The Radicals of Hopf Module Algebras *

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Abstract

The characterization of \( H \)-prime radical is given in many ways. Meantime, the relations between the radical of smash product \( R \# H \) and the \( H \)-radical of Hopf module algebra \( R \) are obtained.

0 Introduction and Preliminaries

In this paper, let \( k \) be a commutative associative ring with unit, \( H \) be an algebra with unit and comultiplication \( \Delta \) (i.e. \( \Delta \) is a linear map: \( H \to H \otimes H \)), \( R \) be an algebra over \( k \) (\( R \) may be without unit) and \( R \) be an \( H \)-module algebra.

We define some necessary concept as follows.

If there exists a linear map \[
\begin{align*}
H \otimes R & \longrightarrow R \\
h \otimes r & \mapsto h \cdot r
\end{align*}
\]
such that \( h \cdot rs = \sum (h_1 \cdot r)(h_2 \cdot s) \) and \( 1_H \cdot r = r \) for all \( r, s \in R, h \in H \), then we say that \( H \) weakly acts on \( R \). For any ideal \( I \) of \( R \), set \( (I : H) := \{ x \in R | h \cdot x \in I \text{ for all } h \in H \} \).

\( I \) is called an \( H \)-ideal if \( h \cdot I \subseteq I \) for any \( h \in H \). Let \( I_H \) denote the maximal \( H \)-ideal of \( R \) in \( I \). It is clear that \( I_H = (I : H) \). An \( H \)-module algebra \( R \) is called an \( H \)-simple module algebra if \( R \) has no non-trivial \( H \)-ideals and \( R^2 \neq 0 \). \( R \) is said to be \( H \)-semiprime if there are no non-zero nilpotent \( H \)-ideals in \( R \). \( R \) is said to be \( H \)-prime if \( IJ = 0 \) implies \( I = 0 \) or \( J = 0 \) for any \( H \)-ideals \( I \) and \( J \) of \( R \). An \( H \)-ideal \( I \) is called an \( H \)-(semi)prime ideal of \( R \) if \( R/I \) is \( H \)-(semi)prime. \( \{a_n\} \) is called an \( H \)-m-sequence in \( R \) with beginning \( a \) if there exist \( h_n, h'_n \in H \) such that \( a_1 = a \in R \) and \( a_{n+1} = (h_n a_n) b_n (h'_n a_n) \) for any

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natural number $n$. If every $H$-$m$-sequence $\{a_n\}$ with $a_{7,1,1} = a$, there exists a natural number $k$ such that $a_k = 0$, then $a$ is called an $H$-$m$-nilpotent element. Set
\[ W_H(R) = \{ a \in R \mid a \text{ is an } H$-$m$-nilpotent element} \].

$R$ is called an $H$-module algebra if the following conditions hold:

(i) $R$ is a unital left $H$-module (i.e. $R$ is a left $H$-module and $1_H \cdot a = a$ for any $a \in R$);

(ii) $h \cdot ab = \sum(h_1 \cdot a)(h_2 \cdot b)$ for any $a, b \in R$, $h \in H$, where $\Delta(h) = \sum h_1 \otimes h_2$.

$H$-module algebra is sometimes called a Hopf module.

If $R$ is an $H$-module algebra with a unit $1_R$, then
\[ h \cdot 1_R = \sum_h (h_1 \cdot 1_R)(h_2 S(h_3) \cdot 1_R) \]
\[ = \sum_h h_1 \cdot (1_R(S(h_2) \cdot 1_R)) = \sum_h h_1 S(h_2) \cdot 1_R = \epsilon(h)1_R, \]
i.e.
\[ h \cdot 1_R = \epsilon(h)1_R \]
for any $h \in H$.

An $H$-module algebra $R$ is called a unital $H$-module algebra if $R$ has a unit $1_R$ such that $h \cdot 1_R = \epsilon(h)1_R$ for any $h \in H$. Therefore, every $H$-module algebra with unit is a unital $H$-module algebra. A left $R$-module $M$ is called an $R$-$H$-module if $M$ is also a left unital $H$-module with $h(am) = \sum(h_1 \cdot a)(h_2 m)$ for all $h \in H, a \in R, m \in M$. An $R$-$H$-module $M$ is called an $R$-$H$-irreducible module if there are no non-trivial $R$-$H$-submodules in $M$ and $RM \neq 0$. An algebra homomorphism $\psi : R \rightarrow R'$ is called an $H$-homomorphism if $\psi(h \cdot a) = h \cdot \psi(a)$ for any $h \in H, a \in R$. Let $r_b, r_j, r_l, r_m$ denote the Baer radical, the Jacobson radical, the locally nilpotent radical, the Brown-MacCoy radical of algebras respectively. Let $I \triangleleft_H R$ denote that $I$ is an $H$-ideal of $R$.

## 1 The $H$-special radicals for $H$-module algebras

J.R. Fisher [7] built up the general theory of $H$-radicals for $H$-module algebras. We can easily give the definitions of the $H$-upper radical and the $H$-lower radical for $H$-module algebras as in [11]. In this section, we obtain some properties of $H$-special radicals for $H$-module algebras.

**Lemma 1.1** (1) If $R$ is an $H$-module algebra and $E$ is a non-empty subset of $R$, then $(E) = H \cdot E + R(H \cdot E) + (H \cdot E)R + R(H \cdot E)R$, where $(E)$ denotes the $H$-ideal generated by $E$ in $R$.

(2) If $B$ is an $H$-ideal of $R$ and $C$ is an $H$-ideal of $B$, then $(C)^3 \subseteq C$, where $(C)$ denotes the $H$-ideal generated by $C$ in $R$.

**Proof.** It is trivial. $\square$
Proposition 1.2 (1) \( R \) is \( H \)-semiprime iff \((H \cdot a)R(H \cdot a) = 0\) always implies \( a = 0 \) for any \( a \in R \).

(2) \( R \) is \( H \)-prime iff \((H \cdot a)R(H \cdot b) = 0\) always implies \( a = 0 \) or \( b = 0 \) for any \( a, b \in R \).

Proof. If \( R \) is an \( H \)-prime module algebra and \((H \cdot a)R(H \cdot b) = 0\) for \( a, b \in R \), then \((a)^2(b)^2 = 0\), where \((a)\) and \((b)\) are the \( H \)-ideals generated by \( a \) and \( b \) in \( R \) respectively. Since \( R \) is \( H \)-prime, \((a) = 0 \) or \((b) = 0 \). Conversely, if \( B \) and \( C \) are \( H \)-ideals of \( R \) and \( BC = 0 \), then \((H \cdot a)R(H \cdot b) = 0\) and \( a = 0 \) or \( b = 0 \) for any \( a \in B, b \in C \), which implies that \( B = 0 \) or \( C = 0 \), i.e. \( R \) is an \( H \)-prime module algebra.

Similarly, part (1) holds. \( \Box \)

Definition 1.4 \( \mathcal{K} \) is called an \( H \)-(weakly )special class if

(S1) \( \mathcal{K} \) consists of \( H \)-(semiprime)prime module algebras.

(S2) For any \( R \in \mathcal{K} \), if \( 0 \neq I \triangleleft_H R \) then \( I \in \mathcal{K} \).

(S3) If \( R \) is an \( H \)-module algebra and \( B \triangleleft_R R \) with \( B \in \mathcal{K} \), then \( R/B^* \in \mathcal{K} \), where \( B^* \) consists of \( a \in R \) such that \((H \cdot a)B = 0 = B(H \cdot a)\).

It is clear that (S3) may be replaced by one of the following conditions:

(S3') If \( B \) is an essential \( H \)-ideal of \( R \)(i.e. \( B \cap I \neq 0 \) for any non-zero \( H \)-ideal \( I \) of \( R \)) and \( B \in \mathcal{K} \), then \( R \in \mathcal{K} \).

