On completely regular codes of covering radius 1 in the halved hypercubes

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Abstract

We consider constructions of covering-radius-1 completely regular codes, or, equivalently, equitable 2-partitions, of halved n-cubes.

Keywords: completely regular code, equitable partition, regular partition, partition design, perfect coloring, halved hypercube.

1. Definitions.

An equitable k-partition of the graph is an ordered partition \( C = (C_0, C_1, \ldots, C_{k-1}) \) of the vertex set such that the number of neighbors in \( C_j \) of a vertex from \( C_i \) is some constant \( S_{ij} \), \( i, j = 0, \ldots, k-1 \); the matrix \( S = \{S_{ij}\}_{i,j \in \{0, \ldots, k-1\}} \) is called the quotient matrix (sometimes, the intersection matrix, or the parameter matrix). The elements of the partition \( (C_0, C_1, \ldots, C_{k-1}) \) are called cells.

We will write \( C(\vec{v}) = i \) if \( \vec{v} \in C_i \) and identify a k-partition with the corresponding k-valued function on the vertices.

A set \( C \) (code) of vertices of a graph is called completely regular if the partition \( (C^{(0)}, C^{(1)}, \ldots, C^{(\rho)}) \) is equitable, where \( C^{(d)} \) is the set of vertices at distance \( d \) from \( C \) and \( \rho \) (the covering radius of \( C \)) is the maximum \( d \) for which \( C^{(d)} \) is nonempty.

A connected graph is called distance regular if every singleton (one-vertex set) is completely regular with the same quotient matrix.

The n-cube (or n-dimensional hypercube) \( Q_n \) is the graph on the binary words of length \( n \) (also treated as vectors over the binary field \( GF(2) \)) where two words are adjacent iff they differ in exactly one position.

The halved n-cube \( \frac{1}{2}Q_n \) (of \( \frac{1}{2}Q_n \)) is the graph on the binary words of length \( n \) with even number of ones (or odd for \( \frac{1}{2}Q_n^* \); the resulting graph is the same up to isomorphism) where two words \( \vec{x}, \vec{y} \) are adjacent (we write \( \vec{x} \sim \vec{y} \)) iff they differ in exactly two positions.

An s-face of the halved n-cube is the set of \( 2^{n-s} \) vertices that have the same values in some \( n-s \) “fixed” coordinates. The remaining coordinates are called free for the s-face.

The eigenvalues of the halved n-cube are denoted \( \theta_0(n), \ldots, \theta_{\lfloor n/2 \rfloor}(n) \), in the decreasing order, \( \theta_i(n) = ((n-2i)^2 - n)/2 \).

In a graph, a set of mutually adjacent vertices is called a clique. In a distance-regular graph with the minimum eigenvalue \( \theta_{\min} \), a clique of size \( 1 - k/\theta_{\min} \) is called a Delsarte clique. In \( \frac{1}{2}Q_n \) with even \( n \), any Delsarte clique have size \( n \). For odd \( n \), \( \frac{1}{2}Q_n \) has no Delsarte cliques.

2. Necessary conditions.

In this section, we collect the necessary conditions for a two-by-two matrix to be a quotient matrix \( S = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) of an equitable 2-partition \( C = (C_0, C_1) \).

1. Firstly, all elements are nonnegative integers, \( b \) and \( c \) are positive, \( a + b \) and \( c + d \) equals the degree of the graph (for \( \frac{1}{2}Q_n \), the degree is \( n(n-1)/2 \)).

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2. Next, by counting the edges between \(C_0\) and \(C_1\), we find \(|C_0|/|C_1| = b/c\). It follows that \(\frac{b+c}{\gcd(b,c)}\) divides the number of vertices of the graph. For \(\frac{1}{2}Q_n\), the number of vertices is \(2^{n-1}\); so, \(\frac{b+c}{\gcd(b,c)}\) is a power of 2.

3. Next, the eigenvalues of the quotient matrix are eigenvalues of the graph. For \(\frac{1}{2}Q_n\), this means that \(a - c = d - b = ((n-2)^2 - n)/2, i \in \{0, \ldots, [n/2]\}\).

