TRANSLATION INVARIANT PURE STATE ON $B = \otimes_{z} M_{d}(\mathbb{C})$ AND ITS SPLİT PROPERTY

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Abstract
We prove that a real lattice symmetric reflection positive translation invariant pure state of $B = \otimes_{z} M_{d}(\mathbb{C})$ is a split state if its two points spatial correlation functions decay exponentially.

1. Introduction

A state $\omega$ on a $C^{*}$-algebra $B$ is called a factor if the center of the von-Neumann algebra $\pi_{\omega}(B)''$ is trivial, where $(H_{\omega}, \pi_{\omega}, \Omega)$ is the GNS space associated with $\omega$ on $B$ [BR1]. A state $\omega$ on $B$ is called pure if $\pi_{\omega}(B)' = B(H_{\omega})$, the algebra of all bounded operators on $H_{\omega}$. Here we fix our convention that Hilbert spaces that are considered here equipped always with inner product $<.,.>$ which is linear in the second variable and conjugate linear in the first variable. In this paper our primary objective is to study states on $C^{*}$-algebra that naturally arise in quantum spin chain models on a lattice $\mathbb{Z}$.

Let $B = \otimes_{z} M_{d}(\mathbb{C})$ be the uniformly hyper-finite $C^{*}$-algebra over the lattice $\mathbb{Z}$, where $M_{d}(\mathbb{C})$ denote the algebra of $d \times d$-matrices over the field of complex number $\mathbb{C}$. Here $B$ is the $C^{*}$-completion of the infinite tensor product of the algebra $M_{d}(\mathbb{C})$ of $d$ by $d$ complex matrices, each component of the tensor product element is indexed by an integer $j$. Let $Q$ be a matrix in $M_{d}(\mathbb{C})$. By $Q(j)$ we denote the element $\ldots \otimes 1 \otimes 1 \otimes Q \otimes 1 \otimes \ldots 1 \otimes \ldots$, where $Q$ appears in the $j$-th component. Given a subset $\Lambda$ of $\mathbb{Z}$, $B_{\Lambda}$ is defined as the $C^{*}$-sub-algebra of $B$ generated by all $Q(j)$ with $Q \in M_{d}(\mathbb{C})$, $j \in \Lambda$. We also set

$$B_{\text{loc}} = \bigcup_{\Lambda: |\Lambda| < \infty} B_{\Lambda}$$

where $|\Lambda|$ is the cardinality of $\Lambda$. Let $\omega$ be a state on $B$. The restriction of $\omega$ to $B_{\Lambda}$ is denoted by $\omega_{\Lambda}$. We also set $\omega_{R} = \omega_{[1, \infty)}$ and $\omega_{L} = \omega_{(-\infty, 0]}$. The translation $\theta_{k}$ is an automorphism of $B$ defined by $\theta_{k}(Q(j)) = Q(j+k)$. Thus $\theta_{1}, \theta_{-1}$ are unital $*$-endomorphisms on $B_{R}$ and $B_{L}$ respectively. We say $\omega$ is translation invariant if $\omega \circ \theta_{k} = \omega$ on $B$ ($\omega \circ \theta_{1} = \omega$ on $B$). In such a case $(B_{R}, \theta_{1}, \omega_{R})$ and $(B_{L}, \theta_{-1}, \omega_{L})$ are two unital $*$-endomorphisms with invariant states. In this paper we will consider only translation invariant states.

If $\omega$ is a translation invariant pure state then Theorem 3.4 in [Mo4] says that $\pi_{\omega_{R}}(B_{R})''$ is either a type-I or a type-III factor state. On the other hand if $\omega_{R}$ is a type-I factor state then a translation invariant state $\omega$ is pure by Theorem 2.8 in [Mo4] (for a different proof see [Ma3]). There are examples [AMa] where $\omega_{R}$ is
a type-III factor state for which \( \omega \) is pure. It is much easier [BJKW] to construct \( \omega \) with \( \omega_R \) as type-I factor state using Popescu’s dilation. A pure mathematical question that arise now: how to assert which type of factor states \( \omega_R(\omega_L) \) are, i.e. type-I or type-III factors, by studying additional symmetry of the state \( \omega \) or asymptotic behavior of the group of automorphisms \( (B, \theta^\alpha, \omega) \)? To that end we first recall [Ma2] a standard definition of a state to be split in the following. For a more general definition of split property we refer to [DL]. For the present problem, we follow definition of split property adopted in [Ma2].

**Definition 1.1.** Let \( \omega \) be a translation invariant state on \( B \). We say that \( \omega \) is *split* if the following condition is valid: Given any \( \epsilon > 0 \) there exists a \( k \geq 1 \) so that

\[
\sup_{||Q|| < 1} |\omega(Q) - \omega_L \otimes \omega_R(Q)| \leq \epsilon
\]

where the above supremum is taken over all local elements \( Q \in B_{(−\infty,−k]} \cup [k,∞) \) with the norm less than 1.

Here we recall few well known facts from [Pow, BR, AMa, Ma1, Ma2]. It is well known since late 60’s [Pow] that a translation invariant state \( \omega \) on \( B \) is a factor state if and only if

\[
\sup_{x \in B_{A_n}, ||x|| \leq 1} |\omega(xy) - \omega(x)\omega(y)| \to 0
\]

for all \( y \in B \) as \( n \to \infty \) where \( A_n = \{ k \in \mathbb{Z} : -n \leq k \leq n \} \). In particular it says for a translation invariant factor state \( \omega \) of \( B \), \( \omega_R \) is also a factor state. The uniform cluster condition (1) is valid if and only if the state \( \omega \) is quasi-equivalent to the product state \( \psi_L \otimes \psi_R \) of a state \( \psi_L \) of \( B_L \) and another state \( \psi_R \) of \( B_R \). Thus a Gibbs state [BR-vol-II] of a finite range interaction is split. If \( \omega \) is a pure translation invariant state, then \( \omega_R(\omega_L) \) is type-I if and only if \( \omega \) is also a split state. The canonical trace of \( B \) is a non-pure split state and unique ground state of XY model [AMa, Ma1, Ma2] is a non-split pure state. Our central aim in this paper is to find a criterion for a pure translation invariant state to be split. To that end we present a precise definition for exponential decay of two points spatial correlation functions.

**Definition 1.2.** Let \( \omega \) be a translation invariant state on one dimensional quantum spin chain algebra \( B \). We say that the two points spatial correlation functions of \( \omega \) decay exponentially if there exists a \( \delta > 0 \) so that

\[
e^{-\delta|k|} |\omega(Q_1 \theta_k(Q_2)) - \omega(Q_1)\omega(Q_2)| \to 0
\]

as \( |k| \to \infty \) for any local elements \( Q_1, Q_2 \in B \).

Taku Matsui, in his paper [Ma3], conjectured that exponential decaying property of two points correlation functions of a translation invariant pure state \( \omega \) will imply split property of the state \( \omega \) i.e. type-I factor state property of \( \omega_R \). We will prove this conjecture under some additional symmetries on \( \omega \) which is describe below and stated as a theorem (Theorem 1.3).

For any translation invariant state \( \omega \) on \( B \) we set translation invariant state \( \tilde{\omega} \) by reflecting around the point \( \frac{1}{2} \) of the lattice \( \mathbb{Z} \) by

\[
\tilde{\omega}(Q_{-l}^{(−1)} \otimes Q_{−l+1}^{(−i+1)} \otimes ... \otimes Q_{−1}^{(−1)} \otimes Q_0^{(0)} \otimes Q_1^{(1)} \otimes ... \otimes Q_n^{(n)})
\]

\[
= \omega(Q_{−n+1}^{i−n+1}) \otimes Q_1^{(0)} \otimes Q_0^{(1)} \otimes Q_{−1}^{(2)} \otimes ... Q_{l−i+1}^{(l)} \otimes Q_{−l}^{(i+1)}
\]
for all $n,l \geq 1$ and $Q_{-1},...Q_{-1},Q_0, Q_1,...,Q_n \in M_n(C)$ where $Q^{(k)}$ is the matrix $Q$ at lattice point $k$. We define $\omega$ on $B$ by extending linearly to any $Q \in B_{loc}$. Thus $\omega \to \tilde{\omega}$ is an affine one to one onto map on the set of translation invariant states on $B$. Thus the state $\tilde{\omega}$ is translation invariant, ergodic, factor state if and only if $\omega$ is translation invariant, ergodic, factor state respectively. We say $\omega$ is lattice reflection symmetric if $\omega = \tilde{\omega}$.

If $Q = Q_0^{(l)} \otimes Q_1^{(l+1)} \otimes ... \otimes Q_m^{(l+m)}$ we set $Q' = Q_0^{(l)} \otimes Q_1^{(l+1)} \otimes ... \otimes Q_m^{(l+m)}$ where $Q_0, Q_1,...,Q_m$ are arbitrary elements in $M_d$ and $Q_0^l, Q_1^l,...$ stands for transpose with respect to an orthonormal basis $(e_i)$ for $C^d$ (not complex conjugate) of $Q_0, Q_1,...$ respectively. We define $Q'$ by extending linearly for any $Q \in B_{loc}$. For a state $\omega$ on UHF$_d C^*$ algebra $\otimes \mathbb{Z}M_d$ we define a state $\hat{\omega}$ on $\otimes \mathbb{Z}M_d$ by the following prescription

$$\hat{\omega}(Q) = \omega(Q')$$

Thus the state $\hat{\omega}$ is translation invariant, ergodic, factor state if and only if $\omega$ is translation invariant, ergodic, factor state respectively. We say $\omega$ is real if $\omega = \hat{\omega}$.

A translation invariant state $\omega$ is said to be in detailed balance if $\omega$ is lattice reflection symmetric and real (for further details see section 3). The canonical trace on $B$ is both real and lattice symmetric. This notion of detailed balance state is introduced as a reminiscence of Onsager’s reciprocal relations explored in recent articles [AM,Mo1,Mo2] on non-commutative probability theory.

Unitary matrices $U_d(C)$ acts naturally on $B$ as group of automorphism defined by

$$\beta_v(Q) = (... \otimes v \otimes v \otimes ...)(... \otimes \delta \otimes \delta \otimes ...)(... \otimes v^* \otimes \delta \otimes \delta \otimes ...)(... \otimes v^* \otimes v^* \otimes v^* \otimes ...)$$

We set map $Q \to \overline{Q}$ for the complex conjugation with respect to a basis $(e_i)$ for $C^d$ defined by extending identity action on elements

$$...I_d \otimes I_d \otimes |e_{j_n} \rangle \rangle < e_{j_n} | \otimes ... |e_{j_2} \rangle \rangle < e_{j_2} | \otimes | e_{j_1} \rangle \rangle < e_{j_1} |$$

anti-linearly. Thus we have $Q^* = \overline{Q}^T$.

Following a well known notion [DLS], a state $\omega$ on $B$ is called reflection positive with a twist $g_0 \in U_d(C)$ if

$$\omega(\mathcal{F}_{g_0}(Q)Q) \geq 0$$

for all $Q \in B_R$ where $g_0^2 = I_d$ and $\mathcal{F}_{g_0}(Q) = \beta_{g_0}(Q)$.

Let $G$ be a compact group and $g \to v(g)$ be a $d$-dimensional unitary representation of $G$. By $\gamma_g$ we denote the product action of $G$ on the infinite tensor product $B$ induced by $v(g)$,

$$\gamma_g(Q) = (... \otimes v(g) \otimes v(g) \otimes v(g)...)(... \otimes \delta \otimes \delta \otimes \delta \otimes ...)(... \otimes \delta \otimes \delta \otimes \delta \otimes ...)(... \otimes \delta \otimes \delta \otimes \delta \otimes ...)(... \otimes v(g)^* \otimes v(g)^* \otimes v(g)^* \otimes v(g)^* \otimes ...)(... \otimes v(g)^* \otimes v(g)^* \otimes v(g)^* \otimes v(g)^* \otimes ...)$$

for any $Q \in B$, i.e. $\gamma_g = \beta_v(g)$. We say $\omega$ is $G$-invariant if $\omega(\gamma_g(Q)) = \omega(Q)$ for all $Q \in B_{loc}$.

Now we state our main theorem proved in this paper.

**Theorem 1.3.** Let $\omega$ be a pure lattice symmetric translation invariant real (with respect to a basis $(e_i)$ of $C^d$) state on $B$. If $\omega$ is also reflection positive with a twist $g_0$ and two points spatial correlation function of $\omega$ decays exponentially then $\omega$ is a split state i.e. $\pi_\omega(B_R)^{\otimes 2}$ is a type-I factor.
Before we describe main ideas behind the proof of Theorem 1.3, we briefly review
the context in which this problem finds its motivation with an aim of application and
some related results. We consider [BR-vol-II,Ru] quantum spin chain Hamiltonian
in one dimensional lattice of the following form

\[ H = \sum_{k \in \mathbb{Z}} \theta^k (h_0) \]

for \( h_0 = h_0 \in \mathcal{B}_{loc} \) where the formal sum gives an auto-morphism \( \alpha = (\alpha_t : t \in \mathbb{R}) \) via the thermodynamic limit of \( \alpha_t^k(x) = e^{itH_{\Lambda}x} e^{-it\Lambda H} \) for a net of finite
subsets of the lattice \( \Lambda \uparrow \mathbb{Z} \) whose surface energies are uniformly bounded, where \( H_{\Lambda} = \sum_{k \in \Lambda} \theta^k (h_0) \). Such a thermodynamic limiting automorphism \( \alpha \) is uniquely
determined by \( H \). In such a case, i.e. translation invariant Hamiltonian \( H \) is
having finite range interaction, KMS state at a given inverse temperature exists
and is always unique [Ara1],[Ara2], [Ki] and thus inherits translation and other
symmetry of the Hamiltonian. Thus low temperature limit points of unique KMS
states give ground states for the Hamiltonian \( H \) inheriting translation and other
symmetry of Hamiltonian. It is a well known fact that ground states of a translation
invariant Hamiltonian form a face in the convex set of states of \( \mathcal{B} \) and its extreme
points are pure. In general the set of ground states need not be a singleton set and
there are other states those are not translation invariant but still a ground state for
a translation invariant Hamiltonian. Ising model admits non translation invariant
ground states known as Neél state [BR vol-II]. However ground states that appear as
low temperature limit of KMS states of a translation invariant Hamiltonian inherit
translation and other symmetry (that we have described above) of the Hamiltonian.
In particular if ground state for a translation invariant Hamiltonian model of type
(9) is unique, then the ground state is a translation invariant pure state. Along the
same line unique ground state is reflection symmetry and real if \( H \) is so. Similarly
reflection positivity property of the unique ground state will follow reflection positive
property of unique \( \beta \)-KMS state. A prime known example of unique \( \beta \)-KMS state
that satisfies reflection positive property with a twist \( g_0 \) of a Hamiltonian of type
(9) is the Heisenberg anti-ferromagnet iso-spin model \( H_{XXX} \) where

\[ h_0 = J(\sigma^x \otimes \sigma^{x+1} + \sigma^y \otimes \sigma^{y+1} + \sigma^z \otimes \sigma^{z+1}) \]

where \( \sigma^x, \sigma^y, \sigma^z \) are Pauli spin matrices located at lattice site \( x \in \mathbb{Z} \) and \( J > 0 \)
constant. Another prime example of \( \beta \)-KMS state that admits reflection positive
property at inverse temperature \( \beta \) includes anti-ferro-magnet XY model namely
\( H_{XY} \) with

\[ h_0 = J(\sigma_x \otimes \sigma_{x+1} + \sigma_y \otimes \sigma_{y+1}) \]

with \( J > 0 \) well studied. In such a case any limiting state at low temperature, inherit
same symmetry namely translation, reflection, real, reflection positive property.

