1}{2s}}}^{+\infty} \frac{C_1}{(1 + |z|^{1+2s})|z|^{1+2s}} dz
\]
In particular, when \( t^{\frac{1}{2s}} x \geq 1 \) we have

\[
v(t, x) \geq C_1 \frac{t}{x^{2s}} \left( \frac{1}{2s} - \frac{t^{\frac{1}{2s}}}{x} \int_1^\infty \frac{dz}{(1 + |z|^{1+2s})} \right),
\]

and thus for any \( t \) we deduce that

\[
\lim_{x \to \infty} \frac{x^{2s}}{t} v(t, x) \geq C_1 \frac{t}{4s}.
\]

### 3.2 Consequence of the flattening property.

Having now this flattening estimate at hand, we can remark that thanks to the definition of \( u \), (2.5), for all \( t \) and \( \kappa \) we have the following asymptotic behaviour as \( x \to \infty \)

\[
\lim_{x \to \infty} x^{2s} \kappa t u(t, x) \leq 2.
\]

So by Proposition 3.1, since for all \( t \)

\[
\lim_{x \to \infty} x^{2s} \kappa t u(t, x) \geq c_0,
\]

we may find \( t' \) such that \( c_0 t' > 2 \kappa t \). Thus we will then achieve \( u(t, x) \leq u(t + t', x) \) for \( x \gg 1 \) says, \( x > x_0 \).

Now by using the above remark that the solution of the linear problem \( v \) satisfies \( v(t, x) \to 1 \) as \( x \to -\infty \), we then readily have

\[
\lim_{x \to -\infty} u(t, x) = 1
\]

for all \( t \) and as a consequence since \( u \leq \varepsilon \), \( u(t + t', x) \geq u(t, x) \) for \( x << -1 \) says for all \( x \leq -x_1 \). By using that \( u \) is monotone in space, we may find \( x \) such that \( u(t + t', x - x_1) \geq u(t, x) \) for all \( x \in \mathbb{R} \).

As a consequence, to prove Theorem 1.2 we only need to prove that \( u \) is a subsolution to (2.2) for \( t \geq t^* \) for some \( t^* \).

### 4 Proof of Theorem 1.2.

Let us quickly recall why proving that \( u \) is a subsolution to (1.1) yields Theorem 1.2.

Start by observing that \( u \) satisfies (2.2) if and only if,

\[
\begin{align*}
0 & \leq D[u] + \varepsilon^\beta (1 - \varepsilon), & x \leq X(t), \\
\phi_t & \leq D[u] + (1 - \varepsilon) \phi^\beta, & \text{else}
\end{align*}
\]

(4.8) (4.9)

As a consequence, the main work is to derive good estimates for \( D[u] \) in both regions \( x \leq X(t) \) and \( x \geq X(t) \). The estimate in the first region will be rather direct to get and will rely mostly on the fact that \( u \) is constant there together with the tails of \( J \). In the latter region, things are more intricate. We have to split it into three zones, as depicted on Figure 3 below, each one being the stage of one specific character of the model and thus demanding a specific way to estimate \( D[u] \).

Let us now show that for the right choice of \( \varepsilon, \kappa \) and \( \gamma \) the function \( u \) is indeed a subsolution to (2.2) for all \( t \geq t^* \) for some explicit \( t^* \).

#### 4.1 Some preliminary estimates.

##### 4.1.1 Facts and formulas on \( X \) and \( w \).

We will use repeatedly in the rest of the proof that for \( t \geq 1 \)

\[
\begin{align*}
\frac{t}{X(t)} & \leq \left( \frac{1}{\kappa \gamma (\gamma(\beta - 1))^{\frac{1}{\gamma(\beta - 1)}}} \right) t^{1 - \frac{\beta}{\gamma(\beta - 1)}} \quad \text{so that} \quad \lim_{t \to \infty} \frac{t}{X(t)} = 0, \\
\frac{\kappa t}{X^{2s}(t)} & \leq \frac{\varepsilon}{(1 + \varepsilon^{\beta - 1} \gamma(\beta - 1)t)^{\frac{1+2s}{\beta - 1}}} \quad \text{so that} \quad \lim_{t \to \infty} \frac{\kappa t}{X^{2s}(t)} = 0.
\end{align*}
\]

(4.10) (4.11)
Figure 3: Schematic view of the sub-solution at a given time $t$. Several zones have to be considered. The exact expression of $X_2(t)$ will appear naturally later. The blue zone is where $u$ is constant, making computations easier. In the orange zone, the fact that $u$ looks like a solution to an ODE $u' = n \beta$ is crucial. In the brown (far-field) zone, the decay imitating a fractional Laplace equation gives the right behaviour. The green zone is subtle and needs a mixture between both surrounding zones.

From the above estimates we can also derive the following useful limit

$$\lim_{t \to \infty} \frac{t \ln t}{X(t)} = 0,$$

(4.12)

From direct computations we have:

$$u_t = u_x = u_{xx} = 0$$

for all $t > 0, x < X(t)$  

(4.13)

$$u_t = 3w_t \left(1 - \frac{w}{\varepsilon}\right)^2$$

for all $t > 1, x > X(t)$  

(4.14)

$$u_x = 3w_x \left(1 - \frac{w}{\varepsilon}\right)^2$$

for all $t > 1, x > X(t)$  

(4.15)

$$u_{xx}(t, x) = 3 \left(1 - \frac{w}{\varepsilon}\right) \left[w_{xx} \left(1 - \frac{w}{\varepsilon}\right) - \frac{2w_x^2}{\varepsilon}\right]$$

for all $t > 1, x > X(t)$  

(4.16)

Note crucially that $u$ is then at a $C^2$ function in $x$ and $C^1$ in $t$. For convenience, let us denote

$$\Phi(t, x) := \frac{\kappa t}{x^{2s}}, \quad U := \frac{w}{\Phi}.$$

We will need repeatedly the following information on derivatives of $w$ at any point $(t, x)$ where $w$ is defined.

$$w_t = w^\beta \left(\gamma + \frac{\Phi_t}{\Phi^\beta}\right)$$

(4.17)

$$w_x = w^\beta \frac{\Phi_x}{\Phi^\beta} = -2sw^\beta x^{2s(\beta-1)-1} \frac{(\kappa t)^{\beta-1}}{(\kappa t)^{\beta-1}} = U^\beta \Phi_x$$

(4.18)

$$w_{xx} = \beta w^{\beta-1} w_x \frac{\Phi_x}{\Phi^\beta} + w^\beta \frac{\Phi_{xx} \Phi^\beta - |\Phi_x|^2 \beta \Phi_{\beta}^{-1}}{\Phi^{2\beta}} = \left[\Phi_{xx} + \beta |\Phi_x|^2 \Phi^{-1} (U^\beta - 1)\right] U^\beta$$

(4.19)
Since $U \geq 1$, we deduce from the latter that $w$ is convex. Moreover by using the definition of $X(t)$ we also deduce that for $t \geq 1$,

$$w_x(t, X(t)) = \frac{-2s \varepsilon^\beta}{X(t)} \left[ \varepsilon^{1-\beta} + \gamma(\beta - 1)t \right] \geq -2s \varepsilon [1 + \varepsilon^{\beta-1} \gamma(\beta - 1)] \frac{t}{X(t)}$$  \tag{4.20}

from which by using (2.4) we get also the following estimate

$$w_x(t, X(t)) = -2s \varepsilon \left( \frac{x}{K} \right)^{\frac{1}{\beta}} \left( \frac{1}{t} + \varepsilon^{\beta-1} [\gamma(\beta - 1)] \right)^{1-\frac{1}{2s(\beta-1)}} \frac{t^{1-\frac{\beta}{2s(\beta-1)}}}{t^{1-\frac{\beta}{2s(\beta-1)}}}.$$  \tag{4.21}

4.1.2 An estimate for $w$ on $[X(t) + 1, 2^{\frac{1}{\beta}} X(t)]$.

**Proposition 4.1.** Let $t_0 > 1$ be such that $X(t) + 1 < 2^{\frac{1}{\beta}} X(t)$ for all $t \geq t_0$. Then for all $\kappa > 0, \gamma > 0$ and all $\varepsilon \in (0, 1)$, there exists a ball $s \in (s, \kappa, \gamma, \varepsilon, \beta)$ such that for all $t \geq 1$,

$$w^\beta(t, x - 1) \leq 2^{s \beta - 1} w^\beta(t, x) \quad \text{for all} \quad x \in [X(t) + 1, 2^{\frac{1}{\beta}} X(t)].$$

