MAPS CONJUGATING HOLOMORPHIC MAPS IN \( C^n \)

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Abstract. If \( \psi \) is a bijection from \( C^n \) onto a complex manifold \( M \), which conjugates every holomorphic map in \( C^n \) to an endomorphism in \( M \), then we prove that \( \psi \) is necessarily biholomorphic or antibiholomorphic. This extends a result of A. Hinkkanen to higher dimensions. As a corollary, we prove that if there is an epimorphism from the semigroup of all holomorphic endomorphisms of \( C^n \) to the semigroup of holomorphic endomorphisms of a complex manifold \( M \) consisting of more than one point, or an epimorphism in the opposite direction for a doubly-transitive \( M \), then it is given by conjugation by some biholomorphic or antibiholomorphic map. We show also that there are two unbounded domains in \( C^n \) with isomorphic endomorphism semigroups but which are neither biholomorphically nor antibiholomorphically equivalent.

1. Introduction

The question of determining a mathematical structure of an object from its semigroup of endomorphisms, i.e. the set of all maps from this object into itself with composition as a semigroup operation, goes back to at least C. J. Everett and S. M. Ulam [5], [15]. In the most general form this question can be formulated as follows. Suppose we have two sets \( A \) and \( B \) with a given structure, whose semigroups of endomorphisms compatible with this structure are isomorphic. Does there exist a bijective map between \( A \) and \( B \), which preserves the structure?

K. D. Magill, L. M. Glusklin, B. M. Schein, L. B. Šneperman, and I. S. Yoroker studied the question of determining a topological space from its semigroup of continuous endomorphisms. See a survey in [12], [6].

To the best of the authors’ knowledge, L. Rubel was the first who raised the question of determining a complex space from its semigroup of holomorphic endomorphisms. In 1993, A. Eremenko [4] proved that every two Riemann surfaces that admit non-constant bounded holomorphic functions, and whose semigroups of holomorphic endomorphisms are isomorphic, are necessarily conformally or anticonformally equivalent. This result was extended by S. Merenkov in [13] to bounded domains in \( C^n \).

On the other hand, A. Hinkkanen [7] proved in 1992 that there exist unbounded domains in \( C \) whose semigroups of holomorphic endomorphisms are isomorphic, but the domains are not even homeomorphic. In the same paper A. Hinkkanen studied another question raised by L. Rubel. Namely, he proved that if \( \psi \) is a one-to-one function of the plane onto itself (but with no assumption of continuity), such that \( \psi \circ f \circ \psi^{-1} \) is entire whenever \( f \) is entire, then \( \psi \) has the form \( \psi(z) = az + b \), or \( \psi(z) = az + b, \) where \( a \) and \( b \) are complex numbers with \( a \neq 0 \); i.e., \( \psi \) is a conformal or anticonformal automorphism.

In higher dimensions, any analog of A. Hinkkanen’s theorem must take into account the fact that the automorphism group of \( C^n \) is quite large, since in \( C^2 \),
for example, there are biholomorphic maps of the form \( \psi(z_1, z_2) = (z_1, z_2 + g(z_1)) \),
where \( g \) is an arbitrary entire function. However, one may hope that arbitrary
\( \psi \) conjugating holomorphic maps to holomorphic maps is still a biholomorphic or
antibiholomorphic automorphism. The main theorem, Theorem 1, of the present
paper asserts that this is indeed the case.

Note that the set of all holomorphic endomorphisms of a complex manifold \( M \)
forms a semigroup (with unit) under composition. We denote this semigroup by
\( E(M) \). If \( M = \mathbb{C}^n \), we denote the semigroup by \( \mathbb{C}^n \).

**Theorem 1.** If \( \psi \) is a bijection of \( \mathbb{C}^n \), \( n \geq 2 \) onto a complex manifold \( M \), such
that \( \psi \circ f \circ \psi^{-1} \in E(M) \) for every map \( f \in E \), then \( \psi \) is biholomorphic or antbiholomorphic.

As in the one-dimensional case [1], it is not sufficient to assume that \( \psi \circ f \circ \psi^{-1} \in E(M) \) for every polynomial map \( f \) in order to conclude that \( \psi \) is a homeomorphism.
The reason is that there are non-continuous field automorphisms of \( \mathbb{C}^n \) [1].
If \( \xi \) is such an automorphism, then we can take \( \psi(z_1, \ldots, z_n) = (\xi(z_1), \ldots, \xi(z_n)) \).
The conjugation by \( \psi \) is an automorphism of semigroups of polynomial maps in \( \mathbb{C}^n \), but
\( \psi \) is not continuous.

