Self-semiconjugation
of piecewise linear unimodal maps

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Abstract

We devote this work to the functional equation $\psi \circ g = g \circ \psi$, where $\psi$ is an unknown function and $g$ is piecewise linear unimodal map, which is topologically conjugated to the tent map. We will call such $\psi$ self-semiconjugations of $g$. Our main results are the following:

1. Suppose that there is a self-semiconjugation of $g$, whose tangent at 0 is not a power of 2, and suppose that all the kinks of $g$ are in the complete pre-image of 0. Then all the self-semiconjugations of $g$ are piecewise linear.

2. Suppose that all self-semiconjugations of $g$ are piecewise linear. Then the conjugacy of $g$ and the tent map is piecewise linear.

1 Introduction

We continue in this work the study of the topological conjugation of piecewise linear unimodal maps, which was started in our previous works [1], [2] and [3].

We call a map $g : [0, 1] \to [0, 1]$ a unimodal map, if it is continuous and is of the form

$$g(x) = \begin{cases} g_l(x), & 0 \leq x \leq v, \\ g_r(x), & v \leq x \leq 1, \end{cases}$$

where $v \in (0, 1)$, the function $g_l$ increase, the function $g_r$ decrease, and

$$g(0) = g(1) = 1 - g(v) = 0.$$

The most studied and popular example of unimodal map is the tent map, i.e. $f : [0, 1] \to [0, 1]$ of the form

$$x \mapsto 2x - |1 - 2x|$$

Theorem 1. [4, p. 53] The tent map (1.1) is topologically conjugated to the unimodal map $g$ if and only if the complete pre-image of 0 under the action of $g$ is dense in $[0, 1]$.  

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Remind, that the set $g^{-\infty}(a) = \bigcup_{n \geq 1} g^{-n}(a)$, where $g^{-n}(a) = \{ x \in [0, 1] : g^n(x) = a \}$ for all $n \geq 1$, is called the complete pre-image of $a$ (under the action of the map $g$).

We will call the map, defined on the set $A \subseteq \mathbb{R}$, with values in $\mathbb{R}$, linear, if its graph is a line (or a line segment). We will call a map piecewise linear, if its domain can be divided to finitely many intervals, such that the map is linear on each of them. A point, where the piecewise linear map is not differentiable, will be called a kink of this map.

We will call piecewise linear unimodal map a carcass map. We will say that a carcass map is firm, if all its kinks are in the complete pre-image of 0.

We have studied the properties of the firm carcass maps in [3]. Precisely, by [3, Theorem 2], the complete pre-image of 0 under the action of firm carcass map is dense in $[0, 1]$, whence, by Theorem 1, any firm carcass map is topologically conjugated to the tent map.

We have described in [2] the continuous solutions $\xi : [0, 1] \rightarrow [0, 1]$ of the functional equation

$$\xi \circ f = f \circ \xi,$$

(1.2)

where $f$ is the tent map.

**Proposition 1.** [2, Theorem 1] 1. Let $\xi$ be an arbitrary continuous solution of the functional equation (1.2). Then $\xi$ is one of the following forms:

a. There exists $t \in \mathbb{N}$ such that

$$\xi = \xi_t : x \mapsto \frac{1 - (-1)^{[tx]}}{2} + (-1)^{[tx]} \{tx\},$$

(1.3)

where $\{\cdot\}$ denotes the function of the fractional part of a number and $[\cdot]$ is the integer part.

b. $\xi(x) = x_0$ for all $x$, where either $x_0 = 0$, or $x_0 = 2/3$.

2. For every $t \in \mathbb{N}$ the function (1.3) satisfies (1.2).

The simplicity of the description of the solution of (1.2) motivates the study of the functional equation

$$\psi \circ g = g \circ \psi,$$

(1.4)

for a unknown (continuous; surjective) $\psi : [0, 1] \rightarrow [0, 1]$ and given carcass map $g$.

The main results of this work is the next two theorems.

**Theorem 2.** Let $g$ be a firm carcass map, which is topologically conjugated to the tent map. Suppose that there is a piecewise linear solution of (1.4) with tangent $t$ at 0, such that $t$ is not a natural power of 2. Then all the solutions of (1.4) are piecewise linear.

