YETTER-DRINFELD MODULES OVER WEAK BIALGEBRAS

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Abstract. We discuss properties of Yetter-Drinfeld modules over weak bialgebras over commutative rings. The categories of left-left, left-right, right-left and right-right Yetter-Drinfeld modules over a weak Hopf algebra are isomorphic as braided monoidal categories. Yetter-Drinfeld modules can be viewed as weak Doi-Hopf modules, and, a fortiori, as weak entwined modules. If \( H \) is finitely generated and projective, then we introduce the Drinfeld double using duality results between entwining structures and smash product structures, and show that the category of Yetter-Drinfeld modules is isomorphic to the category of modules over the Drinfeld double. The category of finitely generated projective Yetter-Drinfeld modules over a weak Hopf algebra has duality.

Introduction

Weak bialgebras and Hopf algebras are generalizations of ordinary bialgebras and Hopf algebras in the following sense: the defining axioms are the same, but the multiplicative of the counit and comultiplicativity of the unit are replaced by weaker axioms. The easiest example of a weak Hopf algebra is a groupoid algebra; other examples are face algebras [10], quantum groupoids [19], generalized Kac algebras [25] and quantum transformation groupoids [18]. Temperley-Lieb algebras give rise to weak Hopf algebras (see [18]). A purely algebraic study of weak Hopf algebras has been presented in [2]. A survey of weak Hopf algebras and their applications may be found in [18]. It has turned out that many results of classical Hopf algebra theory can be generalized to weak Hopf algebras.

Yetter-Drinfeld modules over finite dimensional weak Hopf algebras over fields have been introduced by Nenciu [16]. It is shown in [16] that the category of finite dimensional Yetter-Drinfeld modules is isomorphic to the category of finite dimensional modules over the Drinfeld double, as introduced in the appendix of [1]. It is also shown that this category is braided isomorphic to the center of the category of finite dimensional \( H \)-modules. In this note, we discuss Yetter-Drinfeld modules over weak bialgebras over commutative rings. The results in [16] are slightly generalized and more properties are given.

In Section 2, we compute the weak center of the category of modules over a weak bialgebra \( H \), and show that it is isomorphic to the category of Yetter-Drinfeld modules. If \( H \) is a weak Hopf algebra, then the weak center equals the center. In this situation, properties of the center construction can be applied to show that the four categories of Yetter-Drinfeld modules, namely the left-left, left-right, right-left and...
right-right versions, are isomorphic as braided monoidal categories. Here we apply methods that have been used before in [5], in the case of quasi-Hopf algebras. In [7], it was observed that Yetter-Drinfeld modules over a classical Hopf algebra are special cases of Doi-Hopf modules, as introduced by Doi and Koppinen (see [8, 13]). In Section 3, we will show that Yetter-Drinfeld modules over weak Hopf algebras are weak Doi-Hopf modules, in the sense of Böhm [1], and, a fortiori, weak entwined modules [6], and comodules over a coring [4].

The advantage of this approach is that it leads easily to a new description of the Drinfeld double of a finitely generated projective weak Hopf algebra, using methods developed in [6]: we define the Drinfeld double as a weak smash product of the center construction, and in Section 1.4, we recall the notions of weak Doi-Hopf algebras. Further detail can be found in [4, 2, 18]. In Section 1.3, we recall in Sections 1.1 and 1.2, we recall some general properties of weak bialgebras and projective Yetter-Drinfeld modules over a weak Hopf algebra has duality.

In [7], it was observed that Yetter-Drinfeld modules over a classical Hopf algebra are weak Doi-Hopf modules, in the sense of Böhm [1], and, a fortiori, weak entwined modules [6], and comodules over a coring [4].

1. Preliminary results

1.1. Weak bialgebras. Let \( k \) be a commutative ring. Recall that a weak \( k \)-bialgebra is a \( k \)-module with a \( k \)-algebra structure \((\mu, \eta)\) and a \( k \)-coalgebra structure \((\Delta, \varepsilon)\) such that \( \Delta(hk) = \Delta(h)\Delta(k) \), for all \( h, k \in H \), and

\[
\begin{align*}
\Delta^2(1) &= 1_{(1)} \otimes 1_{(2)} 1_{(1')} \otimes 1_{(2')} = 1_{(1)} \otimes 1_{(1')} 1_{(2)} \otimes 1_{(2')}, \\
\varepsilon(hkl) &= \varepsilon(hk(1))\varepsilon(k(2)l) = \varepsilon(hk(2))\varepsilon(k(1)l),
\end{align*}
\]

for all \( h, k, l \in H \). We use the Sweedler-Heineman notation for the comultiplication, namely

\[
\Delta(h) = h_{(1)} \otimes h_{(2)} = h_{(1')} \otimes h_{(2')}.
\]

We summarize the elementary properties of weak bialgebras. The proofs are direct applications of the defining axioms (see [2, 18]). We have idempotent maps \( \varepsilon_t, \varepsilon_s : H \to H \) defined by

\[
\varepsilon_t(h) = \varepsilon(1_{(1)}h)1_{(2)}; \varepsilon_s(h) = 1_{(1)}\varepsilon(h1_{(2)}).
\]

\( \varepsilon_t \) and \( \varepsilon_s \) are called the target map and the source map, and their images \( H_t = \text{Im}(\varepsilon_t) = \text{Ker}(H - \varepsilon_t) \) and \( H_s = \text{Im}(\varepsilon_s) = \text{Ker}(H - \varepsilon_s) \) are called the target and source space. For all \( g, h \in H \), we have

\[
\begin{align*}
h(1) \otimes \varepsilon_t(h_{(2)}) &= 1_{(1)}h \otimes 1_{(2)} \quad \text{and} \quad \varepsilon_s(h(1)) \otimes h_{(2)} = 1_{(1)} \otimes h1_{(2)}, \\
\end{align*}
\]

and

\[
\begin{align*}
h\varepsilon_t(g) &= \varepsilon(h(1))g1_{(2)} \quad \text{and} \quad \varepsilon_s(g)h = h(1)\varepsilon(gh1_{(2)}).
\end{align*}
\]

From (4), it follows immediately that

\[
\varepsilon(h\varepsilon_t(g)) = \varepsilon(hg) \quad \text{and} \quad \varepsilon(\varepsilon_s(g)h) = \varepsilon(gh).
\]

The source and target space can be described as follows:

\[
\begin{align*}
H_t &= \{ h \in H \mid \Delta(h) = 1_{(1)}h \otimes 1_{(2)} \} = \{ \phi(1_{(1)})1_{(2)} \mid \phi \in H^* \}; \\
H_s &= \{ h \in H \mid \Delta(h) = 1_{(1)} \otimes h1_{(2)} \} = \{ 1_{(1)}\phi(1_{(2)}) \mid \phi \in H^* \}.
\end{align*}
\]
We also have
\[ \varepsilon_t(h)\varepsilon_s(k) = \varepsilon_s(k)\varepsilon_t(h), \]
and its dual property
\[ \varepsilon_s(h_{(1)}) \otimes \varepsilon_t(h_{(2)}) = \varepsilon_s(h_{(2)}) \otimes \varepsilon_t(h_{(1)}). \]
Finally \( \varepsilon_s(1) = \varepsilon_t(1) = 1 \), and
\[ \varepsilon_t(h)\varepsilon_t(g) = \varepsilon_t(\varepsilon_t(h)g) \quad \text{and} \quad \varepsilon_s(h)\varepsilon_s(g) = \varepsilon_s(h\varepsilon_s(g)). \]
This implies that \( H_s \) and \( H_t \) are subalgebras of \( H \).

**Lemma 1.1.** Let \( H \) be a weak bialgebra over a commutative ring. Then \( \Delta(1) \in H_s \otimes H_t \).

**Proof.** Applying \( H \otimes \varepsilon \otimes H \) to (1), we find that \( 1_{(1)} \otimes 1_{(2)} = \varepsilon_s(1_{(1)}) \otimes 1_{(2)} \in H_s \otimes H \) and \( 1_{(1)} \otimes 1_{(2)} = 1_{(1)} \otimes \varepsilon_t(1_{(2)}) \in H \otimes H_t \). Now let \( K_s = \ker(\varepsilon_s) \), \( K_t = \ker(\varepsilon_t) \). Then \( H = H_s \oplus K_s = H_t \oplus K_t \), and
\[ H \otimes H = H_s \otimes H_t \oplus H_s \otimes K_t \oplus K_s \otimes H_t \oplus K_s \otimes K_t, \]
so it follows that \( H_s \otimes H_t = H \otimes H_t \cap H_s \otimes H \). \( \square \)

The target and source map for the weak bialgebra \( H^{op} \) are
\[ \tau_t(h) = \varepsilon(h_{(1)})1_{(2)} \in H_t \quad \text{and} \quad \tau_s(h) = \varepsilon(1_{(2)}h)1_{(1)} \in H_s. \]
\( \tau_t \) and \( \tau_s \) are also projections.

The source and target space are anti-isomorphic, and they are separable Frobenius algebras over \( k \). This was first proved for weak Hopf algebras (see [2]), and then generalized to weak bialgebras (see [22]).

**Lemma 1.2.** [22] Let \( H \) be a weak bialgebra. Then \( \tau_s \) restricts to an anti-algebra isomorphism \( H_t \rightarrow H_s \) with inverse \( \varepsilon_t \), and \( \tau_t \) restricts to an anti-algebra isomorphism \( H_s \rightarrow H_t \) with inverse \( \varepsilon_s \).

**Proposition 1.3.** [22] Let \( H \) be a weak bialgebra. Then \( H_s \) and \( H_t \) are Frobenius separable \( k \)-algebras. The separability idempotents of \( H_t \) and \( H_s \) are
\[ e_t = \varepsilon_t(1_{(1)}) \otimes 1_{(2)} = 1_{(2)} \otimes \tau_t(1_{(1)}); \]
\[ e_s = 1_{(1)} \otimes \varepsilon_s(1_{(2)}) = \tau_s(1_{(2)}) \otimes 1_{(1)}. \]

The Frobenius systems for \( H_t \) and \( H_s \) are respectively \( (\varepsilon_t, \varepsilon_s|H_t) \) and \( (e_s, \varepsilon_t|H_s) \). In particular, we have for all \( z \in H_t \) that
\[ z\varepsilon_t(1_{(1)}) \otimes 1_{(2)} = \varepsilon_t(1_{(1)}) \otimes 1_{(2)}z. \]

It was shown in [17] that the category of modules over a weak Hopf algebra is monoidal; it follows from the results of [22] that this property can be generalized to weak bialgebras. We explain now how this can be done directly.