We say that a complex manifold \( N \) is **doubly-transitive** if \( E(N) \) is doubly-
transitive, i.e. if for every pair \( z_1, z_2 \) of distinct points in \( N \) and every other pair of
points \( w_1, w_2 \) in \( N \), there exists \( f \in E(N) \), such that \( f(z_m) = w_m \), \( m = 1, 2 \).
We say that \( N \) is **weakly doubly-transitive** if in the previous definition we replace the
assumption that \( w_2 \) arbitrary by requiring that it has to be sufficiently close to \( w_1 \).
Clearly, every doubly-transitive complex manifold is weakly doubly-transitive, and
\( \mathbb{C}^n \) is doubly-transitive. As a corollary to Theorem 1 we prove the following

**Theorem 2.** If there exists an epimorphism of semigroups \( \phi : E \to E(M) \), where
\( M \) is a complex manifold consisting of more than one point, then

\[
\phi(f) = \psi \circ f \circ \psi^{-1}, \quad \forall f \in E,
\]

for some biholomorphic or antibiholomorphic map \( \psi : \mathbb{C}^n \to M \).

If there exists an epimorphism of semigroups \( \varphi : E(M) \to E \), where \( M \) is a
weakly doubly-transitive complex manifold, then

\[
\varphi(f) = \eta \circ f \circ \eta^{-1}, \quad \forall f \in E(M),
\]

for some biholomorphic or antibiholomorphic map \( \eta : M \to \mathbb{C}^n \).

We note that the converse to this theorem is trivial. If \( \psi \) is a biholomorphic or
antibiholomorphic map from \( \mathbb{C}^n \) to \( M \), then the map \( f \mapsto \psi \circ f \circ \psi^{-1} \) is an isomorphism
between the semigroups. Similarly, we get an isomorphism of semigroups if there exists an (anti)biholomorphic map \( \eta : M \to \mathbb{C}^n \). In particular, we obtain
the following corollary, which follows immediately from the previous remarks plus the
fact that an antibiholomorphic equivalence from \( \mathbb{C}^n \) to \( M \) implies a biholomorphic
equivalence simply by composing with the involution \( z \mapsto \overline{z} \).

**Corollary 1.** Given a complex manifold \( M \), the endomorphism semigroup of \( M \) is
isomorphic to the endomorphism semigroup of \( \mathbb{C}^n \) if and only if \( M \) is biholomorphic
to \( \mathbb{C}^n \).

The first part of Theorem 2 is in some sense quite surprising because, among
the complex manifolds of dimension \( n \), \( \mathbb{C}^n \) has a large and complicated semigroup
of endomorphisms (compare the simple semigroups in Theorem 3 below). Yet the equivalence given above requires only the existence of an epimorphism from the “large” semigroup $E$ onto $E(M)$.

Also, applying methods used by D. Varolin [16], any Stein manifold $M$ with the (volume) density property is doubly-transitive, and hence can be used in the second part of Theorem 2. Indeed, the fact that the manifold is Stein implies that any single point is a holomorphically convex set. Then for distinct points $p_1, p_2, q_1, q_2$ in $M$, Theorem 0.2 of [16] with $K = \{p_2\}$ implies that there is an automorphism, $f_1$, of $M$ so that $f_1(p_1) = q_1$ and $f_1(p_2) = p_2$. Likewise, there is an automorphism, $f_2$, of $M$ so that $f_2(p_2) = q_2$ and $f_2(q_1) = q_1$. Thus for the map $f = f_2 \circ f_1$ we have $f(p_j) = q_j$. If $p_1, p_2$ are distinct and $q_1, q_2$ are arbitrary (and possibly one or both of them is the same as $p_1$ or $p_2$), then we can first choose $z_1 \neq z_2$, distinct from the previous four points, map $p_j$ to $z_j$, and then $z_j$ to $q_j$ (using a constant map if $q_1 = q_2$). Hence $M$ is doubly-transitive.

We mention also a recent paper by S. G. Krantz [10], where he studies the question of determination of a domain in complex space by its automorphism group. Of course a domain possesses more endomorphisms than automorphisms. Therefore the ability to determine a domain from its automorphism group implies the ability to determine a domain from its endomorphism semigroup. Our Theorem 2 differs from Krantz’s result in that, first of all, we assume the existence of an epimorphism between semigroups, rather than an isomorphism. Secondly, the information we assume has a purely algebraic character, i.e. the existence of an algebraic epimorphism, and not a topological one; i.e., we make no a priori assumptions about continuity. To our knowledge it is an open question if the existence of a purely algebraic isomorphism between the automorphism group of $\mathbb{C}^n$ and the automorphism group of $M$ implies the biholomorphic equivalence of these manifolds. However, one result along these lines is contained in the work of P. Ahern and W. Rudin [1]. They showed that $\text{Aut}(\mathbb{C}^n)$ is sensitive to the dimension, i.e. if $1 \leq m < n$, then the groups $\text{Aut}(\mathbb{C}^m)$ and $\text{Aut}(\mathbb{C}^n)$ are not algebraically isomorphic.