**Proof.** By using (2.3), the definition of $w$, we have

$$\left( \frac{w(t, x-1)}{w(t, x)} \right)^{\beta-1} = \frac{x^{2s(\beta-1)} - \gamma(\beta - 1) \kappa^{\beta-1} t^\beta}{(x - 1)^{2s(\beta-1)} - \gamma(\beta - 1) \kappa^{\beta-1} t^\beta} = \frac{(x-1)^{2s(\beta-1)} \left( 1 + \frac{1}{x-1} \right)^{2s(\beta-1)} - \gamma(\beta - 1) \kappa^{\beta-1} t^\beta}{(x-1)^{2s(\beta-1)} - \gamma(\beta - 1) \kappa^{\beta-1} t^\beta}.$$}

By a standard Taylor expansion, we may find $x_0$ such that for all $x \geq x_0$ we have

$$\left( 1 + \frac{1}{x-1} \right)^{2s(\beta-1)} \leq 1 + \frac{4s(\beta - 1)}{x-1}.$$}

Since $X(t) \to +\infty$, we may then find $t' \geq t_0$ such that

$$\left( 1 + \frac{1}{x-1} \right)^{2s(\beta-1)} \leq 1 + \frac{4s(\beta - 1)}{x-1}, \quad \text{for all} \quad t \geq t', \quad x \geq X(t) + 1.$$}

Therefore, we have

$$\left( \frac{w(t, x-1)}{w(t, x)} \right)^{\beta-1} \leq \frac{(x-1)^{2s(\beta-1)} \left( 1 + \frac{4s(\beta - 1)}{x-1} \right) - \gamma(\beta - 1) \kappa^{\beta-1} t^\beta}{(x-1)^{2s(\beta-1)} - \gamma(\beta - 1) \kappa^{\beta-1} t^\beta} = 1 + \frac{4s(\beta - 1)}{x-1} \frac{(x-1)^{2s(\beta-1)} - \gamma(\beta - 1) \kappa^{\beta-1} t^\beta}{(x-1)^{2s(\beta-1)} - \gamma(\beta - 1) \kappa^{\beta-1} t^\beta}.$$}

Since $X(t) + 1 \leq x \leq 2^{\frac{1}{\beta}} X(t)$, by using (2.4), a short computation shows that for $t \geq t'$

$$\left( \frac{w(t, x-1)}{w(t, x)} \right)^{\beta-1} \leq 1 + 8s(\beta - 1) [1 + \gamma(\beta - 1) \varepsilon^{\beta-1}] \frac{t}{X(t)}.$$}

Thanks to (4.10) the proof is then complete for $t \geq t := \sup \left\{ t', \left( \frac{8s(\beta - 1) [1 + \gamma(\beta - 1) \varepsilon^{\beta-1}]}{\kappa^{\beta} (\gamma(\beta - 1))^{\frac{1}{\gamma(\beta - 1)}}} \right)^{\frac{1}{2s(\beta-1)}} \right\}$. \hfill \square
4.1.3 An estimate for $w$ on $[2^{1/(2s-1)}X(t), +\infty)$.

**Proposition 4.2.** There exists a constant $C_0'$ such that for all $\kappa > 0, \gamma > 0$ and all $\varepsilon \in (0, 1)$, there exists $t^\#(\gamma, \kappa, s, \beta, \varepsilon)$ so that for all $t \geq t^\#$, $x \geq 2^{1/(2s-1)}X(t)$,

$$w(t, x - 1) \leq \frac{C_0' \kappa t}{(x - 1)^{2s}}, \quad w(t, x) \leq \frac{C_0' \kappa t}{x^{2s}}.$$

**Proof.** The argument to obtain the estimate for $w(t, x)$ and $w(t, x - 1)$ being similar, we only show the estimate for $w(t, x - 1)$. By using (2.3), the definition of $w$, since we have

$$w(t, x - 1) = \frac{\kappa t}{(x - 1)^{2s}} \left(1 - \gamma(\beta - 1)t^2\kappa^s - 1\right)^{-\frac{1}{1-\beta}}$$

Since the function $X(t)$ is increasing and tends to $+\infty$, we may find $t'(\kappa)$ such that $X(t) \geq \frac{2}{2^{1/(2s-1)} - 1}$ for all $t \geq t'$ and therefore for $t \geq t'$,

$$2^{1/(2s-1)}X(t) - 1 = \frac{2^{1/(2s-1)} - 1}{2} X(t) + \frac{2^{1/(2s-1)} - 1}{2} X(t) - 1 \geq \frac{2^{1/(2s-1)} - 1}{2} X(t).$$

As a consequence, for all $x \geq 2^{1/(2s-1)}X(t)$ and $t \geq t'$, by using the definition of $X(t)$, (2.4), we have the following estimate for $w$,

$$w(t, x - 1) \leq \frac{\kappa t}{(x - 1)^{2s}} \left(1 - \gamma(\beta - 1)t^2\kappa^s - 1\right)^{-\frac{1}{1-\beta}} \leq \frac{\kappa t}{(x - 1)^{2s}} \left(1 - \frac{1}{C}\right)^{-\frac{1}{1-\beta}}$$

with $C := \left(\frac{2^{1/(2s-1)} - 1}{2}\right)^{2s(\beta - 1)}$. The proof is then ended by taking $C_0'(t) := (1 - \frac{1}{C})^{-\frac{1}{1-\beta}}$.

4.2 Estimating $D[u]$ when $x \leq X(t)$.

On this region, by definition of $u$, we have

$$D[u](t, x) = \int_{y \geq X(t)} [u(t, y) - \varepsilon] J(x - y) \, dy.$$

This section aims at showing (4.8). For the convenience of the reader, we shall state this is the following

**Proposition 4.3.** For all $\varepsilon \leq \frac{1}{2}, \gamma$ and $\kappa$ there exists $t_0(\varepsilon, \kappa, \gamma, \beta, s)$ such that for all $t \geq t_0$

$$D[u](t, x) + \frac{\varepsilon \beta}{2}(1 - \varepsilon) \geq 0 \quad \text{for all} \quad x \leq X(t).$$

**Proof.** We split the interval $(-\infty, X(t)]$ into two sub-intervals $(-\infty, X(t) - B]$ and $(X(t) - B, X(t)]$ with $B > 1$ to be chosen later, and thus estimate $D[u]$ on both subsets.

When $x \leq X(t) - B$: On this subset, then a short computation gives

$$D[u](t, x) = \int_{y \geq X(t)} \frac{u(t, y) - \varepsilon}{|x - y|^{1 + 2s}} J(x - y) \, dy \geq -\varepsilon \mathcal{J}_0 \int_{y \geq X(t)} \frac{dy}{|y - x|^{1 + 2s}} \geq -\varepsilon \mathcal{J}_0 \frac{1}{2s} \left(\frac{1}{|X(t) - x|^{2s}} \geq -\varepsilon \mathcal{J}_0 \frac{1}{2s} \frac{1}{B^{2s}}.

