GALOIS EXTENSIONS, PLUS CLOSURE, AND MAPS ON LOCAL COHOMOLOGY

AKIYOSHI SANNAI AND ANURAG K. SINGH

Abstract. Given a local domain \((R, m)\) of prime characteristic that is a homomorphic image of a Gorenstein ring, Huneke and Lyubeznik proved that there exists a module-finite extension domain \(S\) such that the induced map on local cohomology modules \(H^i_m(R) \rightarrow H^i_m(S)\) is zero for each \(i < \dim R\). We prove that the extension \(S\) may be chosen to be generically Galois, and analyze the Galois groups that arise.

1. Introduction

Let \(R\) be a commutative Noetherian integral domain. We use \(R^+\) to denote the integral closure of \(R\) in an algebraic closure of its fraction field. Hochster and Huneke proved the following:

**Theorem 1.1.** [HH2, Theorem 1.1] If \(R\) is an excellent local domain of prime characteristic, then each system of parameters for \(R\) is a regular sequence on \(R^+\), i.e., \(R^+\) is a balanced big Cohen-Macaulay algebra for \(R\).

It follows that for a ring \(R\) as above, and \(i < \dim R\), the local cohomology module \(H^i_m(R^+)\) is zero. Hence, given an element \([\eta]\) of \(H^i_m(R)\), there exists a module-finite extension domain \(S\) such that \([\eta]\) maps to 0 under the induced map \(H^i_m(R) \rightarrow H^i_m(S)\). This was strengthened by Huneke and Lyubeznik, albeit under mildly different hypotheses:

**Theorem 1.2.** [HL, Theorem 2.1] Let \((R, m)\) be a local domain of prime characteristic that is a homomorphic image of a Gorenstein ring. Then there exists a module-finite extension domain \(S\) such that the induced map
\[
H^i_m(R) \rightarrow H^i_m(S)
\]
is zero for each \(i < \dim R\).

By a generically Galois extension of a domain \(R\), we mean an extension domain \(S\) that is integral over \(R\), such that the extension of fraction fields is Galois; \(\text{Gal}(S/\text{R})\) will denote the Galois group of the corresponding extension of fraction fields. We prove the following:

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Theorem 1.3. Let \( R \) be a domain of prime characteristic.

1. Let \( a \) be an ideal of \( R \) and \([\eta]\) an element of \( H^1_a(R)_{\text{nil}} \) (see Section 2.3). Then there exists a module-finite generically Galois extension \( S \), with \( \text{Gal}(S/R) \) a solvable group, such that \([\eta]\) maps to 0 under the induced map \( H^1_a(R) \rightarrow H^1_a(S) \).

2. Suppose \((R, m)\) is a homomorphic image of a Gorenstein ring. Then there exists a module-finite generically Galois extension \( S \) such that the induced map \( H^i_m(R) \rightarrow H^i_m(S) \) is zero for each \( i < \dim R \).

Set \( R^{+\text{sep}} \) to be the \( R \)-algebra generated by the elements of \( R^{+} \) that are separable over \( \text{frac}(R) \). Under the hypotheses of Theorem 1.3(2), \( R^{+\text{sep}} \) is a separable balanced big Cohen-Macaulay \( R \)-algebra; see Corollary 3.3. In contrast, the algebra \( R^{\infty} \), i.e., the purely inseparable part of \( R^{+} \), is not a Cohen-Macaulay \( R \)-algebra in general: take \( R \) to be an \( F \)-pure domain that is not Cohen-Macaulay; see [HH2, page 77].

For an \( \mathbb{N} \)-graded domain \( R \) of prime characteristic, Hochster and Huneke proved the existence of a \( \mathbb{Q} \)-graded Cohen-Macaulay \( R \)-algebra \( R^{+\text{GR}} \), see Theorem 5.1. In view of this and the preceding paragraph, it is natural to ask whether there exists a \( \mathbb{Q} \)-graded separable Cohen-Macaulay \( R \)-algebra; in Example 5.2 we show that the answer is negative.

In Example 5.3 we construct an \( \mathbb{N} \)-graded domain of prime characteristic for which no module-finite \( \mathbb{Q} \)-graded extension domain is Cohen-Macaulay.

We also prove the following results for closure operations; the relevant definitions may be found in Section 2.1.

Theorem 1.4. Let \( R \) be an integral domain of prime characteristic, and let \( a \) be an ideal of \( R \).

1. Given \( z \in a^F \), there exists a module-finite generically Galois extension \( S \), with \( \text{Gal}(S/R) \) a solvable group, such that \( z \in aS \).

2. Given \( z \in a^{+} \), there exists a module-finite generically Galois extension \( S \) such that \( z \in aS \).

In Example 4.1 we present a domain \( R \) of prime characteristic where \( z \in a^{+} \) for an element \( z \) and ideal \( a \), and conjecture that \( z \notin aS \) for each module-finite generically Galois extension \( S \) with \( \text{Gal}(S/R) \) a solvable group. Similarly, in Example 4.3 we present a 3-dimensional ring \( R \) where we conjecture that \( H^2_a(R) \rightarrow H^2_a(S) \) is nonzero for each module-finite generically Galois extension \( S \) with \( \text{Gal}(S/R) \) a solvable group.

