Superconducting qubit circuit emulation of a vector spin-1/2

Andrew J Kerman
MIT Lincoln Laboratory, Lexington, MA, United States of America
E-mail: ajkerman@ll.mit.edu

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Abstract
We propose a superconducting qubit that fully emulates a quantum spin-1/2, with an effective vector dipole moment whose three components obey the commutation relations of an angular momentum in the computational subspace. Each of these components of the dipole moment also couples approximately linearly to an independently-controllable external bias, emulating the linear Zeeman effect due to a fictitious, vector magnetic field over a broad range of effective total fields around zero. This capability, combined with established techniques for qubit coupling, should enable for the first time the direct, controllable hardware emulation of nearly arbitrary, interacting quantum spin-1/2 systems, including the canonical Heisenberg model. Furthermore, it constitutes a crucial step both towards realizing the full potential of quantum annealing, as well as exploring important quantum information processing capabilities that have so far been inaccessible to available hardware, such as quantum error suppression, Hamiltonian and holonomic quantum computing, and adiabatic quantum chemistry.

Quantum spin-1/2 models serve as basic paradigms for a wide variety of physical systems in quantum statistical mechanics and many-body physics, and are among the most highly studied in the context of quantum phase transitions and topological order [1–4]. In addition, since the spin-1/2 in a magnetic field is one of the simplest realizations of a qubit, many quantum information processing paradigms draw heavily on concepts which originated from or are closely related to quantum magnetism. For example, quantum spin-1/2 language is used to describe nearly all of the constructions underlying quantum error-correction [5–7] and error-suppression [8, 9] methods. It is also the most commonly-used framework for specifying engineered Hamiltonians in many other quantum protocols such as quantum annealing [10, 11], adiabatic topological quantum computing [12], quantum simulation [13–17], Hamiltonian and holonomic quantum computing [9, 18–21], and quantum chemistry [22–24].

The conventional method for simulating vector spin-1/2 Hamiltonians is based on the ‘gate-model’ quantum simulation paradigm, and uses pulsed, high-speed sequences of discrete, non-commuting gate operations [25–27] to approximate time-evolution under a desired Hamiltonian [28, 29]. In this paradigm, simulating a different Hamiltonian simply requires reprogramming the hardware with a different sequence of gates, a desirable property so long as the error introduced by the discretization can be kept sufficiently low. Unfortunately, this becomes increasingly difficult as the required spin interactions become stronger and/or more complex, since for a fixed gate duration, the discretization error grows both with the strength of the spin–spin interactions, and with the number of mutually-noncommuting terms they contain. In addition, gate-based implementation necessarily implies that the system occupies Hilbert space far above its ground state, and the resulting information exchange with the environment via both absorption and emission is at the root of the need for active quantum error correction [5–7].

In light of these considerations, static emulation of a desired Hamiltonian (which suffers from neither of the above problems) becomes a potentially appealing alternative for applications requiring strong, complex spin–
spin interactions. In fact, the resulting intrinsic 'protection' from noise associated with remaining in the ground state of a quasi-static Hamiltonian is precisely the motivation for adiabatic quantum computation protocols [30]. To this end, a wide array of what might be called 'weakly-engineered,' but still fundamentally naturally-occurring, quantum magnetic systems have been explored for their potential quantum information applications, including, for example: the doped ionic crystals used in the original demonstration of quantum annealing [31], ultracold atoms [32] or dipolar molecules [33], chemically-engineered molecular nanomagnets [34, 35], and donor spins in Si [36]. These kind of systems, however, do not have the level of microscopic controllability required in most cases for the applications discussed above.

A major step toward this level of controllability in a static emulation was taken with the advent of the superconducting machines from D-wave systems [16, 17, 37, 38], which are the first examples of large-scale, 'fully-engineered' quantum spin model emulators, and have already been used to great effect in both quantum annealing [10, 11] and quantum simulation [16, 17]. However, even these systems have critical limitations in their ability to emulate spin interactions, which are inherited directly from the persistent-current flux qubits on which their spins are based [37, 39]. In particular, although these qubits are well-suited to emulation of the simpler, transverse-field Ising spin (which interacts with other spins only via its z-component), controllable vector spin interactions of the kind we discuss here are fundamentally beyond their capability. A direct consequence of this is an inability to realize the controllable ‘non-stoquastic’ qubit interactions [40] that have been the subject of intense interest in recent years [41–52], due to the potentially transformative role such interactions may have on the computational power of quantum annealing. Exploring this capability is of paramount importance, in light of the fact that while quantum annealing has many potential applications, its ability to enable beyond-classical capabilities has yet to be fully understood or even conclusively shown. Although there have been multiple experimental efforts aimed at realizing non-stoquastic interactions between conventional flux qubits [51, 52], none have been able to fully realize the desired capability, due to the fundamental limitations of existing qubit hardware (described in detail below in section 1).

In this work, we propose a novel superconducting circuit called the 'Josephson phase-slip qubit' (JPSQ), a device which aims for the first time to directly emulate a fully-controllable, quantum vector spin-$1/2$. When combined with existing techniques for coupling superconducting qubits, this circuit could enable the realization of nearly arbitrary, controllable many-body spin Hamiltonians, without the limitations associated with digital, gate-based quantum simulation methods. Such a capability would provide entirely new modes of access to all of the applications listed above, many of which have never been tried experimentally due to the lack of required capabilities in qubit hardware.

The remainder of the paper is divided as follows: in section 1 we begin by describing how persistent-current flux qubits can be used to emulate quantum spin models, and their fundamental limitations in this context. We then introduce in section 2 our proposed JPSQ, analyzing a simplified version of the circuit in detail, and comparing the results with numerical simulations. This section concludes with a discussion of the coherence of the JPSQ in the context of existing superconducting qubit devices. Section 3 contains discussion and simulations of two multi-JPSQ circuit examples: (i) two JPSQs coupled by independently-controllable $zz$ and $xx$ interactions, and (ii) a four-JPSQ circuit which implements a distance-2 Bacon-Shor logical qubit with quantum error suppression [8, 9]. Finally, section 4 describes a generalization of the JPSQ circuit capable of emulating fully-controllable, anisotropic Heisenberg interactions, and we conclude in section 5. The appendix contains the details of our analytic calculation of fluxon tunneling in the JPSQ, the results of which are used in the discussion of section 2.

1. Persistent-current qubits for quantum spin--$\frac{1}{2}$ model emulation

Superconducting circuits are already among the most engineerable high-coherence quantum systems available, allowing a range of behavior and interactions to be constructed by design [53, 54]. Their capability to emulate complex static spin Hamiltonians is exemplified by the flux-qubit-based machines of D-wave Systems, Inc. [16, 17, 37, 38]. These systems are designed to emulate the quantum transverse-field Ising model, with Hamiltonian:

$$\hat{H}_\text{TIM} = -\sum_i (E_i^x \hat{\sigma}_i^x + E_i^z \hat{\sigma}_i^z) - \sum_{i<j} J_{ij} \hat{\sigma}_i^z \hat{\sigma}_j^z. \quad (1)$$

Each spin has an emulated local magnetic field in the $x-z$ plane with effective Zeeman energies given by the parameters $E_i^x$ and $E_i^z$ in equation (1), and pairwise couplings to other spins parameterized by the classical Ising interaction energies $J_{ij}$. Notably absent from this Hamiltonian are any interactions between the transverse moments of the spins, whose physical realization is the subject of this work. To better understand what follows,
we first describe the physics underlying emulation of equation (1) with two-loop flux qubits [37, 39, 55–57], and why these circuits cannot be used to emulate the vector spin interactions of interest here.

Figure 1 illustrates how flux qubits are currently used to emulate quantum spins. The circuit shown in panel (a) is the basic four-junction flux qubit [37, 39, 55–57], having two loops biased by fluxes $\Phi^x$ and $\Phi^z$, labeled according to the spin moment they are used to emulate. When the $z$ loop is biased with an external flux $\Phi^z = \Phi_0/2 + \delta \Phi^z$ (where $\Phi_0 \equiv h/2e$ is the superconducting fluxoid quantum, and $\delta \Phi^z \ll \Phi_0$), the two lowest-energy semiclassical states of the loop are nearly degenerate, having approximately equal and opposite supercurrents. As shown in figure 1(a), these two semiclassical states correspond to expulsion of the external flux from the loop, or pulling additional flux into it, such that it contains exactly zero or one fluxoid quantum, respectively. They can be identified with two local minima in a magnetic potential (panel (b) in the figure) experienced by a fictitious particle whose ‘position’ corresponds to the gauge-invariant phase difference across the two larger Josephson junctions (which play the role of a loop inductance), and whose ‘momentum’ corresponds to the total charge that has flowed through them. The difference in potential energy between these two minima is controlled by $\delta \Phi^z$ (panel (d)), and approximately corresponds to the interaction energy between an applied field and the two equal and opposite persistent currents (panel (e)). These two states naturally play the role of $|\pm z\rangle$, the eigenstates of $\hat{\sigma}^z$ for the emulated spin. Quantum coupling between these two states can be associated with the operator $\hat{\sigma}^x$: the Zeeman Hamiltonian due to an emulated transverse field. As shown in (a)–(c), this can be understood as tunneling of a ‘fluxon’ between the two states in which it is inside or outside of the larger loop, with a strength controlled by the barrier height via $\Phi^z$ (panel (c)).

In published experimental work to date (with the exception of [51]), this $\Phi^z$ tuning has only been used to adjust the Zeeman energy due to an effective transverse field [37, 39, 55–57]. However, we are concerned here with producing a static interaction between the transverse dipole moments of two such emulated spins. Figure 2(a) shows the circuit which realizes such an interaction between two-loop flux qubits, originally proposed in [58], and demonstrated in [51]. The DC SQUID loops of the two qubits (whose external flux biases

\[ F = F + F \]

Note that we focus here on the flux qubit variant which uses Josephson junctions for its loop inductance [39], instead of the RF-SQUIDs used in quantum annealing applications to date [37, 38, 57]; however, the physics is qualitatively the same, and the conclusions presented here apply equally to either type.