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When \( X(t) - B < x \leq X(t) \): On this subset, by making the change of variable \( z = y - x \), since \( B > 1 \) and \( u(t, x) = \varepsilon \), a short computation gives

\[
\begin{align*}
D[u](t, x) &= \int_{z \geq X(t) - x + B} [u(t, x + z) - \varepsilon] J(z) \, dz + \int_{X(t) - x + B \geq z \geq X(t) - x} [u(t, x + z) - \varepsilon] J(z) \, dz \\
&\geq -\varepsilon J \int_{z \geq X(t) - x + B} \frac{dz}{z^{1 + 2s}} + \int_{X(t) - x + B \geq z \geq X(t) - x} [u(t, x + z) - u(t, x)] J(z) \, dz \\
&\geq -\varepsilon J \int_{X(t) - x + B \geq z \geq X(t) - x} [u(t, x + z) - u(t, x)] J(z) \, dz \\
&\geq -\varepsilon J \int_{X(t) - x + B \geq z \geq X(t) - x} [u(t, x + z) - u(t, x)] J(z) \, dz.
\end{align*}
\]

By using the Taylor formula we have

\[
u(t, x + z) - u(t, x) = z \int_{0}^{1} \frac{d}{dz} u(t, x + \tau z) \, d\tau
\]

and thus we can estimate the last integral by

\[
I := \int_{X(t) - x}^{X(t) - x + B} \frac{dz}{z^{1 + 2s}} + \int_{X(t) - x + B \geq z \geq X(t) - x} [u(t, x + z) - u(t, x)] J(z) \, dz = \int_{X(t) - x}^{X(t) - x + B} \int_{0}^{1} \frac{d}{dz} u(t, x + \tau z) J(z) \, dz \, d\tau.
\]

We now treat separately the two situations \( X(t) - 1 < x \leq X(t) \) and \( X(t) - B < x \leq X(t) - 1 \). In the later case, since \( X(t) - x \geq 1 \) and since \( J \) satisfies (1.1) we have

\[
I \geq J \int_{X(t) - x}^{X(t) - x + B} \int_{0}^{1} \frac{d}{dz} u(t, x + \tau z) z^{-2s} \, dz \, d\tau \geq \min_{\xi > X(t) - x} \int_{X(t) - x}^{X(t) - x + B} \frac{d}{dz} u(t, x + \xi) z^{-2s} \, dz
\]

which by using the definition of \( u(t, x) \), (4.15), and the convexity of \( w \) yields to

\[
I \geq 3 J w(t, X(t)) \int_{X(t) - x}^{X(t) - x + B} z^{-2s} \, dz \geq 3 J w(t, X(t)) \int_{1}^{2B} z^{-2s} \, dz.
\]

In the other situation, since \( u(0, x) = 0 \) for all \( x \leq X(t) \), we can rewrite \( I \) as follows

\[
I = \int_{X(t) - 1}^{X(t) - x + B} \int_{0}^{1} \frac{d}{dz} u(t, x + \tau z) J(z) \, dz \, d\tau = \int_{X(t) - 1}^{X(t) - x + B} \int_{0}^{1} \frac{d}{dz} u(t, x + \tau z) J(z) \, dz \, d\tau.
\]

Let us split the last integral as follows

\[
I = \int_{0}^{1} \int_{0}^{1} \frac{d}{dz} u(t, x + \tau z) J(z) \, dz \, d\tau + \int_{0}^{1} \int_{0}^{1} \frac{d}{dz} u(t, x + \tau z) J(z) \, dz \, d\tau.
\]

The last integral can be treated as above and so we have

\[
I \geq \int_{0}^{1} \int_{0}^{1} \frac{d}{dz} u(t, x + \tau z) J(z) \, dz \, d\tau + 3 J w(t, X(t)) \int_{1}^{2B} z^{-2s} \, dz.
\]

Since \( u(t, x) \) is a \( C^1 \) function and \( \partial_x u(t, x) = 0 \) for \( x < X(t) \), we can reapply a Taylor formula to handle the last integral and thus we get
\[
\int_{-1}^{1} \int_{0}^{1} u_x(t, x + \tau z) z J(z) \, dz d\tau = \int_{-1}^{1} \int_{0}^{1} \left[ u_x(t, x + \tau z) - u_x(t, x) \right] z J(z) \, dz d\tau
= \int_{0}^{1} \int_{0}^{1} \int_{1}^{1} u_{xx}(t, x + \tau \sigma z) J(z) \tau z^2 \, dz d\tau d\sigma.
\]

By using the definition of \( u_{xx} \), (4.16) and the convexity of \( w \), we get for \( x \geq X(t) \)
\[
\int_{-1}^{1} \int_{0}^{1} u_x(t, x + \tau z) z J(z) \, dz d\tau \geq -\frac{3}{J_1} \varepsilon (w_x(t, X(t)))^2,
\]
where we have used Hypothesis 1.1 to estimate the integral. In any case, for all \( X(t) - B \leq x \leq X(t) \), we then get
\[
I \geq -\frac{3J_1}{\varepsilon} (w_x(t, X(t)))^2 + 3J_0 w_x(t, X(t)) \int_{1}^{2B} z^{-2s} \, dz.
\]
As a consequence, we obtain the following estimate:
\[
D[u] \geq -\frac{\varepsilon J_0}{2s} \frac{1}{B^{2s}} + 3w_x(t, X(t)) \left( -\frac{J_1}{\varepsilon} w_x(t, X(t)) + J_0 \int_{1}^{2B} z^{-2s} \, dz \right).
\]

We are now ready to choose \( B := \left( \frac{2J_0}{\varepsilon s \varepsilon - \varepsilon} + 1 \right)^{\frac{1}{\varepsilon}} \) above. We then get
\[
D[u] \geq -\frac{\varepsilon \beta (1 - \varepsilon)}{4} + 3w_x(t, X(t)) \left( -\frac{J_1}{\varepsilon} w_x(t, X(t)) + J_0 \int_{1}^{2B} z^{-2s} \, dz \right).
\]
Recall that by (4.20) we have
\[
w_x(t, X(t)) \geq -2s \varepsilon [1 + \varepsilon^{\beta-1} \gamma (\beta - 1)] \frac{t}{X(t)}.
\]
So by (4.10) we may find \( t' \) such that for all \( t \geq t' \) \( w_x(t, X(t)) \geq -\varepsilon \) and thus we get
\[
D[u] + \frac{\varepsilon \beta}{2} (1 - \varepsilon) \geq \frac{\varepsilon \beta (1 - \varepsilon)}{4} + 3w_x(t, X(t)) \left( -\frac{J_1}{\varepsilon} w_x(t, X(t)) + J_0 \int_{1}^{2B} z^{-2s} \, dz \right),
\]
\[
\geq \frac{1}{4} \varepsilon \beta (1 - \varepsilon) - 6s \varepsilon [1 + \varepsilon^{\beta-1} \gamma (\beta - 1)] \frac{t}{X(t)} \left( J_1 + J_0 \int_{1}^{2B} z^{-2s} \, dz \right).
\]
By using again (4.10), we may then find an explicit \( t'' \) so that
\[
6s \varepsilon [1 + \varepsilon^{\beta-1} \gamma (\beta - 1)] \frac{t}{X(t)} \left( J_1 + J_0 \int_{1}^{2B} z^{-2s} \, dz \right) \leq \frac{1}{4} \varepsilon \beta (1 - \varepsilon).
\]
and thus for \( t \geq t_0 := \sup \{ t', t'' \} \) we then achieve
\[
D[u] + \frac{\varepsilon \beta}{2} (1 - \varepsilon) \geq 0.
\]
4.3 Estimate of $\mathcal{D}[u]$ on $x > X(t)$.

In this region,

$$
\mathcal{D}[u](t, x) := \int_{y \leq X(t)} [\varepsilon - u(t, x)]J(x - y) \, dy + \int_{y \geq X(t)} [u(t, y) - u(t, x)]J(x - y) \, dy, \quad x \geq X(t).
$$

As exposed earlier and shown in Figure 3, we shall estimate $\mathcal{D}[u]$ in three separate intervals

$$
[X(t), \sup\{X(t) + R_0; X_2]\}, \quad [\sup\{X(t) + R_0; X_2\}, 2^{(2s - 3)/(1 - 2s)} X(t)], \quad [2^{(2s - 3)/(1 - 2s)} X(t), +\infty).
$$

The exact expression of $X_2$ is explicit and is such that $w(t, X_2(t)) = \frac{\varepsilon}{2}$.

