Semiclassical Boltzmann magnetotransport theory in anisotropic systems with a nonvanishing Berry curvature

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Abstract. Understanding the transport behavior of an electronic system under the influence of a magnetic field remains a key subject in condensed matter physics. Particularly in topological materials, their nonvanishing Berry curvature can lead to many interesting phenomena in magnetotransport owing to the coupling between the magnetic field and Berry curvature. By fully incorporating both the field-driven anisotropy and inherent anisotropy in the band dispersion, we study the semiclassical Boltzmann magnetotransport theory in topological materials with a nonvanishing Berry curvature. We show that as a solution to the Boltzmann transport equation the effective mean-free-path vector is given by the integral equation, including the effective velocity arising from the coupling between the magnetic field, Berry curvature and mobility. We also calculate the conductivity of Weyl semimetals with an isotropic energy dispersion, and find that the coupling between the magnetic field and Berry curvature induces anisotropy in the relaxation time, showing a substantial deviation from the result obtained assuming a constant relaxation time.

1. Introduction

The effect of a magnetic field on transport behavior has always been a topic of interest in condensed matter physics. Adding another tuning knob (magnetic field) to electronic transport experiments can enhance the understanding of the material of interest. In this regard, magnetotransport measurement can be a useful tool to reveal numerous fascinating features that a material hides. The quantum Hall effect \cite{1}, for example, has been brought to light by magnetoresistance (MR) experiments.

In particular, topological materials with a nonvanishing Berry curvature such as Weyl semimetals or topological insulators, display several interesting magnetotransport behaviors such as negative MR. Recently, the negative MR in Weyl semimetals \cite{2,18} and topological insulators \cite{19,20} has received significant attention. In a strong magnetic field regime, Landau-level-limited quantum magnetotransport is predominant \cite{30,37},
whereas in the weak magnetic field regime, the charge transport can be described by the semiclassical formalism [2, 3, 8, 52].

Most of the studies using the semiclassical approach utilize a simple relaxation time approximation assuming isotropy of the system. However, this approximation could be problematic when the band dispersion of the system is highly anisotropic, and the system is no longer approximated as an isotropic system. Furthermore, in topological materials with a nonvanishing Berry curvature, this isotropic approximation cannot account for the anisotropy that arises from the coupling between the magnetic field and the Berry curvature. In other words, although the system is isotropic in the band dispersion, the magnetic field can induce anisotropy through coupling with the Berry curvature, which cannot be captured by the simple isotropic formalism.

With these motivations, a fully anisotropic Boltzmann magnetotransport equation is formulated that incorporates anisotropy from the energy dispersion as well as anisotropy arising from the coupling between the magnetic field and Berry curvature. Although several studies tried to consider a field-dependent anisotropy [17, 53], our approach utilizes a more generally applicable method. In this work, the integral equation for the effective mean-free-path vector is obtained including the effective velocity due to the coupling between the magnetic field, Berry curvature and mobility. The Boltzmann equation is solved by introducing an ansatz for the nonequilibrium distribution function with a minimal set of assumptions that encompasses the electric field, magnetic field, and Berry curvature, and the corresponding magnetoconductivity was obtained. As an application of our method, we numerically calculate the conductivity of Weyl semimetals and find that the coupling between the magnetic field and Berry curvature is responsible for the field-driven anisotropy in the relaxation time, inducing a significant deviation from the result obtained assuming a constant relaxation time.

The rest of the paper is organized as follows. In section 2, the Boltzmann magnetotransport equation for an isotropic system without a Berry curvature is summarized along with the demonstration of the relaxation time equation in electron gas systems. In section 3, the key results for the Boltzmann transport equation is presented that can be applied to a general system with anisotropy arising from the energy dispersion and the coupling between the magnetic field and the Berry curvature. In section 4, the magnetoconductivity equations are given. In section 5, numerical calculations for the magnetoconductivity of Weyl semimetals are presented. Finally, the conclusions of the study are stated in section 6.

2. Magnetotransport equation in electron gas systems

First, the magnetotransport relaxation time equation for an isotropic electron gas without a Berry curvature is introduced. In this case, the equation of motion for the Bloch electrons with charge $q$ under the influence of an electric field $E$ and a magnetic field $B$ takes a simple form [54]:

$$\dot{r} = v_k, \quad \hbar \dot{k} = qE + \frac{q}{c}v_k \times B,$$

(1)
where $r$ is the position vector, $k$ is the crystal momentum, $v_k = \frac{1}{\hbar} \nabla_k \tilde{\varepsilon}_k$, $\tilde{\varepsilon}_k \equiv \varepsilon_k - m_k \cdot B$, $\varepsilon_k = \frac{\hbar^2 k^2}{2m}$ is the electronic band dispersion relation of an electron gas, $m$ is the effective electron mass, and $m_k$ is the orbital magnetic moment that vanishes for a single-band electron gas with no Berry curvature [38,55–58].

The Boltzmann transport equation can be written as

$$\frac{df}{dt} = \left( \frac{df}{dt} \right)_{\text{coll}},$$

where $f$ is the nonequilibrium distribution function, $\frac{df}{dt}$ is the time derivative of the distribution given by

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{r} \cdot \nabla_r f + \mathbf{k} \cdot \nabla_k f,$$

and $\left( \frac{df}{dt} \right)_{\text{coll}}$ is the collision integral term.

Now, it is assumed that $f$ is spatially homogeneous (in this case, no temperature gradient or nonuniform electric field) and has no explicit time dependence. Hence, (3) becomes $\frac{df}{dt} = \dot{k} \cdot \nabla_k f$. Note that the equilibrium Fermi-Dirac distribution function $f^{(0)}_k$ depends on $k$ via energy dispersion, that is, $f^{(0)}_k = f^{(0)}(\tilde{\varepsilon}_k)$, the nonequilibrium distribution function $f_k$ and its $k$ gradient can be written as

$$\nabla_k f_k = \hbar v_k \frac{\partial f^{(0)}_k}{\partial \tilde{\varepsilon}_k} + \nabla_k g_k,$$

where $g_k = f_k - f^{(0)}_k$ is the part where the field-dependent terms are contained. Then to leading order in $E$,

$$\frac{df}{dt} = \dot{k} \cdot \nabla_k f_k \approx -q E \cdot v_k S^{(0)}(\tilde{\varepsilon}_k) + \frac{q}{\hbar c} (v_k \times B) \cdot \nabla_k g_k,$$

where $S^{(0)}(\tilde{\varepsilon}_k) \equiv -\frac{\partial f^{(0)}_k}{\partial \tilde{\varepsilon}_k}$.

