THE ZETA FUNCTIONS OF COMPLEXES FROM PGL(3): A REPRESENTATION-THEORETIC APPROACH

MING-HSUAN KANG, WEN-CHING WINNIE LI AND CHIAN-JEN WANG

Abstract. The zeta function attached to a finite complex $X_{\Gamma}$ arising from the Bruhat-Tits building for $\text{PGL}_3(F)$ was studied in [KL], where a closed form expression was obtained by a combinatorial argument. This identity can be rephrased using operators on vertices, edges, and directed chambers of $X_{\Gamma}$. In this paper we re-establish the zeta identity from a different aspect by analyzing the eigenvalues of these operators using representation theory. As a byproduct, we obtain equivalent criteria for a Ramanujan complex in terms of the eigenvalues of the operators on vertices, edges, and directed chambers, respectively.

1. Introduction

Let $F$ be a nonarchimedean local field with $q$ elements in its residue field, and let $\pi$ be a uniformizer of $F$. A finite quotient $X_{\Gamma}$ of the Bruhat-Tits building of $G = \text{PGL}_3(F)$ by a cocompact, discrete, and torsion-free subgroup $\Gamma$ of $G$ with $\text{ord}_{\pi}(\det \Gamma) \subseteq 3\mathbb{Z}$ is a 2-dimensional complex. Each vertex of the complex has two kinds of neighboring vertices, of type one and type two. The edges from a vertex to its type $i$ ($i = 1, 2$) neighbors are called type $i$ edges.

The zeta function for the complex $X_{\Gamma}$ was introduced and studied in [KL]. Similar to a graph zeta function, the complex zeta function counts tailless closed geodesics in $X_{\Gamma}$ up to homotopy. More precisely, it is defined as

$$Z(X_{\Gamma}, u) = \prod_{[C]} \frac{1}{(1 - u^{l(C)})},$$

where the product runs through the equivalence classes of primitive tailless closed geodesics $C$ in $X_{\Gamma}$ consisting of solely type one edges or solely type two edges (up to based homotopy), and $l(C)$ denotes the length of $C$. Under the additional assumption that $\Gamma$ is regular, the following closed
form expression of $Z(X, u)$ is obtained in [KL]:

$$Z(X, u) = \frac{(1 - u^3)^{\chi(X, \Gamma)}}{\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I) \det(I + L_B u)},$$

in which $\chi(X, \Gamma)$ is the Euler characteristic of $X$, $A_i$ $(i = 1, 2)$ is the type $i$ vertex adjacency matrix, and $L_B$ is the directed chamber adjacency matrix (cf. [KL]).

As shown in [KL], the zeta function can easily be expressed in terms of the edge adjacency matrices, namely

$$Z(X, u) = \frac{1}{\det(I - L_E u) \det(I - (L_E)^t u^2)},$$

in which $L_E$ is the type one edge adjacency matrix, and $(L_E)^t$, the transpose of $L_E$, is the type two edge adjacency matrix. The proof of the closed form expression (1.1) then boils down to establishing the following zeta function identity:

$$\frac{(1 - u^3)^{\chi(X, \Gamma)}}{\det(I - A_1 u + qA_2 u^2 - q^3 u^3 I) \det(I + L_B u)} = \frac{\det(I + L_B u)}{\det(I - L_E u) \det(I - (L_E)^t u^2)}.$$  

Each determinant in the identity (1.2) has a combinatorial interpretation. The proof given in [KL] follows the combinatorial meaning of each term; the identity is obtained by partitioning the closed geodesics into sets indexed by the conjugacy classes of $\Gamma$, and for each conjugacy class, counting and relating the numbers of closed geodesics of given length, with and without tails.

Since the vertices, type one edges, and directed chambers of the building attached to $G$ are parametrized by cosets of $G$ modulo the standard maximal compact subgroup $K$, a parahoric subgroup $E$, and an Iwahori subgroup $B$, respectively, $A_1$ and $A_2$ can be interpreted as Hecke operators supported on certain $K$-double cosets, $L_E$ as a parahoric operator on an $E$-double coset, and $L_B$ as an Iwahori-Hecke operator on a $B$-double coset. Their precise definitions, originally given in §2, §8 and §7 of [KL], are recalled in §2.1; their actions are described in detail in §2.3. Regarding $A_1$ and $A_2$ as operators on $L^2(\Gamma \backslash G/K)$, $L_E$ as an operator on $L^2(\Gamma \backslash G/E)$, and $L_B$ as an operator on $L^2(\Gamma \backslash G/B)$, in this paper we re-establish the zeta identity by computing the spectrum of each operator using representation theory. Our proof not only provides a totally different, new aspect of the identity (1.2), but also removes the regularity assumption on $\Gamma$.