Let \( M \) be a left \( H \)-module. By restriction of scalars, \( M \) is a left \( H_t \)-module; \( M \) becomes an \( H_t \)-bimodule, if we define a right \( H_t \)-action by
\[ m \cdot z = \tau_s(z)m. \]
Let $M, N \in H\mathcal{M}$, the category of left $H$-modules. We define

$$M \otimes_t N = \Delta(1)(M \otimes N),$$

the $k$-submodule of $M \otimes N$ generated by elements of the form $1_{(1)} \otimes 1_{(2)}$. $M \otimes_t N$ is a left $H$-module, with left diagonal action $h \cdot (m \otimes n) = h_{(1)} m \otimes h_{(2)} n$. It follows from (1) that the tensor product $\otimes$ is associative. Observe that

$$M \otimes_t N \otimes_t P = \Delta^2(1)(M \otimes N \otimes P).$$

$H_t \in H\mathcal{M}$, with left $H$-action $h \cdot z = \varepsilon_t(hz)$. The induced $H_t$-bimodule structure is given by left and right multiplication by elements of $H_t$.

For $M, N \in H\mathcal{M}$, consider the projection

$$\pi : M \otimes N \to M \otimes_t N, \pi(m \otimes n) = 1_{(1)} m \otimes 1_{(2)} n.$$ 

Applying $\pi \otimes H_t$ to (12), we find

$$\pi_s(z \varepsilon_t(1_{(1)})) \otimes 1_{(2)} = 1_{(1)} \pi_s(z) \otimes 1_{(2)} = 1_{(1)} \otimes 1_{(2)} z,$$

hence

$$\pi(mz \otimes n) = \pi(\pi_s(z)m \otimes n) = 1_{(1)} \pi_s(z)m \otimes 1_{(2)} n = 1_{(1)} m \otimes 1_{(2)} zn = \pi(m \otimes zn).$$

So $\pi$ induces a map $\overline{\pi} : M \otimes H_t N \to M \otimes_t N$, which is a left $H_t$-module isomorphism with inverse given by

$$\overline{\pi}^{-1}(1_{(1)} m \otimes 1_{(2)} n) = 1_{(1)} m \otimes H_t 1_{(2)} n = m \otimes H_t n.$$

**Proposition 1.4.** Let $H$ be a weak bialgebra. Then we have a monoidal category $(H\mathcal{M}, \otimes, H_t, a, l, r)$. The associativity constraints are the natural ones. The left and right unit constraints $l_M : H_t \otimes_t M \to M$ and $r_M : M \otimes_t H_t \to M$ and their inverses are given by the formulas

$$l_M(1_{(1)} \rightarrow z \otimes 1_{(2)} m) = zm ; l_M^{-1}(m) = \varepsilon_t(1_{(1)}) \otimes 1_{(2)} m;$$

$$r_M(1_{(1)} m \otimes 1_{(2)} \rightarrow z) = \pi_s(z)m ; r_M^{-1}(m) = 1_{(1)} m \otimes 1_{(2)}.$$

**Proof.** This is a direct consequence of the observations made above. Let us check that

$$l_M^{-1}(l_M(1_{(1)} \rightarrow z \otimes 1_{(2)} m)) = l_M^{-1}(zm) = \varepsilon_t(1_{(1)}) \otimes 1_{(2)} zm$$

$$= \varepsilon_t(1_{(1)} z \otimes 1_{(2)} m) = \varepsilon_t(1_{(1)} z) \otimes 1_{(2)} m = 1_{(1)} \rightarrow z \otimes 1_{(2)} m$$

$$l_M(l_M^{-1}(m)) = l_M(\varepsilon_t(1_{(1)}) \otimes 1_{(2)} m) = m$$

$$r_M^{-1}(r_M(1_{(1)} m \otimes 1_{(2)} \rightarrow z)) = r_M^{-1}(\pi_s(z)m) = 1_{(1)} \pi_s(z)m \otimes 1_{(2)}$$

$$= 1_{(1)} m \otimes 1_{(2)} z = 1_{(1)} m \otimes 1_{(2)} \rightarrow z$$

$$r_M(r_M^{-1}(m)) = r_M(1_{(1)} m \otimes 1_{(2)}) = \pi_s(1)m = m.$$
1.2. **Weak Hopf algebras.** A weak Hopf algebra is a weak bialgebra together with a map \( S : H \to H \), called the antipode, satisfying
\[
S * H = \varepsilon_s, \quad H * S = \varepsilon_t, \quad \text{and} \quad S * H * S = S,
\]
where * is the convolution product. It follows immediately that
\[
S = \varepsilon_s * S = S * \varepsilon_t.
\]
If the antipode exists, then it is unique. We will always assume that \( S \) is bijective; if \( H \) is a finite dimensional weak Hopf algebra over a field, then \( S \) is automatically bijective (see [2, Theorem 2.10]).

**Lemma 1.5.** Let \( H \) be a weak Hopf algebra. Then \( S \) is an anti-algebra and an anti-coalgebra morphism. For all \( h, g \in H \), we have
\[
\begin{align*}
\varepsilon_t(hg) &= \varepsilon_t(h \varepsilon_t(g)) = h(1) \varepsilon_t(g) S(h(2)); \\
\varepsilon_s(hg) &= \varepsilon_s(\varepsilon_s(h) g) = S(g(1)) \varepsilon_s(h) g(2); \\
\Delta(\varepsilon_t(h)) &= h(1) S(h(3)) \otimes \varepsilon_t(h(2)) \\
\Delta(\varepsilon_s(h)) &= \varepsilon_s(h(2)) \otimes S(h(1)) h(3).
\end{align*}
\]

**Lemma 1.6.** Let \( H \) be a weak Hopf algebra. For all \( h \in H \), we have
\[
\varepsilon_t(h) = \varepsilon(S(h)1(1))1(2) = \varepsilon(1(2)h) S(1(1)) S(\varepsilon_t(h)) \\
\varepsilon_s(h) = 1(1) \varepsilon(1(2) S(h)) = \varepsilon(h1(1)) S(1(2)) = S(\varepsilon_t(h)).
\]

**Corollary 1.7.** Let \( H \) be a weak Hopf algebra. For all \( h \in H \), we have
\[
\varepsilon_t(h(1)) \otimes h(2) = S(1(1)) \otimes 1(2) h ; \quad h(1) \otimes \varepsilon_s(h(2)) = h1(1) \otimes S(1(2)).
\]

**Proposition 1.8.** Let \( H \) be a weak Hopf algebra. Then
\[
\varepsilon_t \circ S = \varepsilon_t \circ \varepsilon_s = S \circ \varepsilon_s ; \quad \varepsilon_s \circ S = \varepsilon_s \circ \varepsilon_t = S \circ \varepsilon_t.
\]

**Corollary 1.9.** Let \( H \) be a weak Hopf algebra with bijective antipode. Then \( S|_{H_t} = (\varepsilon_s)|_{H_t} \), and \( S^{-1}|_{H_s} = (\varepsilon_t)|_{H_s} \), so \( S \) restricts to an anti-algebra isomorphism \( H_t \to H_s \).

It follows that the separability idempotents of \( H_t \) and \( H_s \) are \( e_t = S(1(1)) \otimes 1(2) \) and \( e_s = 1(1) \otimes S(1(2)) \). Consequently, we have the following formulas, for \( z \in H_t \) and \( y \in H_s \):
\[
\begin{align*}
z S(1(1)) \otimes 1(2) &= S(1(1)) \otimes 1(2) z; \\
y 1(1) \otimes 1(2) &= 1(1) \otimes S^{-1}(y) 1(2).
\end{align*}
\]
Applying \( S^{-1} \otimes H \) to (23), we find
\[
1(1) S^{-1}(z) \otimes 1(2) = 1(1) \otimes 1(2) z.
\]

1.3. **The center of a monoidal category.** Let \( C = (\mathcal{C}, \otimes, I, a, l, r) \) be a monoidal category. The weak left center \( W_l(C) \) is the category with the following objects and morphisms. An object is a couple \( (M, \sigma_{M,-}) \), with \( M \in \mathcal{C} \) and \( \sigma_{M,-} : M \otimes - \to - \otimes M \) a natural transformation, satisfying the following condition, for all \( X, Y \in \mathcal{C} \):
\[
(X \otimes \sigma_{M,Y}) \circ a_{X,M,Y} \circ (\sigma_{M,X} \otimes Y) = a_{X,Y,M} \circ \sigma_{M,X,Y} \circ a_{M,X,Y},
\]
and such that \( \sigma_{M,I} \) is the composition of the natural isomorphisms \( M \otimes I \cong M \cong I \otimes M \). A morphism between \( (M, \sigma_{M,-}) \) and \( (M', \sigma_{M',-}) \) consists of \( \vartheta : M \to M' \) in \( C \) such that
\[
(X \otimes \vartheta) \circ \sigma_{M,X} = \sigma_{M',X} \circ (\vartheta \otimes X).
\]
The left center $Z_l(C)$ is the full subcategory of $W_l(C)$ consisting of objects $(M, \sigma_{M,-})$ with $\sigma_{M,-}$ a natural isomorphism. $Z_l(C)$ is a braided monoidal category. The tensor product is

$$(M, \sigma_{M,-}) \otimes (M', \sigma_{M',-}) = (M \otimes M', \sigma_{M \otimes M',-})$$

with

$$(27) \quad \sigma_{M \otimes M',X} = a_{X,M,M'} \circ (\sigma_{M,X} \otimes M') \circ a_{M,X,M'}^{-1} \circ (M \otimes \sigma_{M',X}) \circ a_{M,M',X},$$

and the unit is $(I, \sigma_{I,-})$, with

$$(28) \quad \sigma_{I,M} = r_M^{-1} \circ l_M.$$ The braiding $c$ on $Z_l(C)$ is given by

$$(29) \quad c_{M,M'} = \sigma_{M,M'} : (M, \sigma_{M,-}) \otimes (M', \sigma_{M',-}) \to (M', \sigma_{M',-}) \otimes (M, \sigma_{M,-}).$$

$Z_l(C)^{in}$ will be our notation for the monoidal category $Z_l(C)$, together with the inverse braiding $\tilde{c}$ given by $\tilde{c}_{M,M'} = c_{M',M}^{-1} = \sigma_{M',M}^{-1}$.