The collision integral term $\left( \frac{df}{dt} \right)_{\text{coll}}$ is given by [51]

$$\left( \frac{df}{dt} \right)_{\text{coll}} = -\int \frac{d^dk'}{(2\pi)^d} \frac{W_{k'k}}{\tau_k} (g_k - g_{k'}),$$

where $d$ is the dimension of the system and $W_{k'k} = \frac{2\pi}{\hbar} n_{\text{imp}} |V_{k'k}|^2 \delta(\tilde{\varepsilon}_{k'} - \tilde{\varepsilon}_k)$ is the transition rate given by the Fermi’s golden rule with the impurity potential $V_{k'k}$ and the impurity density $n_{\text{imp}}$.

Thus, the Boltzmann equation becomes

$$q E \cdot v_k S^{(0)}(\tilde{\varepsilon}_k) - \frac{q}{\hbar c} (v_k \times B) \cdot \nabla_k g_k = \int \frac{d^dk'}{(2\pi)^d} W_{k'k} (g_k - g_{k'}).$$

The definition and calculation of $g_k$ is the primary concern for solving the Boltzmann equation. When there is no magnetic field, the simple relaxation time equation is often utilized, that is, $\left( \frac{df}{dt} \right)_{\text{coll}} = -\frac{g_k}{\tau_k}$, assuming that the system relaxes back to equilibrium from the impurity scattering with a characteristic time scale given by the relaxation time $\tau_k$. The relaxation time equation is extended to incorporate the additional contribution from the magnetic field, hence, the additional term $g_k$ is given by

$$g_k = q E \cdot v_k \tau_k S^{(0)}(\tilde{\varepsilon}_k) - \frac{q}{\hbar c} \tau_k (v_k \times B) \cdot \nabla_k g_k.$$
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Then, (8) can be iteratively expanded to

\[ g_k = qE \cdot \sum_{N=0}^{\infty} \hat{S}_k^N \left[ v_k \tau_k S^{(0)}(\tilde{\varepsilon}_k) \right], \tag{9} \]

where \( \hat{S}_k \) is the operator defined by

\[ \hat{S}_k \equiv -\frac{q}{\hbar c} \tau_k (v_k \times B) \cdot \nabla_k. \tag{10} \]

It is assumed that \( \tau_k \) depends on \( k \) only through \( \tilde{\varepsilon}_k \). Since \( \nabla_k \tilde{\varepsilon}_k = \hbar v_k \), \( \hat{S}_k \) acting on \( \tau_k \) or \( S^{(0)}(\tilde{\varepsilon}_k) \) vanishes. Thus, (9) transforms into

\[ g_k = \tau_k v_k^{\text{eff}} \cdot G_k^{(0)}, \tag{11} \]

where \( G_k^{(0)} \equiv qES^{(0)}(\tilde{\varepsilon}_k) \) and \( v_k^{\text{eff}} \) is the effective velocity given by

\[ v_k^{\text{eff}} \equiv \sum_{N=0}^{\infty} \hat{S}_k^N v_k = v_k + \sum_{N=0}^{\infty} \hat{S}_k^N \left( \hat{S}_k v_k \right) = v_k - \frac{\mu_k}{c} \left( v_k^{\text{eff}} \times B \right), \tag{12} \]

where \( \mu_k = q\tau_k/m \) is the mobility of the electron gas system. Taking the vector product of \( \frac{\mu_k}{c} B \) for each side of (12),

\[ \frac{\mu_k}{c} v_k^{\text{eff}} \times B = \frac{\mu_k}{c} v_k \times B - \frac{\mu_k^2}{c^2} \left[ (v_k^{\text{eff}} \cdot B) B - B^2 v_k^{\text{eff}} \right]. \tag{13} \]

Using \( \frac{\mu_k}{c} v_k^{\text{eff}} \cdot B = \frac{\mu_k}{c} v_k \cdot B \) and (12), the following equation is obtained

\[ v_k^{\text{eff}} - v_k = -\frac{\mu_k}{c} v_k \times B + \frac{\mu_k^2}{c^2} \left[ (v_k \cdot B) B - B^2 v_k^{\text{eff}} \right]. \tag{14} \]

Thus, a closed form of \( v_k^{\text{eff}} \) is given by

\[ v_k^{\text{eff}} = \frac{v_k - \frac{\mu_k}{c} v_k \times B + \frac{\mu_k^2}{c^2} (v_k \cdot B) B}{1 + \frac{\mu_k^2}{c^2} B^2}. \tag{15} \]

Note that from (12), the effective velocity can be alternatively written as

\[ v_k^{\text{eff}} = v_k N_k, \tag{16} \]

where \( N_k \equiv (1 - \frac{\mu_k}{c} F)^{-1} \), \( F \) is the field strength tensor defined by \( F^{ij} = \sum_k \epsilon_{ijk} B^{(k)} \tag{59} \), and \( \epsilon_{ijk} \) is the Levi-Civita symbol.

Substituting \( g_k \) into (5), \( \frac{df}{dt} = -v_k^{\text{eff}} \cdot G_k^{(0)} \) is obtained. Then the Boltzmann equation can be expressed as

\[ v_k N_k G_k^{(0)} = \int \frac{d^d k'}{(2\pi)^d} W_{k'k} \tau_k \left( v_k N_k G_k^{(0)} - v_{k'} N_{k'} G_{k'}^{(0)} \right). \tag{17} \]

Here, \( N_k = N_{k'} \) and \( G_k^{(0)} = G_{k'}^{(0)} \) were used since \( G_k^{(0)} \) and \( N_k \) depend on \( k \) only through \( \tilde{\varepsilon}_k \). As \( G_k^{(0)} \propto E \) and (17) holds for all \( E \), after canceling out \( G_k^{(0)} \) and multiplying \( N_k^{-1} \) on both sides the following can be obtained

\[ \frac{1}{\tau_k} = \int \frac{d^d k'}{(2\pi)^d} W_{k'k} (1 - \cos \theta_{k'k}), \tag{18} \]
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where $\theta_{k'k}$ is the angle between $\mathbf{k}$ and $\mathbf{k}'$. Note that (18) takes exactly the same form as the $\mathbf{B} = 0$ case [54]. See [Appendix A] for the review of the conventional derivation of the magnetotransport relaxation time equation for electron gas systems [60].

