REGULARITY BOUNDS FOR CURVES BY MINIMAL GENERATORS AND HILBERT FUNCTION

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Abstract. Let $\rho_C$ be the regularity of the Hilbert function of a projective curve $C$ in $\mathbb{P}^n_K$ over an algebraically closed field $K$ and $\alpha_1, \ldots, \alpha_{n-1}$ be minimal degrees for which there exists a complete intersection of type $(\alpha_1, \ldots, \alpha_{n-1})$ containing the curve $C$. Then the Castelnuovo-Mumford regularity of $C$ is upper bounded by $\max\{\rho_C + 1, \alpha_1 + \ldots + \alpha_{n-1} - (n-2)\}$. We study and, for space curves, refine the above bound providing several examples.

1. Introduction

Estimates for the Castelnuovo-Mumford regularity of a standard graded algebra $A$ “give a measure of size of $A$” and it is interesting to predict such estimates from the Hilbert function of $A$ (20, Remark 2.6). In this context, it is well known that the Hilbert polynomial of a graded polynomial ideal gives a bound for the Castelnuovo-Mumford regularity (Gotzmann’s Regularity Theorem), but when the ideal is saturated this bound can be very far from being sharp.

Here we tackle the question of bounding the Castelnuovo-Mumford regularity of the algebra $A = S/I$, where $S = K[x_0, \ldots, x_n]$ and $I$ is the defining ideal of a curve $C$ over an algebraically closed field $K$ with general hyperplane section $Z$. We note that, by well known results about zero-dimensional schemes, an upper bound for the Castelnuovo-Mumford regularity $\text{reg}(C)$ of $C$ arises in terms of the regularity of the Hilbert function of $S/I$ and of the regularity of the Hilbert function of $Z$ (see Theorem 3.4). Then, involving also degrees of minimal generators of $I$, we study upper bounds for $\text{reg}(C)$, by means of upper bounds for $\text{reg}(Z)$ when $C$ is known and $Z$ is not (Proposition 4.1).

Bounds in terms of the degrees of defining equations of projective schemes are given for smooth [2] and locally complete intersection [4] schemes.

To address the above question, let $\mathbb{P}^n_K$ be a projective space of dimension $n$ over $K$ and $C$ be a curve in $\mathbb{P}^n_K$, i.e. a non-degenerate projective scheme of dimension 1. It is well known that, denoting $\rho_C$ the regularity of the Hilbert function of $C$, $\omega_I$ the maximal degree of minimal generators of the defining ideal $I$ of $C$ and $\text{reg}(C)$ the Castelnuovo-Mumford regularity of $C$, then

$$\rho_C + 1 \leq \text{reg}(C), \quad \omega_I \leq \text{reg}(C).$$

In this paper we note how it is possible to “reverse” the inequalities (1.1) for curves basing on Theorem 3.4 and results of [9], thus yielding that, if $\alpha_1, \ldots, \alpha_{n-1}$

1991 Mathematics Subject Classification. Primary 14F17, 14H50; Secondary 14Q05.

Key words and phrases. Castelnuovo-Mumford regularity, hyperplane section, Hilbert function, minimal homogeneous generators.

The authors were supported in part by MURST and GNSAGA.
are minimal degrees for which there exists a complete intersection (c.i. for short) of type $(\alpha_1, \ldots, \alpha_{n-1})$ containing the curve $C$, then (Proposition 4.1)

$$(1.2) \quad \text{reg}(C) \leq \max\{\rho_0 + 1, \alpha_1 + \ldots + \alpha_{n-1} - (n-1) + 1\}.$$  

Degrees of generators of the defining ideal, together with other invariants, are used for bounding regularity of space curves in $[10, 12]$ and, in this context, it is worthwhile to see also $[14]$ (see the references therein too).

The paper is organized as it follows. In section 2 we set notation and recall some definitions and known results on the Castelnuovo-Mumford regularity of a homogeneous ideal. In section 3 we recall and state (Theorem 3.4) some information about the Castelnuovo-Mumford regularity of projective schemes that follows by the study of general hyperplane sections. Then, in section 4, by Proposition 4.1 we note that formula (1.2) raises from well known results about zero-dimensional schemes of $[6]$. Hence, we focus our attention on space curves, refining the above bound (Corollary 4.3) and comparing it with other bounds of the same type, as a bound for integral curves which is described by means of the degree of the curves and based on the shape of Borel ideals (Proposition 4.5).

Many of the computations have been performed by Points $[10]$ and CoCoA $[5]$.