The right center $Z_r(C)$ is defined in a similar way. An object is a couple $(M, \tau_{-,M})$, where $M \in C$ and $\tau_{-,M} : M \to M \otimes -$ is a family of natural isomorphisms such that $\tau_{-,}$ is the natural isomorphism and

$$(30) \quad a_{X,Y,M} \circ \tau_{X \otimes Y,M} \circ a_{X,Y,M}^{-1} = (\tau_{X,M} \otimes Y) \circ a_{X,Y,M}^{-1} \circ (X \otimes \tau_{Y,M}),$$

for all $X, Y \in C$. A morphism between $(M, \tau_{-,M})$ and $(M', \tau_{-,M'})$ consists of $\vartheta : M \to M'$ in $C$ such that

$$(\vartheta \otimes X) \circ \tau_{X,M} = \tau_{X,M'} \circ (X \otimes \vartheta),$$

for all $X \in C$. $Z_r(C)$ is a braided monoidal category. The unit is $(I, l_1^{-1} \circ r_-)$ and the tensor product is

$$(M, \tau_{-,M}) \otimes (M', \tau_{-,M'}) = (M \otimes M', \tau_{-,M \otimes M'})$$

with

$$(31) \quad \tau_{X,M \otimes M'} = a_{M,M',X}^{-1} \circ (M \otimes \tau_{X,M'}) \circ a_{M,X,M'} \circ (\tau_{X,M} \otimes M') \circ a_{X,M,M'}^{-1}.$$ The braiding $d$ is given by

$$(32) \quad d_{M,M'} = \tau_{M,M'} : (M, \tau_{-,M}) \otimes (M', \tau_{-,M'}) \to (M', \tau_{-,M'}) \otimes (M, \tau_{-,M}).$$

$Z_r(C)^{in}$ is the monoidal category $Z_r(C)$ with the inverse braiding $\tilde{d}$ given by $\tilde{d}_{M,M'} = d_{M',M} = \tau_{M',M}^{-1}$.

For details in the case where $C$ is a strict monoidal category, we refer to [12, Theorem XIII.4.2]. The results remain valid in the case of an arbitrary monoidal category, since every monoidal category is equivalent to a strict one. Recall the following result from [5].

**Proposition 1.10.** Let $C$ be a monoidal category. Then we have an isomorphism of braided monoidal categories $F : Z_l(C) \to Z_r(C)^{in}$, given by

$$(33) \quad F(M, \sigma_{M,-}) = (M, \sigma_{M,-}) \text{ and } F(\vartheta) = \vartheta.$$ We have a second monoidal structure on $C$, defined as follows:

$$(34) \quad \overline{C} = (C, \overline{\tau} = \otimes \circ \tau, \overline{I}, \overline{r}, \overline{l})$$

with $\tau : C \times C \to C \times C$, $\tau(M, N) = (N, M)$ and $\overline{r}$ defined by $\overline{r}_{M,N,X} = a_{X,N,M}^{-1}$.

If $c$ is a braiding on $C$, then $\overline{c}$, given by $\overline{c}_{M,N} = c_{N,M}$ is a braiding on $\overline{C}$. In [5], the following obvious result was stated.
Proposition 1.11. Let $C$ be a monoidal category. Then
\[ Z_l(C) \cong Z_r(C) : Z_r(C) \cong Z_l(C) \]
as braided monoidal categories.

1.4. Weak entwining structures and weak smash products. The results in this Section are taken from [6]. Let $A$ be a ring without unit. $e \in A$ is called a preunit if $ea = ae = ae^2$, for all $a \in A$. Then map $p : A \to A$, $p(a) = ae$, satisfies the following properties: $p \circ p = p$ and $p(ab) = p(a)p(b)$. Then $A = \text{Coim}(p)$ is a ring with unit $\tau$ and $A = \text{Im}(p)$ is a ring with unit $e^2$. $p$ induces a ring isomorphism $A \to A$.

Let $k$ be a commutative ring, $A, B$ $k$-algebras with unit, and $R: B \otimes A \to A \otimes B$ a $k$-linear map. We use the notation
\[ R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r, \]
where the summation is implicitly understood. $A \#_R B$ is the $k$-algebra $A \otimes B$ with newly defined multiplication
\[ (a \# b)(c \# d) = ac_R \# bd. \]

$(A, B, R)$ is called a weak smash product structure if $A \#_R B$ is an associative $k$-algebra with preunit $1_A \# 1_B$. The multiplication is associative if and only if
\[ R(bd \otimes a) = a_{Rr} \otimes b_r d_R \quad \text{and} \quad R(b \otimes ac) = a_{Rc_r} \otimes b_{Rr}, \]
for all $a, c \in A$ and $b, d \in B$. $1_A \# 1_B$ is a preunit if and only if
\[ R(1_B \otimes a) = a(1_A)R \otimes (1_B)R \quad \text{and} \quad R(b \otimes 1_A) = (1_A)R \otimes (1_B)Rb. \]

A left-right weak entwining structure is a triple $(A, C, \psi)$, where $A$ is an algebra, $C$ is a coalgebra, and $\psi : A \otimes C \to A \otimes C$ is a $k$-linear map satisfying the conditions
\[ a_\psi \otimes \Delta(c^\psi) = a_\psi \otimes c^{\psi(1)} \otimes c^{\psi(2)} ; \ (ab)_\psi \otimes c^\psi = a_\psi b_\psi \otimes c^{\psi}, \]
\[ 1_\psi \otimes c^\psi = \varepsilon(c^{\psi(1)}) 1_\psi \otimes c^{\psi(2)} ; \ a_\psi \varepsilon(c^\psi) = \varepsilon(c^\psi) a_1. \]

Here we use the notation (with summation implicitly understood):
\[ \psi(a \otimes c) = a_\psi \otimes c^\psi. \]

An entwined module is a $k$-module $M$ with a left $A$-action and a right $C$-coaction such that
\[ \rho(am) = a_\psi m_{[0]} \otimes m_{[1]}^\psi. \]

The category of entwined modules and left $A$-linear right $C$-colinear maps is denoted by $\mathcal{AM}(\psi)^C$.

Let $H$ be a weak bialgebra, and $A$ a right $H$-comodule, which is also an algebra with unit. $A$ is called a right $H$-comodule algebra if $\rho(a)\rho(b) = \rho(ab)$ and $1_{[0]} \otimes \varepsilon_1(1_{[1]}) = \rho(1)$.

From [1], we recall the following definitions. Let $C$ be a left $H$-module which is also a coalgebra with counit. $C$ is called a left $H$-comodule algebra if $\Delta_C(hc) = \Delta_H(h) \Delta_C(c)$ and
\[ \varepsilon_C(hkc) = \varepsilon_H(hk^{(2)}) \varepsilon_C(k^{(1)}c), \]
for all $c \in C$ and $h, k \in H$. Several equivalent definitions are given in [6, Sec. 4]. We then call $(H, A, C)$ a left-right weak Doi-Hopf datum. A weak Doi-Hopf
module over \((H, A, C)\) is a \(k\)-module \(M\) with a left \(A\)-action and a right \(C\)-coaction, satisfying the following compatibility relation, for all \(m \in M\) and \(a \in A\):

\[
(35) \quad \rho(am) = a_0m_0 \otimes a_1m_1.
\]

The category of weak Doi-Hopf modules over \((H, A, C)\) and left \(A\)-linear right \(C\)-colinear maps is denoted by \(\mathcal{A}M(H)^C\).

Let \((H, A, C)\) be a weak left-right Doi-Hopf datum, and consider the map

\[
\psi : A \otimes C \to A \otimes C, \quad \psi(a \otimes c) = a_0 \otimes a_1c.
\]

Then \((A, C, \psi)\) is a weak left-right entwining structure, and we have an isomorphism of categories \(\mathcal{A}M(H)^C \cong \mathcal{A}M(\psi)^C\).

Let \((A, C, \psi)\) be a weak left-right entwining structure, and assume that \(C\) is finitely generated projective as a \(k\)-module, with finite dual basis \(\{c_i, c^*_i\mid i = 1, \ldots, n\}\).

Then we have a weak smash product structure \((A, C^*, R)\), with \(R : C^* \otimes A \to A \otimes C^*\) given by

\[
(36) \quad R(c^* \otimes a) = \sum_i \langle c^*, c^*_i \rangle a \otimes c^*_i.
\]

We have an isomorphism of categories

\[
(37) \quad F : \mathcal{A}M(\psi)^C \to \mathcal{A}M(H)^C,\]

defined also follows: \(F(M) = M\) as a \(k\)-module, with action \([a \# c^*]m = \langle c^*, m_{[1]}\rangle am_{[0]}\).

Details can be found in [6, Theorem 3.4].

\section{Yetter-Drinfeld modules over weak Hopf algebras}

Let \(H\) be a weak bialgebra. A left-left Yetter-Drinfeld module is a \(k\)-module with a left \(H\)-action and a left \(H\)-coaction such that the following conditions hold, for all \(m \in M\) and \(h \in H\):

\[
(38) \quad \lambda(m) = m_{[-1]} \otimes m_{[0]} \in H \otimes_t M;
\]
\[
(39) \quad h_{(1)}m_{[-1]} \otimes h_{(2)}m_{[0]} = (h_{(1)}m_{[-1]}h_{(2)}) \otimes h_{(1)}m_{[0]}.
\]

We will now state some equivalent definitions. First we will rewrite the counit property for Yetter-Drinfeld modules.

\textbf{Lemma 2.1.} Let \(H\) be a weak bialgebra, and \(\lambda : M \to H \otimes_t M, \rho(m) = m_{[-1]} \otimes m_{[0]}\) a \(k\)-linear map. Then

\[
(40) \quad \varepsilon(m_{[-1]})m_{[0]} = \varepsilon_t(m_{[-1]})m_{[0]}.
\]

Consequently, in the definition of a Yetter-Drinfeld module, the counit property \(\varepsilon(m_{[-1]})m_{[0]} = m\) can be replaced by \(\varepsilon_t(m_{[-1]})m_{[0]} = m\).