**Theorem 3.** Let $g$ be a carcass map, which is topologically conjugated to the tent map. If all the solutions of (1.4) are piecewise linear, then the conjugacy of $g$ and the tent map is piecewise linear.
Thus, piecewise linear conjugacy of carcass maps naturally appears in the study of the properties of conjugated carcass maps and firm carcass maps. Remark that for any increasing piecewise linear homeomorphism $h : [0, 1] \to [0, 1]$ we can construct a carcass map $g : [0, 1] \to [0, 1]$ by formula
\[
g = h \circ f \circ h^{-1}, \tag{1.5}
\]
where $f$ is the tent map. Clearly, the constructed map $g$ is piecewise linear as composition of linear maps, and will, by the construction, satisfy
\[
h \circ f = g \circ h. \tag{1.6}
\]
Since $h(0) = 0$, then directly by (1.5) obtain that $g(0) = g(1) = 0$. Conclude that the constructed $g$ is a carcass map, because a topological conjugation preserves the monotonicity of maps.

Thus, there are “a lot” of carcass maps, which are topologically conjugated to the tent-map via the piecewise linear conjugacy, precisely, any increasing piecewise linear conjugacy gives such map. From another hand, the following facts show that the piecewise linearity of the conjugacy of a carcass map and the tent map is a “rare” property.

**Proposition 2.** [1, Lemmas 1 and 13] Let the tent map be topologically conjugated with the carcass map $g$ via a piecewise linear conjugacy. Denote $x_0$ the positive fixed point of $g$. Then $g'(0) = 2$ and $g'(x_0-) \cdot g'(x_0+) = 4$. \[3\]

**Proposition 3.** [1, Theorem 2] For any $v \in (0, 1)$ let $\hat{g}$ be increasing piecewise linear map $\hat{g} : [0, v] \to [0, 1]$ such that $\hat{g}(0) = 1 - \hat{g}(v) = 0$ and $\hat{g}'(0) = 2$. There exists the unique carcass map $g$, which coincides with $\hat{g}$ on $[0, v]$ and is topologically conjugated with a tent map via a piecewise linear conjugacy.

**Proposition 4.** [1, Theorem 3] For any $v \in (0, 1)$ let $\hat{g}$ be decreasing piecewise linear map $\hat{g} : [v, 1] \to [0, 1]$ such that $\hat{g}(v) = 1 - \hat{g}(1) = 1$ and $\hat{g}'(x_0-) \cdot \hat{g}'(x_0+) = 4$, where $x_0$ is the fixed point of $\hat{g}$. There exists the unique carcass map $g$, which coincides with $\hat{g}$ on $[v, 1]$ and is topologically conjugated with a tent map via a piecewise linear conjugacy.

\[3\]Here and below we denote $g'(x_0-)$ and $g'(x_0+)$ the left and the right derivative of $g$ at $x_0$. 
2 Properties of unimodal maps

2.1 The connection of topological conjugacy and semi conjugacy

Lemma 2.1. Suppose that for a unimodal map \( g : [0, 1] \rightarrow [0, 1] \), and the conjugacy \( h : [0, 1] \rightarrow [0, 1] \) the equality (1.6) holds, where \( f \) is the tent-map. For any (continuous; surjective) solution \( \psi : [0, 1] \rightarrow [0, 1] \) of (1.4) there exists a (continuous; surjective) solution \( \xi \) of (1.2) such that

\[
\psi = h \circ \xi \circ h^{-1}.
\]  

(2.1)

Proof. As we have mentioned in [2], the description of the non-homeomorphic (surjective) solutions \( \eta : [0, 1] \rightarrow [0, 1] \) of the functional equation

\[
\eta \circ f = g \circ \eta,
\]  

(2.2)

can be reduced to the description of the non-homeomorphic (surjective) solutions \( \xi : [0, 1] \rightarrow [0, 1] \) of the functional equation (1.2). Indeed, it is clear from the commutative diagram

that there is one-to-one correspondence

\[
\begin{cases}
\xi = h^{-1} \circ \eta, \\
\eta = h \circ \xi
\end{cases}
\]  

(2.3)

between the solutions \( \eta \) of (2.2) and the solutions \( \xi \) of (1.2).

For any continuous (surjective) solution \( \psi \) of (1.4), and the homeomorphism \( h \), which satisfies (1.6), write the commutative diagram

which, similarly to (2.3), defines one-to-one correspondence between the solutions of the functional equations (2.2) and (1.4).
\[
\begin{aligned}
\psi &= \eta \circ h^{-1}, \\
\eta &= \psi \circ h. 
\end{aligned}
\]  

(2.4)

Now (2.1) follows from (2.3) and (2.4).

2.2 Pre-images of 0 under a carcass map and topological conjugation

The next fact is the classical property of the tent map.

Remark 2.2. For every \( n \geq 1 \) we have that

\[
f^{-n}(0) = \left\{ \frac{k}{2^{n-1}}, 0 \leq k \leq 2^{n-1} \right\},
\]

where \( f \) is the tent map.