(S3'') If there exists an \( H \)-ideal \( B \) of \( R \) with \( B^* = 0 \) and \( B \in \mathcal{K} \), then \( R \in \mathcal{K} \).

It is easy to check that if \( \mathcal{K} \) is an \( H \)-special class, then \( \mathcal{K} \) is an \( H \)-weakly special class.

Theorem 1.5 If \( \mathcal{K} \) is an \( H \)-weakly special class, then \( r^\mathcal{K}(R) = \cap \{ I \triangleleft_H R \mid R/I \in \mathcal{K} \} \), where \( r^\mathcal{K} \) denotes the \( H \)-upper radical determined by \( \mathcal{K} \).

Proof. If \( I \) is a non-zero \( H \)-ideal of \( R \) and \( I \in \mathcal{K} \), then \( R/I^* \in \mathcal{K} \) by (S3) in Definition 1.4 and \( I \not\in I^* \) by Proposition 1.3. Consequently, it follows from [7, Proposition 5] that

\[
r^\mathcal{K}(R) = \cap \{ I \mid I \text{ is an } H\text{-ideal of } R \text{ and } R/I \in \mathcal{K} \} .
\]

\( \Box \)
In this section, we give the characterization of \( H \)-radical (i.e. if \( R \) is an \( r \)-\( H \)-module algebra and \( B \) is an \( H \)-ideal of \( R \), then so is \( B \) ) and any nilpotent \( H \)-module algebra is an \( r \)-\( H \)-module algebra, then \( r \) is called a supernilpotent \( H \)-radical.

**Proposition 1.7** \( r \) is a supernilpotent \( H \)-radical, then \( r \) is \( H \)-strongly hereditary, i.e. \( r(I) = r(R) \cap I \) for any \( I \triangleleft_H R \).

**Proof.** It follows from \([7, \text{Proposition 4}] \). \( \square \)

**Theorem 1.8** If \( \mathcal{K} \) is an \( H \)-weakly special class, then \( r^\mathcal{K} \) is a supernilpotent \( H \)-radical.

**Proof.** Let \( r = r^\mathcal{K} \). Since every non-zero \( H \)-homomorphic image \( R' \) of a nilpotent \( H \)-module algebra \( R \) is nilpotent and is not \( H \)-semiprime, we have that \( R \) is an \( r \)-\( H \)-module algebra by Theorem 1.5. It remains to show that any \( H \)-ideal \( I \) of \( r \)-\( H \)-module algebra \( R \) is an \( r \)-\( H \)-ideal. If \( I \) is not an \( r \)-\( H \)-module algebra, then there exists an \( H \)-ideal \( J \) of \( I \) such that \( 0 \neq I/J \in \mathcal{K} \). By (S3), \((R/J)/(I/J)^* \in \mathcal{K} \). Let \( Q = \{ x \in R \mid (H \cdot x)I \subseteq J \) and \( I(H \cdot x) \subseteq J \} \). It is clear that \( J \) and \( Q \) are \( H \)-ideals of \( R \) and \( Q/J = (I/J)^* \). Since \( R/Q \cong (R/J)/(Q/J) = (R/J)/(I/J)^* \) and \( R/Q \) is an \( r \)-\( H \)-module algebra, we have \((R/J)/(I/J)^* \) is an \( r \)-\( H \)-module algebra. Thus \( R/Q = 0 \) and \( I^2 \subseteq J \), which contradicts that \( I/J \) is a non-zero \( H \)-semiprime module algebra. Thus \( I \) is an \( r \)-\( H \)-ideal. \( \square \)

**Proposition 1.9** \( R \) is \( H \)-semiprime iff for any \( 0 \neq a \in R \), there exists an \( H \)-m-sequence \( \{a_n\} \) in \( R \) with \( a_{7,1,1} = a \) such that \( a_n \neq 0 \) for all \( n \).

**Proof.** If \( R \) is \( H \)-semiprime, then for any \( 0 \neq a \in R \), there exist \( b_1 \in R \), \( h_1 \) and \( h'_1 \in H \) such that \( 0 \neq a_2 = (h_1 \cdot a_1)b_1(h'_1 \cdot a_1) \in (H \cdot a_1)R(H \cdot a_1) \) by Proposition 1.2, where \( a_1 = a \). Similarly, for \( 0 \neq a_2 \in R \), there exist \( b_2 \in R \) and \( h_2 \) and \( h'_2 \in H \) such that \( 0 \neq a_3 = (h_2 \cdot a_2)b_2(h'_2 \cdot a_2) \in (H \cdot a_2)R(H \cdot a_2) \), which implies that there exists an \( H \)-m-sequence \( \{a_n\} \) such that \( a_n \neq 0 \) for any natural number \( n \). Conversely, it is trivial. \( \square \)

## 2 \( H \)-Baer radical

In this section, we give the characterization of \( H \)-Baer radical (\( H \)-prime radical) in many ways.

**Theorem 2.1** We define a property \( r^H_b \) for \( H \)-module algebras as follows: \( R \) is an \( r^H_b \)-\( H \)-module algebra iff every non-zero \( H \)-homomorphic image of \( R \) contains a non-zero nilpotent \( H \)-ideal; then \( r^H_b \) is an \( H \)-radical property.
Proof. It is clear that every $H$-homomorphic image of $r_{Hb}H$-module algebra is an $r_{Hb}H$-module algebra. If every non-zero $H$-homomorphic image $B$ of $H$-module algebra $R$ contains a non-zero $r_{Hb}H$-ideal $I$, then $I$ contains a non-zero nilpotent $H$-ideal $J$. It is clear that $(J)$ is a non-zero nilpotent $H$-ideal of $B$, where $(J)$ denotes the $H$-ideal generated by $J$ in $B$. Thus $R$ is an $r_{Hb}H$-module algebra. Consequently, $r_{Hb}$ is an $H$-radical property. \(\square\)

$r_{Hb}$ is called $H$-prime radical or $H$-Baer radical.

**Theorem 2.2** Let

$$E = \{ R \mid R \text{ is a nilpotent } H \text{-module algebra} \},$$

then $r_E = r_{Hb}$, where $r_E$ denotes the $H$-lower radical determined by $E$.

**Proof.** If $R$ is an $r_{Hb}H$-module algebra, then every non-zero $H$-homomorphic image $B$ of $R$ contains a non-zero nilpotent $H$-ideal $I$. By the definition of the lower $H$-radical, $I$ is an $r_EH$-module algebra. Consequently, $R$ is an $r_EH$-module algebra. Conversely, since every nilpotent $H$-module algebra is an $r_{Hb}H$-module algebra, $r_E \leq r_{Hb}$. \(\square\)

**Proposition 2.3** $R$ is $H$-semiprime if and only if $r_{Hb}(R) = 0$.

**Proof.** If $R$ is $H$-semiprime with $r_{Hb}(R) \neq 0$, then there exists a non-zero nilpotent $H$-ideal $I$ of $r_{Hb}(R)$. It is clear that $H$-ideal $(I)$, which the $H$-ideal generated by $I$ in $R$, is a non-zero nilpotent $H$-ideal of $R$. This contradicts that $R$ is $H$-semiprime. Thus $r_{Hb}(R) = 0$. Conversely, if $R$ is an $H$-module algebra with $r_{Hb}(R) = 0$ and there exists a non-zero nilpotent $H$-ideal $I$ of $R$, then $I \subseteq r_{Hb}(R)$. We get a contradiction. Thus $R$ is $H$-semiprime if $r_{Hb}(R) = 0$. \(\square\)

**Theorem 2.4** If $K = \{ R \mid R \text{ is an } H \text{-prime module algebra} \}$, then $K$ is an $H$-special class and $r_{Hb} = r^K$.

