ON BOTT-CHERN COHOMOLOGY OF COMPACT COMPLEX SURFACES

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Abstract. We study Bott-Chern cohomology on compact complex non-Kähler surfaces. In particular, we compute such a cohomology for compact complex surfaces in class VII and for compact complex surfaces diffeomorphic to solvmanifolds.

Introduction

For a given complex manifold $X$, many cohomological invariants can be defined, and many are known for compact complex surfaces.

Among these, one can consider Bott-Chern and Aeppli cohomologies. They are defined as follows:

$$
H^{•,*}_{BC}(X) := \frac{\ker \partial \cap \ker \overline{\partial}}{\text{im } \partial \cap \text{im } \overline{\partial}} \quad \text{and} \quad H^{•,*}_A(X) := \frac{\ker \overline{\partial}}{\text{im } \partial + \text{im } \overline{\partial}}.
$$

Note that the identity induces natural maps

$$
\begin{array}{cccc}
H^{•,*}_{BC}(X) & \downarrow & H^{•,*}_A(X) & \downarrow \\
H^{•,*}_{\alpha}(X) & \downarrow & H^{•,*}_{\partial}(X; \mathbb{C}) & \downarrow \\
H^{•,*}_{\alpha}(X) & \downarrow & H^{•,*}_A(X) & \\
\end{array}
$$

where $H^{•,*}_{\alpha}(X)$ denotes the Dolbeault cohomology and $H^{•,*}_{\alpha}(X)$ its conjugate, and the maps are morphisms of (graded or bi-graded) vector spaces. For compact Kähler manifolds, the natural map $\bigoplus_{p+q=0} H^{p,q}_{BC}(X) \to H^{•,*}_{\partial}(X; \mathbb{C})$ is an isomorphism.

Assume that $X$ is compact. The Bott-Chern and Aeppli cohomologies are isomorphic to the kernel of suitable 4th-order differential elliptic operators, see [19, §2.b, §2.c]. In particular, they are finite-dimensional vector spaces. In fact, fixed a Hermitian metric $g$, its associated $\mathbb{C}$-linear Hodge- operator induces the isomorphism

$$
H^{p,q}_{BC}(X) \cong H^{n-q,n-p}_{A}(X),
$$

for any $p, q \in \{0, \ldots, n\}$, where $n$ denotes the complex dimension of $X$. In particular, for any $p, q \in \{0, \ldots, n\}$, one has

$$
\dim_{\mathbb{C}} H^{p,q}_{BC}(X) = \dim_{\mathbb{C}} H^{p,q}_{A}(X) = \dim_{\mathbb{C}} H^{n-p,n-q}_{A}(X) = \dim_{\mathbb{C}} H^{n-q,n-p}_{A}(X).
$$

For the Dolbeault cohomology, the Frölicher inequality relates the Hodge numbers and the Betti numbers: for any $k \in \{0, \ldots, 2n\}$,

$$
\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\alpha}(X) \geq \dim_{\mathbb{C}} H^{k}_{\partial}(X; \mathbb{C}).
$$

Similarly, for Bott-Chern cohomology, the following inequality à la Frölicher has been proven in [3] Theorem A; for any $k \in \{0, \ldots, n\}$,

$$
\sum_{p+q=k} (\dim_{\mathbb{C}} H^{p,q}_{\alpha}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)) \geq 2 \dim_{\mathbb{C}} H^{k}_{\partial}(X; \mathbb{C}).
$$

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The equality in the Frölicher inequality characterizes the degeneration of the Frölicher spectral sequence at the first level. This always happens for compact complex surfaces. On the other side, in [3] Theorem B], it is proven that the equality in the inequality à la Frölicher for the Bott-Chern cohomology characterizes the validity of the ∂∂-Lemma, namely, the property that every ∂-closed ∂-closed d-exact form is ∂∂-exact too, [3]. The validity of the ∂∂-Lemma implies that the first Betti number is even, which is equivalent to Kählerness for compact complex surfaces. Therefore the positive integer numbers

\[ \Delta^k := \sum_{p+q=k} (\dim \mathcal{H}^{p,q}_{BC}(X) + \dim \mathcal{H}^{p,q}_{A}(X)) - 2b_k \in \mathbb{N}, \]

varying \( k \in \{1, 2\} \), measure the non-Kählerness of compact complex surfaces \( X \).

Compact complex surfaces are divided in seven classes, according to the Kodaira and Enriques classification, see, e.g., [3]. In this note, we compute Bott-Chern cohomology for some classes of compact complex (non-Kähler) surfaces. In particular, we are interested in studying the relations between Bott-Chern cohomology and de Rham cohomology, looking at the injectivity of the natural map \( H^{2,1}_{BC}(X) \rightarrow H^{2,1}_{dR}(X; \mathbb{C}) \). This can be intended as a weak version of the \( \partial \partial \)-Lemma, compare also [10].

More precisely, we start by proving that the non-Kählerness for compact complex surfaces is encoded only in \( \Delta^2 \), namely, \( \Delta^1 \) is always zero. This gives a partial answer to a question by T. C. Dinh to the third author.

**Theorem 1.1.** Let \( X \) be a compact complex surface. Then:

(i) the natural map \( H^{2,1}_{BC}(X) \rightarrow H^{2,1}_{dR}(X; \mathbb{C}) \) induced by the identity is injective;

(ii) \( \Delta^1 = 0 \).

In particular, the non-Kählerness of \( X \) is measured by just \( \Delta^2 \in \mathbb{N} \).

For compact complex surfaces in class VII, we show the following result, where we denote \( h^{p,q}_{BC} := \dim \mathcal{H}^{p,q}_{BC}(X) \) for \( p, q \in \{0, 1, 2\} \).