Let a carcass map \( g \) be fixed till the end of this subsection. By [3, Remark 3.1], the set \( g^{-n}(0) \) consists of \( 2^{n-1} + 1 \) points. Following [3, Notation 3.2], for any \( n \geq 1 \) denote

\[
g^{-n}(0) = \{ \mu_{n,k}(g), 0 \leq k \leq 2^{n-1} \},
\]

(2.5)

where \( \mu_{n,k}(g) < \mu_{n,k+1}(g) \) for all \( k, 0 \leq k < 2^{n-1} \).

Remark 2.3. [3, Remark 3.3] Notice that \( \mu_{n,k}(g) = \mu_{n+1,2k}(g) \) for all \( k, 0 \leq k \leq 2^{n-1} \).

Lemma 2.4. [3, Lemma 3.6] For any unimodal maps \( g_1, g_2 : [0,1] \rightarrow [0,1] \) and the conjugacy \( h \), which satisfies

\[
h \circ g_1 = g_2 \circ h,
\]

(2.6)

the equality

\[h(\mu_{n,k}(g_1)) = \mu_{n,k}(g_2)\]

holds for all \( n \geq 1 \) and \( k, 0 \leq k \leq 2^{n-1} \).

Moreover, by [3, Theorem 7], the conjugacy is unique in the case of Lemma 2.4.

For every \( n \geq 1 \) and \( k, 0 \leq k < 2^{n-1} \) write \( \mu_{n,k} \) instead of \( \mu_{n,k}(g) \).

Notation 2.5. [3, Notation 4.7] For any \( n \geq 1 \) and \( k, 0 \leq k < 2^{n-1} \) denote \( I_{n,k} = (\mu_{n,k}, \mu_{n,k+1}) \) and for any interval \( I = (a, b) \) denote \( \#I = b - a \).

Notation 2.6. [3, Notation 4.9] For any \( n \geq 1 \) and \( k, 0 \leq k < 2^n \) denote

\[
\delta_{n,k} = \frac{\mu_{n+1,2k+1} - \mu_{n,k}}{\mu_{n,k+1} - \mu_{n,k}} = \frac{\#I_{n+1,2k}}{\#I_{n,k}}.
\]
2.3 Pre-images of 0 under a firm carcass map

Let \( g \) be a firm carcass map, which will be fixed till the end of this section. Denote by \( n_0 \) the minimal natural number such that \( g^{n_0}(x) = 0 \) for each kink \( x \) of \( g \).

Write \( \mu_{n,k} \) instead of \( \mu_{n,k}(g) \) for all \( n \geq 0 \) and \( 0 \leq k \leq 2^{n-1} \).

**Notation 2.7.** For every \( a \in \{0; 1\} \) denote

\[
\mathcal{R}(a) = 1 - a.
\]

**Remark 2.8.** [3, Remark 4.19] Let \( n \geq n_0 \) and \( 0 \leq k < 2^{n-1} \) have the binary expansion

\[
k = \sum_{i=0}^{n} x_i 2^{n-i}
\]

for all \( n \geq 1 \). Then \( \delta(I_{n+1,k}) = \mathcal{R}^{x_{n-n_0+1}}(\delta(I_{n_0,p_n})) \), where

\[
p_i = \sum_{j=i-n_0+2}^{i} \mathcal{R}^{x_{i+1-n_0}}(x_j) 2^{i-j}
\]

for all \( i, n_0 + 1 \leq i \leq n \).

It follows from Remark 2.8 that for any \( n \geq n_0 \) and \( k, 0 \leq k < 2^{n-1} \) the number \( \delta(I_{n,k}) \) depends on the last \( n_0 \) binary digits of \( k \) and is independent on \( n \) and other digits of \( k \).

There is an important corollary from Remark 2.8.

**Remark 2.9.** There are \( l_0, \ldots, l_{2^{n_0}-1} \) with the following property. For every \( n \geq n_0 \) and every \( k, 0 \leq k < 2^{n-1} \) let \( k_0 \geq 0 \) and \( j, 0 \leq j < 2^{n_0} \) be such that \( k = 2^{n_0} \cdot k_0 + j \). Then

\[
\#I_{n,k} = l_j \cdot \#I_{n,2^{n_0} \cdot k_0}.
\]

**Notation 2.10.** 1. For any \( k, 0 \leq k < 2^{n_0} \) denote

\[
\delta_k = \delta(I_{n_0,k}).
\]

2. For any \( k \geq 2^{n_0} \) denote by

\[
\delta_k = \delta(I_{n_0,k^*}),
\]

where the last \( n_0 \) binary digits of \( k \) and \( k^* \) coincide and \( 0 \leq k^* < 2^{n_0} \).

The next fact follows from Remark 2.8.