**Proof.** Obviously, $(S1)$ holds. If $I$ is a non-zero $H$-ideal of an $H$-prime module algebra $R$ and $BC = 0$ for $H$-ideals $B$ and $C$ of $I$, then $(B)^3(C)^3 = 0$ where $(B)$ and $(C)$ denote the $H$-ideals generated by $B$ and $C$ in $R$ respectively. Since $R$ is $H$-prime, $(B) = 0$ or $(C) = 0$, i.e. $B = 0$ or $C = 0$. Consequently, $(S2)$ holds. Now we shows that $(S3)$ holds. Let $B$ be an $H$-prime module algebra and be an $H$-ideal of $R$. If $JI \subseteq B$ for $H$-ideals $I$ and $J$ of $R$, then $(BJ)(IB) = 0$, where $B^* = \{ x \in R \mid (H \cdot x)B = 0 = B(H \cdot x) \}$. Since $B$ is an $H$-prime module algebra, $BJ = 0$ or $IB = 0$. Considering $I$ and $J$ are $H$-ideals, we have that $B(H \cdot J) = 0$ or $(H \cdot I)B = 0$. By Proposition 1.3, $J \subseteq B^*$ or $I \subseteq B^*$, which implies that $R/B^*$ is an $H$-prime module algebra. Consequently, $(S3)$ holds and so $K$ is an $H$-special class.
Next we show that $r_{Hb} = r^K$. By Proposition 1.5, $r^K(R) = \cap \{I \mid I$ is an $H$-ideal of $R$ and $R/I \in K\}$. If $R$ is a nilpotent $H$-module algebra, then $R$ is an $r^K$-$H$-module algebra. It follows from Theorem 2.2 that $r_{Hb} \leq r^K$. Conversely, if $r_{Hb}(R) = 0$, then $R$ is an $H$-semiprime module algebra by Proposition 2.3. For any $0 \neq a \in R$, there exist $b_1 \in R$, $h_1, h'_1 \in H$ such that $0 \neq a_2 = (h_1 \cdot a_1)b_1(h'_1 \cdot a_1) \in (H \cdot a_1)R(H \cdot a_1)$, where $a_1 = a$. Similarly, for $0 \neq a_2 \in R$, there exist $b_2 \in R$ and $h_2, h'_2 \in H$ such that $0 \neq a_3 = (h_2 \cdot a_2)b_2(h'_2 \cdot a_2) \in (H \cdot a_3)R(H \cdot a_2)$. Thus there exists an $H$-$m$-sequence $\{a_n\}$ such that $a_n \neq 0$ for any natural number $n$. Let

$$\mathcal{F} = \{I \mid I$ is an $H$-ideal of $R$ and $I \cap \{a_1, a_2, \ldots\} = \emptyset\}.$$ 

By Zorn’s Lemma, there exists a maximal element $P$ in $\mathcal{F}$. If $I$ and $J$ are $H$-ideals of $R$ and $I \not\subseteq P$ and $J \not\subseteq P$, then there exist natural numbers $n$ and $m$ such that $a_n \in I$ and $a_m \in J$. Since $0 \neq a_{n+m+1} = (h_{n+m} \cdot a_{n+m})b_{n+m}(h'_{n+m} \cdot a_{n+m}) \in IJ$, which implies that $IJ \not\subseteq P$ and so $P$ is an $H$-prime ideal of $R$. Obviously, $a \notin P$, which implies that $a \notin r^K(R)$ and $r^K(R) = 0$. Consequently, $r^K = r_{Hb}$. □

**Theorem 2.5** $r_{Hb}(R) = W_H(R)$.

**Proof.** If $0 \neq a \notin W_H(R)$, then there exists an $H$-prime ideal $P$ such that $a \notin P$ by the proof of Theorem 2.4. Thus $a \notin r_{Hb}(R)$, which implies that $r_{Hb}(R) \subseteq W_H(R)$. Conversely, for any $x \in W_H(R)$, let $R = R/\sim_{Hb}(R)$. Since $r_{Hb}(R) = 0$, $R$ is an $H$-semiprime module algebra by Proposition 2.3. By the proof of Theorem 2.4, $W_H(\bar{R}) = 0$. For an $H$-$m$-sequence $\{\bar{a}_n\}$ with $\bar{a}_1 = \bar{x}$ in $\bar{R}$, there exist $\overline{b}_n \in \bar{R}$ and $h_n, h'_n \in H$ such that

$$\overline{a}_{n+1} = (h_n \cdot \overline{a}_n)\overline{b}_n(h'_n \cdot \overline{a}_n)$$

for any natural number $n$. Thus there exists $a'_n \in R$ such that $a'_n = x$ and $a'_{n+1} = (h_n \cdot a'_n)\overline{b}_n(h'_n \cdot a'_n)$ for any natural number $n$. Since $\{a'_n\}$ is an $H$-$m$-sequence with $a'_1 = x$ in $R$, there exists a natural number $k$ such that $a'_k = 0$. It is easy to show that $\overline{a}_n = \overline{a}'_n$ for any natural number $n$ by induction. Thus $\bar{a}_k = 0$ and $\overline{a} \in W_H(\bar{R})$. Considering $W_H(\bar{R}) = 0$, we have $x \in r_{Hb}(R)$ and $W_H(R) \subseteq r_{Hb}(R)$. Therefore $W_H(R) = r_{Hb}(R)$. □

**Definition 2.6** We define an $H$-ideal $N_\alpha$ in $H$-module algebra $R$ for every ordinal number $\alpha$ as follows:

(i) $N_0 = 0$.

Let us assume that $N_\alpha$ is already defined for $\alpha \prec \beta$.

(ii) If $\beta = \alpha + 1$, $N_\beta/N_\alpha$ is the sum of all nilpotent $H$-ideals of $R/N_\alpha$.

(iii) If $\beta$ is a limit ordinal number, $N_\beta = \sum_{\alpha \prec \beta} N_\alpha$.

By set theory, there exists an ordinal number $\tau$ such that $N_\tau = N_{\tau+1}$.

**Theorem 2.7** $N_\tau = r_{Hb}(R) = \cap \{I \mid I$ is an $H$-semiprime ideal of $R\}$. 

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Proof. Let \( D = \cap \{ I \mid I \text{ is an } H\text{-semiprime ideal of } R \} \). Since \( R/N_\tau \) has not any non-zero nilpotent \( H \)-ideal, we have that \( r_{Hb}(R) \subseteq N_\tau \) by Proposition 2.3. Obviously, \( D \subseteq r_{Hb}(R) \). Using transfinite induction, we can show that \( N_\alpha \subseteq I \) for every \( H \)-semiprime ideal \( I \) of \( R \) and every ordinal number \( \alpha \) (see the proof of [12, Theorem 3.7] ). Thus \( N_\tau \subseteq D \), which completes the proof. \( \Box \)

Definition 2.8 Let \( \emptyset \neq L \subseteq H \). An \( H \)-m-sequence \( \{ a_n \} \) in \( R \) is called an \( L \)-m-sequence with beginning \( a \) if \( a_{7,1,1} = a \) and \( a_{n+1} = (h_n, a_n)b_n(h'_n, a_n) \) such that \( h_n, h'_n \in L \) for all \( n \). For every \( L \)-m-sequence \( \{ a_n \} \) with \( a_{7,1,1} = a \), there exists a natural number \( k \) such that \( a_k = 0 \), then \( R \) is called an \( L \)-m-nilpotent element, written as \( W_L(R) = \{ a \in R \mid a \text{ is an } L \text{-m-nilpotent element} \} \).

Similarly, we have

Proposition 2.9 If \( L \subseteq H \) and \( H = kL \), then

(i) \( R \) is \( H \)-semiprime iff \( (L.a)R(L.a) = 0 \) always implies \( a = 0 \) for any \( a \in R \).

(ii) \( R \) is \( H \)-prime iff \( (L.a)R(L.b) = 0 \) always implies \( a = 0 \) or \( b = 0 \) for any \( a, b \in R \).