In case \( H_{XY} \) it is well known that unique ground state once restricted to \( \mathcal{B}_R \)
gives a type-III factor state \( \omega_R^{XY} \). Thus our result says that its two points spatial
correlation function does not decay exponentially i.e. in the language of physical
literature \( \omega \) is strongly correlated.

On the other hand no clear picture has emerged yet so far about ground states of
anti-ferromagnet Heisenburg \( H_{XXX} \) model which serves a more realistic model for
quasi-one dimensional experimentally realized magnetic materials. One standing
conjecture on \( H_{XXX} \) model [AL,Ma3] says that it has unique ground state and admits
a mass gap with two points spatial correlation function decaying exponentially
for odd integer \( d \) (integer spin \( s \), where \( d = 2s + 1 \)). Whereas \( H_{XXX} \) has a unique
ground state but does not admits a mass gap with two points correlation function not decaying exponentially for even $d \left( \frac{d}{2} \text{ odd integer spin } s \right)$, where $d = 2s + 1$).

Hamiltonian $H_{XX}$ admits $SU(2)$ gauge symmetry and $H_{XY}$ admits $S^1 \subset SU(2)$ gauge symmetry. A pure mathematical question that arise here: does this additional symmetry of $H$ helps to understand its low temperature limiting states? We defer our investigation to another paper [Mo6] where we will study $\omega$ satisfying (8) for an irreducible representation of a compact Lie group $G$ and deal with $H$ with additional symmetry such as $G = S^1$ or $G = SU(2)$. Our analysis gives a surprising result for $\frac{d}{2}$ odd integer spin anti-ferromagnets $H_{XX}$: It says that the set of ground states is not a singleton and it is not even a simplex. Furthermore any low temperature limiting state is not pure. However it does not rule out possibility of unique limit point while taking low temperature limit and thus does not rule out possibility of a strongly correlated two points spatial correlation for its low temperature limiting state.

Now we describe basic ideas involved in the proof of Theorem 1.3.

First we recall that the Cuntz algebra $\mathcal{O}_d (d \in \{2, 3, \ldots \})$ [Cun] is the universal unital $\mathbb{C}^*$-algebra generated by the elements $\{s_1, s_2, \ldots, s_d\}$ subject to the following Cuntz relations:

$$ s_i^* s_j = \delta_{ij} I, \quad \sum_{1 \leq i \leq d} s_i s_i^* = I \quad (10) $$

There is a canonical action of the group $U(d)$ of unitary $d \times d$ matrices on $\mathcal{O}_d$ given by

$$ \beta_g(s_i) = \sum_{1 \leq j \leq d} g_{i,j}^* s_j $$

for $g = ((g_{i,j}^j)) \in U(d)$. In particular the gauge action is defined by

$$ \beta_z(s_i) = z s_i, \quad z \in \mathbb{T} = S^1 = \{z \in \mathbb{C} : |z| = 1\}. $$

If UHF$_d$ is the fixed point sub-algebra under the gauge action, then UHF$_d$ is the closure of the linear span of all Wick ordered monomials of the form $s_1 \ldots s_k s_{k+1}^* \ldots s_1^*$ which is also isomorphic to the UHF$_d$ algebra

$$ \mathcal{B}_R = \odot^\infty M_d(\mathbb{C}) $$

so that the isomorphism carries the Wick ordered monomial above into the matrix element

$$ e_{j_1}^1(1) \otimes e_{j_2}^2(2) \otimes \ldots \otimes e_{j_k}^k(k) \otimes 1 \otimes 1 \ldots. $$

and the restriction of $\beta_g$ to UHF$_d$ is then carried into action

$$ Ad(g) \otimes Ad(g) \otimes Ad(g) \otimes \ldots. $$

We also define the canonical endomorphism $\lambda$ on $\mathcal{O}_d$ by

$$ \lambda(x) = \sum_{1 \leq i \leq d} s_i x s_i^* \quad (11) $$

and the isomorphism carries $\lambda$ restricted to UHF$_d$ into the one-sided shift

$$ y_1 \otimes y_2 \otimes \ldots \rightarrow 1 \otimes y_1 \otimes y_2 \ldots. $$

on $\odot^\infty_1 M_d$. Note that $\lambda \beta_g = \beta_g \lambda$ on UHF$_d$. 
Let $d \in \{2, 3, \ldots, \}$ and $\mathbb{Z}_d$ be a set of $d$ elements. $\mathcal{I}$ be the set of finite sequences $I = (i_1, i_2, \ldots, i_m)$ where $i_k \in \mathbb{Z}_d$ and $m \geq 1$. We also include empty set $0 \in \mathcal{I}$ and set $\mathcal{S}_0 = 1 = s^\omega_0, s^\omega_1, \ldots, s^\omega_m \in \mathcal{O}_d$ and $s^\omega_i = s^\omega_{i_1} \ldots s^\omega_{i_m} \in \mathcal{O}_d$.

We fix a translation invariant state $\omega$ on $\mathcal{B}$ and denote by $\omega_R$ to be the restriction of $\omega$ to $\mathcal{B}_R$. Using weak* compactness of the convex set of states on a C*-algebra, standard averaging method ensures that the set

$$K_{\omega} = \{ \psi \in \mathcal{S}(\mathcal{O}_d) : \psi = \psi, \psi_1 \mathcal{UHF}_d = \omega_R \}$$

is a non-empty compact subset of $\mathcal{S}(\mathcal{O}_d)$, where $\mathcal{S}(\mathcal{O}_d)$ is the weak* compact convex set of states on $\mathcal{O}_d$. Further extremal elements in $K_{\omega}$ is a factor state if and only if $\omega_R$ is a factor state and any two such extremal elements $\psi, \psi'$ are related by $\psi' = \psi_\beta_z$ for some $z \in S^1 = \{ z \in C : |z| = 1 \}$ by Lemma 7.4. in [BJKW] where $\beta_z(s_i) = z s_i$ is the automorphism on $\mathcal{O}_d$ determined uniquely by universal property of Cuntz algebra.

Irrespective of factor property of $\omega$, we may choose an element $\psi$ in $K_{\omega}$ and consider the GNS space $(\mathcal{H}, \pi, \Omega)$ associated with state $\psi$ on $\mathcal{O}_d$. We set $P \in \pi(\mathcal{O}_d)^{''}$ to be the support projection of $\pi$ i.e. $P = [\pi(\mathcal{O}_d)]^{''}$. Invariant property of the state $\psi = \psi_\lambda$ will ensure that $P \Lambda(I - P)P = 0$ where

$$\Lambda(X) = \sum_i S_i X S_i^*$$

is the canonical endomorphism on $\pi(\mathcal{O}_d)^{''}$ with $S_i = \pi(s_i)$ and thus verify that

$$(12) \quad S_i^* P = P S_i^* P, \quad 1 \leq i \leq d$$

We define a family of contractions $\{ v_i : 1 \leq i \leq d \}$ in $\mathcal{M}$ by $v_i = P S_i P, 1 \leq i \leq d$ where we set von-Neumann algebra $\mathcal{M} = P \mathcal{O}_d^{''} P$. Thus we get $\mathcal{M} = \{ v_i : 1 \leq i \leq d \}$ and a unital completely positive map $\tau(x) = P \Lambda(P x P)P = \sum_i v_i x v_i^*$ for all $x \in \mathcal{M}$. In Proposition 2.4 in [Mo5] we proved a crucial that the support projection of $\psi$ in $\pi(\mathcal{O}_d)^{''}$ being equal to $P$, by our construction we have $\mathcal{B}^\omega(\mathcal{K}) = \mathcal{M}'$ where set

$$(13) \quad \mathcal{B}^\omega(\mathcal{K}) = \{ x \in \mathcal{B}(\mathcal{K}) : \sum_i v_i x v_i^* = x \}$$

Conversely let $\mathcal{M}$ be a von-Neumann algebra acting on a Hilbert space $\mathcal{K}$. A family of contractions $\{ v_i : 1 \leq i \leq d \}$ in $\mathcal{M}$ is called Popescu’s elements if $\sum_i v_i v_i^* = 1$. Given a Popescu’s elements $P = \{ \mathcal{K}, \mathcal{M}, v_i, 1 \leq i \leq d, \sum_i v_i v_i^* = 1 \}$, the map $s_i s_j^* \rightarrow v_i v_j^*$

is unital completely positive from $\mathcal{O}_d$ to $\mathcal{M}$ and thus Stinespring minimal dilation gives a representation $\pi : \mathcal{O}_d \rightarrow \mathcal{B}(\mathcal{H})$, a Hilbert space $\mathcal{H}$ with a projection $P$ with range equal to $\mathcal{K}$ such that $P \pi(s_i)^* P = \pi(s_i)^* P = v_i^*$ and $\{ \pi(s_i) \mathcal{K} : |i| < \infty \}$ is total in $\mathcal{H}$. For a faithful normal state $\phi$ on $\mathcal{M}$ we define state $\psi$ on $\mathcal{O}_d$ by

$$\psi(s_i s_j^*) = \phi(v_i v_j^*)$$

The crucial point that we arrive at Proposition 2.4 in [Mo5] that $P$ is the support projection for $\pi(\mathcal{O}_d)^{''}$ if and only if

$$(15) \quad \{ x \in \mathcal{B}(\mathcal{K}) : \sum_k v_k x v_k^* = x \} = \mathcal{M}'$$

We verify also with $v_i^* = P S_i^* P$ that

$$(16) \quad \omega_R(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_n} > e_{j_1} \otimes \cdots \otimes e_{j_m}) = \phi(v_i v_j^*)$$
where $v_I = v_{i_1}...v_{i_2}v_{i_1}$ and $v_J^* = v_{j_1}^*...v_{j_n}^*$.

The state $\phi(x) = <\Omega, x\Omega>$ on $M$ being faithful and invariant of $\tau: M \to M'$ we find a unique unit completely positive map $\tilde{\tau}: M' \to M'$ satisfying the duality relation

$$<y\Omega, \tau(x)\Omega> = <\tilde{\tau}(y)\Omega, x\Omega>$$

for all $x \in M$ and $y \in M'$, where $M'$ is the commutant of $M$ in $B(H)$. For a proof we refer to section 8 in the monograph [OP] or [Mo1].

Since $\phi$ is a faithful state, $\Omega \in K$ is a cyclic and separating vector for $M$ and the closure of the close-able operator $S_0: x\Omega \to x^*\Omega$, $S$ possesses a polar decomposition $S = J\Delta^{1/2}$, where $J$ is an anti-unitary and $\Delta$ is a non-negative self-adjoint operator on $K$. Tomita's [BR] theorem says that $\Delta'^tM\Delta^{-it} = M$, $t \in \mathbb{R}$ and $J'FM' = M'$, where $M'$ is the commutant of $M$. We define the modular automorphism group $\sigma = (\sigma_t, t \in \mathbb{T})$ on $M$ by

$$\sigma_t(x) = \Delta'^tx\Delta^{-it}$$

which satisfies the modular relation

$$\phi(x\sigma_t^y(y)) = \phi(\sigma_t^x(y)x)$$

for any two analytic elements $x, y$ for the group of automorphisms $(\sigma_t)$. A more useful form for modular relation here

$$\phi(\sigma_{-\frac{1}{2}}(x^*)\sigma_{-\frac{1}{2}}(y^*)) = \phi(y^*x)$$

which shows that $Jx\Omega = \sigma_{-\frac{1}{2}}(x^*)\Omega$. $J$ and $\sigma = (\sigma_t, t \in \mathbb{R})$ are called Tomita's conjugation operator and modular automorphisms associated with $\phi$. Since $\tau(x) = v_kxv_k^*$ is an inner map i.e. each $v_k \in M$, we have an explicit formula for $\tilde{\tau}$ as follows.

We set $\tilde{v}_k = J\sigma_{-\frac{1}{2}}(v_k^*)J \in M'$. That $\tilde{v}_k$ is indeed well defined as an element in $M'$ given in section 8 in [BJKW]. By the modular relation (18) we have

$$\sum_k \tilde{v}_k\tilde{v}_k^* = 1, \quad \tilde{\tau}(y) = \sum_k \tilde{v}_kgy\tilde{v}_k^*$$

and

$$\phi(v_Iv_J^*) = \phi(\tilde{v}_I\tilde{v}_J^*)$$

where $I = (i_1, i_2, i_1)$ if $I = (i_1, i_2, ..., i_n)$. Moreover $\tilde{v}_I^*\Omega = J\sigma_{-\frac{1}{2}}(v_I^*)J\Omega = J\Delta^t\tau v_I\Omega = v_I\tilde{\tau}\Omega$. We also set $M$ to be the von-Neumann algebra generated by $\{|\tilde{v}_k : 1 \leq k \leq d|\}$. Thus $M \subseteq M'$. The major problem that we will address in the text when do we have the following equality:

$$\{x \in B(K) : \sum_k \tilde{v}_kx\tilde{v}_k^* = x\} = M$$

Equality in (21) will ensure that $P: H \to K$ is also the support projection of $\bar{\pi}(O_d)''$ where $\pi$ is the Popescu's prescription of Stinespring representation $\pi: O_d \to B(H)$ associated with the completely positive map $s_1s_2^* \to \tilde{v}_I\tilde{v}_J^*$, $|I|, |J| < \infty$ so that $P\pi(s_I^*)P = \tilde{\pi}(s_I^*)P = \tilde{v}_I^*$. For details of the proofs we refer to Proposition 2.4 in [Mo5].