\textbf{Proof.}

\[
\varepsilon_t(m_{[-1]})m_{[0]} = \varepsilon(1_{(1)}m_{[-1]})1_{(2)}m_{[0]} = \varepsilon(m_{[-1]})m_{[0]}.
\]

\Box

In the case of a weak Hopf algebra, the compatibility relation (39) can also be restated:
Proposition 2.2. (cf. [16, Remark 2.6]) Let $H$ be a weak Hopf algebra, and $M$ a $k$-module, with a left $H$-action and a left $H$-coaction. $M$ is a Yetter-Drinfeld module if and only if

\[(41) \quad \lambda(hm) = h(1)m[-1]S(h(3)) \otimes h(2)m[0].\]

Proof. Let $M$ be a Yetter-Drinfeld module. Then we compute

\[(42) \quad (h(1)m)[-1]S(h(3)) \otimes h(2)m[0] = (h(1)m)[-1]S(h(3)) \otimes (h(1)m)[0] = (11)hm[-1]1(2) \otimes (11)hm[0] = (hm)[-1] \otimes (hm)[0] = \lambda(hm).\]

Conversely, assume that (41) holds for all $h \in H$ and $m \in M$. Taking $h = 1$ in (41), we find

\[
\lambda(m) = 1(1)m[-1]S(1(3)) \otimes 1(2)m[0] = 1(1)m[-1]S(1(2')) \otimes 1(2)1(1')m[0] \in H \otimes_t M
\]

and

\[
\lambda(m) = m[-1]S(1(2')) \otimes 1(1')m[0].
\]

Now

\[(43) \quad (h(1)m)[-1]h(2) \otimes (h(1)m)[0] = h(1)m[-1]S(h(3))h(4) \otimes h(2)m[0] = h(1)m[-1]S(1(2)) \otimes h(2)1(1)m[0] = (41) h(1)m[-1] \otimes h(2)m[0],
\]

as needed. \[\square\]

Corollary 2.3. Let $M$ be a left-left Yetter-Drinfeld module. For all $y \in H_s$, $z \in H_t$ and $m \in M$, we have

\[(44) \quad \lambda(zm) = zm[-1] \otimes m[0] ; \quad \lambda(ym) = m[-1]S(y) \otimes m[0].\]

Proof. \[
\lambda(zm) = (41) \quad 1(1)zm[-1]S(1(3)) \otimes 1(2)m[0]
\]

\[(8) \quad = z1(1)m[-1]S(1(3)) \otimes 1(2)m[0] = zm[-1] \otimes m[0].\]

The other assertion is proved in a similar way. \[\square\]

Corollary 2.4. Let $M$ be a left-left Yetter-Drinfeld module over a weak Hopf algebra with bijective antipode. Then we have the following identities, for all $m \in M$:

\[(45) \quad S^{-1}(m[-1])m[0] = m[-1] \otimes S^{-1}(m[-1]);\]

\[(46) \quad \varepsilon(S^{-2}(m[-1]))m[0] = m.\]

Proof. Apply $S^{-1}$ to the first factor of (42), and then switch the two tensor factors. Then we obtain (44). (45) is proved as follows:

\[
m = \varepsilon(m[-1])m[0] = \varepsilon(m[-1])m[0] = \varepsilon(S^{-1}(m[-1]))m[0] = \varepsilon(S^{-1}(m[-1]))1(1)m[0] = \varepsilon(S^{-2}(m[-1]))m[0].\]

\[\square\]
The category of left-left Yetter-Drinfeld modules and left $H$-linear, left $H$-colinear maps will be denoted by $H^H\text{YD}$.

**Example 2.5.** Let $G$ be a groupoid, and $kG$ the corresponding groupoid algebra. Then $kG$ is a weak Hopf algebra. Let $M$ be a left-left Yetter-Drinfeld module. Then $M$ is a $kG$-comodule, so $M$ is graded by the set $G$, that is

$$M = \bigoplus_{\sigma \in G_1} M_{\sigma},$$

and $\lambda(m) = \sigma \otimes m$ if and only if $m \in M_{\sigma}$, or $\deg(m) = \sigma$.
Recall that the unit element of $kG$ is $1 = \sum_{x \in G_0} x$, where $x$ is the identity morphism of the object $x \in G_0$. Take $m \in M_{\sigma}$. Using (41), we find

$$\lambda(m) = \lambda(1m) = \sum_{x \in G} x\sigma x \otimes xm = 0,$$

unless $s(\sigma) = \tau(\sigma) = x$. So we have

$$M = \bigoplus_{\sigma \in G_1} M_{\sigma}.$$

Take $m \in M_{\sigma}$, with $s(\sigma) = \tau(\sigma)$, and $\tau \in G_1$. It follows from (41) that $\lambda(\tau m) = \tau \sigma \tau^{-1} \otimes \tau m = 0$, unless $s(\tau) = x$. If $s(\tau) = x$, then $\deg(\tau m) = \tau \sigma \tau^{-1}$.

**Theorem 2.6.** Let $H$ be a weak bialgebra. Then the category $H^H\text{YD}$ is isomorphic to the weak left center $W(\text{H}, M)$ of the category of left $H$-modules. If $H$ is a weak Hopf algebra with bijective antipode, then $H^H\text{YD}$ is isomorphic to the left center $Z(\text{H}, M)$.

**Proof.** We will restrict to a brief description of the connecting functors; for more detail (in the left-right case), we refer to [16, Lemma 4.3]. Take $(M, \sigma_{M,-}) \in W(\text{H}, M)$. For each left $H$-module $V$, we have a map $\sigma_{M,V} : M \otimes_1 V \rightarrow V \otimes_1 M$ in $\text{H}\text{YD}$. We will show that the map

$$\lambda : M \rightarrow H \otimes_1 M, \quad \lambda(m) = \sigma_{M,H}(1_1 m \otimes 1_2) = m_{[-1]} \otimes m_{[0]}$$

makes $M$ into a Yetter-Drinfeld module. Conversely, let $(M, \lambda)$ be a Yetter-Drinfeld module; a natural transformation $\sigma$ is then defined by the formula

$$(46) \quad \sigma_{M,V}(1_1 m \otimes 1_2 v) = m_{[-1]} v \otimes m_{[0]}.$$ 

Straightforward computations show that $(M, \sigma) \in W(\text{H}, M)$. If $H$ is a Hopf algebra with invertible antipode, then the inverse of $\sigma_{M,V}$ is

$$(47) \quad \sigma_{M,V}^{-1}(1_1 v \otimes 1_2 m) = m_{[0]} \otimes S^{-1}(m_{[-1]}) v.$$ 

$\square$

From now on, we assume that $H$ is a weak Hopf algebra with bijective antipode. Since the left center of a monoidal category is a braided monoidal category, it follows from Theorem 2.6 that $H^H\text{YD}$ is a braided monoidal category; a direct but long proof can be given: see [16, Prop. 2.7]. The monoidal structure can be computed using (27). Take $M, N \in H^H\text{YD}$, the $H$-coaction on $M \otimes_1 N$ is given by the formula

$$\lambda(1_1 m \otimes 1_2 n) = ((\sigma_{M,H} \otimes N) \circ (M \otimes \sigma_{N,H}))(1_1 (1_1 m \otimes 1_2 n) \otimes 1_{(2)}).$$
Observe that
\[
x = 1^{(1)}(1^{(1)}m \otimes 1^{(2)}n) \otimes 1^{(2')} = 1^{(1)}1^{(1)}m \otimes 1^{(1')}1^{(2')}l^{(2)}n \otimes 1^{(2'')},
\]
so that
\[
(M \otimes \sigma_{N,H})(x) = 1^{(1)}m \otimes (1^{(2)}n)[-1] \otimes (1^{(2)}n)[0] = 1^{(1)}m \otimes 1^{(1')}1^{(2)}l^{(2)}n \otimes 1^{(2')}.
\]

We compute the left \(H\)-coaction on \(H_t\) using (28) and (46). For any \(z \in H_t\), this gives
\[
\lambda(z) = \sigma_{H_t,H}(1^{(1)}\rightarrow z) \otimes 1^{(2)} = r^{-1}_M(l_M((1^{(1)}\rightarrow z) \otimes 1^{(2)}))
\]
\[
= r^{-1}_M(z) = 1^{(1)}z \otimes 1^{(2)} = \Delta(z).
\]
The braiding and its inverse are given by the formulas
\[
\sigma_{M,N}(1^{(1)}m \otimes 1^{(2)}n) = m_{[-1]}n \otimes m_{[0]}: \sigma^{-1}_{M,N}(1^{(1)}n \otimes 1^{(2)}m) = m_{[0]} \otimes S^{-1}(m_{[-1]}n).
\]

A left-right Yetter-Drinfeld module is a \(k\)-module with a left \(H\)-action and a right \(H\)-coaction such that the following conditions hold, for all \(m \in M\) and \(h \in H\):
\[
\rho(m) = m_{[0]} \otimes m_{[1]} \in M \otimes H;
\]
\[
h^{(1)}m_{[0]} \otimes h^{(2)}m_{[1]} = (h^{(2)}m)_{[0]} \otimes (h^{(2)}m)_{[1]}h^{(1)}.
\]

The category of left-right Yetter-Drinfeld modules and left \(H\)-linear right \(H\)-colinear maps is denoted by \(\mathcal{YD}^H\).

**Proposition 2.7.** Let \(H\) be a weak Hopf algebra with bijective antipode. Then the category \(\mathcal{YD}^H\) is isomorphic to the right center \(Z_r(H,M)\).

**Proof.** Take \((M, \tau_{\cdot,-}, M) \in Z_r(H,M)\). We know from Proposition 1.10 that \((M, \sigma_{M,-} = \tau_{\cdot,M}^{-1}) \in Z_l(H,M)\). Take the corresponding left-left Yetter-Drinfeld \((M, \Lambda)\), as in Theorem 2.6, and define \(\rho : M \rightarrow M \otimes H\) by
\[
\rho(m) = m_{[0]} \otimes m_{[1]} = m_{[0]} \otimes S^{-1}(m_{[-1]}).
\]
It follows from (44) that \(\rho(m) \in M \otimes H\). The coassociativity of \(\rho\) follows immediately from the coassociativity of \(\lambda\) and the anti-comultiplicativity of \(S^{-1}\). Also
\[
\varepsilon(m_{[1]}m_{[0]} \otimes \varepsilon(S^{-1}(m_{[-1]}))m_{[0]} = \varepsilon(m_{[-1]}m_{[0]} = m.
\]
From (47), it follows that
\[
\tau_{\cdot, M}(1^{(1)}m \otimes 1^{(2)}n) = m_{[0]} \otimes m_{[1]}v.
\]
In particular, \(\tau_{M,H}(1^{(1)} \otimes 1^{(2)}m) = \rho(m)\), and the fact that \(\tau_{M,H}\) is left \(H\)-linear implies (51). Hence \((M, \rho)\) is a left-right Yetter-Drinfeld module. Conversely, if \((M, \rho)\) is a left-right Yetter-Drinfeld module, then \((M, \tau_{\cdot,M})\), with \(\tau\) defined by (53) is an object of \(Z_r(H,M)\). \(\square\)
Corollary 2.8. Let $M$ be a $k$-module with a left $H$-action and a right $H$-coaction. Then $M$ is a left-right Yetter-Drinfeld module if and only if
\begin{equation}
\rho(hm) = h(2)m_{[0]} \otimes h(3)m_{[1]}S^{-1}(h(1)).
\end{equation}

Corollary 2.9. Let $M$ be a left-right Yetter-Drinfeld module. For all $y \in H$, $z \in H$, and $m \in M$, we have that
\begin{equation}
\rho(ym) = m_{[0]} \otimes zm_{[1]} ; \rho(zm) = m_{[0]} \otimes m_{[1]}S^{-1}(z).
\end{equation}

Corollary 2.10. Let $M$ be a left-right Yetter-Drinfeld module. Then
\begin{equation}
1_{(2)} m_{[0]} \otimes m_{[1]} S^{-1}(1_{(1)}) = \rho(m),
\end{equation}
for all $m \in M$.