Remark 1.5. The assertion of Theorem 1.2 does not hold for rings of characteristic zero: Let \((R, m)\) be a normal domain of characteristic zero, and \( S \) a module-finite extension domain. Then the field trace map \( \text{tr}: \text{frac}(S) \rightarrow \text{frac}(R) \) provides an \( R \)-linear splitting of \( R \subseteq S \), namely

\[
\frac{1}{[\text{frac}(S) : \text{frac}(R)]} \text{tr}: S \rightarrow R.
\]
It follows that the induced maps on local cohomology $H^i_m(R) \to H^i_m(S)$ are $R$-split. A variation is explored in [RSS], where the authors investigate whether the image of $H^i_m(R)$ in $H^i_m(R^+)$ is killed by elements of $R^+$ having arbitrarily small positive valuation. This is motivated by Heitmann’s proof of the direct summand conjecture for rings $(R, m)$ of dimension 3 and mixed characteristic $p > 0$, [He], which involves showing that the image of $H^2_m(R) \to H^2_m(R^+)$ is killed by $p^{1/n}$ for each positive integer $n$. 

Throughout this paper, a local ring refers to a commutative Noetherian ring with a unique maximal ideal. Standard notions from commutative algebra that are used here may be found in [BH]; for more on local cohomology, consult [ILL]. For the original proof of the existence of big Cohen-Macaulay modules for equicharacteristic local rings, see [Ho].

2. Preliminary Remarks

2.1. Closure operations. Let $R$ be an integral domain. The plus closure of an ideal $a$ is the ideal $a^+ = aR^+ \cap R$.

When $R$ is a domain of prime characteristic $p > 0$, we set

$$R^\infty = \bigcup_{e \geq 0} R^{1/p^e} ,$$

which is a subring of $R^+$. The Frobenius closure of an ideal $a$ is the ideal $a^F = aR^\infty \cap R$. Alternatively, set

$$a^{[p^e]} = (a^{p^e} \mid a \in a) .$$

Then $a^F = \{ r \in R \mid r^{p^e} \in a^{[p^e]} \text{ for some } e \in \mathbb{N} \}$.

2.2. Solvable extensions. A finite separable field extension $L/K$ is solvable if $\text{Gal}(M/K)$ is a solvable group for some Galois extension $M$ of $K$ containing $L$. Solvable extensions form a distinguished class, i.e.,

(1) for finite extensions $K \subseteq L \subseteq M$, the extension $M/K$ is solvable if and only if each of $M/L$ and $L/K$ are solvable;

(2) for finite extensions $L/K$ and $M/K$ contained in a common field, if $L/K$ is solvable, then so is the extension $LM/M$.

A finite separable extension $L/K$ of fields of characteristic $p > 0$ is solvable precisely if it is obtained by successively adjoining

(1) roots of unity;

(2) roots of polynomials $T^n - a$ for $n$ coprime to $p$;

(3) roots of Artin-Schreier polynomials, $T^p - T - a$. 
2.3. Frobenius-nilpotent submodules. Let $R$ be a ring of prime characteristic $p$. A Frobenius action on an $R$-module $M$ is an additive map $F: M \rightarrow M$ with $F(rm) = r^pF(m)$ for each $r \in R$ and $m \in M$. In this case, $\ker F$ is a submodule of $M$, and we have an ascending sequence

$$\ker F \subseteq \ker F^2 \subseteq \ker F^3 \subseteq \ldots$$

The union of these is the $F$-nilpotent submodule of $M$, denoted $M_{\text{nil}}$. If $R$ is local and $M$ is Artinian, then there exists a positive integer $e$ such that $F^e(M_{\text{nil}}) = 0$; see [Ly, Proposition 4.4] or [HS, Theorem 1.12].

3. Proofs

We record two elementary results that will be used later:

**Lemma 3.1.** Let $K$ be a field of characteristic $p > 0$. Let $a$ and $b$ elements of $K$ where $a$ is nonzero. Then the Galois group of the polynomial $T^p + at - b$ is a solvable group.

**Proof.** Form an extension of $K$ by adjoining a primitive $p - 1$ root of unity and an element $c$ that is a root of $T^p - 1 - a$. The polynomial $T^p + at - b$ has the same roots as $(T/c)^p - (T/c) - b/c^p$, which is an Artin-Schreier polynomial in $T/c$. □

**Lemma 3.2.** Let $R$ be a domain, and $\mathfrak{p}$ a prime ideal. Given a domain $S$ that is a module-finite extension of $R_{\mathfrak{p}}$, there exists a domain $T$, module-finite over $R$, with $T_{\mathfrak{p}} = S$.

**Proof.** Given $s_i \in S$, there exists $r_i \in R \setminus \mathfrak{p}$ such that $r_is_i$ is integral over $R$. If $s_1, \ldots, s_n$ are generators for $S$ as an $R$-module, set $T = R[r_1s_1, \ldots, r_ns_n]$. □

**Proof of Theorem L.3.** Since solvable extensions form a distinguished class, (1) reduces by induction to the case where $F([\eta]) = 0$. Compute $H^i\Psi_R(x)$ using a Čech complex $C^\bullet(x; R)$, where $x = x_0, \ldots, x_n$ are nonzero elements generating the ideal $\mathfrak{a}$; recall that $C^\bullet(x; R)$ is the complex

$$0 \rightarrow R \rightarrow \bigoplus_{i=0}^{n} R_{x_i} \rightarrow \bigoplus_{i<j} R_{x_ix_j} \rightarrow \cdots \rightarrow R_{x_0 \cdots x_n} \rightarrow 0.$$ 

Consider a cycle $\eta$ in $C^i(x; R)$ that maps to $[\eta]$ in $H^i_{\Psi_R}(R)$. Since $F([\eta]) = 0$, the cycle $F(\eta)$ is a boundary, i.e., $F(\eta) = \partial(\alpha)$ for some $\alpha \in C^{i-1}(x; R)$.