Thus so far we have taken an arbitrary element $\psi \in K_\omega$ and worked with its support projection to arrive at a representation of $\omega$ given in (16) by Popescu's elements $P = \{(K, v_i \in M, 1 \leq i \leq d : \sum_k v_i^*v_i = I)\}$.
normal state $\phi$ is uniquely determined modulo a unitary conjugation. In other words we find a one-one correspondence between

$$\omega \leftrightarrow \omega_R \leftrightarrow K^*_N \leftrightarrow \mathcal{P}_e$$

modulo unitary conjugation where $K^*_N$ denotes the set of extreme points in $K_\omega$ and $\mathcal{P}_e$ the set of Popescu’s elements associated with extreme points of $K_\omega$ on support projection of the state as described above. Further in such a case $\mathcal{M} = \{e_k : 1 \leq k \leq d\}^\tau$ is a factor and $(\mathcal{M}, \tau, \phi)$ is an ergodic quantum dynamical system \cite{La,Ev}. A unitl completely positive map $\tau$ on a von-Neumann algebra $\mathcal{M}$ with an invariant normal state $\phi$ \cite{BJKW,Mo1} is called ergodic if

$$\frac{1}{N} \sum_{0 \leq k \leq N-1} \tau^k(x) \to \phi(x)I$$

as $N \to \infty$ in weak$^*$ topology for all $x \in \mathcal{M}$. Thus any symmetry of $\omega$ will act on Popescu elements $\mathcal{P}_e$ via this correspondence. This is the precise idea that is followed while investigating $\omega$ with additional symmetries.

To that end for the time being we fix a translation invariant factor state $\omega$ on $\mathcal{B}$ and an extreme point $\psi \in K_\omega$. We consider the GNS space $(\mathcal{H}, \pi, \Omega)$ associated with the state $\psi$ on $\mathcal{O}_d$ and associated Popescu’s elements $\mathcal{P} = (K, \mathcal{M}, e_k, 1 \leq k \leq d, \Omega)$ arises on support projection $P = [\pi(\mathcal{O}_d)\Omega]$. Now consider the dual Popescu’s elements $\tilde{\mathcal{P}} = (K, \bar{\mathcal{M}}, \bar{e}_k; 1 \leq k \leq d, \Omega)$ and the completely positive map from $\tilde{\mathcal{O}}_d$ to $\mathcal{B}(\mathcal{K})$ defined by

$$\tilde{s}_J \tilde{s}_J^* \to \bar{v}_J \bar{v}_J^*, |I|, |J| < \infty$$

Let $\pi : \tilde{\mathcal{O}}_d \to \mathcal{B}(\mathcal{H})$ be the minimal Stinespring representation so that $P \pi(\tilde{s}_J \tilde{s}_J^*)P = \bar{v}_J \bar{v}_J^*$ for all $|I|, |J| < \infty$. In particular we have

$$P \pi(\tilde{s}_J^*)P = \pi(\tilde{s}_J^*)P = \bar{v}_J^*$$

We also consider the state $\tilde{\psi}$ on $\tilde{\mathcal{O}}_d$ given by

$$\tilde{\psi}(\tilde{s}_J \tilde{s}_J^*) = \phi(\bar{v}_J \bar{v}_J^*)$$

By modular relation (18) and (20) we check that $\tilde{\psi}|\text{UHF}_d = \omega|\mathcal{B}_L$. Thus relations are perfectly symmetric while moving from $\psi$ to $\tilde{\psi}$ except the fact that though $P$ is the support projection of $\psi$ in $\pi(\mathcal{O}_d)^\Omega$ same is not guaranteed by our construction i.e. $P$ need not be equal to $[\pi(\tilde{\mathcal{O}}_d)\Omega]$. A non-trivial statement: A translation invariant factor state of $B$ is pure if and only if $P$ is equal to $[\pi(\tilde{\mathcal{O}}_d)\Omega]$ i.e. if and only if (21) holds. This property helps to study lattice symmetric pure state with a twist and relate Popescu elements and dual Popescu elements are inter-twined by an anti-unitary operator modulo a twist.

The paper is organized as follows: In section 2, we will recall canonical amalgamated representation $\pi$ of $\tilde{\mathcal{O}}_d \otimes \mathcal{O}_d$ associated with an state $\psi \in K_\omega$ which extend GNS representation of $\mathcal{B} \equiv \text{UHF}_d \otimes \text{UHF}_d$ associated with state $\omega$. In section 3 we investigate the amalgamated representation $\pi$ with additional symmetry, in particular when $\omega$ is pure. In section 4 we explore the representation $\pi$ to prove Theorem 1.3.

2. Amalgamated representation of $\tilde{\mathcal{O}}_d \otimes \mathcal{O}_d$:

We recall without proof in the following three propositions results proved in \cite{BJKW,Section 6 and Section 7} and \cite{Mo5} for our purpose.
Proposition 2.1. Let \( \psi \) be a \( \lambda \) invariant factor state on \( \mathcal{O}_d \) and \( (\mathcal{H}, \pi, \Omega) \) be its GNS representation. Then the following holds:

(a) The closed subgroup \( H = \{ z \in S^1 : \psi z = \psi \} \) is equal to

\[ \{ z \in S^1 : \beta_z \text{ extends to an automorphism of } \pi(\mathcal{O}_d)'' \} \]

(b) Let \( \mathcal{O}_d^H \) be the fixed point sub-algebra in \( \mathcal{O}_d \) under the gauge group \( \{ \beta_z : z \in H \} \). Then \( \pi(\mathcal{O}_d^H)'' = \pi(\text{UHF}_d)'' \).

(c) If \( H \) is a finite cyclic group of \( k \) many elements and \( \pi(\text{UHF}_d)'' \) is a factor, then \( \pi(\mathcal{O}_d)'' \cap \pi(\text{UHF}_d)' \equiv C^m \) where \( 1 \leq m \leq k \).

Proof. It is a re-statement of Proposition 2.4 in [Mo5].

Let \( \omega' \) be an \( \lambda \)-invariant state on the UHF sub-algebra of \( \mathcal{O}_d \). Following [BJKW, section 7] and \( \omega \) be the inductive limit state \( \omega \) of \( B \equiv \text{UHF}_d \otimes \text{UHF}_d \). We consider the set

\[ K_{\omega} = \{ \psi : \psi \text{ is a state on } \mathcal{O}_d \text{ such that } \psi \lambda = \psi \text{ and } \psi |_{\text{UHF}_d} = \omega' \} \]

By taking invariant mean on an extension of \( \omega' \) to \( \mathcal{O}_d \), we verify that \( K_{\omega} \) is non empty and \( K_{\omega} \) is clearly convex and compact in the weak topology. In case \( \omega' \) is an ergodic state (extremal state) \( K_{\omega} \) is a face in the \( \lambda \) invariant states. Before we proceed to the next section here we recall Lemma 7.4 of [BJKW] in the following proposition.

Proposition 2.2. Let \( \omega' \) be ergodic. Then \( \psi \in K_{\omega} \) is an extremal point in \( K_{\omega} \) if and only if \( \psi \) is a factor state and moreover any other extremal point in \( K_{\omega} \) have the form \( \psi z \beta_z \) for some \( z \in S^1 \).

Now we briefly recall the amalgamated representation \( \pi \) of \( \tilde{\mathcal{O}}_d \otimes \mathcal{O}_d \) developed in [BJKW,Mo5] used to give a powerful criterion for a translation invariant factor state to be pure.

To that end let \( \mathcal{M} \) be a von-Neumann algebra acting on a Hilbert space \( \mathcal{K} \) and \( \{ v_k, 1 \leq k \leq d \} \) be a family of bounded operators on \( \mathcal{K} \) so that \( \mathcal{M} = \{ v_k, v_k^* \} \) and \( \sum_k v_k v_k^* = 1 \). Furthermore let \( \Omega \) be a cyclic and separating vector for \( \mathcal{M} \) so that the normal state \( \phi(x) = \langle \Omega, x \Omega \rangle \) on \( \mathcal{M} \) is invariant for the Markov map \( \tau \) on \( \mathcal{M} \) defined by

\[ \tau(x) = \sum_k v_k x v_k^* \]

for \( x \in \mathcal{M} \). We assume furthermore that \( \mathcal{M}' = B_r(\mathcal{H}) \).

Let \( \omega \) be the translation invariant state on \( \text{UHF}_d = \otimes_z M_d \) defined by

\[ \omega(e^{i_1} (l) \otimes e^{i_2} (l + 1) \otimes ... \otimes e^{i_n} (l + n - 1)) = \phi(v_I v_J^*) \]

where \( e^i(l) \) is the elementary matrix at lattice sight \( l \in \mathbb{Z} \).

We set \( \tilde{v}_k = \mathcal{J} \sigma_k^{-1} v_k \mathcal{J} \in \mathcal{M}' \) (see [BJKW] for details) where \( \mathcal{J} \) and \( \sigma = (\sigma_t, t \in \mathbb{R}) \) are Tomita’s conjugation operator and modular automorphisms associated with \( \phi \). By modular relation (18) we verify that

\[ \sum_k \tilde{v}_k \tilde{v}_k^* = 1 \]
and
\[
\phi(v_f v^*_j) = \phi(\tilde{v}_f \tilde{v}^*_j)
\]
where \( \tilde{I} = (i_1, \ldots, i_d) \) if \( I = (i_1, i_2, \ldots, i_n) \). Moreover \( \tilde{v}_j^* \Omega = J \sigma^*(v_f) J \Omega = J \Delta^* v_f \Omega = v_f^* \Omega \). We also set \( \tilde{M} \) to be the von-Neumann algebra generated by \( \{ \tilde{v}_k : 1 \leq k \leq d \} \). Thus \( \tilde{M} \subseteq M' \).

Let \( (H_t, P, S_k, 1 \leq k \leq d) \) and \( (\tilde{H}_t, P, \tilde{S}_k, 1 \leq k \leq d) \) be the Popescu dilation described as in Theorem 2.1 in [Mo5] associated with \( (K, v_k, 1 \leq k \leq d) \) and \( K, \tilde{v}_k, 1 \leq k \leq d \) respectively. Following [BJKW] we consider the amalgamated tensor product \( H \otimes_K H \) of \( H \) with \( \tilde{H} \) over the joint subspace \( K \). It is the completion of the quotient of the set
\[
C \tilde{I} \otimes CI \otimes K,
\]
where \( \tilde{I}, I \) both consist of all finite sequences with elements in \( \{1, 2, \ldots, d\} \), by the equivalence relation defined by a semi-inner product defined on the set by requiring
\[
< \tilde{I} \otimes I \otimes f, \tilde{I} \otimes I \otimes g >= < f, \tilde{v}_j v^*_j g >,
\]
\[
< \tilde{I} \otimes I \otimes f, \tilde{I} \otimes I \otimes g > = < \tilde{v}_j f, v^*_j g >
\]
and all inner product that are not of these form are zero. We also define two commuting representations \( (S_i) \) and \( (\tilde{S}_i) \) of \( \tilde{O}_d \) on \( H \otimes_{\tilde{K}} H \) by the following prescription:
\[
S_I \lambda(J \otimes J \otimes f) = \lambda(I \otimes I \otimes f),
\]
\[
\tilde{S}_I \lambda(J \otimes J \otimes f) = \lambda(I \otimes I \otimes f),
\]
where \( \lambda \) is the quotient map from the index set to the Hilbert space. Note that the subspace generated by \( \lambda(\emptyset \otimes I \otimes K) \) can be identified with \( H \) and earlier \( S_I \) can be identified with the restriction of \( S_I \) defined here. Same is valid for \( \tilde{S}_I \). The subspace \( K \) is identified here with \( \lambda(\emptyset \otimes \emptyset \otimes K) \). Thus \( K \) is a cyclic subspace for the representation
\[
\tilde{S}_j \otimes s_i \to \tilde{S}_j S_i
\]
of \( \tilde{O}_d \otimes O_d \) in the amalgamated Hilbert space. Let \( P \) be the projection on \( K \). Then we have
\[
S_i^* P = P S_i^* = v_i^*
\]
\[
\tilde{S}_i^* P = P \tilde{S}_i^* = \tilde{v}_i^*
\]
for all \( 1 \leq i \leq d \).

We sum up result required in the following proposition.

**Proposition 2.3.** The following hold:
(a) For any \( 1 \leq i, j \leq d \) and \( |I|, |J| < \infty \) and \( |\tilde{I}|, |\tilde{J}| < \infty \)
\[
< \Omega, \tilde{S}_j \tilde{S}^*_j S_i S_i^* S_j^* \Omega >= < \Omega, \tilde{S}_i \tilde{S}^*_i \tilde{S}^*_j \tilde{S}^*_j S_i S_i^* \Omega >;
\]
(b) The vector state \( \psi_\Omega \) on
\[
UHF_d \otimes UHF_d \equiv \otimes_{\infty \to M_d} M_d \otimes \otimes M_d \equiv \otimes M_d
\]
is equal to \( \omega \);
(c) \( \pi(\tilde{O}_d \otimes O_d)^{\prime\prime} = B(\tilde{H} \otimes_{\tilde{K}} H) \) if and only if \( \omega \) is a factor state;
Furthermore for a factor state \( \omega \) the following hold:
(d) \( \omega \) is pure if and only if \( E = \tilde{F} \) and \( \tilde{E} = E \) where \( E = [\pi(\tilde{O}_d)^{\prime\prime} \Omega], \tilde{E} = [\pi(\tilde{O}_d)^{\prime\prime} \Omega], \)
\( F = [\pi(O_d)^{\prime\prime} \Omega] \) and \( \tilde{F} = [\pi(O_d)^{\prime\prime} \Omega] \);
(e) If \( H = \{ z \in S^1 : z^n = 1 \} \) is a finite subgroup of \( S^1 \) then \( \omega \) is pure.
TRANSLATION INVARIANT PURE STATE ON $B = \otimes_{\mathcal{M}_d(C)}$ AND ITS SPLIT PROPERTY

PROOF. For (a)-(c) we refer to Proposition 3.1 and Proposition 3.2 in [Mo5] and for (d) we refer to Theorem 3.6 in [Mo5].

