Local two-qubit entanglement-annihilating channels

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We address the problem of the robustness of entanglement of bipartite systems (qubits) interacting with dynamically independent environments. In particular, we focus on characterization of so-called local entanglement-annihilating two-qubit channels, which set the maximum permissible noise level allowing to perform entanglement-enabled experiments. The differences, but also subtle relations between entanglement-breaking and local entanglement-annihilating channels are emphasized. A detailed characterization of latter ones is provided for a variety of channels including depolarizing, unital, (generalized) amplitude-damping, and extremal channels. We consider also the convexity structure of local entanglement-annihilating qubit channels and introduce a concept of entanglement-annihilation duality.

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I. INTRODUCTION

Flourishing field of quantum information theory is obliged to the phenomenon of quantum entanglement exhibited by multipartite quantum systems. In the last decades many entanglement-enabled applications of quantum states have been developed and experimentally realized such as quantum key distribution, dense coding, quantum teleportation, etc. (see the detailed review [2]). These quantum information protocols operate efficiently providing that the entanglement between the involved parties (Alice and Bob) is preserved. However, during the protocols the involved systems interact with an environment, which introduces (practically) un-avoidable noise. As a result of these influences, Alice and Bob manipulate modified states whose entanglement can substantially differ from the original one. It may even happen that the systems become disentangled whatever state they start with. Under such circumstances no entanglement-enabled application is implementable.

For purposes of quantum communication protocols it is reasonable to assume that the influence of environments of Alice and Bob are independent. That is, the joint noise applied on the shared state $\rho_{in}$ is of the form $E_1 \otimes E_2$, where $E_1, E_2$ are local channels describing the interaction of Alice’s and Bob’s subsystem, respectively, with their environments. The question of our interest is the robustness of the initial entanglement with respect to local noises, i.e. to characterize the entanglement properties of states $\rho_{out} = (E_1 \otimes E_2)[\rho_{in}]$. Different variations of this problem of so-called entanglement dynamics were addressed in a number of papers (see, e.g., [3,12]), where the time evolution of two-qubit entanglement was studied for different physical systems (initial state, types of interqubit interactions and environments). Many researches have also paid their attention to the phenomena known in the literature as sudden death and sudden birth of the entanglement (see, e.g., [13,14] and references therein). In contrast to the studies, where the time evolution of the entanglement is deduced from the time evolution of the state, an attempt to find a direct relation (inequality) involving the initial and final entanglements of an arbitrary bipartite two-qubit state in presence of local noises has been undertaken in the papers [15–18].

Recently, a related concept of entanglement-annihilating channels has been introduced [10]. These channels destroy any quantum entanglement completely within the system they act on. Following the paper [10], we will refer to a local two-qubit channel $E_1 \otimes E_2$ as entanglement-annihilating (EA) if the output state $(E_1 \otimes E_2)[\rho_{in}]$ is separable for all input states $\rho_{in}$. Therefore, the question whether the channel is EA or not is a question whether the noise level is acceptable for entanglement-enabled quantum applications or not.

It is worth emphasizing the contrast between entanglement-annihilating and entanglement-breaking channels. Let us remind that a channel $E$ (acting on some system) is called entanglement-breaking (EB) if for all its extensions $E \otimes I_{anc}$ it annihilates the entanglement between the system and the ancilla. In particular, a local two-qubit channel $E_1 \otimes E_2$ is EB if the output state $\rho_{out} = (E_1 \otimes E_2 \otimes I_{anc})[\rho_{in}]$ is disentangled with respect to partitioning ‘1+2’|anc for any input state and any dimension of the ancillary system. The entanglement-breaking channels and their properties have been widely discussed in the literature (see, e.g., [20,21]). As it was shown in Ref. [12] even if the channels $E_1 = E$ and $E_2 = E$ are not EB, the channel $E \otimes E$ can cancel any entanglement between quantum subsystems in interest. It means that in order to fulfill an entanglement-enabled protocol it does not suffice to know whether the individual local influences are described by EB channel or not, one has to resort to the concept of EA channel.

Our goal in this paper is to investigate in details the properties of entanglement-annihilating channels for the simplest case of two-qubits system. In Sec. III we briefly review some known properties of EA channels and add
a new one for EA channels of the form $E \otimes E$. Such channels naturally occur in physical experiments, where two parties experience the same influence from the environment and thus undergo the same local transformation $E_1 = E_2 = E$. Although such channels do not describe the general case, they are of significant physical relevance. For the sake of convenience, if $E \otimes E$ is EA, we will refer to a single-qubit channel $E$ as a 2-locally entanglement-annihilating channel (2-LEA).

In order to demonstrate the difference between the EA and EB local two-qubit channels, in Sec. IV we consider the simplest and the most widely used model of quantum noise – depolarizing channels. We find the overall noise level under which the entanglement is destroyed for all input states or can be preserved for some quantum states. Since depolarizing channels belong to the class of unital channels, we further in Sec. V focus our attention on this class. These channels describe important physical processes that do not increase purity of the states.

In Sec. VI we move on to the entanglement-annihilating behavior of non-unital channels. At first, we address the question ‘which extremal single-qubit channels $E_1$ and $E_2$ result in EA channels $E_1 \otimes E_2$?’’ Then, we consider amplitude-damping and generalized amplitude-damping channels as the most prominent representatives of non-unital channels. In Sec. VII we remind that the set of all two-qubit EA channels (including non-local ones) is convex [19]. This fact motivates us to find EA-extremal channels and determine their position with respect to the set of all two-qubit channels and its extreme points. In Sec. VIII we find it quite interesting to reveal and briefly outline an EA-duality between subsets of local channels. Finally, we summarize the obtained results in Sec. VII.

II. PROPERTIES OF EA-CHANNELS

To begin with, we epitomize some basic properties of general EA-channels found in Ref. [19]:

1° The set $T_{EA}$ of all EA channels (including non-local ones) is convex.

2° The channel is EA if and only if it destroys entanglement of all pure input states.

3° If $G_{12}$ is EA (local or non-local), then $G_{12} \cdot F_{12}$ is EA for all two-qubit channels $F_{12}$.

4° The channels $E_1 \otimes E_2$ and $E_2 \otimes E_1$ exhibit the same entanglement-annihilating behavior.

5° If $E_1$ or $E_2$ is EB, then $E_1 \otimes E_2$ is EA. This follows immediately from the definitions of EA and EB channels.

6° The channel $E \otimes I$ is EA if and only if $E$ is EB.

7° If $E$ is 2-LEA and $F$ is EB, then the convex combination $\mu E + (1 - \mu) F$ is 2-LEA, $\mu \in [0, 1]$.