Let $\mu_1, \ldots, \mu_m$ be the square-free monomials of degree $i-2$ in the elements $x_1, \ldots, x_n$, and regard $C^{i-1}(x; R) = C^{i-1}(x_0, \ldots, x_n; R)$ as $R_{x_0\mu_1} \oplus \cdots \oplus R_{x_0\mu_m} \oplus C^{i-1}(x_1, \ldots, x_n; R)$. 

There exist a power $q$ of the characteristic $p$ of $R$, and elements $b_1, \ldots, b_m$ in $R$, such that $\alpha$ can be written in the above direct sum as

$$\alpha = \left( \frac{b_1}{(x_0\mu_1)^q}, \ldots, \frac{b_m}{(x_0\mu_m)^q} \right).$$

Consider the polynomials

$$T^p + x_0^qT - b_i \quad \text{for } i = 1, \ldots, m,$$

and let $L$ be a finite extension field where these have roots $t_1, \ldots, t_m$ respectively. By Lemma 3.1, we may assume $L$ is Galois over frac($R$) with the Galois group being solvable. Let $S$ be a module-finite extension of $R$ that contains $t_1, \ldots, t_m$, and has $L$ as its fraction field; if $R$ is excellent, we may take $S$ to be the integral closure of $R$ in $L$.

In the module $C^{i-1}(x; S)$ one then has

$$\alpha = \left( \frac{t_1^p + x_0^q t_1}{(x_0\mu_1)^q}, \ldots, \frac{t_m^p + x_0^q t_m}{(x_0\mu_m)^q}, \ast, \ldots, \ast \right) = F(\beta) + \gamma,$$

where

$$\beta = \left( \frac{t_1}{(x_0\mu_1)^{q/p}}, \ldots, \frac{t_m}{(x_0\mu_m)^{q/p}}, 0, \ldots, 0 \right)$$

and

$$\gamma = \left( \frac{t_1}{\mu_1^q}, \ldots, \frac{t_m}{\mu_m^q}, \ast, \ldots, \ast \right)$$

are elements of

$$C^{i-1}(x; S) = S_{x_0\mu_1} \oplus \cdots \oplus S_{x_0\mu_m} \oplus C^{i-1}(x_1, \ldots, x_n; S).$$

Since $F(\eta) = \partial(F(\beta) + \gamma)$, we have

$$F(\eta - \partial(\beta)) = \partial(\gamma).$$

But $[\eta] = [\eta - \partial(\beta)]$ in $H^1_\alpha(S)$, so after replacing $\eta$ we may assume that

$$F(\eta) = \partial(\gamma).$$

Next, note that $\gamma$ is an element of $C^{i-1}(1, x_1, \ldots, x_n; S)$, viewed as a submodule of $C^{i-1}(x; S)$. There exits $\zeta$ in $C^{i-2}(1, x_1, \ldots, x_n; S)$ such that

$$\partial(\zeta) = \left( \frac{t_1}{\mu_1^q}, \ldots, \frac{t_m}{\mu_m^q}, \ast, \ldots, \ast \right).$$

Since

$$F(\eta) = \partial(\gamma - \partial(\zeta)),$$

after replacing $\gamma$ we may assume that the first $m$ coordinate entries of $\gamma$ are 0, i.e., that

$$\gamma = \left( 0, \ldots, 0, \frac{c_1}{\lambda_1^Q}, \ldots, \frac{c_l}{\lambda_l^Q} \right),$$

where $Q$ is a power of $p$, the $c_i$ belong to $S$, and $\lambda_1, \ldots, \lambda_l$ are the square-free monomials of degree $i - 1$ in $x_1, \ldots, x_n$. 
The coordinate entries of \( \partial(\gamma) \) include each \( c_i/\lambda^Q_i \). Since \( \partial(\gamma) = F(\eta) \), each \( c_i/\lambda^Q_i \) is a \( p \)-th power in \( \text{frac}(S) \); it follows that each \( c_i \) has a \( p \)-th root in \( \text{frac}(S) \). After enlarging \( S \) by adjoining each \( c_i^{1/p} \), we see that \( \gamma = F(\xi) \) for an element \( \xi \) of \( C^{i-1}(x; S) \). But then
\[
F(\eta) = \partial(F(\xi)) = F(\partial(\xi)).
\]
Since the Frobenius action on \( C^i(x; S) \) is injective, we have \( \eta = \partial(\xi) \), which proves (1).

For (2), it suffices to construct a module-finite generically separable extension \( S \) such that \( \text{H}_i^m(R) \rightarrow \text{H}_i^m(S) \) is zero for \( i < \dim R \); to obtain a generically Galois extension, enlarge \( S \) to a module-finite extension whose fraction field is the Galois closure of \( \text{frac}(S) \) over \( \text{frac}(R) \).

We use induction on \( d = \dim R \), as in [HL]. If \( d = 0 \), there is nothing to be proved; if \( d = 1 \), the inductive hypothesis is again trivially satisfied since \( \text{H}_1^m(R) = 0 \). Fix \( i < \dim R \). Let \((A, \mathfrak{M})\) be a Gorenstein local ring that has \( R \) as a homomorphic image, and set
\[
M = \text{Ext}^d_{A} A^{-i}(R, A).
\]
Let \( p_1, \ldots, p_s \) be the elements of the set \( \text{Ass}_A M \setminus \{\mathfrak{M}\} \).