We will prove first if part of (e). For $H = \{1\}$, a proof follows by (a) of Proposition 2.1 and (c) of the present proposition since $\pi(\tilde{O}_d)$ is also a factor by Proposition 3.2 in [Mo5]. For $H = \{z \in S^1 : z^n = 1\}$, we consider the amalgamated representation of $\pi_H : O_d^H \otimes O_d^H \to B(\tilde{H}_0 \otimes \mathcal{K}_0, \mathcal{H}_0)$ where $\mathcal{K}_0 = [\mathcal{M}_0^\prime]$ with Popescu elements $\{v_I \in \mathcal{M}, \tilde{v}_I \in \tilde{\mathcal{M}}, |I| = n\}$. We have $O_d^H \equiv O_d^\mathcal{K}$ and we claim if for an $x \in B(\mathcal{K}_0)$, $\tau_H(x) = \sum_{|I| = n} v_I x v_I^*$ then $x \in P_0 \mathcal{M}_0 P_0$, where $P_0$ is the projection into $\mathcal{K}_0$. We have for such an element $x \in B(\mathcal{K}_0)$, $\tau_H(P_0 x P_0) = \tau^x(P_0 x P_0) = P_0 x P_0$ and thus $P_0 x P_0 = x P_0 \in M'$ by (15) and so $x P_0 \in \mathcal{M}_0$ by Proposition 2.1 since $\beta_z(P_0) = P_0$ for all $z \in H$. Thus $x \in P_0 \mathcal{M}_0 P_0$. So $P_0$ is the support projection of $\psi_H$ in $\pi_H(O_d)^\mathcal{K}$. Since modular group of $\mathcal{M}$ preserves $\mathcal{M}_0$ [Ta], we get $(P_0 \tilde{v}_I P_0) = P_0 \tilde{v}_I P_0$. Thus we get equality $\omega_H(\hat{O}_d^\mathcal{K} \otimes \hat{O}_d^\mathcal{K}) = \omega|\tilde{\mathcal{M}} \otimes \mathcal{M}_d$ between two amalgamated states. To show $\omega$ is pure, it is enough now to prove that $G_H = \{ z \in S^1 : \psi_H = \psi_H \circ \beta_z^H \}$ is the trivial sub-group where $\psi_H$ is an extremal element in $K_H^\mathcal{K} = \{ \psi \text{ state on } O_d^H : \psi \lambda_H = \psi, \psi_H|B_H = \omega_H \}$ where $B_H = \otimes_2 M_d(C) \equiv \mathcal{B}$ and $\omega_H \equiv \omega$. Since $\beta_z^H(s_I) = \beta_z^{\tilde{I}}(s_I)$, it is a simple observation that $z \in G_H$ if and only if $z \tilde{\pi} \in H$ since Popescu elements for $\psi_H$ are $\{v_I : |I| = n\}$. Thus $G_H$ is trivial. This proves our claim that $\pi_H(U \hat{H}_0^{\mathcal{K}} \otimes \mathcal{K}_0) \mathcal{K}_0 = B(\tilde{H}_0 \otimes \mathcal{K}_0, \mathcal{H}_0)$ by the first part of the argument and so $\omega$ is pure.

3. Symmetries of a translation invariant pure state on $B$

In this section we investigate $\omega$ on $B$ with some additional natural discrete symmetry. Let $\psi$ be a $\lambda$-invariant state on $O_d$ and $\psi$ be the state on $O_d$ defined by $\tilde{\psi}(s^I s^I_j) = \psi(s^I s^I_j)$ for all $|I|, |J| < \infty$ and $(\mathcal{H}_\phi, \pi_\phi, \Omega_\phi)$ be the GNS space associated with $(O_d, \tilde{\psi})$. That $\tilde{\psi}$ is well defined follows once we check by (14) that $\tilde{\psi}(s^I s^I_j) = \phi(v_I v_I^*) = \phi(\tilde{v}_I \tilde{v}_I)$

and appeal to Proposition 2.3 [Mo5] by observing that cyclic condition i.e. the closed linear span $P_0$ of the set of vectors $\{\tilde{v}_I : |I| < \infty\}$ is $\mathcal{K}$, can be ensured if not true already by taking a new set of Popescu elements $\{P_0 \tilde{v}_k P_0 : 1 \leq k \leq d\}$. Otherwise one may also recall Proposition 2.3 [Mo5] that the map $s^I s^I_j \to \tilde{v}_I \tilde{v}_I$ being unital and completely positive [Po] (in particular positive), $\psi$ is a well defined state on $O_d$.

Similarly for any translation invariant state $\omega$ on $B$ we set translation invariant state $\bar{\omega}$ by reflecting around the point $\frac{1}{2}$ on $B$ by

$$\bar{\omega}(Q_{-}^{(-1)} \otimes Q_{l+1}^{-1} \otimes \ldots \otimes Q_{l+1}^{-1} \otimes Q_{0}^{(0)} \otimes Q_{1}^{(1)} \ldots \otimes Q_{n}^{(n)}) = \omega(Q_{-}^{(-1-n+1)} \otimes Q_{1}^{(0)} \otimes Q_{0}^{(1)} \otimes Q_{1}^{(2)} \otimes \ldots \otimes Q_{l+1}^{(l+1)} \otimes Q_{l+1}^{(l+1)})$$

for all $n, l \geq 1$ and $Q_{-}^{(-1)}, \ldots, Q_{l+1}^{(l+1)}, Q_{0}, Q_{1}, \ldots, Q_{n} \in M_n(C)$ where $Q^{(k)}$ is the matrix $Q$ at lattice point $k$. We define $\bar{\omega}$ on $B$ by extending linearly to any $Q \in B_{loc}$.

Note first that the map $\psi \to \tilde{\psi}$ is a one to one and onto affine map in the convex set of $\lambda$ invariant state on $O_d$. In particular the map $\psi \to \tilde{\psi}$ takes an element
from $K_ω$ to $K_ω$ and the map is once more one to one and onto. Hence for any extremal point $ψ ∈ K_ω$, $ψ$ is also an extremal point in $K_ω$. Using Power’s criterion (2) we also verify here that $ω$ is an extremal state if and only if $\tilde{ω}$ is an extremal state. However such a conclusion for a pure state $ω$ is not so obvious. We have the following useful proposition.

**Proposition 3.1.** Let $ω$ be an extremal translation invariant state on $B$ and $ψ → ψ$ be the map defined for $λ$ invariant states on $O_d$. Then the following holds:

(a) $ψ ∈ K_ω$ is a factor state if and only if $ψ \in K_ω$ is a factor state.

(b) $ω$ is pure if and only if $\tilde{ω}$ is pure.

(c) A Popescu systems $(K, M, v, \Omega)$ of $ψ$ satisfies Proposition 2.4 in [Mo5] with $(\pi_ψ(s_k), 1 ≤ k ≤ d)$, $P$, $Ω$ i.e. the projection $P$ on the subspace $K$ is the support projection of the state $ψ$ in $π(O_d)_0$ and $v_i = Pπ_ψ(s_i)P$ for all $1 ≤ i ≤ d$, then the dual Popescu systems $(K, M', \tilde{v}, \Omega)$ satisfies converse part of Proposition 2.4 in [Mo5] with $(\pi_ψ(s_k), 1 ≤ k ≤ d)$, $P$, $Ω$ i.e. the projection $P$ on the subspace $K$ is the support projection of the state $ψ$ in $π_ψ(O_d)_0$ and $v_i = Pπ_ψ(s_i)P$ for all $1 ≤ i ≤ d$, if and only if $\{x ∈ B(K) : \sum_k v_kx_i^* x_k = x\} = M$.

**Proof.** Since $ω$ is an extremal translation invariant state, by Power’s criterion (2) $\tilde{ω}$ is also an extremal state. As an extremal point of $K_ω$ is map to an extremal point in $K_ω$ by one to one property of the map $ψ → \tilde{ω}$, we conclude by Proposition 2.2 that $ψ$ is a factor state if and only if $ψ$ is a factor state. For (b) note that $z \bar{x} y^* = \bar{x} y$ and $x^* = \bar{x}^*$ by our definition. Thus $\tilde{ω}(x^* y) = ω(x^* y) = ω(\bar{x}^* \bar{y})$. Thus one can easily construct a unitary operator between the two GNS spaces associated with $(B, ω)$ and $(\tilde{B}, \tilde{ω})$ intertwining two representation modulo a reflection i.e. $Uπ_ω(x)U^* = π_\tilde{ω}(\bar{x})$ and $UΩ_ω = Ω_\tilde{ω}$. Thus (b) is now obvious. (c) follows by the converse part of the Proposition 2.4 in [Mo5] once applied to the dual Popescu systems $(K, M', \tilde{v}, \Omega)$.

Thus the state $\tilde{ω}$ is translation invariant, ergodic, factor state, pure if and only if $ω$ is translation invariant, ergodic, factor state, pure respectively. We say $ω$ is **lattice symmetric** if $\tilde{ω} = ω$.

For a $λ$ invariant state $ψ$ on $O_d$ we define as before a $λ$ invariant state $\tilde{ψ}$ by

$$\tilde{ψ}(s_is_i^*) = ψ(s_is_i^*)$$

for all $|I|, |J| < ∞$. It is obvious that $ψ ∈ K_ω$ if and only if $\tilde{ψ} ∈ K_ω$ and the map $ψ → \tilde{ψ}$ is an affine map. In particular an extremal point in $K_ω$ is also mapped to an extremal point of $K_ω$. It is also clear that $\tilde{ψ} ∈ K_ω$ if and only if $ω$ is lattice symmetric. Hence a lattice symmetric state $ω$ determines an affine map $ψ → \tilde{ψ}$ on the compact convex set $K_ω$. Furthermore, if $ω$ is also extremal on $B$, then the affine map takes extremal elements to extremal elements of $K_ω$. The set of extremal elements in $K_ω$ can be identified with $S^4/H ≡ S^1$ or $\{1\}$ and the restriction of the affine map on the set of extremal element is continuous in weak topology by Proposition 2.2 the map $z → ψβ_z$ is one to one and onto the set of extremal elements of $K_ω$ for a fixed extremal element $ψ ∈ K_ω$.

Thus there exists $z_0$ in $S^1$ so that $\tilde{ψ} = ψβ_{z_0}$ and as $\beta z_0 = \tilde{β}_z$ for all $z ∈ S^1$, we get the affine map taking $ψβ_z → ψβ_{z_0}β_z$ and thus determines a continuous one to one and onto map on $S^1/H$ and as $ψ = \tilde{ψ}$ its inverse is itself. Thus either the affine map has a fixed point or $z_0^2 = 1$ i.e. it is a rotation map by an angle $2π$.
( Here we have identified $S^1/H$ with $S^1$ in case $H \neq S^1$ ). Thus there exists an extremal element $\psi \in K_\omega$ so that either $\tilde{\psi} = \psi \beta_\zeta$ where $\zeta$ is either 1 or $-1$ where we recall that we have identified $S^1/H = S^1$ when $H = S^1$. Note that if we wish to remove the identification, then for $H = \{ z : z^n = 1 \}$ for some $n \geq 1$, $\zeta$ is either 1 or $exp \pi i/n$. Note that in case $H = S^1$ then $\tilde{\psi} = \psi$ for $\psi \in K_\omega$ as $K_\omega$ is a singleton set by Proposition 2.2.

**Proposition 3.2.** Let $\omega$ be a translation invariant lattice symmetric state on $B$. Then the following holds:
(a) If $\omega$ is also an extremal translation invariant state on $B$ then $H = \{ z \in S^1 : \psi \beta_z = \psi \}$ is independent of $\psi \in K_\omega$.
(b) If $H = \{ z : z^n = 1 \}$ for some $n \geq 0$ then $\tilde{\psi} = \psi \beta_\zeta$ for all $\psi \in K_\omega$ where $\zeta$ is fixed either 1 or $exp \pi i/n$ and there exists a unitary operator $U_\zeta : \mathcal{H} \otimes \tilde{\mathcal{K}} \otimes \tilde{\mathcal{H}}$ so that

$$U_\zeta^* = U_\zeta, U_\zeta \Omega = \Omega, \quad U_\zeta S_k U_\zeta^* = \beta_\zeta(\tilde{S}_k)$$

for all $1 \leq k \leq d$.

Furthermore if $\omega$ is also pure then there exists a unitary operator $u_\zeta : \mathcal{K} \rightarrow \mathcal{K}$ so that

$$u_\zeta \Omega = \Omega, \quad u_\zeta v_k u_\zeta^* = \beta_\zeta(v_k)$$

for all $1 \leq k \leq d$ and $u_\zeta J u_\zeta^* = J$, $u_\zeta \Delta_1^2 u_\zeta^* = \Delta_1^{-2}$, $u_\zeta^* = u_\zeta$ and $u_\zeta M u_\zeta^* = M'$, $u_\zeta^* M u_\zeta = M$. Moreover $M' = \tilde{M}$. Further if $\zeta = 1$ then $u_\zeta$ is self-adjoint and otherwise if $\zeta \neq 1$ then $u_\zeta^{2n}$ is self-adjoint.

(c) If $H = S^1$ then $K_\omega$ is having only one element $\psi$, so $\tilde{\psi} = \psi$ and (29) is valid with $\zeta = 1$. If $\omega$ is also purely then (30) is also valid with $\zeta = 1$.

**Proof.** (a) follows by Proposition 2.2. Now we aim to prove (b). For existence of an extremal state $\psi \in K_\omega$ so that $\tilde{\psi} = \psi \beta_\zeta$ we refer to the paragraph preceding the statement of this proposition. As $(\psi \beta_\zeta) = (\psi \beta_\zeta)$ for all $z \in S^1$, a simple application of Proposition 2.2 says that $\tilde{\psi} = \psi \beta_\zeta$ for all extremal points in $K_\omega$ if it holds for one extremal element. Hence existence part in (b) is true by Krein-Millmann theorem.

$\Omega$ is a cyclic vector for $\pi(O_d \otimes \tilde{O}_d)$ and thus we define $U_\zeta : \mathcal{H} \otimes \tilde{\mathcal{K}} \otimes \tilde{\mathcal{H}} \rightarrow \mathcal{H} \otimes \tilde{\mathcal{K}} \otimes \tilde{\mathcal{H}}$ by

$$U_\zeta : S_1 S_2 \tilde{S}_1 \tilde{S}_2 \Omega \rightarrow \beta_\zeta(S_1 S_2 \tilde{S}_1 \tilde{S}_2) \tilde{\Omega}$$

That $U_\zeta$ is a unitary operator follows from (27) and the dual relation (28) along with our condition that $\tilde{\psi} = \psi \beta_\zeta$. By our construction we also have $U_\zeta S_k = \beta_\zeta(\tilde{S}_k) U_\zeta$ for all $1 \leq k \leq d$. In particular $U_\zeta \pi(O_d)^n U_\zeta^* = \pi(\tilde{O}_d)^n$.