To complete the analogy with Hinkkanen’s results, we show the existence of two unbounded domains in $\mathbb{C}^n$ with isomorphic endomorphism semigroups but which are not (anti)biholomorphically equivalent. This should be compared with Merenkov’s result [13], in which it is shown that for two bounded domains in $\mathbb{C}^n$, an isomorphism between the endomorphism semigroups implies the (anti)biholomorphic equivalence between the two domains.

**Theorem 3.** There exist unbounded domains $D_1$ and $D_2$ in $\mathbb{C}^n$ so that the endomorphism semigroups $E(D_1)$ and $E(D_2)$ are isomorphic but such that there is no biholomorphic or antibiholomorphic map from $D_1$ onto $D_2$.

The paper is organized as follows. In Section 2 we prove that the map $\psi$ in Theorem 1 is a homeomorphism, using the notion of a Fatou-Bieberbach domain and pose a question about Fatou-Bieberbach domains in Stein manifolds with the density property. Section 3 and Section 4 are devoted to the proof that $\psi$ is biholomorphic or antibiholomorphic. In Section 5 we give a proof of Theorem 2, and in Section 6 we prove Theorem 3.

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2. Fatou-Bieberbach domains and continuity of $\psi$

Below we assume that $n \geq 2$.

Let $FB$ denote the set of Fatou-Bieberbach domains, i.e. proper domains in $\mathbb{C}^n$ that are biholomorphic to $\mathbb{C}^n$. A domain from this set will be called an $FB$-domain, and a biholomorphic map from $\mathbb{C}^n$ onto an $FB$-domain will be called an $FB$-map.

We denote by $\Delta(r)$ the disk in $\mathbb{C}$ centered at 0 and of radius $r$, and by $\Delta^k(r)$ the $k$-fold product of $\Delta(r)$. In [2] it was proved that there exists an $FB$-domain in $\mathbb{C}^n$ which is contained in the union of $\Delta(r^2) \times \Delta^{n-1}(r)$ and the set $S_1 = \{z = (z_1, \ldots, z_n): |z| \geq r^2 - 3r + \| (z_2, \ldots, z_n) \|_\infty \}$, for some $r > 4$. This $FB$-domain is a basin of attraction at 0 of a polynomial map that fixes the origin. Therefore 0 is in the $FB$-domain. By using rotations, we deduce that there exists an $FB$-domain which contains the origin and is contained in the union of $\Delta^{k-1}(r) \times \Delta(r^2) \times \Delta^{n-k}(r)$ and the set $S_k = \{z = (z_1, \ldots, z_n): |z| \geq r^2 - 3r + \| (z_1, \ldots, \hat{z}_k, \ldots, z_n) \|_\infty \}$ for some $r > 4$, $\forall k = 1, \ldots, n$, where $\hat{z}_k$ means that $z_k$ is omitted. It follows that in $\mathbb{C}^n$ there are $n$ $FB$-domains whose intersection is non-empty and bounded. By post-composing the corresponding $FB$-maps with contractions, and using translations, we conclude that intersections of $FB$-domains form a base of neighborhoods at each point of $\mathbb{C}^n$.

Now, under the assumptions of Theorem [1] we can prove that $\psi$ is continuous. Let $f_1, \ldots, f_n$ be $FB$-maps as above so that the intersection of their images is bounded. Then, by assumption, $g_i = \psi \circ f_i \circ \psi^{-1}$, $i = 1, \ldots, n$ are holomorphic maps in $\mathcal{M}$. Moreover,

$$\psi(f_1(\mathbb{C}^n) \cap \cdots \cap f_n(\mathbb{C}^n)) = \psi(f_1(\mathbb{C}^n)) \cap \cdots \cap \psi(f(\mathbb{C}^n))$$

and since each $g_i$ is an injective holomorphic map, $\psi(f_i(\mathbb{C}^n)) = g_i(\psi(\mathbb{C}^n)) = g_i(\mathcal{M})$ is an open set. It follows that $\psi$ is an open map. Using this plus the fact that $\psi$ is a bijection of $\mathbb{C}^n$ onto a manifold, we see that $\psi^{-1}(K)$ is compact for each compact $K \subset \mathcal{M}$. With a standard argument, we conclude that $\psi$ is a homeomorphism. In particular [3], the dimension of $\mathcal{M}$ must be equal to $n$.