4.3.1 The region $X(t) \leq x \leq \sup\{X(t) + R_0; X_2\}$

We start this with an estimate

**Lemma 4.4.** For all $B > 1$ and $\delta \geq \sup\{R_0, B + X(t) - x\}$,

$$
\mathcal{D}[u](t, x) \geq -\frac{J_0 u(t, x)}{8B^{2s}} - \frac{3}{\varepsilon} \left( J_1 + J_0 \int_{1}^{B} z^{1-2s} \, dz \right) \sup_{x+\xi > X(t)} (w_x(t, x + \xi))^2. \quad (4.22)
$$

**Proof.** By definition of $u$, for any $\delta \geq R_0$ we have, using Hypothesis 1.1,

$$
\mathcal{D}[u](t, x) = \int_{x+\epsilon \leq X(t) - \delta} \frac{\varepsilon - u(t, x)}{|z|^{1+2s}} J(z) |z|^{1+2s} \, dz + \int_{x+\epsilon \geq X(t) - \delta} [u(t, x + z) - u(t, x)] J(z) \, dz,
$$

$$
= \frac{J_0^{-1}}{2s} \frac{\varepsilon - u(t, x)}{(x - X(t) + \delta)^{2s}} + \int_{x+\epsilon \geq X(t) - \delta} [u(t, x + z) - u(t, x)] J(z) \, dz. \quad (4.23)
$$

We shall now estimate

$$
\int_{x+\epsilon \geq X(t) - \delta} [u(t, x + z) - u(t, x)] J(z) \, dz.
$$

For $B \geq 0$ to be chosen later on, we decompose,

$$
\int_{x+\epsilon \geq X(t) - \delta} [u(t, x + z) - u(t, x)] J(z) \, dz = \int_{x+\epsilon \geq X(t) - \delta, |z| \leq B} [u(t, x + z) - u(t, x)] J(z) \, dz
$$

$$
+ \int_{x+\epsilon \geq X(t) - \delta, |z| \geq B} [u(t, x + z) - u(t, x)] J(z) \, dz. \quad (4.24)
$$

The second integral in the right hand side of the above expression is the easiest. Since $u$ is positive and $J$ satisfies (1.1) we then have for $B > 1$,

$$
\int_{x+\epsilon \geq X(t) - \delta, |z| \geq B} [u(t, x + z) - u(t, x)] J(z) \, dz \geq -u(t, x)J_0 \int_{x+\epsilon \geq X(t) - \delta, |z| \geq B} \frac{dz}{|z|^{1+2s}}.
$$

When $X(t) - \delta \leq x - B$, a short computation shows that

$$
\int_{x+\epsilon \geq X(t) - \delta, |z| \geq B} \frac{dz}{|z|^{1+2s}} = \int_{X(t) - x - \delta \leq z \leq -B} \frac{dz}{|z|^{1+2s}} + \int_{z \geq B} \frac{dz}{|z|^{1+2s}}
$$

$$
= \int_{X(t) - x - \delta \leq z \leq -B} \frac{dz}{z^{1+2s}} + \int_{z \geq B} \frac{dz}{z^{1+2s}}
$$

$$
= \frac{1}{2sB^{2s}} - \frac{1}{2s(x + \delta - X(t))^{2s}} + \frac{1}{2sB^{2s}}.
$$

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On the other hand if \( X(t) - \delta \geq x - B \) then
\[
\int_{x+z \geq X(t) - \delta, |z| \geq B} \frac{dz}{z^{1+2s}} = \int_{z \geq B} \frac{dz}{|z|^{1+2s}} = \int_{z \geq B} \frac{dz}{z^{1+2s}} = \frac{1}{2sB^{2s}}.
\]
In each situation we then have
\[
\int_{x+z \geq X(t) - \delta, |z| \geq B} [u(t, x + z) - u(t, x)] J(z) \, dz \geq -\frac{w(t, x) J_0}{s B^{2s}}. \tag{4.25}
\]

Let us now estimate the first integral of the right hand side of (4.24), that is, let us estimate
\[
I := \int_{x+z \geq X(t) - \delta, |z| \leq B} [u(t, x + z) - u(t, x)] J(z) \, dz.
\]
Since \( u(t, x) \) is \( C^1 \) in \( x \) we have, for all \( t \geq 1 \) and \( x \in \mathbb{R} \),
\[
|u(t, x + z) - u(t, x)| = z \int_0^1 u_x(t, x + \tau z) \, d\tau
\]
and therefore we can rewrite \( I \) as follows:
\[
I = \int_{x+z \geq X(t) - \delta, |z| \leq B} \int_0^1 u_x(t, x + \tau z) z J(z) \, d\tau dz.
\]
For any \( \delta \geq B + X(t) - x \), let us observe that by symmetry we have
\[
\int_{x+z \geq X(t) - \delta, |z| \leq B} \int_0^1 J(z) z d\tau dz = 0.
\]
As a consequence we can rewrite \( I \) as follows:
\[
I = \int_{x+z \geq X(t) - \delta, |z| \leq B} \int_0^1 [u_x(t, x + \tau z) - u_x(t, x)] z J(z) \, d\tau dz.
\]
Since \( u_x \) is a \( C^1 \) function, by using the Taylor expansion
\[
[u_x(t, x + \tau z) - u_x(t, x)] = \tau z \int_0^1 u_{xx}(t, x + \tau\sigma z) \, d\sigma
\]
we have
\[
I = \int_{x+z \geq X(t) - \delta, |z| \leq B} \int_0^1 \int_0^1 u_{xx}(t, x + \sigma \tau z) \tau z^2 \int J(z) \, d\tau d\sigma dz,
\]
\[
\geq \min_{-B < \xi < B} \frac{u_{xx}(t, x + \xi)}{\int_{|z| \leq B} \int_0^1 \tau z^2 J(z) \, d\tau d\sigma dz} \int_{|z| \leq 1} \int_0^1 \tau z^2 J(z) \, d\tau d\sigma dz + \int_{1 \leq |z| \leq B} \int_0^1 \tau z^2 J(z) \, d\tau d\sigma dz.
\]
By using (4.13) and (4.16) and the convexity of \( w \), we deduce that
\[
I \geq \frac{\delta}{\varepsilon} \sup_{-B < \xi < B, \ x + \xi > X(t)} w_x(t, x + \xi)^2 \left( \int_{|z| \leq 1} \int_0^1 \tau z^2 J(z) \, d\tau d\sigma dz + \int_{1 \leq |z| \leq B} \int_0^1 \tau z^2 J(z) \, d\tau d\sigma dz \right).
\]
Hence we have
\[
I \geq -\frac{3}{\varepsilon} \left( J_1 + J_0 \int_1^B z_1^{-2s} \, dz \right) \sup_{-\varepsilon < \zeta < \varepsilon, x+\varepsilon > X(t)} w_x(t, x+\varepsilon)^2. \tag{4.26}
\]

Collecting (4.24), (4.25) and (4.26), we get the following estimate for all \(B > 1\) and \(\delta \geq \sup \{R_0, B + X(t) - x\},\)
\[
\mathcal{D}[u](t, x) \geq \frac{J_0 - \varepsilon - w(t, x)}{2s} - \frac{3}{\varepsilon} \left( J_1 + J_0 \int_1^B z_1^{-2s} \, dz \right) \sup_{-\varepsilon < \zeta < \varepsilon, x+\varepsilon > X(t)} (w_x(t, x+\varepsilon))^2 \tag{4.27}
\]
with ends the proof of the lemma since \(w \leq \varepsilon.\)

With this lemma at hand, we claim that

**Proposition 4.5.** For all \(\varepsilon < \frac{1}{2}, \kappa\) and \(\gamma\) there exists \(t_1\) such that
\[
\mathcal{D}[u](t, x) + (1 - \varepsilon) w^\beta \geq \frac{1}{2} (1 - \varepsilon) w^\beta \quad \text{for all} \quad t \geq t_1, \quad X(t) < x < \sup \{X(t) + R_0, X_2\}.
\]

**Proof.** Use (4.22) from the previous lemma, to get for any \(\delta > \sup \{R_0, X(t) - x + B\},\)
\[
\mathcal{D}[u](t, x) \geq \frac{J_0 - \varepsilon - w(t, x)}{sB^{2s}} - \frac{3}{\varepsilon} \sup_{-\varepsilon < \zeta < \varepsilon, x+\varepsilon > X(t)} (w_x(t, x+\varepsilon))^2 \left( J_1 + J_0 \int_1^B z_1^{-2s} \, dz \right) \quad \text{when} \quad X(t) < x \leq X(t) + R_0.
\]
Specify \(B = \nu \varepsilon^\frac{1-\beta}{2s}\) with \(\nu > 1\) to be chosen later on. Then from the above inequality we deduce that
\[
\mathcal{D}[u](t, x) \geq \frac{J_0 - \varepsilon - w(t, x)}{sB^{2s}} - \frac{3}{\varepsilon} \sup_{-\varepsilon < \zeta < \varepsilon, x+\varepsilon > X(t)} (w_x(t, x+\varepsilon))^2 \left( J_1 + J_0 \int_1^{\nu \varepsilon^\frac{1-\beta}{2s}} z_1^{-2s} \, dz \right) \quad \text{when} \quad x > X(t). \tag{4.28}
\]