Figure 1. Spin-1/2 emulation using a two-loop flux qubit. Panel (a) illustrates the persistent current (fluxon) states of the two-loop flux qubit [37, 39, 55–57]. On the left is the state of fluxon number $N_0 = 0$ where the persistent current expels the externally-applied flux $\Phi^z$, while on the right is the state $N_0 = 1$ in which a persistent current in the opposite direction pulls in the additional flux needed to trap exactly one fluxon in the loop. The path connecting these two semiclassical persistent current states to each other traverses an energy barrier in which the fluxon is stored inside the junctions of the DC-SQUID interrupting the loop, as illustrated in the center panel of (a). Panels (b) and (c) show how this can be viewed in terms of the motion of a fictitious ‘phase’ particle in a double-well potential. In (b), the barrier is high, and the two persistent current states are well-isolated, resulting in the energy levels versus flux shown in (d), which emulate an Ising spin in a $z$ field. Panel (c) illustrates that when the barrier is lowered by threading a flux $\Phi^x$ through the DC-SQUID loop, quantum tunneling between the two fluxon states occurs, producing the energy levels in (f). The tunneling appears as an avoided crossing between the two fluxon states of (d), which emulates the effect of a transverse field along $x$. Panels (e) and (g) show the resulting effective persistent currents $I^x_{\Phi^x}$ and $I^x_{\Phi^z}$ for the cases where the emulated transverse field is small, and large, respectively. Finally, panel (h) illustrates the resulting fundamental asymmetry between emulated $z$ and $x$ fields for the 2-loop flux qubit system, for the parameters: $E_z = h \times 44.7$ GHz, $C_{\Phi^x} = 1.80$ fF, $E_x = h \times 134$ GHz, $C_x = 5.40$ fF (note that the junction capacitances are not shown in panel (a)).
control the height of their fluxon tunnel barriers) are each coupled by a mutual inductance $M$ to a common flux qubit coupler. The latter can be viewed semiclassically as a tunable effective inductance $L_C^{\text{eff}}$ which can assume positive or negative values, depending on how it is biased [38, 59–63]; therefore, we can qualitatively understand the resulting two-qubit interaction using the Hamiltonian:

$$
\hat{H} = \hat{H}_A(\Phi_A^x + ML_C) + \hat{H}_B(\Phi_B^x + ML_C) + \frac{1}{2}L_C^{\text{eff}}I_C^2
$$

$$
\equiv -E_x^A\hat{\sigma}_A^x - E_x^B\hat{\sigma}_B^x - J_{xx}^{\text{eff}}\hat{\sigma}_A^x\hat{\sigma}_B^x,
$$

(2)

where the coupler ground state is described semiclassically in its inductance $L_C^{\text{eff}}$ and current $I_C$. $\Phi_A^x, \Phi_B^x$ are the static flux offsets applied to the two qubits’ DC SQUID loops, and we describe each qubit $q \in \{A, B\}$ in terms of its $\Phi^x$-dependent quantum eigenenergies (in the absence of coupling) at zero effective $z$ field ($\Phi^z = \Phi_0/2$):

$$
\hat{H}_q(\Phi_q^x) \equiv E_q^{+x}(\Phi_q^x)|+x\rangle \langle +x| + E_q^{-x}(\Phi_q^x)|-x\rangle \langle -x|.
$$

(3)

To calculate the effective coupling energy $J_{xx}$ in equation (2), we expand the qubit energies around the points $\Phi_A^x$ and $\Phi_B^x$, and then minimize the total energy with respect to the coupler current $I_C$ (following [64]), to obtain:

$$
J_{xx} = \frac{1}{2}I_C^2\frac{M^2}{L_C^{\text{eff}} + \frac{1}{L_C^{\text{eff}}}} L_A + L_B
$$

(4)

where we have defined the Taylor coefficients (with $q \in \{A, B\}$):

$$
I_q^x \equiv \frac{d}{d\Phi_q^x}\Delta E_q^x,
$$

$$
\Delta E_q^x \equiv \frac{E_q^{-x} - E_q^{+x}}{2},
$$

(5)

$$
I_q^{-1} \equiv \frac{d^2}{d(\Phi_q^x)^2}E_q^x,
$$

$$
E_q^x \equiv \frac{E_q^{-x} + E_q^{+x}}{2},
$$

(6)

and neglected the differential quantum inductance between the $|\pm x\rangle$ states. Equations (4) and (5) show that the semiclassical quantity which plays the role of the qubit magnetic moment along the fictitious $x$ direction is $I_q^x$, the slope of the tunnel splitting energy $\Delta E_q^x$ with respect to $\Phi^x$. This quantity is plotted in figure 2(b) in blue (right axis) as a function of $\Phi^x$ for a single tunable flux qubit, using the full numerical simulation methods of [65] (previously used in [63, 66]). Panel (c) shows in black the corresponding transverse coupling energy $J_{xx}$ obtained by plugging this into equation (4). The crucial point here is that the $xx$ coupling that can be achieved in this way is always much smaller than the local transverse-field Zeeman energies: $J_{xx} \ll \Delta E_A^x, \Delta E_B^x$. The physical reason for this is simple: the $\Phi_A^x$-dependence of the tunneling energy $\Delta E_A^x$ is exponential, since increasing the flux corresponds to lowering the tunnel barrier (see, figure 1(h)). Since the effective $x$ magnetic moment of each qubit (see equation (5)) is the derivative with respect to $\Phi^x$ of this energy, it can only be large when the energy is itself large. In spin language, this constraint on transverse coupling between flux qubits corresponds to the $x$ magnetic

Figure 2. Transverse interaction between two-loop flux qubits, used to emulate an $xx$ interaction. Panel (a) shows a circuit which realizes such an interaction, as in [58, 51]. The two qubits’ DC SQUID loops are coupled magnetically to a common flux qubit coupler, having effective inductance $L_C^{\text{eff}}$. Panel (b) shows the a full simulation [63] of the effective $x$-field Zeeman energy $\Delta E^x$ (left axis) and $\Phi$ (right axis) versus $\Phi$ for a single qubit. The parameters used are: $E_i = h \times 117$ GHz, $C_i = 4.42$ fF, $E_a = h \times 47.8$ GHz, $C_A = 1.80$ fF, $I_1 = 110$ pH, $L_A = 20$ pH, $C_B = 35$ fF. Panel (c) shows the resulting two-qubit coupling energy $J_{xx}$ as a function of $\Delta E^x$, where the black line is obtained from equation (4) using the $I_q^x$ from (b), $M = 40$ pH, and $L_C^{\text{eff}} = 40$ pF from numerical simulation with: $E_{LI} = h \times 376$ GHz, $C_{LI} = 14.2$ fF and $L_{LI} = 575$ pH. For comparison, the red line shows the $J_{xx}$ obtained from numerical simulation of the full three-qubit system [63]. Because of the exponential decrease of $\Phi^x$ with $\Phi^x$ shown in (b), the coupling energy goes exponentially to zero with $\Delta E^x$, so that non-negligible transverse interaction can only be achieved in the presence of large offset $x$ fields for both qubits.
magnets with the exception of the transverse static fashion under discussion here. In section 1, we described how emulation of a large magnetic moment of the emulated spins going exponentially to zero as their local transverse magnetic moment with a persistent-current qubit requires tunable, magnetic interaction of capacitive coupling between both qubits, the electric interaction of [52] is a coupling between the corresponding momenta (charge). However, in either case, the transverse interaction can only be strong if both individual qubit wavefunctions have a strong dependence of their tunneling amplitude (between persistent current states) on the physical quantity which emulates a transverse moment. When the transverse field energy goes to zero, the tunneling amplitude of a flux qubit by definition becomes exponentially insensitive to both flux and charge, causing any two-qubit transverse coupling to go exponentially to zero with the local transverse fields as described above3. This limitation renders present-day superconducting circuits unable to emulate (in the static fashion under discussion here) the majority of the controllable spin models discussed in the introduction, with the exception of the transverse field Ising model of equation (1) emulated by D-wave systems’ machines [16, 17].

2. Josephson phase-slip qubit

In section 1, we described how emulation of a large x magnetic moment with a persistent-current qubit requires its tunneling energy to be sensitive to barrier height, which implies that there must be substantial probability inside the barrier. For this to remain true all the way to zero transverse field (tunneling energy), the wavefunction inside the barrier would need to remain appreciable even when the tunneling itself goes to zero, a self-contradictory requirement that conventional flux qubits cannot satisfy. Figure 3 shows the proposed persistent-current qubit circuit, which can. Here, the tunable fluxon tunnel barrier that provides control of the transverse field (which in figure 1(a) is realized with a DC SQUID) consists of two DC SQUIDs in series, separated by a central superconducting island whose polarization charge can be controlled with an external bias voltage. This object is similar to the so-called ‘quantum phase-slip transistor’ (QPST) [67–70] (with the quantum phase-slip junctions here replaced by DC SQUIDs), hence the name ’Josephson phase-slip qubit’. The key feature which motivates the use of a QPST to control fluxon tunneling in the present context is that it provides two fluxon tunneling paths

3 In addition to having this limitation in common with the scheme of figure 2, the electric coupling of [52] is also non-tunable, such that the aforementioned exponential suppression of transverse coupling with the local transverse fields is the only way the transverse coupling can be controlled in this scheme.
into or out of the loop, as illustrated in (a). If a polarization charge \( Q_b \) is present on the island, the two fluxon tunneling amplitudes acquire a relative phase shift due to the Aharonov–Casher effect \[71]\) when \( Q_b = e \) this phase shift is \( \pi \), and if the magnitudes of the two tunneling amplitudes are equal, total suppression of fluxon tunneling occurs and the transverse field Zeeman energy is zero (figure 3(c)). Crucially, this remains true even when the individual tunneling amplitudes are large and flux-sensitive, allowing a strong, linear flux-sensitivity (magnetic dipole moment; see equation (5)) to persist even around zero field. We note before proceeding that a number of previous works have highlighted and/or experimentally exploited this phenomenon as a means to observe the Aharonov–Casher effect in superconducting circuits \[72–76]\; here, we propose a way to use it to realize a superconducting-circuit-based vector spin-1/2 qubit.

Figure 3(b) shows a simplified JPSQ circuit in more detail, and in particular how a transverse magnetic moment at zero field can be realized. The two DC SQUIDs are biased with an offset flux of the same magnitude \( \Phi_L = \pm \Phi_D = \Phi_{\Delta_1} \) so that their individual fluxon tunneling amplitudes \( \lambda_L (\Phi_D) \) and \( \lambda_R (\pm \Phi_{\Delta_1}) \) have equal magnitude, and are flux-sensitive (the \( \pm \) indicates that there are two possible choices for the relative sign). If we then magnetically couple an input flux \( \delta \phi^x \) to both DC SQUIDs with equal strength, and signs such that the resulting total flux through the two DC SQUIDs is affected oppositely, the two tunneling amplitudes no longer cancel, creating an effective transverse field as illustrated in figure 3(d). If we also change the flux \( \delta \phi^y \), the total effect is analogous to a field in between the \( x \) and \( x \) axes, as shown in Panel (e). Panel (f) shows the effect of charge displacements away from half a Cooper pair, which act as transverse fields in the \( y \) direction.