2. General setting

Let $K$ be an algebraically closed field, $S = K[x_0, \ldots, x_n]$ the polynomial ring over $K$ in $n+1$ variables endowed with the deg-rev-lex term-ordering such that $x_0 > x_1 > \ldots > x_n$, $\mathbb{P}_K^n = \text{Proj } S$ the projective space of dimension $n$ over $K$.

If $I \subset S$ is a homogeneous ideal, $I_t$ denotes the subset of $I$ consisting of the homogeneous polynomials of $I$ of degree $t$ and $I_{\leq t}$ the subset of $I$ consisting of the homogeneous polynomials of $I$ of degree $\leq t$. Moreover, $\alpha_t$ is the initial degree of $I$ and $\omega_I$ is the maximal degree of minimal generators of $I$.

We will use freely the common notation of sheaf cohomology in $\mathbb{P}_K^n$, referring to $[7, 11]$ for notation and basic results. If $\mathcal{F}$ is a coherent sheaf on $\mathbb{P}_K^n$, we will write $H^i(\mathcal{F})$ instead of $H^i(\mathbb{P}_K^n, \mathcal{F})$.

**Definition 2.1.** A finitely generated graded $S$-module $M$ is $m$-regular if the $i$-th syzygy module of $M$ is generated in degree $\leq m + i$, for all $i \geq 0$. The regularity $\text{reg}(M)$ of $M$ is the smallest integer $m$ for which $M$ is $m$-regular.

**Remark 2.2.** The above definition can be applied to a homogeneous ideal $I$ of $S$. So, if $\omega_I$ is the maximal degree of minimal generators of $I$, then $\omega_I \leq \text{reg}(I)$.

**Definition 2.3.** The saturation of a homogeneous ideal $I$ is $I^{\text{sat}} = \{ f \in S \mid \forall j = 0, \ldots, n, \exists r \in \mathbb{N} : x_r^j f \in I \}$. The ideal $I$ is saturated if $I^{\text{sat}} = I$. The ideal $I$ is $m$-saturated if $(I^{\text{sat}})_t = I_t$ for each $t \geq m$. The saturation of $I$ is $\text{sat}(I) = \min \{ t \mid I / I_t \text{ is } t\text{-saturated} \}$.

**Definition 2.4.** A coherent sheaf $\mathcal{F}$ on $\mathbb{P}^n$ is $m$-regular if $H^i(\mathcal{F}(m-i)) = 0$ for all $i > 0$. The Castelnuovo-Mumford regularity (or regularity) of $\mathcal{F}$ is $\text{reg}(\mathcal{F}) = \min \{ m \mid \mathcal{F} \text{ is } m\text{-regular} \}$.

**Proposition 2.5.** ([$7$] Proposition 2.6) A homogeneous ideal $I$ is $m$-regular if and only if $I$ is $m$-saturated and its sheafification $\mathcal{I}$ is $m$-regular. Hence, for a saturated homogeneous ideal $I$, the regularity of $I$ equals the regularity of its sheafification.
Definition 2.6. If $X \subset \mathbb{P}_K^n$ is a closed subscheme, its \textit{regularity} $\text{reg}(X)$ is the regularity $\text{reg}(I)$ of its defining ideal $I$.

For a finitely generated graded $S$-module $M$ we let $H_M(t) := \dim_K M_t$ be the \textit{Hilbert function} of $M$, $\Delta H_M(t) := H_M(t) - H_M(t-1)$, for $t \geq 1$, and $\Delta H_M(0) := 1$.

We recall that, for $t \gg 0$, $H_M(t) = P_M(t)$ where $P_M(z) \in K[z]$ is the \textit{Hilbert polynomial} of $M$. The \textit{regularity of $H_M$} is $\rho_M := \min \{ \ell / H_M(t) = P_M(t), \forall t \geq \ell \}$.

If $I$ is the defining ideal of a closed subscheme $X \subset \mathbb{P}_K^n$ of dimension $k$, instead of $H_{S/I}$, $P_{S/I}$, $\rho_{S/I}$ we can write $H_X$, $P_X$, $\rho_X$. Recall that the Hilbert polynomial of $X$ is also $P_X(t) = \sum_{i=0}^k (-1)^i h^i(O_X(t))$ (see \[8\] Exercise III.5.2).