Let \( q \) be a prime ideal of \( R \) that is not maximal. Since \( R \) is catenary, one has
\[
\dim R = \dim R_q + \dim R/q.
\]
Thus, the condition \( i < \dim R \) may be rewritten as
\[
i - \dim R/q < \dim R_q.
\]
Using the inductive hypothesis and Lemma 3.2, there exists a module-finite extension \( R' \) of \( R \) such that \( \text{frac}(R') \) is a separable field extension of \( \text{frac}(R) \), and the induced map
\[
H_{qR_q}^{i-\dim R/q}(R_q) \rightarrow H_{qR_q}^{i-\dim R/q}(R_q')
\]
is zero. Taking the compositum of finitely many such separable extensions inside a fixed algebraic closure of \( \text{frac}(R) \), there exists a module-finite generically separable extension \( R' \) of \( R \) such that the map (3.2.1) is zero when \( q \) is any of the primes \( p_1R, \ldots, p_sR \). We claim that the image of the induced map \( H_{qR_q}^m(R) \rightarrow H_{qR_q}^m(R') \) has finite length.

Using local duality over \( A \), it suffices to show that
\[
M' = \text{Ext}_{A}^{\dim A^{-i}}(R', A) \rightarrow \text{Ext}_{A}^{\dim A^{-i}}(R, A) = M
\]
has finite length. This, in turn, would follow if
\[
M'_p = \text{Ext}_{A_p}^{\dim A^{-i}}(R'_p, A_p) \rightarrow \text{Ext}_{A_p}^{\dim A^{-i}}(R_p, A_p) = M_p
\]
is zero for each prime ideal \( p \) in \( \text{Ass}_A M \setminus \{\mathfrak{M}\} \). Using local duality over \( A_p \), it suffices to verify the vanishing of
\[
H_{pR_p}^{\dim A_p^{-\dim A+i}}(R_p) \rightarrow H_{pR_p}^{\dim A_p^{-\dim A+i}}(R'_p)
\]
for each \( p \) in \( \text{Ass}_A M \setminus \{M\} \). This, however, follows from our choice of \( R' \) since
\[
\dim A_p - \dim A + i = i - \dim A/p = i - \dim R/pR.
\]

What we have arrived at thus far is a module-finite generically separable extension \( R' \) of \( R \) such that the image of \( H^i_m(R) \to H^i_m(R') \) has finite length; in particular, this image is finitely generated. Working with one generator at a time and taking the compositum of extensions, given \([\eta]\) in \( H^i_m(R') \), it suffices to construct a module-finite generically separable extension \( S \) of \( R' \) such that \([\eta]\) maps to 0 under \( H^i_m(R') \to H^i_m(S) \).

By Theorem 1.2, there exists a module-finite extension \( R_1 \) of \( R' \) such that \([\eta]\) maps to 0 under \( H^i_m(R') \to H^i_m(R_1) \). Setting \( R_2 \) to be the separable closure of \( R' \) in \( R_1 \), the image of \([\eta]\) in \( H^i_m(R_2) \) lies in \( H^i_m(R_2)_\text{nil} \). The result now follows by (1). \( \square \)

**Corollary 3.3.** Let \((R, m)\) be a local domain of prime characteristic that is a homomorphic image of a Gorenstein ring. Then \( H^i_m(R^{+\text{sep}}) = 0 \) for each \( i < \dim R \).

Moreover, each system of parameters for \( R \) is a regular sequence on \( R^{+\text{sep}} \), i.e., \( R^{+\text{sep}} \) is a separable balanced big Cohen-Macaulay algebra for \( R \).

**Proof.** Theorem 1.3 (2) implies that \( H^i_m(R^{+\text{sep}}) = 0 \) for each \( i < \dim R \). The proof that this implies the second statement is similar to the proof of [HL, Corollary 2.3]. \( \square \)

**Proof of Theorem 1.4.** Let \( p \) be the characteristic of \( R \). If \( z \in a^p \), there exists a prime power \( q = p^e \) with \( z^q \in a[q] \). In this case, \( z^{q/p} \) belongs to the Frobenius closure of \( a^{[q/p]} \), and
\[
(z^{q/p})^p \in (a^{[q/p]})^{[p]}.
\]

Since solvable extensions form a distinguished class, we reduce to the case \( e = 1 \), i.e., \( q = p \).

There exist nonzero elements, \( a_0, \ldots, a_m \in a \) and \( b_0, \ldots, b_m \in R \) with
\[
z^p = \sum_{i=0}^m b_i a_i^p.
\]

Consider the polynomials
\[
T^p + a_0^pT - b_i \quad \text{for } i = 1, \ldots, m,
\]
and let \( L \) be a finite extension field where these have roots \( t_1, \ldots, t_m \) respectively. By Lemma 3.1, we may assume \( L \) is Galois over \( \text{frac}(R) \) with the Galois group being solvable. Set
\[
t_0 = \frac{1}{a_0} \left( z - \sum_{i=1}^m t_i a_i \right).
\]

(3.3.1)
Taking $p$-th powers, we have

$$t_0^p = \frac{1}{a_0^p} \left( \sum_{i=0}^m b_i a_i^p - \sum_{i=1}^m t_i^p a_i^p \right) = b_0 + \frac{1}{a_0^p} \sum_{i=1}^m (b_i - t_i^p) a_i^p = b_0 + \sum_{i=1}^m t_i a_i^p.$$ 

Thus, $t_0$ belongs to the integral closure of $R[t_1, \ldots, t_m]$ in its field of fractions. Let $S$ be a module-finite extension of $R$ that contains $t_0, \ldots, t_m$, and has $L$ as its fraction field; if $R$ is excellent, we may take $S$ to be the integral closure of $R$ in $L$. Since (3.3.1) may be rewritten as

$$z = \sum_{i=0}^m t_i a_i,$$

it follows that $z \in \mathfrak{a}S$, completing the proof of (1).