### 2.1. JPSQ potential and semiclassical persistent current states

The Hamiltonian for the JPSQ circuit of figure 3(b) can be written:

\[
\hat{\mathcal{H}} = \frac{1}{2} \hat{Q}^T \cdot \mathbf{C}^{-1} \cdot \hat{Q} - \hat{Q}_b \cdot \hat{V} + \hat{U}_J (\hat{\Phi}),
\]

(7)

where the three terms are: (i) the electrostatic energy; (ii) the interaction with a polarization charge \( \hat{Q}_b \) (supplied by a bias source); and (iii) the Josephson potential energy. Vector notation is used here to capture the three canonical degrees of freedom of the circuit, with \( \hat{Q} \equiv (\hat{Q}_1, \hat{Q}_2, \hat{Q}_3) \) describing the node charges (canonical momenta) conjugate to the node flux variables (canonical coordinates) \( \hat{\Phi} \equiv (\hat{\Phi}_1, \hat{\Phi}_2, \hat{\Phi}_3) \), such that \( \{ \hat{\Phi}_i, \hat{Q}_j \} = i \hbar \). The quantity \( \mathbf{C}^{-1} \) is the inverse node capacitance matrix \[39, 77]\; and \( \hat{Q}_b = (0, 0, Q_b) \) (that is, node 3 is biased with the \( c \)-number polarization charge \( Q_b \)).

In this section, we focus on the properties of the Josephson potential \( \hat{U}_J (\hat{\Phi}) \). To obtain a more intuitive and compact picture of this quantity, we transform to the following dimensionless flux (phase) coordinates:

\[
\begin{align*}
\phi_1 &\equiv \phi_2 - \phi_3, \\
\phi_2 &\equiv \phi_3 + \frac{1}{2} (\phi_{\Delta} - \delta \phi^x - \delta \phi^y), \\
\phi_p &\equiv \frac{1}{2} (\phi_1 + \phi_2),
\end{align*}
\]

(8)

where we use lowercase \( \phi \) to indicate phase quantities according to: \( \phi \equiv 2 \pi \Phi / \Phi_0 \). As will become clear below, the subscripts '1', '2', and 'p' denote 'loop,' 'island,' and 'plasma' modes, respectively. In this representation, the Josephson potential becomes (with operator notation dropped for clarity):

\[
\frac{U_J (\phi_1, \phi_2, \phi_p)}{2E_{\Delta_0}(\phi_{\Delta})} = \beta (\phi_{\Delta}) \left[ 1 - \cos \frac{\phi_1}{2} \cos \phi_2 - \cos \frac{\delta \phi^x}{2} \sin (\phi_1 - \phi_2) \sin \left( \frac{\delta \phi^y - \phi_1}{2} \right) \right]
\]

\[
+ \sec \frac{\phi_{\Delta}}{2} \left[ 1 - \sin \frac{\delta \phi^y}{2} \sin \frac{\phi_2}{2} \cos (\phi_1 - \phi_2) \cos \left( \frac{\delta \phi^y - \phi_1}{2} \right) \right],
\]

(9)

where we have defined the effective DC SQUID Josephson energy \( E_{\Delta_0}(\phi_{\Delta}) \equiv 2E_{\Delta_0}(\phi_{\Delta}/2) \) and the ratio: \( \beta (\phi_{\Delta}) \equiv E_J / E_{\Delta_0}(\phi_{\Delta}) \) between the Josephson inductances of the DC SQUIDs (the flux tunneling elements) and that of the larger loop junctions (functioning as the inductance appearing ‘across’ them)\[8\]. In this circuit, just as in the case of 3- and 4-junction flux qubits, the plasma mode \( \phi_p \) (the symmetric oscillation across the two larger Josephson junctions), can in most cases of interest be treated as a ‘bystander,’ in the sense that the energy barrier between fluxon states along this direction, as well as its characteristic oscillation frequency, are both usually much larger than the energy scale of interest for the qubit, such that to a good approximation the \( \phi_p \) mode does not participate in relevant phenomena at that energy scale.

\[\text{Note that the charge variables must also be transformed to maintain the canonical commutation relations; see appendix.}\]

\[\text{This parameter is related to the quantity } \alpha \text{ which is conventionally used to describe 3- and 4-junction flux qubits according to } \alpha = 1/2 \beta.\]
In this sense the JPSQ, although it is intended to function as a persistent-current flux qubit, has something in common with the well-known charge [75, 79] (or transmon [80–82]) and quantronium [83] qubits, whose physics at a high level is that of a superconducting island with Josephson coupling to ground. As illustrated in figure 4, the JPSQ in fact combines elements of these two classes of superconducting qubits (charge and flux), usually viewed as qualitatively distinct. From a flux qubit point of view, the JPSQ exhibits a double-well potential in the form of two weakly-coupled 1D periodic potentials (assuming tight binding); from a charge qubit point of view, the JPSQ exhibits a 1D periodic Josephson potential whose unit cell contains a double-well. In fact, the JPSQ can be continuously varied between regimes of flux qubit and charge qubit behavior by changing its design and bias parameters.

Figures 4(a), (b) show the potential energy surface of equation (9) obtained by setting \( \Phi_0 = 0 \), for \( \delta \Phi^2 = \delta \Phi_0^2 = 0 \). Whereas the essential physics of a conventional flux qubit can be approximately described in terms a phase particle moving in a one-dimensional double-well potential (see figures 1(b), (c) [39]), whose position corresponds to the gauge-invariant phase across the qubit’s loop inductance, the JPSQ is fundamentally different in that at least two dynamical variables are needed to capture even its qualitative properties. At a very high level, this difference can be associated with the fact that Josephson symmetry (under translations of \( \Phi_0 \)) does not play an essential role in the important low-energy properties of the flux qubit, while it is essential to those of the JPSQ: for the flux qubit of figure 1(a), if we neglect spurious fluxon tunneling through the larger, loop junctions, the only non-trivial tunneling path for a fluxon is the usual one connecting the two persistent current states. However, as shown in figures 3 and 4, for the JPSQ there is a closed fluxon tunneling path which encircles the island, corresponding to a discrete Josephson symmetry. This symmetry is highlighted in figure 4(b) by representing \( \Phi \) (the phase of the island) as an angular coordinate [37].

Figure 4(b) illustrates how the two fluxon tunneling paths between persistent current states shown in figure 3(a) appear on the 2D potential surface, as red and blue arrows, and how they correspond to motion in the two angular directions along \( \Phi \). This coordinate can be viewed as the angular coordinate of a fluxon circling around the island (passing through the DC SQUIDs’ junction barriers), one period of which corresponds to a voltage pulse on the island of area \( \pm \Phi_0 \), with the sign determined by the direction of motion. When there is an offset charge on the island, the system becomes sensitive to this direction, via the second term of equation (7); when

\[
\Phi_0 = 0
\]

is continuously varied between regimes of flux qubit and charge qubit behavior by changing its design and bias parameters.
the island is polarized with exactly half a Cooper pair, the resulting \( \pi \) relative phase shift produces the destructive interference shown in figure 3(c). We will see below that this can also be viewed as a geometric phase shift.

The two local minima of equation (9), illustrated in figure 4(b), correspond semi-classically to the two persistent current states, and are given by (for \( \delta \phi^z = 0 \), and to first order in \( \delta \phi^x \)):

\[
(\phi_{\text{L}}, \phi_{\text{R}}, \phi_p) = \begin{cases} 
(-\phi_{\text{L,m}} - \phi_{\text{R,m}}, 0) & \text{with: } \phi_{\text{L,m}} \equiv \frac{\pi}{2} - \delta x, \quad \phi_{\text{R,m}} \equiv 2\theta 
\end{cases}
\]

and we have defined:

\[
\theta(\phi_\Delta) \equiv \cot^{-1}[\beta(\phi_\Delta)]
\]

\[
\delta x \equiv \frac{\delta \phi^x}{2} \cot \theta \tan \frac{\phi_\Delta}{2},
\]

The saddle-point energy barriers separating these two persistent current states are given by (again for \( \delta \phi^z = 0 \), and to first order in \( \delta \phi^x \)):

\[
E_{BT} \equiv E_b \pm \delta E_b
\]

\[
E_b = 2 \tilde{E}_b(\phi_\Delta) \tan \frac{\theta}{2}
\]

\[
\delta E_b = E_b(1 + \sec \theta) \delta x,
\]

where \( \pm \) here and below refer to the two paths \( T \in \{L, R\} \). From equations (12), and illustrated in figure 4(f), we see that when \( \delta \phi^z \approx 0 \), one of the barriers is lowered and the other is raised, as required to realize the situation pictured in figure 3(d).

In the limit where quantum phase fluctuations about these classical minima and the tunneling between them are negligible, the average magnitude of the corresponding equal and opposite persistent supercurrents circulating in the loop can be written semi-classically as:

\[
\bar{I}_{\text{IPSO}} \approx \tilde{I}_{\text{Ca}}(\phi_\Delta) \cos[\theta(\phi_\Delta)],
\]

where \( \tilde{I}_{\text{Ca}}(\phi_\Delta) \equiv \tilde{E}_b(\phi_\Delta) \times 2\pi / \Phi_0 \) is the effective critical current of each DC SQUID. The corresponding result for the semi-classical persistent current of the flux qubit is:

\[
\bar{I}_{\text{flux}} \approx \tilde{I}_{\text{Ca}}(\phi_\Delta) \sin[2\gamma(\phi_\Delta)],
\]

where the angle \( \gamma(\phi_\Delta) \) is defined by: \( 2 \cos[\gamma(\phi_\Delta)] \equiv \beta(\phi_\Delta) \). Equations (13) and (14) are plotted in figure 5, and they exhibit a qualitative difference between the two circuits when viewed as persistent-current qubits. The flux qubit result is shown in red, and displays the well-known behavior that a double-well potential only exists when \( 0.5 < \beta < 1 \); that is, the Josephson inductance of the small junction must be less than that of the loop. Within this range, the persistent current varies strongly, having a peak at \( \beta = 1/\sqrt{2} \). By contrast, the IPSQ can have a well-defined classical persistent current for any value of \( \beta \), and in fact the current asymptotically approaches \( \tilde{I}_{\text{Ca}} \) as \( \beta \) gets larger, that is, as the loop inductance becomes negligible compared to that of the DC SQUIDs. Note that this is a regime where the flux qubit does not have a double-well potential at all.

This qualitative difference can be intuitively understood by examining the effective inductive division occurring in the two cases. For the IPSQ, the top of the potential barrier between persistent current states
corresponds to a gauge-invariant phase difference of $\pi/2$ across each of the two DC SQUIDs, such that their effective Josephson inductances formally diverge. The SQUIDs therefore look approximately like two series current sources, supplying their critical current $I_{cl}(\phi_b)$ to a ‘load’ consisting of the two larger junctions (as long as $\beta$ is not close to 1). For the tunable flux qubit, however, the top of the potential barrier between persistent current states corresponds to a gauge-invariant phase difference of $\pi$ across the (single) DC SQUID, which is its point of minimum (and negative) Josephson inductance. The result is that unlike the JPSQ, the persistent current of the flux qubit is determined by a balance between the Josephson inductance of the DC SQUID and that of the two loop junctions in series, which are of similar magnitude. Only near the point $\beta = 1/\sqrt{2}$, where the potential minima correspond to $\pm \pi/2$ across the DC SQUID (and where its Josephson inductance diverges) does the persistent current approach $I_{cl}(\phi_b)$.

