ASYMPTOTIC CONE OF SEMISIMPLE ORBITS FOR SYMMETRIC PAIRS

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Abstract. Let $G$ be a reductive algebraic group over $\mathbb{C}$ and denote its Lie algebra by $\mathfrak{g}$. Let $O_h$ be a closed $G$-orbit through a semisimple element $h \in \mathfrak{g}$. By a result of Borho and Kraft [BK79], it is known that the asymptotic cone of the orbit $O_h$ is the closure of a Richardson nilpotent orbit corresponding to a parabolic subgroup whose Levi component is the centralizer $Z_G(h)$ in $G$. In this paper, we prove an analogue on a semisimple orbit for a symmetric pair.

More precisely, let $\theta$ be an involution of $G$, and $K = G^\theta$ a fixed point subgroup of $\theta$. Then we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ which is the eigenspace decomposition of $\theta$ on $\mathfrak{g}$. Let $\{x, h, y\}$ be a normal $\mathfrak{sl}_2$ triple, where $x, y \in \mathfrak{s}$ is nilpotent, and $h \in \mathfrak{k}$ semisimple. In addition, we assume $\overline{x} = y$, where $\overline{\cdot}$ denotes the complex conjugation which commutes with $\theta$. Then $a = \sqrt{-1}(x - y)$ is a semisimple element in $\mathfrak{s}$, and we can consider a semisimple orbit $\text{Ad}(K)a$ in $\mathfrak{s}$, which is closed. Our main result asserts that the asymptotic cone of $\text{Ad}(K)a$ in $\mathfrak{s}$ coincides with $\text{Ad}(G)x \cap \mathfrak{s}$, if $x$ is even nilpotent.

Introduction

Let $G$ be a connected reductive algebraic group over $\mathbb{C}$ and denote its Lie algebra by $\mathfrak{g}$. Let $h \in \mathfrak{g}$ be a semisimple element and denote by $O_h$ the adjoint $G$-orbit through $h$. It is a closed affine subvariety in $\mathfrak{g}$. With this semisimple orbit, we can associate two objects.

One object is a nilpotent orbit called a Richardson orbit. To be more precise, let us consider the centralizer $L := Z_G(h)$ of $h$. Then, there is a parabolic subgroup $P$ whose Levi component is $L$. Let us denote a Levi decomposition of the Lie algebra $\mathfrak{p}$ by $\mathfrak{l} + \mathfrak{u}$, where $\mathfrak{u}$ denotes the nilpotent radical of $\mathfrak{p}$. Then $\text{Ad}(G)\mathfrak{u}$ is the closure of a single nilpotent orbit $O$, which is called the Richardson orbit associated with $P$. The Richardson orbit $O$ in fact does not depend on the choice of the parabolic $P$, and it is determined by $h$.

The other object, which we consider, is the asymptotic cone $\mathcal{C}(O_h)$ of $O_h$, which indicates the asymptotic direction in which the variety $O_h$ spreads out. See §1 for precise definition.

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In [BK79], Borho and Kraft studied Dixmier sheets, and in the course of their study they proved the following theorem.

**Theorem 0.1** (Borho-Kraft). For a semisimple orbit $O_h$, the asymptotic cone $\mathcal{C}(O_h)$ coincides with the closure of the Richardson nilpotent orbit $\mathcal{O}$ above.

This can be interpreted as a generalization of Kostant’s theorem, which asserts that the nilpotent variety $\mathcal{N}(\mathfrak{g})$ is a deformation of the regular semisimple orbits ([Kos63]). Note that $\mathcal{N}(\mathfrak{g})$ is the closure of a principal nilpotent orbit, which is a Richardson orbit associated with a Borel subgroup. In this case, the “deformation” amounts to taking an asymptotic cone of regular semisimple orbits.

In this paper, we prove an analogous theorem for a semisimple orbit for a symmetric pair.

Let us explain it more precisely. Let $\theta$ be an involution of $G$, and $K = G^\theta$ a fixed point subgroup of $\theta$. Then we have a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ of the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ which is the eigenspace decomposition of $\theta$ on $\mathfrak{g}$. We pick a nilpotent element $x$ in $\mathfrak{s}$, and consider a normal $\mathfrak{sl}_2$ triple $\{x, h, y\}$, where $x, y \in \mathfrak{s}$ is nilpotent, and $h \in \mathfrak{k}$ semisimple. In addition, we can assume $\overline{x} = y$ without loss of generality, where $\overline{x}$ denotes the complex conjugation which commutes with $\theta$. Then $a = \sqrt{-1}(x - y)$ is a semisimple element in $\mathfrak{s}_R$, and we can consider a semisimple orbit $O^K_a = \text{Ad}(K)a$ in $\mathfrak{s}$, which is closed.

Our main result asserts that, if $x$ is even nilpotent, the asymptotic cone of $O^K_a$ in $\mathfrak{s}$ coincides with $O^G_x \cap \mathfrak{s}$, where $O^G_x = \text{Ad}(G)x$ is a nilpotent $G$-orbit through $x$. In fact, the intersection $O^G_x \cap \mathfrak{s}$ breaks up into several nilpotent $K$-orbits, $O^G_x \cap \mathfrak{s} = \bigcup_{i=0}^\ell O^K_{x_i}$, each of which is a Lagrangian subvariety of $O^G_x$. So we can state our main theorem as

**Theorem 0.2.** Suppose $x \in \mathfrak{s}$ is an even nilpotent element, and construct a semisimple element $a \in \mathfrak{s}_R$ as explained above. Then the asymptotic cone of the semisimple orbit $O^K_a$ in $\mathfrak{s}$ is given by

$$\mathcal{C}(O^K_a) = \overline{O^G_x \cap \mathfrak{s}} = \bigcup_{i=0}^\ell \overline{O^K_{x_i}}.$$ 

Note that the asymptotic cone is no longer irreducible in the case of symmetric pair. This reflects the reducibility of the nilpotent variety for symmetric pairs as pointed out by [KR71]. Our theorem can be seen as a generalization of Kostant-Rallis’s theorem.