(2) follows from [SII2, Corollary 3.4], though we include a proof using (1). There exists a module-finite extension domain $T$ such that $z \in \mathfrak{a}T$. Decompose the field extension $\text{frac}(R) \subseteq \text{frac}(T)$ as a separable extension $\text{frac}(R) \subseteq \text{frac}(T)^{+\text{sep}}$ followed by a purely inseparable extension $\text{frac}(T)^{+\text{sep}} \subseteq \text{frac}(T)$.

Let $T_0$ be the integral closure of $R$ in $\text{frac}(T)^{+\text{sep}}$.

Since $T$ is a purely inseparable extension of $T_0$, and $z \in \mathfrak{a}T$, it follows that $z$ belongs to the Frobenius closure of the ideal $\mathfrak{a}T_0$. By (2) there exists a generically separable extension $S_0$ of $T_0$ with $z \in \mathfrak{a}S_0$. Enlarge $S_0$ to a generically Galois extension $S$ of $R$. This concludes the argument in the case $R$ is excellent; in the event that $S$ is not module-finite over $R$, one may replace it by a subring satisfying $z \in \mathfrak{a}S$ and having the same fraction field.

The equational construction used in the proof of Theorem 1.4(1) arose from the study of symplectic invariants in [SII].

4. SOME GALOIS GROUPS THAT ARE NOT SOLvable

Let $R$ be a domain of prime characteristic, and let $\mathfrak{a}$ be an ideal of $R$. If $z$ is an element of $\mathfrak{a}^p$, Theorem 1.4(1) states that there exists a solvable module-finite extension $S$ with $z \in \mathfrak{a}S$. In the following example one has $z \in \mathfrak{a}^+$, and we conjecture $z \not\in \mathfrak{a}S$ for any module-finite generically Galois extension $S$ with $\text{Gal}(S/R)$ solvable.

**Example 4.1.** Let $a, b, c_1, c_2$ be algebraically independent over $\mathbb{F}_p$, and set $R$ be the hypersurface

$$\mathbb{F}_p(a, b, c_1, c_2)[x, y, z]$$

$$(zp^2 + c_1(xy)p^2 - p\zeta^2 + c_2(xy)p^2 - 1z + axp^2 + byp^2).$$

We claim $z \in (x, y)^+$. Let $u, v$ be elements of $R^+$ that are, respectively, roots of the polynomials

$$T^p + c_1y^p - pT^p + c_2y^{p^2-1}T + a,$$

and

$$T^p + c_1x^p - pT^p + c_2x^{p^2-1}T + b.$$
Set $S$ to be the integral closure of $R$ in the Galois closure of $\frac{\text{frac}(R)(u,v)}{\text{frac}(R)}$. Then $(z - ux - vy)/xy$ is an element of $S$, since it is a root of the monic polynomial

$$T^p + c_1 T + c_2 T.$$ 

It follows that $z \in (x, y)S$.

We next show that $\text{Gal}(S/R)$ is not solvable for the extension $S$ constructed above. Since $u$ is a root of (4.1.1), $u/y$ is a root of (4.1.2)

$$T^p + c_1 T + c_2 T + \frac{a}{y^{p^2}}.$$ 

The polynomial (4.1.2) is irreducible over $\mathbb{F}_q(c_1, c_2, a/y^{p^2})$, and hence over the purely transcendental extension $\mathbb{F}_q(c_1, c_2, a, x, y, z) = \text{frac}(R)$. Since $\text{frac}(S)$ is a Galois extension of $\text{frac}(R)$ containing a root of (4.1.2), it contains all roots of (4.1.2). As (4.1.2) is separable, its roots are distinct; taking differences of roots, it follows that $\text{frac}(S)$ contains the $p^2$ distinct roots of

$$T^p + c_1 T + c_2 T.$$ 

We next verify that the Galois group of (4.1.3) over $\text{frac}(R)$ is $GL_2(\mathbb{F}_p)$.

Quite generally, let $L$ be a field of characteristic $p$. Consider the standard linear action of $GL_2(\mathbb{F}_p)$ on the polynomial ring $L[x_1, x_2]$. The ring of invariants for this action is generated over $L$ by the Dickson invariants $c_1, c_2$, which occur as the coefficients in the polynomial

$$\prod_{\alpha, \beta \in \mathbb{F}_p} (T - \alpha x_1 - \beta x_2) = T^p + c_1 T + c_2 T,$$

see [Di] or [Be, Chapter 8]. Hence the extension $L(x_1, x_2)/L(c_1, c_2)$ has Galois group $GL_2(\mathbb{F}_p)$.