Let us estimate from below \(u.\) Assume first that \(x \leq X_2(t)\) so that for such \(x\) since \(w(t, x) \geq \frac{\varepsilon}{4}\) then by definition of \(u\) we can check that \(u(t, x) \geq \frac{3\varepsilon}{4}.\) On the other hand when \(x < X(t) + R_0\) since \(u\) is smooth we have
\[
u(t, x) = u(t, X(t) + R_0) = u(t, X(t)) + w(t, X(t) + R_0) - u(t, X(t))
\]
and by using the definition of \(w_x(t, x), (4.15),\) and the convexity of \(w(t, x)\) we deduce that
\[
u(t, x) \geq u(t, X(t)) + 3R_0 w_x(t, X(t)) = \varepsilon + 3R_0 w_x(t, X(t)).
\]
From the estimate of \(w_x(t, X(t)), (4.20),\) it follows that
\[
u(t, x) \geq \varepsilon - \frac{6R_0 \varepsilon t}{X(t)} \left( 1 + \varepsilon^{\beta-1} [\gamma (\beta - 1)] \right),
\]
which thanks to (4.10) enforces \(\nu(t, x) \geq \frac{3\varepsilon}{4}\) for \(t \geq t^*\) for some \(t^*.\) So in both situations for \(t \geq t^*\) we have \(\nu(t, x) \geq \frac{3\varepsilon}{4}\) and therefore
\[
u^\beta(t, x) \geq \frac{3\varepsilon^\beta}{2^2 \beta^2}.
\]

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Plugging this estimate in (4.28), we deduce that for \( X(t) < x < \sup\{X(t) + R_0, X_2(t)\} \) and \( t \geq t^* \),
\[
\mathcal{D}[u](t, x) + \frac{(1 - \varepsilon)}{2} w^{\beta}(t, x) \geq \varepsilon^3 \frac{3\beta(1 - \varepsilon)}{2^{2\beta+1}} - \frac{J_0 \varepsilon^3}{s B^{2s}} \sup_{-\beta < \xi < 0, x + \xi > X(t)} (w_x(t, x + \xi))^2 \left( J_1 + \int_0^{t} \int_{1}^{t} z^{1-2s} dz \right).
\]

Recall that \( w(t, x) \) is convex, so that
\[
\sup_{-\beta < \xi < 0, x + \xi > X(t)} (w_x(t, x + \xi))^2 = w_x(t, X(t))^2,
\]
and choose now \( \nu := \sup \left\{ \left( \frac{x^{2\beta+1}}{s^{\beta+2} B^{2s}} \right)^\frac{1}{\beta} : 1 \right\} \), we then get using (4.20),
\[
\mathcal{D}[u](t, x)(t, x) + \frac{(1 - \varepsilon)}{2} w^{\beta}(t, x) \\
\geq \varepsilon^3 \frac{3\beta(1 - \varepsilon)}{2^{2\beta+1}} - 12 s^2 \varepsilon (1 + \varepsilon) \left( \frac{t}{X(t)} \right)^2 \\
\geq J_1 + \int_0^{t} \int_{1}^{t} z^{1-2s} dz \left( \frac{t}{X(t)} \right)^2.
\]

Finally recalling that \( \lim_{t \to \infty} \frac{t}{X(t)} = 0 \), see (4.10), we may find \( t_1 \geq t^* \) such that for all \( t \geq t_1 \) the right hand side of the above expression is positive ending thus the proof of this proposition. \( \square \)

### 4.3.2 A preliminary estimate in the range \( x \geq \sup\{X(t) + R_0, X_2(t)\} \)

Here we derive a generic estimate of \( \mathcal{D}[u] \) only valid in the range \( x \geq \sup\{X(t) + R_0, X_2(t)\} \).

**Lemma 4.6.** For any time \( t > 1 \) and \( x \geq \sup\{X(t) + R_0, X_2(t)\} \),
\[
\mathcal{D}[u](t, x) \geq \frac{\varepsilon - u(t, x)}{2s J_0 x^{2s}} + \min_{-1 < \xi < 1} \frac{u_{xx}(t, x + \xi)}{2s B^{2s}} - \frac{J_0 u(t, x)}{2s B^{2s}} + \frac{3 J_0}{4} \left( \int_1^{\sup(x, X(t))} z^{1-2s} dz \right) w_x(t, x). \tag{4.29}
\]

**Proof.** Let us go back to the definition of \( \mathcal{D}[u](t, x) \) that we split into three parts:
\[
\mathcal{D}[u](t, x) = \int_{-\infty}^{X(t)} [u(t, x + z) - u(t, x)] J(z) dz + \int_{-1}^{1} [u(t, x + z) - u(t, x)] J(z) dz + \int_{1}^{\infty} [u(t, x + z) - u(t, x)] J(z) dz.
\]

Since \( x \geq X(t) + R_0 \) and \( u \) is decreasing, the first integral can be estimated as follows:
\[
\int_{-\infty}^{X(t)} [u(t, x + z) - u(t, x)] J(z) dz \\
\geq \int_{-\infty}^{X(t)} \frac{u(t, x + z) - u(t, x)}{|z|^{1 + 2s}} J(z) dz + \int_{X(t)}^{\infty} [u(t, x + z) - u(t, x)] J(z) dz \\
\geq \frac{J_0}{2s} \frac{\varepsilon - u(t, x)}{(x - X(t))^{2s}}. \tag{4.30}
\]

To obtain an estimate of the second integral, we actually follow the same steps as several times previously to obtain via Taylor expansion,
\[
\int_{-1}^{1} [u(t, x + z) - u(t, x)] J(z) dz = \int_{-1}^{1} \int_{0}^{1} \int_{0}^{1} w_{xx}(t, x + \tau \sigma) \tau z J(z) d\tau d\sigma dz \\
\geq \int_{-1}^{1} \min_{-1 < \xi < 1} u_{xx}(t, x + \xi). \tag{4.31}
\]

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Finally, let us estimate the last integral
\[
I := \int_1^{+\infty} |u(t, x + z) - u(t, x)| J(z) \, dz
\]
\[
= \int_1^B |u(t, x + z) - u(t, x)| J(z) \, dz + \int_B^{+\infty} |u(t, x + z) - u(t, x)| J(z) \, dz,
\]
for \( B > 1 \) to be chosen later on. Since \( u \) is positive we have
\[
\int_B^{+\infty} |u(t, x + z) - u(t, x)| J(z) \, dz \geq -\frac{\mathcal{J}_0 u(t, x)}{2sB^{2s}}.
\]
(4.32)
By using again a Taylor formula, the last integral rewrites
\[
\int_1^B |u(t, x + z) - u(t, x)| J(z) \, dz = \int_1^B \int_0^1 u_x(t, x + \tau z) J(z) \, d\tau dz.
\]
Observe that since \( x \geq X_2 \) and \( w \) is convex, (4.15) implies
\[
u_x(t, x + \tau z) \geq \frac{3}{4} w_x(t, x).
\]
It follows that
\[
\int_1^B |u(t, x + z) - u(t, x)| J(z) \, dz \geq \frac{3}{4} \left( \int_1^B z J(z) \right) w_x(t, x) \geq \frac{3\mathcal{J}_0}{4} \left( \int_1^B z^{-2s} \, dz \right) w_x(t, x).
\]
(4.33)
using Hypothesis 1.1. Collecting (4.30), (4.31),(4.32), (4.33), we find for \( x \geq X(t) + R_0 \),
\[
D[\mathcal{W}](t, x) \geq \frac{\epsilon - \frac{w(t, x)}{2}}{2s \mathcal{J}_0 x^{2s}} + \mathcal{J}_1 \min_{-1 < \xi < 1} w_x(t, x + \xi) - \frac{3\mathcal{J}_0 u(t, x)}{2s B^{2s}} + \frac{3\mathcal{J}_0}{4} \left( \int_1^B z^{-2s} \, dz \right) w_x(t, x),
\]
which ends the proof of the lemma.

\[\square\]

4.3.3 The region \( \sup \{X(t) + R_0; X_2\} < x < 2^{\frac{1}{\beta - 1}} X(t) \)

The previous lemma at hand, let us now estimate \( D[\mathcal{W}] \) when \( x \geq \sup \{X(t) + R_0, X_2(t)\} \).