2.2. Aharonov–Casher interference between fluxon tunneling amplitudes

Tunneling between the JPSQ’s two persistent current states, denoted here by $|+z\rangle$ and $|-z\rangle$, can be described using the Euclidean (imaginary time) path-integral representation for the transition amplitude between them [84, 85]:

$$\langle -z | e^{-\sum_{\tau}^T} + z \rangle = \oint D[\dot{\phi}(\tau)] e^{-S[\dot{\phi}(\tau)]}$$

where $\tau \equiv it$, $S_{\tau}[\dot{\phi}(\tau)]$ is the Euclidean action associated with a path $\dot{\phi}(\tau) \equiv (\phi_{L}(\tau), \phi_{R}(\tau))$, and $D[\dot{\phi}(\tau)]$ indicates a functional integral over all such paths obeying the instanton boundary conditions:

$$\dot{\phi}_{L}(\tau) \rightarrow \begin{cases} (-\phi_{L_{in}}, -\phi_{L_{in}}) & \tau \to -\infty \\ (\phi_{L_{in}}, \phi_{L_{in}}) & \tau \to \infty, \end{cases}$$

that is, connecting the classical minima of the two potential wells (see, equations (10), (11)). Anticipating the validity of a dilute instanton-gas approximation [84, 85], we can focus on the subset of paths satisfying these boundary conditions which correspond to a single instanton (i.e. those that pass over the barrier only once): $\phi^{(1)}_{L}(\tau)$ and $\phi^{(1)}_{R}(\tau)$, corresponding to those suitably close to one or the other of the two classical, extremal paths $\phi^{cl}_{L}(\tau)$ and $\phi^{cl}_{R}(\tau)$:

$$\langle -z | e^{-\sum_{\tau}^T} + z \rangle^{(1)} = \sum_{T \in \{L, R\}} \oint D[\dot{\phi}_{L}^{(1)}(\tau)] e^{-S_1[\dot{\phi}_{L}^{(1)}(\tau)]},$$

where the superscript (1) indicates a restriction to single-instanton paths.

Based on equation (7), we can write the Euclidean action associated with each of these classical paths as a sum of two distinct contributions:

$$S_{\tau}[\phi^{(1)}_{T}(\tau)] = S_{tb} + S_{tb},$$

where the first is due to interaction with the bias source polarizing the island with charge $Q_b$, and the second due to the tunneling dynamics. The first term is given by:

$$S_{tb} = \frac{i}{\hbar} \int d\tau Q_b \cdot \frac{d\phi^{(1)}_{T}(\tau)}{d\tau} = \frac{i}{\hbar} Q_b \cdot \frac{d\phi^{(1)}_{T}(\tau)}{d\tau},$$

$$= -i\frac{Q_b}{2e}[\pi + 2\delta x] \equiv -id_{tb},$$

where this contribution to the Euclidean action is imaginary because the voltage $V = d\phi/dt \to -iV$ under the Wick rotation to imaginary time, the two signs in the last line refer to the two paths $T \in \{L, R\}$, and we have made use of equation (11). Using the fact that the result of equation (20) depends only on the endpoints and not the specific path connecting them, we can factor these contributions out of the functional integrals in equation (17) to obtain:

$$\langle -z | e^{-\sum_{\tau}^T} + z \rangle^{(1)} = \sum_{T \in \{L, R\}} \oint D[\dot{\phi}_{T}^{(1)}(\tau)] e^{-S_1[\dot{\phi}_{T}^{(1)}(\tau)]},$$

where the phase shifts of each path due to the Aharonov–Casher effect are now apparent in the complex phase factor in front of the functional integral describing the dynamical part of the transition amplitude. In the special, symmetric case where $d\phi^{(1)} = d\phi^{(1)} = 0$, we have $S_{tb} = S_{tb} \equiv S_0$ and this becomes:
\[
(-z|e^{-z^2}| + z)^{(1)} \rightarrow 2 \cos \frac{q_b}{2} \oint D\phi_{\tau}^{(1)}(\tau) e^{-S_{\text{fl}}(\phi_{\tau}^{(1)}(\tau))},
\]

where \(q_b \equiv 2\pi / Q_b\) is the dimensionless polarization charge applied to the island separating the two tunneling paths. As expected from our intuition for the Aharonov–Casher effect, this quantity directly gives the relative phase between the two transition amplitudes for these paths. This relative phase shift can be viewed as a geometric effect \cite{86}, associated with the total area enclosed by their path difference in \((\Phi, Q)\) phase space \cite{64, 87}, as indicated by equation (19)\(^7\).

### 2.3. Dipole moments of the JPSQ

The dynamical part of the functional integrals of equations (15), (16) can be approximately evaluated \cite{appendix}, and combined with the result with equations (13), to obtain the following effective Hamiltonian for the two lowest-energy states (up to an overall energy offset, and a static rotation around \(z\)), valid for small \(\delta \phi^x, \delta \phi^y, \) and arbitrary \(q_b\):

\[
\hat{H} = -\hat{\sigma}^x \delta \phi^x E_{\phi_{\Delta}}(\phi_{\Delta}) \cos \theta + \frac{\hbar \Omega_{\text{ge}}}{2} \left[ \hat{\sigma}^y \cos \frac{q_b}{2} - \hat{\sigma}^x \sin \frac{q_b}{2} \left( S_0 - \frac{1}{2} \right) \delta \phi^z \right],
\]

where the quantities \(\Omega_{\text{ge}}, S_0,\) and \(\delta \phi^z\) are enumerated in the appendix. When \(q_b = e(q_b = \pi)\) and \(\delta \phi^x = \delta \phi^y = 0\), we have \(\hat{H} = 0\), corresponding to the emulated zero field point around which the JPSQ is designed to operate\(^8\). Focusing on the regime near this point, we formally re-write the Hamiltonian of equation (23) in terms of a Zeeman-like interaction between an effective vector dipole moment operator \(\hat{\mu}\), and an effective field \(\hat{\mathcal{E}}\), as follows:

\[
\hat{H} \equiv - \left( \frac{d\hat{H}}{d\hat{\mathcal{E}}} \right) \cdot (\delta \Phi^x, \delta Q^y, \delta \Phi^z),
\]

where we have defined \(\delta Q^y \equiv Q_b - e,\) and \(\hat{\mu}\) is given at zero field by:

\[
\hat{\mu} \equiv (I^x \hat{\sigma}^x, V^y \hat{\sigma}^y, I^z \hat{\sigma}^z).
\]

\[
I^x = e\Omega_{\text{ge}}(S_0 - \frac{1}{2}) \delta \phi^z \mu_v,
\]

\[
V^y = \frac{\phi_0 \Omega_{\text{ge}}}{4}
\]

\[
I^z = I_{\phi_{\Delta}}(\phi_{\Delta}) \cos \theta.
\]

These dipole moments govern the strength with which the JPSQ couples (in the computational space) to external fields, including classical fields used to manipulate it, fields from other qubits that are used to engineer entangling interactions, and fields from its noise environment that are responsible for decoherence. This is, of course, a general feature of nearly any qubit system, that the same interactions with external fields that provide a mechanism for using the qubit also open the door to decoherence processes.

Figure 6 shows a comparison between the predictions of equations (23)–(25) and full numerical simulations of the circuit, performed using a generalization of the methods described in \cite{63, 65, 66}. The abscissa for all of these plots is the parameter \(K(\phi_{\Delta})\), which describes the ratio between the Josephson inductance of the DC SQUIDs and that of the large junctions. Results are shown for different values of the dimensionless DC SQUID admittance \(y_{ja}\), and dimensionless island capacitance \(r_{c}\), defined according to:

\[
y_{ja} \equiv \frac{R_{Q}}{Z_{ja}},
\]

\[
r_{c} \equiv \frac{C_{j}}{4C_{ja}},
\]

where \(Z_{ja} \equiv \sqrt{I_{ja}/2C_{ja}}\), and \(R_{Q} = \hbar/4e^{2}\) is the superconducting resistance quantum. The parameter \(y_{ja}\) describes the importance of quantum phase fluctuations across each DC SQUID, with the large-\(y_{ja}\) limit corresponding to semiclassical behavior (small phase fluctuations). The parameter \(r_{c}\) characterizes the capacitive division between the self-capacitance of the island and that of the four Josephson junctions connected to it. The leftmost column of figure 6, panels (a) and (e), shows the energy splitting \(\Omega_{\text{ge}}/2\pi\) in a more conventional flux-qubit-like regime, where \(\delta \phi^x = \delta \phi^y = Q_b = 0\) (corresponding to zero \(z\) and \(x\) fields, and a maximal \(y\) field). The

\(^7\) Note that since \(\phi_{\Delta}\) can be viewed as an angular coordinate whose corresponding angular momentum is \(q_b\), the geometric phase \(q_b\) can also be viewed as arising from a Sagnac-like effect, where the island charge offset \(q_b\) corresponds to a constant angular ‘rotation’ of the system.

\(^8\) Note that in varying \(\delta \phi^z\) around this zero emulated field point, the effective Hamiltonian can be proportional to either \(-\hat{\sigma}^z\) or \(+\hat{\sigma}^z\) (the latter of which is impossible to realize with a conventional flux qubit \cite{40}).
remaining three columns show the three components of the dipole moment, $I^x, V^y$, and $I^z$, near the qubit’s emulated zero-field point. Both the energy splittings and the dipole moments decrease strongly with increasing $r_C$, a trend that can be understood from equations (A.5) and (A.12); in the $\beta \gg 1$ ($\theta \ll 1$) regime of most interest here, the effective inverse capacitance that acts as a ‘mass’ for fluxon tunneling is mostly controlled by $C_J$. Note that this is somewhat different from an ordinary flux qubit, where the corresponding tunneling ‘mass’ is controlled largely by the capacitance across the (single) small junction (or DC SQUID in the case of a two-loop qubit [55, 56]).

We make two general remarks about the agreement between our analytic results and the full simulations shown in figure 6. First, the agreement is much better for the larger value of $y_J$ (top row) than for the smaller (bottom row). The agreement is also better for larger values of $r_C$. Both of these trends are to be expected, since both of these parameters control the validity of the semiclassical (stationary phase) approximation used to derive equation (23); that is, larger $y_J$ and $r_C$ both result in smaller quantum phase fluctuations. Second, in many cases the agreement is substantially worse for small $\beta$. This is also to be expected, since as $\beta$ is decreased, the relative importance of quantum fluctuations along the $\phi_x$ direction increases, as can be deduced from equation (9) by calculating the effective impedances for small oscillations about the potential minima along the three mode directions.