(iii) \( R \) is \( H \)-semiprime if and only if for any \( 0 \neq a \in R \), there exists an \( L \)-m-sequence \( \{ a_n \} \) with \( a_1 = a \) such that \( a_n \neq 0 \) for all \( n \).

(iv) \( W_H(R) = W_L(R) \).

3 The \( H \)-module theoretical characterization of \( H \)-special radicals

If \( V \) is an algebra over \( k \) with unit and \( x \otimes 1_V = 0 \) always implies that \( x = 0 \) for any right \( k \)-module \( M \) and for any \( x \in M \), then \( V \) is called a faithful algebra to tensor. For example, if \( k \) is a field, then \( V \) is faithful to tensor for any algebra \( V \) with unit.

In this section, we need to add the following condition: \( H \) is faithful to tensor.

We shall characterize \( H \)-Baer radical \( r_{Hb} \), \( H \)-locally nil radical \( r_{Hl} \), \( H \)-Jacobson radical \( r_{Hj} \) and \( H \)-Brown-McCoy radical \( r_{Hbm} \) by \( R \)-\( H \)-modules.

We can view every \( H \)-module algebra \( R \) as a sub-algebra of \( R\#H \) since \( H \) is faithful to tensor. By computation, we have that

\[
h \cdot a = \sum (1\#h_1)a(1\#S(h_2))
\]

for any \( h \in H, a \in R \), where \( S \) is the antipode of \( H \).

Definition 3.1 An \( R \)-\( H \)-module \( M \) is called an \( R \)-\( H \)-prime module if for \( M \) the following conditions are fulfilled:

(i) \( RM \neq 0; \)
(ii) If \( x \) is an element of \( M \) and \( I \) is an \( H \)-ideal of \( R \), then \( I(Hx) = 0 \) always implies \( x = 0 \) or \( I \subseteq (0 : M)_R \), where \( (0 : M)_R = \{a \in R \mid aM = 0\} \).

**Definition 3.2** We associate to every \( H \)-module algebra \( R \) a class \( \mathcal{M}_R \) of \( R-H \)-modules. Then the class \( \mathcal{M} = \bigcup \mathcal{M}_R \) is called an \( H \)-special class of modules if the following conditions are fulfilled:

1. \((M1)\) If \( M \in \mathcal{M}_R \), then \( M \) is an \( R-H \)-prime module.
2. \((M2)\) If \( I \) is an \( H \)-ideal of \( R \) and \( M \in \mathcal{M}_I \), then \( IM \in \mathcal{M}_R \).
3. \((M3)\) If \( M \in \mathcal{M}_R \) and \( I \) is an \( H \)-ideal of \( R \) with \( IM \neq 0 \), then \( M \in \mathcal{M}_I \).
4. \((M4)\) Let \( I \) be an \( H \)-ideal of \( R \) and \( R = R/I \). If \( M \in \mathcal{M}_R \) and \( I \subseteq (0 : M)_R \), then \( M \in \mathcal{M}_R \). Conversely, if \( M \in \mathcal{M}_R \), then \( M \in \mathcal{M}_R \).

Let \( \mathcal{M}(R) \) denote \( \cap \{(0 : M)_R \mid M \in \mathcal{M}_R\} \), or \( R \) when \( \mathcal{M}_R = \emptyset \).

**Lemma 3.3**

1. \((1)\) If \( M \) is an \( R-H \)-module, then \( M \) is an \( R \#H \)-module. In this case, \( (0 : M)_R \cap R = (0 : M)_R \) and \( (0 : M)_R \) is an \( H \)-ideal of \( R \).
2. \((2)\) \( R \) is a non-zero \( H \)-prime module algebra iff there exists a faithful \( R-H \)-prime module \( M \);
3. \((3)\) Let \( I \) be an \( H \)-ideal of \( R \) and \( \bar{R} = R/I \). If \( M \) is an \( \bar{R}-H \)-(resp. prime, irreducible)module and \( I \subseteq (0 : M)_R \), then \( M \) is an \( \bar{R}-H \)-(resp. prime, irreducible)module (defined by \( h \cdot (a + I) = h \cdot a \) and \( (a + I)x = ax \)). Conversely, if \( M \) is an \( \bar{R}-H \)-(resp. prime irreducible)module, then \( M \) is an \( R-H \)-(resp. prime, irreducible)module (defined by \( h \cdot a = h \cdot (a + I) \) and \( ax = (a + I)x \)). In the both cases, it is always true that \( R/(0 : M)_R \cong \bar{R}/(0 : M)_{\bar{R}} \).
4. \((4)\) \( I \) is an \( H \)-prime ideal of \( R \) with \( I \neq R \) iff there exists an \( R-H \)-prime module \( M \) such that \( I = (0 : M)_R \);
5. \((5)\) If \( I \) is an \( H \)-ideal of \( R \) and \( M \) is an \( I-H \)-prime module, then \( IM \) is an \( R-H \)-prime module with \( (0 : M)_I = (0 : IM)_R \cap I \);
6. \((6)\) If \( M \) is an \( R-H \)-prime module and \( I \) is an \( H \)-ideal of \( R \) with \( IM \neq 0 \), then \( M \) is an \( I-H \)-prime module;
7. \((7)\) If \( R \) is an \( H \)-semiprime module algebra with one side unit, then \( R \) has a unit.

**Proof.**

1. Obviously, \( (0 : M)_R = (0 : M)_{R \#H} \cap R \). For any \( h \in H, a \in (0 : M)_R \), we see that \( (h \cdot a)M = \sum(1 \#h_1)a(1 \#S(h_2))M \subseteq \sum(1 \#h_1)aM = 0 \) for any \( h \in H, a \in R \). Thus \( h \cdot a \in (0 : M)_R \), which implies \( (0 : M)_R \) is an \( H \)-ideal of \( R \).

2. If \( R \) is an \( H \)-prime module algebra, view \( M = R \) as an \( R-H \)-module. Obviously, \( M \) is faithful. If \( I(H \cdot x) = 0 \) for \( 0 \neq x \in M \) and an \( H \)-ideal \( I \) of \( R \), then \( I(x) = 0 \) and \( I = 0 \), where \( (x) \) denotes the \( H \)-ideal generated by \( x \) in \( R \). Consequently, \( M \) is a faithful \( R-H \)-prime module. Conversely, let \( M \) be a faithful \( R-H \)-prime module. If \( IJ = 0 \) for two \( H \)-ideals \( I \) and \( J \) of \( R \) with \( J \neq 0 \), then \( JM \neq 0 \) and there exists \( 0 \neq x \in JM \) such
that \( I(Hx) = 0 \). Since \( M \) is a faithful \( R-H \)-prime module, \( I = 0 \). Consequently, \( R \) is \( H \)-prime.

(3) If \( M \) is an \( R-H \)-module, then it is clear that \( M \) is a (left) \( \overline{R} \)-module and \( h(\overline{ax}) = h(ax) = \sum (h_1 \cdot a)(h_2x) = \sum [h_1 \cdot a](h_2x) = \sum (h_1 \cdot a)(h_2x) \) for any \( h \in H \), \( a \in R \) and \( x \in M \). Thus \( M \) is an \( \overline{R} \)-\( H \)-module. Conversely, if \( M \) is an \( \overline{R} \)-\( H \)-module, then \( M \) is an (left) \( R \)-module and

\[
h(ax) = h(\overline{a}x) = \sum (h_1 \cdot \overline{a})(h_2x) = \sum \overline{h_1} \cdot a(h_2x) = \sum (h_1 \cdot a)(h_2x)
\]

for any \( h \in H \), \( a \in R \) and \( x \in M \). This shows that \( M \) is an \( R-H \)-module.