In the presence of magnetic field, even in the systems with isotropic band dispersion, the relaxation time is generally expected to exhibit the anisotropic behavior due to the field-driven anisotropy. However for isotropic electron gas systems, the relaxation time is isotropic, that is, component-independent and depends on $\mathbf{k}$ only via energy, as seen in (18). This interesting feature originates from the quadratic dispersion and the absence of the Berry curvature in the system. Since the corresponding mobility and $N_k$ for the quadratic dispersion depend on $\mathbf{k}$ only via energy, $N_k$ can be canceled off from the magnetotransport equation (17), yielding the absence of preferred direction on the equation and thus the isotropy of the relaxation time. In contrast, if the system has anisotropic or non-quadratic dispersion, or a non-vanishing Berry curvature, the relaxation time is anisotropic in general so that the generalized magnetotransport equation in section 3 must be used to correctly obtain the relaxation time.

3. Magnetotransport equation in anisotropic systems with a nonvanishing Berry curvature

Until this point, only an isotropic single-band system without a Berry curvature was considered, namely an isotropic electron gas. Upon removing this restriction, the anisotropy from the electronic band structure, as well as the anisotropy that arises from the external magnetic field coupled with the Berry curvature of the system can be accounted for.

The semiclassical equation of motion for a Bloch electron in a system with a nonvanishing Berry curvature $\Omega_k$ is given by [61]

$$
\dot{\mathbf{r}} = \mathbf{v}_k - \mathbf{k} \times \Omega_k, \quad \hbar \dot{\mathbf{k}} = q \mathbf{E} + \frac{q}{c} \mathbf{r} \times \mathbf{B}.
$$

Solving (19), the following is obtained

$$
\dot{\mathbf{r}} = \frac{1}{D_k} \left[ \mathbf{v}_k - \frac{q}{\hbar} \mathbf{E} \times \Omega_k - \frac{q}{\hbar c} (\mathbf{v}_k \cdot \Omega_k) \mathbf{B} \right], \tag{20a}
$$

$$
\hbar \dot{\mathbf{k}} = \frac{1}{D_k} \left[ q \mathbf{E} + \frac{q}{c} \mathbf{v}_k \times \mathbf{B} - \frac{q^2}{\hbar c} (\mathbf{E} \cdot \mathbf{B}) \Omega_k \right], \tag{20b}
$$

where $D_k = 1 - \frac{q^2}{\hbar c} (\Omega_k \cdot \mathbf{B})$. In the presence of the Berry curvature, the phase-space volume element changes as $\Delta V_0 \to \Delta V_0 / D_k$ [38,39], modifying the density of states. Therefore, any integral over a Brillouin zone has an additional $D_k$ factor to account for this change.

Then from (3), to the leading order in $\mathbf{E}$ we have

$$
\frac{df}{dt} \approx - \hbar \mathbf{k} \cdot \mathbf{v}_k S^{(0)}(\tilde{\varepsilon}_k) + \mathbf{k} \cdot \nabla_k g_k \\
\approx - \frac{1}{D_k} \left[ q \mathbf{E} - \frac{q^2}{\hbar c} (\mathbf{E} \cdot \mathbf{B}) \Omega_k \right] \cdot \mathbf{v}_k S^{(0)}(\tilde{\varepsilon}_k) + \frac{1}{D_k} \frac{q}{\hbar c} (\mathbf{v}_k \times \mathbf{B}) \cdot \nabla_k g_k
$$
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\[ \begin{align*}
&= -qE \cdot v_k^{\text{mod}} S^{(0)}(\tilde{\epsilon}_k) + \frac{q}{\hbar c} (v_k^{\text{mod}} \times B) \cdot \nabla k g_k,
\end{align*} \]

(21)

where \( v_k^{\text{mod}} = D_k^{-1} \left[ v_k - \frac{q}{\hbar c} (\Omega_k \cdot v_k) B \right] \) is the modified velocity coupled with the magnetic field and Berry curvature.

Then, the Boltzmann equation becomes

\[ qE \cdot v_k^{\text{mod}} S^{(0)}(\tilde{\epsilon}_k) - \frac{q}{\hbar c} (v_k^{\text{mod}} \times B) \cdot \nabla k g_k = \int \frac{d^d k'}{(2\pi)^d} D_{k'k} (g_k - g_{k'}) , \]

(22)

which takes the form similar to (7), however, the velocity \( v_k \) was replaced by the modified velocity \( v_k^{\text{mod}} \) with the additional \( D_k \) factor in the collision integral.

Extending (8) in the previous section, we introduce the modified ansatz for \( g_k \) as

\[ g_k = qE \cdot l_k S^{(0)}(\tilde{\epsilon}_k) - \frac{q}{\hbar c} (l_k \times B) \cdot \nabla n g_k , \]

(23)

where \( l_k \) is the mean-free-path vector analogous to \( v_k \tau_k \) in section 2 and \( \nabla n = \nabla_k - \hbar v_k \partial \tilde{\epsilon}_k \) is the surface gradient. A detailed justification for the ansatz is presented in Appendix B and the consistency with the result in section 2 is discussed in Appendix C.

Unlike isotropic electron gas systems, \( l_k \) is in general not parallel to \( v_k \). Also note that the magnetic field couples only to the surface gradient of the distribution function since the Lorentz force does not affect the energy of the electrons.

Iteratively expanding (23), the following can be obtained

\[ g_k = G_k^{(0)} \cdot \sum_{N=0}^{\infty} \hat{S}_k^N l_k \equiv G_k^{(0)} \cdot L_k , \]

(24)

where \( G_k^{(0)} \equiv qE S^{(0)}(\tilde{\epsilon}_k), \hat{S}_k \) is given by

\[ \hat{S}_k = -\frac{q}{\hbar c} (l_k \times B) \cdot \nabla n , \]

(25)

and \( L_k \) is the effective mean-free-path vector satisfying

\[ L_k \equiv \sum_{N=0}^{\infty} \hat{S}_k^N l_k = l_k + \hat{S}_k L_k , \]

(26)

which describes the relation between the effective mean-free-path vector and conventional mean-free-path vector.