This paper is organized as follows. The eigenvalues of $L_B$, $L_E$, $A_1$, and $A_2$ are computed in §2 using representation theory. As shown in §3, the zeta identity (1.2) follows from comparing these eigenvalues. Recall from [Li] that a finite quotient $X$ of the building of $G$ is called Ramanujan if the nontrivial eigenvalues of $A_1$ and $A_2$ on $X$ fall in the spectrum of those on the building. This definition, which involves only operators on the vertices of the complex, is a natural extension of
Ramanujan graphs. As a byproduct of the spectral information of the operators, we show that a Ramanujan complex can be equivalently characterized in terms of the eigenvalues of the operator $L_B$ on directed chambers or those of the operator $L_E$ on edges (cf. Theorem 2).

2. Eigenvalues of $L_B$, $L_E$, $A_1$, and $A_2$

2.1. The relevant operators. We recall the definition of the operators $L_B$, $L_E$, $A_1$, $A_2$ from [KL]. Following the notation of [KL], denote by $K$ the maximal compact subgroup of $G$, consisting of elements in $G$ with entries in the ring of integers $\mathcal{O}_F$ of $F$ and determinant in $\mathcal{O}_F^\times$. Let $\sigma = \begin{pmatrix} 1 & \pi \\ \pi & 1 \end{pmatrix}$. Define the Iwahori subgroup $B = K \cap \sigma K \sigma^{-1} \cap \sigma^{-1} K \sigma$ and the parahoric subgroup $E = K \cap \sigma K \sigma^{-1}$. Note that $B$ is the set of elements in $K$ congruent to the upper triangular matrices modulo $\pi$, while $E$ consists of elements in $K$ whose third row is congruent to $(0,0,*)$ mod $\pi$. Clearly, $B \subset E \subset K$.

Write $t_2 = \begin{pmatrix} \pi^{-1} \\ \pi \end{pmatrix}$. The operator $L_B$ acts on the space $L^2(\Gamma \backslash G/B)$ by sending a function $f$ in $L^2(\Gamma \backslash G/B)$ to the function $L_B f$, where

$$L_B f(gB) = \sum_{w_iB \in Bt_2^2B/B} f(gw_iB).$$

The operator $L_E$ on $L^2(\Gamma \backslash G/E)$ sends a function $f$ in $L^2(\Gamma \backslash G/E)$ to the function $L_E f$ given by

$$L_E f(gE) = \sum_{w'_jE \in E(t_2^2)^2E/E} f(gw'_jE).$$

The operators $A_1$, $A_2$ on $L^2(\Gamma \backslash G/K)$ are associated to the double cosets $K\text{diag}(1,1,\pi)K = \sqcup g_iK$ and $K\text{diag}(1,\pi,\pi)K = \sqcup g'_jK$ respectively, and they are defined by

$$A_1 f(gK) = \sum_i f(gg_iK),$$

$$A_2 f(gK) = \sum_j f(gg'_jK),$$

for any $f$ in the space $L^2(\Gamma \backslash G/K)$. 
2.2. **Representations containing $B$-invariant vectors.** The group $G$ acts on $L^2(\Gamma \backslash G)$ by right translations. Since $\Gamma$ is a discrete cocompact subgroup of $G$, this representation decomposes into a direct sum of countably many irreducible unitary representations of $G$. The space $L^2(\Gamma \backslash G/B)$ is the direct sum of the functions in each irreducible subspace which are fixed by the Iwahori subgroup $B$, and similar decomposition holds for the spaces $L^2(\Gamma \backslash G/E)$ and $L^2(\Gamma \backslash G/K)$ as well. To understand the actions of $L_B$, $L_E$, and $A_1,A_2$, it then suffices to study the actions of these operators on each direct summand of the corresponding spaces. Recall from Casselman’s paper [Ca] that an irreducible representation of $G$ contains an Iwahori fixed vector if and only if it is an irreducible subquotient of an unramified principal series representation; by the work of Tadić [Ta], such a subquotient is unitary if it is equivalent to one of the following:

(a) The principal series representation $\text{Ind}(\chi_1, \chi_2, \chi_3)$, where $\chi_1, \chi_2, \chi_3$ are unramified unitary characters of $F^\times$ with $\chi_1 \chi_2 \chi_3 = \text{id}$; or the principal series representation $\text{Ind}(\chi^{-2}, |\chi|^a, |\chi|^{-a})$, where $\chi$ is an unramified character of $F^\times$, $0 < a < 1/2$, and $|\cdot|$ is the absolute value of $F$.

(b) The irreducible subrepresentation of $\text{Ind}(\chi|^{-1}, \chi, |\chi|)$, where $\chi$ is an unramified character of $F^\times$ with $\chi^3 = \text{id}$. This is a one-dimensional representation.

(c) The irreducible subrepresentation of $\text{Ind}(\chi|, \chi, |\chi|^{-1})$, where $\chi$ is an unramified character of $F^\times$ with $\chi^3 = \text{id}$. This is the Steinberg representation.

(d) The irreducible subrepresentation of $\text{Ind}(\chi|^{-1/2}, \chi|^{1/2}, \chi^{-2})$, where $\chi$ is an unramified unitary character of $F^\times$.

(e) The irreducible subrepresentation of $\text{Ind}(\chi|^{1/2}, \chi|^{-1/2}, \chi^{-2})$, where $\chi$ is an unramified unitary character of $F^\times$.