**Theorem 2.2.** The Bott-Chern numbers of compact complex surfaces in class VII are:

\[
\begin{align*}
\Delta^0 &= 0, \\
\Delta^1 &= 0, \\
\Delta^2 &= 1.
\end{align*}
\]

Finally, we compute the Bott-Chern cohomology for compact complex surfaces diffeomorphic to solv-manifolds, according to the list given by K. Hasegawa in [11], see Theorem 1.1. More precisely, we prove that the cohomologies can be computed by using just left-invariant forms. Furthermore, for such complex surfaces, we note that the natural map \( H^{2,1}_{BC}(X) \rightarrow H^{2,1}_{db}(X; \mathbb{C}) \) is injective, see Theorem 2.2.

We note that the above classes do not exhaust the set of compact complex non-Kähler surfaces, the cohomologies of elliptic surfaces being still unknown.

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1. **Non-Kählerness of compact complex surfaces and Bott-Chern cohomology**

We recall that, for a compact complex manifold of complex dimension \( n \), for \( k \in \{0, \ldots, 2n\} \), we define the “non-Kählerness” degrees, [3] Theorem A, \[
\Delta^k := \sum_{p+q=k} (h^{p,q}_{BC} + h^{n−q,n−p}_{BC}) - 2b_k \in \mathbb{N},
\]

where we use the duality in [11] §2.c] giving \( h^{p,q}_{BC} := \dim \mathcal{H}^{p,q}_{BC}(X) \) if and only if \( X \) satisfies the \( \partial \partial \)-Lemma, namely, every \( \partial \)-closed \( \partial \)-closed d-exact form is \( \partial \partial \)-exact too. In particular, for a compact complex surface \( X \), the condition \( \Delta^1 = \Delta^2 = 0 \) is equivalent to \( X \) being Kähler, the first Betti number being even, [11] [17] [19], see also [13] Corollaire 5.7, and [5] Theorem 11.

We prove that \( \Delta^1 \) is always zero for any compact complex surface. In particular, a sufficient and necessary condition for compact complex surfaces to be Kähler is \( \Delta^2 = 0 \).

**Theorem 1.1.** Let \( X \) be a compact complex surface. Then:
(i) the natural map $H^{2,1}_{\overline{BC}}(X) \to H^{2,1}_{\overline{\mathcal{D}}}(X)$ induced by the identity is injective;
(ii) $\Delta^1 = 0$.

In particular, the non-Kählerness of $X$ is measured by just $\Delta^2 \in \mathbb{N}$.

Proof. (i) Let $\alpha \in \wedge^{2,1}X$ be such that $[\alpha] = 0 \in H^{2,1}_{\overline{\mathcal{D}}}(X)$. Let $\beta \in \wedge^{2,0}X$ be such that $\alpha = \overline{\partial}\beta$. Fix a Hermitian metric $g$ on $X$, and consider the Hodge decomposition of $\beta$ with respect to the Dolbeault Laplacian $\overline{\partial}$: let $\beta = \beta_h + \overline{\partial}\lambda$ where $\beta_h \in \wedge^{2,0}X \cap \ker \overline{\partial}$, and $\lambda \in \wedge^{2,1}X$. Therefore we have

$$\alpha = \overline{\partial}\beta = \overline{\partial}\partial\lambda = -\overline{\partial}^* (\partial^* \lambda) = -\overline{\partial} (\partial \lambda) = \overline{\partial}\lambda,$$

where we have used that any $(2,0)$-form is primitive and hence, by the Weil identity, is self-dual.

In particular, $\alpha$ is $\overline{\partial}\overline{\partial}$-exact, so it induces a zero class in $H^{2,1}_{\overline{BC}}(X)$.

(ii) On the one hand, note that

$$H^{1,0}_{BC}(X) \begin{array}{c} \ker \overline{\partial} \cap \ker \overline{\partial} \cap \wedge^{1,0}X \\ \subset \ker \overline{\partial} \cap \wedge^{1,0}X \end{array} = \left( \ker \overline{\partial} \cap \wedge^{1,0}X \right) \cap \ker \overline{\partial} \cap \wedge^{1,0}X.$$

It follows that

$$\dim \mathbb{C} H^{1,0}_{BC}(X) = \dim \mathbb{C} H^{1,0}_{\overline{\mathcal{D}}}(X) \leq \dim \mathbb{C} H^{1,0}_{\overline{\mathcal{D}}}(X) = b_1 - \dim \mathbb{C} H^{0,1}_{\overline{\mathcal{D}}}(X),$$

where we use that the Frölicher spectral sequence degenerates, hence in particular $b_1 = \dim \mathbb{C} H^{1,0}_{\overline{\mathcal{D}}}(X) + \dim \mathbb{C} H^{0,1}_{\overline{\mathcal{D}}}(X)$.

On the other hand, by the assumption, we have

$$\dim \mathbb{C} H^{1,2}_{BC}(X) = \dim \mathbb{C} H^{1,2}_{BC}(X) \leq H^{1,2}_{BC}(X) = \dim \mathbb{C} H^{1,2}_{\overline{\mathcal{D}}}(X),$$

where we use the Kodaira and Serre duality $H^{2,1}_{\overline{\mathcal{D}}}(X) \simeq H^1(X; \Omega^2_X) \simeq H^1(X; \mathcal{O}_X) \simeq H^{1,0}_{\overline{\mathcal{D}}}(X)$.

By summing up, we get

$$\Delta^1 = \dim \mathbb{C} H^{1,0}_{BC}(X) + \dim \mathbb{C} H^{1,2}_{BC}(X) + \dim \mathbb{C} H^{1,2}_{BC}(X) + \dim \mathbb{C} H^{2,1}_{BC}(X) - 2 b_1$$

$$= 2 \left( b_1 - \dim \mathbb{C} H^{0,1}_{\overline{\mathcal{D}}}(X) + \dim \mathbb{C} H^{0,1}_{\overline{\mathcal{D}}}(X) - b_1 \right) = 0,$$

concluding the proof. \qed

2. Class VII surfaces

In this section, we compute Bott-Chern cohomology for compact complex surfaces in class VII.