4. Given a completely regular code \(D\), if we know the numbers \(|C_i \cap D|\) for all \(i \in \{0, \ldots, k-1\}\), then we can calculate \(|C_i \cap D^{(d)}|\) for any \(i \in \{0, \ldots, k-1\}\) and \(d \in \{0, \ldots, \rho\}\) (see e.g. [2]) via the quotient matrices of \(C\) and \(D\). The collection \(|C_i \cap D^{(d)}| : i \in \{0, \ldots, k-1\}, d \in \{0, \ldots, \rho\}\) is called the weight distribution of the partition \(C\) with respect to \(D\). An equitable partition \(C\) necessarily possesses at least one weight distribution with nonnegative integer elements. More on calculation of weight distributions of equitable partitions in distance regular graphs could be found in [10]. The most important case is a distance regular graph when \(D\) is a singleton. For an equitable partition with the quotient matrix \(S\) in \(\frac{1}{2}Q_n\), we can conclude that all the matrices \(S^{(0)} = \text{Id}, S^{(2)} = S, S^{(4)}, \ldots, S^{(2^{[n/2]})}\) are related by the recursive identity

\[
S \cdot S^{(i)} = \left(\frac{n-i+2}{2}\right)S^{(i-2)} + i(n-i)S^{(i)} + \left(\frac{i+2}{2}\right)S^{(i+2)}
\]

(this formula is a partial case of [10] (4)) and must consist of nonnegative integers.

**Example 1.** Consider \(n = 12\), \(S = \begin{bmatrix} 4 & 62 \\ 2 & 64 \end{bmatrix}\). We find \(S^{(4)} = S^{(8)} = \begin{bmatrix} -1 & 496 \\ 16 & 479 \end{bmatrix}\). Hence, there is no equitable partition of \(\frac{1}{2}Q_{12}\) with the quotient matrix \(S\).

3. Special cases.

3.1. Odd \(n\).

**Theorem 1.** Let \(C = (C_0, C_1, \ldots, C_{k-1})\) be an equitable \(k\)-partition of \(\frac{1}{2}Q_n\), where \(n\) is odd. Let \(C'_i\) be \(C_i \cup (C_i + 1)\). Then \(C' = (C'_0, \ldots, C'_{k-1})\) is an equitable \(k\)-partition of \(Q_n\).

**Proof.** Consider a vertex \(\bar{x} \in C_i + 1\) for some \(i\) (the case \(\bar{x} \in C_i\) is similar). Its neighbors are exactly the vertices at distance \(n-1\) from \(\bar{x} + 1\) in the graph \(Q_n\); at the same time, the vertices at distance \((n-1)/2\) from \(\bar{x} + 1\) in the graph \(\frac{1}{2}Q_n\). Since \(C\) is an equitable partition and the graph \(\frac{1}{2}Q_n\) is distance regular, the number of such vertices in \(C_j\) for each \(j\) is a constant regardless of choice of \(\bar{x}\) in \(C_i + 1\), see e.g. [11][10]. By the definition, we have an equitable partition of \(Q_n\). \(\blacksquare\)

So, the existence of equitable partitions of the halved \(n\)-cube, where \(n\) is odd \(n\), with given putative quotient matrix \(S\) is equivalent to the existence of equitable partitions of the \(n\)-cube with some quotient matrix, uniquely defined from \(S\) (for the case of 2-paritions, this matrix can be easily found from the eigenvalue and the proportion between the partition cells). We see that in this case, odd \(n\), there is no sense to consider the problem of existence separately for the halved \(n\)-cube.

3.2. Even \(n\), minimum eigenvalue.

**Theorem 2.** A partition \((C_0, C_1)\) of \(\frac{1}{2}Q_n\), \(n\) even, is equitable with the quotient matrix \(S\) and the eigenvalue \(\theta_{n/2}(n)\) if and only if \((C'_0, C'_1)\) is an equitable partition of \(Q_{n-1}\) with the quotient matrix

\[
S' = \begin{bmatrix} c - 1 & c \\ c & n - c \end{bmatrix}
\]

for some \(c\) and \(S = (S'^2 - \text{Id})/2 + S'\).