Proof. Apply $S^{-1} \otimes M$ to $\lambda(m) = 1_{(1)} S(m_{[1]}) \otimes 1_{(2)} m_{[0]}$. \hfill \square

Corollary 2.11. The category $H \mathcal{YD}^H$ is a braided monoidal category, isomorphic to $H \mathcal{YD}^m$.

In a similar way, we can introduce right-right and right-left Yetter-Drinfeld modules. The categories $\mathcal{YD}^H_H$ and $H \mathcal{YD}_H$ of right-right and right-left Yetter-Drinfeld modules are isomorphic to the right and left center of $\mathcal{M}_H$. Let us summarize the results.

A right-right Yetter-Drinfeld module is a $k$-module $M$ with a right $H$-action and a right $H$-coaction such that
\begin{align}
\rho(m) &= m_{[0]} \otimes m_{[1]} \in M \otimes_s H; \\
m_{[0]}h(1) \otimes m_{[1]}h(2) &= (mh(2))_{[0]} \otimes h(1)(mh(2))_{[1]}; \\
or, equivalently, \\
\rho(mh) &= m_{[0]}h(2) \otimes S(h(1))m_{[1]}h(3).
\end{align}

The counit condition $m = \varepsilon(m_{[1]})m_{[0]}$ is equivalent to $m = m_{[0]}\varepsilon(m_{[1]})$.

The natural isomorphism $\tau_{-,M}$ corresponding to $(M, \rho) \in \mathcal{YD}^H_H$ and its inverse are given by the formulas
\begin{equation}
\tau_{M,V}(v1_{(1)} \otimes m1_{(2)}) = m_{[0]} \otimes mv_{[1]} ; \tau_{M,V}^{-1}(m1_{(1)} \otimes v1_{(2)}) = vS^{-1}(m_{[1]}) \otimes m_{[0]}.
\end{equation}

Furthermore
\begin{equation}
m_{[0]} \varepsilon_t(S^{-2}(m_{[1]})) = m,
\end{equation}
and $S^{-1}(m_{[1]}) \otimes m_{[0]} \in H \otimes_s M$.

The monoidal structure on $\mathcal{YD}^H_H$ is given by the formula
\begin{equation}
\rho(m1_{(1)} \otimes n1_{(2)}) = m_{[0]} \otimes n_{[0]} \otimes m_{[1]}n_{[1]}.
\end{equation}

The braiding is given by (60). The category $\mathcal{YD}^H_H$ is isomorphic as a braided monoidal category to $Z_r(\mathcal{M}_H)$.

Let $M$ be a right $H$-module and a left $H$-comodule. $M$ is a right-left Yetter-Drinfeld module if one of the three following equivalent conditions is satisfied, for all $m \in M$ and $h \in H$:
1) $\lambda(m) \in H \otimes_s M$ and
\begin{equation}
h(2)(mh(1))_{[0]} \otimes (mh(1))_{[1]} = m_{[-1]}h(1) \otimes m_{[0]}h(2),
\end{equation}
This isomorphism can be described explicitly as follows:

\[ \lambda(m(1(1) \otimes n(2))) = m(-1) n(-1) \otimes m(0) \otimes n(0); \]

\[ \sigma_{M,N}(m(1(1) \otimes n(2))) = nm(-1) \otimes m(0). \]

As a braided monoidal category, \( \mathcal{H} \mathcal{YD}_H \) is isomorphic to \( Z(H) \) and \( (\mathcal{YD}^H_H)_\text{in} \).

The antipode \( S : H \to H^{\text{op},\text{cop}} \) is an isomorphism of weak Hopf algebras. Observe that the target map of \( H^{\text{op},\text{cop}} \) is \( \varepsilon_s \), and that its source map is \( \varepsilon_l \). Thus \( S \) induces an isomorphism between the monoidal categories \( H \mathcal{M} \) and \( H^{\text{op},\text{cop}} \mathcal{M} \). We also have a monoidal isomorphism \( F : H^{\text{op},\text{cop}} \mathcal{M} \to \overline{\mathcal{M}}_H \), given by

\[ F(M) = M, \quad mh = h^{\text{op},\text{cop}} M. \]

Indeed, in \( H^{\text{op},\text{cop}} \mathcal{M} \), \( M \otimes \mathcal{N} \) is generated by elements of the form \( 1(2)m \otimes 1(1)n \), and \( F(M \otimes N) \) is generated by elements of the form \( m1(2) \otimes n1(1) \). \( F(N) \otimes_s F(M) \) is generated by elements of the form \( n1(1) \otimes m1(2) \), and it follows that the switch map is an isomorphism \( F(M \otimes N) \to F(N) \otimes_s F(M) \). We conclude from Proposition 1.11 that we have isomorphisms of braided monoidal categories

\[ \mathcal{H} \mathcal{YD} \cong Z(H) \mathcal{M} \cong Z(H^{\text{op},\text{cop}} \mathcal{M}) \cong Z(\overline{\mathcal{M}}_H) \cong \mathcal{YD}^H_H. \]

This isomorphism can be described explicitly as follows:

\[ F : \mathcal{H} \mathcal{YD} \to \mathcal{YD}^H_H, \quad F(M) = M, \]

with

\[ m \cdot h = S^{-1}(h)m; \quad \rho(m) = m(0) \otimes S(m(-1)). \]

We summarize our results as follows:

**Theorem 2.12.** Let \( H \) be a weak Hopf algebra with bijective antipode. Then we have the following isomorphisms of braided monoidal categories:

\[ \mathcal{H} \mathcal{YD} \cong \mathcal{YD}^H_H \cong \mathcal{YD}^H_H \cong \mathcal{YD}^H_H \cong \mathcal{YD}^H_H. \]

3. **Yetter-Drinfeld Modules are Doi-Hopf Modules**

It was shown in [7] that Yetter-Drinfeld modules (over a classical Hopf algebra) can be considered as Doi-Hopf modules, and, a fortiori, as entwined modules, and as comodules over a coring (see [4]). Weak Doi-Hopf modules were introduced by Böhm [1], and they are special cases of weak entwined modules (see [6]), and these are in turn examples of comodules over a coring (see [4]). In this Section, we will show that Yetter-Drinfeld modules over weak Hopf algebras are special cases of weak Doi-Hopf modules. We will discuss the left-right case.

**Proposition 3.1.** Let \( H \) be a weak Hopf algebra with a bijective antipode. Then \( H \) is a right \( H \otimes H^{\text{op}} \)-comodule algebra, with \( H \)-coaction

\[ \rho(h) = h(2) \otimes S^{-1}(h(1)) \otimes h(3). \]
Proof. It is easy to verify that $H$ is a right $H \otimes H^{\text{op}}$-comodule and that $\rho(hk) = \rho(h)\rho(k)$. Recall that $H_t = \text{Im}(\varepsilon_t) = \text{Im}(\pi_t)$. The target map of $H^{\text{op}} \otimes H$ is $\pi_t \otimes \varepsilon_t$. We now have
\[
1_{[0]} \otimes (\pi_t \otimes \varepsilon_t)(1_{[1]}) = 1_{(2)} 1_{(1')} \otimes \pi_t(S^{-1}(1_{(1)})) \otimes \varepsilon_t(1_{(2')})
\]
\[
= 1_{(2)} 1_{(1')} \otimes S^{-1}(1_{(1)}) \otimes 1_{(2')} = \rho(1),
\]
where we used the fact that $S^{-1}(1_{(1)}) \otimes 1_{(2)} \in H_t \otimes H_t$. \hfill \Box

**Proposition 3.2.** Let $H$ be a weak Hopf algebra with a bijective antipode. Then $H$ is a left $H^{\text{op}} \otimes H$-module coalgebra with left action
\[
(k \otimes h) \triangleright c = hck.
\]

**Proof.** We easily compute that
\[
\varepsilon((m \otimes l)(k_{(2)} \otimes h_{(2)})) \varepsilon((k_{(1)} \otimes h_{(1)}) \triangleright c)
\]
\[
= \epsilon(k_{(2)}m)\epsilon(lh_{(2)})\epsilon(h_{(1)}ck_{(1)})
\]
\[
= \epsilon(lhc) = \epsilon((m \otimes l)(k \otimes h)) \triangleright c).
\]
The other conditions are easily verified. \hfill \Box

**Corollary 3.3.** Let $H$ be a weak Hopf algebra with bijective antipode. Then we have a weak Doi-Hopf datum $(H^{\text{op}} \otimes H, H, H)$ and the categories $\mathcal{H} \mathcal{M}(H^{\text{op}} \otimes H)^H$ and $\mathcal{H}\mathcal{Y}D^H$ are isomorphic.

**Proof.** The compatibility relation (35) reduces to (54). \hfill \Box

As we have seen in Section 1.4, weak Doi-Hopf modules are special cases of entwined modules. The entwining map $\psi : H \otimes H \to H \otimes H$ corresponding to the weak Doi-Hopf datum $(H^{\text{op}} \otimes H, H, H)$ is given by
\[
\psi(h \otimes k) = h_{(2)} \otimes h_{(3)}kS^{-1}(h_{(1)}).
\]

### 4. The Drinfeld double

Now we consider the particular case where $H$ is finitely generated and projective as a $k$-module, with finite dual basis $\{ (h_i, h_i^*) \mid i = 1, \ldots, n \}$. Then $H^*$ is also a weak Hopf algebra, in view of the selfduality of the axioms of a weak Hopf algebra. Recall that the comultiplication is given by the formula $(\Delta(h^*), h \otimes k) = (h^*, hk)$; the counit is evaluation at 1. Also recall that $H^*$ is an $H$-bimodule, with left and right $H$-action
\[
(h \rightarrow h^* \leftarrow k, l) = (h^*, kh_l),
\]
or
\[
h \rightarrow h^* \leftarrow k = (h^*_{(1)}, k)(h^*_{(3)}, h)h^*_{(2)}.
\]

Using (36), we find a weak smash product structure $(H, H^*, R)$, with $R : H^* \otimes H \to H \otimes H^*$ given by
\[
R(h^* \otimes h) = \sum_i(h^*, h_{(3)}^i) S^{-1}(h_{(1)}^i) h_{(2)} \otimes h_{i}^*
\]
\[
= \sum_i(S^{-1}(h_{(1)}^i) \rightarrow h^* \leftarrow h_{(3)}^i, h_{i}^i) h_{(2)} \otimes h_{i}^*
\]
\[
= h_{(2)} \otimes \left(S^{-1}(h_{(1)}^i) \rightarrow h^* \leftarrow h_{(3)}^i \right).
\]
From Section 1.4, we know that $H \#_R H^*$, which we will also denote by $H \bowtie H^*$, is an associative algebra with preunit $1 \# \varepsilon$. Using (33), we compute the multiplication rule on $H \bowtie H^*$.

\[
(h \bowtie h^*)(k \bowtie k^*) = \sum_i (h^*, k_i) h_i S^{-1}(k_{(1)}) h k_{(2)} \bowtie h^*_i * k^*
\]

(64)

\[
h k_{(2)} \bowtie (S^{-1}(k_{(1)}) \bowtie h^* \prec k_{(3)}) \bowtie k^*
\]

(65)

We have a projection $p : H \bowtie H^* \to H \bowtie H^*$,

\[
p(h \bowtie h^*) = (1 \bowtie \varepsilon)(h \bowtie h^*)(1 \bowtie \varepsilon) = (h \bowtie h^*)(1 \bowtie \varepsilon)^2,
\]

and $D(H) = \overline{H \bowtie H^*} = (H \bowtie H^*)/\text{Ker } p$ is a $k$-algebra with unit $[1 \bowtie \varepsilon]$, which we call the Drinfeld double of $H$. $D(H)$ is also isomorphic to $\overline{H \bowtie H^*} = \text{Im } (p)$, which is a $k$-algebra with unit $(1 \bowtie \varepsilon)^2$. Observe that the multiplication rule (65) is the same as in [1, 16]. We show that the ideal $J$ that is divided out in [1, 16] is equal to $\text{Ker } p$, and this will imply that $D(H)$ is equal to the Drinfeld double introduced in [1, 16]. We first need some Lemmas.