**Remark 2.11.** Notice that \( \delta_k = \delta_{k+2^{n_0}} \) for all \( k \geq 0 \).
Remark 2.12. For every \( n \geq n_0 \) and \( k, 0 \leq k < 2^n - 1 \) we have that
\[
\#I_{n+n_0,2^n} = \frac{\#I_{n,i}}{\sum_{k=1}^{2^n-1} l_k}.
\]

Proof. Since
\[
\bigcup_{i=0}^{2^n-1} I_{n,i} = \bigcup_{k=0}^{2^n-1} \bigcup_{i=0}^{2^n-1} I_{n+n_0,2^n+k},
\]
then, by Remark 2.9
\[
\#I_{n,i} = \sum_{k=0}^{2^n-1} l_k \cdot \#I_{n+n_0,2^n+k}
\]
and the fact follows. \( \square \)

2.4 Self-semiconjugation of carcass maps

Let a carcass map \( g \), such that the complete pre-image of 0 under the action of \( g \) is dense in \([0, 1]\), be fixed till the end of this section. By [3, Theorem 7] let \( h : [0, 1] \to [0, 1] \) be the unique conjugacy, such that (1.6) holds.

Remark 2.13. The map \( \xi_t \), defined by (1.3), can be expressed by the formula
\[
\xi_t : x \to tx
\]
for all \( x \in [0, \frac{1}{t}] \).

Remark 2.14. For the map \( \xi_t \), defined by (1.3),
\[
\xi_t(\mu_{n,k}(f)) = \mu_{n,tk}(f),
\]
whenever \( k \leq \left[ \frac{2^n-1}{t} \right] \).

Proof. The remark follows from Remarks 2.2 and 2.13 \( \square \)

Lemma 2.15. For any self-semiconjugation \( \psi \) of \( g \) there is \( t \in \mathbb{N} \) such that
\[
\psi(\mu_{n,k}(g)) = \mu_{n,tk}(g)
\]
for all \( n \geq 1 \) and \( k \leq \left[ \frac{2^n-1}{t} \right] \).

Proof. Let \( \psi \) be a self-semiconjugation of \( g \), and \( \xi \) be the self-semiconjugation of the tent map, whose existence follows from Lemma 2.1. By Proposition [1] the map \( \xi = \xi_t \) of the form (1.3) for some \( t \geq 1 \) and
\[
\psi = h \circ \xi_t \circ h^{-1}.
\]
Write $\psi_t$ for the function $\psi$, which is defined by (2.7).

By [3, Lemma 3.6] we have that

$$h(\mu_{n,k}(f)) = \mu_{n,k}(g)$$

(2.8)

for all $n \geq 1$ and $k$, $0 \leq k \leq 2^{n-1}$, where $\mu_{n,k}$ are defined in (2.5).

For all $n \geq 1$ and $k \leq \left\lfloor \frac{2^n}{2} \right\rfloor$ it follows from Remark 2.14 that

$$\psi_t(\mu_{n,k}(g)) = (h \circ \xi_t)(h^{-1}(\mu_{n,k}(g))) = (h \circ \xi_t)(\mu_{n,k}(f)) = h(\xi_t(\mu_{n,k}(f))) = h(\mu_{n,k}(f)) = \mu_{n,k}(g),$$

and lemma follows.

2.5 Conjugation of carcass maps via piecewise linear conjugacy

Lemma 2.16. Let $g_1$ and $g_2$ be unimodal maps, which are topologically conjugated to the tent map. If a continuous solution $h$ of (2.6) is linear on some interval, then $h$ is piecewise linear in the entire $[0,1]$.

Proof. Since $g_1$ and $g_2$ are topologically conjugated to the tent map, then, by Theorem [1], there are $n$ and $k_n^-, k_n^+ \in \{0, \ldots, 2^{n-1}\}$ such that $h$ is linear on $[\mu_{n,k_n^-}(g_1), \mu_{n,k_n^+}(g_1)]$.

The formula (2.6) defines $h$ on the interval $g_1 \circ [\mu_{n,k_n^-}, \mu_{n,k_n^+}]$. Since $g_1 \circ g_1^{-n}(0) = g^{n+1}(0)$, then there are $k_{n-1}^-, k_{n-1}^+ \in \{0, \ldots, 2^{n-2}\}$ such that $h$ is piecewise linear on $[\mu_{n-1,k_{n-1}^-}(g_1), \mu_{n-1,k_{n-1}^+}(g_1)] = g_1 \circ [\mu_{n,k_n^-}, \mu_{n,k_n^+}]$.

In this manner lemma follows by induction on $n$, since $[0, 1] = [\mu_{1,0}(g_1), \mu_{1,1}(g_1)]$.

Lemma 2.17. Let $g$ be a carcass map, which is topologically conjugated to the tent map $f$. The following conditions are equivalent.