Note that [1] implies that a Stein manifold $\mathcal{M}$ with the (volume) density property has an injective holomorphic map $F: \mathcal{M} \to \mathcal{M}$ with $F(\mathcal{M}) \neq \mathcal{M}$. Since our proof of the continuity of $\psi$ in Theorem [1] is based on the existence of special maps of this form in $\mathbb{C}^n$, it is an interesting open question whether such $\mathcal{M}$ can be shown to have a base of neighborhoods given by finite intersections of injective images of $\mathcal{M}$. If so, then it should be possible to replace $\mathbb{C}^n$ in Theorem [1] by any manifold with these properties.

3. Local linearization of maps

Having shown that $\psi$ is continuous, we proceed as in [13] to prove that $\psi$ is biholomorphic or antibiholomorphic using a simultaneous linearization of certain commuting maps. Let $a \in \mathbb{C}^n$ be an arbitrary point, and $b = \psi(a)$. It is enough to show that $\psi$ is biholomorphic or antibiholomorphic in a neighborhood of $a$.

A set $P = \{p_i \}_{i=1}^n$ will be called a system of projections at $o$ in a complex manifold $\mathcal{N}$, $o \in \mathcal{N}$, if it consists of holomorphic maps in $E(\mathcal{N})$ that fix $o$, and satisfy:

1. $p_i \neq o$, $\forall i$;
2. $p_i^2 = p_i$, $\forall i$;
3. $p_i \circ p_j = o$, $\forall i \neq j$,
where $p_i^2 = p_i \circ p_i$, or in (1) and (3) stands for the constant map sending $N$ to $o \in N$. Let $f$ be a biholomorphic map of $N$ onto itself, that commutes with all maps of some system of projections $P$ at $o$, and fixes $o$. We also assume that for every neighborhood $U$ of $o$, and every compact set $K$, there exists an iterate of $f$ that brings $K$ into $U$, i.e. there exists a positive integer $l$ such that $f^l(K) \subset U$. Such a map $f$ clearly exists if $N = \mathbb{C}^n$, since we can take it to be a contraction at $o$, and $\{p_i\}$ to be standard projections. Now we introduce a subsemigroup $I_f$ of $E(N)$, consisting of all maps $h$ that satisfy all the properties that $f$ does, with the same system of projections $P$, and such that $h$ commutes with $f$. For reasons that will be clear later, we call the triple $\{f, P, I_f\}$ a linearizing triple. It is immediate to verify that all properties listed for a linearizing triple are preserved under conjugation by $\psi$, i.e. if $\{f, P, I_f\}$ is a linearizing triple in $\mathbb{C}^n$ at $a$, then $\{g, Q, I_g\}$ is a linearizing triple in $\mathcal{M}$ at $b$, where $g = \psi \circ f \circ \psi^{-1}$, $Q = \psi \circ P \circ \psi^{-1}$.

We note here that in general it is impossible to linearize a holomorphic map in a neighborhood of its attracting fixed point due to the presence of resonances among the eigenvalues of its linear part [2], [14]. However, as seen in the following proposition, under the assumption that $h \circ p_i = p_i \circ h$, $\forall i = 1, \ldots, n$, the local linearization of $h \in I_f$ is possible.

**Proposition 1.** For every linearizing triple $\{f, P, I_f\}$ in a complex manifold $N$ at $o$, there exists a biholomorphic map $\theta$ from a neighborhood of $o$ onto a neighborhood of the origin in $\mathbb{C}^n$, such that for every $h \in I_f$, in some neighborhood of $o$,

\[
\theta \circ h = \Lambda_h \circ \theta,
\]

\[
\theta \circ p_i = P_i \circ \theta, \quad \forall i,
\]

where $\Lambda_h$ is a diagonal linear map $(z_1, \ldots, z_n) \mapsto (\lambda_1 z_1, \ldots, \lambda_n z_n)$, $\lambda_i$, $i = 1, \ldots, n$ satisfy $0 < |\lambda_i| < 1$, and are eigenvalues of the linear part of $h$ at $o$, and $P_i$ is a diagonal matrix similar to the linear part of $p_i$ at $o$.