**Lemma 4.1.** Let $H$ a weak bialgebra. For all $h^* \in H^*$, $y \in H_s$ and $z \in H_t$, we have

\[
h^* \bowtie (y \bowtie \varepsilon) = \langle h^*_2, y \rangle h^*_1 = y \bowtie h^*
\]

(66)

\[
h^* \bowtie (\varepsilon \bowtie y) = \langle h^*_1, y \rangle h^*_2 = h^* \bowtie y
\]

(67)

\[
(z \bowtie \varepsilon) * h^* = \langle h^*_2, z \rangle h^*_1 = z \bowtie h^*
\]

(68)

\[
(\varepsilon \bowtie z) * h^* = \langle h^*_1, z \rangle h^*_2 = h^* \bowtie z
\]

(69)

**Proof.** We only prove (68). For all $h \in H$, we have

\[
\langle (z \bowtie \varepsilon) * h^*, h \rangle = \langle \varepsilon, h_{(1)} z \rangle \langle h^*, h_{(2)} \rangle = \langle \varepsilon, h_{(1)} 1_{(1)} z \rangle \langle h^*, h_{(2)} 1_{(2)} \rangle
\]

\[
= \langle \varepsilon * h^*, h z \rangle = \langle h^*, h z \rangle = \langle z \bowtie h^*, h \rangle = \langle h^*_2, z \rangle \langle h^*_1, h \rangle.
\]

\[
\square
\]

**Lemma 4.2.** Let $H$ be a weak Hopf algebra with bijective antipode. For all $y \in H_s$, $z \in H_t$, we have

\[
S^{-1}(z) \bowtie \varepsilon = z \bowtie \varepsilon \quad \text{and} \quad \varepsilon \bowtie y = \varepsilon \bowtie S^{-1}(y).
\]

(70)

**Proof.** For all $h \in H$, we have

\[
\langle S^{-1}(z) \bowtie \varepsilon, h \rangle = \varepsilon(h S^{-1}(z)) \langle 2 \rangle \langle 2 \rangle \varepsilon(h_{1(1)}) \varepsilon(1_{(2)} S^{-1}(z)) \langle 1 \rangle \varepsilon(h_{1(1)}) \varepsilon(z S(1_{(2)})) \langle 2 \rangle = \varepsilon(h_{1(1)}) \varepsilon(z_{s, (h_{2(2)})}) \langle 8 \rangle \varepsilon(z_{s, (h_{2(2)})}) \langle 6 \rangle = \varepsilon(z_{s, (h_{2(2)})}) \langle 5 \rangle = \varepsilon(h z) = \langle z \bowtie \varepsilon, h \rangle.
\]

The second statement can be proved in a similar way. 

\[
\square
\]

**Proposition 4.3.** Let $H$ be a finitely generated projective weak Hopf algebra. Then $\text{Ker } (p)$ is the $k$-linear span $J$ of elements of the form

\[
A = h z \bowtie h^* - h \bowtie (z \bowtie \varepsilon) * h^* \quad \text{and} \quad B = h y \bowtie h^* - h \bowtie (\varepsilon \bowtie y) * h^*.
\]

where $h \in H$, $h^* \in H^*$, $y \in H_s$ and $z \in H_t$.
Proof. $A \in \text{Ker}(p)$ since
\[
(1 \triangleright \varepsilon)(h z \triangleright h^*) = (h(2) 1(2) \triangleright \varepsilon(2) * h^*)(\varepsilon(1), h(3) 1(3)) \langle \varepsilon(3), S^{-1}(h(1) 1(1) z) \rangle
\]
\[
= h(2) \triangleright \varepsilon(2) * h^*(\varepsilon(1), h(3)) \langle \varepsilon(3), S^{-1}(z) \rangle \langle \varepsilon(4), S^{-1}(h(1)) \rangle
\]
\[
= h(2) \triangleright (\varepsilon(2) * (S^{-1}(z) \triangleright \varepsilon) * h^*)(\varepsilon(1), h(3)) \langle \varepsilon(3), S^{-1}(h(1)) \rangle
\]
\[
= (1 \triangleright \varepsilon)(1 \triangleright \varepsilon(2) * h^*) \triangleright (S^{-1}(z) \triangleright \varepsilon) * h^*).
\]
In a similar way, $B \in \text{Ker}(p)$:
\[
(1 \triangleright \varepsilon)(h y \triangleright h^*) = (h(2) 1(2) \triangleright \varepsilon(2) * h^*)(\varepsilon(1), h(3) 1(3) y) \langle \varepsilon(3), S^{-1}(h(1) 1(1)) \rangle
\]
\[
= (h(2) \triangleright \varepsilon(3) * h^*)(\varepsilon(1), h(3)) \langle \varepsilon(2), y \rangle \langle \varepsilon(4), S^{-1}(h(1)) \rangle
\]
\[
= (h(2) \triangleright (\varepsilon(2) \triangleright y) * h^*)(\varepsilon(1), h(3)) \langle \varepsilon(3), S^{-1}(h(1)) \rangle
\]
\[
= (1 \triangleright \varepsilon)(1 \triangleright \varepsilon(2) * h^*).
\]
This shows that $J \subset \text{Ker}(p)$. We now compute for all $h \in H$ and $h^* \in H^*$ that
\[
(h \triangleright h^*)(1 \triangleright \varepsilon) = (h 1(2) 1(1') \triangleright h^*_{(2)})(h^*_{(1)}, 1(2')) \langle h^*_{(3)}, S^{-1}(1(1)) \rangle,
\]
and
\[
\left( h \triangleright (S^{-1}(1(2)) \langle \varepsilon \rangle \ast h^*) \ast h^*_{(2)} \right) \langle h^*_{(1)}, 1(2') \rangle \langle h^*_{(3)}, S^{-1}(1(1)) \rangle
\]
\[
= (h \triangleright \varepsilon_{(1)} * h^*_y) \langle \varepsilon_{(2)}, S^{-1}(1(2)) \rangle \langle \varepsilon_{(1')}, 1(1') \rangle
\]
\[
= (h \triangleright \varepsilon_{(1)} * h^*_y) \langle h^*_{(1)}, 1(2') \rangle \langle h^*_{(3)}, S^{-1}(1(1)) \rangle
\]
\[
= (h \triangleright \varepsilon_{(1)} * h^*_y) \langle \varepsilon_{(2)} * h^*_y, 1 \rangle = h \triangleright (\varepsilon \ast h^*) = h \triangleright h^*.
\]
Observing that
\[
h z y \triangleright h^* - h \triangleright ((S^{-1}(z) \triangleright \varepsilon) \ast (\varepsilon \triangleright y) \ast h^*)
\]
\[
= h z y \triangleright h^* - h \triangleright (\varepsilon \triangleright y) \ast h^*
\]
\[
+ h z \triangleright (\varepsilon \triangleright y) \ast h^* - h \triangleright ((S^{-1}(z) \triangleright \varepsilon) \ast (\varepsilon \triangleright y) \ast h^*) \in J,
\]
it follows that $(h \triangleright h^*)(1 \triangleright \varepsilon) - (h \triangleright h^*) \in J$, for all $h \in H$ and $h^* \in H^*$. If $x \in \text{Ker}(p)$, then $x(1 \triangleright \varepsilon) = 0$, and $x = x - x(1 \triangleright \varepsilon) \in J$. We conclude that $\text{Ker}(p) \subset J$, finishing our proof. $\square$

We now recall the following results from [17]. On $H^* \otimes H$, there exists an associative multiplication
\[
(h^* \otimes h)(k^* \otimes k) = k^*_{(2)} h^* \otimes h(2) k \langle S(h(1)), k^*_{(1)} \rangle \langle h(3), k^*_{(3)} \rangle
\]
\[
= \langle h(3) \rightarrow k^* \rightarrow S(h(1)) \rangle \ast h^* \otimes h(2) k.
\]
The $k$-module $I$ generated by elements of the form
\[
A' = h^* \otimes h z \ast (\varepsilon \triangleright z) h^* \otimes h \text{ and } B' = h^* \otimes y h \ast (y \triangleright \varepsilon) h^* \otimes h
\]
is a two-sided ideal of $H^* \otimes H$. The quotient $D'(H) = (H^* \otimes H)/I$ is an algebra with unit element $\varepsilon \otimes 1$. It is a weak Hopf algebra, with the following comultiplication,
The counit and antipode:

\begin{align}
\Delta[h^* \otimes h] &= [h^*_{(1)} \otimes h_{(1)}] \otimes [h^*_{(2)} \otimes h_{(2)}] \\
\varepsilon[h^* \otimes h] &= \langle h^*, \varepsilon (h) \rangle \\
S[h^* \otimes h] &= [S^-(h^*_{(2)}) \otimes S(h_{(2)})] \langle h^*_{(1)}, h_{(1)} \rangle \langle h^*_{(3)}, S(h_{(3)}) \rangle
\end{align}

**Proposition 4.4.** The $k$-linear isomorphism

\[
f : H \rhd h^* \to H^* \otimes h, \quad f(h \rhd h^*) = h^* \otimes S^-(h)
\]

is anti-multiplicative, and induces an algebra isomorphism $f : D(H) \to D'(H)^{\text{op}}$.