Let \( M \) be an \( R-H \)-prime module and \( I \) be an \( H \)-ideal of \( R \) with \( I \subseteq (0 : M)_R \). If \( \overline{J}(Hx) = 0 \) for \( 0 \neq x \in M \) and an \( H \)-ideal \( J \) of \( R \), then \( J(Hx) = 0 \) and \( J \subseteq (0 : M)_R \). This shows that \( \overline{J} \subseteq (0 : M)_{\overline{R}} \). Thus \( M \) is an \( R-H \)-prime module. Similarly, we can show the other assert.

(4) If \( I \) is an \( H \)-prime ideal of \( R \) with \( R \neq I \), then \( \overline{R} = R/I \) is an \( H \)-prime module algebra. By Part (2), there exists a faithful \( \overline{R} \)-\( H \)-prime module \( M \). By part (3), \( M \) is an \( R-H \)-prime module with \( (0 : M)_R = I \). Conversely, if there exists a \( R-H \)-prime \( M \) with \( I = (0 : M)_R \), then \( M \) is a faithful \( \overline{R} \)-\( H \)-prime module by part (3) and \( I \) is an \( H \)-prime ideal of \( R \) by part (2).

(5) First, we show that \( IM \) is an \( R \)-module. We define

\[
a(\sum_i a_i x_i) = \sum_i (aa_i)x_i
\]

for any \( a \in R \) and \( \sum_i a_i x_i \in IM \), where \( a_i \in I \) and \( x_i \in M \). If \( \sum_i a_i x_i = \sum_i a'_i x'_i \) with \( a_i, a'_i \in R \), \( x_i, x'_i \in M \), let \( y = \sum_i (a_i)x_i - \sum_i (aa'_i)x'_i \). For any \( b \in I \) and \( h \in H \), we see that

\[
b(hy) = \sum_i b\{h[(aa_i)x_i - (aa'_i)x'_i]\} = \sum_i \sum_{(h)} b\{[(h_1 \cdot (aa_i))(h_2x_i)] - [h_1 \cdot (aa'_i)](h_2x'_i)\} = \sum_i \sum_{(h)} \{b[(h_1 \cdot a)(h_2 \cdot a_i)](h_3x_i) - b[(h_1 \cdot a)(h_2 \cdot a'_i)](h_3x'_i)\} = \sum_i \sum_{(h)} b(h_1 \cdot a)[h_2(a_i x_i) - h_2(a'_i x'_i)] = \sum_i b(h_1 \cdot a)h_2 [a_i x_i - a'_i x'_i] = 0.
\]

Thus \( I(Hy) = 0 \). Since \( M \) is an \( I-H \)-prime module and \( IM \neq 0 \), we have that \( y = 0 \). Thus this definition in (1) is well-defined. It is easy to check that \( IM \) is an \( R \)-module. We see that

\[
h(a \sum_i a_i x_i) = \sum_i h[(aa_i)x_i]
\]
for any $h \in H$ and $\sum_i a_i x_i \in IM$. Thus $IM$ is an $R$-$H$-module.

Next, we show that $(0 : M)_I = (0 : IM)_R \cap I$. If $a \in (0 : M)_I$, then $aM = 0$ and $aIM = 0$, i.e. $a \in (0 : IM)_A \cap I$. Conversely, if $a \in (0 : IM)_R \cap I$, then $aIM = 0$. By part (1), $(0 : IM)_R$ is an $H$-ideal of $R$. Thus $(H \cdot a)IM = 0$ and $(H \cdot a)I \subseteq (0 : M)_I$. Since $(0 : M)_I$ is an $H$-prime ideal of $I$ by part (4), $a \in (0 : M)_I$. Consequently, $(0 : M)_I = (0 : IM)_R \cap I$.

Finally, we show that $IM$ is an $R$-$H$-prime module. If $RIM = 0$, then $RI \subseteq (0 : M)_R$ and $I \subseteq (0 : M)_R$, which contradicts that $M$ is an $I$-$H$-prime module. Thus $RIM \neq 0$. If $J(Hx) = 0$ for $0 \neq x \in IM$ and an $H$-ideal $J$ of $R$, then $JI(Hx) \subseteq J(Hx) = 0$. Since $M$ is an $I$-$H$-prime module, $JI \subseteq (0 : M)_I$ and $J(IM) = 0$. Consequently, $IM$ is an $R$-$H$-prime module.

(6) Obviously, $M$ is an $I$-$H$-module. If $J(Hx) = 0$ for $0 \neq x \in M$ and an $H$-ideal $J$ of $I$, then $(J)^3(Hx) = 0$ and $(J)^3 \subseteq (0 : M)_R$, where $(J)$ denotes the $H$-ideal generated by $J$ in $R$. Since $(0 : M)_R$ is an $H$-prime ideal of $R$, $(J) \subseteq (0 : M)_R$ and $J \subseteq (0 : M)_I$.

Consequently, $M$ is an $I$-$H$-prime module.

(7) We can assume that $u$ is a right unit of $R$. We see that

$$
(h \cdot (au - a))b = \sum (1#h) (au - a)(1#S(h_2))b = 0
$$

for any $a, b \in R, h \in H$. Therefore $(H \cdot (au - a))R = 0$ and $au = a$, which implies that $R$ has a unit. □

**Theorem 3.4** (1) If $\mathcal{M}$ is an $H$-special class of modules and $\mathcal{K} = \{ R \mid$ there exists a faithful $R$-$H$-module $M \in \mathcal{M}_R \}$, then $\mathcal{K}$ is an $H$-special class and $r^\mathcal{K}(R) = \mathcal{M}(R)$.

(2) If $\mathcal{K}$ is an $H$-special class and $\mathcal{M}_R = \{ M \mid M$ is an $R$-$H$-prime module and $R/(0 : M)_R \in \mathcal{K} \}$, then $\mathcal{M} = \cup \mathcal{M}_R$ is an $H$-special class of modules and $r^\mathcal{K}(R) = \mathcal{M}(R)$.

**Proof.** (1) By Lemma 3.3(2), $(S1)$ is satisfied. If $I$ is a non-zero $H$-ideal of $R$ and $R \in \mathcal{K}$, then there exists a faithful $R$-$H$-prime module $M \in \mathcal{M}_R$. Since $M$ is faithful, $IM \neq 0$ and $M \in \mathcal{M}_I$ with $(0 : M)_I = (0 : M)_R \cap I = 0$ by (M3). Thus $I \in \mathcal{K}$ and $(S2)$ is satisfied. Now we show that $(S3)$ holds. If $I$ is an $H$-ideal of $R$ with $I \in \mathcal{K}$, then there exists a faithful $I$-$H$-prime module $M \in \mathcal{M}_I$. By (M2) and Lemma 3.3(5), $IM \in \mathcal{M}_R$ and $0 = (0 : M)_I = (0 : IM)_R \cap I$. Thus $(0 : IM)_R \subseteq I^*$. Obviously, $I^* \subseteq (0 : IM)_R$. Thus
\( I^* = (0 : IM)_R \). Using (M4), we have that \( IM \in \mathcal{M}_R \) and \( IM \) is a faithful \( \mathcal{R} \)-\( H \)-module with \( \mathcal{R} = R/I^* \). Thus \( R/I^* \in \mathcal{K} \). Therefore \( \mathcal{K} \) is an \( H \)-special class.

It is clear that

\[
\{ I \mid I \text{ is an } H \text{-ideal of } R \text{ and } R/I \in \mathcal{K} \} = \{(0 : M)_R \mid M \in \mathcal{M}_R \}.
\]

Thus \( r^\mathcal{K}(R) = \mathcal{M}(R) \).