Let $X$ be a compact complex surface. By Theorem 11, the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\overline{\mathcal{D}}}(X)$ is always injective. Consider now the case when $X$ is in class VII. If $X$ is minimal, we prove that the same holds for cohomology with values in a line bundle. We will also prove that the natural map $H^{1,2}_{BC}(X) \to H^{1,2}_{\overline{\mathcal{D}}}(X)$ is not injective.

**Proposition 2.1.** Let $X$ be a compact complex surface in class VII. Let $L \in H^1(X; \mathcal{O}^*) = \text{Pic}^0(X)$. The natural map $H^{1,2}_{BC}(X; L) \to H^{1,2}_{\overline{\mathcal{D}}}(X; L)$ induced by the identity is injective.

**Proof.** Let $\alpha \in \wedge^{1,0}X \otimes L$ be a $\overline{\partial}L$-exact $(2,1)$-form. We need to prove that $\alpha$ is $\partial L\overline{\partial}L$-exact too. Consider $\alpha = \overline{\partial}L\partial$, where $\partial \in \wedge^{1,0}X \otimes L$. In particular, $\partial \overline{\partial} = 0$, hence $\partial$ defines a class in $H^{0,2}_{\overline{\mathcal{D}}}(X; L)$. Note that $H^{1,2}_{\overline{\mathcal{D}}}(X; L) \simeq H^2(X; \mathcal{O}_X(L)) \simeq H^0(X; \mathcal{K}_X \otimes L^{-1}) = \{0\}$ for surfaces of class VII. Thus, it follows that $\overline{\partial} = -\partial L\overline{\partial}L$ for some $\eta \in \wedge^{1,0}X \otimes L$. Hence $\alpha = \overline{\partial}L\overline{\partial}L\eta$, that is, $\alpha$ is $\partial L\overline{\partial}L$-exact. \qed

We now compute the Bott-Chern cohomology of class VII surfaces.

**Theorem 2.2.** The Bott-Chern numbers of compact complex surfaces in class VII are:

| $H^{2,0}_{BC}$ | $H^{1,0}_{BC}$ | $H^{1,1}_{BC}$ | $H^{0,1}_{BC}$ | $H^{0,2}_{BC}$ | $H^{1,2}_{BC}$ | $H^{2,2}_{BC}$ |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| 0             | 1             | $b_2 + 1$     | 0             | 0             | 0             | 1             |
| 0             | 0             | 0             | 1             | 0             | 0             | 1             |
Proof. It holds $H^{1,0}_{BC}(X) = \ker \partial \cap \ker \overline{\partial} \cap \Lambda^{1,0}X = \ker \overline{\partial} \cap \Lambda^{1,0}X \subseteq \ker \overline{\partial} \cap \Lambda^{1,0}X = H^{-1,0}_\overline{\partial}(X) = \{0\}$ hence $h^{1,0}_BC = h^{0,1}_BC = 0$.

On the other side, by Theorem [11] $0 = \Delta^1 = 2 \left(h^{1,0}_{BC} + h^{2,1}_{BC} - b_1 \right) = 2 \left(h^{2,1}_{BC} - 1 \right)$ hence $h^{2,1}_{BC} = h^{1,2}_BC = 1$.

Similarly, it holds $H^{2,0}_{BC}(X) = \ker \partial \cap \ker \overline{\partial} \cap \Lambda^{2,0}X = \ker \overline{\partial} \cap \Lambda^{2,0}X = H^{-2,0}_\overline{\partial}(X) = \{0\}$ hence $h^{2,0}_BC = h^{0,2}_BC = 0$.

Note that, from [3, Theorem A], we have $\Delta^2 = 0 \leq 2 \left(h^{2,0}_{BC} + h^{1,1}_{BC} + h^{0,2}_BC - b_2 \right) = 2 \left(h^{1,1}_BC - b_2 \right)$ hence $h^{1,1}_BC \geq b_2$. More precisely, from [3, Theorem B] and Theorem [11] we have that $h^{1,1}_BC = b_2$ if and only if $\Delta^2 = 0$ if and only if $X$ satisfies the $\overline{\partial}$-Lemma, in fact $X$ is Kähler, which is not the case.

Finally, we prove that $h^{1,1}_BC = b_2 + 1$. Consider the following exact sequences from [21 Lemma 2.3]. More precisely, the sequence

$$0 \to \frac{\text{im} \partial \cap \Lambda^{1,1}X}{\text{im} \overline{\partial}} \to H^{1,1}_{BC}(X) \to \text{im} \left(H^{1,1}_{BC}(X) \to H^{1,1}_{dR}(X; \mathbb{C}) \right) \to 0$$

is clearly exact. Furthermore, fix a Gauduchon metric $g$. Denote by $\omega := g(J \cdot , \cdot )$ the $(1,1)$-form associated to $g$, where $J$ denotes the integrable almost-complex structure. By definition of $g$ being Gauduchon, we have $\overline{\partial}\partial \omega = 0$. The sequence

$$0 \to \frac{\text{im} \partial \cap \Lambda^{1,1}X}{\text{im} \overline{\partial}} \to \mathbb{C}$$

is exact. Indeed, firstly note that for $\eta = \overline{\partial}f \in \text{im} \overline{\partial} \cap \Lambda^{1,1}X$, we have

$$\langle \eta | \omega \rangle = \int_X \overline{\partial}f \wedge \overline{\omega} = \int_X \overline{\partial}f \wedge \omega = \int_X f \overline{\partial} \omega = 0$$

by applying twice the Stokes theorem. Then, we recall the argument in [21 Lemma 2.3(ii)] for proving that the map