1. The conjugacy of $f$ and $g$ is piecewise linear.

2. There is $a \in (0,1)$ and $r, t > 0$ such that for every $n \geq 1$ and $k$, $0 \leq k \leq 2^{n-1}$ the equality

$$\mu_{n,k}(g) = \frac{tk}{2^{n-1}}$$

holds, whenever $\mu_{n,k}(g) < r$.

Proof. It follows from Lemma 2.4 and Remark 2.2 that condition 1. is equivalent to that the conjugacy of $f$ and $g$ is linear in some neighborhood of 0.

Now lemma follows from Lemma 2.16.
**Lemma 2.18.** Let a carcass map \( g \) be linear on some interval \((a, b)\). Then
\[
\#(g \circ (a, b)) = g'(a) \cdot \#(a, b).
\]

**Proof.** This fact is trivial. \( \square \)

## 3 Self-semiconjugations

### 3.1 Carcass maps

**Lemma 3.1.** Let \( g \) be a carcass map and \( a \in (0, 1) \) be the first positive kink of \( g \). If \( \psi'(0) > g'(0) \) for a piecewise linear self-semiconjugation \( \psi \) of \( g \), then \( \frac{a \cdot g'(0)}{\psi'(0)} \) is the first positive kink of \( \psi \).

**Proof.** It follows from Lemma 2.15 that \( \psi'(0) > 1 \).

Since \( \psi \) is piecewise linear, then there is \( \varepsilon > 0 \) such that \( \psi(x) = \psi'(0) \cdot x \) for all \( x \in (0, \varepsilon) \).

Suppose that \( \varepsilon > 0 \) is such that \( \psi'(0) \cdot \varepsilon < a \). Notice that in this case \( x < a \) for all \( x \in (0, \varepsilon) \), because \( \psi'(0) > 1 \). Thus, for all \( x \in (0, \varepsilon) \) we have that \( \psi'(x) = \psi'(0) \cdot x \), \((g \circ \psi)(x) = g'(0) \cdot \psi'(0) \cdot x \) and \( g(x) = g'(0) \cdot x \). Now, by (1.4),
\[
\psi(x) = \psi'(0) \cdot x
\]
for all \( x \in g \circ (0, \varepsilon) \).

Thus, for every \( x \in g \circ \left(0, \frac{a}{\psi'(0)}\right)\) we have that \( \psi(x) = \psi'(0) \cdot x \). Notice that \( g \circ \left(0, \frac{a}{\psi'(0)}\right) = \left(0, \frac{a \cdot g'(0)}{\psi'(0)}\right) \). Remark that \( \frac{a \cdot g'(0)}{\psi'(0)} < a \).

Take an arbitrary \( \delta, 0 < \delta < a \cdot \left((g'(0) - 1)\right) \) is such that \( g \) is linear on \((a, a + \delta)\). Then for every \( x \in \left(\frac{a + \delta}{\psi'(0)}, \frac{a \cdot g'(0)}{\psi'(0)}\right) \) we have that \( \psi(x) = \psi'(0) \cdot x \), because \( x < \frac{a \cdot g'(0)}{\psi'(0)} \); also \( g(x) = g'(0) \cdot x \), because \( x < a \), and \((g \circ \psi)(x) = g'(a+) \cdot (\psi'(0) \cdot x - a) + g(a)\).

Now, it follows from (1.4) that
\[
\psi(g'(0) \cdot x) = g'(a+) \cdot (\psi'(0) \cdot x - a) + g(a). \tag{3.1}
\]
Denote \( u = g'(0) \cdot x \) and remark that \( u \in \left((\frac{(a + \delta) \cdot g'(0)}{\psi'(0)}, \frac{a \cdot (g'(0))^2}{\psi'(0)}\right)\), whenever \( x \in \left(\frac{a + \delta}{\psi'(0)}, \frac{a \cdot g'(0)}{\psi'(0)}\right)\). Rewrite (3.1) as
\[
\psi(u) = g'(a+) \cdot \left(\frac{\psi'(0) \cdot u}{g'(0)} - a\right) + g(a).
\]
Clearly, if \( g'(0) \neq g'(a+) \), then there is \( u \in \left((\frac{(a + \delta) \cdot g'(0)}{\psi'(0)}, \frac{a \cdot (g'(0))^2}{\psi'(0)}\right)\) such that \( \psi(u) \neq \psi'(0) \cdot u \).
Remark that if \( \delta \approx 0 \), then \( \frac{(a + \delta) \cdot g'(0)}{\psi'(0)} \approx \frac{a \cdot g'(0)}{\psi'(0)} \). \( \square \)
Lemma 3.2. Suppose that for any \( t \geq 1 \) the map \( \psi_t \), defined by (2.7), is piecewise linear. Then there is \( w \in \mathbb{R} \) such that for every \( n \geq 1 \) and \( k \), the equality

\[
\mu_{n,k}(g) = \frac{w \cdot k}{2^{n-1}},
\]

holds whenever \( \mu_{n,k}(g) \leq g(a) \), where \( a \in (0,1) \) is the first positive kink of \( g \).