**Proof.** Consider the graph \(Q_{n-1}\) and the graph \(Q_{n-1}^+\) with the same vertex set, where two vertices are adjacent iff the distance between them is 1 or 2 in \(Q_{n-1}\). The adjacency matrices \(A\) and \(A^+\) of \(Q_{n-1}\) and \(Q_{n-1}^+\) are related as follows

\[
A^+ = \frac{A^2 + (n-1)\text{Id}}{2} + A = \frac{(A + \text{Id})^2 + (n-2)\text{Id}}{2}
\]
It is straightforward that an eigenvector of \( A^+ \) corresponding to the minimum eigenvalue is an eigenvector of \( A \) corresponding to the eigenvalue \(-1\) and vice versa. It follows that the equitable 2-partitions of \( Q_{n-1}^- \) and \( Q_{n-1}^+ \) with the eigenvalue \(-1\) and with the minimum eigenvalue, respectively, are also the same.

It remains to note that \( \frac{1}{2}Q_n \) is isomorphic to \( Q_{n-1}^+ \) (an isomorphism is given by removing the last symbol from each vertex word). ▲

So, equitable 2-partitions of \( \frac{1}{2}Q_n \), \( n \) even, with the minimum eigenvalue are in one-to-one correspondence with the equitable 2-partitions of \( Q_{n-1}^\) with the eigenvalue \(-1\), which are constructed in [8] for all admissible parameters.

4. Constructions.

4.1. From equitable partitions of \( Q_n \).

It is known that an equitable partition of an \( n \)-cube restricted to the vertices of \( \frac{1}{2}Q_n \) is also equitable.

In particular, any equitable 2-partition of \( Q_n \) different from the partition into bipartite parts induces an equitable 2-partition of \( \frac{1}{2}Q_n \). The parameters of equitable 2-partition of \( Q_n \) were studied in [2], [3], [5].

Another important case is the even-weight completely regular codes of covering radius \( 3 \) and \( 4 \) in \( Q_n \). In case of radius \( 3 \), the code \( C \) and the set \( C^{(2)} \) form an equitable partition of \( \frac{1}{2}Q_n \). In the case of radius \( 4 \), the sets \( C^{(1)} \) and \( C^{(3)} \) form an equitable partition of \( \frac{1}{2}Q_n \). Below are examples of completely regular codes of covering radius \( 4 \):

- The repetition code \( \{00000000,11111111\} \) in \( Q_8 \).
- The (extended) Preparata codes of minimal distance 6 in \( Q_{2m} \) [4] [12].
- The extended BCH codes of minimum distance 6 in \( Q_{22m+1} \) [4] [12].
- The minimum-distance-6 Hadamard code in \( Q_{12} \) [9].
- The minimum-distance-8 Golay code in \( Q_{24} \).

4.2. Linear partitions.

A 2-partition is called linear if its first cells is a linear code. It is straightforward to show that a linear code induces an equitable partition of \( \frac{1}{2}Q_n \) if and only if it has a parity check matrix of form \( \begin{bmatrix} \bar{1} & B \end{bmatrix} \), where \( \bar{1} \) is the all-one row and the matrix \( B \) is such that every non-zero column of the same weight is represented a constant number of times as the sum of two columns of \( B \) (it is not required that the columns of \( B \) are distinct).

4.3. Union.

**Lemma 1.** If \((C_0, C_1)\) and \((P_0, P_1)\) are equitable 2-partitions of the same regular graph with cospectral quotient matrices \( \begin{bmatrix} a' & b' \\ c' & d' \end{bmatrix} \) and \( \begin{bmatrix} a'' & b'' \\ c'' & d'' \end{bmatrix} \) and \( C_0 \cap P_1 = \emptyset \), then \((C_0 \cup P_0, C_1 \cap P_1)\) is an equitable 2-partition with the quotient matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), \( c = c' + c'' \), \( a = a' + c'' = a'' + c' \), \( b = \ldots \), \( d = \ldots \).