$$\langle \cdot | \omega \rangle : \frac{\text{im} \partial \cap \Lambda^{1,1}X}{\text{im} \overline{\partial}} \to \mathbb{C}$$

is injective. Take $\alpha = \partial \beta \in \text{im} \partial \cap \Lambda^{1,1}X \cap \ker \langle \cdot | \omega \rangle$. Then

$$\langle \Lambda \alpha \rangle = \langle \alpha | \omega \rangle = 0,$$

where $\Lambda$ is the adjoint operator of $\omega \wedge \cdot$ with respect to $\langle \cdot | \cdot \rangle$. Then $\Lambda \alpha \in \ker \langle \cdot | \cdot \rangle = \ker \Lambda \overline{\partial}$, by extending [16 Corollary 7.2.9] by $\mathbb{C}$-linearity. Take $u \in C^\infty(X; \mathbb{C})$ such that $\Lambda \alpha = \Lambda \overline{\partial}u$. Then, by defining $\alpha' := \alpha - \overline{\partial}u$, we have $[\alpha'] = [\alpha] \in \frac{\text{im} \partial \cap \Lambda^{1,1}X}{\text{im} \overline{\partial}}$, and $\Lambda \alpha' = 0$, and $\alpha' = \partial \beta'$ where $\beta' := \beta - \overline{\partial}u$.

In particular, $\alpha'$ is primitive. Since $\alpha'$ is primitive and of type $(1, 1)$, then it is anti-self-dual by the Weil identity. Then

$$\|\alpha'\|^2 = \langle \alpha' | \alpha' \rangle = \int_X \alpha' \wedge \overline{\alpha'} = - \int_X \alpha' \wedge \overline{\alpha'} = - \int_X \partial \beta' \wedge \overline{\partial} \beta' = - \int_X \overline{\partial} (\beta' \wedge \overline{\partial} \beta') = 0$$

and hence $\alpha' = 0$, and therefore $[\alpha] = 0$.

Since the space $\frac{\text{im} \partial \cap \Lambda^{1,1}X}{\text{im} \overline{\partial}}$ is finite-dimensional, being a sub-space of $H^{1,1}_{BC}(X)$, and since the space $\text{im} \left(H^{1,1}_{BC}(X) \to H^{2,1}_{dR}(X; \mathbb{C}) \right)$ is finite-dimensional, being a sub-space of $H^{2,1}_{dR}(X; \mathbb{C})$, we get that

$$\dim_C \frac{\text{im} \partial \cap \Lambda^{1,1}X}{\text{im} \overline{\partial}} \leq \dim_C \mathbb{C} = 1,$$

and hence

$$b_2 < \dim_C H^{1,1}_{BC}(X) = \dim_C \text{im} \left(H^{1,1}_{BC}(X) \to H^{2,1}_{dR}(X; \mathbb{C}) \right) + \dim_C \frac{\text{im} \partial \cap \Lambda^{1,1}X}{\text{im} \overline{\partial}} \leq b_2 + 1.$$

We get that $\dim_C H^{1,1}_{BC}(X) = b_2 + 1$. \hfill $\square$

Finally, we prove that the natural map $H^{1,2}_{BC}(X) \to H^{1,2}_\overline{\partial}(X)$ is not injective.

**Proposition 2.3.** Let $X$ be a compact complex surface in class VII. Then the natural map $H^{1,2}_{BC}(X) \to H^{1,2}_\overline{\partial}(X)$ induced by the identity is the zero map and not an isomorphism.
Proof. Note that, for class VII surfaces, the pluri-genera are zero. In particular, $H^{1,2}_\sigma(X) \cong H^{1,0}_\sigma(X) = \{0\}$, by Kodaira and Serre duality. By Theorem 2.2 one has $H^{1,2}_{BC}(X) \neq \{0\}$. □

2.1. Cohomologies of Calabi-Eckmann surface. In this section, as an explicit example, we compute the cohomologies of a compact complex surface in class VII: namely, we consider the Calabi-Eckmann structure on the differentiable manifolds underlying the Hopf surfaces.

Consider the differentiable manifold $X := S^1 \times S^3$. As a Lie group, $S^3 = SU(2)$ has a global left-invariant co-frame $\{e^1, e^2, e^3\}$ such that $de^1 = -2e^2 \wedge e^3$ and $de^2 = 2e^1 \wedge e^3$ and $de^3 = -2e^1 \wedge e^2$. Hence, we consider a global left-invariant co-frame $\{f, e^1, e^2, e^3\}$ on $X$ with structure equations

\[
\begin{align*}
d f &= 0 \\
d e^1 &= -2e^2 \wedge e^3 \\
d e^2 &= 2e^1 \wedge e^3 \\
d e^3 &= -2e^1 \wedge e^2
\end{align*}
\]

Consider the left-invariant almost-complex structure defined by the $(1,0)$-forms

\[
\begin{align*}
\varphi^1 &:= e^1 + i e^2 \\
\varphi^2 &:= e^3 + i f
\end{align*}
\]

By computing the complex structure equations, we get

\[
\begin{align*}
\partial \varphi^1 &= i \varphi^1 \wedge \varphi^2 \\
\partial \varphi^2 &= 0 \\
\overline{\partial} \varphi^1 &= i \varphi^1 \wedge \overline{\varphi}^2 \\
\overline{\partial} \varphi^2 &= -i \varphi^1 \wedge \overline{\varphi}^1
\end{align*}
\]

We note that the almost-complex structure is in fact integrable.

The manifold $X$ is a compact complex manifold not admitting Kähler metrics. It is bi-holomorphic to the complex manifold $M_{0,1}$ considered by Calabi and Eckmann, [6], see [13] Theorem 4.1.