**Proposition 4.7.** For any \( 0 < \epsilon \leq \frac{1}{2} \) and any \( \gamma, \kappa > 0 \), there exists \( t_3 > 0 \) such that for all \( t \geq t_3 \),
\[
D[\mathcal{W}](t, x) + \frac{\epsilon}{2}(1 - \epsilon) \geq \frac{\epsilon}{8\mathcal{J}_0 x^{2s}} \quad \text{for all} \quad \sup \{X(t) + R_0, X_2(t)\} < x < 2^{\frac{1}{\beta - 1}} X(t).
\]

**Proof.** Let us recall that \( X_2(t) \) is such that \( w(t, X_2(t)) = \frac{\epsilon}{2} \) and consider \( x \geq \sup \{X_2(t), X(t) + R_0\} \). For such \( x \), let us take \( B = (2\mathcal{J}_0)^{\frac{1}{\beta}} x \) in (4.29). Since \( \frac{\epsilon}{2} \leq \epsilon \), we have then
\[
D[\mathcal{W}](t, x) \geq \frac{\epsilon}{8\mathcal{J}_0 x^{2s}} + \mathcal{J}_1 \min_{-1 < \xi < 1} w_x(t, x + \xi) + \frac{3\mathcal{J}_0}{4} \left( \int_1^{(2\mathcal{J}_0)^{\frac{1}{\beta}} x} z^{-2s} \, dz \right) w_x(t, x),
\]
\[
\geq \frac{\epsilon}{8\mathcal{J}_0 x^{2s}} + \mathcal{J}_1 \min_{-1 < \xi < 1} w_x(t, x + \xi) - \frac{3\mathcal{J}_0}{2} \left( \int_1^{(2\mathcal{J}_0)^{\frac{1}{\beta}} x} z^{-2s} \, dz \right) \frac{t_2^{2s(\beta - 1) - 1}}{\kappa^{\beta-1}} \beta \, w^\beta(t, x),
\]
where the last step comes from (4.18).
Let us now estimate from below $\min_{-1<\xi<1} u_{x, \xi}(t, x + \xi)$. Using (4.16) and the convexity of $w$, we see that

$$\min_{-1<\xi<1} u_{x, \xi}(t, x + \xi) \geq -\frac{6}{\varepsilon} w_x(t, x - 1)^2,$$

which thanks to (4.18), Proposition 4.1 and that $w(t, x - 1) \leq w(t, X(t)) = \varepsilon$ leads to

$$\min_{-1<\xi<1} u_{x, \xi}(t, x + \xi) \geq -24s^2 \varepsilon^{\beta - 1} 2^{\frac{\beta}{1 - \beta}} w^\beta(t, x) \frac{(x - 1)^{4s(\beta - 1) - 2}}{(kt)^{2\beta - 2}} \text{ for } t \geq \tilde{t}$$

since $X(t) + 1 \leq X(t) + R_0 \leq x \leq 2^{\frac{1}{2}} X(t)$. Now by using that $x - 1 \leq 2^{\frac{1}{2(\beta - 1)}} X(t)$ and exploiting the definition of $X(t)$, (2.4), we have for $t \geq \tilde{t}$,

$$\frac{(x - 1)^{4s(\beta - 1) - 2}}{(kt)^{2\beta - 2}} \leq \frac{4t^2}{X^2(t)} (\varepsilon^{1 - \beta} + \gamma(\beta - 1))^2,$$

and thus

$$\mathcal{J}_1 \min_{-1<\xi<1} u_{x, \xi}(t, x + \xi) \geq -C_5 w^\beta(t, x) \left( \frac{t}{X(t)} \right)^2 \text{ for } t \geq \tilde{t},$$

where $C_5 := 96\mathcal{J}_1 s^2 \varepsilon^{\beta - 1} 2^{\frac{\beta}{1 - \beta}} (\varepsilon^{1 - \beta} + \gamma(\beta - 1))^2$.

The last step is to estimate $\int_1^{(2\mathcal{J}_0)^{\frac{1}{2}} t} z^{-2s} \, dz$. Start by getting from the definition of $X(t)$ (2.4) that for any $t > 1$,

$$\kappa^{\beta - 1} t^\beta \geq (\varepsilon^{1 - \beta} + \gamma(\beta - 1))^{-1} X(t)^{2s(\beta - 1)},$$

and thus since $X(t) \leq x \leq 2^{\frac{1}{2(\beta - 1)}} X(t)$,

$$\frac{t x^{2s(\beta - 1) - 1}}{\kappa^{\beta - 1} t^\beta} \leq 2 (\varepsilon^{1 - \beta} + \gamma(\beta - 1)) \frac{t}{X(t)}.$$

Moreover,

$$\int_1^{(2\mathcal{J}_0)^{\frac{1}{2}} t} z^{-2s} \, dz \leq \begin{cases} \frac{\left( (2\mathcal{J}_0)^{\frac{1}{2}} t \right)^{\max(0, 1 - 2s)}}{|1 - 2s|} & \text{if } s \neq \frac{1}{2}, \\ \ln \left( (2\mathcal{J}_0)^{\frac{1}{2}} t \right) & \text{when } s = \frac{1}{2}, \\ \max \left( (2\mathcal{J}_0)^{\frac{1}{2}} t, \frac{1}{|1 - 2s|} \right)^{\max(0, 1 - 2s)} & \text{if } s \neq \frac{1}{2}, \\ \ln \left( (2\mathcal{J}_0)^{\frac{1}{2}} t, \frac{1}{|1 - 2s|} \right)^{\max(0, 1 - 2s)} & \text{when } s = \frac{1}{2}. \end{cases}$$

Finally,

$$\frac{t x^{2s(\beta - 1) - 1}}{\kappa^{\beta - 1} t^\beta} \int_1^{(2\mathcal{J}_0)^{\frac{1}{2}} t} z^{-2s} \, dz \leq \begin{cases} 2 (\varepsilon^{1 - \beta} + \gamma(\beta - 1)) \frac{\left( (2\mathcal{J}_0)^{\frac{1}{2}} t, \frac{1}{|1 - 2s|} \right)^{\max(0, 1 - 2s)}}{X(t)^{\max(0, 1 - 2s)}} & \text{if } s \neq \frac{1}{2}, \\ \ln \left( (2\mathcal{J}_0)^{\frac{1}{2}} t, \frac{1}{|1 - 2s|} \right)^{\max(0, 1 - 2s)} & \text{when } s = \frac{1}{2}. \end{cases}$$
Putting all previous steps together, and thanks to the fact that from (4.11) and (4.10) we have
\[ \lim_{t \to \infty} \frac{t}{X(t)} = \lim_{t \to \infty} \frac{t \ln(t)}{X(t)} = \lim_{t \to \infty} \frac{t}{X^{2s}(t)} = 0, \]
we may then find an explicit \( t_2 \geq 0 \) such that for all \( t \geq t_2 \),
\[ D[u](t, x) + (1 - \varepsilon)w^\beta(t, x) \geq \frac{\epsilon}{8J_0^{s}2^{2s}} + \frac{1}{2}(1 - \varepsilon)w^\beta(t, x) \quad \text{for} \quad \sup\{X(t) + R_0; X(t)\} \prec x < 2^{2s}\beta^{-1}X(t). \]

\[ \square \]

### 4.3.4 The region \( x \geq 2^{2s}\beta^{-1}X(t) \)

Let us now obtain an estimate for the last region, \( x \geq 2^{2s}\beta^{-1}X(t) \). In this region we claim

**Proposition 4.8.** For all \( \varepsilon \leq \frac{1}{2}, \gamma \) and \( \kappa \) there exists \( t_3 \) such that for all \( t \geq t_3 \)
\[ D[u](t, x) \geq \frac{\epsilon}{16J_0^{s}2^{2s}} \quad \text{for all} \quad x \geq 2^{2s}\beta^{-1}X(t). \]

**Proof.** We follow the same steps as for the proof of Proposition 4.7 but with some adaptations. Taking \( B = (J_0 2)^{\frac{s}{2}}x \) in (4.29), we have again for \( t \) large enough (but explicit),
\[ D[u](t, x) \geq \frac{\epsilon}{8J_0^{s}2^{2s}} + \frac{1}{2}(1 - \varepsilon)w^\beta(t, x) \geq \frac{3}{8J_0^{s}2^{2s}} + \frac{1}{2}(1 - \varepsilon)w^\beta(t, x), \]
where we have used Proposition 4.2 at the last step since \( x \geq 2^{2s}\beta^{-1}X(t) \).