Reminding that \( G_k^{(0)} \) depends on \( k \) only through \( \tilde{\epsilon}_k \), the following integral equation for the effective mean-free-path vector is obtained by putting \( g_k \) back into (22)

\[ v_k^{\text{mod}} - \frac{q}{\hbar c} [(v_k^{\text{mod}} \times B) \cdot \nabla n] L_k = \int \frac{d^d k'}{(2\pi)^d} D_{k'k} (L_k - L_{k'}) . \]

(27)

Here, \( G_k^{(0)} \) was canceled out since \( G_k^{(0)} \propto E \) and the Boltzmann equation holds for all \( E \). Finding \( L_k \) from (27), the conductivity tensor of the systems would be obtained as discussed in section 4. Note that from the charge conservation \( \int \frac{d^d k}{(2\pi)^d} D_k g_k = 0 \), we have the following constraint for \( L_k \):

\[ \int \frac{d^d k}{(2\pi)^d} D_k L_k = 0 . \]

(28)
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The alternative form of (27) would be useful for conceptual understanding. First, the effective velocity $v^{\text{eff}}_k$ in section 2 can be generalized into

$$v^{\text{eff}}_k \equiv v^{\text{mod}}_k - \frac{q}{\hbar c} \left[ (v^{\text{mod}}_k \times B) \cdot \nabla n \right] L_k,$$

so that (27) can be rewritten as

$$v^{\text{eff}}_k = \frac{d^d k'}{(2\pi)^d} D_{k'} W_{k'k} (L_k - L_{k'}).$$

Notice that (30) has exactly the same form as the anisotropic relaxation time equation at zero magnetic field

$$v_k = \frac{d^d k'}{(2\pi)^d} W_{k'k} (l_k - l_{k'}),$$

with the relaxation time $\tau_{k}^{(i)} \equiv 1/(v^{(i)}_k/l^{(i)}_k)$ [62–66], except that the velocity $v_k$ and the mean-free-path vector $l_k$ are replaced by the effective velocity $v^{\text{eff}}_k$ and the effective mean-free-path vector $L_k$, respectively, with the additional momentum-space volume factor $D_k$ in the integral equation. On the other hand, the effective velocity given by (29) and the effective mean-free-path vector (26) can be rewritten as

$$v^{\text{eff}}_k = v^{\text{mod}}_k P_k, \quad L_k = l_k P_k,$$

where $P_k$ is given by

$$P_{k}^{(ij)} \equiv \delta_{ij} + \frac{q}{\hbar c} \sum_k F^{(ik)} \nabla^{(j)} n L_k. \quad (32)$$

Substituting (32) and iteratively solving (33), the following equation is obtained

$$P_k = \sum_{N=0}^{\infty} \left( F \frac{\hat{\mu}_n}{c} \right)^N \mathbf{1} = \left( 1 - F \frac{\hat{\mu}_n}{c} \right)^{-1} \mathbf{1}, \quad (34)$$

where $\hat{\mu}_n$ is the tensorial surface mobility operator defined by

$$\hat{\mu}_{n}^{(ij)} Y_k = \frac{q}{\hbar} \nabla^{(i)} n (l_k Y_k)^{(j)} \quad (35)$$

for an arbitrary matrix $Y_k$. Note that $P_k$ in (32) describes a transformation of the modified velocity and the mean-free-path vector to the effective velocity and the effective mean-free-path vector, respectively, by the Lorentz force. Recall that a similar transformation appears in section 2 in the form of $N_k$ in (16).

Finally, note that the relaxation time, which linearly relates the mean-free-path vector $l_k$ to the velocity in the absence of the magnetic field, is natural to be extended to the tensorial $\tau_k$ defined by

$$L_k = v^{\text{eff}}_k \tau_k.$$
4. Magnetoconductivity

Given $f_k$, the current density $J$ is given by

$$J = g_s q \int \frac{d^d k}{(2\pi)^d} D_k \hat{r} f_k \quad (37)$$

$$= g_s q \int \frac{d^d k}{(2\pi)^d} \left[ v_k - \frac{q}{\hbar} E \times \Omega_k - \frac{q}{\hbar c} (v_k \cdot \Omega_k) B \right] \left( f_k^{(0)} + g_k \right),$$

where $g_s$ is the spin degeneracy factor. Working out each term up to leading order in $E$, the following is obtained

$$J \approx J_{\text{AHE}} + J_{\text{CME}} + J_{\text{ext}}, \quad (38)$$

where

$$J_{\text{AHE}} = - \frac{g_s q^2}{\hbar} \int \frac{d^d k}{(2\pi)^d} E \times \Omega_k f_k^{(0)} \quad (39)$$

is the anomalous Hall effect (AHE) term,

$$J_{\text{CME}} = - \frac{g_s q^2}{\hbar c} \int \frac{d^d k}{(2\pi)^d} (v_k \cdot \Omega_k) B f_k^{(0)} \quad (40)$$

is the chiral magnetic effect (CME) term, and

$$J_{\text{ext}} = g_s q \int \frac{d^d k}{(2\pi)^d} D_k v_k^{\text{mod}} g_k \quad (41)$$

is the extrinsic current term involving impurity scattering. Note that $v_k f_k^{(0)}$ vanishes after integration.

The focus is on the extrinsic contribution and corresponding conductivity tensor $\sigma_{ij}^{\text{ext}}$ defined by $J^i_{\text{ext}} = \sum_j \sigma_{ij}^{\text{ext}} E_j$. For isotropic electron gas systems, from $g_k$ given by (11) and $v^{\text{eff}}_k = v_k N_k$, the conductivity $\sigma_{ij}^{\text{ext}}$ becomes

$$\sigma_{ij}^{\text{ext}} = g_s q^2 \sum_l \int \frac{d^d k}{(2\pi)^d} S^{(0)}(\tilde{\varepsilon}_k) v_k^{(i)} v_k^{(j)} N_k^{(lj)} \tau_k, \quad (42)$$

for an anisotropic system with a nonvanishing Berry curvature, from (24) and (32). It becomes

$$\sigma_{ij}^{\text{ext}} = g_s q^2 \int \frac{d^d k}{(2\pi)^d} D_k S^{(0)}(\tilde{\varepsilon}_k) v_k^{\text{mod}(i)} L_k^{(j)} , \quad (43)$$

which reduces to (42) in isotropic electron gas systems with $v_k^{\text{mod}} = v_k$, $v_k P_k = v_k N_k$ (see Appendix C), and $L_k = v_k \tau_k$.