$\omega$ being pure we have $EF = E\tilde{E} = \tilde{E}E$ by Proposition 3.2 (d) since $F = \tilde{F}$ and $E = \tilde{E}$. So $U_\zeta P U_\zeta^* = U_\zeta E\tilde{E} U_\zeta^* = \tilde{E}E = P$, which ensures a unitary operator $u_\zeta = PU_\zeta P$ on $\mathcal{K}$ and a routine calculation shows that

$$u_\zeta v_k u_\zeta^* = \beta_\zeta(v_k)$$

for all $1 \leq k \leq d$. As $U_\zeta^* = U_\zeta$ we have $u_\zeta^* = u_\zeta$. If $\zeta \neq 1$, then $\zeta^{2n} = 1$ and thus $U_\zeta^{2n}$ is inverse of its own. Thus $u_\zeta^{2n}$ is self-adjoint. That $M' = \tilde{M}$ proved in Theorem 3.6 (b) in [Mo5].
In the following we consider the case $\zeta = 1$ for simplicity of notation and otherwise for the case $\zeta \neq 1$ very little modification is needed in the symbols or simply reset temporary notation $v_{ik}$ for $\zeta v_{ik}$ i.e. include the phase factor.

We denote $u_1 = u$ in the following for simplicity. It is simple to verify now the following steps $u \overline{\psi} J \overline{\psi}^* \Omega = u \overline{\psi} J \overline{\psi}^* \Omega = \overline{\psi}^* \overline{\psi} \Omega$ where $S x \Omega = x^* \Omega$, $x \in M$ and $F x \Omega = x^* \Omega$, $x \in M'$ are the Tomita’s conjugate operator. Hence $u \overline{\psi} J \overline{\psi}^* = \mathcal{J} \Delta^{-\frac{1}{2}} u$, i.e. $u \overline{\psi} J \overline{\psi}^* u \Delta^{-\frac{1}{2}} u^* = \mathcal{J} \Delta^{-\frac{1}{2}}$. And by uniqueness of polar decomposition we conclude that $u \mathcal{J} u^* = \mathcal{J}$ and $u \Delta^{-\frac{1}{2}} u^* = \Delta^{-\frac{1}{2}}$. That $uM u^* = M$ is obvious. For $uM u^* = M$ we note that by our construction $U \tilde{S}_k U^* = \tilde{S}_k$ and so $U \pi(\tilde{O}_d) U^* = \pi(\tilde{O}_d)$ and hence projecting to its support projection we get the required relation.

Now we introduce another useful symmetry on $\omega$. If $Q = Q_0^{(1)} \otimes Q_1^{(l+1)} \otimes \ldots \otimes Q_m^{(l+m)}$ we set $Q' = Q_0^{(1)} \otimes Q_1^{(l+1)} \otimes \ldots \otimes Q_m^{(l+m)}$ where $Q_0, Q_1, \ldots, Q_m$ are arbitrary elements in $M_2$ and $Q_0, Q_1, \ldots$ stands for transpose with respect to an orthonormal basis $(e_i)$ for $C^d$ (not complex conjugate) of $Q_0, Q_1$, respectively. We define $Q'$ by extending linearly for any $Q \in B_{loc}$. For a state $\omega$ on UHF$_d$ $C^*$ algebra $\otimes M_d$ we define a state $\omega$ on $\otimes M_d$ by the following prescription

$$\omega(Q) = \omega(Q')$$

Thus the state $\omega$ is translation invariant, ergodic, factor state if and only if $\omega$ is translation invariant, ergodic, factor state respectively. We say $\omega$ is real if $\omega = \tilde{\omega}$.

In this section we study a translation invariant real state.

For a $\lambda$ invariant state $\psi$ on $\mathcal{O}_d$ we define a $\lambda$ invariant state $\tilde{\psi}$ by

$$\tilde{\psi}(s_J s_J^*) = \psi(s_J s_J^*)$$

for all $|I|, |J| < \infty$ and extend linearly. That it defines a state follows as for an element $x = \sum c(I, J) s_I s_J^*$ we have $\psi(x^* x) = \psi(y^* y) \geq 0$ where $y = \sum c(I, J) s_I s_J^*$. It is obvious that $\psi \in K_\omega$ if and only if $\tilde{\psi} \in K_\omega$ and the map $\psi \rightarrow \tilde{\psi}$ is an affine map. In particular an extremal point in $K_\omega$ is also mapped to an extremal point in $K_\omega$. It is also clear that $\tilde{\psi} \in K_\omega$ if and only if $\omega$ is real. Hence a real state $\omega$ determines an affine map $\psi \rightarrow \tilde{\psi}$ on the compact convex set $K_\omega$. Furthermore, if $\omega$ is also extremal on $\mathcal{B}$, then the affine map, being continuous on the set of extremal elements in $K_\omega$, which can be identified with $S^1/H \equiv S^1$ or $\{1\}$ (by Proposition 2.2) by fixing an extremal element $\psi_0 \in K_\omega$. In such a case there exists a unique $z_0 \in S^1$ so that $\tilde{\psi}_0 = \psi_0 \beta_{z_0}$.

Now $\psi_0 \beta_z = \tilde{\psi}_0 \beta_z$ for all $z \in S^1$, the affine map takes $\psi_0 \beta_z \rightarrow \psi_0 \beta_{z_0}$. If $z_0 = 1$ we get that the map fixes two points namely $\psi_0$ and $\psi_0 \beta_{-1}$.

Even otherwise we can choose $z \in S^1$ so that $z^2 = z_0$ and for such a choice we get an extremal element namely $\psi_0 \beta_z$ gets fixed by the map. What is also crucial here that we can as well choose $z \in S^1$ so that $z^2 = -z_0$, if so then $\psi_0 \beta_z$ gets mapped into $\psi_0 \beta_{z_0} \beta_z = \psi_0 \beta_{-z} = \psi_0 \beta_{-\beta_{z_0}}$. Thus in any case we also have an extremal element $\psi \in K_\omega$ so that $\tilde{\psi} = \psi_0 \beta_\zeta$ where $\zeta \in \{1, -1\}$.

Thus going back to the original set up, we sum up the above by saying that if $H = \{z : z^2 = 1\} \subseteq S^1$ and $\zeta \in \{1, e^{\pi i}\}$ then there exists an extremal element $\psi \in K_\omega$ so that $\psi = \psi_0 \beta_\zeta$. 


PROPOSITION 3.3. Let $\omega$ be a translation invariant real factor state on $\otimes_\mathbb{Z} M_d$. Then the following holds:

(a) if $H = \{ z : z^n = 1 \} \subseteq S^1$ and $\zeta \in \{ 1, \exp \frac{2\pi i}{n} \}$ then there exists an extremal element $\psi \in K_\omega$ so that $\psi = \psi \beta_\zeta$. Let $(H, \pi_\psi(s_k) = S_k, 1 \leq k \leq d, \Omega)$ be the GNS representation of $(O_\omega, \psi)$, $P$ be the support projection of the state $\psi$ in $\pi(O_\omega)^\prime$ and $(K, M, V_k, 1 \leq k \leq d, \Omega)$ be the associated Popescu systems as in Proposition 2.4 in [Mo5]. Let $v_k = Jv_kJ$ for all $1 \leq k \leq d$ and $(H, S_k, P, \Omega)$ be the Popescu minimal dilation as described in Theorem 2.1 in [Mo5] associated with the systems $(K, M', \tilde{v}_k, 1 \leq k \leq d, \Omega)$. Then there exists a unitary operator $W_\zeta : H \to H$ so that

\[ W_\zeta \Omega = \Omega, \quad W_\zeta S_k W_\zeta^* = \beta_\zeta(S_k) \]

for all $1 \leq k \leq d$. Furthermore $P$ is the support projection of the state $\tilde{\psi}$ in $\tilde{T}(O_\omega)^\prime$ and there exists a unitary operator $w_\zeta$ on $K$ so that

\[ w_\zeta \Omega = \Omega, \quad w_\zeta v_k w_\zeta^* = \beta_\zeta(v_k) J \]

for all $1 \leq k \leq d$ and $w_\zeta J w_\zeta^* = J$ and $w_\zeta \Delta^\frac{1}{2} w_\zeta^* = \Delta^\frac{1}{2}$. $w_\zeta$ is self adjoint if and only if $\zeta = 1$;

(b) If $H = \mathbb{C}^1$, $K_\omega$ is a set with unique element $\psi$ so that $\tilde{\psi} = \psi$ and relations (34) and (35) are valid with $\zeta = 1$.

PROOF. For existence part in (a) we refer the paragraph above preceded the statement of the proposition. We fix a state $\psi \in K_\omega$ so that $\tilde{\psi} = \psi \beta_\zeta$ and define $W : H \to H$ by

\[ W_\zeta : S_I S_J^\prime \Omega \to \beta_\zeta(S_I^\prime S_J^\prime) \Omega \]

That $W_\zeta$ is a unitary operator follows from (33) and thus $W_\zeta S_k = \beta_\zeta(S_k) W_\zeta$ for all $1 \leq k \leq d$. For simplicity of notation we take the case $\zeta = 1$ as very little modification is needed to include the case when $\zeta \neq 1$ or reset Cuntz elements (10) by absorbing the phase factor in the following computation and use notation $W$ for $W_\zeta$.

$P$ being the support projection we have by Proposition 2.4 in [Mo5] that $M' = \{ x \in \mathcal{B}(H) : \sum_k v_k x v_k^* = x \}$ and thus $\mathcal{M} = \{ x \in \mathcal{B}(K) : \sum_k Jv_k J xv_k Jx^* J = x \}$. Hence by the converse part of Proposition 2.4 in [Mo5] we conclude that $P$ is also the support projection of the state $\tilde{\psi}$ in $\tilde{T}(O_\omega)^\prime$. Hence $W_\zeta P W_\zeta^* = P$. Thus we define a unitary operator $w_\zeta : K \to K$ by $w_\zeta = PW_\zeta P$ and verify that

\[ \tilde{v}_k = P \tilde{S}_k^\prime \tilde{\psi} \]

\[ = PW_\zeta \beta_\zeta(S_k^\prime) W_\zeta^* P = PW_\zeta P \beta_\zeta(S_k^\prime) PW_\zeta^* P \]

\[ = PW_\zeta P \beta_\zeta(v_k^\prime) PW_\zeta^* P = w_\zeta \beta_\zeta(v_k^\prime) w_\zeta^*. \]

We recall that Tomita’s conjugate linear operators $S, F$ [BR] are the closure of the linear operators defined by $S : x_1 \Omega \to x_1^* \Omega$ for $x_1 \in \mathcal{M}$ and $F : y_1 \Omega \to y_1^* \Omega$ for $y_1 \in \mathcal{M}'$. We check the following relations for $\zeta = 1$ with simplified notation $w_1 = w$,

\[ wSv_1 v_1^\prime \Omega = wv_1 v_1^\prime \Omega = \tilde{v}_1 v_1^\prime \Omega = F \tilde{v}_1 v_1^\prime \Omega \]

for $|I|, |J| < \infty$. Since such vectors are total, we have $wS = Fw$ on the domain of $S$. Thus $wSw^* = Fw$ on the domain of $F$. We write $S = J \Delta^\frac{1}{2}$ as the unique polar decomposition. Then $F = J^* = \Delta^\frac{1}{2} J = J \Delta^{-\frac{1}{2}}$. Hence $wJw^* \Delta^\frac{1}{2} w^* = J \Delta^{-\frac{1}{2}}$. By the uniqueness of polar decomposition we get $wJw^* = J$ and $w\Delta^\frac{1}{2} w^* = \Delta^{-\frac{1}{2}}$. Same algebra is valid in case $\zeta \neq 1$ if we reset the notations $\tilde{v}_k$ on the right hand.
side absorbing the phase factor. In the following we repeat it for completeness of the proof as the phase factor could be delicate.

\[ w_\zeta S v_I v_J^* \Omega = w_\zeta v_J v_I^* \Omega = w_\zeta v_J v_I^* w_\zeta^* \Omega \]
\[ = \zeta^{|I|-|J|} \bar{v}_J \bar{v}_I^* \Omega = \zeta^{|I|-|J|} F \bar{v}_I \bar{v}_J^* \Omega \]
\[ = F \zeta^{-|I|+|J|} \bar{v}_J \bar{v}_I^* \Omega \text{ for } |I|, |J| < \infty \]
\[ = F w_\zeta v_I v_J^* \Omega \]

for all \(|I|, |J| < \infty\).

Now we are going to show that \(w_\zeta\) is self-adjoint if and only if \(\zeta = 1\). Note that \(\beta_\zeta(w_\zeta x w_\zeta^*) = w_\zeta \beta_\zeta(x) w_\zeta^*\) and thus applying \(\beta_\zeta\) on both side of the following identity

\[ w_\zeta v_k w_\zeta^* = \mathcal{J} \beta_\zeta(v_k) \mathcal{J} \]

for all \(1 \leq k \leq d\), we also get \(w_\zeta \beta_\zeta(v_k) w_\zeta^* = \mathcal{J} \beta_\zeta^2(v_k) \mathcal{J}\) and thus \(w_\zeta^2 v_k (w_\zeta^*)^2 = w_\zeta \mathcal{J} \beta_\zeta(v_k) \mathcal{J} w_\zeta = \beta_\zeta^2(v_k)\) as \(\mathcal{J}\) commutes with \(w_\zeta\).

\[ \zeta^2 = 1 \text{ if and only if } \zeta = 1 \text{ (as } \zeta = 1 \text{ or } \exp(\frac{2\pi i}{n}) \text{ where } n \geq 2\} \). In such a case we get \(w_\zeta^2 \in \mathcal{M}'\) and further as \(w_\zeta\) commutes with \(\mathcal{J}\), \(w_\zeta^2 \in \mathcal{M}\). \(\omega\) being an extremal element in \(\mathcal{K}_\omega\) we have \(\mathcal{M} \vee \mathcal{M} = \mathcal{B}(\mathcal{K})\) by Proposition 3.5 in [Mo5] and as \(\mathcal{M} \subseteq \mathcal{M}'\), we get that \(\mathcal{M}\) is a factor. Thus for a factor \(\mathcal{M}\), \(w_\zeta^2\) is a scalar. Since \(w_\zeta \Omega = \Omega\) we get \(w_\zeta^2 = 1\) i.e. \(w_\zeta^2 = w_\zeta\). This completes the proof. \(\blacksquare\)

A state \(\omega\) on \(\otimes_{\mathbb{Z}} \mathcal{M}_d\) is said be in detailed balance if \(\omega\) is both lattice symmetric and real. In the following proposition as before we identified once more \(S^1/H \equiv S^1\) in case \(H \neq S^1\) and set \(\zeta\) be the least value in \(S^1\) \(H \equiv S^1\) so that \(\zeta^2 \in H\).