From the semisimple element $a \in \mathfrak{s}_R$, we can construct a real parabolic subgroup $P_R$ in a standard way (see §4). The asymptotic cone above is the associated variety of a degenerate principal series representation $\text{Ind}_{P_R}^G \chi$ induced from a character $\chi$ of $P_R$. It seems that the irreducible components $O^K_{x_i}$ of $\mathcal{C}(O^K_a)$ play a important role in the theory of degenerate principal series representations. We discuss what we can expect for this, using an example in the case of $G_R = U(n, n)$ in §5.
1. Asymptotic Cone

Let $V = \mathbb{C}^N$ be a vector space. For a subvariety $X \subset V$, we define the asymptotic cone of $X$, denoted by $\mathcal{C}(X) \subset \mathbb{P}(V)$, as follows. We extend $V$ by the one-dimensional vector space, and denote it by $\tilde{V} = V \oplus \mathbb{C}$. We consider the projective space $\mathbb{P}(\tilde{V})$. Then there is a natural open embedding $\iota : V \hookrightarrow \mathbb{P}(\tilde{V})$ defined by $\iota(v) = [v \oplus 1]$, where $[w]$ denotes the image of $w \in \tilde{V} \setminus \{0\}$ in $\mathbb{P}(\tilde{V})$ under the natural projection. On the other hand, there is a closed embedding $\kappa : \mathbb{P}(V) \hookrightarrow \mathbb{P}(\tilde{V})$ which send $[u] \in \mathbb{P}(V)$ to $\kappa([u]) := [u \oplus 0] \in \mathbb{P}(\tilde{V})$. Thus we have a disjoint decomposition $\mathbb{P}(\tilde{V}) = \iota(V) \cup \kappa(\mathbb{P}(V))$. In the following, we identify $\mathbb{P}(V)$ with $\kappa(\mathbb{P}(V))$ and consider it as a closed subvariety of $\mathbb{P}(\tilde{V})$.

Definition 1.1. Let $X$ be a subvariety of $V$ of positive dimension. We define the asymptotic cone of $X$ by $\mathcal{C}(X) := \overline{\iota(X)} \cap \mathbb{P}(V)$, where $\mathbb{P}(V)$ is identified with $\kappa(\mathbb{P}(V)) \subset \mathbb{P}(\tilde{V})$. Then $\mathcal{C}(X) \subset \mathbb{P}(V)$ is a projective variety of the same dimension as $X$. The affine cone in $V$ associated to $\mathcal{C}(X)$ is denoted by $\mathfrak{C}(X)$, and we call it the affine asymptotic cone, while $\mathcal{C}(X)$ is called the projective asymptotic cone.

If $X$ is 0-dimensional, i.e., if it consists of a finite set of points, we put $\mathcal{C}(X) = \emptyset$ and $\mathfrak{C}(X) = \{0\}$.

The asymptotic cone was introduced by W. Borho and H. Kraft ([BK79]) to study Dixmier sheets of the adjoint representation of a reductive algebraic group. We refer the readers to [BK79] for the details of their properties. Here in this section we only recall some properties of asymptotic cones without proof.

Let $I$ be an ideal of the polynomial ring $\mathbb{C}[V]$. For $f \in I$, let $\text{gr} f$ be the homogeneous part of the maximal degree. We define $\text{gr} I = (\text{gr} f \mid f \in I)$, the homogeneous ideal generated by $\text{gr} f$ ($f \in I$).

Let $\mathbb{I}(X)$ be the annihilator ideal of $X$. Then the annihilator ideal of the asymptotic cone is given by $\mathbb{I}(\mathfrak{C}(X)) = \sqrt{\text{gr} \mathbb{I}(X)}$. Thus the regular function ring $\mathbb{C}[\mathfrak{C}(X)]$ is isomorphic to $\mathbb{C}[V]/\sqrt{\text{gr} \mathbb{I}(X)}$, which is equal to the homogeneous function ring of $\mathcal{C}(X)$.

Let $G$ be a connected reductive algebraic group over $\mathbb{C}$ which acts linearly on $V$ and assume that $X$ is stable under $G$. Then the ring of regular functions $\mathbb{C}[X]$ has a natural $G$-module structure. The asymptotic cone $\mathcal{C}(X)$ as well as $\mathfrak{C}(X)$ is also a $G$-variety, and we have a $G$-action on the regular function ring $\mathbb{C}[\mathfrak{C}(X)]$ in particular.

Lemma 1.2. Let $X$ be a closed affine variety in $V$ which is stable under the action of $G$, and $I = \mathbb{I}(X)$ an annihilator ideal of $X$. Then $\mathbb{C}[X] \simeq \mathbb{C}[V]/I$ is isomorphic to $\mathbb{C}[V]/\text{gr} I$ as a $G$-module. Since $\mathbb{C}[\mathfrak{C}(X)] \simeq \mathbb{C}[V]/\sqrt{I}$, we have a surjective $G$-module morphism $\mathbb{C}[X] \twoheadrightarrow \mathbb{C}[\mathfrak{C}(X)]$.

Let $\mathfrak{N}(V) := \{v \in V \mid f(v) = 0 \ (f \in \mathbb{C}[V])\}$ be the null fiber. It is the zero locus of homogeneous $G$-invariants of positive degree.