At zero magnetic field, $P_k = 1$, $v_k^{\text{mod}} = v_k$, $\tilde{\varepsilon}_k = \varepsilon_k$, $D_k = 1$, and $L_k^{(i)} = v_k^{(i)} \tau_k^{(i)}$; thus,

$$\sigma_{ij}^{\text{ext}} = g_s q^2 \int \frac{d^d k}{(2\pi)^d} S^{(0)}(\varepsilon_k) v_k^{(i)} v_k^{(j)} \tau_k^{(j)} , \quad (44)$$

which is consistent with the conductivity equation obtained for an anisotropic system in the absence of a magnetic field [63,64].
5. Numerical calculations for Weyl semimetals

In this section, the relaxation time and the conductivity of Weyl semimetals are numerically calculated assuming the short-range impurities. In Weyl semimetals, the low-energy effective Hamiltonian for each node with chirality $\chi = \pm 1$ is given by $H_\chi(k) = \chi \hbar v_0 \sigma \cdot k$ which has isotropic linear dispersion, where $\sigma$ is the Pauli matrix vector. The magnetic field is assumed to be applied in the $z$ direction with $B = Bz$ ($B > 0$). Note that the orbital magnetic moment and Zeeman splitting are neglected for simplicity, and the internode scattering is assumed to be negligible. Here we focus on the $xx$ component of the conductivity. For the $zz$ component along with the detailed analysis on the effect of the chiral anomaly, we leave it as a future work.

We obtain the effective mean-free-path vector $L_k$ by numerically solving (27). Then, the relaxation time defined by (36) and the magnetoconductivity given by (43) are directly obtained from $L_k$. For the details, see Appendix D.

S. Woo et al. [59] introduced the dimensionless parameters $b_{BC}$ and $b_\mu$ illustrating the coupling strength of the magnetic field with the Berry curvature and mobility, respectively, at the Fermi energy which are defined by

$$b_{BC} \equiv \frac{q}{\hbar c} \Omega_k |B| = \frac{qB}{2\hbar c k_F^2}, \quad (45)$$

and

$$b_\mu \equiv \frac{\mu c}{|B|} = \frac{q v_0 \tau_{tr} B}{\hbar c k_F}, \quad (46)$$

where $k_F$ is the Fermi wavevector, and $\mu$ and $\tau_{tr}$ are the mobility and relaxation time, respectively, at zero magnetic field. Note that the ratio $b_\mu / b_{BC} \equiv 2k_F l_F$, where $l_F \equiv v_0 \tau_{tr}$ is the mean-free-path at the Fermi energy in the absence of the magnetic field.

Using these parameters, figure 1 illustrates the relaxation time $\tau_{\perp,+}(\theta)$ in the $xy$ plane (see Appendix D for the detailed definition) for the $\chi = +1$ node as a function of $\theta$, the azimuthal angle of the momentum $k$. From figure 1 we find that the anisotropy of the relaxation time induced by the magnetic field becomes larger as the coupling with the Berry curvature, $b_{BC}$, increases. Notice that when $b_{BC} = 0$, the anisotropy in the relaxation time completely vanishes, yielding the same result with the one obtained from the conventional relaxation time approximation with a constant relaxation time [45,59]. This indicates that the coupling between the magnetic field and Berry curvature is responsible for the anisotropy in the relaxation time.

Figure 2 illustrates the $xx$ component of the conductivity. As the coupling between the magnetic field and Berry curvature increases, the result substantially deviates from that obtained assuming a constant relaxation time [59].

6. Discussion

In this study, the semiclassical magnetotransport equations as well as the corresponding mean-free-path equation were derived with a minimal set of assumptions imposed to
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Figure 1. $\tau_{\perp,+}(\theta)$ normalized by $\tau_{tr}$ as a function of $\theta$ in the $\chi = +1$ node for $b_{BC}/b_\mu = 0, 0.5, \text{and } 1$, where $\tau_{tr}$ is the relaxation time in the absence of the magnetic field. Here we set $b_\mu = 0.8$.

Figure 2. The $xx$ component of the conductivity at zero temperature normalized by $\sigma_0$ as a function of $b_{BC}$ for $b_\mu = 0.8$, where $\sigma_0$ is the conductivity in the absence of the magnetic field. The result obtained assuming a constant relaxation time is presented by the dashed lines for comparison.

obtain the compact closed form of the nonequilibrium distribution function. The field-dependent, anisotropic effective mean-free-path vector $L_k$ was obtained as a solution to the nonequilibrium distribution function, which is essential for understanding the effects of impurity scattering and the magnetic field on the transport behavior. We also numerically calculated the conductivity of Weyl semimetals and found that the coupling between the magnetic field and Berry curvature induces the anisotropy in the
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relaxation time, yielding a substantial deviation from the result obtained assuming a constant relaxation time.

As shown in section 5 for Weyl semimetals, our extended Boltzmann transport theory can be applied not only when the system is inherently anisotropic, that is, the band dispersion is anisotropic, but also when the system is made to be anisotropic owing to the magnetic field. The coupling between the magnetic field and nonvanishing Berry curvature makes the distribution of electrons anisotropic, so the movement of electrons can no longer be described within an isotropic formalism, even when the band dispersion of the system is isotropic. Instead, the (effective) mean-free-path is given by the coupled integral equation as the band velocity $v_k$ acquires an additional contribution arising from the magnetic field coupled with the Berry curvature and mobility. Using the proposed formalism, any anisotropy of the system can be properly assessed, regardless of its origin. We stress that previous studies on the magnetotransport did not consider the Berry-curvature-induced anisotropy, which is essential to correctly describe transport in the systems with a nonvanishing Berry curvature.

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Appendix A. Conventional derivation of the magnetotransport equation in electron gas systems

In this section, we review the conventional derivation of the magnetotransport relaxation time equation in electron gas systems in [60] with some revision on mathematically unrigorous parts.

The alternative form of (5) and (8) can be written as $\frac{df}{dt} = -v_k \cdot G_k$ and $g_k = \tau_k v_k \cdot G_k$, where

$$G_k \equiv G_k^{(0)} + \frac{q}{\hbar c} (\nabla_k g_k - A_k v_k) \times B. \tag{A.1}$$

Here $A_k$ is chosen so that $G_k$ depends on $k$ only via $\tilde{\varepsilon}_k$. Substituting $g_k$ into (A.1), the following is obtained

$$G_k \equiv G_k^{(0)} + \frac{\mu_k}{c} G_k \times B, \tag{A.2}$$

where $\mu_k = q\tau_k / m$ is the mobility and $A_k$ is given by

$$A_k = \hbar v_k \cdot \frac{\partial}{\partial \tilde{\varepsilon}_k} (\tau_k G_k). \tag{A.3}$$

Note that we used $v_k = \hbar k / m$ in the electron gas systems. Taking the vector product of $\frac{\mu_k}{c} B$ for each side of (A.2),

$$\frac{\mu_k}{c} G_k \times B = \frac{\mu_k}{c} G_k^{(0)} \times B + \frac{\mu_k^2}{c^2} [(G_k \cdot B) B - B^2 G_k]. \tag{A.4}$$
Using $\frac{\mu_k}{e} G_k \cdot B = \frac{\mu_k}{e} G_k^{(0)} \cdot B$ and (A.2), the following equation is obtained

$$G_k - G_k^{(0)} = \frac{\mu_k}{c} G_k^{(0)} \times B + \frac{\mu_k^2}{c^2} \left[ \left( G_k^{(0)} \cdot B \right) B - B^2 G_k \right].$$  \hspace{1cm} (A.5)

Thus, a closed form of $G_k$ is given by

$$G_k = \frac{G_k^{(0)} + \frac{\mu_k}{c} G_k^{(0)} \times B + \frac{\mu_k^2}{c^2} \left( G_k^{(0)} \cdot B \right) B}{1 + \frac{\mu_k^2}{c^2} B^2},$$  \hspace{1cm} (A.6)

which is consistent with the result in [60], or equivalently $G_k = N_k G_k^{(0)}$.