2.4. Numerical simulation of realistic JPSQ circuits
Although the circuit of figure 3(b) and the corresponding results of equations (23)–(25) capture the most important qualitative features of the JPSQ, full numerical simulation and a more detailed circuit description are needed to go beyond this. Figure 7 shows a JPSQ circuit, now including finite geometric loop inductances in the DC SQUIDs, and figure 8 shows its low-lying energy levels, obtained from numerical simulation [65]. Note that here we simulate an RF-SQUID-like variant of the JPSQ, with the two loop junctions of figure 3(b) replaced by a linear inductor; just as in the case of three-junction and RF SQUID flux qubits, these two methods for realizing a loop inductance produce nearly equivalent results, as can be seen by comparing the filled and open symbols in figure 6. We emphasize that our choice between the two variants is purely a matter of convenience: whereas for our analytic treatment the Josephson loop inductance results in more compact expressions, for numerical simulation the linear inductance produces smaller quantum fluctuations. The simulated energy levels for the circuit of figure 7 are plotted in figure 8 as a function of both $\delta \phi_x$, $\delta \phi_y$, and $Q_b$, for parameters that correspond to those of figure 6, with $\beta = 15$, $r_C = 0.1$. In contrast to the analytic treatment in the previous section, numerical simulation allows us to look in detail at the higher energy levels of the circuit outside the computational subspace, which is important for understanding when these higher levels can safely be neglected. As shown in figures 8(b) and (d), at energies greater than $\sim E_b$ (the height of the fluxon tunnel barrier, labeled in figure 8 with a vertical arrow) above the ground state, the $Q_b$-dependence increasingly looks like that of the simple charging Hamiltonian for the circuit’s island, given by (up to a constant energy offset):

As described in [63, 65, 66], this simulation fully includes all of the electromagnetic modes of a given lumped circuit, and introduces perturbative approximation only in capturing the low-energy effects of high-energy, virtual excitations of these modes.
where $E_{\text{cl}} \equiv \frac{e^2}{2} C_{\text{f}}^{\text{tot}}$ is the island charging energy and $C_{\text{f}}^{\text{tot}} = 4C_{\text{Ja}} + C_{\text{f}}$ for the circuit of figure 7. Referring back to figure 4(a), we can immediately get an intuitive picture for the energy of the lowest excited states above the computational space: as the energy increases above the height of the fluxon tunnel barriers, the double-well potential in each unit cell along the $\phi_1$ direction becomes increasingly unimportant, and the states look more and more like plane waves (charge states) along the $\phi_1$ direction, governed approximately by equation (28). This allows us to identify the next two higher excited states near $Q_b = e$ approximately with the two island charge...
states \((\hat{Q}_1)/2\pi = \{-1, 2\}\), whose energy above the qubit levels is \(\approx 8E_{Cq}\). These higher excited states will act as an upper bound on the energy scale over which we can treat the JPSQ as a two-level quantum system (similar to the so-called ‘anharmonicity,’ which is used to specify the importance of the second excited state in the context of transmon and flux qubits).

Another piece of information evident in figure 8 is the importance of the island parity, indicated by black and blue lines for even and odd island parity states, respectively. Focusing on the region around \(\delta f = 0\), \(Q_{fl} = \epsilon\), we see that the odd-parity ground state is actually lower in energy than the two lowest even-parity states which form the computational subspace. This means that these even parity states can decay into the lower-energy odd parity ground state near this point, if a quasiparticle tunnels onto the island inelastically [88]. Hence, the so-called ‘parity lifetime’ of the island, a quantity well-known in the literature of superconducting and semiconducting qubits [88–92], is crucially important in assessing the potential coherence of the JPSQ. Using the fact that the quasiparticle tunneling operator for a Jl which connects its even and odd-parity charge subspaces is proportional to \(\sin(\hat{\phi}/2)\), where \(\hat{\phi}\) is its gauge-invariant phase difference operator [88, 89], one can readily verify that these inelastic, island-parity-changing quasiparticle tunneling events are strongly allowed in the JPSQ for all bias conditions. As a result of this, both the ground and excited states in the even parity computational subspace will have decay rates \(1/T_1\) of the order of the quasiparticle current spectral density \(S_{qpl}(\omega)\) defined in [88]. For Aluminum junctions, typical quasiparticle densities, and parameters in the range considered here, these lifetimes could be as short as the \(\sim \mu s\) range. Fortunately, there exists a method for almost completely suppressing these processes, which has been used in both superconducting and semiconducting circuits to increase island parity lifetimes to the millisecond range and beyond [90–92]. Referring to the circuit of figure 7, one need only use a higher-gap superconducting material for the island as compared to the rest of the circuit, such that quasiparticles occupying the island see a higher potential energy. Once this potential barrier becomes substantially larger that the sum of the thermal energy and the maximum emulated Zeeman energy, the circuit is effectively protected from noise-induced parity-changing transitions.

2.5. Discussion of JPSQ coherence and comparison with existing superconducting qubits

Assuming that the parity switching discussed above can be circumvented as in previous works [90–92], the dominant intrinsic sources of noise in these circuits will be the same as for other superconducting qubits. These can be described in terms of charge and flux noise, both of which exhibit high-frequency and ‘1/f-like’ low-frequency components that have been observed under a variety of conditions [57, 66, 93–98]. As described above, the sensitivity of the JPSQ to charge and flux noise can be described simply in terms of its dipole moments (see, equation (25)). Table 1 shows these moments calculated for three sets of JPSQ parameters, labeled cases A, B, and C, corresponding to circuit designs with increasing magnetic dipole moments (with \(I^f \sim I^z\) in each case) spanning a range from \(\sim 40\) to \(\sim 250\) nA. Because the JPSQ is engineered to emulate a vector spin-1/2, these moments are, by design, approximately independent of field around the emulated zero-field point, allowing decoherence processes to be viewed in the same simple manner used to describe the response of a spin-1/2 to field noise. In the presence of a non-zero externally-applied offset field, noise fluctuations are naturally divided into their longitudinal and transverse components, relative to the axis of that field. Longitudinal noise fields cause the spin’s Larmor precession frequency (Zeeman energy) to fluctuate, resulting in so-called ‘dephasing’ (\(T_2\) processes) whose magnitude depends mostly on the low-frequency content of the noise power spectrum. These processes are represented in the table by the quantity \(\dot{\Gamma}_{f0}^{\Delta}\), defined in [96, 97], as the rate coefficient in the Gaussian (for 1/f noise) expression for the envelope decay of the system’s response to a spin-echo pulse sequence. Transverse noise fields cause the spin’s precession axis to rotate, resulting in population transfer \(T_1\) processes in the computational basis that depends mostly on the noise power spectral density at the Larmor frequency. These processes are described in the table by the quantity \(T_1\), the lifetime of the qubit’s excited state.

For comparison, table 2 provides typical values for existing, state-of-the art superconducting qubits. Its first three columns show values for three demonstrated examples of tunable flux qubits, to which the JPSQ is most
sensibly compared\textsuperscript{12}, and its final column the well-known transmon qubit used for nearly all gate model applications [80–82]. Broadly speaking, tables 1 and 2 exemplify the fact that unlike the transmon qubit, whose simplicity results in very little design freedom, the dipole moments of JPSQs and flux qubits can vary widely, according to the needs of the designer: tunable flux qubits have been used with $I^{\text{ext}}$ values ranging from the $\sim 50 \, \text{nA}$ in recent capacitively-shunted flux qubits [66] (with coherence times as high as $\sim 50 \, \mu \text{s}$, to $\sim 3 \, \mu \text{A}$ in the machines from D-wave systems [16, 17, 37, 38] (with coherence times in the range of tens of nanoseconds). Correspondingly, unlike the transmon qubit, whose coherence in the ideal case is largely set by fundamental material properties and physical geometry, the coherence of flux qubits and JPSQs also depends very strongly on the specific design requirements set by the system in which it is used (For example, the very large magnetic moments of the D-wave flux qubits, which are responsible for their low coherence, are required to produce the very strong pairwise interactions needed for that system). In spite of this fundamental difference, one simple coherence comparison that can still readily be made between the JPSQ and both flux and transmon qubits is in their sensitivity to high-frequency noise: the data for JPSQ cases A and B in table 1 clearly show that JPSQs can readily be designed with comparable or longer $T_1$ times than the flux qubits and transmon listed in table 2. (Since both flux qubits and the JPSQ have more than one partial $T_1$, the full $T_1$, given an assumed direction of applied field, can be obtained simply from the parallel sum of the rates for the two moments perpendicular to this field. For example, for JPSQ case A in an applied z field, one obtains: $T_1 = 540\times 300 = 190 \, \mu \text{s}$.)

In order to compare the dephasing in these circuits, we first note the following: both flux and transmon qubits have a so-called ’sweet spot,’ a bias point where their energy splitting becomes linearly insensitive to both low-frequency flux and charge noise (the latter is due to a large ratio of Josephson to charging energy, and is not dependent on bias). At such a bias point, we have, by definition: $(e|\hat{\mathbf{f}}|e) - (e|\hat{\mathbf{g}}|g) = 0$; that is, at a sweet spot the qubit’s static dipole moment is zero. This is obviously a useful property in some cases, as it allows the qubit to be decoupled from noise [96, 97]; however, it is incompatible with emulation of a vector spin-$1/2$ (note that this is simply a more general restatement of the conclusion of section 1). Therefore, the JPSQ, or any circuit engineered to emulate a vector spin-$1/2$, must by its very design be open to additional dephasing channels as compared to flux and transmon qubits, which cannot. For the JPSQ, this additional dephasing manifests itself two ways: First, compared to a conventional flux qubit, the JPSQ experiences dephasing due to noise in $\mathcal{F}^x$ at all values of $\Delta E^x$, whereas the flux qubit has a sweet spot when $\Delta E^x$ is at its minimum value, and is only subject to dephasing

\textsuperscript{12} Note that although fluxonium [74] is also a persistent current qubit, we have not included it in the present comparison because its persistent current is, by design, too small to be used for the direct spin emulation discussed here.