The proof of this proposition follows the same arguments as in [13], and therefore we give only an outline here. Because of the property that for every arbitrary compact set and every neighborhood of $o$, some iterate of $f$ brings the compact set into that neighborhood, it follows that the eigenvalues of the linear part of $f$ at $o$ are smaller than 1 in absolute value. Using the fact that projections are locally linearizable [4], and the commutativity relations $h \circ p_i = p_i \circ h$, $\forall i$, the problem about local linearization reduces to the one-dimensional Schröder equation, which is solved [4]. That all maps $h$ are linearized by the same biholomorphic map $\theta$ follows from the uniqueness of the solution to the Schröder equation, and the commutativity relations between $f$, $h$, and $p_i$.

We see in the following lemma that all invertible diagonal linear maps whose entries are smaller than 1 in absolute value appear in $\mathcal{I}$.

**Lemma 1.** The map $\theta$ extends to a biholomorphic map of $N$ onto $\mathbb{C}^n$. Moreover, if $\Lambda$ is a diagonal linear map

\[
(z_1, \ldots, z_n) \mapsto (\lambda_1 z_1, \ldots, \lambda_n z_n), \quad 0 < |\lambda_i| < 1, \quad i = 1, \ldots, n,
\]

then there exists $h \in I_f$, such that

\[
\theta \circ h = \Lambda \circ \theta.
\]
Proof. First we show that the map \( \theta \) extends to a biholomorphic map on the whole \( \mathcal{N} \). This can be seen by using the formula

\[
\theta = \Lambda_f^{-k} \circ \theta \circ f^k, \quad k = 1, 2, \ldots
\]

Because of the property that for every compact subset \( K \) of \( \mathcal{N} \) and every neighborhood \( U \) of \( o \), some iterate of \( f \) brings \( K \) into \( U \), it follows from (6) that \( \theta \) can be extended to larger and larger sets, until its domain fills the whole \( \mathcal{N} \). Since \( f \) and \( \Lambda_f \) are biholomorphisms, \( \theta \) is injective, and hence a biholomorphism on \( \mathcal{N} \). The inverse of \( \theta \) has a representation similar to (6), and therefore \( \theta \) is onto.

Consider a map \( h = \theta^{-1} \circ \Lambda \circ \theta \in E(\mathcal{N}) \). It is a biholomorphism of \( \mathcal{N} \) onto itself, and it commutes with every \( p_i \), which follows from (6). Since all entries of \( \Lambda \) are less than 1 in absolute value, it is clear that for every compact subset \( K \) and a neighborhood \( U \) of \( o \), some iterate of \( h \) brings \( K \) into \( U \). Using (6), we conclude that \( h \) commutes with \( f \), and thus it belongs to \( I_f \).

4. Matrix equation

Using the results of the previous section, we convert the statement of Theorem 4 to a linearized version, thus reducing the problem to determining the exact form of the solution of a matrix equation (7) below. By finding this solution, we obtain an explicit expression for a map \( L \) defined below, which is conjugate to \( \psi \) via biholomorphic maps. This, with some more effort, will lead us to the proof that \( \psi \) is either biholomorphic, or antibiholomorphic.

We denote by \( \mathcal{D}_0 \) the set of invertible diagonal \( n \times n \) matrices whose entries are less than 1 in absolute value, and we denote by \( \mathcal{D}_n \) the set of all diagonal \( n \times n \) matrices. We identify \( \mathcal{D}_0 \) with the set of diagonal linear maps, and \( \mathcal{D}_n \) with a multiplicative semigroup \( \mathbb{C}^n \) in the obvious way, and consider a topology on \( \mathcal{D}_n \) induced by the standard topology on \( \mathbb{C}^n \).

In the previous section, we showed that if \( \{f, P, I_f\} \) is a linearizing triple in \( \mathbb{C}^n \) at \( a \), then \( \theta : \mathbb{C}^n \to \mathbb{C}^n \) conjugates \( I_f \) to the set of diagonal linear maps, which is isomorphic to \( \mathcal{D}_0 \). Similarly, for a linearizing triple \( \{g, Q, I_g\} \) at \( b \in M \), where \( g = \psi \circ f \circ \psi^{-1}, \ Q = \psi \circ P \circ \psi^{-1}, \ I_g \) is conjugated by a biholomorphic map \( \eta : M \to \mathbb{C}^n \) to \( \mathcal{D}_0 \).