The inductions are taken to be normalized and all the irreducible subrepresentations above are uniquely determined. We shall compute the eigenvalues of $L_B$, $L_E$, $A_1$ and $A_2$ for each type of representations listed above. Since the above representations can all be realized in induced spaces, the computation will be carried out using the standard model. Note that $G$ acts by right translation on the induced space, so the Iwahori-Hecke operator $L_B$ is defined by the same formula as in §2.1 in the standard model and so are the operators $L_E$, $A_1$ and $A_2$.

2.3. **Explicit actions of the operators.** We start with a principal series representation $V = \text{Ind}(\chi_1, \chi_2, \chi_3)$ of $G$ induced from three unramified characters $\chi_1, \chi_2, \chi_3$ of $F^\times$ with $\chi_1 \chi_2 \chi_3 = \text{id}$. Write $P$ for the standard Borel subgroup of $G$ and $W$ for the Weyl group. Then $W = \ldots$
the space of Iwahori fixed vectors of $V$, which has dimension six as one can observe from the decomposition $G = \bigsqcup_{\alpha \in W} P\alpha B$. Let $f_\alpha(x)$ be the function in $V$ supported on the coset $P\alpha B$ with $f_\alpha(\alpha) = 1$. Then $V^B$ has a basis consisting of $f_1 := f_{id}, f_2 := f_{\alpha_1}, f_3 := f_{\alpha_2}, f_4 := f_{\alpha_1\alpha_2}, f_5 := f_{\alpha_2}, f_6 := f_{\alpha_2\alpha_1}$.

A straightforward computation using the coset decomposition of $V^B$ gives the following action of $L_B$:

$$L_B(f_1) = \chi_1\chi_3(\pi)f_2, \quad L_B(f_2) = q\chi_2\chi_3(\pi)f_1,$$

$$L_B(f_3) = (q-1)\chi_1\chi_3(\pi)f_2 + \chi_1\chi_2(\pi)f_6, \quad L_B(f_4) = (q-1)q\chi_2\chi_3(\pi)f_3 + q\chi_1\chi_3(\pi)f_5,$$

$$L_B(f_5) = (q-1)q\chi_2\chi_3(\pi)f_1 + \chi_1\chi_2(\pi)f_4, \quad L_B(f_6) = q\chi_2\chi_3(\pi)f_3.$$  \hspace{1cm} (2.1)

This shows that $L_B$ on $V^B$ has eigenvalues $\pm \sqrt{q\chi_1(\pi)}$, $\pm \sqrt{q\chi_2(\pi)}$ and $\pm \sqrt{q\chi_3(\pi)}$.

The operator $L_E$ acts on the space of $E$-fixed vectors $V^E$. As $\alpha_1 \in E$, $V^E$ is 3-dimensional, generated by $g_1 := f_1 + f_2$, $g_2 := f_3 + f_6$, and $g_3 := f_4 + f_5$. Applying the coset decomposition of $V$ to the definition of $L_E$, one obtains the following action of $L_E$:

$$L_E(g_1) = q\chi_3(\pi)g_1 + (q-1)q\chi_3(\pi)g_2 + (q-1)q^2\chi_3(\pi)g_3,$$

$$L_E(g_2) = q\chi_2(\pi)g_2 + (q-1)q\chi_2(\pi)g_3,$$

$$L_E(g_3) = q\chi_1(\pi)g_3.$$  \hspace{1cm} (2.2)

Hence $L_E$ on $V^E$ has eigenvalues $q\chi_1(\pi)$, $q\chi_2(\pi)$ and $q\chi_3(\pi)$.

The Hecke operators $A_1$ and $A_2$ act on the 1-dimensional space of $K$-fixed vectors $V^K$, which is generated by $h = \sum_{i=1}^{6} f_i$. It follows from the coset decomposition
let \[ V \text{ subgroup of } G \]

\[
\begin{pmatrix} 1 & a \\ b & 1 \end{pmatrix} \begin{pmatrix} \pi & a \\ b & 1 \end{pmatrix} \begin{pmatrix} 1 & c \\ \pi & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \pi & 1 \\ 0 & 1 \end{pmatrix}
\]

that the operator \( A_1 \) acts on \( V^K \) via the scalar multiplication by

\[
\lambda := q(\chi_1(\pi) + \chi_2(\pi) + \chi_3(\pi)).
\]

Similarly the action of \( A_2 \) is obtained from the coset decomposition of \( K \text{diag}(1, \pi, \pi)K \), as scalar multiplication by

\[
q(\chi_1\chi_2(\pi) + \chi_2\chi_3(\pi) + \chi_3\chi_1(\pi)) = q(\chi_1(\pi)^{-1} + \chi_2(\pi)^{-1} + \chi_3(\pi)^{-1}).
\]

We remark that the operator \( A_2 \) is the transpose of \( A_1 \) (cf. [Li]), so (2.4) should be the complex conjugation of (2.3). This is indeed the case by checking the representations of types (a)-(e).