Proposition 1.3. Let $\mathcal{O}$ be a $G$-orbit in $V$. Then the affine asymptotic cone $\mathfrak{C}(\mathcal{O})$ is a $G$-stable subvariety of $\mathfrak{N}(V)$, which is equidimensional and $\dim \mathfrak{C}(\mathcal{O}) = \dim \mathcal{O}$. 


Let $\mathfrak{g}$ be a Lie algebra on which $G$ acts by the adjoint action. Then the null fiber $\mathfrak{N}(\mathfrak{g})$ is called the nilpotent variety, which consists of all the nilpotent elements in $\mathfrak{g}$. It is well known that $\mathfrak{N}(\mathfrak{g})$ contains only a finite number of $G$-orbits.

**Corollary 1.4.** For $x \in \mathfrak{g}$, let $O_x = \text{Ad}(G)x$ be the adjoint orbit through $x$. Then the affine asymptotic cone $\mathfrak{C}(O_x)$ is a finite union of the closure of nilpotent orbits, whose dimension is equal to $\dim O_x$.

In the following, we will denote the adjoint action simply by $gx = \text{Ad}(g)x$ for $g \in G, x \in \mathfrak{g}$.

### 2. Richardson Orbit

Let $h \in \mathfrak{g}$ be a semisimple element, and put $\mathfrak{L} := Z_G(h)$ the centralizer of $h$ in $G$. There is a parabolic subgroup $P$ with a Levi decomposition $P = LU$, where $U$ is the unipotent radical. Then $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ is a Levi decomposition of the corresponding Lie algebra.

**Definition 2.1.** Let $\mathfrak{u}$ be the nilpotent radical of a parabolic subalgebra $\mathfrak{p}$. Then adjoint translate $\mathfrak{G}\mathfrak{u} = \{\text{Ad}(g)u \mid g \in G, u \in \mathfrak{u}\}$ of $\mathfrak{u}$ is the closure of a single nilpotent orbit $\overline{O}_x$ ($x$ : nilpotent element). We call $\overline{O}_x$ the Richardson orbit for the parabolic $P$, and $x$ a Richardson element. We often assume $x$ to be taken from $\mathfrak{u}$.

Let us consider a partial flag variety $\mathfrak{B}_P := G/P$ of all parabolics conjugate to $\mathfrak{p}$, and denote by $T^*\mathfrak{B}_P$ the cotangent bundle over $\mathfrak{B}_P$. Then there is a $G$-equivariant map $\mu$ called the moment map defined as follows.

$$\mu : T^*\mathfrak{B}_P \simeq G \times_P \mathfrak{u} \ni (g, z) \rightarrow \text{Ad}(g)z \in \mathfrak{g}$$

The following proposition is well known. See [Jan04] and references therein.

**Proposition 2.2.** Assume that $x$ is a Richardson element for $P$ and that $Z_G(x) = Z_P(x)$ holds.

1. The moment map $\mu : T^*\mathfrak{B}_P \rightarrow \overline{O}_x$ is a resolution of singularities of $\overline{O}_x$.
2. The fiber of $O_x$ is $\mu^{-1}(O_x) = G[e, x]$ and $\mu : G[e, x] \rightarrow O_x$ is an isomorphism.
3. The moment map $\mu$ induces a $G$-equivariant isomorphism $\mathbb{C}[G \times_P \mathfrak{u}] = \mathbb{C}[G \times \mathfrak{u}]^P \simeq \mathbb{C}[O_x]$. In addition, if $O_x$ is normal, then $\mathbb{C}[\overline{O}_x] = \mathbb{C}[O_x]$ holds.

If a reductive group $K$ acts on a variety $\mathfrak{X}$, we get a decomposition of the regular function ring as a $K$-module,

$$\mathbb{C}[\mathfrak{X}] \simeq \bigoplus_{\tau \in \text{Irr}(K)} m_\tau(\mathfrak{X}) \tau \quad \text{(as a $K$-module),}$$

(2.1) where $m_\tau(\mathfrak{X})$ denotes the multiplicity.

**Theorem 2.3 (Borho-Kraft).** Let $h \in \mathfrak{g}$ be a semisimple element and define the parabolic subgroup $P$ and the Richardson orbit $\overline{O}_x$ as above. Then the asymptotic cone of the
semisimple orbit $O_h$ is equal to the Richardson orbit: $\mathcal{C}(O_h) = \overline{O_x}$. In addition, if $Z_G(x)$ is connected and $\overline{O_x}$ is normal, we have

$$\mathbb{C}[O_h] \cong \text{Ind}_G^H 1 \cong \mathbb{C}[O_x] = \mathbb{C}[\overline{O_x}] = \mathbb{C}[\mathcal{C}(O_h)]$$

(as $G$-modules)

i.e., $m_r(O_h) = m_r(O_x) = m_r(\mathcal{C}(O_h)) = \dim \tau^L (\forall \tau \in \text{Irr}(G))$.

Up to this point, we started with a semisimple element, but now we investigate in other ways. So take a nilpotent element $x \in g$, and choose an $\mathfrak{sl}_2$ triple $\{x, h, y\}$, where $h$ is semisimple; $x, y$ are nilpotent; and they satisfy the commutation relations

$$[h, x] = 2x, \quad [h, y] = -2y, \quad [x, y] = h.$$  

Thus $g$ is a representation space of $\mathfrak{sl}_2 = \text{span}_\mathbb{C}\{x, h, y\}$. Therefore the eigenvalues of $\text{ad} \ h$ are integers and we get a $\mathbb{Z}$-grading of $g$ induced by the action of $\text{ad} \ h$.

$$g = \bigoplus_{k \in \mathbb{Z}} g_k \quad g_k := \{X \in g \mid \text{ad} \ (h)X = kX\} \quad (2.2)$$

**Definition 2.4.** If $g_1 = \{0\}$, $x$ is called an *even* nilpotent element. Note that $g_1 = \{0\}$ if and only if $g_k = \{0\}$ ($\forall k : \text{odd}$).