We define a homeomorphism \( L \) on \( \mathbb{C}^n \) by \( L = \eta \circ \psi \circ \theta^{-1} \). For every \( \Lambda \) in \( \mathcal{D}_0 \) we have

\[
L \circ \Lambda \circ L^{-1} = \eta \circ \psi \circ \theta^{-1} \circ \Lambda \circ \theta \circ \psi^{-1} \circ \eta^{-1} = \eta \circ \psi \circ h \circ \psi^{-1} \circ \eta^{-1} = \eta \circ j \circ \eta^{-1} = M,
\]

where \( h = \theta^{-1} \circ \Lambda \circ \theta \in I_f; \ j = \psi \circ h \circ \psi^{-1}, \ M = \eta \circ j \circ \eta^{-1}, \ M \in \mathcal{D}_0 \). Therefore the conjugation by \( L \) defines an injective map \( R \) from \( \mathcal{D}_0 \) to \( \mathcal{D}_0 \), \( R(\Lambda) = L \circ \Lambda \circ L^{-1} \), which is trivially multiplicative, i.e. \( R(\Lambda' \Lambda'') = R(\Lambda') R(\Lambda''), \ \Lambda', \ \Lambda'' \in \mathcal{D}_0 \).

Since \( R \) is continuous, it extends to a multiplicative map, which will also be denoted by \( R \) for convenience, from the subset \( \overline{\mathcal{D}_0} \) of \( \mathcal{D}_n \) that consists of all matrices in \( \mathcal{D}_n \) with entries less than or equal to 1 in absolute value, into itself. Indeed, for every matrix \( \Gamma \) in \( \mathcal{D}_0 \), the image \( R(\Gamma) \) also belongs to \( \overline{\mathcal{D}_0} \), which follows from the continuation process. Now we extend \( R \) to all of \( \mathcal{D}_n \) as follows. Let \( \Gamma \) be an arbitrary matrix in \( \mathcal{D}_n \). We choose \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) in \( \mathcal{D}_0 \) such that \( \sum_{i=1}^n |\lambda_i| \leq 1 \) and \( \Gamma \Lambda \in \overline{\mathcal{D}_0} \). Define

\[
R(\Gamma) = R(\Gamma \Lambda) R(\Lambda)^{-1}.
\]
The extended map $R$ is well defined. Indeed, if $Λ'$ is a different matrix with the same properties as $Λ$, then $R(ΓΛ)R(Λ') = R(ΓΛ')R(Λ)$, and the conclusion follows from the commutativity relations for diagonal matrices. The map $R$ is clearly injective, and multiplicative,

$$R(Λ′Λ'') = R(Λ′)R(Λ''), \quad Λ′, Λ'' ∈ D_n.$$  

We denote by $δ_i$ the diagonal $n \times n$ matrix which has 1 as its $ii$'th entry and all other entries 0. The system \{δ_i\}_{i=1}^n is clearly the only one in $D_n$ which satisfies $δ_i \neq 0$, $δ_i^2 = δ_i$, $δ_i δ_j = 0$, $\forall i \neq j$. Therefore, injectivity of $R$ and (1) imply that $R(δ_i) = δ_j$, $\forall i$, where $j = j(i)$ is a permutation. In particular,

$$R(δ_iΛ) = δ_jR(Λ).$$

If we denote the $jj$'th entry of the diagonal matrix $R(Λ)$ by $r_j(λ_1, ..., λ_n)$, then (8) implies that $r_j$ depends on $λ_i$ only. For convenience, we write $r_j(λ_1, ..., λ_n) = r_j(λ_i)$. We can rewrite (4) as

$$r_j(λ_i'λ_i'') = r_j(λ_i')r_j(λ_i''), \quad i = 1, ..., n, \quad j = j(i).$$

As in (4), for every $j = j(i)$, the equation (9) has either the constant solution $r_j(λ_i) = 1$, or

$$r_j(λ_i) = λ_i^{α_j}λ_i^{β_j}, \quad α_{ij}, β_{ij} ∈ C, \quad α_{ij} - β_{ij} = ±1,$$

where the last relation between $α_{ij}$ and $β_{ij}$ is forced by the injectivity of the map $R$. Using (4), we can obtain an explicit expression for $L$:

$$L(z_1, ..., z_n) = \begin{bmatrix} z_1^{α_{1}(1)}z_i^{α_{1}(1)} & ... & z_i^{α_{1}(n)}z_i^{α_{1}(n)} \\
...
& ...
& ...

\end{bmatrix}L(1, ..., 1) = B(z_1^{α_{1}(1)}z_i^{β_{1}(1)}, ..., z_i^{α_{1}(n)}z_i^{β_{1}(n)}), \quad α_i - β_i = ±1, \quad i = 1, ..., n,$$

where $i = j(i)$ is an inverse permutation to $j = j(i)$, and $B$ is a constant matrix.