Proof. For any \( n \geq 1 \) and \( k \), \( 0 \leq k \leq 2^n - 1 \) we will write \( \mu_{n,k} \) instead of \( \mu_{n,k}(g) \).

Denote \( \tau_t \) the tangent of \( \psi_t \) at 0. Then, by Lemma 3.1,

\[
\tau_t = \frac{\mu_{n,kt}}{\mu_{n,1}}.
\]

Notice that \( \tau_2 = g'(0) \) and \( \mu_{n,k} = \tau_2 \cdot \mu_{n+1,k} \), whenever \( \mu_{n,k} < a \).

Remark that, if \( \mu_{n,k} \leq g(a) \), then

\[
(0, \mu_{n,k}) \xleftarrow{g'} (\mu_{n+1,2^n-k}, 1) \xleftarrow{g'} (\mu_{n+2,2^n-k}, \mu_{2,1}) \xleftarrow{g'} \cdots \xleftarrow{g'} (\mu_{n+2+m,2^n-k}, \mu_{2+m,1})
\]
does not contain any kink of \( g \), where \( m \geq 1 \) is minimal natural number such that \( \mu_{2+m,1} \leq a \).

By Lemma 2.18

\[
\#(\mu_{n+1,2^n-k}, 1) = \frac{1}{g'(-1)} \cdot \#(0, \mu_{n,k}),
\]

\[
\#(\mu_{n+2,2^n-k}, \mu_{2,1}) = \frac{1}{g'(v-)} \cdot \#(\mu_{n+1,2^n-k}, 1)
\]

and there is \( p \in \mathbb{R}_+ \), independent on \( \mu_{n,k} \), such that

\[
\#(0, \mu_{n,k}) = p \cdot \#(\mu_{n+2+m,2^n-k}, \mu_{2+m,1}).
\]

Denote by \( n^* \) a minimal natural number such that \( \mu_{n^*,1} \leq \bar{a} \). Now for any \( n \geq n^* \) and \( k \geq 1 \) such that \( \mu_{n,k} \leq \bar{a} \), we can rewrite (3.3) as

\[
\tau_2^{2+m} \cdot \#(0, \mu_{n+2+m,k}) = p \cdot \#(\mu_{n+2+m,2^n-k}, \mu_{n+2+m,2^n}),
\]

or, by (3.2),

\[
\tau_2^{2+m} \cdot \tau_k = p \cdot (\tau_{2n} - \tau_{2^n-k}).
\]
Denote \( q = \frac{p}{\tau^2 + m} \). Then

\[
\tau_k = q \cdot (\tau_{2^n} - \tau_{2^n-k}).
\] (3.4)

Suppose that \( \mu_{n,k+1} \leq \tilde{a} \). Then, plug \( k + 1 \) instead of \( k \) into (3.4) and obtain

\[
\tau_{k+1} = q \cdot (\tau_{2^n} - \tau_{2^n-k-1}).
\]

After the subtraction of the obtained equality and (3.4), write

\[
\tau_{k+1} - \tau_k = q \cdot (\tau_{2^n-k} - \tau_{2^n-k-1}).
\] (3.5)

Suppose that \( n \geq n^* \) and \( s \geq 1 \) is such that \( \mu_{s,2^n+1} \leq \tilde{a} \). Then rewrite (3.5) as

\[
\#I_{s,k} = q \cdot \#I_{s,2^n-k-1}
\] (3.6)

for all \( k, 0 \leq k \leq 2^n \). Since we can change \( k \) to \( 2^n - k - 1 \) in (3.6), then

\[
\#I_{s,2^n-k-1} = q \cdot \#I_{s,k}
\] (3.7)

for all \( k, 0 \leq k \leq 2^n \). Equalities (3.6) and (3.7) imply that \( q = 1 \) in (3.6), i.e.

\[
\#I_{s,k} = \#I_{s,2^n-k-1}
\] (3.8)

for all \( k, 0 \leq k \leq 2^n - 1 \), all \( s \geq 1 \) and \( n \geq n^* \) such that \( I_{s,2^n} \subseteq (0, \tilde{a}) \).