We put $p = \bigoplus_{k \geq 0} g_k = l \oplus u$, where $l = g_0$ and $u = \bigoplus_{k > 0} g_k$. Then $p$ is a parabolic subalgebra and, if $x$ is even nilpotent, then $O_x$ is a Richardson orbit for $P = N_G(p)$. Even nilpotent elements have good properties (see [Jan04] for example).

**Proposition 2.5.** Assume $x$ is even nilpotent, then $Z_G(x) = Z_P(x)$ holds. Hence the moment map $\mu : T^* \mathfrak{B}_P \to \overline{O_x}$ is a resolution of singularities, and we have an isomorphism of regular function rings $\mathbb{C}[T^* \mathfrak{B}_P] \cong \mathbb{C}[O_x]$.

Moreover, if $\overline{O_x}$ is normal, then $\mathbb{C}[\overline{O_x}] \cong \mathbb{C}[O_x] \cong \mathbb{C}[T^* \mathfrak{B}_P]$.

**Corollary 2.6.** Let $\{x, h, y\}$ be an $\mathfrak{sl}_2$ triple with $x$ even nilpotent and assume that $\overline{O_x}$ is normal. Then the asymptotic cone of a semisimple element $h$ is equal to the closure of the nilpotent orbit through $x$.

$$\mathcal{C}(O_h) = \overline{O_x}$$

Moreover, there is an isomorphism $\mathbb{C}[\mathcal{C}(O_h)] \cong \mathbb{C}[T^* \mathfrak{B}_P]$.

3. **Richardson orbit for symmetric pair**

Let $G_\mathbb{R}$ be a reductive Lie group, which is a real form of a connected complex algebraic group $G$. We fix a Cartan involution $\theta$. Then the fixed point subgroup of $\theta$ is a maximal compact subgroup $K_\mathbb{R} = G_\mathbb{R}^\theta$. We extend $\theta$ to $G$ holomorphically, and put $K = G^\theta$, which is a complexification of $K_{\mathbb{R}}$. We mainly consider a symmetric pair $(G, K)$ in the following.

Let $\mathfrak{g}$ be the Lie algebra of $G$, and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ a (complexified) Cartan decomposition, where $\mathfrak{k}$ is the Lie algebra of $K$ and $\mathfrak{s}$ is the $(-1)$-eigenspace of the differential of $\theta$.

Take a $\theta$-stable parabolic subalgebra $p$ of $\mathfrak{g}$. We denote by $P$ the corresponding parabolic subgroup of $G$, and put $\mathfrak{B}_P = G/P$, the partial flag variety. Then $\mathfrak{B}_P$ can be considered as the totality of the parabolic subalgebras of $\mathfrak{g}$ which is conjugate to $p$ by the adjoint
action of $G$. The $K$-orbit of the $\theta$-stable parabolic $p$ is a closed orbit in $B_P$. Conversely, if there is a $\theta$-stable parabolic, then any closed $K$-orbit in $B_P$ arises as a $K$-conjugacy class of $\theta$-stable parabolic subalgebras.

Let $O$ denote a closed $K$-orbit in $B_P$ generated by $p$. Then the conormal bundle $T_O^*B_P$ over $O$ can be described as follows.

Since $p$ is $\theta$-stable, $q = p \cap \mathfrak{t}$ is a parabolic subalgebra in $\mathfrak{t}$. Let $Q$ be the corresponding parabolic subgroup of $K$. If $p = l \oplus u$ is a $\theta$-stable Levi decomposition, $q = l(\mathfrak{t}) \oplus u(\mathfrak{t})$ with $l(\mathfrak{t}) = l \cap \mathfrak{t}$ and $u(\mathfrak{t}) = u \cap \mathfrak{t}$ gives a Levi decomposition of $q$. Also we put $u(s) = u \cap s$. Then $u(s)$ is $Q$-stable, and we have

$$T_O^*B_P \simeq K \times_Q u(s) = (K \times u(s))/Q$$

where the action of $Q$ on $K \times u(s)$ is given by $q(k, x) = (kq^{-1}, \text{Ad}(q)x)$ for $q \in Q, k \in K, x \in u(s)$. We denote the class of $(k, x) \in K \times u(s)$ in $K \times_Q u(s)$ by $[k, x]$. Then a map

$$\mu : T_O^*B_P \simeq K \times_Q u(s) \to s, \quad \mu([k, x]) = \text{Ad}(k)x$$

is well-defined, and called the moment map. For any $K$-orbit $O$ in $B_P$, the moment map image of the conormal bundle $T_O^*B_P$ is the closure of a single nilpotent $K$-orbit $O^K$ in $s$. The following definition is due to P. Trapa [Tra05] (see also [Tra07]).

**Definition 3.1.** Let $p$ be a $\theta$-stable parabolic subalgebra and $O$ a closed $K$-orbit in $B_P$ through $p$. If a nilpotent $K$-orbit $O^K \subset s$ is dense in the moment map image of $T_O^*B_P$, it is called a **Richardson orbit for the symmetric pair $G/K$** associated to $p$.

The following is a representation theoretic characterization of Richardson orbits.

**Theorem 3.2.** A nilpotent $K$-orbit $O^K \subset s$ is a Richardson orbit for the symmetric pair if and only if its closure is the associated variety of a derived functor module $A_p$ with the trivial infinitesimal character for a certain $\theta$-stable parabolic subalgebra $p$.