In particular,

**Lemma 2.** If \((C_0, C_1)\) is a linear equitable 2-partition of \( \frac{1}{2}Q_n \) with the quotient matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), then we can unify any number \( t < 2^{n-1}/|C_0| \) of cosets of \( C_0 \) to get an equitable partition with the quotient matrix \( \begin{bmatrix} a + (t-1)c & b - (t-1)c \\ tc & d - (t-1)c \end{bmatrix} \).
Another important case is when we have several even-weight completely regular codes \( C_1, C_2, \ldots, C_l \) of covering radius 4 at distance 4 from each other (we imply that the quotient matrix is the same for all these codes). In this case, \((C_1^{(1)} \cap C_2^{(1)} \cap \ldots \cap C_l^{(1)}, C_1^{(3)} \cup C_2^{(3)} \cup \ldots \cup C_l^{(3)})\) is an equitable partition of \( \frac{1}{2}Q_n \). Examples of such collections of completely regular codes can be constructed based on cosets of distance-6 BCH codes or Preparata codes (it is known that the extended Hamming code in \( Q_m \) can be partitioned into cosets of BCH or Preparata codes \([3] \), depending on the parity of \( m \)). Even more interesting example is the Golay code \( G \), when translations of \( G^{(1)} \) can be combined with cosets of some linear code, providing a large family of quotient matrices \([9] \).

### 4.4. \( \times t \) construction.

This simple construction works for many Cayley graphs. For equitable partitions of hypercube, it was implemented in \([7] \). In the case of the halved hypercube, the construction is the same; the difference is only in calculating parameters.

**Lemma 3 \([7] \).** Let \( C = (C_0, C_1, \ldots, C_{k-1}) \) be an equitable partition of \( Q_n \) with the quotient matrix \( S \). Then the partition \( C(t) \) is an equitable partition of \( \frac{1}{2}Q_{tn} \) with the quotient matrix \( tS \), where

\[
C(t)(\bar{x}_1, \ldots, \bar{x}_t) := C(\bar{x}_1 + \ldots + \bar{x}_t).
\]

Further, if \( \theta_t(n) \) is an eigenvalue of \( S \), then \( \theta_t(tn) \) is an eigenvalue of \( tS \).

**Lemma 4.** Let \( C = (C_0, C_1, \ldots, C_{k-1}) \) be an equitable partition of \( \frac{1}{2}Q_n \) with the quotient matrix \( S \). Then \( C(t) \) is an equitable partition of \( \frac{1}{2}Q_{tn} \) with the quotient matrix \( S(t) \), where

\[
C(t)(\bar{x}_1, \ldots, \bar{x}_t) := C(\bar{x}_1 + \ldots + \bar{x}_t), \quad S(t) = t^2S + n\frac{t(t-1)}{2}Id.
\]

**Proof.** The claim follows from the definitions and the following direct facts.

- Each vertex \( \bar{v} \) of \( \frac{1}{2}Q_n \) corresponds to the set
  \[
t(\bar{v}) := \{(\bar{x}_1, \ldots, \bar{x}_t) \mid \bar{x}_1 + \ldots + \bar{x}_t = \bar{v}\}
\]
  of vertices of \( \frac{1}{2}Q_{tn} \); moreover, for all \( \bar{x} \) in \( t(\bar{v}) \) one has \( f(t)(\bar{x}) = f(\bar{v}) \).

- If \( \bar{v} \sim \bar{v}' \), then every vertex from \( t(\bar{v}) \) has exactly \( t^2 \) neighbors in \( t(\bar{v}') \).

- If \( \bar{v} \not\sim \bar{v}' \), then every vertex from \( t(\bar{v}) \) has no neighbors in \( t(\bar{v}') \).

- Every vertex from \( t(\bar{v}) \) has exactly \( n\frac{t(t-1)}{2} \) neighbors in \( t(\bar{v}) \).

The calculation of eigenvalues is straightforward. \( \blacktriangle \)

### 4.5. Splitting construction.

The construction in this section can be considered as a variant of the main construction in \([3] \). However, in the case of the halved hypercube, it gives an equitable 2-partition only if the parameters are connected by some equation. Assume that we have an equitable 2-partition \((P_0, P_1)\) of \( \frac{1}{2}Q_n \) with the quotient matrix \[
\begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]
Moreover, assume that for some \( s \), the cell \( P_1 \) can be partitioned into \( s \)-faces, let \( F(\bar{v}) \) denote the face from the partition such that \( \bar{v} \in F(\bar{v}) \).