By definition, $ψ = η^{-1} ∘ L ∘ θ$. From the expression (4) for $L$ we can conclude that $ψ$ is $R$-differentiable and non-degenerate in $C^n \setminus θ^{-1}(A)$, where $A = \cup_{k=1}^n \{z_1, ..., z_n : z_k = 0\}$. Since the set $θ^{-1}(A)$ is analytic, and using the standard continuation argument for holomorphic maps, we can assume that the map $ψ$ is $R$-differentiable and non-degenerate everywhere in $C^n$. However, this is possible if and only if $α_i + β_i = 1$, $i = 1, ..., n$. Combining this with the equation $α_i - β_i = ±1$, we deduce that either $α_i = 1$, $β_i = 0$, or $α_i = 0$, $β_i = 1$.

It remains to show that either $α_i = 1$, $∀i$, or $α_i = 0$, $∀i$. To get a contradiction, suppose that

$$L(z_1, ..., z_n) = B(z_1, ..., z_i, ..., z_n).$$

Then

$$L^{-1}(w_1, ..., w_n) = (..., l_i(w_1, ..., w_n), ..., l_j(w_1, ..., w_n), ...),$$

where $l_i$, $l_j$ are nonconstant, linear holomorphic functions. Let $θ = (θ_1, ..., θ_n)$. We consider a map $h ∈ E$ in the form

$$h = θ^{-1}(..., θ_iθ_j, ..., θ_j, ...)θ,$$

where $θ_iθ_j$ is the $i$'th coordinate, and $θ_j$ is the $j$'th coordinate. Using the definition of $L$, we obtain

$$η ∘ ψ ∘ h ∘ η^{-1} = L ∘ θ ∘ h ∘ θ^{-1} ∘ L^{-1} = B(..., l_i(w_1, ..., w_n)l_j(w_1, ..., w_n), ..., l_j(w_1, ..., w_n), ...).$$
5. Epimorphism between semigroups

In this section we give a proof of Theorem 2.

For a complex manifold $\mathcal{N}$ we denote by $C(\mathcal{N})$ the subsemigroup of $E(\mathcal{N})$ consisting of constant maps. If $\mathcal{N} = \mathbb{C}^n$, we denote $C = C(\mathbb{C}^n)$. In other words,

\begin{equation}
(12) \quad c \in C(\mathcal{N}) \text{ if and only if } \forall f \in E(\mathcal{N}), \ c \circ f = c.
\end{equation}

There is a natural one-to-one correspondence between the constant maps in $E(\mathcal{N})$ and points of $\mathcal{N}$: for each $z \in \mathcal{N}$ there exists $c_z$ that maps $\mathcal{N}$ to $z$, and conversely, for each $c \in C(\mathcal{N})$ there exists $z \in \mathcal{N}$, such that $c = c_z$.

**Lemma 2.** Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be complex manifolds, with $\mathcal{N}_1$ being weakly doubly-transitive. Let $\Phi : E(\mathcal{N}_1) \to E(\mathcal{N}_2)$ be an epimorphism of semigroups. Then there exists a bijective map $\Psi : \mathcal{N}_1 \to \mathcal{N}_2$ such that

\begin{equation}
(13) \quad \Phi(f) = \Psi \circ f \circ \Psi^{-1}, \quad \forall f \in E(\mathcal{N}_1).
\end{equation}

**Proof.** Because of (12), and the assumption that $\Phi$ is an epimorphism, for every $c \in C(\mathcal{N}_1)$ we have that $\Phi(c) \in C(\mathcal{N}_2)$. Now we can define a map $\Psi : \mathcal{N}_1 \to \mathcal{N}_2$ as follows

\begin{equation}
\Psi(z) = w \text{ if and only if } \Phi(c_z) = c_w.
\end{equation}

Let $f$ be arbitrary map in $E(\mathcal{N}_1)$. Then

\begin{equation}
(14) \quad f \circ c_z = c_{f(z)}.
\end{equation}

Applying $\Phi$ to both sides of (14), we obtain

\begin{equation}
\Phi(f) \circ c_{\Psi(z)} = c_{\Phi(f(z))},
\end{equation}

which is equivalent to

\begin{equation}
(15) \quad \Phi(f) \circ \Psi = \Psi \circ f.
\end{equation}

Equation (15) implies surjectivity of $\Psi$. Indeed, since $\Phi$ is an epimorphism, for every $w \in \mathcal{N}_2$, there exists $f \in E(\mathcal{N}_1)$, such that $\Phi(f) = c_w$. Therefore, by (15), $\Psi \circ f(z) = w$, $\forall z \in \mathcal{N}_1$, which implies that $\Psi$ is onto.