4. **Asymptotic cone for symmetric pair**

Let $x \in s$ be a nilpotent element. Then we can choose $y \in s$ and $h \in \mathfrak{t}$ such that \{x, h, y\} forms a normal $\mathfrak{sl}_2$ triple, where x, y are nilpotent, and h semisimple (see [CM93, §9.4] for example). In addition, after suitable conjugation by $K$, we can assume $\overline{x} = y$, where $\overline{x}$ denotes the complex conjugation with respect to $g_R$. We call a normal $\mathfrak{sl}_2$ triple with this property a $KS$ **triple**. Then

$$a = \sqrt{-1}(x - y) \in s_R$$

is a semisimple element in $s_R$. Also we put

$$e = \frac{1}{2}(x + y + \sqrt{-1}h), \quad f = \frac{1}{2}(x + y - \sqrt{-1}h) = -\theta(e).$$

Then $e$ and $f$ are nilpotent elements belonging to the real form $g_R$, and \{e, a, f\} is a standard $\mathfrak{sl}_2$ triple in $g_R$. We call it a **Cayley triple**. Every standard $\mathfrak{sl}_2$ triple is $G_R$-conjugate to a Cayley triple.
The following theorem is well known.

**Theorem 4.1** (Sekiguchi [Sek87], Vergne [Ver95]). Nilpotent orbits $O_x^K = \text{Ad}(K)x$ and $O_x^{G_k} = \text{Ad}(G_k)e$ are $K_k$-equivariantly diffeomorphic, and moreover they generate the same nilpotent $G$-orbit: $\text{Ad}(G)x = \text{Ad}(G)e$. This correspondence gives a bijection between the set of non-zero nilpotent $K$-orbits in $\mathfrak{s}$ and that of non-zero nilpotent $G_k$-orbits in $\mathfrak{g}_k$.

See [CM93, Theorem 9.5.1 & Remark 9.5.2] and [BS98] for further properties.

Let us denote $O_x^G = \text{Ad}(G)x$. Then the intersection $O_x^G \cap \mathfrak{s}$ breaks up into several nilpotent $K$-orbits $\bigcup_{i=0}^{r} O_x^{K_i}$ where $x = x_0$. It is well known that each $O_x^{K_i}$ is a Lagrangian subvariety for the canonical symplectic structure on $O_x^G$, and consequently they all have the same dimension $\frac{1}{2} \dim O_x^G$ (see [Vog91, Corollary 5.20] for example). We also consider a complex semisimple orbit $O_a^K := \text{Ad}(K)a \subset \mathfrak{s}$, which is closed. Note that $a$ and $h$ generate the same $G$-orbit, $O_a^G = \text{Ad}(G)a = O_h^G$.

Let us consider $\text{ad} h$-eigenspace decomposition $\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k$ as in Equation (2.2). We put

$$\mathfrak{p} = \bigoplus_{k \geq 0} \mathfrak{g}_k = \mathfrak{l} \oplus \mathfrak{u}, \quad \text{where} \quad \mathfrak{l} = \mathfrak{g}_0, \; \mathfrak{u} = \bigoplus_{k > 0} \mathfrak{g}_k. \quad (4.1)$$

Then $\mathfrak{p}$ is a $\theta$-stable parabolic subalgebra, and $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{k}$ is a parabolic in $\mathfrak{k}$. We denote $P$ and $Q$ the parabolic subgroups of $G$ and $K$ respectively corresponding to $\mathfrak{p}$ and $\mathfrak{q}$. We follow the notation in §3.

**Theorem 4.2.** Assume that $x \in \mathfrak{s}$ is an even nilpotent element, and let $\{x, h, y\}$ be a normal $\mathfrak{sl}_2$ triple. After conjugation by $K$, we can assume $\{x, h, y\}$ is a KS triple. Put $a = \sqrt{-1}(x - y) \in \mathfrak{s}_\mathbb{R}$. Then the asymptotic cone of $O_a^K$ is equal to

$$C(O_a^K) = \overline{O_x^G \cap \mathfrak{s}} = \bigcup_{i=0}^{r} \overline{O_x^{K_i}}, \quad (4.2)$$

where $\{x = x_0, x_1, \ldots, x_r\}$ is a complete set of representatives of the $K$-orbits in $O_x^G \cap \mathfrak{s}$, and $\{O_x^{K_i} (0 \leq i \leq \ell)\}$ are Richardson orbits for a symmetric pair $G/K$.

**Proof.** Since $x$ is even nilpotent by assumption, the $K$-orbit $O_x^K$ is a Richardson orbit corresponding to the $\theta$-stable parabolic $\mathfrak{p}$ in (4.1). See [Noé06] for details. For $1 \leq i \leq \ell$, because $x_i$ is a $G$-translate of $x$, they are all even nilpotent. Thus the same reasoning can be applied to the orbits $O_{x_i}^K$ which tells us that they are all Richardson.

Now let us consider $a = \sqrt{-1}(x - y)$. Then we calculate

$$\exp(t \text{ad} h)a = \sqrt{-1}(e^{2t}x - e^{-2t}y) = \sqrt{-1}e^{2t}(x - e^{-4t}y).$$

Therefore we get in $\mathbb{P}(\mathfrak{g} \oplus \mathbb{C})$,

$$[\exp(t \text{ad} h)a \oplus 1] = [(x - e^{-4t}y) \oplus (-\sqrt{-1}e^{-2t})] \rightarrow [x \oplus 0] \in \kappa(\mathbb{P}(\mathfrak{g})) \quad (t \rightarrow \infty).$$

This proves that $x \in C(O_a^K)$ and hence $\overline{O_x} \subset C(O_a^K)$ because $C(O_a^K)$ is a $K$-invariant closed set. By the same reason, we get $\overline{O_{x_i}} \subset C(O_{a_i}^K)$, where $a_i$ is defined similarly as $a$ by using $x_i$ instead of $x$. 
The semisimple elements $a_i$'s are in fact all conjugate to $a$ by the adjoint action of $K$. This follows from the fact that representatives of the little Weyl group (the Weyl group of the restricted root system) can be chosen from the elements in $K$ ([Kna02, Corollary 6.55]).