**Lemma 5.** Define

\[
C^{(2)}(\bar{x}, \bar{y}) := C(\bar{x} + \bar{y})
\]

and for \( \bar{x} \in P_1 \), define

\[
\text{sign}(\bar{x}, \bar{y}) := x_1 + \ldots + x_n + b_1y_1 + \ldots + bNy_n,
\]
where \( \bar{b} = (b_1, \ldots, b_n) \) is defined by the directions of the s-face \( F = F(x + y) \): it has 1s in the free coordinates of \( F \) and 0s in the fixed coordinates. Then

\[
C'(\bar{v}) := \begin{cases} 
0 & \text{if } f^{(2)}(\bar{v}) = 0 \\
1 & \text{if } f^{(2)}(\bar{v}) = 1 \text{ and } \text{sign}(\bar{v}) = 0 \\
2 & \text{if } f^{(2)}(\bar{v}) = 1 \text{ and } \text{sign}(\bar{v}) = 1 
\end{cases}
\]

is an equitable partition of \( \frac{1}{2}Q_m \) with the quotient matrix

\[
S' = \begin{bmatrix}
4a + n & 2b \\
4c & 2d + s^2 \quad 2b \\
4c & 2d + n - s^2 \quad 2d + s^2 
\end{bmatrix}
\]

**Proof.** The claim follows from the definitions and the following direct facts.

- Each vertex \( \bar{v} \) of \( \frac{1}{2}Q_n \) corresponds to the set

\[
t(\bar{v}) := \{(x, y) \mid x + y = \bar{v}\}
\]

of vertices of \( \frac{1}{2}Q_{2n} \); moreover, for all \( x \) in \( t(\bar{v}) \) one has \( f^{(1)}(x) = f(\bar{v}) \).

- If \( \bar{v} \sim \bar{v}' \), then every vertex from \( t(\bar{v}) \) has exactly 4 neighbors in \( t(\bar{v}') \).

- Moreover, if \( f(\bar{v}') = 1 \) and \( \bar{v} \notin F(v') \), then for two of these neighbors sign equals 0 and for two other equals 1.

- Moreover, if \( f(\bar{v}') = 1 \) and \( \bar{v} \in F(v') \), then for all 4 these neighbors the value of sign is the same as for the vertex itself.

- If \( \bar{v} \not\sim \bar{v}' \), then every vertex from \( t(\bar{v}) \) has no neighbors in \( t(\bar{v}') \).

- Every vertex from \( t(\bar{v}) \) has exactly \( n \) neighbors in \( t(\bar{v}) \).

- Moreover, if \( f(\bar{v}) = 1 \), then sign of these neighbors have the same value of \( s \) and the other \( n - s \), the different value.

\[\bullet\]

If \( 2b = 2d + n - s^2 \), then we can merge the first two cells of the obtained 3-partition and get an equitable 2-partition:

**Theorem 3.** Let \( (P_0, P_1) \) be an equitable 2-partition of \( \frac{1}{2}Q_n \) with the quotient matrix \[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \] and \( P_1 \) be partitioned into s-faces, where \( s = \sqrt{2d + n - 2b} \). Then there exists an equitable 2-partition of \( \frac{1}{2}Q_{2n} \) with the quotient matrix \( \begin{bmatrix} 4a + n + 2b \\ 4c + 2b \\
2b \\ 4d + n - 2b \end{bmatrix} \).