Remark 2.7. Let $X \subset \mathbb{P}_K^n$ be a non-degenerate scheme of dimension $k$. From the short exact sequence $0 \to \mathcal{I}_X(t) \to \mathcal{O}_{\mathbb{P}_n}(t) \to \mathcal{O}_X(t) \to 0$ we get the long cohomology exact sequence

$$0 \to H^0(\mathcal{I}_X(t)) \to H^1(\mathcal{O}_{\mathbb{P}_n}(t)) \to H^0(\mathcal{O}_X(t)) \to H^1(\mathcal{I}_X(t)) \to H^1(\mathcal{O}_{\mathbb{P}_n}(t)) \to$$

$$\to H^1(\mathcal{O}_X(t)) \to H^2(\mathcal{I}_X(t)) \to \cdots \to H^n(\mathcal{O}_{\mathbb{P}_n}(t)) \to 0$$

from which it follows that:

\begin{enumerate}
  \item $H_X(t) = h^0(\mathcal{O}_{\mathbb{P}_n}(t)) - h^0(\mathcal{O}_X(t)) - h^1(\mathcal{I}_X(t))$;
  \item $h^i+1(\mathcal{I}_X(t)) = h^i(\mathcal{O}_X(t))$ for $1 \leq i \leq k$ and for $t \geq -n$, $h^i(\mathcal{I}_X(t)) = 0$ for every $i \geq k + 2$;
  \item $\text{reg}(X) \geq \rho_X + 1$;
  \item when $k = 1$ we have $h^2(\mathcal{I}_X(t)) = h^1(\mathcal{I}_X(t))$ for every $t \geq \rho_X$.
\end{enumerate}

3. CASTELNUOVO-MUMFORD REGULARITY AND HYPERPLANE SECTIONS

Let $h$ be a \textit{general} linear form for a homogeneous ideal $I \subset S$, i.e. $h \in S_1$ is a homogeneous polynomial of degree 1 which is not a zero-divisor on $S/I^{\text{sat}}$.

Proposition 3.1. (\[2\], Lemma 1.10) A homogeneous ideal $I$ is \textit{m-regular} if and only if $I$ is $m$-saturated and $(I, h)$ is $m$-regular.

Remark 3.2. From the above proposition it follows immediately that $\text{reg}(I) = \max \{ \text{sat}(I), \text{reg}(I, h) \}$. Thus, if $X$ is an arithmetically Cohen-Macaulay (for short aCM) scheme of dimension $k$, we have $\text{reg}(X) = \rho_X + k + 1$.

Remark 3.3. By the short exact sequence

$$0 \to (S/I)_t \to (S/I)_t \xrightarrow{h} (S/I)_t \to (S/J)_t \to 0$$

it holds that $H_{S/I}(t) = \Delta H_{S/I}(t)$ and then $\rho_{S/I} = \rho_X + 1$.

From now on, let $X \subset \mathbb{P}_K^n$ be a scheme of dimension $k > 0$, $I$ its defining ideal and $J = (I, h)$. Moreover, let $Z \subset \mathbb{P}_K^{n-1}$ be the scheme of dimension $k - 1$ defined by the saturated ideal $J^{\text{sat}} = (I, h)^{\text{sat}}$, i.e. the general hyperplane section of $X$.

Note that, for every $t \geq \max \{ \rho_Z, \rho_{S/I} \}$, it is $H_{S/I}(t) = H_Z(t)$, hence $\text{sat}(J) \leq \max \{ \rho_Z, \rho_{S/I} \}$.

The following result is obtained by an easy computation involving Hilbert functions and the definition of saturated ideal.

Theorem 3.4. \textit{With the above notation}

$$\text{reg}(X) = \max \{ \rho_X + 1, \text{reg}(Z) \}.$$

Namely,
(1) \( \rho_X + 1 \geq \deg(Z) \Rightarrow \text{sat}(J) = \deg(X) = \rho_X + 1 \),
(2) \( \rho_X + 1 = \deg(Z) - 1 \Rightarrow \text{sat}(J) \leq \deg(Z) - 1 \) and \( \deg(X) = \deg(Z) \),
(3) \( \rho_X + 1 < \deg(Z) - 1 \Rightarrow \text{sat}(J) = \deg(Z) - 1 \) and \( \deg(X) = \deg(Z) \).

**Proof.** By Propositions 2.5 and 3.1 it follows that \( \deg(X) = \deg(I) = \deg(J) = \max\{\text{sat}(J), \deg(J)\} \). Hence, \( \deg(X) = \max\{\text{sat}(J), \deg(Z)\} \).

If \( \rho_X + 1 \geq \deg(Z) \) then \( \rho_{S/J} = \rho_X + 1 \geq \rho_Z + 1 \), by Remarks 2.7(3) and 3.3. Hence, for every \( t \geq \rho_{S/J} \geq \rho_Z + 1 \) we have \( H_{S/J}(t) = P_Z(t) = H_Z(t) \) but \( H_{S/J}(\rho_X) \neq P_Z(\rho_X) = H_Z(\rho_X) \). Since \( J \subseteq J^{\text{sat}} \), it follows that, for every \( t \geq \rho_X + 1 \), \( J_t = J_t^{\text{sat}} \), meanwhile \( J_{\rho_X} \neq J_{\rho_X}^{\text{sat}} \). Thus \( \text{sat}(J) = \rho_X + 1 \).