Theorem 3.4. Let \(\omega\) be a translation invariant factor state on \(\mathcal{B} = \otimes_{\mathbb{Z}} \mathcal{M}_d\). Then the following are equivalent:
(a) \(\omega\) is real and lattice symmetric;
(b) There exists an extremal element \(\psi \in \mathcal{K}_\omega\) so that \(\tilde{\psi} = \psi \beta_\zeta\) and \(\tilde{\psi} = \psi \beta_\zeta\epsilon\) where \(\zeta\) is either \(1\) or \(\exp(\frac{2\pi i}{n})\).

Furthermore if \(\omega\) is a pure state then the following holds:
(c) There exists a Popescu elements \((\mathcal{K}, v_k, 1 \leq k \leq d, \Omega)\) for \(\omega\) with relation \(v_k = \mathcal{J}_\nu \delta_k \mathcal{J}_\nu\) for all \(1 \leq k \leq d\), where \(\mathcal{J}_\nu = v \mathcal{J}\) and \(v\) is a self-adjoint unitary operator on \(\mathcal{K}\) commuting with modular operators \(\Delta^\mathcal{K}\) and conjugate operator \(\mathcal{J}\) associated with cyclic and separating vector \(\Omega\) for \(\mathcal{M}\). Further \(\beta_\zeta(v) = v\) for all \(z \in H\) and \(H \subseteq \{1, -1\}\);
(d) The map \(\mathcal{F}_\nu : \mathcal{H} \otimes_{\mathcal{K}} \mathcal{H} \to \mathcal{H} \otimes_{\mathcal{K}} \mathcal{H}\) defined by
\[ \pi(s_1 s_2^* \tilde{s}_1 \tilde{s}_2^*) \Omega = \pi(s_1 s_2^* \tilde{s}_1 \tilde{s}_2^*) \Omega, \]
\(|I|, |J|, |I'|, |J'| < \infty \text{ extends the map } \mathcal{F}_\nu : \mathcal{K} \to \mathcal{K} \text{ to an anti-unitary map so that } \mathcal{F}_\nu \pi(s_i) \mathcal{F}_\nu^{-1}(\pi(\tilde{s}_i) \text{ for all } 1 \leq i \leq d \text{ where } \tilde{s}_i \text{ is the conjugate linear extension of } \pi \text{ from the generating set } (\tilde{s}_i),\]
\(i.e. \pi(\tilde{s}_1 \tilde{s}_2^*) = \pi(\tilde{s}_1 \tilde{s}_2^*) \text{ for } |I|, |J| < \infty \text{ and then extend it anti-linearly for its linear combinations.\}

Proof. Since \(\omega\) is lattice symmetric, by Proposition 3.2 \(\tilde{\psi} = \psi \beta_\zeta\) for all \(\psi \in \mathcal{K}_\omega\) where \(\zeta\) is fixed number either \(1\) or \(\exp(\frac{2\pi i}{n})\) for some \(n \geq 1\). Now we use real property of \(\omega\) and choose by Proposition 3.3 an extremal element \(\psi \in \mathcal{K}_\omega\) so that \(\tilde{\psi} = \psi \beta_\zeta\). This proves that (a) implies (b). That (b) implies (a) is obvious.
Now we aim to prove the last statements which is the main point of the proposition. For simplicity of notation we consider the case $\zeta = 1$ and leave it to reader to check that a little modification needed to include the case $\zeta \neq 1$ and all the algebra stays valid if $\beta_k$ is replaced by $\beta_\zeta(\bar{\beta_k})$. We consider the Popescu system $(K, \mathcal{M}, v_k, 1 \leq k \leq d, \Omega)$ as in Proposition 2.4 in [Mo5] associated with $\psi$. Thus by Proposition 3.2 and Proposition 3.3 there exists unitary operators $u_\zeta, w_\zeta$ on $K$ so that

$$u_\zeta v_k u_\zeta^* = \beta_\zeta(\bar{v_k})$$

and

$$w_\zeta v_k w_\zeta^* = \beta_\zeta(\bar{v_k}) = J \beta_\zeta(v_k)J$$

where $u_\zeta J u_\zeta^* = J$, $w_\zeta J w_\zeta^* = J$ and $u_\zeta \Delta^+ u_\zeta^* = w_\zeta \Delta^+ w_\zeta^* = \Delta^+$. Thus

$$u_\zeta w_\zeta v_k w_\zeta^* u_\zeta^* = w_\zeta \beta_\zeta(\bar{v_k}) J u_\zeta^* = J \beta_\zeta(\bar{v_k}) J u_\zeta^* = J \beta_\zeta(v_k)J = J \bar{v_k} J$$

We also compute that

$$w_\zeta u_\zeta v_k u_\zeta^* w_\zeta^* = w_\zeta \beta_\zeta(\bar{v_k}) w_\zeta^* = J \beta_\zeta(\bar{v_k}) J = J \bar{v_k} J$$

By Proposition 3.2 in [Mo5], for a factor state $\omega$ we also have $\mathcal{M} \vee \bar{\mathcal{M}} = B(K)$. As $\mathcal{M} \subseteq \mathcal{M}'$, in particular we note that $\mathcal{M}$ is a factor. So $u_\zeta^* w_\zeta^* u_\zeta, w_\zeta \in \mathcal{M}'$ commuting also with $\mathcal{J}$ and thus $\mathcal{J}$ is a scalar as $\mathcal{M}$ is a factor. As $u_\zeta \Omega = w_\zeta \Omega = \Omega$, we conclude that $u_\zeta$ commutes with $w_\zeta$.

Now we set $v_\zeta = u_\zeta v_k$ which is a unitary operator commuting with both $\mathcal{J}$ and $\Delta^+$. That $v_\zeta$ commuting with $\Delta^+$ follows as $u_\zeta w_\zeta \Delta^+ = u_\zeta \Delta^+ w_\zeta = \Delta^+ u_\zeta w_\zeta$.

Next claim that we make now that $v_\zeta$ is a self-adjoint element. To that end note that the relations (56) says that $v_\zeta^* \mathcal{M} v_\zeta \subseteq \mathcal{M}$ and so $v_\zeta^* \mathcal{M}' v_\zeta \subseteq \mathcal{M}'$. We check the following identity: $v_\zeta \bar{v_k} v_\zeta^* \Omega = v_\zeta^* v_k \Omega = v_\zeta^* v_k^* v_\zeta^* \Omega = J \bar{v_k} J \Omega = J v_k J \Omega$ and thus separating property we deduce that

$$v_\zeta \bar{v_k} v_\zeta^* = J v_k J$$

for all $1 \leq k \leq d$. So we conclude that $v_\zeta^* \in \mathcal{M}'$ and as $v_\zeta$ commutes with $\mathcal{J}$, $v_\zeta^*$ is an element in the centre of $\mathcal{M}$. The centre of $\mathcal{M}$ being trivial as $\omega$ is a factor state (here we have more namely pure) and $v_\zeta \Omega = \Omega$, we conclude that $v_\zeta^*$ is the unit operator. Hence $v_\zeta$ is a self-adjoint element.

For simplicity of notation we set $v$ for $v_{\zeta}$. $\beta_\zeta(v) = v$ for all $z \in H$ is equivalent to the property that $v$ keeps each subspace $P_k$ invariant where $u_z = \sum_{k \in B} z^k P_k$, is the spectral decomposition of unitary representation $z \to u_z$ and $\pi(\beta_x(x)) = u_z \pi(x) u_z^*$ for $x \in \mathcal{O}_H$ in the GNS space $(\mathcal{H}, \pi, \Omega)$ of the state $\psi$. Since $v_{\psi} v_{\psi}^* = J \bar{v_{\psi}} J$, and $v$ is self-adjoint and $e\Omega = \Omega$ we get $v_{\psi} v_{\psi}^* \Omega = J \bar{v_{\psi}} J \Omega$, $\beta_z$ being an automorphism on $\mathcal{M}$ preserving the state $\phi$, modular elements $J, \Delta^+$ commutes with $u_z$ and in particular $J$ commutes with $P_k$ for all $k \in \bar{H}$. Since $[v_{\psi} v_{\psi}^* : |I| - |J| = k] = P_k$ and $[\bar{v_{\psi}} J \Omega : |I| - |J| = k] = P_k$, we get $(I - P_k)vP_k = 0$ since $\mathcal{J}$ commutes with $P_k$. We can interchange the role of $(v_{\psi})$ and $(\bar{v_{\psi}})$ to conclude that $(I - P_k)vP_k = 0$ for all $k \in \bar{H}$. Otherwise we can as well use the fact that $v$ is self-adjoint to conclude that $v$ commutes with each $P_k$. This shows that $v$ commutes with $u_z$ for all $z \in H$ i.e. $\beta_z(v) = v$.

Fix any $z \in H$. By taking action of $\beta_z$ on both side of the relation $v_{\psi} v_{\psi}^* = J \bar{v_{\psi}} J$, we have $v_{\psi} v_{\psi}^* = z^2 J \bar{v_{\psi}} J = z^2 v_{\psi} v_{\psi}^*$. Thus $z^2 v_k = v_k$ for all $1 \leq k \leq d$. Since $\sum_k v_k v_k^* = 1$, we have $z^2 = 1$. 

TRANSLATION INVARIANT PURE STATE ON $\mathcal{S} = \otimes_{d} \mathcal{M}_d(\mathbb{C})$ AND ITS SPLIT PROPERTY
The last statement (d) follows by a routine calculation as shown below for a special vectors.

\[
\begin{align*}
&\langle \Omega, \pi(s_j s_j^* \tilde{s}^j) \tilde{s}_j \Omega \rangle \\
&= \langle \Omega, v_j v_j^* \tilde{v}_j \tilde{v}_j^* \Omega \rangle \\
&= \langle \Omega, \mathcal{J}_v v_j \tilde{v}_j v_j^* \tilde{v}_j^* \mathcal{J}_v \Omega \rangle \\
&= \langle \pi(s_j s_j^* \tilde{s}_j \tilde{s}_j^*) \Omega, \Omega \rangle
\end{align*}
\]

(as \( \mathcal{J}_v v_i \mathcal{J}_v = \tilde{v}_i \))

\[
= \langle \tilde{v}_j \tilde{v}_j^* v_j^* v_j \Omega, \Omega \rangle
\]

(\( \mathcal{J}_v \) being anti-linear)

\[
= \langle \pi(s_j s_j^* \tilde{s}_j \tilde{s}_j^*) \Omega, \Omega \rangle
\]

For anti-unitary relation involving more general vectors, we use Cuntz relations (10) and the above special cases. The statement is obvious as \( \mathcal{J}_v \) is anti-linear. This completes the proof.

Now we aim to make reflection symmetry a little more general with primarily motivated with reflection symmetry with a twist introduced in [FILS]. To that end we fix any \( g_0 \in U_d(C) \) so that \( g_0^2 = 1 \) and \( \beta_{g_0} \) is the natural action on \( \mathcal{O}_d^r \). We say \( \omega \) is lattice reflection symmetric with twist \( g_0 \) if \( \omega(\beta_{g_0}(r(x))) = \omega(x) \) for all \( x \in \mathcal{B} \) where \( r \) is the reflection automorphism around \(-\half \). So when \( g_0 = 1 \) we get back to our notion of lattice reflection symmetric. We fix now such a lattice reflection \( g_0 \)-twisted factor state \( \omega \). Since \( \beta_{g_0} \beta_\zeta = \beta_\zeta \beta_{g_0} \) for all \( \zeta \in S^1 \), by going along the same line as in Proposition 3.2, any extremal element in \( \psi \in K_\omega \) will admit \( \tilde{\psi}_{g_0} = \psi \circ \zeta \) where \( \zeta = 1 \) or \( \zeta = e^{i \pi z} \) where \( H = \{z \in S^1 : z^n = 1\} \) and \( \tilde{\psi}_{g_0} = \psi_{g_0} \). Thus we can follow the same steps that of Proposition 3.4 to have a modified statements in the proof of Proposition 3.4 with \( v_k \) replaced by \( \beta_{g_0}(v_k) \) for such a pure real state i.e. there exists unitary operators \( u_\zeta, w_\zeta \) on \( K \) so that

\[
u_\zeta \beta_{g_0}(v_k) u_\zeta^* = \beta_\zeta(\tilde{v}_k)
\]

\[
w_\zeta v_k w_\zeta^* = \beta_\zeta(\tilde{v}_k) = \mathcal{J} \beta_\zeta(v_k) \mathcal{J}
\]

where \( u_\zeta \mathcal{J} u_\zeta^* = \mathcal{J}, \ w_\zeta \mathcal{J} w_\zeta^* = \mathcal{J} \) and \( u_\zeta \Delta_\zeta^w u_\zeta^* = w_\zeta \Delta_\zeta^w w_\zeta^* = \Delta^{-\zeta} \). Thus

\[
u_\zeta w_\zeta v_k w_\zeta^* u_\zeta^* = \nu_\zeta \mathcal{J} \beta_\zeta(v_k) \mathcal{J} u_\zeta^* = \mathcal{J} \beta_\zeta(\nu_\zeta v_k u_\zeta^*) \mathcal{J} = \mathcal{J} \beta_\zeta(\beta_{g_0}(\beta_\zeta(\tilde{v}_k))) \mathcal{J} = \mathcal{J} \beta_{g_0}(\tilde{v}_k) \mathcal{J}
\]

We also compute that

\[
u_\zeta w_\zeta v_k w_\zeta^* u_\zeta^* = \nu_\zeta \mathcal{J} \beta_\zeta(v_k) \mathcal{J} u_\zeta^* = \mathcal{J} \beta_\zeta(\nu_\zeta v_k u_\zeta^*) \mathcal{J} = \mathcal{J} \beta_\zeta(\beta_{g_0}(\beta_\zeta(\tilde{v}_k))) \mathcal{J} = \mathcal{J} \beta_{g_0}(\tilde{v}_k) \mathcal{J}
\]

Thus taking \( v_{g_0} = w_\zeta u_\zeta \), as \( g_0^2 = 1 \) we also have

\[
u_{g_0} \beta_{g_0}(v_k) v_{g_0}^* = \mathcal{J} \tilde{v}_k \mathcal{J}
\]

for all \( 1 \leq k \leq d \) where \( v_{g_0} \) is a unitary operator commuting with \( \Delta_\zeta^w \) and \( \mathcal{J} \). Now we check the following identities:

\[
u_{g_0} \beta_{g_0}(\tilde{v}_k) v_{g_0}^* \Omega = \nu_{g_0} \beta_{g_0}(v_k) v_{g_0}^* \Omega \quad \text{(since } \tilde{v}_k \Omega = v_k^* \Omega)\]

\[
= \mathcal{J} \tilde{v}_k \mathcal{J} \Omega \quad \text{(by the relation (41))}
\]

Since \( v_{g_0}(\mathcal{M}) v_{g_0}^* = \mathcal{M} \), we get \( v_{g_0} \mathcal{M} v_{g_0}^* = \mathcal{M} \) and thus separating property of \( \Omega \) for \( \mathcal{M} \) and the above identities say that

\[
u_{g_0} \beta_{g_0}(\tilde{v}_k) v_{g_0}^* = \mathcal{J} \tilde{v}_k \mathcal{J}
\]

for all \( 1 \leq k \leq d \).
Unlike the twist free case, self-adjoint property of \( v_{g_0} \) is not guaranteed in general. In fact we get from the following computation

\[
v_{g_0}^2 v_k (v_{g_0}^*)^2 = v_{g_0} J \beta_{g_0} (v_k) J v_{g_0}^* = J \beta_{g_0} (J \beta_{g_0} (v_k)) J = \beta_{g_0 g_0} (v_k)
\]

Thus \( v_{g_0} \) is self-adjoint if and only if \( g_0 = g_0 \) as \( g_0^2 = 1 \).