Thus we have proved that the right hand side is contained in the asymptotic cone $\mathcal{C}(O^K_a)$.

On the other hand, from Theorem 2.3, we clearly have

$$\mathcal{C}(O^K_a) \subset \mathcal{C}(O^G_a) \subset O^G_x \cap s.$$

Thus we get

$$O^G_x \cap s \subset \mathcal{C}(O^K_a) \subset O^G_x \cap s.$$

Note that $O^G_x \cap s$ is a union of all irreducible components of $O^G_x \cap s$ of maximal dimension $\frac{1}{2} \dim O^G_x$ (cf. Remark 4.3(1) below). Since $\mathcal{C}(O^K_a)$ is equi-dimensional, it must coincide with $O^G_x \cap s$. □

Remark 4.3. (1) The inclusion $O^G_x \cap s \subset O^G_x \cap s$ might be strict. For example, consider a symmetric pair $(G, K) = (\text{GL}_{2n}, \text{GL}_n \times \text{GL}_n)$ which is associated to $U(n, n)$. Take the nilpotent $G$-orbit $O^G_x$ of Jordan type $[3 \cdot 1^{2n-3}]$. Then $O^G_x \cap s$ consists of the $K$-orbits whose signed Young diagrams are

$$[(+ - +) \cdot (+)^{n-2} \cdot (-)^{n-1}], \quad [(- + -) \cdot (+)^{n-1} \cdot (-)^{n-2}],$$

$$[(+ - ) \cdot (+)^{n-2} \cdot (-)^{n-2}], \quad [(- +) \cdot (+)^{n-1} \cdot (-)^{n-1}],$$

$$[(+)^n \cdot (-)^n],$$

while the $K$-orbits $[(+ +)^2 \cdot (+)^{n-2} \cdot (-)^{n-2}]$ and $[(- +)^2 \cdot (+)^{n-2} \cdot (-)^{n-2}]$ are not contained in the closure but contained in $O^G_x \cap s$. See the Hasse diagram of the closure relation below.

(2) The collection of $\{O^K_{x_i} \ (0 \leq i \leq \ell)\}$ is a set of Richardson orbits which are the moment map image of the conormal bundle of closed $K$-orbits in the fixed partial flag variety $B_P$ through $\theta$-stable parabolics (not necessarily all of them). Let us denote a closed $K$-orbit in $B_P$ by $O_i$ which corresponds to the Richardson orbit $O^K_{x_i}$. If $K_{x_i}$ is connected, the moment map $\mu_i : T_{O_i} B_P \to O^K_{x_i}$ is a resolution of the singularities (see Proposition 5.9 and §8.8 of [Jan04]).

Since $a \in s_{\mathbb{R}}$ is a real hyperbolic element, it naturally defines a real parabolic subalgebra $p_{\mathbb{R}}$, which is the non-negative part of the $\mathbb{Z}$-grading similar to (4.1) with respect to $\text{ad} a$ instead of $\text{ad} h$. Let us denote by $P_{\mathbb{R}}$ the corresponding real parabolic subgroup of $G_{\mathbb{R}}$. A parabolically induced representation from a character $\chi$ of $P_{\mathbb{R}}$ is called the degenerate principal series representation, which is denoted by $I_{P_{\mathbb{R}}} (\chi) = \text{Ind}_{G_{\mathbb{R}}}^G \chi$. 
Corollary 4.4. We assume $x \in \mathfrak{s}$ is even nilpotent and use the setting of Theorem 4.2. Let $I_{P_R}(\chi)$ be a degenerate principal series representation of $G_R$, where $P_R$ is obtained from $a \in \mathfrak{s}_R$ as above. Then the associated variety of $I_{P_R}(\chi)$ is equal to the asymptotic cone $\mathcal{C}(O^K_a)$ (see Equation (4.2)).

Proof. It is known that the $G$-hull of the associated variety $\mathcal{A}(I_{P_R}(\chi))$ is the closure of the Richardson $G$-orbit associated to $P$. Thus, by Theorem 4.2, we have $\mathcal{A}(I_{P_R}(\chi)) \subset \mathcal{C}(O^K_a)$. Note that the function ring $\mathbb{C}[\mathcal{A}(I_{P_R}(\chi))]$ is asymptotically isomorphic to the space of $K$-finite vectors in $I_{P_R}(\chi)$ as $K$-modules. If $\chi$ is trivial, we have

$$I_{P_R}(1)|_{K_R} \simeq \text{Ind}^{K_R}_{M_R} 1 \simeq \mathbb{C}[O^K_a], \quad M_R = Z_{K_R}(a).$$

Therefore, asymptotically $\mathbb{C}[\mathcal{A}(I_{P_R}(1))]$ and $\mathbb{C}[\mathcal{C}(O^K_a)]$ are equal. So they must coincide with each other. □

Remark 4.5. The wave front set of $I_{P_R}(\chi)$ is known by the results in [BB99] (see also [Bar00]). Therefore, using Schmid-Vilonen’s theorem [SV00], we basically know the associated variety of $I_{P_R}(\chi)$. Here, in the corollary above, the emphasis is on the coincidence with the asymptotic cone.

The conclusion of Corollary 4.4 does not contain the even nilpotent element $x$ explicitly. In fact, it is plausible to believe the conclusion is always true.