For any \( i, 0 \leq i < s \) write

\[
\overline{I_{s-i,k}} = \bigcup_{j=0}^{2^i-1} \overline{I_{s,k-2^i+j}}
\]

and by (3.8) get

\[
\#I_{s-i,k} = \#I_{s-i,2^n-i-k-1}
\]

for all \( k, 0 \leq k \leq 2^n-i - 1 \). Thus, the condition \( n \geq n^* \) and \( I_{s,2^n} \subseteq (0, \tilde{a}) \) is not important in (3.8), i.e.

\[
\#I_{n,k} = \#I_{n,k+1}
\] (3.9)

for all \( n \geq 1 \) and \( k \) such that \( I_{n,k+1} \subseteq (0, \tilde{a}) \). Now lemma follows from (3.9) by induction on \( n \).

We are now ready to prove Theorem 3.

**Proof of Theorem 3** It follows from Lemma 3.2 that the conjugacy \( h \) has tangent \( w \) on \( \left( 0, \frac{w}{g(a)} \right) \), where \( a \) is the first kink of \( g \).

Now Theorem 3 follows from Lemma 2.17. \( \square \)
3.2 Firm carcass maps

Let a firm carcass map $g$ be fixed till the end of this section and $n_0$ denote the same as in Section 2.3. We will prove Theorem 2 in this section.

We will need the following technical fact.

**Remark 3.3.** For any $n \geq n_0$ and $i, j, k \in \{0, \ldots, 2^{n_0} - 1\}$ we have that

$$
\#I_{n+3n_0,2^{n_0}+2^{n_0}+k} = \frac{l_i \cdot l_j \cdot l_k}{2^{2^{n_0} - 1} \left( \sum_{p=1}^{l_p} \right)} \cdot \#I_{n,0}.
$$

**Proof.** By Remark 2.9 write

$$
\#I_{n+3n_0,2^{n_0}+2^{n_0}+k} = l_k \cdot \#I_{n+3n_0,2^{n_0}+2^{n_0}+j}.
$$

By Remark 2.12 write

$$
\#I_{n+3n_0,2^{n_0}+2^{n_0}+j} = \frac{\#I_{n+2n_0,2^{n_0}+j}}{2^{2^{n_0} - 1} \sum_{k=1}^{l_k}}.
$$

Ones more by Remark 2.9 simplify

$$
\#I_{n+2n_0,2^{n_0}+j} = l_j \cdot \#I_{n+2n_0,2^{n_0}+i}
$$

and by Remark 2.12 obtain

$$
\#I_{n+2n_0,2^{n_0}+i} = \frac{\#I_{n+n_0,i}}{2^{2^{n_0} - 1} \sum_{k=1}^{l_k}}.
$$

Now our remark follows from Remark 2.9. \hfill \square

**Remark 3.4.** Notice that for any $n \geq n_0$ and $k$, $0 \leq k < 2^{n_1} - 1$ we have that

$$
\frac{\#I_{n+1,2^{n_1}+1}}{\#I_{n+1,2^{n_1}}} = 1 - \frac{1}{\delta_k} - 1.
$$

**Proof.** Indeed,

$$
\frac{\#I_{n+1,2^{n_1}+1}}{\#I_{n+1,2^{n_1}}} = \frac{\#I_{n,k} - \#I_{n+1,2^{n_1}}}{\#I_{n+1,2^{n_1}}} = \frac{1}{\delta_{n,k}} - 1 = \frac{1}{\delta_k} - 1.
$$

\hfill \square

**Lemma 3.5.** Assume that there is $a \in (0, 1)$ and a number $t \in \mathbb{N}$, which is not a natural power of 2, the equality

$$
\xi(\mu_{n,k}) = \mu_{n,tk}, \quad (3.10)
$$

whenever $\mu_{n,tk} \leq a$, holds. Then

$$
\delta_0 = \delta_1 = \ldots = \delta_{2^{n_0} - 1}.
$$
Proof. Suppose that $t = s \cdot 2^m$, where $s$ is odd. Thus, by (3.10) and Remark 2.3 obtain that

$$\xi(\mu_{n,k}) = \mu_{n-m,sk}. \quad (3.11)$$

For any $k \geq 0$ such that $\mu_{n,tk} < a$ it follows from (3.10) that

$$\xi \circ [\mu_{n,k}, \mu_{n,k+1}] = [\mu_{n-m,sk}, \mu_{n-m,sk+s}],$$

moreover,

$$\xi(\mu_{n+1,2k+1}) = \mu_{n-m+1,s(2k+1)}. \quad (3.12)$$

Thus, it follows from (3.11) and (3.12) that

$$\frac{\mu_{n,k+1} - \mu_{n+1,2k+1}}{\mu_{n+1,2k+1} - \mu_{n,k}} = \frac{\mu_{n-m,sk+s} - \mu_{n-m+1,s(2k+1)}}{\mu_{n-m+1,s(2k+1)} - \mu_{n-m,sk}}.$$