Nevertheless we have \( \beta_z (v_{g_0}) = v_{g_0} \) for all \( z \in H \). For a proof we can follow the same steps that we did for free case since \( \beta_{g_0} (P_k) = P_k \) for each \( k \in \hat{H} \). For the sake of completeness in the following we give details. For simplicity of notation we set \( \beta_z (v) = v \) for all \( z \in H \) is equivalent to the property that \( v \) keeps each subspace \( P_k \) invariant where \( u_z = \sum_{k \in \hat{H}} z^k P_k \) is the spectral decomposition of unitary representation \( z \rightarrow u_z \) and \( \pi (\beta_z (x)) = u_z \pi (x) u_z^* \) for \( x \in \mathcal{O}_d \) in the GNS space \((\mathcal{H}, \pi, \Omega)\) of the state \( \psi \). Since \( v \beta_{g_0} (v_k) v^* = J \tilde{v}_I \tilde{v}_J^* J \) and \( v \) is self-adjoint and \( v \Omega = \Omega \) we get \( v \beta_{g_0} (v_k) \Omega = J \tilde{v}_I \tilde{v}_J^* \Omega \). \( \beta_z \) being an automorphism on \( \mathcal{M} \) preserving the state \( \phi \), modular elements \( J, \Delta^+ \) commutes with \( u_z \) and in particular \( J \) commutes with \( P_k \) for all \( k \in \hat{H} \). Since \( [v \beta_{g_0} (v_k) \Omega : |I| - |J| = k] = P_k \) and \( [v \beta_{g_0} (v_k) \Omega : |I| - |J| = k] = P_k \) and \( \beta_{g_0} (P_k) = P_k \) we get \( (I - P_k) v P_k = 0 \) since \( J \) commutes with \( P_k \). We can interchange the role of \( (v_k) \) and \( (\tilde{v}_I \tilde{v}_J^* \Omega) \) to conclude that \( (I - P_k) v P_k = 0 \) for all \( k \in \hat{H} \). This shows that \( v \) commutes with \( u_z \) for all \( z \in H \) i.e. \( \beta_z (v) = v \).

Since \( \mathcal{M} \vee \hat{\mathcal{M}} = \mathcal{B} (\mathcal{K}) \) by Proposition 3.1 (c), \( v_{g_0}^* \beta_z (v_{g_0}) \) is a scalar multiple of the identity operator. By our construction we have \( v_0 \Omega = \Omega \) and \( \beta_z (v_0) \Omega = u_z v_0 u_z^* \Omega = \Omega \) as \( u_z \Omega = \Omega \). So we conclude that \( \beta_z (v_{g_0}) = v_{g_0} \) for any \( z \in H \).

For \( z \in H \), taking action \( \beta_z \) on both sides of (41), we get \( z v_{g_0} v_k v_{g_0}^* = z J \beta_{g_0} (v_k) J \) i.e. \( z^2 = 1 \). Thus \( H \subseteq \{1, -1\} \).

We set an anti-linear \(^*\)-automorphism \( J_{g_0} : \mathcal{O}_d \otimes \hat{\mathcal{O}}_d \rightarrow \mathcal{O}_d \otimes \hat{\mathcal{O}}_d \) defined by

\[
J_{g_0} (s_I s_J^* \otimes \tilde{s}_I \tilde{s}_J^*) = \beta_{g_0} (s_I s_J^* \otimes \beta_{g_0} (\tilde{s}_I \tilde{s}_J^*)
\]

for \( |I|, |J|, |I'|, |J'| < \infty \) by extending anti-linearly.

We say a state \( \psi \) on \( \mathcal{O}_d \otimes \hat{\mathcal{O}}_d \) is reflection positive with twist \( g_0 \) if \( \psi (J_{g_0} (x)) \geq 0 \) for all \( x \in \mathcal{O}_d \) and equality holds if and only if \( x = 0 \). Similarly a state \( \omega \) on \( \mathcal{B} \) is called reflection positivity with twist \( g_0 \) if

\[
\omega (\beta_{g_0} (Q) Q) \geq 0
\]

for all \( Q \in \mathcal{B}_R \). Note that this notion extended to \( \hat{\mathcal{O}}_d \otimes \mathcal{O}_d \) is an abstract version of the concept “reflection positivity with twist \( g_0 \)” of a state on \( \mathcal{B} \) introduced in [FILS] for any involution (linear or conjugate linear) taking element from future algebra to past algebra. Such an involution are included within the abstract framework of positive reflection symmetric with twist introduced in [FILS].

The hidden symmetry \( v_{g_0} \), described in (42) will play an important role to determine properties of \( \omega \).

**Theorem 3.5.** Let \( \omega \) be a translation invariant, reflection symmetric with twist \( g_0 \), pure state on \( \mathcal{B} \) and \( \psi \) be an extremal point \( K_\omega \) and \( \pi \) as described as in Theorem 3.4. Then the following statements are true:
(a) $\psi$ is reflection positive with twist $g_0$ on $\pi(\tilde{O}_d \otimes O_d)$ if and only if $v_{g_0}$ in (42) is equal to 1 i.e. we have
\[ J \tilde{v}_k J = \beta_{g_0}(v_k) \]
for all $1 \leq k \leq d$.
(b) $\omega$ is reflection positive with twist on $B$ if and only if
\[ J \tilde{v}_1 \tilde{v}_k^* J = \beta_{g_0}(v_1 v_k^*) \]
for all $|l| = |p| < \infty$. In such a case $v_{g_0}$ commutes with $P_0$ and $v_{g_0} P_0 = P_0$ and $\tilde{\tau}(y) = J \tau(J y J) J$ for $y \in M_0 \subseteq B(\mathcal{K}_0)$ where we recall $P_0 = [M_0 \Omega]$ and $\mathcal{K}_0$ is the Hilbert subspace of $P_0$.
(c) $\Delta = I$ if and only if
\[ v_{g_0} \beta_{g_0}(v_k) v_{g_0}^* = v_k^*, \ 1 \leq k \leq d \]
In such a case $H$ is trivial and $\mathcal{M}$ is finite type-I and spatial correlation functions of $\omega$ decays exponentially. Further if $\omega$ is reflection positive with twist $g_0$, then $v_{g_0} = 1$.

Proof. We recall from Theorem 3.6 in [Mo5] that $P \sigma(\mathcal{O}_d)^\sigma P = \mathcal{M}$ and $P \sigma(\tilde{O}_d)^\sigma P = \mathcal{M} \subseteq \mathcal{M}'$ (we do not need equality here) and $P = E \tilde{E}$. Thus for any $x \in \mathcal{O}_d$ we may write
\[
\psi(J_{g_0}(x) x) = < \Omega, \pi(J_{g_0}(x)) \pi(x) \Omega > = < \Omega, P \pi(J_{g_0}(x)) P \pi(x) P \Omega > = < \Omega, J_{g_0} P \pi(x) P J_{g_0} P \pi(x) P \Omega >
\]
where we have used equality $\pi(J_{g_0}(x)) = J_{g_0} \pi(x) J_{g_0}$ from Theorem 3.4. If $v_{g_0} = 1$ i.e. $J_{g_0} = v_{g_0} J = J$ on $\mathcal{K}$ and thus we have $< \Omega, J \pi(x) P J \pi(x) P \Omega > \geq 0$ by the self-dual property of Tomita’s positive cone $[\mathcal{J}a\mathcal{J}a : a \in \mathcal{M}]$ [BR1]. Thus $\psi$ is a reflection positive map on $\pi(\tilde{O}_d \otimes O_d)$.

Conversely if $\psi$ is reflection positive on $\pi(\tilde{O}_d \otimes O_d)^\sigma$ we have $< \Omega, a J_{g_0} a J_{g_0} \Omega > \geq 0$ where $a \in \mathcal{M} = P \sigma(\mathcal{O}_d)^\sigma P$. Since $v_{g_0}$ commutes with $J$ and $\Delta^\perp$ we may rewrite $< \Omega, a v_{g_0} J a \Omega > = < a^* \Omega, v_{g_0} \Delta^\perp a^* \Omega > \geq 0$ i.e. $v_{g_0} \Delta^\perp$ is a non-negative operator. Since $\Delta^\perp$ is also a non-negative operator commuting with $v_{g_0} \Delta^\perp$, we conclude that $v_{g_0}$ is a non-negative operator. $v_{g_0}$ being unitary we conclude that $v_{g_0} = 1$.

Proof for (b) follows the same route that of (a) replacing the role of $\mathcal{M}$ and $\tilde{\mathcal{M}}$ by $\mathcal{M}_0$ and $\tilde{\mathcal{M}}_0$ respectively with state $\omega = \psi \mid B$.

We will deal with the non-trivial part of (c). Assume $v_{g_0} \beta_{g_0}(v_k) v_{g_0}^* = v_k^*$ for all $1 \leq k \leq d$. So $\Delta$ is affiliated to $\mathcal{M}'$. As $J \Delta^\perp J = \Delta^\perp$, $\Delta$ is also affiliated to $\mathcal{M}$. Hence $\Delta = I$ as $\mathcal{M}$ is a factor and $\Delta \Omega = \Omega$.

In general $\omega$ being a pure state $\mathcal{M}$ is either a type-I or type-III factor [Mo3, Theorem 3.4]. Thus we conclude that $\mathcal{M}$ is a finite type-I factor if $\Delta = 1$ (i.e. $\phi$ is a tracial state on $\mathcal{M}$). This completes the proof of the first part of (c).

The last part of (c) is rather elementary. We note that purity of $\omega$ ensures that the point spectrum of the self-adjoint contractive operator $T$, defined by $T x \Omega = \tau(x) \Omega$ on the KMS Hilbert space, in the unit circle is trivial i.e. $\{ z \in S^1 : T f = z f, \text{ for some non zero } f \in \mathcal{K} \}$ is the trivial set $\{ 1 \}$ (as a consequence of strong mixing property). Thus $T$ being a contractive matrix on a finite dimensional
Hilbert space, the spectral radius of \( T \) is equal to \( \alpha \) for some \( \alpha < 1 \). Now we use Proposition 3.1 for any \( X_t \in \mathcal{B}_L \) and \( X_r \in \mathcal{B}_R \) to verify the following

\[
e^{\delta k} |\omega(X_t\phi(X_r)) - \omega(X_t)\omega(X_r)| = e^{\delta k} |\phi(Jx_tJ\tau_k(x_r)) - \phi(x_t)\phi(x_r)| \to 0
\]
as \( k \to \infty \) for any \( \delta > 0 \) so that \( e^\delta \alpha < 1 \) where \( Jx_tJ = PX_tP \) and \( x_r = PX_rP \) for some \( x_t, x_r \in \mathcal{M} \). As \( \alpha < 1 \) such a \( \delta > 0 \) exists. This completes the proof for (c) as last statement follows from (b).

4. Translation invariant twisted reflection positive pure state and its split property:

Let \( \omega \) be a translation invariant real lattice symmetric with a twist \( g_0 \) pure state on \( \mathcal{B} \). We fix an extremal element \( \psi \in K_\omega \) so that \( \psi = \psi^{g_0} = \psi^{\beta_{\tilde{g}}} \) and consider the Popescu elements \( \{K, M, v_\lambda, \Omega\} \) as in Theorem 3.5. \( P \) being the support projection of a factor state \( \psi \) we have \( \mathcal{M} = P\pi(\Omega)''P = \{v_\delta, v_\delta': 1 \leq k \leq d\}'' \) (Proposition 2.4 in [Mo5]). So the dual Popescu elements \( \{K, M', v_\delta, 1 \leq k \leq d, \Omega\} \) satisfy the relation \( v_\delta v_\delta' v_\delta v_\delta' = J\beta_{\tilde{g}}(v_\delta)J \) (recall that the factor \( \zeta \) won’t show up as two symmetry will kill each other as given in Theorem 3.4) for all \( 1 \leq k \leq d \).