Substituting $\frac{df}{dt}$ and $g_k$ to the Boltzmann equation, the following is obtained

$$v_k \cdot G_k = \int \frac{d^d k'}{(2\pi)^d} W_k' \tau_k \left( v_k \cdot G_k - v_k' \cdot G_k \right),$$  \hspace{1cm} (A.7)

where $G_k = G_k^{(0)}$ was used since $G_k$ depends on $k$ only via $\tilde{\epsilon}_k$ as can be verified in (A.6). As $G_k^{(0)} \propto E$ and the Boltzmann equation holds for all $E$, (A.7) reduces to (18) after canceling out $G_k^{(0)}$ and multiplying $N_k^{-1}$ on both sides.

Appendix B. Justification for the ansatz for $g_k$

Separating $\nabla_k = \nabla_n + \hbar v_k \frac{\partial}{\partial \tilde{\epsilon}_k}$ and using the definition of $v_k^{\text{mod}}$, (21) becomes

$$\frac{df}{dt} \approx -qE \cdot v_k^{\text{mod}} S^{(0)}(\tilde{\epsilon}_k) + \frac{q}{\hbar c} (v_k^{\text{mod}} \times B) \cdot \nabla_n g_k + \frac{q}{c} D_k (v_k^{\text{mod}} \times B) \cdot v_k^{\text{mod}} \frac{\partial g_k}{\partial \tilde{\epsilon}_k}.$$  \hspace{1cm} (B.1)

Note that the last term on the right-hand side of (B.1) actually vanishes. To extend the conventional relaxation time approximation $\frac{df}{dt} = -\frac{q k}{\tau_k}$, we need to focus on the correspondence between the Boltzmann transport theory and many-body diagrammatic theory. In the many-body diagrammatic theory, the relaxation time is introduced from the vertex correction to the current-current response function [65][67][68] transforming velocity $v_k$ into the current vertex $\Lambda^{(i)}(k) = v_k^{(i)} / \tau_k^{\text{up}}$ [65], where $\tau_k^{\text{up}}$ is the quasiparticle lifetime given by

$$\frac{1}{\tau_k^{\text{up}}} = \int \frac{d^d k'}{(2\pi)^d} W_k' k.$$

Thus, it is reasonable to obtain $g_k$ by replacing the (modified) velocity in $-\frac{df}{dt}$ with the mean-free-path vector. Thus, from (B.1), we make an ansatz for $g_k$ as

$$g_k = q E \cdot l_k S^{(0)}(\tilde{\epsilon}_k) - \frac{q}{\hbar c} (l_k \times B) \cdot \nabla_n g_k$$  \hspace{1cm} (B.3)

$$- \frac{q}{2c} D_k \left[ (l_k \times B) \cdot v_k^{\text{mod}} + (v_k^{\text{mod}} \times B) \cdot l_k \right] \frac{\partial g_k}{\partial \tilde{\epsilon}_k},$$

which reduces to the suggested ansatz in [23] since the last term vanishes. For the last term on the right-hand side of (B.3) where two modified velocities appear, we took the average after separately replacing one of $v_k^{\text{mod}}$ with $l_k$. This choice is reasonable in that the vanishing term in (B.3) also originates from the vanishing term in (B.1).
For isotropic electron gas systems where \( \mathbf{v}_k^{\text{mod}} = \mathbf{v}_k, l_k = \mathbf{v}_k \tau_k \) and \( D_k = 1 \), (B.3) becomes
\[
g_k = qE \cdot \mathbf{v}_k \tau_k S^{(0)}(\tilde{\varepsilon}_k) - \frac{q}{\hbar c} \tau_k (\mathbf{v}_k \times \mathbf{B}) \cdot \nabla_n g_k - \frac{q}{c} \tau_k (\mathbf{v}_k \times \mathbf{B}) \cdot \mathbf{v}_k \frac{\partial g_k}{\partial \tilde{\varepsilon}_k},
\]
which becomes consistent with (8) by restoring the vanishing term in (B.4) with \( \nabla_k = \nabla_n + \hbar \mathbf{v}_k \frac{\partial}{\partial \tilde{\varepsilon}_k} \).

Appendix C. Consistency between sections 2 and 3

In this section, the consistency between sections 2 and 3 is verified by showing that the result in section 2 can also be derived using the ansatz in (23).

In isotropic electron gas systems, \( L_k \) obtained from the ansatz in sections 2 and 3 are given by \( L_k = \tau_k \mathbf{v}_k N_k \) and \( L_k = \tau_k \mathbf{v}_k P_k \), respectively. Therefore, the consistency can be shown by verifying \( v_k P_k = v_k N_k \) in isotropic electron gas systems.

Using (35), \( l_k = \mathbf{v}_k \tau_k = (\hbar \tau_k / m_k) k \), and \( \nabla_n = \nabla_k - \hbar \mathbf{v}_k \frac{\partial}{\partial \tilde{\varepsilon}_k} \), we obtain the following
\[
\left( F \frac{\mu_n}{c} \right)^1 = \frac{\mu_k}{c} F - \frac{q \tau_k}{c} F V_k,
\]
where \( V_k \) is defined by
\[
V_k^{(ij)} = v_k^{(i)} \frac{\partial v_k^{(j)}}{\partial \tilde{\varepsilon}_k}.
\]
Note that \( \tau_k \) depends on \( k \) only through \( \tilde{\varepsilon}_k \). Then, using \( v_k F V_k = 0 \), we have
\[
\left( F \frac{\mu_n}{c} \right)^2 = \left( \frac{\mu_k}{c} F \right)^2 - \frac{q \tau_k}{c} F V_k \left( \frac{\mu_k}{c} F \right).
\]
Similarly repeating the process, the following is obtained
\[
\left( F \frac{\mu_n}{c} \right)^N = \left( \frac{\mu_k}{c} F \right)^N - \frac{q \tau_k}{c} F V_k \left( \frac{\mu_k}{c} F \right)^{N-1}
\]
for \( N \geq 1 \). Substituting (C.4) to (34),
\[
P_k = \left( 1 - \frac{q \tau_k}{c} F V_k \right) N_k.
\]
From \( v_k F V_k = 0, v_k P_k = v_k N_k \), which completes the derivation.