If \( \rho_X + 1 = \deg(Z) - 1 \), with analogous arguments as above it holds \( \text{sat}(J) \leq \rho_X + 1 = \rho_Z \) and so \( \deg(X) = \deg(Z) \).

If \( \rho_X + 1 < \deg(Z) - 1 \), similarly we have \( H_{S/J}(\rho_Z - 1) = P_Z(\rho_Z - 1) \) while \( H_Z(\rho_Z - 1) \neq P_Z(\rho_Z - 1) \) and \( H_{S/J}(\rho_Z) = P_Z(\rho_Z) = H_Z(\rho_Z) \), so \( \text{sat}(J) = \rho_Z \). \( \square \)

Note that, by applying Theorem 3.4 to a curve \( C \subseteq \mathbb{P}_K^n \) with general hyperplane section \( Z \), we get \( \deg(C) = \max\{\rho_C + 1, \rho_Z + 1\} = \min\{t \geq \rho_Z + 1 \mid \Delta H_C(t) = 0\} \), being \( \deg(Z) = \rho_Z + 1 \) by Remark 3.2. So, bounds for zero-dimensional schemes’ regularity enter into our aims in section 4.

**Remark 3.5.** In [1] it is proved that if \( C \) is an integral curve of degree \( \deg(C) \) then \( \rho_Z \leq \left\lfloor \frac{\deg(C) - 1}{n - 1} \right\rfloor \).

**Example 3.6.** We give examples of curves for every case of Theorem 3.4. In (2) and (3) we apply the technique used in [10] to construct curves with “high degree generators”, involving basic double linkages. We recall the procedure to obtain a basic double link of type \((a_1, \ldots, a_{n-1})\) from a curve \( C \subseteq \mathbb{P}_K^n \). Consider homogeneous polynomials \( F_1, \ldots, F_{n-1} \) of \( K[x_0, \ldots, x_n] \) of degrees \( a_1, \ldots, a_{n-1} \), respectively, with \( F_1 \) general, \( F_2, \ldots, F_{n-1} \in I \) and \( (F_1, \ldots, F_{n-1}) \) a regular sequence, then the curve \( C' \subseteq \mathbb{P}_K^n \) defined by the ideal \((F_1 I, F_2, \ldots, F_{n-1})\) is called basic double link.

(1) Let \( C \subseteq \mathbb{P}_K^n \) be the rational curve of degree 30 parametrized by

\[
\begin{align*}
x_0 &= u^{30} + v^{30} \\
x_1 &= u^{29}v + u^{19}v^{11} + u^{9}v^{21} \\
x_2 &= u^{29}v + u^{18}v^{12} + u^{8}v^{22} \\
x_3 &= u^{27}v^3 + u^{17}v^{13} + u^{7}v^{23} \\
x_4 &= u^{26}v^4 + u^{16}v^{14} + u^{6}v^{24}
\end{align*}
\]

Over a field of characteristic 31991, we get \( \rho_C + 1 = 21 = \omega_1 \). Since \( \rho_Z \leq \left\lfloor \frac{\deg(C) - 1}{n - 1} \right\rfloor = 10 \), it is \( \deg(C) = 21 \) by Theorem 3.4(1). Further computations tell us that \( \rho_Z = 4 \).

(2) Theorem 3.4(2) holds not only for ACM curves. Let \( C_0 \subseteq \mathbb{P}_K^3 \) be a general elliptic curve of degree 5. By applying a sequence of two basic double linkages to \( C_0 \), respectively of types (1, 5) and (1, 7), we obtain a non-ACM curve \( X \) of degree 17 with \( \deg(X) = 7 = \rho_X + 2 \), both in characteristics 31991 and 0. In fact, \( \rho_Z = 6 \).

(3) Let \( C_0 \subseteq \mathbb{P}_K^3 \) be a general rational curve of degree 4. By applying a sequence of two basic double linkages to \( C_0 \), respectively of types (1, 4) and (1, 6), we obtain a curve \( X \) of degree 14 with \( \deg(X) = 6 = \rho_X + 3 \), both in characteristics 31991 and 0. In fact, \( \rho_Z = 5 \).

**Remark 3.7.** Let \( C \subseteq \mathbb{P}_K^n \) be a connected non-special curve (i.e. \( h^1(O_C(t)) = 0 \) for every \( t \geq 1 \)) over a field of characteristic zero. We have:
(a) If $2 = \omega_I \geq \rho_C + 1$, then $\text{reg}(C) \leq 3$; in particular, $\text{reg}(C) = 2$ if and only if $C$ is the rational normal curve.