Consider the Hermitian metric $g$ whose associated $(1,1)$-form is

\[
\omega := \frac{i}{2} \sum_{j=1}^2 \varphi^j \wedge \overline{\varphi}^j.
\]

As for the de Rham cohomology, from the Künneth formula we get

\[
H^{\bullet}_dR(X; C) = C \langle 1 \rangle \oplus C \langle \varphi^1 - \varphi^2 \rangle \oplus C \langle \varphi^{1\bar2} - \varphi^{2\bar1} \rangle \oplus C \langle \varphi^{12\bar1} \rangle,
\]

where, here and hereafter, we shorten, e.g., $\varphi^{1\bar2} := \varphi^1 \wedge \varphi^2 \wedge \overline{\varphi}^1$.

By [12] Appendix II, Theorem 9.5, one has that a model for the Dolbeault cohomology is given by

\[
H^{\bullet\bullet}_\sigma(X) \cong \bigwedge \langle x_{2,1}, x_{0,1} \rangle,
\]

where $x_{i,j}$ is an element of bi-degree $(i,j)$. In particular, we recover that the Hodge numbers \( \{ h^{p,q}_\sigma := \dim_C H^{p,q}_\sigma(X) \} \) are

\[
\begin{align*}
h^{1,0}_\sigma &= 0 \\
h^{0,1}_\sigma &= 0 \\
h^{2,0}_\sigma &= 1 \\
h^{1,1}_\sigma &= 1 \\
h^{2,2}_\sigma &= 1
\end{align*}
\]

We note that the sub-complex

\[
\iota: \bigwedge \langle \varphi^1, \varphi^2, \overline{\varphi}^1, \overline{\varphi}^2 \rangle \hookrightarrow \wedge^{\bullet\bullet} X
\]

is such that $H^{\bullet\bullet}_\sigma(\iota)$ is an isomorphism. More precisely, we get

\[
H^{\bullet\bullet}_\sigma(X) = C \langle 1 \rangle \oplus C \langle \varphi^{1\bar2} \rangle \oplus C \langle \varphi^{12\bar1} \rangle \oplus C \langle \varphi^{12\bar1} \rangle,
\]

where we have listed the harmonic representatives with respect to the Dolbeault Laplacian of $g$.

By [2] Theorem 1.3, Proposition 2.2, we have also $H^{\bullet\bullet}_{BC}(\iota)$ isomorphism. In particular, we get

\[
H^{\bullet\bullet}_{BC}(X) = C \langle 1 \rangle \oplus C \langle \varphi^{1\bar1} \rangle \oplus C \langle \varphi^{12\bar1} \rangle \oplus C \langle \varphi^{12\bar1} \rangle.
\]
where we have listed the harmonic representatives with respect to the Aeppli Laplacian of $g$.

In particular, the Bott-Chern numbers \( \{ h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X) \}_{p,q \in \{0,1,2\}} \) are

\[
\begin{array}{cccc}
h_{BC}^{2,0} & 0 & h_{BC}^{1,0} & 0 \\
h_{BC}^{0,0} & 0 & h_{BC}^{0,1} & 0 \\
h_{BC}^{1,1} & 1 & h_{BC}^{1,2} & 1 \\
0 & h_{BC}^{2,2} & 1 & 0 \\
\end{array}
\]

By [19, §2.c], we have

\[
H_{A}^{r}(X) = \mathbb{C}(1) \oplus \mathbb{C} \langle [\varphi^2] \rangle \oplus \mathbb{C} \langle [\varphi^3] \rangle \oplus \mathbb{C} \langle [\varphi^{22}] \rangle \oplus \mathbb{C} \langle [\varphi^{12i}] \rangle ,
\]

where we have listed the harmonic representatives with respect to the Aeppli Laplacian of $g$.

In particular, the Aeppli numbers \( \{ h_{A}^{p,q} := \dim_{\mathbb{C}} H_{A}^{p,q}(X) \}_{p,q \in \{0,1,2\}} \) are

\[
\begin{array}{cccc}
h_{A}^{2,0} & 0 & h_{A}^{1,0} & 0 \\
h_{A}^{0,0} & 0 & h_{A}^{0,1} & 0 \\
h_{A}^{1,1} & 1 & h_{A}^{1,2} & 1 \\
0 & h_{A}^{2,2} & 1 & 0 \\
\end{array}
\]

Summarizing, we have the following.

**Proposition 2.4.** Let $X := S^4 \times S^3$ be endowed with the complex structure of Calabi-Eckmann. The non-zero dimensions of the Dolbeault and Bott-Chern cohomologies are the following:

\[
h_{0,0}(X) = h_{0,1}(X) = h_{2,0}(X) = h_{2,2}(X) = 1
\]

and

\[
h_{BC}^{0,0}(X) = h_{BC}^{1,1}(X) = h_{BC}^{2,1}(X) = h_{BC}^{1,2}(X) = h_{BC}^{2,2}(X) = 1.
\]

Note in particular that the natural map $H_{BC}^{2,1}(X) \to H_{BC}^{2,1}(X)$ induced by the identity is an isomorphism, and that the natural map $H_{BC}^{2,1}(X) \to H_{BC}^{2,1}(X)$ induced by the identity is injective.

### 3. Complex Surfaces Diffeomorphic to Solvmanifolds

Let $X$ be a compact complex surface diffeomorphic to a solvmanifold $\Gamma \backslash G$. By [11, Theorem 1], $X$ is (A) either a complex torus, (B) or a hyperelliptic surface, (C) or a Inoue surface of type $S_M$, (D) or a primary Kodaira surface, (E) or a secondary Kodaira surface, (F) or an Inoue surface of type $S^\pm$, and, as such, it is endowed with a left-invariant complex structure.