The last property was not shown in Ref. [19]. In order to prove it we recall the definition of 2-LEA channels from the previous section. Suppose now the composite channel $\mu^2 E \otimes E + \mu(1 - \mu) E \otimes F + \mu(1 - \mu) F \otimes E + (1 - \mu)^2 F \otimes F$. Then the channel $E \otimes E$ is EA by definition of the 2-LEA channel $E$, the rest channels $E \otimes F, F \otimes E$, and $F \otimes F$ are EA in view of property 5°. The convexity property 1° concludes the proof of property 7°.

Is worth mentioning that $E_1 \otimes E_2$ is EB if and only if both $E_1$ and $E_2$ are EB. Thus, if the local channel $E_1 \otimes E_2$ is EB then it is also EA by property 5°.

III. CASE STUDY: DEPOLARIZING CHANNELS

The action of a depolarizing channel on $j$th qubit ($j = 1, 2$) is defined as follows

$$E_j[X] = q_j X + (1 - q_j) \text{tr}[X] \frac{1}{2} I,$$

(1)

where $q_j \in [0, 1]$ and $I$ denotes the identity operator. As a result of such noise the Bloch spheres of individual qubits symmetrically shrink (in all directions). The class of these channels was used in Ref. [19] to show the existence of not entanglement-breaking 2-LEA channels. In this section we will analyze the properties of depolarizing channels of a slightly more general form $E_1 \otimes E_2$. Each channel $E_j$ is known to be EB if and only if $q_j \leq \frac{1}{2}$ (see, e.g., [23]). If $q_j \leq \frac{1}{2}$ then the $j$-th qubit becomes disentangled from arbitrary environment including the rest qubit. Therefore, the two-qubit channel $E_1 \otimes E_2$ is EB if and only if simultaneously $q_1 \leq \frac{1}{2}$ and $q_2 \leq \frac{1}{2}$.

Let us find out when $E_1 \otimes E_2$ is EA and destroys any entanglement between qubits. We resort to property 2° and consider pure input states $\omega = |\psi \rangle \langle \psi|$. We use the Schmidt decomposition of the state vector $|\psi \rangle = \sqrt{p} \phi \otimes \chi + \sqrt{p} \gamma \otimes \chi_{\perp}$, where $\{|\phi \rangle, |\chi_{\perp} \rangle\}$ and $\{|\chi \rangle, |\chi_{\perp} \rangle\}$ are suitable orthonormal bases of the first and the second qubit, respectively. $p$ and $p_\perp$ are real nonnegative numbers such that $p + p_\perp = 1$.

Action of the two-qubit channel $E_1 \otimes E_2$ on the state $\omega$ yields

$$\omega_{\text{out}} = (E_1 \otimes E_2)[\omega] = q_1 q_2 \omega + \frac{1}{2}(1 - q_1)q_2 I \otimes \omega_2$$

$$+ \frac{1}{2} q_1 (1 - q_2) \omega_1 \otimes I + \frac{1}{4}(1 - q_1)(1 - q_2) I \otimes I,$$

(2)

with the reduced states $\omega_1 = p|\phi \rangle \langle \phi| + p_\perp|\gamma \rangle \langle \gamma| \otimes \langle \chi | \otimes | \chi_{\perp} \rangle$ and $\omega_2 = p|\chi \rangle \langle \chi| + p_\perp |\chi_{\perp} \rangle \langle \chi_{\perp}| \otimes | \chi \rangle$. According to the Peres-Horodecki criterion [26, 27], the output state $\omega_{\text{out}}$ is separable if the partially transposed operator $\omega_{\text{out}}^T$ is positive-semidefinite. The condition $\omega_{\text{out}}^{T} \geq 0$ reduces to

$$\begin{vmatrix} A + B & C \\ C & A - B \end{vmatrix} = A^2 - B^2 - C^2 \geq 0,$$

(3)
where $A = 1 - q_1 q_2$, $B = (2p - 1)(q_1 - q_2)$, and $C = 4q_1 q_2 \sqrt{p(1-p)} = 4q_1 q_2 \sqrt{p(1-p)}$. After simplification we obtain

$$(1+q_1 q_2)(1-3q_1 q_2)+4 \left(p - \frac{1}{2}\right)^2 [4q_1^2 q_2^2 - (q_1 - q_2)^2] \geq 0.$$ 

The channel in question is EA if this inequality holds for all $p \in [0,1]$. If $2|q_1 q_2| \geq |q_1 - q_2|$, then the minimum in $p$ is achieved for $p = \frac{1}{2}$ and we end up with the inequality $(1+q_1 q_2)(1-3q_1 q_2) \geq 0$. Taking into account that $-\frac{1}{3} \leq q_1, q_2 \leq 1$, this inequality holds whenever $q_1 q_2 \leq \frac{1}{3}$. If $2|q_1 q_2| < |q_1 - q_2|$, then the minimum is achieved for $p = 0$ or $p = 1$, and for this case the inequality takes the form $(1-q_1^2)(1-q_2^2) \geq 0$, which is always satisfied for all allowed values of $q_1, q_2$.

In summary, the channel $E_1 \otimes E_2$ is entanglement annihilating if and only if $q_1 q_2 \leq \frac{1}{3}$. On the contrary, such a two-qubit channel is EB only if and only if simultaneously $q_1 \leq \frac{1}{3}$ and $q_2 \leq \frac{1}{3}$ (see Fig. 1).

IV. UNITAL CHANNELS

In this section we will focus our investigation on the class of qubit unitary channels. By definition a channel is unital if it preserves the identity operator, i.e. $E[I] = I$. As it was shown in Ref. [28], any such channel can be expressed as a diagonal matrix (acting on Bloch vectors) in a properly chosen basis of self-adjoint operators $\{\sigma_0 \equiv I, \sigma_1, \sigma_2, \sigma_3\}$, where $\text{tr}[\sigma_i \sigma_k] = 2\delta_{jk}$ and $\delta_{jk}$ is the conventional Kronecker delta symbol. Channels of this (diagonal) form are also known as Pauli channels.

It follows that the channel $E_1 \otimes E_2$, where $E_1$ and $E_2$ are unital single-qubit channels, has a diagonal matrix representation in the basis of individual Pauli operators $\{\sigma_m \otimes \sigma_n\}_{m,n=0}^3$. In particular, the entries of the matrix representation of $E_1 \otimes E_2$ read

$$\frac{1}{4} \text{tr} [\sigma_k \otimes \sigma'_l (E_1 \otimes E_2)[\sigma_m \otimes \sigma_n]] = \lambda_m \lambda'_n \delta_{km} \delta_{ln},$$

where $\lambda_0 = \lambda'_0 = 1$ in view of the trace-preserving property, $\{\lambda_m\}_{m=1}^3$ and $\{\lambda'_n\}_{n=1}^3$ are singular values of the channels $E_1$ and $E_2$, respectively. The output state of the local two-qubit unital channel $E_1 \otimes E_2$ takes the form 

$$\rho_{\text{out}} = \frac{1}{4} \sum_{m,n=0}^3 \lambda_m \lambda'_n \text{tr}[\rho_{\text{in}} \sigma_m \otimes \sigma'_n] \sigma_m \otimes \sigma'_n. \quad (4)$$

For each unital qubit channel $E$ we introduce a unital map $\overline{E}$ with $\lambda_0 = 1$ and singular values $\{-\lambda_m\}_{m=1}^3$, where $\{\lambda_m\}_{m=1}^3$ are singular values of the original channel $E$. Note that the map $\overline{E}$ is positive and trace preserving but not necessarily completely positive. This means that $\overline{E}[\rho] \geq 0$ for all qubit density operators $\rho$, whereas $(E \otimes I)[\rho_{\text{in}}]$ can, in principle, have negative eigenvalues for some two-qubit density operators $\rho_{\text{in}}$.