We quickly recall as \( \mathcal{M}_0 \) is the \( \{\beta_z : z \in H\} \) invariant elements of \( \mathcal{M} = P\pi(\Omega)''P \), the normal conditional expectation \( x \to \int_E H \beta_z(x)dz \) from \( \mathcal{M} \) onto \( \mathcal{M}_0 \) preserves the faithful normal state \( \phi \). So by Takesaki’s theorem [Ta] modular group associated with \( \phi \) preserves \( \mathcal{M}_0 \). Further since \( \beta_z(\pi(x)) = \pi(\beta_z(x)) \) for all \( x \in \mathcal{M} \), the restriction of the completely positive map \( \tau(x) = \sum_k v_kx_k^* \) to \( \mathcal{M}_0 \) is a well defined map on \( \mathcal{M}_0 \). Hence the completely positive map \( \tau(x) = \sum_k v_kx_k^* \) on \( \mathcal{M}_0 \) is also KMS symmetric modulo a unitary conjugation by \( v \) i.e.

\[
\langle\langle x, \tau(y) \rangle\rangle = \langle\langle \tau_v(x), y \rangle\rangle
\]
where \( x, y \in \mathcal{M}_0 \) and \( \langle\langle x, y \rangle\rangle = \phi(x^*\sigma_{\tilde{g}}(y)) \) and \( (\sigma_t) \) is the modular automorphism group on \( \mathcal{M}_0 \) associated with \( \phi \) and \( [\mathcal{M}_0\Omega] = P_0 \) and \( \tau_v(x) = v^*\tau(x)v^*v \) for all \( x \in \mathcal{M}_0 \). Thus \( \tau_v = \tau \) if and only if \( \omega \) is reflection positive on \( \mathcal{B} \) with twist \( g_0 \) (Theorem 3.5).

However the inclusion \( \mathcal{M}_0 \subseteq \mathcal{M} \) need not be an equality in general unless \( H \) is trivial. The unique ground state of \( XY \) model in absence of magnetic field give rise to a non-split translation invariant real lattice symmetric pure state \( \omega \) and further \( H = \{1, -1\} \) [Mo6].

We now fix a translation invariant real lattice symmetric pure state \( \omega \) which is also reflection positive with a twist \( g_0 \) on \( \mathcal{B} \) and explore KMS-symmetric property of \( (\mathcal{M}_0, \tau, \phi) \) and the extended Tomita’s conjugation operator \( \mathcal{J}_{g_0} \) on \( \mathcal{H} \otimes \mathcal{K} \mathcal{H} \) defined in Theorem 3.5 to study the relation between split property and exponential decaying property of spatial correlation functions of \( \omega \).

For any fix \( n \geq 1 \) let \( Q \in \pi(\mathcal{B}_{[-k+1, k]}). \) We write

\[
Q = \sum_{|I|=|J|=|I'|=|J'|=n} q(I', J'|I, J)\beta_{g_0}(\tilde{S}_{I'}, \tilde{S}_{J'})S_IS_J^*
\]
and \( q \) be the matrix \( q = ((q(I', J'|I, J))) \) of order \( d^{2n} \times d^{2n} \).

**Proposition 4.1.** The matrix norm of \( q \) is equal to operator norm of \( Q \) in \( \pi(\mathcal{B}_{[-n+1, n]}) \).
and set a notation for simplicity as \( EEXY \). Then 
\[
\hat{P}XYP
\]
use (a) and (c). This completes the proof.

\( \Box \)

Proposition 4.2. Let \( \omega \) be a translation invariant real lattice symmetric with twist \( g_0 \) pure state on \( \otimes_2 \mathcal{M}_d \). Then there exists an extremal point \( \psi \in K_\omega \) so that \( \psi_\beta = \hat{\psi} \beta = \hat{\psi} \) where \( \beta \in \{1, e^{\pi i} \} \) and the associated Popescu systems \( (\mathcal{H}, \mathcal{S}_k, 1 \leq k \leq d(\Omega)) \) and \( (\mathcal{H}, \mathcal{S}_k, 1 \leq k \leq d(\Omega)) \) described in Proposition 3.1 satisfies the following:

(a) For any \( n \geq 1 \) and \( Q \in \pi(B_{[-n+1,n]}) \) we write
\[
Q = \sum_{|I'|=|J'|=|I|=|J|=n} q(I', J'|I, J) \beta_{g_0}(\hat{S}_I^w \hat{S}_J^w) S_I^* S_J
\]
and set a notation for simplicity as
\[
\hat{\theta}_{2k}(Q) = \sum_{|I'|=|J'|=|I|=|J|=n} q(I', J'|I, J) \beta_{g_0}(\hat{A}^k(\hat{S}_I^w \hat{S}_J^w)) \Lambda^k(S_I S_J^*).
\]

Then \( \hat{\theta}_{2k}(Q) \in B_{[-\infty,-b]} \cup [b, \infty] \).
(b) \( Q = \mathcal{J}_{g_0} Q \mathcal{J}_{g_0} \) if and only if \( q(I', J'|I, J) = \overline{q(I, J'I')} \);
(c) If the matrix \( q = ((q(I', J'|I, J))) \) is non-negative then there exists a matrix \( b = (b(I', J'|I, J)) \) so that \( q = b^* b \) and then
\[
q = P Q P = \sum_{|K|=|K'|} \mathcal{J}_e x_{K,K'} \mathcal{J}_e x_{K,K'}
\]
where \( x_{K,K'} = \sum_{I,J : |I|=|J|=n} b(K, K'|I, J) v_I^* v_J \in \mathcal{M}_0 \).

(d) In such a case i.e. if \( Q = \mathcal{J}_{g_0} Q \mathcal{J}_{g_0} \) the following holds:

(i) \( \omega(Q) = \sum_{K,K' : |K|=|K'|} \phi(\mathcal{J}_e x_{K,K'} \mathcal{J}_e x_{K,K'}) \)

(ii) \( \omega(\hat{\theta}_{2k}(Q)) = \sum_{K,K' : |K|=|K'|} \phi(\mathcal{J}_e x_{K,K'} \mathcal{J}_e \tau_{2k}(x_{K,K'})) \).

Proof. Since the elements \( \beta_{g_0}(\hat{S}_I^w \hat{S}_J^w) S_I^* S_J : |I| = |J| = |I'| = |J'| = n \) form an linear independent basis for \( \pi(B_{[-n+1,n]}) \), (a) follows. (b) is also a simple consequence of linear independence of the basis elements and the relation \( \mathcal{J}_{g_0} \beta_{g_0}(\hat{S}_I^w \hat{S}_J^w) S_I S_J^* \mathcal{J}_{g_0} = S_I S_J^* \beta_{g_0}(\hat{S}_I^w \hat{S}_J^w) \) as described in the paragraph followed after Theorem 3.5.

For (c) we write
\[
Q = \sum_{|K|} \mathcal{J}_{g_0}(Q_{K,K'}) \mathcal{J}_{g_0} Q_{K,K'}
\]
where \( Q_{K,K'} = \sum_{I,J : |I|=|J|=n} b(K, K'|I, J) S_I S_J^* \). \( \omega \) being pure we have by Theorem 3.6 in [Mo5] that \( P = E \bar{E} \) where \( E \) and \( \bar{E} \) are support projection of \( \psi \) in \( \pi(O_d)^n \) and \( \pi(\bar{O}_d)^n \) respectively. So for any \( X \in \pi(O_d)^n \) and \( Y \in \pi(\bar{O}_d)^n \) we have
\[
P X Y P = E X Y \bar{E} = E \bar{E} E \bar{E} X E = P X Y P.
\]
Thus (c) follows as \( \omega(Q) = \phi(q) \) by Theorem 3.5 as \( \omega \) is reflection symmetry with twist \( g_0 \). For (d) we use (a) and (c). This completes the proof. \( \Box \)
Proposition 4.3. Let \( \omega \), a translation invariant pure state on \( \mathcal{B} \), be in detailed balance and reflection positive with a twist \( g_0 \). Then the following are equivalent:

(a) \( \omega \) is decaying exponentially.

(b) The spectrum of \( T - |\Omega| < |\Omega| \) is a subset of \([-\alpha, \alpha]\) for some \( 0 \leq \alpha < 1 \) where \( T \) is the self-adjoint contractive operator defined by

\[
T \xi = \tau(x) \Omega, \quad x \in \mathcal{M}_0
\]
on the KMS-Hilbert space \(<< x,y >> = \phi(x^* \sigma_\lambda(y)>>.

Proof. Since \( T^k \xi = \tau(x) \Omega \) for \( x \in \mathcal{M}_0 \) and for any \( L \in \mathcal{B}_L \) and \( R \in \mathcal{B}_R \) we have \( \omega(L \delta_0(R)) = \phi(J \gamma \tau_k(x)) = < < y, T^k x >> \) where \( x = P \pi(R) \) and \( y = J P \pi(L) P J \) are elements in \( \mathcal{M}_0 \). Since \( P \pi(B_R) \by P = \mathcal{M}_0 \) and \( P \pi(B_L) \by P = \mathcal{M}_0 \) as \( \mathcal{M} = \mathcal{M}' \) by Theorem 3.4, we conclude that (a) holds if and only if \( e^{k \delta} < f, T^k g > < f, \Omega, g > | \to 0 \) as \( k \to \infty \) for any vectors \( f, g \) in a dense subset \( \mathcal{D} \) of the KMS Hilbert space.

That (b) implies (a) is now obvious since \( e^{k \delta} \alpha = (e^{\delta} \alpha)^k \to 0 \) whenever we choose \( \alpha > 0 \) so that \( e^{\delta} \alpha < 1 \) where \( \alpha < 1 \).

For the converse suppose that (a) holds and \( T^2 - |\Omega| < |\Omega| \) is not bounded away from 1. Since \( T^2 - |\Omega| < |\Omega| \) is a positive self-adjoint contractive operator, for each \( n \geq 1 \), we find a unit vector \( f_n \) in the Hilbert space so that \( E_{[1/2,1]} f_n = f_n \) and \( f_n \in \mathcal{D} \), where \( s \to E_{[s,1]} \) is the spectral family of the positive self-adjoint operator \( T^2 - |\Omega| < |\Omega| \) and in order to ensure \( f_n \in \mathcal{D} \) we also note that \( E_{[s,1]} \mathcal{D} = \{ E_{s,1} f : f \in \mathcal{D} \} \) is dense in \( E_{[s,1]} \) for any \( 0 < s < 1 \).

Thus by exponential decay there exists a \( \delta > 0 \) so that

\[
e^{2k \delta}(1 - \frac{1}{n})^k \leq e^{2k \delta} \int_{[0,1]} k^k < f_n, dE_s f_n > \leq e^{2k \delta} < f_n, [T^{2k} - |\Omega| < |\Omega|] f_n > \to 0
\]
as \( k \to \infty \) for each \( n \geq 1 \). Hence \( e^{2\delta}(1 - \frac{1}{n}) < 1 \). Since \( n \) is any integer, we have \( e^{2\delta} \leq 1 \). This contradicts that \( \delta > 0 \). This completes the proof.

For an element \( Q \in \pi(B_{loc}) \), \( Q = \frac{1}{2} (Q + J \gamma_0(Q)) + \frac{1}{2} (Q - J \gamma_0(Q)) \) is a sum of an even element in \( \{ Q : J \gamma_0(Q) = Q \} \) and an odd element in \( \{ Q : J \gamma_0(Q) = -Q \} \). Moreover \( iQ \) is an even element if \( Q \) is an odd element. Also note that \( ||Q_{even}|| \leq ||Q|| \) and \( ||Q_{odd}|| \leq ||Q|| \). Hence it is enough if we verify (1) for all even elements for split property. We fix any \( n \geq 1 \) and an even element \( Q \in \mathcal{B}_{[-n,n]} \). We write as in Proposition 4.2

\[
Q = \sum_{|I'|=|J'|=|I|=|J|=n} q(J', I', J, J, J) \beta_{0\otimes J} S_{I', J} S_{J}.
\]

The matrix \( q = (q(J', I', J, J)) \) is symmetric and thus \( q = q_+ - q_- \) where \( q_+ \) and \( q_- \) are the unique non-negative matrix contributing its positive and negative parts of \( q \). Hence \( ||q_+|| \leq ||q|| \) and \( ||q_-|| \leq ||q|| \). We set a notation for simplicity that

\[
\hat{\theta}_k(Q) = \sum_{|I|=|J|=|I'|=|J'|=n} q(J', I', J, J) \beta_{0\otimes J} (S_{I'} S_{J})
\]

which is an element in \( \mathcal{B}_{[-\infty,-k]} \bigcup \{ k, \infty \} \) and by Proposition 4.2 (d)

\[
\omega(\hat{\theta}_k(Q)) = \sum_{|K|=|K'|=n} \phi(J x_{K,K'} J_x_{k,k'})
\]

provided \( q = (q(J', I', J, J)) \) is positive, where \( P Q P = \sum_{|K|=|K'|=n} J x_{K,K'} J_x_{K,K'} \) and \( x_{K,K'} = \sum_{I,J} b(K, K' | I, J) v_I v_J \) and \( q = b^* b \). Thus in such a case we have by
Proposition 4.2 (d) that
\[
|\omega(\hat{\theta}_k(Q)) - \omega_L \otimes \omega_R(\hat{\theta}_k(Q))| = \sum_{|K|=|K'|=n} \phi(Jx_{K,K'}\mathcal{J}(\tau_k - \phi)(x_{K,K'}))
\]
\[
= \sum_{|K|=|K'|=n} <\langle x_{K,K'},(T - |\Omega|<\Omega)\rangle^{2k}x_{K,K'}>
\]
\[
\leq \alpha^{2k} \sum_{|K|=|K'|=n} <\langle x_{K,K'},x_{K,K'}\rangle>
\]
provided \(|T - |\Omega|<\Omega|| \leq \alpha\) and so
\[
\leq \alpha^{2k}\omega(Q) \leq \alpha^{2k}||\hat{q}|| = \alpha^{2k}||q||
\]
In the last identity we have used Proposition 4.1.

Hence for an arbitrary \(Q\) for which \(\mathcal{J}_{g_0}(Q) = Q\) we have
\[
|\omega(\hat{\theta}_k(Q)) - \omega_L \otimes \omega_R(\hat{\theta}_k(Q))| \leq \alpha^{2k}||q_+|| + ||q_-|| \leq 2\alpha^{2k}||q|| = 2\alpha^{2k}||Q||
\]
where in the last identity we have used once more Proposition 4.1. Thus we have arrived at our main result by a well known criterion [BR -vol II] on split property.

**Theorem 4.4.** Let \(\omega\) be a translation invariant pure state. Let \(\omega\) be also real (with respect to a basis for \(C^d\)) and lattice symmetric with twist \(g_0\). If \(\omega\) is reflection positive with twist \(a\) \(g_0\) (\(g_0 \in U_d(C), g_0^2 = 1\)) and the spatial correlation function of \(\omega\) decays exponentially then \(\omega\) is split i.e. \(\pi(\omega(BR))''\) is a type-I factor.

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