2.4. Computation of eigenvalues. Let \( (\rho, V_\rho) \) be an irreducible representation in \( L^2(\Gamma \backslash G) \) with a nontrivial Iwahori-fixed vector, so \( \rho \) is isomorphic to one of the representations described in §2.2, and we identify \( V_\rho \) with its image in the induced space under this isomorphism. We are now ready to determine, type by type, the eigenvalues of \( L_B \) on \( V_\rho^B \), \( L_E \) on \( V_\rho^E \), and \( A_1 \) and \( A_2 \) on \( V_\rho^K \), whenever the underlying space is nontrivial. We will also record the factors in \( \det(I - A_1u + qA_2u^2 - q^3u^3I), \det(I - L_Eu), \) and \( \det(I + L_Bu) \) arising from \( \rho \).

First recall a result of Borel [Bo] and Casselman [Ca] on the dimension of \( V_\rho^B \), which helps to determine the space \( V_\rho^B \). Let \( N \) be the upper triangular maximal unipotent subgroup of \( G \) and let \( V_{\rho,N} = V_\rho/V_\rho(N) \) be the Jacquet module of \( \rho \). Here \( V_{\rho,N} \) is the subspace of \( V_\rho \) generated by vectors of the form \( \rho(n)v - v \) for all \( n \in N \), \( v \in V_\rho \). Write \( M_0 = T \cap K \), where \( T \) is the diagonal subgroup of \( G \). Then the canonical projection \( V_\rho \to V_{\rho,N} \) induces a linear isomorphism between the \( B \)-fixed space \( V_\rho^B \) and the \( M_0 \)-fixed space \( V_{\rho,N}^{M_0} \). In particular, \( \dim V_\rho^B = \dim V_{\rho,N}^{M_0} \).

Case (a). \( \rho \) is a principal series representation with \( (\rho, V_\rho) = \text{Ind}(\chi_1, \chi_2, \chi_3) \), where \( \chi_1, \chi_2, \chi_3 \) are unramified characters of \( F^\times \) and \( \chi_1\chi_2\chi_3 = id \). It follows immediately from §2.3 that the Iwahori-Hecke operator \( L_B \) acting on \( V_\rho^B \) has eigenvalues \( \pm \sqrt{q\chi_1(\pi)} \), \( \pm \sqrt{q\chi_2(\pi)} \) and \( \pm \sqrt{q\chi_3(\pi)} \), which yield the factor \( (1 - q\chi_1(\pi)u^2)(1 - q\chi_2(\pi)u^2)(1 - q\chi_3(\pi)u^2) \) of \( \det(I + L_Bu) \). The operator \( L_E \) on \( V_\rho^E \) has eigenvalues \( q\chi_1(\pi), q\chi_2(\pi), \) and \( q\chi_3(\pi) \) which give rise to the factor \( (1 - q\chi_1(\pi)u)(1 - ...) \).
The operators $A_1$ and $A_2$ on $V^K_\rho$ have eigenvalues $q(\chi_1(\pi) + \chi_2(\pi) + \chi_3(\pi))$ and $q(\chi_1(\pi) + \chi_2(\pi) + \chi_3(\pi))$ respectively, and they yield the factor $(1 - q\chi_1(\pi)u)(1 - q\chi_2(\pi)u)(1 - q\chi_3(\pi)u)$ of det$(I - A_1u + qA_2u^2 - q^3u^3)$.

Case (b). $\rho$ is the 1-dimensional subrepresentation occurring in $V = \text{Ind}(\chi \mid |^{-1}, \chi, \chi \mid |)$ for an unramified character $\chi$ with $\chi^3 = \text{id}$. Then $\rho(g) = \chi(\det g)$ for all $g \in G$ by (cf. [Ze]). Since $\chi$ is unramified, $V_\rho$ is generated by the $K$-invariant function $h = \sum_{i=1}^6 f_i$ in $V$. Therefore $V_\rho = V^K_\rho = V^E_\rho = V^B_\rho = C \cdot h$, and we have $L_B(h) = q\chi(\pi)$ by appealing to (2.1) with $\chi_1 = \chi \mid |^{-1}, \chi_2 = \chi$ and $\chi_3 = \chi \mid |$. This gives rise to the factor $1 + q\chi(\pi)$ of det$(I + L_Bu)$ arising from $\rho$. Similarly, using the action of $L_E$ given in (2.2), we find that $L_E(h) = q^2\chi(\pi)h$, which gives rise to the factor $1 - q^2\chi(\pi)$ of det$(I - L_Eu)$. Applying (2.3) and (2.4) with $\chi_1 = \chi \mid |^{-1}, \chi_2 = \chi$ and $\chi_3 = \chi \mid |$, we see that the actions of $A_1$ and $A_2$ on $V_\rho$ are given by multiplication by $\chi(\pi) + q\chi(\pi) + q^2\chi(\pi)$ and $\chi(\pi) + q\chi(\pi) + q^2\chi(\pi)$, respectively, and hence they yield the factor $(1 - \chi(\pi)u)(1 - q\chi(\pi)u)(1 - q^2\chi(\pi)u)$ of det$(I - A_1u + qA_2u^2 - q^3u^3)$.