In each case, we recall the structure equations of the group $G$, see [11]. More precisely, take a basis \( \{ e_1, e_2, e_3, e_4 \} \) of the Lie algebra $\mathfrak{g}$ naturally associated to $G$. We have the following commutation relations, according to [11]:

(A) differentiable structure underlying a complex torus:

\[
[e_j, e_k] = 0 \quad \text{for any } j, k \in \{1, 2, 3, 4\} ;
\]

(hereafter, we write only the non-trivial commutators);

(B) differentiable structure underlying a hyperelliptic surface:

\[
[e_1, e_4] = e_2 , \quad [e_2, e_4] = -e_1 ;
\]

(C) differentiable structure underlying a Inoue surface of type $S_M$:

\[
[e_1, e_4] = -\alpha e_1 + \beta e_2 , \quad [e_2, e_4] = -\beta e_1 - \alpha e_2 , \quad [e_3, e_4] = 2\alpha e_3 ,
\]

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$;

(D) differentiable structure underlying a primary Kodaira surface:

\[
[e_1, e_2] = -e_3 ;
\]

(E) differentiable structure underlying a secondary Kodaira surface:

\[
[e_1, e_2] = -e_3 , \quad [e_1, e_4] = e_2 , \quad [e_2, e_4] = -e_1 ;
\]

(F) differentiable structure underlying an Inoue surface of type $S^\pm$:

\[
[e_2, e_3] = -e_1 , \quad [e_2, e_4] = -e_2 , \quad [e_3, e_4] = e_3 .
\]
Denote by \( \{ e_1, e_2, e_3, e_4 \} \) the dual basis of \( \{ e_1, e_2, e_3, e_4 \} \). We recall that, for any \( \alpha \in g^* \), for any \( x, y \in g \), it holds \( d \alpha(x, y) = -\alpha([x, y]) \). Hence we get the following structure equations:

(A) differentiable structure underlying a complex torus:
\[
\begin{align*}
    \text{d} e_1 &= 0 \\
    \text{d} e_2 &= 0 \\
    \text{d} e_3 &= 0 \\
    \text{d} e_4 &= 0
\end{align*}
\]

(B) differentiable structure underlying a hyperelliptic surface:
\[
\begin{align*}
    \text{d} e_1 &= e_2 \wedge e_4 \\
    \text{d} e_2 &= -e_3 \wedge e_4 \\
    \text{d} e_3 &= 0 \\
    \text{d} e_4 &= 0
\end{align*}
\]

(C) differentiable structure underlying a Inoue surface of type \( S_M \):
\[
\begin{align*}
    \text{d} e_1 &= \alpha e_1 \wedge e_4 + \beta e_2 \wedge e_4 \\
    \text{d} e_2 &= -\beta e_1 \wedge e_4 + \alpha e_2 \wedge e_4 \\
    \text{d} e_3 &= -2\alpha e_3 \wedge e_4 \\
    \text{d} e_4 &= 0
\end{align*}
\]

(D) differentiable structure underlying a primary Kodaira surface:
\[
\begin{align*}
    \text{d} e_1 &= 0 \\
    \text{d} e_2 &= 0 \\
    \text{d} e_3 &= e_1 \wedge e_2 \\
    \text{d} e_4 &= 0
\end{align*}
\]

(E) differentiable structure underlying a secondary Kodaira surface:
\[
\begin{align*}
    \text{d} e_1 &= e_2 \wedge e_4 \\
    \text{d} e_2 &= -e_3 \wedge e_4 \\
    \text{d} e_3 &= e_1 \wedge e_2 \\
    \text{d} e_4 &= 0
\end{align*}
\]

(F) differentiable structure underlying a Inoue surface of type \( S^\pm \):
\[
\begin{align*}
    \text{d} e_1 &= e_2 \wedge e_3 \\
    \text{d} e_2 &= e_2 \wedge e_4 \\
    \text{d} e_3 &= -e_3 \wedge e_4 \\
    \text{d} e_4 &= 0
\end{align*}
\]

In cases (A), (B), (C), (D), (E), consider the \( G \)-left-invariant almost-complex structure \( J \) on \( X \) defined by
\[
J e_1 := e_2 \quad \text{and} \quad J e_2 := -e_1 \quad \text{and} \quad J e_3 := e_4 \quad \text{and} \quad J e_4 := -e_3.
\]
Consider the \( G \)-left-invariant \((1,0)\)-forms
\[
\begin{align*}
    \varphi^1 &= e_1 + i e_2 \\
    \varphi^2 &= e_3 + i e_4
\end{align*}
\]

In case (F), consider the \( G \)-left-invariant almost-complex structure \( J \) on \( X \) defined by
\[
J e_1 := e_2 \quad \text{and} \quad J e_2 := -e_1 \quad \text{and} \quad J e_3 := e_4 - q e_2 \quad \text{and} \quad J e_4 := -e_3 - q e_1,
\]
where $q \in \mathbb{R}$. Consider the $G$-left-invariant $(1,0)$-forms

\[
\begin{align*}
\varphi^1 &:= e^1 + i e^2 + i q e^4 \\
\varphi^2 &:= e^3 + i e^4
\end{align*}
\]

With respect to the $G$-left-invariant coframe $\{\varphi^1, \varphi^2\}$ for the holomorphic tangent bundle $T^{1,0} \Gamma \backslash G$, we have the following structure equations. (As for notation, we shorten, e.g., $\varphi^{12} := \varphi^1 \wedge \varphi^2$.)