The entanglement-annihilating behavior of local two-qubit unitary channels $E_1 \otimes E_2$ is governed by the following Lemma.

Lemma 1. Let $E_1$ and $E_2$ be unital qubit channels. The two-qubit channel $E_1 \otimes E_2$ is EA if and only if the maps $\overline{E_1} \otimes \overline{E_2}$ and $\overline{E_1}$ are positive.

Proof. Separability of the output state $\overline{E_1}[\rho_{\text{in}}]$ of the channel $E_1 \otimes E_2$ can be checked by the reduction criterion [29], which turns out to be a necessary and sufficient separability condition for two-qubit systems. Assuming that the channel $E_1 \otimes E_2$ is entanglement-annihilating, the two-qubit state $\rho_{\text{out}}$ is separable, thus, in accordance with the reduction criterion the following conditions hold 

$$\text{tr}_2[\rho_{\text{out}}] \otimes I - \rho_{\text{out}} \geq 0 \quad \text{and} \quad I \otimes \text{tr}_1[\rho_{\text{out}}] - \rho_{\text{out}} \geq 0, \quad (5)$$

where $\text{tr}_1[\cdot]$ and $\text{tr}_2[\cdot]$ denote partial traces over the first and the second qubit, respectively. Since, 

$$(E_1 \otimes E_2)[\rho_{\text{in}}] = \text{tr}_2[\rho_{\text{out}}] \otimes I - \rho_{\text{out}}, \quad (\overline{E_1} \otimes \overline{E_2})[\rho_{\text{in}}] = I \otimes \text{tr}_1[\rho_{\text{out}}] - \rho_{\text{out}},$$

the above separability conditions are equivalent with positivity of the maps $\overline{E_1} \otimes \overline{E_2}$ and $\overline{E_1}$, respectively. □

Before we explore the consequences of Lemma 1 and derive some properties of local two-qubit unital channels $E_1 \otimes E_2$, let us make some remarks about single-qubit unitary channels. A qubit unitary map $E$ with singular values $\{\lambda_m\}_{m=1}^3$ is indeed a channel (i.e. completely positive trace-preserving map) if $1 + \lambda_1 + \lambda_2 + \lambda_3 \geq 0$,
$1 + \lambda_1 - \lambda_2 - \lambda_3 \geq 0$, $1 - \lambda_1 + \lambda_2 - \lambda_3 \geq 0$, and $1 - \lambda_1 - \lambda_2 + \lambda_3 \geq 0$. These four inequalities define a tetrahedron in the conventional reference frame $(\lambda_1, \lambda_2, \lambda_3)$ in $\mathbb{R}^3$ (see Fig. 2). The channel $E$ is known to be EB if and only if $|\lambda_1| + |\lambda_2| + |\lambda_3| \leq 1$ [2]. This inequality corresponds to the octahedron in Fig. 2.

Let us note that $E$ is a quantum channel if and only if $E$ is entanglement-breaking. That is, Lemma 1 guarantees that $E_1 \otimes E_2$ is EA if $E_1$ or $E_2$ is EB, which is in agreement with the property 5". Nevertheless, the channel $E_1 \otimes E_2$ can be EA even if neither $E_1$ nor $E_2$ is EB. The following proposition gives a sufficient condition for $E_1 \otimes E_2$ being entanglement-annihilating.

**Proposition 1.** Suppose $E_1, E_2$ are unital qubit channels such that $E_1^2$ and $E_2^2$ are entanglement-breaking channels, i.e. $\sum_{m=1}^3 \lambda_m^2 \leq 1$ and $\sum_{m=1}^3 \lambda_m^4 \leq 1$. Then $E_1 \otimes E_2$ is an entanglement-annihilating channel.

**Proof.** Consider the map $E_1 \otimes E_2$. Let us demonstrate that $(E_1 \otimes E_2)[\rho_{in}] \geq 0$ for all two-qubit input states $\rho_{in}$. In view of convexity of the state space it suffices to show that $(E_1 \otimes E_2)[|\psi\rangle\langle\psi|] \geq 0$ for all pure two-qubit states $|\psi\rangle$.

Any state $|\psi\rangle$ is given by its Schmidt decomposition $|\psi\rangle = \sqrt{p} |\varphi\rangle \otimes |\chi\rangle + \sqrt{p_\perp} |\varphi_\perp\rangle \otimes |\chi_\perp\rangle$, where $p, p_\perp \geq 0$ and $p + p_\perp = 1$, the orthonormal basis $\{|\varphi\rangle, |\varphi_\perp\rangle\}$ can be parameterized by the angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$ as follows:

$$|\varphi\rangle = \begin{pmatrix} \cos(\theta/2) \exp(-i\phi/2) \\ \sin(\theta/2) \exp(i\phi/2) \end{pmatrix},$$

$$|\varphi_\perp\rangle = \begin{pmatrix} -\sin(\theta/2) \exp(-i\phi/2) \\ \cos(\theta/2) \exp(i\phi/2) \end{pmatrix},$$

and the basis $\{|\chi\rangle, |\chi_\perp\rangle\}$ is obtained from formulas (6)–(7) by replacing $|\varphi\rangle \rightarrow |\chi\rangle$, $|\varphi_\perp\rangle \rightarrow |\chi_\perp\rangle$, $\theta \rightarrow \theta'$, and $\phi \rightarrow \phi'$.

The map $E_1 \otimes E_2$ transforms $|\psi\rangle\langle\psi|$ into the operator

$$[I \otimes I - (n \cdot \sigma) \otimes (n' \cdot \sigma') - (p - p_\perp) \{(n \cdot \sigma) \otimes I - I \otimes (n' \cdot \sigma')\}] - 2\sqrt{p_\perp \sqrt{p}} \{[k \cdot \sigma] \otimes [k' \cdot \sigma'] - (1 \cdot \sigma) \otimes (1' \cdot \sigma')\},$$

where $(n \cdot \sigma) = n_1 \sigma_1 + n_2 \sigma_2 + n_3 \sigma_3$ and vectors $n, k, k', l \in \mathbb{R}^3$ are expressed through singular values $\{\lambda_m\}_{m=1}^3$ of the channel $E$ by formulas

$$n = (\lambda_1 \cos \phi \sin \theta, \lambda_2 \sin \phi \sin \theta, \lambda_3 \cos \theta),$$

$$k = (-\lambda_1 \cos \phi \cos \theta, -\lambda_2 \sin \phi \cos \theta, \lambda_3 \sin \theta),$$

$$l = (\lambda_1 \sin \phi, -\lambda_2 \cos \phi, 0).$$

The vectors $n', k', l'$ are obtained from (9), (10), (11), respectively, by replacing $\lambda \rightarrow \lambda'$, $\theta \rightarrow \theta'$, and $\phi \rightarrow \phi'$.