Case (c). $\rho$ is the Steinberg representation, that is, the subrepresentation in the induced space $V = \text{Ind}(\chi \mid |, \chi, \chi \mid |^{-1})$ for an unramified character $\chi$ with $\chi^3 = \text{id}$. It follows from Proposition 2.10 of [Ze] that the Jacquet module of $\rho$ is a one-dimensional unramified character on the diagonal subgroup $T$ in $G$, hence dim$V^M_{\rho,N} = \dim V^N_{\rho,N} = \dim V^B_\rho = 1$, and $V^B_\rho$ is generated by a single function $\phi$.

To determine $\phi$, consider the intertwinings maps $T_{a_1} : \text{Ind}(\chi \mid |, \chi, \chi \mid |^{-1}) \to \text{Ind}(\chi, \chi \mid |^{-1})$ and $T_{a_2} : \text{Ind}(\chi \mid |, \chi, \chi \mid |^{-1}) \to \text{Ind}(\chi \mid |, \chi \mid |^{-1}, \chi)$ defined as in §3 of [Ca]. Applying Theorem 3.4 in [Ca], we see that

\[
T_{a_1}(f_1) = \frac{1}{q}(f'_1 + f''_1), \quad T_{a_1}(f_2) = f'_2 + f''_2, \quad T_{a_1}(f_3) = \frac{1}{q}(f'_3 + f''_3),
\]

\[
T_{a_1}(f_4) = f'_4 + f''_4, \quad T_{a_1}(f_5) = f'_5 + f''_5, \quad T_{a_1}(f_6) = \frac{1}{q}(f'_4 + f''_6);
\]

and

\[
T_{a_2}(f_1) = \frac{1}{q}(f''_1 + f'''_1), \quad T_{a_2}(f_2) = \frac{1}{q}(f''_2 + f'''_2), \quad T_{a_2}(f_3) = f''_1 + f''_3,
\]

\[
T_{a_2}(f_4) = f''_4 + f'''_5, \quad T_{a_2}(f_5) = \frac{1}{q}(f''_4 + f'''_5), \quad T_{a_2}(f_6) = f''_2 + f''_6.
\]

Here $\{f_1, f_2, \cdots, f_6\}$ (resp. $\{f'_1, f'_2, \cdots, f'_6\}$ and $\{f''_1, f''_2, \cdots, f''_6\}$) is a basis of the space of the Iwahori-fixed vectors in $\text{Ind}(\chi \mid |, \chi, \chi \mid |^{-1})$ (resp. $\text{Ind}(\chi, \chi \mid |^{-1})$ and $\text{Ind}(\chi \mid |, \chi \mid |^{-1}, \chi)$) as defined in §2.3. Since the Steinberg representation $\rho$ is irreducible, and it does not appear as a
subrepresentation in \( \text{Ind}(\chi, \chi \mid \chi \mid^{-1}) \), \( T_{\alpha_1} \) must be trivial on \( V_\rho \) (cf. [Ze]). The same reason shows that \( T_{\alpha_2} \) is also trivial on \( V_\rho \). In particular, we have \( T_{\alpha_1}(\phi) = T_{\alpha_2}(\phi) = 0 \), and this condition characterizes \( \phi \). Indeed, writing \( \phi \) as a linear combination of \( f_1, \ldots, f_6 \) and solving \( T_{\alpha_1}(\phi) = 0 \) and \( T_{\alpha_2}(\phi) = 0 \) simultaneously, we find that \( \phi \) is a (nonzero) constant multiple of the function \( f_1 - \frac{1}{q}f_2 - \frac{1}{q}f_3 - \frac{1}{q^2}f_4 + \frac{1}{q}f_5 + \frac{1}{q^2}f_6 \). The action of \( L_B \) as described by (2.11) with \( \chi_1 = \chi \mid \chi_2 = \chi, \chi_3 = \chi \mid^{-1} \) gives \( L_B(\phi) = -\chi^2(\pi)\phi \), which in turn yields the factor \( 1 - \chi^2(\pi)u \) of \( \det(I + L_Bu) \) arising from the Steinberg representation \( \rho \). Note that \( V_\rho^E = V_\rho^K = \{ 0 \} \) since \( \phi \) is not fixed by \( E \).