Appendix D. Details of the numerical calculation in section 5

In this section, the details of the numerical calculations in section 5 are presented. Here we focus on the \( xx \) component of the conductivity assuming that the orbital magnetic moment, Zeeman splitting, and internode scattering are negligible, for simplicity.

In isotropic 3D Weyl semimetals, the Hamiltonian for each node with chirality \( \chi = \pm 1 \) is given by
\[
H_\chi = \chi \hbar v_0 \sigma \cdot \mathbf{k},
\]
where \( \sigma \) is the Pauli matrix.
whose eigenvalues and eigenfunctions (assuming the Fermi energy on the upper band) of the Hamiltonian are given by \( \varepsilon_k = \hbar v_0 k_F r \) and

\[
|u_{k,+}\rangle = \begin{pmatrix} \cos(\theta/2) \\ \sin(\theta/2) e^{i\phi} \end{pmatrix}, \quad |u_{k,-}\rangle = \begin{pmatrix} \sin(\theta/2) \\ -\cos(\theta/2) e^{i\phi} \end{pmatrix},
\]

respectively, with the overlap factors (in a single node)

\[
F_{k'k} = \frac{1}{2} \left[ 1 + \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos (\phi - \phi') \right],
\]

where \((r, \theta, \phi)\) is the spherical coordinate representing \( \hat{k} \equiv k/k_F \), where \( k_F \) is the Fermi wavevector. Thus, the Berry curvature is given by

\[
\Omega_{k,\chi} = -\text{Im} \left[ \langle \nabla_k u_{k,\chi} \times |\nabla_k u_{k,\chi}\rangle \right] = -\chi \frac{k}{2 |k|^3} = -\frac{\chi r}{2k_F^2 r^2}.
\]

With \( B = B\hat{z} \), the phase-space volume factor \( D_{k,\chi} \) is given by

\[
D_{k,\chi} = 1 - \frac{q}{\hbar c} \left( \Omega_{k,\chi} \cdot B \right) = 1 + \frac{\chi b_{BC} \cos \theta}{r^2} \equiv D_\chi(\theta),
\]

where \( b_{BC} \) is defined by (45). Ignoring the angular magnetic moment, the velocities are simply given by

\[
v_k = v_0 \sin \theta \cos \phi \hat{x} + v_0 \sin \theta \sin \phi \hat{y} + v_0 \cos \theta \hat{z},
\]

and the modified velocities are given by \( v_{k,\chi}^{\text{mod}(i)} = v_k^{(i)} / D_{k,\chi} \) for \( i = x, y \), and

\[
v_{k,\chi}^{\text{mod}(z)} = \frac{v_0}{D_{k,\chi}} \left( \cos \theta + \frac{\chi b_{BC}}{r^2} \right).
\]

From the Fermi golden rule, we have

\[
W_{k'k} = \frac{2\pi}{\hbar} n_{\text{imp}} V_0^2 F_{k'k} \delta(\varepsilon_k - \varepsilon_{k'}),
\]

where \( V_0 \) is the impurity potential. Then (27) can be rewritten as

\[
\tilde{v}_{k,\chi}^{\text{mod}} + \frac{b_\mu}{\tau D_{k,\chi}} \frac{\partial}{\partial \phi} \tilde{L}_k = 3 \int \frac{d\theta' d\phi'}{4\pi} r^2 \sin \theta' D_{k',\chi} F_{k'k} \left( \tilde{L}_{k,\chi} - \tilde{L}_{k',\chi} \right),
\]

where \( \tilde{v}_{k,\chi}^{\text{mod}} \equiv v_{k,\chi}^{\text{mod}} / v_0 \), \( \tilde{L}_{k,\chi} \equiv L_{k,\chi} / v_0 \tau_\text{tr} \), \( b_\mu \) is defined by (46), and \( \tau_\text{tr} \) is the relaxation time in the absence of the magnetic field at the Fermi energy given by

\[
\frac{1}{\tau_\text{tr}} = \frac{12}{3} \frac{2\pi}{\hbar} n_{\text{imp}} V_{\text{intra}}^2 \rho_F = \frac{n_{\text{imp}} V_{\text{intra}}^2 k_F^2}{3\pi\hbar^2 v_0},
\]

where \( \rho_F \) is the density of states at the Fermi energy per degeneracy. By the symmetry of the system, each component of \( \tilde{L}_{k,\chi} \) should take the form

\[
\tilde{L}_{k,\chi} = \tilde{L}_{\perp,\chi}(\theta) \{ \cos [\phi - \Theta_\chi(\theta)] \hat{x} + \sin [\phi - \Theta_\chi(\theta)] \hat{y} \} + \tilde{L}_{||,\chi}(\theta) \hat{z}.
\]

Here, \( \Theta_\chi(\theta) \) describes the rotational transform induced by the magnetic field.

Then, the \( x \) component of (D.9) can be rewritten as

\[
J_{\perp,\chi}(\theta) \cos \phi - S_\chi(\theta) \tilde{L}_{\perp,\chi}(\theta) \{} \sin \phi \cos \Theta_\chi(\theta) - \sin \Theta_\chi(\theta) \cos \phi \{
= C_\chi(\theta) \tilde{L}_{\perp,\chi}(\theta) \{ \cos \phi \cos \Theta_\chi(\theta) + \sin \phi \sin \Theta_\chi(\theta) \}
- \sin \theta (\alpha_{\perp,\chi} \cos \phi + \beta_{\perp,\chi} \sin \phi),
\]
where $J_{\perp,\chi}(\theta) \equiv \sin \theta / D_{\chi}(\theta)$, $S_{\chi}(\theta) \equiv b_\mu / (r D_{\chi}(\theta))$, 
\[ C_{\chi}(\theta) = \frac{3}{2} \left( r^2 + \frac{\chi_{\text{BC}}}{3} \cos \theta \right), \] (D.13)
and
\[ \begin{pmatrix} \alpha_{\perp,\chi} \\ \beta_{\perp,\chi} \end{pmatrix} = \frac{3r^2}{8} \int_0^\pi d\theta \sin^2 \theta D_{\chi}(\theta) \begin{pmatrix} \tilde{X}_\chi(\theta) \\ \tilde{Y}_\chi(\theta) \end{pmatrix}. \] (D.14)