Both sets of vectors $\{n, k, l\}$ and $\{n', k', l'\}$ have a particular property

$$|n|^2 + |k|^2 + |l|^2 = \sum_{m=1}^3 \lambda_m^2 \leq 1,$$

$$|n'|^2 + |k'|^2 + |l'|^2 = \sum_{m=1}^3 \lambda_m^4 \leq 1$$

thanks to the statement of the proposition.

The output state $\rho$ is positive semi-definite if and only if its average $\langle\rho_{out}\rangle \geq 0$ for all two-qubit states. In other words, we want to show that the inequality

$$\langle I \otimes I - (n \cdot \sigma) \otimes (n' \cdot \sigma') \rangle \geq (p - p_\perp) \{(n \cdot \sigma) \otimes I - I \otimes (n' \cdot \sigma')\} + 2\sqrt{p_\perp \sqrt{p}} \{[k \cdot \sigma] \otimes [k' \cdot \sigma'] - (1 \cdot \sigma) \otimes (1' \cdot \sigma')\}$$

holds true for any averaging state. Since $(p - p_\perp)^2 + 2(p_\perp)^2 = (p + p_\perp)^2 = 1$, one can treat $(p - p_\perp)$ as $\cos \alpha$ and $2\sqrt{p_\perp \sqrt{p}}$ as $\sin \alpha$. Due to the fact that $\max_\alpha (A \cos \alpha + B \sin \alpha) = \sqrt{A^2 + B^2}$, the inequality (14) if fulfilled whenever

$$\langle I \otimes I - (n \cdot \sigma) \otimes (n' \cdot \sigma') \rangle \geq \langle (n \cdot \sigma) \otimes I - I \otimes (n' \cdot \sigma') \rangle^2 + \langle [k \cdot \sigma] \otimes [k' \cdot \sigma'] - (1 \cdot \sigma) \otimes (1' \cdot \sigma') \rangle^2$$

or, equivalently,

$$\langle (I + (n \cdot \sigma) \otimes (I - (n' \cdot \sigma')) \rangle \langle (I - (n \cdot \sigma)) \otimes (I + (n' \cdot \sigma')) \rangle \geq \langle [k \cdot \sigma] \otimes [k' \cdot \sigma'] - (1 \cdot \sigma) \otimes (1' \cdot \sigma') \rangle^2.$$

Taking into account that $|\varphi \otimes \chi\rangle$, $|\varphi \otimes \chi_\perp\rangle$, $|\varphi_\perp \otimes \chi\rangle$, and $|\varphi_\perp \otimes \chi_\perp\rangle$ are all eigenvectors of the operators in the left hand side of (16) and using the Cauchy-Schwarz inequality (CS) in the form $\langle X \otimes Y \rangle^2 \leq \langle X \rangle^2 \langle Y \rangle^2$, one can readily see that

$$(I + (n \cdot \sigma) \otimes (I - (n' \cdot \sigma')) \rangle \langle (I - (n \cdot \sigma)) \otimes (I + (n' \cdot \sigma')) \rangle \geq$$

$$(|k||k'| + |l||l'|)^2 \geq \langle [k \cdot \sigma] \otimes [k' \cdot \sigma'] - (1 \cdot \sigma) \otimes (1' \cdot \sigma') \rangle^2.$$

Thus, (12) $\land (13) \Rightarrow (17) \Rightarrow (16) \Rightarrow (15) \Rightarrow (14) \Rightarrow E_1 \otimes E_2$ is a positive map. In the same way, $E_1 \otimes E_2$ is also a positive map. According to Lemma 1 the channel $E_1 \otimes E_2$ is EA.

Note that Proposition 1 provides the sufficient but not necessary condition for the channel $E_1 \otimes E_2$ to be EA. For instance, in case of two depolarizing channels $E_1$ and $E_2$ from Sec. III, the channels $E_1^2$ and $E_2^2$ are EB if $q_1 \leq 1/\sqrt{3}$ and $q_2 \leq 1/\sqrt{4}$, respectively. It means that $E_1 \otimes E_2$ is EA if $q_1, q_2 \leq 1/\sqrt{4}$. The corresponding area of parameters $(q_1, q_2)$ is depicted in Fig. 1 which illustrates the ‘power’ of Proposition 1. However, in case of identical environments, i.e. $E_1 = E_2 = E$, the use of Proposition 1 provides sufficient and necessary condition for $E \otimes E$ to be EA.

**Proposition 2.** A unital qubit channel $E$ is 2-LEA if and only if the channel $E^2$ is EB, i.e. $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq 1$.

**Proof.** The condition $\{E^2\}$ is sufficient due to Proposition 1. It turns out to be also necessary if we consider Bell states, e.g., $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. It is readily seen that the operator $(E \otimes E)[|\psi\rangle\langle\psi|]$ is positive semi-definite and if only if $\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq 1$. On the other hand, $\{\lambda_i\}_{i=1}^3$ are eigenvalues of $E^2$ and their sum is less or equal than 1, i.e. $E^2$ is EB [2].
An illustration of 2-LEA unital channels $\mathcal{E}$ in the conventional reference frame $\{\lambda_1, \lambda_2, \lambda_3\}$ is presented in Fig. 2. Points outside the tetrahedron do not satisfy the complete positivity of $\mathcal{E}$. Points outside the sphere do not correspond to 2-LEA channels, which can be easily checked by Bell states. Both the sphere and tetrahedron comprise an octahedron of EB single-qubit channels $\mathcal{E}$ with $|\lambda_1| + |\lambda_2| + |\lambda_3| \leq 1$. Since intersection of the sphere and tetrahedron is a convex body, we have just revealed an interesting property:

**Proposition 3.** The set of 2-LEA unital qubit channels is convex, i.e. if $\mathcal{E}_1$ and $\mathcal{E}_2$ are 2-LEA, then $\mu \mathcal{E}_1 + (1-\mu) \mathcal{E}_2$ is also 2-LEA for all $\mu \in [0,1]$.