**Case (d).** \((\rho, V_\rho)\) is the subrepresentation of \( V = \text{Ind}(\chi \mid \chi \mid^{-1/2}, \chi \mid \chi^{-1/2}, \chi^{-2}) \) for an unramified unitary character \( \chi \). Then the dimension of the Jacquet module \( V_{\rho,N} \) is three (cf. [Ze]), and so is the dimension of \( V_\rho^B \). The intertwining map \( T_{\alpha_1} : \text{Ind}(\chi \mid \chi \mid^{-1/2}, \chi \mid \chi^{-1/2}, \chi^{-2}) \rightarrow \text{Ind}(\chi \mid \chi \mid^{-1/2}, \chi^{-2}) \) when restricted on \( V_B \) is given by

\[
T_{\alpha_1}(f_1) = -f'_1 + \frac{1}{q}f'_2, \quad T_{\alpha_1}(f_2) = f'_1 - \frac{1}{q}f'_2, \quad T_{\alpha_1}(f_3) = -f'_3 + \frac{1}{q}f'_5, \\
T_{\alpha_1}(f_4) = f'_6 - \frac{1}{q}f'_4, \quad T_{\alpha_1}(f_5) = f'_5 - \frac{1}{q}f'_6, \quad T_{\alpha_1}(f_6) = -f'_6 + \frac{1}{q}f'_4,
\]

where \( \{ f_1, f_2, \ldots, f_6 \} \) is a basis for the space of \( B \)-fixed vectors in \( \text{Ind}(\chi \mid \chi \mid^{-1/2}, \chi \mid \chi^{-1/2}, \chi^{-2}) \) and \( \{ f'_1, f'_2, \ldots, f'_6 \} \) that in \( \text{Ind}(\chi \mid \chi \mid^{-1/2}, \chi \mid \chi^{-1/2}, \chi^{-2}) \), defined as in §2.3. The representation \( \rho \) does not occur as a subrepresentation of \( \text{Ind}(\chi \mid \chi \mid^{-1/2}, \chi \mid \chi^{-1/2}, \chi^{-2}) \) (cf. [Ze]), which implies that \( T_{\alpha_1}(f) = 0 \) for all \( f \) in \( V_\rho^B \). (Note that in this case, \( T_{\alpha_2} \) gives an isomorphism between \( \text{Ind}(\chi \mid \chi \mid^{-1/2}, \chi \mid \chi^{-1/2}, \chi^{-2}) \) and \( \text{Ind}(\chi \mid \chi^{-1/2}, \chi^{-2}, \chi \mid \chi^{-1/2}) \), so it adds no condition in determining \( V_\rho^B \) ). Now it is easy to see that \( V_\rho^B \) is a 3-dimensional space generated by \( f_1 + f_2, f_3 + f_5, \) and \( f_4 + f_6 \); \( V_\rho^E \) is a 2-dimensional space generated by \( g_1 = f_1 + f_2 \) and \( g_2 + g_3 = \sum_{3 \leq i \leq 6} f_i \); and \( V_\rho^K \) is a 1-dimensional space generated by \( h = \sum_{1 \leq i \leq 6} f_i \).

Then a routine calculation using (2.1), (2.2), (2.3), and (2.4) shows that the operator \( L_B \) on \( V_\rho^B \) has eigenvalues \( q^{1/2}\chi^{-1}(\pi), -q^{3/4}\chi^{1/2}(\pi), q^{3/4}\chi^{1/2}(\pi) \); the operator \( L_E \) on \( V_\rho^E \) has eigenvalues \( q\chi^{-2}(\pi), q^{3/2}\chi(\pi) \); the Hecke operators \( A_1 \) and \( A_2 \) on \( V_\rho^K \) have eigenvalues \( q^{3/2}\chi(\pi) + q^{1/2}\chi(\pi) + q\chi^{-2}(\pi) \) and \( q^{3/2}\chi^{-1}(\pi) + q^{1/2}\chi^{-1}(\pi) + q\chi^2(\pi) \) respectively. Thus the representation \( \rho \) gives rise to the factor \( (1 + q^{1/2}\chi^{-1}(\pi)u)(1 - q^{3/2}\chi(\pi)u^2) \) in \( \det(I + L_Bu) \), the factor \( (1 - q\chi^{-2}(\pi)u)(1 - q^{3/2}\chi(\pi)u) \) in \( \det(I - L_Eu) \), and \( (1 - q^{3/2}\chi(\pi)u)(1 - q^{1/2}\chi(\pi)u)(1 - q\chi^{-2}(\pi)u) \) in \( \det(I - A_1u + qA_2u^2 - q^3u^3) \).

**Case (e).** \((\rho, V_\rho)\) is a subrepresentation of \( V = \text{Ind}(\chi \mid \chi \mid^{-1/2}, \chi^{-2}) \) with an unramified unitary character \( \chi \). Arguing as before, we see that \( V_\rho^B \) is a 3-dimensional space generated by \( f_2 - qf_1 \),
zeros of $\det(\rho) = f$; while there are no nontrivial $K$-fixed vectors in $V^\rho$.

The eigenvalues of $L_B$ on $V^\rho_B$ are $-q^{1/2}\chi^{-1}(\pi)$, $-q^{1/4}\chi^{1/2}(\pi)$, $q^{1/4}\chi^{1/2}(\pi)$; and the eigenvalue of $L_E$ on $V^\rho_E$ is $q^{1/2}\chi(\pi)$. In this case, $\rho$ gives rise to the factor $(1 - q^{1/2}\chi^{-1}(\pi)u)(1 - q^{1/2}\chi(\pi)u^2)$ in $\det(I + L_Bu)$ and the factor $1 - q^{1/2}\chi(\pi)u$ in $\det(I - L_Eu)$.