(2) It is clear that (M1) is satisfied. If \( I \) is an \( H \)-ideal of \( R \) with \( M \in \mathcal{M}_I \), then \( M \) is an \( I \)-\( H \)-prime module with \( I/(0 : M)_I \in \mathcal{K} \). By Lemma 3.3(5), \( IM \) is an \( R \)-\( H \)-prime module with \( (0 : M)_I = (0 : IM)_R \cap I \). It is clear that

\[
(0 : IM)_R = \{ a \in R \mid (H \cdot a)I \subseteq (0 : M)_I \text{ and } I(H \cdot a) \subseteq (0 : M)_I \}
\]

and

\[
(0 : IM)_R/(0 : M)_I = (I/(0 : M)_I)^*.
\]

Thus \( R/(0 : IM)_R \cong (R/(0 : M)_I)/(0 : IM)_R/(0 : M)_I = (R/(0 : M)_I)/(I/(0 : M)_I)^* \in \mathcal{K} \), which implies that \( IM \in \mathcal{M}_R \) and (M2) holds. Let \( M \in \mathcal{M}_R \) and \( I \) be an \( H \)-ideal of \( R \) with \( IM \neq 0 \). By Lemma 3.3(6), \( M \) is an \( I \)-\( H \)-prime module and \( I/(0 : M)_I = I/(0 : M)_R \cap I \equiv (I + (0 : M)_R)/(0 : M)_R \). Since \( R/(0 : M)_R \in \mathcal{K} \), \( I/(0 : M)_I \in \mathcal{K} \) and \( M \in \mathcal{M}_I \). Thus (M3) holds. It follows from Lemma 3.3(3) that (M4) holds.

It is clear that

\[
\{ I \mid I \text{ is an } H \text{-ideal of } R \text{ and } 0 \neq R/I \in \mathcal{K} \} = \{(0 : M)_R \mid M \in \mathcal{M}_R \}.
\]

Thus \( r^\mathcal{K}(R) = \mathcal{M}(R) \).

\[
\text{Theorem 3.5} \quad \text{Let } \mathcal{M}_R = \{ M \mid M \text{ is an } R \text{-} H \text{-prime module} \} \text{ for any } H \text{-module algebra } R \text{ and } \mathcal{M} = \cup \mathcal{M}_R. \text{ Then } \mathcal{M} \text{ is an } H \text{-special class of modules and } \mathcal{M}(R) = r_{H^b}(R).
\]

\[
\text{Proof.} \text{ It follows from Lemma 3.3(3)(5)(6) that } \mathcal{M} \text{ is an } H \text{-special class of modules. By Lemma 3.3(2),}
\]

\[
\{ R \mid R \text{ is an } H \text{-prime module algebra with } R \neq 0 \} =
\{ R \mid \text{there exists a faithful } R \text{-} H \text{-prime module} \}
\]

Thus \( r_{H^b}(R) = \mathcal{M}(R) \) by Theorem 2.4(1).

\[
\text{Theorem 3.6} \quad \text{Let } \mathcal{M}_R = \{ M \mid M \text{ is an } R \text{-} H \text{-irreducible module} \} \text{ for any } H \text{-module algebra } R \text{ and } \mathcal{M} = \cup \mathcal{M}_R. \text{ Then } \mathcal{M} \text{ is an } H \text{-special class of modules and } \mathcal{M}(R) = r_{H^j}(R), \text{ where } r_{H^j} \text{ is the } H \text{-Jacobson radical of } R \text{ defined in [7].}
\]
Proof. If \( M \) is an \( R\)-\( H \)-irreducible module and \( J(Hx) = 0 \) for \( 0 \neq x \in M \) and an \( H \)-ideal \( J \) of \( R \), let \( N = \{ m \in M \mid J(Hm) = 0 \} \). Since \( J(h(am)) = J(\sum_{h_1 \in H} (h_1 \cdot a)(h_2m)) = 0 \), \( am \in N \) for any \( m \in N, h \in H, a \in R \), we have that \( N \) is an \( R \)-submodule of \( M \). Obviously, \( N \) is an \( H \)-submodule of \( M \). Thus \( N \) is an \( R\)-\( H \)-submodule of \( M \). Since \( N \neq 0 \), we have that \( N = M \) and \( JM = 0 \), i.e. \( J \subseteq (0 : M)_R \). Thus \( M \) is an \( R\)-\( H \)-prime module and \((M_1)\) is satisfied. If \( M \) is an \( I\)-\( H \)-irreducible module and \( I \) is an \( H \)-ideal, then \( IM \) is an \( R\)-\( H \)-module. If \( N \) is an \( R\)-\( H \)-submodule of \( IM \), then \( N \) is also an \( I\)-\( H \)-submodule of \( M \), which implies that \( N = 0 \) or \( N = M \). Thus \((M_2)\) is satisfied. If \( M \) is an \( R\)-\( H \)-irreducible module and \( I \) is an \( H \)-ideal of \( R \) with \( IM \neq 0 \), then \( IM = M \). If \( N \) is a non-zero \( I\)-\( H \)-submodule of \( M \), then \( IN \) is an \( R\)-\( H \)-submodule of \( M \) by Lemma 3.3(5) and \( IN = 0 \) or \( IN = M \). If \( IN = 0 \), then \( I \subseteq (0 : M)_R \) by the above proof and \( IM = 0 \). We get a contradiction. If \( IN = M \), then \( N = M \). Thus \( M \) is an \( I\)-\( H \)-irreducible module and \((M_3)\) is satisfied.

It follows from Lemma 3.3(3) that \((M_4)\) holds. By Theorem 3.4(1), \( \mathcal{M}(R) = r_{Rj}(R) \). □

J.R. Fisher [7, Proposition 2] constructed an \( H \)-radical \( r_H \) by a common hereditary radical \( r \) for algebras, i.e. \( r_H(R) = (r(R) : H) = \{ a \in R \mid h \cdot a \in r(R) \text{ for any } h \in H \} \). Thus we can get \( H \)-radicals \( r_{bH}, r_{1H}, r_{jH}, r_{bnH} \).

Definition 3.7  An \( R\)-\( H \)-module \( M \) is called an \( R\)-\( H \)-BM-module, if for \( M \) the following conditions are fulfilled:

(i) \( RM \neq 0 \);

(ii) If \( I \) is an \( H \)-ideal of \( R \) and \( I \nsubsetneq (0 : M)_R \), then there exists an element \( u \in I \) such that \( m = um \) for all \( m \in M \).

Theorem 3.8 Let \( \mathcal{M}_R = \{ M \mid M \text{ is an } R\)-\( H \)-BM-module \} \) for every \( H \)-module algebra \( R \) and \( \mathcal{M} = \cup \mathcal{M}_R \). Then \( \mathcal{M} \) is an \( H \)-special class of modules.

Proof. It is clear that \( M \) satisfies \((M_1)\) and \((M_4)\). To prove \((M_2)\) we exhibit: if \( I \nsubsetneq H \) \( R \) and \( M \in \mathcal{M}_I \), then \( M \) is an \( I\)-\( H \)-prime module and \( IM \) is an \( R\)-\( H \)-prime module. If \( J \) is an \( H \)-ideal of \( R \) with \( J \nsubsetneq (0 : M)_R \), then \( JI \) is an \( H \)-ideal of \( I \) with \( JI \nsubsetneq (0 : M)_I \). Thus there exists an element \( u \in JI \subseteq J \) such that \( um = m \) for every \( m \in M \). Hence \( IM \in \mathcal{M}_R \).

To prove \((M_3)\), we exhibit: if \( M \in \mathcal{M}_R \) and \( I \) is an \( H \)-ideal of \( R \) with \( IM \neq 0 \). If \( J \) is an \( H \)-ideal of \( I \) with \( J \nsubsetneq (0 : M)_I \), then \( (J) \nsubsetneq (0 : M)_R \), where \( (J) \) is the \( H \)-ideal generated by \( J \) in \( R \). Thus there exists an elements \( u \in (J) \) such that \( um = m \) for every \( m \in M \). Moreover,

\[ m = um = uum = uuum = u^3m \]

and \( u^3 \in J \). Thus \( M \in \mathcal{M}_I \). □
Proposition 3.9 If $M$ is an $R$-$H$-$BM$-module, then $R/(0 : M)_R$ is an $H$-simple module algebra with unit.