For example, take \( n = 6 \), \( S_c = \begin{bmatrix} c - 1 \\ 16 - c \\ c \\ 15 - c \end{bmatrix} \), \( s = 2 \), \( c = 0, 2, \ldots, 14 \). In order to obtain an equitable partition of \( \frac{1}{2}Q_{12} \) with the quotient matrix \( \begin{bmatrix} 34 + 2c \\ 32 + 2c \\ 32 - 2c \\ 34 - 2c \end{bmatrix} \) we construct an equitable partition \( (P_0, P_1) \) of \( \frac{1}{2}Q_6 \) with the quotient matrix \( S_c \) such that \( P_1 \) is partitioned into 2-faces (which are just pairs of adjacent vertices). One can see that \( C = \{0, 1\} \) and its complement gives \( S_1 \). So, the union of \( c \) cosets of \( C \) gives the matrix \( S_c \). If \( c \) is even, then we can take each of \( P_0 \) and \( P_1 \) as the union of \( (c/2 \) and \( 8 - c/2, \) respectively) cosets of \( \{000000, 111111, 000111, 111000, 111011\} \), which is obviously partitioned into edges. If \( c \) is odd, we take \( P_1 \) as the union of \( \{000111, 111100, 001100, 110011, 110000, 001111\} = \{000111, 001111\} \cup \{001100, 111100\} \cup \{110000, 110111\} \) and \((13 - c)/2 \) cosets of \( \{000000, 111111, 000111, 111100\} \). We conclude that equitable partitions of \( \frac{1}{2}Q_{12} \) with the following quotient matrices exist:

\[
\begin{bmatrix}
34 & 32 \\
36 & 30 \\
38 & 28 \\
40 & 26 \\
42 & 24 \\
44 & 22 \\
46 & 20 \\
48 & 18 \\
50 & 16 \\
52 & 14 \\
54 & 12 \\
56 & 10 \\
58 & 8 \\
60 & 6 \\
62 & 4 \\
64 & 12 \\
66 & 10 \\
68 & 8 \\
70 & 6 \\
72 & 4 \\
74 & 2 \\
76 & 0 \\
78 & 8 \\
80 & 6 \\
82 & 4 \\
84 & 2 \\
86 & 0 \\
88 & 8 \\
90 & 6 \\
92 & 4 \\
94 & 2 \\
96 & 0
\end{bmatrix}
\]
5. Computational results.

Let \( C = (C_0, C_1, \ldots, C_{k-1}) \) be a partition of the vertex set of a graph \( G \). Let \( \chi_{C_i} \) denote the characteristic vector of the subset \( C_i \) of the vertex set of \( G \). It is easy to see that \( C \) is equitable with the quotient matrix \( S \) iff

\[
A \begin{bmatrix} \chi_{C_0} & \chi_{C_1} & \cdots & \chi_{C_{k-1}} \end{bmatrix} = \begin{bmatrix} \chi_{C_0} & \chi_{C_1} & \cdots & \chi_{C_{k-1}} \end{bmatrix} S. \tag{2}
\]

The existence problem of an equitable partition could be treated as a binary linear programming problem with the variables \( \chi_{C_0}, \chi_{C_1}, \ldots, \chi_{C_{k-1}} \) and the constrains \( \text{(2)} \). The following fact was settled using GAMS \([1]\).

**Theorem 4.** There are no equitable 2-partitions of \( \frac{1}{2}Q_{10} \) with quotient matrix \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and eigenvalue 13 for any \( a \in \{14, \ldots, 28\} \setminus 21 \).

6. Quotient matrices for small even \( n \)

We list all matrices \( \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) that satisfy \( b \geq c \), conditions 1–3 of Section 2, and, for the eigenvalue \( \theta_{n/2}(n) \), the conclusion of Theorem 2.

For the matrices in red and with index −, equitable partitions do not exist. The index + means the existence, ? means an open question.

Note that for any symmetric \( 2 \times 2 \) matrix with the proper eigenvalues, an equitable partition exists in accordance with Sections 4.1 and 4.2.

6.1. \( n = 4 \)

This case is left as an exercise.

6.2. \( n = 6 \)

In \( \frac{1}{2}Q_6 \), all the equitable partitions with non-symmetric quotient matrices are constructed my merging cosets of the linear perfect code \( \{000000, 111111\} \) (see also the end of Section 4.3).

\[
\theta_1(6) = 5: \begin{bmatrix} 10 & 5 \\ 5 & 10 \end{bmatrix}.
\]

\[
\theta_2(6) = -1: \begin{bmatrix} 7 & 8 \\ 8 & 7 \end{bmatrix}, \begin{bmatrix} 6 & 9 \\ 6 & 9 \end{bmatrix}, \begin{bmatrix} 5 & 10 \\ 5 & 10 \end{bmatrix}, \begin{bmatrix} 4 & 11 \\ 4 & 11 \end{bmatrix}, \begin{bmatrix} 3 & 12 \\ 3 & 12 \end{bmatrix}, \begin{bmatrix} 2 & 13 \\ 2 & 13 \end{bmatrix}, \begin{bmatrix} 1 & 14 \\ 1 & 14 \end{bmatrix}.
\]

\[
\theta_3(6) = -3: \begin{bmatrix} 6 & 9 \\ 6 & 9 \end{bmatrix}.
\]