**Proof.** Let us first prove that $f$ reverses the multiplication. Indeed,

\[
f(k \rhd k^*)f(h \rhd h^*) = (k^* \otimes S^-(k))(h^* \otimes S^-(h))
= (S^-(k_{(1)}) \leftarrow h^* \leftarrow k_{(3)}) * k^* \otimes S^-(k_{(2)}) S^-(h)
= f((h \rhd h^*)(k \rhd k^*)).
\]

Using Lemma 4.2, we easily compute that $f(J) = I$, and the result follows.

Let us now define a comultiplication, counit and antipode on $D(H)$, in such a way that $f : D(H) \to D'(H)$ is an isomorphism of Hopf algebras. Obviously, the comultiplication is given by the formula

\[
\Delta[h \rhd h^*] = [h_{(2)} \rhd h^*_{(1)}] \otimes [h_{(1)} \rhd h^*_{(2)}].
\]

The counit is computed as follows:

\[
\varepsilon[h \rhd h^*] = \varepsilon[h^* \otimes S^-(h)] = \langle h^*, \varepsilon(t(S^-(h))) \rangle = \langle h^*, 1_{(2)} \rangle \langle \varepsilon, h_{1(1)} \rangle.
\]

Since the antipode of $H$ is the inverse of the antipode of $H^{\text{op}}$, the antipode of $D'(H)$ is transported to the inverse of the antipode of $D(H)$. We find

\[
S^-(h \rhd h^*) = f^{-1} \circ f \circ f^{-1}(S[h^* \otimes S^-(h)]) = f^{-1}(f^{-1}(S[h^* \otimes S^-(h)]))
= f^{-1}([S^-(h^*_{(2)}) \otimes h_{(2)}] \langle h^*_{(1)}, S^-(h_{(3)}) \rangle \langle h^*_{(3)}, h_{(1)} \rangle)
= [S(h_{(2)}) \rhd S^-(h^*_{(2)})] \langle h^*_{(1)}, S^-(h_{(3)}) \rangle \langle h^*_{(3)}, h_{(1)} \rangle
\]

The antipode $S$ is then given by the formula

\[
S[h \rhd h^*] = [S^-(h_{(2)}) \rhd h^*_{(2)}] \langle h^*_{(1)}, S^-(h_{(3)}) \rangle \langle h^*_{(3)}, h_{(1)} \rangle
\]

Indeed,

\[
S(S^-(h \rhd h^*))
= [S^-(h_{(3)} \rhd h^*_{(3)}) \langle h^*_{(1)}, S^-(h_{(5)}) \rangle \langle h^*_{(2)}, h_{(4)} \rangle \langle h^*_{(4)}, h_{(5)} \rangle \langle h^*_{(5)}, h_{(1)} \rangle \langle h^*_{(3)}, S^-(h_{(2)}) \rangle]
= [h_{(3)} \rhd h^*_{(2)}] \langle h^*_{(1)}, S^-(h_{(5)}) \rangle \langle h^*_{(2)}, S^-(h_{(5)}) \rangle \langle h^*_{(5)}, h_{(1)} \rangle
= [h_{(2)} \rhd h^*_{(2)}] \langle h^*_{(1)}, \varepsilon_t(S^{-1}(h_{(3)})) \rangle \langle h^*_{(3)}, \varepsilon(S^{-1}(h_{(1)})) \rangle
= \varepsilon([h \rhd h^*]_{(1)}) \langle h \rhd h^* \rangle_{(2)} \varepsilon([h \rhd h^*]_{(3)}) = [h \rhd h^*].
\]

Similar arguments show that $S^{-1}(S[h \rhd h^*]) = [h \rhd h^*]$.

**Proposition 4.5.** Let $H$ be a weak Hopf algebra with bijective antipode, which is finitely generated and projective as a $k$-module. Then $D(H)$ is a weak Hopf algebra, with comultiplication, counit and antipode given by the formulas (74,75,76). As a weak Hopf algebra, $D(H)$ is isomorphic to $D'(H)^{\text{op}}$. 
**Proposition 4.6.** Let \( H \) be a weak Hopf algebra with bijective antipode, which is finitely generated and projective as a \( k \)-module. The functor

\[
F : \mathcal{HYD}^H \to D(H)\overline{\mathcal{M}}, \quad F(M) = M,
\]

with

\[
(h \otimes h^*)m = \langle h^*, m_{[1]} \rangle hm_{[0]},
\]

for all \( h \in H \), \( h^* \in H^* \) and \( m \in M \) is an isomorphism of monoidal categories.

**Proof.** We already know (see (37)) that \( F \) is an isomorphism of categories, so we only have to show that \( F \) preserves the product. Take \( M, N \in \mathcal{HYD}^H \). The right \( H \)-coaction on \( M \otimes_t N \) is given by the formula (use (48) and (52)):

\[
\rho(1_{(1)}m \otimes 1_{(2)}n) = m_{[0]} \otimes n_{[0]} \otimes n_{[1]}m_{[1]},
\]

hence the left \( D(H) \)-action on \( F(M \otimes_t N) \) is the following

\[
(78) \quad [h \otimes h^*](1_{(1)}m \otimes 1_{(2)}n) = \langle h^*, n_{[1]}m_{[1]} \rangle h_{(1)}m_{[0]} \otimes h_{(2)}n_{[0]}.
\]

We now compute

\[
F(N) \otimes_t F(M) = \{ [1 \otimes \varepsilon]X \mid X \in F(N) \otimes F(M) \}.
\]

Observe that

\[
[1 \otimes \varepsilon]_{(1)}n \otimes [1 \otimes \varepsilon]_{(2)}m = \langle \varepsilon_{(1)}, n_{[1]} \rangle 1_{(2)}n_{[0]} \otimes \langle \varepsilon_{(2)}, m_{[1]} \rangle 1_{(1)}m_{[0]},
\]

\[
= \langle \varepsilon, n_{[1]}m_{[1]} \rangle 1_{(2)}n_{[0]} \otimes 1_{(1)}m_{[0]}.
\]

We claim that the switch map \( \tau : M \otimes N \to N \otimes M \) induces an isomorphism \( \tau : F(M \otimes_t N) \to F(N) \otimes_t F(M) \) of \( k \)-modules. Indeed, take \( 1_{(1)}m \otimes 1_{(2)}n \in M \otimes_t N \). Since \( M \otimes_t N \) is a Yetter-Drinfeld module, we have that \( \varepsilon(n_{[1]}m_{[1]}m_{[0]} \otimes n_{[0]} = 1_{(1)}m \otimes 1_{(2)}n \), and

\[
\tau(1_{(1)}m \otimes 1_{(2)}n) = 1_{(2)}n \otimes 1_{(1)}m = 1_{(2')}1_{(2)}n \otimes 1_{(1')}1_{(1)}m
\]

\[
= \varepsilon(n_{[1]}m_{[1]}1_{(2)}n_{[0]} \otimes 1_{(1)}m_{[0]})
\]

\[
= [1 \otimes \varepsilon]_{(1)}n \otimes [1 \otimes \varepsilon]_{(2)}m \in F(N) \otimes_t F(M).
\]

Conversely,

\[
\tau([1 \otimes \varepsilon]_{(1)}n \otimes [1 \otimes \varepsilon]_{(2)}m) = \varepsilon(n_{[1]}m_{[1]}1_{(1)}m_{[0]} \otimes 1_{(2)}n_{[0]} \in F(M \otimes_t N).
\]

Let us now show that \( \tau \) is left \( D(H) \)-linear. To this end, we compute the left \( D(H) \)-action on \( F(N) \otimes_t F(M) \).

\[
[h \otimes h^*]_{(1)}m \otimes 1_{(2)}n = [h \otimes h^*](1_{(2)}n \otimes 1_{(1)}m)
\]

\[
= [h_{(2)} \otimes h^*_{(2)}](1_{(2)}n) \otimes [h_{(1)} \otimes h^*_{(1)}](1_{(1)}m)
\]

\[
= [h^*_{(1)}n_{[1]}S^{-1}(1_{(2)})h_{(2)}n_{[0]} \otimes h^*_{(2)}1_{(1)}m_{[1]}h_{(1)}m_{[0]}
\]

\[
\tau(h_{(2)} \otimes h^*_{(2)})(1_{(2)}n \otimes 1_{(1)}m)
\]

It also follows that \( F(H_t) \) is a unit object in \( D(H)\overline{\mathcal{M}} \). Since the unit object in a monoidal category is unique up to automorphism, we conclude that the target space of \( D(H)_t \) is isomorphic to \( H_t \). This can also be seen as follows: in [17], it is shown that \( D'(H)_t = \varepsilon \otimes H_t \cong H_t \). Since the target spaces of a weak Hopf algebra and its opposite coincide, it follows that \( D(H)_t \cong H_t \). \( \square \)
5. Duality

Let $H$ be a weak Hopf algebra with bijective antipode, and $\mathcal{H}Rep$ the category of left $H$-modules $M$ which are finitely generated projective as a $k$-module. Let $M \in \mathcal{H}Rep$, and let $\{n_i, m_i^*\} | i = 1, \cdots, n\}$ be a finite dual basis of $M$. From [17], we recall the following result. We refer to [12] for the definition of duality in a monoidal category.

**Proposition 5.1.** The category $\mathcal{H}Rep$ has left duality. The left dual of $M \in \mathcal{H}Rep$ is $M^* = \text{Hom}(M, k)$ with left $H$-action defined by

(79) $\langle h \cdot m^*, m \rangle = \langle m^*, S(h)m \rangle$,

for all $h \in H$, $m \in M$ and $m^* \in M^*$. The evaluation map $ev_M : M^* \otimes H \to H$, and the coevaluation map $\text{coev}_M : H \to M^* \otimes H$ are defined as follows:

$ev_M(1(1) \cdot m^* \otimes 1(2)m) = \langle m^*, 1(1)m \rangle 1(2)$;

$\text{coev}_M(z) = z \cdot (\sum_i n_i \otimes n_i^*)$.

Let $M$ be a finitely generated projective left $H$-comodule. Then $M^*$ is also a left $H$-comodule, with left $H$-coaction $\lambda : M^* \to H \otimes M^*$ given by

$\lambda(m^*) = \sum_i \langle m^*, n_i(0) \rangle S^{-1}(n_i[-1]) \otimes n_i^*$.

The definition of $\lambda$ can also be stated as follows: $\lambda(m^*) = m^*[-1] \otimes m^*[0]$ if and only if

(80) $\langle m^*[0], m \rangle S(m^*[-1]) = \langle m^*, m^*[0] \rangle m[-1]$,

for all $m \in M$.

**Proposition 5.2.** Let $M$ be a finitely generated projective left-left Yetter-Drinfel’d module over the weak Hopf algebra $H$. Then $M^*$ with $H$-action and $H$-coaction given by (79) and (80) is also a left-left Yetter-Drinfel’d module.