(A) torus:

\[
\begin{align*}
\{ & d\varphi^1 = 0 \\
& d\varphi^2 = 0
\end{align*}
\]

(B) hyperelliptic surface:

\[
\begin{align*}
\{ & d\varphi^1 = -\frac{1}{2} \varphi^{12} + \frac{i}{2} \varphi^{12} \\
& d\varphi^2 = 0
\end{align*}
\]

(C) Inoue surface $S_{M}$:

\[
\begin{align*}
\{ & d\varphi^1 = \frac{1}{2} \varphi^{12} - \frac{i}{2} \varphi^{12} \\
& d\varphi^2 = -i \alpha \varphi^{22}
\end{align*}
\]

(where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$);

(D) primary Kodaira surface:

\[
\begin{align*}
\{ & d\varphi^1 = 0 \\
& d\varphi^2 = \frac{i}{2} \varphi^{11}
\end{align*}
\]

(E) secondary Kodaira surface:

\[
\begin{align*}
\{ & d\varphi^1 = -\frac{1}{2} \varphi^{12} + \frac{i}{2} \varphi^{12} \\
& d\varphi^2 = \frac{i}{2} \varphi^{11}
\end{align*}
\]

(F) Inoue surface $S_{\pm}$:

\[
\begin{align*}
\{ & d\varphi^1 = \frac{1}{4} \varphi^{12} + \frac{1}{4} \varphi^{21} + \frac{1}{4} \varphi^{22} \\
& d\varphi^2 = \frac{1}{4} \varphi^{22}
\end{align*}
\]

4. Cohomologies of complex surfaces diffeomorphic to solvmanifolds

In this section, we compute the Dolbeault and Bott-Chern cohomologies of the compact complex surfaces diffeomorphic to a solvmanifold.

We prove the following theorem.

**Theorem 4.1.** Let $X$ be a compact complex surface diffeomorphic to a solvmanifold $\Gamma \backslash G$; denote the Lie algebra of $G$ by $\mathfrak{g}$. Then the inclusion $(\wedge^{*} \mathfrak{g}^{*}, \partial, \overline{\partial}) \hookrightarrow (\wedge^{*} X, \partial, \overline{\partial})$ induces an isomorphism both in Dolbeault and in Bott-Chern cohomologies. In particular, the dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies and the degrees of non-Kählerness are summarized in Table 5.

**Proof.** Firstly, we compute the cohomologies of the sub-complex of $G$-left-invariant forms. The computations are straightforward from the structure equations.

| $p$ | $q$ | $H^{0}_{\mathfrak{g}^{*}}$ | $\text{dim} \ H^{0}_{\mathfrak{g}^{*}}$ | $H^{1}_{\mathfrak{g}^{*}}$ | $\text{dim} \ H^{1}_{\mathfrak{g}^{*}}$ | $H^{2}_{\mathfrak{g}^{*}}$ | $\text{dim} \ H^{2}_{\mathfrak{g}^{*}}$ | $H^{0}_{\mathfrak{g}^{*}}$ | $\text{dim} \ H^{0}_{\mathfrak{g}^{*}}$ | $H^{1}_{\mathfrak{g}^{*}}$ | $\text{dim} \ H^{1}_{\mathfrak{g}^{*}}$ | $H^{2}_{\mathfrak{g}^{*}}$ | $\text{dim} \ H^{2}_{\mathfrak{g}^{*}}$ |
|-----|-----|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| (0, 0) | 1 | 1 | 1 | 1 | 1 | 1 |
| (1, 0) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| (0, 1) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| (2, 0) | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| (4, 1) | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 | 4 |
| (0, 2) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| (2, 1) | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 |
| (4, 2) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

**Table 1.** Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 1.
In Tables 1 and 2 and in Tables 3 and 4 we summarize the results of the computations. The subcomplexes of left-invariant forms are depicted in Figure 1 (each dot represents a generator, vertical arrows depict the $\bar{\partial}$-operator, horizontal arrows depict the $\partial$-operator, and trivial arrows are not shown.) The dimensions are listed in Table 5.

On the one side, recall that the inclusion of left-invariant forms into the space of forms induces an injective map in Dolbeault and Bott-Chern cohomologies, see, e.g., [7, Lemma 9], [1, Lemma 3.6]. On
the other side, recall that the Fröhlicher spectral sequence of a compact complex surface $X$ degenerates at the first level, equivalently, the equalities

$$\dim \mathbb{C} H^{1,0}_c(X) + \dim \mathbb{C} H^{0,1}_c(X) = \dim \mathbb{C} H^1_{dR}(X; \mathbb{C})$$

and

$$\dim \mathbb{C} H^{2,0}_c(X) + \dim \mathbb{C} H^{1,1}_c(X) + \dim \mathbb{C} H^{0,2}_c(X) = \dim \mathbb{C} H^2_{dR}(X; \mathbb{C})$$

hold. By comparing the dimensions in Table 5 with the Betti numbers case by case, we find that the left-invariant forms suffice in computing the Dolbeault cohomology for each case. Then, by [1, Theorem 3.7], see also [2, Theorem 1.3, Theorem 1.6], it follows that also the Bott-Chern cohomology is computed using just left-invariant forms.

\[\square\]

**Figure 1.** The double-complexes of left-invariant forms over 4-dimensional solvmanifolds.

| (p, q) | (A) torus | (B) hyperelliptic surface | (C) Inoue surface $S_M$ | (D) primary Kodaira surface | (E) secondary Kodaira surface | (F) Inoue surface $S^\pm$ |
|-------|------------|---------------------------|-----------------------|-----------------------------|-----------------------------|-----------------------------|
| (0, 0) | | | | | | |
| 1 1 1 0 | 1 1 1 0 | 1 1 1 0 | 1 1 1 0 | 1 1 1 0 | 1 1 1 0 |
| (1, 0) | | | | | | |
| 2 2 0 | 2 2 0 | 2 2 0 | 2 2 0 | 2 2 0 | 2 2 0 |
| (2, 0) | | | | | | |
| 1 1 0 | 1 1 0 | 1 1 0 | 1 1 0 | 1 1 0 | 1 1 0 |
| (0, 1) | | | | | | |
| 1 1 2 | 1 1 2 | 1 1 2 | 1 1 2 | 1 1 2 | 1 1 2 |
| (1, 1) | | | | | | |
| 4 4 0 | 4 4 0 | 4 4 0 | 4 4 0 | 4 4 0 | 4 4 0 |
| (2, 1) | | | | | | |
| 2 2 0 | 2 2 0 | 2 2 0 | 2 2 0 | 2 2 0 | 2 2 0 |
| (1, 2) | | | | | | |
| 1 1 1 | 1 1 1 | 1 1 1 | 1 1 1 | 1 1 1 | 1 1 1 |
| (2, 2) | | | | | | |
| 1 1 1 | 1 1 1 | 1 1 1 | 1 1 1 | 1 1 1 | 1 1 1 |