### 3. Main results

The table below summarizes the computations in §2 of the zeros of $\det(I + L_Bu)$, $\det(I - L_Eu)$, and $\det(I - A_1u + qA_2u^2 - q^3u^3)$ arising from each type of the representations in $L^2(\Gamma\backslash G)$. A representation will have no contribution if it does not contain a nontrivial $B$-fixed vector.

| type | zeros of $\det(I + L_Bu)$ | zeros of $\det(I - L_Eu)$ | zeros of $\det(I - A_1u + qA_2u^2 - q^3u^3)$ |
|------|-----------------------------|-----------------------------|-----------------------------------------------|
| (a)  | $\pm q^{-1/2}\chi^{-1/2}(\pi)$, $i = 1, 2, 3$ | $q^{-1}\chi^{-1}(\pi)$, $i = 1, 2, 3$ | $q^{-1}\chi^{-1}(\pi)$, $i = 1, 2, 3$ |
| (b)  | $-q^{-1}\chi^{-2}(\pi)$ | $q^{-2}\chi^{-1}(\pi)$ | $\chi^{-1}(\pi)$, $q^{-1}\chi^{-1}(\pi)$, $q^{-2}\chi^{-1}(\pi)$ |
| (c)  | $\chi^{-2}(\pi)$ | none | none |
| (d)  | $-q^{-1/2}\chi(\pi)$, $\pm q^{3/4}\chi^{-1/2}(\pi)$ | $q^{-1}\chi^2(\pi)$, $q^{-3/2}\chi^{-1}(\pi)$ | $q^{-1}\chi^2(\pi)$, $q^{-1/2}\chi^{-1}(\pi)$, $q^{-3/2}\chi^{-1}(\pi)$ |
| (e)  | $q^{-1/2}\chi(\pi)$, $\pm q^{-1/4}\chi^{-1/2}(\pi)$ | $q^{-1/2}\chi^{-1}(\pi)$ | none |

(Table 1)

Recall that the characters $\chi_1, \chi_2, \chi_3$ in case (a) are all unramified with the product equal to identity; the character $\chi$ in cases (b) and (c) is unramified with order dividing 3; and the character $\chi$ in cases (d) and (e) is unramified and unitary.

Denote by $N_0$, $N_1$, and $N_2$ the number of vertices, edges and chambers in $X_\Gamma$, respectively, so that $\dim L^2(\Gamma\backslash G/K) = N_0$, $\dim L^2(\Gamma\backslash G/E) = N_1$ and $\dim L^2(\Gamma\backslash G/B) = 3N_2$. Denote by $D$ the number of type (d) representations in $L^2(\Gamma\backslash G)$. There are three 1-dimensional representations (type (b)), so the number of representations of types (a) and (d) together is $N_0 - 3$. Comparing the total dimensions of $V^K$ and $V^E$, we conclude that the number of type (e) representations minus the number of type (d) representations is $N_1 - 3N_0 + 6$. Combined with the total dimension of $V^B$, we get that the total number of Steinberg representations (type (c)), counting multiplicities, is equal to $3\chi(X_\Gamma) - 3$, where $\chi(X_\Gamma) = N_0 - N_1 + N_2$ is the Euler characteristic of $X_\Gamma$. The result is summarized below:
| type | dim $V^B$ | dim $V^E$ | dim $V^K$ | number of representations |
|------|----------|----------|----------|--------------------------|
| (a)  | 6        | 3        | 1        | $N_0 - D - 3$            |
| (b)  | 1        | 1        | 1        | 3                        |
| (c)  | 1        | 0        | 0        | $3N_0 - 3N_1 + 3N_2 - 3$ |
| (d)  | 3        | 2        | 1        | $D$                      |
| (e)  | 3        | 1        | 0        | $N_1 - 3N_0 + D + 6$     |
| total| 3$N_2$  | $N_1$   | $N_0$   |

(Table 2)

As there are only three Steinberg representations, which differ by cubic unramified twists, each occurs with multiplicity $\chi(X_\Gamma) - 1$. We have shown

**Proposition 1.** (1) Each Steinberg representation of type (c) occurs in $L^2(\Gamma \backslash G)$ with multiplicity equal to $\chi(X_\Gamma) - 1$.

(2) The number of representations of type (e) occurring in $L^2(\Gamma \backslash G)$ is at least $N_1 - 3N_0 + 6$, where $N_0$ and $N_1$ denote the number of vertices and edges in $X_\Gamma$.

The first statement is in agreement with Garland’s result in [Ga], which is for a general $p$-adic group. The corresponding assertion for graphs is discussed in Hashimoto’s paper [Ha].

The zeros arising from 1-dimensional representations (case (b)) are called trivial zeros of the respective determinant, as they correspond to the trivial eigenvalues of the operators $L_B$, $L_E$, $A_1$ and $A_2$. The remaining zeros are nontrivial.