**Proof.** Despite this fact is evident from geometrical consideration, it also follows from the relation $(\mu \mathcal{E}_1 + (1-\mu) \mathcal{E}_2) \otimes (\mu \mathcal{E}_1 + (1-\mu) \mathcal{E}_2) = \mu^2 \mathcal{E}_1 \otimes \mathcal{E}_1 + \mu(1-\mu) \mathcal{E}_1 \otimes \mathcal{E}_2 + (1-\mu) \mathcal{E}_2 \otimes \mathcal{E}_1 + (1-\mu)^2 \mathcal{E}_2 \otimes \mathcal{E}_2$. As channels $\mathcal{E}_1 \otimes \mathcal{E}_1$ and $\mathcal{E}_2 \otimes \mathcal{E}_2$ are EA by a statement of the involved Proposition, the channels $\mathcal{E}_1^2$ and $\mathcal{E}_2^2$ are EB by Proposition 2. Due to Proposition 1 both $\mathcal{E}_1 \otimes \mathcal{E}_2$ and $\mathcal{E}_2 \otimes \mathcal{E}_1$ are EA. Then we use property 1° to conclude the proof.

One more property immediately follows from Propositions 1 and 2.

**Corollary 1.** If $\mathcal{E}_1$ and $\mathcal{E}_2$ are unital 2-LEA qubit channels, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA.

We can make an interesting observation from Fig. 2, namely, phase-damping channels (forming the edges of the tetrahedron) preserve entanglement unless they contract the whole Bloch sphere into a line.

Let us now analyze unital channels $\mathcal{E}_1 \otimes \mathcal{E}_2$ such that neither $\mathcal{E}_1$, nor $\mathcal{E}_2$ is a 2-LEA channel (i.e. both $\mathcal{E}_1$ and $\mathcal{E}_2$ are outside the sphere in Fig. 2). Surprisingly, it turns out that $\mathcal{E}_1 \otimes \mathcal{E}_2$ can still be EA as it is demonstrated by the following example.

**Example 1.** Suppose $\mathcal{E}_1 = \text{diag}\{1, \frac{1}{2}, 0\}$, $\mathcal{E}_2 = \text{diag}\{1, \frac{1}{2}, \frac{1}{2}, 0\}$. Let us demonstrate that $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA although neither $\mathcal{E}_1$, nor $\mathcal{E}_2$ is 2-LEA. We note that the channel in question can be represented in the form $\mathcal{E}_1 \otimes \mathcal{E}_2 = G \cdot F$, where $G = \frac{1}{3} G_1 \otimes I + \frac{1}{3} I \otimes G_2$, $G_1 = \text{diag}\{1, 0, 0, 1\}$, $G_2 = \text{diag}\{1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$, and the two-qubit map $F$ is defined by its matrix representation $\frac{1}{4} [\sigma_k \otimes \sigma'_l | F | \sigma_m \otimes \sigma'_n] = F_{mn} \delta_{kl} \delta_{ln}$, with $F_{nn} = (4 + \delta_{nn})/5$, $F_{0n} = F_{n0} = (2 - \delta_{nn})/5$. By considering the Choi matrix of the map $F$, it is not hard to see that $F$ is a channel indeed, i.e. completely positive trace-preserving map. As both $G_1$ and $G_2$ are EB, the channels $G_1 \otimes I$ and $I \otimes G_2$ are EA by property 5°. Hence $G$ is EA by property 1° and $G \cdot F$ is EA by property 3°. The equality $\mathcal{E}_1 \otimes \mathcal{E}_2 = G \cdot F$ completes the proof.

A natural question to ask is under which conditions on $\mathcal{E}_1$ and $\mathcal{E}_2$ the channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ is entanglement-annihilating. The following proposition formulates a sufficient condition for the converse statement.

**Proposition 4.** Consider qubit unital channels $\mathcal{E}_1$ and $\mathcal{E}_2$ with singular values $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and $\lambda' = (\lambda'_1, \lambda'_2, \lambda'_3)$, respectively. If $\lambda \cdot \lambda' > 1$, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is not entanglement-annihilating channel.

**Proof.** The channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ preserves entanglement of the Bell state $|\psi_+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ when the matrix

$$\rho_{\text{out}} = ((\mathcal{E}_1 \otimes \mathcal{E}_2) | | \psi_+ \rangle \langle \psi_+ | )^{\Gamma} =$$

$$\begin{pmatrix}
1 + \lambda_3 \lambda'_3 & 0 & 0 & -\lambda_1 \lambda'_1 - \lambda_2 \lambda'_2 \\
0 & 1 - \lambda_3 \lambda'_3 & \lambda_1 \lambda'_1 + \lambda_2 \lambda'_2 & 0 \\
0 & \lambda_1 \lambda'_1 + \lambda_2 \lambda'_2 & 1 - \lambda_3 \lambda'_3 & 0 \\
0 & 0 & 0 & 1 + \lambda_3 \lambda'_3
\end{pmatrix}$$

has negative eigenvalues. It takes place if $\lambda_1 \lambda'_1 + \lambda_2 \lambda'_2 + \lambda_3 \lambda'_3 \equiv \lambda \cdot \lambda' > 1$. In view of the Peres-Horodecki criterion, the output state $\rho_{\text{out}}$ remains entangled.

Let us note that this proposition is very efficient, e.g., for depolarizing channels from Sec. III we immediately have not-EA channel $\mathcal{E}_1 \otimes \mathcal{E}_2$ if $q_1 q_2 > \frac{1}{6}$. Remind that $q_1 q_2 = \frac{1}{6}$ is a boundary between EA and not-EA behavior of depolarizing channels.

Our goal was to give a complete picture of unital two-qubit channels of the form $\mathcal{E}_1 \otimes \mathcal{E}_2$. Our findings are graphically summarized in Fig. 3. Based on this figure one could argue that the property of being entanglement-annihilating is not something rare and the noises should be relatively close to unitary ones (or, equivalently, sufficiently far from complete depolarization) in order to guarantee the conservation of some entanglement.
maximally entangled two-qubit state. Applying PPT criterion to the Choi-Jamiolkowski state \( (E \otimes \mathcal{I})[|\psi_{+}\rangle\langle\psi_{+}|] \), we find that \( E \) is EB if and only if \( \cos u = 0 \) or \( \cos v = 0 \).

In what follows we will use the fact that in formula (18) the entries \( \cos u \) and \( \sin u \sin v \) can simultaneously be made non-negative by an appropriate choice of basis operators \( \{\sigma_1\}_i = 0 \). The singular values \( \cos u \) and \( \cos u \cos v \) have the same sign (either positive or negative).

**Proposition 5.** Suppose \( E_1, E_2 \) are extremal qubit channels. Then \( E_1 \otimes E_2 \) is EA if and only if either \( E_1 \) or \( E_2 \) is EB.

**Proof.** If either \( E_1 \) or \( E_2 \) is EB, then \( E_1 \otimes E_2 \) is EA in view of property \( 5^\circ \). Let us now prove that the channel \( E_1 \otimes E_2 \) is not entanglement-annihilating if neither \( E_1 \) nor \( E_2 \) is EB, i.e. \( \cos u_1 \cos v_1 \cos u_2 \cos v_2 \neq 0 \), where parameters \( (u_j, v_j) \) define the channel \( E_j \) according to formula (18).