It follows from the above that if $c_1, c_2$ are algebraically independent elements over a field $L$ of characteristic $p$, then the polynomial

$$T^p + c_1 T + c_2 T \in L(c_1, c_2)[T]$$

has Galois group $GL_2(\mathbb{F}_p)$.

The group $PSL_2(\mathbb{F}_p)$ is a subquotient of $GL_2(\mathbb{F}_p)$, and, we conjecture, a subquotient of $\text{Gal}(S/R)$ for any module-finite generically Galois extension $S$ of $R$ with $z \in aS$. For $p \geq 5$, the group $PSL_2(\mathbb{F}_p)$ is a nonabelian simple group; thus, conjecturally, $\text{Gal}(S/R)$ is not solvable for any module-finite generically Galois extension $S$ with $z \in aS$.

**Example 4.2.** Extending the previous example, let $a, b, c_1, \ldots, c_n$ be algebraically independent elements over $\mathbb{F}_q$, and set $R$ to be the polynomial ring $\mathbb{F}_q(a, b, c_1, \ldots, c_n)[x, y, z]$ modulo the principal ideal generated by

$$z^{q^n} + c_1(xy)^{q^n-q^{n-1}} z^{q^{n-1}} + c_2(xy)^{q^n-q^{n-2}} z^{q^{n-2}} + \cdots + c_n(xy)^{q^n-1} z + ax^{q^n} + by^{q^n}.$$
Then $z \in (x, y)^+$; imitate the previous example with $u, v$ being roots of

$$T^{q^n} + c_1 y^{q^n-q^{-1}} T^{q^{n-1}} + c_2 y^{q^n-q^{-2}} T^{q^{n-2}} + \cdots + c_n y^{q^n-1} T + a,$$

and

$$T^{q^n} + c_1 x^{q^n-q^{-1}} T^{q^{n-1}} + c_2 x^{q^n-q^{-2}} T^{q^{n-2}} + \cdots + c_n x^{q^n-1} T + b.$$

If $S$ is any module-finite generically Galois extension of $R$ with $z \in aS$, we conjecture that $\text{frac}(S)$ contains the splitting field of

$$T^{q^n} + c_1 T^{q^{n-1}} + c_2 T^{q^{n-2}} + \cdots + c_n T.$$  \hspace{1cm} (4.2.1)

Using a similar argument with Dickson invariants, the Galois group of (4.2.1) over $\text{frac}(R)$ is $\text{GL}_n(\mathbb{F}_q)$. Its subquotient $\text{PSL}_n(\mathbb{F}_q)$ is a nonabelian simple group for $n \geq 3$, and for $n = 2, q \geq 4$.

Likewise, we record conjectural examples $R$ where $H^i_m(R) \rightarrow H^i_m(S)$ is nonzero for each module-finite generically Galois extension $S$ with $\text{Gal}(S/R)$ solvable:

**Example 4.3.** Let $a, b, c_1, c_2$ be algebraically independent over $\mathbb{F}_p$, and consider the hypersurface

$$A = \frac{\mathbb{F}_p(a, b, c_1, c_2)[x, y, z]}{(z^p + c_1(xy)^p + c_2(xy)^p z^p + ax y^2 + by z^2)}.$$  

Let $(R, m)$ be the Rees ring $A[xt, yt, zt]$ localized at the maximal ideal $x, y, z, xt, yt, zt$. The elements $x, yt, y + xt$ form a system of parameters for $R$, and the relation

$$z^2 t \cdot (y + xt) = z^2 t^2 \cdot x + z^2 \cdot yt$$

defines an element $[\eta]$ of $H^2_m(R)$. We conjecture that if $S$ is any module-finite generically Galois extension such that $[\eta]$ maps to 0 under the induced map $H^2_m(R) \rightarrow H^2_m(S)$, then $\text{frac}(S)$ contains the splitting field of

$$T^{p^2} + c_1 T^p + c_2 T,$$

and hence that $\text{Gal}(S/R)$ is not solvable if $p \geq 5$.

5. **Graded rings and extensions**

Let $R$ be an $\mathbb{N}$-graded domain that is finitely generated over a field $R_0$. Set $R^{+\text{GR}}$ to be the $\mathbb{Q}_{\geq 0}$-graded ring generated by elements of $R^+$ that can be assigned a degree such that they then satisfy a homogeneous equation of integral dependence over $R$. Note that $[R^{+\text{GR}}]_0$ is the algebraic closure of the field $R_0$. One has the following:

**Theorem 5.1.** [HH2, Theorem 6.1] Let $R$ be an $\mathbb{N}$-graded domain that is finitely generated over a field $R_0$ of prime characteristic. Then each homogeneous system of parameters for $R$ is a regular sequence on $R^{+\text{GR}}$. 
Let $R$ be as in the above theorem. Since $R^{+\text{GR}}$ and $R^{+\text{sep}}$ are Cohen-Macaulay $R$-algebras, it is natural to ask whether there exists a $\mathbb{Q}$-graded separable Cohen-Macaulay $R$-algebra. The answer to this is negative:

**Example 5.2.** Let $R$ be the Rees ring
\[
\frac{\mathbb{F}_2[x, y, z]}{(x^3 + y^3 + z^3)}[xt, yt, zt]
\]
with the $\mathbb{N}$-grading where the generators $x, y, z, xt, yt, zt$ have degree 1. Set $B$ to be the $R$-algebra generated by the homogeneous elements of $R^{+\text{GR}}$ that are separable over $\text{frac}(R)$. We prove that $B$ is not a balanced Cohen-Macaulay $R$-module.