6.3. \( n = 8 \)

For \( \theta_2(8) \) in \( \frac{1}{2}Q_8 \), partitions with all four matrices can be constructed as \( C^{(1)} \) for \( C = \{00000000, 11111111\} \), and the union of 2, 3, or 4 translations of \( C^{(1)} \), according to Section 4.3.

\[
\theta_1(8) = 14: \begin{bmatrix} 21 & 7 \\ 7 & 21 \end{bmatrix}.
\]

\[
\theta_2(8) = 4: \begin{bmatrix} 16 & 12 \\ 12 & 16 \end{bmatrix}, \begin{bmatrix} 13 & 15 \\ 9 & 19 \end{bmatrix}, \begin{bmatrix} 10 & 18 \\ 6 & 22 \end{bmatrix}, \begin{bmatrix} 7 & 21 \\ 3 & 25 \end{bmatrix}.
\]

\[
\theta_3(8) = -2: \begin{bmatrix} 13 & 15 \\ 15 & 13 \end{bmatrix}.
\]

\[
\theta_4(8) = -4: \begin{bmatrix} 12 & 16 \\ 16 & 12 \end{bmatrix}, \begin{bmatrix} 8 & 20 \\ 12 & 16 \end{bmatrix}, \begin{bmatrix} 4 & 24 \\ 8 & 20 \end{bmatrix}, \begin{bmatrix} 0 & 28 \\ 4 & 24 \end{bmatrix}.
\]
6.4. \( n = 10 \).

The existence results are from equitable 2-partitions of \( Q_{10} \) and, for \( \theta_4(10) \), from a linear code and merging cosets of this code.

\[
\theta_1(10) = 27: \begin{bmatrix} 36 & 9 \\ 9 & 36 \end{bmatrix}, \quad \theta_2(10) = 13: \begin{bmatrix} 29 & 16 \\ 16 & 29 \end{bmatrix}, \quad \theta_3(10) = 3: \begin{bmatrix} 24 & 21 \\ 21 & 24 \end{bmatrix}, \quad \theta_4(10) = -3: \begin{bmatrix} 21 & 24 \\ 24 & 21 \end{bmatrix}, \quad \theta_5(10) = -5: \begin{bmatrix} 20 & 25 \\ 25 & 20 \end{bmatrix}.
\]

6.5. \( n = 12 \).

For \( \theta_2(12) \), the only non-symmetric matrix with known existence comes from an equitable partition of \( Q_{12} \). For \( \theta_4(12) \), all matrices with even \( b \) and \( c \) were considered in Section 6.5 for the nonexistence with \( c = 2 \), see Example 1 for \( c = 1 \), similarly. The only known equitable partition \((H^{(1)}, H^{(3)})\) of \( \frac{1}{2}Q_{12} \) with \( \theta_4(12) \) and odd \( c \) and \( b \) comes from the Hadamard code \( H \) in \( Q_{12} \) as in Section 4.4.

\[
\theta_1(12) = 44: \begin{bmatrix} 55 & 11 \\ 11 & 55 \end{bmatrix}, \quad \theta_2(12) = 26: \begin{bmatrix} 46 & 20 \\ 20 & 46 \end{bmatrix}, \quad \theta_3(12) = 12: \begin{bmatrix} 39 & 27 \\ 27 & 39 \end{bmatrix}, \quad \theta_4(12) = 2: \begin{bmatrix} 34 & 32 \\ 32 & 34 \end{bmatrix}, \quad \theta_5(12) = -4: \begin{bmatrix} 35 & 31 \\ 31 & 35 \end{bmatrix}, \quad \theta_6(12) = -6: \begin{bmatrix} 30 & 36 \\ 36 & 30 \end{bmatrix}.
\]

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