Let us now estimate from below \( \min_{-1 < \xi < 1} u_{xx}(t, x + \xi) \). By using the expression of \( u_{xx}, (4.16) \), and the convexity of \( w \), we see that
\[ \min_{-1 < \xi < 1} u_{xx}(t, x + \xi) \geq -\frac{6}{\epsilon}(w_x(t, x - 1))^2, \]
which thanks to the expression of \( w_x, (4.18) \), leads to
\[ \min_{-1 < \xi < 1} u_{xx}(t, x + \xi) \geq \frac{24}{\epsilon} s^2 w^\beta(t, x - 1) \frac{(x - 1)^{4s(\beta - 1) - 2}}{(kt)^{2\beta - 2}}. \]
Since \( x \geq 2^{2s}\beta^{-1}X(t) \), by Proposition 4.2, there exists a constant \( C_0 \) and \( t' \) such that for all \( t \geq t' \),
\[ w(t, x - 1) \leq \frac{\kappa t}{(x - 1)^{2s}}. \]

As a consequence, since \( x - 1 \geq X(t) \), we have for \( t \geq t' \),
\[ w^\beta(t, x) \leq \frac{\epsilon}{8J_0^{s}2^{2s}} + \frac{1}{2}(1 - \varepsilon)w^\beta(t, x) \geq \frac{3}{8J_0^{s}2^{2s}} + \frac{1}{2}(1 - \varepsilon)w^\beta(t, x), \]
and thus
\[ J_1 \min_{-1 < \xi < 1} u_{xx}(t, x + \xi) \geq -\frac{C_2}{\epsilon} X(t)^2 \] for \( t \geq t' \).
where \( \mathcal{C}_2 := J_1 \frac{24e^2}{e^x} \). 

The last thing to estimate is \( \frac{\kappa t}{x} \int_1^{(2/J_0)^{\frac{1}{2}}} z^{-2s} \, dz \). For this we follow the same steps as for Proposition 4.7,

\[
\frac{\kappa t}{x} \int_1^{(2/J_0)^{\frac{1}{2}}} z^{-2s} \, dz \leq \begin{cases} 
\frac{(2/J_0)^{\frac{1}{2}}}{|1-2s|} \frac{\kappa t}{x} \ln (2/J_0)^{\frac{1}{2}} & \text{if } s \neq \frac{1}{2}, \\
\frac{\kappa t}{x} \ln (2/J_0)^{\frac{1}{2}} & \text{when } s = \frac{1}{2}, \\
\frac{2}{2^{2s(\beta-1)}} \left( \frac{\kappa t}{x} \ln (2/J_0)^{\frac{1}{2}} \right) X(t)^{\min(1,2s)} & \text{if } s \neq \frac{1}{2}, \\
\frac{\kappa t}{x} \ln (2/J_0)^{\frac{1}{2}} X(t)^{\min(1,2s)} & \text{when } s = \frac{1}{2}.
\end{cases}
\]

Thanks to (4.10) and (4.11) we have \( \lim_{t \to \infty} \frac{\kappa t}{x} \ln (2/J_0)^{\frac{1}{2}} X(t)^{\min(1,2s)} = 0 \), so that putting everything above together we may then find an explicit \( t_3 \) large enough such that for all \( t \geq t_3 \),

\[ D(x,t) \geq \frac{\varepsilon}{16/J_0 x^{2s}} \quad \text{for } x > 2^{2s/(\beta-1)} X(t). \]

\[
\square
\]

4.4 Tuning the parameters \( \kappa \) and \( \gamma \)

In this last part of the proof, we choose our parameters \( \gamma \) and \( \kappa \) in order that for some \( t^* > 0 \), \( \bar{u} \) is indeed a sub-solution to (1.1) for \( t \geq t^* \).

Recall that \( \bar{u} \) is a subsolution if and only if (4.8) and (4.9) hold simultaneously. Since (4.8) holds unconditionally for \( t \) sufficiently large, the only thing left to check is that (4.9) holds for a suitable choice of \( \gamma \) and \( \kappa \).

By using (4.14) and (4.17), (4.9) holds if particular

\[
3 \frac{\Phi(t,x)}{\Phi^\beta(t,x)} w^\beta(t,x) \leq D(\bar{u})(t,x) + (1-\varepsilon)w^\beta(t,x) - \gamma w^\beta , \quad x > X(t),
\]

Set \( t^* := \sup\{t_0, t_1, t_2, t_3\} \), where \( t_0, t_1, t_2 \) and \( t_3 \) are respectively determined by Proposition 4.3, Proposition 4.5, Proposition 4.7 and Proposition 4.8. To make our choice, let us decompose the set \( [X(t), +\infty) = I_1 + I_2 \) into two subsets defined as follows

\[ I_1 := [X(t), 2^{\frac{1}{2s(\beta-1)}} X(t)] , \quad I_2 := [2^{\frac{1}{2s(\beta-1)}} X(t), +\infty). \]

On the first interval, we have

**Lemma 4.9.** For all \( \varepsilon < 1 \), there exists \( \gamma^* \) such that for all \( \kappa \) and \( \gamma \leq \gamma^* \), one has, for \( t \geq \sup\{\frac{48}{\varepsilon^{3} - (1-\varepsilon)^{4}}, t^*\} \),

\[
3 \frac{\Phi(t,x)}{\Phi^\beta(t,x)} w^\beta \leq D(\bar{u}) + (1-\varepsilon)w^\beta - \gamma w^\beta , \quad \text{for all } x \in I_1.
\]

**Proof.** By definition of \( \Phi \), we have, at \( (t,x) \),

\[
3 \frac{\Phi(t,x)}{\Phi^\beta(t,x)} w^\beta = 3 \frac{x^{2s(\beta-1)}}{t (\kappa t)^{\beta-1}} w^\beta.
\]

By exploiting the definition of \( X(t) \) it follows that for \( x \leq 2^{\frac{1}{2s(\beta-1)}} X(t) \),

\[
3 \frac{\Phi(t,x)}{\Phi^\beta(t,x)} w^\beta \leq 3 \frac{6}{t} \left( \frac{1}{\varepsilon} \right)^{\beta-1} + 6 \gamma (\beta - 1) \] w^\beta.

\]
Let $\gamma_0 := \frac{1-\varepsilon}{48(\beta-1)}$, then for all $\gamma \leq \gamma_0$ we have

$$3 \frac{\Phi_t}{\Phi_0} w^\beta \leq \left[ \frac{6}{t} \left( \frac{1}{\varepsilon} \right)^{\beta-1} + \frac{1-\varepsilon}{8} \right] w^\beta,$$

which for $t$ large, say $t \geq \frac{48}{\varepsilon^2 - (1-\varepsilon)}$, gives $3 \frac{\Phi_t}{\Phi_0} w^\beta \leq \frac{1-\varepsilon}{8} w^\beta$.

Recall that by Proposition 4.5 and Proposition 4.7, we have for all $x \in I_1$ and $t \geq t^*$,

$$D[u] + (1-\varepsilon)u^\beta - \gamma u^\beta \geq \left( \frac{1-\varepsilon}{2} - \gamma \right) w^\beta,$$

since $u \geq w$ for all $x \geq X(t)$. We then end our proof by taking $\gamma^* := \inf \{ \gamma_0, \frac{1-\varepsilon}{8} \}$ and $t \geq \sup \{ \frac{48}{\varepsilon^2 - (1-\varepsilon)}, t^* \}$.