We prove that $\Psi$ is injective by showing that for every $w \in \mathcal{N}_2$ the full preimage $S_w = \Psi^{-1}(w)$ consists of one point. Assume, by contradiction, that $S_w$ consists of more than one point for some $w$. It cannot be all of $\mathcal{N}_1$, since $\Psi$ is onto. Let $z_1$ be a point in $S_w$, such that in arbitrary neighborhood of it there exist a point in $\mathcal{N}_1 \setminus S_w$. Let $z_2$ be arbitrary point in $S_w$, different from $z_1$. From our assumption that $\mathcal{N}_1$ is weakly doubly-transitive, it follows that there exists $h \in E(\mathcal{N}_1)$, such that $h(z_1) = z_1 \in S_w$, and $h(z_2) \notin S_w$. Evaluating $\Phi(h)$ at $w$, and applying (15) we have

\begin{align*}
\Phi(h)(w) &= \Phi(h) \circ \Psi(z_1) = \Psi \circ h(z_1) = \Psi(z_1) = w, \\
\Phi(h)(w) &= \Phi(h) \circ \Psi(z_2) = \Psi \circ h(z_2) \neq w,
\end{align*}

which is a contradiction. Thus we proved that $\Psi$ is a bijection, and the equation (13) follows from (15). \(\square\)
The first part of Theorem 2 now follows from Lemma 1 and Theorem 1 if we choose \( N_1 = \mathbb{C}^n \) and \( N_2 = M \). The second part follows if we take \( N_1 = M \), \( N_2 = \mathbb{C}^n \), and observe that equation (13) implies that \( \Phi \) is an isomorphism. \( \square \)

6. ISOMORPHIC SEMIGROUPS FOR INEQUIVALENT MANIFOLDS

In this section we prove Theorem 3. We construct the domains \( D_1 \) and \( D_2 \) by taking direct sums of \( n \) copies of domains as in Hinkkanen [7]. From [7], we know that there exist unbounded domains \( U_1, U_2 \in \mathbb{C} \) such that \( U_1 \) is neither conformally nor anticonformally equivalent to \( U_2 \), and such that \( E(U_1) \), and \( E(U_2) \) are isomorphic and consist of the constants plus the identity. One such choice of domains is given by \( U_1 = \mathbb{C} \setminus \{0, 1, 2\} \), and \( U_2 = \mathbb{C} \setminus \{0, 1, 2, \ldots\} \). We set \( D_1 = U_1 \times \cdots \times U_1 \), \( D_2 = U_2 \times \cdots \times U_2 \), and verify that for these domains the conclusion of Theorem 3 holds.

Let \( F \in E(D_m) \), \( m = 1, 2 \). Then each component \( f_j \) of \( F \) maps \( D_m \) holomorphically into \( U_m \). Therefore, by the choice of \( U_m \), if we fix all \( z_k \), \( k = 1, \ldots, n, k \neq i \), then the induced map \( g_j(z_i) \) is in \( E(U_m) \), hence is either a constant map or the identity. Since \( f_j \) is a continuous function in a domain, which is a direct sum of domains in \( \mathbb{C} \), we conclude that it is identically equal to either a constant, or \( z_i \) for some \( i = 1, \ldots, n \). Using this description of the elements in \( E(D_m) \), we can easily show that \( E(D_1) \) and \( E(D_2) \) are isomorphic. Let \( \xi \) be a bijective map from \( U_1 \) onto \( U_2 \). If \( F \) is an endomorphism of \( D_1 \), whose components are \( f_1, \ldots, f_n \), then we set \( \phi(F) \) to be an endomorphism of \( D_2 \), whose \( j \)’th component is \( z_j \) if \( f_j = z_j \), and \( \xi(c) \) if \( f_j = c \), a constant map. It is a simple matter to verify that the map \( \phi \), so defined, is an isomorphism of semigroups.

To show that \( D_1 \) and \( D_2 \) are not biholomorphically or antibiholomorphically equivalent, we argue by contradiction. Suppose first that there exists a biholomorphic map \( F \) from \( D_1 \) onto \( D_2 \). Let \( g \) be a non-constant restriction of a component of \( F \) to a coordinate axis. Such a component exists, since otherwise the map \( F \) would be constant. Since \( g \) omits more than two points, each of the points \( 0, 1, 2, \infty \) must be a removable singularity or a pole. Therefore, \( g \) extends to a rational map. But this is a contradiction, because \( g \) omits infinitely many points. Similarly, we arrive at a contradiction by assuming that there exists an antibiholomorphic map from \( D_1 \) onto \( D_2 \), and applying the same argument to a conjugate map. \( \square \)

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