Without loss of generality it can be assumed that \( |\cos u_j| \geq |\cos v_j|, \quad j = 1, 2 \). Consider the input state \( \psi_{in} = \frac{1}{\sqrt{2}}(|\varphi \rangle \otimes |\chi \rangle + |\varphi \rangle \otimes |\chi \rangle \) \), where the orthogonal qubit states \{\( |\varphi \rangle, |\varphi \rangle \)\} and \{\( |\chi \rangle, |\chi \rangle \)\} are parameterized by angles \( (\theta_1, \phi_1) \) and \( (\theta_2, \phi_2) \), respectively, as in Proposition 4 (see formulas (19)–(27)). We put \( \phi_j = 0 \) and choose the angles \( \theta_j \) such that

\[
\cos \theta_j = \frac{\sin u_j \cos v_j}{\cos u_j \sin v_j}, \quad \sin \theta_j = \frac{(\cos^2 u_j - \cos^2 v_j)^{1/2}}{\cos u_j \sin v_j},
\]

then \( E_1[|\varphi \rangle \langle \varphi |] \) and \( E_2[|\chi \rangle \langle \chi |] \) are known to be pure states [28].

The reduction criterion guarantees that the channel \( E_1 \otimes E_2 \) is not EA if the operator

\[
M = \text{tr}_2[(E_1 \otimes E_2)[|\psi_{in}\rangle\langle\psi_{in}|] \otimes I - (E_1 \otimes E_2)[|\psi_{in}\rangle\langle\psi_{in}|]}
\]

is not positive semi-definite. This takes place if there exists a two-qubit state \( |\psi_{test}\rangle \) such that \( \langle \psi_{test}|M|\psi_{test}\rangle < 0 \).

Let us construct a one-parametric family of candidates \( |\psi_p\rangle, \quad p \in [0, 1] \) for the state \( |\psi_{test}\rangle \). To do that we use the pure states \( |\xi\rangle = E_1[|\varphi \rangle \langle \varphi |] \) and \( |\zeta\rangle = E_2[|\chi \rangle \langle \chi |] \) and write \( |\psi_p\rangle = \sqrt{p} |\xi \rangle + \sqrt{1-p} |\zeta \rangle \). Direct calculation yields

\[
4\langle \psi_p|M|\psi_p\rangle = 1 - \sin^2 u_1 \sin^2 u_2 - \cos^2 u_1 \cos^2 u_2 \\
- (1 - 2p) (\sin^2 u_1 - \sin^2 u_2) - 4\sqrt{p(1-p)} (\cos v_1 \cos v_2 \\
+ \sin u_1 \sin u_2 [(\cos^2 u_1 - \cos^2 v_1)(\cos^2 u_2 - \cos^2 v_2)^{1/2}])
\]

The state \( |\psi_{test}\rangle \) is then equal to such \( |\psi_p\rangle \) that minimizes the expression (20). By a remark before the Proposition involved, \( \sin u_j \geq 0 \) and \( \cos v_j \geq 0, \quad j = 1, 2 \). Since \( (1 - 2p)^2 + (2\sqrt{p(1-p)})^2 = 1 \), the minimum of (20) is achievable and reads

\[
4\langle \psi_{test}|M|\psi_{test}\rangle = 1 - \sin^2 u_1 \sin^2 u_2 - \cos^2 u_1 \cos^2 u_2 \\
- \left( (\sin^2 u_1 - \sin^2 u_2)^2 + 4 (\cos v_1 \cos v_2 \\
+ \sin u_1 \sin u_2 [(\cos^2 u_1 - \cos^2 v_1)(\cos^2 u_2 - \cos^2 v_2)^{1/2}])^2 \right)^{1/2}.
\]
It is not hard to see that the inequality \( \langle \psi_{\text{test}} | M | \psi_{\text{test}} \rangle < 0 \) is equivalent to
\[
\sin u_1 \sin u_2 \sqrt{(\cos^2 u_1 - \cos^2 v_1)(\cos^2 u_2 - \cos^2 v_2)} \\
+ \cos v_1 \cos v_2 > | \sin u_1 \sin u_2 \cos u_1 \cos u_2 |,
\]
which is fulfilled for non-negative \( u_j \) and \( v_j \) whenever \( \cos u_1 \cos u_2 \cos v_1 \cos v_2 \neq 0 \) because \( \sqrt{1-t^2} > 1-t \) for all \( t \in (0,1) \). Thus, the operator (18) is not positive semi-definite and the channel \( \mathcal{E}_1 \otimes \mathcal{E}_2 \) is not EA.

**Corollary 2.** Suppose \( \mathcal{E} \) is an extremal qubit channel. Then \( \mathcal{E} \otimes \mathcal{E} \) is EA, hence \( \mathcal{E} \) is 2-LEA, if and only if \( \mathcal{E} \) is EB.

Using results of Proposition 5, we can present a complete picture (Fig. 4) of factorized extremal channels analogous to that of unital channels (Fig. 3). Comparison of two figures gives a clear insight that factorized extremal channels exhibit the best possible entanglement preserving properties.

**B. Generalized amplitude-damping channels**

Amplitude-damping channels describe how a two-level system approaches the equilibrium due to coupling with its environment, e.g., thanks to a spontaneous emission process at zero temperature. If the environment has a finite temperature, then such a dissipation process is described by the action of a generalized amplitude-damping channel (see, e.g., [30]).

Kraus operators for a generalized amplitude-damping channel \( \mathcal{A}_{p,\gamma} \) have the form
\[
E_0 = \sqrt{\gamma} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad E_1 = \sqrt{\gamma} \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix}, \\
E_2 = \sqrt{1-\gamma} \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_3 = \sqrt{1-\gamma} \begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix},
\]
where \( p \in [0,1] \) determines the amplitude-damping rate and \( \gamma \in [0,1] \) is a parameter that depends on the temperature and defines a fixed (equilibrium) state of \( \mathcal{A}_{p,\gamma} \)
\[
|\psi_{\infty}\rangle = \begin{pmatrix} \gamma & 0 \\ 0 & 1 - \gamma \end{pmatrix}.
\]

If \( \gamma = 0 \) or \( \gamma = 1 \), then \( \mathcal{A}_{p,\gamma} \) is simply an amplitude-damping channel. As it is mentioned above in Sec. V A amplitude-damping channels \( \mathcal{A}_{p,0} \) and \( \mathcal{A}_{p,1} \) are extremal, thus obeying Proposition 5. A channel \( \mathcal{A}_{p_1,0} \otimes \mathcal{A}_{p_2,0} \) is EA if and only if \( p_1 = 1 \) or \( p_2 = 1 \), i.e. when at least one of the constituent channels contracts the whole Bloch sphere into a single pure state.