Finally, let us check what happens on $I_2$,

**Lemma 4.10.** There exists $\kappa^*$ such that for all $\gamma \leq \gamma^*$ and $\kappa \leq \kappa^*$, one has for all $t \geq t^*$,

$$3 \frac{\Phi_t}{\Phi_0} w^\beta \leq D[u] + (1-\varepsilon)u^\beta - \gamma w^\beta, \text{ for all } x \in I_2.$$

**Proof.** As in the above proof, by definition of $\Phi$ we have

$$3 \frac{\Phi_t}{\Phi_0} w^\beta = 3 \kappa \frac{x^{2s(\beta-1)}}{(kt)^\beta} w^\beta.$$

By Proposition 4.2, we have for $x \in I_2$ and $t \geq t^*$,

$$w^\beta(t,x) \leq c_0^\beta \frac{(kt)^\beta}{x^{2s\beta}},$$

with $c_0$ identical as in the proofs of Proposition 4.7 and Proposition 4.8. Therefore, we have

$$3 \frac{\Phi_t}{\Phi_0} w^\beta \leq 3 \kappa \frac{x^{2s(\beta-1)}}{(kt)^\beta} c_0^\beta \frac{1}{x^{2s\beta}} = 3 \kappa c_0^\beta \frac{1}{x^{2s\beta}}.$$

Now recall that by Claim 4.8, we have for all $x \in I_2$ and $t \geq t^*$,

$$D[u] + (1-\varepsilon)u^\beta - \gamma w^\beta \geq \left( \frac{1-\varepsilon}{2} - \gamma \right) w^\beta + \frac{\varepsilon}{16j_0^s x^{2s}},$$

since $u \geq w$ for all $x \geq X(t)$. The claim is then proved by taking $\gamma \leq \gamma_0$ and $\kappa \leq \kappa^* := \frac{\varepsilon c_0^\beta}{16j_0^s}$.

**References**

[1] M. Alfaro. Fujita blow up phenomena and hair trigger effect: the role of dispersal tails. In *Annales de l’Institut Henri Poincare (C) Non Linear Analysis*, volume 34, pages 1309–1327. Elsevier, 2017.

[2] M. Alfaro. Slowing allee effect versus accelerating heavy tails in monostable reaction diffusion equations. *Nonlinearity*, 30(2):687, 2017.

[3] M. Alfaro and J. Coville. Propagation phenomena in monostable integro-differential equations: Acceleration or not? *Journal of Differential Equations*, 263(9):5727 – 5758, 2017.

[4] M. Alfaro and T. Giletti. Interplay of nonlinear diffusion, initial tails and allee effect on the speed of invasions. *arXiv preprint arXiv:1711.10364*, 2017.
[5] M. Alfaro and T. Giletti. When fast diffusion and reactive growth both induce accelerating invasions. *Communications on Pure & Applied Analysis*, 18(6), 2019.

[6] W. C. Allee. *The Social Life of Animals*. Norton, 1938.

[7] D. G. Aronson and H. F. Weinberger. Multidimensional nonlinear diffusion arising in population genetics. *Adv. in Math.*, 30(1):33–76, 1978.

[8] L. Berec, E. Angulo, and F. Courchamp. Multiple allee effects and population management. *Trends Ecol. Evol.*, 22:185–191, 2007.

[9] R. M. Blumenthal and R. K. Getoor. Some theorems on stable processes. *Transactions of the American Mathematical Society*, 95(2):263–273, 1960.

[10] K. Bogdan, T. Byczkowski, T. Kulczycki, M. Ryznar, R. Song, and Z. Vondracek. *Potential analysis of stable processes and its extensions*. Springer Science & Business Media, 2009.

[11] E. Bouin, J. Garnier, C. Henderson, and F. Patout. Thin front limit of an integro-differential fisher-kpp equation with fat-tailed kernels. *SIAM Journal on Mathematical Analysis*, 50(3):3365–3394, 2018.

[12] X. Cabré and J.-M. Roquejoffre. The influence of fractional diffusion in fisher-kpp equations. *Communications in Mathematical Physics*, 320(3):679–722, 2013.

[13] X. Cabré and J.-M. Roquejoffre. Propagation de fronts dans les équations de Fisher-KPP avec diffusion fractionnaire. *C. R. Math. Acad. Sci. Paris*, 347(23–24):1361–1366, 2009.

[14] J. Carr and A. Chmaj. Uniqueness of travelling waves for nonlocal monostable equations. *Proc. Amer. Math. Soc.*, 132(8):2433–2439 (electronic), 2004.

[15] E. Chasseigne, M. Chaves, and J. D. Rossi. Asymptotic behavior for nonlocal diffusion equations. *J. Math. Pures Appl. (9)*, 86(3):271–291, 2006.

[16] F. Courchamp, L. Berec, and J. Gascoigne. Allee effects in ecology and conservation. 2008.

[17] J. Coville. On uniqueness and monotonicity of solutions of non-local reaction diffusion equation. *Ann. Mat. Pura Appl. (4)*, 185(3):461–485, 2006.

[18] J. Coville. Travelling fronts in asymmetric nonlocal reaction diffusion equations: The bistable and ignition cases. *CCSD-Hal e-print*, pages –, May 2007.

[19] J. Coville, J. Davila, and S. Martinez. Nonlocal anisotropic dispersal with monostable nonlinearity. *J. Differential Equations*, 244(12):3080–3118, 2008.

[20] J. Coville and L. Dupaigne. On a non-local equation arising in population dynamics. *Proc. Roy. Soc. Edinburgh Sect. A*, 137(4):727–755, 2007.

[21] J. Coville, C. Gui, and M. Zhao. Propagation acceleration in reaction diffusion equations with anomalous diffusions. *Nonlinearity*, 34(3):1544–1576, Mar. 2021.

[22] B. Dennis. Allee effects: population growth, critical density, and the chance of extinction. *Nat. Resour. Model.*, 3:481–538, 1989.

[23] R. A. Fisher. The wave of advance of advantageous genes. *Ann. Eugenics*, 7:335–369, 1937.

[24] J. Garnier. Accelerating solutions in integro-differential equations. *SIAM J. Math. Anal.*, 43:1955–1974, 2011.

[25] C. Gui and T. Huan. Traveling wave solutions to some reaction diffusion equations with fractional laplacians. *Calculus of Variations and Partial Differential Equations*, 54(1):251–273, 2015.
[26] F. Hamel and L. Roques. Fast propagation for kpp equations with slowly decaying initial conditions. *J. Diff. Equations*, 249:1726–1745, 2010.

[27] J. R. King and P. M. McCabe. On the Fisher-KPP equation with fast nonlinear diffusion. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 459(2038):2529–2546, 2003.

[28] V. Kolokoltsov. Symmetric stable laws and stable-like jump-diffusions. *Proceedings of the London Mathematical Society*, 80(3):725–768, 2000.

[29] J. Medlock and M. Kot. Spreading disease: integro-differential equations old and new. *Math. Biosci.*, 184(2):201–222, 2003.

[30] K. Schumacher. Travelling-front solutions for integro-differential equations. I. *J. Reine Angew. Math.*, 316:54–70, 1980.

[31] D. Stan and J. L. Vázquez. The Fisher-KPP equation with nonlinear fractional diffusion. *SIAM Journal on Mathematical Analysis*, 46(5):3241–3276, 2014.

[32] H. F. Weinberger. Long-time behavior of a class of biological models. *SIAM J. Math. Anal.*, 13(3):353–396, 1982.

[33] H. Yagisita. Existence and nonexistence of travelling waves for a nonlocal monostable equation. *Publ. RIMS, Kyoto Univ.*, 45:925–953, 2009.

[34] G.-B. Zhang, W.-T. Li, and Z.-C. Wang. Spreading speeds and traveling waves for nonlocal dispersal equations with degenerate monostable nonlinearity. *Journal of Differential Equations*, 252(9):5096 – 5124, 2012.

[35] A. Zlatos. Quenching and propagation of combustion without ignition temperature cutoff. *Nonlinearity*, 18(4):1463, 2005.