The elements $x, yt, y + xt$ are a system of parameters for $R$. Suppose, to the contrary, that they form a regular sequence on $B$. Since $z^2t \cdot (y + xt) = z^2t \cdot x + z^2 \cdot yt$, it follows that $z^2t \in (x, yt)B$. Thus, there exist elements $u, v \in B_1$ with
\[
(5.2.1) \quad z^2t = u \cdot x + v \cdot yt.
\]
Since $z^3 = x^3 + y^3$, we also have $z^2 = x\sqrt{xyz} + y\sqrt{yz}$ in $R^{+\text{GR}}$, and hence
\[
(5.2.2) \quad z^2t = t\sqrt{xyz} \cdot x + \sqrt{yz} \cdot yt.
\]
Comparing (5.2.1) and (5.2.2), we see that
\[
(u + t\sqrt{xyz}) \cdot x = (v + \sqrt{yz}) \cdot yt
\]
in $R^{+\text{GR}}$. But $x, yt$ is a regular sequence on $R^{+\text{GR}}$, so there exists an element $c$ in $[R^{+\text{GR}}]_0$ with $u + t\sqrt{xyz} = c yt$ and $v + \sqrt{yz} = cx$. Since $[R^{+\text{GR}}]_0 = \mathbb{F}_2$, it follows that $c \in R$, and hence that $\sqrt{yz} \in B$. This contradicts the hypothesis that elements of $B$ are separable over $\text{frac}(R)$.

The above argument shows that any graded Cohen-Macaulay $R$-algebra must contain the elements $\sqrt{yz}$ and $t\sqrt{xyz}$.

We next show that no module-finite $\mathbb{Q}$-graded extension domain of the ring $R$ in Example 5.2 is Cohen-Macaulay.

**Example 5.3.** Let $R$ be the Rees ring from Example 5.2 and let $S$ be a graded Cohen-Macaulay ring with $R \subseteq S \subseteq R^{+\text{GR}}$. We prove that $S$ is not finitely generated over $R$.

By the previous example, $S$ contains $\sqrt{yz}$ and $t\sqrt{xyz}$. Using the symmetry between $x, y, z$, it follows that $\sqrt{xyz}, \sqrt{xz}, t\sqrt{xy}, t\sqrt{yz}$ are all elements of $S$. We prove inductively that $S$ contains
\[
(5.3.1) \quad x^{1-2/q}(yz)^{1/q}, \quad y^{1-2/q}(xz)^{1/q}, \quad z^{1-2/q}(xy)^{1/q},
\]
\[
tx^{1-2/q}(yz)^{1/q}, \quad ty^{1-2/q}(xz)^{1/q}, \quad tz^{1-2/q}(xy)^{1/q},
\]
for each $q = 2^e$ with $e \geq 1$. The case $e = 1$ has been settled.
Suppose $S$ contains the elements (5.3.1) for some $q = 2^e$. Then, one has
\[
x^{1-2/q}(yz)^{1/q} \cdot ty^{1-2/q}(xz)^{1/q} \cdot (y + xt)
\]
Using as before that
\[
x^{1-2/q}(yz)^{1/q} \cdot ty^{1-2/q}(xz)^{1/q} \cdot x + x^{1-2/q}(yz)^{1/q} \cdot y^{1-2/q}(xz)^{1/q} \cdot yt.
\]
Using as before that, $x, y, z$ is a regular sequence on $S$, we conclude
\[
x^{1-2/q}(yz)^{1/q} \cdot ty^{1-2/q}(xz)^{1/q} = u \cdot x + v \cdot yt
\]
for some $u, v \in S_1$. Simplifying the left hand side, the above reads
\[
(5.3.2) \quad t(xy)^{1-1/q}z^{2/q} = u \cdot x + v \cdot yt.
\]
Taking $q$-th roots in
\[
z^2 = x\sqrt{xz} + y\sqrt{yz}
\]
and multiplying by $t(xy)^{1-1/q}$ yields
\[
(5.3.3) \quad t(xy)^{1-1/q}z^{2/q} = ty^{1-1/q}(xz)^{1/2q} \cdot x + x^{1-1/q}(yz)^{1/2q} \cdot yt.
\]
Comparing (5.3.2) and (5.3.3), we see that
\[
(u + ty^{1-1/q}(xz)^{1/2q}) \cdot x = (v + x^{1-1/q}(yz)^{1/2q}) \cdot yt,
\]
so there exists $c$ in $[R^{+\text{GR}}]_0 = \mathbb{F}_2$ with
\[
u + ty^{1-1/q}(xz)^{1/2q} = cyt \quad \text{and} \quad v + x^{1-1/q}(yz)^{1/2q} = cx.
\]
It follows that $ty^{1-1/q}(xz)^{1/2q}$ and $x^{1-1/q}(yz)^{1/2q}$ are elements of $S$. In view of the symmetry between $x, y, z$, this completes the inductive step. Setting
\[
\theta = \frac{xy}{z^2},
\]
we have proved that
\[
\theta^{1/q} \in \text{frac}(S) \quad \text{for each } q = 2^e.
\]
We claim $\theta^{1/2}$ does not belong to frac$(R)$. Indeed if it does, then $(xy)^{1/2}$ belongs to frac$(R)$, and hence to $R$, as $R$ is normal; this is readily seen to be false. The extension
\[
\text{frac}(R) \subseteq \text{frac}(R)(\theta^{1/q})
\]
is purely inseparable, so the minimal polynomial of $\theta^{1/q}$ over frac$(R)$ has the form $T^Q - \theta^{Q/q}$ for some $Q = 2^e$. Since $\theta^{1/2} \notin \text{frac}(R)$, we conclude that the minimal polynomial is $T^q - \theta$. Hence
\[
\left[\text{frac}(R)(\theta^{1/q}) : \text{frac}(R)\right] = q \quad \text{for each } q = 2^e.
\]
It follows that $[\text{frac}(S) : \text{frac}(R)]$ is not finite.