Let us now move on to generalized amplitude-damping channels \( \mathcal{A}_{p,\gamma} \). We note that \( \mathcal{A}_{p,\gamma} = \gamma \mathcal{A}_{p,1} + (1-\gamma) \mathcal{A}_{p,0} \). Since \( \mathcal{A}_{p,\gamma} \) is a convex combination of two amplitude-damping channels (extremal qubit channels), we expect \( \mathcal{A}_{p_1,\gamma_1} \otimes \mathcal{A}_{p_2,\gamma_2} \) to exhibit worse entanglement preserving properties than factorized extremal channels. We can expect that the closer weights of two channels (the closer \( \gamma \) to \( \frac{1}{2} \)) the stronger the entanglement annihilation is.

Firstly, we note that the channel \( \mathcal{A}_{p,\gamma} \) is entanglement-breaking if \( p \geq (\sqrt{1+4\gamma(1-\gamma)}-1)/2\gamma(1-\gamma) \). Let us remind that a channel is EB if and only if its Choi-Jamiołkowski state is separable and for two qubits one can employ the PPT criterion to verify the separability. In particular, for \( \gamma = 0 \) (zero bath temperature) the channel \( \mathcal{A}_p \) is EB only if \( p = 1 \), hence, it contracts the whole Bloch sphere into the equilibrium state.

Secondly, if we focus on 2-LEA channels, then it turns out that \( \mathcal{E} \otimes \mathcal{E} \) preserves entanglement of the state \( |\psi_{\text{+}}\rangle \) if \( p \leq (1-\sqrt{2\gamma(1-\gamma)})/(1-2\gamma(1-\gamma)) \). These results are shown in Fig. 5a, where the dotted line depicts a symbolic boundary between EA and not EA channels.
The numerical analysis encourages to assume that this boundary coincides with the solid line. Thirdly, if $\gamma = 1/2$, then $\mathcal{E}$ becomes unital. In this case to conclude EA of the channel $\mathcal{E} \otimes \mathcal{E}$ it suffices to consider its action of on the maximally entangled state, e.g., $|\psi_+\rangle$. Illustration of such unital channels with respect to other unital channels is presented in Fig. 5b.

Finally, we can draw a conclusion that a picture of the entanglement-annihilation behavior of generalized amplitude-damping channels takes an intermediate form between the unitary two-qubit channels (Fig. 3) and the factorized extremal channel (Fig. 4).

VI. EA-EXTREMAL LOCAL CHANNELS

Convexity is an important property of many channel sets. Thus, the set of all qubit channels $T_2$ and the set of all EB qubit channels $T_{EB}$ are convex. Extreme points of these sets are studied in Refs. [28] and [24], respectively. Extreme points of the set of EB channels are referred to as EB-extremal. It is shown in the paper [24], that in qubit case all EB-extremal single-qubit channels can be represented in the form

$$\mathcal{F}[\rho] = \langle \psi | \rho | \psi_+ \rangle |\varphi_1\rangle \langle \varphi_1 | + \langle \psi_+ | \rho | \psi \rangle |\varphi_2\rangle \langle \varphi_2 |,$$

(21)

where $|\psi\rangle$, $|\psi_+\rangle$, $|\varphi_1\rangle$, $|\varphi_2\rangle$ are pure qubit states and $\langle \psi | \psi_+ \rangle = 0$.

If we consider general two-qubit channels (not necessarily local), then along with the set of all channels $T_{\text{chan}}$ and the set of two-qubit channels there is a set of all EA channels $T_{\text{EA}}$ which is also convex (property 1). The question arises itself to find extreme points of this set, i.e. channels that are extreme for two-qubit EA channels. We will refer to such channels as EA-extremal. The following proposition provides EA-extremal channels of factorized form.

**Proposition 6.** Let $\mathcal{F}$ be EB-extremal one-qubit channel, then the channels $\mathcal{F} \otimes \mathcal{I}$, $\mathcal{I} \otimes \mathcal{F}$ are EA-extremal.

**Proof.** Let us now prove the statement of Proposition by reductio ad absurdum. Suppose $\mathcal{F} \otimes \mathcal{I}$ is not EA-extremal, i.e. there exist two non-coincident EA channels $\mathcal{G}_1$ and $\mathcal{G}_2$ such that $\mathcal{F} \otimes \mathcal{I} = \mu \mathcal{G}_1 + (1 - \mu) \mathcal{G}_2$, where $\mu \in [0, 1]$. Using the Choi-Jamiołkowski isomorphism [31, 32], the latter equation can be rewritten as $\rho_{\mathcal{F}} \otimes P_+ = \mu \rho_{\omega_1} + (1 - \mu) \rho_{\omega_2}$, where $P_+$ is a 1-rank projector onto maximally entangled state $|\psi_+\rangle$. Taking partial trace over the first subsystem $A$, we get $P_+ = \text{tr}_A[\omega_1] + (1 - \mu) \text{tr}_A[\omega_2]$, from which it follows that $\text{tr}_A[\omega_1] = \text{tr}_A[\omega_2] = P_+$. Consequently, $\omega_j = \rho_j \otimes P_+$, $j = 1, 2$, because if a subspace is in a pure state, then it is necessarily factorized from any other system. As a result we may conclude that $\mathcal{F} \otimes \mathcal{I} = (\mu \mathcal{E}_1 + (1 - \mu) \mathcal{E}_2) \otimes \mathcal{I}$, where $\mathcal{E}_j \otimes \mathcal{I}$ is EA, i.e. $\mathcal{E}_j$ is EB, $j = 1, 2$ (property 6). Since $\mathcal{F}$ is EB-extremal, then the relation $\mathcal{F} = \mu \mathcal{E}_1 + (1 - \mu) \mathcal{E}_2$ can be only fulfilled if $\mathcal{E}_1 = \mathcal{E}_2 = \mathcal{F}$, i.e. $\mathcal{G}_1 = \mathcal{G}_2$. This contradiction concludes the proof. □

Let us stress two points. Firstly, the identity channel in the proposition can be replaced by any unitary channel. Secondly, this proposition is not a special case of Proposition 6 because there we assume channels $\mathcal{E}_1$ and $\mathcal{E}_2$ to be extremal in the set of all qubit channels $T_2$, whereas in Proposition 6 the local channel $\mathcal{F}$ is assumed to be EB-extremal. In particular, if both $\mathcal{E}_1, \mathcal{E}_2$ are extremal and EB, then $\mathcal{E}_1 \otimes \mathcal{E}_2$ is simultaneously extremal, EB-extremal and EA-extremal. The following example demonstrates the existence of EA-extremal channels that are not extreme points of the set of all two-qubit channels $T_{\text{chan}}$.