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**Table 1.** Dipole moments and coherence metrics for three JPSQ parameter cases. Cases A, B, and C are chosen to realize three different magnetic dipole moments, with $I^{\text{ext}} \sim I^{\text{ext}}$ in each case. We have fixed $\beta = 17$ and $\Phi_\Delta = 0.3 \Phi_0$ for all cases. Numerical values for the dipole moments include, for comparison, the analytic results of equations (25), and numerical simulations of the circuits in figures 3 and 7. From the latter, we derive the two coherence metrics $\Gamma_{se}$ and $T_1$, described in the text. For the former, we use the expected noise power spectral densities at $5 \, \text{GHz}$ (due to an effective, applied perpendicular field); $S_N(h \times 5 \, \text{GHz}) = (1.1 \times 10^{-6} \text{e}^2) / \text{Hz}$, and $S_N(h \times 5 \, \text{GHz}) = (4.3 \times 10^{-15} \Phi_0^2) / \text{Hz}$, derived from the results of [66] (obtained using data from a large number of flux qubits of varying design and over a range of bias conditions).

| JPSQ | $I^{\text{ext}}$ (nA) | $\mathcal{V}^x$ (μV) | $I^{\text{ext}}$ (nA) | Parameters |
|------|----------------|-------------|----------------|------------|
| Case A | $I^{\text{ext}} \sim 40 \, \text{nA}$ | | | |
| Analytic | Equation (25) | 43 | 2.6 | 69 | $E_\mu/h = 30 \, \text{GHz}$ |
| Simulated | Figure 3 | 34 | 2.2 | 41 | $C_{\text{se}} = 1.4 \, \text{fF}$ |
| | Figure 7 | 36 | 2.3 | 40 | $C_I = 60 \, \text{fF}$ |
| $\Gamma_{se}$ (μs$^{-1}$) | 1.2 | 2.7 | 1.3 | $y_1 = 5.1$ |
| $T_1$ (μs) | 540 | 300 | 430 | $r_{C} = 10$ |
| Case B | $I^{\text{ext}} \sim 100 \, \text{nA}$ | | | |
| Analytic | Equation (25) | 93 | 5.5 | 150 | $E_\mu/h = 66 \, \text{GHz}$ |
| Simulated | Figure 3 | 75 | 4.9 | 96 | $C_{\text{se}} = 2.6 \, \text{fF}$ |
| | Figure 7 | 79 | 5.1 | 91 | $C_I = 20 \, \text{fF}$ |
| $\Gamma_{se}$ (μs$^{-1}$) | 2.6 | 6.0 | 3.0 | $y_1 = 10$ |
| $T_1$ (μs) | 110 | 62 | 84 | $r_{C} = 1.9$ |
| Case A | $I^{\text{ext}} \sim 250 \, \text{nA}$ | | | |
| Analytic | Equation (25) | 250 | 15 | 410 | $E_\mu/h = 170 \, \text{GHz}$ |
| Simulated | Figure 3 | 200 | 12 | 250 | $C_{\text{se}} = 3.0 \, \text{fF}$ |
| | Figure 7 | 230 | 13 | 240 | $C_I = 0 \, \text{fF}$ |
| $\Gamma_{se}$ (μs$^{-1}$) | 7.4 | 15 | 7.7 | $y_1 = 18$ |
| $T_1$ (μs) | 14 | 9.7 | 13 | $r_{C} = 0$ |
Table 2. Dipole moments and coherence metrics for demonstrated superconducting qubits. The first three columns are examples of persistent-current flux qubits, in order of increasing persistent current, with parameters taken from the indicated references. The final column is for the transmon qubit widely used in gate-model applications, which is included as a point of reference.

| Qubit type       | 1x (nA) | Vx (μV) | 1' (nA) |
|------------------|---------|---------|---------|
| Two-loop flux    | Approx. | Dipole  |         |
| [56]             | 40°     | 7.6°    | 260°    |
|                  | Γx (μs⁻¹) | ~0°     | 1.3°    |
|                  | Tx (μs) |         | 28      |
| Fluxmon          | Approx. | Dipole  |         |
| [57]             | 40°     | 3.4°    | 700°    |
|                  | Γx (μs⁻¹) | ~0°     | 23°     |
|                  | Tx (μs) |         | 140     |
| D-wave flux      | Approx. | Dipole  |         |
| [37]             | 70°     | 2.8°    | 2600°   |
|                  | Γx (μs⁻¹) | ~0°     | 82°     |
|                  | Tx (μs) |         | 210     |
| Transmon         | Approx. | Dipole  |         |
| [60–82]          | N/A     | 5       | 5°      |
|                  | Γx (μs⁻¹) | ~0°     | 0.16°   |
|                  | Tx (μs) |         | 62°     |

* Flux qubits’ 1x and Vx go exponentially to zero as Fx → 0, while for a transmon 1' goes linearly to zero near its maximum energy splitting (often called a ‘sweet spot’). Here, we tabulate values for bias far from these points.

* The very small static charge dispersion of these qubits makes them highly insensitive to dephasing from charge noise.

* Since for flux qubits 1x → 0 as Fx → 0, noise in Fx only produces Tx processes when both Fx and Fl are non-zero, and the resulting rate depends in detail on both of these quantities.

* Viewed as a spin, the transmon can be described as experiencing a large offset field, which points purely along x' if its two junctions are symmetric. Therefore, flux noise in Fx can only produce non-zero transverse Tx processes if this symmetry is broken.

From Fx noise away from this point. As can be seen from the tables, the values for 1x are generally comparable to or smaller than 1'; so, while this additional sensitivity will result in larger total dephasing rates, the difference will be less than a factor of two.

Secondly, unlike both flux qubits and transmons, the JPSQ has a static Vx, making it sensitive to low-frequency charge noise, which is in general a more serious concern. Obviously, one would like therefore to minimize this sensitivity by keeping Vx as small as possible. However, from equations (25) and (A.12) we can see that the value of Vx is closely tied to that of Ix. In fact, the exponential dependence of 1x on the tunneling action S2 implies, for a given Ix (assumed to be set by external system requirements), that β tan(φ2x/2) is the only accessible parameter on which the resulting Vx depends more strongly than logarithmically (for β ≫ 1). This is evident in table 1, where the ratio Vx/Ix (with units of impedance) is nearly identical in all three cases, and leads to the same design conclusion as figure 5: that β should be made as large as possible (in practical cases the maximum permissible size of β will limited by the need to couple other circuits inductively to the qubit loop, and/or the maximum possible junction size if Josephson loop inductance is used). Examining the corresponding JPSQ dephasing rates due to low-frequency charge noise in table 1, we see correspondingly that they are in all three cases about twice as large as those due to flux noise coupling to Ix. So, although this additional charge noise dephasing is unavoidable for the JPSQ, it will at worst increase the total dephasing rate only by this modest factor.
two of the JPSQ circuits detailed in \[ \] have not previously been possible with engineered quantum devices. Figure 9 shows our couplers shown in types of coupling are qualitatively symmetric, as evidenced by the fact that the two panels, one with only coupling turned on and the other with only a result of the JPSQ interaction between qubits 1 and 3. As illustrated in Figure 9, the energies on the boundaries \( \Phi fi \) labeled upper one, labeled lower one, labeled \( \Phi fi \) show the lowest energy levels of this circuit, obtained by full numerical simulation [65] (see footnote 9). For panel (b), the zz coupler is turned on \( (\Phi_{cz} = \Phi_0/2) \), while for panel (c) the reverse is true \( (\Phi_{cz} = \Phi_0/2, \Phi_{cx} = \Phi_0) \). Focusing on the lowest four energy levels, which act as the two-spin computational subspace, we see that strong two-qubit zz and xx coupling are both possible with this circuit just by adjusting the flux controls of the two couplers. Furthermore, the two types of coupling are qualitatively symmetric, as evidenced by the fact that the two panels, one with only xx coupling turned on and the other with only zz, look nearly identical, except that the roles of \( x \) and \( z \) are permuted (compare to the strong non-equivalence seen for a two-loop flux qubit shown in figure 1(h)). This equivalence is a result of the JPSQ’s ability to emulate a true rotational symmetry at its zero-effective-field point while simultaneously maintaining a strong, vector dipole moment.

The circuits we have discussed so far exploited only the \( x \) and \( z \) components of the JPSQ dipole, which are both magnetic. Figure 10 shows a circuit which makes use of all three of its vector components. This circuit contains four coupled JPSQs (each like the one shown in figure 7) which together realize a single logical spin with passive quantum error suppression, based on a distance-2 Bacon–Shor code [8, 9]. The JPSQs are connected by strong, pairwise couplings, with magnetic \( zz \) couplings between the qubit pairs \( (1, 3) \) and \( (2, 4) \), and electric \( yy \) couplings between pairs \( (1, 2) \) and \( (3, 4) \). These interactions correspond to the two-qubit check operators for the code, and together they produce a two-dimensional low-energy logical computational subspace whose effective logical spin operators are products of two single-qubit physical operators, as shown in figure 10(a). Because the logical spin operators are two-local, the two logical states in the ideal case are protected against all single-qubit noise on the four constituent physical qubits; or, more precisely, any single qubit noise process should couple the two logical states only to higher-energy levels, separated by an energy ‘barrier’ whose height is of the order of the strength of the strong pairwise interactions. Therefore, single-qubit noise processes in the four constituent qubits in the ideal case must supply at least this amount of energy to affect the encoded logical spin state.

The circuit of Figure 10(b) also contains two additional two-loop RF SQUID flux qubit couplers (each like the couplers shown in Figure 9), which are used to implement a \( zz \) interaction between qubits 1 and 2, and an \( xx \) interaction between qubits 1 and 3. As illustrated in Figure 10, these two interactions correspond to the logical \( x \) space.
and $z$ operators, respectively. Adjusting the strength of these two-qubit couplings controls the effective field seen by the logical spin. Figure 11(a) shows the resulting dependence of the simulated energy levels (relative to the ground state energy) on these two logical field controls. Around zero field, the resulting energy barrier separating the logical states from the first excited manifold (corresponding to violation of one of the penalty interactions) is $\sim h \times 2.1 \text{GHz} \sim 5.3k_B T$ (for $T = 20 \text{ mK}$), corresponding to a Boltzmann factor of $5 \times 10^{-3}$. This number can be viewed as the relative thermal occupation of environmental photons at the low frequencies which separate the two logical states, and those that connect them both to the next higher set of states.

We now wish to evaluate the sensitivity of this protected qubit to decoherence arising from local, physical flux and charge noise. We define the c-number dipole moment $D_0[\hat{\mathcal{O}}]$ associated with a dipole operator $\hat{\mathcal{O}}$ and a two-dimensional subspace $\{|i\rangle, |j\rangle\}$ as:

$$D_0[\hat{\mathcal{O}}] = \frac{1}{2} \mathcal{W}[\mathcal{O}_{ss'}],$$

where $\mathcal{W}[\mathcal{O}_{ss'}]$ is the numerical range of the $2 \times 2$ matrix $\langle s|\hat{\mathcal{O}}|s'\rangle$, and $s, s' \in \{i, j\}$. We make the simplifying assumption that each circuit node experiences independent charge noise of the same magnitude, and each geometric inductor experiences independent flux noise of the same magnitude. We can then define the following effective total electric and magnetic dipole moments to which these common noise levels can be said to couple in the two-dimensional subspace $\{|i\rangle, |j\rangle\}$:

$$V_{ij} = \sqrt{\sum_{n \in \text{nodes}} (D_0[\hat{V}_n])^2},$$

$$I_{ij} = \sqrt{\sum_{l \in \text{inductors}} (D_0[\hat{I}_l])^2}$$

(30)

and which can be compared to the corresponding dipole moments of physical qubits listed in tables 1 and 2. Figures 11(b) and (c) show the results for these dipole moments in the logical computational subspace ($i, j = 1, 2$) derived from our simulations of the circuit of figure 10(b). The electric dipole moment is in the $\sim \text{nV}$ range,
the logical computational subspace. We estimate the resulting lifetimes of the logical states due to these transitions using the following states in the lowest excited manifold. As expected, these transitions are much stronger. We can readily spurious non-idealities in the circuit realization of the Hamiltonian of the circuit, as opposed to the couplings inside the computational space just discussed, which only occur due to thousands of times smaller than typical superconducting qubits, suggesting that its $T_1$ due to charge noise would be millions of times longer. The magnetic dipole moments shown in figure 11(c) are in the $\sim$nA range, set completely by the fact that we have intentionally engineered logical field controls via the two additional couplers (the shaded red and blue subcircuits in figure 10(b)) and in so doing to have effectively ‘poked a hole’ in the error suppression. The $\sim$nA scale of these magnetic dipoles results simply from choosing to produce an $\sim h \times 1$ GHz logical Zeeman splitting over the maximum coupler control flux tuning range of $\Phi_0/2$. Much smaller magnetic dipole moments could trivially be achieved simply by reducing the accessible tuning range of the Zeeman energy; in the extreme case where we remove the logical field control couplers entirely (reducing this tuning range to zero), our simulations show residual magnetic dipole moments less than 1 pA. The key question here, in assessing the potential coherence of such a logical spin, is the manner in which it would be used. In the context of quantum annealing, one would like to retain a wide tuning range for the Zeeman energy, and the $\sim$nA dipole moments shown in figure 10(e) are therefore likely the smallest possible, if a single loop is used for logical field control (restricting the full flux tuning range to $-\Phi_0/2$). However, this is already $\sim$100–10 000 times smaller than the magnetic dipoles of existing flux qubits (see table 2).