(b) Otherwise, $\text{reg}(C) = \max\{\rho_C + 1, \omega_I\}$. Namely, by Theorem 2.1 of [19] it follows that $\rho_Z \leq \omega_I$. Thus, if $\omega_I < \rho_C + 1$, then $\text{reg}(C) = \rho_C + 1$ by Theorem 3.4(1). If $\omega_I \geq \rho_C + 1$, then $h^1(I_C(\omega_I - 1)) = h^2(I_C(\omega_I - 1)) = h^1(O_C(\omega_I - 1)) = 0$ and $h^2(I_C(\omega_I - 2)) = h^1(O_C(\omega_I - 2)) = 0$ (see Remark 2.7), so $\text{reg}(C) = \omega_I$.

For example, from (b), we get that every general rational curve $C \subset \mathbb{P}_K^n$, with $\deg(C) > n$, being of maximal rank and non-special, has $\text{reg}(C) = \rho_C + 1$ (see also [17] and the references therein).

4. Regularity bounds for curves

From now on, we suppose that the hyperplane section $Z$ of a curve $C \subset \mathbb{P}_K^n$ is obtained by a hyperplane which is general among those intersecting the curve properly. The following statement is based on results of [6] about Hilbert function under liaisons and the symmetry related to Gorenstein rings’ Hilbert function.

**Proposition 4.1.** Let $\alpha_1 \leq \ldots \leq \alpha_{n-1}$ minimal degrees for which there exists a c.i. of type $(\alpha_1, \ldots, \alpha_{n-1})$ containing the curve $C$. Hence

(i) $\text{reg}(C) \leq \max\{\rho_C + 1, \alpha_1 + \ldots + \alpha_{n-1} - (n - 1) + 1\}$;

(ii) if $\deg(C) < \prod_{i=1}^{n-1} \alpha_i$, then $Z$ is not a c.i. of type $(\alpha_1, \ldots, \alpha_{n-1})$ and thus $\text{reg}(C) \leq \max\{\rho_C + 1, \alpha_1 + \ldots + \alpha_{n-1} - (n - 1)\}$.

**Proof.** We will see that the statement is a straightforward consequence of results of [6] and of Theorem 3.4. First of all, recall that such a c.i. exists always (see Theorem 3.14 of Chapter VI of [9]).

Next, let $\beta_1 \leq \ldots \leq \beta_{n-1}$ be minimal degrees for which there exists a c.i. $W \subset \mathbb{P}_K^{n-1}$ of type $(\beta_1, \ldots, \beta_{n-1})$ containing the general hyperplane section $Z \subset \mathbb{P}_K^{n-1}$. Then, it is well known that the regularity of the Hilbert function of $W$ is $\beta_1 + \ldots + \beta_{n-1} - (n - 1)$ and $\text{reg}(W) = \beta_1 + \ldots + \beta_{n-1} - (n - 1) + 1$. Moreover, by Theorem 3(a) of [6], it follows that, for every integer $t$ such that $0 \leq t \leq \text{reg}(W) - 1$, we have $\Delta H_W(t) = \Delta H_W(\text{reg}(W) - 1 - t)$, so in particular $\Delta H_W(\text{reg}(W) - 1) = 1$.

The ideal $(I(W) : I(Z))$ is saturated (Lemma 5.2.1, [11]) and defines a scheme $Y$ algebraically linked to $Z$ by $W$. By Theorem 3(b) of [6], it is $\Delta H_W(t) = \Delta H_Z(t) + \Delta H_Y(\text{reg}(W) - 1 - t)$, for $0 \leq t \leq \text{reg}(W) - 1$, so in particular $\Delta H_W(\text{reg}(W) - 1) = \Delta H_Z(\text{reg}(W) - 1) + \Delta H_Y(0)$. If $Z \neq W$, then $Y$ is not empty and hence $\Delta H_Y(0) = 1$. So, $\Delta H_Z(\text{reg}(W) - 1) = 0$ and

\[ \text{reg}(Z) \leq \text{reg}(W) - 1 = \beta_1 + \ldots + \beta_{n-1} - (n - 1). \]

(4.1)

By the hypotheses, we have minimal degrees $\alpha_1, \ldots, \alpha_{n-1}$ for which there exists a c.i. in $\mathbb{P}_K^n$ of type $(\alpha_1, \ldots, \alpha_{n-1})$ containing the curve $C$. Since regular sequences are preserved under general hyperplane sections, then we have also a c.i. in $\mathbb{P}_K^{n-1}$ of type $(\alpha_1, \ldots, \alpha_{n-1})$ containing $Z$. The degrees of this complete intersection could be minimal or not for $Z$. Anyway we can take a complete intersection containing $Z$ of minimal degrees $\beta_1, \ldots, \beta_{n-1}$ such that $\beta_i \leq \alpha_i$, for every $1 \leq i \leq n - 1$. Hence, by Theorem 3.1, part (i) follows.