**Proof.** We have to show that

$\lambda(h \cdot m^*) = \sum_i \langle m^*, S(h)n_i(0) \rangle S^{-1}(n_i[-1]) \otimes n_i^*$

equals

$h(1)m^*[-1]S(h(3)) \otimes h(2)m^*[-1] = \sum_i \langle m^*, n_i(0)h(1) \rangle S^{-1}(n_i[-1])S(h(3)) \otimes (h(2) \cdot n_i^*)$.

It suffices to show that both terms coincide after we evaluate the second tensor factor at an arbitrary $m \in M$.

$\sum_i \langle m^*, n_i(0)h(1) \rangle S^{-1}(n_i[-1])S(h(3)) \otimes h(2)m$ = $\langle m^*, (S(h(2))m)[0]h(1) \rangle S^{-1}(S(h(2))m)[-1]S(h(3))$ (41) = $\langle m^*, S(h(3))m[0]h(1) \rangle S^{-1}(S(h(4))m[-1]S^2(h(2)))S(h(5))$ = $\langle m^*, S(h(2))m[0]h(1)S(h(3))S^{-1}(m[-1])h(4)S(h(5))$ = $\langle m^*, S(h(2))m[0]\varepsilon(h(1))S^{-1}(m[-1])\varepsilon(h(3))$.
\[(21) \quad \langle m^*, S(1(2)h_{(1)})m_{(0)} S(1(1)) S^{-1}(m_{[-1]}), \varepsilon_t(h_{(2)}) \rangle \]
\[= \langle m^*, S(h_{(1)})1(1)m_{(0)} 1(2) S^{-1}(m_{[-1]}), \varepsilon_t(h_{(2)}) \rangle \]
\[\Rightarrow \quad \langle m^*, S(1(1)h)m_{(0)} S^{-1}(m_{[-1]}), 1(2) \rangle \]
\[= \langle m^*, S(h) S(1(1))m_{(0)} S^{-1}(S(1(2))m_{[-1]}) \rangle \]
\[= \langle m^*, S(h) 1(2)m_{(0)} S^{-1}(1(1)m_{[-1]}) \rangle \]
\[\Rightarrow \quad \langle m^*, S(h)m_{(0)} S^{-1}(m_{[-1]}) \rangle \]
\[= \sum_i \langle m^*, S(h) n_i m_{(0)} S^{-1}(n_i^{*}_{[-1]}), n_i^* \rangle \]

\[\square\]

**Proposition 5.3.** The category of finitely generated projective left-left Yetter-Drinfeld modules has left duality.

**Proof.** In view of the previous results, it suffices to show that the evaluation map \(\text{ev}_M\) and the coevaluation map \(\text{coev}_M\) are left \(H\)-colinear, for every finitely generated projective left-left Yetter-Drinfeld module \(M\). Let us first show that \(\text{ev}_M\) is left \(H\)-colinear.

\[(H \otimes \text{ev}_M)(\lambda(1(1) \cdot m^* \otimes 1(2)m))\]
\[= \langle m_{[-1]}^* m_{[-1]} \otimes m_{(0)}^*, 1(1)m_{(0)} \rangle \otimes 1(2) \]
\[= \langle m^*, 1(1)m_{(0)} \rangle S^{-1}((1(1)m_{(0)} S^{-1}(m_{[-1]})) \otimes 1(2) \]
\[= \langle m^*, m_{(0)} \rangle 1(1)S^{-1}(m_{[-1]}) \otimes 1(2) \]
\[= \langle m^*, m_{(0)} \rangle 1(1)S^{-1}(m_{[-1]}) \otimes 1(2) \]
\[= \langle m^*, 1(1)m_{(0)} \rangle 1(1)S^{-1}(m_{[-1]}) \otimes 1(2) \]
\[= \langle m^*, 1(1)m_{(0)} \rangle 1(1)S^{-1}(m_{[-1]}) \otimes 1(2) \]
\[= \langle m^*, 1(1)m_{(0)} \rangle 1(1)S^{-1}(m_{[-1]}) \otimes 1(2) \]
\[= \lambda(\text{ev}_M(1(1) \cdot m^* \otimes 1(2)m)). \]

To prove that \(\text{coev}_M\) is left \(H\)-colinear, we have to show that, for all \(z \in H_t\),

\[\lambda(\text{coev}_M(z)) = \sum_i \lambda(1(1)z n_i \otimes 1(2) \cdot n_i^*) \]
\[= \sum_i \langle 1(1)z n_i [-1], (1(2) \cdot n_i^*) [-1] \rangle \otimes (1(1)z n_i)_{(0)} \otimes (1(2) \cdot n_i^*)_{(0)} \]

equals

\[(H \otimes \text{coev}_M)(\lambda(z)) = (H \otimes \text{coev}_M)(1(z \circ 1(2))) = \sum_i 1(z \circ 1(2)n_i \otimes 1(3) \cdot n_i^*). \]

It suffices to show that both terms coincide after we evaluate the third tensor factor at an arbitrary \(m \in M\). Indeed

\[\sum_i \langle 1(1)z n_i [-1], (1(2) \cdot n_i^*) [-1] \rangle \otimes (1(1)z n_i)_{(0)} \otimes (1(2) \cdot n_i^*)_{(0)} \]
\[= \sum_i \langle 1(1)z n_i [-1], (1(2) \cdot n_i^*, m_{(0)}) S^{-1}(m_{[-1]}) \otimes (1(1)z n_i)_{(0)} \]
\[= \sum_i \langle 1(1)z n_i [-1], n_i^*, S(1(2))m_{(0)} S^{-1}(m_{[-1]}) \otimes (1(1)z n_i)_{(0)} \]
\[= \langle (1(1)z S(1(2))m_{(0)} [-1], S^{-1}(m_{[-1]}) \otimes (1(1)z S(1(2))m_{(0)}). \]
Recall that an $R$-coring is a triple $(H,\Delta,\varepsilon)$, with $H$ an $R$-bimodule and $\Delta : H \to H \otimes_R H$ and $\varepsilon : H \to R$ $R$-bimodule maps satisfying the usual coassociativity and counit properties; we refer to [4] for a detailed discussion of corings.

Let $k$ be a commutative ring, and $R$ a $k$-algebra. An $R \otimes R^{op}$-ring is a pair $(H,i)$, with $H$ a $k$-algebra and $i : R \otimes R^{op} \to H$. Giving $i$ is equivalent to giving algebra maps $s_H : R \to H$ and $t_H : R \to H^{op}$ satisfying $s_H(a)t_H(b) = t_H(b)s_H(a)$, for all $a, b \in R$. We then have that $i(a \otimes b) = s_H(a)t_H(b)$. Restriction of scalars makes $H$ into a left $R \otimes R^{op}$-module, and an $R$-bimodule:

$$a \cdot h \cdot b = s_H(a)t_H(b)h.$$  

Consider

$$H \times_R H = \{ \sum_i h_i \otimes_R k_i \in H \otimes_R H \mid \sum_i h_i t_H(a) \otimes_R k_i = \sum_i h_i \otimes_R k_i s_H(a), \text{ for all } a \in R \}$$

It is easy to show that $H \times_R H$ is a $k$-subalgebra of $H \otimes_R H$.

Recall that an $R$-coring is a triple $(H,\Delta,\varepsilon)$, with $H$ an $R$-bimodule and $\Delta : H \to H \otimes_R H$ and $\varepsilon : H \to R$ $R$-bimodule maps satisfying the usual coassociativity and counit properties; we refer to [4] for a detailed discussion of corings.

**Definition 6.1.** [14] A left $R$-bialgebroid is a fivetuple $(H,s_H,t_H,\Delta,\varepsilon)$ satisfying the following conditions.

1. $(H,\Delta,\varepsilon)$ is an $R$-coring;
2. $(H,m \circ (s_H \otimes t_H) = i)$ is an $R \otimes R^{op}$-ring;
3. $\text{Im}(\Delta) \subset H \times_R H$;
4. $\Delta : H \to H \times_R H$ is an algebra map, $\varepsilon(1_H) = 1_R$ and

$$\varepsilon(gh) = \varepsilon(\varepsilon(gh)) = \varepsilon(\varepsilon(gh)),$$
for all $g, h \in H$.

Take two left $H$-modules $M$ and $N$; then $M$ and $N$ are $R$-bimodules, by restriction of scalars. $M \otimes_R N$ is a left $H$-module, with

$$h \cdot (m \otimes_R n) = h_{(1)} m \otimes_R h_{(2)} n.$$ 

Also $R$ is a left $H$-module, with

$$h \cdot r = \tilde{\epsilon}(hs_H(r)) = \tilde{\epsilon}(ht_H(r)).$$

$(H, M, \otimes_R, R)$ is a monoidal category, and the restriction of scalars functor $H \cdot M \to R \cdot M$ is strictly monoidal; this can be used to reformulate the definition of a bialgebroid (see [3, 20, 23]).

In [21, Sec. 4], left-left Yetter-Drinfeld modules over $H$ are introduced, and it is shown that $W_1(H \cdot M)$ is isomorphic to the category of Yetter-Drinfeld modules. According to [21], a left-left Yetter-Drinfeld $H$-module is a left comodule $M$ over the coring $H$, together with a left $H$-action on $M$ such that the underlying left $R$-actions coincide, and such that

$$(81) \quad h_{(1)} m_{[-1]} \otimes_R h_{(2)} \cdot m_{[0]} = (h_{(1)} \cdot m)_{[-1]} h_{(2)} \otimes_R (h_{(1)} \cdot m)_{[0]}$$

holds in $H \otimes_R M$, for all $h \in H$ and $m \in M$.

Let $H$ be a weak bialgebra, and consider the maps

$$s_H : H_t \xrightarrow{\subset} H;$$

$$t_H = \tau_{s[H_t]} : H_t \to H_s \subset H;$$

$$\hat{\Delta} = \text{can} \circ \Delta : H \to H \otimes_H \text{can} \rightarrow H \otimes_H H;$$

$$\hat{\epsilon} = \epsilon_t : H \to H_t.$$ 

Then $(H, s_H, t_H, \hat{\Delta}, \hat{\epsilon})$ is a left $H_t$-bialgebroid. The fact that $\text{Im}(\hat{\Delta}) \subset H \times_{H_t} H$ follows from the separability of $H_t$ as a $k$-algebra (cf. Proposition 1.3).

We have seen in Section 1.1 that, for any two left $H$-modules $M$ and $N$, we have an isomorphism $\varpi : M \otimes_{H_t} N \to M \otimes_1 N$. This entails that the monoidal categories $(\mu, \otimes_1, H_t)$ and $(H \cdot M, \otimes_1, H_t)$ are isomorphic, and a fortiori, their weak left centers are isomorphic categories. Consequently, the two corresponding categories of Yetter-Drinfeld modules are isomorphic. This can also be seen directly, comparing the definitions in Section 2 and (81).

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