In addition to the spurious couplings of noise inside the logical computational space, we must also consider processes in which the system is excited out of this space, requiring the environment to supply a photon at an energy equal to that of the energy barrier ($\Delta E_I$ in figure 11(a)). Although these processes are suppressed by the low temperature, they are also ‘allowed’ transitions for single photons coupling locally to the circuit, as opposed to the couplings inside the computational space just discussed, which only occur due to spurious non-idealities in the circuit realization of the Hamiltonian of figure 10(a). Figures 11(d) and (e) show the effective total rms dipole moments for these transitions from the two computational states to all of the states in the lowest excited manifold. As expected, these transitions are much stronger. We can readily estimate the resulting lifetimes of the logical states due to these transitions using the following flux and charge noise amplitudes at the frequency corresponding to the barrier height $\Delta E_I/h \sim 2.1$ GHz, derived from the work of: $S_N(2.1$ GHz $) \sim (7.1 \times 10^{-8} e)^2$/Hz, and $S_N(2.1$ GHz $) \sim (6.4 \times 10^{-11} \Phi_0)^2$/Hz. Assuming a

\[ \text{Figure 11. Simulation of the passive quantum error suppression circuit of figure 10. Numerical parameter values used here for the JPSQs are: } E_{\text{flux}} = h \times 159$ GHz, $C_{\text{J}} = 0.96$ fF, $C_I = 1.0$ fF, $I_J = 850$ pH, $I_{\text{J}} = 100$ pH, $\Phi_0 = 0.4\Phi_0$. Note that smaller capacitances and larger $E_{\text{flux}}$ are used here, as compared to table 1, to increase the JPSQs $I'$ and $V'$ dipole moments and the resulting energy barriers due to the $zz$ and $yy$ penalty interactions (with parameters $M_{zz} = 110$ pH and $C_{yy} = 30$ fF). For the RF SQUID couplers, we used: $E_I = h \times 497$ GHz, $C_I = 6$ fF, $I_{\text{L}} = 120$ pH, $I_{\text{J}} = 10$ pH, $M_{\text{L}} = M_{\text{J}} = 20$ pH. These values were chosen to minimize the flux noise sensitivity induced by the logical field control, for the pictured $h \times -1$ GHz maximum logical Zeeman splitting. Panel (a) shows the low-lying energy levels for the circuit (relative to the ground state energy), obtained using the methods of [65], (see footnote 9) as a function of the two logical field control fluxes. The purple arrow indicates the energy gap between the two computational logical states and the lowest manifold of excited states which violate one of the penalty interactions. Panels (b) and (c) are plots of the total average electric and magnetic dipole moments (see, equation (30)) of the logical states, with respect to physical noise fields, as a function of the logical $z$ and $x$ control fluxes (note that decoherence rates will in general scale with the square of these quantities). Panels (d) and (e) give the total rms electric and magnetic dipole moments, respectively, for excitation out of the logical space (from logical state $|0\rangle$ shown in red, and from logical state $|1\rangle$ in blue) into the first excited manifold of states $|b\rangle$ shown in (a), separated by the energy barrier shown with a magenta arrow.

\[ \text{Note that for the ideal case illustrated in figure 10(a), this electric dipole would be identically zero. However, for the actual circuit of figure 10(b), additional effective interactions (mediated by higher excited states) allow local charge noise to couple to two-qubit operators in the logical computational subspace.} \]
thermal environment at $T = 20$ mK so that the rates of absorption and emission processes at this frequency are related by the Boltzmann factor, we find in the region near zero logical field, a lifetime of $\sim 10$ s due to charge noise, and 1.5–6 ms due to flux noise. The latter, in fact, appears to be the coherence-limiting process for the parameters we have chosen, though it can likely be further improved by additional optimization of circuit parameters beyond that carried out here, with the specific goal of reducing the influence of these high-frequency magnetic transitions.

To employ such logical spins in a quantum annealing machine requires static, pairwise, logical Ising interactions between them. These would correspond to physical four-qubit static interactions, which have not yet been demonstrated experimentally. However, due to their relevance for applications such as adiabatic topological quantum computing [112] and adiabatic quantum chemistry simulation [22–24], there are already several concrete proposals for realizing them [99–102]. A protected qubit like that shown in figure 10 could also be used in gate model applications. In the simplest case, the logical field control could be used to implement single qubit rotations (via Larmor precession around the emulated vector magnetic field), though two-qubit gates would still require four-physical-qubit interactions in some form. However, since only pulsed operations are required in a gate model context, they would not need to be static nor very strong. One possibility would be to use the techniques described in [64], which are based on state-dependent geometric phases accrued by the system when coupled to a resonator and driven, and which are readily scalable to multi-qubit entangling interactions. In this mode of operation, the logical field controls would no longer be necessary (reducing the magnetic dipole moment for physical noise to the $\sim \text{pA}$ level, as mentioned above), and single and two-logical-qubit gates would be realized via selectively modulating the couplings between either two or four physical JPSQs, respectively, and a common resonator.

4. Two-island JPSQ circuit for emulation of a 3D magnetic moment

So far, we have discussed a JPSQ circuit that can be said to emulate a quantum spin-$1/2$, insofar as it has three physical operators that can be engineered to obey the canonical commutation relations in the computational subspace of the lowest two energy levels, while higher levels are kept relatively far away. However, only two of these operators are magnetic, and the remaining one is electric. In the case of figure 10, only strong, fixed electric interactions are needed, which can be realized with the simple capacitive couplings between islands shown in figure 10(b). However, in some of the applications mentioned in the introduction, one needs tunability of the couplings between all three components of the spins’ dipole moments. Although there are proposed methods for engineering tunable electric couplings between superconducting circuits [103], they tend to require the qubits to have sufficiently large electric dipole moments that charge noise is likely to become a major problem. Therefore, it would potentially be of great interest to realize a JPSQ circuit whose magnetic moment has three independent vector components that satisfy the spin-$1/2$ commutation relations, so that the well-established techniques for controllable magnetic coupling could be brought to bear to realize fully-controllable, anisotropic Heisenberg interactions.

Such a circuit is shown in figure 12. Here, we have added an additional fluxon tunneling path by including a third Josephson junction in the loop, and we bias the resulting two islands using separately-controllable voltages. As illustrated in figure 12, if the polarization charges on these two islands are both set to one third of a Cooper pair, and the three tunneling paths have the same amplitude, a completely destructive, three-path Aharonov–Casher interference again occurs, as in the two-path case discussed so far. Because of the $2\pi/3$ relative phase shifts between

![Figure 12. Emulating a Heisenberg quantum spin-1/2 with a JPSQ circuit. Panel(a) shows a circuit having two islands, and therefore three fluxon tunneling paths. The third tunneling path is realized with a single, fixed junction, while the other two are flux-tunable DC SQUIDs. The latter are biased with static flux offsets such that their fluxon tunneling amplitudes have flux sensitivities of equal magnitude. Panel (b) shows how by choosing the island charge offsets to be one third of a Cooper pair, the three tunneling amplitudes can be chosen to form an equilateral triangle in the complex plane, resulting in complete cancellation of tunneling. Panels (c) and (d) show how small changes to the two DC SQUID fluxes can then correspond to two orthogonal magnetic moments, via the common and differential modes of the two fluxes.](image-url)
paths, we need only adjust two of the three amplitudes to control two (emulated) orthogonal transverse field directions, which can be accomplished using differential and common mode bias fluxes coupled to the two DC SQUID loops. As in the circuit of figure 3, where a single island created both a magnetic and an electric transverse dipole moment, the two-island circuit of figure 12(a) has two orthogonal transverse electric dipole moments (in addition to the desired transverse magnetic dipole moments), corresponding to the common and differential modes of the two island voltages.

5. Conclusion

In this work, we have proposed for the first time a superconducting qubit circuit capable of emulating a spin-1/2 quantum object with a true, static, vector dipole moment, whose components can be chosen, to a large extent, by design. We analyzed this circuit in detail, providing some qualitative intuition for its basic properties, and validated this with full numerical simulation of its quantum Hamiltonian. Broadly speaking, our results indicate that JPSQs with reasonable design parameters, when compared to existing circuits of SQUID loops. As in the circuit of SQUID, solving for the extremal paths of equation (17) requires classically integrating the equations of motion for the coordinates $\phi_0, \phi_1$, taking into account the fact that the inverse capacitance matrix has non-degenerate eigenvalues (that is, the fictitious ‘phase particle’ has an anisotropic ‘mass’). Since we are seeking here to obtain simple, analytic expressions useful for understanding the qualitative physics of this circuit, we make the simplifying approximation illustrated in figure A1. For each group of paths $\phi^0_\tau \ (\tau \in \{L, R\})$, we rotate to a new orthogonal set of coordinates $(\phi_T, \phi^*_T)$ such that $\phi_T$ lies along the straight line in $(\phi_0, \phi_1)$-space connecting the initial and final potential minima and the intervening saddle point. We then make the approximation that all of the non-trivial dynamics occurs along $\phi_T$, taking the motion along the perpendicular direction $\phi^*_T$ to be harmonic and separable. The new coordinate $\phi_T$ and its conjugate momentum $q_T$, to leading order in $\delta \phi^0_T$, can be chosen to be:

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Note added in proof. Since acceptance of this paper we became aware of a related proposal: T. Satoh et al. 2015, arXiv:1501.07739.