If $\deg(Z) = \deg(C) < \prod_{i=1}^{n-1} \alpha_i$, then $Z$ is not a c.i. of type $(\alpha_1, \ldots, \alpha_{n-1})$; thus $\text{reg}(Z) \leq \alpha_1 + \ldots + \alpha_{n-1} - (n - 1)$ and also part (ii) holds. \qed
Example 4.2. For every case of Theorem 3.4, we give examples of curves in $\mathbb{P}^4_K$ for which the bound of Proposition 4.1 is sharp.

(1)(a) Let $C$ be a general degree 6 elliptic curve, then $\rho_C = 2$, $\text{reg}(C) = 3$, $\alpha_1 = \alpha_2 = \alpha_3 = 2$ and, so, $\text{reg}(C) = \alpha_1 + \alpha_2 + \alpha_3 - (4 - 1)$.

(b) The ideal $I := (x_0^2, x_1^4, x_2, x_1x_2x_3, x_0x_1^2x_2) \subset K[x_0, \ldots, x_4]$ is saturated and defines a curve $C \subset \mathbb{P}^4_K$ of degree 15 such that $\text{reg}(C) = 6 = \rho_C + 1 = \rho_Z + 2$ and $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 4$.

(2)(a) Let $C$ be a general degree 5 elliptic curve (which is aCM), then $\rho_C = 1$, $\text{reg}(C) = 3$, $\alpha_1 = \alpha_2 = \alpha_3 = 2$ and, so, $\text{reg}(C) = \alpha_1 + \alpha_2 + \alpha_3 - (4 - 1)$.

(b) The ideal $I := (x_0^2, x_1^2, x_0x_1, x_0x_2, x_0x_3) \subset K[x_0, \ldots, x_4]$ is saturated and defines a non-aCM curve $C \subset \mathbb{P}^4_K$ of degree 12 such that $\text{reg}(C) = 6 = \rho_C + 2 = \rho_Z + 1$ and $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 4$. By further computation we note that $Z$ is a degenerated c.i. of type (3,4), hence $Z \subset \mathbb{P}^2_K$.

(3) The ideal $I := (x_0^3, x_1^2, x_0x_1, x_0x_2, x_0x_3) \subset K[x_0, \ldots, x_4]$ is saturated and defines a curve $C \subset \mathbb{P}^4_K$ of degree 9 such that $\text{reg}(C) = 5 = \rho_C + 3 = \rho_Z + 1$ and $\alpha_1 = 2$, $\alpha_2 = 3$, $\alpha_3 = 3$. As in (2), by further computation we note that $Z$ is a degenerated c.i. of type (3,3), hence $Z \subset \mathbb{P}^2_K$.

When $C$ is a curve in $\mathbb{P}^3_K$ with hyperplane section $Z \subset \mathbb{P}^2_K$ we can have further information about $\rho_Z = \text{reg}(Z) - 1$. Indeed, we want to improve and compare the bound of Proposition 4.1 with other bounds of the same type for space curves. So, from now on we suppose that $C$ is a curve in $\mathbb{P}^4_K$ with general hyperplane section $Z \subset \mathbb{P}^2_K$. The following facts are straightforward refinements of Proposition 4.1 for curves $C$ in $\mathbb{P}^3_K$.

Corollary 4.3. (a) If $C \subset \mathbb{P}^3_K$ is integral and $f_1, \ldots, f_s$ are minimal generators of $I$ with $d_1 = \text{deg}(f_1) \leq \ldots \leq d_s = \text{deg}(f_s)$, then Proposition 4.1 holds with $\alpha_1 = d_1$, $\alpha_2 = d_2$.

(b) Let $C$ be equidimensional and locally Cohen-Macaulay over $K$ with characteristic 0 and suppose $C$ to be a non-complete intersection with degree $> 4$. Then, if either $\text{deg}(C)$ is odd or $C$ is not contained in a quadric, we have $\text{reg}(C) \leq \max\{\rho_C + 1, \alpha_1 + \alpha_2 - 2\}$.

Proof. (a) Being $C$ integral, minimal generators of $I$ are always irreducible. Hence $f_i, f_j$ form an $S$-regular sequence for every integers $i, j$ such that $1 \leq i < j \leq s$.

(b) By Theorem 2.3.1 of [11] (that generalizes a result of R. Strano) we have that a general hyperplane section $Z$ cannot be a complete intersection. Then it is enough to apply Proposition 4.1(ii).