Here, $\tilde{X}_\chi(\theta) \equiv \tilde{L}_{\perp,\chi}(\theta) \cos \Theta_{\chi}(\theta)$ and $\tilde{Y}_\chi(\theta) \equiv \tilde{L}_{\perp,\chi}(\theta) \sin \Theta_{\chi}(\theta)$. Comparing both sides of (D.12), we have
\[ \begin{pmatrix} J_{\perp,\chi}(\theta) + \alpha_{\perp,\chi} \sin \theta \\ \beta_{\perp,\chi} \sin \theta \end{pmatrix} = \begin{pmatrix} C_{\chi}(\theta) & -S_{\chi}(\theta) \\ S_{\chi}(\theta) & C_{\chi}(\theta) \end{pmatrix} \begin{pmatrix} \tilde{X}_\chi(\theta) \\ \tilde{Y}_\chi(\theta) \end{pmatrix}. \] (D.15)

Inserting (D.15) to (D.14),
\[ \begin{pmatrix} 1 - u_{\perp,\chi} & -v_{\perp,\chi} \\ v_{\perp,\chi} & 1 - u_{\perp,\chi} \end{pmatrix} \begin{pmatrix} \alpha_{\perp,\chi} \\ \beta_{\perp,\chi} \end{pmatrix} = \begin{pmatrix} m_{\perp,\chi} \\ n_{\perp,\chi} \end{pmatrix}, \] (D.16)
where
\[ \begin{pmatrix} m_{\perp,\chi} \\ n_{\perp,\chi} \end{pmatrix} = \frac{3r^2}{8} \int_0^\pi d\theta \sin^2 \theta \frac{D_{\chi}(\theta) J_{\perp,\chi}(\theta)}{C_{\chi}(\theta)^2 + S_{\chi}(\theta)^2} \begin{pmatrix} C_{\chi}(\theta) \\ S_{\chi}(\theta) \end{pmatrix}, \] (D.17)
and
\[ \begin{pmatrix} u_{\perp,\chi} \\ v_{\perp,\chi} \end{pmatrix} = \frac{3r^2}{8} \int_0^\pi d\theta \sin^3 \theta \frac{D_{\chi}(\theta)}{C_{\chi}(\theta)^2 + S_{\chi}(\theta)^2} \begin{pmatrix} C_{\chi}(\theta) \\ S_{\chi}(\theta) \end{pmatrix}. \] (D.18)

Solving (D.16), we obtain $\tilde{X}_\chi(\theta)$ and $\tilde{Y}_\chi(\theta)$ through (D.15), and thus $\tilde{L}_{\perp,\chi}(\theta)$ and $\Theta_{\chi}(\theta)$.

Considering the rotational symmetry along the $z$ direction, it is reasonable to write the relaxation time on the $xy$ direction satisfying $L_{k,\chi}^{(i)} = \sum_j v_{k,\chi}^{\text{eff}(i) j} (\tau_{k,\chi}^{(ij)} (i, j \in \{x, y\})$ as
\[ \begin{pmatrix} \tau_{\chi}^{(xx)}(\theta) \\ \tau_{\chi}^{(yy)}(\theta) \end{pmatrix} = \tau_{\perp,\chi}(\theta) \begin{pmatrix} \cos \psi_{\chi}(\theta) & -\sin \psi_{\chi}(\theta) \\ \sin \psi_{\chi}(\theta) & \cos \psi_{\chi}(\theta) \end{pmatrix}. \] (D.19)

Noting that the left-hand-side of (D.12) corresponds to $v_{k,\chi}^{\text{eff}(x)}$, we can easily find $\tau_{\perp,\chi}(\theta)$ and $\psi_{\chi}(\theta)$ from (D.11) as follows
\[ \tilde{\tau}_{\perp,\chi}(\theta) = \frac{\tau_{\perp,\chi}(\theta)}{\tau_{\perp,\chi}} = \frac{\tilde{L}_{\perp,\chi}(\theta)}{\sqrt{[J_{\perp,\chi}(\theta) + S_{\chi}(\theta)\tilde{X}_\chi(\theta)]^2 + S_{\chi}(\theta)^2\tilde{X}_\chi^2(\theta)}}, \] (D.20)
\[ \psi_{\chi}(\theta) = \arctan \left( \frac{\tilde{Y}_\chi(\theta)}{\tilde{X}_\chi(\theta)} \right) - \arctan \left( \frac{-S_{\chi}(\theta)\tilde{X}_\chi(\theta)}{J_{\perp,\chi}(\theta) + S_{\chi}(\theta)\tilde{Y}_\chi(\theta)} \right). \] (D.21)

The angle dependence of $\tilde{\tau}_{\perp,\chi}(\theta)$ for $b_{\text{BC}}/b_\mu = 0, 0.5$, and $1$ is illustrated in figure 1 and the corresponding $\psi_{\chi}(\theta)$ is shown in figure D1.

The extrinsic magnetoconductivity given by (43) can be written at zero temperature as follows
\[ \tilde{\sigma}_{ij}^{\text{ext}} = \frac{3}{8\pi} \sum_{\chi} \int d^3k D_{k,\chi} \delta(r - 1) v_{k,\chi}^{\text{mod}(i)} \tilde{L}_{k,\chi}^{(j)}, \] (D.22)
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Figure D1. $\psi_\pm(\theta)$ as a function of $\theta$ in the $\chi = +1$ node for $b_{BC}/b_\mu = 0, 0.5, \text{ and } 1$. Here we set $b_\mu = 0.8$.

where $\tilde{\sigma}_{ij}^{\text{ext}} \equiv \sigma_{ij}^{\text{ext}}/\sigma_0$ and $\sigma_0 \equiv g_n g_0 q^2/\rho_F v_0^2 \tau_3/3$ is the conductivity in the absence of the magnetic field. Here, $g_n = 2$ is the degeneracy for the nodes. Thus, the $xx$ component of the conductivity is given by

$$\tilde{\sigma}_{xx}^{\text{ext}} = \frac{3}{8} \sum_\chi \int d\theta \sin^2 \theta \tilde{X}_{\perp, \chi}(\theta), \quad \text{(D.23)}$$

with $r = 1$. Note that $\tilde{\sigma}_{xx}^{\text{ext}}$ as a function of $b_{BC}$ for $b_\mu = 0.8$ is illustrated in figure 2.

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