Theorem 1.2 and Theorem 1.3(2) discuss the vanishing of the image of $H^i_m(R)$ for $i < \dim R$. In the case of graded rings, one also has the following result for $H^d_m(R)$.
Proposition 5.4. Let $R$ be an $\mathbb{N}$-graded domain that is infinitely generated over a field $\mathbb{Q}$ of prime characteristic. Set $d = \dim R$. Then the submodule $[H^d_m(R)]_{\geq 0}$ maps to zero under the induced map

$$H^d_m(R) \rightarrow H^d_m(R^{GR}).$$

Hence, there exists a module-finite $\mathbb{Q}$-graded extension domain $S$ of $R$ such that the induced map $[H^d_m(R)]_{\geq 0} \rightarrow H^d_m(S)$ is zero.

Proof. Let $F^e : H^d_m(R) \rightarrow H^d_m(R)$ denote the $e$-th iteration of the Frobenius map. Suppose $[\eta] \in [H^d_m(R)]_n$ for some $n \geq 0$. Then $F^e([\eta])$ belongs to $[H^d_m(R)]_{n+e}$ for each $e$. As $[H^d_m(R)]_{\geq 0}$ has finite length, there exists $e \geq 1$ and homogeneous elements $r_1, \ldots, r_e \in R$ such that

$$F^e([\eta]) + r_1 F^{e-1}([\eta]) + \cdots + r_e [\eta] = 0. \tag{5.4.1}$$

We imitate the equational construction from [HL]: Consider a homogeneous system of parameters $\mathbf{x} = x_1, \ldots, x_d$, and compute $H^i_m(R)$ as the cohomology of the Čech complex $C^\bullet(\mathbf{x}; R)$ below:

$$0 \rightarrow R \rightarrow \bigoplus_{i=1}^d R_{x_i} \rightarrow \bigoplus_{i<j} R_{x_i x_j} \rightarrow \cdots \rightarrow R_{x_1 \cdots x_d} \rightarrow 0.$$

This complex is $\mathbb{Z}$-graded; let $\eta$ be a homogeneous element of $C^d(\mathbf{x}; R)$ that maps to $[\eta]$ in $H^d_m(R)$. Equation (5.4.1) implies that

$$F^e(\eta) + r_1 F^{e-1}(\eta) + \cdots + r_e \eta$$

is a boundary in $C^d(\mathbf{x}; R)$, say it equals $\partial(\alpha)$ for a homogeneous element $\alpha$ of $C^{d-1}(\mathbf{x}; R)$. Solving integral equations in each coordinate of $C^{d-1}(\mathbf{x}; R)$, there exists a module-finite extension domain $S$ and $\beta$ in $C^{d-1}(\mathbf{x}; S)$ with

$$F^e(\beta) + r_1 F^{e-1}(\beta) + \cdots + r_e \beta = \alpha.$$

Moreover, we may assume $S$ is a normal ring. Since $\eta - \partial(\beta)$ is an element on $\text{frac}(S)$ satisfying

$$T^e + r_1 T^{e-1} + \cdots + r_e T = 0,$$

it belongs to $S$. But then $\eta - \partial(\beta)$ maps to zero in $H^d_m(S)$. Thus, each homogeneous element of $[H^d_m(R)]_{\geq 0}$ maps to 0 in $H^d_m(R^{GR})$.

For the final statement, note that $[H^d_m(R)]_{\geq 0}$ has finite length. \hfill \qed

The next example illustrates why Proposition 5.4 is limited to $[H^d_m(R)]_{\geq 0}$.

Example 5.5. Let $K$ be a field of prime characteristic, and take $R$ to be the semigroup ring

$$R = K[x_1 \cdots x_d, \ x_1^d, \ldots, x_d^d].$$

It is easily seen that $R$ is normal, and that $[H^d_m(R)]_n$ is nonzero for each integer $n < 0$. We claim that the induced map

$$H^d_m(R) \rightarrow H^d_m(S)$$

...
is injective for each module-finite extension ring $S$. For this, it suffices to check that $R$ is a splinter ring, i.e., that $R$ is a direct summand of each module-finite extension ring; the splitting of $R \subseteq S$ then induces an $R$-splitting of $H^d_m(R) \to H^d_m(S)$.

To check that $R$ is a splinter ring, note that normal affine semigroup rings are weakly $F$-regular by [HH1, Proposition 4.12], and that weakly $F$-regular rings are splinter by [HH3, Theorem 5.25]. For more on splinters, we point the reader towards [Ma, HH3, Si3].

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E-mail address: sannai@ms.u-tokyo.ac.jp
Graduate School of Mathematical Sciences, The University of Tokyo, 3-8-1 Komaba, Meguro, Tokyo, 153-8914 Japan

E-mail address: singh@math.utah.edu

Department of Mathematics, University of Utah, 155 S. 1400 E., Salt Lake City, UT 84112, USA