Proof. Let $I$ be any $H$-ideal of $R$ with $I \not\subseteq (0 : M)_R$. Since $M$ is an $R$-$H$-$BM$-module, there exists an element $u \in I$ such that $uam = am$ for every $m \in M, a \in R$. It follows that $a - ua \in (0 : M)_R$, whence $R = I + (0 : M)_R$. Thus $(0 : M)_R$ is a maximal $H$-ideal of $R$. Therefore $R/(0 : M)_R$ is an $H$-simple module algebra.

Next we shall show that $R/(0 : M)_R$ has a unit. Now $R \not\subseteq (0 : M)_R$, since $RM \neq 0$. By the above proof, there exists an element $u \in R$ such that $a - ua \in (0 : M)_R$ for any $a \in R$. Hence $R/(0 : M)_R$ has a left unit. Furthermore, by Lemma 3.7 (7) it has a unity element. $\square$

Proposition 3.10 If $R$ is an $H$-simple-module algebra with unit, then there exists a faithful $R$-$H$-$BM$-module.

Proof. Let $M = R$. It is clear that $M$ is a faithful $R$-$H$-$BM$-module. $\square$

Theorem 3.11 Let $\mathcal{M}_R = \{ M \mid M$ is an $R$-$H$-$BM$-module$\}$ for every $H$-module algebra $R$ and $\mathcal{M} = \bigcup \mathcal{M}_R$. Then $r_{Hbm}(R) = \mathcal{M}(R)$, where $r_{Hbm}$ denotes the $H$-upper radical determined by $\{ R \mid R$ is an $H$-simple module algebra with unit $\}$.

Proof. By Theorem 3.8, $\mathcal{M}$ is an $H$-special class of modules. Let

$$\mathcal{K} = \{ R \mid \text{there exists a faithful } R$-$H$-$BM$-module $\}.$$ 

By Theorem 3.4(1), $\mathcal{K}$ is an $H$-special class and $r^\mathcal{K}(R) = \mathcal{M}(R)$. Using Proposition 3.9 and 3.10, we have that

$$\mathcal{K} = \{ R \mid R$ is an $H$-simple module algebra with unit $\}.$$ 

Therefore $\mathcal{M}(R) = r_{Hbm}(R). \square$

Assume that $H$ is a finite-dimensional semisimple Hopf algebra with $t \in I_H^1$ and $\epsilon(t) = 1$. Let

$$G_t(a) = \{ z \mid z = x + (t.a)x + \sum (x_i(t.a)y_i + x_iy_i) \text{ for all } x_i, y_i, x \in R \}.$$ 

$R$ is called an $r_{gr}$-$H$-module algebra, if $a \in G_t(a)$ for all $a \in R$.

Theorem 3.12 $r_{gr}$ is an $H$-radical property of $H$-module algebra and $r_{gr} = r_{Hbm}$.

Proof. It is clear that any $H$-homomorphic image of $r_{gr}$-$H$-module algebra is an $r_{gr}$-$H$-module algebra. Let

$$N = \sum \{ I <_H I$ is an $r_{gr}$-$H$-ideal of $R \}.$$ 

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Now we show that $N$ is an $r_{gt}$-$H$-ideal of $R$. In fact, we only need to show that $I_1 + I_2$ is an $r_{gt}$-$H$-ideal for any two $r_{gt}$-$H$-ideals $I_1$ and $I_2$. For any $a \in I_1, b \in I_2$, there exist $x, x_i, y_i \in R$ such that

$$a = x + (t \cdot a)x + \sum_i (x_i(t \cdot a)y_i + x_iy_i).$$

Let

$$c = x + (t \cdot (a + b))x + \sum x_i(t \cdot (a + b))y_i + x_iy_i \in G_t(a + b).$$

Obviously,

$$a + b - c = b - (t \cdot b)x - \sum x_i(t \cdot b)y_i \in I_2.$$

Thus there exist $w, u_j, v_j \in R$ such that

$$a + b - c = w + (t \cdot (a + b - c))w + \sum_{j} (u_j(t \cdot (a + b - c))v_j + u_jv_j).$$

Let $d = (t \cdot (a + b))w + w + \sum_j (u_j(t \cdot (a + b))v_j + u_jv_j)$ and $e = c - \sum_j u_j(t \cdot c)v_j - (t \cdot c)w$.

By computation, we have that

$$a + b = d + e.$$

Since $c \in G_t(a + b)$ and $d \in G_t(a + b)$, we get that $e \in G_t(a + b)$ and $a + b \in G_t(a + b)$, which implies that $I_1 + I_2$ is an $r_{gt}$-$H$-ideal.

Let $\bar{R} = R/N$ and $\bar{B}$ be an $r_{gt}$-$H$-ideal of $\bar{R}$. For any $a \in B$, there exist $x, x_i, y_i \in R$ such that

$$\bar{a} = \bar{x} + (t \cdot \bar{a})\bar{x} + \sum (\bar{x_i}(t \cdot \bar{a})\bar{y}_i + \bar{x}_i\bar{y}_i)$$

and

$$x + (t \cdot a)x + \sum (x_i(t \cdot a)y_i + x_iy_i) - a \in N.$$

Let

$$c = x + (t \cdot a)x + \sum (x_i(t \cdot a)y_i + x_iy_i) \in G_t(a).$$

Thus there exist $w, u_j, v_j \in R$ such that

$$a - c = (t \cdot (a - c))w + w + \sum (u_j(t \cdot (a - c))v_j + u_jv_j)$$

and

$$a = (t \cdot a)w + w + \sum u_j(t \cdot a)v_j + u_jv_j + c - (t \cdot c)w - \sum u_j(t \cdot c)v_j \in G_t(a),$$

which implies that $B$ is an $r_{gt}$-$H$-ideal and $\bar{B} = 0$. Therefore $r_{gt}$ is an $H$-radical property.

\[ \square \]

**Proposition 3.13** If $R$ is an $H$-simple module algebra, then $r_{gr}(R) = 0$ iff $R$ has a unit.
Proof. If $R$ is an $H$-simple module algebra with unit 1, then $-1 \notin G_t(-1)$ since

$$x + (t \cdot (-1))x + \sum (x_i(t \cdot (-1))y_i + x_iy_i) = 0$$

for any $x, x_i, y_i \in R$. Thus $R$ is $r_{gt}$-$H$-semisimple. Conversely, if $r_{gt}(R) = 0$, then there exists $0 \neq a \notin G_t(a)$ and $G_t(a) = 0$, which implies that $ax + x = 0$ for any $x \in R$. It follows from Lemma 9.3.3 (7) that $R$ has a unit. □

**Theorem 3.14** $r_{gt} = r_{Hbn}$.

Proof. By Proposition 3.13, $r_{gt}(R) \subseteq r_{Hbn}(R)$ for any $H$-module algebra $R$. It remains to show that if $a \notin r_{gt}(R)$ then $a \notin r_{Hbn}(R)$. Obviously, there exists $b \in (a)$ such that $b \notin G_t(b)$, where $(a)$ denotes the $H$-ideal generated by $a$ in $R$. Let

$$\mathcal{E} = \{ I \triangleleft_H R \mid G_t(b) \subseteq I, b \notin I \}.$$ 

By Zorn’s Lemma, there exists a maximal element $P$ in $\mathcal{E}$. $P$ is a maximal $H$-ideal of $R$, for, if $Q$ is an $H$-ideal of $R$ with $P \subseteq Q$ and $P \neq Q$, then $b \in Q$ and $x = -bx + (bx + x) \in Q$ for any $x \in R$. Consequently, $R/P$ is an $H$-simple module algebra with $r_{gt}(R/P) = 0$. It follows from Proposition 3.13 that $R/P$ is an $H$-simple module algebra with unit and $r_{Hbn}(R) \subseteq P$. Therefore $b \notin r_{Hbn}(R)$ and so $a \notin r_{Hbn}(R)$. □

**Definition 3.15** Let $I$ be an $H$-ideal of $H$-module algebra $R$, $N$ be an $R$-$H$-submodule of $R$-$H$-module $M$. $(N, I)$ are said to have “$L$-condition”, if for any finite subset $F \subseteq I$, there exists a positive integer $k$ such that $F^k N = 0$.