**Example 2.** Consider a phase damping channel $\mathcal{F}[\rho] = \frac{1}{2} (\rho + \sigma_z \rho \sigma_z)$. Clearly, this channel is unital and not extremal in the set of all channels. Nevertheless, its action can be expressed also as

$$\mathcal{F}[\rho] = \langle 0|\rho| 0 \rangle \langle 0 | + \langle 1|\rho | 1 \rangle \langle 1 |.$$

This means the phase-damping channel is an extreme point of the set of entanglement-breaking channels, hence, it is EB-extremal [24]. Then by Proposition 6 the two-qubit channels $\mathcal{F} \otimes \mathcal{I}$ and $\mathcal{I} \otimes \mathcal{F}$ are EA-extremal.

VII. EA-DUALITY

In this section we will employ the idea of entanglement-annihilation to introduce the concept of EA-duality for local channels. Let us denote by $T_d$ the set of channels on a $d$-dimensional (non-composite) quantum system (qudit). Let $Q \subset T_d$ be an arbitrary subset of qudit channels. A subset $Q_n \subset T_n$ is called $n$-EA-dual to $Q$ if $\mathcal{E}_1 \otimes \mathcal{E}_2$ is EA for all $\mathcal{E}_1 \in Q$ and $\mathcal{E}_2 \in Q_n$. To be more precise,

$$Q_n = \{ \mathcal{E}_2 \in T_n \mid \mathcal{E}_1 \otimes \mathcal{E}_2 \text{ is EA for all } \mathcal{E}_1 \in Q \}. \quad (22)$$

When it is clear from the context we will omit the explicit mention of the dimensions and say $Q$ is EA-dual to $Q$.

Using this concept, we may rephrase the goal of this paper as an identification of the set of channels $T_{\text{ent}} \subset T_2$ such that EA-duals of all its elements coincide with the set of entanglement-breaking channels, i.e. $\{\mathcal{E}\} = T_{EB}$ for all $\mathcal{E} \in T_{\text{ent}}$. Let us note that for any Q its EA-dual $Q$ contains the entanglement-breaking channels, i.e. $T_{EB} \subset Q$. It is also clear that $\overline{Q} \subset \{\mathcal{E}\}$ providing that $\mathcal{E} \in Q$.

**Example 3.** Let us clarify the introduced concept on some examples:

- For unitary channels $U[\rho] = U \rho U^*$ the EA-dual coincides with EB qubit channels, i.e. $\{U\} = T_{EB} \subset T_n$, which means that the system under consideration is maximally robust in sharing the entanglement unless an entanglement-breaking channel is applied on the second system.
• For the set $\mathcal{T}_{\text{EB}}$ of entanglement-breaking channels the EA-dual set equals to the set of all channels, i.e. $\overline{\mathcal{T}}_{\text{EB}} = \mathcal{T}_n$. This only illustrates the fact that entanglement-breaking channels destroy any entanglement between the system under consideration and whatever second system. For each entanglement-breaking channel $\mathcal{E}$ its EA-dual contains all channels, i.e. $\{\mathcal{E}\} = \mathcal{T}_n$.

• The EA-dual of all channels is the set of entanglement-breaking channels, i.e. $\overline{\mathcal{T}}_d = \mathcal{T}_{\text{EB}} \subseteq \mathcal{T}_n$. Providing that $d = n$ we can write $\overline{\mathcal{T}}_d = \mathcal{T}_d$.

Fixing $d$ and $n$ the dual sets obviously satisfy the following properties:

$$Q_1 \subseteq Q_2 \Rightarrow \overline{Q}_1 \supseteq \overline{Q}_2, \quad (23)$$

$$\overline{(Q_1 \cup Q_2)} = \overline{Q}_1 \cap \overline{Q}_2, \quad (24)$$

$$\overline{(Q_1 \cap Q_2)} \supseteq \overline{Q}_1 \cup \overline{Q}_2. \quad (25)$$

The results presented in the previous sections contain partial answers to EA-duality sets for 2-LEA qubit channels, or depolarizing qubit channels, or amplitude-damping qubit channels, etc. However, further analysis is beyond the scope of this paper and the full characterization of EA-duals of these and other interesting subsets of qubit channels remains an open problem.

VIII. SUMMARY

In this paper we have investigated the robustness of entanglement in two-qubit (spatially separated) systems under the influence of independent reservoirs. In particular, we paid attention to characterization of the so-called entanglement-annihilating (EA) channels. These are the channels that completely annihilate any entanglement initially present between the subsystems. In contrast, the so-called entanglement-breaking channels (EB) destroy any entanglement between the system they act on (two-qubit system in our case) and any other system. The dramatic differences, but also subtle relations, between these two concepts were discovered for particular classes of channels.

We succeeded to characterize all unital two-qubits entanglement-annihilating channels of the factorized form $\mathcal{E}_1 \otimes \mathcal{E}_2$ and the results are nicely illustrated in Fig. 2. We derived a sufficient condition (Proposition 4) for a local unital channel not to be EA, which guarantees the entanglement retention and enables to perform entanglement-enabled experiments in the presence of such noise. We gave (Proposition 2) the complete characterization of unital entanglement-annihilating channels $\mathcal{E} \otimes \mathcal{E}$ and showed (Proposition 3) that such (2-LEA) channels $\mathcal{E}$ form a convex subset of the set of all unital single-qubit channels.

For example, in case of depolarizing channels $\mathcal{E}_1$ and $\mathcal{E}_2$ with rates $q_1$ and $q_2$, their product should be kept above $\frac{1}{4}$. Above this critical value, there still exist initial states of two-qubit system for which the entanglement survives the effects of depolarizing noise.

Particular results have been also obtained for the case of extremal non-unital channels, for which the entanglement turns out to be more robust (in comparison with unital channels). In particular, they are entanglement-annihilating only if one of the constituent channels is entanglement-breaking meaning that (in this case) the set of 2-LEA channels coincides with EB channels.

Much attention has been also focused on such a fundamental entity as convexity of the set of all two-qubit channels $\mathcal{T}_{\text{chan}}$ and the set of EA channels $\mathcal{T}_{\text{EA}}$. We have revealed that the sets $\mathcal{T}_{\text{chan}}$ and $\mathcal{T}_{\text{EA}}$ have common extremal points corresponding to local channels. We have constructed a class of purely EA-extremal channels (phase-damping channels), i.e. channels that are extremal for $\mathcal{T}_{\text{EA}}$ but are internal for $\mathcal{T}_{\text{chan}}$.

Finally, we have introduced an important concept of EA-duality between sets of channels defined on individual subsystems. This concept gives another perspective on classification of channels with respect to their entanglement annihilation ‘potential’.

To conclude, the presented analysis contains partial characterization of local entanglement-annihilation two-qubit channels. This class of channels is of particular importance and interest in the domain of quantum information processing. Although the complete understanding of the phenomena of entanglement-annihilation is still missing, the presented results represent important and practical steps towards this direction. Our analysis shows that the phenomenon of being entanglement-annihilating is not a rare one and, hence, deserves further attention. From the practical point of view, we have analyzed in detail classes of physically relevant qubit channels (depolarizing, phase-damping, amplitude-damping) and among them identified the ‘good’ and ‘bad’ ones.

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