Recall also that $X_\Gamma$ is a Ramanujan complex if and only if the nontrivial zeros of $\det(I - A_1u + qA_2u^2 - q^3u^3I)$ have absolute value $q^{-1}$ (cf. [Li], [LSV]). It is natural to ask whether this property is also reflected on other operators. The theorem below answers this question.

**Theorem 2.** The following statements are equivalent.

1. $X_\Gamma$ is Ramanujan.
2. The nontrivial zeros of $\det(I - A_1u + qA_2u^2 - q^3u^3)$ have absolute value $q^{-1}$.
3. The nontrivial zeros of $\det(I - L_Eu)$ have absolute values $q^{-1}$ and $q^{-1/2}$.
4. The nontrivial zeros of $\det(I + L_Bu)$ have absolute values 1, $q^{-1/2}$, and $q^{-1/4}$.

**Proof.** In view of Table 1, $X_\Gamma$ is Ramanujan if and only if the representations of type (d) do not occur and the principal series representations of type (a) are induced from unitary unramified characters. These conditions reflected on $\det(I - A_1u + qA_2u^2 - q^3u^3)$, $\det(I - L_Eu)$, and $\det(I + L_Bu)$ are as stated since representations of types (c) and (e) do occur by Proposition 1. □
Remark. If the group $\Gamma$ arises from an inner form $H$ of $GL_3$ as described in §4 of [KL], then, by strong approximation theorem, the functions on vertices of $X_\Gamma$ can be viewed as automorphic forms on $H$. The underlying automorphic representations then correspond to cuspidal automorphic representations of $PGL_3$ under the Jacquet-Langlands correspondence established in [JPSS]. As such, its local components are generic. In view of Theorem 9.7 of [Ze], representations of type (d) cannot occur.

In particular, when the local field $F$ has positive characteristic, it was shown in [LSV] that $X_\Gamma$ is always Ramanujan for $\Gamma$ arising from an inner form.

We are now ready to prove the main result of this paper.

Theorem 3. There holds the identity

$$(1 - u^3)^{\chi(X_\Gamma)} = \frac{\det(I - A_1 u + qA_2 u^2 - q^3u^3 I) \det(I + L_B u)}{\det(I - LE u) \det(I - (LE)^t u^2)}.$$ 

Proof. Denote by $R(u)$ the right hand side of the identity. As a consequence of Table 1, we have the following list describing the contribution to $R(u)$ arising from each type of the representation:

- Type (a): 1;
- Type (b): $\frac{(1-\chi(\pi)u)(1-q\chi(\pi)u)}{1-q\chi^2(\pi)u}$;
- Type (c): $1 - \chi^2(\pi)u$;
- Type (d): $\frac{1-q^{1/2}\chi^{-1}(\pi)u}{1-q^{1/2}\chi^{-1}(\pi)u}$;
- Type (e): $\frac{1-q^{1/2}\chi^{-1}(\pi)u}{1-q^{1/2}\chi^{1}(\pi)u}$.

Note that the character $\chi$ occurring in representations of types (b) and (e) are unitary. Since each determinant is a polynomial with coefficients in $\mathbb{Z}$, if $\chi$ is not real-valued, then there is a representation of the same type with $\chi$ replaced by $\chi^{-1}$, and the two representations occur with the same multiplicity. Consequently, the contributions from representations of types (a), (d) and (e) all cancel out.

The characters $\chi$ occurring in representations of types (b) and (c) satisfy $\chi^3 = id$. Taking product over all such $\chi$ of the contributions in $R(u)$ from type (b) representations yields $1 - u^3$, while the product of those of type (c) is $(1 - u^3)^{\chi(X_\Gamma)^{-1}}$ by Proposition 1. This shows that the total contribution from all representations in $R(u)$ is equal to $(1 - u^3)^{\chi(X_\Gamma)}$, which is equal to the left hand side of the identity. $\square$

As defined in §4.1 of [KL], an element $\gamma$ in $\Gamma$ is rank-one split if its eigenvalues generate a quadratic extension of the base field $F$. Table 1 shows that the eigenvalues of $L_B$ and $L_E$ are never...
zero, hence the operators $L_B$ and $L_E$ have full rank. Since the rank of $L_B$ is greater than twice the rank of $L_E$, the left hand side of Theorem 26 in §8.3 of [KL] is nonzero, from which we conclude

**Corollary 4.** The group $\Gamma$ contains rank-one split elements.

This is different from the case of graphs. More precisely, the non-identity elements in a regular torsion-free discrete cocompact subgroup of $PGL_2(F)$ are hyperbolic (called split in §4.1 of [KL]), while a regular torsion-free discrete cocompact subgroup of $PGL_3(F)$ contains, in addition to split elements, also rank-one split elements.

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MING-HSUAN KANG, DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802

E-mail address: kang_m@math.psu.edu

WEN-CHING WINNIE LI, DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PA 16802

E-mail address: wli@math.psu.edu
Chian-Jen Wang, Mathematics Division, National Center for Theoretical Sciences, National Tsing Hua University, Hsinchu, Taiwan, R.O.C.

E-mail address: cjwang@math.cts.nthu.edu.tw