Problem 4.6. Let $a \in \mathfrak{s}_R$ be a hyperbolic semisimple element and define the parabolic $p_R$ as above. Does the associated variety of the degenerate principal series $I_{P_R}(\chi)$ coincide with the asymptotic cone $\mathcal{C}(O^K_a)$?
Remark 4.7. (1) For a general $a \in \mathfrak{s}_\mathbb{R}$, it is no longer true that the asymptotic cone $C(O^K_a)$ is equal to the intersection of the closure of the Richardson orbit and $\mathfrak{s}$. For this, we refer to an example in [MT07, Example 3.8].

(2) There is a formula for the asymptotic $K$-support by T. Kobayashi, which is very close to the above problem. His formula ([Kob05, Theorem 6.4.3]) implies

$$\text{AS}_K(I_{P_k}(\chi)|_{K_{K}}) = C^+ \cap \sqrt{-1} \text{Ad}^*(K_{\mathbb{R}})(m_{\mathbb{R}})^{\perp},$$

where $C^+$ denotes the closed Weyl chamber inside $\sqrt{-1} t_{\mathbb{R}}$. However, up to now, we do not know the exact relation of the above formula to our problem.

Corollary 4.8. Suppose that $x \in \mathfrak{s}$ is even nilpotent which satisfies

(1) the fixed point subgroup $K_x$ is connected,

(2) $\overline{O^K_x}$ is normal,

(3) $\text{codim} \partial O^K_x \geq 2$, where $\partial O^K_x = \overline{O^K_x} \setminus O^K_x$ is the boundary of $O^K_x$.

Then the intersection $O^K \cap \mathfrak{s} = O^K_x$ consists of a single $K$-orbit. If we take a KS triple $\{x, h, y\}$ as above, the asymptotic cone of the semisimple orbit $O^K_a (a = \sqrt{-1} (x - y))$ is given by $C(O^K_a) = \overline{O^K_x}$. In this case, we have isomorphisms of algebra

$$\mathbb{C}[T_o \mathfrak{B}_P] \simeq \mathbb{C}[O^K_x] \simeq \mathbb{C}[\overline{O^K_x}],$$

and, as $K$-modules, they are isomorphic to $\mathbb{C}[O^K_a]$.

Proof. We use the following lemma. Let us recall the notation $m_\tau(\mathfrak{x})$ for the multiplicity defined in (2.1).

Lemma 4.9. The following inequality holds.

$$m_\tau(O^K_a) \geq m_\tau(C(O^K_a)) \geq m_\tau(\overline{O^K_x}) \quad (\tau \in \text{Irr}(K)).$$

Proof. Let us denote the annihilator ideal of $O^K_a$ by $I = \mathbb{I}(O^K_a) \subset \mathbb{C}[\mathfrak{s}]$. Then we have $\mathbb{C}[O^K_a] \simeq \mathbb{C}[\mathfrak{s}]/\mathfrak{g} \mathfrak{t} I$ as $K$-modules. Moreover, there is a surjective algebra morphism $\mathbb{C}[\mathfrak{s}]/\mathfrak{g} \mathfrak{t} I \to \mathbb{C}[\mathfrak{s}]/\sqrt{\mathfrak{g} \mathfrak{t} I} = \mathbb{C}[C(O^K_a)]$. Since this morphism is $K$-equivariant, we have the following inequality

$$m_\tau(O^K_a) \geq m_\tau(C(O^K_a)) \quad (\tau \in \text{Irr}(K)).$$

Since $\overline{O^K_x}$ in Theorem 4.2 is an irreducible component of $C(O^K_a)$, we also have an inequality $m_\tau(C(O^K_a)) \geq m_\tau(\overline{O^K_x})$.\qed

Let us return to the proof of the corollary.

By Theorem 4.2, we know $C(O^K_a)$ is the union of $\overline{O^K_x}$’s. By Corollary 4.4, $C(O^K_a)$ is an associated variety of a degenerate principal series $I_{P_k}(\chi)$. For a generic parameter $\chi$, the degenerate principal series representation is irreducible. So by Vogan’s theorem ([Vog91, Theorem 4.6]), if there are more than two irreducible components of the associated variety, they must have a codimension one orbit in its boundary. But by the assumption, there is no such orbit, hence it must be irreducible.
The normality and the codimension-two condition imply the isomorphism \( C[O_x] \cong C[O_x] \). Since \( K_x \) is connected the moment map \( \mu : T^*O_x \to O_x^K \) is a resolution. By [Jan04, Proposition 8.9], we get \( C[T^*O |_{BP}] \cong C[O_K] \).

5. Example: Siegel parabolics

Let \( G_\mathbb{R} = U(n, n) \) and \( K_\mathbb{R} = U(n) \times U(n) \) a maximal compact subgroup. Then \( G = \text{GL}_{2n}(\mathbb{C}) \) is the complexification of \( G_\mathbb{R} \) and \( K = \text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C}) \) is block diagonally embedded into \( G \). \( (G, K) \) is a symmetric pair. The Cartan decomposition \( g = k \oplus s \) is given as follows.

\[
\begin{align*}
k &= \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \mid A, D \in M_n(\mathbb{C}) \right\}, \\
s &= \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B, C \in M_n(\mathbb{C}) \right\}
\end{align*}
\]

Let us consider a nilpotent element

\[ x = \begin{pmatrix} 0 & 1_n \\ 0 & 0 \end{pmatrix} \in s. \]

If we put \( y = \tau x \) and \( h = [x, y] \), then \( \{x, h, y\} \) constitute a KS triple. Note that, in this case, the complex conjugation \( \sigma \) with respect to the real form \( g_\mathbb{R} \) is given by

\[ \sigma(X) = -I_{n, n} X I_{n, n} \quad (X \in g), \quad I_{n, n} = \begin{pmatrix} 1_n & 0 \\ 0 & -1_n \end{pmatrix}. \]

We can check \( \sigma(x) = \tau x = y \) directly.