**Definition 3.16** An $R$-$H$-module $M$ is called an $R$-$H$-$L$-module, if for $M$ the following conditions are fulfilled:

(i) $RM \neq 0$.

(ii) For every non-zero $R$-$H$-submodule $N$ of $M$ and every $H$-ideal $I$ of $R$, if $(N, I)$ has “$L$-condition”, then $I \subseteq (0 : M)_R$.

**Proposition 3.17** If $M$ is an $R$-$H$-$L$-module, then $R/(0 : M)_R$ is an $r_{1H}$-$H$-semisimple and $H$-prime module algebra.

Proof. If $M$ is an $R$-$H$-$L$-module, let $\bar{R} = R/(0 : M)_R$. Obviously, $\bar{R}$ is $H$-prime. If $\bar{B}$ is an $r_{1H}$-$H$-ideal of $\bar{R}$, then $(M, B)$ has ”$L$-condition” in $R$-$H$-module $M$, since for any finite subset $F$ of $B$, there exists a natural number $n$ such that $F^n \subseteq (0 : M)_R$ and $F^n M = 0$. Consequently, $B \subseteq (0 : M)_R$ and $\bar{R}$ is $r_{1H}$-semisimple. □

**Proposition 3.18** $R$ is a non-zero $r_{1H}$-$H$-semisimple and $H$-prime module algebra iff there exists a faithful $R$-$H$-$L$-module.
Proof. If $R$ is a non-zero $r_{IH}$-semisimple and $H$-prime module algebra, let $M = R$. Since $R$ is an $H$-prime module algebra, $(0 : M)_R = 0$. If $(N, B)$ has ”$L$-condition” for non-zero $R$-$H$-submodule of $M$ and $H$-ideal $B$, then, for any finite subset $F$ of $B$, there exists an natural number $n$, such that $F^nN = 0$ and $F^n(NR) = 0$, which implies that $F^n = 0$ and $B$ is an $r_{IH}$-$H$-ideal, i.e. $B = 0 \subseteq (0 : M)_R$. Consequently, $M$ is a faithful $R$-$H$-$L$- module.

Conversely, if $M$ is a faithful $R$-$H$-$L$-module, then $R$ is an $H$-prime module algebra. If $I$ is an $r_{IH}$-$H$-ideal of $R$, then $(M, I)$ has “$L$-condition”, which implies $I = 0$ and $R$ is an $r_{IH}$-semisimple module algebra. □

**Theorem 3.19** Let $\mathcal{M}_R = \{ M \mid M$ is an $R$-$H$-$L$-module$\}$ for any $H$-module algebra $R$ and $\mathcal{M} = \cup \mathcal{M}_R$. Then $\mathcal{M}$ is an $H$-special class of modules and $\mathcal{M}(R) = r_{HI}(R)$, where $K = \{ R \mid R$ is an $H$-prime module algebra with $r_{HI}(R) = 0\}$ and $r_{HI} = r^K$.

Proof. Obviously, $(M1)$ holds. To show that $(M2)$ holds, we only need to show that if $I$ is an $H$-ideal of $R$ and $M \in \mathcal{M}_I$, then $IM \in \mathcal{M}_R$. By Lemma 3.3(5), $IM$ is an $R$-$H$-prime module. If $(N, B)$ has the ”$L$-condition” for non-zero $R$-$H$-submodule $N$ of $IM$ and $H$-ideal $B$ of $R$, i.e. for any finite subset $F$ of $B$, there exists a natural number $n$ such that $F^nN = 0$, then $(N, BI)$ has ”$L$-condition” in $I$-$H$-module $M$. Thus $BI \subseteq (0 : M)_I = (0 : IM)_R \cap I$. Considering $(0 : IM)_R$ is an $H$-prime ideal of $R$, we have that $B \subseteq (0 : IM)_R$ or $I \subseteq (0 : IM)_R$. If $I \subseteq (0 : IM)_R$, then $I^2 \subseteq (0 : M)_I$ and $I \subseteq (0 : M)_I$, which contradicts $IM \neq 0$. Therefore $B \subseteq (0 : IM)_R$ and so $IM$ is an $R$-$H$-$L$- module.

To show that $(M3)$ holds, we only need to show that if $M \in \mathcal{M}_R$ and $I \triangleleft_H R$ with $IM \neq 0$, then $M \in \mathcal{M}_I$. By Lemma 3.3(6), $M$ is an $I$-$H$-prime module. If $(N, B)$ has the ”$L$-condition” for non-zero $I$-$H$-submodule $N$ of $M$ and $H$-ideal $B$ of $I$, then $IN$ is an $R$-$H$-prime module and $(IN, (B))$ has ”$L$-condition” in $R$-$H$-module $M$, since for any finite subset $F$ of $(B)$, $F^3 \subseteq B$ and there exists a natural number $n$ such that $F^{3n}N \subseteq F^{3n} = 0$, where $(B)$ is the $H$-ideal generated by $B$ in $R$. Therefore, $(B) \subseteq (0 : M)_R$ and $B \subseteq (0 : M)_I$, which implies $M \in \mathcal{M}_I$.

Finally, we show that $(M4)$ holds. Let $I \triangleleft_H R$ and $\bar{R} = R/I$. If $M \in \mathcal{M}_R$ and $I \subseteq (0 : M)_R$, then $M$ is an $\bar{R}$-$H$- prime module. If $(N, \bar{B})$ has ”$L$-condition” for $H$-ideal $\bar{B}$ of $\bar{R}$ and $\bar{R}$-$H$-submodule $N$ of $M$, then subset $F \subseteq B$ and there exists a natural number $n$ such that $F^nN = (\bar{F})^nN = 0$. Consequently, $M \in \mathcal{M}_R$. Conversely, if $M \in \mathcal{M}_R$, we can similarly show that $M \in \mathcal{M}_R$.

The second claim follows from Proposition 3.18 and Theorem 3.4(1). □

**Theorem 3.20** $r_{HI} = r_{IH}$. 

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Proof. Obviously, \( r_{lH} \leq r_{HI} \). It remains to show that \( r_{HI}(R) \neq R \) if \( r_{lH}(R) \neq R \). There exists a finite subset \( F \) of \( R \) such that \( F^n \neq 0 \) for any natural number \( n \). Let

\[ \mathcal{F} = \{ I \mid I \text{ is an } H\text{-ideal of } R \text{ with } F^n \not\subseteq I \text{ for any natural number } n \} \]

By Zorn’s lemma, there exists a maximal element \( P \) in \( \mathcal{F} \). It is clear that \( P \) is an \( H \)-prime ideal of \( R \). Now we show that \( r_{lH}(R/P) = 0 \). If \( 0 \neq B/P \) is an \( H \)-ideal of \( R/P \), then there exists a natural number \( m \) such that \( F^m \subseteq B \). Since \( (F^m + P)^n \neq 0 + P \) for any natural number \( n \), we have that \( B/P \) is not locally nilpotent and \( r_{HI}(R/P) = 0 \). Consequently, \( r_{HI}(R) \neq R \). \( \square \)

In fact, all of the results hold in braided tensor categories determined by (co)quasitriangular structure.

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