By Notation 2.5 we can rewrite the last equality as

$$\frac{\#I_{n+1,2k+1}}{\#I_{n+1,2k}} = \frac{2s-1}{\sum_{i=s}^{2s-1} \#I_{n-m+1,2sk+i}} \cdot \frac{\sum_{i=0}^{s-1} \#I_{n-m+1,2sk+i}}{\sum_{i=0}^{s-1} \#I_{n-m+1,2sk+i}}. \quad (3.13)$$

By Remark 3.4 we can rewrite (3.13) as

$$\frac{1}{\delta_k} - 1 = \frac{\sum_{i=s}^{2s-1} \#I_{n-m+1,2sk+i}}{\sum_{i=0}^{s-1} \#I_{n-m+1,2sk+i}}. \quad (3.14)$$

Notice that the right side of (3.14) contains $2s$ intervals, whose the second indices are the consequent numbers

$$\{2sk; 2sk + 1, \ldots, 2sk + 2s - 1\}. \quad (3.15)$$

For every $i \geq 0$ denote $k_i$ the number such that the set (3.15) contains $i \cdot 2^{n_0}$. Next, denote $j_i, 0 \leq j_i \leq 2s - 1$ such that $2sk_i + j_i = i \cdot 2^{n_0}$. Since $s$ and $2^{n_0}$ do not have common divisors, then

$$\{j_0, j_1, \ldots, j_{s-1}\} = \{0, 2, \ldots, 2s - 2\},$$

moreover the set $j_i, i \geq 0$ is periodical with period $s$.

Denote $k^*$ the minimal value $k = k_i \geq 0$ such that $j_i = 2s - 2$. Then there is $w, 0 < w \leq s$ such that plugging $k = k^*$ into (3.14) transforms it to

$$\frac{1}{\delta_{k^*}} - 1 = \frac{\sum_{i=1}^{s-2} \#I_{n-m+1,2^{n_0} \cdot w - i}}{\sum_{i=s-1}^{2s} \#I_{n-m+1,2^{n_0} \cdot w - i}} \quad (3.16)$$
Denote $k_i^* = k^* + i \cdot s \cdot 2^{n_0}$, where $i \geq 0$. Then the plug $k_i^*$ instead of $k^*$ into (3.16) transforms it to

$$
\frac{1}{\delta_{k^*+i\cdot s\cdot 2^{n_0}}} - 1 = \frac{\sum_{j=1}^{s-2} l_{i-s} \cdot l_{i-s} \cdot l_{2^{n_0}-j} + \sum_{j=1}^{s-2} l_{i-s} \cdot l_{2^{n_0}-j}}{\sum_{j=s-1}^{2s} l_{i-s} \cdot l_{2^{n_0}-j}}.
$$

(3.17)

By Remarks 2.11 and 3.3 we can rewrite (3.17) as

$$
\frac{1}{\delta_{k^*}} - 1 = \frac{\frac{l_{i-s}}{l_{i-s-1}} \cdot \sum_{j=1}^{s-2} l_{2^{n_0}-j} + \sum_{j=1}^{s-2} l_{2^{n_0}-j}}{\sum_{j=s-1}^{2s} l_{2^{n_0}-j}},
$$

which can be cancelled to

$$
\frac{1}{\delta_{k^*}} - 1 = \frac{l_{i-s}}{l_{i-s-1}} \cdot \sum_{j=1}^{s-2} l_{2^{n_0}-j} + \sum_{j=1}^{s-2} l_{2^{n_0}-j}
$$

Since the left hand side is independent on $i$, then so is right hand side. Thus, there is $q > 0$ such that

$$
\frac{l_{i-s}}{l_{i-s-1}} = q
$$

for all $i$. Since $s$ and $2^{n_0}$ are pairwise prime, then for every $k$, $0 < k \leq 2^{n_0}$ we have that

$$
\frac{l_k}{l_{k-1}} = q,
$$

which means that

$$
l_0 = l_1 = \ldots = l_{2^{n_0}-1}.
$$

Now Lemma follows from Remark 3.4.

We are now ready to proof Theorem 2.

**Proof of Theorem 2.** By Lemma 3.5 obtain that $\delta_0 = \delta_1 = \ldots = \delta_{2^{n_0}-1}$. Thus, by Lemma 2.17 the conjugacy $h$, which satisfies (1.6), is piecewise linear.

Since any continuous solution $\psi$ of (1.4) can be expressed by (2.1) from some continuous solution $\xi$ of (1.2), then Theorem 2 follows from Proposition 1 because $\xi$ is piecewise linear.

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