The nilpotent element \( x \) generates a nilpotent \( G \)-orbit \( O_x^G \) which has Jordan type \( [2^n] \). Consequently \( x \) is even nilpotent. There are \( (n + 1) \) nilpotent \( K \)-orbits in \( O_x^G \cap s \), which are \( O_{p,q}^K = \left[ (+)^p (-)^q \right] (p, q \geq 0, p + q = n) \) in the notation of signed Young diagram (see [CM93], for example).

Put \( a = \sqrt{-1} (x - y) \in s_\mathbb{R} \). Theorem 4.2 tells us that

\[ C(O_a^K) = \bigcup_{p+q=n} O_{p,q}^K. \]

Let us interpret the meaning of this identity in terms of the representation theory of \( G_\mathbb{R} \).

First, let us see the function ring \( C(O_a^K) \). Put \( M = Z_K(a) \), the stabilizer of \( a \) in \( K \). Then clearly \( M = \Delta \text{GL}_n(\mathbb{C}) \), the diagonal embedding of \( \text{GL}_n(\mathbb{C}) \) into \( K = \text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C}) \).

Thus we have

\[ C(O_a^K) = C[K/M] = C[K]^M \cong \text{Ind}_M^K 1_M, \quad (5.1) \]

where the last isomorphism is an isomorphism as \( K \)-modules, and \( 1_M \) denotes the trivial representation of \( M \). Thus we have

\[ C[O_a^K] \cong \bigoplus_{\rho \in \text{Irr}(\text{GL}_n)} \rho \otimes \rho^* \quad (\text{as a } K \cong \text{GL}_n \times \text{GL}_n \text{-module}), \quad (5.2) \]

which is a multiplicity free \( K \)-module. This is isomorphic to \( C[C(O_a^K)] \) as a \( K \)-module by [NOZ06, Theorem 3.1].
On the other hand, by explicit calculation using the technique in [Nis00] (also see [Nis04]), we have
\[ C[O_{p,q}^K] \simeq \bigoplus_{\alpha \in P_p, \beta \in P_q} \rho_{\alpha \otimes \beta} \otimes \rho_{\alpha \otimes \beta}^*. \]

However, we have the following

**Proposition 5.1.** For any \( p, q \geq 0 \) satisfying \( p + q = n \), there are isomorphisms of \( K \)-modules
\[ C[O_{p,q}^K] \simeq C[\mathcal{C}(O_a^K)] \simeq C[O_a^K], \]
where the first isomorphism is also a morphism of algebras induced by the open embedding \( O_{p,q}^K \hookrightarrow C(O_a^K) \).

Let us denote \( M_R = Z_{K_R}(a) = \Delta U(n) \), and \( L_R = Z_{G_R}(a) \simeq GL_n(C) \). The semisimple element \( a \) naturally defines a maximal parabolic subgroup \( P_R = L_R N_R \), where \( N_R \) is a suitably chosen unipotent radical. Note that \( A_R = \exp R a \) is contained in the center of \( L_R = GL_n(C) \) as the radial part of the complex torus. We consider a degenerate principal series representation induced from a one dimensional character of \( P_R \) (unnormalized induction)
\[ I(\nu) := \text{Ind}_{P_R}^{G_R}(|\det|^\nu \cdot n \otimes 1_{N_R}), \quad (\nu \in \mathbb{C}), \]
where \( \text{det} \) is the determinant character of \( L_R = GL_n(C) \) and the induced character is trivial on \( N_R \). Then we have
\[ I(\nu)|_{K_R} \simeq \text{Ind}_{M_R}^{K_R} 1_{M_R} \simeq \bigoplus_{\rho \in \text{Irr}(U(n))} \rho \otimes \rho^*. \]

Comparing this with (5.2) and (5.1), we conclude that \( O_a^K \) or \( \mathcal{C}(O_a^K) \) carries information of \( K \)-types of degenerate principal series \( I(\nu) \).

**Theorem 5.2** (Sahi, Lee, Johnson, Wallach, ...). Assume that \( \nu \geq 0 \) is even. Then the degenerate principal series \( I(\nu) \) contains precisely \( (n + 1) \) irreducible subrepresentations \( \pi_{p,q}(\nu) (p, q \geq 0, p + q = n) \), which are unitary. If \( \nu > 0 \), then these are only unitarizable irreducible constituents of \( I(\nu) \).

**Remark 5.3.** \( I(\nu) \) is reducible if and only if \( \nu \) is an even integer. If \( \nu \geq 0 \) (and even), then the Hasse diagram of subquotients of \( I(\nu) \) is given below (see [Lee94, §§7&9] and also [Sah93]).

If \( \nu = 0 \), then \( I(\nu) \) contains the trivial representation. In general \( I(\nu) (\nu \geq 0) \) contains a finite dimensional representation as a unique irreducible subrepresentation.

If \( \nu = -n \), then \( I(-n) \) is a direct sum of \( (n + 1) \) irreducible unitary representations \( \{ \pi_{p,q}(-n) \mid p + q = n \} \), which are derived functor modules \( A_{p,q}(-n) \) (see [MT07]). The representations \( \pi_{p,q}(\nu) (p + q = n) \) are translation (or coherent continuation) of these derived functor modules.
\begin{corollary}
The associated variety of $I(\nu)$ is equal to $C(\mathcal{O}_a^K) = \bigcup_{p+q=n} \mathcal{O}_{p,q}^K$. The associated cycle of the largest constituents $\pi_{p,q}(\nu)$ ($p+q = n$) is given by $AC \pi_{p,q}(\nu) = [\mathcal{O}_{p,q}^K]$ with multiplicity one.
\end{corollary}

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