To obtain interesting examples of integral curves, often we will apply known deformation techniques with constant cohomology. In these cases we do not compute the curve explicitly and, so, we do not have further information about the hyperplane section $Z$.

Example 4.4. We give two examples for which the bound of Corollary 4.3(b) is sharp. In the first example we exhibit a family of curves which are integral, while in the second example the given curve is not integral.
(1) Let \( C_0 \) be the curve of Moh \cite{13} parametrized (over a field of characteristic 0) by
\[
\begin{align*}
    x_0 &= y^{31} \\
    x_1 &= u^{25}v^6 + v^{31} \\
    x_2 &= u^{23}v^8 \\
    x_3 &= u^{21}v^{10}
\end{align*}
\]
By applying a sequence of two basic double linkages of type, respectively, (1,5) and (w,7), with \( w \geq 1 \), we obtain a curve \( X_w \) for which, by Proposition 3.5 of \cite{15}, can be deformed with constant cohomology to an integral curve \( C_w \) of degree \( 36 + 7w \) such that \( \rho_{C_w} + 1 = 10 + w = \text{reg}(C_w) \). Moreover, we have \( \alpha_w = 5 + w \) and \( \beta_w = 7 \) if \( w = 1,2 \), whereas \( \alpha_w = 7 \) and \( \beta_w = 5 + w \) otherwise. Thus, \( \text{reg}(C_w) = \max\{\rho_{C_w} + 1, \alpha_w + \beta_w - 2\} < \max\{\rho_{C_w} + 1, \left\lceil \frac{\text{deg}(C_w)}{2} \right\rceil \} + 1 \), for every \( w \geq 1 \).

(2) Let \( C_0 \subset \mathbb{P}_k^3 \) be the following smooth rational curve (over a field of characteristic 0)
\[
\begin{align*}
    x_0 &= u^{12} + v^{12} \\
    x_1 &= u^{11}v + uv^{11} \\
    x_2 &= u^{10}v^2 + v^{12} \\
    x_3 &= u^9v^3
\end{align*}
\]
to which we apply a basic double linkage of type (1,9), obtaining a curve \( C \) of degree 21 such that \( P_C(t) = 21t - 38 \) and \( \text{reg}(C) = 12 = \rho_{C} + 1, \alpha_I = \alpha_1 = 5, \alpha_2 = 9, \omega_I = 12 \). So it happens that \( \max\{\rho_{C} + 1, \alpha_1 + \alpha_2 - 2\} = 12 = \text{reg}(C) \).
In this case we can also compute that \( \rho_{Z} = 8 \) and that minimal degrees of a c.i. containing \( Z \) are also 5 and 9.

For integral space curves over a field of characteristic 0, one can say something more basing on the shape of Borel ideals. Namely, it is known that (see, for example, \cite{7}) the Borel ideal \( \text{gin}(I(Z)) \) is of the following type
\[
(x_0^{\lambda_0}, x_0^{\lambda_{i-1}}x_1^{\lambda_i}, \ldots, x_0x_1^{\lambda_1}, x_1^{\lambda_0}),
\]
where \( \lambda_0 = \rho_{Z} + 1, \lambda_i - 2 \leq \lambda_{i+1} \leq \lambda_i - 1, \) for \( 0 \leq i \leq s - 1, s = \alpha_I(Z), \deg(Z) = \sum_{i=0}^{s-1} \lambda_i \) and (Corollary 4.9 of \cite{7})
\[
\frac{\deg(Z)}{s} + \frac{s - 1}{2} \leq \lambda_0 \leq \frac{\deg(Z)}{s} + s - 1.
\]

**Proposition 4.5.** Let \( \gamma_C := \max\{t \in \mathbb{N} : \deg(C) > t^2 + 1\} \); then
(a) \( \deg(C) > \alpha_I^2 + 1 \Rightarrow \text{reg}(C) \leq \max\{\rho_{C} + 1, \left\lfloor \frac{\deg(C)}{\gamma_C}\right\rfloor + \alpha_I - 1\} \).
(b) \( \deg(C) \leq \alpha_I^2 + 1 \Rightarrow \text{reg}(C) \leq \max\{\rho_{C} + 1, \left\lfloor \frac{\deg(C)}{\gamma_C + 1}\right\rfloor + \gamma_C\} \).

**Proof.** (a) By Theorem 3.4 the thesis is an easy consequence of formula (4.2) and Laudal’s lemma of \cite{18}, because one obtains that \( s = \alpha_J = \alpha_J = \alpha_I \).