Appendix. Approximate analytic calculation of JPSQ tunneling amplitude

Solving for the extremal paths of equation (17) in $2 + 1D$ (already neglecting motion along the $\phi_R$ direction as discussed in the text) requires classically integrating the equations of motion for the coordinates $\phi_0, \phi_1$, taking into account the fact that the inverse capacitance matrix has non-degenerate eigenvalues (that is, the fictitious ‘phase particle’ has an anisotropic ‘mass’). Since we are seeking here to obtain simple, analytic expressions useful for understanding the qualitative physics of this circuit, we make the simplifying approximation illustrated in figure A1. For each group of paths $\phi^0_\tau \ (\tau \in \{L, R\})$, we rotate to a new orthogonal set of coordinates $(\phi_T, \phi^*_T)$ such that $\phi_T$ lies along the straight line in $(\phi_0, \phi_1)$-space connecting the initial and final potential minima and the intervening saddle point. We then make the approximation that all of the non-trivial dynamics occurs along $\phi_T$, taking the motion along the perpendicular direction $\phi^*_T$ to be harmonic and separable. The new coordinate $\phi_T$ and its conjugate momentum $q_T$, to leading order in $\delta \phi^0_T$, can be chosen to be:
where we have chosen the overall scaling such that the sum of the lengths of the two tunneling paths is $2\pi$, such that the resulting effective 1+1D problem can be viewed as occurring on a circle, as illustrated in figure 4(e).

Using equations (A.1) and (9), we can evaluate the resulting 1D potentials along the two paths $U_T(\phi_T) \equiv U_f(\phi_T, \phi_T = 0)$, examples of which are shown in figures 4(f). Along these paths, the potential minima are located at:

\[
\phi_{T_m} \equiv \phi_m \pm \delta \phi_m
\]

\[
\phi_m = \frac{\pi}{N_{+}}
\]

\[
\delta \phi_m = -\varepsilon x \frac{N_{-}}{N_{+}}
\]

and the effective Josephson inductances $L_{T_m}^{-1} \equiv L_{m}^{-1} + \delta L_{m}^{-1}$ at these minima are given by:

\[
L_{m}^{-1} = 2L_{Ja}^{-1} \sin \theta \left(1 + \frac{4\theta^2}{\pi^2 \sin^2 \theta}\right)
\]

\[
\delta L_{m}^{-1} = -\varepsilon x \frac{8\theta L_{Ja}^{-1}}{\pi \cos \theta} \left[1 + \frac{4\theta}{\pi^2 \tan \theta} \left(1 - \frac{2 - 8 \sin^2 \theta}{N_{+}}\right)\right]
\]

\[
L_{Ja}^{-1} \equiv E_{Ja} \left(\frac{2\pi}{\Phi_0}\right)^2
\]

To obtain the corresponding inverse capacitances along the $q_T$ direction, we start from the inverse capacitance matrix in the \{ $q_p$, $q_q$, $q_T$ \} representation, given by:
with the definitions: $C^\text{tot}_f \equiv C_f + 2(2C_d||G_f)$, $C^\text{tot}_m \equiv C_m + C_f/2$, and $C^\text{tot}_r \equiv 2C_f + (G_f||4C_d)$. Transforming this into the $(q_f, q_m)$ representation (see, equation (A.1)), we obtain for the inverse capacitance along the $q_f$ direction $C^{-1}_f \equiv C^{-1} + \delta C^{-1}$:

$$
C^{-1} = \frac{1}{N^2 C_d} \left[ \frac{1}{1 + \frac{1}{\tan \theta \sec \frac{\alpha}{2}} + \frac{64\pi^2}{\pi^2 C_d}} \right],
$$

and

$$
\delta C^{-1} = \delta \alpha \frac{1}{N^2 C_d} \left[ \frac{64\pi^2}{\pi^2 C_d} \right] \left[ 1 + \cot \theta \cos \frac{\alpha}{2} \right],
$$

where $\delta \alpha \equiv C_f/4C_d$.

In order to find a simple analytic solution to the equations of motion in the two double-well potentials, we approximate them using the following sixth-order polynomial form:

$$
U_T \approx E_{fT} \left[ 1 - \left( \frac{\phi_T}{\phi_T} \right)^2 \right] + k_{6T} \left( \frac{\phi_T}{\phi_T} \right)^2.
$$

By construction, equations (A.6) have the same barrier height $E_{fT}$ and potential minima at $\phi_T = \pm \phi_{Tm}$ as the exact potential derived from equations (9) and (A.1). We can match the Josephson inductance at the local minima $\phi_{Tm}$ as well by choosing the (small) sixth order correction coefficient $k_{6T}$ to be:

$$
k_{6T} \equiv \frac{\phi_{Tm}^2 - Tm^{-1}}{8E_{fT}} - 1.
$$

Figure 4(g) illustrates these potentials for the left and right tunneling paths, where solid lines are the exact results, and dashed lines the approximation described by equation (A.7) (note that they are nearly indistinguishable). Using this form for the potential, we can readily find solutions for the imaginary-time equation of motion at zero energy, given by:

$$
\frac{C_T}{2} \left[ \frac{d\phi_T^2(\tau)}{d\tau} \right] - U_T[\phi_T^2(\tau)] = 0
$$

with the boundary conditions $\phi_T(-\infty) = -\phi_{Tm}$ and $\phi_T(\infty) = \phi_{Tm}$ (equivalent to real-time dynamics in the inverted potential $-U_T(\phi_T)$ at zero energy). The simple polynomial form of equation (A.6) allows equation (A.8) to be integrated to obtain:

$$
\phi_T^2(\tau) = \frac{\phi_{Tm} \tanh \left( \frac{\Omega_T(\tau - \eta_0)}{2} \right)}{\sqrt{1 + k_{6T} \sech^2 \left( \frac{\Omega_T(\tau - \eta_0)}{2} \right)}},
$$

where $\eta_0$ is an arbitrary position in imaginary time (known as the collective coordinate of the instanton), and $\Omega_T \equiv \sqrt{L_{Tm}^2 C_T^{-1}}$ is the frequency of small oscillations about the potential minima. Using this solution, we can evaluate the corresponding Euclidean action analytically by expanding in powers of the small parameter $k_{6T}$ and integrating:

Note that this transformation also generates non-zero off-diagonal elements of the inverse capacitance matrix, corresponding to electrostatic coupling between the new charge variables $q_f$ and $q_m$. We neglect this coupling here on the grounds that the transverse motion has a much higher energy scale than the tunneling of interest here, so that their effect will be small.
\[
S_{T_0} = \frac{1}{\hbar} \int_{-\infty}^{\infty} d\tau \left( \frac{G_T}{2} \left[ \frac{d\Phi_T^2(\tau)}{d\tau} \right]^2 + U_T[\Phi_T^2(\tau)] \right)
\]
\[
= \frac{G_T}{\hbar} \int_{-\infty}^{\infty} d\tau \left[ \frac{d\Phi_T^2(\tau)}{d\tau} \right]^2
\approx \frac{2}{3} \frac{\Phi_{Im}^2}{\hbar Z_T} \left[ 1 - \frac{2}{3} k_{T} + \ldots \right]
\]  
(A.10)

where \( Z_T = \sqrt{\frac{C^{-1}}{LCmT}} \). Using equations (A.2), (A.3), (A.5), and (A.7), we can now obtain:

\[
S_{T_0} \equiv S_0(1 + \delta s)
\]
\[
S_0 = \frac{16}{3} \frac{E_b}{\hbar \Omega} \left[ 1 + \frac{3}{5} k_{s} \right]
\]
\[
\delta s \approx \frac{\delta E_{ib}}{E_b} - \frac{1}{2} \left[ \frac{\delta L_{m}^{-1}}{L_{m}^{-1}} + \frac{\delta C^{-1}}{C^{-1}} \right]
\]  
(A.11)

where the two terms in the last line correspond to the \( \delta \phi^x \)-dependence of the barrier height and oscillation frequency, respectively, \( \Omega \equiv 1/\sqrt{L_{m}C} \) is the average single-well oscillation frequency, and we have neglected the small contribution to \( \delta s \) from the \( \delta \phi^x \)-dependence of \( k_{sT} \), by defining: \( k_{s} \equiv (k_{sL} + k_{sR})/2 \) (see equation (A.7)).

To get a more intuitive picture of the important parameter dependencies, we consider the large-\( \beta \) (small-\( \theta \)) limit, obtaining to leading order in \( 1/\beta \):

\[
\Omega \approx \omega_j \sqrt{\frac{\cos \frac{\phi_j}{2}}{(1 + \frac{\pi}{\beta}) \left( \frac{\pi^2 + 4}{\pi^2} \right)}}
\]
\[
S_0 \approx \gamma_{j} \tan \frac{\phi_j}{2} \sqrt{\beta \left( \frac{3}{5} \right)} \left( \frac{3 \pi^2 + 44}{15 \pi^2 + 4} \right)
\]
\[
\delta s \approx \beta \tan \frac{\phi_j}{2} \left[ 1 - \frac{1}{\pi} + \frac{2}{\pi^2 + 4} \right] \delta \phi^x,
\]  
(A.12)

where we have defined the bare junction plasma frequency \( \omega_j \equiv 1/\sqrt{L_{j}C_{j}} \). The dimensionless DC SQUID admittance \( \gamma_{j} \) defined in equation (26), describes the importance of quantum phase fluctuations across each DC SQUID, with the large-\( \gamma_{j} \) limit corresponding to semiclassical behavior (small phase fluctuations). This is evident from that fact that \( S_0 \propto \gamma_{j} \) above. This quantity, in combination with \( \beta \) and \( r_C \), are the fundamental dimensionless parameters of the circuit in this simplified model.

In order to perform the path integrals in equation (17), we generalize well-known results for the quartic potential [84, 85]. In computing the usual fluctuation determinant describing Gaussian fluctuations about each stationary path, we treat the effect of the small sixth-order term in each potential using first-order perturbation theory. We account for the contributions of both stationary paths by viewing our quasi-1D problem as a particle on a circle with two potential minima (see, figure 4(f)). We obtain, in the dilute instanton gas approximation [84, 85], the following result for the Euclidean transition amplitude:

\[
\lim_{\tau \to \infty} \langle -z e^{-\frac{q_0}{\Omega}} + z \rangle = \sqrt{\frac{\Omega}{\pi}} \exp \left[ -\frac{\tau}{\Omega} \Omega_{ge} \left( \cos \frac{q_0}{2} - i \sin \frac{q_0}{2} \left( 1 - \frac{1}{2} \right) \delta s \right) \right]
\]  
(A.13)

where we have defined the fluxon tunnel splitting frequency:

\[
\Omega_{ge} \equiv \Omega \sqrt{\frac{12 S_0}{\pi}} \left( 1 + \frac{4}{5} k_{s} \right) e^{-S_0}.
\]  
(A.14)

From equations (13) and (A.13), we obtain equation (23) (up to an overall energy offset, and a static rotation around \( z \), valid for small \( \delta \phi^x \), \( \delta \phi^s \), and arbitrary \( q_0 \).

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