**Table 5.** Summary of the dimensions of de Rham, Dolbeault, and Bott-Chern cohomologies and of the degree of non-Kählerness for compact complex surfaces diffeomorphic to solvmanifolds.
According to Theorem 4.1, the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\dR}(X; \mathbb{C})$ is injective for any compact complex surface. We are now interested in studying the injectivity of the natural map $H^{2,1}_{BC}(X) \to H^{3}_{\dR}(X; \mathbb{C})$ induced by the identity, at least for compact complex surfaces diffeomorphic to solvmanifolds. In fact, by definition, the property of satisfying the $\partial \bar{\partial}$-Lemma, [8], is equivalent to the natural map $\bigoplus_{p+q=0} H^{p,q}_{BC}(X) \to H^{\bullet}_{\dR}(X; \mathbb{C})$ being injective. Note that, for a compact complex manifold of complex dimension $n$, the injectivity of the map $H^{n,n}_{BC}(X) \to H^{2n-1}_{\dR}(X; \mathbb{C})$ implies the $(n-1,n)$-th weak $\partial \bar{\partial}$-Lemma in the sense of J. Fu and S.-T. Yau, [10, Definition 5].

We prove the following result.

**Theorem 4.2.** Let $X$ be a compact complex surface diffeomorphic to a solvmanifold. Then the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\dR}(X)$ induced by the identity is an isomorphism, and the natural map $H^{2,1}_{BC}(X) \to H^{3}_{\dR}(X; \mathbb{C})$ induced by the identity is injective.

**Proof.** By the general result in Theorem 4.1, the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\dR}(X)$ is injective. In fact, it is an isomorphism as follows from the computations summarized in Tables 1 and 2. As for the injectivity of the natural map $H^{2,1}_{BC}(X) \to H^{3}_{\dR}(X; \mathbb{C})$, it is a straightforward computation from Tables 1 and 2.

As an example, we offer an explicit calculation of the injectivity of the map $H^{2,1}_{BC}(X) \to H^{3}_{\dR}(X; \mathbb{C})$ for the Inoue surfaces of type 0, see [13], see also [22]. We will change a little bit the notation. Recall the construction of Inoue surfaces: let $M \in \text{SL}(3; \mathbb{Z})$ be a unimodular matrix having a real eigenvalue $\lambda > 1$ and two complex eigenvalues $\mu \neq \bar{\mu}$. Take a real eigenvector $(\alpha_1, \alpha_2, \alpha_3)$ and an eigenvector $(\beta_1, \beta_2, \beta_3)$ of $M$. Let $H = \{z \in \mathbb{C} \mid \Im z > 0\}$; on the product $H \times \mathbb{C}$ consider the following transformations defined as

$$f_0(z, w) := (\lambda z, \lambda w), \quad f_j(z, w) := (z + \alpha_j, w + \beta_j) \quad \text{for} \quad j \in \{1, 2, 3\}.$$ 

Denote by $\Gamma_M$ the group generated by $f_0, \ldots, f_3$; then $\Gamma_M$ acts in a proper discontinuous way and without fixed points on $H \times \mathbb{C}$, and $\mathcal{S}_M := H \times \mathbb{C}/\Gamma_M$ is an Inoue surface of type 0, as in case (C) in [11]. Denoting by $z = x + iy$ and $w = u + iv$, consider the following differential forms on $H \times \mathbb{C}$:

$$e^1 := \frac{1}{y} \, dx, \quad e^2 := \frac{1}{y} \, dy, \quad e^3 := \sqrt{y} \, du, \quad e^4 := \sqrt{y} \, dv.$$ (Note that $e^1$ and $e^2$, and $e^3 \wedge e^4$ are $\Gamma_M$-invariant, and consequently they induce global differential forms on $\mathcal{S}_M$.) We obtain

$$\text{d} e^1 = e^1 \wedge e^2, \quad \text{d} e^2 = 0, \quad \text{d} e^3 = \frac{1}{2} e^2 \wedge e^3, \quad \text{d} e^4 = \frac{1}{2} e^2 \wedge e^4.$$ 

Consider the natural complex structure on $\mathcal{S}_M$ induced by $H \times \mathbb{C}$. Locally, we have

$$J e^1 = -e^2 \quad \text{and} \quad J e^2 = e^1 \quad \text{and} \quad J e^3 = -e^4 \quad \text{and} \quad J e^4 = e^3.$$ 

Considering the $\Gamma_M$-invariant $(2,1)$-Bott-Chern cohomology of $\mathcal{S}_M$, we obtain that

$$H^2_{BC}(\mathcal{S}_M) = C \left\langle \left[ e^1 \wedge e^3 \wedge e^4 + i e^2 \wedge e^3 \wedge e^4 \right] \right\rangle.$$ 

Clearly $\bar{\partial} \left( e^1 \wedge e^3 \wedge e^4 + i e^2 \wedge e^3 \wedge e^4 \right) = 0$ and $e^1 \wedge e^3 \wedge e^4 + i e^2 \wedge e^3 \wedge e^4 = -2 \text{Im} e^1 \wedge e^3 \wedge e^4 + 2 \text{d} \left( e^3 \wedge e^4 \right)$, therefore the de Rham cohomology class $\left[ e^1 \wedge e^3 \wedge e^4 + i e^2 \wedge e^3 \wedge e^4 \right]$ is non-zero. 

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