(b) In this case \( s \) is not known. However we obtain that \( s > \gamma_C \), namely, in our hypothesis it can happen that either \( s = \alpha_I > \gamma_C \) or \( 2 \leq s \leq \alpha_I \). In the second case, by \( \deg(C) \leq s^2 + 1 \) and so \( s > \gamma_C \). Therefore \( \deg(C) \geq \lambda_0 + \lambda_1 + \ldots + \lambda_{\gamma_C} \geq \lambda_0 + \lambda_0 - 2 + \ldots + \lambda_0 - 2\gamma_C = (\gamma_C + 1)(\lambda_0 - \gamma_C) \) and we are done. \( \Box \)

**Example 4.6.** We give examples for which the described bounds are sharp.

(1) By Proposition 3.5 of \cite{18}, the curve \( X \) of Example 3.6(2) can be deformed with constant cohomology to an integral curve \( C \) of degree 17, regularity \( \text{reg}(C) = \)
$7 = \rho_C + 2$, initial degree $\alpha_I = \alpha_1 = 5 = \alpha_2$. Thus $\gamma_C = 3$ and $\text{reg}(C) = 7 = \max\{\rho_C + 1, \lfloor \frac{\deg(C)}{4} \rfloor + 3\}$, whereas $\max\{\rho_C + 1, \alpha_1 + \alpha_2 - 2\} = 8$ and $\max\{\rho_C + 1, \lfloor \frac{\deg(C)-1}{2} \rfloor + 1\} = 9$.

(2) By Proposition 3.5 of [15], the curve $X$ of Example 3.6(3) can be deformed with constant cohomology to an integral curve $C$ of degree 14, regularity $\text{reg}(C) = 6 = \rho_C + 3$, initial degree $\alpha_I = \alpha_1 = 4$ and $\alpha_2 = 5$. Thus $\gamma_C = 3$ and $\text{reg}(C) = 6 = \max\{\rho_C + 1, \lfloor \frac{\deg(C)}{4} \rfloor + 3\}$, whereas $\max\{\rho_C + 1, \alpha_1 + \alpha_2 - 2\} = 7$ and $\max\{\rho_C + 1, \lfloor \frac{\deg(C)-1}{2} \rfloor + 1\} = 8$.

(3) By applying a further basic double linkage of type (1, 4) to the curve $C$ in (2) above, we obtain a curve $X'$ which, by Proposition 3.5 of [14], can be deformed with constant cohomology to an integral curve $C'$ (defined by an ideal $I'$) of degree 18 with regularity $\text{reg}(C') = 7 = \rho_{C'} + 3$, initial degree $\alpha_{I'} = \alpha_1 = 4$ and $\alpha_2 = 6$. Thus $\text{reg}(C') = 7 = \max\{\rho_{C'} + 1, \lfloor \frac{\deg(C')}{4} \rfloor + \alpha_{I'} - 1\}$, whereas $\max\{\rho_{C'} + 1, \alpha_1 + \alpha_2 - 2\} = 8$ and $\max\{\rho_{C'}, \lfloor \frac{\deg(C')-1}{2} \rfloor + 1\} = 10$.

(4) In Example 3.2 of [12], over a field of characteristic 0, a family $\{Y_m\}_{m \geq 0}$ of 2-Buchsbaum smooth integral curves is constructed such that $\text{reg}(Y_m) = \max\{m + 4, 2m + 3\}$, the initial degree is $\alpha_m = \alpha_1 = \alpha_2 = m + 3$, $\deg(Y_m) = m^2 + 4m + 6 < \alpha_m^2 + 1$ and $\gamma_m := \gamma_{Y_m} = m + 2$.

If $m = 0$, $Y_0$ is a general elliptic curve of degree 6 and $4 = \text{reg}(Y_0) = \rho_{Y_0} + 1$.

If $m = 1$, then $5 = \rho_{Y_1} + 1 = \text{reg}(Y_1) = \max\{\rho_{Y_1} + 1, \lfloor \frac{\deg(Y_1)}{4} \rfloor + 3\} < 6 = \max\{\rho_{Y_1} + 1, \alpha_1 + \alpha_2 - 2\} < 8 = \max\{\rho_{Y_1} + 1, \lfloor \frac{\deg(Y_1)-1}{2} \rfloor + 1\}$.

If $m \geq 2$, then $\rho_{Y_m} + 2 = \text{reg}(Y_m) = 2m + 3 = \max\{\rho_{Y_m} + 1, \lfloor \frac{\deg(Y_m)}{m+3} \rfloor + m + 2\} < 2m + 4 = \max\{\rho_{Y_m} + 1, \alpha_1 + \alpha_2 - 2\} < \max\{\rho_{Y_m} + 1, \lfloor \frac{\deg(Y_m)-1}{2} \rfloor + 1\}$.

We thank U. Nagel whose suggestions inspired this paper.

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